Gradient Young measures generated by quasiconformal maps in the plane

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Abstract

In this contribution, we completely and explicitly characterize Young measures generated by gradients of quasiconformal maps in the plane. By doing so, we generalize the results of [3] who provided a similar result for quasiregular maps and [12] who characterized Young measures generated by gradients of bi-Lipschitz maps. Our results are motivated by non-linear elasticity where injectivity of the functions in the generating sequence is essential in order to assure non-interpenetration of matter.

Key Words: Orientation-preserving mappings, Gradient Young measures, Quasiconvexity, Quasiconformal maps, Non-interpenetration of matter

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1 Introduction

In non-linear hyperelasticity one finds stable states by minimizing a prescribed stored energy functional over the set of admissible deformations.

It is generally postulated [16, 42] that an element of the set of admissible deformations should be an orientation-preserving and injective map $y : \Omega \rightarrow y(\Omega)$ with a suitably integrable weak gradient; one usually assumes that $y \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $1 < p \leq +\infty$. Here and in the sequel $\Omega$ is a bounded Lipschitz domain, the reference configuration.

In the simplest case, we may consider stored energies $W : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ that depend on the deformation only through its gradient; i.e. in order to find stable states one has to minimize the functional

$$J(y) := \int_{\Omega} W(\nabla y(x)) \, dx ,$$

over the set of admissible deformations, possibly with some prescribed boundary data. It is natural to ask under which conditions on the set of admissible deformations and the stored energy we can find minima of $J(y)$.

We shall take the approach of fixing a suitable set of deformations and we will only be concerned with conditions on the stored energy; in a sense, this is complementary to the standard approach where the growth of the stored energy is fixed, which already determines a suitable set of deformations. Thus, we set admissible deformations to be quasiconformal maps in the plane:

$$QC(\Omega; \mathbb{R}^2) = \left\{ y \in W^{1,2}(\Omega; \mathbb{R}^2) : y \text{ is a homeomorphism and } \exists K > 0 \text{ such that} \right\}
\begin{align*}
|\nabla y|^2 &\leq K \det(\nabla y) \text{ a.e. in } \Omega \right\} \quad (1.2)
\end{align*}$$

Before proceeding, let us motivate this choice from a mechanical point of view: By restricting our attention to homeomorphisms, we aim to model situations in which no failures like cracks or cavities occur. Further, we require in the spirit of [12] that the inverse $y^{-1}$ of a deformation is of the same “quality” as the
deformation itself; here this is fulfilled since the inverse of a quasiconformal map is again quasiconformal (cf. e.g. [11, Theorem 3.1.2]). The idea behind this restriction is that, in elasticity, a body returns to its original shape after the release of all loads. However, since the rôle of the reference and the deformed configuration is arbitrary, we would like to understand this “returning” as a new deformation, corresponding to inverse loads, that takes the deformed configuration to the reference one. Let us mention that the idea of also requiring integrability of the inverse of the deformation already appeared e.g. in [6, 20, 22, 37] and very recently e.g. in [23, 27, 29, 19], the use of quasiconformal maps in hyper-elasticity has been put forward in [26].

Quasiconformal functions map infinitesimally small circles to infinitesimal ellipses of a uniformly bounded eccentricity. This means that in our modelling not even subsets of a vanishingly small measure can get deformed into a flat piece. Finally, quasiconformal mappings form, roughly speaking, the largest class of deformations that is invariant under composition with similarity transformations (i.e. shape preserving transformations in the domain and in the range).\footnote{If a family $F$ of homeomorphisms of domains in $\mathbb{C}$ is normal and invariant under composition with similarity transformations (in the domain and in the range) then it consist of $K$-quasiconformal mappings with some $K \geq 1$ fixed [11].}

On the energy we include the additional requirement that it blows up if the volume of any infinitesimal part of the body shrinks to zero; i.e.

$$W(A) \to +\infty \quad \text{whenever} \quad \det A \to 0_+.$$  \hspace{1cm} (1.3)

Since it is convenient to prove existence of minimizers by the direct method, we characterize the set of energies $W$ satisfying (1.3) such that $J(y)$ from (1.1) is weakly lower semicontinuous on the set $\mathcal{QC}(\Omega; \mathbb{R}^2)$ (with respect to the weak convergence specified in Definition 1.2). It is be expected that this will be connected to some notion of (quasi)convexity.

Recall that we call $W$ quasiconvex [34] if for all $A \in \mathbb{R}^{2 \times 2}$ and all $\varphi \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ such that $\varphi(x) = Ax$ on $\partial \Omega$ it holds that

$$|\Omega|W(A) \leq \int_\Omega W(\nabla \varphi(x)) \, dx.$$  \hspace{1cm} (1.4)

It is well known (cf., e.g., [18]) that if a quasiconvex $W$ additionally satisfies the coercivity/growth condition

$$c(|A|^2 - 1) \leq W(A) \leq c(|A|^2 + 1),$$  \hspace{1cm} (1.5)

it is weakly lower semicontinuous on $W^{1,2}(\Omega; \mathbb{R}^2)$ and so in particular on the set $\mathcal{QC}(\Omega; \mathbb{R}^2)$.

But (1.5) necessarily implies that $W$ is locally finite which is incompatible with (1.3). Indeed, this was recently shown in [31] that $W^{1,p}$-quasiconvexity with $p$ less that dimension is even incompatible with (1.3) at all, so it seems natural to rather consider a natural generalization of quasiconvexity tailored to quasiconformal functions.

To this end, we introduce the concept of quasiconformally quasiconvex functions (cf. Def. 2.2 below) that satisfy (1.4) only for all $\varphi \in \mathcal{QC}(\Omega; \mathbb{R}^2)$, $\varphi = Ax$ on $\partial \Omega$ and all $A \in \mathbb{R}^{2 \times 2}$ with positive determinant. Clearly, this is a weaker notion than the usual $W^{1,2}$ - quasiconvexity in the sense of [10] as well as quasiconvexity in the sense of [34]. In particular, notice that now, since $W$ does not even need to be defined for matrices with negative determinant, we may as well set it to $+\infty$ on there.

The main result of this paper is that for stored energies satisfying the natural coercivity/growth condition compatible with (1.3)

$$c(|A|^2 + |\det (A)|^{-1} - 1) \leq W(A) \leq c(|A|^2 + |\det (A)|^{-1} + 1),$$  \hspace{1cm} (1.6)

$J(y)$ is weakly lower semicontinuous on the set $\mathcal{QC}(\Omega; \mathbb{R}^2)$ (in the sense of Definition 1.2) if and only if it quasiconformally quasiconvex.

To prove this claim, we completely and explicitly characterize gradient Young measures generated by sequences in $\mathcal{QC}(\Omega; \mathbb{R}^2)$ (cf. Section 2). Young measures present a very convenient tool for studying weak lower
semicontinuity when extending the notion of solutions from Sobolev mappings to parameterized measures [7, 21, 36–38, 39, 40, 44]. The idea is to describe the limit behavior of \( \{ J(y_k) \} \) for a minimizing sequence \( \{ y_k \} \). Moreover, Young measures form one of the main relaxation techniques for non-(quasi)convex functionals as appearing when modelling solid-to-solid phase transitions [9, 35].

Generally speaking, the main difficulty in characterizing a class of Young measures generated by invertible mappings is that such a class is non-convex. Thus, many of the classical techniques used in the study of Sobolev functions, like smoothing by a mollifier kernel, fail. In fact, as far as smoothing under the invertibility constraint is concerned, results in the plane were obtained only very recently in [25, 19] and are based on completely different ideas than integral kernels. From the point of view of the problem considered here, it is crucial to design a cut-off technique compatible with the invertibility constraint that will allow us to modify a given function on a set of vanishingly small measure in such a way that it takes some prescribed form on the boundary. Indeed, the necessity of such a technique can be expected already from the very definition of (quasiconformal) quasiconvexity which requires us to verify (1.4) for maps with fixed boundary values and, in fact, cut-off techniques are exploited in all the standard proofs of characterizations of gradient Young measures [29, 30, 36] or weak lower semicontinuity of quasiconvex functionals [18].

To the best of our knowledge, gradient Young measures generated by invertible maps were so far characterized only in [12], where an explicit characterization of Young measures generated by bi-Lipschitz maps in the plane was given. Moreover, the particular case of homogeneous gradient Young measures generated by quasiconformal maps has been treated in [3], but in the non-homogeneous case the global invertibility was given up and only Young measures generated by quasiregular maps were characterized. The present contribution generalizes the above two works in the sense that we adapt the cut-off technique from [12] also to the quasiconformal case and so characterize non-homogeneous gradient Young measures generated by invertible quasiregular maps.

We follow the main strategy of [12] to construct the cut-off, i.e. we modify the given sequence on a set of gradually vanishing measure near the boundary first on a one dimensional grid and then rely on extension theorems for quasiconformal maps going back to [14]. Nevertheless, contrary to the bi-Lipschitz case the modification on the is much more involved and cannot be done by what is essentially an affine interpolation as in the Lipschitz case [12] (cf. Section 5).

Let us note that even if one poses only point-wise constraints on the determinant of the deformation (e.g. by requiring \( \det (\cdot) > 0 \)) similar difficulties to the ones described above arise. However, in some less rigid situations, one may rely on convex integration to construct cut-offs. Such an approach has been taken in [13] as well as in [32, 31], where Young measures generated by orientation preserving maps in \( W^{1,p} \) for \( 1 < p < n \) were characterized.

The plan of the paper is as follows: In the rest of the introduction, we give some background on Young measures and explain some basic concepts that we shall use throughout the contribution. Then we state the main results in Section 2 and recall some facts about quasiconformal maps in Section 3. Proofs of the main theorems are postponed to Section 4 while our cut-off technique is presented in Section 5.

### 1.1 Background on gradient Young measures

We understand Young measures as “generalized functions” that capture the asymptotic behaviour of a non-linear functional along an oscillating sequence \( \{ Y_k \} \). Namely, suppose that \( \{ Y_k \} \) is bounded in \( L^2(\Omega; \mathbb{R}^{2 \times 2}) \) then a classical result [39–14, 33], the fundamental Young measure theorem, states that there exists a sub-sequence (not relabeled) of \( \{ Y_k \} \) and a family of probability measures \( \nu = \{ \nu_x \} \) satisfying

\[
\lim_{k \to \infty} \int_{\Omega} \xi(x)(v \circ Y_k)(x) \, dx = \int_{\Omega} \int_{\mathbb{R}^{2 \times 2}} \xi(x)v(s) \nu_x(ds) \, dx
\]

for all \( \xi \in L^\infty(\Omega) \) and all \( v \in C(\mathbb{R}^{2 \times 2}) \) such that \( \{ v \circ Y_k \} \) is weakly convergent in \( L^1(\Omega) \). We say that \( \{ Y_k \} \) generates the Young measure \( \nu = \{ \nu_x \} \). It is further known that \( \nu_x \in L^\infty(\Omega; \mathcal{M}(\mathbb{R}^{2 \times 2})) \) where \( \mathcal{M}(\mathbb{R}^{2 \times 2}) \) is the set of Radon measures, \( C_0(\mathbb{R}^{2 \times 2}) \) stands for the space of all continuous functions \( \mathbb{R}^{2 \times 2} \to \mathbb{R} \) vanishing at infinity and space \( L^\infty_w(\Omega) \) corresponds to weakly* measurable
uniformly bounded functions. Recall that weakly* measurable means that, for any \(v \in C_0(\mathbb{R}^{n \times n})\), the mapping \(\Omega \rightarrow \mathbb{R}: x \mapsto \langle v, v \rangle = \int_{\mathbb{R}^{n \times n}} v(s)\nu_x(ds)\) is measurable in the usual sense.

An important subset of Young measures are those generated by \(\text{gradients}\) of \(\{y_k\}_{k \in \mathbb{N}} \subset W^{1,2}(\Omega; \mathbb{R}^2)\), i.e., \(Y_k := \nabla y_k\) in (1.7). Let us denote this set \(\mathcal{G}^2(\Omega; \mathbb{R}^{2 \times 2})\). An explicit characterization of this set is due to Kinderlehrer and Pedregal [29, 30]:

**Theorem 1.1** (adapted from [30]). A family of Young measures \(\{\nu_x\}_{x \in \Omega}\) is in \(\mathcal{G}^2(\Omega; \mathbb{R}^{2 \times 2})\) if and only if the following conditions hold:

1. there exists \(z \in W^{1,2}(\Omega; \mathbb{R}^2)\) such that \(\nabla z(x) = \int_{\mathbb{R}^{2 \times 2}} A\nu_x(dA)\) for a.e. \(x \in \Omega\),
2. \(\psi(\nabla z(x)) \leq \int_{\mathbb{R}^{2 \times 2}} \psi(A)\nu_x(dA)\) for a.e. \(x \in \Omega\) and for all \(\psi\) quasiconvex, continuous and such that \(|\psi(A)| \leq c(1 + |A|^2)\),
3. \(\int_\Omega \int_{\mathbb{R}^{2 \times 2}} |A|^2\nu_x(dA)\,dx < \infty\).

### 1.2 Basic notation

We define the set of \(K\)-quasiconformal matrices

\[
\mathbb{R}^{2 \times 2}_K := \{ A \in \mathbb{R}^{2 \times 2} : |A|^2 \leq K \det(A) \},
\]

for \(1 \leq K < \infty\). Note that \(\mathbb{R}^{2 \times 2}_K\) is compact and represents (except for the zero-matrix) the possible values of gradients of affine \(K\)-quasiconformal mappings. Further, we denote the set of matrices in \(\mathbb{R}^{2 \times 2}\) with positive determinant as \(\mathbb{R}^{2 \times 2}_+\).

Next, let us introduce a suitable notion of weak convergence on \(\mathcal{QC}(\Omega; \mathbb{R}^2)\):

**Definition 1.2.** We say that a sequence \(\{u_k\}_{k \in \mathbb{N}} \subset W^{1,2}(\Omega; \mathbb{R}^2)\) of quasiconformal maps converges weakly to \(u \in W^{1,2}(\Omega; \mathbb{R}^2)\) in \(\mathcal{QC}(\Omega; \mathbb{R}^2)\) if \(u_k \rightharpoonup u\) in \(W^{1,2}(\Omega; \mathbb{R}^2)\), there exists a \(K \geq 1\) such that \(\{u_k\}_{k \in \mathbb{N}}\) are all \(K\)-quasiconformal and \(u(x)\) is non-constant. If the weak convergence is given only in \(W^{1,2}_{\text{loc}}(\Omega; \mathbb{R}^2)\), we will speak of local convergence in \(\mathcal{QC}(\Omega; \mathbb{R}^2)\).

**Remark 1.3.** Notice that the set \(\mathcal{QC}(\Omega; \mathbb{R}^2)\) is closed under the weak convergence in \(\mathcal{QC}(\Omega; \mathbb{R}^2)\). Indeed, it follows from Lemma 3.3 (below) that if a sequence converges in \(\mathcal{QC}(\Omega; \mathbb{R}^2)\), it also converges locally uniformly and, thus, the weak limit is either quasiconformal or constant, with the latter possibility however being excluded per definition.

### 2 Main results

We shall denote

\[
\mathcal{GY}^{\mathcal{QC}}(\Omega; \mathbb{R}^{2 \times 2}) = \{ \nu \in \mathcal{G}^2(\Omega; \mathbb{R}^{2 \times 2}) \text{ generated by maps weakly converging in } \mathcal{QC}(\Omega; \mathbb{R}^2) \}. \]

The main results of our paper are formulated in Theorems 2.1, 2.4 and 2.5 below. In Theorem 2.1, we give a complete and explicit characterization of \(\mathcal{GY}^{\mathcal{QC}}(\Omega; \mathbb{R}^{2 \times 2})\). Further, a relation between Young measures in \(\mathcal{GY}^{\mathcal{QC}}(\Omega; \mathbb{R}^{2 \times 2})\) and quasiconformally quasiconvex functions is observed in Theorem 2.4. From this, we readily deduce an equivalent characterization of weak lower semicontinuity of functionals on \(\mathcal{QC}(\Omega; \mathbb{R}^2)\) in Theorem 2.5.

We start with the characterization of measures in \(\mathcal{GY}^{\mathcal{QC}}(\Omega; \mathbb{R}^{2 \times 2})\):

**Theorem 2.1.** Let \(\Omega \subset \mathbb{R}^2\) be a bounded Lipschitz domain. Let \(\nu \in \mathcal{GY}^{\mathcal{QC}}(\Omega; \mathbb{R}^{2 \times 2})\). Then \(\nu \in \mathcal{GY}^{\mathcal{QC}}(\Omega; \mathbb{R}^{2 \times 2})\) if and only if the following conditions hold:

\[
\exists K \geq 1 \text{ such that supp } \nu_x \subset \mathbb{R}^{2 \times 2}_K \text{ for a.a. } x \in \Omega \text{ and } (2.1)
\]

\[
\text{there exists a } K\text{-quasiconformal } y \in W^{1,2}(\Omega; \mathbb{R}^2) \text{ such that } \nabla y(x) = \int_{\mathbb{R}^{2 \times 2}} A\nu_x(A). \tag{2.2}
\]
Notice that similarly to [3], we constrained the support of the given Young measure to a suitable set of quasiconformal matrices. Also, since the set $\mathcal{QC}(\Omega; \mathbb{R}^{2 \times 2})$ is closed under the weak convergence in $\mathcal{QC}(\Omega; \mathbb{R}^{2 \times 2})$ the condition on the first moment (2.2) is natural.

In view of previous results, it might seem surprising that no adaptation of the Jensen inequality is needed. Nevertheless, we will show in Theorem 2.4 that measures in $\mathcal{GY}^2(\Omega; \mathbb{R}^{2 \times 2})$ satisfy, in fact, a more restrictive version of the Jensen perfectly fitted to quasiconformal maps. To this end, let us introduce the following generalized notion of quasiconvexity:

**Definition 2.2.** Suppose $v : \mathbb{R}^{2 \times 2} \to \mathbb{R} \cup \{+\infty\}$ is bounded from below and Borel measurable. We say that $v$ is quasiconformally quasiconvex on $\mathbb{R}^{2 \times 2}$ if

$$|\Omega|v(A) \leq \int_{\Omega} v(\nabla \varphi(x)) \, dx$$

(2.3)

for all $A \in \mathbb{R}^{2 \times 2}$ and $\varphi \in \mathcal{QC}(\Omega; \mathbb{R}^2)$ such that $\varphi(x) = Ax$ on $\partial \Omega$.

**Remark 2.3** (Relation to other notions of quasiconvexity). Notice that quasiconformal quasiconvexity is a weaker condition than $W^{1,2}$-quasiconvexity since all quasiconvex functions are by definition in $W^{1,2}(\Omega; \mathbb{R}^2)$, while the opposite is by far not true. In fact, a quasiconformally quasiconvex function can be completely arbitrary on the set of matrices with negative determinant, while in general this is not true for $W^{1,2}$-quasiconvex functions.

A (in general) even weaker condition than the one from Definition 2.2, so-called bi-quasiconvexity, has been introduced in [12]; this notion is based on verifying the Jensen inequality (2.3) just for bi-Lipschitz maps. In order to prove that these two notions are equivalent, one would need to assure density of bi-Lipschitz maps in quasiconformal ones in a suitable strong convergence respecting the growth of the function $v : \mathbb{R}^{2 \times 2} \to \mathbb{R} \cup \{+\infty\}$ on the set of matrices with positive determinant. For example, one would seek a result showing that for every $K$-quasiconformal function there exists a sequence of $K$-quasiconformal bi-Lipschitz maps that coincide with the original function on $\partial \Omega$ and approximate the given function strongly in the $W^{1,2}$-norm and their inverse Jacobians converge strongly to the inverse Jacobian of the original function in the $L^1$-norm. However, density results on homeomorphisms started to appear only recently in literature [25, 19] and a result of the type mentioned above is currently not available to the authors’ knowledge.

With this definition we have the following theorems:

**Theorem 2.4.** Any Young measure $\nu \in \mathcal{GY}^{\mathcal{QC}}(\Omega; \mathbb{R}^{2 \times 2})$ satisfies the following inequality

$$v(\nabla u(x)) \leq \int_{\mathbb{R}^{2 \times 2}} v(A) \, d\nu(A)$$

(2.4)

for all quasiconformally quasiconvex $v$ in $\mathcal{E}$ and a.a. $x \in \Omega$; where

$$\mathcal{E} := \{ v : \mathbb{R}^{2 \times 2} \to \mathbb{R} \cup \{+\infty\}; v \in C(\mathbb{R}^{2 \times 2}_+), -C \leq v(A) \leq C(1 + |A|^2 + |\det(A)|^{-1}) \}.$$  

This readily yields the following weak lower semicontinuity result

**Theorem 2.5.** Let $v \in \mathcal{E}$ and let $\{y_k\}_{k \in \mathbb{N}} \subset \mathcal{QC}(\Omega; \mathbb{R}^2)$ be a sequence of maps that converge weakly in $\mathcal{QC}(\Omega; \mathbb{R}^2)$. Then $y \rightarrow I(y) := \int_{\Omega} v(\nabla y(x)) \, dx$ is sequentially weakly lower semicontinuous with respect to this convergence if and only if $v$ is quasiconformally quasiconvex.
3 Some auxiliary results on quasiconformal maps

Quasiconformal maps have been studied intensively for several decades now; cf. e.g. the monographs [1, 4] for further details. For the convenience of the reader, let us recall some of their properties that shall be of importance in this work. Note that the classical theory on quasiconformal maps, as presented in e.g. [4], does not treat the class \( QC(\Omega; \mathbb{R}^2) \) but rather the following set

\[
QC_{loc}(\Omega; \mathbb{R}^2) = \left\{ y \in W^{1,2}_{loc}(\Omega; \mathbb{R}^2) : y \text{ is a homeomorphism and } \exists K > 0 \text{ such that } |\nabla y|^2 \leq K \det (\nabla y) \text{ a.e. in } \Omega \right\}
\]

which, in this work, shall be referred to as locally quasiconformal maps. Let us also mention that some of the results given below can be extended even to maps the distortion of which is only integrable [24].

**Lemma 3.1** (Inverse and composition, adapted from Theorem 3.1.2 in [4]). Let \( u : \Omega \rightarrow \mathbb{R}^2 \) be a \( K \)-quasiconformal map. Then its inverse is \( K \)-quasiconformal, too. Moreover, the composition of a \( K \)-quasiconformal map and a \( K_2 \)-quasiconformal map is \( K_1K_2 \)-quasiconformal.

**Lemma 3.2** (Higher integrability, adapted from Theorem 13.2.3 in [4]). Let \( u : \Omega \rightarrow \mathbb{R}^2 \) be a \( K \)-quasiconformal map. Then \( u \in W^{1,p}_{loc}(\Omega; \mathbb{R}^2) \) for all \( p < \frac{2K}{K+1} \), More specifically, take \( x_0 \in \Omega \) and \( r > 0 \) such that the ball \( B_{2r}(x_0) \subset \Omega \). Then there exists a constant \( c \) independent of \( r, x_0 \) and \( K \) such that

\[
\int_{B_r(x_0)} |\nabla u|^p \, dx \leq c.
\]

In particular, a sequence \( \{u_k\}_{k \in \mathbb{N}} \) that converges weakly to \( u \) in \( QC(\Omega; \mathbb{R}^2) \) converges also locally uniformly to \( u \).

It is shown in [15] that gradients of quasiconformal maps have a better integrability than just \( L^2 \), the precise bound is derived in [2].

**Lemma 3.3** (Local quasisymmetry, adapted from Theorem 3.6.2 in [4]). Let \( u : \Omega \rightarrow \mathbb{R}^2 \) be a \( K \)-quasiconformal map; further, take \( x_0 \in \Omega \) and \( r > 0 \) such that the ball \( B_{2r}(x_0) \subset \Omega \). Then \( u_{|_{B_r(x_0)}} \) is quasisymmetric, i.e. it is a homeomorphism and there exists an increasing function \( \eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that for any triple \( x, y, z \in B_r(x_0) \) the following is satisfied

\[
\frac{|u(x) - u(y)|}{|u(x) - u(z)|} \leq \eta\left(\frac{|x - y|}{|x - z|}\right)
\]

and the \( \eta \) depends only on \( K \) and not on \( x_0, r \).

The following is a direct consequence of the famous extension of Beurling and Ahlfors [14]:

**Lemma 3.4** (Extension property). Suppose that \( D \) is a square in \( \mathbb{R}^2 \) and \( u : \partial D \rightarrow u(\partial D) \) is \( \eta \)-quasisymmetric and that \( \text{Int}(u(\partial D)) \) is a bounded domain. Then there exists a \( K \)-quasiconformal map \( \tilde{u} : D \rightarrow \text{Int}(u(\partial D)) \) which coincides with \( u \) on \( \partial D \). Moreover, \( K \) depends only on \( \eta \).

The lemma is obtained by conformally transforming the square as well as \( u(\partial D) \) onto the half-plane. Due to the special geometry of the square and \( u(\partial D) \) (obtained by a quasi-symmetric mapping) the corresponding Carathéodory maps extend to quasisymmetries and allow us to use [14]. By this way, we obtain a \( \tilde{u} \in QC_{loc}(\Omega; \mathbb{R}^2) \), which however is actually in \( W^{1,2}(\Omega; \mathbb{R}^2) \) due to (3.3), since \( \text{Int}(u(\partial D)) \) is bounded.

In fact, the extension property still holds if we replace the square \( D \) in Lemma 3.4 by a quasicircle, that is the image of a circle under a quasisymmetric homeomorphism. We will use the fact that quasicircles can be characterized by a kind of reverse triangle inequality:

\[\text{In fact the second part of the Theorem is obtained by combining Lemma 3.6.1 and Lemma 2.10.9 with the above mentioned theorem.}\]
Lemma 3.5 (Characterization of quasicircles, adapted from Theorem 13.3.1 in [1]). Let $C$ be a closed curve in $\mathbb{R}^2$. Then $C$ is the image of circle under an $\eta$-quasisymmetric homeomorphism if and only if there exists a constant $c$ which depends only on $\eta$ such that for any two points $z_1$ and $z_2$ chosen on the given closed curve and $z_3$ lying on the shorter of the resulting arcs, we have

$$|z_1 - z_3| + |z_2 - z_3| \leq c|z_1 - z_2|.$$  \hspace{1cm} (3.2)

A natural generalization of quasiconformal maps are quasiregular maps; i.e., those that are of bounded distortion but not necessarily homeomorphisms. Let us point out through the following lemma that one of the possibilities to assure that such maps are injective is by imposing the Ciarlet-Nečas condition [17] that is well known in elasticity.

Lemma 3.6 (Quasiregularity and Ciarlet-Nečas condition). A map $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ is $K$-quasiconformal if and only if it is non-constant, $K$-quasiregular and satisfies the Ciarlet-Nečas condition

$$\int_{\Omega} \det(\nabla u) \, dx \leq |u(\Omega)|.$$ \hspace{1cm} (3.3)

Proof. Clearly, in order to be a homeomorphism $u$ cannot be constant. Moreover, since quasiregular maps are continuous (or more specifically have a continuous representative), open (that is map open sets to open sets) and discrete (the set of preimages for any point does not accumulate) (cf. e.g. [4, Corollary 5.5.2], [24]), we only have to prove that the additional condition (3.3) guarantees (and is implied by) injectivity.

The proof of this follows from the area formula. Namely, as both quasiregular and quasiconformal maps satisfy the Lusin $N$-condition (i.e. map sets of zero measure to maps of zero measure) (cf. e.g. [24]), we have that

$$\int_{\Omega} \det(\nabla u) \, dx = \int_{\mathbb{R}^2} N(u, \Omega) \, dy = \int_{u(\Omega)} N(u, \Omega, y) \, dy$$

where $N(u, \Omega, y)$ is defined as the number of preimages of $y$ in $\Omega$. So the Ciarlet-Nečas condition is satisfied if and only if $N(u, \Omega, y) = 1$ almost everywhere on $u(\Omega)$. Also we can immediately see that the reverse inequality to (3.3) always holds.

If $u$ is injective, then $N(u, \Omega, y) = 1$ and (3.3) is satisfied. For the converse, suppose by contradiction, that there is a non-injective quasiregular, non-constant $u$ satisfying (3.3). Then there has to exist a $y \in u(\Omega)$ that has at least two preimages $x_1$ and $x_2$. Now there exists an $\varepsilon > 0$ such that $B_\varepsilon(x_1) \cap B_\varepsilon(x_2) = \emptyset$ and $B_\varepsilon(x_j) \subset \Omega$ for $j = 1, 2$. On the other hand, for the images we have that $u(B_\varepsilon(x_1)) \cap u(B_\varepsilon(x_2)) \neq \emptyset$. In fact, $u(B_\varepsilon(x_1)) \cap u(B_\varepsilon(x_2))$ is of positive measure since both $u(B_\varepsilon(x_1))$ and $u(B_\varepsilon(x_2))$ are open. Therefore, there exists a set of positive measure where $N(u, \Omega, y)$ is at least two; a contradiction to (3.3). \hfill $\square$

Lemma 3.7 (Gluing of quasiconformal maps). Let $\{\Omega_i\}_{i \in \mathbb{N}}$ be mutually disjoint simply connected Lipschitz domains that almost cover $\Omega$, i.e. $\Omega = \bigcup_{i \in \mathbb{N}} \Omega_i \cup N$ with $|N| = 0$. Further, let $u_i : \Omega_i \rightarrow \mathbb{R}^2$ be $K$-quasiconformal maps satisfying $u_i(x) = u(x)$ on $\partial \Omega_i$ with $u : \Omega \rightarrow \mathbb{R}^2$ also $K$-quasiconformal. Then the “glued map”

$$\hat{u}(x) = \begin{cases} 
    u_i(x) & \text{if } x \in \Omega_i, \\
    u(x) & \text{else},
\end{cases}$$

is $K$-quasiconformal as well.

In order to prove the “gluing lemma” we will exploit the characterization by the Ciarlet-Nečas condition from Lemma 3.6. Alternatively, it is known that an open and discrete mapping equal to a homeomorphism near the boundary is already injective [24], which would allow us to show the lemma, too.

Proof. Clearly $\hat{u}$ is non-constant and $K$-quasiregular. To see that it is also $K$-quasiconformal, we verify (3.3). But since $\det(\cdot)$ is a null-Lagrangian or, alternatively, by applying the area formula, we have that

$$\int_{\Omega_i} \det(\nabla \hat{u}) \, dx = \int_{\Omega_i} \det(\nabla u) \, dx,$$
because from construction \( \tilde{u}(x) = u(x) \) on \( \partial \Omega_i \). This implies that \( \tilde{u}(\Omega_i) = u_i(\Omega_i) = u(\Omega_i) \), since each simply connected \( u_i(\Omega_i) \) (recall that \( \Omega_i \) is simply connected) is according to the Jordan curve-theorem completely determined by its boundary curve.

Moreover, since \( u \) is injective, the \( u_i(\Omega_i) \) are mutually disjoint and since \( u \) satisfies Lusin’s N-condition \( \bigcup_{i \in \mathbb{N}} u_i(\Omega_i) \) has full measure in \( u(\Omega) \). Now, as \( u \) fulfills (3.3), the claim follows.

Finally let us mention that homogeneous gradient Young measures with support in quasiconformal matrices can be generated by quasiconformal maps:

**Theorem 3.8** (adapted from Theorem 1.5 in [3]). Let \( \nu \) be a homogeneous \( W^{1,2} \)-gradient Young measure with support contained in \( \mathbb{R}^{2 \times 2} \). Then \( \nu \) can be generated by a sequence of gradients of (uniformly) \( K \)-quasiconformal homeomorphisms \( \{y_k\}_{k \in \mathbb{N}} \subset \text{QC}(\Omega; \mathbb{R}^{2 \times 2}) \).

Let us remark that this theorem is formulated in [3] in the following way: There exists a sequence of quasiconformal mappings \( F_k : \mathbb{R}^2 \to \mathbb{R}^2 \) such that the restriction of their gradients to the unit ball generates \( \nu \). By a linear transformation of variables, we see that the gradients can be restricted to a ball of any radius and so also to \( \Omega \) (recall that \( \Omega \) is a bounded Lipschitz domain).

## 4 Proofs of the main theorems

**Proof of Theorem 2.1 (characterization of quasiconformal gradient Young measures).** For the necessity, take a sequence \( \{y_k\}_{k \in \mathbb{N}} \) of (uniformly) \( K \)-quasiconformal mappings converging weakly to \( y(x) \) in \( \text{QC}(\Omega; \mathbb{R}^2) \). Clearly, \( \{y_k\}_{k \in \mathbb{N}} \) generates a family of gradient Young measures \( \nu_x \in \mathcal{GY}^2(\Omega; \mathbb{R}^{2 \times 2}) \). Moreover, \( \nu_x \) is supported on the set \( \bigcap_{i=1}^{\infty} \{ \nabla y_k(x); \ k \geq i \} \) (cf. [5] [11]) for almost all \( x \in \Omega \); i.e., \( \nu_x \) is supported on \( \mathbb{R}^{2 \times 2} \).

Finally, the equality (2.2) follows from the fundamental theorem of Young measures (cf. e.g. [20] Theorem 6.2).

As for the sufficiency, we rely on a technique of partitioning the domain \( \Omega \), that is routinely used in the analysis of gradient Young measures (cf. [29] Proof of Theorem 6.1], on the result from [3] formulated in Theorem [3.8] and importantly, on our novel cut-off technique that is presented in Section 6.

Take \( \nu \in \mathcal{GY}^2(\Omega; \mathbb{R}^{2 \times 2}) \) and \( y \in \text{QC}(\Omega; \mathbb{R}^2) \) according to (2.1) and (2.2). We aim to construct a sequence \( \{y_k\}_{k \in \mathbb{N}} \subset \text{QC}(\Omega; \mathbb{R}^2) \) converging weakly in \( \text{QC}(\Omega; \mathbb{R}^2) \) to \( y(x) \), satisfying

\[
\lim_{k \to \infty} \int_{\Omega} \nu(\nabla y_k(x)) g(x) \, dx = \int_{\Omega} \int_{\mathbb{R}^{2 \times 2}} \nu(s) \nu_x(ds) g(x) \, dx \quad (4.1)
\]

for all \( g \in \Gamma \) and any \( v \in S \), where \( \Gamma \) and \( S \) are countable dense subsets of \( C(\Omega) \) and \( C(\mathbb{R}^{2 \times 2}_{\text{inv}}) \), respectively. In fact, we may fix \( g \) and \( v \) for the moment and once the generating sequence is found, rely on a diagonalization argument.

We shall proceed, roughly, as follows: We cover \( \Omega \) by its small scaled copies \( a_{i,k} + \varepsilon_{i,k} \Omega \), the exact type of covering is given by an approximation of the integral on the right hand side of (4.1) by suitable “Riemann-sums” in (4.5). On each of these small copies the Young measure is roughly homogeneous, i.e. \( \nu = \nu_{a_{i,k}} \), with \( \nabla y(a_{i,k}) \) being its first moment. For such a measure we may find a quasiconformal generating sequence due to Lemma 3.8. The idea is now to patch all these generating sequences defined on the small scaled copies of \( \Omega \) to obtain the final generating sequence. However, in order for the patched function to be really quasiconformal, we need to assure that all generating sequences on the small copies of \( \Omega \) have the same boundary data – we will need them to be \( y(x) \) on the boundary. To achieve this, we would like to rely on Proposition 5.1, but as a prerequisite we need the generating sequences on the small sets to be (locally) uniformly close to \( y(x) \) and the same holds for the inverses. But we know that the generating sequences converge weakly in \( \text{QC}(\Omega; \mathbb{R}^2) \), and thus are locally uniformly close, to \( \nabla y(a_{i,k})x \) (the same is true for the inverse). Moreover, since \( y \) is differentiable, \( y(a_{i,k}) + \nabla y(a_{i,k})x \) is uniformly close \( y(x) \) on the small set \( a_{i,k} + \varepsilon_{i,k} \Omega \) and a similar argumentation can be performed for the inverse. Let us give the details of the proof:

Since \( y \in \text{QC}(\Omega; \mathbb{R}^2) \), we know from the Gehring-Lehto theorem (cf. [4] Section 3.3) that it differentiable in \( \Omega \) outside a set of measure zero called \( N \). Also, \( y^{-1} : y(\Omega) \to \Omega \) is differentiable almost everywhere and we
may without loss of generality assume that the images of all points where the inverse of \( y \) is not differentiable lie in \( N \) because \( y^{-1} \) maps null sets to null sets. Therefore, we find for every \( a \in \Omega \setminus N \) and every \( k > 0 \) numbers \( r_k(a) > 0 \) such that for any \( 0 < \varepsilon < r_k(a) \) we have

\[
\left| \frac{y(x) - y(a)}{\varepsilon} - \nabla y(a) \left( \frac{x - a}{\varepsilon} \right) \right| \leq \frac{1}{k} \quad \forall x \in a + \varepsilon \Omega \tag{4.2}
\]

and also

\[
\left| \frac{y^{-1}(z) - y^{-1}(y(a))}{\varepsilon} - (\nabla y(a))^{-1} \left( \frac{z - y(a)}{\varepsilon} \right) \right| \leq \frac{1}{k} \quad \forall z \in y(a) + \varepsilon \nabla y(a) \Omega. \tag{4.3}
\]

Notice that in the second inequality we used that \( \nabla y^{-1}(y(a)) = (\nabla y(a))^{-1} \).

Furthermore as \( g \) is continuous, we choose \( r_k(a) \) smaller if necessary to assure that for any \( 0 < \varepsilon < r_k(a) \)

\[
\left| \int_{\Omega} g(a + \varepsilon x) - g(a) \, dx \right| < \frac{1}{k}. \tag{4.4}
\]

Now, we perform the above announced “suitable partitioning” of the domain \( \Omega \) by relying on [36 Lemma 7.9]. Following this lemma, we can find \( a_{ik} \in \Omega \setminus N \), \( \varepsilon_{ik} \leq r_k(a_{ik}) \) such that for all \( v \in \mathcal{S} \) and all \( g \in \Gamma \)

\[
\lim_{k \to \infty} \sum_{i} \nabla(a_{ik}) g(a_{ik}) \varepsilon_{ik} \Omega = \int_{\Omega} \nabla(x) g(x) \, dx, \tag{4.5}
\]

where

\[
\nabla(x) := \int_{\mathbb{R}^2} v(s) \nu_x (ds).
\]

It is well known (see e.g. [36 Proposition 8.18]), that \( \nu_{a_{ik}} \) is a homogeneous \( W^{1,2} \)-gradient Young measure with \( \nabla y(a_{ik}) x \) being its first moment; due to \((2.1)\), we may assume that \( \nu_{a_{ik}} \) is supported on \( \mathbb{R}^{2 \times 2} \) and because the Jacobian of a quasiconformal mapping is strictly positive a.e. (cf. [4 Section 3.7]), we may also assume that \( \det \nabla y(a_{ik}) > 0 \). Thus, in view of Theorem 3.8, this measure can generated by gradients of a sequence of \( K \)-quasiconformal maps denoted \( \{y_{jk}^k\}_{j \in \mathbb{N}} \). In other words we have that

\[
\lim_{j \to \infty} \int_{\Omega} v(\nabla y_{jk}^k(x)) g(x) \, dx = \nabla(a_{ik}) \int_{\Omega} g(x) \, dx \tag{4.6}
\]

and, in addition, \( \{y_{jk}^k\}_{j \in \mathbb{N}} \) converges weakly in \( \mathcal{QC}(\Omega; \mathbb{R}^2) \) to the map \( x \mapsto \nabla y(a_{ik}) x \) for \( j \to \infty \). In view of Lemma 3.2 we know that \( \{y_{jk}^k\}_{j \in \mathbb{N}} \) converges also locally uniformly to \( x \mapsto \nabla y(a_{ik}) x \). Moreover, \( \{\{y_{jk}^k\}^{-1}\}_{j \in \mathbb{N}} \) is also a sequence of \( K \)-quasiconformal maps that converges locally uniformly to some map \( w(z) \). It is easy to identify \( w(z) = (\nabla y(a_{ik}))^{-1} x \). Indeed, take some arbitrary \( x \in \Omega \) and \( j \) large enough so that for some \( \delta > 0 \) we have \( y_{jk}^k(x) \in B_{\delta}(\nabla y(a_{ik}) x) \) and \( B_{2\delta}(\nabla y(a_{ik}) x) \subset y_{jk}^k(\Omega) \) for all such \( j \). Then, \( \{y_{jk}^k\}^{-1}(z) \) converges locally uniformly to \( w(z) \) on \( B_{\delta}(\nabla y(a_{ik}) x) \) and so \( x = y_{jk}^k([y_{jk}^k]^{-1}(x)) \to \nabla y(a_{ik})(w(x)) \) in other words

\[
w(z) = (\nabla y(a_{ik}))^{-1}(z).
\]

Thus, we may, owing to Proposition 5.1, assume that \( y_{jk}^k(x) = \nabla y(a_{ik})(x) \) on \( \partial \Omega \).

Further, consider for \( k \in \mathbb{N} \), the rescaled functions \( y_k \in \mathcal{QC}(a_{ik} + \varepsilon_{ik} \Omega; \mathbb{R}^2) \) defined for \( x \in a_{ik} + \varepsilon_{ik} \Omega \) by

\[
y_k(x) := y(a_{ik}) + \varepsilon_{ik} y_{jk}^k \left( \frac{x - a_{ik}}{\varepsilon_{ik}} \right)
\]

where \( j = j(i, k) \) and \( i = i(k) \) will be chosen later. Note that the above formula defines \( y_k \) almost everywhere in \( \Omega \). Note also that the sequence \( \{y_k\}_{k \in \mathbb{N}} \) is \( K \)-quasiconformal on each set \( x \in a_{ik} + \varepsilon_{ik} \Omega \) and each function maps this set to \( y(a_{ik}) + \varepsilon_{ik} \nabla y(a_{ik}) \Omega \) (due to the fixed boundary data).
We now show that on any compact subset of \(a_{ik} + \varepsilon_{ik}\Omega\) and for \(k\) large enough \(y_k\) is uniformly close to \(y\) and the same holds for the inverses. Take some \(x_0 \in \Omega\) and a radius \(R\), such that \(B_{2R}(x_0) \subset \Omega\); then we have that

\[
\|y(x) - y_k(x)\|_{L^\infty(B_{2R}(a_{ik} + \varepsilon_{ik}x_0); \mathbb{R}^2)} = \|y(x) - y(a_{ik}) + \varepsilon_{ik}y^j_k \left( \frac{x - a_{ik}}{\varepsilon_{ik}} \right)\|_{L^\infty(B_{2R}(a_{ik} + \varepsilon_{ik}x_0); \mathbb{R}^2)}
\]

\[
\leq \left\| y(x) - y(a_{ik}) - \varepsilon_{ik} \nabla y(a_{ik}) \left( \frac{x - a_{ik}}{\varepsilon_{ik}} \right) \right\|_{L^\infty(B_{2R}(a_{ik} + \varepsilon_{ik}x_0); \mathbb{R}^2)} + \varepsilon_{ik} \left\| \nabla y(a_{ik}) \left( \frac{x - a_{ik}}{\varepsilon_{ik}} \right) - y^j_k \left( \frac{x - a_{ik}}{\varepsilon_{ik}} \right) \right\|_{L^\infty(B_{2R}(a_{ik} + \varepsilon_{ik}x_0); \mathbb{R}^2)} \leq \frac{2\varepsilon_{ik}}{k}
\]

if \(j\) is large enough compared to \(k\) and \(i\) (at this point we choose \(j\)), so that we can rely on the locally uniform convergence of \(\{y^j_k\}_{j \in \mathbb{N}}\) to \(\nabla y(a_{ik})x\) for \(j \to \infty\) due to Lemma \[3.2\]. Notice that by precomposing with a similarity mapping (which does not change the \(K\)-quasiconformality), this means that

\[
\|y(a_{ik} + \varepsilon_{ik}x) - y_k(a_{ik} + \varepsilon_{ik}x)\|_{L^\infty(B_R(x_0); \mathbb{R}^2)} \leq \frac{2}{k},
\]

i.e. the two maps are uniformly close to each other. For convenience, let us denote \(\tilde{y}(x) = y(a_{ik} + \varepsilon_{ik}x)\) and \(\tilde{y}_k(x) = y_k(a_{ik} + \varepsilon_{ik}x)\); computing the inverse maps gives

\[
\tilde{y}^{-1}(z) = \left( \frac{z - y(a_{ik})}{\varepsilon_{ik}} \right)
\]

\[
\tilde{y}^{-1}_k(z) = \left( \frac{z - y(a_{ik})}{\varepsilon_{ik}} \right).
\]

Then for any point \(z_0\) and any \(\tilde{R} > 0\) such that \(B_{2\tilde{R}}(z_0) \subset (\tilde{y}(\Omega) \cap \tilde{y}_k(\Omega)) \subset y(a_{ik}) + \varepsilon_{ik} \nabla y(a_{ik})\Omega\), we have that

\[
\|\tilde{y}^{-1}(z) - \tilde{y}^{-1}_k(z)\|_{L^\infty(B_{\tilde{R}}(z_0); \mathbb{R}^2)} = \left\| \frac{\tilde{y}^{-1}(z) - y(a_{ik})}{\varepsilon_{ik}} - \left[ \frac{z - y(a_{ik})}{\varepsilon_{ik}} \right] \right\|_{L^\infty(B_{\tilde{R}}(z_0); \mathbb{R}^2)}
\]

\[
\leq \left\| \left[ \frac{y^j_k}{\varepsilon_{ik}} \right]^{-1} \left( \frac{z - y(a_{ik})}{\varepsilon_{ik}} \right) - \left( \nabla y(a_{ik}) \right)^{-1} \left( \frac{z - y(a_{ik})}{\varepsilon_{ik}} \right) \right\|_{L^\infty(B_{\tilde{R}}(z_0); \mathbb{R}^2)} + \left\| \frac{\tilde{y}^{-1}(z) - y(a_{ik})}{\varepsilon_{ik}} - \left( \nabla y(a_{ik}) \right)^{-1} \left( \frac{z - y(a_{ik})}{\varepsilon_{ik}} \right) \right\|_{L^\infty(B_{\tilde{R}}(z_0); \mathbb{R}^2)} \leq \frac{2}{k}
\]

due to \((4.3)\) and by enlarging \(j\) if necessary.

This puts us again into the situation of Proposition \[5.1\] so that we can modify \(\tilde{y}_k\) so that it has the same trace as \(\tilde{y}\) on the boundary of \(\Omega\). By pre-composing again with the similarity mapping \(\frac{x - a_{ik}}{\varepsilon_{ik}}\) we thus obtain a modification of \(y_k\) that has the same boundary values as \(y(x)\) on \(a_{ik} + \varepsilon_{ik}\Omega\). Let us call this modification \(\tilde{y}_k\). Finally let us set

\[u_k(x) = \begin{cases} 
\tilde{y}_k(x) & \text{if } x \in a_{ik} + \varepsilon_{ik}\Omega, \\
y(x) & \text{else}
\end{cases}\]

and note that by the gluing Lemma \[3.7\] \(u_k\) is a sequence of \(K\)-quasiconformal maps (i.e. in particular homeomorphisms).

To see that \(\{u_k\}\) generates \(\nu_x\), we proceed in the same way as in \[29\] Proof of Th. 6.1. Indeed, by a diagonalization argument (relying on the fact that \(\Gamma\) and \(S\) are countable), we enlarge \(j = j(i, k)\) if necessary so that

\[
\varepsilon_{ik}^n \int_{\Omega} g(a_{ik} + \varepsilon_{ik}y)\nu(\nabla y^j_k(y)) \, dy - \tilde{V}(a_{ik}) \int_{a_{ik} + \varepsilon_{ik}\Omega} g(x) \, dx \leq \frac{1}{2^k}.
\]
for all \((g, v) \in \Gamma \times S\). By summing and in view of \([4.5], [4.4]\) and \([4.7]\), we get that
\[
\lim_{k \to \infty} \int_{\Omega} g(x) v(\nabla y_k(x)) \, dx = \int_{\Omega} \int_{\mathbb{R}^{2 \times 2}} v(s) \nu_s(ds) g(x) \, dx.
\]
Hence, we can pick a sub-sequence of \(\{\nabla y_k\}_k\) generating \(\nu\). The measure \(\nu\) is also generated by \(\{\nabla u_k\}\) because the difference of both sequences vanishes in measure.

Proof. By passing, if necessary, to the maps \(\tilde{y}_k(x) = A^{-1} y_k(x)\), there is no loss in generality by assuming that \(y_k : \Omega \to \Omega\) and \(y(x) = x\) on \(\partial \Omega\). Clearly, such maps can be extended by the identity to \(QC_{loc}(\mathbb{R}^2; \mathbb{R}^2)\), so for simplicity we shall denote these extensions by \(\{y_k\}_k\) as well.

To see that \(\{|\nabla y_k|^2\}_{k \in \mathbb{N}}\) converges weakly in \(L^1(\Omega)\), we notice that the higher integrability obtained in Lemma \(3.2\) holds locally in \(\mathbb{R}^2\) and so in particular \(\{|\nabla y_k|^2\}_{k \in \mathbb{N}}\) is bounded in \(L^{1+\gamma}(\Omega)\) and hence weakly converging in \(L^1(\Omega)\).

To show the same for \(\{(\det \nabla y_k)^{-1}\}_{k \in \mathbb{N}}\) observe that, by the area formula,
\[
\int \frac{1}{\det(\nabla y_k(x))} \, dx = \int_{y_k^{-1}(\Omega)} \, dz = |\Omega|,
\]
so the sequence is necessarily bounded in \(L^1(\Omega)\). To show the equi-integrability, we need to assert that for all \(\varepsilon > 0\) there exists a \(\delta\) such that
\[
\sup_k \int_{B(x_0, \delta)} \frac{1}{\det(\nabla y_k(x))} \, dx \leq \varepsilon \quad \forall B(x_0, \delta) \subset \Omega.
\]
Let us find a \(R > 0\) so that \(\Omega \subset B(0, R)\). Then, since quasiconformal maps are locally Hölder continuous (cf. \([4]\) Corollary 3.10.3)), we know that
\[
|y^{-1}_k(x_1) - y^{-1}_k(x_2)| \leq C(R)|x_1 - x_2|^{1/K}
\]
indeed independently of \(k\) since the Hölder continuity depends only on the quasiconformality constant \(K\), which is fixed through the sequence \(\{y_k\}_k\). Therefore,
\[
\sup_k \int_{B(x_0, \delta)} \frac{1}{\det(\nabla y_k(x))} \, dx = \sup_k |y^{-1}_k(B(x_0, \delta))| \leq \sup \pi \max_k \frac{|y(x_1) - y(x_2)|^2}{\pi C(R) \delta^{2/K}} \leq \pi C(R) \delta^{2/K},
\]
which gives the desired equi-integrability.

We are now ready to prove that measures in \(GQC(\Omega; \mathbb{R}^{2 \times 2})\) satisfy the stricter version of the Jensen inequality given in Theorem \(2.4\). Theorem \(2.5\) will follow from this.

Proof of Theorem \(2.4\). First, we realize that for any \(\nu \in GQC(\Omega; \mathbb{R}^{2 \times 2})\), the homogeneous measure \(\mu := \{\nu_a\}_{a \in \Omega}\) is in \(GQC(\Omega; \mathbb{R}^{2 \times 2})\), too, for a.e. \(a \in \Omega\). To see this, we follow \([36]\) Theorem 7.2; Indeed, if gradients of the sequence \(\{y_k\} \subset QC(\Omega; \mathbb{R}^{2 \times 2})\) generate \(\nu\) then, for almost all \(a \in \Omega\), we construct a localized sequence \(\{y_k(a + x/j)\}_{j,k \in \mathbb{N}}\) (note that this is just a pre-composition with a similarity so it cannot affect \(K\)-quasiconformality) whose gradients generate \(\mu\) as \(j, k \to \infty\).
Fix some $a \in \Omega$. Then, the generating sequence of $\mu = \{\nu_a\}_{x \in \Omega} \in \mathcal{G}\mathcal{Y}^{QC}(\Omega; \mathbb{R}^{2 \times 2})$, denoted $\{\nabla y_k\}_{k \in \mathbb{N}} \subset QC(\Omega; \mathbb{R}^2)$, converges weakly in $QC(\Omega; \mathbb{R}^2)$ to the affine map $x \mapsto (\nabla y(a))x$; notice that, since $y$ is quasiconformal, we may assume that $\det(\nabla y(a)) > 0$.

Using Proposition 5.1, we can without loss of generality suppose that $y_k(x) = \nabla y(a)x$ if $x \in \partial \Omega$ Moreover, due to Lemma 4.1, $\{\nabla y_k\}$ and $\{(\det y_k)^{-1}\}$ are weakly convergent in $L^1(\Omega)$. Therefore, we have

$$\lim_{k \to \infty} \int_{\Omega} v(\nabla y_k) \nu_a(ds) = \lim_{k \to \infty} \int_{\Omega} v(\nabla u_k(x)) \, dx \geq |\Omega|v(\nabla u(a)),$$

for any $v \in \mathcal{E}$ that is quasiconformally quasiconvex.

**Proof of Theorem 2.5.** For showing the weak lower semicontinuity, take a sequence of maps $\{y_k\}_{k \in \mathbb{N}}$ that converges weakly in $QC(\Omega; \mathbb{R}^2)$ to $y$. We know that this sequence generates a measure $\nu \in \mathcal{G}\mathcal{Y}^{QC}(\Omega; \mathbb{R}^{2 \times 2})$ and so we have from Theorem 2.4

$$\int_{\Omega} \nu(x, y) \, dx \leq \int_{\Omega} \int_{\mathbb{R}^{2 \times 2}} v(s) \nu_x(ds) \, dx \leq \liminf_{k \to \infty} \int_{\Omega} \nu(x, y_k) \, dx$$

for any $v \in \mathcal{E}$ that is quasiconformally quasiconvex.

On the other hand, take any $y \in QC(\Omega; \mathbb{R}^2)$ such that $y(x) = Ax$ in $\partial \Omega$. Then this $y$ defines a homogeneous Young measure $\nu \in \mathcal{G}\mathcal{Y}^{QC}(\Omega; \mathbb{R}^{2 \times 2})$ with $A$ being its first moment via setting $\int_{\mathbb{R}^{2 \times 2}} f(s) \nu(ds) := |\Omega|^{-1} \int_{\Omega} f(\nabla y(x)) \, dx$ for every $f \in \mathcal{E}$. Let us find a generating sequence for $\nu$ of gradients of quasiconformal maps $\{y_k\}_{k \in \mathbb{N}}$ which can be taken such that $y_k(x) = Ax$ on $\partial \Omega$; recall that for such sequence a $\{\nabla y_k\}_{k \in \mathbb{N}}$ as well as $\{(\det \nabla y_k)^{-1}\}_{k \in \mathbb{N}}$ are weakly converging in $L^1(\Omega)$. Also notice that $y_k \to Ax$ in $QC(\Omega; \mathbb{R}^2)$ since $A$ is the first moment of $\nu$.

Now, since $I(y) := \int_{\Omega} v(\nabla y(x)) \, dx$ is weakly lower semicontinuous on $QC(\Omega; \mathbb{R}^2)$ we get

$$\int_{\Omega} v(\nabla y(x)) \, dx = \lim_{k \to \infty} I(y_k) \geq I(Ax) = |\Omega|v(A),$$

which shows that $v$ is quasiconformally quasiconvex.

**5 Cut-off technique preserving for quasiconformal maps**

Within this section, we present our cut-off technique that preserves quasiconformality, which is the crucial ingredient to the proofs of Theorems 2.1 and 2.5.

**Proposition 5.1.** Let $\text{diam}(\Omega) >> \varepsilon > 0$. Further let $y_k, y \in QC(\Omega; \mathbb{R}^2)$ be $K$-quasiconformal. Then there exists a $\delta << \varepsilon$ that depends only on $y, K$ and $\varepsilon$ such that if $y_k, y$ satisfy

$$\|y - y_k\|_{L^\infty(B_R(x_0); \mathbb{R}^2)} \leq \delta \quad \text{and} \quad \|y^{-1} - y_k^{-1}\|_{L^\infty(B_R(z_0); \mathbb{R}^2)} \leq \delta \tag{5.1}$$

for all $x_0$ and $R$ such that $B_{2R}(x_0) \subset \Omega$ and all $z_0$ and $R$ such that $B_{2R}(z_0) \subset y(\Omega) \cap y_k(\Omega)$, a $\tilde{K}$-quasiconformal function $\omega \in QC(\Omega; \mathbb{R}^2)$ with the following properties can be constructed:

1. $\|y - \omega\|_{L^\infty(B_R(x_0); \mathbb{R}^2)} \leq C(\varepsilon)$ \quad and \quad $\|y^{-1} - \omega^{-1}\|_{L^\infty(B_R(z_0); \mathbb{R}^2)} \leq C(\varepsilon) \tag{5.2}$

for all $x_0$ such that $B_{2R}(x_0) \subset \Omega$ and all $z_0$ such that $B_{2R}(z_0) \subset y(\Omega)$,

2. $\tilde{K}$ depends only on $K$,

3. $\omega|_{\partial \Omega} = y|_{\partial \Omega}$,

4. $|\{x \in \Omega : y_k(x) \neq \omega\}| < C(\varepsilon)$,
where $C(\varepsilon) \to 0$ for $\varepsilon \to 0$.

We prove Proposition\ref{prop:ball} in the remainder of this section by explicitly constructing the sought function $\omega$. To do so, we will divide the domain $\Omega$ into three parts: An outer shell $\Omega_{\text{outer}}$, which includes all points of $\Omega$ close to $\partial \Omega$, the set $\Omega_{\text{inner}}$, consisting of the bulk of $\Omega$, and a small strip between the two sets $\Omega_{\text{mid}}$; cf. Construction\ref{con:partition} for a formal definition and Figure 1 for a better overview.

We will simply set $\omega = y$ on $\Omega_{\text{outer}}$ to obtain the right the boundary condition (Proposition\ref{prop:ball} item 3) and $\omega = y_k$ on $\Omega_{\text{inner}}$ in order satisfy condition Proposition\ref{prop:ball} item 4. Finally, on the strip $\Omega_{\text{mid}}$ we will join the two parts using the Beurling-Ahlfors extension so that the resulting function still ends up in $QC(\Omega; \mathbb{R}^2)$ with a quasiconformality constant depending only on $K$.

However, as explained after Lemma\ref{lem:quasisymmetry} in order to apply the Beurling-Ahlfors extension on a given domain we need to be able to transform it conformally to the half-plane. This, in particular, is not possible for $\Omega_{\text{mid}}$, since in general not simply connected and so we will further partition $\Omega_{\text{mid}}$ into squares on each of which the extension property can be used. Yet, then we have to define $\omega$ on edges of the squares which lie strictly in $\Omega_{\text{mid}}$ (so-called “bridges”, denoted $G$ in Construction\ref{con:partition} and Figure 1) in a quasisymmetric way with the quasisymmetry modulus $\eta$ determined only by $K$. This will form the heart of our construction and, in fact, the major part of the proof.

Let us start by giving a detailed description of the partition of the domain:

**Construction 5.2 (Partition of $\Omega$).** Fix $\operatorname{diam}(\Omega) \gg \varepsilon > 0$ and consider the grid of points $\alpha \in \varepsilon \cdot \mathbb{Z}^2$. Using this grid, we tile $\Omega$ into

$$S_\alpha := \{ x \in \mathbb{R}^2 \mid \alpha_1 \leq x_1 \leq \alpha_1 + \varepsilon \wedge \alpha_2 \leq x_2 \leq \alpha_2 + \varepsilon \} \cap \Omega$$

and set

\[\begin{align*}
\Omega_{\text{outer}} &:= \bigcup \{ S_\alpha : \alpha \in \varepsilon \mathbb{Z}^2, \operatorname{dist}(S_\alpha, \partial \Omega) < 2\gamma \varepsilon \}, \\
\Omega_{\text{inner}} &:= \bigcup \{ S_\alpha : \alpha \in \varepsilon \mathbb{Z}^2, S_\alpha \cap \Omega_{\text{outer}} = \emptyset \}, \\
\Omega_{\text{mid}} &:= \Omega \setminus \Omega_{\text{inner}} \cup \Omega_{\text{outer}},
\end{align*}\]

where $\gamma$ is the smallest integer satisfying $\gamma \geq 1$ and $\eta(1/\gamma) \leq 1/4$ with $\eta$ being the local quasisymmetry modulus of $y$ and $y_k$ (notice that this function depends only on $K$ due to Lemma\ref{lem:quasisymmetry}). Furthermore, we denote by $G$ all grid-lines $(\alpha, \alpha + e_i \varepsilon) \subset \Omega_{\text{mid}}$ for $i \in \{1, 2\}$.

Finally, we set

$$\omega(x) := \begin{cases} y(x) & \text{for } x \in \Omega_{\text{outer}}, \\ y_k(x) & \text{for } x \in \Omega_{\text{inner}}. \end{cases} \quad (5.3)$$

So, $\Omega_{\text{outer}}$ consists of all those squares that are close to $\partial \Omega$ ($\varepsilon$ is presumed to be small) and $\Omega_{\text{mid}}$ is essentially a one square deep row separating $\Omega_{\text{inner}}$ and $\Omega_{\text{outer}}$; we refer to Figure 1 for an illustration of the situation.

**Remark 5.3.** Note that $\gamma$ determining the distance of $\Omega_{\text{mid}}$ to the boundary is chosen in such a way that \ref{prop:ball} is satisfied and $y$ as well as $y_k$ are $\eta$-quasisymmetric on each of the squares in $\Omega_{\text{mid}}$. For this, we need to verify that every such square lies in a ball $B_R(x_0)$ such that $B_{2R}(x_0) \subset \Omega$ and the image lies in another ball $B_R(z_0)$ such that $B_{2R}(z_0) \subset y(\Omega) \cap y_k(\Omega)$.

Now the first part is easy to see. As for the image of the square, we consider first its image under $y$. Let $x_0$ be the midpoint of the square, then we realize that the image of the square under $y$ has to lie in $y(B_r(x_0))$ which itself lies in a ball $B_{\max\{|x-x_0|\geq\varepsilon\}}|y(x_0) - y(x)| |y(x_0)|$. On the other hand, we know that $B_\min\{|x-x_0|\geq\varepsilon\}|y(x_0) - y(x)| |y(x_0)|$ is contained in $y(\Omega)$. Due to quasisymmetry,

$$\frac{\max_{\{|x-x_0|\leq\varepsilon\}}|y(x_0) - y(x)|}{\min_{\{|x-x_0|\leq\varepsilon\}}|y(x_0) - y(x)|} \leq \eta(1/\gamma).$$
This shows that
\[ B_{2 \max_{\{|x - z_0| = \varepsilon\}} |y(x_0) - y(x)| (y(x_0))} \subset B_{4 \max_{\{|x - z_0| = \varepsilon\}} |y(x_0) - y(x)| (y(x_0))} \subset y(\Omega). \]

Analogously, we obtain that
\[ B_{2 \max_{\{|x - z_0| = \varepsilon\}} |y_k(x_0) - y_k(x)| (y_k(x_0))} \subset B_{4 \max_{\{|x - z_0| = \varepsilon\}} |y_k(x_0) - y_k(x)| (y_k(x_0))} \subset y_k(\Omega). \]

Finally, we need to verify that the ball of radius \( B_{2 \max_{\{|x - z_0| = \varepsilon\}} |y(x_0) - y(x)| (y(x_0))} \subset y_k(\Omega) \) and vice versa. For this we choose
\[ \delta \leq \min_{x_0, \{|x - z_0| = \varepsilon\}} |y(x_0) - y(x)|; \quad (5.4) \]

Note that there is a finite number of \( x_0 \)'s (depending on \( \varepsilon \)) so that the minimum can be found and is positive. But then, since
\[ |y(x_0) - y(x)| - 2\delta \leq |y_k(x_0) - y_k(x)| \leq |y(x_0) - y(x)| + 2\delta, \]

we have that \( B_{2 \max_{\{|x - z_0| = \varepsilon\}} |y_k(x_0) - y_k(x)| (y_k(x_0))} \subset B_{4 \max_{\{|x - z_0| = \varepsilon\}} |y(x_0) - y(x)| (y(x_0))} \) as well as
\[ B_{2 \max_{\{|x - z_0| = \varepsilon\}} |y(x_0) - y(x)| (y(x_0))} \subset B_{4 \max_{\{|x - z_0| = \varepsilon\}} |y_k(x_0) - y_k(x)| (y_k(x_0))} \]
which shows the claim.

Note that \([5.3]\) defines \( \omega \) everywhere except for \( \Omega_{\text{mid}} \). It is trivial to see that \( \omega \) fulfills conditions 2-4 in Proposition \([5.1]\) and that item 1 in this proposition holds so far as \( \omega \) is defined. Furthermore, if \( \delta < \varepsilon/5 \) and \([5.4]\) hold \( \partial \Omega_{\text{inner}} \) and \( \partial \Omega_{\text{outer}} \setminus \partial \Omega \) will not intersect (see Lemma \([5.6]\) below), which makes \( \omega \) injective so far as defined.

It remains to define \( \omega \) on \( \Omega_{\text{mid}} \), which is the non-trivial part of the construction, however. As outlined above, we first define \( \omega \) on the grid segments of \( G \). This can be done for each grid line in \( G \) independently, and so, since all the cases are essentialy equivalent, we may turn our attention to the single the line segment \((\alpha, \alpha + \varepsilon e_1)\) with \( \alpha \in \Omega_{\text{outer}} \) and \( \alpha + \varepsilon e_1 \in \Omega_{\text{inner}} \).

On consider the following construction:

**Construction 5.4 (Building a bridge on \( G \)).** Define
\[ r := \min \{s > 0 : y(\partial B_s(\alpha)) \cap y_k(\partial B_s(\alpha + \varepsilon e_1)) \neq \emptyset\} \]

and take some \( z_0 \in y(\partial B_r(\alpha)) \cap y(\partial B_r(\alpha + \varepsilon e_1)) \). Then we define the affine functions
\[ \phi_1(x) = \frac{2}{\varepsilon} (y^{-1}(z_0) - \alpha)(x_1 - \alpha_1) + \alpha + e_2(x_2 - \alpha_2), \]
\[ \phi_2(x) = \frac{2}{\varepsilon} (y_k^{-1}(z_0) - (\alpha + \varepsilon e_1))(\alpha_1 + \varepsilon - x_1) + \alpha + \varepsilon e_1 + e_2(x_2 - \alpha_2). \]
that are constructed in such a way that
\[ \phi_1(\alpha + s e_2) = \alpha + s e_2 \quad \text{and} \quad \phi_2(\alpha + \varepsilon e_1 + s e_2) = \alpha + \varepsilon e_1 + s e_2 \quad \forall s \in \mathbb{R}, \]
\[ \phi_1 \left( \alpha + \frac{\varepsilon}{2} e_1 \right) = y^{-1}(z_0) \quad \text{and} \quad \phi_2 \left( \alpha + \frac{\varepsilon}{2} e_1 \right) = y_k^{-1}(z_0), \]

see Figure 2 for an illustration of the situation.
We now define
\[ \omega_\alpha : (0, \varepsilon) \to \mathbb{R}^2, s \mapsto \begin{cases} y \circ \phi_1(\alpha + s e_1) & \text{for } s < \varepsilon/2 \\ y_k \circ \phi_2(\alpha + s e_1) & \text{for } s \geq \varepsilon/2 \end{cases} \]

**Remark 5.5.** Note that the maximum of the set \( \{ s > 0 : y(\partial B_s(\alpha)) \cap y_k(\partial B_s(\alpha + \varepsilon e_1)) \neq \emptyset \} \) needed in Construction [5.4] can be found since the set is compact. In our arguments we will also sometimes make use of the equivalent characterizations
\[ r = \min \{ s > 0 : y(B_s(\alpha)) \cap y_k(B_s(\alpha + \varepsilon e_1)) \neq \emptyset \}, \]
\[ r = \max \{ s > 0 : y(B_s(\alpha)) \cap y_k(B_s(\alpha + \varepsilon e_1)) = \emptyset \}. \]

The function \( \omega_\alpha \) defined in Construction [5.4] seems to be a promising candidate for the sought “bridge function”, i.e. a definition of \( \omega \) on \( G \). However, as defined, \( \omega_\alpha \), will not necessarily be quasisymmetric which is essential for Lemma [5.4]. Nevertheless, it will turn out in Lemma [5.7] below that, in fact, the image of \( \omega_\alpha \) is at least a segment of a quasicircle and so the sought bridge on the segment \([\alpha, \alpha + \varepsilon e_1] \subset \Omega_{inner}\) will have to be a reparametrization of \( \omega_\alpha \).

Before proceeding, we will show the well-definedness of Construction [5.4] as well as some bounds on the various quantities involved.
Lemma 5.6. Let δ satisfy (5.4) as well as δ < \frac{\varepsilon}{5}. Let y, y_k be as in Proposition 5.1 and fulfill (5.1). Then the quantities found in Construction 5.4 satisfy the following:

1. \( \frac{\varepsilon}{2} - \delta/2 < r < \frac{\varepsilon}{2} + \delta/2 \)

2. \(|y^{-1}(z_0) - (\alpha + \frac{\varepsilon}{2}e_1)| < \sqrt{5\varepsilon\delta} \) and \(|y_k^{-1}(z_0) - (\alpha + \frac{\varepsilon}{2}e_1)| < \sqrt{5\varepsilon\delta} \),

3. The functions \( \phi_i \) are bi-Lipschitz with constants \( L_{\phi_i} < 1/(1 - \frac{\delta}{\varepsilon}) \), \( i = 1, 2 \).

4. The function \( \omega_\alpha \) is well-defined, injective and \( \omega_\alpha((0, \varepsilon)) \) does neither intersect \( y(\Omega_{\text{outer}}) \), \( y_k(\Omega_{\text{inner}}) \) nor the images of similarly constructed \( \omega_\beta \) on any of the other edges in \( G \).

5. The function \( \omega_\alpha \) is \( \tilde{\eta} \)-quasisymmetric in \([0, \varepsilon/2]\) as well \([\varepsilon/2, \varepsilon]\) with \( \tilde{\eta} \) only dependent on \( \eta \).

Proof.

1. Let us look at the bounds for \( r \) first. We know that \(|y^{-1}(z_0) - y_k^{-1}(z_0)| < \delta \) but \(|y^{-1}(z_0)\) and \(|y_k^{-1}(z_0)\) lie on a circle of equal radius centered at \( \alpha \) and \( \alpha + \varepsilon e_1 \), respectively. Therefore, it has to hold that \( r > \varepsilon/2 - \delta/2 \).

   For the upper bound, consider the point \( \tilde{z} := y(\alpha + \frac{\varepsilon + \delta}{2}e_1) \) in the image. We know that
   \[
   \left| \left( \alpha + \frac{\varepsilon + \delta}{2}e_1 \right) - y_k^{-1}(\tilde{z}) \right| = |y^{-1}(\tilde{z}) - y_k^{-1}(\tilde{z})| < \delta.
   \]

   But then \( y_k^{-1}(\tilde{z}) \in B_\delta(\alpha + \frac{\varepsilon + \delta}{2}e_1) \subset B_{\varepsilon/2+\delta/2}(\alpha + \varepsilon e_1) \), so \( \tilde{z} \in y_k(B_{\varepsilon/2+\delta/2}(\alpha + \varepsilon e_1)) \cap y(B_{\varepsilon/2+\delta/2}(\alpha)) \).

   So this means that \( \varepsilon/2 + \delta/2 \) is an upper bound for \( r \).

2. Using the estimates for \( r \), we can now bound the distances \(|y^{-1}(z_0) - (\alpha + \frac{\varepsilon}{2}e_1)|\) and \(|y_k^{-1}(z_0) - (\alpha + \frac{\varepsilon}{2}e_1)|\); due to symmetry of the two we shall just show the latter. Again, we start with \(|y^{-1}(z_0) - y_k^{-1}(z_0)| < \delta \)

   and so, using the upper bound on \( r \), we have on one hand

   \[
   |\alpha - y_k^{-1}(z_0)| \leq |\alpha - y^{-1}(z_0)| + |y^{-1}(z_0) - y_k^{-1}(z_0)| < r + \delta < \frac{\varepsilon}{2} + \frac{3\delta}{2},
   \]

   and on the other hand

   \[
   |(\alpha + \varepsilon e_1) - y_k^{-1}(x_0)| = r < \frac{\varepsilon}{2} + \frac{\delta}{2} \leq \frac{\varepsilon}{2} + \frac{3\delta}{2}.
   \]

   So \( y_k^{-1}(x_0) \) and \( \alpha + \frac{\varepsilon}{2}e_1 \) are both inside the “lens-like” intersection of the two circles \( B_{\varepsilon/2+3/2\delta}(\alpha) \cap B_{\varepsilon/2+3/2\delta}(\alpha + \varepsilon e_1) \) (cf. Figure 3) and whose diameter is bounded by \( \sqrt{7\varepsilon\delta} \).

3. It is easy to calculate the gradient of \( \phi_1 \):

   \[
   D\phi_1 = \left( \frac{2}{\varepsilon}y^{-1}(z_0) - \alpha \right)e_2,
   \]

   which has eigenvalues 1 and \( \frac{2}{\varepsilon}(y^{-1}(z_0) - \alpha)_1 \), so

   \[
   L_{\phi_1} = \max \left\{ \frac{2}{\varepsilon}(y^{-1}(z_0) - \alpha)_1, \frac{1}{\varepsilon(y^{-1}(z_0) - \alpha)_1} \right\} \leq \max \left\{ 1 + \frac{\delta}{\varepsilon}, \frac{1}{1 - \frac{\delta}{\varepsilon}} \right\} = \frac{1}{1 - \frac{\delta}{\varepsilon}}.
   \]

   The same works for \( \phi_2 \). Also note that \( \delta < \frac{\varepsilon}{5} \) implies \( L_{\phi_i} < \frac{5}{4} \).
Proof. To show this, we will verify the classical arc condition (3.2). Take the image of Lemma 5.7. Since we are in the image of $y$ and on this line we find the points $3_i = k(\omega_4)$. By the bounds from 1., we know that $\omega_\alpha((0, \varepsilon/2)) \subset y(B_\varepsilon(\alpha))$ and $\omega_\alpha((\varepsilon/2, \varepsilon)) \subset y_k(B_\varepsilon(\alpha + \varepsilon/1))$. But since $y(B_\varepsilon(\alpha)) \cap y_k(B_\varepsilon(\alpha + \varepsilon/1)) = \emptyset$ this implies injectivity of $\omega_\alpha$ on all of $(0, \varepsilon)$. By our bounds on the positions of $y^{-1}(z_0)$ and $y_k^{-1}(z_0)$ we have also shown that $\omega_\alpha((0, 1)) \subset y(C_1 \cap B_r(\alpha)) \cup y_k(C_2 \cap B_r(\alpha + \varepsilon/1))$ where $C_1$ and $C_2$ are narrow cones with tips in $\alpha$ and $\alpha + \varepsilon/1$ respectively, opened towards $\alpha + \varepsilon/1$ with opening angle less than $\arccos \left( \frac{\varepsilon}{\varepsilon + 3\delta} \right)$; cf. also Figure 3. It is easy to see that therefore $\omega_\alpha((0, \varepsilon))$ does neither intersect $y(\Omega_{\text{outer}})$, $y_k(\Omega_{\text{inner}})$ nor $\omega_\beta$ on any of the other intervals of $G$.

5. On each of the intervals $[0, \varepsilon/2]$ as well as $[\varepsilon/2, \varepsilon]$ the function $\omega_\alpha$ is a composition of the $\eta$-quasisymmetric functions $y$ and $y_k$ with the $b$-bi-Lipschitz functions $\phi_1$ and $\phi_2$, respectively. This yields the claim.

Lemma 5.7. Let $\delta$ satisfy (5.4) as well as $\delta < \xi$. Let $y, y_k$ be as in Proposition 5.1 and fulfill (5.1). Then the image of $\omega_\alpha$ is a segment of a $K$-quasiconcave where $K$ depends only on $K$.

Proof. To show this, we will verify the classical arc condition (3.2). Take $z_1, z_2, z_3 \in \omega_\alpha((0, \varepsilon))$ and denote $x_i = \omega_\alpha^{-1}(z_i)$. We can immediately assume that $x_1 < x_2 < x_3$. If $\varepsilon/2 < x_1$ or $x_3 < \varepsilon/2$, the result is trivial since we are in the image of $y$ or $y_k$ respectively.

Now, we consider the case when $x_1 < x_2 < \varepsilon/2 < x_3$. We look at the connecting line between $z_1$ and $z_3$ and on this line we find the points $\xi_1$ and $\xi_3$ satisfying $|y^{-1}(z_1) - y^{-1}(\xi_1)| = |\phi_1(x_1) - \phi_1(\varepsilon/2)|$ and $|y_k^{-1}(z_3) - y_k^{-1}(\xi_3)| = |\phi_2(x_3) - \phi_2(\varepsilon/2)|$, respectively.

Then, directly from Construction 5.4 we conclude that $y(B_{|\phi_1(x_1) - \phi_1(\varepsilon/2)|}(\phi_1(x_1))) \cap y_k(B_{|\phi_2(x_3) - \phi_2(\varepsilon/2)|}(\phi_2(x_3))) = \emptyset$

and hence $|z_1 - z_3| \geq |z_1 - \xi_1| + |z_3 - \xi_3|$.

Furthermore, since $y$ and $y_k$ are locally quasisymmetric, we have

$|z_i - \xi_i| > \eta^{-1}(1) |z_i - \omega_\alpha(\varepsilon/2)| \quad i = 1, 3.$
Also, since \( \omega_{\alpha}(\varepsilon/2) = y(\phi_{\varepsilon/2}(x_2)) \) and \( z_2 = y(\phi_{\varepsilon/2}(x_2)) \) lie on image of the quasisymmetric function \( y \circ \phi_{\varepsilon/2} \) of the interval \([0, \varepsilon/2]\), we get directly from the quasisymmetry property that (here we also use that the bi-Lipschitz constant of \( \phi_{\varepsilon/2} \) is bounded by 5)

\[
|z_2 - \omega_{\alpha}(\varepsilon/2)| + |z_2 - z_1| \leq 50\eta(1)|z_1 - \omega_{\alpha}(\varepsilon/2)|
\]

Summing all up,

\[
|z_1 - z_3| > \eta^{-1}(1)(|z_1 - \omega_{\alpha}(\varepsilon/2)| + |z_3 - \omega_{\alpha}(\varepsilon/2)|)
\]

\[
\geq 50\eta^{-1}(1)(|z_1 - z_2| + |z_2 - \omega_{\alpha}(\varepsilon/2)| + |z_3 - \omega_{\alpha}(\varepsilon/2)|)
\]

\[
\geq 50\eta^{-1}(1)(|z_1 - z_2| + C|z_2 - z_3|),
\]

where we used the triangle inequality in the last line.

The final case \( x_1 < \varepsilon/2 < x_2 < x_3 \) follows from symmetry. \( \square \)

**Remark 5.8.** By arguments similar to those used in the previous proof, we can show that if the functions \( y, y_k \) were bi-Lipšitz continuous, so will be \( \omega_{\alpha} \), even with the same exponent. In particular, this applies to the case studied here (recall that quasiconformal maps are locally Hölder continuous (cf. [4, Corollary 3.10.3]).

We know from Lemma 5.7 that there exists a quasisymmetric parametrisation of the image of \( \omega_{\alpha} \). However, what we actually need is not just some parametrisation, but a parametrisation that is still quasisymmetric when connected to the image of \( \Omega_{\text{outer}} \) under \( y \) as well as \( \Omega_{\text{inner}} \) under \( y_k \).

First, we realize why the function \( \omega_{\alpha} \) itself (which does connect in a quasisymmetric way to the images of \( \Omega_{\text{inner}} \) and \( \Omega_{\text{outer}} \)) does not need to be quasisymmetric across the “meeting point” \( \varepsilon/2 \). As noted, \( \omega_{\alpha} \) is at least bi-Lipšitz-continuous, which shows that it cannot form a too sharp angle at \( \varepsilon/2 \). Yet, this does not exclude the possibility that both parts of \( \omega_{\alpha} \) approach the meeting point with “different speeds”. To be more precise, while we know the bi-Lipšitz property shows that \( \omega_{\alpha}(\varepsilon/2 + t) - \omega_{\alpha}(\varepsilon/2) \) and \( \omega_{\alpha}(\varepsilon/2) - \omega_{\alpha}(\varepsilon/2 - t) \) are roughly co-linear for small \( t \), we have no bounds on the quotient

\[
\frac{|\omega_{\alpha}(\varepsilon/2 + t) - \omega_{\alpha}(\varepsilon/2)|}{|\omega_{\alpha}(\varepsilon/2) - \omega_{\alpha}(\varepsilon/2 - t)|}
\]

To fix this issue, we will perform yet another (slight) modification of the construction: We will reparametrize \( \omega_{\alpha} \) around the meeting point \( \varepsilon/2 \) but keep the original parametrization close to 0 and \( \varepsilon \) in order not to run into the same kind of problems when transitioning to \( y \) and \( y_k \) at the endpoints of the interval. This requires a slow passage from one parametrization to another without endangering the quasi-symmetry. Nevertheless, finding such a reparametrization is a one-dimensional problem, which we are able to solve explicitly:

**Lemma 5.9.** Let \( s : [0, a] \rightarrow [0, b] \) be an increasing, \( \eta \)-quasisymmetric homeomorphism. Then there exists a homeomorphism \( \tilde{s} : [0, a] \rightarrow [0, \ell], \ell < 3/2b \) such that \( \tilde{s}|_{[0,a/4]} = s|_{[0,a/4]} \), \( \tilde{s}|_{[3/4,a,a]} = b/a \) and \( \tilde{s} \) is \( \bar{\eta} \)-quasisymmetric, where \( \bar{\eta} \) is only dependent on \( \eta \).

We postpone the proof of this lemma to the end of the section, since it is quite technical and we rather directly state a refinement, which gives the desired passage between parametrizations:

**Proposition 5.10.** Let \( r, s : [0, a] \rightarrow [0, b] \) be \( \eta \)-quasisymmetric homeomorphisms. Then there exists a number \( c \in (0, a/4) \) which is only dependent on \( \eta \) and a homeomorphism \( \tilde{s} : [0, a] \rightarrow [0, b], \) such that \( \tilde{s}|_{[0,c]} = r|_{[0,c]}, \tilde{s}|_{[a-c,a]} = s|_{[a-c,a]} \) and \( \tilde{s} \) is \( \bar{\eta} \)-quasisymmetric, where \( \bar{\eta} \) is only dependent on \( \eta \).

**Proof.** Without loss of generality, we may consider only the case when \( a = b = 1 \) as the general case can be handled by following in verbatim the beginning of the proof of Lemma 5.9.

First of all, we find the largest number \( d \in (0, 1/4] \) such that \( \eta(d) \leq 1/4 \). Then \( r \) maps the interval \([0, d] \) into \([0, 1/4] \) and \( s \) maps \([1 - d, 1] \) into \([3/4, 1] \) since

\[
0 < \eta(d)^{-1} \leq \frac{r(d) - r(0)}{r(1) - r(0)} = r(d) \leq \eta(d) \leq 1/4
\]  

(5.5)
and in the same way
\[ \eta(d)^{-1} \leq 1 - s(d) \leq \eta(d) \leq 1/4. \] (5.6)

Now, we use Lemma 5.9 twice to construct the pieces \( s |_{[0,d]} \) and \( s |_{[1-d,1]} \) in such a way that \( s |_{[0,d]} = r |_{[0,d/4]} \) and \( s |_{[1-d/4,1]} = s |_{[1-d/4,1]} \), as well as \( s |_{[3d/4,1]} = r |_{(d/8)2} \) and \( s |_{[1-d-3d/4]} = s(d/2) \). Then we know from Lemma 5.9 \( \tilde{s}(d) < 3/8 \) and \( s(1-d) > 5/8 \), so we can simply connect the parts of the function affinely without losing injectivity.

The resulting function \( \tilde{s} \) is quasisymmetric on \([0,d] \) and \([1-d,1] \) owing to Lemma 5.9 and moreover, on the interval \([3d/4,1-3d/4] \) it consists of 3 affine segments. Therefore, we have that \( \tilde{s} \) is quasisymmetric on \([0,d] \) and \([1-d,1] \) and that \( \tilde{s} \) is \( \tilde{\eta} \)-quasisymmetric with \( \tilde{\eta} \) only depending on \( \eta \).

Proof. We know from Lemma 5.6 that \( \omega \) is quasisymmetric on \([0,d] \) as well as \([3d/4,1-3d/4] \). Hence, apply Proposition 5.10 to each of those intervals to construct a quasisymmetric \( \tilde{s} : [0,\epsilon] \to [0,\epsilon] \) satisfying
\[
\tilde{s}(t) = \begin{cases} 
  t & \text{for } 0 \leq t \leq \lambda \epsilon \\
  s(t) & \text{for } m - \lambda \epsilon \leq t \leq m + \lambda \epsilon \\
  t & \text{for } \epsilon - \lambda \epsilon \leq t \leq \epsilon 
\end{cases}
\]
for a suitable \( \lambda > 0 \) determined by the number \( e \) in Proposition 5.10 and depending only on the quasisymmetry modulus \( \eta \).

Now define
\[
\omega_\alpha(t) = \omega_\alpha(\tilde{s}(t)) \quad \text{and} \quad \omega(\alpha + e_1 s) = \omega_\alpha(s) \quad \text{on } [\alpha, \alpha + e_1 \epsilon].
\]

We immediately have the following property of \( \omega_\alpha(s) \):

Lemma 5.12. Let \( \delta \) satisfy (5.1) as well as \( \delta < \frac{\epsilon}{2} \). Let \( y_k \) be as in Proposition 5.1 and fulfill (5.1). Then \( \omega_\alpha \) found in Construction 5.11 is well defined and \( \tilde{\eta} \)-quasisymmetric, where \( \tilde{\eta} \) depends only on \( K \).

Proof. We know from Lemma 5.6 that \( \omega_\alpha \) is quasisymmetric on \([0,\epsilon/2] \) as well as \([\epsilon/2,\epsilon] \) with a quasisymmetry modulus depending only on \( K \). Therefore, also \( s \) and whence \( \tilde{s} \) are quasisymmetric on \([0,m] \) and \([m,\epsilon] \) with a quasisymmetry modulus depending only on \( K \) and so is \( \omega_\alpha \) as a composition.

Furthermore, we know that \( \tilde{\omega}_\alpha = \omega_\alpha \circ s \) is quasisymmetric on \([m - \lambda \epsilon, m + \lambda \epsilon] \) per construction. But since those three intervals overlap, \( \tilde{\omega}_\alpha \) is quasisymmetric on all of \([0,\epsilon] \) and the modulus depends only on the moduli of the functions involved, which all derive from \( K \).

Proof of Proposition 5.1. We pick some \( \text{diam}(\Omega) \gg \epsilon > 0 \) and find an appropriate \( \delta \) satisfying simultaneously (5.4) as well as \( \delta < \epsilon/5 \). We perform the partition from Construction 5.2 and define \( \omega \) on \( \Omega_{\text{outer}} \cup \Omega_{\text{inner}} \) as indicated in (5.8). Then on the grid \( G \), we proceed according to Construction 5.11.

\footnote{For the first interval, we obtain this by applying Proposition 5.10 to the piecewise affine function \( r \) satisfying \( r(0) = 0, r(m/2) = m/2, r(m) = \epsilon/2 \) as well as \( r \) constructed above. On the second interval we connect \( s \) to the piecewise affine \( r \) that fulfills \( r(m) = \epsilon/2, r((\epsilon + m)/2) = (\epsilon + m)/2, r(\epsilon) = r(\epsilon) \).}
We know, due to Remark 3.3, that both $y$ and $y_k$ are $\eta$-quasisymmetric on a neighbourhood of each of the squares in $\Omega_{mid}$ with $\eta$ depending only on $K$ due to Lemma 3.3. Therefore, by employing also Lemma 3.12, $\omega$ is quasisymmetric on the boundary of every square $S_{ij} \subset \Omega_{mid}$. So, we may use the Beurling-Ahlfors extension from Lemma 3.4 to extend $\omega$ to a quasiconformal homeomorphism on each square of $\Omega_{mid}$ which makes $\omega$ a homeomorphism defined on all of $\Omega$ satisfying $|\nabla \omega|^2 \leq K \det (\nabla \omega)$ for some $K$ depending only on $K$. Moreover, since $\omega$ coincides with $y$ in a neighbourhood of $\partial \Omega$, it fulfills (3.3) which makes it globally injective. In other words, $\omega$ if $K$-quasiconformal with $K$ depending only on $K$.

Finally, since the image of every square $S \subset \Omega_{inner}$ under $\omega$ is contained in the union of the image of the given square and its neighbours under $y$ and $y_k$, we get that (denote $x_i$ the midpoint of such square)

$\omega(S) \subset y(B_{3\varepsilon}(x_i)) \cup y_k(B_{3\varepsilon}(x_i))$

so that

$\| \omega - y \|_{L^\infty(S_i;\mathbb{R}^2)} < 3\varepsilon + \delta$.

Similarly, we get that

$\| \omega^{-1} - y^{-1} \|_{L^\infty(S_i;\mathbb{R}^2)} < 3\varepsilon + \delta$,

which yields (5.2).

To end this section, we present the proof of Lemma 5.9

Proof of Lemma 5.9 (1-dimensional fitting): First, we realize that it suffices to consider quasisymmetric homeomorphisms $s : [0, 1] \to [0, 1]$ and to then construct $\tilde{s}$ with $\tilde{s}_{[0,1/4]} = s_{[0,1/4]}$ and $\tilde{s}_{[3/4,1]} = 1$. Indeed, in the general case compose with similarities without changing $\eta$ by defining $\tilde{s} : t \mapsto b^{-1}s(at) : [0, 1] \to [0, 1]$. Then, if we can construct $\tilde{s}$ as specified above the function $b\tilde{s} \left( \frac{t}{a} \right)$ will have all the desired properties. Notice also that such a rescaling is equivalent to composing with similarities in the domain and in the image and thus does not change the quasi-symmetry modulus.

Moreover, we may restrict our attention to $s : [0, 1] \to [0, 1]$ that are additionally smooth. While in fact $s$ may not even be absolutely continuous (cf. e.g. [14] Thm. 3) for the construction of a counterexample), in one dimension it can be uniformly approximated by a sequence of smooth $\eta$-quasisymmetric homeomorphisms (cf. [28] Thm. 7). So if $s$ is not smooth, we can approximate it by a sequence of smooth $s_k$ and construct the corresponding functions $\tilde{s}_k$ with $\tilde{s}_k_{[0,1/4]} = s_{[0,1/4]}$ and $\tilde{s}_k_{[3/4,1]} = 1$. Furthermore we know that normalized families of quasisymmetric functions are normal and therefore $\tilde{s}_k \to \tilde{s}$ for a subsequence (cf. [28] Thm. 8 or [4] Cor. 3.9.3)). But then we trivially have $\tilde{s}_{[0,1/4]} = s_{[0,1/4]}$ and $\tilde{s}_{[3/4,1]} = 1$.

Now to start with the actual proof, let us consider the following suitable partition of unity

$$\psi_0(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1/4 \\ e^{-\frac{1}{\sqrt{4-t}}} & \text{for } 1/4 \leq t \leq 3/4 \\ 0 & \text{for } 3/4 \leq t \leq 1 \end{cases}$$

and

$$\psi_1(t) = \begin{cases} 0 & \text{for } 0 \leq t < 1/4 \\ e^{-\frac{1}{\sqrt{4-t}}} & \text{for } 1/4 \leq t \leq 3/4 \\ 1 & \text{for } 3/4 \leq t \leq 1 \end{cases}$$

and define $\tilde{s}$ via the integral of a convex combination of the derivatives of $s$ (recall that we assume that $s$ is smooth) and the identity; i.e.

$$\tilde{s}(t) := \int_0^t \psi_0(x)s'(x) + \psi_1(x) \, dx.$$

The smoothing used in [28] may perturb the end points of the approximation slightly, so we only have $s_k(0) \to 0$ and $s_k(1) \to 1$. Note however that affinely rescaling the image back to $[0,1]$ has no impact on the convergence, so we can assume $s_k(0) = 0$ and $s_k(1) = 1$. 

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Then \( \tilde{s} \) is clearly an absolute continuous, strictly monotone homomorphism with \( \tilde{s}(0) = 0 \) and

\[
\tilde{s}(1) < \int_0^1 s'(x) + \psi_1(x) \, dx = 1 + 1/2.
\]

We now need to show the quasisymmetry of \( \tilde{s} \). For this consider, we verify the well known M-condition \( [1] \), i.e., for all \( t \in [0,1] \) and \( h > 0 \) we have that

\[
\frac{1}{M} \leq \frac{\tilde{s}(t+h) - \tilde{s}(t)}{\tilde{s}(t) - \tilde{s}(t-h)} \leq M,
\]

which reduces to showing that for all \( t \in [0,1] \) and \( h \neq 0 \) there exists a constant \( M \) that is dependent only on \( \eta \) such that

\[
\left| \tilde{s}(t+h) - \tilde{s}(t) \right| = \left| \int_{t-h}^{t} \psi_0(x)s'(x) + \psi_1(x) \, dx \right| \leq M.
\]

(5.7)

In fact, since \( \tilde{s} : [0,1] \to \mathbb{R} \), it suffices to verify (5.7) for \( |h| < 1/8 \) because for larger \( h \) we may proceed by iteration.

Let us first verify (5.7) for \( h > 0 \) and \( t > 1/2 \). In this case, we know that \( \psi_1(t-h) > \psi_1(3/8) > 0 \) and so

\[
\left| \int_{t-h}^{t} \psi_0(x)s'(x) + \psi_1(x) \, dx \right| \leq \left| \int_{t-h}^{t} \psi_0(t)s'(x) + \psi_1(t-h) \, dx \right| + \left| \int_{t-h}^{t} \psi_0(t)s'(x) + \psi_1(t) \, dx \right| < \psi_0(1) \psi_1(3/8).
\]

Similar arguments apply in the case when \( t \leq 1/2 \) and \( h < 0 \) since then \( \psi_0(t-h) \geq \psi_0(5/8) > 0 \) and we estimate

\[
\left| \int_{t-h}^{t} \psi_0(x)s'(x) + \psi_1(x) \, dx \right| \leq \left| \psi_0(t-h) \int_{t-h}^{t} s'(x) \, dx \right| + \left| \int_{t-h}^{t} \psi_1(t) \, dx \right| < \frac{\psi_0(1)}{\psi_0(5/8)} \eta(1) + 1.
\]

Now, we handle the complementary cases beginning with \( t \leq 1/2, h > 0 \) which are slightly more elaborate. To simplify the notation, we will use \( C \) for a generic constant that is independent of the problem parameters and may change from expression to expression.

Relying on monotonicity of \( \psi_0 \) ans \( \psi_1 \) we obtain similarly as above

\[
\left| \int_{t-h}^{t} \psi_0(x)s'(x) + \psi_1(x) \, dx \right| \leq \left| \int_{t-h}^{t} \psi_0(t)s'(x) \, dx \right| + \left| \int_{t-h}^{t} \psi_1(t+h) \, dx \right|. \]

where, even though the denominator of the second term may not vanish, \( \psi_1(t-h) \) can; thus we cannot proceed as in the previous cases. To estimate the second term, we limit ourselves to the situation \( t+h > 1/4 \)
(i.e. $t > 1/8 > h$) since the term vanishes otherwise and still distinguish two cases: $t - 1/4 < \sqrt{h}$ and $t - 1/4 \geq \sqrt{h}$. In the latter case $\psi_1(t-h)$ is strictly positive and so

$$\frac{h \psi_1(t + h)}{\psi_0(t) |s(t) - s(t-h)| + h \psi_1(t-h)} \leq \frac{\psi_1(t + h)}{\psi_1(t-h)} \leq Ce^{-\frac{t-h^1}{4 + t-h} + \frac{t-h}{4}} = Ce^{-\frac{25}{4 + \sqrt{h}}} \leq Ce^{-\frac{2h}{h^{\kappa_1}}},$$

which is bounded for $h \in (0, 1/8)$. When $t - 1/4 < \sqrt{h}$, we write

$$\frac{h \psi_1(t + h)}{\psi_0(t) |s(t) - s(t-h)| + h \psi_1(t-h)} \leq \frac{h \psi_1(t + h)}{\psi_0(t) |s(t) - s(t-h)|} \leq C \frac{h e^{-\frac{t-h^1}{4 + t-h} + \frac{t-h}{4}}}{\psi_0(1/2) |s(t) - s(t-h)|} \leq Ch^{1-\kappa_1} e^{-\frac{1}{2\sqrt{h}}},$$

where in the last estimate we used that (cf. [28, Thm. 5 and Thm. 10]) quasiregular mappings in one dimension are bi-Hölder continuous; i.e. for all $t_1, t_2 \in (0, 1)$

$$8^{\kappa_1} |t_1 - t_2|^{\kappa_2} \geq |s(t_1) - s(t_2)| \geq 8^{-\kappa_1} |t_1 - t_2|^{\kappa_2},$$

where $\kappa_1$ and $\kappa_2$ are solely dependent on $\eta(1)$. However, $\lim_{h \to 0} C h^{1-\kappa_1} e^{-\frac{1}{2\sqrt{h}}} = 0$ which shows that $Ch^{1-\kappa_1} e^{-\frac{1}{2\sqrt{h}}}$ is uniformly bounded for $h \in (0, 1/8)$.

In the remaining case when $t > 1/2, h < 0$ we argue similarly as above:

$$\frac{\int_{t-h}^{t+h} \psi_0(x)s'(x) + \psi_1(x) \, dx}{\int_{t-h}^{t+h} \psi_0(x)s'(x) + \psi_1(x) \, dx} \leq \frac{\psi_0(t + h) |s(t) - s(t+h)|}{\psi_0(t-h) (s(t-h) - s(t) + h \psi_1(t))} + 1$$

Now to bound the first term we can assume $t + h < 3/4$ and again distinguish two cases. Either we have $3/4 - t < \sqrt{h}$ and therefore get that

$$\frac{\psi_0(t + h) |s(t) - s(t+h)|}{\psi_0(t-h) (s(t-h) - s(t) + h \psi_1(t))} \leq \frac{\psi_0(t + h) |s(t) - s(t+h)|}{h \psi_1(t)} \leq Ce^{-\frac{1}{\sqrt{h}}} h^{\kappa_2 - 1}$$

which is bounded, or we have $3/4 - t \geq \sqrt{h}$ and hence

$$\frac{\psi_0(t + h) |s(t) - s(t+h)|}{\psi_0(t-h) (s(t-h) - s(t) + h \psi_1(t))} \leq \frac{\psi_0(t + h) |s(t) - s(t+h)|}{\psi_0(t-h) |s(t-h) - s(t)|} \leq C \eta(1) e^{-\frac{1}{\sqrt{h}}} + \frac{1}{\sqrt{h}} \leq C \eta(1) e^{-\frac{2h}{2h - h^{\kappa_2}}},$$

which is also bounded.

\[\square\]

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