Non-isothermal non-Newtonian flow problem with heat convection and Tresca’s friction law

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\textbf{ABSTRACT}

We consider an incompressible non-isothermal fluid flow with non-linear slip boundary conditions governed by Tresca’s friction law. We assume that the stress tensor is given as

$$\sigma = 2\mu(\theta, u, |D(u)|) |D(u)|^{p-2}D(u) - \pi \text{Id}$$

where $\theta$ is the temperature, $\pi$ is the pressure, $u$ is the velocity and $D(u)$ is the strain rate tensor of the fluid while $p$ is a real parameter. The problem is thus given by the $p$-Laplacian Stokes system with subdifferential type boundary conditions coupled to an $L^1$ elliptic equation describing the heat conduction in the fluid. We establish first an existence result for a family of approximate coupled problems where the $L^1$ coupling term in the heat equation is replaced by a bounded one depending on a parameter $0 < \delta < 1$, by using a fixed point technique. Then we pass to the limit as $\delta$ tends to zero and we prove the existence of a solution $(u, \pi, \theta)$ to our original coupled problem in Banach spaces depending on $p$ for any $p > 3/2$.

\textbf{1. Introduction}

Many industrial applications, like lubrication or extrusion/injection problems, involve complex fluids like molten polymers, oils, or colloidal fluids which do not satisfy the usual linear Newton’s relationship between the stress tensor, the strain rate tensor and the pressure and obey rather a power law of the form

$$\sigma = 2\mu(\theta, u, |D(u)|) |D(u)|^{p-2}D(u) - \pi \text{Id}$$

where $\theta$ is the temperature, $\pi$ is the pressure, $u$ is the velocity and $D(u)$ is the strain rate tensor of the fluid while $p$ is a real parameter. With $p > 2$ we obtain a description of dilatant (or shear thickening) fluids like colloidal fluids and for $p \in (1, 2)$ we recover pseudo-plastic (or shear thinning) fluids like molten polymers (see [1–3] for instance). It follows that both the velocity and the pressure of the fluid will belong to Banach spaces depending on $p$. For a constant function $\mu$ and classical boundary conditions like Dirichlet, Neumann or Navier boundary conditions several existence, uniqueness and regularity results have been already established (see [4,5] and the references therein or more recently [6,7] for unsteady flows).

When $p = 2$ we obtain a generalization of Navier–Stokes Newtonian fluids since the viscosity still depends on the temperature, the velocity and the modulus of the strain rate tensor via an appropriate choice of the function $\mu$ and it allows to consider fluids like oils (see [8] for instance).

Nevertheless several experimental studies have shown that such flows exhibit also non-standard behavior at the boundary with non-linear slip phenomena of friction type (see for instance [9,10]).
The first mathematical results with such boundary conditions have been proposed by H. Fujita et al. in [11–17] for stationary Stokes flows and developed later on for Navier–Stokes flows when $p = 2$ [18,19].

Since friction generates heat, thermal effects cannot be neglected and the energy conservation law leads to an elliptic equation in $L^1$ for the temperature. Moreover, the convection term yields to some compatibility condition between the regularity of the temperature and velocity of the fluid which both will belong to Banach spaces depending on the parameter $p$.

The rest of this paper is devoted to the study of this highly non-linear coupled fluid flow/heat transfer problem and to the proof of an existence result. In Section 2 we introduce the functional framework and starting from the conservation laws of mass, momentum and energy, we derive the mathematical formulation of the problem as a variational inequality describing the fluid flow coupled to an elliptic equation in $L^1$ describing heat conduction into the fluid. In Section 3 we study an auxiliary flow problem where the two first arguments of the viscosity mapping are considered as given parameters. Next in Section 4 we introduce an approximate linearized heat equation depending on a parameter $\delta$, with $0 < \delta << 1$, reminiscent of the technique proposed in [20] for elliptic equations with $L^1$ right-hand side. We establish uniform estimates of the solutions to the auxiliary decoupled fluid flow/heat transfer problems in appropriate Banach spaces. Then in Section 5 we prove the existence of a solution to an approximate coupled problem depending on the parameter $\delta$ by using a fixed point technique and finally we pass to the limit as $\delta$ tends to zero to obtain a solution $(u, \pi, \theta)$ to our original coupled problem.

Then we study in Section 3 the fluid flow problem for a given temperature and in Section 4 the heat transfer equation for a given fluid velocity. Finally we apply an iterative successive approximation technique to establish the existence of a solution $(u, \pi, \theta)$ to the coupled problem.

### 2. Description of the problem

Motivated by lubrication and extrusion/injection phenomena we consider an incompressible fluid flow in the domain

$$\Omega = \{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega, \ 0 < x_3 < h(x')\},$$

where $\omega$ is a non-empty bounded domain of $\mathbb{R}^2$ with a Lipschitz continuous boundary and $h$ is a Lipschitz continuous function which is bounded from above and from below by some positive real numbers.

In the stationary case the conservation of mass and momentum leads to the system

$$\begin{cases}
-\text{div}(\sigma) = f & \text{in } \Omega \\
\text{div}(u) = 0 & \text{in } \Omega 
\end{cases}$$

where $u$ is the velocity of the fluid, $\sigma$ is the stress tensor and $f$ represents the vector of external forces. In order to be able to consider a broad class of fluids like oils, polymers or colloidal fluids we introduce the following very general constitutive law depending on a real parameter $p \in (1, +\infty)$

$$\sigma = 2\mu(\theta, u, |D(u)|)|D(u)|^{p-2}D(u) - \pi \text{Id}_{\mathbb{R}^3}$$

where $\mu$ is a given function, $\theta$ is the temperature, $\pi$ is the pressure and $D(u) = (d_{ij}(u))_{1 \leq i,j \leq 3}$ is the strain rate tensor given by

$$d_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad 1 \leq i,j \leq 3.$$

With $p > 2$ we recover dilatant (or shear thickening) fluids like colloidal fluids and $p \in (1, 2)$ corresponds to pseudo-plastic (or shear thinning) fluids like molten polymers.
We assume that the fluid is subjected to non-homogeneous Dirichlet boundary conditions on a part of $\partial \Omega$ and to non-linear slip boundary conditions of friction type on the other part. More precisely we decompose $\partial \Omega$ as $\partial \Omega = \Gamma_0 \cup \Gamma_L \cup \Gamma_1$ with

$$
\Gamma_0 = \{(x', x_3) \in \overline{\Omega}: x_3 = 0\}, \quad \Gamma_1 = \{(x', x_3) \in \overline{\Omega}: x_3 = h(x')\}
$$

and $\Gamma_L$ is the lateral part of $\partial \Omega$.

We introduce a function $g: \partial \Omega \rightarrow \mathbb{R}^3$ such that

$$
\int_{\Gamma_L} g \cdot n \, dY = 0, \quad g = 0 \text{ on } \Gamma_1, \quad g \cdot n = 0 \text{ on } \Gamma_0, \quad g \neq 0 \text{ on } \Gamma_L,
$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal vector to $\partial \Omega$. We denote here $\nu \cdot w$ the Euclidian inner product in $\mathbb{R}^3$ of two vectors $\nu$ and $w$ and by $| \cdot |$ the Euclidian norm. We define by $u_n = u \cdot n$ and $u_\tau = u - u_n n$ the normal and tangential velocities on $\partial \Omega$. The normal and tangential components of the stress vector on $\partial \Omega$ are given by $\sigma_n$ and $\sigma_\tau$ with

$$
\sigma_n = \sum_{i,j=1}^{3} \sigma_{ij} n_j n_i, \quad \sigma_\tau = \left( \sum_{j=1}^{3} \sigma_{ij} n_j - \sigma_n n_i \right)_{1 \leq i \leq 3}.
$$

As usual in lubrication or extrusion/injection problems the upper part of the boundary is fixed while the lower part is moving with a velocity $s: \Gamma_0 \rightarrow \mathbb{R}^2 \times \{0\}$. So we have

$$
u = 0 \text{ on } \Gamma_1, \quad \nu = g \text{ on } \Gamma_L
$$

and the slip condition

$$u_n = 0 \text{ on } \Gamma_0
$$

combined with a subdifferential condition for the relative velocity $u_\tau - s$ on $\Gamma_0$ given by

$$u_\tau - s \in -\partial \psi_{\mathbb{R}^3}(0,k)(\sigma_\tau)
$$

where $k: \Gamma_0 \rightarrow \mathbb{R}^+_0$ is the friction threshold, $\psi_{\mathbb{R}^3}(0,k)$ is the indicatrix function of the closed ball $\overline{B}_{\mathbb{R}^3}(0,k)$ and $\partial \psi_{\mathbb{R}^3}(0,k)$ its subdifferential (see [21]). This inclusion means that the shear stress $\sigma_\tau$ remains bounded by $k$ and slip occurs only when the threshold $k$ is reached, with a relative velocity in the opposite direction to $\sigma_\tau$ i.e. $u_\tau$ is unknown and satisfies Tresca’s friction law [22]:

$$
\begin{cases}
|\sigma_\tau| < k \Rightarrow u_\tau = s \\
|\sigma_\tau| = k \Rightarrow \exists \lambda \geq 0 \quad u_\tau = s - \lambda \sigma_\tau
\end{cases}
$$

on $\Gamma_0$.

Since friction generates heat we have to take also into account thermal effects in the fluid. By using Fourier’s law for the heat flux we get the following heat equation

$$u \cdot \nabla \theta - \text{div}(K \nabla \theta) = 2\mu(\theta, u, |D(u)|)|D(u)|^p + r(\theta) \quad \text{in } \Omega
$$

where $K$ is the thermal conductivity tensor and $r$ is a real function. We assume mixed Dirichlet–Neumann boundary conditions on $\Gamma_1 \cup \Gamma_L$ and $\Gamma_0$ i.e.

$$\theta = 0 \text{ on } \Gamma_1 \cup \Gamma_L, \quad (K \nabla \theta) \cdot n = \theta^b \text{ on } \Gamma_0
$$

where $\theta^b$ is a given heat flux on $\Gamma_0$. 
Let us introduce now the functional framework. We denote here and throughout this paper by $X$ the functional space $X^3$ and we define

$$W_{Γ_1 ∪ Γ_L}^{1,q}(Ω) = \{ φ ∈ W^{1,q}(Ω) : φ = 0 \text{ on } Γ_1 ∪ Γ_L \} \quad ∀ q ≥ 1$$

endowed with the norm

$$∥φ∥_{1,q} = \left( ∫_Ω |∇φ|^q \, dx \right)^{1/q} \quad ∀ φ ∈ W_{Γ_1 ∪ Γ_L}^{1,q}(Ω)$$

and

$$V_p^0 = \{ φ ∈ W_{Γ_1 ∪ Γ_L}^{1,p}(Ω) : φ · n = 0 \text{ on } Γ_0 \},$$

$$V_p^{0, \text{div}} = \{ φ ∈ V_p^0 : \text{div}(φ) = 0 \text{ in } Ω \} \quad ∀ p > 1.$$ 

Let $p > 1$. We assume that $g$ admits an extension $G$ to $Ω$ such that

$$G ∈ W^{1,p}(Ω), \quad \text{div}(G) = 0 \text{ in } Ω, \quad G = 0 \text{ on } Γ_1, \quad G · n = 0 \text{ on } Γ_0 \quad (9)$$

and that the data $f$, $s$ and $k$ satisfy

$$f ∈ L^{p'}(Ω), \quad s ∈ L^p(Γ_0), \quad k ∈ L^{p'}_+(Γ_0) \quad (10)$$

where $p' = \frac{p}{p-1}$ is the conjugate number of $p$. Moreover we assume that the mapping $μ$ satisfies:

$$(o,e,d) ↦ μ(o,e,d)$$ is continuous on $ℝ × ℝ^3 × ℝ_+$, \quad (11)

$$d ↦ μ(.,.,d)$$ is monotone increasing on $ℝ_+$, \quad (12)

$$∃(μ_0,μ_1) ∈ ℝ^2 \text{ s.t. } 0 < μ_0 ≤ μ(o,e,d) ≤ μ_1 \forall (o,e,d) ∈ ℝ × ℝ^3 × ℝ_+.$$

For all $θ ∈ L^q(Ω)$ with $q ≥ 1$ we let

$$a(θ;v,φ) = ∫_Ω F(θ,v + G,D(v + G)) : D(φ) \, dx \quad ∀ (v,φ) ∈ (V_0^p)^2$$

where $F : ℝ × ℝ^3 × ℝ^{3×3} → ℝ^{3×3}$ is defined by

$$F(λ_0,λ_1,λ_2) = \begin{cases} 2μ(λ_0,λ_1,|λ_2|)|λ_2|^{p-2}λ_2 & \text{if } λ_2 ≠ 0_{ℝ^{3×3}}, \\ 0_{ℝ^{3×3}} & \text{otherwise.} \end{cases} \quad (14)$$

With (13) we have immediately

$$|F(λ_0,λ_1,λ_2)| ≤ 2μ_1|λ_2|^p \quad ∀ (λ_0,λ_1,λ_2) ∈ ℝ × ℝ^3 × ℝ^{3×3}. $$

So $F$ is continuous on $ℝ × ℝ^3 × ℝ^{3×3}$ and the mapping $a(θ;·,·)$ is well defined and continuous on $V_0^p × V_0^p$ for all $p > 1$.

Then we introduce the new unknown function $v = u - G$ in order to deal with homogeneous boundary conditions on $Γ_1 ∪ Γ_L$ and the flow problem (1)-(2) with the boundary conditions (4)-(5)-(6) admits the following variational formulation ([22]):
Find \( \nu \in V^p_{0, \text{div}} \) and \( \pi \in L^p' (\Omega) \) such that

\[
a(\theta; \nu, \varphi - \nu) - \int_\Omega \pi \text{ div} (\varphi) \, dx + \Psi (\varphi) - \Psi (\nu) \geq \int_\Omega f \cdot (\varphi - \nu) \, dx \quad \forall \varphi \in V^p_0
\]

with

\[
\Psi (\varphi) = \int_{\Gamma_0} k |\varphi + G - s| \, dx' \quad \forall \varphi \in V^p_0.
\]

Next we observe that the first term in the right-hand side of the heat equation (7) belongs to \( L^1 (\Omega) \) whenever \( u = \nu + G \in W^{1,p}_{11 \Gamma_1} (\Omega) \). It follows that we may expect only \( \theta \in W^{1,q}_{11 \Gamma_1 \cup \Gamma_L} (\Omega) \) with \( 1 \leq q < 3/2 \) and the convection term \( u \cdot \nabla \theta \) in the left-hand side of (7) belongs to \( L^1 (\Omega) \) only if \( u \in L^{p_*} (\Omega) \) with \( p_* \geq q' \) where \( q' \) is the conjugate number of \( q \). Since \( W^{1,p}_{11 \Gamma_1} (\Omega) \subset L^{p_*} (\Omega) \) with \( p_* = \frac{3p}{3 - p} \) if \( 3 > p > 1 \), any real \( p_* \) if \( p = 3 \) and \( p_* = +\infty \) if \( p > 3 \), we get \( \frac{3}{2} > q \geq \frac{3p}{4p - 3} \) if \( p < 3 \) which implies that \( p \) should be greater than \( \frac{3}{2} \). Thus we obtain a compatibility condition on \( p \) and \( q \) and we look for \( \theta \in W^{1,q}_{11 \Gamma_1 \cup \Gamma_L} (\Omega) \) with

\[
q \in \left[ 1, \frac{3}{2} \right] \text{ if } p > 3, \quad q \in \left( 1, \frac{3}{2} \right) \text{ if } p = 3 \quad \text{and } q \in \left( \frac{3p}{4p - 3}, \frac{3}{2} \right) \text{ if } \frac{3}{2} < p < 3
\]

or equivalently

\[
p > 3 \quad \text{if } q = 1 \quad \text{and } p \geq \frac{3q}{4q - 3} \quad \text{if } q \in \left( 1, \frac{3}{2} \right).
\]

**Remark 2.1:** We may observe that \( t \mapsto \frac{3q}{4q - 3} \) is a decreasing function of \( t \) on \([1, +\infty)\). \( \frac{3}{2} \). So, as expected intuitively, the more regular the temperature, the less regular the fluid velocity. \( \nu + G \in W^{1,3/2}_{11 \Gamma_1} (\Omega) \).

As usual we assume that the thermal conductivity tensor satisfies

\[
K \in (L^\infty (\Omega))^{3 \times 3}
\]

and

\[
\exists k_0 > 0 \quad \text{s.t.} \quad \sum_{i,j=1}^3 K_{ij}(x) \xi_i \xi_j \geq k_0 \sum_{i=1}^3 |\xi_i|^2 \quad \forall \xi \in \mathbb{R}^3, \text{ for a.e. } x \in \Omega.
\]

We assume also that

\[
\text{the mapping } r : \mathbb{R} \to \mathbb{R} \text{ is continuous and uniformly bounded}
\]

and

\[
\theta^b \in L^1 (\Gamma_0).
\]

Then we can derive a weak formulation of the heat equation (7) with the boundary conditions (8) by choosing a test-function \( w \in C^1 (\overline{\Omega}) \) such that \( w = 0 \) on \( \Gamma_1 \cup \Gamma_L \) (see for instance [23]).

Finally for any \( p > 3/2 \) and \( q \in [1, 3/2] \) satisfying (15) the weak formulation of the coupled heat-flow problem (1) – (2) – (4) – (5) – (6) and (7) – (8) is given by
Lemma 3.1: Let $p > 1$. Assume that (12) and (13) hold. Then the mapping $\lambda_2 \mapsto F(\cdot, \cdot, \lambda_2)$ is monotone on $\mathbb{R}^{3 \times 3}$.

Proof: Let us prove now that

$$
(F(\lambda_0, \lambda_1, \lambda_2) - F(\lambda_0, \lambda_1, \lambda_2')) : (\lambda_2 - \lambda_2') \geq 0 \quad \forall (\lambda_2, \lambda_2') \in (\mathbb{R}^{3 \times 3})^2, \quad \forall (\lambda_0, \lambda_1) \in \mathbb{R} \times \mathbb{R}^3.
$$

With (13) the result is obvious if $\lambda_2 = 0_{\mathbb{R}^{3 \times 3}}$ and/or $\lambda_2' = 0_{\mathbb{R}^{3 \times 3}}$. So let us assume from now on that $\lambda_2 \neq 0_{\mathbb{R}^{3 \times 3}}$ and $\lambda_2' \neq 0_{\mathbb{R}^{3 \times 3}}$. We distinguish two cases.

Case 1: $p \geq 2$.

Let

$$
F_1 = \mu(\lambda_0, \lambda_1, |\lambda_2|)|\lambda_2|^{p-2}, \quad F_2 = \mu(\lambda_0, \lambda_1, |\lambda_2'|)|\lambda_2'|^{p-2}.
$$

We have

$$
(F(\lambda_0, \lambda_1, \lambda_2) - F(\lambda_0, \lambda_1, \lambda_2')) : (\lambda_2 - \lambda_2') = 2F_1|\lambda_2|^2 - 2(F_1 + F_2)|\lambda_2| + 2F_2|\lambda_2'|^2.
$$

Observing that $F_1$ and $F_2$ are non-negative and $2\lambda_2 : \lambda_2' \leq |\lambda_2|^2 + |\lambda_2'|^2$, we get

$$
(F(\lambda_0, \lambda_1, \lambda_2) - F(\lambda_0, \lambda_1, \lambda_2')) : (\lambda_2 - \lambda_2') \\
\geq 2F_1|\lambda_2|^2 - (F_1 + F_2)(|\lambda_2|^2 + |\lambda_2'|^2) + 2F_2|\lambda_2'|^2 = (F_1 - F_2)(|\lambda_2|^2 - |\lambda_2'|^2).
$$

3. Fluid flow auxiliary problem

In this section we consider the following auxiliary flow problem

$$
\begin{align*}
\{ & \text{Find } \tilde{\nu} \in V^p_{0, \text{div}} \text{ such that} \\
& \langle A_{\theta, \nu}(\tilde{\nu}), \varphi - \tilde{\nu} \rangle + \Psi(\varphi) - \Psi(\tilde{\nu}) \geq \int_{\Omega} f \cdot (\varphi - \tilde{\nu}) \, dx \quad \forall \varphi \in V^p_{0, \text{div}} \tag{21} \}
\end{align*}
$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket between $V^p_{0, \text{div}}$ and $(V^p_{0, \text{div}})'$ and $A_{\theta, \nu} : V^p_{0, \text{div}} \to (V^p_{0, \text{div}})'$ is the operator defined by

$$
\langle A_{\theta, \nu}(\tilde{\nu}), \varphi \rangle = \int_{\Omega} F(\theta, \nu + G, D(\tilde{\nu} + G)) : D(\varphi) \, dx \quad \forall (\tilde{\nu}, \varphi) \in V^p_{0, \text{div}} \times V^p_{0, \text{div}}
$$

for a given temperature $\theta \in L^q(\Omega)$ and a given velocity $\nu \in L^p(\Omega)$, with $q \geq 1$ and $p > 1$. 


and by replacing $F_1$ and $F_2$

$$(\mathcal{F}(\lambda_0, \lambda_1, \lambda_2) - \mathcal{F}(\lambda_0, \lambda_1, \lambda'_2)) : (\lambda_2 - \lambda'_2)$$

$$\geq (\mu(\lambda_0, \lambda_1, |\lambda_2|)|\lambda_2|^{p-2} - \mu(\lambda_0, \lambda_1, |\lambda'_2|)|\lambda'_2|^{p-2})(|\lambda_2|^2 - |\lambda'_2|^2)$$

$$= |\lambda_2|^{p-2}(\mu(\lambda_0, \lambda_1, |\lambda_2|) - \mu(\lambda_0, \lambda_1, |\lambda'_2|))(|\lambda_2|^2 - |\lambda'_2|^2)$$

$$+ \mu(\lambda_0, \lambda_1, |\lambda'_2|)(|\lambda_2|^{p-2} - |\lambda'_2|^{p-2})(|\lambda_2|^2 - |\lambda'_2|^2).$$

Finally, using (12), (13) and the monotonicity of the function $w \mapsto w^{p-2}$ on $\mathbb{R}^+$ when $p \geq 2$, we infer that

$$(\mathcal{F}(\lambda_0, \lambda_1, \lambda_2) - \mathcal{F}(\lambda_0, \lambda_1, \lambda'_2)) : (\lambda_2 - \lambda'_2) \geq 0.$$

**Case 2:** $1 < p < 2$

Let us denote now

$$\alpha = 2\mu(\lambda_0, \lambda_1, |\lambda_2|), \quad \beta = 2\mu(\lambda_0, \lambda_1, |\lambda'_2|).$$

We have

$$(\mathcal{F}(\lambda_0, \lambda_1, \lambda_2) - \mathcal{F}(\lambda_0, \lambda_1, \lambda'_2)) : (\lambda_2 - \lambda'_2)$$

and using $2\lambda_2 : \lambda'_2 = |\lambda_2|^2 + |\lambda'_2|^2 - |\lambda_2 - \lambda'_2|^2$ we obtain

$$(\mathcal{F}(\lambda_0, \lambda_1, \lambda_2) - \mathcal{F}(\lambda_0, \lambda_1, \lambda'_2)) : (\lambda_2 - \lambda'_2)$$

$$= \frac{1}{2}((\alpha|\lambda_2|^{p-2} + \beta|\lambda'_2|^{p-2})|\lambda_2 - \lambda'_2|^2 + (|\lambda_2|^2 - |\lambda'_2|^2)(\alpha|\lambda_2|^{p-2} - \beta|\lambda'_2|^{p-2})).$$

Without loss of generality we may assume that $|\lambda_2| \geq |\lambda'_2| > 0$. With the triangle inequality $|\lambda_2 - \lambda'_2| \geq |\lambda_2| - |\lambda'_2| \geq 0$, we get

$$(\mathcal{F}(\lambda_0, \lambda_1, \lambda_2) - \mathcal{F}(\lambda_0, \lambda_1, \lambda'_2)) : (\lambda_2 - \lambda'_2)$$

$$\geq \frac{1}{2}((\alpha|\lambda_2|^{p-2} + \beta|\lambda'_2|^{p-2})(|\lambda_2| - |\lambda'_2|)^2 + (|\lambda_2|^2 - |\lambda'_2|^2)(\alpha|\lambda_2|^{p-2} - \beta|\lambda'_2|^{p-2}))$$

$$= \frac{1}{2}(|\lambda_2| - |\lambda'_2|)((\alpha|\lambda_2|^{p-2} + \beta|\lambda'_2|^{p-2})(|\lambda_2| - |\lambda'_2|) + (|\lambda_2| + |\lambda'_2|)(\alpha|\lambda_2|^{p-2} - \beta|\lambda'_2|^{p-2})).$$

Let $t = \frac{|\lambda_2|}{|\lambda'_2|} \geq 1$. Then

$$(\mathcal{F}(\lambda_0, \lambda_1, \lambda_2) - \mathcal{F}(\lambda_0, \lambda_1, \lambda'_2)) : (\lambda_2 - \lambda'_2)$$

$$\geq \frac{1}{2}|\lambda'_2|^p(t - 1)((\alpha t^{p-2} + \beta)(t - 1) + (t + 1)(\alpha t^{p-2} - \beta))$$

$$= |\lambda'_2|^p(t - 1)(\alpha t^{p-1} - \beta)$$

and from (12) we obtain

$$0 < \beta = 2\mu(\lambda_0, \lambda_1, |\lambda'_2|) \leq 2\mu(\lambda_0, \lambda_1, |\lambda_2|) = \alpha.$$

It follows that $\alpha t^{p-1} \geq \alpha \geq \beta$ and finally

$$(\mathcal{F}(\lambda_0, \lambda_1, \lambda_2) - \mathcal{F}(\lambda_0, \lambda_1, \lambda'_2)) : (\lambda_2 - \lambda'_2) \geq 0.$$
By using the monotonicity of $F$ we obtain

**Corollary 3.2:** Let $\theta \in L^q(\Omega)$ with $q \geq 1$ and $\nu \in L^p(\Omega)$ with $p > 1$. Assume that (9), (12)–(13) hold. Then the operator $A_{\theta,\nu}$ is bounded and pseudo-monotone.

**Proof:** The proof is a straightforward adaptation of standard results (see for instance Chapter 8, Theorem 8.9 in [24]) and is left to the reader. ■

Next we prove an existence and uniqueness result for (21).

**Theorem 3.3:** Let $\theta \in L^q(\Omega)$ with $q \geq 1$ and $\nu \in L^p(\Omega)$ with $p > 1$. Assume that (9)–(10), (12)–(13) hold. Then problem (21) admits a unique solution.

**Proof:** The functional $\Psi$ is convex, proper and lower semi-continuous on $V^p_{0, \text{div}}$ and with Corollary 3.2 we already know that the operator $A_{\theta,\nu}$ is bounded and pseudo-monotone. Moreover $A_{\theta,\nu}$ is coercive i.e there exists $\tilde{v}^* \in V^p_{0, \text{div}}$ such that $\Psi(\tilde{v}^*) < +\infty$ and

$$\lim_{\|\tilde{v}\|_{1,p} \to +\infty} \frac{\langle A_{\theta,\nu}(\tilde{v}), \tilde{v} - \tilde{v}^* \rangle + \Psi(\tilde{v})}{\|\tilde{v}\|_{1,p}} = +\infty.$$ 

Indeed let us choose $\tilde{v}^* = 0$. We have

$$\langle A_{\theta,\nu}, \tilde{v}, \tilde{v} \rangle = \int_\Omega F(\theta, \nu + G, D(\tilde{v} + G)) : D(\tilde{v} + G) \, dx$$

and with (13)

$$\langle A_{\theta,\nu}, \tilde{v}, \tilde{v} \rangle \geq 2\mu_0 \int_\Omega |D(\tilde{v} + G)|^p \, dx - 2\mu_1 \int_\Omega |D(\tilde{v} + G)|^{p-1} |D(G)| \, dx$$

$$\geq 2\mu_0 \|D(\tilde{v} + G)\|_{(L^p(\Omega))^3 \times 3}^p - 2\mu_1 \|D(G)\|_{(L^p(\Omega))^3 \times 3}^{p-1} \|G\|_{1,p}$$

$$\geq 2\mu_0 \|D(\tilde{v})\|_{(L^p(\Omega))^3 \times 3}^p - 2\mu_1 \|D(G)\|_{(L^p(\Omega))^3 \times 3}^{p-1} - 2\mu_1 \|G\|_{1,p} \|G\|_{1,p}$$

(22)

So, whenever $\|\tilde{v}\|_{1,p} \neq 0$, we obtain that

$$\|\tilde{v}\|_{1,p}^{-p-1} \left(2\mu_0 \left(\frac{|D(\tilde{v})|_{(L^p(\Omega))^3 \times 3}}{\|\tilde{v}\|_{1,p}}\right)^p - 2\mu_1 \left(1 + \frac{\|G\|_{1,p}}{\|\tilde{v}\|_{1,p}}\right)^{p-1} \frac{\|G\|_{1,p}}{\|\tilde{v}\|_{1,p}}\right).$$

By Korn’s inequality [25], there exists $C_{\text{Korn}} > 0$ such that

$$\|D(u)\|_{(L^p(\Omega))^3 \times 3} = \left(\int_\Omega |D(u)|^p \, dx\right)^{\frac{1}{p}} \geq C_{\text{Korn}} \|u\|_{1,p} \quad \forall u \in V^p_{0, \text{div}}. $$

(23)

Recalling that $\Psi(\tilde{v}) \geq 0$ for all $\tilde{v} \in V^p_{0, \text{div}}$ and $p > 1$ we get

$$\lim_{\|\tilde{v}\|_{1,p} \to +\infty} \frac{\langle A_{\theta,\nu}, \tilde{v}, \tilde{v} \rangle + \Psi(\tilde{v})}{\|\tilde{v}\|_{1,p}} = +\infty.$$ 

Then by applying monotonicity arguments (see Chapter 2, Theorem 8.5 in [4]) we infer that problem (21) admits at least a solution.
Let us prove its uniqueness by a contradiction argument. So let \( \tilde{v}_1 \in \mathcal{V}_{0,\text{div}}^p \) and \( \tilde{v}_2 \in \mathcal{V}_{0,\text{div}}^p \) be two solutions of (21). By choosing \( \varphi = \tilde{v}_2 \) then \( \varphi = \tilde{v}_1 \) we obtain

\[
\langle \mathbf{A}_{\theta, u}(\tilde{v}_1) - \mathbf{A}_{\theta, u}(\tilde{v}_2), \tilde{v}_1 - \tilde{v}_2 \rangle \leq 0. \tag{24}
\]

But

\[
\langle \mathbf{A}_{\theta, u}(\tilde{v}_1) - \mathbf{A}_{\theta, u}(\tilde{v}_2), \tilde{v}_1 - \tilde{v}_2 \rangle \\
= \mu_0 \int_{\Omega} \left( (|D(\tilde{v}_1 + G)|^{p-2}D(\tilde{v}_1 + G) - |D(\tilde{v}_2 + G)|^{p-2}D(\tilde{v}_2 + G)) : D(\tilde{v}_1 - \tilde{v}_2) \right) dx \\
+ \int_{\Omega} \left( \mathcal{F}(\theta, u + G, D(\tilde{v}_1 + G)) - \mathcal{F}(\theta, u + G, D(\tilde{v}_2 + G)) \right) : D(\tilde{v}_1 - \tilde{v}_2) dx \tag{25}
\]

where

\[
\mathcal{F}(\lambda_0, \lambda_1, \lambda_2) = 2\mu(\lambda_0, \lambda_1, \lambda_2)|\lambda_2|^{p-2}\lambda_2 \text{ if } \lambda_2 \neq 0_{\mathbb{R}^{3 \times 3}}, \quad \mathcal{F}(\lambda_0, \lambda_1, \lambda_2) = 0_{\mathbb{R}^{3 \times 3}} \text{ otherwise}
\]

and \( \bar{\mu} = \mu - \frac{\mu_0}{2} \). Since \( \bar{\mu} \) satisfies

\[
d \mapsto \bar{\mu}(\cdot, \cdot, d) \text{ is monotone increasing on } \mathbb{R}_+,
\]

\[
0 < \frac{\mu_0}{2} \leq \bar{\mu}(o, e, d) \leq \mu_1 - \frac{\mu_0}{2} \text{ for all } (o, e, d) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}_+,
\]

we infer with the same arguments as in Lemma 3.1 that \( \lambda_2 \mapsto \mathcal{F}(\cdot, \cdot, \lambda_2) \) is monotone on \( \mathbb{R}^{3 \times 3} \). Similarly we define

\[
\mathcal{F}_0(\lambda_0, \lambda_1, \lambda_2) = \mu|\lambda_2|^{p-2}\lambda_2 \text{ if } \lambda_2 \neq 0_{\mathbb{R}^{3 \times 3}}, \quad \mathcal{F}_0(\lambda_0, \lambda_1, \lambda_2) = 0_{\mathbb{R}^{3 \times 3}} \text{ otherwise}
\]

and we obtain that \( \lambda_2 \mapsto \mathcal{F}_0(\cdot, \cdot, \lambda_2) \) is monotone on \( \mathbb{R}^{3 \times 3} \). Hence

\[
\int_{\Omega} \left( \mathcal{F}(\theta, u + G, D(\tilde{v}_1 + G)) - \mathcal{F}(\theta, u + G, D(\tilde{v}_2 + G)) \right) : D(\tilde{v}_1 - \tilde{v}_2) dx \geq 0
\]

and

\[
\mu_0 \int_{\Omega} \left( (|D(\tilde{v}_1 + G)|^{p-2}D(\tilde{v}_1 + G) - |D(\tilde{v}_2 + G)|^{p-2}D(\tilde{v}_2 + G)) : D(\tilde{v}_1 - \tilde{v}_2) \right) dx \geq 0.
\]

With (24) – (25) we infer that

\[
\int_{\Omega} \left( |D(\tilde{v}_1 + G)|^{p-2}D(\tilde{v}_1 + G) - |D(\tilde{v}_2 + G)|^{p-2}D(\tilde{v}_2 + G) \right) : D(\tilde{v}_1 - \tilde{v}_2) dx = 0 \tag{26}
\]

Let us distinguish now two cases.

**Case 1**: \( p \geq 2 \).
For all \((\lambda, \lambda') \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3}\) we have

\[
(|\lambda|^{p-2} \lambda - |\lambda'|^{p-2} \lambda') : (\lambda - \lambda') \geq \frac{1}{2^{p-1}} |\lambda - \lambda'|^p.
\]  

(27)

Indeed if \(\lambda = \lambda' = 0\) the result is obvious. Otherwise with the same kind of computations as in Lemma 3.1 we get

\[
(|\lambda|^{p-2} \lambda - |\lambda'|^{p-2} \lambda') : (\lambda - \lambda') = \frac{1}{2} \left( |\lambda|^{p-2} + |\lambda'|^{p-2} \right) |\lambda - \lambda'|^2 - \frac{1}{2} (-|\lambda|^{p-2} + |\lambda'|^{p-2}) (|\lambda|^{p-2} - |\lambda'|^{p-2})
\]

\[
\geq \frac{1}{2} (|\lambda|^{p-2} + |\lambda'|^{p-2}) |\lambda - \lambda'|^2.
\]

Since \(|\lambda| + |\lambda'| \neq 0\) we obtain

\[
(|\lambda|^{p-2} \lambda - |\lambda'|^{p-2} \lambda') : (\lambda - \lambda') \geq \frac{1}{2} \left( |\lambda|^{p-2} + |\lambda'|^{p-2} \right) (|\lambda| + |\lambda'|)^{p-2} |\lambda - \lambda'|^2
\]

\[
\geq \frac{1}{2} (|\lambda|^{p-2} + |\lambda'|^{p-2}) |\lambda - \lambda'|^p.
\]

But

\[
\left( \frac{|\lambda| + |\lambda'|}{2} \right)^{p-2} \leq (\max(|\lambda|, |\lambda'|))^{p-2} \leq |\lambda|^{p-2} + |\lambda'|^{p-2}
\]

and thus

\[
(|\lambda|^{p-2} \lambda - |\lambda'|^{p-2} \lambda') : (\lambda - \lambda') \geq \frac{1}{2^{p-1}} |\lambda - \lambda'|^p.
\]

By replacing \(\lambda = D(\tilde{v}_1 + G), \lambda' = D(\tilde{v}_2 + G)\) we infer from (26) that

\[
0 = \int_{\Omega} (|D(\tilde{v}_1 + G)|^{p-2} D(\tilde{v}_1 + G) - |D(\tilde{v}_2 + G)|^{p-2} D(\tilde{v}_2 + G)) : D(\tilde{v}_1 - \tilde{v}_2) \, dx
\]

\[
\geq \frac{1}{2^{p-1}} \|D(\tilde{v}_1 - \tilde{v}_2)\|_{L^p(\Omega)^{3 \times 3}}^p.
\]

and with Korn’s inequality we may conclude that \(\tilde{v}_1 = \tilde{v}_2\).

Case 2: \(1 < p < 2\).

In this case we have

\[
(|\lambda| + |\lambda'|)^{2-p} (|\lambda|^{p-2} \lambda - |\lambda'|^{p-2} \lambda') : (\lambda - \lambda') \geq (p - 1) |\lambda - \lambda'|^2
\]

(28)

for all \((\lambda, \lambda') \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3}\). Indeed if \(|\lambda| = |\lambda'|\) we have

\[
(|\lambda| + |\lambda'|)^{2-p} (|\lambda|^{p-2} \lambda - |\lambda'|^{p-2} \lambda') : (\lambda - \lambda') = 2^{2-p} |\lambda - \lambda'|^2
\]

and the conclusion follows from the inequality \(2^{2-p} > p - 1\) for all \(p \in (1, 2)\). Otherwise, if \(|\lambda| \neq |\lambda'|\) we let

\[
G(\lambda, \lambda') = \frac{(|\lambda| + |\lambda'|)^{2-p} (|\lambda|^{p-2} \lambda - |\lambda'|^{p-2} \lambda') : (\lambda - \lambda')}{|\lambda - \lambda'|^2}.
\]
With the same computations as in Lemma 3.1 we have
\[
G(\lambda, \lambda') = \frac{1}{2} \frac{(|\lambda| + |\lambda'|)^{2-p}}{|\lambda - \lambda'|^2} \times \\
	imes ((|\lambda|^{p-2} + |\lambda'|^{p-2})|\lambda - \lambda'|^2 + (|\lambda|^2 - |\lambda'|^2)(|\lambda|^{p-2} - |\lambda'|^{p-2})) \\
\geq \frac{1}{2} \frac{(\lambda + |\lambda'|)^{2-p}}{|\lambda - \lambda'|^2} \left( (|\lambda|^{p-2} + |\lambda'|^{p-2}) + \frac{(|\lambda| + |\lambda'|)(|\lambda|^{p-2} - |\lambda'|^{p-2})}{|\lambda| - |\lambda'|} \right)
\]

Without loss of generality we may assume that $|\lambda| > |\lambda'|$ and we let $t = \frac{|\lambda|}{|\lambda'|} > 1$. Thus
\[
G(\lambda, \lambda') \geq \frac{(1 + t)^{2-p}}{2} \left( (1 + t^{p-2}) + \frac{(1 + t)(t^{p-2} - 1)}{|t - 1|} \right) \\
= \frac{(1 + t)^{2-p}(t^{p-1} - 1)}{t - 1} = \frac{(p - 1)^p}{t - 1}
\]

But, for all $t > 1$ we have $\frac{t^{p-1} - 1}{t - 1} < 2 - p$ and (28) is satisfied.

Hence
\[
((|\lambda| + |\lambda'|)^{2-p}) \geq (\lambda - \lambda')^p : (\lambda - \lambda')^2 \geq (p - 1)^p |\lambda - \lambda'|^p.
\]

Since $p > 1$ we have also
\[
(|\lambda| + |\lambda'|)^p \leq 2^{p-1}(|\lambda|^p + |\lambda'|^p)
\]
which yields
\[
(|\lambda|^p + |\lambda'|^p)^{\frac{2-p}{p}} \geq (\lambda - \lambda')^p : (\lambda - \lambda') \geq C_p |\lambda - \lambda'|^p \geq 0
\]
for all $(\lambda, \lambda') \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3}$, with $C_p = \frac{(p-1)^p}{2^{p-1}(p-2)}$. By replacing $\lambda = D(\bar{\nu}_1 + G), \lambda' = D(\bar{\nu}_2 + G)$ and using Hölder’s inequality we obtain:
\[
C_p \int_{\Omega} |D(\bar{\nu}_1 - \bar{\nu}_2)|^p \, dx \\
\leq \left( \int_{\Omega} (|D(\bar{\nu}_1 + G)|^{p-2}D(\bar{\nu}_1 + G) - |D(\bar{\nu}_2 + G)|^{p-2}D(\bar{\nu}_2 + G)) : D(\bar{\nu}_1 - \bar{\nu}_2) \, dx \right)^{\frac{p}{2}} \\
\times \left( \int_{\Omega} (|D(\bar{\nu}_1 + G)|^p + |D(\bar{\nu}_2 + G)|^p \, dx \right)^{\frac{2-p}{2}}.
\]

With (26) we get
\[
C_p \|D(\bar{\nu}_1 - \bar{\nu}_2)\|_{L^p(\Omega)}^p = C_p \int_{\Omega} |D(\bar{\nu}_1 - \bar{\nu}_2)|^p \, dx \leq 0
\]
and we may conclude once again that $\bar{\nu}_1 = \bar{\nu}_2$.

Having in mind the study of the coupled problem (P) we establish now a priori estimates, independent of the given temperature and velocity $(\theta, \nu)$ for the solutions of the flow problem (21).
Proposition 3.4: Under the assumptions of Theorem 3.3, there exists a positive real number $C_{\text{flow}}$, independent of $\theta$ and $\nu$ and depending only on $f$, $G$, $k$, $s$, $\mu_0$ and $\mu_1$, such that the solution $\tilde{u}$ of problem (21) satisfies

$$\|\tilde{u}\|_{1,p} \leq C_{\text{flow}}. \quad (29)$$

Proof: Let $\tilde{u}$ be a solution of problem (21). With $\varphi = 0$ we obtain

$$\langle A_{\theta,\nu}\tilde{u}, \tilde{u}\rangle \leq \langle A_{\theta,\nu}\tilde{u}, \tilde{u}\rangle + \Psi(\tilde{u}) \leq \int_\Omega f \cdot \tilde{u} \, dx + \Psi(0).$$

By using (22) and Poincaré’s inequality we get

$$2\mu_0 \|D(\tilde{u})\|_{(L^p(\Omega))^{3 \times 3}} - \|D(G)\|_{(L^p(\Omega))^{3 \times 3}}\|^p \leq C_p \|f\|_{L^p(\Omega)} \|\tilde{u}\|_{1,p} + \Psi(0) + 2\mu_1 (\|\tilde{u}\|_{1,p} + \|G\|_{1,p})^{p-1} \|G\|_{1,p},$$

where $C_p$ denotes the Poincaré’s constant in $W^{1,p}_{\text{div}}(\Omega)$. If $\|\tilde{u}\|_{1,p} \neq 0$ we obtain

$$C_{\text{Korn}} \leq \left[ \frac{C_p}{2\mu_0} \frac{\|f\|_{L^p(\Omega)}}{t^{1-p}} \|\tilde{u}\|_{1,p} + \frac{\Psi(0)}{t^p} \|\tilde{u}\|_{1,p} + \frac{2\mu_1}{t} \left( 1 + \frac{\|G\|_{1,p}}{t} \right)^{p-1} \|G\|_{1,p} \right]^{1/p}$$

$$+ \frac{\|D(G)\|_{(L^p(\Omega))^{3 \times 3}}}{\|\tilde{u}\|_{1,p}}.$$ \quad (30)

By observing that the mapping

$$\Lambda : t \mapsto \left( \frac{1}{2\mu_0} \frac{C_p}{t^{1-p}} \|f\|_{L^p(\Omega)} + \frac{\Psi(0)}{t^p} \|\tilde{u}\|_{1,p} + \frac{2\mu_1}{t} \left( 1 + \frac{\|G\|_{1,p}}{t} \right)^{p-1} \|G\|_{1,p} \right)$$

$$- \frac{\|D(G)\|_{(L^p(\Omega))^{3 \times 3}}}{t} - C_{\text{Korn}}$$

tends to $-C_{\text{Korn}} < 0$ as $t$ tends to $+\infty$ we infer that there exists $C_{\text{flow}} > 0$ such that $\Lambda(t) < 0$ for all $t > C_{\text{flow}}$. With (30) it follows that

$$\|\tilde{u}\|_{1,p} \leq C_{\text{flow}}. \quad \Box$$

Remark 3.1: The reader may wonder why we have not considered the fluid flow problem for a given temperature i.e. the following variational inequality

$$\left\{ \begin{array}{ll}
\text{Find } \nu_0 \in V_{0,\text{div}}^p \text{ such that} \\
ya(\theta; \nu_0, \varphi - \nu_0) + \Psi(\varphi) - \Psi(\nu_0) \geq \int_\Omega f \cdot (\varphi - \nu_0) \, dx, \quad \forall \varphi \in V_{0,\text{div}}^p 
\end{array} \right. \quad (31)$$

for a given temperature $\theta \in L^q(\Omega)$ with $q \geq 1$ and $p > 1$. Indeed with the same arguments as in Corollary 3.2 and Theorem 3.3 we obtain that the operator $A_\theta : V_{0,\text{div}}^p \to (V_{0,\text{div}}^p)'$ given by

$$\langle A_\theta \nu, \varphi \rangle = a(\theta; \nu, \varphi)$$

$$= \int_\Omega \mathcal{F}(\theta, \nu + G, D(\nu + G)) : D(\varphi) \, dx \quad \forall \nu, \varphi \in V_{0,\text{div}}^p \times V_{0,\text{div}}^p$$

is bounded, pseudo-monotone and coercive. Hence problem (31) admits a solution but we can not prove its uniqueness and the fixed point technique introduced in Section 5 fails.
4. Heat transfer auxiliary problem

As already outlined in the Introduction and in Section 2, the right-hand side of the heat equation (7) contains a term belonging to $L^1(\Omega)$, namely $2\mu(\theta, u, |D(u)||D(u)|^p$. It is well known that for elliptic equations of the form

$$\begin{cases} -\text{div} (a(x, \nabla \theta)) = g & \text{in } \Omega \\ \theta = 0 & \text{in } \partial \Omega \end{cases}$$

with $g \in L^1(\Omega)$ and coercivity properties $a(x, \xi) \cdot \xi \geq \alpha_a |\xi|^2$ ($\alpha_a > 0$) for all $\xi \in \mathbb{R}^3$ and for almost every $x \in \Omega$, we may expect solutions $\theta \in W^{1,q}_{0}(\Omega)$ with $1 \leq q < 3/2$ and uniform estimates of $\|\theta\|_{W^{1,q}(\Omega)}$ whenever $g$ remains in a given ball of $L^1(\Omega)$ (see [26] and the references therein). Since we consider also the convection term in (7) and mixed Dirichlet–Neumann boundary conditions (8) we can not apply directly these results. In order to cope with the difficulty due to the $L^1$-term in the right-hand side of (7) we follow the same strategy as in [20] and we replace it by some approximate function $g_\delta(\theta, \nu)$ given by

$$g_\delta(\theta, \nu) = \frac{2\mu(\theta, \nu + G, |D(\nu + G)||D(\nu + G)|^p}{1 + 2\delta \mu(\theta, \nu + G, |D(\nu + G)| |D(\nu + G)|^p}$$

with $\delta > 0$. Obviously for any $\delta > 0$, $\nu \in V^p_{0,\text{div}}$ with $p > 1$ and $\theta \in L^q(\Omega)$ with $q \geq 1$ we have $g_\delta(\theta, \nu) \in L^\infty(\Omega)$ and

$$\|g_\delta(\theta, \nu)\|_{L^\infty(\Omega)} \leq \frac{1}{\delta}. \quad (32)$$

Moreover we may observe that (13) implies that $g_\delta(\theta, \nu)$ is also bounded in $L^1(\Omega)$ uniformly with respect to $\theta$ and $\delta$ and we have

$$\|g_\delta(\theta, \nu)\|_{L^1(\Omega)} \leq 2\mu_1 \|D(\nu + G)\|_{L^p(\Omega)}^{3 \times 3}^{\text{div}} \quad (33)$$

for all $\delta > 0$, for all $\theta \in L^q(\Omega)$ and for all $\nu \in V^p_{0,\text{div}}$ with $p > 1$ and $q \geq 1$. Similarly we define $\theta_\delta^b$ as

$$\theta_\delta^b = \frac{\theta^b}{1 + \delta |\theta^b|} \quad (34)$$

for all $\delta > 0$.

So we consider the following approximate heat equation:

$$(\nu + G) \cdot \nabla \theta_\delta - \text{div} (K \nabla \theta_\delta) = g_\delta(\theta_\delta, \nu) + r(\theta_\delta) \quad \text{in } \Omega \quad (35)$$

with the boundary conditions

$$\theta_\delta = 0 \quad \text{on } \Gamma_1 \cup \Gamma_L, \quad (K \nabla \theta_\delta) \cdot n = \theta_\delta^b \quad \text{on } \Gamma_0. \quad (36)$$

Moreover let us recall that, since we expect $\theta \in W^{1,q}_{1,\Gamma_1 \cup \Gamma_L}(\Omega)$ with $1 \leq q < 3/2$, we have also some condition on $p$ in order to ensure that the convection term $(\nu + G) \cdot \nabla \theta$ belongs to $L^1(\Omega)$ i.e. $\nu + G$ should belong to $L^{p*}(\Omega)$ with $p_*= q'$ which yields the compatibility conditions between $p$ and $q$ given by (15) or equivalently (16). if $1 < p < 3$. Thus if $q = 1$ we should have $p > 3$, and if $q \in (1, 3/2)$ we should have $p \geq \frac{3q}{4q-3}$.

Since problem (35) – (36) is non-linear we consider first a linearized problem and we prove an existence and uniqueness result.
Proposition 4.1: Let \( p \geq 3/2, \ v \in V_{0,\text{div}}^p \) and \( G \) satisfy (9). Let us assume that (11) and (17)–(18)–(19)–(20) hold. Then, for all \( \delta > 0 \) and \( \tilde{\theta} \in L^2(\Omega) \), there exists an unique \( \tilde{\theta}_\delta \in W^{1,2}_{\Gamma_1\cup \Gamma_L}(\Omega) \) such that

\[
B(v, \tilde{\theta}_\delta, w) = L_\delta(v, \tilde{\theta}, w) \quad \forall w \in W^{1,2}_{\Gamma_1\cup \Gamma_L}(\Omega)
\]

(37)

where

\[
B(v, \tilde{\theta}_\delta, w) = \int_\Omega ((v + G) \cdot \nabla \tilde{\theta}_\delta) w \, dx + \int_\Omega (K \nabla \tilde{\theta}_\delta) \cdot \nabla w \, dx,
\]

\[
L_\delta(v, \tilde{\theta}, w) = \int_\Omega g_\delta(\tilde{\theta}, v) w \, dx + \int_\Omega r(\tilde{\theta}) w \, dx + \int_{\Gamma_0} \theta^p_\delta w \, dx'.
\]

Remark 4.1: The first integral term in \( B(v, \tilde{\theta}_\delta, w) \) is well defined: indeed if \( \tilde{\theta}_\delta \) and \( w \) belong to \( W^{1,2}_{\Gamma_1\cup \Gamma_L}(\Omega) \), then \( \nabla \tilde{\theta}_\delta \in L^2(\Omega) \) and \( w \in L^6(\Omega) \) hence \( w\nabla \tilde{\theta}_\delta \in L^{3/2}(\Omega) \) and \( v + G \in L^3(\Omega) \) for all \( v \in V_{0,\text{div}}^p \) with \( p \geq 3/2 \).

Proof: By using the continuous injection of \( W^{1,2}_{\Gamma_1\cup \Gamma_L}(\Omega) \) into \( L^6(\Omega) \) we obtain immediately that \( B(v, \cdot, \cdot) \) is bilinear and continuous on \( W^{1,2}_{\Gamma_1\cup \Gamma_L}(\Omega) \). Moreover, for all \( w \in W^{1,2}_{\Gamma_1\cup \Gamma_L}(\Omega) \) we have

\[
B(v, w, w) = \int_\Omega ((v + G) \cdot \nabla w) w \, dx + \int_\Omega (K \nabla w) \cdot \nabla w \, dx.
\]

yields For any \( w \in D(\Omega) \) Green’s formula yields

\[
\int_\Omega ((v + G) \cdot \nabla w) w \, dx = -\frac{1}{2} \int_\Omega \text{div}(v + G) w^2 \, dx + \frac{1}{2} \int_{\partial\Omega} ((v + G) \cdot n) w^2 \, dY
\]

\[
= \frac{1}{2} \int_{\partial\Omega} ((v + G) \cdot n) w^2 \, dY.
\]

By using the density of \( D(\Omega) \) into \( W^{1,2}(\Omega) \) and the continuity of the trace operator from \( W^{1,2}(\Omega) \) into \( L^4(\partial\Omega) \) we obtain that the same equality holds for any \( w \in W^{1,2}(\Omega) \).

Hence with (18)

\[
B(v, w, w) = \int_\Omega (K \nabla w) \cdot \nabla w \, dx \geq k_0 \|w\|_{1,2}^2 \quad \forall w \in W^{1,2}_{\Gamma_1\cup \Gamma_L}(\Omega).
\]

Finally with (19)–(20) and (32) – (34) we obtain that \( L_\delta(v, \tilde{\theta}, \cdot) \) is linear and continuous on \( W^{1,2}_{\Gamma_1\cup \Gamma_L}(\Omega) \) and the existence of a unique solution \( \tilde{\theta}_\delta \) to (37) follows from Lax–Milgram theorem.

Of course we have a trivial estimate of the solution of problem (37) in \( W^{1,2}_{\Gamma_1\cup \Gamma_L}(\Omega) \). Indeed let \( \tilde{\theta} \in L^2(\Omega) \) and \( v \in V_{0,\text{div}}^p \). By choosing \( w = \tilde{\theta}_\delta \) in (37) we obtain

\[
k_0 \|\tilde{\theta}_\delta\|_{1,2}^2 \leq B(v, \tilde{\theta}_\delta, \tilde{\theta}_\delta) = L_\delta(v, \tilde{\theta}, \tilde{\theta}_\delta) \leq \Lambda |\Gamma_0|^{1/2} \|\tilde{\theta}_\delta\|_{1,2}
\]

(38)

with

\[
\Lambda = C_p \left( \frac{1}{\delta} + \|r\|_{L^\infty(\mathbb{R})} \right) |\Omega|^{1/2} + C_\gamma \frac{1}{\delta}
\]

where \( C_\gamma \) is the norm of the trace operator \( \gamma : W^{1,2}_{\Gamma_1\cup \Gamma_L}(\Omega) \to L^2(\partial\Omega) \) and \( C_p \) denotes the Poincaré’s constant on \( W^{1,2}_{\Gamma_1\cup \Gamma_L}(\Omega) \). Hence

\[
\|\tilde{\theta}_\delta\|_{1,2} \leq R_\delta := \frac{1}{k_0} \left( C_p \left( \frac{1}{\delta} + \|r\|_{L^\infty(\mathbb{R})} \right) |\Omega|^{1/2} + C_\gamma \frac{1}{\delta} |\Gamma_0|^{1/2} \right).
\]
Clearly $R_\delta$ is independent of $\tilde{\theta}$ and $\nu$ but depends on $\delta$. On the other hand, by using the uniform boundedness of $g_\delta(\tilde{\theta}, \nu)$ in $L^1(\Omega)$ with respect to $\tilde{\theta}$ and $\delta$ we may obtain an estimate, independent of $\tilde{\theta}$ and $\delta$ of $\tilde{\theta}_\delta$ in $W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega)$ for all $q \in [1, 3/2]$. More precisely we have

**Proposition 4.2:** Let $q \in [1, 3/2]$ and $p \geq 3/2$. Let $\bar{\theta} \in L^2(\Omega)$, $\nu \in V^p_{0, \text{div}}$ $G$ satisfying (9) and assume that (11)–(13) and (17)–(18)–(19)–(20) hold. Then there exists a positive real number $C_{\text{heat}}$, independent of $\bar{\theta}$ and $\delta$ and depending only on $\mu_1 \|D(\nu + G)\|_{(L^p(\Omega))^{3 \times 3}}^p$, $r$, $\bar{\theta}^b$, $k_0$ and $q$, such that the unique solution $\tilde{\theta}_\delta$ of (37) satisfies

$$\|\tilde{\theta}_\delta\|_{1,q} \leq C_{\text{heat}}. \quad (39)$$

**Proof:** Let $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(t) = sg(t)(1 - \frac{1}{(1 + |t|)^\zeta})$ with $\zeta > 0$ to be chosen later and $sg(t) = \frac{t}{|t|}$ if $t \neq 0$, $sg(0) = 0$. Then $\phi(0) = 0$, $\phi \in C^1(\mathbb{R}; \mathbb{R})$ and

$$|\phi(t)| \leq 1, \quad \phi'(t) = \frac{\zeta}{(1 + |t|)^{\zeta + 1}} \quad \forall t \in \mathbb{R} \quad (40)$$

and $\phi'$ is $\zeta(\zeta + 1)$-Lipschitz continuous on $\mathbb{R}$. It follows that for any $w \in W^{1,2}(\Omega)$ we have $\phi(w) \in W^{1,2}(\Omega)$ and

$$\nabla(\phi(w)) = \phi'(w) \nabla w = \frac{\zeta}{(1 + |w|)^{\zeta + 1}} \nabla w. \quad (41)$$

Moreover if $w \in W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega)$ then $\phi(w) \in W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega)$. Indeed let $(w_n)_{n \geq 1}$ be a sequence of $\mathcal{D}(\overline{\Omega})$ which converges strongly to $w$ in $W^{1,2}(\Omega)$. Then, with (40) – (41), we obtain that $(\phi(w_n))_{n \geq 1}$ converges strongly to $\phi(w)$ in $W^{1,3/2}(\Omega)$. By continuity of the trace operator $\gamma$, the sequences $(\gamma(w_n))_{n \geq 1}$ and $(\gamma(\phi(w_n)))_{n \geq 1}$ converge also strongly to $\gamma(w)$ and $\gamma(\phi(w))$, respectively, in $L^2(\partial \Omega)$. Hence, possibly extracting a subsequence still denoted $(w_n)_{n \geq 1}$, we have

$$\gamma(w_n) = w_n|_{\partial \Omega} \to \gamma(w) \quad \text{a.e. on } \partial \Omega,$$

$$\gamma(\phi(w_n)) = \phi(w_n)|_{\partial \Omega} \to \gamma(\phi(w)) \quad \text{a.e. on } \partial \Omega.$$

Owing that $w \in W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega)$ we get

$$w_n|_{\partial \Omega} \to \gamma(w) = 0 \quad \text{a.e. on } \Gamma_1 \cup \Gamma_L.$$

Thus

$$\phi(w_n)|_{\partial \Omega} \to 0 = \gamma(\phi(w)) \quad \text{a.e. on } \Gamma_1 \cup \Gamma_L$$

and with (40)

$$|\gamma(\phi(w))| \leq 1 \quad \text{a.e. on } \Gamma_0.$$

Let $\Phi : \mathbb{R} \to \mathbb{R}$ given by $\Phi(t) = \int_0^t \phi(s) \, ds$ for all $t \in \mathbb{R}$. For any $w \in W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega)$ we have also $\Phi(w) \in W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega)$ and

$$\nabla(\Phi(w)) = \Phi'(w) \nabla w = \phi(w) \nabla w. \quad (42)$$

Now let $\delta > 0$, $\tilde{\theta} \in L^2(\Omega)$ and $\nu \in V^p_{0, \text{div}}$. By choosing $w = \phi(\tilde{\theta}_\delta)$ as a test-function in (37) we get

$$\int_\Omega ((\nu + G) \cdot \nabla \tilde{\theta}_\delta) \phi(\tilde{\theta}_\delta) \, dx + \zeta \int_\Omega \frac{(K \nabla \tilde{\theta}_\delta) \cdot \nabla \tilde{\theta}_\delta}{(1 + |\tilde{\theta}_\delta|)^{\zeta + 1}} \, dx$$

$$= \int_{\Omega} g_\delta(\tilde{\theta}, \nu) \phi(\tilde{\theta}_\delta) \, dx + \int_\Omega r(\tilde{\theta}) \phi(\tilde{\theta}_\delta) \, dx + \int_{\Gamma_0} \theta_\delta^b \nu(\phi(\tilde{\theta}_\delta)) \, dx'. \quad (43)$$
With Green’s formula we have
\[ \int_{\Omega} (u \cdot \nabla \tilde{\theta}) \phi (\tilde{\theta}) \, dx = \int_{\Omega} u \cdot \nabla (\Phi (\tilde{\theta})) \, dx = - \int_{\Omega} \text{div} (u) \Phi (\tilde{\theta}) \, dx + \int_{\Gamma_0} u \cdot n \gamma (\Phi (\tilde{\theta})) \, dx' \]
for all \( u \in (\mathcal{D}(\mathcal{O}))^3 = \mathcal{D}(\mathcal{O}) \). Since \( p \geq 3/2 \) we have \( W^{1,p}(\Omega) \subset L^3(\Omega) \) with continuous injection and the trace operator maps continuously \( W^{1,p}(\Omega) \) into \( L^2(\partial \Omega) \subset L^4(\partial \Omega) \). By using the density of \( \mathcal{D}(\Omega) \) into \( W^{1,p}(\Omega) \) we obtain that the same equality holds for all \( u \in W^{1,p}(\Omega) \). With \( u = \nu + G \) we get finally
\[ \int_{\Omega} ((\nu + G) \cdot \nabla \tilde{\theta}) \phi (\tilde{\theta}) \, dx = 0. \]

Going back to (43) and using (17) we get
\[ k_0 \xi \int_{\Omega} \frac{|D \tilde{u}|^2}{(1 + |\tilde{\theta}|)^{\xi + 1}} \, dx \leq \int_{\Omega} |g_\delta (\tilde{\theta}, \nu)| \, dx + \int_{\Omega} |r(\tilde{\theta})| \, dx + \int_{\Gamma_0} |\tilde{\theta}|^{\gamma} \, dx' \]
and thus
\[ \int_{\Omega} \frac{|D \tilde{u}|^2}{(1 + |\tilde{\theta}|)^{\xi + 1}} \, dx \leq \frac{1}{k_0 \xi} (2 \mu_1 \|D(\nu + G)\|_{L^p(\Omega))}^p + |\Omega| \|r\|_{L^\infty(\mathbb{R})} + \|\theta^b\|_{L^1(\Gamma_0)}). \] (44)

Then we estimate \( \|\tilde{\theta}\|_{1,q}^q \) as follows
\[ \|\tilde{\theta}\|_{1,q}^q = \int_{\Omega} |D \tilde{u}|^q \, dx = \int_{\Omega} \frac{|D \tilde{u}|^q}{(1 + |\tilde{\theta}|)^{(\xi + 1)q/2} \frac{(\xi + 1)q}{2} \frac{q}{2} \frac{2-q}{2}} \, dx. \]

By using Hölder’s inequality we obtain
\[ \int_{\Omega} |D \tilde{u}|^q \, dx \leq \left( \int_{\Omega} \frac{|D \tilde{u}|^2}{(1 + |\tilde{\theta}|)^{\xi + 1}} \, dx \right)^{\frac{q}{2}} \left( \int_{\Omega} (1 + |\tilde{\theta}|)^{\frac{(\xi + 1)q}{2} \frac{q}{2} \frac{2-q}{2} \frac{2-q}{2}} \right)^{2-q}. \]

Now we observe that since \( q \in [1, 3/2] \) we have \( W^{1,q}(\Omega) \subset L^{q^*}(\Omega) \) with \( q^* = \frac{3q}{3-q} \) and we may choose \( \xi > 0 \) such that \( \frac{(\xi + 1)q}{2} \frac{q}{2} \frac{2-q}{2} \frac{2-q}{2} = q^* \), namely \( \xi = \frac{3-2q}{3-q} \). With (44) we get
\[ \|\tilde{\theta}\|_{1,q} = \left( \int_{\Omega} |D \tilde{u}|^q \, dx \right)^{\frac{1}{q}} \leq C_1 \left( \int_{\Omega} (1 + |\tilde{\theta}|)^{q^*} \, dx \right)^{\frac{2-q}{2q}} \]
with
\[ C_1 = \frac{1}{\sqrt{k_0 \xi}} (2 \mu_1 \|D(\nu + G)\|_{L^p(\Omega))}^p + |\Omega| \|r\|_{L^\infty(\mathbb{R})} + \|\theta^b\|_{L^1(\Gamma_0)})^{1/2}. \] (45)

Since \( q^* > 1 \) we have
\[ (1 + |\tilde{\theta}|)^{q^*} = 2^{q^*} \left( \frac{1}{2} + \frac{|\tilde{\theta}|^2}{2} \right)^{q^*} \leq 2^{q^*} (1 + |\tilde{\theta}|^{q^*}) \]
which yields
\[ \|\tilde{\theta}\|_{1,q} \leq C_1 2^{(q^*-1)\frac{2-q}{2q}} \left( |\Omega| + \int_{\Omega} |\tilde{\theta}|^{q^*} \, dx \right)^{\frac{2-q}{2q}}. \]
Recalling that $\frac{2-q}{2q} \in (0, 1)$ we have

$$1 = \frac{a}{a+b} + \frac{b}{a+b} \leq \left( \frac{a}{a+b} \right)^{\frac{2-q}{2q}} + \left( \frac{b}{a+b} \right)^{\frac{2-q}{2q}} \quad \forall (a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+^*.$$ 

So

$$\|\tilde{\theta}_b\|_{1,q} \leq C_1 2^{(q_s-1)\frac{2-q}{2q}} \left( |\Omega|^{\frac{2-q}{2q}} + \|\tilde{\theta}_b\|^{q_s\frac{2-q}{2q}}_{L^{2s}(\Omega)} \right).$$

But $\frac{2-q}{2q} < \frac{1}{q_s} = \frac{3-q}{2q}$ and $\alpha := q_s \frac{2-q}{2q} \in (0, 1)$. It follows that

$$\|\tilde{\theta}_b\|_{1,q} = \|\tilde{\theta}_b\|^{\alpha}_{1,q} \|\tilde{\theta}_b\|^{1-\alpha}_{1,q} \leq C_1 2^{(q_s-1)\frac{2-q}{2q}} \left( |\Omega|^{\frac{2-q}{2q}} + C_q^\alpha \|\tilde{\theta}_b\|^{\alpha}_{1,q} \right)$$

where $C_q$ denotes the norm of the canonical injection of $W^{1,q}_{\Gamma_1 \cup \Gamma_2} (\Omega)$ into $L^{q_s} (\Omega)$. Thus, if $\|\tilde{\theta}_b\|_{1,q} \geq 1$ we have

$$\|\tilde{\theta}_b\|^{\alpha}_{1,q} \leq C_1 2^{(q_s-1)\frac{2-q}{2q}} \left( \frac{|\Omega|^{\frac{2-q}{2q}}}{\|\tilde{\theta}_b\|^{\alpha}_{1,q}} + C_q^\alpha \right) \leq C_1 2^{(q_s-1)\frac{2-q}{2q}} \left( |\Omega|^{\frac{2-q}{2q}} + C_q^\alpha \right)$$

and finally

$$\|\tilde{\theta}_b\|_{1,q} \leq \max \left( 1, C_1 \left( 2^{\frac{2(3-q)}{q}} \left( \frac{2(q_s-1)(2-q)}{q} \right) \left( |\Omega|^{\frac{2-q}{2q}} + C_q^\alpha \right) \right)^\frac{2(3-q)}{q} \right).$$

**Remark 4.2:** Let us observe that we may prove an existence result for the approximate heat problem (35) – (36) by using a fixed point technique. Indeed, let $\tilde{T} : L^2 (\Omega) \rightarrow L^2 (\Omega)$ be defined by $\tilde{T}(\tilde{\theta}) = \tilde{\theta}_b$ where $\tilde{\theta}_b \in W^{1,2}_{\Gamma_1 \cup \Gamma_2} (\Omega)$ is the unique solution of (37). Starting from (38) we obtain immediately that $\tilde{T}(\tilde{B}_{L^2(\Omega)}(0, C_p R_3)) \subset \tilde{B}_{L^2(\Omega)}(0, C_p R_3)$ and $\tilde{T}(\tilde{B}_{L^2(\Omega)}(0, C_p R_3))$ is relatively compact in $L^2 (\Omega)$, where we recall that $C_p$ is the Poincaré’s constant on $W^{1,2}_{\Gamma_1 \cup \Gamma_2} (\Omega)$. Then, by using the continuity properties and the uniform boundedness of the mappings $\tilde{\theta} \mapsto g_b(\tilde{\theta}, \nu)$ and $\tilde{\theta} \mapsto r(\tilde{\theta})$ we can check the operator $\tilde{T}$ is continuous on $L^2 (\Omega)$. Finally Schauder’s fixed point theorem gives the existence of a solution to problem (35) – (36) as a fixed point of the operator $\tilde{T}$. Unfortunately this technique does not provide uniqueness and the non-linearity of the problem does not allow us to expect uniqueness unless we introduce further assumptions, like for instance some Lipschitz continuity properties for the mappings $\mu$ and $r$ with respect to the temperature.

5. **Existence result for the coupled problem (P)**

We study in this section the coupled problem (P). We proceed in two steps. First we will prove the existence of an approximate coupled problem depending on the parameter $\delta$ where the heat problem (7) – (8) is replaced by (35) – (36) and then we will pass to the limit as $\delta$ tends to zero in order to obtain a solution to problem (P).

More precisely we prove first
Proposition 5.1: \( p \geq \frac{3q}{4q-3} \) otherwise Let \( p \geq 3/2 \) and assume that (9)-(10), (11)-(12)-(13) and (17)-(18)-(19)-(20) hold. Then for any \( \delta > 0 \) there exists \((\bar{\theta}_8, \bar{\nu}_8)\) in \( W^{1,2}_{\Gamma_1 \cup \Gamma_2}(\Omega) \times V^p_{0, \text{div}} \) such that

\[
a(\bar{\theta}_8; \bar{\nu}_8, \varphi - \bar{\nu}_8) + \Psi(\varphi) - \Psi(\bar{\nu}_8) \geq \int_{\Omega} f \cdot (\varphi - \bar{\nu}_8) \, dx \quad \forall \varphi \in V^p_{0, \text{div}}
\]

and

\[
B(\bar{\nu}_8, \bar{\theta}_8, w) = L_\delta(\bar{\nu}_8, \bar{\theta}_8, w) \quad \forall w \in W^{1,2}_{\Gamma_1 \cup \Gamma_2}(\Omega).
\]

Proof: We define the operator \( T : L^2(\Omega) \times L^p(\Omega) \rightarrow L^2(\Omega) \times L^p(\Omega) \) by \( T(\theta, \nu) = (\bar{\theta}_8, \bar{\nu}_8) \) where \( \bar{\nu}_8 \in V^p_{0, \text{div}} \) is the unique solution of problem (21) with \( \nu = \bar{\nu} \) and \( \theta = \bar{\theta} \) i.e.

\[
\langle A_{\bar{\theta}, \bar{\nu}}(\bar{\nu}_8), \varphi - \bar{\nu}_8 \rangle + \Psi(\varphi) - \Psi(\bar{\nu}_8) \geq \int_{\Omega} f \cdot (\varphi - \bar{\nu}_8) \, dx \quad \forall \varphi \in V^p_{0, \text{div}}
\]

and \( \bar{\theta}_8 \in W^{1,2}_{\Gamma_1 \cup \Gamma_2}(\Omega) \) is the unique solution of (37) with \( \nu = \bar{\nu}_8 \) i.e.

\[
B(\bar{\nu}_8, \bar{\theta}_8, w) = L_\delta(\bar{\nu}_8, \bar{\theta}_8, w) \quad \forall w \in W^{1,2}_{\Gamma_1 \cup \Gamma_2}(\Omega).
\]

We already know with (38) that

\[
\|\bar{\theta}_8\|_{L^2(\Omega)} \leq C_p' \|\bar{\theta}_8\|_{1,2} \leq C_p R_\delta
\]

with

\[
R_\delta := \frac{C_p'}{k_0} \left( C_p \left( \frac{1}{\delta} + \|r\|_{L^\infty(\mathbb{R})} \right) |\Omega|^{1/2} + \frac{C_p'}{\delta} |\Gamma_0|^{1/2} \right)
\]

where we recall that \( C_p' \) denotes the Poincaré’s constant on \( W^{1,2}_{\Gamma_1 \cup \Gamma_2}(\Omega) \) and from Proposition 3.4 that

\[
\|\bar{\nu}_8\|_{L^p(\Omega)} \leq C_p \|\bar{\nu}_8\|_{1, p} \leq C_p C_{\text{flow}}
\]

where \( C_p \) is the Poincaré’s constant on \( W^{1,p}_{\Gamma_1 \cup \Gamma_2}(\Omega) \). We infer that \( T(\bar{B}_{L^2}(0, C_p R_\delta) \times \bar{B}_{L^p}(0, C_p C_{\text{flow}})) \subset \bar{B}_{L^2}(0, C_p R_\delta) \times \bar{B}_{L^p}(0, C_p C_{\text{flow}}) \) and \( T(\bar{B}_{L^2}(0, C_p R_\delta) \times \bar{B}_{L^p}(0, C_p C_{\text{flow}})) \) is relatively compact in \( L^2(\Omega) \times L^p(\Omega) \).

Let us prove now that \( T \) is continuous on \( L^2(\Omega) \times L^p(\Omega) \). So let \((\bar{\theta}_n, \bar{\nu}_n)_{n \geq 0}\) be a sequence of \( L^2(\Omega) \times L^p(\Omega) \) which converges strongly to \((\bar{\theta}, \bar{\nu})\). For all \( n \geq 0 \) we define \( (\bar{n}_8^n, \bar{\nu}_8^n) = T(\bar{\theta}_n, \bar{\nu}_n) \) and we have to prove that \((\bar{n}_8^n, \bar{\nu}_8^n)_{n \geq 0}\) converges strongly in \( L^2(\Omega) \times L^p(\Omega) \) to \((\bar{\theta}_8, \bar{\nu}_8) = T(\bar{\theta}, \bar{\nu})\).

For all \( n \geq 0 \) we have

\[
\langle A_{\bar{\theta}_n, \bar{n}_8^n}(\bar{\nu}_8^n), \varphi - \bar{\nu}_8^n \rangle + \Psi(\varphi) - \Psi(\bar{\nu}_8^n) \geq \int_{\Omega} f \cdot (\varphi - \bar{\nu}_8^n) \, dx \quad \forall \varphi \in V^p_{0, \text{div}}
\]

and we have also

\[
B(\bar{\nu}_8^n, \bar{\theta}_n, w) = L_\delta(\bar{\nu}_8^n, \bar{\theta}_n, w) \quad \forall w \in W^{1,2}_{\Gamma_1 \cup \Gamma_2}(\Omega).
\]
By choosing $\varphi = \tilde{v}_\delta$ then $\varphi = \tilde{v}_\delta^n$ we get

$$0 \geq (A_{\tilde{v}_\delta^n}, \tilde{v}_\delta^n, \tilde{v}_\delta^n - \tilde{v}_\delta)$$

$$= \mu_0 \int_{\Omega} \left( |D(\tilde{v}_\delta^n + G)|^{p-2}D(\tilde{v}_\delta^n + G) - |D(\tilde{v}_\delta + G)|^{p-2}D(\tilde{v}_\delta + G) \right) : D(\tilde{v}_\delta^n - \tilde{v}_\delta) \, dx$$

$$+ \int_{\Omega} \left( \mathcal{F}(\tilde{\theta}_n, \tilde{v}_n + G, D(\tilde{v}_\delta^n + G)) - \mathcal{F}(\tilde{\theta}, \tilde{v} + G, D(\tilde{v}_\delta + G)) \right) : D(\tilde{v}_\delta^n - \tilde{v}_\delta) \, dx$$

$$= \mu_0 \int_{\Omega} \left( |D(\tilde{v}_\delta^n + G)|^{p-2}D(\tilde{v}_\delta^n + G) - |D(\tilde{v}_\delta + G)|^{p-2}D(\tilde{v}_\delta + G) \right) : D(\tilde{v}_\delta^n - \tilde{v}_\delta) \, dx$$

$$+ \int_{\Omega} \left( \mathcal{F}(\tilde{\theta}_n, \tilde{v}_n + G, D(\tilde{v}_\delta^n + G)) - \mathcal{F}(\tilde{\theta}_n, \tilde{v}_n + G, D(\tilde{v}_\delta + G)) \right) : D(\tilde{v}_\delta^n - \tilde{v}_\delta) \, dx$$

and thus denoting $\| \cdot \|_{(L^p(\Omega))^3 \times 3}$ by $\| \cdot \|_{L^p}$ and $\| \cdot \|_{(L^{p'}(\Omega))^3 \times 3}$ by $\| \cdot \|_{L^{p'}}$ we obtain

$$\mu_0 \int_{\Omega} \left( |D(\tilde{v}_\delta^n + G)|^{p-2}D(\tilde{v}_\delta^n + G) - |D(\tilde{v}_\delta + G)|^{p-2}D(\tilde{v}_\delta + G) \right) : D(\tilde{v}_\delta^n - \tilde{v}_\delta) \, dx$$

$$\leq \int_{\Omega} \left( \mathcal{F}(\tilde{\theta}, \tilde{v} + G, D(\tilde{v}_\delta + G)) - \mathcal{F}(\tilde{\theta}_n, \tilde{v}_n + G, D(\tilde{v}_\delta + G)) \right) : D(\tilde{v}_\delta^n - \tilde{v}_\delta) \, dx$$

$$\leq \| \mathcal{F}(\tilde{\theta}, \tilde{v} + G, D(\tilde{v}_\delta + G)) - \mathcal{F}(\tilde{\theta}_n, \tilde{v}_n + G, D(\tilde{v}_\delta + G)) \|_{L^{p'}} \| D(\tilde{v}_\delta^n - \tilde{v}_\delta) \|_{L^p}.$$

Let us distinguish two cases.

**Case 1:** $p \geq 2$.

With (27) we get

$$\mu_0 \frac{1}{2p-1} \| D(\tilde{v}_\delta^n - \tilde{v}_\delta) \|_{L^p}^{p-1} \leq \| \mathcal{F}(\tilde{\theta}, \tilde{v} + G, D(\tilde{v}_\delta + G)) - \mathcal{F}(\tilde{\theta}_n, \tilde{v}_n + G, D(\tilde{v}_\delta + G)) \|_{L^{p'}} \| D(\tilde{v}_\delta^n - \tilde{v}_\delta) \|_{L^p}.$$

and thus

$$\mu_0 \frac{1}{2p-1} \| D(\tilde{v}_\delta^n - \tilde{v}_\delta) \|_{L^p}^{p-1} \leq \| \mathcal{F}(\tilde{\theta}, \tilde{v} + G, D(\tilde{v}_\delta + G)) - \mathcal{F}(\tilde{\theta}_n, \tilde{v}_n + G, D(\tilde{v}_\delta + G)) \|_{L^{p'}}.$$

By possibly extracting a subsequence, still denoted $(\tilde{\theta}_n, \tilde{v}_n)_{n \geq 0}$, we have

$$\tilde{\theta}_n \to \tilde{\theta}, \quad \tilde{v}_n \to \tilde{v} \quad \text{a.e. in } \Omega.$$

Then the continuity and uniform boundedness of the mapping $\mathcal{F}$ combined with Lebesgue’s theorem imply that

$$\mathcal{F}(\tilde{\theta}_n, \tilde{v}_n + G, D(\tilde{v}_\delta + G)) \to \mathcal{F}(\tilde{\theta}, \tilde{v} + G, D(\tilde{v}_\delta + G)) \quad \text{strongly in } (L^{p'}(\Omega))^3 \times 3.$$

Thus

$$\tilde{v}_\delta^n \to \tilde{v}_\delta \quad \text{strongly in } V^p_{0, \text{div}}.$$

Recalling that $(\tilde{v}_\delta^n)_{n \geq 0}$ is bounded in $V^p_{0, \text{div}}$, we may conclude that the whole sequence $(\tilde{v}_\delta^n)_{n \geq 0}$ converges strongly to $\tilde{v}_\delta$ in $V^p_{0, \text{div}}$. 

Case 2: $1 < p < 2$.
With (28) and the same computations as in Theorem 3.3 we have
\[
C_p \int_\Omega |D(\tilde{\upsilon}^n_\delta - \tilde{\upsilon}_\delta)|^p \, dx
\]
\[
\leq \left( \int_\Omega (|D(\tilde{\upsilon}^n_\delta + G)|^{p-2}D(\tilde{\upsilon}^n_\delta + G) - |D(\tilde{\upsilon}_\delta + G)|^{p-2}D(\tilde{\upsilon}_\delta + G) : D(\tilde{\upsilon}^n_\delta - \tilde{\upsilon}_\delta)) \, dx \right)^{\frac{p}{2}}
\]
\[
\times \left( \int_\Omega (|D(\tilde{\upsilon}^n_\delta + G)|^p + |D(\tilde{\upsilon}_\delta + G)|^p) \, dx \right)^{\frac{2-p}{2}}
\]
and with Proposition 3.4 we obtain
\[
\frac{C_p}{(2^p(C_{\text{flow}})^p + \|D(G)\|_{L^p})^{\frac{2-p}{2}}} \int_\Omega |D(\tilde{\upsilon}^n_\delta - \tilde{\upsilon}_\delta)|^p \, dx
\]
\[
\leq \left( \int_\Omega (|D(\tilde{\upsilon}^n_\delta + G)|^{p-2}D(\tilde{\upsilon}^n_\delta + G) - |D(\tilde{\upsilon}_\delta + G)|^{p-2}D(\tilde{\upsilon}_\delta + G) : D(\tilde{\upsilon}^n_\delta - \tilde{\upsilon}_\delta)) \, dx \right)^{\frac{p}{2}}
\]
Thus
\[
\left( \frac{C_p}{(2^p(C_{\text{flow}})^p + \|D(G)\|_{L^p})^{\frac{2-p}{2}}} \right)^{\frac{2}{p}} \|D(\tilde{\upsilon}^n_\delta - \tilde{\upsilon}_\delta)\|_{L^p}
\]
\[
\leq \frac{1}{\mu_0} \|\overline{F}(\hat{\theta}, \hat{\upsilon}, D(\tilde{\upsilon}_\delta + G)) - \overline{F}(\hat{\theta}_n, \hat{\upsilon}_n, D(\tilde{\upsilon}_\delta + G))\|_{L^p'}
\]
and we may conclude as in the previous case.

Next for all $n \geq 0$ we have
\[
B(\tilde{\upsilon}^n_\delta, \tilde{\theta}^n_\delta, w) = L_\delta(\tilde{\upsilon}^n_\delta, \tilde{\theta}_n, w) \quad \forall w \in W^{1,2}_{\Gamma_1 \cup \Gamma_2} (\Omega).
\]
(48)
We already know that, possibly extracting a subsequence, still denoted $(\hat{\theta}_n, \hat{\upsilon}_n)_{n \geq 0}$, we have
\[
\hat{\theta}_n \to \hat{\theta}, \quad \hat{\upsilon}_n \to \hat{\upsilon} \quad \text{a.e. in } \Omega.
\]

With the previous result we infer that, possibly extracting another subsequence, we have
\[
\tilde{\upsilon}^n_\delta + G \to \tilde{\upsilon}_\delta + G, \quad D(\tilde{\upsilon}^n_\delta + G) \to D(\tilde{\upsilon}_\delta + G) \quad \text{a.e. in } \Omega.
\]

By using the continuity and the uniform boundedness of the mappings $\mu$ and $r$ combined with Lebesgue’s theorem we obtain that
\[
g_\delta(\tilde{\theta}_n, \tilde{\upsilon}^n_\delta) \to g_\delta(\tilde{\theta}, \tilde{\upsilon}_\delta) \quad \text{strongly in } L^2(\Omega)
\]
and
\[
r(\tilde{\theta}_n) \to r(\tilde{\theta}) \quad \text{strongly in } L^2(\Omega).
\]

On the other hand the sequence $(\tilde{\theta}^n_\delta)_{n \geq 0}$ is bounded in $W^{1,2}_{\Gamma_1 \cup \Gamma_2} (\Omega)$, so possibly extracting another subsequence there exists $\tilde{\theta}^*_\delta \in W^{1,2}_{\Gamma_1 \cup \Gamma_2} (\Omega)$ such that
\[
\tilde{\theta}^n_\delta \to \tilde{\theta}^*_\delta \quad \text{weakly in } W^{1,2}_{\Gamma_1 \cup \Gamma_2} (\Omega) \text{ and strongly in } L^2(\Omega).
\]
By passing to the limit in (48) as \( n \) tends to \(+\infty\) we obtain that

\[
B(\tilde{v}_n, \tilde{\theta}_n^s, w) = L_\delta(\tilde{v}_n, \tilde{\theta}_n^s, w) \quad \forall w \in W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega).
\]

By uniqueness of the solution of (37) with \( v = \tilde{v}_n \) we infer that \( \tilde{\theta}_n^s = \tilde{\theta}_n \) and the whole sequence \( (\tilde{\theta}_n^s)_{n \geq 0} \) converges strongly in \( L^2(\Omega) \) to \( \tilde{\theta}_\delta \).

Finally Schauder’s fixed point theorem allows us to conclude. ■

Let us consider now the sequence \( (\theta_m, \nu_m)_{m \geq 1} \) defined by \( (\theta_m, \nu_m) = (\tilde{\theta}_{1/m}, \tilde{\nu}_{1/m}) \) for all \( m \geq 1 \) i.e.

\[
a(\tilde{\theta}_{1/m}; \tilde{\nu}_{1/m}, \varphi - \tilde{\nu}_{1/m}) + \Psi(\varphi) - \Psi(\tilde{\nu}_{1/m}) \geq \int_{\Omega} f \cdot (\varphi - \tilde{\nu}_{1/m}) \, dx \quad \forall \varphi \in V^p_{0,\text{div}} \quad (49)
\]

and

\[
B(\tilde{\nu}_{1/m}, \tilde{\theta}_{1/m}, w) = L_{1/m}(\tilde{\nu}_{1/m}, \tilde{\theta}_{1/m}, w) \quad \forall w \in W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega). \quad (50)
\]

Let \( q \in [1, 3/2] \) and \( p > 3 \) if \( q = 1 \), \( p \geq \frac{3q}{4q - 3} \) otherwise. With the results of Theorem 3.3, Proposition 3.4 and Proposition 4.2 we know that the sequences \( (\nu_m)_{m \geq 1} \) and \( (\theta_m)_{m \geq 1} \) are bounded in \( V^p_{0,\text{div}} \) and in \( W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega) \), respectively. Reminding that we have to deal with any \( q \in [1, 3/2] \) we can not infer directly any good convergence property for \( (\nabla \theta_m)_{m \geq 1} \) since the unit ball of \( W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega) \) is not weakly compact when \( q = 1 \). So we let \( \zeta \in (0, \frac{3 - 2q}{2q}) \). Then \( \rho = \frac{(\zeta + 1)q}{2 - q} \in (1, q^*) \) and \( W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega) \) is compactly embedded into \( L^\rho(\Omega) \). It follows that, possibly extracting a subsequence, still denoted \( (\nu_m)_{m \geq 1} \) and \( (\theta_m)_{m \geq 1} \), there exist \( \nu \in V^p_{0,\text{div}} \) and \( \theta \in L^\rho(\Omega) \) such that

\[
\nu_m \to \nu \quad \text{weakly in } V^p_{0,\text{div}} \text{ and strongly in } L^\rho(\Omega) \quad (51)
\]

and

\[
\theta_m \to \theta \quad \text{strongly in } L^\rho(\Omega). \quad (52)
\]

Moreover, possibly extracting another subsequence, we have

\[
\nu_m \to \nu, \quad \theta_m \to \theta \quad \text{a.e. in } \Omega. \quad (53)
\]

need also Let us prove now the strong convergence of \( (D(\nu_m + G))_{m \geq 1} \) in \( L^p(\Omega) \).

**Proposition 5.2:** Under the previous assumptions

\[
\nu_m \to \nu \quad \text{strongly in } V^p_{0,\text{div}} \quad (54)
\]

and \( \nu \) satisfies the variational inequality

\[
a(\theta; \nu, \varphi - \nu) + \Psi(\varphi) - \Psi(\nu) \geq \int_{\Omega} f \cdot (\varphi - \nu) \, dx \quad \forall \varphi \in V^p_{0,\text{div}}. \quad (55)
\]
**Proof:** Let \( \tilde{\nu} \in V^p_{0, \text{div}} \) be the unique solution of the following auxiliary problem

\[
\langle A_{\theta, \nu}(\tilde{\nu}), \varphi - \tilde{\nu} \rangle + \Psi(\varphi) - \Psi(\tilde{\nu}) \geq \int_{\Omega} f \cdot (\varphi - \tilde{\nu}) \, dx \quad \forall \varphi \in V^p_{0, \text{div}}.
\]  

(56)

Now we let \( \varphi = \nu_m \) in (56) and \( \varphi = \tilde{\nu} \) in (49). We obtain

\[
0 \geq \langle A_{\theta, \nu}(\nu_m) - A_{\theta, \nu}(\tilde{\nu}), \nu_m - \tilde{\nu} \rangle
\]

\[
= \mu_0 \int_{\Omega} (|D(\nu_m + G)|^{p-2} D(\nu_m + G) - |D(\tilde{\nu} + G)|^{p-2} D(\tilde{\nu} + G)) : D(\nu_m - \tilde{\nu}) \, dx
\]

\[
+ \int_{\Omega} (\mathcal{F}(\theta_m, \nu_m + G, D(\nu_m + G)) - \mathcal{F}(\theta, \nu + G, D(\tilde{\nu} + G))) : D(\nu_m - \tilde{\nu}) \, dx
\]

\[
= \mu_0 \int_{\Omega} (|D(\nu_m + G)|^{p-2} D(\nu_m + G) - |D(\tilde{\nu} + G)|^{p-2} D(\tilde{\nu} + G)) : D(\nu_m - \tilde{\nu}) \, dx
\]

\[
\| \|_{L^p(\Omega)}^p + \int_{\Omega} (\mathcal{F}(\theta_m, \nu_m + G, D(\nu_m + G)) - \mathcal{F}(\theta_m, \nu_m + G, D(\tilde{\nu} + G))) : D(\nu_m - \tilde{\nu}) \, dx
\]

\[
+ \int_{\Omega} (\mathcal{F}(\theta_m, \nu_m + G, D(\tilde{\nu} + G)) - \mathcal{F}(\theta, \nu + G, D(\tilde{\nu} + G))) : D(\nu_m - \tilde{\nu}) \, dx
\]

and thus

\[
\mu_0 \int_{\Omega} (|D(\nu_m + G)|^{p-2} D(\nu_m + G) - |D(\tilde{\nu} + G)|^{p-2} D(\tilde{\nu} + G)) : D(\nu_m - \tilde{\nu}) \, dx
\]

\[
\leq \int_{\Omega} \left( \mathcal{F}(\theta, \nu, G, D(\tilde{\nu} + G)) - \mathcal{F}(\theta_m, \nu_m, G, D(\tilde{\nu} + G)) \right) : D(\nu_m - \tilde{\nu}) \, dx
\]

\[
\leq \left\| \mathcal{F}(\theta, \nu, D(\tilde{\nu} + G)) - \mathcal{F}(\theta_m, \nu_m, D(\tilde{\nu} + G)) \right\|_{L_p^p} \left\| D(\nu_m - \tilde{\nu}) \right\|_{L_p}
\]

Let us distinguish once again two cases.

**Case 1:** \( p \geq 2 \).

With the same computations as in Proposition 5.1 we get

\[
\mu_0 \frac{1}{2^{p-1}} \left\| D(\nu_m - \tilde{\nu}) \right\|_{L_p}^p
\]

\[
\leq \mu_0 \int_{\Omega} (|D(\nu_m + G)|^{p-2} D(\nu_m + G) - |D(\tilde{\nu} + G)|^{p-2} D(\tilde{\nu} + G)) : D(\nu_m - \tilde{\nu}) \, dx
\]

\[
\leq \left\| \mathcal{F}(\theta, \nu, D(\tilde{\nu} + G)) - \mathcal{F}(\theta_m, \nu_m, D(\tilde{\nu} + G)) \right\|_{L_p^p} \left\| D(\nu_m - \tilde{\nu}) \right\|_{L_p}
\]

and thus

\[
\mu_0 \frac{1}{2^{p-1}} \left\| D(\nu_m - \tilde{\nu}) \right\|_{L_p}^{p-1} \leq \left\| \mathcal{F}(\theta, \nu, D(\tilde{\nu} + G)) - \mathcal{F}(\theta_m, \nu_m, D(\tilde{\nu} + G)) \right\|_{L_p}.\]

By using the continuity and uniform boundedness of the mapping \( \mathcal{F} \), the convergence (53) and Lebesgue’s theorem we obtain that

\[
\mathcal{F}(\theta_m, \nu_m + G, D(\tilde{\nu} + G)) \rightarrow \mathcal{F}(\theta, \nu + G, D(\tilde{\nu} + G))
\]

strongly in \((L^p(\Omega))^3 \times 3\).

Thus

\[
\nu_m \rightarrow \tilde{\nu} \quad \text{strongly in } V^p_{0, \text{div}}
\]

and with (51) we infer that \( \nu = \tilde{\nu} \) which yields the announced result.
Case 2: $1 < p < 2$.

Once again with the same computations as in Proposition 5.1

\[
\left( \frac{C_p}{C_{\text{flow}}^{2p}} \right)^{\frac{2}{p}} \| D(u_m - \bar{u}) \|_{L^p}
\leq \frac{1}{\mu_0} \left\| \nabla (\theta, \nu, D(\bar{u} + G)) - \nabla (\theta_m, \nu_m, D(\bar{u} + G)) \right\|_{L^{p'}}
\]

and we may conclude as in the previous case. 

We may construct now a pressure $\pi \in L^p_0(\Omega)$ satisfying

\[
a(\theta; \nu, \varphi - \nu) - \int_\Omega \pi \div (\varphi) \, dx + \Psi(\varphi) - \Psi(\nu) \geq \int_\Omega f \cdot (\varphi - \nu) \, dx \quad \forall \varphi \in V^p_0. \quad (57)
\]

As usual the main tool is De Rham’s theorem.

**Proposition 5.3:** Let $q \in [1, 3/2)$ and $p > 3$ if $q = 1$, $p \geq \frac{3q}{4q - 3}$ otherwise and assume that (9)-(10), (11)-(12)-(13) and (17)-(18)-(19)-(20) hold. Let $(\nu, \theta)$ be obtained as previously. Then there exists a unique $\pi \in L^p_0(\Omega)$ satisfying (57).

**Proof:** We introduce the linear form $\mathcal{L}$ defined on $V^p_0$ by

\[
\mathcal{L}(\psi) = \int_\Omega f \cdot \psi \, dx - a(\theta; \nu, \psi) \quad \forall \psi \in V^p_0. \quad (58)
\]

Clearly $\mathcal{L}$ is continuous on $V^p_0$. Let

\[
W^{1,p}_0(\Omega) = \{ w \in W^{1,p}(\Omega) : w = 0 \text{ on } \partial \Omega \},
\]

\[
W^{1,p}_{0,\text{div}}(\Omega) = \{ w \in W^{1,p}_0(\Omega) : \text{div} \, (w) = 0 \text{ in } \Omega \}.
\]

By choosing $\varphi = \nu \pm \psi$ with $\psi \in W^{1,p}_{0,\text{div}}(\Omega)$ as a test-function in (55) we get $\mathcal{L}(\psi) = 0$ for all $\psi \in W^{1,p}_{0,\text{div}}(\Omega)$. With De Rham’s theorem (see for instance Lemma 2.7 in [27]) we infer that there exists a unique $\pi \in L^p_0(\Omega)$ such that

\[
\mathcal{L}(\psi) = \int_\Omega f \cdot \psi \, dx - a(\theta; \nu, \psi) = \langle \nabla \pi, \psi \rangle_{D'(\Omega),D(\Omega)} \quad \forall \psi \in D(\Omega).
\]

On the other hand, with Green’s formula

\[
a(\theta; \nu, \psi) = \int_\Omega 2\mu (\theta, \nu + G, |D(\nu + G)|) |D(\nu + G)|^{p-2}D(\nu + G) : D(\psi) \, dx \\
= -\left\langle \text{div} \left( 2\mu (\theta, \nu + G, |D(\nu + G)|) |D(\nu + G)|^{p-2}D(\nu + G) \right), \psi \right\rangle_{D'(\Omega),D(\Omega)}
\]

and thus

\[
-\left\langle \text{div} \left( 2\mu (\theta, \nu + G, |D(\nu + G)|) |D(\nu + G)|^{p-2}D(\nu + G) \right), \psi \right\rangle_{D'(\Omega),D(\Omega)} + \langle \nabla \pi, \psi \rangle_{D'(\Omega),D(\Omega)} = \int_\Omega f \cdot \psi \, dx
\]

\[
(59)
\]
for all \( \psi \in \mathcal{D}(\Omega) \). We define the stress tensor as (see (2))

\[
\sigma = 2\mu(\theta, \nu + G, |D(\nu + G)|^p - 2D(\nu + G) - \pi \text{Id}_{\mathbb{R}^3}). \tag{60}
\]

With (59) we have

\[
-\langle \text{div} (\sigma), \psi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \int_\Omega f \cdot \psi \, d\tau \quad \forall \psi \in \mathcal{D}(\Omega).
\]

Recalling that \( f \in L^p(\Omega) \) we obtain that \( \text{div}(\sigma) \in L^{p'}(\Omega) \) and

\[
-\text{div} (\sigma) = f \quad \text{in} \ L^{p'}(\Omega).
\]

With (60) we have also \( \sigma \in L^{p'}(\Omega) \) and with Green’s formula we get

\[
\int_\Omega \sigma : \nabla w \, d\tau - \int_{\partial \Omega} \sum_{i,j=1}^3 \sigma_{ij} n_i n_j \, dY = \int_\Omega f \cdot w \, d\tau \quad \forall w \in W^{1,p}(\Omega).
\]

where we recall that \( n \) denotes the unit outward normal vector to \( \partial \Omega \). We choose \( w = \varphi - \nu \) with \( \varphi \in V^p_0 \). We get

\[
\int_\Omega \sigma : \nabla (\varphi - \nu) \, d\tau - \int_{\Gamma_0} \sum_{i,j=1}^3 \sigma_{ij} n_j (\varphi - \nu)_i \, dx' = \int_\Omega f \cdot (\varphi - \nu) \, d\tau \quad \forall \varphi \in V^p_0.
\]

We observe that the vector \( \sigma \cdot n = (\sum_{j=1}^3 \sigma_{3j} n_j)_{1 \leq i \leq 3} \) can be decomposed as \( \sigma_1 + \sigma_3 n \). Since \( (\varphi - \nu) \cdot n = 0 \) on \( \Gamma_0 \) we obtain

\[
\int_\Omega \sigma : \nabla (\varphi - \nu) \, d\tau - \int_{\Gamma_0} \sigma_1 \cdot (\varphi - \nu) \, dx' = \int_\Omega f \cdot (\varphi - \nu) \, d\tau \quad \forall \varphi \in V^p_0
\]

and by replacing \( \sigma \) by its expression (see (60)) we get

\[
a(\theta; \nu, \varphi - \nu) - \int_\Omega \pi \text{ div} (\varphi) \, d\tau - \int_\Omega f \cdot (\varphi - \nu) \, d\tau = \int_{\Gamma_0} \sigma_1 \cdot (\varphi - \nu) \, dx'
\]

i.e.

\[
a(\theta; \nu, \varphi - \nu) - \int_\Omega \pi \text{ div} (\varphi) \, d\tau - \int_\Omega f \cdot (\varphi - \nu) \, d\tau + \Psi(\varphi) - \Psi(\nu)
\]

\[
= \int_{\Gamma_0} \sigma_1 \cdot (\varphi - \nu) \, dx' + \Psi(\varphi) - \Psi(\nu) \quad \forall \varphi \in V^p_0.
\]

On the other hand, we have

\[
\int_{\partial \Omega} \varphi \cdot n \, dY = 0 \quad \forall \varphi \in V^p_0
\]

so there exists \( \hat{\varphi} \in W^{1,p}(\Omega) \) such that \( \text{div}(\hat{\varphi}) = 0 \) in \( \Omega \) and \( \hat{\varphi} = \varphi \) on \( \partial \Omega \) (see for instance Corollary 3.8 in [27]). Hence \( \hat{\varphi} \in V^p_{0, \text{div}} \) and with (55)

\[
a(\theta; \nu, \hat{\varphi} - \nu) - \int_\Omega \pi \text{ div} (\hat{\varphi}) \, d\tau - \int_\Omega f \cdot (\hat{\varphi} - \nu) \, d\tau + \Psi(\hat{\varphi}) - \Psi(\nu)
\]

\[
= \int_{\Gamma_0} \sigma_1 \cdot (\hat{\varphi} - \nu) \, dx' + \Psi(\hat{\varphi}) - \Psi(\nu) \geq 0.
\]
But
\[ \int_{\Gamma_0} \sigma_\tau \cdot (\varphi - \upsilon) \, dx' + \Psi(\varphi) - \Psi(\upsilon) = \int_{\Gamma_0} \sigma_\tau \cdot (\hat{\varphi} - \upsilon) \, dx' + \Psi(\hat{\varphi}) - \Psi(\upsilon) \]
since \( \hat{\varphi} = \varphi \) on \( \partial \Omega \). Thus
\[ a(\theta; \upsilon, \varphi - \upsilon) \rangle - \int_{\Omega} \pi \, \text{div} (\varphi) \, dx - \int_{\Omega} f \cdot (\varphi - \upsilon) \, dx + \Psi(\varphi) - \Psi(\upsilon) \]
\[ = \int_{\Gamma_0} \sigma_\tau \cdot (\varphi - \upsilon) \, dx' + \Psi(\varphi) - \Psi(\upsilon) \geq 0 \quad \forall \varphi \in V^0_\Omega \]
and the variational inequality (57) is satisfied.

Finally we can state the following existence result

**Theorem 5.4:** Let \( p > 3/2 \) and \( q \in [1, 3/2) \) satisfying the compatibility condition (15) i.e. \( p > 3 \) if \( q = 1 \), \( p \geq \frac{3q}{4q-3} \) otherwise and assume that (9)–(10), (11)–(12)–(13) and (17)–(18)–(19)–(20) hold. Then the coupled problem (P) admits a solution.

**Proof:** Let us prove that the triplet \((\upsilon, \pi, \theta)\) is a solution to the coupled problem (P) i.e. let us prove that
\[ \int_{\Omega} ((\upsilon + G) \cdot \nabla \theta) \, dx + \int_{\Omega} (K \nabla \theta) \cdot \nabla w \, dx \]
\[ = \int_{\Omega} 2\mu(\theta, \upsilon + G, |D(\upsilon + G)|) |D(\upsilon + G)|^p w \, dx \]
\[ + \int_{\Omega} r(\theta) w \, dx + \int_{\Gamma_0} \theta^b w \, dx' \quad \forall w \in C^1(\overline{\Omega}) \text{ s.t. } w = 0 \text{ on } \Gamma_1 \cup \Gamma_L. \]

With Proposition 5.2, by extracting possibly another subsequence, we have
\[ D(\upsilon_m + G) \to D(\upsilon + G) \quad \text{a.e. in } \Omega. \] (61)

By using the continuity and uniform boundedness of the mappings \( \mu \) and \( r \) combined with the convergence properties (53)–(54)–(61) we obtain that
\[ h_{1/m}(\theta_m, \upsilon_m) \to 2\mu(\theta, \upsilon + G, |D(\upsilon + G)|) |D(\upsilon + G)|^p \quad \text{strongly in } L^1(\Omega) \] (62)
and
\[ r(\theta_m) \to r(\theta) \quad \text{strongly in } L^1(\Omega). \] (63)

Moreover
\[ \theta^b_{1/m} \to \theta^b \quad \text{strongly in } L^1(\Gamma_0). \] (64)

These properties allow us to pass to the limit in the right-hand side of (50) for all \( w \in C^1(\overline{\Omega}) \) such that \( w = 0 \) on \( \Gamma_1 \cup \Gamma_L \).

As explained at the beginning of Section 5 the uniform boundedness of the sequence \( (\theta_m)_{m \geq 1} \) in \( W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega) \) does not yield any good convergence property for \( (\nabla \theta_m)_{m \geq 1} \) when \( q = 1 \) but we already know that the (sub-)sequence \( (\theta_m)_{m \geq 1} \) converges strongly to \( \theta \) in \( L^\rho(\Omega) \) with \( \rho = \frac{(c+1)q}{2-q} \in (1, q_*) \) (see (52)).
We may improve this result and we will show that \((\theta_m)_{m \geq 1}\) converges strongly to \(\theta\) in \(W^{1,q}_{\Gamma_1 \cup \Gamma_L} (\Omega)\). Indeed let \(m_1 \geq 1\) and \(m_2 \geq 1\). With (47) we have \(\forall w \in W^{1,2}_{\Gamma_1 \cup \Gamma_L} (\Omega)\)

\[
B(u_{m_1}, \theta_{m_1}, w) - B(u_{m_2}, \theta_{m_2}, w) = L_{1/m_1}(u_{m_1}, \theta_{m_1}, w) - L_{1/m_2}(u_{m_2}, \theta_{m_2}, w)
\]

i.e.

\[
\int_{\Omega} ((u_{m_1} + G) \cdot \nabla(\theta_{m_1} - \theta_{m_2})) \, dx + \int_{\Omega} (K \nabla(\theta_{m_1} - \theta_{m_2})) \cdot \nabla w \, dx
\]

\[
= \int_{\Omega} (u_{m_2} - u_{m_1}) \cdot \nabla\theta_{m_2} \, dx + \int_{\Omega} (g_{1/m_1}(\theta_{m_1}, u_{m_1}) - g_{1/m_2}(\theta_{m_2}, u_{m_2})) \, dx
\]

\[
+ \int_{\Gamma_0} (r(\theta_{m_1}) - r(\theta_{m_2})) \, dx + \int_{\Gamma_0} (\theta_{1/m_1} - \theta_{1/m_2}) \, dx' \quad \forall w \in W^{1,2}_{\Gamma_1 \cup \Gamma_L} (\Omega).
\]

By using the same technique as in Proposition 4.2 we obtain

\[
k_0 \zeta \int_{\Omega} \frac{|\nabla(\theta_{m_1} - \theta_{m_2})|^2}{(1 + |(\theta_{m_1} - \theta_{m_2})|)^{\zeta+1}} \, dx \leq \|(u_{m_2} - u_{m_1}) \cdot \nabla\theta_{m_2}\|_{L^1(\Omega)}
\]

\[
+ \|g_{1/m_1}(\theta_{m_1}, u_{m_1}) - g_{1/m_2}(\theta_{m_2}, u_{m_2})\|_{L^1(\Omega)} + \|r(\theta_{m_1}) - r(\theta_{m_2})\|_{L^1(\Omega)}
\]

\[
+ \|\theta_{1/m_1} - \theta_{1/m_2}\|_{L^1(\Gamma_0)}
\]

and recalling that \(\zeta \in (0, \frac{3-q}{2})\) and \(\rho = \left(\frac{\zeta+1}{2-q}\right) \in (1, q_*)\) we get

\[
\|\theta_{m_1} - \theta_{m_2}\|_{1,q} \leq C_{m_1, m_2} 2^{(\rho-1)\frac{2-q}{2q}} \left(\|\Omega\|_{q_*, q} + \|\theta_{m_1} - \theta_{m_2}\|_{L^p(\Omega)}^{\frac{2-q}{2q}}\right)
\]

(65)

with

\[
C_{m_1, m_2} = \frac{1}{\sqrt{k_0 \zeta}} \left(\|(u_{m_2} - u_{m_1}) \cdot \nabla\theta_{m_2}\|_{L^1(\Omega)}
\]

\[
+ \|g_{1/m_1}(\theta_{m_1}, u_{m_1}) - g_{1/m_2}(\theta_{m_2}, u_{m_2})\|_{L^1(\Omega)}
\]

\[
+ \|r(\theta_{m_1}) - r(\theta_{m_2})\|_{L^1(\Omega)} + \|\theta_{1/m_1} - \theta_{1/m_2}\|_{L^1(\Gamma_0)}\right)^{1/2}.
\]

(66)

If \(q \in (1, 3/2)\) we have

\[
\|(u_{m_2} - u_{m_1}) \cdot \nabla\theta_{m_2}\|_{L^1(\Omega)} \leq \|(u_{m_2} - u_{m_1})\|_{L^q(\Omega)} \|\nabla\theta_{m_2}\|_{L^{q'}(\Omega)}
\]

\[
= \|(u_{m_2} - u_{m_1})\|_{L^q(\Omega)} \|\theta_{m_2}\|_{L^q(\Omega)}
\]

and \(V_{0,\text{div}}^p\) is continuously embedded into \(L^{q'}(\Omega)\) thanks to the compatibility condition \(p \geq \frac{3q}{4q-3}\).

Since \((\theta_m)_{m \geq 1}\) is bounded in \(W^{1,q}_{\Gamma_1 \cup \Gamma_L} (\Omega)\) we infer from (54) that

\[
\lim_{m_1, m_2 \to +\infty} \|(u_{m_2} - u_{m_1}) \cdot \nabla\theta_{m_2}\|_{L^1(\Omega)} = 0.
\]

If \(q = 1\) we have \(p > 3\) and \(V_{0,\text{div}}^p\) is continuously embedded into \(C^0(\overline{\Omega})\) and we have

\[
\|(u_{m_2} - u_{m_1}) \cdot \nabla\theta_{m_2}\|_{L^1(\Omega)} \leq \|(u_{m_2} - u_{m_1})\|_{C^0(\overline{\Omega})} \|\nabla\theta_{m_2}\|_{L^1(\Omega)}
\]

\[
= \|(u_{m_2} - u_{m_1})\|_{C^0(\overline{\Omega})} \|\theta_{m_2}\|_{L^1(\Omega)}
\]
and we obtain once again
\[ \lim_{m_1,m_2 \to +\infty} \| (\nu_{m_2} - \nu_{m_1}) \cdot \nabla \theta_{m_2} \|_{L^1(\Omega)} = 0. \]

Hence with (65) – (66) and (62) – (63) – (64) we may conclude that the (sub-)sequence \((\theta_m)_{m \geq 1}\) is a Cauchy sequence in \(W^{1,q}_{1,1}(\Omega)\) and with (52) we infer that
\[ \theta_m \to \theta \quad \text{strongly in } W^{1,q}_{1,1}(\Omega). \]

Finally we choose \(w \in C^1(\overline{\Omega})\) such that \(w = 0\) on \(\Gamma_1 \cup \Gamma_L\) as a test-function in (50). We can now pass to the limit in all the terms and we get
\[
\int_{\Omega} ( (\nu + G) \cdot \nabla \theta) w \, dx + \int_{\Omega} (K \nabla \theta) \cdot \nabla w \, dx \\
= \int_{\Omega} 2\mu(\theta, \nu + G, |D(\nu + G)|) |D(\nu + G)|^p w \, dx \\
+ \int_{\Omega} r(\theta) w \, dx + \int_{\Gamma_0} \theta^b w \, dx' \quad \forall \ w \in D(\overline{\Omega}) \quad \text{s.t. } w = 0 \quad \text{on } \Gamma_1 \cup \Gamma_L.
\]
Together with Proposition 5.3 we infer that the triplet \((\nu, \pi, \theta)\) is a solution to problem (P). ■

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

**Funding**

This work was supported by Université Jean Monnet de Saint-Etienne [UJM].

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