Effective Construction of a Positive Operator which does not admit Triangular Factorization

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Abstract. We have constructed a concrete example of a non-factorable positive operator. As a result, for the well-known problems (Ringrose, Kadison and Singer problems) we replace existence theorems by concrete examples.

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1 Introduction

We introduce the main notions of the triangular factorization (see [3, 5, 7] and [10, 11, 16]).

In the Hilbert space $L^2_m(a, b)$, $(-\infty \leq a < b \leq \infty)$ we define the orthogonal projectors

$$P_\xi f = f(x), \quad a \leq x < \xi$$

and

$$P_\xi f = 0, \quad \xi < x \leq b,$$

where $f(x) \in L^2_m(a, b)$.

**Definition 1.1.** A bounded operator $S_-$ on $L^2_m(a, b)$ is called lower triangular if for every $\xi$ the relations

$$S_-Q_\xi = Q_\xi S_- Q_\xi,$$  \hspace{1cm} (1.1)

where $Q_\xi = I - P_\xi$, are true. A bounded operator $S_+$ is called upper triangular, if $S_+^*$ is lower triangular.

**Definition 1.2.** A bounded, positive definite and invertible operator $S$ on $L^2_m(a, b)$ is said to admit the left (the right) triangular factorization if it can be represented in the form

$$S = S_- S_+^* \quad (S = S_+^* S_-),$$  \hspace{1cm} (1.2)

where $S_-$ and $S_-^*$ are lower triangular, bounded operators.

In paper [16] (p. 291) we formulated the necessary and sufficient conditions under which a positive definite operator $S$ admits the triangular factorization. The factorizing operator $S_-^*$ was constructed in the explicit form.

Let us introduce the notations

$$S_\xi = P_\xi S P_\xi, \quad (f, g)_\xi = \int_0^\xi f^*(x)g(x)dx, \quad f, g \in L^2_m(0, b).$$  \hspace{1cm} (1.3)

**Theorem 1.1** ([16], p. 291) Let the bounded and invertible operator $S$ on $L^2_m(a, b)$ be positive definite. For the operator $S$ to admit the left triangular factorization it is necessary and sufficient that the following assertions be true.

1. There exists such a $m \times m$ matrix function $F_0(x)$ that

$$Tr \int_a^b F_0^*(x)F_0(x)dx < \infty,$$  \hspace{1cm} (1.4)
and the $m \times m$ matrix function

$$M(\xi) = (F_0(x), S_\xi^{-1} F_0(x))_\xi$$

(1.5)

is absolutely continuous and almost everywhere

$$\det M'(\xi) \neq 0.$$  

(1.6)

2. The vector functions

$$\int_a^x v^*(x,t) f(t) dt$$

are absolutely continuous. Here $f(x) \in L^2_m(a,b)$, 

$$v(\xi,t) = S_\xi^{-1} P_\xi F_0(x),$$

(1.8)

(In (1.8), the operator $S_\xi^{-1}$ is applied columnwise.)

3. The operator

$$Vf = [R^*(x)]^{-1} \frac{d}{dx} \int_a^x v^*(x,t) f(t) dt,$$

(1.9)

is bounded, invertible and lower triangular together with its inverse $V^{-1}$. Here $R(x)$ is a $m \times m$ matrix function such that

$$R^*(x) R(x) = M'(x).$$

(1.10)

**Corollary 1.1.** ([16], p.293) If the conditions of Theorem 1.1 are fulfilled then the corresponding operator $S^{-1}$ can be represented in the form

$$S^{-1} = V^* V.$$ 

(1.11)

The formulated theorem allows us to prove that a wide class of operators admits the triangular factorization [16].

D.Larson proved [7] the existence of positive definite and invertible but non-factorable operators.

In the present article we construct a concrete example of positive definite and invertible but non-factorable operator.

Using this result we have managed to substitute existence theorems by concrete examples in the well-known problems (Ringrose, Kadison and Singer problems).
2 Special class of operators

We consider the operators $S$ of the form

$$Sf = f(x) - \mu \int_{0}^{\infty} h(x-t)f(t)dt, \quad f(x)\in L^2(0,\infty), \quad (2.1)$$

where $\mu = \overline{\mu}$ and $h(x)$ has the representation

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\lambda} \rho(\lambda)d\lambda. \quad (2.2)$$

We suppose that the function $\rho(\lambda)$ satisfies the following conditions

1. $\rho(\lambda) = \overline{\rho(\lambda)} \in L(-\infty, \infty)$. 
2. The function $\rho(\lambda)$ is bounded, i.e. $|\rho(\lambda)| \leq M$, $-\infty < \lambda < \infty$.

Hence the function $h(x)$ $(-\infty < x < \infty)$ is continuous and the operator

$$Hf = \int_{0}^{\infty} h(x-t)f(t)dt \quad (2.3)$$

is self-adjoint and bounded $||H|| \leq M$. We introduce the operators

$$S_{\xi}f = f(x) - \mu \int_{0}^{\xi} h(x-t)f(t)dt, \quad f(x)\in L^2(0,\xi), \quad 0 < \xi < \infty. \quad (2.4)$$

If $|\mu| \leq 1/M$, then the operator $S_{\xi}$ is positive definite, bounded and invertible. Hence we have

$$S_{\xi}^{-1}f = f(x) + \int_{0}^{\xi} R_{\xi}(x,t,\mu)f(t)dt. \quad (2.5)$$

The function $R_{\xi}(x,t,\mu)$ is jointly continuous to the variables $x,t,\xi,\mu$. M.G.Krein (see [4], Ch.IV , section 7) proved that

$$S_{b}^{-1} = (I + V_+)(I + V_-), \quad 0 < b < \infty; \quad (2.6)$$

where the operators $V_+$ and $V_-$ are defined in $L^2(0,b)$ by the relations

$$V_+^*f = V_-f = \int_{0}^{x} R_{x}(x,t,\mu)f(t)dt. \quad (2.7)$$

Remark 2.1 The Krein’s result (2.6) is true for the Fredholm class of operators. The operator $S_b$ belongs to this class but the operator $S$ does not belong. When considering the operator $S$ we use Theorem 1.1.
3 Main example

Let us consider the operator $S$, defined by formula (2.1), where

$$h(x) = \frac{\sin \pi x}{\pi x} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix\lambda} d\lambda, \quad 0 < \mu < 1.$$  \hspace{1cm} (3.1)

In case (3.1) the function $\rho(\lambda)$ has the form

$$\rho(\lambda) = 1, \quad \lambda \in [-\pi, \pi]; \quad \rho(\lambda) = 0 \quad \lambda \notin [-\pi, \pi].$$ \hspace{1cm} (3.2)

Hence we have $M = 1$ and the following statement is true.

**Proposition 3.1.** The operator

$$Sf = f(x) - \mu \int_0^\infty \frac{\sin \pi (x-t)}{\pi (x-t)} f(t) dt, \quad f(x) \in L^2(0, \infty), \quad 0 < \mu < 1$$ \hspace{1cm} (3.3)

is self-adjoint, bounded, invertible and positive definite.

The following assertion is the main result of this paper.

**Theorem 3.1.** The bounded positive definite and invertible operator $S$, defined by formula (3.3), does not admit the left triangular factorization

We shall prove Theorem 3.1 by parts. The key results will be written in the form of lemmas and propositions.

Let us consider the following functions

$$Q_\xi(\xi, \xi, \mu) = R_2(2\xi, 2\xi, \mu), \quad Q_\xi(\xi, -\xi, \mu) = R_2(2\xi, 0, \mu).$$ \hspace{1cm} (3.4)

We use the relation (see [19], p.16, formula (77))

$$\frac{d}{dt}[Q_t(t, t, \mu)] = 2Q_t^2(-t, t, \mu),$$ \hspace{1cm} (3.5)

and the asymptotic representation (see [9], p.189, formulas (1.16) and (1.17))

$$\sigma(x, \mu) = a(\mu)x + b(\mu) + F_{-1}(x)/x + O(1/x^2), \quad x \to \infty.$$ \hspace{1cm} (3.6)

where

$$a(\mu) = \frac{1}{\pi}\log(1 - \mu), \quad b(\mu) = \frac{1}{2}a^2(\mu),$$ \hspace{1cm} (3.7)

$$F_{-1}(x) = \frac{1}{4}a(\mu)^2\sin[2x + x_0 + k\log(x)] + m.$$ \hspace{1cm} (3.8)
Here the fixed numbers $x_0$, $k$, $m$ are real and $k \neq 0$. The functions $Q_t(t, t, \mu)$ and $\sigma(x, \mu)$ are connected by the relation (see [9], p.189, formula (1.18))

$$\sigma(x, \mu) = -2tQ_t(t, t, \mu), \quad \text{where} \quad x = 2\pi t. \quad (3.9)$$

It follows from (3.6) that

$$\frac{d\sigma(x, \mu)}{dx} = a(\mu) + x^{-1}a(\mu)^2 \cos[2x + x_0 + k\log(x)] + O(1/x^2), \quad x \to \infty. \quad (3.10)$$

From (3.6) and (3.9) we deduce that

$$Q_t(t, t, \mu) = -a(\mu)\pi - b(\mu)/(2t) - F_{-1}(2\pi t)/(2\pi t^2) + O(1/t^3). \quad (3.11)$$

Relations (3.5), (3.8) and (3.11) imply

$$Q_t^2(-t, t, \mu) = a^2(\mu)\sin[2\pi t + x_0 + k\log(t)]/(4t^2) + O(1/t^3). \quad (3.12)$$

From (3.4) and (3.12) we deduce the assertion.

**Lemma 3.1** One of the two relations

$$R_t(t, 0, \mu) = \epsilon \frac{a(\mu)}{t} \sin[\pi t + x_0 + k\log(t)] + O(1/t^{3/2}), \quad t \to \infty \quad (3.13)$$

is valid. Here $\epsilon = \pm 1$.

Let us introduce the function

$$q_1(x) = 1 + \int_0^x R_x(x, t, \mu)dt. \quad (3.14)$$

According to the well-known Krein’s formula ([4], Ch.IV, formulas (8.3) and (8.14)) we have

$$q_1(x) = \exp[\int_0^x R_t(t, 0, \mu)dt]. \quad (3.15)$$

**Lemma 3.2** The following relation

$$\lim_{x \to \infty} q_1(x) = \frac{1}{\sqrt{1 - \mu}} \quad (3.16)$$

is true.

**Proof.** Using the asymptotical formula of confluent hypergeometric function (see [1], section 6.13) we deduce that the integral

$$\exp[\int_0^x R_t(t, 0, \mu)dt] = C \quad (3.17)$$
converges. Together with \( q_1(x) \) we shall consider the function

\[
q_2(x) = M(x) + \int_0^x M(t)R_x(x,t,\mu)dt,
\]

where

\[
M(x) = \frac{1}{2} - \mu \int_0^x \sin(s\pi)\frac{ds}{s\pi} ds.
\]

(3.19)

The function \( M(x) \) can be represented in the form

\[
M(x) = (1 - \mu)/2 + q(x).
\]

(3.20)

Using asymptotic of sinus integral (see [2], Ch. 9, formulas (2) and (10)), we have

\[
q(x) = O(1/x), \quad x \to \infty.
\]

(3.21)

From (3.15), (3.17) and (3.20) we deduce

\[
\lim_{x \to \infty} q_1(x) = C, \quad \lim_{x \to \infty} q_2(x) = C(1 - \mu)/2.
\]

(3.22)

Taking into account the relation \( q_1(x)q_2(x) = 1/2 \) (see [15], formulas (53), (56)) we deduce the equality

\[
\lim_{x \to \infty} q_2(x) = 1/(2C)
\]

(3.23)

Relation (3.16) follows directly from formulas (3.22), (3.23) and inequality \( C > 0 \). The lemma is proved.

We note, that relations (3.15) and (3.16) define the sign of \( \epsilon \) in equality (3.13).

The functions \( q_1(x) \) and \( q_2(x) \) generate the \( 2 \times 2 \) differential system

\[
\frac{dW}{dx} = izJH(x)W, \quad W(0, z) = I_2.
\]

(3.24)

Here \( W(x, z), J, H(x) \) are \( 2 \times 2 \) matrix functions and

\[
H(x) = \begin{bmatrix}
q_1^2(x) & 1/2 \\
1/2 & q_2^2(x)
\end{bmatrix}, \quad J = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]

(3.25)

It is easy to see that

\[
JH(x) = T(x)PT^{-1}(x),
\]

(3.26)
where
\[
T(x) = \begin{bmatrix} q_2(x) & -q_2(x) \\ q_1(x) & q_1(x) \end{bmatrix},
\quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\] (3.27)

Let us consider the matrix function
\[
V(x, z) = e^{-ixz/2}T^{-1}(x)W(x, z)T(0).
\] (3.28)

Due to (3.24)-(3.28) we have
\[
\frac{dV}{dx} = (iz/2)jV - Q(x)V, \quad V(0) = I_2,
\] (3.29)
where
\[
Q(x) = \begin{bmatrix} 0 & B(x) \\ B(x) & 0 \end{bmatrix},
\quad j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\] (3.30)
\[
B(x) = \frac{q'_1(x)}{q_1(x)} = R_x(x, 0, \mu).
\] (3.31)

Let us introduce the functions
\[
\Phi_n(x, z) = v_{1,n}(x, z) + v_{2,n}(x, z), \quad (n = 1, 2),
\] (3.32)
\[
\Psi_n(x, z) = i[v_{1,n}(x, z) - v_{2,n}(x, z)], \quad (n = 1, 2),
\] (3.33)
where \(v_{i,n}(x, z)\) are elements of the matrix function \(V(x, z)\). It follows from (3.29) that
\[
\frac{d\Phi_n}{dx} = (z/2)\Psi_n - B(x)\Phi_n, \quad \Phi_1(0, z) = \Phi_2(0, z) = 1
\] (3.34)
\[
\frac{d\Psi_n}{dx} = -(z/2)\Phi_n + B(x)\Psi_n, \quad \Psi_1(0, z) = -\Psi_2(0, z) = i.
\] (3.35)

Let us consider again the differential system (3.24) and the solution \(W(x, z)\) of this system. The element \(w_{2,1}(\xi, z)\) of the matrix function \(W(x, z)\) can be represented in the form (see [13], p.54, formula (2.5))
\[
w_{2,1}(\xi, z) = iz((I - Az)^{-1}1, S^{-1}_\xi 1)_{\xi},
\] (3.36)
where the operator \(A\) has the form
\[
Af = i \int_0^x f(t)dt.
\] (3.37)
It is well-known that
\[ (I - Az)^{-1}1 = e^{izx}. \] (3.38)

Now we need the relations (see [12], Ch.1, formulas (1.37) and (1.44)):
\[ S_{\xi}1 = M(x) + M(\xi - x), \quad S_{\xi} = U_{\xi}S_{\xi}U_{\xi}, \] (3.39)
where the function \( M(x) \) is defined by relation (3.20) and \( U_{\xi}f(x) = f(\xi - x), \quad 0 \leq x \leq \xi \). It follows from (3.39) that
\[ S_{\xi}1 = 1 - \mu + q(x) + U_{\xi}q(x). \] (3.40)

Hence the relation
\[ S_{\xi}^{-1}1 = \frac{1}{(1 - \mu)}[1 - R_{\xi}(x) - U_{\xi}R_{\xi}(x)], \quad q(x) = O(1/x), \] (3.41)
where \( S_{\xi}^{-1}q(x) = R_{\xi}(x) \) is true. According to formulas (3.36) and (3.41) the following statement is true.

**Lemma 3.3** The function \( w_{2,1}(\xi, z) \) has the form
\[ w_{2,1}(\xi,z) = e^{iz\xi}G(\xi,z) - \overline{G(\xi,z)}, \] (3.42)
where
\[ G(\xi,z) = \frac{1}{1 - \mu}[1 - iz \int_{0}^{\xi} e^{-ixx}R_{\xi}(x)dx]. \] (3.43)

We can obtain another representation of \( w_{2,1}(\xi,z) \) without using the operator \( S_{\xi}^{-1} \). Indeed, it follows from (3.28) and (3.32),(3.33) that
\[ W(x,z) = (1/2) e^{ixz/2} T(x) \begin{bmatrix} \Phi_1 - i\Psi_1 & \Phi_2 - i\Psi_2 \\ \Phi_1 + i\Psi_1 & \Phi_2 + i\Psi_2 \end{bmatrix} T^{-1}(0). \] (3.44)

According to equality (3.14) we have \( q_1(0) = 1 \). Due to (3.27) we infer
\[ T(0) = \begin{bmatrix} 1/2 & -1/2 \\ 1 & 1 \end{bmatrix}, \quad T^{-1}(0) = \begin{bmatrix} 1 & 1/2 \\ -1 & 1/2 \end{bmatrix}. \] (3.45)

It follows from (3.16),(3.22) and (3.27) that
\[ T(x) \rightarrow \begin{bmatrix} 1/(2C) & -1/(2C) \\ C & C \end{bmatrix}, \quad x \rightarrow \infty, \quad C = 1/\sqrt{(1 - \mu)}. \] (3.46)
Hence in view of (3.45)-(3.46) the following assertion is true.

**Lemma 3.4.** The function $w_{2,1}(x, z)$ has the form $(x \to \infty)$

$$w_{2,1}(x, z) = Ce^{ixz/2}\phi(x, z)(1 + o(1)), \phi(x, z) = \Phi_1(x, z) - \Phi_2(x, z). \quad (3.47)$$

Further we plan to use one Krein’s result [6]. To do it we introduce the functions

$$P(x, z) = e^{ixz/2}[\Phi(x, z) - i\Psi(x, z)]/2, \quad (3.48)$$
$$P_\ast(x, z) = e^{ixz/2}[\Phi(x, z) + i\Psi(x, z)]/2, \quad (3.49)$$

where

$$\Phi(x, z) = \Phi_1(x, z) + \Phi_2(x, z), \quad \Psi(x, z) = \Psi_1(x, z) + \Psi_2(x, z). \quad (3.50)$$

Using (3.34),(3.35) and (3.48),(3.40) we see that $P(x, z)$ and $P_\ast(x, z)$ is the solution of the following Krein’s system

$$\frac{dP}{dx} = (iz/2)P - B(x)P_\ast, \quad \frac{dP_\ast}{dx} = -B(x)P, \quad (3.51)$$

where

$$P(0, z) = P_\ast(0, z) = 1 \quad (3.52)$$

The coefficient $B(x)$ belongs to $L^2(0, \infty)$ (see (3.13) and (3.31)). Hence the following Krein’s results are true [6].

**Proposition 3.2**

1) There exists the limit

$$\Pi(z) = \lim_{x \to \infty} P_\ast(x, z), \quad (3.53)$$

where the convergence is uniform at any bounded closed set $z$ of the open half-plane $\text{Im} z > 0$.

2) The function $\Pi(z)$ can be represented in the form

$$\Pi(z) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{1 + tz/2}{(t - z/2)(1 + t^2)}(\log \sigma'(t))dt + i\alpha\right], \quad (3.54)$$

where $\alpha = \overline{\alpha}$.

Here $\lambda = z/2$ is the spectral parameter of system (3.52), $\sigma(u)$ is the spectral function of this system and is defined by the relation (see [6], formula (2)):

$$\sigma'(u) = 1/(2\pi) \quad (|u| > \pi), \quad \sigma'(u) = (1 - \mu)/(2\pi) \quad (|u| < \pi). \quad (3.55)$$
In case (3.3) the following conditions are fulfilled:

\[ 1 - \mu \leq |S| \leq 1 + \mu, \quad \int_{0}^{\infty} |h(x)|^2 dx < \infty. \] (3.56)

Therefore in case (3.3) we can use Proposition 1 and formula (2.15) from paper [17]. Hence in formula (3.54) we have

\[ \alpha = 0. \] (3.57)

It follows from (3.55) and (3.57) that (3.54) takes the form

\[ \Pi(z) = \exp \left[ \frac{1}{2i \pi} \int_{-\pi}^{\pi} \frac{\log(1 - \mu) - z/2}{t - z/2} dt \right] \] (3.58)

Now we need the relation

\[ \int_{-\pi}^{\pi} \frac{1}{t - z/2} dt = \frac{1}{2} \log \frac{(2\pi - x)^2 + y^2}{(2\pi + x)^2 + y^2} + iL(x, y), \] (3.59)

where \( z = x + iy \) and

\[ L(x, y) = \arctan \frac{2\pi - x}{y} + \arctan \frac{2\pi + x}{y}. \] (3.60)

Due to (3.50) and (3.60) we obtain that

\[ \lim_{y \to +0} \int_{-\pi}^{\pi} \frac{1}{t - z/2} dt = \log \left| \frac{2\pi - x}{2\pi + x} \right| + i\pi \chi(x), \] (3.61)

where \( \chi(x) = 1 \) when \( |x| < 2\pi \), \( \chi(x) = 0 \) when \( |x| > 2\pi \). Taking into account equalities (3.58) and (3.61) we have

\[ \Pi(x) = \lim_{y \to +0} \Pi(x + iy) = \left| \frac{2\pi - x}{2\pi + x} \right|^{1/(2i\pi)} \sqrt{1 - \mu \chi(x)}, \] (3.62)

\[ \Pi(0) = \lim_{y \to +0} \Pi(iy) = \sqrt{1 - \mu}. \] (3.63)

Using the spectral function \( \sigma(u) \) we can construct the Weyl-Titchmarsh function \( v(z) \) of system (3.51) (see [13]):

\[ v(z) = \int_{-\infty}^{\infty} \frac{1 + tz/2}{(t - z/2)(1 + t^2)} \sigma'(t) dt + \frac{2z\mu}{4\pi^2 - z^2}. \] (3.64)
It follows from (3.59) that
\[
\lim_{y \to +0} v(z) = i \frac{1 - \mu}{2} - \frac{\mu}{\pi} \log \frac{2\pi - x}{2\pi + x} + \frac{2x\mu}{4\pi^2 - x^2}, \quad x \in (-2\pi, 2\pi).
\] (3.65)

Hence the relation
\[
v(0) = \lim_{y \to +0} v(iy) = i(1 - \mu)/2
\] (3.66)
is true. Together with \(P(x, z)\) and \(P_*(x, z)\) let us introduce another solution \(\hat{P}(x, z)\) and \(\hat{P}_*(x, z)\) of system (3.51), where
\[
\hat{P}(0, z) = 1/2, \quad \hat{P}_*(0, z) = -1/2.
\] (3.67)
The following statements are proved in the book ([13], Ch.10).

**Proposition 3.3**
1) There exist the limits
\[
\hat{\Pi}(z) = \lim_{x \to \infty} \hat{P}_*(x, z), \quad \lim_{x \to \infty} \hat{P}(x, z) = 0.
\] (3.68)
where the convergence is uniform at any bounded closed set \(z\) of the open half-plane \(\text{Im} z > 0\).

2) The equality holds
\[
v(z) = -i\Pi^{-1}(z)\hat{\Pi}(z), \quad \exists z > 0.
\] (3.69)
According to (3.60), (3.63) and (3.66) the relations
\[
\hat{\Pi}(z) = iv(z)\Pi(z), \quad \hat{\Pi}(0) = -[(1 - \mu)^{3/2}]/2.
\] (3.70)
are valid. We introduce the function
\[
\psi(x, z) = \Psi_1(x, z) - \Psi_2(x, z).
\] (3.71)
The functions \(\phi(x, z) - i\psi(x, z)\) and \(\phi(x, z) + i\psi(x, z)\) satisfy the the equation (3.51) and \(\phi(0, z) = 0, \psi(0, z) = 2i\). Hence we have
\[
\hat{P}(x, z) = \frac{1}{4} [\phi(x, z) - i\psi(x, z)] e^{ixz/2}, \quad \hat{P}_*(x, z) = \frac{1}{4} [\phi(x, z) + i\psi(x, z)] e^{ixz/2}.
\] (3.72)
It follows from (3.72) that
\[
\hat{P}(x, z) + \hat{P}_*(x, z) = \frac{1}{2} \phi(x, z) e^{ixz/2}.
\] (3.73)
Proof of Theorem 3.1. Let us suppose that the operator $S$ admits the factorization. According to Theorem 1.1 the corresponding operator $S^{-1}$ has the form $S^{-1} = I + V$, where $V$ is defined by (2.7). Hence the operator function $S^{-1}_\xi$ strongly converges to the operator $S^{-1}$ when $\xi \to \infty$. Then the function $R_\xi(x) = S^{-1}_\xi q(x)$ strongly converges to $R(x) = S^{-1} q(x)$ and $R(x) \in L^2(0, \infty)$. Using (3.42) and (3.43) we have

$$e^{-iz\xi/2}w_{2,1}(\xi, z) = e^{iz\xi/2}G(z) - e^{-iz\xi/2\overline{G}(z)} + o(1), \xi \to \infty,$$  \hspace{1cm} (3.74)

where

$$G(z) = \frac{1}{1 - \mu} [1 - iz \int_0^\infty e^{-izx} \overline{R(x)} dx], \quad z = \bar{z}. \hspace{1cm} (3.75)$$

Taking into account Lemma 3.4, Proposition (3.3) and relation (3.73) we have

$$e^{-iz\xi/2}w_{2,1}(\xi, z) = e^{iz\xi/2}H(z) - e^{-iz\xi/2\overline{H}(z)} + o(1), \xi \to \infty,$$  \hspace{1cm} (3.76)

where

$$\overline{H(z)} = -2C\Pi(z), \quad z = \bar{z}. \hspace{1cm} (3.77)$$

Comparing formulas (3.75) and (3.77) we see that

$$G(0) = 1/(1 - \mu) \neq H(0) = (1 - \mu). \hspace{1cm} (3.78)$$

Hence the relation (3.75) is not true, i.e. the operator $S$ does not admit the factorization. The theorem is proved.

4 Examples instead of existence theorems

Let the nest $N$ be the family of subspaces $Q_\xi L^2(0, \infty)$. The corresponding nest algebra $Alg N$ is the algebra of all linear bounded operators in the space $L^2(0, \infty)$ for which every member of $N$ is invariant subspace. Let us denote by $D_N = Alg(N) \cap Alg(N)^*$, the set $N$ has multiplicity one if diagonal $D_N$ is abelian, i.e. $D_N$ is commutative algebra. We can see that the lower triangular operators $S_-$ form the algebra $alg(N)$, the corresponding diagonal $D_N$ is abelian and consists of the commutative operators

$$T_\phi f = \phi(x)f, \quad f \in L^2(0, \infty), \hspace{1cm} (4.1)$$
where $\phi(x)$ is bounded. Hence the introduced nest $N$ has the multiplicity 1.

We obtain the following concrete answer to Ringrose’s question (see [3],[7]).

**Proposition 4.1.** Let the positive definite, invertible operator $S$ is defined by the relation (3.4). The set $S^{1/2}N$ fails to have multiplicity 1.

**Proof.** We use the well-known result (see [3], p.169).

The following assertions are equivalent:
1. The positive definite, invertible operator $T$ admits factorization.
2. $T^{1/2}$ preserves the multiplicity of $N$.

(We stress that in our case the set $N = Q\xi L^2(0,\infty)$ is fixed.) The operator $S$ does not admit the factorization. Therefore the set $S^{1/2}N$ fails to have multiplicity 1. The proposition is proved.

Let us consider the operator

$$Vf = \int_0^x e^{-(x+y)} f(y) dy, \quad f(x) \in L^2(0,\infty) \quad (4.2)$$

An operator is said to be *hyperintransitive* if its lattice of invariant subspaces contains a multiplicity one nest. We note that the lattice of invariant subspaces of the operator $V$ coincides with $N$ (see [8] and [18], Ch.11, Theorem 150). Hence we deduce the answer to Kadison-Singer [5] and to Gohberg and Krein [4] question.

**Corollary 4.1.** The operator $W = S^{1/2}VS^{-1/2}$ is non-hyperintransitive compact operator.

Indeed the lattice of the invariant subspaces of the operator $W$ coincides with $S^{1/2}N$.

**Corollary 4.2.** If $\epsilon > 0$ and $0 < \mu < \epsilon$ then $S = I + K$, where $\|K\| < \epsilon$. The corresponding operator $S$ does not admit the factorization.

**Remark 4.1.** The existence parts of Theorem 3.1, Proposition 4.1, Corollaries 4.1 and 4.2 are proved by D.R.Larson [7].

But we have constructed the concrete simple examples of the operator $S$ of the convolution type and the operator $V$.

**Remark 4.2.** The factorization problems of the convolution type operators were discussed in our papers [10],[11] and [16].

**References**

1. Bateman H. and Erdelyi A., *Higher Transcendental Functions*, New York, v.1, 1953.
2. Bateman H. and Erdelyi A., *Higher Transcendental Functions*, New
3. Davidson K.R., *Nest Algebras. Triangular forms for operator algebras on Hilbert space.*, Pitman Research Notes in Mathematics Series 191, New York, Wiley (1988).

4. Gohberg I. and Krein M.G., *Theory and Applications of Volterra Operators in Hilbert Space*, Amer. Math. Soc., Providence, (1970).

5. Kadison R. and Singer I., *Triangular Operator Algebras, Amer. J. Math. 82, pp. 227-259, (1960)*

6. Krein M.G., *Continuous Analogues of Proposition on Polynomial Orthogonal on the Unit Circle*, Dokl.Akad.Nauk SSSR, 105 (1955), 637-640 (Russian).

7. Larson D.R., *Nest Algebras and Similarity Transformation*, Ann. Math., 125, p. 409-427, (1985).

8. Livshits M.S., *Operators, Oscillations, Waves (Open Systems)*, Amer. Math. Soc., Providence, 1973.

9. McCoy B. and Tang S., *Connection Formulae for Painleve V Functions*, Physica 20D (1986) 187-216.

10. Sakhnovich L.A., *Factorization of Operators in $L^2(a,b)$*, Functional Anal. and Appl., 13, p.187-192, (1979), (Russian).

11. Sakhnovich L.A., *Factorization of Operators in $L^2(a,b)$*, in Linear and Complex Analysis, Problem Book, (Havin V.P., Hruscev S.V. and Nikol'skii N.K. (ed.)) Springer Verlag, 172-174 (1984).

12. Sakhnovich L.A., *Integral Equations with Difference Kernels on Finite Intervals*, Operator Theory, Advances and Appl. v.84, 1996.

13. Sakhnovich L.A., *Spectral Theory of Canonical Differential Systems. Method of Operator Identities*, Operator Theory, Advances and Appl. v.107, 1999.

14. Sakhnovich L.A., *Spectral Theory of a Class of Canonical Systems*, Func.Anal.Appl.34, 119-128, 2000.

15. Sakhnovich L.A., *On Reducing the Canonical System to Two Dual Differential Systems*, Journal of Math. Analysis and Appl., 255, 499-509, 2001.

16. Sakhnovich L.A., *On Triangular Factorization of positive Operators*, Operator Theory: Advances and Appl., vol.179 (2007) 289-308.

17. Sakhnovich L.A., *On Krein’s Differential System and its Generalization*, Integral Equations and Operator Theory, 55, 561-572, 2006.

18. Titchmarsh E.C., *Introduction to the Theory of Fourier Integrals*, Oxford, 1937.
19. Tracy C.A. and Widom H., *Introduction to Random Matrices*, Springer Lecture Notes in Physics, 424, (1993), 103-130.