Correlated spin liquids in the quantum kagome antiferromagnet at finite field: a renormalization group analysis

Santanu Pal, Anirban Mukherjee and Siddhartha Lal

Department of Physical Sciences, Indian Institute of Science Education and Research-Kolkata, WB 741246, India

E-mail: sp13rs010@iiserkol.ac.in, am14rs016@iiserkol.ac.in and slal@iiserkol.ac.in

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Abstract

We analyze the antiferromagnetic spin-1/2 XXZ model on the kagome lattice at finite external magnetic field with the help of a non-perturbative zero-temperature renormalization group (RG) technique. The exact nature of the ground and excited state properties (e.g. gapped or gapless spectrum etc) of this system are still debated. Approximate methods have typically been adopted towards understanding the low-energy spectrum. Following the work of Kumar et al (2014 Phys. Rev. B 90 174409), we use a Jordan–Wigner transformation to map the spin problem into one of spinless fermions (spinons) in the presence of a statistical gauge field, and with nearest-neighbor interactions. While the work of Kumar et al was confined mostly to the plateau at 1/3–filling (magnetization per site) in the XY regime, we analyze the role of inter-spinon interactions in shaping the phases around this plateau in the entire XXZ model. The RG phase diagram obtained contains three spin liquid phases whose position is determined as a function of the exchange anisotropy and the energy scale for fluctuations arising from spinon scattering. Two of these spin liquids are topologically ordered states of matter with gapped, degenerate states on the torus. The gap for one of these phases corresponds to the one-spinon band gap of the Azbel–Hofstadter spectrum for the XY part of the Hamiltonian, while the other arises from two-spinon interactions. The Heisenberg point of this problem is found to lie within the interaction gapped spin liquid phase, in broad agreement with a recent experimental finding. The third phase is an algebraic spin liquid with a gapless Dirac spectrum for spinon excitations, and possess properties that show departures from the Fermi liquid paradigm. The three phase boundaries correspond to critical theories, and meet at a SU(2)–symmetric multicritical point. This special critical point agrees well with the gap-closing transition point predicted by Kumar et al. We discuss the relevance of our findings to various recent experiments, as well as results obtained from other theoretical analyses.

1. Introduction

Geometrically frustrated spin systems have drawn considerable attention ever since Anderson’s seminal work highlighted the connection between the resonating valence bond (RVB) ground states and high temperature superconductivity [1]. These systems are known to show exotic ground and excited state properties, such as liquid like ground states, fractional excitations etc [2, 3]. The nearest-neighbor (n.n.) $S = 1/2$ kagome antiferromagnet (KA) is one such promising model system in the search for unconventional states of matter. Despite sustained interest, the precise nature of the ground state and excitations of this system remain uncertain. Some analytical and numerical approaches have predicted a spin liquid ground state for the $S = 1/2$ Heisenberg KA model Hamiltonian with a gapped excitation spectrum. These include studies using exact diagonalization (ED) [4], density-matrix-renormalization group (DMRG) [5–9], quantum Monte Carlo (QMC) [10], Schwinger fermion mean-field method [11], slave fermion approach [12] and tensor network states (TNS) [13, 14]. On the other hand, some other studies predict a gapless or critical spin liquid ground state. This includes a Gutzwiller

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projected wave function based study [15], effective field theories [15, 16], variational QMC analyses [17–19],
numerical calculations of quantum dimer models [20] as well as TNS [21, 22], ED [23] and DMRG [24] based
studies.

Encouragingly, the material Herbertsmithite (ZnCu$_3$(OH)$_6$Cl$_2$) can be mapped to the n.n. $S = 1/2$ KA
model. Here too, however, conclusive results have remained elusive thus far: some experimental results on this
material appear to show the existence of gapless excitations [25, 26], while some others suggest a gapped spin
liquid ground state with fractional excitations [27, 28]. The materials Volborthite and Vesginite [29–32], as
well as the organic compound Cu-titmb [33], are also believed to be described by the $S = 1/2$ Heisenberg KA.
Further, in the presence of an external magnetic field, frustrated antiferromagnetic spin systems sometimes
display the phenomenon of magnetization plateaux [30, 34–37]. For the $S = 1/2$ KA, a plateau at a
magnetization per site of 1/3 is well studied and predicted to be robust [9, 38, 39]. Indeed, such a robust plateau
at 1/3 has been observed in the Volborthite material system [29–32], as well as in Cu-titmb [33].

Recently, investigations of the XXZ $S = 1/2$ KA model in presence of magnetic field have revealed exotic
possibilities extending across a wide range of the anisotropy parameter (i.e. the ratio of the Ising and XY
couplings) [40–42]. For example, in reference [43], TNS based numerical analyses of the strong Ising limit shows
valence bond crystalline (VBC) order, while a lattice nematic spin ordering is observed at the Heisenberg point
and in the XY limit. Interestingly, a Chern–Simons (CS) gauge field-theoretic approach for the XY dominated
regime of the XXZ KA model revealed signatures of a topological fluid state with spin Hall conductivity
$\sigma_{xy}^f = \frac{1}{4\pi}$ [44]. No firm conclusions could, however, be reached from this analysis for the regime of Ising
anisotropy, with the physics of the Heisenberg point also remaining inaccessible. With evidence for both
symmetry broken ground states as well as symmetry preserved liquid states having been furnished, further
studies of the influence of the Ising correlations is required in order to reach a comprehensive understanding of
the entire problem. This is a task we propose to undertake in this work. Even as our analysis aims to bring insight
on the richer physics of the XXZ model, we will also try to reach some firm conclusions on the Heisenberg point
relevant to various material systems.

A strategy that can be adopted towards understanding the complex physics of the n.n. $S = 1/2$ KA involves a
mapping from the original spin problem to one of fermionic spinons via a Jordan–Wigner (JW) transformation
[45–47]. In two spatial dimensions, this transformation couples spinless fermions (i.e. the spinon excitations of
the spin system) to a CS statistical gauge field [44], allowing for a mean-field treatment of the effects of the gauge
field via the average field approximation. Following Kumar et al [44], for the spinon filling corresponding to a
magnetization per site of 1/3 filling, we begin by computing the dispersion spectrum for the reduced magnetic
Brillouin zone (MBZ) comprising of nine bands. At this special filling, the effective chemical potential for the
spinons is placed in the gap between the third and fourth bands (i.e. the first three bands are completely filled and
the others empty). We then identify two points in the reduced MBZ as possessing the minimum energy gap
between the top–most filled band (3rd band) and lowest empty band (4th band), thereby leading to the
construction of an effective two-patch problem. We then analyze the fate of single spinon gap under two-spinon
interactions via a non-perturbative renormalization group (RG) analysis developed recently by some of us [48].

By studying the XXZ Hamiltonian for the $S = 1/2$ KA, the RG analysis opens new possibilities for the
emergence of non-trivial liquid-like ground states at various values of the anisotropy parameter ($\lambda$ in equation (1)
below) beyond those observed in an earlier study by Kumar et al [44]. Specifically, we have found three different
phases for various ranges of the bare anisotropy $\lambda$. Of these, two are gapped topologically ordered spin-liquid
phases, of which one corresponds to the one-spinon gapped state found in [44], and the other possessing a gap
arising purely from two-spinon scattering. The non-trivial topological features of the two gapped phases can be
distinguished using topological quantum numbers including a Chern number as well as the Volovik invariant
[49, 50]. By connecting the spin–flip part of the effective Hamiltonian (obtained at the stable fixed point of the
RG calculation) with a boundary-condition changing twist operator [31], we are able to show that both gapped
phases possess twofold ground-state degeneracy on the torus, and with fractional excitations interpolating
between them. On the other hand, the third phase obtained from the RG corresponds to an algebraic spin-liquid
state with gapless spinons possessing a Dirac-like dispersion. Here, spinon interactions reduce the effective
Fermi velocity under renormalization, effectively flattening the conical Dirac spectrum somewhat. The spinon
self-energy, quasiparticle residue and lifetime from the RG indicate the existence of an unusual Fermi liquid in
this phase. Finally, we also obtain the effective critical theories for the phase boundaries from the RG, revealing
the existence of a non-trivial multicritical point lying at the intersection of the three spin liquid phases. In this
way, the RG phase diagram obtained (see figure 6) represents a considerable advance in our understanding of the
exotic phases of matter that are emergent in a prototypical geometrically frustrated spin system at finite
magnetic field.

This paper is organized as follows. In section 2, we transform the XXZ–spin Hamiltonian for the KA into that
of fermionic spinons via 2D JW transformation, and discuss the possible magnetization plateaux protected
purely by a one-spinon gap. The computed dispersion spectrum in the reduced MBZ shows the gapped plateaux that are robust in the thermodynamic limit. section 3 will be devoted towards the construction of an effective pseudospin problem enabling the treatment of spinon interactions via a RG analysis. In section 4, we present the RG flow equations for single-spinon gap and unveil the mechanism that leads to the closing of this gap. We extend the RG analysis to also find the physics responsible for the opening of a two-spinon gap opening, as well as a phase that supports gapless spinons with a Dirac dispersion. Finally, by computing the effective Hamiltonians for the critical phase boundaries, we reveal the entire phase diagram obtained from our RG analysis. In section 5, we define the different topological quantities that are employed in distinguishing the different phases. In section 6, we calculate the self-energy, quasiparticle residue and lifetime for spinon excitations of the gapless phase. We conclude in section 7 with some discussions. Finally, we present detailed calculations for various sections in the appendices.

2. Fermionised Kagome XXZ model at finite field

The $S = 1/2$ antiferromagnetic XXZ Hamiltonian for the kagome lattice in the presence of an external magnetic field ($h$) can be written as [15]

$$H = J \sum_{\langle \mathbf{r} \mathbf{r}' \rangle} [S_\mathbf{r}^+ S_{\mathbf{r}'}^- + S_{\mathbf{r}'}^+ S_\mathbf{r}^- + \lambda S_{\mathbf{r}^x} S_{\mathbf{r}'^x}] - h \sum_{\mathbf{r}} S_{\mathbf{r}^z},$$

where $J$ is the exchange constant and $\lambda$ is the magnetic anisotropy between Ising and XY terms ($\lambda = 1$ is the Heisenberg point). The sum is taken over nearest neighbor sites and $\mathbf{r} \in (\mathbf{R}, i)$, where $\mathbf{R} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2$ ($n_1, n_2$ are integer) corresponds to the lattice vector for three sub-lattice unit cell (up triangles). Further, $\mathbf{a}_1$ and $\mathbf{a}_2$ are the basis vectors and $i \in (a, b, c)$ denote the three sub-lattices (see figure 1). The Hamiltonian is invariant under lattice translation along any basis vector of the lattice.

Recently, we developed a non-perturbative approach based on twist operators towards obtaining firm criteria for the nature of the ground state of the kagome Heisenberg quantum antiferromagnet at zero as well as non-zero field [39]. In the present work, we aim to further develop a detailed understanding of the kagome problem at finite magnetic field via a complementary non-perturbative RG analysis. In this way, we will carefully assess the role played by the Ising part of inter-spin interactions in determining the stability, as well as the properties, of the many-body ground state. In doing so, we rely on the results obtained by Fradkin and collaborators, who showed that CS topological gauge-field theories can be obtained for gapped states corresponding to certain rational fractions of the magnetization [44]. Thus, for a microscopic approach to the origin of different ground state properties at finite magnetization, we begin by mapping the problem of interacting quantum spins onto a system of fermionic spinons coupled to a CS gauge field (with statistical angle $\theta = 1/2\pi$) via a JW transformation in two spatial dimensions [46, 44]. For details of the JW transformation, we refer the reader to the detailed discussion in Kumar et al [44].

Upon performing the JW transformation, the fermionic Hamiltonian (1) takes the form

$$H = \frac{J}{2} \sum_{\langle \mathbf{r} \mathbf{r}' \rangle} [\psi_{\mathbf{r}}^\dagger(\mathbf{r}, t) e^{i A(\mathbf{r}, t)} \psi_{\mathbf{r}'}(\mathbf{r}', t) + \text{h.c.}] + J \lambda \sum_{\langle \mathbf{r} \mathbf{r}' \rangle} \left( \frac{1}{2} - n(\mathbf{r}, t) \right) \left( \frac{1}{2} - n(\mathbf{r}', t) \right) + h \sum_{\mathbf{r}} n(\mathbf{r}, t),$$

where $n(\mathbf{r}, t) = \psi_{\mathbf{r}}^\dagger(\mathbf{r}, t) \psi_{\mathbf{r}}(\mathbf{r}, t)$ is the fermion density operator, $A(\mathbf{r}, t)$ is the spatial part of the CS statistical gauge field employed in mapping spins to fermionic spinons. The magnetic field $h$ now tunes the fermionic filling. We first obtain the dispersion spectra of the non-interacting spinons on the kagome lattice from the XY-
part of the Hamiltonian by putting $\lambda = 0$. It is important, however, to remember that the spinons remain coupled to the statistical gauge field in equation (2). A mean-field ansatz then involves considering a uniform flux in every plaquettes with the gauge choice as shown in figure 2, and ignoring the fluctuations of the gauge field. One then obtains plateaux of the magnetization corresponding to $1/3, 2/3, 5/9$ etc. Indeed, the plateaux are associated with a uniform flux of $\phi = 2\pi q/p$ with $p, q \in \mathbb{Z}$, and correspond to an average filling of $\langle n \rangle = \frac{p}{q}$ (where $q$ is the periodicity of magnetic unit cell). From the associated Hofstadter spectrum for the free spinon problem, the plateaux at $1/3$ is found to be the most robust (i.e. it is stabilized by the largest single-spinon gap), and corresponds to $1/3$-filling (average site occupancy) of each sublattice in the unit cell [44]. One can obtain the dispersion spectrum for the $\langle n \rangle = \frac{1}{3}$ state by solving the associated Harper’s equation numerically for the three sub-lattices [44, 52, 53]. Hereafter, we assume a value of the exchange coupling $J = 1$. As the periodicity of the magnetic unit cell is $q = 3$ (i.e. an enlarged unit cell of nine sites), a spinon band associated with a given sub-lattice is further split into three bands, giving a total of nine bands in the spinon spectrum. For a filling of $1/3$, the lowest three bands are filled and the rest empty, i.e. the effective chemical potential (equivalent to the magnetic field $(h)$ of the original spin problem) is placed midway between the third and fourth spinon bands. Therefore, in analyzing the effects of spinon scattering at low energy scales via a RG formalism, it is sufficient to focus on the $3$rd (completely filled) and $4$th (empty) bands.

Figure 3 shows the dispersion spectrum at $1/3$-filling in the 3rd and 4th bands, clearly indicating the one-spinon spectral gap of the free spinon problem. From a finite size scaling analysis, we find that the minimum gap between the third and fourth bands saturates at a value of $\Delta_0 \approx 1.355 \pm 0.001$ (in units of $J$) in the thermodynamic limit at the corresponding momentum coordinates of $(k_x, k_y) = (0.261 55, -1.88 479)$ and $(-0.25 148, 1.91 703)$ (see figure 3). We recall that the effect of Ising interactions on the one-particle spectrum was analyzed in [44] at the level of mean-field theory together with the effects of saddle-point fluctuations of the gauge degrees of freedom. This analysis concluded that the gap would close at $\lambda^* = 0.6$. It, however, was unable to reach any firm conclusions on the nature of Ising dominated phase lying at couplings $\lambda > \lambda^*$. In the following sections, we conduct a RG analysis of the quantum fluctuations arising from the interplay of the XY and Ising terms. This RG phase diagram obtained will clarify the nature of various gapless and gapped phases as well as the transitions between them.
3. Effective two-patch problem for 1/3-plateau

Above, we have identified the existence of a minimum gap between the third and fourth bands at two points in k-space. Although the dispersion spectrum in figure 3 is not symmetric about these two minimum-gap points, we can nevertheless consider the immediate vicinity of these points in constructing an effective two-patch problem. Figure 4 shows the schematic diagram of two-patch problem, where $a$ and $b$ are the two minimum energy-difference patch-center points in the lower band with momentum indices $\vec{k}_{1a}$ and $\vec{k}_{1b}$ respectively, where the suffix 1 denotes the 3rd band in figure 3. In the same way, the momentum indices $\vec{k}_{2a}$ and $\vec{k}_{2b}$ represent the other two minimum energy-difference patch-center points in the upper band, where suffix 2 denotes the 4th band in figure 3. These four momenta are connected: $\vec{k}_{1a} = \vec{k}_{1b}$ and $\vec{k}_{2a} = \vec{k}_{2b}$. In what follows, $\Delta_0$ represents the one-spinon energy gap between two bands, hereafter referred to as the hybridization gap.

The states present at the patch-centers (labeled by the symbols (1, 2) and $(a, b)$) are connected via interband particle–particle (PP) or particle–hole (PH) scattering events between the 3rd and 4th bands. This is induced by the n.n. (Ising) interaction term in the Hamiltonian (1). In this way, the scattering processes in the vicinity of the two patch-centers form a two patch model described in figure 4. A general pair of electronic states taken from these two bands are marked by momentum eigenvalues $(\vec{k}_{1a}, \vec{k}_{1b}, \Lambda - \delta \Lambda)$ (blue and red dot in figure 4) and $(\vec{k}_{2a}, \vec{k}_{2b}, \Lambda' - \delta \Lambda)$ (green and orange dots), where $\Lambda$ and $\Lambda' - \delta \Lambda$ are the momentum displacements from the two patch-centers. $\delta \Lambda$ is the momentum asymmetry of pair $(\vec{k}_{1a}, \vec{k}_{1b}, \Lambda - \delta \Lambda)$ about net pair momentum $\vec{k}_{1a} + \vec{k}_{1b} = 0$. The two-spinon interaction terms in the Hamiltonian, equation (2), involve the scattering of such pairs of states between the 3rd and 4th bands. These scattering processes come with an additional energy cost above the bare band gap $\Delta_0$, $\epsilon_\Lambda \propto \Delta \Lambda$. This energy mismatch then leads to a logarithmic singularity in the second order term of the associated T-matrix for the resonant pairs chosen symmetrically (i.e. with $\delta \Lambda = 0$) from the two patch-centers $(\Delta E_{\Lambda, \delta \Lambda} = \epsilon_{\delta \Lambda} - \epsilon_{\delta \Lambda - \Lambda'})$,

$$T_{\Lambda + \delta \Lambda} = \lim_{\Delta \Lambda \to 0} \left( \epsilon_{\delta \Lambda} - \epsilon_{\delta \Lambda - \Lambda'} \right) \frac{1}{\Omega - \Delta E_{\Lambda, \delta \Lambda}} = \frac{F^2 \lambda^2}{C_0} \log \left| \frac{C_0}{\epsilon_{\delta \Lambda}} \right|, \quad (3)$$

where $C_0 = \Delta_0 - \Delta E_{\Lambda, \delta \Lambda}$. Recall that similar log-divergences appear in the calculation of the T matrix for the Kondo problem [54], and required a careful RG analysis for further insight. For these symmetrically chosen resonant pair states, there exists PP and PH scattering channels acting on the low energy subspaces:

$$\bar{n}_{1/2a} - \bar{n}_{1/2b} = 0 \quad \text{(PP)}, \quad (4)$$

$$\bar{n}_{1/2a} + \bar{n}_{1/2b} = 1 \quad \text{(PH)}. \quad (5)$$

For simplicity, we have redefined $\vec{k}_{1(2)a,b,\Lambda} \equiv (1/2)\vec{k}$ and $\vec{k}_{1(2)b,\Lambda} \equiv (1/2)\hbar$, where 1/2 indicates the 3rd/4th bands respectively. The projection operators $P_{D\Lambda}$ and $P_{S\Lambda}$ lead to the subspaces equations (4) and (5) respectively, where the $D_{\Lambda}$ and $S_{\Lambda}$ indices denote PP and PH pseudospins at momentum $\Lambda$. The number operators are defined in terms of the spinon creation/annihilation operators $\bar{n}_{1/2a} = C_{1/2a}^\dagger C_{1/2a}$, $\bar{n}_{1/2b} = C_{1/2b}^\dagger C_{1/2b}$. The creation/annihilation operators satisfy the usual fermion anti-commutation relations. The selection procedure for the low energy subspaces employed above is analogous to BCS’s construction [55, 56] used to reach the BCS reduced Hamiltonian starting from a general electronic problem. The subspaces equations (4), (5) are spanned by the basis states.

[Figure 4. Schematic diagram of the two-patch problem, where the a and b points (black circles) correspond to the two points with minimum energy gap $\Delta_0$ between the two bands (with the indices 1 and 2 correspond to the 3rd and 4th bands respectively). The right-hand side of the figure represent the problem in pseudospin space, with $(1h, 2h)$ as patch-center and $(1a, 2a)$ as boundaries. (Red and blue)/ (green and orange) dots denote the pair of states $(\vec{k}_{1a}, \vec{k}_{1b}, \Lambda - \delta \Lambda)/(\vec{k}_{2a}, \vec{k}_{2b}, \Lambda' - \delta \Lambda)$ symmetrically placed about the lower/upper band center.]

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respectively. Putting $\Lambda = 0$ gives the basis states of the patch centers. The PP pseudospin vector $\tilde{A}_{1/2|D_\alpha}$ acting on the subspace equation (6) is given as follows [56]

$$A_{1/2|D_\alpha}^\pm = \frac{1}{2}(\hat{n}_{1/2\alpha} + \hat{n}_{1/2\beta})$$

and the pseudo-spin vector $A_{1/2|S_\alpha}$ in the PH channel equation (7) as

$$A_{1/2|S_\alpha}^\pm = \frac{1}{2}(\hat{n}_{1/2\alpha} - \hat{n}_{1/2\beta})$$

We will now present the form of the projected patch-center Hamiltonian: $\mathcal{H} = Q_4 H Q_4$. Here, $H$ is given in equation (2), and the projection operator $Q_4$ is given by $Q_4 = P_{D_\alpha} + P_{S_\alpha}$, where $h$ indexes the states at the patch-center $\Lambda = 0$. $Q_4$ projects onto the subspace ($\Lambda = 0$ in equations (4), (5)) of the four patch-center states $\tilde{h}_{0\alpha}$, $\tilde{k}_{\alpha}$, $\tilde{k}_{\beta}$, $\tilde{k}_{2\beta}$ in the PP and PH scattering channels (see appendix A for detailed derivation). The contribution to the effective Hamiltonian $\mathcal{H}$ from the PP channel is given by

$$H_{PP} = \epsilon_1(\hat{n}_{1\alpha} + \hat{n}_{1\beta}) + \epsilon_2(\hat{n}_{2\alpha} + \hat{n}_{2\beta}) + V_{q=0}(C_{1\alpha}^L C_{1\beta}^L C_{1\beta} + (1 \leftrightarrow 2)) + V_{q=0}(C_{2\alpha}^L C_{2\beta}^L C_{2\beta} + (1 \leftrightarrow 2))$$

Similarly, the contribution from the PH channel is

$$H_{PH} = V_{q=0}(C_{1\alpha}^L C_{2\beta}^L C_{2\beta} + (a \leftrightarrow b)) + V_{q=0}(C_{2\alpha}^L C_{1\beta}^L C_{1\beta} + (a \leftrightarrow b)) + V_{q=0}(C_{1\alpha}^L C_{2\beta}^L C_{2\beta} + (a \leftrightarrow b)) + V_{q=0}(C_{2\alpha}^L C_{1\beta}^L C_{1\beta} + (a \leftrightarrow b))$$

In $H_{PP}$, $\epsilon_1$ and $\epsilon_2$ are the kinetic energies for the patch-centers in the lower (3rd) and upper (4th) bands respectively. As we take the middle of the two bands as the zero energy surface, $\epsilon_1 = -\epsilon_2$. Further, $\hat{n}_{1\alpha}$ is $C_{1\alpha}^L$, and $\hat{n}_{2\alpha}$ is $C_{2\alpha}^L$. The $k$-space spinon number operators at the lower and upper patch-centers respectively. $V_{q=0}$ and $V_{q=0}$ are the zero and non-zero momentum scattering amplitudes. In the PH channel, the two spinons scatter from separate bands. Their kinetic energy terms cancel one another, ensuring the absence of kinetic terms from $H_{PP}$. The origin of various scattering terms in the PP and PH channel are described in detail in appendix A.

We set $\Lambda = 0$ in order to obtain the pseudospins precisely at the patch centers in equations (8) and (9) respectively: $\tilde{A}_{1D\alpha\alpha} = \tilde{A}_{1D}$ and $\tilde{A}_{1S\alpha\alpha} = \tilde{A}_{1S}$. By first writing the Hamiltonians (10) and (11) in terms of pseudospins and then combining both, we have

$$\mathcal{H} = H_{PP} + H_{PH} = U(A_{2D}^L + A_{2D}^R) + \Delta(A_{2D}^L - A_{2D}^R) + 2V(\tilde{A}_{1D} - \tilde{A}_{2D} - \tilde{A}_{1S} - \tilde{A}_{2S})$$

where $U = \epsilon_1 + \epsilon_2 + 2V$, $\Delta = \epsilon_2 - \epsilon_1$. Note that the coupling $V = V_{q=0} - V_{q=0} \equiv \lambda$ (the exchange anisotropy). Further, $V > 0$ as $V_{q=0} \propto \cos \Delta k < 0$ (as the momentum transfer is $\pi/2 \leq \Delta k \leq 2\pi/3$). In the next section, we will consider the effects of scattering between these patch-centers and the states in their vicinity.

4. RG study for two-patch problem

In the section above, we saw the logarithmic instabilities in the low energy subspace of the resonant pairs (i.e. with vanishing kinetic energy) due to PP or PH scattering processes. Based on the calculation equation (3), we confine our interest solely to the resonant pair subspace and project out the off-resonant pair scattering terms from the Hamiltonian equation (2). Akin to the poor man’s scaling approach to the Kondo problem [57], a treatment of such logarithmic singularities arising at leading order in the coupling $J_\lambda$ demands a RG theory for the resonant pairs of the two-patch problem. As the gap between the patch centers of the 3rd and 4th band governs the low energy physics, the PP or PH scattering processes from them to a high energy resonant pair subspace (equations (4), (5)), with the projection operator $Q_4$ projecting onto the states $\tilde{h}_{1\alpha}$, $\tilde{k}_{1\alpha}$, $\tilde{k}_{1\beta}$, $\tilde{k}_{2\beta}$ allow us to formulate a minimal RG scaling theory. Within the pseudospin language, these spinon scattering processes correspond to quantum fluctuations of the $z$-component of the PP and PH pseudospins (equations (8) and (9)) that are induced by pseudospin–flip scattering processes [48]. As we will see below, this Hamiltonian RG formulation will enable us in reaching some firm conclusions on the existence and nature of various ground states.
The RG procedure involves the iteration of two steps. As shown in figure 4, the first step is the partitioning of the Hamiltonian into a low energy resonant pair subspace ($\Lambda = 0$ in equations (4) and (5)) via a patch-center projection operator $Q_h$, and the boundary subspace via a projection operator $Q_d$. The boundary pseudospins are displaced by $\Lambda$ in energy–momentum space from the patch centers of the 3rd and 4th bands. In the second step, off-diagonal terms ($H^X_{h,n} = Q_hHQ_h + \text{h.c.}$) associated with PP or PH scattering between the boundary ($\Lambda$) and the patch-center ($h$) states are removed via Gauss–Jordan block diagonalization, leading to decoupling of the boundary pseudospin. As presented in detail in appendix B, this procedure involves the elimination of pseudospin terms of the kind $A^h_{\Lambda h}A^d_{\Lambda d} + \text{h.c.}$, where $A^h_{\Lambda h}$ represents the pseudo-spin for upper band in the PP channel with momentum at $k_\Lambda$ etc (see figure 4). This RG scheme does not involve a perturbative expansion in any coupling, and is similar to Aoki’s non-perturbative RG [58] procedure of Gaussian elimination of single particle states employed for the Anderson disorder problem. The RG is also connected to Glazek and Wilson’s non-perturbative Hamiltonian RG procedure [59] employed for quantum mechanical Hamiltonians, and has been applied recently by some of us to the study of Mott–Hubbard transitions in the 2D Hubbard model [48].

The iterative removal of one PP or PH pseudospin at every RG step leads to a RG flow equation for the low-energy two-patch subspace Hamiltonian

$$H_{h,n-1} = H_{h,n} + H^X_{h,n}(\omega - H^P_{h,n})^{-1}H^X_{h,n}.$$  

As defined above, $H^X_{h,n}$ is the off-diagonal term coupling states at the energy–momentum boundary ($-\Lambda$ and $\Lambda$) and two patch-centers (1h and 2h). The diagonal operator $H^P_{h,n}$ contains the Ising interaction between boundary pseudospins and all Zeeman-like terms involving the pseudospins. The quantum fluctuation scale $\omega$ is the undetermined energy eigenvalue of the system containing contributions from the quantum dynamics of the inter-pseudospin correlations, as well as the pseudospin self-energy induced by the off-diagonal terms. This quantum dynamics is manifested in the Heisenberg equation of motion for the diagonal operator ($H^P_{h,n}$), arising from the non-commutativity of the diagonal and off-diagonal terms.

By constructing the two-patch problem, we can now write the PP subspace projected (with projection operator $P_{nP}$) Hamiltonian $H$ projected into three parts: patch-center ($H_h = P_{1h}H_{1h}P_{2h}$), boundaries ($H_\Lambda = P_{1h}H_{1h}P_{2h}$), and a patch-center-boundary coupling ($H_{h\Lambda} = P_{1h}H_{1h}P_{2h}$) as

$$H_h = U(A^h_{2h} + A^d_{Dh}) + \Delta A^h_{2h} - A^d_{Dh}) + 2V A^2_{2h}A^d_{Dh},$$

$$H_\Lambda = U(A^h_{2h} + A^d_{Dh}) + \Delta A^h_{2h} - A^d_{Dh}) + 2V A^2_{2h}A^d_{Dh},$$

$$H_{h\Lambda} = 2V[A^2_{2h}A^d_{Dh} + A^d_{Dh}A^d_{Dh} + \frac{1}{2}(A^2_{2h}A^d_{Dh} + A^2_{2h}A^d_{Dh} + \text{h.c.})].$$

We reiterate the fact that, in reaching the Hamiltonians equation (14) for the resonant pair subspace, we have ignored the sub-dominant contributions from all non-resonant scattering terms $P_{1h}H_{1h}P_{2h} + \text{h.c.}$ Further, we have ignored the PH pseudospin scattering processes here, as they include all intraband scattering processes and do not contribute to the RG flow of the hybridization gap. Note that $H_\Lambda$ contains an interaction between the two boundaries, while $H_h$ contains interactions between the patch-center states. Further, as the hybridization energy gap term receives no contribution from the PH channel, we will ignore contributions from the PH channel when analyzing the effect of interactions in leading to a quantum critical point (QCP) (i.e. in closing the single particle gap) in section 4.1. Instead, in section 4.2, we will find their contributions as being critical to the opening of a many-body gap beyond the QCP.

4.1. Renormalization of hybridization gap

The inter-band scattering processes between the patch-centers and the boundaries leads to renormalization of the single particle terms in the two patch Hamiltonian

$$\Delta H^i = \frac{V^2_{2h}A^2_{2h}A^d_{Dh} \langle \uparrow \uparrow \downarrow \downarrow \rangle_{Dh} \langle \uparrow \downarrow \uparrow \downarrow \rangle_{Dh}}{\langle \uparrow \uparrow \downarrow \downarrow \rangle_{Dh}(\omega - H_h)\langle \uparrow \uparrow \downarrow \downarrow \rangle_{Dh}},$$

where $\omega$ quantifies the energy cost for pseudospin fluctuations arising from inter-band spin-flip scattering processes. The renormalization equation (15) implies that a pseudospin scattering between the lower band patch-center (1h) and the boundary of the upper band involves the excitation to an intermediate configuration $\langle \uparrow \downarrow \downarrow \downarrow \rangle_{Dh}$. In the de-excitation process, the pseudospins return to their original configuration from this intermediate configuration (see figure 5(a)). We note that the numerator of the flow equations support a one-loop form similar to that obtained from the poor man scaling RG for the Kondo problem [57]. However, the denominator contains pseudospin (i.e. two-particle) self energies that are also renormalized in the process. This leads to the non-perturbative nature of the flow equations. We note that a similar feedback in the renormalization of the two particle vertices is observed in the functional RG formalism as arising from the single-particle self energy [60].
We now compute the flow equation (15) as follows. We begin by determining the form of the intermediate boundary pseudospin state propagator, and then compute the operation of the spin flip terms on the pseudospin state space. The zero energy surface lies exactly in the middle of the two bands, such that the resonant PP pair possesses zero net kinetic energy, \( \int_0^{2\Lambda} \int_0^{2\Lambda} = -LL \). The coefficient of the effective field-like term in equation (14) simplifies to \( U_0 = 2V_0 \), where \( V_0 \) is the bare interaction strength. \( V_n \) is the renormalised interaction strength/two-particle self-energy obtained from the n-th step of the RG process. The intermediate boundary state's eigenenergy is given by

\[
H_{\text{fl}}^{1/2} = \left( U_0 + \frac{V}{2} \right) | \uparrow \downarrow \rangle_{D_h}. \tag{16}
\]

The action of the pseudospin-flip operators on the intermediate state

\[
A_{2D_h} | \uparrow \downarrow \rangle_{D_h} = | \uparrow \downarrow \rangle_{D_h}
\]

allows determination of the numerator in equation (15). Thus, the renormalization equation takes the form

\[
\Delta H_h^1 = \frac{V^2 A_{2D_h} A_{2D_h} | \uparrow \downarrow \rangle_{D_h} \langle \uparrow \downarrow |_{D_h}(\Omega - H_h)}{\omega - (2V_0 + \frac{V}{2})} = \frac{V^2 (\frac{1}{2} + A_{2D_h}^2) | \uparrow \downarrow \rangle_{D_h} \langle \uparrow \downarrow |_{D_h}}{\omega - (2V_0 + \frac{V}{2})}. \tag{16}
\]

Similarly, the pseudospin scattering between upper band patch-center and the boundary of the lower band (see figure 5(b)) leads to

\[
\Delta H_h^2 = \frac{V^2 A_{2D_h} A_{2D_h} | \uparrow \downarrow \rangle_{D_h} \langle \uparrow \downarrow |_{D_h} A_{2D_h} A_{2D_h}^2 | \uparrow \downarrow \rangle_{D_h}}{\omega - (2V_0 + \frac{V}{2})} \tag{16}
\]

Together, the complete contribution to the Hamiltonian renormalization from the above processes is given by

\[
\Delta H_h^1 + \Delta H_h^2 = \frac{V^2}{\omega - (2V_0 + \frac{V}{2})} (A_{2D_h} | \uparrow \downarrow \rangle_{D_h} \langle \uparrow \downarrow |_{D_h} + A_{2D_h}^2 | \downarrow \uparrow \rangle_{D_h} \langle \downarrow \uparrow |_{D_h} + A_{2D_h} A_{2D_h}^2 | \downarrow \uparrow \rangle_{D_h} \langle \uparrow \downarrow |_{D_h} + A_{2D_h} | \uparrow \downarrow \rangle_{D_h} \langle \uparrow \downarrow |_{D_h} + A_{2D_h}^2 | \downarrow \uparrow \rangle_{D_h} \langle \downarrow \uparrow |_{D_h} + A_{2D_h} | \downarrow \uparrow \rangle_{D_h} \langle \downarrow \uparrow |_{D_h} + A_{2D_h} A_{2D_h} | \downarrow \uparrow \rangle_{D_h} \langle \uparrow \downarrow |_{D_h}) \tag{16}
\]

where \( K = \frac{V^2}{\omega - (2V_0 + \frac{V}{2})} \). By taking a trace of equation (20) with respect to the final pseudospin state | \uparrow \downarrow \rangle_{D_h}, the renormalized hybridization term of the patch-center Hamiltonian (\( H_h \) in equation (14)) attains the form
twofold ground state degeneracy on the torus. The ground state wave function returns the ground state wave function to itself, yielding a gapped energy scale for quantum fluctuations, $2\Delta_0 - \omega$. The region I (pink) represents a one-spinon gapped topological spin liquid phase, while region II (orange-yellow) represents a two-spinon gapped topological spin liquid phase. Region III (white) represents the gapless algebraic spin liquid. The black circle ($V_0 = \Delta_d/2$, $\omega = \Delta_d$) marks the multicritical point lying at the meeting of three critical lines (1, 2 and 3). The blue circle ($V^* = 1$) corresponds to the Heisenberg point.

This leads to the RG flow equation for the hybridization gap $\delta\Delta_n = \Delta_{n+1} - \Delta_n$ and the interaction strength $\delta V_n = V_{n+1} - V_n$ as (for details see appendix B)

$$\delta V_n = \frac{2}{c|\omega - 2V_0| - \frac{\Delta_0}{2}} \equiv \delta V_n,$$

where $c = \text{sgn}(\omega - 2V_0)$ is the sign of the non-interacting two particle Green function.

In the RG equation given above, for $\omega = \Delta_0$, the quantity $c$ will change sign from $+\text{Ve}$ to $-\text{Ve}$ for $\Delta_0 < 2V_0$. This makes the RG flows for both the hybridization gap and inter-band scattering term irrelevant. This marks $\Delta_0/2 = V_0^* = 0.68$ (in units of $J$) as a SU(2)-symmetric QCP across which the hybridization gap-closing transition takes place, revealing a singular Fermi surface with two Dirac points. This is shown in figure 6 as a black filled circle. The critical value $V_0^* = 0.68$ obtained is close to the mean-field value of 0.6 obtained by Kumar et al in [44]. As we will see in section 4.3 below, this special point lies at the meeting of three quantum critical lines, two of which possess SU(2)-symmetry (line-2 and line-3 in figure 6) and the third a U(1) symmetry (line-1 in figure 6). Indeed, this point corresponds to a multicritical point as it is reached at a special value of $\Delta_0$, $V_0$ and $\omega$.

On the other hand, in the regime $V_0 < \frac{\Delta_0}{2}$ (marked as I in figure 6), the interband scattering terms assists the hybridization gap and both the RG flows are relevant (as can be seen from equation (22)). Here, stable fixed point values are reached for the scattering coupling $V_n^* = 2(2V_0 - \omega)$ and the hybridization gap $\Delta_n^* = \Delta_0 + 2\omega - 5V_0$ (see appendix B). The effective two patch-center Hamiltonian at this stable fixed point gives the low-energy theory of the 1/3 plateau

$$H^*(\omega) = 2V_0(A_{Dh}^* + A_{Dd}^*) + \Delta_0^*(\omega)(A_{Dh}^* - A_{Dd}^*) + V^*(\omega)\tilde{A}_{Dh}\tilde{A}_{Dd}^*.$$  

We can now detail some of the properties of the gapped ground state on this magnetization plateau. Following [39], applying the twist operator

$$\hat{O} = \exp \left[ \frac{2\pi i}{N_1} \sum_r \left( n_r + \frac{n_{\parallel}}{2} \right) \tilde{S}_r^z + \sum_{\parallel \parallel} \left( \frac{1}{4} \tilde{S}_{\parallel \parallel}^z + \frac{1}{2} \tilde{S}_{\parallel \parallel}^z \right) \right]$$

twice on the ground state wave function returns the ground state wave function to itself, yielding a gapped twofold ground state degeneracy on the torus [61–63]. Here, the indices $b$ and $c$ of the operator $\tilde{S}^z$ represent the sub-lattices of the kagome system (see figure 1). This gap opening is an outcome of the two particle scattering across the two-patch centers (see figure 4). This scattering process is described by the pseudospin-flip part of the Hamiltonian equation (12), and can be connected to the PP-projected twist operator as follows (for details see...
appendix D)
\[ \mathcal{H}^\pm = V^\pm(\omega)[A_{D_D}^\pm A_{D_D}^\pm + \text{h.c.}] = V^\pm(\omega)P_D(A_D^2 + \hat{\mathcal{O}}^2)P_Dh, \]  
(25)
where \( \mathcal{H}^\pm \) is the PP channel’s spin-flip part of the Hamiltonian (12), and \( P_D = (16/9)\hat{A}_{D_D}^2 \hat{A}_{D_D}^2 \) is the projection operator on the PP channel for the 3rd and the 4th bands. In order to probe the topology of the ground state manifold, we define the center of mass translation operator \( \hat{T}_{D_D} \) as
\[ \hat{T}_{D_D}^\pm = \exp[i \mathcal{E}_m](\text{where } \mathcal{E}_m \text{ is the total momentum}). \]
From the non-commutativity of the twist and translation operators we obtain a \( Z_2 \) invariant
\[ \begin{align*}
P_D \hat{T}_{D_D} \hat{\mathcal{O}}^2 \hat{T}_{D_D}^{-1} \hat{\mathcal{O}} P_D &= \exp[-i\pi], \\
P_D \hat{T}_{D_D} \hat{\mathcal{O}}^2 \hat{T}_{D_D}^{-1} \hat{\mathcal{O}} P_D &= \exp[-i2\pi].
\end{align*} \]
(26)
This \( Z_2 \) invariant indicates that the states \( |\psi_0\rangle \) and \( \hat{\mathcal{O}}|\psi_0\rangle \equiv |\psi_1\rangle \) are orthogonal to one another (\( \langle \psi_1|\psi_0\rangle = 0 \)) and comprise the ground state manifold. Further, that \( \hat{\mathcal{O}}^2|\psi_0\rangle = |\psi_0\rangle \). The topological phase \( \pi \) that arises out of this non-commutativity allows us to extract a fractional charge \( q = \frac{1}{2} \) associated with the spectral flow between the two ground states [64, 65]. Further, we will also see in section 5 below that the many-body gap originating from the operator \( \hat{\mathcal{O}}^2 \) protects the degenerate ground state manifold, resulting in a topological Chern number. Our findings are consistent with the CS gauge field theory developed in Kumar et al [44].

4.2. Gap opening beyond the QCP
We will now study the putative gapped phase lying beyond the QCP \( V_0 > \Delta_0 \) induced by scattering in the PP/PH channels (marked as region II in figure 6). Thus, we begin with the Hamiltonian \( H_{\text{ren}} \) in the neighborhood of the two band-touching Dirac points involving both the PP and PH pseudospins
\[ H_{\text{ren}} = \sum_{\Lambda = -\Lambda_0}^{\Lambda_0} \left[ \epsilon_\Lambda(A_{D_D}^\Lambda - A_{D_D}^\Lambda)^2 + 2\left(V_0 - \frac{\Delta_0}{2}\right)(A_{D_D}^\Lambda + A_{D_D}^\Lambda) \right] + 2V_0 \sum_{\Lambda = -\Lambda_0}^{\Lambda_0} (\hat{A}_{D_D} \cdot \hat{A}_{D_D} - \hat{A}_{2S} \cdot \hat{A}_{2S}), \]
(27)
where \( \epsilon_\Lambda = \nu_{\Lambda} \Lambda, \nu_{\Lambda} \) is the Fermi velocity of the Dirac dispersion, \( \Lambda_0 \) is the momentum space cut-off and \( V_0 \) is the inter-spinon interaction as defined earlier. An effective Hamiltonian for the patch-centers and their couplings to the boundaries can again be constructed as a composition of three parts: patch-center (\( H_0 = Q_0 \mathcal{H}_{\text{ren}} Q_0 \)), boundaries (\( H_1 = Q_\Lambda \mathcal{H}_{\text{ren}} Q_\Lambda \)) and patch-center-boundary (\( H_{\text{boundary}} = Q_\Lambda \mathcal{H}_{\text{ren}} Q_0 + \text{h.c.} \)). Upon closure of the hybridization gap \( (\Delta = 0) \) at the QCP, the lower band patch-center spinons have an energy shifted by \( \frac{1}{2} \Delta_0 \), such that zero energy spinons are now present at the Dirac points. This leads to the renormalization of the Zeeman term from \( 2V_0 \) to \( 2V_0 \) in equation (27). Thus, following the steps given earlier, we write down the effective starting Hamiltonian for the phases lying beyond the QCP \( (V_0 > \frac{1}{2} \Delta_0) \) as
\[ \begin{align*}
H_0 &= 2\left(V_0 - \frac{\Delta_0}{2}\right)(A_{D_D}^\Lambda + A_{D_D}^\Lambda) + 2V_0 \left[\hat{A}_{D_D} \cdot \hat{A}_{D_D} - \hat{A}_{2S} \cdot \hat{A}_{2S}\right], \\
H_1 &= 2\left(V_0 - \frac{\Delta_0}{2}\right)(A_{D_D}^\Lambda + A_{D_D}^\Lambda) + 2V_0 \left[\hat{A}_{D_D} \cdot \hat{A}_{D_D} - \hat{A}_{2S} \cdot \hat{A}_{2S}\right], \\
H_{\text{boundary}} &= 2V_0 \left[(\hat{A}_{D_D} \cdot \hat{A}_{D_D} + A_{D_D}^\Lambda \hat{A}_{D_D}^\Lambda) - (\hat{A}_{2S} \cdot \hat{A}_{2S} + \hat{A}_{2S}^\Lambda \hat{A}_{2S}^\Lambda)\right].
\end{align*} \]
(28)
Now considering intermediate state for PP and PH channel, \( |\psi_D\rangle = |1\ 1\ 2\ 2\ \rangle_D \) and \( |\psi_S\rangle = |1\ 1\ 2\ 2\ \rangle_S \) respectively, the corresponding boundary energies are computed from
\[ \langle \psi|H_{\text{D}}|\psi_D\rangle = 2\left(V_0 - \frac{\Delta_0}{2}\right) + \frac{V_0^D}{2}, \quad \langle \psi|H_{\text{S}}|\psi_S\rangle = -\frac{V_0^S}{2}. \]
(29)
The RG equation for PP/PH channels are then found to be
\[ \begin{align*}
V_{n+1}^D &= V_{n}^D - \frac{(V_0^D)^2}{(\omega - \Delta_0) - \left(\frac{V_0^D}{2}\right)} - \left(\frac{V_0^D}{2}\right), \\
V_{n+1}^S &= V_{n}^S - \frac{(V_0^S)^2}{(\omega - \Delta_0) + \left(\frac{V_0^S}{2}\right)}.
\end{align*} \]
(30)
where the bare values of the couplings \( V_0^D = V_0^S = V_0 \). It is worth noting that, due to the closure of the hybridization gap, the effective energy scale for quantum fluctuations is now given by \( (\omega - \Delta_0) \). From the RG equation (30), we observe that in the regime \( V_0 > \Delta_0 = \frac{1}{2}\omega \), the PP coupling \( V_0^D \) is RG irrelevant whereas the
PH coupling $V^S$ is RG relevant. The relevant RG flow stops at the finite fixed point value $V^{S*} = 2(\Delta_0 - \omega)$ as a function of $\omega$ and $\Delta_0$. Thus the latter forms a gap in the PH pseudospin Hilbert space, and determines the resultant stable fixed point Hamiltonian

$$H^* = -2V^{S*}\tilde{A}_{iS_i} \cdot \tilde{A}_{2S_i}.$$  

(31)

In the fixed point theory equation (31), the PH pseudospin subspace equation (3) condenses at low energies, making the PP pseudospin vector magnitude vanish: $|\tilde{A}_{iD_{1/2}}| = |\tilde{A}_{iD_{3/2}}| = 0$. This leads to the vanishing of the Zeeman term in equation (28). Similar to equation (25) for the PH interband scattering channel, one can show that the spin-flip part in the fixed point Hamiltonian

$$\mathcal{H}^* = -V^{S*}[\tilde{A}_{2S_{\uparrow}}\tilde{A}_{1S_{\downarrow}} + \text{h.c.}] = -V^{S*}P_{S\downarrow}(\hat{O}_{\downarrow}^2 + \hat{O}_{\uparrow}^{12})P_{S\uparrow}$$  

(32)

can be written in terms of the doubled twist operator projected onto the PH pseudospin space (see appendix D for details). This allows us, once again, to predict a gapped, two-fold degenerate ground state on the torus [61]. Further, following the arguments presented earlier, spectral flow between the two degenerate ground states is associated with a fractional charge of $q = \frac{1}{2}$.

In order to place the kagome Heisenberg antiferromagnet in the presence of an external magnetic field along the $z$-direction (1) with $\lambda = 1$, $h \neq 0$ in the RG phase diagram, we recall that $\lambda \equiv V$ in our formalism. Thus, the stable fixed point $V^{S*} = 1$ corresponds to $\lambda^* = 1$. This gives $2\Delta_0 - \omega = \Delta_0 + \frac{1}{2}$. This places the Heisenberg point in region II of the phase diagram at the point $(1, \Delta_0 + \frac{1}{2})$, as indicated by blue filled circle in figure 6. As indicated by the effective Hamiltonian in equation (31) above, the two-particle gapped phase corresponds to a gapped SU(2) symmetric phase comprised of electron–hole pairs from the 3rd and 4th bands. This effective Hamiltonian was reached by an RG analysis respecting the SU(2) symmetry of quantum fluctuation terms (equation (30)) arising from the non-commutativity of various terms in equation (28) (e.g. $[\tilde{A}_{2S_{\uparrow}} \cdot \tilde{A}_{1S_{\downarrow}}, \tilde{A}_{2S_{\downarrow}} \cdot \tilde{A}_{1S_{\downarrow}}] = 0$). This indicates that we could have equivalently studied a SU(2) non-Abelian lattice gauge theory on the kagome lattice associated with such quantum fluctuations [66]. In the continuum limit, such a SU(2)-symmetric non-abelian gauge theory possesses a disordered ground state with a dynamically generated mass gap. Such a gauge theory can be obtained from a fermionic nonlinear sigma model of massive Dirac fermions in $(2 + 1)$ dimensions coupled to a SU(2) order parameter [67], and possesses a topological Hopf term. We will further quantify the topological invariants of the gapped phases in regions I and II in the next section.

4.3. Gapless phase and phase boundaries

In the regime $V_0 > \Delta_0 = \frac{1}{2}\omega$, both PP and PH scattering processes are irrelevant, leading to a state with robust gapless Dirac spinons. This is indicated as region III of the phase diagram in figure 6, and corresponds to an algebraic spin liquid [15, 24]. Here, the RG flows in equation (30) stop at the fixed point values of vanishing interaction strength $V_{0*} = 0$ and hybridization gap $\Delta_{0*} = 0$. This corresponds to the appearance of the Dirac point Fermi surface, with a vanishing effective patch-center Hamiltonian, $H_{III} = 0$. We will further elucidate the properties of this gapless phase in section 6.

Finally, we can define fixed point patch-center Hamiltonians for the various phase boundaries in the phase diagram. The fixed point Hamiltonian for the boundary separating phase-I from phase-II (line-1 in figure 6) is,

$$H_1 = (V_0 - \frac{\Delta_0}{2})(A_{iD_{1/2}} + A_{jD_{3/2}}),$$  

(33)

where 1 in the subscript denotes the critical line-1 in the phase diagram. The absence of the hybridization term, $\Delta(A_{iD_{1/2}} - A_{jD_{3/2}})$, in the above Hamiltonian is consistent with absence of $\Delta$ in the fixed point theories for phase-II. This reinforces the fact that a gapless Dirac-point Fermi surface is revealed along line-1, but with a shift in the effective Zeeman term to the value $V_0 - \frac{\Delta_0}{2}$ (see equation (27)). $H_1$ thus corresponds to a line of quantum critical theories with U(1)-symmetry, and possesses a topological theta-term [68]. On the other hand, for the boundaries separating phase-I from phase-III (line-2) and that separating phase-II from phase-III (line-3) corresponding to SU(2)-symmetric critical theories: $H_2 = H_3 = 0$. The vanishing effective Hamiltonian for such gapless Dirac cones on line-2, line-3 and phase III indicates an emergent PH symmetry. This leads to a SU(2)-symmetric Wess–Zumino–Novikov–Witten topological term with coefficient $S = 1/2$ [68] in the theory.

5. Topological quantum numbers and spin Hall conductivity

We now use various topological quantum numbers to distinguish the different phases in the phase diagram figure 6. We begin by rewriting the RG equation (22) as
\[ \delta \Delta_a = \frac{2(V_a/2)^2}{\text{Sgn}(G_{0,1}^{-1}) \left| G_{0,1}^{-1} \right| - \frac{\Lambda}{2}} \]  

where \( G_{0,1}^{-1}(\omega) = \omega - 2V_0 \), such that \( \text{Sgn}(G_{0,1}^{-1}(\Delta_0)) = \text{Sgn}\left(1 - \frac{2V_0}{\Delta_0}\right) \). Thus, the quantity \( \text{Sgn}(G_{0,1}^{-1}(\Delta_0)) \) decides whether the RG equation for the hybridization gap \( \Delta \) is relevant (+ve) or irrelevant (−ve).

We now define the complex function

\[ \tilde{G}_0(z) = \frac{1}{z - G_{0,1}^{-1}(\Delta_0)} \]  

and the corresponding topological index \([49, 69]\) as

\[ N_\lambda = \frac{1}{2\pi i} \int dz \, \tilde{G}_0(z). \]  

The value of \( N_\lambda \) is then evaluated as (for details see appendix C)

\[ N_\lambda = 1 \quad \text{if} \quad \Delta_0 > 2V_0 \]
\[ = 0 \quad \text{if} \quad \Delta_0 < 2V_0 \]

indicating that \( N_\lambda \) is non-trivial in phase I (the one-spinon gapped phase) and trivial otherwise.

Similarly, we can define a topological index from the RG equation (30) as follows

\[ \tilde{N}_\lambda = \frac{1}{2\pi i} \int dz \, \tilde{G}_0(\xi)(z), \]

where \( G_0(\xi)(z) = 1/(z - G_{0,1}(\omega)\xi) \) and \( G_{0,1}(\omega)\xi = \omega - \Delta_0 \). We then find that \( \tilde{N}_\lambda \) is non-trivial in phase II (the two-spinon gapped phase) and trivial otherwise

\[ \tilde{N}_\lambda = 1 \quad \text{if} \quad \omega > \Delta_0 \]
\[ = 0 \quad \text{if} \quad \omega < \Delta_0. \]

Finally, by mapping the problem in phase I close to the gap-closing point to that of massive Dirac spinons, we find the Chern number \( C = 1 \) (see appendix C). Then, following Kumar et al [44], we can connect this Chern number with an induced CS term in the corresponding gauge-field theory \( (\theta_p, \theta_F) = C/2\pi = 1/2\pi \). Together with the original CS statistical angle \( \theta = 1/2\pi \), this leads to an effective CS coupling [44]

\[ \theta_{\text{eff}} = \frac{\theta}{\theta + \theta_F} = \frac{1}{2\pi}, \]

and, thence, the fractional spin hall conductivity

\[ \sigma_{xy} = \theta_{\text{eff}} \frac{1}{2\pi} = \frac{\nu}{2\pi}. \]

6. Properties of gapless phase in the RG phase diagram

Finally, we turn to a detailed analysis of region III in the phase diagram figure 6. As discussed above, we find here a gapless phase with Dirac spinons, constituting a singular Fermi surface of two Dirac-points. The PP and PH scattering processes between the lower and upper Dirac cones that participate in gap opening were earlier found to be RG irrelevant in this phase (see section 4.2). Nevertheless, the velocity of the Dirac spinons can well undergo a non-trivial renormalization due to residual intercone scattering processes about the Dirac points

\[ H_{\text{spin}} = V_0 \sum_{\Lambda, \Lambda'} (A_{D\Lambda} A_{\Lambda D} + \text{h.c.}), \]

where we have shown the case of \( \Lambda' = \Lambda_0 \) (the boundary state) and \( \Lambda = \Lambda_0 \) as process I and II respectively in figure 7. In turn, such a velocity renormalization will affect the spinon self-energy (\( \Sigma \)), quasiparticle residue (\( Z \)) and lifetime (\( \tau \)) of the gapless spinons. An investigation of these properties is presented below.

To begin with, we define the energy bandwidth (\( W \)) in the problem \( W = \epsilon_{1\Lambda_0} - \epsilon_{2\Lambda_0} = 2\epsilon_{1\Lambda_0} \) (where \( \epsilon_{2\Lambda_0} \) is the highest energy of spinons in the upper Dirac cone), as \( \epsilon_{2\Lambda_0} = -\epsilon_{1\Lambda_0} \) due to the emergent PH symmetry in this phase. \( W \) is the energy difference between the highest energy electrons/holes placed in the lower/upper boundaries of the Dirac cones (see figure 7). The renormalization of the spinon dispersion (and hence the spinon velocity) takes place via the scattering of the pair of spinons in the PP channel. This is shown in figure 7 as processes that lead from the lower Dirac cone boundary \( \epsilon_{1\Lambda_0} \) to the upper Dirac cone window (\( \Lambda < \Lambda_0 \)). The PH pseudospins have zero dispersion magnitude due to patch-center (\( a, b \)) interchange symmetry, as can be seen from the absence of terms involving \( \epsilon_{a\Lambda_0} A_{a\Lambda_0}^2 \) in Hamiltonian \( \mathcal{H} \) equation (12). Thus, the PH pseudospins do not take part in the spinon velocity renormalization. The process indicated as II in figure 7 (via dotted arrows).
proceeds in the same way, but for the holes in the upper Dirac cone. The RG flow proceeds as the bandwidth is lowered iteratively from $\epsilon_{1\Lambda_L}$. The Hamiltonian for the states lying within the cut-off scale $\Lambda_n$ for the lower Dirac cone has the form

$$H_{1,n} = \epsilon_{1\Lambda_n}(\hat{\tau}_{k_{\Lambda_n}} + \hat{\tau}_{-k_{\Lambda_n}}) + V_0\hat{\tau}_{k_{\Lambda_n}}\hat{\tau}_{-k_{\Lambda_n}}$$

$$= (2\epsilon_{1\Lambda_n} + V_0)A^2_{D\Lambda_n} + V[(A^2_{D\Lambda_n})^2 - (A^2_{D\Lambda_n})]$$

$$\simeq (2\epsilon_{1\Lambda_n} + V_0)A^2_{D\Lambda_n},$$

where we have used the notations $\hat{\tau}_{k_{\Lambda_n}}$ and the interaction strength $V_0$, section 3, as well as neglected certain constant terms ($\propto (A^2)^2$). The Hamiltonian $H_{1,n}$ has spinon kinetic energy terms and correlation energy terms that preserve the gapless nature of the spectrum in this phase. As the RG proceeds by lowering the bandwidth, $\epsilon_{1\Lambda_n}$ is the spinon dispersion generated at the nth RG step for states placed at the renormalized lower Dirac cone boundary.

The $A^2$’s are pseudospin operators (see appendix A) defined in the associated Hilbert space of fermions placed at the boundary. The structure of the effective Hamiltonian $H_{1,n}$ above is reminiscent of the Fermi liquid, i.e. it contains a kinetic energy linear in $k_F$ and inter-particle interactions that are quadratic in in $k_F$. Further, $\hat{\tau}_{k_{\Lambda_n}} = 1/2 = A^2_{D\Lambda_n}$ is conserved during the RG transformations. The associated Greens function for pseudospin $A^2_{D\Lambda_n}$ in an intermediate configuration $| 1 \rangle_{D\Lambda_n}$ at renormalised lower Dirac cone boundary $\Lambda_n$ (i.e. with energy $\epsilon_{1\Lambda_n}$) is

$$G_{1,n,\pm}(|\omega_1\rangle) = \frac{1}{\omega_1 + \epsilon_{1\Lambda_n} + \frac{V_0}{2}},$$

where $\omega_1$ is the scale for quantum fluctuations of the spinon dispersion due to two-spinon interactions. Then, the RG equation for process-II in figure 7 arises from scattering processes due to the Hamiltonian $H_{1,n}$ given earlier

$$\epsilon_{2\Lambda_n}^{-1} - \epsilon_{2\Lambda_n}^{1} = -\frac{V_0^2}{2(\omega_1 + \epsilon_{1\Lambda_n} + \frac{V_0}{2})},$$

where $\epsilon_{2\Lambda_n}$’s are the dispersion for the spinons in the upper Dirac cone (upper band). Similarly for process-I

$$\epsilon_{1\Lambda_n}^{-1} - \epsilon_{1\Lambda_n}^{1} = -\frac{V_0^2}{2(\omega_2 - \epsilon_{2\Lambda_n} - \frac{V_0}{2})},$$

where $\epsilon_{2\Lambda_n}$ is the fermion dispersion at the renormalized upper Dirac cone boundary, the $\epsilon_{1\Lambda_n}$’s are the dispersion in the lower band and $\omega_2$ is the fluctuation energyscale for the process (for a detailed derivation, see appendix E). Putting the constraint $|\epsilon_{1\Lambda_n}| = |\epsilon_{2\Lambda_n}|$ (PH symmetry of the massless Dirac spectrum) on equation (46), we have

$$\epsilon_{2\Lambda_n}^{-1} - \epsilon_{2\Lambda_n}^{1} = -\frac{V_0^2}{2(\omega_2 + |\epsilon_{1\Lambda_n}| - \frac{V_0}{2})}.$$  

Comparing equations (45) and (47), we obtain a relation between the two quantum fluctuation energyscale:

$$\omega_2 = V_0 + \omega_1.$$ 

Using this to simplify equation (46), we obtain

$$\epsilon_{1\Lambda_n}^{-1} - \epsilon_{1\Lambda_n}^{1} = -\frac{V_0^2}{2(\omega_1 + |\epsilon_{1\Lambda_n}| + \frac{V_0}{2}).}$$

The energyscales $\omega_1$ and $\omega_2$ are associated with the fluctuations of the z-component of a single pseudospin in the lower and upper band respectively. The RG equations show that they renormalize the pseudospin dispersion.
relation, equations (47) and (48). Recall that in the RG relation for the stability of the gapped phases I and II, the intermediate configuration energy involved the interaction cost of a pseudospin from the lower band and another from upper band (equation (14)). Thus, the fluctuation scale $\omega$ for the RG gapped phases is double that of the scale in the gapless phase: $\omega = 2\omega_1$. Further, note that in the denominator of equation (48), the intermediate configuration energy is negative and the states at Dirac points are fixed at zero energy, such that the fluctuation scale $\omega_1$ for the renormalized lower band states is negative: $\omega_1 < 0$. Thus, in order to put the RG of the spinon dispersion on the same footing as the RG relations that led to the phase diagram figure 6, we will write the RG equation (48) in terms of the fluctuation scale $\omega$ used earlier

$$\epsilon_{1A}^{n-1} - \epsilon_{1A}^n = \frac{V_0^2}{2(\epsilon_{1A} + \frac{\omega - |\omega|}{2})}.$$  \hspace{1cm} (49)

As the dispersion $\epsilon_{1A} < 0$, the RG flow in equation (49) is RG relevant if $|\omega| = V_0 > 2|\epsilon_{1A}|$ and irrelevant for the converse. Recall that $\max(|\epsilon_{1A}|) = 1.0$, as can be seen from figure 3 (by noting that the Dirac points will appear energy $E = -1.0$). Further, in order to stay within the boundaries of the gapless region III in the phase diagram figure 6, the maximum value of $|\omega| = V_0$ is $\max(|\omega| = V_0) = \Delta_0 \sim 1.355$. From this, we can see that $|\omega| < V_0 < 2|\epsilon_{1A}|$. This leads to the RG flow being irrelevant, and to a reduction of the dispersion magnitude until it stops at a stable fixed point value.

At the stable fixed point

$$|\epsilon_{1A}| = (|\omega| - V_0) / 2 = \hbar v_F^* \Lambda^*,$$  \hspace{1cm} (50)

giving the final renormalized Fermi velocity as $v_F^* = (|\omega| - V_0) / 2\hbar \Lambda^*$. In the continuum limit, the difference RG equation (49) is replaced by a differential RG flow equation

$$\frac{d|\epsilon_{1A}|}{d \ln \left(\frac{\Lambda}{\Lambda_0}\right)} = \frac{-V_0^2}{2(\epsilon_{1A} + \frac{|\omega|}{2})},$$  \hspace{1cm} (51)

where $d|\epsilon_{1A}| = |\epsilon_{1A}^n| - |\epsilon_{1A}^0|$ and $\ln \left(\frac{\Lambda}{\Lambda_0}\right)$ is a negative quantity. Integrating this differential equation between the bare $\Lambda_0$ to the final stable fixed point value $\Lambda^*$ obtained in equation (51), we have

$$\frac{\Lambda^*}{\Lambda_0} = e^{\int_{\epsilon_{1A}}^{\epsilon_{1A}} d|\epsilon_{1A}|} = e^{\int_{|\omega| - V_0}^{2|\epsilon_{1A}|} d\epsilon_{1A}} = e^{-\int_{|\omega| - V_0}^{2|\epsilon_{1A}|} d\epsilon_{1A}} = \exp \left[ \frac{1}{V_0^2} \left( \frac{\omega - V_0}{2} - 1 \right)^2 \right].$$  \hspace{1cm} (52)

In the above, $(|\epsilon_{1A}| - |\epsilon_{1A}|)$ is negative, such that for $|\epsilon_{1A}| = 1.0$, $|\omega| = 2\Delta_0$ and $V_0 = \Delta_0$, the expression $(V_0 - |\omega| + 2|\epsilon_{1A}|)$ is positive and the overall exponent negative, leading to $\Lambda^* < \Lambda_0$.

From the above, we can now derive the renormalization of the Fermi velocity $v_F$. In the bare dispersion at a scale $\Lambda^*$ from the Dirac points, the energy expression is $\epsilon_{1A}^0 = v_F^0 \Lambda^*$, where $v_F^0$ is the bare Fermi velocity. From the above discussion, we find that corrections to the dispersion are RG irrelevant, such that $\epsilon_{1A} < \epsilon_{1A}^0$, i.e. the overall magnitude of the dispersion at the stable fixed point is lower in comparison to the bare one. This leads us to conclude that the renormalized Fermi velocity is lowered from its bare value, $v_F^0 = (|\omega| - V_0) / 2\hbar \Lambda^* < v_F^0$, and the Dirac cones are flattened. Below, we will see the effects of the velocity renormalization on the quasiparticle residue as well as the lifetime of the spinon excitations. This will offer further insight on the nature of the gapless spinon liquid phase.

6.1. Spinon self-energy, residue and lifetime

From the stable fixed point single-spinon energy, we compute the real part of the self energy for the gapless phase from the renormalization of the spinon dispersion (i.e. the difference between final fixed point energy and initial energy)

$$\Sigma_{\Lambda^*}^{\text{Re}}(\omega) = \hbar v_F^* \Lambda^* - \hbar v_F^0 \Lambda$$
$$= \frac{\omega - V_0}{2\hbar \Lambda^*} - v_F^0 \Lambda$$
$$= \frac{\Lambda_0(\omega - V_0)}{2\hbar \Lambda_0} - \exp \left[ \frac{1}{V_0^2} \left( \frac{\omega - V_0}{2} - 1 \right)^2 \right] - v_F^0 \Lambda,$$  \hspace{1cm} (53)

where we have used the expressions for $v_F^*$, $\Lambda^*$ and $|\epsilon_{1A}|$ computed above.

For $(\frac{\omega - V_0}{2} - 1) = V_0$, the exponent can be simplified to

$$\Sigma_{\Lambda^*}^{\text{Re}}(\omega) = \frac{\Lambda_0}{2\Lambda_0} (\omega - G),$$  \hspace{1cm} (54)
where $C = V_0 + 2 \nu \Lambda_0$. As $\Delta_0 \leq \omega \leq 2 \Delta_0$, the self energy can be written as

$$\Sigma^\text{Re}_{\Lambda}(\omega) = \frac{\Lambda}{2\Lambda_0}(\omega - C) \Theta(2\Delta_0 - \omega) \Theta(\omega - \Delta_0),$$

(55)

where $\Theta$ is the Heaviside step theta function. We can now compute the quasi-particle residue $Z(\Lambda, \omega)$ for low-energy excitations $\omega \to \Delta_0+$

$$Z^{-1} = 1 - \frac{d\Sigma^{\text{Re}}}{d\omega} = \left(1 - \frac{\Lambda}{2\Lambda_0}\right).$$

(56)

The quasiparticle residue $Z \to 1$ upon approaching the gapless Dirac-point Fermi surface ($\Lambda \to 0$), indicating non-interacting spinon quasiparticles in the gapless phase. However, in order to learn whether this result indicates a Landau–Fermi liquid or not, we should also compute the spinon lifetime $\tau$. Thus, from the Kramers–Kronig relation, we compute the imaginary part of the self-energy and thereby the spinon lifetime $\tau(\Lambda, \omega)$

$$\Sigma^\text{Im}_{\Lambda}(\omega) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Sigma^\text{Re}_{\Lambda}(\omega')}{\omega' - \omega} d\omega'$$

$$= -\frac{1}{\pi} \int_{\Delta_0}^{2\Delta_0} \frac{\Lambda}{\omega' - \omega} d\omega'$$

$$= \frac{1}{\pi} \frac{\Lambda}{2\Lambda_0} \left[\Delta_0 + (\omega - C) \ln \left(\frac{2\Delta_0 - \omega}{\Delta_0 - \omega}\right)\right]$$

$$= \frac{1}{\tau(\Lambda, \omega)} \quad \text{(as lim } Z = 1\text{),}$$

(57)

where $\mathcal{P}$ is the principal part of the contour integral. Then, upon taking the limit $\omega \to \Delta_0+$ in the gapless phase (and away from the quantum critical lines 2 and 3 in figure 6)

$$\lim_{\omega \to \Delta_0+} \lim_{\Lambda \to 0} \tau(\Lambda, \omega)^{-1} \propto \lim_{\omega \to \Delta_0+} \lim_{\Lambda \to 0} \frac{\Lambda}{\Lambda_0} \to 0,$$

(58)

indicating that the lifetime $\tau$ diverges upon approaching the Dirac-point Fermi surface, without matching the faster ($1/\Lambda^2$) divergence of the Landau–Fermi liquid. This departure from the standard Fermi liquid paradigm arises from an interplay of the inter-spinon interactions and the vanishing spinon density of states as the Dirac point Fermi surface is approached.

On the other hand, upon taking the limits in the other way around, we can obtain the lifetime for the excitations on the quantum critical lines 2 and 3

$$\lim_{\Lambda \to 0} \lim_{\omega \to \Delta_0+} \frac{1}{\tau(\Lambda, \omega)} \propto \frac{\Lambda}{\Lambda_0} (C - \Delta_0) \log(\Delta_0 - \omega) \sim \infty,$$

(59)

indicating that the spinons have a vanishing lifetime precisely on the quantum critical line. The spinon Greens function in the gapless phase can thus be written as

$$G(\Lambda, \omega) = \frac{1}{(\Delta_0 - \omega) + \nu^2 \Lambda - \Sigma^{\text{Re}} - i\Sigma^{\text{Im}}}.$$  

(60)

The Fourier transformation of $G(\Lambda, \omega)$ shows that the two-point correlation function decays algebraically in real space, supporting the formation of an algebraic spin liquid ground state with likely long-ranged entanglement in phase III.

7. Discussion and outlook

The RG phase diagram shown in figure 6 encapsulates the major findings of this work. We conclude with a discussion on the relevance of our findings, as well as an outlook on future prospects. Experiments on the Herbertsmithite and Volborthite materials are thought to be described by the physics of the $S = 1/2$ Heisenberg KA. Indeed, recent experiments on Volborthite at high magnetic fields ($\sim 20–160$ T) have revealed the existence of a robust plateau in the magnetization per site of $1/3$ [31]. From our RG analysis, such a gapped ground state for the Heisenberg KA corresponds to a quantum spin liquid driven by spinon correlations (phase-II in figure 6).

It will, therefore, be interesting to search for experimental signatures of topological order within the plateau, e.g. fractional excitations above the gap. Indeed, claims of the observation of such fractional excitations in Herbertsmithite at zero-field from neutron scattering measurements have appeared recently [27]. A magnetization plateau at $1/3$ in the Heisenberg KA at finite field has also been proposed from numerical methods [9, 70].
It is worth noting the liquid-like nature of the ground state obtained by us in the two gapped phases. Given that RG transformations necessarily preserve all symmetries of the Hamiltonian, the spin liquid ground states of Phases-I and II do not exhibit the further breaking of any lattice translation, rotation or spin symmetries. For such topologically ordered spin liquid ground states, we expect the presence of short-ranged RVB type ordering [68]. This can, for instance, be compared with the finding of competing lattice Nematic and VBC orders for the 1/3 plateau in the XXZ [43] and Heisenberg [37] kagome antiferromagnets with nine-site unit cells obtained from tensor network methods. It is tempting, but not straightforward, to relate such symmetry broken orders to instabilities of the liquid ground states obtained by us for an enlarged nine-site unit cell (see section 2). This will involve the incorporation of symmetry broken orders into the RG scheme [48], and we leave this to further investigations.

Equally interesting would be a search for the gapless algebraic spin liquid found by us (phase-III in figure 6). For this, it may be possible to melt the gapped spin liquid in phase-II by enhancing quantum fluctuations through the application of pressure [71, 72]. We note that the existence of a gapless algebraic phase in the KA has just recently been proposed as being accessible via thermal melting of the finite-field ordered gapped phase [22]. It will be interesting to compare such a thermally induced gapless phase with that reached purely from critical quantum fluctuations (line-2 in figure 6). Theoretical proposals of a gapless Dirac spin liquid in the Heisenberg KA at zero field have appeared in the literature [15, 17–19, 21, 24]. It is pertinent to check whether the gapless phase observed by us at finite field is connected in any way to these results. Given that we find the Heisenberg KA at 1/3 magnetization to be in a topological spin liquid phase, it appears plausible that the zero-field algebraic spin liquid found from various analyses is unstable at finite field. A more subtle possibility involves an adiabatic connection between the zero-field gapless spin liquid and that observed by us in Phase III. Confirming this needs, however, an investigation that lies well beyond the purview of the present study. Further, it is promising to investigate whether such a 2D algebraic Dirac spin liquid can exist at the boundaries of a topologically ordered ground state of the three-dimensional pyrochlore $S = 1/2$ Heisenberg antiferromagnet [73, 74].

The non-trivial multicritical point lying at the intersection of the three spin liquid phases (black circle in figure 6) represents an intriguing result. The experimental observation of such an exotic QCP would be highly valuable, as it is likely to be masked by the formation of a symmetry-broken ground state. Finally, given that critical theories describing line-2 and line-3 relate to relativistic fermionic criticality in two-dimensions, it appears pertinent to employ similar RG methods to the investigation of Mott criticality in graphene and its analogs [75–78]), as well as topological transitions in quantum Hall systems [79] and other topological insulators [80].

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Appendix A. Scattering processes in PP and PH channels

The processes labeled as (1–8) and (9–24) in figure A1 present all possible intra- and inter-band scattering responsible for the instability in the PP for PH channels respectively.

The pseudo-spinors for the PP channel are defined as

$$f^+_{(1/2)D} = [C^+_{(1/2)b}, C^+_{(1/2)a}], \ f^-_{(1/2)D} = \begin{bmatrix} C^+_{(1/2)b} \\ C^+_{(1/2)a} \end{bmatrix}$$  \(\text{(A1)}\)

and the corresponding pseudospins are ($\sigma$'s are Pauli matrices)

$$A^+_{(1/2)D} = f^+_{(1/2)D} \frac{\sigma^z}{2} f^-_{(1/2)D} = \frac{1}{2} (\hat{n}_{1/2b} + \hat{n}_{1/2a} - 1)$$

$$A^-_{(1/2)D} = f^+_{(1/2)D} \frac{\sigma^z}{2} f^-_{(1/2)D} = C^+_{(1/2)b} C^+_{(1/2)a}$$

\[ A^+_{(1/2)D} = f^+_{(1/2)D} \frac{\sigma^z}{2} f^-_{(1/2)D} = C^+_{(1/2)b} C^+_{(1/2)a}. \]  \(\text{(A2)}\)
Similarly, the pseudo-spinors for PH channel are defined as

\[ f^+_1 = \begin{bmatrix} C^+_1 b \end{bmatrix} \quad f^-_1 = \begin{bmatrix} C^-_1 b \end{bmatrix} \]

and the corresponding pseudo-spins are

\[ s^+_1 = \begin{bmatrix} \hat{n}_+ \end{bmatrix} \quad s^-_1 = \begin{bmatrix} \hat{n}_- \end{bmatrix} \]

Thus, the pseudo-spinors \( \hat{A} \) and \( \hat{A} \) are spin-1/2 objects in the PP and PH subspaces respectively. The index \( 1/2 \) of the pseudo-spinors (or pseudospins) represents the (lower/upper) band. One can check that the components of the pseudospins (\( \hat{A} \)) satisfy the SU(2) spin algebra. Using these pseudospins in the Hamiltonian for the scattering in the PP channel, equation (10), we obtain

\[ H_{PP} = \epsilon_1 (2A^+ D + 1) + \epsilon_2 (2A^+ D + 1) + V_{q=0} (A^+ D A^- D + (1 \leftrightarrow 2)) - V_{q=0} (A^+ D A^- D - (1 \leftrightarrow 2)) \]

\[ = (\epsilon_1 + \epsilon_2 + V) (A^+ D + A^- D) + (\epsilon_2 - \epsilon_1) (A^+ D - A^- D) + V (A^+ D A^- D + h.c.) \]

where, in the last step, we have used \( A^+ D A^- D = 1/2 + A^z D \). As discussed in the main text, \( V_{q=0} = 0 \), as \( V_{q=0} < 0 \). Similarly, we can use \( \hat{n}_+(1/2) = \hat{A}^+_1 (1/2) - \hat{A}^+_1 (1/2) + \hat{n}_-(1/2) = \hat{A}^+_1 (1/2) + \hat{A}^+_1 (1/2) + \hat{n}_-(1/2) \) in the Hamiltonian for scattering in the PH channel, equation (11), to obtain (upto a constant)

\[ H_{PH} = 2VA^z D A^+ D - 2V\hat{A}^+_S \hat{A}^- S + V (A^+ D + A^- D). \]

Then, the total Hamiltonian for the system is (equation (12) in the main text)

\[ H = H_{PP} + H_{PH} \]

\[ = (\epsilon_1 + \epsilon_2 + 2V) (A^+ D + A^- D) + (\epsilon_2 - \epsilon_1) (A^+ D - A^- D) + 2V (\hat{A}^+_D \hat{A}^- D - \hat{A}^+_S \hat{A}^- S) \]

\[ = U (A^z D + A^- D) + \Delta (A^z D - A^- D) + 2V (\hat{A}^+_D \hat{A}^- D - \hat{A}^+_S \hat{A}^- S), \]

where \( U = \epsilon_1 + \epsilon_2 + 2V \) and \( \Delta = \epsilon_2 - \epsilon_1 \).

**Appendix B. Two particle interaction RG**

Inter-band scattering processes between the patch-centers and the boundaries lead to renormalization of the two particle interaction strength in the two-patch Hamiltonian. The scattering involves both excitation as well as de-excitation processes. In an excitation process, a pseudospin scatters between the lower band patch-center and the boundary of the upper band, giving
\[
\begin{align*}
V_n^2A_{zDh}^+A_{zDan} | \uparrow \downarrow \rangle & \langle \downarrow \uparrow | \Delta \langle \uparrow \downarrow | 2D_n | \downarrow \downarrow \rangle_D, \\
& = \frac{V_n^2A_{zDh}^+A_{zDan} | \uparrow \downarrow \rangle}{\omega - (2V_0 + \frac{V_0}{2})}, \\
& (\uparrow \downarrow | D_h (\omega - H_A) | \downarrow \downarrow \rangle_D, \\
& = \frac{V_n^2A_{zDh}^+A_{zDan} | \downarrow \uparrow \rangle}{\omega - (2V_0 + \frac{V_0}{2})}.
\end{align*}
\]  

On the other hand, in a de-excitation process, a pseudospin scatters between the boundary of the lower band and patch-center of the upper band, yielding

\[
\begin{align*}
V_n^2A_{zDh}^+A_{zDan} | \uparrow \downarrow \rangle & \langle \downarrow \uparrow | \Delta \langle \uparrow \downarrow | 2D_n | \downarrow \downarrow \rangle_D, \\
& = \frac{V_n^2A_{zDh}^+A_{zDan} | \downarrow \uparrow \rangle}{\omega - (2V_0 + \frac{V_0}{2})}, \\
& (\uparrow \downarrow | D_h (\omega - H_A) | \downarrow \downarrow \rangle_D, \\
& = \frac{V_n^2A_{zDh}^+A_{zDan} | \downarrow \uparrow \rangle}{\omega - (2V_0 + \frac{V_0}{2})}.
\end{align*}
\]  

Then, the total contribution from both types of processes gives

\[
\begin{align*}
\frac{V_n^2}{\omega - (2V_0 + \frac{V_0}{2})} & (A_{zDh}^+A_{zDan}^+ | \uparrow \downarrow \rangle D_h | \downarrow \downarrow \rangle + A_{zDh}^+A_{zDan}^+ | \downarrow \uparrow \rangle D_h | \downarrow \downarrow \rangle, \\
& = K \frac{(A_{zDh}^+A_{zDan}^+ + A_{zDh}^+A_{zDan}^+ | \uparrow \downarrow \rangle D_h | \downarrow \downarrow \rangle + 1 | \downarrow \uparrow \rangle D_h | \downarrow \downarrow \rangle, \\
& + (A_{zDh}^+A_{zDan}^+ - A_{zDh}^+A_{zDan}^+ | \uparrow \downarrow \rangle D_h | \downarrow \downarrow \rangle - 1 | \downarrow \uparrow \rangle D_h | \downarrow \downarrow \rangle),
\end{align*}
\]  

where \( K = \frac{V_n^2}{\omega - (2V_0 + \frac{V_0}{2})} \). If we contract equation (B3) with the state \( \frac{1}{\sqrt{2}} (| \uparrow \downarrow \rangle D_h + | \downarrow \uparrow \rangle D_h) \), we obtain

\[
K = \frac{1}{2} (A_{zDh}^+A_{zDan}^+ + A_{zDh}^+A_{zDan}^+),
\]

and the corresponding RG equation

\[
\delta V_n = \frac{2 (V_0/2)^2}{\omega - (2V_0 + \frac{V_0}{2})},
\]

For phase-I of the phase diagram (figure 6), we have numerically integrated the RG relation \( \frac{dV_n}{d\ln \Lambda} = \frac{dA}{d\ln \Lambda} \) with the limits of integration being \( V_0 \leq V \leq V^* \) and \( \Delta_0 \leq \Delta \leq \Delta^* \), where \( V^* = 2(\omega - 2V_0) \) is the interaction at the one-spinon gapped fixed point, and \( \Delta^* \) is the fixed point hybridization gap to be determined. In this way, we find \( \Delta^* = (\Delta_0 + 2\omega - 5V_0) \), and plot it in terms of the (dark-to-light) pink color bar in figure 6. Similarly, phase-II is obtained by using the corresponding final fixed point value of the PH channel interaction (a measure of the strength of the two-spinon gap), \( V^{\text{fix}} = 2(\Delta_0 - \omega) \).

**Appendix C. Topological quantum numbers**

We can use the spinon Greens function \( \tilde{G}_0 \) in the complex plane to define the topological quantum number \( N_A \) \[54\]

\[
N_A = \frac{1}{2\pi i} \int \! dz \tilde{G}_0(z) = \frac{1}{2\pi i} \int \! dz \tilde{G}_0(z) \partial_z \tilde{G}_0^{-1}(z) = \frac{1}{2\pi i} \int \! dz \ln \left( \frac{\tilde{G}_0^{-1}(z)}{\tilde{G}_0^{-1}(0)} \right) \left[ \ln \left( \frac{\tilde{G}_0^{-1}(\infty)}{\tilde{G}_0^{-1}(0)} \right) \right],
\]

where \( \tilde{C} \) is the upper half-circle in the complex plane. In equation (35), for \( \Delta_0 > 2V_0 \) and \( z \to \infty \), \( \tilde{G}_0^{-1}(\infty) \) is always positive (i.e. \( e^{2\pi i} \)) and \( \tilde{G}_0^{-1}(0) \) always negative (i.e. \( e^{3\pi i} \)). On the other hand, for \( \Delta_0 < 2V_0 \) and \( z \to \infty \), both \( \tilde{G}_0^{-1}(\infty) \) and \( \tilde{G}_0^{-1}(0) \) are positive (i.e. \( e^{2\pi i} \)), leading to...
\[ \frac{\Delta_o}{2} = \frac{1}{2\pi i} (4\pi i - 2\pi i) = 1 \quad \text{if} \quad \Delta_o > 2V_0, \]
\[ \frac{\Delta_o}{2} = \frac{1}{2\pi i} (4\pi i - 4\pi i) = 0 \quad \text{if} \quad \Delta_o < 2V_0. \] (C2)

In order to compute the Chern number related to the one-spinon gap, we can write an effective two-level Hamiltonian in the low-energy neighborhood of the gapped two-band problem as follows
\[ H = \begin{bmatrix} \Delta_o/2 & \nu_\phi (p - p_1) \\ \nu_\phi (p - p_2) & -\Delta_o/2 \end{bmatrix} = \vec{p} \cdot \vec{\sigma}, \] (C3)

where the \(\sigma\)'s are Pauli matrices and \(p\) is the momentum (with respect to the Fermi momentum). It is well known that the two-level problem (with a structure of the Hamiltonian given above) possesses a geometric (Berry) phase \(\gamma = \Omega/2\), where \(\Omega\) is the solid angle created by the loop on the Bloch sphere. If we take an integral over the entire Bloch sphere (with a total solid angle of \(4\pi\)), we obtain the topological Chern number \((C)\) \[\[ C = \frac{1}{4\pi} 4\pi = 1. \] (C4)

**Appendix D. Twist operators**

In section 4.1, we have defined the twist operator projected onto the PP subspace (see equations (24) and (25)). The action of the projected doubled twist operator, \(P_D \hat{O}^2 P_D\), on a single particle state gives
\[ P_D \hat{O}^2 P_D |k\rangle = P_D \hat{O}^2 P_D |1\rangle = \frac{1}{\sqrt{\text{Vol}}} \sum \epsilon^{\hat{\rho}} e^{\hat{\rho} |1\rangle}. \]

In the last step we have used the result: \(P_D \hat{O}^2 P_D = e^{-i\mathcal{A}} C_D^2 \hat{O}^2 P_D = e^{-i\mathcal{A}} |0\rangle\). As defined in the main text, \(P_D = (16/9) A_{D_{11}} A_{D_{22}}^2\) is the projection operator on the PP channel for the 3rd and the 4th bands. In this way, the effect of doubled twist operator on the single spinon state \((|0\rangle)\) is to shift its momentum by \(2\pi/N_i\). Defining \(\vec{p}_m\) as the center of mass momentum
\[ P_D \hat{O}^2 P_m \hat{O}^2 P_D = P_D \hat{O}^2 \sum \vec{k} (\bar{n}_{\vec{k},1} + \bar{n}_{\vec{k},2}) \hat{O}^2 P_D = P_D \hat{O}^2 P_m \hat{O}^2 P_D = P_D \hat{O}^2 \sum \vec{k} (n_{\vec{k},1} + n_{\vec{k},2}) \hat{O}^2 P_D = P_D \hat{O}^2 \sum \vec{k} (n_{\vec{k},1} + n_{\vec{k},2}) \hat{O}^2 P_D = P_D \hat{O}^2 \sum \vec{k} (n_{\vec{k},1} + n_{\vec{k},2}) \hat{O}^2 P_D,
\]

where \(\vec{k}_b\) and \(\vec{k}_c\) are the momenta of the two Dirac-points. In the above, we used the fact that \(\vec{k}_b \equiv \vec{k}_i\), i.e. these two points are connected by a reciprocal lattice vector of the magnetic Brillouin zone for the spinon system. The indices 1 and 2 represent, as always, the lower and upper bands respectively. Under the action of the doubled twist operator on the center of mass momentum, \(\vec{k}_i \leftrightarrow \vec{k}_i - 2\pi/N_i\), which is equivalent to the scattering of a spinon between the two patch-centers (i.e. the two Dirac-points). This scattering process is captured by the effective Hamiltonian.
$$\mathcal{H}^\pm = V [A^+_D D A^-_{D m} + \text{h.c.}] = P_{D m} (\hat{O}^{\pm} + \hat{O}^{13}) P_{D m}. \quad (D3)$$

A similar set of arguments is employed in reaching the effective scattering Hamiltonian between the two patch-centers in the PH channel

$$\mathcal{H}^e = V [A^+_S A^-_{S m} + \text{h.c.}] = P_{S m} (\hat{O}^{e} + \hat{O}^{12}) P_{S m}, \quad (D4)$$

where $P_{S m}$ is the projection onto the same bands mentioned above, but for the PH channel. In the projected space of the 3rd and 4th bands the Hamiltonians $\mathcal{H}^e$ have a band-inversion symmetry, as the chemical potential is $\pm$.

**Appendix E. RG relations in the gapless phase**

We can write the RG equation for process-I (see figure 7) from

$$\frac{V^2}{\omega_1 + \epsilon_{A m} + \frac{\nu_m}{2}} \frac{A^+_D D A^-_{D m} A^+_D D A^-_{D m}}{A^+_D D A^-_{D m} A^+_D D A^-_{D m}} = \frac{V^2}{\omega_1 + \epsilon_{A m} + \frac{\nu_m}{2}} \left( \frac{1}{2} + A^2_{D m} \right) \left( \frac{1}{2} - A^2_{D m} \right)$$

$$= \frac{V^2}{\omega_1 + \epsilon_{A m} + \frac{\nu_m}{2}} \left( \frac{1}{2} - A^2_{D m} \right)$$

by taking the trace over the state $|\uparrow\rangle_{D m}$ and $|\downarrow\rangle_{D m}$. Similarly, the RG equation for process-II is obtained from

$$\frac{V^2}{\omega_2 - \epsilon_{2 m} - \frac{\nu_m}{2}} A^+_D D A^-_{D m} A^+_D D A^-_{D m} = \frac{V^2}{\omega_2 - \epsilon_{2 m} - \frac{\nu_m}{2}} \left( \frac{1}{2} + A^2_{D m} \right) \left( \frac{1}{2} - A^2_{D m} \right)$$

$$= \frac{V^2}{\omega_2 - \epsilon_{2 m} - \frac{\nu_m}{2}} \left( \frac{1}{2} + A^2_{D m} \right)$$

by taking the trace over the state $|\uparrow\rangle_{D m}$ and $|\downarrow\rangle_{D m}$.

**ORCID iDs**

Siddhartha Lal  https://orcid.org/0000-0002-5387-6044

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