Abstract

A nonabelian class of massless/massive nonlinear gauge theories of Yang-Mills vector potentials coupled to Freedman-Townsend antisymmetric tensor potentials is constructed in four spacetime dimensions. These theories involve an extended Freedman-Townsend type coupling between the vector and tensor fields, and a Chern-Simons mass term with the addition of a Higgs type coupling of the tensor fields to the vector fields in the massive case. Geometrical, field theoretic, and algebraic aspects of the theories are discussed in detail. In particular, the geometrical structure mixes and unifies features of Yang-Mills theory and Freedman-Townsend theory formulated in terms of Lie algebra valued curvatures and connections associated to the fields and nonlinear field strengths. The theories arise from a general determination of all possible geometrical nonlinear deformations of linear abelian gauge theory for 1-form fields and 2-form fields with an abelian Chern-Simons mass term in four dimensions. For this type of deformation (with typical assumptions on the allowed form considered for terms in the gauge symmetries and field equations), an explicit classification of deformation terms at first-order is obtained, and uniqueness of deformation terms at all higher-orders is proven. This leads to a uniqueness result for the nonabelian class of theories constructed here.
I. INTRODUCTION

Gauge field theories continue to be fundamental in the study of many areas of mathematical physics, ranging from elementary particle interactions, and completely integrable nonlinear differential equations, to topology of three and four dimensional manifolds. Consequently, an effort to generalize the important types of gauge field theories is of natural interest. In recent work [1,2], a new nonlinear gauge theory was found for massless vector fields in three spacetime dimensions, describing a novel type of generalization of nonabelian Yang-Mills theory. Its origin can be understood by considering nonlinear deformations of the abelian linear gauge theory of 1-form potentials in $d$ dimensions.

The deformation process considered here consists of adding linear and higher power terms to the abelian gauge symmetry while also adding quadratic and higher power terms to the linear field equations, such that a gauge invariant action principle exists which is not equivalent to the undeformed linear theory under nonlinear field redefinitions. The property of gauge invariance is very restrictive and can be used to derive determining equations for the allowed form of the deformation terms added order by order in powers of the fields.

Nonabelian Yang-Mills theory describes one type of allowed deformation, which works for 1-form potentials in any dimension $d > 1$. Interestingly, in $d = 3$ dimensions, another type of deformation is allowed, analogous to the Freedman-Townsend theory of antisymmetric tensor gauge fields [3]. The Freedman-Townsend theory was derived originally only for antisymmetric tensor fields in $d = 4$ dimensions but it has a simple geometrical formulation in any dimension $d > 2$ as a nonlinear gauge theory of $d - 2$-form potentials, in particular, 1-form potentials in $d = 3$ dimensions. Moreover, this formulation of the theory has a further natural extension to a coupled tower of $p$-form potentials of all ranks $1 \leq p \leq d - 2$, in particular, coupled 1-form and 2-form potentials in $d = 4$ dimensions.

The novel generalization of nonabelian Yang-Mills theory in Ref. [2] arises by combining the Yang-Mills type and Freedman-Townsend type deformations of the abelian linear 1-form potential gauge theory in $d = 3$ dimensions. In the present paper, a similar nonlinear deformation of the abelian linear gauge theory of 1-form and 2-form potentials in $d = 4$ dimensions is studied, which has been announced in earlier work [2,9]. The resulting nonlinear gauge theory generalizes both nonabelian Yang-Mills theory and Freedman-Townsend theory, describing coupled massless vector and antisymmetric tensor fields in four spacetime dimensions. As a main new result, an interesting extension of this theory to include a Chern-Simons type mass term involving both the vector and antisymmetric tensor fields is presented.

Physically speaking, the field strengths in this nonlinear gauge theory together represent coupled massive spin-one fields in the case with a Chern-Simons term, and otherwise represent massless spin-one fields coupled to massless spin-zero fields in the case with no Chern-Simons term. The construction and features of these two cases of the theory are given in Sec. IV and Sec. V. The theory has a very rich and interesting geometrical structure, mixing and unifying features of Yang-Mills theory and Freedman-Townsend theory in terms of curvatures and connections associated with the fields and field strengths, which is discussed in Sec. IV. In Sec. V, the theory is derived from an analysis of allowed nonlinear geometrical deformations of the abelian linear gauge theory of massless/massive sets of 1-form and 2-form potentials in four dimensions, with the mass determined by a Chern-Simons type

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term. This analysis yields a novel nonlinear gauge theory for coupled massless and massive sets of vector and antisymmetric tensor fields, generalizing the two preceding cases of the new theory from Secs. II and III. Finally, some concluding remarks are made in Sec. VI.

II. DEFORMATION OF NONABELIAN YANG-MILLS/FREEDMAN-TOWNSEND GAUGE THEORY

First consider, as a starting point, the formulation of nonabelian Yang-Mills theory and Freedman-Townsend theory as respective nonlinear gauge theories of massless vector and antisymmetric tensor fields on four dimensional Minkowski spacetime. For simplicity, the gauge groups will be taken to be three dimensional. Recall, in Yang-Mills theory, the Lie algebra underlying the gauge group is required to be compact semisimple, which then fixes it here to be $SU(2)$. In Freedman-Townsend theory, however, no such condition arises on the underlying Lie algebra of the gauge group, and thus here it can be any three dimensional nonabelian Lie algebra, $\mathcal{G}$. From the classification of three dimensional Lie algebras, it then follows that $\mathcal{G}$ either is semisimple and thus $\mathcal{G} = SU(2)$, $\mathcal{G} = SU(1,1)$, or is solvable and thus $\mathcal{G} = U(1) \times U(1)^2$ which is a semi-direct product of abelian Lie algebras $U(1)$ and $U(1)^2$.

To formulate Yang-Mills theory with an $SU(2)$ gauge group, introduce as the field variable a vector potential $A_\mu$ that takes values in the Lie algebra $SU(2)$. Equivalently, with respect to a fixed $SU(2)$ basis $e_a$, $a = 1, 2, 3$, the vector potential components $A_\mu^a e_a$ can be regarded as a set of three ordinary vector fields $A_\mu^a$ on Minkowski spacetime. Let $\varepsilon_{abc}$ denote the $SU(2)$ structure constants, and let $k_{ab}$ denote an $SU(2)$ positive-definite invariant metric, related to Killing metric by $k_{ab} = -\varepsilon_{cde}\varepsilon_{dab}$, and so $\varepsilon_{abc} = \varepsilon_{bce} k_{ae}$ is totally antisymmetric.

The $SU(2)$ Yang-Mills field strength is given by

$$ F_{\sigma\mu}^a = \partial_\sigma A_\mu^a + \frac{1}{2} \varepsilon_{abc} A_\sigma^b A_\mu^c. \quad (2.1) $$

It is convenient in four dimensions to work with the dual field strength

$$ \tilde{F}_{\sigma\mu}^a = \epsilon_{\sigma\mu}^{\tau\nu} F_{\tau\nu}^a, \quad (2.2) $$

which satisfies the Bianchi identity

$$ D^\sigma \tilde{F}_{\sigma\mu}^a = 0 \quad (2.3) $$

where

$$ D_\sigma = \partial_\sigma + \varepsilon_{bc}^a A_\sigma^b \quad (2.4) $$

is the $SU(2)$ covariant derivative operator. The Yang-Mills Lagrangian is given by

$$ L_YM = \frac{1}{4} k_{ab} \tilde{F}_{\sigma\mu}^a \tilde{F}_{\tau\nu}^b \eta^{\sigma\tau} \eta^{\mu\nu} \quad (2.5) $$

yielding the $SU(2)$ Yang-Mills field equation.
\[ E_{A_\tau}^a = \epsilon^\sigma_{\tau \mu} D_\sigma \tilde{F}_{\mu \nu}^a = 0 \] (2.6)

for \( A_\tau^a \). Under the Yang-Mills gauge symmetry on \( A_\mu^a \), given by the field variation

\[ \delta_\xi A_\mu^a = D_\mu \xi^a \] (2.7)

where \( \xi^a \) are arbitrary functions that take values in the Lie algebra \( SU(2) \), the Lagrangian is gauge invariant, \( \delta_\xi L_{YM} = 0 \). These gauge symmetries generate a \( SU(2) \) gauge group with commutator structure \( [\delta_\xi_1, \delta_\xi_2] = \delta_\xi_3 \) such that \( \xi_3^a = \epsilon^a_{\ bc} \xi_1^b \xi_2^c \). The Lagrangian gives rise to a gauge invariant stress-energy tensor

\[ T_{\mu \nu} (\tilde{F}) = k_{ab} \eta^{\alpha \beta} (\tilde{F}_{a \mu \alpha} \tilde{F}_{b \nu \beta} - \frac{1}{4} \eta_{\mu \nu} \eta^{\sigma \tau} \tilde{F}_{a \sigma \alpha} \tilde{F}_{b \tau \beta}) \] (2.8)

which yields a causal energy-momentum for the vector potential \( A_\mu^a \) on spacelike hypersurfaces, i.e. \( T_{\mu \nu} (\tilde{F}) t^\nu \) is timelike or null for all unit timelike vectors \( t^\nu \) on Minkowski spacetime. Gauge invariance of the Yang-Mills Lagrangian relies on the property that \( SU(2) \) is semisimple. The additional property that \( SU(2) \) is compact, corresponding to positive-definiteness of \( k_{ab} \), is essential for causality of the Yang-Mills stress-energy tensor obtained from the Lagrangian.

Next, for formulating Freedman-Townsend theory with gauge group determined by \( G \), introduce as the field variable an antisymmetric tensor potential \( B_{\mu \nu}^a \) that takes values in the Lie algebra \( G \). Hereafter, it is convenient to identify the vector spaces of \( G \) and \( SU(2) \), so the \( SU(2) \) basis provides a vector-space basis \( e_a, \ a = 1, 2, 3, \) for \( G \). Then the components of the antisymmetric tensor potential \( B_{\mu \nu}^a = B_{\mu \nu}^a e_a \) can be regarded equivalently as a set of three ordinary antisymmetric tensor fields \( B_{\mu \nu}^a \) on Minkowski spacetime. Finally, introduce the abelian field strength for \( B_{\mu \nu}^a \) given by the curl

\[ H_{\sigma \mu \nu}^a = \partial_{[\sigma} B_{\mu \nu]}^a, \] (2.9)

along with its dual

\[ \tilde{H}_\sigma^a = \epsilon^\tau_{\sigma \mu \nu} H_{\tau \mu \nu}^a \] (2.10)

which satisfies the divergence identity

\[ \partial^\sigma \tilde{H}_\sigma^a = 0. \] (2.11)

Let \( c^a_{\ bc} \) denote structure constants of \( G \), and let \( c_{\ cd}^e = c_{\ bd}^e k_{ac} k_{cd} \), where the \( SU(2) \) invariant metric provides a positive-definite metric \( k_{ab} \) on \( G \). Note this metric is not invariant with respect to the Lie algebra product in \( G \) unless \( G \simeq SU(2) \).

Now the field strength for Freedman-Townsend theory is defined in terms of \( B_{\mu \nu}^a \) and \( H_{\tau \mu \nu}^a \) by the relation

\[ K_{\sigma \mu \nu}^a + \tilde{K}_{[\sigma}^b B_{\mu \nu]}^c c_{\ bc}^a = H_{\sigma \mu \nu}^a \] (2.12)
where

\[ \tilde{K}^a_\tau = \epsilon^\sigma_{\tau}^{\mu\nu} K^a_{\sigma\mu\nu} \] (2.13)

is the dual field strength. This field strength has a nonpolynomial expression in terms of \( B^c_{\mu\nu} \) given by

\[ \tilde{K}^a_\mu = Y^{-1a\nu}_{\mu b}(B) \tilde{H}^b_\nu \] (2.14)

with \( Y^{-1a\nu}_{\mu b}(B) \) denoting the inverse of the tensor matrix

\[ Y^{-1a\nu}_{\mu b}(B) = \delta^a_{\nu} \delta^b_{\mu} + \epsilon^a_{\nu\sigma\tau} c_{cb}^{\sigma\tau} B^c_{\sigma\tau} \] (2.15)

where \( B^c_{\sigma\tau} \) is restricted to satisfy \( \text{det}(Y^{-1a\nu}_{\mu b}(B)) \neq 0 \). Note the tensor matrix is symmetric \( Y^a_{ab}(B) = Y^b_{ba}(B) \) due to the antisymmetry of volume tensor \( \epsilon^a_{\nu\sigma\tau} \) and the structure constants \( c^a_{cb} \). Then, the Freedman-Townsend Lagrangian is given by

\[ L_{FT} = \frac{1}{2} k_{ab} \tilde{K}^a_\mu \tilde{K}^b_\nu Y^\mu\nu_{ab}(B). \] (2.16)

This yields the field equation for \( B^a_{\sigma\tau} \),

\[ E^a_{B\sigma\tau} = \epsilon^a_{\sigma\tau} (\partial_\nu \tilde{K}^a_\mu + \frac{1}{2} c^a_{bc} \tilde{K}^b_\nu \tilde{K}^c_\mu) = 0. \] (2.17)

The gauge symmetry on \( B^a_{\mu\nu} \) is given by the field variation

\[ \delta_\chi B^a_{\mu\nu} = \partial_{[\mu}^a \chi^a_{\nu]} - c^a_{cb} \tilde{K}^b_{[\mu} \chi^c_{\nu]} \] (2.18)

where \( \chi^a_{\nu} \) are arbitrary covector functions that take values in the Lie algebra \( G \). These gauge symmetries generate an abelian gauge group \( [\delta_\chi_1, \delta_\chi_2] = 0 \) on solutions of the field equation. Off solutions, the commutator structure closes to within a trivial symmetry proportional to the field equation. Finally, the Lagrangian is gauge invariant to within a total divergence, \( \delta_\chi L_{FT} = \partial_\mu (\epsilon^a_{\mu\sigma\tau} \frac{1}{2} c^a_{cd} k^d_{ce} \tilde{K}^c_\sigma \chi^e_\nu) \). In particular, gauge invariance holds without the need for \( \mathcal{G} \) to be semisimple. Moreover, the stress-energy tensor obtained from the Lagrangian

\[ T_{\mu\nu}(\tilde{K}) = k_{ab}(\frac{1}{2} \tilde{K}^a_\mu \tilde{K}^b_\nu - \frac{1}{2} \eta_{\mu\nu} \eta_{\sigma\tau} \tilde{K}^a_\sigma \tilde{K}^b_\tau) \] (2.19)

yields a causal energy-momentum for the antisymmetric tensor potential \( B^a_{\mu\nu} \) on spacelike hypersurfaces, i.e. \( T_{\mu\nu}(\tilde{K})t^\nu \) is timelike or null for all unit timelike vectors \( t^\nu \) on Minkowski spacetime.

A. Nonlinear generalization

We now construct a massless gauge theory with a nonlinear interaction for the fields \( A^a_{\mu}, B^a_{\mu\nu}, a = 1, 2, 3 \), giving a novel generalization of the Yang-Mills/Freedman-Townsend
theories above. The origin of the generalization will be explained by the deformation analysis carried out in Sec. \[\text{IV}\].

To begin, the following algebraic structure \[\text{[10]}\] is needed on the Lie algebras \( SU(2) \) and \( \mathcal{G} \). Let \( f_{ab}^c \) denote a bilinear map \( f \) from \( \mathcal{G} \times SU(2) \) into \( SU(2) \) defining a representation of \( \mathcal{G} \) on \( SU(2) \)

\[
2f_{[d|c}^a f_{|e]b}^c = f_{cb}^a c_{de}^c \tag{2.20}
\]
such that this representation acts as a derivation preserving the \( SU(2) \) commutator

\[
f_{ed}^c \varepsilon_{ab}^d = 2f_{e[a}^d \varepsilon_{|d|b]}^c. \tag{2.21}
\]

Since \( SU(2) \) is semisimple, any derivation is given by an adjoint representation map

\[
f_{eb}^c = \varepsilon_{dc}^b h_{e}^d \tag{2.22}
\]
with \( h_{e}^d \) denoting some linear map \( h \) from \( \mathcal{G} \) into \( SU(2) \). Then, the relation (2.20) implies that \( h \) is a homomorphism (with respect to the Lie algebra product) of \( \mathcal{G} \) into \( SU(2) \).

Consequently, if \( \mathcal{G} \) is semisimple then clearly \( h(\mathcal{G}) = SU(2) \) and so \( \mathcal{G} \simeq SU(2) \) are isomorphic Lie algebras, with the linear map \( h \) being one-to-one. If instead \( \mathcal{G} \) is solvable then the abelian two-dimensional Lie subalgebra \( U(1)^2 \) in \( \mathcal{G} \) is the kernel of \( h \), with \( \mathcal{G}/U(1)^2 \simeq h(\mathcal{G}) = U(1) \) being any one-dimensional Lie subalgebra in \( SU(2) \). Hence there are two different cases allowed for the Lie algebra structures in the construction of the massless nonlinear theory. For the semisimple case when \( \mathcal{G} \simeq SU(2) \), since \( h \) is an isomorphism, then without loss of generality it follows that

\[
h_{a}^b = \kappa \delta_{a}^b, \quad c_{bc}^a = f_{bc}^a = \kappa \varepsilon_{bc}^a, \tag{2.23}
\]
where \( \kappa \) is an arbitrary nonzero constant. Alternatively, for the solvable case when \( \mathcal{G} = U(1) \times U(1)^2 \), the properties of \( h \) and \( \mathcal{G} \) lead to

\[
h_{a}^b = v_{a}^b, \quad c_{bc}^a = c_{[b|v c]}^a, \quad f_{bc}^a = \varepsilon_{dc}^a w_{d}^b v_{b} \tag{2.24}
\]
for some fixed vectors \( v^a, w^a \) in the common vector space of \( \mathcal{G} \) and \( SU(2) \), and for some fixed linear map \( c_{b}^a \) such that

\[
c_{b}^a v_{a} = 0, \quad c_{b}^a w_{b} = 0. \tag{2.25}
\]

To proceed, the construction now follows the pattern of the novel deformation of \( SU(2) \) Yang-Mills theory in three dimensions from Ref. [4]. Let \( c_{ab}^c = c_{d b}^a k_{ad}^c \) and \( f_{ab}^c = f_{d b}^a k_{ad}^c \).

Nonlinear field strengths \( P_{\mu
u}^a, Q_{\mu
u\sigma}^a \) are introduced in terms of \( A_{\mu}^a, B_{\mu
u}^a \) by

\[
P_{\mu
u}^a - f_{bc}^a \tilde{Q}_{[\mu A_{\nu}^c]}^b = \Gamma_{\mu
u}^a, \tag{2.26}
\]
\[
Q_{\mu
u\sigma}^a - f_{cb}^a \tilde{P}_{[\mu A_{\nu\sigma}^c]}^b + c_{cb}^d \tilde{Q}_{[\mu B_{\nu\sigma}^d]}^b = H_{\mu
u\sigma}^a, \tag{2.27}
\]

\[\]
where

\[ \tilde{P}_a^{\sigma \mu} = \epsilon_{\sigma \mu}^{\tau \nu} P_a^{\tau \nu}, \quad \tilde{Q}_a^{\sigma} = \epsilon_{\sigma}^{\tau \mu \nu} Q_a^{\tau \mu \nu} \] (2.28)

are the duals. These field strengths depend nonpolynomially on \( A_a^{\mu}, B_a^{\mu \nu} \) in the following form. Define the tensor matrix

\[ Y(A, B) = Y^T(A, B) = \begin{pmatrix}
\delta_b^a \delta_\mu^\sigma \delta_\nu^\alpha & -f_{bc}^a \epsilon_{\mu}^{\sigma \tau} A_c^\tau \\
-f_{cb}^a \epsilon_{\mu}^{\sigma \tau} A_c^\alpha & \delta_b^a \delta_\mu^\sigma + c_{cb}^a \epsilon_{\mu}^{\sigma \tau \nu} B_c^{\tau \nu}
\end{pmatrix} \] (2.29)

and consider the inverse matrix \( Y^{-1}(A, B) \) satisfying

\[ Y^{-1}(A, B) Y(A, B) = Y(A, B) Y^{-1}(A, B) = \begin{pmatrix}
\delta_b^a \delta_\mu^\sigma \delta_\nu^\alpha & 0 \\
0 & \delta_b^a \delta_\mu^\sigma
\end{pmatrix} \] (2.30)

with \( A_a^{\mu} \) and \( B_a^{\mu \nu} \) restricted by the condition \( \text{det}(Y(A, B)) \neq 0 \) necessary for invertibility of \( Y(A, B) \). Assemble the field strength duals into tensor matrices

\[ N = \begin{pmatrix} \tilde{P}_a^{\mu \nu} \\ \tilde{Q}_a^{\sigma} \end{pmatrix}, \quad M = \begin{pmatrix} \tilde{F}_a^{\mu \nu} \\ \tilde{H}_a^{\sigma} \end{pmatrix}. \] (2.31)

Then \( N = Y^{-1}(A, B) M \), where \( Y^{-1}(A, B) \) is nonpolynomial in terms of \( A_a^{\mu}, B_a^{\mu \nu}. \)

The Lagrangian for the massless nonlinear theory is constructed by

\[ L_N = k_{ab}(\eta^{\sigma \tau} \eta^{\mu \nu} \tilde{P}_a^{\mu \nu} + \eta^{\sigma \tau} \tilde{Q}_a^{\sigma} \tilde{H}_b^{\tau}) \] (2.32)

which can be also expressed in a more symmetrical form \( L = N^T Y(A, B) N = M^T Y^{-1}(A, B) M \). The gauge symmetries in this theory consist of the field variations given by

\[ \delta_\xi A_a^{\mu} = D_\mu \xi^a + f_{bc}^a \tilde{Q}_b^{\tau} \xi_c^c, \quad \delta_\xi B_a^{\mu \nu} = f_{cb}^a \tilde{P}_b^{\mu \nu} \xi^c \] (2.33)

in terms of arbitrary functions \( \xi^a \), and also

\[ \delta_\chi A_a^{\mu} = 0, \quad \delta_\chi B_a^{\mu \nu} = \partial_\mu \chi^a_\nu - c_{cb}^a \tilde{Q}_b^{\mu} \chi^c_\nu \] (2.35)

in terms of arbitrary covector functions \( \chi^a_\nu \). Under both these gauge symmetries the Lagrangian is invariant to within a total divergence,

\[ \delta_\xi L_N = \partial_\mu (\epsilon^{\mu \nu \sigma \tau} 2 f_{ac}^d k_{ba} \tilde{Q}_a^{\nu} \tilde{P}_b^{\mu \sigma \tau} \xi^e), \quad \delta_\chi L_N = \partial_\mu (\epsilon^{\mu \nu \sigma \tau} c_{ab}^c k_{cd} \tilde{Q}_a^{\nu} \tilde{H}_c^{\mu \nu} \chi^d_\tau) \] (2.37)

as shown by results in Sec. [\text{V}].

In this construction, we refer to the underlying Yang-Mills/Freedman-Townsend algebraic structure \((SU(2), G)\) as the structure group of the massless nonlinear theory.
B. Features

The field equations for $A^a_\mu$ and $B^a_{\mu\nu}$ obtained from the Lagrangian are given by

\begin{align}
E^a_\tau &= \epsilon_{\tau \nu}^{\quad \nu \sigma} (D_\nu \tilde{F}^a_{\sigma \mu} + f_{bc} a^{gb} \tilde{P}^c_{\sigma \mu}) = 0, \\
E^a_{\nu \sigma} &= \epsilon_{\nu \sigma}^{\quad \sigma \mu} (\partial_\nu \tilde{Q}^a_{\mu} + \frac{1}{2} c^{ab} c_{bc} a^{gb} \tilde{Q}^c_{\mu}) = 0.
\end{align}

(2.38)

(2.39)

Both these field equations are of second order in derivatives of $A^a_\mu$, $B^a_{\mu\nu}$, with the second derivatives appearing linearly and first derivatives appearing quadratically, while $A^a_\mu$, $B^a_{\mu\nu}$ appear nonpolynomially. As a consequence of gauge invariance, the field equations satisfy nonlinear divergence identities

\begin{align}
D^\tau E^a_\tau &= -\eta_{\mu}^{\quad \nu} f_{bc} a^{gb} \tilde{Q}^c_{\mu} E^c_{\nu} - \eta_{\mu}^{\quad \nu} \eta_{\sigma}^{\quad \rho} f_{\tau \sigma}^{\quad \tau \rho} \tilde{P}^b_{\rho \nu} E^b_{\sigma}, \\
\partial^\nu E^a_{\nu \sigma} &= -\eta_{\mu}^{\quad \nu} c^{bc} a^{gb} \tilde{Q}^c_{\mu} E^c_{\sigma}.
\end{align}

(2.40)

(2.41)

There are also nonlinear divergence identities that arise on the dual field strengths

\begin{align}
\eta_{\mu}^{\quad \nu} (D_\sigma \tilde{F}^a_{\mu \nu} + f_{bc} a^{gb} \tilde{P}^c_{\mu \nu}) = \eta_{\mu}^{\quad \nu} f_{bc} a^{gb} E^b_{\mu \nu} A^c_{\sigma}, \\
\eta_{\mu}^{\quad \nu} (\partial_\sigma \tilde{Q}^a_{\mu} - c_{bc} a^{gb} \tilde{Q}^c_{\mu} - f_{bc} \eta_{\tau \nu}^{\quad \tau \rho} \tilde{P}^b_{\rho \mu} \tilde{Q}^c_{\sigma}) = -\eta_{\mu}^{\quad \nu} f_{bc} E^b_{\mu \nu} A^c_{\sigma} - \eta_{\mu}^{\quad \nu} \eta_{\sigma}^{\quad \rho} f_{\tau \sigma}^{\quad \tau \rho} E^b_{\rho \nu} B^b_{\sigma}.
\end{align}

(2.42)

(2.43)

due to the $SU(2)$ Bianchi identity (2.3) on $\tilde{F}^a_{\mu \nu}$ and the linear divergence identity (2.11) on $\tilde{H}^a_{\mu \nu \sigma}$. Consequently, for solutions of the field equations, the field strengths satisfy a system of divergence and curl equations

\begin{align}
D_{\nu} \tilde{P}^a_{\mu \nu} &= -f_{bc} a^{gb} \tilde{Q}^c_{\nu}, \quad D^{\nu} \tilde{P}^a_{\mu \nu} = -\eta_{\sigma}^{\quad \rho} f_{\tau \sigma}^{\quad \tau \rho} \tilde{Q}^c_{\mu}, \\
\partial_{\nu} \tilde{Q}^a_{\mu \nu} &= -\frac{1}{2} c^{bc} a^{gb} \tilde{Q}^c_{\mu}, \quad \partial^{\nu} \tilde{Q}^a_{\mu \nu} = \eta_{\mu}^{\quad \nu} c^{bc} a^{gb} \tilde{Q}^c_{\nu} + \eta_{\mu}^{\quad \nu} \eta_{\sigma}^{\quad \rho} f_{\tau \sigma}^{\quad \tau \rho} \tilde{P}^b_{\rho \nu} \tilde{Q}^c_{\mu}.
\end{align}

(2.44)

(2.45)

with quadratic source terms. In the divergence equation on $\tilde{Q}^a_{\mu}$, the source terms identically vanish when $c_{(ab)c} = k_{d(a} c^{d b)c} = 0$, which occurs in the case $G \simeq SU(2)$.

In both cases $G \simeq SU(2)$ or $U(1) \times U(1)^2$, the divergence and curl equations (2.44) and (2.45) together with equations (2.26) and (2.27) constitute a first-order nonlinear field theory for $A^a_\mu$, $B^a_{\mu\nu}$, $\tilde{P}^a_{\mu\nu}$, $\tilde{Q}^a_{\mu\nu}$. Moreover, its linearization reduces to the abelian linear gauge theory of vector potentials and antisymmetric tensor potentials (see Sec. (V A)), whose field strengths represent free massless spin-one and spin-zero fields. Hence, in physical terms, solutions of the nonlinear field strength equations (2.44) and (2.45) describe a set of nonlinearly interacting massless fields of spin-one and spin-zero, respectively, in Minkowski spacetime.

Under the gauge symmetries the field strengths have the transformation

\begin{align}
\delta_\xi f^a_{\mu \nu} &= \varepsilon^{a}_{bc} f^{bc}_{\mu \nu} \xi^{c} + (Y^{-1} \xi \cdot E)^{a}_{\mu \nu}, \quad \delta_\xi Q^a_{\mu} = (Y^{-1} \xi \cdot E)^{a}_{\mu}, \\
\delta_\chi \tilde{P}^a_{\mu \nu} &= (Y^{-1} \chi \cdot E)^{a}_{\mu \nu}, \quad \delta_\chi \tilde{Q}^a_{\mu} = (Y^{-1} \chi \cdot E)^{a}_{\mu},
\end{align}

(2.46)

(2.47)

where $Y^{-1} \xi \cdot E$ and $Y^{-1} \chi \cdot E$ are the respective products of the inverse of the tensor matrix (2.29) with the field equation matrices.
\[
\xi \cdot E = \left( f_{bc}^a E_{B_{\mu \nu} c}^b + f_{ab}^c E_{A_{\mu \nu} b}^c \right), \quad \chi \cdot E = \left( \eta \sigma_{\nu a} E_{B_{\sigma \mu} \nu}^c \right). \tag{2.48}
\]

Hence, for solutions of the field equations, \( \bar{P}_{\mu \nu}^a \) and \( \tilde{Q}_\mu^a \) are gauge invariant with respect to \( \delta_\chi \), while with respect to \( \delta_\xi \), \( \tilde{Q}_\mu^a \) is gauge invariant and \( \bar{P}_{\mu \nu}^a \) transforms homogeneously by the adjoint representation of the Lie algebra \( SU(2) \).

The gauge symmetries on solutions of the field equations have the commutator structure

\[
[\delta_\xi, \delta_\xi] = \delta_\xi_3, \quad [\delta_\chi, \delta_\chi] = 0, \quad [\delta_\xi, \delta_\chi] = 0 \tag{2.49}
\]

where \( \xi_3^a = \varepsilon_{bc}^a \xi_1^b \xi_2^c \). Exponentiating these gauge symmetries leads to a group of finite gauge transformations closed on solutions for \( A_{\mu}^a, B_{\mu \nu}^a \). In particular, \( \delta_\chi \) generates a \( U(1)^3 \) abelian group of nonlinear gauge transformations, while \( \delta_\xi \) generates an \( SU(2) \) nonabelian group of nonlinear gauge transformations, with \( \delta_\chi \) and \( \delta_\xi \) commuting. Thus the complete gauge group for the nonlinear theory has the direct product structure \( SU(2) \times U(1)^3 \).

The spin-one field strength equations (2.44) lead to conserved electric and magnetic type currents \( J_{e \mu}^a = \partial^\nu P_{\mu \nu}^a, J_{m \mu}^a = \partial^\nu \bar{P}_{\mu \nu}^a \) in the nonlinear theory. Corresponding sets of electric and magnetic charges are given by

\[
Q_{e}^a = \frac{1}{4\pi} \int_S P_{\nu \mu}^a t^\nu dS^\mu, \quad a = 1, 2, 3 \tag{2.50}
\]
\[
Q_{m}^a = \frac{1}{4\pi} \int_S \bar{P}_{\nu \mu}^a t^\nu dS^\mu, \quad a = 1, 2, 3 \tag{2.51}
\]

for any closed surface \( S \) in a constant time hypersurface in Minkowski spacetime, with surface element \( dS^\mu \) and hypersurface unit normal \( t^\nu \). If the closed surface is taken to be a sphere \( S_\infty \) at spatial infinity, the resulting enclosed total charges are time-independent constants, \( t^\nu \partial_\nu Q_{e}^a = t^\nu \partial_\nu Q_{m}^a = 0 \), provided there is no current flow normal to \( S_\infty \). These total charges are gauge invariant with respect to \( \delta_\chi \) and transform by the adjoint representation of the Lie algebra \( SU(2) \) with respect to \( \delta_\xi \) if the functions \( \xi^a \) are constant on \( S_\infty \),

\[
\delta_\xi Q_{e}^a = \varepsilon_{bc}^a \xi_1^b \xi_2^c, \quad \delta_\xi Q_{m}^a = \varepsilon_{bc}^a \xi_1^b \xi_2^c, \tag{2.52}
\]
\[
\delta_\chi Q_{e}^a = \delta_\chi Q_{m}^a = 0. \tag{2.53}
\]

Similarly, the spin-zero field strength equations (2.43) yield a conserved tensor \( J_{s \sigma \mu}^a = \partial^\nu Q_{s \sigma \mu \nu}^a \), which leads to a set of scalar type charges

\[
Q_{s}^a = \frac{1}{2\pi} \int_C Q_{s \sigma \nu}^a n^\sigma t^\nu ds^\mu, \quad a = 1, 2, 3 \tag{2.54}
\]

for any closed curve \( C \) on a surface \( S \) in a constant time hypersurface in Minkowski spacetime, with line element \( ds^\mu \), surface unit normal \( n^\sigma \), and hypersurface unit normal \( t^\nu \). If the closed curve is taken to be a circle \( C_\infty \) at spatial infinity, the resulting enclosed total charges are time-independent constants, \( t^\nu \partial_\nu Q_{s}^a = 0 \), provided there is no current flow normal to \( C_\infty \). These total charges are gauge invariant with respect to both \( \delta_\xi \) and \( \delta_\chi \).
\[ \delta \zeta Q_a = \delta \chi Q_a = 0. \]  

(2.55)

Note that, due to the source terms in the spin-one and spin-zero field strength equations, the total charges (2.50), (2.51), (2.54) are, in general, nonzero for solutions.

The Lagrangian gives rise in the standard manner (under diffeomorphisms on Minkowski spacetime) to a stress-energy tensor
\[
T_{\mu \nu}(\tilde{P}, \tilde{Q}) = k_{ab}\left( \tilde{P}_a^{\alpha \sigma} \tilde{P}_b^{\nu \tau} \eta^{\sigma \tau} + \frac{i}{2} \tilde{Q}_a^{\alpha \sigma} \tilde{Q}_b^{\nu \tau} \eta^{\sigma \tau} \eta^{\alpha \beta} + \tilde{\eta}_{\alpha \beta} \tilde{Q}_a^{\alpha \sigma} \tilde{Q}_b^{\nu \tau} \right). \tag{2.56}
\]

This tensor is conserved and gauge invariant on solutions. The conservation equation \( \partial_\mu T_{\mu \nu}(\tilde{P}, \tilde{Q}) = 0 \) can be derived in a standard manner from the spacetime covariance of the theory, while gauge invariance \( \delta \zeta T_{\mu \nu}(\tilde{P}, \tilde{Q}) = \delta \chi T_{\mu \nu}(\tilde{P}, \tilde{Q}) = 0 \) manifestly holds due to the gauge transformation properties of the field strengths.

Conserved currents \( J_\mu(\zeta) = \zeta_\nu T_{\mu \nu}(\tilde{P}, \tilde{Q}) \) are obtained from the stress-energy tensor by contraction with a Killing vector field \( \zeta_\nu \) on Minkowski spacetime. These conserved currents define gauge invariant fluxes of energy-momentum and stress carried by the fields on a constant time hypersurface \( \Sigma \), when \( \zeta_\nu \) is taken to be a time translation and space translation, respectively. Fluxes of angular momentum and boost momentum are defined similarly with \( \zeta_\nu \) taken to be a rotation or boost. In particular, for \( \zeta_\nu = t_\nu \) given by the timelike unit normal \( t_\mu \) to \( \Sigma \), a positive energy \( t_\mu t_\nu T_{\mu \nu}(\tilde{P}, \tilde{Q}) \) and a causal energy-momentum \( t_\mu T_{\mu \nu}(\tilde{P}, \tilde{Q}) \) is obtained for solutions. The corresponding total fluxes are given by
\[
Q(\zeta) = \int_\Sigma t_\mu \zeta_\nu T_{\mu \nu}(\tilde{P}, \tilde{Q})dV, \tag{2.57}
\]
where \( dV \) is the volume element on \( \Sigma \).

An extension of this theory from an \((SU(2), \mathcal{G})\) structure group to a general nonabelian structure group is presented in Sec. IV.

### III. EXTENDED DEFORMATION WITH CHERN-SIMONS MASS TERM

The nonlinear generalization of Yang-Mills/Freedman-Townsend gauge theories in Sec. I has an interesting extension to include a Chern-Simons mass term. This construction yields a novel gauge theory for massive vector potentials \( A_\mu^a \) coupled to massive antisymmetric tensor potentials \( B_{\mu \nu}^a \), \( a = 1, 2, 3 \), presented here. For simplicity, the Lie algebra of the underlying Yang-Mills and Freedman-Townsend gauge groups will again be given by the most general 3-dimensional possibilities, respectively, \( SU(2) \) and \( \mathcal{G} \simeq SU(2) \) or \( U(1) \times U(1)^2 \).

The natural starting point is a nonabelian Chern-Simons type term
\[
L_{\text{CS}} = m \epsilon^{\nu \mu \sigma \tau} k_{ab}\left( B_{\sigma \tau}^a \partial_\nu A_\mu^b + \lambda A_\nu^d A_\mu^e B_{\sigma \tau}^a \epsilon^{be} \right), \tag{3.1}
\]
where \( m \neq 0 \) is the Chern-Simons mass, \( \lambda \) is a coupling constant, and, recall, \( k_{ab} = -\epsilon^{c}_{ad} \epsilon^{d}_{bc} \) is a positive definite metric on the common 3-dimensional vector space of the Lie algebras \( SU(2), \mathcal{G} \). In the case \( \mathcal{G} = SU(2) \), the addition of this Lagrangian to the pure Yang-Mills and Freedman-Townsend Lagrangians (2.5) and (2.16) gives a gauge invariant Lagrangian.
$L = L_{\text{YM}} + L_{\text{FT}} + L_{\text{CS}}$ if a $SU(2)$ Yang-Mills-Higgs type coupling is added between the antisymmetric tensor potentials $B^a_{\mu\nu}$ and the Yang-Mills vector potentials $A^a_{\mu}$. Gauge invariance also determines the Chern-Simons coupling to be $\lambda = \frac{1}{2}$. This yields a massive $SU(2)$ Yang-Mills/Freedman-Townsend gauge theory with the mass arising from the nonlinear interaction of the fields $A^a_{\mu}$ and $B^a_{\mu\nu}$ through the Chern-Simons Lagrangian. The origin of the Yang-Mills-Higgs coupling of $B^a_{\mu\nu}$ with $A^a_{\mu}$ will be explained by the deformation analysis in Sec. V. Remarkably, this coupling also allows the Chern-Simons Lagrangian (3.1) to be compatible with the nonlinear generalization of massless Yang-Mills/Freedman-Townsend theory constructed in Sec. II A, as we now carry out.

To begin, we replace the ordinary curl (2.9) of $B^a_{\mu\nu}$ in the nonlinear field strengths (2.26) and (2.27) by the Yang-Mills covariant curl

$$D^\sigma \tilde{H}^a_{\sigma\mu\nu} = F^b_{\mu\nu} B^c_{\sigma\tau} \varepsilon^{a}_{bcd} \eta^\mu_{\sigma} \eta^\nu_{\tau}. \quad (3.3)$$

We also covariantly modify the nonlinear gauge symmetries (2.34) and (2.36) on $B^a_{\mu\nu}$ to involve an $SU(2)$ covariant curl $D_{[\mu} \chi_{\nu]}^{a}$ in $\delta_{\chi} B^a_{\mu\nu}$ and an $SU(2)$ commutator $\varepsilon^{a}_{bcd} B^b_{\mu\nu} \xi^c$ in $\delta_{\xi} B^a_{\mu\nu}$. The nonlinear gauge symmetries (2.33) and (2.35) on $A^a_{\mu}$ remain unchanged. Furthermore, in the algebraic structure used to construct the massless nonlinear theory, the bilinear map defined by $f_{ab}^{c}$ from $\mathcal{G} \times SU(2)$ into $SU(2)$ remains a representation of $\mathcal{G}$ and a derivation of $SU(2)$. However, consistency of the Yang-Mills-Higgs coupling between $B^a_{\mu\nu}$ and $A^a_{\mu}$ requires that the $SU(2)$ commutator needs to act as a derivation of $\mathcal{G}$

$$\varepsilon^{e}_{[cd]} \varepsilon^{d}_{[ab]} = 2 \varepsilon^{d}_{[a|e]} \varepsilon^{c}_{d[b]}. \quad (3.4)$$

This holds only if $\mathcal{G} \simeq SU(2)$, and therefore excludes the possibility $\mathcal{G} \simeq U(1) \times U(1)^2$. Hence, we thereby have

$$c^{c}_{ab} = \kappa \varepsilon^{c}_{ab}, \quad (3.5)$$

where $\kappa$ is a non-zero constant. Since $f_{ab}^{c}$ is then both a derivation of and representation of $\mathcal{G}$, these properties fix $f_{ab}^{c}$ to be the adjoint representation

$$f_{ab}^{c} = c^{c}_{ab}. \quad (3.6)$$

As a result, with the underlying Yang-Mills/Freedman-Townsend algebraic structure $\mathcal{G} \simeq SU(2)$, the nonlinear field strengths are given by

$$P^a_{\mu\nu} - \kappa \varepsilon^a_{bc} \tilde{Q}^b_{[\mu} A^c_{\nu]} = F^a_{\mu\nu}, \quad (3.7)$$

$$Q^a_{\mu\nu\sigma} - \kappa \varepsilon^a_{bc} (\tilde{Q}^b_{[\mu\nu} B^c_{\sigma]} - \tilde{P}^b_{[\mu\nu} A^c_{\sigma]}) = H^a_{\mu\nu\sigma}. \quad (3.8)$$
while the nonlinear gauge symmetries take the form

\[ \delta \xi A^a_\mu = D_\mu \xi^a + \kappa \varepsilon^a_{\ bc} \tilde{Q}^b_\mu \xi^c, \quad (3.9) \]

\[ \delta \chi A^a_\mu = 0, \quad (3.10) \]

and

\[ \delta \xi B^a_\mu = \varepsilon^a_{\ bc} (B^b_\mu + \kappa \tilde{P}^b_\mu) \xi^c, \quad (3.11) \]

\[ \delta \chi B^a_\mu = D_\mu \chi^a + \kappa \varepsilon^a_{\ bc} \tilde{Q}^b_\mu \chi^c, \quad (3.12) \]

in terms of arbitrary scalar functions \( \xi^a \) and covector functions \( \chi^a_\mu \). The complete Lagrangian is then constructed by adding the Chern-Simons Lagrangian (3.1) to the nonlinear field strength Lagrangian (2.32), \( L = L_N + L_{CS} \). This Lagrangian depends on \( A^a_\mu \) and \( B^a_\mu \) in the nonpolynomial form

\[ L = M^T Y^{-1}(A, B) M + m M^T((2 - 2\lambda) B + (2\lambda - 1) A) \quad (3.13) \]

where \( Y(A, B) \) is the symmetric tensor matrix (2.29) constructed linearly from \( A^a_\mu \), \( B^a_\mu \), and \( M \) is the tensor matrix (2.31) of the \( SU(2) \) field strengths of \( A^a_\mu \), \( B^a_\mu \), and where, in the Chern-Simons term,

\[ A = \begin{pmatrix} 0 \\ A^a_\mu \end{pmatrix}, \quad B = \begin{pmatrix} B^a_\mu \\ 0 \end{pmatrix} \quad (3.14) \]

are tensor matrices defined by the fields. Note \( N = Y^{-1}(A, B) M \) yields the tensor matrix (2.31) of the nonlinear field strengths.

Under both gauge symmetries (3.9) to (3.12), the Lagrangian (3.13) is invariant to within a total divergence,

\[ \delta \xi L = \partial_\mu (\varepsilon^{\mu \sigma \rho \tau} \kappa k_{cd} \varepsilon^c_{\ ba} \tilde{Q}^d_\nu (2\tilde{P}^b_\sigma + mB^b_\sigma) \xi^d), \quad (3.15) \]

\[ \delta \chi L = \partial_\mu (\varepsilon^{\mu \sigma \rho \tau} k_{cd} \varepsilon^c_{\ ab} \tilde{Q}^d_\nu \tilde{Q}^a_\sigma + m\tilde{F}^c_\nu \chi^d), \quad (3.16) \]

provided the coupling constants \( \kappa \) and \( \lambda \) are fixed such that

\[ \kappa = 1/m, \quad \lambda = 1/2 \quad (3.17) \]

as shown by results in Sec. V. This gauge theory gives a nonlinear deformation of the massive \( SU(2) \) Yang-Mills/Freedman-Townsend theory from Ref. [6]. We refer to the underlying algebraic structure \( SU(2) \) as the structure group of the massive nonlinear theory.

### A. Features

The Lagrangian (3.13) yields the following field equations for \( A^a_\mu \) and \( B^a_\mu \):

\[ E^a_\Lambda = \varepsilon^{\nu \sigma \mu \lambda} (D_\nu \tilde{P}^{a}_\sigma - \varepsilon^a_{\ bc} (\frac{1}{m} \tilde{Q}^b_\nu A^c_\sigma - A^b_\nu \tilde{P}^c_\sigma)) + m\tilde{Q}^a_\tau = 0, \quad (3.18) \]

\[ E^a_B = \varepsilon^{\nu \mu \tau \rho} (D_\nu \bar{Q}^{a}_\mu + \varepsilon^a_{\ bc} (\frac{1}{2m} \bar{Q}^b_\nu - A^b_\nu \bar{Q}^c_\mu)) + m\bar{P}^a_\tau = 0. \quad (3.19) \]
These field equations are of second order in derivatives of $A^a_\mu$, $B^a_{\mu\nu}$, with the second derivatives appearing linearly and first derivatives appearing quadratically, while $A^a_\mu$, $B^a_{\mu\nu}$ appear nonpolynomially through the nonlinear field strengths. Due to the $SU(2)$ Bianchi identity \ref{2.23} and $SU(2)$ divergence identity \ref{3.3}, these field strengths satisfy nonlinear divergence identities

\begin{align}
\eta^{\nu\sigma} (D_\nu \tilde{P}^a_{\sigma\mu} + \varepsilon_{bc} (\frac{1}{m} \tilde{Q}^b_{\nu\sigma} - A^b_{\nu}) \tilde{P}^c_{\sigma\mu}) &= \frac{1}{m} \eta^{\nu\sigma} \varepsilon_{bc} E^b_{\nu\sigma} A^c_{\sigma}, \\
\eta^{\nu\sigma} (D_\nu \tilde{Q}^a_{\mu\nu} - \varepsilon_{bc} A^b_{\mu} \tilde{Q}^c_{\sigma\nu}) &= \frac{1}{m} \eta^{\nu\sigma} \varepsilon_{bc} (E^b_{\nu\sigma} A^c_{\sigma} + \eta^{\mu\tau} E^b_{\nu\mu} B^c_{\sigma\tau}).
\end{align}

Consequently, for solutions of the field equations, the field strengths satisfy a system of divergence and curl equations. Here, in contrast to the massless nonlinear theory in Sec. \ref{1a}, this system can be written in terms of the field strengths alone

\begin{align}
\partial_{[\nu} \tilde{P}^a_{\mu\nu]} + m Q^a_{\sigma\nu\mu} &= - \frac{1}{m} \varepsilon_{bc} \tilde{Q}^b_{[\nu\sigma} \tilde{P}^c_{\mu\nu]} , \quad \partial^{\nu} \tilde{P}^a_{\nu\mu} = - \frac{1}{m} \eta^{\nu\sigma} \varepsilon_{bc} \tilde{Q}^b_{\sigma\nu} P^c_{\nu\mu}, \\
\partial_{[\nu} \tilde{Q}^a_{\mu\nu]} + m P^a_{\sigma\nu\mu} &= - \frac{1}{2m} \varepsilon_{bc} \tilde{Q}^b_{[\nu\sigma} \tilde{Q}^c_{\mu\nu]} , \quad \partial^{\nu} \tilde{Q}^a_{\nu\mu} = 0,
\end{align}

constituting a first-order nonlinear field theory for $\tilde{P}^a_{\nu\mu}$, $\tilde{Q}^a_{\nu\mu}$.

Linearization of the equations \ref{3.22} and \ref{3.23} produces a system of linear massive spin-one field strength equations, with the mass given by $m$. The corresponding linearization in terms of $A^a_\mu$ and $B^a_{\mu\nu}$ thus reduces to the abelian linear gauge theory of massive vector potentials and antisymmetric tensor potentials (see Sec. \ref{4a}). Hence, solutions of the nonlinear field theory for $A^a_\mu$ and $B^a_{\mu\nu}$, together describe a set of nonlinearly interacting massive spin-one fields in Minkowski spacetime, where the coupling constant of the interaction is proportional to the inverse mass. A connection between this massive nonlinear theory and pure massive $SU(2)$ Yang-Mills theory is discussed in Sec. \ref{5v}.

Some additional features of the massive nonlinear theory will now be highlighted and compared to the massless nonlinear theory from Sec. \ref{1a}.

The gauge symmetries on solutions of the field equations have the commutator structure

\begin{equation}
[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{\xi_3}, \quad [\delta_{\chi_1}, \delta_{\chi_2}] = 0, \quad [\delta_{\xi_1}, \delta_{\chi_1}] = \delta_{\chi_3}
\end{equation}

where $\xi^a = \varepsilon^a_{bc \xi_1^b \xi_2^c}$ and $\chi^a_{\mu} = \varepsilon^a_{bc \chi_1^b \chi_2^c}$. Thus the gauge group generated by exponentiation of these gauge symmetries is the semi-direct product $SU(2) \times U(1)^3$, which differs from the direct product structure in the massless nonlinear theory. Surprisingly, under this gauge group the nonlinear field strengths for solutions of the field equations are gauge invariant

\begin{equation}
\delta_{\xi} \tilde{P}^a_{\mu\nu} = \delta_{\chi} \tilde{P}^a_{\mu\nu} = 0, \quad \delta_{\xi} \tilde{Q}^a_{\nu\mu} = \delta_{\chi} \tilde{Q}^a_{\nu\mu} = 0.
\end{equation}

Off solutions, the gauge symmetries are closed to within trivial gauge symmetries proportional to the field equations.

Conserved electric, magnetic, and scalar type charges are given by the same currents \ref{2.50}, \ref{2.51}, and \ref{2.54} as derived for the massless nonlinear theory. These currents are gauge invariant on solutions of the field equations, due to the gauge transformation properties of the field strengths.
More remarkably, the conserved stress-energy tensor obtained from the Lagrangian (3.13) is of the same form (2.56) as in the massless nonlinear theory. In particular, the Chern-Simons term (3.1) makes no contribution to the stress-energy, as it has no dependence on the spacetime metric \( \eta_{\mu\nu} \) other than through the associated (metric compatible) volume tensor \( \epsilon^{\mu\nu\rho\sigma} \). This stress-energy tensor (2.56) is again conserved and gauge invariant on solutions of the field equations in the massive nonlinear theory. Likewise it again yields a positive energy \( t^\mu t^\nu T_{\mu\nu}(\tilde{P}, \tilde{Q}) \) and a causal energy-momentum \( t^\mu T_{\mu\nu}(\tilde{P}, \tilde{Q}) \) carried by the fields on any constant time hyperplane, with a unit timelike normal \( t^\mu \).

An extension of this theory from an \( SU(2) \) structure group to a general nonabelian structure group is presented in the next section.

IV. GEOMETRICAL ASPECTS

The massless and massive nonlinear deformations of \( SU(2) \) Yang-Mills/Freedman-Townsend gauge theory constructed in Secs. II and III have a straightforward extension from a \( SU(2) \) structure group to a general nonabelian structure group. The resulting nonabelian massless and massive theories of coupled vector and antisymmetric tensor potentials possess a geometrically rich structure involving connections on Lie group bundles and associated covariant derivative operators and curvatures, which blend geometrical features of pure Yang-Mills theory and pure Freedman-Townsend theory, as will be discussed here. In particular, this structure exposes a striking equivalence between the massless/massive Yang-Mills equations for a Lie group connection and the field strength equations in the massless/massive nonlinear deformation. An interesting duality between the massive Yang-Mills equations and massive Freedman-Townsend equations will also be noted.

To begin, recall, the field variables consist of a set of three vector fields \( A_a^\mu \), and three antisymmetric tensor fields \( B_{\mu\nu}^a \), \( a = 1, 2, 3 \), with which is associated an internal three-dimensional real vector space. We fix a basis \( e_a, a = 1, 2, 3 \), for the Lie algebra \( SU(2) \) on this vector space and formulate the field variables geometrically as an \( SU(2) \)-valued 1-form \( A = A_a^\mu e_a dx^\mu \) and an \( SU(2) \)-valued 2-form \( B = B_{\mu\nu}^a e_a dx^\mu dx^\nu \). Similarly, the nonlinear field strengths are represented geometrically as an \( SU(2) \)-valued 2-form \( P = P_{\mu\nu}^\alpha e_a dx^\mu dx^\nu \) and an \( SU(2) \)-valued 3-form \( Q = Q_{\mu\nu\sigma}^\alpha e_a dx^\mu dx^\nu dx^\sigma \), whose duals are the 2-form \( *P = \tilde{P}_{\mu\nu}^\alpha e_a dx^\mu dx^\nu \) and the 1-form \( *Q = \tilde{Q}_{\mu\nu\rho}^\alpha e_a dx^\mu \).

We now introduce the following \( SU(2) \) covariant derivative operators, using \( A \) and \( *Q \) as connection 1-forms:

\[
D_A = d + [A, \cdot], \tag{4.1}
\]
\[
D_{\lambda^*Q} = d + [\lambda^*Q, \cdot], \tag{4.2}
\]
\[
D_{A+\lambda^*Q} = d + [A + \lambda^*Q, \cdot], \tag{4.3}
\]

where \( \lambda \) is a coupling constant, and \([\cdot, \cdot]\) denotes the \( SU(2) \) Lie bracket. The corresponding \( SU(2) \) curvatures are given by the 2-forms

\[
R_A = dA + \frac{1}{2}[A, A], \tag{4.4}
\]
\[ R_{*Q} = \lambda (d*Q + \frac{1}{2}\lambda[*Q, *Q]), \quad (4.5) \]
\[ R_{A++*Q} = R_A + R_{*Q} + \lambda [A, *Q], \quad (4.6) \]

which satisfy
\[ (D_A)^2 = [R_A, \cdot], \quad (D_{*Q})^2 = [R_{*Q}, \cdot], \quad (D_{A++*Q})^2 = [R_{A++*Q}, \cdot]. \quad (4.7) \]

**A. Massless SU(2) Theory**

The nonlinear massless field equations (2.38) and (2.39) for \( A, B \) together with the massless field strength equations (2.26) and (2.27) which define \( P, Q \) are given in geometrical form by
\[ P = R_{A++*Q} - R_{*Q}, \quad (4.8) \]
\[ Q = D_{*Q} B + \lambda [A, *P], \quad (4.9) \]

and
\[ D_{A++*Q} P = 0, \quad (4.10) \]
\[ R_{*Q} = 0. \quad (4.11) \]

Thus, \( *Q \) is a zero-curvature connection, while \( *P \) is covariantly curl-free.

Hence, on solutions, it follows that
\[ P = R_{A++*Q} \]

is a curvature, while
\[ Q = D_{*Q} (B - \lambda *P) \]

is a covariant curl. In addition, the field strength identities (2.42) and (2.43) become
\[ D_{A++*Q} P = 0 \quad (4.14) \]
due to the \( SU(2) \) Bianchi identity, and
\[ D_{*Q} Q = dQ = 0 \quad (4.15) \]

since \([*Q, Q] = 0\) is an identity.

Now, consider the \( SU(2) \)-valued 1-form
\[ A_{SU(2)} = A + \lambda *Q. \quad (4.16) \]

Under the gauge symmetry \( \delta \xi \), \( A_{SU(2)} \) transforms as a \( SU(2) \) connection
\[ \delta \xi A_{SU(2)} = D_{A_{SU(2)}} \xi \quad (4.17) \]

where
\[ D_{A_{SU(2)}} = d + [A_{SU(2)}, \cdot]. \] (4.18)

This connection is invariant under the gauge symmetry \( \delta_\chi \),
\[ \delta_\chi A_{SU(2)} = 0. \] (4.19)

Moreover, in terms of \( A_{SU(2)} \), the nonlinear field strength equations (4.10) and (4.12) involving \( P \) are simply the \( SU(2) \) Yang-Mills equations. In particular,
\[ F_{SU(2)} = P \] (4.20)
is the \( SU(2) \) curvature of \( A_{SU(2)} \), satisfying the Yang-Mills connection equation
\[ D_{A_{SU(2)}} \ast F_{SU(2)} = 0 \] (4.21)
and the Bianchi identity
\[ D_{A_{SU(2)}} F_{SU(2)} = 0. \] (4.22)

Similarly, consider the \( SU(2) \)-valued 2-form
\[ B_{SU(2)} = B - \lambda \ast P. \] (4.23)

From the field strength equation (4.13), note \( Q = D_{\ast Q} B_{SU(2)} \) is equivalent to
\[ \ast Q = Y_{B_{SU(2)}}^{-1} (\ast d B_{SU(2)}) \] (4.24)
where \( Y_{B_{SU(2)}}^{-1} \) is the inverse of the linear map
\[ Y_{B_{SU(2)}} = \mathbb{1} + \lambda \ast [B_{SU(2)}, \cdot] \] (4.25)
acting on \( SU(2) \)-valued 1-forms. Thus,
\[ K_{SU(2)} = \ast Q \] (4.26)
is the \( SU(2) \) Freedman-Townsend 3-form field strength of \( B_{SU(2)} \). In particular, under the gauge symmetry \( \delta_\chi \), \( B_{SU(2)} \) transforms as a Freedman-Townsend antisymmetric tensor potential
\[ \delta_\chi B_{SU(2)} = D_{K_{SU(2)}} \chi \] (4.27)
and is invariant under the gauge symmetry \( \delta_\xi \),
\[ \delta_\xi B_{SU(2)} = 0. \] (4.28)

Here
\[ D_{K_{SU(2)}} = d + [\ast K_{SU(2)}, \cdot] \] (4.29)
is an \( SU(2) \) covariant derivative using the dual field strength as the connection 1-form. Moreover, \( K_{SU(2)} \) satisfies both the Freedman-Townsend field equation

\[ R_{K_{SU(2)}} = 0 \] (4.30)

and field strength identity

\[ dK_{SU(2)} = 0, \] (4.31)

which follow from the field strength equations (4.11) and (4.15) involving \( Q \).

Interestingly, we therefore see that pure \( SU(2) \) Yang-Mills theory for a vector potential \( A_{SU(2)} \) and pure \( SU(2) \) Freedman-Townsend theory for an antisymmetry tensor potential \( B_{SU(2)} \) possess a combined formulation as a massless nonlinear gauge theory given by a nonlinear deformation of \( SU(2) \) Yang-Mills gauge theory for \( A = A_{SU(2)} - \lambda K_{SU(2)} \) and \( SU(2) \) Freedman-Townsend gauge theory for \( B = \lambda * F_{SU(2)} + B_{SU(2)} \).

B. Massive \( SU(2) \) Theory

Compared to the massless case, the massive nonlinear theory has some significant geometrical differences. The nonlinear massive field equations (3.18) and (3.19) for \( A, B \) together with the field strength equations (3.7) and (3.8) which define \( P, Q \) take the geometrical form

\[ P = R_{A*Q} - R_{*Q}, \] (4.32)

\[ Q = D_{A*Q}B + \frac{1}{m}[A, *P], \] (4.33)

with \( \lambda = 1/m \) in the covariant derivatives (4.2) and (4.3), and

\[ \frac{1}{m}D_{*Q}P = -Q, \] (4.34)

\[ R_{*Q} = -P. \] (4.35)

By substitution of equations (4.33) and (4.34) respectively into expressions (1.32) and (1.33), it follows that

\[ R_{A*Q} = 0 \] (4.36)

and

\[ D_{A*Q}(*P + mB) = 0. \] (4.37)

Hence, on solutions, \( A + \frac{1}{m} * Q \) is a zero-curvature connection, while \( *P + mB \) is covariantly curl-free. In addition, the field strength identities (3.20) and (3.21) become

\[ D_{*Q}P = 0 \] (4.38)

due to the \( SU(2) \) Bianchi identity, and

\[ D_{*Q}Q = dQ = 0 \] (4.39)
since $[\ast Q, Q] = [\ast P, P] = 0$ is an identity.

Now, in analogy with the massless case, consider the $SU(2)$-valued 1-form

$$A_{SU(2)} = \frac{1}{m} \ast Q.$$  \hspace{2cm} (4.40)

In terms of $A_{SU(2)}$, the nonlinear field strength equations (4.34), (4.35), and (4.38) involving $P$ are simply the massive $SU(2)$ Yang-Mills equations, in particular,

$$\ast D_{A_{SU(2)}} \ast F_{SU(2)} + m^2 A_{SU(2)} = 0$$  \hspace{2cm} (4.41)

and the Bianchi identity

$$D_{A_{SU(2)}} F_{SU(2)} = 0$$  \hspace{2cm} (4.42)

where

$$F_{SU(2)} = -P$$  \hspace{2cm} (4.43)

is the $SU(2)$ curvature of $A_{SU(2)}$. Moreover, $A_{SU(2)}$ satisfies Lorentz gauge

$$d \ast A_{SU(2)} = 0$$  \hspace{2cm} (4.44)

due to the field strength identity (4.39). Correspondingly, under the $SU(2)$ gauge symmetry $\delta_\xi$, $A_{SU(2)}$ is gauge invariant

$$\delta_\xi A_{SU(2)} = 0.$$  \hspace{2cm} (4.45)

Finally, from the remaining field strength equations (4.36) and (4.37), it follows that the $SU(2)$-valued 1-form

$$A_{flat} = A + \frac{1}{m} \ast Q$$  \hspace{2cm} (4.46)

is a flat connection, with respect to which the $SU(2)$-valued 2-form

$$B_{\text{curl-free}} = B + \frac{1}{m} \ast P$$  \hspace{2cm} (4.47)

is covariantly curl-free. Thus, up to gauge transformations, $B_{\text{curl-free}}$ is an exact 2-form and $A_{flat}$ vanishes. This 2-form has no apparent geometrical relation to Freedman-Townsend theory, in contrast to the situation in the massless case.

Interestingly, however, the nonlinear field strength equations (4.34), (4.35), (4.38), (4.39) exhibit a direct relation to massive $SU(2)$ Freedman-Townsend theory as follows. Consider the $SU(2)$-valued 2-form

$$B_{SU(2)} = -\frac{1}{m} \ast P.$$  \hspace{2cm} (4.48)

From the field strength equation (4.34), we see that

$$K_{SU(2)} = \frac{1}{m} Q$$  \hspace{2cm} (4.49)

is the $SU(2)$ Freedman-Townsend field strength 3-form determined by
\[ D_{K_{SU(2)}} B_{SU(2)} = K_{SU(2)}, \]  
\hspace{1cm} (4.50)\]

and hence
\[ Y_{B_{SU(2)}}^{-1} (\ast dB_{SU(2)}) = \ast K_{SU(2)}. \]  
\hspace{1cm} (4.51)\]

We then see that the field strength equation (4.35) is simply the massive Freedman-Townsend field equation
\[ R_{K_{SU(2)}} = m^2 \ast B_{SU(2)}, \]  
\hspace{1cm} (4.52)\]

while the field strength equation (4.38) yields the \(SU(2)\) Freedman-Townsend field strength identity
\[ dK_{SU(2)} = 0. \]  
\hspace{1cm} (4.53)\]

Finally, from the field strength equation (4.39), we obtain
\[ D_{K_{SU(2)}} \ast B_{SU(2)} = 0 \]  
\hspace{1cm} (4.54)\]

which is a nonlinear \(SU(2)\) Lorentz gauge on \(B_{SU(2)}\).

It now follows through the duality
\[ A_{SU(2)} = \ast K_{SU(2)}, \quad mB_{SU(2)} = \ast F_{SU(2)}, \quad m \neq 0 \]  
\hspace{1cm} (4.55)\]
given by equations (4.40), (4.43), (4.48), and (4.49) that this massive Freedman-Townsend theory for \(B_{SU(2)}\) is equivalent to the massive Yang-Mills theory for \(A_{SU(2)}\).

Consequently, we see that pure massive \(SU(2)\) Yang-Mills theory for a vector potential \(A_{SU(2)}\) (or equivalently pure massive \(SU(2)\) Freedman-Townsend theory for an antisymmetric tensor potential \(B_{SU(2)}\)), along with a \(SU(2)\) theory of a covariantly exact antisymmetric tensor potential \(B_{\text{curl-free}}\) with respect to a flat connection \(A_{\text{flat}}\), together possess a reformulation as a massive nonlinear gauge theory given by a nonlinear deformation of \(SU(2)\) Yang-Mills/Freedman-Townsend theory with a Chern-Simons mass term for \(A = A_{\text{flat}} - A_{SU(2)}\) and \(B = B_{\text{curl-free}} - \frac{1}{m} F_{SU(2)}\).

\section*{C. General Nonabelian Theory}

The \(SU(2)\) massless and massive nonlinear theories are easily generalized so that in place of the \(SU(2)\) structure group we have a nonabelian structure group based on any semisimple Lie algebra, \(\mathfrak{g}\). Geometrically, \(A, \ast Q, B, \ast P\) thereby are generalized to be \(\mathfrak{g}\)-valued 1-forms and 2-forms. The field strength equations in Secs. \(\mathbf{IV}\) and \(\mathbf{V}\) retain the same geometrical form with \([\cdot, \cdot]\) given by the Lie bracket of \(\mathfrak{g}\). As shown by the deformation analysis in Sec. \(\mathbf{V}\), this provides the most general nonabelian massless and massive nonlinear theories representing a geometrical deformation of semisimple Yang-Mills/Freedman-Townsend gauge theory for Lie-algebra valued field variables \(A, B\).

A further type of extension arises from considering non-semisimple structure groups. Recall from Sec. \(\mathbf{I}\), for the massless nonlinear theory an allowed structure group is
massless and massive cases. An explanation for the origin of these properties will be provided.

There are additional properties required to hold on this algebraic structure in the separate cases. A bilinear map \( f \) defined in the natural manner with respect to the inner products on the Lie brackets and bilinear map, introduce the linear maps \( u, v \) for all \( u, v \in A \), and all \( u', v' \in A' \). Let \( h^T(\cdot) \) denote the adjoint map of \( h(\cdot) \) from \( A \) into \( A' \) defined in the natural manner with respect to the inner products on \( A \) and \( A' \). This gives a bilinear map \( f^T(\cdot, \cdot) = [\cdot, h^T(\cdot)]_A \) from \( A \times A' \) into \( A' \). Now, introduce the associated linear maps \( ad_A(\cdot) \) and \( ad_A'(\cdot) \) by

\[
\text{ad}_A(u) = [v, u]_A, \quad \text{ad}_A'(v')u' = [v', u']_{A'}, \quad (4.56)
\]

and the additional linear maps \( ad_A \circ h(\cdot) \) and \( ad_{hA}(\cdot) \) defined via \( f(\cdot, \cdot) \) by

\[
ad_A \circ h(v')u = ad_A(u)h(v') = [u, h(v')]_A = f(u, v'), \quad (4.57)
\]

\[
ad_{hA}(v')u = ad_A \circ h(u')v = f(v, u'), \quad (4.58)
\]

for all \( u, v \in A \), and all \( u', v' \in A' \). Let \( h^T(\cdot) \) denote the adjoint map of \( h(\cdot) \) from \( A \) into \( A' \) defined in the natural manner with respect to the inner products on \( A \) and \( A' \). This gives a bilinear map \( f^T(\cdot, \cdot) = [\cdot, h^T(\cdot)]_{A'} \) from \( A' \times A \) into \( A' \). Now, introduce the associated linear maps \( ad_{A'} \circ h^T(\cdot) \) and \( ad_{hA'}(\cdot) \), defined via \( f^T(\cdot, \cdot) \) by

\[
ad_{A'} \circ h^T(v)u' = ad_{A'}(u')h^T(v) = [u', h^T(v)]_{A'} = f^T(u', v), \quad (4.59)
\]

\[
ad_{hA'}(v')u = ad_{A'} \circ h^T(u)v' = f^T(v', u), \quad (4.60)
\]

for all \( u, v \in A \), and all \( u', v' \in A' \). Similarly, let \( ad_A^R(\cdot) \) and \( ad_A^{TR}(\cdot) \) denote the adjoint maps of \( ad_A(\cdot) \), \( ad_A'(\cdot) \) and define the related adjoints \( ad_A^*(\cdot) \) and \( ad_A'^*(\cdot) \) by

\[
ad_A^*(u) = ad_A^R(u), \quad ad_A'^*(v')u' = ad_A'^{TR}(v', u'), \quad (4.61)
\]

as well as the analogous adjoint maps \( ad_{hA'}^R(\cdot) \), \( ad_{hA'}^{TR}(\cdot) \), \( ad_{hA'}^*(\cdot) \), \( ad_{hA'}'^*(\cdot) \) given via

\[
ad_{hA'}^R(u)v = -h(T(ad_{hA'}^*(u)v) = -h(T(ad_A^*(u))v) = ad_{hA'}^{TR}(v)u, \quad (4.62)
\]

\[
ad_{hA'}^*(u')v' = -h(ad_A'^*(u')v') = -h(ad_A'(u')v') = ad_{hA'}^{TR}(v')u', \quad (4.63)
\]

again for all \( u, v \in A \), and all \( u', v' \in A' \). Since \( h \) is a homomorphism of \( A' \) into \( A \), it follows that

\[
ad_{hA'}^*(\cdot)h = -ad_{hA'}^*(h^T(\cdot)). \quad (4.64)
\]
The appearance of these adjoint maps is an essential feature in the general nonabelian algebraic structure of the massless and massive nonlinear theories. Note, we have $ad_A^* (\cdot) = ad_{A'} (\cdot)$ and $ad_{A'}^* (\cdot) = ad_A (\cdot)$ if and only if the the inner products are invariant with respect to the Lie brackets, which holds whenever $A$ and $A'$ are semisimple and the inner products are given by the Cartan-Killing metrics of $A$ and $A'$.

Next, we take $A, \ast Q$ to be $A,-,A'-\text{valued } 1\text{-forms}$ and $B, \ast P$ to be $A',A\text{-valued } 2\text{-forms}$, respectively. For later use in formulating $\ast Q, \ast P$ geometrically in terms of $A, B$, we first introduce the following inner product norm on pairs $(\alpha, \beta)$ consisting of a $A\text{-valued } 2\text{-form } \alpha$ and a $A'\text{-valued } 1\text{-form } \beta$:

$$ Y_{A,B}((\alpha, \beta), (\alpha, \beta)) = (\alpha, \alpha)_A + (\beta, \beta)_{A'} - 2([h(\beta), A]_A, \ast \alpha)_{A} - ([\beta, \beta]_{A'}, \ast B)_{A'} $$

where $(\cdot, \cdot)_A$ and $(\cdot, \cdot)_{A'}$ are extended to act on $A-,A'-\text{valued forms}$ via the Hodge inner product. Then let $Y_{A,B}(\cdot) = Y_{A,B}^T(\cdot)$ be the associated symmetric linear map on pairs $(\alpha, \beta)$.

Finally, we also introduce the following covariant derivative operators:

$$ D_A = d + ad_A(A), $$
$$ D_{\ast Q} = d + ad_A \circ h(\ast Q), $$
$$ D_{A+\ast Q} = d + ad_A(A + h(\ast Q)), $$

which act on $A\text{-valued functions and forms}$, and

$$ D'_{\ast Q} = d - ad_{A'}^T(\ast Q), $$
$$ D'_{A} = d - ad_{A'}^T(h^{-1}(A)), $$
$$ D'_{A+\ast Q} = d - ad_{A'}(h^{-1}(A) + \ast Q), $$

which act on $A'\text{-valued functions and forms}$, where the last two derivative operators are defined only when $h$ is invertible. The Lie-algebra valued curvature 2-forms associated with these connections are determined by

$$ R_A = dA + \frac{1}{2}[A, A]_A, $$
$$ R'_{\ast Q} = d\ast Q + \frac{1}{2}[\ast Q, \ast Q]_{A'}, $$

which satisfy

$$ (D_A)^2 = ad_A(R_A), $$
$$ (D'_{\ast Q})^2 = -ad_{A'}^T(R'_{\ast Q}). $$

Moreover, note that since $h$ is a homomorphism of $A'$ into $A$, we have

$$ R_{\ast Q} = h(R'_{\ast Q}), $$

while and from the property that $-ad_{A'}^T(\cdot)$ is the coadjoint representation of $A'$, we also have

$$ R'_{A} = -ad_{A'}^T \circ h^{-1}(R_A) $$

since when $h^{-1}$ exists it gives a homomorphism of $A$ onto $A'$. 

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1. Massless Theory

In the massless nonlinear theory, the Lie algebra $\mathcal{A}$ is required to be semisimple, and hence

$$ad^*_{\mathcal{A}}(\cdot) = -ad^T_{\mathcal{A}}(\cdot) = ad_{\mathcal{A}}(\cdot), \quad (4.78)$$

with the inner product $(\cdot, \cdot)_\mathcal{A}$ given by the Cartan-Killing metric of $\mathcal{A}$. Note that, consequently,

$$ad^T_{\mathcal{A'}} \circ h^T(\cdot) = h^T \circ ad_{\mathcal{A}} \circ h(\cdot) \quad (4.79)$$

and so

$$h^T(D_*Q(\cdot)) = D'_{*Q}(h^T(\cdot)). \quad (4.80)$$

No further properties are needed on the Lie algebra structure of $\mathcal{A}, \mathcal{A}'$. Now the entire theory can be constructed geometrically in terms of the covariant derivatives (4.66), (4.68), (4.69) and curvatures (4.72) and (4.73) along with the linear map (4.65). First, the massless nonlinear field strengths are defined by

$$(\ast P, \ast Q) = Y^{-1}_{A,B}(\ast R_A, \ast dB) \quad (4.81)$$

where $Y_{A,B}^{-1}(\cdot)$ is the inverse of the linear map $Y_{A,B}(\cdot)$. In terms of these field strengths, the gauge symmetries on $A, B$ are given by

$$\delta_{\xi} A = D_A + \ast Q\xi, \quad \delta_{\xi} B = \Gamma'_{*P} \xi, \quad (4.82)$$

for arbitrary $\mathcal{A}$-valued functions $\xi$ on $M$, and

$$\delta_{\chi} A = 0, \quad \delta_{\chi} B = D'_{*Q}\chi, \quad (4.83)$$

for arbitrary $\mathcal{A}'$-valued 1-forms $\chi$ on $M$. Here $\Gamma'_{*P}(\cdot)$ is a linear map associated with $\ast P$ by

$$\Gamma'_{*P} = ad^T_{h',\mathcal{A'}}(\ast P). \quad (4.84)$$

Finally, the Lagrangian is given by

$$L = \frac{1}{2}(\ast P, R_A)_\mathcal{A} + \frac{1}{2}(\ast Q, \ast dB)_{\mathcal{A'}} = \frac{1}{2}(\ast P, \ast Q) \cdot Y_{A,B}(\ast P, \ast Q) \quad (4.85)$$

where $(\alpha, \beta) \cdot (\alpha, \beta) = (\alpha, \alpha)_\mathcal{A} + (\beta, \beta)_{\mathcal{A'}}$ for any $\mathcal{A}$-valued forms $\alpha, \mathcal{A}'$-valued forms $\beta$. This yields the field equations for $A, B$:

$$\ast E_A = D_{A+*Q} \ast P = 0, \quad \ast E_B = R'_{*Q} = 0. \quad (4.86)$$

Thus, on solutions, $\ast Q$ is a zero-curvature connection, while $\ast P$ is covariantly curl-free.

From the field strength equation (4.81), $\ast P$ and $\ast Q$ have the form

$$P = R_{A+*Q} - R_{*Q}, \quad Q = D'_{*Q} B - \Gamma'_{*P} A. \quad (4.87)$$
Hence, since $R_{*Q} = h(R'_{*Q}) = 0$ and $ad_A(*P)A = D_{*Q}*P$ on solutions, it respectively follows that

$$P = R_{A+*Q} \quad (4.88)$$

is a curvature, while

$$Q = D'_{*Q}(B - h^T(*P)) \quad (4.89)$$

is a covariant curl, using in addition the algebraic relation

$$ad^T_{h,A}(\cdot) = h^T \circ ad_A(\cdot). \quad (4.90)$$

Then, the covariant exterior derivatives $D'_{*Q}$ of $Q$ and $D_{A+*Q}$ of $P$ yield

$$D_{A+*Q}P = 0 \quad (4.91)$$

and

$$D'_{*Q}Q = 0. \quad (4.92)$$

These are the same geometrical expressions as those in the $SU(2)$ case.

Therefore, geometrically, $ad_A(A + h^T(*Q)) = A_{YM}$ is a Yang-Mills connection 1-form, whose curvature $ad_A(P) = F_{YM}$ satisfies the massless Yang-Mills equations $D_{A_{YM}}*F_{YM} = 0$ and the Bianchi identity $D_{A_{YM}}F_{YM} = 0$, with $D_{A_{YM}} = d + ad_A(A_{YM})$, based on the gauge group associated to the semisimple Lie algebra $A$.

2. Massive Theory

In the massive nonlinear theory, the homomorphism $h$ is required to be a Lie-algebra isomorphism

$$h(\cdot) = \frac{1}{m}\text{id}(\cdot) \quad (4.93)$$

so $A = h(A') \simeq A'$, where $m \neq 0$ is the mass, and $\text{id}$ is a linear map identifying the vector spaces of $A$ and $A'$. But, $A$ and $A'$ need not be semisimple here, and there are no further properties required on the Lie algebra structure of $A, A'$. Thus, surprisingly, compared to massive Yang-Mills/Freedman-Townsend theory \cite{6} as well as to pure massless Yang-Mills theory, a more general structure group is allowed for the massive nonlinear theory.

First, the massive nonlinear field strengths are defined by

$$(*P, *Q) = Y^{-1}_{A,B}(*R_{A}, *D'_{A}B) \quad (4.94)$$

where $Y^{-1}_{A,B}(\cdot)$ is the inverse of the linear map $Y_{A,B}(\cdot)$. In terms of these field strengths, the gauge symmetries on $A, B$ are given by

$$\delta_\xi A = D_{A+*Q}\xi, \quad \delta_\xi B = \Gamma'_{B+*P}\xi, \quad (4.95)$$

for arbitrary $A$-valued functions $\xi$ on $M$, and
\[ \delta \chi A = 0, \quad \delta \chi B = D'_{A + \ast Q} \chi, \]  
for arbitrary \( A' \)-valued 1-forms \( \chi \) on \( M \), where now

\[ \Gamma'_{B + \ast P} = -ad'_{h, A} (\ast P) + ad_{h, A'} (B). \]  

The Lagrangian is given by

\[ L = \frac{1}{2} (\ast P + m^2 h (B), R_A)_A + \frac{1}{2} (\ast Q, \ast D'_{A} B)_{A'}, \]  
which yields the field equations for \( A, B \):

\[ \ast E_A = h (D'_{\ast Q} h^{-1} (\ast P)) + m^2 h (Q) = 0, \quad \ast E_B = R'_{\ast Q} + h^{-1} (P) = 0 \]  
where \( h^{-1} \) is the inverse of the isomorphism \( h \),

\[ h^{-1} (\cdot) = \text{id} (\cdot) = m^2 h^T (\cdot). \]

From the field strength equation (4.94), \( \ast P \) and \( \ast Q \) have the form

\[ P = R_{A + \ast Q} - R_{\ast Q}, \quad Q = D'_{A + \ast Q} B - \Gamma'_{\ast P} A. \]  
Hence, since

\[ R_{\ast Q} = -P \]  
holds on solutions, it follows that

\[ R_{A + \ast Q} = 0 \]
and so \( A + \ast Q \) is a zero-curvature connection. Furthermore, from

\[ D'_{\ast Q} h^T (\ast P) = -Q \]  
on solutions, and from \( \Gamma'_{\ast P} A = ad'_{\ast A} \circ h^{-1} (A) h^T (\ast P) \) through the algebraic relation

\[ ad'_{h, A} (\cdot) = -ad'_{A'} \circ h^{-1} (\cdot) h \]

obtained from the homomorphism equation (4.64), it follows that

\[ D'_{A + \ast Q} (B + h^T (\ast P)) = 0 \]
and so \( B + h^T (\ast P) \) is covariantly curl-free. Then, the covariant exterior derivatives \( D'_{\ast Q} \) of \( Q \) and \( D_{\ast Q} \) of \( P \) yield

\[ D_{\ast Q} P = 0 \]  
and

\[ D'_{\ast Q} Q = h^T (ad'_{\ast A} (\ast P) P). \]
Therefore, geometrically, \( ad_A \circ h(*Q) = A_{YM} \) is a Yang-Mills connection 1-form, whose curvature \(-ad_A(P) = F_{YM}\) satisfies an adjoint version of the massive Yang-Mills equations

\[
*D_{A_{YM}}^T *F_{YM} + m^2 A_{YM} = 0
\]

and the Bianchi identity

\[
D_{A_{YM}}^T F_{YM} = 0,
\]

where

\[
D_{A_{YM}}^T = d - ad_T^T(A_{YM}).
\]

In addition, \( A_{YM} \) satisfies a nonlinear covariant gauge condition

\[
D_{A_{YM}}^T *A_{YM} = -ad_T^T(*F_{YM})F_{YM}.
\]

Interestingly, this adjoint modification is based on having a non-semisimple Lie algebra \( \mathcal{A} \), so that \( ad^*_A(\cdot) \neq ad_A(\cdot) \). Its consistency relies on the property that, for any Lie algebra \( \mathcal{A} \), \(-ad_T^T(\cdot) \) is the coadjoint representation of \( \mathcal{A} \). If \( \mathcal{A} \) is chosen to be semisimple, then note the standard massive Yang-Mills theory is obtained.

Similarly to the \( SU(2) \) case, the non-semisimple massive Yang-Mills theory (4.109) to (4.112) here is equivalent to a non-semisimple massive Freedman-Townsend theory given by the duality

\[
A_{YM} = *K_{YM}, \quad mB_{YM} = *F_{YM}
\]

as follows from the field strength equations (4.102), (4.104), (4.107) and (4.108).

V. DEFORMATION ANALYSIS

Here a systematic determination of the most general nonlinear geometrical deformation will be given for the linear gauge theory of \( n \geq 1 \) vector potentials \( A^a_\mu, a = 1, \ldots, n \), and \( n' \geq 1 \) antisymmetric tensor potentials \( B^{a'}_{\mu\nu}, a' = 1, \ldots, n' \), with a Chern-Simons type mass term, on a 4-dimensional spacetime manifold \( M \). The method used is a geometrical version of the field theoretic approach to deformations developed in Ref. \[3, 1, 12\].

A. Linear theory

We formulate the linear theory geometrically, using a set of 1-forms \( A^a = A^a_\mu dx^\mu \) and 2-forms \( B^{a'} = B^{a'}_{\mu\nu} dx^\mu dx^\nu \). These field variables are regarded as taking values in respective internal vector spaces \( \mathcal{A}, \mathcal{A}' \) of dimensions \( n, n' \).

To proceed, the only structure we require on the spacetime manifold \( M \) is the exterior derivative operator \( d \) and the Hodge dual \(*\) such that \(*^2 = \pm \mathbb{1}\) where \( \mathbb{1} \) is the identity operator. Hereafter, products of fields will be understood to be wedge products of forms on
$M$ (and tensor products with respect to $\mathcal{A}, \mathcal{A}'$). Recall, in terms of $\ast$, there is a standard Hodge inner product on pairs $(\alpha, \beta)$ of 1-forms and 2-forms, $(\alpha, \beta) \cdot (\alpha, \beta) = \ast(\alpha \ast \alpha) - \ast(\beta \ast \beta)$.

The linear field strengths associated with the field variables are given by the $\mathcal{A}$-valued 2-form $F^a = dA^a$ and $\mathcal{A}'$-valued 3-form $H^{a'} = dB^{a'}$. Then the Lagrangian is given by the following real-valued 4-form

$$L^{(2)} = \frac{1}{2} \delta_{ab} F^a \ast F^b - \frac{1}{2} \delta_{a'b'} H^{a'} \ast H^{b'} + m_{aa'} F^a B^{a'}$$

(5.1)

where $\delta_{ab}, \delta_{a'b'}$ represent components of respective inner products on $\mathcal{A}, \mathcal{A}'$, and $m_{aa'}$ represents components of a bilinear form on $\mathcal{A} \times \mathcal{A}'$. We refer to $m_{aa'}$ as the mass tensor.

This Lagrangian is invariant to within an exact 4-form under the separate abelian gauge symmetries given by

$$\delta^{(0)} \xi A^a = d\xi^a, \quad \delta^{(0)} \xi B^{a'} = 0,$$

(5.2)

for arbitrary $\mathcal{A}$-valued functions $\xi^a$, and

$$\delta^{(0)} \chi A^a = 0, \quad \delta^{(0)} \chi B^{a'} = d\chi^{a'},$$

(5.3)

for arbitrary $\mathcal{A}'$-valued 1-forms $\chi^{a'}$. Under variations of the fields $A^a$ and $B^{a'}$, the Lagrangian yields the Euler-Lagrange field equations

$$\ast \varepsilon^{(1)} A^a = d \ast F^a + m_{a''} A^{a''} = 0,$$

(5.4)

$$\ast \varepsilon^{(1)} B^{a'} = d \ast H^{a'} + m_{a'} B^{a'} = 0,$$

(5.5)

where $m_{a''} = \delta^{ab} m_{ba}$ and $m_{a'} = \delta^{a'b'} m_{ab'}$ are components of linear maps $m_{\mathcal{A}}(\cdot)$ from $\mathcal{A}$ into $\mathcal{A}'$, and $m_{\mathcal{A}'}(\cdot)$ from $\mathcal{A}'$ into $\mathcal{A}$.

Note, in the case when the mass tensor vanishes, $m_{aa'} = 0$, the fields $A^a$ and $B^{a'}$ are decoupled and the linear theory reduces to massless abelian Yang-Mills gauge theory for $A^a$ and massless abelian Freedman-Townsend gauge theory for $B^{a'}$. The field strengths $F^a$ and $H^{a'}$ obviously then describe free massless spin-one and spin-zero fields.

In contrast, in the opposite case when the mass tensor is fully nondegenerate, $m_{aa'} = m\delta_{aa'}$ where $\delta_{aa'}$ is a vector-space isomorphism of $\mathcal{A}$ and $\mathcal{A}'$ (and hence $n = n'$), the fields $A^a$ and $B^{a'}$ are coupled through a Chern-Simons mass term. The linear theory then reduces to massive abelian Yang-Mills/Freedman-Townsend gauge theory, which is the linearization of the nonlinear theory given in Ref. [6]. Consequently, the field strengths $F^a$ and $H^{a'}$ together describe free massive spin-one fields, with the mass given by $m$. (In particular, $A^a$ supplies two of the three spin-one helicity components while $B^{a'}$ supplies the third.)

To continue, we consider the general case with no conditions assumed on the mass tensor. Let $\mathcal{A}_0$ and $\mathcal{A}'_0$ denote the kernels of the maps $m_{\mathcal{A}}(\cdot)$ and $m_{\mathcal{A}'}(\cdot)$, and let $\mathcal{A}_m$ and $\mathcal{A}'_m$ denote the orthogonal complements of these kernels. Note there is a direct sum decomposition,
\( \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_m, \mathcal{A}' = \mathcal{A}'_0 \oplus \mathcal{A}'_m, \) with respect to the inner products on the internal vector spaces. Moreover, \( \mathcal{A}_m \) and \( \mathcal{A}'_m \) are isomorphic vector subspaces, with a common dimension denoted by \( 0 \leq k \leq n, n' \).

Fix a basis for these vector subspaces so that the fields \( \mathcal{A}^a \) and \( \mathcal{B}^{a'} \) belong to \( \mathcal{A}_0 \) and \( \mathcal{A}'_0 \) for \( a = a' = 1, \ldots, k \) and belong to \( \mathcal{A}_m \) and \( \mathcal{A}'_m \) for \( a = k + 1, \ldots, n, a' = k + 1, \ldots, n' \). Then, physically speaking, it follows from the linear field equations that the field strengths \( F^a \) and \( H^{a'} \) given by \( a = a' = 1, \ldots, k \), together describe a set of \( k \) free massive spin-one fields with mass equal to the non-zero eigenvalues of \( m_{aa'} \), while the remaining field strengths \( F^a \) and \( H^{a'} \) given by \( a = k + 1, \ldots, n \) and \( a' = k + 1, \ldots, n' \) describe separate sets of \( n - k \) free massless spin-one fields and \( n' - k \) free massless spin-zero fields, respectively.

### B. Determining equations for nonlinear deformations

We now consider nonlinear deformations of the linear abelian gauge theory for \( \mathcal{A}^a \) and \( \mathcal{B}^{a'} \), with the deformation terms being locally constructed in a geometrical manner from the fields by using only the exterior derivative \( d \) and Hodge dual \( * \) on \( M \). Here, a deformation consists of adding linear and higher power terms to the abelian gauge symmetries (5.2) and (5.3),

\[
\delta_\xi A^a = \delta_\xi^{(0)} A^a + \delta_\xi^{(1)} A^a + \cdots, \quad \delta_\xi B^{a'} = \delta_\xi^{(0)} B^{a'} + \delta_\xi^{(1)} B^{a'} + \cdots, \tag{5.6}
\]

and

\[
\delta_\chi A^a = \delta_\chi^{(0)} A^a + \delta_\chi^{(1)} A^a + \cdots, \quad \delta_\chi B^{a'} = \delta_\chi^{(0)} B^{a'} + \delta_\chi^{(1)} B^{a'} + \cdots, \tag{5.7}
\]

while simultaneously adding quadratic and higher power terms to the linear field equations (5.4) and (5.5),

\[
E^a_A = E_{(1)}^a_A + E_{(2)}^a_A + \cdots, \quad E^{a'}_{B} = E_{(1)}^{a'}_B + E_{(2)}^{a'}_B + \cdots, \tag{5.8}
\]

such that there exists a locally constructed Lagrangian 4-form that is gauge invariant to within an exact 4-form. The condition of gauge invariance is expressed by

\[
\delta_\xi L = \delta_\xi^{(0)} L + \delta_\xi^{(1)} L + \delta_\xi^{(2)} L + \cdots = d\Theta_\xi, \tag{5.9}
\]

\[
\delta_\chi L = \delta_\chi^{(0)} L + \delta_\chi^{(1)} L + \delta_\chi^{(2)} L + \cdots = d\Theta_\chi, \tag{5.10}
\]

holding for some locally constructed 3-forms \( \Theta_\xi \) and \( \Theta_\chi \), where the Lagrangian is related to the field equations through

\[
\delta L = \delta A^a * E^b_A \delta_{ab} + \delta B^{a'} * E^{b'}_{B} \delta_{a'b'} + d\Gamma, \tag{5.11}
\]

holding for some locally constructed 3-form \( \Gamma \), under arbitrary variations \( \delta A^a, \delta B^{a'} \).
For writing down deformation terms and analyzing the deformation equations, a precise formal setting is provided by the field space \( S = \{(A^a(x), B^{a'}(x))\} \) defined as the set of all sections of the vector bundle of \( A \)-valued 1-forms and \( A' \)-valued 2-forms on \( M \). Hereafter, geometrically, a field variation \((\delta A^a, \delta B^{a'})\) is regarded as a vector field on \( S \) while field equations \((E^A_a, E^{B}_a)\) are regarded as a covector field on \( S \), where \( E^A_a, E^{B}_a \) are related to \( E^A_A, E^{B}_{A'} \) by

\[
\delta_{ab} * E^a_A \cdot \delta A^b = \delta A^a \{ E^A_A, \delta A^b \} = \delta A_{a} \cdot \delta B^{a'} = \delta B^{a'} \{ E^A_A, \delta B^{a'} \} = E^A_A, \quad \delta_{a'b'} * E^{a'}_A \cdot \delta B^{a'} = \delta B^{a'} \{ E^A_A, \delta B^{a'} \} = E^{B}_{a'}. \tag{5.12}
\]

Here the hook \( \{ \) denotes interior product of a vector field with a covector field on \( S \). Associated to \( S \) is the jet space defined using local coordinates

\[
J^{(\infty)}(S) = (x, A^a, B^{a'}, dA^a, dB^{a'}, d*da^a, d*dB^{a'}, \ldots)
\]

where \( x \) represents a point in \( M \); \( A^a, dA^a, d*da^a \) represent the values of the \( A \)-valued 1-form field \( A^a(x) \) and its exterior derivatives at \( x \); and \( B^{a'}, dB^{a'}, d*dB^{a'}, \ldots \) represent the values of the \( A' \)-valued 2-form field \( B^{a'}(x) \) and its exterior derivatives at \( x \). In this setting, a locally constructed function or \( p \)-form on \( M \) is a function purely of the jet variables \((A^a, B^{a'}, dA^a, dB^{a'}, d*da^a, d*dB^{a'}, \ldots)\) and their Hodge duals \((*A^a, *B^{a'}, *dA^a, *dB^{a'}, *d*da^a, *d*dB^{a'}, \ldots)\), up to some finite order. Let \( \partial_{A^a}, \partial_{B^{a'}} \), and \( \partial_{(ds)^k dA^a}, \partial_{(ds)^k dB^{a'}} \), \( k = 0, 1, 2, \ldots \), denote derivatives with respect to the jet variables. Note the derivatives \( \partial_{A^a}, \partial_{B^{a'}} \) produce covector fields on \( S \). We define contravariant derivatives \( \partial^b_{A^a}, \partial^b_{B^{a'}} \) that produce vector fields on \( S \) via the natural pairing \( \partial^b_{A^a} g \partial_{A^a} f = \delta_{ab} \partial_{A^a} g \partial_{A^b} f \) and \( \partial^b_{B^{a'}} g \partial_{B^{a'}} f = \delta_{a'b'} \partial_{B^{a'}} g \partial_{B^{a'}} f \), for any locally constructed functions \( f, g \). Likewise we define \( \partial^b_{(ds)^k dA^a}, \partial^b_{(ds)^k dB^{a'}} \). Then, we introduce Euler-Lagrange operators given by

\[
E^b_{A^a} = \partial^b_{A^a} - \sum_{k \geq 0} (*d*)^{k+1} \partial^b_{(ds)^k dA^a}, \quad E^b_{B^{a'}} = \partial^b_{B^{a'}} - \sum_{k \geq 0} (*d*)^{k+1} \partial^b_{(ds)^k dB^{a'}}. \tag{5.14}
\]

These operators take locally constructed functions \( f \) into vector fields \( \langle E^b_{A^a}(f), E^b_{B^{a'}}(f) \rangle \) on \( S \) and have the property that \( E^b_{A^a}(f) = E^b_{B^{a'}}(f) = 0 \) annihilates a locally constructed function \( f \) if and only if \( *f = d\Gamma \) for some locally constructed 3-form \( \Gamma \). The related operators

\[
E^z_{A^a} = \partial_{A^a} - \sum_{k \geq 0} (*d*)^{2k+1} \partial_{(ds)^k dA^a}, \quad E^z_{B^{a'}} = \partial_{B^{a'}} - \sum_{k \geq 0} (*d*)^{2k+1} \partial_{(ds)^k dB^{a'}} \tag{5.15}
\]

yield covector fields on \( S \), where \( *z \) and \( dz \) denote the contravariant Hodge dual operator and contravariant exterior derivative operator on vectors and antisymmetric tensors on \( M \).

The relation between the deformation terms in the field equations and Lagrangian is most naturally expressed through the Euler-Lagrange operators by
\[
E^A_a = E^A_{a'}(* L), \quad E^B_{a'} = E^B_{a'}(* L),
\]
which determines
\[
L = -\frac{1}{k+1} (E^A_A A^a \delta_{ab} - E^B_B B^{a'} \delta_{a'b'})
\]
to within an exact 4-form.

In terms of the Euler-Lagrange operators, the condition for existence of a gauge-invariant Lagrangian is equivalent to the equations
\[
E^b_{A'} (\delta \xi A^a \cdot E^A_A \delta_{ab} + \delta \xi B^{a'} \cdot E^B_B \delta_{a'b'}) = 0, \\
E^b_{A'} (\delta \chi A^a \cdot E^A_A \delta_{ab} + \delta \chi B^{a'} \cdot E^B_B \delta_{a'b'}) = 0, \\
E^b_{B'} (\delta \xi A^a \cdot E^A_A \delta_{ab} + \delta \xi B^{a'} \cdot E^B_B \delta_{a'b'}) = 0, \\
E^b_{B'} (\delta \chi A^a \cdot E^A_A \delta_{ab} + \delta \chi B^{a'} \cdot E^B_B \delta_{a'b'}) = 0.
\]
These four equations are the determining system for all allowed deformations.

**Remark** To proceed, we restrict attention to deformations that involve at most one derivative of \(A^a, B^{a'}, \xi^a, \chi^{a'}\) in the gauge symmetries and at most two derivatives of \(A^a, B^{a'}\) in the field equations. Such deformations automatically preserve the number of gauge degrees of freedom and initial-data degrees of freedom in the nonlinear theory to be the same as those in the linear theory. Also, we consider only nontrivial deformations such that the field equations and gauge symmetries in the nonlinear theory are not equivalent to those in the linear theory by a change either of field variables or of gauge symmetry variables (see [3]).

The determining system can be reformulated more usefully and geometrically as Lie derivative equations. We introduce the Lie derivative \(\mathcal{L}_\delta\) with respect to a vector field \((\delta A^a, \delta B^{a'})\) on \(\mathcal{S}\) acting on a locally constructed covector field \((f^A_a, f^B_b)\) by
\[
(\mathcal{L}_\delta f)^A_a = (\delta A^b A^b | \partial A^b | f^A_a + \delta B^{a'} | \partial B^{a'} | f^A_a + \sum_{k \geq 0} (d^*)^k d \delta A^b | \partial (d^*)^k d A^b | f^A_a + (d^*)^k d \delta B^{a'} | \partial (d^*)^k d B^{a'} | f^A_a ) \\
- (\partial A^b | \delta A^b | f^A_b + \partial A^b | \delta B^{a'} | f^B_b - \sum_{k \geq 0} (d^*)^{k+1} (\partial (d^*)^k d A^b | \delta A^b | f^A_b + \partial (d^*)^k d A^b | \delta B^{a'} | f^B_b ) )
\]
\[
(\mathcal{L}_\delta f)^B_{a'} = (\delta A^b A^b | \partial A^b | f^B_{a'} + \delta B^{a'} | \partial B^{a'} | f^B_{a'} + \sum_{k \geq 0} (d^*)^k d \delta A^b | \partial (d^*)^k d A^b | f^B_{a'} + (d^*)^k d \delta B^{a'} | \partial (d^*)^k d B^{a'} | f^B_{a'} ) \\
- (\partial B^{a'} | \delta A^b | f^B_b + \partial B^{a'} | \delta B^{a'} | f^B_b - \sum_{k \geq 0} (d^*)^{k+1} (\partial (d^*)^k d A^b | \delta A^b | f^B_b + \partial (d^*)^k d A^b | \delta B^{a'} | f^B_b ) ).
\]

**Theorem 1** Local gauge invariance holds if and only if the Lie derivative of the field equations with respect to the gauge symmetries vanishes
\[
\mathcal{L}_\xi (E^A_a, E^B_{a'}) = 0, \quad \mathcal{L}_\chi (E^A_a, E^B_{a'}) = 0.
\]
Geometrically, these equations assert that the gauge symmetries are vector fields tangential to the surface in $S$ corresponding to the field equations. Due to gauge invariance, the commutators of these vector fields have the same property.

**Theorem 2** Local gauge invariance holds only if the Lie derivative of the field equations with respect to the gauge symmetry commutators vanishes

$$L_{[\delta \xi_1, \delta \xi_2]}(E^A, E^B) = 0, \quad L_{[\delta \xi_1, \delta \chi]}(E^A, E^B) = 0, \quad L_{[\delta \xi_1, \delta \chi]}(E^A, E^B) = 0. \quad (5.25)$$

An expansion of these equations in powers of $A^a$ and $B^{a'}$ (and their derivatives) gives a hierarchy of determining equations whose solutions yield all allowed deformation terms in the field equations and gauge symmetries. We now find the solution of these determining equations explicitly at the lowest orders to give all first-order deformations and then outline an induction analysis to obtain a uniqueness result for the higher-order deformations.

**C. First-order deformations and uniqueness of higher-order deformations**

Up to a change of field variables and gauge symmetry variables, the most general possible first-order deformation terms for the gauge symmetries are given by

$$(1) \quad \delta_\xi A^a = a_{bc} A^b \chi^c + b_{b'd} H^{b'd} \chi^c + e_{b'd} (B^{b'd} d\chi^c) + \tilde{c}_{b'd} (A^{b'd} d\chi^c), \quad (5.26)$$

and by

$$(1) \quad \delta_\chi B^{a'} = d_{a'b'} A^b \chi^c + d_{a'b'} B^{b'} \chi^c + e_{a'b'} (F^{b'} \chi^c) + e_{a'b'} (F^{b'} d\chi^c) + f_{a'b'} (A^b d\chi^c), \quad (5.27)$$

with

$$e^{a'}_{[bc]} = f^{a'}_{[bc]} = 0, \quad (5.28)$$

and by

$$\delta_\chi A^a = a_{b'd} B^{b'} \chi^c + g_{b'd} (B^{b'} \chi^c) + h_{b'd} (F^{b'} \chi^c) + \tilde{h}_{b'd} (A^{b'd} \chi^c), \quad (5.29)$$

$$\delta_\chi B^{a'} = j_{a'b'} (AB^c \chi^c) + \tilde{j}_{a'b'} (B^{b'} \chi^c) + \tilde{k}_{b'd} (H^{b'd} \chi^c) + \tilde{k}_{b'd} (A^{b'd} \chi^c) + m_{b'd} (B^{b'} \chi^c) + \tilde{m}_{b'd} (B^{b'} \chi^c), \quad (5.30)$$

where the coefficients are constants, which represent the components of bilinear maps from $A \times A$, $A \times A'$, $A' \times A$, $A' \times A'$ into $A$ and $A'$. These coefficients are determined by solving the zeroth-order part of the Lie derivative commutator equation from Theorem 2 (using the methods of Ref. [3][12]). This yields the linear algebraic relations

$$a_{a(bc)} = 0, \quad f_{a'(bc)} = 0, \quad (5.31)$$

$$l_{a'b'} = \tilde{l}_{a'b'} = m_{a'b'} = \tilde{m}_{a'b'} = 0, \quad g_{ab'c} = \tilde{g}_{ab'c} = 0, \quad (5.32)$$

$$c_{ab'c} = \tilde{c}_{ab'c} = 0, \quad \tilde{j}_{a'bc} = \tilde{d}_{a'bc} = 0, \quad d_{a'bc} + j_{a'bc} = 0. \quad (5.33)$$
Additional linear algebraic relations arise from the first-order part of the Lie derivative equation from Theorem 1 applied to the rigid symmetries

\[(\delta \xi)_{\text{rigid}} = \delta \xi|_{d\xi=0}, \quad (\delta \chi)_{\text{rigid}} = \delta \chi|_{d\chi=0}\]  

(5.34)

given by restricting the gauge symmetry variables so that \(d\xi^a = d\chi^{a'} = 0\). This leads to (by the methods of Ref. \[3,12\])

\[\tilde{e}_{a'b'c'} + b_{a'b'c'} = 0, \]  

(5.35)

\[m_a a' j_{a'b'c'} - m_{b'} a_{bac} = 0, \quad a_{(ab)c} - m_{(b')b_a} = 0, \quad j_{(a'|c)b'} - m_{(a')b|b'c} = 0, \]  

(5.36)

\[k_{(a'b')c'} = 0, \quad m_a a' k_{a'b'c'} + j_{b'ac'} = 0, \]  

(5.37)

\[h_{abc'} = \tilde{h}_{abc'} = 0, \quad \tilde{k}_{a'b'c'} = 0. \]  

(5.38)

Then, we return to the first-order part of the Lie derivative equation with \(\xi^a\) and \(\chi^{a'}\) now taken to be arbitrary, which determines the first-order deformation terms in the field equations (by the methods of Ref. \[1\]). This yields

\[\begin{align*}
*E_A^a &= d*(\frac{1}{2}a_{ab} A^b A^c + b^a_{b'c'} H^{b'} A^c) - a_{ab} a^b A^c - b_{b'} a^b H^{b'} F^c \\
&\quad + (2F^{b'} H^{c'} - A_d H^{c'}) e^{a'}_{bc} + j_{b'} a^b H^{b'} B^{c'} - m_{c'} a_{cb} A^b B_{c'}, \\
*E_B^{a'} &= d*(j_{b'c'} A^b B^{c'} - k_{b'c'} H^{b'} B^{c'} + b_{b'} a^b A^c) + \frac{1}{2}k_{b'c'} a^b H^{b'} H^{c'} \\
&\quad - d(A^{b'} F^c) e^{a'}_{bc} - j_{c'b'} a^b H^{b'} + m_{a'} a^a A^b A^c,
\end{align*}\]  

(5.39)

(5.40)

together with the linear algebraic relation

\[m_{(a'|} e_{a'b|c')} + m_{b'} e_{b'ac} = 0. \]  

(5.41)

The corresponding Lagrangian is given by

\[L = \frac{1}{2} a_{abc} A^b A^c + j_{abc} H^{a'} A^b B^{c'} + b_{abc} H^{a'} H^{b'} A^c - \frac{1}{2} k_{a'b'c'} H^{a'} H^{b'} B^{c'} - e_{abc} H^{a'} F^b A^c + \frac{1}{2} m_{a} a_{abc} B^{a'} A^b A^c. \]  

(5.42)

These deformation terms are related to the deformation terms in the gauge symmetries (as follows from general results in Ref. \[3\]) by being the Noether currents of the rigid symmetries associated to the first-order deformed gauge symmetries

\[\begin{align*}
\delta \xi A^a &= a^a_{bc} A^b \xi^c + b^a_{b'c'} * H^{b'} \xi^c, \\
\delta \xi B^{a'} &= -j_{c'b'} B^{b'} \xi^c - b_{b'} a^b \xi^c + e_{b'c'} F^b \xi^c, \\
\delta \chi A^a &= 0, \\
\delta \chi B^{a'} &= j_{b'c'} A^b \chi^{c'} + k_{b'c'} H^{b'} \chi^{c'}. \end{align*}\]  

(5.43)

(5.44)

(5.45)

(5.46)
In particular, we have \( *d(E_A a^b \xi_{ab}) = (\delta_\xi A^a)_{\text{rigid}} E^A_a \) and \( *d(E_B a' b \delta_{a'b'}) = (\delta_\xi B^{a'})_{\text{rigid}} E^B_{a'} \),

where \( d\xi^b = d\chi_{b'} = 0 \).

We next note that, from (5.43) to (5.46),

\[
[\delta^{(0)}_{\xi_1}, \delta^{(0)}_{\xi_2}] = \delta^{(0)}_{\xi_3}, \quad \xi_3^a = a^b_{bc} \xi_1^b \xi_2^c,
\]

(5.47)

\[
[\delta^{(0)}_{\chi_1}, \delta^{(0)}_{\chi_2}] = 0,
\]

(5.48)

\[
[\delta^{(0)}_{\xi_1}, \delta^{(0)}_{\chi_2}] = \delta^{(0)}_{\chi_3}, \quad \chi_3^{a'} = j^{a'}_{bc} \xi_1^b \chi_2^c.
\]

(5.49)

Now we consider the first-order part of the Lie derivative commutator equation from Theorem 2 and subtract the first-order part of the Lie derivative equation from Theorem 1 with the gauge symmetry variables given by the commutators (5.47) to (5.49). This combined equation leads to the result (by the methods of Ref. [3,12]) that the gauge symmetry

Theorem 2 and subtract the first-order part of the Lie derivative equation from Theorem 1 with the gauge symmetry variables given by the commutators (5.47) to (5.49). This combined equation leads to the result (by the methods of Ref. [3,12]) that the gauge symmetry commutators are closed to first-order when \( A^a, B^{a'} \) satisfy the linear field equations \( E_A^a = 0 \) and \( E_B^{a'} = 0 \). Then if the gauge symmetry variables are taken to be rigid, \( d\xi_1^a = d\xi_2^a = 0 \) and \( d\chi_1^{a'} = d\chi_2^{a'} = 0 \), we obtain an integrability condition involving just the first-order deformation terms,

\[
([\delta^{(1)}_{\xi_1}, \delta^{(1)}_{\xi_2}])_{\text{rigid}} A^a - (\delta^{(1)}_{\xi_3})_{\text{rigid}} A^a = d\xi^{(1)}_{(\xi_1, \xi_2)}, \quad ([\delta^{(1)}_{\xi_1}, \delta^{(1)}_{\xi_2}])_{\text{rigid}} B^{a'} - (\delta^{(1)}_{\xi_3})_{\text{rigid}} B^{a'} = d\chi^{(1)}_{(\xi_1, \xi_2)},
\]

(5.50)

\[
([\delta^{(1)}_{\chi_1}, \delta^{(1)}_{\chi_2}])_{\text{rigid}} A^a = d\xi^{(1)}_{(\chi_1, \chi_2)}, \quad ([\delta^{(1)}_{\chi_1}, \delta^{(1)}_{\chi_2}])_{\text{rigid}} B^{a'} = d\chi^{(1)}_{(\chi_1, \chi_2)},
\]

(5.51)

\[
([\delta^{(1)}_{\xi_1}, \delta^{(1)}_{\chi_2}])_{\text{rigid}} A^a - (\delta^{(1)}_{\chi_3})_{\text{rigid}} A^a = d\xi^{(1)}_{(\chi_1, \chi_2)}, \quad ([\delta^{(1)}_{\xi_1}, \delta^{(1)}_{\chi_2}])_{\text{rigid}} B^{a'} = d\chi^{(1)}_{(\chi_1, \chi_2)}.
\]

(5.52)

which hold for some locally constructed \( A \)-valued functions \( \xi^{(1)}_{(\cdot, \cdot)} \) and \( A' \)-valued 1-forms \( \chi^{(1)}_{(\cdot, \cdot)} \) depending linearly on \( \xi_1^a, \xi_2^a, \chi_1^{a'}, \chi_2^{a'} \). The solution of these six equations (using the methods of Ref. [12]) yields the quadratic algebraic relations

\[
a_{ab}a^b_{ec} - 2a_{ab|c}a^b_{d|e} = 0,
\]

(5.53)

\[
2a_{ab|c}b^b_{|d'|e} - b_{ab|c}a^b_{|d|e} + 2b_{ab'|c}b^{b'}_{|e}m_{|d|d'} - 2b_{ab'|c}j^{b'}_{|e}d'| = 0,
\]

(5.54)

\[
j^{a'}_{bc}a^b_{|d|e} - 2j^{a'}_{|d|b}j^{b'}_{|e} = 0,
\]

(5.55)

plus three others that are redundant as a consequence of (5.36) and (5.37).

Another integrability condition arises for the first-order deformation terms if we consider the second-order part of the Lie derivative equation from Theorem 1 under the previous conditions imposed on \( A^a, B^{a'}, \xi^a, \chi^{a'} \). Contracting this equation with \( (A^a, B^{a'}) \), we obtain

\[
(\delta^{(1)}_{\xi})_{\text{rigid}} A^a E^A_a = (\delta^{(1)}_{\xi})_{\text{rigid}} B^{a'} E^B_{a'} = d\Theta^{(3)}_\xi,
\]

(5.56)

\[
(\delta^{(1)}_{\chi})_{\text{rigid}} A^a E^A_a = (\delta^{(1)}_{\chi})_{\text{rigid}} B^{a'} E^B_{a'} = d\Theta^{(3)}_\chi,
\]

(5.57)
holding for some locally constructed 3-forms $\Theta^{(3)}_\xi, \Theta^{(3)}_\chi$ which depend linearly on $\xi^a, \chi^{a'}$. The solution of these two equations (again using the methods of Ref. [12]) yields the additional quadratic algebraic relations

$$k_{[a'b'|e}k^{c'}_{|d']e'} = 0,$$

(5.58)

$$b_{ab'}c'k^{c'}_{|d'|e} - 2b_{a[d'|b|b'}^{b|e']e} = 0,$$

(5.59)

$$2\epsilon_{a'|bc'|d'|e'} - \epsilon_{c'd'|b|b'}\epsilon_{e'c|a|a'} - 2\epsilon_{a'c'|d|m|b}b^{b'|e'}e = 0,$$

(5.60)

plus others that reduce to combinations of (5.53) to (5.55) through (5.36) and (5.37). Moreover, the quadratic relation (5.55) itself is a consequence of (5.58) and (5.37).

It can be shown that the integrability relations (5.53) to (5.55) are necessary and sufficient to allow solving for the second order deformation terms in the gauge symmetries from the first-order part of the Lie derivative commutator equation in Theorem 2. The additional integrability relations (5.58) to (5.60) are necessary in then solving the second-order part of the Lie derivative equation from Theorem 1 for the second order deformation terms in the field equations. However, it is found that a solution exists if and only if the following additional algebraic relation holds on the coefficients of the field equation deformation terms,

$$k_{d'|e'}^{e'}e_{c|a}^{c'} - 2\epsilon_{a'|cb|c'}^{b|e}|b - 2\epsilon_{a'c(d|m|b)\epsilon^{b'}}_{e} = 0.$$

(5.61)

This relation imposes in effect a further integrability relation on allowed first-order deformations.

**Theorem 3** Up to a change of field variables and gauge symmetry variables, all first-order geometrical deformations are given by Eqs. (5.43) to (5.46), Eqs. (5.39) and (5.40), with the coefficients satisfying the linear relations (5.36) and (5.37) and the quadratic relations (5.53) to (5.55) and (5.58) to (5.61). There are no further algebraic obstructions to the existence of second-order geometrical deformations.

These first-order deformations have the following classification: the $a_{abc}$ terms represent a massless Yang-Mills self-coupling of $A^a$, the $k_{a'b'|c'}$ terms represent a massless Freedman-Townsend self-coupling of $B^{a'}$, and the $b_{ab'}c'$ terms represent an extended Freedman-Townsend coupling between $A^a$ and $B^{a'}$, while the $j_{a'bc'}$ terms represent a Higgs type coupling of $B^{a'}$ to $A^a$ which is nontrivial only when the mass tensor $m_{aa'}$ is nonzero. The $e_{a'bc}$ terms, in contrast, represent a different type of coupling between $A^a$ and $B^{a'}$ unrelated to Yang-Mills and Freedman-Townsend type couplings. It is similar in form to the coupling known for 1-form and 2-form fields in extended supergravity theory [13,14]. Moreover, the deformation corresponding to the $e_{a'bc}$ terms is characterized by possessing opposite parity compared to the parity of the other deformation terms. In particular, consider the parity operator $\mathcal{P}$ defined by $d\mathcal{P} = \mathcal{P}d, \mathcal{P}d = -d\mathcal{P}$. If we assign even parity to $A^a$ and odd parity to $B^{a'},

$$\mathcal{P}A^a = A^a, \quad \mathcal{P}B^{a'} = -B^{a'},$$

(5.62)
which thus determines
\[ \mathcal{P} F^a = -F^a, \quad \mathcal{P} H^a = H^a, \] (5.63)
and
\[ \mathcal{P} \xi^a = \xi^a, \quad \mathcal{P} \chi^a = -\chi^a, \] (5.64)
then it follows that all the deformation terms except for the \( c_{a'bc} \) terms have even parity.

To proceed, we now consider the uniqueness of the higher-order deformation terms determined by the first-order terms in Theorem 3. Let \( \Delta E_A^a, \Delta E_B^a, \Delta \delta \chi A^a, \Delta \delta \chi A'^a, \Delta \delta \xi B^a, \Delta \delta \xi B'^a \), \( \Delta \delta \chi B^a \) denote the difference of any two deformations that agree up to some given order \( k \geq 1 \). Then the \( k + 1 \)st order part of the Lie derivative from Theorem 1 yields
\[ \delta_\xi^a (k+2) A^a = \delta \chi^a (k+2) A^a = 0, \quad \delta_\xi^a (k+2) A'^a = \delta \chi^a (k+2) A'^a = 0. \] (5.65)
Similarly, the \( k \)th order part of the Lie derivative commutator equation from Theorem 2 yields the result that, after a change of field variables and gauge symmetry variables,
\[ \delta_\xi^a (k+1) A^a = \delta \chi^a (k+1) A^a = 0, \quad \delta_\xi^a (k+1) A'^a = \delta \chi^a (k+1) A'^a = 0, \] (5.66)
\[ \delta_\xi^a (k+1) B^a = \delta \chi^a (k+1) B^a = 0, \quad \delta_\xi^a (k+1) B'^a = \delta \chi^a (k+1) B'^a = 0. \] (5.67)
Under the assumptions on the number of derivatives considered for possible deformation terms (see Remark in Sec. V B), the solution of equations (5.65) to (5.67) is immediately given by
\[ \Delta E_A^a = 0, \quad \Delta E_B^a = 0, \quad k \geq 1, \] (5.68)
\[ \Delta \delta \chi A^a = \Delta \delta \chi A'^a = 0, \quad \Delta \delta \chi B^a = \Delta \delta \chi B'^a = 0, \quad k \geq 1. \] (5.69)
Hence, we have established the following uniqueness result.

**Theorem 4** If two deformations agree at all orders \( 1 \leq l \leq k \), \( \Delta E_A^{(l)} = \Delta E_B^{(l)} = 0 \), \( \Delta \delta \chi A^a = \Delta \delta \chi A'^a = 0, \Delta \delta \chi B^a = \Delta \delta \chi B'^a = 0 \), then up to a change of field variables and gauge symmetry variables, the deformations also agree at order \( l = k + 1 \).

**D. Deformations to all orders**

Hereafter we restrict attention to parity-invariant and opposite-parity deformations separately and proceed to write down a complete deformation to all orders in each case. A full discussion of the combined parity non-invariant deformations from Theorem 3 is given in Ref. [15].

For a deformation determined at first-order purely by the \( e_{a'bc} \) terms, note that the linear algebraic relations (5.28) and (5.41) imply
and hence the $e_{a'bc}$ terms are incompatible with a nonzero mass tensor. However, in the massless case, the $e_{a'bc}$ terms produce a nontrivial deformation, which we now write down to all orders.

The algebraic structure on $\mathcal{A}, \mathcal{A}'$ associated to $e_{a'bc}$ consists of a symmetric product from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}'$. Then the gauge symmetries are given by

$$\delta_\xi A^a = d\xi^a, \quad \delta_\xi B^{a'} = e_{a'bc} F^b \xi^c,$$

$$\delta_\chi A^a = 0, \quad \delta_\chi B^{a'} = d\chi^{a'},$$

while the Lagrangian is constructed by

$$L = \frac{1}{2} F^a * F^b \delta_{ab} - \frac{1}{2} H^{a'} * H^b \delta_{a'bc} - \ast H^{a'} F^b A^c e_{a'bc} + \frac{1}{2} F^b A^c (F^d A^e) e_{a'bc} c_{a'de'}.$$  (5.73)

It is straightforward to see that this Lagrangian is gauge invariant, $\delta_\xi L = \delta_\chi L = 0$, and that the gauge symmetries commute, $[\delta_\xi, \delta_\chi] = [\delta_\chi, \delta_\xi] = [\delta_\xi, \delta_\chi] = 0$. From the Lagrangian, the field equations are given by

$$E_A^a = d*F^a + (2F^b * H^c - A^b d*H^c)e_{a'bc}$$

$$-(2F^b * (F^c A^d) + A^b d*(F^c A^d))e_{a'bc} = 0,$$

$$E_B^{a'} = d*(H^{a'} - (F^b A^c))e_{a'bc} = 0.$$  (5.75)

**Theorem 5** The massless nonlinear theory $\text{5.71}$) to $\text{5.73}$ is the unique nonlinear geometrical deformation of the abelian linear theory $\text{5.1}$ to $\text{5.3}$ determined by the first-order deformation terms $e_{a'bc}$.

Next we consider a general deformation determined at first-order by all terms except $e_{a'bc}$. This deformation is more general than the massless/massive nonlinear theories constructed in Secs. II and III, since it includes a mixing of massless and massive fields $A^a, B^{a'}$, controlled by the eigenvalues of the mass tensor $m_{aa'}$.

Let $a_{bc}^a, b_{v'c}^a, j_{b'c}^{a'}, k_{a'b'}^{e'}$ be the components of respective bilinear maps from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}, \mathcal{A}' \times \mathcal{A}'$ into $\mathcal{A}, \mathcal{A}' \times \mathcal{A}'$ into $\mathcal{A}'$, fixed to satisfy the linear and quadratic relations (5.36) and (5.37), (5.53) to (5.55), (5.58) and (5.59). Thus, it follows that $a_{bc}^a$ and $k_{a'b'}^{e'}$ define the commutator structure constants (5.53) and (5.58) of respective Lie algebras on $\mathcal{A}, \mathcal{A}'$, while $j_{b'c}^{a'}$ and $b_{v'c}^a$ define linear maps that are representations (5.55) and (5.59) of these Lie algebras on the vector spaces of $\mathcal{A}', \mathcal{A}$, respectively. Further discussion of the additional algebraic structure imposed by the relations (5.36), (5.37), (5.54) is given in the next subsection.

To write down the deformation to all orders, we first define a Yang-Mills field strength 2-form and a related antisymmetric tensor field strength 3-form by

$$F_A^a = dA^a + \frac{1}{2} a_{bc}^a A^b A^c, \quad H_A^{a'} = dB^{a'} + j_{b'c}^{a'} A^b B^{c'}.$$  (5.76)
Geometrically, $F^a_A$ is the curvature of the connection 1-form $a^a_{bc} A^b$, and $H^a_{A'}$ is the covariant curl of $B^a_{A'}$ in terms of the associated connection $j^{a'}_{bc} A^b$. Consequently, using the covariant exterior derivative operators given by

$$D_A = d + a^a_{bc} A^b, \quad D'_A = d + j^{a'}_{bc} A^b,$$

we have

$$D_A B^a = H^a_{A'}, \quad [D_A, D_A] = a^a_{bc} F^b_A, \quad [D'_A, D'_A] = j^{a'}_{bc} F^b_A,$$

due to the algebraic structure (5.53) and (5.55). Next we define a nonlinear field strength 2-form $P^a$ and 3-form $Q^a$ by the equations

$$P^a - * P^{b'} A^a b'_c = F^a_A,$$

$$Q^{a'} - * Q^{b'} A^a b'_c - * P^{b'} B^{c'} k^{a'}_{b'c'} = H^a_{A'}.$$

These field strengths are nonpolynomial expressions in terms of $A^a, B^a_{A'}$, as given by

$$(P^a, Q^{a'}) = Y^{-1}_{A,B}(F^a_A, H^a_{A'})$$

where $Y^{-1}_{A,B}$ is the inverse of the linear map

$$Y_{A,B} = \begin{pmatrix} \text{id} & A^a_{b'c'} b'_{c} * \\ -A^a_{b'c'} b'_{c} * & \text{id} - B^{b'}_{a'c'} k^{a'}_{b'c'} * \end{pmatrix}$$

defined to act on the vector space of pairs of $A$-valued 2-forms, $A'$-valued 3-forms.

Now we write down the deformation, using the previous structure. The gauge symmetries on $A^a, B^a_{A'}$ are given by the field variations

$$\delta_\xi A^a = D_A \xi^a + b^a_{b'c'} * Q^{b'} \xi^c,$$

$$\delta_\xi B^{a'} = -j^{a'}_{cb} B^{c'} \xi^c - b^a_{c} * P^b \xi^c,$$

$$\delta_\chi A^a = 0,$$

$$\delta_\chi B^{a'} = D'_A \chi^{a'} + k^{a'}_{b'c'} * Q^{b'} \chi^{c'},$$

where $\xi^a$ is an arbitrary $A$-valued function, and $\chi^{a'}$ is an arbitrary $A'$-valued 1-form. The Lagrangian 4-form for $A^a, B^a_{A'}$ is constructed by

$$L = \frac{1}{2} P^a_A F^b_A \delta_{ab} + \frac{1}{2} Q^{a'} H^{b'}_{A} \delta_{a'b'} + F^a_A B^{b'} m_{a'b'}.$$

Gauge invariance of this Lagrangian is established as follows.

The variation of $L$ under the gauge symmetry $\delta_\xi$ yields
\[ \delta_\xi L = m_{ab} (D_A \delta_\xi A^a B^{a'} + F^a_A \delta_\xi B^{a'}) + *(P^b_b b_{c'd'}^a \delta_\xi b^{a'}_b c' + Q^{b'}_b \delta_\xi A^e) \delta_{ac} \\
+ \xi^{e'}(D_A \delta_\xi B^{a'} + \xi^{d'} b_{c'd'} \delta_\xi A^b B^{e'} + \frac{1}{2} k^{a'}_{b'c'} Q^{b'}_b \delta_\xi B^{e'}) \delta_{a'e'}. \] (5.89)

To proceed, collecting all terms \( d \xi^c \) we obtain \(-(* P^b b_{c'd'} + B^{a'} m_{bb} b^{a'}_d c') \ast Q^{a'} d \xi^c \). Next we integrate by parts and use the field strength equations (5.76), (5.80), (5.81) to remove \( dA^a, dB^{a'} \) algebraically in terms of \( P^a, Q^{a'}, A^a, B^{a'} \). This yields terms of the type \( P^a \ast P^b \xi^c, Q^{d'}_b \xi^c, Q^{a'}_b \ast Q^{d'}_b \xi^c, P^a \ast Q^{d'}_b A^c \ast d', \ast Q^{a'}_b \ast Q^{b'}_d \xi^d \). Then we find that the coefficients of these terms vanish, respectively, due to the algebraic relations (5.33), (5.39), (5.54), (5.55). Hence, it follows that
\[ \delta_\xi L = d((b_{ac'd'} e + m_{bb} b_{a'd'} B^{a'}) \ast Q^{a'} \xi^c). \] (5.90)

Similarly, under the gauge symmetry \( \chi \), the variation of \( L \) is given by
\[ \delta_\chi L = m_{ab} F^a_A \delta_\chi B^{a'} + * Q^{c'}(D_A \delta_\chi B^{a'} + \frac{1}{2} k^{a'}_{b'c'} Q^{b'}_b \delta_\chi B^{e'}) \delta_{a'e'}. \] (5.91)

Proceeding as before, we find that all \( d \chi^c \) terms yield \(-(* P^a \ast P^b \xi^c, Q^{d'}_b \xi^c, Q^{a'}_b \ast Q^{d'}_b \xi^c, P^a \ast Q^{d'}_b A^c \ast d', \ast Q^{a'}_b \ast Q^{b'}_d \xi^d \). An integration by parts and use of the field strength equations (5.76), (5.80), (5.81) leaves terms of the type \( * Q^{a'}_b \ast P^b \chi^{c'}, Q^{a'}_b \ast Q^{d'}_b \ast Q^{c'} \chi^{d'}, \ast Q^{a'}_b \ast Q^{b'}_d \ast A^c \ast d', \ast Q^{a'}_b \ast Q^{b'}_d \ast B^{a'} \ast \xi^d \). Then we find that the coefficients of these terms vanish, respectively, due to the algebraic relations (5.33), (5.39), (5.54), (5.55). Hence, it follows that
\[ \delta_\chi L = d((\frac{1}{2} k_{a'd'} B^{a'} \ast Q^{a'} \ast Q^{b'} + m_{ac} F^a_A) \chi^{c'}). \] (5.92)

**Proposition** The Lagrangian (5.88) is invariant to within an exact 4-form (5.90) and (5.92) under the gauge symmetries (5.84) to (5.87).

The field equations for \( A^a, B^{a'} \) are given by
\[ E_A^a = D_A \ast P^a + (b_{c'e'} m_{b'}^e \ast A^b - b^a_{b'} Q^{d'}) \ast P^d + Q^{e'} m_{b'}^a = 0, \] (5.93)
\[ E_B^{a'} = D_A \ast Q^{a'} + \frac{1}{2} k_{b'd'} B^{a'} \ast Q^{e'} - b_{e'}^a m_{b'}^e A^b \ast Q^{c'} + P^b m_{b'}^{a'} = 0. \] (5.94)

On solutions of these field equations, the gauge symmetries on \( A^a, B^{a'} \) have the commutator structure
\[ [\delta_1, \delta_2] = \delta_3, \quad [\delta_1, \delta_2] = 0, \quad [\delta_1, \delta_2] = \delta_3 \] (5.95)
where
\[ \xi^a_3 = a^a_{bc} \xi^b_1 \xi^c_2, \quad \chi_3^{a'} = \chi^{a'}_{bc} \xi^b_1 \chi_2^{c'}. \] (5.96)

Off solutions, the commutators structure remains closed to within trivial symmetries proportional to the field equations
\[ \delta E_A^a = 2 b^a_{bc} \xi^c_1 \ast E_A^a - b^a_{bc} B^{a'} \ast \xi^c_1 \ast (\chi_2^{c'} E_B^{d'}), \] (5.97)
\[ \delta E_B^{a'} = 2 b_{c'd'} \ast E_B^{a'} + k_{b'd'} \ast \xi^c_1 \ast (\chi_2^{c'} E_B^{d'}), \] (5.98)
Theorem 6  The nonlinear theory (5.84) to (5.94) is the unique nonlinear geometrical deformation of the abelian linear theory (5.1) to (5.5) determined by the first-order deformation terms $a^a_{bc}, b^a_{\nu c}, j^a_{\nu c}, k^a_{\nu c}$.

We remark that the pure massless/massive $SU(2)$ case of this theory, given by

$$a^a_{bc} = \epsilon^a_{bc}, \quad b^a_{\nu c} = \lambda \epsilon_{\nu c}^a, \quad j^a_{\nu c} = \epsilon_{\nu c}^a, \quad k^a_{\nu c} = \lambda \epsilon_{\nu c}^a \quad (5.99)$$

(where $a, a', \ldots = 1, 2, 3$) with $\lambda = 1/m$ and $m = \text{const.} \neq 0$ in the massive case, and $\lambda = \text{const.} \neq 0$ in the massless case, yields the $SU(2)$ theories from Secs. II and III.

E. Algebraic structure in the nonlinear theory

Finally, we discuss the full algebraic structure on $\mathcal{A}, \mathcal{A}'$ underlying the nonlinear theory (5.84) to (5.94) given by the general parity-invariant deformation.

We start from the vector space decompositions $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_m$ and $\mathcal{A}' = \mathcal{A}'_0 \oplus \mathcal{A}'_m$ into massless and massive subspaces defined by the mass tensor $m_{a'd'}$. Let $\mathcal{P}_0, \mathcal{P}_m, \mathcal{P}'_0, \mathcal{P}'_m$ be the respective projection operators onto these subspaces in $\mathcal{A}, \mathcal{A}'$. Thus we have $m_{\mathcal{A}}(u_0) = m_{\mathcal{A}}(u'_0) = 0$ and $m_{\mathcal{A}}(u_m) \neq 0, m_{\mathcal{A}}(u'_m) \neq 0$, with subscripts denoting subspace projections for all $u, v$ in $\mathcal{A}$ and $u', v'$ in $\mathcal{A}'$. Let $(u, v)_\mathcal{A} = u^a v^b \delta_{ab}$ and $(u', v')_{\mathcal{A}'} = u'^a v'^b \delta_{a'b'}$ denote the vector space inner products on $\mathcal{A}, \mathcal{A}'$. We denote the Lie algebra multiplication $a^a_{bc} u^b v^c$ and $k^a_{\nu c} u'^b v'^c$ on $\mathcal{A}$ and $\mathcal{A}'$ by the brackets $[u, v]_\mathcal{A}$ and $[u', v']_{\mathcal{A}'}$. In addition we denote the Lie algebra representations $j^a_{\nu c} u^b$ and $b^a_{\nu c} u^b$ by the linear maps $\rho'(u)$ and $\rho(u')$. Similarly, $ad_{\mathcal{A}}(u)$ and $ad_{\mathcal{A}'}(u')$ denote the adjoint representations given by $a^a_{bc} u^b$ and $k^a_{\nu c} u'^b$ on $\mathcal{A}$ and $\mathcal{A}'$.

We begin by noting Eqs. (5.58) and (5.59) show that $\rho'$ is a derivation of the Lie algebra $\mathcal{A}'$

$$\rho'(w)[u', v']_{\mathcal{A}'} = [\rho'(w)u', v'_{\mathcal{A}'} + [u', \rho'(w)v']_{\mathcal{A}'} \quad (5.100)$$

for all $w$ in $\mathcal{A}$, $u'v'$ in $\mathcal{A}'$.

We now consider the additional algebraic structure imposed by the algebraic relations (5.36) and (5.37). To proceed, first note that, by Eq. (5.37),

$$\rho'(u_0) = 0, \quad \rho'(u_m) = ad_{\mathcal{A}'}(m_{\mathcal{A}}(u_m)) \quad (5.101)$$

which completely determines $\rho'$ in terms of the adjoint representation of $\mathcal{A}'$. Next, by Eq. (5.36), it follows that

$$m_{\mathcal{A}}([u, v]_\mathcal{A}) = [m_{\mathcal{A}}(u), m_{\mathcal{A}}(v)]_{\mathcal{A}'} \quad (5.102)$$

Hence, $\mathcal{A}'_m$ is a Lie subalgebra of $\mathcal{A}'$, and $\mathcal{A}_0$ is an invariant Lie subalgebra of $\mathcal{A}$, namely $[\mathcal{A}'_m, \mathcal{A}'_m] \subseteq \mathcal{A}'_m$, $[\mathcal{A}, \mathcal{A}_0] \subseteq \mathcal{A}_0$. Furthermore, it also follows from Eq. (5.36) that

$$(u_0, [v_0, w]_\mathcal{A})_\mathcal{A} = -(v_0, [u_0, w]_\mathcal{A})_\mathcal{A} \quad (5.103)$$
and thus the inner product on \( \mathcal{A} \) is an invariant metric with respect to the massless subspace \( \mathcal{A}_0 \). Consequently, since \( \mathcal{A}_0 \) is a Lie subalgebra, it must be a direct sum of an abelian Lie algebra \( \mathcal{A}_0^c \) and a semisimple Lie algebra \( \mathcal{A}_0^s \). However, the inner product is not required to be invariant with respect to the whole Lie algebra \( \mathcal{A} \), since

\[
(u, [v, w]_\mathcal{A})_\mathcal{A} + (v, [u, w]_\mathcal{A})_\mathcal{A} = (u, \rho(m_\mathcal{A}(v))w)_\mathcal{A} + (v, \rho(m_\mathcal{A}(u))w)_\mathcal{A}
\]  
(5.104)

which need not vanish for \( u, v \) in \( \mathcal{A}_m \). Thus, surprisingly, \( \mathcal{A} \) need not be semisimple unless its massive subspace \( \mathcal{A}_m \) is empty. Moreover, the inner product on the Lie algebra \( \mathcal{A}' \) likewise is not required to be invariant except on the massless subspace \( \mathcal{A}'_0 \)

\[
(u'_0, [v'_0, w'_0]_{\mathcal{A}'})_{\mathcal{A}'} = -(v'_0, [u'_0, w'_0]_{\mathcal{A}'})_{\mathcal{A}'},
\]  
(5.105)

since, by Eq. (5.37),

\[
(u', [v', m_\mathcal{A}(w)]_{\mathcal{A}'})_{\mathcal{A}'} + (v', [u', m_\mathcal{A}(w)]_{\mathcal{A}'})_{\mathcal{A}'} = (u', m_\mathcal{A}(\rho(v')w))_{\mathcal{A}'} + (v', m_\mathcal{A}(\rho(u')w))_{\mathcal{A}'}
\]  
(5.106)

which need not vanish for \( u', v' \) in \( \mathcal{A}'_m \). Hence, as \( \mathcal{A}'_m \) is a Lie subalgebra, it need not be semisimple and therefore, again, the whole Lie algebra \( \mathcal{A}' \) is not required to be semisimple unless its massive subspace \( \mathcal{A}'_m \) is empty.

Finally, we consider the remaining algebraic relation (5.34). This imposes further structure on the Lie algebras \( \mathcal{A}, \mathcal{A}' \), and on the representation \( \rho \) as follows. We first examine, separately, the pure massless case \( m_{aa'} = 0 \) and pure massive case \( m_{aa'} = m\delta_{aa'}, m \neq 0 \).

In the massless case,

\[
\mathcal{A} = \mathcal{A}_0, \quad \mathcal{A}' = \mathcal{A}'_0,
\]  
(5.107)

and so \( \mathcal{A} \) and \( \mathcal{A}' \) are each a direct sum of abelian Lie algebras \( \mathcal{A}^c, \mathcal{A}'^c \) and semisimple Lie algebras \( \mathcal{A}^s, \mathcal{A}'^s \), respectively. Now, Eq. (5.54) reduces to

\[
\rho(w')[u, v]_\mathcal{A} = [\rho(w')u, v] + [u, \rho(w')v],
\]  
(5.108)

which states that the linear map \( \rho \) is a derivation of the Lie algebra \( \mathcal{A}' \). Since the quotient of \( \mathcal{A}' \) by its center \( \mathcal{A}'^c \) is semisimple, \( \rho \) must take the form

\[
\rho(w') = \text{ad}_h(w')
\]  
(5.109)

for some linear map \( h \) from \( \mathcal{A}' \) into \( \mathcal{A} \). It then follows that

\[
[h(u'), h(v')]_\mathcal{A} = h([u', v']_{\mathcal{A}'}).
\]  
(5.110)

Hence, the kernel of \( h \) is an invariant Lie subalgebra of \( \mathcal{A}' \). As a consequence of the decomposition \( \mathcal{A}' = \mathcal{A}'^c \oplus \mathcal{A}'^s \), any such subalgebra must belong to \( \mathcal{A}'^c \). Furthermore, the image of \( h \) is a Lie subalgebra of \( \mathcal{A} \) isomorphic to \( \mathcal{A}'^s \). Hence, \( h(\mathcal{A}'^s) \subseteq \mathcal{A}^s \) yields a Lie algebra homomorphism. This now fully describes the structure imposed on \( \mathcal{A} \) and \( \mathcal{A}' \) by the algebraic relations (5.36) and (5.37), (5.53) to (5.55), (5.58) and (5.59) in the massless case.

In the massive case, note
\[ A = A_m, \quad A' = A'_m \quad (5.111) \]

are isomorphic as vector spaces under the map \( m_{A'}(A'_m) = m_A(\mathcal{A}_m) = m_A \). Since this map is a Lie algebra homomorphism by Eq. (5.102), then \( A \) and \( A' \) are isomorphic Lie algebras. Consequently, Eq. (5.54) becomes

\[ [\rho_m(w)u, v]_A + [u, \rho_m(w)v]_A - \rho_m(w)[u, v]_A \\
= \rho_m(\rho_m(w)u)v - \rho_m(\rho_m(w)v)u - [w, u]_A \rho_m([w, v]_A)u + \rho_m([w, v]_A)u \quad (5.112) \]

where \( \rho_m(w) = \rho(m_A(w)) \). Taking into account Eq. (5.36), the relation (5.112) states that the linear map on \( A \) defined by

\[ \tilde{\rho}_m(u) = \rho_m(v)u - ad_A(v)u \quad (5.113) \]

must be a skew-adjoint representation of the Lie algebra \( A \),

\[ (u, \tilde{\rho}_m(w)v)_A = -(v, \tilde{\rho}_m(w)u)_A, \quad [\tilde{\rho}_m(u), \tilde{\rho}_m(v)] = \tilde{\rho}_m([u, v]_A). \quad (5.114) \]

This is satisfied if \( \rho_m = ad_A \), in which case \( \tilde{\rho}_m = 0 \) is a trivial representation, or if \( \rho_m = 0 \), in which case \( \tilde{\rho}_m = -ad_A^* \) is the coadjoint representation. In either case, there is no further algebraic structure imposed by Eq. (5.112). Note then, surprisingly, \( A \) is thus not required to be semisimple in massive case with \( \rho_m = ad_A \).

To conclude the discussion, we return to the general situation when \( A \) and \( A' \) contain both massless and massive nonempty subspaces. In this case, from Eq. (5.54), it follows that

\[ [\rho(w')u, v]_A + [u, \rho(w')v]_A - \rho(w)[u, v]_A \\
= \rho(m_A(\rho(w')u)v - \rho(m_A(\rho(w')v))u - \rho([w', m_A(u)]_A)v + \rho([w', m_A(v)]_A)u. \quad (5.115) \]

We now show that this equation is satisfied by

\[ \rho(w') = ad_A(h(w')) \quad (5.116) \]

for some linear map \( h \) from \( A' \) into \( A \) if

\[ h(m_A(A_m)) = A_m, \quad m_A(h(A'_m)) = A'_m, \quad h(A'_0) \subseteq A_0 \quad (5.117) \]

and if

\[ [A_m, A_m] \subseteq A_m, \quad [A_m, A_0] \subseteq A'_0, \quad [ad_{A_0}(A_m), ad_{A_0}(A_m)] = 0 \quad (5.118) \]

where \( A'_0 \) is the center of the Lie algebra \( A \). To begin the proof, first note that if Eq. (5.116) holds then the left-side of Eq. (5.115) directly vanishes for any \( h \) since \( ad_A \) is a derivation of \( A \). Next, from Eq. (5.59), the last two terms on the right-side of Eq. (5.115) become

\[ -[ad_A(w_m), ad_A(u_m)]v + [ad_A(w_m), ad_A(v_m)]u \\
= -ad_A([w_m, u_m]_A)v + ad_A([w_m, v_m]_A)u = -[[w_m, u_m]_A], v]_A + [[w_m, v_m]_A, v]_A \quad (5.119) \]

where \( w_m = m_A(w') \). Then, by Eq. (5.118), the first two terms on the right-side of Eq. (5.115) reduce to
\[ad_A(\rho'(w_m)v_m) - ad_A(\rho'(w_m)v_m)u = -[[w_m, u], v], A - [[w_m, v], u], A.\]  
(5.120)

Hence, the right-side of Eq. (5.113) vanishes, which completes the proof.

Consequently, note that Eqs. (5.116) to (5.118) determine
\[\rho(w'_m) = ad_{\rho_0}(m_{\mathcal{A}'_m}(w'_m)), \quad \rho(u'_0) = ad_{\rho_0}(h_0(w'_0))\]  
(5.121)
in terms of some linear map \(h_0 = h \circ P'_0\) from \(\mathcal{A}'_0\) into \(\mathcal{A}_0\), and using the inverse \(m_{\mathcal{A}'_m}^{-1}\) from \(\mathcal{A}'_m\) into \(\mathcal{A}'_m\) of the linear map \(m_{\mathcal{A}'_m} \circ P'_m\). It then follows that
\[m_{\mathcal{A}'}([u', v'], \mathcal{A}') = [m_{\mathcal{A}'}(u'), m_{\mathcal{A}'}(v')]_A.\]  
(5.122)

Hence, \(\mathcal{A}'_0\) is an invariant Lie subalgebra of \(\mathcal{A}'\), and \(\mathcal{A}'_m\) is a Lie subalgebra of \(\mathcal{A}\). Consequently, since by Eq. (5.103) the inner product on \(\mathcal{A}'_0\) is an invariant metric with respect to \(\mathcal{A}'\), the Lie subalgebra \(\mathcal{A}'_0\) is a direct sum of an abelian Lie algebra \(\mathcal{A}'_0^s\) and a semisimple Lie algebra \(\mathcal{A}'_0^s\). Furthermore, from Eq. (5.122), it follows that \(\mathcal{A}'_0\) and \(\mathcal{A}'_m\) are isomorphic Lie algebras, but note that they are not required to be semisimple. This now gives a complete description of the algebraic structure imposed by the relations (5.36) and (5.37), (5.53) to (5.58), and (5.59) in the case given by Eqs. (5.110) to (5.118).

Thus, the previous algebraic analysis leads to an interesting generalization of the massless/nonlinear theories in Secs. I and II given by the nonlinear theory (5.84) to (5.94) with the following algebraic structure:

(i) The massless and massive subspaces \(\mathcal{A}_0, \mathcal{A}'_0, \mathcal{A}_m, \mathcal{A}'_m\) are Lie subalgebras of \(\mathcal{A}, \mathcal{A}'\) with \(\mathcal{A}'_m\) and \(\mathcal{A}_m\) being isomorphic under the linear maps \(m_{\mathcal{A}}, m_{\mathcal{A}'}\) given by the mass tensor.

(ii) \(\mathcal{A}_0\) and \(\mathcal{A}'_0\) are semisimple Lie algebras and ideals in \(\mathcal{A}, \mathcal{A}'\), such that \(\mathcal{A}_0\) and \(\mathcal{A}_m\) commute; however, the Lie algebras \(\mathcal{A}_m \simeq \mathcal{A}'_m\) are not restricted to be semisimple (they may be nilpotent or solvable) and \(\mathcal{A}'_m\) is not restricted to commute with \(\mathcal{A}'_0\).

(iii) The representation \(\rho'\) is the adjoint representation of \(\mathcal{A}'_m = m_{\mathcal{A}'}(\mathcal{A}_m)\) on \(\mathcal{A}'\), while the representation \(\rho\) is the sum of the adjoint representations of \(\mathcal{A}_0 = m_{\mathcal{A}}(\mathcal{A}_0)\) and of \(\mathcal{A}_0 = h_0(\mathcal{A}'_0)\) on \(\mathcal{A}\), for any linear map \(h_0\).

In physical terms, the resulting nonlinear theory (5.84) to (5.94) is a novel generalization of Yang-Mills gauge theory for vector potentials \(A^a\) coupled to Freedman-Townsend gauge theory for antisymmetric tensor potentials \(B^{a'}\), involving a Chern-Simons type mass term. It describes a set of nonlinearly interacting massive spin-one fields and massless spin-one and spin-zero fields, with a mutual interaction between the massive and massless fields.

**VI. CONCLUDING REMARKS**

This paper has developed in detail the geometrical, field theoretic, and algebraic aspects of an interesting nonlinear generalization of massless/massive Yang-Mills/Freedman-Townsend gauge theory in four dimensions. The generalization involves an extended Freedman-Townsend coupling between the Yang-Mills 1-form gauge fields and Freedman-Townsend 2-form gauge fields, in addition to a Higgs type coupling tied to a Chern-Simons mass term, and accompanied by a novel form of generalized Yang-Mills and Freedman-Townsend gauge symmetries and field equations in both the massless and massive cases. In particular, the geometrical structure of the resulting nonlinear gauge theory mixes and
unifies well-known features of Yang-Mills theory and Freedman-Townsend theory in terms of Lie algebra valued curvatures and connections associated to the gauge fields and nonlinear field strengths.

This generalization was found by a general determination of the geometrical nonlinear deformations of linear abelian gauge theory for 1-form fields and 2-form fields with an abelian Chern-Simons mass term. The deformation framework used here is a geometrical version of the field theoretic approach developed in Ref. [3,1,12]. It exposes clearly the existence of two integrability conditions on the first-order parts of possible deformations and leads to a simple uniqueness argument for the higher-order parts of allowed deformations.

Another approach to deformations (see Ref. [4] for an overview), which is based on BRST cohomology [5,16,17], has recently yielded important results on the classification of allowed first-order deformations of the free gauge theory for a set of $p$-form fields, $p = 1, \ldots, n-1$, in $n \geq 2$ dimensions [7,18]. While this classification analysis is complete for massless $p$-form fields with $p \geq 2$ and lists the extended Freedman-Townsend and Yang-Mills types of first-order deformations for $p \geq 1$, it did not explicitly treat deformations of massive $p$-form fields with the mass given by an abelian Chern-Simons term in the free gauge theory. Moreover, integrability conditions (i.e. obstructions to the existence of higher-order deformation terms) associated with combining the distinct types of allowed first-order deformations were not obtained for any $p \geq 1$. In the case $p \leq 2$, these gaps are closed by the deformation results obtained in Sec. V. In particular, a complete classification of first-order geometrical deformations of the free gauge theory for a massive/massless set of 1-form and 2-form fields has been obtained in $n = 4$ dimensions, including all integrability conditions that arise on such deformations (with typical assumptions on the allowed number of derivatives considered for terms in the gauge symmetries and field equations). Also, uniqueness results on deformations to all orders in this setting have been proved. (Interestingly, if the restriction to geometrical deformations is relaxed, then an additional type of deformation is known to exist in the case $p = 1$ [19].)

There are several directions in which the main results in this paper could be generalized. First, an extension of the general massless/massive nonlinear theory constructed here for Yang-Mills 1-form gauge fields coupled to Freedman-Townsend 2-form gauge fields with a Chern-Simons mass term in four dimensions is expected to exist in $n$ dimensions, involving a tower of Lie-algebra valued $p$-form fields $A_{(p)}$, $p = 1, \ldots, n-2$, with a Yang-Mills self-coupling on $A_{(1)}$, a Freedman-Townsend self-coupling on $A_{(n-2)}$, and an extended Freedman-Townsend coupling between $A_{(1)}, \ldots, A_{(n-2)}$, in addition to a Higgs coupling of $A_{(2)}, \ldots, A_{(n-2)}$ with $A_{(1)}$ in the massive case.

Second, it is straightforward to couple such a geometrical nonlinear gauge theory to gravity. In particular, on a spacetime with metric tensor $g$, the only structure needed is the Hodge dual operator * determined by $g$, and the exterior derivative $d$ operator (which is independent of $g$). For the case $n = 4$ dimensions, if the Lagrangian of the nonlinear gauge theory given in this paper for $A_{(1)}$ and $A_{(2)}$ is combined with the Einstein gravitational Lagrangian for $g$, then this achieves an interesting generalization of the Einstein-Yang-Mills theory (and there is obvious extension to $n$ dimensions for $A_{(1)}, \ldots, A_{(n-2)}$). Of particular interest would be to consider its field theoretic features, such as black hole solutions, nonabelian monopole solutions, and critical behavior in the initial value problem.
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