SUMS OF MULTIVARIATE POLYNOMIALS IN FINITE SUBGROUPS

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Abstract. Given a finite subgroup \( G \) of the group of units of a commutative unital ring \( R \) and a multivariate polynomial \( f \) in \( R[X_1, \ldots, X_k] \), we evaluate the sum of the \( f(x_1, \ldots, x_k) \) for all choices of pairwise distinct \( x_1, \ldots, x_k \) in \( G \) whenever the subgroup \( G \) satisfies a minimax constraint, which always holds if \( R \) is a field. In particular, let \( p^m \) be a power of an odd prime, \( n \) a positive integer, and \( a_1, \ldots, a_k \) integers with sum divisible by \( \phi(p^m) \) such that \( \gcd(a_i + \cdots + a_j, p(p-1)) \) is smaller than \( (p-1)/\gcd(n, \phi(p^m)) \) for all non-empty proper subsets \( \{i_1, \ldots, i_j\} \) of \( \{1, \ldots, k\} \); then the following congruence holds

\[
\sum_{x_1^{a_1} \cdots x_k^{a_k}} \equiv \frac{\phi(p^m)}{\gcd(n, \phi(p^m))} (-1)^{k-1} (k-1)! \pmod{p^m},
\]

where the summation is taken over all pairwise distinct \( 1 \leq x_1, \ldots, x_k \leq p^m \) such that each \( x_i \) is a \( n \)-th residue modulo \( p^m \) coprime with \( p \).

1. Introduction and Notations

The aim of this article is to provide a characterization, in the framework of commutative rings, of certain types of symmetric sums of distinct elements of a subgroup of the group of units. For instance, it includes the case of sums of multivariate polynomials taking distinct values in the set of \( n \)-th residues modulo a prime. The first result of this type was obtained by Pierce [9], who proved that an integral symmetric homogeneous function of degree \( d \) of the \( n \)-th residues of an odd prime \( p \) is divisible by \( p \) if \( d \) is not divisible by \( (p-1)/\gcd(p-1, n) \). Here, in particular, we evaluate sums of polynomials of \( n \)-th residues in the remaining case.

Moreover, symmetric sums of functions taking distinct values in a given set have been already studied in the literature: Ferrers [6] proved the folklore result that an odd prime \( p \) divides the sum of the products of the numbers \( 1, \ldots, p-1 \), taken \( k \) together, whenever \( k \) is smaller than \( p-1 \). Afterwards, this theorem was increasingly generalized by Glaisher [5], Moritz [8], and Ricci [10]. All these results provide, in turn, generalizations of the well-known Wilson’s theorem. Within this context, the proof of the celebrated Erdős-Ginzburg-Ziv theorem [1] provided by Gao [4] shows a clear connection between zero-sum problems in additive number theory and the study of sums of symmetric functions.

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Here below, let $R$ be a commutative unital non-trivial ring, for which we use the standard notation, and $G = \{x_1, \ldots, x_n\}$ a finite subgroup $G$ of the group of units $R^\times$. Denote the set of non-regular elements with $D$ (in particular $D$ is non-empty since it contains 0).

Given a positive integer $k$ smaller than or equal to $n$, we provide a method to evaluate symmetric sums of the form

$$
\sum_{x_1, \ldots, x_k \in G} f(x_1, \ldots, x_k),
$$

where $f$ is a multivariate polynomial in $R[X_1, \ldots, X_k]$, under a “minimax constraint” on the structure of $G$. Since $f$ can be always written as a finite sum of monomials $r x_1^{a_1} \cdots x_k^{a_k}$ for some $r \in R$ and non-negative integers $a_1, \ldots, a_k$, then it is enough to evaluate symmetric sums of the form

$$
p(A) := \sum_{x_1, \ldots, x_k \in G} x_1^{a_1} \cdots x_k^{a_k}.
$$

(1)

Here, $A$ represents the multiset of integer exponents $\{a_1, \ldots, a_k\}$: the order of the elements of $A$ does not matter since the sum (1) is symmetric in $x_1, \ldots, x_n$. Define also $\lambda$ the exponent of the group $G$, that is, the least common multiple of the orders of all elements of the group (which is well-defined, since $G$ is finite). Let $A^\lambda$ represent the multiset of all the exponents $a$ in $A$ such that $a$ is not divisible by $\lambda$. Then, the symmetric sum (1) verifies

$$
p(A) = (n - |A^\lambda|)(n - |A^\lambda| - 1) \cdots (n + 1)p(A^\lambda).
$$

(2)

According to this representation, we provide a method to evaluate $p(A^\lambda)$. Therefore, from here later, let us suppose that $A$ is equal to $A^\lambda$. It is worth noticing that, according to the Lagrange’s theorem (see e.g. [7]), each order $\text{ord}(g)$ of the elements $g$ in $G$ divides $n$, implying in turn that $\lambda$ divides $n$. At this point, one may expect that $\lambda$ and $n$ are equal “almost always,” but this is not the case: Erdős, Pomerance and Schmutz [2] proved that, choosing $R = \mathbb{Z}/m\mathbb{Z}$ and $G = R^\times$ with $m$ integer, $\lambda = \lambda(m)$ and $n = n(m)$ have different normal orders.

Within this context, define $A$ the collection of subsets $B$ of $A$ such that the exponent $\lambda$ does not divide the sum of elements of $B$, denoted with $s(B)$. Then, a subgroup $G$ is said to be $A$-nice whenever it is finite and the following minimax constraint holds:

$$
\min_{B \in A} \max_{g \in G} \frac{\text{ord}(g)}{\text{gcd}(s(B), \text{ord}(g))} \geq |D| + 1.
$$

(3)

In such case, the set $D$ has to be finite, therefore let us say that $D$ is made by the non-regular elements $d_1, \ldots, d_m$; we will give evidence that condition (3) is really meaningful, providing also some applications in Section 4. Moreover, for each subset $B$ of $A$, let $\mathcal{P}(B)$ represent the collection of the partitions $\mathcal{P}$ of the elements of $B$ such that $\lambda$ divides $s(P)$ for all $P \in \mathcal{P}$; in particular, $\mathcal{P}(B)$ is empty whenever $B$ belongs to $\mathcal{A}$. Finally, define its characteristic number by

$$
\chi(B) := n(-1)^{|B| - 1}(|B| - 1)!.
$$
Then, our main result can be stated as follows:

**Theorem 1.** Let $G$ be a $A$-nice subgroup of a commutative ring $R$ with non-zero identity such that each element of $A$ is not divisible by the exponent of $G$. Then

$$\sum_{x_1, \ldots, x_k \in G, \text{pairwise distinct}} x_1^{a_1} \cdots x_k^{a_k} = \sum_{P \in \mathcal{P}(A) \mid P \in P} \prod \chi(P).$$

As we mentioned before, the assumption that $\lambda$ does not divide each element of $A$ is not restrictive of generality, as that case can be easily obtained from equation (2). To sum up, in the next section we prove some preliminary lemmas related to Theorem 1 and symmetric sums of the type (1), some of which hold under weaker conditions than the ones stated here. Then, the main result is proved in Section 3. Remarks and conclusions follow.

## 2. Preliminary results

Under the standing assumptions, the following result explains why the existence of a $A$-nice subgroup implies that the commutative ring $R$ is necessarily finite, in the case where $R$ is not a domain.

**Remark 2.** Let $G$ be a non-empty $A$-nice subgroup of a commutative unital ring $R$ such that $R$ is not a domain. Then $R$ is finite and has a non-zero multiplicative identity.

Being a straightforward exercise in abstract algebra, we leave the proof to the reader. The underlying idea, which relies on the finiteness of the set $D$, comes back to Ganesan [3].

**Lemma 3.** Let $G = \{x_1, \ldots, x_n\}$ a finite subgroup of the group of units of a unital ring $R$, and $h: G^n \to R$ a symmetric homogeneous function of degree $t$ such that there exists $g$ in $G$ for which $g^t - 1$ is regular. Then $h(x_1, \ldots, x_n) = 0$.

**Proof.** Suppose that $gx_i = gx_j$ for some $1 \leq i, j \leq n$ and $g \in G$; then $g(x_i - x_j) = 0$ implies $x_i = x_j$ since $g \in G$ is invertible. Therefore $\{gx_1, \ldots, gx_n\}$ has to be a permutation of $\{x_1, \ldots, x_n\}$, i.e. there exists a $\sigma$ in the symmetric group of degree $n$ for which $gx_i = x_{\sigma(i)}$ for all $i = 1, \ldots, n$. Moreover, by assumption there exists $g \in G$ such that $g^t - 1$ belongs to $R^\times$, therefore

$$(g^t - 1) h(x_1, \ldots, x_n) = g^t h(x_1, \ldots, x_n) - h(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = g^t h(x_1, \ldots, x_n) - h(gx_1, \ldots, gx_n) = 0.$$  

The claim follows from the assumption that $g^t - 1$ is regular.  

Notice that the subgroup $G$ does not have to be necessarily cyclic; also, the existence of such $g$ in $G$ implies that $\lambda$ does not divide $t$. Moreover, we can replace the existence of $g$ with the assumptions that $\lambda$ does not divide $t$ and $R$ is an integral domain. Pierce’s result [9] is represented by the case where $R$ is the field $\mathbb{Z}/p\mathbb{Z}$ and $G$ is the cyclic subgroup of non-zero $n$-th residues of an odd prime $p$. 


Lemma 4. Let \( n \) be a non-zero integer and \( X \) a non-empty multiset of integers such that \( n \) does not divide any of the elements of \( X \) but divides their sum. Let also \( \mathcal{B}(X) \) be the collection of partitions \( \mathcal{B} \) of \( X \) such that \( n \) divides \( s(B) \) for all \( B \in \mathcal{B} \), and \( \mathcal{C}(X) \) the collection of non-empty subsets \( C \) of \( X \) such that \( n \) divides \( s(C) \). Then for all \( x \) in \( X \) we have that

\[
\mathcal{B}(X) = \{\{X\}\} \cup \bigcup_{Y \in \mathcal{C}(X \setminus \{x\})} \{\{Y\} \cup Z\},
\]

\( Z \in \mathcal{B}(Y) \).

Proof. According to the hypotheses, each \( Y \) in \( \mathcal{C}(X \setminus \{x\}) \) has to be a proper subset of \( X \setminus \{x\} \). It follows that for all suitable partitions \( Z \) of \( Y \) the collection \( \{\{Y\} \cup Z\} \) has at least two elements and belongs to \( \mathcal{B}(X) \). Therefore the right hand side is contained in the left hand side. To show the reverse inclusion, let \( \{P_1, \ldots, P_t\} \) a partition of \( X \) in \( \mathcal{B}(X) \) and assume without loss of generality that \( x \in P_1 \). Then it is enough to set \( Y = P_2 \cup \ldots \cup P_t \) and \( Z = \{P_2, \ldots, P_t\} \).

Everything is finally ready to prove our main result.

3. Proof of Theorem 1

Proof. Given the multiset of integers \( A = \{a_1, \ldots, a_k\} \), none of which is divisible by the exponent of \( G \), let us define the truncated sum

\[
p^\#(A) = \sum_{\mathclap{x \in G \setminus \{1\} \text{ pairwise distinct}}} x_{a_1}^{a_1} \cdots x_{a_k}^{a_k},
\]

so that an addend of \( p(A) \) appears in \( p^\#(A) \) if and only if \( x_i \) is chosen different from 1 for all indices \( i = 1, \ldots, k \). According to Lemma 3, a sum in the form (1) is homogeneous and symmetric in \( x_1, \ldots, x_n \), hence it is equal to 0 whenever \( \lambda \) does not divide \( s(A) \). Indeed, by assumption there exists \( g \) in \( G \) such that \( \text{ord}(g)/\gcd(s(A), \text{ord}(g)) \), the order of the cyclic subgroup generated by \( g^{s(A)} \), is greater than \( m \). It follows that there exists \( g' \) in this subgroup such that \( g' - 1 \) is regular. In particular, the theorem holds in the case that \( \lambda \) does not divide \( s(A) \). Moreover, \( p(A) = 0 \) and \( p^\#(A) = -1 \) whenever \( A \) is a singleton. Then, let us prove by induction on the number of elements of \( A \) that

\[
p^\#(A) = \sum_{B \subseteq A} (-1)^{|B|} (k - |B|)!p(B).
\]

(4)

Let us suppose that it holds for all multiset of integers with at most \( k - 1 \) elements. Notice that we can rewrite the sum \( p^\#(A) \) as the difference \( p(A) - \sum_{a \in A} p^\#(A \setminus \{a\}) \). Applying the
induction hypothesis, we obtain

\[ p^2(A) = p(A) - \sum_{a \in A} \sum_{B \subseteq A \setminus \{a\}} (-1)^{k-1-|B|}(k - 1 - |B|)! p(B) \]

\[ = p(A) - \sum_{a \in A} \left( (-1)^{k-1}(k - 1)! + \sum_{\varnothing \neq B \subseteq A \setminus \{a\}} (-1)^{k-1-|B|}(k - 1 - |B|)! p(B) \right) \]

\[ = p(A) + (-1)^{k} k! - \sum_{\varnothing \neq B \subseteq A} \sum_{a \in A \setminus B} (-1)^{k-1-|B|}(k - 1 - |B|)! p(B) \]

\[ = (-1)^{k-|A|}(k - |A|)! p(A) + \sum_{B \subseteq A} (-1)^{k-|B|}(k - |B|)! p(B), \]

which is equivalent to (4), and the induction is complete. At this point, suppose that \( A \) is multiset of integers such that \( \lambda \) divides \( s(A) \). It is claimed that, independently from the choice of \( a \in A \), we have \( p(A) = np^2(A \setminus \{a\}) \). Indeed, \( G \) is a subgroup of units with orders dividing \( \lambda \) so that, multiplying by \( x_1^{-s(A)} \), we get

\[ p(A) = \sum_{x_1, \ldots, x_k \in G, x_1, \ldots, x_k \text{ pairwise distinct}} \left( \frac{x_2}{x_1} \right)^{a_2} \cdots \left( \frac{x_k}{x_1} \right)^{a_k}. \]  \hspace{1cm} (5)

Notice that the \( k - 1 \)-tupla \( (\frac{x_2}{x_1}, \ldots, \frac{x_k}{x_1}) \) is still a sequence of pairwise distinct elements in \( G \), and each component is different from 1. Moreover, there are exactly \( n \) distinct \( k \)-tuplas \( (x_1, \ldots, x_k) \) producing a given \( k - 1 \)-tupla because \( (gx_1, \ldots, gx_k) \) is mapped in the same \( k - 1 \)-tupla \( (\frac{x_2}{x_1}, \ldots, \frac{x_k}{x_1}) \) for all \( g \in G \). It is also easy to see that the set of suitable sequences of \( k \) elements in \( G \) produces the whole set of suitable sequences with \( k - 1 \) elements in \( G \) such that each component is different from 1. Indeed, the sequence of \( k \)-tuplas \( (x_1, \ldots, x_k) \) can be chosen in \( n(n-1) \cdots (n-k+1) \) ways, while the \( (k-1) \)-tuplas \( (\frac{x_2}{x_1}, \ldots, \frac{x_k}{x_1}) \) in \( (n-1)(n-2) \cdots (n-k+1) \) ways: each one of them appears exactly \( n \) times in the summation (5). Evidently, such reasoning does not depend on the choice of \( x_1^{-s(A)} \), hence the claim follows.

Taking in consideration also the equation (4), we obtain that if \( \lambda \) divide \( s(A) \) then

\[ p(A) = np^2(A \setminus \{a\}) = \chi(A) + \sum_{B \subseteq A \setminus \{a\}} \chi(B)p(B), \]  \hspace{1cm} (6)

independently from the chosen element \( a \) in \( A \). Considering that \( p(B) \) is equal to 0 whenever \( B \) belongs to \( A \), the sum \( p(A) \) can be simplified to \( \chi(A) + \sum_{B \in B} \chi(B)p(B) \), where \( B \) represents the collection of subsets of \( A \setminus \{a\} \) with sum of elements divisible by the exponent \( \lambda \).

Let us finally conclude the proof proving by induction on the elements of \( A \) that if \( \lambda \) divides \( s(A) \) and does not divide each element of \( A \) then \( p(A) = \sum_{P \in \mathcal{P}(A)} \prod_{P \in \mathcal{P}} \chi(P) \). If \( A \) is a singleton then these conditions cannot be satisfied. If \( A \) contains two elements then \( \mathcal{P}(A) \) has to be the singleton \( \{\{\{\}\}\} \), therefore \( p(A) \) would be simply \( \chi(A) \), which is verified according to equation (6). At this point, let us suppose that the claim holds for all proper subsets of \( A \).
Then, independently from the element \( a \) in \( A \), we obtain

\[
p(A) = \chi(A) + \sum_{B \in B} \left( \chi(A \setminus B) \sum_{P \in \mathcal{P}(B)} \prod_{P \in \mathcal{P}} \chi(P) \right).
\]

According to Lemma 4, the collection \( \mathcal{P}(A) \) can be rewritten as the set of partitions of \( A \) in the form \( \{A\} \) or \( \{A \setminus B\} \cup P \) for some \( B \in B \) and \( P \in \mathcal{P}(B) \). Therefore, summing the products of \( \chi(P) \) for subsets \( P \in \mathcal{P} \subseteq \mathcal{P}(A) \), we conclude that \( p(A) \) simplifies to \( \sum_{P \in \mathcal{P}(A)} \prod_{P \in \mathcal{P}} \chi(P) \).

The result is coherent with the special case provided by Pierce [9]: let \( p \) be an odd prime and \( n \) a positive integer such that \( (p - 1)/\gcd(n, p - 1) \) is an even number, let us say \( 2^k \). Then the following congruence holds:

\[
\sum_{\{x_1,\ldots,x_k\} \text{ pairwise distinct } n\text{-th residues}} x_1^2 \cdots x_k^2 \equiv 2(-1)^{k-1} \mod p.
\]

Indeed, we can assume without loss of generality that \( x_1, \ldots, x_k \) belong to cyclic subgroup of non-zero \( n \)-residues, which has order \( 2^k \). Then the multiset \( A \) is identified with \( \{2, \ldots, 2\} \), where the element 2 is repeated \( k \) times, so that \( A = A^* \) and \( \mathcal{P}(A) = \{\{A\}\} \). It follows that \( p(A) = \chi(A) = 2(-1)^{k-1}k! \). The missing constant \( k! \) follows from the fact that the Pierce summation does not take into account the order of the elements \( x_1, \ldots, x_k \).

4. Concluding remarks

Notice also the minimax constraint (3) is easily satisfied in some important cases. Let \( B \) be a set in \( A \), then the maximum of \( \text{ord}(g)/\gcd(s(B), \text{ord}(g)) \) with \( g \in G \) has to be strictly greater than 1: in the opposite case, we would have that \( \text{ord}(g) \) divides always \( s(B) \), so that also \( \lambda \) divides \( s(B) \), against our assumption. According to this argument, such maximum has to be greater or equal than the least prime factor of \( \lambda \), independently from the choice of the subset \( B \). In particular, if \( R \) is a field and \( G \) is a finite subgroup of \( R^\times \) different from \( \{1\} \), then the minimax constraint holds.

Moreover, \( A\text{-nice} \) subgroups exist also in commutative rings which are not domains, as it is shown in the next two examples.

Example 5. Let \( R \) be the ring \( \mathbb{Z}/p^m\mathbb{Z} \) where \( p^m \) is the power of an odd prime \( p \), \( G \) the cyclic subgroup of \( q \)-th residues which are coprime with \( p \), and \( A \) a multiset of integers with sum divisible by \( \lambda = \varphi(p^m)/\gcd(q, \varphi(p^m)) \), where \( \varphi \) represents the Euler’s function. Accordingly, the multiset \( A \) is chosen such that for all non-empty proper subset \( B \) of \( A \) we have

\[
\gcd(s(B), p(p - 1)) < \frac{p - 1}{\gcd(q, \varphi(p^m))}.
\]

This is actually possible, and it becomes really easy in the case where \( k \), the number of elements of \( A \), is “sufficiently small” with respect to \( \lambda \). Indeed, we would have that \( s(B) \) is never divisible
by \( p \) and that \( \gcd(s(B), (p - 1)/\gcd(q, \varphi(p^m))) \) is smaller than \( (p - 1)/\gcd(q, \varphi(p^m)) \), therefore

\[
\gcd\left(s(B), \frac{p - 1}{\gcd(q, \varphi(p^m))}\right) < \frac{p - 1}{\gcd(q, \varphi(p^m))}.
\]

Adding the fact that \( G \) is cyclic, the above inequality implies that \( G \) is \( A \)-nice. Considering that \( A = A^3 \) and that \( B_0(A) = \{\{A\}\} \), we can conclude by Theorem 1 that \( p(A) \) is equal to \( \chi(a) = \lambda(-k-1)^{k-1}(k-1)! \).

**Example 6.** Let \( R \) be the ring \( \mathbb{Z}/pq\mathbb{Z} \) where \( p, q \) are distinct primes greater than 11 such that \( p - 1 \) divides \( q + 1 \), and \( A \) a multiset of integers such that for every non-empty proper subset \( B \) of \( A \) we have

\[
\gcd(s(B), p - 1) \gcd(s(B), q - 1) > \frac{pq}{3(p + q)}
\]

only if \( s(B) \) is divisible by \( \varphi(pq)/2 \). Setting \( G \) equal to the group of units \( R^* \), we have \( n = \varphi(pq) \) and \( \lambda = n/\gcd(p - 1, q - 1) \). The hypothesis \( p - 1 \) divides \( q + 1 \) implies that \( \gcd(p - 1, q - 1) \) is exactly 2, so that \( \lambda = n/2 \). The above inequality is easily satisfied in the case that the primes \( p, q \) are sufficiently “near” each other, i.e. the weakest upper bound on the right hand side is provided when \( p, q \) are twin primes. At this point, this constraint implies that if \( \lambda \) divides \( s(B) \) then \( \gcd(s(B), n) \) is smaller than or equal to \( \frac{pq}{n(p+q)} \). In particular, we obtain that

\[
\frac{\lambda}{\gcd(s(B), \lambda)} \geq \frac{\lambda}{\gcd(s(B), n)} \geq \frac{\lambda}{\frac{3}{(p+q)}} \geq \frac{\lambda}{\frac{1}{2} (\frac{pq}{p+q} - 1)} \geq \frac{pq}{p+q} - 1 + \frac{1}{p+q} = p + q.
\]

Then condition (3) is satisfied, so that the subgroup \( G \) is \( A \)-nice and Theorem 1 holds.

Within this context, it would be interesting to obtain related results which are independent of the combinatorial constraint (3).

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