On Triple Factorisations of Finite Groups

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Abstract

This paper introduces and develops a general framework for studying triple factorisations of the form $G = ABA$ of finite groups $G$, with $A$ and $B$ subgroups of $G$. We call such a factorisation nondegenerate if $G \neq AB$. Consideration of the action of $G$ by right multiplication on the right cosets of $B$ leads to a nontrivial upper bound for $|G|$ by applying results about subsets of restricted movement. For $A < C < G$ and $B < D < G$ the factorisation $G = CDC$ may be degenerate even if $G = ABA$ is nondegenerate. Similarly forming quotients may lead to degenerate triple factorisations. A rationale is given for reducing the study of nondegenerate triple factorisations to those in which $G$ acts faithfully and primitively on the cosets of $A$. This involves study of a wreath product construction for triple factorisations.

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1 Introduction

In this paper we initiate a general theory of triple factorisations $T = (G, A, B)$ of the form $G = ABA$ for finite groups $G$ and subgroups $A, B$. A special case of such factorisations was introduced by Daniel Gorenstein [8] in 1959, namely independent $ABA$-groups in which every element not in $A$ can be written uniquely as $abc$ with $a, c \in A$ and $b \in B$ (see [8], also [9, 10]). Triple factorisations also play a fundamental role as Bruhat decompositions in the theory of Lie type groups (see for example, [4]), and more generally in the study of groups with a $(B, N)$-pair (see [3]). With the extra condition that $AB \cap BA = A \cup B$, Higman and McLaughlin [11] showed in their famous paper that the associated coset geometry is a flag-transitive 2-design, and that $G$ acts primitively on the points of this design (that is, $A$ is a maximal subgroup of $G$). More general factorisations of the form $G = ABC$ arise, for example, as Iwasawa decompositions of semisimple Lie groups (see [6, 12]). Here we focus on the case $A = C$.

Special cases of a triple factorisation $T = (G, A, B)$ occur if one of $A, B$ is equal to $G$, in which case $T$ is said to be trivial, or more generally, if $G$ factorises as $G = AB$, and here we say that $T$ is degenerate. If $G \neq AB$ we call $T$ nondegenerate. The term ‘degenerate’ may seem a misnomer, as group factorisations $G = AB$ arise in many problems involving symmetry in algebra and combinatorics (see [11, 14] and the more than 100 references to these monographs recorded in MathSciNet). As the theory of factorisations $G = AB$ is well developed, both for finite soluble groups and for finite almost simple groups $G$, for the purposes of the present theory we regard them as degenerate. Throughout the paper $G$ will denote a finite group.

It is tempting to replace proper subgroups $A, B$ in a nondegenerate triple factorisation $T = (G, A, B)$ by ‘maximal overgroups’, that is, by subgroups $C, D$ where $A \leq C, B \leq D$, and $C, D$ are maximal subgroups of $G$. This guarantees that $(G, C, D)$ is a triple factorisation, but it may be degenerate. Indeed, in Section 5, we show by a simple example that there may be no choices of maximal overgroups $C, D$ that give a nondegenerate lift $(G, C, D)$, even with $G$ a simple group, see Example 5.4. One of the major questions we
address in this paper is: under what conditions does such a lift \((G, C, D)\) of a nondegenerate triple factorisation \(T = (G, A, B)\) remain nondegenerate?

Important tools for studying triple factorisations come from permutation group theory. For each proper subgroup \(H\) of a group \(G\), the group \(G\) induces a transitive action by right multiplication on the set \(\Omega_H = [G : H]\) of right cosets of \(H\), and the kernel of this action is the \text{core} of \(H\) in \(G\), namely \(\text{core}_G(H) = \cap_{g \in G} H^g\). Thus for a triple \(T = (G, A, B)\) with \(A\) and \(B\) proper subgroups, there are two such transitive actions, on \(\Omega_A\) and \(\Omega_B\). In Section 3 we give two necessary and sufficient conditions for \(T\) to be a triple factorisation, one for each of these actions. The second criterion is in terms of the existence of a certain subset of \(\Omega_B\) with ‘restricted movement’ (see Proposition 3.2). Application of the classification result in [15] yields a non-trivial improvement to the trivially obtained upper bound \(|G| \leq |A|^2 \cdot |B|/|A \cap B|\), and a generalisation to Theorem 1.1 is given in Theorem 3.4.

**Theorem 1.1.** If \(T = (G, A, B)\) is a nontrivial triple factorisation, then

\[
|G| \leq \frac{|A|^2 |B|}{|A \cap B|^2} - \frac{|A| \cdot |B|}{|A \cap B|} + |B|
\]

with equality if and only if \(G\) is a flag-transitive collineation group of a projective plane with point set \(\Omega_B\) and having the \(A\)-orbit containing the coset \(B\) as a line.

We remark that the finite flag-transitive projective planes, and also their flag-transitive collineation groups are known explicitly, see [13]. It turns out that studying the \(G\)-action on \(\Omega_A\) is even more fruitful.

Among the possible simplifications, or ‘reductions’, in studying the class of triple factorisations \(T = (G, A, B)\) are the formation of quotients and restrictions of factorisations, which may yield triple factorisations for groups of order less than \(|G|\). For a normal subgroup \(N\) of \(G\), the \textit{quotient} of \(T\) modulo \(N\) is the triple \(T/N = (G/N, AN/N, BN/N)\). This is always a triple factorisation, and as in the case of lifts, it may be trivial (if \(N\) is transitive on \(\Omega_A\) or on \(\Omega_B\)), degenerate or nondegenerate. In Section 5 we discuss various necessary and/or sufficient conditions under which \(T/N\) is...
nondegenerate. In particular, we prove the following elementary and useful fact (see Lemma 5.2(c)).

**Observation 1.** If $N \leq \text{core}_G(A)\text{core}_G(B)$, then $T$ is nondegenerate if and only if $T/N$ is nondegenerate.

If $T$ is nondegenerate, then it is often sufficient to study $T/N$ with $N = \text{core}_G(A)\text{core}_G(B)$. That is to say, we may assume in our analysis that $G$ acts faithfully on both $\Omega_A$ and $\Omega_B$, or in other words, $A$ and $B$ are core-free in $G$ (see Remark 5.3(a)). Thus problems about triple factorisations may be expressed in the language of transitive permutation groups. In particular if some lift $(G, C, D)$ were nondegenerate, with $C$ or $D$ maximal in $G$, then the $G$-action on $\Omega_C$ or $\Omega_D$ respectively, would be primitive, and we could reduce our analysis to the context of primitive permutation groups. Unfortunately, as we noted above, such a reduction is not always possible simply by considering a lift.

We do however have a rationale for focusing on primitive permutation groups, and to explain this we study the $G$-action on $\Omega_A$ in more detail. If $T = (G, A, B)$ is a nondegenerate triple factorisation and $A$ is not maximal in $G$, then there is a subgroup $H$ such that $A < H < G$. This gives rise to a lift $T' = (G, H, B)$ of $T$ and corresponding quotient $T_1 = T'/N$ (which is also a lift of $T/N$) where $N = \text{core}_G(H)$ (see Proposition 7.5). In Section 6.1 we show that the restriction $T|_H = (H, A, B \cap H)$ is also a triple factorisation, and moreover, that at least one of $T_1$ and $T|_H$ is non-degenerate (see Lemmas 5.2 and 6.3). By Observation 1 we may assume that $G$ induces a faithful and imprimitive action on $\Omega_A$ preserving a block system $\Sigma$ corresponding to the right $H$-cosets. By the Embedding Theorem for imprimitive permutation groups (see Theorem 2.1), we may assume further that $G$ is a subgroup of the wreath product $G_0 \wr G_1$ acting on $\Delta \times \Sigma$, where $G_1 \cong G/N$ is the group induced by $G$ on $\Sigma$ and $G_0 \cong H/\text{core}_H(A)$ is the group induced by $H$ on the block $\Delta$ of $\Sigma$ containing $A$. Moreover, we construct a *wreath triple factorisation* $T_0 \wr T_1$ of $G_0 \wr G_1$, where $T_0$ is the quotient of $T|_H$ modulo $\text{core}_H(A)$ (see Definition 7.1). Our main result is the following imprimitive reduction theorem. Its proof requires an extension of the Embedding Theorem, which we prove in Theorem 2.1.
Theorem 1.2. Let \( T = (G, A, B) \) be a nondegenerate triple factorisation with \( \text{core}_G(A) = 1 \), and suppose that \( A < H < G \), and \( G \leq G_0 \wr G_1 \) with \( G_0 = H/\text{core}_H(A) \), \( G_1 = G/\text{core}_G(H) \) as above. Then there are triple factorisations \( T_0 \) for \( G_0 \), \( T_1 \) for \( G_1 \) and \( T_0 \wr T_1 \) for \( G_0 \wr G_1 \) such that \( T_0 \) is a quotient of \( T|_H \), \( T_1 \) is a quotient of a lift of \( T \), \( T_0 \wr T_1 \) is nondegenerate, and either

(a) \( T_1 \) is nondegenerate, or

(b) the triple factorisations \( T_0 \) and \( T_0 \wr T_1 \) are both nondegenerate, and the restriction of \( T_0 \wr T_1 \) to \( G \) is a nondegenerate lift \( (G, A, D) \) of \( T \).

We prove a more detailed version of Theorem 1.2 in Section 7. If \( B \) is maximal in \( G \), then in Theorem 1.2(b) the restriction of \( T_0 \wr T_1 \) to \( G \) is equal to \( T \) (see Corollary 7.7).

1.1 Rationale for primitive triple factorisations

Observation 1 and Theorem 1.2 provide a reduction pathway to the study of nondegenerate triple factorisations \( T = (G, A, B) \) with \( A \) maximal in \( G \) and both \( A \) and \( B \) core-free in \( G \), as follows. We call such \( T \) primitive. Corollary 1.3 below shows how each nondegenerate \( T \) is associated with at least one primitive triple factorisation.

Corollary 1.3. Let \( T = (G, A, B) \) be a nondegenerate triple factorisation with \( \text{core}_G(A) = 1 \). Then there exist subgroups \( H, K \) such that \( A \leq K < H \leq G \), \( K \) is maximal in \( H \), \( G = HB \), and the quotient of \( (H, K, BN \cap H) \) modulo \( \text{core}_H(K) \) is a primitive triple factorisation, where \( N = \text{core}_G(K) \). In particular, if \( A \) is maximal in \( G \) (so that \( K = A \) and \( H = G \)), then \( T \) is primitive.

From this point of view the basic nondegenerate triple factorisations \( T = (G, A, B) \) to study are the primitive ones, that is, those with \( A \) and \( B \) core-free in \( G \), and \( A \) maximal in \( G \). For this study, we may therefore regard \( G \) as a primitive permutation group on \( \Omega_A \) with point stabiliser \( A \), and study intransitive subgroups \( B \) with the property of Proposition 3.1.
Often the study of primitive permutation groups $G$ focuses on the action of its socle, and indeed, important (Lie type) triple factorisations arising from the ‘Bruhat decomposition’ often have $G$ simple. However reduction of triple factorisations to proper normal subgroups is not straightforward. In our final Section 8 we discuss this problem, giving a rather technical sufficient condition for restriction.

2 An extended Embedding Theorem

In this section we give some preliminary definitions and results, and in particular prove an extension of the imprimitive embedding theorem appropriate for application to triple factorisations.

2.1 Notation and basic definitions

The set of all permutations of a set $\Omega$ is the symmetric group on $\Omega$, denoted by $\text{Sym}(\Omega)$, and a subgroup of $\text{Sym}(\Omega)$ is called a permutation group on $\Omega$. Also we denote by $\text{Alt}(\Omega)$ the alternating group on $\Omega$. If a group $G$ acts on $\Omega$ we denote the induced permutation group of $G$ by $G^\Omega$, a subgroup of $\text{Sym}(\Omega)$. We say that $G$ is transitive on $\Omega$ if for all $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $\alpha^g = \beta$. For a transitive group $G$ on a set $\Omega$, a nonempty subset $\Delta$ of $\Omega$ is called a block for $G$ if for each $g \in G$, either $\Delta^g = \Delta$, or $\Delta^g \cap \Delta = \emptyset$; in this case the set $\Sigma = \{\Delta^g \mid g \in G\}$ is said to be a block system for $G$. The group $G$ induces a transitive permutation group $G^\Sigma$ on $\Sigma$, and the set stabiliser $G_\Delta$ induces a transitive permutation group $G^\Delta_\Delta$ on $\Delta$. If the only blocks for $G$ are the singleton subsets or the whole of $\Omega$ we say that $G$ is primitive, and otherwise $G$ is imprimitive.

2.2 The Embedding Theorem

Let $G \leq \text{Sym}(\Omega)$ be a transitive permutation group, and let

(a) $\Sigma = \{\Delta_1, \ldots, \Delta_\ell\}$ be a block system for $G$ in $\Omega$, set $\Delta = \Delta_1$ and $\alpha \in \Delta_1$;
(b) $G_1 = G^\Sigma \leq \text{Sym}(\Sigma)$, and $G_0 = G^\Delta_\Delta \leq \text{Sym}(\Delta)$.

The group $G_1$ is determined by $G$, but the group $G_0$ may depend on the choice of block $\Delta$ in $\Sigma$. According to the Embedding Theorem, this dependence is only up to permutational isomorphism: the Embedding Theorem gives a permutation embedding $(\varphi, \psi)$ from $(G, \Omega)$ into $(G_0 \wr G_1, \Delta \times \Sigma)$, that is to say, a monomorphism $\varphi : G \rightarrow G_0 \wr G_1$ and a bijection $\psi : \Omega \rightarrow \Delta \times \Sigma$ such that, for all $\delta \in \Delta$, $i \in \Sigma$ and all $g \in G$,

$$(\psi(\delta))^\varphi(g) = \psi(\delta g).$$

(c) Note that if $\alpha \in \Delta$, $A = G_\alpha$, $A_0 = A^\Delta$ and $A_1 := (G_1)_\Delta$, then the proof of the Embedding Theorem in [2, Theorem 8.5] constructs a permutation embedding $(\varphi, \psi)$ such that $\varphi(A) \leq \hat{A}$, where $\hat{A} = (A_0 \times G_0^{\ell-1}) \rtimes A_1$.

When studying triple factorisations $G = ABA$ we need information about the subgroup $B$ as well as $A$. The group $B_1 = B^\Sigma$ is determined by $B$ but the group $B_0 = B^\Delta_\Delta$ induced on $\Delta$ in general varies according to the choice of $\Delta$. However, if $B_1$ is transitive, then $B_0$ is unique up to permutation isomorphism and we have the following refinement of the Embedding Theorem.

**Theorem 2.1** (The Extended Embedding Theorem). Let $G \leq \text{Sym}(\Omega)$ be transitive and let $\Sigma, \Delta, G_1, G_0, A, A_1, A_0$, and $\hat{A}$ be as in (a)-(c) above. Suppose also that $B \leq G$, and let $B_1 = B^\Sigma$, and $B_0 = B^\Delta_\Delta$. Then there is a permutation embedding $(\varphi, \psi)$ of $(G, \Omega)$ into $(G_0 \wr G_1, \Delta \times \Sigma)$ such that $\varphi(A) \leq \hat{A} \cap \varphi(G)$, and if $B_1$ is transitive then $\varphi$ may be chosen such that $\varphi(B) \leq (B_0 \wr B_1)$.

**Proof.** We express the permutation embedding $(\varphi, \psi)$ explicitly, following the treatment in [2, Theorem 8.5]. Here $G_0 \wr G_1$ is identified as the semidirect product $\text{Fun}(\Sigma, G_0).G_1$, where $\text{Fun}(\Sigma, G_0)$ consists of all functions $f : \Sigma \rightarrow G_0$ and acts on $\Delta \times \Sigma$ via

$$(\delta, \Delta_i)^f = (\delta^{f(\Delta_i)}, \Delta_i).$$
for all $\delta \in \Delta$ and $\Delta_i \in \Sigma$, and $G_1$ acts naturally on $\Delta \times \Sigma$ by

$$(\delta, \Delta_i)^g = (\delta, \Delta_i^g),$$

and normalises $\text{Fun}(\Sigma, G_0)$. If $B_1 = B^\Sigma$ is not transitive, then all the assertions follow from [2, Theorem 8.5]. Assume now that $B_1$ is transitive. Then, for each $i$, there is an element $t_i \in B$ such that $\Delta_i^t_i = \Delta_i$, and the family $\{t_i\}_{i=1}^\ell$ is a right transversal for $H := G_{\Delta_1}$ in $G$. For $g \in G$ and $\Delta_i \in \Sigma$, define $i^g$ by $\Delta_i^g = \Delta_{i^g}$. As in [2, Theorem 8.5], the map $\varphi : G \to G_0 \wr G_1$ is defined by

$$\varphi(g) = f_g \Sigma, \quad \text{where} \quad f_g(\Delta_i) = (t_i g t_i^{-1})^\Delta.$$  \hfill (2.1)

Note that for each $i$, by the definition of $t_i$, $t_i g t_i^{-1} \in G_{\Delta_1} = H$, so $f_g(\Delta_i) \in G_0$. Hence $f_g \in \text{Fun}(\Sigma, G_0)$. This map is proved in [2] to be a monomorphism with $\varphi(G) \leq G_0 \wr G_1$, $\varphi(A) = \hat{A} \cap \varphi(G)$, and $(\varphi, \psi)$ is proved to be a permutation embedding, where $\psi : \Omega \to \Delta \times \Sigma$ is given by

$$\psi(\delta) = (\delta t_i^{-1}, \Delta_i), \quad \text{for} \: \delta \in \Delta_i.$$

Since we have chosen all $t_i$ to lie in $B$, we have, for each $g \in B$, that $t_i g t_i^{-1} \in B \cap H$, so $f_g(\Delta_i) \in B_0$. Thus $f_g \in \text{Fun}(\Sigma, B_0)$ and hence $\varphi(g) \in \text{Fun}(\Sigma, B_0).B_1 = B_0 \wr B_1$. Therefore $\varphi(B) \leq B_0 \wr B_1$. \hfill \qed

Note that if $B_1$ is not transitive, then $(B \cap H^t)^{\Delta_i}$ may be very different from $B_0 = (B \cap H)^{\Delta_i}$ for some $i \neq 1$.

**Example 2.2.** Let $G = A_5 \wr S_4$, $A = A_4 \times (A_5 \wr S_3)$, $B = (A_5 \wr S_2) \times (A_4 \wr S_2)$, and $A < H = A_5 \times (A_5 \wr S_3)$. Then $T = (G, A, B)$ is a nondegenerate triple factorisation, and relative to the block system $\Sigma = \{\Delta_1, ..., \Delta_4\}$ corresponding to $H$, $(B \cap H^t)^{\Delta_i}$ on blocks $\Delta_1$, $\Delta_2$ is $A_5$, and on $\Delta_3$, $\Delta_4$ is $A_4$.

### 3 Triple factorisations: two criteria

In what follows, $G$ is a group, $A$ and $B$ are subgroups of $G$, and we consider the triple $T = (G, A, B)$. Recall that $T$ is called a a triple factorisation of $G$.\hfill 8
if $G = ABA$. We use $\Omega_A$ and $\Omega_B$ to denote the set of right cosets of $A$ and $B$, respectively, that is, $\Omega_A := \{Ag \mid g \in G\}$ and $\Omega_B := \{Bg \mid g \in G\}$, and we note that $G$ acts naturally on both $\Omega_A$ and $\Omega_B$ by right multiplication. We call these actions right coset actions. We give two different criteria for $T = (G, A, B)$ to be a triple factorisation based on the actions of $G$ on $\Omega_A$ and $\Omega_B$, respectively. The first criterion is connected to incidence geometries and we call it the geometric interpretation of a triple factorisation. The second criterion relates to the notion of restricted movement of a set under a group action and we call this the restricted movement interpretation of a triple factorisation. We were told of the first criterion by Jan Saxl and it is discussed in [7], while to our knowledge the second interpretation has not appeared in the literature before.

To explain the geometric interpretation, for a group $G$ and proper subgroups $A, B$, call the elements of $\Omega_A$ and $\Omega_B$ points and lines, respectively and define a point $Ax$ and a line $By$ to be incident if and only if $Ax \cap By \neq \emptyset$. An incident point-line pair $(Ax, By)$ is called a flag. It follows from this definition that $G$ preserves incidence and acts transitively on the flags of this geometry. Moreover, $T = (G, A, B)$ is a triple factorisation if and only if any two points lie on at least one line (see Lemma 3 in [11]). With extra conditions on a triple factorisation $T$ making it a Geometric $ABA$-group, this incidence geometry becomes a 2-design (see Proposition 1 in [11]).

**Proposition 3.1** (Geometric criterion). Let $A$ and $B$ be proper subgroups of a group $G$, and consider the right coset action of $G$ on $\Omega_A := \{Ag \mid g \in G\}$. Set $\alpha := A \in \Omega_A$. Then $T = (G, A, B)$ is a triple factorisation if and only if the $B$-orbit $\alpha^B$ intersects nontrivially each $G_\alpha$-orbit in $\Omega_A$.

**Proof.** Set $\Delta := \alpha^B$. Suppose that $T$ is a triple factorisation for $G$, and consider a $G_\alpha$-orbit $\Gamma := \beta^A$ in $\Omega_A$. Since $G$ is transitive on $\Omega_A$, there exists $x \in G$ such that $\beta = \alpha^x$, and since $G = ABA$, $x = abc$, for some $a, c \in A$ and $b \in B$. Therefore, $\beta^{c^{-1}} = \alpha^{xc^{-1}} = \alpha^b \in \Delta$, and also $\beta^{c^{-1}} \in \beta^A = \Gamma$. Thus $\Delta \cap \Gamma \neq \emptyset$. Conversely, suppose that $\Delta$ intersects each $A$-orbit in $\Omega_A$ nontrivially, and let $x \in G$. Let $\Gamma$ be the $A$-orbit containing $\alpha^x$. By assumption, $\Gamma \cap \Delta$ contains a point $\beta$. Since $\beta \in \Delta$, $\beta = \alpha^b$ for some
b \in B$, and since $\beta \in \Gamma$, $\beta = \alpha x^a$ for some $a \in A$. So $\alpha = \alpha x^a b^{-1}$, and hence $c := xab^{-1} \in G_\alpha = A$, that is, $x = cba^{-1} \in ABA$.

The second criterion characterises triple factorisations of $G$ in terms of a subset of $\Omega_B$ having restricted movement. For a finite subset $\Gamma$ of $\Omega$, the movement of $\Gamma$ under the action of a group $G$ on $\Omega$ is defined by $\text{move}(\Gamma) := \max_{g \in G} |\Gamma^g \setminus \Gamma|$. If $\text{move}(\Gamma) < |\Gamma|$, then $\Gamma$ is said to have restricted movement. In other words, $\Gamma^g \cap \Gamma \neq \emptyset$ for all $g \in G$.

**Proposition 3.2** (Restricted movement criterion). Let $A$ and $B$ be proper subgroups of a group $G$. Consider the right coset action of $G$ on $\Omega_B$, and set $\beta := B \in \Omega_B$. Then, $T = (G, A, B)$ is a triple factorization if and only if the $A$-orbit $\beta A$ has restricted movement.

**Proof.** Let $\Gamma := \beta A$. Suppose that $T$ is a triple factorisation and let $x \in G$. Then there exist $a, c \in A$ and $b \in B$ with $x = abc$. Let $\gamma := \beta bc$. Since $G \beta = B$, we have $\beta b = \beta$. Thus $\gamma = \beta bc = \beta c \in \Gamma$. Also, since $\Gamma = \beta A$, we have $\gamma = \beta bc \in (\beta A)^{bc} = (\beta A)^{bc} = \Gamma^x$. Therefore, $\gamma \in \Gamma \cap \Gamma^x$, that is, $\Gamma$ has restricted movement. Conversely, suppose that $\Gamma$ has restricted movement, and $x \in G$. Then $\Gamma \cap \Gamma^x \neq \emptyset$, and hence there exist $a, c \in A$ with $\beta a = \beta c x$. Thus $cxa^{-1} \in G \beta = B$, so there exists $b \in B$ such that $x = c^{-1}ba$. Therefore, $G = ABA$.

**Remark 3.3.** Let $T = (G, A, B)$ be a triple factorisation, and consider $G$ acting on $\Omega_B$. Let $\beta = B \in \Omega_B$, and let $k = |\beta A|$ and $m = \text{move}(\beta A)$. If $m = 0$, then $\beta A = \Omega_B$, that is to say, $G = AB$, and vice versa. Therefore, $T = (G, A, B)$ is nondegenerate if and only if $\beta A \neq \Omega_B$ if and only if $\text{move}(\beta A) > 0$. Hence, by Proposition 3.2, $T$ is nondegenerate if and only if $1 \leq \text{move}(\beta A) \leq k - 1$.

Since $|AB| = |A||B|/|A \cap B|$, the equality $G = ABA$ implies that $|G| \leq |A|^2|B|/|A \cap B|$. A consequence of Proposition 3.2 is an improvement on this upper bound for $|G|$, namely the upper bound in Theorem 1.1 in the introduction and its generalisation in Theorem 3.4 below. A $2 - (v, k, \lambda)$ design is a set $\Omega$ of $v$ points together with a set of $k$-element subsets of $\Omega$, called blocks, such that each pair of points is contained in $\lambda$ blocks. Such
a design is symmetric if it has exactly $v$ blocks. (Note that this usage of the term ‘block’ is different from the blocks of imprimitivity introduced in Section 2.) We apply the main result of [16] to prove Theorem 3.4 below, Theorem 1.1 follows immediately from Theorem 3.4. The special case of [16, Theorem 1.1] needed to prove Theorem 1.1 is proved in [15].

**Theorem 3.4.** Suppose $T = (G, A, B)$ is a nondegenerate triple factorisation, $\beta = B \in \Omega_B$, $k = |\beta A|$ and $m = \text{move}(\beta A)$. Then

$$|G| \leq \frac{|B|}{|A \cap B|} \left( \frac{|A|^2 - m |A \cap B|^2}{|A| - m |A \cap B|} \right)$$

with equality if and only if the $G$-translates of $\beta A$ form the blocks of a symmetric $2 - (\frac{k^2 - m}{k - m}, k, k - m)$ design with point set $\Omega_B$ admitting $G$ as a flag-transitive group of automorphisms. Moreover, Theorem 1.1 holds.

**Proof.** Consider the action of $G$ on $\Omega_B$, and let $k = |\beta A|$. Then $k = |A|/|A \cap B|$, and by [16, Theorem 1.1], we have $|\Omega_B| \leq (k^2 - m)/(k - m)$ with equality if and only if the $G$-translates of $\beta A$ form the blocks of a symmetric $2 - (\frac{k^2 - m}{k - m}, k, k - m)$ design with point set $\Omega_B$ admitting $G$ as a flag-transitive group of automorphisms. Since $|\Omega_B| = |G|/|B|$, the inequality (3.1) follows immediately. To prove Theorem 1.1, note that the function, $f(m) = (k^2 - m)/(k - m)$ increases as $m$ increases. Since $T$ is nondegenerate, it follows from Remark 3.3 that $1 \leq m \leq k - 1$, and so by choosing $m = k - 1$, the inequality in Theorem 1.1 follows from $|\Omega_B| \leq k^2 - k + 1$ with equality if and only if $G$ is a flag-transitive group of collineations of a projective plane with point set $\Omega_B$ such that $\Gamma$ is a line. \qed

4 Isomorphisms of triple factorisations

**Definition 4.1.** Let $T := (G, A, B)$ and $T' := (G', A', B')$ be triple factorisations. We say that $T$ is isomorphic to $T'$ and write $T \cong T'$ if there exists an isomorphism $\varphi : G \rightarrow G'$ such that $\varphi(A) = A'$ and $\varphi(B) = B'$.

Suppose that $T = (G, A, B)$ is a triple factorisation. Then for each $\sigma \in \text{Aut}(G)$, $G = A^\sigma B^\sigma A^\sigma$. However, in general the factorisation property
is not preserved if we apply different automorphisms to $A$ and $B$ (see Lemma 4.2(a)). Moreover, if $T' = (G', A', B')$ is another triple factorisation then $T$ and $T'$ may not be isomorphic even if $G \cong G'$, $A \cong A'$ and $B \cong B'$ (see Lemma 4.2(b)).

**Lemma 4.2.**

(a) There exist infinitely many nondegenerate triple factorisations $T = (G, A, B)$ such that for some $x \in G$, $(G, A, B^x)$ is not a triple factorisation.

(b) There exist infinitely many nondegenerate triple factorisations $T = (G, A, B)$ and $T' = (G, A, B^x)$ where $x \in G$, such that $T$ and $T'$ are not isomorphic.

In order to prove Lemma 4.2(a) we prove a general result about triple factorisations $(G, A, B)$ with $G$ acting 2-transitively on $\Omega_A$, namely Proposition 4.3 and then in Example 4.4 we give an explicit family of examples.

**Proposition 4.3.** Suppose that $G$ is a 2-transitive permutation group on a finite set $\Omega$ of size $n$, let $\alpha \in \Omega$, $A = G_\alpha$, and $B \leq G$. Then $T = (G, A, B)$ is a triple factorisation if and only if $\alpha B \neq \alpha$. Also $T$ is nondegenerate if and only if $B$ is intransitive.

**Proof.** Since $G$ is 2-transitive on $\Omega$, the $A$-orbits on $\Omega$ are $\{\alpha\}$ and $\Omega \setminus \{\alpha\}$. Therefore, by Proposition 3.1 for a subgroup $B$ of $G$, $T = (G, A, B)$ is a triple factorisation if and only if $\alpha B \neq \alpha$. Also $T$ is nondegenerate if and only if $B$ is intransitive.

**Example 4.4.** Let $G = \text{Sym}(\Omega)$ with $\Omega := \{1, \ldots, n\}$, $n \geq 3$, $A := G_1$, $B := \langle (1, 3) \rangle$, and $x := (1, 2)$. Then by Proposition 4.3, $T = (G, A, B)$ is a triple factorisation but $T' = (G, A, B^x)$ is not.

Similarly Lemma 4.2(b) is established by the next family of examples.

**Example 4.5.** Let $G = \text{Sym}(\Omega)$ with $\Omega := \{1, \ldots, n\}$, $n \geq 5$. Let $A := G_n$, $B := \langle (1, 2), (3, 4, n) \rangle$, and $x := (1, 2, n)$. Then both $T = (G, A, B)$ and $T' = (G, A, B^x)$ are nondegenerate triple factorisations by Proposition 4.3. However, $T \not\cong T'$ for if $\varphi \in \text{Aut}(\text{Sym}(\Omega))$ with $\varphi(A) = A$, then
\( \varphi \in \text{Sym}(\Omega) \cap A = A \), and then \( \varphi(B) = \langle (1^\varphi, 2^\varphi), (3^\varphi, 4^\varphi, n) \rangle \) whereas \( B^x = \langle (2, n), (3, 4, 1) \rangle \). Since the \( B^x \)-orbit containing \( n \) is \( \{2, n\} \) while the \( \varphi(B) \)-orbit containing \( n \) contains \( \{3^\varphi, 4^\varphi, n\} \), it follows that \( \varphi(B) \neq B^x \), and hence \( T \neq T' \).

5 Quotients and lifts of a triple factorisation

5.1 Quotients

Let \( T = (G, A, B) \) be a triple factorisation. We study the quotient \( T/N \) of a triple factorisation \( T = (G, A, B) \) modulo a normal subgroup \( N \) of \( G \) as defined in Section 4 to find conditions on \( N \) under which the quotient of a nondegenerate \( T \) remains nondegenerate. We also give a sufficient condition for \( T/N \) to be nondegenerate which, in particular, allows us to assume that \( G \) acts faithfully on \( \Omega_A \) and \( \Omega_B \).

For a subgroup \( H \leq G \), we denote by \( \overline{H} \) the corresponding subgroup \( HN/N \) of \( G/N \). Thus \( T/N = (\overline{G}, \overline{A}, \overline{B}) \) and we sometimes also denote it by \( \overline{T} \). If \( T/N \) is a nondegenerate factorisation, so is \( T \), but the converse is not true in general as is shown by a small example in Example 5.1.

Example 5.1. Let \( G = \text{Alt}(\{1, 2, \ldots, 5\}) \times \text{Sym}(\{6, 7\}) = N \times M \), say, let \( B \) be the Klein 4-subgroup of \( \text{Alt}(\{1, 2, 3, 4\}) \), and let \( A = \langle (1, 2, 3, 4, 5) \rangle \times M \). Then \( T = (G, A, B) \) is a nondegenerate triple factorisation while \( T/N \cong (M, M, 1) \) is trivial.

The following lemma relates the quotient of \( T = (G, A, B) \) modulo \( N \) to the lift \( T' = (G, AN, BN) \) of \( T \). It implies in particular that studying \( T' \) is equivalent to studying \( T/N \).

Lemma 5.2. Let \( T = (G, A, B) \) be a triple factorisation and \( 1 \neq N \triangleleft G \). Then \( T/N \) is a triple factorisation. Moreover,

(a) \( T/N \) is nontrivial if and only if \( (G, AN, BN) \) is nontrivial (that is, if and only if both \( AN \) and \( BN \) are proper subgroups of \( G \)); and \( T/N \) is nondegenerate if and only if \( (G, AN, BN) \) is nondegenerate.

(b) If \( T \) is nondegenerate and \( N \subseteq AB \), then \( T/N \) is nondegenerate.
(c) If $N \subseteq \text{core}_G(A)\text{core}_G(B)$, then $T/N$ is nondegenerate if and only if $T$ is nondegenerate.

(d) For $\alpha = A \in \Omega_A$, $T/N$ is nondegenerate if and only if $\Delta := \alpha^B \subseteq \Omega_A$ is disjoint from at least one $N$-orbit in $\Omega_A$.

Proof. (a) This follows easily from the definitions of $\overline{A} = AN/N$ and $\overline{B} = BN/N$.

(b) Suppose that $T$ is nondegenerate. Then $G \neq AB$. If $N \subseteq AB$, then $G \neq ABN$, or equivalently $\overline{G} \neq \overline{A}\overline{B}$. Therefore, $T/N$ is nondegenerate.

(c) Suppose that $N \subseteq \text{core}_G(A)\text{core}_G(B)$. Then $N \subseteq AB$. Hence if $T$ is nondegenerate, then also $T/N$ is nondegenerate by part (b). On the other hand, if $T/N$ is nondegenerate, then $(G, AN, BN)$ is nondegenerate by part (a), and hence also $T$ is nondegenerate.

(d) Suppose that $T/N$ is degenerate, so $\overline{G} = \overline{A}\overline{B}$, or equivalently, $G = ABN$. This implies that $BN$ is transitive on $\Omega_A = [G : A]$, and hence $\Omega_A = \alpha^{BN} = \{\delta^x : x \in N, \delta \in \Delta\}$. It follows that $\Delta$ contains at least one point from each $N$-orbit in $\Omega_A$. Conversely, if $\Delta$ meets each $N$-orbit in $\Omega_A$, then $\alpha^{BN} = \Omega_A$, and hence $G = ABN$. \qed

Remark 5.3. (a) By Lemma 5.2(c), if $N = \text{core}_G(A)\text{core}_G(B)$, then either both or neither of $T$ and $T/N$ are nondegenerate. Moreover, we have $\text{core}_G(\overline{A}) = \text{core}_G(\overline{B}) = 1$, so that $\overline{G}$ acts faithfully and transitively on both $\Omega_A := [\overline{G} : \overline{A}]$ and $\Omega_B := [\overline{G} : \overline{B}]$. Therefore, we may replace $T$ by $T/N$, and when we do so the group $G$ acts faithfully on both $\Omega_A$ and $\Omega_B$.

(b) It is possible to have a nondegenerate quotient $T/N$ without the condition $N \subseteq AB$ of Lemma 5.2(b). For example, if $T_i = (G_i, A_i, B_i)$ are nondegenerate, for $i = 1, 2$, then taking $G = G_1 \times G_2$, $A = A_1 \times A_2$, $B = B_1 \times B_2$, and $N = G_2$, we have that $T = (G, A, B)$ and $T/N \cong T_1$ are both nondegenerate triple factorisations, while $N \nsubseteq AB$. 

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5.2 Lifts

Let $T = (G, A, B)$ be a triple factorisation, $A \leq C$ and $B \leq D$. Recall from Section I that the triple factorisation $(G, C, D)$ is called a lift of $T$. We give here an example to illustrate that it is not possible in general to obtain a nondegenerate lift $(G, C, D)$ of a given nondegenerate triple factorisation $T = (G, A, B)$, with $C, D$ maximal in $G$.

Example 5.4. Let $G = A_5$, $A = \langle (1, 2, 3, 4, 5) \rangle$, $B = \langle (1, 3)(2, 5), (1, 2)(3, 5) \rangle$. Then $T = (G, A, B)$ is a nondegenerate triple factorisation. Moreover, $A$ and $B$ are each contained in a unique maximal subgroup of $G$, namely $C \cong D_{10}$ and $D \cong A_4$, respectively, and $G = CD$. Thus the only lift $T' = (G, C, D)$ of $T$ with $C, D$ maximal is degenerate.

6 Restrictions of a triple factorisation

Definition 6.1. For a triple factorisation $T = (G, A, B)$, and $H \leq G$, the triple $T|_H = (H, A \cap H, B \cap H)$ is called the restriction of $T$ to $H$. If $T|_H$ is a triple factorisation, we say that $T$ restricts to $H$.

We consider the following question in several contexts, in this section and again in Section 8.

Question 1. Is the restriction $T|_H$ a triple factorisation? If so, when is it nontrivial, or nondegenerate?

Remark 6.2. Let $T := (G, A, B)$ and $T' := (G', A', B')$ be isomorphic triple factorisations via $\varphi$, and let $H \leq G$. Then $T$ restricts to $H$ if and only if $T'$ restricts to $\varphi(H)$. Moreover, $T|_H$ is nondegenerate if and only if $T'|_{\varphi(H)}$ is nondegenerate.

6.1 Restriction to overgroups of $A$

For any triple factorisation $T := (G, A, B)$ restricting to $A$ gives a trivial triple factorisation $T|_A = (A, A, A \cap B)$. However, restricting to proper overgroups of $A$ often yields nondegenerate triple factorisations.
Lemma 6.3. Suppose that $T$ is a triple factorisation and $A < H \leq G$. Then $T|_H$ is a triple factorisation; moreover, $T|_H$ is nontrivial if and only if $B \neq G$, and is nondegenerate if and only if $H \nsubseteq AB$.

Proof. Let $x \in H$. Since $G = ABA$, we have $x = abc$ with $a, c \in A$, $b \in B$. Then $b = a^{-1}xc^{-1} \in B \cap H$, and so $x \in A(B \cap H)A$. Thus $T|_H$ is a triple factorisation. Since $H \neq A$, $T|_H$ is trivial if and only if $B \cap H = H$, or equivalently, $A < H \leq B$, which is equivalent to $G = ABA = B$. If $T|_H$ is degenerate, then $H = A(B \cap H) \subseteq AB$. Conversely, if $H \subseteq AB$, then each $x \in H$ can be written as $x = ab$ with $a \in A$, $b \in B$, and hence $b = a^{-1}x \in B \cap H$, so $H = A(B \cap H)$. \qed

The following lemma is an important special case of Lemma 6.3.

Lemma 6.4. Suppose that $T := (G, A, B)$ is a nondegenerate triple factorisation, and that $N \trianglelefteq G$ is such that $N \nsubseteq A$. Then

(a) $T|_AN$ is a nontrivial triple factorisation, and is nondegenerate if and only if $N \nsubseteq AB$.

(b) At least one of $T/N$ and $T|_AN$ is nondegenerate.

Proof. Part (a) is a special case of Lemma 6.3. For part (b), assume that $T/N$ is degenerate. Then by Lemma 5.2(b), $N \nsubseteq AB$, and hence $T|_AN$ is nondegenerate by part (a). \qed

The proof in Lemma 6.3 that $T|_H$ is a triple factorisation does not use the fact that $B$ is a subgroup. We give this slightly more general statement and apply it in Section 8.

Lemma 6.5. Suppose that $A$ is a subgroup of group $G$, and $S$ is a subset of $G$ such that $G = ASA$. Let $H$ be a subgroup of $G$. Then $A \leq H$ if and only if $H = A(S \cap H)A$.

Proof. Clearly $H = A(S \cap H)A$ implies $A \leq H$. Conversely, suppose that $A \leq H$. Repeating the proof of Lemma 6.3 we get $H = A(S \cap H)A$. \qed
6.2 Restriction to normal subgroups

If \( T := (G, A, B) \) and \( T' := (G', A', B') \) are triple factorisations with \( G = G' \), \( A = A' \), \( B \cong B' \), and if \( N \) has index 2 in \( G \), it is possible for \( T \) but not \( T' \) to restrict to \( N \).

**Lemma 6.6.** There exist infinitely many triple factorisations \( T = (G, A, B) \) and \( T' := (G, A, B^x) \), where \( x \in G \), such that \( G \) has a subgroup \( N \) of index 2 and \( T \) restricts to \( N \) but \( T' \) does not restrict to \( N \).

**Proof.** Consider the factorisations \( T = (G, A, B) \) and \( T' = (G, A, B') \) in Example 4.5 and let \( N = \text{Alt}(\Omega) \). We claim that \( T \) restricts to \( N \) but \( T' \) does not.

Now \( T|_N = (N, G_n \cap N, \langle (3, 4, n) \rangle) \) while \( T'|_N = (N, G_n \cap N, B') \) where \( B' = \langle (3, 4, 1) \rangle \). In the latter case both \( A \cap N \) and \( B' \) fix the point \( n \) and hence \( T' \) does not restrict to \( N \). On the other hand, if \( g \in N \) fixes \( n \), then \( g \in A \cap N \), while if \( j = n^g \neq n \), then there exists \( c \in A \cap N \) such that \( 3^c = j \) and hence setting \( b = (3, 4, n) \in B \cap N \), the element \( a = gc^{-1}b^{-1} \) fixes \( n \) and so lies in \( A \cap N \). Thus \( g = abc \in (A \cap N)(B \cap N)(A \cap N) \), and it follows that \( T \) restricts to \( N \).

We explore this problem further in Section 8.

7 Wreath products of triple factorisations

In this section, we introduce a wreath product construction for triple factorisations, and study its properties. In Subsection 7.1 we prove Theorem 1.2.

Let \( \mathcal{T}_0 = (G_0, A_0, B_0) \) and \( \mathcal{T}_1 = (G_1, A_1, B_1) \) be triples with \( A_i, B_i \) subgroups of \( G_i \), for \( i = 0, 1 \) (not necessarily triple factorisations), and let \( \Delta \) and \( \Sigma \) denote the sets of right cosets of \( A_0 \) and \( A_1 \) in \( G_0 \) and \( G_1 \), respectively. Recall from Section 2 that the set \( \text{Fun}(\Sigma, G_0) \) consisting of all functions from \( \Sigma \) to \( G_0 \) forms a group under pointwise multiplication, that is to say, \((fg)(\Delta) = f(\Delta)g(\Delta)\), and the wreath product \( G_0 \wr G_1 \) is the
semidirect product \( \text{Fun}(\Sigma, G_0) \ltimes G_1 \) acting on \( \Delta \times \Sigma \) via

\[
(\delta, \Delta)^{f\sigma} = (\delta^{f(\Delta)}, \Delta^\sigma)
\]

for all \( \delta \in \Delta \), \( \Delta \in \Sigma \), and \( f\sigma \in G_0 \ltimes G_1 \). Note that \( G_0 \ltimes G_1 \) can also be identified with \( G_0^\ell \times G_1 \), where \( |\Sigma| = \ell \), and for all \( \sigma \in G_1 \) and \( (f_1, \ldots, f_\ell) \in G_0^\ell \), we have

\[
\sigma(f_1, \ldots, f_\ell)^{-1} = (f_1^\sigma, \ldots, f_\ell^\sigma).
\]

We define a wreath product of \( T_0 \) and \( T_1 \) as follows.

**Definition 7.1.** Let \( T_0 = (G_0, A_0, B_0) \) and \( T_1 = (G_1, A_1, B_1) \) be triples as above, and let \( \Delta = [G_0 : A_0] \), \( \Sigma = [G_1 : A_1] \) and \( \ell = |\Sigma| \). Then the wreath product of \( T_0 \) and \( T_1 \) is defined as \( T_0 \wr T_1 = (G_0 \ltimes G_1, \hat{A}, \hat{B}) \), where \( \hat{A} = (A_0 \times G_0^{\ell-1}) \ltimes A_1 \) and \( \hat{B} = B_0 \ltimes B_1 \).

**Lemma 7.2.** Let \( \hat{A} \) and \( \hat{B} \) be as in Definition 7.1 and let \( W = G_0 \ltimes G_1 \). Then \( \hat{A}\hat{B} = (A_0 B_0 \times G_0^{\ell-1})(A_1 B_1) \), \( \text{core}_W(\hat{A}) = \text{core}_{G_0}(A_0) \cap \text{core}_{G_1}(A_1) \) and \( \text{core}_W(\hat{B}) = \text{core}_{G_0}(B_0) \cap \text{core}_{G_1}(B_1) \).

**Proof.** Let \( f\sigma := (a_1, \ldots, a_\ell)\sigma \in \hat{A} \) and \( g\tau := (b_1, \ldots, b_\ell)\tau \in \hat{B} \), where \( a_1 \in A_0, a_i \in G_0 \) for \( i \geq 2 \), \( b_i \in B_0 \) for \( i \geq 1 \), \( \sigma \in A_1 \) and \( \tau \in B_1 \). Since \( \sigma \in A_1 \) and \( b_1^\sigma \in B_0 \), we have

\[
(f\sigma)(g\tau) = [(a_1, \ldots, a_\ell)\sigma][(b_1, \ldots, b_\ell)\tau] = [(a_1, \ldots, a_\ell)(b_1^\sigma, \ldots, b_\ell^\sigma)](\sigma\tau) = (a_1 b_1^\sigma, a_2 b_2^\sigma, \ldots, a_\ell b_\ell^\sigma)(\sigma\tau) \in (A_0 B_0 \times G_0^{\ell-1})(A_1 B_1).
\]

Therefore \( \hat{A}\hat{B} \subseteq (A_0 B_0 \times G_0^{\ell-1})(A_1 B_1) \). The converse is similar and easier. Finally the assertions about \( \text{core}_W(\hat{A}) \) and \( \text{core}_W(\hat{B}) \) follow since \( G_1 \) is transitive on \( \Sigma \).

**Lemma 7.3.** Let \( T_0 \) and \( T_1 \) be as in Definition 7.1. If \( T_0 \) and \( T_1 \) are both triple factorisations, then \( T_0 \wr T_1 \) is a triple factorisation. Moreover, \( T_0 \wr T_1 \) is trivial (nondegenerate) if and only if at least one of the \( T_0 \) and \( T_1 \) is trivial (nondegenerate, respectively).
Proof. Let $\hat{A} := (A_0 \times G_0^{\ell-1}) \rtimes A_1$, $\hat{B} := B_0 \wr B_1$, and $W := G_0 \wr G_1$, so $W = G_0 \wr G_1$. Suppose that $f\sigma \in W$, where $f := (f_1, \ldots, f_\ell) \in G_0^{\ell}$ and $\sigma \in G_1$. Since $T_0$ is a triple factorisation, there exist $a_i, c_i \in A_0$, $b_i \in B_0$ such that $f_i = a_i b_i c_i$, for $i = 1, \ldots, \ell$. Also since $T_1$ is a triple factorisation, there exist $\delta, \nu \in A_1$ and $\tau \in B_1$ such that $\sigma = \delta\tau\nu$. Let $\hat{a} := (a_1, \ldots, a_\ell)\delta \in \hat{A}$, $\hat{b} := (b_1, \ldots, b_\ell)\tau \in \hat{B}$, $\hat{c} := (c_1, \ldots, c_\ell)\nu \in \hat{A}$. Then

\[
    f\sigma = (a_1 b_1 c_1, \ldots, a_\ell b_\ell c_\ell)\delta\tau\nu = (a_1, \ldots, a_\ell)(b_1, \ldots, b_\ell)\delta\tau[c_1, \ldots, c_\ell]\nu = [(a_1, \ldots, a_\ell)\delta][(b_1, \ldots, b_\ell)\tau][(c_1, \ldots, c_\ell)]\nu = \hat{a}\hat{b}\hat{c} \in \hat{A}\hat{B}\hat{A}.
\]

Therefore, $W = \hat{A}\hat{B}\hat{A}$, so $T_0 \wr T_1$ is a triple factorisation.

By the definition, $T_0 \wr T_1$ is trivial if and only if $W = \hat{A}$ or $W = \hat{B}$. Since $\hat{A} := (A_0 \times G_0^{\ell-1}) \rtimes A_1$ and $\hat{B} := B_0 \wr B_1$, $T_0 \wr T_1$ is trivial if and only if either (a) $G_0 = A_0$ and $G_1 = A_1$, or (b) $G_0 = B_0$ and $G_1 = B_1$. Each of the conditions (a) or (b) holds if and only if at least one of $T_0$ and $T_1$ is trivial. By Lemma 7.2, we have $\hat{A}\hat{B} = (A_0B_0 \times G_0^{\ell-1})A_1B_1$, and hence $T_0 \wr T_1$ is nondegenerate if and only if $G_0 \neq A_0B_0$ or $G_1 \neq A_1B_1$, or equivalently, at least one of $T_0$ and $T_1$ is nondegenerate. 

\[\square\]

7.1 Proof of the reduction Theorem 1.2

Suppose that $T := (G, A, B)$ is a nondegenerate triple factorisation of $G$, and that $H$ satisfies $A < H < G$ with $\operatorname{core}_G(A) = 1$. Let $\Omega := \Omega_A = [G : A]$ with $G$ acting by right multiplication. Set $\alpha := A \in \Omega$, $\Delta := \alpha^H$, and let $\Sigma := \{\Delta_1, \ldots, \Delta_\ell\}$ be the block system determined by $H$, where $\Delta = \Delta_1$ and $|G : H| = \ell$. Let $G_0$ and $G_1$ be the permutation groups induced by $H$ on $\Delta$ and $G$ on $\Sigma$, respectively. Let $A_1 = H^\Sigma$, $B_1 = B^\Sigma$, $A_0 = A^\Delta$ and $B_0 = (B \cap H)^\Delta$. By the Embedding Theorem 2.1 we may assume that $\Omega = \Delta \times \Sigma$ and that $G$ is a subgroup of $W = G_0 \wr G_1$ such that $A = \hat{A} \cap G$ where $\hat{A} = (A_0 \times G_0^{\ell-1}) \rtimes A_1$. Moreover, if $B_1$ is transitive, then by Theorem 2.1 we may also assume that $B \leq \hat{B}$ where $\hat{B} = B_0 \wr B_1$. Define

\[
    T_0 := (G_0, A_0, B_0), \quad T_1 := (G_1, A_1, B_1), \quad T_0 \wr T_1 := (W, \hat{A}, \hat{B}) \quad \text{(7.1)}
\]
with \( T_0 \triangleq T_1 \) as in Definition 7.1.

**Lemma 7.4.** Let \( \hat{A} \) and \( \hat{B} \) be as above, and \( \hat{H} := G_0 \triangleleft A_1 \). Then \( \text{core}_{\hat{H}}(\hat{A}) = (1 \times G_0^{\ell-1}) \rtimes A_1 \) and \( \text{core}_{\hat{W}}(\hat{A}) = 1 \).

**Proof.** By the definition of \( G_0 \) and \( G_1 \), it follows that \( \text{core}_{G_0}(A_0) = 1 \) and \( \text{core}_{G_1}(A_1) = 1 \), and hence by Lemma 7.2 \( \text{core}_{\hat{W}}(\hat{A}) = 1 \). Let \( \hat{a} := (a_1, \ldots, a_\ell) \sigma \in \hat{A} \) and \( \hat{h} := h \nu := (h_1, \ldots, h_\ell) \nu \in \hat{H} \), where \( a_1 \in A_0, a_i \in G_0 \) for \( i \geq 2 \), \( h_i \in G_0 \) for \( i \geq 1, \sigma, \nu \in A_1 \). Since \( (h\nu)^{-1} = (h\nu)^{-1}(1) = 1 \sigma = 1 \) and \( 1' = 1 \), we have

\[
\hat{h}^{-1}\hat{a}\hat{h} = ((h\nu)^{-1}1')\hat{a}(h\nu) = [(h_1^{-1}_1, \ldots, h_{\ell'-1}^{-1})\nu^{-1}][(a_1, \ldots, a_\ell)\sigma][h_1, \ldots, h_\ell] \\
= [(h_1^{-1}, h_2^{-1}, \ldots, h_{\ell'-1}^{-1})(a_1, a_2, a_3, \ldots, a_{\ell'-1})] \\
(\nu^{-1}\nu) \\
= (h_1^{-1}a_1, h_2^{-1}a_2, h_3^{-1}a_3, h_4^{-1}a_4, \ldots, h_{\ell'-1}^{-1}a_{\ell'-1})[(\nu^{-1}\nu)].
\]

Therefore, \( (\hat{A})^{\hat{h}} = (A_0^{h_1} \times G_0^{\ell-1}) \rtimes A_1 \), and hence \( \text{core}_{\hat{H}}(\hat{A}) = (\text{core}_{G_0}(A_0) \times G_0^{\ell-1}) \rtimes A_1 = (1 \times G_0^{\ell-1}) \rtimes A_1 \). \( \square \)

**Proposition 7.5.** Suppose that \( T := (G, A, B) \) is a triple factorisation for \( G \) with \( \text{core}_{G}(A) = 1 \), and suppose that \( A < H < G \). Let \( T_0, T_1 \) and \( T_0 \triangleleft T_1 \) be as in (7.1). Then

(a) \( T_0 \) is a triple factorisation, and is nontrivial if and only if \( H \neq \text{core}_{H}(A)(B \cap H) \), and nondegenerate if and only if \( H \nsubseteq AB \).

(b) \( T_1 \) is a triple factorisation, and is the quotient of the lift \( T' := (G, H, B) \) of \( T \) modulo \( \text{core}_{G}(H) \). Moreover, \( T_1 \) is nontrivial if and only if \( G \neq \text{core}_{G}(H)B \), and nondegenerate if and only if \( G \neq HB \).

(c) \( T_0 \triangleleft T_1 \) is a triple factorisation, and is nontrivial if and only if either \( H \neq \text{core}_{H}(A)(B \cap H) \), or \( G \neq \text{core}_{G}(H)B \). Also \( T_0 \triangleleft T_1 \) is nondegenerate if and only if either \( H \nsubseteq AB \), or \( G \neq HB \).

(d) If \( T \) is nondegenerate, then also \( T_0 \triangleleft T_1 \) is nondegenerate.
Proof. (a) Since $A < H$, by Lemma 6.2, $T|_H = (H, A, B \cap H)$ is a nontrivial triple factorisation, and is nondegenerate if and only if $H \not\subseteq AB$. Let $N = \text{core}_H(A)$, so that $T_0 = (T|_H)/N$. It follows from Lemma 5.2(a) that $T_0$ is a triple factorisation for $G_0$, and $T_0$ is nontrivial if and only if the lift $(H, AN, (B \cap H)N)$ of $T|_H$ is nontrivial. Since $N \subset A$, thus latter holds if and only if $(B \cap H)N \neq H$. Moreover, by Observation 1, $T_0$ is nondegenerate if and only if $T|_H$ is nondegenerate, or equivalently, if and only if $H \not\subseteq AB$.

(b) Now let $N = \text{core}_G(H)$ and note that $H(BN) = HB$. Since $A < H$, the triple $T' := (G, H, B)$ is a lift of $T$, and by the definition of a quotient triple factorisation, $T_1 = T'/N$ (and is also a lift of $T/N = (G/N, AN/N, BN/N)$). By Lemma 5.2(a), $T_1$ is a triple factorisation, and is nontrivial if and only if $G \neq BN$. Moreover by Observation 1, $T_1$ is nondegenerate if and only if $T'$ is nondegenerate, that is to say, if and only if $G \neq HB$. As we observed above this is equivalent to $G \neq HB$.

(c) By parts (a) and (b), $T_0$ and $T_1$ are triple factorisations, and so by Lemma 7.3, $T_0 \wr T_1$ is a triple factorisation. Also, by Lemma 7.3, $T_0 \wr T_1$ is nontrivial if and only if at least one of the $T_0$ and $T_1$ is nontrivial and hence, by parts (a) and (b), if and only if either $H \neq \text{core}_H(A)(B \cap H)$, or $G \neq \text{core}_G(H)B$. By Lemma 7.3, $T_0 \wr T_1$ is nondegenerate if and only if at least one of $T_0$ and $T_1$ is nondegenerate and hence, by parts (a) and (b), if and only if either $H \not\subseteq AB$, or $G \neq HB$.

(d) Let $T$ be nondegenerate. By part (c), $T_0 \wr T_1$ is nondegenerate if and only if either $H \not\subseteq AB$, or $G \neq HB$. Suppose that $G = HB$. If also $H \subseteq AB$, then $G = HB \subseteq ABB = AB$, and so $G = AB$, or equivalently, $T$ is degenerate, which is a contradiction. Therefore in this case $H \not\subseteq AB$. Thus at least one of $G \neq HB$ or $H \not\subseteq AB$ holds, and hence by part (c), $T_0 \wr T_1$ is nondegenerate.

We define several subgroups of $W = G_0 \wr G_1$ that correspond to the subgroups $H, \text{core}_H(A)$ and $\text{core}_G(H)$ of $G$ that occur in Proposition 7.5. Recall that $\hat{A} = (A_0 \times G_0^{\ell-1}) \rtimes A_1$ and $\hat{B} = B_0 \wr B_1$.

$$\hat{H} := G_0 \wr A_1, \quad \hat{K} := (1 \times G_0^{\ell-1}) \rtimes A_1, \quad \hat{N} := G_0^\ell. \quad (7.2)$$
Lemma 7.6. Suppose that $\mathcal{T} = (G, A, B)$ is a nondegenerate triple factorisation with $\text{core}_G(A) = 1$, and $A < H < G$. Let $T_0, T_1, T_0 \triangleleft T_1$ and $\ell$ be as in (7.3), and assume that $G \leq W = G_0 \triangleright G_1$ so that $A = \hat{A} \cap G$. Also if $B_1$ is transitive, assume that $B \leq \hat{B} \cap G$. Let $\hat{N}, \hat{H}$ be as in (7.2). Then the following hold.

(a) $\hat{K} = \text{core}_H(\hat{A}), \hat{H}/\hat{K} \cong G_0$, and $\hat{T}'/\hat{K} \cong T_0$, where $\hat{T}' = (T_0 \triangleleft T_1)|_{\hat{H}}$.

(b) $\hat{N} = \text{core}_W(\hat{H})$ and $(T_0 \triangleleft T_1)/\hat{N} \cong T_1$.

(c) If $T_1$ is degenerate, then $T_0$ and $T_0 \triangleleft T_1$ are both nondegenerate and $(T_0 \triangleleft T_1)|_G = (G, A, \hat{B} \cap G)$ is a nondegenerate lift of $\mathcal{T}$.

Proof. (a) By (7.2), we have $\hat{A} < \hat{H}$, and by Lemma 6.3 $(T_0 \triangleleft T_1)|_{\hat{H}} = (\hat{H}, \hat{A}, \hat{B} \cap \hat{H})$ is a nontrivial triple factorisation. Note that $\hat{B} \cap \hat{H} = B_0 \triangleleft (A_1 \cap B_1)$, and also by Lemma 7.4 that $\hat{K} = \text{core}_H(\hat{A})$. Consider the projection map $\varphi : \hat{H} \longrightarrow G_0$ given by $(h_1, \ldots , h_t)\sigma \mapsto h_1$ (for $h_i \in G_0$ and $\sigma \in A_1$). Then $\varphi$ is an epimorphism with kernel $\hat{K}, \varphi(\hat{A}) = A_0, \varphi(\hat{B}) = B_0$, and $\varphi(\hat{H}) \cong \hat{H}/\hat{K}$. Hence $\varphi$ induces an isomorphism from the quotient of $(T_0 \triangleleft T_1)|_{\hat{H}}$ modulo $\hat{K}$.

(b) By (7.2), $(G_0 \triangleleft G_1)/\hat{N} \cong G_1, \hat{A}\hat{N} = G_0 \triangleleft A_1 = \hat{H}$ and $\hat{B}\hat{N} = (B_0 \triangleleft B_1)G_0^\ell = G_0 \triangleleft B_1$. Thus $W/\hat{N} \cong G_1, \hat{A}\hat{N}/\hat{N} \cong A_1$ and $\hat{B}\hat{N}/\hat{N} \cong B_1$, and hence the natural projection map $W \longrightarrow W/\hat{N}$ defines an isomorphism from $(T_0 \triangleleft T_1)/\hat{N}$ to $T_1$.

(c) Suppose that $T_1$ is degenerate, so by Proposition 7.3(b), $G = HB$, or equivalently, $B_1$ is transitive. Thus by our assumption $B \leq \hat{B} \cap G$. Hence the restriction of $T_0 \triangleleft T_1 = (W, \hat{A}, \hat{B})$ to $G$ is $T' := (G, A, \hat{B} \cap G)$, a lift of $\mathcal{T}$. The subset $A(\hat{A} \cap G) \subseteq \hat{A}\hat{B}$, and so by Lemma 7.2 $A(\hat{A} \cap G) \subseteq (A_0B_0 \times G_0^\ell) \times A_1B_1$. We will construct an element $h$ lying in $G$ but not in $(A_0B_0 \times G_0^\ell) \times A_1B_1$, proving that $A(\hat{A} \cap G) \neq G$, and hence that $T'$ is nondegenerate. By Proposition 7.3(d) and Lemma 7.3, and since $T_1$ is degenerate, it follows that $T_0$ and $T_0 \triangleleft T_1$ are both nondegenerate. Thus $G_0 \neq A_0B_0$. Let $g \in G_0 \setminus A_0B_0$. Since $G_0 = H^\Delta, g = h^\Delta$ for some $h \in H = (G_0^\ell \times A_1) \cap G$. This element $h$ is therefore equal to $(g, h_2, \ldots , h_t)\sigma$.
for some $h_2, \ldots, h_\ell \in G_0$ and $\sigma \in A_1$, and therefore $h$ lies in $G$ but not in $(A_0 B_0 \times G_0^{t-1}) \rtimes A_1 B_1$.

**Proof of Theorem 1.2.** Suppose that $T = (G, A, B)$ is a nondegenerate triple factorisation with $\text{core}_G(A) = 1$, that $A < H < G$, and that $G \leq G_0 \wr G_1$ with $G_0$ and $G_1$ as above. Let $T_0$, $T_1$ and $T_0 \wr T_1$ be as in (7.1). By the definitions of $T_0$ and $T_1$, we have that $T_0$ is a quotient of $T|_H$, and $T_1$ is the quotient modulo $\text{core}_G(H)$ of the lift $(G, H, B)$ of $T$. By Proposition 7.5, all three are triple factorisations, and $T_0 \wr T_1$ is nondegenerate. Finally if $T_1$ is degenerate, then $B_1$ is transitive and by Theorem 2.1 we may assume that $B \leq \hat{B}$. Then the assertions of Theorem 1.2(b) all follow from Lemma 7.6.

**Corollary 7.7.** If $B$ is a maximal subgroup of $G$ and $B_1 = B^\Sigma$ is transitive, then $(T_0 \wr T_1)|G = T$.

**Proof.** Since $B_1$ is transitive, by Theorem 1.2(b), $B \leq \hat{B} \cap G$ and $(T_0 \wr T_1)|G = (G, A, D)$ is nondegenerate, for some $D$ such that $B \leq D \leq G$. In particular, $D \neq G$. Then since $B$ is maximal, it follows that $D = B$, so $(T_0 \wr T_1)|G = T$.

**Proof of Corollary 1.3.**

Let $T = (G, A, B)$ be a nondegenerate triple factorisation with $\text{core}_G(A) = 1$.

(a) Suppose first that $A$ is maximal in $G$. Then $G$ acts faithfully and primitively on $\Omega_A$. If $X = \text{core}_G(B) \neq 1$, then since $G$ is primitive, $X$ is transitive on $\Omega_A$. Thus $G = AX \subseteq AB$, contradicting the nondegeneracy of $T$. Hence $X = 1$.

(b) Now let $A = H_1 < H_2 < \ldots < H_r = G$ with $H_i$ maximal in $H_{i+1}$ for $1 \leq i < r$. Note that $G = H_r = BH_r$. Let $j$ be minimal such that $G = BH_j$. Since $G \neq BA$, we have $2 \leq j \leq r$. Let $H = H_j$, $K = H_{j-1}$. By the minimality of $j$, $G \neq BK$. However $G = ABA = KBK$, and hence $T' = (G, K, B)$ is a nondegenerate lift of $T$. By Theorem 1.2 applied to $T'$ modulo $N = \text{core}_G(K)$, we find that $T_0 = (H, K, BN \cap H)$ modulo $M = \text{core}_H(K)$.
is nondegenerate with $K/M$ maximal in $H/M$ and $\text{core}_{H/M}(K/M) = 1$. By part (a) above, $T_0/M$ is primitive. \hfill \square

8 Restriction to transitive normal subgroups

Let $T = (G, A, B)$ be a nondegenerate triple factorisation. By Observation 1 and Theorem 1.2, it is important to study such triple factorisations in which $A$ is maximal and core free in $G$, so that $G$ is faithful and primitive on $\Omega_A = [G : A]$. In this case also $B$ is core-free in $G$ by Corollary 1.3.

The O’Nan-Scott Theorem (see [5, Chapter IV]) describes various types of finite primitive permutation groups $G$ identifying the types by the structure and permutation action of their socles (the socle $\text{Soc}(G)$ of a group $G$ is the product of the minimal normal subgroups of $G$). Thus it is natural to seek conditions under which a primitive triple factorisation $T = (G, A, B)$ restricts to $\text{Soc}(G)$. The examples given for Lemma 6.6 suggest that this may be a difficult problem. In Subsection 8.1, we give a rather technical sufficient condition for restriction. Before that we prove a simple result which explores the role of normal subgroups for primitive $T$.

**Lemma 8.1.** Suppose that $T = (G, A, B)$ is a triple factorisation.

(a) If $1 \neq N \triangleleft G$ and $N$ is transitive on $\Omega_A$, then $T$ is nondegenerate if and only if $N \nsubseteq AB$.

(b) If $\text{core}_G(A) = 1$ and $A$ is maximal in $G$, then $T$ is nondegenerate if and only if the set $AB$ contains no nontrivial normal subgroup of $G$.

**Proof.** Since $N$ is transitive on $\Omega_A$, $G = AN$. Thus $N \subseteq AB$ if and only if $G = AN \subseteq AB$, and the latter holds if and only if $T$ is degenerate. This proves (a) and part (b) follows immediately. \hfill $\square$

Thus a triple factorisation $T = (G, A, B)$ is primitive if and only if $A$ is maximal in $G$ and $AB$ contains no nontrivial normal subgroup of $G$. 24
8.1 Sufficient conditions for restriction to a transitive normal subgroup

Let $G$ be a finite group with subgroups $A$, $B$ and let $N$ be a normal subgroup of $G$ such that $G = AN$.

Let $A_0 = A \cap N$. Then $A = \langle A_0, \sigma_1, \ldots, \sigma_r \rangle$ for some elements $\sigma_i \in A \setminus A_0$. Hence $G = \langle N, \sigma_1, \ldots, \sigma_r \rangle$. Let $A_\sigma := \langle \sigma_1, \ldots, \sigma_r \rangle$. Then $G = AN = A_\sigma N$ and $A = A_0 A_\sigma = A_\sigma A_0$ since $A_0 \leq A$. Moreover, if $B_0 = B \cap N$, then $B = \langle B_0, \tau_1, \ldots, \tau_s \rangle$ for some elements $\tau_j \in B \setminus B_0$. Since $G = A_\sigma N$, each $\tau_j = \delta_j n_j$ for some $\delta_j \in A_\sigma$, $n_j \in N$. Now set $B_\tau := \langle \tau_1, \ldots, \tau_s \rangle$. Then $B = B_0 B_\tau$ because $B_0 \leq B$. We will define a subset $B_\sigma$ of $G$ such that

$$G = ABA \iff G = A_0(B_\sigma A_\sigma)A_0.$$  \hspace{1cm} (8.1)

Note that $B_\sigma A_\sigma$ will not in general be a subgroup.

Suppose that $G = ABA$. If $x \in G$, then $x = abc$, where $a, c \in A$ and $b \in B$. There exist $a_0, c_0 \in A_0$ and $\lambda, \lambda' \in A_\sigma$ such that $a = a_0 \lambda$ and $c = \lambda' c_0$. Then

$$x = abc$$
$$= (a_0 \lambda)b(\lambda' c_0)$$
$$= a_0(\lambda b \lambda^{-1})(\lambda \lambda')c_0$$
$$= a_0 b \lambda^{-1} \lambda'' c_0,$$

where $\lambda'' := \lambda \lambda' \in A_\sigma$. On the other hand, if $x \in A_0 B^\lambda A_\sigma A_0$ for some $\lambda \in A_0$, then $x = a_0 b \lambda^{-1} \lambda'' c_0$, for some $a_0, c_0 \in A_0$, $b \in B$, $\lambda'' \in A_\sigma$ and it follows that $x = abc$, where $a = a_0 \lambda \in A$, $c = \lambda^{-1} \lambda'' c_0 \in A$, that is, $x \in ABA$. Therefore,

$$G = ABA \iff G = \bigcup_{\lambda \in A_\sigma} A_0 B^\lambda A_\sigma A_0.$$ \hspace{1cm} (8.2)

Define $B_\sigma = \cup_{\lambda \in A_\sigma} B^\lambda$. Then (8.1) holds.

Moreover, since $B = B_0 B_\tau$, we have $B_\sigma = \cup_{\lambda \in A_\sigma} B_0^\lambda B_\tau^\lambda$, and also

$$G = ABA \iff G = A_0(\cup_{\lambda \in A_\sigma} B_0^\lambda B_\tau^\lambda)A_\sigma A_0.$$ \hspace{1cm} (8.3)
Lemma 8.2. Let $T = (G, A, B)$ be a triple factorisation for $G$, and let $N$ be a nontrivial normal subgroup of $G$ such that $G = AN$. Suppose furthermore that $A_0$, $A_\sigma$, $B_0$, $B_\tau$ and $B_\sigma$ are as above. Then

(a) $N = A_0(B_\sigma A_\sigma \cap N)A_0$.

(b) If $B_\sigma A_\sigma \cap N \subseteq B_0A_0$, then $T$ restricts to $N$.

(c) If $A_\sigma \subseteq N_G(B_0)$ and $B_\tau \subseteq B_0A$, then $B_\sigma A_\sigma \cap N \subseteq B_0A_0$, so $T$ restricts to $N$.

Proof. (a) By Lemma 8.3, we have $G = A_0(B_\sigma A_\sigma)A_0$. Since $A_0 = A \cap N \subseteq N$, applying Lemma 6.5 with $(G, A, S, H) = (G, A_0, B_\sigma A_\sigma, N)$, we have $N = A_0(B_\sigma A_\sigma \cap N)A_0$ and part (a) follows.

(b) If $B_\sigma A_\sigma \cap N \subseteq B_0A_0$, then by part (a), $N = A_0(B_\sigma A_\sigma \cap N)A_0 \subseteq A_0(B_0A_0)A_0 = A_0B_0A_0 \subseteq N$. Thus $N = A_0B_0A_0$, or equivalently, $T$ restricts to $N$.

(c) Suppose that $A_\sigma \subseteq N_G(B_0)$ and $B_\tau \subseteq B_0A_0$. Then $B_\sigma = \cup_{\lambda \in A_\sigma} (B_0B_\tau)^\lambda = B_0(\cup_{\lambda \in A_\sigma} B_\tau^\lambda)$, so that

$$B_\sigma A_\sigma = B_0(\cup_{\lambda \in A_\sigma} (B_\tau A_\sigma)^\lambda).$$

Since $B_\tau \subseteq B_0A$, we have, for each $\lambda \in A_\sigma$, that $(B_\tau A_\sigma)^\lambda \subseteq (B_0A)^\lambda = B_0A$. So $B_\sigma A_\sigma \subseteq B_0A$, and hence $(B_\sigma A_\sigma) \cap N \subseteq (B_0A) \cap N = B_0(A \cap N) = B_0A_0$. Therefore by part (b), $T$ restricts to $N$. \qed

Using some of the examples from Example 4.5, we illustrate how Lemma 8.2 can be used to prove that triple factorisations restrict to certain normal subgroups.

Example 8.3. Suppose that $G = \text{Sym}(\Omega)$ acting on $\Omega = \{1, \ldots, n\}$ with $n \geq 5$, $N = \text{Alt}(\Omega)$, and $\alpha = n \in \Omega$. Let $A = G_\alpha \cong S_{n-1}$, so that $A_0 := A \cap N = N_\alpha \cong A_{n-1}$, and $A = \langle A_0, \sigma \rangle$, where $\sigma$ is any transposition fixing $\alpha$. As in Example 4.5, let $B = \langle (1, 2), (3, 4, n) \rangle$, so $B_0 := B \cap N = \langle (3, 4, n) \rangle$. To apply Lemma 8.2, we choose $\sigma \in N_G(B_0)$, for example $\sigma = (1, 2)$, so that $A_\sigma = \langle (1, 2) \rangle$. Also if we choose $\tau = (1, 2) = \sigma$, then $B = \langle B_0, \tau \rangle$, $B_\sigma = B$,
and \( B_\tau = \langle \tau \rangle \subseteq B_0 A_\sigma \). Thus by Lemma 8.2(c), \( T \) restricts to \( N \). Note also that \( A_\sigma = \langle \sigma \rangle \cong \mathbb{Z}_2 \) and \( B_\sigma = B \cup B^\sigma = B \).

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**References**

[1] B. Amberg, S. Franciosi, and F. de Giovanni, *Products of groups*, (Oxford University Press, 1992).

[2] M. Bhattacharjee, D. Macpherson, R.G. Möller, and P.M. Neumann, *Notes on infinite permutation groups*, (Hindustan Book Agency, 1997).

[3] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, (Springer-Verlag, 2002).

[4] R. W. Carter, *Simple groups of Lie type*, (John Wiley & Sons Inc., 1989).

[5] J. D. Dixon and B. Mortimer, *Permutation groups*, (Springer-Verlag, 1996).

[6] A. S. Fedenko and A. I. Shtern, Iwasawa decomposition, In *Encyclopaedia of Mathematics*, Edited by Hazewinkel, Michiel., (Kluwer Academic Publishers, 2001). [http://eom.springer.de/I/i053060.htm](http://eom.springer.de/I/i053060.htm).

[7] M. Giudici and J. P. James, Factorisations of groups into three conjugate subgroups, (in preparation).

[8] D. Gorenstein, A class of Frobenius groups, *Canad. J. Math.* 11 (1959), 39–47.

[9] , On finite groups of the form \( ABA \), *Canad. J. Math.* 14 (1962), 195–236.
[10] D. Gorenstein and I. N. Herstein, A class of solvable groups, *Canad. J. Math.* **11** (1959), 311–320.

[11] D. G. Higman and J. E. McLaughlin, Geometric $ABA$-groups, *Illinois J. Math.* **5** (1961), 382–397.

[12] K. Iwasawa, On some types of topological groups, *Ann. of Math. (2)* **50** (1949), 507–558.

[13] W. M. Kantor, Primitive permutation groups of odd degree, and an application to finite projective planes, *J. Algebra* **106** (1987), no. 1, 15–45.

[14] M. W. Liebeck, C. E. Praeger, and J. Saxl, The maximal factorizations of the finite simple groups and their automorphism groups, *Mem. Amer. Math. Soc.* **86** (1990), no. 432, iv+151.

[15] P. M. Neumann and C. E. Praeger, An inequality for tactical configurations, *Bull. London Math. Soc.* **28** (1996), no. 5, 471–475.

[16] C. E. Praeger, Movement and separation of subsets of points under group actions, *J. London Math. Soc. (2)* **56** (1997), no. 3, 519–528.