Approximation and convergence of solutions to semilinear stochastic evolution equations with jumps

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Abstract

We prove that the mild solution to a semilinear stochastic evolution equation on a Hilbert space, driven by either a square integrable martingale or a Poisson random measure, is (jointly) continuous, in a suitable topology, with respect to the initial datum and all coefficients. In particular, if the leading linear operators are maximal (quasi-)monotone and converge in the strong resolvent sense, the drift and diffusion coefficients are uniformly Lipschitz continuous and converge pointwise, and the initial data converge, then the solutions converge.

1 Introduction

Consider the stochastic evolution equation

\[ du(t) + Au(t) \, dt + f(u(t)) \, dt = B(u(t-)) \, dM(t), \quad u(0) = u_0, \tag{1} \]

on a real separable Hilbert space \( H \), where \( A : D(A) \subset H \to H \) is a linear quasi-maximal monotone operator, \( M \) is a Hilbert-space-valued square integrable martingale, and the coefficients \( f, B \) satisfy suitable Lipschitz and linear growth conditions (see below for precise assumptions on all data of the problem). The purpose of this work is to provide sufficient conditions for the (sequential) continuity, in an appropriate topology, of the map \((u_0, A, f, G) \mapsto u\), where \( u \) denotes the mild solution to (1). The same problem is considered also for equations (still with multiplicative noise) driven by compensated Poisson random measures. Our main results are Theorems 2.2 and 2.4 below. The problem we consider, apart of having its own intrinsic interest, is also motivated by several other considerations, such as the study of the stability of models based on stochastic partial differential equations (SPDEs) and the convergence of numerical approximation schemes. Moreover, as is well known, a technique to obtain estimates for mild solutions to SPDEs consists in, first, approximating the unbounded operator \( A \) by a bounded one (such as e.g. the Yosida approximation), so that, roughly speaking, tools from stochastic calculus for semimartingales can be applied to the regularized equation; then, showing

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that the estimates “pass” to the original equation. Such regularization procedure is needed because mild solutions, in general, are not semimartingales, so that tools like Itô’s formula are not directly applicable. Motivated mostly by these considerations, continuous dependence on \( A \) for stochastic convolutions against Hilbert-space-valued Wiener processes was established already in [1] (cf. also [5, Thm. 5.12]), where the authors introduced the by now classical factorization method. Several refinements of this result, all relying on the factorization method, have appeared in the literature, the most sophisticated of which is given in the recent work [13], where stochastic equations on UMD Banach spaces driven by a cylindrical Wiener process are considered. One should also mention the related results due to Gyöngy (see e.g. [8, 9]) for SPDEs in the variational setting driven by finite-dimensional continuous martingales.

If the martingale \( M \) is discontinuous, we are not aware of any results about continuous dependence of the solution on the data (apart of [16], where continuity and (Fréchet) differentiability of the map \( u_0 \mapsto u \) is investigated for equations with Poisson noise). In fact, in the case of jump noise, the factorization method is unfortunately no longer applicable, hence a different approach is needed. The present paper provides such an alternative method, which is in part inspired, perhaps somewhat unexpectedly, by techniques from the theory of \textit{nonlinear} maximal monotone operators on Hilbert spaces (in particular by Brézis’ proof in [2] of a nonlinear version of Trotter-Kato’s theorem). Our method, however, is restricted to operators \( A \) that are quasi-monotone, while the factorization method is not. Therefore, in the context of equations driven by a Wiener process, our method is not a replacement of the “usual” one. Let us also mention that one can find in the literature very satisfactory results on continuous dependence on the coefficients for finite-dimensional stochastic differential equations driven by general semimartingales, see e.g. [7] and [21, pp. 257-ff.]. On the other hand, our continuity results apply to those classes of stochastic evolution equations for which a “decent” well-posedness theory (in the mild sense) is available. In other words, the gap with respect to the finite-dimensional results is mainly due to the less developed well-posedness theory in infinite dimensions.

Before concluding this introductory section with some words about notation, let us give a brief overview of the paper: in Section 2 we state the main results, whose proofs can be found in Section 6. Section 3 collects some facts about (linear) maximal monotone operators on Hilbert spaces and on stochastic integrals (and convolutions) with respect to Hilbert-space-valued square integrable martingales. The core of the paper are Sections 4 and 5 where continuity of stochastic convolutions with respect to the operator \( A \) is established. In particular, first we approximate \( A \) by its Yosida regularization and we prove that the corresponding stochastic convolutions converge. Then we show that the same holds if instead of the Yosida approximation we consider a sequence of maximal quasi-monotone operators converging to \( A \) in the strong resolvent sense. In the last section we briefly comment on the case of equations with additive noise.

**Notation.** Given two normed spaces \( E, F \), we shall denote by \( \dot{C}^{0,1}(E, F) \) the space of Lipschitz continuous functions from \( E \) to \( F \), i.e. the space of functions \( \phi : E \to F \) such that

\[
\|\phi\|_{\dot{C}^{0,1}(E,F)} := \sup_{x \neq y} \frac{\|\phi(x) - \phi(y)\|_F}{\|x - y\|_E} < \infty.
\]

Whenever we write \( \phi \in \dot{C}^{0,1}(E, F) \), it is implicitly assumed, to avoid nonsensical sit-
nations, that there exists $a \in E$ such that $\|\phi(a)\|_F < \infty$. This immediately implies $\|\phi(x)\|_F \leq N(1 + \|x\|_E)$, with $N$ depending only on $\|\phi\|_{C^{0,1}(E,F)}$, $\|a\|_E$, $\|\phi(a)\|_F$. If $E$ and $F$ are complete, the space of linear continuous operators, of trace class, and of Hilbert-Schmidt operators from $E$ to $F$ will be denoted by $\mathcal{L}(E,F)$, $\mathcal{L}_1(E,F)$, and $\mathcal{L}_2(E,F)$, respectively. If $E = F$, we shall simply write $\mathcal{L}(E)$ in place of $\mathcal{L}(E,E)$, and similarly for other spaces. Occasionally we shall drop the indication of the spaces $E$ and $F$ altogether if there is no risk of confusion. We shall write $a \lesssim b$ if there exists a constant $N > 0$ such that $a \leq Nb$. If the constant $N$ depends on parameters $p_1, \ldots, p_n$, we shall also write $N = N(p_1, \ldots, p_n)$ and $a \lesssim_{p_1,\ldots,p_n} b$.

2 Main results

Let $H$ and $K$ two real separable Hilbert spaces. The inner product and norm of $H$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $A : D(A) \subset H \to H$ be a linear (unbounded) maximal quasi-monotone operator, i.e. such that

$$\langle Ax, x \rangle + \eta \|x\|^2 \geq 0 \quad \forall x \in D(A),$$

for some $\eta > 0$, and $R(\lambda I + A) = H$ for all $\lambda > \eta$ (range and domain of operators will be denoted by $R(\cdot)$ and $D(\cdot)$, respectively). The strongly continuous semigroup of quasi-contractions on $H$ generated by $-A$ will be denoted by $S$.

Let $T > 0$ be fixed. All random variables and processes are assumed to be defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, satisfying the “usual” conditions. Statement involving random elements are always meant to hold $\mathbb{P}$-a.s.. The space $L_p(\Omega, H)$, $p > 0$, will be denoted by $\mathbb{L}_p$.

Let $M$ be a $K$-valued square integrable martingale. Further hypotheses on $M$ will be specified when needed. For convenience, we shall say that $M$ satisfies hypothesis $(Q)$ if there exists a deterministic operator $Q \in \mathcal{L}_1(K)$ such that

$$\langle M, M \rangle(t) - \langle M, M \rangle(s) \leq (t - s)Q \quad \forall 0 \leq s \leq t \leq T.$$

Let $(Z, \mathcal{Z}, m)$ be a $\sigma$-finite measure space, and $\mu$ a Poisson random measure on $Z \times [0, T]$ with compensator $m \otimes \text{Leb}$, where Leb stands for the Lebesgue measure on $[0, T]$. The compensated measure $\mu - m \otimes \text{Leb}$ will be denoted by $\tilde{\mu}$. We shall denote the space of functions $\phi : Z \to H$ such that $\|\phi\|_H \in L_p(Z, m)$, $p \geq 2$, by $L_p(Z)$.

For any $0 < t \leq T$, $\mathbb{H}_p(t)$ stands for the Banach space of càdlàg adapted processes $u : \Omega \times [0, t] \to H$ such that

$$\|u\|_{\mathbb{H}_p(t)} := \left(\mathbb{E} \sup_{s \leq t} \|u(s)\|^p\right)^{1/p} < \infty.$$

We shall write $\mathbb{H}_p$ instead of $\mathbb{H}_p(T)$.

Let us consider the equations

$$du(t) + Au(t) \, dt + f(u(t)) \, dt = B(u(t-)) \, dM(t), \quad u(0) = u_0, \quad (2)$$

and, for each $n \in \mathbb{N}$,

$$du_n(t) + A_n u_n(t) \, dt + f_n(u_n(t)) \, dt = B_n(u(t-)) \, dM(t), \quad u(0) = u_0, \quad (3)$$
One has the following well-posedness result in $\mathbb{H}_2$. A proof (of a more general result) can be found for instance in [11].

**Theorem 2.1.** Assume that $M$ satisfies hypothesis (Q) and $u_0 \in L_2(H)$. If $f \in \dot{C}^{0,1}(H)$ and $B \in \dot{C}^{0,1}(H, L_2(Q^{1/2}K, H))$, then (2) admits a unique mild solution $u \in \mathbb{H}_2$, which depends continuously on the initial datum $u_0$.

Clearly, if, for each $n \in \mathbb{N}$, $(u_{0n}, f_n, B_n)$ satisfy the same type of assumptions, then (4) is also well-posed in $\mathbb{H}_2$.

Our first main result, whose proof is postponed to §6.1 is the following.

**Theorem 2.2.** Assume that $M$ satisfies hypothesis (Q). Moreover, assume that

(i) for each $n \in \mathbb{N}$, there exists $\eta_n \leq \eta$ such that $A_n + \eta_n I$ is a linear maximal monotone operator on $H$, and there exists $\lambda_0 > 0$ such that $(I + \lambda A_n)^{-1}h \rightarrow (I + \lambda A)^{-1}h$ as $n \rightarrow \infty$ for all $h \in H$ and $0 < \lambda < \lambda_0$;

(ii) there exists a constant $L_f > 0$ such that $f_n \in \dot{C}^{0,1}(H)$ with $\|f_n\|_{C^{0,1}} + \|f\|_{C^{0,1}} \leq L_f$ for all $n \in \mathbb{N}$ and $f_n \rightarrow f$ pointwise as $n \rightarrow \infty$;

(iii) there exists a constant $L_B > 0$ such that $B_n \in \dot{C}^{0,1}(H, L_2(Q^{1/2}K, H))$ with $\|B_n\|_{C^{0,1}} + \|B\|_{C^{0,1}} \leq L_B$ for all $n \in \mathbb{N}$ and $B_n \rightarrow B$ pointwise as $n \rightarrow \infty$;

(iv) $u_{0n} \in L_2$ for all $n \in \mathbb{N}$ and $u_{0n} \rightarrow u_0$ in $L_2$ as $n \rightarrow \infty$.

Let $u$ and $u_n$ be the mild solutions to (2) and (4), respectively. Then $u_n \rightarrow u$ in $\mathbb{H}_2$ as $n \rightarrow \infty$, that is

$$\lim_{n \rightarrow \infty} \mathbb{E} \sup_{t \leq T} \|u_n(t) - u(t)\|^2 = 0.$$

**Remark 2.3.** (a) The type of convergence of $A_n$ to $A$ assumed in (i) is also called convergence in the strong resolvent sense.

(b) Hypothesis (Q) is satisfied, for instance, if $M$ has stationary independent increments (in particular it $M$ is a Lévy processes without drift, see e.g. [20], p. 69]). One may remove this assumption at the price of assuming that $B$ satisfies a “random” Lipschitz condition, i.e. a condition involving a predictable $L_1$-valued process rather than the (deterministic, time-independent) operator $Q$. Similarly, it would be possible to give a convergence result in $\mathbb{H}_p$, assuming that $B$ satisfies a different “random” Lipschitz condition involving the quadratic variation of $M$. We believe that these conditions are in general too difficult to check, and the corresponding results are of limited interest.

(c) One could allow the coefficients $f$ and $B$ to depend also on $\omega \in \Omega$ and $t \in [0, T]$, assuming that they satisfy suitable measurability conditions and that their Lipschitz constants (with respect to the $H$-valued variable) do not depend on $(\omega, t)$. Details are left to the interested reader.

We now turn to the case of equations driven by compensated Poisson random measures. Consider the equations

$$du(t) + Au(t)\,dt + f(u(t))\,dt = \int_Z G(z, u(t-))\,\mu(dz, dt), \quad u(0) = u_0, \quad (4)$$
and, for each \( n \in \mathbb{N} \),
\[
du_n(t) + A_n u_n(t) \, dt + f_n(u_n(t)) \, dt = \int_Z G_n(z, u_n(t-)) \, d\mu(dz, dt), \quad u(0) = u_0_n. \tag{5}
\]
Recall that (see [16]) if \( f \in \dot{C}^{0,1}(H) \) and
\[
\left( \int_Z \|G(z, u) - G(z, v)\|^2 \, m(dz) \right)^{p/2} + \int_Z \|G(z, u) - G(z, v)\|^p \, m(dz) \lesssim \|u - v\|^p
\]
for all \( u, v \in H \), then (4) is well-posed in \( \mathbb{H}_p \). A completely analogous statement obviously holds for (5). Observing that
\[
\|\Phi\|^p_{L^2(Z)} + \|\Phi\|^p_{L^p(Z)} \lesssim_p \max \left( \|\Phi\|^p_{L^2(Z)}, \|\Phi\|^p_{L^p(Z)} \right) \lesssim_p \|\Phi\|^p_{L^2(Z)} + \|\Phi\|^p_{L^p(Z)}
\]
and recalling that one can turn the intersection of \( L^2(Z) \) with \( L^p(Z) \) into a Banach space with the norm
\[
\|\cdot\|_{L^2(Z) \cap L^p(Z)} := \max \left( \|\cdot\|_{L^2(Z)}, \|\cdot\|_{L^p(Z)} \right)
\]
(see e.g. [12] p. 9), the above Lipschitz condition for \( G \) can be equivalently formulated as \( h \mapsto G(\cdot, h) \in \dot{C}^{0,1}(H, L^2(Z) \cap L^p(Z)) \).

Our second main result is the following.

**Theorem 2.4.** Let \( p \in [2, \infty] \). Assume that \( A_n \) and \( f_n \), \( n \in \mathbb{N} \), satisfy hypotheses (i) and (ii) of the previous theorem, and that
\[
(iii') \text{ there exists a constant } L_G > 0 \text{ such that } h \mapsto G_n(\cdot, h) \in \dot{C}^{0,1}(H, L^2(Z) \cap L^p(Z)) \text{ with } \|h \mapsto G_n(\cdot, h)\|_{\dot{C}^{0,1}} + \|h \mapsto G(\cdot, h)\|_{\dot{C}^{0,1}} \leq L_G \text{ for all } n \in \mathbb{N} \text{ and }
\]
\[
\|G_n(\cdot, h) - G(\cdot, h)\|_{L^2(Z) \cap L^p(Z)} \xrightarrow{n \to \infty} 0 \quad \forall h \in H;
\]
and
\[
(iv') \text{ } u_{0n} \in L^p \text{ for all } n \in \mathbb{N} \text{ and } u_{0n} \to u_0 \text{ in } \mathbb{H}_p \text{ as } n \to \infty.
\]
Let \( u \) and \( u_n \) be the mild solutions to (4) and (5), respectively. Then \( u_n \to u \) in \( \mathbb{H}_p \) as \( n \to \infty \), that is
\[
\lim_{n \to \infty} \mathbb{E} \sup_{t \leq T} \|u_n(t) - u(t)\|^p = 0.
\]

3 Preliminaries

3.1 Linear maximal monotone operators

We are going to recall some definitions and (known) facts about linear maximal (quasi-)monotone operators on Hilbert spaces, referring e.g. to [3] [19] for details.

A linear operator \( A : D(A) \subset H \to H \) is called maximal monotone if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in D(A) \) and \( R(I + \lambda A) = H \) for all \( \lambda > 0 \). An operator \( A \) is called maximal \( \eta \)-monotone if \( A + \eta I \) is maximal monotone. Let \( A \) be maximal \( \eta \)-monotone on \( H \) and, for \( 0 < \lambda < 1/\eta \), let \( J_\lambda := (I + \lambda A)^{-1} \) and \( A_\lambda := \lambda^{-1}(I - J_\lambda) \) (the latter operator is the so-called Yosida regularization, or approximation, of \( A \)). Then
\[
(i) \quad J_\lambda \in \mathcal{L}(H) \text{ for all } 0 < \lambda < 1/\eta \text{ with } \|J_\lambda x\| \leq (1 - \lambda \eta)^{-1} \|x\| \text{ for all } x \in H;
\]
\[
A_{\lambda} \in L(H) \text{ for all } 0 < \lambda < 1/\eta \text{ with } \|A_{\lambda}x\| \leq (1 - \lambda \eta)^{-1}\|Ax\| \text{ for all } x \in D(A);
\]
\[
A_{\lambda}x = AJ_{\lambda}x \text{ for all } x \in H;
\]
\[
J_{\lambda}x \to x \text{ as } \lambda \to 0 \text{ for all } x \in H. \text{ In particular, by (iii), } A_{\lambda}x \to Ax \text{ as } \lambda \to 0 \text{ for all } x \in D(A).
\]

It should be noted that the above properties of the resolvent \(J_{\lambda}\) and of the Yosida approximation \(A_{\lambda}\) continue to hold, mutatis mutandis, for the much more general class of nonlinear (quasi-)\(m\)-accretive operators on Banach spaces (see e.g. [1]).

We shall need for the proofs of the main results the following inhomogeneous version of the Trotter-Kato’s theorem.

**Theorem 3.1.** Let \(A\) and \(A_n\), \(n \in \mathbb{N}\), be maximal monotone operators on \(H\); \(f\) and \(f_n\), \(n \in \mathbb{N}\), be elements of \(L_1([0,T], H)\); \(u_0\) and \(u_{0n}\), \(n \in \mathbb{N}\), be elements of \(H\). Let \(u\) and \(u_n\) denote the mild solutions to the equations

\[
\begin{align*}
u' + Au &= f, & u(0) &= u_0, \\
u_n' + A_nu_n &= f_n, & u_n(0) &= u_{0n},
\end{align*}
\]

respectively. Suppose that, as \(n \to \infty\), \(A_n \to A\) in the strong resolvent sense, \(u_{0n} \to u\) in \(H\) and \(f_n \to f\) in \(L^1([0,T], H)\). Then

\[
\lim_{n \to \infty} \sup_{t \leq T} \|u_n(t) - u(t)\| = 0.
\]

**Proof.** See e.g. [1, p. 241] for a proof (of a much more general result) that uses the theory of \(m\)-accretive operators on Banach spaces, or [13] for a “linear” proof using the factorization method.

\(\square\)

### 3.2 Stochastic integration with jumps and maximal inequalities

We shall use the theory of stochastic integration with respect to Hilbert space-valued martingales, about which we refer to [18] for a detailed treatment. Here we shall essentially limit ourselves to fixing notation.

For a \(K\)-valued square integrable martingale \(M\), let \(Q_M\) be the unique \(L_1(K)\)-valued predictable process \(Q\) such that

\[
\langle \langle M, M \rangle \rangle(t) = \int_0^t Q_M(s) \, d\langle M, M \rangle(s).
\]

We shall denote by \(\Lambda^2_t(K, H)\) the closure of the space of \(L(K, H)\)-valued simple process in the space of processes \(\Phi\) whose values are linear (possibly unbounded) operator from \(K\) to \(H\) such that \(\Phi(t)Q_M^{1/2}(t) \in L_2(K, H)\) for all \((\omega, t) \in \Omega \times [0, T]\), \(\Phi Q_M^{1/2} h\) is predictable for all \(h \in H\), and

\[
\mathbb{E} \int_0^T \|\Phi(t)Q_M^{1/2}(t)\|_{L_2(K, H)}^2 \, d\langle M, M \rangle(t) < \infty.
\]
For any $\Phi \in \Lambda^2_M(K,H)$, the stochastic integral $\Phi \cdot M$ is an $H$-valued square integrable martingale with $\langle \Phi \cdot M, \Phi \cdot M \rangle = \|\Phi Q_{M}^{1/2}\|_{L^2}^2 \cdot \langle M, M \rangle$. Note that, if $M$ satisfies the (Q) hypothesis, then
\[
\mathbb{E} \int_0^T \|\Phi(t)Q_{M}^{1/2}(t)\|^2_{L^2(K,H)} d\langle M, M \rangle(t) \leq \mathbb{E} \int_0^T \|\Phi(t)\|^2_{L^2(K,H)} dt.
\]

In the following proposition we collect some (known) maximal inequalities for stochastic convolutions driven by martingales, of which we sketch a proof for the reader’s convenience. More details can be found e.g. in [10].

**Proposition 3.2.** Let $B$ be a process taking values in the space of linear (not necessarily bounded) operators from $K$ to $H$, and set
\[
Y(t) := \int_0^t S(t-s)B(s) dM(s), \quad 0 \leq t \leq T.
\]
The following holds true:

(i) If $B \in \Lambda^2_f(K,H)$, then $Y \in \mathbb{H}_2$ and
\[
\|Y\|_{\mathbb{H}_2}^2 \equiv \mathbb{E} \sup_{t \leq T} \|Y(t)\|^2 \lesssim_{\eta} \mathbb{E} \int_0^T \|B(t)Q_{M}^{1/2}(t)\|^2_{L^2(K,H)} d\langle M, M \rangle(t); \quad (6)
\]

(ii) If $B : [0,T] \times \Omega \to L^2(K,H)$ is predictable and there exists $p \in [2,\infty]$ such that the right-hand side of (7) below is finite, then $Y \in \mathbb{H}_p$ and
\[
\|Y\|_{\mathbb{H}_p}^p \equiv \mathbb{E} \sup_{t \leq T} \|Y(t)\|^p \lesssim_{\eta} \mathbb{E} \left( \int_0^T \|B(t)\|^2_{L^2(K,H)} d\langle M, M \rangle(t) \right)^{p/2}. \quad (7)
\]

**Proof.** Let $S^0(t) := e^{-\eta t}S(t)$, $t \geq 0$. Then $S^0$ is a strongly continuous contraction semigroup, and, by Sz.-Nagy’s dilation theorem, there exist a (separable) Hilbert space $\tilde{H} \supset H$ and a unitary strongly continuous group $(U(t))_{t \in \mathbb{R}}$ on $\tilde{H}$ such that
\[
S^0(t) = \pi \circ U(t) \circ i \quad \forall t \geq 0,
\]
where $i : H \to \tilde{H}$ is an isometric embedding and $\pi : \tilde{H} \to H$ is an orthogonal projection. We thus have
\[
\mathbb{E} \left\| \int_0^t S(t-s)B(s) dM(s) \right\|^2 \leq e^{2\eta T} \mathbb{E} \left\| \int_0^t S^0(t-s)B(s) dM(s) \right\|^2 \\
\lesssim_{\eta} \mathbb{E} \left\| U(t) \int_0^t U(-s)B(s) dM(s) \right\|^2 \leq \mathbb{E} \left\| \int_0^t U(-s)B(s) dM(s) \right\|^2_H \\
\leq \mathbb{E} \int_0^t \|B(s)\|^2_{Q_M} d\langle M, M \rangle(s).
\]
Then (6) follows by Doob’s inequality for real-valued submartingales. The proof of (7) is completely analogous: it follows from Burkholder’s inequality, rather than from the isometric property of the stochastic integral with respect to $M$, taking into account the easy estimate $[B \cdot M, B \cdot M] \leq \|B\|_2^2 \cdot [M, M]$.

\[ \square \]
Remark 3.3. Unfortunately it is not possible to replace the operator norm of $B$ in (7) with the Hilbert-Schmidt norm of $BQ_{M}^{1/2}$, cf. e.g. [10] for a brief discussion of this issue.

We shall also need a maximal inequality for stochastic convolution with respect to compensated Poisson random measures obtained in [16]. Here $\mathcal{P}$ stands for the predictable $\sigma$-field.

**Proposition 3.4** ([16], Prop. 3.3). Assume that $G : \Omega \times [0,T] \times Z \to H$ is $\mathcal{P} \otimes Z$-measurable and there exists $p \in [2,\infty]$ such that the right-hand side in (8) below is finite. Then, setting

$$Y(t) := \int_{0}^{t} \int_{Z} S(t-s)G(s,z) \bar{\mu}(ds,dz), \quad 0 \leq t \leq T,$$

one has $Y \in \mathbb{H}_{p}$ and

$$\|Y\|_{\mathbb{H}_{p}} \equiv \mathbb{E}\sup_{t \leq T} \|Y(t)\|^{p} \lesssim_{p,n} \mathbb{E} \int_{0}^{T} \left[ \int_{Z} \|G(t,z)\|^{p} m(dz) + \left( \int_{Z} \|G(t,z)\|^{2} m(dz) \right)^{p/2} \right] dt. \quad (8)$$

Note that inequalities (6), (7) and (8) can equivalently be written as

$$\|Y\|_{\mathbb{H}_{2}} \lesssim \left( \|BQ_{M}^{1/2}\|_{L_{2}} \cdot \langle M, M \rangle \right)^{1/2}, \quad (6')$$

$$\|Y\|_{\mathbb{H}_{p}} \lesssim \left( \|B\|_{L_{p}} \cdot \langle M, M \rangle^{1/2} \right)^{1/2}, \quad (7')$$

$$\|Y\|_{\mathbb{H}_{p}} \lesssim \|G\|_{L_{p}(\Omega \times [0,T],L_{2}(Z))}, \quad (8')$$

**Remark 3.5.** (i) A corresponding inequality for stochastic integrals and convolutions with respect to Lévy processes was established in [15]. An analogous estimate holds if the Hilbert space $H$ is replaced by an $L_{q}$ space (see [17] for a basic result, and [6] for far-reaching generalizations).

(ii) The maximal estimates of the previous two propositions continue to hold in the case that $A$ has a bounded $H^{\infty}$-calculus of angle less than $\pi/2$. In fact, exactly the same proofs go through, using a different (and more sophisticated) dilation theorem, cf. e.g. [22] for the case of stochastic convolutions in UMD Banach spaces of type 2 with respect to a Wiener process. In the context of Hilbert spaces, however, the classes of quasi-monotone operators and of operators with bounded $H^{\infty}$-calculus mentioned above essentially coincide (see [14] for a precise result).

### 4 Convergence of stochastic convolutions I

Throughout this and the following section we assume that $\eta = 0$, in particular that $A$ is maximal monotone, rather than just maximal quasi-monotone. That this comes at no loss of generality is showed in Remark 7.2 below.

Let us consider the linear stochastic evolution equation on $H$

$$dy(t) + Ay(t) dt = B(t) dM(t), \quad y(0) = y_{0}, \quad (9)$$
whose mild solution can be written, formally for the time being, as

\[ y(t) = S(t)y_0 + \int_0^t S(t-s)B(s)\,dM(s). \]

It is immediate that \( y_0 \in L_2, B \in \Lambda^2_M \) imply \( y \in H_2 \), and that, for any \( p \in [2, \infty[, y_0 \in L_p, B \) satisfying the hypotheses of Proposition \( 5.2 \) ii) imply \( y \in H_p \).

In the first of the following two subsections we establish convergence to \( y \), in \( H_2 \) and in \( H_p \), of the solutions to the equations obtained replacing \( A \) in \( (9) \) with its Yosida regularization. In the second subsection we consider, more generally, the equations obtained replacing \( A \) by \( A_n \), with \( A_n \) converging to \( A \) in the strong resolvent sense.

### 4.1 Yosida approximation of \( A \)

Let \( A_{\lambda}, \lambda > 0 \), be the Yosida approximation of \( A \), and consider the regularized equation

\[ dy_{\lambda}(t) + A_{\lambda}y_{\lambda} \, dt = B(t) \, dM(t), \quad y_{\lambda}(0) = y_0, \tag{10} \]

whose mild solution can be written, formally for the time being, as

\[ y_{\lambda}(t) = e^{-tA_{\lambda}}y_0 + \int_0^t e^{-(t-s)A_{\lambda}}B(s)\,dM(s). \]

In analogy to the case of equation \( (9) \), \( y_0 \in L_2 \) and \( B \in \Lambda^2_M \) imply that \( y_{\lambda} \in H_2 \), while \( y_0 \in L_p \) and \( B \) predictable with \( (\|B\|_2^2 \cdot [M, M])^{1/2} \in L_p \) imply that \( y_{\lambda} \in H_p \).

We start with an elementary convergence result which will be needed in the proof of Theorem \( 4.2 \) below.

**Lemma 4.1.** Assume that \( y_0 \in L_2 \) and \( B \) satisfies assumption (i) or (ii) of Proposition \( 5.2 \). Let \( y \) and \( y_{\lambda} \in H_2 \) be the solutions to \( (9) \) and \( (10) \), respectively. Then one has \( y_{\lambda}(t) \to y(t) \) in \( L_2 \) for all \( t \in [0, T] \) as \( \lambda \to 0 \), i.e.

\[ \lim_{\lambda \to 0} \mathbb{E}\|y_{\lambda}(t) - y(t)\|^2 = 0 \quad \forall t \in [0, T]. \]

**Proof.** One has

\[ \mathbb{E}\|y_{\lambda}(t) - y(t)\|^2 \leq \mathbb{E}\|e^{-tA_{\lambda}}y_0 - S(t)y_0\|^2 + \mathbb{E}\left\|\int_0^t (e^{-(t-s)A_{\lambda}}B(s) - S(t-s)B(s))\,dM(s)\right\|^2. \tag{11} \]

Let us assume first that \( B \in \Lambda^2_M(K, H) \). Recall that, by the Trotter-Kato’s theorem (see e.g. [19, p. 88]), one has \( e^{-tA_{\lambda}}h \to S(t)h \) as \( \lambda \to 0 \) for all \( t \in [0, T] \) and all \( h \in H \). By the isometric property of the stochastic integral with respect to \( M \), the second term on the right-hand side of the above inequality is equal to

\[ \mathbb{E}\int_0^t \left\|e^{-(t-s)A_{\lambda}}B(s) - S(t-s)B(s)\right\|^2 \left\|Q_M(s)^{1/2}Q_{(M)}(s)^{1/2}\right\|^2 d\langle M, M \rangle(s) \]

\[ = \mathbb{E}\left[\sum_{j=1}^{\infty} \left\|e^{-(t-s)A_{\lambda}}B(s)Q_{(M)}^{1/2}(s)e_j - S(t-s)B(s)Q_{(M)}^{1/2}(s)e_j\right\|^2 d\langle M, M \rangle(s), \right. \]

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where \((e_j)_{j \in \mathbb{N}}\) is an orthonormal basis of \(K\). Then one has
\[
\lim_{\lambda \to 0} \left\| e^{-(t-s)A_\lambda} B(s)Q_M^{1/2}(s) e_j - S(t-s)B(s)Q_M^{1/2}(s) e_j \right\|^2 = 0
\]
for all \(s \leq t\) and for all \(j \in \mathbb{N}\). Since the operator norms of \(S(t)\) and \(e^{-tA_\lambda}\) are not larger than one for all \(t \in [0, T]\), one also has
\[
\left\| e^{-(t-s)A_\lambda} B(s)Q_M^{1/2}(s) e_j - S(t-s)B(s)Q_M^{1/2}(s) e_j \right\|^2 \lesssim \left\| B(s)Q_M^{1/2}(s) e_j \right\|^2
\]
for all \(0 \leq s \leq t\), and
\[
\mathbb{E} \int_0^t \sum_{j=1}^{\infty} \left\| B(s)Q_M^{1/2}(s) e_j \right\|^2 d\langle M, M \rangle(s)
= \mathbb{E} \int_0^t \left\| B(s)Q_M^{1/2}(s) \right\|^2_{\mathcal{L}_2(K, H)} d\langle M, M \rangle(s) < \infty.
\]
The dominated convergence theorem then implies that the second term on the right-hand side of (11) tends to zero as \(\lambda \to 0\). A completely analogous (but simpler) argument shows that the same is true for the first term on the r.h.s. of (11).

If \(B\) satisfies the assumptions of Proposition 3.2(ii) for some \(p \geq 2\), it certainly does for \(p = 2\), in which case we have
\[
\mathbb{E} (\| BQ_M^{1/2} \|_{\mathcal{L}_2}^2 \cdot \langle M, M \rangle) \leq \mathbb{E} (\| B \|_2^2 \cdot Q_M^{1/2}) \cdot \langle M, M \rangle)
\leq \mathbb{E} (\| B \|_2^2 \cdot \langle M, M \rangle) = \mathbb{E} (\| B \|_2^2 \cdot [M, M] < \infty,
\]
where we have used the ideal property of the space of Hilbert-Schmidt operators, the identity \(\| Q_M^{1/2} \|_{\mathcal{L}_2}^2 = \| Q_M \|_{\mathcal{L}_1}\), and the fact that \(\text{Tr} Q_M \leq 1\). We have thus shown that \(B \in \Lambda^2_M\), a condition which has already been proved to imply the claim. \(\square\)

**Theorem 4.2.** Assume that \(y_0 \in \mathbb{L}_2\) and \(B \in \Lambda^2_M(K, H)\). Let \(y\) and \(y_\lambda \in \mathbb{H}_2\) be the solutions to (9) and (10), respectively. Then one has \(y_\lambda \to y\) in \(\mathbb{H}_2\) as \(\lambda \to 0\), i.e.
\[
\lim_{\lambda \to 0} \mathbb{E} \sup_{t \leq T} \left\| y_\lambda(t) - y(t) \right\|^2 = 0.
\]

**Proof.** Let us introduce two auxiliary regularized equations as follows:
\[
dy^\varepsilon(t) + Ay^\varepsilon(t) dt = B^\varepsilon(t) dM, \quad y^\varepsilon(0) = y_0^\varepsilon, \quad (13)
\]
\[
dy^\lambda_\varepsilon(t) + A_\lambda y^\lambda_\varepsilon(t) dt = B^\varepsilon(t) dM, \quad y^\lambda_\varepsilon(0) = y_0^\varepsilon, \quad (14)
\]
with \(y_0^\varepsilon := (I + \varepsilon A)^{-1} y_0\), \(B^\varepsilon := (I + \varepsilon A)^{-1} B\), for \(\varepsilon > 0\). The triangle inequality yields
\[
\| y - y_\lambda \|_{\mathbb{H}_2} \leq \| y - y^\varepsilon \|_{\mathbb{H}_2} + \| y^\varepsilon - y^\lambda_\varepsilon \|_{\mathbb{H}_2} + \| y^\lambda_\varepsilon - y_\lambda \|_{\mathbb{H}_2}.
\]
(15)
By Proposition 3.2(ii) one gets
\[
\mathbb{E} \sup_{t \leq T} \left\| y(t) - y^\varepsilon(t) \right\|^2 \lesssim \mathbb{E} \left\| y_0 - y_0^\varepsilon \right\|^2 + \mathbb{E} \int_0^T \left\| (B(s) - B^\varepsilon(s))Q_M^{1/2}(s) \right\|^2_{\mathcal{L}_2} d\langle M, M \rangle(s),
\]
\[
\mathbb{E} \sup_{t \leq T} \left\| y_\lambda(t) - y^\lambda_\varepsilon(t) \right\|^2 \lesssim \mathbb{E} \left\| y_0 - y_0^\varepsilon \right\|^2 + \mathbb{E} \int_0^T \left\| (B(s) - B^\varepsilon(s))Q_M^{1/2}(s) \right\|^2_{\mathcal{L}_2} d\langle M, M \rangle(s).
\]
Since \((I + \lambda A)^{-1}\) is contracting, the dominated convergence theorem implies that the right-hand sides of the above inequalities converge to zero as \(\varepsilon \to 0\). Let us fix \(\delta > 0\). Then there exists \(\varepsilon > 0\) such that

\[
\|y - y^\varepsilon\|_{H^2} + \|y^\varepsilon - y_\lambda\|_{H^2} < \frac{1}{2}\delta
\]

for all \(\lambda > 0\). We shall keep \(\varepsilon\) fixed from now on. In order to conclude the proof we have to show that \(\|y^\varepsilon - y^\varepsilon_\lambda\|_{H^2} < \delta/2\) for \(\lambda\) small enough. To this purpose note that, for any \(\lambda > 0\), \(y^\varepsilon_\lambda\) is a strong solution (not just a mild solution) to \((A)\) because \(A\lambda\) is a bounded operator. Therefore, for any \(\lambda, \mu > 0\), we infer that \(y^\varepsilon_\lambda - y^\varepsilon_\mu\) is a strong solution to the deterministic evolution equation

\[
(y^\varepsilon_\lambda - y^\varepsilon_\mu)' + A\lambda y^\varepsilon_\lambda - A\mu y^\varepsilon_\mu = 0, \quad y^\varepsilon_\lambda(0) - y^\varepsilon_\mu(0) = 0.
\]

Taking the scalar product of both sides with \(y^\varepsilon_\lambda - y^\varepsilon_\mu\), one has

\[
\frac{1}{2} \frac{d}{dt}\|y^\varepsilon_\lambda - y^\varepsilon_\mu\|^2 + \langle A\lambda y^\varepsilon_\lambda - A\mu y^\varepsilon_\mu, y^\varepsilon_\lambda - y^\varepsilon_\mu \rangle = 0.
\]

Recalling the identity \(\lambda A\lambda = I - J_\lambda\), one has

\[
y^\varepsilon_\lambda - y^\varepsilon_\mu = (J_\lambda y^\varepsilon_\lambda - J_\mu y^\varepsilon_\mu) + (y^\varepsilon_\lambda - J_\lambda y^\varepsilon_\lambda) - (y^\varepsilon_\mu - J_\mu y^\varepsilon_\mu).
\]

This yields, thanks to the identity \(A\lambda = AJ_\lambda\),

\[
\langle A\lambda y^\varepsilon_\lambda - A\mu y^\varepsilon_\mu, y^\varepsilon_\lambda - y^\varepsilon_\mu \rangle = \langle AJ_\lambda y^\varepsilon_\lambda - AJ_\mu y^\varepsilon_\mu, J_\lambda y^\varepsilon_\lambda - J_\mu y^\varepsilon_\mu \rangle
\]

\[
+ \langle A\lambda y^\varepsilon_\lambda - A\mu y^\varepsilon_\mu, \lambda A\lambda y^\varepsilon_\lambda - \mu A\lambda y^\varepsilon_\mu \rangle
\]

\[
\geq \langle A\lambda y^\varepsilon_\lambda - A\mu y^\varepsilon_\mu, \lambda A\lambda y^\varepsilon_\lambda - \mu A\lambda y^\varepsilon_\mu \rangle,
\]

thus also

\[
\|y^\varepsilon_\lambda - y^\varepsilon_\mu\|^2(t) \leq (\lambda + \mu) \int_0^t (\|AJ_\lambda y^\varepsilon_\lambda(s)\|^2 + \|A\mu y^\varepsilon_\mu(s)\|^2)^2 ds,
\]

and

\[
\mathbb{E}\sup_{t \leq T}\|y^\varepsilon_\lambda - y^\varepsilon_\mu\|^2(t) \leq (\lambda + \mu)\mathbb{E}\int_0^T (\|AJ_\lambda y^\varepsilon_\lambda(s)\|^2 + \|A\mu y^\varepsilon_\mu(s)\|^2)^2 ds.
\]

Since it holds

\[
y^\varepsilon_\lambda(s) = e^{-tA\lambda}y^\varepsilon_0 + \int_0^t e^{-(t-s)A\lambda}B^\varepsilon(s) dM(s),
\]

recalling that \(\|A\lambda x\| \leq \|Ax\|\) for all \(x \in D(A)\), one has

\[
\mathbb{E}\|A\lambda y^\varepsilon_\lambda(s)\|^2 \leq \mathbb{E}\|e^{-sA\lambda}A\lambda y^\varepsilon_0\|^2 + \mathbb{E}\int_0^s \|e^{-(s-r)A\lambda}A\lambda B^\varepsilon(r) dM(r)\|^2
\]

\[
\leq \mathbb{E}\|A\lambda y^\varepsilon_0\|^2 + \mathbb{E}\int_0^s \|AB^\varepsilon(r)Q_M^{1/2}(r)\|_{L^2}^2 d(M, M)(r),
\]

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which implies, taking into account that $A(I + \epsilon A)^{-1} = A_\epsilon$ is a bounded operator,

$$
\mathbb{E} \sup_{t \leq T} \|y_\lambda^\epsilon - y_\mu^\epsilon\|^2(t) \lesssim T(\lambda + \mu) \left( \mathbb{E}\|Ay_0\|^2 + \mathbb{E} \int_0^T \|AB^\epsilon(r)Q_M^{1/2}(r)\|^2_{L^2} \, d\langle M, M \rangle(r) \right)
$$

\[ \lesssim \varepsilon T(\lambda + \mu) \left( \mathbb{E}\|y_0\|^2 + \mathbb{E} \int_0^T \|B(r)Q_M^{1/2}(s)\|^2_{L^2} \, d\langle M, M \rangle(r) \right), \]

i.e. $\lambda \mapsto y_\lambda^\epsilon$ is a Cauchy net in $\mathbb{H}_2$. In particular, there exist $z^\epsilon$ such that $y_\lambda^\epsilon \to z^\epsilon$ in $\mathbb{H}_2$ as $\lambda \to 0$. Clearly this implies that $y_\lambda^\epsilon(t) \to z^\epsilon(t)$ in $L_2$ for all $t \in [0, T]$ as $\lambda \to 0$. Since the previous lemma implies that it also holds $y_\lambda^\epsilon(t) \to y^\epsilon(t)$ in $L_2$ for all $t \in [0, T]$ as $\lambda \to 0$, we infer that $z^\epsilon(t) = y^\epsilon(t)$ for all $t \in [0, T]$. Then one has

$$
\|y_\lambda^\epsilon - y^\epsilon\|_{\mathbb{H}_2} = (\mathbb{E} \sup_{t \leq T} \|y_\lambda^\epsilon - y^\epsilon\|^2(t))^{1/2} \lesssim \varepsilon \sqrt{T\lambda},
$$

which is obviously bounded above by $\delta/2$ for $\lambda$ small enough. \qed

An analogous result holds in the case $p > 2$, adapting the assumptions on the coefficient $B$.

**Theorem 4.3.** Let $p > 2$. Assume that $B$ satisfies the hypotheses of Proposition 3.2(ii). Then one has $y_\lambda \to y$ in $\mathbb{H}_p$ as $\lambda \to 0$, i.e.

$$
\lim_{\lambda \to 0} \mathbb{E} \sup_{t \leq T} \|y_\lambda(t) - y(t)\|^p = 0,
$$

where $y_\lambda$ and $y$ denote the mild solutions to (9) and (10), respectively.

**Proof.** In analogy to the argument used in the proof of the previous theorem, one has

$$
\|y - y_\lambda\|_{\mathbb{H}_p} \leq \|y - y^\epsilon\|_{\mathbb{H}_p} + \|y^\epsilon - y_\lambda^\epsilon\|_{\mathbb{H}_p} + \|y_\lambda^\epsilon - y_\lambda\|_{\mathbb{H}_p}, \tag{17}
$$

as well as, by Proposition 3.2(ii),

$$
\mathbb{E} \sup_{t \leq T} \|y(t) - y^\epsilon(t)\|^p \leq \mathbb{E}\|y_0 - y_\epsilon^\epsilon\|^p + \mathbb{E} \left( \int_0^T \|B(s) - B^\epsilon(s)\|^2_{L^2} \, d\langle M, M \rangle(s) \right)^{p/2},
$$

$$
\mathbb{E} \sup_{t \leq T} \|y_\lambda(t) - y_\lambda^\epsilon(t)\|^p \leq \mathbb{E}\|y_0 - y_\epsilon^\epsilon\|^p + \mathbb{E} \left( \int_0^T \|B(s) - B^\epsilon(s)\|^2_{L^2} \, d\langle M, M \rangle(s) \right)^{p/2}.
$$

Let $\delta > 0$ be arbitrary but fixed. Then there exists $\varepsilon > 0$ such that

$$
\|y - y^\epsilon\|_{\mathbb{H}_p} + \|y_\lambda^\epsilon - y_\lambda\|_{\mathbb{H}_p} < \frac{1}{2} \delta
$$

for all $\lambda > 0$. Keeping $\varepsilon$ fixed from now on, let us show that $\|y^\epsilon - y_\lambda^\epsilon\|_{\mathbb{H}_p} < \delta/2$ for $\lambda$ sufficiently small. As in the proof of the previous theorem, exploiting the monotonicity of $A$ and using properties of the Yosida approximation, we arrive at

$$
\|y_\lambda^\epsilon - y_\mu^\epsilon\|^2(t) \leq (\lambda + \mu) \int_0^T \left( \|Ay_\lambda^\epsilon(s)\|^2 + \|A_\mu y_\mu^\epsilon(s)\|^2 \right) \, ds.
$$
Raising both sides to the $p/2$-th power and appealing to Hölder’s inequality we obtain
\[
\|y_\lambda^e(t) - y_\mu^e(t)\|^p \lesssim_T (\lambda + \mu)^{p/2} \int_0^t \left(\|A_\lambda y_\lambda^e(s)\|^p + \|A_\mu y_\mu^e(s)\|^p\right) ds,
\]
hence also
\[
\mathbb{E}\sup_{t \leq T} \|y_\lambda^e(t) - y_\mu^e(t)\|^p \lesssim_T (\lambda + \mu)^{p/2} \mathbb{E} \int_0^T \left(\|A_\lambda y_\lambda^e(s)\|^p + \|A_\mu y_\mu^e(s)\|^p\right) ds.
\]
Note that one has
\[
\mathbb{E}\|A_\lambda y_\lambda^e(s)\|^p \lesssim \mathbb{E}\left| e^{-sA_\lambda} A_\lambda y_0^e \right|^p + \mathbb{E} \left( \int_0^s e^{-(s-r)A_\lambda} A_\lambda B^e(r) dM(r) \right)^p
\]
\[
\quad \leq \mathbb{E}\|A y_0^e\|^p + \mathbb{E} \left( \int_0^s \|AB^e(r)\|_2^2 d[M, M](r) \right)^{p/2},
\]
which implies
\[
\mathbb{E}\sup_{t \leq T} \|y_\lambda^e(t) - y_\mu^e(t)\|^p \lesssim_{T, \varepsilon} (\lambda + \mu)^{p/2} \left( \mathbb{E}|y_0|^p + \mathbb{E} \left( \int_0^T \|B(r)\|_2^2 d[M, M](r) \right)^{p/2} \right).
\]
This shows that $\lambda \mapsto y_\lambda^e$ is a Cauchy net in $H_\mu$, from which we infer that there exists a constant $N$, depending on $\varepsilon$ and $T$, such that
\[
\|y_\lambda^e - y_\varepsilon^e\|_{H_\mu} \leq N \sqrt{\lambda},
\]
the right-hand side of which is clearly bounded above by $\delta/2$ for $\lambda$ small enough. □

The estimates contained in the following corollary are simply extracted from the proofs of the previous two theorems. Since they will be used in the next subsection, we state them explicitly for clarity of exposition.

**Corollary 4.4.** Under the assumptions of Theorem 4.2, the following inequality holds, for any $\varepsilon > 0$ and $\lambda > 0$:
\[
\|y - y_\lambda\|_{H_\mu}^2 \lesssim \mathbb{E}|y_0 - y_0^e|^2 + \mathbb{E} \int_0^T \|B(s) - B^e(s)\|_2^2 d[M, M](s)
\]
\[
+ T\lambda \left( \mathbb{E}\|A y_0^e\|^2 + \mathbb{E} \int_0^T \|AB^e(s)\|_2^2 d[M, M](s) \right).
\]
Assume that there exists $p > 2$ such that the assumptions of Theorem 4.3 are satisfied. Then one has, for any $\varepsilon > 0$ and $\lambda > 0$,
\[
\|y - y_\lambda\|_{H_\mu}^p \lesssim \mathbb{E}|y_0 - y_0^e|^p + \mathbb{E} \left( \int_0^T \|B(s) - B^e(s)\|_2^2 d[M, M](s) \right)^{p/2}
\]
\[
+ (T\lambda)^{p/2} \left( \mathbb{E}\|A y_0^e\|^p + \mathbb{E} \left( \int_0^T \|AB^e(s)\|_2^p d[M, M](s) \right)^{p/2} \right).
\]
4.2 Approximation of A in the strong resolvent sense

For any \( n \in \mathbb{N} \), let \( A_n \) be a linear maximal monotone operator on \( H \), and denote by \( S_n \) the strongly continuous semigroup of contractions generated by \(-A_n\). Then the mild solution to the equation

\[
dy(t) + A_n y(t) \, dt = B(t) \, dM(t), \quad y(0) = y_0,
\]

defined by the variation of constants formula as

\[
y(t) = S_n(t)y_0 + \int_0^t S_n(t-s)B(s) \, dM(s),
\]
is well-defined as a process in \( H \) or in \( H_p \), under the measurability and integrability assumptions on \( B \) of Proposition 3.2(i) or (ii), respectively.

We start with a generalization of Theorem 4.2.

**Theorem 4.5.** Assume that \( A_n \to A \) in the strong resolvent sense as \( n \to \infty \) and that the hypotheses on \( B \) of Proposition 3.2(i) are met. Then one has \( y_n \to y \) in \( H \) as \( n \to \infty \), i.e.

\[
\lim_{n \to \infty} \sup_{t \leq T} \|y_n(t) - y(t)\|^2 = 0.
\]

**Proof.** Let us denote by \( A_n\lambda := A_n(I + \lambda A_n)^{-1}, \lambda > 0 \), the Yosida approximation of \( A_n \), and consider the regularized equations

\[
dy(t) + \lambda y(t) \, dt = B(t) \, dM(t), \quad y_\lambda(0) = y_0, \tag{18}
\]

\[
dy(t) + A_n y(t) \, dt = B(t) \, dM(t), \quad y_{\lambda n}(0) = y_0. \tag{19}
\]

By the triangle inequality one has

\[
\|y - y_n\|_{H_2} \leq \|y - y_\lambda\|_{H_2} + \|y_\lambda - y_{\lambda n}\|_{H_2} + \|y_{\lambda n} - y_n\|_{H_2}. \tag{20}
\]

By Corollary 4.4 we infer that, for any \( \varepsilon > 0 \) and \( \lambda > 0 \), the following inequalities hold true:

\[
\|y - y_\lambda\|_{H_2}^2 \lesssim \mathbb{E}\|y_0 - y_\lambda^0\|^2 + \mathbb{E} \int_0^T \|B(s) - B^\varepsilon(s)\)Q_s^{1/2}(s)\|_{L_2}^2 \, d(M, M)(s)
\]

\[+ T\lambda \left( \mathbb{E}\|Ay_\varepsilon^0\|^2 + \mathbb{E} \int_0^T \|AB^\varepsilon(s)Q_s^{1/2}(s)\|_{L_2}^2 \, d(M, M)(s) \right),
\]

\[
\|y_n - y_{\lambda n}\|_{H_2}^2 \lesssim \mathbb{E}\|y_0 - J_\varepsilon y_0\|^2 + \mathbb{E} \int_0^T \|B(s) - J_\varepsilon B(s)\)Q_s^{1/2}(s)\|_{L_2}^2 \, d(M, M)(s)
\]

\[+ T\lambda \left( \mathbb{E}\|AJ_\varepsilon y_0\|^2 + \mathbb{E} \int_0^T \|AJ_\varepsilon B(s)Q_s^{1/2}(s)\|_{L_2}^2 \, d(M, M)(s) \right),
\]

where we have set, for convenience of notation, \( J_\varepsilon := (I + \varepsilon A_n)^{-1} \). Choosing \( \varepsilon = \lambda^{1/4} \), and recalling that, for any \( n \in \mathbb{N} \), \( A_n\lambda \) is Lipschitz continuous with Lipschitz norm
bounded above by $1/\lambda$, yields
\[
\lambda \left( \mathbb{E}\|A_n J^n_\epsilon y_0\|^2 + \mathbb{E} \int_0^T \|A_n J^n_\epsilon B(s) Q_M^{1/2}(s)\|^2_{L_2} d\langle M, M\rangle(s) \right)
\]
\[
= \lambda \left( \mathbb{E}\|A_n (I + \lambda^{1/4} A_n)^{-1} y_0\|^2 + \mathbb{E} \int_0^T \|A_n (I + \lambda^{1/4} A_n)^{-1} B(s) Q_M^{1/2}(s)\|^2_{L_2} d\langle M, M\rangle(s) \right)
\]
\[
= \lambda \left( \mathbb{E}\|A_n \lambda^{1/4} y_0\|^2 + \mathbb{E} \int_0^T \|A_n \lambda^{1/4} B(s) Q_M^{1/2}(s)\|^2_{L_2} d\langle M, M\rangle(s) \right)
\]
\[
\leq \sqrt{\lambda} \left( \mathbb{E}\|y_0\|^2 + \mathbb{E} \int_0^T \|B(s) Q_M^{1/2}(s)\|^2_{L_2} d\langle M, M\rangle(s) \right).
\]
It goes without saying that the same estimate holds if $A_n$ is replaced by $A$. We are thus left with
\[
\|y - y_\lambda\|^2_{H_2} + \|y_n - y_{n\lambda}\|^2_{H_2} \\
\leq \mathbb{E}\|y_0 - J^{\lambda^{1/4}}_\epsilon y_0\|^2 + \mathbb{E}\|y_0 - J^{\lambda^{1/4}}_\epsilon y_0\|^2 \\
+ \mathbb{E} \int_0^T \|\langle B(s) - J^{\lambda^{1/4}}_\epsilon B(s) \rangle Q_M^{1/2}(s)\|^2_{L_2} d\langle M, M\rangle(s) \\
+ \mathbb{E} \int_0^T \|\langle B(s) - J^{\lambda^{1/4}}_\epsilon B(s) \rangle Q_M^{1/2}(s)\|^2_{L_2} d\langle M, M\rangle(s) \\
+ T\sqrt{\lambda} \left( \mathbb{E}\|y_0\|^2 + \mathbb{E} \int_0^T \|B(s) Q_M^{1/2}(s)\|^2_{L_2} d\langle M, M\rangle(s) \right)
\]
\[
\leq: I_1 + I_2 + I_3 + I_4 + I_5.
\]
Let us now consider the second term on the right-hand side of (21): it is immediately seen that $y_\lambda - y_{n\lambda}$ is the mild solution to the deterministic evolution equation
\[
(y_\lambda - y_{n\lambda})' + A_\lambda y_\lambda - A_{n\lambda} y_{n\lambda} = 0, \quad y_\lambda(0) - y_{n\lambda}(0) = 0.
\]
Since $A_\lambda$ and $A_{n\lambda}$ are bounded operators, it follows that $y_\lambda$ and $y_{n\lambda}$ are actually strong solutions of (14) and (20), respectively. Taking scalar product of both sides with $y_\lambda - y_{n\lambda}$ and integrating (or, equivalently, applying Itô’s formula for the square of the $H$-norm), we obtain
\[
\frac{1}{2} \|y_\lambda(t) - y_{n\lambda}(t)\|^2 + \int_0^t \langle A_\lambda y_\lambda(s) - A_{n\lambda} y_{n\lambda}(s), y_\lambda(s) - y_{n\lambda}(s) \rangle ds = 0.
\]
The monotonicity of $A_{n\lambda}$ implies
\[
\langle A_\lambda y_\lambda(s) - A_{n\lambda} y_{n\lambda}(s), y_\lambda(s) - y_{n\lambda}(s) \rangle
\]
\[
= \langle A_\lambda y_\lambda(s) - A_{n\lambda} y_{n\lambda}(s), y_\lambda(s) - y_{n\lambda}(s) \rangle + \langle A_{n\lambda} y_\lambda(s) - A_{n\lambda} y_{n\lambda}(s), y_\lambda(s) - y_{n\lambda}(s) \rangle
\]
\[
\geq \langle A_\lambda y_\lambda(s) - A_{n\lambda} y_{n\lambda}(s), y_\lambda(s) - y_{n\lambda}(s) \rangle
\]
for all $0 < s \leq T$, hence also
\[
\frac{1}{2} \|y_\lambda(t) - y_{n\lambda}(t)\|^2 \leq - \int_0^t \langle A_\lambda y_\lambda(s) - A_{n\lambda} y_\lambda(s), y_\lambda(s) - y_{n\lambda}(s) \rangle \, ds
\]
\[
\leq \int_0^t \| A_\lambda y_\lambda(s) - A_{n\lambda} y_\lambda(s) \| \| y_\lambda(s) - y_{n\lambda}(s) \| \, ds
\]
\[
\leq \frac{1}{2} \int_0^t \| A_\lambda y_\lambda(s) - A_{n\lambda} y_\lambda(s) \|^2 \, ds + \frac{1}{2} \int_0^t \| y_\lambda(s) - y_{n\lambda}(s) \|^2 \, ds,
\]
which in turn yields, by Gronwall's inequality and obvious estimates,
\[
\| y_\lambda - y_{n\lambda} \|^2_{H^2} \equiv \mathbb{E} \sup_{t \leq T} \| y_\lambda(t) - y_{n\lambda}(t) \|^2 \lesssim T \mathbb{E} \int_0^T \| A_\lambda y_\lambda(s) - A_{n\lambda} y_\lambda(s) \|^2 \, ds =: I_6.
\]
Collecting estimates, we have
\[
\| y - y_n \|^2_{H^2} \lesssim T \sum_{k=1}^6 I_k,
\]
where each $I_k$, $k = 1, \ldots, 6$, depends on $\lambda$ and $n$. We are now going to show that $\lim_{n \to \infty} I_k = 0$ for all $k = 1, \ldots, 6$. Let $\delta$ be any positive real number. Since $J_\lambda$ is a contraction and $J_\lambda x \to x$ as $\lambda \to 0$ for all $x \in H$, the dominated convergence theorem implies that there exists $\lambda_1 > 0$ such that
\[
I_1 \equiv \mathbb{E}\| y_0 - J_{\lambda_1/4} y_0 \|^2 < \frac{\delta}{9} \quad \forall \lambda < \lambda_1.
\]
By exactly the same token, there exists $\lambda_2 > 0$ such that
\[
I_3 \equiv \mathbb{E} \int_0^T \| (B(s) - J_{\lambda_1/4} B(s)) Q^{1/2}_M(s) \|^2_{L^2} \, d\langle M, M \rangle(s) < \frac{\delta}{9} \quad \forall \lambda < \lambda_2.
\]
One also clearly has that there exists $\lambda_3 > 0$ such that
\[
I_5 \equiv T \sqrt{\lambda} \left( \mathbb{E}\| y_0 \|^2 + \mathbb{E} \int_0^T \| B(s) Q^{1/2}_M(s) \|^2_{L^2} \, d\langle M, M \rangle(s) \right) < \frac{\delta}{9} \quad \forall \lambda < \lambda_3.
\]
We can safely assert that $I_1 + I_3 + I_5 < \delta/3$ for $\lambda = \min(\lambda_1, \lambda_2, \lambda_3)/2$. Let $\lambda$ be fixed from now on.

Note that one has, by the triangle inequality and the above estimates,
\[
I_2 + I_4 \equiv \mathbb{E}\| y_0 - J^{n}_{\lambda_1/4} y_0 \|^2 + \mathbb{E} \int_0^T \| (B(s) - J^{n}_{\lambda_1/4} B(s)) Q^{1/2}_M(s) \|^2_{L^2} \, d\langle M, M \rangle(s)
\]
\[
\leq 2\mathbb{E}\| y_0 - J^{n}_{\lambda_1/4} y_0 \|^2 + 2\mathbb{E}\| J^{n}_{\lambda_1/4} y_0 - J^{n}_{\lambda_1/4} y_0 \|^2
\]
\[
+ 2\mathbb{E} \int_0^T \| (B(s) - J_{\lambda_1/4} B(s)) Q^{1/2}_M(s) \|^2_{L^2} \, d\langle M, M \rangle(s)
\]
\[
+ 2\mathbb{E} \int_0^T \| (J_{\lambda_1/4} B(s) - J^{n}_{\lambda_1/4} B(s)) Q^{1/2}_M(s) \|^2_{L^2} \, d\langle M, M \rangle(s)
\]
\[
\leq \frac{4}{9} \delta + 2\mathbb{E}\| J_{\lambda_1/4} y_0 - J^{n}_{\lambda_1/4} y_0 \|^2
\]
\[
+ 2\mathbb{E} \int_0^T \| (J_{\lambda_1/4} B(s) - J^{n}_{\lambda_1/4} B(s)) Q^{1/2}_M(s) \|^2_{L^2} \, d\langle M, M \rangle(s).
\]
Since $A_n \to A$ in the strong resolvent topology, by the dominated convergence theorem we infer that there exists $n_1 > 0$ such that the sum of the last two terms on the right-hand side of the previous inequality is not larger than $\delta/9$ for all $n > n_1$, i.e. that $\sum_{k=1}^{n} I_k < 8\delta/9$ for all $n > n_1$.

In order to conclude the proof, we only have to show that

$$I_6 \equiv \mathbb{E} \int_0^T \|A\lambda y_\lambda(s) - A_n\lambda y_\lambda(s)\|^2 ds$$

can be bounded by $\delta/9$ for $n$ sufficiently large. To this purpose, note that $A_n\lambda x \to A\lambda x$ as $n \to \infty$ for all $x \in H$, because $A_n\lambda = \lambda^{-1}(I - \lambda J^\alpha_n)$. Therefore it is enough to show that the dominated convergence theorem can be applied. Recalling that both $A\lambda$ and $A_n\lambda$ have Lipschitz constant not larger than $1/\lambda$, one has

$$\|A\lambda y_\lambda(s) - A_n\lambda y_\lambda(s)\| \leq \|A\lambda y_\lambda(s)\| + \|A_n\lambda y_\lambda(s)\| \leq \frac{2}{\lambda}\|y_\lambda(s)\|$$

for all $s \in [0,T]$, and $y_\lambda \in L^2(\Omega \times [0,T]) \subset H_2$. There exists then $n_2 > n_1$ such that for all $n > n_2$ one has $I_6 < \delta/9$, hence also, by the above, $\sum_{k=1}^{n} I_k < \delta$ for all $n > n_2$, which is equivalent to $\lim_{n \to \infty} \|y - y_n\|_{H_2} = 0$, thus concluding the proof. 

We now turn to the case $p > 2$, thus providing an extension of Theorem 4.3.

**Theorem 4.6.** Assume that $A_n \to A$ in the strong resolvent sense as $n \to \infty$ and that there exists $p > 2$ such that the hypotheses on $B$ of Proposition 3.2(ii) are met. Then one has $y_n \to y$ in $H_p$ as $n \to \infty$, i.e.

$$\lim_{n \to \infty} \mathbb{E} \sup_{t \leq T} \|y_n(t) - y(t)\|^p = 0.$$  

**Proof.** We follow the reasoning used in the proof of the previous theorem. Denoting the mild solutions to (19) and (20) by $y_\lambda$ and $y_{n\lambda}$, respectively, one has

$$\|y - y_n\|_{H_p} \leq \|y - y_\lambda\|_{H_p} + \|y_\lambda - y_{n\lambda}\|_{H_p} + \|y_{n\lambda} - y_n\|_{H_p}.$$  

By Corollary 4.4, we infer that, for any $\varepsilon > 0$ and $\lambda > 0$, the following inequalities hold true:

$$\|y - y_\lambda\|_{H_p}^p \leq \mathbb{E}\|y_0 - y_\lambda^0\|^p + \mathbb{E}\left(\int_0^T \|B(s) - B^\varepsilon(s)\|_2^2 d[M,M](s)\right)^{p/2} + (T\lambda)^{p/2}\left(\mathbb{E}\|A_\lambda y_\lambda^0\|^p + \mathbb{E}\left(\int_0^T \|AB^\varepsilon(s)\|_2^2 d[M,M](s)\right)^{p/2}\right).$$

$$\|y_n - y_{n\lambda}\|_{H_p}^p \leq \mathbb{E}\|y_0 - J^\varepsilon_n y_\lambda\|^p + \mathbb{E}\left(\int_0^T \|B(s) - J^\varepsilon_n B(s)\|_2^2 d[M,M](s)\right)^{p/2} + (T\lambda)^{p/2}\left(\mathbb{E}\|A_n J^\varepsilon_n y_\lambda^0\|^p + \mathbb{E}\left(\int_0^T \|A_n J^\varepsilon_n B(s)\|_2^2 d[M,M](s)\right)^{p/2}\right).$$
Choosing $\varepsilon = \lambda^{1/4}$, and recalling that, for any $n \in \mathbb{N}$, the Lipschitz constant of $A_{n\lambda}$ is bounded above by $1/\lambda$, we obtain

$$\lambda^{p/2} \left( E\|A_n J^n y_0\|^p + E \left( \int_0^T \|A_n J^n B(s)\|_2^2 \, d[M, M](s) \right)^{p/2} \right) = \lambda^{p/2} \left( E\|A_n \lambda^{1/4} y_0\|^p + E \left( \int_0^T \|A_n \lambda^{1/4} B(s)\|_2^2 \, d[M, M](s) \right)^{p/2} \right) \leq \lambda^{p/4} \left( E\|y_0\|^p + E \left( \int_0^T \|B(s)\|_2^2 \, d[M, M](s) \right)^{p/2} \right).$$

The same estimate holds if $A_n$ is replaced by $A$, therefore we have

$$\|y - y\|_{\mathcal{H}_p}^p + \|y_n - y_n\|_{\mathcal{H}_p}^p \lesssim_p E\|y_0 - J_{\lambda^{1/4}} y_0\|^p + E\|y_0 - J_{\lambda^{1/4}} y_0\|^p + E \left( \int_0^T \|B(s) - J_{\lambda^{1/4}} B(s)\|_2^2 \, d[M, M](s) \right)^{p/2} + T^{p/2} \lambda^{p/4} \left( E\|y_0\|^p + E \left( \int_0^T \|B(s)\|_2^2 \, d[M, M](s) \right)^{p/2} \right) =: I_1 + I_2 + I_3 + I_4 + I_5.$$  

Moreover, as in the proof of the previous theorem, we have

$$\|y_\lambda(t) - y_n\|_{\mathcal{H}_p}^p \leq \int_0^t \|A_\lambda y_\lambda(s) - A_{n\lambda} y_\lambda(s)\|^2 \, ds + \int_0^t \|y_\lambda(s) - y_n\|_{\mathcal{H}_p}^2 \, ds,$$

which in turn yields, by taking $p/2$-th power and applying Gronwall’s inequality,

$$\|y_\lambda - y_n\|_{\mathcal{H}_p}^p \lesssim_{T, p} E \int_0^T \|A_\lambda y_\lambda(s) - A_{n\lambda} y_\lambda(s)\|^p \, ds =: I_6,$$

hence also, collecting estimates, $\|y - y_n\|_{\mathcal{H}_p}^p \lesssim_{T, p} \sum_{k=1}^6 I_k$. Let $\delta$ be an arbitrary but fixed positive real number. By a reasoning already used above, we infer that there exist $\lambda_1$, $\lambda_2$, $\lambda_3 > 0$ such that

$$I_1 \equiv E\|y_0 - J_{\lambda^{1/4}} y_0\|^p < \delta \quad \forall \lambda < \lambda_1,$$

$$I_3 \equiv E \left( \int_0^T \|B(s) - J_{\lambda^{1/4}} B(s)\|_2^2 \, d[M, M](s) \right)^{p/2} < \delta \quad \forall \lambda < \lambda_2,$$

$$I_5 \equiv T^{p/2} \lambda^{p/4} \left( E\|y_0\|^p + E \left( \int_0^T \|B(s)\|_2^2 \, d[M, M](s) \right)^{p/2} \right) < \delta \quad \forall \lambda < \lambda_3,$$

hence $I_1 + I_3 + I_5 < 3\delta$ for $\lambda := \min(\lambda_1, \lambda_2, \lambda_3)/2$, which will remain fixed for the rest
of the proof. Moreover, one has

\[
I_2^{1/p} \equiv \|y_0 - J_{\lambda_1/4}^n y_0\|_{L_p} \leq \|y_0 - J_{\lambda_1/4}^n y_0\|_{L_p} + \|J_{\lambda_1/4}^n y_0 - J_{\lambda_1/4}^n y_0\|_{L_p} \\
\leq \delta^{1/p} + \|J_{\lambda_1/4}^n y_0 - J_{\lambda_1/4}^n y_0\|_{L_p},
\]

\[
I_4^{1/p} \equiv \|B - J_{\lambda_1/4}^n B\|_{L_p}^2 \cdot [M, M]\|_{L_p} \\
\leq 2 \|B - J_{\lambda_1/4}^n B\|_{L_p}^2 \cdot [M, M]\|_{L_p} + 2 \|J_{\lambda_1/4}^n B - J_{\lambda_1/4}^n B\|_{L_p}^2 \cdot [M, M]\|_{L_p} \\
\leq 2 \delta^{1/p} + 2 \|J_{\lambda_1/4}^n B - J_{\lambda_1/4}^n B\|_{L_p}^2 \cdot [M, M]\|_{L_p}.
\]

Since \(J_{\lambda_1/4}^n \to J_{\lambda_1/4}\) as \(n \to \infty\), there exists \(n_0\) such that the sum of the last two terms on the right-hand side of the previous inequalities is not larger than \(\delta^{1/p}\) for all \(n > n_0\), hence \(\sum_{k=1}^n I_k \leq \delta\). The proof is concluded if we show that \(I_6 \leq \delta\) for \(n\) large enough. But this follows by observing that

\[
I_6 \equiv E \int_0^T \|A_{\lambda,y_{\lambda}}(s) - A_{n,\lambda,y_{\lambda}}(s)\|^p ds
\]

converges to zero as \(n \to \infty\) by the dominated convergence theorem, \(A_{n,\lambda,x} \to A_{\lambda,x}\) as \(n \to \infty\) for all \(x \in H, y_{\lambda} \in L_p(\Omega \times [0, T]) \subset L_p^p\), and \(\|A_{\lambda,y_{\lambda}}(s) - A_{n,\lambda,y_{\lambda}}(s)\| \leq 2\lambda^{-1}\|y_{\lambda}(s)\|\). \(\Box\)

5 Convergence of stochastic convolutions II

Let us consider the equation with Poisson random noise

\[
dy(t) + Ay(t) dt = \int_Z G(t, z) \mu(dt, dz), \quad y(0) = y_0,
\]

(22)

where \(\mu\) is a compensated Poisson random measure, as defined in Section 2.

Consider the equations

\[
dy_{\lambda}(t) + A_{\lambda,y_{\lambda}}(t) dt = \int_Z G(t, z) \mu(dt, dz), \quad y(0) = y_0,
\]

(23)

and

\[
dy_n(t) + A_{n,y_n}(t) dt = \int_Z G(t, z) \mu(dt, dz), \quad y(0) = y_0,
\]

(24)

where \(A_{\lambda}\) and \(A_n\) are defined as in the previous sections.

Recall that the mild solutions to (22), (23) and (24) defined by the formula of variations of constants are well-defined processes belonging to \(H_p\), \(p \geq 2\), as soon as \(G \in L_p(\Omega \times [0, T], L_2(Z) \cap L_p(Z))\). To render notation less burdensome, we shall denote the latter space by \(G_p\).

**Theorem 5.1.** Let \(p \geq 2\) and \(G \in G_p\). Denoting the mild solutions to (22) and (23) by \(y\) and \(y_{\lambda}\), respectively, one has \(y_{\lambda} \to y\) in \(H_p\) as \(\lambda \to 0\).
Proof. The proof is similar to the one of Theorem 4.3, and therefore we omit some details. One has

\[ \|y - y_{\lambda}\|_{H^p} \leq \|y - y^\varepsilon\|_{H^p} + \|y_{\lambda}^\varepsilon - y_{\lambda}\|_{H^p}, \]  

(25)

where \(y_{\lambda}\) and \(y_{\lambda}^\varepsilon\) are solutions to the regularized equations

\[
\begin{align*}
    dy^\varepsilon(t) + A y^\varepsilon(t) dt &= \int_Z G^\varepsilon(t, z) \bar{\mu}(dt, dz), \quad y(0) = y_0, \\
    dy_{\lambda}^\varepsilon(t) + A \lambda y_{\lambda}^\varepsilon(t) dt &= \int_Z G^\varepsilon(t, z) \bar{\mu}(dt, dz), \quad y(0) = y_0^\varepsilon,
\end{align*}
\]

where \(G^\varepsilon := (I + \varepsilon A)^{-1} G\). Let \(\delta > 0\) be arbitrary but fixed. By virtue of

\[ \|y - y_{\lambda}\|_{H^p} + \|y_{\lambda}^\varepsilon - y_{\lambda}\|_{H^p} \leq \|y_0 - y_0^\varepsilon\|_{L^p} + \|G - G^\varepsilon\|_{G^p}, \]

there exists \(\varepsilon > 0\) (which will remain fixed for the rest of the proof) such that \(\|y - y_{\lambda}\|_{H^p} + \|y_{\lambda}^\varepsilon - y_{\lambda}\|_{H^p} < \delta/2\) for all \(\lambda > 0\). Recalling the maximal inequality (8'), the same argument used in the proof of Theorem 4.3 yields

\[ \|y_{\lambda} - y_\mu\|_{H^p} \lesssim_{T, \varepsilon, p} \sqrt{\lambda + \mu} \left(\|y_0\|_{L^p} + \|G\|_{G^p}\right), \]

hence that \(\lambda \mapsto y_{\lambda}\) is a Cauchy net in \(H^p\), and that

\[ \|y_{\lambda}^\varepsilon - y^\varepsilon\|_{H^p} \lesssim_{T, \varepsilon, p} \sqrt{\lambda}, \]

which can be made smaller than \(\delta/2\) choosing \(\lambda\) small enough.

In the final result of this section, we prove the analogon of Theorem 4.6 for equations driven by Poisson random noise.

**Theorem 5.2.** Assume that \(A_n \to A\) in the strong resolvent sense as \(n \to \infty\), and that there exists \(p \in [2, \infty)\) such that \(G \in G_p\). Denoting the mild solutions to (22) and (24) by \(y\) and \(y_n\), respectively, one has \(y_n \to y\) in \(H^p\) as \(n \to \infty\).

Proof. The proof follows the same lines of the proofs of Theorems 4.5 and 4.6, therefore we omit some details. Let \(y_{\lambda}\) and \(y_{n \lambda}\) be the solutions to the regularized equations

\[
\begin{align*}
    dy_{\lambda}(t) + A \lambda y_{\lambda}(t) dt &= \int_Z G(t, z) \bar{\mu}(dt, dz), \quad y_{\lambda}(0) = y_0, \\
    dy_{n \lambda}(t) + A \lambda y_{n \lambda}(t) dt &= \int_Z G(t, z) \bar{\mu}(dt, dz), \quad y_{n \lambda}(0) = y_0,
\end{align*}
\]

and observe that

\[
\|y - y_n\|_{H^p} \leq \|y - y_{\lambda}\|_{H^p} + \|y_{\lambda} - y_{n \lambda}\|_{H^p} + \|y_{n \lambda} - y_n\|_{H^p}.
\]

In analogy to Corollary 4.4 one has, for all \(\lambda > 0\), \(\varepsilon > 0\),

\[
\|y - y_{\lambda}\|_{H^p} + \|y_{n \lambda} - y_n\|_{H^p} \lesssim \sum_{k=1}^{5} I_k,
\]

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where
\[
I_1 := \|y_0 - J_{\lambda/4}y_0\|_{\mathbb{H}_p} \xrightarrow{\lambda \to 0} 0,
\]
\[
I_2 := \|y_0 - J_{\lambda/4}^n y_0\|_{\mathbb{H}_p} \leq \|y_0 - J_{\lambda/4}y_0\|_{\mathbb{H}_p} + \|J_{\lambda/4}^n y_0 - J_{\lambda/4}^n y_0\|_{\mathbb{H}_p}
\]
\[
I_3 := \|G - J_{\lambda/4}G\|_{\mathbb{G}_p} \xrightarrow{\lambda \to 0} 0,
\]
\[
I_4 := \|G - J_{\lambda/4}^n G\|_{\mathbb{G}_p} \leq \|G - J_{\lambda/4}G\|_{\mathbb{G}_p} + \|J_{\lambda/4}^n G - J_{\lambda/4}^n G\|_{\mathbb{G}_p}
\]
\[
I_5 := T^{1/2} J_{\lambda/4} \left(\|y_0\|_{\mathbb{H}_p} + \|G\|_{\mathbb{G}_p}\right) \xrightarrow{\lambda \to 0} 0,
\]
and
\[
\|J_{\lambda/4} y_0 - J_{\lambda/4}^n y_0\|_{\mathbb{H}_p} + \|J_{\lambda/4} G - J_{\lambda/4}^n G\|_{\mathbb{G}_p} \xrightarrow{n \to \infty} 0 \quad \forall \lambda > 0,
\]
that is, for any given \( \delta > 0 \), one can choose \( \lambda \) and \( n_0 > 0 \) such that \( \|y - y_\lambda\|_{\mathbb{H}_p} + \|y_n - y_n\|_{\mathbb{H}_p} < \delta \) for all \( n > n_0 \). The proof is finished noting that
\[
\|y_\lambda - y_\lambda\|_{\mathbb{H}_p}^p \leq \mathbb{E} \int_0^T \|A_\lambda y_\lambda(s) - A_n y_\lambda(s)\|^p ds,
\]
which converges to zero as \( n \to \infty \) by the dominated convergence theorem and \( y_\lambda \in \mathbb{H}_p \).

6 Proof of the main results

By definition of mild solution, one has, in the case of Theorem 2.2
\[
u(t) = S(t)u_0 - \int_0^t S(t-s)f(u(s)) \, ds + \int_0^t S(t-s)B(u(s-)) \, dM(s),
\]
\[
u_n(t) = S_n(t)u_0 - \int_0^t S_n(t-s)f_n(u_n(s)) \, ds + \int_0^t S_n(t-s)B_n(u_n(s-)) \, dM(s),
\]
and, in the case of Theorem 2.4
\[
u(t) = S(t)u_0 - \int_0^t S(t-s)f(u(s)) \, ds + \int_0^t \int_Z S(t-s)G(z, u(s-)) \, \bar{\mu}(ds, dz),
\]
\[
u_n(t) = S_n(t)u_0 - \int_0^t S_n(t-s)f_n(u_n(s)) \, ds + \int_0^t \int_Z S_n(t-s)G_n(z, u_n(s-)) \, \bar{\mu}(ds, dz).
\]
We shall set, for notational convenience,
\[
S \ast \phi(t) := \int_0^t S(t-s)\phi(s) \, ds,
\]
\[
S \ast \Phi(t) := \int_0^t S(t-s)\Phi(s) \, dM(s), \quad S \ast \Psi(t) := \int_0^t \int_Z S(t-s)\Psi(z, s) \, \bar{\mu}(ds, dz)
\]
(even though we use the same symbol to denote stochastic convolutions with respect to a martingale and to a compensated Poisson measure, there will be no risk of confusion).

In the case of Theorem 2.2, the triangle inequality yields, for any \( 0 < t \leq T \),
\[
\|u - u_n\|_{\mathbb{H}_p(t)} \leq \|SU_0 - S_n u_0\|_{\mathbb{H}_p(t)} + \|S \ast f(u) - S_n \ast f_n(u_n)\|_{\mathbb{H}_p(t)} + \|S \ast B(u) - S_n \ast B_n(u_n)\|_{\mathbb{H}_p(t)}.
\]
The same estimate holds in the case of Theorem 2.4, replacing $B$ and $B_n$ with $G$ and $G_n$, respectively.

**Lemma 6.1.** Let $p \geq 2$ and $0 \leq t \leq T$. If $\xi_n \to \xi$ in $L_p$ as $n \to \infty$, then $S_n \xi_n \to S \xi$ in $H_p(t)$.

**Proof.** It clearly suffices to prove the claim for $t = T$. By the triangle inequality we can write

$$
\|S_n \xi_n - S \xi\|_{H_p} \leq \|S_n \xi_n - S_n \xi\|_{H_p} + \|S_n \xi - S \xi\|_{H_p}.
$$

Since $\sup_{t \leq T} \|S_n(t) \xi - S(t) \xi\|^p \leq e^{pT}\|\xi_n - \xi\|^p$, taking expectations on both sides and passing to the limit, we get $\|S_n(t) \xi_n - S_n(t) \xi\|_{H_p} \to 0$ as $n \to \infty$. Moreover, by the Trotter-Kato’s theorem, $S_n(\cdot) \xi$ converges to $S(\cdot) \xi$ $P$-a.s. uniformly on compact sets, i.e.

$$
\lim_{n \to \infty} \sup_{t \leq T} \|S_n(t) \xi - S(t) \xi\| = 0,
$$

which implies, together with

$$
\sup_{t \leq T} \|S_n(t) \xi - S(t) \xi\|^p \lesssim e^{pT}\|\xi\|^p
$$

and $E\|\xi\|^p < \infty$, that $\|S_n(t) \xi - S(t) \xi\|_{H_p} \to 0$ as $n \to \infty$, thanks to the dominated convergence theorem.

**Lemma 6.2.** Let $p \geq 2$, $0 < t \leq T$, and $v$, $w \in H_p$. For every $\delta > 0$ there exist $n_0 \in \mathbb{N}$ and $\gamma > 0$, independent of $\delta$ and $n_0$, such that

$$
\|S_n * f_n(v) - S * f(w)\|_{H_p(t)}^p \leq \delta + \gamma \int_0^t \|v - w\|_{H_p(s)}^p \, ds
$$

for all $n > n_0$.

**Proof.** The triangle inequality yields

$$
\|S_n * f_n(v) - S * f(w)\|_{H_p(t)}^p \leq 3^p\|S_n * f_n(v) - S_n * f_n(w)\|_{H_p(t)}^p + 3^p\|S_n * f_n(w) - S_n * f(w)\|_{H_p(t)}^p + 3^p\|S_n * f(w) - S * f(w)\|_{H_p(t)}^p.
$$

Recalling that the operator norm of $S_n(t)$ is bounded by $e^{qt}$, by Hölder’s inequality one has that there exists a constant $N = N(T, p, \eta)$ such that

$$
\|S_n * f_n(v) - S_n * f_n(w)\|_{H_p(t)}^p = \mathbb{E} \sup_{s \leq t} \left\| \int_0^s S_n(s - r) (f_n(v(r)) - f_n(w(r))) \, dr \right\|^p 
\leq N \mathbb{E} \sup_{s \leq t} \int_0^s \|f_n(v(s)) - f_n(w(s))\|^p \, dr 
\leq NL^p f \int_0^t \mathbb{E} \sup_{r \leq s} \|v(r) - w(r)\|^p \, dr 
= NL^p f \int_0^t \|v - w\|_{H_p(s)}^p \, ds,
$$

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Therefore there exists \( n_1 \) such that \( \|S_n * f_n(w) - S_n * f(w)\|_{L_p(T)}^p < 3^{-p}\delta/2 \) for all \( n > n_1 \). Theorem 3.1 implies
\[
\limsup_{n \to \infty} \mathbb{E} \left( \sup_{t \leq T} \| (S_n * f(w))(t) - (S * f(w))(t) \| \right) = 0,
\]
which implies
\[
\|S_n * f(w) - S * f(w)\|_{L_p(T)}^p \to 0
\]
as \( n \to \infty \) by the dominated convergence theorem. In fact, one has
\[
\mathbb{E} \sup_{t \leq T} \| (S_n * f(w))(t) - (S * f(w))(t) \|^p \leq \mathbb{E} \sup_{t \leq T} \|S_n * f(w))(t)\|^p + \mathbb{E} \sup_{t \leq T} \|S * f(w))(t)\|^p \leq n \|f\|_{C_{\delta/1}} \|w\|_{L_p(T)}^p < \infty.
\]
Therefore there exists \( n_2 \in \mathbb{N} \) such that \( \|S_n * f(w) - S * f(w)\|_{L_p(T)}^p < 3^{-p}\delta/2 \) for all \( n > n_2 \). The proof is completed taking \( n_0 = \max(n_1, n_2) \) and \( \gamma = 3^pNL_f^p \).

6.1 **Proof of Theorem 2.2**

The following estimate is crucial for the proof of the theorem.

**Lemma 6.3.** Let \( 0 < t \leq T \). For every \( \delta > 0 \) there exist \( n_0 \in \mathbb{N} \) and \( \gamma > 0 \), independent of \( \delta \) and \( n_0 \), such that
\[
\|S_n \circ B_n(u_{n-}) - S \circ B(u_-)\|_{L^2_H(t)}^2 \leq \delta + \gamma \int_0^t \|u_n - u\|_{L^2_H(s)}^2 ds
\]
for all \( n > n_0 \).

**Proof.** The triangle inequality yields
\[
\|S_n \circ B_n(u_{n-}) - S \circ B(u_-)\|_{L^2_H(t)}^2 \leq 9 \|S_n \circ B_n(u_{n-}) - S_n \circ B_n(u_-)\|_{L^2_H(t)}^2 + 9 \|S_n \circ B_n(u_-) - S_n \circ B(u_-)\|_{L^2_H(t)}^2 + 9 \|S_n \circ B(u_-) - S \circ B(u_-)\|_{L^2_H(t)}^2.
\]
Therefore there exists a constant $N = N(\eta)$ such that
\[
\|S_n \circ B_n(u_\cdot) - S_n \circ B_n(u_\cdot)\|_{H_2(t)}^2
\]
\[
= \mathbb{E} \sup_{s \leq t} \left( \int_0^s |S_n(s-r)(B_n(u_\cdot(r))-B_n(u_\cdot))| \, dM(r) \right)^2
\]
\[
\leq N \mathbb{E} \int_0^t \| (B_n(u(s))-B_n(u(s))) Q_M^{1/2}(s) \|^2 \, d(M,M)(s)
\]
\[
\leq NL_B^2 \mathbb{E} \int_0^t \sup_{r \leq s} \| u_n(r) - u(r) \|^2 \, ds
\]
\[
= NL_B^2 \int_0^t \| u_n - u \|^2_{H_2(s)} \, ds,
\]
as well as
\[
\|S_n \circ B_n(u_\cdot) - S_n \circ B(u_\cdot)\|_{H_2(t)}^2
\]
\[
\lesssim \eta \mathbb{E} \int_0^t \| (B_n(u(s))-B(u(s))) Q_M^{1/2}(s) \|^2 \, d(M,M)(s)
\]
\[
\leq \mathbb{E} \int_0^T \| B_n(u(s)) - B(u(s)) \|^2_Q \, ds,
\]
which converges to zero by pointwise convergence of $B_n$ to $B$ and the dominated convergence theorem, taking into account that
\[
\| B_n(u) - B(u) \|^2_Q \lesssim 2\| u \| \in L_2(\Omega \times [0,T]).
\]
Therefore there exists $n_1 \in \mathbb{N}$ such that $\|S_n \circ B_n(u_\cdot) - S_n \circ B(u_\cdot)\|_{H_2(t)}^2 < \delta/18$ for all $n > n_1$. Finally, Theorem 4.3 implies that there exists $n_2 \in \mathbb{N}$ such that
\[
\|S_n \circ B(u_\cdot) - S \circ B(u_\cdot)\|_{H_2(t)}^2 < \frac{\delta}{18}
\]
for all $n > n_2$. The proof is completed setting $n_0 = \max(n_1, n_2)$ and $\gamma = 9NL_B^2$. \(\square\)

**Proof of Theorem 2.2.** By (26) we have, for any $0 < t \leq T$,
\[
\|u - u_n\|_{H_2(t)}^2 \leq 9\|Su_0 - S_n u_0n\|_{H_2(t)}^2 + 9\|S \ast f(u) - S_n \ast f_n(u_\cdot)\|_{H_2(t)}^2
\]
\[
+ 9\|S \circ B(u) - S_n \circ B_n(u_n)\|_{H_2(t)}^2.
\]
Let $\varepsilon > 0$ be arbitrary and $\varepsilon' = e^{-\gamma T} \varepsilon$, where $\gamma$ is a constant whose value will be specified later. By Lemma 6.1 there exists $n_1 \in \mathbb{N}$, independent of $t$, such that
\[
9\|Su_0 - S_n u_0n\|_{H_2(t)}^2 < \frac{\varepsilon'}{3} \quad \forall n > n_1.
\]
Similarly, by Lemmata 6.2 and 6.3 there exist $n_2, n_3 \in \mathbb{N}$, independent of $t$, and $\gamma_1, \gamma_2 > 0$, depending only on the Lipschitz constants of $f$ and $B$, such that
\[
9\|S \ast f(u) - S_n \ast f_n(u_\cdot)\|_{H_2(t)}^2 < \frac{\varepsilon'}{3} + \gamma_1 \int_0^t \| u_n - u \|_{H_2(s)}^2 \, ds \quad \forall n > n_2,
\]
\[
9\|S \circ B(u) - S_n \circ B_n(u_n)\|_{H_2(t)}^2 < \frac{\varepsilon'}{3} + \gamma_2 \int_0^t \| u_n - u \|_{H_2(s)}^2 \, ds \quad \forall n > n_3.
\]
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Therefore, setting \( n_0 = n_1 + n_2 + n_3 \) and \( \gamma = \gamma_1 + \gamma_2 \), we are left with
\[
\|u_n - u\|^2_{\mathcal{H}_2(t)} < \varepsilon' + \gamma \int_0^t \|u_n - u\|^2_{\mathcal{H}_2(s)} ds \quad \forall t \in [0,T],
\]
which implies, by Gronwall’s inequality,
\[
\|u_n - u\|^2_{\mathcal{H}_2(T)} < e^{\gamma T} \varepsilon' = \varepsilon \quad \forall n > n_0.
\]
Since \( \varepsilon \) is arbitrary, this is equivalent to asserting that
\[
\lim_{n \to \infty} \|u_n - u\|_{\mathcal{H}_2(T)} = 0 \quad \square
\]

6.2 Proof of Theorem 2.4

As in the previous subsection, we establish first a key estimate.

**Lemma 6.4.** Let \( 2 \leq p < \infty \) and \( 0 < t \leq T \). For every \( \delta > 0 \) there exist \( n_0 \in \mathbb{N} \) and \( \gamma > 0 \), independent of \( \delta \) and \( n_0 \), such that
\[
\|S_n \circ G_n(u_{n-}) - S \circ G(u_-)\|_{\mathcal{H}_p(t)}^p \leq \delta + \gamma \int_0^t \|u_n - u\|_{\mathcal{H}_p(s)}^p ds
\]
for all \( n > n_0 \).

**Proof.** The triangle inequality yields
\[
\|S_n \circ G_n(u_{n-}) - S \circ G(u_-)\|_{\mathcal{H}_p(t)}^p \lesssim_p \|S_n \circ G_n(u_{n-}) - S_n \circ G_n(u_-)\|_{\mathcal{H}_p(t)}^p + \|S_n \circ G_n(u_-) - S_n \circ G(u_-)\|_{\mathcal{H}_p(t)}^p + \|S_n G(u_-) - S \circ G(u_-)\|_{\mathcal{H}_p(t)}^p,
\]
where the implicit constant in the inequality can be taken equal to \( N_1 = N_1(p) = 3^p \).

Thanks to the maximal inequality (3), there exists a constant \( N_2 = N_2(\eta,p) \) such that
\[
\|S_n \circ G_n(u_{n-}) - S_n \circ G_n(u_-)\|_{\mathcal{H}_p(t)}^p \leq N_2 \mathbb{E} \int_0^t \|G_n(u_{n}(s)) - G_n(u(s))\|_{L_p(Z) \cap L_2(Z)}^p ds \leq N_2 \delta
\]
for all \( n > n_0 \).

Similarly, one also has
\[
\|S_n \circ G_n(u_-) - S_n \circ G(u_-)\|_{\mathcal{H}_p(t)}^p \lesssim_{\eta,p} \mathbb{E} \int_0^t \|G_n(u(s)) - G(u(s))\|_{L_p(Z) \cap L_2(Z)}^p ds
\]
which converges to zero by pointwise convergence of \( G_n \) to \( G \) and the dominated convergence theorem. In particular, there exists \( n_1 \in \mathbb{N} \) such that
\[
\|S_n \circ G_n(u_-) - S_n \circ G(u_-)\|_{\mathcal{H}_p(t)}^p \leq \frac{1}{2} \frac{\delta}{3^p} \quad \forall n > n_1.
\]
Finally, Theorem 5.2 implies that there exists \( n_2 \in \mathbb{N} \) such that
\[
\|S_n \circ G(u_-) - S \circ G(u_-)\|_{\mathcal{H}_p(t)}^p \leq \frac{1}{2} \frac{\delta}{3^p} \quad \forall n > n_2.
\]
The proof is completed setting \( n_0 = \max(n_1, n_2) \) and \( \gamma = N_1 N_2 \delta \). \quad \square
Proof of Theorem 2.4. By (26) we have, for any $0 < t \leq T$,

$$
\|u - u_n\|_{H^p(t)}^p \leq N_1\|Su_0 - S_n u_{0n}\|_{H^p(t)}^p + N_1\|S \ast f(u) - S_n \ast f_n(u_n)\|_{H^p(t)}^p + N_1\|S \circ G(u) - S_n \circ G_n(u_n)\|_{H^p(t)}^p,
$$

where $N_1 = 3^p$. Let $\varepsilon > 0$ be arbitrary and $\varepsilon' = e^{-\gamma T} \varepsilon$, where $\gamma$ is a constant whose value will be specified later. By Lemma 6.1 there exists $n_1 \in \mathbb{N}$, independent of $t$, such that

$$
N_1\|S u_0 - S_n u_{0n}\|_{H^p(t)}^p < \frac{\varepsilon'}{3} \quad \forall n > n_1.
$$

Similarly, by Lemmata 6.2 and 6.4 there exist $n_2, n_3 \in \mathbb{N}$, independent of $t$, and $\gamma_1$, $\gamma_2 > 0$, depending only on the Lipschitz constants of $f$ and $G$, such that

$$
N_1\|S \ast f(u) - S_n \ast f_n(u_n)\|_{H^p(t)}^p < \frac{\varepsilon'}{3} + \gamma_1 \int_0^t \|u_n - u\|_{H^p(s)}^p \, ds \quad \forall n > n_2,
$$

$$
N_1\|S \circ G(u) - S_n \circ G_n(u_n)\|_{H^p(t)}^p < \frac{\varepsilon'}{3} + \gamma_2 \int_0^t \|u_n - u\|_{H^p(s)}^p \, ds \quad \forall n > n_3.
$$

Therefore, setting $n_0 = n_1 + n_2 + n_3$ and $\gamma = \gamma_1 + \gamma_2$, we are left with

$$
\|u_n - u\|_{H^p(t)}^p < \varepsilon' + \gamma \int_0^t \|u_n - u\|_{H^p(s)}^p \, ds \quad \forall t \in [0, T],
$$

which implies, by Gronwall’s inequality,

$$
\|u_n - u\|_{H^p(T)}^p < e^{\gamma T} \varepsilon' = \varepsilon \quad \forall n > n_0,
$$

i.e.

$$
\lim_{n \to \infty} \|u_n - u\|_{H^p(T)} = 0
$$

because $\varepsilon$ is arbitrary. \hfill \Box

7 On equations with additive martingale noise

It was observed in Remark 2.3 that the reason for considering equations driven by $M$ only in $\mathbb{H}_2$ is that we do not know whether it is possible to find a Lipschitz condition involving only $B$ (and at most a “deterministic” quantity depending on $M$, such as $Q$) such that a fixed point argument could be used to prove well-posedness in $\mathbb{H}_p$. In the case of equations with additive noise the problem disappears, and it is immediate to deduce the following result.

Theorem 7.1. Let $p > 2$. Assume that hypotheses (i) and (ii) of Theorem 2.2 are satisfied, and that

(iii’) $B_n$ are predictable $\mathcal{L}(K, H)$-valued processes such that

$$
\left(\|B\|_{L^2}^2 \cdot [M, M]\right)^{1/2} \in L_p, \quad \left(\|B_n - B\|_{L^2}^2 \cdot [M, M]\right)^{1/2} \overset{n \to \infty}{\to} 0 \quad \text{in } L_p;
$$

(iv’) $u_{0n} \to u_0$ in $\mathbb{L}_p$ as $n \to \infty$. 

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Denoting by $u$ and $u_n$ the mild solutions to
\begin{equation*}
du + A u \, dt + f(u) \, dt = B \, dM, \quad u(0) = u_0,
\end{equation*}
and
\begin{equation*}
du_n + A_n u_n \, dt + f_n(u_n) \, dt = B_n \, dM, \quad u(0) = u_{0n},
\end{equation*}
respectively, one has $u_n \to u$ in $\mathbb{H}_p$.

Remark 7.2. As mentioned at the beginning of Section 4, Theorems 4.2, 4.3, 4.5 and 4.6 remain true also assuming that $A + \eta I$ is maximal monotone. Let us consider the case of $A_n \to A$ in the strong resolvent sense. Then $y_n$ is the mild solution to
\begin{equation*}
dy_n + \tilde{A}_n y_n \, dt - \eta y_n \, dt = B \, dM, \quad y_n(0) = y_0,
\end{equation*}
where $\tilde{A}_n := A_n + \eta I$. Setting $f = f_n := \eta I$ for all $n \in \mathbb{N}$, the previous theorem implies that $y_n \to y$ in $\mathbb{H}_p$, with $p$ depending on the hypotheses on $B$ and $M$.

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