Condensate Heating by Atomic Losses

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Atomic Bose-Einstein condensate is heated by atomic losses. Predicted depletion ranges from 1% for a uniform 3D condensate to around 10% for a quasi-1D condensate in a harmonic trap.

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Heating by atomic losses.— An ideal Bose-Einstein condensate (BEC) is a state where all bosons occupy the same single particle state \( \phi_0 \). So far the atomic BEC is the closest to this ideal \[1\]. However, even the atomic BEC is not perfect because atoms are depleted from \( \phi_0 \) by atom-atom interactions (quantum depletion) and thermal fluctuations (thermal depletion). Moreover, atomic condensates last only for tens of seconds before atomic losses empty the trap. This process is dominated by 3-body losses, where 3 atoms collide to form a bi-atomic molecule and an atom with large kinetic energy, and then both of them escape from the trap. There are also 1-body losses, where individual atoms are kicked out from the trap by external heating agents, see e.g. Ref. [2]. While the 1-body losses can, at least in principle, be minimized, so far the more intrinsic 3-body losses have not been eliminated.

Atomic losses from a trap can be modeled by repeated application of the annihilation operator \( \hat{\psi}(\vec{x}) \). An ideal condensate \( |N : \phi_0\rangle \) is a “fixed point” of the annihilation operator: action of \( \hat{\psi}(\vec{x}) \) neither depletes an ideal condensate nor changes its wave function \( \hat{\psi}(\vec{x})|N : \phi_0\rangle = \sqrt{N}\phi_0(\vec{x})|N - 1 : \phi_0\rangle \). A destructive measurement of atomic positions in a trap can also be described by repeated application of \( \hat{\psi}(\vec{x}) \). In Refs. [3] it was shown on a few examples that such a repeated annihilation can gradually increase a condensed fraction of atoms remaining in a trap. Annihilation is driving the remaining atoms toward an ideal condensate. This phenomenon is a foundation for a quite popular believe that “atomic losses improve a condensate”. This is not a sound foundation.

In the destructive measurement all annihilations happen at the same time, or in a very short measurement time. Hamiltonian of atoms has no time to do anything before all atoms are annihilated. In contrast, in a typical experiment it is the Hamiltonian that is much faster than atomic losses. An ideal condensate \( |N : \phi_0\rangle \) is not an eigenstate of the Hamiltonian: atom-atom interactions are depleting atoms from the condensate wave function \( \phi_0 \). On the other hand, eigenstates of the Hamiltonian \( \hat{H} \) are not “fixed points” of the annihilation operator: \( \hat{\psi}(\vec{x}) \) applied to an eigenstate of \( \hat{H} \) gives a non-stationary state because \( \hat{\psi}(\vec{x}) \) does not commute with \( \hat{H} \), \([\hat{\psi}(\vec{x}), \hat{H}] \neq 0\). A competition between the Hamiltonian depleting a condensate and the atomic losses increasing the condensed fraction leads to a state that is neither an ideal condensate nor the N-body ground state. A condensate is heated by atomic losses — this phenomenon will be more quantified in the following.

In Ref. [4] Timmermans demonstrated that heating of fermionic atoms by atomic losses is a serious obstacle on the way to atomic superconductors. In the case of fermions the mechanism is simple: atomic losses create holes in the Fermi sea. In the case of bosons the heating is a conceptually more subtle effect. One has to realize the interplay between the losses and the Hamiltonian to see that losses are in fact heating a condensate. However, as we will see below, a conceptually subtle effect is not necessarily a quantitatively subtle effect. It is a serious and rather fundamental limitation on quantum coherence of atomic BEC.

Master equation.— The non-unitary (due to atomic losses) evolution of trapped atoms is described by a master equation \[5\]

\[
\frac{d\rho}{dt} = \frac{1}{i\hbar}[\hat{H}, \rho] + \sum_i \gamma_i \int d^3x \, \mathcal{D}[\hat{\psi}^i(\vec{x})] \rho.
\]

Here \( \hat{H} \) is a Hamiltonian of trapped atoms

\[
\hat{H} = \int d^3x \left[ \frac{\hbar^2}{2m} \nabla^2 \hat{\psi}^i \nabla \hat{\psi} + V(\vec{x}) \hat{\psi}^i \hat{\psi} + \frac{g}{2} \hat{\psi}^i \hat{\psi}^i \hat{\psi}^j \hat{\psi}^j \right],
\]

with \( g = 4\pi\hbar^2 a/m \), where \( a \) is the s-wave scattering length. \( \mathcal{D}[\hat{\psi}^i(\vec{x})] \) is a Lindblad superoperator \[6\]

\[
\mathcal{D}[\hat{a}^\dagger \rho \hat{a}] \equiv \hat{a} \rho \hat{a}^\dagger - \frac{1}{2} \hat{a}^\dagger \hat{a} \rho - \frac{1}{2} \rho \hat{a}^\dagger \hat{a},
\]

describing l-body losses.

Bogoliubov theory.— We assume that almost all \( N \) atoms occupy a common condensate wave function \( \phi_0(\vec{x}) \), which solves a stationary Gross-Pitaevskii equation \[7\]

\[
\mu \phi_0 = -\frac{\hbar^2}{2m} \nabla^2 \phi_0 + V(\vec{x}) \phi_0 + N g |\phi_0|^2 \phi_0.
\]

The annihilation operator can be split into a condensed part and a non-condensed part which is then approximated by an expansion in Bogoliubov modes \[8\],

\[
\hat{\psi}(\vec{x}) \approx N^{1/2} \phi_0(\vec{x}) + \delta \hat{\psi}(\vec{x}) \approx N^{1/2} \phi_0(\vec{x}) + \sum_{m=1}^{\infty} \left[ \hat{b}_m u_m(\vec{x}) + \hat{b}_m^\dagger v_m(\vec{x}) \right].
\]
Here $\hat{b}$'s are bosonic quasiparticle annihilation operators, and the wave functions $u_m$ and $v_m$ satisfy Bogoliubov-de Gennes equations [3].

The operator of a number of atoms depleted from $\phi_0$ is $d\hat{N} = \int d^3x \, \delta \hat{\psi}^\dagger \delta \hat{\psi}$. In the Bogoliubov vacuum state $|0_b\rangle$ without any quasiparticles, $\hat{b}_m |0_b\rangle = 0$, the number of depleted atoms is

$$d\hat{N}^{(0)} = \sum_m \int d^3x \, |u_m|^2 = \sum_m dN_m^{(0)}. \quad (6)$$

More generally, in a state with exactly $n_m$ quasiparticles in a mode $m$ the number of depleted atoms is

$$d\hat{N} = \sum_m \left[ dN_m^{(0)} + \left( 1 + 2dN_m^{(0)} \right) n_m \right]. \quad (7)$$

Here we used $\int d^3x \, (|u_m|^2 - |v_m|^2) = 1$.

**Master equation in the quasiparticle representation.** — Bogoliubov expansion [3] can be used to rewrite the master equation (11) in the quasiparticle representation. Expansion to second order in $\hat{b}$'s in Eqs. (11) results in a Bogoliubov Hamiltonian [3], which is a sum of harmonic oscillators $\hat{H} \approx \sum_n \hbar \omega_n \hat{b}^\dagger_n \hat{b}_n$, and an approximate master equation

$$\frac{d\hat{\rho}}{dt} = \sum_m -i\hbar \omega_m [\hat{b}_m^\dagger \hat{b}_m, \hat{\rho}] + \sum_{ml} \lambda_{ml} \alpha_{lm} N_l^{-1} \left[ (1 + n_{lm}) D[\hat{b}_m] \rho + n_{lm} D[\hat{b}_m^\dagger] \rho \right]. \quad (8)$$

The coefficients are defined by integrals

$$\int d^3x \, |\phi_0|^{2l-2} |u_m|^2 = \alpha_{lm} (1 + n_{lm}), \quad (9)$$

$$\int d^3x \, |\phi_0|^{2l-2} |v_m|^2 = \alpha_{lm} n_{lm}. \quad (10)$$

In addition to small depletion, derivation of the master equation (8) requires the rotating wave approximation (RWA). In the RWA we neglect all terms of the form $\hat{b} \hat{b}^\dagger$, $\hat{b}^\dagger \hat{b}$, or $\hat{b}_m \hat{b}_n^\dagger$ for $m \neq n$, but keep all terms like $\hat{b}_m \hat{b}_n$ or $\hat{b}_m^\dagger \hat{b}_n^\dagger$. The RWA is accurate when the Hamiltonian evolution is much faster than atomic losses, or more precisely $\omega_m \gg \sum_l \lambda_{lm} \alpha_{lm} N_l^{-1} n_{lm}$. This condition is satisfied in all present day experiments.

**A thermal state.** — Due to atomic losses the coefficients in (5) are not constant. However, we fix them (for a while) and analyze a stationary state of the resulting master equation. Later on we will see that such an analysis allows us to predict a lower bound for a stationary depletion of a condensate caused by atomic losses.

A remarkable thing is that the master equation (8), with the coefficients fixed, can be recognized to describe a set of harmonic oscillators (numbered by $m$) coupled to external heat reservoirs (numbered by $l$). Every oscillator relaxes to a thermal state. When atomic losses are dominated by only one of the channels $l$, then average numbers of quasiparticles in the thermal states are

$$n_m = \text{Tr} \hat{b}_m^\dagger \hat{b}_m \rho (t \to \infty) = n_{lm}. \quad (11)$$

When many channels $l$ are involved, then the averages $n_m$ can be obtained from equations

$$\frac{1 + n_m}{n_m} = \frac{\sum_l \lambda_{lm} \alpha_{lm} N_l^{-1} (1 + n_{lm})}{\sum_l \lambda_{lm} \alpha_{lm} N_l^{-1} n_{lm}}. \quad (12)$$

Every oscillator $m$ can be assigned to an inverse temperature $\beta_m$ which follows from a textbook formula $n_m = (e^{\beta_m \hbar \omega_m} - 1)^{-1}$. The thermal state is

$$\rho (t \to \infty) = \otimes_m e^{- \beta_m \hbar \omega_m \hat{b}_m^\dagger \hat{b}_m}. \quad (13)$$

**A thermal state of a uniform BEC.** — For a 3D condensate with $\phi_0(\vec{x}) = \text{const}$, the $n_{lm} = dN_m^{(0)}$ are independent of $l$, compare Eqs. (11) and (13). In a uniform condensate of density $\rho_c$ a phonon of momentum $\hbar k$ has energy $\hbar \omega_k$

$$\hbar \omega_k = \sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2g\rho_c \right) + \frac{k^2 \xi^2}{m}} \approx \frac{c}{\hbar k} \sqrt{\frac{m}{\hbar}}. \quad (14)$$

Here $c = \sqrt{g\rho_c/m}$ is a velocity of sound. In the thermal state (13) there are on average $\hbar \omega_k$

$$n_k = \int d^3x \, |v_k|^2 = \frac{k^2 \xi^2}{2m} + g\rho_c \frac{1}{2} \frac{1}{\xi^2} \frac{c}{\hbar k} \approx \frac{c}{\hbar k} \frac{1}{2 \hbar k} \quad (15)$$

phonons of momentum $\hbar k$. With the general formula (7) we can calculate a fraction of depleted atoms in the thermal state $d = d\hat{N}/N = \frac{1}{4\pi \rho_c a^3}$ for a typical condensate density of $\rho_c = 10^{20} m^{-3}$ we find depletions $d_{2Na} = 0.44\%$ and $d_{37Rb} = 1.55\%$ at the scattering lengths of $a = 2.5$ nm and $a = 5.8$ nm respectively.

The depletion is dominated by a contribution from small $k$ where the number of quasiparticles $n_k$ in Eq. (13) is divergent. A remarkable thing is that for small $k$ the equipartition of energy $n_k \hbar \omega_k = \beta^{-1}$ yields the same temperature $T$ for all phonons. The temperature is $T_{2Na} = 76$ nK and $T_{37Rb} = 37$ nK.

These estimates are valid for a 3D uniform condensate. In less than 3D the infrared divergence $n_k \sim k^{-1}$ results in a divergent depleted fraction $d$. Anticipating a much larger but finite $d$ we now turn to effectively one-dimensional harmonic traps.

**A thermal state of a BEC in a 1D harmonic trap.** — In a sufficiently anisotropic trap $V(x,y,z) = \frac{\hbar}{2m} \omega^2 [x^2 + \kappa^2 (y^2 + z^2)]$ with $\kappa \gg 1$ the $y - z$ state of all atoms is frozen in the ground state. The condensate wave function $\phi_0(x)$ solves a 1D Gross-Pitaevskii equation [4] with an effective $g_{1D} = g/\xi^2$, where $\xi$ is the size of the ground state in the $y - z$ plane. We solved the 1D Bogoliubov-de Gennes equations [3] to get $u_m$ and
TABLE I: Lower bound for the stationary depletion due to atomic losses for parameters corresponding to quasi-1D \cite{9} and 1D \cite{10} experiments. In the calculations the values of $g_{1D}/\hbar \omega$ have been estimated to be 7500 (where $N = 1.5 \cdot 10^6$) and 500 ($N = 10^3$), respectively.

|                | 1-body losses | 3-body losses |
|----------------|--------------|---------------|
| quasi-1D       | 6%           | 10%           |
| 1D             | 2%           | 4%            |

$v_m$ for two sets of parameters relevant to the quasi-1D \cite{9} and strictly 1D \cite{10} experiments. Relative depletions 

d = \frac{dN}{N}$ corresponding to the thermal state \cite{19} are listed in Table I. These values give lower bounds for the 
stationary depletions in these experiments. The 10% depletion in the 3-body losses for the quasi-1D condensate 
is actually close to the thermal cloud fraction estimated in the Hannover experiment \cite{9}. In other words, this 
experiment is close to the minimal stationary depletion set by atomic losses.

Relaxation time.— It takes time to reach the thermal state \cite{19}. To simplify notation we assume here 
that losses are dominated by only one of the channels $l$. Time evolution of the average number of quasiparticles $n_m(t) = \text{Tr} \rho(t) \hat{b}_m \hat{b}_m^\dagger$ directly follows from the master equation \cite{8} and satisfies a differential equation

$$\frac{d}{dt} n_m(t) = -\gamma l \alpha_l m N^{-1}(t)[n_m(t) - n_{lm}] \ . \quad (16)$$

$n_m(t)$ is relaxing toward its equilibrium value $n_{lm}$. Eq. (16) has to be compared with the decay law for the 
total number of atoms $N(t) = \text{Tr} \rho(t) \int d^3x \hat{\psi}^\dagger \hat{\psi}$

$$\frac{d}{dt} N(t) = -\gamma l \alpha_l N(t) \ . \quad (17)$$

Here $\alpha_l = \int d^3x |\phi_l|^2$. This equation is valid for small depletion, when almost all atoms occupy $\phi_l(x)$.

As dominant Bogoliubov modes are localized on the condensate, we can approximate $\alpha_l m \approx \alpha_l$. Consequently the 
relaxation and decay rates in Eqs. (16) and (17) are comparable, $d n_{lm}/n_m \approx d N/N$. As the equilibrium value 
n_{lm} in Eq. (16) depends on the time-dependent $N(t)$ and the rates are comparable, $n_m(t)$ will never quite reach the 
instantaneous equilibrium value $n_{lm}(t)$. In the following we will consider 1- and 3-body losses in the two modes 
(double well) toy model to see that the estimated equilibrium depletion, we have considered so far, is a lower 
bound for a stationary depletion.

Double well model.— The model is described by Hubbard Hamiltonian

$$\hat{H}_2 = -\Omega (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1) \ . \quad (18)$$

Here we use rescaled dimensionless units such that $\hbar = 1$.

The master equation (1) becomes

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar} [\hat{H}_2, \hat{\rho}] + \gamma_l \sum_{j=1,2} D[\hat{a}_j^\dagger \hat{a}_j] \hat{\rho} \ . \quad (19)$$

We assume that $l$-body losses dominate.

The condensate wavefunction $\phi_0 = (1, 1)/\sqrt{2}$ and $\phi_1 = (1, -1)/\sqrt{2}$ span the single particle Hilbert space. There 
is only one Bogoliubov mode with $u_1 = X \phi_1/\sqrt{X^2 - 1}$ and $v_1 = -\phi_1/\sqrt{X^2 - 1}$, where $X = (1 + \frac{4O}{N}) + \sqrt{(1 + \frac{4O}{N})^2 - 1}$. As $\phi_0$ is “uniform”, the equilibrium number of quasiparticles is independent of $l$,

$$n_{l1} = dN_{l1}(0) = \nu_1 v_1 = \sqrt{\frac{N}{32\Omega}} - \frac{1}{2} + O(N^{-1/2}) \ . \quad (20)$$

Depletion in the equilibrium thermal state \cite{18} is

$$d = \frac{dN(0)}{N} + \left(1 + 2dN(0)\right)n_{l1} = \frac{1}{16\Omega} + O(N^{-1}) \ . \quad (21)$$

1-body losses.— We begin with $l = 1$ and a condensate initially in the Bogoliubov vacuum state, i.e. the 
number of quasiparticles $n_{l1}(0) = 0$. A formal solution of Eq. (20), where $\alpha_l = 1$, is $n_{l1}(t) = \gamma_1 \int_0^t d\tau \text{exp}[-\gamma_1 (t - \tau)] n_{l1}(\tau)$. $N(t) = N_0 e^{-\gamma_1 t}$, which solves Eq. (17), and Eqs. (20) give (time-dependent) depletion

$$d^{l=1}(t) = \frac{1 - \sqrt{f}}{8\Omega} + \frac{f}{\sqrt{32\Omega} N} + O(N^{-1}) \ . \quad (22)$$

Here $f = N(t)/N_0$ is a fraction of atoms remaining in the trap. When $f \to 0$ (the system is forgetting about the 
initial conditions) the depletion (22) becomes roughly twice the equilibrium value (21). At small $f$ the system is 
reaching a stationary state with twice the equilibrium depletion.

When we initially prepare the system with $n_{l1}(0) = 2\sqrt{\frac{N_0}{32\Omega}} - \frac{1}{2}$, instead of the rather arbitrary $n_{l1}(0) = 0$, 
then the depletion is stationary — it does not depend on $f$ (and through $f$ on the initial $N_0$),

$$d^{l=1}_{\text{stat}} = \frac{1}{8\Omega} + O(N^{-1}) \ . \quad (23)$$

This stationary depletion (23) is roughly twice the equilibrium value (21).

3-body losses.— When 3-body losses dominate, the system gets much closer to the equilibrium than in the 
life of 1-body losses. For a condensate initially in the Bogoliubov vacuum state, $n_{l1}(0) = 0$, a similar procedure 
as for $l = 1$ leads to a depletion

$$d^{l=3}(t) = \frac{3(1 - f^{5/2})}{40\Omega} + \frac{f^3}{\sqrt{32\Omega} N} + O(N^{-1}) \ . \quad (24)$$

When $f \to 0$ $d^{l=3}(t)$ is approaching a stationary value

$$d^{l=3}_{\text{stat}} = \frac{3}{40\Omega} + O(N^{-1}) \ , \quad (25)$$
that is only 1.2 higher than the equilibrium value \[21\]. Starting with the initial \[n_1(0) = \frac{6}{\pi} \sqrt{\frac{N_0}{32\pi}} - \frac{1}{2}\] results in a depletion independent of \(f\) (and thus also on \(N_0\)) and equal to the stationary value \[20\].

We conclude that our estimates of depletion based on the equilibrium values are lower bounds for stationary depletions.

**Numerical experiment.**— We verified the predictions \[22\] of the Bogoliubov theory in numerical simulations. For a large \(N_0\) a direct solution of the master equation \[19\] is not the most efficient. It is better to replace the deterministic \(\rho(t)\) by an ensemble of stochastic pure states \(|\Psi(t)\rangle\), such that \(\rho(t)\) is reproduced as an average over many stochastic realizations, \([\Psi(t)]|\Psi(t)\rangle = \rho(t)\). A stochastic “unraveling” of the master equation \[19\] is given by Ito stochastic nonlinear Schrödinger equation \[11\],

\[
\begin{align*}
    d|\Psi\rangle &= -idt\hat{H}_2|\Psi\rangle - \frac{d\gamma_l}{2} \sum_{j=1,2} \left( (\hat{a}^+_j)^\dagger \hat{d}^+_j \hat{d}_j - (\hat{a}_j)^\dagger \hat{a}_j \right) |\Psi\rangle \\
    &+ \sum_{j=1,2} dN_j(t) \left[ \frac{\hat{d}_j|\Psi\rangle}{\sqrt{\langle (\hat{a}_j)^\dagger \hat{a}_j \rangle}} - |\Psi\rangle \right],
\end{align*}
\]

where \(\langle \hat{A} \rangle \equiv \langle \Psi|\hat{A}|\Psi\rangle\), \(dN_j(t) \in \{0,1\}\) is a stochastic process \([dN_j(t) = 1\) when \(l\) atoms escape from a well \(j\) between \(t\) and \(t + dt\), and 0 otherwise\]. A probability that \(l\) atoms will escape between \(t\) and \(t + dt\) is \(dt \langle (\hat{a}_j)^\dagger \hat{a}_j \rangle\).

In Fig.1 we compare the predictions \[22\] with corresponding averages over many stochastic realizations. As they compare quite well, the Bogoliubov theory \[8\] passes the test on the double-well model.

**Conclusion.**— A condensate is heated by atomic losses. The depletion of the system approaches a stationary value that ranges from around 1% for a uniform 3D condensate to around 10% for a quasi-1D harmonic trap. As atomic losses cannot be easily eliminated, this depletion is a serious limitation on quantum coherence of atomic BEC. We only note here that outcoupling in the atom laser is a non-markovian process of atomic losses. Its influence on laser coherence will be addressed elsewhere.

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