Subalgebras, subgroups, and singularity

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Abstract
This paper is concerned with the noncommutative analog of the normal subgroup theorem for certain groups. Inspired by Kalantar and Panagopoulos (arXiv:2108.02928, 2021, 16), we show that all $\Gamma$-invariant subalgebras of $L^\Gamma$ and $C_0^\ast(\Gamma)$ are ($\Gamma$-)coamenable. The groups we work with satisfy a singularity phenomenon described by Bader et al. (Invent. Math. 229 (2022), 929–985). The setup of singularity allows us to obtain a description of $\Gamma$-invariant intermediate von Neumann subalgebras $L^\infty(X, \xi) \subset \mathcal{N} \subset L^\infty(X, \xi) \rtimes \Gamma$ in terms of the normal subgroups of $\Gamma$.

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1 | INTRODUCTION

The notion of “singularity” has been used to prove rigidity results for $\Gamma$-operator algebras in various settings, where $\Gamma$ is a discrete countable group. It appears in the works of [4, 15, 18, 23], and
so on, where the authors put singular states into use. More recently, [2, 3] used singularity in the context of $\Gamma$-equivariant ucp maps $\Phi : M \to L^\infty(B, \nu)$, where $M$ is a $\Gamma$-von Neumann algebra and $(B, \nu)$ is a nonsingular probability $\Gamma$-space. Such $\Phi$ is called singular if the states on $M$ given by the dual map for almost every $b \in B$ are singular with respect to their $\Gamma$-translations.

In this paper, we further highlight the role of the singularity of ucp maps for rigidity phenomena. We first make the following definition. Let $E$ denote the canonical conditional expectation on $L^\infty(B, \nu) \rtimes \Gamma$. We say that the action has the “singular-hereditary” property (abbreviated as SH) if for every $\Gamma$-invariant von Neumann algebra $M \subset L^\infty(B, \nu) \rtimes \Gamma$, either $E|_M$ is $\Gamma$-singular as a ucp map or $\nu \circ E|_M$ is a $\Gamma$-invariant state. In our first main result, we use the SH property, combined with Zimmer amenability, to conclude that all the invariant subalgebras of $C^*_\nu(\Gamma)$, or of $L(\Gamma)$ are coamenable.

**Theorem 1.1.** Let $\Gamma$ be a countable discrete group. Assume that there exists a nonsingular $\Gamma$-space $(B, \nu)$ that has the SH-property and is Zimmer amenable. Then, every nontrivial $\Gamma$-$C^*$-subalgebra $A \subset C^*_\nu(\Gamma)$ is coamenable. Similarly, every $\Gamma$-invariant von Neumann subalgebra $M \subset L(\Gamma)$ is coamenable.

Here, coamenability of $A$ is in the sense of [24], namely, the commutant $A' \subset B(\ell^2(\Gamma))$ admits a $\Gamma$-invariant state (similarly for $M \subset L(\Gamma)$). Kalantar–Panagopoulos proved the conclusion of Theorem 1.1 for higher rank lattices using “noncommutative Nevo–Zimmer” theorem [5]. This neoteric result of Kalantar–Panagopoulos stirred our interest in this problem.

It is worth pointing out that the noncommutative Nevo–Zimmer theorem [5, Theorem B] made up one of the key ingredients in the work of [24]. It is known that such a structure theorem cannot hold for semisimple Lie groups admitting a rank one factor. However, there are examples of groups that are a product of rank one factors and yet have an action on a nonsingular $\Gamma$ space that has SH-property (see Example 2.4).

Let us also note that the conclusion of Theorem 1.1 can be considered a noncommutative strengthening of Margulis’ normal subgroup theorem. Indeed, any subgroup for which the conclusion holds is just-nonamenable (i.e., all its normal subgroups are coamenable). If we assume in addition that the group $\Gamma$ has property (T), then all $\Gamma$-invariant subalgebras are cofinite, and all normal subgroups are of finite index. We remark that the key to the normal subgroup theorem (NST) lies in understanding the structure of the Furstenberg–Poisson boundary.

In addition, it follows from Theorem 1.1 that if $\Gamma$ is a nonamenable group admitting a Zimmer amenable SH-space, then $\Gamma$ has trivial amenable radical. Examples of groups that satisfy the conditions of Theorem 1.1 can be found in [2, 3], where NST was shown. In these examples, the source of such $\Gamma$-actions is the Furstenberg–Poisson boundary of a random walk on a locally compact group $(G, \mu)$ associated with $\Gamma$. The setup in these examples contrasts with the structure of higher rank lattices [24], where the space is a Furstenberg–Poisson boundary of the group $\Gamma$ itself.

Since our classification in Theorem 1.1 is dependent on the structure of the subalgebras of the crossed product $L^\infty(B, \nu) \rtimes \Gamma$, we conjecture the following for higher rank lattices.

**Conjecture.** Let $\Gamma$ be an irreducible lattice in a higher rank semisimple Lie group $G$ with a finite center and no nontrivial compact factor, all of whose simple factors have real rank of at least two. Let $(G/P, \nu_P)$ be the Furstenberg–Poisson boundary associated with a random walk $\mu$ on $\Gamma$. Then, every $\Gamma$-invariant subalgebra of the crossed product $L^\infty(G/P, \nu_P) \rtimes \Gamma$ is of the form $L^\infty(G/Q, \nu_Q) \rtimes \Lambda$, where $\Lambda < \Gamma$ and $Q$ is a parabolic subgroup of $G$. 
An affirmative answer to the above conjecture would completely describe the \(\Gamma\)-invariant subalgebras of the crossed product. We provide a sufficient condition (Proposition 5.6) that implies this conjecture.

Almost all the known results in this direction deal with “intermediate algebras” \(M\) of the form \(L(\Gamma) \subset M \subset N \rtimes \Gamma\) (see, e.g., [1, 19, 25]). At the same time, there has been considerable work to describe intermediate algebras \(M\) of the form \(N \subset M \subset \mathbb{K} \rtimes \Gamma\), where \(N\) is a von Neumann algebra (see, e.g., [10–12, 20] to name a few). In this paper, we provide a similar kind of classification for \(\Gamma\)-invariant subalgebras of \(L^\infty(X, \xi) \rtimes \Gamma\) containing \(L^\infty(X, \xi)\), where \((X, \xi)\) is a nontrivial factor of an ergodic nonsingular \(\Gamma\)-space \((B, \nu)\) satisfying the SH-property.

**Theorem 1.2.** Let \((B, \nu)\) be an ergodic nonsingular \(\Gamma\)-space with the SH-property and let \((X, \xi)\) be a nontrivial factor of \((B, \nu)\). Then, every \(\Gamma\)-invariant von Neumann algebra \(M\) with \(L^\infty(X, \xi) \subset M \subset L^\infty(X, \xi) \rtimes \Lambda\) for a normal subgroup \(\Lambda \leq \Gamma\) is a crossed product of the form \(L^\infty(X, \xi) \rtimes \Lambda\).

Notice that since \(\mathbb{K}\) is not assumed to contain \(L(\Gamma)\), it is not automatically \(\Gamma\)-invariant. Moreover, we cannot say anything about intermediate algebras that are not \(\Gamma\)-invariant.

## 2   PRELIMINARIES

Let \(\Gamma\) be a discrete countable group and \(A\) be an unital \(\Gamma\)-\(C^\ast\)-algebra. By this, we mean a \(C^\ast\)-algebra \(A\) endowed with the action \(\Gamma \rtimes A\) by \(\star\)-automorphisms such that the map \(\Gamma \times A \to A\) which sends \((g, x) \to g.x\) is continuous. For a von Neumann algebra \(M\) with separable predual \(M^\ast\), we endow \(M\) with the ultraweak (i.e., weak\(^\ast\)) topology coming from the canonical identification \(M = (M^\ast)^\ast\). Via this identification, \(M^\ast\) (as a subset of \(M^\ast\)) consists of all ultraweakly continuous linear functionals, also called normal linear functionals. By a \(\Gamma\)-von Neumann algebra \(M\), we mean a von Neumann algebra \(M\) equipped with an action \(\Gamma \rtimes M\) by \(\star\)-automorphisms such that the map \(\Gamma \times M \to M\) which sends \((g, x) \to g.x\) is continuous. We briefly recall the notion of boundary structure as defined in [3]. We denote by \(S(A)\), the set of all states on \(A\).

Let us recall the notion of singular states. The states \(\tau, \tilde{\tau} \in S(A)\) are said to be singular \((\tau \perp \tilde{\tau})\) if there exists a net \(0 \leq a_i \leq 1 \in A\) such that \(\lim_i \tau(a_i) = 1\) and \(\lim_i \tilde{\tau}(a_i) = 0\).

Let \(\Phi : M \to L^\infty(B, \nu)\) be a \(\Gamma\)-equivariant ucp map. Upon restricting to an ultraweakly dense \(\Gamma\)-invariant separable \(\mathbb{C}^\ast\)-subalgebra \(\tilde{A}\), we obtain a \(\Gamma\)-equivariant map \(\tilde{\Phi} : B \to S(\tilde{A})\). Moreover, for \(\nu\)-almost every \(b \in B, \vartheta(b) \in S(\tilde{A})\) is defined by \(\Phi(a)(b) = \theta(b)(a)\) for \(a \in \tilde{A}\). We say that \(\Phi\) is \(\Gamma\)-singular if \(s.\vartheta(b) \perp \vartheta(b)\) for almost every \(b \in B\) and \(s \in \Gamma \setminus \{e\}\) (see, e.g., [19, Definition 3.6]). In particular, that there exists a net \(\tilde{a}_i \in \tilde{A} \cap M\) with \(0 \leq \tilde{a}_i \leq 1\) such that \(\lim_i \vartheta(b)(\tilde{a}_i) = 1\) and \(\lim_i g.\vartheta(b)(\tilde{a}_i) = 0\). It follows from [3, Proposition 4.10] that the notion of \(\Gamma\)-singularity of \(\Phi\) is independent of the choice of \(\tilde{A}\). Moreover, we say that \(\Phi\) is invariant if \(\Phi(M) = \mathbb{C}\).

### 2.1   Weak topologies

We now turn to recall the notions of weak topology and ultraweak topology on the set of \(\mathbb{B}(H)\) of bounded linear maps on \(H\). The readers can refer to [26] for more details on these. The weak operator topology (WOT) is generated by open sets of the form

\[ \{ T \in \mathbb{B}(H) : |\langle (T - T_0)\xi, \eta \rangle | < \varepsilon \}, \]
where $T_0 \in \mathbb{B}(\mathcal{H})$, $\xi, \eta \in \mathcal{H}$ and $\varepsilon > 0$. The ultraweak topology (also known as $\sigma$-weak topology) is the topology induced by the open sets of the form

$$\left\{ T \in \mathbb{B}(\mathcal{H}) : \left| \sum_i \langle (T - T_0)\xi_i, \eta_i \rangle \right| < \varepsilon \right\},$$

where $T_0 \in \mathbb{B}(\mathcal{H})$, $\xi_i, \eta_i \in \mathcal{H}$ with $\sum_i \|\xi_i\|^2, \sum_i \|\eta_i\|^2 < \infty$ and $\varepsilon > 0$.

On the closed unit ball of $\mathbb{B}(\mathcal{H})$, the ultraweak topology and the WOT coincide (see [26, Chapter-II, Lemma 2.5]).

### 2.2 Ultraweakly dense $C^*$-subalgebra of a von Neumann algebra

Given a von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space $\mathcal{H}$, we can find an ultraweakly dense $C^*$-subalgebra $\mathcal{A} \subset \mathcal{M}$. We can also choose $\mathcal{A}$ to be separable (in the norm). We shall refer to such a $C^*$-subalgebra as a “separable model” of $\mathcal{M}$. We include a proof of this fact.

**Proposition 2.1.** Let $\mathcal{M}$ be a von Neumann algebra action on a separable Hilbert space $\mathcal{H}$. Then, there exists a unital $C^*$-algebra $\mathcal{A} \subset \mathcal{M}$ such that $\mathcal{A}$ is ultraweakly dense inside $\mathcal{M}$. Moreover, $\mathcal{A}$ is separable in the norm-topology. In particular, $\mathcal{M}$ has a separable model.

**Proof.** Let $\mathcal{B}_1$ denote the closed unit ball of $\mathbb{B}(\mathcal{H})$. It is compact in WOT, and hence, in the ultraweak topology. Since $\mathcal{H}$ is separable, the ultraweak topology on $\mathcal{B}_1$ is metrizable (see [26, Chapter-II, Proposition 2.7]). Therefore, $\mathcal{B}_1$ is separable in the ultraweak topology. Since subsets of separable sets are separable in metric spaces, the unit ball $\mathcal{M}_1$ of $\mathcal{M}$ is separable in the ultraweak topology. Let $\mathcal{A} = \{ a_n : n \in \mathbb{N} \}$ be a countable dense (in the ultraweak topology) subset of $\mathcal{M}_1$. By adjoining the unit of $\mathcal{M}$ to $\mathcal{A}$ if required, we can assume that $\mathcal{A}$ contains the unit of $\mathcal{M}$. Let $\mathcal{A}$ be the unital $C^*$-algebra generated by $\mathcal{A}$. We observe that

$$\left\{ \sum_{j=1}^m (c_j + id_j)a_{1j} \ldots a_{nj} : n, m \in \mathbb{N}, c_j, d_j \in \mathbb{Q} \text{ and } a_{1j}, \ldots, a_{nj} \in \mathcal{A} \cup \mathcal{A}^* \right\}$$

is a countable dense subset of $\mathcal{A}$ in the norm topology. Hence, $\mathcal{A}$ is separable. We now show that $\mathcal{A}$ is ultraweakly dense in $\mathcal{M}$. Let $x \in \mathcal{M}$ and $\varepsilon > 0$ be given. Consider a basic open set $W^x_{\varphi_1, \ldots, \varphi_n}$ around $x$. Note that

$$W^x_{\varphi_1, \ldots, \varphi_n} = \{ y \in \mathcal{M} : |\varphi_i(x - y)| < \varepsilon, \ i = 1, 2, \ldots, n \}.$$

Moreover, for each $i = 1, 2, \ldots, n$, $\varphi_i \in \mathbb{B}(\mathcal{H})_s$ is given by

$$\varphi_i(\cdot) = \sum_j \langle (\cdot)\xi_j, \eta_j^i \rangle, \ \xi_j, \eta_j^i \in \mathcal{H}, \ \sum_j \|\xi_j\|^2, \ \sum_j \|\eta_j\|^2 < \infty.$$

Let $m \in \mathbb{N}$ be such that $\|x\| < m$. Then, $\frac{1}{m} x \in \mathcal{M}_1$. Since $\mathcal{A}$ is ultraweakly dense inside $\mathcal{M}_1$, there exists $n_0 \in \mathbb{N}$ such that

$$a_{n_0} \in W^x_{\varphi_1, \ldots, \varphi_n} = \{ y \in \mathcal{M} : |\varphi_i\left(\frac{1}{m} x - y\right)| < \frac{\varepsilon}{m}, \ i = 1, 2, \ldots, n \}. $$
This, in particular, implies that for each \( i = 1, 2, \ldots, n \),
\[
\left| \varphi_i \left( x - ma_{n_0} \right) \right| = m \left| \varphi_i \left( \frac{1}{m} x - a_{n_0} \right) \right| < m \frac{\varepsilon}{m} = \varepsilon.
\]
Therefore, \( ma_{n_0} \in W_{\varphi_1, \ldots, \varphi_n}^\varepsilon \). Since \( ma_{n_0} \in A \), it follows that \( A \) is ultraweakly dense in \( M \). \( \square \)

### 2.3 Crossed product von Neumann algebra

We briefly recall the construction of the crossed product von Neumann algebra. Let \( \mathcal{M} \) be a \( \Gamma \)-von Neumann algebra. Given a Hilbert space \( \mathcal{H} \), let \( \ell^2(\Gamma, \mathcal{H}) \) be the space of square summable \( \mathcal{H} \)-valued functions on \( \Gamma \), that is,
\[
\ell^2(\Gamma, \mathcal{H}) = \left\{ \xi : \Gamma \to \mathcal{H} \text{ such that } \sum_{h \in \Gamma} \| \xi(h) \|^2_{\mathcal{H}} < \infty \right\}.
\]
There is a natural action \( \Gamma \rtimes \ell^2(\Gamma, \mathcal{H}) \) by left translation:
\[
\lambda_g \xi(h) := \xi(g^{-1}h), \xi \in \ell^2(\Gamma, \mathcal{H}), g, h \in \Gamma
\]
Given a faithful \(*\)-representation \( \pi : \mathcal{M} \to \mathcal{B}(\mathcal{H}) \) of a \( \Gamma \)-von Neumann algebra \( \mathcal{M} \) into the space of bounded operators on the Hilbert space \( \mathcal{H} \), let \( \sigma \) be the \(*\)-representation
\[
\sigma : \mathcal{M} \to B(\ell^2(\Gamma, \mathcal{H}))
\]
defined by
\[
\sigma(a)\xi(h) := \pi(h^{-1}a)\xi(h), a \in \mathcal{M},
\]
where \( \xi \in \ell^2(\Gamma, \mathcal{H}), h \in \Gamma \). The von Neumann crossed product \( \mathcal{M} \rtimes \Gamma \) is generated (as a von Neumann algebra inside \( \mathcal{B}(\ell^2(\Gamma, \mathcal{H})) \)), by the left regular representation \( \lambda \) of \( \Gamma \) and the faithful \(*\)-representation \( \sigma \) of \( \mathcal{M} \) in \( \mathcal{B}(\ell^2(\Gamma, \mathcal{H})) \). Moreover, this representation translates the action \( \Gamma \rtimes \mathcal{M} \) into an inner action by the unitaries \( \{ \lambda(g), g \in \Gamma \} \). It follows from the construction that \( \mathcal{M} \rtimes \Gamma \) contains \( L(\Gamma) \) as a von Neumann subalgebra. The von Neumann crossed product \( \mathcal{M} \rtimes \Gamma \) comes equipped with a \( \Gamma \)-equivariant faithful normal conditional expectation \( \mathbb{E} : \mathcal{M} \rtimes \Gamma \to \mathcal{M} \) defined by
\[
\mathbb{E}(\sigma(a_g)\lambda_g) = \begin{cases} 0 & \text{if } g \neq e \\ \sigma(a_e) & \text{otherwise} \end{cases}
\]
We are now ready to define an SH-space.

**Definition 2.2** (Singular hereditary space). Let \( (B, \nu) \) be an ergodic nonsingular \( \Gamma \)-space. We say that the action \( \Gamma \rtimes (B, \nu) \) has “singular hereditary property” if for every \( \Gamma \)-invariant von Neumann algebra \( \mathcal{M} \subset L^\infty(B, \nu) \rtimes \Gamma \), either \( \mathbb{E}|\mathcal{M} \) is \( \Gamma \)-singular or \( \mathbb{E}(\mathcal{M}) = \mathbb{C} \). In this case, we say that \( (B, \nu) \) is an SH-space.
One can view the definition of SH-spaces as a noncommutative analog of the case where the action on \((B, \nu)\), and on all of its nontrivial factors, is essentially free. Examples of SH-spaces originate from the works of [3] and [2].

**Example 2.3.** Let \(\Gamma\) be a discrete group having trivial amenable radical that satisfies the condition (a) in [3, Proposition 4.17]. We point out that concrete examples of such groups have been provided in Example 2.4. We now claim that the space \((B, \nu)\) mentioned there is an SH-space. Indeed, let \(M\) be a \(\Gamma\)-invariant subalgebra of \(L^\infty(B, \nu) \rtimes \Gamma\) and \(E\) be the canonical conditional expectation associated with the crossed product. Then, letting \(M = M\) and \(E = E\), it follows from condition (a) that either \(E\) is \(\Gamma\)-singular or invariant. Suppose that \(E|_M\) is not \(\Gamma\)-singular. Then, \(E|_M\) being invariant in the sense of [3, Definition 4.1] means that \(E(M) \subset L^\infty(B, \nu)\). Since \((B, \nu)\) is an ergodic space (even metrically ergodic), it follows that \(E(M) = \mathbb{C}\).

We now discuss an example of a group for which the noncommutative Nevo–Zimmer theorem does not hold and admits an SH-space.

**Example 2.4** [3, Theorem D]. Let \(T\) be a biregular tree. We denote by \(\text{Aut}^+(T)\), the group of bicoloring preserving automorphisms of \(T\) that acts 2-transitively on the boundary \(\partial T\). Assume that \(n \geq 2\). For each \(i = 1, 2, \ldots, n\), let \(G_i\) be a closed subgroup of the biregular tree \(\text{Aut}^+(T_i)\). Moreover, let \(\Gamma\) be a cocompact lattice in \(G = G_1 \times \cdots \times G_n\) with dense projections. Note that the noncommutative Nevo–Zimmer theorem does not hold for \(\Gamma\). Now, for each \(i = 1, 2, \ldots, n\), let \(B_i = \partial T_i\). Moreover, equipped with the right measure \(\nu_i\), \((B_i, \nu_i)\) is the Furstenberg–Poisson boundary of \(G_i\) for some generating measure \(\mu_i\) on \(G_i\) (see the discussion in the proof of [3, Theorem D] and [7, Theorem 5.1]). It follows from [7, Corollary 3.2] that \((B, \nu) = (\prod_{i=1}^n B_i, \otimes_{i=1}^n \nu_i)\) is the Furstenberg–Poisson boundary of \(G\). Arguing similarly as in the proof of [3, Theorem D], we obtain that the action \(\Gamma \acts (B, \nu)\) is ergodic and Zimmer-amenable. It follows from the 2-transitivity assumption that the group \(\Gamma\) has a trivial amenable radical. Now, it is shown in [3, Theorem D] that the group \(\Gamma\) satisfies the condition (a) in [3, Proposition 4.17]. As a consequence, it follows from Example 2.3 that \((B, \nu)\) is an SH-space.

We also provide an example of a group to which the noncommutative Nevo–Zimmer theorem applies and, as an upshot, accedes an SH-space.

**Example 2.5** [2]. Let \(k\) be any local field. Let \(G\) be any almost \(k\)-simple connected algebraic group with real rank \(\text{rank}_k(G) \geq 2\). Let \(P < G\) be a minimal parabolic \(k\)-subgroup. Set \(G = G(k)\) and \(P = P(k)\). Let \(\Gamma < G\) be a lattice, equipped with a Furstenberg measure, that is, a measure for which there exists a measure \(\nu_p\) on \(G/P\) such that \((G/P, \nu_p)\) is a Furstenberg–Poisson boundary. We shall argue that \((G/P, \nu_p)\) is an SH-space. Let us begin by observing that the action \(\Gamma \acts (G/P, \nu_p)\) is essentially free and ergodic (3, Lemma 6.2]). Let \(M \subset L^\infty(G/P, \nu_p) \rtimes \Gamma\) be an invariant subalgebra. Arguing similarly as in [24, Lemma 2.16], we see that the action \(\Gamma \acts M\) is ergodic, that is, \(M^\Gamma = \mathbb{C}\). We can now appeal to [2, Theorem 5.4] to conclude that either \(E(M) = \mathbb{C}\) or \(E|_M\) is \(\Gamma\)-singular.

Let us also note that we shall identify \(\mathbb{B}(\ell^2(\Gamma))\) as a \(\Gamma\)-invariant subalgebra of \(\mathbb{B}(\ell^2(\Gamma, H))\). Under this identification, it immediately follows that for any \(\Gamma\)-invariant subalgebra \(A \subset \mathbb{B}(\ell^2(\Gamma))\), the relative commutant \(A' \cap \mathbb{B}(\ell^2(\Gamma))\) is contained inside \(\mathbb{A}'\), the commutant of \(A\) inside \(\mathbb{B}(\ell^2(\Gamma), H)\).
We end this section with the following easy observation that allows us to relate the commutant of \( A \) (or, \( M \)) inside \( \mathbb{B}(\ell^2(\Gamma, H)) \) for \( H = L^2(B, \nu) \) to that of the relative commutant inside \( L^\infty(B, \nu) \rtimes \Gamma \).

**Lemma 2.6.** Let \((B, \nu)\) be a nonsingular \( \Gamma \)-space. Suppose that \( A \) (or, \( M \)) is a \( \Gamma \)-invariant \( C^* \)-subalgebra (or, von Neumann subalgebra) of \( L^\infty(B, \nu) \rtimes \Gamma \) such that there exists a ucp map \( \Phi : \mathbb{B}(\ell^2(\Gamma, L^2(B, \nu))) \to L^\infty(B, \nu) \rtimes \Gamma \) such that \( \Phi|_{L^\infty(B, \nu) \rtimes \Gamma} = \text{id} \).

Then, \( \Phi \) maps \( A' \cap \mathbb{B}(\ell^2(\Gamma, L^2(B, \nu))) \) to the respective relative commutants inside \( L^\infty(B, \nu) \rtimes \Gamma \). Moreover, the map \( \Phi|_A \) (similarly, for \( \Phi|_{M'} \)) is surjective.

**Proof.** Let \( M \) be a \( \Gamma \)-invariant von Neumann subalgebra of \( L^\infty(B, \nu) \rtimes \Gamma \). Let \( T \in \mathbb{B}(\ell^2(\Gamma, L^2(B, \nu))) \) such that \( Tx = xT \) for all \( x \in M \). Then, applying \( \Phi \) on both sides, we obtain that \( \Phi(Tx) = \Phi(xT) \) for all \( x \in M \). Since \( \Phi|_{L^\infty(B, \nu) \rtimes \Gamma} = \text{id} \), \( L^\infty(B, \nu) \rtimes \Gamma \) falls in the multiplicative domain of \( \Phi \) (see [6, Proposition 1.5.7]). Therefore, for all \( x \in M \), we obtain that

\[
\Phi(T)x = \Phi(T)\Phi(x) = \Phi(Tx) = \Phi(xT) = \Phi(x)\Phi(T) = x\Phi(T).
\]

Consequently, it follows that \( \Phi(T) \in M' \cap (L^\infty(B, \nu) \rtimes \Gamma) \). The proof for a \( \Gamma \)-invariant \( C^* \)-subalgebra follows vis-à-vis to the above argument. The surjectivity of the map \( \Phi|_A \) (similarly, for \( \Phi|_{M'} \)) follows from the fact that \( \Phi|_{L^\infty(B, \nu) \rtimes \Gamma} = \text{id} \).

\[\square\]

### 3 | THE SINGULAR HEREDITARY PROPERTY

The key ingredient in the proof of [24] is the deep structural noncommutative-Nevo–Zimmer theorem (see [5, Theorem B]). However, such a phenomenon is only observed in the case of higher-rank lattices. To prove coamenability in our setup, we needed to use instead, the singular hereditary property. The following proposition establishes the link between an invariant algebra and its relative commutant in the crossed product if we know that the second object is singular (also see [18, Lemma 2.2] and [19, Proposition 3.7]).

**Proposition 3.1.** Let \((X, \nu)\) be a nonsingular \( \Gamma \)-space. Let \( M \subset L^\infty(X, \nu) \rtimes \Gamma \) be a \( \Gamma \)-invariant subalgebra. Suppose that the relative commutant \( \hat{M} \) of \( M \) in \( L^\infty(X, \nu) \rtimes \Gamma \) is \( \Gamma \)-singular (i.e., \( E|_{\hat{M}} \) is \( \Gamma \)-singular). Then, \( E(\alpha \lambda(g)) = 0 \) for all \( a \in M \) and for all \( g \in \Gamma \setminus \{e\} \).

**Proof.** We shall complete the proof in three steps.

**Step-I:** Choose a separable model \( \hat{A} \subset L^\infty(X, \nu) \rtimes \Gamma \) such that \( \hat{A} \) contains \( \lambda(\Gamma) \), a separable model \( A \) of \( M \), and a separable model \( A_1 \) of \( \hat{A} \).

Let \( \hat{M}_1 \) denote the unit ball of \( \hat{M} \). It follows from the first part of the proof of Proposition 2.1 that \( \hat{M}_1 \) is separable in the ultraweak topology. Let \( \hat{A}_1 \) be a countable dense (in the ultraweak topology) subset of \( \hat{M}_1 \). By adjoining \( \hat{A}_1 \) with \( \{ \lambda(s)\alpha_1(s)^* : a \in \hat{A}_1, s \in \Gamma \} \), we shall assume that \( \hat{A}_1 \) is \( \Gamma \)-invariant. Likewise, we can find a \( \Gamma \)-invariant countably dense (in the ultraweak topology) subset \( M_1 \) of the unit ball of \( M \). Similarly, we can also find a countably dense (in the ultraweak topology) subset \( A_1 \) of the unit ball of \( L^\infty(X, \nu) \rtimes \Gamma \). Let \( \hat{A} \) be the \( C^* \)-algebra generated by \( \hat{A}_1, M_1, A_1 \), and \( \lambda(\Gamma) \), that is,

\[
\hat{A} = C^*(\hat{A}_1 \cup M_1 \cup A_1 \cup \lambda(\Gamma)).
\]
Moreover, let $\mathcal{A}$ be the $C^*$-algebra generated by $\bar{A}$ and $\lambda(e)$, and $\mathcal{A}_1$, the $C^*$-algebra generated by $M_1$ and $\lambda(e)$. It follows from the later part of the proof of Proposition 2.1 that $\bar{A}$, $\mathcal{A}$, and $\mathcal{A}_1$ are separable models for $L^\infty(X, \nu) \rtimes \Gamma$, $\mathcal{M}$, and $\tilde{\mathcal{M}}$ respectively. Moreover, it is evident from the construction that $\bar{A}$ contains $\mathcal{A}$, $\mathcal{A}_1$, and $\lambda(\Gamma)$.

**Step-2:** \( \mathbb{E}(a\lambda(g)) = 0 \) for all \( a \in \mathcal{A} \) and \( g \in \Gamma \setminus \{e\} \).

Note that $\Gamma \rtimes \bar{A}$ by conjugation. Restrict $\mathbb{E}$ to $\bar{A}$ and denote by $\tilde{\theta} : X \to S(\bar{A})$ the corresponding $\Gamma$-equivariant measurable map. Since $\mathcal{A}_1$ is a separable model for $\mathcal{M}$, using the uniqueness of the map, we see that $\tilde{\theta} : X \to S(\mathcal{A}_1)$ is given by $\tilde{\theta}(b)(a) = \theta(b)(a)$ for $a \in \mathcal{A}_1$. Let $g \in \Gamma \setminus \{e\}$. Since $\tilde{\mathcal{M}}$ is $\Gamma$-singular, we can find $\tilde{X} \subset X$ a conull measure subset such that for every $x \in \tilde{X}$, $\tilde{\theta}(x) \perp g.\tilde{\theta}(x)$. Fix $x \in \tilde{X}$. It follows that there exists a net $\bar{a}_i \in \mathcal{A}_1$ with $0 \leq \bar{a}_i \leq 1$ such that $\lim_i \tilde{\theta}(x)(\bar{a}_i) = 1$ and $\lim_i g.\tilde{\theta}(x)(\bar{a}_i) = 0$. This, in particular, shows that $\tilde{\theta}(x) \perp g.\tilde{\theta}(x)$.

We first note that $\theta(x)$ is a state on $\bar{A}$. Since $\mathcal{A}$ and $\lambda(\Gamma)$ are both contained in $\bar{A}$, $\theta(x)(a\lambda(g))$ makes sense for all $a \in \mathcal{A}$ and for all $g \in \Gamma \setminus \{e\}$. Let $\tau = \theta(x)$. Now, since $\bar{a}_i a = a \bar{a}_i$, we see that

$$|	au(\bar{a}_i a \lambda(g))|^2 = |\tau\left(a\bar{a}_i a^\pi \lambda(g)\right)|^2 \leq \tau(a\bar{a}_i a^*)\tau(\lambda(g^{-1})\bar{a}_i \lambda(g)) = \tau(a\bar{a}_i a^*)g.\tau(\bar{a}_i).$$

Therefore, we obtain that

$$\lim_i \tau(\bar{a}_i a \lambda(g)) = 0.$$

On the other hand,

$$\lim_i \tau((1 - \bar{a}_i) a \lambda(g)) = \lim_i \tau\left((1 - \bar{a}_i) \frac{1}{2} (1 - \bar{a}_i) \frac{1}{2} a \lambda(g)\right) \leq \lim_i \|\tau((1 - \bar{a}_i))\|^\frac{1}{2} \|\tau(\lambda(g^{-1})a^*(1 - \bar{a}_i)a \lambda(g))\|^\frac{1}{2} = 0.$$

Now, combining the above two identities, we see that

$$\tau(a \lambda(g)) = \lim_i \tau(\bar{a}_i a \lambda(g)) + \lim_i \tau((1 - \bar{a}_i) a \lambda(g)) = 0.$$

In particular, we obtain that $\theta(x)(a \lambda(g)) = 0$ for all $x \in \tilde{X}$. This, in turn, implies that $\mathbb{E}(a \lambda(g)) = 0$ for all $a \in \mathcal{A}$ and $g \in \Gamma \setminus \{e\}$.

**Step-3:** $\mathbb{E}(a \lambda(g)) = 0$ for all $a \in \mathcal{M}$ and for all $g \neq e$.

Let $a \in \mathcal{M}$ and $g \in \Gamma \setminus \{e\}$ be given. Since $\mathcal{A}$ is ultraweakly dense in $\mathcal{M}$, we can find a net $a_i \in \mathcal{A}$ such that $a_i \xrightarrow{\text{ultraweakly}} a$. Therefore, $a_i \lambda(g) \xrightarrow{\text{ultraweakly}} a \lambda(g)$. Since $\mathbb{E}$ is normal, it follows that $\mathbb{E}(a \lambda(g)) = \lim_i \mathbb{E}(a_i \lambda(g)) = 0$. \(\square\)
Let us now discuss our proof strategy for Theorem 1.1. Let \( \mathcal{A} \subset C^*_r(\Gamma) \) (or \( \mathcal{M} \subset L(\Gamma) \)) be a \( \Gamma \)-invariant subalgebra. We shall use Proposition 3.1 combined with the SH-property of the action \( \Gamma \acts (B, \nu) \) to conclude that the relative commutant of the subalgebra in the crossed product \( L^\infty(B, \nu) \rtimes \Gamma \) has a \( \Gamma \)-invariant state. From this point onward, our method varies from our predecessors in [24].

Since \((B, \nu)\) is not the Poisson boundary associated with a random walk on \( \Gamma \), we can no longer use Izumi’s isomorphism theorem [21, Theorem 4.1]. Instead, we shall use the Zimmer amenability of the action \( \Gamma \acts (B, \nu) \) to conclude that the commutant of \( \mathcal{A} \) (or \( \mathcal{M} \)) inside \( \mathbb{B}(\ell^2(\Gamma, L^2(B, \nu))) \) has a \( \Gamma \)-invariant state.

**Proof of Theorem 1.1.** Let \((B, \nu)\) be a Zimmer amenable SH-space. We first prove the result in the setting of \( C^*_r(\Gamma) \). Assume that \( \mathcal{A} \subset C^*_r(\Gamma) \) is a \( \Gamma \)-invariant nontrivial subalgebra. We shall show that the commutant \( \mathcal{A}' \) contained in \( \mathbb{B}(\ell^2(\Gamma, L^2(B, \nu))) \) has a \( \Gamma \)-invariant state.

Denote by \( \mathcal{A}'_1 \), relative commutant of \( \mathcal{A} \) inside \( L^\infty(\Gamma) \rtimes \Gamma \). Since \((B, \nu)\) is an SH-space, it follows that either \( \mathbb{E}|_{\mathcal{A}'_1} \) is \( \Gamma \)-singular or \( \mathbb{E}(\mathcal{A}'_1) = \mathbb{C} \). Let us now argue that the former cannot happen, that is, we shall show that \( \tau|_{\mathcal{A}'_1} \) is \( \Gamma \)-invariant, where \( \tau = \nu \circ \mathbb{E} \). We would like to point out that whenever we write \( \nu \circ \mathbb{E} \), we think of \( \nu \) as a state on \( L^\infty(B, \nu) \) given by the integration with respect to \( \nu \). For the sake of contradiction, let us assume that the relative commutant \( \mathcal{A}'_1 \) is \( \Gamma \)-singular. We denote by \( \tau_0 \) the canonical trace on \( C^*_r(\Gamma) \). Fix \( g \neq h \in \Gamma \setminus \{e\} \) and \( a \in \mathcal{A} \). Since \( \mathbb{E}|_{C^*_r(\Gamma)} = \tau_0 \), using Proposition 3.1, we see that

\[
\langle \tilde{a} \delta_g, \delta_h \rangle = \langle \tilde{a} \lambda(g) \delta_e, \lambda(h) \delta_e \rangle = \langle \lambda(h^{-1}) \tilde{a} \lambda(g) \delta_e, \delta_e \rangle = \tau_0(\lambda(h^{-1}) \tilde{a} \lambda(g)) = \tau_0(\tilde{a} \lambda(gh^{-1})) = 0.
\]

Let \( \mathcal{E} : \mathbb{B}(\ell^2(\Gamma)) \to \ell^\infty(\Gamma) \) be the projection onto the diagonal part, that is,

\[
\mathcal{E}(T)(\delta_g) = \langle T(\delta_g), \delta_g \rangle \delta_g, \quad T \in \mathbb{B}(\ell^2(\Gamma)), \quad g \in \Gamma.
\]

Considering \( \tilde{a} \) as an element in \( \mathbb{B}(\ell^2(\Gamma)) \), we can write

\[
\tilde{a}(\delta_g) = \sum_{h \in \Gamma} \langle \tilde{a}(\delta_g), \delta_h \rangle \delta_h = \langle \tilde{a}(\delta_g), \delta_g \rangle \delta_g = \mathcal{E}(\tilde{a})(\delta_g).
\]

Therefore, it follows that \( \tilde{a} \in \ell^\infty(\Gamma) \cap C^*_r(\Gamma) = \mathbb{C} \). Since \( \tilde{a} \) is an arbitrary element, it follows that \( \mathcal{A} = \mathbb{C} \). This, in turn, leads to a contradiction because we assumed \( \mathcal{A} \) to be nontrivial in the beginning. As a result, it follows that \( \tau|_{\mathcal{A}'_1} \) is invariant. A similar argument applies to a nontrivial \( \Gamma \)-invariant von Neumann subalgebra \( \mathcal{M} \subset L(\Gamma) \). Alternatively, we can also argue the following. Let \( \mathcal{M}'_1 \) denote the relative commutant of \( \mathcal{M} \) inside \( L^\infty(B, \nu) \rtimes \Gamma \). Suppose that \( \mathbb{E}|_{\mathcal{M}'_1} \) is \( \Gamma \)-singular.
Let \( \tilde{a} \in \mathcal{M} \). We can now appeal to Proposition 3.1 to conclude that \( \mathbb{E}(a\lambda(g^{-1})) = 0 \) for all non-identity elements \( g \in \Gamma \). Since the family \( \{\mathbb{E}(\tilde{a}\lambda(g^{-1})) : g \in \Gamma\} \) completely determines \( \tilde{a} \) (see, e.g., the discussion following Lemma 7.5 in [26]), it follows that \( \tilde{a} \in \mathbb{C} \). Since \( \tilde{a} \in \mathcal{M} \) is arbitrary, this implies that \( \mathcal{M} = \mathbb{C} \) which contradicts the nontriviality of \( \mathcal{M} \). Hence, we obtain that \( \tau|_{\tilde{\mathcal{M}}_1} \) is invariant.

Now, since \( \Gamma \rtimes (B, \nu) \) is Zimmer amenable, we obtain a projection \( \Phi : \ell^2(\Gamma, L^2(B, \nu)) \to L^\infty(B, \nu) \rtimes \Gamma \) (cf. [28, Theorem 2.1]). Since \( \Phi|_{L^2(\Gamma)} = \text{id} \), using Lemma 2.6, we obtain that \( \Phi \) maps \( \mathcal{A}' \) (similarly, \( \mathcal{M}' \)) to the respective relative commutants inside \( L^\infty(B, \nu) \rtimes \Gamma \). Consequently, the composition of the restriction of \( \tau|_{\tilde{\mathcal{M}}_1} \) (or, \( \tau|_{\tilde{\mathcal{M}}_1} \)) with \( \Phi|_{\tilde{\mathcal{A}}'} \) (or, \( \Phi|_{\tilde{\mathcal{M}}'} \)) gives us an invariant state on \( \tilde{\mathcal{A}}' \) (or, \( \tilde{\mathcal{M}}' \), respectively). □

**Remark 3.2.** We can also deduce the coamenability of the \( C^* \)-algebra case by arguing similarly as in the proof of [9, Corollary 5.7]. We include the proof that was kindly provided to us by the anonymous reviewer. For a \( \Gamma \)-invariant \( C^* \)-algebra \( \mathcal{A} \subset C^*_r(\Gamma) \), consider \( \mathcal{M} = \mathcal{A}'' \cap L(\Gamma) \), the von Neumann algebra generated by \( \mathcal{A} \) inside \( L(\Gamma) \). Now, it follows from the von Neumann algebra case above that there is a \( \Gamma \)-invariant state on \( \mathcal{M}' \cap \mathbb{B}(\ell^2(\Gamma)) \). Since \( \mathcal{M}' \cap \mathbb{B}(\ell^2(\Gamma)) = \mathcal{A}' \cap \mathbb{B}(\ell^2(\Gamma)) \), the claim follows.

## 4 CORRESPONDENCE OF INVARIANT ALGEBRAS FOR SH-ACTIONS

In this section, we give a description of the \( \Gamma \)-invariant intermediate algebras \( \mathcal{M} \) associated with \( L^\infty(X, \nu) \subset \mathcal{M} \subset L^\infty(X, \nu) \rtimes \Gamma \) for essentially free \( \Gamma \)-space \((X, \nu)\) with the singular hereditary property. We begin with the following definition.

We would like to point out that in [12], a correspondence was obtained for intermediate von Neumann algebras \( \tilde{\mathcal{N}} \) of the form \( \tilde{\mathcal{M}} \subset \mathcal{N} \subset \mathcal{M} \rtimes \Gamma \) for a \( \Gamma \)-von Neumann algebras \( \mathcal{M} \) on which the action \( \Gamma \rtimes \mathcal{M} \) is by properly outer \( * \)-automorphisms (also see [11, Corollary 4.5] and the remark thereafter).

We begin with the following observation, which is essentially contained in [8, Theorem 3.7].

**Lemma 4.1.** Let \( \tilde{\mathcal{M}} \subset \mathcal{M} \) be an inclusion of von Neumann algebras with expectation, and let \( u \) be a unitary element in \( \mathcal{M} \) such that \( \tilde{\mathcal{M}} \) is invariant under the conjugation by \( u \). Let \( E_{\tilde{\mathcal{M}}} : \mathcal{M} \to \tilde{\mathcal{M}} \) be a conditional expectation, then \( E_{\tilde{\mathcal{M}}}(u)u^* \in \tilde{\mathcal{M}} \cap \mathcal{M} \).

**Proof.** For \( x \in \tilde{\mathcal{M}} \), we need to show that \( xE_{\tilde{\mathcal{M}}}(u)u^* = E_{\tilde{\mathcal{M}}}(u)u^*x \). Indeed, let us observe that

\[
E_{\tilde{\mathcal{M}}}(u)u^*x = E_{\tilde{\mathcal{M}}}(u)u^*xuu^* = E_{\tilde{\mathcal{M}}}(uu^* x u)u^* = E_{\tilde{\mathcal{M}}}(xu)u^* = xE_{\tilde{\mathcal{M}}}(u)u^*.
\]

In general, given an inclusion of unital von Neumann algebras \( \tilde{\mathcal{M}} \subset \mathcal{M} \), there may not be a conditional expectation from \( \mathcal{M} \) onto \( \tilde{\mathcal{M}} \). However, if the inclusion \( \tilde{\mathcal{M}} \subset \mathcal{M} \) is Cartan, then every intermediate von Neumann algebra \( \tilde{\mathcal{M}} \) of the form \( \tilde{\mathcal{M}} \subset \tilde{\mathcal{M}} \subset \mathcal{M} \) is in the image of a normal (even faithful) conditional expectation [27].
In our context, we only need to deal with intermediate von Neumann algebra $\mathcal{N}$ of the form $L^\infty(X,\nu) \subset \mathcal{N} \subset L^\infty(X,\nu) \rtimes \Gamma$ for nonsingular essentially free $\Gamma$-spaces $(X,\nu)$. It is well known that the inclusion $L^\infty(X,\nu) \subset L^\infty(X,\nu) \rtimes \Gamma$ is Cartan if the action $\Gamma \curvearrowright (X,\nu)$ is essentially free. And hence, every intermediate von Neumann algebra $L^\infty(X,\nu) \subset \mathcal{N} \subset L^\infty(X,\nu) \rtimes \Gamma$ lies in the image of a faithful normal conditional expectation.

In fact, for $\Gamma$-von Neumann algebras $\mathcal{M}$, where the action $\Gamma \curvearrowright \mathcal{M}$ is by properly outer $\ast$-automorphisms, every intermediate von Neumann algebra $\mathcal{N} \subset \mathcal{M} \subset L^\infty(X,\nu) \rtimes \Gamma$ lies in the image of a faithful normal conditional expectation [12, Theorem 3.2]. The notion of properly outer $\ast$-automorphisms coincides with that of essential freeness for commutative von Neumann algebras.

Now, let $(\mathcal{B},\nu)$ be an SH-space. Let us further assume $L^\infty(X,\xi)$ to be a $\Gamma$-invariant subalgebra of $L^\infty(B,\nu)$ with the property that the action $\Gamma \curvearrowright (B,\nu)$ restricted to $(X,\xi)$ is essentially free. Let $\mathcal{M}$ be an intermediate von Neumann algebra of the form $L^\infty(X,\xi) \subset \mathcal{M} \subset L^\infty(X,\xi) \rtimes \Gamma$ lying in the image of a faithful normal conditional expectation $\mathbb{E}_\mathcal{M}$. Then,

$$\tau(x) := \nu|_{L^\infty(X,\xi)} \circ \mathbb{E}_\mathcal{M}(x), \quad x \in L^\infty(X,\xi) \rtimes \Gamma$$

is a faithful normal state on $L^\infty(X,\xi) \rtimes \Gamma$. We can then define the $\|\cdot\|_2$-norm on $L^\infty(X,\xi) \rtimes \Gamma$ associated with $\tau$, defined by

$$\|x\|_2 = \sqrt{\tau(x^*x)}, \quad x \in L^\infty(X,\xi) \rtimes \Gamma.$$ 

The $\|\cdot\|_2$-norm is continuous with respect to the $\sigma$-strong topology, and induces the $\sigma$-strong topology on any bounded (in the operator norm) subset of $L^\infty(X,\xi) \rtimes \Gamma$.

**Remark 4.2.** In the above setup, the $\|\cdot\|_2$-norm is continuous with respect to $\mathbb{E}_\mathcal{M}$, that is, $\|\mathbb{E}_\mathcal{M}(x - y)\|_2 \leq \|x - y\|_2$, $x, y \in L^\infty(X,\xi) \rtimes \Gamma$. Indeed, for $x, y \in L^\infty(X,\xi) \rtimes \Gamma$, which follows easily using the Kadison–Cauchy–Schwartz inequality for the ucp map $\mathbb{E}$.

We now proceed to give a complete description of intermediate von Neumann algebras $\mathcal{M}$ of the form $L^\infty(X,\xi) \subset \mathcal{M} \subset L^\infty(X,\xi) \rtimes \Gamma$.

**Proposition 4.3.** Let $(\mathcal{B},\nu)$ be an SH-space, $L^\infty(X,\xi) \subset L^\infty(B,\nu)$ a $\Gamma$-invariant subalgebra with the property that the action $\Gamma \curvearrowright (B,\nu)$ restricted to $(X,\xi)$ is essentially free. Then, every intermediate $\Gamma$-invariant von Neumann algebras $\mathcal{M}$ of the form $L^\infty(X,\xi) \subset \mathcal{M} \subset L^\infty(X,\xi) \rtimes \Gamma$ is a crossed product of the form $L^\infty(X,\xi) \rtimes \Lambda$ for a normal subgroup $\Lambda \lhd \Gamma$.

**Proof.** Let $\mathcal{M}$ be an intermediate $\Gamma$-invariant von Neumann algebra of the form $L^\infty(X,\xi) \subset \mathcal{M} \subset L^\infty(X,\xi) \rtimes \Gamma$. Since the action $\Gamma \curvearrowright (X,\xi)$ is nonsingular and essentially free, we can use [27, Section 4] or [12, Theorem 3.2] to conclude the existence of a faithful normal conditional expectation $\mathbb{E}_\mathcal{M} : L^\infty(X,\xi) \rtimes \Gamma \to \mathcal{M}$. Let $\tilde{\mathcal{M}} = \mathcal{M}' \cap (L^\infty(X,\xi) \rtimes \Gamma)$ be the relative commutant of $\mathcal{M}$ inside $L^\infty(X,\xi) \rtimes \Gamma$. Observe that $\tilde{\mathcal{M}} \subset L^\infty(X,\xi)' \cap (L^\infty(X,\xi) \rtimes \Gamma)$. Since the action $\Gamma \curvearrowright (X,\xi)$ is essentially free, the latter intersection coincides with $L^\infty(X,\xi)$. Hence, $\tilde{\mathcal{M}} \subset L^\infty(X,\xi)$, and therefore, $\mathbb{E}(\tilde{\mathcal{M}}) = \tilde{\mathcal{M}}$. Now, since $(\mathcal{B},\nu)$ is an SH-space, $\mathbb{E}|_{\tilde{\mathcal{M}}}$ (in this case, we view $\tilde{\mathcal{M}}$ as a subalgebra of $L^\infty(B,\nu) \rtimes \Gamma$) is either $\Gamma$-singular or $\mathbb{E}(\tilde{\mathcal{M}}) = \mathbb{C}$. We consider each of these cases one by one. In the case when $\mathbb{E}(\tilde{\mathcal{M}}) = \mathbb{C}$, since $\mathbb{E}(\tilde{\mathcal{M}}) = \tilde{\mathcal{M}}$, we obtain that $\tilde{\mathcal{M}} = \mathbb{C}$. Let $\Lambda = \{g \in \Gamma : \lambda(g) \in \mathcal{M}\}$.
Since $\mathcal{M}$ is $\Gamma$-invariant, it is easy to see that $\Lambda \triangleleft \Gamma$. Moreover, it is clear from the construction that $L^\infty(X, \xi) \rtimes \Lambda \subseteq \mathcal{M}$. All that remains to show is that $\mathcal{M} \subseteq L^\infty(X, \xi) \rtimes \Lambda$. Since $\mathcal{M}$ is $\Gamma$-invariant, it follows from Lemma 4.1 that

$$E_{\mathcal{M}}(\lambda(g))\lambda(g)^* \in \mathcal{M}' \cap (L^\infty(X, \xi) \rtimes \Gamma) = \mathbb{C}, \ \forall g \in \Gamma.$$ 

Therefore, we obtain that $E_{\mathcal{M}}(\lambda(g)) = a_g\lambda(g)$ for some $a_g \in \mathbb{C}$. Moreover, if $a_g \neq 0$, we see that $E_{\mathcal{M}}(\lambda(g)) \in L(\Lambda)$ just by construction. Let $\tau = \nu|_{L^\infty(X, \xi)} \circ E\circ E_{\mathcal{M}}$, and consider the $\| \cdot \|_2$ norm on $L^\infty(X, \xi) \rtimes \Gamma$ associated with $\tau$. Now, for $x \in \mathcal{M}$ and an arbitrary $\varepsilon > 0$, we can find $f_1, f_2, \ldots, f_n \in L^\infty(X, \xi)$ and $s_1, s_2, \ldots, s_n \in \Gamma$ such that

$$\left\| x - \sum_{i=1}^n f_i\lambda(s_i) \right\|_2 < \varepsilon.$$ 

Since $E_{\mathcal{M}}|_{L^\infty(X, \xi)} = \text{id}$, it follows from remark 4.2 that

$$\left\| E_{\mathcal{M}}(x) - \sum_{i=1}^n f_iE_{\mathcal{M}}(\lambda(s_i)) \right\|_2 < \varepsilon.$$ 

Moreover, since $a \in \mathcal{M}$ and $E_{\mathcal{M}}|_{\mathcal{M}} = \text{id}$, we see that

$$\left\| x - \sum_{i=1}^n f_iE_{\mathcal{M}}(\lambda(s_i)) \right\|_2 < \varepsilon.$$ 

Let us now observe that $E_{\mathcal{M}}(\lambda(s_i)) \in L(\Lambda)$ for each $i = 1, 2, \ldots, n$. As a consequence, we obtain that $\sum_{i=1}^n f_iE_{\mathcal{M}}(\lambda(s_i)) \in L^\infty(X, \xi) \rtimes \Lambda$. Since $\varepsilon > 0$ is arbitrary, it is evident that $x \in L^\infty(X, \xi) \rtimes \Lambda$. This finishes the proof for the case when $\nu|_{\mathcal{M}}$ is invariant. If $E|_{\mathcal{M}}$ is $\Gamma$-singular, it follows from Proposition 3.1 that $\mathcal{M} = E(\mathcal{M})$. Since $L^\infty(X, \xi) \subset \mathcal{M} \subset L^\infty(X, \xi) \rtimes \Gamma$, it follows that $E(\mathcal{M}) = L^\infty(X, \xi) = \mathcal{M}$. $\square$

5 | TOWARD THE CONJECTURE

Let $\Gamma$ be an irreducible lattice in a higher rank connected semisimple Lie group $G$ with a trivial center and no nontrivial compact factor, all of whose simple factors have real rank of at least two. It is known that $\Gamma$ admits a Furstenberg measure $\mu$, that is, a random walk on $\Gamma$ such that the Furstenberg–Poisson boundary associated with a random walk $\mu$ is realized on $G/P$. We denote by $\nu_p$ the corresponding Poisson measure.

Let us now put Proposition 4.3 along with [19, Corollary F] in perspective. The first result gives us a description of the intermediate invariant subalgebras $\mathcal{M}$ of the form $L^\infty(G/Q, \nu_Q) \subset \mathcal{M} \subset L^\infty(G/Q, \nu_Q) \rtimes \Gamma$, where $P \leq Q \leq G$ is a closed subgroup. On the other hand, the second result gives a description of the intermediate algebras $\mathcal{M}$ with $L(\Gamma) \subset \mathcal{M} \subset L^\infty(G/P, \nu_P) \rtimes \Gamma$. Observe that such a $\mathcal{M}$ is automatically $\Gamma$-invariant. At the same time, let us also observe that the invariant algebras $\mathcal{M}$ considered above either share the same group algebra part or the commutative algebra part with those of their upper and lower bounds.

Consequently, considering all of the above, we make the following conjecture.
**Conjecture.** Let $\mathcal{M}$ be a $\Gamma$-invariant subalgebra of $L^\infty(G/P, \nu_P) \rtimes \Gamma$. Then, $\mathcal{M}$ is a crossed product of the form $L^\infty(G/Q, \nu_Q) \rtimes \Lambda$, where $\Lambda \triangleleft \Gamma$.

We can only address the above conjecture under a certain technical assumption. We briefly recall the notion of Poisson transform and some related properties for our later use.

**Definition 5.1.** Let $\mathcal{A}$ be a unital $\Gamma$-$C^*$-algebra and $\varphi$, a state on $\mathcal{A}$. The Poisson transform associated with $\varphi$ is the map $P_\varphi : \mathcal{A} \to \ell^\infty(\Gamma)$ defined by

$$P_\varphi(a)(s) = \varphi(s^{-1}a), \; a \in \mathcal{A}, \; s \in \Gamma.$$  

**Remark 5.2.** Let $\mathcal{A}^\Gamma$ denote the invariant elements in $\mathcal{A}$, that is, $\mathcal{A}^\Gamma = \{a \in \mathcal{A} : s.a = a \; \forall s \in \Gamma\}$. It is clear that $\mathcal{A}^\Gamma \subseteq \{a \in \mathcal{A} : \varphi(sa) = \varphi(a) \forall s \in \Gamma\}$. We observe below that the other inclusion holds if $P_\varphi$ is an isometry. Indeed, assume that this is the case. Let $a \in \mathcal{A}$ be such that $\varphi(sa) = \varphi(a)$ for all $s \in \Gamma$. Fix $g \in \Gamma$. Then, $P_\varphi(a - ga)(s) = \varphi(s^{-1}a - s^{-1}ga) = 0$ for all $s \in \Gamma$. Therefore, $P_\varphi(a - ga) = 0$. Since $P_\varphi$ is an isometry, we see that $a - ga = 0$ for all $g \in \Gamma$. Consequently, $a \in \mathcal{A}^\Gamma$.

**Remark 5.3.** In the case that $\mathcal{A}$ is commutative, namely, $\mathcal{A} = C(X)$, the Poisson transform $P_\varphi$ is an isometry if and only if the measure $\varphi$ is contractible in the sense of Azencott ([14, Chapter-V, Proposition 2.1]).

If $\mathcal{A} = \mathcal{M}$ is a commutative von Neumann algebra, namely, $\mathcal{M} = L^\infty(X, \varphi)$, then the Poisson transform is an isometry if and only if the measure $\varphi$ is SAT in the sense of Jaworski [22].

For the relation between topological models of SAT-measures and contractible measures, see [13, Theorem 8.9].

We now briefly recall the notion of stationary states in the context of unital $C^*$-algebras and refer the readers to [18] for more details.

**Definition 5.4.** Let $\mathcal{A}$ be a unital $\Gamma$-$C^*$-algebra. Let $\mu \in \text{Prob}(\Gamma)$. A state $\varphi \in \mathcal{S}(\mathcal{A})$ is called $\mu$-stationary if

$$\mu \ast \varphi(a) = \sum_{s \in \Gamma} \mu(s)\varphi(s^{-1}a) = \varphi(a), \; \forall a \in \mathcal{A}.$$  

For the canonical conditional expectation $E : \mathcal{A} \rtimes_\tau \Gamma \to \mathcal{A}$, $\tau = \varphi \circ E$ is a $\mu$-stationary state on $\mathcal{A} \rtimes_\tau \Gamma$ for any $\mu$-stationary state $\varphi$ on $\mathcal{A}$. Indeed, for any $a \in \mathcal{A} \rtimes_\tau \Gamma$, it follows from the $\Gamma$-equivariance of $E$ that

$$\mu \ast \tau(a) = \sum_{s \in \Gamma} \mu(s)\tau(s^{-1}a) = \sum_{s \in \Gamma} \mu(s)\varphi(s^{-1}E(a)) = \varphi(E(a)) = \tau(a).$$  

We now proceed to prove the following lemma. Unless otherwise stated, $\tau$ denotes $\nu_P \circ E$. Here, $E : L^\infty(G/P, \nu_P) \rtimes \Gamma \to L^\infty(G/P, \nu_P)$ is the canonical conditional expectation. We think about $\nu_P$ as a state on $L^\infty(G/P, \nu_P)$ given by the integration with respect to $\nu_P$. Recall that $(G/P, \nu_P)$ is the Furstenberg–Poisson boundary associated with a random walk $\mu$ on $\Gamma$. In particular, $\nu_P$ is a $\mu$-stationary state on $L^\infty(G/P, \nu_P)$, and hence, it follows from the above observation that $\tau$ is a $\mu$-stationary state on $L^\infty(G/P, \nu_P) \rtimes \Gamma$.  

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Lemma 5.5. Let \( \mathcal{M} \) be a \( \Gamma \)-invariant subalgebra of the crossed product \( L^\infty(G/P, \nu_p) \rtimes \Gamma \). Then, \( E(\mathcal{M}) \subset \mathcal{M} \).

Proof. Let us first consider the case when \( \tau \) is \( \Gamma \)-invariant. For this case, since \( \tau|_{\mathcal{M}} \) is \( \Gamma \)-invariant, we see that the restriction of \( \nu_p \) on \( E(\mathcal{M}) \) is invariant. Since \( (G/P, \nu_p) \) is the \((\Gamma, \mu)\)-Furstenberg-Poisson boundary, the Poisson transform \( P_{\nu_p} : L^\infty(G/P, \nu_p) \to \ell^\infty(\Gamma) \) is an isometry (see [22, Proposition 2.2]). Since \( \nu_p \) is \( \Gamma \)-ergodic, using Remark 5.2, we see that the only functions \( f \in L^\infty(G/P, \nu_p) \) on which \( \nu_p \) is invariant are the constant functions. Hence, \( E(\mathcal{M}) \) consists of constant functions only, and therefore, \( \mathbb{C} = E(\mathcal{M}) \subset \mathcal{M} \).

Now, assume that \( \tau \) is not \( \Gamma \)-invariant. Let us observe that the action \( \Gamma \rtimes \mathcal{M} \) is ergodic (cf. [24, Lemma 2.16]). Therefore, using [5, Theorem B], we see that there exists a closed subgroup \( P \leq Q \leq G \) and a \( \Gamma \)-equivariant von Neumann algebra embedding \( \Phi : L^\infty(G/Q, \nu_Q) \to \mathcal{M} \), such that \( \nu_Q \) is the push forward measure of \( \nu_p \) under the canonical quotient map \( G/P \to G/Q \). Moreover, the action \( \Gamma \rtimes (G/Q, \nu_Q) \) is essentially free (see, e.g., [5, Lemma 6.2]). From the proof of [25, Theorem 3.6], for each \( a \in L^\infty(G/P, \nu_p) \rtimes_{\text{alg}} \Gamma \), we can find \( p_1, p_2, \ldots, p_n \in L^\infty(G/Q, \nu_Q) \) such that \( \sum_{i=1}^n p_i a p_i = E(a) \). Hence, if \( a \in \mathcal{M} \cap L^\infty(G/P, \nu_p) \rtimes_{\text{alg}} \Gamma \), then \( E(a) \in \mathcal{M} \). Now, a standard approximation argument yields \( E(\mathcal{M}) \subset \mathcal{M} \).

Proposition 5.6. Let \( \mathcal{M} \) be a \( \Gamma \)-invariant subalgebra of \( L^\infty(G/P, \nu_p) \rtimes \Gamma \). Suppose that there exists a \( \Gamma \)-equivariant normal ucp map \( \Phi : \langle \mathcal{M}, L(\Gamma) \rangle \to E(\mathcal{M}) \rtimes \Gamma \). Then, \( \mathcal{M} \subset E(\mathcal{M}) \rtimes \Gamma \).

Proof. It follows from [19, Corollary F] that \( \langle \mathcal{M}, L(\Gamma) \rangle \) is of the form \( L^\infty(G/Q, \nu_Q) \rtimes \Gamma \), where \( P \leq Q \leq G \) is a closed subgroup. We claim that

\[
L^\infty(G/Q, \nu_Q) \rtimes \Gamma = E(\mathcal{M}) \rtimes \Gamma,
\]

which, in turn, will imply that \( \mathcal{M} \subset E(\mathcal{M}) \rtimes \Gamma \).
In order to show this, we first argue that \( C(G/Q) \hookrightarrow L^\infty(G/Q, \nu_Q) \) is \( \Gamma \)-tight. This follows from the fact that the compact space \( G/Q \) has a unique \( \mu \)-stationary measure \( \nu_Q \). Therefore, using [17, Corollary 2.13], we see that \( C(G/Q) \hookrightarrow L^\infty(G/Q, \nu_Q) \rtimes \Gamma \) is \( \Gamma \)-tight. Since \( L^\infty(G/Q, \nu_Q) \rtimes \Gamma \) is generated as a von Neumann algebra by \( C(G/Q) \) and \( L(\Gamma) \), we see that the inclusion \( L(\Gamma) \hookrightarrow L^\infty(G/Q, \nu_Q) \rtimes \Gamma \) is cotight in the sense of [17, Definition 4.1]. In particular, the inclusion \( E(M) \rtimes \Gamma \hookrightarrow L^\infty(G/Q, \nu_Q) \rtimes \Gamma \) is cotight. By our assumption, \( \Phi : L^\infty(G/Q, \nu_Q) \rtimes \Gamma \to E(M) \rtimes \Gamma \) is a \( \Gamma \)-equivariant conditional expectation. Equation (1) is now a consequence of [17, Lemma 4.5], where \( C = E(M) \rtimes \Gamma \) and \( B = L^\infty(G/Q, \nu_Q) \rtimes \Gamma \).

\[ \square \]

Remark 5.7. Let us note that if \( M \subset L^\infty(G/P, \nu_P) \rtimes \Gamma \) is a \( \Gamma \)-invariant subalgebra, then \( E(M) \subset M \) (cf. Lemma 5.5). Moreover, it is enough to construct a \( \Gamma \)-equivariant normal ucp map \( \Psi : \langle M, L(\Gamma) \rangle \to M \). Indeed, if such a \( \Psi \) exists, then \( \Phi \) (of Proposition 5.6) can be constructed by composing \( \Psi \) with the canonical conditional expectation \( E \), that is, \( \Phi = E \circ \Psi \). We also note that \( M \cap L(\Gamma) = L(\Lambda) \), where \( \Lambda \triangleleft \Gamma \). There exists a normal faithful conditional expectation \( \tilde{E}_\Lambda : L^\infty(G/P, \nu_P) \rtimes \Lambda \to L^\infty(G/P, \nu_P) \rtimes \Lambda \) defined by

\[ \tilde{E}_\Lambda(f_g \gamma_g) = \begin{cases} 0 & \text{if } g \notin \Lambda \\ f_g \gamma_g & \text{otherwise} \end{cases} \]

We refer the reader to [10, Proposition 2] for proof of the above. Since \( E(M) \subset M \) and \( M \cap L(\Gamma) = L(\Lambda) \), it follows that \( \tilde{E}_\Lambda(M) \subset M \). Therefore, to prove the conjecture, it is enough to show that \( \tilde{E}_\Lambda(\langle M, L(\Gamma) \rangle) \subset M \). Once this is established, then \( \Phi = E \circ \tilde{E}_\Lambda \) will be the required map. However, we do not know how to show this.

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