ON THE NON-INHERITANCE OF SYMMETRIES OF PARTIAL DIFFERENTIAL EQUATIONS

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The inheritance of symmetries of partial differential equations occurs in a different manner from
that of ordinary differential equations. In particular, the Lie algebra of the symmetries of a partial
differential equation is not sufficient to predict the symmetries that will be inherited by a resulting
reduced partial (or ordinary) differential equation. We show how this suggests a possible source of
Type I hidden symmetries of partial differential equations as well as provide interesting consequences
for solutions of partial differential equations.

Keywords: Lie symmetries; hidden symmetries; Lie algebras; partial differential equations; group
invariant solutions.

1. Introduction

Nonlinear partial differential equations are notoriously difficult to solve. Indeed, no tech-
nique has been devised to find their general solution for most equations (beyond applications
of the the inverse scattering transform [10]). It is usual to try to find exact solutions using
a variety of methods. The most successful method is due to Lie — his method generates
(usually physically important [7]) solutions by exploiting the group invariant properties of
the equations [14, 5].

In the route to finding these group invariant solutions of partial differential equations (PDEs),
one needs to reduce the original PDE to a new PDE (or ordinary differential equation (ODE))
using symmetries of the original equation. In order to solve the reduced equation,
it is useful to determine the symmetries of this equation. As a result, it is important to
understand the fate of symmetries of the original PDE. Those symmetries (other than the one
used for the reduction variables) that are lost for the reduced PDE are called Type I hidden
symmetries of the original PDE. Any new symmetries that are gained are termed Type II
hidden symmetries of the reduced PDE (See [1–3, 8] for examples of this phenomenon. It is
nontrivial to determine the origins of symmetries (hidden or otherwise) in the reduction of PDEs to ODEs [12].

We illustrate the phenomenon via the well–known shallow water wave (SWW) equation [6]

\[ u_{xxxt} + \alpha u_x u_{xt} + \beta u u_{xx} - u_{xt} - u_{xx} = 0 \quad u = u(x,t) \]  

with the four Lie point symmetries

\begin{align*}
V_1 &= x\partial_x - \left( u - \frac{2x\alpha - \beta}{\alpha + \beta} \right) \partial_u \\
V_2 &= \partial_x \\
V_3 &= \partial_u \\
V_4 &= g(t) \left( \partial_t + \frac{1}{\beta} \partial_u \right).
\end{align*}

If we reduce (1.1) via the combination

\[ V_a = V_2 + V_4, \]

i.e.

\[ z = x - \int \frac{1}{g(t)}, \quad w(z) = u - \frac{t}{\beta} \]

we obtain

\[ w_{zzzz} + (\alpha + \beta) w_3 w_{zz} - w_{zz} = 0. \]

This reduced equation has the symmetries

\begin{align*}
U_1 &= \partial_z \\
U_2 &= \partial_w \\
U_3 &= z\partial_z - \left( w - \frac{2z}{\alpha + \beta} \right) \partial_w.
\end{align*}

\( U_3 \) is a Type II hidden symmetry as it does not arise from any of (1.2)–(1.4) and \( V_a \). Often here, is that, in addition to \( V_a \) being “used up” in the reduction, \( V_1 \) also has no relevance to the reduced equation.

We can indicate one possible origin of this hidden symmetry (though others are possible [13]). If we consider the equation

\[ w_{zzzz} + (\alpha + \beta) w_3 w_{zz} - w_{zz} = 0, \]

where we take \( w = w(z, y) \) then we find its symmetries are

\begin{align*}
X_1 &= f(y) \partial_z \\
X_2 &= g(y) \partial_y.
\end{align*}
\[ X_3 = h(y)\partial_w \]  
\[ X_4 = t(g)\left( z\partial_z - \left( w - \frac{2z}{\alpha + \beta} \right) \partial_w \right). \]

Setting all the arbitrary functions above to unity, we have that the Type II hidden symmetry \( U_3 \) could have arisen from \( X_4 \) when we reduce (1.12) via \( X_2 \).

One could argue for a most systematic approach to finding possible origins of hidden symmetries. However, such an approach is difficult to determine due to some surprising observations (some of which were first indicated in [12]). In the next section we indicate exactly what the complications are and the implications thereof.

### 2. Loss of Symmetry

The reduction of order of ODEs is governed by the Lie algebra of the equation under analysis. If the Lie bracket relationship of two symmetries of the ODEs, say \( U_1 \) and \( U_2 \), is given by

\[ [U_1, U_2] = \lambda U_1, \]

where \( \lambda \) is a nonzero constant, it is well-known that reduction of order of the ODE by \( U_1 \) will result in \( U_2 \) (transformed) being a point symmetry of the reduced equation [14]. In the case of PDEs, this is not the case, as has been implicitly pointed out in [11].

Consider the Lie algebra of symmetries [17]

\[ G_1 = \partial_y \]
\[ G_2 = y\partial_y + t\partial_t \]
\[ G_3 = \partial_w \]
\[ G_4 = t\partial_y, \]

where \( w = w(y, t) \), which was a (failed) candidate to determine the origin of symmetries of a PDE obtained from a reduction of the Korteweg-de Vries equation [12]. As the Lie brackets of \( G_1 \) and \( G_4 \) are

\[ [G_1, G_4] = 0 \]

one would expect that reduction via either \( G_1 \) or \( G_4 \) would result in the other symmetry being a symmetry (suitably transformed) of the new equation. However, the reduction variables defined by \( G_1 \) are simply \( t \) and \( w \). Since \( G_4 \) only has the \( \partial_y \) operator, it has no relevance to the new equation. Thus two symmetries are unexpectedly (based on the Lie algebra) ‘used up’ in the reduction. As a result, we could not utilise this Lie algebra of symmetries to construct an appropriate PDE that had a common reduced equation with the Korteweg-de Vries equation.

To take a more general case, we define \( U_1 \) and \( U_2 \) via

\[ U_1 = \partial_x \]
\[ U_2 = f(t, u) + \lambda \partial_y + g(t, u)\partial_t + h(t, u)\partial_u, \]

where \( u = u(t, x) \), and so (2.1) holds. The reduction variables defined by \( U_1 \), viz.

\[ p = t, \quad q = u \]
ensure that $U_2$ transforms to

$$
\bar{U}_2 = f(p, q) + \lambda x \partial_x + g(p, q) \partial_y + h(p, q) \partial_y. \tag{2.10}
$$

Since the reduced equation must be in the variables $p$ and $q$, the first part of the generator is not relevant and we have that

$$
\bar{U}_2 = g(p, q) \partial_y + h(p, q) \partial_y. \tag{2.11}
$$

However, what this result hides is the fact that, if $g = h = 0$, then $U_2$ in (2.11) does not manifest itself as a point symmetry of the reduced equation as it is now zero. Thus, if we have

$$
U_1 = \partial_x, \quad U_2 = f(t, u) + \lambda x \partial_x \tag{2.12}
$$

then (2.1) holds, but we lose both symmetries after reducing the number of variables in the pde via $U_1$.

While the above example is instructive, it does, by its simplicity, obscure the true dependence of $U_1$ and $U_2$ for this result to hold. Consider now the symmetry

$$
X_1 = \partial_x, \quad X_2 = [\lambda x + f(u - xt, t)] \partial_x + g(u - xt, t) \partial_u + h(u - xt, t) \partial_u. \tag{2.14}
$$

The reduction variables defined by (2.14) are

$$
p = u - xt, \quad q = t. \tag{2.16}
$$

Using these variables, $X_2$ transforms to

$$
\bar{X}_2 = [-qf(p, q)] \partial_y + g(p, q) \partial_y \tag{2.17}
$$

which is a point symmetry of the reduced pde. In the event that

$$
h = qf, \quad g = 0 \tag{2.18}
$$

$\bar{X}_2$ is annihilated, i.e. when

$$
X_2 = [f(u - xt, t) + \lambda x] \partial_x + t[f(u - xt, t) + \lambda x] \partial_u \tag{2.19}
$$

and $X_1$ is given by (2.14) then reduction via (2.16) will cause $X_2$ to be lost for the reduced equation in spite of (2.1) holding.
We observe that $X_2$ in (2.19) can be written as

$$X_2 = [f(u - xt, t) + \lambda x]X_1,$$

(2.20)
a result that was not immediately apparent in (2.13). The following proposition then holds:

**Proposition 2.1.** Define two Lie point symmetries as

$$Y_1 = \xi \partial_x + \tau \partial_t + \eta \partial_u$$

(2.21)

and

$$Y_2 = [f(p, q) + \lambda g]Y_1,$$

(2.22)

where $p$ and $q$ are reduction variables (in the original variables) defined by (equivalently zeroth order invariants of) $Y_1$ and $g$ is given by any one of the following three (equivalent) functions:

$$\int \frac{1}{\xi(p, q, x)} dx; \quad \int \frac{1}{\tau(p, q, t)} dt; \quad \int \frac{1}{\eta(p, q, u)} du.$$  

(2.23)

These symmetries satisfy

$$[Y_1, Y_2] = \lambda Y_1$$

(2.24)

but $Y_2$ will not be a symmetry of the reduced equation in the new variables $p$ and $q$.

This is in contrast to the case of ODEs where (2.24) ensures that $Y_2$ (transformed) will always be a point symmetry of the reduced ODE. (Note that, in (2.23) the integrals must only be evaluated after substituting for the non-integration variables via the reduction variables. After evaluating the integral, a back substitution must be effected to obtain $g$ in the original variables.)

3. Discussion

An important consequence of Proposition 2.1 is that, when one seeks to obtain a PDE which admits a symmetry of the form $Y_1$ by increasing the number of variables, then $Y_2$ also arises as a symmetry of this new PDE. This is one source of Type I hidden symmetries of PDEs.

Let us examine some equations which admit symmetries of the form given in Proposition 2.1. We begin with the simplest commuting case, i.e.

$$[U_1, U_2] = 0$$

(3.1)

with

$$U_1 = \partial_x$$

(3.2)

$$U_2 = f(t, u) \partial_u$$

(3.3)

and $u = u(t, x).$ The invariants of $U_1$ are calculated with little effort and yield

$$F(t, u, u_t, u_x, u_{tx}, u_{tt}, u_{tt}, \ldots) = 0$$

(3.4)

which is the general form of the PDE invariant under $U_1.$
We now impose $U_2$ suitably extended. We find (using SYM [9]) that $F$ is now restricted further to

\[ G(t, u, k_1, k_2, k_3, k_4, \ldots) = 0, \quad (3.5) \]

where

\[ k_1 = \frac{u_t}{f_t u_t + f_t} \quad (3.6) \]

\[ k_2 = \frac{f_u u_t^2 + f_u u_{xx}}{f_u (f_t u_t + f_t)^2} \quad (3.7) \]

\[ k_3 = \frac{f_u^3 u_t u_{xx} - f_u^2 u_t u_{xxx} - f_u f_u f_u u_t u_{xx} - f_u f_u f_t u_t u_{xx} + f_u u_t^2 f_u - 2 f_t f_u u_t^2}{f_u^2 (f_t u_t + f_t)^2} \quad (3.8) \]

\[ k_4 = \left( \frac{u_{xx} (f_u u_t + f_t)^2}{f_u^2 u_t^2} - \frac{2 (f_u u_t + f_t) u_{xx}}{f_u u_t} + \frac{3 f_u^2 f_u u_{xx}}{f_u^3} + \frac{2 f_t f_u u_t^2}{f_u^2} \right. \]

\[ - \frac{4 f_t f_u f_u}{f_u^2} + \frac{f_u u_t + u_{xx}}{(f_u u_t + f_t)} \right) (3.9) \]

If we set the first two invariants to zero, we essentially have ODEs. However, in the case of $k_3$ we can construct a proper PDE via

\[ k_3 = 0, \quad (3.10) \]

i.e. (when we set $f(t, u) = u$)

\[ u_t u_{xx} - u_t u_{xt} = 0. \quad (3.11) \]

Interestingly, reducing (3.11) utilising $U_1$ results in the identity

\[ 0 = 0. \quad (3.12) \]

Thus any function of the invariants of $U_1$ will satisfy (3.11). Indeed, given the form of (3.5) we believe that any function of the invariants of $U_1$ will satisfy the equation.

As another example, let us consider a case where the equation admits more than just two symmetries. Thus we first impose

\[ U_1 = \partial_t \quad (3.13) \]

\[ U_2 = f(t, u) \partial_x \quad (3.14) \]

in order to have the Lie bracket relationship (3.1) and satisfy the requirements of Proposition 2.1. We now choose the third symmetry as

\[ U_3 = \partial_x \quad (3.15) \]

(both for simplicity and to allow the equation to admit travelling wave solutions). Taking the Lie bracket of $U_2$ and $U_3$ we obtain

\[ [U_3, U_2] = f_t \partial_x. \quad (3.16) \]
Requiring the Lie algebra to close requires $f(t, u)$ to take on one of the following forms:

$$f(u); \quad f(u) + t; \quad e^t f(u)$$

so that the Lie algebra formed is Abelian, the solvable Lie algebra $A_1 \oplus A_2$ or the nilpotent algebra $A_3$ which is the algebra of the Weyl group [15, 16] respectively. We take the third form to illustrate our example. Imposing $U_1$ and $U_2$ yields an equation of the form

$$F(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0.$$  \hspace{1cm} (3.18)

Imposing $U_3$ next results in

$$G(u, k_1, k_2, k_3, k_4, \ldots) = 0,$$  \hspace{1cm} (3.19)

where

$$k_1 = \frac{u_x}{u_t f' + f}$$  \hspace{1cm} (3.20)

$$k_2 = \frac{u_x^2 f'' + u_x u_{xx} f'}{f(u_t f' + f)^2}$$  \hspace{1cm} (3.21)

$$k_3 = -\frac{u_x^2 f'' + (u_x^2 + u_x u_{xx} - u_{xx} u_t) f'' - u_x^2 u_t f''}{u_x f''(u_t f' + f)^2}$$  \hspace{1cm} (3.22)

$$k_4 = \frac{u_x^2 f' f'' + (u_{xx} - 2u_x) u_x^2 - 2u_x u_{xx} u_t + u_x u_t}{u_x f''(u_t f' - f)}.$$  \hspace{1cm} (3.23)

Again, it seems clear that any function of the invariants of $U_1$ will satisfy (3.19).

This leads us to the following conjecture:

**Conjecture 3.1.** Let a PDE admit two symmetries satisfying the requirements of Proposition 2.1. Then any function of the invariants of symmetry $Y_1$ will satisfy the PDE identically.

As a final observation, we note that hidden symmetries in PDEs, notwithstanding the common origins indicated with ODEs, are indeed different objects. This is clearly illustrated in the following equation:

$$u_{xx} + u_t f(\cdot) = 0, \quad u = u(x, t)$$  \hspace{1cm} (3.24)

where $f(\cdot)$ is an arbitrary function of dependent and independent variables and all derivatives of the dependent variable. As a result (3.24) cannot admit any point symmetries. If we now look for steady state solutions of this equation (i.e., independent of $t$) then the PDE reduces to the ODE

$$u_{xx} = 0, \quad u = u(x)$$  \hspace{1cm} (3.25)

which has eight Lie point symmetries, all of which are Type II hidden symmetries. Such examples are easy to generate and we have given in [2] a different reduction to (3.25) as well as a source of these hidden symmetries. Thus it would seem that Type II hidden symmetries proliferate in the study of PDEs. We are able to find symmetries via reductions of equations that do not admit any Lie point symmetries. This is not entirely surprising as nonclassical symmetries [4] exhibit exactly this behaviour. However, the origin of transformations that generate these Type II hidden symmetries is still unclear.
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