Existence and Scattering for Solutions to Semilinear Wave Equations on High Dimensional Hyperbolic Space

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Abstract

We prove small-data global existence to semi-linear wave equations on hyperbolic space of dimension $n \geq 3$, for nonlinearities that have the form of a sufficiently high integer power of the solution. We also prove the existence and asymptotic completeness of wave operators in this setting.

1 Introduction

The semilinear wave equation

$$\Box u = F(u)$$

(1.1)

with Cauchy data

$$u(0, \cdot) = f(x), \partial_t u(0, \cdot) = g(x)$$

(1.2)

where $F(u)$ has the form

$$F(u) = a|u|^b$$

(1.3)

has been extensively studied on $\mathbb{R}^{n+1}$. In a number of works including [4], [6], [9], [11], and [12], it was proved that (1.1) has a small-data global solution when $b$ exceeds $\frac{p}{2}$ the positive square root of the quadratic

$$(n-1)p^2 - (n+1)p - 2 = 0.$$ 

(1.4)

Recent work has been done in examining this problem on $\mathbb{R} \times M$, where $M$ is hyperbolic space of dimension $n$. In [1], Anker and Pierfelice obtain a wider range of dispersive and Strichartz estimates than in the Euclidean case, owing to the better dispersion on hyperbolic space. The resulting global existence results proved first in [10] for dimension 3 and then improved and expanded in [1] are as follows:

When $3 \leq n$ and $1 < b < 1 + \frac{4}{n-1}$, (1.1) has a global solution given sufficiently small initial data $(f,g) \in H^{\gamma,2}(M) \oplus H^{\gamma-1,2}(M)$ for $\gamma = \frac{n+1}{2} - \frac{1}{b+1}$.

When $3 \leq n \leq 5$ and $1 + \frac{4}{n-1} \leq b \leq 1 + \frac{4}{n-2}$, (1.1) has a global solution given sufficiently small initial data $(f,g) \in H^{\gamma,2}(M) \oplus H^{\gamma-1,2}(M)$ for $\gamma = \frac{n}{2} - \frac{2}{b+1}$.

When $n \geq 6$ and $1 + \frac{4}{n-4} \leq b \leq \frac{n-1}{2} + \frac{3}{n+1} - \sqrt{(\frac{n-3}{2} + \frac{3}{n+1})^2 - 4\frac{n-1}{n+2}}$, (1.1) has a global solution given sufficiently small initial data $(f,g) \in H^{\gamma,2}(M) \oplus H^{\gamma-1,2}(M)$ for $\gamma = \frac{n}{2} - \frac{2}{b+1}$.

When $n = 3$ and $b \geq 5$, (1.1) has a global solution given sufficiently small initial data $(f,g) \in H^{\gamma,2}(M) \oplus H^{\gamma-1,2}(M)$ for $\gamma = \frac{3}{2} - \frac{2}{b+1}$.

In this paper we will add to this picture results for large $b$ and large $n$, obtained by using the approach of Lindblad and Sogge in [1] adapted to this setting. This requires using the Leibniz rule for fractional derivatives, which leads to the additional restriction that $b \in \mathbb{Z}$. We finish by demonstrating the existence and asymptotic completeness of wave operators, allowing us to conclude that the solution obtained scatters to a linear solution over time.
2 Strichartz Estimates

We will need to make use of the Strichartz estimates already known in this setting. In all that follows let $M = H^n$ unless otherwise specified. Set

$$Tf(t, x) = e^{it\sqrt{-\Delta}}f(x)$$

$$T^*g(x) = \int_{-\infty}^{\infty} e^{-it\sqrt{-\Delta}}g(t, x)dt$$

and a relevant theorem, proved in [10] and [1], is:

**Theorem 2.1** We have the mapping properties $T : H^{\gamma, 2}(M) \rightarrow L^q(\mathbb{R}, L^p(M))$ and $T^* : L^{p'}(\mathbb{R}, L^{q'}(M)) \rightarrow H^{-\gamma, 2}(M)$, whenever $(p, q, \gamma) \in \mathcal{R} \cup \mathcal{E}$, where

$$\mathcal{R} = \{(p, q, \gamma)| 2 < q < \frac{2(n - 1)}{n - 3}, 2 \leq p \leq \frac{4q}{(n - 1)(q - 2)}, \gamma = \frac{1}{2}(n + 1)(\frac{1}{2} - \frac{1}{q})\}$$

and

$$\mathcal{E} = \{(p, q, \gamma)| \frac{1}{p} \leq \frac{1}{2}(n - 1)(\frac{1}{2} - \frac{1}{q}), \gamma = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{p}\}.$$

Setting

$$Vf(t, x) = \int_0^t \sin(t - s)\sqrt{-\Delta}f(s, x)ds,$$

so that $Vf = u$ solves the zero-data inhomogeneous equation

$$\Box u = f, u(0) = \partial_t u(0) = 0$$

on $\mathbb{R} \times M$, we also have

**Theorem 2.2** For $(p, q, \gamma), (\tilde{p}, \tilde{q}, \tilde{\gamma}) \in \mathcal{R} \cup \mathcal{E}, (p, \tilde{p}) \neq (2, 2)$, we have

$$V : L^{\tilde{p}}(\mathbb{R}, H^{\tilde{\gamma}, \tilde{\beta}}(M)) \rightarrow L^p(\mathbb{R}, H^{1-\gamma, q}(M)).$$

(2.5)

From Theorem 2.2, together with the commutativity of $V$ with $(-\Delta)^{-\frac{1}{2}}$, we also deduce:

**Corollary 2.1** In the setting of Theorem 2.2, we have for each $\sigma \in \mathbb{R}$

$$\|Vf\|_{L^p(\mathbb{R}, H^{\sigma+1-\gamma, q}(M))} \leq C\|f\|_{L^{\tilde{p}}(\mathbb{R}, H^{\tilde{\gamma}, \tilde{\beta}}(M))}$$

(2.6)

3 Existence of Solutions

Here we will use the theorems of the previous section to prove the following:

**Theorem 3.1** Assume $M = H^n$, $n \geq 3$, and take $b \in [1 + \frac{1}{n - 1}, \infty) \cap \mathbb{Z}$. Then there exists $\epsilon_0 > 0$ such that, if the initial data $(f, g)$ satisfy

$$\|f\|_{H^{\gamma, 2}(M)}, \|g\|_{H^{\gamma-1, 2}(M)} < \epsilon_0,$$

for

$$\gamma = \frac{n}{2} - \frac{2}{b - 1}$$

(3.2)

the equation (1.1) is globally solvable.
Proof Using the technique of Lindblad and Sogge in [9], the method of proof will be Picard iteration on the space

$$\mathcal{X} = \{ u \in L^{\frac{2(n+1)}{n-1}}(\mathbb{R}, H^{\frac{1}{2} - \frac{1}{n+1}}(M)) \cap L^q(\mathbb{R} \times M) : \|u\|_{L^q(\mathbb{R} \times M)} \leq \delta \}$$

with $\gamma$ as in (3.2) and

$$q = \frac{(n+1)(b-1)}{2}. \quad (3.4)$$

Note that

$$b \geq 1 + \frac{4}{n-1} \Rightarrow q \geq \frac{2(n+1)}{n-1}, \quad (3.5)$$

so that in this setting it is possible to have $(q, q, \gamma) \in \mathcal{E}$. Also

$$\gamma = \frac{n}{2} - \frac{n+1}{q}. \quad (3.6)$$

The distance function we put on $\mathcal{X}$ is:

$$d(u, v) = \|u - v\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R} \times M)}. \quad (3.7)$$

It is an important observation that $\mathcal{X}$ is complete with respect to this distance. Now, following the standard iteration scheme, we define a sequence $\{u_i\}$ by setting $u_i$ to solve:

$$\pm u_{i-1}^{b-1} = \Box u_i \quad (3.8)$$

with

$$u_i(0, x) = f, \partial_t u_i(0, x) = g \quad (3.9)$$

and

$$u_{-1} \equiv 0. \quad (3.10)$$

Our task is now to demonstrate that the non-linear mapping

$$u_i \rightarrow u_{i+1} \quad (3.11)$$

is:

(i) well-defined

(ii) a contraction on $\mathcal{X}$ under the norm (3.7).

Further, we will need to demonstrate that for $u = \lim_{i \to \infty} u_i$, we have

(iii) $F(u_i) \to F(u)$ in $D'(\mathbb{R} \times M)$.

To begin, define

$$N_i = \sup_{\frac{2(n+1)}{n-1} \leq q \leq \frac{2(n+1)}{n-1}} \|u_i\|_{L^q(\mathbb{R}, H^{\frac{1}{2} - \frac{1}{n+1}}(M))}. \quad (3.12)$$

We pause to note some facts about $N_i$. First, $N_0$ is finite: Set $u_0(t) = \Xi_0(f, g)(t) = \cos t\sqrt{-\Delta}f + \frac{\sin t\sqrt{\Delta}}{\sqrt{-\Delta}}g$, and observe that Thorem 2.1 and the commutativity of $\Xi_0$ with $(\lambda I - \Delta)^{\frac{b}{2}}$ together imply:

$$\|u_0\|_{L^q(\mathbb{R}, H^{\frac{1}{2} - \frac{1}{n+1}}(M))} \lesssim \|f\|_{H^{\gamma, 1}(M)} + \|g\|_{H^{\gamma, 1}(M)} \quad (3.13)$$

provided $(q, q, \frac{n}{2} - \frac{n+1}{q}) \in \mathcal{E}$ and $\gamma$ is as in (3.2). Thus $N_0$ is finite in this setting, and bounded above by the (small) norm of the initial data. Second, it is also true that, for initial data sufficiently small, we have

$$N_m \leq 2N_0. \quad (3.14)$$
One proves this by induction on \( m \), writing:

\[
 u_{i+1} = u_0 + \int_0^t \sin(t - s) \sqrt{-\Delta} F(u_i)(s) \, ds,
\]

(3.15)

This gives

\[
 N_{m+1} \leq N_0 + ||V(F(u_i))||_{L^\infty(\mathbb{R},H^{n-1+\frac{n+1}{n+3} - \frac{2(n+1)}{n+3}})}.
\]

(3.16)

We then use Corollary 2.1 with \( \sigma = \frac{n}{2} - \frac{2}{b-1} + 1 \) to deduce that, as \((q,q, \frac{n}{2} - \frac{n+1}{q})\) and \((\frac{2(n+1)}{n-1}, \frac{2(n+1)}{n-1}, \frac{1}{q}\) are in \( \mathcal{E} \),

\[
 ||V(F(u_i))||_{L^\infty(\mathbb{R},H^{n-1+\frac{n+1}{n+3} - \frac{2(n+1)}{n+3}})} \lesssim ||u_m||_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R},H^{n-1+\frac{2(n+1)}{n+3}})}.
\]

(3.17)

and hence

\[
 N_{m+1} \leq N_0 + ||u_m||_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R},H^{n-1+\frac{2(n+1)}{n+3}})}.
\]

(3.18)

At this point we will need:

**Lemma 3.1** For \( \sigma \in (0,1) \) and \( M \) a Riemannian manifold with \( C^\infty \) bounded geometry,

\[
 ||uv||_{H^\sigma,\mathcal{F}(M)} \leq C ||u||_{H^\sigma_1(\mathcal{F}(M))} ||v||_{L^2(\mathcal{F}(M))} + C ||u||_{L^1(\mathcal{F}(M))} ||v||_{H^\sigma_2(\mathcal{F}(M))}
\]

(3.19)

where \( \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{p} \).

We will prove Lemma 3.1 presently, but before that let us see how it implies (3.14). If we apply Lemma 3.1 and the standard Leibniz rule to the last term of (3.18), we see that it is bounded by a finite sum of terms that look like:

\[
 \Pi^b_{j=1} ||u_m||_{L^{p_j}(\mathbb{R},H^{\alpha_j,p_j}(\mathcal{F}(M)))}
\]

(3.20)

where

\[
 0 \leq \alpha_j \leq \frac{n-1}{2} - \frac{2}{b-1} \tag{3.21}
\]

and

\[
 \sum_{j=1}^b \alpha_j = \frac{n-1}{2} - \frac{2}{b-1} \tag{3.22}
\]

and

\[
 \frac{2(n+1)}{n-1} \leq p_j \leq \infty \tag{3.23}
\]

and

\[
 \sum_{j=1}^b \frac{1}{p_j} = \frac{n+3}{2(n+1)} \tag{3.24}
\]

Fixing the \( \alpha_j \)'s to meet the above conditions and considering the definition of \( N_m \), we take \( p_j \) in (3.20) to satisfy:

\[
 \frac{n+1}{p_j} - \frac{2}{b-1} = \alpha_j \tag{3.25}
\]

Summing over these quantities yields

\[
 \sum_{j=1}^b \frac{1}{p_j} = \frac{n+3}{2(n+1)} \tag{3.26}
\]

and (3.22) gives

\[
 \frac{2(n+1)}{n-1} \leq p_j \leq \frac{(b-1)(n+1) - 2}{2(n+1)} \tag{3.27}
\]

Then for each term in (3.20) we have

\[
 ||u_m||_{L^{p_j}(\mathbb{R},H^{\alpha_j,p_j}(\mathcal{F}(M)))} \leq N_m, \tag{3.28}
\]
and hence that (3.20) is bounded above by $N_{m+1}^b$. Plugging this into (3.18) gives
\[ N_{m+1} \leq N_0 + N_m, \]  
which by induction yields (3.14) for $N_0$ sufficiently small. Then since $\frac{(n+1)(b-1)}{2}$ and $\frac{2(n+1)}{n-1}$ are in $\left[\frac{2(n+1)}{n-1}, \frac{(b-1)(n+1)}{2}\right]$, we see that $\|u_m\|_{L^b(U \times M)}$ and $\|u_m\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n \to \frac{2(n+1)}{n-1} (M))}$ are also bounded above by $2N_0$. Hence (3.11) is well-defined on $X$ for initial data sufficiently small.

We must now demonstrate that (3.11) is a contraction under the norm (3.27). Write
\[ ||u_{m+1} - u_{k+1}||_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R} \times M)} = (3.30) \]
\[ ||V(F(u_m) - F(u_k))||_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R} \times M)} \lesssim \|F(u_m) - F(u_k)||_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R} \times M)}, \]
the last line of course coming from Theorem 2.2. Then using
\[ ||u|^b - |v|^b|| \lesssim |u - v|(||u||^{b-1} - ||v||^{b-1}) \]
(3.31)
and
\[ \frac{2}{n+1} + \frac{n-1}{2(n+1)} = \frac{n+3}{2(n+1)}, \]  
(3.32)
Holder’s inequality tells us that the last term in (3.30) is bounded above by
\[ ||u_m - u_k||_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R} \times M)} \left[ ||u_m||^{b-1}_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R} \times M)} + ||u_k||^{b-1}_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R} \times M)} \right] = (3.33) \]
\[ ||u_m - u_k||_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R} \times M)} \left[ ||u_m||^{b-1}_{L^b(\mathbb{R} \times M)} + ||u_k||^{b-1}_{L^b(\mathbb{R} \times M)} \right]. \]
The second term is bounded above by $2\delta^{b-1}$, giving us the contractivity property.

Finally we need to show that $F(u_i) \to F(u)$ in $D'(\mathbb{R} \times M)$, where $u$ is the limit of $\{u_i\}_i$ in $X$. This step is implicit in our previous arguments:
\[ ||F(u_i) - F(u)||_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R} \times M)} \lesssim (3.34) \]
\[ ||u_i - u||_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R} \times M)} \left[ ||u_i||^{b-1}_{L^b(\mathbb{R} \times M)} + ||u||^{b-1}_{L^b(\mathbb{R} \times M)} \right]. \]
The second term here is finite given $u$, $u_i \in X$, while the first term goes to zero as $i \to \infty$.

We return now to the proof of Lemma 3.1. This result is already established on Euclidean space; see for instance [7] and [2]. We will use the Euclidean version in conjunction with:

**Lemma 3.2** For $M$ a Riemannian manifold with $C^\infty$ bounded geometry, with $s > 0$ and $p \in (1, \infty)$,
\[ ||u||_{H^{s,p}(M)}^p \approx \sum_j ||\phi_j u||_{H^{s,p}(M)}^p + ||u||_{L^p}^p \]  
(3.35)
where $\{\phi_j : j \in \mathbb{N}\}$ is a tame partition of unity as defined in (1.27) of [13].

**Proof** The proof of this lemma may be found in Lemma 6.7 of [10]. The term $C^\infty$ bounded geometry is defined in (1.19) - (1.23) of [13] as follows: First, there exists $R_0 \in \mathbb{R}$ such that for all $p \in M$, the exponential map $\text{Exp}_p : T_p M \to M$ maps $B_{R_0}(0)$ to $B_{R_0}(p)$ diffeomorphically. Second, the pull-back of the metric tensor from $B_{R_0}(p)$ to $B_{R_0}(0)$ yields a collection of $n \times n$ matrices $G_p(x)$ such that $\{G_p : p \in M\}$ is bounded in $C^\infty(B_{R_0}(0), \text{End}(\mathbb{R}^n))$. Finally, for all $p \in M$, $x \in B_{R_0}(0)$, and $\xi \in \mathbb{R}^n$, we have that $\xi \cdot G_p(x)\xi \geq \frac{1}{2}||\xi||^2$ and $B_{R_0}(p)$ is geodesically convex. Then, a tame partition of unity is one whose supports have a bounded number of overlaps and whose elements $\phi_k$ have the property that $\phi_k \circ \text{Exp}$ is bounded in $C^\infty_0$ of a ball in $\mathbb{R}^n$. 

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Also of use will be the fact that:
\[ \| \phi_j u \|_{H^{s,p}(M)} \approx \| \phi_j u \circ \text{Exp} \|_{H^{s,p}(\mathbb{R}^n)}. \]  
(3.36)

We then write:
\[ \| uv \|_{H^{s,p}(M)} = \left( \| uv \|_{H^{s,p}(M)}^p \right)^{\frac{1}{p}} \]
\[ \approx \left( \sum_j \| \phi_j(uv) \|_{H^{s,p}(M)}^p + \| uv \|_{L^p(M)} \right)^{\frac{1}{p}} \]
\[ \leq C \left( \sum_j \| \phi_j(uv) \|_{H^{s,p}(M)}^p \right)^{\frac{1}{p}} + \| uv \|_{L^p(M)} \]
\[ \leq C \left( \sum_j \| \phi_j(uv) \|_{H^{s,1}(M)}^p \| \phi_j v \|_{L^{s_2}(M)}^p \right)^{\frac{1}{p}} + \| uv \|_{L^p(M)} \]
\[ \leq C \left( \sum_j \| \phi_j(uv) \|_{H^{s,1}(M)}^p \right)^{\frac{1}{p}} \]
\[ + C \left( \sum_j \| \phi_j(v) \|_{H^{s,2}(M)}^p \right)^{\frac{1}{p}} + \| uv \|_{L^p(M)} \]
\[ \leq C \left( \sum_j \| u \|_{H^{s,2}(M)}^p \| v \|_{L^{s_2}(M)}^p \right)^{\frac{1}{p}} + \| uv \|_{L^p(M)} \]
\[ \leq C \| u \|_{H^{s,1}(M)} \| v \|_{L^{s_2}(M)} + C \| u \|_{L^{s_1}(M)} \| v \|_{H^{s,2}(M)} + \| uv \|_{L^p(M)}. \]

The last term will be dealt with via Holder’s inequality, to write
\[ \| uv \|_{L^p(M)} \leq \| u \|_{L^{s_1}} \| v \|_{L^{s_2}}. \]  
(3.38)

This concludes the proof of Lemma 3.1.

4 Scattering

In this section we examine the asymptotic behavior of the solution \( u \) to (1.1) in the setting of Theorem 3.1.

First, we define:
\[ w = \begin{pmatrix} u \\ u_t \end{pmatrix}, \quad h = \begin{pmatrix} f \\ g \end{pmatrix}, \quad G(w) = \begin{pmatrix} 0 \\ F(u) \end{pmatrix}, \quad iL = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \]
(4.1)
so that (1.1) may then be rewritten:
\[ w(t) = e^{itL}h + \int_0^t e^{i(t-s)L}G(w(s))ds, \]  
(4.2)
or
\[ e^{-itL}w(t) = h + \int_0^t e^{-isL}G(w(s))ds, \]  
(4.3)
where \( e^{itL} = \begin{pmatrix} \cos tA & A^{-1} \sin tA \\ -A \sin tA & \cos tA \end{pmatrix}, \quad A = \sqrt{-\Delta} \).

We will investigate the convergence of (1.3) as \( t \to +\infty \) and \( t \to -\infty \). (4.3) implies:
\[ e^{-it_2L}w(t_2) - e^{-it_1L}w(t_1) \]
\[ = \int_{t_1}^{t_2} e^{-isL}G(w(s))ds \]
\[ = \int_{t_1}^{t_2} \begin{pmatrix} -A^{-1} \sin (sA)F(u(s)) \\ \cos (sA)F(u(s)) \end{pmatrix}ds \]
\[ = \begin{pmatrix} \phi_{t_1,t_2} \\ \psi_{t_1,t_2} \end{pmatrix}. \]
Now set
\[ H_{t_1 t_2}(s) = F(s) \chi_{[t_1, t_2]}(s), \quad F(s) = F(s, x). \]

Then
\[
\int_{t_1}^{t_2} e^{-isA} F(s) ds = \int_{-\infty}^{\infty} e^{-isA} H_{t_1 t_2}(s) ds = T^* H_{t_1 t_2}
\]
for \( T^* \) as in Section 2. We note that \( T^* \) commutes with powers of \( A \), and this together with Theorem 2.1 yields
\[
T^* : L^{\tilde{q}}(\mathbb{R}, H^{\tilde{\sigma}}(M)) \rightarrow H^{\sigma - 5,2}(M)
\]
for all \( \sigma \in \mathbb{R} \) and \((\tilde{p}, \tilde{q}, \tilde{\sigma}) \in \mathbb{R} \cup \mathcal{E}\). Taking \((2(n+1)/n-1, 2(n+1)/n-1, 1/2) \in \mathcal{E} \) and \( \sigma = \gamma - 1/2 \), we obtain
\[
T^* : L^{2(n+1)/(n+3)}(\mathbb{R}, H^{\gamma - 1/2, 2(n+1)/(n+3)}(M)) \rightarrow H^{\gamma - 1/2}(M).
\]

This yields
\[
\left\| \int_{t_1}^{t_2} e^{-isA} F(s) ds \right\|_{H^{\gamma - 1,2}(M)} \lesssim \left\| T^* H_{t_1 t_2} \right\|_{H^{\gamma - 1,2}(M)} \lesssim \left\| F \right\|_{L^{2(n+1)/(n+3)}([t_1, t_2], H^{\gamma - 1/2, 2(n+1)/(n+3)}(M))}.
\]

From Section 3 we know that the right hand side is bounded above by \( 2N_0 \) which is in turn bounded above by the small norm of the initial data. Hence, we may say that
\[
\left\| \phi_{t_1 t_2} \right\|_{H^{\gamma - 2}(M)}, \left\| \psi_{t_1 t_2} \right\|_{H^{\gamma - 1,2}(M)} \rightarrow 0
\]
as \( t_1, t_2 \rightarrow \pm \infty \). Thus
\[
e^{-itL} w(t) \text{ is Cauchy in } H^{\gamma - 2}(M) \oplus H^{\gamma - 1,2}(M) \text{ as either } t \rightarrow \infty \text{ or } t \rightarrow -\infty.
\]

From (4.8) and the fact that \( \{e^{itL} : t \in \mathbb{R}\} \) is a uniformly bounded family of operators on \( H^{\gamma - 2}(M) \oplus H^{\gamma - 1,2}(M) \), we have the following scattering result:

**Theorem 4.1** In the setting of Theorem 3.1, with \( \gamma \) as in (3.2), \((f, g) \in H^{\gamma - 2}(M) \oplus H^{\gamma - 1,2}(M) \) with sufficiently small norm, and \( u \) the solution to (1.1), there exist
\[
(f_+, f_-) \in H^{\gamma - 2}(M) \oplus H^{\gamma - 1,2}(M)
\]
such that
\[
\left\| u(t) - e^{itL} \begin{pmatrix} f_+ \\ g_+ \end{pmatrix} \right\|_{H^{\gamma - 2}(M) \oplus H^{\gamma - 1,2}(M)} \rightarrow 0 \text{ as } t \rightarrow \pm \infty.
\]

**5 Wave Operators**

Having analyzed the asymptotic behavior of solutions to (1.1) as \( t \rightarrow \pm \infty \), we will now define wave operators and prove their existence in this context.

From the previous section, we know that, given the Cauchy problem (1.1), it is possible to find initial data \( \begin{pmatrix} \phi_+ \\ \psi_+ \end{pmatrix} \) that, when acted upon by the linear operator
\[
S_n(t) = e^{itL} = \begin{pmatrix} \cos tA & A^{-1} \sin tA \\ -A \sin tA & \cos tA \end{pmatrix},
\]

A = \sqrt{-\Delta},

yields a solution asymptotically close to that of (1.1) as $t \to \pm \infty$. Now we posit an inverse problem: Given $(\phi_\pm, \psi_\pm)$ as initial data, is it possible to obtain a solution to (1.1)? In other words, we ask if there exist well-defined operators

$$W_-(\phi_-, \psi_-) \to (u, u_t)$$

and

$$W_+ (\phi_+, \psi_+) \to (u, u_t).$$

If $W_-$ and $W_+$ exist, we call them wave operators. It turns out that in this context we can indeed find wave operators, provided $(\phi_\pm, \psi_\pm)$ lie in the space $H^{\gamma, 2} \oplus H^{\gamma-1, 2}$ and have sufficiently small norm. The relevant theorem is as follows:

**Theorem 5.1** In the setting of Theorem 3.1, there exists an $\epsilon_0$ with the following property: For $\phi_- \in H^{\gamma, 2}(M)$ and $\psi_- \in H^{\gamma-1, 2}(M)$ with

$$||\phi||_{H^{\gamma, 2}(M)}, ||\psi||_{H^{\gamma-1, 2}(M)} \leq \epsilon_0$$

the equation

$$w(t) = e^{itL}(\phi_- - \psi_-) + \int_{-\infty}^{t} e^{i(t-s)L}G(w(s))ds$$

has global solution, satisfying $w = (u, \partial_t u)$, with

$$u \in L^{2(n+1)(b-1)}(\mathbb{R}, H^{\gamma-1, 2}) \cap L^q(\mathbb{R} \times M)$$

where $q = \frac{(n+1)(b-1)}{2}$.

Proof Solving (5.4) is equivalent to solving

$$u(t) = (\cos tA)\phi_- + A^{-1}(\sin tA)\psi_- + \int_{-\infty}^{t} A^{-1}\sin(t-s)AF(u(s))ds. \tag{5.6}$$

As before we can find a solution via an iteration argument on the space $X$ in (3.3), making use of the Leibniz rule for fractional derivatives and the Strichartz estimates of section 2. The only difference is that here we have

$$\int_{-\infty}^{t} A^{-1}\sin(t-s)AF(u(s))ds = VF(v)(t) \tag{5.7}$$

where this $V$ is like the $V$ in (2.3), but with $\int_{0}^{t}$ replaced by $\int_{-\infty}^{t}$. The proof of Theorem 2.2 may be trivially extended to include this case, giving the desired result.

Having $w = (u, u_t)$, we now estimate the difference:

$$e^{-itL}w(t) - (\phi_- - \psi_-) = \int_{-\infty}^{t} \left( -A^{-1}\sin sAF(u(s)) \right) ds. \tag{5.8}$$

Parallel to (4.4), we have, for all real $\sigma$,

$$\left\| \int_{-\infty}^{t} e^{isA} F(s)ds \right\|_{H^{\sigma, 2}} = \left\| T^\ast H_\sigma \right\|_{H^{\sigma, 2}} \tag{5.9}$$

where

$$H_\sigma(s, x) = \chi_{(-\infty, t]}(s)F(s, x). \tag{5.10}$$
Setting $\sigma = \gamma - 1$ and noting again that $(\frac{2(n+1)}{n-1}, \frac{2(n+1)}{n-3}, \frac{1}{l}) \in \mathcal{E}$, we apply (4.4) to obtain
\[
\left\| \int_{-\infty}^{t} e^{-isA} F(s) ds \right\|_{\dot{H}^{\gamma-1,2}} \lesssim \left\| F(u) \right\|_{L^{2(\frac{n+1}{n-1})}((-\infty,t], \dot{H}^{\gamma-\frac{2(n+1)}{n-3}, \frac{2(n+1)}{n-3} + 1}(M))}.
\]  
(5.11)

The right-hand side here may be bounded above (using Lemma 3.1) by the norm of the initial data. We may then write the right-hand side of (5.8) as
\[
\left( \begin{array}{c} \phi(t) \\ \psi(t) \end{array} \right),
\]
and we have
\[
\left\| \phi(t) \right\|_{\dot{H}^{\gamma-2}(M)} + \left\| \psi(t) \right\|_{\dot{H}^{\gamma-1,2}(M)} \lesssim \left\| F(u) \right\|_{L^{2(\frac{n+1}{n-1})}((-\infty,t], \dot{H}^{\gamma-\frac{2(n+1)}{n-3}, \frac{2(n+1)}{n-3} + 1}(M))} \to 0,
\]
(5.13)
as $t \to -\infty$. Hence we have the conclusion:

**Theorem 5.2** In the setting of Theorem 5.1, there exists $\epsilon_0 > 0$ such that if $\phi_-$ and $\psi_-$ are chosen satisfying $\left\| \phi_- \right\|_{\dot{H}^{\gamma-2}(M)} \leq \epsilon_0$ and $\left\| \psi_- \right\|_{\dot{H}^{\gamma-1,2}(M)} \leq \epsilon_0$, then (5.3) has a solution $w = (u,u_t)$, with $u \in L^{\gamma-1,2}((\mathbb{R}, \dot{H}^{\gamma-\frac{2(n+1)}{n-3}, \frac{2(n+1)}{n-3} + 1} \cap L^q((\mathbb{R} \times M), \text{with } \gamma = \frac{n}{2} - \frac{2}{q}, \text{and } q = \frac{(n+1)(\gamma - 1)}{2})$, and
\[
\left\| \begin{array}{c} u(t) \\ u_t(t) \end{array} - e^{itL} \begin{array}{c} \phi_- \\ \psi_- \end{array} \right\|_{\dot{H}^{\gamma-2}(M) \cap \dot{H}^{\gamma-1,2}(M)} \to 0 \text{ as } t \to -\infty.
\]
(5.14)

We can, of course, obtain a similar result for $t \to \infty$, through a trivial modification of the preceding arguments.

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