Partial Hasse invariants for Shimura varieties of Hodge-type

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Abstract

For a connected reductive group $G$ over a finite field, we define partial Hasse invariants on the stack of $G$-zip flags. We obtain similar sections on the flag space of Shimura varieties of Hodge-type. They are mod $p$ automorphic forms which cut out a single codimension one stratum. We study their properties and show that such invariants admit a natural factorization through higher rank automorphic vector bundles. We define the socle of an automorphic vector bundle, and show that partial Hasse invariants lie in this socle.

1 Introduction

Originally, partial Hasse invariants were defined for Hilbert–Blumenthal Shimura varieties by Goren and Andreatta–Goren in [Gor01, AG05] as certain Hilbert modular forms in characteristic $p$. The vanishing locus of each partial Hasse invariant coincides with the closure of a codimension one Ekedahl–Oort stratum. The product of all these sections is the classical Hasse invariant, which vanishes exactly outside the ordinary locus. Similar constructions can be found in [Her18, Bij18, BH17]. In the case of Hilbert modular varieties, Diamond–Kassaei have shown in [DK17, DK20] that partial Hasse invariants play an important role in the theory of mod $p$ automorphic forms. View the weight of a Hilbert modular form as a tuple $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ (where $n$ is the degree of the totally real extension defining the corresponding Hilbert–Blumenthal Shimura variety). Diamond–Kassaei define the minimal cone $C_{\text{min}} \subset \mathbb{Z}^n$ and show the following:

1. Any automorphic form whose weight lies outside of $C_{\text{min}}$ is divisible by (a specific) partial Hasse invariant.

2. The cone $C_{\text{Hasse}}$ generated by the weights of all partial Hasse invariants spans (over $\mathbb{Q}_{\geq 0}$) all possible weights of Hilbert modular forms in characteristic $p$.

In [GK18], Goldring and the second named author showed that (2) also holds for Picard modular surfaces and Siegel threefolds (conditionnally on a Koecher principle for strata that should follow from [LS18]). The present paper provides a general theory of partial Hasse invariants. It serves as a preliminary to a follow-up paper ([GIK21]) in which we discuss generalizations of Diamond–Kassaei’s result to other Shimura varieties. In another joint paper by Goldring and the second author, we show that both (1) and (2) generalize naturally to Picard Shimura varieties (at split or inert primes), Siegel Shimura varieties of rank 2 and 3.

Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum of Hodge-type and $\text{Sh}_K(\mathbf{G}, \mathbf{X})$ the corresponding Shimura variety with level $K$ over a number field $E$. Let $\mu: \mathbb{G}_{m, \mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$ be the cocharacter deduced from $\mathbf{X}$. Write $\mathbf{L} \subset \mathbf{G}_{\mathbb{C}}$ for the Levi subgroup centralizing $\mu$. Choose a Borel pair $(\mathbf{B}, \mathbf{T})$ such that $\mu$ factors through $\mathbf{T}$. Denote by $I := \Delta_{\mathbf{L}}$ the simple roots of
Let $L$ with respect to the opposite Borel of $B$. For any $\lambda \in X^*(T)$, there is a vector bundle $V_\lambda$ on $\text{Sh}_K(G, X)$, modeled on the $L$-representation $\text{Ind}_B^L(\lambda)$, where $B_L := B \cap L$. Let $p$ be a prime of good reduction, and let $S_K$ be the smooth canonical model over $\mathcal{O}_{\mathbb{F}_q}$ (where $p|p$) constructed by Kisin (Kis10) and Vasiu (Vas99). The vector bundle $V_\lambda$ extends naturally to $S_K$. We define partial Hasse invariants as certain sections of $V_\lambda$ over $S_K := S_K \otimes_{\mathcal{O}_{\mathbb{F}_q}} \mathbb{Z}_p$. More precisely, let $G$ be the special fiber of a reductive $\mathbb{Z}_p$-model of $G_{\mathbb{Q}_p}$, and write $B, T, L$ for the subgroups of $G$ naturally induced by $B, T, L$. Moonen–Wedhorn and Pink–Wedhorn–Ziegler define the stack of $G$-zips of type $\mu$ denoted by $G\text{-Zip}^\mu$. By Zhang [Zha18], there is a smooth, surjective map $\zeta: S_K \rightarrow G\text{-Zip}^\mu$. We construct partial Hasse invariants on this stack.

More generally, one can start with an arbitrary connected, reductive group $G$ over $\mathbb{F}_q$, endowed with a cocharacter $\mu: G_\mathbb{F}_q \rightarrow G_{\mathbb{F}_q}$ (in the context of Shimura varieties, we take $q = p$). Goldring and the second named author defined in [GK19a] the stack of $G\text{-Zip}^\mu$. One can define similarly the flag space $\text{Flag}(S_K)$ of $S_K$. There is a natural map $\pi: G\text{-ZipFlag}^\mu \rightarrow G\text{-Zip}^\mu$ whose fibers are flag varieties. For $\lambda \in X^*(T)$, there is a line bundle $V_{\text{flag}}(\lambda)$ on $G\text{-ZipFlag}^\mu$ and one has $\pi_*(V_{\text{flag}}(\lambda)) = V_\lambda$. In particular, we may identify global sections of $V_\lambda$ and $V_{\text{flag}}(\lambda)$. Furthermore, $G\text{-ZipFlag}^\mu$ admits a natural stratification $(C_w)_{w \in W}$ (where $W = W(G, T)$ is the Weyl group), similar to the Bruhat decomposition of $G$. The codimension one strata are $(C_{s_a}w_0)_a \in \Delta$, where $\Delta$ is the set of simple roots, $s_a$ is the reflection along $\alpha$, and $w_0 \in W$ is the longest element. Then, a (flag) partial Hasse invariant for $\alpha$ is a section $\text{Ha}_\alpha \in H^0(G\text{-ZipFlag}^\mu, V_{\text{flag}}(\lambda_\alpha))$ whose vanishing locus is the Zariski closure of $C_{w_0s_a}$. A simple argument (Proposition 5.2.7) shows that such sections always exist. More generally, for any dominant $\chi \in X^*(T)$, we define a section $\text{Ha}_\chi$ of $V_{\text{flag}}(\lambda_\chi)$ where $\lambda_\chi = \chi - qw_0I(\sigma^{-1}\chi)$ (here $\sigma$ denotes the action of Frobenius and $w_0I$ is the longest element in $W_L = W(L, T)$). We call $\text{Ha}_\chi$ a Schubert section. If $\chi_\alpha$ is orthogonal to $\Delta \setminus \{\alpha\}$ and $\langle \chi_\alpha, \alpha^\vee \rangle > 0$, then $\text{Ha}_\alpha := \text{Ha}_\chi_\alpha$ is a partial Hasse invariant for $\alpha$.

We now explain the main results of the paper. First, we define Verschiebung homomorphisms for vector bundles on $G\text{-Zip}^\mu$. Assume that $L$ is defined over $\mathbb{F}_q$. For an $L$-representation $(V, \rho)$, we define a subspace $V_{G_S} \subset V$ (see §6.1). When $V = V_{G_S}$, we show that the attached vector bundle $V(\rho)$ on $G\text{-Zip}^\mu$ admits a natural map $\psi: V(\rho) \rightarrow V(\rho[1])$, where $\rho[1]$ is the composition of $\rho$ with the Frobenius homomorphism $\varphi: L \rightarrow L$. In particular, when $\chi \in X^*(T)$ is dominant, $V_I(w_0Iw_{0\chi})$ admits a Verschiebung $\psi_\chi$. By Proposition 6.3.1, $V_I(w_0Iw_{0\chi})$ is naturally a sub-$L$-representation of the $G$-representation $V_\Delta(\chi) = \text{Ind}_B^L(\lambda)$. Denote by $V_\chi(\lambda)$ the attached vector bundle on $G\text{-Zip}^\mu$.

**Theorem 1** (Theorem 6.3.5). Let $\chi \in X^*(T)$ be a dominant character. There exists a map of vector bundles $\psi_\chi: V_\Delta(\chi) \rightarrow V_\Delta(\chi)[1]$ over $G\text{-Zip}^\mu$ with image $V_I(w_0Iw_{0\chi})[1]$ and such that $\psi_\chi$ extends the Verschiebung map of $V_I(w_0Iw_{0\chi})$.

$$
V_\Delta(\chi) \xrightarrow{\psi_\chi} V_I(w_0Iw_{0\chi})[1] \subset V_\Delta(\chi)[1].
$$

We call $\psi_\chi$ the Verschiebung homomorphism of $V_\Delta(\chi)$. Schubert sections admit a factorization in terms of Verschiebung homomorphisms. Specifically, let $\chi$ be dominant and set $V = V_I(-w_0I\chi)$. It admits a Verschiebung map $\psi: V \rightarrow V[1]$. We show (Corollary 7.1.2) that there are natural maps of vector bundles over $G\text{-ZipFlag}^\mu$:

$$
V_{\text{flag}}(-\chi) \rightarrow \pi^*V \xrightarrow{\pi^*\psi} \pi^*V[1] \rightarrow V_{\text{flag}}(-w_0I(\sigma^{-1}\chi))^\otimes q.
$$
such that the composition coincides with $H_{\alpha}$ as a section of $\mathcal{V}_{flag}(\lambda_{\alpha})$. In particular, this factorization applies to partial Hasse invariants.

Lastly, we prove that partial Hasse invariants are primitive automorphic forms. For an $L$-dominant $\lambda$, denote by $L_I(\lambda) \subset V_I(\lambda)$ the socle of $V_I(\lambda)$. Write $V^*_I(\lambda)$ for the vector bundle attached to $L_I(\lambda)$. We call sections of $V^*_I(\lambda)$ primitive automorphic forms of weight $\lambda$. For each $\alpha \in \Delta$, suppose there exists a character $\chi_{\alpha} \in X^*(T)$ satisfying

(a) $0 < \langle \chi_{\alpha}, \alpha^v \rangle < q$ and $\langle \chi_{\alpha}, \beta^v \rangle = 0$ for all $\beta \in \Delta \setminus \{\alpha\}$,
(b) one has $L_I(\chi_{\alpha}) = V_I(\chi_{\alpha})$ and $L_I(-w_0, I(\sigma^{-1}\chi_{\alpha})) = V_I(-w_0, I(\sigma^{-1}\chi_{\alpha})).$

Such characters $\chi_{\alpha}$ often exist. For example, for $G = \text{Sp}(2n)_{q}$, the fundamental weights satisfy (a) and (b) (even for $q = 2$).

**Theorem 2** (Theorem 7.2.5). Let $\alpha \in \Delta$ and let $\chi_{\alpha} \in X^*(T)$ satisfying (a) and (b) above. Set $\lambda_{\alpha} := \chi_{\alpha} - qw_0, I(\sigma^{-1}\chi_{\alpha})$. Then there exists a section $H_{\alpha}$ over $G$-ZipFlag$^u$ of the line bundle $\mathcal{V}_{flag}(\lambda_{\alpha})$, such that

1. $H_{\alpha}$ is a flag partial Hasse invariant for $\alpha$.
2. $H_{\alpha}$ is a primitive automorphic form on $G$-Zip$^u$.

Finally, we give in §8 the modular interpretation of partial Hasse invariants in the case of Siegel-type and unitary Shimura varieties (at inert and split primes).

## 2 Preliminaries and reminders on the stack of G-zips

### 2.1 Notation

Let $p$ be a prime number and $q$ a power of $p$. Let $\mathbb{F}_q$ denote a finite field with $q$ elements. We write $k$ for an algebraic closure of $\mathbb{F}_q$ of $\mathbb{F}_q$. Let $G$ be a connected reductive group over $\mathbb{F}_q$. For a $k$-scheme $X$, we denote by $X^{(q)}$ its $q$-th power Frobenius twist and by $\varphi : X \rightarrow X^{(q)}$ its relative Frobenius morphism. Write $\sigma \in \text{Gal}(k/\mathbb{F}_q)$ for the $q$-power Frobenius. We write $(B, T)$ for a Borel pair of $G$ defined over $\mathbb{F}_q$, i.e. $T$ is a maximal torus, $B$ a Borel subgroup in $G$ and $T \subset B$. We do not assume that $T$ is split over $\mathbb{F}_q$. Let $B^+$ be the Borel subgroup of $G_k$ opposite to $B$ with respect to $T$, which is the unique Borel subgroup such that $B^+ \cap B = T$. We will use the following notations:

- $X^*(T)$ (resp. $X_*(T)$) denotes the group of characters (resp. cocharacters) of $T$. The group $\text{Gal}(k/\mathbb{F}_q)$ acts naturally on these groups. Let $W = W(G_k, T)$ be the Weyl group of $G_k$. Similarly, $\text{Gal}(k/\mathbb{F}_q)$ acts on $W$, compatibly with the action of $W$ on characters and cocharacters.
- $\Phi \subset X^*(T)$ is the set of $T$-roots of $G$, and $\Phi_+ \subset \Phi$ is the set of positive roots with respect to $B^+$ (i.e. $\alpha \in \Phi_+$ if the $\alpha$-root group $U_{\alpha}$ is contained in $B^+$). This convention differs from other authors. We use it to match conventions of [GK19a], [Kos19].
- $\Delta \subset \Phi_+$ is the set of simple roots.
- For $\alpha \in \Phi$, let $s_{\alpha} \in W$ be the corresponding reflection. Then $(W, \{s_{\alpha} \mid \alpha \in \Delta\})$ is a Coxeter system. Write $\ell : W \rightarrow \mathbb{N}$ for the length function and $\leq$ for the Bruhat order on $W$. We have $\ell(s_{\alpha}) = 1$ for all $\alpha \in \Delta$. Let $w_0$ denote the longest element of $W$.
- Let $K$ be a subset of $\Delta$. We write $W_K$ for the subgroup of $W$ generated by $\{s_{\alpha} \mid \alpha \in K\}$. Let $w_{0, K}$ be the longest element in $W_K$. Let $K^W$ (resp. $W^K$) denote the subset of elements $w \in W$ which have minimal length in the coset $W_K w$ (resp. $W_k w$). Then $K^W$ (resp. $W^K$) is a set of representatives of $W_k \backslash W$ (resp. $W/W_K$). The map $g \mapsto g^{-1}$ induces a bijection $K^W \rightarrow W^K$. The longest element in the set $K^W$ is $w_{0, K} w_0$. 

3
Let $W$ recall some facts about the stack of $G$.

2.2 The stack of $G$ is a parabolic subgroup with Levi subgroup $L$ as restricting this action to $E$. Then $E$ is the category of tuples $M = (G,P,L,Q,M,\varphi)$.

For a subset $I \subset \Delta$, let $X^*_{\text{sl}}(T)$ denote the set of characters $\lambda \in X^*(T)$ such that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$.

For a subset $I \subset \Delta$, let $X^*_{\text{sl}}(T)$ denote the set of characters $\lambda \in X^*(T)$ such that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in I$. We call them $I$-dominant characters.

Let $P \subset G_k$ be a parabolic subgroup containing $B$ and let $L \subset P$ be the unique Levi subgroup of $P$ containing $T$. Define $I_P \subset \Delta$ as the unique subset such that $W(L,T) = W_{I_P}$. For an arbitrary parabolic subgroup $P \subset G_k$ containing $T$, define $I_P \subset \Delta$ by $I_P := I_{P'}$ where $P'$ is the unique conjugate of $P$ containing $B$. Moreover, set $\Delta^P := \Delta \setminus I_P$.

For all $\alpha \in \Phi$, choose an isomorphism $u_\alpha : \mathfrak{g}_a \to U_\alpha$ so that $(u_\alpha)_{\alpha \in \Phi}$ is a realization in the sense of [Spr98] 8.1.4. In particular, we have

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x), \quad \forall x \in \mathfrak{g}_a, \forall t \in T.$$ 

Let $\phi_\alpha : \text{SL}_2 \to G$ denote the map attached to $\alpha$, as in [Spr98] 9.2.2. It satisfies

$$\phi_\alpha \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = u_\alpha(x), \quad \phi_\alpha \left( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) = u_{-\alpha}(x).$$

Fix a $B$-representation $(V,\rho)$. For $j \in \mathbb{Z}$ and $\alpha \in \Phi$, we define a map $E^{(j)}_\alpha : V \to V$ as follows. Let $V = \bigoplus_{\nu \in X^*(T)} V_\nu$ be the weight decomposition of $V$. For $v \in V_\nu$, we can write uniquely

$$u_\alpha(x)v = \sum_{j \geq 0} x^j E^{(j)}_\alpha(v), \quad \forall x \in \mathfrak{g}_a,$$

for elements $E^{(j)}_\alpha(v) \in V_{\nu + j\alpha}$ ([Don55] Proposition 3.3.2]). Extend $E^{(j)}_\alpha$ by additivity to a map $V \to V$. For $j < 0$, we put $E^{(j)}_\alpha = 0$.

### 2.2 The stack of $G$-zips

We recall some facts about the stack of $G$-zips of Pink–Wedhorn–Ziegler in [PWZ11].

#### 2.2.1 Definitions

Let $G$ be a connected, reductive group over $\mathbb{F}_q$. A zip datum means a tuple $Z := (G,P,L,Q,M,\varphi)$ consisting of the following objects:

1. $P,Q \subset G_k$ are parabolic subgroups of $G_k$.
2. $L \subset P$ and $M \subset Q$ are Levi subgroups such that $L^{(q)} = M$.

For an algebraic group $H$, let $R_u(H)$ denote the unipotent radical of $H$. If $P' \subset G_k$ is a parabolic subgroup with Levi subgroup $L' \subset P'$, any $x \in P'$ can be written uniquely as $x = \overline{x} u$ with $\overline{x} \in L'$ and $u \in R_u(P')$. Denote by $\theta^{P'}_{L'} : P' \to L'$ the map $x \mapsto \overline{x}$. Since $M = L^{(q)}$, we have a Frobenius isogeny $\varphi : L \to M$. The zip group is the subgroup of $P \times Q$ defined by

$$E := \{(x,y) \in P \times Q \mid \varphi(\theta^{P}_{L'}(x)) = \theta^{Q}_{M}(y)\}. \quad (2.2.1)$$

Then $E$ is the subgroup of $P \times Q$ generated by $R_u(P) \times R_u(Q)$ and elements of the form $(a,\varphi(a))$ with $a \in L$. Let $G \times G$ act on $G$ by $(a,b) \cdot g := agb^{-1}$, and let $E$ act on $G$ by restricting this action to $E$. We define the stack of $G$-zips of type $Z$ ([PWZ11], [PWZ15]) as the quotient stack

$$G\text{-}\text{Zip}^Z = [E \backslash G_k].$$

The stack $G\text{-}\text{Zip}^Z$ is the stack over $k$ such that for any $k$-scheme $S$, the groupoid $G\text{-}\text{Zip}^Z(S)$ is the category of tuples $\underline{Z} = (I, I_P, I_Q, \iota)$, where $I$ is a $G$-torsor over $S$, $I_P \subset I$ and $I_Q \subset I$. 

are respectively a $P$-subtorus and a $Q$-subtorus of $I$, and $\iota: (I_P/R_u(P))^{(p)} \to I_Q/R_u(Q)$ is an isomorphism of $M$-torsors.

### 2.2.2 Cocharacter datum

A cocharacter datum is a pair $(G, \mu)$ where $G$ is a connected, reductive group over $\mathbb{F}_q$ and $\mu: \mathbb{G}_{m,k} \to G_k$ is a cocharacter. One can attach to $(G, \mu)$ a zip datum $Z_{\mu}$ as follows. Let $P_+(\mu)$ (resp. $P_-(\mu)$) denote the unique parabolic subgroup of $G_k$ such that $P_+(\mu)(k)$ (resp. $P_-(\mu)(k)$) consists of the elements $g \in G(k)$ satisfying the condition that the map $G_{m,k} \to G_k; \ t \mapsto \mu(t)g(t)^{-1}$ (resp. $t \mapsto \mu(t)^{-1}g(t)$) extends to a morphism of varieties $\mathbb{A}^1 \to G_k$. We obtain a pair of opposite parabolics $(P_+(\mu), P_-(\mu))$ whose intersection $L(\mu) = P_+(\mu) \cap P_-(\mu)$ is the centralizer of $\mu$ (it is a common Levi subgroup of $P_+(\mu)$ and $P_-(\mu)$). We put $P := P_-(\mu)$, $Q := (P_+(\mu))^{(q)}$, $L := L(\mu)$ and $M := L(\mu)^{q}$. The tuple $Z_{\mu} := (G, P, L, Q, M, \varphi)$ is a zip datum, which we call the zip datum attached to the cocharacter datum $(G, \mu)$. We write simply $G\text{-Zip}^\mu$ for $G\text{-Zip}^{\varphi^*}$. We will always consider zip data arising in this way from a cocharacter datum.

### 2.2.3 Frames

Let $Z = (G, P, Q, L, M)$ be a zip datum. In this paper, a frame for $Z$ is a triple $(B, T, z)$ where $(B, T)$ is a Borel pair of $G_k$ defined over $\mathbb{F}_q$ such that $B \subset P$ and $z \in W$ is an element such that

$$2B \subset Q \quad \text{and} \quad B \cap M = 2B \cap M. \quad (2.2.2)$$

A frame (as defined here) may not always exist. However, if $(G, \mu)$ is a cocharacter datum and $Z_{\mu}$ is the associated zip datum by §2.2.2, then we can find $g \in G(k)$ such that $Z_{\mu'}$ for $\mu' = ad(g) \circ \mu$ admits a frame. Hence, it is harmless to assume the existence of a frame, and we consider only zip data which admit frames. For a zip datum $(G, P, L, Q, M, \varphi)$, we define subsets $I, J, \Delta^P \subset \Delta$ as follows:

$$I := I_P, \quad J := I_Q, \quad \Delta^P = \Delta \setminus I.$$

**Lemma 2.2.1** ([GK191] Lemma 2.3.4). Let $\mu: \mathbb{G}_{m,k} \to G_k$ be a cocharacter, and let $Z_{\mu}$ be the attached zip datum. Assume that $(B, T)$ is a Borel pair defined over $\mathbb{F}_q$ such that $B \subset P$. Define the element

$$z := w_0w_{0,I} = \sigma(w_{0,1})w_0.$$

Then $(B, T, z)$ is a frame for $Z_{\mu}$.

### 2.2.4 Parametrization of the E-orbits in G

Recall that the group $E$ defined in (2.2.1) acts on $G_k$. By [PWZ11], Proposition 7.1, there are finitely many $E$-orbits in $G_k$. The $E$-orbits are smooth and locally closed in $G$. This gives a stratification of $G_k$, in the sense that the closure of an $E$-orbit is a union of $E$-orbits. We review below the parametrization of $E$-orbits following [PWZ11].

For $w \in W$, fix a representative $w \in N_G(T)$, such that $(w_1w_2)^p = \bar{w}_1\bar{w}_2$ whenever $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ (this is possible by choosing a Chevalley system, [ABD66], XXIII, §6). For $w \in W$, define $G_w$ as the $E$-orbit of $w\bar{z}^{-1}$. If no confusion occurs, we write $w$ instead of $\bar{w}$. For $w, w' \in W$, write $w' \preceq w$ if there exists $w_1 \in W_L$ such that $w' \leq w_1w\sigma(w_1)^{-1}$. This defines a partial order on $W$ ([PWZ11 Corollary 6.3]).
Theorem 2.2.2 (PWZ11, Theorem 7.5, Theorem 11.2]). We have two bijections:

\[ \begin{align*}
\mathcal{I} W & \rightarrow \{ E\text{-orbits in } G_k \}, \quad w \mapsto G_w \quad (2.2.3) \\
W^J & \rightarrow \{ E\text{-orbits in } G_k \}, \quad w \mapsto G_w. \quad (2.2.4)
\end{align*} \]

For \( w \in \mathcal{I} W \), one has \( \dim(G_w) = \ell(w) + \dim(P) \) and the Zariski closure of \( G_w \) is

\[ \overline{G_w} = \bigsqcup_{w' \in \mathcal{I} W, \ w' \prec w} G_{w'}. \quad (2.2.5) \]

There is a unique open \( E \)-orbit \( U_{\mathcal{I}} \subset G \) corresponding to the longest elements \( w_{0,1}w_0 \in \mathcal{I} W \) via (2.2.3) and to \( w_0w_{0,J} \in W^J \) via (2.2.4). If \( \mathcal{Z} \) arises from a cocharacter datum (§2.2.2), we write \( U_\mu \) for \( U_{\mathcal{I}_\mu} \). In this case we can choose \( z = w_0w_{0,J} = \sigma(w_{0,1})w_0 \) (Lemma 2.2.1), hence (2.2.4) shows that \( 1 \in U_\mu \). Using the terminology pertaining to Shimura varieties, \( U_\mu \) is called the \( \mu \)-ordinary stratum of \( G \). The corresponding substack \( U_\mu := [E \backslash U_\mu] \) is called the \( \mu \)-ordinary locus. It corresponds to the \( \mu \)-ordinary reduction of Shimura varieties, studied for example in [Moo04] and [HN17]. For more details about relations between Shimura varieties and stacks of \( E \)-zips, we refer to §2.7.

For an \( E \)-orbit \( G_w \) (with \( w \in \mathcal{I} W \) or \( w \in W^J \)), we write \( \mathcal{X}_w := [E \backslash G_w] \) for the corresponding locally closed substack of \( G\text{-Zip}^2 = [E \backslash G_k] \). We obtain similarly a stratification

\[ G\text{-Zip}^2 = \bigsqcup_{w \in \mathcal{I} W} \mathcal{X}_w \]

and one has closure relations between strata similar to (2.2.5).

2.3 Representation theory

For an algebraic group \( G \) over a field \( K \), denote by \( \text{Rep}(G) \) the category of algebraic representations of \( G \) on finite-dimensional \( K \)-vector spaces (we will mostly consider the case \( K = k = \overline{\mathbb{F}_q} \)). We denote such a representation by \( (V, \rho) \), or sometimes simply \( \rho \) or \( V \). For an algebraic group \( G \) over \( \overline{\mathbb{F}_q} \), a \( G_k \)-representation \( (V, \rho) \) and a positive integer \( m \), we denote by \( (V^m, \rho^m) \) the representation such that \( V^m = V \) and

\[ \rho^m \colon G_k \overset{\varphi^m}{\longrightarrow} G_k \overset{\rho}{\rightarrow} \text{GL}(V). \]

Let \( H \) be a split connected, reductive \( K \)-group and choose a Borel pair \((B_H, T)\) defined over \( K \). The isomorphism classes of irreducible representations of \( H \) are in 1-to-1 correspondence with the dominant characters \( X^*_+ (T) \) of \( T \), where the positivity is defined with respect to the Borel subgroup opposite to \( B_H \) (§2.1). This bijection is given by the highest weight of a representation. For a dominant character \( \lambda \), we denote by \( L_H(\lambda) \) the corresponding irreducible representation of highest weight \( \lambda \). If there is no confusion, we simply write \( L(\lambda) \) for \( L_H(\lambda) \). If \( K \) has characteristic zero, \( \text{Rep}(H) \) is semisimple. In characteristic \( p \) however, this is no longer true in general. For \( \lambda \in X^*_+ (T) \), let \( \mathcal{L}_\lambda \) be the line bundle on the flag variety \( H/B_H \) attached to \( \lambda \) ([Jan03, §5.8]). We define an \( H \)-representation by

\[ V_H(\lambda) := H^0(H/B_H, \mathcal{L}_\lambda) \]

Equivalently, \( V_H(\lambda) \) is the induced representation \( \text{Ind}_{B_H}^H \lambda \). The representation \( V_H(\lambda) \) is of highest weight \( \lambda \). If \( \text{Char}(K) = 0 \), \( V_H(\lambda) \) is irreducible, hence \( V_H(\lambda) = L_H(\lambda) \). In positive
characteristic, this is not true in general, but \( L_H(\lambda) \) is always the socle of \( V_H(\lambda) \). We view elements of \( V_H(\lambda) \) as functions \( f: H \to \mathbb{A}^1 \) satisfying
\[
f(hb) = \lambda(b^{-1})f(h), \quad h \in H, \ b \in B_H. \tag{2.3.1}
\]
For dominant characters \( \lambda, \lambda' \), we have a natural surjective map
\[
V_H(\lambda) \otimes V_H(\lambda') \to V_H(\lambda + \lambda')
\]
sending \( f \otimes f' \) (where \( f \in V_H(\lambda) \), \( f' \in V_H(\lambda') \) as in (2.3.1)) to the function \( ff' \in V_H(\lambda + \lambda') \).

Let \( W_H := W(H,T) \) be the Weyl group and \( w_{0,H} \in W_H \) the longest element. Then there is a unique \( B_H \)-stable subline of \( V_H(\lambda) \), which is a weight space for the lowest weight \( w_{0,H} \lambda \).

Let \( H \) be a connected, reductive group over \( \mathbb{F}_q \) with a Borel pair \((B,T)\) of \( H_k \) defined over \( \mathbb{F}_q \). We recall the following well-known lemma:

**Lemma 2.3.1.** Let \( \lambda \in X^*_+(T) \). Let \( v_{\text{high}} \in V_H(\lambda) \) be a nonzero element in the highest weight line, and let \( p_\lambda: V_H(\lambda) \to kv_{\text{high}} \) be the projection onto \( kv_{\text{high}} \). Then \( p_\lambda \in V_H(\lambda)^\vee \) is a \( B \)-eigenvector for the weight \(-\lambda\). In other words, \( p_\lambda \) is a \( B \)-equivariant map \( V_H(\lambda) \to \lambda \).

**Proof.** This follows from [IK21, Lemma 3.3.1], as \( B \) is generated by \( T \) and \( (U_{-\alpha})_{\alpha \in \Phi_+} \). \( \square \)

We put
\[
X_1^+(T) = \{ \lambda \in X^+(T) \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < q \text{ for all } \alpha \in \Delta \}.
\]
The Steinberg’s tensor product theorem (cf. [Ste63]) is usually stated in the split case. We give a statement of the theorem in the general case.

**Theorem 2.3.2.** Let \( \lambda = \sum_{i=0}^m q^i \sigma^{-i}(\lambda_i) \) with \( \lambda_0, \lambda_1, \ldots, \lambda_m \in X_1^+(T) \). Then we have
\[
L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)[1] \otimes \cdots \otimes L(\lambda_m)[m].
\]

**Proof.** We take a split form \( T' \subset B' \subset H' \) over \( \mathbb{F}_q \) of \( T \subset B \subset H_k \). Put \( \lambda'_i = \sigma^{-i} \lambda_i \). By [Jan03, II, 3.17], we have \( L(\lambda) \simeq L(\lambda'_0) \otimes L(q \lambda'_1) \otimes \cdots \otimes L(q^m \lambda'_m) \). Since \( L(\lambda_i)[i] = L(q^i \lambda'_i) \) for \( 0 \leq i \leq m \) the claim follows. \( \square \)

### 2.4 Vector bundles on the stack of \( G \)-zips

#### 2.4.1 General theory

For an algebraic stack \( \mathcal{X} \), we write \( \mathfrak{VB}(\mathcal{X}) \) for the category of vector bundles on \( \mathcal{X} \). Let \( X \) be a \( k \)-scheme and \( H \) an affine \( k \)-group scheme acting on \( X \). We have a functor
\[
\mathcal{V}_{H,X}: \text{Rep}(H) \to \mathfrak{VB}([H \setminus X]).
\]
sending \((V, \rho)\) to the vector bundle defined geometrically as \([H \setminus (X \times_k V)]\), where \( H \) acts diagonally on \( X \times_k V \). For \((V, \rho) \in \text{Rep}(H)\), we have a natural identification
\[
H^0([H \setminus X], \mathcal{V}_{H,X}(\rho)) = \{ f: X \to V \mid f(h \cdot x) = \rho(h) f(x) \text{ for } h \in H, \ x \in X \}. \tag{2.4.1}
\]
2.4.2 Vector bundles on $G$-Zip$^\mu$

Fix a cocharacter datum $(G, \mu)$. Let $Z = (G, P, L, Q, M, \varphi)$ be the attached zip datum. Fix a frame $(B, T)$ as in [22.2.3]. By [24.1] we obtain a functor $\mathcal{V}_{E, G}: \text{Rep}(E) \to \mathfrak{V}(G-\text{Zip}^\mu)$, that we simply denote by $\mathcal{V}$. For $(V, \rho) \in \text{Rep}(E)$, the global sections of $\mathcal{V}(\rho)$ are

$$H^0(G-\text{Zip}^\mu, \mathcal{V}(\rho)) = \{ f : G_k \to V \mid f(\epsilon \cdot g) = \rho(\epsilon) f(g) \text{ for } \epsilon \in E, \ g \in G_k \}.$$  

Since $G$ admits an open dense $E$-orbit $U_\mu$ (see discussion below Theorem [22.2.2]), the space $H^0(G-\text{Zip}^\mu, \mathcal{V}(\rho))$ is finite-dimensional by [Kos19] Lemma 1.2.1 (see also (2.5.1) below). The first projection $p_1 : E \to P$ induces a functor $p_1^* : \text{Rep}(P) \to \text{Rep}(E)$. If $(V, \rho) \in \text{Rep}(P)$, we write again $\mathcal{V}(\rho)$ for $\mathcal{V}(p_1^*(\rho))$. In this paper, we will only consider $E$-representations coming from $P$ in this way.

2.5 Global sections over $G$-Zip$^\mu$

We review some results of [IK21] regarding the space of global sections of $\mathcal{V}(\rho)$ over $G$-Zip$^\mu$ for a $P$-representation $\rho$. We view $H^0(G-\text{Zip}^\mu, \mathcal{V}(\rho))$ as a subspace of $V$, as follows. By (2.4.1), such a global section is a map $h : G \to V$ satisfying $h(ab^{-1}) = \rho(a) h(g)$ for all $(a, b) \in E$ and all $g \in G$. Since 1 lies in the open dense $E$-orbit $U_\mu \subset G$ (see paragraph after Theorem [22.2.2]), the map $h \mapsto h(1)$ is an injection

$$\text{ev}_1 : H^0(G-\text{Zip}^\mu, \mathcal{V}(\rho)) \to V. \quad (2.5.1)$$

We describe the image of the map ev$_1$. First, we recall the space of sections over the open subset $U_\mu \subset G$-Zip$^\mu$. Recall that $U_\mu = [E \setminus U_\mu]$ and $1 \in U_\mu$ (see [22.4]). In particular, we can extend (2.5.1) to an injection $\text{ev}_1 : H^0(U_\mu, \mathcal{V}(\rho)) \to V$. The scheme-theoretical stabilizer of 1 in $E$ is given by

$$L_\varphi = E \cap \{(x, x) \mid x \in G_k \},$$

which is a 0-dimensional algebraic group (in general non-smooth). The first projection $E \to P$ induces a closed immersion $L_\varphi \to P$. Hence we identify $L_\varphi$ with its image and view it as a subgroup of $P$. Denote by $L_0 \subset L$ the largest algebraic subgroup defined over $\mathbb{F}_q$. In other words,

$$L_0 = \bigcap_{n \geq 0} L(\mathbb{F}_q^n).$$

**Lemma 2.5.1 ([KW18] Lemma 3.2.1]).**

1. One has $L_\varphi \subset L$.
2. The group $L_\varphi$ can be written as a semidirect product $L_\varphi = L^\circ_\varphi \rtimes L_0(\mathbb{F}_q)$ where $L^\circ_\varphi$ is the identity component of $L_\varphi$. Furthermore, $L^\circ_\varphi$ is a finite unipotent algebraic group.
3. Assume that $P$ is defined over $\mathbb{F}_q$. Then $L_0 = L$ and $L_\varphi = L(\mathbb{F}_q)$, viewed as a constant algebraic group.

**Lemma 2.5.2 ([IK21] Corollary 3.2.3]).** One has an identification

$$H^0(U_\mu, \mathcal{V}(\rho)) = V^{L_\varphi}.$$  

More precisely, we mean that the image of $\text{ev}_1 : H^0(U_\mu, \mathcal{V}(\rho)) \to V$ coincides with $V^{L_\varphi}$. Here, the notation $V^{L_\varphi}$ denotes the space of scheme-theoretical invariants, i.e. the set of $v \in V$ such that for any $k$-algebra $R$, one has $\rho(x)v = v$ in $V \otimes_k R$ for all $x \in L_\varphi(R)$. We now move on to the space of global sections over $G$-Zip$^\mu$. Note that restriction of sections
via the open dense inclusion $U_\alpha \subset G$-$\text{Zip}^\mu$ induces an injective map $H^0(G$-$\text{Zip}^\mu, V(\rho)) \to H^0(U_\alpha, V(\rho)) = V^{L_\alpha}$. We only explain a partial result, for the general result see [IK21] Theorem 3.4.1. The Lang torsor map $\varphi: T \to T, g \mapsto g \varphi(g)^{-1}$ induces an isomorphism:

$$\varphi_\ast: X_\ast(T)_R \simto X_\ast(T) \circ \delta \mapsto \varphi \circ \delta = \delta - q \sigma(\delta).$$

Set $\delta_\alpha := \varphi^{-1}(\alpha^\vee)$. Define a subspace $V_{\geq 0}^\rho \subset V$ as follows:

$$V_{\geq 0}^\rho = \bigoplus_{(\nu, \delta_\alpha) \geq 0, \nu \in \Delta^P} V_\nu. \tag{2.5.2}$$

If $T$ is split over $\mathbb{F}_q$, then $\delta_\alpha = -\alpha^\vee/(q-1)$, and $V_{\geq 0}^\rho$ is the direct sum of the weight spaces $V_\nu$ for those $\nu \in X_\ast(T)$ satisfying $(\nu, \alpha^\vee) \leq 0$ for all $\alpha \in \Delta^P$.

**Proposition 2.5.3.** Assume that $P$ is defined over $\mathbb{F}_q$ and furthermore that $(V, \rho) \in \text{Rep}(P)$ is trivial on the unipotent radical $R_u(P)$. Then one has an equality

$$H^0(G$-$\text{Zip}^\mu, V(\rho)) = V^{L(\mathbb{F}_q)} \cap V_{\geq 0}^\rho.$$

### 2.6 Category of $L$-vector bundles on $G$-$\text{Zip}^\mu$

Let $\theta_\rho^P: P \to L$ denote the natural projection modulo the unipotent radical $R_u(P)$ as \[\text{(2.2.1)}\]

It induces a fully faithful functor

$$(\theta_\rho^P)^*: \text{Rep}(L) \to \text{Rep}(P)$$

by composition, and its image is the full subcategory of $\text{Rep}(P)$ of $P$-representations which are trivial on $R_u(P)$. By this functor, we view $\text{Rep}(L)$ as a full subcategory of $\text{Rep}(P)$. If $(V, \rho) \in \text{Rep}(L)$, we write again $\mathcal{V}(\rho) := \mathcal{V}(\theta_\rho^P)^* \rho$. For $\lambda \in X_\ast(T)$, we write $B_L := B \cap L$ and define an $L$-representation $(V_\lambda(\lambda), \rho_{I, \lambda})$ by

$$V_\lambda(\lambda) = \text{Ind}_{B_L}^{L}(\lambda), \quad \rho_{I, \lambda}: L \to \text{GL}(V_\lambda(\lambda)). \tag{2.6.1}$$

We note that if $\lambda \in X_\ast(T)$ is not $I$-dominant, then we have $V_\lambda(\lambda) = 0$. Let $V_\lambda(\lambda)$ denote the vector bundle on $G$-$\text{Zip}^\mu$ attached to $V_\lambda(\lambda)$. We call it an automorphic vector bundle on $G$-$\text{Zip}^\mu$ associated to $\lambda$. This terminology stems from its relation to Shimura varieties (cf. \[\text{(2.2.1)}\]). For $\lambda \in X_\ast(L)$, viewing it as an element of $X_\ast(T)$ by restriction to $T$, the vector bundle $\mathcal{V}_\lambda(\lambda)$ is a line bundle.

Denote by $\mathfrak{VB}_L(G$-$\text{Zip}^\mu)$ the essential image of the functor $\mathcal{V}: \text{Rep}(L) \to \mathfrak{VB}(G$-$\text{Zip}^\mu)$. An object of $\mathfrak{VB}_L(G$-$\text{Zip}^\mu)$ is called an $L$-vector bundle. Assume that $P$ is defined over $\mathbb{F}_q$, hence $L_\varphi = L(\mathbb{F}_q)$. We describe the category $\mathfrak{VB}_L(G$-$\text{Zip}^\mu)$. Define the category $L_\varphi$-$\text{MF}_{\Delta^P}$ of $\Delta^P$-filtered $L_\varphi$-modules over $k$ as follows. Its objects are pairs $((V, \tau), \mathcal{F})$ where $\tau: L_\varphi \to \text{GL}_k(V)$ is a finite-dimensional representation of $L_\varphi$ and $\mathcal{F} := \{V_{\geq 0}^\rho\}_{\rho \in \Delta^P}$ is a set of filtrations on $V$, one for each $\alpha \in \Delta^P$. Here, $V_{\geq 0}^\rho$ denotes a descending filtration $(V_{\geq 0}^\rho)_{r \in \mathbb{R}}$. Morphisms between two $\Delta^P$-filtered $L_\varphi$-modules $((V, \tau), \mathcal{F})$ and $((V', \tau'), \mathcal{F}')$ over $k$ are $k$-linear, $L_\varphi$-equivariant maps $f: V \to V'$ which map $V_{\geq 0}^\rho$ to $V_{\geq 0}^\rho$ for all $r \in \mathbb{R}$ and all $\alpha \in \Delta^P$.

If $(V, \rho)$ is an $L$-representation, we can attach to it a $\Delta^P$-filtered $L_\varphi$-module $\text{F_{MF}}(V, \rho)$ as follows. First, define $\tau$ as the restriction of $\rho$ to $L_\varphi$. Then, for $\alpha \in \Delta^P$, define the filtration $V_{\geq 0}^\rho$ as follows. Let $V := \bigoplus \rho V_\nu$ be the $T$-weight decomposition of $V$. For $\alpha \in \Delta^P$ and $r \in \mathbb{R}$, let $V_{\geq r}^\alpha$ be the direct sum of $V_\nu$ for all $\nu$ satisfying $(\nu, \alpha) \geq r$. This construction gives rise to a functor

$$\text{F_{MF}}: \text{Rep}(L) \to L_\varphi$-$\text{MF}_{\Delta^P},$$
and we say that a $\Delta^P$-filtered $L_\varphi$-module is admissible if it is in the essential image of $F_{MF}$. Let $L_\varphi \cdot \text{MF}^{adm}_{\Delta^P}$ denote the full subcategory of $L_\varphi \cdot \text{MF}^{}_{\Delta^P}$ consisting of the admissible $\Delta^P$-filtered $L_\varphi$-modules.

**Theorem 2.6.1** ([IK21 Theorem 5.2.3]). Assume that $P$ is defined over $\mathbb{F}_p$. The functor $V: \text{Rep}(L) \to \mathfrak{M} \mathfrak{B}_L(G \cdot \text{Zip}^\mu)$ factors through the functor $F_{MF}: \text{Rep}(L) \to L_\varphi \cdot \text{MF}^{adm}_{\Delta^P}$ and induces an equivalence of categories

$$L_\varphi \cdot \text{MF}^{adm}_{\Delta^P} \to \mathfrak{M} \mathfrak{B}_L(G \cdot \text{Zip}^\mu).$$

### 2.7 Shimura varieties and G-zips

We explain the connection between the stack of $G$-zips and Shimura varieties. Let $(G, X)$ be a Shimura datum of Hodge-type (cf. [Del79, 2.1.1]). In particular, $G$ is a connected, reductive group over $\mathbb{Q}$, and $X$ is a $G(\mathbb{Q})$-conjugacy class of cocharacters $\{\mu\}$ of $G_{\mathbb{Q}}$. We write $E = E(G, X)$ for the reflex field of $(G, X)$ and $\mathcal{O}_E$ for its ring of integers. For an open compact subgroup $K \subset G(\mathbb{A}_f)$, we write $\text{Sh}(G, X)_K$ for Deligne’s canonical model at level $K$ over $E$ (see [Del79]). For $K \subset G(\mathbb{A}_f)$ small enough, the canonical model $\text{Sh}(G, X)_K$ is a smooth, quasi-projective scheme over $E$. We fix a prime number $p$ of good reduction. In particular, $G_{\mathbb{Q}_p}$ is unramified, so there exists a reductive $\mathbb{Z}_p$-model $\mathcal{G}$, such that $G := \mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is connected. For any place $v$ above $p$ in $E$, Kisin ([Kis10]) and Vasin ([Vas99]) constructed a smooth canonical model $\mathcal{S}_K$ over $\mathcal{O}_{E_v}$-schemes. Write $S_K := \mathcal{S}_K \otimes_{\mathcal{O}_{E_v}} \mathbb{F}_p$.

For $\mu \in \{\mu\}$, let $P = P_-(\mu)$ be the parabolic of $G_{\mathbb{C}}$ defined as in [IK21 §2.5]. As explained in [IK21 §2.5], we can find $\mu \in \{\mu\}$ which extends to a cocharacter of $G_{\mathbb{C}_v}$. Write again $\mu$ for its special fiber. Then $(G, \mu)$ is a cocharacter datum, and yields a zip datum $(G, P, L, Q, M, \varphi)$ (we always take $q = p$ in the context of Shimura varieties). Zhang ([Zha18 4.1]) constructed a smooth morphism

$$\zeta: S_K \to G \cdot \text{Zip}^\mu,$$

which is surjective by [SYZ19 Corollary 3.5.3(1)]. Let $\mathcal{P}$ be the unique parabolic of $G_{\mathbb{C}_v}$ which extends $P$. Then, we have a commutative diagram

$$\begin{array}{ccc}
\text{Rep}_{\mathbb{F}_p}(\mathcal{P}) & \simeq & \mathfrak{M} \mathfrak{B}(\mathcal{S}_K) \\
\downarrow & & \downarrow \\
\text{Rep}_{\mathbb{F}_p}(P) & \simeq & \mathfrak{M} \mathfrak{B}(S_K).
\end{array}$$

The vector bundles of this form on $\mathcal{S}_K$ and $S_K$ are called automorphic vector bundles in [Mil90 III, Remark 2.3]. In particular, let $\lambda \in X^*(T)$ be an $L$-dominant character and let $V_L(\lambda) = H^0(P/B, L_\lambda)$ denote the unique irreducible representation of $P$ over $\mathbb{Q}_p$ of highest weight $\lambda$. It admits a natural model over $\mathbb{Z}_p$, namely $V_L(\lambda)_{\mathbb{F}_p} := H^0(\mathcal{P}/\mathcal{B}, L_\lambda)$ where $L_\lambda$ is the line bundle attached to $\lambda$ viewed as a character of $\mathcal{F}$. Its reduction modulo $p$ is the $P$-representation $V_I(\lambda) = H^0(P/B, L_\lambda)$ over $k = \mathbb{F}_p$ as in [2.6.1].

### 3 Vector bundles on the stack of G-zip flags

#### 3.1 The stack of G-zip flags

Let $(G, \mu)$ be a cocharacter datum with attached zip datum $Z = (G, P, L, Q, M, \varphi)$ (see §2.2). Fix a frame $(B, T, z)$ with $z = \sigma(w_{0,i})w_0 = w_0 w_{0,j}$ (Lemma 2.2.1). The stack of
zip flags ([GK19a, Definition 2.1.1]) is defined as

$$G\text{-ZipFlag}^\mu = \left[ E\backslash (G_k \times P/B) \right]$$

where the group $E$ acts on the variety $G_k \times (P/B)$ by the rule $(a,b) \cdot (g, hB) := (agb^{-1}, ahB)$ for all $(a, b) \in E$ and all $(g, hB) \in G_k \times P/B$. The first projection $G_k \times P/B \to G_k$ is $E$-equivariant, and hence yields a natural morphism of stacks

$$\pi: G\text{-ZipFlag}^\mu \to G\text{-Zip}^\mu.$$  \hspace{1cm} (3.1.1)

Similarly to the stack $G\text{-Zip}^\mu$, there is an interpretation of this stack in terms of torsors, given in [GK19a, Definition 2.1.1]. For any $k$-scheme $S$, the $S$-points of $G\text{-ZipFlag}^\mu$ are pairs $(I,J)$ where $I = (I, I_P, I_Q, \iota)$ is an $E$-zip over $S$ (see the end of §2.2.1) and $J \subset I_P$ is a $B$-torsor. The map (3.1.1) is given by forgetting the $B$-torsor $J$.

Set $E' := E \cap (B \times G_k)$. Then the injective map $G_k \to G_k \times P/B; g \mapsto (g, B)$ yields an isomorphism of stacks $[E'\backslash G_k] \simeq G\text{-ZipFlag}^\mu$ (see [GK19a (2.1.5)]). One of the main features of the stack $G\text{-ZipFlag}^\mu$ is the existence of a stratification (indexed by $W$). First define the Schubert stack as the quotient stack

$$\text{Sbt} := [B\backslash G_k/B].$$

This stack is finite and smooth. Its topological space is isomorphic to $W$, endowed with the topology induced by the Bruhat order on $W$. This follows easily from the Bruhat decomposition of $G$. For $w \in W$, put

$$\text{Sbt}_w := [B\backslash BwB/B].$$  \hspace{1cm} (3.1.2)

It is a locally closed substack of $\text{Sbt}$. $\text{Sbt}_w$ is the unique open stratum of $\text{Sbt}$. One has the inclusion $E' \subset B \times z^2 B$. In particular, there is a natural projection map $[E'\backslash G_k] \to [B\backslash G_k/z^2 B]$. To obtain a map to $\text{Sbt}$, we compose with the isomorphism of stacks $[B\backslash G_k/z^2 B] \to [B\backslash G_k/B]$ induced by $G_k \to G_k; g \mapsto gz$. In the end, we obtain a smooth, surjective morphism of stacks

$$\psi: G\text{-ZipFlag}^\mu \to \text{Sbt}.$$  \hspace{1cm} (3.1.3)

The flag strata of the stack $G\text{-ZipFlag}^\mu$ are defined as the fibers of $\psi$. They are locally closed substacks (endowed with the reduced structure). This gives a stratification of $G\text{-ZipFlag}^\mu$ indexed by $W$. For $w \in W$, put

$$C_w := B(wz^{-1})^2 B = BwBz^{-1},$$

which is the $B \times z^2 B$-orbit of $wz^{-1}$. The set $C_w$ is locally closed in $G_k$, and one has the dimension formula $\dim(C_w) = l(w) + \dim(B)$. The Zariski closure $\overline{C_w}$ is normal ([RRS5, Theorem 3]) and coincides with the union $\bigcup_{w' \leq w} C_{w'}$, where $\leq$ is the Bruhat order of $W$. Via the isomorphism $G\text{-ZipFlag}^\mu \simeq [E'\backslash G_k]$, the flag strata of $G\text{-ZipFlag}^\mu$ are given by

$$C_w := [E'\backslash C_w], \quad w \in W.$$

The set $C_w \subset G_k$ is the unique open $B \times z^2 B$-orbit in $G_k$ and similarly the flag stratum $C_{w_{\alpha}}$ is open in $G\text{-ZipFlag}^\mu$. The $B \times z^2 B$-orbits of codimension 1 are given by $C_{s_{\alpha}w_{\alpha}}$ for $\alpha \in \Delta$. 

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3.2 Vector bundles on $G$-$\text{ZipFlag}^\mu$

Let $\rho: B \to \text{GL}(V)$ be an algebraic representation. Since $G$-$\text{ZipFlag}^\mu$ classifies pairs $(L, J)$ where $J$ is a $B$-torsor, applying $\rho$ to $J$ yields a vector bundle $\mathcal{V}_{\text{flag}}(\rho)$ on $G$-$\text{ZipFlag}^\mu$. We can also construct $\mathcal{V}_{\text{flag}}(\rho)$ as follows. View $\rho$ as a representation of $E'$ via the first projection $E' \to B$. Then $\rho$ induces a vector bundle on $[E' \setminus G_k] \simeq G$-$\text{ZipFlag}^\mu$ by §2.4.1. In this description, we see that the functor $\mathcal{V}_{\text{flag}}$ extends to a functor $\text{Rep}(E') \to \mathcal{V}'(G$-$\text{ZipFlag}^\mu$).

This fact will briefly be used in equation (4.1.1). However, we will mainly consider vector bundles on $G$-$\text{ZipFlag}^\mu$ coming from $\text{Rep}(B)$.

Note that the rank of $\mathcal{V}_{\text{flag}}(\rho)$ is the dimension of $\rho$. In particular, if $\lambda \in X^*(B) = X^*(T)$, then $\mathcal{V}_{\text{flag}}(\lambda)$ is a line bundle. Let $(V, \rho) \in \text{Rep}(P)$ and let $\mathcal{V}(\rho)$ be the attached vector bundle on $G$-$\text{Zip}^\mu$. One has

$$\pi^*(\mathcal{V}(\rho)) = \mathcal{V}_{\text{flag}}(\rho|_B).$$

For $(V, \rho) \in \text{Rep}(B)$, write $\mathcal{V}_{P/B}(\rho)$ for the vector bundle on $P/B$ attached to $\rho$ by §2.4.1.

Then $H^0(P/B, \mathcal{V}_{P/B}(\rho)) = \text{Ind}^P_B(\rho)$ is a $P$-representation, where Ind denotes induction. Concretely, we have:

$$\text{Ind}^P_B(\rho) = \{ f: P \to V \mid f(xb) = \rho(b^{-1})f(x), \forall b \in B, \forall x \in P \}.$$

Furthermore, for $y \in P$ and $f \in \text{Ind}^P_B(\rho)$, the element $y \cdot f$ is the function $x \mapsto f(y^{-1}x)$.

**Proposition 3.2.1.** For $(V, \rho) \in \text{Rep}(B)$, we have the identification

$$\pi^*(\mathcal{V}_{\text{flag}}(\rho)) = \mathcal{V}(\text{Ind}^P_B(\rho)).$$

In particular, $\pi^*(\mathcal{V}_{\text{flag}}(\rho))$ is a vector bundle on $G$-$\text{Zip}^\mu$.

**Proof.** Consider the natural map $\text{Ind}^P_B(\rho) \to \rho$ defined by $f \mapsto f(1)$ (where $f \in \text{Ind}^P_B(\rho)$ is viewed as a regular function $f: P \to V$). This defines a map of $B$-representations. Hence, it induces a morphism $\pi^*(\mathcal{V}(\text{Ind}^P_B(\rho))) = \mathcal{V}_{\text{flag}}(\text{Ind}^P_B(\rho)|_B) \to \mathcal{V}_{\text{flag}}(\rho)$ of vector bundles on $G$-$\text{ZipFlag}^\mu$. By adjunction, we obtain a morphism of vector bundles $\mathcal{V}(\text{Ind}^P_B(\rho)) \to \pi^*(\mathcal{V}_{\text{flag}}(\rho))$ on $G$-$\text{Zip}^\mu$. Since this map induces an isomorphism on the stalks, it is itself an isomorphism. □

As a special case of Proposition 3.2.1 consider the case when $\rho$ is a character $\lambda \in X^*(T)$. Then $\mathcal{V}_{\text{flag}}(\lambda)$ is a line bundle and one has the formula $\pi^*(\mathcal{V}_{\text{flag}}(\lambda)) = \mathcal{V}_\lambda(\lambda)$ where the vector bundle $\mathcal{V}_\lambda(\lambda)$ was defined in §2.10. This recovers the formula (1.3.3) of [Kos19] (where $\mathcal{V}_{\text{flag}}(\lambda)$ was denoted by $\mathcal{L}(\lambda)$ and $\mathcal{V}_\lambda(\lambda)$ by $\mathcal{X}(\lambda)$). In particular, we have

$$H^0(G$-$\text{Zip}^\mu, \mathcal{V}_\lambda(\lambda)) = H^0(G$-$\text{ZipFlag}^\mu, \mathcal{V}_{\text{flag}}(\lambda)).$$

(3.2.2)

If $f: G_k \to k$ is a section of the right hand side of (3.2.2), then the corresponding function $f_1: G_k \to V_\lambda(\lambda)$ on the left hand side of (3.2.2) is given by

$$(f_1(g))(x) = f((x^{-1}, \varphi(x)^{-1}) \cdot g) = f(x^{-1}g\varphi(x))$$

for $g \in G_k$ and $x \in L$ by the construction of the identification. Note also that the line bundles $\mathcal{V}_{\text{flag}}(\lambda)$ satisfy the identity

$$\mathcal{V}_{\text{flag}}(\lambda + \lambda') = \mathcal{V}_{\text{flag}}(\lambda) \otimes \mathcal{V}_{\text{flag}}(\lambda'), \quad \forall \lambda, \lambda' \in X^*(T).$$
4 Vector bundles on Sbt

4.1 Definition

Any algebraic representation $\rho: B \times B \to \text{GL}(V)$ is given by two commuting $B$-representations $\rho_1, \rho_2: B \to \text{GL}(V)$. We write $\rho = (\rho_1, \rho_2)$. By §2.4.1 it induces a vector bundle $\mathcal{V}_{\text{Sbt}}(\rho_1, \rho_2)$ on $\text{Sbt} = [B \setminus G_k/B]$. Let $m_z: B \times zB \to B \times B; (x, y) \mapsto (x, z^{-1}yz)$. Recall that we have a map $\psi: G\text{-ZipFlag}^n \to \text{Sbt}$, as in (3.1.3). We have

$$\psi^*\mathcal{V}_{\text{Sbt}}(\rho_1, \rho_2) = \mathcal{V}_{\text{flag}}((m_z^*(\rho_1, \rho_2))|_{E'})$$

(4.1.1)

where $m_z^*(\rho_1, \rho_2)$ is the pullback via $m_z$ of $(\rho_1, \rho_2)$ to a $B \times zB$-representation. Since we are mainly interested in vector bundles on $G\text{-ZipFlag}^n$ attached to $B$-representations, we define the following condition:

**Condition 4.1.1.** The representation $\rho_2$ is trivial on $z^{-1}R_u(Q)z$.

Note that $z^{-1}R_u(Q)z \subset B$ by (2.2.2). If Condition 4.1.1 is satisfied, the $E'$-representation $m_z^*(\rho_1, \rho_2)|_{E'}$ arises from a $B$-representation via the first projection $E' \to B$. Specifically, we have in this case

$$\psi^*\mathcal{V}_{\text{Sbt}}(\rho_1, \rho_2) = \mathcal{V}_{\text{flag}}(R(\rho_1, \rho_2)),$$

(4.1.2)

where $R(\rho_1, \rho_2): B \to \text{GL}(V)$ is defined by

$$R(\rho_1, \rho_2)(b) = \rho_1(b)\rho_2(z^{-1}\varphi(\theta^B_1(b))z), \quad \forall b \in B.$$

Let $\mathcal{A}$ be the subcategory of $\text{Rep}(B)$ of representations which are trivial on $z^{-1}R_u(Q)z$. Put $B_M := B \cap M$. For $\rho \in \text{Rep}(B)$, denote by $\rho[z^{-1}]$ the $B_M$-representation $b \mapsto \rho(z^{-1}bz)$ (note that $z^{-1}B_Mz \subset B$ by (2.2.2)). This gives an equivalence

$$[z^{-1}]: \mathcal{A} \simeq \text{Rep}(B_M); \rho \mapsto \rho[z^{-1}].$$

Similarly, for $\rho \in \text{Rep}(B_M)$, write $\rho[z]$ for the $B$-representation in $\mathcal{A}$ given by $b \mapsto \rho(zbz^{-1})$ for $b \in z^{-1}B_Mz$. We view both sides as full subcategories of $\text{Rep}(B)$. We will always assume that $\rho_2 \in \mathcal{A}$ when considering $\mathcal{V}_{\text{Sbt}}(\rho_1, \rho_2)$.

4.2 Global sections of vector bundles on Sbt

Let $(V, \rho) \in \text{Rep}(B \times B)$ with $\rho = (\rho_1, \rho_2)$, as in §4.1, and let $\mathcal{V}_{\text{Sbt}}(\rho_1, \rho_2)$ be the attached vector bundle on Sbt. We determine the space of global sections of the vector bundle $\mathcal{V}_{\text{Sbt}}(\rho_1, \rho_2)$ on Sbt. As a first step, we study section over the open stratum $\text{Sbt}_{w_0} \subset \text{Sbt}$ (see (3.1.2)). Consider the action of $B \times B$ on $G_k$ given by $(a, b) \cdot g = agb^{-1}$. Define $S_{w_0} \subset B \times B$ as the stabilizer of $w_0$ for this action.

**Lemma 4.2.1.**

(1) One has $S_{w_0} = \{(t, w_0tw_0) \mid t \in T\}$.

(2) The open stratum $\text{Sbt}_{w_0}$ is naturally isomorphic to $[1/S_{w_0}]$.

(3) We have an identification $H^0(\text{Sbt}_{w_0}, \mathcal{V}_{\text{Sbt}}(\rho_1, \rho_2)) = V^{S_{w_0}}$.

**Proof.** The first assertion is clear. For the second assertion, note that $B \times B \to G_k; (a, b) \mapsto aw_0b^{-1}$ induces an isomorphism $(B \times B)/S_{w_0} \to Bw_0B$. Thus $\text{Sbt}_{w_0} = [B \times B \setminus Bw_0B] \simeq [B \times B \setminus (B \times B)/S_{w_0}] \simeq [1/S_{w_0}]$. The last assertion is an immediate consequence. \(\square\)
Let \((H_\alpha)_{\alpha \in \Phi}\) and let \(\phi_\alpha : SL_2 \to G\) be the map attached to \(\alpha\) (see [2.1]). Put \(A(t) = \left( \begin{array}{cc} t & 1 \\ -1 & 0 \end{array} \right) \in SL_2\). Note that \(\phi_\alpha(A(0)) = s_\alpha\) and \(\phi_\alpha(\text{diag}(t, t^{-1})) = \alpha^\vee(t)\). As in [IK21 (3.1.3)], one has a decomposition
\[
A(t) = \left( \begin{array}{cc} 1 & 0 \\ -t^{-1} & 1 \end{array} \right) \left( \begin{array}{cc} t & 1 \\ 0 & t^{-1} \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ -t^{-1} & 1 \end{array} \right) \left( \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right). \tag{4.2.1}
\]

Let \(\alpha \in \Delta\) be a simple root. Set \(Y = B \times B \times \mathbb{A}^1\) and let \(\psi_\alpha : Y \to G\) be the map
\[
\psi_\alpha : ((b, b'), t) \mapsto b\phi_\alpha(A(t))w_b^{-1}.
\]

We have \(\phi_\alpha(A(t)) \in BB^+ = BW_0BW_0\) for \(t \neq 0\). We deduce that \(\phi_\alpha(b, b', t) \in BW_0B\) if \(t \neq 0\) and \(\phi_\alpha(b, b', 0) \in BS_\rho B\). Write \(Y_0 \subset Y\) for the open set where \(t \neq 0\). Using a similar argument as [IK21 Corollary 3.1.5], we deduce that \(\tilde{f}\) extends to \(G\) if and only if \(\tilde{f} \circ \psi_\alpha : Y_0 \to V\) extends to a regular map \(Y \to V\) for all \(\alpha \in \Delta\). We have
\[
\tilde{f} \circ \psi_\alpha(b, b', t) = \rho_1(b)\phi_\alpha \left( \begin{array}{cc} 1 & 0 \\ -t^{-1} & 1 \end{array} \right) \alpha^\vee(t) \phi_\alpha(b)\phi_\alpha \left( \begin{array}{cc} 1 & 0 \\ -t^{-1} & 1 \end{array} \right) \rho_2(b)\phi_\alpha \left( \begin{array}{cc} 1 & 0 \\ -t^{-1} & 1 \end{array} \right) \rho_2\phi_\alpha(b)\phi_\alpha \left( \begin{array}{cc} 1 & 0 \\ -t^{-1} & 1 \end{array} \right) \cdot f.
\]

Write \(F_\alpha(t) = \rho_1(u_{-\alpha}(-t^{-1})\alpha^\vee(t))\rho_2(u_{\alpha_a}(t^{-1})) \cdot f\). It is an element of \(V \otimes_k k[t, t^{-1}]\). It follows that \(\tilde{f} \circ \psi_\alpha\) extends to \(Y\) if and only if \(F_\alpha(t)\) lies in \(V \otimes_k k[t]\). Write \(f = \sum_{\chi_1} f_{\chi_1}\) for the weight decomposition with respect to the action of \(T \times \{1\}\) on \(V\). We deduce
\[
F_\alpha(t) = \sum_{\chi_1 \in X^*(T)} \chi_1(t) \rho_1(u_{-\alpha}(-t^{-1})) \rho_2(u_{\alpha_a}(t^{-1})) \cdot f_{\chi_1}.
\]

Here \(E_{\bullet, \bullet}\) are the maps attached to the \(B \times B\)-representation \(V\) (see [2.1]). The \(\chi_1\)-component \(F_{\alpha, \chi_1}(t)\) of \(F_\alpha(t)\) is
\[
F_{\alpha, \chi_1}(t) = \sum_{j_1, j_2 \geq 0} (-1)^{j_1 + j_2} t^{(\chi_1, \alpha^\vee) + j_1 - j_2} E_{\alpha, \chi_1, j_1, j_2}^{(j_1, j_2)} \rho_1(u_{\alpha_a}(t^{-1})) \cdot f_{\chi_1}.
\]

Hence \(F_\alpha(t) \in V \otimes_k k[t]\) if and only if for all \(d \in \mathbb{Z}\) and all \(\chi_1\) such that \(d + (\chi_1, \alpha^\vee) < 0\), one has \(\sum_{j_1 \geq d} E_{\alpha, \chi_1, j_1, j_2}^{(j_1, j_2)} f_{\chi_1} = 0\). Therefore, we showed the following proposition.

**Proposition 4.2.2.** The space \(H^0(Sbt, V_{Sbt}(\rho_1, \rho_2))\) identifies with the subspace of \(f \in V^{S_{\rho_0}}\) satisfying: For all \(d \in \mathbb{Z}\) and all \(\chi_1 \in X^*(T)\) such that \(d + (\chi_1, \alpha^\vee) < 0\), one has
\[
\sum_{j_2 \geq d} E_{\alpha, \chi_1, j_1, j_2}^{(j_1, j_2)} f_{\chi_1, j_1, j_2} = 0.
\]

There is a similar result for maps of vector bundles on \(Sbt\), which we now explain. Let \((\rho_1, \rho_2)\) and \((\rho'_1, \rho'_2)\) be \(B \times B\)-representations with underlying vector spaces \(V, V'\) respectively. Then a morphism \(V_{Sbt}(\rho_1, \rho_2) \to V_{Sbt}(\rho'_1, \rho'_2)\) is equivalent to a \(k\)-linear map \(f : V \to V'\) satisfying the following conditions.
(1) $f$ is $S_{w_0}$-equivariant.
(2) For all $\alpha \in \Delta$, for all $d \in \mathbb{Z}$ and all character $\chi_1$ such that $d + \langle \chi_1, \alpha \rangle < 0$, the map
$$
\sum_{j_1 \geq d} \sum_{a=0}^{j_1-d} \sum_{b=0}^{j_1-a} (-1)^{a+b} E_{-a,0}^n E_{0,0}^n f E_{d-b}^n E_{-a,0}^n
$$
maps to zero under the projection
$$
\text{Hom}(V, V) \simeq \bigoplus_{\nu_1, \nu_1'} \text{Hom}(V_{\nu_1}, V_{\nu_1'}) \to \bigoplus_{\nu_1} \text{Hom}(V_{\nu_1}, V_{\nu_1 + \chi_1}).
$$

5 Schubert sections, partial Hasse invariants

5.1 Definition

Let $(\chi, \nu) \in X^*(T) \times X^*(T)$. Then $H^0(Sbt_{w_0}, \mathcal{V}_{\text{Sbt}}(\chi, \nu)) \neq 0$ if and only if $\nu = -w_0 \chi$. This follows from Lemma 12.13 (see also [GK19a, Theorem 2.2.1(a)]). If this condition is satisfied, the space $H^0(Sbt_{w_0}, \mathcal{V}_{\text{Sbt}}(\chi, \nu))$ is one-dimensional, and the divisor of any nonzero element
$$
h_{\chi} \in H^0(Sbt_{w_0}, \mathcal{V}_{\text{Sbt}}(\chi, -w_0 \chi))
$$
is given by Chevalley’s formula (loc. cit., Theorem 2.2.1(c)):
$$
\text{div}(h_{\chi}) = \sum_{\alpha \in \Delta} \langle \chi, \alpha \rangle B w_0 s_\alpha B.
$$

In particular, $h_{\chi}$ extends to $\text{Sbt}$ if and only if each $\chi \in X^+_+(T)$ (this also follows from Proposition 4.2.2). For all $\chi, \nu \in X^+(T)$, one has $\psi^* \mathcal{V}_{\text{Sbt}}(\chi, \nu) = \mathcal{V}_{\text{flag}}(R(\chi, \nu))$ by (4.1.2), where $R(\chi, \nu)$ is the character $t \mapsto \chi(t)\nu(z^{-1}\varphi(t)z)$. Hence, we obtain
$$
\psi^* \mathcal{V}_{\text{Sbt}}(\chi, \nu) = \mathcal{V}_{\text{flag}}(\chi + q\sigma^{-1}(z\nu)).
$$

In particular, for the pair $(\chi, -w_0 \chi)$, we obtain
$$
\psi^* \mathcal{V}_{\text{Sbt}}(\chi, -w_0 \chi) = \mathcal{V}_{\text{flag}}(\chi - qw_0 f(\sigma^{-1} \chi))
$$
where we used that $z = \sigma(w_0 f)w_0$.

Recall that we have a locally closed stratification $(C_w)_{w \in W}$ on $G\text{-ZipFlag}^\mu$ defined as fibers of the map $\psi$: $G\text{-ZipFlag}^\mu \to \text{Sbt}$. In particular, $C_{w_0}$ is the unique open flag stratum (see §3.1). By the previous discussion, it follows that for any $\chi \in X^+(T)$, $H_{\chi} := \psi^*(h_{\chi})$ is a section of $\mathcal{V}_{\text{flag}}(\chi - qw_0 f(\sigma^{-1} \chi))$ over $C_{w_0}$. Furthermore, for $\chi \in X^+_+(T)$, the section $H_{\chi}$ extends to a global section over $G\text{-ZipFlag}^\mu$.

**Definition 5.1.1.** We say that $f \in H^0(G\text{-ZipFlag}^\mu, \mathcal{V}_{\text{flag}}(\lambda))$ is a Schubert section if there exists $\chi \in X^+_+(T)$ such that $\lambda = \chi - qw_0 f(\sigma^{-1} \chi)$ and $f = H_{\chi}$ up to a nonzero scalar.

Recall that $H^0(G\text{-ZipFlag}^\mu, \mathcal{V}_{\text{flag}}(\lambda)) = H^0(G\text{-Zip}^\mu, \mathcal{V}_I(\lambda))$, and this space injects into the representation $V_I(\lambda)$ by $ev_1$ (see (2.5.1)). Write
$$
V_{I,\text{Sbt}}(\lambda) \subseteq V_I(\lambda)
$$
for the subspace of Schubert sections. By unicity of $h_{\chi}$ up to scalar, we have $\dim(V_{I,\text{Sbt}}(\lambda)) \leq 1$, and $V_{I,\text{Sbt}}(\lambda) \neq 0$ if and only if there exists $\chi \in X^+_+(T)$ such that $\lambda = \chi - qw_0 f(\sigma^{-1} \chi)$. We will call $V_{I,\text{Sbt}}(\lambda)$ the Schubert line of $V_I(\lambda)$.  

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Lemma 5.1.2. Let $H$ be a connected, smooth algebraic group over $k$ acting on an irreducible normal $k$-variety $X$. Let $f : X \to \mathbb{A}^1$ be a nonzero function such that the vanishing locus of $f$ is $H$-stable. Then there exists a character $\lambda \in X^*(H)$ such that $f$ satisfies $f(h \cdot x) = \lambda(h) f(x)$ for all $(h, x) \in H \times X$.

Proof. First, note that since $H$ is connected, each irreducible component of $\text{div}(f)$ is stable under the action of $H$. Hence we have $\text{div}(h \cdot f) = \text{div}(f)$ for all $h \in H$. Consider the rational map $\psi : H \times X \dashrightarrow \mathbb{A}^1$, $(h, x) \mapsto \frac{(h \cdot x)}{f(x)}$. Its divisor is zero, so $\psi$ extends to a non-vanishing map $H \times X \to \mathbb{G}_m$. By [Ros61, Theorem 2], we can write $\psi(h, x) = A(h)B(x)$ for two non-vanishing functions $A : H \to \mathbb{G}_m$ and $B : X \to \mathbb{G}_m$. Then the claim follows from [Ros61, Theorem 3] (cf. arguments in the proof of [Kos19, Lemma 5.1.2]).

Proposition 5.1.3. For a regular map $f : G_k \to \mathbb{A}^1$, the following are equivalent:
(i) There is $\lambda \in X^*(T)$ such that $f \in H^0(G\text{-ZipFlag}^\alpha, \mathcal{V}_{\text{flag}}(\lambda))$ and $f$ is a Schubert section.
(ii) $f$ is non-vanishing on $C_{w_0}$.

Proof. (i) $\Rightarrow$ (ii) is easy since $f$ arises from a section over Sbt, and all such sections are of the form $h_\chi$ as in (5.1.1). Conversely, assume $f$ is non-vanishing on $C_{w_0}$. Then the vanishing locus of $f$ is necessarily a union of $B \times zB$-orbits. By Lemma 5.1.2, $f$ is an $B \times zB$-eigenfunction. Twisting by $z$, the function $g \mapsto f(gz)$ is a $B \times B$-eigenfunction, hence it is of the form $h_\chi$ for some character $\chi \in X^*(T)$. This shows the result.

5.2 Partial Hasse invariants

We define the notion of partial Hasse invariant. There is an ambiguity between partial Hasse invariants for $G\text{-Zip}^\alpha$ and $G\text{-ZipFlag}^\alpha$, therefore we use a different terminology for each case.

5.2.1 Zip partial Hasse invariants

Consider the stratification $G\text{-Zip}^\alpha = \bigsqcup_{w \in W} X_w$ defined at the end of §2.2.4. The codimension one strata are $X_{w_0, t_{s_n}w_0}$ for $\alpha \in \Delta^P$. Also recall that for each $\lambda \in X^*(L)$, the vector bundle $\mathcal{V}_\lambda(\lambda)$ on $G\text{-Zip}^\alpha$ is a line bundle.

Definition 5.2.1. A zip partial Hasse invariant $h$ is a section of a line bundle $\mathcal{V}_\lambda(\lambda)$ (for $\lambda \in X^*(L)$) over $G\text{-Zip}^\alpha$, whose vanishing locus is the Zariski closure of a codimension one stratum $X_{w_0, t_{s_n}w_0}$. We say that $h$ is a strict zip partial Hasse invariant if furthermore it has multiplicity one.

Viewing a zip partial Hasse invariant as a map $G_k \to \mathbb{A}^1$ as in (2.4.1), its vanishing locus is the closure of an $E$-orbit of codimension one. Conversely, we have the following:

Lemma 5.2.2. Assume that $h : G_k \to \mathbb{A}^1$ is a regular map whose vanishing locus is the closure of an $E$-orbit of codimension one. Then, there exists a character $\lambda \in X^*(L)$ such that under the identification (2.4.1), $h$ identifies with an element of $H^0(G\text{-Zip}^\alpha, \mathcal{V}_\lambda(\lambda))$, and it is thus a zip partial Hasse invariant.

Proof. By Lemma 5.1.2 and the fact that any character of $E$ factors through $L$, the function $h$ satisfies $h(\text{agb}^{-1}) = \lambda(a)h(g)$ for some character $\lambda \in X^*(L)$. Hence the claim holds.

Proposition 5.2.3. Let $X_{w_0, t_{s_n}w_0}$ be a codimension one stratum. There exists a zip partial Hasse invariant $h_0$ for $X_{w_0, t_{s_n}w_0}$. Furthermore, if we choose $h_0$ such that $\text{deg}(\text{div}(h_0))$ is minimal, the set of all zip partial Hasse invariants for that stratum is given by

$$\{s\chi h_0^m \mid s \in k^\times, \chi \in X^*(G), m \geq 1\}.$$
Proof. Consider the Zariski closure $Z := \overline{G_{w_{\alpha}, I_{s_{\alpha}, w_{\lambda}}}}$. By [KKLV89, Proposition 4.5], the Picard group $\text{Pic}(G)$ is finite, hence there exists a function $h_0: G_k \to \mathbb{A}^{1}$ whose divisor is $d_0[Z]$ for some $d_0 \geq 1$. Hence there exists $\lambda_0 \in X^*(L)$ such that $h_0 \in H^0(G\text{-Zip}^\alpha, V_I(\lambda_0))$ by Lemma 5.2.2 and it is a zip partial Hasse invariant. Now, take $d_0 \geq 1$ minimal for this property. It is easy to see that any other function $G_k \to \mathbb{A}^{1}$ with vanishing locus $Z$ is of the form $fh_0^m$ where $f: G_k \to \mathbb{G}_m$ is non-vanishing. By [Ros61, Theorem 3], the function $f$ is of the form $f = s\chi$ for $s \in k^\times$ and $\chi \in X^*(G)$.

\begin{lemma}
Assume $h$ is a non-zero section of $V_I(\lambda)$ for some $\lambda \in X^*(G)$. Then $h$ is non-vanishing on $G$.
\end{lemma}

Proof. We can see that $h^{q-1}\lambda$ is a section of $V_I(0)$. But since the action of $E$ is dense in $G$, any section of $V_I(0)$ must be constant. Hence we obtain the claim.

\begin{corollary}
For a given codimension one stratum and a given $\lambda \in X^*(L)$, there is up to scalar at most one zip partial Hasse invariant of weight $\lambda$ for that stratum.
\end{corollary}

Proof. We use notations in Proposition 5.2.3. Assume that $\chi h_0^m$ and $\chi' h_0^{m'}$ have the same weight for $\chi, \chi' \in X^*(G)$ and $m \geq m' \geq 1$. By Proposition 5.2.3 it suffices to show that $\chi = \chi'$ and $m = m'$. Writing $\lambda_0$ for the weight of $h_0$, we have $(1-q)\chi + m\lambda_0 = (1-q)\chi' + m'\lambda_0$. Hence $(m-m')\lambda_0 \in X^*(G)$. Then we have $m = m'$ by Lemma 5.2.4. Hence we also have $\chi = \chi'$.

\subsection{Flag partial Hasse invariants}

Similarly, we define flag partial Hasse invariant. Recall that we have a stratification $(C_{w_{\alpha}})_{w \in W}$ on $G\text{-ZipFlag}^\alpha$. The codimension one strata are $C_{s_{\alpha}, w_{\lambda}}$ for $\alpha \in \Delta$. For each $\lambda \in X^*(T)$, we have a line bundle $V_{\text{flag}}(\lambda)$ on $G\text{-ZipFlag}^\alpha$.

\begin{definition}
A flag partial Hasse invariant $h$ is a global section of a line bundle $V_{\text{flag}}(\lambda)$ over $G\text{-ZipFlag}^\alpha$ for $\lambda \in X^*(T)$, whose vanishing locus is the Zariski closure of a codimension one flag stratum $\overline{C_{s_{\alpha}, w_{\lambda}}}$.

We say that $h$ is a strict flag partial Hasse invariant if furthermore it has multiplicity one.
\end{definition}

A flag partial Hasse invariant identifies with a map $G_k \to \mathbb{A}^{1}$ whose vanishing locus is the closure of a $B \times B$-orbit of codimension one. Conversely, similarly to the discussion in 5.2.1 any such map $G_k \to \mathbb{A}^{1}$ is a flag partial Hasse invariant. Similarly to Proposition 5.2.3 we have

\begin{proposition}
Let $C_{s_{\alpha}, w_{\lambda}}$ be a codimension one stratum. There exists a flag partial Hasse invariant $h_0'$ for $C_{s_{\alpha}, w_{\lambda}}$. Furthermore, if we choose $h_0'$ such that $\deg(\text{div}(h_0'))$ is minimal, the set of all flag partial Hasse invariants for that stratum is given by
\[
\{s\chi h_0^m \mid s \in k^\times, \chi \in X^*(G), m \geq 1\}.
\]

Again, a flag partial Hasse invariant of weight $\lambda$ is uniquely determined up to scalar by $\alpha \in \Delta$ and $\lambda$. Finally, we note that if Pic($G$) = 0, then all zip (resp. flag) strata admit strict zip (resp. flag) Hasse invariants, by the proof of Proposition 5.2.3

\begin{lemma}
Assume that $P$ is defined over $\mathbb{F}_q$. If $h \in H^0(G\text{-Zip}^\alpha, V_I(\lambda))$ (for $\lambda \in X^*(L)$) is a zip partial Hasse invariant, then $\pi^*(h) \in H^0(G\text{-ZipFlag}^\alpha, V_{\text{flag}}(\lambda))$ is a flag partial Hasse invariant.
\end{lemma}

Proof. When $P$ is defined over $\mathbb{F}_q$, the unique open $E$-orbit in $G$ is stable by $B \times B$ by [Wed11, Corollary 2.15]. Hence, the closure of any $E$-orbit of codimension 1 coincides with the closure of a $B \times B$-orbit of codimension one. The result follows immediately.

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6 Verschiebung homomorphism

Let $\mathcal{M}$ be a vector bundle on an algebraic stack (or scheme) $S$ over $\mathbb{F}_q$, and let $\mathcal{M}^{(q)} := F^q_S \mathcal{M}$ be the pullback by the absolute Frobenius $F_S : S \to S$. In general, there is no natural map $\mathcal{M} \to \mathcal{M}^{(q)}$. However, if $\mathcal{A} \to S$ is an abelian variety over a scheme of characteristic $p$, then $\mathcal{H} := \mathcal{H}_{\text{dR}}(\mathcal{A}/S)$ admits a Verschiebung map $\mathcal{V} : \mathcal{H} \to \mathcal{H}^{(q)}$. We will see that certain vector bundles $\mathcal{V}(\rho)$ (for a $P$-representation $(V, \rho)$) on $G$-$\text{Zip}^p$ are endowed with a Verschiebung homomorphism $\nu_\rho : \mathcal{V} \to \mathcal{V}^{[1]}$, where $\mathcal{V}^{[1]}$ is an analogue of the Frobenius-twist. We avoid the notation "$V^{[1]}$" for the Verschiebung. We will later that partial Hasse invariants admit a factorization in terms of the Verschiebung of certain vector bundles $\mathcal{V}(\rho)$.

6.1 Verschiebung homomorphism for $L$-representations

Assume that $P$ is defined over $\mathbb{F}_q$. For a $P$-representation $(V, \rho)$, denote by $(V^{[1]}, \rho^{[1]})$ the representation such that $V^{[1]} = V$ and

$$\rho^{[1]} : P \xrightarrow{\varphi} P \rightarrow GL(V)$$

where $\varphi$ denotes the Frobenius homomorphism of $P$. If $\mathcal{V} = \mathcal{V}(\rho)$ denotes the vector bundle attached to $\rho$, we sometimes write $\mathcal{V}^{[1]} := \mathcal{V}(\rho^{[1]})$.

Recall (Theorem 2.6.1) that for any two $L$-representations $(V_1, \rho_1)$, $(V_2, \rho_2)$, there is a bijection between morphisms of vector bundles $\mathcal{V}(\rho_1) \to \mathcal{V}(\rho_2)$ on $G$-$\text{Zip}^p$ and maps $F_{\text{MF}}(\rho_1) \to F_{\text{MF}}(\rho_2)$ of $\Delta^p$-filtered $L_{\varphi}$-modules. If $V = \bigoplus \nu V_\nu$ denotes the weight decomposition of $V$, then one has $V^{[1]} = \bigoplus \nu V_\nu^{[1]}$ and $V_\nu^{[1]} = V_\nu$. Recall from §2.6 that for $r \in \mathbb{R}$ and $\alpha \in \Delta^p$, the filtration $V^{[1]}_\alpha$ is defined as the direct sum of $V_\nu$ for all $\nu$ satisfying $\langle \nu, \delta_\alpha \rangle \geq r$. Therefore, we find for all $\alpha \in \Delta^p$ and all $r \in \mathbb{R}$:

$$(V^{[1]})^\alpha_{\geq r} = V^{[1]}_{\gamma \geq \frac{r}{q}}. \quad (6.1.1)$$

Let $\rho \in \text{Rep}(L)$ be an arbitrary $L$-representation. Similarly to the subspace $V_{\alpha \geq 0}^{\Delta^p}$ (see 2.5.2), define a subspace $V_{\text{GS}} \subset V$ as follows. Let $V = \bigoplus \nu \in \mathbb{X}^*(T) V_\nu$ be the $T$-weight decomposition of $V$. Define $V_{\text{GS}}$ as the direct sum of the weight spaces $V_\nu$ such that $\langle \nu, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Delta^p$.

Lemma 6.1.1. If $\sigma \alpha = \alpha$ for all $\alpha \in \Delta^p$, then $V_{\text{GS}} = V_{\gamma \geq 0}^{\Delta^p}$.

Proof. This was already pointed out in [IK21] (sentence after equation (3.4.3)). It follows from the fact that if $\sigma \alpha = \alpha$, then $\delta_\alpha = -\frac{\alpha}{q^r}$. $\square$

Proposition 6.1.2. Let $(V, \rho)$ be an $L$-representation such that $V_{\text{GS}} = V$. Then the identity map $\text{Id} : V \to V = V^{[1]}$ is a morphism of $\Delta^p$-filtered $L_{\varphi}$-modules $F_{\text{MF}}(\rho) \to F_{\text{MF}}(\rho^{[1]})$, hence induces a morphism of vector bundles

$$\nu_\rho : \mathcal{V}(\rho) \to \mathcal{V}(\rho^{[1]})$$

We call this map the Verschiebung homomorphism of $\mathcal{V}(\rho)$.

Proof. Since $\rho^{[1]} = \rho \circ \varphi$, it is clear that $\text{Id}$ is $L(\mathbb{F}_q)$-equivariant. We show that it is compatible with $\alpha$-filtrations. Using (6.1.1), we must prove $V^{\alpha}_{\geq r} \subset (V^{[1]})^\alpha_{\geq r} = V^{[1]}_{\gamma \geq \frac{r}{q}}$ for all $\alpha \in \Delta^p$. Since $\delta_\alpha = \varphi^{-1}(\alpha^\vee)$, we have $\alpha^\vee = \varphi_\ast(\delta_\alpha) = \delta_\alpha - q\delta_\alpha$ (see 2.5). Suppose that $\langle \nu, \delta_\alpha \rangle \geq r$ for a weight $\nu$. We have $q\langle \nu, \delta_\alpha \rangle = \langle \nu, \delta_\alpha \rangle - \langle \nu, \alpha^\vee \rangle$ and $\langle \nu, \alpha^\vee \rangle \leq 0$ by assumption. Hence $\langle \nu, \delta_\alpha \rangle \geq \frac{r}{q}$. This shows that $V^{\alpha}_{\geq r} \subset V^{[1]}_{\gamma \geq \frac{r}{q}}$ and terminates the proof. $\square$
Remark 6.1.3. On the $\mu$-ordinary locus, one has $\mathcal{V}(\rho)|_{\mathcal{U}_0} = \mathcal{V}(\rho^{[1]})|_{\mathcal{U}_0}$, since $\rho^{[1]}|_{L(S)} = \rho|_{L(S)}$. Proposition 6.1.2 shows that this isomorphism $\mathcal{V}(\rho)|_{\mathcal{U}_0} \to \mathcal{V}(\rho^{[1]})|_{\mathcal{U}_0}$ extends to a map of vector bundles $v_p: \mathcal{V}(\rho) \to \mathcal{V}(\rho^{[1]})$ over $G$-Zip (which need not be an isomorphism).

Recall the Griffiths–Schmid cone ([Kos19 Definition 1.8.1]) in a relation with Griffiths–Schmid manifolds, a generalization of Shimura varieties.

**Lemma 6.1.5.** A character $\lambda$ is in $C_{GS}$ if and only if $-w_{0,I} \lambda$ is $G$-dominant.

**Proof.** This follows from the fact that $w_{0,I}$ preserves $\Phi_+ \setminus \Phi_{L,+}$ and swaps $I$ and $-I$. \qed 

The conditions defining $C_{GS}$ were first understood by Griffiths–Schmid (cf. [GS69 p.275]) in a relation with Griffiths–Schmid manifolds, a generalization of Shimura varieties to non-minuscule cocharacters. The following is [Kos19 Proposition 3.7.5(1)] (where the notation $V(\lambda)|_{\leq 0}$ was used instead of $V(\lambda)|_{GS}$):

**Lemma 6.1.6.** Assume $\lambda \in C_{GS}$. Then one has $V_I(\lambda)|_{GS} = V_I(\lambda)$. In particular, for all $\lambda \in C_{GS}$, the vector bundle $V_I(\lambda)$ admits a Verschiebung homomorphism.

**Example 6.1.7.** Set $G := GL_{2n,F}$, endowed with the cocharacter $\mu: \mathbb{G}_m \to G$; $x \mapsto \text{diag}(xI_n, I_n)$. Let $P, L, Q$ be the subgroups of $G$ attached to $\mu$. Let $(u_1, \ldots, u_{2n})$ be the canonical basis of $V_0 = k^{2n}$ and set $V_{0,P} = \text{Span}_k(u_{n+1}, \ldots, u_{2n})$ (hence $P$ is the stabilizer of $V_{0,P}$). Let $T \subset L$ be the diagonal torus and $B \subset G$ the lower-triangular Borel subgroup. Identify $X^*(T) = \mathbb{Z}^{2n}$ such that $(a_1, \ldots, a_{2n})$ corresponds to the character $\text{diag}(x_1, \ldots, x_{2n}) \mapsto \prod_{i=1}^{2n} x_i^{a_i}$. Write also $(e_1, \ldots, e_{2n})$ for the canonical basis of $\mathbb{Z}^{2n}$. In this case, a $G$-zip of type $\mu$ over an $F_p$-scheme $S$ is equivalent a tuple $\langle \mathcal{M}, \Omega, F, V \rangle$ where

1. $\mathcal{M}$ is a locally free $O_S$-module of rank $2n$ and $\Omega \subset \mathcal{M}$ is a locally free, locally direct factor $O_S$-submodule of rank $n$.
2. $F: \mathcal{M}^{(q)} \to \mathcal{M}$ and $V: \mathcal{M} \to \mathcal{M}^{(q)}$ are maps of vector bundles satisfying $\text{Im}(F) = \text{Ker}(V)$ and $\text{Im}(V) = \text{Ker}(F) = \Omega^{(q)}$.

Let $f: A \to S$ be an abelian scheme of rank $g$ over an $F_p$-scheme $S$. We can attach to $A$ a $G$-zip by setting $\mathcal{M} = H^1_{dR}(A/S)$ and $\Omega$ as the Hodge filtration of $\mathcal{M}$. The Frobenius and Verschiebung maps of $A$ give rise to similar maps on $\mathcal{M}$. Thus, we get a map of stacks $\zeta: S \to G$.

The $P$-representation $V_{0,P}$ has highest weight $\lambda_{\Omega} = -e_n$, which lies in $C_{GS}$. Hence, $V_I(\lambda_{\Omega})$ admits a Verschiebung $v_{\Omega}: V_I(\lambda_{\Omega}) \to V_I(\lambda^{[1]})$ by Lemma 6.1.6. The pullback by $\zeta$ of this map coincides with the Verschiebung $V: \Omega \to \Omega^{(q)}$.

### 6.2 Twisting homomorphism on Sbt

We show that the Verschiebung map constructed in the previous section arises from a similar map the stack Sbt.

We start with a connected, reductive group $G$ over $k$, a Borel pair $(B,T)$ such that $T \subset B \subset G$ and a parabolic subgroup $P$ containing $B$. Write $L$ for the unique Levi subgroup of $P$ containing $T$. Put $z_1 = w_{0,I}w_0$. Let $V$ be an $L$-representation, which we view
as a $P$-representation trivial on $R_q(P)$. Note that $z_1^{-1}B_1z_1 \subset B$. Define a $B$-representation $\rho[z_1]$ as follows: First, for $b \in z_1^{-1}B_1z_1$, put $\rho[z_1](b) = \rho(z_1b_1z_1^{-1})$. Then, extend $\rho[z_1]$ to a $B$-representation trivial on $R_0(z_1^{-1}Lz_1)$. We may consider the $B \times B$-representations $(\rho, \rho_0)$ and $(\rho_0, \rho[z_1])$, both with underlying vector space $V$. Put $u_\rho := \rho(w_0, 1)$, viewed as a map $u_\rho : V \to V$. It is easy to see that $u_\rho$ is $S_{w_0}$-equivariant (see [4.2] for the definition of $S_{w_0}$). Indeed, for all $t \in T$, and $x \in V$ we have

$$(t, w_0tw_0) \cdot u_\rho(x) = \rho[z_1](w_0tw_0)\rho(w_0t)(\rho(t)^{-1}x) = \rho(w_0t^2w_0^{-1})\rho(w_0t)(\rho(t)^{-1}x) = u_\rho(x).$$

Hence, $u_\rho$ defines a morphism of vector bundles over the open substack $Sbt_{w_0} \subset Sbt$:

$$u_\rho : \mathcal{V}_{Sbt}(\rho, \rho_0)|_{Sbt_{w_0}} \to \mathcal{V}_{Sbt}(\rho_0, \rho[z_1])|_{Sbt_{w_0}}. \tag{6.2.1}$$

**Theorem 6.2.1.** Assume that $V = V_{GS}$. Then $u_\rho$ extends to a map of vector bundles $u_\rho : \mathcal{V}_{Sbt}(\rho, \rho_0) \to \mathcal{V}_{Sbt}(\rho_0, \rho[z_1])$ over Sbt.

Before we prove Theorem 6.2.1, we discuss the following situation. We may choose $\mathbb{F}_q \subset k$ such that $G$, $B$, $T$ and $P$ are defined over $\mathbb{F}_q$. Let $Q := LB^+$ be the opposite parabolic subgroup of $P$ with respect to $L$. Hence $Z = (G, P, L, Q, L)$ is a zip datum for the $q$-th power Frobenius, and we have $Z = Z_\mu$, for any dominant cocharacter $\mu : G_{m,k} \to G_k$ with centralizer $L$. Hence, if we define $z = z_1 = w_0, tw_0$, then $(B, T, z)$ is a frame of $Z$ (Lemma 2.2.1). Furthermore, by (4.1.2), we have

$$\psi^\ast \mathcal{V}_{Sbt}(\rho, \rho_0) = \mathcal{V}_{flag}(\rho),$$

$$\pi^\ast \mathcal{V}_{Sbt}(\rho_0, \rho[z_1]) = \mathcal{V}_{flag}(\rho[1]).$$

Now, let $(V, \rho)$ be an $L$-representation such that $V_{GS} = V$. By Proposition 6.1.2, we have a Verschiebung map $v_\rho : \mathcal{V}(\rho) \to \mathcal{V}(\rho)[1]$ on $G$-$Zip^\mu$. Pulling back via the natural projection $\pi : G$-$ZipFlag^\mu \to G$-$Zip^\mu$ (see (3.1.1)), we obtain a map $\pi^\ast v_\rho : \mathcal{V}_{flag}(\rho) \to \mathcal{V}_{flag}(\rho[1])$ using (3.2.1). Then, Theorem 6.2.1 above is a consequence of the following, more precise result:

**Proposition 6.2.2.** Assume that $V_{GS} = V$. There exists a unique morphism of vector bundles $u_\rho : \mathcal{V}_{Sbt}(\rho, \rho_0) \to \mathcal{V}_{Sbt}(\rho_0, \rho[z_1])$ on the stack Sbt such that $\psi^\ast u_\rho = \pi^\ast v_\rho$. Furthermore, $u_\rho$ extends the map defined in equation (6.2.1) over Sbt.

**Proof.** The uniqueness is clear, since $\psi$ is surjective. Using (2.4.1), view $\pi^\ast v_\rho$ as a map $f : G_k \to \text{Hom}(V, V[1])$. By construction, $f$ is the only such map satisfying: for all $(a, b) \in E, f(ab^{-1}) = a \cdot \text{Id}_V = \rho(\phi(a)a^{-1})$. We obtain that $f$ satisfies

$$f(r_1xr_2) = \rho(x^{-1}), \tag{6.2.2}$$

for all $x \in L, r_1 \in R_u(P)$ and $r_2 \in R_u(Q)$. Define $h : G_k \to \text{Hom}(V, V)$, by $h(g) := f(gz_1^{-1})$. By (6.2.2), we have

$$h(bxb^{-1}) = \rho(\theta_L^0(z_1b_1z_1^{-1}) \circ h(x) \circ \rho(b^{-1}) = \rho(z_1)[b] \circ h(x) \circ \rho(b^{-1})$$

for all $b, b' \in B$ and all $x \in Lz_1$. The same equality is true for all $b, b' \in B$ and all $x \in G_k$, since $BLz_1B$ is dense in $G_k$. Thus, $h$ defines a morphism $u_\rho : \mathcal{V}_{Sbt}(\rho, \rho_0) \to \mathcal{V}_{Sbt}(\rho_0, \rho[z])$ as claimed. Finally, we show that $u_\rho$ coincides with (6.2.1) over $Sbt_{w_0}$. Viewed as a map $V \to V$, the map $\pi^\ast v_\rho$ corresponds to $h(w_0) = f(w_0^{-1}) = \rho(w_0, 1)$, hence the result. $\square$
Remark 6.2.3. The statement of Theorem 6.2.1 also makes sense and holds for groups over an arbitrary algebraically closed field $F$. To explain this, we first write explicitly conditions (i) and (ii) for the map $u_p = \rho(w_{0,t})$. Using §4.2, one sees that the map (6.2.1) extends to Sbt if and only if for all $d \in \mathbb{Z}$ and $\chi_1 \in X^+(T)$ such that $d + \langle \chi_1, \alpha \rangle < 0$, the map

$$
\sum_{j_1 \geq d} (-1)^{j_1} E^{(j_1-d)}_{w_{0,t}^{-1}\alpha} \rho(w_{0,t}) E^{(j_1)}_{-\alpha} = \rho(w_{0,t}) \sum_{j_1 \geq d} (-1)^{j_1} E^{(j_1-d)}_{\alpha} E^{(j_1)}_{-\alpha}
$$

maps to zero under the projection (4.2.2). We deduce that $u_p$ extends to Sbt if and only if for all $\alpha \in \Delta$ and all $d \in \mathbb{Z}$ such that $d + \langle w_{0,t}^{-1}\nu - \nu - dw_{0,t}^{-1}\alpha, \alpha \rangle < 0$ we have

$$
\sum_{j \geq d} (-1)^j E^{(j-d)}_{\alpha} E^{(j)}_{-\alpha}(V_\rho) = 0.
$$

(6.2.3)

Now, assume that $F$ has characteristic zero. By the classification of reductive groups, there exists a model of $G$ over $\mathbb{Q}$, and hence over a number field $K$, and even over $\mathcal{O}_K[\frac{1}{N}]$ for some integer $N \geq 1$. After possibly changing $\mathcal{O}_K$ and $N$, we may assume that all other objects ($P$, $B$, the representation $V$, etc.) admit a model over $\mathcal{O}_K[\frac{1}{N}]$. Therefore, for infinitely many primes $p$, the equation (6.2.3) holds in $V \otimes \mathcal{O}_K[\frac{1}{N}] \mathbb{F}_p$, since we showed the result in characteristic $p$. Hence it also holds in $V$, and we deduce that $u_p$ extends to Sbt. In particular, formula (6.2.3) above holds in general.

Example 6.2.4. For $G = \text{Sp}(4)$, $L = \text{GL}_2$ and $V = \text{Sym}^n(\text{Std})$, formula (6.2.3) amounts to the following: For all $d \in \mathbb{Z}$ and $0 \leq i \leq n$ such that $4i - 2n + 3d < 0$, we have

$$
\sum_{j=0}^{n-i-d} (-1)^j \binom{n-i}{j+d} \binom{j+d+i}{d+i} = 0.
$$

The above discussion shows that this formula holds (in $\mathbb{Z}$, not only in $\mathbb{F}_q$). It is also possible to show this formula directly using the binomial transform.

6.3 Verschiebung homomorphism for $G$-representations

We study the existence of Verschiebung maps on $G$-representations. We restrict to representations of the form $V_\Delta(\chi) = \text{Ind}_{G}^B(\lambda)$ for a dominant $\chi \in X^+_1(T)$. There is a map of $L$-representations:

$$
\Pi_\chi : V_\Delta(\chi)|_L \rightarrow V_L(w_{0,t}w_0\chi)
$$

defined as follows. Let $f \in V_\Delta(\chi)$, viewed as a function $f : G \rightarrow \mathbb{A}^1$ satisfying $f(xb) = \chi^{-1}(b)f(x)$ for all $x \in G$ and $b \in B$. Define $\Pi_\chi(f)$ as the function $L \rightarrow \mathbb{A}^1$ mapping $x \in L$ to $f(xw_{0,t}w_0)$. Clearly $\Pi_\chi(f)$ lies in $V_L(w_{0,t}w_0\chi)$ and $\Pi_\chi$ is $L$-equivariant.

Proposition 6.3.1. The map $\Pi_\chi$ induces an isomorphism of $L$-representations

$$
V_\Delta(\chi)^{R_{0}(P)} \rightarrow V_L(w_{0,t}w_0\chi).
$$

In particular, $V_\Delta(\chi)|_L$ decomposes as $V_\Delta(\chi)|_L = V_L(w_{0,t}w_0\chi) \oplus \text{Ker}(\Pi_\chi)$.

Proof. We first show that $\Pi_\chi$ is injective on $V_\Delta(\chi)^{R_{0}(P)}$. Suppose $f \in V_\Delta(\chi)^{R_{0}(P)}$ satisfies $\Pi_\chi(f) = 0$. Then $f(xw_{0,t}b) = 0$ for all $x \in P$ and all $b \in B$. Since $Pw_{0,t}B$ is Zariski dense in $G$, we deduce $f = 0$. We now show that $\Pi_\chi|_{V_\Delta(\chi)^{R_{0}(P)}}$ is surjective. Let $f' \in V_L(w_{0,t}w_0\chi)$, viewed as a function $f' : L \rightarrow \mathbb{A}^1$. We must define $f(xw_{0,t}w_0b) = f'(\theta_L^x(b))\chi(b)^{-1}$ for all
(x, b) ∈ P × B. We claim that f is well-defined on Pw_0B. Indeed, suppose xw_0,tw_0b = x'w_0,tw_0b' for x, x' ∈ P and b, b' ∈ B. We obtain x'^{-1}x = w_0,tw_0b'b^{-1}w_0^{-1}w_0,t ∈ B_L, hence

\[ f'(\theta_L^p(x)) \chi(b)^{-1} = f'(\theta_L^p(x')x'^{-1}x) \chi(b)^{-1} = f'(\theta_L^p(x'))(w_0,tw_0\chi)^{-1}(w_0,tw_0b'b^{-1}w_0^{-1}w_0,t) \chi(b)^{-1} = f'(\theta_L^p(x')) \chi(b')^{-1}. \]

Next, we show that f extends to a regular function on G. For this, we consider the map ψ_α: P × A^1 → G for α ∈ Δ \ I given by ψ_α: (x, t) ↦ xφ_α(A(t))w_0 for (x, t) ∈ P × A^1. It suffices to show that for all α ∈ Δ \ I, the map f ◦ ψ_α: P × G_m → A^1 extends to a map P × A^1 → A^1. For this, we write A(t) as in \ ([2.2.1]), and we find for (x, t) ∈ P × G_m:

\[ f(\psi_α(x, t)) = f'(\theta_L^p(x)\alpha^\vee(t)w_0, t) = f'(\theta_L^p(x)w_0, t)^{-(w_0,tw_0\chiw_0\alpha^\vee)} = f'(\theta_L^p(x)w_0, t)^{-(\chiw_0\alpha^\vee)}. \]

Since χ is dominant, we have −⟨χ, w_0α⟩ ≥ 0, which terminates the proof.

Let (G, μ) be a cocharacter datum over F_q with attached zip datum Z_μ = (G, P, L, Q, M, φ). Assume in the rest of this section that P is defined over F_q. Let again χ ∈ X^+_*(T). We identify V_L(w_0,tw_0χ) with V_Δ(χ)^{R_P} and view Π_χ as a map V_Δ(χ) → V_Δ(χ) with image V_L(w_0,tw_0χ). It is clear that Π_χ: V_Δ(χ) → V_Δ(χ)^{[1]} is L(F_q)-equivariant. Therefore, by Lemma \ ([2.5.2]), the map Π_χ gives rise to a morphism of vector bundles

\[ v_χ: V_Δ(χ)|_{μ} → V_Δ(χ)^{[1]}|_{μ}. \]

In the following, we investigate whether this map extends to a morphism v_χ: V_Δ(χ) → V_Δ(χ)^{[1]} over the whole stack G-Zip^n. When it does, we call the unique extension the Verhoeff isomorphism of V_Δ(χ)

**Lemma 6.3.2.** Let α ∈ Δ^P, β ∈ Δ_L and assume iα + jβ ∈ Φ_+ for i ≥ 0, j ≥ 1. Then we have i ≤ −j⟨β, α^\vee⟩.

**Proof.** The claim is equivalent to (n − j⟨β, α^\vee⟩)α + jβ ∉ Φ_+ for any n ≥ 1. This is equivalent to s_n(jβ − nα) ∉ Φ_+ for any n ≥ 1. If s_n(jβ − nα) ∈ Φ_+ for some n ≥ 1, then jβ − nα is a root, but this contradicts that α and β are different simple roots.

**Lemma 6.3.3.** Let α ∈ Δ^P. For (x, t) ∈ R_α(B_L) × G_m, define

\[ Γ(x, t) = φ_α \begin{pmatrix} t & 0 \\ 1 & t^{-1} \end{pmatrix} xφ_α \begin{pmatrix} t & 0 \\ 1 & t^{-1} \end{pmatrix}^{-1} ∈ G. \]

Then Γ extends to a regular map R_α(B_L) × A^1 → G.

**Proof.** We can view Γ as an element of R[t, t^{-1}] where R = k[R_α(B_L)] is the ring of functions of R_α(B_L). We need to show that Γ ∈ R[t]. For this, it suffices to check that for any x ∈ R_α(B_L), the map t ↦ Γ(x, t) extends to A^1. Furthermore, we may assume that x is of the form x = u_−β(s) for s ∈ A^1 and β ∈ Δ_L, as elements of this form generate R_α(B_L). We may write Γ(x, t) = α^\vee(t)F(t, s)α^\vee(t)^{-1}, where F(t, s) = u_−α(t)u_−β(s)u_−α(t)^{-1}. Write [u_−α(t), u_−β(s)] for the commutator of u_−α(t) and u_−β(s). By \ ([Spr98], Proposition 8.2.3], we can write

\[ F(t, s)u_−β(s)^{-1} = [u_−α(t), u_−β(s)] = \prod_{i,j > 0} φ_α+jβ \left( \begin{array}{cc} 1 & 0 \\ u_i j^t s^j & 1 \end{array} \right). \]
for some \( u_{i,j} \in k \), and for an arbitrarily chosen order on the set of positive roots \( \Phi_+ \). Returning to \( \Gamma(x,t) \), we deduce

\[
\Gamma(x,t) = \left( \prod_{i,j > 0} \phi_{ia+j\beta} \left( u_{i,j} t^{i-(ia+j\beta,\alpha)} s^j \right) \frac{1}{0} \right) \phi_\beta \left( \frac{1}{st^{-\beta,\alpha^\vee}} \frac{1}{0} \right)
\]

Since \( \alpha \) and \( \beta \) are distinct simple roots, \(-\langle \beta, \alpha^\vee \rangle \geq 0\). Moreover, \( i - (ia + j\beta, \alpha^\vee) = -i - j\langle \beta, \alpha^\vee \rangle \geq 0 \) by Lemma 6.3.2. Hence the result follows.

\[\square\]

**Lemma 6.3.4.** Let \( \chi \in X^+_*(T) \) and \( f \in V_\Delta(\chi) \), viewed as a function \( G \rightarrow \mathbb{A}^1 \). For \( \alpha \in \Delta^P \), consider the function \( L \times \mathbb{G}_m \rightarrow \mathbb{A}^1 \)

\[
F_{f,\alpha}: (x,t) \mapsto f \left( \phi_\alpha \left( \begin{array}{cc} t & 0 \\ 1 & t^{-1} \end{array} \right) x w_0 \right).
\]

Then \( F_{f,\alpha} \) extends to a function \( L \times \mathbb{A}^1 \rightarrow \mathbb{A}^1 \).

**Proof.** We may view \( F_{f,\alpha} \) as an element of \( R_1[t, t^{-1}] \) where \( R_1 = k[L] \) is the ring of functions of \( L \) over \( k \). We must show that \( F_{f,\alpha} \) lies in \( R_1[t] \). Furthermore, it suffices to show that this is true for \( x \) in a dense subset of \( L \) (because we can write \( F_{f,\alpha} = \sum_{j \geq 0} F_j(x)t^j \) for some \( F_j \in R_1 \) and \( N \in \mathbb{N} \), and we deduce by density that \( F_j = 0 \) for all \( j < 0 \)). The elements of the form \( x = yz \) for \( y \in R_u(B_L) \) and \( z \in B^+_L \) form an open dense subset of \( L \). Note that

\[
F_{f,\alpha}(yz,t) = \chi(w_0^{-1}zw_0)^{-1}F_{f,\alpha}(y,t).
\]

Hence, it suffices to show that \( t \mapsto F_{f,\alpha}(x,t) \) extends to \( \mathbb{A}^1 \) for \( x \in R_u(B_L) \). We have

\[
\phi_\alpha \left( \begin{array}{cc} t & 0 \\ 1 & t^{-1} \end{array} \right) x = \Gamma(x,t) \phi_\alpha \left( \begin{array}{cc} t & 0 \\ 1 & t^{-1} \end{array} \right) \phi_\alpha \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
\]

Using that \( w_0^{-1} \phi_\alpha \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) w_0 \in R_u(B) \), we deduce \( F_{f,\alpha}(x,t) = f \left( \Gamma(x,t) \phi_\alpha \left( \begin{array}{cc} t & 1 \\ 0 & 1 \end{array} \right) w_0 \right) \).

The result then follows from Lemma 6.3.3. \(\square\)

Recall that we view \( V_f(w_0, Iw_0\chi) \) as a subspace of \( V_\Delta(\chi) \) as in Proposition 6.3.1. Note that since \( \chi \) is dominant, we have \( w_0, Iw_0 \chi \in C_{GS} \) (Lemma 6.1.5). Thus by Proposition 6.1.2, we have a map \( \psi: V_f(w_0, Iw_0\chi) \rightarrow V_f(w_0, Iw_0\chi)^{[1]} \) (where \( \rho = \rho_{I, w_0, Iw_0\chi} \)).

**Theorem 6.3.5.** Let \( \chi \in X^+_*(T) \) be a dominant character. The map \( \psi_\chi \) extends to a morphism of vector bundles \( \psi_\chi: V_\Delta(\chi) \rightarrow V_\Delta(\chi)^{[1]} \) over \( G \cdot \text{Zip}^\mu \). Furthermore, the image of \( \psi_\chi \) is \( V_f(w_0, Iw_0\chi)^{[1]} \) and \( \psi_\chi \) extends the Verschiebung map of \( \rho = \rho_{I, w_0, Iw_0\chi} \). In other words, there is a commutative diagram

\[
\begin{array}{ccc}
V_\Delta(\chi) & \xrightarrow{\psi_\chi} & V_f(w_0, Iw_0\chi)^{[1]} \\
\subset & \nearrow \psi_\psi & \subset V_f(w_0, Iw_0\chi)
\end{array}
\]

We call \( \psi_\chi \) the Verschiebung homomorphism of \( V_\Delta(\chi) \).
Proof. We need only show that $\nu_\chi$ extends to a morphism of vector bundles $\nu_\chi : V_\Delta(\chi) \to V_\Delta(\chi)^[i]$. The other assertions follow immediately from the fact that $\Pi_\chi$ extends the identity map of $V_I(w_0t,w_0\chi)$. As explained in [IK21, Remark 5.1.3], we need to show that for all $\alpha \in \Delta^P$, all $\eta \in X^*(T)$ and $j \in \mathbb{N}$ such that $j(1-\langle \alpha, \delta_\alpha \rangle) > \langle \eta, \delta_\alpha \rangle$, we have
\[
pr_\eta \circ \sum_{0 \leq i \leq \frac{j}{\delta_\alpha}} (-1)^i E_{-\sigma_\alpha}^{(i)} \Pi_\chi E_{-\alpha}^{(j-i)} = 0
\] (6.3.1)
where $pr_\eta$ denotes the projection.

\[
\text{Hom}(V_\Delta(\chi), V_\Delta(\chi)^[i]) \simeq \bigoplus_{\nu, \nu' \in X^*(T)} \text{Hom}(V_\Delta(\chi)_\nu, V_\Delta(\chi)^[i]) \to \bigoplus_{\nu \in X^*(T)} \text{Hom}(V_\Delta(\chi)_\nu, V_\Delta(\chi)^[i]).
\]

Since the image of $\Pi_\chi$ is $V_I(w_0t,w_0\chi) = V_\Delta(\chi)^{Ra(P)}$, we have $E_{-\sigma_\alpha}^{(i)} \Pi_\chi = 0$ except for $i = 0$. Hence, we can write condition (6.3.1) as $pr_\eta(\Pi_\chi E_{-\alpha}^{(j)}) = 0$ for all $j \in \mathbb{Z}$ such that $j(1-\langle \alpha, \delta_\alpha \rangle) > \langle \eta, \delta_\alpha \rangle$. The map $\Pi_\chi E_{-\alpha}^{(j)}$ takes $V_\Delta(\chi)_\nu$ to $V_\Delta(\chi)^[i]_{\eta' \sigma_\alpha^{-1}(\nu-j\alpha)}$, so it suffices to check that $\Pi_\chi E_{-\alpha}^{(j)}$ is zero on $V_\Delta(\chi)_\nu$ for $\nu$ such that $\eta = q\sigma^{-1}(\nu-j\alpha) - \nu$. We have $q(\sigma^{-1}\nu, \delta_\alpha) - \langle \nu, \delta_\alpha \rangle = -\langle \nu, \alpha^\vee \rangle$, hence $\langle \eta, \delta_\alpha \rangle = -\langle \nu, \alpha^\vee \rangle - j\langle \sigma^{-1}\alpha, \delta_\alpha \rangle$. Therefore, the condition $j(1-\langle \alpha, \delta_\alpha \rangle) > \langle \eta, \delta_\alpha \rangle$ becomes $j(1-\langle \alpha, \delta_\alpha \rangle + q\langle \alpha, \delta_\alpha \rangle) + \langle \nu, \alpha^\vee \rangle > 0$. Using again $q(\alpha, \delta_\alpha) = \langle \alpha, \delta_\alpha \rangle - \langle \alpha, \alpha^\vee \rangle$, it is also equivalent to $j(1-\langle \alpha, \alpha^\vee \rangle) = -j > -\langle \eta, \alpha^\vee \rangle$. Hence, it suffices to show that for $j < \langle \nu, \alpha^\vee \rangle$, one has $\Pi_\chi E_{-\alpha}^{(j)}(V_\Delta(\chi)_\nu) = 0$. Furthermore, we can assume that $\langle \nu, \alpha^\vee \rangle > 0$ (since $E_{-\alpha}^{(j)} = 0$ for $j \leq 0$). Now, let $f \in V_\Delta(\chi)_\nu$, viewed as a function $f : G \to \mathbb{A}^1$. Recall that $\text{Ker}(\Pi_\chi)$ consists of functions satisfying $f(xw_0) = 0$ for all $x \in L$. By the definition of $E_{-\alpha}^{(j)}$, we must show that for all $x \in L$, the expression
\[
f\left(\phi_\alpha \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} xw_0 \right) \in k[t, t^{-1}]
\]
has $t$-valuation $\geq \langle \nu, \alpha^\vee \rangle$. Since $f$ is an eigenvector for $\nu$, this amounts to saying that
\[
f\left(\alpha(t)\phi_\alpha \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} xw_0 \right) = f\left(\phi_\alpha \begin{pmatrix} 1 & 0 \\ 1 & t^{-1} \end{pmatrix} xw_0 \right)
\]
is a polynomial in $k[t]$. Hence, the result follows from Lemma 6.3.4.

Example 6.3.6. Continuing Example 6.1.7, the vector bundle $\mathcal{M} = H_{dR}^1(\mathbb{A}/\mathbb{S})$ is given by $\mathcal{M} = \mathcal{V}(\text{Std}_{\mathbb{G}_{2\mathbb{F}}}) = V_\Delta(\lambda_{\mathcal{M}})$ with $\lambda_{\mathcal{M}} = -e_{2\mathbb{F}}$. The Verschubung $V : \mathcal{M} \to \mathcal{M}^[\mathbb{F}]$ corresponds to the map $\nu_{\lambda_{\mathcal{M}}}$ afforded by Theorem 6.3.2.

7 Consequences

7.1 Decomposition of Schubert sections

We return to the following setting: Let $G$ be a connected, reductive group over $\mathbb{F}_q$ and $\mu : \mathbb{G}_{m,k} \to G_k$ a cocharacter. Let $\mathbb{Z}_\mu = (G, P, L, Q, M)$ be the attached zip datum. Fix a frame $(B, T, z)$ where $z = \sigma(w_0t)w_0$ (see 2.2.3). Furthermore, we put $z_1 = w_0t$. Hence, if $L$ is defined over $\mathbb{F}_q$, then $z = z_1$.

Let $\chi \in X^*_+(T)$ be a dominant character. Let $h_\chi \in H^0(\text{Sbt}, \mathcal{V}_{\text{Sbt}}(\chi, -w_0\chi))$ be a non-zero section as in (5.1.1). Multiplication by $h_\chi$ gives a map $\mathcal{V}_{\text{Sbt}}(-\chi, 0) \to \mathcal{V}_{\text{Sbt}}(0, -w_0\chi)$. By Lemma 6.1.5, the character $-w_0t\chi$ lies in $C_{\mathbb{G}_{2\mathbb{F}}}$, in particular it is $I$-dominant. To
simplify, we write \( V := V_I(-w_{0,I}\chi) \) and \( \rho := \rho_{I,-w_{0,I}\chi}|_B \). Fix a nonzero element \( v_{\text{low}} \in V \) in the lowest weight line of \( V \). This choice identifies the \( B \)-subrepresentation \( kv_{\text{low}} \) with \(-\chi\). We obtain a natural map \( \iota_\chi \) of \( B \)-representations \( \iota_\chi : -\chi \to \rho \) given by the inclusion \( kv_{\text{low}} \subset V \). This induces a morphism \[
abla_{\text{Sbt}}(\iota_\chi) : \nabla_{\text{Sbt}}(-\chi,0) \to \nabla_{\text{Sbt}}(\rho,\rho_0).
\]

Recall that \( \rho_0 \) denotes the trivial representation of \( B \) on \( V \). Similarly, there is a projection map of \( B \)-representations \( p_\chi : \rho \to -w_{0,I}\chi \) given by projection onto the highest weight line of \( V \). Twisting by \( z_1 \), we have \( p_\chi[z_1] : \rho[z_1] \to -w_0\chi \), which gives a map:

\[
\nabla_{\text{Sbt}}(p_\chi[z_1]) : \nabla_{\text{Sbt}}(\rho_0,\rho[z_1]) \to \nabla_{\text{Sbt}}(0,-w_0\chi).
\]

**Proposition 7.1.1.** Let \( \rho = \rho_{I,-w_{0,I}\chi}|_B \). If we put \( u_1 := \nabla_{\text{Sbt}}(\iota_\chi) \), \( u_2 := \nabla_{\text{Sbt}}(p_\chi[z_1]) \) and \( u_\rho \) is the map afforded by Theorem [6,2,2], then the composition \[
\nabla_{\text{Sbt}}(-\chi,0) \xrightarrow{u_1} \nabla_{\text{Sbt}}(\rho,\rho_0) \xrightarrow{u_\rho} \nabla_{\text{Sbt}}(\rho_0,\rho[z_1]) \xrightarrow{u_2} \nabla_{\text{Sbt}}(0,-w_0\chi)
\]

coincides with the multiplication by \( h_\chi \) up to a nonzero scalar.

**Proof.** It suffices to show that the composition \( u := u_2 \circ u_\rho \circ u_1 \) is nonzero. Indeed, in this case \( u \) gives rise to a section in \( H^0(\text{Sbt},\nabla_{\text{Sbt}}(-w_0\chi)) \). Since this space is one-dimensional (see §5.1), it coincides with \( h_\chi \) up to a nonzero scalar. If \( (\rho_1,\rho_2) \) and \( (\rho'_1,\rho'_2) \) are \( B \times B \)-representations with underlying vector spaces \( V \), \( V' \) respectively, we may view a map \( \nabla_{\text{Sbt}}(\rho_1,\rho_2) \to \nabla_{\text{Sbt}}(\rho'_1,\rho'_2) \) as a \( k \)-linear map \( V \to V' \) satisfying properties (i), (ii) of section §2.3. First, the map \( u_1 \) corresponds to the inclusion \( kv_{\text{low}} \subset V \). Choose a nonzero vector \( v_{\text{high}} \) in the highest weight line of \( V \). Then, \( u_2 \) is the projection \( p_\chi : V \to kv_{\text{high}} \). Finally, by construction the map \( u_\rho \) corresponds to \( \rho(w_0,I) : V \to V \). Since \( \rho(w_0,I)v_{\text{low}} \in kv_{\text{high}} \), it is clear that the composition is nonzero. \( \square \)

By §5.1 there is an associated Schubert section \( H_{\chi} = \psi^*(h_\chi) \), which is a section of \( \nabla_{\text{flag}}(-v_{\text{low}})(\sigma^{-1}\chi) \). Multiplication by \( \psi^*(h_\chi) \) gives a map \( \nabla_{\text{flag}}(-\chi) \to \nabla_{\text{flag}}(-v_{\text{low}})(\sigma^{-1}\chi) \).

We obtain a decomposition of \( H_{\chi} \), as follows.

**Corollary 7.1.2.** Assume that \( P \) is defined over \( \mathbb{F}_q \). Write \( V := V_I(-w_{0,I}\chi) \). There exist natural maps of vector bundles on \( G\text{-Zip}^h \):

\[
\nabla_{\text{flag}}(-\chi) \xrightarrow{\xi_1} \pi^*V \xrightarrow{\pi^*v_\rho} \pi^*V[I] \xrightarrow{\xi_2} \nabla_{\text{flag}}(-v_{\text{low}})(\sigma^{-1}\chi)
\]

such that the composition coincides with multiplication with \( H_{\chi} \).

**Proof.** Since \( P \) is defined over \( \mathbb{F}_q \), we may apply Proposition [6,2,2] to get \( \pi^*v_\rho = \psi^*u_\rho \). \( \square \)

### 7.2 Primitiveness of partial Hasse invariants

For \( \lambda \in X^*(T) \), we write \( L_I(\lambda) \subset V_I(\lambda) \) for the unique irreducible \( L \)-representation of highest weight \( \lambda \) (see §2.3). Denote by \( V_I^L(\lambda) \subset V_I(\lambda) \) the subbundle attached to the irreducible \( L \)-subrepresentation \( L_I(\lambda) \subset V_I(\lambda) \).

**Definition 7.2.1.** We say that \( f \in H^0(G\text{-Zip}^h,V_I(\lambda)) \) is primitive if it lies in the subspace \( H^0(G\text{-Zip}^h,V_I^L(\lambda)) \).

Recall that there is an injective map \( \text{ev}_1 : H^0(G\text{-Zip}^h,V_I(\lambda)) \to V_I(\lambda) \) defined in §2.5.1.
Lemma 7.2.2. A section \( f \in H^0(G\text{-}Zip^\mu, V_I(\lambda)) \) is primitive if and only \( \text{ev}_1(f) \in L_I(\lambda) \).

Proof. More generally, let \( V_1 \subset V_2 \) two \( P \)-representations and let \( \mathcal{V}_1 \subset \mathcal{V}_2 \) be the corresponding vector bundles on \( G\text{-}Zip^\mu \). We have a commutative diagram

\[
\begin{array}{c}
H^0(G\text{-}Zip^\mu, V_I) \xrightarrow{\text{ev}_1} V_1 \\
\downarrow \\
H^0(G\text{-}Zip^\mu, V_2) \xrightarrow{\text{ev}_1} V_2
\end{array}
\]

We claim that \( H^0(G\text{-}Zip^\mu, V_I) \) is the intersection of \( H^0(G\text{-}Zip^\mu, V_2) \) and \( V_1 \) inside \( V_2 \). Assume that \( f \in H^0(G\text{-}Zip^\mu, V_2) \) satisfies \( \text{ev}_1(f) \in V_1 \). Since \( V_1 \cap V_2^{I_\varphi} = V_1^{I_\varphi} \), we obtain by Lemma 2.5.2 that \( f \in H^0(U_\mu, V_1) \). The claim follows by density of \( U_\mu \).

We study the primitiveness of (flag) partial Hasse invariants. We do not assume that \( P \) is defined over \( \mathbb{F}_q \). For an \( L \)-representation \((V, \rho)\), define \((V^{(1)}, \rho^{(1)})\) as follows. Let \( L_1 \) be the split form of \( L \) over \( \mathbb{F}_q \) and let \( \varphi_1 : L_k \to L_k \) be the Frobenius homomorphism of \( L_1 \). We set \( V^{(1)} = V \) and \( \rho^{(1)} = \rho \circ \varphi_1 \). Let \( \chi \in X^*_\lambda(T) \). Then \( V_I(\chi)^{(1)} \subset V_I(\chi) \) is the sub-\( L \)-representation whose underlying space is the image of the (non-linear) map \( V_I(\chi) \to V_I(\chi)^{(1)} \), \( f \mapsto f^1 \). For \( \lambda := \chi - \rho w_0, I(\sigma^{-1} \chi) \), we obtain maps

\[
V_I(\chi) \otimes V_I(-w_0, I(\sigma^{-1} \chi)) = V_I(\chi) \otimes V_I(-w_0, I(\sigma^{-1} \chi)) \to V_I(\chi)
\]

which are morphisms of \( L \)-representations.

Proposition 7.2.3. Let \( \chi \in X^*_\lambda(T) \) and set \( \lambda = \chi - q w_0, I(\sigma^{-1} \chi) \). The Schubert line \( V_{I,SBT}(\lambda) \) is contained in the image of \( V_I(\chi) \otimes V_I(-w_0, I(\sigma^{-1} \chi)) \) by the map (7.2.1).

Proof. Let \( \chi \in X^*_\lambda(T) \) and write \( \rho \) for a \( G \)-representation \( V_\Delta(\chi) \). Let \( p_\chi \in V_\Delta(\chi)^\vee \) be a \( B \)-eigenvector in \( V_\Delta(\chi)^\vee \) for the weight \( -\chi \) afforded by Lemma 2.3.1. It satisfies \( p_\chi(\rho(b)x) = \chi(b)p_\chi(x) \) for all \( x \in G \) and all \( b \in B \). Let \( v_{low} \subset V_\Delta(\chi) \) be a nonzero vector in the lowest weight line. Consider the function \( \Delta_\chi : G \to A^1 \) defined by

\[
\Delta_\chi(x) = p_\chi(\rho(x)v_{low}), \quad x \in G.
\]

We have for all \( x \in G \) and all \( (b, b') \in B \times B \):

\[
\Delta_\chi(bxb'^{-1}) = p_\chi(\rho(bxb'^{-1})v_{low}) = \chi(b)p_\chi(\rho(x)((w_0, \chi)(b'^{-1})v_{low})) = \chi(b)(w_0, \chi)(b'^{-1})\Delta_\chi(x).
\]

It follows that \( \Delta_\chi \) is a \( B \times B \)-eigenfunction and it lies in the space \( H^0(SBT, V_{SBT}(\chi - w_0, \chi)) \). Put again \( \lambda = \chi - w_0, I(\sigma^{-1} \chi) \). Then \( f_\lambda := \psi^*(\Delta_\chi) \) is a Schubert section of weight \( \lambda \). Explicitly, \( f_\lambda \) is a function \( G_k \to k \) satisfying \( f_\lambda(x) = \Delta_\chi(xz) \) for all \( x \in G_k \). Let \( f_{\lambda, I} \) be the element of \( H^0(G\text{-}Zip^\mu, V_I(\lambda)) \) corresponding to \( f_\lambda \) under (3.2.2). Viewing \( f_{\lambda, I} \) in \( V_I(\lambda) \) via \( \text{ev}_1 \), it is given as follows: For all \( x \in L \), one has

\[
\text{ev}_1(f_{\lambda, I})(x) = f_\lambda(x^{-1} \varphi(x)) = p_\chi(\rho(x^{-1} \varphi(x))z)v_{low}.
\]

Let \( \langle - , - \rangle : V_\Delta(\chi) \times V_\Delta(\chi)^\vee \to k \) be the natural pairing. We can write for all \( x \in L \):

\[
\text{ev}_1(f_{\lambda, I})(x) = \langle \rho(x^{-1} \varphi(x))zv_{low}, p_\chi \rangle = \langle \rho(\varphi(x))zv_{low}, \rho^\vee(x)p_\chi \rangle.
\]
Choose a basis $\mathcal{B} := (e_1, \ldots, e_r)$ of $V_\Delta(\chi)$ and let $\mathcal{B}' := (e'_1, \ldots, e'_r)$ denote the dual basis. For all $x \in G$, we define $A(x) := \rho(x)v_{\text{low}}$ and write $A(x) = (A_1(x), \ldots, A_r(x))$ for the coordinates of $A(x)$ with respect to $\mathcal{B}$. Similarly, $C(x) := \rho'(x)p_{\chi}$ and $C(x) = (C_1(x), \ldots, C_r(x))$ with respect to $\mathcal{B}'$. Hence we have for all $x \in L$, $ev_1(f_{\lambda,i})(x) = \sum_{i=1}^r A_i(\varphi(x))C_i(x)$. Since $G$ is defined over $\mathbb{F}_q$, we may also consider the functions $\sigma^{-1}A_i: \ G \to \mathbb{A}^1$. We obtain

$$ev_1(f_{\lambda,i})(x) = \sum_{i=1}^r \sigma^{-1}A_i(x)^g C_i(x). \quad (7.2.2)$$

Note that for all $x \in L$ and all $b \in B_L$, we have:

$$A(xb) = \chi(\sigma(w_{0,i})b\sigma(w_{0,i})^{-1})A(x),$$

$$C(xb) = \chi(b)^{-1}C(x).$$

Hence for $1 \leq i \leq r$, the functions $x \mapsto C_i(x)$ lie in $V_I(\chi)$ and the functions $x \mapsto \sigma^{-1}A_i(x)$ lie in $V_I(-w_{0,i}\sigma^{-1}\chi)$. This shows that $ev_1(f_{\lambda,i})$ lies in the image of the map $(7.2.1)$. \hfill \Box

Let $\alpha \in \Delta$. Consider the following condition on a character $\chi_\alpha$:

**Condition 7.2.4.**

(a) One has $0 < \langle \chi_\alpha, \alpha' \rangle < q$ and $\chi_\alpha$ is orthogonal to $\beta'$ for all $\beta \in \Delta \setminus \{\alpha\}$.

(b) One has $L_I(\chi_\alpha) = V_I(\chi_\alpha)$ and $L_I(-w_{0,i}(\sigma^{-1}\chi_\alpha)) = V_I(-w_{0,i}(\sigma^{-1}\chi_\alpha)).$

**Theorem 7.2.5.** Let $\alpha \in \Delta$ and assume that $\chi_\alpha \in X^*(T)$ satisfies Condition 7.2.4. Denote by $f_\alpha := \psi^*(h_\chi_\alpha)$ the corresponding Schubert section, of weight $\lambda_\alpha := \chi_\alpha - qw_{0,i}(\sigma^{-1}\chi_\alpha).$

We have the following properties:

1. $f_\alpha$ is a flag partial Hasse invariant for the flag stratum $C_{s_\alpha\sigma_0}$.

2. $f_\alpha$ is a primitive automorphic form on $G$-$\text{Zip}^\mu$.

**Proof.** The first statement follows from (a) of Condition 7.2.4. We now show (2). View $f_\alpha$ as an element of $V_I(\lambda_\alpha)$. By part (b), we deduce from Proposition 7.2.3 that $f_\alpha$ lies in the image of the map $L_I(\chi_\alpha) \otimes L_I(-w_{0,i}(\sigma^{-1}\chi_\alpha))^{(1)} \to V_I(\lambda_\alpha)$ given by $(7.2.1)$. We put

$$X^*_{I,I}(T) = \{ \chi \in X^*(T) \mid 0 \leq \langle \chi, \alpha' \rangle < q \text{ for all } \alpha \in I \}$$

(cf. [Jan03 II, §3.15]). By part (a) of Condition 7.2.4, $\chi_\alpha$ and $-w_{0,i}(\sigma^{-1}\chi_\alpha)$ are in $X^*_{I,I}(T)$. By Theorem 2.3.2, we have $L_I(\chi_\alpha) \otimes L_I(-w_{0,i}(\sigma^{-1}\chi_\alpha))^{(1)} = L_I(\lambda_\alpha)$. The result follows. \hfill \Box

**Remark 7.2.6.** Let $\chi \in X^*(T)$. In [GGN17, Theorem 1.1 and §2], it is studied when $L_I(\chi) = V_I(\chi)$ holds for all $p$. On the other hand, $L_I(\chi) = V_I(\chi)$ holds for sufficiently large $p$ by [Jan03 II, 5.6 Corollary].

8 Examples

8.1 The symplectic case

The Siegel Shimura variety $\mathcal{A}_n$ of rank $n$ is attached to the reductive group $G = GSp(V_0, \psi)$, where $(V_0, \psi)$ is a symplectic space of rank $n$ over $\mathbb{Q}$. The special fiber $\mathcal{A}_{n,F_p}$ at a place of good reduction admits a smooth surjective map to $G$-$\text{Zip}^\mu$ where $G$ is the special fiber
of a reductive \( \mathbb{Z}_p \)-model of \( G_{Q_p} \), and \( \mu \) is the usual cocharacter attached to the Siegel-type Shimura datum. We may define the flag space of \( \mathcal{A}_{n,F_p} \) as the fiber product:

\[
\begin{array}{ccc}
\text{Flag}(\mathcal{A}_{n,F_p}) & \longrightarrow & G\text{-ZipFlag}^\mu \\
\downarrow & \Downarrow & \downarrow \\
\mathcal{A}_{n,F_p} & \xrightarrow{\zeta} & G\text{-Zip}^\mu.
\end{array}
\tag{8.1.1}
\]

We obtain by pullback a stratification on \( \text{Flag}(\mathcal{A}_{n,F_p}) \) as well as (flag) partial Hasse invariants for each of the codimension one strata of \( \text{Flag}(\mathcal{A}_{n,F_p}) \). In this section, we compute these partial Hasse invariants explicitly in the case \( G = \text{Sp}(V_0, \psi) \) (where \( (V_0, \psi) \) is a symplectic space over \( F_p \)). The case of \( G = \text{GSp}(V_0, \psi) \) is completely similar. Furthermore, we give the modular interpretation of these sections in section 8.1.4.

### 8.1.1 The group \( G \)

We consider the case when \( G \) is the reductive \( F_q \)-group \( \text{Sp}(V_0, \psi) \), where \( (V_0, \psi) \) is a non-degenerate symplectic space over \( F_q \) of dimension \( 2n \), for some integer \( n \geq 1 \). After choosing an appropriate basis \( B \) for \( V_0 \), we assume that \( \psi \) is given by the matrix

\[
\begin{pmatrix}
J & -J \\
-1 & J
\end{pmatrix}
\]

where \( J := \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & 1 \\ 0 & \cdots & 1 \end{pmatrix} \).

Define \( G \) as follows:

\[
G(R) = \{ f \in \text{GL}_{F_q}(V_0 \otimes F R) \mid \psi_R(f(x), f(y)) = \psi_R(x, y), \ \forall x, y \in V_0 \otimes F_q R \}
\]

for all \( F_q \)-algebras \( R \). Identify \( V_0 = k^{2n} \) and view \( G \) as a subgroup of \( \text{GL}_{2n,F_q} \). Fix the \( F_q \)-split maximal torus \( T \) given by diagonal matrices in \( G \), i.e.

\[
T(R) := \{ \text{diag}_{2n}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}) \mid x_1, \ldots, x_n \in R^\times \}.
\]

Define \( B \) as the Borel subgroup of \( G \) consisting of the lower-triangular matrices in \( G \). For a tuple \( (a_1, \ldots, a_n) \in \mathbb{Z}^n \), define a character of \( T \) by mapping \( \text{diag}_{2n}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}) \) to \( x_1^{a_1} \cdots x_n^{a_n} \). From this, we obtain an identification \( X^*(T) = \mathbb{Z}^n \). Denoting by \( (e_1, \ldots, e_n) \) the standard basis of \( \mathbb{Z}^n \), the \( T \)-roots of \( G \) and the \( B \)-positive roots are respectively

\[
\Phi := \{ \pm e_i \pm e_j \mid 1 \leq i \neq j \leq n \} \cup \{ \pm 2e_i \mid 1 \leq i \leq n \},
\]

\[
\Phi_+ := \{ e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{ 2e_i \mid 1 \leq i \leq n \}
\]

and the \( B \)-simple roots are \( \Delta := \{ \alpha_1, \ldots, \alpha_{n-1}, \beta \} \) where

\[
\alpha_i := e_{i+1} - e_i \quad \text{for } i = 1, \ldots, n-1,
\]

\[
\beta := 2e_n.
\]

The Weyl group \( W := W(G, T) \) can be identified with the group of permutations \( \sigma \in \mathfrak{S}_{2n} \) satisfying \( \sigma(i) + \sigma(2n + 1 - i) = 2n + 1 \) for all \( 1 \leq i \leq 2n \).

Define a cocharacter \( \mu : G_{m,F_q} \to G \) by \( z \mapsto \text{diag}(zI_n, z^{-1}I_n) \). Write \( Z := (G, P, L, Q, M, \varphi) \) for the associated zip datum (since \( \mu \) is defined over \( F_q \), we have \( M = L \)). Concretely, if we denote by \( (u_i)_{i=1}^{2n} \) the canonical basis of \( k^{2n} \), then \( P \) is the stabilizer of \( V_{0,P} = \text{Span}_k(u_{n+1}, \ldots, u_{2n}) \) and \( Q \) is the stabilizer of \( V_{0,Q} = \text{Span}_k(u_1, \ldots, u_n) \). The intersection \( L := P \cap Q \) is the common Levi subgroup, which is isomorphic to \( \text{GL}_{n,F_q} \).
8.1.2 Partial Hasse invariants

There is a unique stratum of codimension one in $G$-$\text{Zip}^\mu$. It is the vanishing locus of the (ordinary) Hasse invariant $H_{\mu} \in H^0(G$-$\text{Zip}^\mu, \omega^q)$. Concretely, this section is given by

$$H_{\mu}: \left( \begin{array}{cc} A & * \\ * & * \end{array} \right) \rightarrow \text{det}(A).$$

The (flag) partial Hasse invariants are given as follows. For $1 \leq d \leq n$, define a function $\Delta_d: \text{GL}_n \rightarrow \mathbb{A}^1$ by

$$\Delta_d(A) = \begin{vmatrix} a_{1,n+1-d} & a_{1,n+2-d} & \cdots & a_{1,n} \\ a_{2,n+1-d} & a_{2,n+2-d} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,n+1-d} & a_{d,n+2-d} & \cdots & a_{d,n} \end{vmatrix}$$

for $A = (a_{i,j})_{1 \leq i,j \leq n}$. (8.1.2)

Define $H_d: G \rightarrow \mathbb{A}^1$ by

$$\left( \begin{array}{cc} A & * \\ * & * \end{array} \right) \rightarrow \Delta_d(A).$$

Then $H_d$ is a section in $H^0(G$-$\text{ZipFlag}^\mu, \nu_{\text{flag}}(\lambda_d))$ for the weight

$$\lambda_d = (1, \ldots, 1, 0, \ldots, 0) - (0, \ldots, 0, q, \ldots, q).$$

The family $\{H_d\}_{1 \leq d \leq n}$ is a complete set of flag partial Hasse invariants. Note that $H_n$ coincides with the classical Hasse invariant $H_{\mu}$. This illustrates Lemma 5.2.8 which states that any zip partial Hasse invariant becomes a flag partial Hasse invariant when pulled back to $G$-$\text{ZipFlag}^\mu$ (since $P$ is defined over $\mathbb{F}_q$ in this example).

8.1.3 Primitiveness

Write $\chi_d = (1, \ldots, 1, 0, \ldots, 0)$, where 1 appears $d$ times and 0 appears $n - d$ times. Then we have $\chi_d = qw_{0,1}^d\chi_d$. One can show easily that $\chi_d$ satisfies Condition 7.2.4. Hence, by Theorem 7.2.5, the section $H_d$ is a primitive automorphic form on $G$-$\text{Zip}^\mu$, in the sense that it corresponds to an element of $L_I(\lambda_d) \subset V_I(\lambda_d)$. We check this in the case $d = 1$ (the computation for larger $d$ is slightly tedious). The element $\text{ev}_1(H_d) \in V_I(\lambda_d)$ corresponding to $H_d$ via (2.5.1) is the function

$$\text{ev}_1(H_d): L \rightarrow \mathbb{A}^1, \quad A \mapsto \Delta_d(A^{-1}\varphi(A)).$$

When $d = 1$, we can write this function as

$$\text{ev}_1(H_1)(A) = \frac{1}{\delta} \sum_{i=1}^{n} (-1)^i a^q_{i,n},$$

where $\delta = \begin{vmatrix} a_{2,1} & \cdots & a_{2,i-1} & a_{2,i+1} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots \\ a_{d,1} & \cdots & a_{n,i-1} & a_{n,i+1} & \cdots & a_{d,n} \end{vmatrix}$ (8.1.3)

and $\delta$ is the determinant $\text{GL}_n \rightarrow \mathbb{C}_m$. One shows easily that the function $\delta_i$ lies in $L_I(\chi_1)$ and that $a_{i,n}$ lies in $L_I(-w_{0,1}\chi_1)$. Thus, $\text{ev}_1(H_1)$ lies in $L_I(\chi_1) \otimes L_I(-w_{0,1}\chi_1)[1] = L_I(\lambda_1)$. For $d > 1$, the function $\text{ev}_1(H_d)$ admits a similar expansion in terms of minors of the matrix $A$. Using the formula given in equation (7.2.2) in the proof of Proposition 7.2.3, there is an expansion with $\text{dim}(V_\Delta(\chi))$ summands (for $\chi = \chi_d$). Actually, when $P$ is defined
over $\mathbb{F}_q$, one can modify the proof of Proposition 7.2.3 by using the representation $V_I(\chi)$ instead of $V_{\Delta}(\chi)$. Hence, in this case we expect an expansion with $\dim(V_I(\chi_d)) = (n+d-1)$ summands (for $d = 1$, we indeed have $n$ terms in (8.1.3)).

We give a counter-example showing that in general, Schubert sections do not lie in $L_I(\lambda)$. Take $G = \text{Sp}(4)$ and consider $H^s_{\lambda} \mathbb{P}^1$ with $m := q^2 - q + 1$. Set $f_1 = \text{ev}_1(H^1_{\lambda}) \in L_I(1,-q)$. We claim that the section $f_1^m \in V_I(m,-qm)$ does not lie in $L_I(m,-qm)$. We can write $m(1,-q) = m(1,1) + (0,-1) + q^3(0,-1)$. By Theorem 2.3.2, we see that

$$L_I(m,-qm) \simeq \delta^{-m} \otimes L_I(0,-1) \otimes L_I(0,-1)^{[3]}$$

where $\delta$ is the determinant $\text{GL}_2 \to \mathbb{G}_m$. By equation (8.1.3), we can write $f_1 = \delta^{-1}(xy^3 - yx^3)$ for a certain basis $(x, y)$ of $L_I(0,-1)$. Then $L_I(m,-qm)$ has dimension 4, with basis

$$\delta^{-m}x^3y^1, \delta^{-m}xy^3, \delta^{-m}xy^3, \delta^{-m}y^3x^1.$$

It is easy to see that the expansion of $f_1^m = \delta^{-m}(xy^3 - yx^3)^m$ involves also other monomials, hence it does not lie in $L_I(m,-qm)$.

### 8.1.4 Modular interpretation

We now explain the modular interpretation of partial Hasse invariants. We refer to [PWZ15, §8] for the modular interpretation of symplectic zips. Let $S$ be an $\mathbb{F}_q$-scheme and $n \geq 1$ an integer. Define a symplectic zip of rank $n$ as a tuple $\mathcal{M} = (\mathcal{M}, F, V, \Omega, (-,-), \iota)$ where

(i) $\mathcal{M}$ is a locally free $\mathcal{O}_S$-module of rank $2n$ and $\Omega \subset \mathcal{M}$ is a locally free $\mathcal{O}_S$-submodule of rank $n$ which is Zariski locally a direct factor of $\mathcal{M}$.

(ii) $(-,-) : \mathcal{M} \times \mathcal{M} \to \mathcal{O}_S$ is a perfect $\mathcal{O}_S$-bilinear pairing.

(iii) $F : \mathcal{M}^{(q)} \to \mathcal{M}$ and $V : \mathcal{M} \to \mathcal{M}^{(q)}$ are maps of vector bundles.

(iv) We have $\text{Im}(F) = \text{Ker}(V)$ and $\text{Im}(V) = \text{Ker}(F) = \Omega^{(q)}$.

(v) We have $\langle Fx, y \rangle = \langle x, Vy \rangle$ for all $x, y \in \mathcal{M}$.

Note that [PWZ15, §8] uses a slightly different formulation. Instead of maps $(F, V)$, the authors define an $F$-zip using two filtrations of $\mathcal{M}$ and isomorphisms between the graded pieces. In the above description, the corresponding filtrations are given by $\Omega$ and $\text{Im}(F)$. The isomorphisms between the graded pieces are precisely induced by $F$ and $V$.

**Example 8.1.1.** Let $(A, \chi)$ be an abelian scheme over $S$ together with a principal polarization. Then we obtain a symplectic $F$-zip by defining $\mathcal{M} = H^1_{dR}(A/S)$ and $\Omega$ as the Hodge filtration of $\mathcal{M}$. The polarization induces a perfect pairing on $\mathcal{M}$, and the Frobenius and Verschiebung maps of $A$ give rise to similar maps on $\mathcal{M}$.

There is an obvious notion of morphisms of symplectic $F$-zips over $S$. The symplectic zips form a stack over $\mathbb{F}_q$, which is isomorphic to $G$-$\text{Zip}^\mu$ for $G = \text{Sp}(2n)_{\mathbb{F}_q}$ and $\mu : z \mapsto \text{diag}(z, z^{-1})$.

We define a flagged symplectic zip of rank $n$ over $S$ as a pair $(\mathcal{M}, F_\bullet)$ where $\mathcal{M} = (\mathcal{M}, F, V, \Omega, (-,-))$ is a symplectic zip of rank $n$ over $S$ and $0 \subset F_1 \subset \cdots \subset F_n = \Omega$ is a filtration by locally free, locally direct factor $\mathcal{O}_S$-submodules such that $\text{rank}(F_i) = i$. This flag extends uniquely to a symplectic flag of $\mathcal{M}$ by defining $F_i = F_i^\perp$ for all $n < i \leq 2n$.
For all $1 \leq i \leq n$, we define a line bundle $L_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ (with the convention $\mathcal{F}_0 = 0$). For a tuple $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we define a line bundle

$$L(a) = \bigotimes_{i=1}^n L_i^{\otimes -a_i}.$$ 

The minus signs in the above formula account for our choice of $B$ as the lower-triangular Borel subgroup. Indeed, $B$ is the stabilizer of the flag $0 \subset \text{Span}_k(e_{2n}) \subset \text{Span}_k(e_{2n-1}, e_{2n}) \subset \cdots \subset \text{Span}_k(e_{n+1}, \ldots, e_{2n}) \subset V_{0, p}$, and $T$ acts on the graded pieces of this flag by $t_1^{-1}, \ldots, t_n^{-1}$ in this order.

The stack of flagged symplectic zips is isomorphic to the stack $G\text{-ZipFlag}^\mu$. We now explain the modular interpretation of partial Hasse invariants. Let $1 \leq d \leq n$. Consider the composition

$$\mathcal{F}_d \longrightarrow \Omega \underset{V}{\longrightarrow} \Omega^{(q)} \underset{q}{\longrightarrow} \Omega^{(q)}/\mathcal{F}^{(q)}_{n-d}.$$ 

(8.1.4)

The first map is the inclusion $\mathcal{F}_d \subset \Omega$, and the third map is the natural projection. Denote by $f_d$ the composition of the maps (8.1.4). We have

$$\det(\mathcal{F}_d) = L_1 \otimes \cdots \otimes L_d, \quad \det(\Omega^{(q)}/\mathcal{F}^{(q)}_{n-d}) = L_{n-d+1}^q \otimes \cdots \otimes L_n^q.$$

Therefore, $f_d$ induces a map $\wedge^d(f_d): L_1 \otimes \cdots \otimes L_d \to L_{n-d+1}^q \otimes \cdots \otimes L_n^q$. Hence, we obtain a section $H_{\text{ad}} = \wedge^d(f_d) \in H^0(S, \mathcal{L}(\lambda_0))$ which corresponds to the flag partial Hasse invariant given by (8.1.2).

### 8.2 The unitary inert case

This example arises in the theory of Shimura varieties attached to general unitary groups. More precisely, let $E/\mathbb{Q}$ be a quadratic totally imaginary extension and $(V_0, \psi)$ a hermitian space over $E$ of dimension $n$. Then there is a Shimura variety of PEL-type attached to the group $G = \text{GU}(V_0, \psi)$. If we write $n = \text{dim}_E(V_0)$, then the attached Shimura variety parametrizes abelian varieties of dimension $n$ with a polarization, an action of $\mathcal{O}_E$ and a level structure. Let $p$ be a prime of good reduction, and let $X$ be the special fiber of the Kisin–Vasiu (canonical) integral model of the Shimura variety. By (2.4.11), we have a smooth, surjective morphism $\zeta: X \to G\text{-Zip}^\mu$, where $G$ is the special fiber of a reductive $\mathbb{Z}_p$-model of the group $G_{\mathbb{Q}_p}$. If $p$ is inert in $E$, then $G$ is a unitary group over $\mathbb{F}_p$. If $p$ is split in $E$, then $G \simeq \text{GL}_{m,p} \times \mathbb{G}_m, \mathbb{F}_p$. Define again the flag space $\text{Flag}(X)$ of $X$ as the fiber product of $X$ and $G\text{-ZipFlag}^\mu$ over $G\text{-Zip}^\mu$, similarly to (8.1.1). We explain the modular interpretation of partial Hasse invariants on $\text{Flag}(X)$. We will first assume that $p$ is an inert prime in $E$. To simplify, we consider the case of a unitary group $G = U(V_0, \psi)$ (the case of $G = \text{GU}(V_0, \psi)$ is very similar).

### 8.2.1 Group theory

Let $(V_0, \psi)$ be an $n$-dimensional $\mathbb{F}_q$-vector space endowed with a non-degenerate hermitian form $\psi: V_0 \times V_0 \to \mathbb{F}_q$ (in the context of Shimura varieties, we always take $q = p$). Choose a basis $B$ of $V_0$ where $\psi$ is given by the matrix

$$J = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$
Define a reductive group $G$ by

$$G(R) = \{ f \in \text{GL}_{q^2}(V_0 \otimes F_q R) \mid \psi_R(f(x), f(y)) = \psi_R(x, y), \forall x, y \in V_0 \otimes F_q R \}$$

for any $F_q$-algebra $R$. There is an isomorphism $G_{F_q^2} \simeq \text{GL}(V_0)$. It is induced by the $F_q$-algebra isomorphism $F_q^2 \otimes F_q^2 \to R \to \text{GL}(V_0)$, $a \otimes x \mapsto (ax, \sigma(a)x)$ (where $\text{Gal}(F_q^2/F_q) = \{1, \sigma\}$).

The action of $\sigma$ on the set $\text{GL}_n(k)$ is given by $\sigma \cdot A = J\sigma(A)^{-1}J$. Let $T$ denote the maximal diagonal torus and $B$ the lower-triangular Borel subgroup of $G_k$. By our choice of the basis $B$, the groups $B$ and $T$ are defined over $F_q$. Identify $X^*(T) = \mathbb{Z}^n$ such that $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ corresponds to the character $\text{diag}(x_1, \ldots, x_n) \mapsto \prod_{i=1}^n x_i^{a_i}$. The simple roots are given by $\Delta = \{ e_i - e_{i+1} \mid 1 \leq i \leq n - 1 \}$, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{Z}^n$.

Choose non-negative integers $(r, s)$ such that $n = r + s$. We will assume that $r \geq s$. Define a cocharacter $\mu: G_{m,k} \to G_k$ by $x \mapsto \text{diag}(x_i, I_s)$ via the identification $G_k \simeq \text{GL}_{m,k}$. Let $Z_\mu = (G, P, L, Q, M, \varphi)$ be the associated zip datum. Concretely, if we denote by $(u_i)_{i=1}^n$ the canonical basis of $k^n$, then $P$ is the stabilizer of $V_0, F := \text{Span}_k(u_{r+1}, \ldots, u_n)$. Note that $P$ is not defined over $F_q$ unless $r = s$. We may also identify $L = \text{GL}_r \times \text{GL}_s$. One has $\Delta^P = \{ \alpha \}$ with $\alpha = e_r - e_{r+1}$. The element $z = (\sigma(u_0), w_0)$ is given by the matrix

$$z = \begin{pmatrix} 0 & I_s \\ I_r & 0 \end{pmatrix}.$$ 

Consider again the function $\Delta_d: G_k \to A^1$ defined in (8.1.2). It is clear that $\Delta_d$ is a $B \times B$-eigenfunction on $G_k$. It is a section of $V_{\text{Sh}}(\chi_d, -w_0 \chi_d)$ for the character $\chi_d = (1, \ldots, 1, 0, \ldots, 0)$, where 1 appears $d$ times and 0 appears $n - d$ times. Specifically, $\Delta_d$ is the function $h_\chi$ for $\chi = \chi_d$, as defined in (5.1.1). The function $G_k \to A^1$, $g \mapsto \Delta_d(gz)$ corresponds to the pull-back $\text{Ha}_d = \psi^*(\Delta_d)$. The sections $\{\text{Ha}_d\}_{1 \leq d \leq n-1}$ form a complete set of partial Hasse invariants for $G$-$\text{ZipFlag}^n$. If $d \leq s$, the section $\text{Ha}_d$ lies in $H^0(G$-$\text{ZipFlag}^n, V_{\text{flag}}(\lambda_d))$, where

$$\lambda_d = \chi_d - q w_{0,l}(\sigma^{-1} \chi_d) = (1, \ldots, 1, 0, \ldots, 0) + (0, \ldots, 0, q, \ldots, q, 0, \ldots, 0).$$

Similarly, in the case $s < d \leq n-1$, the weight of $\text{Ha}_d$ is

$$\lambda_d = \chi_d - q w_{0,l}(\sigma^{-1} \chi_d) = (1, \ldots, 1, 0, \ldots, 0) + (q, \ldots, q, 0, \ldots, 0, q, \ldots, q).$$

Note also that the determinant function $\text{det}: G_k \to \mathbb{G}_m$ defines a non-vanishing section of $H^0(G$-$\text{Zip}^n, V_I(\lambda_{\text{det}}))$ where $\lambda_{\text{det}} = (q+1, \ldots, q+1)$. In particular, the line bundles $V_I(\lambda_{\text{det}})$ and $V_{\text{flag}}(\lambda_{\text{det}})$ are trivial. Hence, the line bundles $\{V_{\text{flag}}(\lambda)\}_\lambda$ on $G$-$\text{ZipFlag}^n$ are parametrize up to isomorphism by the group $\mathbb{Z}^n/\mathbb{Z}_{\lambda_{\text{det}}}$.

### 8.2.2 Modular interpretation

We explain the modular interpretation of the sections $\text{Ha}_d$. We first define the notion of unitary zip of type $(r, s)$. Let $S$ be an $F_q$-scheme and let $M = (M, F, V, \Omega, (-, -))$ be a symplectic zip over $S$ of rank $n$. Consider a ring homomorphism $\iota: F_q \to \text{End}_{O_S}(M)$. (Here, $\text{End}_{O_S}(M)$ denotes the ring of endomorphisms of the symplectic zip $M$.) Let $k$ be an algebraic closure of $F_q$ and write $\iota_0, \iota_1$ for the two embeddings $F_q \to k$. Then $M_k$ splits naturally as

$$M_k = M_0 \oplus M_1$$

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where $\alpha \in \mathbb{F}_2^*$ acts on $\mathcal{M}_i$ (for $i = 0, 1$) by multiplication with $t_i(\alpha)$. One sees easily that $\mathcal{M}_0$, $\mathcal{M}_1$ are totally isotropic. The Frobenius and Verschiebung interchange the components. More precisely, they induce maps $V: \mathcal{M}_0 \to \mathcal{M}_1^{(q)}$, $V: \mathcal{M}_1 \to \mathcal{M}_0^{(q)}$, $F: \mathcal{M}_1^{(q)} \to \mathcal{M}_1$ and $F: \mathcal{M}_1^{(q)} \to \mathcal{M}_0$. Similarly, we have a decomposition $\Omega_k = \Omega_0 \oplus \Omega_1$ since by definition $\Omega$ is stable by the action of $\mathbb{F}_2^*$.

**Definition 8.2.1.** A unitary zip of type $(r, s)$ is a tuple $(\mathcal{M}, F, V, \Omega, (\langle - , - \rangle), \iota)$ where

1. $\mathcal{M} = (\mathcal{M}, F, V, \Omega, (\langle - , - \rangle), \iota)$ is a symplectic zip over $S$ of rank $n = r + s$.
2. $\iota: \mathbb{F}_2^* \to \text{End}_{\mathcal{O}_S}(\mathcal{M})$ is a ring homomorphism.
3. We have $\text{rank}_{\mathcal{O}_S}(\Omega_0) = r$, $\text{rank}_{\mathcal{O}_S}(\Omega_1) = s$.

Unitary zips of type $(r, s)$ form an algebraic stack over $\mathbb{F}_2$ which is isomorphic to $G$-Zip$^n$. By our choice of convention, the standard representation $\text{Std}: G \to \text{GL}_{n, \mathbb{F}_q}$ corresponds to the vector bundle $\mathcal{M}_1$. Furthermore, the sub-$P$-representation $V_{0, P} \subset V_0 = k^n$ corresponds to the vector bundle $\Omega_1 \subset \mathcal{M}_1$. We define a flagged unitary zip as a tuple $(\mathcal{M}, \iota, \mathcal{F}_\bullet)$ where $(\mathcal{M}, \iota)$ is a unitary zip, and

$$0 = \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \cdots \subset \mathcal{F}_r = \Omega_1 \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{M}_1$$  \hspace{1cm} (8.2.1)

is a filtration by locally free, locally direct factor $\mathcal{O}_S$-submodules such that $\text{rank}(\mathcal{F}_i) = n - i$ for $1 \leq i \leq n$. Furthermore, put $\mathcal{L}_i = \mathcal{F}_{i-1}/\mathcal{F}_i$ (for $1 \leq i \leq n$), and define

$$\mathcal{L}(\mathbf{a}) = \bigotimes_{i=1}^n \mathcal{L}_i^a_i \quad \text{for} \quad \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n.$$  

Our convention for the ordering of the flag (8.2.1) is consistent with the choice of the Borel subgroup $B$. Indeed, $B$ stabilizes the filtration $\text{Span}_k(e_n) \subset \text{Span}_k(e_{n-1}) \subset \cdots \subset \text{Span}_k(e_1, \ldots, e_2) \subset k^n$ and the torus $T$ acts on the graded pieces respectively by $t_n, \ldots, t_1$ in this order. We can extend the filtration (8.2.1) uniquely to a symplectic flag of $\mathcal{M}$ by taking the orthogonals

$$\mathcal{M}_1 = \mathcal{M}_1^{\perp} \subset \mathcal{F}_1^{\perp} \subset \cdots \subset \mathcal{F}_r^{\perp} \subset \cdots \subset \mathcal{F}_{n-1}^{\perp} \subset \mathcal{F}_n^{\perp} = \mathcal{M}.$$  

Furthermore, we have $\mathcal{F}_r^{\perp} = \Omega_1^{\perp} = \Omega_0 \oplus \mathcal{M}_1$. Intersecting the filtration with $\mathcal{M}_0$, we obtain a full flag of $\mathcal{M}_0$ as follows:

$$0 \subset \mathcal{F}_1^{\perp} \cap \mathcal{M}_0 \subset \cdots \subset \mathcal{F}_r^{\perp} \cap \mathcal{M}_0 \subset \cdots \subset \mathcal{F}_{n-1}^{\perp} \cap \mathcal{M}_0 \subset \mathcal{M}_0$$

and have $\mathcal{F}_r^{\perp} \cap \mathcal{M}_0 = \Omega_0$. For $d \leq s$, consider the composition

$$\mathcal{F}_d^{\perp} \cap \mathcal{M}_0 \longrightarrow \mathcal{M}_0 \overset{V}{\longrightarrow} \Omega_1^{(p)} \longrightarrow \Omega_1^{(q)}/F_{r+d}^{(q)}.$$  

It is a map of vector bundles of rank $d$. We have $\mathcal{F}_d^{\perp} \cap \mathcal{M}_0 = \mathcal{F}_d^{\perp} = \left(\mathcal{M}_d^{\perp}\right)^V$ and hence $\det(\mathcal{F}_d^{\perp} \cap \mathcal{M}_0) = \det(\mathcal{M}_d/\mathcal{F}_d)^{-1} = (\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_d)^{-1}$. Similarly, we find $\det(\Omega_1/F_{r+d}) = \mathcal{L}_{r+1} \otimes \cdots \otimes \mathcal{L}_{r+d}$. Thus, we obtain a map

$$\left(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_d\right)^{-1} \to \left(\mathcal{L}_{r+1} \otimes \cdots \otimes \mathcal{L}_{r+d}\right)^q$$  

which is the same as a section of $H^0(S, \mathcal{L}(\lambda_d))$. This section is the partial Hasse invariant $\text{Ha}_d$ of weight $\lambda_d$ explained in the previous section. We now assume $s < d \leq n - 1$. In this case, consider the composition map

$$\frac{\mathcal{F}_d^{(q)}}{\Omega_1^{(q)}} \subset \frac{\mathcal{M}_1^{(q)}}{\Omega_1^{(q)}} \overset{F}{\longrightarrow} \mathcal{M}_0 \longrightarrow \frac{\mathcal{M}_0}{\mathcal{F}_d^{\perp} \cap \mathcal{M}_0}. \hspace{1cm} (8.2.2)$$

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Taking determinants, we obtain a map $(\mathcal{L}_{d-s+1} \otimes \cdots \otimes \mathcal{L}_r)^q \to (\mathcal{L}_{d+1} \otimes \cdots \otimes \mathcal{L}_n)^{-1}$. Thus, we constructed a section of weight

$$\left(0, \ldots, 0, -1, \ldots, -1\right) + \left(0, \ldots, 0, -q, \ldots, -q, 0, \ldots, 0\right) = \lambda_d - \lambda_{\det}.$$  

Up to multiplication by the determinant, the section given by (8.2.2) corresponds to the partial Hasse invariant $\text{Ha}_d$ (in particular, the vanishing loci coincide).

Lastly, we give a modular interpretation for the $\mu$-ordinary Hasse invariant $\text{Ha}_\mu$. This is a section of a line bundle on $G$-Zip$^\mu$ whose vanishing locus is exactly the closure of the unique codimension one stratum. The existence and properties of such sections were proved for general groups in [GN17]. In the general unitary case, Goldring–Nicole have constructed unique codimension one stratum. The existence and properties of such sections were proved directly from the modular interpretation (8.2.3). We see on this example that $\text{Ha}_\mu$ must be a codimension one closed substack of $G$-Zip$^\mu$. Since there is a unique stratum of codimension 1, it shows that the section $\text{Ha}_\mu$ is a $\mu$-ordinary Hasse invariant. This would be difficult to prove directly from the modular interpretation (8.2.3). We see on this example that $\text{Ha}_\mu$ is different from all flag partial Hasse invariants $\{\text{Ha}_d\}_{1 \leq d \leq n-1}$. This is in contrast to the case when $P$ is defined over $\mathbb{F}_q$ (see Lemma 5.2.8).

In the case of $r = 2$, $s = 1$, we gave in [IK21] Lemma 6.3.1 a group-theoretical representation of $\text{Ha}_\mu$. In [IK21] Proposition 6.3.2, the section $\text{Ha}_\mu$ has weight $(q + 1, q + 1, q^2 + q) = \lambda_{\det} + (0, 0, q^2 - 1)$. It coincides with the section constructed above up to multiplication by the determinant.

### 8.3 The unitary split case

We now consider the case of a unitary Shimura variety at a prime $p$ of good reduction which is split in the totally imaginary quadratic field $E$. In this case, $G$ is isomorphic to $\text{GL}_{n, \mathbb{F}_p}$.

#### 8.3.1 Group theory

Define $G = \text{GL}_{n, \mathbb{F}_q}$ (take $q = p$ in the context of Shimura varieties). Define a cocharacter $\mu: \mathbb{G}_{m,k} \to G_k$ as in the previous section by $\mu(x) = \text{diag}(xI_r, I_s)$. Write again $\mathbb{Z}_\mu = (G, P, L, Q, M, \varphi)$ for the attached zip datum. Let again $B$ denote the lower-triangular Borel and $T$ the diagonal torus. Again, we identify $X^*(T) = \mathbb{Z}^n$ as in §8.2.1 and define $V_{0,P} \subset V_0$ similarly. In this case, the Galois action is trivial, so the element $z = w_{0,I}w_0$ is given by the matrix

$$z = \begin{pmatrix} 0 & I_r \\ I_s & 0 \end{pmatrix}.$$
Consider again the function $\Delta_d$ defined in (5.1.1). The function $G_k \to A^1_k$, $g \mapsto \Delta_d(gz)$ corresponds to the pull-back $H_a_d = \psi^*(\Delta_d)$. The sections $\{H_a_d\}_{1 \leq d \leq n-1}$ form a complete set of partial Hasse invariants for $G$-ZipFlag$^\mu$. If $d \leq r$, the section $H_{a_d}$ lies in $H^0(G$-ZipFlag$^\mu, \mathcal{V}_{flag}(\lambda_d))$, where

$$\lambda_d = \chi_d - qw_{0,1}x_d = (1, \ldots, 1, 0, \ldots, 0) + (0, \ldots, 0, -q, \ldots, -q, 0, \ldots, 0).$$

Similarly, in the case $r < d \leq n - 1$, the weight of $H_{a_d}$ is

$$\lambda_d = \chi_d - qw_{0,1}x_d = (1, \ldots, 1, 0, \ldots, 0) + (-q, \ldots, -q, 0, \ldots, 0, -q, \ldots, -q).$$

The determinant function $\det : G_k \to \mathbb{G}_m$ defines a nowhere vanishing section $\det \in H^0(G$-ZipFlag$^\mu, \mathcal{V}_I(\lambda_{det}))$ where $\lambda_{det} = -(q - 1), \ldots, -(q - 1)$. In particular, the line bundles $\mathcal{V}_I(\lambda_{det})$ and $\mathcal{V}_{flag}(\lambda_{det})$ are trivial. Hence, the line bundles $\{\mathcal{V}_{flag}(\lambda)\}_\lambda$ on G-ZipFlag$^\mu$ are parametrized up to isomorphism by the group $\mathbb{Z}^n/\mathbb{Z}\lambda_{det}$. The section $H_{a_r}$ is the ordinary Hasse invariant of G-Zip$^\mu$, which illustrates again Lemma 5.2.8

### 8.3.2 Partial Hasse invariants

We give a modular interpretation for partial Hasse invariants. In this case, the corresponding notion of a unitary zip ($\mathcal{M}_i, \iota$) is similar to Definition 8.2.1 except that $\iota$ is now an action of $\mathbb{F}_q \times \mathbb{F}_q$ on $\mathcal{M}_i$. This implies that $\mathcal{M}$ splits again as $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$, but this time the two components are stable under the Frobenius and Verschiebung maps. As a consequence, ($\mathcal{M}_i, \iota$) is completely and uniquely determined up to isomorphism by the datum ($\mathcal{M}_1, \Omega_1, F, V$) where $F : \mathcal{M}_1^{(q)} \to \mathcal{M}_1$ and $V : \mathcal{M}_1 \to \mathcal{M}_1^{(q)}$. Define the flag space as the stack of triples ($\mathcal{M}_i, \iota, \mathcal{F}_i$) where $\mathcal{F}_i$ is a flag of $\mathcal{M}_i$ as in (8.2.1). Define also $\mathcal{L}_i$ for $1 \leq i \leq n$ and $\mathcal{L}(a)$ for $a \in \mathbb{Z}^n$ as in (8.2.2). For $d \leq r$, consider the composition

$$\mathcal{F}_{r-d}/\Omega_1^{(q)} \subset \mathcal{M}_1^{(q)}/\Omega_1^{(q)} \xrightarrow{E} \mathcal{M}_1 \longrightarrow \mathcal{M}_1/\mathcal{F}_d.$$  

We have $\det(\mathcal{F}_{r-d}/\Omega_1) = \mathcal{L}_r \otimes \cdots \otimes \mathcal{L}_{r-d+1}$ and $\det(\mathcal{M}_1/\mathcal{F}_d) = \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_d$. Taking the determinant, we obtain a section of weight $\lambda_d$, and it corresponds to the Hasse invariant $H_{a_d}$. Similarly, for $r < d \leq n - 1$, consider the composition

$$\mathcal{F}_d \subset \mathcal{M}_1 \xrightarrow{V} \Omega_1^{(q)} \longrightarrow \Omega_1^{(q)}/\mathcal{F}_{r+n-d}.$$  

We have $\det(\mathcal{F}_d) = \mathcal{L}_{d+1} \otimes \cdots \otimes \mathcal{L}_n$ and $\det(\Omega_1/\mathcal{F}_{r+n-d}) = \mathcal{L}_{r+1} \otimes \cdots \otimes \mathcal{L}_{r+n-d}$. Taking the determinant, we obtain a section of weight

$$\lambda_d = \chi_d - \omega_{0,1}x_d = (0, \ldots, 0, -1, \ldots, -1) + (0, \ldots, 0, q, \ldots, q, 0, \ldots, 0) = \lambda_d - \lambda_{det}.$$

This section coincides with the partial Hasse invariant $H_{a_d}$ up to multiplication by the determinant.

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