Connected Sums of Special Lagrangian Submanifolds

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Abstract

Let $M_1$ and $M_2$ be special Lagrangian submanifolds of a compact Calabi-Yau manifold $X$ that intersect transversely at a single point. We can then think of $M_1 \cup M_2$ as a singular special Lagrangian submanifold of $X$ with a single isolated singularity. We investigate when we can regularize $M_1 \cup M_2$ in the following sense: There exists a family of Calabi-Yau structures $X_\alpha$ on $X$ and a family of special Lagrangian submanifolds $M_\alpha$ of $X_\alpha$ such that $M_\alpha$ converges to $M_1 \cup M_2$ and $X_\alpha$ converges to the original Calabi-Yau structure on $X$. We prove that a regularization exists in two key cases: (1) when $\dim \mathbb{C}X = 3$, $\text{Hol}(X) = \text{SU}(3)$, and $[M_1]$ is not a multiple of $[M_2]$ in $H_3(X)$, and (2) when $X$ is a torus with $\dim \mathbb{C}X \geq 3$, $M_1$ is flat, and the intersection of $M_1$ and $M_2$ satisfies a certain angle criterion. One can easily construct examples of the second case, and thus as a corollary we construct new examples of non-flat special Lagrangian submanifolds of Calabi-Yau tori.

1 Introduction

One of the fundamental problems in special Lagrangian geometry is to understand moduli spaces of special Lagrangian submanifolds (SLags). Much interest in this problem arises from the study of mirror symmetry since it is related to the SYZ Conjecture [19]. McLean’s deformation theorem [17] together with some work by Hitchin [8] provide some understanding of these moduli spaces locally near nonsingular SLags, but in order to understand these moduli spaces globally, we need to understand singular SLags. Recently, some research has focused on the more modest goal of understanding SLags with isolated conical singularities. For example, see [10, 11, 12, 13]. In order to study SLags with isolated conical singularities, we need to know something about the SLag cones in $\mathbb{C}^n$ on which these singularities are modelled. Some researchers have undertaken the study of these SLag cones. See [4].
In this paper we restrict our attention to isolated conical singularities modelled on a very simple type of SLag cone, namely, the union of two transversely intersecting SLag planes in \( \mathbb{C}^n \). By the work of Lawlor \[15\], we know that such a SLag cone can be deformed through a family of nonsingular SLags in \( \mathbb{C}^n \) if the two planes meet a certain angle criterion to be described later. (This criterion is always satisfied when \( n \leq 3 \).) This local regularization holds out hope that if we have a singular SLag with this simple type of singularity, then it can be globally regularized, that is, it can be deformed through a family of nonsingular SLags. This simple type of singularity arises when a connected, immersed SLag intersects itself, and when two embedded SLags intersect. The first case has already been treated by Yng-Ing Lee \[16\] who answered the question in the affirmative: A compact, connected, immersed SLag with an isolated point of transverse self-intersection satisfying the angle criterion can be regularized. The second case is more difficult, and that is the case which we consider in this paper. Simply put, our problem is to try to regularize the union of two compact embedded SLags with an isolated point of transverse intersection satisfying the angle criterion. A problem related to ours has been solved by Butscher \[1\]: The union of two embedded SLags with boundary in \( \mathbb{C}^n \) with \( n \geq 3 \) with an isolated point of transverse intersection satisfying the angle criterion can be regularized. In Butscher’s paper, the regularization takes advantage of the freedom to deform the boundary of the singular SLag. In our problem, we have no boundaries, and therefore we cannot use the added degrees of freedom. Indeed, our problem as stated above probably cannot be solved. We must introduce another degree of freedom, and we do this by deforming the Calabi-Yau structure of the ambient manifold.

Before we state our results, we recall some basic definitions and facts.

**Definition.** A **Calabi-Yau structure** (or CY structure) on a compact \( 2n \)-fold \( X \) is a 3-tuple \((J, \omega, \Omega)\) such that \(J\) is a complex structure on \( X\), \(\omega\) is a Kähler form with respect to \(J\), and \(\Omega\) is a holomorphic \((n,0)\) form with respect to \(J\) such that

\[
\frac{\omega^n}{n!} = (-1)^{\frac{1}{2}n(n-1)} \left( \frac{i}{2} \right)^n \Omega \wedge \overline{\Omega}
\]

(1)

It is a fact that \(\text{Re} \, \Omega\) is a calibration with respect to the Kähler metric. We say that a submanifold \(M\) of \(X\) is **special Lagrangian** iff \(M\) is calibrated by \(\text{Re} \, \Omega\).

The special Lagrangian condition on \(M\) is equivalent to the vanishing of both \(\omega\) and \(\text{Im} \, \Omega\) on \(M\). If \((X,J)\) admits a Calabi-Yau structure at all, then in each Kähler class there is a unique Kähler form \(\omega\) such that \((X,J,\omega)\) admits a Calabi-Yau structure. (In this case, the Kähler metric corresponding to \(\omega\) is Ricci-flat.) Also, if \((X,J)\) admits a Calabi-Yau structure, then \(\Omega\) is uniquely determined up to a complex constant. If we also choose \(\omega\), then the normalization \[1\] uniquely determines \(\Omega\) up to a phase. Because of these facts, a choice of Calabi-Yau structure amounts to a choice of complex structure, a Kähler class, and a phase. For more general background on Calabi-Yau manifolds and special Lagrangian geometry, see \[13, 9, 14, 6\].

We now describe the angle criterion.
The one in the Main Theorem. Since \([\Omega]\) spans and \([\text{Im }\Omega]\) span \(H^\bullet(M)\) such that

\[0 \leq \theta_1 \leq \cdots \leq \theta_{n-1} \leq \frac{\pi}{2} \text{ and } \theta_{n-1} \leq \theta_n \leq \pi - \theta_{n-1}\]

while

\[\eta = e_1 \wedge \ldots \wedge e_n\]

and

\[\xi = [(\cos \theta_1)e_1 + (\sin \theta_1)e_{n+1}] \wedge \ldots \wedge [(\cos \theta_n)e_n + (\sin \theta_n)e_{2n}].\]

We say that \(\eta\) and \(\xi\) satisfy the angle criterion iff the characterizing angles between \(\eta\) and \(-\xi\) satisfy \(\sum_{i=1}^{n} \theta_i = \pi\).

The Lawlor-Nance Angle Theorem states that a pair of oriented planes \((\eta, \xi)\) is minimizing iff the characterizing angles between \(\eta\) and \(-\xi\) satisfy \(\sum_{i=1}^{n} \theta_i \geq \pi\). Therefore the angle criterion may be thought of as describing the “borderline case” of minimizing pairs of planes. See [3] for more on characterizing angles and the Angle Theorem.

We are now ready to state our main theorem.

**Theorem 1 (Main Theorem).** Let \(M_1\) and \(M_2\) be two embedded special Lagrangian submanifolds of a Calabi-Yau manifold \((X, J, \omega, \Omega)\) such that \(n = \dim_{\mathbb{C}} X \geq 3\) and the holonomy of the Kähler metric is exactly \(SU(n)\). Assume that \(M_1\) and \(M_2\) intersect transversely at a single point \(p\) such that the tangent cone of \(M_1 \cup M_2\) at \(p\) satisfies the angle criterion. Further assume that \(\Re[H^{n-1, 1}(X) \oplus H^{1, n-1}(X)]\) is not contained in the kernel of \(\frac{[M_1]}{\text{Vol}(M_1)} - \frac{[M_2]}{\text{Vol}(M_2)}\), thought of as a functional on \(H^n(X)\). Then there exists a family of Calabi-Yau structures \((J_\alpha, \omega_\alpha, \Omega_\alpha)\) on \(X\) converging to \((J, \omega, \Omega)\) and a family of embedded submanifolds \(M_\alpha \subset X\) converging to \(M_1 \cup M_2\) such that \(M_\alpha\) is special Lagrangian in \((X, J_\alpha, \omega_\alpha, \Omega_\alpha)\).

When \(n = 3\), the theorem reduces to the following nice result.

**Corollary 2.** Let \(M_1\) and \(M_2\) be two embedded special Lagrangian submanifolds of a Calabi-Yau manifold \((X, J, \omega, \Omega)\) such that \(\dim_{\mathbb{C}} X = 3\) and the holonomy of the Kähler metric is exactly \(SU(3)\). Assume that \(M_1\) and \(M_2\) intersect transversely at a single point and that \([M_1]\) is not a multiple of \([M_2]\) in \(H_3(X)\). Then there exists a family of Calabi-Yau structures \((J_\alpha, \omega_\alpha, \Omega_\alpha)\) on \(X\) converging to \((J, \omega, \Omega)\) and a family of embedded submanifolds \(M_\alpha \subset X\) converging to \(M_1 \cup M_2\) such that \(M_\alpha\) is special Lagrangian in \((X, J_\alpha, \omega_\alpha, \Omega_\alpha)\).

**Proof.** As mentioned earlier, the angle criterion is automatically satisfied when \(n = 3\). It suffices to show that the homology condition in the Corollary implies the one in the Main Theorem. Since \([\Omega]\) spans \(H^{3,0}(X)\), it follows that \([\Re \Omega]\) and \([\Im \Omega]\) span \(H^{3,0}(X) \oplus H^{0,3}(X)\). By assumption, \(\frac{[M_1]}{\text{Vol}(M_1)} - \frac{[M_2]}{\text{Vol}(M_2)} \neq 0\), and by the special Lagrangian condition, this real homology class is zero on both \([\Re \Omega]\) and \([\Im \Omega]\). So there exists some element of \(\Re[H^{2,1}(X) \oplus H^{1,2}(X)]\) which is not in the kernel of \(\frac{[M_1]}{\text{Vol}(M_1)} - \frac{[M_2]}{\text{Vol}(M_2)}\). \(\square\)
Since a CY torus has trivial holonomy, our Main Theorem does not apply to this important case. However, we can still prove a version of the theorem in this setting.

**Theorem 3 (Torus Version).** Let $M_1$ and $M_2$ be two embedded special Lagrangian submanifolds of a Calabi-Yau torus $(T, J, \omega, \Omega)$ such that $\dim_C T \geq 3$ and $M_1$ is flat. Assume that $M_1$ and $M_2$ intersect transversely at a single point $p$ such that the tangent cone of $M_1 \cup M_2$ at $p$ satisfies the angle criterion. Then there exists a family of Calabi-Yau structures $(J_\alpha, \omega, \Omega_\alpha)$ on $T$ converging to $(J, \omega, \Omega)$ and a family of embedded submanifolds $M_\alpha \subset T$ converging to $M_1 \cup M_2$ such that $M_\alpha$ is special Lagrangian in $(T, J_\alpha, \omega, \Omega_\alpha)$.

The Main Theorem and the Torus Version share the hypothesis that $M_1$ and $M_2$ must intersect at a single point, but this condition is somewhat artificial. In light of the proof to follow, as long as there exists an isolated transverse intersection point $p$, we can still regularize the singularity at $p$, but the $M_\alpha$’s will only be immersed rather than embedded. However, if $M_1 \cap M_2$ is a finite set of isolated transverse intersection points, all of which satisfy the angle criterion, then we can recover the embeddedness as follows: We first apply our result to one of these intersection points, and then we apply Yng-Ing Lee’s result on immersed SLags to each of the other intersection points. This procedure is possible because the property of being a transverse intersection point satisfying the angle criterion is an open condition with respect to the relevant topology. See Lemma 5.

It is a simple matter to construct infinitely many distinct pairs of flat SLag tori satisfying the angle criterion in the standard CY torus, $\mathbb{C}^n/\mathbb{Z}^n$. Applying the discussion in the previous paragraph, we immediately obtain the following result.

**Corollary 4.** There exist non-flat embedded special Lagrangian submanifolds of Calabi-Yau tori.

With some extra work, the methods of this paper can probably be used to prove that our results hold in dimension two also. The methods of algebraic geometry should also apply in the dimension two case.

Concurrent with the writing of this paper, Joyce has produced some results on the general problem of desingularizing special Lagrangians with isolated conical singularities in almost Calabi-Yau manifolds. In particular, Theorem 7.11 of combined with Lemma of this paper and an understanding of the Lawlor necks can be used to prove the results of this paper. Note that Lemma is the main ingredient of the Key Lemma of this paper. The methods used by Joyce are different from those presented here, and because of the added generality, the proofs are also more complicated.

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1If either $M_1$ or $M_2$ is not connected, then we apply our result multiple times.
2 Preliminaries

The Main Theorem and the Torus Version share certain assumptions: We have two embedded special Lagrangian submanifolds $M_1$ and $M_2$ of a Calabi-Yau manifold $(X, J, \omega, \Omega)$ with $\dim C X \geq 3$. We also assume that $M_1$ and $M_2$ intersect transversely at a single point $p$ such that the tangent cone of $M_1 \cup M_2$ at $p$ satisfies the angle criterion. This is the situation we assume from now until the proofs of the Key Lemma, which will depend on the additional assumptions in the two cases. We also assume without loss of generality that $M_1$ and $M_2$ are connected.

We now explain the idea behind these results. We wish to construct a family of approximate solutions $M_\alpha$ such that $M_\alpha$ converges to $M = M_1 \cup M_2$, $M_\alpha$ is very close to being special Lagrangian. Once we have these $M_\alpha$'s, we can construct small Hamiltonian deformations of them and hope that at least one of them is exactly special Lagrangian. This is actually too much to hope for, but we can add another degree of freedom to this deformation by simultaneously deforming the Calabi-Yau structure $(J, \omega, \Omega)$ and hope that at least one of them is exactly special Lagrangian. Using these deformations we define a deformation operator whose solutions correspond together with some estimates, we obtain the desired solutions. The work lies in obtaining the appropriate estimates.

First we construct our family of approximate solutions $M_\alpha$. This is where the angle criterion is relevant. Given $\phi_1, \ldots, \phi_n \in \mathbb{R}$, we define $P(\phi_1, \ldots, \phi_n)$ to be the oriented plane $[(\cos \phi_1) \frac{\partial}{\partial x_1} + (\sin \phi_1) \frac{\partial}{\partial y_1}] \wedge \ldots \wedge [(\cos \phi_n) \frac{\partial}{\partial x_n} + (\sin \phi_n) \frac{\partial}{\partial y_n}]$.

By the work of Lawlor [13], we know that for any $\theta_1, \ldots, \theta_n \in (0, \pi)$ satisfying $\sum_{j=1}^n \theta_j = \pi$, there exists a special Lagrangian submanifold $N$ of $\mathbb{C}^n$ that is asymptotic in an oriented sense to the two planes $P(0, \ldots, 0)$ and $-P(-\theta_1, \ldots, -\theta_n)$.

This submanifold $N$ has the property that $\epsilon N$ converges to $[P(0, \ldots, 0)] \cup [-P(-\theta_1, \ldots, -\theta_n)]$ in an appropriate sense as $\epsilon \to 0$. These $N$'s, as well as their images under $\text{SU}(n) \ltimes (\text{dilations})$ are called Lawlor necks. Note that there is also a Lawlor neck asymptotic in an oriented sense to the two planes $P(0, \ldots, 0)$ and $-P(\theta_1, \ldots, \theta_n)$.

Lemma 5. If $\eta$ and $\xi$ are two special Lagrangian planes in $\mathbb{C}^n$, then there exists a Lawlor neck asymptotic in an oriented sense to $\eta$ and $\xi$ if and only if $\eta$ and $\xi$ are transverse planes satisfying the angle criterion. Moreover, both of these equivalent conditions are open conditions in the space of pairs of special Lagrangian planes. Finally, when $n \leq 3$, every pair of special Lagrangian planes satisfies the angle criterion.

Proof. Let $\eta$ and $\xi$ be two transverse special Lagrangian planes in $\mathbb{C}^n$. The necessity of the angle criterion for the existence of a Lawlor neck is trivial, so we need only prove sufficiency. Assume that $\eta$ and $\xi$ satisfy the angle criterion. Without loss of generality, we may assume that $\eta = P(0, \ldots, 0)$ by performing an $\text{SU}(n)$ change of coordinates. Since $-\xi$ is Lagrangian and transverse to $\eta$, we have $-\xi = P(\phi_1, \ldots, \phi_n)$ for some $\phi_1, \ldots, \phi_n \in (-\pi, 0) \cup (0, \pi)$, after an
we can choose a Darboux and normal coordinate system in a ball $B^C$ in $[1, 2, 16]$. Here we only give a broad overview. Near the singular point $p$ approximate global regularization. The details of this construction are described to those two planes. For sufficiently small $\alpha > 0$ and certain constants $C_\delta$ and $C_\epsilon$, choose

$$\delta = \frac{\alpha}{C_\delta} \text{ and } \epsilon = \frac{\alpha^{1+1/n}}{C_\epsilon}.$$
Moreover, on $M$ we always use $S_X$ such that $\omega$ is approximately special Lagrangian in the mean curvature field $H$. On a special Lagrangian submanifold, $e^{i\theta} = 1$ and $H = 0$. One can show that $M_\alpha$ is approximately special Lagrangian in following sense [1, 2, 16].

Lemma 6. Given the construction above, the Lagrangian angle function and the mean curvature field on $M_\alpha$ satisfy

$$
|\sin \theta|_0 + \alpha^\beta |\sin \theta|_\beta + \alpha |\nabla \sin \theta|_0 \leq C\alpha
$$

$$
|1 - \cos \theta|_0 + \alpha^\beta |\cos \theta|_\beta + \alpha |\nabla \cos \theta|_0 \leq C\alpha^2
$$

$$
|H|_0 + \alpha^\beta |H|_\beta + \alpha |\nabla H|_0 \leq C.
$$

Moreover, on $M_\alpha - B_\delta$, $e^{i\theta} = 1$ and $H = 0$.

### 3 The Deformation Operator

Now that we have our approximate solutions $M_\alpha$, we can define the relevant deformation operator. Suppose that we have a smooth deformation $(J_t, \omega_t, \Omega_t)$ of the CY structure $(J, \omega, \Omega) = (J_0, \omega_0, \Omega_0)$ such that $\omega_t$ is always cohomologous to $\omega$ and $\langle [M_1] + [M_2], [\text{Im } \Omega_t] \rangle = 0$. We will choose the appropriate deformation of CY structure later in this paper; for now assume that we have already chosen it. By Moser’s Theorem, there exists a smooth path of diffeomorphisms $\Psi_t$ of $X$ such that

$$
\Psi_t^* \omega_t = \omega. \tag{2}
$$

By the Lagrangian Neighborhood Theorem, let $U$ be a tubular neighborhood of $M_\alpha$ symplectomorphic to $T^* M_\alpha$ so that we have a projection map $\pi : U \to M_\alpha$. Let $\tau$ be a smooth cutoff function supported in $U$ such that $\tau = 1$ on $\frac{1}{2} U$, where $\frac{1}{2} U$ is defined using the structure of $T^* M$. Observe that we can choose $U$

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[1] [2] [10] for the definitions of $C_5$ and $C_7$. These constants are chosen so that the following construction works and has the stated properties.

We can cut out a small ball $B_\delta(0)$ from $M$, glue a rescaled Lawlor neck $\epsilon N$ into $B_{\frac{3}{2}}(0)$, and then interpolate in the annular region.$^3$ This gives us $M_\alpha$ which we can think of as $M'_1 \cup T_1 \cup N' \cup T_2 \cup M'_2$, where $M'_1 = M_\alpha - B_\delta(0)$, $N'$ is the rescaled Lawlor neck, and the $T_i$’s are the interpolated regions connecting $M'_i$ to $N'$. It is evident that $M_\alpha$ converges to $M$ by construction, and since each Lawlor neck has the topology of a cylinder, $S^{n-1} \times \mathbb{R}$, $M_\alpha$ is topologically the connected sum of $M_1$ and $M_2$. We can choose the interpolation so that $M_\alpha$ is exactly Lagrangian. Since $M_\alpha$ is Lagrangian, it is a fact that at each point of $M_\alpha$, $\Omega|_{M_\alpha} = e^{i\theta} \text{Vol}_{M_\alpha}$ for some $\theta$. We call the multi-valued function $\theta$ the Lagrangian angle function. This “function” has the property that $J \nabla \theta$ is the mean curvature field $H$. On a special Lagrangian submanifold, $e^{i\theta} = 1$ and $H = 0$. One can show that $M_\alpha$ is approximately special Lagrangian in following sense $[1, 2, 16]$.  

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$^2$Throughout this paper $\alpha$ will be the parameter upon which most of our constructions depend. Because of this, we will explicitly write out the $\alpha$ dependence of all of our constants, with the only exceptions being $\delta$ and $\epsilon$. We will use the letter $C$ without subscript as a generic constant independent of $\alpha$ whose value may change even in a single chain of inequalities. For consistency we always use $C$ as an upper bound.

$^3$From now on we will write $B_\delta$ for $B_\delta(0)$ where there is no chance of confusion.
to have width greater than $\frac{1}{r}$ over $M_\alpha \cap B_{r0}$ for some $r_0$ and width greater than $1$ over $M_1 \cup M_2'$, with an inverse linear interpolation in between. Now extend any function $h \in C^{2,\beta}(M_\alpha)$ to a function $\tilde{h} \in C^{2,\beta}(X)$ by defining $\tilde{h}(q) = \tau(q)h(\pi(q))$ on $U$ and $\tilde{h} = 0$ outside $U$. Now define $\Phi_h$ to be the symplectomorphism generated by the Hamiltonian function $\tilde{h}$.

**Definition.** The deformation operator $F_\alpha : C^{2,\beta}(M_\alpha) \times \mathbb{R} \to C^{0,\beta}(M_\alpha)$ is defined by

$$F_\alpha(h, t) = \langle (\Psi_t \circ \Phi_h)^*(\text{Im } \Omega), \text{Vol}_{M_\alpha} \rangle_{M_\alpha}$$

where the metric on $M_\alpha$ is the one induced by the Kähler metric on $(X, J, \omega)$, independent of $t$.

Since $M_\alpha$ is a Lagrangian submanifold of $(X, \omega)$ and $\Phi_h$ is a symplectomorphism it follows that $\Phi_h(M_\alpha)$ is a Lagrangian submanifold of $(X, \omega)$. Then by (2), it follows that $(\Psi_t \circ \Phi_h)(M_\alpha)$ is a Lagrangian submanifold of $(X, \omega_t)$. Clearly, $F_\alpha(h, t) = 0$ if $\text{Im } \Omega_t$ restricted to $(\Psi_t \circ \Phi_h)(M_\alpha)$ is identically zero. Therefore a solution of the equation $F_\alpha(h, t) = 0$ corresponds to a special Lagrangian submanifold of $(X, J, \omega_t, \Omega_t)$, and a small solution corresponds to a nearby special Lagrangian. So our goal is to show that for sufficiently small $\alpha$, $F_\alpha$ has a small solution. Our method of constructing such solutions is the following version of the Inverse Function Theorem.

**Theorem 7 (Inverse Function Theorem).** Let $F : B \to B'$ be a $C^1$ map between Banach spaces and suppose that the linearization $DF(0)$ is an isomorphism. Moreover, assume that for some constants $C_1, C_N, \alpha_1$ and $r_1$, we have

1. $\|DF(0)x\|_{B'} \geq \frac{1}{C_1}\|x\|_B$ for all $x \in B$, and
2. $\|DF(0)x - DF(y)x\|_{B'} \leq C_N\|x\|_B\cdot\|y\|_B$ for all $x, y \in B$ with $\|y\|_B < r_1$.

Then there exist neighborhoods $U$ of 0 and $V$ of $F(0)$ such that $F : U \to V$ is a $C^1$-diffeomorphism. Moreover, if $r \leq \min(r_1, (2C_1C_N)^{-1})$, then $B_{r/2C_1}(F(0)) \subset V$ and $B_{r/2C_1}(F(0)) \subset F(B_r(0))$.

In particular, when the hypotheses of the theorem are satisfied and additionally, $\|F(0)\|_{B'} < \frac{1}{2C_1}$, we can solve the equation $F(y) = 0$ for some $\|y\|_B < r$.

In order to invoke the Inverse Function Theorem in our situation, we need to choose our Banach spaces carefully. We define a smooth weight function $\rho$ on $M_\alpha$ with the key property that the ball of radius $\rho(x)$ in $M_\alpha$ centered at $x$ has uniformly bounded geometry. That is, in geodesic normal coordinates at $x$, we have $|g_{ij} - \delta_{ij}|_{1, \beta, B_{\rho(x)}}(x) \leq 1$ where the norm here is the local scale-invariant Schauder norm on $B_{\rho(x)}(x)$. We also require that $\tau = 1$ on the ball $B_{\rho(x)}(x, X)$. We can construct such a $\rho$ with the following additional properties. See (1) [2] [10].

- For some $r_0$ and $R$ independent of $\alpha$,

$$\rho(x) = \begin{cases} \epsilon R & \text{for } x \in N' = M_\alpha \cap B_{r\rho_0} \\ \text{interpolation} & \text{for } x \in M_\alpha \cap (B - B_{r\rho_0}) \\ R & \text{for } x \in M_\alpha - B \end{cases}$$
\( \rho(x) \leq C|x| \) for \( x \in M_\alpha \cap (B - B_{\delta/2}) \).

- \( |\nabla \rho|_0 \leq C \).

- \( \|\rho^{-1}\|_{L^2(M_\alpha)} \leq C \).

**Definition.** For any \( 0 < \beta < 1 \), the \( \rho \)-weighted \( (k, \beta) \)-Schauder norm on \( C^{k, \beta}(M_\alpha) \) is given by

\[
|u|_{C^{k, \beta}(M_\alpha)} = |u|_{0, M_\alpha} + |\rho \nabla u|_{0, M_\alpha} + \cdots + |\rho^k \nabla^k u|_{0, M_\alpha} + [\rho^{k+\beta} \nabla^k u]_{\beta, M_\alpha}.
\]

Let \( S \) be the first eigenfunction of the Laplacian on \( M_\alpha \), normalized so that \( \|S\|_{L^2(M_\alpha)} = 1 \). Then we define the Banach spaces \( \mathcal{B}_{1, \alpha}, \mathcal{B}_\alpha, \) and \( \mathcal{B}'_\alpha \) as vector spaces

\[
\mathcal{B}_{1, \alpha} = \left\{ u \in C^{2, \beta}(M_\alpha) \left| \int_{M_\alpha} u = \int_{M_\alpha} uS = 0 \right. \right\}
\]

\[
\mathcal{B}_\alpha = \mathcal{B}_{1, \alpha} \times \mathbb{R}
\]

\[
\mathcal{B}'_\alpha = \left\{ u \in C^{0, \beta}(M_\alpha) \left| \int_{M_\alpha} u = 0 \right. \right\}
\]

with the norms

\[
\|u\|_{\mathcal{B}_{1, \alpha}} = |u|_{C^{2, \beta}(M_\alpha)}
\]

\[
\|(u, a)\|_{\mathcal{B}_\alpha} = |u|_{C^{2, \beta}(M_\alpha)} + |a|
\]

\[
\|f\|_{\mathcal{B}'_\alpha} = |\rho^2 f|_{C^{0, \beta}(M_\alpha)}.
\]

(The integrations above are taken with respect to the \( t \)-independent Kähler metric on \( M_\alpha \).)

Since \( \Psi_t \) and \( \Phi_h \) are isotopies, \( \langle ([\Psi_t \circ \Phi_h](M_\alpha)), [\text{Im } \Omega_t] \rangle = \langle ([M_\alpha], [\text{Im } \Omega_t]) = \langle ([M_1] + [M_2], [\text{Im } \Omega_t]) = 0 \), and therefore \( F_\alpha(\mathcal{B}_\alpha) \subset \mathcal{B}'_\alpha \). From now on we think of the **deformation operator** \( F_\alpha \) as an operator from \( \mathcal{B}_\alpha \) to \( \mathcal{B}'_\alpha \).

The choice of \( \beta \) is not particularly important; it is simply a small constant independent of \( \alpha \). The purpose of the weighted norm is to achieve estimates that scale nicely with respect to \( \alpha \). The reason why we take the orthogonal complement of the functions 1 and \( S \) is that 1 lies in the kernel of the linearization, and \( S \) lies in the approximate kernel of the linearization.

Let us summarize what we need to prove in order to invoke the Inverse Function Theorem argument:

- We need an injectivity estimate on \( D F_\alpha(0, 0) \); we must establish the existence of a constant \( C_I(\alpha) \) as in condition 1 of the Inverse Function Theorem and find its dependence on \( \alpha \).

- We need to show that \( D F_\alpha(0, 0) \) is surjective.

\[\text{In contrast to our use of constants, many geometric objects such as functions and operators will depend on } \alpha, \text{but we will suppress this dependence in the notation for the purpose of readability. The loss of clarity should be minimal since these objects are all defined on } M_\alpha.\]
We need a nonlinear estimate; we must establish the existence of a constant $C_N(\alpha)$ as in condition 2 of the Inverse Function Theorem and find its dependence on $\alpha$.

We first bound $F_\alpha(0,0)$ in terms of $\alpha$.

**Proposition 8.**

$$DF_\alpha(0,0)(u,a) = \Delta u + Pu + a\psi$$ \hspace{1cm} (3)

where $P : B_{1,\alpha} \rightarrow B'_{\alpha}$ is given by

$$Pu = (\cos \theta - 1)\Delta u - (\sin \theta)\langle H, J \nabla u \rangle$$

and $\psi \in B'_{\alpha}$ is given by

$$\psi = \langle L_{V}(\text{Im } \Omega) + \text{Im } \dot{\Omega}, \text{Vol}_{M_{\alpha}} \rangle$$

where $V$ is the vector field generating the flow $\Psi_{t}$ at time $t = 0$, and $\dot{\Omega} = \frac{\partial}{\partial t} \Omega|_{t=0}$.

The calculation of $\psi$ is self-evident. The rest of the calculation is straightforward and can be found in [1] and [10]. The reason we write $\Delta$ and $P$ separately in equation (3) is that the $P$ term turns out to be negligible, and therefore it suffices to understand $\Delta$ and $\psi$. The unimportance of $P$ is expressed in the following lemma.

**Lemma 9.** For sufficiently small $\alpha$, for any $u \in B_{1,\alpha}$,

$$\|Pu\|_{B'_{\alpha}} \leq C\alpha^{1-\beta}\|u\|_{B_{1,\alpha}}.$$ 

**Proof.** The proof essentially follows directly from the bounds given in Lemma 6:

$$|\rho^2(1 - \cos \theta)\Delta u|_0 \leq |1 - \cos \theta|_0 |\rho^2 \Delta u|_0 \leq C\alpha^2 |u|_{C^{2,\alpha}}.$$ 

$$[\rho^{2+\beta}(1 - \cos \theta)\Delta u]_\beta \leq [1 - \cos \theta]_\beta |\rho^{2+\beta} \Delta u|_0 + |1 - \cos \theta|_0 |\rho^{2+\beta} \Delta u|_\beta \leq (C\alpha^{2-\beta} R^\beta + C\alpha^2) |u|_{C^{2,\alpha}}.$$ 

$$|\rho^2(\sin \theta)\langle H, J \nabla u \rangle|_0 \leq |\rho(\sin \theta)H|_0 |\rho \nabla u|_0 \leq RC|u|_{C^{2,\alpha}}.$$ 

For the final inequality, we use the fact that $|v|_\beta \leq |v|_0 + |\nabla v|_0$.

$$[\rho^{2+\beta}(\sin \theta)\langle H, J \nabla u \rangle]_\beta \leq |\sin \theta|_\beta |H|_0 |\rho^{2+\beta} J \nabla u|_0 + |\sin \theta|_0 |H|_\beta |\rho^{2+\beta} J \nabla u|_\beta$$ 

$$\leq (C\alpha^{1-\beta})CR^{1+\beta}|\rho \nabla u|_0 + (C\alpha)(C\alpha^{1-\beta})R^{1+\beta}|\rho \nabla u|_0 + (C\alpha)C(|\rho^{2+\beta} \nabla u|_0 + |(\nabla |\rho^{2+\beta}(\nabla u)|_0 + |\rho^{2+\beta} \nabla^2 u|_0$$ 

$$\leq C\alpha^{1-\beta} |u|_{C^{2,\alpha}} + C\alpha(R^{1+\beta} + CR^\beta + R^\beta) |u|_{C^{2,\alpha}}$$

where the last line uses the bound on $|\nabla \rho|_0$. Now combine the previous four inequalities to deduce the desired result. \qed
4 Analysis of the Laplacian on $M_\alpha$

The first step in establishing an injectivity estimate for the linearized deformation operator is finding a lower bound for the second eigenvalue of the Laplacian. The second step is to combine this lower bound with an elliptic estimate to obtain an injectivity estimate for $\Delta$:

$$B^{1,\alpha}_1 \rightarrow B'_\alpha.$$

It is a fact that on any Riemannian manifold $M$, for any $f \in L^2(M)$ with one derivative in $L^2(M)$ such that $\int_M f = 0$, we have $\int_M |\nabla f|^2 \geq \lambda_1(M) \int_M f^2$, where $\lambda_1(M)$ is the first eigenvalue of the Laplacian. From this it follows easily that if we drop the condition $\int_M f = 0$, then we have

$$\lambda_1(M) \leq \frac{\int_M |\nabla f|^2}{\int_M f^2 - \frac{1}{\text{Vol}(M)} (\int_M f)^2}. \quad (4)$$

We define a smooth cutoff function $\varphi$ on $M_\alpha$ with the following properties: $\varphi = 0$ in $B_\delta$, $\varphi = 1$ outside $B_{2\delta}$, and $|\nabla \varphi| \leq \frac{C}{\delta}$ for some $C$. Recall that $M_\alpha - B_\delta = (M_1 \cup M_2) - B_\delta$, and therefore we may think of $\varphi$ as a function on either $M_\alpha$ or on $M_1 \cup M_2$. Observe that because we have uniformly bounded mean curvature, the Monotonicity Formula provides the following bounds which we will use repeatedly:

- $\text{Vol}(M_\alpha \cap B_{2\delta}) = O(\delta^n)$
- $\text{Vol}(M_i \cap B_{2\delta}) = O(\delta^n)$
- $\text{Vol}(M_i - B_{2\delta}) \leq C$
- $\text{Vol}(M_i - B_{2\delta}) \geq \frac{1}{C}$

**Lemma 10.** For small enough $\alpha$,

$$\lambda_1(M_\alpha) \leq C\delta^{n-2}.$$

**Proof.** Let

$$f = \varphi \left[ \frac{\chi_{M_1 - B_\delta}}{\text{Vol}(M_1 - B_{2\delta})} - \frac{\chi_{M_2 - B_\delta}}{\text{Vol}(M_2 - B_{2\delta})} \right],$$

where $\chi_A$ denotes the characteristic function of $A$. Clearly, $\int_{M_\alpha} f = O(\delta^n)$. Using inequality (4), we see that

$$\lambda_1(M_\alpha) \leq \frac{\int_{M_\alpha} |\nabla f|^2}{\int_{M_\alpha} f^2 - \frac{1}{\text{Vol}(M_\alpha)} (\int_{M_\alpha} f)^2} \leq \frac{C \int_{M_\alpha \cap B_{2\delta}} \delta^{-2}}{\text{Vol}(M_1 - B_{2\delta}) + \text{Vol}(M_2 - B_{2\delta})} - O(\delta^{2n}) \text{ by the properties of } \varphi \leq C\delta^{n-2} \text{ since the denominator is bounded below.} \qed
We would like to have some idea of what $S$ looks like. By the previous Lemma together with Lemma 5 of Yng-Ing Lee’s paper [16], we know that $|S|_0$ is bounded independently of $\alpha$. This fact allows us to use our knowledge of the kernel of the Laplacian on $M_1 \cup M_2$ to construct a function that approximates $S$ in the $L^2$ sense.

**Lemma 11.** Define

$$\bar{S} = a_1 \chi_{M_1} + a_2 \chi_{M_2}$$

where

$$a_1 = \frac{1}{\text{Vol}(M_1)} \sqrt{\frac{\text{Vol}(M_1) \text{Vol}(M_2)}{\text{Vol}(M_1) + \text{Vol}(M_2)}}$$

and

$$a_2 = \frac{-1}{\text{Vol}(M_2)} \sqrt{\frac{\text{Vol}(M_1) \text{Vol}(M_2)}{\text{Vol}(M_1) + \text{Vol}(M_2)}}.$$ 

Then for small enough $\alpha$,

$$\|S - \varphi \bar{S}\|_{L^2(M_\alpha)} \leq C\delta^{(n-2)/2}$$

and

$$\|\bar{S} - \varphi \bar{S}\|_{L^2(M_\alpha \cup M_\beta)} \leq C\delta^{n/2}.$$ 

**Proof.** First, the bound on $S$ implies that

$$\|S - \varphi S\|_{L^2(M_\alpha)}^2 \leq \|S\|_{L^2(M_\alpha \cap B_{2\delta})}^2 = O(\delta^n). \quad (5)$$

Since $\varphi S$ is defined on $M_1 \cup M_2$, we have

$$\int_{M_1 \cup M_2} |\nabla (\varphi S)|^2 = \int_{M_\alpha} |\nabla (\varphi S)|^2$$

$$\leq 2 \int_{M_\alpha \cap B_{2\delta}} |\nabla \varphi|^2 S^2 + 2 \int_{M_\alpha} \varphi^2 |\nabla S|^2$$

$$\leq O(\delta^{n-2}) + 2 \int_{M_\alpha} |\nabla S|^2$$

arguing as in Lemma 10

$$= O(\delta^{n-2}) \text{ by Lemma 10 and the normalization of } S.$$ 

Note that $M_1 \cup M_2$ has a two-dimensional kernel spanned by $\chi_{M_1}$ and $\chi_{M_2}$, and its first non-zero eigenvalue is obviously a constant independent of $\alpha$. Therefore the estimate above shows that if $a'_1 \chi_{M_1} + a'_2 \chi_{M_2}$ is the orthogonal projection of $\varphi S$ onto the kernel, then

$$\|\varphi S - (a'_1 \chi_{M_1} + a'_2 \chi_{M_2})\|_{L^2(M_1 \cup M_2)}^2 \leq C\|\nabla (\varphi S) - (a'_1 \chi_{M_1} + a'_2 \chi_{M_2})\|_{L^2(M_1 \cup M_2)}^2$$

$$= C\|\nabla (\varphi S)\|_{L^2(M_1 \cup M_2)}^2$$

$$= O(\delta^{n-2}) \text{ by the previous calculation.} \quad (6)$$

The bounds (5) and (6) show that

$$(a'_1)^2 \text{Vol}(M_1) + (a'_2)^2 \text{Vol}(M_2) = \|a'_1 \chi_{M_1} + a'_2 \chi_{M_2}\|_{L^2(M_1 \cup M_2)}^2 = 1 + O(\delta^{n-2})$$

This Lemma depends on the Michael-Simon Inequality and uses the fact that the mean curvature of $M_\alpha$ is bounded independently of $\alpha$. 

12
and
\[ a_1' \text{Vol}(M_1) + a_2' \text{Vol}(M_2) = \int_{M_1 \cup M_2} a_1' \mathcal{X}_{M_1} + a_2' \mathcal{X}_{M_2} = O(\delta^{(n-2)/2}). \]

Solving these equations, we find that
\[ a_1' = a_1 + O(\delta^{(n-2)/2}) \quad \text{and} \quad a_2' = a_2 + O(\delta^{(n-2)/2}) \]
where \( a_1 \) and \( a_2 \) were defined above.\(^6\) It now follows that
\[ \| a_1' \mathcal{X}_{M_1} + a_2' \mathcal{X}_{M_2} - \bar{S} \|^2_{L^2(M_1 \cup M_2)} = O(\delta^{n-2}). \tag{7} \]

Finally, similar to the bound (5), we see that
\[ \| \bar{S} - \varphi \bar{S} \|^2_{L^2(M_1 \cup M_2)} = O(\delta^n). \tag{8} \]

Putting together the bounds (5), (6), (7), and (8), we obtain the desired result. \( \Box \)

We can now use our knowledge of \( S \) to help us show that the second eigenvalue of the Laplacian on \( M_\alpha \) is bounded below.

**Proposition 12 (Second Eigenvalue Estimate).** For small enough \( \alpha \), the second eigenvalue of the Laplacian of \( M_\alpha \), \( \lambda_2(M_\alpha) \), is bounded below. In particular, for each \( u \in B_{1,\alpha}, \)
\[ \| \Delta u \|^2_{L^2(M_\alpha)} \geq \frac{1}{C} \| u \|^2_{L^2(M_\alpha)} \]

**Proof.** Let \( f \) be an eigenfunction for the second eigenvalue of the Laplacian, normalized so that \( \| f \|^2_{L^2(M_\alpha)} = 1 \). Using the min-max characterization of \( \lambda_2(M_\alpha) \) one can show that \( \lambda_2(M_\alpha) \) is bounded above independently of \( \alpha \). This fact allows us to apply Lemma 5 of Yng-Ing Lee’s paper \([16]\) to show that \( |f|_0 \) is bounded independently of \( \alpha \). We compute

\[
\lambda_2(M_\alpha) \quad = \quad \int_{M_\alpha} |\nabla f|^2 \\
\quad = \quad \int_{M_\alpha} |\nabla (\varphi f) + \nabla ((1 - \varphi) f)|^2 \\
\quad \geq \quad \int_{M_1} |\nabla (\varphi f)|^2 + \int_{M_2} |\nabla (\varphi f)|^2 - 2 \int_{M_\alpha \cap B_{\varphi \delta}} |\nabla (1 - \varphi)|^2 f^2 - 2 \int_{M_\alpha} (1 - \varphi)^2 |\nabla f|^2 \\
\quad \geq \quad \int_{M_1} |\nabla (\varphi f)|^2 + \int_{M_2} |\nabla (\varphi f)|^2 - O(\delta^{n-2}) - 2 \int_{M_\alpha} |\nabla f|^2.
\]

\(^6\)Of course, the solution is only determined up to a sign, but there was a sign ambiguity in our original definition of \( S \), so we can simply define \( S \) to have the sign consistent with these equations.
Therefore,
\[
3\lambda_2(M_\alpha) \geq \int_{M_1} |\nabla (\varphi f)|^2 + \int_{M_2} |\nabla (\varphi f)|^2 - O(\delta^{n-2}) \\
\geq \lambda_1(M_1) \left[ \int_{M_1} \varphi^2 f^2 - \frac{1}{\text{Vol}(M_1)} \left( \int_{M_1} \varphi f \right)^2 \right] + \lambda_2(M_2) \left[ \int_{M_2} \varphi^2 f^2 - \frac{1}{\text{Vol}(M_2)} \left( \int_{M_2} \varphi f \right)^2 \right] - O(\delta^{n-2}) \text{ by inequality (4)} \\
\geq C \int_{M_\alpha} f^2 - O(\delta^n) - \frac{\lambda_1(M_1)}{\text{Vol}(M_1)} \left( \int_{M_1} \varphi f \right)^2 - \frac{\lambda_2(M_2)}{\text{Vol}(M_2)} \left( \int_{M_2} \varphi f \right)^2 - O(\delta^{n-2}) \\
\geq C - C \left[ \left( \int_{M_1} \varphi f \right)^2 + \left( \int_{M_2} \varphi f \right)^2 \right] - O(\delta^{n-2}).
\]

Now we must bound the terms in brackets in the previous line.
\[
\left| \int_{M_1} \varphi f \right| = \left| \int_{M_1 \cup M_2} \left( \tilde{S} - a_2 \right) \varphi f \right| \text{ by the definition of } \tilde{S} \text{ from Lemma 11} \\
= \frac{1}{a_1 - a_2} \left| \int_{M_\alpha} (\varphi \tilde{S}) f - a_2 \int_{M_\alpha} \varphi f \right| \\
= \frac{1}{a_1 - a_2} \left| \int_{M_\alpha} S f + \int_{M_\alpha} (\varphi \tilde{S} - S) f - a_2 \int_{M_\alpha} f + O(\delta^n) \right| \\
\leq C \| \varphi \tilde{S} - S \|_{L^2(M_\alpha)} + O(\delta^n) \text{ since } \int_{M_\alpha} S f = \int_{M_\alpha} f = 0 \\
= O(\delta^{(n-2)/2}) + O(\delta^n) \text{ by Lemma 11.}
\]

The estimate for $M_2$ is similar.

Recall that the weight function $\rho$ was chosen so that $M_\alpha$ has uniformly bounded geometry in $\rho(x)$ neighborhoods of $x$ in $M_\alpha$. This allows us to use the local scale-invariant elliptic Schauder estimate to deduce a global elliptic Schauder estimate independent of $\alpha$. We omit the proof which is standard and straightforward.

**Proposition 13 (Global Elliptic Schauder Estimate).** For sufficiently small $\alpha$, for any $u \in C^{2,\beta}(M_\alpha)$,
\[
|u|_{C^{2,\beta}} \leq C(\| \rho^2 \Delta u \|_{C^{0,\beta}} + |u|_0).
\]

The following lemma, proved in [16], translates the second eigenvalue estimate from the $L^2$ setting to the Schauder setting.\(^7\) Choose any $\nu > 0$ independent of $\alpha$.

\(^7\) The proof of this lemma follows easily from a De Giorgi-Nash estimate, which in turn depends on the Michael-Simon inequality and bounded mean curvature.
Lemma 14. For sufficiently small $\alpha$,
\[ |u|_0 \leq \epsilon^{-\nu} |\rho^2 \Delta u|_{C^{0,\beta}} \]
for all $u \in B_{1,\alpha}$.

Combining this Lemma with Global Elliptic Estimate immediately leads us to an injectivity estimate for $\Delta : B_{1,\alpha} \to B'_{\alpha}$.

Proposition 15 (Laplacian Injectivity Estimate). For sufficiently small $\alpha$, for all $u \in B_{1,\alpha}$,
\[ |u|_{C^{2,\beta}} \leq C \epsilon^{-\nu} |\rho^2 \Delta u|_{C^{0,\beta}}. \]

5 Proofs of the Key Lemma

It is well-known that $\Delta$ is an isomorphism from $B_{1,\alpha} \oplus \langle S \rangle$ to $B'_{\alpha}$, but because of the small first eigenvalue, we had to remove $S$ in order to obtain a good injectivity estimate. Of course, removing $S$ costs us surjectivity. We added the extra degree of freedom in order to restore surjectivity. The essential requirement of the extra degree of freedom is that its linearization $\psi$ must have a significant $S$ component. This is the content of our Key Lemma. We now construct a deformation $(J_t, \omega_t, \Omega_t)$ so that $\psi$ has the desired property.

Lemma 16. Under the assumptions of the Main Theorem or the Torus Version, there exists a deformation of Calabi-Yau structure $(J_t, \omega_t, \Omega_t)$ such that $\omega_t$ is cohomologous to $\omega$, $\langle [M_1] + [M_2], [\text{Im } \Omega_t] \rangle = 0$, and $\langle \frac{[M_1]}{\text{Vol}(M_1)} - \frac{[M_2]}{\text{Vol}(M_2)}, [\text{Im } \Omega] \rangle \neq 0$ for sufficiently small $\alpha$.

Proof of Main Case. By a result of Tian [20] and Todorov [21], the first-order deformation space of complex structures on $(X,J)$ is exactly the space of harmonic $(n-1,1)$ forms, $\mathcal{H}^{n-1,1}(X)$, and all of these first-order deformations extend to actual deformations. By the hypotheses of the Main Theorem, we can find $\lambda \in \text{Re}[\mathcal{H}^{n-1,1}(X) \oplus \mathcal{H}^{1,n-1}(X)]$ such that $\left\langle \frac{[M_1]}{\text{Vol}(M_1)} - \frac{[M_2]}{\text{Vol}(M_2)}, [\lambda] \right\rangle \neq 0$. Since $\mathcal{H}^{n-1,1}(X)$ and $\mathcal{H}^{1,n-1}(X)$ are complex conjugate to each other, we can certainly find $\chi \in \mathcal{H}^{n-1,1}(X)$ such that $\text{Im } \chi = \lambda$. Choose a complex structure deformation $J_t$ whose first-order deformation is $\chi$. Recall that the choice of $J_t$ determines the holomorphic $(n,0)$-form $\Omega_t$ up to a constant. It is a fact that the effect of the first-order deformation $\chi$ on $\Omega$ is expressed by the formula
\[ \hat{\Omega} = c \Omega + \chi. \]

See Candelas and de la Ossa [3] for details. Since $\text{Im } \Omega$ vanishes on $M_1$ and $M_2$, and $\text{Re } \Omega$ calibrates $M_1$ and $M_2$, it follows that $\left\langle \frac{[M_1]}{\text{Vol}(M_1)} - \frac{[M_2]}{\text{Vol}(M_2)}, [\Omega] \right\rangle = 0$, and therefore $\left\langle \frac{[M_1]}{\text{Vol}(M_1)} - \frac{[M_2]}{\text{Vol}(M_2)}, [\text{Im } \hat{\Omega}] \right\rangle \neq 0$. 


We know that \([\omega]\) lies in the Kähler cone of \((X,J)\). Since \((X,J)\) admits a Calabi-Yau structure with holonomy equal to \(SU(n)\), so does \((X,J_t)\). In this case, it is a fact that \(H^{2,0}(X,J_t) = H^{0,2}(X,J_t) = 0\). See Joyce \[14\] for details. Therefore the Kähler cone of \((X,J_t)\) is open in \(H^2(X)\) for each \(t\). So for small \(t\), \([\omega]\) lies in the Kähler cone of \((X,J_t)\). Therefore we can find a unique \(\omega_t \in [\omega]\) such that \((X,J_t,\omega_t)\) admits a CY structure. This determines \(\Omega_t\) up to a phase, and we choose the phase so that \([\langle [M_1] + [M_2], [\text{Im}\Omega_t] \rangle] = 0\). \(\square\)

Before we discuss the Torus Case, let us study Calabi-Yau tori and the moduli space of CY tori.

**Proposition 17.** We can characterize all Calabi-Yau tori and all flat special Lagrangian submanifolds of them as follows.

1. **Given a rank \(2n\) lattice \(\Gamma \subset \mathbb{C}^n\), we define a Calabi-Yau structure on \(\mathbb{C}^n/\Gamma\) using the standard Calabi-Yau structure on \(\mathbb{C}^n\). Every Calabi-Yau torus \(T\) is given by this construction.**

2. **Given a special Lagrangian plane \(\eta\) invariant with respect to some rank \(n\) sublattice of \(\Gamma\), we obtain a special Lagrangian torus \(\eta/\Gamma\) in \(\mathbb{C}^n/\Gamma\). Every flat special Lagrangian submanifold of \(T\) is a union of special Lagrangian tori given by this construction.**

**Proof.** Let \((T,J,\omega,\Omega)\) be a CY torus. Clearly, we have an induced CY structure on the universal cover \((\mathbb{R}^{2n},\tilde{J},\tilde{\omega},\tilde{\Omega})\). Since the Kähler metric on \(T\) has zero Ricci curvature, we can apply Cheeger and Gromoll’s theorem on manifolds with non-negative Ricci curvature \[3\] to see that the induced metric on the universal cover is flat. Choose a point \(q \in \mathbb{R}^{2n}\). We can find a basis \(e_1,\ldots,e_{2n}\) of \(T_q\mathbb{R}^{2n}\) such that \(e^1,\ldots,e^{2n}\) is the dual basis, then \(\tilde{J}e_j = e_{j+n}\) for \(j \leq n\), \(\tilde{J}e_j = -e_{j-n}\) for \(j > n\), \(\tilde{\omega} = \sum_{j=1}^{n} e^j \wedge e^{j+n}\), and \(\tilde{\Omega} = (e_1 + ie_{n+1}) \wedge \ldots \wedge (e_n + ie_{2n})\) at the point \(q\). Since the metric is flat, we can extend \(e_1,\ldots,e_{2n}\) to a frame field over all of \(\mathbb{R}^{2n}\) in a unique way via parallel translation. Note that \([e_j,e_k] = 0\) everywhere, so we can find a global coordinate system \(x^1,\ldots,x^n,y^1,\ldots,y^n\) such that \(\frac{\partial}{\partial x^j} = e_j\) and \(\frac{\partial}{\partial y^j} = e_{j+n}\). Since \(\nabla J = \nabla \omega = \nabla \Omega = 0\), it follows that \(\tilde{J}(\frac{\partial}{\partial x^j}) = \frac{\partial}{\partial x^j}, \tilde{J}(\frac{\partial}{\partial y^j}) = -\frac{\partial}{\partial y^j}, \tilde{\omega} = \sum_{j=1}^{n} dx^j \wedge dy^j\), and \(\tilde{\Omega} = dz^1 \wedge \ldots \wedge dz^n\) everywhere. That is, the induced CY structure on the universal cover is the standard CY structure on \(\mathbb{C}^n\). Moreover, since each Deck transformation must be an orientation-preserving isometry, the Deck transformations are translations.

Let \(M\) be a flat SLag in \(T\). Since \(M\) is flat and minimal in \(T\), which is also flat, it is a simple consequence of the Gauss equation that \(M\) is totally geodesic in \(T\). In fact, Ricci-flatness would have been sufficient. See \[5\]. Therefore, locally, the lift of \(M\) up to \(\mathbb{C}^n\) is a piece of a SLag plane. It follows that \(M\) is the quotient of a union of SLag planes which are invariant under some rank \(n\) sublattice of \(\Gamma\). \(\square\)

We know that the CY structure of a torus is determined by a lattice. The space of all rank \(2n\) lattices in \(\mathbb{C}^n\) is \(GL(2n,\mathbb{R})/SL(2n,\mathbb{Z})\) where \(SL(2n,\mathbb{Z})\) acts
on the right. Since the group of CY structure preserving automorphisms of \( \mathbb{C}^n \) is \( \text{SU}(n) \times \text{translations} \), it follows that the global moduli space of Calabi-Yau tori is precisely \( \text{GL}(2n, \mathbb{R})/\text{SL}(2n, \mathbb{Z}) \) modulo the action of \( \text{SU}(n) \) on the left. Therefore, locally, the deformation space is simply a neighborhood of the identity in \( \text{GL}(2n, \mathbb{R})/\text{SU}(n) \) where \( \text{SU}(n) \) acts on the left. Ignoring the (discrete) redundancies arising from the lattice automorphisms, \( \text{SL}(2n, \mathbb{Z}) \), the space of possible complex structures corresponds to \( \text{GL}(2n, \mathbb{R})/\text{GL}(n, \mathbb{C}) \), and after this choice is made, the space of compatible symplectic structures corresponds to \( \text{GL}(n, \mathbb{C})/\text{U}(n) \). On the other hand, we can choose the symplectic structure from the space \( \text{GL}(2n, \mathbb{R})/\text{Sp}(2n, \mathbb{R}) \) and then choose a compatible complex structure from the space \( \text{Sp}(2n, \mathbb{R})/\text{U}(n) \). Finally, of course, we choose a phase for \( \Omega \) from \( \text{U}(n)/\text{SU}(n) \). Finally, observe that changing the lattice by \( A \in \text{GL}(2n, \mathbb{R}) \) has the same effect as keeping the lattice and the canonical local coordinate systems on \( T \) fixed, but changing the CY structure to \( (A^{-1}JA, A^*\omega, A^*\Omega) \) with respect to those coordinates. This is the point of view we adopt in the following proof, and we will no longer mention lattices.

**Proof of Torus Case of Lemma 16.** By Proposition 17, any connected, flat \( \text{SLag} \) \( M_1 \) in \( (T, J, \omega, \Omega) \) is actually a \( \text{SLag} \) torus with constant tangent plane with respect to the canonical local coordinates, so we can perform an \( \text{SU}(n) \) change of coordinates taking \( M_1 \) to a \( \text{SLag} \) torus with tangent plane \( \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \) at each point of \( M_1 \). Since the \( \text{SU}(n) \) change of coordinates preserves the CY structure, we may assume without loss of generality that \( T_q M_1 = \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \) at each \( q \in M_1 \).

Consider a deformation \( A_t^{-1}JA_t \) of the complex structure. Then the holomorphic \((n, 0)\) form is \( A_t^*\Omega \) up to a constant. A simple calculation then shows that \( \text{Im} \dot{\Omega} = \chi + \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \) and \( \chi = \sum_{j,k=1}^n B_{jk} \text{d}\bar{z}^j \wedge \ldots \wedge \frac{\partial}{\partial x^n} \). Another simple calculation shows that if \( A_t \in \text{Sp}(2n, \mathbb{R}) \), then the corresponding \( B \) must be (complex) symmetric, and conversely, for any (complex) symmetric \( B \), we can find \( A_t \in \text{Sp}(2n, \mathbb{R}) \) such that \( B_{jk} = \dot{A}_{kj} \).

In particular, we can find \( A_t \in \text{Sp}(2n, \mathbb{R}) \) and a phase \( \phi_t \in \mathbb{R} \) such that \( (A_t^{-1}JA_t, \omega, \Omega_t = e^{i\phi_t} A_t^*\Omega) \) is a CY structure with \( \langle [M_1] + [M_2], [\text{Im} \Omega_t] \rangle = 0 \), and

\[
\chi = i \sum_{j=1}^n \text{d}\bar{z}^j \wedge \ldots \wedge \frac{\partial}{\partial x^n}.
\]

Note that

\[
\text{Im} \chi = \sum_{j=1}^n \text{Re}(\text{d}\bar{z}^j \wedge \ldots \wedge \frac{\partial}{\partial x^n}).
\]
We now consider the second term. Observe that since \( \text{Im } \dot{\Omega} \) is a form on \( \alpha \) for each \( j \), and that \( \frac{\partial}{\partial x^j} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \) is calibrated by each of these calibrations. Therefore \( \frac{1}{\text{Vol}(M_1)} \int_{M_1} \text{Im } \chi = n \) and \( \frac{1}{\text{Vol}(M_2)} \int_{M_2} \text{Im } \chi \leq n \) with equality iff \( M_2 \) is also calibrated by each of the calibrations \( \text{Re}(dz^1 \wedge \ldots \wedge d\bar{z}^j \wedge \ldots \wedge dz^n) \). We momentarily consider the “torus” planes \( \xi \) of the form

\[
[(\cos \theta_1) \frac{\partial}{\partial x^1} + (\sin \theta_1) \frac{\partial}{\partial y^1}] \wedge \ldots \wedge 
[(\cos \theta_n) \frac{\partial}{\partial x^n} + (\sin \theta_n) \frac{\partial}{\partial y^n}].
\]

It is easy to verify that if \( \xi \) is calibrated by \( \text{Re } \Omega = \text{Re}(dz^1 \wedge \ldots \wedge dz^n) \) and \( \text{Re}(dz^1 \wedge \ldots \wedge d\bar{z}^j \wedge \ldots \wedge dz^n) \) for each \( j \), then \( \xi = \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \). Then by Morgan’s Torus Lemma, it follows that \( \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \) is the only plane simultaneously calibrated by \( \text{Re}(dz^1 \wedge \ldots \wedge dz^n) \) and \( \text{Re}(dz^1 \wedge \ldots \wedge d\bar{z}^j \wedge \ldots \wedge dz^n) \) for each \( j \). See [10] for more on Morgan’s Torus Lemma. Therefore if \( M_2 \) is calibrated by \( \text{Re } \Omega = \text{Re}(dz^1 \wedge \ldots \wedge dz^n) \) and \( \text{Re}(dz^1 \wedge \ldots \wedge d\bar{z}^j \wedge \ldots \wedge dz^n) \) for each \( j \), it follows that \( T_{pM_2} = \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \), violating the transversality assumption. \( \square \)

**Lemma 18 (Key Lemma).** For small \( \alpha \),

\[ \left| \int_{M_n} \psi S \right| \geq \frac{1}{C}. \quad (9) \]

**Proof.** Recall that \( \psi = \langle L_V (\text{Im } \Omega), \text{Vol}_{M_n} \rangle + \langle \text{Im } \dot{\Omega}, \text{Vol}_{M_n} \rangle \). We’ll show that the first term integrated against \( S \) is small while the second term integrated against \( S \) is bounded below. Since \( \Omega \) is closed, \( L_V (\text{Im } \Omega) = d(V \cdot \text{Im } \Omega) \). Note that the two differential forms \( V \cdot \text{Im } \Omega \) and \( d(V \cdot \text{Im } \Omega) \) are defined on \( X \) independently of \( \alpha \). From this it follows that \( |V \cdot \text{Im } \Omega|_{M_n,0} \) and \( |d(V \cdot \text{Im } \Omega)|_{M_n,0} \) are bounded independently of \( \alpha \), where these are the induced norms on \( M_n \). Thus

\[
\left| \int_{M_n} (d(V \cdot \text{Im } \Omega)|_{M_n}) S \right| \leq \left| \int_{M_n} (d(V \cdot \text{Im } \Omega)|_{M_n}) \phi S \right| + C \delta^{(n-2)/2} \text{ by Lemma 11} \leq \left| \int_{M_n} (V \cdot \text{Im } \Omega)d(\phi S) \right| + C \delta^{(n-2)/2} \leq \left| \int_{M_{n} \cap B_{2\delta}} |V \cdot \text{Im } \Omega|_{M_n,0} \cdot C d(\phi) \right| + C \delta^{(n-2)/2} \leq C \delta^{n-1} + C \delta^{(n-2)/2}.
\]

We now consider the second term. Observe that since \( \text{Im } \dot{\Omega} \) is a form on \( X \) defined independently of \( \alpha \), \( |\text{Im } \dot{\Omega}|_{M_n,0} \) is bounded independently of \( \alpha \). And
obviously $|\text{Im} \hat{\Omega}|_{M_1 \cup M_2}[0, M_1 \cup M_2]$ is bounded independently of $\alpha$. Therefore

$$\int_{M_\alpha} (\text{Im} \hat{\Omega}|_{M_\alpha}) S = \int_{M_\alpha} (\text{Im} \hat{\Omega}|_{M_\alpha}) \varphi S + O(\delta^{(n-2)/2}) \text{ by Lemma } \text{[11]}$$

$$= \int_{M_1 \cup M_2} (\text{Im} \hat{\Omega}|_{M_1 \cup M_2}) \varphi S + O(\delta^{(n-2)/2})$$

$$= \int_{M_1 \cup M_2} (\text{Im} \hat{\Omega}|_{M_1 \cup M_2}) S + O(\delta^{n/2}) + O(\delta^{(n-2)/2}) \text{ by Lemma } \text{[11]}$$

$$= \sqrt{\text{Vol}(M_1) \text{Vol}(M_2)} \left\langle \frac{[M_1]}{\text{Vol}(M_1)} - \frac{[M_2]}{\text{Vol}(M_2)}, [\text{Im} \hat{\Omega}] \right\rangle$$

$$+ O(\delta^{(n-2)/2}) \text{ by definition of } \bar{S}.$$}

Now the result follows from Lemma \text{[16]} \hfill \Box

6 \hspace{1em} \text{The Full Linearized Deformation Operator}

We are now ready to prove the full injectivity estimate.

\textbf{Proposition 19 (Full Injectivity Estimate).} For small enough $\nu$ independent of $\alpha$, $D F_\alpha(0, 0)$ satisfies the injectivity estimate $C_\epsilon(\alpha) = C \epsilon^{-\nu}$ for sufficiently small $\alpha$. That is,

$$\|D F_\alpha(0, 0)(u, a)\|_{g_\alpha} \geq \frac{1}{C} \epsilon^\nu \|(u, a)\|_{g_\alpha}.$$

\textbf{Proof.}

$$|a| \leq C \left| \int_{M_\alpha} a \psi S \right| \text{ by the Key Lemma}$$

$$= C \left| \int_{M_\alpha} (\Delta u + a \psi) S \right| \text{ since } \Delta u \text{ is orthogonal to } S$$

$$= C \left| \int_{M_\alpha} \rho^2 (\Delta u + a \psi) \rho^{-2} S \right|$$

$$\leq C \|\Delta u + a \psi\|_{g_\alpha} \int_{M_\alpha} |\rho^{-2} S|$$

$$\leq C \|\Delta u + a \psi\|_{g_\alpha}$$

where the last line follows from the bounds on $|S|_0$ and $\|\rho^{-1}\|_{L^2(M_\alpha)}$. On the other hand,

$$\|u\|_{g_{1, \alpha}} \leq C \epsilon^{-\nu} \|\Delta u\|_{g_\alpha} \text{ by the Laplacian Injectivity Estimate}$$

$$\leq C \epsilon^{-\nu} (\|\Delta u + a \psi\|_{g_\alpha} + \|a \psi\|_{g_\alpha})$$

$$\leq C \epsilon^{-\nu} \|\Delta u + a \psi\|_{g_\alpha}$$

19
where the last line follows from the previous calculation and the fact that $|\psi|_{C^{0,\beta}(M)}$ is bounded independently of $\alpha$, by the definition of $\psi$. Finally, we deal with the $Pu$ term.

\[\|(u, a)\|_{B_{\alpha}} \leq C\epsilon^{-\nu}|\Delta u + a\psi|_{B_{\alpha}}\] by combining the previous two calculations

\[\leq C\epsilon^{-\nu}(|DF_\alpha(0, 0)(u, a)|_{B_{\alpha}} + \|Pu\|_{B_{\alpha}})\]

\[\leq C\epsilon^{-\nu}|DF_\alpha(0, 0)(u, a)|_{B_{\alpha}} + C\epsilon^{-\nu}\alpha^{1-\beta}|(u, a)|_{B_{\alpha}}\] by Lemma \[\]

For sufficiently small $\nu$, $\epsilon^{-\nu}\alpha^{1-\beta} \to 0$ as $\alpha \to 0$. So for small enough $\alpha$, we can absorb the last term into the left-hand side. \[\Box\]

**Proposition 20.** For small $\nu$, $DF_\alpha(0, 0)$ is surjective for sufficiently small $\alpha$.

**Proof.** Consider the map $A : B_{\alpha} \to B_{\alpha}'$ defined by by $A : (u, a) \mapsto \Delta u + a\psi$. By the proof of the Full Injectivity Estimate together with the Key Lemma, it is evident that $A$ is an isomorphism with $|A^{-1}| \leq C\epsilon^{-\nu}$. By Lemma \[\] $\|P\| \leq C\alpha^{1-\beta}$, therefore $|A^{-1}P| \leq C\alpha^{1-\beta}\epsilon^{-\nu}$. For small enough $\nu$, $\alpha^{1-\beta}\epsilon^{-\nu} \to 0$ as $\alpha \to 0$, therefore $I + A^{-1}P$ is invertible, and it follows that $A + P = DF_\alpha(0, 0)$ is surjective. \[\Box\]

### 7 Solving the Deformation Problem

The following Proposition can be found in [16].

**Proposition 21 (Nonlinear Estimate).** For small $\alpha$, $F_\alpha$ satisfies a nonlinear estimate with $C_2 = C\epsilon^{-2}$ and $r_1 = \frac{1}{C}\epsilon^2$. That is, for $(h, t) \in B_{\alpha}$ with $\|(h, t)\|_{B_{\alpha}} \leq r_1$,

\[|DF_\alpha(h, t)(u, a) - DF_\alpha(0, 0)(u, a)|_{B_{\alpha}} \leq C\epsilon^{-2}|(h, t)|_{B_{\alpha}} \cdot |(u, a)|_{B_{\alpha}}\]

for all $(u, a) \in B_{\alpha}$.

The bound on $r_1$ is needed so that we can always assume that $\tau = 1$ in our definition of $F_\alpha$. Finally, we have the following simple estimate.

**Proposition 22 (Estimate of $F_\alpha(0, 0)$).** For small enough $\alpha$,

\[|F_\alpha(0, 0)|_{B_{\alpha}} \leq C\alpha^3.\]

**Proof.** Note that $F_\alpha(0, 0) = (\langle \text{Im} \Omega, \text{Vol} M_\alpha \rangle_{M_\alpha} = \sin \theta$. Recall that $\rho(x) \leq C\delta$ for $x \in M_\alpha \cap B_\delta$ and $\delta = \frac{\alpha}{C\epsilon}$. Then since $\sin \theta$ is supported in $M_\alpha \cap B_\delta$,

\[|\rho^2 \sin \theta| \leq C\delta^2 |\sin \theta| \leq C\alpha^3\]

where the second inequality follows from the bound on $\sin \theta$ from Lemma \[\]

Now we will estimate $|\rho^{2+\beta} \sin \theta|_{\beta}$ by interpolation. As in \[10\], we see that

\[|\rho^{2+\beta} \sin \theta| \leq C\alpha^{2+\beta}.\]
We also have
\[
|\nabla(\rho^{2+\beta}\sin \theta)|_0 \leq |(2 + \beta)\rho^{1+\beta}(\nabla \rho)\sin \theta|_0 + |\rho^{2+\beta}\nabla(\sin \theta)|_0
\leq C\delta^{1+\beta}|\sin \theta|_0 + C\delta^{2+\beta}|\nabla(\sin \theta)|_0
\leq C\alpha^{2+\beta}
\]
where the second line follows from the bound on $\nabla \rho$ and the same reasoning as in [10], and the last line uses the bounds on $\sin \theta$ and $\nabla(\sin \theta)$ from Lemma 6. Combining the two previous inequalities, we see that
\[
|\rho^{2+\beta}\sin \theta|_\beta \leq C\alpha^3.
\]

Finally, let $r = (2C_1C_N)^{-1} = \frac{1}{C}\epsilon^{2+\nu}$, which is less than $r_1$ for sufficiently small $\alpha$. For small enough $\nu$, we have $\frac{\nu}{2(C_1 C_N)} = \frac{1}{C}\epsilon^{2+2\nu} > C\alpha^3 \geq \|F_\alpha(0,0)\|_{\mathcal{B}}$ for sufficiently small $\alpha$. We can now invoke the Inverse Function Theorem to find a solution $F_\alpha(h,t) = 0$ with $\|(h,t)\|_{\mathcal{B}} \leq r$, and by elliptic regularity, $h$ is smooth. Since $\|\nabla h\|_0 \leq C\epsilon^{1+\nu}$, it follows that there exists an embedded special Lagrangian submanifold of $(X,J_t,\omega_t,\Omega_t)$ in a $C\epsilon^{1+\nu}$-neighborhood of $M_\alpha$ for some $t < r$. Finally, since the construction of $M_\alpha$ and $F_\alpha$ can be made to depend smoothly on $\alpha$, and there is a unique solution to $F_\alpha(h,t) = 0$ in $B_r(0,0)$, we can also say that the embedded SLags we constructed, as well as $t$, depend smoothly on $\alpha$. This concludes the proof of the Main Theorem and the Torus Case.

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