A reconstruction formula for an inverse problem for a one-dimensional multilayer medium

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Abstract. We consider a multilayer rod such that waves through this medium are described by the one-dimensional wave equation. We give a reconstruction formula for identifying the physical properties of the rod by a measurement at its one side of the boundary. Our reconstruction formula is based on the explicit solution formula to the mixed problem.

1. Introduction
We consider an inverse boundary value problem for a one-dimensional multilayer rod to identify its physical properties by a measurement at its one side of the boundary. This problem arises from an inverse boundary value problem for a layered elastic plate (see [1], [4]). The inverse problem for a one-dimensional multilayer rod has been investigated by many authors (for example, [1], [4]). In this paper, we approach this inverse problem by obtaining the explicit solution formula to the equations. It is not easy to obtain the explicit solution due to the existence of multiple reflections and transmission at the interfaces of the multilayer medium. We achieve this by using the theory of the boundary value problem for hyperbolic equations (see [5], [2]).

In order to formulate our inverse problem, we introduce several notations. Put \( h_0 := 0 \) and let \( h_k \) be a positive constant and \( h_k > h_{k-1} \). The positive numbers \( a_k \) and \( b_k \) describe the speed of the waves in the \( k \)-th layer and impedance at the \( k \)-th transmission boundary, respectively. An impedance describes the ratio of amplitudes of incident waves and reflected waves at an interface of the layered medium.

Now, we consider the following initial boundary value problems:

\[
\begin{align*}
(\partial_t^2 - a_k^2 \partial_x^2)u(t, x) &= 0, & h_{k-1} < x < h_k, & k = 1, \ldots, N-1, \\
& x > h_{N-1}, & k = N, \\
u(t, x) &= 0, & t < 0, \\
\partial_x u(t, x)|_{x=0+0} &= \varphi(t), \\
u(t, x)|_{x=h_k-0} &= u(t, x)|_{x=h_k+0}, & k = 1, \ldots, N-1, & \text{(continuity of the displacement)} \\
a_k b_k \partial_x u(t, x)|_{x=h_k-0} &= a_{k+1} b_{k+1} \partial_x u(t, x)|_{x=h_k+0}, & k = 1, \ldots, N-1, & \text{(continuity of the stress)}
\end{align*}
\]
or

\[
\begin{cases}
\partial^2_t w^\sharp (t, x) = 0, & h_{k-1} < x < h_k, \quad k = 1, \ldots, N \\
w^\sharp (t, x) \equiv 0, & t < 0, \\
\partial_x w^\sharp (t, x)_{|x=0} = \varphi (t), \\
w^\sharp (t, x)_{|x=h_k} = w^\sharp (t, x)_{|x=h_{k+1}}, & k = 1, \ldots, N - 1, \quad \text{(continuity of the displacement)} \\
a_k b_k \partial_x w^\sharp (t, x)_{|x=h_k} = a_{k+1} b_{k+1} \partial_x w^\sharp (t, x)_{|x=h_{k+1}}, & k = 1, \ldots, N - 1, \\
w^D (t, x)_{|x=h_N} = 0 \quad \text{or} \quad \partial_x w^N (t, x)_{|x=h_N} = 0,
\end{cases}
\]

where ‘\(\sharp\)’ is ‘D’ or ‘N’. The equations (1) and (2) correspond to the case that the medium is a half line (see Figure 1) and the case that the medium is a finite interval (see Figure 2), respectively.

Our main result is as follows:

**Main result.** We can provide an explicit solution formulas in the first layer to the problems (1) (see the equation (3) and Proposition 2) and (2) (see the equation (10) and Proposition 5).

Our inverse problem is to give a reconstruction formula for extracting information on the unknowns:

- \(N\) (the number of layers),
- \(a_k, b_k \ (k \geq 2)\); \(h_k \ (k \geq 1)\)

by assuming that we know

- \(a_1, b_1\),
- an impulse \(\varphi(t)\),
- the observation data \(v(t) (= u(t, 0) \text{ or } w^\sharp (t, 0))\).

By the main result, we will see below in Theorem 12 that we can reconstruct the impedances \(b_{k+1}\) and the ratios \((h_k - h_{k-1})/a_k\) between the width and the speeds of the waves from known data.

We remark that the proof of Theorem 12 is an alternative proof of Theorem 1 in [4]. The first author studied an analogous problem with an input \(\delta(t, x - y) \ (y \in (0, h_1))\) in [3]. This input is
too much, because it can generate all inputs. We remark here that we use only one input with a very weak restriction in Theorem 12.

The rest of this paper is organized as follows. In Section 2, we construct the solution formulas of the equations (1) and (2). In Section 3, we state our reconstruction procedure in a more precise form and give its proof.

2. The solution formulas
In this section, we construct the explicit solution formulas in the first layer of the equations (1) and (2). In order to clarify the dependence of the solutions on the coefficients, we denote the solutions by

\[ u(t, x) = u_N(t, x; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N-1}; \varphi), \]

\[ u^+(t, x) = u_N^+(t, x; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N}; \varphi). \]

In order to avoid stating the compatibility condition on \( \varphi \), we simply assume that \( \text{supp} \varphi \cap (-\infty, 0) = \emptyset. \)

2.1. The solution formula of \( u_1(t, x) \)
The equations for \( u_1(t, x) \) are

\[
\begin{cases}
(\partial_t^2 - a_1^2 \partial_x^2)u_1(t, x) = 0, & t \in \mathbb{R}, \ x > 0, \\
u_1(t, x) \equiv 0, & t < 0, \\
\partial_x u_1(t, x)|_{x=0+} = \varphi(t),
\end{cases}
\]

and its solution is

\[ u_1(t, x; \{a_1\}; \{b_1\}; \cdot; \varphi) = -a_1 \int_{-\infty}^{t-\frac{x}{a_1}} \varphi(s) \, ds. \]

2.2. The solution formula of \( u_N(t, x) \) \((N \geq 2)\)
Based on the argument in [3], we construct the solutions to the problem (1). We first define \( F_k^{(N)}(t, x) = F_k^{(N)}(t, x; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N-1}; \varphi) \) by

\[
\begin{cases}
F_k^{(N)}(t, x) = u_{N-1}(t, x) - u_N(t, x), & h_{k-1} < x < h_k, \ k = 1, \ldots, N-1, \\
F_N^{(N)}(t, x) = u_N(t, x), & x > h_{N-1},
\end{cases}
\]

where

\[ u_{N-1}(t, x) = u_{N-1}(t, x; \{a_j\}_{j=1}^{N-1}; \{b_j\}_{j=1}^{N-1}; \{h_j\}_{j=1}^{N-2}; \varphi), \]

\[ u_N(t, x) = u_N(t, x; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N-1}; \varphi). \]

The function \( F_k^{(N)}(t, x) \) describes the behavior of the waves in the \( k \)-th layer which are affected by the \( N \)-th layer. We rewrite the equations (1) in terms of \( F_k^{(N)} \), and then apply the Fourier-Laplace transformation with respect to \( t \) to the equations (ref. [2], [3]). Then, by the wave equation in (1), we can write

\[ \hat{F}_k^{(N)}(\rho, x) = \Phi_k^{(N)}(\rho) e\left(-\frac{x}{a_k}\right) + \Psi_k^{(N)}(\rho) e\left(\frac{x}{a_k}\right), \ k = 1, \ldots, N-1, \]

\[ \hat{F}_N^{(N)}(\rho, x) = \Phi_N^{(N)}(\rho) e\left(-\frac{x}{a_N}\right), \]
where we take \( \rho = \tau - \im \log(2 + |\tau|) \) (\( m > 0 \) is large enough) and we put \( e(s) := e(s; \rho) := \exp(\im \rho s) \). Likewise \( F^{(N)}_k \), we write

\[
\Phi_k^{(N)}(\rho) = \Phi_k^{(N)}(\rho; \{ a_j \}_{j=1}^N; \{ b_j \}_{j=1}^N; \{ h_j \}_{j=1}^{N-1}; \varphi).
\]

Before we continue constructing the explicit solution formula of \( u_N(t, x) \) for general \( N \), we state the way of obtaining the explicit solution formula of \( u_2(t, x) \) in the first layer in order to explain ideas of the proof, which is same as in [2] and [3]. We want the explicit solution formula of \( F^{(2)}_1(t, x) \). In order to obtain it, we need the explicit formula of \( \Phi^{(2)}_1 \) and \( \Psi^{(2)}_1 \). We substitute (4) and (5) into the Fourier-Laplace transforms of the boundary or transmission conditions in (1) and simplify them. Then we obtain

\[
\begin{pmatrix}
1 & -1 & 0 \\
b_1 e^{\left(-\frac{h_1}{a_1}\right)} & e^{\left(-\frac{h_1}{a_1}\right)} & e^{\left(-\frac{h_2}{a_2}\right)} \\
\end{pmatrix}
\begin{pmatrix}
\Phi^{(2)}_1 \\
\Psi^{(2)}_1 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{a_1}{\im \rho} e^{\left(-\frac{h_1}{a_1}\right)} \hat{\varphi}(\rho) \\
-\frac{a_1 b_1}{\im \rho} e^{\left(-\frac{h_1}{a_1}\right)} \hat{\varphi}(\rho) \\
\end{pmatrix}.
\]

The determinant of the matrix which appears in the left-hand side of the above equation does not vanish because we take \( \rho = \tau - \im \log(2 + |\tau|) \). Hence we can solve this linear equation and we obtain

\[
\Phi^{(2)}_1(\rho) = \Psi^{(2)}_1(\rho) = \frac{b_1 - b_2}{b_1 + b_2} \left( -\frac{2h_1}{a_1} \right) \frac{1}{1 - b_1 - b_2} \left( -\frac{2h_1}{a_1} \right) \frac{a_1}{\im \rho} \hat{\varphi}(\rho)
\]

in particular. We remark that we have

\[
|e^{\left(-\frac{2h_1}{a_1}\right)}| = (2 + |\tau|)^{-2mh_1/a_1} \ll 1
\]

since \( m > 0 \) is large enough. Then we have

\[
\frac{1}{1 - b_1 - b_2} e^{\left(-\frac{2h_1}{a_1}\right)} = \sum_{j=0}^{\infty} \left( \frac{b_1 - b_2}{b_1 + b_2} \right)^j e^{\left(-\frac{2jh_1}{a_1}\right)}.
\]

Hence we have

\[
\hat{F}^{(2)}_1(\rho, x) = \Phi^{(2)}_1(\rho) e^{\left(-\frac{x}{a_1}\right)} + \Psi^{(2)}_1(\rho) e^{\left(\frac{x}{a_1}\right)}
\]

\[
= a_1 \sum_{j=0}^{\infty} \left( \frac{b_1 - b_2}{b_1 + b_2} \right)^j \sum_{\nu = \pm 1} \frac{1}{\im \rho} e^{\left(-\nu \frac{x}{a_1} + 2(j + 1) \frac{h_1}{a_1}\right)} \hat{\varphi}(\rho)
\]

Applying the inverse Fourier-Laplace transformation to it, we have

\[
F^{(2)}_1(t, x) = a_1 \sum_{j=0}^{\infty} \left( \frac{b_1 - b_2}{b_1 + b_2} \right)^j \sum_{\nu = \pm 1} \int_{-\infty}^{t} e^{-\nu \frac{x}{a_1} + 2(j + 1) \frac{h_1}{a_1}} \varphi(s) ds.
\]
The difficulty in obtaining the explicit solution formula of conditions in (1) and simplify them. Then we obtain a linear equation for $\Phi$

where we define

Lemma in the same way as the proofs of Lemmas 1–3 and 6 in [3], we state the key formulas in Lemma 1.

We have

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Now, we continue constructing the explicit solution formula of $u_N(t, x)$ for general $N$. We substitute (4) and (5) into the Fourier-Laplace transforms of the boundary or transmission conditions in (1) and simplify them. Then we obtain a linear equation for $\Phi^{(N)}_k$ and $\Psi^{(N)}_k$:

$\mathcal{Z}_N \left[ \Phi^{(N)}_1, \Psi^{(N)}_1, \Phi^{(N)}_2, \Psi^{(N)}_2, \ldots, \Phi^{(N)}_{N-1}, \Psi^{(N)}_{N-1}, \Phi^{(N)}_N \right]^T = [0, 0, \ldots, 0, K_{N-1}, L_{N-1}]^T$, (7)

where we define

$$K_M \left( \rho; \{a_j\}_{j=1}^M; \{b_j\}_{j=1}^M; \{h_j\}_{j=1}^M; \varphi \right) := \tilde{\alpha}_M \left( \rho, x; \{a_j\}_{j=1}^M; \{b_j\}_{j=1}^M; \{h_j\}_{j=1}^M; \varphi \right) \bigg|_{x=b_M}, \quad (8)$$

$$L_M \left( \rho; \{a_j\}_{j=1}^M; \{b_j\}_{j=1}^M; \{h_j\}_{j=1}^M; \varphi \right) := -\frac{a_M b_M}{\rho p} \partial_x \tilde{\alpha}_M \left( \rho, x; \{a_j\}_{j=1}^M; \{b_j\}_{j=1}^M; \{h_j\}_{j=1}^M; \varphi \right) \bigg|_{x=b_M}, \quad (9)$$

and $\mathcal{Z}_N = \mathcal{Z}_N \left( \rho; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N-1} \right)$ is the $(2N - 1) \times (2N - 1)$ matrix which is defined as Figure 3 and in (7) we write

$K_{N-1}[\text{resp. } L_{N-1}] = K_{N-1}[\text{resp. } L_{N-1}] \left( \rho; \{a_j\}_{j=1}^{N-1}; \{b_j\}_{j=1}^{N-1}; \{h_j\}_{j=1}^{N-1}; \varphi \right)$

for simplicity. It is convenient to define $Z_1$ by $Z_1(\rho; \{a_1\}; \{b_1\}; \cdot) := (-1)$.

Our aim in this section is to express $\Phi^{(N)}_1$ and $\Psi^{(N)}_1$ explicitly. Since we can prove the following lemma in the same way as the proofs of Lemmas 1–3 and 6 in [3], we state the key formulas in the following lemma without details of their proofs, where

$\mathcal{Z}_N = \mathcal{Z}_N \left( \rho; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N-1} \right)$,

$\mathcal{Z}_{N-1} = \mathcal{Z}_{N-1} \left( \rho; \{a_j\}_{j=1}^{N-1}; \{b_j\}_{j=1}^{N-1}; \{h_j\}_{j=1}^{N-2} \right)$,

$K_N[\text{resp. } L_N] = K_N[\text{resp. } L_N] \left( \rho; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^N; \varphi \right)$,

$\Phi^{(N)}_k[\text{resp. } \Psi^{(N)}_k] = \Phi^{(N)}_k[\text{resp. } \Psi^{(N)}_k] \left( \rho; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N-1}; \varphi \right)$.

Lemma 1. We have

$$\det \mathcal{Z}_N = (-1)^N e \left( -\frac{h_{N-1}}{a_N} \right) \mathcal{Z}_N \left( \rho; \{b_j\}_{j=1}^N; \left\{ \frac{h_j - h_{j-1}}{a_j} \right\}_{j=1}^{N-1} \right),$$

$$K_N = \Phi^{(N)}_N e \left( -\frac{h_N}{a_N} \right), \quad L_N = b_N K_N, \quad \Phi^{(N)}_N = (-2)^{N-1} a_1 \frac{1}{i p} \left( \prod_{j=1}^{N-1} b_j \right) \frac{1}{\det \mathcal{Z}_N} \tilde{\varphi}(\rho),$$

$$\Phi^{(N)}_1 = \Psi^{(N)}_1 = -\frac{2^{N-1} a_1}{i p} \frac{b_N - b_{N-1}}{\det \mathcal{Z}_N \det \mathcal{Z}_{N-1}} \left\{ \prod_{j=1}^{N-2} (b_j b_{j+1}) \right\} \tilde{\varphi}(\rho) e \left( -\frac{h_{N-1}}{a_N} \frac{h_{N-1}}{a_{N-1}} \right).$$
Figure 3. The definition of $Z_N$. 

\[ Z_N \left( \rho; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N-1} \right) = \left[ \begin{array}{cccc} 1 & -1 & 0 & \cdots \\ e(-\frac{b_1}{a_1}) & e(\frac{h_1}{a_1}) & -e(\frac{h_2}{a_2}) & -e(\frac{b_2}{a_2}) \\ b_1 e(-\frac{h_1}{a_1}) & -b_1 e(\frac{h_1}{a_1}) & b_2 e(\frac{h_2}{a_2}) & -b_2 e(\frac{h_2}{a_2}) \\ e(\frac{b_2}{a_2}) & e(\frac{b_2}{a_2}) & -e(\frac{b_3}{a_3}) & -e(\frac{b_3}{a_3}) \\ b_2 e(-\frac{b_2}{a_2}) & -b_2 e(\frac{b_2}{a_2}) & b_3 e(-\frac{b_3}{a_3}) & b_3 e(\frac{b_3}{a_3}) \\ \vdots & \vdots & \vdots & \vdots \\ e(-\frac{b_{N-2}}{a_{N-2}}) & e(\frac{h_{N-2}}{a_{N-2}}) & -e(\frac{h_{N-2}}{a_{N-1}}) & -e(\frac{b_{N-2}}{a_{N-1}}) \\ b_{N-2} e(-\frac{h_{N-2}}{a_{N-2}}) & -b_{N-2} e(\frac{h_{N-2}}{a_{N-2}}) & -b_{N-1} e(-\frac{h_{N-2}}{a_{N-1}}) & b_{N-1} e(\frac{h_{N-2}}{a_{N-1}}) \\ e(-\frac{h_{N-1}}{a_{N-1}}) & e(\frac{h_{N-1}}{a_{N-1}}) & -e(\frac{b_{N-1}}{a_{N-1}}) & b_{N} e(-\frac{b_{N-1}}{a_{N-1}}) \\ b_{N-1} e(-\frac{h_{N-1}}{a_{N-1}}) & -b_{N-1} e(\frac{h_{N-1}}{a_{N-1}}) & b_{N} e(-\frac{b_{N-1}}{a_{N-1}}) & \end{array} \right] \]
where we denote
\[
Z_N(\rho; \{b_j\}_{j=1}^N; \{\Theta_j\}_{j=1}^{N-1}) := \sum_{\alpha_k=\pm 1} \alpha_1 \left( \prod_{j=1}^{N-2} (b_j + \alpha_j \alpha_{j+1} b_{j+1}) \right) \left( b_{N-1} + \alpha_{N-1} b_N \right) e \left( \sum_{j=1}^{N-1} \alpha_j \Theta_j \right)
\]
for \( N \geq 2 \), and \( Z_1(\rho; \{b_1\}; \cdot) := 1 \).

Hence, in the same way as the proof of Proposition 7 in [3], we have the following proposition by using the equation (6).

**Proposition 2.** For \( N \geq 2 \),
\[
F_1^{(N)}(t, x; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N-1}; \phi) = f^{(N)}(t, x; \{b_j\}_{j=1}^N; \{\Theta_j\}_{j=1}^{N-1}; a_1; \phi)
\]
holds, where
\[
f^{(N)}(t, x; \{b_j\}_{j=1}^N; \{\Theta_j\}_{j=1}^{N-1}; a_1; \phi) := a_1 \sum_{0 \leq m_k < \infty} \psi_N \left( \{m_j\}_{j=1}^{N-1}; \{b_j\}_{j=1}^N \right) \sum_{\nu=\pm 1} \int_{-\infty}^t \left( \nu \frac{\tau^2}{\pi^2} + 2 \sum_{j=1}^{N-1} (m_{j+1} \Theta_j) \right) \varphi(s) ds
\]
and \( \psi_N \) is given in Appendix.

### 2.3. The solution formulas of \( w^2(t, x) \)

Based on the argument in [3], we also construct the solution to the problem (2). We define \( G_k^{(N)}(t, x) = G_k^{(N)}(t, x; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N-1}; \phi) \) by
\[
G_k^{(N)}(t, x) = u_N(t, x) - w_N^2(t, x), \quad h_{k-1} < x < h_k, \quad k = 1, \ldots, N, \tag{10}
\]
where
\[
u(t, x) = u_N(t, x; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N-1}; \phi),
\]
\[
\nu_N^2(t, x) = w_N^2(t, x; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^{N-1}; \phi).
\]

Rewriting the equations (2) in terms of \( G_k^{(N)} \) and applying the Fourier-Laplace transformation with respect to the equations as in Section 2.2, we can write
\[
\hat{G}_k^{(N)}(\rho, x) = \Phi_k^{(N)}(\rho) e \left( -\frac{x}{a_k} \right) + \Psi_k^{(N)}(\rho) e \left( \frac{x}{a_k} \right) \quad \text{(We take } \rho = \tau - i m \log(2 + |\tau|)).
\]
and obtain a linear equation for \( \Phi_k^{(N)} \) and \( \Psi_k^{(N)} \):
\[
\mathcal{X}_N^2 \left[ \Phi_1^{(N)}, \Phi_2^{(N)}, \Psi_1^{(N)}, \Psi_2^{(N)}, \ldots, \Phi_N^{(N)}, \Psi_N^{(N)} \right]^T = [0, 0, \ldots, 0, K_N]^T
\]
by using also \( L_N = b_N K_N \) (proved in Lemma 1) in the case that ‘\( \ast \)’ is ‘\( N \)’, where \( \mathcal{X}_N^2 = \mathcal{X}_N^2 \left( \rho; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^N \right) \) is a \( 2N \times 2N \) matrix which is defined as Figure 4 and \( K_N = K_N \left( \rho; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^N; \phi \right) \) is defined by (8). We first express \( \det \mathcal{X}_N^2 \) explicitly.
\begin{equation}
\Delta_N \left( p \left( a_j \right) \right) = \left( \begin{array}{cccc}
\frac{1}{a_j} & \frac{1}{a_j} & \cdots & \frac{1}{a_j} \\
\frac{1}{a_j - 1} & \frac{1}{a_j - 1} & \cdots & \frac{1}{a_j - 1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{a_j - N + 1} & \frac{1}{a_j - N + 1} & \cdots & \frac{1}{a_j - N + 1}
\end{array} \right)

\end{equation}

where \( \mu = \left\{ \begin{array}{ll}
1 & \text{if } \sharp \in D \\
-1 & \text{if } \sharp \in N
\end{array} \right. \)

\text{Figure 4. The definition of } \Delta_N.
Lemma 3. For $N \geq 1$, we have

\begin{align*}
\det \mathcal{X}_N^D = (-1)^{N-1} \sum_{\alpha_k = \pm 1}^{(k=1, \ldots, N)} \alpha_1 \alpha_N \left\{ \prod_{j=1}^{N-1} (b_j + \alpha_j \alpha_{j+1} b_{j+1}) \right\} e \left( \sum_{j=1}^N \frac{\alpha_j h_j - h_{j-1}}{a_j} \right), \quad (11) \\
\det \mathcal{X}_N^N = (-1)^N \sum_{\alpha_k = \pm 1}^{(k=1, \ldots, N)} \alpha_1 \left\{ \prod_{j=1}^{N-1} (b_j + \alpha_j \alpha_{j+1} b_{j+1}) \right\} e \left( \sum_{j=1}^N \frac{\alpha_j h_j - h_{j-1}}{a_j} \right). \quad (12)
\end{align*}

Proof. We prove this lemma by induction on $N$. By a direct computation, we can show the equations (11) and (12) for $N = 1$. Then we assume that the equations (11) and (12) for $N(\geq 1)$ hold, and we show the equations (11) and (12) for $N + 1$. We first expand $\det \mathcal{X}_{N+1}^D$ along the $(2N + 2)$-nd row, and expand them along the $(2N + 1)$-st column. Then we have

\begin{align*}
\det \mathcal{X}_{N+1}^D (\rho; \{a_j\}_{j=1}^{N+1}; \{b_j\}_{j=1}^{N+1}; \{h_j\}_{j=1}^{N+1}) &= -e \left( -\frac{h_{N+1} - h_N}{a_{N+1}} \right) (b_N \det \mathcal{X}_N^N + b_{N+1} \det \mathcal{X}_N^D) + \mu e \left( \frac{h_{N+1} - h_N}{a_{N+1}} \right) (b_N \det \mathcal{X}_N^N - b_{N+1} \det \mathcal{X}_N^D) \\
&= b_{N+1} \det \mathcal{X}_N^D \left\{ -\mu \left( \frac{h_{N+1} - h_N}{a_{N+1}} \right) - e \left( -\frac{h_{N+1} - h_N}{a_{N+1}} \right) \right\} + b_N \det \mathcal{X}_N^N \left\{ \mu \left( \frac{h_{N+1} - h_N}{a_{N+1}} \right) - e \left( -\frac{h_{N+1} - h_N}{a_{N+1}} \right) \right\} \\
&= b_{N+1} (-1)^{N-1} \sum_{\alpha_k = \pm 1}^{(k=1, \ldots, N)} \alpha_1 \alpha_N \left\{ \prod_{j=1}^{N-1} (b_j + \alpha_j \alpha_{j+1} b_{j+1}) \right\} e \left( \sum_{j=1}^N \frac{\alpha_j h_j - h_{j-1}}{a_j} \right) \times \left\{ -\mu \left( \frac{h_{N+1} - h_N}{a_{N+1}} \right) - e \left( -\frac{h_{N+1} - h_N}{a_{N+1}} \right) \right\} \\
&\quad + b_N (-1)^N \sum_{\alpha_k = \pm 1}^{(k=1, \ldots, N)} \alpha_1 \left\{ \prod_{j=1}^{N-1} (b_j + \alpha_j \alpha_{j+1} b_{j+1}) \right\} e \left( \sum_{j=1}^N \frac{\alpha_j h_j - h_{j-1}}{a_j} \right) \times \left\{ \mu \left( \frac{h_{N+1} - h_N}{a_{N+1}} \right) - e \left( -\frac{h_{N+1} - h_N}{a_{N+1}} \right) \right\} \\
&= (-1)^N \sum_{\alpha_k = \pm 1}^{(k=1, \ldots, N)} \alpha_1 \left\{ \prod_{j=1}^{N-1} (b_j + \alpha_j \alpha_{j+1} b_{j+1}) \right\} e \left( \sum_{j=1}^N \frac{\alpha_j h_j - h_{j-1}}{a_j} \right) \times \left\{ \mu (b_N + \alpha_N b_{N+1}) e \left( \frac{h_{N+1} - h_N}{a_{N+1}} \right) - (b_N - \alpha_N b_{N+1}) e \left( -\frac{h_{N+1} - h_N}{a_{N+1}} \right) \right\},
\end{align*}

where $\mu; = 1$ if ‘$+$’ is ‘$D$’ and $\mu := -1$ if ‘$+$’ is ‘$N$’, and

\begin{align*}
\mathcal{X}_N^D[\text{resp. } \mathcal{X}_N^N] = \mathcal{X}_N^D[\text{resp. } \mathcal{X}_N^N](\rho; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^N)
\end{align*}

for simplicity, and we use the inductive hypothesis at (\ref{inductive}). This means the equations (11) and (12) for $N + 1$. \qed
On the other hand, we obtain the following lemma in the same way as the proof of Lemma 6 in [3].

Lemma 4. For $N \geq 1$, we have

$$
\Phi_1^{\sharp(N)}(\rho) = \Psi_1^{\sharp(N)}(\rho) = \frac{2^{2N-2}a_1}{i\rho} \frac{1}{\det Z_N(\rho) \det X_N^{\sharp}(\rho)} \left\{ \prod_{j=1}^{N-1} (b_j b_{j+1}) \right\} \tilde{\varphi}(\rho) e \left( -\frac{h_N}{a_N} \right).
$$

Proof. This follows from

$$
\Phi_1^{\sharp(N)}(\rho) = \Psi_1^{\sharp(N)}(\rho) = \frac{1}{\det X_N^{\sharp}(\rho)} (-2)^{N-1} \left( \prod_{j=1}^{N-1} b_{j+1} \right) K_N(\rho)
$$

and Lemma 1.

Hence we can obtain the explicit solution formula of $G_1^{\sharp(N)}$ as the following proposition.

Proposition 5. For $N \geq 1$,

$$
G_1^{\sharp(N)}(t, x; \{a_j\}_{j=1}^N; \{b_j\}_{j=1}^N; \{h_j\}_{j=1}^N; \varphi) = g^{\sharp(N)}(t, x; \{b_j\}_{j=1}^N; \{\frac{h_j - h_{j-1}}{a_j}\}_{j=1}^N; a_4; \varphi)
$$

holds, where

$$
g^{\sharp(N)}(t, x; \{b_j\}_{j=1}^N; \{\Theta_j\}_{j=1}^N; a_1; \varphi)
$$

$$
:= a_1 \sum_{0 \leq m_k < \infty \atop (k=1, \ldots, N)} \tilde{\psi}_{N+1}^{1} \left( \{m_j\}_{j=1}^N; \{b_j\}_{j=1}^N \right) \left( -1 \right)^{m_{N+1}} \sum_{\nu = \pm 1} \int_{-\infty}^{t-\left( \nu \frac{\rho}{\pi} + 2 \sum_{j=1}^{N} (m_j + 1) \Theta_j \right)} \varphi(s) ds,
$$

$$
g^{\sharp(N)}(t, x; \{b_j\}_{j=1}^N; \{\Theta_j\}_{j=1}^N; a_1; \varphi)
$$

$$
:= a_1 \sum_{0 \leq m_k < \infty \atop (k=1, \ldots, N)} \tilde{\psi}_{N+1}^{1} \left( \{m_j\}_{j=1}^N; \{b_j\}_{j=1}^N \right) \sum_{\nu = \pm 1} \int_{-\infty}^{t-\left( \nu \frac{\rho}{\pi} + 2 \sum_{j=1}^{N} (m_j + 1) \Theta_j \right)} \varphi(s) ds
$$

and $\tilde{\psi}_p$ is given in Appendix.

Proof. By using Lemma 4, the proof can be done in the same way as the proof of Proposition 2.

We remark that there is a formal argument to reduce the finite interval case to the half line case.

Remark 6. We have

$$
G_1^{\sharp(N)}(t, x) = F_1^{(N+1)}(t, x)\big|_{b_{N+1} = -\infty}, \quad G_1^{\sharp(N)}(t, x) = F_1^{(N+1)}(t, x)\big|_{b_{N+1} = 0}
$$
3. The reconstruction procedure and its proof

In this section, we state the reconstruction procedure and prove it. We first discuss the behavior of the functions $f^{(p)}(t, 0)$ and $g^{(N)}(t, 0)$ near $t = 0$ in the following lemmas.

**Lemma 7.** Let $N \geq k + 1 \geq 2$. Let $\Theta_j > 0$ ($j = 1, \ldots, N - 1$). Suppose that $\varphi \in L^1(-\infty, T)$ satisfies supp $\varphi \cap (-\infty, 0) = \emptyset$. Then, we have

$$
\sum_{p=k+1}^{N} f^{(p)}(t, 0; \{b_j\}_{j=1}^{N}; \{\Theta_j\}_{j=1}^{p-1}; a_1; \varphi)
$$

$$
= 2^{k-1}a_1 \left\{ \prod_{j=1}^{k-1} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} \right\} \frac{b_k - b_{k+1}}{b_k + b_{k+1}} \int_{-\infty}^{t} \varphi(s) \, ds,
$$

where

$$
\delta_k^{(N)} := 2 \min_{1 \leq j \leq \min\{k+1, N-1\}} \Theta_j.
$$

**Proof.** This follows from

$$
f^{(p)}(t, 0; \{b_j\}_{j=1}^{N}; \{\Theta_j\}_{j=1}^{p-1}; a_1; \varphi)
$$

$$
= 2^{2p-3}a_1 \left\{ \prod_{j=1}^{p-2} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} \right\} \frac{b_{p-1} - b_p}{b_{p-1} + b_p} \int_{-\infty}^{t} \varphi(s) \, ds,
$$

where

$$
2 \sum_{j=1}^{p-1} \Theta_j < t < 2 \sum_{j=1}^{p-1} \Theta_j + 2 \min_{1 \leq j \leq p-1} \Theta_j \text{ and } t < T.
$$
for \( p \geq 2 \).

In the same way as the proof of Lemma 7, we have the next two lemmas.

**Lemma 8.** Let \( \Theta_j > 0 \) (\( j = 1, \ldots, N \)). Suppose that \( \varphi \in L^1(\infty, T) \) satisfies \( \text{supp} \varphi \cap (-\infty, 0) = \emptyset \). Then, we have

\[
\Theta_j > 0, \quad 0 \leq t \leq 2 \sum_{j=1}^{N} \Theta_j \text{ and } t < T,
\]

\[
-2^{2N-1} a_1 \left\{ \prod_{j=1}^{N-1} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} \right\} \int_{\infty}^{t-2} \sum_{j=1}^{N} \Theta_j \varphi(s) ds,
\]

\[
2 \sum_{j=1}^{N} \Theta_j < t < 2 \sum_{j=1}^{N} \Theta_j + 2 \min_{1 \leq j \leq N} \Theta_j, \quad t < T, \quad \text{and } \varphi = \text{D},
\]

\[
2 \sum_{j=1}^{N} \Theta_j < t < 2 \sum_{j=1}^{N} \Theta_j + 2 \min_{1 \leq j \leq N} \Theta_j, \quad t < T, \quad \text{and } \varphi = \text{N}.
\]

**Lemma 9.** Let \( N \geq k + 1 \geq 2 \). Let \( \Theta_j > 0 \) (\( j = 1, \ldots, N \)). Suppose that \( \varphi \in L^1(\infty, T) \) satisfies \( \text{supp} \varphi \cap (-\infty, 0) = \emptyset \). Then, we have

\[
\sum_{p=k+1}^{N} f^{(p)}(t, 0; \{b_j\}_{j=1}^{p}; \{\Theta_j\}_{j=1}^{p}; a_1; \varphi) + g^{(N)}(t, 0; \{b_j\}_{j=1}^{N}; \{\Theta_j\}_{j=1}^{N}; a_1; \varphi)
\]

\[
= \begin{cases} 
0, & 0 \leq t \leq 2 \sum_{j=1}^{k} \Theta_j \text{ and } t < T, \\
2^{2k-1} a_1 \left\{ \prod_{j=1}^{k-1} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} \right\} \frac{b_k - b_{k+1}}{b_k + b_{k+1}} \int_{\infty}^{t-2} \sum_{j=1}^{k} \Theta_j \varphi(s) ds, & 2 \sum_{j=1}^{k} \Theta_j < t < 2 \sum_{j=1}^{k} \Theta_j + 2 \min_{1 \leq j \leq k+1} \Theta_j \text{ and } t < T.
\end{cases}
\]

Now, we state the propositions which are the keys to the proof of our reconstruction procedure.

**Proposition 10.** Let \( N \geq k + 1 \geq 2 \). Let \( \Theta_j > 0 \). Suppose \( b_k \neq b_{k+1} \). Let \( T > 0 \) and we suppose that \( \varphi \in L^1(\infty, T) \) satisfies \( \text{supp} \varphi \subset [0, T] \) and \( \int_{\infty}^{\infty} \varphi(s) ds \neq 0 \) \((\varepsilon \in (0, \varepsilon_0))\) for some \( \varepsilon_0 > 0 \). Put

\[
\tilde{v}(t) := \sum_{p=k+1}^{N} f^{(p)}(t, 0; \{b_j\}_{j=1}^{p}; \{\Theta_j\}_{j=1}^{p}; a_1; \varphi)
\]

or

\[
\tilde{v}(t) := \sum_{p=k+1}^{N} f^{(p)}(t, 0; \{b_j\}_{j=1}^{p}; \{\Theta_j\}_{j=1}^{p}; a_1; \varphi) + g^{(N)}(t, 0; \{b_j\}_{j=1}^{N}; \{\Theta_j\}_{j=1}^{N}; a_1; \varphi).
\]

Then, the following holds:

\[\text{12}\]
We obtain this proposition from Lemma 8.

Proof. We obtain this proposition from Lemmas 7 and 9. Proof.

Then, we have the followings:

- If \( \tilde{v}(t) \equiv 0 \) on \([0, T]\) then
  \[
  \Theta_k \geq \frac{T}{2} - \sum_{j=1}^{k-1} \Theta_j.
  \]

- Assume that \( \tilde{v}(t) \not\equiv 0 \) on \([0, T]\). Put \( t_* := \inf\{t \in [0, T) : \tilde{v}(t) \neq 0\} \). Then,
  \[
  \Theta_k = \frac{t_*}{2} - \sum_{j=1}^{k-1} \Theta_j,
  \]

  Then, we have the followings:

  \[
  b_{k+1} = \frac{2^{2k-1}a_1 \prod_{j=1}^{k-1} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} - \frac{\tilde{v}(t)}{\int_{-\infty}^{t_*} \varphi(s) \, ds} b_k}{2^{2k-1}a_1 \prod_{j=1}^{k-1} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} + \frac{\tilde{v}(t)}{\int_{-\infty}^{t_*} \varphi(s) \, ds}}, \quad t_* < t < t_* + \delta_*,
  \]

  where

  \[
  \delta_* := \begin{cases} \min\{2\Theta_j (j = 1, \ldots, N - 1), \varepsilon_0\}, & \text{if } \tilde{v}(t) = f^{(N)}(t, 0), \\
  \min\{2\Theta_j (j = 1, \ldots, k + 1), \varepsilon_0\}, & \text{otherwise}. \end{cases}
  \]

  In particular

  \[
  \tilde{v}(t) \neq \pm 2^{2k-1}a_1 \left\{ \prod_{j=1}^{k-1} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} \right\} \int_{-\infty}^{t_*} \varphi(s) \, ds, \quad t_* < t < t_* + \delta_*.
  \]

Proof. We obtain this proposition from Lemmas 7 and 9.

Proposition 11. Let \( T > 0 \). Let \( \Theta_j > 0 \) (\( j = 1, \ldots, N \)). Suppose that \( \varphi \in L^1(-\infty, T) \) satisfies \( \supp \varphi \subset [0, T) \) and \( \int_{-\infty}^{\varepsilon} \varphi(s) \, ds \neq 0 \) (\( \varepsilon \in (0, \varepsilon_0) \)) for some \( \varepsilon_0 > 0 \). Put

  \[
  \tilde{v}(t) := g^{(N)}(t, 0; \{b_j\}_{j=1}^{N}; \{\Theta_j\}_{j=1}^{N}; a_1; \varphi).
  \]

Then, we have the followings:

- If \( \tilde{v}(t) \equiv 0 \) on \([0, T]\), then
  \[
  \Theta_N \geq \frac{T}{2} - \sum_{j=1}^{N-1} \Theta_j.
  \]

- Assume that \( \tilde{v}(t) \neq 0 \) on \([0, T]\). Put \( t_* := \inf\{t \in [0, T) : \tilde{v}(t) \neq 0\} \). Then,
  \[
  \Theta_N = \frac{t_*}{2} - \sum_{j=1}^{N-1} \Theta_j,
  \]

  where \( \nu := -1 \) if ‘\( \varphi \)’ is ‘D’ and \( \nu := 1 \) if ‘\( \varphi \)’ is ‘N’.

Proof. We obtain this proposition from Lemma 8.
Now, we state the reconstruction procedure precisely.

**Theorem 12.** Assume that \( b_j \neq b_{j+1} \ (j = 1, \ldots, N - 1) \). Let \( T > 0 \) and we suppose that the positive constants \( a_1 \) and \( b_1 \) are known. We also assume that a given impulse \( \varphi \in L^1(-\infty, T) \) satisfies \( \text{supp } \varphi \subset [0, T) \) and \( \int_{-\infty}^{T} \varphi(s) \, ds \neq 0 \) for some \( \varepsilon_0 > 0 \). Then, observing \( v(t) \) on \([0, T]\) given by \( v(t) \) or \( w(t) \) \((\# \text{ is } 'D' \text{ or } 'N')\), \( b_{k+1} \) and \( (h_k - h_{k-1})/a_k \) are reconstructed by the following procedure:

1. **The first step:** Put \( v_1(t) := -a_1 \int_{-\infty}^{t} \varphi(s) \, ds - v(t) \).
   - **The \((k+1)\)-st step** \((k = 1, 2, \ldots)\):
     - If \( v_k(t) \equiv 0 \) on \([0, T]\) then we finish the procedure. Put \( N_0 := k \) (that is, \( v_{N_0}(t) \equiv 0 \) on \([0, T]\)). By the above procedure, we have reconstructed \( b_{k+1} \) and \( (h_l - h_{l-1})/a_l \) for \( l = 1, \ldots, N_0 - 1 \). Then we have either
       \[
       v(t) = u(t, 0) \text{ and } N_0 = N
       \]
   or
       \[
       N_0 < N \text{ and } h_{N_0} - h_{N_0-1} \geq \frac{T}{2} - \sum_{j=1}^{N_0-1} \frac{h_j - h_{j-1}}{a_j}.
       \]

2. Assume \( v_k(t) \neq 0 \) on \([0, T]\) and put \( t_k := \inf \{ t \in [0, T] : v_k(t) \neq 0 \} \). Then, the ratio \( (h_k - h_{k-1})/a_k \) is given by
   \[
   \frac{h_k - h_{k-1}}{a_k} = \frac{t_k}{2} \sum_{j=1}^{k-1} \frac{h_j - h_{j-1}}{a_j} = \frac{t_k}{2} - \frac{t_{k-1}}{2}, \text{ where put } t_0 := 0,
   \]
   and
   - if
     \[
     \lim_{t \to t_k} \frac{v_k(t)}{\int_{-\infty}^{t} \varphi(s) \, ds} = -2^{k-1} a_1 \prod_{j=1}^{k-1} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2}
     \]
     then \( k = N \) and \( v(t) = w^D(t, 0) \),
   - if
     \[
     \lim_{t \to t_k} \frac{v_k(t)}{\int_{-\infty}^{t} \varphi(s) \, ds} = 2^{k-1} a_1 \prod_{j=1}^{k-1} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2}
     \]
     then \( k = N \) and \( v(t) = w^N(t, 0) \),
   - otherwise we have \( k < N \) and \( b_{k+1} \) is reconstructed by
     \[
     b_{k+1} = \frac{2^{k-1} a_1 \prod_{j=1}^{k-1} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} - \lim_{t \to t_k} \frac{v_k(t)}{\int_{-\infty}^{t} \varphi(s) \, ds}}{2^{k-1} a_1 \prod_{j=1}^{k-1} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} + \lim_{t \to t_k} \frac{v_k(t)}{\int_{-\infty}^{t} \varphi(s) \, ds}} b_k.
     \]

We define
\[
v_{k+1}(t) := v_k(t) - 2a_1 \sum_{0 \leq m_j < \infty} \psi_{k+1}(\{m_j\}_{j=1}^{k}; \{b_j\}_{j=1}^{k+1}) \int_{-\infty}^{t} \sum_{j=1}^{k+1} \frac{h_j - h_{j-1}}{a_j} \varphi(s) \, ds \]
   and go the next step, where \( \psi_p \) is defined in Appendix.
Proof. We remark that we have \( u_1(t, 0) = -a_1 \int_{-\infty}^{t} \varphi(s) \, ds \). By the definitions of \( F_{1}^{(N)} \) and \( G_{1}^{(N)} \), we have
\[
\begin{align*}
  u_N(t, 0) &= -a_1 \int_{-\infty}^{t} \varphi(s) \, ds - \sum_{p=2}^{N} f^{(p)} \left( t, 0; \{ b_j \}_{j=1}^{p}; \left\{ \frac{h_j - h_{j-1}}{a_j} \right\}_{j=1}^{p-1}; a_1; \varphi \right), \\
  w_N^2(t, 0) &= -a_1 \int_{-\infty}^{t} \varphi(s) \, ds - \sum_{p=2}^{N} f^{(p)} \left( t, 0; \{ b_j \}_{j=1}^{p}; \left\{ \frac{h_j - h_{j-1}}{a_j} \right\}_{j=1}^{p-1}; a_1; \varphi \right) \\
  &\quad - g^{(N)} \left( t, 0; \{ b_j \}_{j=1}^{N}; \left\{ \frac{h_j - h_{j-1}}{a_j} \right\}_{j=1}^{N}; a_1; \varphi \right).
\end{align*}
\]
Hence we obtain this theorem from Propositions 10 and 11.

Remark 13. Although we take the limit in order to reconstruct \( b_{k+1} \) as (13) in Theorem 12, we have the equality without taking the limit:
\[
\begin{align*}
  b_{k+1} &= \frac{2^{2k-1}a_1 \prod_{j=1}^{k-1} b_j b_{j+1}}{2^{2k-1}a_1 \prod_{j=1}^{k-1} b_j b_{j+1}} - \frac{v_k(t)}{\int_{-\infty}^{t-t_k} \varphi(s) \, ds} b_k \\
  &= \frac{2^{2k-1}a_1 \prod_{j=1}^{k-1} b_j b_{j+1}}{2^{2k-1}a_1 \prod_{j=1}^{k-1} b_j b_{j+1}} + \frac{v_k(t)}{\int_{-\infty}^{t-t_k} \varphi(s) \, ds} b_k.
\end{align*}
\]
for \( t_k < t < t_k + \delta_k \), where we define \( \delta_k \) by
\[
\delta_k := \begin{cases} 
  \min \left\{ \frac{h_j - h_{j-1}}{a_j} (1 \leq j \leq N - 1), \, \varepsilon_0 \right\}, & \text{if } N = k + 1 \text{ and } v(t) = u(t, 0), \\
  \min \left\{ \frac{h_j - h_{j-1}}{a_j} (1 \leq j \leq k + 1), \, \varepsilon_0 \right\}, & \text{otherwise}.
\end{cases}
\]
However, the positive constant \( \delta_k \) depends on \( (h_{k+1} - h_k)/a_k \), which is not reconstructed at the \((k + 1)\)-st step yet. This is the reason why we take the limit in (13).

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Appendix
In this appendix, we state the definition of \( \psi_N \) and \( \tilde{\psi}_N \). We remark that \( \psi_N \) is the same as the one in [3, Proposition 7]. We define \( \tilde{\psi}_N \) by
\[
\tilde{\psi}_2(\{m_1\}; \{b_1\}) := 1
\]
for \( N = 2 \) and for \( N \geq 3 \):
\[
\tilde{\psi}_N \left( \{m_j\}_{j=1}^{N-1}; \{b_j\}_{j=1}^{N-1} \right)
\]
\[ := \sum_{\{j_\alpha\}_{\alpha \in C_N}, \{i_\beta\}_{\beta \in A_{N-1}} \in G_N} 2^{2N-4} \]
\[ \times (-1)^{\text{#}(k : \alpha_k = -1)j_\alpha} \times \sum_{\beta \in A_{N-1}} (1-\text{#}(k : \beta_k = -1)i_\beta) \]
\[ \times \left( \sum_{k=1}^{N-1} m_k + \sum_{\alpha \in C_N} (1-\text{#}(k : \alpha_k = -1))j_\alpha - \sum_{\beta \in A_{N-1}} \text{#}(k : \beta_k = -1)i_\beta \right) \]
\[ \times \left( \sum_{\beta \in A_{N-1}} i_\beta \right) \]
\[ \times \prod_{\alpha \in C_N} (j_\alpha!) \prod_{\beta \in A_{N-1}} (i_\beta!) \]
\[ \times \prod_{J=1}^{N-3} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} \left( \frac{b_j - b_{j+1}}{b_j + b_{j+1}} \right)^{m_j + m_{j+1} - 2} \]
\[ \times \sum_{\beta \in A_{N-1}} \left( \sum_{\alpha \in C_N} j_\alpha \right) \]
\[ \times \frac{b_{N-2} b_{N-1}}{(b_{N-2} + b_{N-1})^2} \left( \frac{b_{N-2} - b_{N-1}}{b_{N-2} + b_{N-1}} \right)^{m_{N-2} + m_{N-1} - 2} \sum_{\alpha \in C_{N-1}} \frac{1}{j_\alpha}. \]

Here, we put
\[ A_N := \{ \alpha = (\alpha_1, \ldots, \alpha_{N-1}) : \alpha_k = \pm 1, \alpha \neq (1, 1, \ldots, 1) \}, \]
\[ B_N := \{ \alpha \in A_N : \text{#}(k : \alpha_k = -1) = 1 \}, \]
\[ C_N := A_N \setminus B_N, \]
\[ A_{(k_1, \ldots, k_v)^\pm} := \{ \alpha \in A_N : \alpha_{k_1} = \cdots = \alpha_{k_v} = \pm 1 \}, \]
\[ C_{(k_1, \ldots, k_v)^\pm} := \{ \alpha \in C_N : \alpha_{k_1} = \cdots = \alpha_{k_v} = \pm 1 \}, \]
\[ G_N = G_N(m_1, \ldots, m_{N-1}) \]
\[ := \left\{ \{j_\alpha\}_{\alpha \in C_N}, \{i_\beta\}_{\beta \in A_{N-1}} : \right. \]
\[ \left. \begin{array}{l}
  j_\alpha \geq 0, \quad i_\beta \geq 0, \\
  \sum_{\alpha \in C_N} j_\alpha + \sum_{\beta \in A_{N-1}} i_\beta \leq m_k (1 \leq k \leq N-2), \\
  \sum_{\alpha \in C_{N-1}} j_\alpha \leq m_{N-1}. 
\end{array} \right\} \]

Then, we define \( \psi_N \) by
\[ \psi_N \left( \{m_j\}_{j=1}^{N-1}; \{b_j\}_{j=1}^N \right) := \tilde{\psi}_N \left( \{m_j\}_{j=1}^{N-1}; \{b_j\}_{j=1}^N \right) \left( \frac{b_{N-1} - b_N}{b_{N-1} + b_N} \right)^{m_{N-1} + 1}. \]

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