Trainable Nonlinear Reaction Diffusion: A Flexible Framework for Fast and Effective Image Restoration

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Abstract—Image restoration is a long-standing problem in low-level computer vision with many interesting applications. We describe a flexible learning framework to obtain simple but effective models for various image restoration problems. The proposed approach is based on the concept of nonlinear reaction diffusion, but we extend conventional nonlinear reaction diffusion models by highly parametrized linear filters as well as highly parametrized influence functions. In contrast to previous nonlinear diffusion models, all the parameters, including the filters and the influence functions, are learned from training data through a loss based approach. We call this approach TNRD – Trainable Nonlinear Reaction Diffusion. The TNRD approach is applicable for a variety of image restoration tasks by incorporating appropriate reaction force. We demonstrate its capabilities with three representative applications, Gaussian image denoising, single image super resolution and JPEG deblocking. Experiments show that our trained nonlinear diffusion models largely benefit from the training of the parameters and finally lead to the best reported performance on common test datasets with respect to the tested applications. Our trained models retain the structural simplicity of diffusion models and take only a small number of steps, thus are highly efficient. Moreover, they are also well-suited for parallel computation on GPUs, which makes the inference procedure extremely fast.

Index Terms—nonlinear reaction diffusion, loss specific training, image denoising, image super resolution, JPEG deblocking

1 INTRODUCTION

Image restoration is the process of estimating uncorrupted images from noisy or blurred ones. It is one of the most fundamental operations in image processing, video processing, and low-level computer vision. For several decades, image restoration remains an active research topic and hence new approaches are constantly emerging. There exists a huge amount of literature addressing the topic of image restoration problems, see for example [39] for a survey.

In recent years, the predominant approaches for image restoration are non-local methods based on patch modeling, for example, image denoising with (i) Gaussian noise [38], [26], [15], [44], (ii) multiplicative noise [14], or (iii) Poisson noise [24], image super resolution [45], image deconvolution [20], etc. Generally speaking, most state-of-the-art techniques mainly concentrate on achieving utmost image restoration quality, with little consideration on the computational efficiency, e.g., [38], [26], [45], despite the fact that it is a critical factor for real applications. However, there are a few exceptions. For example, there are two notable exceptions for the task of Gaussian denoising, BM3D [15] and the recently proposed Cascade of Shrinkage Fields (CSF) [49] model, which simultaneously offer high efficiency and high image restoration quality.

It is well-known that BM3D is a highly engineered Gaussian image denoising algorithm. It involves a block matching process, which is challenging for parallel computation on GPUs, alluding to the fact that it is not straightforward to accelerate BM3D algorithm on parallel architectures. In contrast, the recently proposed CSF model offers high levels of parallelism, making it well suited for GPU implementation, thus owning high computational efficiency.

In this paper, we propose a flexible learning framework to train fast and effective models for a variety of image restoration problems. Our approach is based on the concept of nonlinear diffusion with a reaction force. The trained nonlinear reaction diffusion models preserve the structural simplicity of diffusion-based approaches. As a consequence, it is straightforward to implement the corresponding algorithms on GPUs for parallel computation.

1.1 Previous nonlinear diffusion models

Among the approaches to tackle the problem of image restoration, nonlinear diffusion [43], [52] defines a class of efficient approaches, as each diffusion step merely contains the convolution operation with a few linear filters. A nonlinear diffusion process usually corresponds to a certain Partial Differential Equation (PDE) formulation. In the 1990s, PDEs once led to an entire new field in image processing and computer vision and a large amount of publications appeared, as PDE-based methods are mathematically sound techniques. However, this type of methods fell out of favor in recent years mainly because of limited performance. Unfortunately, the image restoration quality of diffusion based
approaches is usually far away from state-of-the-art algorithms for specific tasks, even though many improvements [28], [17], [27].

We give a brief review of nonlinear diffusion based approaches and then introduce our proposed diffusion model. In the seminal work [43], Perona and Malik (PM) demonstrated that nonlinear anisotropic diffusion models yield very impressive results for image processing. This gave rise to many revised models with various formulations. A notable variant is the so-called biased anisotropic diffusion (also known as reaction diffusion) proposed by Nordström [41], which introduces a bias term (forcing term) to free the user from the difficulty of specifying an appropriate stopping time for the PM diffusion process. This additional term reacts against the strict smoothing effect of the pure PM diffusion, therefore resulting in a nontrivial steady-state.

Subsequent works consider modification to the diffusion term or the reaction term for the reaction diffusion model [21], [13], [1], e.g., Acton et al.[1] exploited a more complicated reaction term to enhance oriented textures; [4] proposed to replace the ordinary diffusion term with a flow equation based on mean curvature. A notable work is forward and backward diffusion process proposed by Gilboa et al.[23], which incorporates explicit inverse diffusion with negative diffusivity coefficient by carefully choosing the diffusivity function. The resulting diffusion processes can adaptively switch between forward and backward diffusion processes. In a latter work [53], the theoretical foundations for discrete forward-and-backward diffusion filtering were investigated. Researchers also propose to exploit higher-order nonlinear diffusion filtering, which involves larger linear filters, e.g., fourth-order diffusion models [28], [17], [27]. Meanwhile, theoretical properties about the stability and local feature enhancement of higher-order nonlinear diffusion filtering are established in [16].

It should be noted that all the above mentioned diffusion processes are handcrafted models, which include elaborate selections of diffusivity coefficients, or optimal stopping time or proper reaction forces. It is generally a difficult task to design a good-performing PDE for a specific image processing problem because good insights into this problem and a deep understanding of the behavior of the PDEs are usually required. Therefore, some researchers propose to learn PDEs from training data via an optimal control approach [37]. Unfortunately, at present [37] is the sole previous work we can find in this direction. The basic idea of our approach is similar to [37], but our proposed model will be much more expressive. More details are presented in Section 2.4.

1.2 Motivations and Contributions

In this paper we concentrate on nonlinear diffusion process due to its high efficiency and propose a trainable nonlinear diffusion model, which is parameterized by the linear filters and the influence functions. The trained diffusion model contains many special influence functions (see Fig. 5 for an illustration), which greatly differ from usual influence functions employed in conventional diffusion models. It turns out that the trained diffusion processes can lead to effective image restoration with state-of-the-art performance, while preserve the property of high efficiency of diffusion based approaches. To our best knowledge, we are not aware of any previous works that simultaneously optimize the linear filters and influence functions of a nonlinear diffusion process.

Our proposed nonlinear diffusion process has several remarkable benefits as follows:

1) It is conceptually simple as it is merely a standard nonlinear diffusion model with trained filters and influence functions;
2) It has broad applicability to a variety of image restoration problems. In principle, all the diffusion models can be revisited with appropriate training;
3) It yields excellent results for several tasks in image restoration, including Gaussian image denoising, single image super resolution and JPEG deblocking;
4) It is highly computationally efficient, and well suited for parallel computation on GPUs.

A shorter paper has been presented as a conference version [12]. In this paper, we incorporate additional contents listed as follows

1) We investigate more details of the training phase, such as the influence of (a) initialization, (b) the model capacity and (c) the number of training samples;
2) We consider more detailed analysis of trained models, such as how the trained models generate the patterns;
3) We exploit an additional application of single image super resolution to further illustrate the potential breadth of our proposed learning framework.

2 PROPOSED REACTION DIFFUSION PROCESS

In this section, we start with conventional nonlinear diffusion processes, then propose a learning based reaction diffusion model for image restoration. Finally we show the relations between the proposed model and existing image restoration models.

2.1 Perona and Malik diffusion model

In the continuous formulation, the well-know Perona-Malik type nonlinear diffusion model [43] is given as the following PDE

\[ \frac{\partial u}{\partial t} = \text{div} \left( g(|\nabla u|) \nabla u \right) \]

where \( \nabla \) is the gradient operator and the function \( g \) is known as edge-stopping function [7] or diffusivity function [52], a typical \( g \) function given as \( g(z) = 1/(1 + z^2) \).

In our work, we stick to the fully discrete setting, where images are represented as column vectors, i.e., \( u \in \mathbb{R}^N \). Therefore, the discrete version of the PM model can be reformulated as the following discrete PDE with an explicit finite difference scheme

\[ \frac{u_{t} + 1 - u_{t}}{\Delta t} = - \sum_{i=(x,y)} \nabla^T_{x} \Lambda(u_{t}) \nabla_{x} u_{t} \pm \sum_{i=(x,y)} \nabla^T_{y} \phi(\nabla_{y} u_{t}), \]

where matrices \( \nabla_{x} \) and \( \nabla_{y} \in \mathbb{R}^{N \times N} \) are finite difference approximation of the gradient operators in \( x \)-direction and \( y \)-direction, respectively and \( \Delta t \) denotes the time step. \( \Lambda(u_{t}) \in \mathbb{R}^{N \times N} \) is defined as a diagonal matrix

\[ \Lambda(u_{t}) = \text{diag} \left( g \left( \sqrt{\left( \nabla_{x} u_{t} \right)^2 + \left( \nabla_{y} u_{t} \right)^2} \right) \right)_{p=1,\ldots,N}, \]

1. Anisotropic diffusion in this paper is understood in the sense that the smoothing induced by PDEs can be favored in some directions and prevented in others. The diffusivity is not necessary to be a tensor. It should be note that this definition is different from Weickert’s terminology [52], where anisotropic diffusion always involves a diffusion tensor, and the PM model is regarded as an isotropic model.

2. Even though the linear filters and penalty functions in the image prior model [46], [10] can be trained simultaneously, the penalty function is optimized only in the sense that the weight \( \alpha \) of certain fixed function (e.g., \( \alpha \log(1 + z^2) \)) can be tuned. Our approach can exploit much more generalized penalty functions, some of which are intractable in those previous models.
where function $g$ is the edge-stopping function. If ignoring the coupled relation between $\nabla_x u$ and $\nabla_y u$, the PM model can also be written as the second formula on the right side in (2), where $\phi_i(\nabla u) = (\phi(\nabla u)_1, \ldots, \phi(\nabla u)_N)^T \in \mathbb{R}^N$ with function $\phi(z) = zg(z)$, known as influence function [7] or flux function [32]. In the upcoming subsection, we will stick to this decoupled formulation, as it is the starting point of our approach.

2.2 Proposed nonlinear diffusion model

Recall that the matrix-vector product, $\nabla_x u$ can be interpreted as a 2D convolution of $u$ with the linear filter $k_x = [-1, 1] \otimes \nabla_y$ corresponds to the linear filter $k_y = [-1, 1]^T$). Intuitively, in order to improve the capacity of the diffusion model, we can consider the following possible changes:

a) We can employ more filters of larger kernel size, in contrast to previous models that typically involve few filters with relatively small kernel size, usually pair-wise kernels;

b) We can additionally consider different influence functions for different filters, rather than unique function $\phi$;

c) Moreover, the parameters of each iteration can vary across iterations, i.e., time varying linear operators and influence functions.

Finally, we additionally incorporate a reaction term in order to apply our model for different image processing problems, as shown later. As a consequence, our proposed nonlinear reaction diffusion model is formulated as

$$\frac{u_t - u_{t-1}}{\Delta t} = -\sum_{i=1}^{N_k} K_i^T (K_i^T u_{t-1}) - \psi(u_{t-1}, f), \quad (3)$$

where $K_i \in \mathbb{R}^{N \times N}$ is a highly sparse matrix, implemented as 2D convolution of the image $u$ with the filter kernel $k_i$, i.e., $K_i u \leftrightarrow k_i * u$. $K_i$ is a set of linear filters and $N_k$ is the number of filters. In practice, we set $\Delta t = 1$, as we can freely scale the functions $\phi_i^T$ and $\psi$ on the right side. Note that in our proposed diffusion model, the influence functions are adjustable and can be different from each other.

The specific formulation for the reaction term $\psi(u)$ depends on applications. For classical image restoration problems, such as Gaussian denoising, image deblurring, image super resolution and image inpainting, we can set the reaction term to be the gradient of a data term, i.e., $\psi(u) = \nabla_y D(u)$.

For example, if $D(u, f) = \frac{1}{2} \|Au - f\|_2^2$, $\psi(u) = \lambda A^T (Au - f)$, where $f$ is the input degraded image, $A$ is the associated linear operator, and $\lambda$ is related to the strength of the reaction term. In the case of Gaussian denoising, $A$ is the identity matrix; for image super resolution, $A$ is related to the down sampling operation and for image deconvolution, $A$ corresponds to the linear blur kernel.

2.3 A more general formulation of the proposed diffusion model

Note that the data term $D(u)$ related to (3) should be differentiable. In order to handle the problems involving a non-differentiable data term, e.g., the JPEG deblocking problem investigated in Section 6, we consider a more general form of our proposed diffusion model as follows

$$u_t = \text{Prox}_{\tau G}(u_{t-1} - \sum_{i=1}^{N_k} K_i^T (K_i^T u_{t-1}) + \psi(u_{t-1}, f))) \quad (4)$$

where $\text{Prox}_{\tau G}(\hat{u})$ is the proximal mapping operation [42] related to the non-differentiable function $G$, given as

$$\text{Prox}_{\tau G}(\hat{u}) = \min_u \frac{\|u - \hat{u}\|_2^2}{2} + \tau G(u).$$

In our work, instead of making use of well-chosen filters and influence functions, we train the nonlinear diffusion process for specific image restoration problem, including both the linear filters and the influence functions. As the diffusion process is an iterative approach, typically we run it for a few iterations. In order to make our proposed diffusion process more flexible, we consider varying parameters in each diffusion iteration. Finally, we arrive at certain optimized diffusion process with several iterations (referred to as stages).

2.4 Relations to existing image restoration models

Previous works [48], [41] show that in the nonlinear diffusion framework, there exist natural relations between reaction diffusion and regularization based energy functional. First of all, we can interpret (4) as one gradient descent step at $u_{t-1}$ of a certain energy functional given by

$$E(u, f) = \sum_{i=1}^{N_k} R_i(u) + D(u, f) + G(u, f), \quad (5)$$

where $R_i(u) = \sum_{p=1}^{N_k} \rho_i^p ((K_i^T u)_p)$ are the regularizers and the functions $\rho_i^p$ are the so-called penalty functions. Note that $\rho(z) = \psi(z)$. Since the parameters $\{K_i^T, \rho_i^p\}$ vary across the stages, (5) is a dynamic energy functional, which changes at each iteration.

In the case of fixed $\{K_i^T, \rho_i^p\}$ across the stages $t$, it is easy to check that functional (5) is exactly the fields of experts (FoE) image prior regularized variational model for image restoration [46], [11], [10]. In our work, we do not exactly solve this minimization problem anymore, but in contrast, we run the gradient descent step for several stages, and each gradient descent step is optimized by training. More importantly, we are allowed to investigate more generalized penalty functions with the proposed framework.

In a very recent work [49], Schmidt et al. exploited an additive function of half-quadratic optimization to solve the same problem (5), which finally leads to a fast and effective image restoration model called cascade of shrinkage fields (CSF). The CSF model makes an assumption that the data term in (5) is quadratic and the operator $A$ can be interpreted as a convolution operation, such that the corresponding subproblem has fast closed-form solution based on discrete Fourier transform (DFT). This restrains its applicability to many other problems such as image super resolution. However, our proposed diffusion model does not have this restriction on the data term. In principle, any smooth data term is appropriate. Moreover, as shown in the following sections, we can easily handle the case of non-smooth data term.

A few previous works, e.g., [3], [18], also attempted to train an optimized gradient descent algorithm for the energy functional similar to (5). In their works, the Gaussian denoising problem is considered. However, their model is much more constrained in the sense that, they exploited the same filters for each gradient descent step. More importantly, the influence function in their model is fixed to be a unique one. This clearly restricts the model capacity, as demonstrated in Sec. 4.

As mentioned in the Introduction, there are few preliminary works, e.g., [37] to go beyond traditional PDEs of the form (2), and
propose to learn optimal PDEs for image restoration via optimal control. The PDEs considered in [37] have the form of
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \kappa(u) + a(t)^T \mathcal{O}(u) \\
u_{1:T} &= f ,
\end{align*}
\]
where coefficients \( a(t) \) is the parameters to be optimized. \( \kappa(u) \) is related to the Total Variation regularization [47] and \( \mathcal{O}(u) \) denotes a few operations, called invariants over \( u \), e.g., \( u \) and \( \| \nabla u \|_2^2 = u_x^2 + u_y^2 \). Basically, this learning approach is to optimize the linear combination coefficients of a few predefined terms, which depend on selected derivative filters. One can see that this learning model is a simplified version of our model (3) with fixed linear filters and influence functions, and only the weight of each term is optimized. Numerical results in [37] demonstrate that their investigated PDE model is too simple to generate state-of-the-art performance.

The proposed diffusion model also bears an interesting link to convolutional networks (CNs) applied to image restoration problems in [32]. One can see that each iteration (stage) of our proposed diffusion process involves convolution operations with a set of linear filters, and thus it can be treated as a convolutional network. The architecture of our proposed diffusion model is shown in Figure 1, where it is represented as a common feed-forward network. We refer to this network in the following as diffusion network.

However, we can introduce a feedback step to explicitly illustrate the special architecture of our diffusion network that we subtract “something” from the input image. Therefore, our diffusion model can be represented in a more compact way in Figure 2, where one can see that the structure of our CN model is different from conventional feed-forward networks. Due to this feedback step, it can be categorized into recurrent networks [25]. It should be noted that the nonlinearity (i.e., influence functions in the context of nonlinear diffusion) in our proposed network are trainable. However, conventional CNs make use of fixed activation function, e.g., the ReLU function [40] or sigmoid functions [32].

### 3 Learning Framework

We train our diffusion networks in a supervised manner, namely we firstly prepare the input/output pairs for a certain image processing task, and then exploit a loss minimization scheme to learn the model parameters \( \Theta_t \) for each stage \( t \) of the diffusion process. The training dataset consists of \( S \) training samples \( \{ f^s, u^s_{gt} \}_{s=1}^S \), where \( f^s \) is a noisy observation and \( u^s_{gt} \) is the corresponding ground-truth clean image. The model parameters \( \Theta_t \) of each stage include the parameters of (1) the reaction force weight \( \lambda \), (2) linear filters and (3) influence functions, i.e., \( \Theta_t = \{ \lambda^t, \phi^t, k^t \} \).

#### 3.1 Overall training model

In the supervised manner, a training cost function is required to measure the difference between the output of the diffusion network and the ground-truth image. As our goal is to train a diffusion network with \( T \) stages, the cost function is formulated as
\[
\mathcal{L}(\Theta_1, \ldots, T) = \sum_{s=1}^S \ell(u^s_T, u^s_{gt}) ,
\]
where \( u_T \) is the output of the final stage \( T \). In our work we exploit the usual quadratic loss function, defined as
\[
\ell(u^s_T, u^s_{gt}) = \frac{1}{2} \| u^s_T - u^s_{gt} \|_2^2 .
\]

As a consequence, the training task is generally formulated as the following optimization problem
\[
\min_{\Theta} \mathcal{L}(\Theta) = \sum_{s=1}^S \frac{1}{2} \| u^s_T - u^s_{gt} \|_2^2 ,
\]
\[
\begin{align*}
u^0_T &= I^s \\
u^s_T &= \text{Prox}_{\tau G} \left( u^s_{t-1} - \sum_{i=1}^N (K^t_i)^T \phi^t(K^t_i u^s_{t-1} + \psi(u^s_{t-1}, f^s)) \right) ,
\end{align*}
\]
\[
\begin{align*}
t &= 1 \cdots T,
\end{align*}
\]
3. This loss function is related to the PSNR quality measure. Note that as shown in [33], other quality measures, such as structural similarity (SSIM) and mean absolute error (MAE) can be chosen to define the loss function. At present we only consider the quadratic loss function due to its simplicity.
where $\Theta = \{\Theta^T\}_{t=1}^T$ and $I_0$ is the initial status of the diffusion process. Note that the above loss function only depends on the output of the final stage $T$, i.e., the parameters in all stages are simultaneously trained such that the output of the diffusion process $u_T$ is optimized. We call this training scheme **joint training**. The joint training strategy is a minimization problem with respect to the parameters in all stages $\{\Theta_1, \Theta_2, \ldots, \Theta_T\}$.

One can see that our training model is also a deep model with many stages (layers). It is well-known that deep models are usually sensitive to initialization, and therefore training from scratch is prone to getting stuck at bad local minima. As a consequence, people usually consider a greedy layer-wise pre-training [5] to provide a good initialization for the joint training (fine tune).

In our work, we also consider a **greedy training** scheme to pre-train our diffusion network stage-by-stage, where each stage is greedily trained such that the output of each stage is optimized, i.e., for stage $t$, we minimize the cost function

$$L(\Theta_t) = \sum_{s=1}^{S} \ell(u^s_t, u^s_{gt}),$$

where $u^s_t$ is the output of stage $t$ of the diffusion process. Note that this is a minimization problem only with respect to the parameters $\Theta_t$ in stage $t$.

### 3.2 Parameterizing the influence functions $\phi_t^i$ and linear filters $k_t^i$

In this paper, we aim to investigate arbitrary influence functions. In order to conduct a fast and accurate training, an effective function parameterization method is required. We parameterize the influence function via standard radial basis functions (RBFs), i.e., each function $\phi$ is represented as a weighted linear combination of a family of RBFs as follows

$$\phi_t^i(z) = \sum_{j=1}^{M} u^i_j \varphi\left(\frac{|z - \mu_j|}{\gamma_j}\right),$$

where $\varphi$ represents different RBFs. In this paper, we exploit RBFs with equidistant centers $\mu_j$ and unified scaling $\gamma_j$. We investigate two typical RBFs [31]: (1) Gaussian radial basis $\varphi_g$ and (2) triangular-shaped radial basis $\varphi_t$, given as

$$\varphi_g(z) = \varphi\left(\frac{|z - \mu|}{\gamma}\right) = \exp\left(-\frac{(z - \mu)^2}{2\gamma^2}\right)$$

and

$$\varphi_t(z) = \varphi\left(\frac{|z - \mu|}{\gamma}\right) = \begin{cases} 1 - \frac{|z - \mu|}{\gamma}, & |z - \mu| \leq \gamma \\ 0, & |z - \mu| > \gamma \end{cases}$$

respectively. The basis functions are shown in Figure 3, together with an example of the function approximation by using two different RBF methods. In our work, we have investigated both function approximation methods, and we find that they lead to similar results. We only present the results obtained by the Gaussian RBF in this paper.

In our work, the linear kernels $k_t^i$ related to the linear operators $K_t^i$ are defined as a linear combination of Discrete Cosine Transform (DCT) basis kernels $b_r$, i.e.,

$$k_t^i = \frac{\sum_{r} w^i_{t,r} b_r}{\|w^i_t\|_2},$$

where the kernels $k_t^i$ are normalized to get rid of an ambiguity appearing in the proposed diffusion model. More details can be found in the **supplemental material**. The kernels are formed in this way in order to keep the expected property of zero-mean.

### 3.3 Computing gradients

For both greedy training and joint training, we make use of gradient-based algorithms (e.g., the L-BFGS algorithm [36]) for optimization. The key point is to compute the gradients of the loss function with respect to the training parameters. In greedy training, the gradient of the loss function at stage $t$ with respect to the model parameters $\Theta_t$ is computed using standard chain rule, given as

$$\frac{\partial \ell(u_t, u_{gt})}{\partial \Theta_t} = \frac{\partial u_t}{\partial \Theta_t} \cdot \frac{\partial \ell(u_t, u_{gt})}{\partial u_t},$$

where $\frac{\partial \ell(u_t, u_{gt})}{\partial u_t} = u_t - u_{gt}$ is directly derived from (8). $\frac{\partial u_t}{\partial \Theta_t}$ is computed from the diffusion process for specific task. For the applications exploited in this paper, such as image denoising with Gaussian noise, single image super resolution and JPEG deblocking, we present the detailed derivations of $\frac{\partial u_t}{\partial \Theta_t}$ in the **supplemental material**.

In the joint training, we compute the gradients of the loss function with respect to $\Theta_t$ by using the standard back-propagation technique widely used in the neural networks learning [34], namely, $\frac{\partial F}{\partial \Theta_t}$ is computed by using

$$\frac{\partial F(u_{T}, u_{gt})}{\partial \Theta_t} = \frac{\partial u_t}{\partial \Theta_t} \cdot \frac{\partial F(u_{T}, u_{gt})}{\partial u_{T}} \cdot \frac{\partial u_{T}}{\partial \Theta_{T}} \cdots \frac{\partial F(u_{T}, u_{gt})}{\partial u_{T}}.$$

Compared to the greedy training, we additionally need to calculate $\frac{\partial \Theta_{T}}{\partial \Theta_t}$. For the investigated image processing problems in this paper, we provide all necessary derivations in the **supplemental material**.
3.4 Experimental setup and implementation details

3.4.1 Boundary condition of the convolution operations
In our convolution based diffusion network, the image size stays the same when an image goes through the network, and we use the symmetric boundary condition for convolution calculation. In our original diffusion model (4), there is matrix transpose $K^T v$, which exactly corresponds to the convolution operation $\bar{K} * v$ ($\bar{K}$ is obtained by rotating the kernel $k$ 180 degrees) in the cases of periodic and zero-padding boundary conditions. It should be noted that in the case of symmetric boundary condition used in this paper, this result holds only in the central image region. However, we still want to explicitly use the formulation $\bar{K} * v$ to replace $K^T v$, because the former can significantly simplify the derivation of the gradients required for training.

We find that the direct replacement introduces some artifacts at the image boundary. In order to avoid these artifacts, we symmetrically pad the input image before it is sent to the diffusion network, and then we discard those padding pixels in the final output. More details are found in the supplemental material.

3.4.2 RBF kernels
Images exploited in this paper have the dynamic range in $[0, 255]$, and the filters have unit norm. In order to cover most of the filter response, we consider influence functions in the range $[-310, 310]$. We use 63 Gaussian RBFs with equidistant centers at $[-310 : 10 : 310]$, and set the scaling parameter $\gamma = 10$.

3.4.3 Experimental setup
Model capacity: In our work, we train the proposed diffusion network with at most 8 stages to observe its saturation behavior after certain stages. We first greedily train $T$ stages of our model with specific model capacity, then conduct a joint training to refine the parameters of the whole $T$ stages.

In this paper, we mainly consider four different diffusion networks with increasing capacity:

- TNRD$^T_{3\times3}$: Fully trained model with 8 filters of size $3 \times 3$,
- TNRD$^T_{5\times5}$: Fully trained model with 24 filters of size $5 \times 5$,
- TNRD$^T_{7\times7}$: Fully trained model with 48 filters of size $7 \times 7$,
- TNRD$^T_{9\times9}$: Fully trained model with 80 filters of size $9 \times 9$,

where TNRD$^T_{m\times m}$ denotes a nonlinear diffusion process of stage $T$ with filters of size $m \times m$. The filters number is $m^2 - 1$, if not specified.

Training and test dataset: In order to make a fair comparison to previous works, we make use of the same training datasets used in previous works for our training, and then evaluate the trained models on commonly used test datasets. For image processing problems investigated in this paper, i.e., Gaussian denoising, single image super resolution and JPEG deblocking, we consider the following training and test datasets, respectively.

a) Gaussian denoising. Following [49], we use the same 400 training images, and crop a $180 \times 180$ region from each image, resulting in a total of 400 training samples of size $180 \times 180$, i.e., roughly 13 million pixels. We then evaluate the denoising performance of a trained model on a standard test dataset of 68 natural images, which is suggested by [46], and later widely used for Gaussian denoising testing. Note that the test images are strictly separate from the training datasets.

b) Single image super resolution. The publicly available framework of Timofte et al.[51] provides a perfect base to compare single image super resolution algorithms. It includes 91 training images and two test datasets Set5 and Set14. Many recent state-of-the-art learning based image super resolution approaches [50], [19] accomplish their comparison based on this framework. Therefore, we also use the same 91 training images. We crop 4-5 sub-images of size $150 \times 150$ from each training image, according to its size, and this finally gives us 421 training samples. We then evaluate the performance of the trained models on the Set5 and Set14 dataset.

c) JPEG deblocking. We train the diffusion models using the same training images as in the case of Gaussian denoising. In the test phase, we follow the test procedure in [33] for performance evaluation. The test images are converted to gray-value, and scaled by a factor of 0.5, resulting 200 images of size $240 \times 160$.

3.4.4 Approximate training time
Note that the calculation of the gradients of the loss function in (12) is very efficient even using a simple Matlab implementation, since it mainly involves 2D convolutions. The training time varies greatly for different configurations. Important factors include (1) model capacity, (2) number of training samples, (3) number of iterations taken by the L-BFGS algorithm, and (4) number of Gaussian RBF kernels used for function approximation. We report the most time consuming cases as follows.

In training, computing the gradients $\frac{\partial E}{\partial u}$ with respect to the parameters of one stage for 400 images of size $180 \times 180$ takes about 35s (TNRD$^{3\times3}_{5\times5}$), 75s (TNRD$^{7\times7}_{5\times5}$) or 165s (TNRD$^{9\times9}_{5\times5}$) using Matlab implementation on a server with CPUs: Intel(R) Xeon E5-2680 @ 2.80GHz (eight parallel threads using parfor in Matlab, 63 Gaussian RBF kernels for the influence function parameterization). We typically run 200 L-BFGS iterations for optimization. Therefore, the total training time, e.g., for the TNRD$^{3\times3}_{5\times5}$ model is about $5 \times (200 \times 75)/3600 = 20.8h$. Code for learning and inference is available on the authors’ homepage www.GPU4Vision.org. For the training of the Gaussian denoising task, we have also accomplished a GPU implementation, which is about 4-5 times faster than our CPU implementation.

4 Training for Gaussian denoising
For the task of Gaussian denoising, we consider the following energy functional

$$\min_u E(u) = \sum_{i=1}^{N_k} \rho_i(k_i * u) + \frac{\lambda}{2} \|u - f\|^2_2.$$ 

By setting $D(u) = \frac{\lambda}{2} \|u - f\|^2_2$ and $G(u) = 0$, we arrive at the following diffusion process with $u_0 = f$

$$u_t = u_{t-1} - \left( \sum_{i=1}^{N_k} \tilde{k}_i \phi_i\left( \tilde{k}_i * u_{t-1} \right) + \lambda (u_{t-1} - f) \right),$$

(14)

where we explicitly use a convolution kernel $\tilde{k}_i$ (obtained by rotating the kernel $k_i$, 180 degrees) to replace the $K^T_i$ for the sake of model simplicity, but we have to pad the input image. The gradients $\frac{\partial u_t}{\partial \tilde{k}_i}$ and $\frac{\partial u_t}{\partial u_{t-1}}$ required in training are computed from this equation. Detailed derivations are presented in the supplemental material.

We started with the training for TNRD$^T_{5\times5}$. We first considered the greedy training phase to train a diffusion process up to 8 stages.
in our training model: (1) the linear filters and (2) the influence functions. In order to have a better understanding of the trained models, we went through a series of experiments to investigate the impact of these two aspects.

Concentrating on the model capacity of 24 filters of size $5 \times 5$, we considered the training of a diffusion process with 10 steps, i.e., $T = 10$ for the Gaussian denoising of noise level $\sigma = 25$. We exploited two main classes of configurations: (A) the parameters of every stage are the same and (B) every diffusion stage is different from each other. In both configurations, we consider two cases: (I) only train the linear filters with fixed influence function $\phi(z) = 2z/(1 + z^2)$ and (II) simultaneously train the filters and influence functions.

Based on the same training dataset and test dataset, we obtained the following results: (A.I) every diffusion step is the same, and only the filters are optimized with fixed influence function. This is a similar configuration to previous works [3], [18]. The trained model achieves a test performance of 28.47dB. (A.II) with additional tuning of the influence functions, the resulting performance is boosted to 28.60dB. (B.I) every diffusion step can be different, but only the linear filters are trained with fixed influence functions. The corresponding model obtains a result of 28.56dB, which is equivalent to the variational model [10] with the same model capacity. Finally (B.II) with additional optimization of the influence functions, the trained model leads to a significant improvement with the result of 28.86dB.

The analytical experiments demonstrate that without the training of the influence functions, there is no chance to achieve significant improvements over previous works, no matter how hard we tune the linear filters. Therefore, we believe that the additional freedom to tune the influence functions is the critical factor of our proposed training model. After having a closer look at the learned influence functions of the $\text{TNRD}_5$ model, these functions strengthen our argument.

4.2 Learned influence functions

The form of 120 learned penalty functions $\rho$ in the $\text{TNRD}_5$ model can be divided into four classes (see the corresponding sub-figures in Figure 5):

(a) Truncated convex penalty functions with low values around zero to promote smoothness.

4. The implementation of the RTF$_5$ model is not available, and we quoted its result from [49].

5. The penalty function $\rho(z)$ is integrated from the influence function $\phi(z)$ according to the relation $\phi(z) = \rho'(z)$.
Fig. 5. The figure shows four characteristic influence functions (left plot in each subfigure) together with their corresponding penalty functions (right plot in each subfigure), learned by our proposed method in the TNRD$_{5,5}$ model. A major finding in this paper is that our learned penalty functions significantly differ from the usual penalty functions adopted in partial differential equations and energy minimization methods. In contrast to their usual robust smoothing properties which is caused by a single minimum around zero, most of our learned functions have multiple minima different from zero and hence are able to enhance certain image structures. See Sec. 4.3 for more information.

(a) Truncated convex
(b) Negative Mexican hat
(c) Truncated concave
(d) Double-well penalty

Fig. 6. Patterns synthesized from uniform noise using our learned diffusion models. (a) is generated by (15) using the parameters (linear filters and influence functions) in a stage of our learned TNRD$_{5,5}$ for image denoising, (b) is generated by (15) using the parameters in a stage of our learned TNRD$_{5,5}$ for image super resolution and (c) is also from a stage of our learned TNRD$_{5,5}$ for image super resolution.

We also find that this penalty function is exactly the type of bimodal expert functions for texture synthesis employed in [30].

Now it is clear that the diffusion process involving the learned influence functions does not perform pure image smoothing any more for image processing. In contrast, it leads to a diffusion process for adaptive image smoothing and sharpening, distinguishing itself from previous commonly used image regularization techniques.

4.3 Pattern formation using the learned influence functions

In the previous work on Gibbs reaction diffusion [56], it is shown that those unconventional penalty functions such as Figure 5(c) have significant meaning in visual computation, as they can produce patterns. We also find that those unconventional penalty functions learned in our models can produce some interesting image patterns.

We consider the following diffusion process involving our learned linear filters and the corresponding influence functions

$$\frac{u_t - u_{t-1}}{\Delta t} = - \sum_{i=1}^{N_c} k_i \ast \phi_i (k_i \ast u_{t-1}),$$

where the filters $k_i$ and influence functions $\phi_i$ are chosen from a certain stage of the learned models. Note that we do not incorporate a reaction term in this diffusion model. We run (15) from starting points $u_0$ (uniform noise images in the range $[0, 255]$), and it converges to a local minimum$^7$. Some synthesized patterns are shown in Figure 6. One can see that the diffusion model with our learned influence functions and filters can produce edge-like image structure and repeated patterns from fully random images. This kind of diffusion model is known as Gibbs reaction diffusion in [56]. We provide another example in Figure 13 to demonstrate how our learned diffusion models can generate meaningful patterns for image super resolution.

4.4 Important aspects of the training framework

4.4.1 Influence of initialization

Our training model is also a deep model with many stages (layers), but we find that it is not very sensitive to initialization. In the case

6. The terminology of “reaction diffusion” in [56] is a bit different from ours. In our formulation, “reaction term” is related to the data term, while in [56], it means the diffusion term controlled by those downright penalty functions.

7. The corresponding diffusion processes are unstable, and therefore we have to restrict the image dynamic range to $[0, 255]$. 
of Gaussian denoising, we have experiments with fully random initializations in range $[-0.5, 0.5]$, the trained models lead to a deviation within 0.01dB in the test phase and (2) joint training. Fully random initializations lead to models with inferior results, e.g., TNRD$_{5 \times 5}$ (28.61 vs. 28.78). However, a plain initialization (all stages with DCT filters, influence function $\phi(z) = 2z/(1 + z^2)$) works almost the same, e.g., TNRD$_{5 \times 5}$ (28.75 vs. 28.78) and TNRD$_{7 \times 7}$, (28.91 vs. 28.92).

We believe that this appealing property of our training framework is attributed to the well-distributed gradients across stages. We show in Figure 7 an example to illustrate the gradients of the training loss function with respect to the parameter of all stages. One can see that the well-known phenomenon of “vanishing gradient” [6] in the back-propagation phase of a usual deep model does not appear in our training model. We believe that the reason for the well-distributed gradients is that our training model is more constrained. For example, in a more general sense, the rotated kernel $k_i$ in our formulation is not necessary to be the rotated version of the kernel $k_i$, and it can be an arbitrary kernel. However, we stick to this form, as it has a clear meaning derived from energy minimization.

4.4.2 Influence of number of training samples
In our training, we do not consider any regularization for the training parameters, and we finally reach good-performing models. A probable reason is that we have exploited sufficient training samples (400 samples of size $180 \times 180$). Thus an interesting question arises: how many samples are sufficient for our training?

In order to answer this question, we re-train the TNRD$_{5 \times 5}$ model using different size of training dataset, and then evaluate the test performance of trained models. We summarize the results in Figure 8. One can see that (1) too less training samples (e.g., 40 images) will clearly lead to over-fitting, thus inferior test performance, and (2) 200 images are typically enough to prevent over-fitting.

4.4.3 Influence of filter size
In our model, the size of involved filters is a free parameter. In principle, we can exploit filters of any size, but in practice, we need to consider the trade-off between run time and accuracy.

In order to investigate the influence of the filter size, we increase the filter size to $7 \times 7$ and $9 \times 9$. We find that increasing the filter size from $5 \times 5$ to $7 \times 7$ brings a significant improvement of 0.14dB (TNRD$_{7 \times 7}$ vs. TNRD$_{5 \times 5}$) as show in Table 1. However, if we further increase the filter size to $9 \times 9$, the resulting TNRD$_{9 \times 9}$ leads to a performance of 28.96dB (a slight improvement of 0.05dB relative to the TNRD$_{7 \times 7}$ model). We can conjecture that continue to increase the filter size to $11 \times 11$ will bring negligible improvements. However, it will significantly increase particularly the training time. We also consider a model with smaller filters, $3 \times 3$. We summarize the results of different model capacities in Figure 9. In practice, we prefer the TNRD$_{7 \times 7}$ model as it provides the best trade-off between performance and computation time. Therefore, in later applications, we only consider TNRD$_{7 \times 7}$ models.

Fig. 10 shows the trained filters of the TNRD$_{7 \times 7}$ model in the first and last stage for the task of Gaussian denoising. One can find many edge and image structure detection filters along different directions and in different scales.
model for the noise level $\sigma$ = 25. We can find first, second and higher-order derivative filters, as well as rotated derivative filters along different directions. These filters are effective for image structure detection, such as image edge and texture.

![Example Image](image.png)

**Fig. 10.** Trained filters (in the first and last stage) of the TNRD$_{7 \times 7}$ model for the noise level $\sigma$ = 25. We can find first, second and higher-order derivative filters, as well as rotated derivative filters along different directions. These filters are effective for image structure detection, such as image edge and texture.

**TABLE 1**

Average PSNR (dB) on 68 images from [46] for image denoising with $\sigma$ = 15, 25 and 50.

| Method     | $\sigma$ | $\sigma$ | $\sigma$ | $\sigma$ |
|------------|----------|----------|----------|----------|
|            | 15       | 25       | 50       | 15       |
| BM3D       | 31.08    | 28.56    | 25.62    | 31.14    |
| LSSC       | 31.27    | 28.70    | 25.72    | 31.30    |
| EPLL-GMM   | 31.19    | 28.68    | 25.67    | 31.34    |
| opt-MRF    | 31.18    | 28.66    | 25.70    | 31.34    |
| RTF$_{5 \times 5}$ | $-$ | 28.75    | $-$ | 28.75    |
| WNNM       | 31.37    | 28.83    | 25.83    | 28.78    |
| CSF$_{5 \times 5}$ | 31.14    | 28.60    | $-$ | 28.83    |
| CSF$_{7 \times 7}$ | 31.24    | 28.72    | $-$ | 28.87    |

| $\sigma$  | $\sigma$ | $\sigma$ |
|-----------|----------|----------|
| 5         | 25.54    | 25.78    |
| 25        | 25.80    | 25.96    |
| 50        | 25.87    | 26.01    |

**4.5 Training for different noise levels and comparison to recent state-of-the-art**

The above training experiments are based on Gaussian noise of level $\sigma$ = 25. We also trained diffusion models for the noise levels $\sigma$ = 15 and $\sigma$ = 50. The test performance is summarized in Table 1, together with comparison to very recent state-of-the-art denoising algorithms. In experiments, we observed that joint training can always gain an improvement of about 0.1dB over the greedy training for the cases of $T \geq 5$.

From Table 1, one can see that for all noise levels, the resulting TNRD$_{7 \times 7}$ model achieves the highest average PSNR. The TNRD$_{7 \times 7}$ model outperforms the benchmark - BM3D method by 0.35dB in average. This is a notable improvement as few methods can surpass BM3D more than 0.3dB in average [35]. Moreover, the TNRD$_{7 \times 7}$ model also surpasses the best-reported algorithm - WNNM method, which is quite slow as shown in Table 2.

**4.6 Run time**

The algorithm structure of our TNRD model is similar to the CSF model, which is well-suited for parallel computation on GPUs. We implemented our trained models on GPU using CUDA programming to speed up the inference procedure, and finally it leads to significantly improved performance, see Table 2. We make a run time comparison to other denoising algorithms based on strictly enforced single-threaded CPU computation (e.g., start Matlab with -singleCompThread) for a fair comparison, see Table 2. We only present the results of some selective algorithms, which either have the best denoising result or run time performance. We refer to [49] for a comprehensive run time comparison of various algorithms.

We see that our TNRD model is generally faster than the CSF model with the same model capacity. It is reasonable, because in each stage the CSF model involves additional DFT and inverse DFT operations, i.e., our model only requires a portion of the computation of the CSF model. Even though the BM3D is a non-local model, it still possesses high computational efficiency. In contrast, another non-local model - WNNM achieves compelling denoising results at the expense of huge computation time. Moreover, the WNNM algorithm is hardly applicable for high resolution images (e.g., 10 mega-pixels) due to its huge memory requirements. Note that our model can be also easily implemented with multi-threaded CPU computation.

In summary, our TNRD$_{7 \times 7}$ model outperforms these recent state-of-the-arts, meanwhile it is the fastest method even with a CPU implementation. We present an illustrative denoising example in Figure 11 on an image from the test dataset. More denoising examples can be found in the supplemental material based on images from the test dataset and a megapixel-size natural image of size $1050 \times 1680$.

**5 SINGLE IMAGE SUPER RESOLUTION (SISR)**

As demonstrated in the last section that our trained diffusion model can lead to explicit backward diffusion process, which sharpens image structures like edges. This is the very property demanded for the task of image super resolution. Therefore, we are motivated to investigate the SISR problem with our proposed approach.

We start with the following energy functional

$$\min_u E(u) = \sum_{i=1}^{N_s} \rho_i (k_i \ast u) + \lambda \frac{1}{2} \| Au - f \|_2^2,$$

where the linear operator $A$ is a bicubic interpolation which links the high resolution (HR) image $h$ to the low resolution (LR) image $f$.

8. LSSC, EPLL, opt-MRF and RTF$_{5 \times 5}$ methods are much slower than BM3D on the CPU, cf. [49].
energy functional (16) suggests the following diffusion process
\[ u_t = u_{t-1} - \left( \sum_{i=1}^{N_k} \phi_i^k (k_i^u u_{t-1}) + \lambda^t A^T (Au_{t-1} - f) \right), \]
where the starting point \( u_0 \) is given by the direct bicubic interpolation of the LR image \( f \). Computing the gradients \( \frac{\partial u}{\partial t} \) and \( \frac{\partial u}{\partial u_{t-1}} \) with respect to (17) can be done with little modifications to the derivations for image denoising. Detailed derivations are presented in the supplemental material.

We considered the model capacity of TNRD\(_{5 \times 7}^2\), and trained diffusion models for three upscaling factors \( \times2, \times3 \) and \( \times4 \), using exactly the same 91 training images as in previous works [51], [50]. The trained models are evaluated on two different test data sets: Set5 and Set14. Following previous works [51], [19], [50], the trained models are only applied to the luminance component of an image, and a regular bicubic upscaling method is applied to the color components.

The test results are summarized in Table 3 and Table 4. One can see that in terms of average PSNR, our trained diffusion model TNRD\(_{5 \times 7}^2\) leads to significant improvements over very recent state-of-the-arts in all cases, meanwhile it is still among the fast algorithms.9 A SISR example is shown in Figure 12 to illustrate its effectiveness. One can see that our approach can obtain more natty

9. Note that our approach is a Matlab implementation, while some of other algorithms are based on C++ implementations, such as SR-CNN.

**Fig. 11.** Denoising results on a test image of size \( 481 \times 321 \) \((\sigma = 25)\) by different methods (compared with BM3D [15], WNNM [26] and CSF model [49]), together with the corresponding computation time either on CPU or GPU. Note the differences in the highlighted region.
image edges, as shown in the zoom-in parts. More SISR examples can be found in the supplemental material.

We apply the learned diffusion parameters to the diffusion equation (15). It turns out that the diffusion process can also generate some interesting patterns from random images, as shown in Figure 6. We believe that this ability to generate image patterns from weak evidence is the main reason for the superiority of our trained model for the SISR task. In order to further validate our argument, we carry out a toy SISR experiment based on a synthesized image with repeated hexagons. The results are shown in Figure 13, where one can see that our trained model can better reconstruct those repeated image structures.

### 6 JPEG DEBLOCKING EXPERIMENTS

In order to further demonstrate the applicability of our proposed framework for those problems with a non-smooth data term, we investigate the JPEG deblocking problem - suppressing the block artifacts in the JPEG compressed images, which is formulated as a non-smooth optimization problem. Motivated by [8], we consider the following variational model based on the FoE image prior

\[
\min_u E(u) = \sum_{i=1}^{N_k} \rho_i(k_i \ast u) + I_Q(Du),
\]

where \(I_Q\) is a indicator function over the set \(Q\) (quantization constraint set). In JPEG compression, information loss happens in the quantization step, where all possible values in the range \([d-0.5, d+0.5]\) (\(d\) is an integer) are quantized to a single number \(d\). Given a compressed data, we only know \(d\). Therefore, all possible values in the interval \([d-0.5, d+0.5]\) define a convex set \(Q\) which is a box constraint. The sparse matrix \(D \in \mathbb{R}^{N \times N}\) denotes the block DCT transform. We refer to [8] for more details.

By setting \(D(u) = 0\) and \(G(u) = I_Q(Du)\), we obtain the following diffusion process

\[
u_t = D^\top \text{proj}_Q \left( D \left( u_{t-1} - \sum_{i=1}^{N_k} \tilde{k}_i \ast \phi_i^{\ast}(k_i \ast u_{t-1}) \right) \right),
\]

| Set14 images | Bicubic | K-SVD [55] | ANR [51] | SR-CNN [19] | RFL [50] | TNRD_{7/7}^2 |
|--------------|---------|------------|----------|-------------|----------|-------------|
| baboon       | 23.21   | 23.52      | 23.56    | 23.60       | 23.57    | 23.62       |
| barbara      | 26.25   | 26.76      | 26.69    | 26.66       | 26.63    | 26.25       |
| bridge       | 24.40   | 25.02      | 25.01    | 25.07       | 25.11    | 25.29       |
| comic        | 26.55   | 27.15      | 27.08    | 27.20       | 27.16    | 27.12       |
| face         | 23.12   | 23.96      | 24.04    | 24.39       | 24.27    | 24.67       |
| flowers      | 32.82   | 33.53      | 33.62    | 33.58       | 33.65    | 33.82       |
| foreman      | 27.23   | 28.43      | 28.49    | 28.97       | 28.86    | 29.55       |
| lemm          | 31.18   | 33.19      | 33.23    | 33.35       | 33.37    | 34.65       |
| man           | 31.68   | 33.00      | 33.08    | 33.39       | 33.38    | 33.77       |
| monarch      | 27.01   | 27.90      | 27.92    | 28.18       | 28.20    | 28.52       |
| pepper        | 29.43   | 31.10      | 31.09    | 32.39       | 32.10    | 33.61       |
| ppt3         | 32.39   | 34.07      | 33.82    | 34.35       | 34.55    | 35.06       |
| zebra         | 23.71   | 25.23      | 25.03    | 26.02       | 25.84    | 27.08       |
| average performance | 27.54 | 28.67 | 28.65 | 29.00 | 29.02 | 29.46 |

| | PSNR | Time | PSNR | Time | PSNR | Time | PSNR | Time | PSNR | Time |
|---|------|------|------|------|------|------|------|------|------|------|
| baboon       | 29.43 | -    | 29.43 | -    | 29.43 | -    | 29.43 | -    | 29.43 | -    |
| barbara      | 31.10 | -    | 31.10 | -    | 31.10 | -    | 31.10 | -    | 31.10 | -    |
| bridge       | 32.39 | -    | 32.39 | -    | 32.39 | -    | 32.39 | -    | 32.39 | -    |
| comic        | 33.61 | -    | 33.61 | -    | 33.61 | -    | 33.61 | -    | 33.61 | -    |
| face         | 33.77 | -    | 33.77 | -    | 33.77 | -    | 33.77 | -    | 33.77 | -    |
| flowers      | 35.06 | -    | 35.06 | -    | 35.06 | -    | 35.06 | -    | 35.06 | -    |
| foreman      | 33.61 | -    | 33.61 | -    | 33.61 | -    | 33.61 | -    | 33.61 | -    |
| lemm          | 33.77 | -    | 33.77 | -    | 33.77 | -    | 33.77 | -    | 33.77 | -    |
| man           | 28.52 | -    | 28.52 | -    | 28.52 | -    | 28.52 | -    | 28.52 | -    |
| monarch      | 1.66  | -    | 1.66  | -    | 1.66  | -    | 1.66  | -    | 1.66  | -    |
| pepper        | 1.20  | -    | 1.20  | -    | 1.20  | -    | 1.20  | -    | 1.20  | -    |
| ppt3         | 1.48  | -    | 1.48  | -    | 1.48  | -    | 1.48  | -    | 1.48  | -    |
| zebra         | 1.04  | -    | 1.04  | -    | 1.04  | -    | 1.04  | -    | 1.04  | -    |

**Fig. 12.** A super resolution example for the “Monarch” image from Set14 with an upsampling factor ×3. Note the differences in the highlighted region that our model achieves more clean and sharp image edges. *Best viewed on screen and zoom in.*
Meanwhile, the structure of trained models is very simple and well-suited for parallel computation on GPUs. As a consequence, the resulting algorithms are significantly faster than all competing algorithms and hence are also applicable to the restoration of high resolution images.

### 7.2 Discussion

One possible limitation of the proposed TNRD approach is that one has to define the ground truth - the expected output of the diffusion network during training. For image restoration applications in this paper, this is not a problem as we have a clear choice for the ground truth. However, for those applications with ambiguous ground truth, e.g., image structure extraction [54], we will have to make efforts to define the ground truth.

Furthermore, the trained diffusion networks will only perform well in the way they are trained. For example, the trained model based on noise level \( \sigma = 25 \) will break for an input image with noise \( \sigma = 50 \), and the trained model for upsampling factor \( \times 3 \) will also lead to inferior performance when it is applied to the SISR problem of upsampling factor \( \times 2 \). It is generally impossible to train a universal diffusion model to handle all the noise levels or all upsampling factors.

Our approach is to optimize a time-discrete PDE, which is inspired by FoE based model, but we do not aim to minimize a series of FoE based energies. Our model directly learns an optimal trajectory for a certain possibly unknown energy functional, the minimizer of which provides a good estimate of the demanded solution. Probably, such a functional cannot be modeled by a single FoE energy, while our learned gradient descent steps provide good approximation to the local gradients of this unknown functional.

### 7.3 Future work

From an application point of view, we think that it will be interesting to consider learned nonlinear reaction diffusion based models also for other image processing tasks such as image inpainting, blind image deconvolution, optical flow. Moreover, since learning the influence functions turned out to be crucial, we believe that learning optimal nonlinearities (i.e., activation functions) in standard CNs could lead to a similar performance increase. There are actually two recent works [2], [29] to investigate a parameterized ReLU function in standard deep convolutional networks, which indeed brings improvements even with little freedom to tune the activation functions. Finally, it will also be interesting to investigate the unconventional penalty functions learned by our approach in usual energy minimization approaches.

### References

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