Joint Frequency- and Image-Space Learning for Fourier Imaging

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Abstract

We propose a neural network layer structure that combines frequency and image feature representations for robust Fourier image reconstruction. Our work is motivated by the challenges in magnetic resonance imaging (MRI) where the acquired signal is a corrupted Fourier transform of the desired image. The proposed layer structure enables both correction of artifacts native to the frequency-space and manipulation of image-space representations to reconstruct coherent image structures. This is in contrast to the current deep learning approaches for image reconstruction that manipulate data solely in the frequency-space or solely in the image-space. We demonstrate the advantages of the proposed joint learning on three diverse tasks including image reconstruction from undersampled acquisitions, motion correction, and image denoising in brain MRI. Unlike purely image based and purely frequency based architectures, the proposed joint model produces consistently high quality output images. The resulting joint frequency- and image-space feature representations promise to significantly improve modeling and reconstruction of images acquired in the frequency-space. Our code is available at [https://github.com/nalinimsingh/interlacer](https://github.com/nalinimsingh/interlacer).

1 Introduction

Imaging modalities that acquire frequency-space data and convert these measurements to images for visualization and downstream analysis include magnetic resonance imaging [1], Fourier space optical coherence tomography [2], Fourier ptychography [3], and synthetic aperture radar [4]. Practical imaging considerations often affect these data acquisition processes. For example, sub-Nyquist undersampling is routinely used to speed up data acquisition [5], motion occurs during acquisition [6], and noise affects sensor readings [7]. The acquired frequency-space data is often converted to image-space reconstructions via an inverse Fourier transform, with each individual frequency-space measurement contributing to all output pixels in the image-space. As a result, local changes in the acquired frequency-space data induce global effects on the entire output image. To produce accurate image reconstructions, modeling tools for Fourier imaging must correct these global image-space artifacts in addition to performing fine-scale image-space processing to produce coherent structure.

Consider the correlation structure for both frequency- and image-space representations of MNIST and brain MRI for a particular pixel (Fig. 1). In all cases, local neighborhoods around the pixel exhibit strong correlations, suggesting that local convolution operations, which have proven successful on image-space computer vision tasks, might also be useful when applied to frequency-space data. Convolutional frequency-space processing enables direct correction of local frequency-space changes...
Figure 1: Maps of correlation coefficients between a single pixel (circled) and other pixels in image-space representations and frequency-space representations (‘spectra’) of MNIST (a,b) and a brain MRI dataset (c,d). All maps exhibit strong local correlations useful for inferring missing or corrupted data. Frequency-space correlations (b,d) also display the conjugate symmetry characteristic of frequency-space data since they represent Fourier transforms of real images.

corresponding to global image-space effects. However, imprecisions in a frequency-space representation yield jarring visual artifacts in the corresponding image-space representation. Convolutional image-space processing offers complementary correction of these artifacts.

We combine the strengths of these two strategies via a neural network layer that learns weighted combinations of frequency-space and image-space convolutions. We demonstrate the proposed network in the context of brain MRI reconstruction in the presence of undersampling, motion, and noise. Our network performs as well as or better than pure image-space or pure frequency-space convolutional architectures for all artifacts, at varying levels of image corruption. In contrast, frequency-space and image-space architectures each perform poorly on at least one of the three tasks. These results suggest that joint image-space and frequency-space feature representations are a useful, consistent building block for modeling and reconstructing Fourier imaging data.

2 Background and Related Work

In 2D MRI, the goal is to reconstruct a 2D image $I$ from the acquired 2D discrete Fourier transform measurements $\hat{F} = \mathcal{F}\{\hat{I}\}$. Classically, this reconstruction is computed via a 2D inverse Fourier transform, producing estimated image $\hat{I} = \mathcal{F}^{-1}\{\hat{F}\}$. We consider Cartesian sampling, where measurement coordinates $k_x$ and $k_y$ are evenly sampled across the 2D Fourier plane. In practice, all of the points within a single line $F[k_x, k_y]$ in frequency-space are acquired rapidly together. In this section, we describe three observation processes in which corrupted measurements $\tilde{F}$ are acquired instead of $F$ and previous methods for estimating the desired image $I$ from $\tilde{F}$.

**Undersampling.** To speed up image acquisition, a common technique is to only acquire data at a subset $S_y$ of "lines", i.e., values of $k_y \in S_y$:

$$\hat{F}[k_x, k_y] = \begin{cases} F[k_x, k_y] & k_y \in S_y \\ 0 & k_y \notin S_y. \end{cases}$$

Classical image reconstruction techniques for undersampled data vary in their choice of operating domain for performing reconstruction. SENSE reconstruction reduces the problem to least-squares estimation of the image-space from undersampled frequency-space data [8]. In contrast, GRAPPA applies convolutions in Fourier space to estimate missing lines, and then applies the inverse Fourier transform to reconstruct the image [9]. Most recent deep learning methods apply convolutions to image-space reconstructions of frequency-space data [10]-[15]. Others apply convolutional architectures directly to frequency-space data [16]-[18]. An alternative strategy, AUTOMAP, uses fully-connected layers to effectively convert frequency-space data to the image-space and then applies further image-space convolutions [19]. The size of such a network is quadratic in the size of the image, incurring immense memory costs for reconstructing larger images.

In this paper, we propose a flexible, fundamental layer structure that combines the advantages of frequency-space and image-space strategies. Insights from previous work (e.g., data consistency via cascading [11]) can be further applied to this basic architecture. Because our proposed layer structure only uses convolutions, it incurs a memory cost that is linear in the image size.

**Motion.** We consider rigid-body subject motion that occurs between acquisition of successive frequency-space lines. If a line $k_y$ is affected by a rotation $\phi_{k_y}$ about the origin, horizontal translation $\Delta x_{k_y}$, and vertical translation $\Delta y_{k_y}$, the acquired signal $\tilde{F}[, k_y] = \mathcal{F}\{\tilde{I}_{k_y}\}[, k_y]$ is determined by
Figure 2: Diagram of our proposed layer (dotted box), embedded within a full network structure. Weighted combinations of frequency- and image-space features undergo Batch Normalization (BN) and convolution, and pass through an activation function at each layer.

the moved image $\tilde{I}_{k_y}$ during acquisition of line $k_y$:

$$\tilde{I}_{k_y}[x, y] = I \left[ (x - \Delta x_{k_y}) \cos \phi_{k_y} - (y - \Delta y_{k_y}) \sin \phi_{k_y}, \right.$$

$$\left. (x - \Delta x_{k_y}) \sin \phi_{k_y} + (y - \Delta y_{k_y}) \cos \phi_{k_y} \right].$$  \hspace{1cm} (2)

Eq. (2) forms a translated and rotated version of the original image $I$. A pure translation without rotation in the image-space corresponds to a phase shift in the frequency-space:

$$\tilde{F}_{t}[k_x, k_y] = F[k_x, k_y] \exp \left\{ -j2\pi \left( \frac{k_x \Delta x_{k_x}}{N} + k_y \Delta y_{k_y} \right) \right\},$$  \hspace{1cm} (3)

for an image of size $N \times N$ pixels. A pure rotation about the center of the image-space without translation corresponds to a rotation by the same angle in the frequency-space:

$$\tilde{F}_{r}[k_x, k_y] = F \left[ k_x \cos \phi_{k_y} - k_y \sin \phi_{k_y}, k_x \sin \phi_{k_y} + k_y \cos \phi_{k_y} \right].$$  \hspace{1cm} (4)

In the above development, we consider the discrete image and Fourier transform corresponding to a continuous-valued image estimated via a spline filter of order 3.

Previous retrospective motion correction strategies \cite{20, 21} are cast as large, non-convex optimization problems with iterative solutions that are slow to compute. More recent deep learning approaches \cite{22–25} frame the motion correction problem as an image-to-image prediction task, with the neural network operating purely on image-space representations, even though motion artifacts are induced directly in the frequency-space during the MRI data acquisition process.

Noise. Noisy MRI data can be modeled via an additive i.i.d. complex Gaussian distribution:

$$\tilde{F}[k_x, k_y] = F[k_x, k_y] + \epsilon_1 + j\epsilon_2, \text{ where } \epsilon_1, \epsilon_2 \sim \mathcal{N}(0, \sigma^2 I_{N \times N}), \epsilon_1 \perp \epsilon_2.$$  \hspace{1cm} (5)

This noise distribution gives rise to the standard Rician distribution on MRI image-space pixel magnitudes \cite{26}. Previous work on MRI denoising applies classical signal processing techniques including filtering \cite{27} and wavelet-based methods \cite{28, 29}. Recent deep learning methods for denoising apply learned convolutional networks solely on image-space data \cite{30–32}.

3 Joint Frequency- and Image-Space Networks

We use a neural network to estimate a complex ground truth image $I$ from a complex, corrupted acquired signal $\tilde{F}$. In this section, we introduce the novel structure of the layers in the proposed network, referred to as JOINT, and specify the overall network architecture and learning procedure.

3.1 Proposed Layer Structure

A diagram of the proposed network architecture is shown in Fig. 2. We use $u_n$ to denote the frequency-space input and $v_n$ to denote the image-space input of layer $n$. Following this notation, $u_0 = \tilde{F}$ and $v_0 = \mathcal{F}^{-1} \{ u_0 \}$ are the frequency-space and image-space inputs to the network, respectively. These inputs are combined via learned, layer-specific mixing parameters $\alpha_n$ and $\beta_n$; $\alpha_n$ and $\beta_n$
We choose an $\ell$ (where $\ell$ is the set of learned frequency-space convolution weights and biases), which is the set of learned frequency-space parameters.

$\hat{u}_n = s(\alpha_n) u_n + (1 - s(\alpha_n)) \mathcal{F} \{v_n\}$, 
$\hat{v}_n = s(\beta_n) v_n + (1 - s(\beta_n)) \mathcal{F}^{-1} \{u_n\}$. 

(6)

Real and imaginary parts of layer inputs are represented as separate channels at each stage of computation and are joined appropriately to form complex numbers before computing the Fourier transform $\mathcal{F} \{\cdot\}$ or its inverse. Next, the layer applies a combination of batch normalization (BN), convolution, and activation functions with a skip connection to produce the outputs:

$u_{n+1} = \sigma(w_n \odot \text{BN}(\hat{u}_n) + b_n) + u_0$,
$v_{n+1} = \sigma'(w_n' \odot \text{BN}(\hat{v}_n') + b_n') + v_0$. 

(7)

where $(w_n, b_n)$ is the set of learned frequency-space convolution weights and biases, $(w_n', b_n')$ are their image-space counterparts, and $\sigma(\cdot)$ and $\sigma'(\cdot)$ are activation functions specific to the frequency-space and image-space network components. Our choices for these functions are described below.

This layer structure is a generalization of networks that operate purely in frequency-space, obtained by choosing $s(\alpha_n) = 1$ and $s(\beta_n) = 0$, and of networks that operate purely in image-space, that arise by choosing $s(\alpha_n) = 0$ and $s(\beta_n) = 1$. When $0 < s(\alpha_n) < 1$ and $0 < s(\beta_n) < 1$, this layer represents a function that cannot be expressed solely via purely image-space or purely frequency-space convolutional layers that do not invoke the Fourier transform or its inverse. Although our layers include explicit Fourier transforms and their inverses, no parameters are learned in association with those transforms. Thus, our network architecture requires learning only convolutional weights and biases and mixing coefficients, and has linear space complexity in the size of the image.

### 3.2 Activation Functions

Standard image-space neural networks typically use the ReLU nonlinearity; we accordingly set $\sigma'(x) = \text{ReLU}(x)$. However, the zero-gradient of this nonlinearity for negative values is ill-suited for networks that operate on frequency-space data as individual inputs can take on a large range of positive and negative values. Normalizing the values to a range such as [-1,1] can induce vanishing gradients for the smallest of the absolute values. We introduce an alternative nonlinear activation function that we apply to both the real and imaginary channels of each layer:

$\sigma(x) = x + \text{ReLU} ((x - 1)/2) + \text{ReLU} ((-x - 1)/2)$. 

(8)

This nonlinearity has non-zero gradient for all inputs where the gradient is defined. We found that using this nonlinearity produced superior results for networks using frequency-space convolutional layers, as shown in Figure 3.

### 3.3 Learning

We choose an $\ell_1$ metric to quantify reconstruction quality and seek $\hat{I}$ that minimizes $|I - \hat{I}|$, although the proposed architecture can be integrated with other differentiable error measures. Training a deep network $f(\cdot; \theta_f, \theta_i)$ for image reconstruction involves optimizing a set of frequency-space parameters $\theta_f$ and a set of image-space parameters $\theta_i$ over the training image dataset $\mathcal{I} = \{(\hat{F}_m, I_m)\}$:

$(\theta_f^*, \theta_i^*) = \arg \min_{\theta_f, \theta_i} \sum_{m=1}^{\mathcal{|I|}} |I_m - \mathcal{F}^{-1} \left(f(\hat{F}_m; \theta_f, \theta_i)\right)|$. 

(9)
Here, $\theta_f = \{(w_n, b_n, \alpha_n]\}$ and $\theta_i = \{(w'_n, b'_n, \beta_n]\}$. We employ stochastic gradient descent-based strategies to identify the optimal set of parameters $(\theta_f^*, \theta_i^*)$.

## 4 Implementation Details and Baseline Models

The proposed network contains 6 joint frequency- and image-space layers. A single 2D convolutional layer acts on the frequency-space output $u_0$ of the final joint layer to produce the final 2-channel complex output $\hat{F}$. The estimated image $I$ is the inverse Fourier transform of the network’s output, i.e., $I = \mathcal{F}^{-1}\{\hat{F}\}$. Each joint layer contains dual frequency-space and image-space convolution blocks with kernel size 9x9 and 32 output features, resulting in a total of 763,581 parameters.

We aim to evaluate the utility of the joint frequency- and image-space layer as a network building block for manipulating Fourier imaging data. To that end, we focus on comparing performance of our JOINT architecture to similarly structured baseline architectures with only frequency- or only image-space operations, instead of comparing our network to task-specific architectures. Task-specific network design strategies, such as cascading [17, 18], could further augment our proposed architecture.

First, we created an architecture that performs convolutions on only frequency-space data (FREQ). Training this network reduces to the optimization:

$$\theta_f^* = \arg \min_{\theta_f} \sum_{m=1}^{\|I\|} \left| I_m - \mathcal{F}^{-1} \left( g(\mathcal{F}_m; \theta_f) \right) \right|. \tag{10}$$

The network contains 12 convolution blocks to match the JOINT network’s 6 pairs of 2 convolution blocks. As in the JOINT network, each convolution block has kernel size 9x9 and 32 output features, followed by the final, two-feature 2D convolutional layer, resulting in 924,554 parameters. Each layer operates solely on frequency-space data. By operating directly on frequency-space representations, this architecture emulates the related approaches in [17,18].

We also implemented an image-space architecture (IMAGE), trained by optimizing $\theta_i$:

$$\theta_i^* = \arg \min_{\theta_i} \sum_{m=1}^{\|I\|} \left| I_m - g \left( \mathcal{F}^{-1}(\mathcal{F}_m); \theta_i \right) \right|. \tag{11}$$

This network’s architecture is identical to that of FREQ and also contains 924,554 parameters, but it operates on and outputs image-space data. We noticed that this architecture consistently landed in local minima, so we also tested versions of this architecture without residual connections to $u_0$ and $v_0$ and report the best result. By applying convolutions to image-space representations, the network emulates the related approaches in [17,18,22,24,30].

We initialized all convolution weights using the He normal initializer [33] and used the Adam optimizer [34] (learning rate 0.001) until convergence. We initialized $s(\alpha)$ and $s(\beta)$ to 0.5. Training each model required between 12 hours and 3 days on an NVIDIA RTX 2080 Ti GPU. Our code is available at [https://github.com/nalinimsingh/interlacer](https://github.com/nalinimsingh/interlacer).

## 5 Experiments

### Data.

We demonstrate the advantages of our method on a large collection of T1-weighted brain MRI images from patients aged 55-90 collected as part of the Alzheimer’s Disease Neuroimaging Initiative (ADNI) [35]. For training and evaluation, we selected the central 2D axial image of each volume, resampled to a 128x128 grid of 2mm resolution, and rescaled it to axial intensity range [0,1]. To simulate acquired data, we applied the 2D Fourier transform to each image. We split the dataset into 4,115 training images, 2,061 validation images, and 96 test images such that that no subjects were shared across the training, validation, and test sets. Preliminary experiments and hyperparameters were evaluated on the validation dataset; the test set was only used for computing the final results.

### Evaluation.

We compare the reconstructed images to the ground truth using mean absolute error as a metric of reconstruction quality. Dataset-wide and subjectwise error metrics are shown in Fig. 4 and Fig. 5, respectively. Each individual experiment is discussed below.

### Undersampling.

To simulate undersampling as described in Section 2, we set the sampling frequency $\gamma_s$ to be 33%, 25%, or 10% (equivalent to an acceleration factor of 3, 4, and 10, respectively). The selected line indices $S_y$ were sampled at random, without a bias toward the low-frequency
lines at the center of the Fourier plane. This process degrades the low frequency structure of the corresponding images with varying degrees of severity, ensuring that the trained network is robust in the face of varying acquisition schemes, and contrasts with previous work [11, 13], where the center rows of the Fourier plane are sampled more densely than the rest of the image across training and test examples. The ground truth data used in this work has conjugate symmetry in the frequency-space, so in the hypothetical case of $\gamma_s=50\%$ (acceleration factor 2) with our random sampling scheme it is possible, but not guaranteed, that all of the data required to perfectly reconstruct the image is present in the input. This is impossible for the acceleration factors considered here. Performance statistics of all networks are reported in Fig. 4a and Fig. 5a. The proposed JOINT architecture outperforms all baseline architectures at all three levels of undersampling (Fig. 3a), and consistently across nearly all test subjects (Fig. 4a). Fig. 6 illustrates sample reconstruction results.

**Motion.** To simulate motion during image acquisition as described in Section 2, we sampled a horizontal translation $\Delta x$, vertical translation $\Delta y$, and rotation $\phi$ for the fraction $\gamma_m$ of the total number of frequency-space lines, and applied the sampled translation to contiguous lines in frequency-space between consecutive motion line samplings. We set $\gamma_m$ to be 0.01, 0.05, or 0.1. Translation parameter values were drawn uniformly from the range $[-20px, 20px]$. Rotation parameter values were drawn uniformly from the range $[-15^\circ, 15^\circ]$. For a Cartesian, fully-sampled acquisition, the combined frequency-space data at the end of this process represents the signal acquired when the imaging subject shifts according to the sampled motion parameters at each of the randomly sampled lines in frequency-space. Fig. 4b and Fig. 5b report performance statistics of all networks. The proposed JOINT architecture performs as well as or better than the FREQ architecture, and both outperform the IMAGE architecture at all three levels of motion (Fig. 4b). Further, the JOINT architecture performs the best across nearly all test subjects, except for a small number where all methods perform relatively poorly (Fig. 5b). Fig. 7 illustrates sample reconstruction results.

**Denoising.** To simulate noise during image acquisition as described in Section 2, we added independent noise to the real and imaginary parts of each element in frequency-space, sampled from a zero-mean Gaussian distribution with standard deviation $\gamma_n$ as either 10.0, 20.0, or 30.0. Fig. 4c and Fig. 5c report the performance statistics of all networks. Our JOINT architecture performs as well as or better than the IMAGE and FREQ architectures (Fig. 4c). Further, the JOINT architecture performs the best or equal to the IMAGE architecture across nearly all test subjects (Fig. 5c). Fig. 8 illustrates sample reconstruction results.

**Cross-Experiment Discussion.** Results in Fig. 4 suggest that neural networks based on our joint frequency- and image-space layers perform on-par or better than the purely image- and purely frequency-space architectures tested at every corruption level for every task. This is in contrast to purely frequency- or purely image-space architectures, both of which do poorly compared to the other two networks on at least one task. Fig. 5 further demonstrates that networks comprised of our proposed layer consistently outperform the other networks in terms of absolute reconstruction error.

**Figure 4:** Mean absolute error across corruption levels for all three tasks.

**Figure 5:** Subjectwise mean absolute error comparison for all three tasks, for the middle value of $\gamma$. 

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for nearly every subject, except in a few cases for which all architectures perform poorly. Across all corruption models, the JOINT architecture was statistically significantly superior to the FREQ and IMAGE architectures (pairwise t-test: \( p < 10^{-6} \) for all experiments). The learned mixing coefficients \( s(\alpha) \) and \( s(\beta) \) at each layer of the JOINT architecture are provided in Fig. 9. Different mixing parameter configurations are learned for different tasks, suggesting that flexibility in learning the mixtures is important for accommodating different imaging artifacts.

6 Discussion and Conclusion

We present a fundamental neural network architecture building block for reconstructing corrupted Fourier imaging data. Unlike previous strategies, our approach provides both correction of artifacts native to the frequency-space and reconstruction of coherent structures in the image-space. Further, the network learns the degree to which image- and feature-space representations should be mixed at individual stages of computation.

We demonstrate that this strategy performs as well as or better than pure image- or frequency-space baselines on image reconstruction tasks under three diverse data corruption mechanisms. This finding holds even in the regime of extreme data corruption. We find that the learned mixing parameters vary with the method and degree of induced corruption, suggesting that the representation flexibility afforded by this architecture enables generalization to different tasks. This is particularly in the setting of real-world MRI reconstruction, where several of the corruption models analyzed here coexist.

In the future, we plan to combine our proposed layer architecture with task-specific reconstruction techniques developed on single-space networks. The flexibility of the proposed layer architecture and its ease of embedding into a variety of network architectures makes it easy to combine with, for example, imposition of a hard or soft data consistency constraint that has been an effective strategy for undersampled reconstruction [11,13]. We also plan to investigate local operations
beyond convolutions that more directly capitalize on properties frequency-space data, for use in that arm of our proposed layer. Further, we plan to investigate applications of our architecture to challenging image reconstruction problems in modalities beyond MRI. The combination of these advances promises significantly improved reconstruction and analysis of Fourier imaging data.

**Broader Impact**

Our work shows promise to significantly improve the quality, applicability, cost, and accessibility of clinical MRI scans. First, our method is able to reconstruct images accurately from aggressively undersampled data, enabling accelerated scan time. This fact, combined with the ability of our method to correct motion-corrupted data, improves imaging of dynamic, moving anatomy, as in cardiac and fetal MRI. Further, the ability of our method to reconstruct data in the presence of significant noise would improve the quality of images generated via low-field MRI, decreasing the cost and improving the accessibility of MRI. Via improved quality of diagnostic imaging, application of our method thus has the potential to improve outcomes in cardiac care, fetal and maternal health, and resource-constrained settings where only low-field MRI is available.

For proper application of this method to a particular imaging application, care must be taken to collect training and evaluation data that reflects subjects with a wide variety of demographic and physiological characteristics to enable accurate generalization to future imaging subjects.

**Acknowledgments**

NMS is supported by an NSF Graduate Research Fellowship. This research was supported by NIBIB, NICHD, NIA, and NINDS of the National Institutes of Health under award numbers 5T32EB1680, P41EB015902, R01EB017337, R01HD100009, 1R01AG064027-01A1 and 5R01NS105820-02, and European Research Council Starting Grant 677697, project “BUNGEE-TOOLS”.
Figure 8: Example reconstructions with pixelwise Gaussian noise of standard deviation 20.0, zoomed-in image patches, difference patches between reconstructions and ground truth data, and frequency-space reconstructions.

Figure 9: Mixing coefficients for all JOINT networks. The amount of mixing varies across different tasks and data corruption levels.
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Figure 10: Additional example 4x undersampled reconstruction.

Figure 11: Additional example 4x undersampled reconstruction.
Figure 12: Additional example reconstruction of image corrupted with motion at 5% of lines.

Figure 13: Additional example reconstruction of image corrupted with motion at 5% of lines.
Figure 14: Additional example reconstruction of image corrupted with noise of variance 20.0.

Figure 15: Additional example reconstruction of image corrupted with noise of variance 20.0.