The renormalization group for correlated scales:

one-stage versus two-stage running

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Abstract

Nonrelativistic bound states have two low energy scales, a soft scale $\mu_S$ of order $mv$ and an ultrasoft scale $\mu_U$ of order $mv^2$. In two-stage running, the soft and ultrasoft scales are lowered from $m$ to $mv$, and then the ultrasoft scale is lowered from $mv$ to $mv^2$. In one-stage running, the two scales are lowered in a correlated way using a subtraction velocity. We compare these two methods of summing logarithms and show that only the correlated running in velocity space of the one-stage method correctly reproduces the logarithms in nonrelativistic bound states in QED. The argument for one-stage running is general, and should apply to any system with correlated scales.
Nonrelativistic bound states have two important scales, a soft scale $\mu_S$ of order $mv$ and an ultrasoft scale $\mu_U$ of order $mv^2$, which are of order the typical momentum and energy of the bound state \[1,2\]. In a Coulombic bound state such as Hydrogen, the typical velocity $v$ is of order the coupling constant $\alpha$. Logarithms of $v$ or equivalently $\alpha$ can be summed using renormalization group equations (RGE) [3–5]. This is accomplished by constructing an effective theory below the scale $m$ that contains soft and ultrasoft modes [1–17], and then scaling the coefficients in the effective Lagrangian to low energies using the RGE. In a properly constructed effective theory, all the large logarithms should arise from renormalization group running, and there should be no large logarithms in the matching conditions. In QCD the summation of logarithms is important because $\alpha_s \ln v \sim 1$ and the RGE systematically accounts for the running of $\alpha_s$. In QED, $\alpha \ln \alpha$ is small but higher order corrections are necessary to confront precision experiments.

In two-stage running, one first lowers the subtraction scale $\mu$ from $m$ to $mv$. At this point the soft modes, with energy-momentum of order $mv$, are integrated out of the theory. The ultrasoft modes are then lowered from $mv$ to $mv^2$. This is the definition of two-stage running that we will use in this article. In one-stage running [3], one introduces a subtraction velocity $\nu$, with $\mu_S = m\nu$ and $\mu_U = m\nu^2$. The subtraction velocity $\nu$ is then lowered from $\nu = 1$ to $\nu = v$, so that $\mu_S = mv$ and $\mu_U = mv^2$ at the end of the renormalization group evolution. The difference between the two methods can be seen in Fig. 1, where the path in the $\mu_S$–$\mu_U$ plane is shown. In contrasting the two methods we will consider NRQED, where the analysis is simplified by the fact that the coupling constant $\alpha$ does not run below the electron mass. Both the two-stage and one-stage methods correctly reproduce observables with a single $\ln \alpha$, such as the $\alpha^5 \ln \alpha$ Lamb shifts in Hydrogen and positronium [10–17,5]. In Ref. [5] one-stage running was shown to correctly reproduce the $\alpha^7 \ln^2 \alpha$ hyperfine splittings for Hydrogen, positronium and muonium [10–20], the $\alpha^3 \ln^2 \alpha$ ortho and para-positronium widths [18–20], and the $\alpha^8 \ln^3 \alpha$ Hydrogen Lamb shift [18–21]. This is a highly non-trivial check of the one-stage method, and is the first time higher-order logarithms in QED bound states were computed using the renormalization group. In previous computations, higher order logarithms were obtained by direct computation of multi-loop diagrams [18–22]. In this article, we explain why the two-stage method fails to reproduce the $\ln^2 \alpha$ and $\ln^3 \alpha$ terms, and why the one-stage method is essential for correctly summing all the logarithms 1,2

1See, for example, Sec. 1.4 of Ref. [15]. The running in Ref. [2] corresponds to the $m$ to $mv$ running of the two-stage method.

2Note that the $\alpha^8 \ln^3 \alpha$ term has also been calculated numerically in Refs. [23–25] and there is general agreement on the value of the 4-loop diagrams calculated analytically by Karshenboim in Ref. [18]. However, Refs. [23–25] have an additional contribution from another graph.
FIG. 1. Paths in the $\mu_U - \mu_S$ plane for one-stage and two-stage running using the renormalization group equations.

A loop graph computed in the effective theory below $m$ typically contains logarithms of the form $\ln p/\mu_S$, $\ln E/\mu_U$ and $\log \sqrt{mE}/\mu_S$, where $E$ and $p$ are the typical energy and momentum in the bound state. We will define the soft and ultrasoft anomalous dimensions $\gamma^S$ and $\gamma^U$ by differentiating with respect to $\mu_S$ and $\mu_U$ respectively. Introducing both $\mu_S$ and $\mu_U$ allows us to simultaneously discuss the one and two-stage running. In the effective theory the potential is the Wilson coefficient of a four-fermion operator and has an expansion in $v$. The coefficients $\{U_i\}$ in the potential have anomalous dimensions:

$$\mu_S \frac{dU_i}{d\mu_S} = \gamma^S_i, \quad \mu_U \frac{dU_i}{d\mu_U} = \gamma^U_i,$$

(1)

where $\gamma^S$ and $\gamma^U$ are functions of $\alpha$ and potential coefficients $\{U_j\}$.

In two-stage running, one sets $\mu = \mu_S = \mu_U$ and lowers $\mu$ from $m$ to $mv$. Thus, one runs the potentials between $m$ and $mv$ using the anomalous dimension $\Gamma^S$:

$$\mu \frac{dU_i}{d\mu} = \Gamma^S_i = \gamma^S_i + \gamma^U_i \quad mv < \mu < m.$$  

(2)

One then integrates out the soft modes at $\mu = mv$, and runs to $mv^2$ with the ultrasoft anomalous dimension.

In pNRQED, $\Gamma^S_i$ is called the soft anomalous dimension because it governs the running between the hard scale $m$ and the soft scale $mv$. In the one-stage approach, it is $\gamma^S_i$ which is called the soft anomalous dimension. This is merely a difference in terminology with no physical consequences.
\[
\mu \frac{dU_i}{d\mu} = \gamma_i^U \quad \text{for} \quad mv^2 < \mu < mv.
\] (3)

In one-stage running the subtraction scales \(\mu_S\) and \(\mu_U\) are different: \(\mu_S = mv\) and \(\mu_U = m\nu^2\).

With the velocity renormalization group one runs from \(\nu = 1\) to \(\nu = v\) with the anomalous dimension

\[
\nu \frac{dU_i}{d\nu} = \gamma_i^S + 2\gamma_i^U \quad v < \nu < 1.
\] (4)

As an example, consider the running of the \(\delta\)-function contact potential for Hydrogen in the limit \(m_{\text{proton}} \to \infty\). The coefficient of this potential will be defined as \(U_2\), using the notation of Ref. [5]. \(U_2\) plays a key role in NRQED since it is the lowest order coefficient that runs. With one-stage running \(U_2\) has a velocity renormalization group equation of the form [5]

\[
\nu \frac{dU_2}{d\nu} = 2\alpha^2 \lambda^U + \alpha \lambda^S U_2^2 + \ldots.
\] (5)

Here \(\lambda^U = -4/(3m^2)\), \(\lambda^S = m^2/\pi\), and the matching value \(U_2(\nu = 1) = \pi\alpha/(2m^2)\), where \(m\) is the electron mass. The coefficient \(U_2\) is of order \(\alpha\), so the first term in Eq. (5) is the leading order (LO) anomalous dimension, and the second term belongs to the next-to-leading order (NLO) anomalous dimension. For Hydrogen, the first term is purely ultrasoft and the second term is purely soft, so

\[
\gamma^U = \alpha^2 \lambda^U, \quad \gamma^S = \alpha \lambda^S U_2^2.
\] (6)

In Eq. (5), we have only retained the NLO term relevant for the \(\alpha^8 \ln^3 \alpha\) Lamb shift in Hydrogen. The full equation is given in Ref. [5] and gives additional contributions to \(\alpha^7 \ln^2 \alpha\) and \(\alpha^6 \ln \alpha\), but the simplified version in Eq. (5) will suffice for this paper.

In our example, integrating the renormalization group equation using the LO anomalous dimension and one-stage running gives

\[
U_2(\nu = v) = U_2(1) + 2\alpha^2 \lambda^U \ln v,
\] (7)

which is the leading order series \(\alpha^{n+1} \ln^n v\) for \(U_2(\nu)\) and determines the \(\alpha^5 \ln \alpha\) Lamb shift correction. There is only a single-log term since the LO anomalous dimension is a constant. To next-to-leading log order only the leading log value of \(U_2\) is needed on the RHS of Eq. (5). Substituting Eq. (6) for \(U_2\) into Eq. (5) and reintegrating gives

\[4\] In dimensional regularization where \(d = 4 - 2\epsilon\), factors of \(\mu_S^\epsilon\) and \(\mu_U^\epsilon\) appear along with interaction vertices involving soft and ultrasoft modes respectively [3,4].
\[ U_2(\nu = v) = U_2(1) + 2\alpha^2\lambda U \ln v \]
\[ + \alpha \lambda S U_2(1)^2 \ln v + 2 \alpha^3 \lambda U \lambda S U_2(1) \ln^2 v + \frac{4}{3} \alpha^5 \left( \lambda U \right)^2 \lambda S \ln^3 v, \]

which includes the LO series \( \alpha^{n+1} \ln^n v \) and the next-to-leading order series \( \alpha^{n+2} \ln^n v \). The highest logarithm in the NLO series is \( \ln^3 v \), since the NLO anomalous dimension has a \( U_2^2 \) term that contains \( \ln^2 v \) terms when the LO running for \( U_2 \) is used. The \( \ln^3 v \) term gives an \( \alpha^8 \ln^3 \alpha \) Lamb shift for Hydrogen in agreement with Karshenboim [18].

Next we redo this example using two-stage running. In the first step, one integrates
\[ \mu \frac{dU_2}{d\mu} = \alpha^2\lambda U + \alpha \lambda S U_2^2, \]

to give
\[ U_2(\mu = mv) = U_2(1) + \alpha^2\lambda U \ln v \]
\[ + \alpha \lambda S U_2(1)^2 \ln v + \alpha^3 \lambda U \lambda S U_2(1) \ln^2 v + \frac{1}{3} \alpha^5 \left( \lambda U \right)^2 \lambda S \ln^3 v, \]

where \( U_2(1) = U_2(\mu = m) \). In the second step one integrates
\[ \mu \frac{dU_2}{d\mu} = \alpha^2\lambda U. \]

Since the \( \lambda S \) term comes solely from the soft scale it does not appear in the RGE for the second step. Integrating Eq. (11) using Eq. (10) as the starting point gives the final result
\[ U_2(\mu = mv^2) = U_2(1) + 2\alpha^2\lambda U \ln v \]
\[ + \alpha \lambda S U_2(1)^2 \ln v + \alpha^3 \lambda U \lambda S U_2(1) \ln^2 v + \frac{1}{3} \alpha^5 \left( \lambda U \right)^2 \lambda S \ln^3 v. \]

Comparing Eqs. (8), and Eqs. (12), we see that the final expressions agree in the single-log term, but disagree in the higher-order logarithms.

One can study the difference between one- and two-stage running in the general case, by integrating the renormalization group equations perturbatively. The single-log terms (i.e. proportional to \( \ln v \)) are obtained by integrating the renormalization group equations using the values for the anomalous dimensions at the matching scale, i.e. by substituting the tree-level values for \( \{U_i\} \) in \( \gamma(\{U_i\}) \), which will be denoted by \( \gamma(0) \). In the two-stage approach, one finds that
\[ U_i(\mu = mv^2) = \Gamma_i^S(0) \ln v + \gamma_i^U(0) \ln v = \left[ \gamma_i^S(0) + \gamma_i^U(0) \right] \ln v + \gamma_i^U(0) \ln v, \]

where the first term is the contribution between \( m \) and \( mv \), and the second term is the contribution between \( mv \) and \( mv^2 \). In the one-stage approach, one finds
\[ U_i(\nu = v) = \left[ \gamma_i^S(0) + 2\gamma_i^U(0) \right] \ln v. \]
Both methods give the same answer for the $\ln v$ term, since

$$
\Gamma_i^S(0) + \gamma^U_i(0) = \left[\gamma_i^S(0) + \gamma^U_i(0)\right] + \gamma^U_i(0) = \left[\gamma_i^S(0) + 2\gamma^U_i(0)\right].
$$

(15)

This is to be expected, since the anomalous dimensions in both methods should reproduce the single logarithms in the graphs from which they were calculated. This is true of our Hydrogen example, cf. Eq. (8) and Eq. (12). We have also checked that this is true for the $\alpha^5 \ln \alpha$ Lamb shift for positronium in which case the LO anomalous dimension has both an ultrasoft and soft contribution.

The non-trivial terms are the higher order logarithms, which are generated when one includes the running potentials in the expression for the anomalous dimensions $\gamma^{S,U}$. The anomalous dimensions depend on $t = \ln v$ through their dependence on $U_i$. Consider expanding $\gamma$ in powers of $t$,

$$
\gamma_j(t) = \gamma_j(0) + t \left(\frac{d\gamma_j}{dt}\right)_{t=0}^2 + \frac{t^2}{2} \left(\frac{d^2\gamma_j}{dt^2}\right)_{t=0}^2 + \ldots,
$$

(16)

where the derivatives with respect to $t$ can be evaluated using the chain rule,

$$
\gamma_j(t) = \gamma_j(0) + t \left(\frac{\partial\gamma_j}{\partial U_k} \frac{dU_k}{dt}\right)_{t=0}^2 + \frac{t^2}{2} \left(\frac{\partial^2\gamma_j}{\partial U_k \partial U_l} \frac{dU_k}{dt} \frac{dU_l}{dt}\right)_{t=0}^2 + \ldots
$$

$$
= \gamma_j(0) + t \left(\frac{\partial\gamma_j}{\partial U_k} \frac{\partial\gamma_k}{\partial U_l} \gamma_l\right)_{t=0}^2 + \frac{t^2}{2} \left(\frac{\partial^2\gamma_j}{\partial U_k \partial U_l} \gamma_k \gamma_l + \frac{\partial\gamma_j}{\partial U_k} \frac{\partial\gamma_k}{\partial U_l} \gamma_l\right)_{t=0}^2 + \ldots.
$$

(17)

For Hydrogen, the $\ln^2 v$ and $\ln^3 v$ terms are generated from cross-terms of the form $\gamma^U \partial \gamma^S / \partial U_2$ and $\left(\gamma^U\right)^2 \partial^2 \gamma^S / \partial U_2^2$ that involve both soft and ultrasoft anomalous dimensions. In the one-stage approach, the renormalization group equations are integrated with the anomalous dimension in Eq. (8) between $\nu = 1$ and $\nu = v$, so the cross-terms have the structure

$$
(2\gamma^U) \frac{\partial \gamma^S}{\partial U_2} \ln^2 v, \quad (2\gamma^U)^2 \frac{\partial^2 \gamma^S}{\partial U_2^2} \ln^3 v.
$$

(18)

In the two-stage approach, one integrates Eq. (8) from $m$ to $mv$, after which $\gamma^S = 0$, so the corresponding terms have the structure

$$
(\gamma^U) \frac{\partial \gamma^S}{\partial U_2} \ln^2 v, \quad (\gamma^U)^2 \frac{\partial^2 \gamma^S}{\partial U_2^2} \ln^3 v,
$$

(19)

which differ by a factor of two and four, respectively, from the one-stage running values.\footnote{Recall that there are terms in the NLO anomalous dimension that are not shown in Eq. (8) which contribute to terms involving $\ln^2 \alpha$ in the energy. Including these additional terms makes the first term in Eq. (18) and in Eq. (19) more complicated, however the one- and two-stage $\ln^2 v$ results still differ by an overall factor of 2.}
In our Hydrogen example, $\gamma^U = \alpha^2 \lambda^U$ and $\gamma^S = \alpha \lambda^S U^2_2$, so that
\[
\mu_U \frac{d\gamma^S}{d\mu_U} = 2\alpha \lambda^S U_2 \mu_U \frac{dU_2}{d\mu_U} = 2\alpha^3 \lambda^S \lambda^U U_2,
\]
\[
\mu_S \frac{d\gamma^U}{d\mu_S} = 0,
\]
and the integral is path-dependent.

In Ref. [5], one-stage running was shown to correctly give the $\alpha^8 \ln^3 \alpha$ Lamb shift and $\alpha^7 \ln^2 \alpha$ hyperfine splitting for Hydrogen, muonium and positronium, as well as the $\alpha^3 \ln^2 \alpha$ correction to the decay width for positronium. Since the answer differs from two-stage running, this implies that two-stage running does not sum all the logarithms of $\nu$. The reason can be seen by considering a diagram with mixed soft and ultrasoft divergences, such as the four-loop graph in Fig. 2. If we calculate logarithms of $\alpha$ directly by calculating matrix elements (using vertices at the hard scale), then this graph contributes to the $\alpha^8 \ln^3 \alpha$ Lamb shift. Calculating either ultrasoft subgraph in Fig. 2 gives a term of the form $\ln(k_0^2/\mu_U)$, whereas the two-loop potential subgraph gives a term of the form $\ln(\sqrt{m k^0}/\mu_S)$, where $k^0$ is the time-component of a loop momentum. In the RGE, to make sure that both logarithms are small it is essential that $\mu_U = \mu^2_S/m$, so that the soft and ultrasoft scales are correlated. If one attempts to set $\mu_S = \mu_U = \mu$, then one cannot simultaneously minimize both soft and ultrasoft logarithms inside this four-loop graph. In the two-stage trajectory shown in Fig. 1, the logarithms $\ln(k^0/\mu_U)$ and $\ln(\sqrt{m k^0}/\mu_S)$ are made small at the endpoint of the path. However, both logarithms are not small everywhere along the integration path, so that the logarithms are not minimized inside subgraphs of the full graph. As a result, two-stage running gives the correct single log, but not the higher order logarithms. In the one-stage approach the logarithms are small along the entire integration path and the higher order logarithms are correctly reproduced.
Although two-stage running seems natural within the pNRQED framework [10,16,17], a single stage running can be implemented as well. The discussion above suggests that even though matching from QED to pNRQED can be done in a two-stage procedure through NRQED, the renormalization group improvement should be done in one stage. However, it may well be that a more sophisticated version of two-stage running does properly take into account the fact that the soft and ultrasoft scales are correlated.

The difference between one- and two-stage running is a generic feature of systems with correlated scales. In nonrelativistic bound states, the soft and ultrasoft scales are determined in terms of the velocity \( v \), with \( \mu_S = mv \) and \( \mu_U = mv^2 \), and one needs to use one-stage running in velocity space. We have illustrated this point using QED, but the arguments are obviously also valid for weak coupling nonrelativistic QCD bound states. Correlated scales also occur in the problem of Sudakov logarithms near kinematic endpoints (for effective theory approaches see Refs. [26]). In \( B \to X_s \gamma \) decay, there are two correlated scales, the collinear scale \( m_b \sqrt{1 - x} \) and the soft scale \( m_b (1 - x) \) (here \( x = 2E_\gamma / m_b \)) that are important as \( x \to 1 \). To sum subleading Sudakov logarithms using the renormalization group one needs to run in \( x \) (or equivalently moment) space. One-stage running might also be applicable to finite temperature field theory, where one has the correlated scales \( T, gT \) and \( g^2 T \). This will be discussed elsewhere.

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