Accelerated Message Passing for Entropy-Regularized MAP Inference

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Abstract

Maximum a posteriori (MAP) inference in discrete-valued Markov random fields is a fundamental problem in machine learning that involves identifying the most likely configuration of random variables given a distribution. Due to the difficulty of this combinatorial problem, linear programming (LP) relaxations are commonly used to derive specialized message passing algorithms that are often interpreted as coordinate descent on the dual LP. To achieve more desirable computational properties, a number of methods regularize the LP with an entropy term, leading to a class of smooth message passing algorithms with convergence guarantees. In this paper, we present randomized methods for accelerating these algorithms by leveraging techniques that underlie classical accelerated gradient methods. The proposed algorithms incorporate the familiar steps of standard smooth message passing algorithms, which can be viewed as coordinate minimization steps. We show that these accelerated variants achieve faster rates for finding \(\epsilon\)-optimal points of the unregularized problem, and, when the LP is tight, we prove that the proposed algorithms recover the true MAP solution in fewer iterations than standard message passing algorithms.

1. Introduction

Discrete undirected graphical models are extensively used in machine learning since they provide a versatile and powerful way of modeling dependencies between variables (Wainwright & Jordan, 2008). In this work we focus on the important class of discrete-valued pairwise models. Efficient inference in these models has multiple applications, ranging from computer vision (Jegelka & Bilmes, 2011), to statistical physics (Mezard & Montanari, 2009), information theory (MacKay, 2003) and even genome research (Torada et al., 2019).

In this paper we study and propose efficient methods for maximum a posteriori (MAP) inference in pairwise, discrete-valued Markov random fields. The MAP problem corresponds to finding a configuration of all variables achieving a maximal probability and is a key problem that arises when using these undirected graphical models. There exists a vast literature on MAP inference spanning multiple communities, where it is known as constraint satisfaction (Schiex et al., 1995) and energy minimization (Kappes et al., 2013). Even in the binary case, the MAP problem is known to be NP-hard to compute exactly or even to approximate (Kolmogorov & Zabin, 2004; Cooper, 1990).

As a result, there has been much emphasis on devising methods that may work on settings under which the problem becomes tractable. A popular way to achieve this goal is to express the problem as an integer program and then relax this to a linear program (LP). If the LP constraints are set to the convex hull of marginals corresponding to all global settings, also known as the marginal polytope (Wainwright & Jordan, 2008), then the LP would yield the optimal integral solution to the MAP problem. Unfortunately, writing down this polytope would require exponentially many constraints and therefore it is not tractable. We can consider larger polytopes defined over subsets of the constraints required to define the marginal polytope. This is a popular approach that underpins the family of LP relaxations known as the Sherali-Adams (SA) hierarchy (Sherali & Adams, 1990). Instead of enforcing global consistency, we enforce only pairwise consistency via the local polytope, thus yielding pseudo-marginals that are pairwise consistent but may not correspond to any true global distribution. Despite the local polytope requiring a number of constraints that is linear in the number of edges of the input graph, the runtime required for solving this linear program for large graphs may be prohibitive in practice (Yanover et al., 2006). These limitations have motivated the design and theoretical analysis of message passing algorithms that exploit the structure of the problem. In this paper we study a class of smooth message passing algorithms, derived from a regularized version of the
local polytope LP relaxation (Ravikumar et al., 2010; Meshi et al., 2012; Savchynskyy et al., 2011; Hazan & Shashua, 2008).

The technique of using entropy penalties to regularize linear programs has a long and successful history. It has been observed, both practically and in theory that, in some problems, solving a regularized linear program yields algorithms with computational characteristics that make them preferable to simply using an LP solver, particularly with large scale problems. Previous work has studied and analyzed convergence rates, and even rounding guarantees for simple message passing algorithms (Ravikumar et al., 2010; Lee et al., 2020; Meshi et al., 2012) based on iterative Bregman projections onto the constraints. These algorithms are sometimes described as being smooth, as the dual objective is smooth in the dual variable as a result of the entropy regularization. Inspired by accelerated methods in optimization (Lee & Sidford, 2013; Nesterov, 2012) we propose and analyze two new variants of accelerated message passing algorithms Accel-EMP and Accel-SMP. In this paper, we are able to show our methods drastically improve upon the convergence rate of previous message passing algorithms.

1.1. Related Work

MAP Inference The design and analysis of convergent message passing algorithms has attracted a great deal of attention over the years. Direct methods of deriving asymptotically convergent algorithms have been extensively explored. Examples include tree-reweighted message passing (Kolmogorov, 2006), max-sum diffusion (Werner, 2007), MP-LP (Globerson & Jaakkola, 2008), and other general block-coordinate ascent methods (Kappes et al., 2013; Sonntag et al., 2011). Our work builds upon regularized inference problems that directly regularize the linear objective with strongly convex terms, often leading to “smooth” message passing (Savchynskyy et al., 2011; 2012; Hazan & Shashua, 2008; Weiss et al., 2007; Meshi et al., 2015). This formalization has led to a number of computationally fast algorithms, but often with asymptotic guarantees.

The focus of this paper is non-asymptotic convergence guarantees for families of these algorithms. Ravikumar et al. (2010) provided one of the first directions towards this goal leveraging results for proximal updates, but ultimately the rates were not made explicit due to the approximation at every update. Meshi et al. (2012) provided a comprehensive analysis of a message passing algorithm derived from the entropy-regularized objective and later a gradient-based one for the quadratically regularized objective (Meshi et al., 2015). However, the convergence rates were only given for the regularized objective, leaving the guarantee on the unregularized problem unknown. Furthermore, the objective studied by Meshi et al. (2015) did not ultimately yield a message passing (or coordinate minimization) algorithm, which could be more desirable from a computational standpoint. Lee et al. (2020) provided rounding guarantees for a related message passing scheme derived from the entropy regularized objective, but did not consider convergence on the unregularized problem either. Savchynskyy et al. (2011) studied the direct application of the acceleration methods of Nesterov (2018); however, this method also forwent the message passing scheme and convergence on the unregularized problem was only shown asymptotically. Jojic et al. (2010) gave similar results for a dual decomposition method that individually smooths subproblems. Acceleration was applied to get fast convergence but on the dual problem.

In addition to problem-specific message passing algorithms, there are numerous general purpose solvers that can be applied to MAP inference to solve the LP with strong theoretical guarantees. Notably, interior point methods (Karmarkar, 1984; Renegar, 1988; Gondzio, 2012) offer a promising alternative for faster algorithms. For example recent work provides a $O(\sqrt{\text{rank}})$ iteration complexity by Lee & Sidford (2014), where rank is the rank of the constraint matrix. In this paper, we only consider comparisons between message passing algorithms; however, it would be interesting to compare both empirical and theoretical differences between message passing and interior point methods in the future.

Accelerating Entropy-Regularized Linear Programs We also highlight a connection with similar problems in other fields. Notably, optimal transport also admits an LP form and has seen a surge of interest recently. As in MAP inference, these approximations are conducive to empirically fast algorithms, such as the celebrated Sinkhorn algorithm, that outperforms generic solvers (Cuturi, 2013; Benamou et al., 2015; Genevay et al., 2016). In theory, Altschuler et al. (2017) showed convergence guarantees for Sinkhorn and noted that it can be viewed as a block-coordinate descent algorithm on the dual, similar to the MAP problem. Since this work, several methods have striven to obtain faster rates (Lin et al., 2019; Dvurechensky et al., 2018), which can be viewed as building on the seminal acceleration results of Nesterov (2005) and maximum flow (Sidford & Tian, 2018).

1.2. Contributions

For the case of MAP inference from entropy-regularized objectives, we address the question: is it possible to directly accelerate message passing algorithms with faster non-asymptotic convergence and improved rounding guarantees? We answer this question affirmatively from a theo-
We prove a convergence guarantee for standard schemes on the unregularized MAP objective over \( L_2 \), showing convergence on the order of \( O(m^3/\epsilon^3) \) iterations where \( m \) is the number of edges, assuming the number of vertices and labels and the potential functions are fixed. We then prove that the accelerated variants converge in expectation on the order of \( O(m^{9/2}/\epsilon^2) \) iterations. We conclude by showing that the accelerated variants recover the true MAP solution with high probability in fewer iterations compared to prior message passing analyses when the LP relaxation is tight and the solution is unique (Lee et al., 2020).

1.3. Notation

Let \( \mathbb{R}_+ \) denote the set of non-negative reals. The \( d \)-dimensional probability simplex over the finite set \( \chi \) is \( \Sigma^d := \left\{ p \in \mathbb{R}_+^d : \sum_{x \in \chi} p(x) = 1 \right\} \). A joint distribution, \( P \in \Sigma^{d \times K} \), is indexed by \( x \in \chi \). The transportation polytope of \( p, q \in \Sigma^d \) is defined as the set of pairwise joint distributions that marginalize to \( p \) and \( q \), written as \( \mathcal{U}_d(p,q) = \left\{ P \in \mathbb{R}_+^{d \times d} : \sum_{x} P(x,y) = p(x), \sum_{y} P(x,y) = q(y) \right\} \). For any vector \( p \in \mathbb{R}_+^d \), we write the entropy as \( H(p) := -\sum_{x} p(x) \log p(x) - 1 \). This is a somewhat unusual definition, it simplifies terms later and has been used by (Benamou et al., 2015; Lee et al., 2020). We will use \( \langle \cdot, \cdot \rangle \) generally to mean the sum of the elementwise products between two equal-length indexable vectors. For any two vectors \( p, q \in \mathbb{R}_+^d \), the Hellinger distance is \( h(p,q) := \frac{1}{\sqrt{2}} \| p - q \|_2 \). The vector \( 1_d \in \mathbb{R}_+^d \) consists of all ones.

2. Coordinate Methods for MAP Inference

2.1. Smooth Approximations

For the pairwise undirected graph \( G = (V, E) \) with \( n := |V| \) and \( m := |E| \), we associate each vertex \( i \) with the random variable \( X_i \) on the finite set of labels \( \chi \) of size \( d = |\chi| \geq 2 \) and a distribution that factorizes as \( p(x_V) = \frac{1}{Z(\phi)} \prod_{e \in E} \phi_e(x_e) \prod_{i \in V} \phi_i(x_i) \) where \( \phi_i \in \mathbb{R}^d \) and \( \phi_e \in \mathbb{R}^{2d} \). For any edge \( e \in E \), we use \( i \in e \) to denote that \( i \) is one of the two endpoints of \( e \). For each variable \( i \), we define \( N_i := \{ e \in E : i \in e \} \), as the set of all incident edges to \( i \). We assume that each vertex has at least one edge. MAP inference refers to the combinatorial problem of identifying the configuration that maximizes this probability distribution. In our case, we cast it as the following minimization:

\[
\text{minimize} \quad \sum_{i \in V} C_i(x_i) + \sum_{e \in E} C_e(x_e), \quad \text{(MAP)}
\]

where \( C = -\log \phi \), i.e., we view \( C \) as indexable by vertices, edges, and their corresponding labels. It can be shown (Wainwright & Jordan, 2008) that (MAP) is equivalent to the following linear program:

\[
\min_{\mu \in M} \langle C, \mu \rangle \quad \text{s.t.} \quad \mu \in M,
\]

where \( \mu \in \mathbb{R}^r \) for \( r_p = nd + md^2 \) is known as a marginal vector, and \( \langle C, \mu \rangle = \sum_{i \in V} \sum_{x_i \in \chi} C_i(x_i) \mu_i(x_i) + \sum_{e \in E} \sum_{x_e \in \chi^2} C_e(x_e) \mu_e(x_e) \), and \( M \) is the marginal polytope defined as

\[
M := \left\{ \mu : \exists \mathbb{P} \text{ s.t. } \mathbb{P}_{X_i}(x_i) = \mu_i(x_i), \forall i, x_i \right\}.
\]

Here, \( \mathbb{P} \) is any valid distribution over the random variables \( \{X_i\}_{i \in V} \). Since \( M \) is described by exponentially many constraints in the graph size, outer-polytope relaxations are a standard paradigm to approximate the above problem by searching instead over the local polytope:

\[
L_2 := \left\{ \mu : \mu_i \in \Sigma^d, \forall i \in V \right\},
\]

The local polytope \( L_2 \) enforces only pairwise consistency between variables while \( M \) requires the marginal vector to be generated from a globally consistent distribution of \( \{X_i\}_{i \in V} \), so that \( M \subseteq L_2 \). We refer the reader to the survey of Wainwright & Jordan (2008, §3) for details. Thus, our primary objective throughout the paper will be finding solutions to the approximate problem

\[
\text{minimize} \quad \langle C, \mu \rangle \quad \text{s.t.} \quad \mu \in L_2. \quad \text{(P)}
\]

Let \( \epsilon > 0 \). We say that a point \( \hat{\mu} \in L_2 \) is \( \epsilon \)-optimal for (P) if it satisfies \( \langle C, \hat{\mu} \rangle \leq \min_{\mu \in L_2} \langle C, \mu \rangle + \epsilon \). For a random \( \hat{\mu} \), we say that it is expected \( \epsilon \)-optimal if

\[
\mathbb{E} \left[ \langle C, \hat{\mu} \rangle \right] \leq \min_{\mu \in L_2} \langle C, \mu \rangle + \epsilon.
\]

Despite the simple form of the linear program, it has been observed to be difficult to solve in practice for large graphs even with state-of-the-art solvers (Yanover et al., 2006), motivating researchers to study an approximate version with entropy regularization:

\[
\text{minimize} \quad \langle C, \mu \rangle + \frac{1}{\eta} H(\mu) \quad \text{s.t.} \quad \mu \in L_2, \quad \text{(Reg-P)}
\]

where \( \eta \in \mathbb{R}_+ \) controls the level of regularization. Intuitively, the regularization encourages \( \mu_i \) and \( \mu_e \) to be closer to the uniform distribution for all vertices and edges.
The dual problem takes on the succinct form of an unconstrained log-sum-exp optimization problem. Thus, when combined, the local polytope relaxation and entropy-regularizer result in a smooth approximation.

Proposition 1. The dual objective of (Reg-P) can be written as

$$\min_{\lambda} \text{L}(\lambda), \quad \text{(Reg-D)}$$

where \( \text{L} \) is defined as

$$\text{L}(\lambda) = \frac{1}{\eta} \sum_{i \in V} \sum_{x \in \chi} \log \sum_{e \in \chi} \exp \left( -\eta C_e(x) + \sum_{i \in N_i} \lambda_{e,i}(x) \right)$$

$$+ \frac{1}{\eta} \sum_{e \in \Theta} \sum_{x \in \chi} \exp \left( -\eta C_e(x) - \sum_{i \in N_i} \lambda_{e,i}(x_i) \right).$$

Furthermore, primal variables can be recovered directly by

$$\mu^\lambda_e(x_i) \propto \exp \left( -\eta C_e(x_i) + \sum_{e \in \Theta} \lambda_{e,i}(x_i) \right)$$

$$\mu^\lambda_e(x_i) \propto \exp \left( -\eta C_e(x_i) - \sum_{i \in N_i} \lambda_{e,i}(x_i) \right).$$

For convenience we let \( r_D = 2md \) denote the dimension of the dual variables \( \lambda \in \mathbb{R}^{rd} \). This is in contrast to the dimension \( r_p \) of the primal marginal vectors defined earlier. We use \( \lambda^* \subseteq \mathbb{R}^{rd} \) to denote the set of solutions to (Reg-D).

There is a simple interpretation to dual optimality coming directly from the Lagrangian: a variable \( \lambda \) is optimal if the candidate primal solution is primal feasible: \( \mu^\lambda \in \mathbb{L}_2 \).

It can be seen that the derivative of the dual function \( \text{L}(\lambda) \) captures the slack of a \( \mu^\lambda \):

$$\frac{\partial \text{L}(\lambda)}{\partial \lambda_{e,i}(x_i)} = \mu^\lambda_e(x_i) - \text{S}^\lambda_e(x_i)$$

where we define \( \text{S}^\lambda_e(x_i) := \sum_{j \in \chi} \mu^\lambda_e(x_i, j) \). The gradient captures the amount and direction of constraint violation in \( \mathbb{L}_2 \) by \( \mu^\lambda \). In order to discuss this object concisely and intuitively, we formally define the notion of a slack vector, which is simply the negative of the gradient, and a slack polytope (Lee et al., 2020), which describes the same polytope as \( \mathbb{L}_2 \) if the constraints were offset by exactly the amount by which \( \mu^\lambda \) is offest.

Definition 1 (Slack vector and slack polytope). For \( \lambda \in \mathbb{R}^{rd} \), the slack vector \( \mu^\lambda \in \mathbb{R}^{rd} \) of \( \lambda \) is defined as \( \nu^\lambda_{e,i}(x) = \text{S}^\lambda_e(x) - \mu^\lambda_e(x_i) \) for all \( e \in E \), \( i \in e \), and \( x \in \chi \).

The slack polytope for a slack vector \( \nu \) is defined as

$$\mathbb{L}^{\nu} := \left\{ \mu \in \mathbb{R}^{rd} : \mu_i \in \sum_{e \in \epsilon} \left( \nu_{e,i} + \mu_{e,i} \right), \mu_j + \nu_{e,j} \right\}.$$ 

2.2. Entropy-Regularized Message Passing

The results in this paper will be primarily concerned with algorithms that approximately solve (MAP) by directly solving (Reg-D). For solving this objective, message passing algorithms can effectively be viewed as block-coordinate descent, except that a full minimization is typically taken at each step. Here we outline two variants.

2.2.1. Edge Message Passing

Edge message passing (EMP) algorithms reduce to block-coordinate methods that minimize (Reg-D) for a specific edge \( e = \{i, j\} \in E \) and endpoint vertex \( i \in e \), while keeping all other variables fixed. Let \( L_{e,i}(:, \lambda) : \mathbb{R}^d \to \mathbb{R} \) denote the block-coordinate loss of \( L \) fixed at \( \lambda \) except for free variables \( \{\lambda_{e,i}(x)\}_{x \in \chi} \). For each \( i \in V \), \( e \in N_i \), we define the EMP operator for \( \lambda \in \mathbb{R}^{rd} \):

$$\text{EMP}_{e,i}^\eta(\lambda) \in \arg \min_{\lambda} L_{e,i}(\lambda_{e,i}(:); \lambda).$$

Proposition 2. The operator \( \text{EMP}_{e,i}^\eta : \lambda \mapsto \lambda_{e,i}(:) \in \mathbb{R}^d \) is satisfied by \( \lambda_{e,i}(:) = \lambda_{e,i}(x_i) + \frac{1}{\eta} \log \frac{S^\lambda_{e,i}(x_i)}{\mu^\lambda_{e,i}(x_i)} \).

In the entropy-regularized setting, this update rule has been studied by Lee et al. (2020); Ravikumar et al. (2010). Non-regularized versions based on max-sum diffusion have much earlier roots and also been studied by Werner (2007; 2009); however, we do not consider these particular unregularized variants. EMP offers the following improvement on \( L \).

Lemma 1. Let \( \mathcal{N} \) be the result of applying \( \text{EMP}_{e,i}^\eta(\lambda) \) to \( \lambda \), keeping all other coordinates fixed. Then, \( L(\lambda) - L(\lambda') \geq \frac{1}{4\eta} \| \mu^\lambda - \mu^\lambda' \|_2^2 \).

2.2.2. Star Message Passing

Star message passing (SMP) algorithms consider block-coordinates that include all edges incident to a particular vertex \( i \in V \). For \( \lambda \in \mathbb{R}^{rd} \) and \( i \in V \) let \( L_i(:, \lambda) : \mathbb{R}^d \to \mathbb{R} \) denote the block-coordinate loss of \( L \) fixed at \( \lambda \). For a given \( i \in V \) and arbitrary \( \lambda \in \mathbb{R}^{rd} \), we define the SMP operator:

$$\text{SMP}_{i}^\eta(\lambda) \in \arg \min_{\lambda} L_i(\lambda_{i}(:); \lambda).$$

That is, SMP is the minimization over the block-coordinate at vertex \( i \) for all edges incident to \( i \) in \( N_i \) and all possible labels in \( \chi \).
Algorithm 2 Accel-EMP($G, C, \eta, K$)
1: $\lambda^{(0)} = 0, \mathbf{v}^{(0)} = 0, \theta_{-1} = 1$
2: for $k = 0, 1, \ldots, K - 1$ do
3: \hspace{1em} $\theta_k = -\frac{\theta_{k-1}}{2} + \sqrt{\frac{\lambda_{k-1}}{4} + 4\phi_k}$
4: \hspace{1em} $\mathbf{y}^{(k)} = \theta_k \mathbf{v}^{(k)} + (1 - \theta_k) \lambda^{(k)}$
5: \hspace{1em} Sample $(e_k, i_k) \sim \text{Unif}\{ \{(e, i) : e \in E, i \in v\} \}$
6: \hspace{1em} Set $\lambda^{(k+1)} = \lambda^{(k)}$
7: \hspace{1em} $\lambda_{e,i}^{(k+1)}(\cdot) = \text{EMP}^\eta_{e,k,i}(\mathbf{y}^{(k)})$
8: \hspace{1em} $\mathbf{v}_{e,k,i}^{(k+1)} = \mathbf{v}_{e,k,i}^{(k)} + \frac{1}{2\eta} \delta_{\lambda_{e,i}^{(k+1)}}(\mathbf{v}_{e,k,i}^{(k)})$
9: end for
10: return $\lambda^{(K)}$

Algorithm 3 Accel-SMP($G, C, \eta, K$)
1: $\lambda^{(0)} = 0, \mathbf{v}^{(0)} = 0, \theta_{-1} = 1$
2: for $k = 0, 1, \ldots, K - 1$ do
3: \hspace{1em} $\theta_k = -\frac{\theta_{k-1}}{2} + \sqrt{\frac{\lambda_{k-1}}{4} + 4\phi_k}$
4: \hspace{1em} $\mathbf{y}^{(k)} = \theta_k \mathbf{v}^{(k)} + (1 - \theta_k) \lambda^{(k)}$
5: \hspace{1em} Sample $i_k \sim \{p_i\}_i \in \mathcal{V}$
6: \hspace{1em} Set $\lambda^{(k+1)} = \lambda^{(k)}$
7: \hspace{1em} $\lambda_{e,i}^{(k+1)}(\cdot) = \text{SMP}^\eta_{e,i}(\mathbf{y}^{(k)})$
8: \hspace{1em} $\mathbf{v}_{e,i}^{(k+1)} = \mathbf{v}_{e,i}^{(k)} + \min_{\nu_{e,i} \in \mathcal{V}} \frac{\nu_{e,i} \mathbf{y}^{(k)}}{2p_{e,i} \theta_k \eta} + \mathbf{y}^{(k)}$
9: end for
10: end for
11: return $\lambda^{(K)}$

3. Accelerating Entropy-Regularized Message Passing

We now present a scheme for accelerating message passing algorithms in the entropy-regularized formulation. The key idea is to leverage the block-coordinate nature of standard message passing algorithms. We draw inspiration from both the seminal work of Nesterov (2018) on accelerating gradient methods and accelerating randomized coordinate gradient methods similar to Lee & Sidford (2013); Lu & Xiao (2015); however, the presented method is specialized to incorporate full block-coordinate minimization at each round, so as to be consistent with existing message passing algorithms used in practice. Furthermore, we can leverage the same simple sampling procedures for selecting the block-coordinates. The presented scheme can thus be viewed as a direct method of acceleration in that standard message passing algorithms can be plugged in.

The scheme is presented in Algorithm 2 for EMP and Algorithm 3 for SMP. In Accel-EMP, at each round $k$, a random coordinate block is sampled uniformly. That coordinate block for $\lambda$ is then updated with a step of EMP$^\eta_{e_k,i_k}$ evaluated at $\mathbf{y}^{(k)}$. A block-coordinate gradient step evaluated at $\mathbf{y}^{(k)}$ in the form of the slack vector $\nu_{e,i}^{(k)}$ is also applied to $\mathbf{v}^{(k)}$. Accel-SMP works similarly but we instead sample from the non-uniform distribution defined in (3). The choice of distributions ultimately determines the step size for $\mathbf{v}$. As in the case of the standard smooth message passing algorithms, which optimize the dual function (Reg-D), the returned solutions may not be primal feasible in finite iterations. To obtain feasible solutions, we consider a projection operation $\text{Proj}$, shown in Algorithm 4, that is simply a repeated application of Algorithm 2 of Altschuler et al. (2017), originally designed for optimal transport. The method ef-
We now present iteration complexities for the above algorithm by Lee et al. (2020) for Algorithm 4 vertex marginals but we do not consider those here. Be handy for comparing to our primary contribution on the antecedes for the true relaxed problem, which will ultimately purpose of this extension is to standardize convergence guarantees for the standard randomized algorithms, specifying exactly the dependence on graph parameters. The optimality for (P) for the standard randomized algorithms, in contrast to these prior works, we give a guarantee on parameters and

depend on the tightness of the relaxation, rounding heuristics required to round to the optimal MAP solution when it is available. Meshi et al. (2012) gave a guarantee on both the primal and dual regularized problems (Reg-P) and (Reg-D), but not the original (P) and without rounding or tuning of \( \eta \). Additionally, both works focused mostly on the \( \epsilon \) dependence in the rate rather than actually specifying the graph parameters \( m, n \), and \( d \).

In contrast to these prior works, we give a guarantee on optimality for (P) for the standard randomized algorithms, specifying exactly the dependence on graph parameters. The purpose of this extension is to standardize convergence guarantees for the true relaxed problem, which will ultimately be handy for comparing to our primary contribution on the accelerated algorithms.

\textbf{Theorem 1.} Let \( \hat{\lambda} \) be the result of running Algorithm 1 with EMP, uniform sampling distribution (2) and \( \eta = \frac{4}{\epsilon^6} \log d \). Let \( \hat{\lambda} = \proj(\lambda^0, 0) \) be its projection onto \( \mathbb{L}_2 \). Then, the number of iterations sufficient for \( \hat{\mu} \) to be expected \( \epsilon \)-optimal is

\[ O \left( m d^2 (m + n)^4 \|C\|_\infty^3 \epsilon^{-3} \log d \right). \]

If \( \hat{\lambda} \) is the output of Algorithm 1 using SMP and sampling distribution (3), and \( \hat{\mu} := \proj(\lambda^0, 0) \) then \( \hat{\mu} \) is expected \( \epsilon \)-optimal in the same order of iterations.

The rate harbors a \( O(1/\epsilon^3) \) dependence, which at first appears to be worse than that of Meshi et al. (2012) and Lee et al. (2020); however, their convergence guarantees hold only for the regularized objective. The extra \( O(1/\epsilon) \) in our guarantee occurs in the conversion to the original unregularized problem (P), which is a stronger result. It is interesting to observe that the guarantees are effectively the same for both variants, despite having somewhat different analyses. We hypothesize that this is due to the fact that the “smoothness” constant for SMP in Lemma 2 is greater than that of EMP in Lemma 1. Therefore, the larger block-coordinate size is effectively canceled by the smaller improvement per step.

We now describe the proof briefly here since the first part appears to be fairly standard while the second will be covered in the proof of our main acceleration result. The full proof is found in Appendix D. The basic idea is to use Lemma 1 to lower bound the expected improvement each iteration, which can be done in terms of the average squared norms of the slack \( \|v_{e,i}\|^2 \). We can guarantee improvement on \( L \) by at least \( (\epsilon')^2 \) until the norms are on average below \( \epsilon' \). Knowing that the slack norms are small, we can prove that the projection \( \hat{\mu} \) of \( \lambda^0 \) onto \( \mathbb{L}_2 \) is not too far from \( \mu^0 \) and so the expected value of \( \langle C, \hat{\mu} \rangle \) is not much worse than that of \( \langle C, \mu^0 \rangle \). We then prove that \( \langle C, \mu^0 \rangle \) is small with respect to the slack norms up to some entropy term, and we set \( \eta \) so the entropy term is sufficiently small with respect to a given \( \epsilon > 0 \).

\textbf{4. Main Results}

We now present iteration complexities for the above algorithms for finding \( \epsilon \)-optimal solutions to the original unregularized problem (P) over \( \mathbb{L}_2 \). These guarantees make it easy to compare various algorithms as they do not inherently depend on the tightness of the relaxation, rounding heuristics to find integral solutions, or arbitrary choices of \( \eta \).

\textbf{4.1. Standard Algorithms}

Our first result bounds the number of iterations required to compute \( \epsilon \)-optimal solutions to (P). To recapitulate, prior work by Lee et al. (2020) for EMP only provided an iteration guarantee for bounding the norm of the slack vector. They also gave guarantees for the number of iterations required to round to the optimal MAP solution when it is available. Meshi et al. (2012) gave a guarantee on both the primal and dual regularized problems (Reg-P) and (Reg-D), but not the original (P) and without rounding or tuning of \( \eta \). Additionally, both works focused mostly on the \( \epsilon \) dependence in the rate rather than actually specifying the graph parameters \( m, n \), and \( d \).

In contrast to these prior works, we give a guarantee on optimality for (P) for the standard randomized algorithms, specifying exactly the dependence on graph parameters. The purpose of this extension is to standardize convergence guarantees for the true relaxed problem, which will ultimately be handy for comparing to our primary contribution on the

\textbf{Algorithm 4} \( \proj(\mu, \nu) \)

1: Set \( \hat{\mu}_i = \mu_i \) for all \( i \in V \)
2: for \( (i, j) = e \in E \) do
3: \( \text{Compute } \hat{\mu}_e \) by applying Algorithm 2 of (Altschuler et al., 2017) on \( \mu_e \) with transportation polytope \( \mathcal{U}_d(\mu_i + \nu_{e,i}, \mu_j + \nu_{e,j}) \)
4: end for
5: return \( \hat{\mu} \)

\footnote{Ambiguity could arise for more sophisticated rounding schemes but we do not consider those here.}
Theorem 2. Let $\hat{\lambda}$ be the output of Algorithm 2 with $\eta = \frac{4(m + n) \log d}{d}$. Let $\mu = \text{Proj}(\mu \hat{\lambda}, 0)$ be its projection onto $\mathbb{L}_2$. Then, the number of iterations sufficient for $\mu$ to be expected $\epsilon$-optimal is

$$O \left( m^{3/2} d^2 (m + n)^3 \|C\|_2^2 \epsilon^{-2} \log d \right).$$

If $\hat{\lambda}$ is the output of Algorithm 3 and $\hat{\mu} := \text{Proj}(\mu \hat{\lambda}, 0)$, then $\hat{\mu}$ is expected $\epsilon$-optimal in the same order of iterations.

The primary difference between the iteration complexities for the standard algorithms and the accelerated ones is the dependence on $\epsilon$. For the accelerated algorithms, we are left with only a $O(1/\epsilon^2)$ dependence versus the $O(1/\epsilon^3)$ dependence for the standard algorithms. This can lead to far fewer iterations to get the same level of accuracy on the original relaxed problem (P). In addition, the bounds in Theorem 2 are strictly better in dependence on the number of edges as well for both EMP and SMP. For $m + n \approx m$, the accelerated algorithms shave off a $m^{1/2}$ factor. It is interesting to observe that these improved guarantees come with virtually no extra computation per iteration. For example, to update the sequences $\lambda^{(k)}$, $\nu^{(k)}$, and $\gamma^{(k)}$ in EMP at each iteration, we need only compute the primal variables $\mu_{\lambda_{e,i}}^{(k)}$ and $\mu_{\gamma_{e,i}}^{(k)}$ once to use in both the slack vector $\nu^{(k)}_{\lambda_{e,i}}$ and the update rule $\text{EMP}_{\nu_{\lambda_{e,i}}^{(k)}}^{(k)}$.

We will give the proof for Accel-EMP to convey the main idea. The analogous Accel-SMP case can be found in the appendix. First, we will derive a faster convergence rate on the dual objective, which in turn implies that we can bound the slack norms by the same $\epsilon^\prime$ > 0 in fewer iterations. In the second part, we will bound the approximation error caused by the entropy regularization. Finally, we put these pieces together to determine the appropriate choice of $\epsilon^\prime$ and $\eta$ in terms of $\epsilon$ to recover the final rate.

4.2.1. Faster Convergence on the Dual

The first steps will involve defining a randomized estimate sequence to work with and then using this sequence to prove the faster convergence rate on the dual objective.

Definition 2. Let $\phi_0 : \mathbb{R}^{d} \rightarrow \mathbb{R}$ be an arbitrary deterministic function. A sequence $\{\phi_k, \delta_k\}_{k=0}^K$ of random functions $\phi_k : \mathbb{R}^{d} \rightarrow \mathbb{R}$ for $k \geq 1$ and deterministic real values $\delta_k \in \mathbb{R}_+$ is a randomized sequence for $L(\lambda)$ if it satisfies $\delta_k \xrightarrow{K} 0$ and, for all $k$, $E[\phi_k(\lambda)] \leq (1 - \delta_k) L(\lambda) + \delta_k \phi_0(\lambda)$.

If we are given a random estimate sequence $\{\phi_k, \delta_k\}_{k=0}^K$ and a random sequence $\{\lambda^{(k)}\}_{k=0}^K$ that satisfies $E[L(\lambda^{(k)})] \leq \min_{\lambda} E[\phi_k(\lambda)]$, then

$$\mathbb{E}[L(\lambda^{(k)})] - L(\lambda^*) \leq \min_{\lambda} \mathbb{E}[\phi_k(\lambda)] - L(\lambda^*)$$

$$\leq \delta_k (\phi_0(\lambda^*) - L(\lambda^*))$$

This expected error converges to zero since $\delta_k \xrightarrow{K} 0$. We now identify a candidate estimate sequence. Let $\lambda^{(0)} = 0$, $\phi_0 = 1$, $\lambda^* \in \Lambda^*$, and $q := 2m$. Let the sequence $\{\phi_k\}_{k=0}^K$ be as it is defined in Algorithm 2 and let $\{\gamma^{(k)}\}_{k=0}^K \subset \mathbb{R}^{d}$ be arbitrary. Consider $\{\phi_k, \delta_k\}_{k=0}^K$ defined recursively as

$$(e_k, i_k) \sim \text{Unif}\{(e, i) : e \in E, i \in e\}$$

$$\delta_{k+1} = (1 - \theta_k) \delta_k$$

$$\phi_{k+1} = \lambda^* + \theta_k L(y^{(k)})$$

$$- \frac{q \theta_k}{2} \|\nu_{\gamma_{e,i}}^{(k)}\|_2^2$$

where $\phi_0(\lambda) = L(\lambda^{(0)}) + \frac{m}{2} \|\lambda^* - \gamma^{(0)}\|_2^2$ for $\gamma_0 = 2 q^2 \eta$.

Lemma 3. The sequence $\{\phi_k, \delta_k\}_{k=0}^K$ defined in (5) is a random estimate sequence. Furthermore, it maintains the form $\phi_k(\lambda) = \omega_k + \frac{m}{2} \|\lambda - \gamma^{(k)}\|_2$ for all $k$ where

$$\gamma_{k+1} = (1 - \theta_k) \gamma_k$$

$$\gamma^{(k+1)} = \begin{cases} \gamma^{(k)} + \frac{q \theta_k}{2} \|\gamma^{(k)}\|_2 & \text{if } (e, i) = (e_k, i_k) \\ \gamma^{(k)} & \text{otherwise} \end{cases}$$

$$\omega_{k+1} = (1 - \theta_k) \omega_k + \theta_k L(y^{(k)}) - \frac{(\theta_k q)^2}{2} \|\nu_{\gamma_{e,i}}^{(k)}\|_2^2$$

The proof is similar to what is given by Lee & Sidford (2013), but since we consider the non-strongly convex case and slightly different definitions, we give a full proof in Appendix C for completeness. We can use this fact to show a rate of convergence on the dual objective.

Lemma 4. For the random estimate sequence in (5), let $\{\lambda^{(k)}\}_{k=0}^K$ and $\{\gamma^{(k)}\}_{k=0}^K$ be defined as in Algorithm 2 with $\lambda^{(0)} = 0$. Then, the dual objective error in expectation can be bounded as $E[L(\lambda^{(k)}) - L(\lambda^*)] \leq \frac{G(\eta)}{\epsilon^{(s+2)}}$, where $G(\eta) := 24md(m + n)(\sqrt{\eta}) \|C\|_\infty + \log d / \sqrt{\eta}$.

Proof. It suffices to show that the sequence in (5) with the definitions of $\{\lambda^{(k)}\}_{k=0}^K$ and $\{\gamma^{(k)}\}_{k=0}^K$ satisfies $E[L(\lambda^{(k)})] \leq \min_{\lambda} E[\phi_k(\lambda)]$. To do this, we will use induction and Lemma 1. Note that $\min_{\lambda} \phi_k(\lambda) = \omega_k$. The base case holds trivially with $E[\omega_0] = L(\lambda^{(0)})$. Suppose $E[L(\lambda^{(k)})] \leq E[\omega_k]$ at iteration $k$. For $k + 1$, we have

$$E[\omega_{k+1}] \geq (1 - \theta_k) E[L(\lambda^{(k)})] + \theta_k E[L(y^{(k)})]$$

$$- \frac{(\theta_k q)^2}{2} \|\nu_{\gamma_{e,i}}^{(k)}\|_2^2 - \theta_k q \nu_{\gamma_{e,i}}^{(k)} \nu_{\gamma_{e,i}}^{(k)} - \nu_{\gamma_{e,i}}^{(k)}.$$
The above inequality uses the inductive hypothesis. Then,
\[
\begin{align*}
\geq & \mathbb{E} \left[ L(y^{(k)}) - \frac{\theta_k q^2}{2\gamma k+1} \|\nabla L(y^{(k)})\|_2^2 \right] \\
+ & (1 - \theta_k) \mathbb{E} \left[ \langle \nabla L(y^{(k)}), \lambda^{(k)} - y^{(k)} \rangle \right] \\
+ & \theta_k \mathbb{E} \left[ \langle \nabla L(y^{(k)}), \nabla L(y^{(k)}) \rangle \right] \\
= & \mathbb{E} \left[ L(y^{(k)}) - \frac{\theta_k q^2}{2\gamma k+1} \|\nabla L(y^{(k)})\|_2^2 \right].
\end{align*}
\]
This second inequality uses convexity of \( L \) and then applies the law of total expectation to the dot products, conditioning on randomness \( \{s_i, i_s\}_{s=1}^{k+1} \). The last identity uses the definition of \( y^{(k)} \) and total expectation again on the norm.

Using Lemma 1 and the definition of \( \lambda^{(k+1)} \) from Algorithm 2, we have
\[
\begin{align*}
E[L(\lambda^{(k+1)})] & \leq E[L(y^{(k)}) - \frac{1}{4\eta} \|\nu_{e,i}^{(k)}\|_2^2] \\
& = E[L(y^{(k)}) - \frac{1}{4\eta} \|\nabla L(y^{(k)})\|_2^2].
\end{align*}
\]
The equality uses the law of total expectation. Therefore, taking \( \theta^2_0 := \frac{\gamma_0}{2q^2\eta} \) ensures that \( \min_{\lambda} \mathbb{E}[\theta_0 \|\lambda - \lambda^{(k+1)}\|] \geq \mathbb{E}[\omega_{k+1}] \geq E(L(\lambda^{(k+1)})) \). By setting \( \gamma_0 = 2q^2\eta \), we ensure that \( \theta_k \) need only satisfy \( \theta_k^2 = (1 - \theta_k)\theta_{k-1}^2 \), which occurs in Algorithm 2. From Nesterov (2018, Lemma 2.2.4), this choice of \( \gamma_0 \) and \( \theta_k \) ensures \( \delta_k \leq \left(1 + \frac{k}{q} \sqrt{\frac{2m}{8\eta}} \right)^{-2} = \frac{4}{(k+2)^2} \). From (4) and the definition of \( \phi_0 \), we get
\[
E[L(\lambda^{(k)}) - L(\lambda^*)] \leq \frac{4L(0) - 4L(\lambda^*) + 16m^2\eta\|\lambda^*\|_2^2}{(k + 2)^2}.
\]
The proof that the numerator can be bounded by \( G(\eta)^2 \) is deferred to the appendix, Lemma 7.

\[\Box\]

4.2.2. Approximation Error due to Entropy

Recall that our end goal is to ensure that \( \hat{\mu} \), the projection of \( \mu^\lambda \) onto \( L_2 \), is expected \( \epsilon \)-optimal for \( \epsilon > 0 \). To show this, we need to develop some relations which we outline here for brevity but state formally and prove in Appendix C.2. The first relation is how close \( \hat{\mu} \in L_2 \) is to the algorithm’s output \( \hat{\mu}^\lambda \), which is in the slack polytope \( L_2^\mathcal{A} \). This is essentially a direct extension of Altschuler et al. (2017, Lemma 7), which tells us that we can bound the projection by the norm of the slacks. Then, we show that there exists a point \( \hat{\mu}^* \) in the slack polytope \( L_2^\mathcal{A} \) that is close to the optimal point \( \mu^* \in L_2 \) with respect to the norm of the slacks also. We use the relations to conclude a bound on the original relaxed problem \( (P) \) for any realization of \( \hat{\mu} \).

**Proposition 4.** Let \( \mu^* \in L_2 \) be optimal, \( \lambda \in \mathbb{R}^\mathcal{A} \), \( \hat{\mu} = \text{Proj}(\mu^\lambda, 0) \in L_2 \), and \( \delta = \max_{e \in E, i \in e} \|\nu_{e,i}^\lambda\|_1 \). The following inequality holds:
\[
\langle C, \hat{\mu} - \mu^* \rangle \leq 16(m + n)\|C\|_\infty \delta \\
+ 4\|C\|_\infty \sum_{e \in E, i \in e} \|\nu_{e,i}^\lambda\|_1 + \frac{n \log d + 2m \log d}{\eta}.
\]

4.2.3. Completing the Proof

**Proof of Theorem 2 for EMP.** Let \( \hat{\lambda} \) be the output from Algorithm 2 after \( K \) iterations. From Lemma 1, we can lower bound the result in Lemma 4 with \( \frac{1}{m} \mathbb{E}[\|\mu_{e,i}^\lambda\|^2] \leq \frac{G(\eta)}{K+2} \) for all \( e \in E, i \in e \). Then, for \( \epsilon' > 0 \), we can ensure that
\[
\mathbb{E}[\|\nu_{e,i}^\lambda\|_1] \leq \epsilon' \quad \text{and} \quad \mathbb{E} \sum_{e \in E, i \in e} \|\nu_{e,i}^\lambda\|_1^2 \leq 2m(\epsilon')^2
\]
in \( K = \frac{\sqrt{n}G(\eta)}{\epsilon} \) iterations. Let \( \hat{\mu} \in L_2 \) be the projected version of \( \mu^\lambda \). Taking the expectation of both sides of the result in Proposition 4 gives us
\[
\mathbb{E}[\langle C, \hat{\mu} - \mu^* \rangle] \leq \|C\|_\infty (16(m + n)\|\delta\| + 8m\epsilon') \\
+ \frac{n \log d + 2m \log d}{\eta},
\]
where \( \|\delta\|^2 \leq \mathbb{E}[\|\delta\|^2] \leq \mathbb{E} \sum_{e \in E, i \in e} \|\nu_{e,i}^\lambda\|_1^2 \leq 2m(\epsilon')^2 \). Then we can conclude
\[
\mathbb{E}[\langle C, \hat{\mu} - \mu^* \rangle] \leq 16\sqrt{2m(m + n)d}\|C\|_\infty \epsilon' \\
+ 8m\|C\|_\infty \epsilon' + \frac{n \log d + 2m \log d}{\eta} \\
\leq 24\sqrt{2m(m + n)d}\|C\|_\infty \epsilon' \\
+ \frac{n \log d + 2m \log d}{\eta}.
\]
Therefore, \( \hat{\mu} \) is expected \( \epsilon \)-optimal with \( \eta \) as defined in the statement and \( \epsilon' = \frac{\epsilon}{48\sqrt{2m(m + n)d}\|C\|_\infty} \). Substituting these values into \( K \) and \( G(\eta) \) yields the result. \( \Box \)

5. Rounding to Integral Solutions

Inspired by recent results regarding the approximation error achieved by entropy regularization in linear programming (Weed, 2018), we are able to derive rounding guarantees for our algorithms under the assumption that the LP relaxation is tight and the solution is unique. We use a simple rounding scheme: for any \( \mu \) that may not lie in \( L_2 \), \( \text{round}(\mu) = \arg \max_{x \in \mathbb{R}^n} \mu(x) \). The main challenge in achieving the results we present here is surpassing the difficulty in obtaining bounds for the \( l_1 \) distance between \( \hat{\mu} = \mu^\lambda \), the candidate solution obtained from the final iterate \( \hat{\lambda} \) resulting from our algorithms, and \( \mu^* \), the optimal solution of \( (P) \). Define \( \mu^\mathcal{A} \) be the solution to the regularized problem where \( \mathcal{A} \in \mathcal{A}^\star \).
We proceed in two steps. First, we bound the approximation error \( \|\mu^* - \hat{\mu}\|_1 \) using recent results on the quality of solutions for entropy regularized \( L_2 \) (Lee et al., 2020). Then we bound the optimization error of \( \hat{\mu} \) using the results derived in the previous section. The proof and a comparison to standard EMP are in Appendix F. Let \( \text{deg} \) denote the maximum degree of the graph \( G \) and define \( \Delta \) as the suboptimality gap of the LP over \( L_2 \).

**Theorem 3.** Let \( \delta \in (0, 1) \). If \( L_2 \) is tight for potential vector \( C \) and there exist a unique solution to the MAP problem and \( \eta = \frac{16(m+n)(\log(m+n)+\log(d))}{\Delta} \), then with probability \( 1 - \delta \) in at most
\[
O\left( \frac{d^2m^7\text{deg}^2\|C\|_2^2\log^2\frac{dm}{\Delta}}{\delta} \right)
\]
iterations of Accel-EMP, \( \text{round}(\hat{\mu}) \) is the optimal MAP assignment.

### 6. Numerical Experiments

Our goal now is to understand the empirical differences between the above algorithms and also where certain theoretical guarantees can likely be improved, if at all. To this end, we compare the convergence rates of EMP and SMP and their accelerated variants on several synthesized Erdős-Rényi random graphs. First, we constructed a graph with \( n = 100 \) vertices and then generated edges between each pair of vertices with probability \( 1.1 \frac{\log n}{n} \). We considered the standard multi-label Potts model with \( d = 3 \) labels. The cost vector \( C \) was initialized randomly in the following way: \( C_{ij}(x_i) \sim \text{Unif}([-0.01, 0.01]), \forall x_i \in \chi \) and \( C_{ij}(x_i, x_j) \sim \text{Unif}([-1.0, 1.0]), \forall x_i, x_j \in \chi. \)

We consider two different metrics. (1) The first is the original primal objective value (P). This metric computes the objective value of the projection \( \hat{\mu} = \text{Proj}(\mu^\lambda, 0) \). (2) The second reports the log-competitive ratio between the standard and accelerated variants. The competitive ratio is computed as \( \log \left( \frac{C(\text{Proj} - \hat{\mu}^*)}{C(\hat{\mu}^{\text{Accel-EMP}} - \mu^*)} \right) \), where \( \hat{\mu}_{\text{EMP}} \) and \( \hat{\mu}^{\text{Accel-EMP}} \) are the projections due to \( \text{Proj} \). Positive values indicate that the accelerated variant has lower error. We implemented the four message passing algorithms exactly as they are described in Algorithms 1, 2, and 3. We also implemented block-coordinate descent and its accelerated variant as baselines (Lee & Sidford, 2013) with a step size of \( \frac{1}{\eta} \). Each algorithm used \( \eta = 1000 \) over 10 trials, measuring means and standard deviations. We computed the ground-truth optimal value of (P) using a standard solver in CVXPY.

Figure 1 depicts convergence on the primal objective in the left column. SMP achieves convergence in the fewest iterations, and EMP converges faster than coordinate descent. Interestingly, we find that the accelerated variants, including accelerated coordinate descent, appear to have marginal improvement on this metric. However, the competitive ratio figures confirm that the accelerated variants are consistently faster, especially for SMP. These results suggest that, at least for this particular problem, the upper bounds for standard algorithms may be overly conservative. It would be interesting to investigate tighter bounds for the standard algorithms in future work. Further details can be found in the appendix.

### 7. Conclusion

We analyze the convergence of message passing algorithms on the MAP inference problem over \( L_2 \). In addition to providing a novel rate of convergence rate for standard schemes derived from entropy regularization, we show that they can be directly accelerated in the sense of Nesterov with significant theoretical improvement. In future work it would be interesting to consider accelerating greedy message passing algorithms; however, Lu et al. (2018) suggest that, despite empirical success, proving accelerated rates for greedy methods is an open question even in the basic coordinate descent case. The tightness of the presented guarantees is also an open question, motivated by the empirical results here. Finally, we conjecture that reductions from the active area of optimal transport could yield novel, faster algorithms.
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A. Numerical Experiments Details

All experiments were run on an Intel Core i7 processor with 16 GB RAM. We consider two different metrics.

1. The first metric is the original primal objective value \( P \), which is the actual value we wish to minimize. Since the optimization is done in the dual variables, we use \( \text{Proj} \) to project the point \( \mu^* \) onto \( \mathbb{L}_2 \) and report the projection’s function value, additively normalized with respect to the optimal value.

2. The second metric emphasizes the first by reporting the log-competitive ratio between the standard and accelerated variants of the algorithms. The competitive ratio is computed as \( \log \left( \frac{\langle C, \hat{\mu}_{\text{EMP}} \rangle - \mu^*}{\langle C, \hat{\mu}_{\text{Accel-EMP}} \rangle - \mu^*} \right) \), where \( \hat{\mu}_{\text{EMP}} \) and \( \hat{\mu}_{\text{Accel-EMP}} \) are the projections due to \( \text{Proj} \) of the outputs of EMP and Accel-EMP, respectively, and \( \mu^* \) is a minimizer over \( \mathbb{L}_2 \). The same is computed for SMP and Accel-SMP. Thus, positive values at a given time indicate that the accelerated variant has lower error on the original objective.

Message Passing Algorithms

We implemented the message passing algorithms with their update rules exactly as prescribed in Algorithms 1, 2, and 3. The algorithms are compared with respect to the number of updates (i.e., iterations); however, we note that the cost of each update is greater for SMP and Accel-SMP since they both require computing slacks of the entire neighborhood surrounding a given vertex.

Block-Coordinate Methods

In addition to studying the empirical properties of the message passing algorithms, we present a supplementary empirical comparison with block-coordinate descent and its accelerated variant (Lee & Sidford, 2013). The purpose of this inclusion is to standardize how much we expect acceleration to improve the algorithms. We chose a stepsize of \( 1/\eta \). We note that each update in block-coordinate descent is essentially as expensive as an update of EMP.

This choice of cost vectors \( C \) ensures that vertices cannot be trivially set to their minimal vertex potentials to achieve a reasonable objective value; the MAP algorithm must actually consider pairwise interactions. We evaluated each of the four algorithms on the same graph with \( \eta = 1000 \). Due to the inherent stochasticity of the randomized algorithms, we ran each one 10 times and took the averages and standard deviations. Since the graphs are small enough, we computed the ground-truth optimal value of \( P \) using a standard LP solver in CVXPY.

In order to understand the effect of the graph size on the competitive ratio between the standard and accelerated algorithms, we generated random graphs of sizes \( n = 9, 36 \), and 81 with the same randomly drawn edges and cost vectors. We ran SMP and Accel-SMP for a fixed number of iterations over 10 random trials and computed the average log competitive ratio. Again, we used \( \eta = 1000 \). Figure 2 shows that Accel-SMP runs faster in all regimes, especially at beginning, and then the performance improvement eventually tapers after many iterations.

B. Omitted Proofs and Derivations of Section 2

B.1. Proof of Proposition 1

Recall that the primal objective is to solve the following:

\[
\text{minimize} \quad \langle C, \mu \rangle - \frac{1}{\eta} H(\mu) \quad \text{s.t.} \quad \mu \in \mathbb{L}_2,
\]

where

\[
H(\mu) = -\sum_{i \in V} \sum_{x_i \in \chi} \mu_i(x_i)(\log \mu_i(x_i) - 1) - \sum_{e \in E} \sum_{x_e \in \chi^2} \mu_e(x_e)(\log \mu_e(x_e) - 1).
\]
Though it is not strictly necessary, we will also include a normalization constraint on the pseudo-marginal edges, which amounts to $\sum_{e} \mu_e(x_e) = 1$ for all $e \in E$. The Lagrangian is therefore

$$L(\mu, \lambda, \xi) = \langle C, \mu \rangle - \frac{1}{\eta} H(\mu) + \sum_{e \in E, i \in e} \lambda_{e,i}^T (S_{e,i} - \mu_i) + \sum_i \xi_i (\sum_i \mu_i(x_i) - 1) + \sum_{e \in E} \xi_e (\sum_e \mu_e(x_e) - 1)$$

Taking the derivative w.r.t. $\mu$ yields

$$\frac{\partial L(\mu, \lambda, \xi)}{\partial \mu_i(x_i)} = C_i(x_i) + \frac{1}{\eta} \log \mu_i(x_i) + \xi_i - \sum_{e \in N_i} \lambda_{e,i}(x_i)$$

(6)

$$\frac{\partial L(\mu, \lambda, \xi)}{\partial \mu_e(x_e)} = C_e(x_e) + \frac{1}{\eta} \log \mu_e(x_e) + \xi_e + \sum_{i \in e} \lambda_{e,i}(x_e)$$

(7)

Here we are using $(x_e)_i$ to denote selecting the label associated with endpoint $i \in V$ from the pair of labels $x_e \in \chi^2$. The necessary conditions for optimality imply that

$$\mu_i^{\lambda, \xi}(x_i) = \exp \left( -\eta C_i(x_i) - \eta \xi_i + \eta \sum_{e \in N_i} \lambda_{e,i}(x_i) \right)$$

(8)

$$\mu_e^{\lambda, \xi}(x_e) = \exp \left( -\eta C_e(x_e) - \eta \xi_e - \eta \sum_{i \in e} \lambda_{e,i}(x_e) \right)$$

(9)

where we use the superscripts to show that these optimal values are dependent on the dual variables, $\lambda$ and $\xi$. The dual problem then becomes

$$\text{maximize} \quad -\frac{1}{\eta} \sum_i \sum_{x_i} \mu_i^{\lambda, \xi}(x_i) - \frac{1}{\eta} \sum_e \sum_{x_e} \mu_e^{\lambda, \xi}(x_e) - \sum_i \xi_i - \sum_e \xi_e$$

Note that we can solve exactly for $\xi$ as well, which simply normalizes the individual pseudo-marginals for each edge and vertex so that

$$\xi_i = \frac{1}{\eta} \log \sum_{x_i} \exp \left( -\eta C_i(x_i) - \eta \xi_i + \eta \sum_{e \in N_i} \lambda_{e,i}(x_i) \right)$$

$$\xi_e = \frac{1}{\eta} \log \sum_{x_e} \exp \left( -\eta C_e(x_e) - \eta \xi_e - \eta \sum_{i \in e} \lambda_{e,i}(x_e) \right)$$

Plugging this into $\mu^{\lambda, \xi}$ ensures that each local vertex and edge distribution is normalized to 1. Therefore the final objective becomes

$$\text{minimize} \quad \frac{m + n}{\eta} + \frac{1}{\eta} \sum_i \log \sum_{x_i} \exp \left( -\eta C_i(x_i) - \eta \xi_i + \eta \sum_{e \in N_i} \lambda_{e,i}(x_i) \right)$$

$$+ \frac{1}{\eta} \sum_e \log \sum_{x_e} \exp \left( -\eta C_e(x_e) - \eta \xi_e - \eta \sum_{i \in e} \lambda_{e,i}(x_e) \right),$$

and we can ignore the constant.

### B.2. Entropy-Regularized Message Passing Derivations

In this section, we derive the standard message passing algorithms that will be the main focus of the paper. Both come from simply computing the gradient and choosing additive updates to satisfy the optimality conditions directly.

**Proposition 2.** The operator $\text{EMP}_{c,i}^\eta : \lambda \mapsto \lambda'_{c,i}(\cdot) \in \mathbb{R}^d$ is satisfied by $\lambda'_{c,i}(x_i) = \lambda_{c,i}(x_i) + \frac{1}{2\eta} \log \frac{S_{\lambda}(x_i)}{\mu^\eta(x_i)}$. 


Proof. From (1), the partial gradient of $L$ with respect to coordinate $(e, i, x_i)$ yields the following necessary and sufficient optimality condition:

$$S^\lambda_{e,i}(x_i) = \mu^\lambda_i(x_i).$$

Suppose that $\lambda'$ satisfies this condition, and thus minimizes $L_{e,i}(\cdot; \lambda)$. We can decompose $\lambda'$ at coordinate $(e, i, x_i)$ additively as $\lambda'_{e,i}(x_i) = \lambda_{e,i}(x_i) + \delta_{e,i}(x_i)$. From the definition of $\mu^\lambda$, the optimality condition becomes

$$\exp(2\eta\delta_{e,i}(x_i)) = \frac{\sum_{x_j} \mu^\lambda_i(x_i, x_j)}{\mu^\lambda_i(x_i)}$$

Rearranging to find $\delta_{e,i}(x_i)$ and then substituting into $\lambda'_{e,i}(x_i)$ yields the desired result.

Now, we can derive a lower bound on the improvement on the dual objective $L$ from applying an update of EMP.

**Lemma 1.** Let $X$ be the result of applying EMP$_{e,i}(\lambda)$ to $\lambda$, keeping all other coordinates fixed. Then, $L(\lambda) - L(X) \geq \frac{1}{4\eta}\|\nu_{e,i}\|^2_1$.

**Proof.** Let $\tilde{\mu}$ denote the unnormalized marginals. From the definition of $L$,

$$L(\lambda) - L(X) = \frac{1}{\eta} \log \left( \sum_{x_i} \exp \left( -\eta C_i(x_i) + \eta \sum_{e \in N_i} \lambda_{e,i}(x_i) \right) \right) + \frac{1}{\eta} \log \left( \sum_{x_e} \exp \left( -\eta C_e(x_e) - \eta \sum_{i \in e} \lambda_{e,i}(x_i) \right) \right)$$

$$- \frac{1}{\eta} \log \sum_{x_i} \exp \left( -\eta C_i(x_i) + \eta \delta_{e,i}(x_i) + \eta \sum_{e \in N_i} \lambda_{e,i}(x_i) \right)$$

$$- \frac{1}{\eta} \log \sum_{x_e} \exp \left( -\eta C_e(x_e) - \eta \delta_{e,i}(x_i) - \eta \sum_{i \in e} \lambda_{e,i}(x_i) \right)$$

Define $\tilde{\mu}^\lambda_i(x_i) = \exp \left( -\eta C_i(x_i) + \sum_{e \in E} \lambda_{e,i}(x_i) \right)$ and $\tilde{\mu}_e(x_e) = \exp \left( -\eta C_e(x_e) - \sum_{i \in e} \lambda_{e,i}(x_i) \right)$. The cost difference can then be written as

$$L(\lambda) - L(X) = \frac{1}{\eta} \log \left( \sum_{x_i} \frac{\tilde{\mu}^\lambda_i(x_i) e^{\eta \delta_{e,i}(x_i)}}{\sum_{x'_i} \tilde{\mu}^\lambda_i(x'_i)} \right) - \frac{1}{\eta} \log \left( \sum_{x_e} \frac{\tilde{\mu}^\lambda_e(x_e) e^{-\eta \delta_{e,i}(x_e)}}{\sum_{x'_e} \tilde{\mu}^\lambda_e(x'_e)} \right)$$

$$= -\frac{1}{\eta} \log \left( \sum_{x_i} \tilde{\mu}^\lambda_i(x_i) \right) - \frac{1}{\eta} \log \left( \sum_{x_e} \tilde{\mu}^\lambda_e(x_e) \right)$$

$$= -\frac{2}{\eta} \log \left( \sum_{x_i} \sqrt{\frac{S^\lambda_{e,i}(x_i)}{\mu^\lambda_i(x_i)}} \right)$$

Note that the right-hand contains the Bhattacharyya coefficient $BC(p, q) := \sum \sqrt{pq_i}$ which has the following relationship with the Hellinger distance: $BC(p, q) = 1 - h^2(p, q)$.

The inequality then follows from exponential inequalities:

$$L(\lambda) - L(X) = -\frac{2}{\eta} \log (1 - h^2(S^\lambda_{e,i}, \mu^\lambda_i)) \geq -\frac{2}{\eta} \log \exp(-h^2(S^\lambda_{e,i}, \mu^\lambda_i)) = \frac{2}{\eta} h^2(S^\lambda_{e,i}, \mu^\lambda_i)$$

Furthermore, the Hellinger inequality gives us

$$\frac{1}{4} \|p - q\|^2_1 \leq 2h^2(p, q).$$

We conclude the result by applying this inequality with $p = S^\lambda_{e,i}$ and $q = \mu^\lambda_i$. 

Proposition 3. The operator $\text{SMP}_\eta^f : \lambda \mapsto \lambda' \in \mathbb{R}^{d|N_i|}$ is, for all $e \in N_i$ and $x_i \in \chi$, satisfied by

$$\lambda'_{e,i}(x_i) = \lambda_{e,i} + \frac{1}{\eta} \log S^\lambda_{e,i}(x_i) - \frac{1}{\eta(|N_i| + 1)} \log \left( \mu^\lambda_{e,i}(x_i) \prod_{e' \in N_i} S^\lambda_{e',i}(x_i) \right),$$

where $\lambda_{e,i} = \lambda'_{e,i}(x_i) - \lambda_{e,i}(x_i)$. Then, let $e_1, e_2 \in N_i$. At optimality, it holds that

$$S^\lambda_{e_1,i}(x_i) \prod_{e' \in N_i} S^\lambda_{e',i}(x_i) \exp(\eta \lambda_{e_1,i}(x_i) - \eta \lambda_{e_2,i}(x_i)).$$

Substituting each $\delta_{e_2,i}(x_i)$ in terms of $\delta_{e_1,i}(x_i)$, we then have

$$S^\lambda_{e_1,i}(x_i) = \mu^\lambda_{e_1,i}(x_i) \prod_{e' \in N_i} S^\lambda_{e',i}(x_i) \exp(\eta \lambda_{e_1,i}(x_i) - \eta \lambda_{e_2,i}(x_i)).$$

Collecting and then rearranging the above results in

$$\exp(|N_i| + 1) \lambda_{e_1,i}(x_i)) = (S^\lambda_{e_1,i}(x_i))\prod_{e' \in N_i} S^\lambda_{e',i}(x_i) \exp(\eta \lambda_{e_1,i}(x_i) - \eta \lambda_{e_2,i}(x_i)).$$

In additive form, the update equation is

$$\delta_{e_1,i}(x_i) = \frac{1}{\eta} \log S^\lambda_{e_1,i}(x_i) - \frac{1}{\eta(|N_i| + 1)} \log \left( \mu^\lambda_{e_1,i}(x_i) \prod_{e' \in N_i} S^\lambda_{e',i}(x_i) \right).$$

\[\square\]

C. Omitted Proofs of Technical Lemmas of Section 4

C.1. Proof of Random Estimate Sequences Lemma 3

Lemma 3. The sequence $\{\phi_k, \delta_k\}_{k=0}^K$ defined in (5) is a random estimate sequence. Furthermore, it maintains the form $\phi_k(\lambda) = \omega_k + \frac{2\eta}{\gamma_k} \eta \lambda - \nu^{(k)}$ for all $k$ where

$$\gamma_{k+1} = 1 - \theta_k \gamma_k$$

$$\nu^{(k+1)}_{e,i} = \begin{cases} \nu^{(k)}_{e,i} + \frac{\theta_k q}{\gamma_k+1} \nu^{(k)}_{e,i} & \text{if } (e, i) = (e_k, i_k) \\ \nu^{(k)}_{e,i} & \text{otherwise} \end{cases}$$

$$\omega_{k+1} = (1 - \theta_k) \omega_k + \theta_k \gamma_k \lambda - \frac{(\theta_k q)^2}{\gamma_k+1} \nu^{(k)}_{e_k,i_k}^2$$

$$- \theta_k q(q \nu^{(k)}_{e_k,i_k}, \nu^{(k)}_{e_k,i_k} - \nu^{(k)}_{e_k,i_k})$$
Algorithm 2 of Altschuler et al. (2017) to generate The first line uses the definition of \( E \) from definition of \( \phi_{k+1} \) and the second line uses the law of total expectation and the fact that \((\epsilon_k, i_k)\) is sampled uniformly. The inequality leverages the inductive hypothesis and convexity of \( L \). From Nesterov (2018, §2), we know that the definition of \( \delta_k \) from \( \theta_k \) ensures that \( \delta_k \to 0 \). Therefore, \( \{\phi_k, \delta_k\}_{k=0}^K \) is a random estimate sequence.

As noted, the identities are fairly standard (Nesterov, 2018; Lee & Sidford, 2013). We prove each claim in order.

- From definition of \( \phi_{k+1} \), computing the second derivative of the combination shows that it is constant at \((1 - \theta_k)\gamma_k \).
- Computing the gradient with respect to block-coordinate \( \lambda_{e_k, i_k} \) of the combination shows, at optimality, we have
  \[
  (1 - \theta_k) \gamma_k (\lambda_{e_k, i_k} - v^{(k)}_{e_k, i_k}) - \theta_k q y^{(k)}_{e_k, i_k} = 0
  \]
  which implies
  \[
  \lambda_{e_k, i_k} = v^{(k)}_{e_k, i_k} + \frac{\theta_k y^{(k)}_{e_k, i_k}}{\gamma_k + 1}
  \]
  For any other block-coordinate \((e, i)\), the optimality condition simply implies \( \lambda_{e, i} = v^{(k)}_{e, i} \).
- The last claim can be show by inserting the minimizer, \( v^{(k)}_{e_k, i_k} \), into \( \phi_{k+1} \). Therefore, we have

\[
\omega_{k+1} := \min_{\lambda} \phi_{k+1} = \phi_{k+1}(v^{(k+1)})
\]

\[
= (1 - \theta_k) \omega_k + \frac{\gamma_k + 1}{2} \left\| v^{(k)} - v^{(k+1)} \right\|_2^2 + \theta_k L(y^{(k)}) - \theta_k q y^{(k)}_{e_k, i_k}, v^{(k+1)}_{e_k, i_k} - y^{(k)}_{e_k, i_k} \right)
\]

\[
= (1 - \theta_k) \omega_k + \left(\frac{\theta k \gamma k^2}{2(\gamma k + 1)}\right) \left\| v^{(k)}_{e_k, i_k} \right\|_2^2 + \theta_k L(y^{(k)}) - \theta_k q y^{(k)}_{e_k, i_k}, v^{(k+1)}_{e_k, i_k} - y^{(k)}_{e_k, i_k} \right)
\]

\[
= (1 - \theta_k) \omega_k + \theta_k L(y^{(k)}) - \left(\frac{\theta k}{\gamma k + 1}\right) \left\| y^{(k)}_{e_k, i_k} \right\|_2^2 - \theta_k q y^{(k)}_{e_k, i_k}, v^{(k+1)}_{e_k, i_k} - y^{(k)}_{e_k, i_k} \right)
\]

\[
\square
\]

C.2. Proof of L2 Projection Lemma 5 and \( L^\lambda \) Projection Lemma 6

**Lemma 5.** For \( \lambda \in \mathbb{R}^D \) and \( \mu^\lambda \in L_2^\lambda \), Algorithm 4 returns a point \( \widehat{\mu} = \text{Proj}(\mu^\lambda, 0) \) such that \( \widehat{\mu} = \mu^\lambda \) for all \( i \in V \) and

\[
\sum_{e \in E} \|\mu_e^\lambda - \widehat{\mu}_e\|_1 \leq 2 \sum_{e \in E, i \in e} \|\mu^\lambda_{e, i}\|_1.
\]

**Proof.** Since \( \nu = 0 \), we know that \( \mu^\lambda_e + \nu_{e, i} = \mu^\lambda_e \in \Delta_n \) for all \( i \in V \) and \( e \in N_i \). For any \( (i, j) = e \in E \), \( \text{Proj} \) applies Algorithm 2 of Altschuler et al. (2017) to generate \( \widehat{\mu}_e \) from \( \mu^\lambda_e \) with the following guarantee due to Altschuler et al. (2017, Lemma 7): \(
\|\mu^\lambda_e - \mu^\lambda_e\|_1 \leq 2\|S_{e, i}^\lambda - \mu^\lambda_{e, i}\|_1 + 2\|S_{e, j}^\lambda - \mu^\lambda_{e, j}\|_1
\)

and \( \widehat{\mu}_e \in \mathcal{U}_d(\mu^\lambda_e, \mu^\lambda_e) \). Applying this guarantee for all edges in \( E \) gives the result. \( \square \)
Lemma 6. Let \( \mu \in \mathbb{L}_2 \) and \( \lambda \in \mathbb{R}^p \). Define \( \delta = \max_{e \in E, i \in e} \| \nu_{e,i}^\lambda \|_1 \) There exists \( \tilde{\mu} \) in the slack polytope \( \mathbb{L}^\lambda_2 \) such that
\[
\| \mu - \tilde{\mu} \|_1 \leq 16(m+n)d\delta + 2 \sum_{e \in E, i \in e} \| \nu_{e,i}^\lambda \|_1
\]

Proof. For convenience, we just write \( \nu \) for the slack, dropping the notational dependence on \( \lambda \). We will proceed with this proof by constructing such a \( \tilde{\mu} \) in two cases. We would like to show that the edge marginals \( \mu_e \) can be modified to give \( \tilde{\mu} \in \mathbb{L}^\lambda_2 \). To do this, we aim to use Algorithm 4 to match \( \tilde{\mu}_e \) to the modified marginals \( \mu_i + \nu_{e,i} \) for every \( e \in E \) and \( i \in e \). As long as \( \mu_i + \nu_{e,i} \in \Delta_d \), setting \( \mu_i' = \mu_i \) and \( \mu_e' = \mu_e \) and computing \( \tilde{\mu} = \text{Proj}(\tilde{\mu}, \nu) \) would return \( \tilde{\mu} \in \mathbb{L}^\lambda_2 \) that satisfies the condition by Lemma 5.

However, if \( \mu_i + \nu_{e,i} \notin \Delta_d \), then \( \exists x \in \chi \) such that \( \mu_i(x) + \nu_{e,i}(x) \notin [0, 1] \). Consider the case where \( \delta \leq \frac{1}{2d} \). We aim to create a temporary marginal vector \( \mu_i' \) which is made by modifying \( \mu_i \) appropriately until the slack can be added to \( \mu_i' \) while maintaining a valid distribution. To do this, we set \( \mu_i' \) as the convex combination with the uniform distribution
\[
\mu_i' = (1 - \theta) \mu_i + \theta \text{Unif}(\chi)
\]
for some \( \theta \in [0, 1] \). Choosing \( \theta = d\delta \) ensures that
\[
\delta \leq \mu_i'(x) \leq 1 - \delta \quad \forall x \in \chi,
\]
Furthermore, \( \mu_i' \in \Delta_d \) because \( \Delta_d \) is convex and we have
\[
\| \mu_i' - \mu_i \|_1 = \sum_{x \in \chi} \delta |1 - d\mu_i(x)| \leq \sum_{x} \delta + \delta d\mu_i(x) = 2d\delta
\]
Then we set \( \mu_e' = \mu_e \) for all \( e \in E \). Using Algorithm 4, we compute \( \tilde{\mu} = \text{Proj}(\mu', \nu) \in \mathbb{L}^\lambda_2 \). Together with Lemma 5, we have that
\[
\| \tilde{\mu} - \mu \|_1 = \sum_{i \in V} \| \tilde{\mu}_i - \mu_i \|_1 + \sum_{e \in E} \| \tilde{\mu}_e - \mu_e \|_1 \leq 2nd\delta + 2 \sum_{e \in E, i \in e} \| \nu_{e,i} \|_1 + \| \mu_i' - \mu_i \|_1 \leq (n + 8m)d\delta + 2 \sum_{e \in E, i \in e} \| \nu_{e,i} \|_1
\]
On the other hand, consider the case where \( \delta > \frac{1}{2d} \). Then instead we choose the temporary marginal vector as \( \mu_i' = \mu_i^\lambda \) for all \( i \in V \) and \( \mu_e' = \mu_e \) for all \( e \in E \), which ensures that \( \mu_i' + \nu_{e,i} \in \Delta_d \) by definition of \( \nu \). We then compute \( \tilde{\mu} = \text{Proj}(\mu', \nu) \), which ensures
\[
\| \tilde{\mu} - \mu \|_1 \leq \sum_{i \in V} \| \mu_i - \mu_i^\lambda \|_1 + 2 \sum_{e \in E, i \in e} \| \mu_i - \mu_i^\lambda \|_1 + \| \nu_{e,i} \|_1 \leq 2n + 8m + 2 \sum_{e \in E, i \in e} \| \nu_{e,i} \|_1 \leq 4nd\delta + 16md\delta + 2 \sum_{e \in E, i \in e} \| \nu_{e,i} \|_1
\]
where the second inequality uses the fact that the \( l_1 \) distance is bounded by 2 and the last inequality uses the assumption that \( \delta > \frac{1}{2d} \). We take the worst of these two cases for the final result. \( \square \)
C.3. Proof of Proposition 4

**Proposition 4.** Let $\mu^* \in \mathbb{L}_2$ be optimal, $\lambda \in \mathbb{R}^d$, $\hat{\mu} = \text{Proj}(\mu^\lambda, 0) \in \mathbb{L}_2$, and $\delta = \max_{e \in E, i \in e} \|\nu^\lambda_{e,i}\|_1$. The following inequality holds:

$$\langle C, \hat{\mu} - \mu^* \rangle \leq 16(m + n)d\|C\|_\infty \delta + 4\|C\|_\infty \sum_{e \in E, i \in e} \|\nu^\lambda_{e,i}\|_1 + \frac{n \log d + 2m \log d}{\eta}.$$ 

**Proof.** Consider $\mu^\lambda$, which may not lie in $\mathbb{L}_2$. It does, however, lie within its own slack polytope $\mathbb{L}^\lambda$ from Definition 1. Therefore, it can be seen that $\mu^\lambda$ is a solution to

$$\minimize \quad \langle C, \mu \rangle - \frac{1}{\eta}H(\mu) \quad \text{s.t.} \quad \mu \in \mathbb{L}^\lambda \quad (10)$$

Then, consider the point $\hat{\mu} = \text{Proj}(\mu^\lambda, 0) \in \mathbb{L}_2$. Let $\mu^* \in \arg \min_{\mu \in \mathbb{L}_2} \langle C, \mu \rangle$. We have

$$\langle C, \hat{\mu} - \mu^* \rangle = \langle C, \hat{\mu} - \mu^\lambda + \mu^\lambda - \mu^* \rangle$$

$$\leq \|C\|_\infty \|\hat{\mu} - \mu^\lambda\|_1 + \langle C, \mu^\lambda - \mu^* \rangle. \quad (11)$$

Note that the last term in the right-hand side can be written as

$$\langle C, \mu^\lambda - \mu^* \rangle = \langle C, \mu^\lambda - \mu^* + \hat{\mu}^* - \mu^* \rangle$$

$$\leq \|C\|_\infty \|\hat{\mu}^* - \mu^*\|_1 + \langle C, \mu^\lambda - \hat{\mu}^* \rangle, \quad (12)$$

where $\hat{\mu}^* \in \mathbb{L}^\lambda$ is the existing vector from Lemma 6 using $\mu^* \in \mathbb{L}_2$ and slack from $\lambda$. Because $\mu^\lambda$ is the solution to (10), we further have

$$\langle C, \mu^\lambda - \hat{\mu}^* \rangle \leq \frac{1}{\eta}(H(\mu^\lambda) - H(\hat{\mu}^*))$$

$$\leq \frac{n \log d + 2m \log d}{\eta} \quad (13)$$

Combining inequalities (11), (12), and (13) shows that

$$\langle C, \hat{\mu} - \mu^* \rangle \leq \|C\|_\infty (\|\hat{\mu} - \mu^\lambda\|_1 + \|\hat{\mu}^* - \mu^*\|_1) + \frac{n \log d + 2m \log d}{\eta}.$$ 

Using Lemma 5 and 6, we can further bound this as

$$\langle C, \hat{\mu} - \mu^* \rangle \leq \|C\|_\infty \left( 16(m + n)d\delta + \sum_{e,i} 4\|\nu^\lambda_{e,i}\|_1 \right) + \frac{n \log d + 2m \log d}{\eta}.$$ 

\[ \square \]

C.4. Proof of $G(\eta)$ Upper Bound

In the proof of Lemma 4, we used the fact that the numerator of the final convergence rate can be bounded by $G(\eta)^2$. Here, we formally state this result and prove it.

**Lemma 7.** It holds that

$$4L(0) - 4L(\lambda^*) + 16m^2\eta\|\lambda^*\|_2^2 \leq G(\eta)^2,$$

where $G(\eta) := 24md(m + n)\sqrt{\eta}\|C\|_\infty + \frac{\log d}{\sqrt{\eta}}$. 

The proof requires bounding both $L(0) - L(\lambda^*)$, which we have already done in Lemma 10, and bounding the norm $\|\lambda^*\|^2_2$.

We rely on the following result from Meshi et al. (2012).

**Lemma 8.** There exists $\lambda^* \in \Lambda^*$ such that

$$\|\lambda^*\|_1 \leq 2d(n + m)\|C\|_{\infty} + \frac{4d(m + n)}{\eta} \log d$$

$$\leq \frac{4d(m + n)}{\eta} (\|C\|_{\infty} + \log d)$$

**Proof.** Modifying Meshi et al. (2012, Supplement Lemma 1.2) for our definition of $H$ gives us

$$\|\lambda^*\|_1 \leq 2d(L(0) - n - m - \langle C, \mu^{\lambda^*} \rangle + \frac{1}{\eta} H(\mu^{\lambda^*})).$$

Using Cauchy-Schwarz and maximizing over the entropy yields the result.

**Proof of Lemma 7.** Using these results, we can prove the claim. We bound the square root of the numerator, multiplied by $\sqrt{\eta}$:

$$\sqrt{\eta (4L(0) - 4L(\lambda^*) + 16m^2\eta\|\lambda^*\|_2^2)} \leq \sqrt{8(m + n)(\eta\|C\|_{\infty} + \log d) + 16m^2 (4d(m + n) (\eta\|C\|_{\infty} + \log d))^2}$$

$$\leq \sqrt{8(m + n)(\eta\|C\|_{\infty} + \log d) + 16md(m + n)(\eta\|C\|_{\infty} + \log d)}$$

$$\leq 24md(m + n)(\eta\|C\|_{\infty} + \log d)$$

The first inequality used Lemma 10 and Lemma 8. The second inequality uses the fact that $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$. The last inequality uses the fact that the first term is greater than 1 under the assumption $d \geq 2$. Dividing through by $\sqrt{\eta}$ gives the result.

**D. Proof of Theorem 1**

We begin with a complete proof of Theorem 1 for EMP and then show how to modify it slightly for SMP. The result also builds on some of the same technical lemmas used in the proof of Theorem 2.

**D.1. Edge Message Passing**

The cornerstone of the proof is showing that the expected slack norms can be bounded over iterations.

**Lemma 9.** Let $L^* = \min_\lambda L(\lambda)$ and let $\hat{\lambda}$ be the output of Algorithm 1 after $K$ iterations with EMP and a uniform distribution. For any $e \in E$ and $i \in e$, the expected norm of the constraint violation in $\mathbb{L}_2$ is bounded as

$$\mathbb{E} \sum_{e \in E, i \in e} \|S_{e,i}^{\hat{\lambda}} - \mu_i^{\hat{\lambda}}\|_2^2 \leq \frac{8m\eta(L(0) - L^*)}{K}$$

**Proof.** From Lemma 1, we have that the expected improvement is lower bounded at each iteration

$$\mathbb{E} \left[ L(\lambda^{(k)}) - L(\lambda^{(k+1)}) \right] \geq \frac{1}{4\eta} \mathbb{E} \left[ \|\nabla_{e,i} L(\lambda^{(k)})\|_2^2 \right],$$
Accelerated Message Passing for Entropy-Regularized MAP Inference

Then, using that $\nabla_{e,i} L(\lambda) = \mu_i^{(k)} - S_{e,i}^{(k)}$, we apply the bound $k = 1, 2, \ldots, K$:

$$L(0) - L^* \geq \frac{1}{4\eta} \sum_{k=0}^{K-1} \mathbb{E} \left[ \| S_{e,i}^{(k)} - \mu_i^{(k)} \|_1^2 \right]$$

$$= \frac{1}{8m\eta} \sum_{k=0}^{K-1} \sum_{e \in E, i \in e} \mathbb{E} \left[ \| S_{e,i}^{(k)} - \mu_i^{(k)} \|_1^2 \right]$$

$$\geq \frac{K}{8m\eta} \sum_{e \in E, i \in e} \mathbb{E} \left[ \| \hat{S}_{e,i}^{(k)} - \hat{\mu}_i^{(k)} \|_1^2 \right]$$

The equality uses the law of total expectation, conditioning on $\lambda^{(k)}$. The second inequality uses the fact that $\hat{\lambda}$ is chosen to minimize the sum of squared constraint violations.

Next, we provide a bound on the initial function value gap.

**Lemma 10.** For $\lambda^* \in \Lambda^*$ it holds that $L(0) - L(\lambda^*) \leq 2(m + n)\|C\|_\infty + \frac{2}{\eta}(m + n) \log d$.

**Proof.** We will bound both $L(0)$ and $L(\lambda^*)$ individually. First, from the definition

$$L(0) = \frac{1}{\eta} \sum_i \log \sum_{x_i} \exp(-\eta C_i(x_i)) + \frac{1}{\eta} \sum_e \log \sum_{x_e} \exp(-\eta C_e(x_e))$$

$$\leq \frac{1}{\eta} \sum_i \log \sum_{x_i} \exp(\eta \|C\|_\infty) + \frac{1}{\eta} \sum_e \log \sum_{x_e} \exp(\eta \|C\|_\infty)$$

$$= (n + m)\|C\|_\infty + \frac{n}{\eta} \log d + \frac{2m}{\eta} \log d.$$ 

For $L(\lambda^*)$, we recognize that $L$ is simply the negative of the primal problem (Reg-P), shifted by a constant amount. In particular, we have

$$-L(\lambda^*) - \frac{1}{\eta}(n + m) = \langle C, \mu^* \rangle - \frac{1}{\eta} H(\mu^*)$$

For some $\mu^*$ that solves (Reg-P). Note that $H$ is offset with a linear term (different from the usual definition of the entropy) that exactly cancels the $-\frac{1}{\eta}(n + m)$ on the left-hand side. We then conclude $-L(\lambda^*) \leq (m + n)\|C\|_\infty$ by Cauchy-Schwarz. Summing these two gives the desired result. 

**Proof of Theorem 1 for EMP.** Fix $\epsilon' > 0$. Lemma 9 and Lemma 10 ensure that

$$\mathbb{E} \sum_{e \in E, i \in e} \| \hat{\nu}_{e,i} \|_1^2 \leq (\epsilon')^2$$

after

$$K = \frac{16m(m + n)(\eta \|C\|_\infty + \log d)}{(\epsilon')^2}$$

(14) iterations. By Jensen’s inequality and recognizing that the norms are non-negative, this also implies that

$$\mathbb{E}[\| \hat{\nu}_{e,i} \|_1] \leq \epsilon' \forall e \in E, i \in e$$

after the same number of iterations.
Now, we use the upper bound due to the approximation from Proposition 4 and take the expectation, giving
\[
\mathbb{E}[(C, \tilde{\mu} - \mu^*)] \leq \|C\|_{\infty} (8m\epsilon' + 16(m + n)d\mathbb{E}[\delta])
= n\log d + 2m\log d
\]
where
\[
\mathbb{E}[\delta]^2 \leq \mathbb{E}[^2] \leq \sum_{e \in E, i \in e} \|\nu_{e,i}^{\lambda}\|_2 \leq (\epsilon')^2.
\]
Therefore, the bound becomes
\[
\mathbb{E}[(C, \tilde{\mu} - \mu^*)] \leq 24(m + n)d\|C\|_{\infty}\epsilon'
+ n\log d + 2m\log d
\]
We conclude the result from substituting into (14), using the definition of \(\eta\) and choosing \(\epsilon' = \frac{\epsilon}{48(m + n)d\|C\|_{\infty}}\).\]

D.2. Star Message Passing

The significant difference between the SMP proof and the EMP proof is that there is variable improvement at each update, dependent on the node being updated. Using the distribution from (3), we ensure that the improvement becomes uniform in expectation. This analysis is similar to weighting coordinates by their coordinate-wise smoothness coefficients in coordinate gradient algorithms (Nesterov, 2012).

A slight modification of the proof of (Meshi et al., 2012) is required the get the tighter \(l_1\)-norm lower bound.

**Lemma 2.** Let \(X\) be the result of applying SMP\(_{\eta}^\gamma\) to \(\lambda\), keeping all other coordinates fixed. Then, \(L(\lambda) - L(X) \geq \frac{1}{8|N_i|\eta} \sum_{e \in N_i} \|\nu_{e,i}^{\lambda}\|_2^2\).

**Proof.** Meshi et al. (2012) show that
\[
L(\lambda) - L(X) = -\frac{1}{\eta} \log \left( \sum_{x_i} \left( \mu_i^{\lambda} \prod_{e \in N_i} S_{e,i}^{\lambda}(x_i) \right)^{\frac{1}{|N_i|+1}} \right),
\]
and further
\[
|N_i| - |N_i| \left( \sum_{x_i} \left( \mu_i^{\lambda} \prod_{e \in N_i} S_{e,i}^{\lambda}(x_i) \right)^{\frac{1}{|N_i|+1}} \right)^{\frac{1}{|N_i|+1}} \geq \sum_{e \in N_i} \left( 1 - \left( \sum_{x_i} \sqrt{\mu_i^{\lambda}(x_i)S_{e,i}^{\lambda}(x_i)} \right)^2 \right)
\]
We recognize the inner term of the square as the Bhattacharyya coefficient which satisfies \(BC \in [0, 1]\). Therefore,
\[
\sum_{e \in N_i} \left( 1 - \left( \sum_{x_i} \sqrt{\mu_i^{\lambda}(x_i)S_{e,i}^{\lambda}(x_i)} \right)^2 \right) \geq \sum_{e \in N_i} \left( 1 - \left( \sum_{x_i} \sqrt{\mu_i^{\lambda}(x_i)S_{e,i}^{\lambda}(x_i)} \right) \right)
= \sum_{e \in N_i} h^2(\mu_i^{\lambda}, S_{e,i}^{\lambda})
\]
Then,
\[
\left( \sum_{x_i} \left( \mu_i^{\lambda} \prod_{e \in N_i} S_{e,i}^{\lambda}(x_i) \right)^{\frac{1}{|N_i|+1}} \right)^{|N_i|+1} \leq 1 - \frac{1}{N_i} \sum_{e \in N_i} h^2(\mu_i^{\lambda}, S_{e,i}^{\lambda})
Finally, we lower bound the original difference of values

\[ L(\lambda) - L(\lambda') \geq -\frac{1}{\eta} \log \left( 1 - \frac{1}{N_i} \sum_{e \in N_i} h^2(\mu^e, S_{e,i}) \right) \]

\[ \geq \frac{1}{N_i \eta} \sum_{e \in N_i} h^2(\mu^e, S_{e,i}) \]

\[ \geq \frac{1}{8N_i \eta} \sum_{e \in N_i} ||S^e_{e,i} - \mu^e||_1^2 \]

\[ \square \]

**Lemma 11.** Let \( L^* = \min_{\lambda} L(\lambda) \) and let \( \hat{\lambda} \) be the output of Algorithm 1 after \( K \) iterations with SMP and distribution (3). Define \( N = \sum_{j \in V} |N_j| \). For any \( e \in E \) and \( i \in e \), the expected norm of the constraint violation in \( L_2 \) is bounded as

\[ \mathbb{E} \sum_{e \in E, i \in e} ||S^e_{e,i} - \mu||_1^2 \leq \frac{8N\eta(L(0) - L^*)}{K} \]

**Proof.** Lemma 2 gave us the following lower bound on the improvement:

\[ \mathbb{E} \left[ L(\lambda^{(k)}) - L(\lambda^{(k+1)}) \right] \geq \mathbb{E} \left[ \frac{1}{8} |N_{ik}| \eta \sum_{e \in N_{ik}} ||\mu^e_{i,i}||_1^2 \right] \]

Then, since \( i_k \) is chosen with probability \( p_i = \frac{|N_i|}{N} \), we can apply the bound for \( k = 1, 2, \ldots, K \) and expand the expectations:

\[ L(0) - L^* \geq \sum_{k=0}^{K-1} \mathbb{E} \left[ \frac{1}{8} |N_{ik}| \eta \sum_{e \in N_{ik}} ||\mu^e_{i,i}||_1^2 \right] \]

\[ = \frac{1}{8N\eta} \sum_{k=0}^{K-1} \mathbb{E} \sum_{e \in E, i \in e} ||\mu^e||_1^2 \]

\[ \geq \frac{1}{8N\eta} \sum_{k=0}^{K-1} \mathbb{E} \left[ ||p^e||_1^2 \right] \]

The equality uses the law of total expectation, conditioning on \( \lambda^{(k)} \). The second inequality uses the fact that \( \hat{\lambda} \) is chosen to minimize the sum of squared constraint violations. \( \square \)

The rest of the proof for SMP proceeds in an identical manner to the case for EMP; however, we simply replace the \( 8m\eta \) with \( 8N\eta \) everywhere. This stems from the fact that we can now guarantee

\[ \mathbb{E} \sum_{e \in E, i \in e} ||S^e_{e,i} - \mu||_1^2 \leq (\epsilon')^2 \]

in \( \frac{8N\eta(L(0)-L^*)}{(\epsilon')^2} \) iterations instead. We can then use the same upper bound from Lemma 10 and substitute in the same choices of \( \epsilon' \) and \( \eta \) as in EMP to get the result.

**E. Proof of Theorem 2 for SMP**

The proof for SMP essentially follows the same structure, but it requires defining the estimate sequence in slightly different way. Define the probability distribution \( \{p_i\}_{i \in V} \) over \( V \) with \( p_i = \frac{|N_i|}{\sum_{j \in V} |N_j|} \). We propose the candidate:

\[ i_k \sim \text{Cat}(V, \{p_i\}_{i \in V}) \]

\[ \delta_{k+1} = (1 - \theta_k)\delta_k \]

\[ \phi_{k+1}(\lambda) = (1 - \theta_k)\phi_k(\lambda) + \theta_k L(y^{(k)}) - \frac{\theta_k}{p_{i_k}} \langle \nu^{(k)}_{i_k}, \lambda_{i_k} - y_{i_k}^{(k)} \rangle \]  (15)

\[ \square \]
We now provide a faster convergence guarantee on the dual objective function for the random estimate sequence in (15), letting \( \phi_k(\lambda) = \omega_k + \frac{3k^2}{2} \| \lambda - \nu^{(k)} \| \) for all \( k \) where

\[
\gamma_{k+1} = (1 - \theta_k) \gamma_k \\
\nu_{i,k}^{(k+1)} = \begin{cases} 
\nu_{i,k}^{(k)} + \frac{\theta_k}{\gamma_{k+1}} \nu^{(k)}_i & \text{if } i = i_k \\
\nu_{i,k}^{(k)} & \text{otherwise}
\end{cases}
\]

\[
\omega_{k+1} = (1 - \theta_k) \omega_k + \theta_k L(\nu^{(k)}) - \frac{\theta_k^2}{2 \gamma_{k+1}^2 \nu_{i,k}^{(k)}} \| \nu^{(k)}_i - \nu^{(k)}_i \|_2^2 + \frac{\theta_k}{\gamma_{k+1}^2 \nu_{i,k}^{(k)}} \| \nu^{(k)}_i - \nu^{(k)}_i \|_2^2 - \frac{\theta_k}{\gamma_{k+1}^2 \nu_{i,k}^{(k)}} \| \nu^{(k)}_i - \nu^{(k)}_i \|_2^2 - \frac{\theta_k}{\gamma_{k+1}^2 \nu_{i,k}^{(k)}} \| \nu^{(k)}_i - \nu^{(k)}_i \|_2^2 - \theta_k \nu^{(k)}_i
\]

**Proof.** To show that this is an estimate sequence, the proof is essentially identical to the EMP case. The only exception is that we take expectation over \( V \) with distribution \( \{ p_i \} \) \( i \in V \). However, this ensures that

\[
E[\frac{\theta_k}{\gamma_{k+1}^2 \nu_{i,k}^{(k)}} \| \nu^{(k)}_i - \nu^{(k)}_i \|_2^2] = \theta_k E[\langle \nabla L(\nu^{(k)}), \lambda - \nu^{(k)} \rangle]
\]

by the law of total expectation. So the the proof that this is an estimate sequence remains the same.

To show that it retains the desired quadratic structure, we again analyze all terms of interest

- \( \gamma_{k+1} \) is identical to the EMP case so the result holds.
- Taking the gradient with respect to \( \lambda_{i,k} \), we have that the optimality conditions, for \( i = i_k \), are

\[
\gamma_{k+1} (\lambda_{i,k} - \nu_{i,k}^{(k)}) - \frac{\theta_k}{\gamma_{k+1}^2 \nu_{i,k}^{(k)}} \nu^{(k)}_i = 0
\]

and, for all other \( i \), they are

\[
\gamma_{k+1} (\lambda_{i,k} - \nu_{i,k}^{(k)}) = 0.
\]

These conditions imply the given construction for \( \nu_{i,k}^{(k+1)} \).
- We can then compute \( \omega_{k+1} \) by plugging in the choice for \( \nu_{i,k}^{(k+1)} \) again:

\[
\omega_{k+1} := \min_{\lambda} \phi_{k+1} = \phi_{k+1}(\nu^{(k+1)}) = (1 - \theta_k) \omega_k + \gamma_{k+1} \| \nu^{(k)} - \nu^{(k+1)} \|_2^2 + \gamma_{k+1} L(\nu^{(k)}) - \frac{\theta_k^2}{2 \gamma_{k+1}^2 \nu_{i,k}^{(k)}} \| \nu^{(k)}_i - \nu^{(k)}_i \|_2^2 + \frac{\theta_k}{\gamma_{k+1}^2 \nu_{i,k}^{(k)}} \| \nu^{(k)}_i - \nu^{(k)}_i \|_2^2 + \frac{\theta_k}{\gamma_{k+1}^2 \nu_{i,k}^{(k)}} \| \nu^{(k)}_i - \nu^{(k)}_i \|_2^2 - \theta_k \nu^{(k)}_i
\]

We now provide a faster convergence guarantee on the dual objective function for SMP which depends on \( N = \sum_{j \in V} |N_j| \).

**Lemma 12.** The sequence \( \{ \phi_k, \delta_k \} \) defined in (15) is a random estimate sequence. Furthermore, it maintains the form \( \phi_k(\lambda) = \omega_k + \frac{3k^2}{2} \| \lambda - \nu^{(k)} \| \) for all \( k \).

**Lemma 13.** For the random estimate sequence in (15), let \( \{ \lambda^{(k)} \} \) \( k \geq 0 \) and \( \{ \nu^{(k)} \} \) \( k \geq 0 \) be defined as in Algorithm 3 with \( \lambda^{(0)} = 0 \). Then, the dual objective error in expectation can be bounded as

\[
E[\| L(\lambda^{(k)}) - L(\lambda^*) \|] \leq \frac{G_{SMP}(\eta)}{(k + 2)^2}
\]

where \( G_{SMP}(\eta) := 2^4 d (m + n) \sqrt{T} \| C \|_\infty + \frac{\log d}{\sqrt{T}} \) and \( N = \sum_{j \in V} |N_j| \).
Proof. As in the EMP proof, it suffices to show that \( \mathbb{E}[\omega_{k+1}] \geq \mathbb{E}[L(\lambda^{(k+1)})] \) by induction. As before we have

\[
\mathbb{E}[\omega_{k+1}] \geq (1 - \theta_k)\mathbb{E}[L(\lambda^{(k)})] + \theta_k \mathbb{E}[L(y^{(k)})] - \mathbb{E} \left[ \frac{\theta_k^2}{2\gamma_{k+1}p_{ik}} \| \nu_{i_{ik}}^{(k)} \|_2^2 - \frac{\theta_k}{p_{ik}} (\nu_{i_{ik}}^{(k)}, y^{(k)} - y^{(k)}) \right]
\]

\[
\geq \mathbb{E} \left[ L(y^{(k)}) - \frac{\theta_k^2}{2\gamma_{k+1}p_{ik}} \| \nu_{i_{ik}}^{(k)} \|_2^2 \right] + (1 - \theta_k) \mathbb{E} \left[ \langle \nabla L(y^{(k)}), \lambda^{(k)} - y^{(k)} \rangle \right] + \theta_k \mathbb{E} \left[ \langle \nabla L(y^{(k)}), y^{(k)} - y^{(k)} \rangle \right]
\]

\[
= \mathbb{E} \left[ L(y^{(k)}) - \sum_{i \in V} \frac{\theta_k^2}{2\gamma_{k+1}p_{ik}} \| \nu_{i_{ik}}^{(k)} \|_2^2 \right],
\]

where the last line comes from the definition of \( y^{(k)} \). Choosing \( \theta_k \) such that \( \theta_k^2 = \frac{\gamma_{k+1} \min_j |N_j|}{4N^2\eta} \) results in

\[
\mathbb{E}[\omega_{k+1}] \geq \mathbb{E} \left[ L(y^{(k)}) - \sum_{i \in V} \frac{1}{8N\eta} \| \nu_{i_{ik}}^{(k)} \|_2^2 \right]
\]

\[
= \mathbb{E} \left[ L(y^{(k)}) - \frac{1}{8N\eta} \| \nabla L(y^{(k)}) \|_2^2 \right]
\]

Recall, from the improvement in Lemma 2, we have

\[
\mathbb{E}[L(\lambda^{(k+1)})] \leq \mathbb{E}[L(y^{(k)})] - \mathbb{E} \left[ \frac{1}{8N\eta} \| \nu_{i_{ik}}^{(k)} \|_2^2 \right]
\]

\[
= \mathbb{E}[L(y^{(k)})] - \mathbb{E} \left[ \frac{1}{8N\eta} \| \nabla L(y^{(k)}) \|_2^2 \right]
\]

Therefore, by this induction, the inequality \( \mathbb{E}[L(\lambda^{(k+1)})] \leq \mathbb{E}[\omega_{k+1}] \) holds for all \( k \). Furthermore, by choosing \( \gamma_0 = \frac{4N^2\eta}{\min_j |N_j|} \), we ensure that \( \theta_k \) can be updated recursively as in Algorithm 3 and the update equation for \( v \) is simplified to

\[
v^{(k+1)}_{i_{ik}} = \begin{cases} v^{(k)}_{i_{ik}} + \min_j |N_j| \nu_{i_{ik}}^{(k)} & \text{if } i = i_k \\ v^{(k)}_{i_{ik}} & \text{otherwise} \end{cases}
\]

Using the property of randomized estimate sequences derived in Section 4, we can bound the expected error in the dual norm as

\[
\mathbb{E}(L(\lambda^{(k)})) - L^* \leq \frac{4}{(k+2)^2} \left( \mathbb{E}(L(0)) - L^* + \frac{\gamma_0}{2} \| \lambda \|_2^2 \right)
\]

\[
= \frac{4}{(k+2)^2} \left( \mathbb{E}(L(0)) - L^* + \frac{2N^2\eta}{\min_j |N_j|} \| \lambda \|_2^2 \right)
\]

\[
\leq \frac{4}{(k+2)^2} \left( \mathbb{E}(L(0)) - L^* + 2N^2\eta \| \lambda \|_2^2 \right)
\]

The numerator can then be bounded in an identical manner to the EMP proof by replacing \( 4m^2 \) with \( 2N^2 \) in Lemma 7, instead yielding \( G_{\text{SMP}}(\eta) = 40md(m+n)(\sqrt{\eta}C\|C\|_\infty + \frac{\log(d)}{\sqrt{\eta}}) \), which is only different by a constant. We then have

\[
\mathbb{E}(L(\lambda^{(k)})) - L^* \leq \frac{G_{\text{SMP}}(\eta)}{(k+2)^2}
\]

With these tools, we are ready to present the proof of Theorem 2 for SMP.
Proof of Theorem 2 for SMP. Let $\hat{\lambda}$ be the output from Algorithm 3 after $K$ iterations. From Lemma 2, we can lower bound the result in Lemma 13 with

\[
\frac{1}{8\eta|N_i|} \mathbb{E} \left[ \sum_{e \in N_i} \| \nu_{e,i}^\lambda \|_1^2 \right] \leq \mathbb{E}[L(\hat{\lambda})] - L^*
\]

for all $i \in V$. This further implies that

\[
\frac{1}{8\eta|N_i|} \mathbb{E} \left[ \| \nu_{e,i}^\lambda \|_1^2 \right] \leq \frac{G_{\text{SMP}}(\eta)}{(K + 2)^2}
\]

for all $e \in E$ and $i \in e$. Then, for $\epsilon' > 0$, we can ensure that

\[
\mathbb{E}[\| \nu_{e,i}^\lambda \|_1] \leq |N_i|\epsilon'
\]

\[
\mathbb{E} \sum_{i \in V, e \in N_i} \| \nu_{e,i}^\lambda \|_1^2 \leq N(\epsilon')^2
\]

in $K = \frac{\sqrt{G(\eta)}}{\epsilon'}$ iterations. Letting $\hat{\mu} \in \mathbb{L}_2$ be the projected version of $\mu^\lambda$, we have

\[
\langle C, \hat{\mu} - \mu^* \rangle \leq \| C \|_\infty \left( 16(m + n)d\delta + \sum_{e, i} 4\| \nu_{e,i}^\lambda \|_1 \right) + \frac{n \log d + 2m \log d}{\eta}.
\]

Taking the expectation of both sides gives us

\[
\mathbb{E}[\langle C, \hat{\mu} - \mu^* \rangle] \leq \| C \|_\infty \left( 16(m + n)d\mathbb{E}[\delta] + 4N\epsilon' \right) + \frac{n \log d + 2m \log d}{\eta},
\]

where

\[
\mathbb{E}[\delta^2] \leq \mathbb{E}[\delta^2] \leq \mathbb{E} \sum_{e \in E, i \in e} \| \nu_{e,i}^\lambda \|_1^2 \leq N(\epsilon')^2.
\]

Then we can conclude

\[
\mathbb{E}[\langle C, \hat{\mu} - \mu^* \rangle] \leq 16\sqrt{N}(m + n)d\| C \|_\infty \epsilon' + 4\mathbb{E}[\delta] + 4N\| C \|_\infty \epsilon' + \frac{n \log d + 2m \log d}{\eta}
\]

\[
\leq 24\sqrt{N}(m + n)d\| C \|_\infty \epsilon' + \frac{n \log d + 2m \log d}{\eta}.
\]

The last inequality uses the fact that $N = 2m$. Therefore, $\hat{\mu}$ is expected $\epsilon$-optimal with $\eta$ as defined in the statement and $\epsilon' = \frac{24\sqrt{N}(m + n)d\| C \|_\infty}{\eta}$. Substituting these values into $K$ and $G(\eta)$ yields the result.

\[\square\]

F. Rounding to Integral Solutions Proofs

In this section, we prove the bound on the number of iterations sufficient to recover the MAP solution using Accel-EMP and rounding the output of the algorithm. We then compare with standard EMP.

F.1. Approximation Error

Let $V_2$ be the set of vertices of $\mathbb{L}_2$ and $V_2^\ast$ be the set of optimal vertices with respect to $C$. Denote by $\Delta = \min_{V_1 \in V_2 \setminus V_2^\ast} \langle C, V_1 - V_2 \rangle$ the suboptimality gap. Let $R_L = \max_{\mu \in L_2} \| \mu \|_1$, and $R_H = \max_{\mu, \mu' \in L_2} H(\mu) - H(\mu')$. Define $\text{deg}$ to be the maximum degree of the graph. The following holds:

**Theorem 4** (Theorem 1 of (Lee et al., 2020)). If $L_2$ is tight, $|V_2^\ast| = 1$ and $\eta \geq \frac{2R_L}{\Delta}$, then $\| \mu^\ast - \mu^\ast \|_1 \leq \frac{1}{8}$ and therefore the rounded solution $\text{round}(\mu^\ast)$ is a MAP assignment.
F.2. Estimation Error for Accelerated Message Passing

To bound the estimation error, we invoke the accelerated convergence guarantees presented in the previous section. In particular, we showed that

$$E \left[ \| \nu_{e,i}^\lambda \|_1 \right] \leq \epsilon' \quad \forall e \in E, i \in e$$

after $$K = \frac{\sqrt{4\eta G(\eta)}}{\epsilon'}$$ iterations for Accel-EMP. Markov’s inequality implies that with probability $$1 - \delta$$, $$\| \nu_{e,i}^\lambda \|_1 \leq \frac{2m\epsilon'}{\delta} := \epsilon$$ for all $$e \in E$$ and $$i \in e$$. From Theorem 3 of Lee et al. (2020), we require

$$\epsilon < O \left( d^{-2}m^{-2} \delta^{-2} \max(1, \eta \| C \|_\infty)^{-1} \right)$$

Furthermore, the theorem of the previous subsection implies we can set

$$\eta = \frac{16(m + n)(\log(m + n) + \log(d))}{\Delta}$$

Then, by setting

$$\epsilon' \leq O \left( d^{-2}m^{-4} \delta \deg^{-2} \max(1, \| C \|_\infty/\Delta)^{-1} (\log dm)^{-1} \right)$$

the condition is satisfied. Therefore, plugging into $$\sqrt{4\eta G(\eta)}$$ yields

$$\sqrt{4\eta G(\eta)} = O \left( \frac{dm^3\| C \|_\infty \log dm}{\Delta} \right)$$

which implies, with probability $$1 - \delta$$,

$$K = O \left( \frac{d^3m^7 \deg^2 \| C \|_\infty^2 \log^2 dm}{\delta \Delta} \right)$$

These conditions of $$\epsilon'$$ and $$\eta$$ guarantee that the round($$\mu^\lambda$$) is the MAP solution by invoking Theorem 3 of Lee et al. (2020).

F.3. Comparison to Standard Methods

Using standard EMP, we require the same conditions be satisfied on $$\epsilon'$$ and $$\eta$$ to guarantee recover of the MAP solution. However, the rate of convergence differs, requiring $$K = \frac{L(0) - L(\lambda^*)}{(\epsilon')^2}$$ iterations, as seen previously. Note that

$$L(0) - L(\lambda^*) = O \left( \frac{m^3\| C \|_\infty \log dm}{\Delta} \right)$$

Note that there is no additional $$d$$ dependence. It holds that with probability $$1 - \delta$$, the MAP solution is recovered by EMP in at most

$$K = O \left( \frac{d^4m^{11} \deg^4 \| C \|_\infty^3 \log^3 dm}{\delta^2 \Delta} \right)$$

iterations. We emphasize that this iteration bound is only a sufficient condition by directly applying the technique developed in this section. We suspect it can be greatly improved.