Global Well-posedness of the Chemotaxis-Navier-Stokes Equations in two dimensions

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Abstract

We consider two dimensional Keller-Segel equations coupled with the Navier-Stokes equations modelled by Tuval et al. [32]. Assuming that the chemotactic sensitivity and oxygen consumption rate are nondecreasing and differentiable, we prove that there is no blow-up in a finite time for solutions with large initial data to chemotaxis-Navier-Stokes equations in two dimensions. In addition, temporal decays of solutions are shown, as time tends to infinity.

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1 Introduction

In this paper we consider mathematical models describing the dynamics of oxygen, swimming bacteria, and viscous incompressible fluids in $\mathbb{R}^2$. Bacteria or microorganisms often live in fluid, in which the biology of aerotaxis is intimately related to the surrounding physics. Tuval et al. proposed the mathematical model to describe the dynamics of swimming bacteria, *Bacillus subtilis* in [32], which is given in two dimensions model as follows:

\[
\begin{align*}
\partial_t n + u \cdot \nabla n - \Delta n &= -\nabla \cdot (\chi(c)n\nabla c), \\
\partial_t c + u \cdot \nabla c - \Delta c &= -k(c)n, \\
\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p &= -n\nabla \phi, \\
\nabla \cdot u &= 0
\end{align*}
\]

where $c \geq 0$, $n \geq 0$, $u$ and $p$ denotes the oxygen concentration, cell concentration, fluid velocity, and scalar pressure, respectively. The nonnegative function $k(c)$ denotes the oxygen consumption rate, and the nonnegative function $\chi(c)$ denotes chemotactic sensitivity. Initial data are given by $(n_0(x), c_0(x), u_0(x))$ with $n_0(x)$, $c_0(x) \geq 0$ and $\nabla \cdot u_0 = 0$. To describe the fluid motions, we use Boussinesq approximation to denote the effect due to heavy bacteria. The function $\phi = \phi(x)$ is usually given as a time-independent one, which denotes the potential function such as the gravitational force or centrifugal force.

In this paper we prove that if initial data are sufficiently smooth, solutions for (1.1) become regular globally in time under suitable assumptions (see the assumption (A) in (1.6)) on the chemotactic sensitivity and oxygen consumption rate.

The classical model to describe the motion of cells was suggested by Patlak [24] and Keller-Segel [13, 14], which is given as

\[
\begin{align*}
\tau c_t &= \Delta c - \alpha c + \beta n, \\
n_t &= \Delta n - \nabla \cdot (n\chi\nabla c),
\end{align*}
\]

\quad $\tau \geq 0$, 

(1.2)
where $n = n(t, x)$ is the cell density and $c = c(t, x)$ is the concentration of chemical attractant substance. Here, $\chi$ is the chemotactic sensitivity, and $\alpha \geq 0$ and $\beta \geq 0$ are the decay and production rate of the chemical, respectively. The system (1.2) has been comprehensively studied and we will not try to give list of results here (see e.g. [8, 20, 22, 33] and the survey papers [10, 12]). In the absence of effect of fluids, i.e., $u = 0$, the system (1.1) is reduced to Keller-Segel equations (1.2) ($\tau = 1$), but with opposite sign $\beta \leq 0$. It is due to different biological contexts; the oxygen concentration in (1.1) is consumed and the chemical substance, respectively. The system (1.2) has been comprehensively studied and we will not try to give list of results here (see e.g. [8, 20, 22, 33] and the survey papers [10, 12]). In [34] and [35] Winkler proved the global existence of regular solutions and its asymptotic behavior in general, or some initial data would develop a blow-up. We review some known results related to the global existence issue on (1.1). [2], [6], [29] showed the global existence of regular solution with various smallness conditions on initial data. See the references therein for more extensive results on weak solutions and the nonlinear diffusion models (see also [5]). Asymptotic behaviors of the small global regular solution were studied in [2], [3]. Up to authors’ knowledge, there are a few results allowing large initial data. In [34] and [35] Winkler proved the global existence of regular solutions and its asymptotic profile for bounded domains with boundary conditions $\partial_{\nu} n = \partial_{\nu} c = u = 0$ under the following sign conditions on $\chi(\cdot)$ and $k(\cdot)$:

$$
\left(\frac{k(\cdot)}{\chi(\cdot)}\right)'>0, \quad (\chi(\cdot)k(\cdot))' \geq 0, \quad \left(\frac{k(\cdot)}{\chi(\cdot)}\right)^{''} \leq 0. \quad (1.3)
$$

In [11] the authors of the paper established global existence of smooth solutions assuming the relations of $\chi(\cdot)$ and $k(\cdot)$ such that

$$
\chi(c), k(c), \chi'(c), k'(c) \geq 0, \quad k(0) = 0 \quad \text{and} \quad \sup_c |\chi(c) - \mu k(c)| < \epsilon \quad \text{for some } \mu > 0. \quad (1.5)
$$

The assumption is based on experimental results in [11] and [32]. The initial data in [11] need not to be small and it, however, requires that chemotactic parameter functions $\chi(\cdot)$ and $k(\cdot)$ are almost proportional each other.

The goal of the paper is to get rid of the condition (1.5) and to show a global in time existence of smooth solution under only the assumption (1.4), that is

$$
(A) \quad \chi(c) \geq 0, \quad \chi'(c) \geq 0, \quad k(c) \geq 0, \quad k'(c) \geq 0, \quad k(0) = 0. \quad (1.6)
$$

From now on we denote $L^p_{x,t} = L^p(0, T; L^p(\mathbb{R}^2))$ and $L^p_{x,t} = L^p(0, T; L^p(\mathbb{R}^2))$ with $T$ given in the context. We mostly suppress the spatial domain $\mathbb{R}^2$ in $L^p(\mathbb{R}^2)$.

We are ready to state our main result.

**Theorem 1** (Global existence of regular solution and temporal decay in 2D) Let $m \geq 3$ and $\|\nabla^l \phi\|_{L^\infty} < \infty$ for $1 \leq |l| \leq m$. Assume $\chi, k \in C^m(\mathbb{R}^+)$ and satisfy (A). If the initial data
(n_0, c_0, u_0) \in H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2), then, for any finite time T > 0, there exists a unique regular solution \((n, c, u)\) of \((1.1)\) satisfying
\[(n, c, u) \in L^\infty(0, T; H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)), \]
\[(\nabla n, \nabla c, \nabla u) \in L^2(0, T; H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)).\]

Furthermore, the following estimates of temporal decay hold: For any \(p\) with \(1 < p \leq \infty\)
\[\|n(t)\|_{L^p} + \|c(t)\|_{L^p} \leq Ct^{-1+\frac{1}{p}}, \quad \|\omega(t)\|_{L^2} \leq Ct^{-\frac{1}{2}}, \]
\[\|\nabla n(t)\|_{L^2} + \|\nabla c(t)\|_{L^2} \leq Ct^{-1},\]
where \(\omega = \nabla^\perp u\) is the vorticity field. In addition, if it is assumed that \(\|\nabla \phi(\cdot, t)\|_{L^\infty} \leq Ct^{-\frac{1}{2}},\)
then
\[\|\nabla \omega(t)\|_{L^2} \leq Ct^{-1}.\]

We remark that time decay rates of \(n\) and \(c\) in Theorem 1 can be seen optimal in the regard of that of a linear heat solution in two dimensions. Such result improves [2], where it was proved that if \(\|c_0\|_{L^\infty}\) is sufficiently small,
\[\|n(t)\|_{L^\infty(\mathbb{R}^2)} + \|c(t)\|_{L^\infty(\mathbb{R}^2)} \leq Ct^{-\frac{1}{2}.} \quad (1.7)\]
On the other hand, it was shown in [3] that if smallness on \(\|n_0\|_{L^1}\) and \(\|\omega_0\|_{L^1}\) is additionally assumed, \(n\) and \(\omega\) converge asymptotically to the two dimensional heat kernels, as time tends to infinity.

Followings are brief summary of main idea for proof of Theorem 1. Due to the negative sign of the second equation in \((1.1)\) we compare it to the drift equation
\[\theta_t - \Delta \theta + u \cdot \nabla \theta = 0\]
in a parabolic cylinder with the same boundary data as the oxygen concentration \(c\). It is known that \(\theta\) is Hölder continuous, and so is \(c\) by comparison, provided that the drift is \(u\) in \(L^4_{x,t}\) (see [23] and [27]). Once \(c\) becomes Hölder continuous, we rewrite \(\chi(c)\) and \(k(c)\) as small perturbation of constant values in a local parabolic cylinder centered at \((x_0, t_0)\), i.e.
\[\chi(c) = \chi(c) - \chi(c(x_0, t_0)) + \chi(c(x_0, t_0)), \quad k(c) = k(c) - k(c(x_0, t_0)) + k(c(x_0, t_0)).\]
We then confirm that the relation in \((1.5)\) holds in a local parabolic cylinder, which enables us to show that \((x_0, t_0)\) is a regular point (Proposition 1), due to a local regularity criterion of Proposition 3.

On the other hands, the fluid velocity \(u\) satisfies the two dimensional Navier-Stokes equations with the forcing term \(-n \nabla \phi\), which is a priori known only in \(L^1\) due to mass conservation of \(n\). Thus, it is not immediate to see that \(u\) is in \(L^4_{x,t}\). To get around it, we decompose the fluid equation by a Stokes system with the forcing term \(-n \nabla \phi\) and the perturbed Navier-Stokes equations. Using the regularizing property of Stokes operator, we manage to show that \(u\) is indeed in \(L^4_{x,t}\). We provide the details at Lemma 4.

This paper is organized as follows. In Section 2, we recall some preliminaries and state main propositions, and give the proof of the first part of Theorem 1 regarding the global existence of regular solution as well as Lemma 4. In Section 3, we provide the proof of Proposition 4 and Section 4 is devoted to proving the proof of the second part of Theorem 1 regarding the temporal decay. The proof of Proposition 3, local regularity criterion, is presented in the appendix.
2 Preliminaries

Local existence of smooth solution in time was established in [12] to chemotaxis-Navier-Stokes equations in $\mathbb{R}^d \times [0, T), d = 2, 3$. We briefly recall the result of local existence in [12] for the reader’s convenience. Let $X_T$ be the class of functions defined as

$$X_T := (C([0, T); H^s) \cap L^2(0, T; H^{s+1})) \times C([0, T); H^{s+1}) \times (C([0, T); H^s) \cap L^2(0, T; H^{s+1})).$$

**Proposition 1** ([2, Proposition 2]) Let $m \geq 3$ and $d = 2, 3$. Assume that $\chi(\cdot), k(\cdot) \in C^m(\mathbb{R})$, $\|\nabla^l \phi\|_{L^\infty} < \infty$ for $1 \leq |l| \leq m$. Then, there exists $T > 0$ depending on $\|n_0\|_{H^m}$, $\|c_0\|_{H^{m+1}}$, $\|u_0\|_{H^m}$ such that a unique solution $(n, c, u)$ exists in $X_T$.

**Definition 2** We say that $T_0$ the maximal time of existence of the unique solution $(n, c, u)$ in Proposition 1 if $T_0 = \sup\{t \in \mathbb{R}_+ : (n, c, u) \in X_t^m\}$. We say that classical solutions globally exist if $T_0 = \infty$. In case that $T_0 < \infty,$ we say that solutions blow up in a finite time.

**Remark 1** It is worth reminding that if solutions blow up in a finite time, i.e. $T_0 < \infty$, then $\limsup_{t \to T_0} \|n(t)\|_{L^\infty} = \infty$ (see [2]).

Our aim in this paper is to show $T_0 = \infty$, that is the solutions in Proposition 1 are global. For the time being, we assume that $T_0 < \infty$ and in the end we lead to a contradiction.

As mentioned in Introduction, we will work with local regularity criteria for the solutions of (1.1). In the below we present the notion of regular point and a parabolic embedding result related to Hölder continuity in spatial and temporal variables. For a point $z = (x, t)$ in $\mathbb{R}^2 \times (0, T)$, we denote

$$B_{x,r} := \{y \in \mathbb{R}^2 : |y - x| < r\}, \quad Q_{x,r} := B_{x,r} \times (t - r^2, t).$$

**Definition 3** A solution $(n, c, u)$ is said to be regular at $z$ if $(n, c, u)$ is continuous in $Q_{x,r}$ for some $r > 0$ and such point is called a regular point.

Next, we recall a parabolic embedding theorem involving Hölder continuity. Since its verification is standard, we skip the details (see e.g. [15]).

**Proposition 2** Let $2 < s < 4$ and $\mu = 2 - \frac{4}{s}$. Assume that $v \in C_0^\infty(Q_{x,r})$ for a point $z = (x, t)$ in $\mathbb{R}^2 \times (0, T)$. Then, for all $z_1 = (x_1, t_1)$ and $z_2 = (x_2, t_2)$ in $Q_{x,r}$

$$|v(z_1) - v(z_2)| \leq C \left( |x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}} \right)^{\mu} \left( \|D^2 v\|_{L_2^{s,\mu}(Q_{x,r})} + \|\partial_t v\|_{L_2^{s,\mu}(Q_{x,r})} \right).$$

We say that $v$ in Proposition 2 is Hölder continuous with exponent $\mu$.

In next proposition, we present a local regularity criteria. Its proof will be given in the Appendix.

**Proposition 3** Under the assumptions given in Proposition 1, let $T_0$ be the maximal time of existence for solution $(n, c, u)$ to (KSNS). Suppose that $T_0 < \infty$. If $n \in L^2(Q_{x_0,r})$ for a point $z_0 = (x_0, T_0), x_0 \in \mathbb{R}^2$, then $z_0$ is a regular point.
Next we state the crucial proposition from which Theorem 1 follows in consequence. Its proof is given in the section 3. Proposition 3 and 4 consist of main body of the paper.

**Proposition 4** Under the assumptions given in Proposition 1, let $T_0$ be the maximal time of existence for solution $(n, c, u)$ to (KSNS). Suppose that $T_0 < \infty$. Then, every point $z = (x, t)$ for $t \leq T_0$ is a regular point, i.e. $(n, c, u)$ is Hölder continuous in $Q_{z,r}$ for some $r > 0$. Moreover, the radius, $r$, of the cylinder $Q_{z,r}$ can be chosen uniformly, independent of $z$.

Assuming Proposition 4 for the moment, let us conclude the proof of the first part of Theorem 1 regarding the global regularity. The proof of the second part regarding the time decay will be put off until section 4.

**Proof of Theorem 1** (Global existence of regular solution)

Let $T_0$ be a maximal time of existence introduced in Proposition 1. We claim that $T_0 = \infty$. Suppose that this is not the case, i.e. $T_0 < \infty$. Then, $\limsup_{t \to T_0} \|n(t)\|_{L^\infty} = \infty$ (see Remark 1).

It implies there is a sequence \( \{z_k = (x_k, T_k) : T_k \geq T_0, k = 1, 2, \cdots \} \) such that $n(z_k) \geq k$. On the other hand, for any $z_0 = (x_0, T_0)$ there exists $r > 0$ independent of $z_0$ such that $(n, c, u)$ is Hölder continuous in $Q_{z_0,r}$ by Proposition 1. If the sequence $\{x_k\}$ has a limit point, say $\tilde{z} = (\tilde{x}, T_0)$, then, because $(\tilde{x}, T_0)$ is a regular point, there is a uniform constant $C$ and $0 < \mu < 1$ such that

\[
|n(x', T_0 - \delta) - n(\tilde{x}, T_0 - \frac{r^2}{2})| < C r_\mu
\]

for any $x' \in B_{\tilde{r},r}$ and $0 < \delta < r^2$. Since $\delta > 0$ is arbitrary, we have

\[
\infty = \limsup_{k \to \infty} n(z_k) \leq \sup_{Q(\tilde{z},r)} n(z) \leq C r_\mu + n(\tilde{x}, T_0 - \frac{r^2}{2}).
\]

It leads to a contradiction. On the other hand, if $\{x_k\}$ is unbounded, we set $w_k = (x_k, T_0)$ and assume that without loss of generality $T_0 - T_k < r^2$ and $B_{z_k,r}$ are mutually disjoint, i.e. $B_{z_i,r} \cap B_{z_j,r} = \emptyset$ when $i \neq j$. Again using $n$ is Hölder continuous in $Q_{w_k,r}$,

\[
|n(x, t) - n(x_k, T_k)| < C r_\mu
\]

for $(x, t) \in Q(w_k,r)$ and $0 < \mu < 1$, we observe there is $0 < \tilde{r} < r$ independent of $k$ such that $n(z_k) > \frac{k}{2}$ in $Q_{w_k,\tilde{r}}$. We find that

\[
r^2 \|n_0\|_{L^1(\mathbb{R}^2)} = \int_{T_0 - r^2}^{T_0} \int_{\mathbb{R}^2} n(x, t)dxdt \geq \sum_{k=1}^{\infty} \int_{Q(w_k,\tilde{r})} n(z)dz \geq \sum_{k=1}^{\infty} \frac{k}{2} \frac{r^4}{4\pi} = \infty.
\]

It leads to a contradiction. \hfill \Box

We finish this section by establishing that the drift $u$ has a better regularity than studied before e.g. 1. The improved regularity is crucial for exploiting Hölder regularity of $c$. In the below, we consider the fluid equations in a time interval, where $t$ is not yet the maximal time of existence $T_0$ in the below.

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p &= -n \nabla \phi, \quad \nabla \cdot u = 0 &\text{ in } \mathbb{R}^2 \times (0, T_0) \\
u(x, 0) &= u_0(x) &\text{ in } \mathbb{R}^2.
\end{align*}
\]
Lemma 4 Under the assumptions given in Proposition 4, the solution \( u \) to (2.1) belongs to \( L^\infty(0, T_0; L^2) \cap L^2(0, T_0; W^{1,q}) \cap L^4(0, T_0; L^4) \) for any \( q \in [1, 2) \).

Proof. We remind that total mass of \( n \) is preserved, that is \( \| n(t) \|_{L^1(\mathbb{R}^2)} = \| n_0 \|_{L^1(\mathbb{R}^2)} \) for all \( t < T_0 \). Thus, \( n \nabla \phi \) belong to \( L^\infty([0, T_0); L^1(\mathbb{R}^2)) \), since \( \phi \) is assumed to satisfies \( \| \nabla^l \phi \|_{L^\infty} < \infty \) for \( 1 \leq |l| \leq m \). Let \( Q := \mathbb{R}^2 \times (0, T_0) \). We decompose the solution \( u \) to the equations (2.1) to \( v + w \) in \( Q \), where \( v \) satisfies the Stokes system:

\[
\begin{align*}
\{ \partial_t v - \Delta v + \nabla p_1 &= -n \nabla \phi, & \nabla \cdot v &= 0 & \text{in } Q, \\
v(x, 0) &= u_0(x) & \text{in } \mathbb{R}^2,
\end{align*}
\]

and \( w \) satisfies a perturbed homogeneous Navier-Stokes equation with zero initial data:

\[
\begin{align*}
\{ \partial_t w - \Delta w + \nabla p_2 &= -(v + w) \cdot \nabla v - ((v + w) \cdot \nabla) w, & \nabla \cdot w &= 0 & \text{in } Q, \\
w(x, 0) &= 0 & \text{in } \mathbb{R}^2.
\end{align*}
\]

For convenience, we denote \( f := -n \nabla \phi \). Let \( G_t \) be the Stokes operator. Then the solution \( v \) can be represented as

\[
v(x, t) = G_t \ast u_0 + \int_0^t G_{t-s} \ast f(s) ds.
\]

It is well known that in two dimensions \( G_t \) satisfies

\[
\| G_t \ast f \|_{L^p} \leq C t^{\frac{1}{p} - 1} \| f \|_{L^1}, \quad \| \nabla G_t \ast f \|_{L^p} \leq C t^{\frac{1}{p} - \frac{3}{2}} \| f \|_{L^1}
\]

for \( 1 \leq p \leq \infty \). Hence, for any \( p \in [1, \infty) \), we have

\[
\| v \|_{L^{p, \infty}_x(Q)} \leq C \| u_0 \|_{L^p} + C \left( \int_0^{T_0} t^{\frac{1}{p} - 1} dt \right) \| f \|_{L^{1, \infty}_x(Q)} < \infty.
\]

Similarly, for any \( 1 \leq q < 2 \) we have

\[
\| \nabla v \|_{L^{q, \infty}_x(Q)} \leq C \| \nabla u_0 \|_{L^q} + C \left( \int_0^{T_0} t^{\frac{1}{q} - \frac{3}{2}} dt \right) \| f \|_{L^{1, \infty}_x(Q)} < \infty.
\]

Note that \( \| f \|_{L^{1, \infty}_x(Q)} \leq C \| n_0 \|_{L^1(\mathbb{R}^2)} \). Summing up, we obtain

\[
\| v \|_{L^{p, \infty}_x(Q)} + \| \nabla v \|_{L^{q, \infty}_x(Q)} \leq C = C(T_0), \quad p \in [1, \infty), \quad q \in [1, 2),
\]

which is more than \( v \in L^\infty(0, T_0; L^2) \cap L^2(0, T_0; W^{1,q}) \cap L^4(0, T_0; L^4) \) for any \( q \in [1, 2) \). For the Navier-Stokes part \( w \), we first compute

\[
\frac{1}{2} \frac{d}{dt} \| w \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 \leq \left| \int_{\mathbb{R}^2} ((v + w) \cdot \nabla) w \cdot \nu dx \right|
\]

\[
\leq \| v \|_{L^4}^2 \| \nabla w \|_{L^2} + \| w \|_{L^4} \| \nabla w \|_{L^2} \| v \|_{L^4}
\]

\[
\leq \frac{1}{2} \| \nabla w \|_{L^2}^2 + C \| v \|_{L^4}^4 (\| w \|_{L^2}^2 + 1).
\]

By the Gronwall’s inequality, we have \( w \in L_{x,t}^{2, \infty}(Q) \cap L^2(0, T_0; H^1_0) \). It remains to show that \( w \in \bigcap_{1 \leq q < 2} L^2(0, T_0; W^{1,q}) \). Using the Stokes operator, \( w \) can be written as

\[
\nabla w(x, t) = -\int_0^t \nabla G_{t-s} \ast ((v \cdot \nabla) v + (v \cdot \nabla) w + (w \cdot \nabla) v + (w \cdot \nabla) w) (s) ds
\]
What it follows, we separately compute $I_i$, $i = 1, 2, 3, 4$.

$$
\|I_1(t)\|_{L^4} \leq \int_0^t \|\nabla G_{t-s} \ast ((v \cdot \nabla)v)(s)\|_{L^4} ds \leq C \int_0^t (t-s)^{\frac{4}{q} - \frac{3}{2}} \|v \nabla v\|_{L^1(\mathbb{R}^2)}(s) ds \\
\leq C \int_0^t (t-s)^{\frac{1}{q} - \frac{3}{2}} \|v\|_{L^4(\mathbb{R}^2)}(s) \|\nabla v\|_{L^2(\mathbb{R}^2)}(s) ds
$$

Similarly,

$$
\|I_2(t)\|_{L^4} \leq C \int_0^t (t-s)^{\frac{1}{q} - \frac{3}{2}} \|v \nabla w\|_{L^1(\mathbb{R}^2)}(s) ds \\
\leq C \|v\|_{L^{2,\infty}_{x,t}(Q)} \int_0^t (t-s)^{\frac{1}{q} - \frac{3}{2}} \|\nabla w\|_{L^2(\mathbb{R}^2)}(s) ds.
$$

Therefore, using Young’s inequality again, we have

$$
\|I_2\|_{L^{4,2}_{x,t}(Q)} \leq C(T_0) \|v\|_{L^{2,\infty}_{x,t}(Q)} \|\nabla w\|_{L^2(Q)}.
$$

For $I_3$, using $w \in L^4(Q)$, we observe that

$$
\|I_3(t)\|_{L^4} \leq C \int_0^t (t-s)^{\frac{1}{q} - \frac{3}{2}} \|w \nabla v\|_{L^1(\mathbb{R}^2)}(s) ds \\
\leq C \int_0^t (t-s)^{\frac{1}{q} - \frac{3}{2}} \|w\|_{L^4(\mathbb{R}^2)}(s) \|\nabla v\|_{L^4(\mathbb{R}^2)}(s) ds \\
\leq C \|\nabla v\|_{L^{4,\infty}_{x,t}(Q)} \int_0^t (t-s)^{\frac{1}{q} - \frac{3}{2}} \|w\|_{L^4(\mathbb{R}^2)}(s) ds.
$$

Using Young’s inequality again, we obtain

$$
\|I_3\|_{L^{4,2}_{x,t}(Q)} \leq C(T_0) \|\nabla v\|_{L^{4,\infty}_{x,t}(Q)} \|\nabla w\|_{L^4_{x,t}(Q)}.
$$

Finally, we compute

$$
\|I_4(t)\|_{L^4} \leq C \int_0^t (t-s)^{\frac{1}{q} - \frac{3}{2}} \|w \nabla w\|_{L^1(\mathbb{R}^2)}(s) ds \\
\leq C \int_0^t (t-s)^{\frac{1}{q} - \frac{3}{2}} \|w\|_{L^2(\mathbb{R}^2)}(s) \|\nabla w\|_{L^2(\mathbb{R}^2)}(s) ds \\
\leq C \|w\|_{L^{2,\infty}_{x,t}(Q)} \int_0^t (t-s)^{\frac{1}{q} - \frac{3}{2}} \|\nabla w\|_{L^2(\mathbb{R}^2)}(s) ds.
$$

As in above cases, we get

$$
\|I_4\|_{L^{4,2}_{x,t}(Q)} \leq C(T_0) \|w\|_{L^{2,\infty}_{x,t}(Q)} \|\nabla w\|_{L^2(Q)}.
$$

Summing up estimates, we obtain that $\nabla w \in \bigcap_{1 \leq q < 2} L^2(0,T_0; L^2(\mathbb{R}^2))$. We complete the proof. \qed
Remark 2 The assumption in Lemma 4 can be relaxed. To be more precisely, if \( \nabla \phi \in L^\infty(\mathbb{R}^2) \) and \( u_0 \in L^2(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2) \), the result in Lemma 4 can be deduced.

Finally we introduce a special type of a cut-off function \( \varphi \in C_0^\infty \) to be used throughout the paper. For given \( \varrho \) and \( r \) with \( 0 < \varrho < r \ll 1 \) we define

\[
\varphi(x) = \begin{cases} 
1, & x \in B_{x_0,\varrho} \\
0, & x \in \mathbb{R}^2 \setminus B_{x_0,r}
\end{cases}
\]

such that \( \varphi \) satisfies the pointwise bound

\[
|\nabla \varphi| \leq C \varphi^{\frac{5}{6}}, \quad |\Delta \varphi| \leq C \varphi^{\frac{2}{3}},
\]

where constant \( C \) is determined by \( \varrho \) and \( r \) (see e.g. [26]).

## 3 Proof of Proposition 4

In this section, we provide the proof of Proposition 4. We start by showing \( c \) is Hölder continuous.

**Lemma 5** Let \( z = (x, T_0), \ x \in \mathbb{R}^2 \) and \( r_0 > 0 \). There exists \( \gamma = \gamma(\|u\|_{L^4(Q)}) \in (0, 1) \), independent of \( z \), such that solution \( c \) to (3.1) is Hölder continuous in \( Q_{z,r} \), i.e., \( c \in C^{\gamma,\gamma}(Q_{z,r}) \) for any \( r < r_0/2 \).

**Proof.** Consider the equation of \( c \) in (KSNS)

\[
c_t - \Delta c + (u \cdot \nabla)c = -k(c)n, \quad \text{in } \mathbb{R}^2 \times (0, T_0) = Q.
\]

In the above, \( n \geq 0 \) and \( k(c) \) satisfies the assumption (A). We recall first, due to Lemma 4, that \( u \in L^4(x,t)(Q) \). We compare (3.1) to the following drift equation;

\[
\theta_t - \Delta \theta + u \cdot \nabla \theta = 0 \quad \text{in } Q_r
\]

with boundary data as the same as \( c \), i.e.

\[
\theta = c \quad \text{on } \partial Q_r.
\]

Due to Maximum principle, \( \theta \) is positive in \( Q_r \) and its maximum attains at the boundary. Moreover, since \( u \in L^4(x,t) \), it is known that \( \theta \) is Hölder continuous and satisfies the Harnack inequality (see [23] and [27]), that is there exists \( \mu = \mu(\|u\|_{L^4(Q)}) < 1 \) such that for any \( \rho \leq r \)

\[
\sup_{Q_{\rho}} \theta \leq \mu \inf_{Q_{\rho}} \theta.
\]

Here we remark that the constant \( \mu \) is independent of \( z \), since boundary value of \( \theta \) in (3.2) is uniformly bounded above by \( \|c_0\|_{L^\infty(\mathbb{R}^2)} \) and \( \|u\|_{L^4(Q_{z,r})} \) is uniformly bounded by Lemma 4.

Next, we observe that \( c \) is the sub-solution, namely

\[
c_t - \Delta c + u \cdot \nabla c = -k(c)n \leq 0.
\]
Thus, we note that by comparison principle \( c \leq \theta \) in \( Q_r \) and therefore, we can show that \( c \) is Hölder continuous. Indeed, for any \( \rho < r \)

\[
\text{osc } c = \sup_{Q_{\rho}} c - \inf_{Q_{\rho}} c \leq \sup_{Q_{\rho}} \theta - \inf_{Q_{\rho}} c \leq \mu \inf_{Q_{\rho}} \theta - \inf_{Q_{\rho}} c \\
\leq \mu (\sup_{Q_{\rho}} \theta - \inf_{Q_{\rho}} c) = \mu (\sup_{Q_{\rho}} c - \inf_{Q_{\rho}} c) = \mu (\sup_{Q_{\rho}} c - \inf_{Q_{\rho}} c) = \mu \text{osc } c.
\]

Therefore, we obtain

\[
\text{osc } c \leq \mu \text{osc } c.
\]

The above estimate implies that \( c \) is Hölder continuous in \( Q_\delta \) for some \( \delta \leq r/2 \). Since \( \mu \) is independent of \( z \), so is \( \delta \). This completes the proof. \( \square \)

Now it remains to show \( n \in L^2(Q_{2r}, r) \) for a given point \( z_0 = (x_0, T_0) \) with regard to the local blow up criteria Proposition 5.

From now on, we fix a point \( z_0 = (x_0, T_0) \). Let \( \varphi \in C_0^\infty \) be the cutoff function satisfying (2.5). We note first that

\[
\left( \int_{\mathbb{R}^2} |\nabla (n^{\frac{1}{2}}\varphi)|^2 \, dx \right)^2 \leq 2 \left( \int_{\mathbb{R}^2} |\nabla n| \sqrt{\varphi} \, dx \right)^2 + 2 \left( \int_{\mathbb{R}^2} n |\nabla \sqrt{\varphi}| \, dx \right)^2 \leq 2 \left( \int_{\mathbb{R}^2} n \, dx \right) \left( \int_{\mathbb{R}^2} |\nabla n|^2 \, dx \right) + C \|n\|^2_{L^1} = C \left( \int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \varphi \, dx \right) + C,
\]

where we used the mass conservation, i.e. \( \|n(t)\|_{L^1} = \|n_0\|_{L^1} \). Via the Sobolev inequality \( \|n^{\frac{1}{2}}\varphi\|^2_{L^2} \leq C \|n^{\frac{1}{2}}\varphi\|^2_{W^{1,1/2}} \), we obtain

\[
\int_{\mathbb{R}^2} n^2 \varphi \, dx \leq C \left( \int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \varphi \, dx + 1 \right). \tag{3.3}
\]

Reminding by Lemma 5 that \( c \) is Hölder continuous at \( z_0 = (x_0, T_0) \) for every \( x_0 \in \mathbb{R}^2 \), we set \( k_0 := k(c(x_0, T_0)) \) and \( \chi_0 := \chi(c(x_0, T_0)) \). We take \( Q_{2r} \) with sufficiently small \( r > 0 \) such that \( c \) is Hölder continuous with exponent \( \gamma \) in \( Q_{2r} \), where \( r \) will be specified later.

We write the equations of \( n \) and \( c \) with \( \chi(c) \) and \( k(c) \) interpolated at \( \chi_0 \) and \( k_0 \) respectively,

\[
\begin{align*}
\partial_t n - \Delta n + (u \cdot \nabla)n &= -\chi_0 \nabla \cdot (n \nabla c) - \nabla \cdot ((\chi(c) - \chi_0)n \nabla c) \\
\partial_t c - \Delta c + (u \cdot \nabla)c &= -k_0 n - (k(c) - k_0) n.
\end{align*} \tag{3.4}
\]

We remind that

\[
|\chi(c) - \chi_0| + |k(c) - k_0| \leq Cr^\gamma \quad \text{for } (x, t) \in Q_{2r}.
\tag{3.5}
\]

Testing \( \log n \varphi \) on the \( n \) equation, we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^2} (n \log n) \varphi \, dx + 4 \int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \varphi \, dx
= \chi_0 \int_{\mathbb{R}^2} \nabla c \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^2} (\chi(c) - \chi_0) \nabla c \cdot \nabla n \varphi \, dx + \int_{\mathbb{R}^2} \chi(c)(n \log n) \nabla c \cdot \nabla \varphi \, dx
+ \int_{\mathbb{R}^2} (u \cdot \nabla) \varphi (n \log n) \, dx + \int_{\mathbb{R}^2} n \Delta \varphi \, dx + \int_{\mathbb{R}^2} \chi(c) n \nabla c \cdot \nabla \varphi \, dx
\]

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On the other hands, testing $-\Delta c\varphi$ on the $c$ equation, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx + \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx = -k_0 \int_{\mathbb{R}^2} \nabla n \cdot \nabla c \varphi dx - k_0 \int_{\mathbb{R}^2} n \nabla c \nabla \varphi dx
\]
\[+ \int_{\mathbb{R}^2} (k(c) - k_0) n\Delta c \varphi dx - \int_{\mathbb{R}^2} \partial_c \nabla c \cdot \nabla \varphi dx + \int_{\mathbb{R}^2} (u \cdot \nabla) c \Delta c \varphi dx
\]
\[:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\]
We note that $I_1$ and $J_1$ are of opposite signs. Together with the global bound of $\|u\|_{L_{x,t}^2}$ and (3.5), we will obtain the energy estimate for $\|\sqrt{\nabla n} \varphi^\frac{1}{2}\|_{L^2(Q_{x_0},r)}$, which yields $\|n \varphi^\frac{1}{2}\|_{L^2(Q_{x_0},r)}$ by (3.8). We summarize the estimates on $I_1 - I_6$ and $J_1 - J_6$ in the lemma below.

**Lemma 6** Let $(n, c, u)$ be the local solution constructed in Proposition [7]. For a fixed $(x_0, T_0)$ and a cut-off function $\varphi$ aforementioned, the following inequalities holds with $0 < \epsilon < 1$:

\[
\frac{d}{dt} \int_{\mathbb{R}^2} (n \log n) \varphi dx + 2 \int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \varphi dx + \int_{\mathbb{R}^2} \chi'(c) |\nabla c|^2 n \varphi dx
\]
\[\leq \chi_0 \int_{\mathbb{R}^2} \nabla c \cdot \nabla n \varphi dx + C \left( \int_{\mathbb{R}^2} |u|^4 dx + \int_{\mathbb{R}^2} |u|^2 dx + \int_{\mathbb{R}^2} n dx \right)
\]
\[+ C(r^\gamma + \epsilon) \left( \int_{\mathbb{R}^2} n^2 \varphi dx + \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx \right) + C \|\nabla c\|_{L^2}^2 \left( 1 + \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx \right). \tag{3.6}
\]

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx + \frac{3}{4} \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx \leq -k_0 \int_{\mathbb{R}^2} \nabla n \cdot \nabla c \varphi dx + (\epsilon + C r^\gamma) \int_{\mathbb{R}^2} n^2 \varphi dx
\]
\[+ C \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx + C \left( \|u\|_{L^4}^4 + \|u\|_{L^1}^{\frac{4}{3}} \right) \left( \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx \right). \tag{3.7}
\]

**Proof.** $I_2$ can be rewritten as
\[
I_2 = -\int_{\mathbb{R}^2} \chi'(c) |\nabla c|^2 n \varphi dx - \int_{\mathbb{R}^2} (\chi(c) - \chi_0) \Delta n \varphi dx - \int_{\mathbb{R}^2} (\chi(c) - \chi_0) n \nabla c \cdot \nabla \varphi dx
\]
\[:= I_2^1 + I_2^2 + I_2^3.
\]
For the term $I_2^1$, we use the positivity of $\chi'(c)$ so that $I_2^1$ is in fact “good” term. By (3.5), the term $I_2^3$ can be estimated as
\[
|I_2^3| \leq C r^\gamma \left( \|\Delta c\varphi\|_{L^2}^2 + \|n\varphi\|_{L^2}^2 \right)
\]
and
\[
|I_2^3| \leq C \|\nabla c\|_{L^2} \|n\varphi\|_{L^2} \leq C \|\nabla c\|_{L^2}^2 + \epsilon \|n\varphi\|_{L^2}^2.
\]
Reminding that for any $\delta > 0$
\[
n(x) |\log n(x)| \leq C_{\delta}(\sqrt{n} + n^{1+\delta}), \tag{3.8}
\]
we take $\delta = \frac{1}{9}$ for $I_3$ and $\delta = \frac{1}{2}$ for $I_4$. Also we use the interpolation inequality for $\|\nabla c \varphi^{\frac{1}{2}}\|_{L^3}$,
\[
\int_{\mathbb{R}^2} |\nabla c|^3 \varphi^\frac{2}{3} dx \leq C \left( \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx + \int_{\mathbb{R}^2} |\nabla c|^2 dx \right)^\frac{2}{3} \left( \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx \right).
\]

It is proved by combination of Gagliardo-Nirenberg inequality $\|\nabla c \varphi^{\frac{1}{2}}\|_{L^3} \leq C \|\nabla c \varphi^{\frac{1}{2}}\|_{L^2}^\frac{1}{3} \|\nabla c \varphi^{\frac{1}{2}}\|_{L^2}^\frac{2}{3}$ and (2.4). $I_3$ and $I_4$ are estimated as follows:
\[
|I_3| \leq C \int_{\mathbb{R}^2} \sqrt{n} |\nabla c| \varphi^\frac{5}{6} dx + C \int_{\mathbb{R}^2} n^\frac{1}{2} |\nabla c| \varphi^\frac{5}{6} dx
\]
\[
\leq C \int_{\mathbb{R}^2} n dx + C \int_{\mathbb{R}^2} |\nabla c|^2 dx + \left( \int_{\mathbb{R}^2} n^2 \varphi dx \right)^\frac{2}{3} \left( \int_{\mathbb{R}^2} |\nabla c|^3 \varphi^\frac{2}{3} dx \right)^\frac{1}{3}
\]
\[
\leq \epsilon \int_{\mathbb{R}^2} n^2 \varphi dx + \epsilon \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx + C \|\nabla c\|_{L^2}^2 \left( 1 + \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx \right).
\]

\[
|I_4| \leq C \int_{\mathbb{R}^2} |u| \sqrt{n} \varphi^\frac{5}{6} dx + C \int_{\mathbb{R}^2} |u| n^\frac{1}{2} \varphi^\frac{5}{6} dx
\]
\[
\leq C \int_{\mathbb{R}^2} |u|^2 dx + C \int_{\mathbb{R}^2} n dx + C \int_{\mathbb{R}^2} |u|^4 dx + \epsilon \int_{\mathbb{R}^2} n^2 \varphi dx.
\]

$I_5$ and $I_6$ are controlled as follows:

\[
|I_5| \leq C \int_{\mathbb{R}^2} n dx = C,
\]

\[
|I_6| \leq C \int_{\mathbb{R}^2} n |\nabla c| \varphi^\frac{2}{3} dx \leq \epsilon \int_{\mathbb{R}^2} n^2 \varphi dx + C \|\nabla c\|_{L^2}^2.
\]

Adding up all estimates and using the estimate (3.3), we have (3.6). Next we estimate $J_2, J_3, J_4$ and $J_5$.

\[
|J_2| \leq \epsilon \int_{\mathbb{R}^2} n^2 \varphi dx + C \int_{\mathbb{R}^2} |\nabla c|^2 dx,
\]

\[
|J_3| \leq C r^{-\gamma} \int_{\mathbb{R}^2} n^2 \varphi dx + \frac{1}{16} \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx,
\]

\[
|J_4| \leq \int_{\mathbb{R}^2} |\Delta c| |\nabla c| |\nabla \varphi| dx + \int_{\mathbb{R}^2} |u| |\nabla c|^2 |\nabla \varphi| dx + \int_{\mathbb{R}^2} k(c) n |\nabla c| |\nabla \varphi| dx
\]
\[
\leq \frac{1}{16} \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx + C \int_{\mathbb{R}^2} |\nabla c|^2 dx + \|u\|_{L^4} \left( \int_{\mathbb{R}^2} |\nabla c|^4 \varphi^2 dx \right)^\frac{1}{12} \|\nabla c\|_{L^2}^\frac{1}{2} + \epsilon \int_{\mathbb{R}^2} n^2 \varphi dx
\]
\[
\leq \frac{1}{8} \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx + \epsilon \int_{\mathbb{R}^2} n^2 \varphi dx + C \|\nabla c\|_{L^2}^2 + C \|u\|_{L^4}^{12} \|\nabla c\| \varphi^\frac{1}{2} \|_{L^2}^2,
\]

where we use the fact $u \in L^4_{x,t}$ shown in Lemma 4 and the interpolation inequality for $\|\nabla c \varphi^{\frac{1}{2}}\|_{L^4}$:

\[
\int_{\mathbb{R}^2} |\nabla c|^4 \varphi^2 dx \leq \left( \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx + \int_{\mathbb{R}^2} |\nabla c|^2 dx \right) \left( \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx \right).
\]
Similarly $J_5$ is estimated by

$$|J_5| \leq \int_{\mathbb{R}^2} |u| |\nabla c| \varphi^\frac{1}{2} |\Delta c| \varphi^\frac{1}{2} dx \leq \|u\|_{L^4} \|\nabla c| \varphi^\frac{1}{2} \|_{L^4} \|\Delta c| \varphi^\frac{1}{2} \|_{L^2}$$

$$\leq C \|u\|_{L^4} \left( \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx \right) + \frac{1}{16} \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx + C \|\nabla c\|_{L^2}^2.$$

Summing up all estimates, we obtain (3.7).

When (3.6) is integrated in $t$, the negative part of the integral $\int_{\mathbb{R}^2} (n \log n) \varphi$ is treated by

$$\left| \int_{\{n(x) \leq 1\}} (n \log n) \varphi dx \right| \leq C (1 + \|n\|_{L^1}),$$

(3.10)

since $n(x) |\log n(x)| \leq C \sqrt{n(x)}$ when $0 \leq n(x) \leq 1$.

To close the estimate of $\|n^2 \varphi^\frac{1}{2}\|_{L^2}$, via (3.6), we consider three cases depending on whether $k_0$ and $\chi_0$ vanish. What it follows, $\lambda > 0$ is a positive number to be specified later.

- **Case 1: $k_0 \neq 0$ and $\chi_0 \geq \lambda$**
  In this case, there exists a constant $C_0 > 0$ such that $k_0 - C_0 \chi_0 = 0$, which leads that $J_1 - C_0 I_1$ vanishes. Indeed, multiplying $C_0$ on the estimate (3.7) of $n$ and adding it with (3.6), we obtain

$$\frac{d}{dt} \left( C_0 \int_{\mathbb{R}^2} (n \log n) \varphi + |\nabla c|^2 \varphi \right) + C \int_{\mathbb{R}^2} (|\nabla \sqrt{n}|^2 \varphi + |\Delta c|^2 \varphi) dx + C_0 \int_{\mathbb{R}^2} \chi'(c) |\nabla c|^2 n \varphi dx$$

$$\leq C \left( \|\nabla c\|_{L^2}^2 + \|u\|_{L^4}^4 + 1 \right) \left( \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx \right) + C \left( \|\nabla c\|_{L^2}^2 + \|u\|_{L^4}^4 + \|u\|_{L^2}^2 + 1 \right),$$

(3.11)

where $\epsilon$ and $r$ are taken sufficiently small. We remark that $C_0 \leq \frac{k_0}{\lambda}$ and thus, $r$ can be taken uniformly, independent of $z_0$. For convenience, we set $T := T_0 - \sigma$, where $\sigma$ is any positive number with $0 < \sigma < r^2$. Reminding (3.10) and integrating above inequality (3.11) in time variable over $(T_0 - r^2, T)$, we have

$$C_0 \int_{\mathbb{R}^2} n(T) |\log n(T)| \varphi + |\nabla c|^2(t) \varphi dx + C \int_0^T \int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \varphi + |\Delta c|^2 \varphi dx dt$$

$$+ C_0 \int_0^T \int_{\mathbb{R}^2} \chi'(c) |\nabla c|^2 n \varphi dx dt \leq C \int_0^T (\|\nabla c\|_{L^2}^2 + \|u\|_{L^4}^4 + 1) \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx dt$$

$$+ C \int_0^T \int_{\mathbb{R}^2} (|\nabla c|^2 + |u|^4 + |u|^2) dx dt + C T.$$

Using Gronwall’s inequality, we have

$$\sup_{T_0 - \sigma < t < T_0} \left( C_0 \int_{\mathbb{R}^2} n |\log n| \varphi + |\nabla c|^2 \varphi dx \right) + C \left( \|\nabla \sqrt{n}| \varphi^\frac{1}{2} \|_{L^2}^2, t \right)$$

$$\leq C (\|\nabla c\|_{L^2}^2 + \|u\|_{L^4}^4 + 1) \exp \left( C \int_0^T (\|\nabla c\|_{L^2}^2 + \|u\|_{L^4}^4 + 1) dt \right).$$
Since $\sigma > 0$ is arbitrary and
\[
\int_0^{T_0} (\|\nabla c\|_{L^2}^2 + \|u\|_{L^4}^4) dt < \infty,
\]
we conclude via the estimate (3.3) that $n \in L^2(Q_{z_0,T})$, which implies from Proposition 8 that $z_0 = (x_0, T_0)$ is a regular point.

- (Case 2: $k_0 = 0$) We rewrite (3.6) in Lemma 6 by estimating $I_1$ via (3.3)
\[
\chi_0 \int_{\mathbb{R}^2} \nabla \cdot \nabla n \varphi dx \leq \chi_0 \|n \varphi^{\frac{1}{2}}\|_{L^2} \|\Delta c \varphi^{\frac{1}{2}}\|_{L^2} + C \chi_0 \|n \varphi^{\frac{1}{2}}\|_{L^2} \|\nabla c\|_{L^2}
\leq \frac{C_1 \chi_0^2}{2} \|\Delta \varphi^{\frac{1}{2}}\|_{L^2}^2 + \frac{1}{2} \|\nabla \sqrt{n} \varphi^{\frac{1}{2}}\|_{L^2}^2 + C \chi_0 \|n \varphi^{\frac{1}{2}}\|_{L^2} \|\nabla c\|_{L^2} + C.
\]
Multiplying $C_1 \chi_0^2$ to (3.7) and adding it to the newly obtained (3.6), we have
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^2} (n \log n) \varphi + \frac{C_1 \chi_0^2}{2} \|\nabla \varphi\|_{L^2}^2 \right) + \int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \varphi dx + \frac{C_1 \chi_0^2}{4} |\Delta c|^2 \varphi dx
\leq (\epsilon + Cr^2) \left( \int_{\mathbb{R}^2} n^2 \varphi + |\Delta c|^2 \varphi dx \right) + C \left( \|\nabla c\|_{L^2}^2 + \|u\|_{L^4}^4 + 1 \right) \int_{\mathbb{R}^2} |\nabla \varphi|^2 dx + C(n \|n\|_{L^1} + \|u\|_{L^2}^2 + 1).
\]
We take $\epsilon$ and $r$ sufficiently small to absorb the first term in the right hand side to the left hand side. Following similar arguments as the Case 1, we can show that $L^2(Q_{z_0,T})$ is bounded. Here we emphasize that $\epsilon$ and $r$ are independent of $z_0$ depending on the upper bound of $\chi_0$ and generic constants.

- (Case 3: $k_0 \neq 0$ and $0 \leq \chi_0 < \lambda$) We use $\chi(c) \leq \lambda + r^\gamma$ by (3.5) to have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} (n \log n) \varphi dx + 4 \int_{\mathbb{R}^2} |\nabla n|^2 \varphi dx + \int_{\mathbb{R}^2} \chi'(c) |\nabla c|^2 n \varphi dx
= - \int_{\mathbb{R}^2} c \chi(c) \Delta \varphi dx - \int_{\mathbb{R}^2} \chi(c) \nabla c \cdot \nabla \varphi dx + \int_{\mathbb{R}^2} \chi(c) (n \log n) \nabla c \cdot \nabla \varphi dx
+ \int_{\mathbb{R}^2} (u \cdot \nabla) n \varphi dx + \int_{\mathbb{R}^2} (u \cdot \nabla) \varphi (n \log n) dx + \int_{\mathbb{R}^2} \partial_t n \varphi dx
\leq C(\lambda + r^\gamma)(|\Delta c \varphi^{\frac{1}{2}}|_{L^2}^2 + |n \varphi^{\frac{1}{2}}|_{L^2}^2) + C \|\nabla c\|_{L^2}^2 + C \|u\|_{L^2}^2 + C \|u\|_{L^4}^4 + C \|n\|_{L^1} + C \|\nabla c\|_{L^2}^2 \|\nabla \varphi^{\frac{1}{2}}\|_{L^2}^2.
\]
We replace (3.7) in Lemma 6 by estimating $J_1 + J_2$ by
\[
J_1 + J_2 = k_0 \int n \Delta c \varphi dx \leq k_0 \|n \varphi^{\frac{1}{2}}\|_{L^2} \|\Delta c \varphi^{\frac{1}{2}}\|_{L^2} \leq \frac{1}{2} \|\Delta c \varphi^{\frac{1}{2}}\|_{L^2}^2 + \frac{C_1 k_0^2}{2} \|\nabla \sqrt{n} \varphi^{\frac{1}{2}}\|_{L^2}^2 + C,
\]
where we use (3.3). Then the newly obtained (3.7) is the form of
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx + \frac{1}{4} \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx \leq \frac{C_1 k_0^2}{2} \|\nabla \sqrt{n} \varphi \|_{L^2}^2 + (\epsilon + C r^2) \int_{\mathbb{R}^2} n^2 \varphi dx + C \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx + C \left( \|u\|_{L^4}^4 + \|u\|_{L^2}^2 \right) \left( \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx + 1 \right).
\]  
(3.13)

Choosing \(\lambda\) and \(r\) to be sufficiently small such that \(\lambda + r^2 \leq \frac{1}{2C_1 k_0^2}\), multiplying the both sides of (3.12) by \(C_1 k_0^2\) and also adding inequalities (3.13), we obtain
\[
\frac{d}{dt} \left( C_1 k_0^2 \int_{\mathbb{R}^2} (n \log n) \varphi dx + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx \right) + C_1 k_0^2 \int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \varphi dx + \frac{1}{4} \int_{\mathbb{R}^2} |\Delta c|^2 \varphi dx 
\leq C \left( \|\nabla c\|_{L^2}^2 + \|u\|_{L^4}^4 + 1 \right) \int_{\mathbb{R}^2} |\nabla c|^2 \varphi dx + C(\|\nabla c\|_{L^2}^2 + \|u\|_{L^4}^4 + 1).
\]

Here again we note that the choice of \(\lambda\) and \(r\), depending on the upper bound of \(k_0\) and generic constants, can be taken independent of \(z_0\). By the similar reasoning as in Case 1, we see \(\|n \varphi^2\|_{L^2(Q_{r_0, \epsilon})}\) to be bounded.

Summing up Case 1-Case 3, we conclude that \(z_0\) is a regular point from Proposition \(\ref{prop:regularity}\) which completes the proof of Proposition \(\ref{prop:regularity}\).

## 4 Temporal decay

In this section, we provide the proof of the second part of Theorem \(\ref{thm:temporal_decay}\).

**Proof of Theorem \(\ref{thm:temporal_decay}\) (Temporal decay)**

Since we have a global existence of regular solution, for any \(\epsilon\) there exists \(T_1\) such that \(\|c(t)\|_{L^\infty(\mathbb{R}^2)} < \epsilon\) for all \(t > T_1\). This automatically implies that \(k(c(t))\) becomes sufficiently small for any \(t > T_1\). With the aid of decaying result (1.7) in [2], we also note that there exists \(T_2 > 0\) such that \(\|n(t)\|_{L^\infty(\mathbb{R}^2)} < \epsilon\). Testing \(n^{p-1}\), \(p \geq 2\), to the equation of \(n\), for all \(t > \max\{T_1, T_2\}\) we have
\[
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} |n(t)|^p + \frac{4(p - 1)}{p^2} \int_{\mathbb{R}^2} |\nabla n^{\frac{p}{2}}|^2 \leq C \int_{\mathbb{R}^2} n^{\frac{p}{2}} |\nabla n^{\frac{p}{2}}| |\nabla c| \leq C \|n(t)\|_{L^\infty} \left( \|\nabla n^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)} \|\nabla c\|_{L^2(\mathbb{R}^2)} \right) \left( \|\nabla n^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla c\|_{L^2(\mathbb{R}^2)}^2 \right).
\]  
(4.1)

We recall that
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |c(t)|^2 + \int_{\mathbb{R}^2} |\nabla c|^2 \leq 0.
\]  
(4.2)

Combining (4.1) and (4.2) and taking a sufficiently small \(\epsilon\),
\[
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} |n(t)|^p + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |c(t)|^2 + \frac{2(p - 1)}{p^2} \int_{\mathbb{R}^2} |\nabla n^{\frac{p}{2}}|^2 + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla c|^2 \leq 0.
\]  
(4.3)
Changing variable $\tilde{c} = \left(\frac{p}{2}\right)^{\frac{1}{p}} c^\frac{2}{p}$, i.e. $c = \sqrt{\frac{p}{2}} \tilde{c}^p$, we see that (4.3) becomes

$$
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} \left( |n(t)|^p + |\tilde{c}(t)|^p \right) + \frac{2(p-1)}{p^2} \int_{\mathbb{R}^2} \left| \nabla n \right|^2 + \frac{1}{p} \int_{\mathbb{R}^2} \left| \nabla \tilde{c} \right|^2 \leq 0. \tag{4.4}
$$

Reminding that $\|f\|_{L^2(\mathbb{R}^2)} \leq C \|f\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}$, we note that

$$
\|f\|_{L^p(\mathbb{R}^2)}^{\frac{p}{2}} \leq C \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \left\| \nabla f \right\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}.
$$

Using the above inequality, we observe

$$
\frac{d}{dt} \int_{\mathbb{R}^2} \left( |n(t)|^p + |\tilde{c}(t)|^p \right) + \frac{2}{C} \left( \|n\|_{L^2(\mathbb{R}^2)}^{2p} + \|\tilde{c}\|_{L^2(\mathbb{R}^2)}^{2p} \right) \leq 0, \tag{4.5}
$$

where $C$ is independent of $p$.

For convenience, we denote $y_p(t) := \|n(t)\|_{L^p(\mathbb{R}^2)}$ and $z_p(t) := \|\tilde{c}(t)\|_{L^p(\mathbb{R}^2)}$. We first show that for sufficiently large $t > T_3$ and $p = 2^k$ with $k = 1, 2, \ldots$

$$
y_{2k}(t) + z_{2k}(t) \leq \frac{C_k}{t^{1-\frac{2}{2^k}}}, \quad C_k \leq \frac{C_k}{t^{1-\frac{2}{2^k}}}, \quad t \geq T_3, \tag{4.6}
$$

where $C_k = C_1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} = 2 \sum_{l=0}^{k} 2^{-2l}$.

Indeed, in case that $k = 1$, the (4.5) turns out that

$$
\frac{d}{dt} \left( y_2^2(t) + z_2^2(t) \right) + C \left( y_2^2(t) + z_2^2(t) \right)^2 \leq \frac{d}{dt} \left( y_2^2(t) + z_2^2(t) \right) + 2C \left( y_2^2(t) + z_2^2(t) \right) \leq 0.
$$

Direct computations show the validity of (4.6) for $k = 1$. Suppose that (4.6) is true up to $k = m - 1$ with $m > 1$. We then have

$$
\frac{d}{dt} \left( y_{2m}^2(t) + z_{2m}^2(t) \right) + \frac{C}{C_{2m}^2} t^{2m-2} \left( y_{2m}^2(t) + z_{2m}^2(t) \right)^2 \\
\leq \frac{d}{dt} \left( y_{2m}^2(t) + z_{2m}^2(t) \right) + C \left( y_{2m-1}^2(t) y_{2m+1}^2(t) + z_{2m-1}^2(t) z_{2m+1}^2(t) \right) \leq 0.
$$

Solving the above inequality of ODE type, it is straightforward that for sufficiently large $t$

$$
y_{2m}^2(t) + z_{2m}^2(t) \leq \frac{CC_{2m-1}^2 (2m - 1)}{t^{2m-1}} \leq \frac{CC_{2m-1}^2 (2m - 1)^2}{t^{2m-1}},
$$

which automatically implies that (4.6) is true for $k = m$. Next, interpolation argument implies that we can show that

$$
y_p(t) + z_p(t) \leq \frac{C_p}{t^{1-\frac{2}{p}}}, \quad C_p \leq C(p-1)^{\frac{2}{p}}, \quad t \geq T_3. \tag{4.7}
$$

Since the constant $C_p$ is uniformly bounded, independent of $p$, we also obtain by passing $p$ to the limit

$$
\|n(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{t}, \quad t \geq T_3. \tag{4.8}
$$
Since it is direct that \( c \) satisfies
\[
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} |c(t)|^p + \frac{4(p-1)}{p^2} \int_{\mathbb{R}^2} |\nabla c|^2 \leq 0,
\]
we also have
\[
\|c(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{t}, \quad t \geq T_3.
\] (4.10)

For vorticity field \( \omega \), we note that
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\omega|^2 dx + \int_{\mathbb{R}^2} |\nabla \omega|^2 dx \leq \frac{1}{16} \|\nabla \omega\|_{L^2}^2 + C_1 \|\nabla n\|_{L^2}^2.
\] (4.11)

Multiplying \( 2C_1 \) to the equation \( n \) and combining (4.1)-(4.2) and (4.11), we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \left( |\omega|^2 + C_1 |n(t)|^2 + |c(t)|^2 \right) dx + \int_{\mathbb{R}^2} \left( |\nabla \omega|^2 + C_1 |\nabla n|^2 + |\nabla c|^2 \right) \leq 0.
\] (4.12)

Thus, it is direct due to (4.12) that
\[
\|\omega(t)\|_{L^2} \leq \frac{C}{t^\frac{1}{2}}.
\] (4.13)

Next, testing \(-\Delta c\) to the equation of \( c \) and integrating it by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla c|^2 dx + \int_{\mathbb{R}^2} |\Delta c|^2 dx + \int k'(c) n |\nabla c|^2 dx
\]
\[
= -\sum_{k=1}^2 \int (\partial_k u \cdot \nabla) c \partial_k c dx - \int k(c) \nabla n \cdot \nabla c dx
\]
\[
\leq \|\nabla u\|_{L^2} \|\nabla c\|_{L^4}^2 + (\text{sup } k(c))\|\nabla n\|_{L^2} \|\nabla c\|_{L^2}
\]
\[
\leq C \|\omega\|_{L^2} \|\nabla c\|_{L^2} \|\nabla^2 c\|_{L^2} + (\text{sup } k(c))\|\nabla n\|_{L^2} \|\nabla c\|_{L^2}.
\]

Here we note that \( \text{sup } k(c) \leq C t^{-1} \) for sufficiently large \( t \). Indeed, since \( k(0) = 0 \) and \( c(t) \) becomes sufficiently small with rate \( 1/t \) for sufficiently large \( t \), we have \( \text{sup } k(c) \leq C c \leq C t^{-1} \).

Thus, together with (4.13) we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla c|^2 dx + \int_{\mathbb{R}^2} |\Delta c|^2 dx \leq \frac{C}{t^\frac{1}{2}} \left( \|\nabla c\|_{L^2} \|\nabla^2 c\|_{L^2} + \|\nabla n\|_{L^2} \|\nabla c\|_{L^2} \right).
\] (4.14)

Similarly, testing \(-\Delta \omega\) to vorticity equations, we can see that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} |\nabla \omega|^2 dx + \int_{\mathbb{R}^2} |\Delta \omega|^2 dx \leq C \|\omega\|_{L^2} \|\nabla \omega\|_{L^2} \|\nabla^2 \omega\|_{L^2} + (\text{sup } |\nabla \phi|)\|\nabla n\|_{L^2} \|\Delta \omega\|_{L^2}
\]
\[
\leq \frac{C}{t^\frac{1}{2}} \|\nabla \omega\|_{L^2} \|\nabla^2 \omega\|_{L^2} + (\text{sup } |\nabla \phi|)\|\nabla n\|_{L^2} \|\Delta \omega\|_{L^2}.
\] (4.15)

Thirdly, testing \(-\Delta n\) to equation of \( n \), we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^2} |\nabla n|^2 dx + \int_{\mathbb{R}^2} |\Delta n|^2 dx \leq C \|\omega\|_{L^2} \|\nabla n\|_{L^2} \|\nabla^2 n\|_{L^2} + \int_{\mathbb{R}^2} \nabla (\chi n \nabla c) \Delta n dx.
\]
Since $\nabla(\chi(c)n\nabla c) = \chi'(c)n|\nabla c|^2 + \chi(c)\nabla c\nabla c + \chi(c)n\Delta c$, we have

$$
\int_{\mathbb{R}^2} \nabla(\chi n) \nabla c \Delta n dx \leq C \|n\|_{L^\infty} \|\nabla c\|_{L^2}^2 \|\Delta n\|_{L^2} + C \|\nabla n\|_{L^4} \|\nabla c\|_{L^4} \|\Delta n\|_{L^2} + C \|n\|_{L^\infty} \|\Delta c\|_{L^2} \|\Delta n\|_{L^2}
$$

$$
\leq C \|n\|_{L^\infty} \|c\|_{L^\infty} \|\nabla^2 c\|_{L^2} \|\Delta n\|_{L^2} + C \|n\|_{L^\infty} \|c\|_{L^\infty} \|\nabla^2 c\|_{L^2}^2 \|\Delta n\|_{L^2}^2 + C \|\Delta c\|_{L^2} \|\Delta n\|_{L^2} \|\Delta n\|_{L^2},
$$

where we used (4.8) and (4.10). Using the above estimate, we obtain

$$
\frac{d}{dt} \int_{\mathbb{R}^2} |\nabla n|^2 dx + \int_{\mathbb{R}^2} |\Delta n|^2 dx \leq \frac{C}{t^2} \|\nabla n\|_{L^2} \|\nabla^2 n\|_{L^2}^2 + \frac{C}{t} \|\nabla^2 c\|_{L^2} \|\Delta n\|_{L^2} + \frac{C}{t} \|\Delta c\|_{L^2} \|\Delta n\|_{L^2}. \tag{4.16}
$$

Combining (4.14) and (4.16), we get

$$
\frac{d}{dt} \int_{\mathbb{R}^2} |\nabla n|^2 dx + \frac{d}{dt} \int_{\mathbb{R}^2} |c|^2 dx + \int_{\mathbb{R}^2} |\Delta n|^2 dx + \int_{\mathbb{R}^2} |\Delta c|^2 dx \leq \frac{C}{t} \left( \|\nabla n\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 \right). \tag{4.17}
$$

Recalling the estimate (4.3) for the case $q = 2$, i.e.

$$
\frac{d}{dt} \int_{\mathbb{R}^2} |n|^2 dx + \frac{d}{dt} \int_{\mathbb{R}^2} |c|^2 dx + \int_{\mathbb{R}^2} |\nabla n|^2 dx + \int_{\mathbb{R}^2} |\nabla c|^2 dx \leq 0,
$$

we can obtain a slightly modified version of the above estimate

$$
\frac{d}{dt} \left( \frac{1}{t} \int_{\mathbb{R}^2} |n|^2 dx \right) + \frac{d}{dt} \left( \frac{1}{t} \int_{\mathbb{R}^2} |c|^2 dx \right) + \frac{1}{t} \int_{\mathbb{R}^2} |\nabla n|^2 dx + \frac{1}{t} \int_{\mathbb{R}^2} |\nabla c|^2 dx \leq 0. \tag{4.18}
$$

Multiplying (4.21) with a big constant $C$, we obtain

$$
\frac{d}{dt} \left( \frac{C}{t} \int_{\mathbb{R}^2} |n|^2 dx \right) + \frac{d}{dt} \left( \frac{C}{t} \int_{\mathbb{R}^2} |c|^2 dx \right) + \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla n|^2 dx + \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla c|^2 dx \tag{4.19}
$$

We note that

$$
\frac{\|\nabla n\|_{L^2}^4}{\|n\|_{L^2}^4} \leq C \frac{\|\nabla^2 n\|_{L^2}^2}{\|n\|_{L^2}^2}, \quad \frac{\|\nabla c\|_{L^2}^4}{\|c\|_{L^2}^4} \leq C \frac{\|\nabla^2 c\|_{L^2}^2}{\|c\|_{L^2}^2}.
$$

Reminding $\|n\|_{L^2} + \|c\|_{L^2} \lesssim t^{-1/2}$, we have

$$
\frac{d}{dt} \left( \frac{C}{t} \int_{\mathbb{R}^2} |n|^2 dx \right) + \frac{d}{dt} \left( \frac{C}{t} \int_{\mathbb{R}^2} |c|^2 dx \right) + \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla n|^2 dx + \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla c|^2 dx \tag{4.20}
$$

$$
+ \frac{C}{t} \left( \int_{\mathbb{R}^2} |n|^2 dx \right)^2 + \frac{C}{t} \left( \int_{\mathbb{R}^2} |c|^2 dx \right)^2 + Ct \left( \int_{\mathbb{R}^2} |\nabla n|^2 dx \right)^2 + Ct \left( \int_{\mathbb{R}^2} |\nabla c|^2 dx \right)^2 \leq 0.
$$
We let \( y(t) = C_t \| n \|^2_{L^2} + C_t \| c \|^2_{L^2} + \| \nabla n \|^2_{L^2} + \| \nabla c \|^2_{L^2} \). From (4.20), we observe that
\[
\frac{d}{dt} y(t) + C_t y^2(t) \leq 0. \tag{4.21}
\]
This gives \( y(t) \leq C/t^2 \), which implies
\[
\| \nabla n(t) \|_{L^2} + \| \nabla c(t) \|_{L^2} \leq \frac{C}{t}. \tag{4.22}
\]
Furthermore, if \( \nabla \phi \) satisfies a certain decay condition, namely \( |\nabla \phi| \leq \frac{C}{t^2} \), we can have from (4.15) that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} |\nabla \omega|^2 dx + \int_{\mathbb{R}^2} |\Delta \omega|^2 dx \leq \frac{C}{t} \| \nabla \omega \|^2_{L^2} + \frac{C}{t} \| \nabla n \|^2_{L^2}. \tag{4.23}
\]
Therefore, similarly as above, we compute
\[
\frac{d}{dt} \left( \frac{1}{t} \int_{\mathbb{R}^2} |n|^2 dx + \frac{1}{t} \int_{\mathbb{R}^2} |c|^2 dx + \frac{1}{t} \int_{\mathbb{R}^2} |\omega|^2 dx \right)
+ \frac{1}{t} \left( \int_{\mathbb{R}^2} (|\nabla n|^2 + |\nabla c|^2 + |\nabla \omega|^2) dx \right) \leq 0. \tag{4.24}
\]
Following the same procedure as (4.22), we also obtain
\[
\| \nabla \omega(t) \|_{L^2} \leq \frac{C}{t}. \tag{4.25}
\]
This completes the proof of the second part of Theorem 1. \( \square \)

**Appendix**

In this Appendix, we prove local regularity criteria given in Proposition 3.

**Proof.** Fix \( z_0 = (x_0, T_0 - \sigma) \) for any \( \sigma > 0 \). We shall show that there exist a parabolic cylinder \( Q_{z_0, r} \) such that
\[
\| \Delta (n, c, u) \|_{L^p_{t,x}(Q_{z_0, r})} + \| \partial_t (n, c, u) \|_{L^p_{t,x}(Q_{z_0, r})} < \infty, \quad p = 2, 4. \tag{4.25}
\]
Here the bound in (4.25) is independent of \( \sigma \), which implies that (4.25) also holds in \( B_{x_0, r} \times (T_0 - r^2, T_0) \). Then, by Proposition 2, we conclude the solution \( (n, c, u) \) is regular at any given \( (x_0, T_0) \). In the course of proof, we take various cylinders \( Q_{z_0, r} \) decreasing the size \( r \) which depend on some generic constants, independent of choice of \( z_0 \).

First, we show that there exists a constant \( r > 0 \) such that
\[
\| \Delta c \|_{L^2_{t,x}(Q_{z_0, r})} + \| \nabla c \|_{L^2_{t,x}(Q_{z_0, r})} + \| \nabla c \|_{L^2_{t,x}(Q_{z_0, r})} < \infty. \tag{4.26}
\]
Without loss of generality, we assume that \( r < 1 \) and we choose a cut-off function \( \varphi \in C_0^\infty \) satisfying (2.5) with \( q = r_1 < r \). We note first that
\[
\| (\nabla c) \varphi^{\frac{1}{4}} \|_{L^4} \leq C (\| \nabla (c \varphi^{\frac{1}{4}}) \|_{L^4} + \| c \|_{L^4(B_{x_0, r})}),
\]
and

\[ \|\nabla (c\varphi^{\frac{1}{2}})\|_{L^2}^4 \leq C\|\nabla (c\varphi^{\frac{1}{2}})\|_{L^2}^2\|\Delta (c\varphi^{\frac{1}{2}})\|_{L^2}^2 \]

\[ \leq C \left(\|\nabla c\|_{L^2}^2 + \|c\|_{L^2(B_{x_0,r})}^2\right) \left(\|\Delta c\|_{L^2}^2 + \|\nabla c\|_{L^2(B_{x_0,r})}^2 + \|c\|_{L^2(B_{x_0,r})}^2\right). \]

Testing \(-\Delta c\varphi\) to \(1.1_2\), we obtain

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla c|^2 \varphi \, dx + \int_{\mathbb{R}^2} |\Delta c|^2 \varphi \, dx \]

\[ = - \int_{\mathbb{R}^2} \partial_t c \nabla c \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^2} k(c) n(\Delta c) \varphi \, dx + \int_{\mathbb{R}^2} (u \cdot \nabla c)(\Delta c) \varphi \, dx \]

\[ = - \int_{\mathbb{R}^2} \Delta c \nabla c \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^2} (u \cdot \nabla c) \nabla c \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^2} k(c) n \nabla c \cdot \nabla \varphi \, dx \]

\[ + \int_{\mathbb{R}^2} k(c) n(\Delta c) \varphi \, dx + \int_{\mathbb{R}^2} (u \cdot \nabla c)(\Delta c) \varphi \, dx \]

\[ \leq \frac{\epsilon}{2} \int_{\mathbb{R}^2} |\Delta c|^2 \varphi \, dx + C \int_{B_{x_0,r}} |\nabla c|^2 \, dx + C\|u\|_{L^4}\|n\|_4\|\nabla c\|^2_{L^2(B_{x_0,r})} \]

\[ + C\int_{B_{x_0,r}} n^2 \, dx + \|u\|_{L^4}\|\nabla c\|_{L^4}\|\Delta c\|_{L^2} \varphi^{\frac{1}{2}}_L \]

\[ \leq \epsilon \int_{\mathbb{R}^2} |\Delta c|^2 \varphi \, dx + C\|u\|_{L^4}^4 \int_{\mathbb{R}^2} |\nabla c|^2 \varphi \, dx \]

\[ + C\left(\|n\|_{L^2(B_{x_0,r})}^2 + \|\nabla c\|_{L^2(B_{x_0,r})}^2 + \|u\|_{L^4}^4\right). \]

Using Gronwall’s lemma, we obtain that

\[ \|\nabla c\|_{L^2,\infty}^{2}(Q_{x_0,t}) \leq \]

\[ \left(\|\nabla c(t_0 - r_1^2)\|_{L^2(B_{x_0,r_1})}^2 + C(\|n\|_{L^2,\infty}^{2}(Q_{x_0,t})) + \|\nabla c\|_{L^2,\infty}^{2}(Q_{x_0,t}) + \|u\|_{L^4}^4\right) \exp \left(C\|u\|_{L^4}^4\right). \]

Note that \(\|\nabla c\|_{L^2(0,T_0;\mathbb{R}^2)}\) is bounded. We have shown \((4.26)\).

Next, we consider the equation \((1.1)_1\) and prove that there exists a constant \(r > 0\) such that

\[ \|n\|_{L^2,\infty}(Q_{x_0,t}) + \|\nabla n\|_{L^2}(Q_{x_0,t}) < \infty. \] (4.27)

We choose a cut-off function \(\varphi \in C_0^\infty\) satisfying \((2.5)\) with \(\rho = r_2 < r = r_1\) (this cut-off function has a smaller support than the one introduced at the beginning, but we still use the same notation as \(\varphi\), unless there is any confusion to be expected). Choosing \(n\varphi\) as a test function, we note that

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |n|_2^2 \varphi \, dx + \int_{\mathbb{R}^2} |\nabla n|^2 \varphi \, dx = - \int_{\mathbb{R}^2} n \nabla n \cdot \nabla \varphi \, dx \]

\[ + \frac{1}{2} \int_{\mathbb{R}^2} (u \cdot \nabla \varphi) n^2 \, dx + \int_{\mathbb{R}^2} \chi(c)n \nabla c \cdot (\nabla n) \varphi + \int_{\mathbb{R}^2} \chi(c)n^2 \nabla c \cdot \nabla \varphi \, dx \]

\[ := J_1 + J_2 + J_3 + J_4. \]

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Each $J_1$, $J_2$, $J_3$ and $J_4$ can be estimated as follows.

\[
|J_1| \leq C \|n\|_{L^2(B_{2x_0}r_1)} \left( \int_{\mathbb{R}^2} |\nabla n|^2 \varphi \, dx \right)^{\frac{1}{2}} \leq C \|n\|^2_{L^2(B_{2x_0}r_1)} + \epsilon \int_{\mathbb{R}^2} |\nabla n|^2 \varphi \, dx, \tag{4.28}
\]

\[
|J_2| \leq C \|u\|_{L^4} \left( \int_{\mathbb{R}^2} |n|^4 \varphi^2 \, dx \right)^{\frac{1}{4}} \|n\|^2_{L^2(B_{2x_0}r_1)}
\leq C \|u\|_{L^4} (\|\nabla n\|_{L^2}^2 + \|n\|^2_{L^2(B_{2x_0}r_1)})^{\frac{1}{2}} \|n\|^2_{L^2(B_{2x_0}r_1)} + \epsilon \int_{\mathbb{R}^2} |\nabla n|^2 \varphi \, dx,
\]

\[
|J_3| \leq \left( \int_{B_{2x_0}r_1} |\nabla c|^4 \, dx \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}^2} |n|^4 \varphi^2 \, dx \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}^2} |\nabla n|^2 \varphi \, dx \right)^{\frac{1}{2}}
\leq C \|\nabla c\|_{L^4(B_{2x_0}r_1)} (\|\nabla n\|_{L^2}^2 + \|n\|^2_{L^2(B_{2x_0}r_1)})^{\frac{1}{2}} \|n\|^2_{L^2(B_{2x_0}r_1)} + \epsilon \int_{\mathbb{R}^2} |\nabla n|^2 \varphi \, dx,
\]

and

\[
|J_4| \leq C \|\nabla c\|_{L^4(B_{2x_0}r_1)} (\|\nabla n\|_{L^2}^2 + \|n\|^2_{L^2(B_{2x_0}r_1)})^{\frac{1}{2}} \|n\|^2_{L^2(B_{2x_0}r_1)} + \epsilon \int_{\mathbb{R}^2} |\nabla n|^2 \varphi \, dx.
\tag{4.30}
\]

Hence collecting all the estimates (4.28)-(4.31) and using Gronwall’s inequality, we obtain

\[
\|n\varphi^\frac{1}{2}\|^2_{L^2_t,\infty(Q_{x_0}r_1)} \leq \left( \|n\varphi(t_0 - r_1^2)\|^2_{L^2(B_{2x_0}r_1)} + C \|n\|^2_{L^2_t(Q_{x_0}r)} \right) \times \exp \left( C \|u\|^4_{L^4_{x,t}} + C \|\nabla c\|^4_{L^4_{x,t}(Q_{x_0}r_1)} \right). \tag{4.32}
\]

It is immediate that (4.32) implies (4.27).

Next, we consider the vorticity equation

\[
\partial_t \omega - \Delta \omega + (u \cdot \nabla) \omega = -\partial_{x_1} (n \partial_{x_2} \phi) + \partial_{x_2} (n \partial_{x_1} \phi),
\]

where $\omega = \nabla^\perp u = \partial_{x_1} u_2 - \partial_{x_2} u_1$. We will prove that there exists a constant $r > 0$ such that

\[
\|\omega\|_{L^2_t,\infty(Q_{x_0}r)} + \|\nabla \omega\|_{L^2_t(Q_{x_0}r)} < \infty. \tag{4.33}
\]

Testing $\omega \varphi$ with the similar cut-off function $\varphi$ as above (with $\rho = r_3 < r = r_2$), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\omega|^2 \varphi \, dx + \int_{\mathbb{R}^2} |\nabla \omega|^2 \varphi \, dx
= -\int_{\mathbb{R}^2} \omega \nabla \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^2} (u \cdot \nabla \varphi) |\omega|^2 \, dx - \int_{\mathbb{R}^2} (\nabla^\perp (n \nabla \phi)) \omega \varphi \, dx
\]

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\[
\begin{align*}
&= \frac{1}{2} \int_{\mathbb{R}^2} \omega^2 \Delta \varphi \, dx + \int_{\mathbb{R}^2} u \cdot \nabla \varphi |\omega|^2 \, dx - \int_{\mathbb{R}^2} \nabla^2 (n \nabla \phi) \omega \varphi \, dx \\
&:= \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3.
\end{align*}
\]

Note that if \( f \) has compact support in \( B_{R, r_2} \), then \( \|f\|_{L^p} \leq C \|f\|_{L^2} \). Also by Lemma 4 we have
\[
\|u\|_{L^{\frac{9}{2}, \frac{18}{5}}(Q_{0, r})} + \|\omega\|_{L^{\frac{5}{2}, \frac{2}{2}}(Q_{0, r})} < \infty.
\]

Each \( \tilde{J}_1, \tilde{J}_2 \) and \( \tilde{J}_3 \) can be estimated as follows.
\[
|\tilde{J}_1| \leq C \|\omega \varphi\|_{L^4} \|\omega\|_{L^\infty(B_{R, r_2})} \leq C \|\omega \varphi\|_{L^2} \|\nabla (\omega \varphi)\|_{L^2} \|\omega\|_{L^2} \leq C \|\nabla \varphi\|_{L^2} + C \|\omega\|_{L^2}^2 + C \|\omega \varphi\|_{L^2}^2,
\]
\[
|\tilde{J}_2| \leq C \int_{\mathbb{R}^2} |u| |\omega\|_{L^4} \|\varphi\|_{L^2} \|\omega\|_{L^2} \, dx \leq C \|u\|_{L^{\frac{9}{2}, \frac{18}{5}}(Q_{0, r_2})} \|\omega \varphi\|_{L^2} \|\omega\|_{L^2} \leq C \|u\|_{L^{\frac{9}{2}, \frac{18}{5}}(Q_{0, r_2})} \|\omega \varphi\|_{L^2} \|\omega\|_{L^2} + C \|\omega \varphi\|_{L^2}^2 + C \|u\|_{L^{\frac{9}{2}, \frac{18}{5}}(Q_{0, r_2})} \|\omega \varphi\|_{L^2}^2 + C \|u\|_{L^{\frac{9}{2}, \frac{18}{5}}(Q_{0, r_2})} \|\omega \varphi\|_{L^2}^2 + C \|u\|_{L^{\frac{9}{2}, \frac{18}{5}}(Q_{0, r_2})} \|\omega \varphi\|_{L^2}^2 + 1,
\]
and
\[
|\tilde{J}_3| \leq C \|\nabla (n \nabla \phi)\|_{L^2} \|\omega \varphi\|_{L^2} \leq C \|\nabla n\|_{L^2(B_{R, r_2})} + C \|\omega \varphi\|_{L^2} + C \|\omega \varphi\|_{L^2}^2.
\]

Then collecting all the estimates (4.34)–(4.36) and using Gronwall’s inequality, we have
\[
\|\omega \varphi\|_{L^{\frac{9}{2}, \frac{18}{5}}(Q_{0, r_2})} \leq \left( \|\varphi\|_{L^2(B_{R, r_2})} + C \|\omega\|_{L^2} \|\omega \varphi\|_{L^2} + C \|\omega \varphi\|_{L^2}^2 \right) \exp \left( C \|u\|_{L^{\frac{9}{2}, \frac{18}{5}}(Q_{0, r_2})} \right).
\]
Therefore, we deduce (4.33).

Next we prove that there exists a constant \( r > 0 \) such that
\[
\|c\|_{L^4_{\infty}(Q_{0, r})} + \|\Delta c\|_{L^4_{\infty}(Q_{0, r})} < \infty.
\]
Take a cut-off function \( \varphi \in C_0^{\infty} \) satisfying (2.5) with \( \varphi = r_3 \) (we use again the same notation). We then consider the equation of \( c \) localized by \( \varphi \), namely
\[
\partial_t (e \varphi) - \Delta (e \varphi) = -2 \nabla c \cdot \nabla \varphi - c \Delta \varphi - (u \cdot \nabla c) \varphi - k(c)n \varphi.
\]
Using maximal regularity of the heat equation, we obtain that
\[ \|\nabla^2 (c\varphi)\|_{L^4_{x,t}(Q_{z_0,r_3})}^4 \leq C \left( \|c\varphi(t_0 - r_3)\|_{W^{2,4}(B_{z_0,r_3})}^4 + \|\nabla c\|_{L^4_{x,t}(Q_{z_0,r_3})}^4 \right) + \|c\|_{L^4_{x,t}(Q_{z_0,r_3})}^4 + \|(u \cdot \nabla)c\|_{L^4_{x,t}(Q_{z_0,r_3})}^4 + \|n\|_{L^4_{x,t}(Q_{z_0,r_3})}^4. \] (4.39)

We estimate \(\|u \cdot \nabla c\|_{L^4_{x,t}(Q_{z_0,r_3})}^4\) as follows: Recalling that \(u \in L^{p,\infty}_{x,t}(Q_{z_0,r})\) for all \(p < \infty\) because of \(\omega \in L^{2,\infty}_{x,t}(Q_{z_0,r})\), we observe for small \(r_2 > 0\),
\[ \|u \cdot \nabla c\|_{L^4_{x,t}}^4 \leq \|u \nabla c\|_{L^4_{x,t}}^4 + \|u \nabla \varphi\|_{L^4_{x,t}}^4 \leq C \|u\|_{L^4_{x,t}}^4 \|\nabla \varphi\|_{L^4_{x,t}}^4 + C \|u\|_{L^4_{x,t}}^4 \|c\|_{L^4_{x,t}}. \]

Since \(\|\nabla (c\varphi)\|_{L^8} \leq C \|\nabla (c\varphi)\|_{L^4_{x,t}}^3 \|\nabla^2 (c\varphi)\|_{L^4_{x,t}}^4\), we have
\[ \|\nabla (c\varphi)\|_{L^8}^4 \leq C \|\nabla (c\varphi)\|_{L^4_{x,t}}^3 \|\nabla^2 (c\varphi)\|_{L^4_{x,t}}^4. \]

Combining estimates, we obtain
\[ \|u \cdot \nabla c\|_{L^4_{x,t}}^4 \leq \epsilon \|\nabla^2 (c\varphi)\|_{L^4_{x,t}}^4 + C \|u\|_{L^{\infty}_{x,t}}^2 \|\nabla (c\varphi)\|_{L^4_{x,t}}^4 + C \|u\|_{L^4_{x,t}}^4 \|c\|_{L^4_{x,t}}^4. \] (4.40)

Via (4.39) and (4.40) with small choice of \(\epsilon\), we obtain
\[ \|\partial_t (c\varphi)\|_{L^4_{x,t}(Q_{z_0,r_3})}^4 + \|\nabla^2 (c\varphi)\|_{L^4_{x,t}(Q_{z_0,r_3})}^4 \leq C \left( \|c\varphi(t_0 - r_3)\|_{W^{2,4}(B_{z_0,r_3})}^4 + \|\nabla c\|_{L^4_{x,t}(Q_{z_0,r_3})}^4 \right) + \|c\|_{L^4_{x,t}(Q_{z_0,r_3})}^4 + \|u\|_{L^{\infty}_{x,t}}^2 \|\nabla (c\varphi)\|_{L^4_{x,t}}^4 + \|u\|_{L^4_{x,t}}^4 \|c\|_{L^4_{x,t}}^4 + \|n\|_{L^4_{x,t}(Q_{z_0,r_3})}^4. \] (4.41)

Furthermore, we can obtain
\[ c_t, \nabla^2 c \in L^{q,p}_{x,t}(Q_{z_0,r_4}), \text{ with } \frac{2}{q} + \frac{2}{p} = 1, \quad 2 < p < \infty. \]

Since its verification is similar to (4.41), we skip its details.

Next we prove that there exists a constant \(r > 0\) such that
\[ \|n_t\|_{L^p_{x,t}(Q_{z_0,r})} + \|\nabla^2 n\|_{L^p_{x,t}(Q_{z_0,r})} = p = 2,4. \] (4.42)

Choose a cut-off function \(\varphi \in C_0^\infty\) satisfying (2.5) with \(q = r_5 < r = r_4\) and taking the spatial derivative of the equation of \(n\) in \(x_i, i = 1, 2\), we can see that
\[ \partial_t \partial_i n - \Delta \partial_i n + \partial_i((u \cdot \nabla)n) = -\nabla \cdot \partial_i (\chi(c)n \nabla c), \]
where \(\partial_i = \partial_{x_i}\). Testing the above equation with \((\partial_i n)\varphi\) and summing over \(i = 1, 2\), we obtain
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla n|^2 \varphi dx + \int_{\mathbb{R}^2} |\nabla^2 n|^2 \varphi dx. \]
\[ = - \int_{\mathbb{R}^2} \nabla^2 n : \nabla n \otimes \nabla \varphi \, dx + \int_{\mathbb{R}^2} (u \cdot \nabla n) \nabla n \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^2} (u \cdot \nabla n) \Delta n \varphi \, dx \]

\[ + \int_{\mathbb{R}^2} (\nabla \cdot (\chi(c)n \nabla c)) \Delta n \varphi \, dx + \int_{\mathbb{R}^2} (\nabla \cdot (\chi(c)n \nabla c)) \nabla n \cdot \nabla \varphi \, dx := I_1 + I_2 + I_3 + I_4 + I_5. \]

Each term can be estimated as before:

\[ |I_1| \leq C \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})}^2 - \epsilon \|\nabla^2 n\|_{L^2}^2, \quad (4.43) \]

\[ |I_2| \leq C \|u\|_{L^4} \left( \int_{\mathbb{R}^2} \|\nabla n\|^4 \varphi^2 \, dx \right)^{\frac{1}{4}} \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})} \]

\[ \leq C \|u\|_{L^4} \left( \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})} \right)^{\frac{1}{4}} \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})} \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})} \leq C \|u\|_{L^4} \|\nabla n\|^2_{L^2} + C \|n\|^2_{L^2(B_{2\epsilon r_0, r_0})} + \epsilon \int_{\mathbb{R}^2} \|\nabla n\|^2 \varphi \, dx, \quad (4.44) \]

\[ |I_3| \leq C \|u\|_{L^4} \left( \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})} \right)^{\frac{1}{4}} \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})} \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})} \leq C \|u\|_{L^4} \|\nabla n\|^2_{L^2} + C \|n\|^2_{L^2(B_{2\epsilon r_0, r_0})} + \epsilon \int_{\mathbb{R}^2} \|\nabla n\|^2 \varphi \, dx. \quad (4.45) \]

After computing the derivatives, \( I_4 \) and \( I_5 \) can be estimated as follows:

\[ |I_4| \leq C \int_{\mathbb{R}^2} n |\nabla c|^2 |\Delta n| \varphi \, dx + C \int_{\mathbb{R}^2} |\nabla n| |\nabla c| |\Delta n| \varphi \, dx + C \int_{\mathbb{R}^2} n |\Delta c| |\Delta n| \varphi \, dx \]

\[ := I_{41} + I_{42} + I_{43}. \]

We separately estimate \( I_{41}, I_{42} \) and \( I_{43} \).

\[ I_{41} \leq C \|n\|_{L^1(B_{2\epsilon r_0, r_0})}^4 + C \|\nabla c\|_{L^8(B_{2\epsilon r_0, r_0})}^8 + \|\nabla c\|_{L^4(B_{2\epsilon r_0, r_0})}^2 + \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})}^2, \quad (4.46) \]

\[ I_{42} \leq C \|\nabla n\|_{L^1} \left( \|\nabla n\|_{L^1} \right) \|\nabla n\|^2_{L^2} + C \|\nabla c\|_{L^4(B_{2\epsilon r_0, r_0})}^4 + 1 \|\nabla n\|^2_{L^2} \]

\[ + C \|\nabla n\|^2_{L^2(B_{2\epsilon r_0, r_0})} + \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})}^2, \quad (4.47) \]

\[ I_{43} \leq C \|\nabla c\|_{L^4(B_{2\epsilon r_0, r_0})}^4 + C \|n\|_{L^1(B_{2\epsilon r_0, r_0})}^4 + \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})}^2. \quad (4.48) \]

Finally, we estimate \( I_5 \).

\[ |I_5| \leq C \int_{\mathbb{R}^2} n |\nabla c|^2 |\nabla \varphi| \, dx + C \int_{\mathbb{R}^2} |\nabla c| |\nabla n|^2 |\nabla \varphi| \, dx + C \int_{\mathbb{R}^2} n |\Delta c| |\nabla n| |\nabla \varphi| \, dx \]

\[ := I_{51} + I_{52} + I_{53}. \]

We separately estimate \( I_{51}, I_{52} \) and \( I_{53} \).

\[ I_{51} \leq C \|n\|^4_{L^1(B_{2\epsilon r_0, r_0})} + C \|\nabla c\|^8_{L^8(B_{2\epsilon r_0, r_0})} + C \|\nabla n\|^2_{L^2(B_{2\epsilon r_0, r_0})}^2, \quad (4.49) \]

\[ I_{52} \leq C \|\nabla c\|_{L^1(B_{2\epsilon r_0, r_0})} \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})} \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})} \|\nabla n\|_{L^2(B_{2\epsilon r_0, r_0})}^2. \]
In the similar manner, we can even show that
\[ I_{53} \leq C \|n\|_{L^4_t(B_{2r_0}, r_4)}^4 + C \|\nabla c\|_{L^4_t(B_{2r_0}, r_4)}^4 + C \|\nabla c\|_{L^4_t(B_{2r_0}, r_4)}^4 \|\nabla n\|_{L^2_t(B_{2r_0}, r_4)}^2, \tag{4.50} \]

Hence using Gronwall’s lemma, we obtain
\[ \|\nabla n\|_{L^2_{x,t}(Q_{2r_0}, r_4)}^2 \leq \left( \|\nabla \varphi\|_{L^2_{x,t}(B_{2r_0}, r_4)}^2 + C \|n\|_{L^4_t(B_{2r_0}, r_4)}^4 + C \|\nabla c\|_{L^4_t(B_{2r_0}, r_4)}^4 \|\Delta n\|_{L^2_t(B_{2r_0}, r_4)} \right) \times \exp \left( C \left( \|u\|_{L^4_t}^4 + \|\nabla c\|_{L^4_t}^4 Q_{2r_0} + r_4^2 \right) \right). \tag{4.51} \]

Furthermore, we have
\[ \nabla n \in L^2_{x,t}(Q_{2r_0, r_5}), \quad \text{and} \quad \nabla^2 n \in L^2_{x,t}(Q_{2r_0, r_5}). \]

Lastly, choosing a cut-off function \( \varphi \in C_0^\infty \) satisfying \( \varphi = r_6 < r = r_5 \), we consider the equation of \( n \) localized by \( \varphi \), that is
\[ \partial_t (n \varphi) - \Delta(n \varphi) = -2\nabla n \cdot \nabla \varphi - n \Delta \varphi - (u \cdot \nabla) n \varphi - \nabla (\chi(c) n \nabla c) \varphi. \]

Using maximal regularity of the heat equation and similar estimates \( 4.40 - 4.41 \), we obtain
\[ \|\partial_t (n \varphi)\|_{L^4_t(Q_{2r_0, r_5})}^4 + \|\nabla^2 (n \varphi)\|_{L^4_t(Q_{2r_0, r_5})}^4 \leq C \left( \|n\varphi\|_{W^{2,4}(B_{2r_0}, r_5)}^4 + \|\nabla n\|_{L^4_t(Q_{2r_0, r_5})}^4 + \|\nabla (\chi(c) n \nabla c) \varphi\|_{L^4_t(Q_{2r_0, r_5})}^4 \right) < \infty. \tag{4.53} \]

In the similar manner, we can even show that
\[ n_t, \quad \nabla^2 n \in L^{p,q}_{x,t}(Q_{2r_0, r_5}), \quad \text{with} \quad \frac{2}{q} + \frac{2}{p} = 1. \]

For \( \omega \) we can also prove
\[ \|\omega_t\|_{L^p_{x,t}(Q_{2r_0, r})} + \|\nabla^2 \omega\|_{L^p_{x,t}(Q_{2r_0, r})} \quad p = 2, 4, \tag{4.54} \]

and even more generally we can show
\[ \omega_t, \quad \nabla^2 \omega \in L^{p,q}_{x,t}(Q_{2r_0, r_3}), \quad \frac{2}{q} + \frac{2}{p} = 1. \]

Since arguments are again similar, we omit the details.

By interpolating \( L^4_{x,t}(Q_{2r_0, r}) \) and \( L^4_{x,t}(Q_{2r_0, r}) \), we finally obtain that for \( 2 \leq p \leq 4 \)
\[ \|\Delta(n, c, u)\|_{L^p_{x,t}(Q_{2r_0, r})} + \|\partial_t(n, c, u)\|_{L^p_{x,t}(Q_{2r_0, r})} < \infty. \]

As mentioned at the beginning, all above estimates such as \( 4.25, 4.32, 4.37, 4.52, 4.53 \) and \( 4.37 \) are independent of \( \sigma \), and thus, passing \( \sigma \) to zero, we conclude via Proposition \( 2 \)
\[ (n, c, u) \in C^{\alpha, \frac{2}{p}}(B_{2r_0, r} \times [T_0 - r^2, T_0]) \text{ for small } r > 0 \text{ and any } 0 < \alpha < 1. \]

This completes the proof.
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