Sudden braking and turning in the single/multi-stream inflation: primordial black hole formation

Chengjie Fu\textsuperscript{a} Chao Chen\textsuperscript{b}
\textsuperscript{a}Department of Physics, Anhui Normal University, Wuhu, Anhui 241000, P.R.China
\textsuperscript{b}Jockey Club Institute for Advanced Study, The Hong Kong University of Science and Technology, Hong Kong S.A.R., P.R.China

E-mail: fucj@ahnu.edu.cn, iaschao@ust.hk

Abstract.

We study a two-field inflation model with a Gaussian bump on the potential, also known as the multi-stream inflation, which can give rise to multiply inflationary trajectories with various interesting phenomena. With a shifted Gaussian bump, the multiply streams are approximately reduced to a single stream. We find that when inflaton rounds the Gaussian potential, its speed is easily slowed down, and thus the slow-roll parameter can be largely reduced. Consequently, the original decaying modes of comoving curvature perturbations during the slow-roll phase start growing, and lead to enhanced small-scale density perturbations which can produce amounts of primordial black holes (PBHs) and associated scalar-induced gravitational waves. In addition, inflaton also undergoes sudden turnings at the encounter of the Gaussian potential, which is insignificant to the overall curvature power spectrum since their durations are quite short. Our work gives a simple example of the extension of a bump-like potential for PBH formation in a single-field inflation to a two-field case, which can relax the fine-tuning of initial conditions to some extent.

Keywords: primordial black holes, multi-stream inflation, ultra-slow roll, sudden turning, scalar-induced gravitational waves

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1 Introduction

As a type of potentially existed compact objects in the early Universe, the primordial black holes (PBHs) have been attracting an amount of attentions over decades, due to their relevance with enormous astronomical and cosmological observational phenomena, in particular, they are a reasonable candidate for the whole or a fraction of dark matter [1, 2]. With more and more stringent constraints on PBH current energy fraction \( f_{\text{PBH}} \equiv \Omega_{\text{PBH}}/\Omega_{\text{DM}} \) (here \( \Omega_{\text{PBH}} \) and \( \Omega_{\text{DM}} \) are the normalized PBH and dark matter energy densities, respectively. See Refs. [2, 3] for the up-to-date constraints), there is also a growing interest in the evidence of PBHs from LIGO/Virgo gravitational wave (GW) events [4, 5], and the potential correlation between PBHs and high-redshift supermassive BHs [6, 7]. PBHs are usually considered to be formed from the direct collapse of overdense regions that may existed in the early Universe [8–10], as a consequence, they display an extensive mass distribution which is a distinct property in contrast to astrophysical BHs which follow a lower mass bound [11]. Especially, PBHs can be light enough for Hawking radiation becomes significant [12, 13], and these Hawking-radiated particles may have an impact on the early Universe, including Big Bang Nucleosynthesis [14–16], cosmic microwave background (CMB) [17–19] and cosmic-ray experiments [20–22], etc. With those robust observational data, one can impose constraints on the initial abundance of such light PBHs that have already evaporated at present [3].

PBH formation mechanism has been considered within a huge number of inflationary models in past decades, the central task is to amplify the small-scale primordial curvature perturbations to around \( 10^7 \) times larger than the CMB observed spectrum, while keep the scale-invariant large-scale spectrum. Until now, PBH formation mechanism can be roughly classified into the following categories: additional contributions to curvature perturbations from other fields in multi-field models [23–31], the enhancement of perturbations arose from non-attractor evolutions (e.g., in presence of ultra-slow-roll (USR) phases) [32–46], parametric
resonance or tachyonic instability of curvature perturbations [47–54], the non-Gaussianity of distribution probability [55–60] and the growth of perturbations in a contracting Universe [61–63]. Recently, it has been noticed in Ref. [64] that inflaton’s speed can be slowed down when it climbs over a bump-like potential or passes a dip-like potential, leading to the amplification of comoving curvature perturbations. Similar models involving a step-like potential have been investigated in the literature [40, 58, 59, 65]. Ref. [40] pointed out that the amplification of perturbations can be simply understood from the energy conservation, such that whether inflation climb up or down the potential, the energy transfer between its kinematic and potential energies resulting in a rapid conservation of the incoming positive modes of comoving curvature perturbations into a superposition of the positive and negative modes, implying the particle productions. More interestingly, Refs. [58, 59] reported non-perturbative effects on the tail of probability distribution of curvature perturbations even from a tiny step on inflation’s potential, which may have a huge impact on the resulting PBH abundance. Inspired by these works, we develop a two-field inflation model and replace the bump-like potential by a smooth Gaussian potential, the enhanced comoving curvature spectrum can be also realized. The greatest merit of considering two-field model is that the inflaton’s speed can be naturally reduced when it climbs up the potential, without fine tuning the initial condition to ensure that the inflaton is able to pass the bump in a single-field case.

Figure 1. An illustration of the multi-stream inflation scenario. Inflaton rolls down the potential $V(\phi, \chi)$ and bifurcates into two trajectories $A$ and $B$ at the encounter of a potential barrier, i.e., at the scale $k_1$, and two trajectories then converge into a single trajectory at the scale $k_2$.

The existence of a Gaussian potential can lead to multiply inflationary trajectories in the multi-field case, which is the so-called multi-stream inflation and was firstly proposed in Ref. [66]. The inflaton will rolls down along different trajectories with the associated probabilities, resulting in multiply patches in the Universe. The local physics within patches maybe different from each other, i.e., depending on the local shape of the potential along each trajectory. Hence, a number of interesting observational phenomena are predicted in the multi-stream inflation scenario: (i) the trajectories typically have different e-folding numbers, and this difference $\delta N$ contributes the additional curvature perturbations [66, 67]. Depending on the scale of a bifurcation, these additional curvature perturbations can explain the CMB cold spot [68] or trigger the formation of PBHs [69]; (ii) Physical environments along trajectories may be different, which can account for the spectrum asymmetry in the CMB [66, 70], the initial cluster of PBHs [71], the stellar bubbles [72], the cosmological timer [73] and reconcile
cosmic dipolar tensions as well [74]; (iii) The domain wall in between $A$ and $B$ may have observational effects [75], and the thickness of the domain wall can help evade the Sunyaev-Zeldovich effect constraint to ease the Hubble tension [76]. The lattice simulation performed in Ref. [77] shows that the temporary domain wall with the time-varying tension will occur from the oscillatory lower-probability trajectory.

In this paper, a shifted two-dimensional Gaussian potential barrier is considered. Since the symmetry is broken in one field direction (i.e., the $\chi$-direction shown in Fig. 1), the inflaton will go along one trajectory with a high probability (the probability for another trajectory is exponentially suppressed [77]), the multiply streams become a “single stream”. We find that inflaton undergoes several quasi-constant-roll (quasi-CR) phases in this model, the ordinary “decaying modes” of comoving curvature perturbations start growing after transitions and give rise to the enhanced spectrum at the end of inflation. Several resulting features of the comoving curvature perturbations can be understood by the matching method studied for USR inflation [34, 78]. Moreover, inflaton also makes sudden turnings when it falls off from the Gaussian potential, which results in a tachyonic instability of isocurvature modes, and this instability transfers to adiabatic modes through their kinematic coupling. Since the duration of each sudden turning is quite short, this type of contribution is insignificant to the overall curvature spectrum compared with quasi-CR phases. A large number of PBHs can be produced by this enhanced curvature spectrum, and they can explain the LIGO/Virgo GW events and the associated scalar-induced gravitational waves (SIGWs) also account for the NANOGrav’s suspected signal of stochastic gravitational wave background [79].

The organization of this paper is as follows. In Sec. 2, we present the details of the model construction and the dynamics of background and perturbations in a generic setting. Then, we perform analytical estimates of the comoving curvature perturbations by the matching calculations of five successive CR phases, and the sudden-turning phase as well, which are shown to be consistent with the numerical results. In Sec. 3, we calculate the PBH abundance by the enhanced curvature perturbations derived in Sec. 2, which is allowed by the current PBH constraints. The associated SIGWs are found to be likely detected by the current or forthcoming multi-frequency GW experiments. Finally, we summarize the results in Sec. 4.

2 Sudden braking and turning in the single/multi-stream inflation

2.1 Dynamics of background and perturbations

We consider a simple concrete model of the multi-stream inflation scenario, which consists of two canonical scalar fields minimally coupled to Einstein gravity,

$$ S = \int d^4x \sqrt{-g} \left( \frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} \nabla_\mu \chi \nabla^\mu \chi - V(\phi, \chi) \right), $$

where $M_{\text{Pl}}$ is the reduced Planck mass, $R$ is the Ricci scalar constructed out of the spacetime metric $g_{\mu\nu}$. We consider the following potential with a Gaussian bump:

$$ V(\phi, \chi) = \lambda M_{\text{Pl}}^{18/5} \phi^{2/5} + \frac{1}{2} m^2 \chi^2 + \Lambda^4 \exp \left[ -\frac{(\phi - \phi_c)^2}{2\sigma_\phi^2} - \frac{(\chi - \chi_c)^2}{2\sigma_\chi^2} \right], $$

where the field $\phi$ has a fractional power-law potential with power $2/5$ [80]. Here, the dimensionless parameter $\lambda$ controls the mass of the field $\phi$ and $m$ is the bare mass of the field $\chi$. 

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The constant \( \Lambda \) with the dimension of mass controls the height of the Gaussian bump, \( \phi_c \) and \( \chi_c \) control the peak position of potential, while \( \sigma_\phi \) and \( \sigma_\chi \) determine its width.

The dynamics of double fields are directly derived from the action (2.1) with the potential (2.2). Decomposing the fields into the background and perturbation: \( \phi(t, x) = \phi_0(t) + \delta \phi(t, x) \), \( \chi(t, x) = \chi_0(t) + \delta \chi(t, x) \), the EoMs for background fields in the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric are given by,

\[
\frac{\dot{\phi}_0}{3H} + \frac{2}{3} \lambda M_{\text{Pl}}^{18/5} \phi_0^{-3/5} = \frac{\phi_0 - \phi_c}{\sigma_\phi^2} \Lambda^4 \exp \left[ -\frac{(\phi_0 - \phi_c)^2}{2\sigma_\phi^2} - \frac{(\chi_0 - \chi_c)^2}{2\sigma_\chi^2} \right] = 0 \tag{2.3},
\]

\[
\frac{\dot{\chi}_0}{3H} + m^2 \chi_0 = \frac{\chi_0 - \chi_c}{\sigma_\chi^2} \Lambda^4 \exp \left[ -\frac{(\phi_0 - \phi_c)^2}{2\sigma_\phi^2} - \frac{(\chi_0 - \chi_c)^2}{2\sigma_\chi^2} \right] = 0 \tag{2.4},
\]

where \( H = \dot{a}/a \) is the Hubble parameter, and the Friedmann equation gives

\[
H^2 = \frac{1}{3M_{\text{Pl}}^2} \left[ \frac{1}{2} \phi_0^2 + \frac{1}{2} \chi_0^2 + V(\phi_0, \chi_0) \right]. \tag{2.5}
\]

The above background equations (2.3), (2.4) and (2.5) determine the trajectory of inflaton in the field space spanned by \{\( \phi, \chi \)\}, as shown in the top-left panel of Fig. 2. Including metric perturbations, it is convenient to apply the covariant formalism to the multifield perturbations. The dynamics is given by

\[
\ddot{Q}_I + 3H \dot{Q}_I + \frac{k^2}{a^2} Q_I + \sum_J \left[ \frac{8\pi G}{a^3} \partial_t \left( \frac{a^3}{H} \dot{\phi}_I \dot{\phi}_J \right) \right] Q_J = 0 \tag{2.6},
\]

where \( Q_I \equiv \delta \phi_I + \dot{\phi}_I \psi \), with \( \phi_I = 1, 2 = \{ \phi_0, \chi_0 \} \) and \( \delta \phi_{I=1,2} = \{ \delta \phi, \delta \chi \} \), are Mukhanov-Sasaki variables which are gauge-invariant field perturbations [81, 82]. In the spatially flat gauge \( \psi = 0 \), we have \( Q_I = \delta \phi_I \), here \( \psi \) is the curvature perturbations of the spatial hypersurface of the perturbed FLRW metric. It is convenient to apply the covariant formalism to the multifield inflation [83–87], which treats multifield perturbations systematically and geometrically (also see Appendix A for the detailed discussions). One decomposes field perturbations \( Q_I \) along the tangent and perpendicular directions of the inflaton’s trajectory, and define the comoving curvature (adiabatic) and entropy perturbations as

\[
\mathcal{R} \equiv \frac{H}{\dot{\phi}_0} T_a Q^a = \frac{H}{\dot{\phi}_0} T_a \delta \phi^a, \quad \mathcal{F} \equiv \dot{N}_a Q^a = N_a \delta \phi^a, \tag{2.7}
\]

respectively. Here \( \dot{\phi}_0 = \sqrt{\dot{\phi}_0^2 + \dot{\chi}_0^2} \). Note that we set \( \psi = 0 \) for the comoving curvature perturbation \( \mathcal{R} \). The unit tangent and normal vectors are given by \( T^a = \frac{1}{\dot{\phi}_0}(\dot{\phi}_0, \chi_0) \) and \( N^a = \frac{1}{\dot{\phi}_0}(-\chi_0, \dot{\phi}_0) \), which satisfy the orthogonality and normalization: \( T^a N_a = 0, T^a T_a = N^a N_a = 1 \), as illustrated in the top-right panel of Fig. 2. The Latin indices are raised or lowered by the Kroneck delta function. It is customary to define dimensionless slow-roll (SR) parameters \( \epsilon \) and \( \eta^a \) to account for the background dynamics of inflation, such that \( \epsilon \equiv -\frac{H}{\dot{H}} = \frac{\dot{\phi}_0^2}{2M_{\text{Pl}}^2 H^2} \) and \( \eta^a \equiv -\frac{1}{H \dot{\phi}_0} \frac{\partial \phi^a}{\partial t} \), where we introduce the field space covariant derivative \( D_t \equiv \phi_0^a \nabla_a \) with respect to the cosmic time \( t \). Notice that the vector \( \eta^a \) could
tell us that how fast $\dot{\phi}_0$ varies in time. We may decompose $\eta^a$ along the tangent and normal directions as

$$\eta^a = \eta_\parallel T^a + \eta_\perp N^a, \quad \eta_\parallel = -\frac{\dot{\phi}_0}{H\phi_0}, \quad \eta_\perp \equiv \frac{V_N}{H\dot{\phi}_0}. \quad (2.8)$$

Note that $\eta_\parallel$ is recognized as the usual SR parameter in a single-field inflation, while $\eta_\perp$ indicates how fast $T^a$ rotates and therefore describes the turning rate of the trajectory $\dot{\phi}_0$. The quadratic action for curvature and entropy perturbations is directly derived from the action (2.1) as [85, 86]

$$S^{(2)} = \int \! dt d^3x a^3 \left[ M_{\text{Pl}}^2 \epsilon R^2 - M_{\text{Pl}}^2 \epsilon (\nabla R)^2 + 2\sqrt{2} \epsilon M_{\text{Pl}}^2 \Omega R F + \frac{1}{2} F^2 - \frac{1}{2} M^2 \Omega F^2 \right], \quad (2.9)$$

where we have defined the entropic mass of the entropy field $F$ as

$$M^2 = V_{NN} + \epsilon H^2 M_{\text{Pl}}^2 \Omega - \Omega^2, \quad (2.10)$$

where $V_{NN} \equiv N^a N^b \nabla_a \nabla_b V$ and $\Omega$ is the Ricci scalar of the field space. The quantity $\Omega \equiv H \eta_\perp$ is the turning rate describing the bends of the trajectory. It is clear from Eq. (2.10) that the entropic mass is also affected by the turning rate of the trajectory, due to the fact that the potential receives a correction coming from the centrifugal force experienced by the turning. In our model (2.1), the geometry of field space is trivial, the bare entropic mass $m^2_{\text{bare}} \equiv V_{NN} + \epsilon H^2 M_{\text{Pl}}^2 \Omega$ is merely contributed by the potential $V(\phi, \chi)$, i.e., $m_{\text{bare}}^2 = V_{NN}$. Notice that at the quadratic level, the interaction strength between adiabatic and entropic modes is controlled by a dimensionless quantity $\eta_\perp$. From the action (2.9), we derive coupled dynamical equations for curvature and entropy perturbations,

$$\ddot{R} + (3 + 2\epsilon - 2\eta_\parallel) H \dot{R} + \frac{k^2}{a^2} R = -2\Omega \frac{H}{\phi_0} \left[ \dot{F} + \left( 3 - \eta_\parallel - \epsilon + \frac{\dot{\Omega}}{\Omega} \right) \frac{H}{F} \right], \quad (2.11)$$

$$\ddot{F} + 3H \dot{F} + \left( \frac{k^2}{a^2} + M^2 \right) F = 2\Omega \frac{H}{\phi_0} \dot{R}, \quad (2.12)$$

which are consistent with Eq. (2.6) with the definitions (2.7). In practice, one usually resorts to the numerical calculations of these two coupled equations, however, on the superhorizon scales, they can be simplified and one can make some estimates for their evolutions.

### 2.2 Anatomy of curvature and entropy perturbations

From the top-left panel of Fig. 2, it is clearly see that the dynamics of our model can be divided into three phases: (a) **Single-field phase.** Initially, the inflaton rolls down the potential along the $\phi$-direction while strapped to the local minimum on $\chi$-direction (i.e., $\chi = 0$). During this phase, the dynamics is entirely described by $\phi^{2/3}$ inflation model [80], which is allowed by the Planck experiment [88] for around $30 \sim 40$ e-folds; (b) **Braking-Turning phase.** When inflaton encounters the shifted Gaussian barrier with a SR speed, it will climb up the barrier with a quickly-decreasing speed $\dot{\phi}$, and reach the highest potential point when inflaton’s speed is nearly zero (The corresponding parameters are calculated as $N_c \approx 31.26$, $\phi \approx 1.998$, $\chi \approx -3.27 \times 10^{-6}$, $\dot{\phi} \approx -4.97 \times 10^{-7}$, $\dot{\chi} \approx -3.49 \times 10^{-5}$ and $V/\lambda M_{\text{Pl}}^4 \approx 1.33$ in terms of the parameter set 1 as listed in Table 1). And then, the inflation starts entering the braking-turning phase, which mixes the quasi-CR (i.e., the second SR parameter $\eta \equiv \epsilon/(\epsilon H)$ is a constant) and sudden-turning processes. The SR parameter decreases rapidly as shown in
Figure 2. The background evolutions of our model (2.1) and (2.2). *Top-left panel:* The red curve denotes the inflationary trajectory in the presence of the Gaussian potential, and the colored shadow refers to the height of the potential in Eq. (2.2). The existence of sudden turnings is clearly shown in the *top-right panel*, while the quasi-CR phases are displayed in the *middle-left & -right panels* for the Hubble parameter and SR parameter, respectively. The red dashed curve denotes the smoothed SR parameter in order to make analytical estimates via matching calculations of quasi-CR phases. *Bottom-left & -right panels* show the evolutions of background fields and their speeds during the whole inflationary stage. The values of parameters are taken as the parameter set 1 listed in Table 1.
the middle-right panel of Fig. 2, while the Hubble parameter is nearly a constant during this phase. Meanwhile, due to the asymmetry of the shifted Gaussian potential along the \(\chi\)-direction, inflaton will gain an initially tiny but slowly-increasing speed \(\dot{\chi}\), which is shown by the cyan curve in the bottom-right panel of Fig. 2. After reaching the apex on the Gaussian potential, the inflationary trajectory bends significantly, and then oscillates around the border of the Gaussian potential, the true inflation direction turns to the \(\chi\)-direction gradually. Then, inflaton is able to round this Gaussian potential, and the braking-turning phase ends. Subsequently, there is a second (c) single-field phase, in which the inflationary trajectory returns back to the \(\phi\)-direction, and ends inflation when the SR parameter \(\epsilon\) reaches unity. For two single-field phases (a) and (c) with \(\phi^{2/5}\) potential, the curvature perturbation is expected to be nearly scale-invariant and consistent with the Planck data, as shown in the Fig. 4. The major results of this paper come from the phase (b), so that we focus on this braking-turning phase in the following analysis.

2.2.1 Braking phase

USR inflation was originally studied by Ref. [89] (and similar topic was also discussed in Refs. [90, 91]), while the CR inflation was firstly investigated in Ref. [92]. The CR and USR phases are typically non-attractor\(^1\) and thus possess distinct features compared to the ordinary SR phase, in particular, the CR or USR is a single-field inflation but not a single-clock inflation, since the decaying mode of curvature perturbations could be dominated over the constant mode, and the long-wavelength curvature perturbations can no longer be regarded as a local time reparametrization of background. To be more precise, the long-wavelength comoving curvature perturbations evolve as \(R_k \simeq C_k + D_k \int_{-\infty}^{\tau} \frac{d \tau'}{a(\tau')}\), and thus the second term — “decaying mode” no longer decays once \(\eta \leq -3\). The direct consequence of this decaying mode is the growth of superhorizon curvature perturbations, and the “steepest” sustained growth rate\(^2\) \(k^4\) of curvature power spectrum in a single-field inflation has been found in Ref. [34]. Considering an additional CR phase \(\eta = -1\) in between the SR and USR phases, a steeper sustained growth rate \(k^5(\ln k)^2\) was reported in Ref. [78]. Recently, it has been shown in Ref. [44] that, the mass function of PBHs is insensitive to the steepness of the power spectrum when it is steeper than \(k^2\), but other relevant detections for PBHs (e.g., via SIGWs) may be sensitive to the shape of curvature spectrum which are needed to investigate further. In addition, Ref. [44] also pointed out several artefacts arose from unphysical instant transitions among SR, CR and USR phases that are usually considered in the literature. For example, the reduction of oscillations and the peak amplitude of the curvature power spectrum, and the latter one have a significant impact on the resulting PBH mass function. Moreover, the super-\(k^4\) sustained growth may be erased by considering more realistic smoothed transitions.

In our case, the quasi-CR phases occurred when the inflationary trajectory is almost along the one field direction (either \(\phi\) or \(\chi\)), as shown in the bottom panel of Fig. 2. Hence, we can apply the similar analytical calculations for the USR phase in the single-field inflation to our model. In what follows, we adopt the matching calculations presented in Ref. [34] to approximate the curvature power spectrum by considering instant transitions for simplicity (see Appendix B for details). To capture the essential features of this analytical approximation, we smooth the SR parameter \(\epsilon\) shown by the red dashed

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\(^1\)The conditions for the existence of an attractor behavior during the USR phase has been investigated in Ref. [93].

\(^2\)The “sustained” here means that such special growth can last multiply e-folds. The transient super-\(k^4\) growth were reported in Refs. [41, 94].
curve in the middle-right panel of Fig. 2, and we plot the second SR parameter $\eta$ in the left panel of Fig. 3. As a rough estimate, we consider five successive CR phases: $\eta = 0 \rightarrow \eta = -3 \rightarrow \eta = 0.5 \rightarrow \eta = 2 \rightarrow \eta = 0$, where the values inside the parentheses refer to the e-folding numbers of the corresponding CR phases. The estimated spectrum of the comoving curvature perturbations is shown by the thin green curve in Fig. 4, which is close to the numerical result (the black curve). This fact also implies that the braking phase is the major contribution to the comoving curvature perturbations in our model.

Moreover, one can also make estimates on the growth and decay slopes of spectra shown in Fig. 4 by following the treatments in Ref. [78]. At the end of inflation, all observable modes exit the horizon, and the growth and decay slopes of the final spectrum can therefore be estimated from its asymptotic expansion [i.e., from Eq. (B.4)] on the superhorizon scales in a single $\eta$-phase [78],

$$R_k \simeq Ak^{-\frac{3+\eta}{2}} + B(-\tau)^2 k^{\frac{1-\eta}{2}} + C(-\tau)^{3+\eta} k^{\frac{3+\eta}{2}},$$  \hspace{1cm} (2.13)$$

where $A, B, C$ refer to normalization constants determined by the initial conditions or matching conditions, the $A$- or $C$-term represents the constant mode that depends on the value of $\eta$, while the $B$-term always decays during inflation. In the SR phase, $\eta \simeq 0$, the $B$-term is the subleading decaying mode and would lead to $k^4$ scaling if it is dominated after the transition. This simple asymptotic expansion shows the origin of the $k^4$ growth [78]. The key point here is that, if the dominated mode in a $\eta$-phase before the transition is not constant, and then only this leading term could be imprinted on the solution after the transition, which will determine the growth or decay slope. On the other hand, if the dominated mode is constant before the transition, the final growth or decay slope after the transition is affected by the interplay among this constant mode and other growing modes. For $\eta \leq -3$, the dominated mode in the $\eta$-phase before the transition is not constant (i.e., the $C$ term, the growing mode in this case), and thus we can directly derive the index of the curvature power spectrum as

$$n_s - 1 = 3 - |3 + \eta|,$$

which is clearly to see the bound on the slope $n_s - 1 \leq 3$, and the scale-invariant spectra for $\eta = 0, -6^3$. For $\eta > -3$, the dominated mode in the expansion (2.13) is the $A$-term, and thus the decaying modes $B$- or $C$-term would be dominated after the transition, the growth rate can be determined as $n_s - 1 = 5 - |1 + \eta|$. Evidently, for $\eta = 0$, we would have $k^4$ growth after the transition, which is consistent with the above discussion. The above analyses based on the superhorizon asymptotic expansion (2.13) tell us two important things about curvature perturbations during the CR and USR phases such that, (i) the growing modes arise from subleading terms (usually decaying modes) before the transition; (ii) there is superhorizon growth of curvature perturbations. In Fig. 4, the growth slopes for the red, blue, black curves are around $n_s - 1 \simeq 3$, which are mainly determined by the interplay among three terms in the expansion (2.13) and other higher-order terms that are not shown explicitly in Eq. (2.13). For the scales of interest, the decay slopes are shown to be around $n_s - 1 \simeq 0.5$ in Fig. 4, which can be derived from the relation (2.14) by setting $\eta = 0.5$, since before the $\eta = 2$
Another generic feature stemmed from the cancellation between constant and decaying modes in Eq. (2.13) after transitions is the appearance of a dip before the growth in the power spectrum, as shown in Fig. 4. It has been shown in Ref. [34] that the dip position \( k_{\text{dip}} \) is given by \( k_{\text{dip}} = \frac{\tau_{\text{USR}}^{-3/2}}{\tau_{\text{USR}}} k_{\text{USR}} \), here \( \tau_{\text{USR}} = \tau_{\text{USR,start}}/\tau_{\text{USR,end}} \) is the duration of the USR phase, and \( k_{\text{USR}} \) is the scale exits horizon when the USR phase starts which is also regarded as the peak position roughly. This dip structure indeed implies the following superhorizon growth before the start of the USR phase. In our case, the dip position does not satisfy this relation, since the growth is not merely due to the individual decaying mode shown in the expansion (2.13), but their interplay. The numerical results tell us the relation \( k_{\text{dip}} \approx e^{-N_{\text{CR}}} k_{\text{USR}} \), where \( N_{\text{CR}} \) is the e-folding number of the CR phase following the first SR phase. In our case, the dip essentially comes from the cancellation mentioned above, but it is more physically transparent to understand via the “overshoot” argument [33]. We have to emphasize that, in a general case, the appearance of the dip can only be explained in part by the “overshoot” argument [34], and it works well in our model. Overshoot means that inflaton rolls down from the SR phase to the USR phase, its acceleration need to grow to some extent to overcome the potential gradient, so its speed and SR parameter will increase before their reduction (as shown in the middle-right panel of Fig. 2) and the resulting curvature power spectrum over that range of scales would be reduced. In order to suppress this dip, one direct way is to reduce the inflaton’s speed at the transition [33], which is consistent with the result shown in Fig. 4. When the initial position of inflaton is more closer to the Gaussian potential, and its speed is thus more lower at the transition, the corresponding dip will be more suppressed. In the USR case, the peak amplitude can be simply estimated by comparing the constant and growing modes [34, 78]. As we mentioned above, the growth of spectrum is not simply
determined in our model, we need to apply numerical calculations to obtain $k_{\text{dip}}$ at first, and then the peak amplitude is approximately to $P_R(k_{\text{USR}}) \sim P_{\text{CMB}} e^{\alpha N_{\text{CR}}}$, where $\alpha \simeq 3$ is growth rate of spectrum.

In this paper, we ignore the non-Gaussianity that may be generated in our model. In fact, one distinct feature of USR is the violation of Maldacena’s non-Gaussianity consistency relation in any attractor-single-field inflation, $f_{\text{NL}} = 5(1-n_s)/12^5$, which connects the squeezed limit of the bispectrum to the power spectrum [95]. Hence, the non-Gaussianity is negligible for the SR single-field inflation where $n_s \simeq 1$. In contrast, in the USR limit $\eta = -6$, the non-linearity parameter is found to be $f_{\text{NL}} = 5(3-n_s)/4$ even for any configuration [96], and it becomes $5/2$ when the power spectrum is scale-invariant [97]. However, the USR phase needs to be followed by a SR phase in order to solve the horizon and flatness problems without fine-tuning of an initial condition, and explain CMB observed primordial density perturbations. Considering the transition phase to an attractor phase after USR, Ref. [98] shown that the non-Gaussianity generated during the USR phase can be erased to some extent or even vanishes completely by this transition phase, while Maldacena’s consistency relation is still violated. However, some discrepancies of the non-Gaussianity associated with an non-attractor phase have been reported in the literature [35, 99–101], in addition, the quantum stochastic effect associated with USR phase may also have an impact on the enhancement of curvature perturbations [102–104]. In light of the above mentioned uncertainties and the complexity in two-field inflation, we ignore the non-Gaussianity and quantum stochastic effect in this paper for simplicity, and focus on the growth of curvature perturbations from transitions in between different constant-$\eta$ phases.

| Parameter Sets | Results |
|----------------|---------|
| $m^2/(\lambda M_{\text{Pl}}^2)$ | $\Lambda^3/(\lambda M_{\text{Pl}}^2)$ | $\phi_c/M_{\text{Pl}}$ | $\chi_c/M_{\text{Pl}}$ | $\sigma_{\phi}/M_{\text{Pl}}$ | $\sigma_{\chi}/M_{\text{Pl}}$ | $\lambda$ | $n_s$ | $r$ |
| 1 | 2 | 2 | 1.90 | 2.63 x 10^{-5} | 0.030 | 0.030 | 6.89 x 10^{-10} | 0.9674 | 0.044 |
| 2 | 2 | 2 | 2.73 | 3.10 x 10^{-5} | 0.092 | 0.024 | 7.36 x 10^{-10} | 0.9645 | 0.048 |
| 3 | 2 | 2 | 3.55 | 1.8 x 10^{-5} | 0.120 | 0.020 | 7.80 x 10^{-10} | 0.9639 | 0.049 |

Table 1. Parameter sets in our model for a fixed peak amplitude of curvature spectra $P_R \sim 10^{-2}$ as shown in Fig. 4. The corresponding results of dimensionless parameter $\lambda$, the power index $n_s$ and the tensor-to-scalar ratio $r$ are also presented.

2.2.2 Turning phase

It can be clearly seen from the right panel of Fig. 3 (corresponding to the top-right panel of Fig. 2) that, there exist sudden turnings during $\eta \simeq -3$ phase. This strongly non-geodesic motion with $\eta_1^2 \gg 1$ in the field space has been investigated in an extensive literature [105–114], mainly motivated by ultraviolet completions of inflation scenario. On the superhorizon scales, the master equations (2.11) and (2.12) can be greatly simplified as [115]

$$\dot{R} \simeq -2\Omega \frac{H}{\dot{\varphi}_0} F, \quad \frac{\chi}{\dot{\varphi}_0} F + 3H\dot{F} + M_{\text{eff}}^2 F \simeq 0,$$

where the effective mass of the entropy field is defined as $M_{\text{eff}}^2 = M^2 + 4\Omega^2$. From the above two equations, we clearly see that adiabatic modes are sourced by entropy modes on the

\[ ^5 \text{Note that the nonlinearity parameter } f_{\text{NL}} \text{ here is defined as } \left< R_{k_1} R_{k_2} R_{k_3} \right> \equiv (2\pi)^3 \delta(k_1 + k_2 + k_3) \frac{1}{2} f_{\text{NL}} P_R(k_1) P_R(k_3) \text{ in the squeezed limit } k_1 \ll k_2 = k_3. \]
superhorizon scales while entropy modes evolve independently. A sudden turning would give rise to a negative large entropy mass square $M^2/H^2$ defined in Eq. (2.10), which signifies a transient tachyonic instability of subhorizon entropy modes $F_k$ for $k^2/a^2 \lesssim |M^2|$. Through the kinematic coupling, shown in the action (2.9), this exponential growth of $F_k$ would transfer to the adiabatic modes $R_k$, resulting in an exponential growth of curvature perturbations on relevant scales. On the other hand, this sudden turning contributes a positive large effective entropy mass $M_{\text{eff}}^2/H^2$ on the superhorizon scales, implying a stable background and a rapid decay of superhorizon entropy modes $F_k$, so that the adiabatic perturbation would be still conserved on the superhorizon scales. This fact can be understood from the effective field theory (by integrating out the heavy entropy field) with an imaginary sound speed $c_s^2 \equiv M^2/M_{\text{eff}}^2$ [106, 110, 114, 116], which is a manifestation of the exponential growth of curvature perturbations before the sound horizon crossing. One can design a turning rate to generate PBHs via the resulting enhanced curvature spectrum [26–28, 117]. Hence, in our model, several sudden turnings occurred during $\eta \approx -3$ phase would contribute the spikes around the peak position $k_{\text{USR}}$ of spectrum as shown in Fig. 4, and do not have significant contributions to spectrum on other scales.

![Figure 4](image_url)

**Figure 4.** The numerical results of power spectra of the comoving curvature perturbations at the end of inflation. The black, blue, and red curves respectively refer to parameter sets 1-3 listed in Table 1. The current constraints on $P_R(k)$ from Planck [88], Lyman-α [118], FIRAS [119] and PTA [34] are shown by the shadowed regions, while the grey dashed line refers to $P_R \sim 10^{-2}$ in order to produce an abundance of PBHs. The thin green curve denotes the analytical results from the matching calculations.

3 Primordial black holes and scalar-induced gravitational waves

3.1 PBH abundance

The large density contrast $\delta \equiv \delta \rho/\bar{\rho}$ on the small scales can cause PBH formation and the SIGWs at the horizon reentry. As we mentioned above, the comoving curvature perturbation
\( \mathcal{R} \) is assumed to follow the Gaussian distribution in our model, the same is true for the density contrast on the comoving slicing,

\[
\delta_k = \frac{2}{3} \left( \frac{k}{aH} \right)^2 \Phi_k \simeq 4 \left( \frac{k}{aH} \right)^2 \mathcal{R}_k ,
\]

where \( \Phi \) is the Bardeen potential in the Newtonian gauge, and we ignore the anisotropic stress of matter sector for simplicity. We have used the relation \( \Phi \simeq \frac{2}{3} \mathcal{R} \) on the superhorizon scales in Eq. (3.1). According to the gravitational instability, when the density contrast \( \delta \) exceeds the threshold \( \delta_c \), the overdense region will stop expansion and collapse to a BH. There are continuous studies on the typical value of the density contrast and it is suggested by the recent study [120] that \( 0.4 \lesssim \delta_c \lesssim 0.7 \), which depends on the shape of the superhorizon power spectrum. Here, we take a conservative value \( \delta_c = 0.6 \) in this paper. The initial mass function \( \beta(M) \) at formation epoch is calculated as

\[
\int \beta(M) d\ln M = \frac{\rho_{\text{PBH}}}{\rho_c} ,
\]

where \( \rho_{\text{PBH}} \) and \( \rho_c \) are energy densities of PBHs and background at formation epoch, respectively. The simplest and straightforward way to estimate the PBH abundance is to use the Press-Schechter formalism [121], and we obtain\(^6\)

\[
\beta(M) = 2 \int_{\delta_c}^{\infty} P(\delta_R) d\delta_R = \text{erfc} \left[ \frac{\delta_c}{\sqrt{2} \sigma_R} \right] ,
\]

where \( \text{erfc} \) is the complementary error function, and we use the Gaussian distribution function of density fluctuations \( P(\delta_R) = \frac{1}{\sqrt{2\pi} \sigma_R} \exp \left( -\frac{\delta_R^2}{2\sigma_R^2} \right) \) in Eq. 3.3. Note that \( \delta_R \) is the smoothed density field, \( \delta_R(x) \equiv \int d^3x' W(x - x'; R) \delta(x') \), where \( W(x - x'; R) \) is a window function with a characteristic comoving smoothing scale \( R \), and the associated mass is given by \( M \equiv \frac{4\pi}{3} \rho_c (aR)^3 \). In the case of PBH formation, the comoving smoothing scale is usually chosen as the comoving Hubble radius, \( R = (aH)^{-1} \), and the PBH formation is thus related to the Horizon-crossing \( k \) mode of density perturbations, \( M \simeq M_\odot (k/1.9 \times 10^6 \text{Mpc}^{-1})^{-2} \) [125], where \( M_\odot \simeq 2 \times 10^{33} \text{ g} \) is the solar mass. In this paper, we choose the spherically symmetric real-space top-hat window function\(^7\):

\[
W(r; R) = \frac{3}{4\pi R^3} \Theta(R - r) ,
\]

\[
W(k; R) = \frac{1}{(2\pi)^{3/2}} \frac{3 (\sin(kR) - kR \cos(kR))}{(kR)^3} ,
\]

where \( \Theta \) is the Heaviside step function. It is known that the required amplitude of density perturbations for a fixed PBH abundance is smallest for the real-space top-hat window function, compared with the Gaussian and k-space top-hat window functions [126]. The variance of the smoothed density field is calculated as

\[
\sigma_R^2 \equiv \langle \delta_R^2 \rangle = \frac{1}{(2\pi)^3} \int \frac{dk}{k} W(kR)^2 \mathcal{P}_\delta(k) ,
\]

\(^6\)Note that what we calculate here is actually the cumulative probability for a PBH formed with mass \( > M \), and the “rigorous” expression for the mass function is given by \( \beta(M) = \frac{\rho_{\text{PBH}}}{\rho_c} \sqrt{\frac{2}{\pi}} \nu_R \exp \left( -\nu^2/2 \right) \frac{4 \sin \nu_R}{\nu_R} \frac{\sin \nu_R}{\nu_R} \) [122–124], where \( \nu_R \equiv \delta_c/\sigma_R \). However, the mass function of PBHs is always calculated using Eq. (3.2) in PBH community, and we follow this convention in this paper for a conservative estimate for PBH abundance.

\(^7\)Note that the factor \((2\pi)^{-3/2}\) appearing here is due to our Fourier transformation convention.
where \( \mathcal{P}_\delta(k) \equiv \frac{k^3}{2\pi^2} \int d^3r e^{-i k \cdot r} (\delta(x) \delta(x+r)) \), and we have the relation \( \mathcal{P}_\delta(k) = \frac{16}{81} \left( \frac{k}{\pi H} \right)^4 \mathcal{P}_R(k) \) by using Eq. (3.1). Assuming the adiabatic background expansion after PBH formation, one can relate the initial PBH abundance to the current energy fraction \([125]\):

\[
 f_{PBH}(M) \simeq 2.7 \times 10^8 \left( \frac{M}{M_\odot} \right)^{-1/2} \beta(M) .
\]  

(3.7)

We plot \( f_{PBH}(M) \) in Fig. 5 in terms of three parameter sets listed in Table 1. For the parameter set 1, PBHs produced in our model could be the whole dark matter, while as for the parameter set 3, PBHs can account for the LIGO/Virgo GW events. The produced PBHs in terms of the parameter set 2 can explain OGLE ultrashort-timescale microlensing events [127].

\[0.55\text{cm} \times 0.55\text{cm}

Figure 5. Three current energy fractions of PBHs \( f_{PBH}(M) \) in our model in terms of parameter sets listed in Table 1 with various constraints on \( f_{PBH}(M) \) adopted from Ref. [3]. The brown shaded region represents the PBH abundance inferred by the OGLE ultrashort-timescale microlensing events [127].

3.2 SIGWs and multi-frequency GW experiments

The overlarge density perturbations also induce the SIGWs when their modes re-enter the Hubble horizon during the radiation domination (RD). There is an extensive literature on this direction, thus we here present the master equations and the details of derivation can be found, e.g., in Refs. [30, 133–143]. In the Newtonian gauge, the perturbed metric is written as

\[
ds^2 = a(\tau)^2 \left\{ -(1 + 2\Phi) d\tau^2 + \left[ (1 - 2\Phi) \delta_{ij} + \frac{1}{2} h_{ij} \right] dx^i dx^j \right\} ,
\]

(3.8)

where the first-order scalar perturbation is described by \( \Phi \) and we ignore the anisotropic stress at the linear order for simplicity. The secondary tensor modes \( h_{ij} \), namely the SIGWs, can be induced by the nonlinear couplings between \( \Phi \). The dynamics of SIGWs in Fourier space is given by

\[
h_k^{\mu'}(\tau) + 2 \mathcal{H} h_k^{\mu}(\tau) + k^2 h_k^{\lambda}(\tau) = S_k^{\lambda}(\tau) ,
\]

(3.9)
where the source term $S^λ_κ(τ)$ during RD is given by

$$S^λ_κ(τ) = \frac{4}{(2\pi)^{3/2}} \int \delta^3(p) \exp(i(k \cdot p)) \left[ 3\Phi^λ_κ(τ)\Phi_{κ-p}(τ) + \mathcal{H}^{-2}\Phi^′_κ(τ)\Phi_{κ-p}(τ) + \mathcal{H}^{-1}\Phi^′_κ(τ)\Phi_{κ-p}(τ) + \mathcal{H}^{-1}\Phi_κ(τ)\Phi^′_{κ-p}(τ) \right].$$

Figure 6. The current energy spectra $Ω_{GW}(τ_0,f)$ of the SIGWs in terms of three parameter sets listed in Table 1, with various expected sensitivity curves of the forthcoming GW experiments, e.g. SKA [128], LISA [129], Taiji [130], DECIGO [131] and BBO [132]. The gray vertical bars refer to the NANOGrav detection data [79].

where the source term $S^λ_κ(τ)$ during RD is given by

$$S^λ_κ(τ) = \frac{4}{(2\pi)^{3/2}} \int \delta^3(p) \exp(i(k \cdot p)) \left[ 3\Phi^λ_κ(τ)\Phi_{κ-p}(τ) + \mathcal{H}^{-2}\Phi^′_κ(τ)\Phi_{κ-p}(τ) + \mathcal{H}^{-1}\Phi^′_κ(τ)\Phi_{κ-p}(τ) + \mathcal{H}^{-1}\Phi_κ(τ)\Phi^′_{κ-p}(τ) \right].$$

where $\Phi^λ_κ(τ)$ is a shorthand for $\exp(i(k \cdot p))$. Using the Green function method, $h^λ_κ(τ) = \int_τ^\infty dτ' g_κ(τ,τ') S^λ_κ(τ')$, to solve the EoM (3.9), one obtains the total power spectrum (i.e., $\mathcal{P}_h = \sum_{λ=+,−} \mathcal{P}^λ_h$) for SIGWs as

$$\mathcal{P}_h(τ,k) = \int_0^∞ dv \int_{[1+v]}^{[1-v]} du \left[ \frac{4v^2 - (1 + v^2 - u^2)^2}{4uv} \right]^2 I^2(v,u,x) \mathcal{P}_κ(κu)\mathcal{P}_κ(κv).$$

where we define $u \equiv |k - p|/k, v \equiv p/k$ and $x \equiv κτ$. The kernel function $I^2(v,u,x)$ describes the evolution information on the source and can be calculated as [137, 138],

$$I^2(v,u,x) = \overline{I^2(v,u,x)} = 4\left(\frac{3(u^2 + v^2 - 3)}{4u^3v^3} \right)^2 \left[ -4uv + (u^2 + v^2 - 3) \ln \left( \frac{3 - (u + v)^2}{3 - (u - v)^2} \right) \right]^2 + \pi^2(u^2 + v^2 - 3)^2 \Theta(v + u - \sqrt{3}) \right],$$

where the overline denotes the time average. Note that the large limit $x \to ∞$ is taken above, since we are interested in the present GW spectrum. With the canonical definition of GW’s energy density [144–150], $ρ_{GW}(τ,x) = \frac{M^2}{16\pi} (h^′_i(τ,x)h^′j(τ,x))^8$, one can relate the GW energy spectrum to its power spectrum in the following form,

$$Ω_{GW}(τ,k) = \frac{1}{48} \left( \frac{k}{aH(τ)} \right)^2 \mathcal{P}_h(τ,k).$$

Note that the prefactor $1/2$ in the metric perturbations (3.8) is counted in the total GW energy.
The GW energy density starts to decay relative to matter energy density after the radiation-matter equality $\tau_{eq}$, the GW spectrum observed today is given by [151]

$$\Omega_{GW}(\tau_0, f) h^2 \simeq 1.6 \times 10^{-5} \left( \frac{g_{*,s}}{106.75} \right)^{-1/3} \left( \frac{\Omega_{r,0} h^2}{4.1 \times 10^{-5}} \right) \Omega_{GW}(\tau_{eq}, f), \quad (3.14)$$

where $\Omega_{GW}(\tau_{eq}, f)$ can be calculated by Eq. (3.13) at the radiation-matter equality, with the physical frequency $f = k/(2\pi a_0) \simeq 1.5 \times 10^{-9}(k/1\text{ pc}^{-1})$ Hz. Since the scalar perturbations damp quickly inside the subhorizon during radiation, a majority of SIGWs is thus produced just after the source reenters the horizon [138]. $\Omega_{GW}(\tau_0, f)$ are shown in Fig. 6 in terms of three parameter sets listed in Table 1. The predicted SIGWs could be detectable by the future GW observations. Note that as for the parameter set 3, the consequent SIGWs (red) can be matching with the NANOGrav detection data [79].

4 Conclusions

PBHs are served as a promising tool to test the early Universe physics, especially for detecting the small-scale nontrivial physical phenomena. The enhancement of PBH formation can be realized within inflationary Universe by introducing extra fields or non-trivial background dynamics. One direct way is to slow down inflaton’s speed in a short period, which is the case for a flatten or a bump-like potential. In this paper, we develop a two-field inflation with a smooth Gaussian potential which acts as a two-dimensional bump on the top of the $\phi^{2/5}$ potential, the braking and turning of inflationary trajectory can be easily realized when inflaton rounds this potential bump. Compared with previous works on similar topics, one merit of this two-dimensional Gaussian potential is that without too many fine-tunings of initial conditions, inflaton generally undergoes the braking phases that can result in the growth of comoving curvature perturbations. We perform the matching calculations to the braking phase (i.e., approximated by five successive CR phases), and obtain the estimates of several features of the curvature power spectrum at the end of inflation, including the growth rate, dip scale and peak amplitude. It turns out that the braking phase provides the major contribution to the spectrum, while the turning phase contributes the additional spikes around the peak amplitude since their durations are too short to have a significant impact on overall influence.

Our paper provides a simple example of the generalization of a single-field bump-like potential to a two-field inflation, which can be free from the fine-tuning issue to some extent. There are also several extensions of this work that need to be investigated further. For example, there are several works appeared recently and show the potential significant influences on PBH formation model and SIGWs by considering quantum [152, 154, 155] or classical [153] one-loop corrections in detail. Similar one-loop corrections can also be considered in our model. In additional, we ignore the non-Gaussianity and quantum stochastic effects in our paper for simplicity, one may include these effects to give a more precise result for PBH formation with our Gaussian bump potential. Moreover, if the Gaussian bump is symmetric or slightly asymmetric with respect to the inflationary trajectory, one has to consider both trajectories $A$ and $B$ shown in Fig. 1, which may also affect the PBH abundance and also their spatial clustering. We leave these interesting questions in the follow-up works.
A Covariant formalism of multifield dynamics during inflation

Here, we briefly review the so-called covariant formalism of the multifield dynamics (in particular of two fields) during inflation, which is a covariant treatment of field perturbations and applicable to the curved field space, one can find the detailed discussions in Refs. [83–87]. We start from a generic action consisting of gravity and \( n \) scalar fields \( \phi^a \) (\( a = 1, \ldots, n \)):

\[
S = \int d^4x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R - \frac{1}{2} G_{ab}(\phi) \phi^{\mu \nu} \partial_\mu \phi^a \partial_\nu \phi^b - V(\phi) \right],
\]

where \( G_{ab}(\phi) \) is the symmetric and positive-defined metric of the scalar manifold \( \mathcal{M} \) spanned by \( \phi^a \). Note that Latin indices \( a, b, \ldots \) and the Greek indices \( \mu, \nu, \ldots \) refer to the field space metric \( G_{ab}(\phi) \) and the spacetime metric \( g_{\mu \nu} \), respectively. The set of scalar fields \( \phi^a \) can be thought of coordinates in the \( n \)-dimensional manifold \( \mathcal{M} \), and this allows us to define several standard geometrical quantities related to \( \mathcal{M} \). The Christoffel symbol of field space is analogously defined as \( \Gamma^a_{bc} = \frac{1}{2} G^{ad} (\partial_c G_{db} + \partial_b G_{dc} - \partial_d G_{bc}) \), where \( G^{ad} \) is the reversed metric satisfying \( G^{ad} G_{ab} = \delta^a_b \), and \( \partial_a \) denotes a partial derivative with respect to \( \phi^a \). Naturally, the associated covariant derivative \( \nabla_a \) is defined as \( \nabla_a X_b = \partial_a X_b - \Gamma^c_{ab} X_c \). The Ricci scalar can be derived from Eq. (A.1) with respect to \( \phi^a \):

\[
\nabla_\mu \nabla^\mu \phi^a + \Gamma^a_{bc} \nabla_\mu \phi^b \nabla^\mu \phi^c - V^a = 0 ,
\]

where \( V^a \equiv G^{ab} V_b \) and \( V_b \equiv \partial_b V \).

In the spatially flat FLRW Universe,

\[
ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j ,
\]

and the EoMs for homogeneous fields \( \phi_0^a(t) \) can be derived from Eq. (A.2),

\[
D_t \dot{\phi}_0^a + 3H \dot{\phi}_0^a + V^a = 0 ,
\]

where \( D_t X^a \equiv \dot{X}^a + \Gamma^a_{bc} X^b \dot{\phi}_0^c \). The Friedmann equation tells the evolution of the scalar factor \( a(t) \) in terms of the energy density of scalar fields \( \phi_0^a \):

\[
H^2 = \frac{1}{3M_{Pl}^2} \left[ \frac{1}{2} \dot{\phi}_0^2 + V(\phi_0) \right],
\]

where \( \dot{\phi}_0^2 = G_{ab}(\phi_0) \dot{\phi}_0^a \dot{\phi}_0^b \) is the change rate of the scalar field vacuum expectation value along the trajectory. Once the initial values of scalar fields \( \dot{\phi}_0^a \) are chosen, the trajectory of \( \{ \phi_0^a \} \)

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on the manifold $\mathcal{M}$ are determined through the above mentioned equations. A key feature of multifield models is that the trajectories are not necessarily the geodesics of $\mathcal{M}$, which usually deviate due to the presence of potential $V(\phi)$.

A given background evolution of $\phi(t)$ defines a trajectory in the field space parametrized by time $t$. Before going into the details of solving these coupled equations, it is very useful to define the unit tangent vector $T^a(t)$ along the trajectory as

$$T^a = \frac{\dot{\phi}_0}{\dot{\phi}_0},$$

while the unit normal vector is defined as

$$N_a = s_N(t)\frac{D_t T^a}{\sqrt{G_{bc} D_t T^b D_t T^c}},$$

where $s_N(t) = \pm 1$, denoting the orientation of $N^a$ with respect to $D_t T^a$. For the two-field case, we obtain

$$N_a = \sqrt{\det G} \varepsilon_{ab} T_b,$$

such that $N^a T_a = 0, N_a N^a = 1$. $\varepsilon_{ab}$ is the Levi-Civita symbol with $\varepsilon_{11} = \varepsilon_{22} = 0, \varepsilon_{12} = -\varepsilon_{21} = -1$. This choice fixes the orientation (right-handed) of $N_a$ with respect to the trajectory. The explicit expressions for $T^a$ and $N^a$ can be expressed as

$$T^a = \frac{1}{\dot{\phi}_0}(\dot{\phi}_0, \dot{\phi}_0^2), \quad N^a = \frac{1}{\sqrt{\det G}}(-G_{22} T^2 - G_{12} T^1, G_{21} T^2 + G_{11} T^1).$$

Projecting the EoM (A.4) onto the tangent and normal directions, which gives two independent equations,

$$\ddot{\phi}_0 + 3H \dot{\phi}_0 + V_T = 0, \quad D_t T^a = -\frac{V_N}{\dot{\phi}_0} N^a,$$

where $V_T = T^a V_a$ and $V_N = N^a V_a$ are tangent and normal projection of the derivative $V_a$, respectively. One can simply verify that $V_a$ is entirely spanned by $T_a$ and $N_a$, i.e., $V_a = V_T T_a + V_N N_a$. One can derive the following relations,

$$D_t T^a = -\Omega N^a, \quad D_t N^a = \Omega T^a,$$

It is obvious that if $\eta_\perp = 0$ then the vectors $T^a$ and $N^a$ would remain constant along the path, the trajectory would not turn; if $\eta_\perp > 0$, the turn towards right and if $\eta_\perp < 0$, the turn towards left, and we have sign$(\eta_\perp) = -s_N$.

**B The matching calculations for constant-$\eta$ phases**

In what follows, we present analytical matching calculations of sudden transitions among various constant-$\eta$ phases, mainly follow the treatments in Ref. [34], and similar calculations can be found in Refs. [41, 78]. The Mukhanov-Sasaki equation for comoving curvature perturbations is written as $v''_k + (k^2 - z''/z) v_k = 0$, where $v_k = z \mathcal{R}_k$ with $z^2 = 2a^2 M_{Pl}^2 \epsilon$. It can be shown that

$$\frac{z''}{z} = (aH)^2 \left(2 - \epsilon + \frac{3}{2} \eta + \frac{1}{4} \dot{\eta}^2 - \frac{1}{2} \epsilon \eta + \frac{1}{2} \frac{\dot{\eta}}{H}\right),$$

- 17 -
which is the exact expression to all orders. Considering \( \epsilon \ll 1 \) and \( \eta \) a constant, we drop higher-order terms, the Mukhanov-Sasaki equation becomes

\[
v''_k + \left( k^2 - \frac{\nu^2 - 1/4}{\tau^2} \right) v_k = 0 , \tag{B.2}\]

where \( \nu = \frac{3+\eta}{2} \). Thus, the general solution of \( R_k(\tau) \) is given by

\[
R_k(\tau) = \frac{\sqrt{-\tau}}{a(\tau) M_{\text{Pl}} \sqrt{2} \epsilon} \left[ A H^{(1)}_{\nu}(\tau) + B H^{(2)}_{\nu}(\tau) \right] , \tag{B.3}\]

with the constants \( A \) and \( B \) that are determined by initial conditions. For example, the initial condition is chosen be the Bunch-Davies vacuum state, \( \lim_{\tau \to -\infty} v_k \simeq \frac{1}{\sqrt{2\epsilon}} e^{-ik\tau} \), and then we obtain

\[
R_k(\tau) = \frac{\sqrt{\pi}}{2a(\tau) M_{\text{Pl}} \sqrt{2} \epsilon} e^{i(\nu+\frac{1}{2})\frac{\tau}{2}} \sqrt{-\tau} H^{(1)}_{\nu}(\tau) , \tag{B.4}\]

where it is appropriate to use the de Sitter approximation \( a(\tau) = -\frac{1}{H\tau} \). In each constant-\( \eta \) phase, we have the solution (B.3) and the coefficients \( A \) and \( B \) are determined by the matching condition with the previous phase. We expect that there is no energy jump in the background between two phases, so that the comoving curvature perturbation and its first time derivative are continuous at the transition point [34],

\[
[R_k]_\pm = 0 , \quad [R'_k]_\pm = 0 , \tag{B.5}\]

which is also called the Israel junction condition [156, 157]. The final analytical results of curvature power spectrum for the transitions \( \eta = 0 \to \eta = -3 \to \eta = 0.5 \to \eta = 2 \to \eta = 0 \), considered in this paper, are tedious long and is meaningless to present exact expressions. Without loss of generality, we present the formalism for two typical sudden transitions \( 0 \to \eta \) and \( \eta_1 \to \eta_2 \).

**Transition 0 \to \eta:** In the first SR phase, the evolution of comoving curvature perturbation derived from Eq. (B.4) is given by

\[
R_k(\tau) = i \frac{H}{M_{\text{Pl}} \sqrt{\epsilon_1 k^3}} e^{-ik\tau} (1 + ik\tau) , \tag{B.6}\]

where \( \epsilon_1 \) is the SR parameter in this SR phase. The solution of \( R_k \) in the second \( \eta \)-phase is given by Eq. (B.3) with using the corresponding SR parameter \( \epsilon_2(\tau) = \epsilon_1 (\tau_1/\tau)^\eta \), where \( \tau_1 \) is the time of the sudden transition. Using the matching condition (B.5), we calculate the coefficients as

\[
A = -\frac{e^{-ik\tau_1} \pi}{4\sqrt{2\sqrt{-\tau_1} k}} \left[ (1 + ik\tau_1) H^{(2)}_{\frac{\nu_1}{2}}(-k\tau_1) + k\tau_1 H^{(2)}_{\frac{\nu_1}{2}-2}(-k\tau_1) \right] , \tag{B.7}\]

\[
B = \frac{e^{-ik\tau_1} \pi}{4\sqrt{2\sqrt{-\tau_1} k}} \left[ (1 + ik\tau_1) H^{(1)}_{\frac{\nu_1}{2}}(-k\tau_1) + k\tau_1 H^{(1)}_{\frac{\nu_1}{2}-1}(-k\tau_1) \right] . \tag{B.8}\]

**Transition \( \eta_1 \to \eta_2 \):** In these two phases, their solutions are both given by Eq. (B.3), and we use subscripts to distinguish each pair of coefficients, i.e., \( (A_1, B_1) \) and \( (A_2, B_2) \). Similarly,
using the junction condition (B.5), we obtain

\[
A_2 = \frac{1}{k \tau_1} \left[ \left( \frac{H^{(1)}_{\eta + \frac{2}{3}}}{\eta_{1/5}} (k \tau_1) - H^{(1)}_{\eta + \frac{2}{5}} (k \tau_1) \right) H^{(2)}_{\eta + \frac{2}{3}} (k \tau_1) + \frac{H^{(1)}_{\eta + \frac{2}{3}}}{\eta_{1/5}} (k \tau_1) \left( H^{(2)}_{\eta + \frac{2}{5}} (k \tau_1) - H^{(2)}_{\eta + \frac{2}{5}} (k \tau_1) \right) \right]^{-1} \\
\times \left[ +k_1 H^{(2)}_{\eta + \frac{2}{5}} (k \tau_1) \left( A_1 H^{(1)}_{\eta + \frac{2}{3}} (k \tau_1) - A_1^{-1} H^{(1)}_{\eta + \frac{2}{5}} (k \tau_1) + B_1 H^{(2)}_{\eta + \frac{2}{3}} (k \tau_1) - B_1^{-1} H^{(2)}_{\eta + \frac{2}{5}} (k \tau_1) \right) \right] .
\] (B.9)

\[
B_2 = \frac{1}{k \tau_1} \left[ \left( H^{(1)}_{\eta + \frac{2}{3}} (k \tau_1) - H^{(1)}_{\eta + \frac{2}{5}} (k \tau_1) \right) H^{(2)}_{\eta + \frac{2}{3}} (k \tau_1) + \frac{H^{(1)}_{\eta + \frac{2}{3}}}{\eta_{1/5}} (k \tau_1) \left( H^{(2)}_{\eta + \frac{2}{5}} (k \tau_1) - H^{(2)}_{\eta + \frac{2}{5}} (k \tau_1) \right) \right]^{-1} \\
\times \left[ -k_1 B_1 H^{(2)}_{\eta + \frac{2}{5}} (k \tau_1) \left( H^{(1)}_{\eta + \frac{2}{3}} (k \tau_1) - \eta_{2} B_1 H^{(2)}_{\eta + \frac{2}{3}} (k \tau_1) H^{(1)}_{\eta + \frac{2}{5}} (k \tau_1) + k_B \eta_{1} H^{(2)}_{\eta + \frac{2}{5}} (k \tau_1) \right) \right] .
\] (B.10)

With the choice of parameters \((A_1, B_1) = (\frac{\sqrt{\pi}}{2}, 0)\) [i.e., referring to the SR solution (B.6)], the above two coefficients indeed reduce to the expressions (B.7) and (B.8). As we mentioned in the text, inflation need to return back to the SR phase before its end. For this last SR phase, it is conventional to rewrite the solution (B.3) as

\[
\mathcal{R}_k(\tau) = i \frac{H}{M_{Pl}} \frac{1}{\sqrt{4\epsilon k^3}} \left[ A(1 + ik\tau)e^{-ik\tau} - B(1 - ik\tau)e^{ik\tau} \right] ,
\] (B.11)

where \(A\) and \(B\) are determined through the above matching calculations, \(\epsilon\) is the corresponding SR parameter. Hence, the final power spectrum can be expressed as

\[
\mathcal{P}_R(k) = \lim_{\tau \to 0^+} \frac{k^3}{2\pi} |\mathcal{R}_k|^2 = \frac{H^2}{8\pi^2 M_{Pl}^2 \epsilon} |A - B|^2 .
\] (B.12)

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