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Durability Reliability Demonstration Test Methods

Yung-Li Leea,*, Sandeep Makama, Sean McKelveya, Ming-Wei Lua

aFCA US LLC, 800 Chrysler Drive, Auburn Hills MI 48326, USA

Abstract

A complete product design cycle should include the following three stages: target setting, product design and development, and product validation. In the first target setting stage, it is required to understand the voice of customers (VOC), define the extreme customer usage conditions, acquire the duty cycle or customer usage data for the target, and cascade the top-down VOC targets from vehicle to system and component levels. In the second stage of product design and development, the multi-objective design optimization in the virtual analysis domain needs to be performed, where the analytical tools to design the products for various design criteria should have been developed and in place for analyses. In the final product validation stage, the test methods with a limited sample size are required to demonstrate the reliability of the product population for meeting the specific design functional and performance requirements. Since the test results exhibit significant statistical variations, these test methods (named the reliability demonstration test methods) are the most important mechanism to ensure a successful product launch for durability, reliability and quality. Again, due to the limited sample size for testing, a specific reliability and confidence target should be predetermined to be demonstrated by these test methods.

The focus of this paper is to review the theoretical background and discuss the pros and cons of commonly used reliability demonstration test methods. Depending on the failure criterion in product validation testing, these reliability demonstration test methods can be categorized into the methods for non-repairable and repairable systems. For a non-repairable system, the test is considered complete, censored or suspended as the system fails, while a repairable system allows the actions required to restore/renew a failed system to operational status. The methods of interest for non-repairable systems include the test-to-failure method, the test-to-bogey method, the extended life test bogey method, and the step-stress accelerated life test method. Repairable systems are often modeled with the non-homogeneous Poisson process which includes the Duane and the Crow reliability growth models.

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* Corresponding author. Tel.: +1-248-877-5180.
E-mail address: yung.lee@fcagroup.com
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1. Introduction

| Nomenclature | Description |
|--------------|-------------|
| AMSAA | Army Material Systems Analysis Activity |
| C | Lower bound confidence |
| CB | Two-sided confidence bounds on a variable |
| C(t) | Cumulative failure rate at time or life t |
| cov(β,θ) | Covariance of β and θ |
| E[N(t)] | Expected (average) number of failures [N(t)] within the test interval |
| E(G) | Mean of a function G |
| [F] | Fisher information matrix |
| [F]^{-1} | Inverse of Fisher information matrix = Covariance matrix |
| F(t) | Cumulative distribution function of time or life t |
| F(i) | Discrete cumulative distribution formula |
| L(β,θ|t_1,t_2,...,t_n) | Likelihood function |
| L_q | Likelihood function of the q^{th} system |
| LR | Likelihood ratio function |
| M(t) | Cumulative mean time between failures at time or life t |
| MTBF | Mean time between failures |
| NHPP | Non-homogeneous Poisson process |
| N_q | Total number of failure experienced by the q^{th} system |
| N(t) | Cumulative number of failure up to time t |
| 2N_f | Number of reversals to failure |
| R(t) | Reliability function of time or life t |
| S_{a_i} | m levels of stress amplitudes; \( i = 1,2,\ldots,m \) |
| S_f | Fatigue strength coefficient |
| T_q | Ending (or current) time of the q^{th} system |
| T_i | Suspended life data; \( i = 1,2,\ldots,k \) |
| var(β) | Variance of a variable β |
| b | Fatigue strength exponent |
| c(t) | Instantaneous failure intensity or rate at time or life t |
| f(t) | Probability density function of time or life t |
| f(t_1,t_2,...,t_n | β,θ) | Joint probability density function of \( t_1,t_2,...,t_n \) |
A successful product design development process needs to serve as the foundation of how a company will develop products that will delight our customers and improve their loyalty by providing value through optimization of quality, reliability and durability. A good product should be designed based on the voice of customers and be met with customers’ satisfaction. Many different product development processes have been developed over the decades, specifying a series of activities that make up the system engineering approaches. Among these models, the system V model first developed in the 1980s is actually a systematic engineering process, and is merging as a standard product development roadmap for complex products. A simplification for this V model can be illustrated in Figure 1, showing the process is composed of four key processes:

- **Cap** – System engineering management over the entire life cycle for integration management and configuration control.
- **Left leg** – Requirements and architecture (top-down activity), defining requirement and the product to be built.
- **Rungs** – Design and development (horizontal activity), managing the system element development process.

---

| Symbol | Description |
|-------|-------------|
| \( f(t_{i,q} | t_{i-1,q}) \) | Probability density function of \( i^{th} \) event at \( t_{i,q} \) given that the \((i-1)^{th}\) event survived \( t_{i-1,q} \) for \( q^{th} \) system |
| \( h(t) \) | Weibull hazard function |
| \( k \) | Number of suspended data |
| \( m(t) \) | Instantaneous MTBF at time or life \( t \) |
| \( n \) | Number of total data |
| \( P(r) \) | Possibility of \( r \) failures out of \( n \) tests |
| \( r \) | Number of failures |
| \( t_p \) | Time at which the unreliability is \( p = F(t_p; \beta, \theta) \) |
| \( t_i \) | Failure life data; \( i = 1, 2, \cdots, r \) |
| \( t_{i,q} \) | System time of the \( q^{th} \) system at the \( i^{th} \) occurrence of failure; \( i = 1, 2, \cdots, N_q \) |
| \( \frac{z_{1-\alpha}}{2} \) | Standard normal variable with the cumulative probability of failure of \( \frac{1-\alpha}{2} \) |
| \( z \) | Standard normal variable |
| \( z_p \) | Standard Weibull variable |
| \( \binom{n}{r} \) | Number of combination of \( r \) failures out of \( n \) tests |
| \( \chi^2_{\alpha,1} \) | Chi-squared distribution with one degree of freedom for a two-sided \( \alpha \) confidence interval |
| \( \beta \) | Shape parameter (or Weibull slope) |
| \( \theta \) | Scale parameter (or characteristic life) |
| \( \theta_C \) | True Weibull scale parameter \( \theta \) corresponding to one-sided lower-bound confidence \( C \) |
| \( \lambda \) | Duane’s curve-fitting “positive” parameter |
| \( \alpha \) | Duane’s curve-fitting “positive” parameter |
| \( \Lambda \) | Log-likelihood function |
| \( \frac{\partial \Lambda}{\partial \lambda} \) | Differentiating the log likelihood function with respect to \( \lambda \) |
| \( \frac{\partial \Lambda}{\partial \beta} \) | Differentiating the log likelihood function with respect to \( \beta \) |
• Right leg – Integration and validation (bottom-up activity), managing the assembly or integration of system elements.

The left leg of requirement and architecture addresses the creative process of partitioning the initial product concept into a structure of system elements and allocating and cascading customer, company, and regulatory requirement, level by level to lowest level system elements. It is crucial in this stage to decompose a complex product into an architecture of smaller, simpler, definable system elements and to understand the voice of customers (VOC), define the extreme customer usage conditions, acquire the duty cycle or customer usage data for the target, and cascade the top-down VOC targets from product to system, subsystem, component and parts levels. Once the requirement and architecture of system elements is defined, activities shift to the rungs of development and optimization of designs. This is a design verification and launch stage, where there is a need to define, refine, or create interface and system element designs and to perform the multi-objective design optimization in the virtual analysis domain. Once the system elements are optimized, the right leg of integration and validation provides the opportunity to test the interfaces and to mitigate interactions between system elements for all the requirements. In this leg, the test methods with a limited sample size are required to demonstrate the reliability of the product population for meeting the specific design functional and performance requirements. Since the test results exhibit significant statistical variations, these test methods (named the reliability demonstration test methods) are the most important mechanism to ensure a successful product launch for durability, reliability and quality. Again due to the limited sample size for testing, a specific reliability and confidence target should be predetermined to be demonstrated by these test methods.

There are two unique test phases within a reliability development process: reliability development testing and reliability demonstration testing. Reliability development testing is performed to identify problems and evaluate the effectiveness of corrective redesign actions. The primary purpose is to improve reliability performance rather than measure the reliability. The most important actions are to log all failures generated during testing and assure corrective actions are taken. This is verified through testing to ensure that failure prone aspects of the design are fixed. Reliability quantification during this development phase is intent on measuring improvement. The purpose of a reliability demonstration test is to determine whether a designed product meets or exceeds the established minimum reliability requirement.
Testing material samples or components for fatigue life or strength is a common procedure. However, practically no two nominally identical samples would produce identical test results. This is because of inherent variation in the material which induces variation in the creation, location and propagation of dislocations and cracks as well as due to inevitable variations in manufacturing and assembly. However, overall, the population of specimens that are nominally identical do exhibit certain properties or traits that can be statistically characterized. Practically, one cannot test the entire population and hence typically only certain random samples are tested. Engineering statistics helps one understand the behavior of the population by examining data from sample sets much smaller than the population. The objective of statistics is to make inferences about a population based on information contained in a sample. Two types of estimations are in common use: (1) point estimation and (2) interval estimation. The point estimate is named because a single number represents the estimate. The interval estimate is to calculate the region that is intended to enclose the true value of the population parameter estimated using the data in the sample.

It should be recognized that when population parameters are estimated on the basis of finite samples, errors of estimation are unavoidable. The significance of such errors is not reflected in the point estimation (estimation of a parameter that defines the population’s distribution). A point estimate of a parameter is not very meaningful without some measure of the possible error in the estimate. This estimated value rarely coincides with the true value of the parameter being estimated. Therefore, an interval that is expected to include the true value of the parameter with some specified odds with a prescribed confidence level is necessary. This interval is called the confidence interval. Thus, a 90% confidence interval for a given parameter implies that in the long sequence of replications of an experiment, the computed limits of the interval will include the true value of the parameter about 90% of the time. The fraction of 90% is called the confidence level. The confidence interval can be one-sided or two-sided. It is true that higher the degree of confidence, larger the resulting interval. In reliability life testing with a demonstrated target, for example of R95C90 requirement, it means that the 90% confidence lower limit of a product reliability estimate at a measurement point should be greater than 95%. The reliability and confidence are related when one attempts to project reliability determined from a sample to the population.

This paper intends to review the theoretical background and discuss the pros and cons of the commonly used points analysis methods, confidence interval estimation methods and reliability demonstration test methods. Depending on the failure criterion in product validation testing, the reliability demonstration test methods can be categorized into methods for non-repairable and repairable systems. For non-repairable systems, the commonly used reliability demonstration test methods include (1) the test-to-failure method (the Weibull analysis method), (2) the Weibull analysis of reliability data with few or no failures, (3) the attribute test method, (4) the extended life test method, and (5) the step-stress accelerated life test method. For repairable systems, the two non-homogeneous Poisson process approaches (the Duane model and the Crow-AMSAA model) are discussed.

2. Points analysis methods and confidence interval estimation techniques

2.1. Points Analysis Methods

Point estimation is concerned with the calculation of a single number/value from a set of observed data to represent a statistical parameter of the underlying population. For a Weibull distribution, this would include the estimation of its shape and slope parameters. This section describes two basic parameter estimation methods (the median rank regression method and maximum likelihood estimation method) which are used in most commercially available software packages.
Presented below are the four important Weibull functions. The Weibull cumulative distribution function for the product population life failing by time or life \( t \) is

\[
F(t) = 1 - \exp \left[ -\left( \frac{t}{\theta} \right)^\beta \right] \tag{1}
\]

where \( \beta \) is the shape parameter (or Weibull slope) and \( \theta \) is the scale parameter (or characteristic life).

The characteristic life defined as life/time at the 63.2\(^{rd}\) percentile of the cumulative distribution function. For \( \beta = 1 \), the Weibull distribution becomes the exponential distribution. The Weibull reliability function for the population life surviving beyond time or life \( t \) is

\[
R(t) = 1 - F(t) = \exp \left[ -\left( \frac{t}{\theta} \right)^\beta \right]. \tag{2}
\]

The Weibull probability density function is

\[
f(t) = \frac{\beta}{\theta} \left( \frac{t}{\theta} \right)^{\beta-1} \exp \left[ -\left( \frac{t}{\theta} \right)^\beta \right]. \tag{3}
\]

The Weibull hazard function is the instantaneous failure rate; defined as

\[
h(t) = \frac{f(t)}{R(t)} = \left( \frac{\beta}{\theta} \right) \left( \frac{t}{\theta} \right)^{\beta-1}. \tag{4}
\]

2.1.1. Median Rank Regression (MRR) Method

Median rank regression is a very popular method for estimating Weibull scale and shape parameters, as evidenced by this being the default method in numerous commercial statistical software packages. It gained popularity due to the ease of calculations and programming as compared to the Maximum Likelihood method.

Essentially, this procedure fits a least squares regression line through the test data (failure) points on a probability plot. The Weibull cumulative distribution function can be rearranged as

\[
1 - F(t) = \exp \left[ -\left( \frac{t}{\theta} \right)^\beta \right]. \tag{5}
\]

If natural logarithms are applied twice to both sides of Equation (5), it yields

\[
\ln(t) = \ln(\theta) + \frac{1}{\beta} \ln \left( \frac{1}{1 - F(t)} \right). \tag{6}
\]

Equation (6) can be construed in the form of \( Y = \hat{A} + \hat{B}X \) and solved by the linear regression analysis, where

\[
Y = \ln(t), \quad X = \ln \left( \frac{1}{1 - F(t)} \right), \quad \hat{A} = \ln(\hat{\theta}) \quad \text{and} \quad \hat{B} = \frac{1}{\beta}.
\]

Therefore, the Weibull parameter estimates are \( \hat{\beta} = 1/\hat{B} \) and \( \hat{\theta} = \exp(\hat{A}) \). Please note that the parameter estimates are written with a ^ above them, indicating that they are estimated from a sample of limited size and the actual \( \beta \) and \( \theta \) represent the parameters for the population.
Per Abernethy [1], the procedures for calculating MRR estimates are listed as follows:

1. Group \( n \) number of life data in ascending order, where \( r \) failures and \((n-r)\) suspension data are listed.
2. Assign reverse ranks for each one of the data points from \( n \) to 1.
3. Use the following equation developed by Johnson [4] to adjust the rank orders of failures due to the presence of suspensions:

\[
\text{adjusted rank}_i = \frac{\text{rank}_{\text{reverse}}(i) \times \text{rank}_{\text{adjusted}}(i-1) + (n+1)}{\text{rank}_{\text{reverse}}(i) + 1} \quad (7)
\]

4. Determine the median rank for each adjusted failure data order, using the following discrete cumulative distribution formula \( F(i) \) developed by Benard and Bosi-Levenbach [5]:

\[
F(i) = \frac{i - 0.3}{n + 0.4} \quad (8)
\]

5. Estimate the Weibull parameters by a linear regression method on the \( r \) failures and corresponding median rank values.

2.1.2. Maximum Likelihood Estimation (MLE) Method

The maximum likelihood estimation (MLE) method is a statistical procedure for parameter estimation by maximizing the likelihood or probability that a certain set of failures or suspensions occur as observed, assuming an underlying Weibull distribution for the population. Further details of the maximum likelihood estimation method can be found elsewhere [1-2].

A general concept description of the maximum likelihood estimation method is provided here. For example, the Weibull joint probability density function for all identical independent random failures observations (such as \( t_1, t_2, ..., t_n \)) is given below

\[
f(t_1, t_2, ..., t_n | \beta, \theta) = f(t_1 | \beta, \theta) \cdot f(t_2 | \beta, \theta) \cdots f(t_n | \beta, \theta) = \prod_{i=1}^{n} f(t_i | \beta, \theta) \quad (9)
\]

The Likelihood function is the function that the observed values are to be fixed and the functional parameters are variables, defined as

\[
L(\beta, \theta | t_1, t_2, ..., t_n) = f(t_1, t_2, ..., t_n | \beta, \theta) = \prod_{i=1}^{n} f(t_i | \beta, \theta) \quad (10)
\]

In practice, the logarithm of the likelihood function, called the log-likelihood function \( \Lambda \) is used because it enables easier computations. Log-likelihood function \( \Lambda \) is expressed as follows:

\[
\Lambda = \ln L(\beta, \theta | t_1, t_2, ..., t_n) = \sum_{i=1}^{n} \ln f(t_i | \beta, \theta) \quad (11)
\]

The Weibull parameters are estimated by maximizing the log-likelihood function, since log is a monotonic transformation. So it can be written as the argument of the maximum such that the set of values of \( t_i, i = 1, ..., n \) for which the log-likelihood function has the largest value

\[
\hat{\beta}, \hat{\theta} = \arg \max \Lambda.
\]

Determine the two statistical parameters which will maximize the log-likelihood function by solving \( \partial \Lambda / \partial \beta = 0 \) and \( \partial \Lambda / \partial \theta = 0 \).
Presented below is the Weibull parameters estimation method with censored data. Given a total sample size \( n \) where \( r \) is the number of failure data \( (t_i, i=1,...,r) \) and \( k = n - r \) is the number of suspension data \( (T_i, i=1,...,k) \), the likelihood function can be written as

\[
L(\beta, \theta | t_1, t_2, ..., t_r, T_1, T_2, ..., T_k) = \prod_{i=1}^{r} f(t_i | \beta, \theta) \prod_{j=1}^{k} R(T_j | \beta, \theta) .
\] (13)

With the introduction of the Weibull probability density and reliability functions, the log-likelihood function becomes

\[
\Lambda = \sum_{i=1}^{r} \ln \left[ \left( \frac{\beta}{\theta} \right)^{t_i} \left( \frac{t_i}{\theta} \right)^{\beta-1} e^{-\left( \frac{t_i}{\theta} \right)^{\beta}} \right] + \sum_{j=1}^{k} \ln \left[ e^{-\left( \frac{T_j}{\theta} \right)^{\beta}} \right] .
\] (14)

The MLE method differentiates the logarithm of the log-likelihood function with respect to \( \beta \) and \( \theta \), equates the resulting expression to zero, and simultaneously solves for both \( \beta \) and \( \theta \). It is expressed below

\[
\frac{\partial \Lambda}{\partial \beta} = \sum_{i=1}^{r} \left[ \left( \frac{1}{\beta} \right) + \ln \left( \frac{t_i}{\theta} \right) \right] - \sum_{j=1}^{k} \left[ \left( \frac{T_j}{\theta} \right)^{\beta} \right] = 0
\] (15)

\[
\frac{\partial \Lambda}{\partial \theta} = \sum_{i=1}^{r} \left[ \left( \frac{\beta}{\theta} \right) \ln \left( \frac{t_i}{\theta} \right) \right] + \sum_{j=1}^{k} \left[ \left( \frac{\beta}{\theta} \right) \left( \frac{T_j}{\theta} \right)^{\beta} \right] = 0 .
\] (16)

Therefore, the maximum likelihood estimate of \( \hat{\beta} \) is obtained by solving the following equation:

\[
\frac{\sum_{i=1}^{r} t_i^{\hat{\beta}} \ln t_i}{\sum_{i=1}^{r} t_i^{\hat{\beta}}} - \frac{1}{r} \sum_{i=1}^{r} \ln t_i - \frac{1}{\hat{\beta}} = 0
\] (17)

and the maximum likelihood estimate of \( \hat{\theta} \) is obtained from

\[
\hat{\theta} = \left( \frac{\sum_{i=1}^{r} t_i^{\hat{\beta}}}{r} \right)^{\frac{1}{\hat{\beta}}} .
\] (18)

2.2. Confidence Interval Estimation Methods

Interval estimates can be contrasted with point estimates. A point estimate is a single value given as the estimate of a population parameter or a reliability value that is of interest. A confidence interval estimate specifies a range within which the true population parameter or a reliability value is likely to occur for a certain percentage of the time or the population. For example, a 90% confidence interval for a true reliability value implies that in a long sequence of replications of an experiment, the calculated limits of the interval will include the true reliability value for 90% of the time or 90% of the population. This fraction 90% is called the confidence level, which is usually chosen to be 90, 95, or 99%. The desired level of confidence is determined by each company’s policy. The confidence interval can be one-sided or two-sided. In reliability engineering, one usually uses one-sided lower confidence intervals rather than two-sided ones, but both can be easily related to each other.

The confidence interval width is influenced by the

- Size of sample: A larger sample size will lead to a better estimate of the population parameter because large samples are more similar to each other and have more information leading to narrow confidence intervals.
• Level of confidence: A higher level of required confidence drives the width of the confidence interval wider to ensure that the true population parameter lies within the estimated confidence interval.
• Population variability: A population with large variation leads to samples with high variations, resulting in wider confidence intervals.

There are many methods used to establish (both one-sided and two-sided) confidence intervals for a given data set. This section will cover three important methods used to estimate confidence intervals – the likelihood ratio method, Fisher matrix method, and Monte Carlo pivotal statistics methods.

2.2.1. The Likelihood Ratio Method

The method is described in detail by Abernethy [1] and Nelson [2]. If \( L(\beta, \theta) \) is the likelihood function for the population and \( L(\hat{\beta}, \hat{\theta}) \) is the likelihood function for the sample dataset, then the likelihood ratio function is defined as

\[
LR = \frac{L(\beta, \theta)}{L(\hat{\beta}, \hat{\theta})}.
\]

(19)

Taking logarithms and multiplying by a constant (-2) gives

\[
-2\ln(LR) = -2\left(\ln(L(\beta, \theta)) - \ln(L(\hat{\beta}, \hat{\theta}))\right) = -\left(\ln(L(\beta, \theta)) - \ln(L(\hat{\beta}, \hat{\theta}))\right)^2.
\]

(20)

Equation (20) represents the square of the likelihood ratio function, which is the square of the error estimates between the likelihood function for the population and sample. This can be best described by a chi-squared \( \chi^2_{\alpha,1} \) distribution with one degree of freedom for a two-sided \( \alpha \) confidence interval as follows:

\[
-2\left(\ln(L(\beta, \theta)) - \ln(L(\hat{\beta}, \hat{\theta}))\right) = \chi^2_{\alpha,1}.
\]

(21)

Equation (21) can be further reduced to

\[
\ln(L(\beta, \theta)) = \ln(L(\hat{\beta}, \hat{\theta})) - \frac{1}{2} \chi^2_{\alpha,1}.
\]

(22)

Equation (22) indicates that slices off of the sample’s log-likelihood surface sample at different depths from the maxima provide for \( \alpha \) confidence bounds on the parameters \( \beta \) and \( \theta \). The depth of the slices (parallel to \( \beta - \theta \) the plane) is determined by the term \( \frac{1}{2} \chi^2_{\alpha,1} \).

The confidence bounds on a parameter such as time \( t \) or reliability \( R \) can be determined in a similar fashion as described above. But it requires the parameter \( \theta \) in the likelihood function \( L(\beta, \theta) \) needs to be replaced by \( t \) or \( R \). Given the Weibull reliability function, the following relation exists:

\[
\theta = \frac{t}{(-\ln R)^{1/\beta}}.
\]

(23)

Therefore the likelihood ratio functions defined by \( L(\beta, t) \) and \( L(\beta, R) \) can be used estimate the confidence bounds on time \( t \) and reliability \( R \), respectively.

2.2.2. The Fisher Matrix Method

A very popular method for estimating confidence bounds is through the use of Fisher’s information matrix, sometimes called the information matrix. The Fisher matrix was named after Sir Ronald Fisher to honor all his contributions to the field of modern statistics. Like all estimation methods it has its advantages and disadvantages.
This method has the advantage of being computationally very simple allowing for quick estimations of confidence bounds. However, it produces inaccurate results for small sample sizes. In addition, it is not recommended for data sets containing suspensions and it is recommended that this analysis be used for populations having ten or more failures [1]. The method is described in detail by Nelson [2].

This matrix, essentially the negative Hessian, consists of negative second partial derivatives of the log-likelihood function as follows:

$$
[F] = \begin{bmatrix}
\frac{\partial^2 \Lambda}{\partial \beta^2} & -\frac{\partial^2 \Lambda}{\partial \beta \partial \theta} \\
-\frac{\partial^2 \Lambda}{\partial \beta \partial \theta} & -\frac{\partial^2 \Lambda}{\partial \theta^2}
\end{bmatrix}
$$  \hspace{1cm} (24)

Based on the Weibull log-likelihood function and its first derivatives with respect to $\theta$ and $\beta$ described previously in Equations (18-20), the second and cross derivatives of the log-likelihood function in the Fisher matrix can be expressed as follows:

$$
\frac{\partial^2 \Lambda}{\partial \theta^2} = \sum_{j=1}^{k} \left[ \frac{1}{\theta} - \frac{T_j}{\theta} \right] \left( \ln \left( \frac{T_j}{\theta} \right) \right)^2 + \sum_{j=1}^{k} \left( \frac{T_j}{\theta} \right) \left( \ln \left( \frac{T_j}{\theta} \right) \right) + 1
$$  \hspace{1cm} (25)

$$
\frac{\partial^2 \Lambda}{\partial \beta^2} = \sum_{j=1}^{k} \left[ \frac{1}{\theta} - \frac{T_j}{\theta} \right] \left( \ln \left( \frac{T_j}{\theta} \right) \right)^2 + \sum_{j=1}^{k} \left( \frac{T_j}{\theta} \right) \left( \ln \left( \frac{T_j}{\theta} \right) \right) + 1
$$  \hspace{1cm} (26)

$$
\frac{\partial^2 \Lambda}{\partial \beta \theta} = \sum_{j=1}^{k} \left[ \frac{1}{\theta} - \frac{T_j}{\theta} \right] \left( \ln \left( \frac{T_j}{\theta} \right) \right) \left( \beta \ln \left( \frac{T_j}{\theta} \right) + 1 \right) + \sum_{j=1}^{k} \left( \frac{T_j}{\theta} \right) \left( \ln \left( \frac{T_j}{\theta} \right) \right) + 1
$$  \hspace{1cm} (27)

It should be noted

$$
\frac{\partial^2 \Lambda}{\partial \beta \partial \theta} = \frac{\partial^2 \Lambda}{\partial \theta \partial \beta}.
$$  \hspace{1cm} (28)

By taking the inverse of the Fisher matrix, it is possible to calculate the variance and covariance of the statistical parameters $\beta$ and $\theta$. The inverse of the Fisher information matrix, also referred to as the covariance matrix, is written as

$$
[F]^{-1} = \begin{bmatrix}
\text{Var}(\beta) & \text{Cov}(\beta, \theta) \\
\text{Cov}(\theta, \beta) & \text{Var}(\theta)
\end{bmatrix}.
$$  \hspace{1cm} (29)

Evaluated at $\beta = \hat{\beta}$ and $\theta = \hat{\theta}$, Equations (24) and (29) are the local estimated Fisher information matrix and covariance matrix, respectively. The local estimates will be used to obtain approximate confidence bounds. The concept of confidence bounds estimation is described here. In general, $G(\beta, \theta)$ is a function representing a statistical distribution with two parameters $\beta$ and $\theta$. The local mean $E(G)$ and variance $\text{Var}(G)$ of this function at $\beta = \hat{\beta}$ and $\theta = \hat{\theta}$ can be approximated by

$$
E(G) = G(\hat{\beta}, \hat{\theta})
$$  \hspace{1cm} (30)

$$
\text{Var}(G) \approx \left( \frac{\partial G}{\partial \beta} \right)_{\beta = \hat{\beta}} \cdot \text{Var}(\hat{\beta}) + \left( \frac{\partial G}{\partial \theta} \right)_{\theta = \hat{\theta}} \cdot \text{Var}(\hat{\theta}) + 2 \left( \frac{\partial G}{\partial \beta} \right)_{\beta = \hat{\beta}} \left( \frac{\partial G}{\partial \theta} \right)_{\theta = \hat{\theta}} \cdot \text{Cov}(\hat{\beta}, \hat{\theta})
$$  \hspace{1cm} (31)

where the estimated variance and covariance of the statistical parameters $\beta$ and $\theta$ are determined by the local Fisher information matrix. Therefore the approximate two-sided confidence bounds $\alpha$ on the function $G$ can be written as
CB = E(G) ± z_{1-α} \sqrt{Var(G)} \quad (32)

where $z_{1-α}$ is the standard normal variable with the cumulative probability of failure of $\frac{1-α}{2}$.

Presented below are some examples of estimated confidence bounds on the Weibull parameters, reliability and time or life $t$.

2.2.2.1. Confidence Bounds on Weibull Parameters

Both the shape parameter estimate $\hat{β}$ and scale parameter $\hat{θ}$ for a Weibull distribution must be positive, therefore $\ln(\hat{β})$ and $\ln(\hat{θ})$ are assumed to be normally distributed. Based on the assumption of log-normally distributed $\hat{β}$ and $\hat{θ}$, the two-sided confidence bounds on $θ$ can be written as

$$CB = \ln(\hat{θ}) ± \frac{z_{1-α}}{\sqrt{T}} \sqrt{Var(\ln(\hat{θ}))}. \quad (33)$$

Equation (33) can be rewritten as

$$CB = \hat{θ} \cdot \exp \left( ± \frac{z_{1-α}}{\sqrt{T}} \sqrt{Var(\ln(\hat{θ}))} \right). \quad (34)$$

Since the variance of the natural log of $\hat{θ}$ can be approximated by

$$\sqrt{Var(\ln(\hat{θ}))} \approx \frac{\sqrt{Var(\hat{θ})}}{\hat{θ}}, \quad (35)$$

the two-sided upper and lower bounds on $θ$ become

$$CB = \hat{θ} \cdot \exp \left( ± \frac{z_{1-α}}{\sqrt{\hat{θ}}} \right). \quad (36)$$

Likewise, the two-sided upper and lower bounds on $β$ can be derived as

$$CB = \hat{β} \cdot \exp \left( ± \frac{z_{1-α}}{\sqrt{\hat{β}}} \right). \quad (37)$$

2.2.2.2. Confidence Bounds on Reliability

In order to estimate the bounds on reliability, the Weibull reliability function needs to be rearranged as follows

$$R(t) = \exp \left( -\left( \frac{t}{\hat{θ}} \right)^{\hat{β}} \right) = \exp \left( -\exp(\hat{β}\ln(t) - \ln(\hat{θ})) \right). \quad (38)$$

Next a new parameter $u$ is introduced, where

$$u = \hat{β}\ln(t) - \ln(\hat{θ}). \quad (39)$$

Therefore, the reliability equation becomes

$$R(t) = \exp(-\exp(u)), \quad (40)$$

and the two-sided confidence bounds $α$ on the parameter $u$ are given by:
The local estimated variance of $u$ is calculated as follows:

$$\text{Var}(\hat{u}) = \left(\frac{\partial u}{\partial \beta}\right)^2 \text{Var}(\hat{\beta}) + \left(\frac{\partial u}{\partial \theta}\right)^2 \text{Var}(\hat{\theta}) + 2\left\langle\frac{\partial u}{\partial \beta}\right\rangle\left\langle\frac{\partial u}{\partial \theta}\right\rangle \text{Cov}(\hat{\beta}, \hat{\theta}).$$

(42)

Since the partial derivatives of Equation (39) are

$$\frac{\partial u}{\partial \beta} = \ln(x) - \ln(\theta) = \frac{u}{\beta}$$

(43)

$$\frac{\partial u}{\partial \theta} = \frac{\beta}{\theta},$$

(44)

the local estimated variance of $u$ can be obtained by

$$\text{Var}(\hat{u}) = \left(\frac{\hat{u}}{\hat{\beta}}\right)^2 \text{Var}(\hat{\beta}) + \left(\frac{\hat{\beta}}{\hat{\theta}}\right)^2 \text{Var}(\hat{\theta}) + 2\left\langle\frac{\hat{u}}{\hat{\beta}}\right\rangle\left\langle\frac{\hat{\beta}}{\hat{\theta}}\right\rangle \text{Cov}(\hat{\beta}, \hat{\theta}).$$

(45)

Finally, the two-sided confidence bounds on reliability can be determined by the two-sided confidence bounds on $u$.

### 2.2.2.3. Confidence Bounds on Time $t$

The Weibull reliability function needs to be written so that time $t$ is a function of $R$, $\beta$ and $\theta$, expressed as follows

$$\ln(t) = \frac{1}{\beta} \ln(-\ln(R)) + \ln(\theta).$$

(46)

A new parameter $v$ is introduced below:

$$v = \ln(t),$$

(47)

and, alternatively,

$$t = \exp(v).$$

(48)

The two-sided confidence bounds on the new parameter $v$ are

$$CB = \hat{v} \pm z_{\alpha/2} \sqrt{\text{Var}(\hat{v})}$$

(49)

where the local estimated variance of $v$ is determined by

$$\text{Var}(\hat{v}) = \left(\frac{\partial v}{\partial \beta}\right)^2 \text{Var}(\hat{\beta}) + \left(\frac{\partial v}{\partial \theta}\right)^2 \text{Var}(\hat{\theta}) + 2\left\langle\frac{\partial v}{\partial \beta}\right\rangle\left\langle\frac{\partial v}{\partial \theta}\right\rangle \text{Cov}(\hat{\beta}, \hat{\theta}).$$

(50)

Because the partial derivatives of $v$ from Equation (46) are

$$\frac{\partial v}{\partial \beta} = -\frac{1}{\beta^2} \ln(-\ln(R))$$

(51)

$$\frac{\partial v}{\partial \theta} = \frac{1}{\theta},$$

(52)

the local estimated variance of $v$ can then be obtained by

$$\text{Var}(\hat{v}) = \frac{1}{\hat{\beta}^2} \left(\ln(-\ln(R))\right)^2 \text{Var}(\hat{\beta}) + \left(\frac{1}{\hat{\theta}}\right)^2 \text{Var}(\hat{\theta}) + 2\left\langle\frac{1}{\hat{\beta}^2}\right\rangle\left\langle\frac{\ln(-\ln(R))}{\hat{\theta}}\right\rangle \text{Cov}(\hat{\beta}, \hat{\theta}).$$

(53)

Finally, the two-sided confidence bounds on time $t$ at a specific reliability can be obtained by the two-sided confidence bounds on $v$.

### 2.2.3. The Monte Carlo Pivotal Statistics Method
Both the Fisher matrix method and likelihood ratio method provide for very narrow (i.e. optimistic or non-conservative) confidence bounds for sample sizes that are less than 10. Therefore, it is recommended that one use the Monte Carlo pivotal statistics method to estimate the confidence bounds for a sample size smaller than 10. This method was in part developed by Wes Fulton and Robert Abernethy with Chrysler LLC. A pivotal statistic is a quantity that is parameter-free, meaning it is independent of the properties/characteristics (such as mean, standard deviation or slope, shape parameters) of the distribution while representing the data points of the sample. It is essentially a function of the sample data but does not depend on the unknown parameters of the distribution [6].

For a normal distribution with mean \( \mu \), standard deviation \( \sigma \), and an observation \( t \), a standard normal variable \( z \) can be defined as:

\[
z = \frac{t - \mu}{\sigma}
\]  
(54)

It is noted that the \( z \) follows a normal distribution with a mean of 0 and a standard deviation of 1, irrespective of the values of \( \mu \) and \( \sigma \) calculated from the samples. This is said to be following a “standard normal distribution”. Thus, the standard normal variable \( z \) is effectively a pivotal statistic for any data that fits a normal distribution.

Extending the idea of a pivotal statistic quantity to a Weibull distribution, a standard Weibull variable \( z_p \) can be defined as:

\[
z_p = \frac{\log(\hat{\theta}) - \log(t_p)}{\frac{1}{\hat{\beta}}}
\]  
(55)

where \( t_p \) is the time at which the unreliability is \( p = F(t_p; \beta, \theta) \) and \( t_p = F^{-1}(p; \beta, \theta) \).

The above equation can be written as

\[
z_p = \frac{\hat{u}_o - w_p}{\hat{b}_o}
\]  
(56)

where \( \hat{u}_o = \log(\hat{\theta}) \), \( w_p = \log(t_p) \), and \( \hat{b}_o = \frac{1}{\hat{\beta}} \).

By definition, \( z_p \) does not depend on unknown parameters, meaning that its distribution stays the same regardless of the actual values of \( \beta \) and \( \theta \). Therefore, the pivotal statistics can be arbitrarily set to \( \beta = \theta = 1 \) for further calculations.

Based on the previous work [6-8], the following Monte Carlo simulations procedures are described to generate the empirical distribution of \( z_p \):

1. Given a specific reliability \( R \), calculate \( w_p = \log(F^{-1}(p;1,1)) \) where \( p = 1 - R \).
2. Create \( N \) sets of synthetic samples by randomly generating synthetic observations from the standard Weibull distribution with \( \beta = \theta = 1 \). Each set of synthetic samples should have the same sample size as the original ones.
3. Find the \( N \) sets of synthetic standard Weibull parameter estimates \( \left( \hat{u}_o, \hat{\beta}_o \right) \) using Median Rank Regression.
4. Calculate the \( N \) synthetic z-scores \( \left( z_{pi} \right) \) using Equation (56).
5. Determine 2-sided confidence bounds \( \alpha \) on the z-scores, \( z_{p,\alpha} \), including \((\alpha \times 100)\% \) of the true \( z_p \) population.
6. The two-sided confidence bound $\alpha$ on the time $t$, $t_{p, CB}$, can be estimated as follows, using Equation (57),

$$\log(t_{p, CB}) = \log(\hat{\theta}) - \frac{z_{p, CB}}{\hat{\beta}}$$

where $\hat{\theta}$ and $\hat{\beta}$ are the two estimated Weibull parameters from the test samples.

Ready to use tables are provided in [8] for easy of confidence interval calculations for Weibull distributed samples up to a size of 20, with no suspensions.

3. Reliability assessment methods for non-repairable systems

It should be mentioned that the materials presented here have been drawn heavily from Abernethy [1], Nelson [2] and Lu [3].

3.1. The Test-to-Failure Method (the Weibull Analysis Method)

The test-to-failure method has the advantage of knowing the product life distribution and failure modes under the same duty cycle test loads. Given the fact that the fatigue life failures follow the Weibull distribution, the failure data is analyzed by fitting a Weibull distribution to the data. This method relies on testing as many specimens as possible until failure occurs. The primary purpose is to learn how and when components fail in order to identify weakest links to improve reliability/durability.

With such a process, there must be at least two failures to estimate the Weibull parameters and confidence bounds. For statistically significant confidence interval estimates, there must be a much larger sample size. However, testing larger sample sizes would be more expensive as well as time consuming. Availability and cost of the parts, test rigs/setups and the test engineers required to run the tests are some of the issues with such a method. Therefore, there is a tradeoff between accuracy and time that needs to be addressed – some of the methods discussed hereafter provide alternate solutions for this problem.

3.2. Weibull Analysis of Reliability Data with Few or No Failures

For the methods described thus far, there must be at least two failures to estimate the Weibull parameters and confidence bounds. However, one may have few or zero failures in real life, resulting in inaccurate estimations or inability to estimate parameters for practical purposes. An accurate estimation method for point parameters and confidence bounds, which can apply to very few or no failures is presented here.

For a given sample of $n$ units where $r$ is the number of failures and $k (= n - r)$ is the number of suspensions, the failure times or lives are denoted as $t_1, t_2, ..., t_n$. It is also assumed that the Weibull shape parameter $\beta$ is given (or known from experience or historical data), the failure times are independent and identically distributed, and the failure times may be intermixed among the censored times.
Nelson [9] derived the following formula to estimate corresponding one-sided lower-bound confidence limit $C\%$ for the true Weibull scale parameter $\theta$ as follows:

$$\theta_C = \left[ \frac{2 \sum_{i=1}^{n} t_i^\beta}{\chi^2_{(C,2r+2)}} \right]^\frac{1}{\beta}$$

(58)

where $\chi^2_{(C,2r+2)}$ is the $C^{th}$ percentile of the Chi-square distribution with $(2r+2)$ degrees of freedom. Hence the reliability at $t$ cycles with one-sided lower $C90$ is given by

$$R(t) = \exp \left[ -\left( \frac{t}{\theta_C} \right)^\theta \right] = \exp \left\{ -\frac{t^\beta - \chi^2_{(C,2r+2)}}{2 \sum_{i=1}^{n} t_i^\beta} \right\}.$$  

(59)

Since $\chi^2_{(C,2)} = -2\ln(1-C)$ for a special case of no failures where $r = 0$, thus the reliability function reduces to

$$R(t) = \exp \left\{ \frac{t^\beta \ln(1-C)}{\sum_{i=1}^{n} t_i^\beta} \right\}.$$  

(60)

### 3.3. The Attribute Test Method

The attribute test method is a “success/failure”, “go/no go”, or “acceptable/not acceptable” type of test, and is often referred to as the binomial test, or the test-to-bogey method. A product is submitted to a minimum durability test or performance criterion or bogey. If a test sample makes it to the bogey, it is a success, and if it does not, it is a failure. For the “success/failure” type of situation, the binomial distribution is applied. The following are some limitations of using the binomial method: (1) it requires numerous test samples, (2) no test failures are allowed, and (3) failure modes and variability are not disclosed.

For a given sample size $n$, it is assumed that the $n$ tests are independent and repeated under identical test conditions. For each individual test, the reliability or probability of success denoted by $R$ is the same. Since each individual trial results in either success or failure, the probability of failure is calculated by $(1-R)$. The binomial probability formula is used to find the possibility of $r$ failures out of $n$ tests as follows:

$$P(r) = \binom{n}{r}(1-R)^r R^{n-r}$$

(61)

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$  

Also the probabilities of at most $r$ failures out of $n$ tests can be related to the lower bound confidence level $C$ with the following relationship:

$$P(x \leq r) = \sum_{x=0}^{n} \binom{n}{x}(1-R)^x R^{n-x} \leq 1 - C.$$  

(62)

The lower confidence level is a random quantity that falls above the corresponding population value with the probability of $C$, which can be chosen to be 90%, 95%, or 99%. In generally the confidence limit based on few failures will generally be quite wide, indicating the fact that the estimate has great uncertainty.
For a special success-run case where no failures occur \((r = 0)\), Equation (62) becomes

\[
R^* = 1 - C
\]  \hspace{1cm} (63)

The is the commonly used equation for the attribute test method to estimate the minimum number of samples required with no failures in order to demonstrate the product meeting a \(R\) reliability and a \(C\) confidence level. For example, the use of Equation (63) with \(R = 0.9\) and \(C = 0.9\) gives \(n = 22\). Therefore, all the 22 samples need to be tested and suspended at the bogey without any failures in order to demonstrate the fact that 90% of the estimated R90 quantify based on any 22 samples from the product population will satisfy the test criterion or bogey.

The attribute test method is an easy reliability demonstration test method, but could be costly for many samples required for prototype testing. Most importantly, the product life distribution and failure modes are not revealed.

3.4. The Extended Life Test Method

One of the situations encountered frequently in testing involves a tradeoff between sample size and testing time. If the test product is expensive, the number of test products can be reduced by extending the time of testing on fewer products. Extended testing is a method to reduce sample size by testing the samples to a time that is higher than the test bogey requirement without failures. It is also referred to as the test to extended bogey method.

Assuming that there are no failures in the sample set and an estimate of the Weibull slope is known, the theory is derived on the equivalence of the Weibull distribution and the success-run theory (Equation (63)). For example, the reliability at \(t_1\) with a lower bound confidence \(C\) and sample size \(n_1\) is expressed by

\[
R(t_1) = (1 - C)^{1/n_1} = \exp\left[-\left(\frac{t_1}{\theta}\right)^\beta\right].
\]  \hspace{1cm} (64)

Similarly, the reliability at \(t_2\) with a lower bound confidence \(C\) and sample size \(n_2\) is expressed by

\[
R(t_2) = (1 - C)^{1/n_2} = \exp\left[-\left(\frac{t_2}{\theta}\right)^\beta\right].
\]  \hspace{1cm} (65)

Equating the above two equations yields

\[
\frac{t_1}{t_2} = \left(\frac{n_2}{n_1}\right)^{1/\beta}.
\]  \hspace{1cm} (66)

Let \(n_1\) be the number of test products suspended at the extended time \(t_1\) and \(n_2\) be the number of test products suspended at the test bogey time \(t_2\). Since the success run theory satisfies the following relationship among the sample size, lower confidence bound and reliability at the bogey time:

\[
n_2 = \frac{\ln(1 - C)}{R(t_2)}
\]  \hspace{1cm} (67)

Equation (66) can be reduced to

\[
\frac{t_1}{t_2} = \left(\frac{\ln(1 - C)}{n_1 \ln R(t_1)}\right)^{1/\beta}.
\]  \hspace{1cm} (68)

3.5. The Step-Stress Accelerated Life Test Method

In general, the test-to-failure method provides the life distribution for a product, but the test-time-to-failure is usually longer. The binomial test method is a good testing method for pass and fail criterion at a bogey, but requires a larger sample size and does not reveal the life distribution. The extended testing method allows for trading additional test time for smaller sample size requirement, provided the Weibull slope of the life distribution is given.
However, this method cannot provide the life distribution of a product. There is a need for a new test method called the step-stress accelerated life testing (SSALT) method which can provide the life distribution of a product and can also accelerate the test-time-to-failure. This method of accelerated life testing is a good way to obtain time-to-failures distribution in a relatively short amount of time. But the SSALT method needs to be used with caution because there is likelihood that the failure mode may be changed due to a higher stress amplitude level. This test is achieved by testing the product at the derived stress level for a fixed amount of time. At the end of that time, if there are products surviving, the stress level is increased in a stepwise fashion and held for another amount of time. This process is repeated until all of the test parts are failed.

The cumulative exposure model [1] has been widely used to determine the reliability of the test products under SSALT. The model assumes that the cumulative life distribution at any constant stress amplitude level follows a two-parameter Weibull distribution and the Weibull shape parameter is the same at each stress level. Since 1990, many extensions of Nelson’s cumulative exposure models [10-14] have been developed.

Presented below is the introduction to the Nelson cumulative exposure model. A step-stress accelerated test can be described in the following mathematical expressions. Let $S_{a_1}, S_{a_2}, \ldots, S_{a_m}$ denote $m$ levels of stress amplitudes, $S_{a_i}, i = 1, 2, \ldots, m$. Let $S_{a_j}$ be the developed test stress amplitude for a laboratory bogey, which is usually about 80% of the maximum stress amplitude in the service loading condition. So the SSALT is defined as

\[
S = S_{a_1} \quad 0 \leq n < n_1 \\
S = S_{a_2} \quad n_1 \leq n < n_2 \\
\vdots \\
S = S_{a_m} \quad n_{m-1} \leq n < n_m 
\]

(69)

where $n$ is the accumulated life cycles.

The Weibull life cumulative distribution function at $S_{a_j}$ is defined as

\[
F_i(n) = 1 - \exp \left(-\left(\frac{n}{\theta_i}\right)^\alpha\right), \quad i = 1, 2, \ldots, m
\]

(70)

where $\beta$ is the Weibull shape parameter and $\theta_i$ is the Weibull scale parameter at $S_{a_j}$. The $\beta$ value is assumed to be constant for each of the stress amplitude. It is also found that the $\theta_i$ value can be obtained by

\[
\theta_i = \left(\frac{\alpha}{S_{a_j}}\right)^\gamma, \quad i = 1, 2, \ldots, m
\]

(71)

where $\alpha$ and $\gamma$ are positive parameters.

Let $n_0 = 0$ and $\Delta_i = n_i - n_{i-1}$ for $i = 1, 2, \ldots, m$. Based on Nelson’s cumulative exposure model [1], the life cumulative distribution $F_i(n)$ of the accumulated test time $n$ can be shown as

\[
F_i(n) = 1 - \exp\left[-\left(\frac{\Delta_i}{\theta_i}\right)^\alpha\right], \quad n_{i-1} \leq n < n_i, i = 1, 2, \ldots, m
\]

(72)

where

\[
\Delta(n) = \frac{\Delta_1}{\theta_1} + \frac{\Delta_2}{\theta_2} + \cdots + \frac{n-n_i}{\theta_i}.
\]

(73)

The concept of Nelson’s cumulative exposure model can be schematically illustrated in Figure 2.
Here is the median rank regression method to estimate the three parameters \( (\alpha, \beta, \gamma) \) involved in Nelson’s cumulative exposure model. If an S-N curve is given as follows:

\[
S_n = S_f\left(2N_f\right)^b,
\]

where \( S_f \) is the fatigue strength coefficient, \( b \) is the fatigue strength exponent, and \( 2N_f \) is the number of reversals to failure. \( \gamma \) can be found as

\[
\gamma = \frac{1}{b}.
\]

Given \( n \) samples (\( r \) failures and \( n-r \) censored) from a population having \( F_o(n) \), let \( M_i, i = 1, 2, ..., r \) be the corresponding median ranks of \( r \) failure times. The least squares estimators \( \hat{\alpha}, \hat{\beta}, \hat{\gamma} \) can be obtained by minimizing \( \Omega \),

\[
\Omega = \sum_{i=1}^{r} (F_o(n) - M_i)^2.
\]

Once the estimated parameters \( \hat{\alpha}, \hat{\beta}, \hat{\gamma} \) were determined on the observed failure data, the Weibull life cumulative distribution at \( S_{a,i} \) would be known from Equations (70) and (71).

**4. RELIABILITY ASSESSMENT METHODS FOR REPAIRABLE SYSTEMS**

The reliability growth test planning and management strategy assumes that, as the product design matures, potential failure modes for the product will be identified through controlled testing in a series of phases. The corrective fixes for some of all of the identified failure modes are to be implemented to reduce the likelihood that the revised product design will fail. The implementation of the corrective actions will determine the reliability growth management strategy. There are three basic approaches that will affect the analysis and decision-making process. They are:
Test-Fix-Test: Fixes are implemented during the test after the failure modes and the corrective actions have been identified. In this case, the test may be stopped until the correction action is implemented, and the reliability growth is tracked in the given test phase.

Test-Find-Test: Failure modes are identified but the fixes are not implemented until after the completion of the test phase. In this case, the reliability growth is tracked after the completion of a given test phase and the improved product design will be in place for the beginning of the next test phase.

Test-Fix-Test with Delayed Fixes: Some fixes are implemented during the test while other corrective actions are delayed until the completion of the test phase. In this case, the reliability growth will be tracked during and after the completion.

The ultimate goal of these approaches is to ensure that the data are captured from the first test phase through subsequent test phases until reliability goals have been achieved and the product can be released.

This section describes the reliability assessment techniques for repairable systems based on the test-fix-test strategy. Assuming that failures are not necessarily independent or identically distributed, one may use the non-homogenous Poisson process to assess reliability of the system, describing failures or incident events (such as failures or incidents) in a continuum such as time or mileage. Two popular non-homogeneous Poisson process models are discussed and they are known as the reliability growth models as the Duane model [15] and the Crow-AMSAA model [16-18]. Please note that the test-find-test and the test-fix-find-test methods can be found in Reference [19].

4.1. The Duane Model

The first reliability growth model was developed by James T. Duane in 1964 [15], a GE reliability engineer who observed the cumulative failure rate \( C(t) \) is linear and decreasing with the cumulative operating time \( t \) on a log-log plot as illustrated in Figure 3. This relationship is known as the “Duane postulate”. If \( N(t) \) is the cumulative number of failure up to time \( t \), the cumulative failure rate \( C(t) \) by time \( t \) is defined as

\[
C(t) = \frac{N(t)}{t}.
\]  

(77)

Based on Duane’s postulate, the following empirical power law function exists:

\[
C(t) = \lambda t^{-\alpha}
\]  

(78)

where \( \lambda \) and \( \alpha \) are the two curve-fitting “positive” parameters.

Taking logarithms in Equation (78) yields

\[
\log C(t) = \log \lambda - \alpha \log t.
\]  

(79)

Thus the value of \(-\alpha\) is the slope of the line between \( C(t) \) and \( t \) when plotted in a log-log coordinate. Typically, the \( \alpha \) value for vehicle reliability tests ranges from 0.2 to 0.4 [20].
The Duane postulate can be expressed in terms of cumulative mean time between failures $M(t)$ as

$$M(t) = \frac{1}{C(t)} = \frac{1}{\lambda t^\alpha}.$$  \hfill (80)

Generally the cumulative mean time between failures (MTBF) expression is preferred because the upward slope reflects reliability improvement. The positive slope $\alpha$ is sometimes referred to as the “growth rate”. Figure 4 shows the cumulative MTBF versus test time relationship.
Also, by substituting Equation (78) into Equation (77) and introducing a new variable $\beta = 1 - \alpha$, the cumulative number of failures $N(t)$ by time $t$, can be obtained as follows:

$$N(t) = \lambda t^{1-\alpha} = \lambda t^\beta.$$  \hfill (81)

The instantaneous failure intensity or rate $c(t)$ at time $t$ is the derivative of the cumulative number of failures $N(t)$ with respect to time $t$. It is also called the recurrence failure rate. Therefore,

$$c(t) = \frac{dN(t)}{dt} = \lambda(1-\alpha)t^{-\alpha} = \lambda t^{\beta-1}$$ \hfill (82)

or

$$c(t) = (1-\alpha)c(t).$$ \hfill (83)

And, the instantaneous MTBF $m(t)$ is given by

$$m(t) = \frac{1}{c(t)} = \left(\frac{1}{\lambda\beta}\right)t^{\beta-\alpha}. \hfill (84)$$

Assuming reliability growth occurs, the instantaneous MTBF will be greater than the cumulative MTBF because improvements have been introduced. Comparing Equations (80) and (84), one can determine the following relationship:

$$m(t) = \frac{M(t)}{1-\alpha}. \hfill (85)$$

Finally, on the basis of assumption of an exponential failure distribution, one can predict the instantaneous reliability of a system on test as follows:

$$R(t) = \exp(-c(t)t). \hfill (86)$$
Both Equations (78) and (80) are not valid when the start time equals 0 because they yield an infinite failure rate or zero MTBF. Also zero does not appear on a log-log plot. To overcome this difficulty, a start point (or datum) such as the starting cumulative MTBF $M(t_s)$ at time $t_s$ must be selected. The growth rate in the log-log plot can be determined in the following:

$$\alpha = \frac{\log M(t) - \log M(t_s)}{\log t - \log t_s}$$  \hspace{1cm} (87)

or

$$\log M(t) = \log M(t_s) + \alpha (\log t - \log t_s).$$  \hspace{1cm} (88)

Alternatively, Equation (88) can be written as:

$$M(t) = \left(\frac{M(t_s)}{t_s^\alpha}\right)t^\alpha.$$  \hspace{1cm} (89)

With the given datum data $M(t_s)$ and the growth rate $\alpha$, the above equation can be used to estimate the test time $t_i$ required for the system to meet the target cumulative MTBF $M(t_i)$ as

$$t_i = t_s \left(\frac{M(t_i)}{M(t_s)}\right)^{\frac{1}{\alpha}}.$$  \hspace{1cm} (90)

Applications

The reliability growth model allows one to predict reliability of a product by analyzing failure data in the development phase of the product. And as illustrated in Figure 5, the reliability growth prediction tries to generate answers to the following questions:

- How much reliability improvement is obtained by each design change?
- Is design change improving the product reliability or is it degrading it?
- How much reliability improvement can be achieved from the last design change that was done in “good faith” and did not get tested?

![Figure 5 - MTBF growth vs. design development stages](image-url)
Also there is a need to set the target reliability growth curve of a system in order to quantify the current reliability achievement at any time during a design development stage and to predict the likely achievement by the end of the program. Here is an example to demonstrate how to set the target MTBF curve. Consider a test program for a specific design stage with 4000 hours of test time and the target instantaneous MTBF of 500 hours. Based on the assumption that the growth rate $\alpha$ equals to 0.4, the target cumulative and instantaneous MTBF curves can be established and illustrated in Figure 6.

Figure 6 – Development of the Duane reliability growth target curve

4.2. The Crow-AMSAA Model

Larry H. Crow [16-18] noted that the Duane model could be statistically represented as a non-homogeneous Poisson process (NHPP) model with a Weibull failure intensity function. This statistical extension became what is known as the Crow-AMSAA model which was first employed by the U.S. Army Material Systems Analysis Activity (AMSAA) and allows for statistical procedures to be used in the application of this model in reliability growth including a goodness of fit test. The Crow-AMSAA model is designed for tracking the reliability within a test phase and not across test phases.

Crow assumes that the instantaneous failure intensity $c(t)$ can be approximately by the Weibull hazard function, which can be expressed as follows:

$$c(t) = \frac{\beta}{\gamma^\beta} t^{\beta-1}$$  \hspace{1cm} (91)
where \( \theta \) is the Weibull scale parameter and \( \beta \) is the shape parameter. If \( \lambda = \frac{1}{\theta^{\beta}} \), the instantaneous failure intensity at time \( t \) can be rewritten as

\[
c(t) = \lambda \beta \theta^{\beta-1} t^{\beta-1}.
\]

Therefore, the expected (average) number of failures \( N(t) \) within the test interval \([0,t]\) is given by

\[
E[N(t)] = \int_{0}^{t} c(y) dy = \lambda \theta^\beta.
\]

The Crow-AMSAA model is a probabilistic interpretation of the Duane postulate. Duane observed that the cumulative number of failures for a system at the total operating time \( t \) can be approximately by \( \lambda t^\beta \). However, for the Crow-AMSAA model which assumes the actual number of failures observed up to the total operating time \( t \) is a random variable described by the Weibull process, where the average number of failures by time \( t \) is expressed by \( \lambda t^\beta \).

Also the probability that that number of failures equals to \( n \) in a time interval \([0, t]\) is given by the following Poisson distribution:

\[
P[N(t) = n] = \left( \lambda t^\beta \right)^n \exp(-\lambda t^\beta) / n!.
\]

This can be used for the development and use of statistical procedures for reliability growth assessments. So the Crow-AMSAA model is referred to as the non-homogeneous Poisson process with the mean of \( \lambda t^\beta \). For \( \beta = 1 \), the instantaneous failure intensity \( c(t) = \lambda \) is a constant, a sign of reliability static, and the number of failures follows a homogeneous (constant) Poisson process with the mean of \( \lambda T \). The period where a system exhibits constant failure intensity is often called the “useful life” of a system, corresponding to the horizontal part of the bathtub curve. For \( \beta > 1 \), the instantaneous failure intensity \( c(t) \) is increasing, a sign of system reliability deterioration, and this is the characteristic of a wear-out situation in the bathtub curve. For \( \beta < 1 \), the instantaneous failure intensity \( c(t) \) is decreasing, an indication of system reliability growth as occurring in the infant mortality portion of the bathtub curve.

### 4.2.1. Point estimation using the maximum likelihood method (MLE)

The procedures described here are to be used to analyze data from tests, which are terminated at a predetermined time. The parameters \( \lambda \) and \( \beta \) in the instantaneous failure intensity function can be determined by the maximum likelihood method as follows. Let \( k \) be the total number of systems of interest. For the \( q \)-th system, \( q = 1, 2, \ldots, k \), let

- \( T_q \) = ending (or current) time of the \( q \)-th system
- \( N_q \) = total number of failures experienced by the \( q \)-th system
- \( t_{i,q} \) = system time of the \( q \)-th system at the \( i \)-th occurrence of failure, \( i = 1, 2, \ldots, N_q \).

So the successive failure time occurrence is \( 0 < t_{1,q} < t_{2,q} < \ldots < t_{N_q,q} = T_q \). The maximum likelihood estimates of \( \lambda \) and \( \beta \) are values of \( \hat{\lambda} \) and \( \hat{\beta} \), which were derived by Crow [16-17] and Lu and Rudy [18].

The probability density function of the \( i \)-th event at \( t_{i,q} \) given that the \((i-1)\)-th event survived at \( t_{i-1,q} \) for a \( q \)-th system for time duration \([0,T_q]\) is given by

\[
f(t_{i,q} \mid t_{i-1,q}) = f(t_{i,q}) = \frac{\lambda \beta t_{i,q}^{\beta-1} \exp(-\lambda t_{i,q}^\beta)}{R(t_{i-1,q})} = \frac{\lambda \beta t_{i,q}^{\beta-1} \exp(-\lambda t_{i,q}^\beta)}{\exp(-\lambda t_{i-1,q}^\beta)} = \frac{\lambda \beta t_{i,q}^{\beta-1} \exp(-\lambda (t_{i,q}^\beta - t_{i-1,q}^\beta))}{\exp(-\lambda t_{i-1,q}^\beta)},
\]

and the likelihood function of the \( q \)-th system is
Thus the likelihood function for the entire systems can be expressed as follows:

\[
L = \left( \lambda^N \beta^N \exp\left(-\lambda T_q^\beta\right) \prod_{i=1}^{N_q} t_{i,q}^{\beta-1} \right)^k. 
\]

(96)

By taking the nature log on both sides of the above equation, the log likelihood function becomes

\[
\Lambda = \sum_{q=1}^k N_q \ln \lambda + N_q \ln \beta - \lambda T_q^\beta + (\beta - 1) \sum_{i=1}^{N_q} \ln t_{i,q}. 
\]

(97)

(98)

Differentiating the log likelihood function with respect to \( \lambda \) and setting its value equal to zero yields

\[
\frac{\partial \Lambda}{\partial \lambda} = \sum_{q=1}^k \frac{N_q}{\lambda} - \sum_{q=1}^k T_q^\beta = 0. 
\]

(99)

The estimate of \( \lambda \) is then obtained by

\[
\hat{\lambda} = \frac{k}{\sum_{q=1}^k T_q^\beta}. 
\]

(100)

Differentiating the log likelihood function with respect to \( \beta \) and setting its value equal to zero has

\[
\frac{\partial \Lambda}{\partial \beta} = \sum_{q=1}^k \frac{N_q}{\beta} - \lambda \sum_{q=1}^k T_q^\beta \log T_q + \sum_{q=1}^k \sum_{i=1}^{N_q} \log t_{i,q} = 0. 
\]

(101)

The estimate of \( \beta \) can be determined by

\[
\hat{\beta} = \frac{k}{\sum_{q=1}^k T_q^\beta \ln T_q - \sum_{q=1}^k N_q \sum_{i=1}^{N_q} \ln t_{i,q}}. 
\]

(102)

For a special case where all the entire systems have the same ending time, \( T_q = T, q = 1, 2, \cdots, k \), then the estimate of \( \beta \) in Equation (102) can be reduced to

\[
\hat{\beta} = \frac{\sum_{q=1}^k N_q}{\sum_{q=1}^k \sum_{i=1}^{N_q} \ln t_{i,q}}. 
\]

(103)

Obtaining \( \hat{\lambda} \) and \( \hat{\beta} \), one may also estimate the expected number of failures at system time \( t \), instantaneous reliability at system time \( t \), and reliability of a system going another interval \( d \) without failure at system time \( t \) by using the following equations, respectively:

\[
E[N(t)] = \lambda t^\beta, 
\]

(104)

\[
R(t) = \exp\left(-\lambda t^\beta\right). 
\]

(105)
\[ R[(t+d)|t] = \frac{R(t+d)}{R(t)} = \exp \left( -\left[ \hat{\lambda} (t+d)^\hat{\beta} - \hat{\lambda} t^\hat{\beta} \right] \right). \]  

(106)

**4.2.2. Confidence bounds estimation using the Fisher matrix**

Given a function of two parameter statistical distribution \( G(\lambda, \beta) \), its estimated mean and variance can be approximated, respectively, by

\[ \hat{\mathbf{G}} = G(\hat{\lambda}, \hat{\beta}) \]  

(107)

\[ \text{Var}(G) = \left( \frac{\partial G}{\partial \lambda} \right)^2 \text{Var} (\hat{\lambda}) + \left( \frac{\partial G}{\partial \beta} \right)^2 \text{Var} (\hat{\beta}) + 2 \left( \frac{\partial G}{\partial \lambda} \right) \left( \frac{\partial G}{\partial \beta} \right) \text{Cov} (\hat{\lambda}, \hat{\beta}). \]  

(108)

It is assumed that the statistical function \( \hat{\mathbf{G}} \) at the local parameters estimated by the maximum likelihood method (MLE) follows a log normal distribution (i.e. \( \ln(\lambda) \) and \( \ln(\beta) \) follow a normal distribution). Then, with the calculated mean and variance of the function, the approximate two-sided confidence bounds \( \alpha \) on the function can be given as

\[ CB = \hat{\mathbf{G}} \cdot \exp \left( \pm z_{\frac{1-\alpha}{2}} \sqrt{\text{Var}(\ln \hat{\mathbf{G}})} \right) \]  

(109)

where \( z_{\frac{1-\alpha}{2}} \) is the standard normal variable with the cumulative probability of failure of \( \frac{1-\alpha}{2} \).

Because of the following approximation,

\[ \sqrt{\text{Var}(\ln \hat{\mathbf{G}})} \approx \sqrt{\frac{\text{Var}(\hat{\mathbf{G}})}{\hat{\mathbf{G}}}}, \]  

(110)

the above two-sided confidence bounds of the function can be rewritten as

\[ CB = \hat{\mathbf{G}} \cdot \exp \left( \pm z_{\frac{1-\alpha}{2}} \sqrt{\frac{\text{Var}(\mathbf{G})}{\hat{\mathbf{G}}}} \right). \]  

(111)

The determination of the variance and covariance of the parameters is accomplished by the use of the Fisher information matrix. The Fisher information matrix is given by

\[ [F] = \begin{bmatrix} \frac{\partial^2 \Lambda}{\partial \lambda^2} & -\frac{\partial^2 \Lambda}{\partial \lambda \partial \beta} \\ -\frac{\partial^2 \Lambda}{\partial \beta \partial \lambda} & \frac{\partial^2 \Lambda}{\partial \beta^2} \end{bmatrix} \]  

(112)

where \( \Lambda \) is the natural log-likelihood function. The partial derivatives for the Fisher information matrix are:

\[ \frac{\partial^2 \Lambda}{\partial \lambda^2} = -\sum_{q=1}^{l} N_q \]  

(113)

\[ \frac{\partial^2 \Lambda}{\partial \beta^2} = -\sum_{q=1}^{l} \frac{N_q}{\beta^2} - \hat{\lambda} \sum_{q=1}^{l} T_q^\beta (\ln T_q)^2 \]  

(114)

\[ \frac{\partial^2 \Lambda}{\partial \beta \partial \lambda} = -\sum_{q=1}^{l} T_q^\beta \ln T_q \]  

(115)
It should be noted that \( \frac{\partial^2 \Lambda}{\partial \beta \partial \lambda} = \frac{\partial^2 \Lambda}{\partial \lambda \partial \beta} \). The local estimate of the covariance matrix can be obtained by inverting the Fisher information matrix as
\[
\begin{bmatrix}
\text{Var}(\hat{\lambda}) & \text{Cov}(\hat{\lambda}, \hat{\beta}) \\
\text{Cov}(\hat{\beta}, \hat{\lambda}) & \text{Var}(\hat{\beta})
\end{bmatrix} = [F]^{-1}.
\] (116)

**Two-sided confidence bounds for the instantaneous failure intensity at \( t \)**

The mean and variance of the instantaneous failure intensity at \( t \) are calculated
\[
\hat{\dot{c}}(t) = \hat{\lambda} \hat{t}^{\hat{\beta} -1} \quad \text{(117)}
\]
\[
\text{Var}(\dot{c}(t)) = \left( \frac{\partial \dot{c}(t)}{\partial \lambda} \right)^2 \text{Var}(\hat{\lambda}) + \left( \frac{\partial \dot{c}(t)}{\partial \beta} \right)^2 \text{Var}(\hat{\beta}) + 2 \frac{\partial \dot{c}(t)}{\partial \lambda} \frac{\partial \dot{c}(t)}{\partial \beta} \text{Cov}(\hat{\lambda}, \hat{\beta})
\] (118)
where
\[
\frac{\partial \dot{c}(t)}{\partial \lambda} = \hat{\beta} t^{\hat{\beta} -1}
\] (119)
\[
\frac{\partial \dot{c}(t)}{\partial \beta} = \hat{\lambda} t^{\hat{\beta} -1} + \hat{\lambda} \hat{t}^{\hat{\beta} -1} \ln(t).
\] (120)

The approximate two-sided confidence bounds for the instantaneous failure intensity at \( t \) with a confidence level \( \alpha \) are given
\[
\text{CB} = \hat{\dot{c}}(t) \cdot \exp \left( \pm z_{1-\alpha} \frac{\sqrt{\text{Var}(\dot{c}(t))}}{\hat{\dot{c}}(t)} \right).
\] (121)

**Two-sided confidence bounds for the cumulative failure intensity at \( t \)**

The mean and variance of the cumulative failure intensity at \( t \) are calculated
\[
\hat{\dot{C}}(t) = \hat{\lambda} t^{\hat{\beta} -1} \quad \text{(122)}
\]
\[
\text{Var}(C(t)) = \left( \frac{\partial C(t)}{\partial \lambda} \right)^2 \text{Var}(\hat{\lambda}) + \left( \frac{\partial C(t)}{\partial \beta} \right)^2 \text{Var}(\hat{\beta}) + 2 \frac{\partial C(t)}{\partial \lambda} \frac{\partial C(t)}{\partial \beta} \text{Cov}(\hat{\lambda}, \hat{\beta})
\] (123)
where
\[
\frac{\partial C(t)}{\partial \lambda} = t^{\hat{\beta} -1}
\] (124)
\[
\frac{\partial C(t)}{\partial \beta} = \hat{\lambda} t^{\hat{\beta} -1} \ln(t).
\] (125)

The approximate two-sided confidence bounds for the cumulative failure intensity at \( t \) with a confidence level \( \alpha \) are given
\[
\text{CB} = \hat{\dot{C}}(t) \cdot \exp \left( \pm z_{1-\alpha} \frac{\sqrt{\text{Var}(\dot{C}(t))}}{\hat{\dot{C}}(t)} \right).
\] (126)
5. Summary

5.1. Non-Repairable Systems

- **The test-to-failure method (the Weibull analysis method)**

  The test-to-failure method presented in section 2.1 is one of the most common reliability demonstration methods. It has the advantage of providing product life distribution and the failure modes under realistic loading conditions. Point estimation methods such as Median Rank Regression (MRR) and Maximum Likelihood Estimation (MLE) produce point estimates of the Weibull distribution. Depending upon sample size and presence/absence of suspensions, one can use multiple methods to estimate confidence intervals on the parameters such as the likelihood ratio method, the Fisher matrix method and the Monte Carlo pivotal statistics method.

  It must be noted that the likelihood ratio method is best used for medium to large samples, with few or no suspensions. The Fisher matrix method is recommended for medium to large samples as well. It can be easily used to also estimate confidence bounds on fatigue life/time as well as Reliability. The Monte Carlo pivotal statistics method is suggested for small samples with at least 3 failures. The test-to-failure method does present accuracy of problems when there none or very few failures observed via testing.

- **The Weibull analysis of reliability data with few or no failures**

  The Nelson model presented in Section 2.2 is very useful in reliability demonstration for trade-off of sample size, reliability, statistical confidence bounds, and total test time. The Weibull slope should be known before the testing, which can be estimated from historical data, experience, or engineering knowledge. All products on test must complete the planned test time without failure in order to successfully demonstrate the reliability/confidence target.

- **The attribute test method**

  The attribute test method is a binomial test method which can be interpreted as the “success-failure”, “go-no go”, or “acceptable-not acceptable” type of testing. It is typical in verifying minimum reliability levels for new products prior to production release. The test requires multiple test samples, does not allow for any failures occur during testing, and does not disclose failure modes and variability.

- **The extended life test method**

  The extended life test method is derived on the Weibull distribution and the success-run theorem from the binomial distribution. It assumes that there are no failures in the sample set during testing and an estimate of the Weibull slope is known. This is a nice method involving a trade-off between sample size and test time, but like the attribute test method, the extended life test method does not reveal any failure mode and life distribution.

- **The step-stress accelerated life test method**

  This is an accelerated test method to obtain time-to-failure distribution in a relatively short time, but it should be used with caution for there is likelihood that the failure model may be changed due to a higher stress amplitude level.

5.2. Repairable Systems

The reliability growth model allows one to predict reliability of a repairable system by analyzing failure data in the development phase of the system. If a failure can occur at any instant, it may occur more than once in a given time interval. In such a case, the number of failures in any time interval (the cumulative reoccurrence rate) follows a Poisson process. If the instantaneous reoccurrence rate is not constant, then the Poisson process is called the non-homogeneous Poisson process (NHPP), otherwise, it is named the homogenous Poisson process. The non-homogeneous Poisson process can be applied for the infant mortality and the wear out portion of the bathtub curve in a product development life cycle, and the homogeneous Poisson process, for the design life portion. Both of the Duane and the Crow-AMSAA models are the two popular generic non-homogenous Poisson processes that will cover all the three portions of the bathtub product development life curve. Both models were presented in Sections 3.1 and 3.2.
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