Quantum computation with Josephson-qubits by using a current-biased information bus

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We propose an effective scheme for manipulating quantum information stored in a superconducting nanocircuit. The Josephson qubits are coupled via their separate interactions with an information bus, a large current-biased Josephson junction treated as an oscillator with adjustable frequency. The bus is sequentially coupled to only one qubit at a time. Distant Josephson qubits without any direct interaction can be indirectly coupled with each other by independently interacting with the bus sequentially, via exciting/de-exciting vibrational quanta in the bus. This is a superconducting analog of the successful ion trap experiments on quantum computing. Our approach differs from previous schemes that simultaneously coupled two qubits to the bus, as opposed to their sequential coupling considered here. The significant quantum logic gates can be realized by using these tunable and selective couplings. The decoherence properties of the proposed quantum system are analyzed within the Bloch-Redfield formalism. Numerical estimations of certain important experimental parameters are provided.

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I. INTRODUCTION.

The coherent manipulation of quantum states for realizing certain potential applications, e.g., quantum computation and quantum communication, is attracting considerable interest [1]. In principle, any two-state quantum system works as a qubit, the fundamental unit of quantum information. However, only a few real physical systems have worked as qubits, because of requirements of a long coherent time and operability. Among various physical realizations, such as ions traps (see, e.g., [2, 3, 4]), QED cavities (see, e.g., [5, 6]), quantum dots (see, e.g., [7, 8]) and NMR (see, e.g., [9, 10]), etc., superconducting qubits with Josephson junctions offer one of the most promising platforms for realizing quantum computation (see, e.g., [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31]). The nonlinearity of Josephson junctions can be used to produce controllable qubits. Also, circuits with Josephson junctions combine the intrinsic coherence of the macroscopic quantum state and the possibility to control its quantum dynamics by using voltage and magnetic flux pulses. In addition, present-day technologies of integration allow scaling to large and complex circuits. Recent experiments have demonstrated quantum coherent dynamics in the time domain in both single-qubit (see, e.g., [12, 13, 14]) and two-qubit Josephson systems [15].

There are two basic types of Josephson systems used to implement qubits: charge qubits [12] and flux qubits [13], depending on the ratio of two characteristic energies: the charging energy $E_C$ and the Josephson energy $E_J$. The charge qubit is a Cooper-pair box with a small Josephson coupling energy, $E_J \ll E_C$, and a well defined number of Cooper pairs is well defined. The flux qubit operates in another extreme limit, where $E_J \gg E_C$ and the phase is well defined. A “quantonium” circuit operating in the intermediate regime of the former two has also been proposed [14]. Voltage-biased superconducting quantum interference devices (SQUIDs), which work in the charge regime and with controllable Josephson energies, form the SQUID-based charge qubits that we will consider in this work. Our results can be extended to flux and flux-charge qubits.

The key ingredient for computational speedup in quantum computation is entanglement, a property that does not exist in classical physics. Thus, manipulating coupled qubits plays a central role in quantum information processing (QIP). Heisenberg-type qubit-couplings are common for the usual solid state QIP systems, e.g., the real spin states of the electrons in quantum dots [7, 8]. However, the interbit couplings for Josephson junctions involve Ising-type interactions, as superconducting qubits with two macroscopic quantum states provide pseudo-spin-1/2 states. Recently, either the current-current interaction, by connecting to a common inductor, or the charge-charge coupling, via sharing a common capacitor, have been proposed to directly couple two Josephson charge qubits: the $i$th and $j$th ones. These interactions implement $\sigma_z^{(i)} \otimes \sigma_z^{(j)}$-type [15, 16], $\sigma_y^{(i)} \otimes \sigma_y^{(j)}$-type [17], and the $\sigma_z^{(i)} \otimes \sigma_x^{(j)}$-type [18] Ising couplings, respectively. Compared to the single-qubit operations, the two-qubit operations based on these second-order interactions are more sensitive to the environment. Thus, quantum decoherence can be more problematic. In addition, capacitive coupling between qubits is not easily tunable [15]. Thus adjusting the physical parameters for realizing two-qubit operation is not easy. In order to ensure that the quanta of the relevant $LC$ oscillator is not excited during the desired quantum operations, the time scales of manipulation in the inductively coupled circuit should be much slower than the eigenfrequency of the $LC$-circuit [17].

Alternatively, the Josephson qubits may also be coupled together by sequentially interacting with a data bus, instead of simultaneously. This is similar to the techniques used for trapped ions [2, 3], wherein the trapped ions are entangled by exciting and de-exciting quanta of their shared center-of-mass vibrational mode (i.e., the data bus). This scheme allows for...
faster two-qubit operations and possesses longer decoherent times. In fact, an externally connected LC-resonator \[20\] and a cavity QED mode \[21\] were chosen as alternative data buses. However, it is not always easy to control all the physical properties, such as the eigenfrequencies and decoherence, of these data buses.

A large (e.g., up to 10\(\mu\)m) current-biased Josephson junction (CBJJ) \[21\] is very suitable to act as information bus for coupling Josephson qubits. This because: i) the CBJJ is an easily fabricated device \[22\] and may provide more effective immunities to both charge and flux noise; ii) due to its large junction capacitance, the CBJJ can enable to be capacitively coupled over relatively long distances; iii) the quantum properties, e.g., quantum transitions between the junction energy levels, of the current-biased Josephson junction are well established \[23, 24\]; and iv) its eigenfrequency can be controlled by adjusting the applied bias-current. In fact, a CBJJ itself can be an experimentally realizable qubit, as demonstrated by the recent observations of Rabi oscillations in them \[25, 26\]. Two logic states of such a qubit are encoded by the two lowest zero-voltage metastable quantum energy levels of the CBJJ. The decoherent properties of this CBJJ-qubit were discussed in detail in \[27\]. Experimentally, the entangled macroscopic quantum states in two CBJJ-qubits coupled by a capacitor were created \[28\]. Also, by numerical integration of the time-dependent Schrödinger equation, a full dynamical simulation of two-qubit quantum logic gates between two capacitively coupled CBJJ-qubits was given in \[29\].

In this paper, we propose a convenient scheme to selectively couple two Josephson charge-qubits. Here, a large CBJJ acts only as the information bus for transferring the quantum information between the qubits. Thus, hereafter the CBJJ will not be a qubit, as in \[21, 25, 26, 27, 28, 29\]. Two chosen distant SQUID-based charge qubits can be indirectly coupled by sequentially interacting these with the bus. This coupling method provides a repeatable way to generate entangled states, and thus can implement elementary quantum logic gates between arbitrarily selected qubits. Our proposal shares some features with the circuits proposed in \[17, 18, 19, 21\], but also has significant differences. Our proposal might be more amenable to experimental verification.

The outline of the paper is as follows. In Sec. II we propose a superconducting nanocircuit and its elementary quantum evolutions. Conclusions and some discussions are given in Sec. V.

II. A SUPERCONDUCTING NANOCIRCUIT AND ITS ELEMENTARY QUANTUM EVOLUTIONS.

The circuit considered here is sketched in Fig. 1. It consists of \(N\) voltage-biased SQUIDs connected to a large CBJJ. The \(k\)th \((k = 1, 2, \ldots, N)\) qubit consists of a gate electrode of capacitance \(C_{gk}\) and a single-Cooper-pair box with two ultrasmall Josephson junctions of capacitance \(C_{Jk}\) and Josephson energy \(E_{Jk}^0\), forming a DC-SQUID ring. The inductances of these DC-SQUID rings are assumed to be very small and can be neglected. The SQUIDs work in the charge regime with \(k_B T \ll E_j \ll E_C \ll \Delta\), in order to suppress quasiparticle tunneling or excitation. Here, \(k_B, \Delta, E_C, T, \text{and } E_J\) are the Boltzmann constant, the superconducting gap, charging energy, temperature, and the Josephson coupling energy, respectively.

The connected large CBJJ biased by a dc current works in the phase regime with \(E_j \gg E_C\). It acts as a tunable anharmonic LC-resonator with a nonuniform level spacing and works as a data bus for transferring quantum information between the chosen qubits. The mechanism for manipulating quantum information in the present approach is different from that in \[17, 18, 19, 21\], although the circuit proposed here might seem similar to those there. The differences are:

1. A large CBJJ, instead of \(LC\)-resonator \[17, 18, 19\] formed by the externally connected inductance \(L\) and the capacitances in circuit, works as the data bus;
2. We modulate the applied external flux, instead of the bias-current \[21\], to realize the perfect coupling/decoupling between the chosen qubit and the bus; and especially
3. The free evolution of the bus during the operational delays will be utilized for the first time to control the dynamical phases for implementing the expected quantum gates.

The Hamiltonian for the present circuit can be written as

\[
\hat{H} = \sum_{k=1}^{N} \left[ \frac{2e^2}{C_k} \left( \hat{n}_{g_k} - n_{g_k} \right)^2 - E_{J_k} \cos \left( \hat{\theta}_k - \frac{C_{g_k}}{C_k} \frac{\hat{\theta}_b}{k_B} \right) \right] + \hat{H}_r, (1)
\]
with
\[ \hat{H}_r = \frac{(2\pi \hat{p}_b/\Phi_0)^2}{2 \tilde{C}_b} - E_b \cos \hat{\theta}_b - \frac{\Phi_0 I_b}{2\pi} \hat{\theta}_b. \]  
(2)

Here, \( n_{gk} = C_{gk} V_k/(2e) \), \( C_b = C_{gk} + C_{jk} \), \( C_{jk} = 2C^0_{jk} \), \( \tilde{C}_b = C_b + \sum_{k=1}^{N} C_{jk} C_{gk}/C_b, \) \( E_{jk} = 2E_{jk} \cos(\pi \Phi_k/\Phi_0), \) and \( \theta_k = (\theta_{k_1} + \theta_{k_2})/2 \) with \( \theta_{k_1} \) and \( \theta_{k_2} \) being the phase drops across two small Josephson junctions in the \( k \)th qubit, respectively. Also, \( C_{gk}, \Phi_k, \) and \( V_k \) are the gate capacitance, flux quantum, external flux, and gate voltage applied to the \( k \)th qubit, respectively. Correspondingly, \( C_b, \theta_b, E_b, \) and \( I_b \) are the capacitance, phase drops, Josephson energy, and the bias-current of the large CBJJ, respectively. Above, the number operator \( \hat{n}_k \) of excess Cooper-pair charges in the superconducting island and the phase operator \( \hat{\theta}_k \) of the order parameter of the \( k \)th charge qubit are a pair of canonical variables and satisfy the commutation relation:
\[ [\hat{\theta}_k, \hat{n}_k] = i. \]

The operators \( \hat{\theta}_b \) and \( \hat{p}_b \) are another pair of canonical variables and satisfy the commutation relation:
\[ [\hat{\theta}_b, \hat{p}_b] = i\hbar, \]

with \( 2\pi p_b/\Phi_0 = 2n_b e \) representing the charge difference across the CBJJ.

The CBJJ works in the phase regime. Thus, \( E_{Cs} = e^2/(2\tilde{C}_b) \ll E_b \) and the quantum motion ruled by the Hamiltonian \( \hat{H}_r \) equals that of a particle with mass \( m = C_b(\Phi_0/2\pi)^2 \) in a potential \( U(\theta_b) = -E_b(\cos \theta_b + I_b \theta_b/I_r), \) \( I_r = 2\pi E_b/\Phi_0. \) For the biased case \( I_b < I_r, \) there exists a series of minima of \( U(\theta_b), \) where \( \partial^2 U(\theta_b)/\partial \theta_b^2 > 0. \) Near these points, \( U(\theta_b) \) approximates to a harmonic oscillator potential with a characteristic frequency
\[ \omega_b = \sqrt{2\pi I_r} \sqrt{C_b \Phi_0} \left[ 1 - \left( \frac{I_b}{I_r} \right)^2 \right]^{1/4}, \]

depending on the applied bias-current \( I_b. \) Correspondingly, the Hamiltonian \( \hat{H}_r \) reduces to
\[ \hat{H}_b = \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \hbar \omega_b, \]  
(3)

with
\[ \hat{a} = \frac{1}{\sqrt{2}} \left[ \left( \frac{\Phi_0}{2\pi} \right) \sqrt{\frac{C_b \omega_b}{\hbar}} \hat{\theta}_b + i \left( \frac{2\pi}{\Phi_0} \right) \frac{\hat{p}_b}{\sqrt{\hbar \omega_b C_b}} \right], \]

and
\[ \hat{a}^\dagger = \frac{1}{\sqrt{2}} \left[ \left( \frac{\Phi_0}{2\pi} \right) \sqrt{\frac{C_b \omega_b}{\hbar}} \hat{\theta}_b - i \left( \frac{2\pi}{\Phi_0} \right) \frac{\hat{p}_b}{\sqrt{\hbar \omega_b C_b}} \right]. \]

The approximate number of quantum metastable bound states \([33]\) of the quantum oscillator is
\[ N_s = 2^{3/4} \sqrt{E_b/E_{Cs}} (1 - I_b/I_r)^{5/4}. \]

The energy scale of the quantum oscillator \((3)\) is \( \omega_b/(2\pi) \approx 10 \text{ GHz} [25], \) which is of the same order of the Josephson energy in the SQUID. Therefore, the oscillating quantum of the information bus will be really excited, even if only one of the qubits is operated quantum mechanically. This is different from the case considered in \([17]\), wherein the \( LC \)-oscillator shared by all charge qubits are not really excited, as the eigenfrequency of the \( LC \)-circuit is much higher than the typical frequencies of the qubits dynamics. For operational convenience, we assume that the bus is coupled to only one qubit at a time. The coupling between any one of the qubits (e.g., the \( k \)th one) and the bus can, in principle, be controlled by adjusting the applied external flux (e.g., \( \Phi_k \)). In this case, any direct interaction does not exist between the qubits, and the dynamics of the CBJJ can be safely restricted to the Hilbert space spanned by the two Fock states: \( |0_k\rangle \) and \( |1_k\rangle \), which are the lowest two energy eigenstates of the harmonic oscillator of Eq. (3). Furthermore, we assume that the applied gate voltage of any chosen \((k)\) qubit works near its degeneracy point with \( n_{gk} = 1/2, \) and thus only two charge states: \(|n_k = 0\rangle = |\uparrow_k\rangle \) and \(|n_k = 1\rangle = |\downarrow_k\rangle \), play a role during the quantum operation. All other charge states with a higher energies can be safely ignored. Therefore, the Hamiltonian
\[ \hat{H}_{kb} = \hat{H}_k + \hat{H}_b + \lambda_k (\hat{a}^\dagger + \hat{a}) \sigma_y^{(k)}, \]  
(4)

with
\[ \hat{H}_k = \left[ \delta E_{Cs} \sigma_z^{(k)} - E_{jk} \sigma_x^{(k)} \right], \]

describes the interaction between any one of the qubits (e.g., the \( k \)th one) and the bus, and provides the basic dynamics for the present network. Here, \( \delta E_{Cs} = 2e^2(1 - 2n_{gk})/C_k, \) \( \lambda_k = C_{gk} (2\pi/\Phi_0) \sqrt{\hbar/(2C_b \omega_b)}, \) and the pseudospin operators are defined by:
\[ \begin{cases} 
\sigma_z^{(k)} &= |\uparrow_k\rangle \langle \downarrow_k| + |\downarrow_k\rangle \langle \uparrow_k|, \\
\sigma_y^{(k)} &= -i |\uparrow_k\rangle \langle \downarrow_k| + i |\downarrow_k\rangle \langle \uparrow_k|, \\
\sigma_x^{(k)} &= |\uparrow_k\rangle \langle \uparrow_k| - |\downarrow_k\rangle \langle \downarrow_k|.
\end{cases} \]

Above, when the first cosine-term in Hamiltonian (2) was expanded, only the single-quantum transition process approximated to the first-order of \( \hat{\theta}_b \) was considered. The higher order nonlinearities have been neglected as their effects are very weak. In fact, for the lower number states of the bus, we have \( C_{gk} \sqrt{\langle \theta^2 \rangle}/C_b \lesssim 10^{-2}, \) for the typical experimental parameters \([14, 15, 24, 27]\): \( C_b \approx 1pF, \) \( \omega_b/2\pi \approx 10 \text{ GHz}, \) and \( C_{gk}/C_b \approx 10^{-2}. \)

Notice that the coupling strength \( \lambda_k \) between the qubit and the bus is tunable by controlling the flux \( \Phi_k, \) applied to the selected qubit, and the bias-current \( I_b, \) applied to the information bus. For example, such a coupling can be simply turn off by setting the flux \( \Phi_k = \Phi_0/2. \) This allows various elemental operations for quantum manipulations to be realizable in a
Josephson network. Without loss of generality, we assume different Hamiltonians can be induced from Eq. (6) and on the different settings of the controllable external parameters: the fluxes applied to the qubits and the circulating currents in the $|_{1}^{\uparrow}$ and $|_{1}^{\downarrow}$ states of the Gaussian persistent circulating currents in the $k$th SQUID-loop, respectively.

We now discuss the quantum dynamics of the above Josephson network. Without loss of generality, we assume in what follows that the bias-current $I_b$ applied to the CBJ doesn’t change, once it is set up properly beforehand. The quantum evolutions of the system are then controlled by other external parameters: the fluxes applied to the qubits and the voltages across the gate capacitances of the qubits. Depending on the different settings of the controllable external parameters, different Hamiltonians can be induced from Eq. (6) and thus different time-evolutions are obtained. Obviously, during any operational delay $\tau$ with $\Phi_{X_{k}} = \Phi_{0}/2$ and $V_{k} = e/C_{g_{k}}$, the $i$th qubit remains in its idle state because the Hamiltonian vanishes (i.e., $H_{0}^{(i)} = 0$) as $E_{J_{i}} = 0$, $n_{g_{i}} = 0$. However, the data bus still undergoes a free time-evolution:

$$U_{d}(t) = \exp \left( \frac{-it}{\hbar} H_{b} \right).$$

This evolution is useful for controlling the dynamical phase of the qubits to exactly realize certain quantum operations. For the other cases, the dynamical evolutions of the chosen qubit depend on the different settings of the experimental parameters.

1) For the case where $\Phi_{k} = \Phi_{0}/2$ and $V_{k} \neq e/C_{g_{k}}$, the $i$th qubit and the bus separately evolve with the Hamiltonians $\tilde{H}_{b}^{(i)} = -\delta E_{C_{k}} \hat{\sigma}_{x}^{(k)}/2$ and $\tilde{H}_{b}$ determined by Eq. (3), respectively. The relevant time-evolution operator of the whole system reads

$$\tilde{U}_{d}(t) = \exp \left( \frac{-it}{\hbar} \tilde{H}_{b}^{(i)} \right) \otimes \exp \left( \frac{-it}{\hbar} \tilde{H}_{b} \right).$$

2) If the $k$th qubit works at its degenerate point and couples to the bus, i.e., $V_{k} = e/C_{g_{k}}$ and $\Phi_{k} \neq \Phi_{0}/2$, then we have the Hamiltonian

$$\tilde{H}_{kb} = E_{J_{k}} \hat{\sigma}_{z}^{(k)}/2 + \tilde{H}_{b} + i\lambda_{k} \left[ \hat{\sigma}_{+}^{(k)} - \hat{\sigma}_{-}^{(k)} \right].$$

from (6). The corresponding dynamical evolutions are

$$\left\{ \begin{array}{l}
|0_{b}\rangle|0_{k}\rangle \rightarrow \tilde{U}_{kb} e^{i\Delta_{k} t/2}|0_{b}\rangle|0_{k}\rangle, \quad \tilde{U}_{kb} = \exp(-i\tilde{H}_{kb}t), \quad \Delta_{k} = E_{J_{k}}/h - \omega_{b}, \\
|0_{b}\rangle|1_{k}\rangle \rightarrow \tilde{U}_{kb} e^{-i\omega_{b}t} \left\{ \cos \left( \frac{\Omega_{k} t}{2} \right) - i \frac{\Delta_{k}}{\Omega_{k}} \sin \left( \frac{\Omega_{k} t}{2} \right) \right\} |0_{b}\rangle|1_{k}\rangle - \frac{2\lambda_{k}}{\hbar} \sin \left( \frac{\Omega_{k} t}{2} \right) |1_{b}\rangle|0_{k}\rangle, \\
|1_{b}\rangle|0_{k}\rangle \rightarrow \tilde{U}_{kb} e^{-i\omega_{b}t} \left\{ \cos \left( \frac{\Omega_{k} t}{2} \right) + i \frac{\Delta_{k}}{\Omega_{k}} \sin \left( \frac{\Omega_{k} t}{2} \right) \right\} |1_{b}\rangle|0_{k}\rangle + \frac{2\lambda_{k}}{\hbar} \sin \left( \frac{\Omega_{k} t}{2} \right) |0_{b}\rangle|1_{k}\rangle,
\end{array} \right.$$

with $\Omega_{k} = \sqrt{\Delta_{k}^{2} + (2\lambda_{k}/\hbar)^{2}}$.

Specifically, we have the time-evolution operator

$$\tilde{U}_{d}^{(b)}(t) = \tilde{A}(t) \left( \begin{array}{cc}
\cos \left( \frac{2\lambda_{k} t}{\hbar} \sqrt{n + 1} \right) & -\frac{1}{\sqrt{n + 1}} \sin \left( \frac{2\lambda_{k} t}{\hbar} \sqrt{n + 1} \right) \hat{a} \\
\frac{1}{\sqrt{n}} \sin \left( \frac{2\lambda_{k} t}{\hbar} \sqrt{n} \right) & \cos \left( \frac{2\lambda_{k} t}{\hbar} \sqrt{n} \right)
\end{array} \right),$$

with

$$\tilde{A}(t) = \exp \left[ -it \left( \frac{\tilde{H}_{b}}{\hbar} + \frac{E_{J_{k}} \hat{\sigma}_{z}^{(k)}}{2\hbar} \right) \right],$$

for the resonant case: $\Delta_{k} = 0$. This reduces Eq. (10) to the time evolutions:
with persisive regime (far from the resonant point): $2\lambda_k/(\hbar \Delta_k) \ll 1$, we have the time evolution operator
\[
\tilde{U}_2^{(k)}(t) = \hat{A}(t) \exp \left(-i \frac{\tilde{H}_{kb} t}{\hbar} \right), \quad (12)
\]
with
\[
\tilde{H}_{kb} = \lambda_k^2 (|1_k\rangle \langle 1_k| - |0_k\rangle \langle 0_k|)/(\hbar \Delta_k).
\]
It reduces to the following time evolutions:

\[
\begin{align*}
|0_b\rangle|0_k\rangle & \xrightarrow{\tilde{U}_2^{(k)}(t)} \exp \left(it \frac{\lambda_k I}{2}\right) |0_b\rangle|0_k\rangle, \\
|0_b\rangle|1_k\rangle & \xrightarrow{\tilde{U}_2^{(k)}(t)} \exp \left[-it \left(\omega_b + \frac{\lambda_k^2}{\hbar \Delta_k}\right)\right] |0_b\rangle|1_k\rangle, \\
|1_b\rangle|0_k\rangle & \xrightarrow{\tilde{U}_2^{(k)}(t)} \exp \left[-it \left(\omega_b - \frac{\lambda_k^2}{\hbar \Delta_k}\right)\right] |1_b\rangle|0_k\rangle, \\
|1_b\rangle|1_k\rangle & \xrightarrow{\tilde{U}_2^{(k)}(t)} \exp \left[-it \left(2\omega_b + \frac{\lambda_k^2}{\hbar \Delta_k}\right)\right] |1_b\rangle|1_k\rangle.
\end{align*}
\]

3) Generally, if $\Phi_k \neq \Phi_0/2$ and $V_{gb} \neq e/C_{gb}$, then the Hamiltonian (6) can be rewritten as
\[
\tilde{H}_{kb} = \frac{E_k}{2} \tilde{\sigma}_z^{(k)} + \hat{H}_b + i\lambda_k (\hat{a}^{\dagger} \tilde{\sigma}_-^{(k)} - \hat{a} \tilde{\sigma}_+^{(k)}), \quad (13)
\]
with
\[
\begin{align*}
\tilde{\sigma}_x^{(k)} &= -\sin \eta_k \tilde{\sigma}_x^{(k)} - \cos \eta_k \tilde{\sigma}_y^{(k)}, \\
\tilde{\sigma}_y^{(k)} &= -\eta_k \tilde{\sigma}_x^{(k)}, \\
\tilde{\sigma}_z^{(k)} &= \cos \eta_k \tilde{\sigma}_x^{(k)} - \sin \eta_k \tilde{\sigma}_y^{(k)},
\end{align*}
\]
and $\tilde{\sigma}_\pm^{(k)} = (\tilde{\sigma}_x^{(k)} \pm \tilde{\sigma}_y^{(k)})/2$. Here, $\cos \eta_k = E_{J_k}/E_k$, and $E_k = \sqrt{(\delta E_{C_k})^2 + E_{J_k}^2}$. If the bias-current $I_b$ and the flux $\Phi_k$ are set properly beforehand such that $E_{J_k} \sim \hbar \omega_b \ll \delta E_{C_k}$, then the detuning $\hbar \Delta_k = E_k - \hbar \omega_b$ is very large (compared to the coupling strength $\lambda_k \lesssim 10^{-1} E_{J_k}$). Therefore, the time-evolution operator of the system can be approximated as
\[
\tilde{U}_4^{(k)}(t) = \hat{B}(t) \exp \left\{-i \frac{\lambda_k^2 t}{\hbar^2 \Delta_k} \left[ \tilde{\sigma}_x^{(k)} \left( \hat{a}^{\dagger} \hat{a} + \frac{1}{2} \right) + \frac{1}{2} \right] \right\} \quad (14)
\]
with
\[
\hat{B}(t) = \exp \left[-it \left( \frac{\hat{H}_b}{\hbar} + \frac{E_k \tilde{\sigma}_x^{(k)}}{2\hbar} \right) \right].
\]
This implies the following evolutions
\[
\begin{align*}
|0_b\rangle|0_k\rangle & \xrightarrow{\tilde{U}_4^{(k)}(t)} \exp \left(-it \frac{\lambda_k^2}{\hbar^2 \Delta_k} \right) \exp \left(it \frac{\lambda_k}{\hbar} \right) |0_b\rangle|0_k\rangle, \\
|0_b\rangle|1_k\rangle & \xrightarrow{\tilde{U}_4^{(k)}(t)} \exp \left[-it \left(\omega_b + \frac{\lambda_k^2}{\hbar \Delta_k}\right)\right] \exp \left(it \frac{\lambda_k}{\hbar} \right) |0_b\rangle|1_k\rangle, \\
|1_b\rangle|0_k\rangle & \xrightarrow{\tilde{U}_4^{(k)}(t)} \exp \left[-it \left(\omega_b - \frac{\lambda_k^2}{\hbar \Delta_k}\right)\right] \exp \left(it \frac{\lambda_k}{\hbar} \right) |1_b\rangle|0_k\rangle, \\
|1_b\rangle|1_k\rangle & \xrightarrow{\tilde{U}_4^{(k)}(t)} \exp \left[-it \left(2\omega_b + \frac{\lambda_k^2}{\hbar \Delta_k}\right)\right] \exp \left(it \frac{\lambda_k}{\hbar} \right) |1_b\rangle|1_k\rangle.
\end{align*}
\]
cuit can be effectively implemented by selectively using the
nipulating the quantum information stored in the present cir-
and making use of the data bus interacting sequentially with these-
son qubits by using the direct interactions between them. By
plementing one of the universal two-qubit gates with Joseph-
set. Several schemes \[17, 18, 29\] have been proposed for im-
any quantum computing circuit comprises only gates from this

\[
\begin{align*}
|0_b\rangle|0_k\rangle & \rightarrow e^{-i\zeta_k t} \left\{ \left[ \cos(\xi_k t) + i \cos \eta_k \sin(\xi_k t) \right] |0_b\rangle |0_k\rangle + i \sin \eta_k \sin(\xi_k t) |0_b\rangle |1_k\rangle \right\}, \\
|0_b\rangle|1_k\rangle & \rightarrow e^{-i\zeta_k t} \left\{ \left[ \cos(\xi_k t) - i \cos \eta_k \sin(\xi_k t) \right] |0_b\rangle |0_k\rangle + i \sin \eta_k \sin(\xi_k t) |0_b\rangle |0_k\rangle \right\}, \\
|1_b\rangle|0_k\rangle & \rightarrow e^{-i(\zeta_k + \omega_b) t} \left\{ \left[ \cos(\xi'_k t) + i \cos \eta_k \sin(\xi'_k t) \right] |1_b\rangle |0_k\rangle + i \sin \eta_k \sin(\xi'_k t) |1_b\rangle |1_k\rangle \right\}, \\
|1_b\rangle|1_k\rangle & \rightarrow e^{-i(\zeta_k + \omega_b) t} \left\{ i \sin \eta_k \sin(\xi'_k t) |1_b\rangle |0_k\rangle + \left[ \cos(\xi'_k t) - i \cos \eta_k \sin(\xi'_k t) \right] |1_b\rangle |1_k\rangle \right\},
\end{align*}
\]

with
\[
\zeta_k = \omega_k/2 + \lambda_k^2/(2\hbar^2 \Delta_k), \quad \xi_k = E_k/(2\hbar) + \lambda_k^2/(2\hbar^2 \Delta_k),
\]
and
\[
\xi'_k = \xi_k + \lambda_k^2/(h^2 \Delta_k).
\]

In what follows we shall show that any process for ma-
nipulating the quantum information stored in the present cir-
cuit can be effectively implemented by selectively using the
above elementary time-evolutions: \(\tilde{U}_0(t), \tilde{U}_1^{(k)}(t), \tilde{U}_2^{(k)}(t), \tilde{U}_3^{(k)}(t)\).

### III. QUANTUM MANIPULATIONS OF THE SUPERCONDUCTING NANOCIRCUIT.

It is well known that any valid quantum transformation can be
decomposed into a sequence of elementary one- and two-
qubit quantum gates. The set of these gates is universal, and
any quantum computing circuit comprises only gates from this
set. Several schemes \[17, 18, 29\] have been proposed for im-
plementing one of the universal two-qubit gates with Joseph-
son qubits by using the direct interactions between them. By
making use of the data bus interacting sequentially with the se-
lective qubits, Blais et al. \[21\] showed that the two-qubit gate
may be effectively realized. Two important problems will be
solved in our indirect-coupling approach:

i) when one of two qubits is selected to couple with the data
bus, how we can let the remainder qubit decouple completely
from the bus; and

ii) the phase changes of the bus’ and qubit’s states during
the operations are very complicated, how we can control these
phase changes in order to precisely implement the desired
quantum gate.

The scheme in \[21\] assumed that, when one of the two
qubits is tuned to resonance with the bus, then the other qubit
is hardly affected because of its different Rabi frequency. Ob-
viously, this decoupling is not complete and thus it is not easy
to assure that the bus couples only one qubit at a time. By
controlling the external flux \(\Phi_k\) applied to the qubits, the net-
work proposed here provides an effective method for making
the remainder qubit completely decouple from the bus. All the
desired elementary operations for quantum computing can be
exactly implemented by properly setting the experimentally
controllable parameters, e.g., the external \(\Phi_k\), the gate volt-
age \(V_k\), the bias-current \(I_b\), and the duration \(t\) of each selected
quantum evolution, etc.

Hereafter, we assume that each of the selected time-
evolutions can be switched on/off very quickly.

### A. single-qubit operations

First, we show how to realize the single-qubit operations
on each SQUID-qubit. This will be achieved by simply turn-
ning on/off the relevant experimentally controllable param-
ters. For example, if \(n_{\Phi_k} \neq 1/2\) and \(E_{J_b} = 0\) for a time
span \(t\), then the time-evolution \(\tilde{U}_1^{(k)}(t)\) in equation (8) is
realized. This operation is the single-qubit rotation around the \(x\)
axis:

\[
\tilde{R}_x^{(k)}(\phi_k) = \left( \begin{array}{cc} \cos \frac{\phi_k}{2} & i \sin \frac{\phi_k}{2} \\ i \sin \frac{\phi_k}{2} & \cos \frac{\phi_k}{2} \end{array} \right),
\]

with \(\phi_k = \delta E_{C_k} t / \hbar\). Rotations by \(\phi_i = \pi\) and \(\phi_k = \pi/2\)
produce a spin flip (i.e., a NOT-gate operation) and an equal-
weight superposition of logic states, respectively.

The rotation around the \(z\) axis can be implemented by using
the evolution (12). This operation is conditional and depend-
ent on the state of the bus. If the bus is in the ground state
\(|0_b\rangle\), the rotation reads

\[
\tilde{R}_z^{(k)}(\phi_k) = e^{-i\phi_k t} \left( \begin{array}{cc} e^{-i\phi_k} & 0 \\ 0 & e^{i\phi_k} \end{array} \right),
\]

with \(\phi_k = \omega_k/2 + \lambda_k^2/(2\hbar^2 \Delta_k), \phi_k = E_{J_b} t / (2\hbar) + \lambda_k^2 t / (2\hbar^2 \Delta_k)\).
With a sequence of \(x\)- and \(z\)-rotations, any rotation
on the single-qubit can be performed. For example, the Hadamard gate applied to the \(k\)th qubit:

\[
\tilde{H}_g = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right),
\]

can be implemented by a three-step rotation:

\[
\tilde{R}_z^{(k)} \left( \frac{\pi}{4} \right) \otimes \tilde{R}_x^{(k)} \left( \frac{\pi}{2} \right) \otimes \tilde{R}_z^{(k)} \left( \frac{\pi}{4} \right) = -i e^{-i\eta_k(t_1 + t_3)} \tilde{H}_g^{(k)}.
\]
Here, the relevant durations $t_1$, $t_2$, and $t_3$ are set properly to satisfy the conditions

\[
\cos\left(\frac{\delta E_C t_2}{\hbar}\right) = -\sin\left(\frac{\delta E_C t_2}{\hbar}\right) = \sin\left[\frac{E_{jk} t_1}{2\hbar} + \frac{(\lambda_k/\hbar)^2 t_1}{2\Delta_k}\right] = \sin\left[\frac{E_{jk} t_3}{2\hbar} + \frac{(\lambda_k/\hbar)^2 t_3}{2\Delta_k}\right] = \frac{1}{\sqrt{2}}.
\]

B. Two-qubit operations

Second, we show how to realize two-qubit gates by letting a pair of qubits (the $k$th- and $j$th ones) interact separately with the bus. Before the quantum operation, the chosen qubits decouple from the bus. At the end of the desired gate operation the bus should be disentangled again from the qubits, and returned to its ground state. For operational simplicity, we assume that the bus resonates with the control qubit, the $k$th one, i.e., $\Delta_k = 0$. We now consider the following three-step operational process:

i) Couple the control qubit to the bus (i.e., the applied external flux $\Phi_k$ is varied to $\Phi_0$) and realize the evolution $\hat{U}_1^{(k)}(t_1)$ for the duration $t_1$:

\[
\sin\left(\frac{\lambda_k t_1}{\hbar}\right) = -1.
\]  

Then, by returning the $\Phi_k$ to its initial value, i.e., $\Phi_k = \Phi_0/2$, the $k$th qubit can be decoupled from the bus exactly. Before the next step operation, there is an operational delay $\tau_1$. During this delay the state of the qubits does not evolve, while the data bus still undergoes a time-evolution $\hat{U}_0(\tau_1)$.

ii) Couple the target qubit (the $j$th one) to the bus and realize the time-evolution $\hat{U}_2^{(j)}(t_2)$. This is achieved by letting the chosen qubit work near its degenerate point (i.e., $n_{0j} \neq 1/2$) and switching on its Josephson energy (i.e., $\Phi_j \neq \Phi_0/2$). After the time $t_2$ determined by the condition

\[
\cos(\xi j t_2) = -\sin(\xi j t_2) = 1,
\]

we decouple the $j$th qubit from the bus and let it be in the idle state by returning its gate-voltage $V_j$ to the degenerate point ($n_{0j} = 1/2$), and simultaneously switching off the relevant Josephson energy. During another operational delay $\tau_2$ before the next step operation, the bus undergoes another free-evolution $\hat{U}_0(\tau_2)$.

iii) Repeat the first step and realize the evolution $\hat{U}_1^{(k)}(t_3)$ with

\[
\sin\left(\frac{\lambda_k t_3}{\hbar}\right) = 1.
\]  

Diagrammatically, the above three-step operational process with two delays can be represented as follows:

\[
\begin{align*}
|0_k 0_j\rangle &\xrightarrow{\hat{U}_0(\tau_1)} |0_k 0_j\rangle e^{-i\omega_b \tau_1/2} |0_k 0_j\rangle &\xrightarrow{\hat{U}_0(\tau_2) \hat{U}_2^{(j)}(t_2)} &\xrightarrow{e^{-ix} |0_k 0_j\rangle} &\xrightarrow{\hat{U}_2^{(j)}(t_3)} &\xrightarrow{e^{-ix} |0_k 0_j\rangle} |0_k 0_j\rangle, \\
|0_k 1_j\rangle &\xrightarrow{\hat{U}_0(\tau_1)} |0_k 1_j\rangle e^{-i\omega_b \tau_1/2} |0_k 1_j\rangle &\xrightarrow{\hat{U}_0(\tau_2) \hat{U}_2^{(j)}(t_2)} &\xrightarrow{e^{-ix} |0_k 1_j\rangle} &\xrightarrow{\hat{U}_2^{(j)}(t_3)} &\xrightarrow{e^{-ix} |0_k 1_j\rangle} |0_k 1_j\rangle, \\
|0_k 0_j\rangle &\xrightarrow{\hat{U}_0(\tau_1) \hat{U}_2^{(k)}(t_1)} e^{-i\omega_b (t_1 + 3\tau_1/2)} |0_k 0_j\rangle &\xrightarrow{\hat{U}_0(\tau_2) \hat{U}_2^{(j)}(t_2)} &\xrightarrow{ie^{-ix-i\omega_b (t_1 + t_2 + \tau_1 + \tau_2)} (\cos\eta_j |1_0 0_j\rangle + \sin\eta_j |1_0 1_j\rangle)} &\xrightarrow{\hat{U}_2^{(j)}(t_3)} &\xrightarrow{ie^{-ix-i\omega_b T} (\cos\eta_j |0_1 0_j\rangle + \sin\eta_j |0_1 1_j\rangle)} |0_k 0_j\rangle, \\
|0_k 0_j\rangle &\xrightarrow{\hat{U}_0(\tau_1) \hat{U}_2^{(k)}(t_1)} e^{-i\omega_b (t_1 + 3\tau_1/2)} |0_k 0_j\rangle &\xrightarrow{\hat{U}_0(\tau_2) \hat{U}_2^{(j)}(t_2)} &\xrightarrow{ie^{-ix-i\omega_b (t_1 + t_2 + \tau_1 + \tau_2)} (\sin\eta_j |1_0 0_j\rangle - \cos\eta_j |1_0 1_j\rangle)} &\xrightarrow{\hat{U}_2^{(j)}(t_3)} &\xrightarrow{ie^{-ix-i\omega_b T} (\sin\eta_j |0_1 0_j\rangle - \cos\eta_j |0_1 1_j\rangle)} |0_k 0_j\rangle,
\end{align*}
\]

with $T = t_1 + t_2 + t_3 + \tau_1 + \tau_2$ being the total duration of the process, and $\chi = \xi_j t_2 + \omega_b (\tau_1 + \tau_2)/2$. Obviously, the information bus remains in its ground state $|0_b\rangle$ after the operations. If the total duration $T$ is satisfied as

\[
\sin(\omega_b T) = 1,
\]  

the above three-step process with two delays yields a two-
gate approximates the well-known controlled-NOT (CNOT) gate. Then the two-qubit operation
lays out the durations of the first- and third-step operations have been
arbitrary rotations of single qubits. Obviously, if the system
works in the strong charge regime:
which above forms a universal set. Any quantum manipulation can
be implemented. If using one of them, accompanied by arbitrary rotations of single qubits. Obviously, if the system
works in the strong charge regime: \( E_{C}\) is much smaller, and \( \cos \eta_j \approx 0, \sin \eta_j \approx 1 \), then the two-qubit gate \( \hat{U}_1^{(kj)}(\eta_j) \) in (22) approximates the well-known controlled-Phase (CROT) gate

\[
\hat{U}_{CROT}^{(kj)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

Also, if the duration \( t_2 \) of the evolution \( \hat{U}_2^{(j)}(t_2) \) and the delays \( \tau_1, \tau_2 \) are further set properly such that

\[
\cos(\gamma_j t_2) = \sin(\gamma_j' t_2) = \sin(2\Delta t) = 1,
\]

then the two-qubit operation \( \hat{U}_2^{(kj)} \) in (23) reduces to the well-known controlled-phase (CROT) gate

\[
\hat{U}_{CROT}^{(kj)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

with \( \Gamma_j = \exp(i\gamma_j t_2), \Lambda_j = \exp(i\gamma_j' t_2), \gamma_j = E_j/(2\hbar) + \lambda_j^2/(2\hbar^2 \Delta_j), \gamma_j' = \gamma_j + \lambda_j^2/(2\hbar^2 \Delta_j) \), can be implemented. This three-step operational process can similarly be represented diagrammatically as

IV. DECOHERENCE OF THE QUBIT-BUS SYSTEM DUE TO THE BIASED VOLTAGE- AND CURRENT-NOISES

An ideal quantum system preserves quantum coherence, i.e., its time evolution is determined by deterministic reversible unitary transformations. Quantum computation requires a long phase coherent time-evolution. In practice, any physical quantum system is subject to various disturbing factors which destroy phase coherence. In fact, solid-state systems are very sensitive to decoherence, as they contain a macroscopic number of degrees of freedom and interact with the environment. However, coherent quantum manipulations of the qubits are still possible if the decoherence time is finite but not too short. Hence, it is important to investigate the effects of the environmental noise on the present quantum circuit.
The typical noise sources in Josephson circuits consist of the linear fluctuations of the electromagnetic environments (e.g., circuitry and radiation noises) and the low-frequency noise due to fluctuations in various charge/current channels (e.g., the “background charge” and “critical current”). Usually, the former one behaves as Ohmic dissipation [35] and the latter one produces a 1/ν spectrum [36]. Within the present work, we will consider the case of Ohmic dissipation due to linear fluctuations of the external circuit parameters: the bias-current Í applied to the CBJJ and the gate voltages applied to the qubits. The effect of gate-voltage noise on a single charge qubit and that of bias-current noise on a single CBJJ is treated as a quantum system with many degrees of freedom and modeled by a bath of harmonic oscillators. For example, introducing the impedance \( I_\nu \) and bias-current \( I_b \) applied to the CBJJ and the gate voltages applied to the qubits. The well-established Bolch-Redfield formalism [32, 37] offers a systematic way to obtain a generalized master equation for the reduced density matrix of the system, weakly influenced by dissipative environments. A subtle Markov approximation is also made in this theory such that the resulting master equation is local in time. Of course, in the regime of weak bath coupling and low temperatures, this theory is numerically equivalent to a full non-Markovian path-integral approach [38]. For the present qubit-bus system and in the basis spanned by the eigenstates \( \{|g\}, |u_n\rangle, |v_n\rangle, n = 1, 2, \ldots \) of the non-dissipative Hamiltonian \( \hat{H}_{kb} \), the Bloch-Redfield theory leads to the following master equations

\[
\frac{d\sigma_{\alpha\beta}}{dt} = -i\omega_{\alpha\beta} \sigma_{\alpha\beta} + \sum_{\mu\nu} (R_{\alpha\beta\mu\nu} + S_{\alpha\beta\mu\nu}) \sigma_{\mu\nu},
\]

with

\[
\begin{align*}
\hat{H} &= \hat{H}_{kb} + \hat{H}_B + \hat{V}, \\
H_B &= \sum_{j=1,2} \sum_{\omega_j} \left[ \frac{p_{\omega_j}^2}{2m_{\omega_j}} + \frac{m_{\omega_j} \omega_j^2 x_{\omega_j}^2}{2} \right] \\
&= \sum_{j=1,2} \sum_{\omega_j} \left( \hat{a}_{\omega_j}^\dagger \hat{a}_{\omega_j} + \frac{1}{2} \right) \hbar \omega_j, \\
\hat{V} &= -\sin \alpha \sigma_z^{(k)} + \cos \alpha \sigma_x^{(k)} \left( \hat{R}_1 + \hat{R}_1^\dagger \right) - (\hat{a}_{\omega_1}^\dagger R_2 + \hat{a}_{\omega_1} R_2^\dagger),
\end{align*}
\]

being the Hamiltonians of the two baths and their interactions with the non-dissipative qubit-bus system \( \hat{H}_{kb} \), respectively. Above, \( \hat{a}_{\omega_j}, \hat{a}_{\omega_j}^\dagger \) are the Boson operators of the \( j \)th bath, and

\[
\hat{R}_1 = \frac{eC_g}{C_k} \sum_{\omega_1} g_{\omega_1} \hat{a}_{\omega_1}, \quad R_2 = \sqrt{\frac{\hbar}{2C_{\omega_1}}} \sum_{\omega_2} g_{\omega_2} \hat{a}_{\omega_2},
\]

with \( g_{\omega_j} \) being the coupling strength between the oscillator of frequency \( \omega_j \) and the non-dissipative system. The effects of these noises can be characterized by their power spectra, which in turn depend on the corresponding “impedance” (or “inductance”) and the temperature of the relevant circuits. For example, the spectral density for the bias-current source can be approximated as

\[
G(\omega) = \pi \sum_{\omega_1} |g_{\omega_1}|^2 \delta(\omega - \omega_1) \sim R_V \omega. \quad (26)
\]

Similarly, the spectral density for the gate-voltage source can be approximated as

\[
F(\omega) = \pi \sum_{\omega_2} |g_{\omega_2}|^2 \delta(\omega - \omega_2) \sim Y_I \omega, \quad (27)
\]

with \( Y_I \) being the dissipative part of the admittance of the current bias.
\[ R_{\alpha\beta\mu
u} = -\frac{1}{\hbar^2} \int_0^\infty d\tau \times \left[ g_1(\tau) \left( \delta_{\beta\nu} \sum_\kappa A_{\alpha\kappa} A_{\kappa\mu} e^{i\omega_{\mu\nu}\tau} \right) - A_{\alpha\mu} A_{\nu\beta} e^{i\omega_{\mu\nu}\tau} \right] + g_1(-\tau) \delta_{\alpha\mu} \sum_\kappa A_{\nu\kappa} A_{\kappa\beta} e^{i\omega_{\mu\nu}\tau} - A_{\alpha\mu} A_{\nu\beta} e^{i\omega_{\mu\nu}\tau} \right], \tag{29} \]

and

\[ S_{\alpha\beta\mu
u} = -\frac{1}{\hbar^2} \int_0^\infty d\tau \times \left[ g_2^+(\tau) \left( \delta_{\beta\nu} \sum_\kappa B_{\alpha\kappa}^\dagger B_{\kappa\mu} e^{i\omega_{\mu\nu}\tau} - B_{\alpha\mu} B_{\nu\beta}^\dagger e^{i\omega_{\mu\nu}\tau} \right) + g_2^-(\tau) \delta_{\alpha\mu} \sum_\kappa B_{\nu\kappa}^\dagger B_{\kappa\beta} e^{i\omega_{\mu\nu}\tau} - B_{\alpha\mu} B_{\nu\beta}^\dagger e^{i\omega_{\mu\nu}\tau} \right], \tag{30} \]

with

\[
\begin{align*}
&g_1(\pm \tau) = \left( \frac{\epsilon C_0}{C_k} \right)^2 \sum_{\omega_1} |g_{\omega_1}|^2 \left[ \langle n(\omega_1) + 1 \rangle e^{\mp i\omega_1 \tau} + \langle n(\omega_1) \rangle e^{\pm i\omega_1 \tau} \right], \\
&g_2^+(\pm \tau) = \left( \frac{\hbar}{2C_0\omega_b} \right) \sum_{\omega_2} |g_{\omega_2}|^2 \langle n(\omega_2) + 1 \rangle e^{\mp i\omega_2 \tau}, \\
&g_2^-(\pm \tau) = \left( \frac{\hbar}{2C_0\omega_b} \right) \sum_{\omega_2} |g_{\omega_2}|^2 \langle n(\omega_2) \rangle e^{\pm i\omega_2 \tau}.
\end{align*}
\]

Above, each one of the states \(|\alpha\rangle, |\beta\rangle, \ldots\) can be equal to one of the eigenstates of \(\tilde{H}_{kb}\). \(\langle n(\omega_j) \rangle = 1/\left[ \exp(\hbar\omega_j/k_BT) - 1 \right]\) is the average number of thermal photons in the mode of frequency \(\omega_j\). The denotation \(x_{ab} = \langle \alpha| \hat{x} |\beta\rangle\) accounts for the matrix element of operator \(\hat{x}\), i.e.,

\[
A_{\alpha\beta} = \langle \alpha| \hat{A}_k |\beta\rangle, \quad \hat{A}_k = \vec{\sigma}_x \sin \alpha_k + \vec{\sigma}_y \cos \alpha_k = \sigma_z^{(k)},
\]

and

\[
B_{\alpha\beta} = \langle \alpha| \hat{a} |\beta\rangle, \quad B_{\alpha\beta}^\dagger = \langle \alpha| \hat{a}^\dagger |\beta\rangle.
\]

Also, \(\omega_{\alpha\beta} = (E_{\alpha} - E_{\beta})/\hbar\) with \(E_{\alpha} (E_{\beta})\) being one of eigenvalues of the non-dissipative Hamiltonian \(\tilde{H}_{kb}\), corresponding to the eigenstate \(|\alpha\rangle (|\beta\rangle\). The spectrum of \(\tilde{H}_{kb}\) includes the ground state \(|g\rangle = |\pm k, 0\rangle\), corresponding to the energy \(E_g = -\hbar \Delta_k/2\), and a series of dressed doubled states

\[
\begin{align*}
|u_n\rangle &= \cos \theta_n |+ k, n\rangle - i \sin \theta_n |- k, n + 1\rangle, \\
|v_n\rangle &= -i \sin \theta_n |+ k, n\rangle + \cos \theta_n |- k, n + 1\rangle
\end{align*}
\]

corresponding to the eigenvalues

\[
E_{u_n} = \hbar \omega_b (n + 1) - \frac{\rho_n}{2}, \quad E_{v_n} = \hbar \omega_b (n + 1) + \frac{\rho_n}{2},
\]

with

\[
\cos \theta_n = \frac{\rho_n - \hbar \Delta_k}{\sqrt{(\rho_n - \hbar \Delta_k)^2 + 4 \Delta_k^2 (n + 1)}},
\]
Here, $|\pm k\rangle$ and $|n\rangle$ are the eigenstates of the operators $\sigma_z^{(k)}$ and $\hat{H}_0$ with eigenvalues $\pm 1$ and $\hbar \omega_b (n + 1/2)$, respectively.

Under the secular approximation, the evolution of the non-diagonal element $\sigma_{\alpha\beta}$ of the reduced density matrix $\sigma$ is determined by

$$\frac{d}{dt} \sigma_{\alpha\beta} = \{ i \omega_{\alpha\beta} + \text{Im}(R_{\alpha\beta\bar{\alpha}\bar{\beta}}) + \text{Im}(S_{\alpha\beta\bar{\alpha}\bar{\beta}}) \} \sigma_{\alpha\beta} = 0. \quad (31)$$

Here, $R_{\alpha\beta\mu\nu}$ and $S_{\alpha\beta\mu\nu}$ are calculated respectively from $R_{\alpha\beta\mu\nu}$ and $S_{\alpha\beta\mu\nu}$ by setting $\mu = \alpha$ and $\nu = \beta$. $\text{Re}(x)$ and $\text{Im}(x)$ represent the real- and imaginary parts of the complex number $x$. The formal solution of the above differential equation (31) reads

$$\sigma_{\alpha\beta}(t) = \sigma_{\alpha\beta}(0) \exp \left( -T_{\alpha\beta}^{-1} t \right) \exp \left( -i \Theta_{\alpha\beta} t \right), \quad (32)$$

with $\Theta_{\alpha\beta} = \omega_{\alpha\beta} + \text{Im}(R_{\alpha\beta\alpha\beta}) + \text{Im}(S_{\alpha\beta\alpha\beta})$ being the effective oscillating frequency (the original Bohr frequency $\omega_{\alpha\beta}$ plus the Lamb shift $\Delta \omega_{\alpha\beta} = \text{Re}(R_{\alpha\beta\alpha\beta}) + \text{Im}(S_{\alpha\beta\alpha\beta})$, and

$$T_{\alpha\beta}^{-1} = -[\text{Re}(R_{\alpha\beta\alpha\beta}) + \text{Re}(S_{\alpha\beta\alpha\beta})] \quad (33)$$

describing the rate of decoherence between the states $|\alpha\rangle$ and $|\beta\rangle$.

In the present qubit-bus system operating near the resonant point: $E_k/\hbar \sim \hbar \omega_b$, the decoherences relating to the lowest three energy eigenstates, i.e., $|g\rangle, |u_0\rangle = |u\rangle$, and $|v_0\rangle = |v\rangle$, are specially important for the desired quantum manipulations. The decoherences outside these three states are negligible. After a long but direct derivation, we obtain the decoherence rates of interest:

$$T_{gu}^{-1} = \alpha V \left\{ 4 \left( \sin \alpha \cos^2 \theta_0 \right)^2 \frac{2 \hbar B T}{\hbar} + 2 \left( \cos \alpha \cos \theta_0 \right)^2 \coth \left( \frac{\hbar \omega_{ug}}{2 \hbar B T} \right) \omega_{ug} \right. \right.$$  
$$\left. + \left( \cos \alpha \sin \theta_0 \right)^2 \left[ \coth \left( \frac{\hbar \omega_{ug}}{2 \hbar B T} \right) - 1 \right] \omega_{ug} + \left( \sin \alpha \sin \theta_0 \right)^2 \left[ \coth \left( \frac{\hbar \omega_{vu}}{2 \hbar B T} \right) - 1 \right] \omega_{vu} \right\} + \omega_{ug} + \omega_{vu} \quad (34)$$

$$T_{gv}^{-1} = \alpha V \left\{ 4 \left( \sin \alpha \sin^2 \theta_0 \right)^2 \frac{2 \hbar B T}{\hbar} + 2 \left( \cos \alpha \sin \theta_0 \right)^2 \coth \left( \frac{\hbar \omega_{vg}}{2 \hbar B T} \right) \omega_{vg} \right. \right.$$  
$$\left. + \left( \cos \alpha \sin \theta_0 \right)^2 \left[ \coth \left( \frac{\hbar \omega_{vg}}{2 \hbar B T} \right) - 1 \right] \omega_{vg} + \left( \sin \alpha \sin \theta_0 \right)^2 \left[ \coth \left( \frac{\hbar \omega_{vv}}{2 \hbar B T} \right) + 1 \right] \omega_{vv} \right\} + \omega_{vg} + \omega_{vv} \quad (35)$$

and

$$T_{uv}^{-1} = \alpha V \left\{ 4 \left( \sin \alpha \cos \theta_0 \right)^2 \frac{2 \hbar B T}{\hbar} + 2 \left( \sin \alpha \sin \theta_0 \right)^2 \coth \left( \frac{\hbar \omega_{uv}}{2 \hbar B T} \right) \omega_{uv} \right. \right.$$  
$$\left. + \left( \cos \alpha \cos \theta_0 \right)^2 \left[ \coth \left( \frac{\hbar \omega_{uv}}{2 \hbar B T} \right) + 1 \right] \omega_{uv} + \left( \cos \alpha \sin \theta_0 \right)^2 \left[ \coth \left( \frac{\hbar \omega_{vv}}{2 \hbar B T} \right) + 1 \right] \omega_{vv} \right\} + \omega_{uv} + \omega_{vv} \quad (36)$$

Above, the various Bohr frequencies read

$$\omega_{ug} = \omega_b/2 + E_k/(2\hbar) - \sqrt{(\hbar \omega_b - E_k)^2 + 4\lambda_b^2/(2\hbar)},$$

$$\omega_{vg} = \omega_b/2 + E_k/(2\hbar) + \sqrt{(\hbar \omega_b - E_k)^2 + 4\lambda_b^2/(2\hbar)},$$

and

$$\omega_{uv} = \omega_b/2 + E_k/(2\hbar) + \sqrt{(\hbar \omega_b - E_k)^2 + 4\lambda_b^2/(2\hbar)}.$$
Thus, the minimum decoherent rates \( a \) single SQUID-qubit is sufficiently weak: CBJJ can be engineered \([25]\) to be the dimensionless parameter  

\[
\alpha = \frac{\pi R_V C_{g_u}^2}{|R_K C_{g_u}^2|}, \quad R_K = \frac{h}{e^2} \approx 25.8 \, k\Omega \text{ and } \alpha_I = \frac{Y_I}{(\bar{C}_g \omega_b)} \text{ characterize the coupling strengths between the environments and the system.}
\]

Specially, if the system works far from the resonant point (with \( \lambda_k \sim 0 \), achieved by switching off the Josephson energy), the above results (shown in Eqs. \((34-36)\)) reduce to those \([11, 27, 35]\) for the case when the qubit and the bus independently decohere. Namely, \( T_{g^{-1}} \) reduces to the rate \([11]\)

\[
T_{g}^{-1} = 8\alpha_V k_B T/h,
\]

which describes the decoherence between two charge states \(| \downarrow \rangle \) and \(| \uparrow \rangle \) of the superconducting box with zero Josephson energy. Also, \( T_{g^{-1}} \) reduces to the decoherent rate \([27]\)

\[
T_{01}^{-1} = \alpha_I [\coth(\hbar \omega_b / (2k_B T)) + 1] \omega_b,
\]

between the ground and first excited states of the data bus. However, for the strongest coupling case (i.e., when the system works at the resonant point), we have \( E_k = E_{J \delta} = \hbar \omega_b, \cos \alpha_k = 1, \cos \theta_0 = \sin \theta_0 = 1/\sqrt{2}, \) and \( \coth[\hbar \omega_{ug} / (2k_B T)] - 1 \approx \coth[\hbar \omega_{ug} / (2k_B T)] - 1 \approx 0 \) (\(< 10^{-7} \)), for the typical experimental parameters \([12]\): \( \lambda_k \approx 0.1 E_{J \delta}, \quad E_{J \delta} = \hbar \omega_b \approx 50 \mu eV \gg k_B T \approx 3 \mu eV \). Thus, the minimum decoherent rates

\[
\bar{T}_{g}^{-1} = (\alpha_V + \alpha_I) \omega_{ug}, \tag{37}
\]

\[
\bar{T}_{g}^{-1} = (\alpha_V + \alpha_I) \omega_{ug}, \tag{38}
\]

and

\[
\bar{T}_{u}^{-1} = \bar{T}_{g}^{-1} + \bar{T}_{g}^{-1}, \tag{39}
\]

are obtained for the above three dressed states, respectively.

It has been estimated in Ref. \([11]\) that the dissipation for a single SQUID-qubit is sufficiently weak: \( \alpha_V \sim 10^{-6} \) for \( R_V = 500 \Omega, \quad C_{g_u}/C_{g_b} \sim 10^{-2}, \) which allows, in principle, for \( 10^5 \) coherent single-qubit manipulations. For a single CBJJ the dimensionless parameter \( \alpha_I \) only reaches \( 10^{-3} \) for typical experimental parameters \([25]\): \( \lambda_k \sim 100 \Omega, \quad C_{g_b} \sim 6 \, pF, \quad \omega_b / 2\pi \sim 10 \, GHz \). This implies that the quantum coherence of the present qubit-bus system is mainly limited by the bias-current fluctuations. Fortunately, the impedance of the above CBJJ can be engineered \([25]\) to be \( 1/Y_I \sim 560 \, k\Omega \). This lets \( \alpha_I \) reach up to \( 10^{-3} \) and allow about \( 10^5 \) coherent manipulations of the qubit-bus system.

### V. CONCLUSIONS AND DISCUSSIONS

In summary, we have proposed an effective scheme to couple any pair of selective Josephson charge qubits by letting them sequentially couple to a common CBJJ, which can be treated as an oscillator with adjustable frequency. Two logic states of the present qubit are encoded by the clockwise and anti-clockwise persistent circuiting currents in the dc SQUID-loop. At most one qubit can be set to interact with the bus at any moment. The interaction between the selected qubit and the data bus is tunable by controlling the flux applied to the qubit and the bias-current applied to the data bus. This selective coupling provides a simple way to manipulate the quantum information stored in the connected SQUID-qubits. Indeed, any pair of selective qubits without any direct interaction can be entangled by using a three-step coupling process. Furthermore, if the total duration is set up properly, the desired two-qubit universal gates, which are very similar to the CNOT- and CROT gates, can be implemented via such three-step operational processes. During this operation, the mode of the data bus is unchanged, although its vibrational quantum is really excited/absorbed. After the desired quantum operation is performed on the chosen qubits, the data bus disentangles from the qubits and returns to its ground state.

In previous schemes, the distant Josephson qubits are coupled directly by either the charge-charge interaction, via connecting to a common capacitor, or by a current-current interaction, via sharing a common inductor. The present indirect coupling scheme offers some advantages: i) the coupling strength is tunable and thus easy to be controlled for realizing the desired quantum gate, ii) this first-order interaction is more insensitive to the environment, and thus possesses a longer decoherence time. Also, compared to previous data buses, the externally connected \( LC \)-resonator \([19]\) and cavity QED mode \([20]\), the present CBJJ bus might be easier to control for coupling the chosen qubit. For example, its eigenfrequency can be controlled by adjusting the applied dc bias-current. In addition, the CBJJ is easy to fabricate using current technology \([22]\) and may provide more effective immunities to both charge and flux noise.

By considering the decoherence due to the linear fluctuations of the applied voltage \( V_k \) and current \( I_k \), we have analyzed the experimental possibility of the present scheme within the Bloch-Redfield formalism. A simple numerical estimate showed that the quantum manipulations of the present qubit-bus system are experimentally possible, once the impedance \( Y_I \) of the CBJJ can be engineered to have a sufficient low value, i.e., \( 1/Y_I \) can be enlarged sufficiently (e.g., \( 1/Y_I \sim 560 \, k\Omega \) \([25]\)). Of course, this possibility, like those in previous schemes \([17, 18, 19, 20, 21]\), is also limited by other technological difficulties, e.g., suppress the low-frequency \( 1/f \) noise, and fast switch on/off the external flux to couple/decouple the chosen qubit, etc.. For example, a very high sweep rate of magnetic pulse (e.g., up to \( 10^8 \) Oe/s \([39]\)), is required to change half of flux quantum through a SQUID-loop (with the size e.g., \( 50 \mu m \)) in a sufficiently short time (e.g., the desired \( \sim 40 \) ps). This and other obstacles pose a challenge that motivates the exploration of novel circuit designs that might minimize some of the problems that lie ahead in the future.
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