Electron transport in a two-terminal Aharonov-Bohm ring with impurities

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Abstract

Electron transport in a two-terminal Aharonov-Bohm ring with a few short-range scatterers is investigated. An analytical expression for the conductance as a function of the electron Fermi energy and magnetic flux is obtained using the zero-range potential theory. The dependence of the conductance on positions of scatterers is studied. We have found that the conductance exhibits asymmetric Fano resonances at certain energies. The dependence of the Fano resonances on magnetic field and positions of impurities is investigated. It is found that collapse of the Fano resonances occurs and discrete energy levels in the continuous spectrum appear at certain conditions. An explicit form for the wave function corresponding to the discrete level is obtained.

Key words: zero-range potential, conductance, Fano resonance, quantum ring, scattering

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Introduction

Quantum ring interferometers of various geometry are intensively studied theoretically [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and experimentally [12, 13, 14, 15, 16, 17, 18, 19] for several decades starting from pioneering works [1, 2]. The interest to the systems is stipulated by several new phenomena such as Aharonov-Bohm oscillations [20, 21], persistent current [22, 23], electron trapping effect in magnetic field [24] and so forth. Recently, the interest to the system has been enhanced due to the experimental detection [18].
of the Fano resonances in the electron transport. These resonances consist of asymmetric peak and dip on the energy dependence of transmission coefficient. They are caused by interference of bound states and continuum of propagating electron waves. Fano resonances in electron transport through various systems are widely studied in literature.

The simplest and most popular model of quantum interferometer is the one-dimensional ring with two wires attached to it. This model has been studied in a lot of papers. In particular, the periodic dependence of conductance on the magnetic field was obtained in Ref. using the scattering matrix approach. The dependence of the transmission probability on the phase shift of the transmitted wave for different degrees of coupling between the current leads and the ring in the presence of a flux was investigated in Ref. It was found that sharp resonances of the Breit–Wigner type exist in the transmission probability. Two mechanisms giving rise to sharp resonances were considered. A one-dimensional quantum waveguide theory for mesoscopic structures was proposed in Ref. In particular, the conductance of two-terminal Aharonov-Bohm ring as a function of the magnetic flux, the arm lengths, and the electron wave vector was found. The resonant-transport properties of the two-terminal ring and multiply connected ring systems threaded by the magnetic flux was studied using the tight-binding model. An explicit form for the conductance of the one-dimensional two-terminal ring threaded by the magnetic flux was found in Ref. using the zero-range potential theory. The charge carrier interference in mesoscopic semiconductor rings formed by two quantum wires in self-organizing silicon quantum wells was investigated in Ref. The periodic dependence of the transmission coefficient and the phase shift on source-drain bias in the presence of several δ-shaped barriers in wires was obtained. The model of one-dimensional quantum ring was used to explain the results in this work.

The presence of an additional scatterer, for instance a quantum dot or an impurity, in the interferometer provides new possibilities to control the electron transport. Therefore the effect of scatterers on the conductance is considered in a number of papers. The electron transport and the persistent current in a mesoscopic ring connected to current leads was investigated in Ref. using an S-matrix approach. The case of one impurity in the ring and diametrically opposite contacts was considered. A contact between wires and ring was defined using an a priori given energy-independent scattering matrix. It was shown that in the presence of an impurity transmission
probability of the ring may exhibit resonances of the Fano type in addition
to symmetric Breit-Wigner peaks. It was found that the transition from the
weak to the strong coupling regime leads to Fano line shapes with gradually
smaller asymmetry parameters, but the positions of the transmission zeros
and ones remain unaffected. The Breit-Wigner like shape is sensitive to the
coupling and disappears in the strong coupling limit. The presence of an
Aharonov-Bohm flux alters the amplitudes of the Fano resonances, turning
them progressively into broad oscillations. It was found that placing the
impurity in a special positions in the arm and certain values of flux causes
systematic collapse of certain Fano resonances.

However, the effect of impurities on the electron transport in the quantum
ring requires further analysis. In spite of the large number of theoretical
works the case of many impurities has not been studied in detail yet. At the
same time, strong dependence of the conductance on positions of impurities
might be expected in the system due to quantum interference phenomena.

The main purpose of the present paper is the investigation of the electron
transport in the Aharonov-Bohm ring containing several short-range
scatterers. The role of scatterers may be played by impurities or small quan-
tum dots. We consider the model of one-dimensional ring with two one-
dimensional wires attached to it. The advantage of the model is the possibil-
ity to obtain an analytical expression for the transmission coefficient of the
device. In the case of one-dimensional wires the one-mode transport regime
takes place and the conductance does not exceed a unit of the conductance
quantum. In this case the transport properties of the device are determined
by the single transmission coefficient. To obtain this coefficient we use an
approach based on the zero-range potential theory that was used before in
Refs. [32, 33, 34, 35, 36]. In this method, contacts and impurities are de-
scribed by boundary conditions at the points of perturbations. The general
form of the conditions is defined with the help of the self-adjoint extensions
theory for symmetric operators [39, 40, 41, 42, 37, 38]. In the framework of
the approach, the system is described by the single Hamiltonian and parts
of the device are connected to each other via energy independent boundary
conditions. At the same time, the scattering matrix of each contact is energy
dependent in this case. Since the Hamiltonian of the device should be the
energy-independent operator our method seems to be more accurate then the
approach based on a priori defined energy-independent scattering matrix of
each contact.

It should be mentioned that non-zero width rings have been studied nu-
merically \[9, 10, 11\]. Results of those papers are in qualitative increment with the results obtained in the framework of the one-dimensional model. In particular, the one-dimensional model describes correctly quantum interference phenomena if the width of the ring is much smaller then its radius \[11\].

1. Hamiltonian and transmission coefficient

Let us consider a quantum ring \(S_{\rho}\) of radius \(\rho\) with two one-dimensional wires \(W_1\) and \(W_2\) attached to it at points \(A_1\) and \(A_2\) respectively. The scheme of the device is shown in Fig. 1.

![Figure 1: Two-terminal Aharonov-Bohm ring \(S_{\rho}\) with impurities located at points \(P_i\). The wires are denoted by \(W_1\) and \(W_2\). Points \(A_1\) and \(A_2\) define positions of contacts, \(r\) and \(t\) are transmission and reflection amplitudes for the electron wave.](image)

We assume that the ring contains \(N\) short-range impurities at points \(P_i\) \((i = 1, \ldots N)\). All perturbations are described by the same method using the boundary conditions for the wave function. Angles defining positions of perturbations we denote by \(\varphi_j\), where indexes \(j = 1, 2\) correspond to contacts while indexes \(j = 3, \ldots N + 2\) correspond to impurities. It is convenient to use continuous numeration for all perturbations (contacts and impurities) since they all are described by the same way. Without loss of generality we can put \(\varphi_1 = 0\). The wires \(W_j\) are modeled by semiaxes \(x \geq 0\). We suppose
that the ring is placed in the magnetic field $B$ perpendicular to the plane of the system. The magnetic flux through the ring is denoted by $\Phi = \pi \rho^2 B$.

The electron Hamiltonian of the unperturbed ring is given by the following equation \[5\]

$$H_\rho = \frac{\hbar^2}{2m^* \rho^2} \left(-i \frac{d}{d\varphi} + \eta\right)^2.$$ (1)

Here $m^*$ is the electron effective mass, $\eta = \Phi / \Phi_0$ is the number of magnetic flux quanta and $\Phi_0 = 2\pi \hbar c / |e|$ is the magnetic flux quantum.

The eigenvalues of the Hamiltonian $H_\rho$ are well known
$$E_m^\eta = \frac{\hbar^2}{2m^* \rho^2} (m + \eta)^2,$$ (2)

where $m$ is the magnetic quantum number.

The electron Hamiltonian $H_j$ of each wire $W_j$ has the form

$$H_j = -\frac{\hbar^2}{2m^*} \frac{d^2}{dx^2}.$$ (3)

The electron wave function of the device may be represented in the form of one-column matrix

$$\psi = \begin{pmatrix} \psi_\rho \\ \psi_1 \\ \psi_2 \end{pmatrix},$$ (4)

where $\psi_\rho$ is the wave function on the ring $S_\rho$, and $\psi_j$ are wave functions in wires $W_j$.

The impurities are described by the potential

$$V(\varphi) = \sum_{j=3}^{N+2} V_j \delta(\varphi - \varphi_j),$$ (5)

where $\delta(\varphi)$ is the Dirac $\delta$-function, and coefficients $V_j$ define the strength of the point perturbations. It should be mentioned that the potential $V(\varphi)$ is equivalent to the boundary conditions of the following form

$$\psi_\rho(\varphi_j) = \frac{1}{v_j}[\psi'_\rho(\varphi_j + 0) - \psi'_\rho(\varphi_j - 0)], \quad j = 3, \ldots N + 2,$$ (6)

where $\psi'_\rho$ means the derivative of $\psi_\rho$ on angle $\varphi$ and $v_j = 2m^* \rho^2 V_j / \hbar^2$ is the dimensionless parameter determining the strength of the point perturbation.
The contacts are modeled with the help of the zero-range potential theory [32, 33, 34, 35, 36, 37]. If there were no contact between different parts of the system then the Hamiltonian $H_0$ should have the form

$$H_0 = H_ρ ⊕ H_1 ⊕ H_2.$$  \hspace{1cm} (7)

The direct sum in Eq. (7) means that each operator acts on its own part of the wave function and those parts are independent. Hamiltonian $H$ of the systems is obtained from the Hamiltonian $H_0$ by applying linear boundary conditions for the wave function at points of contacts.

The most general form of the conditions may be obtained from the operator extension theory [37]

$$\begin{align*}
\psi_ρ(\varphi_j) &= b_j [\psi_ρ'(\varphi_j + 0) - \psi_ρ'(\varphi_j - 0)] + a_j \psi_j'(0), \\
\psi_j(0) &= a_j [\psi_ρ'(\varphi_j + 0) - \psi_ρ'(\varphi_j - 0)] + c_j \psi_j'(0),
\end{align*}$$  \hspace{1cm} (8)

where $\psi_j'$ is the derivative of $\psi_j$ on x, $a_j$ are complex parameters of dimension of length, while $b_j$ and $c_j$ are real parameters of the same dimension (here $j = 1, 2$). In the framework of the approach each contact is characterized by four real parameters. We note that the same general form of boundary conditions may be obtained from the current conservation law for each contact.

In the present paper, we will restrict ourselves by the case of continuous one-dimensional wave function at the point of contact that corresponds to equal effective width of the ring and the wires. One can see from Eqs. (8) that parameters $a_j$, $b_j$ and $c_j$ should be equal in this case. It is convenient to represent them in terms of the dimensionless coupling constant $u_j$ by the equation

$$a_j = b_j = c_j = \rho/u_j.$$  \hspace{1cm} (9)

Then the boundary conditions are written in the form

$$\psi_j(0) = \psi_ρ(\varphi_j) = \frac{1}{u_j} \{\psi_ρ'(\varphi_j + 0) - \psi_ρ'(\varphi_j - 0) + \rho \psi_j'(0)\},$$  \hspace{1cm} (10)

where $j = 1, 2$.

To obtain the transmission coefficient we have to find a solution of the Schrödinger equation that is the superposition of incident and reflected wave in the first wire and corresponds to propagated wave in the second wire. Consequently, the wave function in the wire $W_1$ has the form

$$\psi_1(x) = \exp(-ikx) + r \exp(ikx),$$  \hspace{1cm} (11)
where $r$ is the reflection amplitude and $k = \sqrt{2m^*E}/\hbar$ is the electron wave number. The wave function in the second wire is given by
\[
\psi_2(x) = t \exp(ikx),
\]
where $t$ is the transmission amplitude.

Since the Hamiltonian $H_0$ is perturbed by the zero-range potentials the wave-function $\psi_\rho(\varphi)$ can be represented in terms of the Green function $G_\rho(\varphi, \varphi_j; E)$ of the operator $H_\rho$
\[
\psi_\rho(\varphi, E) = \sum_{j=1}^{N+2} A_j(E) G_\rho(\varphi, \varphi_j; E).
\]
Here $A_j(E)$ are coefficients which should be determined from the boundary conditions.

The Green function of the Hamiltonian $H_\rho$ is well-known
\[
G_\rho(\varphi, \varphi_j; E) = \frac{m^*}{2\hbar^2k} \left[ \frac{\exp(i(\varphi_j - \varphi \pm \pi)(\eta - k\rho))}{\sin \pi(\eta - k\rho)} - \frac{\exp(i(\varphi_j - \varphi \pm \pi)(\eta + k\rho))}{\sin \pi(\eta + k\rho)} \right],
\]
where “plus” sign corresponds to $\varphi \geq \varphi_j$ and “minus” sign should be used otherwise.

We denote $Q_{ij}(E) = \frac{\hbar^2}{m^*_\rho} G_\rho(\varphi_i, \varphi_j; E)$ and $\alpha_j = \frac{m^*_\rho}{\hbar^2} A_j$. The matrix $Q_{ij}(E)$ is called Krein’s $Q$-matrix [37, 38]. Applying the boundary conditions (6) and (10) to the wave function given by Eqs. (11)–(13), we obtain the system of $N + 4$ equations
\[
\begin{align*}
\alpha_1 Q_{11} + \alpha_2 Q_{12} + \ldots + \alpha_{N+2} Q_{1,N+2} &= \frac{2\alpha_1 + ik\rho(r-1)}{u_1}, \\
\alpha_1 Q_{21} + \alpha_2 Q_{22} + \ldots + \alpha_{N+2} Q_{2,N+2} &= \frac{2\alpha_2 + ik\rho t}{u_2}, \\
\alpha_1 Q_{31} + \alpha_2 Q_{32} + \ldots + \alpha_{N+2} Q_{3,N+2} &= \frac{2\alpha_3}{u_1}, \\
&\ldots \\
\alpha_1 Q_{N+2,1} + \alpha_2 Q_{N+2,2} + \ldots + \alpha_{N+2} Q_{N+2,N+2} &= \frac{2\alpha_{N+2}}{u_N}, \\
1 + r &= \frac{2\alpha_1 + ik\rho(r-1)}{u_1}, \\
t &= \frac{2\alpha_2 + ik\rho t}{u_2}.
\end{align*}
\]
Using two last equations we can represent $r$ and $t$ in terms of $\alpha_j$. Then the system takes the form
\[
\sum_{l=1}^{N+2} [Q_{jl} - P_j \delta_{jl}] \alpha_l = D \delta_{j1}, \quad j = 1 \ldots, N + 2.
\]
Here we use the following notations

\[ P_j(E) = \begin{cases} 
    2/(u_j - i k \rho), & j = 1, 2 \\
    2/v_j, & j = 3, \ldots, N + 2, 
\end{cases} \]

\[ D(E) = \frac{2i k \rho}{i k \rho - u_1}. \]

The solution of system (16) may be represented in the form

\[ \alpha_n = \frac{\Delta_n}{\Delta}, \quad (17) \]

where \( \Delta \) is the principal determinant of the system

\[ \Delta = \det [Q_{jl} - P_j \delta_{jl}], \quad (18) \]

and \( \Delta_n \) is the determinant of the matrix which is obtained from the basic matrix by replacing of \( n \)-th column with the column of absolute terms

\[ \Delta_n = \det [(Q_{jl} - P_j \delta_{jl}) (1 - \delta_{nl}) + D \delta_{j1} \delta_{nl}]. \quad (19) \]

Taking into account Eq. (17), we can represent the transmission amplitude in the form

\[ t(E) = \frac{2}{u_2 - i k \rho} \frac{\Delta_2}{\Delta}. \quad (20) \]

We note that Eq. (20) is valid for arbitrary values of magnetic field and contact parameters. To study the effect of impurity positions on the electron transport we will restrict ourselves by the case of equal contacts \( (u_1 = u_2 = u) \) and impurities \( (v_3 = \ldots = v_{N+2} = v) \).

2. Case of one impurity and diametrically opposite contacts

Let us consider at first the case of one impurity. Then the system (16) consist of three equations. Using Eq. (20) and the explicit form (14) of the Green function \( G_{\rho}(\varphi, \varphi_j; E) \), we obtain the following equation for transmission amplitude \( t \):

\[ t(k, \eta) = \frac{F_1(k, \eta) + v F_2(k, \eta)}{F_3(k, \eta) + v F_4(k)}. \quad (21) \]

Here

\[ F_1(k, \eta) = -16i k^3 \rho^3 \cos \pi \eta \sin \pi k \rho, \quad (22) \]
\[ F_2(k, \eta) = 8ik^2 \rho^2 e^{-i\pi \eta} \sin[(\pi - \Delta \varphi)k\rho] \sin(\Delta \varphi k\rho), \tag{23} \]
\[ F_3(k, \eta) = 4k\rho \{-u^2 + k\rho 2iu + 5k^2 \rho^2\} \sin^2 \pi k\rho \\
-4ik\rho (iu + k\rho) \sin 2\pi k\rho \\
-16k^3 \rho^3 \sin \pi \eta, \tag{24} \]
\[ F_4(k) = 2k\rho (u - ik\rho) \{2 \cos 2\pi k\rho - \cos(2\Delta \varphi k\rho) \\
-\cos[2(\pi - \Delta \varphi)k\rho]\} \\
+2(u - ik\rho)^2 \sin \pi k\rho \cos[(\pi - 2\Delta \varphi)k\rho] \\
+(5k^2 \rho^2 + 2ik\rho u - u^2) \sin 2\pi k\rho, \tag{25} \]

and \( \Delta \varphi = \varphi_3 - \varphi_2 \).

Equation (21) coincides with the corresponding equation for transmission amplitude of the ring without impurities if we take \( v = 0 \) and represent parameter \( u \) in terms of the scattering length \( \lambda \) by the equation \( u = -2\rho/\lambda \). As it follows from Eqs. (21)–(25) transmission coefficient is a periodic function of the magnetic flux with the period equal to the flux quantum. It should be mentioned that the behavior of the transmission coefficient changes considerably at integer and half-integer values of the magnetic flux \( \eta \) since the eigenvalues of the Hamiltonian \( H_0 \) are degenerated in that case.

We will start with the general case of non-zero magnetic field. The dependence of the transmission coefficient on the dimensionless parameter \( k\rho \) is shown in Fig. 2. One can see that the transmission coefficient oscillates as a function of the electron energy. Oscillations are caused by interference of electron waves multiply scattered by contacts. In the absence of the impurity the dependence \( T(k\rho) \) exhibits asymmetrical resonances in vicinities of \( k\rho = m \) (Fig. 2a). The maximal values of resonance peaks reach unity while the minimal values of dips are equal to zero. The impurity leads to destruction of absolute reflection and transmission that results in decrease of peaks and increase of dip values (Fig. 2b). This result may be explained with help of Eq. (21). If the impurity is absent \((v = 0)\) then the numerator in Eq. (21) contains only one term \( F_1 \) which vanishes at integer values of \( k\rho \). On the other hand, the denominator of Eq. (21) contains non-trivial real and imaginary parts and does not vanish at non-integer values of the magnetic flux. Therefore transmission coefficients has zeros at \( k\rho = m \) \((m \text{ is integer})\) if the ring does not contain the impurity. In presence of the impurity, zeros
disappear. However, some of them are conserved at specific positions of impurity. In particular, the zero at \( k\rho = m \) is conserved if the following relation holds \( \sin m\Delta \varphi = 0 \).

Let us consider the behavior of the transmission coefficient in vicinities of energy values \( E_m \) where the asymmetric Fano resonances can appear. To study the energy dependence of the transmission coefficient in the vicinity of \( E = E_m \) we expend numerator and denominator of Eq. (21) in the Taylor series up to the first-order terms. After some simply algebra we obtain the following asymptotic expression for the transmission amplitude in the vicinity of \( E = E_m \)

\[
t(E) \simeq 2i(-1)^m \frac{\pi m^2 \cos \pi \eta \Delta E + v E_m \cos \frac{\pi m}{2} \sin^2 m \Delta \varphi}{\pi m (2im - 2u - v) \Delta E + 4m E_m \sin^2 \pi \eta}.
\]

where \( \Delta E = E - E_m \).

Equation (26) shows that the transmission coefficient does not vanish at \( E = E_m \) in presence of the impurity if \( \sin m\Delta \varphi \neq 0 \). However, the zero of

Figure 2: Transmission probability \( T \) versus the dimensionless wave vector \( k\rho \) in the presence of the magnetic flux \( \eta = 0.14 \). (a) The case of the ring without impurities. (b) The case of an impurity located at point \( \varphi_3 = 4\pi/3 \). All figures are plotted for \( u = v = 10 \).
transmission at $E = E_0^m$ is conserved if the position of impurity satisfies the condition $\sin m\Delta\varphi = 0$ (Fig. 2b). The transmission amplitude near $E = E_0^m$ may be represented in the form

$$t(E) \simeq \mu_m \frac{E - E_0^m}{E - E_m^{(r)} - i\Gamma_m}. \quad (27)$$

Here we use the following notations

$$\mu_m = \frac{(-1)^m 2im \cos \pi \eta}{2im - 2u - v}, \quad (28)$$

$$E_m^{(r)} = E_0^m \left(1 + \frac{4(2u + v) \sin^2 \pi \eta}{\pi[(2u + v)^2 + 4m^2]}\right), \quad (29)$$

$$\Gamma_m = \frac{8mE_0^m \sin^2 \pi \eta}{\pi[(2u + v)^2 + 4m^2]}. \quad (30)$$

Equation (27) shows that transmission coefficient has the form of asymmetric Fano resonance [25] in the vicinity of $E_0^m$.

The peak of the Fano resonance corresponds to the pole $E_m^{(r)} + i\Gamma_m$ on the complex plane. Here $E_m^{(r)}$ determines the position of the resonance and $\Gamma_m$ determines the half-width of the resonance curve. The zeros associated with the Fano resonances are situated on the real axis at the points $E_0^m$.

One can see from Eq. (30) that the width $\Gamma_m$ of the Fano resonance is proportional to $\sin^2 \pi \eta$ therefore the width tends to zero as the magnetic flux approaches to integer values. In this case, the pole $E_m^{(r)} + i\Gamma_m$ and the zero $E_0^m$ on the complex plane coincide and cancel each other that corresponds to collapse of the Fano resonance.

Let us consider now the dependence of $T$ on $k$ in the case of integer magnetic flux. The influence of the impurity on the transmission coefficient is shown in (Fig. 3). One can see that transmission coefficient has no zeros in absence of the impurity (Fig. 3a), as it was shown in previous studies [5]. The presence of the impurity leads to decrease of oscillating maxima and appearing of Fano resonances (Fig. 3b).

Zeros associated with the Fano resonances does not coincide with the values $E_0^m$. We denote by $\kappa_m$ the values of the electron wave vector corresponding to the zeros of transmission coefficient. According to equations (22) and (23) values of $\kappa_m$ are determined by the equation

$$2\kappa \rho \sin(\pi \kappa \rho) - v \sin[(\pi - \Delta\varphi)\kappa \rho] \sin(\Delta\varphi \kappa \rho) = 0. \quad (31)$$
Figure 3: Transmission probability $T$ versus the dimensionless wave vector $k\rho$ at the zero magnetic field. (a) The case of the ring without impurities. Transmission coefficient has no zeros in this case. (b) The case of an impurity located at $\varphi_3 = 1.475\pi$. One can see sharp Fano resonances in vicinities of $k\rho = 2n$, where $n$ is integer.

The following approximate expression for $\kappa_m$ may be obtained if $|\sin(n\Delta \varphi)| \ll 1$:

$$\kappa_n \approx \frac{n}{\rho} - \frac{v \sin^2 n\Delta \varphi}{2\pi n\rho},$$  \hspace{1cm} (32)

The corresponding electron energy is given by

$$E_n^{(z)} = \frac{\hbar^2 \kappa_n^2}{2m^*} \approx \frac{\hbar^2}{2m^*\rho^2} \left( n - \frac{v \sin^2 n\Delta \varphi}{2\pi n} \right)^2.$$  \hspace{1cm} (33)

Let us consider the behavior of the transmission amplitude in the vicinity of $E_n^{(z)}$. For this purpose we expand the numerator and the denominator of $t(k)$ in Eq. (21) in the Taylor series near $k_m = m/\rho$ neglecting the second order terms with regard to $\sin m\Delta \varphi$. Then we get the following approximate equation for $t(k)$:

$$t(k, 0) \simeq \frac{2i\pi(-1)^{m+1}m^2\rho(k - \kappa_m)}{\pi m\rho(v - 2i)(k - k_m) + v(u - im)\sin^2 m\Delta \varphi}.$$  \hspace{1cm} (34)
Now we introduce the following notations:
\[
\mu_m = -\frac{2i(-1)^m m}{(v-2i)},
\]  
(35)
\[
E_m^{(r)} \simeq E_m^0 \left( 1 - \frac{2v(vu + 2m) \sin^2 m\Delta \varphi}{\pi m^2 (v^2 + 4)} \right)
\]  
(36)
and
\[
\Gamma_m \simeq \frac{2E_m^0(vm - 2u) \sin^2 m\Delta \varphi}{\pi m^2 (v^2 + 4)}.
\]  
(37)

Then we obtain the following equation for the transmission amplitude in the vicinity of \(\kappa_m\)
\[
t(E) \simeq \mu_m \frac{E - E_m^{(z)}}{E - E_m^{(r)} - i\Gamma_m}.
\]  
(38)

One can see from Eq. (38) that the transmission amplitude has the form of the Fano resonance in the vicinity of \(E_m^{(z)}\). The width of the resonance curve is determined by the position of the impurity. If the position satisfies the condition \(\sin m\Delta \varphi = 0\) then the collapse of the Fano resonance in the vicinity of \(E_m^{(r)}\) occurs (Fig. 4). It should be mentioned that the similar result was obtained in Ref. [7] in the framework of the scattering matrix approach.

We note that the Fano resonance near \(E_m^0\) disappear if the magnetic field is absent and the impurity position is determined by the equation \(\varphi_3 = l\pi/n\) where \(n\) is the integer divisor of \(m\). In the case of the non-zero magnetic field the resonances at those \(E_m^0\) appear again.

Let us consider in detail the case of half-integer magnetic flux. The transmission coefficient vanishes for all electron energies if there is no impurity on the ring and contacts are located in diametrically opposite points. The presence of the impurity leads to appearance of the non-zero transmission as follows from Eq. (21). The phenomenon might be explained in terms of the electron wave phase. The electron waves spreading on the ring from the point \(A_1\) in different directions get phase shifts \(\pi(k\rho + \eta)\) and \(\pi(k\rho - \eta)\) at the point \(A_2\) (Fig. 1). The phase difference is equal to \(2\pi\eta\) therefore those waves have opposite phases in the second lead and cancels each other for all energies at half-integer values of the magnetic flux \(\eta\). Consequently the reflection coefficient equals to unity and the device may be considered as an ideal electron mirror. The impurity breaks destructive interference and the non-zero transmission coefficient appears.
Figure 4: Transmission probability $T$ versus the dimensionless wave vector $k\rho$ at the zero magnetic field. (a) The impurity is located at point $\varphi_3 = 1.475\pi$. One can see the Fano resonance in the vicinity of $k\rho = 4$. (b) Collapse of the Fano resonance occurs if the impurity is located at $\varphi_3 = 1.5\pi$.

Let us consider the dependence of the transmission coefficient on the magnetic field that is represented in Fig. 5. It follows from Eq. (21) that $T(k, \eta)$ is a periodic function of $\eta$ with the period equal to 1. The dependence of the transmission coefficient on $\eta$ contains one or two maxima on the period (Fig. 5a) depending on the electron energy and the impurity position. Figure 5b) shows that maxima of $T(\eta)$ can conjugate and diverge with variation of the electron energy. The similar features are present on the dependence of the transmission coefficient on the magnetic field and the impurity position.

3. Non-opposite position of contacts

Let us consider now the system with contacts attached at arbitrary points ($\varphi_2 - \varphi_1 \neq \pi$). The transmission amplitude in this case is given by

$$t(k, \eta) = \frac{\tilde{F}_1(k, \eta) + v\tilde{F}_2(k, \eta)}{\tilde{F}_3(k, \eta) + v\tilde{F}_4(k)},$$

(39)
Figure 5: (a) Transmission probability $T$ versus the magnetic flux $\eta$ for the case of diametrically opposite contacts for several impurity positions: $\varphi_3 = 1.02\pi$ (solid line 1), $\varphi_3 = 1.033\pi$ (dashed line 2) and $\varphi_3 = 1.06\pi$ (dotted line 3). (b) Transmission probability $T$ as a function of magnetic flux $\eta$ and dimensionless wave vector $kr$ for the case of diametrically opposite contacts. The impurity is located at the point $\varphi_3 = 3\pi/2$.

where

$$
\tilde{F}_1(k, \eta) = -8ik^3\rho^3e^{-i\varphi_2\eta}\{\sin[(2\pi - \varphi_2)k\rho] + e^{2i\pi\eta}\sin(\varphi_2k\rho)\},
$$

(40)

$$
\tilde{F}_2(k, \eta) = 8ik^2\rho^2e^{-i\varphi_2\eta}\sin[(2\pi - \varphi_3)k\rho]\sin(\Delta\varphi k\rho),
$$

(41)

$$
\tilde{F}_3(k, \eta) = 2k\rho\{u^2\cos 2\pi k\rho
+ (iu + k\rho)^2 \cos [2(\pi - \varphi_2)k\rho]
+ k\rho[4k\rho \cos 2\pi\eta - (2iu + 5k\rho) \cos 2\pi k\rho
+ 4(u - ik\rho) \sin 2\pi k\rho]\},
$$

(42)

and

$$
\tilde{F}_4(k) = 2ik\rho(iu + k\rho)\{\cos[2(\pi - \Delta\varphi)k\rho] + 
$$

15
$$+ \cos [2(\pi - \varphi_3)k\rho] - 2 \cos 2\pi k\rho \right) + 
abla^2 + (i u + k \rho)^2 \left\{ \sin [2(\pi - \varphi_3)k\rho] - 
abla^2 - \sin [2(\pi - \varphi_3)k\rho] - 2 \sin [2(\pi - \Delta \varphi)k\rho] \right\} + 
abla^2 + (5k^2 \rho^2 + 2ik\rho u - u^2) \sin 2\pi k\rho. \quad (43)$$

If the ring has no impurities and the magnetic field is absent then Eq. (39) takes the form

$$t(k, 0) = \tilde{F}_1(k, 0)/\tilde{F}_3(k, 0), \quad (44)$$

where $\tilde{F}_1(k, 0)$ is given by

$$\tilde{F}_1(k, 0) = -16ik^3\rho^3 \sin k\pi \rho \cos [(\pi - \varphi_2)k\rho]. \quad (45)$$

Equation (45) shows that the transmission coefficient vanishes if $\sin k\pi \rho = 0$ or $\cos [(\pi - \varphi_2)k\rho] = 0$ and the function $\tilde{F}_3$ remains non-zero.

Figure 6: Transmission probability $T$ versus the dimensionless wave vector $k\rho$ for the case of non-opposite contacts ($\varphi_1 = 0$, $\varphi_2 = 14\pi/15$). The impurity is located at the point $\varphi_3 = 1.1\pi$. (a) The magnetic field is absent. (b) The magnetic flux is equal to $\eta = 0.05$.

The presence of the impurity leads to shift of transmission zeros from the values $k\rho = n$ (Fig. 6 a). The positions of zeros are determined by the
Equation (46) shows that if the impurity position is defined by \( \Delta \varphi = \pm l \pi / m \) (\( l \) and \( m \) are integer) then zeros are situated at \( k \rho = n \) where \( n \) is integer multiply \( m \). The magnetic field leads to disappearing of zeros at \( k \rho = n \) (Fig. 6 b). In contrast to the collapse of the Fano resonances the depth of dips decreases while the width remains finite. Transmission zeros on the complex plane are shifted from the real axis and corresponding dips do not reach zero in the case of the applied magnetic field. The energy determining the position of the zero on the complex plane acquire an imaginary part which increases with the magnetic field. Thus there are two different mechanisms of disappearing of zeros in the system: the first one is collapse of the Fano resonance and the second one is shift of the transmission zero from the real axis in the complex plane. We mention that the collapse of the Fano resonance is accompanied by increase in symmetry of the system while the second mechanism is caused by decrease in symmetry.

We mention that the transmission reaches maximal value when the system has mirror symmetry. In particular, the height of the conductance peak is maximal if the impurity is located in the middle of an arc connecting contacts and parameters of contacts are equal.

4. Discrete levels in the continuum

It is known that the collapse of the Fano resonance is accompanied by appearance of the discrete level in the continuous spectrum. This level corresponds to localized state on the ring. The wave function \( \psi^d_\rho \) corresponding to this state has to satisfy the boundary conditions (6) and (10) and vanish at the points of contacts

\[
\psi^d_\rho (\varphi_1) = \psi^d_\rho (\varphi_2) = 0.
\]

If the electron energy coincides with the eigenvalue \( E^\eta_m \) of the Hamiltonian \( H_\rho \) then the wave function is a linear combination of the corresponding eigenfunctions. The eigenvalues \( E^\eta_m \) are non-degenerated if \( 2\eta \) is not integer. In this case, the eigenfunction has the form

\[
\psi^m_\rho (\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}.
\]
The function $\psi^m_\rho(\varphi)$ has no zeros and therefore it does not satisfy the condition (47).

At integer and half-integer values of the magnetic flux $\eta$ the eigenvalues $E^\eta_m$ are double-degenerated. The general form of the wave function for these cases is given by

$$\psi_\rho(\varphi) = C_1 e^{im\varphi} + C_2 e^{-i(m+2\eta)\varphi}, \quad (49)$$

where $C_1$ and $C_2$ are some coefficients. It is easy to obtain the function of the form (49) which vanishes at $\varphi_1 = 0$

$$\psi^d_\rho(\varphi) = \frac{1}{\sqrt{\pi}} \exp(-i\eta\varphi) \sin[(m + \eta)\varphi]. \quad (50)$$

According to Eq. (47) the function $\psi^d_\rho(\varphi)$ has to vanish at the point $\varphi_2$. Since the wave function (50) is smooth it has to vanish at the point $\varphi_3$ to satisfy the boundary condition (6). Therefore the discrete level appears in the continuous spectrum if the wave function given by Eq. (50) has zeros at all perturbation points on the ring. Positions of the impurity and the second contact corresponding to the collapse of the Fano resonance at $E^\eta_m$ are given by the following equations

$$\varphi_2 = \frac{n_2}{m + \eta}\pi, \quad \varphi_3 = \frac{n_3}{m + \eta}\pi, \quad (51)$$

where $n_2$ and $n_3$ are arbitrary integer numbers. Equation (51) coincides with the condition of the collapse of the Fano resonance obtained in previous sections.

In the case of $E \neq E^\eta_m$ the wave function corresponding to the discrete level should have the form

$$\psi^d_m(\varphi) = A_3 G(\varphi, \varphi_3, E). \quad (52)$$

Applying the boundary condition (6) to the function (52) we obtain the equation

$$Q_{33}(E) + \frac{2}{v} = 0. \quad (53)$$

determining the energy of the discrete level associated with the impurity. In the case of the zero magnetic field Eq. (53) is written in the form

$$\cot \pi k\rho + \frac{2k\rho}{v} = 0. \quad (54)$$
However Eq. (53) is not enough for appearance of the discrete level in the case of attached leads. The condition (47) has to be satisfied in addition to Eq. (53). That means the contacts should be situated at zeros of the function (52). Applying the boundary conditions (47) to the wave function (52) we obtain the conditions of appearance of the discrete level

$$Q_{31}(E) = Q_{32}(E) = 0.$$  

(55)

This equation is valid only at special positions of the impurity and at special energies. We note that the function (52) has no zeros at irrational values of the magnetic flux $\eta$. Furthermore the position of zeros depends on the strength of the impurity potential. Therefore the positions of contacts have to be coordinated with the value of impurity potential. The coordination does not take place for all levels at the same time. Hence only one discrete level can appear in the continuous spectrum at $E \neq E_m^\eta$. By virtue of the symmetrical property of the function (52), the appearance of this level is more likely if the contacts are situated on equal distance from the impurity.

5. Case of two impurities

In the general case of arbitrary location of impurities and contacts, the dependence of transmission coefficient on the electron energy contains different type of overlapping resonances and oscillations. The dependence is irregular in this case and hardly analyzable, therefore we will restrict ourselves by the configuration of diametrically opposite contacts.

If the angles defining positions of the impurities are incommensurable and magnetic field is absent then the dependence $T(k\rho)$ exhibits Fano resonances (Fig. 7a) in the vicinity of the values $k\rho = m$ where $m$ is integer. If the impurities are located at equal distances from the contacts ($\varphi_4 = 2\pi - \varphi_3$) then the simultaneous collapse of all Fano resonances occurs (Fig 7b).

The wave function of the discrete level associated with the collapse in the case of two impurities at $E \neq E_m^\eta$ has the form

$$\psi^d_m(\varphi) = A_3G(\varphi, \varphi_3, E) + A_4G(\varphi, \varphi_4, E).$$  

(56)

The energy of the discrete level is defined by the equation

$$\begin{vmatrix} Q_{33}(E) + \frac{2}{v} & Q_{34}(E) \\ Q_{43}(E) & Q_{44}(E) + \frac{2}{v} \end{vmatrix} = 0.$$  

(57)
Figure 7: Transmission probability $T$ versus the dimensionless wave vector $k\rho$ for the case of diametrically opposite contacts and the zero magnetic field at $u = 10$ and $v = 5$. (a) Two impurities are located at points $\varphi_3 = 0.8\pi$ and $\varphi_4 = 1.15\pi$. (b) The impurities are located on equal distance from each contact ($\varphi_3 = 0.83\pi$ and $\varphi_4 = 1.17\pi$).

The condition (47) should be satisfied in addition to Eq. (57). At zero magnetic field, the Green function $G(\varphi, \varphi_j, E)$ has the property

$$G(\varphi_j - \Delta \varphi, \varphi_j, E) = G(\varphi_j + \Delta \varphi, \varphi_j, E).$$

(58)

Taking into account Eq. (58), one can see that the wave function

$$\psi_m^{\text{d}}(\varphi) = A_3[G(\varphi, \varphi_3, E) - G(\varphi, \varphi_4, E)].$$

(59)

satisfies the boundary condition (47) at arbitrary energy. Therefore the presence of the mirror symmetry in the system leads to simultaneous collapse of all Fano resonances. The Fano resonances appear again if the symmetry is broken by shift of any impurity or by different values of point potentials ($v_1 \neq v_2$). Collapse of several Fano resonances in vicinities of $k\rho = m$ occurs if the positions of the impurities are given by $\varphi_3 = n_3\pi/m$ and $\varphi_4 = n_4\pi/m$ where $n_3$, $n_4$ and $m$ are integer.
Figure 8: Transmission probability $T$ versus the dimensionless wave vector $k\rho$ for the case of diametrically opposite contacts and zero magnetic field at $u = 10$ and $v = 5$ in presence of two impurities located diametrically opposite at points $\varphi_3 = \pi/15$ and $\varphi_4 = 16\pi/15$.

If the impurities are located in diametrically opposite points ($\varphi_4 = \varphi_3 + \pi$) then the maximal values of transmission peaks reach unity (Fig. 8) in contrast to the case of single impurity. Hence the addition of the second impurity at the diametrically opposite point leads to increase of the conductance due to constructive interference.

It should be mentioned that the destructive interference at half-integer values of the magnetic flux leads to the perfect reflection if the system has mirror or inverse symmetry. Therefore the transmission coefficient vanishes at the half-integer magnetic flux if two identical impurities are located at diametrically opposite points ($\varphi_4 = \varphi_3 + \pi$) or at the same distance from contacts ($\varphi_4 = 2\pi - \varphi_3$).

Zeros of transmission coefficient disappear in the magnetic field with magnetic flux $\eta \neq n/2$ where $n$ is integer. The similar effect has been considered in the case of one impurity and illustrated by Fig. 6.
Conclusion

We have investigated the electron transport in the two-terminal Aharonov–Bohm ring with impurities. Using the zero-range potential theory we have obtained analytical expressions for the electron transmission coefficient as a function of the electron energy. The effect of the magnetic field and position of impurities on the electron transport has been studied. We have found that the dependence of the transmission coefficient on the electron energy exhibits oscillations caused by the interference of electron waves on the ring. The presence of impurities can lead either to decrease or to increase in the transmission coefficient. In particular, the transmission increases in presence of impurities due to breaking of the destructive interference if the value magnetic flux through the ring equals to half-integer number of the magnetic flux quanta. The ring without impurities may be considered as ideal electron mirror in this case.

Our analysis shows that the dependence of the transmission coefficient on the electron energy exhibits Fano resonances at certain energies. These resonances consist of the sharp transmission peak and the nearby transmission zero. In the complex plane of energy, the Fano resonance is represented by pole and nearby zero of the transmission amplitude. Resonances arise as a result of interaction between discrete level and continuous spectrum. The necessary condition of appearing of the Fano resonance in the quantum ring is partial symmetry breaking by the magnetic field or the asymmetrical location of impurities or contacts. We have found that the collapse of Fano resonances occurs at certain parameters of the system. In this case, the pole and the zero of the transmission amplitude coincide and cancel each other. The discrete level immersed in the continuous spectrum arises in this case. In particular, collapse of the Fano resonance in the vicinity of the discrete level $E_{m}^{\eta}$ occurs if all perturbations on the ring are situated at zeros of the wave function defined by Eq. (50).

Another mechanism of disappearing of transmission zeros is possible in the system in addition to the collapse of Fano resonances. The zero may be shifted from the real axis in the complex plane. In this case, the dip of transmission coefficient is conserved but the minimum value does not reach zero. We mention that the collapse of the Fano resonance is accompanied by increase in symmetry of the system while the shift of the zero is accompanied by decrease in symmetry.

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