PROJECTIVE BUNDLE FORMULA FOR HELLER’S RELATIVE $K_0$

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Abstract. In this article, we study the Heller relative $K_0$ group of the map $\mathbb{P}^r_X \to \mathbb{P}^r_S$, where $X$ and $S$ are quasi-projective schemes over a commutative ring. More precisely, we prove that the projective bundle formula holds for Heller’s relative $K_0$, provided $X$ is flat over $S$. As a corollary, we get a description of the relative group $K_0(\mathbb{P}^r_X \to \mathbb{P}^r_S)$ in terms of generators and relations, provided $X$ is affine and flat over $S$.

1. Introduction

Throughout this article, we shall assume that all schemes have an ample family of line bundles. For example, every quasi-projective scheme over a commutative ring has an ample family of line bundles. Usually, $K$-theory of vector bundles behaves well under this assumption. By a quasi-projective scheme will always mean a scheme which is quasi-projective over some commutative ring.

Let $\mathcal{E}$ be a vector bundle of rank $r+1$ over a quasi-projective scheme $X$, and $\mathbb{P}(\mathcal{E})$ denote the associated projective space bundle with the structure map $\pi : \mathbb{P}(\mathcal{E}) \to X$. Then the projective bundle theorem for algebraic $K$-theory says that (see Theorem V.1.5 of [8]) for all $q \in \mathbb{Z}$, there is an isomorphism of groups

$$K_q(X)^{r+1} \xrightarrow{\sim} K_q(\mathbb{P}(\mathcal{E})).$$

The goal of this article is to present a relative version of the above stated result for Heller’s relative $K_0$.

Given a map of schemes $f : X \to S$, let $K(f)$ denote the homotopy fibre of $K(S) \to K(X)$. Here $K(X)$ denotes the non-connective Bass $K$-theory spectrum of the scheme $X$. Then $K_n(f)$, the $n$-th relative $K$-group of $f$, is defined as $\pi_n K(f)$.

In [3], Heller defined the relative $K_0$-group for a functor between certain categories. In this article, we are interested in Heller’s relative $K_0$-groups for the pullback functor between the categories of vector bundles associated with a map of schemes. Given a map of schemes $f : X \to S$, Heller’s relative $K_0$ of $f^*$ is generated by the triples $(V_1, \alpha, V_2)$.
with suitable relations, where \( V_1, V_2 \) are vector bundles over \( S \) and \( \alpha : f^*V_1 \to f^*V_2 \) is an isomorphism (see Section 3). We write \( K_0^{He}(f) \) for the Heller relative \( K_0 \) of \( f^* \).

It is natural to wonder the following: Is there any group isomorphism between \( K_0(f) \) and \( K_0^{He}(f) \)? It is known that there is a group isomorphism if \( f : X \to S \) is a map of schemes with \( X \) affine. More precisely, if \( f : X \to S \) is a map of schemes with \( X \) affine then there is an isomorphism of groups \( K_0^{He}(f) \cong K_0(f) \) (see [4, Theorem 1.5 and Example 1.16]). For an arbitrary \( X \), we do not know whether the above mentioned result is true or not. Therefore, it is interesting to study the group \( K_0^{He}(P^r X \to P^r S) \).

Given a map of quasi-projective schemes \( f : X \to S \), we have the following commutative diagram

\[
\begin{array}{ccc}
P(E) \times_S X = P(f^*E) & \xrightarrow{p(f)} & P(E) \\
\pi_X & \downarrow \pi_S & \\
X & \xrightarrow{f} & S,
\end{array}
\]

where \( E \) is a vector bundle over \( S \). Note that if \( E = O_{S}^{r+1} \) then \( P(f) \) is just the map \( P_X^r \to P_S^r \). Under this situation, it is not hard to see that there is a group isomorphism (see Lemma 3.2)

\[ K_n(f)^{r+1} \cong K_n(P(f)). \]

We prove a similar result for relative \( K_0^{He} \) in section 6. More precisely, we prove the following theorem (see Theorem 6.3).

**Theorem 1.1.** Let \( P(f) : P(f^*E) \to P(E) \) be a map as in diagram (1.1). Assume that \( f : X \to S \) is a flat map of quasi-projective schemes and \( \text{rank}(E) = r+1 \). Then there is an isomorphism of groups

\[ K_0^{He}(f)^{r+1} \cong K_0^{He}(P(f)). \]

Using the Theorem 1.1 we also prove the following (see Corollary 6.4):

**Corollary 1.2.** Suppose \( P(f) \), \( f \) and \( E \) are as in Theorem 1.1. Further, we assume that \( X \) is an affine scheme. Then there is an isomorphism of groups \( K_0^{He}(P(f)) \cong K_0(P(f)) \).

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## 2. Preliminaries

For a scheme \( X \), let \( \text{Mod}(X) \) denote the category of \( \mathcal{O}_X \)-modules and \( \text{VB}(X) \) denote the category of vector bundles on \( X \). It is well known that \( \text{Mod}(X) \) is an abelian category and \( \text{VB}(X) \) is an exact subcategory of \( \text{Mod}(X) \).
Given a map of schemes $f : X \to S$, we define a category $\text{Mod}(f)$ whose objects are triples $(\mathcal{F}_1, \alpha, \mathcal{F}_2)$ with $\mathcal{F}_1, \mathcal{F}_2 \in \text{Mod}(S)$ and $\alpha : f^*\mathcal{F}_1 \cong f^*\mathcal{F}_2$ in $\text{Mod}(X)$. A morphism $(\mathcal{F}_1, \alpha, \mathcal{F}_2) \to (\mathcal{F}_1', \alpha', \mathcal{F}_2')$ is a pair of maps $u : \mathcal{F}_1 \to \mathcal{F}_1'$, $v : \mathcal{F}_2 \to \mathcal{F}_2'$ in $\text{Mod}(S)$ such that $\alpha' f^*u = f^*v \alpha$. In a similar way, we can define $\text{VB}(f)$ by replacing $\mathcal{O}_S$-modules with vector bundles on $S$.

If $f$ is a flat map then $(\ker(u), \tilde{\alpha}, \ker(v)) \in \text{Mod}(f)$, where

$$\tilde{\alpha} : f^*\ker(u) \cong \ker(f^*u) \cong \ker(f^*v) \cong f^*\ker(v).$$

Hereafter, we assume that $f : X \to S$ is a flat map of schemes.

Given a morphism $(u, v) : (\mathcal{F}_1, \alpha, \mathcal{F}_2) \to (\mathcal{F}_1', \alpha', \mathcal{F}_2')$ in $\text{Mod}(f)$, we define

$$\ker(u, v) := (\ker(u), \tilde{\alpha}, \ker(v)) \text{ and } \coker(u, v) := (\coker(u), \tilde{\alpha}', \coker(v)),$$

where $\tilde{\alpha}'$ induces from $\alpha'$. Under this definition, $\text{Mod}(f)$ is an abelian category.

The following lemma is an easy observation.

**Lemma 2.1.** Let

$$\begin{align*}
(\mathcal{F}_1, \alpha, \mathcal{F}_2) & \xrightarrow{(u, v)} (\mathcal{F}_1', \alpha', \mathcal{F}_2') & \xrightarrow{(u', v')} & (\mathcal{F}_1'', \alpha'', \mathcal{F}_2'')
\end{align*}$$

be a sequence in $\text{Mod}(f)$. Then \(\text{(2.1)}\) is exact if and only if

$$\mathcal{F}_1 \xrightarrow{\mu} \mathcal{F}_1' \xrightarrow{\mu'} \mathcal{F}_1''$$

and

$$\mathcal{F}_2 \xrightarrow{\nu} \mathcal{F}_2' \xrightarrow{\nu'} \mathcal{F}_2''$$

both are exact in $\text{Mod}(S)$.

Suppose that the sequence

$$0 \to (\mathcal{F}_1, \alpha, \mathcal{F}_2) \xrightarrow{(u, v)} (\mathcal{F}_1', \alpha', \mathcal{F}_2') \xrightarrow{(u', v')} (\mathcal{F}_1'', \alpha'', \mathcal{F}_2'') \to 0$$

is exact in $\text{Mod}(f)$ with $(\mathcal{F}_1, \alpha, \mathcal{F}_2), (\mathcal{F}_1'', \alpha'', \mathcal{F}_2'') \in \text{VB}(f)$. Since $\text{VB}(S)$ is an exact subcategory of $\text{Mod}(S)$, $p : \mathcal{F}_1' \cong V_1$ and $q : \mathcal{F}_2' \cong V_2$, where $V_1, V_2 \in \text{VB}(S)$ (using Lemma 2.1). Then $(p, q) : (\mathcal{F}_1', \alpha', \mathcal{F}_2') \cong (V_1, \beta, V_2)$ in $\text{VB}(f)$, where $\beta = f^*q.\alpha'.f^*p^{-1}$.

This proves the following:

**Lemma 2.2.** $\text{VB}(f)$ is an exact subcategory of $\text{Mod}(f)$.
3. Relative K-theory

Let \( f : X \to S \) be a map of schemes. Let \( K(f) \) be the homotopy fibre of \( K(S) \to K(X) \). Here \( K(X) \) denotes the non-connective Bass K-theory spectrum of the scheme \( X \). Then

\[
K_n(f) := \pi_n K(f)
\]

for \( n \in \mathbb{Z} \).

In [3], Heller introduced the relative \( K_0 \)-groups for a functor between certain categories (see also [1]). Following Heller [3], very recently R. Iwasa in [4] define the relative \( K_0 \)-groups for an exact functor between small exact categories. We now recall the definition from [4] in a special situation. For more details, we refer to section 1 of [4]. Consider \( VB(f) \) as the relative category associated to the pullback functor \( f^* : VB(S) \to VB(X) \). We define \( K_0(f^*) \) to be the abelian group generated by \([V_1, \alpha, V_2]\), where \((V_1, \alpha, V_2) \in VB(f)\). The relations are:

- \([V'] + [V''] = [V]\) for every exact sequence \(0 \to V' \to V \to V'' \to 0\) in \( VB(f)\);
- \([V_1, \alpha, V_2] + [(V_2, \beta, V_3)] = [(V_1, \beta \alpha, V_3)]\) for every pair \((V_1, \alpha, V_2), (V_2, \beta, V_3)\) of objects in \( VB(f)\).

We prefer to write \( K_0^{He}(f) \) for \( K_0(f^*) \).

Next, we discuss the relationship between \( K_0(f) \) and \( K_0^{He}(f) \). By applying Theorem 1.5 of [4] to the functor \( f^* : VB(S) \to VB(X) \) with \( X \) affine, we get the following:

**Lemma 3.1.** Let \( f : X \to S \) be a map of schemes with \( X \) affine. Then there is an isomorphism \( K_0^{He}(f) \xrightarrow{\sim} K_0(f) \).

The next result is the projective bundle formula for relative K-theory.

**Lemma 3.2.** Let \( P(f) : P(f^*E) \to P(E) \) be a map as in diagram \((\mathcal{D})\). Assume that the rank of \( E \) is \( r + 1 \). Then there is a natural isomorphism of groups \( K_n(f)^{r+1} \xrightarrow{\sim} K_n(P(f)) \) for all \( n \in \mathbb{Z} \).

**Proof.** By Theorem V.1.5 of [8], we have an equivalence \( K(S)^{r+1} \simeq K(P(E)) \). Consider the following commutative diagram of K-theory spectra

\[
\begin{array}{ccc}
K(f)^{r+1} & \to & K(S)^{r+1} \\
\downarrow & & \downarrow \simeq \\
K(P(f)) & \to & K(P(E))
\end{array}
\]

This induces a commutative diagram of long exact sequences

\[
\begin{array}{cccccccc}
\ldots & \to & K_{n+1}(X)^{r+1} & \to & K_n(f)^{r+1} & \to & K_n(S)^{r+1} & \to & K_n(X)^{r+1} & \to & \ldots \\
\downarrow \simeq & & \downarrow & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
\ldots & \to & K_{n+1}(P(f^*E)) & \to & K_n(P(f)) & \to & K_n(P(E)) & \to & K_n(P(f^*E)) & \to & \ldots
\end{array}
\]
Hence the assertion. □

The rest of the paper is dedicated to proving the projective bundle formula for $K_{\text{He}}^0$.

4. Mumford-regular bundles

Let $\mathcal{E}$ be a vector bundle of finite rank over a quasi-projective scheme $X$. Let $\mathbb{P} = \mathbb{P}(\mathcal{E})$ be the associated projective space bundle. There is a natural map $\pi = \pi_X : \mathbb{P}(\mathcal{E}) \rightarrow X$. Some relevant details can be found in [2, Chapter 8].

A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is said to be Mumford-regular if for all $q > 0$ the higher derived sheaves $R^q\pi_* (\mathcal{F}(-q)) = 0$. Here $\mathcal{F}(n)$ is the twisted sheaf $\mathcal{F} \otimes \mathcal{O}_X(n)$.

We now recall some known results pertaining to Mumford-regular modules. For details, we refer to [6, Section 8] and [8, Chapter II.8].

Lemma 4.1. If $\mathcal{F}$ is Mumford-regular, then:

1. The twist $\mathcal{F}(n)$ are Mumford-regular for all $n \geq 0$.
2. The canonical map $\varepsilon : \pi_* \pi^*(\mathcal{F}) \rightarrow \mathcal{F}$ is onto.

Proof. See Proposition II.8.7.3 of [8]. □

Lemma 4.2. The functor $\pi_*$ is exact from Mumford-regular modules to $\mathcal{O}_X$-modules.

Proof. See Lemma II.8.7.4 of [8]. □

Lemma 4.3. Let $\mathcal{F}$ be a vector bundle on $\mathbb{P}$.

1. $\mathcal{F}(n)$ is a Mumford-regular vector bundle on $\mathbb{P}$ for all large enough $n$.
2. If $\mathcal{F}$ is Mumford-regular, then $\pi_* \mathcal{F}$ is a vector bundle on $X$.

Proof. See Lemma II.8.7.5 of [8]. □

Let $f : X \rightarrow S$ be a map of quasi-projective schemes and $\mathcal{E}$ be a vector bundle of finite rank on $S$. Then we have a commutative diagram (1.1). Let $\text{MR}(\mathbb{P}(\mathcal{E}))$ denote the category of Mumford-regular vector bundles. Now, we define a category $\text{MR}(\mathbb{P}(f))$ whose objects are triples $(\mathcal{F}_1, \alpha, \mathcal{F}_2)$ with $\mathcal{F}_1, \mathcal{F}_2 \in \text{MR}(\mathbb{P}(\mathcal{E}))$ and $\alpha : \mathbb{P}(f)^* \mathcal{F}_1 \cong \mathbb{P}(f)^* \mathcal{F}_2$ in $\text{MR}(\mathbb{P}(\mathcal{E}))$. A morphism $(\mathcal{F}_1, \alpha, \mathcal{F}_2) \rightarrow (\mathcal{F}_1', \alpha', \mathcal{F}_2')$ is a pair of maps $u : \mathcal{F}_1 \rightarrow \mathcal{F}_1', v : \mathcal{F}_2 \rightarrow \mathcal{F}_2'$ in $\text{MR}(\mathbb{P}(\mathcal{E}))$ such that $\alpha' \mathbb{P}(f)^* u = \mathbb{P}(f)^* v \alpha$.

Assume that $f : X \rightarrow S$ is a flat map. Let $(\mathcal{F}_1, \alpha, \mathcal{F}_2) \in \text{MR}(\mathbb{P}(f))$. By Lemma 4.1(2), there are canonical onto maps $\varepsilon_i : \pi^*_S \pi_* \mathcal{F}_i \rightarrow \mathcal{F}_i$ for $i = 1, 2$. We also have

$$\mathbb{P}(f)^* \pi^*_S \pi_* \mathcal{F}_1 = \pi^*_X f^* \pi_* \mathcal{F}_1 \cong \pi^*_X \pi_* \mathcal{F}_1 \mathbb{P}(f)^* \mathcal{F}_1$$

for $i = 1, 2$, where the first equality by the commutativity of the diagram (1.1) and the second isomorphism by the flat base change theorem (see Lemma 30.5.2 of [7]). Thus, we get an isomorphism $\mathbb{P}(f)^* \pi^*_S \pi_* \mathcal{F}_1 \cong \mathbb{P}(f)^* \pi^*_S \pi_* \mathcal{F}_2$ and is denoted by $\pi^*_X \pi_* \alpha$. Since $\pi_S$
is quasi-compact and separated, \( \pi_S^* \pi_{S*}(\mathcal{F}_i) \in \text{MR} (\mathbb{P}(\mathcal{E})) \) for \( i = 1, 2 \) by Example II.8.7.2 of [8]. This implies that

\[
(\pi_S^* \pi_{S*} \mathcal{F}_1, \pi_X^* \pi_{X*} \alpha, \pi_S^* \pi_{S*} \mathcal{F}_2) \in \text{MR} (\mathbb{P}(f)).
\]

Note that the canonical map \( \varepsilon : \pi^* \pi_*(\mathcal{F}) \to \mathcal{F} \) is natural in \( \mathcal{F} \). So, the diagram

\[
\begin{align*}
\mathbb{P}(f)^* \pi_S^* \pi_{S*} \mathcal{F}_1 & \cong \pi_X^* \pi_{X*} \mathbb{P}(f)^* \mathcal{F}_1 & \mathbb{P}(f)^* \mathbb{P}(f)^* \mathcal{F}_2 & \cong \pi_X^* \pi_{X*} \mathbb{P}(f)^* \mathcal{F}_2 \\
\pi_X^* \pi_{X*} \alpha & \downarrow \alpha & \downarrow \alpha
\end{align*}
\]

(4.1)

is commutative by the naturality of \( \mathbb{P}(f)^* \varepsilon \). Hence we get the following (by Lemma [2.1]):

**Lemma 4.4.** \((\varepsilon_1, \varepsilon_2) : (\pi_S^* \pi_{S*} \mathcal{F}_1, \pi_X^* \pi_{X*} \alpha, \pi_S^* \pi_{S*} \mathcal{F}_2) \to (\mathcal{F}_1, \alpha, \mathcal{F}_2) \) is a morphism in \( \text{MR} (\mathbb{P}(f)) \) and it is onto. Here \( f : X \to S \) is a flat map.

## 5. Relative version of Quillen’s Resolution Theorem

In this section, we prove a relative version of Quillen’s resolution theorem which will play an important role in the later part of the paper. Throughout this section, \( f : X \to S \) is a flat map of quasi-projective schemes.

First, we recall some notations from [8] and [8]. Given a Mumford-regular \( \mathcal{O}_P \)-module \( \mathcal{F} \), we define a sequence of \( \mathcal{O}_X \)-modules \( T_n = T_n \mathcal{F} \) and \( \mathcal{O}_P \)-modules \( Z_n = Z_n \mathcal{F} \) as follows. Starting with \( T_0 \mathcal{F} = \pi_* \mathcal{F} \) and \( Z_{-1} = \mathcal{F} \). Since \( \mathcal{F} \) is Mumford-regular, there is a canonical onto map \( \varepsilon : \pi^* \pi_*(\mathcal{F}) \to \mathcal{F} \) (see Lemma [4.1]). Let \( Z_0 \mathcal{F} = \ker \varepsilon \). So, we get an exact sequence

\[
0 \to Z_0 \mathcal{F} \to \pi^* T_0 \mathcal{F} \to Z_{-1} \mathcal{F} \to 0.
\]

Inductively, we define

\[
T_n \mathcal{F} = \pi_* Z_{n-1}(n), \quad Z_n \mathcal{F} = \ker(\varepsilon)(-n),
\]

where \( \varepsilon \) is the canonical map \( \pi^* \pi_* Z_{n-1}(n) \to Z_{n-1}(n) \). Therefore, we have sequences

\[
0 \to Z_n(n) \to \pi^* T_n \mathcal{F} \to Z_{n-1}(n) \to 0,
\]

which are exact except possibly at \( Z_{n-1}(n) \). Now we state a result which is known as the Quillen Resolution theorem.

**Theorem 5.1.** Let \( \mathcal{F} \) be a vector bundle on \( \mathbb{P}(\mathcal{E}) \), \( \text{rank}(\mathcal{E}) = r + 1 \). If \( \mathcal{F} \) is Mumford-regular, then \( Z_r = 0 \), and the sequences (5.1) are exact for \( n \geq 0 \), so there is an exact sequence

\[
0 \to (\pi^* T_r \mathcal{F})(-r) \overset{\varepsilon(-r)}{\longrightarrow} \cdots \to (\pi^* T_i \mathcal{F})(-i) \overset{\varepsilon(-i)}{\longrightarrow} \cdots \overset{\varepsilon(-1)}{\longrightarrow} \pi^* T_0 \mathcal{F} \to Z_{-1} \mathcal{F} \to 0.
\]

**Proof.** See Theorem 8.7.8 of [8]. \( \square \)
Next, our goal is to prove a relative version of the above theorem. To do this let us first fix some notation.

For \( n \in \mathbb{Z} \), we define the relative twist functor \((n)^{rel} : \text{Mod}(\mathbb{P}(f)) \to \text{Mod}(\mathbb{P}(f))\) by
\[
(n)^{rel}(\mathcal{F}_1, \alpha, \mathcal{F}_2) = (\mathcal{F}_1, \alpha, \mathcal{F}_2)(n) := (\mathcal{F}_1(n), \alpha(n), \mathcal{F}_2(n)),
\]
where \( \mathbb{P}(f) \) as in diagram (1.1) and \( \alpha(n) := \alpha \otimes \text{id} : \mathbb{P}(f)^* \mathcal{F}_1 \otimes \mathcal{O}_{\mathbb{P}(f)}(n) \cong \mathbb{P}(f)^* \mathcal{F}_2 \otimes \mathcal{O}_{\mathbb{P}(f)}(n) \).

**Lemma 5.2.** For \( n \in \mathbb{Z} \), \((n)^{rel}\) is an exact functor on \( \text{Mod}(\mathbb{P}(f))\).

**Proof.** Since \( \mathcal{O}_{\mathbb{P}(f)}(n) \) is flat over \( \mathcal{O}_{\mathbb{P}(f)} \), the twist \((n)\) is an exact functor on \( \text{Mod}(\mathbb{P}(f))\). Hence the result follows by Lemma 2.4. \(\square\)

Let \( \mathcal{F} := (\mathcal{F}_1, \alpha, \mathcal{F}_2) \in \text{MR}(\mathbb{P}(f)) \). Since \( f \) is a flat map,
\[
\pi_{X^*} \alpha : f^* \pi_{S^*} \mathcal{F}_1 \cong \pi_{X^*} \mathbb{P}(f)^* \mathcal{F}_1 \cong \pi_{X^*} \mathbb{P}(f)^* \mathcal{F}_2 \cong f^* \pi_{S^*} \mathcal{F}_2.
\]

Let
\[
T_0 \alpha = \pi_{X^*} \alpha, \ Z_{-1} \alpha = \alpha.
\]

Then we define
\[
T_0(\mathcal{F}) = (T_0 \mathcal{F}_1, T_0 \alpha, T_0 \mathcal{F}_2) \quad \text{and} \quad Z_{-1} \mathcal{F} = (Z_{-1} \mathcal{F}_1, Z_{-1} \alpha, Z_{-1} \mathcal{F}_2) = \mathcal{F},
\]
where \( T_0 \mathcal{F}_i = \pi_{S^*} \mathcal{F}_i \) for \( i = 1, 2 \). Clearly, \( T_0(\mathcal{F}) \in \text{Mod}(f) \). Let \( Z_0 \mathcal{F} = (Z_0 \mathcal{F}_1, Z_0 \alpha, Z_0 \mathcal{F}_2) \), where \( Z_0 \alpha = \pi_{X^*} T_0 \alpha \). Since \( \mathbb{P}(f) \) is a flat map, \( Z_0 \mathcal{F} \in \text{Mod}(\mathbb{P}(f)) \). Inductively, we define
\[
T_n(\mathcal{F}) = (T_n \mathcal{F}_1, T_n \alpha, T_n \mathcal{F}_2) \quad \text{and} \quad Z_n \mathcal{F} = (Z_n \mathcal{F}_1, Z_n \alpha, Z_n \mathcal{F}_2),
\]
where \( T_n \alpha = \pi_{X^*}((Z_{-1} \alpha(n)) \text{ and } Z_n \alpha = (\pi_{X^*} T_n \alpha)(-n) \). One can easily check that for each \( n \in \mathbb{N} \), \( T_n(\mathcal{F}) \in \text{Mod}(f) \) and \( Z_n \mathcal{F} \in \text{Mod}(\mathbb{P}(f)) \). Define
\[
\pi_{S^*} T_n(\mathcal{F}) := (\pi_{S^*} T_n \mathcal{F}_1, \pi_{X^*} T_n \alpha, \pi_{S^*} T_n \mathcal{F}_2).
\]

Thus we have sequences
\[
(5.3) \quad 0 \to Z_n \mathcal{F}(n) \to \pi_{S^*} T_n \mathcal{F} \xrightarrow{(\varepsilon_1, \varepsilon_2)} Z_{n-1} \mathcal{F}(n)
\]
in \( \text{Mod}(\mathbb{P}(f)) \). We are now ready to prove a relative version of Theorem 5.1.

**Theorem 5.3.** Let \( \mathcal{F} := (\mathcal{F}_1, \alpha, \mathcal{F}_2) \in \text{MR}(\mathbb{P}(f)) \), where \( \mathbb{P}(f) \) as in diagram (1.1) with \( \text{rank}(\mathcal{E}) = r + 1 \). Then there is an exact sequence
\[
(5.4) \quad 0 \to (\pi_{S^*} T_n \mathcal{F})(-r) \xrightarrow{(\varepsilon_1(-r), \varepsilon_2(-r))} \cdots \to (\pi_{S^*} T_n \mathcal{F})(-i) \xrightarrow{(\varepsilon_1(-i), \varepsilon_2(-i))} \cdots \xrightarrow{(\varepsilon_1(-1), \varepsilon_2(-1))} \pi_{S^*} T_0 \mathcal{F} \to \mathcal{F} \to 0
\]
in \( \text{Mod}(\mathbb{P}(f)) \). Moreover, each \( \mathcal{F} \mapsto T_n \mathcal{F} \) is an exact functor from \( \text{MR}(\mathbb{P}(f)) \) to \( \text{VB}(f) \).
Proof. Since $F_1, F_2$ both are Mumford-regular, so are $Z_{n-1}F_1(n), Z_{n-1}F_2(n)$ (see the proof of Theorem II. 8.7.8 of [8]). By Lemma 4.4 the sequences (5.3) are exact at $Z_{n-1}F(n)$, i.e., we have exact sequences

\[
0 \to Z_nF(n) \to \pi_S^*T_nF \xrightarrow{(\varepsilon_1, \varepsilon_2)} Z_{n-1}F(n) \to 0.
\]

Now the twists of the sequences (5.3) fit together into the sequence of the form (5.4).

For the second part, each $F \mapsto T_1F$ is an exact functor from $\text{MR}(\mathbb{P}(E))$ to $\text{VB}(X)$ by Corollary II.8.7.9 of [8]. Hence the assertion follows from Lemma 2.1.

6. Projective bundle formula for relative $K_0^{He}$

In this section, we prove that the projective bundle formula holds for Heller’s relative $K_0^{He}$ of a flat map. Throughout this section, $f : X \to S$ is a flat map of quasi-projective schemes. Also, $\mathbb{P}(f)$ always mean the map $\mathbb{P}(f^*E) \to \mathbb{P}(E)$ as in diagram (1.1) with $\text{rank}(E) = r + 1$.

We observe in Lemma 2.2 that $\text{VB}(f)$ is an exact subcategory of $\text{Mod}(f)$. So we can define $K_0(\text{VB}(f))$ in the sense of Quillen absolute $K_0$ of exact categories. By definition, $K_0(\text{VB}(f))$ is the abelian group generated $[(V_1, \alpha, V_2)], (V_1, \alpha, V_2) \in \text{VB}(f)$, and relations $[V'] + [V''] = [V]$ for every exact sequence $0 \to V' \to V \to V'' \to 0$ in $\text{VB}(f)$. It is denoted by $K_0^Q(f)$. Clearly, there is a natural surjection

\[
\eta^f : K_0^Q(f) \to K_0^{He}(f)
\]

and $\ker(\eta^f)$ is generated by $[(V_1, \alpha, V_2)] + [(V_2, \beta, V_3)] - [(V_1, \beta \alpha, V_3)]$ for every pair $(V_1, \alpha, V_2), (V_2, \beta, V_3)$ of objects in $\text{VB}(f)$.

The $n$-th twist of $\text{MR}(\mathbb{P}(f))$, notation $\text{MR}(\mathbb{P}(f))(n)$, is a category consisting of objects $(\mathcal{F}_1, \alpha, \mathcal{F}_2)$ of $\text{VB}(\mathbb{P}(f))$ such that $(\mathcal{F}_1(-n), \alpha(-n), \mathcal{F}_2(-n))$ is in $\text{MR}(\mathbb{P}(f))$. Each $\text{MR}(\mathbb{P}(f))(n)$ is an exact category because the relative twisting is an exact functor (see Lemma 5.2) and the Mumford-regular modules are closed under extensions (see Lemma II.8.7.4 of [8]). By Lemma 4.1(1), we have

\[
\text{MR}(\mathbb{P}(f)) = \text{MR}(\mathbb{P}(f))(0) \subseteq \text{MR}(\mathbb{P}(f))(-1) \subseteq \cdots \subseteq \text{MR}(\mathbb{P}(f))(n) \subseteq \text{MR}(\mathbb{P}(f))(n-1) \subseteq \cdots
\]

**Theorem 6.1.** For all $n \leq 0$, $K_0^Q \text{MR}(\mathbb{P}(f)) \cong K_0^Q \text{MR}(\mathbb{P}(f))(n) \cong K_0^Q(\mathbb{P}(f))$ induced by the inclusion $\text{MR}(\mathbb{P}(f))(n) \subset \text{VB}(\mathbb{P}(f))$.

**Proof.** Let $(\mathcal{F}_1, \alpha, \mathcal{F}_2) \in \text{VB}(\mathbb{P}(f))$. By Lemma 4.3(1), $\mathcal{F}_1(n), \mathcal{F}_2(n) \in \text{MR}(\mathbb{P}(E))$ for $n \geq 0$ large enough. Then

\[
(\mathcal{F}_1(n), \alpha(n), \mathcal{F}_2(n)) \in \text{MR}(\mathbb{P}(f))(-n)
\]

for $n \geq 0$ large enough. So, it is clear that $\bigcup_{n \leq 0} \text{MR}(\mathbb{P}(f))(n) = \text{VB}(\mathbb{P}(f))$. Since $K_0^Q$ commutes with filtered colimits (see Example II. 7.1.7 of [8]), we have $K_0^Q \text{VB}(\mathbb{P}(f)) = \text{VB}(\mathbb{P}(f))$. 
Where $F$ denotes the triples $(F_1, \alpha, F_2) \mapsto (F_1(\alpha) \otimes \pi^*_S \wedge^i E, \alpha(\alpha) \otimes \text{id}, F_2(\alpha) \otimes \pi^*_S \wedge^i E)$ defines an exact functor from $\text{MR}(\mathbb{P}(f))((n-1)$ to $\text{MR}(\mathbb{P}(f))(n)$. It induces a homomorphism $\lambda_i : K^Q_0 \text{MR}(\mathbb{P}(f))(n-1) \rightarrow K^Q_0 \text{MR}(\mathbb{P}(f))(n)$. For a vector bundle $F$ in $\text{VB}(\mathbb{P}(E))$, we have the Koszul resolution (see the proof of Lemma 1.3 in the section 8 of [6])

$$0 \rightarrow F \rightarrow F(1) \otimes \pi^* \wedge E^\vee \rightarrow \cdots \rightarrow F(r + 1) \otimes \pi^* \wedge^{r+1} E^\vee \rightarrow 0.$$ 

Here $E^\vee$ denotes the dual of $E$. Similarly, a relative version of Koszul resolution is (by Lemma 2.1)

$$0 \rightarrow (F_1, \alpha, F_2) \rightarrow (F_1(1) \otimes \pi^*_S \wedge E^\vee, \alpha(1) \otimes \text{id}, F_2(1) \otimes \pi^*_S \wedge E^\vee) \rightarrow \cdots \rightarrow (F_1(r + 1) \otimes \pi^*_S \wedge^{r+1} E^\vee, \alpha(r + 1) \otimes \text{id}, F_2(r + 1) \otimes \pi^*_S \wedge^{r+1} E^\vee) \rightarrow 0.$$ 

By the additivity theorem (see Theorem 2, Cololary 3 of [6]), the map $\sum_{i>0}(-1)^{i-1}\lambda_i$ is an inverse to the map $l_n$. Hence the assertion.

Let $(F_1, \alpha, F_2) \in \text{VB}(f)$. Then the assignment

$$u_i : (F_1, \alpha, F_2) \mapsto (\pi^*_S F_1, \pi^*_X \alpha, \pi^*_S F_2)(-i)$$

defines an exact functor from $\text{VB}(f)$ to $\text{VB}(\mathbb{P}(f))$. Let $u_{i*}$ denote the induced map $K^Q_0(f) \rightarrow K^Q_0(\mathbb{P}(f))$.

For notational convenience, we prefer to write $F_k$ (resp. $\pi^*_S F_k$) instead of $(F_{k1}, \alpha_k, F_{k2})$ (resp. $(\pi^*_S F_{k1}, \pi^*_X \alpha_k, \pi^*_S F_{k2})$). We can now define a group homomorphism

$$u^Q : K^Q_0(f)^{r+1} \rightarrow K^Q_0(\mathbb{P}(f))$$

by sending $([F_k])_{k=0,1,\ldots,r}$ to $\sum_{k=0}^r u_{k*}[F_k] = \sum_{k=0}^r [u_k F_k] = \sum_{k=0}^r [\pi^*_S F_k(-k)]$.

Note that each $u_i$ also induces a map $K^H_0(f) \rightarrow K^H_0(\mathbb{P}(f))$. Therefore, in a similar way we can define a group homomorphism

$$u^H : K^H_0(f)^{r+1} \rightarrow K^H_0(\mathbb{P}(f)).$$

**Theorem 6.2.** The map $u^Q : K^Q_0(f)^{r+1} \rightarrow K^Q_0(\mathbb{P}(f))$ is an isomorphism.

**Proof.** By Theorem 5.3 each $\mathcal{T}_n$ is an exact functor from $\text{MR}(\mathbb{P}(f))$ to $\text{VB}(f)$. Hence we can define a group homomorphism

$$\varphi : K^Q_0 \text{MR}(\mathbb{P}(f)) \rightarrow K^Q_0(f)^{r+1}, [F] \mapsto ([\mathcal{T}_0 F], -[\mathcal{T}_1 F], \ldots, (-1)^r[\mathcal{T}_r F]),$$

where $F$ denotes the triples $(F_1, \alpha, F_2)$. Then the composition map

$$u^Q \varphi : K^Q_0(\mathbb{P}(f)) \xrightarrow{\xi} K^Q_0 \text{MR}(\mathbb{P}(f)) \xrightarrow{\varphi} K^Q_0(f)^{r+1} \xrightarrow{u^Q} K^Q_0(\mathbb{P}(f))$$

is an isomorphism.
Using these $\pi$'s, we can define a group homomorphism

$$v_i : F := (F_1, \alpha, F_2) \mapsto \pi_{S_i}(F(i)) := (\pi_{S_i}(F_1(i)), \pi_{X_i}(\alpha(i)), \pi_{S_i}(F_2(i)))$$

is also an exact functor from $\text{MR}(\mathbb{P}(f))$ to $\text{VB}(f)$ by Lemmas 4.2 and 4.3. Let $v_i$ denote the induced map

$$K_0^Q \text{MR}(\mathbb{P}(f)) \to K_0^Q(\mathbb{P}(f)), [F] \mapsto [v_iF].$$

Using these $v_i$'s, we can define a group homomorphism

$$v^Q : K_0^Q \text{MR}(\mathbb{P}(f)) \to K_0^Q(f)^{r+1}, [F] \mapsto ([v_0F], [v_1F], \ldots, [v_rF]).$$

Then the composition map (using Theorem 6.1)

$$v^Q u^Q : K_0^Q(f)^{r+1} \to K_0^Q(f)^{r+1}$$

is given by the matrix $(v_i u_j)$. Recall from Example II. 8.7.2 of [8] that for a quasi-coherent $O_X$-module $N$, we have $\pi_*\pi^*N = N$, $\pi_*\pi^*N(n) = 0$ for $n < 0$ and $\pi_*\pi^*N(n) = \text{Sym}_n E \otimes N$ for $n > 0$. Thus

$$v_{i*, u_{j*}}[(F_1, \alpha, F_2)] = [(\pi_{S_i}((\pi_{S_i}F_1)(i - j)), \pi_{X_i}((\pi_{S_i}X)(i - j)), \pi_{S_i}((\pi_{S_i}F_2)(i - j))].$$

Since the diagram

$$\begin{array}{ccc}
\pi_{X_i} & \xrightarrow{\alpha} & \pi_{X_i} \\
\downarrow & & \downarrow \\
\pi_{S_i} & \xrightarrow{\alpha} & \pi_{S_i}
\end{array}$$

is commutative, $[(\pi_{S_i}((\pi_{S_i}F_1)(i - j)), \pi_{X_i}((\pi_{S_i}X)(i - j)), \pi_{S_i}((\pi_{S_i}F_2)(i - j))] \in K_0^Q(f)$. This implies that $v_{i*, u_{j*}} = 0$ for $i < j$ and $v_{i*, u_{j*}} = \text{id}$ for $i = j$. We get that $(v_{i*, u_{j*}})$ is a lower triangular matrix with all of its diagonal entries equal to $\text{id}$. Therefore $v^Q u^Q$ is an isomorphism and hence $u^Q$ is one-one.

Next, we prove a similar result for $K_0^{He}$. Theorem 6.3. The map $u^{He} : K_0^{He}(f)^{r+1} \to K_0^{He}(\mathbb{P}(f))$ is an isomorphism.

Proof. We consider the following commutative diagram

$$
\begin{array}{ccccccc}
0 & \rightarrow & \ker(\eta^{MR}) & \rightarrow & K_0^Q(\text{MR}(\mathbb{P}(f))) & \rightarrow & K_0^{He}(\mathbb{P}(f)) & \rightarrow & 0 \\
& & \downarrow \cong & \downarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \\
0 & \rightarrow & \ker(\eta^{P(f)}) & \rightarrow & K_0^Q(\mathbb{P}(f)) & \rightarrow & K_0^{He}(\mathbb{P}(f)) & \rightarrow & 0,
\end{array}
$$

where the middle map is an isomorphism by Theorem 6.1. So we get $\ker(\eta^{MR}) \cong \ker(\eta^{P(f)})$. Let $F_{12} := (F_1, \alpha, F_2), F_{23} := (F_2, \beta, F_3)$ and $F_{13} := (F_1, \beta\alpha, F_3)$ be in $\text{MR}(\mathbb{P}(f))$. Note that $\varphi(F_{12} + [F_{23}] - [F_{13}]) \in \ker((\eta^f)^{r+1})$ whenever $([F_{12}] + [F_{23}] - [F_{13}]) \in \ker(\eta^{MR})$ because
each $T_n$ is an exact functor from $\text{MR}(\mathbb{P}(f))$ to $\text{VB}(f)$ (see Theorem 6.2 for the map $\varphi$). Then the composition map
\[
u_Q \varphi : \ker(\eta^P(f)) \xrightarrow{\cong} \ker(\eta^MR) \xrightarrow{\varphi} \ker((\eta^f)^{r+1}) \xrightarrow{\nu_Q} \ker(\eta^P(f))
\]
sends $[\mathcal{F}_{12}] + [\mathcal{F}_{23}] - [\mathcal{F}_{13}]$ to
\[
\sum_{k=0}^{r} (-1)^k [((\pi_0^* S T_k) \mathcal{F}_{12})(-k)] + \sum_{k=0}^{r} (-1)^k [((\pi_0^* S T_k) \mathcal{F}_{23})(-k)] - \sum_{k=0}^{r} (-1)^k [((\pi_0^* S T_k) \mathcal{F}_{13})(-k)]
\]
which is equal to $[\mathcal{F}_{12}] + [\mathcal{F}_{23}] - [\mathcal{F}_{13}]$ by the additivity theorem (see Theorem 2, Corollary 3 of [6]). This shows that $u_Q|_{\ker((\eta^f)^{r+1})}$ is onto. Therefore, we get the desired isomorphism from the following commutative diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker((\eta^f)^{r+1}) & \longrightarrow & K_0^Q(f)^{r+1} & \xrightarrow{(\eta^f)^{r+1}} & K_0^{He}(f)^{r+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow u_Q & & \downarrow u_{He} & & \\
0 & \longrightarrow & \ker(\eta^P(f)) & \longrightarrow & K_0^Q(\mathbb{P}(f)) & \xrightarrow{\eta^P(f)} & K_0^{He}(\mathbb{P}(f)) & \longrightarrow & 0,
\end{array}
\]
where $u_Q$ is an isomorphism by Theorem 6.2 and the first vertical arrow is also an isomorphism by the above observation.

\[\square\]

**Corollary 6.4.** Suppose $\mathbb{P}(f)$, $f$ and $\mathcal{E}$ are as in Theorem 6.3. Further, we assume that $X$ is an affine scheme. Then there is an isomorphism of groups $K_0^{He}(\mathbb{P}(f)) \cong K_0(\mathbb{P}(f))$.

**Proof.** By Lemma 3.2 $K_0(f)^{r+1} \xrightarrow{\cong} K_0(\mathbb{P}(f))$. We also have an isomorphism $K_0^{He}(f)^{r+1} \xrightarrow{\cong} K_0(f)^{r+1}$ by Lemma 3.1. Hence we get
\[
K_0(\mathbb{P}(f)) \xrightarrow{\cong} K_0(f)^{r+1} \xrightarrow{\cong} K_0^{He}(f)^{r+1} \xrightarrow{\cong} K_0^{He}(\mathbb{P}(f)),
\]
where the last isomorphism by Theorem 6.3.

\[\square\]

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