Elliptic surfaces over $\mathbb{P}^1$ and large class groups of number fields

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Abstract

Given a non-isotrivial elliptic curve over $\mathbb{Q}(t)$ with large Mordell-Weil rank, we explain how one can build, for suitable small primes $p$, infinitely many fields of degree $p^2 - 1$ whose ideal class group has a large $p$-torsion subgroup. As an example, we show the existence of infinitely many cubic fields whose ideal class group contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{11}$.

1 Introduction

Our initial motivation for the present paper is the following conjecture on class groups of number fields, which belongs to folklore, and is a consequence of the Cohen-Lenstra heuristics.

Conjecture 1.1. Let $k$ be a number field, let $p$ be a prime number, and let $d > 1$ be an integer. Then $\dim_{\mathbb{F}_p} \text{Cl}(L)[p]$ is unbounded when $L/k$ runs through all extensions of degree $[L : k] = d$.

When $d = p$, and more generally when $p$ divides $d$, this conjecture follows from class field theory [Bru65, RZ69] (or see [Lev07, Th. 3.4] for a proof aligned with the techniques of the present paper). On the other hand, when $p$ and $d$ are coprime, there is not a single case where Conjecture 1.1 is known to hold. For example, given a prime $p$, it is known [Yam70] that there exist infinitely many imaginary quadratic fields $L/\mathbb{Q}$ such that $\dim_{\mathbb{F}_p} \text{Cl}(L)[p] \geq 2$. For $p \geq 7$ this is currently the best known result on $p$-ranks of class groups of quadratic fields.

In the present paper, we investigate this conjecture in the specific case when $d = p^2 - 1$. Our strategy is closely related to the techniques developed in [GL12] and [BG18]. The main new ingredient is the following result of [GL18]: given a non-isotrivial elliptic surface over $k(t)$ with large rank, for almost all primes $p$ one is able to produce a curve $C$ of gonality $p^2 - 1$ whose Picard group has large $p$-rank (see Theorem 2.1). This construction allows one to use large Mordell-Weil rank over $\mathbb{Q}(t)$ to produce number fields of degree $p^2 - 1$ whose ideal class group has large $p$-rank.

In particular, we show the existence of infinitely many cubic fields whose ideal class group has 2-rank $r \geq 11$. This improves on a result of Kulkarni [Kul18], who proved this statement with $r \geq 8$ (see Remark 1.7).

1.1 The main result

Let $k$ be a number field, and let $E$ be an elliptic curve over $k(t)$. We denote by $\mathcal{E} \to \mathbb{P}^1_k$ the Néron model of $E$ over $\mathbb{P}^1_k$, by which we mean the group scheme model, which is the smooth locus of Néron’s minimal regular model (see [BLR90 §1.5] or [Liu02 §10.2]).

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By abuse of notation, we identify closed points of \( \mathbb{P}^1_k \) and discrete valuations of \( k(t) \). If \( v \in \mathbb{P}^1_k \) is such a point, we denote by \( \mathcal{E}_v \) the special fiber of \( \mathcal{E} \) at \( v \), and by \( \Phi_v \) the group of connected components of \( \mathcal{E}_v \). The Tamagawa number of \( E \) at \( v \) is by definition the order of \( \Phi_v(k_v) \), and we denote it by \( c_v \). If \( E \) has good reduction at \( v \), then \( \Phi_v = 0 \) and \( c_v = 1 \). The set of places of bad reduction being finite, we have that \( c_v = 1 \) for all but finitely many \( v \).

**Theorem 1.2.** Let \( k \) be a number field and let \( E \) be an elliptic curve over \( k(t) \). Assume that \( E \) does not have universal bad reduction at any prime of \( k \) (Definition 2.5). Let \( p \) be a prime number such that \( E[p] \) is irreducible as a Galois module over \( \overline{k}(t) \).

1) If \( p \geq 3 \), there exist infinitely many field extensions \( L/k \) of degree \( p^2 - 1 \) such that

\[
\dim_{\mathbb{F}_p} \text{Cl}(L)[p] - \dim_{\mathbb{F}_p} \text{Cl}(k)[p] \geq \text{rk}_Z E(k(t)) - \# \{ v \in \mathbb{P}^1_k, p|c_v \} - \text{rk}_Z \mathcal{O}_L^\times + \text{rk}_Z \mathcal{O}_k^\times. \tag{1}
\]

2) If \( p = 2 \), there exist infinitely many field extensions \( L/k \) of degree 3 such that

\[
\dim_{\mathbb{F}_2} \text{Cl}(L)[2] - \dim_{\mathbb{F}_2} \text{Cl}(k)[2] \geq \text{rk}_Z E(k(t)) - \# \{ v \in \mathbb{P}^1_k, 2|c_v \} - \text{rk}_Z \mathcal{O}_L^\times + \text{rk}_Z \mathcal{O}_k^\times
- \# \{ v \in \mathbb{P}^1_k, \text{the red. type of } \mathcal{E} \text{ at } v \text{ is } I_{2n}^* \text{ for some } n \geq 0 \}. \tag{2}
\]

The proof is given in [2.3]. The assumption that \( E \) does not have universal bad reduction at any prime of \( k \) is a technical condition detailed in [2.4]. Given a Weierstrass equation for the elliptic curve \( E \), this condition can be verified numerically.

If \( E \) is not isotrivial, then \( E[p] \) is irreducible as a Galois module over \( \overline{k}(t) \) for all but finitely many primes \( p \). If in addition \( E \) does not have universal bad reduction at any prime of \( k \), then the conclusion of Theorem 1.2 holds for all but finitely many primes \( p \).

**Remark 1.3.** When \( p \) gets large, the term \( \text{rk}_Z \mathcal{O}_L^\times \) also gets large: according to Dirichlet’s unit theorem, it is bounded below by \( \frac{p^2-1}{2} - 1 \). Therefore, given an elliptic surface \( E \), Theorem 1.2 should be applied to primes \( p \) which are small with respect to the rank of \( E \).

**Remark 1.4.** When \( p = 2 \), the condition that \( E \) does not have universal bad reduction at any prime of \( k \) can be replaced by the condition that its Néron model \( \mathcal{E} \) has a fiber of type II, II*, IV or IV* at some \( v \in \mathbb{P}^1(k) \). In fact, we obtain in this case a stronger version of (2), in which the contribution from the units is removed; see Proposition 2.4.

**Remark 1.5.** The fields \( L/k \) of degree \( p^2 - 1 \) in (1) have in fact a canonical subfield \( L' \) with \([L' : k] = p + 1 \). In specific situations, the equation of \( E \) being given, one may perform additional computations in order to obtain a lower bound on \( \dim_{\mathbb{F}_p} \text{Cl}(L')[p] \). See Remark 2.2 for details.

### 1.2 Example: cubic fields

Applying Theorem 1.2 to an elliptic curve over \( \mathbb{Q}(t) \) with large rank constructed by Kihara [Kih01], we obtain the following result (see [2.4] for the proof).

**Theorem 1.6.** There exists a trigonal curve \( C \), defined over \( \mathbb{Q} \), such that

\[
\dim_{\mathbb{F}_2} \text{Pic}(C)[2] \geq 12.
\]

Moreover, there exists a degree three morphism \( \phi : C \to \mathbb{P}^1 \) such that, for all but \( O(\sqrt{N}) \) integers \( t \in \mathbb{Z} \) with \(|t| \leq N \), the field of definition of \( \phi^{-1}(t) \) is a cubic field \( K_t \) with exactly one real place, satisfying

\[
\dim_{\mathbb{F}_2} \text{Cl}(K_t)[2] \geq 11.
\]
Remark 1.7. Nakano [Nak88] proved that there are infinitely many cubic fields $K$ whose ideal class group has 2-rank at least 6. Recently, Kulkarni [Kul18] has improved on this result, obtaining infinitely many cubic fields $K$ whose ideal class group has 2-rank at least 8. To our knowledge, this is the best previously known result on the existence of an infinite family of cubic fields whose class group has large 2-rank. In fact, our method is closely related to Kulkarni’s, in the sense that in both cases one considers trigonal curves which come from the 2-torsion of an elliptic fibration.

1.3 Picard groups of curves over finite fields

Picard groups of curves over finite fields are considered natural analogues of class groups of number fields. Thus, we include in this section a remark on the construction of curves over finite fields whose Jacobian contains a large torsion subgroup.

Ulmer has shown [Ulm02] that, given a prime $q$, one can find non-isotrivial elliptic curves over $\mathbb{F}_q(t)$ with arbitrarily large Mordell-Weil rank. More precisely, given an integer $n \geq 1$, he considers the elliptic curve $E_n$ defined by the equation

$$y^2 + xy = x^3 + t^d,$$

where $d = q^n + 1$, (E_n)

and he proves that the rank of $E_n$ over $\mathbb{F}_q(t)$ is at least $(q^n - 1)/2n$.

Let us assume that $q \geq 5$. Then the trigonal curve $C_{2,n} \rightarrow \mathbb{P}^1$ describing the 2-torsion of $E_n$ is defined by the equation

$$4x^3 + x^2 - 4t^d = 0.$$  (C_{2,n})

We claim that $C_{2,n}$ is geometrically irreducible. Indeed, if the equation above had a root $x$ in $\mathbb{F}_q(t)$, then by Gauss’s lemma this root would be a polynomial which divides $t^d$, and this leads to a contradiction.

Applying Theorem 2.1 to the elliptic curves $E_n$, we obtain in §2.1 the following result.

Theorem 1.8. Let $q \geq 5$ be a prime. Then

1) The family of trigonal curves $C_{2,n}$ defined above (over $\mathbb{F}_q$) satisfies

$$\dim_{\mathbb{F}_q} \text{Pic}(C_{2,n})[2] \geq \frac{q^n - 1}{2n} - 3.$$

2) Given an integer $n \geq 1$, for all but finitely many primes $p$, there exists a curve $C$ of gonality $p^2 - 1$ over $\mathbb{F}_q$ such that

$$\dim_{\mathbb{F}_p} \text{Pic}(C)[p] \geq \frac{q^n - 1}{2n} - 2.$$

Remark 1.9. Using similar techniques, it has been proved in [ˇCes15, Section 5] that, if $p$ and $q$ are two distinct primes, there exists a constant $m$ depending only on $p$ such that the size of $\text{Pic}(C)[p]$ is unbounded when $C$ runs through hyperelliptic curves over $\mathbb{F}_{q^m}$.

2 Proofs

2.1 Elliptic surfaces over $\mathbb{P}^1_k$

In this section, we briefly recall the main result of [GL18] in the setting of elliptic surfaces over $\mathbb{P}^1_k$. We refer the reader to loc. cit. for further details and comments.
Let $k$ be a perfect field of characteristic not 2 or 3. In the applications we have in mind, $k$ may be a number field, or a finite field, or the algebraic closure of such fields.

Let $E$ be an elliptic curve over $k(t)$, and let $E \to \mathbb{P}^1_k$ be its Néron model. If $v$ is a closed point of $\mathbb{P}^1_k$, we let $E_v$, $\Phi_v$, and the Tamagawa number $c_v$ be as in \[11\]. We denote by $\Phi$ the skyscraper sheaf over $\mathbb{P}^1_k$ whose fiber at $v$ is $\Phi_v$, and by $E^{\Phi}$ the inverse image of $p\Phi$ by the natural map $E \to \Phi$.

If $p \neq \text{char}(k)$ is a prime number, we denote by $E[p] \to \mathbb{P}^1_k$ the group scheme of $p$-torsion points of $E$. The map $E[p] \to \mathbb{P}^1_k$ is étale, because $p \neq \text{char}(k)$. We shall denote by $C$ the smooth compactification of $E[p] \setminus \{0\}$, endowed with its canonical finite map $C \to \mathbb{P}^1_k$ of degree $p^2 - 1$.

The following result is a special case of [GL18, Theorem 1.1]. Assuming that $C$ is geometrically integral, it provides an upper bound on the rank of $E$ in terms of the Tamagawa numbers $c_v$ and the $p$-torsion in the Picard group of $C$. Its proof relies on $p$-descent techniques, analogous to the number field case.

\textbf{Theorem 2.1.} Let $p \neq \text{char}(k)$ be a prime number, and let $C$ be the smooth compactification of $E[p] \setminus \{0\}$. Assume that $C$ is geometrically integral, or equivalently that $E[p]$ is irreducible as a Galois module over $\overline{k}(t)$. Then

1) There is an injective map

$$E^{\Phi}(\mathbb{P}^1_k)/pE(\mathbb{P}^1_k) \longrightarrow \text{Pic}(C)[p].$$

2) If $p \geq 3$, we have

$$\dim_{F_p} \text{Pic}(C)[p] \geq \dim_{F_p} E^{\Phi}(\mathbb{P}^1_k)/pE(\mathbb{P}^1_k) \geq \text{rk}_{\mathbb{Z}} E(k(S)) - \#\{v \in \mathbb{P}^1_k, p \mid c_v\},$$

where $c_v$ denotes the Tamagawa number of $E$ at $v$.

3) If $p = 2$, then

$$\dim_{F_2} \text{Pic}(C)[2] \geq \dim_{F_2} E^{2\Phi}(\mathbb{P}^1_k)/2E(\mathbb{P}^1_k) \geq \text{rk}_{\mathbb{Z}} E(k(S)) - \#\{v \in \mathbb{P}^1_k, 2 \mid c_v\}$$

$$- \#\{v \in \mathbb{P}^1_k, \text{the red. type of } E \text{ at } v \text{ is } I_{2n} \text{ for some } n \geq 0\}.$$ 

In fact, the injective morphism (3) is obtained by composing maps

$$E^{\Phi}(\mathbb{P}^1_k)/pE(\mathbb{P}^1_k) \longrightarrow H^1(C, \mu_p) \longrightarrow \text{Pic}(C)[p]$$

the first being obtained by Kummer theory on $E$, and the second being the natural one.

A comment on the terminology: when we say that $E$ has a fiber of type $I_{2n}$ at $v$, we mean it over $k_v$, and not just over $\overline{k}$. More precisely, this means that the Kodaira type of $E_v$ over $\overline{k}$ is $I_{2n}^*$, and that the four components of $E_v$ are rational over $k_v$, in other terms $\Phi_v(k_v) \simeq (\mathbb{Z}/2)^2$. In general, the reduction type at $v$ can be described by the data of the reduction type over $\overline{k}$ together with the action of the absolute Galois group of $k_v$ on $\Phi_v$. See Liu’s book [Liu02, §10.2].

\textbf{Remark 2.2.} The curve $C$ in the statement of Theorem 2.1 corresponds to the field over which $E$ acquires one rational $p$-torsion point. One can also introduce a curve $C'$ corresponding to the field over which $E$ has a rational cyclic subgroup of order $p$; then we have canonical maps

$$C \to C' \to \mathbb{P}^1_k$$
of degree $p-1$ and $p+1$, respectively. Given a specific example of a curve $E$, one can find equations for $C$ and $C'$, and compute the kernel of the norm map $\text{Pic}(C)[p] \to \text{Pic}(C')[p]$. If this kernel is small enough then, using techniques of [2.3] one could use the map $C' \to \mathbb{P}^1_k$ of degree $p+1$ in order to build extensions $L'/k$ of degree $p+1$ with $\dim_{\mathbb{F}_p} \text{Cl}(L')[p]$ large.

**Proof of Theorem 1.8.** Ulmer has checked that $E_n$ has reduction $I_1$ at places dividing $(1-2^4 3^2 t/2^d)$, and has split multiplicative reduction $I_d$ at $t=0$. The last place of bad reduction is $t=\infty$, where the possible reduction types are $I_1$, $II$, $II^*$, $IV$, $IV^*$ or $I_0^*$ depending on $q^n+1 \pmod{6}$. Then 1) is a consequence of (5) in Theorem 2.1, the error term $-3$ being obtained as $-1-2$, where $-1$ corresponds to the fiber at $t=0$ whose Tamagawa number is divisible by two, and $-2$ corresponds to the worst case when the reduction type at infinity is $I_0^*$. The second statement follows similarly from (4) in Theorem 2.1, combined with the following observation: the elliptic curve $E_n$ is not isotrivial, hence according to the geometric version of Shafarevich’s theorem, $E_n[p]$ is $\overline{k}$-irreducible for all but finitely many primes $p$.

### 2.2 Fibers of type $II$, $II^*$, $IV$, $IV^*$

Let $k$ be a number field, and let $E$ be an elliptic curve over $k(t)$ defined by a Weierstrass equation

$$y^2 = x^3 + a(t)x + b(t).$$

Then the smooth compactification of $E[2]\setminus\{0\}$ is none other than the smooth, projective curve $C$ defined by the affine equation

$$x^3 + a(t)x + b(t) = 0,$$

and the canonical map $C \to \mathbb{P}^1_k$ is just the $t$-coordinate map, which has degree 3.

**Lemma 2.3.** Assume $E \to \mathbb{P}^1_k$ has a fiber of type $II$, $II^*$, $IV$ or $IV^*$ at some $v \in \mathbb{P}^1_k$. Then the $t$-coordinate map $C \to \mathbb{P}^1_k$ is totally ramified above $v$.

**Proof.** This follows from the proof of Lemma 2.8 in [GL18].

**Proposition 2.4.** Let $E$ be an elliptic curve over $k(t)$ such that $E(\overline{k}(t))[2] = 0$. Assume in addition that its Néron model has a fiber of type $II$, $II^*$, $IV$ or $IV^*$ at some $v \in \mathbb{P}^1(k)$. Then there exist infinitely many cubic field extensions $L/k$ such that

$$\dim_{\mathbb{F}_2} \text{Cl}(L)[2] - \dim_{\mathbb{F}_2} \text{Cl}(k)[2] \geq \text{rk}_2 E(\mathbb{Q}(t)) - \#\{v \in \mathbb{P}^1_k, 2|c_v\} - \#\{v \in \mathbb{P}^1_k, \text{the red. type of } E \text{ at } v \text{ is } I_{2n}^* \text{ for some } n \geq 0\}.$$  \(7\)

**Proof.** Let $C$ be the trigonal curve defining 2-torsion points as above. According to Lemma 2.3 the natural trigonal map $C \to \mathbb{P}^1_k$ is totally ramified over some rational point. According to [BG18, Theorem 1.4], there exist infinitely many number fields $L/k$ of degree 3 such that

$$\dim_{\mathbb{F}_2} \text{Cl}(L)[2] - \dim_{\mathbb{F}_2} \text{Cl}(k)[2] \geq \dim_{\mathbb{F}_2} \text{Pic}(C)[2].$$

The result then follows from the last statement of Theorem 2.1.
2.3 Avoiding universal bad reduction

We now examine proving inequalities as in Proposition 2.4 in the absence of fibers of type II, II*, IV or IV*, and working with any prime number $p$.

For that purpose, we first introduce the notion of universal bad reduction for an elliptic family defined over a number field.

Let $k$ be a number field, and let $E$ be an elliptic curve over $k(t)$ defined by a Weierstrass equation

$$y^2 = x^3 + a(t)x + b(t)$$

with $a(t), b(t) \in \mathcal{O}_k[t]$. Let $E \to \mathbb{P}^1_k$ be the Néron model of $E$, and let $\Sigma \subset \mathbb{P}^1_k$ be the set of places of bad reduction of $E$.

**Definition 2.5.** Let $p$ be a prime ideal in $\mathcal{O}_k$. We will say that $p$ is a prime of universal bad reduction for $E$ if $p$ is a prime of bad reduction for every elliptic fiber $E_t$, $t \in \mathbb{P}^1(k) \setminus \Sigma$.

**Remark 2.6.** Let $\Delta(t) := -16(4a(t)^3 + 27b(t)^2)$ be the discriminant of the Weierstrass equation defining $E$. If $p$ is a prime of universal bad reduction for $E$, then $\Delta(t) \pmod{p}$ is divisible by $t^q - t$, where $q = N(p)$ is the (absolute) norm of $p$. In particular, if $\Delta(t) \pmod{p}$ is not identically zero, then we must have $N(p) \leq \deg \Delta$.

**Theorem 2.7.** Suppose that $E$ does not have universal bad reduction at any prime of $k$. Assume that $E[p]$ is an irreducible Galois module over $\overline{k}(t)$. Let $C$ be the smooth compactification of $E[p] \setminus \{0\}$, and let $H$ denote the image of the injective map from $E$

$$E^{p \Phi}(\mathbb{P}^1_k)/pE(\mathbb{P}^1_k) \hookrightarrow H^1(C, \mu_p).$$

Then there exists a map $\psi : C \to \mathbb{P}^1_k$ of degree $p^2 - 1$ such that, for all but $O(\sqrt{N})$ integers $t \in \mathbb{Z}$ with $|t| \leq N$, we have that:

1) $P_{t,\psi} := \psi^{-1}(t)$ is the spectrum of a field $k(P_{t,\psi})$, with $[k(P_{t,\psi}) : k] = p^2 - 1$;

2) the image of $H$ under the specialization map $P_{t,\psi}^* : H^1(C, \mu_p) \to H^1(k(P_{t,\psi}), \mu_p)$ lands into the subgroup $H^1(\mathcal{O}_{k(P_{t,\psi})}, \mu_p)$;

3) the specialization map $P_{t,\psi}^*$ above is injective on $H$.

**Proof.** Let $X_1, \ldots, X_r$ be independent $\mu_p$-torsors over $C$ generating $H$. Then by the Chevalley-Weil theorem, there exists a finite set $T$ of places of $k$ such that the $X_i$ can be extended to $\mu_p$-torsors in $H^1(C_T, \mu_p)$, where $C_T$ denotes a smooth projective model of $C$ over $\text{Spec}(\mathcal{O}_{k,T})$. In particular, $T$ contains the set of bad places of $C$.

We denote by $\phi : C \to \mathbb{P}^1_k$ the natural map $(x, t) \mapsto t$, of degree $p^2 - 1$. For each $t \in \mathbb{P}^1(k)$, we let $P_t := \psi^{-1}(t)$. It follows from the construction of $T$ and the projectivity of $C_T$ that, for each $t \in \mathbb{P}^1(k)$, the image of $H$ under the specialization map $P_t^* : H^1(C, \mu_p) \to H^1(k(P_t), \mu_p)$ lands into the subgroup $H^1(\mathcal{O}_{k(P_t), T}, \mu_p)$.

Let us now pick $p \in T$. By assumption, $E$ does not have universal bad reduction at $p$, and hence there exists $t_p \in \mathbb{P}^1(k)$ such that $E_{t_p}$ has good reduction at $p$.

Let $N_{t_p}$ be the Néron model of $E_{t_p}$ over $\text{Spec}(\mathcal{O}_{k,p})$. Then $N_{t_p}$ is an abelian scheme and it follows that the map $[p] : N_{t_p} \to N_{t_p}$ is an epimorphism for the fppf topology, regardless of the residue characteristic of $p$. 

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Let $q|p$ be a place of $k(P_t)$ above $p$, and let $P_q \in C(k(P_t)_q)$ be the corresponding localization of $P_t$. More geometrically, $P_t \otimes k_p$ is the disjoint union of the $P_q$, hence $(E_t\{p\} \setminus \{0\}) \otimes k_p$ is the union of the $P_q$. The Weil pairing induces a map

$$w : \mathcal{N}_t(p) \to \prod_{q|p} \text{Res}_{k\langle p_t \rangle,q/O_{k\langle p \rangle}} H^1_{\text{gp}}.$$

We have a commutative diagram

\[
\begin{array}{ccc}
E_t(k_p)/p & \longrightarrow & \mathcal{N}_t(O_{k,p})/p \\
\uparrow & & \downarrow \\
E^{\psi}(k_p)/pE(k_p) & \longrightarrow & H^1(C,\mu_p) \\
\end{array}
\]

in which the upper right map comes from fpf Kummer theory on the abelian scheme $\mathcal{N}_t$, combined with the Weil pairing map. The vertical left map is obtained by specializing the elliptic family at $t_p \in \mathbb{P}^1(k)$.

This proves that, under the map $P^*_t$, all elements of $H$ (and, in fact, all $\mu_p$-torsors built from Kummer theory on $E$) specialize to torsors which are integral at all places of $k(P_t)$ above $p$. By a variant of Krasner’s lemma, similar to [BG18, Lemma 2.3], the same holds for any $t \in \mathbb{P}^1(k)$ which is $p$-adically close enough to $t_p$.

Finally, let us recall that an element of $H^1(O_{k(P_t),T},\mu_p)$ belongs to $H^1(O_{k(P_t)},\mu_p)$ if and only if it belongs to $H^1(O_{k(P_t),T},\mu_p)$ for each prime $q \in T_{k(P_t)}$. Putting everything together, we may conclude that, if $t \in \mathbb{P}^1(k)$ is $p$-adically close enough to $t_p$ for every $p$ in $T$, then the image of $H$ under $P^*_t$ lands into $H^1(O_{k(P_t)},\mu_p)$.

Let $T_\mathbb{Z}$ denote the set of prime numbers lying below primes in $T$, and let $\psi : C \to \mathbb{P}^1_k$ be the map defined by

$$\psi := \frac{1}{(\prod_{p \in T_\mathbb{Z}} p)} (\phi - t_0),$$

where $t_0 \in \phi$ is $p$-adically close enough to $t_p$ for every $p \in T$, and $M$ is a sufficiently large positive integer. The conclusion of the theorem follows from a quantitative version of Hilbert’s irreducibility Theorem (see [GL12, Theorem 2.1]) applied to the composite map

$$X_1 \times C \cdots \times C X_r \longrightarrow C \longrightarrow \mathbb{P}^1_k.$$ (8)

In this final step we implicitly use the fact that $X_1 \times C \cdots \times C X_r$ is geometrically irreducible, which can be proved as follows: $H$ injects into Pic($C$) according to the first statement of Theorem 2.1, and this remains true over $\overline{k}$ by injectivity of the map Pic($C$) $\to$ Pic($C \times k, \overline{k}$). Hence $H$ injects into $H^1(C \times k, \overline{k}, \mu_p)$ via the natural base-change map; in other terms, $X_1, \ldots, X_r$ remain independent over $\overline{k}$.

**Proof of Theorem 1.2** Given the inequalities (4) and (5) from Theorem 2.1 combined with the following fact: when applying Hilbert’s irreducibility theorem to the cover (8), it is always possible to ensure that the field extension obtained is linearly disjoint from a given extension fixed in advance. If one fixes the latter extension to be the compositum of the fields that correspond to torsors in $H^1(O_{k},\mu_p)$, then the natural map $H^1(O_{k},\mu_p) \to H^1(O_{k(P_t),\psi},\mu_p)$ is injective, and it follows from Kummer theory (see the proof of Theorem 1.3 in [BG18] that

$$\dim_{\mathbb{P}} H^1(O_{k(P_t),\psi},\mu_p)/H^1(O_{k},\mu_p) = \dim_{\mathbb{P}} (k(P_t,\psi))[p] - \dim_{\mathbb{P}} Cl(k)[p]$$

$$+ \operatorname{rk}_\mathbb{Z} O^\times_{k(P_t,\psi)} - \operatorname{rk}_\mathbb{Z} O^\times_k.$$
To conclude, it suffices to point out that the image of the subgroup $H$ from Theorem 2.7 by the map $P_{t,\psi}^*$ injects into the quotient group on the left.

2.4 Proof of Theorem 1.6

Let $E$ be an elliptic curve over $\mathbb{Q}(t)$ without a rational 2-torsion point over $\mathbb{Q}(t)$. We may assume that $E$ is defined by an equation

$$y^2 = x^3 + a(t)x + b(t)$$

where $a(t)$ and $b(t)$ belong to $\mathbb{Z}[t]$.

We denote by $C$ the smooth projective curve with affine equation $x^3 + a(t)x + b(t) = 0$, which is the smooth compactification of $\mathcal{E}[2] \setminus \{0\}$. We also let $\Delta(t) := -16(4a(t)^3 + 27b(t)^2)$ be the discriminant of the Weierstrass equation (9).

Using an appropriately chosen elliptic curve $E$ of large $\mathbb{Q}(t)$-rank, we shall apply Theorem 1.2 to obtain infinitely many cubic fields with a class group of large 2-rank.

In [Kih01], Kihara gives an example of an elliptic curve $E$ over $\mathbb{Q}(t)$ of rank at least 14. A calculation shows that in Kihara’s example, $E$ contains singular fibers (over $\mathbb{P}^1_{\mathbb{Q}}$) of types $I_2$, $I_4$, and $I_6$. The elliptic curve $E$ is obtained as a specialization of a 3-dimensional family of elliptic curves having rank at least 12. By using a different specialization of this family, we may obtain a more advantageous singular fiber configuration (with Theorem 1.2 in mind) at the expense of (possibly) lowering the $\mathbb{Q}(t)$-rank.

Specifically, let $E'$ be the elliptic curve over $\mathbb{Q}(p, q, u)$ of rank at least 12 constructed in [Kih01] (in fact, Kihara gives a genus one quartic with 13 $\mathbb{Q}(p, q, u)$-points; one must choose one of the 13 points as a base point for the elliptic curve). Let $E$ be the elliptic curve over $\mathbb{Q}(t)$ obtained from $E'$ by the specialization $p = t$, $q = t + 6$, $u = t + 1$. Then one can check numerically that there is no prime of universal bad reduction for $E$.

We may write the equation of $E$ in the form (9), where the discriminant $\Delta(t)$ is irreducible in $\mathbb{Q}[t]$ of degree 96. There is exactly one bad place in $\mathbb{P}^1_{\mathbb{Q}}$, where $E$ has a singular fiber of type $I_1$.

By specializing Kihara’s points, at say $t = 1$, and computing the associated height pairing, it is easily verified that $E$ has rank at least 12 over $\mathbb{Q}(t)$.

Then, according to Theorem 2.1, we have

$$\dim_{\mathbb{F}_2} \text{Pic}(C)[2] \geq \text{rk}_{\mathbb{Z}} E(\mathbb{Q}(t)) \geq 12.$$  

One also checks that there exists a rational value of $t$ for which the equation $x^3 + a(t)x + b(t) = 0$ has a single real root. By variants of Theorem 2.7 (resp. Theorem 1.2) taking into account the places at infinity, it follows that there exists a trigonal morphism $\phi : C \to \mathbb{P}^1$ such that for all but $O(\sqrt{N})$ integers $t \in \mathbb{Z}$ with $|t| \leq N$, $\mathbb{Q}(P_t)$ is a cubic number field with exactly one real place and

$$\dim_{\mathbb{F}_2} \text{Cl}(\mathbb{Q}(P_t)) \geq \text{rk}_{\mathbb{Z}} E(\mathbb{Q}(t)) - \text{rk}_{\mathbb{Z}} \mathcal{O}_{\mathbb{Q}(P_t)}^{\times} \geq 11.$$  

Remark 2.8. One may possibly go further using famous constructions of Elkies. More precisely, Elkies describes in [Elk07] constructions of elliptic curves over $\mathbb{Q}(t)$ of ranks 17 and 18. The details of these constructions (e.g., explicit equations for the curves) remain unpublished. This example is likely to lead to further applications of our techniques.
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