Abstract. A word \( u \) is a scattered factor of \( w \) if \( u \) can be obtained from \( w \) by deleting some of its letters. That is, there exist the (potentially empty) words \( u_1, u_2, \ldots, u_n, \) and \( v_0, v_1, \ldots, v_n \) such that \( u = u_1u_2\ldots u_n \) and \( w = v_0u_1v_1u_2v_2\ldots u_nv_n. \) We consider the set of length-\( k \) scattered factors of a given word \( w \), called here \( k \)-spectrum and denoted \( \text{ScatFact}_k(w) \). We prove a series of properties of the sets \( \text{ScatFact}_k(w) \) for binary strictly balanced and, respectively, \( c \)-balanced words \( w \), i.e., words over a two-letter alphabet where the number of occurrences of each letter is the same, or, respectively, one letter has \( c \)-more occurrences than the other. In particular, we consider the question which cardinalities \( n = |\text{ScatFact}_k(w)| \) are obtainable, for a positive integer \( k \), when \( w \) is either a strictly balanced binary word of length \( 2k \), or a \( c \)-balanced binary word of length \( 2k - c \). We also consider the problem of reconstructing words from their \( k \)-spectra.

1 Introduction

Given a word \( w \), a scattered factor (also called scattered subword, or simply subword in the literature) is a word obtained by removing one or more factors from \( w \). More formally, \( u \) is a scattered factor of \( w \) if there exist \( u_1, \ldots, u_n \in \Sigma^* \), \( v_0, \ldots, v_n \in \Sigma^* \) such that \( u = u_1u_2\ldots u_n \) and \( w = v_0u_1v_1u_2v_2\ldots u_nv_n. \) Consequently a scattered factor of \( w \) can be thought of as a representation of \( w \) in which some parts are missing. As such, there is considerable interest in the relationship of a word and its scattered factors from both a theoretical and practical point of view. For an introduction to the study of scattered factors, see Chapter 6 of [8]. On the one hand, it is easy to imagine how, in any situation where discrete, linear data is read from an imperfect input – such as when sequencing DNA or during the transmission of a digital signal – scattered factors form a natural model, as multiple parts of the input may be missed, but the rest will remain unaffected and in-sequence. For instance, various applications and connections of this model in verification are discussed in [13,5] within a language theoretic framework, while applications of the model in DNA sequencing are discussed in [3] in an algorithmic framework. On the other hand, from a more algebraic perspective, there have been efforts to bridge the gap between the non-commutative field of combinatorics on words with traditional commutative mathematics via Parikh matrices (cf. e.g., [10,12]) which are closely related to, and influenced by the topic of scattered factors.
The set (or also in some cases, multi-set) of scattered factors of a word w, denoted ScatFact(w) is typically exponentially large in the length of w, and contains a lot of redundant information in the sense that, for \( k' < k \leq |w| \), a word of length \( k' \) is a scattered factor of \( w \) if and only if it is a scattered factor of a scattered factor of \( w \) of length \( k \). This has led to the idea of \( k \)-spectra: the set of all length-\( k \) scattered factors of a word. For example, the 3-spectrum of the word \( ababb \) is the set \{aab, aba, abb, bab, bbb\}. Note that unlike some literature, we do not consider the \( k \)-spectra to be the multi-set of scattered factors in the present work, but rather ignore the multiplicities. This distinction is non-trivial as there are significant variations on the properties based on these different definitions (cf. e.g., [9]). Also, the notion of \( k \)-spectrum is closely related to the classical notion of factor complexity of words, which counts, for each positive integer \( k \), the number of distinct factors of length \( k \) of a word. Here, the cardinality of the \( k \)-spectrum of a word gives the number of the word’s distinct scattered factors of length \( k \).

One of the most fundamental questions about \( k \)-spectra of words, and indeed sets of scattered factors in general, is that of recognition: given a set \( S \) of words (of length \( k \)), is \( S \) the subset of a \( k \)-spectrum of some word. In general, it remains a long standing goal of the theory to give a “nice” descriptive characterisation of scattered factor sets (and \( k \)-spectra), and to better understand their structure [8].

Another fundamental question concerning \( k \)-spectra, and one well motivated in several applications, is the question of reconstruction: given a word \( w \) of length \( n \), what is the smallest value \( k \) such that the \( k \)-spectrum of \( w \) is uniquely determined? This question was addressed and solved successively in a variety of cases. In particular, in [2], the exact bound of \( \frac{n}{2} + 1 \) is given in the general case. Other variations, including for the definition of \( k \)-spectra where multiplicities are also taken into account, are considered in [9], while [6] considers the question of reconstructing words from their palindromic scattered factors.

In the current work, we consider \( k \)-spectra in the restricted setting of a binary alphabet \( \Sigma = \{a, b\} \). For such an alphabet, we can always identify the natural number \( c \in \mathbb{N}_0 \) which describes how balanced a word is: \( c \) is the difference between the amount of as and bs. Thus, it seems natural to categorise all words over \( \Sigma \) according to this difference: a binary word where one letter has exactly \( c \) more occurrences than the other one is called \( c \)-balanced. In Section 3 the cardinalities of \( k \)-spectra of \( c \)-balanced words of length \( 2k - c \) are investigated. Our first results concern the minimal and maximal cardinality ScatFact\(_k\) might have. We show that the cardinality ranges for 0-balanced (also called strictly balanced words) between \( k + 1 \) and \( 2^k \), and determine exactly for which words of length \( 2k \) these values are reached. In the case of \( c \)-balanced words, we are able to replicate the result regarding the minimal cardinality of ScatFact\(_k\), but the case of maximal cardinality seems to be more complicated. To this end, it seems that the words containing many alternations between the two letters of the alphabet have larger sets ScatFact\(_k\). Therefore, we first investigate the scattered factors of the words which are prefixes of \((ab)^\omega\) and give a precise description of all scattered factors of any length of such words. That is, not only we compute
the cardinality of $\text{ScatFact}_k(w)$, for all such words $w$, but also describe a way to obtain directly the respective scattered factors, without repetitions. We use this to describe exactly the sets $\text{ScatFact}_k$ for the word $(ab)^{k-c}a^c$, which seems a good candidate for a $c$-balanced word with many distinct scattered factors.

Further, in Section 4 we explore more the cardinalities of $\text{ScatFact}_k(w)$ for strictly balanced words $w$ of length $2k$. We obtain for these words that the smallest three numbers which are possible cardinalities for their $k$-spectra are $k + 1$, $2k$, and $3k - 3$, thus identifying two gaps in the set of such cardinalities. Among other results on this topic, we show that for every constant $i$ there exist a word $w$ of length $2k$ such that $|\text{ScatFact}_k(w)| \in \Theta(n^i)$; we also show how such a word can be constructed.

Finally, in Section 5 we also approach the question of reconstructing strictly balanced words from $k$-spectra in the specific case that the spectra are also limited to strictly balanced words only. While we are not able to resolve the question completely, we conjecture that the situation is similar to the general case: the smallest value $k$ such that the $k$-spectrum of $w$ is uniquely determined is $k = \frac{|w|}{2} + 1$ if $\frac{|w|}{2}$ is odd and $k = \frac{|w|}{2} + 2$, otherwise, in the case when $w$ contains at most two blocks of $bs$.

After introducing a series of basic definitions, preliminaries, and notations, the organisation of the paper follows the description above.

2 Preliminaries

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\mathbb{N}_{\geq k}$ be all natural numbers greater than or equal to $k$. Let $[n]$ denote the set $\{1, \ldots, n\}$ and $[n]_0 = [n] \cup \{0\}$ for an $n \in \mathbb{N}$.

We consider words $w$ over the alphabet $\Sigma = \{a, b\}$. $\Sigma^*$ denotes the set of all finite words over $\Sigma$, also called binary words. $\Sigma^\omega$ the set of all infinite words over $\Sigma$, also called binary infinite words. The empty word is denoted by $\varepsilon$ and $\Sigma^+$ is the free semigroup $\Sigma^* \setminus \{\varepsilon\}$. The length of a word $w$ is denoted by $|w|$. Let $\Sigma_{\leq k} := \{w \in \Sigma^* \mid |w| \leq k\}$ and $\Sigma^k$ be the set of all words of length exactly $k \in \mathbb{N}$. The number of occurrences of a letter $a \in \Sigma$ in a word $w \in \Sigma^*$ is denoted by $|w|_a$. The $i$th letter of a word $w$ is given by $w[i]$ for $i \in [|w|]$. For a given word $w \in \Sigma^n$ the reversal of $w$ is defined by $w^R = w[n]w[n-1] \ldots w[2]w[1]$. The powers of $w \in \Sigma^*$ are defined recursively by $w^0 = \varepsilon$, $w^n = ww^{n-1}$ for $n \in \mathbb{N}$.

A word $w \in \Sigma^*$ is called $c$-balanced if $|w|_a - |w|_b = c$ for $c \in \mathbb{N}_0$. 1-balanced words are also called balanced and 0-balanced words are also called strictly balanced. Thus strictly balanced words have the same number of $as$ and $bs$. Let $\Sigma_{sb}$ be the set of all strictly balanced words over $\Sigma$. For example, $abaa$ is 2-balanced, $aba$ is balanced, while $ababa$ is strictly balanced ($0$-balanced).

A word $u \in \Sigma^*$ is a factor of $w \in \Sigma^*$, if $w = xuy$ holds for some words $x, y \in \Sigma^*$. Moreover, $u$ is a prefix of $w$ if $x = \varepsilon$ holds and a suffix if $y = \varepsilon$ holds. The factor of $w$ from the $i$th to the $j$th letter will be denoted by $w[i..j]$ for $0 \leq i \leq j \leq |w|$. Given a letter $a \in \Sigma$ and a word $w \in \Sigma^*$, a block of $a$ is a factor $u = w[i..j]$ with $u = a^{|i-j|}$, such that either $i = 1$ or $w[i-1] = b \neq a$ and either
\[ j = |w| \text{ or } w[j + 1] = b \neq a. \] For example the word \textit{ababaabbb} has 3 \textit{a}-blocks and 3 \textit{b}-blocks. Scattered factors and \( k \)-spectra are defined as follows.

**Definition 1.** A word \( u = a_1 \ldots a_n \in \Sigma^n \), for \( n \in \mathbb{N} \), is a scattered factor of a word \( w \in \Sigma^+ \) if there exists \( v_0, \ldots, v_n \in \Sigma^* \) with \( w = v_0a_1v_1 \ldots v_{n-1}a_nv_n \).

Let \( \text{ScatFact}(w) \) denote the set of \( w \)'s scattered factors and consider additionally \( \text{ScatFact}_k(w) \) and \( \text{ScatFact}_{\leq k}(w) \) as the two subsets of \( \text{ScatFact}(w) \) which contain only the scattered factors of length \( k \in \mathbb{N} \) or the ones up to length \( k \in \mathbb{N} \).

The sets \( \text{ScatFact}_{\leq k}(w) \) and \( \text{ScatFact}_k(w) \) are also known as full \( k \)-spectrum and, respectively, \( k \)-spectrum of a word \( w \in \Sigma^* \) (see [1], [9], [11]) and moreover, scattered factors are often called subwords or scattered subwords. Obviously the \( k \)-spectrum is empty for \( k > |w| \) and contains exactly \( w \)'s letters for \( k = 1 \) and only \( w \) for \( k = |w| \). Considering the word \( w = \text{abba} \), the other spectra are given by \( \text{ScatFact}_2(w) = \{a^2, b^2, \text{ab}, \text{ba}\} \) and \( \text{ScatFact}_3(w) = \{ab^2, \text{aba}, b^2\text{a}\} \).

It is worth noting that if \( u \) is a scattered factor of \( w \), and \( v \) is a scattered factor of \( u \), then \( v \) is a scattered factor of \( w \). Additionally, notice two important symmetries regarding \( k \)-spectra. For \( w \in \Sigma^* \) and the renaming morphism \( \tau : \Sigma \rightarrow \Sigma \) with \( \overline{a} = b \) and \( \overline{b} = a \) we have \( \text{ScatFact}(w^R) = \{u^R \mid u \in \text{ScatFact}(w)\} \) and \( \text{ScatFact}(\overline{\Sigma}) = \{\overline{\sigma} \mid \sigma \in \text{ScatFact}(\Sigma)\} \). Thus, from a structural point of view, it is sufficient to consider only one representative from the equivalence classes induced by the equivalence relation where \( w_1 \) is equivalent to \( w_2 \) whenever \( w_2 \) is obtained by a composition of reversals and renamings from \( w_1 \). Considering \( w.l.o.g. \) the order \( a < b \) on \( \Sigma \), we choose the lexicographically smallest word as representative from each class. As such, we will mostly analyse the \( k \)-spectra of words starting with \( a \). We shall make use of this fact extensively in Section 3.

### 3 Cardinalities of \( k \)-Spectra of \( c \)-Balanced Words

In the current section, we consider the combinatorial properties of \( k \)-spectra of \( c \)-balanced, finite words. In particular, we are interested in the cardinalities of the \( k \)-spectra and in the question: which cardinalities are (not) possible? Since the \( k \)-spectra of \( a^n \) and \( b^n \) are just \( a^k \) and \( b^k \) respectively for all \( n \in \mathbb{N}_0 \) and \( k \in [n]_0 \), we assume \( |w|_a, |w|_b > 0 \) for \( w \in \Sigma^* \). It is a straightforward observation that not every subset of \( \Sigma^k \) is a \( k \)-spectrum of some word \( w \). For example, for \( k = 2 \), \( \text{aa} \) and \( \text{bb} \) can only be scattered factors of a word containing both \( \text{a} \)s and \( \text{b} \)s, and therefore having either \( \text{ab} \) or \( \text{ba} \) as a scattered factor as well. Thus, there is no word \( w \) such that \( \text{ScatFact}_2(w) = \{\text{aa}, \text{bb}\} \).

In general, for any word containing only \( \text{a} \)’s or only \( \text{b} \)’s, there will be exactly one scattered factor of each length, while for words containing both \( \text{a} \)’s and \( \text{b} \)’s, the smallest \( k \)-spectra are realised for words of the form \( w = a^n\text{b} \) (up to renaming and reversal), for which \( \text{ScatFact}_k(w) = \{a^k, a^{k-1}\text{b}\} \) for each \( k \in [\|w\|] \). On the other hand, as Proposition 5 shows, the maximal \( k \)-spectra are those containing all words of length \( k \) – and hence have size \( 2^k \), achieved by e.g. \( w = (\text{ab})^n \) for \( n \geq k \). Note that when strictly balanced words are considered, the same
maximum applies, since \((ab)^n\) is strictly balanced, while the minimum does not, since \(a^nb\) is not strictly balanced.

It is straightforward to enumerate all possible k-spectra, and describe the words realising them for \(k \leq 2\), hence we shall generally consider only k-spectra in the sequel for which \(k \geq 3\). Our first result generalises the previous observation about minimal-size k-spectra.

**Theorem 2.** For \(k \in \mathbb{N}_{\geq 3}\), \(c \in \lfloor k - 1 \rfloor_0\), \(i \in [c]_0\), and a c-balanced word \(w \in \Sigma^{2k-c}\), we have \(|\text{ScatFact}_{k-i}(w)| \geq k - c + 1\), where equality holds if and only if \(w \in \{a^k b^{k-c}, a^{k-c} b^k, b^k a^{k-c}, b^{k-c} a^k\}\). Moreover, if \(w \in \Sigma_{ab}^{2k} \setminus \{a^k b^k\}\), then \(|\text{ScatFact}_k(w)| \geq k + 3\).

**Proof.** Consider firstly only strictly balanced words, i.e. \(c = 0\) and w.l.o.g. only \(w = a^k b^k\). The cases \(k = 1\) and \(k = 2\) are the induction basis.

Consider now a word \(w \in \Sigma_{ab}^{2k} \setminus \{a^k b^k, b^k a^k\}\). Since \(w\) is not \(a^k b^k\), \(w\) contains a factor \(aba\) or \(bab\). Assume w.l.o.g. that \(w = xab\) holds for \(x, y \in \Sigma^*\) with \(|x| + |y| = 2k - 3\). By \(w \in \Sigma_{ab}^{2k}\) follows that \(|x|b\) or \(|y|b\) is not zero. Choose w.l.o.g. \(z_1, z_2 \in \Sigma^*\) with \(y = z_1 b z_2\) which implies \(w = xab z_1 b z_2\). Consequently \(|x z_1 z_2\) is \(k - 2\) holds.

**Case 1:** \(x z_1 z_2 = a^{k-2} b^{k-2}\)

By induction \(|\text{ScatFact}_{k-2}(x z_1 z_2)| = (k - 2) + 1 = k - 1\). Let \(u\) be a scattered factor of \(x z_1 z_2\) of length \(k - 2\). Then there exist \(u_1, u_2\), and \(u_3\) such that \(u_1\) is a scattered factor of \(x\), \(u_2\) of \(z_1\), and \(u_3\) of \(z_3\) respectively. Consequently

\[u_1 aau_2 u_3, \quad u_1 bau_2 u_3, \quad \text{and} \quad u_1 bau_2 u_3\]

are different elements of \(\text{ScatFact}_k(w)\). Each scattered factor of \(x z_1 z_2\) is of the form \(a^r b^s\) for \(r, s \in \lfloor k - 2 \rfloor_0\). We will know prove in which cases the aforementioned scattered factors are different. Consider \(u = u_1 u_2 u_3 = a^r b^s\) and \(u' = u_1' u_2' u_3 = a^{r'} b^{s'}\) to be different scattered factors of this form, i.e. \(r \neq r'\) and \(s \neq s'\). Set

\[
\alpha_1 = u_1 aau_2 u_3, \quad \beta_1 = u_1' aau_2' u_3, \\
\alpha_2 = u_1 bau_2 u_3, \quad \beta_2 = u_1' bau_2' u_3, \\
\alpha_3 = u_1 bau_2 u_3, \quad \beta_3 = u_1 bau_2 u_3.
\]

If \(u_1 = a^r_1, u_2 u_3 = a^r 2 b^s\) and \(u_1' = a^{r'}_1, u_1' u_2' u_3 = a^{r'} 2 b^{s'}\) with \(r_1 + r_2 = r\) and \(r_1' + r_2' = r'\), we get because of \(r \neq r'\), \(r_1 \neq -1\),

\[
\alpha_1 = a^{r+2} b^s \neq a^{r'+2} b^{s'} = \beta_1, \\
\alpha_1 = a^{r+2} b^s \neq a^{r'} b^{r'+1} b^s' = \beta_2, \\
\alpha_2 = a^{r'} b^{r'+1} b^s \neq a^{r'} b^{r'+1} b'^{s'} = \beta_2.
\]
If \( u_1 = a^r \), \( u_2 u_3 = a^{r+2} b^s \) and \( u'_1 = a^{r'} b^{s'}, u'_2 u'_3 = b^{s'} \) with \( r_1 + r_2 = r, s'_1 + s'_2 = s' \), and \( s'_1 \neq 0 \) (already in the previous case) we get because of \( s'_1 \neq 0, r' \neq r \)

\[
\begin{align*}
\alpha_1 &= a^{r+2} b^s \neq a^{r'} b^{s'} b^{s'}, \\
\alpha_2 &= a^{r'} b^{s'} b^{s'} b^{s'} \neq a^{r'} b^{s'} b^{s'} b^{s'}.
\end{align*}
\]

If \( u_1 = a^r b^{s'}, u_2 u_3 = b^{s'} \) and \( u'_1 = a^{r'} b^{s'}, u'_2 u'_3 = b^{s'} \) with \( r_1 + r_2 = r, s'_1 + s'_2 = s' \), and \( s'_1, s'_1 \neq 0 \) (already in the previous case) we get because of \( r' \neq r \) and \( s'_1 \neq 0, r' \neq r + 1 \)

\[
\begin{align*}
\alpha_1 &= a^{r+2} b^s \neq a^{r'} b^{s'} b^{s'}, \\
\alpha_2 &= a^{r'} b^{s'} b^{s'} b^{s'} \neq a^{r'} b^{s'} b^{s'} b^{s'}.
\end{align*}
\]

Consequently \( \alpha_1 \) and \( \alpha_2 \) are all different and we get \( 2(k - 1) \) many different scattered factors. Assume now additionally \( |r - r'| = 3 \). If \( u_1 = a^r \), \( u_2 u_3 = a^{r+2} b^s \) and \( u'_1 = a^{r'} \), \( u'_2 u'_3 = a^{r'2} b^{s'} \) with \( r_1 + r_2 = r \) and \( r'_1 + r'_2 = r' \), we get because of \( s'_1 \neq 0, r' \neq r, r' \neq r + 1 \)

\[
\begin{align*}
\alpha_1 &= a^{r+2} b^s \neq a^{r'} b^{s'} b^{s'}, \\
\alpha_2 &= a^{r'} b^{s'} b^{s'} b^{s'} \neq a^{r'} b^{s'} b^{s'} b^{s'}.
\end{align*}
\]

If \( u_1 = a^r \), \( u_2 u_3 = a^{r+2} b^s \) and \( u'_1 = a^{r'} b^{s'}, u'_2 u'_3 = b^{s'} \) with \( r_1 + r_2 = r, s'_1 + s'_2 = s' \), and \( s'_1 \neq 0 \) (already in the previous case) we get because of \( s'_1 \neq 0, r' \neq r + 2 \)

\[
\begin{align*}
\alpha_1 &= a^{r+2} b^s \neq a^{r'} b^{s'} b^{s'}, \\
\alpha_2 &= a^{r'} b^{s'} b^{s'} b^{s'} \neq a^{r'} b^{s'} b^{s'} b^{s'}.
\end{align*}
\]

If \( u_1 = a^r b^{s'}, u_2 u_3 = b^{s'} \) and \( u'_1 = a^{r'} b^{s'}, u'_2 u'_3 = b^{s'} \) with \( r_1 + r_2 = r, s'_1 + s'_2 = s' \), and \( s'_1, s'_1 \neq 0 \) (already in the previous case) we get because of \( r' \neq r \) and \( s'_1 \neq 0, r' \neq r + 2 \)

\[
\begin{align*}
\alpha_1 &= a^{r+2} b^s \neq a^{r'} b^{s'} b^{s'}, \\
\alpha_2 &= a^{r'} b^{s'} b^{s'} b^{s'} \neq a^{r'} b^{s'} b^{s'} b^{s'}.
\end{align*}
\]

Consequently we have another \( \frac{k}{4} + 1 \) different scattered factors. This sums up to \( |\text{ScatFact}_k(w)| \geq \frac{2k-3}{4} > k + 1 \). An immediate result is that the \( k \)-spectrum
has at least \( k + 3 \) elements for \( k \geq 5 \). For \( k = 3 \) and \( k = 4 \) the results can be easily verified by testing.

**Case 2**: \( x_{1}z_{2} \neq a^{k-2}b^{k-2} \)

In this case all words of the form \( a^{r}b\text{abaa}^{s} \) for \( r + s = k - 3 \), \( r \in [ [ x ]_{a} ]_{0} \) and \( s \in [ [ y ]_{a} ]_{0} \) are \( |x|_{a} + 1 \) different scattered factors of length \( k \) of \( w \). Analogously all \( b^{r'}\text{abab}^{s'} \) with \( r' + s' = k - 3 \), \( r' \in [ [ x ]_{b} ]_{0} \), \( s' \in [ [ y ]_{b} ]_{0} \) are \( |x|_{b} + 1 \) different scattered factors of length \( k \) of \( w \). All these factors are different and additionally \( w \) has \( a^{k} \) and \( b^{k} \) as scattered factors. Hence \( |\text{ScatFact}_{k}(w)| \geq |x|_{a} + |x|_{b} + 4 = |x| + 4 \) holds. Since the length of \( w \) is \( 2k \), the length of \( xy \) is \( 2k - 3 \) and consequently \( x \) and \( y \) have different lengths. Assume w.l.o.g. \( |x| > |y| \), i.e. \( |x| \geq k - 1 \). This implies \( |\text{ScatFact}_{k}(w)| \geq k + 3 \) follows. This proves the claim for \( c = 0 \).

Assume now \( c > 0 \) and let \( w = a^{k}b^{k-c} \). By the previous part we know \( |\text{ScatFact}_{k-c}(w)| = k - c + 1 \) if and only if \( w = a^{k-c}b^{k-c} \). The claim about the \( (k-c) \)-spectrum follows immediately by \( \text{ScatFact}_{k-c}(w) = \text{ScatFact}_{k-c}(a^{k}b^{k-c}) \) since the prepended \( a \) do not change the \( (k-c) \)-spectrum. For \( i \in [ c - 1 ]_{0} \) notice that \( x \in \text{ScatFact}_{k-i}(\text{a}^{k}b^{k-c}) \) implies that \( ax \) (resp. \( xb, xa, bx \)) is a scattered factor of \( \text{a}^{k}b^{k-c} \) of length \( k - i + 1 \). Thus \( |\text{ScatFact}_{k-i}(w)| \geq k - c + 1 \) follows. On the other hand a scattered factor of \( \text{a}^{k}b^{k-c} \) of length \( k - i + 1 \) is exactly of this form, so it can neither start with \( b \) \( (\text{a}^{k}b^{k-c} \) has only \( (k-c) \) occurrences of \( b \) nor contain \( ba \) resp. \( ab \) (this would be the implication of a scattered factor being of the form \( ax' \) with \( |x'| = k - i \), \( x' \notin \text{ScatFact}_{k-i}(\text{a}^{k}b^{k-c}) \)). \( \square \)

**Remark 3.** Theorem 2 answers immediately the question, whether a given set \( S \subseteq \Sigma^{k} \), with \( |S| < k + 1 \) or \( |S| = k + 2 \), is a \( k \)-spectrum of a word \( w \in \Sigma_{ab}^{2k} \) in the negative.

Theorem 2 shows that the smallest cardinality of the \( k \)-spectrum of a word \( w \) is reached when the letters in \( w \) are *nicely ordered*, both for strictly balanced words as well as for \( c \)-balanced words with \( c > 0 \). The largest cardinality is, not surprisingly, reached for words where the alternation of \( a \) and \( b \) letters is, in a sense, maximal, e.g., for \( w = (ab)^{k} \). To this end, one can show a general result.

**Theorem 4.** For \( w \in \Sigma^{*} \), the \( k \)-spectrum of \( w \) is \( \Sigma^{k} \) if and only if

\[
\{ab, ba\}^{k} \cap \text{ScatFact}_{2k}(w) \neq \emptyset.
\]

**Proof.** We will show this result by induction. For \( k = 1 \), the equivalence is:

\[
\text{ScatFact}_{1}(w) = \Sigma \iff \{ab, ba\} \cap \text{ScatFact}_{2}(w) \neq \emptyset.
\]

If both \( a \) and \( b \) are scattered factors of \( w \), \( ab \) or \( ba \) has to be a factor and thus a scattered factor of \( w \). On the other hand if \( w \) has \( ab \) or \( ba \) as a scattered factor, it has \( a \) and \( b \) as scattered factors.

Assume now that the equivalence holds for an arbitrary but fixed \( k - 1 \in \mathbb{N} \). We will show it holds for \( k \).

For the \( \iff \)-direction consider \( u \in \{ab, ba\}^{k} \cap \text{ScatFact}_{2k}(u) \). Thus, \( u \in \{ab, ba\}^{k-1}\{ab, ba\} \) and hence there exists \( u' \in \{ab, ba\}^{k-1} \) with \( u \in u'\{ab, ba\} \).
By induction we have have ScatFact_{k-1}(u') = \Sigma^{k-1}. For any \( x \in \Sigma^k \) exists \( x' \in \Sigma^{k-1} \) with \( x \in x'\{a,b\} \). This implies that there exist \( a_0, \ldots, a_{k-1} \in \Sigma^* \) with \( u' = a_0x'[1]a_1 \ldots x'[k-1]a_{k-1} \) since \( x' \in \text{ScatFact}_{k-1}(u') \). By
\[
u \in a_0x'[1]a_1 \ldots x'[k-1]a_{k-1}\{ab, ba\}
\]
it follows in both cases, namely \( x = x'a \) or \( x = x'b \), that \( x \in \text{ScatFact}_k(w) \). This proves the inclusion \( \Sigma^k \subseteq \text{ScatFact}_k(w) \). By \( \text{ScatFact}_k(w) \subseteq \Sigma^k \) the first direction is proven.

For the \( \Rightarrow \)-direction assume \( \text{ScatFact}_k(w) = \Sigma^k \). Assume w.l.o.g. \( w[|w|] = a \). Choose \( x, y \in \Sigma^* \) with \( w = xy \) and \( x[|x|] = b \), and \( y \in a^* \). As \( \Sigma^{k-1}b \subseteq \text{ScatFact}_k(x) \), clearly, this means that \( \Sigma^{k-1} \subseteq \text{ScatFact}_{k-1}(x[1..|x|-1]) \). By the induction hypothesis, we get that \( \{ab, ba\}^{k-1} \cap \text{ScatFact}_{2(k-1)}(x[1..|x|-1]) \neq \emptyset \). Thus, \( \{ab, ba\}^{k-1} \cap \text{ScatFact}_k(w[1..|x|+1]) \neq \emptyset \), because \( w[1..|x|+1] = x[1..|x|]b \). Hence, \( \{ab, ba\}^{k-1}ba \cap \text{ScatFact}_k(w) \neq \emptyset \). The conclusion follows.

The previous theorem has an immediate consequence, which exactly characterises the strictly balanced words of length \( 2k \) for which the maximal cardinality of \( \text{ScatFact}_k(w) \) is reached.

**Proposition 5.** For \( k \in \mathbb{N}_{\geq 3} \) and \( w \in \Sigma^{2k}_b \) we have \( w \in \{ab, ba\}^k \) if and only if \( \text{ScatFact}_k(w) = \Sigma^k \).

**Proof.** If \( w \in \{ab, ba\}^k \), then \( \{ab, ba\}^k \cap \text{ScatFact}_k(w) \neq \emptyset \) and the claim follows by Theorem 4. On the other hand if \( \text{ScatFact}_k(w) = \Sigma^k \) then \( \{ab, ba\}^k \cap \text{ScatFact}_k(w) \neq \emptyset \) and since \(|w| = 2k\) we get \( w \in \{ab, ba\}^k \). \( \square \)

To see why from \( w \in \{ab, ba\}^k \) it follows that \( \text{ScatFact}_k(w) = \Sigma^k \), note that, by definition, a word \( w \in \{ab, ba\}^k \) is just a concatenation of \( k \) blocks from \{ab, ba\}. To construct the scattered factors of \( w \), we can simply select from each block either the \( a \) or the \( b \). The resulting output is a word of length \( k \), where in each position we could choose freely the letter. Consequently, we can produce all words in \( \Sigma^k \) in this way. The other implication follows by induction.

Generalising Proposition 4 for \( c \)-balanced words requires a more sophisticated approach. A natural generalisation would be to consider \( w \in \{ab, ba\}^{k-c}a^c \). By Theorem 4 we have \( \text{ScatFact}_{k-c}(w) = \Sigma^{k-c} \). But the size of \( \text{ScatFact}_{k-i}(w) \) for \( i \in [c]_0 \) depends on the specific choice of \( w \). To see why, consider the words \( w_1 = baabba \) and \( w_2 = (ba)^3 \). Then by Proposition 4 \(|\text{ScatFact}_3(w_1)| = 8 = |\text{ScatFact}_3(w_2)|\). However, when we append \( a \) to the end of both \( w_1 \) and \( w_2 \), we see that in fact \(|\text{ScatFact}_4(w_1a)| = 10 \neq 12 = |\text{ScatFact}_4(w_2a)|\). The main difference between strictly and \( c \)-balanced words for \( c > 0 \), regarding the maximum cardinality of the scattered factors-sets, comes from the role played by the factors \( a^2 \) and \( b^2 \) occurring in \( w \).

In the remaining part of this section we present a series of results for \( c \)-balanced words. Intuitively, the words with many alternations between \( a \) and \( b \) have more distinct scattered factors. So, we will focus on such words mainly. Our
first result is a direct consequence from Theorem 4. The second result concerns words avoiding \(a^2\) and \(b^2\) gives a method to identify efficiently the \(\ell\)-spectra of words which are prefixes of \((ab)^\omega\), for all \(\ell\). Finally, we are able to derive a way to efficiently enumerate (and count) the scattered factors of length \(k\) of \((ab)^{k-c}a^c\).

**Corollary 6.** For \(k \in \mathbb{N}_{\geq 3}\), \(c \in [k]_0\), and \(w \in \Sigma^{2k-c}\) -balanced, the cardinality of \(\text{ScatFact}_{k-c}(w)\) is exactly \(2^{k-c}\) if and only if \(\text{ScatFact}_{2(k-c)}(w) \cap \{ab, ba\}^{k-c} \neq \emptyset\).

**Proof.** The claim follows directly by Theorem 4. \(\square\)

As announced, we further focus our investigation on the words \(w = (ab)^{k-c}a^c\). By Theorem 4 we have \(|\text{ScatFact}_w| = \Sigma^*\) for all \(i \in [k-c]_0\). For all \(i\) with \(k - c < i \leq k\), a more sophisticated counting argument is needed. Intuitively, a scattered factor of length \(i\) of \((ab)^{k-c}a^c\) consists in a part that is a subword (of arbitrary length) of \((ab)^{k-c}\) followed a (possibly empty) suffix of \(a\). Thus, a full description of the \(\ell\)-spectra of words that occur as prefixes of \((ab)^\omega\), for all appropriate \(\ell\), is useful. To this end, we introduce the notion of a deleting sequence: for a word \(w\) and a scattered factor \(u\) of \(w\) the deleting sequence contains (in a strictly increasing order) \(w\)'s positions that have to be deleted to obtain \(u\).

**Definition 7.** For \(w \in \Sigma^n\), \(\sigma = (s_1, \ldots, s_l) \in [|w|]^{\ell}\), with \(\ell \leq |w|\) and \(s_i < s_{i+1}\) for all \(i \in [\ell - 1]\), is a deleting sequence. The scattered factor \(u_\sigma\) associated to a deleting sequence \(\sigma\) is \(u_\sigma = u_1 \ldots u_{\ell+1}\), where \(u_1 = w[1..s_1 - 1]\), \(u_{i+1} = w[s_i + 1..|w|]\), and \(u_i = w[s_i-1 + 1..s_i - 1]\) for \(2 \leq i \leq \ell\). Two sequences \(\sigma, \sigma'\) with \(u_\sigma = u_{\sigma'}\) are called equivalent.

For the word \(w = abbaa\) and \(\sigma = (1, 3, 4)\) the associated scattered factor is \(u_\sigma = ba\). Since \(ba\) can also be generated by \((1, 3, 5), (1, 2, 4)\) and \((1, 2, 5)\), these sequences are equivalent.

In order to determine the \(\ell\)-spectrum of a word \(w \in \Sigma^n\) for \(\ell, n \in \mathbb{N}\), we can determine how many equivalence classes does the equivalence defined above have, for sequences of length \(k = n - \ell\). The following three lemmata characterise the equivalence of deleting sequences.

**Lemma 8.** Let \(w \in \Sigma^n\) be a prefix of \((ab)^\omega\). Let \(\sigma = (s_1, \ldots, s_k)\) be a deleting sequence for \(w\) such that there exists \(j \geq 2\) with \(s_{j-1} < s_j - 1\) and \(s_j + 1 = s_{j+1}\). Then \(\sigma\) is equivalent \(\sigma' = (s_1, \ldots, s_{j-1}, s_j - 1, s_j + 1 - 1, s_{j+2}, \ldots, s_k)\), i.e., \(\sigma'\) is the sequence \(\sigma\) where both \(s_j\) and \(s_{j+1}\) were decreased by 1.

**Proof.** Since \(s_{j-1} < s_j - 1\), the factor \(u_\sigma\) contains the letter \(w[s_j - 1]\). If \(w[s_j] = a\) then \(w[s_{j+1}] = w[s_j + 1] = b\) and \(w[s_j - 1] = b\). Clearly, when deleting \(w[s_j - 1]\) and \(w[s_j]\) according to the sequence \(\sigma'\), the \(b\) that was corresponding to \(w[s_j - 1]\) will be replaced by a letter \(b\) corresponding to \(w[s_{j+1}]\), which is not deleted. So, in the end, \(u_{\sigma'} = u_\sigma\). The case \(w[s_j] = b\) is analogous. \(\square\)

**Lemma 9.** Let \(w \in \Sigma^n\) be a prefix of \((ab)^\omega\). Let \(\sigma = (s_1, \ldots, s_k)\) be a deleting sequence for \(w\). Then there exists an integer \(j \geq 0\) such that \(\sigma\) is equivalent to the
deleting sequence \( (1, 2, \ldots, j, s'_{j+1}, \ldots, s'_k) \), where \( s'_{j+1} > j + 1 \) and \( s'_i > s'_{i-1} + 1 \), for all \( j < i \leq k \). Moreover, \( j \geq 1 \) if and only if \( \sigma \) contained two consecutive positions or \( \sigma \) started with 1.

Proof. Let \( \sigma_0 = \sigma \). For \( i \geq 0 \), we iteratively transform \( \sigma_i \) into \( \sigma_{i+1} \) as follows: if \( \sigma_i \) contains on consecutive positions the numbers \( g, t, t+1, h \), such that \( g < t - 1 \) and \( h > t + 2 \), we replace them by \( g, t - 1, t, h \) and obtain the sequence \( \sigma_{i+1} \). By Lemma 10, \( \sigma_i \) is equivalent to \( \sigma_{i+1} \). It is clear that in \( O(n^2) \) steps we will reach a sequence \( \sigma' \) which cannot be transformed anymore. We take \( \sigma' = \sigma_f \) and it is immediate that it will have the required form. \( \square \)

**Lemma 10.** Let \( w \in \Sigma^n \) be a prefix of \( (ab)^\omega \). Let \( \sigma_1 = (1, 2, \ldots, j, s'_{j+1}, \ldots, s'_k) \), where \( s'_{j+1} > j + 1 \) and \( s'_i > s'_{i-1} + 1 \), for all \( j < i \leq k \), and \( \sigma_2 = (1, 2, \ldots, j, s''_{j+1}, \ldots, s''_k) \), where \( s''_{j+1} > j + 1 \) and \( s''_i > s''_{i-1} + 1 \), for all \( j < i \leq k \). If \( \sigma_1 \neq \sigma_2 \) then \( \sigma_1 \) and \( \sigma_2 \) are not equivalent (i.e., \( u_{\sigma_1} \neq u_{\sigma_2} \)).

Proof. We first consider the case \( j_1 = j_2 \). Let \( \ell \) to be minimum such that \( s'_\ell \neq s''_\ell \). We can assume without losing generality that \( s'_\ell < s''_\ell \). Then \( u_{\sigma_1} \) and \( u_{\sigma_2} \) share the same prefix of length \( t = (s'_\ell - 1) - (\ell - 1) \). This prefix ends with \( w[s'_\ell - 1] \) and is followed by \( w[s'_{\ell} + 1] \) in \( u_{\sigma_1} \), and, respectively, by \( w[s''_{\ell}] \) in \( u_{\sigma_1} \). But \( w[s'_\ell + 1] \neq w[s''_{\ell}] \), so \( u_{\sigma_1} \neq u_{\sigma_2} \).

Further, we consider the case when \( j_1 < j_2 \) (the case \( j_2 < j_1 \) is symmetric): assume, as a convention, that \( s''_{j+1} = 0 \) and let \( d = j_2 - j_1 \). Clearly, \( j_1 \) and \( j_2 \) must have the same parity, or \( u_{\sigma_1} \) and \( u_{\sigma_2} \) would start with different letters, so they would not be equal. Let \( \ell \) to be minimum integer such that \( s'_\ell - j_1 \neq s''_{\ell+d} - j_2 \): because \( s''_{j+1} = 0 \) by convention, we have \( \ell \leq k \). If both \( \ell \) and \( \ell + d \) are at most \( k \), then we get similarly to the case \( j_1 = j_2 \) that \( u_{\sigma_1} \neq u_{\sigma_2} \). In the case when \( \ell \leq k < \ell + d \), then, by length reasons, all positions \( j > s'_\ell \) (so, including \( s'_{\ell} + 1 \)) in \( w \) should belong to \( \sigma_1 \), a contradiction. This concludes our proof. \( \square \)

Lemmas 8, 9, 10 show that the representatives of the equivalence classes w.r.t. the equivalence relation between deleting sequences, introduced in Definition 4, are the sequences \( (1, 2, \ldots, j, s'_{j+1}, \ldots, s'_k) \), where \( s'_{j+1} > j + 1 \) and \( s'_i > s'_{i-1} + 1 \), for all \( j < i \leq k \). For a fixed \( j \geq 1 \), the number of such sequences is \( \binom{(n-j-1)-(k-j)+1}{k-j} = \binom{n-k}{k-j} \). For \( j = 0 \), we have \( \binom{(n-1)-k+1}{k} = \binom{n-k}{k} \) nonequivalent sequences (note that none starts with 1, as those were counted for \( j = 1 \) already). In total, we have, for a word \( w \) of length \( n \), which is a prefix of \( (ab)^\omega \), exactly \( \sum_{j \in [k]} \binom{n-k}{k-j} \) nonequivalent deleting sequences of length \( k \), so \( \sum_{j \in [k]} \binom{n-k}{k-j} \) different scattered factors of length \( n - k \). In the above formula, we assume that \( \binom{n-k}{k} = 0 \) when \( a < b \).

Moreover, the distinct scattered factors of length \( \ell = n - k \) of \( w \) can be obtained efficiently as follows. For \( j \) from 0 to \( \ell \), delete the first \( j \) letters of \( w \). For all choices of \( \ell - j \) positions in \( w[j+1..n] \), such that each two of these positions are not consecutive, delete the letters on the respective positions. The resulted word is a member of ScatFact\( _\ell (w) \), and we never obtain the same word twice by this procedure. The next theorem follows from the above.
Theorem 11. Let \( w \) be a word of length \( n \) which is a prefix of \((ab)^n\). Then \( |\text{ScatFact}_{\ell}(w)| = \sum_{j \in [n-\ell]} (n-\ell-j) \).

A straightforward consequence of the above theorem is that, if \( \ell \leq n-\ell \) then \( |\text{ScatFact}_{\ell}(w)| = 2^\ell \). With Theorem 11 we can now completely characterise the cardinality of the \( \ell \)-spectra of the \( c \)-balanced word \((ab)^{k-c}a^c\) for \( \ell \leq k \).

Theorem 12. Let \( w = (ab)^{k-c}a^c \) for \( k \in \mathbb{N}, c \in [k]_0 \). Then, for \( i \leq k - c \) we have \( |\text{ScatFact}_{i}(w)| = 2^i \). For \( k \geq i > k - c \) we have \( |\text{ScatFact}_{i}(w)| = 1 + 2^{k-c} + \sum_{j \in [(i+c)-k-1]_0} |\text{ScatFact}_{i-j-1}((ab)^{k-c-1}a)| \).

Proof. We will need to show the proof for \( k \geq i > k - c \), as the other part follows immediately from Theorem 11.

We give a method to count the scattered factors of \( w = (ab)^{k-c}a^c \). To begin with, we have the scattered factor \( a^i \). All the other scattered factors must contain a letter \( b \). Thus, we count separately the scattered factors of the form \( uba^j \), for each \( j \in [i-1]_0 \). This is equivalent to counting in how many ways we can choose \( u \). For each such \( u \) we will just have to append \( ba^j \) at the end to get the desired scattered factors of length. Thus, \( |u| = i - j - 1 \). If \( j \geq c \) then \( u \) should occur as a scattered factor of \((ab)^{k-j-1}a\) (in order to be able to append \( ba^j \) at its end and still stay as a scattered factor of \( w \)), while if \( j < c \) then \( u \) should occur as a scattered factor of \((ab)^{k-c-1}a\). In the first case, the length of the scattered factor \( u \) we want to generate is less than half of the length of the word \((ab)^i a\) from which we generate it. So, there are \( 2^{i-j-1} \) choices for \( u \). In the second case, if \( j \geq (i+c) - k \), again, the length of the scattered factor \( u \) we want to generate is less than half of the length of the word \((ab)^{k-c-1}a\) from which we generate it. So, there are \( 2^{i-j-1} \) choices for \( u \) again. Finally, if \( j < (i+c) - k \), then there \( i - j - 1 > k - c - 1 \), and we need Theorem 11 to generate \( u \). There are \( |\text{ScatFact}_{i-j-1}((ab)^{k-c-1}a)| \) ways to choose \( u \) in this case. Summing all these up, we get the result from the statement:

\[
1 + \sum_{j=i+c-k}^{i-1} 2^{i-j-1} + \sum_{j \in [(i+c)-k-1]_0} \text{ScatFact}_{i-j-1}((ab)^{k-c-1}a) = 1 + 2^{k-c} + \sum_{j \in [(i+c)-k-1]_0} \text{ScatFact}_{i-j-1}((ab)^{k-c-1}a).
\]

This concludes our proof. \( \square \)

As in the case of the scattered factors of prefixes of \((ab)^n\), we have a precise and efficient way to generate the scattered factors of \( w = (ab)^{k-c}a^c \). For scattered factors of length \( i \leq k - c \) of \( w \), we just generate all possible words of length \( i \). For greater \( i \), on top of \( a^i \), we generate separately the scattered factors of the form \( uba^j \), for each \( j \in [i-1]_0 \). It is clear that, in such a word, \( |u| = i - j - 1 \), and if \( j \geq c \) then \( u \) must be a scattered factor of \((ab)^{k-j-1}a\), while if \( j < c \) then \( u \) must be a scattered factor of \((ab)^{k-c-1}a\). If \( j \geq (i+c) - k \) then, by Theorem 11 \( u \) can take all \( 2^{i-j-1} \) possible values. For smaller values of \( j \), we need to
generate $u$ of length $i - j - 1$ as a scattered factor of $(ab)^{k-c-1}a$, by the method described after Proposition 5.

Nevertheless, Theorems 11 and 12 are useful to see that in order to determine the cardinality of the sets of scattered factors of words consisting of alternating $a$s and $b$s or, respectively, of $(ab)^{k-c}a^c$, it is not needed to generate these sets effectively.

4 Cardinalities of $k$-Spectra of Strictly Balanced Words

In the last section a characterisation for the smallest and the largest $k$-spectra of words of a given length are presented (Proposition 2 and 5). In this section the part in between will be investigated for strictly balanced words (i.e. words of length $2k$ with $k$ occurrences of each letter). As before, we shall assume that $k \in \mathbb{N}_{\geq 3}$. In the particular case that $k = 3$, we have already proven that the $k$-spectrum with minimal cardinality has 4 elements and that the maximal cardinality is 8. Moreover as mentioned in Remark 3 a $k$-spectrum set of cardinality 5 does not exist for strictly balanced words of length $2k$. The question remains if $k$-spectra of cardinalities 6 and 7 exist, and if so, for which words.

Before showing that a $k$-spectrum of cardinality $2^k - 1$ for strictly balanced words of length $2k$ also exists for all $k \in \mathbb{N}_{\geq 3}$, we prove that only scattered factors of the form $b^{i+1}a^{k-i-1}$ for $i \in [k - 2]_0$ (up to renaming, reversal) can be “taken out” from the full set of possible scattered factors independently, without additionally requiring the removal of additional scattered factors as well. In particular, if a word of length $k$ of another form is absent from the set of scattered factors of $w$, then $|\text{ScatFact}_k(w)| < 2^k - 1$ follows.

**Lemma 13.** If for $w \in \Sigma_{2k}^{ab}$ there exists $u \notin \text{ScatFact}_k(w)$ with $u \notin \{b^ia^{k-i} \mid i \in [k - 1]\} \cup \{a^ib^{k-i} \mid i \in [k - 1]\}$, then $|\text{ScatFact}_k(w)| < 2^k - 1$.

**Proof.** Let be $i \in [k - 2]_0$. Consider firstly $u = b^ia^s$ for $r + s = k$ and $r \notin \{i\} \cup \{k - i, \ldots, k\}$ and $\Sigma_k \setminus \{u\} \supset \text{ScatFact}_k(w)$ for a word $w \in \Sigma_{2k}^{ed}$. If $b^{r+1}a^{s-1}$ is also not a scattered factor of $w$, the claim is proven (in this case two elements of $\Sigma_k^r$ are missing in $\text{ScatFact}_k(w)$). Assume $b^{r+1}a^{s-1} \in \text{ScatFact}(w)$. This implies that (possibly intertwined) $(s - 1)$ occurrences of $a$ follow $(r + 1)$ occurrences of $b$. Since $u$ is not a scattered factor of $w$, after these $(s - 1)$ as only $b$s may occur. If $b^{r-1}a^s$ is not a scattered factor, the claim is again proven and so suppose that it is one. This implies that the $(r - 1)$ $b$s are preceded by $a$s and not by $b$s. This implies that $b^{r+1}a^{s-1}$ is not a scattered factor and that contradicts the assumption. Consider now $u = u_1b^r a^s b^{r+1}u_2$ with $|u| = k$ not to be a scattered factor of $w$ for $r, s, t \in \mathbb{N}$. Following the same arguments as before, the claim is proven if $u_1b^{r-1}a^t b^{r+1}u_2$ is not a scattered factor and hence it is assumed to be one. This implies that exactly $|u_1|b$s occur before $b^{r-1}$. This implies that $u_1b^{r+1}a^t b^{r-1}u_2$ is not a scattered factor of $w$ of length $k$. Analogously it can be proven that scattered factors containing the switch from $a$ to $b$ and back to $a$ cannot lead to the cardinality $2^k - 1$. \(\square\)
Proposition 14. For $k \in \mathbb{N}_{\geq 3}$ and $w \in \Sigma_{ab}^{2k}$, the set $\text{ScatFact}_k(w)$ has $2^k - 1$ elements if and only if $w \in \{(ab)^i a^2 b^2 (ab)^{k-i-2} \mid i \in [k-2]_0\}$ (up to renaming and reversal). In particular $\text{ScatFact}_k(w) = \Sigma^k \setminus \{b^{i+1} a^{k-i-1}\}$ holds for $w = (ab)^i a^2 b^2 (ab)^{k-i-2}$ with $i \in [k-2]_0$.

Proof. Let be $i \in [k-2]_0$. First "$\Leftarrow\$" will be proven and for that consider $w = (ab)^i a^2 b^2 (ab)^{k-i-2}$. By Lemma 5 follows

$\text{ScatFact}_i((ab)^i) = \Sigma^i$ and $\text{ScatFact}_{k-i-2}((ab)^{k-i-2}) = \Sigma^{k-i-2}$.

With $\text{ScatFact}_2(a^2 b^2) = \{aa, ab, bb\}$ the $k$-spectrum of $w$ has at least $3 \cdot 2^i \cdot 2^{k-i-2} = 3 \cdot 2^{k-2} = 2^{k-2}$ elements. Notice that by this construction, scattered factors with a $ba$ at the middle position cannot be reached. For this reason we have to have a look at $w$'s remaining scattered factors not being gained by the above construction. This means that not only $i$ letters are allowed to be taken of the first part and not only $k-i-2$ letters from the last part.

Having a deeper look into $(ab)^i$ one can notice that all binary numbers (encoded by $a, b$) of length $i$ are scattered factors of $(ab)^{i+1} a$. This implies that nearly all binary numbers concatenated with $b$ are in the $i+1$-spectrum of $ab^i$. Appending now a $a$ from the middle part and then each of the words from the last part leads to nearly all remaining scattered factors of the $k$-spectrum of $w$. The only missing word is $b^{i+1}$, since the last $b$ cannot be reached within the first part. This implies that the word $b^{i+1} a^{k-i-1}$ is not in the $k$-spectrum of $w$ since with the $(i+1)$th $b$ the middle part is reached and the last part contains only $k-i-2$ as. This concludes $|\text{ScatFact}_k(w)| = 2^k - 1$.

On the other hand if $|\text{ScatFact}_k(w)| = 2^k - 1$ an element of the form $b^{i+1} a^{k-i-1}$ for an $i \in [k-2]_0$ is missing in the $k$-spectrum of $w$. Moreover this is exactly the only element missing. Fix an $i \in [k-2]_0$ and set $u = b^{i+1} a^{k-i-1}$. The proof will be very technically and exclude step by step all other possibilities than $w$ being $(ab)^i a^2 b^2 (ab)^{k-i-2}$. Firstly consider $i = k-2$. This implies $u = b^{k-1} a$. In this case $w$ has to end in $b^2$ but not in $b^3$ since otherwise $b^{k-2} a^2$ would not be a scattered factor. If $w$ were of the form $w_1 b a b^2$, $|w_1|_a = k-1$ and $|w_1|_b = k-3$ would hold which would imply that $b^{k-2} a^2$ is not a scattered factor. If $w$ ended in $a^3 b^2$, $a^{k-2} b a$ would be excluded. Hence, $w$ ends in $a^2 b^2$. Suppose at last that $w = (ab)^i a^2 b^2 w_2$ holds for $\ell < k-2$ and $w_2 \in \Sigma^\ast$. Then $w_2$ has each $(k-\ell-2)$ $a$ and $b$. Thus $b^{i+1} a^{k-\ell-1}$ is not a scattered factor of length $k$. This proofs that for $i = k-2$ $w = (ab)^{k-2} a^2 b^2$ is implied by $b^{k-2} a^2$ being the only excluded scattered factor from $\Sigma^k$. Hence assume $i \in [k-3]_0$.

Supposition: $w$ ends in $b^\ell$ for $\ell \geq 2$
If $i < k-2$ holds, then $b^{k-1} a \not\in \text{ScatFact}_k(w)$ follows and since $i + 1 < k-1$ holds, this element is different from $u$.
In the next step it will be shown that exactly $k-i-1$ repetitions of $ab$ are a suffix of $w$.

Supposition: $w = w_1 b^2 (ab)\ell$
If $\ell > k-i-2$ held, $b^{i+1} a^{k-i-1}$ would not be a scattered factor of $w$. If $\ell < k-i-2$ held, $b^{k-\ell-1} a^{i+1}$ would not be a scattered factor since $w_1$ has $(k-1)$ $a$ and $(k-\ell-2)$ $b$. 
Supposition: \( w = w_1a^2(ba)^i b \)

In this case \(|w_1|_a = k - 2 - \ell \) and \(|w_1|_b = k - \ell - 1 \) holds. This implies that \( a^{k-2-\ell}b^{\ell+1}a \) is not in the \( k \)-spectrum of \( w \).

Consequently there exists a \( w_1 \) such that \( w = w_1b^2(ab)^{k-i-2} \) holds. In the next it will be shown that \( b^2 \) has to be preceded by \( a^2 \).

Supposition: \( w = w_1b^2(ab)^{k-i-2} \)

Here \( w_1 \) has \((i+2)\) \( a \) and \((i-1)\) \( b \) and hence \( b^i a^{k-i-2} b^2 \) is not a scattered factor of length \( k \) of \( w \).

Supposition: \( w = w_1bab^2(ab)^{k-i-2} \)

This implies \( a^{i+2}bab^{k-i} \not\in \text{ScatFact}_k(w) \) since \( w_1 \) has \( i+1 \) occurrences of \( a \) and \( i-1 \) occurrences of \( b \).

This proofs that \( a^2b^2(ab)^{k-i-2} \) is a suffix of \( w \). The case that this is preceded by another \( a \) is excluded since then \( a^2b^{k-i-1} \) would not be in the \( k \)-spectrum of \( k \). In the last step it will be shown that the first occurrence of \( a^2 \) is at the point \( 2\ell \).

Supposition: \( w = (ab)^\ell a^2w_2 \) for \( \ell \neq i \)

If \( \ell \) is smaller than \( i \), \(|w_2|_a = k - \ell - 2 \) and \(|w_2|_b = k - \ell \) hold and \( b^{\ell+1}a^{k-\ell-1} \not\in \text{ScatFact}_k(w) \) follows. If \( \ell \) is greater than \( i \), in contradiction to the main assumption \( b^{i+1}a^{k-i-1} \) is a scattered factor, because \( b^{i+1} \) is a scattered factor of \( (ab)^\ell \) and \( k - \ell + (i + 1) = k - i - 1 \) \( a \) are left in the rest of \( w \).

Combining \( w = (ab)^i a^2w_2 \) and \( w = w_1a^2b^2(ab)^{k-i-2} \) the claim that \( w \) is of the form \( (ab)^i a^2b^2(ab)^{k-i-2} \) is proven. \( \square \)

By Proposition 14 we get that 7 is a possible cardinality of the set of scattered factors of length 3 of strictly balanced words of length 6 and, moreover, that exactly the words \( a^3b^2ab \) and \( ab^2a^2b^2 \) (and symmetric words obtained by reversal and renaming) have seven different scattered factors. The following lemma demonstrates that there always exists a strictly balanced word \( w \) of length \( 2k \) such that \(|\text{ScatFact}_k(w)| = 2k \). Thus, for the case \( k = 3 \) also the question if six is a possible cardinality of \( \text{ScatFact}_3(w) \) can be answered positively.

**Theorem 15.** The \( k \)-spectrum of a word \( w \in \Sigma_{ab}^{2k} \) has exactly \( 2k \) elements if and only if \( w \in \{a^{k-1}bab^{k-1}, a^{k-1}b^ka\} \) holds (up to renaming and reversal).

Moreover, there does not exist a strictly balanced word \( w \in \Sigma_{ab}^{2k} \) with a \( k \)-spectrum of cardinality \( 2k - i \) for \( i \in [k - 2] \).

**Proof.** Consider first \( w = a^{k-1}bab^{k-1} \). By Lemma 22 follows \( w \) has at least \( k + 1 \) elements since the \( k \)-spectrum of \( a^ib^k \) is a subset of the \( k \)-spectrum of \( w \).

Additionally \( w \) has the scattered factors of the form \( a^i bab^{k-2-i} \), which sum up to \( k-1 \). Hence \(|\text{ScatFact}_k(w)| = k + 1 + k - 1 = 2k \) holds. Again by Lemma 22, \( a^{k-1}b^ka \) has all elements of \( a^kb^k \)'s \( k \)-spectrum as scattered factors. Here the word has in addition all words of the form \( a^i b^{k-1-i} a \) as scattered factors which sum up to \( k-1 \) as well. This proves that both words have a scattered factor set of cardinality 2k.

The other direction will be proven by contraposition following the two main cases

\[ a^{k-1}bab^{k-1} \quad \text{and} \quad a^{k-1}b^ka. \]
Assume first \( w = a^b a^r x \) for \( \ell \in [k - 2]_{\geq 2}. \) Notice that it does not have to be considered that the word starts with one \( a, \) since this is symmetric to the reversal of the case \( a^b a^r b^k a. \) This implies \( |x|_a = k - \ell \) and \( |x|_b = k - 1. \) Notice here \( k - \ell < k - 1. \) Thus, there exists a scattered factor \( x' \) of \( x \) of length \( 2(k - \ell) \) with \( |x'|_a = |x'|_b = k - \ell. \) By Lemma 2 follows

\[
|\text{ScatFact}_{k-\ell}(y)| = k - \ell + 1 \iff y \in \{a^{k-\ell}b^{k-\ell}, b^{k-\ell}a^{k-\ell}\}
\]

and \( |\text{ScatFact}_{k-\ell}(y)| > k - \ell + 1 \) otherwise. This implies that the \((k - \ell)\)-spectrum of \( x' \) is minimal with respect to cardinality if \( x' \) is either \( a^{k-\ell}b^{k-\ell} \) or \( b^{k-\ell}a^{k-\ell}. \) For giving a lower bound of the cardinality of \( w \)'s scattered factor set of length \( k, \) it is sufficient to only take these both options into consideration. This implies that it is not necessary to examine the cases where \( x \) contains other scattered factors with both \( k - \ell \) \( a \) and \( b. \)

**case 1:** \( x' = a^{k-\ell}b^{k-\ell} \)

Thus \( x \) contains \( \ell - 1 \) \( b \) which are not in \( x'. \)

**case a:** \( x = b^{\ell-1}a^{k-\ell}b^{k-\ell} \)

In this case \( w = a^b a^r a^{k-\ell}b^{k-\ell} \) holds and that the \( k \)-spectrum of \( a^k b^k \) is a subset of \( \text{ScatFact}_k(w) \) follows.

**case i:** \( \ell < k - \ell \)

For all \( s \in [\ell] \) the words \( a^{\ell-s}b^s a^{k-\ell}, \ldots, a^b a^r a^{k-\ell-s} \) are well-defined and sum up to \( s + 1. \) Moreover for every \( s_2 \in [k - \ell] \) exists \( r_1 \in \mathbb{N}_0 \) and exist \( r_2, s_2 \in \mathbb{N} \) such that the words \( a^{r_1}b^{s_1}a^{r_2}b^{s_2} \) with \( s_1 + r_1 + s_2 + r_2 = k \) are all distinct and distinct to the aforementioned. Thus, in this case

\[
k + 1 + \sum_{s=1}^{\ell} (s + 1) + k - \ell = 2k + 1 - \ell + \frac{\ell(\ell + 1)}{2} + \ell \geq 2k + 4
\]

is a lower bound for \( \text{ScatFact}_k(w). \)

**case ii:** \( \ell > k - \ell \)

Consider here for \( r \in [k - \ell] \) the words \( b^{\ell-r}a^r b^{k-\ell}, \ldots, b^a b^{r}b^{k-\ell-r}. \) For fixed \( r \) these are \( r + 1. \) Moreover in this case for all \( r_1 \in [\ell] \) exist \( s_1, r_2 \in \mathbb{N} \) and \( s_2 \in \mathbb{N} \) such that the words \( a^{r_1}b^{s_1}a^{r_2}b^{s_2} \) with \( s_1 + r_1 + s_2 + r_2 = \ell \) are all distinct and distinct to the aforementioned. In total this sums up to

\[
k + 1 + \sum_{r=1}^{k-\ell} (r + 1) + k - \ell = k + 1 + \frac{(k - \ell)(k - \ell + 1)}{2} + (k - \ell) + \ell \geq 2k + 4
\]

different scattered factors.

**case b:** \( x = a^{k-\ell}b^{k-1} \)

Thus, \( w = a^b a^b a^{k-\ell}b^{k-1} \) holds. Here it holds as well that the \( k \)-spectrum of \( a^k b^k \) is a subset of \( \text{ScatFact}_k(w). \) Moreover all words of the form \( b^a b^a \) for \( r + s = k - 1 \) and \( r \in [k - \ell] \) are different scattered factors, i.e. \( k - \ell \) many. Additionally the words \( a^r b^a b^a \) for \( r + s = k - 2 \) and \( r, s > 0 \) are different scattered factors and distinct to the aforementioned. This sums up to \( k + 1 + k - 1 + k - 2 = 3k - 2 \) for the cardinality of \( \text{ScatFact}_k(w). \) This proves the claim for \( k \geq 3. \)
Lemma 16. For \( k \) the more unlikely it is to find a not reachable, i.e. \( 3^k \) a (witnessed by, e.g. elements. 

\[
\ell \in w \quad \text{of} \quad x \quad \text{case a:}
\]

Hence \( k \) factors are different, the \( \ell \) is determined analogously to case 1a.

\[
\text{case b: } x = b^{k-\ell}a^{k-\ell} \quad \text{In this case } w = a^b b^{k-\ell} a^{k-\ell} \quad \text{holds. Here the cardinality of the } k\text{-spectrum of } w \text{ is determined analogously to case 1a.}
\]

By Lemma 14 and Theorem 15 the possible cardinalities of \( \text{ScatFact}_3(w) \) for strictly balanced words \( w \) of length 6 are completely characterized.

**Fig. 1.** For \( k = 3 \) there exist strictly balanced words of length \( 2k \), such that their \( k \)-spectra have cardinality \( k + 1, 2k, 2^{k-1} \) and \( 2^k \).

Theorem 15 determines the first gap in the set of cardinalities of \( |\text{ScatFact}_k(w)| \) for \( w \in \Sigma_{2}^{k} \); there does not exist a word \( w \in \Sigma_{2}^{k} \) with \( |\text{ScatFact}_k(w)| = k + i + 1 \) for \( i \in [k - 2] \) and \( k \geq 3 \), since all words that are not of the form \( a^b b^i, b^i a^k, a^{k-1} b a^{k-1} b^i \), or \( a^{k-1} b^i a^k \) have a scattered factor set of cardinality at least \( 2k + 1 \). As the size of this first gap is linear in \( k \), it is clear that the larger \( k \) is, the more unlikely it is to find a \( k \)-spectrum of a small cardinality.

In the following we will prove that the cardinalities \( 2k + 1 \) up to \( 3k - 4 \) are not reachable, i.e. \( 3k - 3 \) is the thirtysmallest cardinality after \( k + 1 \) and \( 2k \) (witnessed by, e.g. \( a^{k-2} b^k a^2 \)).

Lemma 16. For \( i \in \left[ \left( \frac{k}{2} \right) \right] \) and \( j \in [k-1] \)

- \(|\text{ScatFact}_k(a^{k-1} b^i a^t)| = k(i + 1) - i^2 + 1 \quad \text{for } k \geq 4, \)
- \(|\text{ScatFact}_k(a^{k-1} b^2 a^{k-2})| = 3k - 2, \)
- \(|\text{ScatFact}_k(a^{k-2} b^2 a^{k-2} a)| = k(2j + 2) - 6j + 2 \quad \text{for } k \geq 5 \text{, and} \)
- \(|\text{ScatFact}_k(a^{k-2} b^{j+1} a^{k-2} b^j)| = k(2j + 1) - 4j + 2. \)

Proof. For the first part, let \( i \in \left[ \left( \frac{k}{2} \right) \right]_2 \). The \( k \)-spectrum of \( a^{k-i} b^k a^i \) contains exactly all words of the form \( a^r b^s a^t \) with \( r + s + t = k, \ t \in [i]_0, \ r \in [k-i]_0, \text{ and } s \in [k]_0. \) If \( t \) and \( r \) are fixed, \( s \) is uniquely determined. Since all these scattered factors are different, the \( k \)-spectrum has \( (i + 1)(k - i + 1) = k(i + 1) - i^2 - 1 \) elements.
For the second part, notice that the scattered factors \(a^{k-1}b^2ab^{k-2}\) are of four different forms: \(b^r bab^s, a^tb^r a, ab^s,\) and \(a^s bab^r\). All these scattered factors are different if in the second one \(s\) is chosen greater than or equal to 1 and in the last one \(r, s_1, s_2 \geq 1\) holds. The first and second one lead to two scattered factors, since for every \(s \in [2]\) there are enough a at the beginning for padding from the left. The third form leads to \(k + 1\) different scattered as shown in Lemma [2]. The last one is a little bit more complicated. Notice firstly that \(r\) is at most \(k - 3\) since \(s_1, s_2 \geq 1\) holds. In this case there exists only one possibility for choosing \(s_1\) and \(s_2\), namely as 1. If \(r = k - 4\) there exist two possibilities, namely \(s_1 = 1\) and \(s_2 = 2\) or vice versa. For \(r \in [k - 5]\) there exist always 2 possibilities for the bs between the as. This leads to \(2(k - 5)\) possibilities. Allover it sums up to \(2 + 2 + k + 1 + 1 + 2 + 2(k - 5) = 8 + 3k - 10 = 3k - 2\).

As in the proof of the second part for the third and fourth part the scattered factors can be categorized in the form \(a^r b^s, b^r a^s, a^r b^s,\) and \(a^s b^r\), where with appropriate chosen exponents no factors is counted twice. Also as before, \(i\) can be chosen in \([\left\lfloor \frac{k}{2} \right\rfloor]\), since otherwise the proof is analogous for \(k - i\).

The first form contributes \(k + 1\) elements. The second and third form contribute \(2i\) each, since \(s\) resp. \(r\) range in \([2]\). For the last form a distinction is necessary. If \(r = k - 3\) holds, \(a^{k-3}bab\) is the only scattered factor. If \(r\) is smaller than \(k - 3\), \(2i\) possibilities for each \(r \in [k - 3]\) lead to scattered factors. Allover this sums up to \(k + 1 + 2i + 1 + 2i(k - 4) = k(2i + 1) - 4i + 2\). By this the first claim is proven.

For the fourth claim again scattered factors of different forms will be distinguished. Since also here the minimal \(k\)-spectrum is a subset of the \(k\)-spectrum of \(w\), these \(k + 1\) elements counts for the cardinality. There exists \(i\) many scattered factors of the form \(a^r b^s a^2\) and \(k - 2\) of the form \(a^r b^s a\), since with the last a all occurrences of b are before it. Assuming w.l.o.g. again that \(i\) is at most \(\left\lfloor \frac{k}{2} \right\rfloor\) only \(b^{k-1}a\) is a scattered factor of the form \(b^s a^t\). The scattered factors of the form \(b^r a^s\) contribute \(i\) many. The remaining two forms need again a case analysis. 

There exists exactly one scattered factor of the form \(a^r b^s ab^{s_2}\) for \(r = k - 3\) and exactly one scattered factor of the form \(a^r b^s ab^{s_2}\) for \(r_1 = k - 4\). If \(r\) resp. \(r_1\) are smaller there exists \(i\) different scattered factors for each choice of \(r \in [k - 4]\) resp. \(r_1 \in [k - 5]\). This sums up to \(k + 1 + k - 2 + i + i + 1 + 1 + 1 + 1 + (k - 5) + 1 + k(i - 4) = 2k + 2 + 3i + ik - 5i + ik - 4i = k(2 + 2i) - 6i + 2.\)

Notice that for \(i \in [\left\lfloor \frac{k}{2} \right\rfloor]\) the sequence \((k(2i + 1) - 4i + 2)\), is increasing and its minimum is \(3k - 2\) while for \(i \in [\left\lfloor \frac{k}{2} \right\rfloor]\) the sequence \((k(2i + 2) - 6i + 2)\), is increasing and its minimum is \(4k - 4\). The following lemma only gives lower bounds for specific forms of words, since, on the one hand, it proves to be sufficient for the Theorem [18] which describes the second gap, and, on the other hand, the proofs show that the formulas describing the exact number of scattered factors of a specific form are getting more and more complicated. It has to be shown that also words starting with \(i\) letters \(a\), for \(i \in [k - 3]\), have a \(k\)-spectrum of greater (as lower is already excluded) cardinality. By Lemma [16] only words with another transition from a’s to b’s need to be considered, \((* = a^r b^s w_1 a^s b^{s_2}).\) W.l.o.g. we can assume \(s_1\) to be maximal, such that \(w_1\) starts with an a, and similarly,
by maximality of $r_2$, ends with a $b$, thus only words of the form $a^{r_1}b^{s_1} \ldots a^{r_n}b^{s_n}$ have to be considered, and by Proposition 5 it is sufficient to investigate $n < k$.

**Lemma 17.** $|\text{ScatFact}_k(a^{k-2}b^iab^jabb^{k-i-j})| \geq 3k - 3$ for $i, j \in [k - 2]$, $i + j \leq k - 1$.

$|\text{ScatFact}_k(a^{k-2}b^ia^{r_1}b^{s_2}a^{r_2}b^{s_3})| \geq 3k - 4$ for $s_1 + s_2 + s_3 = k$, $r_1 + r_2 = 2, s_1 > 0, r_1, r_2, s_2, s_3 \geq 0$.

$|\text{ScatFact}_k(a^{r_1}b^{s_1} \ldots a^{r_n}b^{s_n})| \geq 3k - 3$ for $r_1 \leq k - 3, \sum_{i \in [n]} r_i = \sum_{i \in [n]} s_i = k$, and $r_i, s_i \geq 1$.

**Proof.** Choose for the first claim $i, j \in [k - 2]$. Then all words of the form $a^rb^s$ for $r, s \in [k]$ are scattered factors of $w_{ij}$ and by Lemma 2 follows that $w_{ij}$ has $k + 1$ scattered factors of this form. Scattered factors of the form $a^{r_1}b^{s_2}a^{r_2}$ can occur in three variants. In the first variant only the second block of $a$ is involved after the first block of $b$, namely the second single $a$ is not involved. Since $i \in [k - 2]$ holds, for each $s \in [i]$ exists $r_1, r_2$ ($r_2 = 1$) such that $a^{r_1}b^{r_2}$ is a scattered factor of $w_{ij}$, i.e. $w_{ij}$ has additionally $i$ scattered factors. The second variant uses the $a$ of each the second and the third $a$-block. This only scattered factors of the form $a^{r_1}b^ia^{r_2}$ are of interest, the second $b$-block is not involved. If $i + j = k - 1$ holds only $i - 1$ scattered factors of this form occurs, otherwise again $i$ new elements are in the $k$-spectrum. If only the $a$ from the third block is involved then $j$ (resp. $j - 1$) new elements are in the spectrum. This sums up to at least $2i + j - 2$ elements of the form $a^{r_1}b^ia^{r_2}$. A similar distinction leads to the number of scattered factors of the form $a^{r_1}b^ia^{r_2}b^{s_3}$. Assume first $r_2 = 1$ and for this only the $a$ from the second $a$-block. This implies that either only $b$ from the second block or from the second and third block can be taken for the last $b$-block in the scattered factor. Moreover $r_1, s_1, s_2$ are at most $k - 3$. For each choice of $r_1$ in $[k - 3]$ there are $\min\{j, k - 2 - i\}$ possibilities, which leads to

$$i \left((k - j - 2)j + \sum_{\ell=1}^{k-j-2} k - 2 - \ell\right) = 6i + \frac{1}{2}k^2i - kji - 3\frac{1}{2}ki + 9\frac{1}{2}j + \frac{1}{2}ij.$$  

If $b$ from the second and third block are allowed, all of the second block have to occur for obtaining different scattered factors to the previous ones. Thus,

$$i \left((k - j - i - 2)j + \sum_{\ell=1}^{k-j-i-2} k - 2 - \ell\right) = kij + \frac{1}{2}k^2i - \frac{1}{2}k^2i - ik - jk - i^2j - ij + \frac{1}{2}ij + \frac{1}{2}j^2 + \frac{1}{2}j^2.$$  

If both, the second and the third $a$-block, are involved $ik - \frac{1}{2}j^2 - ij - \frac{1}{2}j$ additional scattered factors are in the $k$-spectrum. This all sums up to

$$k + 1 + 9i - 2 + \frac{1}{2}k^2i - \frac{1}{2}k^2i - 3\frac{1}{2}ki + \frac{1}{2}j^2i - \frac{1}{2}j^2i - \frac{1}{2}ji + \frac{1}{2}j^2 + \frac{1}{2}j^2.$$
Since either $i^2 \geq ij$ or $j^2 \geq ij$ and $i, j \in [k - 3]$ hold, this is greater than or equal to
\[
\frac{1}{2}k^2 - 2 \frac{1}{2}k + \frac{9}{2} \geq 3k - 3.
\]
Notice that additionally there exist scattered factors of other forms, which enlarge the concrete $k$-spectrum.

For the second claim, consider first the case, when $s_2 = 0$, $r_1 = 0$, or $r_2 = 0$. This leads to words of the form matching Lemma 16 and consequently the $k$-spectrum has $k(2i+1) - 4i + 2 \geq 3k - 2 > 3k - 4$ elements. Consider now the case that $s_3 = 0$ holds and all other exponents are at least 1. By Lemma 16 follows again that each such word has at least $k(2i + 2) - 6i + 2 \geq 4k - 4 > 3k - 4$ elements. Finally by Lemma 17 follows that the remaining words of the given form have at least $3k - 3$ scattered factors.

For the third claim, obviously $a^k$ is a scattered factor and $a^{k-1}b^j$ for $s_n$ also. Notice here, that the proof leads to $s_{n-1}$ scattered factors, if in the claim $s_n = 0$ would be allowed. With a similar argumentation the number of occurrences of the form $a^i b^j a^{k-i-j}$ will be shown. If for a specific $i, j$-combination $a^i b^j a^{k-i-j}$ is not a scattered factor, then choose $m_1, m_2$ such that the $i$th $a$ is in block $m_1$ and the $j$th $b$ after that is in block $m_2$. Thus in the blocks $m_2 + 1$ to $n$ are less than $k - i - j$ a. Let $r_{m_2}$ be the a in the $m_1$th block which don’t belong to $a'$. Then $r_{m_1} + \cdots + r_{m_2}$ contains more than $k - j - i$ a since $k - j - i$ a occur in the $m_1$th to the $n$th block. Thus $a^i a^{m_1} b^{m_1} \cdots a^{m_2} b^{m_2}$ is a scattered factor of length at least $k + 1$ where $s_{m_2}$ describes the part of the $m_2$th block until the $j$th $b$. If $1 < m_1, m_2 < n$ holds, $ba^{k-j-2}b^{j-1}$ is a scattered factor of $w$. If $m_1 = m_2 = 1$ holds, $a^{k-j}bab$ is a scattered factor. If both are equal to $n$, $ba^{k-j-1}b^{j-2}$ is a scattered factor. In both cases the last $b$ exist even if $s_m = 0$ holds, since the scattered factor ends in the examined block $m_2$. If $m_1 < m_2$ holds, there exists a factor of length $> k$ which can be narrowed to a factor starting in $a$, ending in $b$, and having at least one switch from $b$ back to $a$ and back to $b$. This concludes to at least $(k - 2)^2$ scattered factors of the form $a^i b^j a^{k-i-j}$ (or a different one in exchange). By $k^2 - k + 3 \geq 3k - 3$ for $k \geq 5$ follows the claim.

By Lemmas 16 and 17 we are able to prove the following theorem, which shows the second gap in the set of cardinalities of ScatFact$_k$ for words in $Σ_{ab}^{2k}$. 
**Theorem 18.** For $k \geq 5$ there does not exist a word $w \in \Sigma_{sb}^{2k}$ with $k$-spectrum of cardinality $2k + i$ for $i \in [k - 4]$. In other words, i.e. between $2k + 1$ and $3k - 4$ is a cardinality-gap.

**Proof.** The Lemmas 14 and 15 show that exactly the words $a^k b^k$, $a^{k-1} b a b^{k-1}$, and $a^k b^k a$ have $k$-spectrum of cardinality less than or equal to $2k$. By Lemma 16 and 17 follows that $a^{k-2} b^k a^2$ has a $k$-spectrum of cardinality $3k - 3$. Assume a $w \in \Sigma_{sb}^{2k} \setminus \{a^k b^k, a^{k-1} b a b^{k-1}, a^k b^k a, a^{k-2} b^k a^2\}$. Since renaming and reversal do not influence the cardinality, it can be assumed w.l.o.g. that $w$ starts with $a$. By assumption $w$ does not start with $a^k$. If $w$ starts with $a^{k-1}$, $w = a^{k-1} b a b^{k-1}$ follows with $i \in [k - 1] \geq 2$ and by Lemma 16 the $k$-spectrum has $(i + 1) k - 4 i + 6 \geq 3 k - 2 > 3 k - 4$ elements. By Lemma 17 the claim follows for words starting with $(k - 2)$ a. and it is shown that words starting with at least two and at most $k - 3$ a lead to $k$-spectra of cardinality greater than $3k - 3$. \[\square\]

Going further, we analyse the larger possible cardinalities of ScatFact$_k$, trying to see what values are achievable (even if only asymptotically, in some cases).

**Corollary 19.** All square numbers, greater or equal to four, occur as the cardinality of the $k$-spectrum of a word $w \in \Sigma_{sb}^{2k}$; in particular $|\text{ScatFact}_k(a^i b^i a^j)| = (\frac{k}{2} + 1)^2$ holds for $k$ even.

**Proof.** Apply Lemma 16 to $i = \frac{k}{2}$. This implies that the cardinality of the $k$-spectrum of $a^i b^k a^j$ is

$$k \left( \frac{k}{2} + 1 \right) - \frac{k^2}{4} - 1 = \frac{1}{4} k^2 + k - 1 = \left( \frac{k}{2} + 1 \right)^2 \cdot \square$$

Inspired by the previous Corollary, we can show the following result concerning the asymptotic behaviour of the cardinality of ScatFact$_k$ for words of length $2k$.

**Proposition 20.** Let $i > 1$ be a fixed (constant) integer. Let $d = \lfloor \frac{k}{i} \rfloor$ and $r = k - d i$, and $d' = \lfloor \frac{k}{i-1} \rfloor$ and $r' = k - d' (i - 1)$. Then the following hold:

- the word $a^i b^i (a^d b^d)^i$ has $\Theta(k^{2i-1})$ scattered factors of length $n$;
- the word $a^i b^i (a^d b^d)^{i-1} a^d$ has $\Theta(k^{2i-2})$ scattered factors of length $n$.

**Proof.** Let us first show the upper bounds. The following algorithm can be used to find the scattered factors of length $k$ of $a^i b^i (a^d b^d)^i$. Choose 2 numbers $q_1$ and $q_2$ from $[i]_0$, and $2i - 1$ integers $r_1, \ldots, r_{2i-1}$ from $[d]_0$. Let $r_{2i} = k - (q_1 + q_2 + \sum_{j \in [2i-1]} r_j)$. If $r_{2i} \geq 0$ then the word $w' = a^{q_1} b^{q_2} (a^{r_1} b^{r_2}) (a^{r_3} b^{r_4}) \cdots (a^{r_{2i-1}} b^{r_{2i}})$ is a scattered factor of $a^i b^i (a^d b^d)^i$, and all scattered factors of length $k$ of this word have this form. From the construction of $w'$, because $d \leq \frac{k}{i}$, it follows
that there are at most $O(\frac{i^2 k^{2i-1}}{i^{2i-1}})$ possible ways to obtain it. As $i$ is seen as a constant, this means that $a^i b^r (a^d b^d)^i$ has $O(n^{2i-1})$ scattered factors of length $k$.

In the same way one can show that $a^i b^r (a^d b^d)^i a^d$ has $O(n^{2i-2})$ scattered factors of length $n$.

Let us now show the lower bounds. We first consider the word $a^i b^r (a^d b^d)^i$. As $i$ is constant, let us assume that $k > \frac{i(2i-1)}{i+1}$. Clearly, $\frac{k(i-1)}{i(2i-1)} \leq \frac{k}{2i-1} \leq \frac{k}{2i-1} - 1 \leq i \leq \frac{k}{2i-1}$ and $d + r \geq \frac{k}{2i-1}$. We generate scattered factors of the word $a^i b^r (a^d b^d)^i$ as follows. We firstly choose $2i - 1$ integers $r_1, \ldots, r_{2i-1}$ between $\frac{k(i-1)}{i(2i-1)}$ and $\frac{k}{2i-1}$.

Under our assumptions, the word

$$w'' = b^r (a^r a^r) \cdots (a^{2i-2} b^{2i-1})$$

is a scattered factor of the suffix $b^d (a^d b^d)^{i-1}$ of $a^i b^r (a^d b^d)^i$. Let $r_0 = k - \sum_{j \in [2i-1]} r_j$. We have $r_0 \leq \frac{k}{2i-1} \leq d + r$, so $a^{r_0} w''$ is a scattered factor of $a^i (a^d b^d)^i$, so also of $a^i b^r (a^d b^d)^i$. Moreover, each choice of a tuple $(r_1, \ldots, r_{2i-1})$ leads to a different scattered factor of $a^i b^r (a^d b^d)^i$. The total number of scattered factors of length $k$ of $a^i b^r (a^d b^d)^i$ is at least

$$\left( \frac{k}{2i-1} - \frac{k(i-1)}{i(2i-1)} \right)^{2i-1} \geq \left( \frac{k}{i(2i-1)} \right)^{2i-1}.$$

So the total number of scattered factors of length $k$ of $a^i b^r (a^d b^d)^i$ is at least

$$\left( \frac{k}{i(2i-1)} \right)^{2i-1}.$$

As the total number of scattered factors of length $k$ of $a^i b^r (a^d b^d)^i$ is also $O(k^{2i-1})$, we get that $a^i b^r (a^d b^d)^i$ has $\Theta(k^{2i-1})$ scattered factors of length $k$.

The proof that $a^i b^r (a^d b^d)^i a^d$ has $\Theta(n^{2i-2})$ scattered factors of length $k$ follows in a very similar manner. 

**Remark 21.** Let $i$ be an integer, and consider $k$ another integer divisible by $i$. Consider the word $w_k = (a^1 b^1)^i$. The exact number of scattered factors of length $k$ of $w_k$ equals to the number $C(k, 2i, \frac{k}{i})$ of weak $2i$-compositions of $k$, whose terms are bounded by $\frac{k}{i}$, i.e., the number of ways in which $k$ can be written as a sum $\sum_{j \in [2i]} r_j$ where $r_j \in [\frac{k}{i}, \frac{k}{i}]$. From Lemma 20 we also get that this number is $\Theta(\frac{n^{2k-1}}{k})$, but we also have:

$$C(k, 2i, \frac{k}{i}) = \sum_{0 \leq j < M} (-1)^j \binom{2i}{j} \binom{k + 2i - j (\frac{k}{i} + 1) - 1}{2i - 1},$$

for $M = \frac{i (k + 2i - 1)}{k + i}$. It is known that there exists a constant $E > 0$ such that

$$C(k, 2i, \frac{k}{i}) \leq E \sum_{0 \leq j < M} (-1)^j \binom{2i}{j} \binom{k + 2i - j (\frac{k}{i} + 1) - 1}{2i - 1}.$$

The coefficient of $k^{2i-1}$ in the right hand side of this inequality has to be positive. Consequently $\sum_{0 \leq j < M} (-1)^j \binom{2i}{j} (i-j)^{2i-1} > 0$. This seems to be an interesting combinatorial inequality in itself.
One can also show as in Lemma \[20\] that the number of scattered factors of length $k$ of $w_k$, which have, at their turn, $(ab)^i$ as a scattered factor, is $\Theta(2^{2i-1})$.

The number also equals the number $C'$ \left( k; 2i, \frac{1}{2} \right)$ of $2i$-compositions of $k$ whose terms are strictly positive integers upper bounded by $\frac{k}{i}$, i.e., the number of ways in which $k$ can be written as a sum $\sum_{j \in [2i]} r_j$ where $r_j \in \left[ \frac{k}{i} \right]$. Just as above, from this we get $\sum_{0 \leq j < i} (-1)^j \binom{2i}{j}(i-j)^{2i-1} > 0$. Again, this inequality seems interesting to us. \[
\]

We will end this analysis with the conjecture that, in contrast to the first gap, which always starts immediately after the first obtainable cardinality, the last gap ends earlier the larger $k$ is. More precisely, if $w = a^2b^2(ab)^{k-3-i}ba(ab)^i$ for $k \in \mathbb{N}_{\geq 4}$, $i \in [k-2]$ then $|\text{ScatFact}_k(w)| = 2^k - 2 - i$.

At the end of this section, we will briefly introduce $\theta$-palindromes in this specific setting. Let $\theta: \Sigma^* \to \Sigma^*$ be an antimorphic involution, i.e., $\theta(uv) = \theta(v)\theta(u)$ and $\theta^2$ is the identity on $\Sigma^*$. By $\Sigma = \{a, b\}$ only the identity and renaming are such mappings. The fixed points of $\theta$ are called $\theta$-palindromes $(ab^i, \theta(b)^i\theta(a))$ and exactly the words where $w^R = \varnothing$ holds. They were studied in different fields well (see e.g., \[11\], \[17\]). A word $w \in \Sigma^{2k}_{ab}$ is a $\theta$-palindrome iff either $w \in \{aw'b, bw'a\}$ for some $\theta$-palindrome $w' \in \Sigma^{2(k-1)}_{ab}$ or additionally $w = a^ib^k\theta\theta^i$ in the case that $k$ is even. Two cardinality results for $\theta$-palindromes are presented in Lemma \[16\] and Corollary \[19\]. We believe that persuing the $k$-spectra of $\theta$-palindromes may lead to a deeper insight of which cardinalities can be reached, but due to space restrictions we will only mention one conjecture here, which may already show that cardinalities are somehow propagating for $\theta$-palindromes. Notice that this conjecture implies that indeed similar to the second gap where $4k - 4$ is always reached but that in contrast to the second gap, the third gap is not of the form $4k - 4 - i$ for $i \in [k - 4]$. (see Figure \[6\]).

**Conjecture 22.** The $k$-spectrum of $w = ab^{k-1}a^{k-1}b$ has $4(k - 1)$ elements and moreover if $w' = w^R$ with a $k$-spectrum of cardinality $\ell \in \mathbb{N}_{\geq 12}$ then the scattered factor set of $awb$ has cardinality $2 \frac{1}{4} \ell - 5$.

5 Reconstructing Strictly Balanced Words from their $k$-Spectra

In the final section we consider the slightly different problem of reconstructing a word from its scattered factors, or more specifically in this case, $k$-spectra. More generally, we are interested in how much information about a (strictly balanced) word $w$ is contained in its scattered factors, and more precisely, which scattered factors are, respectively, are not necessary or useful for reconstructing the word $w$, or distinguishing it from others. Since $w$ is a scattered factor of itself, it is trivial that the scattered factor of length $|w|$ is sufficient to uniquely reconstruct $w$. On the other hand, all words over $\{a, b\}^*$ containing both letters will have the same 1-spectrum. Thus we see that the length of the scattered factors of a word $w$ plays a role in how much information about $w$ they contain. This relationship
is described more precisely by the following result of Dress and Erdős [2] along with the fact that (cf. e.g. Proposition 5) a word of length $2k$ is not uniquely determined by its scattered factors of length $k$.

**Proposition 23** (Dress and Erdős [2]). If $\text{ScatFact}_{k+1}(w) = \text{ScatFact}_{k+1}(w')$ holds for $w, w' \in \Sigma^{\leq 2k}$ then $w = w'$ follows.

**Proof of Proposition 23** in the case that $w, w'$ are strictly balanced.

Proof. We give a procedure for uniquely reconstructing $w$ from $\text{ScatFact}_k w$. For all $i, j \in \mathbb{N}_0$ such that $i + j = k$, ask whether $a^i b a^j \in \text{ScatFact}_k w$. Since there are exactly $i + j$ occurrences of $a$ in $w$, all are accounted for in the (potential) scattered factor $a^i b a^j$, and thus the answer is ‘yes’ if and only if there are one or more bs between the $i$th and $i + 1$th occurrences of $a$ in $w$. Hence after these queries, we know exactly which as are consecutive (i.e. do not have a b between them) in $w$. Similarly we ask for all $i, j \in \mathbb{N}_0$ such that $i + j = k$, ask whether $b^i a b^j \in \text{ScatFact}_k w$. By symmetry, this tells us exactly which bs are consecutive. This is sufficient information to specify $w$ completely. \(\square\)

In the proof of Proposition 23 a pivotal role is played by particularly unbalanced scattered factors, which contain many a’s and a few b’s or vice-versa. The question arises as to whether this is due to the fact that these unbalanced scattered factors contain inherently more information about the structure of the whole word than the strictly balanced ones? In the general case, the answer is, sometimes at least, yes: we cannot distinguish between e.g. two words in $\{a\}^*$ by their strictly balanced scattered factors, as the only such factor is $\epsilon$. The same problem arises for all words which are sufficiently unbalanced. However, if in addition we consider only strictly balanced words $w$, then the situation changes. We conjecture that in fact, for strictly balanced words $w$, the strictly balanced scattered factors are just as informative about the $w$ as the unbalanced ones. More formally, we believe the following adaptation of Proposition 23 holds:

**Conjecture 24.** Let $k \in \mathbb{N}$. Let $k' = k+1$ for odd $k$, and $k' = k+2$ for even $k$. Let $w, w' \in \Sigma_{sb}^{2k}$ such that $\text{ScatFact}_{k'}(w) \cap \Sigma_{sb}^{k'} = \text{ScatFact}_{k'}(w') \cap \Sigma_{sb}^{k'}$. Then $w = w'$.

While we do not resolve the conjecture, we give an example of a subclass of words for which it holds true, namely when there are at most two ‘blocks’ of b’s (and therefore by symmetry if there are at most two ‘blocks’ of a’s).

**Proposition 25.** Let $k \in \mathbb{N}$. If $k$ is odd, then each word $w \in a^* b^r a^* b^s a^* \cap \Sigma_{sb}^{2k}$ is uniquely determined by the set $\text{ScatFact}_{k+1}(w) \cap \Sigma_{sb}^{k+1}$. Similarly, if $k$ is even, then each word $w \in a^* b^r a^* b^s a^* \cap \Sigma_{sb}^{2k}$ is uniquely determined by the set $\text{ScatFact}_{k+2}(w) \cap \Sigma_{sb}^{k+2}$.

Proof. As in the proof of Proposition 23 we give an algorithm for uniquely reconstructing $w$. W.l.o.g., let $k$ be odd. The case that $k$ is even is easily adapted. Let $w = a^1 b^r a^1 b^s a^1$ and let $S = \text{ScatFact}_{k+1}(w) \cap \Sigma_{sb}^{k+1}$. Firstly, we shall deal with the case that $\ell = 0$. Note that we can decide whether $\ell = 0$ by querying
whether there exists a scattered factor \( u \in S \) such that \( u \in a^i b^j a^k b^l a^m \). Now, if \( \ell = 0 \), we have \( w = a^i b^j a^k \). Since \( k \) is odd, exactly one of \( i, k - i \) will be at most \( \frac{k-1}{2} \). We can decide which one by querying whether \( a^{k-1} b^i \in S \). Wlog., suppose \( i \leq \frac{k-1}{2} \) (so the query returns “no”). The other case is symmetric. Then note that \( a^i b^{k-1-i} a^k \in S \) but \( a^{k-1} b^i a^{k-1-i} \notin S \). Thus the exact value of \( i \) (and therefore \( k-i \)) can be inferred directly from observing scattered factors of this form in \( S \).

Now consider the the case that \( \ell \neq 0 \). Note that there exists \( u \in b^+ a^{k-1} b^\ell \cap S \) if and only if \( \ell \geq \frac{k-1}{2} \). Suppose firstly that \( \ell \geq \frac{k-1}{2} \). Then \( i + (k - i - \ell) \leq \frac{k-1}{2} \). Thus we can determine \( i \) and \( (k - i - \ell) \) (and therefore \( \ell \)) by looking for the maximum \( m_1, m_2 \) such that there exists \( u \in a^m b^i a^k b^j a^m \) with \( u \in S \) (\( i \) is the maximum \( m_1 \) while \( k - i - \ell \) is the maximum \( m_2 \)). Moreover, exactly one of \( j, k - j \) will be less than \( \frac{k-1}{2} \). We can decide which one by querying whether \( a^{k-1} b^i \in S \). If so, it must be that \( k - j \geq \frac{k-1}{2} \). Suppose that this is the case (the other case is symmetric). Then as before, we can determine the exact value of \( j \) by looking at the scattered factors of the form \( b^m a^i b^j a^m \) (i.e., \( j \) is the maximum \( m \)) and we are done.

Finally, we consider the case that \( 0 < \ell < \frac{k-1}{2} \). Then \( \ell \) can be uniquely determined as the maximum \( m \) such that there exists \( u \in a^m b^i a^k b^j a^m \) with \( u \in S \). In order to determine \( i \) (or equivalently \( k-i-\ell \)), we look for the maximum \( m_1, m_2 \) such that there exist \( u_1 \in a^m b^i a^k b^j a^m \) and \( u_2 \in a^m b^i a^k b^j a^m \) with \( u_1, u_2 \in S \). In particular at least one of \( m_1, m_2 \) must be strictly less than \( \frac{k-1}{2} \). If \( m_1 < \frac{k-1}{2} \), then \( j = m_1 \) and if \( m_2 < \frac{k-1}{2} \) then \( k - \ell - i = m_2 \). In either case, since \( \ell \) is already known, this uniquely determines both \( i \) and \( k - i - \ell \).

It remains to determine \( j \) (or equivalently \( k - j \)). Recall that exactly one of \( j, k - j \) will be less than \( \frac{k-1}{2} \). Let \( m_1 \) be the maximum \( m \) such that there exists \( u \in a^m b^i a^k b^j a^m \) with \( u \in S \) and let \( m_2 \) be the maximum \( m \) such that there exists \( u \in a^m b^i a^k b^j a^m \) with \( u \in S \). Note that \( m_1, m_2 \leq \frac{k-1}{2} \). If \( m_1 < \frac{k-1}{2} \) (resp. \( m_2 < \frac{k-1}{2} \)), then \( j = m_1 \) (resp \( k-j = m_2 \)), and thus \( j \) and \( k-j \) can be inferred. If \( m_1 = m_2 = \frac{k-1}{2} \), then either \( j = \frac{k-1}{2} \) or \( k-j = \frac{k-1}{2} \). Now, if \( k - i - \ell < \frac{k-1}{2} \), there exists \( u \in a^m b^i a^k b^j a^m \) with \( u \in S \) if and only if \( j = \frac{k-1}{2} \) (in which case \( k - j = \frac{k-1}{2} \)). On the other hand, if \( k - i - \ell \geq \frac{k-1}{2} \), then \( j < \frac{k-1}{2} \) and there exists \( u \in a^m b^i a^k b^j a^m \) with \( u \in S \) if and only if \( k-j \geq \frac{k-1}{2} \) (in which case \( j \leq \frac{k-1}{2} \)). In either case, all exponents are known and we have uniquely reconstructed \( w \).

The difficulty in proving Conjecture 24 seems to arise from the fact that, for different pairs of words \( w, w' \in \Sigma_{sb} \), the set of scattered factors which distinguish them, namely the symmetric difference of \( \text{ScatFact}_k(w) \cap \Sigma_{sb}^k \) and \( \text{ScatFact}_k(w') \cap \Sigma_{sb}^k \) (for appropriate \( k \)), varies considerably, unlike with the proof(s) of Proposition 23 where the set of distinguishing scattered factors is always made up words of the same form, regardless of the choice of \( w \) and \( w' \). As an example, consider the words \( w = ababab, w' = bababa, \) and \( w'' = ababba \). Then the symmetric difference of \( \text{ScatFact}_4(w) \cap \Sigma_{sb}^4 \) and \( \text{ScatFact}_4(w') \cap \Sigma_{sb}^4 \) is \{aabb, bbaa\}. On
the other hand, considering \( \text{ScatFact}_4(w') \cap \Sigma^4_{sb} \) and \( \text{ScatFact}_4(w'') \cap \Sigma^4_{sb} \), the symmetric difference is \( \{baab\} \).

6 Conclusions

We have considered properties of \( k \)-spectra of strictly balanced words. In particular, in Section 3 we give several insights into the structure of the set of all \( k \)-spectra of strictly balanced words of length \( 2k \) by considering for which numbers \( n \) there exists \( w \) such that the \( k \)-spectrum of \( w \) has cardinality \( n \). In particular, we characterise the first two gaps in the possibilities for each \( k \) which are regular (in the sense that the first and second gaps are always from \( k + 2 \) to \( 2k - 1 \) and \( 2k + 1 \) to \( 3k - 4 \) (inclusive). On the other hand, we see that the third gap is considerably less regular and thus resists a natural characterisation.

In Section 4, we consider the task of reconstructing strictly balanced words from their \( k \)-spectra. We note that this is, in a sense, as hard as in the general case, however, we also conjecture that even if we consider only the scattered factors which are also strictly balanced, then the situation remains the same, in the sense that it can be achieved for the same choices of \( k \). Resolving this conjecture appears to require some new approach however since the techniques for the general case are not easily adapted.

As mentioned at the end of Section 3 some of the strictly balanced words are \( \theta \)-palindromes. Since the \( \theta \)-palindromes of length \( 2k \) are constructible from the ones of length \( 2(k - 1) \) (except for each even \( k \) exactly one \( \theta \)-palindrome) we surmised that the structure and properties propagate. Moreover we expected that the knowledge of the word’s second half helps in finding the cardinalities of the \( k \)-spectra. Nevertheless we were only able to get results for \( \theta \)-palindromes in the same manner as for the other words, but we still believe that the structure of the \( \theta \)-palindromes can reveal more insights with further work.

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Fig. 2. On the left hand side the lower part of possible cardinalities is shown. The numbers surrounded by the thicker borders are proven in the paper, the others are calculated by a computer program. The both grey areas are the first and the second gap. On the right hand side the upper part is shown, where the grey area is a conjecture. There the line indicates a jump of unknowns and then the first (again as a result of a computer program) and the last missing number is given. In both figures grey numbers indicate that there is not a $k$-spectrum of this cardinality and the black ones are reached.

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