CONFORMAL FOURTH-RANK GRAVITY

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Abstract. We consider the consequences of describing the metric properties of space-time through a quartic line element $ds^4 = G_{\mu\nu\lambda\rho}dx^\mu dx^\nu dx^\lambda dx^\rho$. The associated "metric" is a fourth-rank tensor $G_{\mu\nu\lambda\rho}$. We construct a theory for the gravitational field based on the fourth-rank metric $G_{\mu\nu\lambda\rho}$ which is conformally invariant in four dimensions. In the absence of matter the fourth-rank metric becomes of the form $G_{\mu\nu\lambda\rho} = g(\mu\nu g_{\lambda\rho})$ therefore we recover a Riemannian behaviour of the geometry; furthermore, the theory coincides with General Relativity. In the presence of matter we can keep Riemannianicity, but now gravitation couples in a different way to matter as compared to General Relativity. We develop a simple cosmological model based on a FRW metric with matter described by a perfect fluid. Our field equations predict that the entropy is an increasing function of time. For $k_{obs} = 0$ the field equations predict $\Omega \approx 4y$, where $y = \frac{\rho}{p}$, for $\Omega_{small} = 0.01$ we obtain $y_{pred} = 2.5 \times 10^{-3}$. $y$ can be estimated from the mean random velocity of typical galaxies to be $y_{random} = 1 \times 10^{-5}$. For the early universe there is no violation of causality for $t > t_{class} \approx 10^{19}t_{planck} \approx 10^{-24}s$.

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"The next case in simplicity includes those manifoldnesses in which the line-element may be expressed as the fourth-root of a quartic differential expression."

B. Riemann, 1854

1. Introduction

If we adopt the materialist vision that the physical world is an objective reality then, necessarily, our geometrical conception of the universe is limited by our psychological perception of it. There is in fact a self-consistency in that physical laws generate the very mathematics necessary to make those laws understandable. In other words, we can conceive what nature allows us to conceive. In the scale of distances of our daily life, i.e., distances much greater than the Planck length, the universe behaves quite smoothly and one hopes that this behaviour might be extrapolated to very large, cosmological, and also to very small, even subnuclear, distances. This smooth behaviour would allow the universe to be mathematically modeled by a differentiable manifold. Of course, the very concept of a differentiable manifold is possible only because our perception of space allows us to conceive it, and one can wonder how our mathematical conceptions are restricted by this kind of anthropic principle.

It seems that the problem of determining the geometry realised in nature was first addressed by Riemann in his famous, but little read, thesis in 1854. He pointed out that this geometry has to be determined by purely empirical, experimental and observational, means and cannot be decided upon a priori. The first indirect statements about themetrical properties of our universe can be found in the Pythagoras theorem which, in a modern language, is equivalent to Riemannian geometry

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu.$$ (1.1)

The only thing we can try to understand now is the Riemannian, or Pythagorean, nature of the geometry. Here we take recourse to the classical argumentation by Riemann. The infinitesimal element of distance $ds$ should be a function of the coordinates $x$’s and their differentials $dx$’s

$$ds = (x, dx).$$ (1.2)

This function must satisfy the single requirement

$$f(x, \lambda dx) = |\lambda| f(x, dx).$$ (1.3)

Of course, the possibilities are infinitely many. Let us restrict our considerations to monomial functions

$$ds = (G_{\mu_1...\mu_r} dx^{\mu_1} ... dx^{\mu_r})^{1/r}.$$ (1.4)

In order for this quantity to satisfy (1.3) $r$ must be an even number. The simplest choice is $r = 2$, which corresponds to Riemannian geometry.
As pointed out by Riemann, the next possibility is $r = 4$. In this case the line element is given by

$$ds^4 = G_{\mu\nu\lambda\rho} \, dx^\mu \, dx^\nu \, dx^\lambda \, dx^\rho. \quad (1.5)$$

Riemann went no farther in exploring the above geometry, and gave no justification for that omission. Of course, at first sight, a space with a line element of the form (1.5) may seem bizarre. However, such geometry cannot be excluded \textit{a priori} and its exclusion must be done in a mathematically educated way.

This was partially done by Helmholtz.\textsuperscript{2} He showed that the existence of rigid bodies, which do not change their shapes and therefore the metric relations under translations and rotations, leaves us with Riemannian geometry as the only possibility. The Helmholtz result seemed quite satisfactory and therefore no more concern for higher-rank geometries appeared. It seems that the arrival of General Relativity, with its underlying Riemannian geometry, caused this important problem to be forgotten. However, the problem merits further attention, not only from a mathematical point of view, but also for the applications it found in theoretical and mathematical physics. It is here that the introductory considerations come into play. In fact, the difficulty of conceiving geometries other than the Riemannian limited their developments. We are therefore going to develop that chapter of differential geometry in which Riemann and Helmholtz stopped their scientific enquiries.

To close these historical comments. An indirect verification of the Riemannian structure of the universe at our daily life scales was performed by Gauss\textsuperscript{3} in 1826. The experiment was intended to verify departures from flatness, but as a side result he also verified no departures from Riemannianicity.

At the scale of distances of our daily life fourth-rank geometry is not realised in nature and the only place where it can play some role is in high-energy, or short distances, physics. In fact, at high energies, a regime to which we do not have direct experimental access, the very concept of rigid body may be no longer valid and the Helmholtz argumentation no longer applicable.

The natural question now is: why we would like to work with fourth-rank geometry, and no other of the infinitely many possible generalisations of Riemannian geometry, to describe the physics at high energies. The answer is provided by experiments, such as deep inelastic scattering, which show that, at very high energies, physical processes are scale, or conformally, invariant. Therefore, high-energy physics is associated to a geometry exhibiting, in a model independent way, conformal invariance in 4 dimensions. In another work\textsuperscript{4} we show that the critical dimension, for which field theories are integrable, is equal to the rank of the metric. Therefore, if we want to construct an integrable field theory in 4 dimensions showing agreement with the observed conformal invariance at high energies we must take recourse to fourth-rank geometry. This result also explains why, if one relies only on Riemannian geometry, integrable conformal models can be constructed only in 2 dimensions (strings).

We arrive therefore to the following scheme: at short distances, high-energies, the geometry is of fourth-rank while at large distances, low energies, the geometry is of second-rank, Riemannian. It is clear furthermore that the Riemannian behaviour of the geometry must be recovered as the low-energy limit of the high-energy theory. This would be possible if at low-energies the fourth-rank metric tensor $G_{\mu\nu\lambda\rho}$ becomes of the form
\[ G_{\mu\nu\lambda\rho} = g_{(\mu\nu} g_{\lambda\rho)} \cdot \tag{1.6a} \]

In this case the line element factors and one is back to the Riemannian case

\[ ds^4 = (ds^2)^2. \tag{1.6b} \]

Our next task is to construct a geometric invariant to be used as the Lagrangian describing the dynamics of the geometry, i.e., of the gravitational field. From the metric alone it is impossible to construct any invariant, apart from the trivial solution: a constant. Therefore, we must take recourse to a further geometrical object: the Ricci tensor for an arbitrary connection \( \Gamma^\lambda_{\mu\nu} \)

\[ R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\lambda\mu} + \Gamma^\lambda_{\lambda\sigma} \Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\lambda\nu}. \tag{1.7} \]

The simplest invariants which can be constructed with the metric \( G_{\mu\nu\lambda\rho} \) and the Ricci tensor \( R_{\mu\nu} \) are

\[ \langle R^2 \rangle = G^{\mu\nu\lambda\rho} R_{\mu\nu} R_{\lambda\rho}, \]

\[ \langle R^4 \rangle = G^{\mu\nu\lambda\rho} G^{\alpha\beta\gamma\delta} R_{\mu\alpha} R_{\nu\beta} R_{\lambda\gamma} R_{\rho\delta}, \quad \text{etc.} \tag{1.8} \]

The Lagrangian therefore will be of the form

\[ \mathcal{L} = L(\langle R^2 \rangle, \langle R^4 \rangle, \cdots) G^{1/4}, \tag{1.9} \]

where \( G \) is the determinant of \( G_{\mu\nu\lambda\rho} \). The scalar function \( L \) to be put in (1.9) should make the Lagrangian a conformally invariant function. Under rescalings of the metric

\[ G_{\mu\nu\lambda\rho} \rightarrow \lambda G_{\mu\nu\lambda\rho}, \tag{1.10} \]

the inverse metric \( G^{\mu\nu\lambda\rho} \) and \( G^{1/4} \) transform as

\[ G^{\mu\nu\lambda\rho} \rightarrow \lambda^{-1} G^{\mu\nu\lambda\rho}, \tag{1.11a} \]

\[ G^{1/4} \rightarrow \lambda G^{1/4}. \tag{1.11b} \]

Therefore the Lagrangian should be of the form

\[ \mathcal{L} = [\alpha \langle R^2 \rangle + \beta \frac{\langle R^4 \rangle}{\langle R^2 \rangle} + \cdots] G^{1/4}. \tag{1.12} \]

However, all the terms after the first one are highly non-local. Therefore, the only sensible solution is

\[ \mathcal{L} = \kappaCG \langle R^2 \rangle G^{1/4}, \tag{1.13} \]

where
\[ \kappa_{CG} \approx \kappa E L_{Planck}^2 = \frac{\hbar c}{8\pi}, \quad (1.14) \]

is the Einstein gravitational constant \( \kappa E = \frac{c^4}{8\pi G} \), times a constant of the order of \( L_{Planck}^2 \).

The total Lagrangian must also consider the contributions of matter. Now we must apply a Palatini-like variational principle in which the connection and the metric are varied independently. However, in all known cases of physical interest the matter Lagrangian does not depend on the affine connection.\(^5\) In this case the variation of the gravitational Lagrangian with respect to the connection leads to a metricity condition for which the solution is

\[ \Gamma^\lambda_{\mu\nu} = \{ \lambda_{\mu\nu} \} (\gamma). \quad (1.15) \]

I.e., the connection is the Christoffel symbol of the second kind of the tensor \( \gamma^{\mu\nu} \) given by

\[ \gamma^{\mu\nu} = G^{\mu\nu\lambda\rho} R_{\lambda\rho}, \quad (1.16) \]

which we have assumed to be regular. Equations (1.15) and (1.16) are a metricity condition since they give the relation between \( \Gamma^\lambda_{\mu\nu} \) and \( G_{\mu\nu\lambda\rho} \). Therefore

\[ R_{\mu\nu}(\Gamma) = R_{\mu\nu}(\gamma). \quad (1.17) \]

One easily verifies then that

\[ \langle R^2 \rangle = G^{\mu\nu\lambda\rho} R_{\mu\nu} R_{\lambda\rho} = \gamma^{\mu\nu} R_{\mu\nu}(\gamma) = R(\gamma). \quad (1.18) \]

Variation of the Lagrangian with respect to the metric \( G_{\mu\nu\lambda\rho} \) gives

\[ \kappa_{CG} [R_{(\mu\nu} R_{\lambda\rho)} - \frac{1}{4} \langle R^2 \rangle G_{\mu\nu\lambda\rho}] = T_{\mu\nu\lambda\rho}, \quad (1.19) \]

where \( T_{\mu\nu\lambda\rho} \) is the energy-momentum tensor of matter, to be defined below.

The field equations (1.19) exhibit three energy regimes: low, medium, and high. In the low-energy regime there is no matter and therefore the fourth-rank metric is separable, \( G_{\mu\nu\lambda\rho} = g_{(\mu\nu} g_{\lambda\rho)} \), as can be read from (1.19). Then the line element would factor, as in (1.6b), and one would be back to the Riemannian case. In the medium-energy regime the geometry is still Riemannian, \( G_{\mu\nu\lambda\rho} = g_{(\mu\nu} g_{\lambda\rho)} \), but there is matter involved in the game. This possibility is not excluded as a closer analysis of eqs. (1.19) reveals. In this case the gravitational field couples in a different way, as compared to General Relativity, to matter. Lastly, we have the true high-energy regime in which there is matter and the geometry is truly fourth-rank.

Let us further analyse these energy regimes. In vacuum, the field equations (1.19) are equivalent to

\[ R_{\mu\nu}(\gamma) - \frac{1}{4} R(\gamma) \gamma_{\mu\nu} = 0. \quad (1.20) \]

For a spherically symmetric field the solution is the Kottler metric\(^6\) which contains the Schwarzschild solution as a special case. Therefore the predictions based on the
Schwarzschild metric, which agree with observation by 1 per cent or better, will be con-
tained in this theory.

The large scale geometry of the universe seems to be Riemannian and, since there is
matter present in it, this corresponds to the medium-energy regime mentioned above. In
this context we develop a cosmological model based on the Friedman-Robertson-Walker
metric coupled to cosmic matter described by a perfect fluid.

The theory predicts an increasing total entropy such that the expansion of the universe
is an adiabatic non-isentropic process. Therefore, the evolution of the universe, in the
framework of fourth-rank cosmology is, as expected on physical grounds, an irreversible
process.

For \( k_{\text{obs}} = 0 \), as imposed by the observed flatness of the universe, the field equations
give, for the present Universe,

\[
\Omega = \frac{4y}{1 - 4y - y^2},
\]

where \( y = \frac{\rho}{\rho} \). For \( \Omega_{\text{small}} = 0.01 \) [7] we obtain \( y_{\text{pred}} \approx 2.5 \times 10^{-3} \) which corresponds to
an almost pressureless perfect fluid. This must be compared with the observed value of \( \frac{\rho}{\rho} \),
which can be determined from the mean random velocity of typical galaxies and is given
by \( y_{\text{random}} = 1 \times 10^{-5} \). Therefore, our prediction differs by two orders of magnitude with
respect to the observed value. We hope to improve this situation since the estimation
of \( y \) from the random motion of galaxies is a quite rough one. Furthermore, eq. (1.21)
was obtained under the strong assumption that \( y \) behaves like a constant. Therefore,
there are hopes that this theory shows a better agreement with the observed values of the
cosmological parameters.

For the early universe we find that causality is not violated for \( t > t_{\text{class}} \approx 10^{19}t_{\text{Planck}} \)
\( \approx 10^{-24} \) s. At earlier times quantum mechanical effects dominate the scene. In fact,
the radius of the universe is exactly the Compton wavelength associated to its mass.
Our classical approach breaks down so that the very concept of causality is meaningless.
Therefore, there is no violation of causality, or horizon problem.

Some final introductory comments. It is a popular view that the gravitational field is
correctly described by General Relativity. This is true of the pure gravitational field, i.e.,
when no coupling to matter, or other fields, is present. In fact, Einstein field equations are
in excellent agreement, 1 per cent or better, with observation when applied, for example, to
the solar system. However, when matter is coupled to gravity the observational agreement
is not so good. This is the case when General Relativity is applied to cosmology where the
gravitational field get coupled to cosmic matter described by a perfect fluid. One obtains
qualitatively good predictions, as the evolution of the universe from an initial singularity
and some good quantitative predictions as the temperature of the microwave background
and the relative abundance of elements. However, the quantitative agreement is weaker
in other aspects. In fact, flatness, \( k_{\text{obs}} = 0 \), implies \( \Omega_{GR} = 1 \), which is hardly observed.
Furthermore, the Standard Model of Cosmology predicts a constant entropy, something
which is difficult to accept on physical grounds. These are some of the reasons to look for
an improved theory for the gravitational field.

In previous works\(^{10,11,12}\) we developed a similar model based on the Lagrangian
\[ \mathcal{L} = \kappa_E \langle R^2 \rangle^{1/2} G^{1/4}. \]  

This Lagrangian was chosen in order to have only the Einstein gravitational constant for dimensional purposes. Later on we became convinced that the appearance of $\hbar$ in the Lagrangian (1.13) creates no conflict between the classical character of the Lagrangian and the quantum origin of $\hbar$.

The paper is organised as follows: In Section 2 we start by giving some mathematical considerations. In Section 3 we develop the fundamentals of fourth-rank gravity. In Section 4 we consider the low energy regime and the Schwarzschild solution. In Section 5 we apply fourth-rank gravity to cosmology. Section 6 study the high-energy regime and the coupling to conformal matter. Section 7 is dedicated to the conclusions. The Appendices A, B and C, collect some standard results on Cosmography, General Relativity and the Standard Model of Cosmology, respectively.

To our regret, due to the nature of this approach, in the Appendices we must bore the reader by exhibiting some standard and well known results, but this is necessary in order to illustrate where the new approach departs from the standard one.

2. Mathematical Preliminaries. Differentiable Manifolds

Here we consider some elementary results for differentiable manifolds. Let us start by considering the metric properties, which are related to the way in which distances are measured. In what follows we take recourse to the classical argumentation by Riemann.\(^1\)

Let $M$ be a $d$-dimensional differentiable manifold, and let $x^\mu, \mu = 0, \ldots, d - 1$, be local coordinates. The infinitesimal element of distance $ds$ should be a function of the coordinates $x$ and their differentials $dx$’s

\[ ds = f(x, dx), \]  

which is homogeneous of the first-order in $dx$’s

\[ f(x, \lambda dx) = \lambda f(x, dx), \]  

for $\lambda > 0$, and is positive definite

\[ f \geq 0. \]  

Condition (2.2b1) was written in a time in which distances were, so to say, positive. However, with the arrival of General Relativity one got used to line elements with undefined signature. Condition (2.2b1) was there to guarantee the invariance under the change $dx \rightarrow -dx$, i.e., to assure that distances measured when going in one direction are the same as measured when going in the opposite direction. Therefore, we can replace (2.2b1) by the weaker condition

\[ f(x, -dx) = f(x, dx). \]  

Conditions (2.2a) and (2.2b2) can now we resumed into the single condition
\[ f(x, \lambda dx) = |\lambda| f(x, dx), \quad (2.2) \]

with no restriction over the sign of \( \lambda \).

Of course the possible solutions to (2.2) are infinitely many. Let us restrict our considerations to monomial functions. Then we will have

\[ ds = (G_{\mu_1\ldots\mu_r}(x) dx^\mu_1 \ldots dx^\mu_r)^{1/r}. \quad (2.3) \]

In order for this quantity to be positive definite \( r \) must be an even number.

The simplest choice is \( r = 2 \)

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2.4) \]

which corresponds to Riemannian geometry. The coefficients \( g_{\mu\nu} \) are the components of the covariant metric tensor. The determinant of the metric is defined by

\[ g = \frac{1}{d!} \varepsilon^{\mu_1\ldots\mu_d} \varepsilon_{\nu_1\ldots\nu_d} g_{\mu_1\nu_1} \cdots g_{\mu_d\nu_d}. \quad (2.5) \]

If \( g \neq 0 \) we can define the inverse metric by

\[ g^{\mu\nu} = \frac{1}{(d-1)!} \frac{1}{g} \varepsilon^{\mu_1\ldots\mu_{d-1}} \varepsilon_{\nu_1\ldots\nu_{d-1}} g_{\mu_1\nu_1} \cdots g_{\mu_{d-1}\nu_{d-1}}, \quad (2.6) \]

and satisfies

\[ g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu. \quad (2.7) \]

Densities of weight one can be constructed in terms of the quantity \( g^{1/2} \).

As pointed out by Riemann, the next possibility is \( r = 4 \). In this case the line element is given by

\[ ds^4 = G_{\mu\nu\lambda\rho} dx^\mu dx^\nu dx^\lambda dx^\rho. \quad (2.8) \]

The coefficients \( G_{\mu\nu\lambda\rho} \) are the components of a covariant fourth-rank tensor. Since it is related to the metric properties of the given manifold it is not an error to call it a “metric”. The determinant of the metric \( G_{\mu\nu\lambda\rho} \) is defined as

\[ G = \frac{1}{d!} \varepsilon^{\mu_1\ldots\mu_d} \cdots \varepsilon_{\rho_1\ldots\rho_d} G_{\mu_1\nu_1\lambda_1\rho_1} \cdots G_{\mu_d\nu_d\lambda_d\rho_d}. \quad (2.9) \]

where the \( \varepsilon \)'s can be chosen as the usual completely antisymmetric Levi-Civita symbols. If \( G \neq 0 \) we can define the inverse metric by

\[ G^{\mu\nu\lambda\rho} = \frac{1}{(d-1)!} \frac{1}{G} \varepsilon^{\mu_1\ldots\mu_{d-1}} \cdots \varepsilon_{\rho_1\ldots\rho_{d-1}} G_{\mu_1\nu_1\lambda_1\rho_1} \cdots G_{\mu_{d-1}\nu_{d-1}\lambda_{d-1}\rho_{d-1}}. \quad (2.10) \]

This inverse metric satisfies the relations
\[ G^{\mu \alpha \beta \gamma} G_{\nu \alpha \beta \gamma} = \delta^\mu_\nu. \] (2.11)

That eq. (2.11) holds true for \( G^{\mu \nu \lambda \rho} \) as defined in (2.10) can be verified by hand in the two-dimensional case and with computer algebraic manipulation for three and four dimensions.\(^8\) Now, densities of weight one can be constructed in terms of the quantity \( G^{1/4} \).

It is clear that fourth-rank geometry is observationally excluded at the scale of distances of our daily life. However, a Riemannian behaviour can be obtained for separable spaces. A space is said to be separable if \( G_{\mu \nu \lambda \rho} \) is of the form

\[ G_{\mu \nu \lambda \rho} = \frac{1}{3} (g_{\mu \nu} g_{\lambda \rho} + g_{\mu \lambda} g_{\nu \rho} + g_{\mu \rho} g_{\nu \lambda}). \] (2.12)

In this case formula (2.8) reduces to (2.4). Separable metrics can also be used as a quality control of later formal developments. In fact, all the results and developments obtained for a generic metric \( G_{\mu \nu \lambda \rho} \) must reduce to those for Riemannian geometry when applied to separable metrics.

In the case of a separable metric the determinant and the inverse metric are given by

\[ G = g^2, \] (2.13a)

\[ G^{\mu \nu \lambda \rho} = \frac{3}{d+2} g^{(\mu \nu} g^{\lambda \rho)}. \] (2.13b)

Let us finish this Section with some considerations on the curvature properties of manifolds. Curvature properties are described by the curvature tensor

\[ R^\lambda_{\rho \mu \nu} = \partial_\mu \Gamma^\lambda_{\nu \rho} - \partial_\nu \Gamma^\lambda_{\rho \mu} + \Gamma^\lambda_{\mu \sigma} \Gamma^\sigma_{\nu \rho} - \Gamma^\lambda_{\nu \sigma} \Gamma^\sigma_{\rho \mu}, \] (2.14)

constructed in terms of a connection \( \Gamma^\lambda_{\mu \nu} \).

The metric and the connection are, in general, independent objects. They can be related through a metricity condition. In Riemannian geometry the metricity condition reads

\[ \nabla_\lambda g_{\mu \nu} = \partial_\lambda g_{\mu \nu} - \Gamma^\rho_{\lambda \mu} g_{\rho \nu} - \Gamma^\rho_{\nu \lambda} g_{\mu \rho} = 0. \] (2.15)

The number of unknowns for a symmetric connection \( \Gamma^\lambda_{\mu \nu} \) and the number of equations (2.15) are the same, \( \frac{1}{2} d^2(d + 1) \). Therefore, since this is an algebraic linear system, the solution is unique and is given by the familiar Christoffel symbols of the second kind

\[ \Gamma^\lambda_{\mu \nu} = \{^\lambda_{\mu \nu}\} (g) = \frac{1}{2} g^{\lambda \rho} (\partial_\mu g_{\nu \rho} + \partial_\nu g_{\mu \rho} - \partial_\rho g_{\mu \nu}). \] (2.16)

Therefore, in Riemannian geometry one can talk of the curvature properties of a metric \( g_{\mu \nu} \). This can be done because there exist a natural connection, the Christoffel symbol of the second kind, in terms of which we can construct a curvature tensor.

In the case of a fourth-rank metric a condition analogous to (2.15) would read
\[ \nabla_\mu G_{\alpha\beta\gamma\delta} = \partial_\mu G_{\alpha\beta\gamma\delta} - \Gamma^\nu_{\mu\alpha} G_{\nu\beta\gamma\delta} - \Gamma^\nu_{\mu\beta} G_{\alpha\nu\gamma\delta} - \Gamma^\nu_{\mu\gamma} G_{\alpha\beta\nu\delta} - \Gamma^\nu_{\mu\delta} G_{\alpha\beta\gamma\nu} = 0. \] (2.17)

However, in this case, the number of unknowns \( \Gamma^\lambda_{\mu\nu} \) is, as before, \( \frac{1}{2} d^2 (d + 1) \), while the number of equations is

\[ \frac{1}{24} d^2 (d + 1) (d + 2) (d + 3) > \frac{1}{2} d^2 (d + 1). \] (2.18)

Therefore the system is overdetermined and some differentio-algebraic conditions must be satisfied by the metric. Since, in general, such restrictions will not be satisfied by a generic metric, one must deal with \( \Gamma^\lambda_{\mu\nu} \) and \( G_{\mu\nu\lambda\rho} \) as independent objects. Therefore, for physical applications, the connection and the metric must be considered as independent fields.

A metricity condition can be imposed consistently only if the number of independent components of the metric is less than that naively implied by (2.18). The maximum acceptable number of independent components is \( \frac{1}{2} d(d + 1) \). This can be achieved, for instance, if the metric is a separable one. Furthermore, one can verify that in this case the metricity condition (2.17) reduces to the usual metricity condition (2.15) for the metric \( g_{\mu\nu} \) and therefore \( \Gamma^\lambda_{\mu\nu} \) is precisely that for Riemannian geometry, i.e., the Christoffel symbol of the second kind.

3. Conformal Fourth-Rank Gravity

In this Section we develop a theory for the gravitational field based on fourth-rank geometry. The use of fourth-rank geometry is motivated by the following considerations. At very high energies the masses of particles involved in physical processes become negligible as compared to the energies, in fact they can be set equal to zero. Therefore, there is no fundamental mass setting the scale of energies, and all physical processes must be scale invariant. This speculation is confirmed by experiments, such as deep inelastic scattering, which show that, in fact, at very high energies, physical processes are scale invariant. It can furthermore be shown, from a mathematical point of view, that scale invariance is equivalent to conformal invariance. Therefore, high-energy physics is associated to a geometry exhibiting, in a model independent way, conformal invariance in 4 dimensions. In another work, we show that the critical dimension, for which field theories are integrable, is equal to the rank of the metric. Therefore, if we want to construct a field theory in 4 dimensions showing agreement with the observed conformal invariance at high energies we must take recourse to fourth-rank geometry.

We arrive therefore to the following scheme: at short distances, high-energies, the geometry is of fourth-rank while at large distances, low energies, the geometry is of second-rank, Riemannian. It is clear furthermore that the Riemannian behaviour of the geometry must be recovered as the low-energy limit of the high-energy theory. This would be possible if at low-energies the fourth-rank metric tensor \( G_{\mu\nu\lambda\rho} \) becomes separable. In this case the line element factors and one is back to the Riemannian case. This would explain why the universe, even when described by a fourth-rank metric, looks Riemannian at large, low
energy, scales. The problem is now to obtain this Riemannian behaviour as the low-energy regime of some field theory.

The conformal invariance requirement determines, almost uniquely the geometrical invariant to be used as Lagrangian. The field equations exhibit three energy regimes: low, medium, and high. In the low-energy regime there is no matter and the fourth-rank metric is separable, $G_{\mu\nu\lambda\rho} = g_{(\mu\nu}g_{\lambda\rho)}$. Then the line element factors and one is back to the Riemannian case. In the medium-energy regime the geometry is still Riemannian, $G_{\mu\nu\lambda\rho} = g_{(\mu\nu}g_{\lambda\rho)}$, but there is matter involved in the game. In this case the gravitational field couples in a different way, as compared to General Relativity, to matter. Lastly, we have the true high-energy regime in which there is matter and the geometry is truly fourth-rank. These energy regimes, and their observational consequences, are further analysed in Section 4, 5, and 6.

### 3.1 Fourth-Rank Gravitational Equations

As in General Relativity, in order to describe the dynamics of the gravitational field we need to construct a geometrical invariant. From the metric alone it is impossible to construct any invariant, apart from the trivial solution: a constant. Therefore we must take recourse to another geometrical object. The necessary object is the Ricci tensor for an arbitrary connection $\Gamma^\lambda_{\mu\nu}$, which is obtained as a contraction of the Riemann tensor, defined in (2.14),

$$ R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = \partial_{\lambda}\Gamma^\lambda_{\mu\nu} - \partial_{\nu}\Gamma^\lambda_{\lambda\mu} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\lambda\nu}. \tag{3.1} $$

The simplest invariants which can be constructed with the metric $G_{\mu\nu\lambda\rho}$ and the Ricci tensor $R_{\mu\nu}$ are

$$ \langle R^2 \rangle = G^{\mu\nu\lambda\rho} R_{\mu\nu} R_{\lambda\rho}, $$

$$ \langle R^4 \rangle = G^{\mu\nu\lambda\rho} G^{\alpha\beta\gamma\delta} R_{\mu\alpha} R_{\nu\beta} R_{\lambda\gamma} R_{\rho\delta}, \text{ etc.} \tag{3.2} $$

The Lagrangian therefore will be of the form

$$ \mathcal{L} = L(\langle R^2 \rangle, \langle R^4 \rangle, \ldots) G^{1/4}, \tag{3.3} $$

where $G$ is the determinant of $G_{\mu\nu\lambda\rho}$. The scalar function $L$ to be put in (3.3) should make the Lagrangian a conformally invariant function. Under rescalings of the metric

$$ G_{\mu\nu\lambda\rho} \rightarrow \lambda G_{\mu\nu\lambda\rho}, \tag{3.4} $$

the inverse metric $G^{\mu\nu\lambda\rho}$ and $G^{1/4}$ transform as

$$ G^{\mu\nu\lambda\rho} \rightarrow \lambda^{-1} G^{\mu\nu\lambda\rho}, \tag{3.5a} $$

$$ G^{1/4} \rightarrow \lambda G^{1/4}. \tag{3.5b} $$
Therefore the Lagrangian should be of the form
\[ L = [\alpha \langle R^2 \rangle + \beta \langle R^4 \rangle + \cdots] G^{1/4}. \] (3.6)
However, all the terms after the first one, are highly non-local. Therefore, the only sensible solution is
\[ L_{CG} = \kappa_{CG} \langle R^2 \rangle G^{1/4}, \] (3.7)
where the coupling constant
\[ \kappa_{CG} \approx \kappa_E L_{Planck}^2 = \frac{\hbar c}{8\pi}, \] (3.8)
is the Einstein gravitational constant \( \kappa_E = \frac{c^4}{8\pi G} \), times a constant of the order of \( L_{Planck}^2 \).

The above is the analogue of the Palatini Lagrangian for General Relativity. But now, since there is no metricity condition, a Lagrangian analogous to the Einstein-Hilbert one simply does not exist.

The total Lagrangian must consider also the contributions of matter and is given by
\[ L = L_{CG} + L_{\text{matter}}. \] (3.9)
Variation with respect to the connection gives
\[ \frac{\delta L}{\delta \Gamma^\lambda_{\mu\nu}} = \frac{\delta L_{CG}}{\delta \Gamma^\lambda_{\mu\nu}} + \frac{\delta L_{\text{matter}}}{\delta \Gamma^\lambda_{\mu\nu}} = 0, \] (3.10)
where
\[ \frac{\delta L_{CG}}{\delta \Gamma^\lambda_{\mu\nu}} = \frac{\partial L_{CG}}{\partial \Gamma^\lambda_{\mu\nu}} - \frac{\partial}{\partial \rho} \left( \frac{\partial L_{CG}}{\partial (\partial_\rho \Gamma^\lambda_{\mu\nu})} \right) \]
\[ = \gamma^{\alpha\beta} \left[ \frac{1}{2} (\delta_\lambda^\nu \Gamma^\mu_{\alpha\beta} + \delta_\lambda^\mu \Gamma^\nu_{\alpha\beta}) + \delta_\mu^\alpha \delta_\nu^\beta \Gamma^\sigma_{\lambda\sigma} - \delta_\beta^\nu \Gamma^\mu_{\lambda\alpha} - \delta_\beta^\mu \Gamma^\nu_{\lambda\alpha} \right] G^{1/4} \]
\[ - \frac{d}{d\rho} \left[ \gamma^{\alpha\beta} \left( \delta^\rho_\lambda \delta^\mu_\alpha \delta^\nu_\beta - \frac{1}{2} \delta^\rho_\beta \left( \delta^\mu_\lambda \delta^\nu_\alpha + \delta^\nu_\lambda \delta^\mu_\alpha \right) \right) \right] G^{1/4}. \] (3.11)
with
\[ \gamma^{\alpha\beta} = G^{\alpha\beta\gamma\delta} R_{\gamma\delta}, \] (3.12)
(for simplicity, we have omitted \( \kappa_{CG} \)). In all known cases of physical interest the matter Lagrangian does not depend on the connection. Therefore the second term in (3.10) vanishes and one remains with a metricity condition which has the solution
\[ \Gamma^\lambda_{\mu\nu} = \{^\lambda_{\mu\nu}\}(\gamma) = \frac{1}{2} \gamma^{\lambda\rho} (\partial_\mu \gamma_{\nu\rho} + \partial_\nu \gamma_{\rho\mu} - \partial_\rho \gamma_{\mu\nu}), \] (3.13)
i.e., the connection is the Christoffel symbol of the second kind for the tensor $\gamma_{\mu\nu}$, which we have assumed to be regular. We can therefore write

$$R_{\mu\nu}(\Gamma) = R_{\mu\nu}(\gamma).$$

(3.14)

Furthermore

$$\langle R^2 \rangle = G_{\mu\nu\lambda\rho} R_{\mu\nu} R_{\lambda\rho} = \gamma_{\mu\nu} R_{\mu\nu}(\gamma) = R(\gamma).$$

(3.15)

Variation with respect to $G_{\mu\nu\lambda\rho}$

$$\frac{\delta \mathcal{L}}{\delta G_{\mu\nu\lambda\rho}} = \frac{\partial \mathcal{L}}{\partial G_{\mu\nu\lambda\rho}} = d_\sigma \left( \frac{\partial \mathcal{L}}{\partial (\partial_\sigma G_{\mu\nu\lambda\rho})} \right) = 0,$$

(3.16)

gives

$$\kappa_{CG} [R_{(\mu\nu} R_{\lambda\rho)} - \frac{1}{4} \langle R^2 \rangle G_{\mu\nu\lambda\rho}] = T_{\mu\nu\lambda\rho}.$$

(3.17)

where

$$T_{\mu\nu\lambda\rho} = -G^{-1/4} \frac{\delta \mathcal{L}_{\text{matter}}}{\delta G_{\mu\nu\lambda\rho}},$$

(3.18)
is the fourth-rank energy-momentum tensor.

More information can be obtained from eq. (3.17) by observing that the energy-momentum tensor must decompose into one part proportional to the metric and another part which is a separable tensor. In order to accommodate all the symmetries is necessary to have

$$T_{\mu\nu\lambda\rho} = \frac{L_{\text{Planck}}}{\kappa_{CG}} [S_{4,\mu\nu} S_{4,\lambda\rho} - \frac{1}{4} \langle S_4^2 \rangle G_{\mu\nu\lambda\rho}],$$

(3.19)

where

$$\langle S_4^2 \rangle = G_{\mu\nu\lambda\rho} S_{4,\mu\nu} S_{4,\lambda\rho}.$$  

(3.20)

In this case the field equations reduce to the simple form

$$\kappa_E R_{\mu\nu}(\gamma) = \pm S_{4,\mu\nu};$$

(3.21)

and, as a further consequence we have

$$\kappa_E R(\gamma) = \kappa_E \langle R^2 \rangle = \langle S_4^2 \rangle = S_4^2(\gamma),$$

(3.22)

where $S_4(\gamma) = \gamma_{\mu\nu} S_{4,\mu\nu}(\gamma)$. One would be tempted to replace $S_{4,\mu\nu}$ by the reduced energy-momentum tensor appearing in (B.14). However, that tensor is derived from a Lagrangian containing a metric $g_{\mu\nu}$, an object which is, in principle, absent in fourth-rank geometry. Concerning the $\pm$ sign in (3.21), this must be determined by taking recourse to some application, as will be done in Section 4.
3.2 The Different Energy Regimes

The field equations (3.17) exhibit three energy regimes: low, medium and high. In the low-energy regime there is no matter and therefore the geometry is Riemannian, $G_{\mu\nu\lambda\rho} = g_{(\mu\nu}g_{\lambda\rho)}$, as can be read from (3.17). In this case the field equations do not reduce to the Einstein field equations in vacuum. In the medium-energy regime the geometry is still Riemannian, $G_{\mu\nu\lambda\rho} = g_{(\mu\nu}g_{\lambda\rho)}$, but now there is matter in the game. This possibility is not excluded as a closer analysis of eqs. (3.17) reveals. Finally, we have the true high-energy regime in which there is matter and the geometry is truly fourth-rank.

3.2.1. The Low-Energy Regime

In the low-energy regime $\mathcal{L}_{\text{matter}} = 0$ and then the field equations reduce to

$$R_{\mu\nu}R_{\lambda\rho} - \frac{1}{4} \langle R^2 \rangle G_{\mu\nu\lambda\rho} = 0.$$  

(3.23)

The only sensible solution is

$$G_{\mu\nu\lambda\rho} = g_{(\mu\nu}g_{\lambda\rho)},$$  

(3.24a)

$$R_{\mu\nu} = \frac{1}{2} \langle R^2 \rangle^{1/2} g_{\mu\nu},$$  

(3.24b)

and therefore the geometry is Riemannian.

The tensor $\gamma^{\mu\nu}$ is given by

$$\gamma^{\mu\nu} = \frac{1}{2} \langle R^2 \rangle^{1/2} g^{\mu\nu},$$  

(3.25)

$$\gamma_{\mu\nu} = 2 \langle R^2 \rangle^{-1/2} g_{\mu\nu} = 2 R^{-1/2} g_{\mu\nu}.$$  

(3.26)

Then, eq. (3.24b) is rewritten as

$$R_{\mu\nu}(\gamma) - \frac{1}{4} R(\gamma) g_{\mu\nu} = 0.$$  

(3.27)

One must therefore compute equations (3.27) for a tensor $\gamma_{\mu\nu}$, and then the physical metric $g_{\mu\nu}$ is obtained from (3.26).

Let us furthermore observe that the dimensions of $\gamma^{\mu\nu}$, $\gamma_{\mu\nu}$, $R_{\mu\nu}(\gamma g)$ and $R(\gamma)$ are given by

$$\text{dim}(\gamma^{\mu\nu}) = L^{-2},$$

$$\text{dim}(\gamma_{\mu\nu}) = L^{2},$$

$$\text{dim}(R_{\mu\nu}(\gamma)) = L^{-2},$$

$$\text{dim}(R(\gamma)) = L^{2}.$$
Let us now rewrite the field equations (3.27) in terms of the metric $g_{\mu\nu}$. Let us start by rewriting eq. (3.26) as

$$\gamma_{\mu\nu} = \lambda^2 e^\psi g_{\mu\nu}, \quad (3.29)$$

where

$$\lambda^2 e^\psi = 2 R^{-1/2}(\gamma), \quad (3.30)$$

and $\lambda$ has dimensions of length. Therefore

$$R(\gamma) = \frac{4}{\lambda^4} e^{-2\psi}. \quad (3.31)$$

The Ricci tensors are related by

$$R_{\mu\nu}(\gamma) = R_{\mu\nu}(g) + \nabla_\mu \psi_\nu - \frac{1}{2} \psi_\mu \psi_\nu + \frac{1}{2} g_{\mu\nu} (\nabla_g^2 \psi + g^{\alpha\beta} \psi_\alpha \psi_\beta), \quad (3.32)$$

while the scalar curvatures are related by

$$R(\gamma) = \frac{1}{\lambda^2} e^{-\psi} [R(g) + 3 \nabla_g^2 \psi + \frac{3}{2} g^{\alpha\beta} \psi_\alpha \psi_\beta]. \quad (3.33)$$

The field equations are rewritten as

$$R_{\mu\nu}(g) + \nabla_\mu \psi_\nu - \frac{1}{2} \psi_\mu \psi_\nu - \frac{1}{4} g_{\mu\nu} [R(g) + \nabla_g^2 \psi - \frac{1}{2} g^{\alpha\beta} \psi_\alpha \psi_\beta] = 0. \quad (3.34)$$

Combining (3.31) and (3.33) we obtain the differential equation for the conformal factor $\psi$

$$e^{-\psi} [R(g) + 3 \nabla_g^2 \psi + \frac{3}{2} g^{\alpha\beta} \psi_\alpha \psi_\beta] = \frac{4}{\lambda^2} e^{-2\psi}. \quad (3.35)$$

As mentioned in the introduction, in vacuum, General Relativity is in excellent agreement with observation. Therefore, in this regime, our theory must coincide with General Relativity. This is not evident from eqs. (3.27) and in fact they are not equivalent. Therefore, the equivalence must be established at the level of the solutions rather than of the field equations. This regime is further explored in Section 4.

3.2.2. The Medium-Energy Regime

In the medium-energy regime the metrics $G_{\mu\nu\lambda\rho}$ and $g_{\mu\nu}$ are still related by (3.24a). In this case it is therefore reasonable to replace $S_{4,\mu\nu}$ with that appearing in (B.14)
\[ \kappa E R_{\mu\nu}(\gamma) = \pm S_{2,\mu\nu}(g), \tag{3.36} \]

However, the field equations (3.36) are not equivalent to Einstein field equations since the Ricci tensor appearing here is for the tensor \( \gamma_{\mu\nu} \) and not for the metric \( g_{\mu\nu} \). The above choice is a delicate point since other mechanisms of coupling the fourth-rank geometry with "second-rank" matter can be conceived. For example one can consider

\[ \kappa E G_{\mu\nu}(\gamma) = \pm T_{2,\mu\nu}(g), \tag{3.37} \]

which is not equivalent to (3.36). We have tested this and other possibilities and we have concluded that (3.36) is the correct choice. Which sign is to be chosen in (3.36) must be decided by considering some application.

The large scale geometric structure of our universe seems to be well described by Riemannian geometry, and since matter is involved, its description belongs to medium-energy regime. We explore this possibility in Section 5.

### 3.2.3. The High-Energy Regime

In this case \( G_{\mu\nu\lambda\rho} \) is not a separable metric. Furthermore, the energy-momentum tensor of matter is traceless, a property which is equivalent to scale invariance. Therefore, scale invariance is present at very high-energies, and one can confidently consider this as the high-energy regime of the theory. This regime is further explored in Section 6.

### 4. The Low-Energy Regime

Now we explore the consequences of our field equations in the absence of matter. Our field equations do not reduce to the vacuum Einstein field equations. Therefore, the observational equivalence must be established at the level of the solutions rather than at the level of the field equations.

#### 4.1. Static Spherically Symmetric Fields

In order to check the validity of the field equations (3.27) we consider the standard test of a spherically symmetric field. The line element is given by

\[ ds^2 = A(r) \, dt^2 - B(r) \, dr^2 - r^2 \, d\Omega^2, \tag{4.1} \]

where

\[ d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2. \tag{4.2} \]

One can certainly assume that \( R(\gamma) \) will be a function of \( r \) only. The physical metric \( g_{\mu\nu} \) and the tensor \( \gamma_{\mu\nu} \) are related by
\[ \gamma_{\mu\nu} = f(R(\gamma)) g_{\mu\nu} = f(r) g_{\mu\nu}. \]  

(4.3)

Therefore, the line element associated to the tensor \( \gamma_{\mu\nu} \) is

\[ ds_{\gamma}^2 = f(r) [A(r) \, dt^2 - B(r) \, dr^2 - r^2 \, d\Omega^2], \]

(4.4)

and, by a redefinition of \( r \), it can be rewritten as in (4.1)

\[ ds_{\gamma}^2 = \lambda^2 [\bar{A}(r) \, dt^2 - \bar{B}(r) \, dr^2 - r^2 \, d\Omega^2], \]

(4.5)

where \( \lambda \) has dimensions of length.

The solution is the Kottler metric\(^6\)

\[ \gamma_{00} = \lambda^2 (a + \frac{b}{r} + cr^2), \]

\[ \gamma_{11} = - \lambda^2 (a + \frac{b}{r} + cr^2)^{-1}. \]

(4.6)

The associated scalar curvature \( R(\gamma) \) is constant

\[ R(\gamma) = 12 \frac{c}{a} \lambda^{-2}. \]

(4.7)

We can now write the field equations (3.27) in terms of the metric \( g_{\mu\nu} \)

\[ R_{\mu\nu}(g) - \frac{1}{4} R(g) g_{\mu\nu} = 0. \]

(4.8)

This is possible due to the fact that the Ricci tensor is homogeneous of order zero in \( g_{\mu\nu} \). Therefore, the constant conformal factor, essentially (4.7), will cancel from the field equations. Then, the metric \( g_{\mu\nu} \) will be the Kottler metric as given in (4.6)

\[ g_{00} = \left( a + \frac{b}{r} + cr^2 \right), \]

\[ g_{11} = - \left( a + \frac{b}{r} + cr^2 \right)^{-1}. \]

(4.9)

Now we can put \( c = 0 \) and obtain the Schwarzschild metric.

This long detour was necessary in order to check that the limit \( c \rightarrow 0 \) was a consistent procedure.

4.2. Comments

This is the weakest energy regime and coincides with General Relativity. Therefore, the Schwarzschild solution, the Newtonian limit and the properties of gravitational radiation will be the same as for General Relativity. One must therefore not check that the
proper limit exists but that the observables departures agree with observation. For this we must turn our attention to the two following regimes.

5. The Medium-Energy Regime. Fourth-Rank Cosmology

Now we explore the consequences of our field equations in the medium-energy regime. The ideal laboratory is the universe. The large scale geometry of the universe seems to be Riemannian. Furthermore there is matter present. Therefore, in the context of fourth-rank gravity, the description of the universe belongs to the medium-energy regime. The metric $G_{\mu\nu\lambda\rho}$ will be separable in terms of a metric $g_{\mu\nu}$ which we assume to be the FRW metric. Matter is described by a perfect fluid; therefore we use the energy-momentum tensor appearing in (A.7).

When fourth-rank gravity is applied to cosmology one should deal with equations analogous to the Einstein-Friedman equations of the Standard Model of Cosmology. In fourth-rank gravity however, matter enters the field equations in a non-linear way. An essential difference with respect to General Relativity is the fact that the equations determining the evolution of the universe involve not only the energy density and the pressure but also their time derivatives. Therefore, in order to correctly deal with these equations, one should provide a time dependent state equation. As a first approach we restrict our considerations to the case of a time independent state equation. Of course, this is a quite strong assumption.

The theory predicts an increasing total entropy such that the expansion of the universe is an adiabatic non-isoentropic process. Therefore, the evolution of the universe, in the framework of fourth-rank cosmology is, as expected on physical grounds, an irreversible process.

The following conclusions are obtained after incorporating $k_{obs} = 0$.

For the early universe matter is described by a state equation in which $y \approx \frac{1}{3}$. In this case $q > 0$ and from this fact one deduces the existence of a very dense state of matter at some time in the past. Causality is not violated for $t > t_{class} \approx 10^{19} t_{planck} \approx 10^{-24}$ s. At earlier times quantum mechanical effects dominate the scene. In fact, the radius of the universe is exactly the Compton wavelength associated to its mass. Our classical approach breaks down so that the very concept of causality is meaningless. Therefore, there is no violation of causality, or horizon problem.

For the present Universe it is necessary to assume $q < 0$; this does not contradict the observed expansion of the universe from an initial hot ball. In General Relativity $q > 0$ and $a(t)$ is a convex function of $t$. Due to this fact one is used to think of the evolution of the universe with $q > 0$. However, an evolution from an initial singularity may also be conceived with a concave function, $q < 0$. The field equations predict $\Omega \approx 4y$, where $y = \frac{p}{\rho}$.

For the present universe we use $\Omega_{small} = 0.01$ and we obtain $y_{pred} = 2.5 \times 10^{-3}$. $y$ can be estimated from the mean random velocity of typical galaxies to be $y_{random} = 1 \times 10^{-5}$. 
5.1. The Field Equations

The field equations are

\[ \kappa E \, R_{\mu \nu}(\gamma) = \pm [(\rho + p) u_\mu \, u_\nu - \frac{1}{2} (\rho - p) g_{\mu \nu}] . \] (5.1)

On the other hand we have

\[ \gamma^{\mu \nu} = G^{\mu \nu \lambda \rho} \, R_{\lambda \rho} = \frac{1}{3\kappa E} \, \left[(\rho + p) \, u^\mu \, u^\nu - (\rho - 2p) g^{\mu \nu}\right] . \] (5.2)

The inverse of (5.2) is given by

\[ \gamma_{\mu \nu} = \frac{3}{\kappa E} \, \left[(\rho + p) \, u_\mu \, u_\nu - \frac{1}{(\rho - 2p)} \, g_{\mu \nu}\right] . \] (5.3)

There is a global \( \pm \) sign in (5.2) and (5.3) which we have fixed at will since the final result is independent of this choice.

The first step is to calculate the Ricci tensor for the metric \( \gamma_{\mu \nu} \). Let us start by writing the associated line element

\[ ds_{\gamma}^2 = \frac{1}{p} \, dt^2 + \frac{3}{(\rho - 2p)} \, a^2 \, d\ell^2 . \] (5.4)

We have omitted the constant in front of (5.3) since the final result is independent of this factor. In what follows we assume \( p > 0 \) and \( \rho - 2p > 0 \). We assume furthermore that \( \rho \) and \( p \) are functions of \( t \) only. Then we can introduce the new time coordinate

\[ d\tau = \frac{1}{p^{1/2}} \, dt . \] (5.5)

Then the line element (5.5) is rewritten as

\[ ds^2 = d\tau^2 + A^2 \, d\ell^2 , \] (5.6)

with

\[ A = \left[\frac{3}{(\rho - 2p)}\right]^{1/2} a(\tau) . \] (5.7)

The above is nothing more than a FRW line element with Euclidean signature. We can therefore use eqs. (A.3) with \( a^2 \to -A^2 \). The Ricci tensor is then given by

\[ R_{\mu \nu}(\gamma) = -\frac{2}{A^2} \, [A \, A'' - (-k + A^2)] \, u_\mu \, u_\nu - \frac{1}{A^2} \, [A \, A'' + 2 (-k + A^2)] \, \gamma_{\mu \nu} , \] (5.8)

where primes denote derivatives with respect to \( \tau \). In the system of coordinates involving \( t \) the above expression is given by
\[
R_{\mu\nu}(\gamma) = -\frac{2}{pA^2} [A A'' - (-k + A'^2)] \delta^0_\mu \delta^0_\nu - \frac{1}{A^2} [A A'' + 2(-k + A'^2)] \gamma_{\mu\nu} . \tag{5.9}
\]

Comparison with the Ricci tensor obtained from the field equations, eq. (5.1), gives

\[
-3 \kappa_E \frac{1}{p} \frac{A''}{A} = \pm \frac{1}{2} (1 + 3y) \rho , \tag{5.10a}
\]
\[
-3 \kappa_E \frac{1}{(\rho - 2p)} \frac{1}{A^2} [A A'' + 2(-k + A'^2)] = \pm \frac{1}{2} (1 - y) \rho . \tag{5.10b}
\]

For the applications the field equations are better rewritten as

\[
6 \kappa_E \frac{1}{(\rho - 2p)} \frac{1}{A^2} (-k + A'^2) = \mp \frac{1}{2} \frac{(1 - 4y - y^2)}{(1 - 2y)} \rho , \tag{5.11a}
\]
\[
- \frac{(1 - 4y - y^2)}{(1 - 2y)} \frac{1}{p} A A'' + 2(1 + 3y) \frac{1}{(\rho - 2p)} (-k + A'^2) = 0 . \tag{5.11b}
\]

The field equations written in this form are of practical use since the first one allows us to determine the value of \(k\) when evaluated at the present time. The second one allows us to determine the evolution of the early universe.

### 5.2. The Entropy of the Universe

The entropy variation is governed by

\[
T dS = dE + p dV = d(\rho a^3) + p d(a^3)
\]

\[
= \frac{(1 + 3y)(1 - y^2)}{(1 - 4y - y^2)} \rho a^2 da + \frac{(1 + 8y^2 + 16y^3 - 5y^4)}{(1 - 2y)(1 - 2y + 5y^2)(1 - 4y - y^2)} dy . \tag{5.12}
\]

Since the radius of the universe grows at a rate much larger than that by which \(y\) decreases, the above quantity is positive. Hence the theory predicts, in a natural way, an increasing total entropy of the universe. Thus, fourth-rank cosmology predicts an adiabatic non-isentropic, and therefore irreversible, expansion of the universe.

The next step is to go back to the time coordinate \(t\). This contains the time dependence of \(p\) on \(t\). We assume that the almost pressureless regime has lasted for such a long time that we can confidently work under the assumption that \(p\) and \(\rho\) are constant, i.e., a time independent state equation. This is done now.
5.3. Constant $y$

We assume that the almost pressureless regime has lasted for such a long time that we can confidently work under the assumption that $p$ and $\rho$ are constant, i.e., a time independent state equation. In this case the relevant equations are obtained with the simple replacements

$$A \rightarrow \left( \frac{3}{(\rho - 2p)} \right)^{1/2} a, \quad (5.13a)$$

$$()' \rightarrow p^{1/2} ('). \quad (5.13b)$$

In this case eq. (5.10a) is rewritten like

$$-3 \kappa_E \frac{\ddot{a}}{a} = \pm \frac{1}{2} \left(1 + 3y \right) \rho . \quad (5.14)$$

Equations (5.11) reduce to

$$6 \kappa_E \frac{1}{a^2} (-k + \frac{3y}{(1-2y)} \dot{a}^2) = \mp \frac{3}{2} \frac{(1-4y-y^2)}{(1-2y)} \rho , \quad (5.15a)$$

$$-3 \frac{(1-4y-y^2)}{(1-2y)} a \ddot{a} + 2 (1+3y) \left(-k + \frac{3y}{(1-2y)} \dot{a}^2 \right) = 0 . \quad (5.15b)$$

5.4. Incorporating Flatness

Let us now incorporate the observed fact $k_{obs} = 0$. Then, eqs. (5.15) reduce to

$$18 \kappa_E \frac{y}{(1-2y)} \frac{\dot{a}^2/a^2}{\dot{a}^2} = \mp \frac{3}{2} \frac{(1-4y-y^2)}{(1-2y)} \rho , \quad (5.16a)$$

$$-3 \frac{(1-4y-y^2)}{(1-2y)} a \ddot{a} + 6 \frac{y(1+3y)}{(1-2y)} \dot{a}^2 = 0 . \quad (5.16b)$$

For the physically interesting cases $0 < y < \frac{1}{3}$. Therefore, the previous equations can be simplified to

$$12 \kappa_E y \frac{a^2/\dot{a}^2}{\dot{a}^2} = \mp (1-4y-y^2) \rho , \quad (5.17a)$$

$$-(1-4y-y^2) a \ddot{a} + y (1+3y) \dot{a}^2 = 0 . \quad (5.17b)$$

In terms of the cosmological parameters these equations are rewritten as

$$\Omega = \mp \frac{4y}{1-4y-y^2} . \quad (5.18a)$$
\[ q = \pm \frac{1}{2}(1 + 3y) \Omega. \quad (5.18b) \]

The only positive root of \((1 - 4y - y^2)\) is \(\sqrt{5} - 2 \approx 0.236\). Therefore we can distinguish two regimes: I. \(0.236 < y < \frac{1}{3}\). In this case we must choose the upper sign. The resulting equations can be applied to the description of the early universe. II. \(0 < y < 0.236\). In this case we must choose the lower sign. The resulting equations can be applied to the description of the present universe.

Let us observe that eqs. (5.18a) becomes singular for \(y = 0.236\). There is no contradiction here since \(\Omega = \frac{\rho}{\rho_c}\), \(\rho_c = 3\kappa E H^2 = 3\kappa E \frac{\dot{a}^2}{a^2}\), and for \(y = 0.236\) we have \(\dot{a} = 0\) as can be read from eq. (5.17b).

### 5.5 The Early Universe

In this case \(0.236 < y < \frac{1}{3}\) and eqs. (5.18) reduce to

\[ \Omega = -\frac{4y}{1 - 4y - y^2}. \quad (5.19a) \]

\[ q = \frac{1}{2}(1 + 3y) \Omega. \quad (5.19b) \]

Let us observe that eq. (5.19b) is the same than that we obtain in General Relativity, viz. (C.2a). As in General Relativity one concludes the existence of a singularity in the past. As explained in the Appendix A, since matter cannot be compressed beyond the Planck density, it is more reasonable to consider an initial ball with finite radius. We call this the "inflationary" stage of our model.

For the early universe matter is described by the state equation \(y = \frac{1}{3}\). Then, eqs. (5.19) reduce to

\[ 9\kappa E \frac{\dot{a}^2}{a^2} = \rho, \quad (5.20a) \]

\[ a \ddot{a} + 3 \dot{a}^2 = 0. \quad (5.20b) \]

The solution to eq. (5.20b) is

\[ a = a_0 \left(1 + 4 \frac{\dot{a}_0}{a_0} \right)^{1/4} \approx a_0 + \dot{a}_0 t. \quad (5.21) \]

In this approximation the horizon radius is

\[ r_H(t) = \frac{a_0}{\dot{a}_0} \left(1 + \frac{\dot{a}_0}{a_0} t\right) ln(1 + \frac{\dot{a}_0}{a_0} t). \quad (5.22) \]

Causality is not violated when \(r_H(t) > a(t)/c\). This condition is satisfied for
\[ t > t_{\text{class}} \approx \frac{a_0}{c} \approx 10^{19} t_{\text{Planck}} \approx 10^{24} \text{s}. \] (5.23)

At earlier times quantum mechanical effects dominate the scene. In fact, the radius of the Universe is exactly the Compton length associated to its mass. Our classical approach breaks down so that the very concept of causality is meaningless. Therefore, there is no violation of causality, or horizon problem.

5.6. The Present Universe

In this case \( 0.236 < y < \frac{1}{2} \) and eqs.(5.18) reduce to

\[ \Omega = \frac{4y}{1 - 4y - y^2}. \] (5.24a)

\[ q = -\frac{1}{2} (1 + 3y) \Omega. \] (5.24b)

For the present universe matter is described by the state equation \( y \approx 0 \), i.e., almost pressureless matter. In this case eq. (5.17b) reduces to

\[ \ddot{a} \approx 0. \] (5.25)

Therefore

\[ a = \alpha t + \beta, \] (5.26)

where \( \alpha \) and \( \beta \) are integration constants. Therefore, in the present time the radius of the universe grows linearly with time. In this case eq. (5.24a) can be inverted to

\[ y = \frac{1}{\Omega} \left[ \sqrt{4(1 + \Omega)^2 + \Omega^2} - 2(1 + \Omega) \right]. \] (5.27)

Since today matter is almost pressureless small values of \( \Omega \) are favoured by our equation. For the smallest reported value \( \Omega_{\text{small}} = 0.01 \) we obtain

\[ y_{\text{small}} = 2.48 \times 10^{-3}. \] (5.28)

This should be compared with the observed value of \( \frac{p}{\rho} \). This can be determined from the mean random velocity of typical galaxies, \( \langle v \rangle = 1 \times 10^3 \text{km/s} \), and gives \( y_{\text{random}} = 1 \times 10^{-5} \). Therefore, our prediction differs by two orders of magnitude with respect to the observed value. We hope to improve this situation since the estimation of \( y \) from the the random motion of galaxies is a quite rough one. Furthermore, eq. (5.27) was obtained under the assumption of a time independent state equation.
5.7. Comments

The cosmological model we have developed here shows a reasonable agreement with observational results. In fact, we obtain field equations which among others: predict an increasing entropy of the universe; are almost consistent with the observed flatness of the universe; and do not violate causality (horizon problem). The model is still incomplete in that we have not yet considered in details the effects of a time dependent state equation for matter and how this modify the relation (5.18).

The calculation for $p < 0$ is almost identical to that for $p > 0$. Since one is used to thinking of the evolution of the universe in terms of $q > 0$ this was the case we favoured in our previous works.\textsuperscript{10,11,12} The entropy is again an increasing function of time. In this case one has also a linear growing of the radius of the universe.

6. The High-Energy Regime

Here we explore the high-energy regime of our theory. The form of the Lagrangian, and of the field equations, for conformal fourth-rank gravity, puts several strong restrictions, on the kind of matter which can be coupled consistently to it. The first condition is that matter fields must be described by conformally invariant Lagrangians in four dimensions. In fact the trace of eq. (3.27) gives

\[ T_4 = G^{\mu\nu\lambda\rho} T_{4,\mu\nu\lambda\rho} = 0, \quad (6.1) \]

which is always satisfied by conformal fields.

The situation is similar to that for Einstein gravity in 2 dimensions. The field equations are

\[ \kappa_E [R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}] = T_{\mu\nu}. \quad (6.2) \]

The trace of this equation gives

\[ T_2 = g^{\mu\nu} T_{2,\mu\nu} = 0. \quad (6.3) \]

However, the previous equation collapses to a useless identity since

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \equiv 0. \quad (6.4) \]

In our case, however, this collapse does not occur.

As a second property of our field equations let us observe that the coupling to conformal fields automatically excludes the existence of a cosmological constant. In fact, from the Lagrangian

\[ L_4 = \kappa_{CG} (\langle R^2 \rangle + \Lambda) G^{1/4}, \quad (6.5) \]

we obtain
\[ \kappa_{CG} [R_{(\mu \nu} R_{\lambda \rho)} - \frac{1}{4} \langle R^2 \rangle + \Lambda) G_{\mu \nu \lambda \rho}] = T_{\mu \nu \lambda \rho}. \]  
(6.6)

The trace of this equation is

\[ -\kappa_{CG} \Lambda = T_4, \]  
(6.7)

but the conformal invariance, eq. (6.1), impose

\[ \Lambda = 0. \]  
(6.8)

This situation is again similar to that for Einstein gravity in 2 dimensions. We have

\[ \mathcal{L}_{GR} = \kappa_E (R + \Lambda) g^{1/2}. \]  
(6.9)

The field equations are

\[ \kappa_E [R_{\mu \nu} - \frac{1}{2} (R + \Lambda) g_{\mu \nu}] = T_{\mu \nu}. \]  
(6.10)

The trace of this equation gives

\[ -\kappa_E \Lambda = T_2, \]  
(6.11)

but conformal invariance, eq. (6.3), gives

\[ \Lambda = 0. \]  
(6.12)

Therefore the high-energy regime of our theory exhibits all the properties relevant to a conformal model. In fact it can be consistently coupled to conformal fields; cf. Ref. 4 for further details. Furthermore, predicts a cosmological constant which is exactly zero.

7. Conclusions

The results reported here are the product of more than one year effort. We elaborated many previous versions which were corrected once and again, and our work was often plagued by false starts.

The conception of new geometries has taught us the importance of observation in physics and the close relation existing between physics and geometry.

The fourth-rank geometry combined with the observed scale invariance of physical processes at high-energies leaves us with an almost unique choice for a gravitational Lagrangian. What we have done here was just to explore the consequences of this theory.

We would like to emphasize that we did not construct this theory in order to solve specific problems. We just started from simple principles and explored their consequences.

In fact, it was unexpected for us to find that the field equations in vacuum, even when differing from Einstein field equations, give the same solution for a static spherically symmetric field, namely, the Schwarzschild metric. More surprising was the fact that our
field equations started to differ from those of General Relativity exactly where they are in disagreement with observation. It was also unexpected that our field equations provided solution for long unsolved problems. In fact, entropy is created in the universe. Causality is not violated, etc.

Of course, from the few results we have presented here one cannot establish the validity of this theory. It is our purpose to explore further consequences of our field equations.

Since we began this work with a quotation, it seems convenient to close it in the same way by including three more quotations.\textsuperscript{13,14,15} Even when they refer to other historical moments, they can be reread even today with changes which are obvious. We think they speak by themselves so that no more comments are necessary.

"The danger of asserting dogmatically that an axiom based on the experience of a limited region holds universally will now be to some extent apparent to the reader. It may lead us to entirely overlook, or when suggested at once reject, a possible explanation of phenomena. The hypothesis that space is not homaloidal, and again, that its geometrical character may change with the time, may or may not be destined to play a great part in the physics of the future; yet, we cannot refuse to consider them as possible explanations of physical phenomena, because they may be opposed to the popular dogmatic belief in the universality of certain geometrical axioms- a belief which has arisen from centuries of indiscriminating worship of the genius of Euclid."

\textit{W.K. Clifford, 1885}

"[Saccheri’s] brilliant failure is one of the most remarkable instances in the history of mathematical thought of the mental inertia induced by an education in obedience and orthodoxy, confirmed in mature life by an excessive reverence for the perishable works of the immortal dead [Euclid]. With two geometries, each as valid as Euclid’s in his hand, Saccheri threw both away because he was willfully determined to continue in the obstinate worship of his idol, despite the insistent promptings of his own sane reason."

\textit{E.T. Bell, 1947}

"People have often tried to figure out ways of getting these new concepts. Some people work on the idea of the axiomatic formulation of the present quantum mechanics. I don’t think that will help at all. If you imagine people having worked on the axiomatic formulation of the Bohr orbit theory, they would never have been lead to Heisenberg’s quantum mechanics. They would never have thought of non-commutative multiplication as one of their axioms which could be challenged. In the same way, any future development must involve changing something which people have never challenged up to the present, and which will we not be shown up by an axiomatic formulation."

\textit{P.A.M. Dirac, 1973}
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Appendix A. Cosmography

In this Section we collect the observational results concerning the structure of the universe and its evolution. Further details can be found in refs. 16 and 17.

The observed isotropy and homogeneity of the universe gives as the only possible Riemannian geometry for the universe a Friedman-Robertson-Walker (FRW) geometry. FRW spaces are characterised by the cosmic radius \( a(t) \) and by the constant \( k = 1, 0, -1 \), corresponding to a closed, spatially flat, and open universe, respectively. The curvature properties of a FRW geometry can be rewritten in terms of the Hubble constant, \( H \), and the deacceleration parameter, \( q \). These cosmological parameters can, in principle, be determined from the observed distance versus velocity Hubble diagram.

At large scales cosmic matter can be described as a perfect fluid which is characterised by the energy density, \( \rho \), and the pressure, \( p \), and they are related by the state equation of matter, \( \frac{p}{\rho} = y \). For \( y = \frac{1}{3} \) one has a radiation dominated, or ultrarelativistic, perfect fluid; for \( y \approx 0 \) one has instead a non-relativistic, or almost pressureless, perfect fluid.

Associated to the FRW geometry, with the use of the Einstein gravitational constant, there is a critical density parameter, \( \rho_c = 3\kappa E H^2 \), which sets the scale of energy densities. One can then introduce the cosmic density parameter \( \Omega = \frac{\rho}{\rho_c} \).

The cosmological parameters \( H \), \( q \) and \( \Omega \) are observable, however, they are quite difficult to determine with accuracy. For the Hubble constant \( H \) there are two preferred values close to 50 and 100 km/sec/Mpc. However the observed data does not allow to determine the value of the deacceleration parameter \( q \).\(^{17}\) According to ref. 17, \( \Omega_{obs} \approx 0.1 - 0.3 \), with an upper safety bound \( \Omega_{safe} \approx 0.18 \). However, early reports give smaller values such as \( \Omega_{small} = 0.01.\(^7\)\)

The observed redshift of galaxies shows that the universe is expanding. Therefore, it was very dense in the past and it is almost diluted today. Since matter cannot be compressed beyond the Planck density one must consider the universe as evolving from an initial hot ball. From the conservation of mass one can furthermore estimate the radius of the initial hot ball to be \( a_0 = 10^{19} L_{Planck} \).

Further observations, such as the galaxy count-volume test, show that to a very big extent our universe is spatially quite flat. Therefore, the parameter \( k \) characterising the FRW geometries must be zero, \( k_{obs} = 0 \), i.e., its geometry is Euclidean.

The last important observation is the fact that the universe is quite isotropic and
homogeneous at large scales. This is an indication that matter was in causal contact in the very remote past. This condition roughly translates into \( r_H > a \), where \( r_H \) is the horizon radius. However, for very small times, \( t < t_{\text{class}} \approx 10^{20} t_{\text{Planck}} \), quantum mechanical effects dominate the scene. In fact, the radius of the universe is exactly the Compton wavelength associated to its mass and the very concept of causality is meaningless. Therefore, causality should not be violated only for \( t > t_{\text{class}} \).

Any proposed cosmological model must agree with the previously described observational results. The next task is to develop a gravitational theory fitting the above observations. The first candidate is General Relativity (Appendix B).

### A.1. Isotropy, the Cosmological Principle and FRW Spaces

Observation shows that the universe is isotropic and homogeneous. These properties give as the only possible Riemannian geometry a FRW metric. In this case the line element is

\[
  ds^2 = dt^2 - a^2(t) d\ell^2, \tag{A.1}
\]

where

\[
  d\ell^2 = (1 - k r^2)^{-1} dr^2 + r^2 d\Omega^2, \tag{A.2a}
\]

\[
  d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2. \tag{A.2b}
\]

In the above \( a(t) \) is the cosmic scale factor and is interpreted as the radius of the universe; \( d\ell^2 \) is the line element of a maximally symmetric three-dimensional space-like section. The radial coordinate \( r \) is written in units such that the constant \( k \) takes the values 1, 0 or -1. The parameter \( k \) characterises the geometry of the space-like sections of the universe. For \( k = 1 \) the universe is closed; for \( k = 0 \) it is flat; for \( k = -1 \) it is open.

The Ricci tensor is given by

\[
  R_{\mu\nu} = -2 \left[ a \ddot{a} - (k + \dot{a}^2) \right] \delta_{\mu}^0 \delta_{\nu}^0 - \frac{1}{a^2} \left[ a \ddot{a} + 2 (k + \dot{a}^2) \right] g_{\mu\nu}. \tag{A.3}
\]

Hence, the scalar curvature is

\[
  R = -\frac{6}{a^2} \left[ a \ddot{a} + (k + \dot{a}^2) \right]. \tag{A.4}
\]

These quantities can be parametrised in terms of the cosmological parameters

\[
  H = \frac{\dot{a}}{a}, \tag{A.5a}
\]

which is the Hubble "constant", and it is a true constant only for a de Sitter space; and

\[
  q = -\frac{a \ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2}, \tag{A.5b}
\]
which is the deacceleration parameter.

In a FRW universe the luminosity distance \(d_L\) and the redshift \(z\) of a galaxy are related by

\[
d_L \approx \frac{1}{H} (z + \frac{1}{2} (q - 1) z^2) .
\] (A.6)

The distance to a galaxy can be determined by different means. The redshift is determined by simple spectral techniques. This constitutes the distance versus redshift Hubble diagram. If \(z > 0\) one talks of redshifts and galaxies are receding, while if \(z < 0\) one talks of blueshifts and galaxies are approaching. Therefore \(H\) can be determined from the slope of the Hubble diagram while \(q\) is related to its convexity.

### A.2. The Matter Content of the Universe. The Perfect Fluid

In order to be compatible with the observed homogeneity and isotropy of the universe cosmic matter must be described as a perfect fluid.

A perfect fluid is characterised by the energy-momentum tensor

\[
T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu} ,
\] (A.7)

where \(\rho\) and \(p\) are the energy density and pressure of cosmic matter, and

\[
u_\mu = (g^{00})^{-1/2} \delta^0_\mu ,
\] (A.8a)

such that

\[
g^{\mu\nu} u_\mu u_\nu = 1 .
\] (A.8b)

The reduced energy-momentum tensor is

\[
S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} = (\rho + p) u_\mu u_\nu - \frac{1}{2} (\rho - p) g_{\mu\nu} .
\] (A.9)

In order to relate the energy density \(\rho\) and the pressure \(p\) one needs a state equation. Two well understood regimes are the radiation dominated regime in which \(y = \frac{p}{\rho} = \frac{1}{3}\), and the matter dominated regime in which \(y\) approaches to zero for incoherent matter.

The coupling of gravity to matter needs the Einstein gravitational constant \(\kappa_E\). Combining this constant with the functions characterising the FRW geometry we obtain the critical density

\[
\rho_c = 3 \kappa_E H^2 = 1.96 \times 10^{-29} h^2 \ g/cm^3 ,
\] (A.10)

where

\[
h = \frac{H}{100 \ km/sec/Mpc} .
\] (A.11)

This leads to the introduction of the cosmic energy density parameter
\[ \Omega = \frac{\rho}{\rho_c}, \]  

(A.12)

in such a way that \( \rho_c \) sets the scale of energy densities.

### A.3. Observed Values of the Cosmological Parameters

Due to several practical difficulties the observed values of the cosmological parameters are quite inaccurate. In some cases this inaccuracy does not allow even to have a reliable value for some parameters. There exists a wide range of reported values for the cosmological parameters depending on both the nature of the performed observation and the interpretation of the observed data. We use the values reported in ref. 17.

#### A.3.1. The Hubble Diagram

In principle, the Hubble diagram should provide, at the same time, the Hubble constant, \( H \), and the deacceleration parameter, \( q \). \( H \) is related to the slope of such diagram while \( q \) is related to its convexity. However, the Hubble diagram shows a large dispersion for large values of \( H \) such that no reliable value for \( q \) exists today. The reported values are

\[ h_{\text{obs}} = 0.5 - 1.0, \]  

(A.13)

with preferred values closer to 0.5 and to 1.0. The determination of \( q \) from deviations from the linear Hubble law is almost impossible with the present day accuracy of the existing observations.

Since \( H \) is positive one can conclude that the universe is expanding. Let us observe that eq. (A.5a) can be rewritten as

\[ a = \frac{\dot{a}}{H} \approx \dot{a}T. \]  

(A.14)

This equation tells us that the radius of the universe is approximately its velocity of expansion times the period of expansion. Therefore, \( H^{-1} \) can be interpreted as the age of the universe.

#### A.3.2. The Energy Density

The energy density is determined from the cosmic virial theorem and the infall to the Virgo cluster. According to ref. 17 the observed value for the energy density ratio is in the range

\[ \Omega_{\text{obs}} \approx 0.1 - 0.3. \]  

(A.15)

There is furthermore a safety upper bound
\[ \Omega_{safe} \approx 0.18. \quad (A.16) \]

However, early reports\textsuperscript{7} give smaller values

\[ \Omega_{small} = 0.01. \quad (A.17) \]

### A.3.3. The Pressure of the Universe

Up to our knowledge there is no direct determination of the pressure of the universe. However, an upper bound can be put based on the mean random velocity, \( \langle v \rangle \), of typical galaxies

\[ \frac{p}{\rho} \approx \frac{\langle v \rangle^2}{c^2}. \quad (A.18) \]

The proper velocity can be determined from direct measurements and is given approximately by \( \langle v \rangle \approx 1 \times 10^3 \) km/s. Therefore we obtain

\[ y_{\text{random}} \approx 1 \times 10^{-5}. \quad (A.19) \]

The previous figure put an upper bound to the \( \frac{p}{\rho} \) ratio. It must be

\[ \frac{p}{\rho} < 1 \times 10^{-5}. \quad (A.20) \]

### A.3.4. The Radius of the Initial Universe

The expansion of the universe shows that it has evolved from a very dense regime in the past and it is almost diluted today. Since matter cannot be compressed beyond the Planck density it is more reasonable to consider the universe as evolving from an initial ball.

The radius of the initial universe can be estimated as follows. Let us assume that the mass of the universe is a conserved quantity. At \( t = 0 \) we assume that mass was compressed at Planck density. This allows determining \( a_0 \) to be

\[ a_0 = \left[ \frac{M_{\text{Univ}}}{M_{\text{Planck}}} \right]^{1/3} L_{\text{Planck}}. \quad (A.21) \]

The mass of the universe is given by

\[ M_{\text{Univ}} \approx \frac{4\pi}{3} \rho R_{\text{Univ}}^3, \quad (A.22) \]

where \( R_{\text{Univ}} \) is the radius of the universe. This is bounded by the maximum velocity by which the universe can expand, the velocity of light \( c \), and the time of expansion \( H^{-1} \). Therefore

\[ R_{\text{Univ}} \approx \frac{c}{H}. \quad (A.23) \]
Finally
\[ M_{Univ} \approx \frac{4\pi}{3} \frac{\rho c^3}{H^3} \approx 10^{57} M_{Planck}. \] (A.24)

Therefore
\[ a_0 \approx 10^{19} L_{Planck}. \] (A.25)

Our estimation contains an error of a few orders of magnitude which is not relevant to our analysis.

A.3.5. Entropy, Flatness and Causality

From the microwave background radiation one observes that the present value of the total entropy in the universe is so large as to be of order \(10^{87}\), in some convenient units.\(^{18,19,20}\) On the other hand one would expect entropy to be governed by the second thermodynamical principle, the statement that the entropy is always a non-decreasing function of time. One is therefore faced with the problem of determining whether the entropy of the universe has always been as large as it is today, \(dS = 0\), or if it has evolved from a smaller value. From an intuitive point of view it is quite improbable that \(dS = 0\).

Further observations show that our present day universe is almost flat, i.e., its geometry is almost Euclidean; this means that
\[ k_{obs} = 0. \] (A.26)

Another observation concerns the observed isotropy of the universe over large regions of space; this means that all regions were causally connected in the past. For this to be the case one should have
\[ r_H > a, \] (A.27)

where \(r_H\) is the horizon radius which sets the size of the region in which causal contact can be achieved.

For a FRW space the horizon radius is
\[ r_H(t) = a(t) \int_0^t \frac{du}{a(u)}, \] (A.28)

which is the maximum distance that light signals can travel during the age \(t\) of the universe.

Let us introduce a time scale
\[ t_{class} = \frac{a_0}{c} \approx 10^{19} t_{Planck} \approx 10^{-24} \text{ s}. \] (A.29)

For times smaller than \(t_{class}\) quantum mechanical effects dominate the scene. In fact, the radius of the universe is exactly the Compton wavelength associated to its mass. Therefore
the very concept of causality is meaningless. Therefore, causality should be required only for times greater than \( t_{\text{class}} \).

**Appendix B. General Relativity**

In General Relativity space-time is conceived as a Riemannian manifold and the metric \( g_{\mu\nu} \) is identified with the gravitational field.

In order to describe the dynamics of the gravitational field we need to construct an invariant which might be used as Lagrangian. In Riemannian geometry the simplest invariant which can be constructed is

\[
R(g, \Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma),
\]

which in the case of a metric space is rewritten as

\[
R(g) = g^{\mu\nu} R_{\mu\nu}(g).
\]

The analytical formulation of General Relativity takes as its starting point the Einstein-Hilbert Lagrangian

\[
\mathcal{L}_{\text{EH}}(g) = \kappa_E R(g) g^{1/2},
\]

where \( \kappa_E = \frac{c^4}{8\pi G_N} \) is the Einstein gravitational constant; \( G_N \) being the Newton constant. The full Lagrangian must consider also the contributions of matter

\[
\mathcal{L}_1 = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{matter}},
\]

Variation of the Lagrangian with respect to the metric

\[
\frac{\delta \mathcal{L}_1}{\delta g^{\mu\nu}} = 0,
\]

gives the Einstein field equations

\[
\kappa_E [R_{\mu\nu}(g) - \frac{1}{2} R(g) g_{\mu\nu}] = T_{2,\mu\nu},
\]

where

\[
T_{2,\mu\nu} = \frac{1}{g^{1/2}} \frac{\delta \mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}},
\]

is the energy-momentum tensor of matter; the 2 stands for the fact that the energy-momentum tensor is related to Riemannian, second-rank, geometry.

As a starting point for General Relativity one can also consider the ”Palatini” Lagrangian

\[
\mathcal{L}_P(g, \Gamma) = \kappa_E g^{\mu\nu} R_{\mu\nu}(\Gamma) g^{1/2}.
\]
In this case one must also consider the contributions of matter
\[ \mathcal{L}_2 = \mathcal{L}_P + \mathcal{L}_{\text{matter}}. \] (B.9)

Now the connection and the metric are varied independently in a procedure known as the Palatini variational principle. Variation of the Lagrangian with respect to \( \Gamma \) gives
\[ \frac{\delta \mathcal{L}_2}{\delta \Gamma^\lambda_{\mu\nu}} = \frac{\delta \mathcal{L}_P}{\delta \Gamma^\lambda_{\mu\nu}} + \frac{\delta \mathcal{L}_{\text{matter}}}{\delta \Gamma^\lambda_{\mu\nu}} = 0. \] (B.10)

In all known cases of physical interest one has\(^5\)
\[ \frac{\delta \mathcal{L}_{\text{matter}}}{\delta \Gamma^\lambda_{\mu\nu}} = 0. \] (B.11)

In this case eq. (B.10) reduces to a metricity condition equivalent to (2.15). Therefore the connection is given by the Christoffel symbol of the second kind for the metric \( g_{\mu\nu} \).

Variation with respect to the metric
\[ \frac{\delta \mathcal{L}_2}{\delta g^\mu\nu} = 0, \] (B.12)
gives
\[ \kappa E \left[ R_{\mu\nu}(\Gamma) - \frac{1}{2} g^{\lambda\rho} R_{\lambda\rho}(\Gamma) g_{\mu\nu} \right] = T_{2,\mu\nu}. \] (B.13)

If we now use the previously obtained metricity condition these equations reduce to the original Einstein field equations (B.6). Therefore, the procedures of imposing the metricity condition and of applying the variational principle commute.

Einstein field equations can be rewritten in the Landau form
\[ \kappa E R_{\mu\nu} = S_{2,\mu\nu}, \] (B.14)
where
\[ S_{2,\mu\nu} = T_{2,\mu\nu} - \frac{1}{2} T_2 g_{\mu\nu}, \] (B.15)
with
\[ T_2 = g^{\mu\nu} T_{2,\mu\nu}, \] (B.16)
is the reduced energy-momentum tensor.

Einstein field equations have been applied to many physical situations. The first classical test of any theory of gravitation is in the solar system. In this case one needs to solve Einstein field equations in vacuum for a spherically symmetric field. The solution is the exterior Schwarzschild metric. Using this metric one can account for the anomalous shift of the perihelion of inner planets and for the bending of light rays near the solar surface to an accuracy of 1 per cent or better. In this case one is describing the effects of the gravitational field alone.
The next test concerns the coupling of gravity to matter. This is achieved, for instance, when considering the large scale structure of the universe where gravity becomes coupled to a perfect fluid. As shown in the next Appendix the agreement with observation is qualitatively good. One obtains qualitatively good predictions, as the evolution of the universe from an initial singularity and some good quantitative predictions as the temperature of the microwave background and the relative abundance of elements. However, the quantitative agreement is weaker in other aspects. In fact, flatness, $k_{\text{obs}} = 0$, implies $\Omega_{GR} = 1$, which is hardly observed. Furthermore, the Standard Model of Cosmology predicts a constant entropy, something which is difficult to accept on physical grounds. These are some of the reasons to look for an improved theory for the gravitational field.

Therefore one must consider the possibility that General Relativity is an incomplete theory. However, it is in good agreement with observation in the vacuum case: the Schwarzschild solution. Therefore General Relativity describes well the dynamics of the gravitational field alone, but it fails when coupled to matter.

Some hints, on how this problem can be approached, come from high-energy physics. When one tries to quantise General Relativity one discovers that there are irremovable ultraviolet divergences. This is taken as indicative that at small distances the geometry of space-time may be different from the Riemannian one. The current view is that General Relativity, with its Riemannian structure, is only the low-energy, large distance, manifestation of a more general theory at small distances. One must therefore construct a field theory for a more general geometry. The field theory one constructs for this new geometry must produce, in the absence of matter, a Riemannian geometry. Furthermore, gravitation, in the form of a theory equivalent to General Relativity must be recovered. Many possibilities have been explored mainly in the direction of modifying the affine structure of space-time. Up to our knowledge, modifications of the metric structure of space-time have not yet been attempted. As stated in the Introduction, the purpose of this work is to explore this possibility.

Appendix C. The Standard Model of Cosmology

The Standard Model of Cosmology is based on the application of the Einstein field equations to the universe. They provide the coupling of gravity, or geometry, given by a FRW metric, to cosmic matter, described by a perfect fluid.

The Einstein-Friedman equations are equivalent to

$$\rho = 3 \kappa_E \frac{1}{a^2} (k + \dot{a}^2) > 0, \quad (C.1a)$$

$$p = -\kappa_E \frac{1}{a^2} [2 a \ddot{a} + (k + \dot{a}^2)]. \quad (C.1b)$$

In terms of the cosmological parameters eqs. (C.1) can be rewritten as

$$q = \frac{1}{2} (1 + 3 y) \Omega, \quad (C.2a)$$
The first equation shows that \( q > 0 \) and from here one is used to conceive the universe as expanding from an initial singularity. This is in agreement with observation.

However, the Standard Model of Cosmology is in disagreement with some observations as we will show in detail now.

C.1. The Entropy Problem

One question the Standard Model of Cosmology is unable to answer concerns the problem of the large total entropy in the universe.\(^{18,19,20} \) One of the predictions of the Standard Model of Cosmology is that the expansion of the universe is an adiabatic isoentropic process. In fact, from the field equations (C.1) one can easily deduce that

\[
T \, dS = dE + p \, dV = d(ra^3) + p \, d(a^3) = 0. \quad (C.3)
\]

Therefore, the Standard Model of Cosmology predicts that the expansion of the universe is an adiabatic isoentropic process. There is no entropy production and the entropy of the universe has always been as large as it is today, something which is hard to accept based on physical grounds.

C.2. The Flatness Problem

The observed flatness of the universe implies \( k_{obs} = 0 \). If we put this value in eq. (C.2b) we obtain \( \Omega_{pred} = 1 \). However, this is incompatible with the reported values for \( \Omega_{obs} \), (A.14), (A.15) and (A.16). There exist two possibilities in the face of this impasse. The first one consists in assuming \( k_{obs} = 0 \), \( \Omega_{pred} = 1 \). This is preferred by some authors for ”aesthetic or philosophical reasons”.\(^{17} \) This takes us to the ”missing mass” problem. The second possibility is more difficult to implement. If we accept \( \Omega_{obs} < 1 \), then one should have \( k_{pred} = -1 \), as deduced from (C.2a), which corresponds to an open universe. This possibility is more or less excluded by the cosmological data\(^{17} \) indicating that for the large scale structure of the universe \( k \) is rather close to zero. This ambiguous situation is known as the flatness problem.

The above inconsistencies can be removed if the true value of the energy density parameter \( \Omega_{obs} \) is greater than that observed. But dark matter is hardly observed.

C.3. The Early Universe

At early times matter is described by the state equation \( y = \frac{1}{3} \). In this case the Einstein field equations reduce to

\[
a \ddot{a} + (k + \dot{a}^2) = 0. \quad (C.4)
\]
The solution is

\[ a = (\alpha t + a_0^2)^{1/2}, \quad (C.5a) \]

\[ a = (c^2 t^2 + a_0^2)^{1/2}, \quad (C.5b) \]

for \( k = 0, -1 \), respectively. The horizon radius are given by

\[ r_H = 2 \frac{c}{\alpha} (\alpha t + a_0^2)^{1/2} \left[ (\alpha t + a_0^2)^{1/2} - a_0 \right], \quad (C.6a) \]

\[ r_H = a_0 \left( 1 + x^2 \right)^{1/2} \ln \left[ x + (1 + x^2)^{1/2} \right], \quad (C.6b) \]

where \( x = \frac{ct}{a_0} \). In both cases causality is not violated for \( t > t_{\text{class}} \).

In the Standard Model of Cosmology one usually assumes that \( a_0 = 0 \). In this case causality would be violated. This is called the horizon problem. Our result differs from the standard one since we have considered a universe evolving from an initial hot ball with a finite radius rather than from an initial singularity.

C.4. Comments

Hence, it is clear that the observed cosmological data do not fit into the field equations of the Standard Model of Cosmology, and that even under strong assumptions on the observed values of the cosmological parameters the situation cannot be much improved.

We must therefore conclude that the Standard Model of Cosmology is in disagreement with some cosmological observations. In order to solve the above problems one can consider inflation.\(^{18,19,20}\) Inflation is intended to solve the entropy, the flatness and the horizon problems. However, inflationary cosmology will be sound only if later observations will show that \( \Omega_{\text{obs}} = 1 \).

Therefore, our conclusion is that General Relativity is an incomplete theory. In fact, it describes well, to a very high accuracy, the effects of the gravitational field alone: the shift of the perihelion of inner planets, the bending of light in strong gravitational fields, etc., however it fails to describe the coupling of the gravitational field to matter, for example, the Standard Model of Cosmology.

We must look therefore for an improved theory for the gravitational field coinciding with General Relativity in the vacuum case and with a different way of coupling the gravitational field to matter. The theory of fourth-rank gravity we have developed satisfies these requirements.
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a. We thank J. Russo for clarifying us this point.
b. The fact of calling $G_{\mu\nu\lambda\rho}$ a metric is a purely linguistic issue completely unrelated to the mathematical properties of this object.

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