Vortex stabilization in a small rotating asymmetric Bose-Einstein condensate

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We use a variational method to investigate the ground-state phase diagram of a small, asymmetric Bose-Einstein condensate with respect to the dimensionless interparticle interaction strength $\gamma$ and the applied external rotation speed $\Omega$. For a given $\gamma$, the transition lines between no-vortex and vortex states are shifted toward higher $\Omega$ relative to those for the symmetric case. We also find a re-entrant behavior, where the number of vortex cores can decrease for large $\Omega$. In addition, stabilizing a vortex in a rotating asymmetric trap requires a minimum interaction strength. For a given asymmetry, the evolution of the variational parameters with increasing $\Omega$ shows two different types of transitions (sharp or continuous), depending on the strength of the interaction. We also investigate transitions to states with higher vorticity; the corresponding angular momentum increases continuously as a function of $\Omega$.

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I. INTRODUCTION

The first experimental creation and detection of a vortex in a dilute trapped Bose-Einstein condensate relied on two hyperfine components in a nonrotating symmetric trap, with an applied electromagnetic field coupling the two components [1,2]. In other recent related experiments, several groups have used rotating nonaxisymmetric traps [3–7], focusing, respectively, on the irrotational flow induced by the nonaxisymmetric shape [3,4] and on the stabilization of vortices in such geometries [5,6,8]. Indeed, the Paris group [6,8] also measured the angular momentum of the rotating Bose condensate. In all these experiments, the condensate is large and generally well-described by the Thomas-Fermi approximation [9–12].

In contrast, the present work focuses on a small rotating nonaxisymmetric condensate, which requires a small dimensionless interaction strength $\gamma$. Although current experiments have $\gamma \sim 10-1000$, this value, in principle, can be decreased by reducing the particle number and/or tuning the scattering length with the help of a Feshbach resonance [13,14]. So far, this limit has been studied for rotationally symmetric traps with a fixed angular momentum $L_z$ [15–19]. However, since experiments do not fix the angular momentum and since we focus on nonaxisymmetric traps, we work at a fixed applied rotation $\Omega$.

We first consider a noninteracting system in a rotating anisotropic harmonic trap. Although the energy eigenvalues for this problem have been studied previously [20,23], the relevant structure of the corresponding quantum-mechanical eigenstates has not been considered in detail. A subset of these low-lying eigenstates of the anisotropic harmonic trap becomes nearly degenerate for increasing trap rotation, and conventional perturbation theory fails (as it does for a symmetric trap [24]). Therefore, we use a variational approach to study the ground state of the interacting system. This method permits us to determine the phase diagram with respect to the interaction strength $\gamma$ and the applied rotation frequency $\Omega$. Monitoring the evolution of the variational ground state provides insight into the character of the transition to states containing one or more vortices. Moreover, we discuss the resulting angular momentum carried by the condensate, which increases quasi-continuously for rotations beyond that required for stabilization of a few vortices.

One striking new feature of a small condensate in a rotating anisotropic trap is the possibility of increasing the angular momentum by elongating (which increases the moment of inertia) and simultaneously decreasing the number of vortices (because fewer cores can fit in the stretched form). In this way, the anisotropic condensate mimics solid-body rotation in a qualitatively different way from the familiar addition of more singly quantized vortices [25]. Our investigations illustrate the changing relative importance of rotational and asymmetry effects in different areas of the $\Omega-\gamma$ space and different values of the anisotropy parameter $\omega_y/\omega_x$.

The basic formalism is presented in Sec. II along with the variational trial wave function. For a fixed anisotropy, the phase diagram is discussed in Sec. III as a function of the interaction parameter and the externally applied rotation, showing asymmetric vortex states and in some cases re-entrant behavior. Since angular momentum is not conserved in an asymmetric trap, its behavior is analyzed in Sec. IV.
II. BASIC FORMALISM

In this section, we introduce the geometric configuration, construct the exact eigenstates of the asymmetric noninteracting Bose condensate in a rotating anisotropic harmonic trap, and investigate their behavior for different rotation speeds and asymmetries. These states provide the basis of our variational trial function for the interacting system. This ansatz, along with its numerical implementation, is described at the end of this section.

A. Hamiltonian for a rotating anisotropic harmonic trap

To study the stationary states of a low-temperature Bose-Einstein condensate in the rotating frame (when the trap potential becomes time-independent), we start with the time-independent Gross-Pitaevskii (GP) equation

\[ \left( \mathcal{H}^{(0)} + V_T - \mu \right) \Psi = 0. \]  

(1)

Here \( \mathcal{H}^{(0)} = T + V_T - \Omega L_z \) is the Hamiltonian for a single particle with kinetic energy \( T = -\hbar^2 \nabla^2 / 2M \) in an anisotropic harmonic trap \( V_T = \frac{1}{2} M (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \) that rotates about the \( z \) axis with an angular velocity \( \Omega \), and \( V_H = g |\Psi|^2 \) is the self-consistent Hartree interaction term (the \( s \)-wave scattering length \( a \) determines the coupling constant \( g = 4 \pi a \hbar^2 / M \)). For definiteness, we assume \( \omega_x \leq \omega_y \).

In the absence of rotation, the noninteracting Hamiltonian \( \mathcal{H}^{(0)} \) separates into a sum of three cartesian terms \( \mathcal{H}_j = \frac{1}{2} \hbar \omega_j (a_j^\dagger a_j + a_j a_j^\dagger) \), where \( j = x, y, z \). It is convenient to use the three oscillator lengths \( d_j = \sqrt{\hbar / M \omega_j} \) to scale the three cartesian coordinates, in which case the harmonic-oscillator operators have the dimensionless form

\[ a_j = \frac{1}{\sqrt{2}} \left( x_j + \frac{\partial}{\partial x_j} \right) \quad \text{and} \quad a_j^\dagger = \frac{1}{\sqrt{2}} \left( x_j - \frac{\partial}{\partial x_j} \right). \]  

(2)

When the trap potential \( V_T \) rotates, the term \( -\Omega L_z = -\Omega (xp_y - yp_x) \) couples \( H_x \) and \( H_y \), and the unperturbed Hamiltonian becomes \( \mathcal{H}^{(0)} = H_\perp + H_z \). Use of Eq. (2) shows that

\[ H_\perp = \frac{1}{2} \hbar \omega_x \left( a_x^\dagger a_x + a_x a_x^\dagger \right) + \frac{1}{2} \hbar \omega_y \left( a_y^\dagger a_y + a_y a_y^\dagger \right) \]

\[ + \frac{i \hbar \Omega}{2 \sqrt{\omega_x \omega_y}} \left[ (\omega_x + \omega_y) (a_x^\dagger a_y - a_y^\dagger a_x) + (\omega_x - \omega_y) (a_x^\dagger a_y^\dagger - a_y^\dagger a_x) \right]. \]

(3)

In addition to the usual “diagonal” (number-conserving) terms proportional to \( a_j a_j^\dagger \), this operator also has an “off-diagonal” (number-violating) term proportional to \( a_j a_j^\dagger - a_j^\dagger a_j \).

As in the familiar case of the two-component Bogoliubov transformation for a dilute Bose gas, \( H_\perp \) can be diagonalized with a generalized Bogoliubov transformation that couples all four operators. It is convenient to define a four-component vector \( \mathbf{a} = (a_x, a_y, a_x^\dagger, a_y^\dagger) \); its elements obey the commutation relations \( [a_j, a_k^\dagger] = J_{jk} \), where \( J \) is a diagonal matrix with elements \((1, 1, -1, -1)\). The Hamiltonian can now be written in matrix form as \( H_\perp = \frac{i}{2} \hbar a^\dagger \mathcal{H} a \)

where \( \mathcal{H} \) is a \( 4 \times 4 \) hermitian matrix given by

\[ \mathcal{H} = \begin{pmatrix} \omega_x & ic & 0 & id \\ -ic & \omega_y & id & 0 \\ 0 & -id & \omega_x & -ic \\ -id & 0 & ic & \omega_y \end{pmatrix}, \]

(4)

with \( c = (\Omega / \sqrt{\omega_x \omega_y}) \frac{1}{2} (\omega_x + \omega_y) \) and \( d = (\Omega / \sqrt{\omega_x \omega_y}) \frac{1}{2} (\omega_x - \omega_y) \).

The quadratic form \( H_\perp \) can now be diagonalized with a linear canonical transformation to a new set of four “quasiparticle” operators \( \alpha_k \), defined by the matrix relation \( \mathbf{a} = \mathcal{U} \alpha \), where the quasiparticle operators \( \alpha_k \) obey the same boson commutation relations \( [\alpha_j, \alpha_k^\dagger] = J_{jk} \). The transformation matrix \( \mathcal{U} \) follows from the eigenvalue problem

\[ \mathcal{H} u^{(k)} = \lambda_k \mathcal{U} u^{(k)}, \]

(5)

where \( u^{(k)} \) is the \( k \)th eigenvector and \( \lambda_k \) is the corresponding \( k \)th eigenvalue obtained from the determinantal condition \( |\mathcal{H} - \lambda J| = 0 \). The four roots are the eigenvalues \( \omega_\perp \)

\[ \omega_\perp^2 = \omega_1^2 + \Omega^2 \mp \sqrt{\frac{1}{4} (\omega_y^2 - \omega_x^2)^2 + 4 \omega_1^2 \Omega^2}; \]

(6)

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where $\omega_\perp^2 \equiv \frac{1}{2}(\omega_x^2 + \omega_y^2)$; these eigenvalues are identical with the classical normal-mode frequencies \[20\].

To understand our choice of notation $\omega_\perp$, note that the eigenvalues reduce to $\omega_\perp^2 = (\omega_\perp + \Omega)^2$ for an axisymmetric trap ($\omega_x = \omega_y = \omega_\perp$). In the absence of rotation ($\Omega = 0$), these two modes are degenerate. As is familiar from degenerate perturbation theory, applying an infinitesimal rotation breaks the degeneracy and selects out the circularly polarized helicity states with unnormalized wave functions $\psi_\pm(r, \phi) \propto (x \pm iy) \exp(-\frac{1}{2}r^2)$ in contrast to the expression of $\psi_\pm(r, \phi) \propto \exp(-\frac{1}{2}r^2)$. In particular, the $+$ mode rotates in the positive sense defined by the right-hand rule and its frequency $\omega_+ = \omega_\perp + \Omega$ decreases with increasing angular velocity as is obvious when viewed in the rotating frame \[25\].

In the general case of an anisotropic trap with $\omega_x < \omega_y$, the two modes $\omega_\pm$ are nondegenerate even for $\Omega = 0$, when the $+$ mode with $\omega_+ = \omega_x$ is linearly polarized along $x$ and the $-$ mode with $\omega_- = \omega_y$ is linearly polarized along $y$. For nonzero rotation speed $\Omega$, the dispersion relation \[6\] exhibits a crossover near $\omega = \Omega$ for moderate $\Omega$; in the large-$\Omega$ regime ($\Omega/\omega \lesssim 1$), however, $\omega_+$ vanishes like $\sqrt{(\omega_y - \omega_x)(\omega_x - \Omega)}$ with an infinite slope as $\Omega \to \omega_x$. In contrast, the larger positive eigenvalue $\omega_-^\prime$ increases with increasing $\Omega$. Figure \[8\] illustrates the positive eigenvalues $\omega_\pm$ as functions of $\Omega$ for two asymmetries. For the small asymmetry $\omega_y = \Omega \omega_x$, the deviation of $\omega_x$ from a straight line is seen at both ends of the range of rotation speed $0 \leq \Omega \leq \omega_x$.

The four eigenvalues in Eq. \[8\] of $H_\perp$ in Eq. \[8\] can be taken to constitute a diagonal matrix $\Lambda$ with elements $(\omega_+, \omega_-, -\omega_+, -\omega_-)$ for the two positive eigenvalues $\omega_\pm$, the eigenvectors have the form

$$u^{(\pm)} = \begin{pmatrix}
\cosh \chi_\pm \cos \theta_\pm \\
\pm i \cosh \chi_\pm \sin \theta_\pm \\
- \sinh \chi_\pm \sin \eta_\pm \\
\pm i \sinh \chi_\pm \cos \eta_\pm
\end{pmatrix}. \quad (7)
$$

These two eigenvectors satisfy the normalization condition $u^{(j)\dagger} J u^{(k)} = \delta_{jk}$, where + and $-$ correspond to $j$ and $k = 1$ and 2, respectively. It is not difficult to obtain explicit expressions for the (real) hyperbolic parameters $\chi_\pm$ and for the (real) trigonometric parameters $\theta_\pm$ and $\eta_\pm$; they depend on the trap frequencies $\omega_x$ and $\omega_y$ and on the external rotation speed $\Omega$. Symmetry considerations readily relate the remaining two eigenvectors (those for the two negative eigenvalues $-\omega_\pm$) to $u^{(\pm)}$. The resulting four eigenvectors $u^{(j)}$ with $j = 1, \ldots, 4$ form a complete basis set and obey the normalization condition

$$u^{(j)\dagger} J u^{(k)} = \delta_{jk}. \quad (8)$$

The transformation to the quasiparticle operators is determined by the matrix $U_{jk} = u^{(j)}_\dagger$ of the four eigenvectors written in successive columns (it is the analog of the “modal matrix” that plays a central role in the theory of small oscillations of mechanical systems about stationary configurations \[28\]). In this way, the matrix $U$ satisfies the eigenvalue equation [compare Eq. \[8\]]

$$H U = J U \Lambda, \quad (9)$$

and the normalization \[8\] for the eigenvectors implies that $U^\dagger J U = J$. Equivalently, multiplication of the eigenvalue equation \[8\] by $U^\dagger$ gives $U^\dagger H U = J \Lambda$, showing that the transformation matrix $U$ indeed diagonalizes the Hamiltonian matrix $H$ with the appropriate boson metric $J$. Correspondingly, the quasiparticle operators follow from the matrix equation

$$\alpha = J U^\dagger J a, \quad (10)$$

and it is easy to see that the transformed unperturbed Hamiltonian has the expected diagonal form

$$H_\perp = \frac{\hbar}{2} a_\dagger \Lambda a = \frac{\hbar}{2} \omega_+ (a_+^\dagger a_+ + a_+ a_+^\dagger) + \frac{\hbar}{2} \omega_- (a_-^\dagger a_- + a_- a_-^\dagger) \quad (11)$$

that represents a set of uncoupled harmonic oscillators. Evidently, the spectrum of allowed states has the eigenvalues

$$\epsilon_{n_+, n_-} = \hbar(n_+ + \frac{1}{2}) \omega_+ + \hbar(n_- + \frac{1}{2}) \omega_- \quad (12)$$

where $n_\pm$ is a nonnegative integer. As is clear from Fig. \[4\], the set of lowest eigenvalues $\epsilon_{n_+, n_-} = \hbar(n_+ + \frac{1}{2}) \omega_+$ vanishes as $\Omega$ approaches the confinement limit ($\Omega \to \omega_x$). This degeneracy precludes a simple perturbation approach for the interacting system.
Equation (10) yields explicit expressions for the quasiparticle operators

$$\alpha_\pm = \cosh \chi_\pm (\cos \theta_\pm a_x \mp i \sin \theta_\pm a_y) + \sinh \chi_\pm (\sin \eta_\pm a_\pm^\dagger \pm i \cos \eta_\pm a_\pm^\dagger), \quad (13)$$

along with their adjoints. Here, the hyperbolic parameters $\chi_\pm$ and the trigonometric parameters $\theta_\pm$ and $\eta_\pm$ guarantee the correct commutation relations for the quasiparticle operators. In particular, the parameters $\chi_\pm$ determine the number-violating “Bogoliubov” coupling between the $a$ and $a^\dagger$ operators. As expected from the form of Eq. (3), these parameters vanish for $\Omega = 0$, and $\chi_-$ remains small and positive for all allowed $\Omega > 0$, whereas $\chi_+$ is negative and decreases rapidly as $\Omega \to \omega_x$. For an axisymmetric trap, $\chi_\pm$ vanish identically for all $\Omega$, and the quasiparticle operators $\alpha_\pm$ then reduce to the familiar helicity operators $a_\pm = (a_x \mp ia_y)/\sqrt{2}$.

We can now analyze the noninteracting eigenstates. The rotating ground state $\varphi_{00}(x, y)$ is determined from the pair of conditions $\alpha_\pm \varphi_{00} = 0$; it has the form

$$\varphi_{00}(x, y) = \left(\frac{\ln \pi^2}{\pi^2}\right)^{1/4} \exp \left[-\frac{1}{2}(lx^2 + 2imxy + ny^2)\right], \quad (14)$$

where the real quantities $l$, $m$, and $n$ depend on the trap frequencies and the rotation speed [21]. For an axisymmetric trap with $\omega_x = \omega_y = \omega_\perp$, the ground state has the expected isotropic structure with $l = n = 1$ and $m = 0$. This state remains isotropic for all $\Omega < \omega_\perp$.

For an anisotropic trap with $\omega_x < \omega_y$, however, the noninteracting ground state has a nontrivial phase $\propto xy$ that represents the irrotational flow induced by the rotating trap [22][23][10][12][30]. For small rotation speeds and small asymmetry, the ground-state density is indistinguishable from that of the symmetric trap, so that it has an essentially circular shape fitting wholly into the elliptical trap geometry. For fast rotations, when the angular momentum per particle is considerable, the condensate experiences a torque that stretches the ground state density along the axis of lesser confinement (the $x$ axis in our choice). For 10% trap asymmetry, the central peak also decreases appreciably.

The normalized excited states are given by the familiar harmonic-oscillator construction

$$\varphi_{n_+ n_-}(x, y) = \frac{(\alpha_\pm^\dagger)^{n_+}}{\sqrt{n_+!}} \frac{(\alpha_\mp^\dagger)^{n_-}}{\sqrt{n_-!}} \varphi_{00}(x, y). \quad (15)$$

In the axisymmetric, static limit ($\omega_x = \omega_y = \Omega = 0$), $\varphi_{10}$ and $\varphi_{01}$ reduce to the degenerate pair $\psi_\pm$ discussed below Eq. (4). For $\omega_x < \omega_y$, the first excited noninteracting state $\varphi_{10}$ has an excitation energy $\hbar \omega_x$ and represents a nonaxisymmetric vortex with unit positive circulation and a node at the trap center. The other noninteracting singly quantized vortex state $\varphi_{01}$ also has a node at the trap center, with higher excitation energy $\hbar \omega_x$ and unit negative circulation.

A recent study [13] of a small rotating axisymmetric Bose-Einstein condensate works at fixed angular momentum $L_z = \hbar l$ (which is appropriate only for an axisymmetric trap). The resulting equilibrium configuration is then determined by minimizing the total energy $NE_{lab}(l)$ subject to the constraint of fixed $l$. The corresponding angular velocity $\Omega$ then follows from the relation $\Omega = \partial E_{lab}/\hbar \partial l$.

The constraint of fixed $L_z = \hbar l$ is analogous to the constraint of fixed total $N$ in the canonical ensemble. As in the transition to the grand canonical ensemble, however, it is often advantageous to eliminate the constraint by making a Legendre transformation from fixed $l$ to fixed $\Omega$, which here merely means transforming to a rotating frame. Since the resulting Hamiltonian $H = E^{(0)} + V_H$ then contains the term $-\Omega L_z$, the expectation value $\langle L_z \rangle$ for the angular momentum as a function of $\Omega$ follows directly from the Hellmann-Feynman theorem [31] $\langle L_z \rangle = -\partial E(\Omega)/\partial \Omega$, where $E = \langle H \rangle$ is the energy in the rotating frame. For example, the expectation value of the angular momentum for the noninteracting eigenstate $\varphi_{n_+ n_-}$ is simply

$$L_{n_+ n_-}/\hbar = -(n_+ + \frac{1}{2})\partial \omega_+ / \partial \Omega - (n_- + \frac{1}{2})\partial \omega_- / \partial \Omega, \quad (16)$$

as follows directly from Eq. (12). Figure 3 shows the dependence of $\omega_\pm$ on the external rotation $\Omega$, and the resulting $L_{n_+ n_-}$ also depends on $\Omega$ for any nonzero trap anisotropy. With the definition $L_\pm = -\hbar \partial \omega_\pm / \partial \Omega$, we have $L_{00} = \frac{1}{2}(L_+ + L_-)$ and $L_{10} = L_{00} + L_+$. These angular momenta per particle for the two lowest noninteracting states $\varphi_{00}$ and $\varphi_{10}$ are included in Figure 3. For both anisotropies shown ($\omega_y/\omega_x = 1.014$ and 1.1), $L_{00}$ remains small until $\Omega$ approaches $\omega_x$; in contrast, $L_{10}$ rises rapidly and linearly for small $\Omega$, remains close to one quantum of angular momentum for most of the allowed range, and then grows rapidly as $\Omega \to \omega_x$. 

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The asymmetric noninteracting excited states possess a rich structure and the complete density distribution is needed to characterize them. As an example, Fig. 2 presents density contours of $|\varphi_{30}|^2$ across the whole $xy$ plane, for slow, medium and fast rotation and for two different values of $\omega_y/\omega_x > 1$. The asymmetry aligns the vortex cores along the axis of weak confinement. At slow rotation ($\Omega/\omega_x = 0.01$), most of the condensate density accumulates at the ends on the weak axis. For the case of $\omega_y/\omega_x = 1.1$, the depletion along the tight axis essentially splits the condensate at the position of each vortex “core.” This result is plausible because this small $\Omega$ is in the asymmetry-dominated regime where $\Omega \omega_x \ll \omega_y^2 - \omega_x^2$. For medium rotation ($\Omega/\omega_x = 0.5$), the effect of the asymmetry smears out, and the densities resemble rotation-distorted symmetric trap eigenstates, with the vortex cores located close to the trap center and surrounded by a region of nearly constant density toward the edge. For fast rotation ($\Omega/\omega_x = 0.95$), the condensate stretches along the weak axis, particularly pronounced for $\omega_y/\omega_x = 1.1$. There, the non-ellipsoidal shape of the inner most contour indicates that two of the three overlapping vortex cores are now off-center on the $x$-axis. The corresponding loss in angular momentum is again compensated by having a larger density at the ends of the weak axis. Note that the depletion along the tight axis is here much less than for slow rotation so that the core area is still fully surrounded by the condensate.

B. Variational ansatz

The noninteracting eigenstates now allow us to treat the interacting problem. With the previous dimensionless spatial variables, the GP equation (4) becomes

$$H(0) - \mu + 4\pi\gamma \hbar \sqrt{\omega_x \omega_y} \psi^* \psi \psi = 0,$$

where $H(0) = H_L + H_z$, with $H_L$ in diagonal form taken from Eq. (11), $\gamma = Na/d_z$ is the small interaction parameter, and $\psi$ is the condensate wave function normalized to 1. Note that we assume a nonaxisymmetric trap rotating with a fixed angular speed $\Omega$. This approach describes a “helium-bucket-like” experiment and complements the theoretical work on vortices in small axisymmetric condensates (for example, Refs. [15,19]), where the angular momentum (a good quantum number) is fixed.

As is obvious from the vanishing of the noninteracting eigenfrequency $\omega_L(\Omega)$ for $\Omega \rightarrow \omega_x$, the lowest eigenstates $\varphi_{n0}$ become nearly degenerate for asymmetric traps, and straightforward perturbation theory is not applicable. Because we are interested in small condensates with $\gamma \lesssim 1$, however, the noninteracting eigenfunctions discussed in Sec. II.A provide a suitable basis set for an expansion of the planar part of the interacting condensate wave function. Our strategy is thus to use a linear combination of the lowest eigenstates as a trial function [15],

$$\psi_n(x,y) = \sum_{s=0}^{n} c_s \varphi_{s0}(x,y),$$

where $n$ indicates the cutoff of the $n$th-order trial function at the excited state $\varphi_{n0}$. This cutoff makes the calculation tractable; it can be chosen so that the trial function captures the essential physics for rotation speeds at which higher excited states do not contribute. This trial function restricts the region in the phase diagram that we can investigate.

We assume that the bosons are in their ground state along the axis of rotation $\varphi_0(z) = \pi^{-1/4} \exp(-z^2/2)$. The variational ground state follows by minimizing the free-energy functional

$$\frac{E}{\hbar} = \sum_{s} |s| \omega_{s}^{(0)} + \sqrt{2} \omega_x \omega_y \sum_{ijkl} c_i^* c_j^* c_k c_l I_{ijkl},$$

with respect to the set of variational parameters $c_s$. Here $\omega_{s}^{(0)} = (s + \frac{1}{2}) \omega_x + \frac{1}{2} (\omega_y - \omega_x)$ and $I_{ijkl} = \int d^2 r \varphi_{i0}^* \varphi_{j0}^* \varphi_{k0} \varphi_{l0}$. The variation is constrained by the normalization condition

$$\int d^2 r \psi^* \psi = \int d^2 r \sum_{s'=0}^{n} c_{s'}^* c_{s'} \varphi_{s0}^* \varphi_{s0}(x,y) = \sum_{s=0}^{n} |c_s|^2 = 1.$$
C. Numerical implementation

We use a well-known simplex algorithm [2] to minimize the energy (19) in terms of the \( c_s \) as variational variables. The algorithm basically determines the function to be minimized with respect to \( 2n \) variational parameters at the corners of a \( 2n+1 \)-dimensional simplex in parameter space. We reduce the number of independent variables by choosing the phase of one of the \( c_s \) to be equal to zero and rewriting its modulus explicitly as a function of the other variational parameters \( |c_k| = \sqrt{1 - \sum_{s \neq k} |c_s|^2} \). The energy functional is restricted to the unit sphere. We filled the space outside by taking the values of the energy on the surface of the unit sphere and magnifying them with increasing distance from the sphere. This procedure allows us to minimize in an unrestricted space and yet ensures that the variational equilibrium state satisfies the norm condition. In principle, it makes no difference which \( c_k \) we exclude from the minimization procedure. However, since we have to find the minimum with respect to many parameters and since the energy functional has many local minima, we swept all \( k = 0, \ldots, n \) for a given point in the phase diagram. This way, we repeated the minimization on \( n+1 \) different representations of the energy and thus drastically enhanced the likelihood of finding the true global minimum.

We monitored the reliability of our results by comparing the contributions of higher excited states. We conclude that we have captured all important ingredients when the distribution of weights among the \( c_s \) was not affected by including higher excited states into the trial function. For reasons of symmetry, this test required increasing \( n \) by at least 2. The minimization was very robust and reliable for most rotation speeds and interaction strengths that we examined. The behavior of the resulting \( c_s \) with increasing rotation \( \Omega \) was generally very smooth apart from the transition lines between equilibrium states with different numbers of vortices. Another check was the convergence of the equilibrium energies for different \( n \). If a higher basis state was important, the energy clearly decreased when it was included. If the additional dimensions in the minimization space were irrelevant, the resulting minimal energy was the same within the numerical errors.

III. THE PHASE DIAGRAM

In this section, we determine the lowest transition lines between variational equilibrium states in \( \Omega-\gamma \) space for symmetric traps and for weakly and moderately asymmetric traps. In discussing the variational equilibrium condensate, we clarify our criterion for the transition with increasing rotation speed \( \Omega \) when the interaction and the trap geometry are fixed. Moreover, we investigate the change in the character of the transition in different regions of the phase diagram.

A. Symmetric geometry

First, we review the phase diagram for an axisymmetric trap that serves as a comparison for the new features induced by the asymmetry. The transition lines between equilibrium states are determined by minimizing the energy (19) for many sets of parameters \((\omega_y, \Omega, \gamma)\) and comparing the resulting states. We call a state a \( q \) vortex if \( \varphi_{q0} \) dominates the complete trial function \( \psi \), namely \( |c_q|^2 > |c_s|^2 \), for all \( s \neq q \). A transition between two states is identified by a change in the dominant weight from \( |c_q|^2 \) to \( |c_q'|^2 \). This criterion works well for “clear” states, where one particular \( |c_q|^2 \) dominates strongly over the other contributions.

From previous theoretical studies [15,33] and from the experiments of the Paris and MIT groups [6,7], we expect that the axisymmetric condensate accommodates the angular momentum associated with the rotation by a sequence of transitions to states with more vortices. For the axisymmetric trap potential, our results agree completely with the earlier theoretical phase diagram [15]. As seen in the dot-dashed lines in Fig. 3, the first two transition lines are strictly linear in \( \gamma \) and pass through \( \Omega = \omega_x \) in the noninteracting limit \((\gamma \to 0)\). As expected, the line \( \Omega_\gamma(\gamma) \) for the first transition to a state with a single vortex agrees exactly with the critical frequency \( \Omega_c/\omega_x = 1 - \gamma/\sqrt{8\pi} \) obtained with first-order perturbation theory [15,34]. Most of our data are obtained with an 8th-order trial function (the area very close to \( \Omega \to \omega_x \) has been tested by using \( n = 10 \)). For the phases with up to two vortices, only the lowest five \( \varphi_{q0} \) contribute significantly.

In order to characterize the various phases more precisely, it is instructive to monitor the behavior of the variational parameters along a vertical cut in the phase diagram, namely with increasing \( \Omega \) for a fixed interaction strength \( \gamma \). Such a cut is included in Fig. 4 for an axisymmetric trap with \( \gamma = 1 \) (we measure \( \Omega \) in units of \( \omega_x \)). We plot the \( |c_0|^2 \) starting in the no-vortex phase. Below the \( \Omega \) range shown in Fig. 4, we always find \( |c_0|^2 = 1 \), confirming that the noninteracting ground state is also the variational equilibrium state for these slow rotations. At \( \Omega_c = 0.80053 = 1 - 1/\sqrt{8\pi} \), there
is a sharp transition to a new state that consists purely of the noninteracting \( \varphi_{10} \) state. This behavior identifies the critical frequency for thermodynamic stability of a singly quantized central vortex. The next three transitions to other combinations are also clearly seen, indicating a sequence of transitions to states with an increasing number of singly quantized vortices, each in a well-separated range of \( \Omega \). Beyond the one-vortex phase, the variational states are mixtures of various noninteracting states. In particular, the two-vortex phase involves mixing with other states of two-fold symmetry (\( \varphi_{00}, \varphi_{40} \)) and the three-vortex phase similarly contains other states with three-fold symmetry (\( \varphi_{00}, \varphi_{60} \)). For rotation speeds higher than \( \approx 0.98 \omega_x \), we are too close to the degeneracy limit \( \Omega = \omega_x \) to exclude the possibility that higher-order trial functions might change the distribution of weights. In the present symmetric case (dot-dashed lines in Fig. 3), the order of appearance of the different phases is the same for any vertical cut at fixed \( \gamma \) in the phase diagram.

We have included the angular momentum \( L_z(\Omega) = -\partial E/\partial \Omega \) in Fig. 4 for \( \gamma = 1 \). Each transition that adds one more vortex induces a discontinuous upward jump in the angular momentum. Evidently, not all values of angular momentum are allowed. Specifically, values in the range \( 0 < L_z < 1 \) are absent, which is consistent with the character of the equilibrium functions before and after the first transition (they consists purely of \( \varphi_{00} \) or \( \varphi_{10} \), carrying exactly zero or one quantum of angular momentum respectively). For the allowed ranges of \( L_z \), direct comparison with Ref. [19] shows that we found the same mixtures of \( \varphi_{10} \), although they fix the angular momentum. Since the transition lines are linear in \( \gamma \), we can also make contact with the results from Ref. [15]. In particular, the first three transition frequencies indicated in their Fig. 2 for fixed angular momentum are the same as those found here for fixed rotation speed \( \Omega \). Furthermore, the phases have the same symmetry, the lowest two being pure \( \varphi_{00} \) and \( \varphi_{10} \) and thus rotationally symmetric, whereas the equilibrium state of the \( q = 2 \) and \( q = 3 \) phases have off-center vortex cores arranged to give a two-fold or three-fold symmetric structure.

B. Asymmetric geometry

We investigate two specific trap asymmetries in detail. A weakly distorted trap with \( \omega_y/\omega_x = 1.014 \) already shows some new features and reflects a delicate balance between the symmetric (rotation-dominated) and asymmetric (trap-dominated) influences. A second trap geometry of \( \omega_y/\omega_z = 1.1 \) displays more pronounced effects of the asymmetry and thus provides a clearer picture of the basic physics. Figure 3 shows the first two transitions in the \( \gamma-\Omega \) plane for both asymmetries, along with the corresponding curves for the symmetric trap with \( \omega_y/\omega_x = 1 \).

1. Transition lines

First, note that the transitions for the asymmetric traps occur at higher rotation speeds \( \Omega \) than in the symmetric case; in addition, the shift increases with increasing asymmetry. In contrast to the linear behavior (\( \Omega_c \propto \gamma \)) of the symmetric system, the transition lines curve significantly for small interaction strengths and high rotation speeds. Most remarkably, there is a critical threshold coupling constant \( \gamma_c \) below which a ground state with a singly quantized vortex is never favorable. This behavior is understandable because the vortex core size decreases with increasing interaction parameter and only fits into the trapped condensate for not too weak interactions. Since the semi-minor axis of the ellipsoidal trap fixes the size available for a vortex core, a smaller asymmetry allows the introduction of a vortex at lower \( \gamma \).

Second, the term \( -\Omega L_z \) in the Hamiltonian tends to favor states with large angular momentum. In addition to increasing the number of vortices, a greater trap asymmetry makes the condensate more susceptible to rotation-induced elongation, placing a greater part of the condensate farther away from the rotation axis, increasing the moment of inertia and hence the angular momentum. In this way, the condensate can accommodate a higher angular momentum without introducing vortices.

For fixed interaction strength \( \gamma \) not too far above \( \gamma_c \), we even find a re-entrant region where the one-vortex phase is followed for higher rotation speeds by a no-vortex phase. The tip of the one-vortex phase surrounded by the no-vortex phase for the small asymmetry (\( \omega_y/\omega_x = 1.014 \)) is illustrated by the occupancies \( |c_{s}|^2 \) along a cut through that tip (at \( \gamma = 0.08935 \), Fig. 4). The occupancies in the re-entrant no-vortex phase continue as if it had never been interrupted by the one-vortex phase. For illustration, Fig. 5 also includes typical density contours for \( \Omega \) around the lobe tip. The elongation along the horizontal \( x \) axis for the vortex-free states is pronounced. The thinner waist of the condensate on the vertical axis results from the admixture of \( \varphi_{20} \). The one-vortex state again has only \( \varphi_{10} \) as a constituent and illustrates how the circular ring of maximal density in the symmetric case deforms to two pronounced density peaks on the \( x \) axis for this range of \( \Omega \). This re-entrant behavior reflects the singular character of the limit \( \Omega \to \omega_x \) for asymmetric geometries, when the confinement parameter \( l \) in Eq. (4) tends to zero. Below the threshold
interaction strength for vortex stabilization, the density contours always represent elongated no-vortex states; their width is smaller for smaller interactions strengths and increases slowly with increased rotation. This elongation seems to hinder vortex formation.

In order to study the detailed structure of the phases in the asymmetric cases, we considered again the intermediate interaction strength $\gamma = 1$ and determined the occupancies $|c_1|^2$ for small ($\omega_y/\omega_x = 1.014$) and moderate ($\omega_y/\omega_x = 1.1$) asymmetry, as shown in Fig. 3 and Fig. 4 respectively. As the applied rotation increases, the smaller asymmetry shows phases with increasing number of vortices, just in the case of a symmetric trap (compare Fig. 2 for a symmetric trap). In contrast to the re-entrant behavior for $\gamma = 0.09$ (Fig. 3), we conclude that these relatively strong interactions $\gamma = 1$ eliminate the effect of (small) asymmetry, in part because the transitions occur at slower angular velocity. For the moderate asymmetry ($\omega_y = 1.1 \omega_x$), however, the phase diagram in Fig. 4 still exhibits re-entrant behavior; thus a 10%-asymmetry dominates the behavior for this interaction strength ($\gamma = 1$) and precludes more than two vortex cores. Note that the one-vortex phase continues for $\Omega/\omega_x \geq 0.99$ as if it had not been interrupted by the two-vortex phase.

Typical density contours for the two asymmetries are included in Fig. 8. Several features differ significantly from the symmetric geometry. The no-vortex phase has a considerable $\varphi_{20}$ admixture, producing a constriction along the $y$ axis. In the one-vortex phase, the condensate has a central vortex, but the elliptical trap and rotation-induced elongation deform the condensate noticeably. For $\omega_y = 1.1 \omega_x$, the contributions from $\varphi_{20}$ (and other odd states) grow as $\Omega$ approaches the transition to two vortices, deforming the surface region because of four vortex cores that move in from infinity. The density contours for the interacting two-vortex state show the admixture of the noninteracting ground state because the vortex cores are pushed further from the center of the trap than in the symmetric case. The small admixtures of other basis states with an even number of vortex cores favors the accumulation of density closer to the center (outside the core regions).

For the smaller asymmetry, the subsequent figures display three and four separate cores, respectively. In the three-vortex phase, however, the cores are not symmetrically distributed around the center (the appreciable occupation of $\varphi_{20}$ enhances the two vortices along the $x$ axis, placing the third core on the horizontal axis and further away from the trap center). From Fig. 6, note that the mixture of noninteracting states differs from the symmetric case (Fig. 3), where the states contained only noninteracting states with the same rotational symmetry. The last two density contours in the right column illustrate that the re-entrant phases can indeed be classified as states with one or zero vortices. For such fast rotations, the condensate is very elongated and very flat (as can be seen from the fewer density contours). It is energetically favorable to reduce the number of vortices (which need a wider condensate) and to compensate the loss of vorticity in the vortex cores by expanding the condensate along the horizontal axis. In the re-entrant $\varphi_{000}$-phase, we have several small contributions from basis states with an even number of cores. This causes small ripples on a thin extended Gaussian density. Basis states with an odd number of cores would put a density minimum at the center of the trap, but the dominant $\varphi_{000}$ suppresses this tendency.

In recent experiments, the Paris group measured the nucleation of vortices in their large, cigar-shaped condensate and the corresponding angular momentum resulting from the vortices alone. When ramping up the rotation, they eventually find vortex-free states again, which is a re-entrant phenomenon similar to what we find in our analysis for much smaller condensates. Moreover, they also measure the nucleation of vortices when sweeping the asymmetry $\epsilon$ and keeping the rotation fixed, leading again to a window of vortex stabilization. For the small condensate, we also determined the transition to a one-vortex state in the $\Omega-\omega_y$ plane, as illustrated for several $\gamma$ and asymmetries up to 20% (Fig. 5). For asymmetries larger than the rightmost end of the graphs, there is no vortex state for rotations up to $\Omega/\omega_x = 1$. In agreement with the $\Omega-\gamma$ phase diagram (Fig. 3), we here see the re-entrance in that $\Omega-\omega_y$ phase diagram, leading to a qualitatively similar restricted rotation window for stabilizing a vortex.

2. Types of transition

Having discussed the typical phases, we can now consider the details of the transitions themselves. The occupations of the separate constituent noninteracting states (compare Fig. 3 and Fig. 4) have an important new feature. There is a smooth transition between the no-vortex and the one-vortex state for $\gamma = 1$ for both asymmetric geometries, reminiscent of a second-order transition. This phenomenon can already be found for a very small asymmetry, as illustrated for $\omega_y = 1.001 \omega_x$ in Fig. 10. The critical frequency $\Omega_c = (0.80125 \pm 0.00005) \omega_x$ is a few per cent larger than $\Omega_c = 0.80053 \omega_x$ for the symmetric trap. Nevertheless, the density contours close to the transition reveal that a vortex core gradually enters the condensate along the $y$ axis (as seen in Fig. 10). This behavior is qualitatively distinct from that for the symmetric case (for the first transition at $\gamma = 1$, we examined points as close as rotation speeds $\Delta(\Omega/\omega_x) = \pm 10^{-7}$). Thus we infer that the character of the transition in asymmetric traps differs fundamentally from the symmetric case, for we now have a cross-over region where both $c_0$ and $c_1$ are nonzero. In this situation,
the meaning of the critical transition frequency $\Omega_c$ becomes somewhat blurred. For both asymmetric geometries ($\omega_y/\omega_x = 1.014$ and 1.1), this cross-over region shrinks for smaller interactions. In fact, we find a sharp transition below $\gamma < 0.1$ for the smaller asymmetry and below $\gamma < 0.8$ for the moderate asymmetry. For all other subsequent equilibrium phases, we found spontaneous jumps in the occupancies, similar to first-order transitions.

The change in character can be understood as follows: for parameters that lead to a sharp transition, the energy functional has two competing, well-separated main minima, one lying in the $c_0$-dominated sector and the other in the $c_1$-dominated sector. Indeed, depending on whether we choose $c_0$ or $c_1$ to implement the norm condition in our minimization procedure, we find one or the other minimum in the neighborhood of the transition. The comparison of the energies then gives the true global minimum. The depth of these two minima gradually changes with $\Omega$; at the transition, the minimum representing the one-vortex state becomes deeper. This picture allows for hysteresis in stabilizing a single vortex with increasing rotation. In fact, hysteresis is the favored explanation for the deviation of the measured $\Omega_c$ from the Thomas-Fermi predictions (see [33,34] and references therein). Moreover, the fact that the phase after re-entrance appear to be the continuation of the phase before the previous transition means that the corresponding minimum still exists and again lies below the energy of the intervening higher-vortex state. For a continuous transition, in contrast, the minimum energy functional must lie in a valley connecting the two sectors. Here, the different implementations of the norm condition lead to the same minimal state. With increasing $\Omega$, this global minimum gradually moves along this valley from the $c_0$-dominated sector through a cross-over region to the $c_1$-dominated sector.

From Fig. 3 for the smaller asymmetry with $\omega_y = 1.014 \omega_x$, we can also observe that the three-vortex phase undergoes much stronger changes across its range than other phases. Although there is a sharp transition to a phase with $\varphi_{30}$, an appreciable amount of the two-vortex basis-state remains and $\varphi_{20}$ dies out only gradually. This behavior yields a very complicated picture for smaller interaction strengths (roughly at $\gamma \approx 0.6$), where the $\varphi_{30}$-dominated phase disappears completely leading to a direct transition from two to four vortices. For even smaller $\gamma$, the two-vortex phase is followed by a one-vortex phase before the four-vortex phase develops. In part, this complicated picture arises from the suppression of the $\varphi_{30}$ contribution to the one-vortex phase (which in the moderately asymmetric trap with $\omega_y = 1.1 \omega_x$ leads to an contribution of $\varphi_{50}$ rather than $\varphi_{30}$ to the one-vortex phase).

**IV. THE ANGULAR MOMENTUM**

The asymmetry in the trap geometry breaks the cylindrical symmetry, so that the angular momentum $L_z$ around the axis of rotation is no longer a good quantum number. We therefore investigate the effect of the asymmetry on $L_z$ in some detail.

The angular momentum (in units of $\hbar$) as a function of the trap rotation $\Omega$ is shown in Fig. 11 for all three geometries ($\omega_y/\omega_x = 1.0$, 1.014, and 1.1) for the fixed interaction $\gamma = 1$. The intimate relation to the occupancies within the various phases for each geometry can be seen in the corresponding plots (Figs. 2, 3 and 4). The sharp kinks in $L_z$ for the symmetric trap arise from the sudden changes in the occupancies and the different angular momentum carried by the noninteracting states. The finite slope of the plateaus beyond the second transition reflects the mixing of the various noninteracting states in the variational ground state and the off-center positions of the vortex cores.

For the asymmetric rotating trap, even the noninteracting eigenstates $\varphi_{s0}$ with a central $s$-fold vortex carry an angular momentum different from $s$ (in fact, the angular momentum diverges for $\Omega \to \omega_x$). The divergence comes from the contribution of the circulating quanta with positive helicity, $L_+(\Omega)$, occurring in the angular momentum of every basis state $\varphi_{s0}$. This is easily seen by expanding $L_+ = -\partial \omega_x/\partial \Omega$ for small $\omega_x - \Omega$ and small asymmetries $\omega_y^2 = 1 + 2 \epsilon^2$, leading to

$$L_+ \approx \frac{\epsilon}{\sqrt{\omega_x - \Omega}}$$

for $\Omega \to \omega_x$. (21)

In asymmetric traps, this divergence dominates the angular momentum in all noninteracting eigenstates $\varphi_{s0}$ for high rotation speeds and hence in any linear combinations of them. Thus, even discrete changes in the structure of the equilibrium state (as at a transition) will have a far less dramatic effect on the angular momentum at large rotation speeds near $\omega_x$. Moreover, asymmetric condensates are significantly elongated along the axis of weaker confinement as $\Omega \to \omega_x$. The resulting redistribution of density (and mass) with respect to the axis of rotation produces an additional, nonquantized contribution to the angular momentum.

Compared to the symmetric condensate, the angular momentum for the trap with smaller asymmetry at $\gamma = 1$ has its jumps smoothed over a small but finite range of $\Omega$ because of the continuous transition where one off-center vortex gradually moves towards the center (see Fig. 3); in addition, the transition is shifted toward higher rotation speeds (see Fig. 11). The subsequent sequence of phases corresponds to an increasing number of vortex cores that
cause nearly vertical jumps just as in the symmetric case. The divergence of the angular momentum dominates only at Ω/ωx ≈ 1.

For the moderate asymmetry, the stronger admixture of ϕ20 in the no-vortex phase yields a faster rise of the angular momentum below the first transition. The first kink is even more tilted because of the extended range for the transition to a one-vortex state. Inside this one-vortex phase, there is a broad region with a pure ϕ10 state (up to Ω/ωx ≈ 0.92). The angular momentum, however, exceeds one because of both the initial growth of Lz with Ω and the deformation of the condensate density. The rapid rise of Lz below the second transition reflects the growing admixture of ϕ50. That second transition, seen as a discontinuous derivative in Lz, occurs at an angular momentum Lz ≈ 2.4, already exceeding the value for a symmetric trap with two vortices. Beyond that point, the angular momentum increases smoothly with Ω, even beyond the re-entrant transitions at Ω ≈ 0.9905ωx and Ω ≈ 0.9945ωx respectively (see inset in Fig. 11). In this limit, the divergent Lz completely dominates the behavior.

V. CONCLUSION

The behavior of a small Bose condensate in a rotating anisotropic trap differs significantly from that of a symmetric condensate, even for very small asymmetries. In large part, this difference arises from the stretching of the condensate along the axis of weak confinement, especially for Ω → ωx. In addition, it reflects the irrotational flow induced by the rotating confining potential that pushes on the gas when viewed from the laboratory frame [29,9,10,12].

For any specific asymmetry and sufficiently small interaction strength, a one-vortex state can never be stabilized, as illustrated in Fig. 3. For somewhat larger values of the coupling constant γ, the first transition, from a no-vortex state to a one-vortex state, is sharp, but the subsequent transitions are re-entrant, back to a no-vortex state with significant elongation that carries the relevant angular momentum (see Figs. 5 and 8). For these weak interactions, the transitions all occur close to Ω ≈ ωx, so that the stretching dominates.

For larger interaction strength (γ ∼ 1), the transitions occur at lower values of Ω, and they become quasicontinuous in that the occupation |c0|2 of the noninteracting ground state vanishes smoothly (see Figs. 6, 7, and 10). As in the case of a classical fluid in a rotating elliptic cylinder [29], the angular momentum is finite even below the transition to a state with one vortex because of the irrotational flow induced by the rotating asymmetric trap (see Fig. 11).

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FIG. 1. The positive eigenvalues $\omega_{\pm}$ and the angular momenta $L_{00}$ and $L_{10}$ of the two lowest noninteracting eigenstates $\varphi_{00}$ and $\varphi_{10}$ as a function of the rotation speed $\Omega$ for asymmetries $\omega_y = 0.014$ (dashed lines) and $\omega_y = 1.1$ (solid lines). All quantities are in dimensionless units (scaled with respect to the lower oscillator frequency $\omega_x$).

FIG. 2. Density contours of the noninteracting $|\varphi_{30}|^2$ across the $xy$ plane for the traps with $\omega_y/\omega_x = 1.014$ (left) and $\omega_y/\omega_x = 1.1$ (right) for $\Omega/\omega_x = 0.01$, 0.5 and 0.95 (top to bottom). Distances are scaled in units of $d_x$ in both directions and the width shown is $6d_x$ across.

FIG. 3. The two lowest transition lines for $\omega_y = 1.0 \omega_x$ (dot-dashed lines), $\omega_y = 1.014 \omega_x$ (dotted lines) and $\omega_y = 1.1 \omega_x$ (solid lines), determined with trial functions up to 10th order. The lowest transition lines (a,b,c) represent the critical rotation $\Omega_c$ for the stabilization of a single vortex; at the second transition lines (d,e,f) the condensate starts to be dominated by the $\varphi_{20}$. There are unresolved higher vortex-phases beyond the second transition for sufficiently large interaction strength.

FIG. 4. $\omega_y = 1.0 \omega_x$, $\gamma = 1$ : Occupancies (left scale) and $L_z$ (right scale) for axisymmetric condensate as function of $\Omega$ for fixed interaction $\gamma = 1$. The numbers above the lines denote the corresponding $|c_s|^2$ and the starred solid line represents the angular momentum $L_z$. 
FIG. 5. Occupancies $|c_s|^2$ for the basis-states for $\omega_y = 1.014 \omega_x$, $\gamma = 0.08935$, calculated with an 8th-order trial function and typical density contours (at $\Omega = 0.99515$, 0.9952, 0.9953). Contributions for $s > 2$ are negligible. The tip of the pure $\varphi_{10}$-lobe cuts sharply into the no-vortex phase, which has an appreciable $\varphi_{20}$ admixture. Note the narrow range in $\Omega$. The density contours are shown over 6 oscillator lengths $d_x$ in each direction.

FIG. 6. $\omega_y = 1.014 \omega_x$, $\gamma = 1$: Occupancies (left scale) and $L_z$ (right scale, note upward shift). The numbers above the lines are short for the correspondent $|c_s|^2$ and small contributions from $|c_6|^2 \ldots |c_{10}|^2$ are not separately labeled. The starred solid line represents the angular momentum $L_z$.

FIG. 7. $\omega_y = 1.1$, $\gamma = 1$ : Occupancies (left scale) and $L_z$ (right scale). The numbers above the lines are short for the correspondent $|c_s|^2$ and small contributions from $|c_3|^2$, $|c_6|^2 \ldots |c_{10}|^2$ are not separately labeled. The starred solid line represents the angular momentum $L_z$.

FIG. 8. Density contours for variational equilibrium states for small (1.014 left) and moderate (1.1 right) asymmetry and $\gamma = 1$. The pictures represent typical states within the different phases (cf. Fig. 5 and 6 respectively). For the small asymmetry, the number of vortices increases from zero to four ($\Omega = 0.805, 0.935, 0.95, 0.97, 0.982$) whereas in the moderate asymmetry, we find re-entrant behavior into one and zero-vortex states ($\Omega = 0.87, 0.9, 0.97, 0.993, 0.997$). The maximum width shown is 6 oscillator lengths $d_x$.

FIG. 9. Lines of critical rotation for the stabilization of one vortex versus the asymmetry, for $\gamma = 0.8, 1, 1.5$ (left to right), from 8th-order trial functions. The one-vortex states are to the left of each graph, the vortex-free ones to the right. All quantities are given in dimensionless units.

FIG. 10. Occupancies and density contours for $\omega_y = 1.001$, $\gamma = 1$, from an 8th-order trial function. The occupancies for $\varphi_{30}$ and higher are not labeled for clarity. The density contours are taken at rotations $\Omega = 0.8007, 0.8011, 0.8012, 0.8013$ and are about 4 oscillator lengths $d_x$ across.

FIG. 11. The angular momentum $L_z$ in units of $\hbar$ as a function of the angular velocity $\Omega$ for the three geometries $\omega_y/\omega_x = 1.0, 1.014, 1.1$ (dot-dashed, dotted and solid lines respectively). The inset shows the magnified re-entrance range for $\omega_y/\omega_x = 1.1$. 

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