Certified Equational Reasoning via Ordered Completion
Christian Sternagel and Sarah Winkler

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28 August 2019, Natal
Automated Reasoning Systems

- sophisticated pieces of software
Automated Reasoning Systems

- sophisticated pieces of software

\[
\begin{align*}
x - 0 & \approx x & s(x) - s(y) & \approx x - y \\
0 - y & \approx 0 & s(x) > s(y) & \approx x > y \\
x ÷ y & \approx (0, y) & x ÷ y & \approx (s(q), r) \\
s(x) > 0 & \approx \text{true} & s(x) \preceq s(y) & \approx x \preceq y \\
0 \preceq x & \approx \text{true}
\end{align*}
\]
Automated Reasoning Systems

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\[
\begin{align*}
    x - 0 & \approx x \\
    0 - y & \approx 0 \\
    x \div y & \approx \langle 0, y \rangle \\
    s(x) \succ 0 & \approx \text{true} \\
    0 \preceq x & \approx \text{true} \\
    s(x) & \approx s(y) \approx x - y \\
    s(x) \succ s(y) & \approx x \succ y \\
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heuristics
Motivation

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s(x) \succ s(y) & \approx x \succ y \\
\langle s(q), r \rangle \approx x \div y & \approx (s(q), r) \\
s(x) \preceq s(y) & \approx x \preceq y \\
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Motivation

Automated Reasoning Systems

- sophisticated pieces of software
- producing complex derivations: trustworthy?

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Automated Reasoning Systems

▶ sophisticated pieces of software
▶ producing complex derivations: trustworthy? better check!

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\[
\begin{align*}
x - 0 & \rightarrow x \\
s(x) - s(y) & \rightarrow x - y \\
x \div y & \rightarrow 0, y \\
s(x) & \rightarrow \text{true} \\
s(x) - s(y) & \rightarrow x \preceq y \\
0 - x & \rightarrow 0 \\
\langle 0, x \rangle & \approx \langle 0, y \rangle \\
\langle s(x), y \rangle & \approx \langle s(q), r \rangle \\
0 \preceq x & \rightarrow \text{true} \\
s(x) \preceq s(y) & \rightarrow x \preceq y \\
s(x) \succ s(y) & \rightarrow x \succ y \\
0 - x & \rightarrow 0 \\
\langle s(q), r \rangle & \approx \langle 0, y \rangle \\
\langle 0, x \rangle & \approx \langle 0, y \rangle
\end{align*}
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Motivation

Automated Reasoning Systems

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\[
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s(x) > s(y)) \approx x > y \\
x \div y \approx \langle s(q), r \rangle \\
s(x) > 0 \approx \text{true} \\
s(x) \leq s(y) \approx x \leq y \\
0 - x \rightarrow 0 \\
\langle 0, x \rangle \approx \langle 0, y \rangle
data:image/png;base64,iVBORw0KGgoAAAANSUhEUgAAAIgAAAAgCAYAAADAQsX1AAAABGd7o6WAAAAABJRU5ErkJggg==}

Ordered Completion

Input: set of input equalities \( \mathcal{E}_0 \)
Output: ground complete TRS \( \mathcal{E}^\succ \cup \mathcal{R} \)
Motivation

Automated Reasoning Systems

- sophisticated pieces of software
- producing complex derivations: trustworthy? better check!

Ordered Completion

Input: set of input equalities $\mathcal{E}_0$
Output: ground complete TRS $\mathcal{E}^> \cup \mathcal{R}$

Applications

- decide ground equational theory
- used by confluence tool ConCon to decide infeasibility of CPs
Contributions

- Extend formal library IsaFoR with
- Finite ordered completion runs
- Ground joinability criteria
- Add proof checks to CeTA for
- Ordered completion runs
- Satisfiability (TPTP) proofs in equational logic
- Infeasibility of conditional critical pairs
- Respect output in equational theorem prover MædMax and ConCon
Contributions

- Extend formal library IsaFoR with finite ordered completion runs
- Ground joinability criteria
- Add proof checks to certificate CeTA for ordered completion runs
- Satisfiability (TPTP) proofs in equational logic
- Infeasibility of conditional critical pairs
- Respectively output in equational theorem prover MædMax and ConCon

The IsaFoR/CeTA Framework
The IsaFoR/CeTA Framework

Contributions

- extend formal library IsaFoR with
  - finite ordered completion runs
  - ground joinability criteria
The IsaFoR/CeTA Framework

Contributions

- extend formal library IsaFoR with
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- extend formal library IsaFoR with
  - finite ordered completion runs
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  - satisfiability (TPTP) proofs in equational logic
  - infeasibility of conditional critical pairs
- respective output in equational theorem prover MædMax and ConCon
Outline

Preliminaries

Ordered Completion

Ground Joinability Criteria

Proof Checking

Conclusion
Definition

term rewrite system $\mathcal{R}$ is

- terminating if $\not\exists \ t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow_{\mathcal{R}} t_3 \rightarrow_{\mathcal{R}} \cdots$

Definition (Ordered Rewriting)

$E >\mathcal{=} = \{ s_{\sigma} \rightarrow t_{\sigma} | s \approx t \in E \pm$ and $s_{\sigma} > t_{\sigma} \}$
Definition

term rewrite system \( \mathcal{R} \) is

- terminating if \( \not\exists \ t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow_{\mathcal{R}} t_3 \rightarrow_{\mathcal{R}} \cdots \)

- ground confluent if for all ground terms \( s, t, u \) such that \( s \overset{\ast}{\rightarrow}_{\mathcal{R}} u \overset{\ast}{\rightarrow}_{\mathcal{R}} t \) there is some \( v \) such that \( s \overset{\ast}{\leftarrow}_{\mathcal{R}} v \overset{\ast}{\rightarrow}_{\mathcal{R}} t \)
Definition

term rewrite system $\mathcal{R}$ is

- terminating if $\not\exists t_1 \xrightarrow{\mathcal{R}} t_2 \xrightarrow{\mathcal{R}} t_3 \xrightarrow{\mathcal{R}} \cdots$

- ground confluent if for all ground terms $s, t, u$ such that $s \xrightarrow{\mathcal{R}} u \xrightarrow{\mathcal{R}} t$ there is some $v$ such that $s \xleftarrow{\mathcal{R}} v \xrightarrow{\mathcal{R}} t$
Definition

A term rewrite system \( \mathcal{R} \) is

- **terminating** if there is no infinite sequence \( t_1 \rightarrow_\mathcal{R} t_2 \rightarrow_\mathcal{R} t_3 \rightarrow_\mathcal{R} \cdots \)

- **ground confluent** if for all ground terms \( s, t, u \) such that \( s \stackrel{\mathcal{R}}{\leftarrow} u \rightarrow_\mathcal{R}^* t \) there is some \( v \) such that \( s \rightarrow_\mathcal{R}^* v \stackrel{\mathcal{R}}{\leftarrow} t \)

- **ground complete** if terminating and ground confluent
Definition

term rewrite system $\mathcal{R}$ is

- terminating if $\not\exists t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow_{\mathcal{R}} t_3 \rightarrow_{\mathcal{R}} \cdots$

- ground confluent if for all ground terms $s$, $t$, $u$ such that $s \xrightarrow{\mathcal{R}} u \rightarrow_{\mathcal{R}}^* t$ there is some $v$ such that $s \xrightarrow{\mathcal{R}}^* v \rightarrow_{\mathcal{R}}^* t$

- ground complete if terminating and ground confluent

- terms $s$ and $t$ are ground joinable in $\mathcal{R}$, denoted $s \xrightarrow{\mathcal{R}}^g t$ if $s\sigma \xrightarrow{\mathcal{R}}^* t\sigma$ for all ground $s\sigma$, $t\sigma$
Definition

term rewrite system $\mathcal{R}$ is

- terminating if $\not\exists t_1 \rightarrow_{\mathcal{R}} t_2 \rightarrow_{\mathcal{R}} t_3 \rightarrow_{\mathcal{R}} \cdots$
- ground confluent if for all ground terms $s$, $t$, $u$ such that $s \rightarrow_{\mathcal{R}}^* u \rightarrow_{\mathcal{R}}^* t$ there is some $v$ such that $s \rightarrow_{\mathcal{R}}^* v \rightarrow_{\mathcal{R}}^* t$
- ground complete if terminating and ground confluent
- terms $s$ and $t$ are ground joinable in $\mathcal{R}$, denoted $s \downarrow^{g}_{\mathcal{R}} t$ if $s\sigma \downarrow_{\mathcal{R}} t\sigma$ for all ground $s\sigma$, $t\sigma$
- reduction order is ground-total if $s > t$ or $t > s$ for all ground $s \neq t$
Definition

term rewrite system \( \mathcal{R} \) is

- terminating if \( \nexists \; t_1 \rightarrow \mathcal{R} \; t_2 \rightarrow \mathcal{R} \; t_3 \rightarrow \mathcal{R} \cdots \)
- ground confluent if for all ground terms \( s, t, u \) such that \( s \xleftarrow{\mathcal{R}} u \rightarrow^{*} \mathcal{R} t \) there is some \( v \) such that \( s \rightarrow^{*} \mathcal{R} v \xleftarrow{\mathcal{R}} t \)
- ground complete if terminating and ground confluent
- terms \( s \) and \( t \) are ground joinable in \( \mathcal{R} \), denoted \( s \downarrow_{\mathcal{R}}^{g} t \) if \( s \sigma \rightarrow_{\mathcal{R}} t \sigma \) for all ground \( s \sigma, t \sigma \)
- reduction order is ground-total if \( s > t \) or \( t > s \) for all ground \( s \neq t \)

Definition (Ordered Rewriting)

\[ \mathcal{E}_{\succ} = \{ s \sigma \rightarrow t \sigma \mid s \approx t \in \mathcal{E}^{\pm} \; \text{and} \; s \sigma > t \sigma \} \]
Outline

Preliminaries

Ordered Completion

Ground Joinability Criteria

Proof Checking

Conclusion
Definition (oKB)
for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $\succ$ have six inference rules:
Definition (oKB)
for equations $E$, rules $R$, reduction order $\succ$ have six inference rules:

\[
\frac{E, R}{E \cup \{ s \approx t \}, R}
\]

\[
\text{if } s \not\rightarrow_{E \cup R} \cdot \not\rightarrow_{E \cup R} \text{ then } t
\]
Ordered Completion

Definition (oKB)
for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $\succ$ have six inference rules:

**deduce**

$$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$$

if $s \mathcal{R} \mathcal{E} \leftrightarrow \cdot \leftrightarrow \mathcal{R} \mathcal{E} t$

$$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}}$$

**orient**

if $s \succ t$

$$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}}$$

$$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}}$$
Definition (oKB)
for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $\succ$ have six inference rules:

**deduce**

\[
\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{ s \approx t \}, \mathcal{R}}
\]

if $s \mathcal{R} \mathcal{E} \leftrightarrow \cdot \leftrightarrow \mathcal{R} \mathcal{E} t$

\[
\frac{\mathcal{E} \cup \{ s \approx t \}, \mathcal{R}}{\mathcal{E} \cup \{ s \rightarrow t \}}
\]

orient

\[
\frac{\mathcal{E} \cup \{ t \approx s \}, \mathcal{R}}{\mathcal{E} \cup \{ s \rightarrow t \}}
\]

if $s \succ t$
Ordered Completion

Definition (oKB)
for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $\succ$ have six inference rules:

**deduce**

$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$

if $s \mathcal{R} \mathcal{E} \leftrightarrow \cdot \leftrightarrow \mathcal{R} \mathcal{E} t$

$\frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}}$

**orient**

$\frac{\mathcal{E} \cup \{t \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}}$

if $s \mathcal{R} \mathcal{E} \succ t$

**delete**

$\frac{\mathcal{E} \cup \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$
Ordered Completion

Definition (oKB)
for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $\succ$ have six inference rules:

- **deduce**: $\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{ s \approx t \}, \mathcal{R}}$
  if $s \mathcal{R} \mathcal{E} \iff \mathcal{R} \mathcal{E} t$

- **compose**: $\frac{\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}}{\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow u \}}$
  if $t \rightarrow_{\mathcal{R} \mathcal{E} \succ} u$

- **orient**: $\frac{\mathcal{E} \cup \{ s \approx t \}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}}$
  $\frac{\mathcal{E} \cup \{ t \approx s \}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}}$
  if $s \succ t$

- **delete**: $\frac{\mathcal{E} \cup \{ s \approx s \}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$
Ordered Completion

Definition (oKB)
for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $>$ have six inference rules:

| Inference Rule | Premise | Conclusion |
|----------------|---------|------------|
| deduce         | $\mathcal{E}, \mathcal{R}$ | $\mathcal{E} \cup \{ s \approx t \}, \mathcal{R}$ |
|                | if $s \mathcal{R} \mathcal{E} \leftrightarrow t$ | if $s \mathcal{R} \mathcal{E} \leftrightarrow t$ |
|                | $\mathcal{E} \cup \{ s \approx t \}, \mathcal{R}$ | $\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}$ |
|                | $\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}$ | $\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow u \}$ |
|                | if $s \mathcal{R} \mathcal{E} \leftrightarrow t$ | if $t \mathcal{R} \mathcal{E} \rightarrow u$ |
| orient         | $\mathcal{E} \cup \{ s \approx s \}, \mathcal{R}$ | $\mathcal{E}, \mathcal{R}$ |
|                | $\mathcal{E} \cup \{ t \approx s \}, \mathcal{R}$ | $\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}$ |
|                | $\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}$ | $\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow u \}$ |
|                | if $s \mathcal{R} \mathcal{E} \leftrightarrow t$ | if $t \mathcal{R} \mathcal{E} \rightarrow u$ |
| delete         | $\mathcal{E} \cup \{ s \approx s \}, \mathcal{R}$ | $\mathcal{E}, \mathcal{R}$ |
|                | $\mathcal{E}, \mathcal{R}$ | $\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}$ |
|                | if $s \mathcal{R} \mathcal{E} \leftrightarrow t$ | if $t \mathcal{R} \mathcal{E} \rightarrow u$ |

ordered rewriting
Ordered Completion

Definition (oKB)
for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $>\text{ have six inference rules:}$

- **Deduce**
  
  \[
  \frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{ s \rightarrow t \}, \mathcal{R}}
  \]
  
  if $s \not\in \mathcal{E} \rightarrow \mathcal{E}$

- **Compose**
  
  \[
  \frac{\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}}{\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow u \}}
  \]
  
  if $t \not\in \mathcal{E} \rightarrow \mathcal{E}$

- **Orient**
  
  \[
  \frac{\mathcal{E} \cup \{ s \rightarrow t \}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}, \mathcal{R}}
  \]
  
  if $s > t$

- **Simplify**
  
  \[
  \frac{\mathcal{E} \cup \{ s \rightarrow t \}, \mathcal{R}}{\mathcal{E} \cup \{ u \rightarrow t \}, \mathcal{R}}
  \]
  
  if $s \not\in \mathcal{E} \rightarrow \mathcal{E}$

- **Delete**
  
  \[
  \frac{\mathcal{E} \cup \{ s \rightarrow s \}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}
  \]
Definition (oKB)
for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $\succ$ have six inference rules:

- **deduce**
  $$\frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{ s \approx t \}, \mathcal{R}}$$
  if $s \mathcal{R} \mathcal{E} \leftrightarrow \cdots \leftrightarrow \mathcal{R} \mathcal{E} t$

- **orient**
  $$\frac{\mathcal{E} \cup \{ s \approx t \}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}}$$
  $$\frac{\mathcal{E} \cup \{ t \approx s \}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}}$$
  if $s > t$

- **compose**
  $$\frac{\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow t \}}{\mathcal{E}, \mathcal{R} \cup \{ s \rightarrow u \}}$$
  if $t \mathcal{R} \mathcal{E} \succ u$

- **simplify**
  $$\frac{\mathcal{E} \cup \{ s \approx t \}, \mathcal{R}}{\mathcal{E} \cup \{ u \approx t \}, \mathcal{R}}$$
  $$\frac{\mathcal{E} \cup \{ t \approx s \}, \mathcal{R}}{\mathcal{E} \cup \{ t \approx u \}, \mathcal{R}}$$
  if $s \mathcal{R} \mathcal{E} \succ u$

- **delete**
  $$\frac{\mathcal{E} \cup \{ s \approx s \}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$$
Definition (oKB)
for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $\succ$ have six inference rules:

- **deduce**
  $\frac{s \approx t \ \mathcal{E}, \mathcal{R}}{s \approx t \ \mathcal{E}, \mathcal{R}}$
  if $s \mathcal{R} \mathcal{E} \leftrightarrow \cdot \leftrightarrow \mathcal{R} \mathcal{E} t$

- **orient**
  $\frac{s \approx t \ \mathcal{E}, \mathcal{R}}{s \rightarrow t \ \mathcal{E}, \mathcal{R} \ \mathcal{R}}$
  if $s \succ t$

- **simplify**
  $\frac{s \approx t \ \mathcal{E}, \mathcal{R}}{t \approx s \ \mathcal{E} \ \mathcal{R}}$
  if $s \rightarrow_{\mathcal{R} \mathcal{E}} u$

- **compose**
  $\frac{s \rightarrow t \ \mathcal{E}, \mathcal{R}}{t \rightarrow u \ \mathcal{E} \ \mathcal{R}}$
  if $t \rightarrow_{\mathcal{R} \mathcal{E}} u$

- **delete**
  $\frac{s \approx s \ \mathcal{E}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$

- **collapse**
  $\frac{t \rightarrow s \ \mathcal{E}, \mathcal{R}}{u \approx s \ \mathcal{E}, \mathcal{R}}$
  if $t \rightarrow_{\mathcal{R} \mathcal{E}} u$
**Definition (oKB)**

for equations $\mathcal{E}$, rules $\mathcal{R}$, reduction order $\succ$ have six inference rules:

- **deduce**
  
  $$
  \frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s_{\pi} \approx t_{\pi}\}, \mathcal{R}}
  $$

  if $s \mathcal{R} \mathcal{E} \leftrightarrow \mathcal{R} \mathcal{E} t$

- **compose**
  
  $$
  \frac{\mathcal{E}, \mathcal{R} \cup \{s_{\pi} \rightarrow t_{\pi}\}}{\mathcal{E}, \mathcal{R} \cup \{s_{\pi} \rightarrow u_{\pi}\}}
  $$

  if $t \rightarrow \mathcal{R} \mathcal{E} u$

- **orient**
  
  $$
  \frac{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s_{\pi} \rightarrow t_{\pi}\}}
  $$

  if $s \succ t$

- **simplify**
  
  $$
  \frac{\mathcal{E} \cup \{u_{\pi} \approx t_{\pi}\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}
  $$

  if $t \rightarrow \mathcal{R} \mathcal{E} u$

- **delete**
  
  $$
  \frac{\mathcal{E} \cup \{s \approx s\}}{\mathcal{E}, \mathcal{R}}
  $$

- **collapse**
  
  $$
  \frac{\mathcal{E} \cup \{u_{\pi} \approx s_{\pi}\}, \mathcal{R}}{\mathcal{E} \cup \{t_{\pi} \approx u_{\pi}\}, \mathcal{R}}
  $$

**Relaxations**

- allow variants for renaming $\pi$

---

L. Bachmair, N. Dershowitz, and D. Plaisted. *Completion Without Failure.* In *Resolution of Equations in Algebraic Structures*, 1989.
Ordered Completion

**Definition (oKB)**

for equations $E$, rules $R$, reduction order $\succ$ have six inference rules:

- **deduce**
  \[
  \frac{E, R}{E \cup \{s\pi \approx t\pi\}, R}
  \]
  if $s \stackrel{R\cup E}{\leftrightarrow} t \cdot \stackrel{R\cup E}{\leftrightarrow} t$

- **orient**
  \[
  \frac{E \cup \{s \approx t\}, R}{E, R \cup \{s\pi \rightarrow t\pi\}}
  \]
  if $s \succ t$

- **compose**
  \[
  \frac{E, R \cup \{s\pi \rightarrow u\pi\}}{E, R \cup \{s\pi \approx t\pi\}, R}
  \]
  if $t \rightarrow_{R\cup E} u$

- **simplify**
  \[
  \frac{E \cup \{s \approx t\}, R}{E \cup \{t \approx s\}, R}
  \]
  if $s \rightarrow_{R\cup E} u$

- **delete**
  \[
  \frac{E \cup \{s \approx s\}, R}{E, R}
  \]

- **collapse**
  \[
  \frac{E, R \cup \{t \rightarrow s\}}{E \cup \{u\pi \approx s\pi\}, R}
  \]
  if $t \rightarrow_{R\cup E} u$

**Relaxations**

- allow variants for renaming $\pi$
- no encompassment condition
Ordered Completion

Definition (oKB)
for equations $E$, rules $R$, reduction order $\succ$ have six inference rules:

- **deduce**
  \[
  \frac{E, R}{E \cup \{s \pi \approx t \pi\}, R}
  \]
  if $s \xrightarrow{R \cup E} t$

- **compose**
  \[
  \frac{E, R \cup \{s \pi \rightarrow t \pi\}}{E, R \cup \{s \pi \rightarrow u \pi\}}
  \]
  if $t \xrightarrow{R \cup E} u$

- **orient**
  \[
  \frac{E \cup \{s \approx t\}, R}{E, R \cup \{s \pi \rightarrow t \pi\}}
  \]
  if $s \succ t$

- **simplify**
  \[
  \frac{E \cup \{s \approx t\}, R}{E \cup \{t \approx s\}, R}
  \]
  \[
  \frac{E \cup \{t \pi \approx u \pi\}, R}{E \cup \{t \pi \approx u \pi\}, R}
  \]
  if $s \xrightarrow{R \cup E} u$

- **delete**
  \[
  \frac{E \cup \{s \approx s\}, R}{E, R}
  \]

- **collapse**
  \[
  \frac{E, R \cup \{t \rightarrow s\}}{E \cup \{u \pi \approx s \pi\}, R}
  \]
  if $t \xrightarrow{R \cup E} u$

Notation
write $(E, R) \vdash_{oKB} (E', R')$ if there is oKB step from $(E, R)$ to $(E', R')$
Example (Ordered Completion)

$E : \quad x - 0 \approx x$

$R :$

$s(x) - s(y) \approx x - y$

$0 - y \approx 0$

$s(x) \succ s(y) \approx x \succ y$

$x \div y \approx \langle 0, y \rangle$

$x \div y \approx \langle s(q), r \rangle$

$s(x) \succ 0 \approx \text{true}$

$s(x) \preceq s(y) \approx x \preceq y$

$0 \preceq x \approx \text{true}$

Remark
generated by conditional confluence tool ConCon from Cops #361:
ground complete system used to show infeasibility of critical pairs
Example (Ordered Completion)

\[ \mathcal{E} : \quad x - 0 \approx x \]

\[ \mathcal{R} : \]

\[ s(x) - s(y) \approx x - y \]
\[ 0 - y \approx 0 \]

\[ s(x) \succ s(y) \approx x \succ y \]
\[ x \div y \approx \langle 0, y \rangle \]
\[ x \div y \approx \langle s(q), r \rangle \]

\[ s(x) \succ 0 \approx \text{true} \]

\[ s(x) \preceq s(y) \approx x \preceq y \]
\[ 0 \preceq x \approx \text{true} \]

▶ fix some KBO (...)
Example (Ordered Completion)

\[ \mathcal{E} : \]
\[
\begin{align*}
x - 0 & \approx x \\
s(x) - s(y) & \approx x - y \\
0 - y & \approx 0 \\
s(x) \succ s(y) & \approx x \succ y \\
x \div y & \approx \langle 0, y \rangle \\
x \div y & \approx \langle s(q), r \rangle \\
s(x) \succ 0 & \approx \text{true} \\
s(x) \preceq s(y) & \approx x \preceq y \\
0 \preceq x & \approx \text{true}
\end{align*}
\]

\[ \mathcal{R} : \]

- fix some KBO (...) 
- orient \hspace{1cm} x - 0 \geq_{\text{kbo}} x
Example (Ordered Completion)

\[ E : \]
\[
\begin{align*}
  s(x) - s(y) & \approx x - y \\
  0 - y & \approx 0 \\
  s(x) \succ s(y) & \approx x \succ y \\
  x \div y & \approx \langle 0, y \rangle \\
  x \div y & \approx \langle s(q), r \rangle \\
  s(x) \succ 0 & \approx true \\
  s(x) \preceq s(y) & \approx x \preceq y \\
  0 \preceq x & \approx true
\end{align*}
\]

\[ R : \]
\[
\begin{align*}
  x - 0 & \rightarrow x
\end{align*}
\]

\[ \text{fix some KBO (...) } \]
Example (Ordered Completion)

\[ s(x) - s(y) \approx x - y \]

\[ 0 - y \approx 0 \]

\[ s(x) \succ s(y) \approx x \succ y \]

\[ x \div y \approx \langle 0, y \rangle \]

\[ x \div y \approx \langle s(q), r \rangle \]

\[ s(x) \succ 0 \approx \text{true} \]

\[ s(x) \preceq s(y) \approx x \preceq y \]

\[ 0 \preceq x \approx \text{true} \]

- fix some KBO (...) 

- orient \[ s(x) - s(y) \succ_{\text{kbo}} x - y \]
Example (Ordered Completion)

\[ E : \quad R : \]

\[ x - 0 \rightarrow x \]
\[ s(x) - s(y) \rightarrow x - y \]

\[ 0 - y \approx 0 \]
\[ s(x) \triangleright s(y) \approx x \triangleright y \]
\[ x \div y \approx \langle 0, y \rangle \]
\[ x \div y \approx \langle s(q), r \rangle \]
\[ s(x) \triangleright 0 \approx \text{true} \]
\[ s(x) \preceq s(y) \approx x \preceq y \]
\[ 0 \preceq x \approx \text{true} \]

- fix some KBO (...)
Example (Ordered Completion)

\[ \mathcal{E} : \]

\[ \mathcal{R} : \]

\[ x - 0 \rightarrow x \]

\[ s(x) - s(y) \rightarrow x - y \]

\[ 0 - y \approx 0 \]

\[ s(x) \succ s(y) \approx x \succ y \]

\[ x \div y \approx \langle 0, y \rangle \]

\[ x \div y \approx \langle s(q), r \rangle \]

\[ s(x) \succ 0 \approx \text{true} \]

\[ s(x) \preceq s(y) \approx x \preceq y \]

\[ 0 \preceq x \approx \text{true} \]

\[ \text{fix some KBO (...) } \]

\[ \text{orient } 0 - y \geq_{\text{kbo}} 0 \]
Example (Ordered Completion)

\[ E : \quad R : \]

\[ x - 0 \rightarrow x \]
\[ s(x) - s(y) \rightarrow x - y \]
\[ 0 - y \rightarrow 0 \]

\[ s(x) \succ s(y) \approx x \succ y \]
\[ x \div y \approx \langle 0, y \rangle \]
\[ x \div y \approx \langle s(q), r \rangle \]
\[ s(x) \succ 0 \approx \text{true} \]
\[ s(x) \preceq s(y) \approx x \preceq y \]
\[ 0 \preceq x \approx \text{true} \]

\[ \blacktriangleright \text{fix some KBO (...) \blacktriangleright} \]
Example (Ordered Completion)

\( \mathcal{E} : \)

\( \mathcal{R} : \)

\[ x - 0 \rightarrow x \]
\[ s(x) - s(y) \rightarrow x - y \]
\[ 0 - y \rightarrow 0 \]

\[ s(x) \succ s(y) \approx x \succ y \]
\[ x \div y \approx \langle 0, y \rangle \]
\[ x \div y \approx \langle s(q), r \rangle \]
\[ s(x) \succ 0 \approx \text{true} \]
\[ s(x) \preceq s(y) \approx x \preceq y \]
\[ 0 \preceq x \approx \text{true} \]

- fix some KBO (...)
- orient \( s(x) \succ s(y) >_{\text{kbo}} x \succ y \)
Example (Ordered Completion)

$\mathcal{E}$:  

$\mathcal{R}$:  

\[
\begin{align*}
    x - 0 & \rightarrow x \\
    s(x) - s(y) & \rightarrow x - y \\
    0 - y & \rightarrow 0 \\
    s(x) \succ s(y) & \rightarrow x \succ y
\end{align*}
\]

\[
\begin{align*}
    x \div y & \approx \langle 0, y \rangle \\
    x \div y & \approx \langle s(q), r \rangle \\
    s(x) \succ 0 & \approx \text{true} \\
    s(x) \preceq s(y) & \approx x \preceq y \\
    0 \preceq x & \approx \text{true}
\end{align*}
\]

▶ fix some KBO (...)

Example (Ordered Completion)

\[\mathcal{E} :\]

\[\mathcal{R} :\]

\[x - 0 \rightarrow x\]

\[s(x) - s(y) \rightarrow x - y\]

\[0 - y \rightarrow 0\]

\[s(x) > s(y) \rightarrow x > y\]

\[x \div y \approx \langle 0, y \rangle\]

\[x \div y \approx \langle s(q), r \rangle\]

\[s(x) > 0 \approx true\]

\[s(x) < s(y) \approx x < y\]

\[0 < x \approx true\]

- fix some KBO (...)

- simplify \[x \div y \rightarrow_{\varepsilon} \langle 0, y \rangle\] (no encompassment!)
Example (Ordered Completion)

\( E : \)

\[ x \div y \approx \langle 0, y \rangle \]
\[ \langle 0, y \rangle \approx \langle s(q), r \rangle \]
\[ s(x) \succ 0 \approx \text{true} \]
\[ s(x) \preceq s(y) \approx x \preceq y \]
\[ 0 \preceq x \approx \text{true} \]

\( R : \)

\[ x - 0 \rightarrow x \]
\[ s(x) - s(y) \rightarrow x - y \]
\[ 0 - y \rightarrow 0 \]
\[ s(x) \succ s(y) \rightarrow x \succ y \]

\( \triangleright \) fix some KBO (...)
Example (Ordered Completion)

$\mathcal{E} : \quad \mathcal{R} :$

$x - 0 \rightarrow x$

$s(x) - s(y) \rightarrow x - y$

$0 - y \rightarrow 0$

$s(x) \succ s(y) \rightarrow x \succ y$

$x \div y \approx \langle 0, y \rangle$

$\langle 0, y \rangle \approx \langle s(q), r \rangle$

$s(x) \succ 0 \approx true$

$s(x) \preceq s(y) \approx x \preceq y$

$0 \preceq x \approx true$

- fix some KBO (...) 
- orient $s(x) \succ 0 >_{kbo} true$
Example (Ordered Completion)

$\mathcal{E}$:

$x \div y \approx \langle 0, y \rangle$

$\langle 0, y \rangle \approx \langle s(q), r \rangle$

$s(x) \preceq s(y) \approx x \preceq y$

$0 \preceq x \approx \text{true}$

$\Rightarrow$ fix some KBO (…)
**Example (Ordered Completion)**

\[\mathcal{E} : \quad \mathcal{R} :\]

\[x \div y \approx \langle 0, y \rangle\]

\[\langle 0, y \rangle \approx \langle s(q), r \rangle\]

\[s(x) \preceq s(y) \approx x \preceq y\]

\[0 \preceq x \approx \text{true}\]

- fix some KBO (...) 
- orient \( s(x) \preceq s(y) \geq \text{KBO} x \preceq y \)
Example (Ordered Completion)

$\mathcal{E}:$

\[ x \div y \approx \langle 0, y \rangle \]
\[ \langle 0, y \rangle \approx \langle s(q), r \rangle \]

\[ 0 \preceq x \approx \text{true} \]

$\mathcal{R}:$

\[ x - 0 \rightarrow x \]
\[ s(x) - s(y) \rightarrow x - y \]
\[ 0 - y \rightarrow 0 \]
\[ s(x) \succ s(y) \rightarrow x \succ y \]

\[ s(x) \succ 0 \rightarrow \text{true} \]
\[ s(x) \preceq s(y) \rightarrow x \preceq y \]

- fix some KBO (...)
Example (Ordered Completion)

$\mathcal{E} : \quad \mathcal{R} :$

$x \div y \approx \langle 0, y \rangle$

$\langle 0, y \rangle \approx \langle s(q), r \rangle$

$0 \preceq x \approx \text{true}$

- fix some KBO (...)
- orient \quad 0 \preceq x \succ_{\text{kbo}} \text{true}$

$x - 0 \rightarrow x$

$s(x) - s(y) \rightarrow x - y$

$0 - y \rightarrow 0$

$s(x) \succ s(y) \rightarrow x \succ y$

$s(x) \succ 0 \rightarrow \text{true}$

$s(x) \preceq s(y) \rightarrow x \preceq y$
Example (Ordered Completion)

\[ E : \]

\[ \frac{x}{y} \approx \langle 0, y \rangle \]

\[ \langle 0, y \rangle \approx \langle s(q), r \rangle \]

\[ \rightarrow \]

\[ \text{fix some KBO (…)} \]
Example (Ordered Completion)

$E : \quad R :$

$x - 0 \rightarrow x$

$s(x) - s(y) \rightarrow x - y$

$0 - y \rightarrow 0$

$s(x) \succ s(y) \rightarrow x \succ y$

$x \div y \approx \langle 0, y \rangle$

$\langle 0, y \rangle \approx \langle s(q), r \rangle$

$s(x) \succ 0 \rightarrow \text{true}$

$s(x) \preceq s(y) \rightarrow x \preceq y$

$0 \preceq x \rightarrow \text{true}$

➤ fix some KBO (...)  

➤ deduce  $\langle 0, x \rangle \leftarrow \langle s(u), v \rangle \rightarrow \langle 0, y \rangle$
Example (Ordered Completion)

$\mathcal{E} : \langle 0, x \rangle \approx \langle 0, y \rangle$

$\mathcal{R} :$

$x - 0 \rightarrow x$

$s(x) - s(y) \rightarrow x - y$

$0 - y \rightarrow 0$

$s(x) \succ s(y) \rightarrow x \succ y$

$x \div y \approx \langle 0, y \rangle$

$\langle 0, y \rangle \approx \langle s(q), r \rangle$

$s(x) \succ 0 \rightarrow \text{true}$

$s(x) \preceq s(y) \rightarrow x \preceq y$

$0 \preceq x \rightarrow \text{true}$

$\rightarrow$ fix some KBO (...)
Example (Ordered Completion)

\[ \mathcal{E} : \quad \langle 0, x \rangle \approx \langle 0, y \rangle \]

\[ \mathcal{R} : \quad x - 0 \rightarrow x \]

\[ s(x) - s(y) \rightarrow x - y \]

\[ 0 - y \rightarrow 0 \]

\[ s(x) \succ s(y) \rightarrow x \succ y \]

\[ s(x) \succ 0 \rightarrow \text{true} \]

\[ s(x) \preceq s(y) \rightarrow x \preceq y \]

\[ 0 \preceq x \rightarrow \text{true} \]

- fix some KBO (...)
- deduce \[ \langle s(x), y \rangle \leftarrow \langle 0, u \rangle \rightarrow \langle s(q), r \rangle \]
Example (Ordered Completion)

$\mathcal{E}$:

\[
\begin{align*}
\langle 0, x \rangle & \approx \langle 0, y \rangle \\
\langle s(x), y \rangle & \approx \langle s(q), r \rangle
\end{align*}
\]

\[
x \div y \approx \langle 0, y \rangle
\]

\[
\langle 0, y \rangle \approx \langle s(q), r \rangle
\]

$\mathcal{R}$:

\[
\begin{align*}
x - 0 & \rightarrow x \\
s(x) - s(y) & \rightarrow x - y \\
0 - y & \rightarrow 0 \\
s(x) \succ s(y) & \rightarrow x \succ y
\end{align*}
\]

\[
s(x) \succ 0 \rightarrow \text{true}
\]

\[
s(x) \preceq s(y) \rightarrow x \preceq y
\]

\[
0 \preceq x \rightarrow \text{true}
\]

$\blacktriangleright$ fix some KBO (...)
Example (Ordered Completion)

\( E : \)
\[ \langle 0, x \rangle \approx \langle 0, y \rangle \]
\[ \langle s(x), y \rangle \approx \langle s(q), r \rangle \]

\[ x \div y \approx \langle 0, y \rangle \]
\[ \langle 0, y \rangle \approx \langle s(q), r \rangle \]

\( R : \)
\[ x - 0 \rightarrow x \]
\[ s(x) - s(y) \rightarrow x - y \]
\[ 0 - y \rightarrow 0 \]
\[ s(x) \succ s(y) \rightarrow x \succ y \]
\[ s(x) \succ 0 \rightarrow true \]
\[ s(x) \preceq s(y) \rightarrow x \preceq y \]
\[ 0 \preceq x \rightarrow true \]

- fix some KBO (...) 
- orient \( x \div y \gtrsim_{kbo} \langle 0, y \rangle \)
Example (Ordered Completion)

\( \mathcal{E} : \)

\[ \langle 0, x \rangle \approx \langle 0, y \rangle \]

\[ \langle s(x), y \rangle \approx \langle s(q), r \rangle \]

\[ \langle 0, y \rangle \approx \langle s(q), r \rangle \]

\( \mathcal{R} : \)

\[ x - 0 \rightarrow x \]

\[ s(x) - s(y) \rightarrow x - y \]

\[ 0 - y \rightarrow 0 \]

\[ s(x) \succ s(y) \rightarrow x \succ y \]

\[ x \div y \rightarrow \langle 0, y \rangle \]

\[ s(x) \succ 0 \rightarrow \text{true} \]

\[ s(x) \preceq s(y) \rightarrow x \preceq y \]

\[ 0 \preceq x \rightarrow \text{true} \]

▶ fix some KBO (...)
Example (Ordered Completion)

\[ E : \quad \langle 0, x \rangle \approx \langle 0, y \rangle \quad R : \quad x - 0 \rightarrow x \]
\[ \langle s(x), y \rangle \approx \langle s(q), r \rangle \quad s(x) - s(y) \rightarrow x - y \]
\[ \langle 0, y \rangle \approx \langle s(q), r \rangle \quad 0 - y \rightarrow 0 \]
\[ \langle 0, y \rangle \approx \langle s(q), r \rangle \quad s(x) \succ s(y) \rightarrow x \succ y \]
\[ x \div y \rightarrow \langle 0, y \rangle \quad s(x) \succeq 0 \rightarrow true \]
\[ s(x) \preceq s(y) \rightarrow x \preceq y \quad 0 \preceq x \rightarrow true \]

- fix some KBO (...)
- deduce \( s(s(x)) \succ s(0) \leftarrow s(x) \succ 0 \rightarrow true \)
Example (Ordered Completion)

$\mathcal{E} : \langle 0, x \rangle \approx \langle 0, y \rangle$

$\langle s(x), y \rangle \approx \langle s(q), r \rangle$

$s(s(x)) \succ s(0) \approx true$

$\langle 0, y \rangle \approx \langle s(q), r \rangle$

$\mathcal{R} :$

$x - 0 \rightarrow x$

$s(x) - s(y) \rightarrow x - y$

$0 - y \rightarrow 0$

$s(x) \succ s(y) \rightarrow x \succ y$

$x \div y \rightarrow \langle 0, y \rangle$

$s(x) \succ 0 \rightarrow true$

$s(x) \preceq s(y) \rightarrow x \preceq y$

$0 \preceq x \rightarrow true$

▶ fix some KBO (...)
Example (Ordered Completion)

\[ E : \langle 0, x \rangle \approx \langle 0, y \rangle \]
\[ \langle s(x), y \rangle \approx \langle s(q), r \rangle \]
\[ s(s(x)) \succ s(0) \approx \text{true} \]

\[ \langle 0, y \rangle \approx \langle s(q), r \rangle \]

\[ R : \]
\[ x - 0 \rightarrow x \]
\[ s(x) - s(y) \rightarrow x - y \]
\[ 0 - y \rightarrow 0 \]
\[ s(x) \succ s(y) \rightarrow x \succ y \]
\[ x \div y \rightarrow \langle 0, y \rangle \]
\[ s(x) \succ 0 \rightarrow \text{true} \]
\[ s(x) \preceq s(y) \rightarrow x \preceq y \]
\[ 0 \preceq x \rightarrow \text{true} \]

▶ fix some KBO (...)
▶ orient \[ s(s(x)) \succ s(0) \succ_{\text{kbo}} \text{true} \]
Example (Ordered Completion)

\( \mathcal{E} : \)
\[
\langle 0, x \rangle \approx \langle 0, y \rangle \\
\langle s(x), y \rangle \approx \langle s(q), r \rangle
\]

\( \mathcal{R} : \)
\[
x - 0 \rightarrow x \\
s(x) - s(y) \rightarrow x - y \\
0 - y \rightarrow 0 \\
s(x) \succ s(y) \rightarrow x \succ y \\
x \div y \rightarrow \langle 0, y \rangle \\
s(s(x)) \succ s(0) \rightarrow \text{true} \\
s(x) \succ 0 \rightarrow \text{true} \\
s(x) \preceq s(y) \rightarrow x \preceq y \\
0 \preceq x \rightarrow \text{true}
\]

- fix some KBO (...)
Example (Ordered Completion)

\[ \mathcal{E} : \quad \langle 0, x \rangle \approx \langle 0, y \rangle \quad \quad \mathcal{R} : \quad x - 0 \rightarrow x \]

\[ \langle s(x), y \rangle \approx \langle s(q), r \rangle \]

\[ x - 0 \rightarrow x \]

\[ s(x) - s(y) \rightarrow x - y \]

\[ 0 - y \rightarrow 0 \]

\[ s(x) \succ s(y) \rightarrow x \succ y \]

\[ x \div y \rightarrow \langle 0, y \rangle \]

\[ s(s(x)) \succ s(0) \rightarrow \text{true} \]

\[ s(x) \succ 0 \rightarrow \text{true} \]

\[ s(x) \preceq s(y) \rightarrow x \preceq y \]

\[ 0 \preceq x \rightarrow \text{true} \]

➤ fix some KBO (...)

➤ collapse \[ s(s(x)) \succ s(0) \rightarrow_{\mathcal{R}} s(x) \succ 0 \]
Example (Ordered Completion)

\[ \mathcal{E} : \quad \langle 0, x \rangle \approx \langle 0, y \rangle \]  
\[ \langle s(x), y \rangle \approx \langle s(q), r \rangle \]  
\[ s(x) \succ 0 \approx \text{true} \]

\[ \langle 0, y \rangle \approx \langle s(q), r \rangle \]

\[ \mathcal{R} : \quad x - 0 \rightarrow x \]  
\[ s(x) - s(y) \rightarrow x - y \]  
\[ 0 - y \rightarrow 0 \]  
\[ s(x) \succ s(y) \rightarrow x \succ y \]  
\[ x \div y \rightarrow \langle 0, y \rangle \]

\[ s(x) \succ 0 \rightarrow \text{true} \]  
\[ s(x) \preceq s(y) \rightarrow x \preceq y \]  
\[ 0 \preceq x \rightarrow \text{true} \]

- fix some KBO (...)
Example (Ordered Completion)

\[\mathcal{E} : \quad \langle 0, x \rangle \approx \langle 0, y \rangle \quad \mathcal{R} : \quad x - 0 \rightarrow x\]
\[\langle s(x), y \rangle \approx \langle s(q), r \rangle \quad s(x) - s(y) \rightarrow x - y\]
\[s(x) \succ 0 \approx \text{true} \quad s(x) \succ s(y) \rightarrow x \succ y\]

\[\langle 0, y \rangle \approx \langle s(q), r \rangle \quad s(x) \preceq s(y) \rightarrow x \preceq y\]
\[x \div y \rightarrow \langle 0, y \rangle \quad 0 \preceq x \rightarrow \text{true}\]

▶ fix some KBO (...)  
▶ simplify \quad s(x) \succ 0 \rightarrow_\mathcal{R} \text{true}
Example (Ordered Completion)

\[\mathcal{E} : \quad \langle 0, x \rangle \approx \langle 0, y \rangle \quad \mathcal{R} : \quad x - 0 \rightarrow x\]
\[\langle s(x), y \rangle \approx \langle s(q), r \rangle \quad s(x) - s(y) \rightarrow x - y\]
\[\text{true} \approx \text{true} \quad 0 - y \rightarrow 0\]
\[\langle 0, y \rangle \approx \langle s(q), r \rangle \quad s(x) \succ s(y) \rightarrow x \succ y\]
\[\langle 0, y \rangle \approx \langle s(q), r \rangle \quad x \div y \rightarrow \langle 0, y \rangle\]
\[s(x) \succ 0 \rightarrow \text{true} \quad s(x) \preceq s(y) \rightarrow x \preceq y\]
\[0 \preceq x \rightarrow \text{true}\]

\[\quad \text{fix some KBO (...)\text{}}\]
Example (Ordered Completion)

\[ E : \begin{align*}
\langle 0, x \rangle & \approx \langle 0, y \rangle \\
\langle s(x), y \rangle & \approx \langle s(q), r \rangle \\
\text{true} & \approx \text{true}
\end{align*} \]

\[ R : \begin{align*}
x - 0 & \rightarrow x \\
s(x) - s(y) & \rightarrow x - y \\
0 - y & \rightarrow 0 \\
s(x) \triangleright s(y) & \rightarrow x \triangleright y \\
x \div y & \rightarrow \langle 0, y \rangle \\
s(x) \triangleright 0 & \rightarrow \text{true} \\
s(x) \less g s(y) & \rightarrow x \less s y \\
0 \less x & \rightarrow \text{true}
\end{align*} \]

- fix some KBO (⋯)
- delete true \approx true
Example (Ordered Completion)

$\mathcal{E}$: \begin{align*}
\langle 0, x \rangle & \approx \langle 0, y \rangle \\
\langle s(x), y \rangle & \approx \langle s(q), r \rangle
\end{align*}

$\mathcal{R}$: \begin{align*}
x - 0 & \rightarrow x \\
s(x) - s(y) & \rightarrow x - y \\
0 - y & \rightarrow 0 \\
s(x) \succ s(y) & \rightarrow x \succ y \\
x \div y & \rightarrow \langle 0, y \rangle \\
s(x) \succ 0 & \rightarrow \text{true} \\
s(x) \preceq s(y) & \rightarrow x \preceq y \\
0 \preceq x & \rightarrow \text{true}
\end{align*}

$\quad \triangleright$ fix some KBO (...)
Example (Ordered Completion)

$\mathcal{E} :$

\[ \langle 0, x \rangle \approx \langle 0, y \rangle \]
\[ \langle s(x), y \rangle \approx \langle s(q), r \rangle \]
\[ \langle 0, y \rangle \approx \langle s(q), r \rangle \]

$\mathcal{R} :$

\[ x - 0 \rightarrow x \]
\[ s(x) - s(y) \rightarrow x - y \]
\[ 0 - y \rightarrow 0 \]
\[ s(x) \succ s(y) \rightarrow x \succ y \]
\[ x \div y \rightarrow \langle 0, y \rangle \]
\[ s(x) \succ 0 \rightarrow \text{true} \]
\[ s(x) \preceq s(y) \rightarrow x \preceq y \]
\[ 0 \preceq x \rightarrow \text{true} \]

- fix some KBO (...) 
- oKB run produced ground complete system
Really, it's ground complete!
Lemma

If \( (\mathcal{E}, \mathcal{R}) \vdash_{oKB}^* (\mathcal{E}', \mathcal{R}') \) and \( \mathcal{R} \subseteq > \) then \( \mathcal{R}' \subseteq > \).

We stick to the order . . .
Lemma

If \((E, \mathcal{R}) \vdash^{\ast}_{oKB} (E', \mathcal{R}')\) and \(\mathcal{R} \subseteq >\) then \(\mathcal{R}' \subseteq >\).

Lemma

If \((E, \mathcal{R}) \vdash^{\ast}_{oKB} (E', \mathcal{R}')\) then \(\leftrightarrow_{E \cup \mathcal{R}}^{\ast} = \leftrightarrow_{E' \cup \mathcal{R}'}^{\ast}\).

...don't change the equational theory...
Lemma
If $(\mathcal{E}, \mathcal{R}) \vdash^*_{oKB} (\mathcal{E}', \mathcal{R}')$ and $\mathcal{R} \subseteq >$ then $\mathcal{R}' \subseteq >$.

Lemma
If $(\mathcal{E}, \mathcal{R}) \vdash^*_{oKB} (\mathcal{E}', \mathcal{R}')$ then $\leftrightarrow^*_{\mathcal{E} \cup \mathcal{R}} = \leftrightarrow^*_{\mathcal{E}' \cup \mathcal{R}'}$.

Lemma
If $\forall s \approx t \in \text{CP}_>(\mathcal{E})$ have $s \Downarrow^g_{\mathcal{E} >} t$ then $\mathcal{E} >$ is ground complete.

...and check ground-joinability of critical pairs, see?
**Lemma**

If \((\mathcal{E}, \mathcal{R}) \vdash^{*}_{\text{KB}} (\mathcal{E}', \mathcal{R}')\) and \(\mathcal{R} \subseteq >\) then \(\mathcal{R}' \subseteq >\).

**Lemma**

If \((\mathcal{E}, \mathcal{R}) \vdash^{*}_{\text{KB}} (\mathcal{E}', \mathcal{R}')\) then \(\leftrightarrow^{*}_{\mathcal{E} \cup \mathcal{R}} = \leftrightarrow^{*}_{\mathcal{E}' \cup \mathcal{R}'}\).

**Lemma**

If \(\forall s \approx t \in \text{CP}_{>}(\mathcal{E})\) have \(s \downarrow^{g}_{\mathcal{E}>} t\) then \(\mathcal{E}>\) is ground complete.

**Lemma**

If \(>\) is total precedence then \(>_{\text{Ipo}}\) and \(>_{\text{kbo}}\) are total on ground terms.

Our favorite orders are ground total.
Theorem (Correctness)

Suppose \((E_0, \emptyset) \vdash^\ast_{oKB} (E, R)\)

- using LPO or KBO as ground-total reduction order \(\triangleright\), and
- \(\forall s \approx t \in CP_{\triangleright}(E \cup R)\) have \(s \downarrow^g_{R \cup E} t\)

Then \(\leftrightarrow^\ast_{E_0} = \leftrightarrow^\ast_{R \cup E}\) and \(R \cup E >\) is ground complete.

So, altogether our procedure is correct!
Theorem (Correctness)

Suppose \((\mathcal{E}_0, \emptyset) \vdash^{*}_{oKB} (\mathcal{E}, \mathcal{R})\)  

- using LPO or KBO as ground-total reduction order \(\gg\), and  
- \(\forall s \approx t \in \text{CP}_{\mathcal{R} \cup \mathcal{E}}(\mathcal{E} \cup \mathcal{R})\) have \(s \downarrow^{g}_{\mathcal{R} \cup \mathcal{E}} t\)

Then \(\leftrightarrow^{*}_{\mathcal{E}_0} = \leftrightarrow^{*}_{\mathcal{R} \cup \mathcal{E}}\) and \(\mathcal{R} \cup \mathcal{E} \gg\) is ground complete.

So, altogether our procedure is correct!

Ok, alright . . .
Theorem (Correctness)

Suppose \((\mathcal{E}_0, \emptyset) \vdash^{*}_{\text{oKB}} (\mathcal{E}, \mathcal{R})\)

- using LPO or KBO as ground-total reduction order \(\succ\), and
- \(\forall s \approx t \in \text{CP}_{\succ}(\mathcal{E} \cup \mathcal{R})\) have \(s \downarrow^{g}_{\mathcal{R} \cup \mathcal{E}} t\)

Then \(\leftrightarrow^{*}_{\mathcal{E}_0} = \leftrightarrow^{*}_{\mathcal{R} \cup \mathcal{E}}\) and \(\mathcal{R} \cup \mathcal{E} \succ\) is ground complete.

So, altogether our procedure is correct!

Ok, alright . . .

. . . but you need to find ground joinability criteria
Outline

Preliminaries

Ordered Completion

Ground Joinability Criteria

Proof Checking

Conclusion
Lemma (Criterion 1)

relationship $s \xrightarrow{g} t$ holds if

- $s \xrightarrow{\mathcal{E}} t$, or
- $s \approx t$ is instance of equation in $\mathcal{E}^\pm$
Lemma (Criterion 1)

relationship $s \downarrow_{E}^{g} t$ holds if

- $s \downarrow_{E} t$, or
- $s \approx t$ is instance of equation in $E^{\pm}$

Example

for $\mathcal{R}$ and $\mathcal{E}$ derived by ConCon from Cops 361:

\[
\begin{align*}
-x \cdot 0 & \rightarrow x & -0 \cdot x & \rightarrow 0 & -s(x) \cdot s(y) & \rightarrow -x \cdot y \\
0 & \preceq x \rightarrow \text{true} & s(x) & \preceq s(y) \rightarrow x \preceq y & x \div y & \rightarrow \langle 0, y \rangle \\
s(x) & \succ 0 \rightarrow \text{true} & s(x) & \succ s(y) \rightarrow x \succ y & \\
\langle s(x), y \rangle & \approx \langle s(q), r \rangle & \langle 0, y \rangle & \approx \langle s(q), r \rangle & \langle 0, x \rangle & \approx \langle 0, y \rangle
\end{align*}
\]
Lemma (Criterion 1)

relationship $s \downarrow_{g, E}^{g, t}$ holds if

- $s \downarrow_{E}^{t}$, or
- $s \approx t$ is instance of equation in $E^\pm$

Example

for $R$ and $E$ derived by ConCon from Cops 361:

\[-x \cdot 0 \rightarrow x \quad -0 \cdot x \rightarrow 0 \quad -s(x) \cdot s(y) \rightarrow -x \cdot y\]

\[0 \preceq x \rightarrow \text{true}\quad s(x) \preceq s(y) \rightarrow x \preceq y\]

\[s(x) \succ 0 \rightarrow \text{true}\quad s(x) \succ s(y) \rightarrow x \succ y\]

\[\langle s(x), y \rangle \approx \langle s(q), r \rangle\quad \langle 0, y \rangle \approx \langle s(q), r \rangle\quad \langle 0, x \rangle \approx \langle 0, y \rangle\]

ground confluence can be established by Criterion 1:

- critical overlap between first two equations:
  \[\langle 0, y \rangle \leftarrow \langle s(q), r \rangle \rightarrow \langle s(x), y \rangle\]

- critical overlap between first two rules:
  \[0 \leftarrow -0 \cdot 0 \rightarrow 0\]
Example (AC)

set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \quad x \cdot y \approx y \cdot x \quad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$
Example (AC)

set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \quad x \cdot y \approx y \cdot x \quad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$

gives rise to extended overlap

$$s = z \cdot (x \cdot y) \leftrightarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t$$
Example (AC)

set of equations $\mathcal{E}$:

\[(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \quad x \cdot y \approx y \cdot x \quad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)\]

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\[s = z \cdot (x \cdot y) \leftarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t\]

- Criterion 1 fails to show $s \vdash_{\mathcal{E}}^g t$
Example (AC)

set of equations $\mathcal{E}$:

\[(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \quad x \cdot y \approx y \cdot x \quad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)\]

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\[s = z \cdot (x \cdot y) \leftrightarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t\]

- Criterion 1 fails to show $s \Downarrow_{\mathcal{E}>}^g t$

Observation

for any grounding substitution $\sigma$ terms $x\sigma$, $y\sigma$, and $z\sigma$ are totally ordered
Example (AC)

set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \quad x \cdot y \approx y \cdot x \quad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$

gives rise to extended overlap

$$s = z \cdot (x \cdot y) \leftarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t$$

- Criterion 1 fails to show $s \not\Downarrow^g_{\mathcal{E}} t$
- if $\approx$ is “extended” by $x\sigma > z\sigma > y\sigma$ then joining sequence exists:

$$\begin{align*}
z\sigma \cdot (x\sigma \cdot y\sigma) & \rightarrow z\sigma \cdot (y\sigma \cdot x\sigma) \\
& \rightarrow y\sigma \cdot (z\sigma \cdot x\sigma) \\
x\sigma \cdot (y\sigma \cdot z\sigma) & \leftarrow y\sigma \cdot (x\sigma \cdot z\sigma) \\
& \leftarrow x\sigma \cdot (y\sigma \cdot z\sigma)
\end{align*}$$

Observation

for any grounding substitution $\sigma$ terms $x\sigma$, $y\sigma$, and $z\sigma$ are totally ordered
Example (AC)
set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \quad x \cdot y \approx y \cdot x \quad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$

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- Criterion 1 fails to show $s \Downarrow^{g}_{\mathcal{E}} t$
- if $>$ is “extended” by $x\sigma > z\sigma > y\sigma$ then joining sequence exists:

$$z\sigma \cdot (x\sigma \cdot y\sigma) \quad x\sigma \cdot (y\sigma \cdot z\sigma)$$
$$\rightarrow z\sigma \cdot (y\sigma \cdot x\sigma) \quad y\sigma \cdot (x\sigma \cdot z\sigma)$$
$$\rightarrow y\sigma \cdot (z\sigma \cdot x\sigma)$$

- can verify $s\sigma \Downarrow^{g}_{\mathcal{E}} t\sigma$ for all 13 orderings of $x\sigma$, $y\sigma$, $z\sigma$

Observation
for any grounding substitution $\sigma$ terms $x\sigma$, $y\sigma$, and $z\sigma$ are totally ordered
Example (AC)

set of equations $\mathcal{E}$:

\[
(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \quad x \cdot y \approx y \cdot x \quad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)
\]

gives rise to extended overlap

\[
s = z \cdot (x \cdot y) \leftrightarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t
\]

- Criterion 1 fails to show $s \downarrow_{\mathcal{E}>}^{g} t$
- if $>$ is “extended” by $x\sigma > z\sigma > y\sigma$ then joining sequence exists:

\[
\begin{align*}
&z\sigma \cdot (x\sigma \cdot y\sigma) \\
&\quad \quad \rightarrow z\sigma \cdot (y\sigma \cdot x\sigma) \\
&\quad \quad \quad \rightarrow y\sigma \cdot (z\sigma \cdot x\sigma) \\
&\quad \quad \quad \quad \quad \rightarrow y\sigma \cdot (x\sigma \cdot z\sigma)
\end{align*}
\]

- can verify $s\sigma \downarrow_{\mathcal{E}>} t\sigma$ for all 13 orderings of $x\sigma$, $y\sigma$, $z\sigma$
- repeat this check for all CPs: $\mathcal{E}>$ is ground complete

Observation

for any grounding substitution $\sigma$ terms $x\sigma$, $y\sigma$, and $z\sigma$ are totally ordered
**Example (AC)**

set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \quad x \cdot y \approx y \cdot x \quad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$

gives rise to extended overlap

$$s = z \cdot (x \cdot y) \leftarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t$$

- Criterion 1 fails to show $s \Downarrow_{\mathcal{E}}^g t$
- if $>$ is “extended” by $x\sigma > z\sigma > y\sigma$ then joining sequence exists:

$$z\sigma \cdot (x\sigma \cdot y\sigma) \rightarrow z\sigma \cdot (y\sigma \cdot x\sigma) \rightarrow y\sigma \cdot (z\sigma \cdot x\sigma)$$

- can verify $s\sigma \Downarrow_{\mathcal{E}} \ t\sigma$ for all 13 orderings of $x\sigma$, $y\sigma$, and $z\sigma$
- repeat this check for all CPs: $\mathcal{E}_>$ is ground complete

**Observation**

for any grounding substitution $\sigma$ terms $x\sigma$, $y\sigma$, and $z\sigma$ are totally ordered

---

U. Martin and T. Nipkow. *Ordered Rewriting and Confluence*. Proc. 10th CADE, 1990.
Example (AC)

set of equations $\mathcal{E}$:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \quad x \cdot y \approx y \cdot x \quad x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$$

gives rise to extended overlap

$$s = z \cdot (x \cdot y) \leftrightarrow (x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z) = t$$

- Criterion 1 fails to show $s \downarrow^{\mathcal{E}}_{g \Rightarrow} t$
- if $\Rightarrow$ is “extended” by $x\sigma > z\sigma > y\sigma$ then joining sequence exists:

  $z\sigma \cdot (x\sigma \cdot y\sigma) \rightarrow z\sigma \cdot (y\sigma \cdot x\sigma) \rightarrow y\sigma \cdot (z\sigma \cdot x\sigma) \leftarrow y\sigma \cdot (x\sigma \cdot z\sigma) \leftarrow x\sigma \cdot (y\sigma \cdot z\sigma)$

- can verify $s\sigma \downarrow_{\mathcal{E}} \Rightarrow t\sigma$ for all 13 orderings of $x\sigma$, $y\sigma$, $z\sigma$
- repeat this check for all CPs: $\mathcal{E}_{\Rightarrow}$ is ground complete

Definition (Joinable wrt Closure)

write $s \downarrow^{C}_{\mathcal{E}} t$ if $\forall$ equivalence relations $\equiv$ on $\mathcal{V}ar(s, t) \quad \forall$ order $\Rightarrow$ on $\equiv$

$$\hat{\equiv}(s) \xrightarrow{*} \cdot \xleftarrow{*} \hat{\equiv}(t)$$

\[ \varepsilon_{C(\Rightarrow)} \]
Definiton
inductively defined ground joinability predicate gj(·, ·)

delete

gj(t, t)

closure

s \downarrow_{\mathcal{E}} t \implies gj(s, t)

step

s \leftrightarrow t \implies gj(s, t)

rewrite left

s \rightarrow_{\mathcal{E}} u \text{ and } gj(u, t) \implies gj(s, t)

rewrite right

s \rightarrow_{\mathcal{E}} u \text{ and } gj(s, u) \implies gj(s, t)

congruence

gj(s_i, t_i) \text{ for all } 1 \leq i \leq n \implies gj(f(\bar{s}), f(\bar{t}))
Definition

inductively defined ground joinability predicate $g_j(\cdot, \cdot)$

- delete
  \[ g_j(t, t) \]

- closure
  \[ s \downarrow^C_{\mathcal{E}} t \implies g_j(s, t) \]

- step
  \[ s \leftrightarrow_{\mathcal{E}} t \implies g_j(s, t) \]

- rewrite left
  \[ s \rightarrow_{\mathcal{E}} u \text{ and } g_j(u, t) \implies g_j(s, t) \]

- rewrite right
  \[ t \rightarrow_{\mathcal{E}} u \text{ and } g_j(s, u) \implies g_j(s, t) \]

- congruence
  \[ g_j(s_i, t_i) \text{ for all } 1 \leq i \leq n \implies g_j(f(\bar{s}), f(\bar{t})) \]
Definition
inductively defined ground joinability predicate $gj(\cdot, \cdot)$

- **delete** $gj(t, t)$
- **closure** $s \downarrow^C \varepsilon t \implies gj(s, t)$
- **step** $s \leftrightarrow \varepsilon t \implies gj(s, t)$
- **rewrite left** $s \xrightarrow{\varepsilon>} u$ and $gj(u, t) \implies gj(s, t)$
- **rewrite right** $t \xrightarrow{\varepsilon>} u$ and $gj(s, u) \implies gj(s, t)$
- **congruence** $gj(s_i, t_i)$ for all $1 \leq i \leq n \implies gj(f(\vec{s}), f(\vec{t}))$

**Lemma (Criterion 2)**

$gj(s, t)$ implies $s \downarrow^g_{\varepsilon>} t$
Definition
inductively defined ground joinability predicate \( g_j(\cdot, \cdot) \)

- delete: \( g_j(t, t) \)
- closure: \( s \downarrow^C_E t \implies g_j(s, t) \)
- step: \( s \leftarrow^E t \implies g_j(s, t) \)
- rewrite left: \( s \rightarrow^E u \) and \( g_j(u, t) \implies g_j(s, t) \)
- rewrite right: \( t \rightarrow^E u \) and \( g_j(s, u) \implies g_j(s, t) \)
- congruence: \( g_j(s_i, t_i) \) for all \( 1 \leq i \leq n \)

Lemma (Criterion 2)
\( g_j(s, t) \) implies \( s \downarrow^g_E t \)

gain flexibility/efficiency over Martin & Nipkow criterion
Definition

inductively defined ground joinability predicate \( gj(\cdot, \cdot) \)

- delete \( \rightarrow \)
  \( gj(t, t) \)

- closure \( \downarrow \)
  \( s \downarrow \mathcal{E} t \implies gj(s, t) \)

- step \( \leftrightarrow \)
  \( s \leftrightarrow t \implies gj(s, t) \)

- rewrite left \( \rightarrow \)
  \( s \rightarrow u \text{ and } gj(u, t) \implies gj(s, t) \)

- rewrite right \( \rightarrow \)
  \( t \rightarrow u \text{ and } gj(s, u) \implies gj(s, t) \)

- congruence \( \rightarrow \)
  \( gj(s_i, t_i) \text{ for all } 1 \leq i \leq n \implies gj(f(\overline{s}), f(\overline{t})) \)

Example

for set of equations \( \mathcal{E} \):

\[ f(x) \approx f(y) \quad \text{and} \quad g(x, y) \approx f(x) \]
Definition

inductively defined ground joinability predicate $g_j(\cdot, \cdot)$

- **delete**
  \[ g_j(t, t) \]

- **closure**
  \[ s \downarrow^C_\mathcal{E} t \implies g_j(s, t) \]

- **step**
  \[ s \leftrightarrow^\mathcal{E} t \implies g_j(s, t) \]

- **rewrite left**
  \[ s \rightarrow^\mathcal{E} u \text{ and } g_j(u, t) \implies g_j(s, t) \]

- **rewrite right**
  \[ t \rightarrow^\mathcal{E} u \text{ and } g_j(s, u) \implies g_j(s, t) \]

- **congruence**
  \[ g_j(s_i, t_i) \text{ for all } 1 \leq i \leq n \implies g_j(f(\overline{s}), f(\overline{t})) \]

Example

for set of equations $\mathcal{E}$:

\[ f(x) \approx f(y) \quad g(x, y) \approx f(x) \]

can show $g(x, y) \downarrow^g_R g(z, w)$:

\[ \begin{align*}
  f(x) \leftrightarrow^\mathcal{E} f(z) & \quad \text{step} \quad gj(f(x), f(z)) \\
  g(x, y) \rightarrow^\mathcal{E} f(x) & \quad \text{rewrite left} \quad gj(g(x, y), f(z)) \\
  g(z, w) \rightarrow^\mathcal{E} f(z) & \quad \text{rewrite right} \quad gj(g(x, y), g(z, w))
\end{align*} \]
Definition

inductively defined ground joinability predicate \( gj(\cdot, \cdot) \)

- delete \( gj(t, t) \)
- closure \( s \downarrow^C t \implies gj(s, t) \)
- step \( s \leftrightarrow t \implies gj(s, t) \)
- rewrite left \( s \xrightarrow{\varepsilon} u \text{ and } gj(u, t) \implies gj(s, t) \)
- rewrite right \( t \xrightarrow{\varepsilon} u \text{ and } gj(s, u) \implies gj(s, t) \)
- congruence \( gj(s_i, t_i) \text{ for all } 1 \leq i \leq n \implies gj(f(\bar{s}), f(\bar{t})) \)

Example

for set of equations \( E \):

\[ f(x) \approx f(y) \]

\[ g(x, y) \downarrow^g g(z, w) : \]

\[ f(x) \leftrightarrow_{\varepsilon} f(z) \]

\[ g(f(x), f(z)) \]

\[ g(x, y) \xrightarrow{\varepsilon} f(x) \]

\[ gj(g(x, y), f(z)) \]

\[ g(z, w) \xrightarrow{\varepsilon} f(z) \]

\[ gj(g(x, y), g(z, w))) \]

MN90 approach would need to check 81 relations
Outline

Preliminaries

Ordered Completion

Ground Joinability Criteria

Proof Checking

Conclusion
Ordered Completion

Certificate Components

- initial equations $\mathcal{E}_0$
- reduction order $>$
- resulting system $(\mathcal{E}, \mathcal{R})$
- oKB steps from $\mathcal{E}_0$ to $(\mathcal{E}, \mathcal{R})$
Ordered Completion

Certificate Components

- initial equations $\mathcal{E}_0$
- reduction order $>$
- resulting system $(\mathcal{E}, \mathcal{R})$
- oKB steps from $\mathcal{E}_0$ to $(\mathcal{E}, \mathcal{R})$

Checks Done in CeTA

1. valid run $(\mathcal{E}_0\pi, \emptyset) \vdash^{*}_{oKB} (\mathcal{E}, \mathcal{R})$, termination of $\mathcal{R}$, $\leftrightarrow^{*}_{\mathcal{E}_0} = \leftrightarrow^{*}_{\mathcal{R} \cup \mathcal{E}}$
2. ground confluence of $\mathcal{R} \cup \mathcal{E} >$ according to Correctness Theorem
3. ground-totality and admissibility of $>$
Ordered Completion

Certificate Components

- initial equations $\mathcal{E}_0$
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- resulting system $(\mathcal{E}, \mathcal{R})$
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Checks Done in CeTA

1. valid run $(\mathcal{E}_0 \pi, \emptyset) \vdash^{*}_{\text{oKB}} (\mathcal{E}, \mathcal{R})$, termination of $\mathcal{R}$, $\leftrightarrow^{*}_{\mathcal{E}_0} = \leftrightarrow^{*}_{\mathcal{R} \cup \mathcal{E}}$
2. ground confluence of $\mathcal{R} \cup \mathcal{E} >$ according to Correctness Theorem
3. ground-totality and admissibility of $>$

Certified Ordered Completion Proofs

- 🎟️ 94% of MædMax oKB proofs with KBO (58% of all oKB proofs)
- missing: LPO + trick to ignore CPs with ground joinable equations
Equational Satisfiability

Certificate Components

- initial equations $E_0$
- reduction order $>$
- ground inequality $s \not\approx t$

- resulting system $(E, R)$
- oKB steps from $E_0$ to $(E, R)$

Checks Done in CeTA

1. valid run $(E_0\pi, \emptyset) \vdash^{*}_{\text{oKB}} (E, R)$, termination of $R$, $\leftrightarrow_{E_0}^* = \leftrightarrow_{R \cup E}^*$
2. ground confluence of $R \cup E>$ according to Correctness Theorem
3. ground-totality and admissibility of $>$
4. $s$ and $t$ are not joinable in $R \cup E>$
Equational Satisfiability

Certificate Components
- initial equations $\mathcal{E}_0$
- reduction order $>$
- ground inequality $s \not\approx t$
- resulting system $(\mathcal{E}, \mathcal{R})$
- oKB steps from $\mathcal{E}_0$ to $(\mathcal{E}, \mathcal{R})$

Checks Done in CeTA
1. valid run $(\mathcal{E}_0, \emptyset) \vdash^*_{\text{oKB}} (\mathcal{E}, \mathcal{R})$, termination of $\mathcal{R}$, $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{R} \cup \mathcal{E}}$
2. ground confluence of $\mathcal{R} \cup \mathcal{E}>$ according to Correctness Theorem
3. ground-totality and admissibility of $>$
4. $s$ and $t$ are not joinable in $\mathcal{R} \cup \mathcal{E}>$

Certified Satisfiability Proofs
- ⭐ 100% of MædMax proofs with KBO (79% of all proofs)
- missing: LPO
Infeasibility

Certificate Components

- initial equations $\mathcal{E}_0$
- reduction order $>$
- ground inequality $s \not\approx t$
- resulting system $(\mathcal{E}, \mathcal{R})$
- oKB steps from $\mathcal{E}_0$ to $(\mathcal{E}, \mathcal{R})$
- ...  

Checks Done in CeTA

1. valid run $(\mathcal{E}_0 \pi, \emptyset) \vdash^*_\text{oKB} (\mathcal{E}, \mathcal{R})$, termination of $\mathcal{R}$, $\leftrightarrow^*_\mathcal{E}_0 = \leftrightarrow^*_\mathcal{R} \cup \mathcal{E}$
2. ground confluence of $\mathcal{R} \cup \mathcal{E}^>$ according to Correctness Theorem
3. ground-totality and admissibility of $>$
4. $s$ and $t$ are not joinable in $\mathcal{R} \cup \mathcal{E}^>$

Certified Conditional Confluence Proofs

- previously: 112 ConCon proofs, 109 certified
- with infeasibility checks using MædMax: 114 proofs, 111 certified
Conclusion

Summary

- formalized finite ordered completion:
  - allowing variants: greatly simplifies output for tools
  - no encompassment

Future Work

- support other orders
- support equational disproofs with narrowing
- certify more TPTP proofs (Instgen-Eq?)
Summary

- formalized finite ordered completion:
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  - no encompassment
- formalized more flexible version of ground joinability criterion by Martin & Nipkow (+ added missing assumption on order closure)
Conclusion

Summary

- formalized finite ordered completion:
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- formalized more flexible version of ground joinability criterion by Martin & Nipkow (+ added missing assumption on order closure)
- CeTA is first (and certified) proof checker for
  - ordered completion
  - equational satisfiability
  - infeasibility of critical pairs

Future Work

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- formalized finite ordered completion:
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