Exact solution of a generalized version of the Black-Scholes equation

L.-A. Cotfas\textsuperscript{a,*}, C. Delcea\textsuperscript{a}, N. Cotfas\textsuperscript{b}

\textsuperscript{a}Faculty of Economic Cybernetics, Statistics and Informatics, Bucharest University of Economic Studies, 6 Piata Romana, 010374 Bucharest, Romania

\textsuperscript{b}University of Bucharest, Physics Department, P.O. Box MG-11, 077125 Bucharest, Romania

Abstract

We analyze a generalized version of the Black-Scholes equation depending on a parameter $a \in (-\infty, 0)$. It satisfies the martingale condition and coincides with the Black-Scholes equation in the limit case $a \rightarrow 0$. We show that the generalized equation is exactly solvable in terms of Hermite polynomials and numerically compare its solution with the solution of the Black-Scholes equation.

Keywords: econophysics, quantum finance, Black-Scholes equation, option pricing

2010 MSC: 91B80, 91G80

1. Introduction

The mathematical model based on the Black-Scholes equation

$$\frac{\partial C}{\partial t} = -\frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} - rS \frac{\partial C}{\partial S} + rC$$

(1)

anticipates rather well the observed prices for options in the case of a strike price that is not too far from the current price of the underlying asset \cite{8}. The price of an option at a moment of time $t$ depends on the current price $S$, the volatility $\sigma$, the risk-free interest rate $r$, the strike price $K$ and the

*Corresponding author

Email addresses: lcotfas@gmail.com (L.-A. Cotfas), camelia.delcea@yahoo.com (C. Delcea), ncotfas@yahoo.com (N. Cotfas)

Preprint submitted to Physica A November 12, 2014
maturity time $T$. In the case of an European option, the price is described by the solution $C(S, t)$ of equation (1) satisfying the condition

$$C(S, T) = \begin{cases} S - K & \text{if } S \geq K \\ 0 & \text{if } S < K \end{cases}$$

in the case of a call option, and

$$C(S, T) = \begin{cases} 0 & \text{if } S \geq K \\ K - S & \text{if } S < K \end{cases}$$

in the case of a put option. The alternative version of the equation (1)

$$\frac{\partial \tilde{C}}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial^2 \tilde{C}}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right) \frac{\partial \tilde{C}}{\partial x} + r \tilde{C}$$

obtained by using the change of independent variable

$$S = e^x$$

allows one to use the formalism of quantum mechanics in option pricing [1, 2, 4, 7]. In the new variable the conditions (2) and (3) become

$$\tilde{C}(x, T) = \begin{cases} e^x - K & \text{if } x \geq \ln K \\ 0 & \text{if } x < \ln K \end{cases}$$

and respectively,

$$\tilde{C}(x, T) = \begin{cases} 0 & \text{if } x \geq \ln K \\ K - e^x & \text{if } x < \ln K. \end{cases}$$

The more general version of (1) depending on a function $V(x)$

$$\frac{\partial \tilde{C}}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial^2 \tilde{C}}{\partial x^2} + \left(\frac{\sigma^2}{2} - V(x)\right) \frac{\partial \tilde{C}}{\partial x} + V(x) \tilde{C}$$

satisfies the martingale condition [1] and hence can be used for studying processes in finance. Our purpose is to investigate the particular case

$$V(x) = ax + r$$
that is, the equation
\[
\frac{\partial \tilde{C}}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial^2 \tilde{C}}{\partial x^2} + \left( \frac{\sigma^2}{2} - ax - r \right) \frac{\partial \tilde{C}}{\partial x} + (ax + r) \tilde{C}
\]
(10)

where \( a \in (-\infty, 0) \) is a parameter. The equation (10) is exactly solvable in terms of Hermite polynomials and coincides in the limit case \( a \rightarrow 0 \) with the equation (4) which corresponds to the standard Black-Scholes equation.

2. A shifted oscillator

Let \( \alpha \in (-\infty, 0) \) and \( \beta \in \mathbb{R} \) be two constants. By using the Hermite polynomial
\[
H_n(s) = (-1)^n e^{s^2} \frac{d^n}{ds^n}(e^{-s^2})
\]
(11)
we define for each \( n \in \{0, 1, 2, \ldots\} \) the function
\[
\psi_n(x) = \frac{1}{\sqrt{n!} 4^{n/2}} e^{\frac{x^2}{2} + \frac{\beta^2}{4n}} H_n \left( \sqrt{-\frac{\alpha}{2}} x - \frac{\beta}{\sqrt{-2\alpha}} \right)
\]
(12)
If we denote
\[
s = \sqrt{\frac{-\alpha}{2}} x - \frac{\beta}{\sqrt{-2\alpha}}
\]
(13)
then the previous relation can be written as
\[
H_n(s) = \sqrt{n!} 4^{n/2} e^{\frac{s^2}{2} - \frac{\beta^2}{4n}} \psi_n \left( \sqrt{\frac{2}{-\alpha}} s - \frac{\beta}{\alpha} \right)
\]
(14)
By substituting this relation into the differential equation
\[
H_n''(s) - 2sH'_n(s) + 2nH_n(s) = 0
\]
(15)
satisfied by the Hermite polynomial \( H_n \) we get the equality
\[
-\psi_n'' \left( \sqrt{\frac{2}{-\alpha}} s - \frac{\beta}{\alpha} \right) + \left( -\frac{\alpha}{2} s^2 + \frac{\alpha}{2} + \alpha n \right) \psi_n \left( \sqrt{\frac{2}{-\alpha}} s - \frac{\beta}{\alpha} \right) = 0
\]
(16)
which can be written in the form
\[
\left( -\frac{\partial^2}{\partial x^2} + \frac{(ax + \beta)^2}{4} + \frac{\alpha}{2} \right) \psi_n = -\alpha n \psi_n.
\]
(17)
This means that $\psi_n$ is an eigenfunction of the shifted oscillator $[3, 5, 6]$.

$$H = -\frac{\partial^2}{\partial x^2} + \frac{(\alpha x + \beta)^2}{4} + \frac{\alpha}{2} \quad (18)$$
corresponding to the eigenvalue $\lambda_n = -\alpha n$, for any $n \in \{0, 1, 2, \ldots\}$. Since

$$\int_{-\infty}^{\infty} e^{-s^2} H_n(s) H_k(s) \, ds = \begin{cases} n! 2^n \sqrt{\pi} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases} \quad (19)$$

the system of functions $\{\psi_n\}_{n=0,1,2,\ldots}$ is orthonormal, that is,

$$\int_{-\infty}^{\infty} \psi_n(x) \psi_k(x) \, dx = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases} \quad (20)$$

One can prove that it is complete in the space of square integrable functions.

3. A generalized version of the Black-Scholes equation

A straightforward generalization of the equation

$$\frac{\partial \tilde{C}}{\partial t} = -\frac{\sigma^2}{2} S^2 \frac{\partial^2 \tilde{C}}{\partial S^2} - r S \frac{\partial \tilde{C}}{\partial S} + r \tilde{C} \quad (21)$$
is the equation

$$\frac{\partial \tilde{C}}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial^2 \tilde{C}}{\partial x^2} + \left( \frac{\sigma^2}{2} - V(x) \right) \frac{\partial \tilde{C}}{\partial x} + V(x) \tilde{C} \quad (22)$$
satisfying the martingale condition $[1]$. It can be written in the form

$$\frac{\partial \tilde{C}}{\partial t} = H_V \tilde{C} \quad (23)$$
by using the Hamiltonian

$$H_V = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{\sigma^2}{2} - V(x) \right) \frac{\partial}{\partial x} + V(x). \quad (24)$$
The Hamiltonian $H_V$ is equivalent with the Hermitian Hamiltonian

$$H_{\text{eff}} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \frac{\partial V}{\partial x} + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} + \frac{1}{2} V + \frac{\sigma^2}{8} \quad (25)$$
by the similarity transformation

$$H_{\text{eff}} = e^{-u} H V e^u$$  \hspace{1cm} (26)

where

$$u = \frac{1}{2} x - \frac{1}{\sigma^2} \int_0^x V(y) dy. \hspace{1cm} (27)$$

In the particular case $V(x) = ax + r$ considered in this article

$$H_V = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{\sigma^2}{2} - ax - r \right) \frac{\partial}{\partial x} + ax + r$$

$$H_{\text{eff}} = \frac{\sigma^2}{2} \left[ -\frac{\partial^2}{\partial x^2} + \frac{1}{4} \left( \frac{2a}{\sigma^2} x + \frac{2r}{\sigma^2} + 1 \right)^2 + \frac{a}{\sigma^2} \right]$$  \hspace{1cm} (28)

$$u = -\frac{a}{2\sigma^2} x^2 - \left( \frac{r}{\sigma^2} - \frac{1}{2} \right) x. \hspace{1cm}$$

The Hamiltonian $H_{\text{eff}}$ is up to the multiplicative factor $\frac{\sigma^2}{2}$ the Hamiltonian of a shifted oscillator, namely

$$H_{\text{eff}} = \frac{\sigma^2}{2} H$$  \hspace{1cm} (29)

where

$$H = -\frac{\partial^2}{\partial x^2} + \frac{(\alpha x + \beta)^2}{4} + \frac{a}{2}$$  \hspace{1cm} (30)

with

$$\alpha = \frac{2a}{\sigma^2} \quad \text{and} \quad \beta = \frac{2r}{\sigma^2} + 1. \hspace{1cm}$$

Therefore, for each $n \in \{0, 1, 2, \ldots\}$ the function

$$\psi_n(x) = \frac{1}{\sqrt{n!}} 2^{\frac{n}{2}} \sigma \sqrt{\frac{\pi}{\sigma}} e^{\frac{\alpha}{\sigma} x^2 + \left( \frac{\beta}{\sigma} + \frac{1}{2} \right) x + \frac{a^2}{4\sigma^2} (\frac{2r}{\sigma^2} + 1)^2} H_n \left( \frac{\sqrt{a}}{\sigma} x - \frac{\sigma}{2\sqrt{a} - \sigma} (\frac{2r}{\sigma^2} + 1) \right)$$  \hspace{1cm} (31)

is an eigenfunction of $H_{\text{eff}}$ corresponding to the eigenvalue $-an$

$$H_{\text{eff}} \psi_n = -an \psi_n. \hspace{1cm} (32)$$

The equation (23) can be written as

$$\frac{\partial \tilde{C}}{\partial t} = e^u H_{\text{eff}} e^{-u} \tilde{C} \hspace{1cm} (33)$$

or in the form

$$\frac{\partial}{\partial t} e^{-u} \tilde{C} = H_{\text{eff}} e^{-u} \tilde{C}. \hspace{1cm} (34)$$
A function $\tilde{C}(x,t)$ is the solution of (23) satisfying (6) if and only if

$$\psi(x,t) = e^{-u}\tilde{C}(x,t)$$

that is, the function

$$\psi(x,t) = e^{\frac{a}{2}\sigma^2 x^2 + \left(\frac{r}{2\sigma^2} - \frac{1}{4}\right)x} \tilde{C}(x,t)$$

is the solution of the equation

$$\frac{\partial \psi}{\partial t} = H_{\text{eff}} \psi$$

satisfying the condition

$$\psi(x,T) = e^{\frac{a}{2}\sigma^2 x^2 + \left(\frac{r}{2\sigma^2} - \frac{1}{2}\right)x} \left\{ \begin{array}{ll} e^x - K & \text{if } x \geq \ln K \\ 0 & \text{if } x < \ln K \end{array} \right.$$  (38)

But, the solution of (37) satisfying (38) is

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n e^{-ant} \psi_n(x)$$

with the coefficients $c_n$ determined from the relation

$$\sum_{n=0}^{\infty} c_n e^{-anT} \psi_n(x) = e^{\frac{a}{2}\sigma^2 x^2 + \left(\frac{r}{2\sigma^2} - \frac{1}{2}\right)x} \left\{ \begin{array}{ll} e^x - K & \text{if } x \geq \ln K \\ 0 & \text{if } x < \ln K \end{array} \right.$$  (40)

namely,

$$c_n = e^{anT} \int_{\ln K}^{\infty} e^{\frac{a}{2}\sigma^2 x^2 + \left(\frac{r}{2\sigma^2} - \frac{1}{2}\right)x} (e^x - K) \psi_n(x) dx.$$  (41)

In the case of the call option, the solution of the generalized Black-Scholes equation expressed in terms of Hermite polynomials is

$$C_a(S,t) = S \sqrt{\frac{\sigma^2}{2\pi t}} e^{\frac{S^2}{2\sigma^2 t^2} + \frac{2r}{\sigma^2 t} + 1} \sum_{n=0}^{\infty} c_n e^{-ant} \psi_n(x) \ln S - \frac{\sigma^2}{2\sqrt{2\pi t}} \sqrt{2\pi t} + 1 \right).$$  (42)

The case of a put option can be analyzed in a very similar way.

The Black-Scholes equation is exactly solvable. By denoting

$$\Phi(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta} e^{-\eta^2/2} d\eta = \frac{1}{2} \left( 1 + \text{erf}(x/\sqrt{2}) \right)$$

(43)
the formulas for the values of a European option can be written in the form

\[ C_{\text{call}}(S, t) = S \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2) \]
\[ C_{\text{put}}(S, t) = Ke^{-r(T-t)} \Phi(d_1) - S \Phi(d_2) \]  \hspace{1cm} (44)

where

\[ d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \]  \hspace{1cm} (45)

In Fig. 1 we present \( C_{\alpha}(S, t) \) (thick line) versus \( C_{\text{call}}(S, t) \) (thin line) and \( C_{\text{call}}(S, T) \) (dashed line) for \( t = 3 \) (left side), \( t = 4 \) (right side) and \( a = -0.03 \) (first row), \( a = -0.02 \) (second row), \( a = -0.01 \) (third row) by choosing the following values of the parameters: \( \sigma = 0.25, r = 0.03, K = 3 \) and \( T = 5 \).

4. Concluding remarks

In quantum mechanics as well as in econophysics, exact solutions are known only in a small number of particular cases and, generally, they play an important role. The generalized version of the Black-Scholes equation investigated in this article:
- is exactly solvable for any \( a \in (-\infty, 0) \),
- satisfies the martingale condition for any \( a \in (-\infty, 0) \),
- coincides with the Black-Scholes equation in the limit case \( a \uparrow 0 \).

In practice, there are some deviations of prices from those described by the solution of the Black-Scholes equation. We think that the solution of the generalized Black-Scholes equation might describe some observed prices, and the parameter \( a \) might have a certain financial meaning.

References

[1] B.E. Baaquie, Quantum Finance, Cambridge University Press, 2004.
[2] F. Bagarello, A quantum statistical approach to simplified stock markets, Physics A 388 (2009) 4397.
[3] F. Cooper, A. Khare, U. Sukhatme, Supersymmetry and quantum mechanics, Phys. Rep. 251 (1995) 267.
[4] L.-A. Cotfas, A finite-dimensional quantum model for the stock market, Physics A 392 (2013) 371.
[5] N. Cotfas and L.A. Cotfas, Hypergeometric type operators and their supersymmetric partners, J. Math. Phys. 52 (2011) 052101.

[6] M.A. Jafarizadeh and H. Fakhri, Parasupersymmetry and shape invariance in differential equations of mathematical physics and quantum mechanics, Ann. Phys. NY 262 (1998) 260.

[7] T.K. Jana and P. Roy, Supersymmetry in option pricing, Physica A 390 (2011) 2350-55.

[8] Ö. Uğur, An Introduction to Computational Finance, Imperial College Press, London, 2009.
Figure 1: The solution $C_a(S, t)$ (thick line) of the generalized Black-Scholes equation versus the solution $C_{\text{call}}(S, t)$ (thin line) of the Black-Scholes equation and the payoff function $C_{\text{call}}(S, T)$ (dashed line) for $t = 3$ (left hand side), $t = 4$ (right hand side) and $a = -0.03$ (first row), $a = -0.02$ (second row), $a = -0.01$ (third row).