LIMIT CYCLE BIFURCATIONS FOR PIECEWISE SMOOTH INTEGRABLE DIFFERENTIAL SYSTEMS

JIHUA YANG∗
School of Mathematics and Computer Science, Ningxia Normal University
Guyuan 756000, China

LIQIN ZHAO
School of Mathematical Sciences, Beijing Normal University
Beijing 100875, China

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ABSTRACT. In this paper, we study a class of piecewise smooth integrable non-Hamiltonian systems, which has a center. By using the first order Melnikov function, we give an exact number of limit cycles which bifurcate from the above periodic annulus under the polynomial perturbation of degree n.

1. Introduction and the main results. Limit cycle theory of piecewise smooth differential systems which raised from nonlinear oscillations, mechanics, electrical engineering and control systems [1, 2, 8] has been extensively studied for many years and many methodologies have been developed. In particular, it can be seen as an extension of the Hilbert’s 16th problem, which has not been solved since Hilbert proposed the 23 mathematical problems in 1990 [6]. For more details on the Hilbert’s 16th problem, see for instance [7, 9, 10, 17, 18] and the references therein.

Consider a piecewise smooth differential system

$$\begin{align*}
\frac{dx}{d\tau}, \frac{dy}{d\tau} &= \begin{cases}
(p^+(x,y), q^+(x,y)), & x \geq 0, \\
(p^-(x,y), q^-(x,y)), & x < 0.
\end{cases}
\end{align*}$$

(1.1)

Suppose that system (1.1) has first integrals $H^+(x,y) = h$ (resp. $H^-(x,y) = \tilde{h}$) for $x \geq 0$ (resp. $x < 0$), and has integrating factors $\mu_1(x,y)$ (resp. $\mu_2(x,y)$) for $x \geq 0$ (resp. $x < 0$). We call (1.1) a piecewise smooth integrable differential system. For a perturbed piecewise smooth integrable differential system

$$\begin{align*}
\frac{dx}{d\tau}, \frac{dy}{d\tau} &= \begin{cases}
(p^+(x,y) + \varepsilon f^+(x,y), q^+(x,y) + \varepsilon g^+(x,y)), & x \geq 0, \\
(p^-(x,y) + \varepsilon f^-(x,y), q^-(x,y) + \varepsilon g^-(x,y)), & x < 0,
\end{cases}
\end{align*}$$

(1.2)

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∗ Corresponding author: Jihua Yang.
where \( f^\pm, g^\pm \in C^\infty \), since
\[
\frac{\partial H^+(x, y)}{\partial y} = \mu_1(x, y)p^+(x, y), \quad \frac{\partial H^+(x, y)}{\partial x} = -\mu_1(x, y)q^+(x, y),
\]
\[
\frac{\partial H^-(x, y)}{\partial y} = \mu_2(x, y)p^-(x, y), \quad \frac{\partial H^-(x, y)}{\partial x} = -\mu_2(x, y)q^-(x, y),
\]

multiplying (1.2) by \( \mu_1(x, y) \) (resp. \( \mu_2(x, y) \)) for \( x \geq 0 \) (resp. \( x < 0 \)) and letting \( dt = \mu_1(x, y)dt \) \((i = 1, 2)\), we have
\[
(\dot{x}, \dot{y}) = \begin{cases}
\left( \frac{\partial H^+(x, y)}{\partial y} + \varepsilon \mu_1(x, y)f^+(x, y), -\frac{\partial H^+(x, y)}{\partial x} + \varepsilon \mu_1(x, y)g^+(x, y) \right), & x \geq 0, \\
\left( \frac{\partial H^-(x, y)}{\partial y} + \varepsilon \mu_2(x, y)f^-(x, y), -\frac{\partial H^-(x, y)}{\partial x} + \varepsilon \mu_2(x, y)g^-(x, y) \right), & x < 0,
\end{cases}
\tag{1.3}
\]

where the dot denotes derivative with respect to an independent variable \( t \). Systems (1.2) and (1.3) have qualitatively the same orbit structure.

As in the smooth case, a very important issue associated with (1.3) is to find the number of limit cycles and their distributions. Up to now most of the results on this problem have been concerned with the perturbation of the Hamiltonian system, i.e. the case \( \mu_1(x, y) = \mu_2(x, y) = 1 \), see [3, 5, 13, 15, 16]. For the integrable but non-Hamiltonian system, since \( \mu(x, y) \) and \( \mu_2(x, y) \) are no longer constants, the study of Melnikov functions in this case is much more difficult than in the Hamiltonian case. As far as we know, few papers study piecewise smooth integrable non-Hamiltonian systems [11].

In this paper, we first give the first order Melnikov function of system (1.3). Hence, we make the following assumptions for system (1.3) \( \varepsilon = 0 \):

Assumption (I). There exist an interval \( J = (\alpha, \beta) \), and two points \( A(h) = (0, a(h)) \) and \( B(h) = (0, b(h)) \) such that for \( h \in J \)
\[
H^+(A(h)) = H^+(B(h)) = h, \quad H^-(A(h)) = H^-(B(h)) = \tilde{h}, \quad a(h) \neq b(h).
\]

Assumption (II). The system (1.3) \( \varepsilon = 0 \) has an orbital arc \( L_h^+ \) starting from \( A(h) \) and ending at \( B(h) \) defined by \( H^+(x, y) = h \), \( x \geq 0 \); the system (1.3) \( \varepsilon = 0 \) has an orbital arc \( L_h^- \) starting from \( B(h) \) and ending at \( A(h) \) defined by \( H^-(x, y) = H^-(B(h)), x < 0 \).

Under the above assumptions (I) and (II), system (1.3) \( \varepsilon = 0 \) has a family of periodic orbits \( L_h = L_h^+ \cup L_h^- \) for \( h \in J \). Each of the closed curves \( L_h \) is piecewise smooth, in general. Further, without loss of generality, suppose that \( L_h \) has a clockwise orientation.

From Theorem 1.1 in Liu and Han [14], we get the first order Melnikov function \( M(h) \) of system (1.3) as follow.

Proposition 1.1. Under the conditions (I) and (II), the first order Melnikov function of system (1.3) has the following form
\[
M(h) = \frac{H_y^+(A)}{H_y^-(A)} \cdot \frac{H_y^+(B)}{H_y^-(B)} \oint_{L_h^+} \mu_1(x, y)(g^+(x, y)dx - f^+(x, y)dy) + \oint_{L_h^-} \mu_2(x, y)(g^-(x, y)dx - f^-(x, y)dy), \quad h \in J.
\tag{1.4}
\]

Remark 1. By Lemma 2.2 in [12], we have \( \frac{H_y^+(A)}{H_y^-(A)} \cdot \frac{H_y^+(B)}{H_y^-(B)} = 1 \).
In the following, we consider the following piecewise smooth integrable differential system
\[
\begin{align*}
(\dot{x}, \dot{y}) = \begin{cases} 
-\varepsilon f^+(x, y), & x > 0, \\
-\varepsilon f^-(x, y), & x < 0,
\end{cases}
\end{align*}
\] (1.5)
where \(ab \neq 0\),
\[
f^\pm(x, y) = \sum_{i+j=0}^n a_{i,j}^\pm x^i y^j, \quad g^\pm(x, y) = \sum_{i+j=0}^n b_{i,j}^\pm x^i y^j.
\] (1.6)

System (1.5)\(\varepsilon=0\) has the first integral \(H^\pm(x, y) = x^2 + y^2\) with respect to \(x \geq 0\) and \(x < 0\). The origin is a center. It is worth noting that system (1.5)\(\varepsilon=0\) has an invariant straight line \(ax^2 + 1 = 0\) (resp. \(bx^2 + 1\)) for \(a < 0\) (resp. \(b < 0\)). Denote
\[
h_1 = \begin{cases} 
\sqrt{-\frac{1}{a}}, & a < 0, \\
+\infty, & a \geq 0,
\end{cases} \quad h_2 = \begin{cases} 
\sqrt{-\frac{1}{b}}, & b < 0, \\
+\infty, & b \geq 0.
\end{cases}
\]

Let \(H(n)\) denote the maximum number of limit cycles of (1.5) bifurcated from the period annulus \(\bigcup_{0<h<h_0} L_h\) for all possible \(f^\pm(x, y)\) and \(g^\pm(x, y)\) satisfying (1.6) up to the first order Melnikov function in \(\varepsilon\), where \(h_0 = \min\{h_1, h_2\}\). Our main result is the following theorem.

**Theorem 1.1.** Suppose that \(ab \neq 0\), then for any \(n \geq 1\), we have

(i) If \(a \neq b\), then \(H(n) = n + 2\left[\frac{n+1}{2}\right] - 1\).

(ii) If \(a = b\), then \(H(n) = n + \left[\frac{n+1}{2}\right] - 1\).

2. **The first order Melnikov function.** In this section, we give an expression of the first order Melnikov function \(M(h)\) of system (1.5) for \(0 < h < h_0\). Note that \(H^+(0, y) = H^-(0, y)\). Then by Proposition 1.1, we have the first order Melnikov function of system (1.5) satisfying
\[
M(h) = \oint_{\overline{AB}} \mu_1(g^+ dx - f^+ dy) + \oint_{\overline{BA}} \mu_2(g^- dx - f^- dy),
\] (2.1)
where \(0 < h < h_0\), and
\[
\overline{AB} = \{(x, y)|H^+(x, y) = h, \ x \geq 0\}, \quad \overline{BA} = \{(x, y)|H^-(x, y) = h, \ x < 0\}.
\]

Letting \(x = \sqrt{h} \cos \theta, \ y = \sqrt{h} \sin \theta\), we obtain
\[
M(h) = -\sum_{i+j=0}^n h^{i+j} b_{i,j}^+ \frac{\int \frac{\pi}{2} \cos^i \theta \sin^{i+1} \theta d\theta}{1 + ah \cos^2 \theta} - \sum_{i+j=0}^n h^{i+j} a_{i,j}^- \frac{\int \frac{\pi}{2} \cos^{i+1} \theta \sin^i \theta d\theta}{1 + ah \cos^2 \theta}
\]
(2.2)
\[
-\sum_{i+j=0}^n h^{i+j} b_{i,j}^- \frac{\int \frac{\pi}{2} \cos^i \theta \sin^{i+1} \theta d\theta}{1 + bh \cos^2 \theta} - \sum_{i+j=0}^n h^{i+j} a_{i,j}^+ \frac{\int \frac{\pi}{2} \cos^{i+1} \theta \sin^i \theta d\theta}{1 + bh \cos^2 \theta}
\]
\[
= \sum_{i+j=1}^{n+1} \sigma_{i,j} \frac{\int \frac{\pi}{2} \cos^i \theta \sin^{i+1} \theta d\theta}{1 + ah \cos^2 \theta} + \sum_{i+j=1}^{n+1} \tau_{i,j} \frac{\int \frac{\pi}{2} \cos^{i+1} \theta \sin^i \theta d\theta}{1 + bh \cos^2 \theta},
\]
where \(\sigma_{i,j} = -b_{i,j}^+ - a_{i+1,j}^-\) and \(\tau_{i,j} = -b_{i,j}^- - a_{i+1,j}^+\), here we assume that \(b_{i-1,j}^+ = a_{i-1,j}^+\), \(b_{i-1,j}^- = a_{i-1,j}^-\), \(\sigma_{0,0} = \tau_{0,0} = 0\). Since the coefficients \(a_{i,j}^\pm\)
and $b_{i,j}^k$ are arbitrary for $i, j \in \mathbb{N}$, $\sigma_{i,j}$ and $\tau_{i,j}$ are also arbitrary. Furthermore, for the sake of convenience, we denote

$$I_{i,j}(h) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^i \theta \sin^j \theta}{1 + ah \cos^2 \theta} d\theta,$$

$$J_{i,j}(h) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^i \theta \sin^j \theta}{1 + bh \cos^2 \theta} d\theta,$$

$$(2.3) \quad N(k) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^k \theta d\theta.$$

**Lemma 2.1.** Suppose that $ab \neq 0$, the following equalities hold:

(i) $I_{1,2j+1}(h) = J_{1,2j+1}(h) = 0$.

(ii) $I_{i,2l}(h) = \sum_{k=0}^{l} (-1)^k C_k^i I_{i+2k,0}(h), \quad J_{i,2l}(h) = \sum_{k=0}^{l} (-1)^k C_k^i J_{i+2k,0}(h)$.

(iii) $h^i I_{2i,0}(h) = \left( -\frac{1}{a} \right)^i I_{0,0}(h) - \sum_{k=1}^{i} \left( -\frac{1}{a} \right)^k h^{-k} N(2i - 2k)$.

(iv) $h^i J_{2i,0}(h) = \left( -\frac{1}{b} \right)^i J_{0,0}(h) - \sum_{k=1}^{i} \left( -\frac{1}{b} \right)^k h^{-k} N(2i - 2k)$.

(v) If $a \neq -\frac{1}{h}, (1 + ah) I_{0,0}(h) = -\frac{1}{2} J_{0,0}(h)$. If $b \neq -\frac{1}{h}, (1 + bh) J_{0,0}(h) = -\frac{1}{2} J_{0,0}(h)$.

**Proof.** (i) and (ii) can be got by straight calculation.

(iii) From

$$(ah \cos^2 \theta)^i - (-1)^i = (1 + ah \cos^2 \theta) \sum_{k=1}^{i} (-1)^{k-1}(ah \cos^2 \theta)^{i-k},$$

we have

$$\frac{(ah \cos^2 \theta)^2}{1 + ah \cos^2 \theta} = \frac{(-1)^i}{1 + ah \cos^2 \theta} + \sum_{k=1}^{i} (-1)^{k-1}(ah \cos^2 \theta)^{i-k}.$$

Integrating the above equality from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we obtain the conclusion (iii). Conclusion (iv) can be proved similarly.

(v) Noting that

$$I_{0,0}(h) = \begin{cases} \frac{\pi}{\sqrt{1+ah}}, & a \geq 0, \ h \in (0, +\infty) \ or \ a < 0, \ h \in (0, -\frac{1}{a}), \\ 0, & a < 0, \ h \in (-\frac{1}{a}, +\infty), \\ \infty, & a < 0, \ h = -\frac{1}{a}, \end{cases}$$

and

$$J_{0,0}(h) = \begin{cases} \frac{\pi}{\sqrt{1+bh}}, & b \geq 0, \ h \in (0, +\infty) \ or \ b < 0, \ h \in (0, -\frac{1}{b}), \\ 0, & b < 0, \ h \in (-\frac{1}{b}, +\infty), \\ \infty, & b < 0, \ h = -\frac{1}{b}, \end{cases}$$

we obtain the conclusion (v). \qed

**Lemma 2.2.** Suppose that $ab \neq 0$.

(i) If $a \neq b$, then $M(h)$ can be expressed by the following $2[n + 1] + n + 1$ functions:

$$a_0 I_{0,0}(h) - \alpha \pi + \hat{b}_0, \ h I_{0,0}(h), \ h^2 I_{0,0}(h), \cdots, \ h^{[a+b]} I_{0,0}(h),$$
where \( I_{0,0}(h) \) and \( J_{0,0}(h) \) are given by (2.3), \( a_0, c_0, \tilde{b}_0, \tilde{b}_0, \alpha \) and \( \beta \) are constants.

Proof. (i) From (2.2), (2.3) and (i)-(iv) in Lemma 2.1, we have

\[
M(h) = \sum_{i+j=1}^{n+1} h^{\frac{i+j}{2}} I_{i,j}(h) + \sum_{i+j=1}^{n+1} h^{\frac{i+j}{2}} \tau_{i,j} \langle J_{i,j}(h) \rangle
= \sum_{i=1}^{n+1} h^\frac{i}{2} \sum_{j=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \sigma_{i-j,j} I_{i-j,j}(h) + \sum_{i=1}^{n+1} h^\frac{i}{2} \sum_{j=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \tau_{i-j,j} J_{i-j,j}(h)
= \sum_{i=0}^{n+1} \sum_{j=0}^{\left\lfloor \frac{i}{2} \right\rfloor} S_{i,j} h^\frac{i+j}{2} I_{i,0}(h) + \sum_{i=0}^{n+1} \sum_{j=0}^{\left\lfloor \frac{i}{2} \right\rfloor} T_{i,j} h^\frac{i+j}{2} J_{i,0}(h)
= a_0 I_{0,0}(h) - \alpha \pi + \sum_{i=1}^{\left\lceil \frac{n+1}{2} \right\rceil} a_i h^i I_{0,0}(h) + \sum_{i=0}^{n-1} \tilde{b}_i h^{\frac{i}{2}}
+ c_0 J_{0,0}(h) - \beta \pi + \sum_{i=1}^{\left\lceil \frac{n+1}{2} \right\rceil} c_i h^i J_{0,0}(h) + \sum_{i=0}^{n-1} \tilde{b}_i h^{\frac{i}{2}}
= a_0 I_{0,0}(h) - \alpha \pi + \tilde{b}_0 + \sum_{i=1}^{\left\lceil \frac{n+1}{2} \right\rceil} a_i h^i I_{0,0}(h) + \sum_{i=1}^{\left\lceil \frac{n+1}{2} \right\rceil} \tilde{b}_i h^{\frac{i}{2}}
+ c_0 J_{0,0}(h) - \beta \pi + \tilde{b}_0 + \sum_{i=1}^{\left\lceil \frac{n+1}{2} \right\rceil} c_i h^i J_{0,0}(h),
\]

where

\[
S_{i,j} = \sum_{k=0}^{\left\lfloor \frac{i}{2} \right\rfloor} (-1)^k C^k_{j+k} \sigma_{i-2k,2k+2k}, \quad T_{i,j} = \sum_{k=0}^{\left\lfloor \frac{i}{2} \right\rfloor} (-1)^k C^k_{j+k} \tau_{i-2k,2k+2k},
\]
Remark 2. Since $\sigma_{i,j}$ and $\tau_{i,j}$ are arbitrary, $S_{i,j}$ and $T_{i,j}$ are also arbitrary. It is easy to get that

$$A = \frac{\partial(a_1, a_2, \cdots, a_{\left[\frac{n+1}{2}\right]}, a_0, b_1, b_2, \cdots, b_n)}{\partial(S_{0,1}, S_{0,2}, \cdots, S_{0,\left[\frac{n+1}{2}\right]}, S_{1,0}, S_{2,0}, S_{3,0}, \cdots, S_{n+1,0})}
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -\frac{1}{a} & \left(-\frac{1}{a}\right)^2 & \cdots & \left(-\frac{1}{a}\right)^{n+1} \\
0 & 0 & \cdots & 0 & N(1)\frac{1}{a} & -N(1)\left(-\frac{1}{a}\right)^2 & \cdots & N(1)\left(-\frac{1}{a}\right)^{n+1} \\
0 & 0 & \cdots & 0 & 0 & N(2)\frac{1}{a} & \cdots & -N(2)\left(-\frac{1}{a}\right)^{n+1} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & N(n)\frac{1}{a}
\end{pmatrix},
$$

and $\det A = -\frac{1}{a^{n+1}} \prod_{i=1}^{n} N(i) \neq 0$. So $a_i, b_i$ can be chosen arbitrary. In a similar way, we can prove that $c_i, \tilde{b}_i$ are arbitrary.

In the following, we show that the functions defined in (2.6) and (2.7) are linearly independent and suppose that $h \in \mathbb{C}$. From Lemma 2.1 (v), we have $I_{0,h}(h) = \frac{a}{\sqrt{1+b}}$ with $a > 0$. Therefore, $I_{0,h}(h)$ can be analytically extended to the complex domain $D_1 = \mathbb{C} \setminus \{h \in \mathbb{R}, h \leq -h_1\}$ with $a > 0$. In a similar way, $J_{0,h}(h)$ can be analytically extended to the complex domain $D_2 = \mathbb{C} \setminus \{h \in \mathbb{R}, h \geq h_2\}$ with $b > 0$. For $h < -h_1$, we denote $I_{0,h}^+(h)$ by the analytic continuation of $I_{0,h}(h)$ along an arc such that $\operatorname{Im}(h) > 0$ (resp. $\operatorname{Im}(h) < 0$). For other functions, we will use similar notations.
Lemma 2.3. (i) If $a > 0$, then the functions $I_{0,0}^\pm(h)$, which are defined in $(-\infty, -h_1)$, satisfy

$$I_{0,0}^+(h) - I_{0,0}^-(h) = \frac{c_1i}{\sqrt{1 + ah}}. \quad (2.9)$$

(ii) If $b > 0$, then the functions $J_{0,0}^\pm(h)$, which are defined in $(h_2, +\infty)$, satisfy

$$J_{0,0}^+(h) - J_{0,0}^-(h) = \frac{c_2i}{\sqrt{1 + bh}}. \quad (2.10)$$

where $c_1$ and $c_2$ are nonzero real number and $i^2 = -1$.

Proof. From Lemma 2.1 (v) and noting that $I_{0,0}^\pm(h)$ are both analytic continuation of $I_{0,0}(h)$, we have $(1 + ah)(I_{0,0}^\pm(h))' = -\frac{a}{2}I_{0,0}^\pm(h)$, which implies

$$(1 + ah)(I_{0,0}^+(h) - I_{0,0}^-(h))' = -\frac{a}{2}(I^+0.0(h) - I^-0.0(h)).$$

Solving the above equation we obtain

$$I_{0,0}^+(h) - I_{0,0}^-(h) = \frac{c}{\sqrt{1 + ah}},$$

where $c \in \mathbb{C}$ is a constant. Since $I_{0,0}^\pm(h)$ conjugate each other, $I_{0,0}^+(h) - I_{0,0}^-(h)$ is a pure imaginary number, so we can suppose $c = c_1i$ ($c_1 \in \mathbb{R}$). We claim that $c_1$ is nonzero, else $I_{0,0}(h)$ is global single-valued, $I_{0,0}(h)$ will be analytic at $h = -h_1$ or is $-h_1$ a pole of $I_{0,0}(h)$, which is a contradiction with $I_{0,0}(h) = \frac{e^{-\pi h}}{\sqrt{1 + ah}}$. In a similar way, we can get the conclusion (ii). \qed

Now we begin to prove the independence of functions defined in (2.6). Suppose that

$$G(h) = \tilde{a}_0(a_0I_{0,0}(h) - \alpha \pi + \tilde{b}_0) + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \tilde{a}_i h^i I_{0,0}(h) + \sum_{i=1}^{n-1} \tilde{b}_i h^i \quad \quad (2.11)$$

$$+ \tilde{c}_0(c_0I_{0,0}(h) - \beta \pi + \tilde{b}_0) + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \tilde{c}_i h^i I_{0,0}(h) \equiv 0.$$

We only prove that the coefficients $\tilde{a}_i = \tilde{c}_i = 0$, $i = 0, 1, \cdots, \lfloor \frac{n+1}{2} \rfloor$ and $\tilde{b}_i = 0$, $i = 1, 2, \cdots, n - 1$.

If $a > 0, b > 0$, then $G(h)$ can be analytically extended to the domain $D = D_1 \cap D_2$. By Lemma 2.3, when $h < -h_1$, then

$$G^+(h) - G^-(h) = \frac{c_1i}{\sqrt{1 + ah}} \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \tilde{a}_i h^i = 0,$$

which implies that $\tilde{a}_i = 0$, $i = 0, 1, \cdots, \lfloor \frac{n+1}{2} \rfloor$. Similarly, we can obtain $\tilde{c}_i = 0$, $i = 0, 1, \cdots, \lfloor \frac{n+1}{2} \rfloor$. Thus $G(h) = \sum_{i=1}^{n-1} \tilde{b}_i h^i = 0$, which implies that $\tilde{b}_i = 0$, $i = 1, \cdots, n - 1$. The cases $a > 0$ and $b < 0$, $a < 0$ and $b > 0$, and $a < 0$ and $b < 0$ can be proved similarly.
3. Proof of the main results. We only prove the case that \(a > 0\) and \(b > 0\). To obtain the lower bound of \(H(n)\), we need

**Lemma 3.1.** [4] Consider \(p + 1\) linearly independent analytical functions \(f_i : U \rightarrow \mathbb{R}, \ i = 1, 2, \cdots, p\), where \(U \in \mathbb{R}\) is an interval. Suppose that there exists \(j \in \{0, 1, \cdots, p\}\) such that \(f_j\) has constant sign. Then there exists \(p + 1\) constants \(\delta_i, \ i = 0, 1, \cdots, p\) such that \(f(x) = \sum_{i=0}^{p} \delta_i f_i(x)\) has at least \(p\) simple zeros in \(U\).

From the above Lemma 3.1, we immediately obtain

(i) If \(a \neq b\), then \(H(n) \geq 2[\frac{n+1}{2}] + n\).

(ii) If \(a = b\), then \(H(n) \geq [\frac{n+1}{2}] + n - 1\).

Let \(0 < \varepsilon \ll 1 \ll R\) and denote by \(D_{\varepsilon,R}\) the domain obtained by removing two small discs \(\{|h + h_1| \leq \varepsilon\}, \{|h - h_2| \leq \varepsilon\}\) and two real intervals \([-R, -h_1 - \varepsilon], [h_2 + \varepsilon, R]\) from \(\{|h| \leq R\}\). Now we will use argument principle to estimate the number of zeros of \(M(h)\) in the domain \(D_{\varepsilon,R}\).

Recall that \(M(h)\) has the form

\[
M(h) = a_0 I_{0,0}(h) - \alpha \pi + \hat{b}_0 + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} a_i h^i I_{0,0}(h) + \sum_{i=1}^{n-1} b_i h^i \\
+ c_0 J_{0,0}(h) - \beta \pi + \hat{b}_0 + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} c_i h^i J_{0,0}(h),
\]

On \(\{|h + h_1| = \varepsilon\}\), \(M(h) \sim \frac{\varepsilon}{\sqrt{1 + ah}}\). So the argument of \(M(h)\) increases \(\pi + o(1)\). On \([-R, -h_1 - \varepsilon]\), \(M(h)\) is real if and only if \(\text{Im}(M(h)) = 0\), that is,

\[
M^+(h) - M^-(h) = \frac{c_1 i}{\sqrt{1 + ah}} \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} a_i h^i = 0.
\]

Note that we must count \([-R, -h_1 - \varepsilon]\) twice, so together with \(\{|h + h_1| = \varepsilon\}\) and \([-R, -h_1 - \varepsilon]\), the argument of \(M(h)\) increases at most \((2 + 2[\frac{n+1}{2}])\pi\). Similarly, together with \(\{|h - h_2| = \varepsilon\}\) and \([h_2 + \varepsilon, R]\), the argument of \(M(h)\) also increases at most \((2 + 2[\frac{n+1}{2}])\pi\).

Let \(r^2 = h\), then \(M(r) \sim r^{n-1}\). On \(\{|h| = R\}\), the argument of \(r^{n-1}\) increases \(2(n-1)\pi\), thus the argument of \(M(r)\) increases at most \(2(n-1)\pi + o(1)\). That is, the argument of \(M(h)\) increases at most \(2(n-1)\pi + o(1)\).

Along the boundary of \(D_{\varepsilon,R}\), the argument of \(M(h)\) increases at most \(2[\frac{n+1}{2}]\pi + 2(n-1)\pi + o(1)\). By the argument principle, \(M(h)\) has at most \(n + 2[\frac{n+1}{2}] + 1\) zeros in \(D_{\varepsilon,R}\). Noting that \(M(0) = 0\), we have

\[
H(n) \leq n + 2[\frac{n+1}{2}] + 1.
\]

So \(H(n) = n + 2[\frac{n+1}{2}]\), the proof of Theorem 1.2 for \(a > 0\) and \(b > 0\) is finished. For other \(a, b\), the proofs are similar.

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REFERENCES

[1] S. Banerjee and G. Verghese, Nonlinear Phenomena in Power Electronics: Attractors, Bifurcations, Chaos, and Nonlinear Control, Wiley-IEEE Press, New York, 2001.

[2] M. Bernardo, C. Budd, A. Champneys and P. Kowalczyk, Piecewise-smooth Dynamical Systems, Theory and Applications, Springer-Verlag, London, 2008.

[3] W. Chen and W. Zhang, Isochronicity of centers in a switching Bautin system, J. Differential Equations, 252 (2012), 2877–2899.

[4] B. Coll, A. Gasull and R. Prohens, Bifurcation of limit cycles from two families of centers, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 12 (2005), 275–287.

[5] L. Dieci and C. Elia, Periodic orbits for planar piecewise smooth systems with a line of discontinuity, J. Dyn. Diff. Equat., 26 (2014), 1049–1078.

[6] D. Hilbert, Mathematical problems (M. Newton, Transl.), Bull. Am. Math. Soc., 8 (1902), 437–479.

[7] Y. Ilyashenko, Centennial history of Hilbert’s 16th problem, Bull. Amer. Math. Soc., 39 (2002), 301–354.

[8] M. Kunze, Non-Smooth Dynamical Systems, Springer-Verlag, Berlin, 2000.

[9] C. Li, Abelian integrals and limit cycles, Qual. Theory Dyn. Syst., 11 (2012), 111–128.

[10] J. Li, Hilbert’s 16th problem and bifurcations of planar polynomial vector fields, Int. J. Bifurcation Chaos, 13 (2003), 47–106.

[11] S. Li and C. Liu, A linear estimate of the number of limit cycles for some planar piecewise smooth quadratic differential system, J. Math. Anal. Appl., 428 (2015), 1354–1367.

[12] F. Liang and M. Han, Limit cycles near generalized homoclinic and double homoclinic loops in piecewise smooth systems, Chaos Solitons Fractals, 45 (2012), 454–464.

[13] F. Liang, M. Han and V. G. Romanovsky, Bifurcation of limit cycles by perturbing a piecewise linear Hamiltonian system with a homoclinic loop, Nonlinear Anal., 75 (2012), 4355–4374.

[14] X. Liu and M. Han, Bifurcation of limit cycles by perturbing piecewise Hamiltonian systems, Internat. J. Bifur. Chaos Appl. Sci. Engrg, 20 (2010), 1379–1390.

[15] J. Llibre, A. Mereu and D. Novaes, Averaging theory for discontinuous piecewise differential systems, J. Differential Equations, 258 (2015), 4007–4032.

[16] Y. Xiong, Limit cycle bifurcations by perturbing piecewise smooth Hamiltonian systems with multiple parameters, J. Math. Anal. Appl., 421 (2015), 260–275.

[17] Y. Xiong, The number of limit cycles in perturbations of polynomial systems with multiple circles of critical points, J. Math. Anal. Appl., 440 (2016), 220–239.

[18] J. Yang and L. Zhao, Zeros of Abelian integrals for a quartic Hamiltonian with figure-of-eight loop through a nilpotent saddle, Nonlinear Analysis: Real World Applications, 27 (2016), 350–365.

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E-mail address: jihua1113@163.com
E-mail address: zhaoliqin@bnu.edu.cn