A NEW GEOMETRIC STRUCTURE ON TANGENT BUNDLES

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Abstract. For a Riemannian manifold \((N, g)\), we construct a scalar flat metric \(G\) in the tangent bundle \(TN\). It is locally conformally flat if and only if either, \(N\) is a 2-dimensional manifold or, \((N, g)\) is a real space form. It is also shown that \(G\) is locally symmetric if and only if \(g\) is locally symmetric. We then study submanifolds in \(TN\) and, in particular, find the conditions for a curve to be geodesic. The conditions for a Lagrangian graph to be minimal or Hamiltonian minimal in the tangent bundle \(T\mathbb{R}^n\) of the Euclidean real space \(\mathbb{R}^n\) are studied. Finally, using the cross product in \(\mathbb{R}^3\) we show that the space of oriented lines in \(\mathbb{R}^3\) can be minimally isometrically embedded in \(T\mathbb{R}^3\).

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1. Introduction

The geometry of the tangent bundle \(TN\) of a Riemannian manifold \((N, g)\) has been a topic of great interest for the last 60 years. In the celebrated article [12], Sasaki used the Levi-Civita connection of \(g\) to split the tangent bundle \(TTN\) of \(TN\) into a horizontal and a vertical part, constructing the first geometric structure of \(TN\). Namely, one can obtain a splitting \(TTN = HN \oplus VN\), where the subbundles \(HN\) and \(VN\) of \(TTN\) are both isomorphic to the tangent bundle \(TN\) - for more details.

Date: 14th June 2018.
see section 2. For \( X \in TTN \), we write \( X \simeq (\Pi X, KX) \), where \( \Pi X \in HN \) and \( KX \in VN \). Sasaki defined the following metric on \( TN \) [8]:
\[
(\tilde{X}, \tilde{Y}) \mapsto g(\Pi \tilde{X}, \Pi \tilde{Y}) + g(K \tilde{X}, K \tilde{Y}),
\]
The Sasaki metric is "rigid" in the following sense: it is scalar flat if and only if \( g \) is flat [9].
In the years since, several new geometries on the tangent bundle \( TN \) have been constructed using the splitting of \( TTN \) - see for example [10] and [13]. When the base manifold \( N \) admits additional structure one can use it to define other geometries in the tangent bundle. H. Anciaux and R. Pascal constructed in [3] a canonical pseudo-Riemannian metric in \( TN \) derived from a Kähler structure on \( N \). One example, the canonical neutral metric in \( T \mathbb{S}^2 \) defined by the standard Kähler structure of the round 2-sphere \( \mathbb{S}^2 \), has been used to study classical differential geometry in \( \mathbb{R}^3 \) - see [1], [6] and [7].

Using the Riemannian metric \( g \) one can define a canonical symplectic structure \( \Omega \) in \( TN \). This can be achieved by using the musical isomorphism between the tangent bundle and the cotangent bundle. In this article we use the existence of an almost paracomplex structure \( J \) on \( TN \) compatible with \( \Omega \) to construct a neutral metric \( G \) on \( TN \).

If the base manifold \( N \) admits a Kähler structure, the neutral metric \( G \) and the pseudo-Riemannian metric, derived from the Kähler structure, are isometric - see Proposition 5. In other words, the neutral metric \( G \) is a natural extension of the Kähler metric constructed by H. Anciaux and R. Pascal in [3] to the case where the base manifold does not admit a Kähler structure.

The purpose of this article is to study the geometric properties and submanifolds of the neutral metric \( G \). We first prove the following:

**Theorem 1.** The neutral metric \( G \) has the following properties:

1. \( G \) is scalar flat,
2. \( G \) is Einstein if and only if \( g \) is Ricci flat,
3. \( G \) is locally conformally flat if and only if either \( n = 2 \) or, \( g \) is of constant sectional curvature,
4. \( G \) is locally symmetric if and only if \( g \) is locally symmetric.

We then focus our attention on submanifolds. In particular, the geodesics of \( G \) are characterized by:

**Theorem 2.** A curve \( \gamma(t) = (x(t), V(t)) \) in \( TN \) is a geodesic with respect to the metric \( G \) if and only if the curve \( x \) is a geodesic on \( N \) and \( V \) is a Jacobi field along \( x \).

It was shown in [2], that the existence of a minimal Lagrangian graph in \( TN \), where \( (N, g) \) is a 2-dimensional Riemannian manifold, implies that \( g \) is flat. A generalization of this result is given by the following Theorem:
Theorem 3. If $TN$ contains a Lagrangian graph with parallel mean curvature then the neutral metric $G$ is Ricci flat.

To continue with submanifolds, we need to introduce some more terminology. A vector field $X$ in a symplectic manifold $(N, \omega)$, is called a Hamiltonian field if $\omega(X,.) = dh$, where $h$ is a smooth function on $N$. A Lagrangian immersion will be called Hamiltonian minimal if the variations of its volume along all Hamiltonian vector fields are zero. Let $\Sigma$ be a Lagrangian submanifold in a (para-) Kähler manifold $(N, g, \omega, j)$, and $H$ be its mean curvature. The first variation formula shows that $\Sigma$ is Hamiltonian minimal if and only if the tangential vector field $jH$ is divergence free [14].

For Lagrangian graphs in $T\mathbb{R}^n$ we prove the following:

Theorem 4. Suppose that $u$ is a $C^4$-smooth function in an open set of $\mathbb{R}^n$ and let $f$ be the corresponding Lagrangian graph $f : \mathbb{R}^n \to T\mathbb{R}^n : p \mapsto (p, Du(p))$ in $T\mathbb{R}^n$. Then the following two statements hold true:

1. If $u$ is functionally related of second order then the induced metric $f^*G$ is flat.
2. The graph $f$ is Hamiltonian minimal if and only if $\log |\det Hess u|$, is an harmonic function with respect to the metric $G$ induced by $f$.
3. The graph $f$ is minimal if and only if $u$ satisfies the following Monge-Ampère equation:

$$\det Hess(u) = c_0,$$

where $c_0$ is a positive real constant. Furthermore, $f$ is totally geodesic if and only if $u$ is of the following form:

$$u(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} (a_{ij}x_i x_j + b_i x_i + c).,$$

where $a_{ij}, b_i$ and $c$ are all real constants and $\det a_{ij} > 0$. In particular, every totally geodesic Lagrangian graph is flat.

As an example of Theorem 4, we focus our attention to the graph of a vector field in $\mathbb{R}^n$ of spherical symmetry, called source fields (see Definition 3). Such graphs are Lagrangian submanifolds in $T\mathbb{R}^n$.

Theorem 5. Suppose that $f : \mathbb{R}^n - \{0\} \to T(\mathbb{R}^n - \{0\}) : p \mapsto (p, V(p))$ is the graph of a source field $V(p) = H(R) \partial / \partial R$, where $R = |p|$. Then the following two statements hold:

1. $f$ is minimal if and only if the field intensity $H$ is given by

$$H = (c_0 R^n + c_1)^{1/n},$$

where $c_0, c_1$ are constants with $c_0 > 0$. Furthermore, $f$ is totally geodesic if and only if $c_1 = 0$. 

(2) \( f \) is Hamiltonian minimal (but non minimal) if and only if the field intensity \( H \) is given by

\[
H = \left( c_2 + c_1 e^{c_0 R} \sum_{k=0}^{n-1} \frac{k!(-1)^k}{c_0^k} \binom{n-1}{k} R^{n-k-1} \right)^{1/n},
\]

where \( c_0 \neq 0, c_1, c_2 \) are real constants.

It is well known that the space \( L(\mathbb{R}^3) \) of oriented lines in \( \mathbb{R}^3 \) is identified with the tangent bundle \( TS^2 \). Guilfoyle and Klingenberg in [6] and M. Salvai in [11], studied the geometry of \((L(\mathbb{R}^3), G, J)\) derived by the standard Kähler structure \((TS^2, g, J)\).

Using this identification we show the following:

**Theorem 6.** There exists a minimal isometric embedding of \((L(\mathbb{R}^3), G)\) in \((T\mathbb{R}^3, G)\).

The paper is organized as follows. In the next section the new geometry is introduced in the context of almost Kähler and almost para-Kähler structures. Theorem 1 is proven in section 3, while proofs of Theorems 2, 3, 4 and 5 are contained in sections 4.1, 4.2, 4.3, and 4.4, respectively. Theorem 6 is proven in the final section.

### 2. Almost (para-) Kähler structures on TN

Let \( N \) be an \( n \)-dimensional differentiable manifold and \( \pi : TN \to N \) be the canonical projection from the tangent bundle \( TN \) to \( N \). We define the vertical bundle \( VN \) as the subbundle \( \text{Ker}(d\pi) \) of \( TTN \). If \( N \) is equipped with an affine connection \( D \), then we may define the horizontal bundle \( HN \) of \( TTN \) as follows:

If \( \bar{X} \) is a tangent vector of \( TN \) at \((p_0, V_0)\), there exists a curve \( a(t) = (p(t), V(t)) \subset TN \) such that \( a(0) = (p_0, V_0) \) and \( a'(0) = \bar{X} \). Define the connection map (see [5] and [9], for further details) \( K : TTN \to TN \) by \( K\bar{X} = D_{p'(0)} V(0) \) if \( \bar{X} \neq VN \) (i.e., \( p'(0) \neq 0 \)) and if \( \bar{X} \in VN \) (in this case, \( \bar{X} \) is said to be a vertical vector field) then \( K\bar{X} = V'(0) \). The horizontal bundle \( HN \) is simply \( \text{Ker}(K) \) and therefore we obtain the direct sum:

\[
TTN = HN \oplus VN \simeq TN \oplus TN
\]

\[
\bar{X} \simeq (\Pi \bar{X}, K\bar{X}),
\]

**Proposition 1.** [9] Given a vector field \( X \) on \((N, D)\) there exist unique vector fields \( X^h, X^v \) on \( TN \) such that \( (\Pi X^h, KX^h) = (X, 0) \) and \( (\Pi X^v, KX^v) = (0, X) \).

In addition, if \( X, Y \) are vector fields of \( N \), we have at \((p, V) \in TN:\)

\[
[X^v, Y^v] = 0, \quad [X^h, Y^v] = (DX Y)^v \simeq (0, DX Y), \quad [X^h, Y^h] \simeq ([X, Y], -R(X, Y)V),
\]

where \( R \) denotes the curvature of \( D \).
If \( g \) is a Riemannian metric on \( N \), we identify the cotangent bundle \( T^*N \) with \( TN \) by the following bundle isomorphism:

\[
g(p, X) = g_p(X, \cdot) \quad \text{for any } X \in T_p N.
\]

Using the canonical projection \( \pi^* : T^*N \to N \), we define the Liouville form \( \xi \in \Omega^1(T^*N) \) by:

\[
\xi(p, \beta)(\eta) = \beta(d\pi^* \eta) \quad \text{where,} \quad \beta \in T^*_p N \quad \text{and} \quad \eta \in T_{(p, \beta)} T^*N.
\]

The derivative of the Liouville form defines a canonical symplectic structure, \( \Omega^* := -d\xi \), on \( T^*N \) and using the isomorphism \( g \) we define a symplectic structure \( \Omega \) on \( TN \) by \( \Omega = g^* \Omega^* \). The symplectic structure \( \Omega \) is given by

\[
\Omega(\bar{X}, \bar{Y}) = g(K\bar{X}, \Pi\bar{Y}) - g(\Pi\bar{X}, K\bar{Y}).
\]

An almost complex structure (resp. almost paracomplex structure) in \( TN \) is an endomorphism \( J \) of \( TTN \) such that \( J^2 = -\text{Id} \) (resp. \( J^2 = \text{Id} \) and \( J \) is not the identity), for every \( \bar{X} \in TTN \). and is said to be compatible with \( \Omega \) if, \( \Omega(J., J.) = \Omega(., .) \) (resp. \( \Omega(J., J.) = -\Omega(., .) \)).

**Proposition 2.** Let \((N, g)\) be a Riemannian manifold and let \( J_0, J_1, J_2 \) be the following \((1, 1)\)-tensors in \( TN \):

\[
J_0\bar{X} \simeq (K\bar{X}, \Pi\bar{X}), \quad J_1\bar{X} \simeq (\Pi\bar{X}, -K\bar{X}), \quad J_2\bar{X} \simeq (-K\bar{X}, \Pi\bar{X}).
\]

Then \( J_0, J_1 \) are almost paracomplex structures on \( TN \) while, \( J_2 \) is an almost complex structure all compatible with \( \Omega \). The trio \((J_0, J_1, J_2)\) defines an almost paraquaternionic structure on \( TN \).

**Proof.** A straightforward computation shows that

\[
J_0^2 = J_1^2 = \text{Id}, \quad J_2^2 = -\text{Id}, \quad J_0 J_1 = J_2,
\]

and for any \( k \neq l \in \{0, 1, 2\} \), we have

\[
J_k J_l = -J_l J_k.
\]

Furthermore, we have

\[
\Omega(J_0., J_0.) = \Omega(J_1., J_1.) = -\Omega(J_2., J_2.) = -\Omega(., .),
\]

which shows the compatibility conditions. \(\square\)

Consider now the metrics \( G_0, G_1 \) and \( G_2 \), defined by

\[
G_k(., .) := \Omega(., J_k.), \quad k = 0, 1, 2.
\]

The metric \( G_2 \) is the Riemannian Sasaki metric while, \( G_0 \) is the neutral Sasaki metric. The neutral metric \( G_1 \) is given by

\[
G_1(\bar{X}, \bar{Y}) = g(\Pi\bar{X}, K\bar{Y}) + g(K\bar{X}, \Pi\bar{Y}). \quad (3)
\]

The Sasaki metrics \( G_0 \) and \( G_2 \) are very well known and have been studied extensively by several authors - see for example \([4, 8, 13]\). In this article we fill the gap by
studying the geometry of \((TN,G_1)\). From now on and throughout this article, we simply write \(G\) for \(G_1\).

3. CURVATURE OF THE NEUTRAL METRIC \(G\)

Consider the neutral metric \(G\) constructed in Section 2. We now study the main geometric properties of \((TN,G)\).

Denote the Levi-Civita connection of \(G\) by \(\nabla\). For a vector field \(X\) in \(N\) we use Proposition 1, to consider the unique vector fields \(X^h\) and \(X^v\) in \(TN\) such that \(\Pi X^h = X\), \(KX^h = 0\) and \(\Pi X^v = 0\), \(KX^v = X\). We do the same for the vector fields \(Y, Z\) on \(N\). Since all quantities of type \(G(Y^h,Z^v)\) are constant on the fibres, we have that \(X^vG(Y^h,Z^v) = 0\).

Using the Koszul formula:

\[
2G(\nabla_X \bar{Y}, \bar{Z}) = \bar{X}G(\bar{Y}, \bar{Z}) + \bar{Y}G(\bar{X}, \bar{Z}) - \bar{Z}G(\bar{X}, \bar{Y}) + G([\bar{X}, \bar{Y}], \bar{Z}) - G([\bar{X}, \bar{Z}], \bar{Y}) - G(\bar{Y}, [\bar{Z}, \bar{X}]),
\]

one may obtain the following relations

\[
\nabla_{X^v}Y^v = \nabla_{X^v}Y^h = 0, \quad \nabla_{X^h}Y^v \simeq (0, D_X Y).
\]  

(4)

We now have at \((p, V) \in TN\),

\[
2G(\nabla_{X^h} Y^h, Z^h) = X^h G(Y^h, Z^h) + Y^h G(X^h, Z^h) - Z^h G(X^h, Y^h) + G([X^h, Y^h], Z^h) - G([X^h, Z^h], Y^h) - G([Y^h, Z^h], X^h)
\]

\[
= -g(R(X, Y) V, Z) + g(R( X, Z) V, Y) + g(R(Y, Z) V, X)
\]

and using the first Bianchi identity we finally get,

\[
G(\nabla_{X^h} Y^h, Z^h) = g(R( V, X) Y, Z).
\]  

(5)

Similar calculations give,

\[
G(\nabla_{X^h} Y^h, Z^v) = g(D_X Y, Z).
\]  

(6)

Using (5) and (6) we obtain

\[
\nabla_{X^h} Y^h(p, V) \simeq (D_X Y, R( V, X) Y),
\]

(7)

where \(X = \Pi X^h\) and \(Y = \Pi Y^h\). Putting all together we find that

\[
\Pi \nabla_{X^h} \bar{Y}(x, V) = D_{\Pi X} \Pi \bar{Y}, \quad K \nabla_{X^h} \bar{Y}(x, V) = D_{\Pi X} K \bar{Y} + R( V, \Pi \bar{X}) \Pi \bar{Y}.
\]  

(8)

Proposition 3. The Riemann curvature tensor \(\overline{Rm}\) of the metric \(G\) is given by

\[
\overline{Rm}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W})|_{(p, V)} = Rm(K\bar{X}, \Pi \bar{Y}, \Pi \bar{Z}, \Pi \bar{W}) + Rm(\Pi \bar{X}, K\bar{Y}, \Pi \bar{Z}, \Pi \bar{W})
\]

\[
+ Rm(\Pi \bar{X}, \Pi \bar{Y}, K\bar{Z}, \Pi \bar{W}) + Rm(\Pi \bar{X}, \Pi \bar{Y}, \Pi \bar{Z}, K\bar{W})
\]

\[
+ g((D_V R)(\Pi \bar{X}, \Pi \bar{Y}) \Pi \bar{Z}, \Pi \bar{W}),
\]
where \( Rm \) is the Riemann curvature tensor of \( g \) and

\[
(D_u R)(v, w)(z) = D_u R(v, w)z - R(D_u v, w)z - R(v, D_u w)z - R(v, w)D_u z
\]

**Proof.** Using the second Bianchi identity, we have at the point \((p, V) \in TN\):

\[
\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{Z} - \nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{Z} - \nabla_{[\bar{X}, \bar{Y}]}\bar{Z} \\
\simeq (D_{\Pi\bar{X}}D_{\Pi\bar{Y}}\Pi\bar{Z} - D_{\Pi\bar{Y}}D_{\Pi\bar{X}}\Pi\bar{Z} - D_{[\Pi\bar{X}, \Pi\bar{Y}]}\Pi\bar{Z}, \\
D_{\Pi\bar{X}}D_{\Pi\bar{Y}}K\bar{Z} - D_{\Pi\bar{Y}}D_{\Pi\bar{X}}K\bar{Z} - D_{[\Pi\bar{X}, \Pi\bar{Y}]}K\bar{Z} + D_{\Pi\bar{X}}R(V, \Pi\bar{Y})\Pi\bar{Z} \\
+ R(V, \Pi\bar{X})D_{\Pi\bar{Y}}\Pi\bar{Z} - D_{\Pi\bar{Y}}R(V, \Pi\bar{X})\Pi\bar{Z} - R(V, \Pi\bar{Y})D_{\Pi\bar{X}}\Pi\bar{Z} \\
- R(V, D_{\Pi\bar{X}}\Pi\bar{Y})\Pi\bar{Z} + R(V, D_{\Pi\bar{Y}}\Pi\bar{X})\Pi\bar{Z}) \\
= (R(\Pi\bar{X}, \Pi\bar{Y})\Pi\bar{Z}, R(\Pi\bar{X}, \Pi\bar{Y})K\bar{Z} + (D_V R)(\Pi\bar{X}, \Pi\bar{Y})(\Pi\bar{Z}) \\
+ R(D_{\Pi\bar{X}}V, \Pi\bar{Y})\Pi\bar{Z} - R(D_{\Pi\bar{Y}}V, \Pi\bar{X})\Pi\bar{Z}).
\]

The proposition follows by using the fact that

\[
Rm(X, Y, Z, W) = g(R(X, Y)Z, W).
\]

\[\square\]

We are now in position to calculate the Ricci tensor:

**Proposition 4.** The Ricci tensor \( \overline{\text{Ric}} \) of the metric \( G \) is given by

\[
\overline{\text{Ric}}(\bar{X}, \bar{Y}) = 2\text{Ric}(\Pi\bar{X}, \Pi\bar{Y}),
\]

where \( \text{Ric} \) denotes the Ricci tensor of \( g \).

**Proof.** Following a similar method as in the proof of the main theorem of [3], we consider the orthonormal frame \((e_1, \ldots, e_n)\) of \((N, g)\). Define the following frame \((\bar{e}_1, \ldots, \bar{e}_n, \bar{e}_{n+1}, \ldots, \bar{e}_{2n})\) of \( TN \) to be the unique vector fields such that

\[
\Pi\bar{e}_k = e_k, \quad K\bar{e}_k = 0
\]

\[
\Pi\bar{e}_{n+k} = 0, \quad K\bar{e}_{n+k} = e_k
\]

\[\square\]
For $i, j = 1, \ldots, n$ we have $G_{ij} = 0$, and $G_{i,n+j} = \delta_{ij}$. Observe that $\overline{\text{Ric}}(X^v, Y^v) = 0$. Using Proposition 3, we have
\[
\overline{\text{Ric}}(X^h, Y^h) = \sum_{i,j=1}^{n} G^{i,n+j}(\overline{\text{Rm}}(X^h, \bar{e}_i, Y^h, \bar{e}_{n+j}) + \overline{\text{Rm}}(X^h, \bar{e}_{n+j}, Y^h, \bar{e}_i))
\]
\[
= \sum_{i,j=1}^{n} \delta_{ij}(\overline{\text{Rm}}(X, e_i, Y, e_j) + \overline{\text{Rm}}(X, e_j, Y, e_i))
\]
\[
= 2 \sum_{i=1}^{n} \overline{\text{Rm}}(X, e_i, Y, e_i)
\]
\[
= 2 \text{Ric}(X, Y),
\]
and this completes the Proposition.

**Proof of Theorem 1.** 1. For the first part consider, as before, the frame $(\bar{e}_1, \ldots, \bar{e}_n, \bar{e}_{n+1}, \ldots, \bar{e}_{2n})$ of $TN$. The scalar curvature $\bar{S}$ of the metric $G$ is
\[
\bar{S} = \sum_{i,j=1}^{2n} G^{ij}\overline{\text{Ric}}(\bar{e}_i, \bar{e}_j) = \sum_{i,j=1}^{2n} G^{ij}\overline{\text{Ric}}(\Pi \bar{e}_i, \Pi \bar{e}_j) = 0.
\]
2. The second part follows directly from the Proposition 4.
3. We now prove the third part. Let $\overline{W}$ be the Weyl tensor of $G$. Using Proposition 5 of [3] we have
\[
\overline{W}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = \overline{\text{Rm}}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) - \frac{\text{Ric}(\Pi \bar{Y}, \Pi \bar{W})G(\bar{X}, \bar{Z}) + \text{Ric}(\Pi \bar{X}, \Pi \bar{Z})G(\bar{Y}, \bar{W})}{n-1} + \frac{\text{Ric}(\Pi \bar{Y}, \Pi \bar{Z})G(\bar{X}, \bar{Z}) + \text{Ric}(\Pi \bar{X}, \Pi \bar{W})G(\bar{Y}, \bar{W})}{n-1}.
\]
Assume that $G$ is locally conformally flat and $n \geq 3$. We then have
\[
\overline{\text{Rm}}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = \frac{1}{n-1}(\text{Ric}(\Pi \bar{Y}, \Pi \bar{W})G(\bar{X}, \bar{Z}) + \text{Ric}(\Pi \bar{X}, \Pi \bar{Z})G(\bar{Y}, \bar{W}))
\]
\[
- \frac{1}{n-1}(\text{Ric}(\Pi \bar{Y}, \Pi \bar{Z})G(\bar{X}, \bar{W}) + \text{Ric}(\Pi \bar{X}, \Pi \bar{W})G(\bar{Y}, \bar{Z})).
\]
Let $X, Y, Z, W$ be vector fields on $N$ with corresponding unique vector fields $X^h, X^v, Y^h, Y^v, Z^h, Z^v, W^h, W^v$ on $TN$. Using Proposition 3, we have
\[
\overline{\text{Rm}}(X^h, Y^h, Z^h, W^v) = \text{Rm}(X, Y, Z, W),
\]
and thus, (10) becomes,

$$Rm(X, Y, Z, W) = -\frac{1}{n-1}(\text{Ric}(X, Z)g(Y, W) - \text{Ric}(Y, Z)g(X, W))$$

We now have,

$$Rm(X, Y, X, Y) = -\frac{1}{n-1}(\text{Ric}(X, X)|X|^2 - \text{Ric}(X, Y)g(X, Y)).$$

On the other hand,

$$Rm(Y, X, Y, X) = -\frac{1}{n-1}(\text{Ric}(Y, Y)|X|^2 - \text{Ric}(X, Y)g(X, Y)).$$

which implies

$$\frac{\text{Ric}(X, X)}{|X|^2} = \frac{\text{Ric}(Y, Y)}{|Y|^2},$$

for any vector fields $X, Y$ on $N$. Thus, there exists a smooth function $\lambda$ on $N$ such that

$$\text{Ric}(X, X) = \lambda|X|^2,$$

which shows that $g$ is Einstein. Since $n \geq 3$, the function $\lambda$ must be constant. Let $P$ be the plane spanned by $\{e_1, e_2\}$. Then the sectional curvature

$$K(P) = Rm(e_1, e_2, e_1) = \frac{\lambda}{n-1},$$

which is constant.

Assume the converse, that is, $n \geq 3$ and $g$ is of constant sectional curvature $K$. Thus,

$$K = \frac{R}{n(n-1)},$$

(11)

where $R$ denotes the scalar curvature. Also $g$ is locally symmetric and therefore from Proposition 3 we have

$$\overline{Rm}(X, Y, Z, W) = \text{Rm}(KX, K\bar{\bar{Y}}, K\bar{\bar{Z}}, K\bar{\bar{W}}) + \text{Rm}(\Pi\bar{\bar{X}}, K\bar{\bar{Y}}, K\bar{\bar{Z}}, K\bar{\bar{W}}) + \text{Rm}(\Pi\bar{\bar{X}}, \Pi\bar{\bar{Y}}, K\bar{\bar{Z}}, K\bar{\bar{W}}) + \text{Rm}(\Pi\bar{\bar{X}}, \Pi\bar{\bar{Y}}, \Pi\bar{\bar{Z}}, K\bar{\bar{W}})$$

Hence, $\overline{Rm}(X^h, Y^h, Z^h, W^h) = 0$ and using (9) we have,

$$\overline{W}(X^h, Y^h, Z^h, W^h) = -\frac{1}{n-1}(\text{Ric}(Y, W)G(X^h, Z^h) + \text{Ric}(X, Z)G(Y^h, W^h))$$

$$+ \frac{1}{n-1}(\text{Ric}(Y, Z)G(X^h, W^h) + \text{Ric}(X, W)G(Y^h, Z^h)) = 0.$$
\[ Rm(X, Y, Z, W) = \frac{1}{n-1} (Ric(X, Z)g(Y, W) - Ric(Y, Z)G(X, W)) \]
\[ = Rm(X, Y, Z, W) - \frac{R}{n(n-1)} (g(X, Z)g(Y, W) - g(Y, Z)G(X, W)). \]

Since \( g \) is of constant sectional curvature \( K \), we have
\[ Rm(X, Y, Z, W) = K(g(X, Z)g(Y, W) - g(Y, Z)G(X, W)), \]
and therefore, using (11) we have
\[ \overline{W}(X^h, Y^h, Z^h, W^v) = \left( K - \frac{R}{n(n-1)} \right) (g(X, Z)g(Y, W) - g(Y, Z)G(X, W)) = 0. \]

Similarly, we prove that all coefficients of the Weyl tensor vanish and that \( \overline{W} = 0 \). Thus, for \( n \geq 3 \), the metric \( G \) is locally conformally flat if and only if \( g \) is of constant sectional curvature.

For \( n = 2 \), the Riemann curvature tensor is given by
\[ Rm(X, Y, Z, W) = K(g(X, Z)g(Y, W) - g(Y, Z)G(X, W)), \]
where \( K \) is the Gauss curvature of \( g \). Hence, following a similar argument as before, one can prove that for every 2-manifold \( (N, g) \) the neutral metric \( G \) of \( TN \) is locally conformally flat.

4. We now proceed with the last part of the proof. Assume first that \( g \) is locally symmetric. Then for any vector fields \( \xi, X, Y, Z \) on \( N \) we have, by definition,
\[ D_\xi(R(X, Y)Z) = R(D_\xi X, Y)Z + R(X, D_\xi Y)Z + R(X, Y)D_\xi Z. \]

Using (12), a brief computation shows
\[ R(\xi, V)(R(X, Y)Z) = R(R(\xi, V)X, Y)Z + R(X, R(\xi, V)Y)Z + R(X, Y)(R(\xi, V)Z). \]

We want to prove that \( G \) is locally symmetric, that is, \( \nabla \bar{R} = 0 \). Proposition 3 tells us that
\[ \bar{R}(X, Y)\bar{Z}|_{(\xi, V)} \simeq (R(\Pi X, \Pi Y)\Pi \bar{Z}, R(K X, \Pi Y)\Pi \bar{Z} + R(\Pi X, K \bar{Z})\Pi \bar{Z} + R(\Pi X, \Pi Y)K \bar{Z} \]
\[ + (D_\xi R)(\Pi X, \Pi Y)\Pi \bar{Z}), \]
and using the fact that \( g \) is locally symmetric we have
\[ \bar{R}(X, Y)\bar{Z} \simeq (R(\Pi X, \Pi Y)\Pi \bar{Z}, R(K X, \Pi Y)\Pi \bar{Z} + R(\Pi X, K \bar{Z})\Pi \bar{Z} + R(\Pi X, \Pi Y)K \bar{Z}. \]

We thus obtain the following
\[ \bar{R}(X^v, Y^v)Z^v = R(X^v, Y^v)Z^h = R(X^h, Y^v)Z^v = 0, \]
\[ \bar{R}(X^v, Y^h)Z^h \simeq (0, R(X, Y)Z), \]
and
\[ \bar{R}(X^h, Y^h)Z^h \simeq (R(X, Y)Z, 0). \]
Applying the relations (4) and (7) we get
\[
\nabla_{\xi}^v (\bar{R}(X^v, Y^h) Z^h) = \nabla_{\xi}^h (\bar{R}(X^v, Y^h) Z^h) = 0,
\]
\[
\nabla_{\xi}^v (\bar{R}(X^v, Y^h) Z^h) \simeq (0, D_\xi (R(X, Y) Z)),
\]
\[
\nabla_{\xi}^h (\bar{R}(X^h, Y^h) Z^h) \simeq (D_\xi (R(X, Y) Z), R(V, \xi) (R(X, Y) Z)),
\]
\[
\bar{R}(\nabla_{\xi}^h X^h, Y^h) Z^h \simeq (R(D_\xi X, Y) Z, R(R(V, \xi) X, Y) Z),
\]
\[
\bar{R}(X^h, \nabla_{\xi}^h Y^h) Z^h \simeq (R(X, D_\xi Y) Z, R(X, R(V, \xi) Y) Z),
\]
\[
\bar{R}(X^h, Y^h) \nabla_{\xi}^h Z^h \simeq (R(X, Y) D_\xi Z, R(X, Y) R(V, \xi) Z),
\]

We now use all relations above and together with (13) finally obtain,
\[
(\nabla_{\xi}^h \bar{R})(X^h, Y^h, Z^h) = \nabla_{\xi}^h (\bar{R}(X^h, Y^h) Z^h) - \bar{R}(\nabla_{\xi}^h X^h, Y^h) Z^h - \bar{R}(X^h, \nabla_{\xi}^h Y^h) Z^h
\]
\[
- \bar{R}(X^h, Y^h) \nabla_{\xi}^h Z^h = 0.
\]

Similar arguments shows that this relation holds:
\[
(\nabla_{\xi}^v \bar{R})(X^h, Y^h, Z^h) = 0,
\]

showing that \( \nabla \bar{R} = 0 \), which means \( G \) is locally symmetric.

Conversely, assume that \( G \) is locally symmetric. Then the following holds true:
\[
\nabla_{\xi}^h (\bar{R}(X^h, Y^h) Z^v) = \bar{R}(\nabla_{\xi}^h X^h, Y^h) Z^v + \bar{R}(X^h, \nabla_{\xi}^h Y^h) Z^v + \bar{R}(X^h, Y^h) \nabla_{\xi}^h Z^v,
\]

implying,
\[
D_\xi (R(X, Y) Z) = R(D_\xi X, Y) Z + R(X, D_\xi Y) Z + R(X, Y) D_\xi Z,
\]

which means that \( g \) is locally symmetric, completing the proof of the Theorem. \( \Box \)

Suppose that the manifold \( N \) is equipped with a Kähler structure \((j, g)\). An almost complex structure \( \mathbb{J} \) on \( TN \) can be defined by
\[
\mathbb{J} \bar{X} = (j\Pi \bar{X}, j\bar{K} \bar{X}).
\]

It has been proved that \( \mathbb{J} \) is integrable and one can check easily that is compatible with \( G \), that is,
\[
G(\mathbb{J} \mathbb{J}, \mathbb{J}) = G(\mathbb{J}, \mathbb{J}).
\]

It can be easily proved that \( \mathbb{J} \) is also parallel with respect to \( \nabla \). Namely,
\[
\nabla_X \mathbb{J} \bar{Y}(p, V) = (D_{\Pi \bar{X}} j \Pi \bar{Y}, D_{\Pi \bar{X}} j \bar{K} \bar{Y} + R(V, \Pi \bar{X}) j \Pi \bar{Y})
\]
\[
= (j D_{\Pi \bar{X}} \Pi \bar{Y}, j(D_{\Pi \bar{X}} \bar{K} \bar{Y} + R(V, \Pi \bar{X}) \Pi \bar{Y}))
\]
\[
= \mathbb{J} (D_{\Pi \bar{X}} j \Pi \bar{Y}, D_{\Pi \bar{X}} j \bar{K} \bar{Y} + R(V, \Pi \bar{X}) j \Pi \bar{Y})
\]
\[
= \mathbb{J} \nabla_X \bar{Y}(p, V).
\]

The complex structure \( \mathbb{J} \) is compatible with \( \Omega \) and together with the metric \( \mathbb{G} \) given by
\[
\mathbb{G} = \Omega(\mathbb{J}, \mathbb{J}).
\]
defines another Kähler structure \((G, \Omega, J)\) on \(TN\) which it has been introduced by H. Anciaux and R. Pascal in [3]. In particular,
\[
G(\bar{X}, \bar{Y}) = g(K\bar{X}, J\Pi\bar{Y}) - g(\Pi\bar{X}, JK\bar{Y}).
\]
(14)
The following Proposition shows that the neutral metric \(G\) is an extension of \(G\) for the non-Kähler structures.

**Proposition 5.** The metrics \(G\) and \(G\), defined respectively in (3) and (14), are isometric.

**Proof.** Let \(N\) be a smooth manifold equipped with a Kähler structure \((j, g)\). Let \(G\) and \(G\) be the Kähler metrics defined as above and define the following diffeomorphism:

\[
f : TN \to TN : (p, V) \mapsto (p, -jV).
\]
If \(\bar{X} \in T_{(p, V)}TN\) then \(\Pi df(\bar{X}) = \Pi\bar{X}\) and using the fact that \(j\) is parallel, we have that \(Kdf(\bar{X}) = -jK\bar{X}\). Thus,
\[
f^*G(\bar{X}, \bar{Y}) = g(\Pi\bar{X}, -jK\bar{Y}) + g(\Pi\bar{Y}, -jK\bar{X})
\]
\[
= -g(\Pi\bar{X}, jK\bar{Y}) + g(j\Pi\bar{Y}, K\bar{X})
\]
\[
= G(\bar{X}, \bar{Y}),
\]
which shows that \(f\) is an isometry. \(\square\)

4. **Submanifold theory**

We now investigate the submanifold theory of \((TN, G)\). In particular, we will study geodesics and the Lagrangian graphs.

4.1. **Geodesics.** We are now in position to characterize the geodesics of the neutral metric \(G\). In fact, we prove our second result:

**Proof of Theorem 2.** Let \(\bar{X}(t) := \gamma'(t)\). Then, using (8), we have \(\Pi\nabla_\gamma\gamma' = D_{\Pi\bar{X}}\Pi\bar{X}\), and \(K\nabla_\gamma\gamma' = D_{\Pi\bar{X}}K\bar{X} + R(V, \Pi\bar{X})\Pi\bar{X}\). If \(\gamma(t) = (x(t), V(t))\) is a geodesic then, \(D_{\Pi\bar{X}}\Pi\bar{X} = 0\), and thus \(D_{x'}x' = 0\), which implies that \(x(t)\) is a geodesic. On the other hand, \(K\bar{X} = D_{x'}V\) and therefore, \(D_{\Pi\bar{X}}K\bar{X} = D_{x'}^2V\). Using the fact that \(D_{\Pi\bar{X}}K\bar{X} + R(V, \Pi\bar{X})\Pi\bar{X} = 0\), we have \(D_{x'}^2V + R(V, x')x' = 0\), which shows that \(V\) is a Jacobi field along the geodesic \(x(t)\).

Conversely, when \(V(t)\) is a Jacobi field along the geodesic \(x(t)\) then, obviously, \(\gamma(t) = (x(t), V(t))\) is a geodesic. \(\square\)
4.2. Graph submanifolds. Let \((N, g)\) be a \(n\)-dimensional Riemannian manifold and \(U\) be an open subset of \(N\). Considering a vector field \(V\) in \(N\), we obtain a \(n\)-dimensional submanifold \(V \subset TN\) which is a section of the canonical bundle \(\pi : TN \to N\). Such submanifolds are immersed as graphs, that is, \(V = f(N)\), where \(f(p) = (p, V(p))\).

The following proposition gives a relation between the null points of the graph \(V\) with the critical points of the length function of an integral curve of \(V\).

**Proposition 6.** Let \(V\) be a vector field of \(N\) and \(V\) be the corresponding graph. Then the following two statements hold true:

1. If an integral curve of \(V\) is a geodesic in \((N, g)\) then \(V\) admits a null curve.
2. If \(V\) admits a closed integral curve then \(V\) must contain a null point.

**Proof.** Let \(p = p(t)\) be an integral curve of \(V\), that is, \(V(t) := V(p(t)) = p'(t)\). The corresponding curve in \(TN\) is given by \(f(t) = (p(t), V(t))\).

Then \(f'(t) = (p'(t), D_{p'(t)}V(t))\) and thus,

\[
G(f'(t), f'(t)) = 2g(p'(t), D_{p'(t)}V(t)) = 2g(p'(t), D_{p'(t)}p'(t)) = D_{p'(t)}\left(g(p'(t), p'(t))\right) = \frac{d}{dt}|p'|^2 = \frac{d}{dt}|V|^2.
\]

(1) When the curve \(p = p(t)\) is a geodesic then \(D_{p'(t)}p'(t)\) must vanish and thus \(G(f'(t), f'(t)) = 0\) for any \(t\). Therefore \(f(t)\) is a null curve.
(2) Assuming that the integral curve \(p(t)\) is closed, there exists \(t_0 \in S^1\) such that,

\[
\frac{d}{dt}|p'(t_0)|^2 = \frac{d}{dt}|V(t_0)|^2 = 0,
\]

which means that \(f\) is null at the point \(t_0\). \(\square\)

We now study Lagrangian graphs in \(TN\). We need first to recall the definition of a Lagrangian submanifold:

**Definition 1.** Let \(N\) be a \(2n\)-dimensional manifold equipped with a symplectic structure \(\omega\). An immersion \(f : \Sigma^n \to N\) is said to be Lagrangian if \(f^*\omega = 0\).

For Lagrangian graphs we have the following:

**Proposition 7.** Let \(V\) be a vector field in the open subset \(U \subset N\). The graph \(V\) is Lagrangian if and only if \(V\) is locally the gradient of a real smooth function \(u\) on \(U \subset N\), i.e, \(V = Du\).

We now prove our third result.
Proof of Theorem 3. Let $g$ be the Riemannian metric in $N$ and $V$ be the submanifold of $TN$ obtained by the image of the graph:

$$f : U \subset N \to TN : p \mapsto (p, V(p)),$$

where $V$ is a vector field defined on the open subset of $N$. The fact that $f$ is Lagrangian implies that the almost paracomplex structure $J$ is a bundle isomorphism between the tangent bundle $TV$ and the normal bundle $NV$. We then consider the Maslov form $\eta$ on $V$ defined by,

$$\eta = G(JH, .),$$

where $H$ is the mean curvature vector of $f$. The Lagrangian condition implies the following relation:

$$d\eta = \frac{1}{2} Ric(J., .)|_V,$$

where $\overline{Ric}$ denotes the Ricci tensor of $G$. Assuming that $H$ is parallel, the Maslov form is closed and therefore,

$$\overline{Ric}(J\bar{X}, \bar{Y}) = 0,$$

for every tangential vector fields $\bar{X}, \bar{Y}$. If $Ric$ denotes the Ricci tensor of $g$, the Proposition 4, gives

$$\overline{Ric}(\bar{X}, \bar{Y}) = 2Ric(\Pi\bar{X}, \Pi\bar{Y}),$$

If $X, Y$ are vector fields in $U$, the fact that $f$ is a graph, implies that $\Pi df(X) = X$ and $\Pi df(Y) = Y$. On the other hand, using the definition $J$, we have

$$J(df(X)) = (\Pi df(X), -Kdf(X)).$$

Thus,

$$0 = \overline{Ric}(Jdf(X), df(Y))$$

$$= \overline{Ric}((\Pi df(X), -Kdf(X)), (\Pi df(Y), Kdf(Y))$$

$$= 2Ric(\Pi df(X), \Pi df(Y))$$

$$= 2Ric(X, Y),$$

and the Theorem follows. \qed

Corollary 1. Let $(N, g)$ be a non-flat Riemannian 2-manifold and $\Sigma$ be a Lagrangian surface of $(TN, G, \Omega)$. Then $\Sigma$ has parallel mean curvature if and only if it is a set of lines that are orthogonal to a geodesic $\gamma$ of $N$.

Proof. Suppose that $g$ is non-flat. Using Theorem 3, the Lagrangian surface $\Sigma$ can’t be the graph of a smooth function on $N$. Following a similar argument as the proof of Proposition 2.1 of [11], $\Sigma$ can be parametrized by:

$$f : U \subset \mathbb{R}^2 \to TN : (s, t) \mapsto (\gamma(s), a(s)\gamma'(s) + tj\gamma'(s)),$$
where \( j \) denotes the canonical complex structure on \( N \) defined as a rotation on \( TN \) about \( \pi/2 \) and \( \gamma = \gamma(s) \) is a curve in \( N \). The mean curvature \( \mathbb{H} \) of \( f \) is
\[
\mathbb{H} = (0, k(s)j\gamma'(s)),
\]
where \( k \) denotes the curvature of \( \gamma \). Obviously, we have that \( \nabla_{\partial_t}\mathbb{H} = 0 \) and
\[
\nabla_{\partial_s}\mathbb{H} = (0, -k^2\gamma' + k_sj\gamma'),
\]
which shows that \( \Sigma \) has parallel mean curvature is equivalent to the fact that \( \gamma \) is a geodesic. \( \square \)

4.3. **Lagrangian graphs in the Euclidean space.** In this subsection we study Lagrangian graphs in \( T\mathbb{R}^n \).

**Definition 2.** A smooth function \( u \) on \( \mathbb{R}^n \) is said to be functionally related of second order if for every two pairs \( (i_1, i_2) \) and \( (j_1, j_2) \) there exists a function \( F \) on \( \mathbb{R}^2 \) such that
\[
F(u_{x_{i_1}x_{i_2}}, u_{x_{j_1}x_{j_2}}) = 0.
\]

**Example:** The following functions in \( \mathbb{R}^n \) are functionally related of second order:
\[
u(x_1, \ldots, x_n) := f(a_1x_1 + \ldots + a_nx_n)
\]
and
\[
v(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} (a_{ij}x_i x_j + b_i x_i + c).
\]

Consider the \( n \)-dimensional Euclidean space \( (\mathbb{R}^n, ds^2) \), where \( ds^2 \) is the usual inner product and let \( G \) be the neutral metric in \( T\mathbb{R}^n \) derived by \( ds^2 \). For Lagrangian graphs in \( T\mathbb{R}^n \) we prove the Theorem 4.

**Proof of Theorem 4.** Let \( (x_1, \ldots, x_n) \) be the standard Cartesian coordinates of \( \mathbb{R}^n \) and let
\[
f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, u_{x_1}, \ldots, u_{x_n}),
\]
be the local expression of the Lagrangian submanifold.

1. We then have,
\[
f_i := f_{x_i} = (\partial/\partial x_i, D_{\partial/\partial x_i} Du) = (\partial/\partial x_i, u_{x_1x_1}, \ldots, u_{x_nx_n})
\]
The first fundamental form has coefficients:
\[
g_{ij} = G(f_i, f_j) = 2 \langle \partial/\partial x_i, D_{\partial/\partial x_j} Du \rangle = 2 u_{x_ix_j},
\]
which yields,
\[
g = f^*G = 2\text{Hess}(u).
\]
For tangential vector fields \( X, Y, Z, W \) the Gauss equation for \( f \) is
\[
G(R(X, Y)Z, W) = G(\tilde{B}(X, W), \tilde{B}(Y, Z)) - G(\tilde{B}(X, Z), \tilde{B}(Y, W)),
\]
(15)
where $\bar{B}$ denotes the second fundamental form of $f$. The Levi-Civita connection $\nabla$ of $g$ is:

$$\nabla f_i f_j = \sum_{l,k} g^{lk} u_{ijl} f_k.$$ 

Thus,

$$\bar{B}(f_i, f_j) = f_{ij} - \sum_{l,k} g^{lk} u_{ijl} f_k. \quad (16)$$

The equation $(15)$ gives

$$G(R(f_i, f_j) f_k) = \sum_{s,t} g^{st} (\partial_x u_{xs} \partial_x u_{xt} u_{xk})$$

Assuming now that $u$ is functionally related of second order implies

$$\partial_x u_{xs} \partial_x u_{xt} u_{xk} - \partial_x u_{xs} \partial_x u_{xk} = 0,$$

for any indices $i, j, k, l, s, t$, and this completes the first statement of the Theorem.

2. Let $GL(n, \mathbb{R})$ be the space of all invertible $n \times n$ real matrices and let $g : \mathbb{R}^n \to GL(n, \mathbb{R}) : (x_1, \ldots, x_n) \mapsto (g_{ij}(x_1, \ldots, x_n))$ be a smooth mapping. Using Jacobi’s formula,

$$\frac{\partial}{\partial x_i} (\det g) = \text{Tr} \left( \text{Adj}(g) \frac{\partial}{\partial x_i} g \right),$$

where Adj denotes the adjoint matrix of $h$, then one can prove the following:

**Lemma 1.** Let $g : \mathbb{R}^n \to GL(n, \mathbb{R})$ be a smooth mapping. If $g_{ij}$ and $g^{-1}$ are the entries of $g$ and $g^{-1}$, respectively, then

$$\sum_{i,j} g^{ij} \partial_{x_i} g_{ij} = \partial_{x_i} \log |\det g|.$$  

Using the expression $(16)$, the second fundamental form becomes

$$\bar{B}(f_i, f_j) = (0, (u_{1ij}, \ldots, u_{nij})) - \sum_{l,k} g^{lk} u_{ijl} (\partial/\partial x_k, (u_{1k}, \ldots, u_{nk}))$$

$$= \left( - \sum_{l,k} g^{lk} u_{ijl} \partial/\partial x_k, \frac{1}{2} \sum_{l} u_{ijl} \partial/\partial x_l \right) \quad (17)$$

The mean curvature vector $H$ is

$$H = \left( - \sum_{i,j,l,k} g^{ij} g^{lk} u_{ijl} \partial/\partial x_k, \frac{1}{2} \sum_{i,j,l} g^{ij} u_{ijl} \partial/\partial x_l \right)$$

Using the expression $(16)$, the second fundamental form becomes

$$\bar{B}(f_i, f_j) = (0, (u_{1ij}, \ldots, u_{nij})) - \sum_{l,k} g^{lk} u_{ijl} (\partial/\partial x_k, (u_{1k}, \ldots, u_{nk}))$$

$$= \left( - \sum_{l,k} g^{lk} u_{ijl} \partial/\partial x_k, \frac{1}{2} \sum_{l} u_{ijl} \partial/\partial x_l \right) \quad (17)$$

The mean curvature vector $H$ is

$$H = \left( - \sum_{i,j,l,k} g^{ij} g^{lk} u_{ijl} \partial/\partial x_k, \frac{1}{2} \sum_{i,j,l} g^{ij} u_{ijl} \partial/\partial x_l \right)$$

A straightforward computation gives

$$H = - \sum_{i,j,l,k} g^{ij} g^{lk} u_{ijl} J f_k$$
We then have from Lemma 1,

$$J = -\sum_{l,k} g^{lk} \partial_{x_l} \log |\det \text{Hess}(u)| f_k.$$  \hfill (18)

Using (18), we get

$$G(J, f_s) = \partial_{x_s} \log |\det \text{Hess}(u)|^{-2},$$

and thus,

$$J = \nabla \log |\det \text{Hess}(u)|^{-2}.$$  

Denote the divergence with respect to the induced metric \(g\) by \(\text{div}\). Then,

$$\text{div} J = \Delta \log |\det \text{Hess}(u)|^{-2},$$

where \(\Delta\) denotes the Laplacian with respect to \(g\). Therefore, \(f\) is Hamiltonian minimal if and only if \(\Delta \log |\det \text{Hess}(u)| = 0\).

3. The minimal condition follows easily from (18).

Assuming that \(f\) is totally geodesic, the relation (17) gives

$$u_{i,jl} = 0,$$

for every indices \(i, j, k\) and thus \(u\) must satisfy (2).

Conversely, assume that \(u\) is defined by (2). Then all third derivatives vanish and obviously the second fundamental form \(B\) is identically zero. The fact that totally geodesic Lagrangian graphs are flat comes from part (1) of the Theorem. \(\square\)

As an application of Theorem 4, we give the following corollary:

**Corollary 2.** Let \(\Omega\) be an embedded ball in \(\mathbb{R}^n\) such that \(\partial \Omega\) is an embedded sphere. Suppose that \(f\) is a Hamiltonian minimal graph in \(\Omega\) such that the induced metric is Riemannian. If \(f\) is minimal at \(\partial \Omega\) then it is minimal in \(\Omega\).

**Proof.** Let \(g\) be the induced metric of \(G\) in \(\Omega\) through \(f\) and let \(\eta\) be the outward unit vector field normal to \(\partial \Omega\) with respect to \(g\). Since \(f\) is a Lagrangian graph, there exists a smooth function \(u : \Omega \to \mathbb{R}\) such that \(f(p) = (p, Du(p)), p \in \Omega\). Therefore, \(g = 2\text{Hess}(u)\) and let \(h = \det \text{Hess}(u)\). The immersion \(f\) is minimal at \(\partial \Omega\) and thus the Theorem 4 tells us that \(h\) is constant in \(\partial \Omega\). Using Stokes Theorem and the fact that \(\partial \Omega\) is a sphere we have,

$$\int_{\Omega} \text{div}(h\nabla h) = \int_{\partial \Omega} g(h\nabla h, \eta) = \int_{\partial \Omega} g(\nabla h, \eta) = h \int_{\partial \Omega} \frac{\partial h}{\partial \eta} = 0,$$

where \(\text{div}\) and \(\nabla\) denote the divergence and the gradient of \(g\). This, implies

$$\int_{\Omega} g(\nabla h, \nabla h) = -\int_{\Omega} \Delta h,$$  \hfill (19)

where \(\Delta\) is the Laplacian of \(g\).
On the other hand, \( f \) is Hamiltonian minimal. Then, Theorem 4 says that \( \log h \) must be harmonic. In other words,

\[ \Delta \log h = 0. \]

That means,

\[ g(\nabla h, \nabla h) = h \Delta h, \]

and by integrating over \( \Omega \), we then have

\[ \int_{\Omega} g(\nabla h, \nabla h) = \int_{\Omega} \Delta h. \quad (20) \]

Using (19) and (20) we have

\[ \int_{\Omega} g(\nabla h, \nabla h) = 0, \]

and since \( g \) is Riemannian it follows that

\[ \nabla h(p) = 0, \]

for every \( p \) in \( \Omega \) and the Corollary is completed. \( \square \)

4.4. **Source fields in \( \mathbb{R}^n \).** As an application of Theorem 4, we explore the geometry of source vector fields in \( \mathbb{R}^n \).

**Definition 3.** A vector field \( V \) in \( \mathbb{R}^n - \{0\} \) is said to be an \( SO(n) \) invariant source field (or simply a source field) if it is given by

\[ V = H(R) \frac{\partial}{\partial R} \]

where \( R \) is the distance to the origin (the source). The function \( H \) is called the field intensity.

We now study submanifolds in \( T\mathbb{R}^n \) that are graphs of source fields in \( \mathbb{R}^n \). If \( V \) is the source field, the corresponding graph in \( \mathbb{R}^n \) will be denoted by \( V \).

**Proposition 8.** Let \( f : \mathbb{R}^n - \{0\} \rightarrow T(\mathbb{R}^n - \{0\}) \) : \( p \mapsto (p, V(p)) \) be the graph of a source field \( V \) in \( \mathbb{R}^n \) with field intensity \( H \). Then \( f \) is Lagrangian and in particular \( V = \nabla u \), where \( u = u(R) \) is the anti-derivative of \( H \).

**Proof.** Observe that if \( u'(R) = H(R) \), then

\[ \frac{\partial u}{\partial x_i} = \frac{du}{dR} \frac{\partial R}{\partial x_i} = \frac{H(R)}{R} x_i. \]
Thus,
\[
V = H(R) \frac{\partial}{\partial R} = \frac{H(R)}{R} \sum_i x_i \frac{\partial}{\partial x_i} = \sum_i u_{x_i} \frac{\partial}{\partial x_i},
\]
and the proposition follows. \(\square\)

We now prove our next result:

**Proof of Theorem 5.** Let \(H\) be the field intensity of a source field \(V\). Then
\[
V = H(R) \frac{\partial}{\partial R}
\]
where \(R = \sqrt{x_1^2 + \ldots + x_n^2}\). If \(h = H/R\), then \(V\) can be also written as
\[
V(x_1, \ldots, x_n) = h(R) (x_1, \ldots, x_n).
\]
It is not hard to see that there exists a smooth function \(u\) such that \(V = \nabla u\), which means that \(u_{x_i} = h(R) x_i\). The induced metric \(g = f^*G\) has coefficients
\[
g_{ij} = 2u_{x_i x_j} = 2h \left( \frac{x_i x_j}{R} \frac{d}{dR} (\log h) + \delta_{ij} \right). \tag{21}
\]
Then,
\[
g = 2\text{Hess}(u) = 2h \left( \frac{1}{R} \frac{d}{dR} (\log h) A + I \right),
\]
where \(I\) is the \(n \times n\) diagonal matrix and,
\[
A = \begin{pmatrix}
x_1^2 & x_1 x_2 & \ldots & x_1 x_n \\
x_1 x_2 & x_2^2 & \ldots & x_2 x_n \\
\vdots & \vdots & \ddots & \vdots \\
x_1 x_n & x_2 x_n & \ldots & x_n^2
\end{pmatrix}.
\]
We now have,
\[
\frac{R}{h \frac{d}{dR} (\log h)} \text{Hess}(u) = A + \frac{R}{\frac{d}{dR} (\log h)} I,
\]
which yields,
\[
\det \left( \frac{R}{h \frac{d}{dR} (\log h)} \text{Hess}(u) \right) = \det \left( A + \frac{R}{\frac{d}{dR} (\log h)} I \right) = \prod_{i=1}^n \left( \frac{R}{\frac{d}{dR} (\log h)} - \lambda_i \right),
\]
where $\lambda_i$ are the eigenvalues of $-A$, which are $\lambda_1 = \ldots = \lambda_{n-1} = 0$ and $\lambda = -R^2$. Thus,

$$\det\text{Hess}(u) = h^n + Rh^{n-1}h' = h^n + \frac{R}{n}(h^n)' ,$$

where $h'$ denotes differentiation with respect to $R$. In terms of the field intensity, we have

$$\det\text{Hess}(u) = \frac{H^{n-1}H'}{R^{n-1}}.$$  

The nondegeneracy of $g = f^*G$ implies that the function $H$ monotone.

1. Using Theorem 4, the immersion $f(p) = (p, \nabla u(p))$ is minimal if and only if $\det\text{Hess}(u) = c_0$, where $c_0$ is a positive constant. This is equivalent to solving the following differential equation:

$$h^n + \frac{R}{n}(h^n)' = c_0 .$$

Solving this ODE, we get,

$$h(R) = (c_0 + c_1R^{-n})^{1/n} , \quad (22)$$

where $c_1$ is a real constant and thus the field intensity $H$ is

$$H(R) = Rh(R) = (c_0R^n + c_1)^{1/n} .$$

Suppose now that $c_1 = 0$. Then $h(R) = c_0^{1/n}$ and using (21) we have that $u_{x_ix_j} = c_0^{1/n} \delta_{ij}$. Hence, one can see easily that

$$u(x_1, \ldots, x_n) = \frac{c_0^{1/n}}{2}(x_1^2 + \ldots + x_n^2) + d ,$$

where $d$ is a real constant. Using Theorem 4, it follows that the graph $f$ is totally geodesic. If, conversely, $f$ is totally geodesic, the function $u$ must satisfy (2). Since $\nabla u = h(R)(x_1, \ldots, x_n)$, we have

$$u_{x_k} = h(R)x_k .$$

On the other hand, taking the derivative of (2) with respect to $x_k$ we have

$$2 \sum_{i=1}^{n} a_{ik}x_i + b_k = h(R)x_k ,$$

which yields,

$$(2a_{kk} - h)x_k + 2 \sum_{i \neq k} a_{ik}x_i + b_k = 0 .$$

Therefore, $b_k = 0$ and $a_{ik} = 0$ for any $i \neq k$. Thus, $h(R) = a_{kk}$ for any $k$, which implies that $a_{kk} = a$ for some constant $a$. Then $h(R) = a$ and it can be obtained by using (22) by setting $c_1 = 0$. 

2. Suppose that the immersion $f$ is Hamiltonian minimal and set $\Phi = \det\text{Hess}(u)$. We have seen from part 1, that

$$\Phi = \frac{H^{n-1}H'}{R^{n-1}}.$$ 

Note that $f$ is minimal if and only if $\Phi$ is positive constant. Therefore, we assume that $\Phi' \neq 0$ except to some isolated points. Using Theorem 4, we know that $f$ is Hamiltonian minimal when

$$\Delta \log \Phi = 0,$$

where $\Delta$ stands for the Laplacian with respect to $g$. This implies,

$$g(\tilde{\nabla} \Phi, \tilde{\nabla} \Phi) = \Phi \Delta \Phi,$$

where $\tilde{\nabla}$ denotes the gradient with respect to $g$. One can show easily that the coefficients $g^{ij}$ of the inverse of $g$ is given by

$$g^{ij} = -\frac{x_i x_j}{2RH'} \frac{d}{dR} \log \left( \frac{H}{R} \right), \quad \text{if } i \neq j$$

$$g^{ii} = \frac{1}{2H'} \left( 1 + \frac{R^2 - x_i^2}{R} \frac{d}{dR} \log \left( \frac{H}{R} \right) \right).$$

Using (24), a brief computation gives

$$g(\tilde{\nabla} \Phi, \tilde{\nabla} \Phi) = \frac{(\Phi')^2}{2H'},$$

and

$$\Phi \Delta \Phi = -(n-1) \frac{\Phi \Phi'}{2H'} \frac{d}{dR} \log \left( \frac{H}{R} \right) + R\Phi \frac{d}{dR} \left( \frac{\Phi'}{2RH'} \right) + \frac{\Phi'}{2RH'} + \frac{(\Phi')^2}{2H'}.$$

The Hamiltonian minimal condition (23) now becomes

$$-(n-1) \frac{\Phi'}{2H'} \frac{d}{dR} \log \left( \frac{H}{R} \right) + R\Phi' \frac{d}{dR} \left( \frac{\Phi'}{2RH'} \right) + \frac{\Phi'}{2RH'} + \frac{(\Phi')^2}{2H'} = 0,$$

and away from possible minimal points (i.e. $\Phi' = 0$), we get

$$\frac{\Phi''}{\Phi'} = (n-1) \frac{d}{dR} \log \left( \frac{H}{R} \right) + \frac{d}{dR} \log H'.$$

This implies

$$\Phi' = c_0 \Phi,$$

where $c_0$ is a nonzero real constant. Then

$$\frac{H^{n-1}H'}{R^{n-1}} = ke^{c_0 R},$$

and solving the differential equation we obtain the Hamiltonian minimal condition of the Proposition. $\square$
5. Special isometric embeddings

It is well known that the space $L(R^3)$ of oriented lines in the Euclidean 3-space $R^3$ is identified with the $TS^2$, where $S^2$ denotes the round 2-sphere. Consider the Kähler metric $G$ of $L(R^3)$ derived from the standard Kähler structure endowed on $S^2$. We then prove:

**Proof of Theorem 6.** Consider the round 2-sphere $S^2$ and let $f : TS^2 \rightarrow TR^3 : (p, V) \mapsto (p, −p \times V)$ be the embedding, where $\times$ is the cross product in $R^3$. For $X \in T_{(p, V)}TS^2$, the derivative $df(X)$ is given by

$$\Pi df(X) = \Pi X,$$

$$K df(X) = −\Pi X \times V − p \times KX. \quad (25)$$

The metric $G$ in $TS^2$ is

$$G_{(p, V)}(X, Y) = g(KX, p \times \Pi Y) − g(\Pi X, p \times KY).$$

For $X, Y \in T_{(p, V)}TS^2$, we have

$$(f^*G)_{(p, V)}(X, Y) = G_{(p, V)}(f_*X, f_*Y)$$

$$= g((\Pi df X, K df Y) + g(\Pi df Y, K df X)$$

$$= g(\Pi X, −\Pi Y \times V − p \times KY) + g(\Pi Y, −\Pi X \times V − p \times KX)$$

$$= −g(\Pi X, p \times KY) − g(\Pi Y, p \times KX),$$

which shows that $f^*G = G$ and thus $f$ is an isometric embedding.

We now show that $f$ is minimal. Denote by $\nabla, \overline{\nabla}$ the Levi-Civita connections of $(R^3, \langle ., . \rangle)$ and $(TR^3, G)$, respectively and also denote respectively by $D, \overline{D}$ the Levi-Civita connections of $(S^2, g)$ and $(TS^2, G)$.

A brief computation gives

$$\nabla dfX dfY = (\nabla_{\Pi X} \Pi Y, −\nabla_{\Pi X} \Pi Y \times V − p \times \nabla_{\Pi X} KY − \Pi Y \times KX + (\Pi X, V) p \times \Pi Y)$$

and using the fact that $\overline{D}X Y = (D_{\Pi X} \Pi Y, D_{\Pi X} KY − (V, \Pi X) \Pi Y)$, we have

$$df(\overline{D}X Y) = (\nabla_{\Pi X} \Pi Y − (\Pi X, \Pi Y) p, \Pi X) p \times V + (\Pi X, \Pi Y) p \times V$$

$$= −p \times \nabla_{\Pi X} KY + (V, \Pi X) p \times \Pi Y)$$

The second fundamental form $h$ of $f$ is given by

$$h(dfX, dfY) = \overline{\nabla}_{dfX} dfY − df(\overline{D}X Y)$$

$$= (\langle \Pi X, \Pi Y \rangle p, \langle \Pi X, \Pi Y \rangle p \times V − (\Pi X, V) p \times (KX − \Pi X × KY)$$

Suppose that $(p, V) \in TS^2$ and $|V| \neq 0$. Consider the following orthogonal basis of $T_{(p, V)}TS^2$:

$$E_1 = (V, p \times V), \quad E_2 = (p \times V, V), \quad E_3 = (V, −p \times V), \quad E_4 = (−p \times V, V).$$

Thus,

$$|E_1|^2 = −|E_2|^2 = −|E_3|^2 = |E_4| = 2|V|^2.$$
Using (25) we have
\[ df E_1 = (V, V), \quad df E_2 = (p \times V, |V|^2 p - p \times V), \quad df E_3 = (V, -V), \]
\[ df E_4 = (-p \times V, -|V|^2 p - p \times V). \]

The second fundamental form for this basis is
\[ h(df E_1, df E_1) = h(df E_3, df E_3) = (|V|^2 p, -|V|^2 p \times V), \]
and
\[ h(df E_2, df E_2) = h(df E_4, df E_4) = (|V|^2 p, -|V|^2 p \times V - 2V), \]

The mean curvature \( H \) of \( f \) is
\[ H = \frac{1}{2|V|^2} \left( h(df E_1, df E_1) - h(df E_2, df E_2) - h(df E_3, df E_3) + h(df E_4, df E_4) \right), \]
which clearly vanishes and therefore the embedding \( f \) is minimal.

Suppose that \( V = 0 \) and let \( \xi \) be a unit vector in \( T_p \mathbb{S}^2 \). Then we obtain the following orthonormal frame for \( T_{(p,0)}T\mathbb{S}^2 \):
\[ E_1 = (\xi, p \times \xi), \quad E_2 = (p \times \xi, \xi), \quad E_3 = (\xi, -p \times \xi), \quad E_4 = (-p \times \xi, \xi). \]
We then have
\[ df E_1 = (\xi, \xi), \quad df E_2 = (p \times \xi, -p \times \xi), \quad df E_3 = (\xi, -\xi), \]
\[ df E_4 = (-p \times \xi, -p \times \xi). \]

The second fundamental form for this basis is
\[ h(df E_1, df E_1) = h(df E_3, df E_3) = h(df E_2, df E_2) = h(df E_4, df E_4) = (p, 0), \]
and thus,
\[ H = h(df E_1, df E_1) - h(df E_2, df E_2) - h(df E_3, df E_3) + h(df E_4, df E_4) = 0, \]
and the Theorem follows. \( \Box \)

Note that \((T\mathbb{S}^2, \mathbb{G})\) can’t be isometrically embedded in \((T\mathbb{R}^3, G_0)\), where \( G_0 \) is the Sasakian metric.

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