Localization Trajectory and Chern-Simons axion coupling for Bilayer Quantum Anomalous Hall Systems

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Quantum anomalous Hall (QAH) multilayers provide a platform of topological materials with high Chern numbers. We investigate the localization routes of bilayer QAH systems with Chern number \( C = 2 \) under strong disorder, by numerical simulations on their quantum transport properties and the Chern-Simons axion coupling. Compared to the single layer counterpart with \( C = 2 \), the localization trajectories present much richer behaviors, for example, the existence of the stable intermediate state with \( C = 1 \) can be tuned by model parameters. This state was always unstable in the single layer case. Furthermore, the two parameter scaling trajectories also exhibit multiple patterns, some of which were not captured by the standard Pruisken picture. During the process towards localization, the Chern-Simons axion coupling shows a surprisingly remarkable peak which is even higher and sharper in the large size limit. Therefore the disordered bilayer QAH system can be a good candidate for this nontrivial magnetoelectric coupling mediated by orbital motions.

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I. INTRODUCTION

The research of topological materials recovers the interests of searching the quantum anomalous Hall (QAH) effects, i.e., Chern insulators without an external magnetic field[1]. QAH effects in different materials have been theoretically proposed[2–6] and experimentally observed[7–11]. The Chern insulator, as a topological state, is robust against weak disorder, manifested as a plateau of Hall conductance \( \sigma_{xy} = \frac{e^2}{2\pi} C \) with \( C \) the Chern number[12–13]. Nevertheless, it can still be localized into Anderson insulators with \( C = 0 \) ultimately, by sufficiently strong disorder. This can be regarded as a disorder induced topological transition with Chern number from nonzero to zero[14–16]. On the other hand, this trajectory towards localization can also be understood in the context of renormalization group (RG), i.e., the Pruisken formalism[17], where all the quantized Hall conductances (associated with all nonzero Chern numbers) correspond to stable fixed points attracting all nearby RG flows in the conductance plane \((\sigma_{xy}, \sigma_{xx})\). Besides the analytical RG treatment, these trajectories along scaling transformations can also be observed by experiments[18–20] or simulated by numerical calculations[16–22].

QAH states with different Chern numbers are topologically distinct from each other. The localization trajectory of the single-layer (two-band) QAH state with Chern number \( C = 2 \) has been numerically investigated[11,21]. The novelty is that the intermediate state with \( C = 1 \) is not stable during the process towards localization, no matter how the model parameters are changed. Microscopically this has been attributed to a \( C = 1 \) state rooted in broad statistical tails, which cannot manifests itself after statistical averaging and size scaling[16].

In additional to transport properties, the orbital mediated magnetoelectric coupling attracts many attentions in topological materials. The axion electrodynamics[25] introduces a non-Abelian Lagrangian term[26–30],

\[
L_\theta = \frac{e^2}{h} \frac{\theta}{2\pi} E \cdot B, \tag{1}
\]

where \( \theta \) is a dimensionless parameter which is determined by the electronic structure of the material, and \( E \) and \( B \) are the electric and magnetic fields respectively. This quantity reflects a deep magnetoelectric connection mediated by the electronic orbital motions in solids, especially remarkable in topological materials. Due to the involvement of three dimensional motions [see Eq. (1) below], a bilayer system, which will be investigated in this manuscript, is the simplest lattice with a nonzero Chern-Simons axion coupling[31].

An important knowledge from (quasi) two-dimensional (2D) systems is that a bilayer system may possess quite different properties from its single layer counterpart[31–32]. Few layer QAH systems have been experimentally fabricated recently by molecular beam epitaxy[33]. Among other QAH systems with Chern number 2, the
bilateral QAH system we will discuss is unique in at least two aspects. First, it is a four band model, or, a model with two pairs of bands, with each pair contributing one Chern number. Second, each pair corresponds to one layer which is distinguishable in the real space, and their coupling is tunable.

In this manuscript, we will investigate the trajectory towards localization of such a bilayer system with increasing disorder, by performing numerical simulations on the two terminal conductance, two-parameter scaling, and the axion magnetoelectric coupling. We find that, starting from some appropriate model parameters, there can be a stable Hall conductance plateau with \( C = 1 \), in a region of intermediate disorder strength. The existence of this intermediate plateau has been ruled out in a region of intermediate disorder strength. The existence of this intermediate plateau has been ruled out for single layer QAH systems\(^\text{16, 21}\). Through numerical scaling, we also find that some trajectories cannot be perfectly fitted into the Pruisken flows, especially in the limit of strong inter-layer coupling. Finally, the disorder induced collapse of the quantum Hall plateau is shown to correspond to a peak of axion magnetoelectric coupling.

II. THE MODEL

The general form of the spinless bilayer QAH system can be expressed as

\[
\mathcal{H}_{\text{III}} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_c, \tag{2}
\]

where \( \mathcal{H}_L \) \( (L = 1, 2) \) is the Hamiltonian for the \( L \)-th layer, and \( \mathcal{H}_c \) is the coupling between them. Here, we choose each layer to be the spin up component of the Bernevig-Hughes-Zhang (BHZ) model\(^\text{34}\) defined on a square lattice, with one \( s \) orbital and one \( p \) orbital on each site. This is one of the minimum models of the QAH state. In the \( k \)-space, the layer Hamiltonian reads

\[
\mathcal{H}_L = \sum_{k, \alpha \beta} c_{L, \alpha \beta}^\dagger(k)c_{L, \alpha \beta}(k), \tag{3}
\]

where \( c_{L, \alpha \beta}^\dagger \) (\( c_{L, \alpha \beta} \)) creates (annihilates) an electron with \( k \)-wavevector and orbital \( \alpha \in \{s, p\} \) in layer \( L \in \{1, 2\} \). For BHZ model, \( \mathcal{H}_{L, \alpha \beta}(k) \) is a \( 2 \times 2 \) matrix as\(^\text{34}\)

\[
\begin{align*}
H_L(k) &= \varepsilon_L(k)I_{2 \times 2} + \sum_i d_i^L(k)\sigma_i, \tag{4} \\
\varepsilon_L(k) &= -2D_L[2 - \cos(k_x - q_L^x) - \cos(k_y - q_L^y)] \\
d_i^L(k) &= A_L^+ \sin(k_x - q_L^x), \quad d_i^L(k) = A_L \sin(k_y - q_L^y) \\
d_i^L(k) &= M_L - 2B_L[2 - \cos(k_x - q_L^x) - \cos(k_y - q_L^y)],
\end{align*}
\]

where \( \sigma_i \) are the Pauli matrices acting on the orbital space \( \{s, p\} \). We have included a layer dependent momentum shift \( \mathbf{q}_L = (q_L^x, q_L^y) \) of the band structure in the Brillouin zone, so that two groups of edge states (originated from two layers) can be better distinguished visually, as can be seen in Fig. 1 (a). We have checked that this shift does not affect the transport properties investigated in the following. The real space version of the layer Hamiltonian \( \mathcal{H}_L = \sum_{i, j, \alpha \beta} c_{L, i \alpha}^\dagger H_{L, \alpha \beta}(i, j) c_{L, j \beta} \) can be obtained from Eqs. (2) and (4) by performing a straightforward inverse Fourier transformation \( c_{L, \alpha \beta} = \frac{1}{\sqrt{N}} \sum_i c_{L, \alpha \beta} e^{-i k \cdot q_i} \), where \( i \) is the site index. The size of such a finite sample is characterized by the length \( N_x \) and the width \( N_y \) measured in units of the lattice constant.

\[
\mathcal{H}_c = \sum_{i, \alpha} (t c_{1, i \alpha}^\dagger c_{2, i \alpha} + \text{H.c.}), \tag{6}
\]

![Figure 1](image-url) (Color online) The band structure of the quasi-one dimensional ribbon with width \( N_y = 80 \) in the clean limit, for (a) \( A_1 = 1 \) and (b) \( A_1 = 0.3 \). The rest model parameters are identical for both panels: \( A_1 = 0.3, B_1 = 0.2, B_2 = 0, M_1 = 0.2, A_2 = 1.0, B_2 = 0.6, D_2 = 0, M_2 = 1.0, \) and \( t = 0.3 \). The red lines are edge states.

In the absence of the inter-layer coupling \( \mathcal{H}_c \), the band structure and Chern number of each layer can be tuned independently by varying the parameters. For example, the band gap is \( 2|M_L| \), and the layer Chern number

\[
C_L = \begin{cases} +1, & 0 < M_L/2B_1 < 2 \\ 0, & M_L/2B_1 < 0 \\ -1, & \text{otherwise} \end{cases} \tag{5}
\]

The inter-layer coupling is considered to be the simple form in real space as

\[
\mathcal{H}_c = \sum_{i, \alpha} (t c_{1, i \alpha}^\dagger c_{2, i \alpha} + \text{H.c.}), \tag{6}
\]
where \( t \) is the strength of the coupling. If the adiabatic turning on of the inter-layer coupling term \( \Theta \) does not close the bulk gap, the Chern number of the bilayer system is just the sum of those of each layer, \( C = C_1 + C_2 \). Throughout this manuscript, we focus on the case of \( C = 1 + 1 \).

In Fig. 1 we present the band structures of the bilayer systems in the ribbon geometry, with two typical groups of model parameters. Two groups of edge states (red lines) in the bulk gap can be clearly seen, reflecting the Chern number \( C = 1 + 1 \) at half filling. To have a direct visual comparison with the case of a single layer,[16, 24], we have appropriately tuned the model parameter \( q_L \) [see Eq. (4)], so that two groups of edge states (originated from two layers respectively) are centered at different locations with two local gaps respectively. Furthermore, the magnitudes of these two local gaps can be similar [Fig. 1 (a), with \( A_1 = 1.0 \)] or different [Fig. 1 (b) \( A_1 = 0.3 \)]. In the following, we will see that these two patterns correspond to different quantum transport properties.

The effect of disorder is modeled in real space by adding a random onsite potential term

\[
\sum_{L, \alpha, \delta} c_{L, \alpha, \delta}^\dagger U_{L, \alpha} c_{L, \alpha, \delta}
\]

(7)

to the Hamiltonian \( \mathcal{H}_0 \), where \( U_{L, \alpha} \) are independently random numbers uniformly distributed in \((-W, W)\), and \( W \) is the disorder strength.

III. METHODS

At zero temperature, the two-terminal conductance of a finite sample can be expressed by Green’s functions as[32, 37]

\[
G = \frac{e^2}{h} \text{Tr} \left[ \Gamma_S G_r \Gamma_D G_a \right],
\]

(8)

where \( G^{r/a} \) is the retarded/advanced Green’s function, and \( \Gamma_{S(D)} = i(\Sigma_{S(D)} - \Sigma_{S(D)}^a) \) with \( \Sigma_{S(D)}^{r/a} \) being retarded/advanced self energies due to the source (drain) lead, respectively. For a Chern insulator, this two-terminal conductance is just the Hall conductance, \( G = \sigma_{xy} \). To plot the two-parameter scaling trajectories, longitudinal and transverse conductances should be calculated respectively, which will be described below.

For a finite 2D sample with disorder, the concept of the wavenumber \( \mathbf{k} = (k_x, k_y) \) can be restored if twisted boundary conditions are adopted along both directions[11, 14, 38, 39]. This is simply equivalent to treating this finite sample as the supercell of an infinite superlattice. In this sense, hereafter, we call these finite samples as supercells and the energy bands of this superlattice as subbands. Now, the longitudinal conductance \( \sigma_{xx} \) of the supercell is calculated as the Thouless conductance[40, 41]

\[
\sigma_{xx}(E_n) = \frac{e^2}{h} \pi \rho(E_n) \frac{\partial^2 E_n}{\partial k_x^2} |_{k_x=0},
\]

(9)

where \( \rho \) is the density of states, and \( E_n \) is the subband at the Fermi energy. Moreover, the Hall conductance \( \sigma_{xy} \) can be calculated as an integration of the Berry curvature \( F_n^\gamma \) under the Fermi energy \( E_F \) [22, 39, 42, 43]

\[
\sigma_{xy}(E_n) = \frac{e^2}{h} \sum_{n} \frac{1}{2\pi i} \int_{BZ} d^2k F_{n}^\gamma (k) f_0 [E_n(k)]
\]

(10)

\[
F_n = \nabla_k \times A_n
\]

\[
A_n = \langle n(k) | \nabla_k | n(k) \rangle,
\]

where \( f_0 \) is the Fermi distribution at zero temperature, and \( \langle n(k) \rangle \) is the normalized wave function of the \( n \)-th subband which satisfies \( H(k)|n(k)\rangle = E_n(k)|n(k)\rangle \). For a given \( E_F \), \( \sigma_{xy} = \frac{e^2}{h} C \) with \( C \) being the Chern number, if all subbands below the Fermi energy are fully occupied[13], but \( \sigma_{xy} \) may not be quantized if some subbands are only partly filled. The numerical evaluations of Eqs. (9) and (10) are based on the methods used in previous works[16, 22, 41]. Hereafter, all conductances will be expressed in units of \( \frac{e^2}{h} \).

In the presence of disorder, an ensemble average of these transport quantities over disorder configurations should be performed to obtain physically meaningful results, for a certain group of model parameters and supercell size. Then, by enlarging the supercell size, the ensemble averaged conductance vector \( \bar{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) \) forms a flow in this 2D parameter space[16, 22]. Throughout this manuscript, we will call this a “scaling flow” (or a “scaling trajectory”), while that calculated from RG theories[17] an “RG flow” (or an “RG trajectory”). These flows offer the information of transport properties in the thermodynamic limit.

Now we introduce the calculation of orbital magnetoelectric coupling. Due to the close relation between \( \theta \) in Eq. (4) and the linear orbital magnetoelectric response \( \alpha_{ij} = (\partial P_i / \partial B_j)_{E} = (\partial M_j / \partial E_i)_{B} \), with \( P \) and \( M \) the macroscopic polarization and magnetization[31], it is adequate to focus on the Chern-Simons axion (CSA) coupling[20, 30]

\[
\alpha_{CS} = \frac{e^2}{h} \frac{\theta_{CS}}{2\pi},
\]

(11)

where \( \theta_{CS} \) is a part of \( \theta \) with a \( 2\pi \) ambiguity. For an infinite three-dimensional (3D) crystalline material, the CSA coupling can be expressed as[42]

\[
\theta_{CS} = -\frac{1}{4\pi} \int d\mathbf{k} \epsilon_{ijl} \text{Tr}(A_k^l \partial_{k_i} A_k^j - i \frac{2}{3} A_k^l A_k^j A_k^l),
\]

(12)
where $A_{ik}$ is the matrix of Berry connection between sub-bands in direction $i$, and the trace is over occupied bands.

Moreover, the expression of the CSA coupling has been carefully generalized to an infinite 2D slab as $\theta_0$, and to a finite (zero dimensional) crystal as $\theta_0 = -8\pi^2 \text{Im} \text{Tr} [P x P y P z]$, (13)

where $P = \sum_n |\psi_n\rangle \langle \psi_n|$ is the projection operator onto the occupied subspace in the ground state. It has been verified that in the thermodynamic limit $N_x N_y \to \theta_2$, (14)

This $\theta_0$ defined by Eq. (13) is convenient for treating disordered and finite samples which we are interested here. The CSA coupling [Eqs. (12) and (13)] involves responses in all three directions and thus it is meaningless for single layer systems.

III. RESULTS

Appearance of the intermediate state with $C = 1$

A Chern insulator (with $C \neq 0$) manifests itself as a robust Hall conductance plateau at weak disorder, but it can be localized at strong disorder. For a single layer QAH system (with two bands) with Chern number $C = 2$, it has been found that, the intermediate state with $C = 1$ is not a stable plateau on the route towards localization [16, 24]. Here we try to test the stability of the intermediate state $C = 1$ of the bilayer system [3]. Since each layer contributes one Chern number, a natural idea is to localize the edge state associated with a certain layer first. To this end, we can make these two layers (especially their gaps) quite asymmetric to each other, by tuning model parameters distinct on two layers.

To have a first glance at the quantum transport property in the presence of disorder, we calculate the two-terminal conductance by using Eq. (3), which is equal to the Hall conductance for a Chern insulator [33]. In Fig. 2, the disorder averaged conductance $G$ is plotted as a function of disorder strength $W$, at Fermi energy $E_F = 0.05$. (a) Fixing $t = 0.3$ while changing $A_1$. (b) Fixing $A_1 = 0.3$ while changing $t$. (c) Fixing $A_1 = 1$ while changing $t$. The rest model parameters are identical to Fig. 1.

FIG. 2: (Color online) Disorder averaged two-terminal conductance $G$ as a function of disorder strength $W$, at Fermi energy $E_F = 0.05$. (a) Fixing $t = 0.3$ while changing $A_1$. (b) Fixing $A_1 = 0.3$ while changing $t$. (c) Fixing $A_1 = 1$ while changing $t$. The rest model parameters are identical to Fig. 1.

as one can imagine, changing $t$ will never give rise to a stable plateau with $\sigma_{xy} = 1$.

Scaling Flows

To test the stability of this intermediate state in Fig. 2 in the thermodynamic limit, we plot the supercell size driven scaling flows on the $\sigma \equiv (\sigma_{xy}, \sigma_{xx})$ plane, through numerical simulations by using Eqs. (9) and (10), as in previous works [16, 22, 23]. The results for different $A_1$ are displayed in Fig. 3 as red arrows, where each data point is an average over at least 10000 disorder samples (30000 samples around transitions). For comparison, RG flows of the Pruisken pattern [17] are also illustrated as
FIG. 3: (Color online) Red arrows: The supercell size driven scaling flows for \( t = 0.3 \), with arrows beginning at size \( N_x = N_y = 10 \) and ending at size \( N_x = N_y = 15 \). Each arrow is an average over 10000 disorder samples (30000 samples around the transition region). The disorder strength \( W \) varies from 0 to 12, and at the Fermi energies \( E_F = 0.05 \) near the center of the gap. (a) \( A_1 = 1.0 \); (b) \( A_1 = 0.7 \); (c) \( A_1 = 0.5 \); (d) \( A_1 = 0.3 \); the other parameters are identical to Fig. 1. Grey curves are illustrative configurations of Pruisken RG trajectories\[17\], just as a fitting reference (see text for details).

FIG. 4: (Color online) Same as Fig. 3 but for different \( t \): (a) \( t = 0.1 \); (b) \( t = 0.3 \); (c) \( t = 0.5 \); (d) \( t = 1.0 \); the other parameters are identical to Fig. 1.
grey curves in Fig. 3. These RG flows are just plotted by fitting the original RG equations [17] with a few free parameters (including zooming factors in two directions), instead of by a rigorous RG treatment of our bilayer model. Therefore they are only presented here as a visual reference, rather than a conclusive result. Furthermore, due to the critical slowing down and large fluctuations near the transition, the numerical computation of scaling flows is extremely resource consuming (e.g., almost 100,000 samples should be averaged around the transition region to obtain a stable pattern). As a result, it is rather difficult to have a precise and global reproduction (or prediction) of the RG flows from numerical scaling flows. However, intuitive insights can still be drawn.

When the two local gaps are similar \((A_1 = 1)\), the scaling trajectory is a single semicircle connecting two attracting fixed points \(\sigma = (0, 0)\) and \(\sigma = (2, 0)\), as shown in Fig. 3 (a). This is consistent with the blue line in Fig. 2 (a), i.e., the absence of a stable \(\sigma_{xy} = 1\) plateau. When \(A_1\) is decreased, in Fig. 3 (b) to (d), with two local gaps different, the state with \(\sigma_{xy} = 1\) manifests itself and becomes more and more attractive for scaling flows, suggesting that it is a stable state during the route towards localization. When \(A_1 = 0.5\), the trajectories almost become two semicircle touching at \(\sigma_{xy} = 1\). In the language of modular symmetry, when the two layers become more and more asymmetric, the system experiences a transition from type \(\Gamma_R\) to \(\Gamma_T\) [49].

The appearance of this stable intermediate plateau with \(\sigma_{xy} = 1\) is the first important finding of this manuscript. Since this happens when the magnitudes of two local gaps (each accommodating a group of edge states) are extremely different [Fig. 3 (b)], this can be illustrated as disorder induced band inversion [15, 16] at the smaller local gap. This is completely different from the single-layer QAH effect with \(C = 2\) [16, 24]. In that case, changing model parameters, e.g., making two local gaps (each of which carries a group of edge states) different will never give rise to a stable plateau with \(G = 1\) [14].

Another notable feature can be seen in the most asymmetric case with small \(A_1 = 0.3\) [Fig. 3 (d)]: the shape of scaling trajectories between \(\sigma_{xy} = 1\) and \(\sigma_{xy} = 2\) is remarkably deformed from a perfect semicircle, and these scaling trajectories do not seem to follow typical Pruisken RG flows [17, 49]. This tendency becomes more remarkable when the inter-layer coupling \(t\) is increased, as can be seen in Fig. 3 (c) and (d). For example, the directions of the scaling flows remarkably deviate from those of the standard Pruisken RG flows, especially when \(\sigma_{xy} \gtrsim 1.5\). When \(\sigma\) is high above the semicircle, RG trajectories (grey lines) should be flowing down straightly, but the scaling flows (red arrows) in Fig. 3 (d) are not the case. Such a remarkable deviation from the Pruisken picture for Bilayer QAH systems is the second finding of this work. We conjecture that when the interlayer coupling is much larger than the intra-layer hopping, the in-plane electronic motions could be suppressed. This may have an non-ignorable impact on the in-plane transport, which was, however, only understood in a purely 2D regime in traditional RG analysis.

\[\text{FIG. 5: Distribution of the Hall conductance } \sigma_{xy} \text{ corresponding to the case of Fig. 3 (a), where the state with } \sigma_{xy} = 1 \text{ is not a stable fixed point. Four panels are for different disorder strengths } W: \ (a) \ W = 2.0; \ (b) \ W = 5.0; \ (c) \ W = 7.0; \ (d) \ W = 9.0.\]

\[\text{FIG. 6: Same as Fig. 5 but corresponding to the case of Fig. 3 (a), where the state with } \sigma_{xy} = 1 \text{ is a stable fixed point.}\]

**Distributions of \(\sigma_{xy}\)**

The above results were based on transport quantities averaged over disorder ensembles. To obtain more insights from another angle, now we investigate the statistical distributions of the Hall conductance at different disorder strength \(W\). Fig. 4 is the statistical histograms of \(\sigma_{xy}\) at different disorder strength, corresponding to the scaling flows of Fig. 3 (a), with just one semicircle and with an unstable \(\sigma_{xy} = 1\) state. On the other hand,
Fig. 6 corresponds to the case of Fig. 4 (a), with two semicircles and with a stable $\sigma_{xy} = 1$ state. When the disorder is weak, $W = 1.5$ [panels (a) in both figures], both cases show isolated narrow peaks with quantized Hall conductances, among which the $\sigma_{xy} = 2$ peak is much higher than others. This suggests that the system is in a well defined insulating state with an extremely high probability of $\sigma_{xy} = 2$. Such a distribution gives rise to a stable $\sigma_{xy} = 2$ plateau after ensemble average, and a stable fixed point of scaling flows, as we have seen above. The most remarkable difference between Figs. 5 and 6 appears at the intermediate disorder strength $W = 5$ [panels (b)]. In Fig. 6 (b), similarly, an isolated and prominent narrow peak at $\sigma_{xy} = 1$ gives rise to stable transport plateau of $\sigma_{xy} = 1$ which is a stable fixed point in Fig. 4 (a). On the other hand, the pattern is different in Fig. 5 (b). Here although the $\sigma_{xy} = 1$ peak is still very high, it is rooted in continuous and broad tails, where $\sigma_{xy} = 1$ is not quantized. This suggests that the energy location of the bulk gap is fluctuating strongly from sample to sample, making $\sigma_{xy} = 1$ only a “hidden state” which is not stable in the thermodynamic limit, similar to what was observed in the single layer case[10].

![Chern-Simons axion coupling](image)

**FIG. 7:** (Color online) Solid curves: The area normalized finite size CSA coupling $\theta_0/(N_xN_y)$ as a function of disorder for different $A_1$. Each data point is an average over 4000 samples with size $N_x = N_y = 20$. For comparison with the transport property, the corresponding two-terminal conductances are also plotted in dashed curves. The interlayer coupling $t = 0.3$, and the other parameters are same as Fig. 4.

Finally, we explore how CSA coupling strength $\theta_{cs}$ behaves during the process towards localization. Due to the size relation, Eq. (14), what makes sense for different sizes is the finite size CSA coupling $\theta_0/(N_xN_y)$ normalized by the sample area, $\theta_0/(N_xN_y)$. In Fig. 7 solid curves are the $\theta_0/(N_xN_y)$ as a function of disorder strength $W$ for different $A_1$. For a comparison with the transport properties, we also plot the corresponding two-terminal conductance [identical to Fig. 2 (a)] as dashed curves. It is interesting to notice that $\theta_0$ starts to increase sharply just before the collapse of the conductance plateau with $G = \sigma_{xy} = 2$. Then it climbs to a peak after which the tendency of localization begins to dominate. However, the decay of $\theta_0$ is slower than that of $G$ after $W > 10$. Another distinct feature is that there is also only one peak for all these cases, even for the case of $A_1 = 0.3$ where there are two transitions of the Hall conductance: $\sigma_{xy} = 2 \rightarrow 1$ and $1 \rightarrow 0$. These suggest that during the process of localization, the variation of CSA coupling is closely related to the disorder induced topological phase transition, but not a simple “one to one” correspondence with transport properties.

![Fig. 8](image)

**FIG. 8:** (Color online) Similar to Fig. 7 but for different sample size $N = N_x = N_y$, (a): $A_1 = 0.3$, (b): $A_1 = 1$. The insets are the corresponding peak value $\theta_0(\text{max})$ as a function of sample size.

In order to obtain the information in the thermodynamic limit, we present the CSA coupling $\theta_0/(N_xN_y)$ for increasing sample sizes in Fig. 3 for $A_1 = 0.3$ and $A_1 = 1$ respectively. In both cases, the peak shapes of CSA become sharper with the increasing of the sample sizes. The increasing of the peak value $\theta_0(\text{max})/(N_xN_y)$
with the sample size is plotted in the corresponding inset respectively. These peak values do not seem to converge until the largest size we can calculate, $N_x \times N_y = 30 \times 30$. With the size axis in the logarithmic scale, one can see that its dependence on sample size $N$ is as fast as (or a little bit slower than) $\theta_{0(\text{max})}/(N^2) \sim \ln N$. This scaling growth makes this CSA coupling experimental observation possible. Therefore, besides intrinsic materials previously proposed \cite{50–52}, we find that bilayer QAH systems with disorder are also candidates for high $\theta$ materials.

V. SUMMARY

In summary, the localization trajectories of the bilayer QAH systems with Chern number 2 are investigated numerically. Contrary to the case of the single layer counterpart, there can be a stable state with Chern number 1 before the complete localization. The stability of this intermediate state can be controlled by the model parameters on two layers. This picture can be confirmed by numerical simulations of two-parameter scaling flows on the conductance plane. However, some of the scaling flows deviate from the standard Pruisken pattern, especially in the case of strong inter-layer coupling. Microscopically, different stabilities of this intermediate state with $C = 1$ correspond to different types of distribution of the Hall conductance. These can be tested by state of the art experimental techniques and raise new topics for renormalization group theories. Finally, we find the CSA coupling of such systems can be highly remarkable in the transition window with medium disorder strength. Therefore we propose a new way of finding candidate axion insulators.

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