PROOF OF A MASS SINGULARITY FREE PROPERTY

IN

HIGH TEMPERATURE QCD

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ABSTRACT

It is shown that three series of diagrams entering the calculation of some hot QCD process, are mass (or collinear) singularity free, indeed. This generalizes a result which was recently established up to the third non trivial order of (thermal) Perturbation Theory.

PACS: 12.38.Cy, 11.10.Wx

Keywords: Hot QCD, Resummation Program, infrared, mass/collinear singularities,..
I. Introduction

During the past fourteen years, a considerable amount of work has been devoted to the study of quantized fields at high temperature and/or chemical potential (high temperature, e.g., means higher than any bare or renormalized mass involved in the theory). The inherent non-perturbative character of thermal quantum field theories has been recognized, and naive perturbation theories accordingly reorganized. This is achieved by means of a Resummation Program (RP), which, in the high temperature limit, must be used whenever one is calculating processes involving Green’s functions with soft external/internal lines. The soft scale is defined to be on the order of $gT$ where $T$ is the temperature and $g$ some relevant and small enough coupling constant, so as to decide, at least formally, of two separate hard (on the order of $T$) and soft energy scales. The RP is given by effective Feynman rules, consisting of effective field propagators and $n$-points proper vertices, all at a given leading order of approximation which turns out to be $g^2T^2$, and is referred to as HTL (Hard Thermal Loops). While HTL vertices are purely perturbative objects, effective propagators are not, as they give rise to pole residues and dispersion laws that do not admit perturbative series expansions in the coupling constant. In the course of practical calculations, effective propagators are easily handled, relying on analyticity properties and Cauchy’s theorem.

Endowed with most beautiful symmetries, the RP is an effective theory that has led to a number of satisfying results, but has also met two serious obstructions, emanating both from the infrared (IR) sector.

Now, Resummations can in general be defined a number of consistent, still different ways. In this article, we take advantage of a so called Perturbative Resummation scheme, hereafter denoted PR for short, previously introduced in the context of the first obstruction, to address the problem of the soft real photon emission rate of thermal QCD. This problem is the following. When use is made of the Resummation Program to calculate the soft real photon emission rate, out of a Quark-Gluon plasma in thermal equilibrium, the answer comes out affected with a collinear singularity. In the context of massless Quantum Field Theories, it may be worth recalling that collinear singularities manifest themselves as singularities of the angular integration, or equivalently, of the integration on the virtuality, $P^2 = p_0^2 - p^2$, and are thus also called mass singularities. Though of a different nature, mass/collinear singularities
are regrouped with singularities of the integration on three-momenta $|\vec{p}|$, under the common spell of *Infrared singularities*.

Several attempts to cure that *IR* disease have been proposed ever since$^9$, which, though consistent at one loop order, encounter further important difficulties when extended to higher number of loop calculations$^{10}$. Our present work is motivated by a recent study of the problem, projected out on a toy model, with the conclusion that things come out very different according to the Resummation scheme in use, *RP* or *PR* $^{11}$. Then, the case of interest, that is *hot* *QCD*, has been studied in its first three non trivial perturbative orders: Again, no collinear singularity did show up in a *PR* resummation scheme. Moreover, both the questionable nature of that singularity$^{12}$, and the very mechanism through which it comes about in an *RP* scheme, have been discovered thanks to an original comparison with the *PR* scheme$^{13}$.

However encouraging, this first analysis of the *QCD* case has only been performed up to three loop order. In order to see if a *PR* resummation scheme has any chance to avoid that serious problem, it is crucial to extend the proofs of Ref.$^{13}$ to any loop order, and this sets in, we think, the strategical interest of the present analysis. By the same token, we note that the quite as much important *collinear enhancement* problem$^{10}$, which comes out of the latter if one tries to solve the difficulty by introducing a so called *asymptotic thermal mass*, $m_\infty$ $^9$, is circumvented also.

The paper is organized as follows. Section 2 is a reminder of the collinear singularity problem under consideration, while introducing elements and notations necessary to the next sections. In Section 3, topologies involving only bare vertices, with $N(N')$ *HTL*-self energy insertions along the $P(P')$ internal fermionic lines, denoted by $(N, N'; 0)$, are investigated in details. To do so, the matter of Ref.$^7$ is exploited so as to show that any $(N, N'; 0)$ imaginary part is mass singularity free, or msf for short. The same property can be established concerning the contributions attached to $(N, N'; 1)$ topologies, with one *HTL*-vertex correction included, and this is Section 4. The results of both previous sections are obtained on the basis of purely technical calculations, but it seems almost impossible to proceed further along this line of approach: For contributions of type $(N, N'; 2)$, involving two *HTL*-vertex corrections, functions at play are so complicated that they preclude any control of the ensuing integrals. Remarkably enough, though, Section 3 is able to provide enough information so as to initiate an efficient, global and conclusive approach by induction. On the other hand, used right from the onset, an induction procedure does not appear to be, by itself, fully conclusive.
This efficient articulation of Section 3 calculational approach, to an induction process is the matter of Section 5. All three series of diagrams are definitely shown to possess mass/collinear singularity free imaginary parts, in the end.

Our conclusions are gathered in Section 6, whereas an Appendix displays the technical complexities encountered by a calculational approach to \((N, N'; 2)\)-type diagrams.

Throughout the article, we work in the \(R/A\) real time formalism, which is based on Retarded/Advanced free field functions\(^{14}\). Also, we will be using the convention of upper case letters for quadrimomenta and lower case ones for their components, writing, for example, \(P = (p_0, \vec{p})\). Our conventions for labelling internal and external momenta can be read off Fig.1.

## II. The soft real photon emission rate of hot QCD

It is convenient to work in the real time formalism with retarded/advanced \((R/A)\) field functions, where a concise and elegant derivation of the famous collinear singularity can be achieved\(^{15}\). The soft real photon emission rate is essentially related to the imaginary part of the quantity \(\Pi_{RR}^{\mu \mu}(Q)\), trace of the soft real photon polarization tensor, hereafter written as \(\Pi_R(Q)\). At pure one loop order, this imaginary part is zero. However, when the photon is soft, this result is incomplete and the \textit{Resummation Program} must be used instead of bare thermal Perturbation Theory. This amounts to keep the one loop diagram of ordinary Perturbation Theory, while replacing bare vertices and propagators by their HTL-dressed counterparts. In Feynman gauge, the resulting expression reads (with \(n_F\), the Fermi-Dirac statistical factor, defined without absolute value),

\[
\Pi_R(Q) = i \int \frac{d^4P}{(2\pi)^4} (1 - 2n_F(p_0)) \text{ disc} \left\{ *S_R(P) \ *\Gamma_\mu(P_R, Q_R, -P'_A) \right. \\
\left. \ *S_R(P') \ *\Gamma^\mu(P_R, Q_R, -P'_A) \right\} \quad (2.1)
\]

The discontinuity is to be taken in the energy variable \(p_0\), by forming the difference of \(R\) and \(A\)-indiced \(P\)-dependent quantities. Within standard notations, the fermionic HTL self energies, effective propagators and vertices are respectively given by

\[
\Sigma_\alpha(P) = m^2 \int \frac{d\vec{K}}{4\pi} \frac{\vec{K}}{K \cdot P + i\epsilon_\alpha}, \quad m^2 = C_F \frac{g^2 T^2}{8}, \quad \alpha = R, A \quad (2.2)
\]
\[ \star S_\alpha(P) = \frac{i}{\slashed{P} - \Sigma_\alpha(P)} \]  

(2.3)

\[ \star \Gamma_\mu(P_\alpha, Q_\beta, P'_\delta) = -ie \left( \gamma_\mu + \Gamma_{\mu}^{HTL}(P_\alpha, Q_\beta, P'_\delta) \right) \]  

(2.4)

\[ \Gamma_{\mu}^{HTL}(P_\alpha, Q_\beta, P'_\delta) = m^2 \int \frac{d\hat{K}}{4\pi} \frac{\hat{K}_\mu \hat{K}}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)} \]  

(2.5)

where \( \hat{K} \) is the lightlike four vector \( (1, \hat{k}) \). In the sequel, it will reveal extremely useful to introduce a ”self energy four vector” (of course, not a genuine Lorentz-4-vector!), by writing, instead of standart expression (2.2),

\[ \Sigma_\alpha(P) = \gamma \cdot \Sigma_\alpha(P) = \gamma_\mu m^2 \int \frac{d\hat{K}}{4\pi} \frac{\hat{K}_\mu}{\hat{K} \cdot P + i\epsilon} \]  

(2.6)

The \( RP \) basic steps entering the soft real photon emission rate calculation of thermal \( QCD \) are as follows. In view of (2.1) and (2.4), one gets three types of terms: A term with two bare vertices \( \Gamma_{\mu}^{(0)} \), two terms with one bare vertex \( \Gamma_{\mu}^{(0)} \) and the other \( \Gamma_{\mu}^{HTL} \), and a term with two \( HTL \) vertices \( \Gamma_{\mu}^{HTL} \). In \( QCD \), the first three terms pose no problem: Terms of second type entail a collinear singularity which, thanks to a \( U(1) \)-Ward identity, cancels out with a similar singularity coming from the last term. A residual collinear singularity remains though, induced by the latter, and we therefore focus on that particular contribution including two vertices \( \Gamma_{\mu}^{HTL} \). One gets,

\[ \Pi_R(Q) = i \int \frac{d^4P}{(2\pi)^4} (1 - 2n_F(p_0)) \text{ disc} \left\{ \star S_R(P) \Gamma_{\mu}^{HTL}(P_R, Q_R, -P'_A) \right\} \]  

(2.7)

Then substituing the relevant \( QCD \) expressions, (2.2)-(2.5), one can write, with the convention \( \epsilon_R = +\epsilon \),

\[ \Pi_R(Q) = -ie^2 m^4 \int \frac{d^4P}{(2\pi)^4} (1 - 2n_F(p_0)) \int \frac{d\hat{K}}{4\pi} \int \frac{d\hat{K}'}{4\pi} \text{ disc} \frac{\hat{K} \cdot \hat{K}' \text{ Tr} \left( \star S_R(P) \hat{K} \star S_R(P') \hat{K}' \right)}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)(\hat{K}' \cdot P + i\epsilon)(\hat{K}' \cdot P' + i\epsilon)} \]  

(2.8)
Because of the factor $\hat{K} \cdot \hat{K'}$ appearing in the numerator, there is no double pole but a simple collinear one at $\hat{K} = \hat{Q}$, whose residue just involves the $U(1)$ Ward identity alluded to above, that is,

$$m^2 \int \frac{d\hat{K'}}{4\pi} \frac{[\hat{Q} \cdot \hat{K}']}{(\hat{K'} \cdot P + i\epsilon)(\hat{K'} \cdot P' + i\epsilon)} = \frac{1}{q'} \left[ \Sigma_R(P) - \Sigma_R(P') \right]$$

(2.9)

and yields for $\Pi_R(Q)$ the expression

$$-i\frac{e^2 m^2}{q} \int \frac{d^4 P}{(2\pi)^4} (1 - 2n_F(p_0)) \int \frac{d\hat{K}}{4\pi} \frac{\delta(\hat{K} \cdot P)}{\hat{K} \cdot Q + i\epsilon}
\times Tr \left( *S_A(P) \hat{Q} \cdot S_R(P') \left[ \Sigma_A(P) - \Sigma_R(P') \right] \right)$$

(2.10)

The discontinuity in $p_0$ can be taken, and an appropriate choice of the integration contour in the $p_0$-complex plane allows to write

$$\Pi_R(Q) = -2 \frac{e^2 m^2}{q} \left( \int \frac{d\hat{K}}{4\pi} \frac{1}{\hat{Q} \cdot \hat{K} + i\epsilon} \right) \int \frac{d^4 P}{(2\pi)^3} \frac{\delta(P \cdot \hat{Q})}{(2\pi)^3} \delta(P \cdot \hat{Q}) (1 - 2n_F(p_0))
\times Tr \left( *S_A(P) \hat{Q} \cdot S_R(P') \left[ \Sigma_A(P) - \Sigma_R(P') \right] \right)$$

(2.11)

where a factor of 2 accounts for the two possibilities $\hat{K} = \hat{Q}$ and $\hat{K'} = \hat{Q}$, and where the relation $P' = P + Q$ has been used. The angular integration develops a collinear singularity at $\hat{K} = \hat{Q}$, and is responsible for that singular part of $\Pi_R(Q)$ which can be expressed as

$$-2 \frac{e^2 m^2}{q} \left( \int \frac{d\hat{K}}{4\pi} \frac{1}{\hat{Q} \cdot \hat{K} + i\epsilon} \right) \int \frac{d^4 P}{(2\pi)^3} \frac{\delta(P \cdot \hat{Q})}{(2\pi)^3} \delta(P \cdot \hat{Q}) (1 - 2n_F(p_0))
\times Tr \left( *S_A(P) \hat{Q} \cdot S_R(P') \left[ \Sigma_A(P) - \Sigma_R(P') \right] \right)$$

(2.12)

The two terms involving one bare vertex $\gamma_\mu$ and a one loop HTL correction $\Gamma^{HTL}_\mu$, entail a similar singularity which, when combined with (2.12), leave unc cancelled the $\Pi_R(Q)$ singular contribution

$$-2i \frac{e^2 m^2}{q^2} \left( \int \frac{d\hat{K}}{4\pi} \frac{1}{\hat{Q} \cdot \hat{K} + i\epsilon} \right) \int \frac{d^4 P}{(2\pi)^3} \frac{\delta(P \cdot \hat{Q})}{(2\pi)^3} \delta(P \cdot \hat{Q}) (1 - 2n_F(p_0))
\times \left[ Tr \left( *S_A(P) \hat{Q} \right) - Tr \left( *S_R(P') \hat{Q} \right) \right]$$

(2.13)

It is this result which, in the literature\textsuperscript{6} is most usually written in the form

$$\frac{C^{st}}{\epsilon} \int \frac{d^4 P}{(2\pi)^4} \delta(\hat{Q} \cdot P) (1 - 2n_F(p_0)) \sum_{s=\pm 1, V=P,P'} \pi(1 - s \frac{u_0}{v}) \beta_s(V)$$

(2.14)
where the overall $1/\varepsilon$ results of a dimensionally regularized evaluation of the factored out angular integration of (2.13), and where $\beta_s(V)$ is related to the effective fermionic propagator usual parametrization\textsuperscript{16},

\[ *S_{R,A}(P) = i \sum_{s=\pm 1} \frac{\hat{P}_s}{D^s_{R,A}(p_0, \vec{p})} \]  

(2.15)

with $\hat{P}_s = (1, s\hat{p})$, the label $s$ referring to the two dressed fermion propagating modes. Then one has,

\[ \frac{1}{D^s_{R,A}(V)} = \alpha_s(V) \mp i\pi \beta_s(V) \]  

(2.16)

III. Self Energy Diagrams, of type $(N, N'; 0)$

The imaginary part of a general term of type $(N, N'; 0)$, depicted in Fig.1, can be written

\[ 2\epsilon^2 \int \frac{d^4 P}{(2\pi)^4} (1 - 2n_F(p_0)) Tr p \text{disc}_P \left( \frac{\Sigma_R(P)p}{p^2 N+1} \right) p' \text{disc}_{P'} \left( \frac{\Sigma_R(P')p'}{p'^2 N'+1} \right) \]  

(3.1)

where the "Self Energy four-vector" (2.2), has components,

\[ \Sigma_0^\alpha(P) = \frac{m^2}{2p} \ln \left( \frac{p_0 + p}{p_0 - p} \right) , \quad \Sigma_i^\alpha(P) = \left( \frac{\hat{P}^i}{p} \equiv \hat{p}^i \right) \frac{m^2}{p} Q_1 \left( \frac{P_0}{p} \right) \]  

(3.2)

with $Q_1$ standing for the Legendre function of the second kind

\[ Q_1(x) = xQ_0(x) - 1 , \quad Q_0(x) = \frac{1}{2} \ln \left( \frac{x+1}{x-1} \right) \]  

(3.3)

The label $\alpha = \{ R, A \}$ denoting one of the two Retarded or Advanced specifications of the real time formalism being used, in the right hand sides of (3.2) these specifications are encoded in the logarithmic determinations.

It is elementary to prove that one has

\[ (\Sigma_R p)^N = a_N \Sigma_R p + b_N \mathbb{I}_4 \]  

(3.4)

where $\mathbb{I}_4$ is the $4 \times 4$ identity matrix, and the coefficients $a_N$ and $b_N$ are polynomials in the variables $p \cdot \Sigma_R(P) = m^2$ and $-P^2\Sigma_R^2$ whose formation laws can be found to be given by

\[ a_N = (m^2)^{N-1} \sum_{j=0}^{j_{N+1}} C^j_N \left( 1 - \frac{p^2 \Sigma^2}{m^4} \right)^{k} \]  

(3.5)
\[ b_N = (m^2)^N \left( -\frac{P^2 \Sigma^2}{m^2} \right)^{j_M(N-1)} \sum_{j=0}^{j_M(N-1)} C_{N-1}^{2k+1} \left( 1 - \frac{P^2 \Sigma^2}{m^4} \right)^k \]  

(3.6)

The \( C_{N}^{2k+1} \) are the binomial coefficients, and \( j_M \), the maximal value of \( j \) can be expressed as

\[ j_M(N) = \frac{(N - 1 - \Theta((-1)^N))}{2} \]  

(3.7)

where \( \Theta(x) \) is the usual Heaviside step function.

Because of the decomposition law (3.4), four types of trace factors are found, that are

\[ 4P \cdot P' \]  

(3.8)

\[ 8m^2P \cdot P' - 4P^2P' \cdot \Sigma_R(P) \quad (P \leftrightarrow P') \]  

(3.9)

\[ (3.9) \quad (P \leftrightarrow P') \]  

(3.9')

\[ (2m^2)^2 4P \cdot P' - \left( 8m^2P^2P \cdot \Sigma_R(P') + (P \leftrightarrow P') \right) + 4P^2P'^2 \Sigma_R(P) \cdot \Sigma_R(P') \]  

(3.10)

We will therefore begin with proving that integrals of the generic type

\[ \int \frac{d^4P}{(2\pi)^4} (1 - 2n_F(p_0)) \text{disc}_P \left( \frac{(-P^2 \Sigma^2 R(P))^n}{(P^2 + i\epsilon p_0)^{N+1}} \right) \text{disc}_{P'} \left( \frac{(-P'^2 \Sigma^2 R(P'))^{n'}}{(P'^2 + i\epsilon p'_0)^{N'+1}} \right) \]  

(3.11)

are mass singularity free, or msf, for short. Then, since all of the trace factors (3.8)-(3.10) come into play as multiplicative functions of the integrands appearing in (3.11), we will check that they leave unaltered its msf character.

Integrals of generic type (3.11) : With \( y = \hat{q} \cdot \hat{p} \), where \( \hat{q} \) and \( \hat{p} \) are the unit three-vectors in the directions of \( \vec{q} \) and \( \vec{p} \) respectively, integration on \( y \) can be traded for an integration on the virtuality \( P'^2 = -x'p'^2(y) \) by writing,

\[ \int_{P'^2 + 2q_0p_0}^{1} \frac{dy}{2q_0} y \int_{0}^{1} \frac{dx'}{(1 - x')^2} \]  

(3.12)

where the restrictions on \( y \) and \( x' \) come from the \( \Theta(-P'^2) \) support of the distribution to be folded in (3.11). Now, particular to the thermal context\(^\dagger\), so called Lebesgue non-integrable mass (and/or IR) singularities do arise, which cannot be taken care of by means of a standard dimensional regularization procedure, and require that an extra IR regulator be introduced\(^\ddagger\).

This is achieved by proceeding to the following replacement

\[ \frac{1}{(P'^2 + i\epsilon p'_0)^{N'+1}} \mapsto \frac{1}{(P'^2 - \mu^2 + i\epsilon p'_0)^{N'+1}} \]  

(3.13)

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that is also,
\[ \delta^{(N')}(P^2) \mapsto \delta^{(N')}(P'^2 - \mu^2), \quad \frac{P}{(P^2)^{N'+1}} \mapsto \frac{P}{(P'^2 - \mu^2)^{N'+1}} \]  
(3.14)

where, as shown in Ref. 7, Appendix B, the auxiliary IR regulator $\mu^2$, is chosen a small, negative parameter, to be taken to zero in the end. Gathering pieces, integration on $y$ can eventually be written as
\[
\frac{(m^4)^{n'}}{(-\mu^2 + (p_0 + q)^2)^{N'+1}} \left( -1 \right)^{N'+1} \int_0^1 \frac{dx'}{(1-x')^2} \left( 1 - x' \right)^{N'+1} \left( x' \right)^{n'} \nonumber
\]
\[
\left\{ \frac{P}{(x' - \lambda)^{N'+1}} \right\} \Im \left( -1 + \frac{x'}{4} [\epsilon(p_0') \ln X']^2 + \sqrt{1 - x'} [\epsilon(p_0') \ln X'] \right)^{n'} \nonumber
\]
\[
+ \pi \epsilon(p_0') \frac{(-1)^N}{N!} \delta^{(N')} \left( x' - \lambda' \right) \Re \left( -1 + \frac{x'}{4} [\epsilon(p_0') \ln X']^2 + \sqrt{1 - x'} [\epsilon(p_0') \ln X'] \right)^{n'} \}
\]  
(3.15)

where we have defined
\[
\lambda' \overset{\text{def}}{=} \frac{-\mu^2}{-\mu^2 + (p_0 + q)^2}, \quad X' \overset{\text{def}}{=} \frac{\epsilon(p_0') \sqrt{1 - x' + 1}}{\epsilon(p_0') \sqrt{1 - x' - 1}} \]  
(3.16)

and where $\epsilon(p_0')$ is the distribution "sign of $p_0'". The remaining two integrations are on $p = |\vec{p}|$, and $p_0$, and the latter can be translated into an integration on the virtuality variable $x = -P^2/p^2$.

Now, if we consider the integral,
\[
\int_{-p}^{+p} dp_0 \left( 1 - 2n_F(p_0) \right) \text{disc}_P \left( \frac{(-P^2 \Sigma_R^2(P))^{n'}}{(P^2 + i\epsilon p_0)^{N+1}} \right) \]  
(3.17)

which enters (3.11) as a building block:
\[
(3.11) = \int \frac{p^2 dp}{(2\pi)^3} \int_{-p}^{+p} dp_0 \left( 1 - 2n_F(p_0) \right) \text{disc}_P \left( \frac{(-P^2 \Sigma_R^2(P))^{n'}}{(P^2 + i\epsilon p_0)^{N+1}} \right) \times (3.15) \]  
(3.18)

we get for (3.17) the expression,
\[
\frac{1}{2} p \left( \frac{-1}{p^2} \right)^{N+1} (m^4)^n \sum_{\epsilon(p_0) = \pm 1} \int_0^1 \frac{dx}{\sqrt{1 - x}} \left( 1 - 2n_F(\epsilon(p_0)p\sqrt{1 - x}) \right) x^n \left\{ \frac{P}{(x - \lambda)^{N+1}} \right\} \nonumber
\]
\[
\times \Im \left( -1 + \frac{x}{4} [\epsilon(p_0) \ln X]^2 + \sqrt{1 - x} [\epsilon(p_0) \ln X] \right)^n \nonumber
\]
\[
+ \pi \epsilon(p_0) \frac{(-1)^N}{N!} \delta^{(N)}(x - \lambda) \nonumber
\]
\[
\times \Re \left( -1 + \frac{x}{4} [\epsilon(p_0) \ln X]^2 + \sqrt{1 - x} [\epsilon(p_0) \ln X] \right)^n \}
\]  
(3.19)
with the definitions,
\[
\lambda \overset{\text{def}}{=} -\mu^2 / p^2, \quad X \overset{\text{def}}{=} \frac{\epsilon(p_0)\sqrt{1 - x} + 1}{\epsilon(p_0)\sqrt{1 - x} - 1}
\]

Eventually, an integration on \(p\) must be performed, which can symbolically be written as
\[
\int_{p_m}^{p^*} \frac{p^2 dp}{(2\pi)^3} \times G(p, q, m)
\]

An upper bound of integration on \(p\) is introduced so as to avoid the hard region \(p = O(T)\) (a customary choice consists in taking \(p^*\) on the order of an intermediate scale, say, on the order of \(\sqrt{gT}\)), whereas the lower boundary, \(p_m\), not relevant to our concern here, will be discussed elsewhere\(^\text{19}\). Note that in (3.15) and (3.19), we have written between brackets expressions of the form \([\epsilon(p_0)\ln X]\). This is because, irrespective of the sign of \(p_0\), these expressions can be written with the help of a most efficient representation\(^\text{7,11,13}\)
\[
\epsilon(p_0)\ln X(p_0) = \epsilon(-p_0)\ln X(-p_0) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( 1 - \frac{x^\varepsilon e^{i\pi \varepsilon}}{(1 + \sqrt{1 - x})^{2\varepsilon}} \right)
\]

Thanks to the \(x^\varepsilon\) factor, this representation is able to provide mass/collinear singularities with the same regularization as a dimensional one would operate, while being far simpler. It is also endowed with interesting regularity properties, since, in particular, the limit \(\varepsilon = 0\), commutes with both the sum over \(N\) and integral on \(p_0\)\(^7\).

Using (3.22), one obtains an expansion
\[
\left( -1 + \frac{x}{4}[\epsilon(p_0)\ln X]^2 + \sqrt{1 - x}[\epsilon(p_0)\ln X] \right)^n = \sum_{i=0}^{n} C_n^i (-1)^i \left( \frac{x}{4} \right)^{n-i} \sum_{k=0}^{i} C_i^k (-1)^k \sqrt{1 - x}^k \times \frac{1}{\varepsilon^{2(n-i)+k}} \sum_{m=0}^{2(n-i)+k} \frac{C_{2(n-i)+k}^m (-1)^m e^{i\pi m \varepsilon} x^m \varepsilon}{(1 + \sqrt{1 - x})^{2m \varepsilon}}
\]

which can be put back into (3.19). Introducing the following family of functions
\[
\mathcal{F}_{k-1, m}(m \varepsilon, x) = \frac{\sqrt{1 - x}^{k-1}}{(1 + \sqrt{1 - x})^{2m \varepsilon}} (1 - 2n_F(\epsilon(p_0)p\sqrt{1 - x}))
\]

we note that it is convenient to proceed as for the moving fermion damping rate problem\(^7\), keeping the leading order term of the statistical weight high temperature expansion
\[
\mathcal{F}_{k-1, m}(m \varepsilon, x) = \frac{p \epsilon(p_0)}{2T} \frac{\sqrt{1 - x}^k}{(1 + \sqrt{1 - x})^{2m \varepsilon}} (1 + O(g^2)) \quad \overset{\text{def}}{=} \frac{p \epsilon(p_0)}{2T} F_{km}(m \varepsilon, x) (1 + O(g^2))
\]
Though no way mandatory (the same results being obtained otherwise), this simplification is consistent with the leading order calculation we are concerned with, preserves the correct parity in $p_0$, and allows to recognize in $F_{km}$ the same expression as defined in Ref.7, Eq.(7.2). Whereof we know that (3.19) is rigorously \textit{integrable} and \textit{non-integrable} mass singularity free: The expression (3.19) can effectively be written as

$$
\frac{1}{4T} p^2 \left( \frac{-1}{p^2} \right)^{N+1} (m^4)^n \sum_{\epsilon(p_0) = \pm 1} \sum_{i=0}^{n} C_n^i (-1)^i \left( \frac{1}{4} \right)^{n-i} \sum_{k=0}^{i} C_i^k (-1)^k \frac{1}{\epsilon^{2(n-i)+k}} \sum_{m=0}^{2(n-i)+k} C_{2(n-i)+k}^m (-1)^m \int_0^1 dx \left\{ \text{Im}(e^{i\pi m \epsilon}) \right\} x^{2n-i+m \epsilon} \times F_{km}(m \epsilon, x)
$$

which is Eq.(D.3) of Ref.7. Mass singularities of the \textit{non-integrable} type, $\mathcal{O}(1/\lambda)^k$, cancel out

$$
\int_0^1 dx \left\{ \text{Im}(e^{i\pi m \epsilon}) \right\} x^{2n-i+l+m \epsilon} \times F_{km}(m \epsilon, x)
$$

whereas \textit{integrable} mass singularities obey arithmetical cancellation patterns thanks to the identities

$$
\left\{ \frac{\epsilon^p}{\epsilon^j} \right\} \times \sum_{m=0}^{j} C_j^m (-1)^m m^p = 0, \quad 1 \leq p \leq j - 1
$$

$$
\sum_{m=0}^{j} C_j^m (-1)^m m^j = (-1)^j j!
$$

It can even be shown (Appendix D of Ref.7) that (3.26) defines a mapping of $C \times C$ into $C$ which is analytic for $(\epsilon, \lambda)$ choosen in the product of discs $D(0, \frac{1}{2N}) \times D(0, \frac{1}{2})$. The limit $\epsilon = 0, \lambda = 0$ therefore exists and is independent of the sequence along which it is taken.

The entirely new feature is of course that the integrand appearing in (3.19), gets supplied, now, with the extra function (3.15). Considered as a function of $x$, the properties of (3.15) are therefore crucial in order to address the ensuing behaviour of generic type (3.11) integrals, and this is what we now turn to examine in the particular case of positive energies, $p_0 > 0$, for the sake of a simpler illustration.
As made obvious by inspection, (3.15) is essentially relevant of the same structure as displayed by (3.19): Up to an overall multiplicative function of \( p_0 \),

\[
\frac{(m^4)^{n'}}{(-\mu^2 + (p_0 + q)^2)^{N'+1}} \frac{(p_0 + q)^2}{2q} \quad (3.30)
\]

the difference is entirely in the integration range

\[
0 \leq x' \leq x'_M(x) \overset{def}{=} 1 - \frac{(p_0(x) + q)^2}{(p + q)^2} \quad (3.31)
\]

instead of \( 0 \leq x \leq 1 \). Using the representation (3.22) for the expression \([\epsilon(p'_0)\ln X']\), and the binomial expansion (3.23), the same functions as in (3.25) can effectively be identified, with accordingly, the same properties

\[
F_{2(N'-1)+k', m'} (m'\epsilon', x') = \frac{\sqrt{1 - x'}^{2(N'-1)+k'}}{(1 + \sqrt{1 - x'})^{2m'\epsilon'}} \quad (3.32)
\]

In the limit of \( \lambda' = 0 \), we learn out of Ref.7, that Lebesgue non-integrable mass singularities cancel out: Up to the overall multiplicative factor (3.30), one is left for the full expression (3.15), with an expression which still displays a finite series of Lebesgue integrable mass singularities:

\[
\sum_{l'=0}^{\infty} \frac{1}{l'!} \sum_{n'=0}^{n'} \frac{C_{n'}^{l'} (1)^{n'}}{m'=2(n'-i'+l')} \sum_{k'=0}^{l'} C_{k'}^{l'} (-1)^{k'} \left( g(p_0) \frac{(q + p)^2}{-P^2} \right)^{N'-(2n'-i'+l')}
\]

\[
\times \frac{1}{\epsilon^{2(n'-i')+k'}} \sum_{m'=0}^{m'=2(n'-i')+k'} C_{2(n'-i')+k'}^{m'} (-1)^{m'} \frac{\sin(\pi m' \epsilon')}{N'-(2n'-i'+l')-m'\epsilon'}
\]

\[
\times \left( g(p_0) \frac{(q + p)^2}{-P^2} \right)^{-m'\epsilon'} F_{2(n'-1)+k', m'} (m'\epsilon', 0) \quad (3.33)
\]

where we have defined the function,

\[
g(p_0) \overset{def}{=} \frac{p + p_0}{p + p_0 + 2q} = 1 - \frac{2q}{2q + p} \left( \frac{p}{2q + p} \sqrt{1 - x} \right)^{-1} \quad (3.34)
\]

Type \( O(\epsilon'^{-r}) \)-mass singularities are thus controlled by the finite sum,

\[
\left( \frac{p}{p + q} \right)^{m'\epsilon'} \left( g(p_0) \right)^{-m'\epsilon'} F_{2(n'-1)+k', m'} (m'\epsilon', 0) \quad (3.35)
\]
where the last term appearing in the right hand side of (3.35), stands for the \( l^{th} \)-order derivative of the function (3.32), taken at \( x' = 0 \). This means that, in deriving (3.35), we have interchanged the sum on \( l' \), in the Taylor expansion of (3.32),

\[
F_{2(N'-1)+k', \ m'} (m' \varepsilon', x') = \sum_{l'=0}^{\infty} \frac{(x')^{l'}}{l'!} F_{2(N'-1)+k', \ m'} (m' \varepsilon', 0) \tag{3.36}
\]

with the integration on \( x' \). Such a permutation is proven to be licit in Ref.7, Appendix C (note that in the present situation, this permutation is the more licit, as the integration range (3.31) lies within the unit convergence radius of the series expansion for the functions \( F_{2(N'-1)+k', \ m'} (m' \varepsilon', x') \)). Likewise, it is demonstrated (Eqs. (C.7)-(C.9), (C.12) of Ref.7) that each of the coefficients \( F_{2(N'-1)+k', \ m'} (m' \varepsilon', 0) \) admits a Taylor series expansion in the parameter \( m' \varepsilon' \). Now, whatever \( N' \) and \( 2n' - i' + l' \), the same property holds clearly true, for any of the other three factors of (3.35) which, with \( F_{2(N'-1)+k', \ m'} (m' \varepsilon', 0) \), enter the sum over \( m' \).

Then, forming the Cauchy’s product of their \( m' \varepsilon' \)-Taylor series expansions, and relying on the set of arithmetical identities (3.28) and (3.29), we conclude that the \( \varepsilon' = 0 \)-limit of (3.35) is msf, and reduces to a polynomial of degree \( 2(n' - i') + k' \) in the variable \( \ln(x/g(p_0)) \),

\[
\lim_{\varepsilon' \to 0} (3.35) = (-1)^{2(n' - i') + k'} \sum_{j'=0}^{2(n'-i')+k'} C_{2(n'-i')+k'}^{j'} H_{2(n'-i')+k'-j'}^{j'}(0) \ln^{j'} \left( \frac{x}{g(p_0)} \right) \tag{3.37}
\]

where \( H^{2(n'-i')+k'-j'}(0) \), a pure (real) number, is a shorthand notation for the derivative of order \( 2(n' - i') + k' - j' \), taken at \( m' \varepsilon' = 0 \), of the product

\[
\frac{\sin(\pi m' \varepsilon')}{N' - (2n' - i' + l') - m' \varepsilon'} \times \left( \frac{p}{p + q} \right)^{m' \varepsilon'} \times F_{2(N'-1)+k', \ m'} (m' \varepsilon', 0) \tag{3.38}
\]

Gathering all pieces, the whole expression (3.15) can eventually be written as

\[
\frac{(m^4)^{n'}}{(-\mu^2 + (p_0 + q)^2)^{N'+1}} \frac{(p_0 + q)^2}{2qp} \times \sum_{l'=0}^{\infty} \frac{(-1)^l'}{l'!} \sum_{i'=0}^{n'} C_{n'}^{i'} \left( g(p_0) \frac{(q + p)^2}{p^2} \right)^{N' - (2n' - i' + l')} \times \sum_{k'=0}^{i'} C_{k'}^{i'} \sum_{j'=0}^{2(n'-i')+k'} H^{2(n'-i')+k'-j'}(0) \ln^{j'} \left( \frac{x}{g(p_0)} \right) \tag{3.39}
\]
Getting back to (3.18), one is now in a position so as to estimate the incidence on (3.19) of any of the extra $x$-dependences which are introduced by (3.39).

For positive (as well as negative) energies, the auxiliary IR regulator $-\mu^2$ can safely be taken to zero in the prefactor of (3.39), and the latter expanded as

$$\frac{-(m^4)^{n'}}{(-\mu^2 + (p_0 + q)^2)^{N'+1}} \frac{(p_0 + q)^2}{2qp} = -\frac{(m^4)^{n'}}{2pq} \sum_{r=0}^{\infty} C_k(N', p, q) \sqrt{1 - x^r} \quad (3.40)$$

where it easy to check that the existence of (3.40) does not depend on the relative magnitude of $p$ and $q$, contrarily, of course, to the explicit form of the coefficients $C_k(N', p, q)$. The same property is obviously shared by the function

$$g(p_0)^{N'-(2n'-i'+l')} = \left(1 - \left(\frac{2q}{2q + p}\right) \frac{1}{1 + \frac{p}{2q + p} \sqrt{1 - x}}\right)^{N'-(2n'-i'+l')} \quad (3.41)$$

Eventually, such is also the case of factors like

$$\left(\ln \frac{1}{g(p_0)}\right)^{r'} = (-1)^r' \ln r' \left(1 - \left(\frac{2q}{2q + p}\right) \frac{1}{1 + \frac{p}{2q + p} \sqrt{1 - x}}\right) \quad (3.42)$$

The product of terms (3.40)-(3.42) can therefore be written as a series in the variable $\sqrt{1 - x}$, whose general term, no matters how complicated, just redefines the integer power $k$ of the function $F_{km}(m\varepsilon, x)$ introduced in (3.25). The properties of (3.25) are thus left the same, and the extra $x$-dependences introduced through (3.40)-(3.42) preserve the msf character of (3.19).

The extra factors of (3.39),

$$\left(\frac{(q + p)^2}{P^2}\right)^{N'-(2n'-i'+l')} \quad (3.43)$$

redefine the power $N + 1$ of the scalar propagator appearing in (3.11), according to the replacement

$$\frac{1}{(P^2 - \mu^2 + i\epsilon p_0)^{N+1}} \rightarrow \frac{1}{(P^2 - \mu^2 + i\epsilon p_0)^{N+1+N'-(2n'-i'+l')}}$$

This splits into the distributions

$$\delta^{(N+N'-(2n'-i'+l'))}(P^2 - \mu^2), \quad \frac{P}{(P^2 - \mu^2)^{N+1+N'-(2n'-i'+l')}}$$

which, with respect to the previous power of $N + 1$, require extra differentiability of the $x$-dependences they act upon. Now, this condition is clearly met thanks both to a full identification of the new $x$-dependences brought about by (3.39), and to the introduction of the
auxiliary $IR$ regulator $\lambda = -\mu^2/p^2$ of (3.20). Since, at $N \geq 2n - i + l + 1$ (which is just the condition for the occurrence of mass singularities), the overall compensation of mass singularities does not depend on the relative magnitude of the integers $N + 1$ and $2n - i + l$, in mass singularity compensation patterns of the generic type\textsuperscript{7},

$$
\lim_{\varepsilon,\lambda=0} \frac{1}{\varepsilon^{2(n-i)+k}} \sum_{m=0}^{2(n-i)+k} C_{2(n-i)+k}^{m} (-1)^{m} F_{kw}^{(l)} (m\varepsilon,0) \int_0^1 \frac{dx}{x-\lambda} \left\{ \Im \left( \frac{1}{(x-\lambda)^{N+1}} \right) \right\} + \Re \left( \frac{1}{x-\lambda} \right) \pi \varepsilon (p_0) \frac{(-1)^N}{N!} \delta (N) (x-\lambda) \right] x^{2n-i+l+m\varepsilon} = \mathcal{O}(1) \quad (3.44)
$$

we deduce that the same mass singularity compensations hold true of (3.44), with $N + 1 + N' - (2n' - i' + l')$ replacing $N + 1$, and that extra factors of type (3.43) are $msf$-preserving.

Eventually, the last extra $x$-dependences introduced into (3.18) by (3.39), are the functions

$$
\ln r' x, \quad r' \in N, \quad 0 \leq r' \leq 2(n' - i') + k'
$$

Previous patterns (3.44) are now taken to the form

$$
\lim_{\varepsilon,\lambda=0} \frac{1}{\varepsilon^{2(n-i)+k}} \sum_{m=0}^{2(n-i)+k} C_{2(n-i)+k}^{m} (-1)^{m} F_{kw}^{(l)} (m\varepsilon,0) \int_0^1 \frac{dx}{x-\lambda} \left\{ \Im \left( \frac{1}{(x-\lambda)^{N+1}} \right) \right\} + \Re \left( \frac{1}{x-\lambda} \right) \pi \varepsilon (p_0) \frac{(-1)^N}{N!} \delta (N) (x-\lambda) \right] x^{2n-i+l+m\varepsilon} \ln r' x \quad (3.46)
$$

As in (3.22), we introduce the representation

$$
\ln r' x = \lim_{\varepsilon=0} \frac{1}{\varepsilon^{r'}} \sum_{s'=0}^{r'} C_{r'}^{s'} (-1)^s' (x)^{s'} \quad (3.47)
$$

and interchange the sum on $s'$, which is finite, with the integration on $x$. In the limit $\lambda = 0$, we get first the expression\textsuperscript{7},

$$
\frac{(-1)^{r'}}{\varepsilon^{r'}} \sum_{s'=0}^{r'} C_{r'}^{s'} (-1)^{s'} \frac{-\sin (\pi m \varepsilon)}{N - (2n - i + l) + m \varepsilon + s' \varepsilon} + \mathcal{O}(\lambda) \quad (3.48)
$$

At $N \geq 2n - i + l + 1$, such a factor admits an $s' \varepsilon$-Taylor series expansion, so that relying on arithmetical identities (3.28) and (3.29), the $\varepsilon = 0$-limit of (3.48) is readily obtained to be given by

$$
\left( \frac{-\sin (\pi m \varepsilon)}{N - (2n - i + l) + m \varepsilon + s' \varepsilon} \right)^{(r')} \bigg|_{s' \varepsilon = 0} = (-1)^{r'} r! \left( \frac{-\sin (\pi m \varepsilon)}{N - (2n - i + l) + m \varepsilon} \right)^{r'+1} \quad (3.49)
$$
At its turn, (3.49) admits itself a Taylor series expansion in the variable \( m\varepsilon \). Since the whole expression (3.46) factors out a global factor of

\[
\frac{1}{\varepsilon^{2(n-i)+k}} \sum_{m=0}^{2(n-i)+k} C_{2(n-i)+k}^m (-1)^m F_{km}^{(l)}(m\varepsilon,0)
\]

the \( \varepsilon = 0 \)-limit of (3.46) is finite in view, again, of arithmetical identities (3.28) and (3.29), and extra factors of type (3.45) are \( msf \)-preserving too.

We thus reach the conclusion that generic type (3.11) integrals are \( msf \). Now, getting back to the mass singularity issue of \((N,N';0)\) self energy diagrams, it is immediate to realize that all of the trace factors (3.8)-(3.10) only involve \( x(x'), \sqrt{1-x}(\sqrt{1-x'}) \), and \([\epsilon(p_0)\ln X]/[\epsilon(p'_0)\ln X']\) \( msf \)-preserving dependences (some of them will be treated in full details in next Section 5), as, for example, the most involved piece of (3.10)

\[
8(2m^2)^2 (\hat{K} \cdot P)(\hat{K} \cdot P') - 8(2m^2)^2 (P^2 \hat{K} \cdot P' \hat{K} \cdot \Sigma_R(P)) + (P \leftrightarrow P')
\]

Since the same analysis can be carried through in the case of negative energies, we can conclude that \((N,N';0)\) self energy diagrams have collinear singularity free imaginary parts.

**IV. Diagrams of type \((N,N';1)\), with one effective vertex**

Using (3.4), the trace factors associated with diagrams of type \((N,N';1)\), depicted in Fig.2, are easily obtained to be,

\[
8 (\hat{K} \cdot P)(\hat{K} \cdot P')
\]

\[
16m^2 (\hat{K} \cdot P)(\hat{K} \cdot P') - 8P^2 \hat{K} \cdot P' \hat{K} \cdot \Sigma_R(P)
\]

\[
(4.2) (P \leftrightarrow P')
\]

\[
8(2m^2)^2 (\hat{K} \cdot P)(\hat{K} \cdot P') - 8(2m^2)^2 \left( P^2 \hat{K} \cdot P' \hat{K} \cdot \Sigma_R(P) + (P \leftrightarrow P') \right)
\]

\[
+ 8P^2P'^2 \hat{K} \cdot \Sigma_R(P) \hat{K} \cdot \Sigma_R(P')
\]
In \((N, N'; 1)\)-type diagrams, each of the trace factors (4.1)-(4.3) must be integrated over \(\hat{K}\) with the "measure",

\[
m^2 \int \frac{d\hat{K}}{4\pi} \frac{1}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)} \tag{4.4}
\]

Then the above trace factors yield respectively

\[
8m^2 \tag{4.5}
\]

\[
16m^4 - 8P^2 \Sigma_R^2(P) \tag{4.6}
\]

\[
(4.6) \quad (P \leftrightarrow P') \tag{4.6'}
\]

\[
8m^2 \left( (2m^2)^2 - 2 \left( P^2 \Sigma_R^2(P) + (P \rightarrow P') \right) + P^2 P'^2 \int \frac{d\hat{K}}{4\pi} \frac{\hat{K} \cdot \Sigma_R(P) \hat{K} \cdot \Sigma_R(P')}{\hat{K} \cdot P + i\epsilon \hat{K} \cdot P' + i\epsilon} \right) \tag{4.7}
\]

One may observe that the trace factors of \((N, N'; 0)\) diagrams are more involved than those attached to diagrams of type \((N, N'; 1)\). In particular, it should be clear that at the exception of the last term of (4.7), all of the factors appearing in (4.5)-(4.7) will preserve the msfcharacter of type (3.11) integrals.

That is, the whole mass singularity issue of \((N, N'; 1)\) contributions is entirely in the incidence, upon generic type (3.11) integrals, of the very function

\[
\int \frac{d\hat{K}}{4\pi} \frac{\hat{K} \cdot \Sigma_R(P)}{\hat{K} \cdot P + i\epsilon} \frac{\hat{K} \cdot \Sigma_R(P')}{\hat{K} \cdot P' + i\epsilon} \tag{4.8}
\]

As it stands, (4.8) can be calculated with the help of the three angular identities,

\[
\int \frac{d\hat{K}}{4\pi} \frac{\hat{K}^0 \hat{K}^0}{(\hat{K} \cdot R + i\epsilon)^2} = \frac{1}{R^2 + i\epsilon r_0} \tag{4.9}
\]

\[
\int \frac{d\hat{K}}{4\pi} \frac{\hat{K}^0 \hat{K}^i}{(\hat{K} \cdot R + i\epsilon)^2} = \bar{r}^i \left( \frac{-1}{2r^2} \ln \left( \frac{r_0 + r}{r_0 - r} \right) + \frac{r_0}{r} \frac{1}{R^2 + i\epsilon r_0} \right) \tag{4.10}
\]

\[
\int \frac{d\hat{K}}{4\pi} \frac{\hat{K}^i \hat{K}^j}{(\hat{K} \cdot R + i\epsilon)^2} = -\frac{g^{ij}}{r^2} Q_1 \left( \frac{r_0}{r} \right) - \bar{r}^i \bar{r}^j \left( \frac{3}{r^2} Q_1 \left( \frac{r_0}{r} \right) - \frac{1}{R^2 + i\epsilon r_0} \right) \tag{4.11}
\]

where a Feynman parameter, \(s\), has been introduced so as to re-write (4.8) as

\[
\Sigma_\mu(P) \Sigma(P') \nu \int_0^1 ds \int \frac{d\hat{K}}{4\pi} \frac{\hat{K}^\mu \hat{K}^\nu}{(\hat{K} \cdot R(s) + i\epsilon)^2} \tag{4.12}
\]

with,

\[
R(s) = P + sQ \tag{4.13}
\]
At this point, and though not immediately relevant to our concern, the following remark may be in order.

Some years ago, the use of a Feynman parametrization in Thermal Quantum Field Theories has been questioned\textsuperscript{20}. Feynman parametrization was suspected delicate, using, for example, bare propagator determinations different from the usual $+i\epsilon$-Feynman’s one. Of course, passing from (4.8) to (4.12), this situation is not encountered, but the difficulty may come about, in particular in a real time formalism using Retarded/Advanced propagator prescriptions. The solution to this difficulty has been given in Ref.\textsuperscript{21}. Later on, it has even been suggested that using a Feynman parametrization in a hot quantum field context could lead to non gauge invariant results\textsuperscript{22}. This latter statement however was erroneous, based on incorrect calculations, and indeed, taking the modification of Ref.\textsuperscript{21} into account, it must be stated that there is definitely no problem in dealing with Feynman parametrizations in non zero Temperature Quantum Field Theories.

Getting back to (4.8), a shortcut to its calculation consists in writing,
\begin{equation}
\hat{K} \cdot \Sigma_R(P) = \frac{m^2}{p^2} Q_1 \left( \frac{p_0}{p} \right) + \frac{m^2}{p} \left( \frac{p_0}{p} - \frac{1}{2} \frac{P^2}{p^2} \ln X \right) \frac{1}{\hat{K} + i\epsilon} \tag{4.14}
\end{equation}
from which a remarkable relation may be deduced,
\begin{equation}
m^2 \int \frac{d\hat{K}}{4\pi} \frac{\hat{K} \cdot \Sigma_R(P)}{\hat{K} + i\epsilon} = \Sigma_R^2(P) \tag{4.15}
\end{equation}
and likewise, in obvious notations,
\begin{equation}
(4.8) = \frac{m^2}{p^2} Q_1 \frac{m^2}{p^2} Q_1' + \left( \frac{m^2}{p^2} Q_1 \frac{m^2}{p'} Q_1' \left( \frac{p_0}{p'} - \frac{1}{2} \frac{P'^2}{p'^2} \ln X' \right) \frac{1}{2p'} \ln X' + (P \leftrightarrow P') \right) + \frac{m^2}{p} \left( \frac{p_0}{p} - \frac{1}{2} \frac{P^2}{p^2} \ln X \right) \frac{m^2}{p'} \left( \frac{p_0'}{p'} - \frac{1}{2} \frac{P'^2}{p'^2} \ln X' \right) \frac{1}{2Q \cdot P + i\epsilon q_0} \ln \left( \frac{P'^2 + i\epsilon q_0'}{P^2 + i\epsilon p_0} \right) \tag{4.16}
\end{equation}
where identity (4.9) only, has been used. Noting that $2Q \cdot P = P'^2 - P^2$, one recovers in (4.16) the full original symmetry of (4.8) under the exchange $P \leftrightarrow P'$. Since the terms appearing in (4.16) just redefine the integer numbers $k(k')$, $n(n')$ and $2(n - i) + k (2(n' - i') + k')$, they leave totally unaffected the $msf$ structures of previous Section 3. The only new feature is the factor $(2Q \cdot P)^{-1} \ln P'^2/P^2$. As observed in Ref.\textsuperscript{13} for the topology $(1, 1; 1)$, this factor is reminiscent of the collinear singularity plagued $(N, N'; 1)$ diagrams at the light cone, when an $RP$ treatment of the problem is adopted.
In the end, recalling that (4.8) comes out affected with a multiplicative factor of \(8m^2P^2P'^2\), this means that expressions

\[
\int \frac{d^4P}{(2\pi)^4} (1 - 2n_F(p_0)) \left(\frac{(-P'^2\Sigma_R^2(P')^n')}{(P'^2 + i\epsilon p'_0)^{N'}}\right) \text{disc} \left(\frac{(-P^2\Sigma_R^2(P))^n}{(P^2 + i\epsilon p_0)^N}\right) \int \frac{d^4K}{4\pi} \frac{\hat{K} \cdot \Sigma_R(P)}{\hat{K} \cdot P + i\epsilon} \frac{\hat{K} \cdot \Sigma_R(P')}{\hat{K} \cdot P' + i\epsilon} \tag{4.17}
\]

do have msf imaginary parts, if integrals

\[
\int \frac{d^4P}{(2\pi)^4} (1 - 2n_F(p_0)) \left(\frac{(-P'^2\Sigma_R^2(P'))^n'}{(P'^2 + i\epsilon p'_0)^{N'}}\right) \text{disc} \left(\frac{(-P^2\Sigma_R^2(P))^n}{(P^2 + i\epsilon p_0)^N}\right) \frac{1}{2Q \cdot P + i\epsilon q_0} \ln \frac{P^2 + i\epsilon p'_0}{P^2 + i\epsilon p_0} \tag{4.18}
\]

have msf imaginary parts either. That it is so can be demonstrated quite easily. However, a byproduct of the next section will provide this statement with a systematic derivation, so that we can here content ourselves with a heuristic, still instructive argument.

The potential collinear singularity due to the HTL vertex comes from the factor \((1/2Q \cdot P)\), as \(Q \cdot P\) reaches zero. For example, in the \(RP\) calculation of Sec.2, we learn out of Eqs.(2.12)-(2.14), that the collinear singularity expression effectively involves a \(\delta(P \cdot Q)\) constraint. Now, as \(Q \cdot P\) tends to zero, one has indeed

\[
\frac{1}{2Q \cdot P + i\epsilon q_0} \ln \frac{P^2 + 2Q \cdot P + i\epsilon p'_0}{P^2 + i\epsilon p_0} \simeq \frac{1}{P^2 + i\epsilon p_0} \tag{4.19}
\]

and this light cone potentially singular behaviour obviously gets mixed with partial effective propagator \(S_R^{(N)}(P)\) own light cone potentially singular behaviour,

\[
P^2 S_R^{(N)}(P) = P^2 \frac{dP \Sigma_R(P) P^N}{(P^2 + i\epsilon p_0)^{N+1}} \tag{4.20}
\]

the whole just boiling down to a simple shift of power,

\[
P^2 \frac{1}{(P^2 + i\epsilon p_0)^{N+1}} \rightarrow P^2 \frac{1}{(P^2 + i\epsilon p_0)^{N+2}}
\]

From previous Sec.3, Eqs. (3.43)-(3.44), we know that the overall detailed balance compensation of mass singularities is preserved by such a shift, and this is how we can see that \((N, N'; 1)\) contributions to the soft real photon emission rate are msf.

This generalizes to any \((N, N'; 1)\) contribution, the observation first made in Ref.13, Sec.5, for the diagram \((1, 1; 1)\), and simply enforces the conclusion we drew then, that HTL vertex
collinear singularities should not be desentangled from partial effective propagator mass singularities, as they all mix up into structural patterns which grant their overall compensations. In an \( RP \) resummation scheme, unfortunately, a dissociation of Eqs.(4.19) and (4.20) potentially singular behaviours is achieved right from the onset. There, in effect, the sum over \( N \) being performed before the integration on \( p_0 \), partial effective propagators \( S_R^{(N)}(P) \), get replaced by full effective ones, \( *S_R(P) \), whose poles, contrarily to \( S_R^{(N)}(P) \)-poles, are no longer lightlike at \( P^2 \simeq 0 \), but timelike, at \( p_0 = \pm \omega_s(p) \). It results that the light cone singular behaviour of (4.19) remains isolated, with no other singular behaviour to cancel with.

V. Two effective vertex diagrams \((N, N'; 2)\)

We now turn to the analysis of \((N, N'; 2)\) topologies depicted in Fig.3, which are the most important to consider, the famous collinear problem of hot QCD being induced by these double effective vertex insertions.

In Ref.13, it was shown that \((1, 0; 2)\) is singularity free. While an encouraging result, it would certainly be preposterous to take it for granted that the property trivially extends to any \((N, N'; 2)\) diagram, and in our opinion, this is why the present analysis had to be undertaken.

The contribution to \( \Pi_R(Q) \) of a diagram \((N, N'; 2)\) reads,

\[
\Pi_R^{(N,N';2)}(Q) = i e^2 m^4 \int \frac{d^4 P}{(2\pi)^4} (1 - 2n_F(p_0)) \int \frac{d\hat{K}}{4\pi} \int \frac{d\hat{K'}}{4\pi} \hat{K} \cdot \hat{K'} \cdot \text{disc} \cdot \text{Tr} \left( \frac{(\Sigma_R(P)P)^N}{(P^2 + i\epsilon p_0)^{N+1}} \hat{K} P' \frac{(\Sigma_R(P')P')^{N'}}{(P'^2 + i\epsilon p'_0)^{N'+1}} \hat{K}' \right) \times \frac{1}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)(\hat{K}' \cdot P + i\epsilon)(\hat{K}' \cdot P' + i\epsilon)} \tag{5.1}
\]

Expanding the \((\Sigma_R(P)P)^N\) factors as in (3.4), four types of traces come about,

\[
4 \left( 2\hat{K} \cdot P \hat{K}' \cdot P' - \hat{K} \cdot \hat{K}' P \cdot P' \right) \tag{5.2}
\]
\[
8m^2 \left( 2\hat{K} \cdot P \hat{K}' \cdot P' - \hat{K} \cdot \hat{K}' P \cdot P' \right) - 4P^2 \left( 2\hat{K} \cdot \Sigma \hat{K}' \cdot P' - \hat{K} \cdot \hat{K}' P' \cdot \Sigma \right) \tag{5.3}
\]
\[
8m^2 \left( 2\hat{K} \cdot P \hat{K}' \cdot P' - \hat{K} \cdot \hat{K}' P \cdot P' \right) - 4P'^2 \left( 2\hat{K} \cdot \Sigma' \hat{K}' \cdot P - \hat{K} \cdot \hat{K}' P \cdot \Sigma' \right) \tag{5.4}
\]
\[
(2m^2)^2 \left( 8\hat{K} \cdot P \hat{K}' \cdot P' - 4\hat{K} \cdot \hat{K}' P \cdot P' \right) - 8m^2 P^2 \left( 2\hat{K} \cdot \Sigma \hat{K}' \cdot P - \hat{K} \cdot \hat{K}' P \cdot \Sigma \right) - 8m^2 P^2 \left( 2\hat{K} \cdot \Sigma' \hat{K}' \cdot P' - \hat{K} \cdot \hat{K}' P' \cdot \Sigma' \right) + 4P^2 P'^2 \left( 2\hat{K} \cdot \Sigma \hat{K}' \cdot \Sigma' - \hat{K} \cdot \hat{K}' \Sigma \cdot \Sigma' \right) \tag{5.5}
\]
Integrated on both light-like vectors $\hat{K}, \hat{K}'$, the first trace (5.2) yields the expression

$$\frac{8}{m^4} \Sigma: \Sigma' - 4P\cdot P' W_2(P, P')$$

where $W_2(P, P')$ is the double vertex function met in Ref. 13,

$$W_2(P, P') = \int \frac{d\hat{K}}{4\pi} \int \frac{d\hat{K}'}{4\pi} \frac{(\hat{K} \cdot \hat{K}')^2}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)(\hat{K}' \cdot P + i\epsilon)(\hat{K}' \cdot P' + i\epsilon)}$$

The second trace (5.3) gives

$$2m^2 \times (5.6) - 4P^2 \left\{ \frac{2}{m^2} \int \frac{d\hat{K}}{4\pi} \frac{(\hat{K} \cdot \Sigma)^2}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)} - P \cdot \Sigma W_2(P, P') \right\}$$

The third trace (5.4) gives the same as (5.8) with $P$ and $P'$ interchanged,

$$2m^2 \times (5.6) - 4P'^2 \left\{ \frac{2}{m^2} \int \frac{d\hat{K}'}{4\pi} \frac{(\hat{K}' \cdot \Sigma')^2}{(\hat{K}' \cdot P + i\epsilon)(\hat{K}' \cdot P' + i\epsilon)} - P' \cdot \Sigma' W_2(P, P') \right\}$$

Eventually, the fourth trace (5.5) yields

$$-(2m^2)^2 \times (5.6) + 2m^2 \times (5.8) + 2m^2 \times (5.9) - 4P^2 P'^2 \Sigma \cdot \Sigma' W_2(P, P')$$

$$+ 8P^2 P'^2 \int \frac{d\hat{K}}{4\pi} \int \frac{d\hat{K}'}{4\pi} \frac{(\hat{K} \cdot \hat{K}')(\hat{K} \cdot \Sigma')(\hat{K}' \cdot \Sigma')}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)(\hat{K}' \cdot P + i\epsilon)(\hat{K}' \cdot P' + i\epsilon)}$$

To summarize, the whole expression (5.1) reads

$$\Pi_R^{(N, N'; 2)}(Q) = i\epsilon^2 m^4 \int \frac{d^4 P}{(2\pi)^4} \frac{1}{(P^2 + i\epsilon p_0)^{N'}} \frac{1}{(P'^2 + i\epsilon p'_0)^{N'+1}} \left\{ b_N b_{N'} \times (5.6) + a_N b_{N'} \times (5.8) + b_N a_{N'} \times (5.9) + a_N a_{N'} \times (5.10) \right\}$$

where the coefficients $a_N, b_N$, polynomials of degree $j_M(N)$ and $j_M(N) + 1$ in the variable $-P^2 \Sigma^2 / m^4$, respectively, are given in (3.5) and (3.6). That is, one would have again to investigate the incidence upon generic type (3.11) integrals, of the new multiplicative functions appearing through (5.6)-(5.10). For example, we quote that an expression like (5.6) will contribute $(N, N'; 2)$ a quantity,

$$\left( \frac{m^2}{m^4} \right)^{N+N'} \sum_{j=0}^{j_M(N)} \sum_{j'=0}^{j_M(N')} \sum_{n=0}^{C_{j-N-1}} \sum_{n'=0}^{C_{j'-N'-1}} C_j^{n+1} C_{j'-n'} C_{j'-n'} \times i\epsilon^2 m^4 \int \frac{d^4 P}{(2\pi)^4} \frac{1}{(P^2 + i\epsilon p_0)^{N'+1}} \left( \frac{8}{m^4} \Sigma \cdot \Sigma' - 4P \cdot P' W_2(P, P') \right)$$

$$(m^2)^{N+N'} \left( \frac{-P^2 \Sigma^2}{P^2 + i\epsilon p_0} \right)^{N'+1} \text{disc} \left( \frac{(-P^2 \Sigma^2)^n}{(P^2 + i\epsilon p_0)^{N'+1}} \right) \left( \frac{8}{m^4} \Sigma \cdot \Sigma' - 4P \cdot P' W_2(P, P') \right)$$

(5.12)
where the sums over \( j \) and \( n \), which are finite, have been interchanged with the integral on \( P \).

Actually, things may be further reduced, and this helps identifying the new mutliplicative functions that come out to be specific to the double effective vertex diagrams. To do so, we can make use of the relation

\[
\int \frac{d\hat{K}}{4\pi} \frac{(\hat{K}\Sigma)^2}{(\hat{K}\cdot P + i\epsilon)(\hat{K}\cdot P' + i\epsilon)} = \frac{1}{p^2} Q_1\left(\frac{p_0}{p}\right) \Sigma \cdot \Sigma' + m^2 p^2 Q_1\left(\frac{p_0}{p}\right) \frac{1}{2p'} \ln X' + \left( \frac{m^2}{p} \left( \frac{p_0}{p} - \frac{1}{2} \frac{P^2}{p^2} \ln X \right) \right)^2 \frac{1}{2Q\cdot P + i\epsilon_0} \ln \frac{P^2 + 2Q\cdot P + i\epsilon_0'}{P^2 + i\epsilon_0} \tag{5.13}
\]

and of a similar one, with \( P' \) and \( P \) interchanged, and likewise,

\[
\int \frac{d\hat{K}}{4\pi} \int \frac{d\hat{K}'}{4\pi} \frac{(\hat{K}\cdot \hat{K}') (\hat{K}\cdot \Sigma)(\hat{K}'\cdot \Sigma')}{(\hat{K}\cdot P + i\epsilon)(\hat{K}\cdot P' + i\epsilon)(\hat{K}'\cdot P + i\epsilon)(\hat{K}'\cdot P' + i\epsilon)} = \frac{1}{p^2} Q_1\left(\frac{p_0}{p}\right) \frac{1}{p'^2} Q_1\left(\frac{p_0'}{p'}\right) \Sigma \cdot \Sigma' + \frac{m^2}{p^2} Q_1\left(\frac{p_0}{p}\right) \frac{m^2}{p'^2} Q_1\left(\frac{p_0'}{p'}\right) \left( \frac{1}{p} \left( \frac{p_0}{p} - \frac{1}{2} \frac{P^2}{p^2} \ln X \right) \right)^2 \frac{1}{2Q\cdot P + i\epsilon_0} \ln \frac{P^2 + 2Q\cdot P + i\epsilon_0'}{P^2 + i\epsilon_0} + (P \leftrightarrow P') \tag{5.14}
\]

where \( W_1(P, P') \) is another double effective vertex function, not encountered in Ref.13,

\[
W_1(P, P') = \int \frac{d\hat{K}}{4\pi} \int \frac{d\hat{K}'}{4\pi} \frac{(\hat{K}\cdot \hat{K}')}{(\hat{K}\cdot P + i\epsilon)(\hat{K}\cdot P' + i\epsilon)(\hat{K}'\cdot P + i\epsilon)(\hat{K}'\cdot P' + i\epsilon)} \tag{5.15}
\]

Deriving (5.13) and (5.14), identities (4.9) and (4.14) only, have been used. As expected on the basis of general gauge invariance arguments\(^{15}\), (5.13) and (5.14) entail some potential collinear structures similar those found in the case of one effective vertex diagrams, (4.16). Now, a comparison with the previous cases of \((N, N'; 0)\) and \((N, N'; 1)\), also allows to identify with \( W_1 \) and \( W_2 \), the new extra mutiplicative functions that come out to be specific to the double effective vertex diagrams, \((N, N'; 2)\).

But it seems difficult to proceed further : As shown in the Appendix, an explicit calculation of functions \( W_1 \) and \( W_2 \) is doable, relying this time, on the full set of angular identities (4.9)-(4.11). Results, however, come out so cumbersome that controlling the ensuing integrals on \( x' \) and on \( x \), is rendered extremely hazardous. In order to proof that (5.11) does have an msf imaginary part, we must therefore proceed differently, and construct a proof by induction.
We consider the contribution to $\Pi_R(Q)$ of the diagram $(N + 2, N'; 2)$. It is,

\[
\Pi_R^{(N+2, N'; 2)}(Q) = 8ie^2 m^4 \int \frac{p^2 dp}{(2\pi)^2} \sum_{\epsilon(p_0)} \int_0^1 \frac{dx}{2\pi} \frac{p_0(x)}{2T} \text{ disc} \frac{1}{(P^2)_R^{N+3}} \times \int_0^{x_M} \frac{dx'}{(P^2)_R^{N+1}} \int_{\hat{K}, \hat{K}'} \hat{K} \cdot \hat{K}' \frac{Tr \left( \left( \Sigma_R P \right)^{N+2} \hat{K} P' \left( \Sigma_R P' \right)^{N'} \hat{K}' \right)}{(\hat{K} \cdot P)_R \left( \hat{K} \cdot P' \right)_R \left( \hat{K}' \cdot P' \right)_R}
\]

where some obvious shorthand notations have been introduced so as to alleviate too large expressions,

\[
\int \frac{d\hat{K}}{4\pi} \int \frac{d\hat{K}'}{4\pi} \equiv \int_{\hat{K}, \hat{K}'}, \quad \frac{1}{P^2 + i\epsilon p_0} \equiv \frac{1}{(P^2)_R}, \quad \Sigma_R(P) \equiv \Sigma_R, \quad \Sigma_R(P') \equiv \Sigma_R'
\]

Having,

\[(\Sigma_R P)^2 = 2m^2 \Sigma_R P - P^2 \Sigma_R I_4\]

Eq.(5.16) may be written as,

\[
8ie^2 m^4 \int \frac{p^2 dp}{(2\pi)^2} \sum_{\epsilon(p_0)} \int_0^1 \frac{dx}{2\pi} \frac{p_0(x)}{2T} \left\{ \text{ disc} \frac{2m^2}{(P^2)_R^{N+3}} \times \int_0^{x_M} \frac{dx'}{(P^2)_R^{N+1}} \int_{\hat{K}, \hat{K}'} \hat{K} \cdot \hat{K}' \frac{Tr \left( \left( \Sigma_R P \right)^{N+1} \right)}{(\hat{K} \cdot P)_R \left( \hat{K}' \cdot P' \right)_R} \right. \vphantom{\frac{\Sigma^2_R}{(P^2)_R^{N+2}}} \left. \right. \right.
\]

\[
\left. \frac{\Sigma^2_R}{(P^2)_R^{N+2}} \int_0^{x_M} \frac{dx'}{(P^2)_R^{N+1}} \int_{\hat{K}, \hat{K}'} \hat{K} \cdot \hat{K}' \frac{Tr \left( \left( \Sigma_R P \right)^{N-1} \right)}{(\hat{K} \cdot P)_R \left( \hat{K}' \cdot P' \right)_R} \right\}
\]

where the dots stand for all of those factors which are left the same as in (5.16). Next, we can form the difference of (5.16) with the first term of (5.19), obtaining

\[
8ie^2 m^4 \int \frac{p^2 dp}{(2\pi)^2} \sum_{\epsilon(p_0)} \int_0^1 \frac{dx}{2\pi} \frac{p_0(x)}{2T} \text{ disc} \frac{1}{(P^2)_R^{N+3}} \times \int_0^{x_M} \frac{dx'}{(P^2)_R^{N+1}} \int_{\hat{K}, \hat{K}'} \hat{K} \cdot \hat{K}' \frac{Tr \left( \left\{ \Sigma_R P \right)^{N+2} - 2m^2 \Sigma_R P \right)}{(\hat{K} \cdot P)_R \left( \hat{K}' \cdot P' \right)_R}
\]

This difference is, of course, also given by,

\[
-8ie^2 m^4 \int \frac{p^2 dp}{(2\pi)^2} \sum_{\epsilon(p_0)} \int_0^1 \frac{dx}{2\pi} \frac{p_0(x)}{2T} \text{ disc} \frac{\Sigma^2_R}{(P^2)_R^{N+2}} \times \int_0^{x_M} \frac{dx'}{(P^2)_R^{N+1}} \int_{\hat{K}, \hat{K}'} \hat{K} \cdot \hat{K}' \frac{Tr \left( \left\{ \Sigma_R P \right)^{N-1} \right)}{(\hat{K} \cdot P)_R \left( \hat{K}' \cdot P' \right)_R}
\]
Our induction hypothesis is that \((N + 1, N'; 2)\) is endowed with an \(msf\) imaginary part, that is, the expression

\[
8ie^2 m^4 \int \frac{p^2 dp}{(2\pi)^2} \sum_{\epsilon(p_0)} \int_0^1 \frac{dx}{2\pi} \frac{p_0(x)}{2T} \text{disc} \frac{1}{(P^2)_{N+1}^{R}} \times \int_{0}^{x'M} \frac{dx'}{(P^2)_{N+1}^{R}} \int_{\hat{K}, \hat{K}'} \hat{T} \left( \ldots \right) \frac{\hat{K} \cdot \hat{K}}{(K \cdot P)^{N+1}} \cdot (K' \cdot P')_{R} (5.22)
\]

We will need the following result: Let there be \(S^{(2)}(q, p; p_0)\) some function, such that, for all integers \(k\) and \(l\), and all integers \(i\) and \(n\), with \(0 \leq n - i\), the following finite sum of integrals,

\[
\frac{1}{\epsilon^{2(n-i)+k}} \sum_{m=0}^{2(n-i)+k} C_{2(n-i)+k}^m (-1)^m F_{km}^{(l)}(m\epsilon, 0) \int_0^1 \frac{dx}{x-\lambda} \left\{ \text{Im} \left( e^{i\pi m \epsilon} S^{(2)}(q, p; p_0) \right) \right\} x^{2n-i+l+m\epsilon} (5.23)
\]

has \(msf\) imaginary part in the limits \(\lambda = 0\) and \(\epsilon = 0\), with no further specifications required\(^7\).

Then we claim that so is the case of the finite sum of integrals,

\[
\frac{1}{\epsilon^{2(n-i)+k}} \sum_{m=0}^{2(n-i)+k} C_{2(n-i)+k}^m (-1)^m F_{km}^{(l)}(m\epsilon, 0) \int_0^1 \frac{dx}{x-\lambda} \left\{ \text{Im} \left( e^{i\pi m \epsilon} S^{(2)}(q, p; p_0) \times \Sigma_{R}^2(P) \right) \right\} x^{2n-i+l+m\epsilon} \text{Re} \left( e^{i\pi m \epsilon} S^{(2)}(q, p; p_0) \times \Sigma_{R}^2(P) \right) (5.24)
\]

with,

\[
\Sigma_{R}^2(P) = \frac{m^4}{p^2} \left( -1 + \frac{x}{4} [\epsilon(p_0) \ln X]^2 + \sqrt{1 - x} [\epsilon(p_0) \ln X] \right) (5.25)
\]

Before we proceed further, the relation of structural patterns (5.23) (and (5.24)), with a general term \((N, N'; 2)\) is worth making explicit. This is achieved by noting that the imaginary part of \((N, N'; 2)\) can be written as,

\[
8e^2 m^4 \int \frac{p^2 dp}{(2\pi)^2} \sum_{\epsilon(p_0)} \int_0^1 \frac{dx}{2\pi} \frac{p_0(x)}{2T} \text{disc} \frac{a_N S_a^{(2)} + b_N S_b^{(2)}}{(P^2)_{N+1}^{R}} (5.26)
\]

with \(p_0(x) = \epsilon(p_0)p\sqrt{1-x}\), and \(S^{(2)}(q, p; p_0)\) the distributions,

\[
S_a^{(2)}(q, p; p_0) = \frac{1}{2} \text{disc}_{P'} \int_{0}^{x'M(x)} \frac{dx'}{(P^2)_{N+1}^{R}} \int_{\hat{K}, \hat{K}'} \hat{T} \left( \ldots \right) \frac{\hat{K} \cdot \hat{K}}{(K \cdot P)^{N+1}} \cdot (K' \cdot P')_{R} (5.27)
\]
$S^{(2)}_b(q,p;p_0) = \frac{1}{2} \text{disc}_P \int_0^x \frac{dx'}{(P^2)^{N+1}} \int_{K,K'} \text{Tr} \left( \mathbb{P} \{ \mathbb{I}_4 \} \hat{K} \hat{K}' (\mathbb{N}_R^P \mathbb{P}'_R)^{N'} \hat{K}' \right) (K \cdot P)_{R} \ldots (K' \cdot P')_{R}$ \hspace{1cm} (5.28)

In the R/A real time formalism we are using, the imaginary part of $(N,N';2)$ is effectively obtained out of (5.1), by forming the difference (divided by a factor of 2) of retarded and advanced $P'$-lines. Whereof results (after integrating on $x'$) functions of $P$ which exhibit the features of distributions rather than of ordinary functions. As displayed for example by (3.43) in the $(N,N';0)$ case, the discontinuities in $p_0$ of the $S^{(2)}(q,p;p_0)$ may develop imaginary parts, and this is why they appear inside the discontinuity prescription of (5.26), and not simply factored out, as would be overall real valued multiplicative functions.

The connection with patterns (5.23) and (5.24) is made complete by recalling that, in virtue of (3.5) and (3.6), the coefficients $a_N$ and $b_N$ are polynomials of degree $j_M(N)$ in the variable $(-P^2 \Sigma_R^2/m^4)$. We have then, for all $n, 0 \leq n \leq j_M(N)$,

$$
\int_0^1 dx \text{ disc}_P \left( \frac{(-P^2 \Sigma_R^2)^n S^{(2)}}{(P^2)^{N+1}} \right) = \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{i=0}^{n} C^i_n (-1)^i (\frac{1}{4})^{n-i} \sum_{k=0}^{i} C^k_i (-1)^k \times (5.23) \hspace{1cm} (5.29)
$$

Now, the statement (5.23)-(5.24) is rather obvious indeed, because expression (5.25) is nothing but a linear combination of terms whose general form reads

$$
x^a \sqrt{1 - x^b} \left[ e(p_0) \ln X \right]^c, \hspace{0.5cm} 0 \leq a, b \leq 1, \hspace{0.5cm} 0 \leq c \leq 2 \hspace{1cm} (5.30)
$$

The first contribution of (5.25) to (5.24), is at $a = b = c = 0$, and up to an overall multiplicative factor of $-m^4/p^2$, leaves (5.23) the same as it is. The second contribution, is at $a = 1, b = 0, c = 2$. Up to an overall multiplicative factor of $m^4/4p^2$, this contribution leaves (5.23) unchanged, but for the only modification brought about by the shift of integer number $n - i$,

\[ (n - i) \mapsto (n - i) + 1 \hspace{1cm} (5.31) \]

The third contribution is at $a = 0, b = 1, c = 1$, and up to an overall multiplicative factor of $m^4/p^2$, it is entirely contained in the shift of integer number $k$, with

\[ k \mapsto k + 1 \hspace{1cm} (5.32) \]

It results that, if (5.23) has an msf imaginary part, then, so does (5.24). Somehow conversely, the very structure of mass singularity compensation patterns (5.23), makes it clear that if (5.29) is msf, then so is the case of the same whole expression, but taken at $n - 1$ instead of $n$.  

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Next we need to prove that if (5.22) has an msf imaginary part, then, so does

$$-8ie^2m^4 \int \frac{p^2dp}{(2\pi)^2} \sum_{\epsilon(p_0)} \int_0^1 \frac{dx}{2\pi} \frac{p_0(x)}{2T} \text{disc} \frac{1}{(P^2)_R^{N+2}} \times \int_0^{x_M} \frac{dx'}{(P^2)_R^{N'+1}} \int_{\hat{K}, \hat{K}'} \frac{\hat{K} \cdot \hat{K}'}{(\hat{K} \cdot P)_R \cdots (\hat{K}' \cdot P')_R}$$

(5.33)

where the difference with (5.22) is that, inside the trace, we have now a power of $N$ instead of $N + 1$. This statement is again rather obvious if one considers that the only change is in the substitution of the couple of polynomials $(a_{N+1}, b_{N+1})$ by the couple $(a_N, b_N)$. However, the proof of the above statement can be obtained by induction either, assuming first that the involvement holds true at $(N + 1, N'; 2)$, that is between (5.22) and (5.33). At next order, $(N + 2, N'; 2)$ is given by (5.16), which decomposes into the sum (5.19). Now, it has just been assumed that (5.22), and hence (5.33) have msf imaginary parts. In view of statement (5.23)-(5.24), the same is therefore true of the second term in (5.19), which only differs (5.33) a multiplicative function of $\Sigma^2_R$. It results that if $(N + 2, N'; 2)$ has an msf imaginary part, then so does have the expression

$$8ie^2m^4 \int \frac{p^2dp}{(2\pi)^2} \sum_{\epsilon(p_0)} \int_0^1 \frac{dx}{2\pi} \frac{p_0(x)}{2T} \text{disc} \frac{1}{(P^2)_R^{N+3}} \times \int_0^{x_M} \frac{dx'}{(P^2)_R^{N'+1}} \int_{\hat{K}, \hat{K}'} \frac{\hat{K} \cdot \hat{K}'}{(\hat{K} \cdot P)_R \cdots (\hat{K}' \cdot P')_R}$$

(5.34)

That is, the involvement extends from $(N + 1, N'; 2)$ to $(N + 2, N'; 2)$. Eventually, we learn out of Ref.13, that $(0, 0; 2)$ and $(1, 0; 2)$ have msf imaginary parts. Whereof it is immediate to check that the property of involvement under consideration is verified at $N = 1$; and thus, at all $N$.

Getting back to our central induction hypothesis that $(N + 1, N'; 2)$ has an msf imaginary part, the above two statements allow to conclude that (5.21) does have an msf imaginary part either, and this establishes that the imaginary part of the difference (5.20) is msf.

Two possibilities have to be considered whereupon: Either mass singular behaviours of both members compensate for each others in the difference (5.20), or both members of (5.20) have, separately, msf imaginary parts.
Let us suppose that a compensation of singularities is at the origin of the difference \( msf \) imaginary part. The trace of (5.20) can be written as

\[
Tr \left( \mathcal{P} \left( \Sigma R \mathcal{P}^\prime \right)^{N+1} \left\{ \Sigma R \mathcal{P}^2 - 2m^2 \right\} \hat{K} \mathcal{P}^\prime \left( \Sigma R \mathcal{P}^\prime \right)^N \hat{K}^\prime \right)
\]  

(5.35)

As (5.35) stands, however, inspection shows that nothing conclusive can be derived. Relying again on (3.4)-(3.6), it is interesting to decompose (5.35) into a sum of terms

\[
(\Delta a) a_N T r \left( \mathcal{P} \left( \Sigma R \mathcal{P}^\prime \right)^N \hat{K} \mathcal{P}^\prime \left( \Sigma R \mathcal{P}^\prime \right)^{N} \hat{K}^\prime \right) + (\Delta a) b_N T r \left( \mathcal{P} \left( \Sigma R \mathcal{P}^\prime \right)^N \hat{K} \mathcal{P}^\prime \hat{K}^\prime \right)
\]  

(5.36)

where we have defined

\[
\Delta a = a_{N+2} - 2m^2 a_{N+1}, \quad \Delta b = b_{N+2} - 2m^2 b_{N+1}
\]  

(5.37)

At its turn, the first trace of (5.36), decomposes into a sum of terms,

\[
Tr \left( \mathcal{P} \left( \Sigma R \mathcal{P}^\prime \right)^N \hat{K} \mathcal{P}^\prime \left( \Sigma R \mathcal{P}^\prime \right)^{N} \hat{K}^\prime \right) = m^2 Tr \left( \mathcal{P} \left( \Sigma R \mathcal{P}^\prime \right)^N \hat{K} \mathcal{P}^\prime \hat{K}^\prime \right) + m^2 Tr \left( \mathcal{P} \left( \Sigma R \mathcal{P}^\prime \right)^N \hat{K} \mathcal{P}^\prime \hat{K}^\prime \right) + ..
\]  

(5.38)

where the two traces of the right hand side are the second and third traces of (5.36), respectively, whereas the dots stand for terms which belong, in proper, to the trace under consideration. If the latter induce further singular imaginary parts, the imaginary part \( msf \) character of (5.20) indicates that mass singularity compensations are taking place, that is,

\[
\Delta a = 0
\]  

(5.39)

If \( N \) is an even number, then, \( j_M(N+1) = j_M(N+2) = N/2 \), and one has

\[
\Delta a = \left( m^2 \right)^{N+1} \sum_{j=0}^{N/2} \left( \begin{array}{c} C_{N+1}^{2j+1} - 2C_{N+1}^{2j+1} \end{array} \right) \sum_{n=0}^{j} C_n^j \left( \frac{P^2 \Sigma R^2}{m^4} \right)^n
\]  

(5.40)

For some given powers \( n \) to induce singular integrations to be further compensated in the difference, the following condition must therefore be satisfied,

\[
\sum_{j=n}^{N/2} C_n^j \left( \begin{array}{c} C_{N+1}^{2j+1} - 2C_{N+1}^{2j+1} \end{array} \right) = 0
\]  

(5.41)

Binomial coefficients are positive definite, and if \( n > N/4 \), then, the terms in the sum (5.41) are positive definite either, precluding any compensations of possible singular contributions.
That is, contributions attached to the range of powers $N/4 < n \leq N/2$ are necessarily \textit{msf} in imaginary parts, separately. Now, so are also all of the other powers, $1 \leq n \leq N/2$, in virtue of the statement "somehow reciprocal" to (5.24). Since a similar argument can be developed in case of an odd number $N$, it results that the dots of (5.38) induce, in both members of the difference (5.20), contributions whose imaginary parts are \textit{msf}, separately.

We consider the second trace, and note that in view of (5.36) and (5.38), it has coefficient

$$ (b_{n'} + m^2 a_{n'}) \Delta a $$

(5.42)

If, when plugged into (5.20), the second trace of (5.36) generates non \textit{msf} imaginary parts, then the latter have to compensate each others in the difference. Now, selecting a power of $n'$ in the variable $\left(-P^2 \Sigma_R^2/m^4\right)$, its coefficient reads

$$ \sum_{j'=n'}^{N'/2} \left(C_{n'+1}^{2j'+1} + C_{n'-1}^{2j'+1}\right) $$

(5.43)

where an even value of $N'$ is choosen, for the sake of illustration. It is clear that for all $n' \in \{1, 2, \ldots N'/2\}$, (5.43) is a never vanishing quantity. A compensation of possible singular subsequent integrations on $x$, can only come from (5.39), with the conclusion that for this second trace of (5.36), both members of the difference (5.20) have, separately, \textit{msf} imaginary parts.

The third trace of (5.36) comes into play with a coefficient of

$$ a_{n'}(\Delta b + m^2 \Delta a) $$

(5.44)

which may be explicit as

$$ a_{n'}(m^2)^{N/2 + 2} \left\{ \sum_{j=0}^{N/2} (C_{N+2}^{2j+1} - 2C_{N+1}^{2j+1})(1 - \frac{P^2 \Sigma_R^2}{m^4})^j \right\} $n$$

$$ + \left( -\frac{P^2 \Sigma_R^2}{m^4} \right)^{N/2} \sum_{j=0}^{N/2 - 1} (C_{N+1}^{2j+1} - 2C_{N}^{2j+1})(1 - \frac{P^2 \Sigma_R^2}{m^4})^j + \left( -\frac{P^2 \Sigma_R^2}{m^4} \right)(1 - \frac{P^2 \Sigma_R^2}{m^4})^{N/2} \right\} $$

(5.45)

The higher power in the variable $-P^2 \Sigma_R^2/m^4$ is a power of $N/2 + 1$, with coefficient 1. There is no available compensation for this isolated term of (5.45) which has accordingly to yield a regular subsequent integration on $x$. So is therefore the case of all of the powers $n \in \{1, 2, \ldots N/2+1\}$,
because of the statement reciprocal to (5.24). Whereof results that, irrespective of possible compensations among the other (regular) terms of (5.45), both members of the difference (5.20) have msf imaginary parts attached to the third trace of (5.36).

The fourth trace of (5.36) has coefficient,

\[
(m^2)^2 a_{N'} \Delta a + m^2 (a_{N'} \Delta b + b_{N'} \Delta a) + b_{N'} \Delta b = (b_{N'} + m^2 a_{N'})(\Delta b + m^2 \Delta a) \tag{5.46}
\]

If this trace generates any singular subsequent integration on \(x\), when put back into both members of (5.20), the msf character of the imaginary part of (5.20) requires that \(\Delta b + m^2 \Delta a\) vanishes. This condition turns out to be the one just dealt with, and it results that, when put into both members of the difference (5.20), the fourth trace of (5.36) induce subsequent \(x\)-integrations that have, separately, msf imaginary parts.

To summarize, both \((0, 0; 2)\) and \((1, 0; 2)\) diagrams, have been shown to possess msf imaginary parts in Ref.13. Then, assuming that a diagram \((N + 1, N'; 2)\) has msf imaginary part, we have been able to prove that the next diagram \((N + 2, N'; 2)\), with one more HTL-self energy insertion, has an msf imaginary part either. We can therefore conclude that any of the two effective vertex diagrams contribute msf imaginary parts to the soft real photon emission rate.

The power and simplicity of the proof just developed appears the more clearly as one realizes that the distributions \(S^{(2)}(q, p; p_0)\) introduced in full generality in (5.23), entail the double vertex function \(W_2(P, P')\) and \(W_1(P, P')\) defined in (5.7) and (5.15) respectively. We have in effect, for the \(S^{(2)}(q, p; p_0)\) the expressions

\[
\frac{1}{2} \text{disc}_{P'} \int_0^{x_{M}(x)} dx' a_{N'}(-P'^2 \Sigma^2 / m^4) \int_{\hat{K}, \hat{K}'} \frac{\hat{K} \cdot \hat{K}'}{(K \cdot P)_{R} \cdots (K' \cdot P')_{R}} T r \left( P \{ \Sigma_{R} P, \ II_4 \} \hat{K} \ P' \{ \Sigma'_{R} P' \} \hat{K}' \right) \tag{5.47}
\]

and whereas the second term entails \(W_2(P, P')\), the first one entails both \(W_2(P, P')\) and \(W_1(P, P')\) which are so complicated functions of \(P, P'\), that they practically exclude any control of the ensuing integrations on \(x'\), and \textit{a posteriori} on \(x\), contrarily to what could be achieved in Sections 3 and 4, for the topologies \((N, N'; 0)\) and \((N, N'; 1)\).

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An interesting byproduct of this analysis is obtained by writing,

\[
S^1_a(q,p; p_0) = \frac{1}{2} \text{disc}_P \int_0^{x_M(x)} \frac{dx'}{(P'^2)^{N'_R+1}} \int \frac{d\hat{K}}{4\pi} \frac{Tr \left( P \{ \mathbb{S}_R \hat{P} \} \hat{K} P' \{ \mathbb{S}'_R \hat{P}' \}^{N'_R} \hat{K} \right)}{(\hat{K} \cdot P)_{R}(\hat{K} \cdot P')_{R}} \tag{5.48}
\]

\[
S^1_b(q,p; p_0) = \frac{1}{2} \text{disc}_P \int_0^{x_M(x)} \frac{dx'}{(P'^2)^{N'_R+1}} \int \frac{d\hat{K}}{4\pi} \frac{Tr \left( P \{ \mathbb{II}_A \} \hat{K} P' \{ \mathbb{S}'_R \hat{P}' \}^{N'_R} \hat{K} \right)}{(\hat{K} \cdot P)_{R}(\hat{K} \cdot P')_{R}} \tag{5.49}
\]

and by recognizing that the imaginary part of a diagram \((N, N'; 1)\) is hereby expressed as,

\[
8e^2 m^2 \int \frac{p^2 dp}{(2\pi)^2} \sum_{\epsilon(p_0)=\pm 1} \int_0^1 \frac{dx}{2\pi} \frac{p_0(x)}{2T} \text{disc}_P \frac{a_N S^1_a + b_N S^1_b}{(P'^2)^{N'_R+1}} \tag{5.50}
\]

Then, knowing from Ref.13 that \((0, 0; 1), (1, 0; 1)\) and \((1, 1; 1)\) have \(msf\) imaginary parts, the same steps as followed throughout this section can be taken, and allow to conclude by induction that \((N, N'; 1)\) diagrams contribute \(msf\) parts to the soft photon emission rate. This is the more systematic derivation which was advertised in the end of Sec.4: It encompasses all of the terms (4.5)-(4.7) of the \((N, N'; 1)\) situation, and not solely the peculiar one, (4.8), which was treated there.

Likewise, identifying now \(S^0_a(q,p; p_0)\) and \(S^0_b(q,p; p_0)\) the distributions,

\[
S^0_a(q,p; p_0) = \frac{1}{2} \text{disc}_P \int_0^{x_M(x)} \frac{dx'}{(P'^2)^{N'_R+1}} \frac{dx'}{(P'^2)^{N'_R+1}} \cdot Tr \left( P \{ \mathbb{S}_R \hat{P} \} \hat{K} P' \{ \mathbb{S}'_R \hat{P}' \}^{N'_R} \hat{K} \right) \tag{5.51}
\]

\[
S^0_b(q,p; p_0) = \frac{1}{2} \text{disc}_P \int_0^{x_M(x)} \frac{dx'}{(P'^2)^{N'_R+1}} \cdot Tr \left( P \{ \mathbb{II}_A \} \hat{K} P' \{ \mathbb{S}'_R \hat{P}' \}^{N'_R} \hat{K} \right) \tag{5.52}
\]

and observing that the imaginary part of a diagram \((N, N'; 0)\) can be expressed as

\[
8e^2 \int \frac{p^2 dp}{(2\pi)^2} \sum_{\epsilon(p_0)=\pm 1} \int_0^1 \frac{dx}{2\pi} \frac{p_0(x)}{2T} \text{disc}_P \frac{a_N S^0_a + b_N S^0_b}{(P'^2)^{N'_R+1}} \tag{5.53}
\]

we can make use of the \(msf\) character of \((1, 0; 0)\) and \((2, 0; 0)\) imaginary parts as established in Ref.13 so as to follow the same steps as taken throughout this section and conclude, in agreement with the calculational approach of Sec.3, that \((N, N'; 0)\) diagrams contribute \(msf\) parts to the soft photon emission rate.
VI. Conclusion

Some years ago\textsuperscript{23}, we had suggested that the collinear problem met in hot QCD when the Resummation Program (RP) is used, could be traced back to the particular perturbative series re-arrangement the RP amounts to. Strictly speaking though (and contrarily to what can be read off the existing literature) the RP should not be mistaken for any Feynman diagrams resummation, possibly infinite. This happens to be so, simply because of the effective propagators non perturbative character: Pole residues and dispersion relations, in effect, cannot be derived out of pure thermal Perturbation Theory.

This suggestion has motivated our construction of a coherent Perturbative Resummation scheme (PR) of the leading thermal effects (the so-called Hard Thermal Loops, HTL) enjoying by construction the same symmetry properties as the usual RP, with the hope that things could come out at variance with the troublesome (undefined!) RP results.

In the case of the so called rapid fermion damping rate problem of both QED and QCD, a first obstruction met by the RP\textsuperscript{5}, this hope revealed itself non deceptive indeed\textsuperscript{6,7}, whereas the collinear problem under consideration was subsequently projected out on a simpler toy model, with promising results\textsuperscript{11}.

In a recent publication\textsuperscript{13}, the physically interesting case of hot QCD has been analyzed through its first non trivial perturbative orders, with very instructive new insights. As stated in the introduction, not only did the PR analysis allowed to elucidate the so far questionable nature\textsuperscript{12} of the collinear singularity encountered in hot QCD, when the RP is used, but a tight and original comparison of both RP and PR calculations made it possible to understand how the collinear singularity unavoidably shows up in an RP treatment.

Now, a PR calculation of the soft real photon emission rate, involves the infinite resum-mations on $N$ and $N'$, of any perturbative contribution of type $(N,N';0)$, $(N,N';1)$ and $(N,N';2)$, describing one loop photonic self energy diagrams at $N(N')$ HTL self energy insertions along the $P(P')$-fermionic line, respectively endowed with zero, one and two HTL effective vertex corrections. In order to set our PR calculation a sound, significant result, in contradistinction to the yet confused RP situation, it was therefore crucial to check that the properties that could be derived for the perturbative orders of $m^2$, $m^4$ and $m^6$, extended indeed to any contribution of order $m^{2n}$, and this is the task which has been achieved throughout the present article.
However tedious the calculational developments of Section 3, they eventually revealed extremely useful so as to provide a proof by induction with sound enough a basis. In other words, a proof by induction is not reliable until enough information is gained concerning the mass singularity cancellation patterns, and not before.

Physically, a salient aspect comes out to be the one of propagator’s pole migration. This migration in effect, from the light-like to the time-like region, appears to be at the very origin of the $RP$ dramatic consequence under consideration, and in contrast to the $T = 0$ situation, is really peculiar to the thermal context$^7$. This is because pole displacements involve a decoupling of partial effective propagators (potential) mass singularities, from effective vertices collinear ones: Whereas all singularities mix up into patterns which grant their overall compensations in a $PR$ calculation, effective vertices mass/collinear singularities remain isolated in an $RP$ calculation, with no singular counterpart to cancel against. This mechanism, first guessed in Ref.11, then discovered in Ref.13 for the perturbative orders of $m^2$, $m^4$ and $m^6$, is easily seen here to spread to any perturbative orders, $(m^2)^n$.

As we have long been suspecting$^{23}$, the collinear singularity plaguing the soft real photon emission rate $RP$ calculation is likely to be nothing but an artefact, peculiar to the $RP$ resummation scheme itself. Here may be the place where to recall a part of our conclusion in Ref.13: After all, whenever resummation is required by the context, a guiding principle could very well be that it be conceived and taken out of finite, well defined elements, and in particular, of mass singularity free terms. In this respect, it is instructive to come back to the original article where the $RP$ was mostly founded, and to realize that the authors were conscious of difficulties that could be inherited from the fact that the $RP$ did not necessarily comply with this requirement$^{24}$.

Acknowledgement

It is a pleasure to thank B.Candelpergher for encouraging discussions.

Appendix: Calculating the $W_i(P, P')$

In this appendix account is given of the difficulty inherent to the explicit calculations of the double effective vertex functions $W_i(P, P')$, $i = 1, 2$. The first function, $W_1(P, P')$ is given
in (5.15). It is given by the integration on a Feynman parameter $s$, of the squared norm of a would be 4-vector with components the right hand sides of (4.9) and (4.10),

$$m^2 \int \frac{d\hat{K}}{4\pi} \frac{\hat{K}^\mu}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)} m^2 \int \frac{d\hat{K}'}{4\pi} \frac{\hat{K}'^\mu}{(\hat{K}' \cdot P + i\epsilon)(\hat{K}' \cdot P' + i\epsilon)} = m^4 \int_0^1 ds \frac{r_0}{r^3} \ln X_r - \frac{1}{4r^2} \ln^2 X_r - \frac{1}{r^2 (R^2 + i\epsilon r_0)}$$  \hspace{1cm} (A.1)

As it stands however, the remaining integration on $s$ is not very easy. A more economic way to proceed consists in writing

$$m^4 \int_0^1 ds \int_0^1 ds' m^2 \int \frac{d\hat{K}}{4\pi} \frac{\hat{K}^\mu}{(\hat{K} \cdot R(s) + i\epsilon)^2} m^2 \int \frac{d\hat{K}'}{4\pi} \frac{\hat{K}'^\mu}{(\hat{K}' \cdot R(s') + i\epsilon')^2}$$  \hspace{1cm} (A.2)

This allows to write (A.1) as,

$$(- \frac{d}{d\epsilon})(- \frac{d}{d\epsilon'}) \int_0^1 ds \int_0^1 ds' \Sigma (R(s)) \cdot \Sigma (R(s')) \int \frac{m^2}{R^2(s) + i\epsilon r_0(s)} \frac{m^2}{R^2(s') + i\epsilon r_0(s')} (1 - \frac{r_0}{r} (s) r_0(s') \hat{r} \cdot \hat{r}')$$  \hspace{1cm} (A.3)

The term +1 in the right hand side last parenthesis yields simply,

$$\int_0^1 ds \int_0^1 ds' \frac{m^2}{R^2(s) + i\epsilon r_0(s)} \frac{m^2}{R^2(s') + i\epsilon r_0(s')} = \left( \frac{m^2}{2P \cdot Q} \ln \frac{P^2}{P^2} \right)^2$$  \hspace{1cm} (A.4)

The contribution of the term involving the cosine $\hat{r} \cdot \hat{r}'$ is of course more involved. It is

$$\int_0^1 ds \int_0^1 ds' \frac{m^2}{R^2(s) + i\epsilon r_0(s)} \frac{m^2}{R^2(s') + i\epsilon r_0(s')} \frac{p_0 + qs}{r^2(s)} \frac{p_0 + qs'}{r^2(s')} (p(p + qys) + q(py + qs)s')$$  \hspace{1cm} (A.5)

where,

$$r^2(s) = p^2 + 2pqys + q^2 s^2, \hspace{1cm} R^2(s) = P^2 + 2P \cdot Qs$$  \hspace{1cm} (A.6)

Introducing the three functions,

$$F_1(P, Q) = \frac{1}{2Q \cdot P} \ln \frac{P^2}{P^2}$$  \hspace{1cm} (A.7)

$$F_2(P, Q) = \frac{1}{qq \sqrt{1 - y^2}} \arctan \frac{q \sqrt{1 - y^2}}{p + qy}$$  \hspace{1cm} (A.8)
\[ F_3(P, Q) = \frac{1}{(p^2 - \bar{p}^2)^2 \left\{ \frac{1}{2} (2Q \cdot P)^2 F_1(P, Q) - (q^2 P^2 - qP_0(2Q \cdot P) + \frac{1}{2} (2Q \cdot P)^2) F_2(P, Q) \right\} } \]

one verifies that (A.4) cancels out, and that the integration of (A.1) can be given the interesting

\[
m^2 \int \frac{d\hat{K}}{4\pi} \frac{\hat{K}^\mu}{(\hat{K} \cdot P + i\epsilon)(\hat{K} \cdot P' + i\epsilon)} m^2 \int \frac{d\hat{K}'}{4\pi} \frac{\hat{K}'^\mu}{(\hat{K}' \cdot P + i\epsilon)(\hat{K}' \cdot P' + i\epsilon)} = -m^4 \sum_{i,j=1}^3 \left( \sum_{k=-2}^{+1} a_{ij}^k (2Q \cdot P)^k \right) F_iF_j \]  

(A.10)

where the non vanishing \(a_{ij}^k\) coefficients are given by

\[
a_{22}^{-2} = -q^2 P^2, \quad a_{22}^{-1} = qP_0 \]  

(A.11)

\[ a_{33}^{-2} = -q^2 (P^2)^3, \quad a_{33}^{-1} = \frac{5}{2} qP_0 (P^2)^2, \quad a_{33}^0 = -\frac{9}{4} (P^2)^2 - \frac{5}{2} qP_0 P^2, \quad a_{33}^1 = \frac{p_0 (3P^2 + 4q^2)}{4q} \]  

(A.12)

\[ a_{12}^0 = 1 \]  

(A.13)

\[ a_{13}^{-1} = -qP_0 P^2, \quad a_{13}^0 = \frac{3}{2} P^2, \quad a_{13}^{-1} = \frac{p_0}{q} \]  

(A.14)

\[ a_{23}^{-2} = 2q^2 (P^2)^2, \quad a_{23}^{-1} = -4qP_0 P^2, \quad a_{23}^0 = \frac{11}{4} P^2 + \frac{3}{2} qP_0 P^2, \quad a_{23}^1 = \frac{p_0}{q} \]  

(A.15)

Calculating \(W_2(P, P')\), given in (5.7), is the most tedious angular integration to be coped with, ”an order of magnitude” more difficult than the latter. One has,

\[
W_2(P, P') = \int_0^1 ds \int_0^1 ds' \int \frac{d\hat{K}}{4\pi} \int \frac{d\hat{K}'}{4\pi} \frac{1 - 2\hat{K}' \hat{K}' + \hat{K} \hat{K}' \hat{K}' \hat{K}'}{(\hat{K}' \cdot R(s') + i\epsilon)^2 (\hat{K}' \cdot R(s') + i\epsilon)^2} \]  

(A.16)

The full set of angular identities (4.9)-(4.11) must be used, but since the first two terms in the numerator of (A.16) are those which have just been dealt with in the calculation of \(W_1(P, P')\), we may focus on the contribution due to the third term, \(\{\hat{K}' \hat{K}' \hat{K}' \hat{K}'\}\). Relying on the angular identity (4.11), one finds, with \(Q_1\) as defined in (3.3),

\[
-3 \left( \int_0^1 ds \frac{Q_1(R(s))}{r^2(s)} \right)^2 + 6 \left( \int_0^1 ds \frac{Q_1(R(s))}{r^2(s)} \right) \left( \int_0^1 ds' \frac{r^2(s')}{R^2(s') + i\epsilon_0(s')} \right) \]  

\[ + 9 \left( \int_0^1 ds \frac{Q_1(R(s))}{r^2(s)} \right) \left( \int_0^1 ds' [\hat{r}(s) \hat{r}(s')]^2 \frac{Q_1(R(s'))}{r^2(s')} \right) - 6 \left( \int_0^1 ds \frac{Q_1(R(s))}{r^2(s)} \right) \left( \int_0^1 ds' \frac{[\hat{r}(s) \cdot \hat{r}(s')]^2}{R^2(s') + i\epsilon_0(s')} \right) \]

\[ + \int_0^1 \frac{ds}{\left( R^2(s) + i\epsilon_0(s) \right)} \int_0^1 ds' \frac{[\hat{r}(s) \cdot \hat{r}(s')]^2}{\left( R^2(s') + i\epsilon_0(s') \right)} \]  

(A.17)

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The second term gives

\[ 6F_1 \int_0^1 ds \frac{Q_1(R(s))}{r^2(s)} = \frac{3F_1}{pq(1-y^2)} \left( (p_0 y - p + \frac{Z}{2p} \ln X' - (p_0 y - p) \frac{\ln X}{p} \right) - \frac{3}{2} \frac{(ZF_1)^2}{p^2 q^2(1-y^2)} \]

(A.18)

where the shorthand notation \( Z = 2Q \cdot P \) has been introduced. The first term of (A.17) gives

\[ -3 \left( \int_0^1 ds \frac{Q_1(R(s))}{r^2(s)} \right)^2 \]

\[ = \frac{-3}{4p^2 q^2(1-y^2)^2} \left( -\frac{1}{2} Z^2 F_1 + (p_0 y - p + \frac{Z}{2p} \ln X' - (p_0 y - p) \frac{\ln X}{p} \right)^2 \]

(A.19)

The fifth term of (A.17) gives

\[ \sum_{i,j=1}^{3} \left( \sum_{k=-2}^{+1} c^k_{ij} Z^k \right) F_i F_j \]

(A.20)

where the non vanishing \( c^k_{ij} \) are given by,

\[ c^0_{11} = 1 \]

(A.21)

\[ c^{-2}_{22} = 2q^2 p^2 (1-y^2) \]

(A.22)

\[ c^{-2}_{33} = -2q^2 p^4 (1-y^2)^2 p^2 , \quad c^{-1}_{33} = -2qp_0 p^2 (1-y^2) p^2 , \quad c^0_{33} = \frac{p^2 + 3p_0^2}{2} \]

(A.23)

\[ c^0_{13} = -2p^2 (1-y^2) \]

(A.24)

\[ c^{-2}_{23} = 2q^2 p^2 (1-y^2)(-P^2 + p^2 (1-y^2)) , \quad c^{-1}_{23} = 2qp_0 p^2 (1-y^2) , \quad c^0_{23} = \frac{3p^2 (1-y^2)}{2} \]

(A.25)

The fourth term of (A.17) can first be expressed as,

\[ -6 (F_1 - p^2 (1-y^2) F_3) \left( \int_0^1 ds \frac{Q_1(R(s))}{r^2(s)} \right) + 6p^2 (1-y^2) (F_1 - 2p^2 F_3) \left( \int_0^1 ds \frac{Q_1(R(s))}{r^4(s)} \right) \]

\[ -12 p_0 q^2 y (1-y^2) F_3 \left( \int_0^1 ds \frac{Q_1(R(s))}{r^4(s)} \right) \]

(A.26)

allowing to see that the second term contribution, (A.18), cancels out with an identical part in (A.26). In terms of more elementary integrals, the remaining parts of (A.26) may be written,

\[ 6F_1 F_2 - 6F_1 p^2 (1-y^2) \int_0^1 \frac{ds}{r^4(s)} - 6F_1 \left( \frac{1}{2} \int_0^1 \frac{ds}{r^3(s)} r_0 \ln X_R(s) \right) \]

\[ + 6p^2 (1-y^2) (F_1 - F_3) \left( \frac{1}{2} \int_0^1 \frac{ds}{r^3(s)} r_0 \ln X_R(s) \right) \]

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To summarize, the third term of (A.17) is the more involved one, and may be written as,

\[-6F_3p^2(1-y^2) \left( F_2 - 2p^2 \int_0^1 \frac{ds}{r^4(s)} - 2qpy \int_0^1 \frac{ds}{r^4(s)} \right)\]

\[+12q(1-y^2) \frac{F_2 - P^2F_3}{Z} \left( p^3y \int_0^1 \frac{ds}{r^4(s)} + qP^2 \int_0^1 ds \frac{s}{r^4(s)} \right)\]

\[-6qP^2(1-y^2) \frac{F_2 - P^2F_3}{Z} \left\{ \int_0^1 ds \frac{\ln X_{R(s)}}{r^3} + p(yP_0 - p) \int_0^1 ds \frac{\ln X_{R(s)}}{r^5} \right\}\]

\[+ \frac{Z}{2} \int_0^1 ds \frac{s \ln X_{R(s)}}{r^5} \} \] (A.27)

The third term of (A.17) is the more involved one, and may be written as,

\[9 \left( \int_0^1 ds \frac{Q_1(R(s))}{r^2(s)} \right)^2 - 18p^2(1-y^2) \left( \int_0^1 ds \frac{Q_1(R(s))}{r^2(s)} \right) \left( \int_0^1 ds \frac{Q_1(R(s))}{r^4(s)} \right)\]

\[+18p^4(1-y^2) \left( \int_0^1 ds \frac{Q_1(R(s))}{r^4(s)} \right)^2 + 36qP^3y(1-y^2) \left( \int_0^1 ds \frac{Q_1(R(s))}{r^4(s)} \right) \left( \int_0^1 ds \frac{Q_1(R(s))}{r^4(s)} \right)\]

\[+18q^2P^2(1-y^2) \left( \int_0^1 ds \frac{s Q_1(R(s))}{r^4(s)} \right)^2 \] (A.28)

To summarize, the last piece of \(W_2(P, P')\) can eventually be expressed in the form,

\[\sum_{i,j=1}^3 \left( \sum_{k=-2}^{+1} c_{ij}^k Z^k \right) F_i F_j + 6p^2(1-y^2) \left\{ F_3 \left( \int_0^1 ds \frac{Q_1}{r^2} \right)\right.\]

\[+ (F_1 - 2p^2F_3) \left\{ \int_0^1 ds \frac{Q_1}{r^4} \right\} \}

\[+12qp^3y(1-y^2) \left( \int_0^1 ds \frac{Q_1}{r^4} \right) \left( -F_3 + 3 \int_0^1 ds \frac{Q_1}{r^4} \right)\]

\[+6 \left( \int_0^1 ds \frac{Q_1}{r^2} \right)^2 - 18p^2(1-y^2) \left( \int_0^1 ds \frac{Q_1}{r^2} \right) \left( \int_0^1 ds \frac{Q_1}{r^4} \right)\]

\[+18p^4(1-y^2) \left( \int_0^1 ds \frac{Q_1}{r^4} \right)^2 + 18q^2P^2(1-y^2) \left( \int_0^1 ds \frac{s Q_1}{r^4} \right)^2 \] (A.29)

We will not proceed further, giving for example the more elementary integrals displayed in (A.27) and (A.29), as it should already appear clear that the statement concerning "so cumbersome calculations that they practically preclude any peer control of ensuing integrations on \(x'\) and then on \(x\)" is not exaggerated.

Before concluding this appendix we may stress that the calculation of \(W_1(P, P')\) and \(W_2(P, P')\) does not display singularities other than (potentially) collinear ones, showing up by the light cone boundary, at \(P^2 \simeq 0\). For the function \(F_3\) of (A.9), this property may be not so
easy to see. However, it is straightforward to check that the denominator of (A.9) reads as,

\[(P^2 - \bar{p}^2)^2 = 4q^2 p^2 p_0^2 (y - \frac{p_0^2 + p^2}{2p_0 p})^2 \quad (A.30)\]

It vanishes at the light cone only, at \(p_0 = \pm p\), and this corresponds effectively to a collinear singularity, at \(y = \pm 1\).
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**Figure caption**

**Fig.1**: A graph denoted by \((N, N'; 0)\), with \(N(N')\) insertions of HTL self energy along the \(P(P')\)-line, and two bare vertices \(-ie\gamma_\mu\) of (2.4).
**Fig. 2:** A graph denoted by \((N, N'; 1)\), with \(N(N')\) insertions of \(HTL\) self energy along the \(P(P')\)-line, one bare vertex \(-ie\gamma_\mu\), and one \(HTL\) vertex correction (2.5).

**Fig. 3:** A graph denoted by \((N, N'; 2)\), with \(N(N')\) insertions of \(HTL\) self energy along the \(P(P')\)-line, and two \(HTL\) vertex corrections (2.5).