METRIC HYPERGRAPHS AND METRIC-LINE EQUIVALENCES

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Abstract

In a metric space $M = (X, \text{dist})$, we say that $v$ is between $u$ and $w$ if $\text{dist}(u, w) = \text{dist}(u, v) + \text{dist}(v, w)$. Taking all triples $\{u, v, w\}$ such that $v$ is between $u$ and $w$, one can associate a 3-uniform hypergraph with each finite metric space $M$. An effort to solve some basic open questions regarding finite metric spaces has motivated an endeavor to better understand these associated hypergraphs. In answer to a question posed in [2], we present an infinite family of hypergraphs that are non-metric, i.e., they don’t arise from any metric space.

Another basic structure associated with a metric space is a binary equivalence on the vertex set, where two pairs are in the same class if they induce the same line. An equivalence that comes from some metric space is a metric-line equivalence. We present an infinite family of so-called obstacles, that is, binary equivalences that prevent an equivalence from being a metric-line equivalence.

1 Introduction

Given a metric space $(V, \text{dist})$, we follow [1] in writing $[uvw]$ to signify that $u, v, w$ are pairwise distinct points of $V$ and $\text{dist}(u, v) + \text{dist}(v, w) = \text{dist}(u, w)$. With $M$ standing for the metric space, it will be convenient to write

$$\mathcal{E}_M = \{\{x, y, z\}: [zxy] \text{ or } [xzy] \text{ or } [xyz].\}.$$  

Following [2], we say that a 3-uniform hypergraph $(V, \mathcal{E})$ is metric if there is a metric space $M$ such that $\mathcal{E} = \mathcal{E}_M$. All induced subhypergraphs of metric hypergraphs are metric, and so metric hypergraphs can be characterized as hypergraphs without certain induced subhypergraphs, namely, the minimal non-metric ones. Section 3 of [2] presents three minimal non-metric hypergraphs along with the following comment:
If there are only finitely many minimal non-metric hypergraphs, then metric hypergraphs can be recognized in polynomial time. However, it is conceivable that there are infinitely many minimal non-metric hypergraphs and it is not clear whether metric hypergraphs can be recognized in polynomial time.

Our first main result shows that there are indeed infinitely many minimal non-metric hypergraphs. To formulate it, we need the following definition: When $G$ is a graph with vertex set $V$ and edge set $E$, the hypergraph based on $G$ is the 3-uniform hypergraph with vertex set $V \cup \{x\}$, where $x \notin V$, and hyperedge set consisting of all three-point subsets of $V$ and all three-point sets $\{x, u, v\}$ such that $\{u, v\} \in E$.

**Theorem 1.** For every even integer $n$ greater than four, the hypergraph based on the cycle $C_n$ is minimal non-metric.

In 1943, Erdős [7] proved that a set of $n$ points in the Euclidean plane determines at least $n$ distinct lines unless these $n$ points are collinear. In 2006, Chen and Chvátal [5] asked whether the same statement holds true more generally in all metric spaces $M$ with the line $L_M(xy)$ determined by two points $x$ and $y$ defined by

$$L_M(xy) = \{x, y\} \cup \{z : \{x, y, z\} \in E_M\}.$$

Early progress toward the conjecture that this generalization does hold true is surveyed in [6]; contributions too recent to be included there are [3], [4], [8], [11], [12].

When $W$ is a set, we let $\binom{W}{2}$ denote the set of all 2-point subsets of $W$. We say that an equivalence relation $\equiv$ on $\binom{W}{2}$ is a metric-line equivalence if there is a metric space $M$ on ground set $W$ such that

$$L_M(ab) = L_M(cd) \iff \{a, b\} \equiv \{c, d\}$$

for every choice of two-point subsets $\{a, b\}$ and $\{c, d\}$ of $W$.

How difficult is it to tell which equivalence relations on $\binom{W}{2}$ are metric-line equivalences and which of them are not? Attempts at answering the Chen-Chvátal question could be only helped by an efficient algorithm for their recognition. Until now, all we have had here was a polynomial-time
algorithm that, given an equivalence relation \( \equiv \) on \( (W_2) \), will in some cases certify that \( \equiv \) is not a metric-line equivalence [6, Algorithms G and H]. We are going to offer another such certificate.

We say that an equivalence relation \( \equiv \) on \( (V_2) \) is an obstacle if no metric space \( M \) on a superset \( W \) of \( V \) satisfies (1) for every choice of two-point subsets \( \{a, b\} \) and \( \{c, d\} \) of \( V \). It may not be obvious that there exist any obstacles at all; our second main result shows that there are infinitely many genuinely different ones. To formulate this result, we need additional definitions again.

We say that an obstacle \( \equiv \) on \( (V_2) \) is minimal if there is no proper subset \( U \) of \( V \) such that the restriction of \( \equiv \) on \( (U_2) \) is an obstacle. Given a graph \( G \) with vertex set \( V \) and edge set \( E \), we define the equivalence relation \( G \equiv \) on \( (V_2) \) by

\[
e^G \equiv f \iff (e \in E, f \in E) \quad \text{or} \quad (e \notin E, f \notin E)
\]

**Theorem 2.** If \( G = C_n \) with \( n \) an even integer greater than four, then the equivalence relation \( G \equiv \) is a minimal obstacle.

## 2 Metric and non-metric hypergraphs

A corollary of the following lemma is one of the ingredients of our proof of Theorem 1.

**Lemma 1.** For every odd integer \( n \) greater than one, the hypergraph based on \( C_n \) is metric.

**Proof.** Writing \( n = 2s + 1 \), consider the metric space on the ground set \( \{0, 1, \ldots, 2s\} \cup \{x\} \) with \( \text{dist}(i, j) = j - i \) whenever \( 0 \leq i < j < n \) and

\[
\text{dist}(x, k) = \begin{cases} s & \text{if } k \text{ is even,} \\ s + 1 & \text{if } k \text{ is odd.} \end{cases}
\]

The next lemma comes from [10] and has been generalized as [2, Lemma 3.2].
Lemma 2. Let $M$ be a metric space and let $V$ be a subset of its ground set such that $|V| \geq 5$. If every three-point subset of $V$ belongs to $\mathcal{E}_M$, then the elements of $V$ can be renamed as $0, 1, \ldots, n - 1$ with $n = |V|$ in such a way that

$$0 \leq u < v < w < n \Rightarrow [uvw] \quad (2)$$

Lemma 3. Given a metric space $M$ on a ground set $V \cup \{x\}$, where $x \not\in V$ and $V = \{0, 1, \ldots, n - 1\}$, set

$$D_1 = \{(j, \ell) : j < \ell \text{ and } [jx\ell]\},$$
$$D_2 = \{(j, \ell) : j < \ell \text{ and } [jx\ell]\},$$
$$D_3 = \{(j, \ell) : j < \ell \text{ and } [jx\ell]\}.$$

If (2) holds true, then

$$\begin{align*}
(j, \ell) \in D_1, \; j < k < \ell &\Rightarrow (j, k) \in D_1, \; (k, \ell) \in D_1, \quad (3) \\
(j, \ell) \in D_3, \; j < k < \ell &\Rightarrow (j, k) \in D_3, \; (k, \ell) \in D_3, \quad (4) \\
(j, \ell) \in D_2, \; i < j < \ell &\Rightarrow (i, j) \in D_3, \; (i, \ell) \in D_2, \quad (5) \\
(j, \ell) \in D_2, \; j < k < m &\Rightarrow (j, m) \in D_2, \; (\ell, m) \in D_1, \quad (6) \\
(i, j) \in D_1, \; (j, k) \in D_1 &\Rightarrow (i, k) \in D_1, \quad (7) \\
(i, j) \in D_3, \; (j, k) \in D_3 &\Rightarrow (i, k) \in D_3, \quad (8) \\
(i, k) \in D_2, \; (j, k) \in D_1 &\Rightarrow (i, j) \in D_2 \text{ or } (j, i) \in D_2, \quad (9) \\
(i, k) \in D_2, \; (i, j) \in D_3 &\Rightarrow (j, k) \in D_2 \text{ or } (k, j) \in D_2, \quad (10)
\end{align*}$$

Proof. Implications (3) — (9) are special cases of the easily verifiable

$$[abd], [bcd] \Rightarrow [abc], [acd], \quad (11)$$

which has been pointed out first by Menger [9].

Actually, the conclusion of (9) can be strengthened to $(i, j) \in D_2$ since $j < i$ would contradict (3) with $(j, i, k)$ in place of $(j, k, \ell)$. Similarly, the conclusion of (10) can be strengthened to $(j, k) \in D_2$ since $k < j$ would contradict (3) with $(i, k, j)$ in place of $(j, k, \ell)$. These niceties are irrelevant to our purpose.

Lemma 4. Let $G$ be a graph with the vertex set $V = \{0, 1, \ldots, n - 1\}$ and an edge set $E$. Let $x$ be a point outside $V$ and let $M$ be a metric space on the ground set $V \cup \{x\}$ such that, for every $u, v, w \in V,$
\[ u < v < w \Rightarrow [uvw] \]
\[ \{u, v\} \in E \iff \{x, u, v\} \in E_M. \]

If \( G \) contains no triangle, then
\[ E \subseteq \{(0, 1), (1, 2), \ldots (n - 2, n - 1), (n - 1, 0)\}. \]

**Proof.** Lemma 3 guarantees that (3) — (9) are satisfied. Since a two-point subset \( \{u, v\} \) of \( V \) belongs to \( E \) if and only if \( \{x, u, v\} \in E_M \), we have
\[ E = \{\{u, v\} : (u, v) \in D_1 \cup D_2 \cup D_3\} \]
If \( G \) contains no triangle, then (3), (4) show that every \((j, \ell)\) in \( D_1 \cup D_3 \) has \( \ell = j + 1 \) and (5) and (6) show that every \((j, \ell)\) in \( D_2 \) has \( j = 0, \ell = n - 1 \). Therefore
\[ D_1 \cup D_3 \subseteq \{(i, i + 1) : 0 \leq i < n - 1\} \text{ and } D_2 \subseteq \{(0, n - 1)\}. \]

**Proof of Theorem 1**  First, we shall deduce a contradiction from the assumption that the hypergraph \( H \) based on a \( C_n \) with \( n \) even and greater than four is metric. For this purpose, let \( V \) denote the vertex set of the \( C_n \) and let \( E \) denote its edge set. If \( H \) is metric, then Lemma 2 guarantees that (possibly after renaming the vertices) the hypothesis of Lemma 4 with \( G = C_n \) is satisfied, and so
\[ E = \{(0, 1), (1, 2), \ldots (n - 2, n - 1), (n - 1, 0)\}. \]
By (10) with \( i = 0, j = 1, k = n - 1 \), we have
\[ (0, 1) \in D_1; \]
by (9) with \( i = 0, j = n - 2, k = n - 1 \), we have
\[ (n - 2, n - 1) \in D_3; \]
by (7) and (8) with \( j = i + 1, k = i + 2 \), we have
\[ (i, i + 1) \in D_1 \iff (i + 1, i + 2) \in D_3 \quad \text{whenever } 0 \leq i < n - 2. \]
Therefore the ordered pairs \((0,1), (1,2), (2,3), \ldots, (n-2, n-1)\) alternate between \(D_1\) and \(D_3\), beginning with \((0,1)\) in \(D_1\) and ending with \((n-2, n-1)\) in \(D_3\). This contradicts the assumption that \(n\) is even.

It remains to be proved that every proper induced subhypergraph \(H_0\) of \(H\) is metric. For this purpose, note that \(H_0\) is an induced subhypergraph of the complete 3-uniform hypergraph on \(n\) vertices, which is trivially metric, or an induced subhypergraph of the hypergraph based on the path of order \(n-1\), which (being a subhypergraph of the hypergraph based on any larger cycle) is metric by Lemma 1.

The lower bound on \(n\) in Theorem 1 is essential: the hypergraph based on \(C_4\) is metric. To see this, note that the metric space on the ground set \(\{a, b, c, d, x\}\) with metric defined by the chart

\[
\begin{array}{cccc|c}
  a & b & c & d & x \\
  a & 0 & 1 & 2 & 1 & 2 \\
  b & 1 & 0 & 1 & 2 & 3 \\
  c & 2 & 1 & 0 & 1 & 2 \\
  d & 1 & 2 & 1 & 0 & 3 \\
  x & 2 & 3 & 2 & 3 & 0 \\
\end{array}
\]

has

\([abc], [bcd], [cda], [dab], [xab], [xad], [xcb], [xcd], [xac], [axc], [acx], [xbd], [bxd], [bdx].]

But none of \([xac], [axc], [acx], [xbd], [bxd], [bdx].]

The following lemma is used in the next section in our proof of Theorem 2:

**Lemma 5.** The hypergraph based on the complement \(\overline{P_5}\) of the path of order five is non-metric.

**Proof.** We shall deduce a contradiction from the assumption that the hypergraph \(H\) based on \(\overline{P_5}\) is metric. For this purpose, let \(V\) denote the vertex set of our \(\overline{P_5}\) and let \(E\) denote its edge set. If \(H\) is metric, then there is a metric space \(M\) on the ground set \(V \cup \{x\}\), where \(x \not\in V\), such that all three-point subsets of \(V\) belong to \(E_M\) and such that a two-point subset \(\{u, v\}\) of \(V\) belongs to \(E\) if and only if \(\{x, u, v\} \in E_M\). By Lemma 2, the elements of \(V\) can be renamed as 0, 1, \ldots, 4 in such a way that

\[0 \leq u < v < w < 5 \Rightarrow [uvw]\]
and Lemma 3 guarantees that (3) — (10) are satisfied. Since a two-point subset \( \{u, v\} \) of \( V \) belongs to \( E \) if and only if \( \{x, u, v\} \in \mathcal{E}_M \), we have
\[
E = \{\{u, v\} : (u, v) \in D_1 \cup D_2 \cup D_3\}.
\]
Since \( P_5 \) contains only one triangle, (3) and (4) show that every \((j, \ell)\) in \( D_1 \cup D_3 \) has \( \ell \leq j + 2 \) and (5), (6) show that every \((j, \ell)\) in \( D_2 \) has \( \ell - j \geq 3 \). Explicitly, we have
\[
D_1 \cup D_3 \subseteq \{(0, 1), (1, 2), (2, 3), (3, 4), (0, 2), (1, 3), (2, 4)\},
D_2 \subseteq \{(0, 3), (0, 4), (1, 4)\}.
\]
Next, let \( T \) denote the unique triangle in our \( P_5 \) and let \( P \) denote the three-edge path resulting when the three edges of \( T \) are deleted.

Since no edge of \( P \) extends to a triangle, implications (3), (4) show that none of \( \{0, 2\}, \{1, 3\}, \{2, 4\} \) can belong to \( P \) and implications (5), (6) show that neither of \( \{0, 3\}, \{1, 4\} \) can belong to \( P \). Hence each of the three edges of \( P \) must be one of \( \{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{0, 4\} \).

If the vertex set of \( T \) is \( \{0, 1, 3\} \), then \( (1, 3) \in D_1 \cup D_3 \) and by (3) or (4), \( E \) contains also \( \{1, 2\} \) and \( \{2, 3\} \). However, \( E \) contains only one triangle, so this won’t happen. We rule out the possibility of \( T \) being one of \( \{0, 2, 3\}, \{0, 2, 4\}, \{1, 2, 4\}, \{1, 3, 4\} \) similarly. The remaining options are \( \{0, 1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}, \{0, 1, 4\}, \) and \( \{0, 3, 4\} \). Flip symmetry \( i \leftrightarrow 4 - i \) reduces these to the following three.

**Option 1:** the vertex set of \( T \) is \( \{0, 1, 2\} \),
**Option 2:** the vertex set of \( T \) is \( \{1, 2, 3\} \),
**Option 3:** the vertex set of \( T \) is \( \{0, 3, 4\} \).

We are going to eliminate these three options one by one.

**Option 1:** \( E = \{\{0, 1\}, \{1, 2\}, \{0, 2\}, \{2, 3\}, \{3, 4\}, \{0, 4\}\} \).
\((0, 4) \in D_2 \) and \( \{2, 4\} \notin E \) force \((0, 2) \notin D_3 \) [and so \((0, 2) \in D_1 \)] by (10),
\((0, 2) \in D_1 \) and \( \{0, 3\} \notin E \) force \((2, 3) \notin D_1 \) [and so \((2, 3) \in D_3 \)] by (7),
\((0, 4) \in D_2 \) and \( \{0, 3\} \notin E \) force \((3, 4) \notin D_1 \) [and so \((3, 4) \in D_3 \)] by (9),
\((2, 3) \in D_3 \) and \( \{3, 4\} \in D_3 \) force \((2, 4) \in D_3 \) by (8).

However, \( (2, 4) \in D_3 \) is incompatible with \( \{2, 4\} \notin E \).

**Option 2:** \( E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{0, 1\}, \{3, 4\}, \{0, 4\}\} \).
\((0, 4) \in D_2 \) and \( \{1, 4\} \notin E \) force \((0, 1) \notin D_3 \) [and so \((0, 1) \in D_1 \)] by (10),
\((0, 1) \in D_1 \) and \( \{0, 3\} \notin E \) force \((1, 3) \notin D_1 \) [and so \((1, 3) \in D_3 \)] by (7),
(0, 4) ∈ D_2 and \{0, 3\} \not\in E \text{ force } (3, 4) \not\in D_1 \text{ [and so } (3, 4) \in D_3 \text{] by (9),}
(1, 3) \in D_2 \text{ and } (3, 4) \in D_3 \text{ force } (1, 4) \in D_3 \text{ by (8).}
However, (1, 4) \in D_3 \text{ is incompatible with } \{1, 4\} \not\in E.

**Option 3:** \(E = \{\{0, 3\}, \{0, 4\}, \{3, 4\}, \{0, 1\}, \{1, 2\}, \{2, 3\}\}.
(0, 3) \in D_2 \text{ and } \{1, 3\} \not\in E \text{ force } (0, 1) \not\in D_3 \text{ [and so } (0, 1) \in D_1 \text{] by (10),}
(0, 1) \in D_1 \text{ and } \{0, 2\} \not\in E \text{ force } (1, 2) \not\in D_1 \text{ [and so } (1, 2) \in D_3 \text{] by (7),}
(0, 3) \in D_2 \text{ and } \{0, 2\} \not\in E \text{ force } (2, 3) \not\in D_1 \text{ [and so } (2, 3) \in D_3 \text{] by (9),}
(1, 2) \in D_3 \text{ and } (2, 3) \in D_3 \text{ force } (1, 3) \in D_3 \text{ by (8).}
However, (1, 3) \in D_3 \text{ is incompatible with } \{1, 3\} \not\in E.

By the way, the hypergraph based on \(P_5\) is minimal non-metric. To verify this, enumerate the vertices of \(P_5\) as \(a, b, c, d, e\) in such a way that the edges of this \(P_5\) are
\[
\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{e, b\}, \{e, c\}.
\]
Now each of \(P_5 - b\) and \(P_5 - c\) is a \(P_4\); by Lemma 1, the hypergraph based on \(P_4\) is metric. Next, \(P_5 - e\) is a \(C_4\); by the comment following our proof of Theorem 1, the hypergraph based on \(C_4\) is metric. Finally, \(P_5 - a\) and \(P_5 - d\) are isomorphic; to see that the hypergraph based on \(P_5 - a\) is metric, note that the metric space on the ground set \(\{e, b, c, d, x\}\) with metric defined by the chart
\[
\begin{array}{c|ccccc}
 & e & b & c & d & x \\
\hline
e & 0 & 1 & 2 & 3 & 2 \\
b & 1 & 0 & 1 & 2 & 3 \\
c & 2 & 1 & 0 & 1 & 4 \\
d & 3 & 2 & 1 & 0 & 3 \\
x & 2 & 3 & 4 & 3 & 0
\end{array}
\]
has
\[
[ebc], [ebd], [ecd], [bcd], [xeb], [xec], [xdc], [xbc],
\]
but none of \([xed], [edx], [ebx], [xbd], [bxd], [bxd]\).

### 3 Metric-line equivalences

**Lemma 6.** If \(G\) is a graph such that neither the hypergraph based on \(G\) nor the hypergraph based on its complement \(\overline{G}\) is metric, then the equivalence
relation $\equiv$ is an obstacle.

Proof. Assuming that $\equiv$ is not an obstacle, we will prove that at least one of the two hypergraphs based on $G$ and on $\overline{G}$ is metric. This conclusion is immediate when $G$ or $\overline{G}$ is a complete graph, and so we may assume that each of $G$ and $\overline{G}$ has at least one edge. Since $\equiv$ is not an obstacle, there is a metric space $M$ on a superset $W$ of the vertex set $V$ of $G$ such that

$$L_M(ab) = L_M(cd) \iff \{a, b\} \equiv \{c, d\}$$

for every choice of two-point subsets $\{a, b\}$ and $\{c, d\}$ of $V$. Let $L$ denote the common value of $L_M(uv)$ with $\{u, v\}$ ranging over the edge set of $G$ and let $L'$ denote the common value of $L_M(uv)$ with $\{u, v\}$ ranging over the edge set of $\overline{G}$. Since $L$ and $L'$ are distinct, their symmetric difference $(L - L') \cup (L' - L)$ is nonempty. Switching $G$ and $\overline{G}$ if necessary, we may assume that $L - L'$ is nonempty. Now we are going to prove that the hypergraph based on $G$ is metric. More precisely, with $x$ standing for an arbitrary but fixed element of $L - L'$, we will prove that

(i) all three-point subsets of $V$ belong to $E_M$,
(ii) $x \not\in V$,
(iii) if $\{u, v\}$ is an edge of $G$, then $\{x, u, v\} \in E_M$,
(iv) if $\{u, v\}$ is an edge of $\overline{G}$, then $\{x, u, v\} \not\in E_M$.

To prove (i), consider an arbitrary three-point subset $T$ of $V$. Since at least two of the three two-point subsets of $T$ belong to the same class of $\equiv$, we may label the elements of $T$ as $u, v, w$ in such a way that $\{u, v\} \equiv \{v, w\}$. This means that $L_M(uv) = L_M(vw)$, and so $w \in L_M(uv)$, and so $\{u, v, w\} \in E_M$. To prove (ii), we rely on the assumption that each of $G$ and $\overline{G}$ has at least one edge; this assumption along with (i) guarantees that both $L$ and $L'$ contain $V$; now $x \not\in V$ follows from $x \not\in L'$. With (ii) established, (iii) and (iv) follow from $x \in L - L'$.

Proof of Theorem 2. Let $n$ be an even integer greater than four and let $G$ be the cycle $C_n$. By Theorem 1, the hypergraph based on $G$ is not metric; by Lemma 5, the hypergraph based on $\overline{G}$ is not metric; hence, by Lemma 6, the equivalence relation $\equiv$ is an obstacle.
To see that $G \equiv$ is a minimal obstacle, consider any proper subset $U$ of $V$. The restriction of $\equiv$ on $(U^2)$ is $F \equiv$, where $F$ is the subgraph of $G$ induced by $U$. Since $F$ is an induced subgraph of $P_{n-1}$, Lemma 1 guarantees that there is a metric space $M$ on the ground set $U \cup \{x\}$, where $x \not\in U$, such that all three-point subsets of $U$ belong to $E_M$ and such that a two-point subset $\{u, v\}$ of $U$ is an edge of $F$ if and only if $\{x, u, v\} \in E_M$. Whenever $\{u, v\}$ is a two-point subset of $U$, we have

$$L_M(uv) = \begin{cases} U \cup \{x\} & \text{if } \{u, v\} \text{ is an edge of } F, \\ U & \text{otherwise}, \end{cases}$$

and so $F \equiv$ is not an obstacle. $\square$

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