GEOMETRIC SYZYGIES OF CANONICAL CURVES OF EVEN GENUS LYING ON A K3-SURFACE

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Contents

1. Introduction 1
2. Syzygies of Low Rank 4
3. Syzygy Schemes 5
4. Generic Syzygy Schemes 5
5. \( p \)-th Syzygies of Rank \( p + 1 \) 9
6. \( p \)-th Syzygies of Rank \( p + 2 \) 10
7. \( p \)-th Syzygies of Rank \( p + 3 \) 12
8. Syzygies of \( \mathbb{G} \) 13
9. Linear sections of \( \mathbb{G} \) 19
10. Syzygies of \( S \) 22
11. Syzygies of \( C \) 24
12. Cohomology of \( S^jL(-j) \) 26
References 28

1. Introduction

In this paper we study the syzygies of canonical curves \( C \subset \mathbb{P}^{g-1} \) for \( g = 2k \).

In [GL84] Green and Lazarsfeld construct low-rank-syzygies of \( C \) from special linear systems on \( C \). More precisely linear systems of Clifford index \( c \) give a \((g - c - 3)\)-rd syzygy. We call these syzygies geometric syzygies. Green’s conjecture [Gre84a] paraphrased in this way is

1.1. Conjecture (Green). Let \( C \) be a canonical curve, then

\[ C \text{ has no geometric } p \text{-th syzygies } \iff \text{C has no } p \text{-th syzygies at all} \]
This conjecture as received a lot of attention in the last years and it is now known in many cases [Pet23], [Gre84a], [Sch86], [Sch88], [Voi88], [Sch91], [Ehb94], [HR98], [Tei99], [Voi01].

Much less is known about the following natural generalization of Green’s conjecture:

1.2. Conjecture (Geometric Syzygy Conjecture). Let $C$ be a canonical curve, then the scheme of geometric $p$-th syzygies spans the space of all $p$-th syzygies.

For general canonical curves both conjectures are equivalent in the range $p \geq \frac{g-3}{2}$ since a general canonical curve has no linear systems of Clifford index $c \leq \frac{g-3}{2}$.

The case $p = 0$ (geometric quadrics) of the geometric syzygy conjecture was proved by [AM67] for general canonical curves, and by [Gre84b] for all canonical curves. The case $p = 1$ was done for general canonical curves of genus $g \geq 9$ in [vB00].

In this paper, based on Voisin’s recent result [Voi01], we clarify the cases with $g = 2k$ and $p = k-2$ for canonical curves lying on general $K3$ surfaces.

![Figure 1. The geometric syzygy conjecture for general curves.](image)

We proceed as follows:

By a construction of Mukai [Muk89] a general $K3$-surface $S$ of sectional genus $g = 2k$ can be embedded in a Grassmannian $G = \text{Gr}(k+2,2)$, such that

$$S \subset G \cap \mathbb{P}^g$$

for a particular $\mathbb{P}^g \not\subset G$. We reconstruct this embedding from a $(k-2)$-nd Grassmannian syzygy of $S$.

The minimal free resolution of $G$ is known by work of Józefiak, Pragacz and Weyman [JPW81]. It has the form:
using the MACAULAY-notation \([GS]\). The minimal free resolution of the \(K3\) surface is also known by a recent result of Voisin \([Voi01]\):

Our main observation is that both resolutions have a linear strand of the same length and that the dimensions of the last nonzero linear syzygy spaces (*) are equal.

Using Voisin’s theorem we show that the natural map between the two spaces is an isomorphism even though the intersection \(G \cap \mathbb{P}^g\) is not of expected dimension. It turns out to be enough, that \(S\) is irreducible.

We go on to describe the space of minimal rank syzygies \(Y_{min}\) of \(G\) and \(S\) as a \((k-2)\)-uple embedded \(\mathbb{P}^{k+1}\).

Cutting down one more dimension to a canonical curve \(C \subset \mathbb{P}^{g-1}\) we obtain a finite number of lines of geometric syzygies in \(\mathbb{P}^{k+1}\) whose image under the \((k-2)\)-uple embedding spans the space of all \((k-2)\)-nd syzygies of \(C\).

As a main tool of our work we associate two geometric objects to a syzygy \(s\):

1. The space of linear forms \(L_s\) involved in \(s\). This space and its associated vector bundle of linear forms allows us to control the rank of \(s\) after we have cut down to \(S\) or \(C\).
2. The syzygy scheme \(\text{Syz}(s)\) of \(s\) which is in a certain sense the vanishing locus of \(s\). It is used to prove that certain syzygies do survive the restriction to the linear subspaces \(\mathbb{P}^{g}\) and \(\mathbb{P}^{g-1}\).

Several of our arguments also appear in Voisin’s proof of her theorem, but with different aims. Most notably she also shows that \(\mathbb{P}^{g} \cap G^* = \emptyset\) and that the above rational normal curves span the space of the \((k-2)\)uple embedding. As far as we know our corollary \(9.3\) is new.

I would to thank Frank-Olaf Schreyer for introducing me to this subject and for the many helpful discussions leading to this paper. Also I am grateful to Thomas Eckl, for the numerous discussions which clarified many details of this work.
2. Syzygies of Low Rank

Let \( X \subseteq \mathbb{P}^n \cong \mathbb{P}(V) \) be an irreducible non-degenerate variety, \( I_X \) generated by quadrics and

\[
I_X \leftarrow V_0 \otimes \mathcal{O}(-2) \xleftarrow{\varphi_1} V_1 \otimes \mathcal{O}(-3) \xleftarrow{\varphi_2} \ldots \xleftarrow{\varphi_m} V_m \otimes \mathcal{O}(-m - 2)
\]

the linear strand of its minimal free resolution

2.1. Definition. An element \( s \in V_p \) is called a \( p \)-th (linear) syzygy of \( X \). \( \mathbb{P}(V^*_p) \) is called the space of \( p \)-th syzygies.

Every linear syzygy \( s \) involves a well-defined number of linearly independent linear forms. This number is called the rank of \( s \). In a more formal way we have:

2.2. Definition. Let \( s \in V_p \) be a syzygy,

\[
\tilde{\varphi} : V_p \to V_{p-1} \otimes V
\]

the map of vector spaces induced by \( \varphi_p \). Then the image of \( s \) under \( \tilde{\varphi} \) can be interpreted as a linear map:

\[
\tilde{\varphi}(s) \in V_{p-1} \otimes V \cong \text{Hom}(V^*_{p-1}, V).
\]

With this

\[
L_s := \text{Im} \tilde{\varphi}(s) \subset V
\]

is called the space of linear forms involved in \( s \), and

\[
\text{rank } s := \text{rank } \tilde{\varphi}(s) = \dim L_s
\]

the rank of \( s \).

To apply geometric methods to the study of low rank syzygies we projectivize the space of \( p \)-th syzygies to \( \mathbb{P}(V^*_p) \) and give a determinantal description of the space \( Y_{\text{min}} \) of minimal rank syzygies. The linear forms involved in these syzygies define a vector bundle on \( Y_{\text{min}} \):

2.3. Definition. On the space of \( p \)-th syzygies \( \mathbb{P}(V^*_p) \) the map of vector spaces \( \tilde{\varphi}_p \) induces a map of vector bundles

\[
\psi : V^*_{p-1} \otimes \mathcal{O}_{\mathbb{P}(V_p^*)}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}(V_p^*)}
\]

that satisfies

\[
\psi|_{Y_{\text{min}}} = \tilde{\varphi}_p(s) \in \text{Hom}(V^*_{p-1}, V)
\]

The determinantal loci \( Y_r(\psi) \subset \mathbb{P}(V^*_p) \) of \( \psi \) are called schemes of rank \( r \) syzygies, since the syzygies in their support have rank \( \leq r \).

On the scheme of minimal rank syzygies \( Y_{\text{min}} := Y_{\text{min}}(\psi) \) the restricted map \( \psi|_{Y_{\text{min}}} \) has constant rank \( r_{\text{min}} \). Therefore the image \( \mathcal{L} := \text{Im} (\psi|_{Y_{\text{min}}}) \) is a vector bundle. We call it the vector bundle of linear forms, since

\[
\mathcal{L}|_s = L_s \subset V
\]

for all minimal rank syzygies \( s \in Y_{\text{min}} \).
3. Syzygy Schemes

A second geometric object associated to a syzygy $s$ is obtained by calculating $V_p$ via Koszul cohomology:

3.1. Lemma. 

$$V_p \cong \ker (\Lambda^p V \otimes (I_X)_2 \to \Lambda^{p-1} V \otimes (I_X)_3) \cong H^0(\mathbb{P}(V), \Omega^p(p + 2) \otimes I_X)$$

Proof. [Gre84a], [Ehb94]

So linear syzygies are twisted $p$-forms that vanish on $X$. Often these $p$-forms vanish on a larger variety:

3.2. Definition. Let $s \in V_p$ be a $p$-th syzygy of $X$. Then the vanishing set $\text{Syz}(s)$ of the corresponding twisted $p$-form is called the syzygy scheme of $s$.

The ideal of a syzygy scheme can be calculated via:

3.3. Lemma and Definition. Let $\{v_\alpha\}$ be a basis of $\Lambda^p V$. Then every syzygy $s \in V_p$ can be uniquely written as

$$s = \sum_\alpha v_\alpha \otimes Q_\alpha$$

where $Q_\alpha$ are quadrics in the ideal of $X$, and the ideal of $\text{Syz}(s)$ is generated by the $Q_\alpha$. This ideal is also called the ideal of quadrics involved in $s$.

Proof. Since $V_q = \ker (\Lambda^p V \otimes (I_X)_2 \to \Lambda^{p-1} V \otimes (I_X)_3)$ every syzygy can be written as above. Since the $v_\alpha$ are linearly independent, $s = \sum_\alpha v_\alpha \otimes Q_\alpha$ vanishes if and only if all $Q_\alpha$ vanish.

Often the syzygies of low rank have the most interesting syzygy varieties. Some of them can be calculated with the methods of the next section.

4. Generic Syzygy Schemes

We now consider syzygies $s$ of low rank $r$ and their syzygy schemes. At it turns out, these syzygy schemes are always cones over linear sections of certain generic syzygy schemes:

4.1. Definition. Let $L$ be an $r$-dimensional vector space. Then

$$\text{Gensyz}_p(L) = \{(l^*, a^*) \in L^* \oplus \Lambda^{r-p-1} L^* | l^* \wedge a^* = 0\} \subset \mathbb{P}(L \oplus \Lambda^{r-p-1} L)$$

is called the $p$-th generic syzygy scheme of $L$.

This definition goes back to Ensen and Schreyer [Ens94]. We will now recall some well known facts about generic syzygy schemes.

First of all, the ideal of a generic syzygy variety can be easily calculated:

4.2. Proposition. $\text{Gensyz}_p(L)$ is cut out by the quadrics in the image of

$$\Lambda^{r-p} L \to L \otimes \Lambda^{r-p-1} L \to S_2(L \oplus \Lambda^{r-p-1} L).$$
Proof. Consider the natural map
\[
\varphi: \quad L^* \otimes \Lambda^{r-p-1}L^* \to \Lambda^{r-p}L^*,
\]
\[
l^* \otimes a^* \mapsto l^* \wedge a^*
\]
Notice that \((l^*, a^*)\) is in \(\text{Gensyz}_p(L)\) if and only if \((l^* \otimes a^*)\) is in the kernel of \(\varphi\). Now \(\ker \varphi\) is cut out by the linear forms in the image of the dual map
\[
\varphi^*: \Lambda^{r-p}L \to L \otimes \Lambda^{r-p-1}L
\]
On \(\mathbb{P}(L \oplus \Lambda^{r-p-1}L)\) these linear forms become quadrics that cut out \(\text{Gensyz}_p(L)\).

We now identify an up to a constant canonically defined \(p\)-th syzygy \(s_{\text{gen}}\) of \(\text{Gensyz}_p(L)\) with \(L_{s_{\text{gen}}} = L\):

**4.3. Proposition and Definition.** Let \(L\) be an \(r\)-dimensional vector space, and \(s_{\text{gen}} \in \Lambda^r L\) an orientation. Then \(s_{\text{gen}}\) is a natural \(p\)-th syzygy of \(\text{Gensyz}_p(L)\) via the inclusion
\[
\Lambda^r L \hookrightarrow \Lambda^p L \otimes (L \otimes \Lambda^{r-p-1}L).
\]
Furthermore
\[
\text{Syz}(s_{\text{gen}}) = \text{Gensyz}_p(L).
\]
We call \(s_{\text{gen}}\) a generic \(p\)-th syzygy.

Proof. First we show, that \(s_{\text{gen}}\) is really a \(p\)-th syzygy of \(\text{Gensyz}_p(L)\). For this we have the following maps

\[
\begin{array}{ccccccc}
s \in \Lambda^r L &\xrightarrow{1} & \Lambda^p \otimes \Lambda^{r-p}L &\xrightarrow{2} & \Lambda^{p+1} L \otimes \Lambda^{r-p-1}L &\xrightarrow{4} & \Lambda^p L \otimes (L \otimes \Lambda^{r-p-1}L) &\xrightarrow{5} & \Lambda^{p-1} \otimes (S_2(L) \otimes \Lambda^{r-p-1}L) \\
& & & & & & & & \Lambda^p V \otimes S_2(V) &\xrightarrow{6} & \Lambda^{p-1} V \otimes S_3(V) \\
\end{array}
\]

Mapping \(s\) via \(1\) to \(s'\) and then via \(3\) to \(s_{\text{gen}}\) shows
\[
s_{\text{gen}} \in \Lambda^p L \otimes (I_{\text{Gensyz}_p(L)})_2
\]
since
\[
(I_{\text{Gensyz}_p(L)})_2 = \text{Im}(\Lambda^{r-p}L \to L \otimes \Lambda^{r-p-1}L)
\]
by proposition 4.2.
Mapping $s$ via 2 and 4 to $s_{gen}$ shows that $s_{gen}$ is in the kernel of 5 since the middle row of the above diagram is a complex. Now 5, which is the restriction of 6, restricts further to

$$
\varphi: \Lambda^p L \otimes (I_{\text{Gensyz}_p(L)})_2 \to \Lambda^p L \otimes (I_{\text{Gensyz}_p(L)})_3
$$

and $s_{gen}$ is consequently also in the kernel of $\varphi$. This proves that $s_{gen}$ is a $p$-th syzygy of $\text{Gensyz}_p(L)$.

For $\text{Syz}(s_{gen}) = \text{Gensyz}_p(L)$ notice that $s'$ is a trace element

$$
s' = \sum w^*_\alpha \otimes w_\alpha \in (\Lambda^{r-p} L)^* \otimes \Lambda^{r-p} L
$$

with $\{w_\alpha\}$ a basis of $\Lambda^{r-p} L$ and $(\Lambda^{r-p} L)^* \cong \Lambda^p L$ via the orientation $s$. Now since $\{w_\alpha\}$ is a basis of $\Lambda^{r-p} L$, the quadrics involved in image $s_{gen}$ of $s'$ form a basis of

$$
\text{Im}(\Lambda^{r-p} L \to L \otimes \Lambda^{r-p-1} L) = (I_{\text{Gensyz}_p(L)})_2.
$$

This proves $\text{Syz}(s_{gen}) = \text{Gensyz}_p(L)$. \hfill \Box

We are now ready to stated the main result of this section:

4.4. Theorem. Let $X \subset \mathbb{P}(V)$ be a non degenerate, possibly reducible variety, $I_X$ generated by quadrics and $s \in V_p$ a $p$-th syzygy of $X$. If we denote the space of linear forms involved in $s$ by $L_s$, then there exists a linear map

$$
\pi^*: L_s \oplus \Lambda^{r-p-1} L_s \to V
$$

and a generic $p$-th syzygy $s_{gen}$ of $\text{Gensyz}_p(L_s)$ such that

$$
s = \pi^*(s_{gen})
$$

More geometrically consider the linear projection

$$
\pi: \mathbb{P}(V) \dashrightarrow \mathbb{P}^n \subset \mathbb{P}(L_s \oplus \Lambda^{r-p-1} L_s)
$$

corresponding to $\pi^*$. Then

$$
\text{Syz}(s) = \pi^{-1}(\mathbb{P}^n \cap \text{Gensyz}_p(L_s)).
$$

Proof. To construct $\pi^*$ we have to make the isomorphism

$$
V_p \cong \ker(\Lambda^p V \otimes (I_X)_2 \to \Lambda^{p-1} V \otimes (I_X)_3)
$$

from lemma 3.1 more explicit.

Let

$$
(*) \quad O_{\mathbb{P}(V)} \leftarrow V_0 \otimes O(-2) \leftarrow \ldots \leftarrow V_{p-1} \otimes O(-p - 1) \leftarrow V_p \otimes O(-p - 2)
$$

be the linear strand of the minimal free resolution of $X$. As in definition 2.2 $\varphi$ induces a map

$$
\tilde{\varphi}: V_p \to V_{p-1} \otimes V \cong \text{Hom}(V_{p-1}^*, V)
$$

by taking global sections. $\tilde{\varphi}(s)$ is a map from $V_{p-1}^*$ to $V$ with image $L_s$.

$s$ and $\tilde{\varphi}(s)$ induce a map between the dual of $(*)$ and the Koszul-complex associated to $L_s$. 
All liftings of $\tilde{\varphi}(s)$ except $\alpha$ are of degree 0. Since the Koszul-complex is a minimal free resolution, all these liftings are uniquely determined. Consequently $\sigma$ is also uniquely determined. $\sigma$ is a section in

$$H^0(\Lambda^p L_s \otimes O(2)) = \Lambda^p L_s \otimes S_2(V)$$

and since it factors over $V^*_0 \otimes O(2)$ all quadrics involved in $\sigma$ are in $(I_X)_2$.

On the other hand $\sigma$ factors over $\Lambda^p L_s \otimes (I_X)_2$ and therefore $\sigma \in \ker(\Lambda^p V \otimes (I_X)_2 \rightarrow \Lambda^{p-1} V \otimes (I_X)_3)$. This defines a map

$$V_p \rightarrow \ker(\Lambda^p V \otimes (I_X)_2 \rightarrow \Lambda^{p-1} V \otimes (I_X)_3).$$

For the inverse map we take

$$\sigma \in \ker(\Lambda^p V \otimes (I_X)_2 \rightarrow \Lambda^{p-1} V \otimes (I_X)_3)$$

and dualize to

$$\sigma^* : \Lambda^p L_s \otimes O(-2) \rightarrow V_0 \otimes O(-2).$$

Now $\sigma^*$ lifts to a map of complexes from the Koszul complex associated to $L^*_s$ to the linear strand of $X$. The last map

$$\Lambda^0 L^*_s \otimes O(-p - 2) \rightarrow V_p \otimes O(-p - 2)$$

produces a unique $p$-th syzygy.

Our projection $\pi$ is now constructed from the section $\alpha$ above. We have $\alpha \in H^0(\Lambda^{p+1} L_s \otimes O(1)) \cong \Lambda^{p+1} L_s \otimes V \cong \text{Hom}(\Lambda^{p+1} L^*_s, V) \cong \sigma^* \text{Hom}(\Lambda^{r-p-1} L, V)$ where the last isomorphism is obtained by choosing an orientation $\sigma^' \in \Lambda^r L_s$. Together with the inclusion

$$\iota : L_s \hookrightarrow V$$

we can define

$$\pi^* = \iota + \alpha \in \text{Hom}(L_s \oplus \Lambda^{r-p-1} L_s, V).$$

We denote the induced map on quadrics by the same letter

$$\pi^* \in \text{Hom}(L_s \otimes \Lambda^{r-p-1} L_s, S_2(V)).$$

With this we have a commutative diagram
\[
\begin{align*}
\Lambda^{p+1} L_s \otimes V & \xrightarrow{id \otimes \alpha} \Lambda^{p} L_s \otimes (I_X)_2 \\
\Lambda^{p+1} L_s \otimes \Lambda^{r-p-1} & \xrightarrow{\sigma'' \in \Lambda^{p} L_s \otimes \Lambda^{r-p-1}} \Lambda^{p} L_s \otimes (L_s \otimes \Lambda^{r-p-1}) \\
\sigma' & \in \Lambda^{r} L_s
\end{align*}
\]

where \( \sigma' \) maps via \( \sigma'' \) and \( \alpha \) to \( \sigma \) by the construction of \( \alpha \). Mapping \( \sigma' \) the other way yields a generic \( p \)-th syzygy \( \sigma_{\text{gen}} \) of \( \text{Gensyz}_p(L_s) \) by proposition/definition. The commutativity of the diagram shows

\[(id \otimes \pi^*){(\sigma_{\text{gen}})} = \sigma\]

Since \( \sigma = s \) via the natural isomorphism described above, this proves the theorem.

4.5. Corollary. With the notations above, \( \mathbb{P}^n \not\subset \text{Gensyz}_p(L) \).

Proof. Let \( s \neq 0 \) be a syzygy, and

\[\pi : \mathbb{P}(V) \dashrightarrow \mathbb{P}^n\]

the corresponding linear projection.

Suppose \( \mathbb{P}^n \) is contained in \( \text{Gensyz}_p(L) \). Then by the above theorem we have

\[\text{Syz}(s) = \pi^{-1}(\mathbb{P}^n \cap \text{Gensyz}_p(L)) = \pi^{-1}\mathbb{P}^n = \mathbb{P}(V)\]

Consequently all quadrics involved in \( s \) must vanish on \( \mathbb{P}(V) \), which is not possible for \( s \neq 0 \).

5. \( p \)-th Syzygies of Rank \( p + 1 \)

The lowest possible rank of a \( p \)-th linear syzygy is \( p + 1 \). As it will turn out, only reducible varieties can have such a syzygy. To prove this we need:

5.1. Proposition. Let \( L \) be a \((p + 1)\)-dimensional vector space. Then

\[\text{Gensyz}_p(L) \cong \mathbb{P}^p \cup \mathbb{P}^0 \subset \mathbb{P}^{p+1}\]

Proof. We have \( r - p - 1 = 0 \), so

\[\text{Gensyz}_p(L) \subset \mathbb{P}(L \oplus \Lambda^0 L) \cong \mathbb{P}^{p+1}\]

Let \( \{l_1, \ldots, l_{p+1}\} \) be a basis of \( L \) and \( a_0 \) a generator of \( \Lambda^0 L \). The ideal of \( \text{Gensyz}_p(L) \) is generated by the the image of

\[
\begin{align*}
\Lambda^{1} L & \to L \otimes \Lambda^{0} L \to S_2(L \oplus \Lambda^{0} L) \\
l_i & \mapsto l_i \otimes a_0 \mapsto l_i \cdot a_0
\end{align*}
\]
and consequently
\[ \text{Gensyz}_p(L) = V(l_1a_0, \ldots, l_{p+1}a_0) = V(l_1, \ldots, l_{p+1}) \cup V(a_0) = \mathbb{P}^0 \cup \mathbb{P}^p \subset \mathbb{P}^{p+1} \]

5.2. Corollary. Let \( X \subset \mathbb{P}(V) \) be a non degenerate scheme, \( I_X \) generated by quadrics and \( s \in V_p \) a \( p \)-th syzygy of rank \( p + 1 \). Then \( X \) is reducible.

Proof. Let \( L = L_s \) be the space of linear forms involved in \( s \). By theorem \[4.4\] there exists a linear projection
\[
\pi : \mathbb{P}(V) \to \mathbb{P}^n \subset \mathbb{P}(L \oplus \Lambda^1 L)
\]
such that
\[
\pi(X) \subset \pi(\text{Syz}(s)) \subset \mathbb{P}^n \cap \text{Gensyz}_p(L) = \mathbb{P}^n \cap (\mathbb{P}^p \cup \mathbb{P}^0)
\]
Since \( \mathbb{P}^n \not\subset \mathbb{P}^p \) by corollary \[4.5\] and \( \mathbb{P}^p \) is a hypersurface in \( \mathbb{P}^{p+1} \) we have
\[
\mathbb{P}^n \cap \mathbb{P}^p = \mathbb{P}^{n-1}
\]
Now \( X \) is non degenerate in \( \mathbb{P}(V) \) so \( \pi(X) \) is non degenerate in \( \mathbb{P}^n \). Therefore
\[
\pi(X) \not\subset \mathbb{P}^n \cap \mathbb{P}^p = \mathbb{P}^{n-1} \quad \text{and} \quad \pi(X) \not\subset \mathbb{P}^n \cap \mathbb{P}^0.
\]
Consequently \( X \) has to be reducible. \[\square\]

5.3. Definition. \( p \)-th syzygies of rank \( p + 1 \) are called reducible syzygies.

6. \( p \)-th Syzygies of Rank \( p + 2 \)

If \( X \) is a non degenerate irreducible variety, the lowest possible rank of a \( p \)-th syzygy is \( p + 2 \). As noted by Green and Lazarsfeld these syzygies are closely connected to linear systems on \( X \). We will recall the corresponding facts in this section.

Let’s start by calculating the relevant generic syzygy variety:

6.1. Proposition. Let \( L \) be a \((p+2)\)-dimensional vector space. Then
\[
\text{Gensyz}_p(L) \cong \mathbb{P}^1 \times \mathbb{P}^{p+1} \subset \mathbb{P}^{2p+3}
\]
where the inclusion is the Segre-embedding.

Proof. \( r - p - 1 = 1 \). Therefore
\[
\text{Gensyz}_p(L) \subset \mathbb{P}(L \oplus \Lambda^1 L) \cong \mathbb{P}^{2p+3}
\]
Let \( \{l_i\} \) be a basis of \( L \) and \( \{a_i\} \) a basis of \( \Lambda^1 L \). By proposition \[4.2\] \( \text{Gensyz}_p(L) \) is cut out by the image of
\[
\Lambda^2 L \to L \otimes \Lambda^1 L \to S_2(L \oplus \Lambda^1 L)
\]
\[
l_i \wedge l_j \mapsto l_i \otimes a_j - l_j \otimes a_i \mapsto l_ia_j - l_ja_i.
\]
Notice that this image is also generated by the \( 2 \times 2 \)-minors of
\[
M = \begin{pmatrix}
l_1 & \cdots & l_{p+2} \\
a_1 & \cdots & a_{p+2}
\end{pmatrix}.
\]
Now these are the equations of the Segre embedded \( \mathbb{P}^1 \times \mathbb{P}^{p+1} \). This proves the proposition. \[\square\]
6.2. Corollary. Let $X \subset \mathbb{P}(V)$ be a non degenerate irreducible variety, $I_X$ generated by quadrics and $s \in V_p$ a $p$-th syzygy of rank $p + 2$. Then $X$ is contained in a scroll $S$ of degree $p + 2$ and codimension $p + 1$.

Proof. Let $L = L_s$ be the space of linear forms involved in $s$. By theorem 4.4 there exists a linear projection

$$\pi: \mathbb{P}(V) \rightarrow \mathbb{P}^n \subset \mathbb{P}(L \oplus \Lambda^1 L)$$

such that

$$\pi(X) \subset \pi(Syz(s)) \subset \mathbb{P}^n \cap \text{Gensyz}_p(L) = \mathbb{P}^n \cap (\mathbb{P}^1 \times \mathbb{P}^{p+1})$$

Since $\mathbb{P}^1 \times \mathbb{P}^{p+1}$ has codimension $p + 1$ and degree $p + 2$ in $\mathbb{P}(L \oplus \Lambda^1 L)$ we only have to prove that this intersection is of expected codimension. By Eisenbud [Eis95, Ex. A2.19] this is the case if the matrix

$$M = \begin{pmatrix} l_1 & \cdots & l_{p+2} \\ a_1 & \cdots & a_{p+2} \end{pmatrix}$$

whose $2 \times 2$-minors cut out $\mathbb{P}^1 \times \mathbb{P}^{p+1}$ remains 1-generic when restricted to $\mathbb{P}^n$. Now if $M|_{\mathbb{P}^n}$ wasn’t 1-generic, we would have, after some row and column operations, a $2 \times 2$-minor of the form

$$\det \begin{pmatrix} l & l' \\ 0 & a \end{pmatrix} = l \cdot a$$

The pullback of this quadric to $\mathbb{P}(V)$ is involved in the syzygy $s$ and therefore contained in the ideal of $X$. But this is impossible, since $X$ is irreducible and non degenerate. \qed

This suggests the following definition:

6.3. Definition. The $p$-th syzygies of rank $p + 2$ are called scrollar syzygies. The total space of these syzygies is called the $p$-th space of scrollar syzygies.

We can construct special linear syzygies on $X$ from scrollar syzygies by intersecting the fibers of the corresponding scroll $S$ with $X$. For a canonical curve we have the following well known fact:

6.4. Proposition. Let $C \subset \mathbb{P}^{g-1}$ be a non hyperelliptic canonical curve of genus $g$, $s \in V_p$ a $p$-th scrollar syzygy and $D = C \cap \mathbb{P}^{g-p-3}$ with $\mathbb{P}^{g-p-3} = V(L_s)$. Then $|D|$ is a special linear system with Clifford index $\text{cliff}(D) \leq g - p - 3$.

Proof. see for example [vBR01] \qed

6.5. Remark. In [GL84] Green and Lazarsfeld use linear systems $|D|$ of Clifford index $\text{cliff}(D) = g - p - 3$ to construct geometric $p$-th syzygies of $C$. If $|D|$ is a $g_{g-p-1}$ these syzygies are scrollar.

We can now make a precise statement of the geometric syzygy conjecture for general canonical curves.
6.6. Conjecture (Generic Geometric Syzygy Conjecture). Let $C \subset \mathbb{P}^{g-1}$ be a general canonical curve of genus $g$. Then all minimal rank syzygies are scrollar, and all spaces of scrollar syzygies are non degenerate.

6.7. Remark. For special canonical curves it is important to consider the scheme structure on the space of scrollar syzygies as can be seen in the case of a curve of genus 6 with only one $g_1^4$ [AH81, p. 174].

Also there are geometric $p$-th-syzygies in the sense of Green and Lazarsfeld [GL84] which are not scrollar. These must also be considered in the case of special curves. The easiest example of this phenomenon is exhibited by the plane quintic curve of genus 6 [vB00].

7. $p$-th Syzygies of Rank $p + 3$

We now consider syzygies whose rank is slightly larger than the rank of scrollar syzygies. In the remainder of this paper we will see, that these syzygies imply just enough structure on the minimal free resolution of a general $K3$ surface with even sectional genus, to prove the geometric syzygy conjecture for canonical curves on these surfaces.

We start again by calculating the relevant generic syzygy variety:

7.1. Proposition. Let $L$ be a $(p + 3)$-dimensional vector space. Then

$$\text{Gensyz}_p(L) \cong \mathbb{G} \cup \mathbb{P}^{N-p-3} \subset \mathbb{P}^N$$

where $N = (\frac{p+4}{2}) - 1$, $\mathbb{G}$ is the Grassmannian $\text{Gr}(p+4, 2)$ and the inclusion is the Plücker embedding.

Proof. $r - p - 1 = 2$. Therefore

$$\text{Gensyz}_p(L) \subset \mathbb{P}(L \otimes \Lambda^2 L) \cong \mathbb{P}^{p+3+(\frac{p+3}{2})-1} \cong \mathbb{P}^N$$

Let $\{l_i\}$ be a basis of $L$ and $\{a_{ij}\}$ a basis of $\Lambda^2 L$. By proposition 4.2 $\text{Gensyz}_p(L)$ is cut out by the image of

$$\Lambda^3 L \quad \rightarrow \quad L \otimes \Lambda^2 L \quad \rightarrow \quad S_2(L \oplus \Lambda^2 L)$$

$$l_i \wedge l_j \wedge l_k \quad \mapsto \quad l_i \otimes a_{jk} - l_j \otimes a_{ik} + l_k \otimes a_{ij} \quad \mapsto \quad l_i a_{jk} - l_j a_{ik} + l_k a_{ij}$$

Notice that these quadrics are also generated by the $4 \times 4$-Pfaffians of the skew-symmetric matrix

$$M = \begin{pmatrix}
0 & l_1 & \cdots & l_{p+3} \\
-l_1 & & & \\
\vdots & & a_{ij} & \\
-l_{p+3} & & & 
\end{pmatrix}$$

that involve the first row and column. $\mathbb{G}$ is cut out by all $4 \times 4$-Pfaffians of $M$. Therefore

$$\mathbb{G} \subset \text{Gensyz}_p(L)$$

On the other hand the above Pfaffians also vanish, if $l_1 = \cdots = l_{p+3} = 0$. This shows

$$\mathbb{P}^{N-p-3} \subset \text{Gensyz}_p(L)$$
where $\mathbb{P}^{N-p-3} = V(l_1, \ldots, l_{p+3})$. To prove the proposition we have to show, that every point of $\text{Gensyz}_p(L)$ outside of $\mathbb{P}^{N-p-3}$ lies on $\mathbb{G}$.

Let $x$ be such a point. Since $x \not\in \mathbb{P}^{N-p-3}$, there is at least one linear form of $L$ that doesn’t vanish in $x$. Without restriction we can assume $l_1(x) \neq 0$.

After the appropriate row and column transformations $M(x)$ has the form

$$M(x) = \begin{pmatrix} 0 & l_1(x) & 0 & \ldots & 0 \\ -l_1(x) & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & M' \\ 0 & 0 \\ \end{pmatrix}$$

with $M' = (m_{ij})_{i,j \geq 3}$ skew symmetric. Now consider the Pfaffian that involves the rows and columns 1, 2, $i$ and $j$:

$$P(x) = P_{12ij}(x) = l_1(x)m_{ij}$$

Since $P$ involves the first row and $x$ is in the generic syzygy scheme, we have

$$l_1(x)m_{ij} = P(x) = 0.$$

Because $l_1(x) \neq 0$, this implies $m_{ij}(x) = 0$. Consequently $M' = 0$ and $M(x)$ is of rank 2. Therefore $x$ is in $\mathbb{G}$.  

7.2. Corollary. Let $X \subset \mathbb{P}(V)$ be a non degenerate irreducible variety, $I_X$ generated by quadrics and $s \in V_p$ a $p$-th syzygy of rank $p+3$. Then $\pi(X)$ is contained in the Grassmannian $\mathbb{G} = \text{Gr}(p+4, 2)$.

Proof. Let $L = L_s$ be the space of linear forms involved in $s$. By theorem 7.1 there exists a linear projection

$$\pi : \mathbb{P}(V) \rightarrow \mathbb{P}^n \subset \mathbb{P}(L \oplus \Lambda^2 L)$$

such that

$$\pi(X) \subset \pi(\text{Syz}(s)) = \mathbb{P}^n \cap \text{Gensyz}_p(L)$$

$$= \mathbb{P}^n \cap (\mathbb{G} \cup \mathbb{P}^{N-p-3})$$

$$= (\mathbb{P}^n \cap \mathbb{G}) \cup (\mathbb{P}^n \cap \mathbb{P}^{N-p-3}).$$

With $\mathbb{P}^n \cap \mathbb{P}^{N-p-3} \neq \mathbb{P}^n$ since $\mathbb{P}^n \not\subset \text{Gensyz}_p(L)$ by corollary 7.1.5.

Now $X$ is non degenerate and irreducible in $\mathbb{P}(V)$ and $\pi(X)$ has therefore the same properties in $\mathbb{P}^n$. Consequently $\pi(X) \subset \mathbb{G}$.  

7.3. Definition. The $p$-th syzygies of rank $p+3$ are called Grassmannian syzygies.

8. Syzygies of $\mathbb{G}$

We will now study the minimal free resolution of the Grassmannian $\mathbb{G}$ that occurs in the generic syzygy variety of a Grassmannian $p$-th syzygy.

I.e. let $\mathbb{G} = \text{Gr}(U, 2) \subset \mathbb{P}(\Lambda^2 U)$ be the Grassmannian of 2-dimensional quotient spaces of the vector space $U$ with basis $\{u_1, \ldots, u_{p+4}\}$. 
8.1. Proposition. The equations of $G \subset \mathbb{P}(\Lambda^2 U)$ are generated by the $4 \times 4$-Pfaffians of

$$M_U = \begin{pmatrix}
  0 & u_{12} & \ldots & u_{1,p+4} \\
  -u_{12} & 0 & \ldots & u_{2,p+4} \\
  \vdots & \vdots & \ddots & \vdots \\
  -u_{1,p+4} & -u_{2,p+4} & \ldots & 0
\end{pmatrix}$$

where the $u_{ij} = u_i \wedge u_j$ are linear forms on $\Lambda^2 U^\ast$.

Proof. For example [Har92, Ex. 9.20]. \hfill \Box

8.2. Proposition. The linear strand of the minimal resolution of $G$ is

$$I_G \leftarrow \Lambda^4 U \otimes O(-2) \leftarrow \Lambda^5 U \otimes O(-3) \leftarrow \ldots \leftarrow \Lambda_{p+4,1} U \otimes O(-p-2).$$

Proof. The minimal free resolution of an ideal generated by the $(2q+2) \times (2q+2)$-Pfaffians of a generic skew symmetric matrix $M$ is calculated by Józefiak, Pragacz and Weyman in [JPW81, Thm 3.14]. In our case $q = 1$ and we are only interested in the linear strand of the resolution ($k = 1$ in the notation of Józefiak, Pragacz and Weyman). The $i$th step of the linear strand is then the Schur functor corresponding to a Young diagram of the form

![Young diagram]

where the total number of squares is equal to $2(i + 1)$. This proves the proposition. \hfill \Box

We will now focus our attention on the space $U_p$ of $p$-th syzygies of $G$. First we describe the syzygies of minimal rank and their spaces of linear forms:

8.3. Proposition. The scheme of minimal rank $p$-th syzygies of $G$ contains the $p$-uple embedding of $\mathbb{P}^{p+3} \cong \mathbb{P}(U^\ast) =: Y_{\text{min}} \hookrightarrow \mathbb{P}(U_p^\ast)$. The space of linear forms $L_u$ involved in a minimal rank syzygy $u \in U$ is given by

$$L_u = u \wedge U \subset \Lambda^2 U.$$ The vector bundle of linear forms $L$ on $Y_{\text{min}}$ is $T_{\mathbb{P}^{p+3}}(-2)$.

Proof. Consider the action of $\text{GL}(p + 4)$ of the space of $p$-th syzygies $U_p \cong \Lambda_{p+4,1} U$. Now the rank of a syzygy is invariant under this action and the space of minimal rank syzygies is compact. Therefore it has to contain the minimal orbit

$$Y_{\text{min}} = G/P \cong \text{Flag}(\mathbb{P}^0 \subset \mathbb{P}^{p+3}) \cong \mathbb{P}^{p+3} \xrightarrow{p\text{-uple}} \mathbb{P}(U_p^\ast).$$
Let $u$ be a syzygy in $Y_{min}$. Since $Y_{min}$ is the minimal orbit, we can without restriction assume $u$ to be the maximal weight vector 

$$u = \begin{array}{c}
p \downarrow \\
1 & 1 & \cdots & 1 \\
2 \\
\vdots \\
p + 3 \\
p + 4
\end{array}$$

To determine the linear forms involved in $u$ we restrict the map

$$\psi: \Lambda_{p+3,1} U^* \otimes \mathcal{O}_P(\Lambda_{p+4,1} U^*) \rightarrow \Lambda^2 U \otimes \mathcal{O}_P(\Lambda_{p+4,1} U^*)$$

from definition 2.3 to the syzygy above. This gives

$$\psi|_u = \tilde{\phi}(u) \in \text{Hom}(\Lambda_{p+4,1} U^*, \Lambda^2 U) \cong \Lambda_{p+4,1} U \otimes \Lambda^2 U$$

where

$$\tilde{\phi}: \Lambda_{p+4,1} U \hookrightarrow \Lambda_{p+4,1} U \otimes \Lambda^2 U$$

is the map induced by the last step in the linear strand of the minimal free resolution. Using Young diagrams as in [Ful97] we get:

Consequently the linear forms involved in $u$ are

$$L_u = \text{Im} \tilde{\phi}(u) = \langle \frac{1}{p+4}, \ldots, \frac{1}{2} \rangle = \langle u_1 \wedge u_{p+4}, \ldots, u_1 \wedge u_2 \rangle = u \wedge U$$

as claimed.

We will now determine the vector bundle of linear forms $L$ on $Y_{min}$. For this notice, that the above description of the fibers $L_u$ of $L$ exhibits $L$ as the image of the map

$$U \otimes \mathcal{O}_{pp+3}(-1) \xrightarrow{\Lambda^U} \Lambda^2 U \otimes \mathcal{O}_{pp+3}.$$

which is part of a Koszul complex and factors over $T_{pp+1}(-2)$:
This completes the proof of the proposition. □

8.4. Corollary. $G$ has no scrollar $p$-th syzygies.

Proof. The proposition above shows $p$-th syzygies have rank greater or equal to rank $T_{p+3} = p + 3$. Scrollar $p$-th syzygies would have rank $p + 2$. □

Next we will determine the syzygy varieties of the minimal rank syzygies:

8.5. Proposition. Let $s \in Y_{\min}$ be a minimal rank syzygy. Then

$$\text{Syz}(s) = G \cup \mathbb{P}^{N-p-3} \subset \mathbb{P}^N$$

where $\mathbb{P}^{N-p-3} = V(L_s)$ is cut out by the linear forms involved in $s$.

Proof. The space $U_p$ of $p$-th syzygies of $G$ is isomorphic to $\Lambda_{p+4,1,p}U$ by proposition 3.2. Furthermore lemma 3.1 exhibits $U_p$ as a subspace of $\Lambda^p(\Lambda^2 U) \otimes (I_G)_2$:

where the $\Lambda_{\lambda_i}$ are the irreducible representations of $\Lambda^{p+4}(\Lambda^2 U)$. We will now show, that $U_p$ is contained in only one $\Lambda_{\lambda_i} \otimes \Lambda_4$.

To do this, observe, that there are only two ways of coloring 4 squares of $\Lambda_{p+4,1,p}$ which are compatible with the Littlewood-Richardson rule for $\Lambda_{\lambda_i} \otimes \Lambda_4$:

\begin{itemize}
  \item $p$
  \item $p+1$
\end{itemize}
Since the summands of $\Lambda^p(\Lambda^2U)$ have at most $p$ squares in each row, only the second possibility can occur, i.e.

$$
\begin{align*}
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix} \\
\begin{bmatrix}
p-1 \\
p-1 \\
p-1 \\
\end{bmatrix}
\end{align*}
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix}
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix}
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix}
\end{align*}
$$

By lemma 3.3, the syzygy scheme $\text{Syz}(s)$ is cut out by the quadrics involved in the image of $s$ under the above inclusion. Without loss of generality we can assume $s$ to be the maximal weight vector

$$
\begin{align*}
s = \\
\begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
2 & \\
\vdots & \\
p+1 & \\
p+2 & \\
p+3 & \\
p+4 &
\end{bmatrix}
\end{align*}
\begin{align*}
\mapsto \\
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
p-1 & \\
p-1 & \\
p-1 & \\
1 & \\
2 & \\
p+2 & \\
p+3 & \\
p+4 & \\
p+4 &
\end{bmatrix}
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
1 & 2 & \\
3 & \\
4 &
\end{bmatrix}
\end{align*}
$$

Consequently the syzygy scheme $\text{Syz}(s)$ of $s$ is cut out by the $4 \times 4$-Pfaffians that involve the first row and column of $M$. The same argument as in the proof of proposition 7.1 shows

$$G \cup \mathbb{P}^{N-p-3} = \text{Syz}(s)$$

If $s \in U_p$ is any $p$-th syzygy, the syzygy scheme might be more complicated, but its ideal still contains certain special quadrics:

**8.6. Definition.** A quadric $Q$ is called a generalized $4 \times 4$-Pfaffian of a skew symmetric matrix $M$, if there exists an invertible matrix $B$ such that $Q$ is a $4 \times 4$-Pfaffian of $B^tMB$.

**8.7. Remark.** In our case the generalized $4 \times 4$-Pfaffians of $MU$ are the decomposable elements in $(I_G)_2 = \Lambda^4U$.

**8.8. Lemma.** Let $s \in U_p$ a $p$-th syzygy of $G$. Then the ideal of $\text{Syz}(s)$ contains a generalized $4 \times 4$ Pfaffian.
Proof. Recall the inclusion

\[
\begin{array}{c}
p + 1 \\
\vdots \\
1
\end{array}
\subset \bigotimes \left( \Lambda^p \right)
\]

from the last proof. For simplicity we will call the first Young diagram a big hook and the second one a small hook. The third one we will call a line.

Now \( U_p \) has a basis \( \{ s_\beta \} \) enumerated by big hooks of the form

\[
s_\beta = \begin{array}{c}
1^* \ldots 1^*
\end{array}
\]

The image of such a big hook \( s_\beta \) under the above map is a sum of tensor products of small hooks and lines, where each product contains the same entries as \( s_\beta \). Notice that the small hook \( d \) of maximal weight can therefore only occur in the image of the following big hooks:

\[
s_l = \begin{array}{c}
1^* \ldots 1^* \\
2 \\
\vdots \\
p + 3 \\
p + 4
\end{array}
\Rightarrow \bigotimes \left( \Lambda^p \right)
\]

Let now \( s = \sum \mu_\beta s_\beta \) be any \( p \)-th syzygy of \( G \), i.e a linear combination of big hooks. Then the image of \( s \) can be written as \( \sum d_\alpha \otimes Q_\alpha \) with \( \{ d_\alpha \} \) a basis of \( \Lambda_{p+1, 1} \otimes U \) enumerated by small hooks. By the argument above the small hook \( d \) of maximal weight occurs with a quadric \( Q \) of the form

\[
Q = \sum \mu_l \begin{array}{c}
l \\
p + 2 \\
p + 3 \\
p + 4
\end{array} = \sum \mu_l (u_1 \wedge u_{p+2} \wedge u_{p+3} \wedge u_{p+4}) = \left( \sum \mu_l u_l \right) \wedge u_{p+2} \wedge u_{p+3} \wedge u_{p+4}
\]

Consequently \( Q \) is either zero or a generalized \( 4 \times 4 \)-Pfaffian.
We will now check, that we can assume $Q \not\equiv 0$. For this consider the minimal orbit $G/P$ of $GL(p+4)$ in the space $\Lambda_{p+1,p+1}U$ of small hooks. Since this representation is irreducible, $G/P$ is non degenerate. We can therefore choose a basis $\{d_j\}$ of this space consisting only of points in the minimal orbit. Now write the image of $s$ in this basis:

$$s \mapsto \sum_j d_j \otimes Q_j$$

Since $s \not\equiv 0$, there is at least one $Q_j \not\equiv 0$. Since the corresponding $d_j$ is in the minimal orbit of $GL(p)$ we can after a coordinate change of $U$ assume it to be the maximal weight vector $d$ above. Applying the above argument to the transformed $Q_j \not\equiv 0$ shows that $Q_j$ is a generalized $4 \times 4$-Pfaffian.

8.9. Remark. The argument above even shows, that the ideal of $\text{Syz}(s)$ is generated by generalized Pfaffians.

9. Linear sections of $\mathbb{G}$

Let $X \subset \mathbb{P}(V)$ be a non degenerate irreducible variety, $s \in V_p$ a Grassmannian syzygy and

$$\pi: \mathbb{P}(V) \longrightarrow \mathbb{P}^n \subset \mathbb{P}(\Lambda^2 U)$$

the induced projection with

$$\pi(X) \subset \mathbb{P}^n \cap \mathbb{G}.$$ 

In this situation we have a natural map from $p$-th syzygies of $\mathbb{G}$ to $p$-th syzygies of $X$

$$\alpha_p: H^0(\mathbb{P}(\Lambda^2 U), \Omega^p(p + 2) \otimes I_G) \to H^0(\mathbb{P}(V), \Omega^p(p + 2) \otimes I_X)$$

given by restriction of the corresponding twisted $p$-forms to $\mathbb{P}^n$ and pulling them back to $\mathbb{P}(V)$. If a syzygy $s$ is not in the kernel of $\alpha_p$ we say “$s$ survives the restriction to $X$”.

In this section we want to prove that $\alpha_p$ is injective, i.e. that all $p$-th syzygies of $\mathbb{G}$ survive the restriction to $X$.

The first step is

9.1. Proposition. All minimal rank syzygies $u \in Y_{min}$ survive the restriction to $X$.

Proof. By theorem 4.4 there exist a generic $p$-th syzygy $u_{gen} \in Y_{min}$ of $\text{Gensyz}_p(L_s) = \mathbb{G} \cup \mathbb{P}_s$ with

$$\mathbb{P}_s = V(L_s)$$

and

$$s = \pi^*(u_{gen}).$$

Consequently $u_{gen}$ survives the restriction to $X$. Furthermore we have $L_s = L_{u_{gen}}$, i.e. no linear form in $L_{u_{gen}}$ vanishes on $\mathbb{P}^n$. 


Suppose \( u \in Y_{\text{min}} \) is a minimal rank syzygy that doesn’t survive the restriction to \( X \), i.e.
\[
P^n \subset \text{Syz}(u) = G \cup \mathbb{P}_u
\]
where \( \mathbb{P}_u = V(L_u) \). Now \( P^n \not\subset G \) so we must have \( P^n \subset \mathbb{P}_u \). In particular all linear forms in \( L_u \) must vanish on \( P^n \). But this is impossible, since \( u \wedge u_{\text{gen}} \) is a linear form in \( L_u \) and \( L_{u_{\text{gen}}} \) that doesn’t vanish on \( P^n \) by the above argument.

To extend this result to arbitrary \( p \)-th syzygies, we need

**9.2. Proposition.** Let \( X \) be an irreducible variety as above. Then all generalized Pfaffians of \( M_U \) survive the restriction to \( X \)

*Proof.* Suppose \( P \) is a generalized Pfaffian that doesn’t survive the restriction to \( X \), i.e. \( P^n \subset \mathbb{P}(P) \). Without loss of generality we can assume 
\[
P = u_1 \wedge u_2 \wedge u_3 \wedge u_4 = \text{paff}(M_{1234}) = \text{paff} \left( \begin{array}{cccc} 0 & u_{12} & u_{13} & u_{14} \\ -u_{12} & 0 & u_{23} & u_{24} \\ -u_{13} & -u_{23} & 0 & u_{34} \\ -u_{14} & -u_{24} & -u_{34} & 0 \end{array} \right)
\]

First of all we will prove that \( P^n \subset \mathbb{P}(P) \) implies that the restriction of \( M_{1234} \) to \( P^n \) can, after a suitable coordinate change, be written in the form
\[
M_{1234}|_{P^n} = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ 0 & * & 0 & * \\ * & * & * & 0 \end{pmatrix}.
\]

To see this consider the vertex \( \mathbb{P}^{N-6} = V(u_{12}, \ldots, u_{34}) \) of \( V(P) \) and project from there:
\[
\phi: \mathbb{P}(\Lambda^2 U) \rightarrow \mathbb{P}^5
\]

The image of \( V(P) \) is the Grassmannian \( \text{Gr}(2, 4) \). \( \text{Gr}(2, 4) \) is also cut out by the Pfaffian \( P \). Now \( P^n \subset V(P) \) so either \( P^n \subset \mathbb{P}^{N-6} \) and
\[
M_{1234}|_{P^n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
or \( \phi(P^n) \subset \text{Gr}(2, 4) \). Since \( \text{Gr}(2, 4) \) is a quadric of dimension 4, \( \dim \phi(P^n) \leq 2 \) [GH78, p. 735].

If \( \phi(P^n) \cong \mathbb{P}^0 \) is a point in \( \text{Gr}(2, 4) \), \( M_{1234}|_{P^0} \) is a matrix of rank 2. Therefore after a suitable coordinate change we have
\[
M_{1234}|_{P^n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix}.
\]

If \( \phi(P^n) \cong \mathbb{P}^1 \) is a line in \( \text{Gr}(2, 4) \), then this line is a Schubert-cycle
\[
\mathbb{P}^1 = \{ l \in \text{Gr}(2, 4) \mid p_0 \in l \subset H_0 \}
\]
with \( p_0 \in \mathbb{P}^3 \) a point and \( H_0 \subset \mathbb{P}^3 \) a hyperplane [GH78, p. 757]. Consequently

\[
M_{1234}|_{\mathbb{P}^n} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & * \\
0 & 0 & 0 & * \\
0 & * & * & 0
\end{pmatrix}
\]

after a suitable coordinate change.

If \( \phi(\mathbb{P}^n) \cong \mathbb{P}^2 \) lies in \( \text{Gr}(2, 4) \), then there are two possibilities. Firstly

\[
\mathbb{P}^2 = \{ l \in \text{Gr}(2, 4) \mid l \subset H_0 \}
\]

with \( H_0 \subset \mathbb{P}^3 \) a hyperplane, and

\[
M_{1234}|_{\mathbb{P}^n} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & * & * \\
0 & * & 0 & * \\
0 & * & * & 0
\end{pmatrix}
\]

or

\[
\mathbb{P}^2 = \{ l \in \text{Gr}(2, 4) \mid p_0 \in l \}
\]

with \( p_0 \in \mathbb{P}^3 \) a point, and

\[
M_{1234}|_{\mathbb{P}^n} = \begin{pmatrix}
0 & 0 & 0 & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & * \\
* & * & * & 0
\end{pmatrix}
\]

This proves out claim about the shape of \( M_{1234}|_{\mathbb{P}^n} \). For the whole matrix \( M_U \) we have therefore, after a coordinate change

\[
M_U|_{\mathbb{P}^n} = \begin{pmatrix}
0 & 0 & 0 & * & \ldots & * \\
0 & \quad & \quad & \quad & \quad & \quad \\
0 & \quad & \quad & \quad & \quad & \quad \\
0 & \quad & \quad & \quad & \quad & \quad \\
\vdots & \quad & \quad & \quad & \quad & \quad \\
* & \quad & \quad & \quad & \quad & \quad
\end{pmatrix}
\]

Now the linear forms in the first row of \( M_U \) are the linear forms of \( L_u \) for a particular \( u \in Y_{\text{min}} \). During the restriction to \( X \) at least two of these linear forms vanish because of the shape of \( M_U|_{\mathbb{P}^n} \). Therefore

\[
\text{rank } u|_X \leq p + 3 - 2 = p + 1.
\]

But this is impossible for irreducible \( X \) by corollary 5.2. Consequently

\( \mathbb{P}^n \not\in V(\mathcal{P}) \) for all generalized Pfaffians \( \mathcal{P} \) of \( M_U \).

9.3. Corollary. If \( X \) is irreducible, then the restriction \( \alpha_p \) of \( p \)-th syzygies of \( G \) to \( p \)-th syzygies of \( X \) is injective.

Proof. Let \( s \in V_p \) be a \( p \)-th syzygy of \( G \). By lemma 8.8 the ideal of \( \text{Syz}(s) \) contains at least one generalized Pfaffian \( \mathcal{P} \) of \( M_U \). Since \( \mathcal{P} \) survives the restriction to \( X \) by proposition 7.2 \( \text{Syz}(s) \) cannot contain \( \mathbb{P}^n \). Consequently \( s \) also survives the restriction. \( \square \)
Finally we will describe which syzygies $u \in Y_{\text{min}}$ of $G$ drop rank during the restriction to $X$. For this let $\mathbb{P}^\perp \subset \mathbb{P}(\Lambda^2 U^*)$ be the orthogonal space of $\mathbb{P}^n$ and $G^* = \text{Gr}(U^*, 2)$ the dual Grassmannian.

**9.4. Lemma.** Let

$$u \in G^* \cap \mathbb{P}^\perp$$

be a decomposable linear form in $\mathbb{P}^\perp$.

Then all syzygies in the line of $Y_{\text{min}} \cong \mathbb{P}(U^*)$ corresponding to $u = u' \land u''$ drop rank when restricted to $X$.

**Proof.** Consider a syzygy $s \in Y_{\text{min}}$, and its space of linear forms $L_s \subset \mathbb{P}(\Lambda^2 U^*)$.

When we restrict $s$ to $\mathbb{P}^n$ all linear forms $l$ in $\mathbb{P}^\perp \cap L_s$ vanish. So all syzygies whose space of linear forms $L_s$ intersects $\mathbb{P}^\perp$ drop rank when restricted to $X$.

In our case consider the line spanned by the minimal rank syzygies $u'$ and $u''$ in $Y_{\text{min}} \cong \mathbb{P}(U^*)$. The space of linear forms of a syzygy $\lambda u' + \mu u''$ on this line is

$$L_{\lambda, \mu} = (\lambda u' + \mu u'') \land U$$

and therefore contains $u = u' \land u'' \subset \mathbb{P}^\perp$. Consequently all syzygies on this line drop rank during restriction.

**10. SYZYGIES OF $S$**

Let $S \subset \mathbb{P}^9$ be a $K3$-Surface whose Picard group is generated by $\mathcal{O}(C)$ where $C$ is a smooth curve of even genus $g = 2k$.

We first prove some standard facts about $S$:

**10.1. Lemma.** The ideal of $S$ contains no rank 4 quadric.

**Proof.** Suppose $Q$ is a rank 4 quadric with $S \subset Q$. Then the rulings of $Q$ cut out two linear series $|C_1|$ and $|C_2|$ with $|C_1 + C_2| = |C|$. This is impossible, since the Picard group of $S$ is generated by $|C|$.

**10.2. Corollary.** $S$ has no scrollar syzygies.

**Proof.** By corollary 6.2 a scrollar syzygy implies the existence of a scroll $S$ containing $S$. These scrolls are cut out by rank 4 quadrics. Since $S$ is contained in no rank 4 quadric this is impossible.

**10.3. Corollary.** $S$ has a Grassmannian $(k - 2)$-nd syzygy

**Proof.** Consider a general hyperplane section $C \cap \mathbb{P}^{g-1}$ with $\mathbb{P}^{g-1} = V(l)$. Since $S$ is arithmetically Cohen Macaulay, the restriction maps

$$\alpha_p: H^0(\mathbb{P}^g, \Omega^p(p + 2) \otimes I_S) \to H^0(\mathbb{P}^{g-1}, \Omega^p(p + 2) \otimes I_C)$$

are isomorphisms by [Gre84a, Thm 3.b.7].
Let $s$ be a $p$-th syzygy of $S$ and

$$\check{\varphi}(s) : V_{p-1}^* \to V$$

the map from definition 2.2, where $V_{p-1}$ is the space of $(p-1)$-st syzygies of $S$, and $\mathbb{P}(V) = \mathbb{P}^g$. Then $\text{Im}(\check{\varphi}(s)) = L_s$. Restriction to $C$ gives the diagram

$$
\begin{array}{ccc}
V_{p-1} & \xrightarrow{\check{\varphi}(s)} & V \\
\alpha_{p-1} & \downarrow & \downarrow \iota^* \\
V_{p-1} & \xrightarrow{\check{\varphi}(\alpha_p(s))} & V'
\end{array}
$$

where $\mathbb{P}(V') = \mathbb{P}^{g-1}$, and $\iota^*$ is the natural projection induced by the inclusion $\iota : \mathbb{P}^{g-1} \hookrightarrow \mathbb{P}^g$.

The kernel of $\iota^*$ is generated by $l$. The space of linear forms involved in $\alpha_p(s)$ is therefore

$$L_{\alpha_p(s)} = \text{Im}(\check{\varphi}(\alpha_p(s))) = \iota^*(\text{Im}(\check{\varphi}(s))) = \iota^*L_s$$

In particular we have

$$\text{rank} \alpha_p(s) = \begin{cases} \text{rank } s - 1 & \text{if } l \in L_s \\ \text{rank } s & \text{if } l \notin L_s \end{cases}$$

Now $C$ has at least finitely many $g_{k+1}$'s which induce scrollar $(k-2)$-nd syzygies by the construction of Green and Lazarsfeld [GL84].

By prop 10.2 these cannot come from scrollar syzygies of $S$, so they have to come from syzygies with higher rank. Since the difference in rank can be at most 1 by the above argument these syzygies are Grassmannian.

Consider now such a Grassmannian $(k-2)$-nd syzygy of $S$. It induces a linear projection

$$\pi : \mathbb{P}^g \dashrightarrow \mathbb{P}^n \subset \mathbb{P}^N$$

with $N = \binom{k+2}{2} - 1$ and

$$\pi(S) \subset \mathbb{G} = \text{Gr}(k+2, 2) \subset \mathbb{P}^N.$$ 

We even have

**10.4. Proposition.** $\pi$ is an embedding, i.e. $\mathbb{P}^g \cong \mathbb{P}^n$ via $\pi$.

**Proof.** Consider the orthogonal space $\mathbb{P}^\perp$ of $\mathbb{P}^n$ and the dual Grassmannian $\mathbb{G}^* = \text{Gr}(2, k+2)$. Since $S$ has no scrollar syzygies, $\mathbb{P}^\perp$ does not intersect $\mathbb{G}^*$ by proposition 3.4.

Now $\dim \mathbb{G}^* = 2k = g$, and consequently

$$1 + \dim \mathbb{P}^n = \text{codim} \mathbb{P}^\perp \geq \dim \mathbb{G}^* + 1 = g + 1.$$ 

Since $\pi$ is surjective on $\mathbb{P}^n$ this proves the proposition. \qed
10.5. Remark. Notice that this is the embedding constructed by Mukai in [Muk89] with vector bundle methods. His rank 2 vector bundle $E$ is the restriction of the universal quotient bundle $Q$ on $G\equiv \pi(S)\subset G$.

Notice also that our construction also gives an algorithm to determine $\pi$ explicitly from a Grassmannian syzygy $s$ by lifting $\tilde{\varphi}(s)$ to a map of complexes as in theorem 4.4.

Using the theorem of Voisin we can now describe the $(k-2)$-nd syzygies of $S$ geometrically:

10.6. Proposition. The space of minimal rank syzygies of $S$ contains a $(k-2)$-uple embedded $\mathbb{P}^{k+1}$. Further more the space of all $(k-2)$-nd syzygies of $S$ is isomorphic to the ambient space of this embedding.

Proof. Let $U_{k-2}$ be the space of $(k-2)$-nd syzygies of $G=\text{Gr}(U,2)$ and $V_{k-2}$ the corresponding space of $(k-2)$-nd syzygies of $S$.

By the corollary 9.3 the map

$$\alpha_{k-2}: U_{k-2} \to V_{k-2}$$

induced by restriction of syzygies is injective. Consequently

$$\dim V_{k-2} \geq \dim U_{k-2} = \dim \Lambda_{k+2,1}^{-1}U = \dim S^{k-2}U = \binom{2k-1}{k-2}.$$  

On the other hand, the Hilbert function of $S$ gives:

$$\dim W_{p} - \beta_{p,\nu} = (p+1)\binom{g-2}{p+2} - (g-p+2)\binom{g-2}{g-p-1}$$

(see for example [Sch91]). In our case $p=k-2$, $g=2k$ and $\beta_{p,\nu}=0$ by Voisin’s theorem. Consequently

$$\dim V_{k-2} = (k-1)\binom{2k-2}{k} - \binom{2k-2}{k+1} = \binom{2k-1}{k+1} = \binom{2k-1}{k-2}$$

and $V_{k-2} \cong U_{k-2}$.

Since $\mathbb{P}^1$ doesn’t intersect $G^*$ all syzygies in

$$Y_{\min} \cong \mathbb{P}^{k+1} \xrightarrow{(k-2)-uple} \mathbb{P}(U_{k-2}^*)$$

remain of rank $k+1$ during restriction. They also remain minimal, since $S$ has no scrollar $(k-2)$-nd syzygies of rank $k$ by corollary 10.2.

11. Syzygies of $C$

Now consider a general linear section

$$C = S \cap \mathbb{P}^{g-1}.$$  

During this further restriction some syzygies drop rank and become scrollar syzygies:

11.1. Proposition. The scrollar $(k-2)$-nd syzygies of $C$ from a configuration of $\binom{2k}{k}$ lines in $\mathbb{P}^{k+1}$ that are embedded in the space of $(k-2)$-nd syzygies of $C$ as rational normal curves of degree $(k-2)$ on a $(k-2)$-uple embedding of $\mathbb{P}^{k+1}$. 

**Proof.** Since $C$ is a general linear section of $S$, their spaces of syzygies are isomorphic. We now determine which syzygies do drop rank. With lemma 9.4 we find some of these, when the orthogonal space $P_C$ of $P^{g−1}$ intersects $G^*$. $P_C$ contains the orthogonal space $P_S$ of $P^g$. Since $P_S$ doesn’t intersect $G^*$, $P_C$ can only intersect in finitely many points. On the other hand $\dim G^* = 2k = g = \text{codim}P_C$ so this is also the expected intersection dimension. Consequently the number $r$ of intersection points is equal to the degree of $G^*$. A formula for the degree of Grassmannians can be found in [Har92, p. 247]:

$$r = \text{deg } G^* = (2k)! \prod_{i=0}^{1} \frac{i!}{(k+i)!} = \frac{(2k)!}{k!(k+1)!} = \frac{1}{k+1} \binom{2k}{k}$$

Now each point of the intersection corresponds to a line of scroller syzygies in $P^{k+1} \cong P(U^*)$. This gives our configuration of lines.

To prove these are all scroller syzygies of $C$, recall that the syzygy variety of a scroller $(k−2)$-nd syzygy is a scroll $S$ whose fibers cut out linear equivalent divisors. These divisors are part of a linear system with Clifford index

$$\text{cliff}(D) \geq g − (k − 2) − 3 = k − 1$$

by proposition 5.4.

Since $C$ is general in the sense of Brill-Noether-Theory by [Laz86], the only linear systems on $C$ with Clifford index $k − 1$ are the $g_{k+1}^1$’s. In other words: all scroller $(k−2)$-nd syzygies of $C$ come from $g_{k+1}^1$’s. Now each $g_{k+1}^1$ induces a $P^1$ of scroller $(k−2)$-nd syzygies. See for example [vBR01, Lem 2.2.8] for a proof of this elementary fact.

A formula for the number of $g_{k+1}^1$’s in $W_{k+1}^1(C)$ for a Brill-Noether-general curve $C$ is given in [ACGH85, p. 211]:

$$\deg W_{k+1}^1 = g! \prod_{i=0}^{1} \frac{i!}{(g − (k + 1) − 1 + i)!} = 2k! \prod_{i=0}^{1} \frac{i!}{(k + i)!} = \text{deg } G^*$$

So there are no scroller $(k−2)$-nd syzygies of $C$ except the ones in the configuration above. \qed

We now deduce the desired special case of the geometric syzygy conjecture

**11.2. Theorem.** Let $C$ be a general hyperplane section of a K3 surface $S$ whose Picard group is generated by $C$. Then the space of $(k−2)$-nd scroller syzygies of $C$ is non degenerate.

**Proof.** Let $Z$ the configuration of lines in $P^{k+1}$ that correspond to scroller syzygies by the previous proposition. The $(k−2)$-uple embedding of $P^{k+1}$ is

$$\iota : P^{k+1} \hookrightarrow P^N$$

with $N = \binom{2k−1}{k−2} − 1$. We have to show, that $\iota(Z)$ spans $P^N$. 
For this notice that $Z$ is the locus of syzygies $s$ where the dimension of the space of linear forms $L_s$ is $k$. $Z$ can therefore be described by the determinantal locus where the composition map of vector bundles

$$\beta: L \to \Lambda^2 U \otimes \mathcal{O}_{\mathbb{P}^{k+1}} \to V' \otimes \mathcal{O}_{\mathbb{P}^{k+1}}$$

drops rank. Here $V'$ denotes the space of linear forms on $\mathbb{P}^{g-1}$ and $L \cong T_{\mathbb{P}^{k+1}}(-2)$ the vector bundle of linear forms on $Y_{\text{min}} \cong \mathbb{P}^{k+1}$. Notice that $\beta$ drops rank in expected dimension

$$\dim \mathbb{P}^{k+1} - (\text{rank } \mathcal{L} - k)(\dim V' - k) = k + 1 - (k + 1 - k)(2k - k) = 1$$

Therefore the ideal of the degeneracy locus $Z$ is resolved by the Eagon-Northcott complex

$$I_{Z/\mathbb{P}^{k+1}} \leftarrow \Lambda^{k+1} V'' \otimes \Lambda^{k+1} \mathcal{L} \leftarrow \Lambda^{k+2} V'' \otimes \Lambda^{k+1} \mathcal{L} \otimes \mathcal{L} \leftarrow \ldots \leftarrow \Lambda^{2k} V'' \otimes \Lambda^{k+1} \mathcal{L} \otimes S_{k-1} \mathcal{L} \leftarrow 0$$

To show that $\iota(Z)$ is non degenerate in $\mathbb{P}^N$ we have to prove

$$h^0(I_{Z/\mathbb{P}^{k+1}}(k - 2)) = 0$$

since $\iota$ is the $(k - 2)$-uple embedding of $\mathbb{P}^{k+1}$.

This follows if the cohomology groups

$$H^i(\Lambda^{k+1} \mathcal{L} \otimes S^j \mathcal{L} \otimes \mathcal{O}(k - 2))$$

vanish for $0 \leq i \leq k + 1, 0 \leq j < k - 1$ and $0 \leq i \leq k, j = k - 1$. We will prove this in the next section.

12. Cohomology of $S^i \mathcal{L}(-j)$

We well calculate the cohomology of the needed homogeneous bundles on $\mathbb{P}^{k+1}$ using the theorem of Bott. We start by fixing some notation.

Let $G = \text{GL}(k + 2)$ and $P \subset G$ the parabolic subgroup with elements of the form

$$\begin{pmatrix}
* & * & \cdots & * \\
0 & & & \\
\vdots & & & * \\
0 & & & 
\end{pmatrix}$$

Then $G/P \cong \mathbb{P}^{k+1}$. Let $H \subset P \subset G$ be the subgroup of diagonal matrices, $H_i := E_{i,i}$ the natural basis of $H$ and $\{L_i\}$ the dual basis of $H^*$. Then the $L_i$ span the weight lattice of $G$. The positive roots of $G$ are $L_i - L_j$ with $k + 2 \geq i > j \geq 1$ and the fundamental weights are $\omega_j = \sum_{j=1}^{i} L_j$.

If $\rho$ is a representation of $P$, it induces a vector bundle $E_\rho$ on $\mathbb{P}^{k+1}$ with $P$ acting on the fibers of $E_\rho$ via $\rho$. Sometimes we write

$$E_\rho = E(\lambda) = E(\lambda_1, \ldots, \lambda_{k+2})$$

with $\lambda = \lambda_1 L_1 + \cdots + \lambda_{k+2} L_{k+2}$ the maximal weight vector of $\rho$.

Often it is sufficient to consider the semisimple part $S_P$ of $P$:
12.1. Theorem (Classification of irreducible bundles over $G/P$). Let $P(\Sigma) \subset G$ be a parabolic subgroup and $\omega_1, \ldots, \omega_k$ the fundamental weights corresponding to the subset of simple roots $\Sigma \subset \Delta$. Then all irreducible representations of $P(\Sigma)$ are

$$V \otimes L_{\omega_1}^{n_1} \otimes \cdots \otimes L_{\omega_k}^{n_k}$$

where $V$ is a representation of $S_P$ and $n_i \in \mathbb{Z}$. $L_{\omega_i}$ are the one dimensional representations of $S_P$ induced by the fundamental weights.

The weight lattice of $S_P$ is embedded in the weight lattice of $G$. If $\lambda$ is the highest weight of $V$, we will call $\lambda + \sum n_i w_i$ the highest weight of the irreducible representation of $P(\Sigma)$ above.

Proof. [Ott95, Proposition 10.9 and remark 10.10]

In our case the semisimple part $S_P$ of $P$ is $GL(1) \times GL(k+1)$. Notice that the weight lattice of $GL(1) \times GL(k+1)$ is embedded in the weight lattice of $GL(k+2)$. In particular $L_1$ belongs to $GL(1)$ and $(L_2, \ldots, L_{k+2})$ belongs to $GL(k+1)$.

12.2. Remark. We have $O(1) = E(1,0,\ldots,0)$ since $GL(1)$ acts on the fibers of $O(1)$ in the standard way. In particular this representation has maximal weight vector $L_1$. Similarly we have $\Omega^2(1) = E(0,1,0,\ldots,0)$ since $GL(k+1)$ acts on the fibers of $\Omega^1(1)$ with maximal weight vector $L_2$. Consequently

$$\mathcal{L} = \mathcal{T}_{p^{k+1}}(-2) = [\Omega^1(2)]^* = E(1,1,0,\ldots,0)^*$$

With this we are ready to use

12.3. Theorem (Bott). Consider the homogeneous vector bundle $E(\lambda)$ on $X = G/P$ and $\delta$ the sum of fundamental weights of $G$. Then

- $H^i(X, E(\lambda))$ vanishes for all $i$ if there is a root $\alpha$ with $(\alpha, \delta + \lambda) = 0$
- Let $i_0$ be the number of positive roots $\alpha$ with $(\alpha, \delta + \lambda) < 0$. Then $H^i(X, E(\lambda))$ vanishes for $i \neq i_0$ and $H^{i_0}(X, E(\lambda)) = \rho_{w(\delta + \lambda) - \delta}$

where $(.,.)$ denotes the Killing form on $\mathfrak{h}^*$, $w(\delta + \lambda)$ is the unique element of the fundamental Weyl chamber which is congruent to $\delta + \lambda$ under the action of the Weyl group, and $\rho_{w(\delta + \lambda) - \delta}$ is the corresponding representation of $G$.

Proof. [Ott95, Theorem 11.4]

12.4. Corollary. The cohomology groups

$$H^i(\Lambda^{k+1} \mathcal{L} \otimes S^j \mathcal{L} \otimes \mathcal{O}(k-2))$$

vanish

(a) for all $i$ if $0 \leq j \leq k-2$ and
(b) for all $i \neq k$ if $j = k-1$. 
Proof. First of all we have
\[ \Lambda^{k+1}L^* = \Lambda^{k+1} \Omega^1(2) = \omega_{\mathfrak{p}^{k+1}}(2k + 2) = \mathcal{O}(k) \]
and therefore
\[ \Lambda^{k+1}L \otimes \mathcal{O}(k-2) = \mathcal{O}(-2) = E(-2, 0, \ldots, 0). \]

Similarity we get
\[ S^j \mathcal{L} = (S^j \mathcal{L}^*)^* = (S^j E(1, 1, 0, \ldots, 0))^* = E(j, j, 0, \ldots, 0)^* = E(-j, 0, \ldots, 0, -j). \]

Consequently
\[ \Lambda^{k+1}L \otimes S^j \mathcal{L} \otimes \mathcal{O}(k-2) = E(-j - 2, 0, \ldots, 0, -j) =: E(\lambda) \]
Now the sum \( \delta \) of the fundamental weights of \( GL(k+2) \) is
\[ \delta = (k + 2)L_1 + \cdots + L_{k+2} = (k + 2, k + 1, \ldots, 2, 1) \]
so
\[ \lambda + \delta = (k - j, k + 1, \ldots, 2, 1 - j). \]
If \( 0 \leq j \leq k - 2 \) then \( \alpha = L_1 - L_{j+1} \) is a positive root with \( (\alpha, \lambda + \delta) = 0 \).
Therefore, by the theorem of Bott, all cohomology groups vanish in this case. This proves (a).
If \( j = k - 1 \) we have
\[ \lambda + \delta = (1, k + 1, \ldots, 2, -k) \]
and \( (\alpha, \lambda + \delta) < 0 \) for \( \alpha = L_1 - L_l \) with \( 2 \leq l \leq k + 1 \). All other positive roots \( \alpha \) satisfy \( (\alpha, \lambda + \delta) > 0 \). Consequently
\[ H^i(\Lambda^{k+1}L \otimes S^j \mathcal{L} \otimes \mathcal{O}(k-2)) \]
vanesishes for \( i \neq k \) by the theorem of Bott. This proves (b).

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