The bending of light and the cosmological constant

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The bending of light in Kottler space (the Schwarzschild vacuum with cosmological constant) is examined. Unlike the advance of the perihelion, the cosmological constant produces no change in the bending of light. In this note we examine the conditions under which this statement holds.

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I. INTRODUCTION

The deflection of light is one of the “classical” tests of general relativity. With the continuing interest in the cosmological constant one would assume that the effect of a cosmological constant on the deflection of light is well known. Certainly the effect of Λ on the advance of general relativity is well known. This appears not to be the case and is the subject of this note.

Gravitational lensing is now a common tool in astrophysics. There are two approaches to gravitational lensing: the thin lens approximation and the exact approach. These differ fundamentally in that in the thin lens approach there is a background lens split whereas in the exact approach one uses the full null geodesic equations in an exact solution of Einstein’s equations. In a cosmological context, even for strong-lensing calculations, some variant of the thin lens approximation is adequate and used exclusively. For the purposes of the present calculation, however, only the exact approach is appropriate. Recent studies of lensing in the Schwarzschild field are available. Virbhadra and Ellis have examined a strong-field version of the thin lens approximation and Newman and coworkers have compared the exact approach with various approximations.

The present note involves exact lensing in Kottler space, the Schwarzschild vacuum with a cosmological constant (Λ). In terms of familiar curvature coordinates (r, θ, φ, t) the line element is given by

\[ ds^2 = \frac{dr^2}{f(r)} + r^2(dθ^2 + \sin^2 θ dφ^2) - f(r)dt^2, \]

where

\[ f(r) = 1 - \frac{2m}{r} - \frac{Λr^2}{3}. \]

The associated generalization of the Birkhoff theorem is well known. It is interesting to note that the Λ generalization of the black hole uniqueness theorems is not known. Geodesically complete forms of the metric along with Penrose-Carter diagrams are now well known.

II. NULL GEODESICS

The coordinates (r, θ, φ, t) are adapted to two Killing vectors and so geodesics of the metric have two constants of motion. The orbits are stably planar and we choose the plane to be θ = π/2. The momentum conjugate to φ is the orbital angular momentum \( l \), \( r^2 \dot{φ} = l \), and the momentum conjugate to t is the energy \( γ \), \( f(r) \dot{t} = γ \) where \( · = \frac{d}{dt} \) and \( λ \) is an affine parameter. Since we are interested here only in null geodesics, it is convenient to reparametrize them with \( \lambda = tλ \) so that

\[ r^2 \dot{γ} = 1, \]

and

\[ f(r) \dot{t} = \frac{1}{b}, \]

where \( b = \frac{l}{r} \). If \( Λ = 0 \) then \( b \) is the impact parameter. From (1), (3) and (4) it follows that

\[ \left( \frac{d \lambda}{d \lambda} \right)^2 = \frac{1}{b^2} - \frac{f(r)}{r^2} \]

where \( r_+^2 \equiv b^2 f(r_+) \) defines the turning points in r. If \( r_+ \) is considered derived from a source then \( r_+ \) is the boundary of that source (see below). From (3) and (4) we have

\[ \left( \frac{d u}{d \phi} \right)^2 = \left( \frac{m}{b} \right)^2 + \frac{Λm^2}{3} - u^2 + 2u^3 \]

where \( u = \frac{\dot{φ}}{\dot{t}} \). Differentiation of (6) with respect to \( ϕ \) of course eliminates \( Λ \), but this does not prove that \( Λ \) has no effect since (3) is the first integral of the motion. In the usual way yields, in first order, the deflection angle

\[ δ = 4 \sqrt{\frac{m}{b}^2 + \frac{Λm^2}{3}}. \]

Consider the cases \( Λ = 0 \) and \( Λ \neq 0 \) separately and assume that \( r_+ \) and \( m \) are fixed, assumptions we examine below. With these conditions it is necessary to distinguish two values of \( b \) and \( δ \), say \( (b, δ) \) for \( Λ = 0 \) and \( (b, δ) \) for \( Λ \neq 0 \). Note that from (5), since \( δ \) is measurable, if
where, from the $t$ component of the Einstein equations ($G^t_t = -8\pi\rho(r) - \Lambda$),

$$4\pi\rho(r) + \Lambda/2 = \frac{M'(r)}{r^2}.$$  

Despite that fact that the T-OV equation has been known for over sixty years, only recently [13] has its mathematical structure been fully appreciated, even for $\Lambda = 0$.

\section*{B. Junction Conditions}

The junction of a static perfect fluid onto vacuum in spherical symmetry by way of the Darmois - Israel conditions is well understood [14]. Here we follow Musgrave and Lake [14]. To summarize, the continuity of the first fundamental form (intrinsic metric) associated with the boundary $(\Sigma)$ ensures that the continuity of $\theta$ and $\phi$ in metrics (11) and (11) is allowed and that the history of the boundary is given by

$$r_S = r_\Sigma.$$  

In terms of intrinsic coordinates $(\tau, \theta, \phi)$, the continuity of the extrinsic curvature component $K_{\tau\tau}$ along with the T-OV equation (12) gives

$$p(r_\Sigma) = 0.$$  

Equation (15) defines $r_\Sigma$ and clearly $\Lambda$ plays no role in that definition. The continuity of the extrinsic curvature components $K_{\theta \theta}$ and $K_{\phi \phi}$ give

$$M(r_\Sigma) = \frac{m}{r_\Sigma} + \frac{\Lambda r_\Sigma^2}{6}.$$  

It follows from equations (13) and (14) that

$$m = \int_0^{r_\Sigma} 4\pi r^2 \rho(r) dr$$  

independent of the value of $\Lambda$.

\section*{IV. DISCUSSION}

Equations (14) and (13) show that the turning point (at the boundary of the configuration) $r_S = r_\Sigma$ is unaffected by $\Lambda$. With equation (14) then we conclude that if the Kottler-Schwarzschild field is considered generated by a source of specified energy density and isotropic pressure $(\rho(r), p(r))$ then $u_\Sigma$ is fixed and by virtue of (14) $\Lambda$ has no effect on the bending of light. Clearly, it is the use of $m$ as opposed to $M(r_\Sigma)$ that is crucial here. When we say “the mass” it is $m$, and not $M(r_\Sigma)$, that we refer to. For example, for timelike orbits in the weak field approximation, $m$ is “the mass”, not $M(r_\Sigma)$. Indeed, $M(r_\Sigma)$, though invariantly defined, cannot be given without a
priori knowledge of $\Lambda$. With the choice that $M(r_\Sigma)$ alone specifies the configuration, it follows from equation (16) that $\Lambda$ would alter the bending of light, but only by way of one’s choice as to how the Kottler-Schwarzschild field was specified. The results of this paper can be understood in a more general context. For an irrotational null geodesic congruence there are two types of “focusing”: Ricci focusing and Weyl focusing [20]. It is well known, and easy to show, that $\Lambda$ has no effect on Ricci focusing. For the case studied here it is easy to see that $\Lambda$ has no effect on the Weyl focusing either since for the metric (1) with (2) $\Lambda$ does not enter the $C_{\gamma\delta}$ components of the Weyl tensor.

Acknowledgments

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[12] See, for example, S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, (Wiely, New York, 1972).

Appendix

The exact solution to (10) is given, up to sign, by

$$\phi(u) = \sqrt{\frac{k^2(u) - \Theta}{l(u)A}} B(2\sqrt{\frac{u_\Sigma - u}{A}}, \sqrt{\frac{A}{B}}),$$

(18)

where,

$$k(u) = 4u - 1 + 2u_\Sigma,$$

(19)

$$l(u) = -2u_\Sigma^2 + u_\Sigma - 2u_\Sigma u + u - 2u^2,$$

(20)

$$A = 6u_\Sigma - 1 + \sqrt{\Theta},$$

(21)

$$B = 6u_\Sigma - 1 - \sqrt{\Theta},$$

(22)

$$\Theta = (1 - 2u_\Sigma)(1 + 6u_\Sigma),$$

(23)

and $F$ is the incomplete elliptic integral of the first kind.