Collapse and dispersal of a homogeneous spin fluid in Einstein-Cartan theory

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We study the collapse process of a massive star whose matter content is a Weyssenhoff fluid and show that the spin of matter, in the context of a negative pressure, acts against the pull of gravity. Such a mechanism and decelerates the collapse dynamics to finally replace the spacetime singularity by a bounce after which an expanding phase starts. We analyze the solutions in the large and small scale factor regimes and show that the scale factor never vanishes but reaches a minimum in the later one. Depending on the model parameters, there can be found a minimum value for the boundary of the collapsing star or correspondingly a threshold value for the mass content below which the formation of a dynamical horizon can be avoided. Our results are supported by a thorough numerical analysis.

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I. INTRODUCTION

The process of gravitational collapse of a massive star with the mass many times than the size of the Sun after it exhausts its thermonuclear fuel, and the issue of singularity formation is one of the most fundamental open problems in the contemporary general relativity. Such a study was first started by the pioneering work of Oppenheimer and Snyder (OS) [1] who considered spherically symmetric distribution of pressure-less matter collapsing under its own gravity. The result was the formation of spacetime singularity covered by an event horizon. According to the celebrated theorems proved by Hawking and Penrose [2], under physically reasonable circumstances, the spacetimes describing the solutions to the Einstein field equations in a typical collapse scenario would inevitably admit singularities hidden either behind a horizon (black hole) or visible to faraway observers (naked singularities).

However, the OS model is faraway to be realized as a physical collapse setting for a realistic star, since for such an object there exists many physical processes that could possibly alter the collapse dynamics and its end-state. There exists in the literature a variety of works dealing with this issue. Among them we quote the role of inhomogeneities within the matter distribution on the final fate of gravitational collapse [3], collapse of a perfect fluid with heat conduction [4], effects of shear on the collapse end product [5], and collapse within different gravitational theories [6] (see also [7] for recent reviews). In the study of gravitational collapse of massive real stars, the inclusion of intrinsic angular momentum of fermions and thus its possible effects on the collapse dynamics could be crucial, specially at the final stages where the fermion degeneracy pressure could go against the gravitational attraction to ultimately balance it. In such a scenario the collapse may no longer terminate in a spacetime singularity and instead is replaced by a bounce, a point at which the collapse stops and an expanding phase begins. The research of the recent years has shown that in the final stages of a typical collapse scenario where a high energy regime governs, the effects of quantum gravity would regularize the singularity that happens in the classical model [8]. In cosmological settings, it is shown that non-perturbative quantum geometric effects in loop quantum cosmology would replace the classical singularity by a quantum bounce in the high energy regime where the loop quantum modifications are dominant [9]. However, since the full quantum theory of gravity has not yet been discovered, investigating the repulsive spin effects of fermions, which is more physically reasonable and confirmed observationally, on the final state of collapse could be well-motivated.

The inclusion of spin effects of particles within a collapsing cloud can be worked out in the frame work of the Einstein-Cartan (EC) theory. This is the simplest generalization of general relativity (GR) in which the intrinsic angular momentum of spinning particles is coupled to a geometric object which is related to the rotational degrees of freedom of the spacetime, the so called spacetime torsion. Within this context, many cosmological models have been found in which the unphysical big bang singularity is replaced with a bounce at a finite amount the scalar factor [10]. The organization of this paper is as follows: In Sec. II we give a brief review on the EC theory and energy momentum-tensor of the Weyssenhoff fluid. In Sec. III we study the dynamics of the collapse together with numerical
analysis and the possibility of singularity removal in the presence of a spin fluid. Finally, Conclusions are drawn in Sec. IV.

II. EINSTEIN-CARTAN THEORY

The action integral for Einstein-Cartan (EC) theory is given by

$$S = \int d^4x \sqrt{-g} \left( -\frac{\hat{R}}{\kappa^2} + \mathcal{L}_m \right),$$

(2.1)

where \( \kappa^2 = 16\pi G \) (we set \( c = 1 \)) is the gravitational coupling constant and \( \hat{R} \) being the EC curvature scalar constructed out of the general asymmetric connection (the connection of Riemann-Cartan manifold) that the antisymmetric part of which is the torsion tensor given by

$$T^\mu_{\alpha\beta} := \hat{\Gamma}^\mu_{\alpha\beta} - \hat{\Gamma}^\mu_{\beta\alpha}. \quad (2.2)$$

From the metricity condition, \( \hat{\nabla}_\alpha g_{\mu\nu} = 0 \) we can find the affine connection as

$$\hat{\Gamma}^\mu_{\alpha\beta} = \{^\mu_{\alpha\beta}\} + K^\mu_{\alpha\beta}, \quad (2.3)$$

where the first part being the Christoffel symbols and the second part being the contorsion tensor defined as

$$K^\mu_{\alpha\beta} = \frac{1}{2} \left( Q^\mu_{\alpha\beta} - Q^\mu_{\beta\alpha} - Q^\mu_{\alpha\beta} \right), \quad (2.4)$$

$$Q^\mu_{\alpha\beta} := \hat{\Gamma}^\mu_{[\alpha\beta]}.$$

\( \mathcal{L}_m = \mathcal{L}_m(g_{\mu\nu}, \hat{\nabla}_\alpha, \Phi) \) is the Lagrangian for the material fields \( \Phi \). Extremizing the action with respect to the independent variables, metric field and contorsion, gives the field equations in EC theory as [12]

$$\begin{cases}
\hat{G}_{\mu\nu} - \left( \hat{\nabla}_{\alpha} + 2Q_{\alpha\beta}^\beta \right)(T^{\mu\alpha\beta} - T^{\nu\alpha\mu} + T^{\alpha\mu\nu}) = 8\pi G T_{\mu\nu}, \\
T^{\mu\nu} = 8\pi G T^{\mu\nu},
\end{cases} \quad (2.5)$$

where

$$T^{\mu\alpha\beta} = Q^{\mu\alpha\beta} + \delta^\alpha_{\mu} Q^{\nu\beta} - \delta^\alpha_{\nu} Q^{\mu\beta}. \quad (2.6)$$

and the Einstein tensor and covariant derivative operator depend on both metric and affine connection. The metrical energy-momentum tensor and the spin tensor are obtained as

$$\begin{cases}
T^{\mu\nu} := \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}}, \\
\tau^{\mu\nu\alpha\beta} := \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta K_{\alpha\beta\mu\nu}}. \quad (2.7)
\end{cases}$$

From the second part of equation (2.5) we observe that the equation governing torsion tensor is an equation of pure algebraic type, the torsion is not allowed to propagate and can be only nonzero inside the matter distribution. Therefore, we can substitute for it in the first part to finally achieve the modified field equations as

$$G^{\mu\nu} \left( \mathcal{L}_m \right) = 8\pi G \left( T^{\mu\nu} + \theta^{\mu\nu} \right), \quad (2.8)$$

where the left hand side denotes the usual symmetric Einstein tensor and

$$\theta^{\alpha\beta} = \left[ -4\tau^{\alpha\mu}_{\mu} T^{\beta\nu}_{\nu} - 2\tau^{\alpha\mu}_{\mu} T^{\beta\mu}_{\nu} + \tau^{\mu\nu\alpha}_{\mu} T^{\beta\nu}_{\nu} + \frac{1}{2} \theta^{\alpha\beta} \left( 4\tau^{\mu\nu}_{\lambda} T^{\lambda\nu}_{\nu} + \tau^{\mu\nu\lambda}_{\nu} T^{\nu\lambda}_{\lambda} \right) \right]. \quad (2.9)$$
A. Classical description of spin fluid

In the model presented in this paper we employ an ideal Weyssenhof fluid which is considered as a continuous medium whose elements are characterized by the intrinsic angular momentum (spin) of particles. Following [13], the energy-momentum tensor for the Weyssenhof fluid is defined as

$$\tau_{\mu\nu} = \frac{1}{2} S_{\nu\mu} u^\alpha,$$

(2.10)

where $S_{\mu\nu}$ is the antisymmetric spin density tensor and $u^\alpha$ is the four velocity of fluid. The Frenkel condition which arises by varying the Lagrangian of the sources [14] requires

$$S_{\mu\nu} u^\nu = 0.$$

(2.11)

This condition further restricts the torsion tensor to be traceless. From the microscopical viewpoint a randomly oriented gas of fermions is the source for the spacetime torsion. However we have to treat this issue from a macroscopic viewpoint, that means we need to perform suitable spacetime averaging. In this respect, the average of the spin density tensor vanishes, $\langle S_{\mu\nu} \rangle = 0$. But even with vanishing this term at macroscopic level, the square of spin density tensor $S^2 = \frac{1}{2} (S_{\mu\nu} S_{\mu\nu})$ contributes to the total energy-momentum tensor. Then, the Einstein field equations with additional spin term become

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}^{\text{eff}},$$

(2.12)

where

$$T_{\mu\nu}^{\text{eff}} = (\rho^{\text{eff}} + p^{\text{eff}}) u^\mu u^\nu - p^{\text{eff}} g_{\mu\nu},$$

(2.13)

is the effective energy-momentum tensor with $\rho^{\text{eff}} = \rho - 2\pi G S^2$ and $p^{\text{eff}} = p - 2\pi G S^2$ where $\rho$ and $p$ being the usual energy density and pressure of the fluid satisfying a barotropic equation of state $p = w\rho$. The spin propagation equation that comes from the antisymmetric part of the Einstein-Cartan equations (2.5) is given by

$$\nabla_\alpha (u^\alpha S_{\mu\nu}) = 2 u^\sigma u_{[\mu} \nabla_{\alpha]} (u^\alpha S_{\sigma\nu]),$$

(2.14)

where the indexes between the vertical lines are excluded from antisymmetrization.

III. SPIN EFFECTS ON COLLAPSE DYNAMICS AND SINGULARITY REMOVAL

The study of gravitational collapse process of a compact object and its importance in relativistic astrophysics was initiated since the work of Datt [12] and OS [1], where they used general relativity to investigate the dynamics of the collapse of a homogeneous dust cloud under its own weight. For this idealized collapse setting which necessarily gives rise to the formation of a black hole, the only evolving portion of the spacetime is the interior of the collapsing object while the exterior one remains that of Schwarzschild solution with a dynamical boundary. However, an initial homogeneous density profile of a collapsing star may become inhomogeneous at later times and finally alters the collapse end-state. The spin effects within a more realistic collapsing body may do so as we shall see in this section.

For the collapse setting we present here, the matter content is taken as a homogeneous and isotropic Weyssenhoff fluid that collapses under its own gravity. The interior line element is parametrized as

$$ds^2 = dt^2 - \frac{a(t)^2 dr^2}{1 - kr^2} - R^2(t, r) d\Omega^2,$$

(3.1)

where $R(t, r) = ra(t)$ is the physical radius of the collapsing star with $a(t)$ is the scalar factor, $k$ is a constant that is related to the curvature of spatial metric and $d\Omega^2$ is the standard line element on the unit two-sphere. The field equations then read

$$\begin{cases}
(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho - \frac{(4\pi G)^2}{3} S^2, \\
\ddot{a} = - \frac{4\pi G}{3} (\rho + 3p) + \frac{2}{3} (4\pi G)^2 S^2.
\end{cases}$$

(3.2)
The contracted Bianchi identities give rise to the continuity equation as

\[ \dot{\rho}_{\text{eff}} = -3 \frac{\dot{a}}{a} (\rho_{\text{eff}} + \rho_{\text{eff}}), \quad (3.3) \]

whence we have

\[ \dot{\rho} = -3 \frac{\dot{a}}{a} (\rho + p), \quad (S^2) = -6 \frac{\dot{a}}{a} S^2. \quad (3.4) \]

We note that the second part of the above equation is nothing but the spin propagation equation (2.14) written in terms of spin density scalar \( S^2 \) \[16\]. The first part of the above equations give

\[ \rho = \rho_i \left( \frac{a}{a_i} \right)^{3(1+w)}, \quad (3.5) \]

where \( \rho_i \) is the initial energy density profile. A suitable averaging gives \[17\]

\[ S^2 = \frac{\hbar^2}{8 A_w^{-\frac{2}{3+w}} \rho_i^{-\frac{2}{3+w}}}, \quad (3.6) \]

where \( A_w \) is a dimensional constant dependent on \( w \). It should be noticed that substituting (3.5) into the above expression leads to \( S^2 \propto a^{-6} \) which is nothing but the solution of the second part of (3.4). Defining

\[ C := \frac{4\pi G}{3} \rho_i a_i^{3(1+w)}, \quad D := \frac{(4\pi G)^2}{24} \hbar^2 A_w^{-\frac{2}{3+w}} \rho_i^{-\frac{2}{3+w}} a_i^6, \quad (3.7) \]

we finally get

\[
\begin{align*}
\left( \dot{\frac{a}{a}} \right)^2 + k a^3 &= 2 C a^{-3(1+w)} - D a^{-6}, \\
\frac{\ddot{a}}{a} &= -C (1 + 3w) a^{-3(1+w)} + 2Da^{-6}.
\end{align*}
\]

(3.8)

Next we proceed to study the collapse evolution for the different values of the spatial curvature. We assume that the star begins its contraction phase from a stable situation, i.e. \( \dot{a}(t_i) = 0 \), where \( t_i \) is the initial time at which the collapse commences. Thus, from the first part of (3.8) we find

\[ k = \left[ \frac{2C}{D} - a_i^{3(w-1)} \right] D a_i^{-(1+3w)}, \quad (3.9) \]

where \( a_i \) is the initial value of the scalar factor at the initial epoch. Depending on the sign of the expression in double brackets, the constant \( k \) may be either positive, negative or zero. Therefore we may write

\[
\begin{align*}
\begin{cases}
  k > 0 & \frac{2C}{D} > a_i^{3(w-1)}, \\
  k \leq 0 & \frac{2C}{D} \leq a_i^{3(w-1)}.
\end{cases}
\end{align*}
\]

(3.10)

Let us consider the dust fluid \((w = 0)\) for which the solution of \((k = 0)\) clearly represents an expanding solution. For the case \((k < 0)\) the collapse velocity is non-real which is physically implausible \[18\]. Thus the only remained case is \( k > 0 \) for which we are to investigate the collapse dynamics for large and small values of the scale factor, i.e., the early and late stages of the collapse process, respectively.

In the early stages of the collapse, the spin contribution is negligible and thus the first part of (3.8) can be approximated as

\[ a^2 \approx -k + \frac{2C}{a}. \quad (3.11) \]

Performing the transformation \( ad\xi = \sqrt{k} dt \) we get the solution as

\[
\begin{align*}
\begin{cases}
a(\xi) = \frac{C}{k} (1 \pm \cos(\xi)), \\
t(\xi) = \frac{Ci}{k} (1 \pm \sin(\xi)) + t_i,
\end{cases}
\end{align*}
\]

(3.12)
where according to equation (3.11) \( a_i \cong \frac{2C}{D} \).

For the collapse evolution where the scale factor has become small enough and the spin effects are dominant, the \( k \) term in the first Friedmann equation (3.8) can be neglected as compared to the rest and we get

\[
a^2 \cong \frac{2C}{a} - \frac{D}{a^4},
\]  

(3.13)

for which the solution is given by

\[
a(t) = \left\{ a_i^3 + \frac{9}{2}C(t - t_s)^2 - \sqrt{18C(t - t_s)}\sqrt{a_i^3 - \frac{D}{2C}} \right\}^{\frac{1}{3}},
\]  

(3.14)

where \( t_s > t_i \) represents the time at which the small scale factor regime starts at a finite value of the scale factor \( a_s < a_i \). The solution (3.14) exhibits a bounce at a finite time, say \( t = t_b \), where the collapse halts (\( \dot{a}(t_b) = 0 \)) at a minimum value of the scale factor given by

\[
a_{\text{min}} = \left[ \frac{D}{2C} \right]^{\frac{1}{3}} = \left[ \frac{\pi G \hbar^2 \rho_i}{4A_0^3} \right]^{\frac{1}{3}} a_i.
\]  

(3.15)

We note that for \( a > a_{\text{min}} \), equation (3.14) is always real.

For a physically reasonable collapse setting the weak energy condition (WEC) must be studied. This condition states that for any non-spacelike vector field \( T_{\alpha\beta}V^\alpha V^\beta \geq 0 \), which for our model amounts to \( \rho_{\text{eff}} \geq 0 \) and \( \rho_{\text{eff}} + p_{\text{eff}} \geq 0 \). The first inequality suggests that \( \rho \geq 2\pi G S^2 \) while the latter implies \( \rho \geq 4\pi G S^2 \). The first inequality with the use of (3.6) gives

\[
\rho \leq \frac{4A_0^3}{\pi G \hbar^2} = \rho_i \left( \frac{a_i}{a_{\text{min}}} \right)^3, 
\]  

(3.16)

whereby considering (3.5) we arrive at \( \frac{a}{a_{\text{min}}} \geq 1 \). Since the scalar factor never reaches the values smaller than \( a_{\text{min}} \), this inequality is always held implying the satisfaction of positive energy density condition. Moreover, the second inequality with similar calculations for dust tells us \( \frac{a}{a_{\text{min}}} \geq 2^\frac{1}{2} \). This means that in the later stages of the collapse as governed by a spin dominated regime, WEC is violated. Such a violation of the weak energy condition can be compared to the counterpart models where the effects of quantum gravity on the final fate of the collapse scenario has been discussed [19]. In brief, we have WEC violation for the following interval

\[
a_{\text{min}} \leq a < 2^\frac{1}{2} a_{\text{min}}.
\]  

(3.17)

A. Numerical analysis

In order to get a better understanding of the situation we perform a numerical simulation of the quantities, the time behavior of the scalar factor, its time derivative, collapse acceleration and Kretschmann scalar. In figure 1 we have plotted for these quantities by solving the second part of (3.8) numerically and taking the first part as the initial constraint. As the full curve for the scale factor shows, it begins from its initial value and reaches a minimum value \( a_{\text{min}} \) at the bounce time, while the solution for the dashed curve for which the spin effects are absent \( (D = 0) \) represents the formation of a spacetime singularity at a finite amount of time. The diagram for the speed of collapse indeed verifies such a behavior (see the full curve) where the collapse begins at rest with the speed changing its sign from negative to positive values at the point we called the bounce time. The behavior of the collapse acceleration gives us more interesting results. We see that \( \ddot{a} \) changes its sign at two inflection points. Before the first inflection point \( \ddot{a} < 0 \) and \( \dot{a} < 0 \) signaling an inflationary contracting phase. The collapse acceleration then passes its first inflection point entering a decelerated contracting phase to finally brake at the bounce time. After the bounce occurs the collapse begins an inflationary expanding phase which as we see in the diagrams is accompanied by the positive values of the collapse acceleration and its velocity. Finally a decelerating expanding phase commences once the collapse acceleration crosses its second inflection point (the same cosmological scenario has been discussed in [20]). The collapse then disperses at later times. The Kretschmann scalar remains finite (full curve) and thus no singularity happens. Now, what would happen to the formation of apparent horizon during the entire evolution of the star and specially whether the bounce is hidden within a dynamical horizon or not. In order to answer this question we proceed by recasting the metric (3.1) into the double-null form as

\[
ds^2 = -2d\zeta^+d\zeta^- + R^2 d\Omega^2.
\]  

(3.18)
with the null one-forms defined as
\[ d\zeta^+ = -\frac{1}{\sqrt{2}} \left[ dt - \frac{a}{\sqrt{1 - kr^2}} dr \right], \]
\[ d\zeta^- = -\frac{1}{\sqrt{2}} \left[ dt + \frac{a}{\sqrt{1 - kr^2}} dr \right], \] (3.19)
whereby we can easily find the null vector fields as
\[ \partial_+ = \frac{\partial}{\partial \zeta^+} = -\sqrt{2} \left[ \partial_t - \frac{\sqrt{1 - kr^2}}{a} \partial_r \right], \]
\[ \partial_- = \frac{\partial}{\partial \zeta^-} = -\sqrt{2} \left[ \partial_t + \frac{\sqrt{1 - kr^2}}{a} \partial_r \right]. \] (3.20)
The condition for the radial null geodesics, \( ds^2 = 0 \), leaves us with the two kinds of null geodesics characterized by \( \zeta^+ = \text{constant} \) and \( \zeta^- = \text{constant} \). The expansions along these two congruences are given by
\[ \theta_\pm = \frac{2}{R} \partial_\pm R. \] (3.21)
In a spherically symmetric spacetime, the Misner-Sharp quasi-local mass which is the total mass within the radial coordinate \( r \) at the time \( t \) is defined as [21]
\[ m(t, r) = \frac{R(t, r)}{2} \left( 1 + g^{\mu\nu} \partial_\mu R(t, r) \partial_\nu R(t, r) \right) \]
\[ = \frac{R(t, r)}{2} \left( 1 + \frac{R(t, r)^2}{2} \theta_+ \theta_- \right). \] (3.22)
Therefore, it is the ratio \( 2m(t, r)/R(t, r) = \ddot{R}(t, r)^2 + kr^2 \) that controls the formation or otherwise of trapped surfaces so that the apparent horizon defined as the outermost boundary of the trapped surfaces is given by the condition \( \theta_+ \theta_- = 0 \) or equivalently \( 2m(t, r_{ah}(t)) = R(t, r_{ah}(t)) \). The equation for the apparent horizon curve then reads
\[ R(t, r_{ah}(t))^{-2} = \left( \frac{\dot{a}(t)}{a(t)} \right)^2 + \frac{k}{a(t)^2}, \] (3.23)
or by the virtue of the first part of 3.8
\[ r_{ah}(a(t)) = \left[ \frac{2C}{a(t)} - \frac{D}{a(t)^2} \right]^{-\frac{1}{2}}. \] (3.24)
In order to find the minimum value for the apparent horizon curve we can easily extremize the above equation to get
\[ a_* = \left( \frac{2D}{C} \right)^{\frac{1}{2}}. \] (3.25)
Therefore there exists a minimum radius
\[ r_{min} = r_{AH}(a_*) = \frac{1}{\sqrt{2}a_i} \left( \frac{\hbar}{\pi G \rho_i A_0} \right)^{\frac{1}{2}}, \] (3.26)
for which if the boundary is taken as \( r_b < r_{min} \), then no horizon forms throughout the collapsing and expanding phases. The existence of a minimum value for the boundary implies that there exists a minimum value for the total mass contained within the collapsing star. Let us be more precise. Using equation (3.22) we can rewrite the dynamical interior field equations (3.8) as
\[ \partial_t m(t, r) = 4\pi G \rho_{\text{eff}} R(t, r)^2 \partial_t R(t, r), \]
\[ \partial_t m(t, r) = -4\pi G \rho_{\text{eff}} R(t, r)^2 \partial_t R(t, r), \] (3.27)
The unit spacelike vector fields normal to the interior and exterior hypersurfaces are given by

\[ \text{interior line element given by (3.1). The interior coordinates are labeled as} \]

\[ \text{Matching the induced metrics give} \]

\[ m(t, r) = \frac{4\pi G}{3} r^3 \rho_i a_i^3 \left[ 1 - \frac{\pi G\hbar^2}{4A_0^2} \hat{\rho}_i \left( \frac{a_i}{a(t)} \right)^3 \right]. \tag{3.28} \]

We then get a threshold mass confined within \( r_{\text{min}} \) as

\[ m_* = m(a_*, r_{\text{min}}) = \frac{\hbar}{\sqrt{2} A_0}. \tag{3.29} \]

Thus, if the total mass is chosen so that \( m < m_* \), there would not exist enough mass within the collapsing cloud at the later stages of the collapse to trap the light and as a result the formation of apparent horizon is avoided. Furthermore, the time derivative of the mass function (or correspondingly using the second part of (3.27))

\[ \partial_t m(t, r) = \frac{\pi^2 G^2 \hbar^2 r^2 a(t)^6}{A_0^2 a(t)^4} r^3 \dot{a}(t), \tag{3.30} \]

is negative throughout the contracting phase. This may be interpreted as if some mass may be thrown away from the star till the bounce time is approached, where the whole evaporation of the star occurs, \( m(a_{\text{min}}, r) = 0 \). After this time, when the expanding phase begins, the ejected mass may be regained since \( m|_{(t > t_b)} > 0 \). We note that such a behavior is due to the homogeneity of the model since all the shells of matter collapse or expand simultaneously. For the case of dust fluid considered here, the exterior region of the configuration is Schwarzschild spacetime since the spin effects are negligible at the early stages of the collapse. However, as the collapse advances, the mass profile is no longer constant due to the presence of negative pressure originating from spin of fermions and decreases toward the bounce point. Hence, at the very late stages of the collapse, The Schwarzschild spacetime may not be suitable candidate for the matching process and instead the interior region should be smoothly matched to the exterior generalized Vaidya metric [22]. Let us take the exterior spacetime in retarded null coordinates as

\[ ds^2_{\text{out}} = f(u, r_v) du^2 + 2dudr_v - r_v^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{3.31} \]

where \( f(u, r_v) = 1 - 2M(r_v, u)/r_v \) with \( M(r_v, u) \) being the Vaidya mass. We label the exterior coordinates as \( \{ X^\mu_{\text{out}} \} \equiv \{ u, r_v, \theta, \phi \} \) where \( u \) is the retarded null coordinate labeling different shells of radiation and \( r_v \) is the Vaidya radius. The above metric is to be matched through the timelike hypersurface \( \Sigma \) given by the condition \( r = r_b \) to the interior line element given by (3.31). The interior coordinates are labeled as \( \{ X^\mu_{\text{in}} \} \equiv \{ t, r, \theta, \phi \} \). The induced metrics from the interior and exterior spacetimes close to \( \Sigma \) then read

\[ ds^2_{\Sigma_{\text{out}}} = dt^2 - a^2(t) \hat{\rho}_i a_i^3 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \tag{3.32} \]

and

\[ ds^2_{\Sigma_{\text{out}}} = \left[ f(u(t), r_v) \right] \hat{u}^2 + 2\hat{r}_v \hat{u}] \hat{u}^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right). \tag{3.33} \]

Matching the induced metrics give

\[ f \hat{u}^2 + 2\hat{r}_v \hat{u} = 1, \quad r_v(t) = r_b a(t). \tag{3.34} \]

The unit spacelike vector fields normal to the interior and exterior hypersurfaces are given by

\[ n^\mu_{\text{in}} = \left[ \begin{array}{c} 0 \\ \sqrt{1 - kr^2} a(t) \\ 0 \\ 0 \end{array} \right], \]

\[ n^\mu_{\text{out}} = \frac{1}{f(u, r_v) \hat{u}^2 + 2\hat{r}_v \hat{u}] \hat{u}^2} \left[ -\hat{r}_v, \hat{u}, 0, 0 \right]. \tag{3.35} \]

The extrinsic curvature tensors for the interior and exterior spacetimes are given by

\[ K_{\text{in}}^{ab} = -n^\mu_{\text{in}} \left[ \frac{\partial^2 X^\mu_{\text{in}}}{\partial y^a \partial y^b} + \hat{\Gamma}^{\mu}_{\nu \sigma} \frac{\partial X^\nu_{\text{in}}}{\partial y^a} \frac{\partial X^\sigma_{\text{in}}}{\partial y^b} \right], \tag{3.36} \]

and

\[ K_{\text{out}}^{ab} = -n^\mu_{\text{out}} \left[ \frac{\partial^2 X^\mu_{\text{out}}}{\partial y^a \partial y^b} + \{ \nu \sigma \}^{\text{out}} \frac{\partial X^\nu_{\text{out}}}{\partial y^a} \frac{\partial X^\sigma_{\text{out}}}{\partial y^b} \right]. \tag{3.37} \]
respectively where \( y^a = \{t, \theta, \phi\} \) are coordinates on the boundary. We note that in computing the components of the interior extrinsic curvature, the general affine connection should by utilized. However, from equations together with (2.10), we see that the affine connection is finally obtained linearly with respect to the spin density tensor. Therefore, for a suitable spacetime averaging only the Christoffel symbols would remain to be used in (3.36). The non-vanishing components of the extrinsic curvature tensors then read

\[
K^\theta_\theta^{\text{in}} = 0, \quad K^\phi_\phi^{\text{in}} = \frac{\sqrt{1 - kr_r^2}}{r_0 a(t)},
\]

\[
K^{\theta \text{out}}_{tt} = -\frac{\dot{\theta}^2}{r_v^2} \left[ ff, r_v \dot{\theta} + f, u \dot{\theta} + 3 f, r_v \dot{r}_v \right] + 2 (\dot{u} r_v - \dot{r}_v \ddot{u}),
\]

\[
K^{\phi \text{out}}_{\theta} = K^{\phi \text{out}}_{\theta} = \frac{\ddot{u} + \ddot{r}_v}{r_v \sqrt{\dot{f}^2 + 2 \dot{r}_v}}.
\]  

Matching the components of extrinsic curvatures on the boundary give

\[
f \ddot{u} + \ddot{r}_v = \sqrt{1 - kr_r^2}, \quad (3.39)
\]

\[
\dot{u}^2 \left[ ff, r_v \dot{u} + f, u \dot{u} + 3 f, r_v \dot{r}_v \right] + 2 (\dot{u} r_v - \dot{r}_v \ddot{u}) = 0.
\]  

A straightforward but lengthy calculation reveals that results in \( f(r_v, u) = f(r_v) \) on the boundary. Furthermore from (3.39) and the first part of (3.34) we get

\[
\dot{r}_v = -(1 - f - kr_r^2)^{\frac{1}{2}}, \quad (3.41)
\]

whence using the second part of (3.34) we readily arrive at the following equality

\[
M(r_v) = m(t, r_b).
\]  

Thus from exterior view point and equation (3.29), there can be found a threshold value for the mass so that the apparent horizon fails to meet the boundary of the collapsing star. In view of (3.28), we observe that as the scale factor increases in the post-bounce regime the second term decreases to finally vanish asymptotically. This leaves us the exterior spacetime to be Schwarzschild with a constant mass. Finally we would like to mention that the above considerations can be also illustrated in terms of the collapse velocity using equation (3.29). It then translates into saying that if the collapse velocity is bounded as the full curve shows in figure 1, the boundary surface of the collapsing star can be chosen arbitrary small so that the horizon equation is never satisfied and thus as a result the bounce can be observable. However, as the dashed curve shows the speed of collapse diverges in the limit of approach to the singularity and thus the apparent horizon converges to finally cover the singularity. Figure 2 better illustrates the time behavior of the apparent horizon. As the full curve shows, in the presence of spin effects, the apparent horizon curve decreases for a while to a minimum value and diverges at the bounce time. It then converges again at the post-bounce regime to the same minimum value as that of the contracting phase and then goes to infinity. There exists a minimum radius (the dashed red curve) below which no horizon forms to meet the boundary of the collapsing object. When the spin effects are absent the apparent horizon decreases monotonically (see the full curve) and finally vanishes to cover the singularity. There can not be found any minimum for the boundary of the collapsing cloud in order to avoid the formation of the horizon.

IV. CONCLUDING REMARKS

We studied the process of gravitational collapse of a homogeneous star with the Weyssenhoff fluid as its matter content. Such a fluid is considered as a perfect fluid with spin corrections due to the presence of intrinsic angular momentum of fermionic particles within a real star. The contribution of spin effects may be negligible at the early stages of the collapse, while as the collapse proceeds further, these effects become important just as in the very early universe. We showed that in contrast to the homogeneous dust models which lead to the a spacetime singularity, the formation of such a singularity is avoided and instead a bounce occurs at the end of the contracting phase. We found that there exists a critical threshold value for the mass below which no horizon would form. The same picture can be found in where the non-minimal coupling of gravity to fermions is allowed. The whole evolution of the star
experiences four phases, the two of which are in the contracting regime and the rest in the post-bounce regime. While in the homogeneous dust case without spin correction terms, the singularity is necessarily dressed by an event horizon, formation of such a horizon can be always prevented by suitably choosing the surface boundary of the collapsing star.

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FIG. 1: Time behavior of the scale factor, the collapse velocity, its acceleration and the Kretschmann scalar for $C = 1.11$, $D = 0.08$, $a_i = 1$ and $\dot{a}(t_i) = 0$.

FIG. 2: Time behavior of the apparent horizon curve for $a_i = 1$ and $\dot{a}(t_i) = 0$, $C = 1.11$, $D = 0.08$ (full curve) and $C = 1.11$, $D = 0$ (dashed curve).