On the determination of log-normal flux distributions for astrophysical systems

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ABSTRACT
Determining whether the flux distribution of an Astrophysical source is a Gaussian or a log-normal, provides key insight into the nature of its variability. For lightcurves of moderate length (<10^3), a useful first analysis is to test the Gaussianity of the flux and logarithm of the flux, by estimating the skewness and applying the Anderson-Darling (AD) method. We perform extensive simulations of lightcurves with different lengths, variability, Gaussian measurement errors and power spectrum index β (i.e. P(f) ∝ f^{-β}), to provide a prescription and guidelines for reliable use of these two tests. We present empirical fits for the expected standard deviation of skewness and tabulated AD test critical values for β = 0.5 and 1.0, which differ from the values given in the literature which are for white noise (β = 0). Moreover, we show that for white noise, for most practical situations, these tests are meaningless, since binning in time alters the flux distribution. For β ≳ 1.5, the skewness variance does not decrease with length and hence the tests are not reliable. Thus, such tests can be applied only to systems with β ≳ 0.5 and β ≲ 1.0. As an example of the prescription given in this work, we reconfirm that the Fermi data of the blazar, 3FGL J0730.2-1141, shows that its γ ray flux is consistent with a log-normal distribution and not with a Gaussian one.

Key words: methods: statistical – galaxies: active – galaxies: jets – X-rays: binaries – radiation mechanisms: thermal, non-thermal– gamma-rays: galaxies.

1 INTRODUCTION
Recent advancements in observational astronomy have now made long-term continuous observation of astrophysical sources possible. These observations are crucial for obtaining consistent and conclusive interpretation of flux distributions. The study of long-term flux distributions is an important tool for analysing and characterising the stochastic variations in astrophysical sources. Characterisation of flux distribution can provide insight into the underlying physical process which drives the variability in the source. For example, the central limit theorem states that the addition of a larger number of independent random variables will result in a Gaussian distribution. Therefore, a Gaussian distribution of flux would imply additive models, where a linear summation of components contributes to the emission. On the other hand, a log-normal distribution (which is a Gaussian distribution when the flux is taken in log scale) would suggest that the mechanism driving the emission is of a multiplicative nature (Lyubarskii 1997; Uttley et al. 2005).

Over the past decade, log-normal flux distributions have been observed in X-ray light curves of compact accreting system like X-ray binaries and Active Galactic Nuclei (AGN) (Uttley & McHardy 2001; Vaughan et al. 2003; Gaskell 2004; Uttley et al. 2005). The emission in these sources is mainly produced by accretion discs. This possibly connects the log-normal behaviour of observed flux distributions to the accretion disc. The log-normal flux variations in the accretion disc have been explained by the propagation fluctuation model (Lyubarskii 1997). In this model, fluctuations in the mass accretion rate are produced on timescale corresponding to the local viscous time scales. They propagate inward and couple together to produce a multiplicative behaviour in the inner part of disk (Lyubarskii 1997; Uttley & McHardy 2001; Uttley et al. 2005; Arévalo & Uttley 2006; McHardy 2010). Log-normal behaviour of flux distribution has also been observed in sources which are dominated by the non-thermal jet emission (Giebels & Degrange 2009; Ackermann et al. 2015; Sinha et al. 2016; Shah et al. 2018; Meyer et al. 2019; Khatoon et al. 2020). One possible realisation for such observations is that disk fluctuations are possibly imprinted on the jet emission (Giebels & Degrange 2009; McHardy 2010; Shah et al. 2018) i.e. the fluctuations from the disk are efficiently transmitted to the jet and thereby modulating the jet emission accordingly. However, the minute time scale variation observed in high energy light curves (Gaidos et al. 1996; Aharonian et al. 2007; Paliya et al. 2015) reflect that the jet emission should be independent of accretion disc fluctuations (Narayan & Piran 2012). In such cases, the log-normal distribution in flux can be produced by the linear Gaussian perturbations in the intrinsic particle acceleration time-scales in the acceleration region (Sinha et al. 2018). Moreover, the fluctuations in the diffuse
escape time scales of the emitting electrons would produce flux distribution shapes other than Gaussian or log-normal. Alternatively, Biteau & Giebels (2012) have shown that under specific conditions, the additive shot noise-model can also produce a log-normal flux distribution. For example the Doppler boosting of emission from a large number of randomly oriented mini-jets results in a flux distribution with features similar to that of a log-normal distribution.

Two of the common tests used to differentiate between a Gaussian and a log-normal distribution are the skewness (Zwillinger & Kokoska 2000) and the Anderson-Darling (AD) (Stephens 1974). These tests can be used to determine at some confidence level (a) that the distribution is not consistent with a Gaussian one and (b) that it is consistent with a log-normal one. Thus, they are first step prelude for more detailed investigations such as direct fitting of the binned distributions. Also, they can be implemented on data with a relatively small number of points. However, the reliability and effectiveness of the tests depend on several aspects of the flux light curve such as the amplitude of the variability, the length of the data and the percentage of measurement error that the data has. Further, the slope of the power spectral density which characterizes the variability also has an important impact on the reliability of these tests. In this work, using simulated data, we study these effects on the effectiveness and reliability of the tests. Our aim is to quantitatively understand these effects and provide guidelines to users as to the minimum number of points (as a function of the variability strength and measurement errors) required for an effective implementation of AD and skewness test. The parameter values corresponding to different confidence levels for the AD test, have been numerically computed in the literature for the case when the lightcurve has an underlying power spectrum independent of frequency i.e. for white noise. However, lightcurves often have power spectrum with a power-law frequency dependence and we tabulate the corresponding parameter values for different confidence levels as a function of the power-law index. Thus we facilitate the correct use of the test to realistic lightcurves. Moreover, there are subtle and important issues regarding the effect of time-binning of the data on the nature of the resultant flux distribution. In general the binning changes nature of non-Gaussian distribution such as log-normal one, i.e. an intrinsic log-normal distribution may not remain a log-normal when the light curve is binned. Since the observed lightcurves are always binned to some time-bin, it is not clear why several astrophysical systems display log-normal distributions? In this work, we address this paradoxical issue by analysing the simulated lightcurves with the skewness and AD test.

The framework of this paper is as following:- In section (§2), we carry a detailed investigation of the lightcurves derived from white noise, for the effective and reliable use of skewness and AD test to check the Gaussianity/log-normality of flux distribution. In section (§3), we explore the efficiency of the skewness and AD tests for light curves with power-law power spectra. We applied the derived results of our simulation to observed flux distribution of one of bright Fermi blazar in section §4. In section §5, we provide summary and discussion of the work with important guidelines for the effective use of skewness and AD test.

2 SYSTEMS WITH WHITE NOISE

Skewness is measure of the asymmetry of a distribution and its value can be zero, positive or negative. For a flux distribution of $N$ points, the sample skewness, which is third standardised moment is computed as the Fisher-Pearson coefficient i.e.,

$$\kappa_F = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^3}{s^3}$$

(1)

where $\bar{x}$ and $s^2$ are mean and variance of the sample, these two quantities are first and second statistical moments, respectively. In case of symmetric distribution such as a Gaussian one, the skewness value will be zero. Thus, if the skewness of a flux distribution is shown to be inconsistent with zero, while the skewness of its log flux distribution is shown to be consistent with zero, then one can ascertain that the underlying distribution is consistent with a log-normal one and inconsistent with a Gaussian one.

In addition to skewness test, Anderson Darling (AD) test is also used to check the Gaussianity or log-normality of the flux distribution. AD test is a statistical test which checks the null hypothesis that a sample is drawn from a particular distribution (e.g. normal distribution). The AD test returns the test statistic ($T^2$-value) and critical value ($\nu$). The $\nu$ are function of the number of points in the lightcurve and are obtained numerically at significance levels of 15%, 10%, 5%, 2.5%, 1%. The null hypothesis that the sample is drawn from normal distribution is rejected at a significance level, if the returned $T^2$ values is greater than the critical value at that significance level. We have used $\nu$ at 5% significance level in this work, which implies that we accept the null hypothesis at 95% confidence level. In this section, we consider the case when the distribution is derived from white noise i.e. when the power spectrum is independent of frequency.

For a finite length lightcurve from a Gaussian flux distribution, the measured skewness, $\kappa_F$ will deviate from zero with a standard deviation of $\sqrt{6/n}$ (Tabachnick & Fidell 1996), where $n$ is the number of bins in the lightcurve. A user would then check if the skewness difference of a given lightcurve

$$\Delta \kappa_F = |\kappa_F| - L \sqrt{\frac{6}{n}}$$

(2)

exceeds zero for a certain confidence, $L$ then its true $\kappa_F$ is nonzero. Since it is sensible to use 95% confidence level, therefore we carried the analysis for $L = 2$ (i.e. 2-sigma confidence). For a lightcurve whose intrinsic distribution is log-normal, $\Delta \kappa_F$ would exceed zero only for a sufficiently large number of points, $n$. To quantify the minimum number of points required, we simulated lightcurves corresponding to a log-normal distribution. In practice, we generated a time-series of the logarithm flux from a Gaussian distribution with centroid $\mu_{LF} = 0$ and some values of width $\sigma_{LF}$ and length $n$, and then converted it to a flux lightcurve by taking the exponential of each value. Given the values of $\mu_{LF}$ and $\sigma_{LF}$, one can obtain the theoretical estimate of mean of log-normal distribution $\mu_F$, using $\mu_F = \exp(\mu_{LF} + \frac{\sigma_{LF}^2}{2})$. For different sets of $\sigma_{LF}$ and $n$, we simulated 3000 such flux lightcurves, where $\sigma_{LF}$ ranges from 0.1–1. For each of the 3000 simulated lightcurves corresponding to a fixed value of $\sigma_{LF}$ and length $n$, we computed the mean $\mu_F$, square root of the variance $\sigma_F$ and the skewness difference $\Delta \kappa_F$. The fraction, $f$ of the 3000 lightcurves for which $\Delta \kappa_F > 0$ was noted, which increased as the number of bins in the lightcurve, $n$ increased. Figure 1 shows the minimum number of bins required, $N_{50}$ and $N_{90}$, for 50% and 90% of the lightcurves to have $\Delta \kappa_F > 0$, as a function of the average normalised standard deviation, defined as

$$\sigma_{F,norm} = \frac{\sigma_F}{\mu_F}$$

(3)

The Figure 1 should be interpreted as giving an estimate of
the minimum number of bins a lightcurve should have in order to use skewness to reject the hypothesis that its flux distribution is Gaussian. The minimum number of bins required for 90% of the lightcurves to have $\Delta \kappa_F > 0$ ranges from 25 for $\sigma_{F,\text{norm}} \sim 1.1$ to 730 for $\sigma_{F,\text{norm}} \sim 0.1$.

While, $\Delta \kappa_F$ can be used to reject the hypothesis that the lightcurve is from a Gaussian distribution, in principle one can go further and compare the measured skewness $\kappa_F$ with the theoretical value for a log-normal distribution,

$$\kappa_T = \frac{\sigma_F^2}{\mu_F} + 3 \frac{\sigma_F}{\mu_F}$$

(4)

However, from our simulation we found that this is not reliable. The measured value of skewness deviates significantly from the theoretical value for finite number of data points and large $\sigma_F$. The comparison between the observed and theoretical skewness values are given in Appendix A.

Measurement errors of the flux can have a significant impact on the skewness test. If the errors are significant, it would require a larger number of flux points for a distribution obtained from an intrinsically log-normal one, to reject the hypothesis that the distribution is Gaussian. To quantify this effect, we introduce measurement errors which are normally distributed to the simulated lightcurves. We define a parameter, error ratio, $R$ as the ratio of the sigma of the error distribution $\sigma_{err}$ to that of the intrinsic flux $\sigma_F$. The measurement error is generated as random normal distribution with standard deviation, $\sigma_{err} = R \times \sigma_F$. We have added this random normal distribution to the intrinsic simulated log-normal light curves. The AD and skewness tests are then applied to the modified log-normal distribution (combination of random noise and intrinsic log-normal distribution). Figure 2 shows the variation of $N_{90}$ (left panel) as a function of average normalised standard deviation $\sigma_{F,\text{norm}}$, for $R = 0.1, 0.5$ and 1.0. The Figures shows that if the error fraction is larger than 0.5, the number of points in the lightcurve required to reject the Gaussianity of the distribution is significantly larger than the case when there are no measurement errors.

Moreover, in some circumstances the measurement error on the flux depends on the flux value, $F_i$, i.e. $\sigma_{err} \propto F_i^n$. We have simulated lightcurves for such cases and show the result for $\alpha = 0.1$ and 1.0 in Figure 3. Here $N_{90}$ is plotted against $\sigma_{F,\text{norm}}$, for $R = 0.1$ and 1.0. For $\alpha = 1.0$, the number of points required to reject the Gaussianity is less than for $\alpha = 0$.

Similarly, we repeated the above analysis for the AD test. For a lightcurve, we define a measure

$$TS_{adiff} = TS - c_{\text{blue}}$$

(5)

where $c_{\text{blue}}$ is the critical value such that the probability that $TS$ is larger for a Gaussian distribution is 5%. Analogous to the skewness test, we use simulations to estimate $N_{90}$ ($N_{95}$), the minimum number of bins for which 90% (50%) of the simulated lightcurves have a $TS_{adiff} > 0$. The results are presented in the right panels of Figures 1, 2 and 3.

For a log-normal distribution, the skewness and AD test will accept the normality of the logarithm of the flux at more than 95% confidence level by definition. However, if there is measurement error in the flux, then this may not be so and for sufficiently large error and large number of data points, the test would indicate that the log of the flux distribution is not Gaussian. We simulated lightcurves with measurement errors and find that when the number of points exceeds $10^5$ ($10^6$), the skewness test fails when the error ratio is 0.2 (0.5). Similar results are obtained for the AD test.

As a thumb rule, our results show that both skewness and AD tests can be used to determine that the distribution is not Gaussian, for lightcurves having more than 100 points and with a normalised standard deviation (fractional r.m.s of variability) greater than 30%. The ratio of the sigma of the measurement error to that of the intrinsic variability should be less 0.2. The Figures presented can be used to gauge the reliability of the tests for other values.

The measurement errors for a lightcurve can be decreased by binning the lightcurve in time. Moreover, most measurement processes involves summing the fluxes for a minimum time which is the time resolution of the instrument. While a Gaussian distribution remains a Gaussian one with binning in time, this is not the case for an intrinsically log-normal one. We have performed simulations to illustrate this point quantitatively. We binned simulated lightcurves from a log-normal distributions such that the resultant binned lightcurve has 400 data points and a $\sigma_{err} \sim 0.3$. The loga-rithm of the fluxes of these binned lightcurves were then subjected to the skewness and AD test, and the fraction of the lightcurves which failed the tests were noted. This fraction for the skewness and AD test are plotted as a function of the number points averaged in Figure 4. It is clear from the Figure that binning changes the intrinsic log distribution to some other distribution.

Thus, an intrinsically “true” log-normal distribution which has a white noise power spectrum cannot be determined to be such, if the lightcurve has been binned. If there is a cutoff in the power spectrum such that there is no intrinsic variability on time-scales shorter than a particular value and if the time resolution is smaller than this value, then the distribution maybe determined to be log-normal. This is further discussed in the last section.

3 SYSTEMS WITH POWER-LAW NOISE

For many physical systems, the power spectra generated from the lightcurves can be described as a power-law i.e. $P(f) \propto f^{-\beta}$. In this section, we explore the efficiency of the skewness and AD tests for such systems. Random time series representing power-law power spectra and whose distribution are Gaussian can be generated using the algorithm proposed by Timmer & Koenig (1995). Using this algorithm, we generated a large sample of light curves for different sets of $n$ and $\beta$, the minimum and maximum frequency of the corresponding power law power spectra are chosen as $f_{\text{min}} = \frac{1}{T}$ and $f_{\text{max}} = 0.5$, respectively. For $\beta \leq 1$, the time-series generated with Timmer & Koenig (1995) algorithm will have Gaussian PDF for all values of $n$ (Morris et al. 2019). However, for steeper power spectral index ($\beta > 1$), the light curves becomes more non-stationary and generated time-series are less likely to be described by Gaussian PDF (Alston 2019; Morris et al. 2019). Therefore, in this section we mainly focus on the results for $\beta \leq 1$.

In the previous section, the analysis was based on systems with white noise i.e. for the special case of $\beta = 0$. It should be noted that the standard deviation of skewness as well as the critical values for the AD tests are valid only for white noise and cannot be used for the general non-zero $\beta$ cases. Since we did not find these values in the Literature, we first undertook a simulation study to estimate them.

In order to estimate standard error of skewness for nonzero $\beta$-cases, we simulated lightcurves having power law PSD for different sets of $n$ and $\beta$. For each pair of $n$ and $\beta$, we generated 2000 random light curves and measured the skewness value in each trial. We plotted the standard deviation of skewness, $\sigma_{\kappa}$ (standard deviation obtained from 2000 skewness values) with $n$ in Figure 5. For $\beta = 0$, the variation in $\sigma_{\kappa}$ with $n$ is as expected i.e. $\sigma_{\kappa} = \sqrt{(6/n)}$.
and this is shown as a solid line. For $\beta < 0.5$ the standard deviation of skewness is close to that for $\beta = 0$, but for larger $\beta$ there is significant deviation. We empirically fit the variation with $n$ using a generic function

$$\sigma_{\kappa}(n) = \frac{a}{n^b}$$

(6)

and tabulate the best fit values of $a$ and $b$ for different values of $\beta$ in Table 1. The fits are shown by dotted lines in Figure 5. For a stationary time-series, the error (i.e. the standard deviation of the skewness) should decrease with the number of elements, such that for a long enough series, the skewness is well defined and convergent. This is indeed confirmed here for white noise ($\beta = 0$) which is expected to be stationary. For $\beta \lesssim 1$, the slope of standard deviation of skewness with $n$ gets flattened (i.e. its dependence on $n$ decreases) showing that the system is approaching non-stationarity. Further, for $\beta > 1$, Morris et al. (2019) have shown that the light curves generated with the Timmer & Koenig (1995) algorithm are less likely to be Gaussian due non-stationarity. This is further established here as in such cases the standard deviation in skewness
Log-normal flux distributions

Figure 3. Minimum number of flux bins, \( N_{90} \), as function of normalised standard deviation of flux distribution \( \sigma_{F,\text{norm}} \), when the measurement error depend on the flux value. The left and right panel are obtained using skewness and AD tests respectively. In both panels, black and red curves correspond to \( \alpha = 0.1 \) and 1.0 respectively. The dashed and solid representation of curves are for \( R = 0.1 \) and 1.0 respectively.

Figure 4. Top panel: The fraction for which log-normality is rejected in the binned flux distribution \( f(N) \) as function of the number of flux points over which averaging is done. The solid red curve with filled square points corresponds to skewness test, while the dotted blue curve with open circles corresponds to AD test.

Figure 5. The standard deviation of skewness if plotted against the number of flux points for different values of the power spectrum index \( \beta \). The solid line is the expected standard deviation \( \sigma_{\kappa} = \sqrt{6/n} \) valid for the case when \( \beta = 0 \). The dotted lines are the fitted lines using Equation 6, with best fit values tabulated in Table 1.

remains almost constant with \( n \). For example, as shown in Figure 5, for \( \beta \geq 1.5 \), \( \sigma_{\kappa} \) is independent of \( n \). Thus for such cases the tests are not reliable. The divergence of tests for \( \beta > 1 \) can thus be mainly attributed to increase in non-stationarity, where the variance of individual lightcurve segments is not constant in time but instead varies about a mean value determined by the underlying PSD (Uttley et al. 2005).

Similarly using 2000 simulated lightcurves, we estimated the critical values for the AD test for different values of \( \beta \) and significance levels as a function of the number of bins and the results are tabulated in Table 2. As a check we note that the critical values obtained for \( \beta = 0 \) are similar to those given in Literature (written in parenthesis in Table 2).

Having obtained the standard deviation for skewness and the critical values of AD test applicable for lightcurves having power-law power spectra, we perform similar analysis as was done in the previous section for white noise. In the end of the previous section it was shown that if a lightcurve having a log-normal distribution and white noise, is binned, the resultant distribution is no longer a log-normal one and both the skewness and AD test will reject the hypotheses that it is log-normal. We first test the effect of binning on lightcurves generated from a power-law distribution. We simulated a sample of Gaussian time-series using Timmer & Koenig (1995) algorithm, which were exponentiated in order to obtain lightcurves having log-normal distribution. The exponential model reproduces the observed properties for both stationary and non-stationary light curves (Alston et al. 2019; Alston 2019). It should be noted that the shape of power spectra may not be preserved by exponentiating
a time-series. However, Uttley et al. (2005) showed that the effect is negligible for PSD’s observed in astrophysical system and it is acceptable to consider the shape of power spectra of log-normal distribution same as that of input linear time-series. The generated log-normal lightcurves are binned in order to have lightcurves with 400 points and a $\sigma_L \sim 0.3$. The distribution of the logarithm of the fluxes were tested using skewness and AD tests. In Figure 6, the fraction of the lightcurves which failed the tests are plotted as a function of the binning factor. As expected for no binning ($N_{\text{average}} = 1$), the fraction $f_s$ and $f_{\text{AD}}$ is 0.05 since a 95% confidence has been taken for the tests. As shown earlier, for white noise ($\beta = 0$), the fractions increase with binning showing that the distribution is no longer consistent with being a log-normal. However, for $\beta = 0.5$ and $\beta = 1.0$, the fractions are more or less independent of binning indicating that the log-normal nature is maintained after binning for systems where the power spectrum has a power-law shape with index greater than 0.5. A plausible reason that binning preserves the log-normality could be related to low power in high frequency components for lightcurve having steeper PSD. The maximum frequency in case of binned lightcurve can be written as $f_{\text{max}} = \frac{\Delta t}{N_{\text{average}}}$, where $\Delta t$ is the time bin of unbinned (or original) light curve, the maximum frequency decreases as the binning (or $N_{\text{average}}$) increases. In case of steeper PSD, most power is contained in low frequency components and less power will be at higher frequencies. In such cases, the effect of truncation of high frequencies will be less, hence the properties of the original light curve (e.g. log-normal) are preserved. However, in case of white noise, the variability at higher frequencies strongly influence the behaviour of the time-series, hence removal of high frequencies from the PSD will change the time series property effectively.

We continue the analysis by estimating the minimum number of bins required to reject the hypothesis that the flux distribution of such lightcurves are Gaussian. The left panel of Figure 7, displays the minimum number of points required for a log-normal lightcurve, such that for 90% of them $\Delta x_P > 0$, $N_{\text{90}}$, as a function of the normalised standard deviation. The plots are for $\beta = 1.0, 0.5$ and 0.0, where $\beta = 0.0$ is a repetition of the results obtained in the previous section for white noise. The right panel of Figure 7, is the same as the left except that it is for the case when $T_{\text{Soff}} > 0$ for the AD test. We note that the results are similar for $\beta = 0.5$ and white noise, but there is marked difference for $\beta = 1.0$, where a substantially larger number of bins is required to reject Gaussianity for normalised standard deviations less than 0.5 as compared to white noise. The effect of measurement error is shown in Figure 8, where $N_{\text{90}}$ as a function of the normalised standard deviation is shown for different values of the error ratio, $R$ for $\beta = 1.0$. As expected increasing the ratio of measurement errors to the intrinsic variation, $R = \sigma_{\text{err}}/\sigma_F$, requires a larger number of data points to reject Gaussianity.

Thus, the skewness and AD tests to identify a system has a log-normal distribution can only be effectively used if the power spectra has an index, $\beta \gtrsim 0.5$, otherwise flux binning would modify the distribution. On the other hand, for large $\beta \sim 1$, the number of bins required to reject the hypothesis that the distribution is Gaussian, is significantly larger ($\gtrsim 1000$), than for lower values of $\beta$. The most effective analysis can be undertaken when $\beta \sim 0.5$, error ration $R < 0.2$, the normalised standard deviation of the lightcurve is $\gtrsim 0.05$ and the number of bins is $> 100$.

### 4 APPLICATION OF SIMULATION RESULTS ON BLAZAR $\gamma$-RAY OBSERVATIONS

The $\gamma$-ray flux distribution for a sample of bright Fermi-blazars has been studied in detail using skewness, AD test and histogram fitting by Shah et al. (2018). To check the application of our results derived in section 3, we consider one of the bright Fermi blazar viz. 3FGL J0730.2-1141 from the sample studied by Shah et al. (2018). Using the skewness and AD test, Shah et al. (2018) showed that the monthly binned $\gamma$-ray flux distribution of this source is log-normal distribution. The number of flux points (after applying quality cuts) in the considered distribution are 99 with normalised standard deviation, $\sigma_{F, \text{norm}} \sim 0.68$. The skewness, $\kappa$ and AD TS values of the logarithm of flux distribution are obtained as -0.35 and 0.62 respectively. In order to check the reliability of the AD and skewness test for confirming the log-normality of flux distribution, we computed the error ratio ‘$R$’ and power spectra of the monthly binned $\gamma$-ray light curve. The R-value for the observed light curve is obtained as 0.0417. We computed the averaged power spectrum by dividing the monthly binned light curve into four segments, each segment of duration 24 months with a binning time resolution of one month. The power law fit ($P(f) \propto f^{\beta}$) to the averaged power spectrum, shown in Figure 9, results in $\beta = 1.02 \pm 0.22$. Corresponding to $\beta \sim 1.0$ and $n=99$, the standard deviation on skewness $\sigma_{\kappa}$ and $\sigma_{\text{AD}}$ are obtained as $\sim 0.25$ and $\sim 0.98$ respectively (see Table 1 and 2). On using the $\gamma$, $\kappa_{\gamma}$, $\sigma_{\gamma}$ and $\sigma_{\text{AD}}$, the values of $\Delta \kappa$ and $T_{\text{Soff}}$ of the logarithm of flux distribution are obtained as -0.15 and -0.36 respectively, both values suggesting the log-normal behaviour of the flux distribution. Further, as shown in Figure 8, for $\beta \sim 1$, $\sigma_{F, \text{norm}} \sim 0.68$ and $R < 0.1$, the available number of flux points, $n=99$ are sufficient to satisfy the condition required to reject the Gaussianity of flux distribution at 90% confidence.

### 5 SUMMARY AND DISCUSSION

In this work, using simulations, we have studied the efficiency and reliability of the skewness and AD tests to study whether a lightcurve corresponds to a log-normal distribution. The tests are used to ascertain whether the lightcurve’s flux distribution is inconsistent with a Gaussian one, while the distribution of the log of flux is consistent with being a normal distribution. Our motivation is to provide a prescription or guidelines for the effective use of these tests for this endeavour. The recommended prescription is described below, while an example of its implementation is given in the previous section.

- For an intrinsically log-normal distribution, the efficiency

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**Table 1.** Skewness error fit parameters ‘a’ and ‘b’ obtained by fitting the equation (6) to the variation of skewness standard deviation with $n$ for different values of $\beta$.

| $\beta$ | $a$     | $b$     |
|---------|---------|---------|
| 0.0     | 5.96±0.17 | 0.499±0.003 |
| 0.1     | 5.63±0.21 | 0.494±0.003 |
| 0.2     | 6.86±0.25 | 0.513±0.004 |
| 0.3     | 5.89±0.27 | 0.497±0.004 |
| 0.4     | 6.04±0.33 | 0.496±0.005 |
| 0.5     | 5.58±0.24 | 0.482±0.004 |
| 0.6     | 4.54±0.23 | 0.451±0.004 |
| 0.7     | 3.92±0.19 | 0.418±0.004 |
| 0.8     | 1.81±0.18 | 0.345±0.005 |
| 0.9     | 0.58±0.07 | 0.269±0.004 |
| 1.0     | 0.06±0.02 | 0.189±0.005 |
Table 2. Critical values of AD test for power law noise with $\beta = 0.0, 0.5, 1.0$ obtained from simulations. The critical values are obtained at significance level of 15%, 10%, 5%, 2.5% and 1%.

| Slope ($\beta$) | number of data points (n) | significance level % | CV calculated (CV given) |
|-----------------|---------------------------|-----------------------|--------------------------|
| 0.0             | 4096                      | 15                    | 0.56 (0.575)             |
| 0.0             | 2048                      | 15                    | 0.57 (0.575)             |
| 0.0             | 1024                      | 15                    | 0.56 (0.574)             |
| 0.0             | 512                       | 15                    | 0.56 (0.572)             |
| 0.0             | 100                       | 15                    | 0.55 (0.555)             |
| 0.0             | 4096                      | 10                    | 0.62 (0.655)             |
| 0.0             | 2048                      | 10                    | 0.63 (0.655)             |
| 0.0             | 1024                      | 10                    | 0.63 (0.653)             |
| 0.0             | 512                       | 10                    | 0.63 (0.651)             |
| 0.0             | 100                       | 10                    | 0.68 (0.632)             |
| 0.0             | 4096                      | 5                     | 0.75 (0.786)             |
| 0.0             | 2048                      | 5                     | 0.74 (0.785)             |
| 0.0             | 1024                      | 5                     | 0.75 (0.784)             |
| 0.0             | 512                       | 5                     | 0.75 (0.781)             |
| 0.0             | 100                       | 5                     | 0.75 (0.759)             |
| 0.0             | 4096                      | 2.5                   | 0.88 (0.917)             |
| 0.0             | 2048                      | 2.5                   | 0.87 (0.916)             |
| 0.0             | 1024                      | 2.5                   | 0.89 (0.914)             |
| 0.0             | 512                       | 2.5                   | 0.87 (0.911)             |
| 0.0             | 100                       | 2.5                   | 0.86 (0.885)             |
| 0.0             | 4096                      | 1.0                   | 1.02 (1.091)             |
| 0.0             | 2048                      | 1.0                   | 1.04 (1.09)              |
| 0.0             | 1024                      | 1.0                   | 1.06 (1.088)             |
| 0.0             | 512                       | 1.0                   | 1.03 (1.084)             |
| 0.0             | 100                       | 1.0                   | 1.02 (1.053)             |
| 0.5             | 4096                      | 15                    | 0.58                    |
| 0.5             | 2048                      | 15                    | 0.58                    |
| 0.5             | 1024                      | 15                    | 0.58                    |
| 0.5             | 512                       | 15                    | 0.57                    |
| 0.5             | 100                       | 15                    | 0.56                    |
| 0.5             | 4096                      | 10                    | 0.66                    |
| 0.5             | 2048                      | 10                    | 0.65                    |
| 0.5             | 1024                      | 10                    | 0.65                    |
| 0.5             | 512                       | 10                    | 0.65                    |
| 0.5             | 100                       | 10                    | 0.68                    |
| 0.5             | 4096                      | 5                     | 0.80                    |
| 0.5             | 2048                      | 5                     | 0.79                    |
| 0.5             | 1024                      | 5                     | 0.79                    |
| 0.5             | 512                       | 5                     | 0.78                    |
| 0.5             | 100                       | 5                     | 0.76                    |
| 0.5             | 4096                      | 2.5                   | 0.94                    |
| 0.5             | 2048                      | 2.5                   | 0.93                    |
| 0.5             | 1024                      | 2.5                   | 0.92                    |
| 0.5             | 512                       | 2.5                   | 0.91                    |
| 0.5             | 100                       | 2.5                   | 0.88                    |
| 0.5             | 4096                      | 1.0                   | 1.11                    |
| 0.5             | 2048                      | 1.0                   | 1.12                    |
| 0.5             | 1024                      | 1.0                   | 1.09                    |
| 0.5             | 512                       | 1.0                   | 1.04                    |
| 0.5             | 100                       | 1.0                   | 1.05                    |
| 1.0             | 4096                      | 15                    | 2.62                    |
| 1.0             | 2048                      | 15                    | 2.09                    |
| 1.0             | 1024                      | 15                    | 1.57                    |
| 1.0             | 512                       | 15                    | 1.15                    |
| 1.0             | 100                       | 15                    | 0.69                    |
| 1.0             | 4096                      | 10                    | 3.30                    |
| 1.0             | 2048                      | 10                    | 2.65                    |
| 1.0             | 1024                      | 10                    | 1.95                    |
| 1.0             | 512                       | 10                    | 1.40                    |
| 1.0             | 100                       | 10                    | 0.67                    |
| 1.0             | 4096                      | 5                     | 4.70                    |
| 1.0             | 2048                      | 5                     | 3.72                    |
| 1.0             | 1024                      | 5                     | 2.68                    |
| 1.0             | 512                       | 5                     | 1.64                    |
| 1.0             | 100                       | 5                     | 0.98                    |
| 1.0             | 4096                      | 2.5                   | 5.85                    |
| 1.0             | 2048                      | 2.5                   | 4.83                    |
| 1.0             | 1024                      | 2.5                   | 3.47                    |
| 1.0             | 512                       | 2.5                   | 2.42                    |
| 1.0             | 100                       | 2.5                   | 1.17                    |
| 1.0             | 4096                      | 1.0                   | 7.76                    |
| 1.0             | 2048                      | 1.0                   | 6.67                    |
| 1.0             | 1024                      | 1.0                   | 4.90                    |
| 1.0             | 512                       | 1.0                   | 3.26                    |
| 1.0             | 100                       | 1.0                   | 1.43                    |
Figure 6. Left panel: The fraction $f_\kappa(N)$ for which log-normality of the binned light curves is rejected ($\Delta \kappa > 0$), plotted against the number of averaged bins. The length of each binned light curve is 400. The three curves viz. violet curve with open circle points, red curve with filled square points and blue curve with filled triangle points corresponds to the light curves which are generated from power law noise with $\beta = 0.0, 0.5$ and 1.0 respectively. Right panel: Same as the left one except the condition used is for AD test.

Figure 7. The minimum number of flux points required, $N_{90}$ such that 90% of the lightcurve generated will have $\Delta \kappa_F > 0$ (left panel) and $TS_{diff} > 0$ (right panel) as a function of normalised $\sigma_{F,norm}$. In both panels, the three curves viz. green curve with filled squares, blue curve with open circles and red curve with filled circles corresponds to $\beta=0.0, 0.5$ and 1.0 respectively.

of the skewness and AD tests to reject that the distribution is a Gaussian one, is similar. The minimum number of bins required in a lightcurve to reject Gaussianity is nearly equal for both the tests.

- It is important to ascertain the nature of the power spectrum of the lightcurve by estimating the index $\beta$ of the power-law describing the spectrum. If $\beta \sim 0$ (i.e. white noise) then binning changes the distribution and since all lightcurves are binned to some extent, estimating its flux distribution is meaningless. If $\beta \gtrsim 1.5$, then the variance of the skewness is large and independent of $n$, and again the skewness and AD tests are not meaningful.

- If $0.5 \lesssim \beta \lesssim 1$, then we have provided empirical fits to the standard deviations of the skewness and tabulated critical values for the AD test. These are different from the ones for white noise and we computed them using simulations since we could not find these values in the Literature. The skewness standard deviation and critical values can be used to determine significance by which the tests reject that the distribution is normal.
Log-normal flux distributions

Figure 8. Left panel: Minimum number of flux points, $N_{90}$, required for 90% of the light curves generated from power law noise with $\beta = 1.0$, to have $\Delta \kappa_F > 0$, is plotted as function of normalised standard deviation $\sigma_{F,\text{norm}}$. The three curve’s with points shown by colours red (filled square), green (open circle) and blue (filled circle) are for error fraction, $R=0.1$, 0.5 and 1 respectively. Right Panel: Same as left one except the condition $T_{S,\text{diff}} > 0$ used is for the case AD test.

Figure 9. Power spectrum of the monthly binned $\gamma$-ray light curve of 3FGL J0730.2-1141. The red solid line is the power law fit with $\beta = 1.02 \pm 0.22$.

- The minimum number of bins required to show that a log-normal distribution is not consistent with a Gaussian one, $N_{90}$, has been computed as a function of normalised standard deviation and shown in Figures 7 and 8 for different power spectrum index $\beta$ and levels of measurement error. These can be used a priori to indicate whether a particular observation plan will produce lightcurves where these tests can be applied. Moreover, if a lightcurve having larger number of points shows that its flux distribution is consistent with a Gaussian one, then one can say with some confidence that its distribution is not a log-normal one.

- The presence of Gaussian measurement errors increases $N_{90}$ and we recommend that the error ratio, $R$ which is the ratio of the standard deviation of error to the intrinsic standard deviation of flux, to be $< 0.2$. We suggest that the data should be binned in time till $R < 0.2$. One needs to take care while constructing a histogram for characterising a PDF shape. The histogram bin width should be chosen such that the width of bin is larger than the measurement error in that bin. Smaller histogram bin-width will introduce bias in the shape of PDF, as the flux point with large error cannot be located in single bin. Larger bin-width in the histogram will be required for the larger measurement errors, which some times produces histogram bins with empty counts. Therefore, in order to reduce the measurement error, one should bin the data till the error ratio, $R$, is optimised. Our work suggest $R < 0.2$.

- For a system having an intrinsic log-normal distribution, the skewness and AD test will accept the normality of the logarithm of the flux at more than 95% confidence level. However if there is significant Gaussian measurement error in the flux, the skewness and AD test may not accept the normality of the logarithm of the flux at 95% confidence level. We show that for an error fraction of $R \lesssim 0.2$, this would require more than $10^5$ data points. Thus for most practical considerations this effect would be negligible.

- The average skewness of the flux of a log normal distribution differs from its theoretical value for finite number of bins. Moreover, its variance is large. Hence, We do not recommend that the value of the skewness of the flux to be used to show that the distribution is a log-normal one.

This analysis is pertinent to the case when one has a lightcurve with moderate number of bins say of the order of hundreds. For a larger data set, one can directly fit the histogram of the flux (or the log of the flux) with different distributions to ascertain its nature. Naturally, for a data set with moderate number of bins, the power spectra estimate will not be precise and the power spectra index $\beta$ would only be known approximately. That is the reason why we have only considered $\beta \sim 0.5$ and 1.0, and not undertaken numerically expensive analysis for finer values of $\beta$. This implies that any result obtained from the skewness and AD tests should come with the caveat that they depend on the uncertainty of estimating the
power spectrum. For similar reasons, we have not considered the more generic situation when the power spectrum can be described as a broken power-law. We note that the variance of the skewness and AD critical values listed in this work are for power-law power spectra and are not applicable for a broken power-law. For typical cases, where a flat power spectrum (e.g. $\beta \sim 0.5$) breaks into a steeper one (e.g. $\beta \sim 1.5$) beyond a break frequency, we recommend that the lightcurve be rebinned to a bin size corresponding to the break frequency and then apply the skewness and AD tests. Since the power-law index is typically steep beyond the break frequency, rebinning will not lead to significant loss of information and in general will decrease the measurement noise level.

As an important aside, we have addressed a conceptually paradoxical issue regarding non-Gaussian flux distributions. While Gaussian distributions remain Gaussian on addition, i.e. Gaussian flux distribution remain Gaussian when the lightcurve is rebinned, this is in general not true for other distributions such as a log-normal one. Thus, it is not clear why several astrophysical systems display log-normal distributions when the lightcurves they are estimated from, are always binned to some time-bin? We resolve this issue by showing that if the power spectra of the lightcurves can be described by a power law with index $\beta \gtrsim 0.5$, then the nature of the distribution is invariant to rebinning. Thus for a system to be scale free in time and have a log-normal flux distribution its power spectra must be steeper than 0.5. This insight may have important consequences for the model development and understanding of systems with log-normal distributions.

Understanding the nature of flux distribution using first analysis tools like skewness and AD that can be applied to moderate size data sets, is an important step laying the foundation for more elaborate studies. The results presented here provide a prescription and guidelines for the effective and reliable use.

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7 DATA AVAILABILITY

The Python codes and data used in this article will be shared on reasonable request to the corresponding author, Zahir Shah (email: zahir@iucaa.in or shahzahir4@gmail.com).

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APPENDIX A: SKEWNESS

A possible way to test the log-normality of a flux distribution is to compare the computed skewness of the distribution with the theoretical value given by Equation 4. However, we find that for a moderate number of data points of hundreds, this is not a reliable method. We simulated light curves corresponding to a log-normal flux distribution for different sets of $\sigma_{LF}$ and $n$ (see section 2 for details), 10000 light curves for each set of $\sigma_{LF}$ and $n$. We measured the skewness value $\kappa_F$ for all such light curve, and calculated their average, $\bar{\kappa}_F$. Figure (A1) shows $\bar{\kappa}_F$ as a function of $\sigma_{F,norm}$ using dotted lines with symbols representing different lengths of the lightcurve, $n$. The solid curve is the theoretical skewness calculated using Equation (4). The Figure clearly shows that $\kappa_F$ is significantly smaller than the theoretical values for small $n$ and this deviation increases with the standard deviation of the lightcurve. Only asymptotically with large $n$ does $\kappa_F$ match with the theoretical values. Thus, comparison of the computed skewness with the theoretical value cannot be done, for moderate length lightcurves. While it is possible to empirically fit the expected $\bar{\kappa}_F$ as a function of $n$ and normalised flux standard deviation, $\sigma_{F,norm}$, this may not be warranted because of the large variation of $\kappa_F$. This is shown in Figure (A2), where the standard deviation of the skewness (normalised by the average) is shown as function of $\sigma_{F,norm}$ for different $n$. For $n < 1000$, the variation of $\kappa_F$ is always more than 20%, which will not allow for any meaningful inference.
Figure A1. Average of the skewness of flux distribution ($\bar{\kappa}_F$) against normalised standard deviation of flux distribution $\sigma_{F,\text{norm}}$. The solid curve with black colour corresponds to the theoretical curve. The dotted curves are for different values of length of light curve, $n$.

Figure A2. The standard deviation of the skewness divided by the mean of skewness ($\sigma_{\kappa}/\bar{\kappa}_F$) vs the normalised standard deviation of flux distribution $\sigma_{F,\text{norm}}$ for $n=100$, 1000, 10000