What exactly is a Skyrmion?

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Skyrmions are well known to be baryons because their topological charge has been positively identified with the baryon number. Beyond that their identity has never been clear. In view of the possibility of skyrmion production through Disoriented Chiral Condensates in heavy ion collisions, the exact identity of the skyrmion must be resolved before they can be identified in experiments. It is shown that skyrmions are not individual baryons but coherent states of known baryons and higher resonances on a compact manifold associated with the spin and flavor symmetry group. An outline of how to calculate exactly the probability amplitudes of the superposition of physical baryon and excited baryon states that make up the skyrmion is given.

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INTRODUCTION

Over a decade ago it was suggested that in heavy ion collisions classical pion field may be formed, this would lead to coherent pion production [1]. Motivated by the Centauro events large fluctuation in the ratio of the number of charged-to-neutral pions was predicted to occur from event to event in heavy ion collisions [1, 2, 3]. More careful investigation showed that large domains with different chiral orientations were necessary for any observably large fluctuations in the pion ratios to occur [4]. Such a phenomenon was coined Disoriented Chiral Condensates (DCC) [4]. A number of papers were devoted to looking for DCC at the Tevatron [5, 6] and a even greater number were devoted to finding them in heavy ion collisions. Unfortunately so far all searches for them ended in failure at the Super Proton Synchrotron (SPS) [7] and at the Fermilab Tevatron from the Minimax experiment [8]. Facing these results it is all too easy to conclude that there is no DCC formation at all in these experiments. Nevertheless one has to bear in mind that in heavy ion collisions pions are the most copiously produced hadrons. In order to observe DCC a large number of charged-to-neutral pions was predicted to occur [5, 7] and a even greater number were devoted to finding them in heavy ion collisions. Unfortunately so far all searches for them ended in failure at the Super Proton Synchrotron (SPS) [8] and at the Fermilab Tevatron from the Minimax experiment [9]. Facing these results it is all too easy to conclude that there is no DCC formation at all in these experiments. Nevertheless one has to bear in mind that in heavy ion collisions pions are the most copiously produced hadrons. In order to observe DCC a large number of charged pions must be produced so that they can rise above this background. Large domains with fixed chiral orientation are required for this to happen [4].

Recently the possibility of baryon-antibaryon productions through small domain DCC was raised and connected to the Ω and ¯Ω data from the SPS [10]. This possibility in the context of heavy ion collisions was previously raised in [11, 12, 13]. However it was first connected to DCC and data only recently in [14]. In it a problem on how to confront experimental data was encountered. Although a skyrmion is generally known to be a baryon or nucleon, its exact identity has never been clear. However for serious phenomenological applications such inexactitude cannot be tolerated. In this paper the mystery to the exact identity of the skyrmion is revealed. They are coherent states of baryons and excited baryons. An outline will be given on how to connect and obtain the physical states for a skyrmion. The details will be presented elsewhere [15].

THE SKYRME MODEL

In 1962 Skyrme introduced the following lagrangian

\[ \mathcal{L}_S = \frac{f_\pi^2}{4} \text{tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32g^2} \text{tr}[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2 \] (1)

where

\[ U = \exp\{i\tau \cdot \phi/f_\pi\} = (\sigma + i\tau \cdot \pi)/f_\pi, \] (2)

\[ f_\pi \] is the pion decay constant and \( g \) is now known to be the \( p-\pi-\pi \) coupling. The first term is the usual non-linear sigma model and the second part was introduced by Skyrme [11]. This model belongs to a family of lagrangians that are known to approximate QCD at low energies. Skyrme found a family of classical static solutions to the equation of motion derived from \( \mathcal{L}_S \). They can be written in the form

\[ U = U_S = \exp\{i\tau \cdot \hat{r} F(r)\} \] (3)

where \( F(r) \) is a radial function which must satisfy certain specific boundary conditions. They are

\[ F(r \to \infty) \to 0 \quad \text{and} \quad F(r = 0) = N\pi \] (4)

[11, 16, 17]. \( N \) in the last expression is the integral valued topological charge or winding number. It has been identified as the baryon number [11, 18] provided that the Wess-Zumino effective action [13] is included to eliminate the unphysical symmetry inherent in the non-linear sigma model. Only \( N = 1 \) will be considered in this paper which corresponds to the case of one baryon. This connection of skyrmion to baryon has generated much applications: from being the model for studying baryon mass spectrum [21, 22] to that for studying the quark,
spin content and form factors of the nucleon \(E^3\), from being a non-linear mechanism for baryon production in jets \(E^3\) to that in heavy ion collisions \(E^3\). There are many more works for generalizing to more than two quark flavors. A nice review of many of these applications and generalization can be found for example in \(E^3\) and in references therein.

Although the identification of a skyrmion as a baryon was made early \(E^3\) and confirmed in \(E^3\), there was no direct link of a skyrmion to a nucleon. It was speculated in \(E^3\) that the ground state of the skyrmion is no direct link of a skyrmion to a nucleon. It was made early \(E^3\) and confirmed in \(E^3, 25\), there was no quantum numbers whereas baryons have spin and isospin. To obtain these quantum numbers, a skyrmion has to be quantized.

### Quantization of the \(N = 1\) Skyrme Hamiltonian

In this paper only flavor \(SU(2)\) skyrmion will be considered. Quantization of skyrmion was done in \(E^3\) using the techniques of collective coordinate. This amounts to finding time-evolution about the static solution. Substituting the solution Eq. \(E^3\) for \(N = 1\) into the lagrangian density Eq. \(E^3\) and integrating over space gives the energy of the static skyrmion \(E^3\). The collective coordinate is introduced by observing that if \(U_S\) is a solution of the Euler-Lagrange equation to Eq. \(E^3\) so is \(U'_S = A(t)U_S A'(t)\). Here \(A(t)\) is an element of the group \(SU(2)\), which may be more conveniently written as \(A = a_0 + i\mathbf{a} \cdot \mathbf{r}\). The \(a_i\)'s, \(i = 0, 1, 2, 3\) satisfies the constraint \(a_0^2 + \mathbf{a}^2 = 1\) as for any element of \(SU(2)\). Substituting now the new \(U'_S\) into Eq. \(E^3\) and integrating over space gives the lagrangian in terms of the \(a_i\)’s as

\[
L = 2\lambda \sum_{i=0}^{3} (a_i)^2 - M .
\]

The Hamiltonian commutes with any of these \(\{H, j_i\} = \{H, j_k\} = 0\), so simultaneous eigenstates of energy, spin and isospin can be found. Favoring the third direction as usual

\[
j_3(a_1 + ia_2)^l = j_3(a_1 + ia_2)^l = \frac{\pm l}{\pi}(a_1 + ia_2)^l \quad (9)
\]

so odd integer \(l\) gives half-integral spin and isospin states while even integer \(l\) gives integral spin and isospin states. This system permits both bosonic and fermionic states. Since baryons are fermions so one can have only polynomials of odd degree \(l\) for physical states. For example labeling states by their total \(j\), spin \(m\) and isospin \(n\) quantum numbers \(|jmn\rangle\), the simplest proton and neutron spin up wavefunctions are polynomials of degree 1

\[
|p \uparrow\rangle = \langle j|1/2, 1/2, 1/2 = \frac{1}{\pi}(a_1 + ia_2) \quad (10)
\]

\[
|n \uparrow\rangle = \langle j|1/2, 1/2, -1/2 = \frac{i}{\pi}(a_0 + ia_3) .
\]

where the Dirac ket notation \(|a\rangle\) is used to represent the position vector on \(S^3\). In general the wavefunctions are holomorphic functions on \(S^3\) in terms of the coordinate \(a_i\). One could easily write down the wavefunctions for the \(\Delta\)'s and other excited baryon states. With the collective coordinate, the quantization of this system is quite straightforward.

Although the physical states have been produced by quantization and they possess the required quantum numbers, this does not furnish a connection to the classical skyrmion. Quantization by itself does not connect the quantum mechanical to the classical but if one recalls the problem of the simple harmonic oscillator, Schrödinger showed a long time ago that classical-like solutions could be found among the quantum states. These are the so-called coherent states. They are the closest quantum equivalence to the classical solutions. Our task is now reduced to finding the coherent states which are superposition of the physical quantum states.

### The Criteria for Coherent States

Very unlike the \(n\)-dimensional harmonic oscillator where the space is \(R^n\), our problem resides on the surface of the four-dimensional sphere which is a compact manifold. The techniques used in the harmonic oscillator problem cannot be applied straightforwardly. One should however be able to draw close analogy and use that as a guide. There are many works in the literature on coherent states, in curved space-time, compact manifold with
in the usual harmonic oscillator problem. From quantum state transform [31, 32].

According to [33], coherent states on a circle and a sphere have been done in S^2 and generalized to coherent state transform [31, 32]. The construction of coherent states can be summarized in analogy to Eq. (12). The angular momentum or spin J dependent coefficients σ(J) are a feature of the compact manifold S^n [32, 33]. Equipped with the annihilation operators, simultaneous eigenstates of them can be constructed from a fixed position state vector |x⟩ on the manifold. By definition X_i|x⟩ = x_i|x⟩, then if we let |ψ⟩ = e^{-C}|x⟩, this must be a simultaneous eigenstate of A_i because A_i|ψ⟩ = x_i|ψ⟩. The coherent state construction is almost complete. Because x_i is real, one must now analytically continued |x⟩ and x_i to complex coordinates |x^C⟩ and x^C. Once this last step is done, |ψ⟩ is now the coherent state labeled by x^C. Therefore there is one coherent state per phase space point as in the usual harmonic oscillator.

The actual application of this method to our problem is not so simple. Remember that on S^3 the coordinates are the a’s and we are interested in coherent states expanded in terms of physical states. This can be done by inserting a complete set of states, both integral and half integral quantum numbers (j, m, n), in the above |ψ⟩

\[ |ψ⟩ = e^{-C}|a^C⟩ = e^{-C} \sum_{j, m, n} |j, m, n⟩⟨j, m, n|a^C⟩ . \]

\( ⟨j, m, n|a^C⟩ \) is the complex conjugate of the wavefunctions evaluated at the complex coordinate a^C and C is related to the kinetic term of the Hamiltonian Eq. (6) [15]. This is highly undesirable because a skyrmion should not involve unphysical states. The simplest solution is to discard these from Eq. (16). Unfortunately this must fail. To see this, let us momentarily use a representation that states are wavefunctions and operators act by multiplication and/or differentiation. Then with only physical states, we have

\[ ψ = e^{-C} \sum_{j, m, n ∈ ℤ + \frac{1}{2}} ψ_{jmn} . \]

Acting on this with the position operator ˆa_i gives ˆa_iψ = a_iψ. Remember that physical state wavefunctions are even degree polynomials. The above operator turns all odd into even polynomials. |ψ⟩ without the unphysical states is no longer eigenstates of A_i and therefore not a coherent state. So straightforward application of the method to our physical problem fails immediately.

A SKYRMION IS A COHERENT STATE OF BARYONS AND EXCITED BARYONS

If we write the complete Hilbert space as a sum of fermionic and bosonic space \( H = F + B \), then the problem with the appearance of non-physical states is directly linked to the fact that ˆa_i maps F to B and B to F: ˆa_i(F) = |B⟩, ˆa_i(B) = |F⟩. If only one could use the operator products ˆa_i^2 or ˆa_i ˆa_j as coordinate operators instead then the problem would disappear. These map F to itself and likewise for B. This should allow us to discard
\( B \). How can one justify using such product operators? In any case instead of four \( \hat{a}_i \)'s, there would be nine independent combinations of \( \hat{a}_i \hat{a}_j \). The solution to this is instead of using the coordinate on \( SU(2) \), one should use instead those of \( SO(3) \). In other words, one first use the map of \( SU(2) \) to \( SO(3) \). An element of the former \( A \) can be mapped to one of the latter using

\[
A \tau_i A^\dagger = \tau_j R_{ji}(A) \tag{18}
\]

Here \( R \) is the \( 3 \times 3 \) rotation matrix. This map gives

\[
R_{ij} = 2 a_i a_j + \delta_{ij} (2a_0^2 - 1) - 2 \epsilon_{ijk} a_0 a_k \tag{19}
\]

where \( i, j = 1, 2, 3 \). The new coordinate with nine components represents points on the \( SO(3) \) manifold and is now combinations of products of \( a_i \) in disguise.

One will have to quantize the Hamiltonian again in terms of \( R \) instead of the \( a_i \)'s. With these new coordinates, one can now follow the above prescription to construct the coherent state involving only physical baryon and excited baryon states. The relative probability of the physical states depends on the operator \( e^{-C} \) which acts on the physical states and the wavefunction evaluated at a complex point \( R \). From the above discussion, these can be calculated. The make up of a skyrmion as well as the relative probability a physical baryon state will be produced from skyrmion formation through DCC in heavy ion experiments can now be determined.

The solution to the identity of the skyrmions naturally generates a lot of questions in light of the many existing literature and applications. The detail construction of a skyrmion in terms of baryonic coherent state, their application to heavy ion collisions, as well as other related questions will be pursued elsewhere. This completes the outline of how to arrive at a solution to the identity of a skyrmion. In brief a skyrmion is not a single baryon, but a quantum mechanical superposition of baryon and resonance states.

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