String theory derivation of RR couplings to D-branes

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Abstract

We derive the couplings of D-branes to the RR fields from the first principles, i.e. from the nonlinear $\sigma$-model. We suggest a procedure to extract poles from string amplitudes before the eventual formula for the couplings is obtained. Using the fact that the parts of amplitudes with poles are irrelevant for the effective action this procedure ultimately simplify the derivation since it allows us to omit these poles already from the very beginning. We also use the vertex for the RR fields in the $(\frac{3}{2}, -\frac{1}{2})$ ghost picture as the useful tool for the calculation of such string amplitudes. We carry out the calculations in all orders of brane massless excitations and obtain the Myers-Chern-Simons action. Our goal is to present the calculations in full technical details and specify the approximation in which one obtains the action in question.

1 Introduction

The couplings of the RR fields to the D-branes have been intensively studied recently from the various points of view. In several cases they were determined using anomaly inflow arguments [1, 2, 3, 4, 5]. But this approach is inconsistent when transversal scalars are nonzero. In this case the couplings were derived using properties of the string theory with respect to T-duality transformations [6] or using some mathematical tools [7]. Unfortunately when scalars are nonzero the interaction between the RR fields and excitations of branes becomes more complicated, but it still preserves a nontrivial gauge symmetry [8]. Several of these results were checked in the leading orders by calculating string-scattering amplitudes [9, 10, 11].

In this paper we study the couplings of RR fields to a stack of $N$ D-branes in the type II superstring theory by calculating string-scattering amplitudes. We restrict ourself to the tree-level amplitudes in the low-energy limit. We also assume that the only nontrivial fields are massless RR fields $C$, gauge fields $A_\mu$ and scalar fields $\Phi^i$.

The main difficulty in this approach is the calculation of integrals over locations of vertex operators. This problem become more and more complicated when the number of vertexes increases. However in order to determine the effective action we are looking for only a part of all integrals is needed. In fact, in order to determine the effective action for fields rather than sources one has to make the Legendre transformation. In the case of tree-level scattering amplitudes this transformation becomes simple — one just has to remove all terms with poles in momentum and leave finite in arbitrary momentum part. If this extraction of singularities (poles in amplitudes) could be done before the calculation of integrals the derivation of couplings simplifies drastically. Because most of complicated, but irrelevant expressions can be omitted. In the next section we suggest the procedure of such extraction and discuss it’s application in details. Note that the integrals for the finite parts of the amplitudes are simple and can be calculated for arbitrary number of vertex operators. The result is presented in appendix.

Before proceeding with the general case with both nontrivial $A_\mu$ and $\Phi_i$ in section 4 we consider the simplified case, when all matrix-valued scalars vanish. We use the RR vertex

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These quadratic in $A$ following. We are interested in the on-shell amplitudes. The correct way to obtain them is to take momentum. After this one should put the momentum on-shell. Unfortunately carrying out $\sigma$ take from (3) only linear part and then calculate $\mu, \nu,...$ indexes and with ghost number 0 in the section 4 have already fixed it. In other case the result would depend on symmetry-fixing condition.

The paper is organized as follows: in the next section we discuss the separation of the amplitudes into the finite part and the part with poles. In sections 3 and sections 4 we calculate the contribution from the first part and see that it gives us the full Myers action. Section 5 ends the paper by conclusions.

2 Separation of amplitudes

Our aim now is to devide the string amplitudes, given by the following $\sigma$-model correlator

$$Tr \left( \int_{\Sigma} V_{RR} \ P \exp \left( \oint_{\Sigma} \Phi A, \Phi \right) \right)$$  \hfill (1)

with vertexes $V_{RR}$ and $V_{A,\Phi}$ for RR and gauge/massless matter fields respectively, into two parts. The first should contain finite terms (1PI graphs) and the second one — all poles from 1PR graphs.

It is clear that poles in amplitudes arise when two vertex operators merge (come close to each other). We could consider the Wick pairing between those vertices, which never merge. These pairings contribute to finite (1PI) part and so to the effective action.

In order to avoid $SL_2$ divergence one should fix the location of several vertex operators. If we fix $V_{RR}$ then there is a hole set of Wick pairings between $V_{RR}$ and all others, which are finite in the above sense.

We assume that besides those, finite at first glance terms, all others terms contribute only to S-matrix poles and do not contain part finite in all momentum. According to our assumption we are going to consider only Wick pairings between $V_{RR}$ and $V_{A,\Phi}$, but omit all pairings between $V_{A,\Phi}$ themselves, because all pairs of $V_{A,\Phi}$ meet together during the integration over $\partial \Sigma$. We also conclude that the separation should preserve $SL_2$ since we have already fixed it. In other case the result would depend on symmetry-fixing condition.

We will use the following vertex operators: with ghost number $-1$ in the next section

$$V_{A}^{-1}(z) = A_{\mu}(P)c(z)e^{-\phi(z)}\psi^\mu(z)e^{2iP_{\mu}X^\nu(z)}, \quad z \in \partial \Sigma$$ \hfill (2)

and with ghost number 0 in the section 4

$$V_{A,\Phi}^{0}(z) = c(z)\ e^{2iP_{\mu}X^\nu(z)} \left[ A(P)_{\mu}\partial X^{\mu}(z) + \alpha' F_{\mu\nu}(P)\psi^\mu\psi^\nu(z) + \Phi(P)\partial X^{i}(z) + i\alpha' [\Phi_i, \Phi_j](P)\psi^i\psi^j(z) + 2\alpha' D_{\mu}\Phi_j(P)\psi^\mu\psi^j(z) \right]$$ \hfill (3)

Indexes $\mu, \nu,...$ correspond to dimensions along $D_p$-brane, $i, j,...$— along transversal directions and $A, B,...$— along all. We use doubling trick $[2]$ in (3) and therefore all the fields on the boundary $\partial \Sigma$ depend on $z$ but are independent on $\bar{z}$. Note that the conformal ghost anomaly on the disk is $-2$. Hence to have non-zero correlation function the total ghost charge of vertex operators in (3) should be 2.

It is important that vertices (3) contain terms, which are nonlinear in external fields. These quadratic in $A_{\mu}$ and $\Phi^\ell$ expressions are so-called contact terms (3). Their origin is the following. We are interested in the on-shell amplitudes. The correct way to obtain them is to take from (3) only linear part and then calculate $\sigma$-model correlator with arbitrary external momentum. After this one should put the momentum on-shell. Unfortunately carrying out
this program in all orders in $A_\mu$ and $\Phi^i$ is complicated and have never been done. But if
one adds contact terms to vertexes before the calculation starts, one can calculate correlator
with on-shell momentum. These results should coincide. Thus contact terms are very useful
in simplification of our calculations.

The RR vertexes are

$$V_{RR}^{(-\frac{1}{2},-\frac{1}{2})}(w, \bar{w}) =$$

$$= \int d^{10} P \langle P, \vec{\Sigma} \alpha \beta \rangle (P) : c(z) e^{-\phi(w)/2} e^{iP_A X^A(w)} S_\alpha (w) : c(\bar{z}) e^{-\phi(\bar{w})/2} e^{iF_A X^A(\bar{w})} S_\beta (\bar{w}) :$$

$$\vec{\Sigma} \alpha \beta (P) = \frac{i2\sqrt{2\pi} \alpha'}{32(p' + 2)!} \int d^{10} X e^{-iP_A X^A} \Phi_{\alpha_0 ... \alpha_{p' + 1}} (X) \left[ \Gamma^{A_0 ... A_{p' + 1}} \Gamma^{p' + 1} ... \Gamma^{9} \Gamma^{11} \right]^{\alpha}_{\beta'} C^{\beta' \beta} \partial^{p} \partial_{p'} \partial_{\alpha_0} ... \partial_{\alpha_{p' + 1}} \partial_{p'} X^{A} = D_{AB}^{\alpha} X^{B}$$

$$P_{\pm} = (1 \pm \Gamma^{11}), \quad \Phi_{\alpha_0 ... \alpha_{p' + 1}} = (p' + 2) \delta_{\alpha_0 A_1 ... A_{p' + 1}}, \quad \Phi_{\alpha_0 ... \alpha_{p' + 1}} = D_{AB}^{\alpha} X^{B}$$

$$D_{AB}^{\alpha} = \begin{cases} \delta_{BA}^{A}, & A \leq p \\ -\delta_{BA}^{A}, & A > p \end{cases}$$

$$\text{and}$$

$$V_{RR}^{(-\frac{1}{2},-\frac{1}{2})}(w, \bar{w}) =$$

$$= \int d^{10} P \vec{\Sigma} \alpha \beta (P) : c(z) e^{-\phi(w)/2} e^{iP_A X^A(w)/2} S_\alpha (w) : c(\bar{z}) e^{-\phi(\bar{w})/2} e^{iP_A X^A(\bar{w})} S_\beta (\bar{w}) :$$

$$\vec{\Sigma} \alpha \beta (P) = \frac{i2\sqrt{2\pi} \alpha'}{32(p' + 2)!} \int d^{10} X e^{-iP_A X^A} C_{\alpha_0 ... \alpha_{p' + 1}} (X) \left[ \Gamma^{A_0 ... A_{p' + 1}} \Gamma^{p' + 1} ... \Gamma^{9} \Gamma^{11} \right]^{\alpha}_{\beta'} C^{\beta' \beta} \partial^{p} \partial_{p'} \partial_{\alpha_0} ... \partial_{\alpha_{p' + 1}} \partial_{p'} X^{A} = D_{AB}^{\alpha} X^{B}$$

The coefficient $\alpha' = \frac{\sqrt{2\pi} \alpha'}{32(p' + 2)!}$ is taken for convenience here. The origin of \[4\] will be discussed in section 4.

Now we are going to consider Wick pairing between $X$ fields. In the low-energy limit
$P_{\mu} P^\mu \to 0$ Wick pairings between $e^{2iP_{\mu} X^\mu(z)}$ are proportional to $|z_1 - z_2|^2 P_{\mu} P^\mu \to 1$ and
finite, even when vertexes merge.

The pairing between $\partial_{\tau} X^{A}(z_1) e^{iP_{\mu} X^\mu(z_1)}$ and $e^{iP_{\mu} X^\mu(z_2)}$ in the $P^2 \to 0$ limit goes to
\[\frac{1}{z_1 - z_2}\] and is singular when $z_1 \to z_2$.

Note that there is a significant difference between operators $A_\mu \partial X^\mu$ and $\Phi_i \partial X^i$. In fact,
operator $A_\mu \partial X^\mu$ could be contracted with all other vertexes and $SL_2$ covariant expression
is the sum over all these contractions (see example below). As we mentioned above, among
the latter there are divergent terms which we omit in our calculations. So all the sum—all
pairings should be omitted to respect $SL_2$. While $\Phi_i \partial X^i$ can be contracted separately with
$C_{\alpha_0 ... \alpha_{p' + 1}} (X^\mu, X^i)$, giving rise to the following $SL_2$ covariant expression

$$\left\langle X^i(w, \bar{w}) | \partial_{\tau} X^i(z, \bar{z} = z) \right\rangle = -i \alpha' \frac{g^{ij} (w - \bar{w})}{(w - z)(\bar{w} - z)}$$

That is why $\Phi_i \partial X^i$ should be taken into account in the course of our calculations.

Now recall how to calculate the CFT correlator between $\psi\psi$ and some others operators
$O_k^{\alpha_0 ... \alpha_n} (x)$:

$$\left\langle \psi A \psi B (z) : O^\alpha_{1} (x_1) : \ldots : O^\alpha_n (x_n) \left\rangle = \right.$$\n
$$\sum_{m=1}^{n} f_m(z, x_m) \rho(M_{AB})^{a_m}_{b_m} \left\langle \psi A^{a_m} (x_1) : \ldots : O^\alpha_m (x_m) : \ldots : O^\alpha_n (x_n) \right\rangle$$

Here $a_m$ denotes the index in any representation of Lorentz group $O(9, 1)$, $\rho(M_{AB})^{a_m}_{b_m}$ is the
representation of the generator $M_{AB}$ and $f_m(z, x_m)$ is a function corresponding to operator
O_m. For example,

\[ O =: \psi^C : \quad f = \frac{1}{z - x} \quad \rho(M_{\mu\nu})_{\mu}^\nu = \delta_{D}^B \cdot g_{AC} - \delta_{D}^A \cdot g_{BC} \quad \text{fundamental representation} \]

In the case of spin operator \( S_\alpha \):

\[ O = S_\alpha \quad f = \frac{1}{2(z - x)} \quad \rho(M_{\mu\nu})_{\beta}^\nu = \frac{1}{2} [\Gamma^\mu, \Gamma^\nu]_{\beta} - \text{spinor representation} \]

Formula (8) should be modified a little when \( O_m =: \psi^C \psi^{C'} : \). In fact, in this case

\[ f = \frac{1}{z - x} \quad \rho(M_{\mu\nu})_{CC'}^{DD'} = (\delta_{D}^B \cdot g_{AC} \delta_{D'}^C - \delta_{D}^A \cdot g_{BC}) + \delta_{D}^C (\delta_{D'}^A \cdot g_{BC'} - \delta_{D'}^B \cdot g_{AC'}) \]

\( (\rho \text{ is the generator of the Lorentz group in the tensor square of the fundamental representation) and we should also add} \]

\[ \frac{1}{(z - x_m)^2} (g^{AC'} g^{BC} - g^{AC} g^{BC'}) \left\langle : O_1^{O_1} (x_1) : \ldots : O_m^{O_m} : \ldots : O_n^{O_n} (x_n) : \right\rangle \]

to the RHS of (8). Hat means that \( O_m \) is omitted in (12).

If one wants to calculate the correlator with more than one \( \psi \psi \) : operator one has to continue by induction: calculating step by step, decreasing the number of operators \( \psi \psi \) : inside the correlator by one per one step. If the correlator is complicated it is convenient to represent graphically all items in the final sum. This technic is developed and discussed in (13).

To complete the evaluation of correlators like (8) we use the following results [15, 16]

\[ \left\langle S_\alpha (w) \ S_\beta (\bar{w}) \right\rangle = C_{\alpha \beta} (w - \bar{w})^{-\frac{4}{4}} \]

and

\[ \left\langle S_\beta (\bar{w}) \ S_\alpha (w) \ \psi^A (z) \right\rangle = \frac{1}{\sqrt{2}} \Gamma_{\alpha \beta}^{\mu} (w - z)^{-\frac{1}{2}} (\bar{w} - z)^{-\frac{1}{2}} (w - \bar{w})^{-\frac{1}{2}} \]

Now we are ready to discuss the separation of fermion fields correlators. We start with considering an example which can help us to realize the specific features of \( SL_2 \) invariance of such correlation functions. An important point here is that both parts, finite one (1PI) and the one with poles in momentum, should be represented as the integrals of \( SL_2 \) covariant expressions.

The example is (all vertexes are taken without c-ghost part here)

\[ \left\langle V_{RR}^{-1} (w, \bar{w}, p) V_{A}^{-1} (z, k) V_{A}^{0} (x, k)_1 \right\rangle = \frac{-1}{(w - z)(\bar{w} - z)(w - \bar{w})} \times \]

\[ \frac{1}{\sqrt{2}} \left\{ -i\sqrt{2} \alpha \beta H^{\alpha \beta} (p) A_{\mu}(k) A_{\nu}(k_1) \Gamma_{\alpha \beta}^{\mu} \left\{ \frac{p^\nu}{(x - w)} + \frac{p^\nu}{(x - \bar{w})} + \frac{2k^\nu}{(x - z)} \right\} + \right. \]

\[ + \frac{1}{2} H^{\alpha \beta} (p) A_{\mu}(k) F_{\sigma \rho}(k_1) \left\{ (\Sigma^\tau \Gamma^\tau C_{\alpha \beta}) \frac{1}{(x - w)} + (\Gamma^\tau C\Sigma^\tau \rho \Gamma^\tau \rho_{\alpha \beta}) \frac{1}{(x - \bar{w})} \right\} + \right. \]

\[ + H^{\alpha \beta} (p) A_{\mu}(k) F_{\rho \sigma}(k_1) \left\{ g^{\alpha \rho \tau, \beta} - g^{\alpha \rho \beta} g^{\sigma \tau \rho} \frac{1}{(x - z)} \right\}, \quad H^{\alpha \beta} = (P_\gamma)_{\alpha \beta} \Gamma^{\gamma}_{\alpha \beta} \Gamma^{\gamma}_{\alpha \beta}, \quad \Sigma^{A,B} = 1 2 [\Gamma^A, \Gamma^B], \quad p^2 = k^2 = k_1^2 = 0, \quad p + k + k_1 = 0 \]

We want now to separate eq. (15) into two parts, preserving action of \( SL_2 \) group. The first term is covariant due to on-shell condition \( k_\mu A_{\mu}(k) = 0 \). Second term is also covariant due to the following property of the gamma-matrices
\[(\Sigma^{\sigma\rho})_{\alpha}^{\delta'} (\Gamma^\mu)_{\delta'}^{\beta} C_{\alpha\beta} = - (\Gamma^\mu)_{\alpha}^{\gamma} C_{\gamma\beta} (\Sigma^{\sigma\rho})_{\beta}^{\delta'} \]  \hspace{1cm} (16)

This property could be derived easily with the convenient choice of matrix of charge conjugation \( C_{\alpha\beta} = C_{\beta\alpha} \) and its main property \( C_{\alpha\gamma}(\Gamma^\mu)_{\gamma}^{\delta} C_{\epsilon\zeta}(\Gamma^\nu)_{\zeta}^{\eta} C_{\eta\beta} = (\Gamma^\mu)_{\alpha}^{\nu} (\Gamma^\nu)_{\gamma}^{\beta} C_{\epsilon\delta} = -(\Sigma^{\mu\nu}) C_{\gamma\delta} \)  \hspace{1cm} (17)

and

\[(\Gamma^\mu)_{\epsilon}^{\alpha} (\Sigma^{\rho\sigma})_{\delta}^{\beta} = (\Sigma^{\rho\sigma})_{\epsilon}^{\alpha} (\Gamma^\mu)_{\delta}^{\beta} \]  \hspace{1cm} (18)

The identity \( (18) \) is true in the case, when \( \mu \neq \rho \) and \( \mu \neq \sigma \). This completes the derivation of \( (16) \).

The last term in \( (15) \) seems to be non-covariant. But if \( \mu \) is equal either \( \rho \) or \( \sigma \) then

\[(\Sigma^{\rho\sigma} \Gamma^\mu C)_{\alpha\beta} + (\Gamma^\mu C \Sigma^{\sigma\rho})_{\alpha\beta} \]  \hspace{1cm} (19)

is equal to

\[-g^{\rho\mu} \Gamma^\mu_{\alpha\beta} + g^{\rho\mu} \Gamma^\mu_{\alpha\beta} \]  \hspace{1cm} (20)

and together two last terms from \( (13) \) are covariant.

Thus, since the first and the last terms are singular when \( x \to z \), by assumption, only the second term contributes to the effective action in question. As well we also have obtained a nontrivial constraint that all three indexes \( \mu, \rho, \sigma \) in the second term should be different.

This consideration could be easily generalized to arbitrary correlation function in question.

In another words we see that separation of the correlators such as \( (1) \) could be done in two steps. First, it is necessary to skip \( A\mu \partial X^\mu \) from \( V^0 \). Second, it is necessary to consider Wick pairings (items in \( (8) \), or graphs in another words) only between \( V_{RR} \) and any other vertexes. As was described in the above example, we also have constraint that all operators \( \psi^A \) in vertexes, connected by Wick pairings, have all distinct vector indexes.

### 3 RR couplings to gauge fields only

In this section we carry out all necessary calculations when matter fields \( \Phi_i \) are absent and derive RR couplings in this case.

We use \( V_{RR}^{-1} \) and one \( V_{A}^{-1} \) in our calculus and preserve transformations \( C \to C + dA \) explicitly. Really, as mentioned in introduction \( V_{RR}^{-1} \) depends only on \( \delta = dC \) and does not change under such transformations. On the other hand the gauge invariance under \( A \mu \to A \mu + D \mu \alpha \) is not explicit in this approach until the eventual result is obtained.

Thus, we change

\[ P \exp \left( \oint_{\partial \Sigma} V^0_A \right) \]  \hspace{1cm} (21)

from \( (1) \) to

\[ 1 + \left\{ P \sum_{n=0}^{\infty} \oint_{\partial \Sigma} V_{A}^{-1} \left( \oint_{\partial \Sigma} V^0_A \right)^n \right\} = 1 + \int_0^1 dt P \left\{ \oint_{\partial \Sigma} t V_{A}^{-1} \exp \left( \oint_{\partial \Sigma} t V^0_A \right) \right\} \]  \hspace{1cm} (22)

in order not to change total ghost number of correlator.
First item — 1 — in this expression corresponds to the fact that D-brane is the source for the R-R field under which it is charged:

\[
\int d^{p+1}X \frac{1}{(p+1)!} C_{\mu_0...\mu_p}^{\rho_0...\rho_p} \tag{23}
\]

This coupling is non-perturbative from the first-quantized string theory point of view [2] and we will add it by hands in the end of our calculations. At this step we remove 1 from (22).

Unfortunately the naive expression (22) does not preserve SUSY. We see that all vertexes \( V_0^A \) have been changed by multiplication by \( t \). Recall that the contact terms are simply the integrals of total derivative from two-point correlation function of the linear in \( A_{\mu} \) vertexes [13]. Hence, if \( V_l = A_{\mu}(X) \partial_{\nu}X_\mu + 2\alpha'\partial_{[\mu}A_{\nu]}(X) : \psi^{\mu}\psi^{\nu} : \) change into \( tV_c \) then \( V_c = \alpha'[A_{\mu},A_{\nu}](x) : \psi^{\mu}\psi^{\nu} : \) should become proportional to \( t^2 \). This means that correct contact term is \( t^2V_c \) and we change

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_{\mu},A_{\nu}] \tag{24}
\]

from [3] by

\[
F_{\mu\nu} = t\partial_\mu A_\nu - t\partial_\nu A_\mu + it^2[A_{\mu},A_{\nu}] \tag{25}
\]

Now the action in question is given by

\[
S = Tr \int_0^1 dt \left\langle V_{RR}^{-1} P \left\{ V_{A}^{-1} \exp \left( \oint_{\partial \Sigma} V_{A,t}^0 \right) \right\} \right\rangle_{\text{CFT}} \tag{26}
\]

where

\[
V_{A,t}^0 = \alpha' F_{\mu\nu}^t(P) \psi^{\mu}\psi^{\nu}(z)e^{2itP_\mu X^\mu(z)} \tag{27}
\]

and we consider only that Wick pairings in (26), which are shown in graph 1

\[
\text{graph 1}
\]

Before our calculus starts note that P-ordering could be dropped. This is so because traces over one matrix \( A_\lambda \) and several matrices \( F_{\mu_1\nu_1}^t \) in arbitrary order are equal to each other, if these traces are multiplied by some tensor \( T_{\lambda\mu_1\nu_1...} \) with property

\[
T_{\lambda_1\mu_1\nu_1...\mu_k\nu_k...\mu_1\nu_1...} = T_{\lambda_1\mu_1\nu_1...\mu_k\nu_k...\mu_1\nu_1...} \tag{28}
\]

In our case tensor \( T \) appears from the correlation function (30) (see below) —the fermion functions correlator from \( \text{24} \). The property \( \text{28} \) is guaranteed by bose-symmetry.
This leads us to the following expression for the action

\[
S = \int d^{p+1}X \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_k \, \Im \delta_{A_0 \ldots A_{p+1}}(X^\mu, X^i = 0) \times \\
\times \frac{i 2^{p+2} \pi \alpha'}{32(p' + 2)!} (w - z)(\bar{w} - \bar{z})(w - \bar{w}) \times \\
\times \left[ P_-, \Gamma^{A_0 \ldots A_{p' + 1}} \Gamma^{p' + 1} \ldots \Gamma^{9} \right]_{\gamma}^{\alpha} (C^{-1})_{}^{\gamma \beta} \left\langle S_\alpha(w) S_\beta(\bar{w}) \psi^\lambda(z) : \psi^{\mu_1} \psi^{\nu_1}(x_1) : \ldots : \psi^{\mu_k} \psi^{\nu_k}(x_k) : \right\rangle_{\text{gr. 1}} \\
\times (\alpha')^k \int_0^1 dt \, Tr \left[ A_\lambda(X^\mu) F^{t}{}_{\mu_1 \nu_1}(X^\mu) \ldots F^{t}{}_{\mu_k \nu_k}(X^\mu) \right],
\]

The integral over D-brane world-volume appears from the integral over zero modes of string coordinates in the functional integral for \( (26) \). Here

\[
\left\langle S_\alpha(w) S_\beta(\bar{w}) \psi^\lambda(z) : \psi^{\mu_1} \psi^{\nu_1}(x_1) : \ldots : \psi^{\mu_k} \psi^{\nu_k}(x_k) : \right\rangle_{\text{gr. 1}} = \frac{(C^\Lambda \lambda)_{\alpha_0}^{\alpha} \beta_0}{\sqrt{2} (w - z)(\bar{w} - \bar{z})(w - \bar{w})} \\
\times \frac{1}{2} \left[ (\Sigma^{\mu_1 \nu_1})_{\alpha_0}^{\alpha} \delta_1^{\beta_1} \right] (x_1 - w) + \frac{\delta_0^{\alpha_0} (\Sigma^{\mu_1 \nu_1})_{\beta_1}^{\beta}}{(x_1 - w)} \times \ldots \times \frac{1}{2} \left[ (\Sigma^{\mu_k \nu_k})_{\alpha_k}^{\alpha} \delta_{k-1}^{\beta_{k-1}} \right] (x_k - w) + \frac{\delta_0^{\alpha_k} (\Sigma^{\mu_k \nu_k})_{\beta_{k-1}}^{\beta}}{(x_k - w)}
\]

In a full analogy with the example from section 2 the equal indexes in the set \( \mu_1, \nu_1, \ldots, \mu_k, \nu_k, \lambda \) correspond to singular expressions, when some of \( x_i, z \) coincide. Assuming that all \( \mu_1, \nu_1, \ldots, \mu_k, \nu_k, \lambda \) are distinct we rewrite \((30)\) as

\[
\frac{1}{\sqrt{22^k}} (\Gamma^\lambda \Sigma^{\mu_1 \nu_1} \ldots \Sigma^{\mu_k \nu_k} C)_{\alpha_0}^{\alpha} \beta_0 (w - z)(\bar{w} - \bar{z})(w - \bar{w})^{k} (x_1 - w)(x_k - w)(x_1 - \bar{w})(x_k - \bar{w})
\]

Combining it with \((29)\) and performing the integration over \( x_i \) one gets

\[
S = \int d^{p+1}X \sum_{k=0}^{\infty} \Im \delta_{A_0 \ldots A_{p' + 1}}(X^\mu, X^i = 0) \times \\
\times \frac{2}{32(p' + 2)!} \left[ P_-, \Gamma^{A_0 \ldots A_{p' + 1}} \Gamma^{p' + 1} \ldots \Gamma^{9} \right]_{\gamma}^{\alpha} (C^{-1})_{}^{\gamma \beta} \left[ \Gamma^\lambda \Sigma^{\mu_1 \nu_1} \ldots \Sigma^{\mu_k \nu_k} C \right]_{\alpha}^{\alpha} \times \\
\times \frac{(i \pi \alpha')^{k+1}}{k!} Tr \int_0^1 dt \, A_\lambda(X^\mu) F^{t}{}_{\mu_1 \nu_1}(X^\mu) \ldots F^{t}{}_{\mu_k \nu_k}(X^\mu)
\]

The \( \Gamma^{11} \) from \( P_- \) in \((32)\) has a simple interpretation, due to identity

\[
\frac{1}{(p' + 2)!} \Gamma^{11} \Gamma^{A_0 \ldots A_{p' + 1}} \Im \delta_{A_0 \ldots A_{p' + 1}} = \frac{1}{(9 - p')!} \Gamma^{B_{p' + 1} \ldots B_9} \Im \delta_{B_{p' + 2} \ldots B_9},
\]

\[\text{This means that besides the action with the RR fields } C \text{ one gets the action with dual RR fields } \widetilde{C} (\widetilde{C} = \dbar C). \text{ In order to make calculus simple we remove } \Gamma^{11} \text{ from } (32) \text{ and remember that the action for dual RR fields should be added at the end.}
\]

Using the explicit expression for the trace over gamma matrices one finds that

\[\text{This is done in appendix A.}\]
\[ S = \int_0^1 dt \int d^{p+1}X \sum_{k=0}^{\infty} \frac{2(i\pi\alpha')^{k+1}}{k!} (-1)^{p'} \delta_{p'+2+2k,p} \times \]
\[ \times \Im \alpha_{\mu_0...\mu_{p'+1}}(X^\mu, X^1 = 0) \text{Tr} \left[ A_\lambda(X^\mu)F^t_{\mu_1\nu_1}(X^\mu)...F^t_{\mu_k\nu_k}(X^\mu) \right] e^{\alpha_0...\alpha_{p'+1}\lambda_\mu_1\nu_1...\mu_k\nu_k} \]
\[ = (-1)^{p'} (2\pi\alpha') \int_0^1 dt \int \text{Tr} \left[ \Im \wedge A \wedge e^{2\pi\alpha' F^t} \right]_{\text{top}} \]  \hspace{1cm} (34)

Here \( e^{\alpha_0...\nu_k} \) is a \((p+1)\)-dimensional absolutely antisymmetric tensor.

Our result (34) has a remarkable structure: it is the product of external derivative \( \Im = dC \) and the Chern-Simons term \( \text{CS} \).

\[ \text{CS}_n = \int_0^1 dt \text{Tr} \left[ A \wedge F^t \wedge ... \wedge F^t \right] \] \hspace{1cm} (35)

The Chern-Simons term is changed by full derivative under the gauge transformations \( A_\mu \rightarrow A_\mu + D_\mu a \) and this proves the gauge invariance of the action \( S \). Its derivative is proportional to a gauge invariant Chern character \( \text{CS} \).

\[ d\text{CS}_n = \frac{1}{n+1} \text{Tr} \left[ F \wedge ... \wedge F \right] \] \hspace{1cm} (36)

After integration by parts the final answer is \( \Im \) \hspace{1cm} (\lambda = 2\pi\alpha')

\[ S = \int \text{Tr} \left( C \wedge e^{\lambda F} \right)_{\text{top}} \] \hspace{1cm} (37)

4 RR couplings to both gauge and matter fields

4.1 RR vertex with \( \left(-\frac{3}{2}, -\frac{1}{2}\right) \) ghost number

In the end of previous section we showed that the use of \( V_{RR}^{-1} \) leads to necessity to integrate by parts in the final expression \( \Im \). From the answer \( \Im \) we know that when nonabelian fields \( \Phi \) are present the RR field \( C \) becomes the function of \( \Phi \) rather than \( X^i \) and the integration by parts is much more complicated then. Thus in this section instead of \( V_{RR}^{-1} \) \( \Im \) we will use \( V_{RR}^{-2} \) \( \Im \). Vertex operator \( V_{RR}^{-2} \) could be constructed by the analogy with operator \( V_{RR}^{-1} \) as the product of two spin vertices \( \Im \) \hspace{0.5cm} (in the left and right sectors) with defined ghost numbers \( \Im \). Moreover \( V_{RR}^{-2} \) and \( V_{RR}^{-1} \) are connected by the picture-changing procedure. For further consideration it is convenient to represent \( V_{RR}^{-1} \) as

\[ V_{RR}^{(-\frac{1}{2}, -\frac{1}{2})} = (P^{-1})^{\alpha\beta} V_{\alpha}^{\frac{1}{2}}(w) V_{\beta}^{\frac{1}{2}}(\bar{w}), \]
\[ V_{RR}^{(-\frac{1}{2}, -\frac{1}{2})} = \hat{C}^{\alpha\beta} V_{\alpha}^{\frac{1}{2}}(w) V_{\beta}^{\frac{1}{2}}(\bar{w}), \]
\[ V_{\alpha}^{\frac{1}{2}}(\bar{w}) = c(w) V_{\alpha}^{\frac{1}{2}}(\bar{w}) = c(\bar{w}) e^{-\Phi(\bar{w})} S_{\alpha}(\bar{w}) e^{iP_{\alpha} \hat{X}(\bar{w})}, \]
\[ \bar{w} = w, \bar{w}, \hat{X}(w) = X(w), \hat{X}(\bar{w}) = \bar{X}(\bar{w}), \] \hspace{1cm} (38)

The BRST operator \( \Im \) \hspace{0.5cm} (39)

\[ Q_{\text{BRST}} = Q_0 + Q_1 + Q_2, \]

\hspace{1cm} (39)

\hspace{1cm} (39)

\[ \]
where $Q_0$ is built from the generators of the Virasoro algebra, $Q_1$—from the supersymmetry generator and $Q_2$ is needed for $Q_{\text{BRST}}^2 = 0$. Since both $\tilde{V}_\alpha^{\frac{1}{2}}$ and $\bar{V}_\alpha^{\frac{1}{2}}$ have conformal dimension 1, both commutators $[Q_0, V_\alpha^{\frac{1}{2}}], [Q_1, V_\alpha^{\frac{1}{2}}]$ = 0. It is also straightforward to see that commutators $[Q_2, V_\alpha^{\frac{1}{2}}]$, for $S = 1, 3$ are equal to zero as well. But

$$
\left[ Q_1, V_\alpha^{-\frac{1}{2}} (w) \right] = \frac{1}{\sqrt{2}} i P_A (\Gamma^A)_{\alpha}^{\beta} V_\beta^{1-\frac{3}{2}} \eta(w) \neq 0
$$

(40)

On-shell condition $d \Im = d \ast \Im = 0$ leads to $P_A (\Gamma^A)_{\gamma}^{\alpha} \hat{S} \gamma \beta = P_A (\Gamma^A)_{\beta}^{\alpha} \hat{S} \alpha \gamma = 0$. However for the $\hat{C} \alpha \beta$ this is not true, because $dC = \Im \neq 0$ ($\ast C = 0$ is the gauge-fixing condition — the analog of $\partial_\mu A_\mu = 0$). The consequence is that when $V_{RR}^{(-\frac{1}{2}, -\frac{1}{2})}$ is BRST closed the vertex $V_{RR}^{(-\frac{1}{2}, -\frac{1}{2})}$ is not closed. It could be demonstrated by straightforward calculations, that

$$
\left[ Q_{\text{BRST}}, \xi V_\alpha^{-\frac{1}{2}} (w) \right] = \frac{1}{\sqrt{2}} i P_A (\Gamma^A)_{\alpha}^{\beta} V_\beta^{1-\frac{3}{2}} (w)
$$

(41)

when

$$
\left[ Q_{\text{BRST}}, \xi V_{A, \gamma}^{-1} (z) \right] = V_A^0
$$

(42)

where $\xi$ is the superconformal ghost. Note that there is no $P_-$ in the definition for $V^{-2}$ and therefore after $\xi$ manipulation [14] in the left sector we obtain $V^{-1}$ with removed $P_-$. This means that in the end of calculations with $V^{-2}$ one should simply add the action for the dual RR fields. But this is not the end of the story. Since $V^{-2}$ is not BRST closed correlation function with $V^{-2}$ and with $V^{-1}$ are not equal! However they are proportional to each other. More carefully

$$
\frac{1}{2 \sqrt{2}} \int d \xi_0 \left\langle V_{RR}^{(-\frac{1}{2}, -\frac{1}{2})} (w, \bar{w}) V_{A, \phi}^{-1} (z) \left( \prod V_{A, \phi}^0 \right) \right\rangle =
$$

$$
\frac{1}{2 \sqrt{2}} \hat{S}^{\alpha \beta} \int d \xi_0 \left\langle \bar{V}_\alpha^{\frac{1}{2}} V_\beta^{-\frac{1}{2}} (w, \bar{w}) \xi (z) V_{A, \phi}^{-1} (z) \left( \prod V_{A, \phi}^0 \right) \right\rangle =
$$

$$
\hat{C}^{\alpha \beta} \int d \xi_0 \left\langle [Q_{\text{BRST}}, \xi V_\alpha^{-\frac{1}{2}} (w)] V_\beta^{\frac{1}{2}} (\bar{w}) \xi (z) V_{A, \phi}^{-1} (z) \left( \prod V_{A, \phi}^0 \right) \right\rangle =
$$

$$
= \hat{C}^{\alpha \beta} \left\langle \bar{V}_\alpha^{-\frac{1}{2}} (w) V_\beta^{\frac{1}{2}} (\bar{w}) V_\alpha^{-1} (z) \left( \prod V_{A, \phi}^0 \right) \right\rangle +
$$

$$
\frac{1}{2 \sqrt{2}} \hat{C}^{\alpha \beta} i P_A (\Gamma^A)_{\gamma}^{\alpha} \int d \xi_0 \left\langle \xi V_\alpha^{-\frac{1}{2}} (w) \left( \prod V_{A, \phi}^0 \right) \right\rangle
$$

(43)

Noting that

$$
\int d \xi_0 \left\langle \xi (w) e^{-\frac{1}{2} \phi(w)} \left( \prod V_{A, \phi}^0 \right) \right\rangle = \left\langle e^{-\frac{1}{2} \phi(w)} \left( \prod V_{A, \phi}^0 \right) \right\rangle
$$

(44)

we conclude that

$$
\frac{1}{\sqrt{2}} \left\langle V_{RR}^{(-\frac{1}{2}, -\frac{1}{2})} (w, \bar{w}) V_{A, \phi}^{-1} (z) \left( \prod V_{A, \phi}^0 \right) \right\rangle = \left\langle V_{RR}^{(-\frac{1}{2}, -\frac{1}{2})} (w, \bar{w}) V_\alpha^{0} (z) \left( \prod V_{A, \phi}^0 \right) \right\rangle
$$

(46)

when $P_-$ removed from $V^{-1}$. Therefore the action in question is given by

$$
S = \sqrt{2} Tr \left\langle V_{RR}^{-2} P \exp \left( \oint_{\partial \Sigma} \frac{1}{2} V_{A, \phi}^{0} \right) \right\rangle_{CFT}
$$

(47)
4.2 The origin of the symmetric trace

Let us explain now the appearance of the symmetrized trace \( \text{Str} \) in the action we are looking for. For a moment let us consider a generic correlator

\[
\langle V(x_1) \ldots V(x_n) D(w) \rangle.
\]

Using the general result (51) we present the action (47) without the term (23) in the form

\[
\int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n \sum_{l_a + \ldots + l_c = n} \frac{n!}{l_a! \ldots l_c!} \text{Str} \left[ A(x_1) \ldots A(x_{l_a}) \ldots C(x_{l_b}) \ldots C(x_{l_c}) \right] \times
\]

\[
\times \left\langle a(x_1) \ldots a(x_{l_a}) \ldots c(x_{l_b}) \ldots c(x_{l_c}) D(w) \right\rangle.
\]

Thus we see that \( \text{Str} \) appears from the string-scattering amplitudes in the natural way.

4.3 Calculus

Using the general result (51) we present the action (47) without the term (23) in the form

\[
S = \sqrt{2} \sum_{k=1}^{\infty} \sum_{l,m,n,r} \int d^{p+1} X \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_k \times
\]

\[
\times \frac{\pi \sqrt{2}}{32(p'+1)!} 2^{r}(a')^{m+n+r} (x_1-w)(x_1-w) \delta(x_1-z) \frac{1}{(w-w)^{2}} \times
\]

\[
\times \frac{1}{k! l! m! n! r!} \langle CA_{\alpha_0} \ldots A_{\alpha_r}(X^{\mu}, X^{\nu}) \partial_{\alpha_0} X^{\nu}(x_1) \ldots \partial_{\alpha_r} X^{\nu}(x_1) \rangle \left[ \Gamma^{\gamma_1} \ldots \Gamma^{\gamma_r} \Gamma^{\mu+1} \ldots \Gamma^{\nu} \right]_{\gamma} (C^{-1})_{\gamma} \times
\]

\[
\times \left\langle S_{\alpha}(w) S_{\beta}(\bar{w}) : \psi^{\mu_1} \psi^\nu_1(x_1) : \ldots : \psi^{\mu_n} \psi^\nu_n(x_1) : \right\rangle_{gr,1} \times
\]

\[
\times \text{Str} \left\{ \Phi_{1}^{1} (X^{\mu}) \ldots \Phi_{1}^{i} (X^{\mu}) F_{\mu_1 \nu_1} (X^{\mu}) \ldots F_{\mu_n \nu_n} (X^{\mu}) \right. \times
\]

\[
\left. \cdot [\Phi_{i_1}, \Phi_{j_1}] (X^{\mu}) \ldots [\Phi_{i_m}, \Phi_{j_m}] (X^{\mu}) D_{\lambda_1} \Phi_{s_1} (X^{\mu}) \ldots D_{\lambda_r} \Phi_{s_r} (X^{\mu}) \right\}
\]

(52)
Fermion correlator was already discussed and here we present only the result

\[
\left\langle S_\alpha(w_1)S_\beta(w_2) : \psi^{\mu_1}\psi^{\nu_1}(x_1) : \ldots : \psi^{\mu_n}\psi^{\nu_n}(x_n) : \right\rangle_{gr.1} = \frac{1}{2g} \left[ \Sigma_{\mu_1\nu_1} \ldots \Sigma_{\mu_n\nu_n} \Sigma_{i_1j_1} \ldots \Sigma_{i_mj_m} \Sigma_{\lambda_1s_1} \ldots \Sigma_{\lambda_rs_r} C \right]_{\alpha\beta} \times \frac{(w-w)^q}{(x_1-w)(x_1-w)\ldots(x_q-w)(x_q-w)}, \quad q = m + n + r
\]

(53)

Correlator between \(X^i\) is also easy to calculate using Wick’s rule and Fourier image of (52)

\[
\left\langle C(X^i)(w)\partial_n X^{i_1}(x_1)\ldots\partial_n X^{i_l}(x_l) \right\rangle = \frac{1}{l!} \frac{\partial^l C}{\partial X^{i_1}\ldots\partial X^{i_l}}(X^i = 0) \left\langle X^{j_1} \ldots X^{j_l}(w) : \partial_n X^{i_1}(x_1)\ldots\partial_n X^{i_l}(x_l) \right\rangle = \frac{1}{l!} \frac{\partial^l C}{\partial X^{i_1}\ldots\partial X^{i_l}}(X^i = 0) (-ia)^l \left\{ g^{i_1j_1} \ldots g^{i_lj_l} (w-w)^l \right\}
\]

(54)

Substituting last two formulae into (52) and adding (23) one obtains the RR couplings to D-branes

\[
S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \int dp+1 X^{i_1} \frac{(i\pi\alpha')^{m+n+r+2r}}{32(p'+1)!} \times S_{p'} \Gamma^{A_0\ldots A_{p'}R_0\ldots R_1} \Sigma_{\mu_1\nu_1} \ldots \Sigma_{\mu_n\nu_n} \Sigma_{i_1j_1} \ldots \Sigma_{i_mj_m} \Sigma_{\lambda_1s_1} \ldots \Sigma_{\lambda_rs_r} \times \frac{1}{n!m!r!} \mathrm{STr} \left\{ C(X^\mu, X^i = |\lambda|\Phi^i) A_0 \ldots A_{p'} F_{\mu_1\nu_1}(X^\mu) \ldots F_{\mu_n\nu_n}(X^\mu) \right\} \times [\Phi_{i_1}, \Phi_{j_1}](X^\mu) \ldots [\Phi_{i_m}, \Phi_{j_m}](X^\mu) D_{\lambda_1} \Phi_{s_1}(X^\mu) \ldots D_{\lambda_r} \Phi_{s_r}(X^\mu) \right\} = \int dp+1 X^{i_1} \frac{C(X^\mu, \Phi^i)_{j_1i_1} \ldots j_m i_m k_1 \ldots k_r \sigma_1 \ldots \sigma_s \times \frac{\lambda^r}{r!} D_{\lambda_1} \Phi^{k_1} \ldots D_{\lambda_r} \Phi^{k_r} \frac{\lambda^n}{2^n r!} \Gamma_{\mu_1\nu_1} \ldots F_{\mu_n\nu_n} \times \delta_{2m,p+2m-p} \epsilon^{\lambda_1 \ldots \lambda_r \sigma_1 \ldots \sigma m_{\mu_1 \ldots \mu_n}} \}
\]

(55)

(56)

Using the following notations

\[
e^{\mu_1 \ldots \mu_n} P\left[ C_{\mu_1 \ldots \mu_n} = e^{\mu_1 \ldots \mu_n} \sum_{m=0}^{m=n} \left( \frac{\lambda^m}{2^m} \right) C_{n m} D_{\mu_1} \Phi^{i_1} \ldots D_{\mu_m} \Phi^{i_m} C_{i_1 \ldots i_m \mu_{m+1} \ldots \mu_n}, \quad m = n-k \right)
\]

(57)

and

\[
e^{I_{\Phi} I_{\Phi}} C = C_{A_0 \ldots A_{p'}} + \Phi^i \Phi^j C_{j i A_0 \ldots A_{p'-2}} + \frac{1}{2} \Phi^i \Phi^j \Phi^k \Phi^\ell C_{j i j'k' A_0 \ldots A_{p'-4}} + \ldots
\]

(58)

the final expression can be represented as:

\[
S = \int \mathrm{STr} \left( P \left[ e^{\lambda I_{\Phi} I_{\Phi}} C \left( X^\mu, X^i = |\lambda|\Phi^i \right) \right] \wedge e^{\lambda F} \right)_{top}
\]

(59)
5 Conclusions

In this paper we obtain low-energy effective action which describes the interaction between massless Ramond-Ramond and gauge/matter fields by calculating tree-level string scattering amplitudes. Many results were already derived in this way. Among them open-string low-energy action (so-called Born-Infeld) and its nonabelian generalizations [20, 20, 21, 22]. This approach was already used in order to check the results, derived in other ways (see introduction).

Unfortunately this approach usually is not useful since the calculations in all orders are very complicated.

Moreover after calculations it is necessary to carefully extract all singularities (poles in external momentum) and this leads to additional difficulties. In the general case this extractions could not be done before the explicit result for string amplitudes is obtained.

Our goal was to find the procedure, which extracts the poles before the explicit result is obtained. We do not know how to prove our proposal carefully and leave it as assumption, but considering only the Wick pairings between vertexes, which never merge leads to correct result.

This assumption allow us to avoid tedious calculations, which ultimately difficult to carry out in all orders. Really these calculations were successfully carried out only in leading orders [1] for RR couplings.

It will be very interesting to prove this assumption and also generalize it to the case of arbitrary fields. For instance the calculations with tachyon could not be done using the assumption in the described form because the off-shell amplitudes (and therefore $SL_2$ non-invariant expressions) are relevant for this interactions [1].

We also derive the vertex operator for RR filed in the ($-\frac{3}{2}, -\frac{1}{2}$) picture. It depends on RR fields themselves rather than their fields strength and this simplify the derivation of low-energy action. Note that this vertex is not BRST closed but it still leads to the right result inside the correlators.

Thus we demonstrate that Myers-Chern-Simons action is really the consequence of the superstring theory in the low-energy limit.

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7 Appendix A

In this appendix we are going to calculate the following integral

$$ I = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_k \frac{(w - \bar{w})^k}{(x_1 - w)(x_1 - \bar{w}) \cdots (x_k - w)(x_k - \bar{w})} $$

(60)

The most convenient way to do this is to make analytical change of variables

$$ x \rightarrow z = \frac{x - i}{x + i}, \quad x = i \frac{1 + z}{1 - z} $$

(61)

This is the transformation of the string world-sheet from upper-half plane to the unit disk. In fact, if $x \in R$ then $|z| = 1$ and therefore $z = e^{it}$. The constraint $+\infty > x_1 > \cdots > x_k > -\infty$ transforms to $2\pi > t_1 > \cdots > t_k > 0$. It is straightforward to check that
\[ dx = \frac{2idz}{(1 - z)^2}, \quad (x_1 - x_2) = \frac{2i(z_1 - z_2)}{(1 - z_1)(1 - z_2)} \quad (62) \]

so

\[ I = \oint |z_1| = 1 dz_1 \ldots \oint |z_k| = 1 dz_k \frac{(z_w - z_{\bar{w}})^k}{(z_1 - z_{\bar{w}})(z_2 - z_{\bar{w}}) \ldots (z_k - z_{\bar{w}})(z_k - z_{\bar{w}})}, \quad \forall i = 1, \ldots, k \quad |z_i| = 1 \quad (63) \]

Since \(|z_w| < 1\) and \(|z_{\bar{w}}| > 1\) it is easy to evaluate (63) with the help of Cauchy theorem:

\[ I = (2\pi i)^k \quad (64) \]

We see that \(I\) does not depend on \(w\). This is the consequence of the \(SL_2\) invariance of (60) under

\[ x \to x' = \frac{ax + b}{cx + d}, \quad (x_1 - x_2) \to (x'_1 - x'_2) = \frac{(x_1 - x_2)}{(cx_1 + d)(cx_2 + d)} \quad (65) \]

Note that \(SL_2\) transforms the points \(+\infty\) and \(-\infty\) into the same point and (60) is explicitly invariant under linear change of variables \(x \to x' = x + const\). The latter transformation belongs to \(SL_2\) as well. All these considerations allow us to put \(w\) to any complex number in the upper half plane without change of the result.
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