Regularity of Solutions to Second-Order Integral Functionals in Variational Calculus

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ABSTRACT

We obtain regularity conditions of a new type of problems of the calculus of variations with second-order derivatives. As a corollary, we get non-occurrence of the Lavrentiev phenomenon. Our main result asserts that autonomous integral functionals of the calculus of variations with a Lagrangian having superlinearity partial derivatives with respect to the higher-order derivatives admit only minimizers with essentially bounded derivatives.

Keywords: optimal control, calculus of variations, higher order derivatives, regularity of solutions, non-occurrence of the Lavrentiev phenomenon.

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1 Introduction

Let \( L(t, x^0, \ldots, x^m) \) be a given \( C^1([a, b] \times \mathbb{R}^{(m+1)\times n}) \) real valued function. The problem of the calculus of variations with high-order derivatives consists in minimizing an integral functional

\[
J^m[x(\cdot)] = \int_a^b L\left(t, x(t), \dot{x}(t), \ldots, x^{(m)}(t)\right) dt
\]

over a certain class \( X \) of functions \( x : [a, b] \to \mathbb{R}^n \) satisfying the boundary conditions

\[
x(a) = x_a^0, x(b) = x_b^0, \ldots, x^{(m-1)}(a) = x_a^{m-1}, x^{(m-1)}(b) = x_b^{m-1}.
\]

Often it is convenient to write \( x^{(1)} = x', x^{(2)} = x'', \) and sometimes we revert to the standard notation used in mechanics: \( x' = \dot{x}, x'' = \ddot{x} \). Such problems arise, for instance, in connection with the theory of beams and rods [18]. Further, many problems in the calculus of variations with higher-order derivatives describe important optimal control problems with linear dynamics [17].
Regularity theory for optimal control problems is a fertile field of research and a source of many challenging mathematical issues and interesting applications [5, 22, 21]. The essential points in the theory are: (i) existence of minimizers and (ii) necessary optimality conditions to identify those minimizers. The first systematic approach to existence theory was introduced by Tonelli in 1915 [19], who showed that existence of minimizers is guaranteed in the Sobolev space \( W^m_m \) of absolutely continuous functions. The direct method of Tonelli proceeds in three steps: (i) regularity and convexity with respect to the highest-derivative of the Lagrangian \( L \) guarantees lower semi-continuity, (ii) the coercivity condition (the Lagrangian \( L \) must grow faster than a linear function) insure compactness, (iii) by the compactness principle, one gets the existence of minimizers for the problem \( (P_m) \). Typically, Tonelli’s existence theorem for \( (P_m) \) is formulated as follows [5, 9]: under hypotheses (H1)-(H3) on the Lagrangian \( L \),

(H1) \( L(t,x^0,\ldots,x^m) \) is locally Lipschitz in \((t,x^0,\ldots,x^m)\);

(H2) \( L(t,x^0,\ldots,x^m) \) is convex as a function of the last argument \( x^m \);

(H3) \( L(t,x^0,\ldots,x^m) \) is coercive in \( x^m \), i.e. \( \exists \Theta : [0, \infty) \to \mathbb{R} \) such that

\[
\lim_{r \to \infty} \frac{\Theta(r)}{r} = +\infty ,
\]

\[
L(t,x^0,\ldots,x^m) \geq \Theta(|x^m|) \text{ for all } (t,x^0,\ldots,x^m) ,
\]

there exists a minimizer to problem \( (P_m) \) in the class \( W^m_m \).

The main necessary condition in optimal control is the famous Pontryagin maximum principle, which includes all the classical necessary optimality conditions of the calculus of variations [14]. It turns out that the hypotheses (H1)-(H3) do not assure the applicability of the necessary optimality conditions, being required more regularity on the class of admissible functions [11]. For \( (P_m) \), the Pontryagin maximum principle [14] is established assuming \( x \in W^\infty_m \subset W^m_m \).

In the case \( m = 1 \), extra information about the minimizers was proved, for the first time, by Tonelli himself [19]. Tonelli established that, under the hypotheses (H2) and (H3) of convexity and coercivity, the minimizers \( x \) have the property that \( \dot{x} \) is locally essentially bounded on an open subset \( \Omega \subset [a,b] \) of full measure. If

\[
\left| \frac{\partial L}{\partial x} \right| + \left| \frac{\partial L}{\partial \dot{x}} \right| \leq c|L| + r , \quad \tag{1.2}
\]

for some constants \( c \) and \( r, c > 0 \), then \( \Omega = [a,b] \) \( (\dot{x}(t) \text{ is essentially bounded in all points } t \text{ of } [a,b], \text{i.e. } x \in W^\infty_1) \), and the Pontryagin maximum principle, or the necessary condition of Euler-Lagrange, hold. Condition (1.2) is now known in the literature as the Tonelli-Morrey regularity condition [6, 8, 17]. Since L. Tonelli and C. B. Morrey, several Lipschitzian regularity conditions were obtained for the problem \( (P_m) \) with \( m = 1 \): S. Bernstein (for the scalar case \( n = 1 \)), F. H. Clarke and R. B. Vinter (for the vectorial case \( n > 1 \)) obtained [7] the condition

\[
\left| \left( \frac{\partial^2 L}{\partial \dot{x}^2} \right)^{-1} \left( \frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial \dot{x} \partial t} \right) - \frac{\partial^2 L}{\partial \dot{x} \partial x} \right| \leq c \left( |\dot{x}|^3 + 1 \right) , \quad \frac{\partial^2 L}{\partial \dot{x}^2} > 0 ;
\]

F. H. Clarke and R. B. Vinter [7] the regularity conditions

\[
\left| \frac{\partial L}{\partial t} \right| \leq c|L| + k(t) , \quad k(\cdot) \in L_1 \, , \quad \tag{1.3}
\]
and

\[ \left| \frac{\partial L}{\partial x} \right| \leq c \left| L \right| + k(t) \left| \frac{\partial L}{\partial \dot{x}} \right| + m(t), \quad k(\cdot), m(\cdot) \in L_1; \]

and A. V. Sarychev and D. F. M. Torres [16] the condition

\[
\left( \left| \frac{\partial L}{\partial t} \right| + \left| \frac{\partial L}{\partial x} \right| \right) \left| \dot{x} \right|^\mu \leq \gamma \mathcal{L}^\beta + \eta, \quad \gamma > 0, \beta < 2, \mu \geq \max \{\beta - 1, -1\}. \quad (1.4)
\]

Lipschitzian regularity theory for the problem of the calculus of variations with \( m = 1 \) is now a vast discipline (see e.g. [2, 3, 10, 13, 22] and references therein). Results for \( m > 1 \) are scarcer: we are aware of the results in [9, 16, 20]. In 1997 A.V. Sarychev [15] proved that the second-order problems of the calculus of variations may show new phenomena non-present in the first-order case: under the hypotheses (H1)-(H3) of Tonelli’s existence theory, autonomous problems \((P_m)\) with \( m = 2 \) may present the Lavrentiev phenomenon [12]. This is not a possibility for \( m = 1 \), as shown by the Lipschitzian regularity condition (1.3). Sarychev’s result was recently extended by A. Ferriero [11] for the case \( m > 2 \). It is also shown in [11] that, under some standard hypotheses, the problems of the calculus of variations \((P_m)\) with Lagrangians only depending on two consecutive derivatives \( x^{(\gamma)} \) and \( x^{(\gamma+1)} \), \( \gamma \geq 0 \), do not exhibit the Lavrentiev phenomenon for any boundary conditions (1.4) (for \( m = 1 \), this follows immediately from (1.3)). In the case in which the Lagrangian only depends on the higher-order derivative \( x^{(m)} \), it is possible to prove more [16, Corollary 2]: when \( \mathcal{L} = \mathcal{L} \left( x^{(m)} \right) \), all the minimizers predicted by the existence theory belong to the space \( W^m_m \subset W^\infty_m \) and satisfy the Pontryagin maximum principle (regularity). As to whether this is the case or not for Ferriero’s problem with Lagrangians only depending on consecutive derivatives \( x^{(\gamma)} \) and \( x^{(\gamma+1)} \), seems to be an open question.

The results of Sarychev [15] and Ferriero [11] on the Lavrentiev phenomenon show that the problems of the calculus of variations with higher-order derivatives are richer than the problems with \( m = 1 \), but also show, in our opinion, that the regularity theory for higher-order problems is underdeveloped. One can say that the Lipschitzian regularity conditions found in the literature for the higher-order problems of the calculus of variations are a generalization of the above mentioned conditions for \( m = 1 \): [9] generalizes (1.2) for \( m > 1 \), [16] generalizes (1.4) for problems of optimal control with control-affine dynamics, [20] generalizes (1.2) for optimal control problems with more general nonlinear dynamics.

To the best of our knowledge, there exist no regularity conditions for the higher-order problems of the calculus of variations of a different type from those also obtained (also valid) for the first-order problems. We give here what we claim to be a new regularity condition which is of a different nature than those appearing for the first-order problems. For the sake of simplicity, we restrict ourselves to second-order problems \( (m = 2) \). The results of the paper can be naturally extended to derivatives of higher order than two, but the proofs become rather technical. While existence follows by imposing coercivity to the Lagrangian \( \mathcal{L} \) (hypothesis (H3)), we prove (cf. Theorem 4.1) that for the autonomous second-order problems of the calculus of variations, regularity follows by imposing a superlinearity condition to the partial derivatives \( \partial L / \partial x_i \) of the Lagrangian. We observe that our condition is intrinsic to the higher-order problems: for autonomous problems of the calculus of variations with \( m = 1 \) (1.3) is trivially satisfied and no superlinearity on the partial derivatives of \( \mathcal{L} \) is needed, while such conditions are required in the higher-order case as a consequence of Sarychev’s results [15].
2 Outline of the paper and hypotheses

We shall limit ourselves to the second order problems of the calculus of variations, i.e. to the problem of minimizing

$$\int_a^b \mathcal{L}(t, x(t), \dot{x}(t), \ddot{x}(t)) \, dt \tag{P_2}$$

for some given Lagrangian $\mathcal{L}(\cdot, \cdot, \cdot, \cdot)$, assumed to be a $C^1$ function with respect to all arguments. In this case it is appropriate to choose the admissible functions $x$ to be twice continuously differentiable with derivatives $\dot{x}$ and $\ddot{x}$ in $L^2$, i.e. $\mathcal{X} = W^2_2$. In Section 3, we establish generalized integral forms of duBois-Reymond and Euler-Lagrange necessary conditions valid for $\mathcal{X} = W^2_2$ (the optimal solutions $x$ may have unbounded derivatives $\dot{x}$, $\ddot{x}$). Then, in Section 4, we obtain regularity conditions under which all the minimizers of $(P_2)$ are in $W^{2,\infty}_2 \subset W^2_2$ and thus satisfy the classical necessary conditions. In general terms, the techniques used here are extensions of those appearing in [4] for one-derivative problems.

In the sequel we shall assume the following hypotheses:

$(S_0)$ There exists a continuous function $S(t, s, v, w) \geq 0$, $t, s, v, w \in \mathbb{R}^{1+3n}$, and some $\delta > 0$, such that the function $t \to S(t, x(t), x'(t), x''(t))$ is $L^2$-integrable in $[a, b]$ and

$$\left| \frac{\partial \mathcal{L}}{\partial t}(t, x, x', x'') \right| \leq S(t, x, x', x''),$$

for all $t \in [a, b], |\tau - t| < \delta, x = x(t)$.

$(S_1)$ There exists a nonnegative continuous function $G(\cdot, \cdot, \cdot, \cdot)$, and some $\delta > 0$, such that the function $t \to G(t, x(t), x'(t), x''(t))$ is $L^2$-integrable on $[a, b]$, and

$$\left| \frac{\partial \mathcal{L}}{\partial x_i}(t, y, x', x'') \right| \leq G(t, x, x', x''),$$

$$\left| \frac{\partial \mathcal{L}}{\partial \dot{x}_i}(t, x, y, x'') \right| \leq G(t, x, x', x''),$$

$$\left| \frac{\partial \mathcal{L}}{\partial \ddot{x}_i}(t, x, x', y) \right| \leq G(t, x, x', x''),$$

for all $t \in [a, b], x, x', x'' \in \mathbb{R}^n, x = (x_1, \ldots, x_n) \in \mathbb{R}^n, y = (y_1, \ldots, y_n) \in \mathbb{R}^n, y_j = x_j^{(k)}(t)$ for $j \neq i, |y_i - x_i^{(k)}(t)| \leq \delta, i = 1, \ldots, n$ and $k = 0, 1, 2$, where $x_i^{(k)}(t)$ is the $i$th component of the $k^{th}$ derivative with the convention $x_i^{(0)}(t) = x_i(t)$.

Remark 2.1. Hypothesis $(S_0)$ is certainly verified if $\mathcal{L}(t, x, \dot{x}, \ddot{x})$ does not depend on $t$: $(S_0)$ holds trivially in the autonomous case. Conditions $(S_i), i = 0, \ldots, n$, are needed in the proof of Theorems 3.1 and 3.2, to justify the usual rule of differentiation under the sign of an integral.

3 Generalized integral form of duBois-Reymond and Euler-Lagrange equations

In this section we prove integral forms of duBois Reymond and Euler-Lagrange equations (see (3.1) and (3.5) below, respectively). For this, we consider an arbitrary change of the independent variable $t$. Let $s$ be the arc length parameter on the curve $C_0 : x = x(t), a \leq t \leq b$, so that the Jordan length of $C_0$ is $s(t) = \int_a^t \sqrt{1 + |x'(\tau)|^2} \, d\tau$ with $s(a) = 0, s(b) = l$ and $s(t)$ is absolutely continuous with $s'(t) \geq 1$. 

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Thus $s(t)$ and its inverse $t(s)$, $0 \leq s \leq l$, are absolutely continuous with $t'(s) > 0$ a.e. in $[0,l]$. If $X(s) = x(t(s))$, $0 \leq s \leq l$, then $t(s)$ and $X(s)$ are Lipschitzian of constant one in $[0,l]$. By change of variable,

$$I[x] = \int_0^b \mathcal{L}(t, x(t), \dot{x}(t), \ddot{x}(t)) dt$$

$$= \int_0^l \mathcal{L} \left( t(s), X(s), \frac{X'(s)}{t'(s)}, \frac{1}{t'(s)^2} \left( X''(s) - \frac{X'(s)}{t'(s)} t'''(s) \right) \right) t'(s) ds .$$

Setting $F(t, x, t', x', t'', x'') = \mathcal{L}(t, x, \frac{x'}{t'}, \frac{1}{t^2} (x'' - \frac{x'}{t} t''')) t'$, we have:

$$I[x] = J[C] = J[X] = \int_0^l F \left( t(s), X(s), t'(s), X'(s), t''(s), X''(s) \right) ds .$$

### 3.1 Generalized duBois-Reymond equations

The following necessary condition will be useful to prove our regularity theorem (Theorem 4.1 on Section 4).

**Theorem 3.1.** Under hypotheses $(S_i)_{0 \leq i \leq n}$, if $x(\cdot) \in W^2_b$ is a minimizer of problem (P2), then the following integral form of duBois-Reymond necessary condition holds:

$$\phi_0(s) = \frac{\partial F}{\partial t'}(\theta(s)) - \int_0^s \frac{\partial F}{\partial t'}(\theta(\sigma)) d\sigma + \int_0^s \int_0^\tau \frac{\partial F}{\partial \tau'}(\theta(\sigma)) d\sigma d\tau = c_0, \quad 0 \leq \tau \leq s \leq l,$$  

(3.1)

where functions $\frac{\partial F}{\partial t'}, \frac{\partial F}{\partial \tau'}, \frac{\partial F}{\partial t}$ are evaluated at $\theta(s) = (t(s), X(s), t'(s), X'(s), t''(s), X''(s))$ and $c_0$ is a constant.

**Proof.** It is to be noted that $(t(s), X(s), t'(s), X'(s), t''(s))$ may not exist in a set of null-measure of all $s$. The proof is done by contradiction. Suppose that (3.1) is not true. Then, there exist constants $d_1 < d_2$ and disjoints sets $E_1^*$ and $E_2^*$ of non-zero measure such that

$$\phi_0(s) \leq d_1 \text{ for } s \in E_1^*,$$

$$\phi_0(s) \geq d_2 \text{ for } s \in E_2^*,$$

while $t'(s) > 0$ a.e in $[0,l]$. Hence there exist some constant $k > 0$ and two subsets $E_1, E_2$ of positive measure of $E_1^*, E_2^*$, such that

$$t'(s) \geq k, \quad \phi_0(s) \leq d_1 \text{ for } s \in E_1, \quad |E_1| > 0 ,$$

(3.2)

$$t'(s) \geq k, \quad \phi_0(s) \geq d_2 \text{ for } s \in E_2, \quad |E_2| > 0 .$$

(3.3)

Let us consider

$$\psi(s) = \int_0^s \int_0^\tau \left( |E_2| \chi_1(\sigma) - |E_1| \chi_2(\sigma) \right) d\sigma d\tau , \quad 0 \leq \tau \leq s \leq l ,$$

where $\chi_i$ denotes the indicator function defined by

$$\chi_i(s) = \begin{cases} 1 & \text{for } s \in E_i, \\ 0 & \text{for } s \in \{0,l\}/E_i, \quad i = 1,2 \text{ and } 0 \leq s \leq l. \end{cases}$$
We have that \( \psi' \) is an absolutely continuous function in \([0, l]\) with \( \psi'(0) = \psi'(l) = 0 \). Moreover,

\[
\psi''(s) = \begin{cases} 
-|E_1| & \text{a.e } s \in E_2, \\
|E_2| & \text{a.e } s \in E_1, \\
0 & \text{a.e } s \in [0, l] - E_1 \cup E_2.
\end{cases}
\]

We also define \( C_\alpha : t = t_\alpha(s), x = X_\alpha(s), 0 \leq s \leq l \), by setting

\[
t_\alpha(s) = t(s) + \alpha \psi(s) + \alpha^2 \psi'(s),
\]

\[
X_\alpha(s) = X(s), \quad 0 \leq s \leq l, \quad |\alpha| \leq 1.
\]

Let \( \rho > 0 \) be chosen in such a way that \( t, \tau \in [a, b] \) and \( |t - \tau| < \rho \) imply \( |x(t) - x(\tau)| \leq \delta \), where \( \delta \) is the constant in condition \((S_0)\). We have \( |\psi''(s)| < l \) putting \( N = \max |\psi'(s)| \) and choosing \( \alpha \leq \alpha_0 = \min \{1, \frac{k}{N^2 + 1}, \frac{\rho}{N^2 + 1}\} \). Then we have, for \( |\alpha| \leq \alpha_0 \), that

\[
t'_\alpha(s) = t'(s) + \alpha \psi'(s) + \alpha^2 \psi''(s) \geq k - (N + l) \alpha_0 \geq k - \frac{k}{2} > 0,
\]

\( s \in E_1 \cup E_2 \), and \( C_\alpha \) has an absolutely continuous representation \( x = x_\alpha(t), a \leq t \leq b \). We also have \( |t_\alpha(s) - t(s)| < |\alpha|(N + l^2) < \rho \). Hence \( |x_\alpha(t) - x(t)| = |x(t_\alpha(s)) - x(t(s))| < \delta \) and we conclude that \( J[C_\alpha] \geq J[C] \). On the other hand, by setting \( \beta(\alpha, s) = F(t, X, t', X', t'', X'') \), we have by differentiation that

\[
\frac{\partial \beta}{\partial \alpha} \biggr|_{\alpha=0} = \frac{\partial F}{\partial t} \psi + \frac{\partial F}{\partial t'} \psi' + \frac{\partial F}{\partial t''} \psi'',
\]

where

\[
\frac{\partial F}{\partial t} = \frac{\partial L}{\partial t'}, \quad \frac{\partial F}{\partial t'} = L - \frac{\partial L}{\partial x} \frac{\partial x}{\partial t'} + \frac{1}{t''} \frac{\partial L}{\partial \dot{x}} \left( -\frac{2 \ddot{x}}{t''} + \frac{3 \dot{x}}{t''} \right), \quad \frac{\partial F}{\partial t''} = \frac{1}{t''} \frac{\partial L}{\partial \dot{x}} \dot{x}.
\]

By hypotheses \((S_i)_{0 \leq i \leq n}\) both terms \( \frac{\partial F}{\partial \psi}, \frac{\partial F}{\partial \psi'}, \frac{\partial F}{\partial \psi''} \) are bounded in \( E_1 \cup E_2 \) by a function which is \( L \)-integrable in \([0, l]\). Then, we can differentiate under the sign of the integral to obtain:

\[
0 = \frac{\partial J(C_\alpha)}{d\alpha} \biggr|_{\alpha=0} = \int_0^l \left( \frac{\partial F}{\partial t} \psi + \frac{\partial F}{\partial t'} \psi' + \frac{\partial F}{\partial t''} \psi'' \right) ds.
\]

Integration by parts, and using \((3.2)\)–\((3.3)\), yields

\[
0 = \int_0^l \phi_0(s) \psi'' ds = \int_{E_1} \phi_0(s) \psi'' ds + \int_{E_2} \phi_0(s) \psi'' ds \leq |E_1||E_2|(d_1 - d_2) < 0
\]

which is a contradiction. Equality \((3.1)\) is now proved. \(\square\)
3.2 Generalized Euler-Lagrange equations

Arguments similar to those used to prove Theorem 3.1 can be utilized to prove a generalized Euler-Lagrange equation. This condition is not necessary in the proof of our regularity theorem, but is given here because of its significance: necessary conditions for \( P_2 \) in the class \( W_2 \) have an interest of their own (cf. Example 4.2).

**Theorem 3.2.** Under the hypotheses \( (S_i)_{1 \leq i \leq n} \), if \( x(\cdot) \in W_2 \) is a minimizer of problem \( P_2 \), then we have the following integral form of the Euler-Lagrange equations:

\[
\phi_i(s) = \frac{\partial F}{\partial x_i}(\theta(s)) - \int_0^s \frac{\partial F}{\partial x_i}(\theta(\sigma))d\sigma + \int_0^s \frac{\partial F}{\partial x_i}(\theta(\sigma))d\sigma d\tau = c_i, \quad 1 \leq i \leq n, \tag{3.5}
\]

where functions \( \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_{ij}}, \frac{\partial F}{\partial x_{ii}} \) are evaluated at \( \theta(s) = (t(s), X(s), t'(s), X'(s), t''(s), X''(s)) \) and \( c_i, i \in \{1, \ldots, n\} \), denote constants.

**Proof.** The proof is also by contradiction and is similar to that of Theorem 3.1. Suppose that (3.5) is not satisfied. For \( i = 1 \ldots n \) and \( |\alpha| \leq 1 \), we consider the curve \( C_\alpha : t = t_\alpha(s), x = X_\alpha(s), 0 \leq s \leq l, \) with

\[
X_{i\alpha}(s) = X_i(s) + \alpha \psi(s) + \alpha^2 \psi'(s),
\]

\[
X_{\alpha j}(s) = X_j(s), \quad j \neq i.
\]

We have \( |\psi''(s)| \leq l \) a.e and, if we put \( N = \max |\psi'(s)| \), then for

\[
|\alpha| \leq \alpha_0 = \min \left\{ 1, \frac{\delta}{(N + 1)l}, \frac{\delta}{N + l}, \frac{\delta}{l} \right\}
\]

we can write that

\[
|X_{i\alpha}(s) - X_i(s)| = |\alpha \psi + \alpha^2 \psi'| \leq \alpha_0(N + 1)l \leq \delta,
\]

\[
|X_{i\alpha}(s) - \dot{X_i}(s)| = |\alpha \psi' + \alpha^2 \psi''| \leq \alpha_0(N + l) \leq \delta,
\]

\[
|\dot{X}_{i\alpha}(s) - \dot{X}_i(s)| = |\alpha \psi''| \leq \alpha_0 l \leq \delta,
\]

and thus \( J[C] \leq J[C_\alpha] \) for all \( |\alpha| \leq \alpha_0 \). Setting, as before,

\[
\beta(\alpha, s) = F(t(s), X(s), t'(s), X'(s), t''(s), X''(s))
\]

we have

\[
\frac{\partial \beta}{\partial \alpha} \bigg|_{\alpha=0} = \frac{\partial F}{\partial x_i} \psi + \frac{\partial F}{\partial x_{ij}} \psi' + \frac{\partial F}{\partial x_{ii}} \psi'', \text{ for } s \in [0, l] \text{ a.e.}
\]

Note that by the hypotheses \( (S_i)_{1 \leq i \leq n} \)

\[
\left| \frac{\partial F}{\partial x_i} \right| = \left| \frac{\partial L}{\partial x_i} \right| \leq G \left( t(s), X, \frac{X'}{t'}, \frac{1}{t'^2} (X'' - \frac{X'}{t'}) \right) t',
\]

\[
\left| \frac{\partial F}{\partial x_{i1}} \right| \leq G \left( t(s), X, \frac{X'}{t'}, \frac{1}{t'^2} (X'' - \frac{X'}{t'}) \right) + G \left( t, X, \frac{X'}{t'}, \frac{1}{t'^2} (X'' - \frac{X'}{t'}) \right) \frac{t''}{t'^2},
\]

\[
\left| \frac{\partial F}{\partial x_{i2}} \right| \leq G \left( t(s), X, \frac{X'}{t'}, \frac{1}{t'^2} (X'' - \frac{X'}{t'}) \right) \frac{1}{t'}
\]

are \( L \)-integrable in \([0, l]\). Thus, the terms \( \frac{\partial F}{\partial x_i} \psi, \frac{\partial F}{\partial x_{ij}} \psi', \frac{\partial F}{\partial x_{ii}} \psi'' \) are bounded in \( E_1 \cup E_2 \) by a fixed \( L \)-integrable function. For \( s \in (E_1 \cup E_2)^c \), we have \( \psi''(s) = 0 \) and \( \frac{\partial F}{\partial x_i} \psi = 0 \). The proof is continued in the same lines as in the end of the proof of Theorem 3.1 applying the usual rule of differentiation under the integral sign and integration by parts, which leads to a contradiction.
4 Regularity result for autonomous problems

We shall present now a regularity result for $\{P_2\}$ under certain additional requirements on the Lagrangian $L$.

**Theorem 4.1.** In addition to the hypotheses $(S_i)_{0 \leq i \leq n}$, let us consider the autonomous problem $\{P_2\}$, i.e. let us assume that $L$ does not depend on $t$: $L = L(x, \dot{x}, \ddot{x})$. If $\frac{\partial L}{\partial \dot{x}}$ is superlinear, i.e. there exist constants $a > 0$ and $b > 0$ such that

$$a|w| + b \leq \left| \frac{\partial L}{\partial \dot{x}}(s, v, w) \right| \quad \text{for all } (s, v, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \quad (4.1)$$

then every minimizer $x \in W^2_2$ of the problem is on $W^\infty_2$.

**Remark 4.1.** Theorem 4.1 remain valid if instead of the superlinearity condition (4.1) we impose the stronger quadratically coercive condition: there exist constants $a > 0$ and $b > 0$ such that

$$a|w|^2 + b \leq \left| \frac{\partial L}{\partial \dot{x}}(s, v, w) \right| \quad \text{for all } (s, v, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n.$$

**Example 4.1.** A trivial example of a Lagrangian satisfying all the conditions $(S_i)_{0 \leq i \leq n}$ and (4.1) is $L(x, \dot{x}, \ddot{x}) = a|\dot{x}|^2 + b\ddot{x}$ with $a$ and $b$ strictly positive constants (one can choose $g(t, x, \ddot{x}, \dddot{x}) = 2a|\dot{x}| + b \in L^2$ in $(S_i)$). It follows from Theorem 4.1 that all minimizers of the problem

$$I[x(\cdot)] = \int_{t_0}^{t_1} \left[a\dddot{x}(t)^2 + b\dddot{x}(t)\right] dt \rightarrow \min$$

$$x(\cdot) \in W^2_2, \quad a, b > 0$$

$$x(t_0) = \alpha, \quad x(t_1) = \beta$$

are $W^\infty_2$ functions.

As an immediate corollary to our Theorem 4.1 we obtain conditions of non-occurrence of the Lavrentiev phenomenon for the autonomous second-order variational problems.

**Corollary 4.2.** Under the hypotheses of Theorem 4.1 the autonomous problems do not admit the Lavrentiev gap $W^2_2 - W^\infty_2$:

$$\inf_{x(\cdot) \in W^2_2} \int_a^b L(x(t), \dot{x}(t), \ddot{x}(t)) dt = \inf_{x(\cdot) \in W^\infty_2} \int_a^b L(x(t), \dot{x}(t), \ddot{x}(t)) dt.$$

**Example 4.2.** Let us consider the autonomous problem proposed in [5, 9] ($n = 1, m = 2$): $L(s, v, w) = |s^2 - v^5|^2|w|^{22} + \epsilon|w|^2$, $t \in [0, 1]$. The problem satisfies hypotheses (H1)-(H3) of Tonelli’s existence theorem. Function $\dddot{x}(t) = k t^{\frac{4}{3}}$ verifies the integral form of the Euler-Lagrange equations (3.5). However, $\dot{x}$ belongs to $W^2_2$ but not to $W^\infty_2$. The regularity condition (4.1) of Theorem 4.1 is not satisfied.

**Proof.** (of Theorem 4.1) Using (3.1) and (3.4) we get

$$\frac{1}{t^2} \frac{\partial L}{\partial \dot{x}} \dddot{x} + \int_0^s \left\{ L - \frac{1}{t'} \frac{\partial L}{\partial \dot{x}} \dddot{x} + \frac{1}{t^2} \frac{\partial L}{\partial \dot{x}} \left( -\frac{2\ddot{x}}{t'} + \frac{3t''}{t'} \dddot{x} \right) \right\} dt = c_0$$

and since we are in the autonomous case,

$$\frac{1}{t^2} \frac{\partial L}{\partial \dot{x}} \dddot{x} + \int_0^s \left\{ L - \frac{1}{t'} \frac{\partial L}{\partial \dot{x}} \dddot{x} + \frac{1}{t^2} \frac{\partial L}{\partial \dot{x}} \left( -\frac{2\ddot{x}}{t'} + \frac{3t''}{t'} \dddot{x} \right) \right\} = c_0.$$
Therefore,
\[
\frac{1}{t^2} \frac{\partial L}{\partial \dot{x}} \ddot{x} = c_0 - \int_0^s \left\{ \mathcal{L} - \frac{1}{t'} \frac{\partial L}{\partial \dot{x}} \ddot{x} + \frac{1}{t^2} \frac{\partial L}{\partial x} \left( \frac{-2\ddot{x}}{t'} + 3t'' \dddot{x} \right) \right\} dt
\]
\[
= c_0 - \int_0^s \mathcal{L} + \int_0^s \frac{1}{t'} \frac{\partial L}{\partial \dot{x}} \ddot{x} + 2 \int_0^s \frac{1}{t^3} \frac{\partial L}{\partial x} \dot{x} - \int_0^s \frac{3t''}{t^3} \frac{\partial L}{\partial \dot{x}} \dddot{x}.
\]

Applying the Holder’s inequality, we obtain
\[
\left| \frac{1}{t^2} \frac{\partial L}{\partial \dot{x}} \ddot{x} \right| \leq |c_0| + \|\mathcal{L}\|_1 + k_1 \left( \frac{\partial L}{\partial \dot{x}} \right)_2 \|\ddot{x}\|_2 + k_2 \left( \frac{\partial L}{\partial x} \right)_2 \|\dddot{x}\|_2 + \int_0^s \frac{1}{t'} \left( \frac{1}{t^2} \frac{\partial L}{\partial \dot{x}} \right) dt,
\]
where \(k_1, k_2\) are positive constants. Then, using the fact that \(\mathcal{L} \in C^1, \mathcal{L}, \frac{\partial L}{\partial x}, \frac{\partial L}{\partial \dot{x}} \in L^2\) and \(x \in W^2_2\) (in other terms, \(x, \dot{x}, \ddot{x} \in L^2\)), it follows that \(\frac{1}{t^2} \frac{\partial L}{\partial \dot{x}} \dddot{x}\) satisfies a condition of the form
\[
\left| \frac{1}{t^2} \frac{\partial L}{\partial \dot{x}} \ddot{x} \right| \leq k_3 + \int_0^s \left| \frac{3t''}{t'} \right| \left( \frac{1}{t^2} \frac{\partial L}{\partial \dot{x}} \right) dt,
\]
for a certain positive constant \(k_3\). Now, Gronwall’s Lemma leads to the following uniform bound:
\[
\left| \frac{1}{t^2} \frac{\partial L}{\partial \dot{x}} \ddot{x} \right| \leq k_4
\]
with a positive constant \(k_4\). Since \(t' \leq 1\), we deduce that \(\frac{\partial L}{\partial \dot{x}} \ddot{x}\) is uniformly bounded. Besides, since \(\frac{\partial L}{\partial \dot{x}}\) verifies (4.1), we have
\[
|\dddot{x}| (a|\dddot{x}| + b) \leq \left| \frac{\partial L}{\partial \dot{x}} \ddot{x} \right| \leq k_4 \quad (b > 0).
\]

Therefore we get for a positive constant \(k_5\)
\[
|\dddot{x}| \leq \frac{k_4}{a|\dddot{x}| + b} \leq k_5.
\]

Then \(\frac{\partial L}{\partial \dot{x}}\) is uniformly bounded. Since \(\frac{\partial L}{\partial \dot{x}}(s, v, w)\) goes to \(+\infty\) with \(|w|\) (by superlinearity), this implies a uniform bound on \(|\dddot{x}|\) which leads to the intended conclusion that \(\dddot{x}\) is essentially bounded.

\[\square\]

Theorems 3.1, 3.2 and 4.1 admit a generalization for problems of an order higher than two. This is under study and will be addressed in a forthcoming paper.

5 Conclusions

The search for appropriate conditions on the data of the problems of the calculus of variations with higher-order derivatives, under which we have regularity of solutions or under which more general necessary conditions hold, is an important area of study. In this paper we have obtained necessary optimality conditions of duBois-Reymond and Euler-Lagrange type, valid in the class of functions where the existence is proved. Minimizers in this class may have unbounded derivatives and fail to satisfy the classical necessary conditions of duBois-Reymond or Euler-Lagrange. We prove that if the derivatives of the Lagrangian function with respect to the highest derivatives verify a superlinear condition, then all the minimizers have essentially bounded derivatives. This imply non-occurrence of the Lavrentiev phenomenon and validity of classical necessary optimality conditions.
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