THREE LECTURES ON THE Riemann Zeta-Function

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INTRODUCTION

These lectures were delivered at the “International Conference on Subjects Related to the Clay Problems” held at Chonbuk National University, Chonju, Korea in July, 2002. My aim was to give mathematicians and graduate students unfamiliar with analytic number theory an introduction to the theory of the Riemann zeta–function focusing, in particular, on the distribution of its zeros. Professor Y. Yildirim of the University of Ankara, who also delivered a set of lectures at the conference, concentrated on the distribution of prime numbers.

A few general remarks about the lectures are in order before I summarize their contents. First, since I could only cover a small part of the subject in the time allotted, my choices about what to include and exclude were necessarily personal. Second, I have glossed over a number of technical details in order to keep the focus on the main ideas. Finally, there is almost nothing new in the lectures. The exception is the description of a new random matrix model due to C. Hughes, J. Keating, and the author at the end of the third lecture. I should also add that this manuscript is a very close record of the lectures I delivered and this, I think, accounts for the somewhat breezy style.

In the first lecture I presented the basic background material on the zeta–function, sketched a proof of the Prime Number Theorem, explained how the Riemann Hypothesis (RH) comes into the picture, and briefly summarized the evidence for it.

In the second lecture I wanted to explain how one studies the distribution of the zeros and chose mean–value estimates as a unifying theme. I described what mean–value estimates are, gave several examples, and explained in a general way their connection with the zeros. I then sketched the ideas behind two applications – the most primitive zero–density estimate (due to H. Bohr and E. Landau) and the proof of N. Levinson’s famous result that at least one–third of the zeros of the zeta–function lie on the critical line. Both results were cited in Lecture I as evidence for the Riemann Hypothesis. I had also intended to present the conditional

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result of J. B. Conrey, A. Ghosh, and the author that more than seventy percent of the zeros are simple, but there was not enough time. However, I have included that application here.

The third lecture began with the observation that the Riemann Hypothesis does not answer all our questions about the primes; one also needs detailed information about the vertical distribution of the zeros on the critical line. I then presented H. Montgomery’s pioneering work on the pair correlation of the zeros. In the remainder of the lecture I stated the GUE hypothesis and described the most recent work on modeling the zeta–function by characteristic polynomials of random matrices from the Circular Unitary Ensemble (CUE).

For those wishing to study the zeta–function in more depth, the most important books are by H. Davenport [D], H. M. Edwards [E], A. E. Ingham [I2], A. Ivic [Iv], and E. C. Titchmarsh [T1], [T2]. For a background in random matrix theory the reader should consult M. L. Mehta [M] and P. Deift [Df].

I take this opportunity to thank the organizers and the many other fine Korean mathematicians I got to meet for the first time at the conference. Thanks also to the mathematicians and students who so warmly hosted us visiting mathematicians and made the conference such an enjoyable and memorable one.
Lecture I

The Zeta–Function, Prime Numbers, and the Zeros

Although most mathematicians are aware that the prime numbers, the Riemann zeta–function, and the zeros of the zeta–function are intimately connected, very few know why. In this first lecture I will outline the basic properties of the zeta–function, sketch a proof of the prime number theorem, and show how the location of the zeros of the zeta–function directly influences the distribution of the primes. I will then explain why the Riemann Hypothesis (RH) is important and the evidence for it.

1 The Riemann zeta–function

The Riemann zeta–function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which can also be written

$$\zeta(s) = \prod_p (1 + p^{-s} + p^{-2s} + \cdots) = \prod_p (1 - p^{-s})^{-1},$$

where \(s = \sigma + it\) is a complex variable. We immediately see that the zeta–function is built out of the prime numbers. Observe that the series and product both converge absolutely in the half–plane \(\sigma > 1\). Their equality in this region may be regarded as an analytic equivalent of the Fundamental Theorem of Arithmetic. For the Fundamental Theorem assures us that each term \(n^{-s}\) in the series occurs once, and only once, among the terms resulting from multiplying out the Euler product. Conversely, if we know the equality of the sum and product, the Fundamental Theorem follows. From the equality of the sum and product we can also deduce the well known fact that there are an infinite number of primes. For if there were not, the product would remain bounded as \(\sigma \to 1^+\), whereas we know that the sum tends to infinity.

Since no factor in the Euler product equals zero when \(\sigma > 1\), we deduce that \(\zeta(s) \neq 0\) when \(\sigma > 1\). Also, since the series converge absolutely when \(\sigma > 1\), it converges uniformly in compact subsets there. It follows that \(\zeta(s)\) is analytic in the half–plane \(\sigma > 1\).

The most fundamental properties of the zeta–function are:

1. **Analytic continuation:** \(\zeta(s)\) has an analytic continuation to \(\mathbb{C}\) except for a simple pole at \(s = 1\).
2. **Functional equation:** The zeta–function satisfies the functional equation

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-(1-s)/2} \Gamma((1-s)/2) \zeta(1-s).$$

3. **Trivial zeros:** The only zeros of \(\zeta(s)\) in \(\sigma < 0\) are simple ones at \(s = -2, -4, -6, \ldots\).
4. **Nontrivial zeros:** \(\zeta(s)\) has infinitely many zeros \(\rho = \beta + i\gamma\) in the “critical strip” \(0 \leq \sigma \leq 1\). These lie symmetrically about the “critical line” \(\sigma = 1/2\), and about the real axis.
(5) **Density of zeros in the critical strip:** If \( N(T) \) denotes the number of zeros \( \rho = \beta + i\gamma \) in the critical strip with ordinates \( 0 < \gamma \leq T \), then
\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)
\]
as \( T \to \infty \).

Since the zeros of \( \zeta(s) \) are symmetric about the critical line, the simplest possible assumption is that they all lie on the line. This is the famous

**Riemann Hypothesis:** If \( \rho = \beta + i\gamma \) is a nontrivial zero of the zeta–function, then \( \beta = 1/2 \).

I will discuss the evidence for the truth of the Riemann Hypothesis later. First, however, I want to explain the most direct connection between the primes and zeros of the zeta–function.

## 2 The Prime Number Theorem

The Prime Number Theorem is the fundamental statistical fact about the primes.

**Prime Number Theorem.** Let \( \pi(x) = \sum_{p \leq x} 1 \). Then we have
\[
\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty.
\]

One interpretation of the theorem is that the probability that a positive integer chosen at random in the interval \([1, x]\) is a prime equals \( 1/\log x \). Another is that the average distance between consecutive primes in the interval \([1, x]\) is \( \log x \).

For technical reasons, it is more convenient to express the theorem in the following form, which can be shown to be equivalent by partial summation.

**Prime Number Theorem (second version).** Set
\[
\Lambda(x) = \begin{cases} 
\log p & \text{if } x = p^k, \\
0 & \text{if } x \neq p^k.
\end{cases}
\]

Then we have
\[
\psi(x) = \sum_{n \leq x} \Lambda(n) \sim x \quad \text{as} \quad x \to \infty.
\]
The proof I’ll sketch here is based on the “explicit formula”, which is called that because it explicitly shows the relationship between the zeros and primes.

We begin by assuming that $\Re s > 1$. From the Euler product representation for the zeta–function we see that

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s}) = \sum_{k=1}^{\infty} \frac{1}{kp^{ks}}.$$ 

Differentiating, we find that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{k=1}^{\infty} \frac{\log p}{p^k s} = \sum_n \frac{\Lambda(n)}{n^s}.$$ 

Here we have used a consequence of the fact that $\zeta(s)$ has an Euler product, namely, that its logarithm and, therefore, its logarithmic derivative also have Dirichlet series representations.

The idea now is to express the sum up to $x$ of the coefficients $\Lambda(n)$ of the last series (that is, $\psi(x)$) as an integral transform. This is analogous to writing the Fourier coefficients of a periodic function as an integral.

We break the argument into steps.

**StepI.** Note that

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{y^s}{s} ds = \begin{cases} 1 & \text{if } y > 1, \\ 1/2 & \text{if } y = 1, \\ 0 & \text{if } 0 < y < 1. \end{cases}$$

This is a standard exercise in complex function theory. If $y > 1$ we may pull the contour left to $-\infty$. In doing so we pass a simple pole of the integrand with residue 1. If $y < 1$ we pull the contour right to $+\infty$. This time we pass no poles, so the value of the integral is 0. When $y = 1$, we can calculate the Cauchy principal value of the integral directly, and it turns out to be 1/2.

**StepII.**

We use the formula above to evaluate

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left( -\frac{\zeta'}{\zeta}(s) \right) \frac{x^s}{s} ds = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left( \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \right) \frac{x^s}{s} ds$$

$$= \sum_{n=2}^{\infty} \Lambda(n) \left( \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{(x/n)^s}{s} ds \right)$$

$$= \psi(x) - \frac{1}{2} \Lambda(x).$$

The interchange of summation and integration is not quite justified here. We should really truncate the integral first and keep track of the error terms. But we will ignore this technical point so as not to obscure the main idea.

**StepIII.**

Evaluate the integral in Step II in a different way by pulling the contour left to $-\infty$. We pick up residues from the simple poles of $-\frac{\zeta'}{\zeta}(s) \frac{x^s}{s}$ at i) the trivial and nontrivial zeros of $\zeta(s)$, ii) the pole of $\zeta(s)$ at $s = 1$, and
iii) the pole at $s = 0$. Calculating and summing the residues, and then equating the result to $\psi(x) - \frac{1}{2}\Lambda(x)$, we find that

$$\psi(x) - \frac{1}{2}\Lambda(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'}{\zeta}(0) - \sum_{m=1}^{\infty} \frac{x^{-2m}}{-2m}.$$  

This is the “explicit formula”. Had we worked with a truncated integral over the interval $[2 - iT, 2 + iT]$, say, rather than the integral over $[2 - i\infty, 2 + i\infty]$ (as we should have done to overcome the convergence problem in Step II), the sum over $\rho$’s would also be truncated. The analysis is more complicated, but leads to a more useful form of the explicit formula, namely,

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho} + E(x,T),$$  

where $E(x,T)$ is a known error term. For the applications we have in mind, one can show that it is possible to choose $T$ as a function of $x$ in such a way that the error term is not significant. We therefore will not bother with the exact form of $E(x,T)$.

From the last form of the explicit formula one can almost see the Prime Number Theorem. Since $|x^{\rho}| = |x^{\beta + i\gamma}| = x^{\beta}$, the term involving the sum over zeros should be $o(x)$ as long as the $\beta$ are not too close to 1. Indeed, using the estimate $N(T) << T \log T$, we see that the sum is

$$<< \left( \max_{0<\gamma \leq T} x^{\beta} \right) \sum_{0<\gamma \leq T} |\rho|^{-1} << \left( \max_{0<\gamma \leq T} x^{\beta} \right) \log^2 T. $$  

Now, one can show that the inequality $\beta \leq 1 - \frac{c}{\log \gamma}$ holds, where $c$ is a positive constant, for every zero $\rho = \beta + i\gamma$. This leads to the Prime Number Theorem with an error term:

$$\psi(x) = x + O \left( x e^{-b \sqrt{\log x}} \right),$$  

with $b$ a positive constant. Clearly the farther left the zeros all lie from the line $\Re s = 1$, the better the error term. Since the zeros are symmetric about the line $\sigma = 1/2$, the farthest left they can be is on the critical line, and in this case one can show that the sum over zeros in the explicit formula is $O(x^{1/2+\epsilon})$. Thus the Riemann Hypothesis implies that

$$\psi(x) = x + O(x^{1/2+\epsilon}).$$  

In fact, this statement also implies the Riemann Hypothesis.

Why do we care about the error term? Because the main term just tells us the large scale behavior of the sequence of primes; all the detailed fluctuations in the counting function for the primes is hidden in the $O$-term. To illustrate this point, let us assume RH and consider the problem of how large the gaps between consecutive primes can be. From $\psi(x) = x + O(x^{1/2+\epsilon})$ we easily see that

$$\psi(x+h) - \psi(x) = h + O(x^{1/2+\epsilon}), \quad 1 \leq h \leq x.$$
Now suppose that there are no primes in \([x, x + h]\) (it can easily be shown that we may ignore the prime powers). Then \(0 = h + O(x^{1/2 + \epsilon})\), so we have \(h = O(x^{1/2 + \epsilon})\). Thus, on RH there is a positive constant \(C\) such that the interval \((x, x + Cx^{1/2 + \epsilon}]\) always contains a prime. Hence, the error term in the Prime Number Theorem has a bearing on the size of the maximal gap between primes. Had we not assumed RH, the same analysis would have only led to the assertion that every interval \((x, x + Cxe^{-b\sqrt{\log x}}]\) contains a prime. This of course is much weaker.

3 The evidence for the Riemann Hypothesis

I will conclude by indicating why we believe the Riemann Hypothesis. The main evidence supporting it is the following.

1. **Zero–free regions:** There is a region to the left of the line \(\Re s = 1\) that is free of zeros. More specifically, there is a positive constant \(c\) such that the region in the critical strip bounded by the curve \(\sigma = 1 - c/\log(|t| + 2)\) on the left, and \(\sigma = 1\) on the right, contains no zero of \(\zeta(s)\). (We used this fact when we deduced the Prime Number Theorem with error term.) The region has been widened slightly, but no one has been able to extend it to a vertical strip. The conjecture that there is such a strip is refered to as the Quasi–Riemann Hypothesis.

2. **Zero–density estimates:** Let \(N(\sigma, T)\) denote the number of zeros \(\rho = \beta + i\gamma\) of the zeta–function such that \(\sigma \leq \beta \leq 1\) and \(0 < \gamma \leq T\). Many estimates have been proved of the type \(N(\sigma, T) \leq T^{\lambda(\sigma)}\) with \(\lambda(\sigma) < 1\) and \(\lambda(\sigma)\) decreasing for \(1/2 < \sigma \leq 1\).

3. **Calculations of zeros:** The first fifty billion zeros of the zeta–function above the real axis have been shown to be simple and to lie on the critical line. Also, A. M. Odlyzko [O] has performed extensive computations showing, among many other things, that the nearest several hundred million zeros to the \(10^{20}\)th zero lie on the critical line. Zeros of many other \(L\)–functions have also been computed and all of these have been shown to lie on the (corresponding) critical line.

4. **Estimates of zeros on the critical line:** Let \(N_0(T)\) denote the number of zeros of \(\zeta(s)\) on the critical line whose ordinates \(\gamma\) satisfy \(0 < \gamma \leq T\). In 1914, Hardy [H] showed that \(N_0(T) \rightarrow \infty\) with \(T\). In 1921, Hardy and Littlewood [HL2] showed that \(N_0(T) > T\). Then, in 1942, A. Selberg [S] proved that \(N_0(T) \geq cN(T)\), for some positive constant \(c\). Thus, a positive proportion of the zeros lie on the critical line. The constant \(c\) was quite small, but in 1974 N. Levinson [L], using a different method, showed that \(N_0(T) > 1/3N(T)\). In 1989, B. Conrey [C], increased the proportion to more than \(2/5\).

5. **The finite field case:** It is possible to define analogues of the zeta–function for curves and varieties over finite fields. It has been shown that the analogous Riemann Hypotheses for these zeta–functions are true.
Mean-Value Theorems and the Zeros: Three Applications

1 An introduction to mean-value formulas

In this lecture I will explain what mean–value estimates are, give a sampling of some of the most important ones, and present three applications to the study of the zeros of the zeta–function. These should make it clear why they play such a central role in the theory.

Let’s begin with some general remarks on mean–value theorems.

By a mean–value theorem we mean an estimate for an integral of the type

$$\int_0^T |F(\sigma + it)|^2 \, dt$$

or

$$\int_0^T F(\sigma + it) \, dt$$

as $T \to \infty$, where $F(s)$ is a function representable by a convergent Dirichlet series in some half–plane $\Re s > \sigma_0$ of the complex plane. The path of integration here need not lie in this half–plane. For example, we would like to know the size of the integrals

$$I_k(\sigma, T) = \int_0^T |\zeta(\sigma + it)|^{2k} \, dt,$$

for $\sigma \geq 1/2$ and $k$ a positive integer. Here $F(s) = \zeta(s)^k$ and its Dirichlet series converges only for $\sigma > 1$.

There are many variations on this theme. For example, one might also consider a discrete version, namely an estimate for a sum of the form

$$\sum_{r=1}^R |F(\sigma_r + it_r)|^2,$$

where the points $\sigma_r + it_r$ lie in $\mathbb{C}$. Another possibility is for $F(s)$ to involve a parameter $N$, say. We then desire as uniform an estimate as possible in both $N$ and $T$. The simplest case is when

$$F(s) = F_N(s) = \sum_{n=1}^N a_n n^{-s}$$
is a Dirichlet polynomial. Here one can calculate the mean–value in a straightforward way. We have

\[
\int_0^T |F_N(\sigma + it)|^2 dt = \int_0^T \left| \sum_{n=1}^N a_n n^{-\sigma-it} \right|^2 dt
\]

\[
= \int_0^T \sum_{n=1}^N \sum_{m=1}^N a_n \overline{a}_m n^{-\sigma-it} m^{-\sigma+it} dt
\]

\[
= \sum_{n=1}^N \sum_{m=1}^N \frac{a_n \overline{a}_m}{(nm)^{\sigma}} \int_0^T (m/n)^{\sigma + it} dt
\]

\[
= T \sum_{n=1}^N \frac{|a_n|^2}{n^{2\sigma}} + \sum_{1 \leq m,n \leq N \atop m \neq n} \frac{a_n \overline{a}_m}{(nm)^{\sigma}} \left( \frac{(n/m)^{\sigma T} - 1}{i \log n/m} \right).
\]

It is not difficult to show that the second term on the last line is

\[
\ll N \log N \sum_{n=1}^N \frac{|a_n|^2}{n^{2\sigma}}.
\]

Hence, we find that

\[
\int_0^T \left| \sum_{n=1}^N a_n n^{-\sigma-it} \right|^2 dt = (T + O(N \log N)) \sum_{n=1}^N \frac{|a_n|^2}{n^{2\sigma}}.
\]

From this we see that, as long as \( N \ll T^{1-\epsilon} \) for some small positive \( \epsilon \), the asymptotic estimate

\[
\int_0^T \left| \sum_{n=1}^N a_n n^{-\sigma-it} \right|^2 dt = T \sum_{n=1}^N \frac{|a_n|^2}{n^{2\sigma}}
\]

holds. On the other hand, when \( N \gg T \) the mean-value can be about as large as

\[
N \log N \sum_{n=1}^N \frac{|a_n|^2}{n^{2\sigma}}.
\]

Thus, the size of the mean-value is dominated by the contribution of “diagonal” terms when \( N \) is smaller than \( T \) but, in the opposite case, the main contribution may be from the “off–diagonal” terms. Goldston and Gonek [GG] have given a much more precise version of the mean–value formula for such “long” Dirichlet polynomials in terms of the size of the coefficient correlations sums \( \sum_{n=1}^N a_n a_{n+h} \).

## 2 Connections between zeros and mean-values

Mean–value estimates are used in many ways to study the zeros of the zeta–function; indeed, this is one of the reasons that so much effort has been expended on them. Why should there be a connection? One direct link is the general relationship between the zeros of an analytic function and its average size as given by Jensen’s Formula in classical function theory.
Jensen’s Formula: Let \( f(z) \) be analytic for \( |z| \leq R \) and suppose that \( f(0) \neq 0 \). If \( r_1, r_2, \ldots, r_n \) are the moduli of all the zeros of \( f(z) \) inside \( |z| \leq R \), then
\[
\log\left( \frac{|f(0)| R^n}{r_1 r_2 \cdots r_n} \right) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| \, d\theta.
\]
Here we see that the size of the mean–value of \( \log |f(z)| \), this time around a circle, is related to the distribution of the zeros of \( f(z) \) inside that circle. There is an analogous result for rectangles, which is often more useful when working with Dirichlet series, namely,

Littlewood’s Lemma: Let \( f(s) \) be analytic and nonzero on the rectangle \( C \) with vertices \( \sigma_0, \sigma_1, \sigma_1 + iT, \) and \( \sigma_0 + iT, \) where \( \sigma_0 < \sigma_1 \). Then
\[
2\pi \sum_{\rho \in \mathcal{C}} \text{Dist}(\rho) = \int_0^T \log |f(\sigma_0 + it)| \, dt - \int_0^T \log |f(\sigma_1 + it)| \, dt
\]
\[
+ \int_{\sigma_0}^{\sigma_1} \arg f(\sigma + iT) \, d\sigma - \int_{\sigma_0}^{\sigma_1} \arg f(\sigma) \, d\sigma,
\]
where the sum runs over the zeros \( \rho \) of \( f(s) \) in \( C \) and \( \text{Dist}(\rho) \) is the distance from \( \rho \) to the left edge of the rectangle.

When we use Littlewood’s Lemma below, it will turn out that only the first term on the right–hand side is significant. So in order to not get too technical, I will always use the result in the form
\[
2\pi \sum_{\rho \in \mathcal{C}} \text{Dist}(\rho) = \int_0^T \log |f(\sigma_0 + it)| \, dt + \mathcal{E},
\]
where \( \mathcal{E} \) is an error term that can be ignored and may be different on different occasions.

The Integral of the logarithm usually cannot be dealt with directly, so we often use the following trick. We have
\[
\frac{1}{T} \int_0^T \log |f(\sigma + it)| \, dt = \frac{1}{2T} \int_0^T \log(|f(\sigma + it)|^2) \, dt \leq \frac{1}{2} \log\left( \int_0^T |f(\sigma + it)|^2 \, dt \right),
\]
where the inequality follows from the arithmetic–geometric mean inequality. In this way we see a direct connection between the location of the zeros within a rectangle and the type of mean–values we have been considering.

### 3 A Sampling of Mean–value Results

A great deal of work has been devoted to estimating the means
\[
I_k(\sigma, T) = \int_0^T |\zeta(\sigma + it)|^{2k} \, dt.
\]
When \( k = 1 \) we know that for each fixed \( \sigma > 1/2 \)
\[
I_1(\sigma, T) \sim c(\sigma) T,
\]
as $T \to \infty$, where $c(\sigma)$ is a known function of $\sigma$. In 1918 Hardy and Littlewood [HL1] proved that, on the critical line itself,

$$I_1(1/2, T) \sim T \log T.$$ 

What can such estimates tell us about the zeta–function? Comparing the result for $\sigma$ greater than $1/2$ with that for $\sigma = 1/2$, we see that the zeta–function tends to assume, on average, much larger values on the critical line than to the right of it. Since it also has many zeros on the critical line, we should expect the zeta–function to behave rather erratically there.

The next higher moment was determined in 1926 by Ingham [I1], who proved that

$$I_2(1/2, T) \sim \frac{T}{2\pi^2} \log^4 T.$$ 

Unfortunately, no asymptotic estimate for any $k$ greater than 2 has ever been proven. Ramachandra [R] has shown that

$$I_k(1/2, T) \gg T\log^{k^2} T,$$

and we expect that

$$I_k(1/2, T) \ll T\log^{k^2} T.$$ 

Conrey and Ghosh [CG1] have conjectured that

$$I_k(1/2, T) \sim \frac{a_k g_k}{\Gamma(k^2 + 1)} T\log^{k^2} T,$$

where

$$a_k = \prod_p \left( 1 - \frac{1}{p} \right)^{(k-1)^2} \frac{1}{\sum_{r=0}^{k-1} \binom{k-1}{r} p^{-r}}$$

and $g_k$ is an unknown constant. Not only does a proof of the conjecture seem far off, but it is only recently that anyone been able to suggest a plausible value for $g_k$. I will return to the problem of $g_k$ in the final lecture.

Another type of mean–value important in applications is

$$\int_0^T |\zeta(\sigma + it)M_N(\sigma + it)|^2 dt,$$

where

$$M_N(s) = \sum_{1 \leq n \leq N} \frac{\mu(n)}{n^s} P\left( \frac{\log n}{\log N} \right)$$

and $P(x)$ is a polynomial. Since

$$\frac{1}{\zeta(s)} = \sum_{n=1}^\infty \frac{\mu(n)}{n^s} \quad (\Re s > 1),$$

we can view $M_N(s)$ as an approximation to the reciprocal of $\zeta(s)$ in $\Re s > 1$. We might then expect the approximation to hold (in some sense) inside the critical strip as well. If that is the case, we should also expect that multiplying the zeta–function by $M_N$ dampens, or mollifies, the large values of zeta. Below we
will see two applications of this idea. The most general estimates known for such integrals are due to Conrey, Ghosh, and Gonek [CGG2], who obtained asymptotic estimates for them when the length of the Dirichlet polynomial $M_N(s)$ is $N = T^\theta$ with $\theta < 1/2$. Later, Conrey [C] used Kloosterman sum techniques to show that these formulas also hold for $\theta < 4/7$.

Assuming the Riemann Hypothesis and the Generalized Lindeloff Hypothesis are true, Conrey, Ghosh, and Gonek [CGG2] also proved discrete versions of such mollified mean–values, including estimates for sums of the type

$$\sum_{0 < \gamma < T} |\zeta'(\rho) M_N(\rho)|^2,$$

where $\gamma$ runs over the ordinates of the zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. The first result of this type, but without the polynomial $M_N$, were proved by Gonek [G] under the assumption of the Riemann Hypothesis alone.

Having presented a brief catalogue of mean–value estimates, I will now turn to a few of their applications.

### 4 A simple zero–density estimate

We want to show that there are relatively few zeros of the zeta–function in the right half of the critical strip. Let $\sigma_0$ be a fixed real number strictly between $1/2$ and 1 and let $C$ be the rectangle in the complex plane with vertices at 2, $2 + iT$, $\sigma_0 + iT$, $\sigma_0$. Applying our (simplified) version of Littlewood’s Lemma, we see that

$$\sum_{\rho \in C} \text{Dist}(\rho) = \frac{1}{2\pi} \int_0^T \log(|\zeta(\sigma_0 + it)|) \, dt + \mathcal{E},$$

where $\text{Dist}(\rho)$ is the distance of the zero $\rho$ from the line $\Re s = \sigma_0$. Now let $\sigma$ be a fixed real number with $\sigma_0 < \sigma < 1$ and write $N(\sigma, T)$ for the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\sigma < \beta \leq 2$ and $0 < \gamma < T$.

On the one hand, we have

$$\sum_{\rho \in C} \text{Dist}(\rho) \geq \sum_{\rho \in C, \sigma \leq \beta} \text{Dist}(\rho) \geq (\sigma - \sigma_0) N(\sigma, T).$$

On the other hand,

$$\frac{1}{2\pi} \int_0^T \log(|\zeta(\sigma_0 + it)|) \, dt = \frac{1}{4\pi} \int_0^T \log(|\zeta(\sigma_0 + it)|^2) \, dt$$

$$\leq \frac{T}{4\pi} \log\left(\frac{1}{T} \int_0^T |\zeta(\sigma_0 + it)|^2 \, dt\right)$$

by the arithmetic–geometric mean inequality, as before. The integral on the last line is $I_k(\sigma_0, T)$, which we have seen is $\sim c(\sigma_0) T$, where $c(\sigma_0)$ is positive and independent of $T$. Thus, the last expression is $O(T)$.

It follows that

$$N(\sigma, T) \ll T.$$

Since $N(T) \sim \frac{T}{\log T} \log T$, we see that

$$N(\sigma, T)/N(T) = O\left(\frac{1}{\log T}\right).$$
for any fixed $\sigma > 1/2$. We may interpret this as saying that the proportion of zeros to the right of any line $\Re s = \sigma > 1/2$ is infinitesimal.

This, the first zero–density estimate, was proved by H. Bohr and E. Landau [BL] in 1914. Since then much stronger results have been proven, typically of the form

$$N(\sigma, T) << T^{\lambda(\sigma)},$$

where $\lambda(\sigma) < 1$ for $\sigma > 1/2$. Nevertheless, the underlying idea in the proof of many (but not all) of these results already appears here.

5 Levinson’s method

Zero–density theorems tell us there are (relatively) few zeros to the right of the critical line. Our goal here is to sketch the metod of Levinson [L], which shows that there are many zeros on it.

Recall that

$$N(T) = \# \{ \rho = \beta + i\gamma \mid \zeta(\rho) = 0, \ 0 < \gamma < T \} \sim \frac{T}{2\pi} \log T$$

and let

$$N_0(T) = \# \left\{ \rho = \frac{1}{2} + i\gamma \mid \zeta(\rho) = 0, \ 0 < \gamma < T \right\}$$

denote the number of zeros on the critical line up to height $T$. The important estimations of $N_0(T)$ were:

- G. H. Hardy (1914) : $N_0(T) \to \infty$ (as $T \to \infty$)
- G. H. Hardy-J. E. Littlewood (1921) : $N_0(T) > cT$
- A. Selberg (1942) : $N_0(T) > c'N(T)$
- N. Levinson (1974) : $N_0(T) > \frac{1}{5}N(T)$
- J. B. Conrey (1989) : $N_0(T) > \frac{2}{5}N(T)$

In keeping with the theme of this lecture, I should point out that each of the last four results requires the use of mean–value theorems.

Levinson’s method begins with the following fact first proved by Speiser [Sp].

**Theorem (Speiser).** The Riemann Hypothesis is equivalent to the assertion that $\zeta'(s)$ does not vanish in the left half of the critical strip.

In the early seventies, N. Levinson and H. L. Montgomery [LM] proved a quantitative version of this. Let

$$N'_-(T) = \# \{ \rho' = \beta' + i\gamma' \mid \zeta'(\rho') = 0, \ -1 < \beta' < 1/2, \ 0 < \gamma' < T \}$$

and

$$N_-(T) = \# \{ \rho = \beta + i\gamma \mid \zeta(\rho) = 0, \ -1 < \beta < 1/2, \ 0 < \gamma < T \}.$$
Theorem (Levinson-Montgomery). We have \(N_-(T) = N'_-(T) + O(\log T)\).

The idea behind the proof is as follows. Let \(0 < a < 1/2\) and let \(C\) denote the positively oriented rectangle with vertices \(a + iT/2, a + iT, -1 + iT, \) and \(-1 + iT/2\). By a standard method it is not difficult to show that

\[
\Delta \arg \frac{\zeta'}{\zeta}(s) \bigg|_C = O(\log T),
\]

independently of \(a\). Given this, we see that

\[
2\pi (\# \text{ zeros of } \zeta'(s) \text{ in } C - \# \text{ zeros of } \zeta(s) \text{ in } C) = O(\log T).
\]

The theorem now follows on observing that \(a\) was arbitrary, and by “adding” rectangles with top and bottom edges, respectively, at \(T\) and \(T/2, T/2\) and \(T/4, \ldots\).

We now sketch Levinson’s method. We have just seen that \(N_-(T) = N'_-(T) + O(\log T)\). Now, the nontrivial zeros of \(\zeta(s)\) are symmetric about the critical line. Hence, the number of them lying to the right of the critical line, to the left of the line \(\sigma = 2\), and above the real axis up to height \(T\) is also \(N_-(T)\). Therefore

\[
N(T) = N_0(T) + 2N_-(T)
= N_0(T) + 2N'_-(T) + O(\log T),
\]

or

\[
N_0(T) = N(T) - 2N'_-(T) + O(\log T).
\]

The size of the first term on the left hand side of the last line is known, namely, \((1 + o(1))\frac{T}{2\pi} \log T\). Hence, if we can determine a sufficiently small upper bound for \(N'_-(T)\), we can deduce a lower bound for \(N_0(T)\).

To find such an upper bound it is convenient to first note that the zeros of \(\zeta'(s)\) in the region \(-1 < \sigma < 1/2, 0 < t < T\), are identical to the zeros of \(\zeta'(1 - s)\) in the reflected region \(1/2 < \sigma < 2, 0 < t < T\) . One can also show, by the functional equation of the zeta–function, that \(\zeta'(1 - s)\) and \(G(s) = \zeta(s) + \zeta'(s)/L(s)\), where \(L(s)\) is essentially \(\frac{1}{2\pi} \log T\), have the same zeros in \(1/2 < \sigma < 2, 0 < t < T\) . It turns out to be technically advantageous to count the zeros of \(G(s)\) rather than those of \(\zeta'(1 - s)\).

To bound the number of zeros of \(G(s)\) in this region, we apply Littlewood’s Lemma. Let \(a = \frac{1}{2} - \frac{\delta}{\log T}\), with \(\delta\) a small positive number, and let \(R_a\) denote the rectangle whose vertices are at \(a, 2, 2 + iT,\) and \(a + iT\).

It would be natural to apply our abbreviated form of the lemma to obtain

\[
\sum_{\rho^* \in R_a} \text{Dist}(\rho^*) = \frac{1}{2\pi} \int_0^T \log |G(a + it)| dt + \mathcal{E},
\]
where $\rho^*$ denotes a zero of $G(s)$ and $\text{Dist}(\rho^*)$ is its distance to the left edge of $\mathcal{R}_a$. However, in the next step, when we apply the arithmetic–geometric mean inequality to the integral, we would lose too much. To avoid this loss, we first dampen, or mollify, $G(s)$ and apply Littlewood’s Lemma in the form

$$\sum_{\rho^* \in \mathcal{R}_a} \text{Dist}(\rho^*) = \frac{1}{2\pi} \int_0^T \log |G(a + it)M(a + it)| dt + \mathcal{E}.$$ 

Here

$$M(s) = \sum_{n \leq T^\theta} \frac{a_n}{n^s}, \quad a_n = \mu(n)n^{-1/2} \left(1 - \frac{\log n}{\log T^\theta}\right),$$

approximates $1/\zeta(s)$ and $\theta > 0$. Note that included among the zeros of $G(s)M(s)$ in $\mathcal{R}_a$ are all the zeros of $G(s)$ in $\mathcal{R}_a$. Therefore we have

$$\sum_{\rho^* \in \mathcal{R}_a} \text{Dist}(\rho^*) \geq \sum_{\rho^* \in \mathcal{R}_a, G(\rho^*)=0} \text{Dist}(\rho^*)$$

$$\geq \sum_{\rho^* \in \mathcal{R}_a, \Re \rho^* > 1/2} \text{Dist}(\rho^*)$$

$$\geq (\frac{1}{2} - a)N'_-(T).$$

We now see that

$$(1/2 - a)N'(T) \leq \frac{1}{2\pi} \int_0^T \log |GM(a + it)| dt + \mathcal{E}$$

$$= \frac{1}{4\pi} \int_0^T \log |GM(a + it)|^2 dt + \mathcal{E}$$

$$\leq \frac{T}{4\pi} \log \left(1 + \frac{1}{T} \int_0^T |GM(a + it)|^2 dt\right) + \mathcal{E}.$$ 

Thus, we require an estimate for

$$\int_0^T |GM(a + it)|^2 dt.$$ 

This is similar to a mean–value we saw in Section 3. Levinson was able prove an asymptotic estimate for this integral when $\theta = 1/2 - \epsilon$ with $\epsilon$ arbitrarily small. The resulting upper bound for $N'_-(T)$ then led to the lower bound

$$N_0(T) > \left(\frac{1}{3} + o(1)\right) N(T).$$

Much later, Conrey was able to establish an asymptotic estimate when $\theta = 4/7 - \epsilon$, which led to

$$N_0(T) > \left(\frac{2}{5} + o(1)\right) N(T).$$

The form of the asymptotic estimate in both cases is the same as a function of $\theta$, and D. Farmer [F] has given various heuristic arguments that suggest it should remain true even when one takes $\theta$ arbitrarily large. From Farmer’s conjecture it follows that
Before concluding this section, we remark that had we introduced a mollifier into our proof of the Bohr–Landau result in the previous section, we would have obtained a much stronger zero–density estimate.

6 The number of simple zeros

Our final application demonstrates the use of discrete mean–value theorems.

Let

\[ N_s(T) = \# \{ \rho = \beta + i\gamma \mid \zeta(\rho) = 0, \zeta'(\rho) \neq 0, \ 0 < \gamma < T \} \]

denote the number of simple zeros of the zeta–function in the critical strip with ordinates between 0 and \( T \). It is believed that all the nontrivial zeros are on the critical line and simple, in other words, that \( N(T) = N_0(T) = N_s(T) \) for every \( T > 0 \). In 1973, H. Montgomery [Mo], used his pair correlation method to show that if the Riemann Hypothesis is true, then at least \( \frac{2}{3} \) of the zeros are simple. In other words,

\[ \frac{N_s(T)}{N(T)} > \frac{2}{3} \]

provided that \( T \) is sufficiently large. We will present his argument in the third lecture. Now, however, we briefly describe a different method of Conrey, Ghosh, and Gonek [CGG1], which shows that on the stronger hypotheses of RH and the Generalized Lindeloff Hypothesis, one can replace the \( \frac{2}{3} \) above by \( \frac{19}{27} = 0.703\ldots \).

By the Cauchy–Schwarz inequality, we have

\[
\left| \sum_{0 < \gamma < T} \zeta'(1/2 + i\gamma)M_N(1/2 + i\gamma) \right|^2 \leq \left( \sum_{0 < \gamma < T} 1 \right) \left( \sum_{0 < \gamma < T} |\zeta'(\rho)M_N(\rho)|^2 \right),
\]

where \( M_N(s) \) is a Dirichlet polynomial of length \( N \) with coefficients similar, but not identical, to those of \( M(s) \) in the last section. Its purpose is also similar: to mollify \( \zeta'(1/2 + i\gamma) \) so as to minimize the loss in applying the Cauchy–Schwarz inequality. If one assumes RH, the sum on the left–hand side is easy to compute and turns out to be \( \sim \frac{19}{27} N(T) \log T \). The sum on the right–hand side is much more difficult to treat, but one can show that if RH and GLH are true, then it is \( \sim \frac{57}{24} N(T)\log^2 T \). Inserting these estimates into the inequality above and solving for \( N_s(T) \) leads to the result stated. An elaboration of the method leads to the conclusion that, on the same hypotheses, at least 95.5% of the zeros of \( \zeta(s) \) are either simple or double.
Lecture III

Beyond the Riemann Hypothesis

1 Gaps between primes again

In the first lecture we saw that the Prime Number Theorem with error term implies that if \( \psi(x+h) - \psi(x) = 0 \), then there is a positive constant \( c_1 \) such that \( h \ll xe^{-c_1 \sqrt{\log x}} \). We also saw that if the Riemann Hypothesis is true, then \( h \ll x^{1/2+\epsilon} \) for any positive \( \epsilon \). The prime powers higher than the first contribute at most \( O(x^{1/2}) \) to \( \psi(x+h) - \psi(x) \), so another way to phrase this is that the size of the gap between any two consecutive primes \( p \) and \( p' \) is \( O(pe^{-c_1 \sqrt{\log p}}) \) unconditionally, and \( O(p^{1/2+\epsilon}) \) on RH. On the other hand, the Prime Number Theorem tells us that the size of the average gap between \( p \) and \( p' \) is \( \sim \log p \). This suggests that if the primes behave “randomly”, then \( p' - p \ll p' \), and the numerical evidence does indeed support this.

Here we have a problem for which even the assumption of the Riemann Hypothesis does not seem to give the right answer. The question I want to begin with here is: Why?

The answer is not difficult to find. Consider again the explicit formula

\[
\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + \mathcal{E}(x,T),
\]

where \( \mathcal{E}(x,T) \) is a known error term and, from now on, we assume the Riemann Hypothesis. If we apply the formula with the arguments \( x+h \) and \( x \) and subtract, we obtain

\[
\psi(x+h) - \psi(x) = h - \sum_{|\gamma| \leq T} \frac{(x+h)^\rho - x^\rho}{\rho} + \mathcal{E}(x+h,T) - \mathcal{E}(x,T)
\]

\[
= h - \sum_{|\gamma| \leq T} \left( \int_x^{x+h} u^{\rho-1} du \right) + \mathcal{E}'(x,h,T)
\]

\[
= h - \int_x^{x+h} \left( \sum_{|\gamma| \leq T} u^{i\gamma} \right) \frac{du}{\sqrt{u}} + \mathcal{E}'(x,h,T).
\]

There is likely to be a lot of cancellation in the sum in the integrand. However, when we estimated the error term in the Prime Number Theorem, we lost it all by putting absolute values around the individual terms. Clearly this cancellation depends completely on the distribution of the sequence of ordinates \( \gamma \). In other words, on the vertical distribution of the zeros of the zeta–function.

This example is not unique; it often happens that the strength of the Riemann Hypothesis, or even of the Generalized Riemann Hypothesis, is not sufficient to establish what we think is the ultimate truth in important arithmetical questions. We also often find that we need to understand the vertical distribution of the zeros of the zeta–function and \( L \)–functions.
Prior to the early seventies, such an understanding seemed beyond reach. Then, in 1973, Hugh Montgomery [Mo] found a way to study the distribution of the differences between all pairs of ordinates of zeros of the Riemann zeta–function, assuming RH is true.

Montgomery’s Theorem: Assume the Riemann Hypothesis. Set \( w(u) = \frac{4}{1+u^2} \), and for \( \alpha \) real and \( T \geq 2 \) write

\[
F(\alpha) = F(\alpha, T) = \left( \frac{T}{2\pi} \log T \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} w(\gamma - \gamma') T^{i\alpha(\gamma - \gamma')},
\]

where \( \gamma \) and \( \gamma' \) run over ordinates of zeros of the Riemann zeta–function. Then \( F(\alpha) \) is real and an even function of \( \alpha \). Moreover, for any \( \epsilon > 0 \) we have

\[
F(\alpha) = (1 + o(1)) T^{-2\alpha} \log T + \alpha + o(1) \quad (as T \to \infty)
\]

uniformly for \( 0 \leq \alpha \leq 1 - \epsilon \).

It was later observed that \( F(\alpha) \) is nonnegative.

Integrating \( F(\alpha) \) against a kernel \( \hat{r}(\alpha) \), we see that

\[
\left( \frac{T}{2\pi} \log T \right) \int_{-\infty}^{\infty} F(\alpha) \hat{r}(\alpha) \, d\alpha = \int_{-\infty}^{\infty} \sum_{0 < \gamma, \gamma' \leq T} w(\gamma - \gamma') T^{i\alpha(\gamma - \gamma')} \hat{r}(\alpha) \, d\alpha
\]

\[
= \sum_{0 < \gamma, \gamma' \leq T} w(\gamma - \gamma') \int_{-\infty}^{\infty} T^{i\alpha(\gamma - \gamma')} \hat{r}(\alpha) \, d\alpha
\]

\[
= \sum_{0 < \gamma, \gamma' \leq T} r((\gamma - \gamma') \log T) w(\gamma - \gamma'),
\]

where \( r \) is the inverse Fourier transform of \( \hat{r} \), that is,

\[
r(u) = \int_{-\infty}^{\infty} \hat{r}(\alpha) e^{2\pi i u \alpha} \, d\alpha.
\]

Thus, the integral of \( F(\alpha) \) against a kernel \( \hat{r}(\alpha) \) produces a sum involving the inverse transform \( r \) evaluated at the differences of pairs of ordinates. Since Montgomery’s Theorem is only valid in the range \(-1 < \alpha < 1\), one can only use kernels \( \hat{r}(\alpha) \) supported on \((-1, 1)\). For example, assuming RH and taking \( \hat{r}(\alpha) = \max\{0, \beta^{-1}(1 - |\alpha/\beta|)\} \) with \( 0 < \beta < 1 \), one obtains

\[
\sum_{0 < \gamma, \gamma' \leq T} \left( \frac{\sin((\beta/2)(\gamma - \gamma') \log T)}{(\beta/2)(\gamma - \gamma') \log T} \right)^2 w(\gamma - \gamma') \sim \left( \frac{1}{\beta} + \frac{\beta}{3} \right) \frac{T}{2\pi} \log T.
\]

Montgomery used this to obtain a lower bound for the number of simple zeros of the zeta-function as follows. First observe that

\[
\sum_{0 < \gamma, \gamma' \leq T} \frac{1}{\gamma - \gamma'} \leq \sum_{0 < \gamma, \gamma' \leq T} \left( \frac{\sin((\beta/2)(\gamma - \gamma') \log T)}{(\beta/2)(\gamma - \gamma') \log T} \right)^2 w(\gamma - \gamma').
\]
Taking $\beta = 1 - \epsilon$ in this, we obtain

$$\sum_{0 < \gamma, \gamma' \leq T, \gamma = \gamma'} 1 \leq \left( \frac{4}{3} + o(1) \right) \frac{T}{2\pi} \log T.$$ 

Now, if the zero $\frac{1}{2} + i\gamma$ has multiplicity $m(\gamma)$, then each $\gamma$ occurs $m(\gamma)$ times in the sum on the left. Thus, we have

$$\sum_{0 < \gamma, \gamma' \leq T, \gamma = \gamma'} 1 = \sum_{0 < \gamma < T} m(\gamma)$$

and therefore

$$\sum_{0 < \gamma < T} m(\gamma) \leq \left( \frac{4}{3} + o(1) \right) \frac{T}{2\pi} \log T.$$ 

Finally, we easily see that

$$\sum_{0 < \gamma < T, \frac{1}{2} + i\gamma \text{ is simple}} 1 \geq \sum_{0 < \gamma < T} (2 - m(\gamma)) \geq (2 - \frac{4}{3} + o(1)) \frac{T}{2\pi} \log T.$$ 

Hence, if the Riemann Hypothesis is true, then at least two-thirds of the zeros are simple. Although this is not quite as strong a result as that obtained in Lecture II, namely $(19/27 + o(1)) \frac{T}{2\pi} \log T$, the hypotheses are also not as strong. For there we needed to assume the Generalized Lindeloff Hypothesis in addition to RH.

Since we have focused so much on mean–value theorems, I should point out that Montgomery proved his theorem by relating $F(\alpha)$ to the mean–value of a Dirichlet series, namely,

$$\frac{1}{x} \int_0^T \left| \sum_{n \leq x} \Lambda(n) \left( \frac{x}{n} \right)^{-1/2 + it} + \sum_{n > x} \Lambda(n) \left( \frac{x}{n} \right)^{3/2 + it} \right|^2 dt,$$

where $x = T^{\alpha}$. Here we see a different explicit connection between the zeros and the primes. Indeed, Montgomery’s starting point was a generalization of the explicit formula we saw in Lecture I (and again at the beginning of this lecture). The restriction $\alpha < 1$ in Montgomery’s Theorem arises for a familiar reason: when $\alpha \geq 1$, the off–diagonal terms in the integral above contribute to the main term in the mean–value estimate. To determine this contribution (heuristically), Montgomery used a strong form of the Hardy–Littlewood twin prime conjecture. In this way he arrived at

**Montgomery’s Conjecture:** We have

$$F(\alpha, T) = (1 + o(1)) \quad (as T \to \infty)$$

for $\alpha \geq 1$, uniformly in bounded intervals.

This together with Montgomery’s theorem determines $F(\alpha)$ on all of $\mathbb{R}$. Thus, one may use the conjecture to integrate $F(\alpha)$ against a much wider class of kernels than just those supported in $(-1, 1)$. Using an
appropriate kernel he arrived at the

**Pair Correlation Conjecture:** For any fixed $\alpha$ and $\beta$ with $\alpha < \beta$, we have

$$\sum_{0 < \gamma, \gamma' \leq T} 1 \sim \left( \int_{0}^{\beta} 1 - \left( \frac{\sin \pi x}{\pi x} \right)^2 dx + \delta(\alpha, \beta) \right) \frac{T}{2\pi \log T}$$

as $T$ tends to infinity, where $\delta(\alpha, \beta) = 1$ if $0 \in [\alpha, \beta]$, and $\delta(\alpha, \beta) = 0$ otherwise.

The Pair Correlation Conjecture is an assertion about the distribution of the set of all differences between pairs of ordinates of the zeros. An enormous amount of data concerning the zeros has been collected and analyzed by A. M. Odlyzko [O], and the fit with the conjecture is remarkable.

As an example of the type of information we can deduce from it, let $0 < \alpha < \beta$ with $\alpha$ arbitrarily small. Then we find that

$$\sum_{0 < \gamma, \gamma' \leq T \atop 0 < \gamma' - \gamma \leq 2\pi \beta / \log T} 1 \sim \left( \int_{0}^{\beta} 1 - \left( \frac{\sin \pi x}{\pi x} \right)^2 dx \right) \frac{T}{2\pi \log T}.$$ 

This shows that an infinite number of the zeros have another zero no farther away than $2\pi \beta / \log T$, no matter how small $\beta$ is. We also deduce that

$$\sum_{-2\pi \beta / \log T \leq \gamma' - \gamma \leq 2\pi \beta / \log T} 1 \sim \left( \int_{-\beta}^{\beta} 1 - \left( \frac{\sin \pi x}{\pi x} \right)^2 dx + 1 \right) \frac{T}{2\pi \log T}.$$

Combining this with the previous formula, we obtain

$$\sum_{0 < \gamma, \gamma' \leq T \atop \gamma' = \gamma} 1 \sim \frac{T}{2\pi \log T}.$$

By our earlier discussion, we may write this as

$$\sum_{0 < \gamma < T} m(\gamma) \sim \frac{T}{2\pi \log T}.$$

On the other hand, von Mangoldt’s formula tells us that

$$\sum_{0 < \gamma < T} 1 \sim \frac{T}{2\pi \log T}.$$

It therefore follows that

$$\sum_{0 < \gamma < T \atop \frac{1}{2} + i\gamma \text{ is simple}} 1 \sim \frac{T}{2\pi \log T}.$$

In other words, almost all the zeros are simple.

Before moving on we mention that D. Goldston and H. Montgomery [GM] have shown that the Pair Correlation Conjecture is equivalent to a certain estimate of the variance of the number of primes numbers
in short intervals. D. Goldston, S. Gonek, and H. Montgomery [GGM] have shown that it is also equivalent
to an estimate for the mean–value
\[ \int_0^T \left| \frac{\zeta'(\sigma + it)}{\zeta} \right|^2 dt , \]
for \( \sigma \) near 1/2. Estimates of \( F(\alpha, T) \) when \( \alpha \geq 1 \) remain elusive. The only progress in this
direction so far is the lower bound \( F(\alpha, T) \geq 3/2 - \alpha + o(1) \) on the interval \( (1, 3/2) \) under the assumption of the Generalized
Riemann Hypothesis. This is due to D. Goldston, S. Gonek, A. E. Özlük, and C. Snyder [GGOS].

3 Random matrix theory

Shortly after completing the work described above, Montgomery was told by F. Dyson that the “form
factor” \( 1 - \left( \frac{\sin \pi x}{\pi x} \right)^2 \) in the distribution law he had conjectured for pairs of zeros of the zeta–function is the
same one that holds for pairs of eigenvalues of large random Hermitian matrices from the Gaussian Unitary
Ensemble, or GUE, which we describe below. This and other matrix ensembles had been studied by physici-
sists for decades because they can be used to model the Hamiltonians of complicated physical systems. The
spectra, or energy levels, of such systems are given by the eigenvalues of the corresponding Hamiltonian. But
in complicated situations, the Hamiltonian, let alone its eigenvalues, may not be known with any certainty.
In such cases the Hamiltonian can be modeled by large random Hermitian matrices with symmetry properties
dictated by the physical situation. It is found that the average behavior of the eigenvalues of such families of
matrices is often in agreement with the experimental data. Physicists are particularly interested in knowing
various statistics of the energy levels, and pair correlation is merely one of these. They had also worked out
“n–level” correlations of the eigenvalues, and Montgomery conjectured that the analogous law (there is a nor-
malization one has to take into account) holds for the “n–level” correlations of the zeros. Specifically, we have

**Montgomery’s GUE Hypothesis:** The distribution of all \((n - 1)\)-tuples \((\gamma_2 - \gamma_1, \gamma_3 - \gamma_1, \ldots, \gamma_n - \gamma_1)\),
with the \(\gamma_i\) ordinates of the zeros, has the form factor \(\det K(x_1, \ldots, x_n)\), where

\[
K(x_1, \ldots, x_n) = \det (k_{ij})_{i,j=1}^n, \quad k_{ii} = 1, \quad k_{ij} = \frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)}.
\]

The Pair Correlation Conjecture is the \(n = 2\) case. Odlyzko [O] also used his data (alluded to above) to
check this prediction, and the evidence is again compelling. Moreover, so is the theoretical support (see, for
example, E. Bogomolny and J. Keating [BK], D. Hejhal [He], and Z. Rudnık and P. Sarnak [RS]).

Finally, the Gaussian Unitary Ensemble of order \(N\) is the set of all \(N \times N\) Hermitian matrices \(H = (H_{j,k})_{1 \leq j, k \leq N}\) made into a probability space by equipment it with a probability measure \(p(H) dH\), invariant
under conjugation by all \(N \times N\) Unitary matrices, where

\[
dH = \prod_{j \leq k} d\Re H_{j,k} \prod_{j < k} d\Im H_{j,k}
\]
and
\[ p(H) = \prod_{j \leq N} \frac{1}{\sqrt{\pi}} e^{-H^2_j} \prod_{j<k} \frac{2}{\pi} e^{-2((\Re H_{j,k})^2+(\Im H_{j,k})^2)}. \]

In practice it is often easier to work with the so called Circular Unitary Ensemble, or CUE, rather than the GUE. This is the compact group of $N \times N$ unitary matrices equipped with Haar measure normalized so that the measure of the group is 1. All eigenvalues have modulus one and the statistics of the eigenangles are known to be the same as those for the GUE eigenvalues.

4 Applications of random matrix theory to the zeta–function.

Another remarkable development in the application of random matrix theory to analytic number theory has been the discovery by J. Keating and N. Snaith [KS] that the characteristic polynomial of a large random matrix from the Gaussian Unitary Ensemble or Circular Unitary Ensemble can be used to model the Riemann zeta–function and other L–functions.

The idea is as follows. Since Riemann’s function $\xi(s)$ is entire, it has a Hadamard product representation. Moreover, $\zeta(s)$ and $\xi(s)$ are the same up to well understood multiplicative factors. Therefore, one might plausibly assume that at a large height $t$ in the critical strip, $\zeta(s)$ (with $s = \sigma + it$) should behave like a polynomial with the same zeros near $t$. If the zeros are distributed like the eigenangles of matrices from the Circular Unitary Ensemble, one might then expect
\[ Z_N(U, \theta) = \prod_{n=1}^{N} (1 - e^{i(\theta_n - \theta)}), \]
where the $\theta_n$ are the eigenangles of a random $N \times N$ unitary matrix $U$ from CUE, to model $\zeta(1/2 + it)$. For scaling reasons one takes $N = \frac{1}{2\pi} \log t$.

Keating and Snaith conjecture that the average of $|Z_N(U, \theta)|^{2k}$ over the full Circular Unitary Ensemble, with respect to Haar measure on the group, should be directly related to the $2k$th moment
\[ I_k(1/2, T) = \int_0^T |\zeta(1/2 + it)|^{2k} dt \]
of the zeta–function. Similarly, the distribution of values of $\log Z_N(U, \theta)$, say, should be the same as that of $\log \zeta(1/2 + it)$. The agreement with known results in both cases is remarkable.

Consider the case of $I_k$. Recall from Lecture II that it had long been conjectured that there is a constant $c_k$ such that
\[ I_k(1/2, T) \sim c_k T \log^{k^2} T \]
as $T \to \infty$. J. B. Conrey and A. Ghosh [CG] have recast the conjecture into a more precise form, namely that
\[ c_k = g_k a_k / \Gamma(k^2 + 1), \]
where
\[ a_k = \prod_p \left(1 - \frac{1}{p}\right)^{(k-1)^2} \sum_{r=0}^{k-1} \binom{k-1}{r} p^{-r} \]
and \( g_k \) is an integer. Thus the question comes down to the value of \( g_k \). The only proven values are the classical ones due to Hardy and Littlewood [HL1] and Ingham [I1] of \( g_1 = 1 \) and \( g_2 = 2 \), respectively. Conrey and Ghosh [CG2] conjectured that \( g_3 = 42 \) and, using long Dirichlet polynomials to approximate \( \zeta(s)^k \), Conrey and Gonek [CGo] conjectured that \( g_4 = 24024 \). At about the same time, Keating and Snaith [KS] calculated arbitrary complex moments of the characteristic polynomials \( Z_N(U, \theta) \) averaged over all \( N \times N \) matrices \( U \) in the CUE, and when \( k = 1, 2, 3, \) and \( 4 \) they obtained the same values for the numbers corresponding to \( g_k \) as those above. They argued that one could therefore model the moments \( I_k(1/2, T) \) by the average of \( |Z_N(U, \theta)|^{2k} \) over CUE and conjectured that
\[ g_k = (k^2!) \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}. \]

Interestingly, the Keating–Snaith and Conrey–Gonek conjectures were first publicly announced at the Riemann Hypothesis Conference in Vienna, just moments after it was checked that the Keating–Snaith conjecture in fact predicts that \( g_4 = 24024 \).

The characteristic polynomial model has proven to be extremely powerful for predicting other behavior of the zeta–function and L–functions that once seemed hopelessly beyond reach. In fact, to a large extent it has been responsible for an explosion of activity in the field and of collaboration between number theorists and theoretical physicists.

Impressive as the characteristic polynomial model has proven to be, it has the obvious drawback that it contains no arithmetical information. The prime numbers do not appear in this model of the zeta–function! In the moment problem, this is reflected by the absence of the arithmetical factor \( a_k \) in the Keating–Snaith conjecture. They had to insert it in an ad hoc way. Fortunately, in the moment problem, it was only the factor \( g_k \) and not \( a_k \) that proven elusive. A precise and more satisfactory model for the zeta–function (and other L–functions) clearly has to include such relevant information.

In work in progress with J. Keating and C. Hughes, we have now succeeded in finding such a model for \( \zeta(s) \) and it can easily be generalized to model any L–function. I will conclude this lecture by describing the new model.

Roughly, we have proven that if the Riemann Hypothesis is true, then for \( t \in \mathbb{R} \) and \( x > 1 \) we have
\[ \zeta\left(\frac{1}{2} + it\right) = \exp \left( \sum_{2 \leq n \leq x} \Lambda(n)/(n^{1/2+it} \log n) \right) \prod_n \exp \left( E_1((t - \gamma_n) \log x) \right) \]
(times an error term that is essentially 1), where \( E_1(z) = \int_{\infty}^{\infty} \frac{e^{-w}}{w} dw \) is the exponential integral. I say “roughly” because one also has to include smooth weights in the various factors. A similar formula holds
throughout the critical strip. Since we expect the ordinates $\gamma_n$ of the zeros to behave like the eigenangles $\theta_n$ of $N \times N$ random matrices in CUE, and "scaling" suggests that we take $N$ to be the nearest integer to $\frac{1}{2 \pi} \log t$, we take as our model for zeta

$$\exp \left( \sum_{2 \leq n \leq x} \Lambda(n)/(n^{1/2+it} \log n) \right) \prod_{n \leq N} \exp \left( E_1((\theta - \theta_n) \log x) \right).$$

The presence of the exponential integral makes it a little complicated to compare this with the previous model,

$$Z_N(U, \theta) = \prod_{n=1}^{N} (1 - e^{i(\theta_n - \theta)}).$$

We note, however, that if $\theta_n$ is not too near $\theta$, then the new model looks approximately like

$$\prod_{p \leq x} \left(1 - p^{-(1/2+it)}\right)^{-1} \prod_{n \leq N} \exp \left(1 - x^{i(\theta - \theta_n)}\right).$$

Here we clearly see both the primes and the zeros, and how the parameter $x$ serves to connect them. The moments $I_k(1/2, T)$ should now be given by the product of two moments– one being the $2k$th power of the modulus of the product over primes integrated with respect to $t$, the other being the $2k$th power of the modulus of the product over the eigenangles averaged over the Circular Unitary Ensemble. We call the conjecture that the mean can be computed this way, that is, as a product of two different types of means, the “Splitting Conjecture”.

The new model seems promising for many other investigations as well. To give just one example, we hope to use it to understand the horizontal distribution of the zeros of $\zeta'(s)$ in the right half of the critical strip, a problem that has long defied us. We also expect it to give us more insight into the connection between primes and zeros. If we are extremely lucky, perhaps we will even find explicit and useful connections between primes in special sequences, such as twin primes or primes of the form $n^2 + 1$, and the zeros.

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