THE JUMPING PHENOMENON OF THE DIMENSIONS OF BOTT-CHERN COHOMOLOGY GROUPS AND AEPPLI COHOMOLOGY GROUPS

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ABSTRACT. Let $X$ be a compact complex manifold, and let $\pi : X \to B$ be a small deformation of $X$, the dimensions of the Bott-Chern cohomology groups $H_{BC}^{p,q}(X(t))$ and Aeppli cohomology groups $H_{A}^{p,q}(X(t))$ may vary under this deformation. In this paper, we will study the deformation obstructions of a $(p, q)$ class in the central fiber $X$. In particular, we obtain an explicit formula for the obstructions and apply this formula to the study of small deformations of the Iwasawa manifold.

1. Introduction

Let $X$ be a compact complex manifold and $\pi : X \to B$ be a family of complex manifolds such that $\pi^{-1}(0) = X$. Let $X_t = \pi^{-1}(t)$ denote the fibre of $\pi$ over the point $t \in B$. In [9], the author has studied the jumping phenomenon of hodge numbers $h^{p,q}$ of $X$ by studying the deformation obstructions of a $(p, q)$ class in the central fiber $X$. In particular, the author obtained an explicit formula for the obstructions and apply this formula to the study of small deformations of the Iwasawa manifold. Besides the Hodge numbers, the dimensions of Bott-Chern cohomology groups and the dimensions of Aeppli cohomology groups are also important invariant of complex structures. In [2], D. Angella has studied the small deformations of the Iwasawa manifold and found that the dimensions of

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Bott-Chern cohomology groups and the dimensions of Aeppli cohomology groups are not deformation invariants.

In this paper, we will study the Bott-Chern cohomology and Aeppli cohomology by studying the hypercohomology of a complex $B_{p,q}^\bullet$ constructed in [7]. In [7], M. Schweitzer proved that

$$H_{BC}^{p,q}(X) \cong \mathbb{H}^{p+q}(X, B_{p,q}^\bullet),$$

and

$$H_A^{p,q}(X) \cong \mathbb{H}^{p+q+1}(X, B_{p+1,q+1}^\bullet).$$

As the author did in [9], we will such study the jumping phenomenons from the viewpoint of obstruction theory. More precisely, for a certain small deformation $X'$ of $X$ parameterized by a basis $B$ and a certain class $[\theta]$ of the hypercohomology group $H^l(X, B_{p,q}^\bullet)$, we will try to find out the obstruction to extend it to an element of the relative hypercohomology group $\mathbb{H}^l(X, B_{p,q}^\bullet; X/B)$. We will call those elements which have non trivial obstruction the obstructed elements. And then we will see that these elements will play an important role when we study the jumping phenomenon. Because we will see that the existence of the obstructed elements is a sufficient condition for the variation of the dimensions of Bott-Chern cohomology and Aeppli cohomology.

In §2 we will summarize the results of M. Schweitzer about Bott-Chern cohomology and Aeppli cohomology, from which we can define the relative Bott-Chern cohomology and Aeppli cohomology on $X_n$ where $X_n$ is the $n$th order deformation of $\pi : X \to B$. We will also introduce some important maps which will be used in the calculation of the obstructions in §4. In §3, we will try to explain why we need to consider the obstructed elements. The relation between the jumping phenomenon of Bott-Chern cohomology and Aeppli cohomology and the obstructed elements is the following.

**Theorem 1.1.** Let $\pi : X \to B$ be a small deformation of the central fibre compact complex manifold $X$. Now we consider $\dim \mathbb{H}^l(X(t), B_{p,q}^\bullet)$ as a function of $t \in B$. It jumps at $t = 0$ if there exists an element $[\theta]$ either in $\mathbb{H}^l(X, B_{p,q}^\bullet)$ or in
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$H^{l-1}(X, B_{p,q}^\bullet)$ and a minimal natural number $n \geq 1$ such that the $n$-th order obstruction

$$o_n([\theta]) \neq 0.$$ 

In §4 we will get a formula for the obstruction to the extension we mentioned above.

**Theorem 1.2.** Let $\pi : X' \to B$ be a deformation of $\pi^{-1}(0) = X$, where $X$ is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the $n$-th order deformation of $X$. For arbitrary $[\theta]$ belongs to $H^l(X, B_{p,q}^\bullet)$, suppose we can extend $[\theta]$ to order $n - 1$ in $H^l(X_{n-1}, B_{p,q}^\bullet \cap B_{X_{n-1}/B_{n-1}})$. Denote such element by $[\theta_{n-1}]$. The obstruction of the extension of $[\theta]$ to $n$th order is given by:

$$o_n([\theta]) = -\partial_{X_{n-1}/B_{n-1}}^\bar{\partial} \circ \kappa_n \circ \partial_{X_{n-1}/B_{n-1}}^{B, \bar{\partial}} ([\theta_{n-1}]) - \bar{\partial}_{X_{n-1}/B_{n-1}}^{B, \partial} \circ \bar{\kappa}_n \circ \bar{\partial}_{X_{n-1}/B_{n-1}}^{B, \partial} ([\theta_{n-1}]),$$

where $\kappa_n$ is the $n$th order Kodaira-Spencer class and $\bar{\kappa}_n$ is the $n$th order Kodaira-Spencer class of the deformation $\bar{\pi} : X' \to \bar{B}$. $\partial_{X_{n-1}/B_{n-1}}^\bar{\partial}$, $\bar{\partial}_{X_{n-1}/B_{n-1}}^\partial$, $\partial_{X_{n-1}/B_{n-1}}^{B, \partial}$ and $\bar{\partial}_{X_{n-1}/B_{n-1}}^{B, \partial}$ are the maps defined in §2.

In §5 we will use this formula to study carefully the example given by Iku Nakamura and D. Angella, i.e. the small deformation of the Iwasama manifold and discuss some phenomenons.

2. THE RELATIVE BOTT-CHERN COHOMOLOGY AND AEPPLI COHOMOLOGY OF $X_n$ AND THE REPRESENTATION OF THEIR COHOMOLOGY CLASSES

2.1. The Bott-Chern(Aeppli) Cohomology and Bott-Chern(Aeppli) Hypercohomology.

All the details of this subsection can be found in [7]. Let $X$ be a compact complex manifold. The Dolbeault cohomology groups $H^{p,q}_\partial(X)$, and more generally the terms $E^{p,q}_r(X)$ in the Frölicher spectral sequence [5], are well-known finite
dimensional invariants of the complex manifold $X$. On the other hand, the Bott-Chern and Aeppli cohomologies define additional complex invariants of $X$ given, respectively, by [1, 3]

$$H^{p,q}_{BC}(X) = \frac{\ker\{d: \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+q+1}(X)\}}{\text{im}\{\partial\bar{\partial}: \mathcal{A}^{p-1,q-1}(X) \to \mathcal{A}^{p,q}(X)\}}$$

and

$$H^{p,q}_{A}(X) = \frac{\ker\{\partial\bar{\partial}: \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q+1}(X)\}}{\text{im}\{\partial: \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X)\} + \text{im}\{\partial: \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X)\}}.$$ 

By the Hodge theory developed in [7], all these complex invariants are also finite dimensional and one has the isomorphisms $H^{p,q}_{A}(X) \cong H^{n-q,n-p}_{BC}(X)$. Notice that $H^{q,p}_{BC}(X) \cong H^{p,q}_{BC}(X)$ by complex conjugation. For any $r \geq 1$ and for any $p, q$, there are natural maps

$$H^{p,q}_{BC}(X) \to E^{p,q}_r(X) \quad \text{and} \quad E^{p,q}_r(X) \to H^{p,q}_{A}(X).$$

Recall that $E^{p,q}_1(X) \cong H^{p,q}_0(X)$ and that the terms for $r = \infty$ provide a decomposition of the de Rham cohomology of the manifold, i.e. $H^{k}_{dR}(X, \mathbb{C}) \cong \bigoplus_{p+q=k} E^{p,q}_\infty(X)$. From now on we shall denote by $h^{p,q}_{BC}(X)$ the dimension of the cohomology group $H^{p,q}_{BC}(X)$. The Hodge numbers will be denoted simply by $h^{p,q}(X)$ and the Betti numbers by $b_k(X)$. For any given $p \geq 1, q \geq 1$, we define the complex of sheaves $\mathcal{L}^{•}_{p,q}$ by

$$\mathcal{L}^k_{p,q} = \bigoplus_{r+s=k \atop r<p<s<q} \mathcal{A}^{r,s} \quad \text{if} \quad k \leq p+q-2,$$

$$\mathcal{L}^{k-1}_{p-1,q-1} = \bigoplus_{r+s=k \atop r \geq p, s \geq q} \mathcal{A}^{r,s} \quad \text{if} \quad k \geq p+q,$$

and the differential:

$$\mathcal{L}^0_{p,q} \xrightarrow{pr_{p,q}} \mathcal{L}^1_{p,q} \xrightarrow{od_{p,q}} \mathcal{L}^2_{p,q} \xrightarrow{od_{p,q}} \ldots \xrightarrow{\partial\bar{\partial}} \mathcal{L}^{k-2}_{p,q} \xrightarrow{\partial\bar{\partial}} \mathcal{L}^{k-1}_{p,q} \xrightarrow{d_{p,q}} \mathcal{L}^k_{p,q} \xrightarrow{d_{p,q}} \ldots$$

Then by the above construction, we have the following isomorphisms

$$H^{p,q}_{BC}(X) = H^{p+q-1}(\mathcal{L}^{•}_{p,q}(X)) \cong H^{p+q-1}(X, \mathcal{L}^{•}_{p,q}),$$

$$H^{p,q}_{A}(X) = H^{p+q}(\mathcal{L}^{•}_{p+1,q+1}(X)) \cong H^{p+q}(X, \mathcal{L}^{•}_{p+1,q+1}).$$
because $L^k_{p,q}$ are soft.

We define a sub complex $S^{•}_{p,q}$ of $L^{•}_{p,q}$ by:

$$(S'_{p,q}, ∂) : O → Ω^1 → ... → Ω^{p-1} → 0, \quad (S''_{q}, ∂_q) : O → \bar{Ω}^1 → ... → \bar{Ω}^{q-1} → 0,$$

$S^{•}_{p,q} = S^{•}_{p} + S^{•}_{q} : O + \bar{O} → Ω^1 \oplus \bar{Ω}^1 → ... → Ω^{p-1} \oplus \bar{Ω}^{p-1} → Ω^p → ... → \bar{Ω}^{q-1} → 0.$

Note that the inclusion $S^{•} ⊂ L^{•}$ is an quasi-isomorphism [7]. There is another complex $B^{•}_{p,q}$ used in [7] which is defined by:

$$B^{•}_{p,q} : \mathbb{C} \xrightarrow{(+,−)} O + \bar{O} → Ω^1 \oplus \bar{Ω}^1 → ... → Ω^{p-1} \oplus \bar{Ω}^{p-1} → Ω^p → ... → \bar{Ω}^{q-1} → 0.$$

and the following morphism of from $B^{•}_{p,q}$ to $S^{•}_{p,q}[1]$ is a quasi-isomorphism [7]:

$$\mathbb{C} \xrightarrow{(+,−)} O + \bar{O} → \Omega^1 + \bar{Ω} \rightarrow ... \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow$$

$$0 → O + \bar{O} → \Omega^1 + \bar{Ω} → ...$$

Therefore we have:

$$H^{p,q}_{BC}(X) \cong \mathbb{H}^{p+q}(X, L^{•}_{p,q}[1]) \cong \mathbb{H}^{p+q}(X, S^{•}_{p,q}[1]) \cong \mathbb{H}^{p+q}(X, B^{•}_{p,q}),$$

and

$$H^{p,q}_{A}(X) \cong \mathbb{H}^{p+q}(X, L^{•}_{p+1,q+1}) \cong \mathbb{H}^{p+q}(X, S^{•}_{p+1,q+1}) \cong \mathbb{H}^{p+q+1}(X, B^{•}_{p+1,q+1}),$$

### 2.2. The Relative Bott-Chern Cohomology and Aeppli Cohomology of $X_n$.

Let $π : \mathcal{X} → B$ be a deformation of $π^−1(0) = X$, where $X$ is a compact complex manifold. For every integer $n ≥ 0$, denote by $B_n = \text{Spec} \mathcal{O}_{B,0}/m_0^{n+1}$ the $n$th order infinitesimal neighborhood of the closed point $0 ∈ B$ of the base $B$. Let $X_n ⊂ \mathcal{X}$ be the complex space over $B_n$. Let $π_n : X_n → B_n$ be the $n$th order deformation of $X$. Denote $π^\ast(m_0)$ by $\mathcal{M}_0$. If we take the complex conjugation, we have another complex structure of the differential manifold of $\mathcal{X}$, we denote this manifold by $\bar{\mathcal{X}}$ and $π$ induce a deformation $\bar{π} : \bar{\mathcal{X}} → B$ of $X$. Then we have $\bar{X}_n$ and $\bar{π}_n : \bar{X}_n → \bar{B}_n$. Let $\mathcal{C}^\ast_{\bar{B}}$ be the sheaf of $\mathbb{C}$-valued real analytic functions
on \( B \). Denote \( \mathcal{O}_X^\omega = \pi^*(\mathcal{O}_B^\omega) \), \( \bar{\mathcal{O}}_X^\omega = \bar{\pi}^*(\mathcal{O}_B^\omega) \). Let \( m_0^\omega \) be the maximal idea of \( \mathcal{O}_{B,0}^\omega \) and \( \mathcal{M}_B^\omega = \pi^*(m_0^\omega) \), \( \bar{\mathcal{M}}_B^\omega = \bar{\pi}^*(m_0^\omega) \). For any sheaf of \( \mathcal{O}_X \) module \( \mathcal{F} \).

Denote \( \mathcal{F}^\omega = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^\omega \) (resp. \( \mathcal{F}^\bar{\omega} = \mathcal{F} \otimes_{\mathcal{O}_X} \bar{\mathcal{O}}_X^\omega \)). Let \( \mathcal{O}_X = \mathcal{O}_{X,0}^\omega / (\mathcal{M}_B^\omega)^{n+1} \mathcal{O}_X^\omega = \bar{\mathcal{O}}_{X,0}^\omega / (\bar{\mathcal{M}}_B^\omega)^{n+1} \). For any sheaf of \( \mathcal{O}_{X,n} \) module \( \mathcal{F} \). Denote \( \mathcal{F}^\omega = \mathcal{F} \otimes_{\mathcal{O}_{X,n}} \mathcal{O}_{X,n}^\omega \) (resp. \( \mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_{X,n}} \bar{\mathcal{O}}_{X,n}^\omega \)).

For any given \( p \geq 1, q \geq 1 \), We define the complex \( S_{X_n/B_n} = S_{p,q;X_n/B_n}^{•} \) by:

\[
\left( S_{p,q;X_n/B_n}^{•} : \mathcal{O}_{X_n}^\omega \to \Omega_{X_n/B_n}^{1,\omega} \to \ldots \to \Omega_{X_n/B_n}^{p-1,\omega} \to 0, \right)
\]

\[
\left( S_{p,q;X_n/B_n}^{•} : \mathcal{O}_{X_n}^\omega \to \bar{\Omega}_{X_n/B_n}^{1,\omega} \to \ldots \to \bar{\Omega}_{X_n/B_n}^{q-1,\omega} \to 0, \right)
\]

\[
S_{p,q;X_n/B_n}^{•} = S_{p,q;X_n/B_n}^{•} + S_{p,q;X_n/B_n}^{•} : \]

\[
\mathcal{O}_{X_n}^\omega + \mathcal{O}_{X_n}^\omega \to \Omega_{X_n/B_n}^{1,\omega} \oplus \bar{\Omega}_{X_n/B_n}^{1,\omega} \to \ldots \Omega_{X_n/B_n}^{p-1,\omega} \oplus \bar{\Omega}_{X_n/B_n}^{p-1,\omega} \to \bar{\Omega}_{X_n/B_n}^{p,\omega} \to \ldots \to \bar{\Omega}_{X_n/B_n}^{q-1,\omega} \to 0.
\]

We can also define \( \mathcal{B}_{p,q;X_n/B_n}^{•} \) by:

\[
\mathcal{B}_{p,q;X_n/B_n}^{•} : \omega_{B_n}^{(1,-)} \to \mathcal{O}_{X_n}^\omega \oplus \mathcal{O}_{X_n}^\omega \to \Omega_{X_n/B_n}^{1,\omega} \oplus \bar{\Omega}_{X_n/B_n}^{1,\omega} \to \ldots \Omega_{X_n/B_n}^{p-1,\omega} \oplus \bar{\Omega}_{X_n/B_n}^{p-1,\omega} \]

\[
\to \bar{\Omega}_{X_n/B_n}^{p,\omega} \to \ldots \to \bar{\Omega}_{X_n/B_n}^{q-1,\omega} \to 0,
\]

where \( \omega_{B_n} = \pi^{-1}(\mathcal{O}_{B,0}^\omega / (m_0^\omega)^{n+1}) \).

Then the Relative Bott-Chern cohomology and Aeppli cohomology of \( X_n \) is defined by

\[
H_{BC}^{p,q}(X_n/B_n) \cong \mathbb{H}^{p+q}(X, S_{p,q;X_n/B_n}^{•} [1]) \cong \mathbb{H}^{p+q}(X, B_{p,q;X_n/B_n}^{•}).
\]

and

\[
H_{A}^{p,q}(X_n/B_n) \cong \mathbb{H}^{p+q}(X, S_{p+1,q+1;X_n/B_n}^{•}) \cong \mathbb{H}^{p+q+1}(X, B_{p+1,q+1;X_n/B_n}^{•}).
\]
2.3. The Representation of the Relative Bott-Chern Cohomology and Aeppli Cohomology Classes.

In this subsection we will follow [7] to construct a hypercocycle in \( \hat{Z}^{p+q}(X, B^*_{p,q}) \) to represent the relative Bott-Chern cohomology classes. Let \( [\theta] \) be an element of \( H^{p,q}_{BC}(X) \), represented by a closed \((p,q)\)-form \( \theta \). It is defined in \( \mathbb{H}^{p+q}(X, \mathcal{L}_{p,q}[1]^*) \) by a hypercocycle, still denoted by \( \theta \) and defined by \( \partial^p \theta = \theta |_{U_j} \) and \( \partial^r \theta = 0 \) otherwise. For given \( p \geq 1 \) and \( q \geq 1 \), there exists a hypercocycle \( \theta = (c; u^p, v^q) \in \hat{Z}^{p+q}(X, B^*_{p,q}) \) and a hypercochain \( \alpha = (\alpha^r, s) \in \hat{C}^{p+q-1}(X, \mathcal{L}_{p,q}[1]^*) \) such that \( \theta = \delta \alpha + w \). We represent the data in the following table:

\[
\begin{array}{c|c|c}
\theta & \theta^0_{00} & \alpha^r, s \\
\theta^0_{01} & \theta^0_{10} & \theta^1_{00} \\
\theta^r, s & \theta^0_{u0} & \theta^p_{u0} \\
\theta^0_{v0} & \theta^0_{01} & \theta^1_{01} \\
\theta^r, s & \theta^0_{v0} & \theta^p_{v0} \\
\theta^0_{c} & \theta^0_{u0} & \theta^1_{u0} \\
\theta^r, s & \theta^0_{c} & \theta^p_{c} \\
\end{array}
\]

The equality \( \theta = \delta \alpha + w \) corresponds to the following relations:

\[
\begin{align*}
\theta^{p, q} &= \bar{\partial} \alpha^{p-1, q-1} \\
(-1)^{r+s} \delta \alpha^{r, s} &= \bar{\partial} \alpha^{r, s-1} + \partial \alpha^{r-1, s} \quad \forall 1 \leq r \leq p - 1, 1 \leq s \leq q - 1 \\
(-1)^s \delta \alpha^{0, s} &= \bar{\partial} \alpha^{0, s-1} + \theta^{0, s} \quad \forall 1 \leq s \leq q - 1 \\
(-1)^r \delta \alpha^{r, 0} &= \theta^{r, 0} + \partial \alpha^{r-1, 0} \quad \forall 1 \leq r \leq p - 1 \\
\xi \alpha^{0, 0} &= \theta^{0, 0} + \theta^{0, 0} \\
\bar{\delta} \theta^{0, 0} &= \theta^c \\
\end{align*}
\]

Note that these relations involve relations of the hypercocycles for \( \theta_u \) and \( \theta_v \):

\[
(-1)^r \delta \theta^{r, 0} = \partial \theta^{r-1, 0} \quad \forall 1 \leq r \leq p - 1, \quad (-1)^s \delta \theta^{0, s} = \bar{\partial} \theta^{0, s-1} \quad \forall 1 \leq s \leq q - 1.
\]

If \( q = 0 \), we simply have:

\[
\theta \longleftrightarrow (\theta^c, \theta^0_{u0}, \ldots, \theta^p_{u0})
\]

with the relations:

\[
\begin{align*}
\theta^{p, 0} &= \partial \theta^{p-1, 0}, \quad (-1)^r \delta \theta^{r, 0} = \partial \theta^{r-1, 0} \quad \forall 1 \leq r \leq p - 1, \quad \delta \theta^{0, 0} = \theta^c.
\end{align*}
\]
Similarly, if \( p = 0 \), we have

\[
\theta \longleftrightarrow (\theta^0, \theta^{0,0}, \ldots, \theta^{0,q-1})
\]

with the relations:

\[
\theta^{0,q} = -\delta \theta^{0,q-1}, \quad (\theta^{0,0})^* = \partial \theta^{0,s-1} \quad \forall 1 \leq r \leq p - 1, \quad -\delta \theta^{0,0} = \theta_c.
\]

Similarly, let \([\theta]\) be an element of \( H^{p,q}_{BC}(X_n/B_n)\), then it can be represented by a Čech hypercocycle \( \theta_u, \theta_v \) and \( \theta_c \) of \( \tilde{Z}^{p+q}(X, \mathcal{B}_{p,q}^{*}X_n/B_n)\) with the relations:

\[
(\theta^{0,0})^* = \partial \theta^{0,s-1} \quad \forall 1 \leq s \leq q - 1.
\]

\[
\theta^{0,0} = \theta_c, \quad -\delta \theta^{0,0} = \theta_c;
\]

while for an element \([\theta]\) of \( H^{p,q}_{A}(X_n/B_n)\), it can be represented by a Čech hypercocycle \( \theta_u \) and \( \theta_v \) of \( \tilde{Z}^{p+q+1}(X, \mathcal{B}_{p+1,q+1}^{*}X_n/B_n)\) with the relations:

\[
(\theta^{0,0})^* = \partial \theta^{0,s-1} \quad \forall 1 \leq s \leq q,
\]

\[
\theta^{0,0} = \theta_c, \quad -\delta \theta^{0,0} = \theta_c.
\]

Before the end of this subsections, we will introduce some important maps which will be used in the computation in §4.

Define

\[
\partial_{X_n/B_n}^{\mathcal{B}} : H^{\bullet}(X_n, \Omega_{X_n/B_n}^{p-1,\omega}) \rightarrow H^{\bullet+1}(X_n, \mathcal{B}_{p,q}^{*}X_n/B_n)
\]

in the following way:

Let \([\theta]\) be an element of \( H^{\bullet}(X_n, \Omega_{X_n/B_n}^{p-1,\omega})\) then \( \theta \) can be represented by a cocycle of \( \tilde{Z}^{\bullet}(X, \Omega_{X_n/B_n}^{p-1,\omega})\), we define \( \partial_{X_n/B_n}^{\mathcal{B}}([\theta]) \) to be the cohomology class associated to the hypercocycle in \( \tilde{Z}^{\bullet+1}(X, \mathcal{B}_{p,q}^{*}X_n/B_n)\) given by \( \theta^{p-1,0} = \theta, \theta^{r,0} = 0 \quad \forall 0 \leq r \leq p - 2 \) and \( \theta^{0,r} = 0 \quad \forall 0 \leq r \leq q - 1, \theta_c = 0 \).

**Lemma 2.1.** \( \partial_{X_n/B_n}^{\mathcal{B}} \) is well defined.

**Proof.** It is easy to check that the hypercochain given by \( \theta_u, \theta_v \) and \( \theta_c \) is a hypercocycle. On the other hand if there exists a cochain \( \alpha' \) in \( C^{\bullet-1}(X, \Omega_{X_n/B_n}^{p-1,\omega})\) such that \( \bar{\delta} \alpha' = \theta \), then if we take a hypercochain \( \alpha \) in \( \bar{C}^{\bullet+1}(X, \mathcal{B}_{p,q}^{*}X_n/B_n)\)
given by \( \alpha_u^{p-1,0} = (-1)^{p-1} \alpha' \), \( \alpha_u^{0,0} = 0 \) \( \forall 0 \leq r \leq p - 2 \), \( \alpha_u^{0,r} = 0 \) \( \forall 0 \leq r \leq q - 1 \) and \( \alpha_c = 0 \) we have \( \delta \alpha = \partial B_{X_n/B_n} (\theta|\theta) \). Therefore \( \partial B_{X_n/B_n} (\theta|\theta) = 0 \). \( \square \)

Similarly, we can define

\[
\partial B_{X_n/B_n} : H^\bullet (\bar{X}_n, \Omega^{p-1,\omega}) \to \mathbb{H}^{\bullet+q} (X_n, \mathcal{B}_{p,q;X_n/B_n}^\omega)
\]

in the following way:

Let \( [\theta] \) be an element of \( H^\bullet (\bar{X}_n, \Omega^{p-1,\omega}) \) then \( \theta \) can be represented by a cocycle of \( \bar{Z}^\bullet (\bar{X}, \Omega^{p-1,\omega}_{X_n/B_n}) \), we define \( \partial B_{X_n/B_n} (\theta|\theta) \) to be the cohomology class associated to the hypercocyle in \( \bar{Z}^{q+\bullet} (X, \mathcal{B}_{p,q;X_n/B_n}^\omega) \) given by \( \theta^0_q - 1 = \theta, \theta^0_r = 0 \) \( \forall 0 \leq r \leq q - 2 \), \( \theta_u^0 = 0 \) \( \forall 0 \leq r \leq p - 1 \) and \( \theta_c = 0 \). This map is also well defined and the proof is just as lemma 2.1.

Define

\[
\partial B_{X_n/B_n} : \mathbb{H}^{\bullet+q} (X_n, \mathcal{B}_{p,q;X_n/B_n}^\omega) \to H^\bullet (X_n, \Omega^{p,\omega}_{X_n/B_n})
\]

in the following way:

Let \( [\theta] \) be an element of \( \mathbb{H}^{\bullet+q} (X_n, \mathcal{B}_{p,q;X_n/B_n}^\omega) \) then \( \theta \) can be represented by a hypercocycle of \( \bar{Z}^{\bullet+\bullet} (X, \mathcal{B}_{p,q;X_n/B_n}^\omega) \), we define \( \partial B_{X_n/B_n} (\theta|\theta) \) to be the cohomology class associated to the cocycle in \( \bar{Z}^\bullet (X, \Omega^{p,\omega}_{X_n/B_n}) \) given by \( \partial X_n/B_n \theta_u^{p-1,0} \).

**Lemma 2.2.** \( \partial B_{X_n/B_n} \) is well defined.

**Proof.** At first we need to check the cochain given by \( \partial X_n/B_n \theta_u^{p-1,0} \) is a cocycle. In fact, since \( \theta \) is a hypercocycle in \( \bar{Z}^{\bullet+\bullet} (X, \mathcal{B}_{p,q;X_n/B_n}^\omega) \), we have \((-1)^{p-1} \partial \theta_u^{p-1,0} = \partial X_n/B_n \theta_u^{p-2,0} \), therefore, \( \delta \partial X_n/B_n \theta_u^{p-1,0} = (-1)^p \partial X_n/B_n \circ \partial X_n/B_n \theta_u^{p-2,0} = 0 \).

On the other hand if there exists a cochain \( \alpha \) in \( \bar{C}^{p-1,0} (X, \mathcal{B}_{p,q;X_n/B_n}^\omega) \) such that \( \delta \alpha = \theta \), then if we take a cochain \( \alpha' \) in \( \bar{C}^{p+1,0} (X, \mathcal{B}_{p,q;X_n/B_n}^\omega) \) given by \( \alpha' = (-1)^p \partial X_n/B_n \alpha_u^{p-1,0} \), we have \( \delta \alpha' = (-1)^p \delta \partial X_n/B_n \alpha_u^{p-1,0} = (-1)^{p+1} \partial X_n/B_n \delta \alpha_u^{p-1,0} = (-1)^{p+1+p-1} \partial X_n/B_n \theta_u^{p-1,0} = \partial B_{X_n/B_n} (\theta|\theta) \). Therefore \( \partial B_{X_n/B_n} (\theta|\theta) = 0 \). \( \square \)

Similarly, we can define

\[
\bar{\partial} B_{X_n/B_n} : \mathbb{H}^{\bullet+q} (X_n, \mathcal{B}_{p,q;X_n/B_n}^\omega) \to H^\bullet (\bar{X}_n, \Omega^{p,\omega}_{X_n/B_n})
\]
in the following way:

Let \([\theta]\) be an element of \(\mathbb{H}^{*+q}(X_n, \mathcal{B}^\bullet_{p,q}; X_n/B_n)\) then \(\theta\) can be represented by a hypercocycle of \(\tilde{Z}^{*+q}(X, \mathcal{B}^\bullet_{p,q}; X_n/B_n)\), we define \(\bar{\partial}^{\mathcal{B},\theta}_{X_n/B_n}([\theta])\) to be the cohomology class associated to the cocycle in \(\tilde{Z}^\bullet(X, \Omega^{p,\omega}_{X_n/B_n})\) given by \(\bar{\partial}^{\mathcal{B},\theta}_{X_n/B_n} \theta^0,q-1\). This map is also well defined and the proof is just as lemma 2.2.

**Remark 2.3.** The natural maps from \(H^{p,q}_{BC}(X_n/B_n)\) to \(H^q(X_n, \Omega^{p,\omega}_{X_n/B_n})\) and from \(H^q(X_n, \Omega^{p,\omega}_{X_n/B_n})\) to \(H^{p,q}_A(X_n/B_n)\) mentioned in \(\S 2.1\) respectively are exactly the map:

\[
\partial^{\mathcal{B},\theta}_{X_n/B_n} : \mathbb{H}^{q+p}(X_n, \mathcal{B}^\bullet_{p,q}; X_n/B_n) (\cong H^{p,q}_{BC}(X_n/B_n)) \to H^q(X_n, \Omega^{p,\omega}_{X_n/B_n}),
\]

\[
\partial^{\mathcal{B},\theta}_{X_n/B_n} : H^q(X_n, \Omega^{p,\omega}_{X_n/B_n}) \to \mathbb{H}^{q+p+1}(X_n, \mathcal{B}^\bullet_{p+1,q+1}; X_n/B_n) (\cong H^{p,q}_A(X_n/B_n)).
\]

and we denote these maps by \(r^{BC,\theta}\) and \(r^{\partial,A}\).

We also denote the following two maps:

\[
\partial^{\mathcal{B},\theta}_{X_n/B_n} : H^q(X_n, \Omega^{p-1,\omega}_{X_n/B_n}) \to \mathbb{H}^{q+p}(X_n, \mathcal{B}^\bullet_{p,q}; X_n/B_n) (\cong H^{p,q}_{BC}(X_n/B_n)),
\]

\[
\partial^{\mathcal{B},\theta}_{X_n/B_n} : \mathbb{H}^{q+p+1}(X_n, \mathcal{B}^\bullet_{p+1,q+1}; X_n/B_n) (\cong H^{p,q}_A(X_n/B_n)) \to H^q(X_n, \Omega^{p+1,\omega}_{X_n/B_n}).
\]

by \(\partial^{\mathcal{B},\partial}_{X_n/B_n}\) and \(\partial^{A,\theta}_{X_n/B_n}\).

The following lemma is an important observation which will be used for the computation in \(\S 4\).

**Lemma 2.4.** Let \([\theta]\) be an element of \(\mathbb{H}^l(X_n, \mathcal{B}^\bullet_{p,q}; X_n/B_n)\) which is represented by an element \(\theta\) in \(\tilde{Z}^l(X, \mathcal{B}^\bullet_{p,q}; X_n/B_n)\) given by \(\theta_u, \theta_v\) and \(\theta_c\), then \(\partial_{X_n/B_n}(\theta - \theta_{u}^{p-1,0})\) is a hypercoboundary.

**Proof.** The hypercochian \(\partial_{X_n/B_n}(\theta - \theta_{u}^{p-1,0})\) is given by \((\partial_{X_n/B_n}(\theta - \theta_{u}^{p-1,0}))r,0 = \partial_{X_n/B_n}(\theta_{u}^{p-1,0}, \forall 0 < r < p - 1, (\partial_{X_n/B_n}(\theta - \theta_{u}^{p-1,0}))0,0 = 0, (\partial_{X_n/B_n}(\theta - \theta_{u}^{p-1,0}))0,r = 0, \forall 0 \leq r \leq q - 1\) and \((\partial_{X_n/B_n}(\theta - \theta_{u}^{p-1,0}))c = 0\). Let \(\alpha\) be the hypercochian in \(\mathcal{C}^l(X, \mathcal{B}^\bullet_{p,q}; X_n/B_n)\) given by \(\alpha_u^{2r,0} = 0, \forall 0 \leq r < p/2, \alpha_u^{2r-1,0} = \theta_{u}^{2r-1,0}, \forall 0 < r \leq p/2, \alpha_c^{0,r} = 0, \forall 0 \leq r \leq q - 1\) and \(\alpha_c = 0\) and it is easy to see that \(\tilde{\alpha} = \partial_{X_n/B_n}(\theta - \theta_{u}^{p-1,0})\). Therefore \(\partial_{X_n/B_n}(\theta - \theta_{u}^{p-1,0})\) is a hypercoboundary. \(\square\)
Similarly, we have

**Lemma 2.5.** Let $[\theta]$ be an element of $\mathbb{H}^l(X_n, \mathcal{B}_{p,q}^\bullet X_n/B_n)$ which is represented by an element $\theta$ in $\check{Z}^l(X, \mathcal{B}_{p,q}^\bullet X_n/B_n)$ given by $\theta_u$, $\theta_v$ and $\theta_c$, then $\bar{\partial}X_n/B_n(\theta - \theta^0_{v,q-1})$ is a hypercoboundary.

3. The Jumping Phenomenon and Obstructions

There is a Hodge theory also for Bott-Chern and Aeppli cohomologies, see [7]. More precisely, fixed a Hermitian metric on $X$, one has that

$$H_{BC}^{\bullet,\bullet}(X) \simeq \ker \tilde{\Delta}_{BC} \quad \text{and} \quad H_A^{\bullet,\bullet}(X) \simeq \ker \tilde{\Delta}_A,$$

where

$$\tilde{\Delta}_{BC} := (\partial \bar{\partial}) (\partial \bar{\partial})^* + (\partial \bar{\partial})^* (\partial \bar{\partial}) + \left( \bar{\partial}^* \partial \right) \left( \bar{\partial}^* \partial \right)^* + \left( \bar{\partial}^* \partial \right)^* \left( \bar{\partial}^* \partial \right) + \partial^* \partial + \partial^* \partial$$

and

$$\tilde{\Delta}_A := \partial \partial^* + \partial \partial^* + (\partial \bar{\partial})^* (\partial \bar{\partial}) + (\partial \bar{\partial}) (\partial \bar{\partial})^* + (\partial \partial^*)^* (\partial \partial^*) + (\partial \partial^*) (\partial \partial^*)^*$$

are 4-th order elliptic self-adjoint differential operators. In particular, one gets that

$$\dim_{\mathbb{C}} H_z^{\bullet,\bullet}(X) < +\infty \quad \text{for } z \in \{ \bar{\partial}, \partial, BC, A \}.$$

Let $\pi: \mathcal{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where $X$ is a compact complex manifold and $B$ is a neighborhood of the origin in $\mathbb{C}$. Note that $h_{BC}^{p,q}(X(t))$ and $h_A^{p,q}(X(t))$ are semi-continuous functions of $t \in B$ where $X(t) = \pi^{-1}(t)$ [7]. Denote the $\tilde{\Delta}_{BC}$ operator and the $\tilde{\Delta}_A$ on $X(t)$ by $\tilde{\Delta}_{BC,t}$ and $\tilde{\Delta}_{A,t}$. From the prove of the semi-continuity of $h_{BC}^{p,q}(X(t))$ ($h_A^{p,q}(X(t))$) on [7], we can see that $h_{BC}^{p,q}(X(t))$ ($h_A^{p,q}(X(t))$) does not jump at the point $t = 0$ if and only if all the $\tilde{\Delta}_{BC,0}(\tilde{\Delta}_{A,0})$-harmonic forms on $X$ can be extended to relative $\tilde{\Delta}_{BC,t}(\tilde{\Delta}_{A,t})$-harmonic forms on a neighborhood of $0 \in B$ which is real analytic in the direction of $B$, since the $\tilde{\Delta}_{BC,t}(\tilde{\Delta}_{A,t})$ varies real analytic on $B$. The above condition is equivalent to the following: all the cohomology classes $[\theta]$ in $H_{BC}^{p,q}(X)(H_A^{p,q}(X))$ can be extended to a relative $d_t-closed( \partial_t \bar{\partial}_t-closed)$ forms $\theta(t)$ such that $[\theta(t)] \neq 0$ on a neighborhood
of $0 \in B$ which is real analytic on the direction of $B$. Therefore in order to study the jumping phenomenon, we need to study the extension obstructions. So we need to study the obstructions of the extension of the cohomology classes in $\mathbb{H}^*(X, \mathcal{B}_{p,q}^*)$ to a relative cohomology classes in $\mathbb{H}^*(X_n, \mathcal{B}_{p,q,X_n/B_n}^*)$. We denote the following complex

$$
\pi^{-1}(m_0^\omega/(m_0^\omega)^{n+1}) \xrightarrow{\rightarrow} M_0^\omega/(M_0^\omega)^{n+1} \otimes \mathcal{O}_{X_n}^\omega \oplus \tilde{M}_0^\omega/(\tilde{M}_0^\omega)^{n+1} \otimes \mathcal{O}_{X_n}^\omega
$$

$$
\rightarrow M_0^\omega/(M_0^\omega)^{n+1} \otimes \Omega^1_{X_n/B_n} \oplus \tilde{M}_0^\omega/(\tilde{M}_0^\omega)^{n+1} \otimes \Omega^1_{X_n/B_n}
$$

$$
\rightarrow \ldots M_0^\omega/(M_0^\omega)^{n+1} \otimes \Omega^{p-1}_{X_n/B_n} \oplus \tilde{M}_0^\omega/(\tilde{M}_0^\omega)^{n+1} \otimes \Omega^{p-1}_{X_n/B_n}
$$

$$
\rightarrow M_0^\omega/(M_0^\omega)^{n+1} \otimes \Omega^{n-1}_{X_n/B_n} \rightarrow \ldots \rightarrow M_0^\omega/(M_0^\omega)^{n+1} \otimes \Omega^{n-1}_{X_n/B_n} \rightarrow 0,
$$

by $M_0^\omega/(M_0^\omega)^{n+1} \otimes B_{p,q,X_n/B_n}^*$

Now we consider the following exact sequences:

$$
0 \rightarrow M_0/M_0^{n+1} \otimes B_{p,q,X_n/B_n}^* \rightarrow B_{p,q,X_n/B_n}^* \rightarrow B_{p,q,X_0/B_0}^* \rightarrow 0
$$

which induces a long exact sequence

$$
0 \rightarrow \mathbb{H}^0(X_n, M_0/M_0^{n+1} \otimes B_{p,q,X_n/B_n}^*) \rightarrow \mathbb{H}^0(X_n, B_{p,q,X_n/B_n}^*) \rightarrow \mathbb{H}^0(X, B_{p,q,X_0/B_0}^*)
$$

$$
\rightarrow \mathbb{H}^1(X_n, M_0/M_0^{n+1} \otimes B_{p,q,X_n/B_n}^*) \rightarrow \ldots.
$$

Let $[\theta]$ be a cohomology class in $\mathbb{H}^0(X, B_{p,q,X_0/B_0}^*)$. The obstruction for the extension of $[\theta]$ to a relative cohomology classes in $\mathbb{H}^0(X_n, B_{p,q,X_n/B_n}^*)$ comes from the non trivial image of the connecting homomorphism $\delta^*: \mathbb{H}^0(X, B_{p,q,X_0/B_0}^*) \rightarrow \mathbb{H}^1(X_n, M_0/M_0^{n+1} \otimes B_{p,q,X_n/B_n}^*)$. We denote this obstruction by $o_n([\theta])$. On the other hand, for a given real direction $\frac{\partial}{\partial x}$ on $B$, if there exits $n \in \mathbb{N}$, such that $o_i([\theta]) = 0$, $\forall i \leq n$ and $o_n([\theta]) \neq 0$. Let $\theta_{n-1}$ be a $n-1$ th order extension of $\theta$ to a relative cohomology classes in $\mathbb{H}^0(X_{n-1}, B_{p,q,X_{n-1}/B_{n-1}}^*)$. $\tilde{\delta}\theta_{n-1} = 0$ up to order $n-1$. Now if we consider $\hat{\delta}\theta_{n-1}/x^n$, it is easy to check that $\hat{\delta}\theta_{n-1}/x^n$ is an extension of a non trivial cohomology classes $[\hat{\delta}\theta_{n-1}/x^n(0)]$ in $\mathbb{H}^0(X, B_{p,q})$ while $[\hat{\delta}\theta_{n-1}/x^n(x_0)]$ is trivial in $X(x_0)$ as a cohomology classes in $\mathbb{H}^0(X(x_0), B_{p,q,x_0})$ if $x_0 \neq 0$. From the above discussion, we have the following theorem:
Theorem 3.1. Let \( \pi : X \rightarrow B \) be a small deformation of the central fibre compact complex manifold \( X \). Now we consider \( \dim \mathbb{H}^l(X(t), B_{p,q}) \) as a function of \( t \in B \). It jumps at \( t = 0 \) if there exists an element \([\theta]\) either in \( \mathbb{H}^l(X, B_{p,q}) \) or in \( \mathbb{H}^{l-1}(X, B_{p,q}) \) and a minimal natural number \( n \geq 1 \) such that the \( n \)-th order obstruction

\[
o_n([\theta]) \neq 0.
\]

4. The Formula for the Obstructions

Since these obstructions we discussed in the previous section are so important when we consider the problem of jumping phenomenon of Bott-Chern cohomology and Aeppli cohomology, we try to find out an explicit calculation for such obstructions in this section. As we had done in [G], we need some preparation. Cover \( X \) by open sets \( U_i \) such that, for arbitrary \( i \), \( U_i \) is small enough. More precisely, \( U_i \) is stein and the following exact sequence splits

\[
0 \rightarrow \pi^*(\Omega_{B_n})^\omega(U_i) \rightarrow \Omega_{X_n}^\omega(U_i) \rightarrow \Omega_{X_n/B_n}^\omega(U_i) \rightarrow 0;
\]

\[
0 \rightarrow \pi^*(\Omega_{B_n})^\omega(U_i) \rightarrow \Omega_{X_n}^\omega(U_i) \rightarrow \Omega_{X_n/B_n}^\omega(U_i) \rightarrow 0.
\]

So we have a map \( \varphi_i : \Omega_{X_n/B_n}^\omega(U_i) \oplus \Omega_{X_n/B_n}^\omega(U_i) \rightarrow \Omega_{X_n}^\omega(U_i) \oplus \Omega_{X_n}^\omega(U_i) \), such that \( \varphi_i|_{\Omega_{X_n/B_n}^\omega(U_i)}(\Omega_{X_n/B_n}^\omega(U_i)) \oplus \pi^*(\Omega_{B_n})^\omega(U_i) \cong \Omega_{X_n}^\omega(U_i) \) and \( \varphi_i|_{\Omega_{X_n/B_n}^\omega(U_i)}(\Omega_{X_n/B_n}^\omega(U_i)) \oplus \pi^*(\Omega_{B_n})^\omega(U_i) \cong \Omega_{X_n}^\omega(U_i) \). This decomposition determines a local decomposition of the exterior differentiation \( \partial_X \) in \( \Omega_{X_n}^\omega \) (resp. \( \Omega_{X_n}^\omega \)) on each \( U_i \)

\[
\partial_X = \partial^i_{X_n/B_n} + \partial^i_{X_n/B_n}(\text{resp. } \partial_X = \partial^i_{X_n/B_n} + \partial^i_{X_n/B_n}).
\]

By definition, \( \partial_{X_n/B_n} \) and \( \partial_{X_n/B_n} \) are given by \( \varphi_i^{-1} \partial_{X_n/B_n} \varphi_i \) and \( \varphi_i^{-1} \partial_{X_n/B_n} \varphi_i \).

Denote the set of alternating \( q \)-cochains \( \beta \) with values in \( F \) by \( C^q(U, F) \), i.e. to each \( q + 1 \)-tuple, \( i_0 < i_1 < \ldots < i_q \), \( \beta \) assigns a section \( \beta(i_0, i_1, \ldots, i_q) \) of \( F \) over \( U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_q} \).
Let us still using $\varphi_i$ denote the following map,

$$
\varphi_i : \pi_n^*(\Omega_{B_n})^\omega \land \Omega^{p+1}\omega_{X_n/B_n}(U_i) \oplus \pi_n^*(\Omega_{B_n})^\omega \land \bar{\Omega}^{p+1}\omega_{X_n/B_n}(U_i) \rightarrow \Omega^{p+1}\omega_{X_n}(U_i) \oplus \Omega^{p+1}\omega_{X_n}(U_i)
$$

$$
\varphi_i(\omega_1 \land \beta_1 \land \ldots \land \beta_{i_p} + \omega_2 \land \beta_j^{'} \land \ldots \land \beta_{j_p}^{'}) = \omega_1 \land \varphi_i(\beta_1 \land \ldots \land \varphi_i(\beta_{i_p}^{'})) + \omega_2 \land \varphi_i(\beta_j^{'} \land \ldots \land \varphi_i(\beta_{j_p}^{'})).
$$

Define $\varphi : \check{C}_q(U, \pi_n^*(\Omega_{B_n})^\omega \land \Omega^{p+1}\omega_{X_n/B_n} \oplus \pi_n^*(\Omega_{B_n})^\omega \land \bar{\Omega}^{p+1}\omega_{X_n/B_n}) \rightarrow \check{C}_q(U, \Omega^{p+1}\omega_{X_n} \oplus \Omega^{p+1}\omega_{X_n})$ by

$$
\varphi(\beta)(i_0, i_1, \ldots, i_q) = \varphi_{i_0}(\beta(i_0, i_1, \ldots, i_q)) \quad \forall \beta \in \check{C}_q(U, \pi_n^*(\Omega_{B_n})^\omega \land \Omega^{p+1}\omega_{X_n/B_n} \oplus \pi_n^*(\Omega_{B_n})^\omega \land \bar{\Omega}^{p+1}\omega_{X_n/B_n}),
$$

where $i_0 < i_1 \ldots < i_q$.

Define the total Lie derivative with respect to $B_n$

$$
L_{B_n} : \check{C}_q(U, \Omega^{p+1}\omega_{X_n} \oplus \Omega^{p+1}\omega_{X_n}) \rightarrow \check{C}_q(U, \Omega^{p+1}\omega_{X_n} \oplus \Omega^{p+1}\omega_{X_n})
$$

by

$$
L_{B_n}(\beta)(i_0, i_1, \ldots, i_q) = \partial_{B_n}^{i_0}(\beta(i_0, i_1, \ldots, i_q)) \quad \forall \beta \in \check{C}_q(U, \Omega^{p+1}\omega_{X_n}),
$$

where $i_0 < i_1 \ldots < i_q$ (see [4]).

Define, for each $U_i$ the total interior product with respect to $B_n$,

$$
I^i : \Omega^{p+1}\omega_{X_n}(U_i) \oplus \Omega^{p+1}\omega_{X_n}(U_i) \rightarrow \Omega^{p+1}\omega_{X_n}(U_i) \oplus \Omega^{p+1}\omega_{X_n}(U_i)
$$

by

$$
I^i(\mu_1 \partial_{X_n} g_1 \land \partial_{X_n} g_2 \land \ldots \land \partial_{X_n} g_p + \mu_2 \partial_{X_n} g_1^{'} \land \partial_{X_n} g_2^{'} \land \ldots \land \partial_{X_n} g_p^{'} ) = \mu_1 \sum_{j=1}^{p} \partial_{X_n} g_1 \land \ldots \land \partial_{X_n} g_{j-1} \land \partial_{B_n}^{i_j}(g_j) \land \partial_{X_n} g_{j+1} \land \ldots \land \partial_{X_n} g_p + \mu_2 \sum_{j=1}^{p} \partial_{X_n} g_1^{'} \land \ldots \land \partial_{X_n} g_{j-1}^{'} \land \partial_{B_n}^{i_j}(g_j^{'}) \land \partial_{X_n} g_{j+1}^{'} \land \ldots \land \partial_{X_n} g_p^{'} .
$$

When $p = 0$, we put $I^i = 0$ (see [4]).

Define $\lambda : \check{C}_q(U, \Omega^{p+1}\omega_{X_n} \oplus \Omega^{p+1}\omega_{X_n}) \rightarrow \check{C}_q(U, \Omega^{p+1}\omega_{X_n} \oplus \Omega^{p+1}\omega_{X_n})$ by

$$
(\lambda/\beta)(i_0, \ldots, i_{q+1}) = (I^{i_0} - I^{i_1})\beta(i_1, \ldots, i_{q+1}) \quad \forall \beta \in \check{C}_q(U, \Omega^{p+1}\omega_{X_n} \oplus \Omega^{p+1}\omega_{X_n}).
$$
An we have the following lemma, the prove is completely the same as lemma 3.1 in [9]:

**Lemma 4.1.**

\[ \lambda \circ \varphi \equiv \delta \circ \varphi - \varphi \circ \delta. \]

With the above preparation, we are ready to study the jumping phenomenon of the dimensions of Bott-Chern or Aeppli cohomology groups, for arbitrary \([\theta]\) belongs to \(H^l(X, \mathcal{B}_{p,q})\), suppose we can extend \([\theta]\) to order \(n - 1\) in \(H^l(X_{n-1}, \mathcal{B}^*_{p,q;X_{n-1}/B_{n-1}})\). Denote such element by \([\theta_{n-1}]\). In the following, we try to find out the obstruction of the extension of \([\theta_{n-1}]\) to \(n\)th order. Consider the exact sequence

\[
0 \to \mathcal{M}^n_0/\mathcal{M}^{n+1}_0 \otimes \mathcal{B}^*_{p,q;X_0/B_0} \to \mathcal{B}^*_{p,q;X_0/B_0} \to \mathcal{B}^*_{p,q;X_{n-1}/B_{n-1}} \to 0
\]

which induces a long exact sequence

\[
0 \to H^0(X_n; \mathcal{M}^n_0/\mathcal{M}^{n+1}_0 \otimes \mathcal{B}^*_{p,q;X_0/B_0}) \to H^0(X_n; \mathcal{B}^*_{p,q;X_0/B_0}) \to H^0(X_{n-1}; \mathcal{B}^*_{p,q;X_{n-1}/B_{n-1}}) \to H^1(X_n; \mathcal{M}^n_0/\mathcal{M}^{n+1}_0 \otimes \mathcal{B}^*_{p,q;X_0/B_0}) \to \ldots
\]

Let \([\theta]\) be a cohomology class in \(H^l(X, \mathcal{B}^*_{p,q;X_0/B_0})\).

The obstruction for \([\theta_{n-1}]\) comes from the non trivial image of the connecting homomorphism \(\delta^*: H^l(X_{n-1}, \mathcal{B}^*_{p,q;X_{n-1}/B_{n-1}}) \to H^{l+1}(X_n; \mathcal{M}^n_0/\mathcal{M}^{n+1}_0 \otimes \mathcal{B}^*_{p,q;X_0/B_0})\).

Now we are ready to calculate the formula for the obstructions. Let \(\tilde{\theta}\) be an element of \(\tilde{C}^l(U, \mathcal{B}^*_{p,q;X_n/B_n})\) such that its quotient image in \(\tilde{C}^l(U, \mathcal{B}^*_{p,q;X_{n-1}/B_{n-1}})\) is \(\theta_{n-1}\). Then \(\delta^*([\theta_{n-1}]) = [\tilde{\delta}(\tilde{\theta})]\) which is an element of

\[
H^{l+1}(X_n; \mathcal{M}^n_0/\mathcal{M}^{n+1}_0 \otimes \mathcal{B}^*_{p,q;X_0/B_0}) \cong m^n_0/m^{n+1}_0 \otimes H^{l+1}(X, \mathcal{B}^*_{p,q;X_0/B_0}).
\]

Denote \(r_n\) the restriction to the space \(X^n_n\) (topological space \(X\) with structure sheaf \(\mathcal{O}^{\omega}_{X_n}\)) and denote the following complex

\[
\pi^{-1}(\Omega^{\omega}_{B_n/B_{n-1}}) \xrightarrow{(+, -)} \pi^{-1}(\Omega_{B_n/B_{n-1}})^{\omega} \otimes \mathcal{O}^{\omega}_{X_n} \oplus \pi^{-1}(\overline{\Omega}_{B_n/B_{n-1}})^{\omega} \otimes \mathcal{O}^{\omega}_{X_n}
\]

\[
\to \pi^{-1}(\Omega_{B_n/B_{n-1}})^{\omega} \wedge \Omega^{1,\omega}_{X_n/B_n} \oplus \pi^{-1}(\overline{\Omega}_{B_n/B_{n-1}})^{\omega} \wedge \overline{\Omega}^{1,\omega}_{X_n/B_n}
\]
\[
\pi_n^*(\Omega_{B_n|B_{n-1}}) \rightarrow \ldots \rightarrow \pi_{n-1}^*(\Omega_{B_n|B_{n-1}}) \rightarrow \Omega_{X_n|X_{n-1}}^\omega \rightarrow \Omega_{X_n|X_{n-1}/B_{n-1}}^\omega \rightarrow 0,
\]

by \(\pi_n^*(\Omega_{B_n|B_{n-1}}) \cap \mathcal{B}_{p,q}^\bullet \). 

In order to give the obstructions an explicit calculation, we need to consider the following map 

\[
\rho : \mathbb{H}^l(X, \mathcal{M}_0^n / \mathcal{M}_0^{n+1} \otimes \mathcal{B}_{p,q}^\bullet ; X_0 / \mathcal{B}_0) \rightarrow \mathbb{H}^l(X_{n-1}, \pi_{n-1}^*(\Omega_{B_n|B_{n-1}}) \cap \mathcal{B}_{p,q}^\bullet ; X_{n-1}/B_{n-1})
\]

which is defined by \(\rho[\sigma] = [\varphi^{-1} \circ r_{X_{n-1}} \circ (L_{B_n} + L_{\tilde{B}_n}) \circ \varphi(\sigma)]\), where \(\varphi^{-1}\) is the quotient maps: \(\mathcal{C}^\bullet(U, \pi_{n-1}^*(\Omega_{B_n|B_{n-1}}) \cap \Omega_{X_n|X_{n-1}}^\omega) \rightarrow \mathcal{C}^\bullet(U, \pi_{n-1}^*(\Omega_{B_n|B_{n-1}}) \cup (\Omega_{X_{n-1}/B_{n-1}}^\omega \cup \Omega_{X_{n-1}/B_{n-1}}^\omega)).\) An we have the following lemmas, the proof is completely the same as lemma 3.2 and lemma 3.3 in [9].

**Lemma 4.2.** The map \(\rho\) is well defined.

**Lemma 4.3.** \(\rho([\delta(\tilde{\theta})])\) is exactly \(\omega_n([\theta])\) in the previous section.

Now consider the following exact sequence. The connecting homomorphism of the associated long exact sequence gives the Kodaira-Spencer class of order \(n\) [1.3.2],

\[
(3.1) \quad 0 \rightarrow \pi_{n-1}^*(\Omega_{B_n|B_{n-1}}) \rightarrow \Omega_{X_n|X_{n-1}}^\omega \rightarrow \Omega_{X_n|X_{n-1}/B_{n-1}}^\omega \rightarrow 0.
\]

By wedge the above exact sequence with \(\Omega_{X_{n-1}/B_{n-1}}^{p-1}\), we get a new exact sequence. The connecting homomorphism of such exact sequence gives us a map from \(H^q(X_{n-1}, \Omega_{X_{n-1}/B_{n-1}}^{p-q})\) to \(H^{q+1}(X_{n-1}, \pi^*(\Omega_{B_n|B_{n-1}}) \cap \Omega_{X_{n-1}/B_{n-1}}^{p-1})\). Denote such map by \(\kappa_{n,q}\), for such map is simply the inner product with the Kodaira-Spencer class of order \(n\). With the above preparation, we are ready to proof the main theorem of this section.

**Theorem 4.4.** Let \(\pi : X \rightarrow B\) be a deformation of \(\pi^{-1}(0) = X\), where \(X\) is a compact complex manifold. Let \(\pi_n : X_n \rightarrow B_n\) be the \(n\)th order deformation of \(X\). For arbitrary \([\theta]\) belongs to \(\mathbb{H}^l(X, \mathcal{B}_{p,q}^\bullet)\), suppose we can extend \([\theta]\) to order \(n - 1\).
in $H^1(X_{n-1}, \mathcal{B}^\bullet_{p,B,B_{X^n_{n-1}/B_{n-1}}})$. Denote such element by $[\theta_{n-1}]$. The obstruction of the extension of $[\theta]$ to $n$th order is given by:

$$o_n([\theta]) = -\bar{\partial}^B_{X_{n-1}/B_{n-1}} \circ \kappa_n \circ \partial^B_{X_{n-1}/B_{n-1}}([\theta_{n-1}]) - \bar{\partial}^B_{X_{n-1}/B_{n-1}} \circ \kappa_n \circ \partial^B_{X_{n-1}/B_{n-1}}([\theta_{n-1}]),$$

where $\kappa_n$ is the $n$th order Kodaira-Spencer class and $\bar{\kappa}_n$ is the $n$th order Kodaira-Spencer class of the deformation $\tilde{\pi} : \mathcal{X} \to \mathcal{B}$. $ar{\partial}^B_{X_{n-1}/B_{n-1}}$, $\partial^B_{X_{n-1}/B_{n-1}}$ and $\partial^B_{X_{n-1}/B_{n-1}}$ are the maps defined in $\S$ 2.

Proof. Note that $o_n([\theta]) = \rho \circ \delta(\bar{\theta}) = [\varphi^{-1} \circ r_{X_{n-1}} \circ (L_{B_n} + L_{B_n}) \circ \varphi \circ \delta(\bar{\theta})]$. Because

$$(L_{B_n} + L_{B_n} + \partial_{X_{n}/B_n} + \bar{\partial}_{X_{n}/B_n}) \circ \bar{\delta} = -\bar{\delta} \circ (L_{B_n} + L_{B_n} + \partial_{X_{n}/B_n} + \bar{\partial}_{X_{n}/B_n}).$$

$$r_{X_{n-1}} \circ (L_{B_n} + L_{B_n}) \circ \varphi \circ \bar{\delta}(\bar{\theta}) \equiv r_{X_{n-1}} \circ (L_{B_n} + L_{B_n}) \circ (\bar{\delta} \circ \varphi - \lambda \circ \varphi)(\bar{\theta})$$

$$\equiv r_{X_{n-1}} \circ (L_{B_n} + L_{B_n}) \circ \bar{\delta} \circ \varphi(\bar{\theta})$$

$$\equiv -r_{X_{n-1}} \circ \bar{\delta} \circ r_{X_{n}} \circ \partial_{X_{n}/B_n} +$$

$$\bar{\partial}_{X_{n}/B_n} \circ \bar{\delta} + \bar{\delta} \circ \bar{\partial}_{X_{n}/B_n} + \bar{\delta} \circ (L_{B_n} + L_{B_n}) \circ \varphi(\bar{\theta})$$

$$\equiv -r_{X_{n-1}} \circ \bar{\delta} \circ r_{X_{n}} \circ \partial_{X_{n}/B_n} +$$

$$\bar{\partial}_{X_{n}/B_n} \circ \varphi(\bar{\theta}) - \bar{\delta} \circ r_{X_{n-1}} \circ (L_{B_n} + L_{B_n}) \circ \varphi(\bar{\theta}).$$

Therefore

$$[\varphi^{-1} \circ r_{X_{n-1}} \circ (L_{B_n} + L_{B_n}) \circ \varphi \circ \delta(\bar{\theta})] = [-\varphi^{-1} \circ r_{X_{n-1}} \circ (\partial_{X_{n}/B_n} + \bar{\delta} \circ \partial_{X_{n}/B_n})$$

$$\circ \bar{\delta} \circ r_{X_{n}} \circ \partial_{X_{n}/B_n}) +$$

$$\bar{\partial}_{X_{n}/B_n} \circ \varphi^{-1} \circ r_{X_{n-1}} \circ \bar{\delta} \circ \varphi(\bar{\theta}) +$$

$$\bar{\partial}_{X_{n}/B_n} \circ \varphi^{-1} \circ r_{X_{n-1}} \circ \bar{\delta} \circ \varphi(\bar{\partial}_{X_{n-1}/B_{n-1}}(\theta_{n-1})) +$$

$$\varphi^{-1} \circ r_{X_{n-1}} \circ \bar{\delta} \circ \varphi(\bar{\partial}_{X_{n-1}/B_{n-1}}(\theta_{n-1})).$$
Since \((\varphi^{-1} \circ r_{X_{n-1}} \circ \tilde{\delta} \circ \varphi(\tilde{\theta}))_{u}^{p-1,0} = 0\) and \((\varphi^{-1} \circ r_{X_{n-1}} \circ \tilde{\delta} \circ \varphi(\tilde{\theta}))_{v}^{0,q-1} = 0\), by Lemma 2.4 and Lemma 2.5, we know that \([\partial_{X_{n-1}/B_{n-1}} \circ \varphi^{-1} \circ r_{X_{n-1}} \circ \tilde{\delta} \circ \varphi(\tilde{\theta})] = 0\) and \([\partial_{X_{n-1}/B_{n-1}} \circ \varphi^{-1} \circ r_{X_{n-1}} \circ \tilde{\delta} \circ \varphi(\tilde{\theta})] = 0\). And from Lemma 2.4 and Lemma 2.5, we also know that

\[
[\varphi^{-1} \circ r_{X_{n-1}} \circ \tilde{\delta} \circ \varphi(\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1}))] = [\varphi^{-1} \circ r_{X_{n-1}} \circ \tilde{\delta} \circ \varphi(\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1} - \theta_{n-1}^{p-1,0})) + \partial_{X_{n-1}/B_{n-1}} \theta_{n-1}^{p-1,0})]
\]

\[
= [\varphi^{-1} \circ r_{X_{n-1}} \circ \tilde{\delta} \circ \varphi(\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1}^{p-1,0}))]
\]

and

\[
[\varphi^{-1} \circ r_{X_{n-1}} \circ \tilde{\delta} \circ \varphi(\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1}^{0,q-1})) + \partial_{X_{n-1}/B_{n-1}} \theta_{n-1}^{0,q-1})]
\]

\[
= [\varphi^{-1} \circ r_{X_{n-1}} \circ \tilde{\delta} \circ \varphi(\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1}^{0,q-1}))]
\]

By the definition of the maps: \(\partial_{X_{n-1}/B_{n-1}}^{\tilde{B}}\), \(\partial_{X_{n-1}/B_{n-1}}^{R,\tilde{\delta}}\) and Lemma 3.4 in [9], we have

\[
[\varphi^{-1} \circ r_{X_{n-1}} \circ \tilde{\delta} \circ \varphi(\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1}^{p-1,0})) = \partial_{X_{n-1}/B_{n-1}}^{\tilde{B}} \circ \tilde{\kappa}_{n-1} \circ \partial_{X_{n-1}/B_{n-1}}^{R,\tilde{\delta}}(\theta_{n-1})]
\]

and similarly, we have

\[
[\varphi^{-1} \circ r_{X_{n-1}} \circ \tilde{\delta} \circ \varphi(\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1}^{0,q-1})) = \partial_{X_{n-1}/B_{n-1}}^{\tilde{B}} \circ \tilde{\kappa}_{n-1} \circ \partial_{X_{n-1}/B_{n-1}}^{R,\tilde{\delta}}(\theta_{n-1})]
\]

So we have:

\[
[\varphi^{-1} \circ r_{X_{n-1}} \circ (L_{B_{n}} + L_{B_{n}}) \circ \varphi \circ \delta(\tilde{\theta})] = -\partial_{X_{n-1}/B_{n-1}}^{\tilde{B}} \circ \tilde{\kappa}_{n-1} \circ \partial_{X_{n-1}/B_{n-1}}^{R,\tilde{\delta}}(\theta_{n-1})]
\]

\[
-\partial_{X_{n-1}/B_{n-1}}^{\tilde{B}} \circ \tilde{\kappa}_{n-1} \circ \partial_{X_{n-1}/B_{n-1}}^{R,\tilde{\delta}}(\theta_{n-1}).
\]

\(\square\)

Apply the above theorem and theorem 3.1 to study the jumping phenomenon of the dimensions of Bott-Chern(Aeppli) cohomology groups. We have the following theorems.
Theorem 4.5. Let $\pi : \mathcal{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where $X$ is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the $n$th order deformation of $X$. If there exists an element $[\theta^1]$ in $H^{p,q}_{BC}(X)$ or an element $[\theta^2]$ in $H^{p-1,q-1}_{\Lambda}(X)$ and a minimal natural number $n \geq 1$ such that the $n$th order obstruction $o_n([\theta^1]) \neq 0$ or $o_n([\theta^2]) \neq 0$, then the $h^{BC}_{p,q}(X(t))$ will jump at the point $t = 0$. And the formulas for the obstructions are given by:

$$o_n([\theta^1]) = -\partial_{X_n/B_n} \circ \kappa_n \circ \partial_{BC,0}([\theta_n]) - \bar{\partial}_{X_n/B_n} \circ \kappa_n \circ \partial_{BC,0}([\theta_n])$$

$$o_n([\theta^2]) = -\partial_{X_n/B_n} \circ \kappa_n \circ \partial_{A,0}([\theta_n]) - \bar{\partial}_{X_n/B_n} \circ \kappa_n \circ \partial_{A,0}([\theta_n])$$

Theorem 4.6. Let $\pi : \mathcal{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where $X$ is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the $n$th order deformation of $X$. If there exists an element $[\theta^1]$ in $H^{p,q}_{\Lambda}(X)$ or an element $[\theta^2]$ in $H^{p+q}_{BC}(X)$ and a minimal natural number $n \geq 1$ such that the $n$th order obstruction $o_n([\theta^1]) \neq 0$ or $o_n([\theta^2]) \neq 0$, then the $h^{BC}_{p,q}(X(t))$ will jump at the point $t = 0$. And the formulas for the obstructions are given by:

$$o_n([\theta^1]) = -\partial_{X_n/B_n} \circ \kappa_n \circ \partial_{BC,0}([\theta_n]) - \bar{\partial}_{X_n/B_n} \circ \kappa_n \circ \partial_{BC,0}([\theta_n])$$

$$o_n([\theta^2]) = -r_{A,0} \circ \kappa_n \circ \partial_{X_n/B_n}([\theta_n]) - \bar{\partial}_{X_n/B_n} \circ \kappa_n \circ \partial_{A,0}([\theta_n])$$

By these theorems, we can get the following corollaries immediately.

Corollary 4.7. Let $\pi : \mathcal{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where $X$ is a compact complex manifold. Suppose that up to order $n$, the maps $r_{BC,0} : H^{p,q}_{BC}(X_n/B_n) \to H^q(X_n, \Omega_{X_n/B_n}^{p,0})$ and $r_{BC,0} : H^{p,q}_{BC}(X_n/B_n) \to H^p(X_n, \Omega_{X_n/B_n}^{0,q})$ is 0. For arbitrary $[\theta]$ that belongs to $H^{p,q}_{BC}(X)$, it can be extended to order $n + 1$ in $H^{p,q}_{BC}(X_{n+1}/B_{n+1})$.

Proof. This result can be shown by induction on $k$.

Suppose that the corollary is proved for $k - 1$, then we can extend $[\theta]$ to and element $[\theta_{k-1}]$ in $H^{p,q}_{BC}(X_{k-1}/B_{k-1})$. By Theorem 4.5, the obstruction for the extension of $[\theta]$ to $k$th order comes from:

$$o_k([\theta]) = -\partial_{X_{k-1}/B_{k-1}} \circ \kappa_n \circ \partial_{BC,0}([\theta_{k-1}]) - \bar{\partial}_{X_{k-1}/B_{k-1}} \circ \kappa_n \circ \partial_{BC,0}([\theta_{k-1}])$$
By the assumption, \( r_{BC,\bar{\partial}} : H^{p,q}_{BC}(X_{k-1}/B_{k-1}) \to H^q(X_{k-1}, \Omega^{p+1}_{X_{k-1}/B_{k-1}}) \) and \( r_{BC,\partial} : H^{p,q}_{BC}(X_{k-1}/B_{k-1}) \to H^p(X_{k-1}, \bar{\Omega}^{p+1}_{X_{k-1}/B_{k-1}}) \) is 0, where \( k \leq n + 1 \). So we have 

\[
\theta \in \mathbb{R}
\]

\[
\theta \in \mathbb{C}
\]

we omit the proof.

Since we have 

\[
\partial_A^{X_n/B_n} : H^q_A(X_n/B_n) \to H^q(X_n, \Omega^{p+1+\omega})
\]

is the composition of \( \partial^{X_n/B_n} : H^q_A(X_n/B_n) \to H^q_{BC}(X_n/B_n) \) and \( r_{BC,\bar{\partial}} : H^{p+1,q}_{BC}(X_n/B_n) \to H^q(X_n, \Omega^{p+1+\omega}) \). With the same proof of the above corollary, we have the following result and we omit the proof.

**Corollary 4.8.** Let \( \pi : X \to B \) be a deformation of \( \pi^{-1}(0) = X \), where \( X \) is a compact complex manifold. Suppose that up to order \( n \), the maps \( r_{BC,\bar{\partial}} : H^{p+1,q}_{BC}(X_n/B_n) \to H^q(X_n, \Omega^{p+1+\omega}) \) and \( r_{BC,\partial} : H^{p+1,q}_{BC}(X_n/B_n) \to H^q(X_n, \Omega^{p+1+\omega}) \) is 0. For arbitrary \( \theta \) that belongs to \( H^{p,q}_A(X) \), it can be extended to order \( n + 1 \) in \( H^{p,q}_A(X_{n+1}/B_{n+1}) \).

5. An Example

In this section, we will use the formula in previous section to study the jumping phenomenon of the dimensions of Bott-Chern cohomology groups \( h_{BC}^{p,q} \) and Aeppli cohomology groups \( h_{A}^{p,q} \) of small deformations of Iwasawa manifold. It was Kodaira who first calculated small deformations of Iwasawa manifold \([6]\). In the first part of this section, let us recall his result.

Set

\[
G = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C} \right\} \cong \mathbb{C}^3,
\]

\[
\Gamma = \left\{ \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} : \omega_i \in \mathbb{Z} + \mathbb{Z}\sqrt{-1} \right\}.
\]
The multiplication is defined by

\[
\begin{pmatrix}
1 & z_2 & z_3 \\
0 & 1 & z_1 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & \omega_2 & \omega_3 \\
0 & 1 & \omega_1 \\
0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & z_2 + \omega_2 & z_3 + \omega_1 z_2 + \omega_3 \\
0 & 1 & z_1 + \omega_1 \\
0 & 0 & 1
\end{pmatrix}.
\]

\(X = G/\Gamma\) is called Iwasawa manifold. We may consider \(X = \mathbb{C}^3/\Gamma\). \(g \in \Gamma\) operates on \(\mathbb{C}^3\) as follows:

\[
z'_1 = z_1 + \omega_1, \quad z'_2 = z_2 + \omega_2, \quad z'_3 = z_3 + \omega_1 z_2 + \omega_3,
\]

where \(g = (\omega_1, \omega_2, \omega_3)\) and \(z' = z \cdot g\). There exist holomorphic 1-forms \(\varphi_1, \varphi_2, \varphi_3\) which are linearly independent at every point on \(X\) and are given by

\[
\varphi_1 = dz_1, \quad \varphi_2 = dz_2, \quad \varphi_3 = dz_3 - z_1 dz_2,
\]

so that

\[
d\varphi_1 = d\varphi_2 = 0, \quad d\varphi_3 = -\varphi_1 \wedge \varphi_2.
\]

On the other hand we have holomorphic vector fields \(\theta_1, \theta_2, \theta_3\) on \(X\) given by

\[
\theta_1 = \frac{\partial}{\partial z_1}, \quad \theta_2 = \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_3}, \quad \theta_3 = \frac{\partial}{\partial z_3}.
\]

It is easily seen that

\[
[\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3, \quad [\theta_1, \theta_3] = [\theta_2, \theta_3] = 0.
\]

In view of Theorem 3 in [6], \(H^1(X, \mathcal{O}_X)\) is spanned by \(\varphi_1, \varphi_2\). Since \(\Theta\) is isomorphic to \(\mathcal{O}^3\), \(H^1(X, T_X)\) is spanned by \(\theta_i \varphi_\lambda, i = 1, 2, 3, \lambda = 1, 2\).
Consider the small deformation of $X$ given by

\[ \psi(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} t_{i\lambda} \bar{\varphi}_\lambda t - (t_{11} t_{22} - t_{21} t_{12}) \theta_3 \bar{\varphi}_3 t^2. \]

We summarize the numerical characters of deformations. The deformations are divided into the following three classes, the classes and subclasses of this classification are characterized by the following values of the parameters (all the details can be found in [2]):

**class (i):** $t_{11} = t_{12} = t_{21} = t_{22} = 0$;

**class (ii):** $D(t) = 0$ and $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)$:

- **subclass (ii.a):** $D(t) = 0$ and rank $S = 1$;
- **subclass (ii.b):** $D(t) = 0$ and rank $S = 2$;

**class (iii):** $D(t) \neq 0$:

- **subclass (iii.a):** $D(t) \neq 0$ and rank $S = 1$;
- **subclass (iii.b):** $D(t) \neq 0$ and rank $S = 2$;

The matrix $S$ is defined by

\[ S := \begin{pmatrix} \sigma_{11} & \sigma_{22} & \sigma_{12} & \sigma_{21} \\ \sigma_{11} & \sigma_{22} & \sigma_{21} & \sigma_{12} \end{pmatrix} \]

where $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22} \in \mathbb{C}$ and $\sigma_{12} \in \mathbb{C}$ are complex numbers depending only on $t$ such that

\[ d\varphi^3_t := \sigma_{12} \varphi^1_t \wedge \varphi^2_t + \sigma_{11} \varphi^1_t \wedge \bar{\varphi}^1_t + \sigma_{12} \varphi^1_t \wedge \bar{\varphi}^2_t + \sigma_{21} \varphi^2_t \wedge \bar{\varphi}^1_t + \sigma_{22} \bar{\varphi}^2_t \wedge \bar{\varphi}^2_t. \]
The first order asymptotic behaviour of $\sigma_{12}, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$ for $t$ near 0 is the following:

\[
\begin{align*}
\sigma_{12} &= -1 + o(|t|) \\
\sigma_{11} &= t_{21} + o(|t|) \\
\sigma_{12} &= t_{22} + o(|t|) \quad \text{for} \quad t \in \text{classes (i), (ii) and (iii)}. \\
\sigma_{21} &= -t_{11} + o(|t|) \\
\sigma_{22} &= -t_{12} + o(|t|)
\end{align*}
\]

The following tables are given by D. Angella in [2].

Dimensions of the cohomologies of the Iwasawa manifold and of its small deformations:

| $H^*_{\text{DR}}$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ |
|-------------------|-------|-------|-------|-------|-------|
| $\mathbb{I}_3$ and (i), (ii), (iii) | 4     | 8     | 10    | 8     | 4     |

| $H^*_{\partial}$ | $h^0_{\partial}$ | $h^1_{\partial}$ | $h^2_{\partial}$ | $h^3_{\partial}$ | $h^0_{\partial}$ | $h^1_{\partial}$ | $h^2_{\partial}$ | $h^3_{\partial}$ | $h^0_{\partial}$ | $h^1_{\partial}$ | $h^2_{\partial}$ | $h^3_{\partial}$ |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $\mathbb{I}_3$ and (i) | 3 | 2 | 3 | 6 | 2 | 1 | 6 | 6 | 1 | 2 | 6 | 3 | 2 | 3 |
| (ii) | 2 | 2 | 2 | 5 | 2 | 1 | 5 | 5 | 1 | 2 | 5 | 2 | 2 | 2 |
| (iii) | 2 | 2 | 1 | 5 | 2 | 1 | 4 | 4 | 1 | 2 | 5 | 1 | 2 | 2 |

| $H^*_{\text{BC}}$ | $h^0_{\text{BC}}$ | $h^1_{\text{BC}}$ | $h^2_{\text{BC}}$ | $h^3_{\text{BC}}$ | $h^0_{\text{BC}}$ | $h^1_{\text{BC}}$ | $h^2_{\text{BC}}$ | $h^3_{\text{BC}}$ | $h^0_{\text{BC}}$ | $h^1_{\text{BC}}$ | $h^2_{\text{BC}}$ | $h^3_{\text{BC}}$ |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $\mathbb{I}_3$ and (i) | 2 | 2 | 3 | 4 | 3 | 1 | 6 | 6 | 1 | 2 | 8 | 2 | 3 | 3 |
| (ii.a) | 2 | 2 | 2 | 4 | 2 | 1 | 6 | 6 | 1 | 2 | 7 | 2 | 3 | 3 |
| (ii.b) | 2 | 2 | 2 | 4 | 2 | 1 | 6 | 6 | 1 | 2 | 6 | 2 | 3 | 3 |
| (iii.a) | 2 | 2 | 1 | 4 | 1 | 1 | 6 | 6 | 1 | 2 | 7 | 2 | 3 | 3 |
| (iii.b) | 2 | 2 | 1 | 4 | 1 | 1 | 6 | 6 | 1 | 2 | 6 | 2 | 3 | 3 |
From the tables above, we know that the jumping phenomenon happens in $h_{BC}^{2,0}$, $h_{BC}^{0,2}$, and $h_{BC}^{2,2}$ of Bott-Chern cohomology and symmetrically happens in $h_{A}^{3,1}$, $h_{A}^{1,3}$ and $h_{A}^{1,1}$ of Aeppli cohomology. Now let us explain the jumping phenomenon of the dimensions of Bott-Chern cohomology and Aeppli cohomology by using the obstruction formula. From §4 in [2], it follows that the Bott-Chern cohomology groups in bi-degree $(2, 0)$, $(0, 2)$, $(2, 2)$ are:

$$H_{BC}^{2,0}(X) = \text{Span}_C \{[\varphi_1 \wedge \varphi_2], [\varphi_2 \wedge \varphi_3], [\varphi_3 \wedge \varphi_1]\};$$

$$H_{BC}^{0,2}(X) = \text{Span}_C \{[\overline{\varphi}_1 \wedge \overline{\varphi}_2], [\overline{\varphi}_2 \wedge \overline{\varphi}_3], [\overline{\varphi}_3 \wedge \overline{\varphi}_1]\};$$

$$H_{BC}^{2,2}(X) = \text{Span}_C \{[\varphi_2 \wedge \varphi_3 \wedge \overline{\varphi}_1 \wedge \overline{\varphi}_2], [\varphi_3 \wedge \varphi_1 \wedge \overline{\varphi}_2 \wedge \overline{\varphi}_3],$$
$$[\varphi_1 \wedge \varphi_2 \wedge \overline{\varphi}_2 \wedge \overline{\varphi}_3], [\varphi_2 \wedge \varphi_3 \wedge \overline{\varphi}_2 \wedge \overline{\varphi}_3], [\varphi_3 \wedge \varphi_1 \wedge \overline{\varphi}_2 \wedge \overline{\varphi}_3],$$
$$[\varphi_1 \wedge \varphi_2 \wedge \overline{\varphi}_3 \wedge \overline{\varphi}_1], [\varphi_2 \wedge \varphi_3 \wedge \overline{\varphi}_3 \wedge \overline{\varphi}_1], [\varphi_3 \wedge \varphi_1 \wedge \overline{\varphi}_3 \wedge \overline{\varphi}_1]\},$$

and the Aeppli cohomology groups in bi-degree $(3, 1)$, $(1, 3)$, $(1, 1)$ are:

$$H_{A}^{3,1}(X) = \text{Span}_C \{[\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \overline{\varphi}_1], [\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \overline{\varphi}_2], [\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \overline{\varphi}_3]\};$$

$$H_{A}^{1,3}(X) = \text{Span}_C \{[\varphi_1 \wedge \overline{\varphi}_1 \wedge \overline{\varphi}_2 \wedge \overline{\varphi}_3], [\varphi_2 \wedge \overline{\varphi}_1 \wedge \overline{\varphi}_2 \wedge \overline{\varphi}_3], [\varphi_3 \wedge \overline{\varphi}_1 \wedge \overline{\varphi}_2 \wedge \overline{\varphi}_3]\};$$

$$H_{A}^{1,1}(X) = \text{Span}_C \{[\varphi_1 \wedge \overline{\varphi}_1], [\varphi_1 \wedge \overline{\varphi}_2], [\varphi_1 \wedge \overline{\varphi}_3], [\varphi_2 \wedge \overline{\varphi}_1],$$
$$[\varphi_2 \wedge \overline{\varphi}_2], [\varphi_2 \wedge \overline{\varphi}_3], [\varphi_3 \wedge \overline{\varphi}_1], [\varphi_3 \wedge \overline{\varphi}_2]\}. $$
For example, let us first consider $h_{BC}^{2,0}$, in the ii) class of deformation. The Kodaira-Spencer class of the this deformation is $\psi_1(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} t_{i\lambda} \theta_i \varphi_\lambda$, and $\tilde{\psi}_1(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} t_{i\lambda} \tilde{\theta}_i \varphi_\lambda$, with $t_{11} t_{22} - t_{21} t_{12} = 0$. It is easy to check that $o_1(\varphi_1 \wedge \varphi_2) = -\partial(int(\psi_1(t))(\varphi_1 \wedge \varphi_2)) - \partial(int(\tilde{\psi}_1(t))(\varphi_1 \wedge \varphi_2)) = 0$, $o_1(t_{11} \varphi_2 \wedge \varphi_3 - t_{21} \varphi_1 \wedge \varphi_3) = -\partial((t_{11} t_{22} - t_{21} t_{12}) \varphi_3 \wedge \bar{\varphi}_2) = 0$, and $o_1(\varphi_2 \wedge \varphi_3) = t_{21} \varphi_1 \wedge \varphi_2 \wedge \bar{\varphi}_1 + t_{22} \varphi_1 \wedge \varphi_2 \wedge \bar{\varphi}_2$, $o_1(\varphi_1 \wedge \varphi_3) = t_{11} \varphi_1 \wedge \varphi_2 \wedge \bar{\varphi}_1 + t_{12} \varphi_1 \wedge \varphi_2 \wedge \bar{\varphi}_2$. Therefore, we have shown that for an element of the subspace $\text{Span}_C\{[\varphi_1 \wedge \varphi_2], [t_{11} \varphi_2 \wedge \varphi_3 - t_{21} \varphi_1 \wedge \varphi_3]\}$, the first order obstruction is trivial, while, since $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)$, at least one of the obstruction $o_1(\varphi_2 \wedge \varphi_3)$, $o_1(\varphi_1 \wedge \varphi_3)$ is non trivial which partly explain why the Hodge number $h_{BC}^{2,0}$ jumps from 3 to 2. For another example, let us consider $h_{A}^{1,1}$, in the ii) class of deformation. It is easy to check that all the first order obstructions of the cohomology classes are trivial. However, if we want to study the jumping phenomenon, we also need to consider the obstructions come from $H^2(X, \mathcal{B}_{2,2}^*)$. It is easy to check that:

$$H^2(X, \mathcal{B}_{2,2}^*) = \text{Span}_C\{[\varphi_3], [\bar{\varphi}_3]\}.$$

and

$$o_1(\varphi_3) = -t_{11} \varphi_2 \wedge \bar{\varphi}_1 - t_{12} \varphi_2 \wedge \bar{\varphi}_1 + t_{21} \varphi_1 \wedge \varphi_1 + t_{22} \varphi_1 \wedge \bar{\varphi}_1;$$

$$o_1(\bar{\varphi}_3) = -\bar{t}_{11} \bar{\varphi}_2 \wedge \varphi_1 - \bar{t}_{12} \bar{\varphi}_2 \wedge \varphi_1 + \bar{t}_{21} \bar{\varphi}_1 \wedge \varphi_1 + \bar{t}_{22} \bar{\varphi}_1 \wedge \bar{\varphi}_1.$$

Note that the first order of $S$ is

$$\begin{pmatrix}
-\bar{t}_{21} & -\bar{t}_{12} & \bar{t}_{22} & \bar{t}_{11} \\
-t_{21} & -t_{12} & t_{11} & t_{22}
\end{pmatrix}$$

If rank of the first order of $S = 1$, then there exists $c_1, c_2$ such that

$$o_1(c_1 \varphi_3 + c_2 \bar{\varphi}_3) \neq 0.$$

If rank of the first order of $S = 2$, then for all $c_1, c_2$

$$o_1(c_1 \varphi_3 + c_2 \bar{\varphi}_3) = 0.$$

and exactly these obstructions make $h_{A}^{1,1}$ jumps from 8 to 7 in (ii.a) and from 8 to 6 in (ii.b).
In the end of the section, we want to give the following observation as an application of the formula.

**Proposition 5.1.** Let $X$ be a non-Kähler nilpotent complex parallelisable manifold whose dimension is more than 2, and $\pi : \mathcal{X} \to B$ be the versal deformation family of $X$. Then the number $h^{1,1}_A$ will jump in any neighborhood of $0 \in B$.

**Proof.** From the proof of [9] proposition 4.2, we know there exists an element $[\theta]$ in $H^0(X, \Omega_X)$ whose $o_1([\theta]) \neq 0$. It is easy to check that $\theta$ also represents an element in $\mathbb{H}^2(X, \mathcal{B}_{2,2}^*)$, let us denote it by $[\theta]_B$ and it is also easy to check that $o_1([\theta]) = o_1([\theta]_B)$ in this case. Therefore the number $h^{1,1}_A$ will jump in any neighborhood of $0 \in B$. 

□

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