SO(3) nonlinear $\sigma$ model for a doped quantum helimagnet

S. Klee$^{1,2}$, A. Muramatsu$^2$

$^1$Institut für Theoretische Physik, Universität Würzburg, 97074 Würzburg, Germany
$^2$Institut für Physik, Universität Augsburg, 86135 Augsburg, Germany

Abstract

A field theory describing the low-energy, long-wavelength sector of an incommensurate, spiral magnetic phase is derived from a spin-fermion model that is commonly used as a microscopic model for high-temperature superconductors. After integrating out the fermions in a path-integral representation, a gradient expansion of the fermionic determinant is performed. This leads to an $O(3)\otimes O(2)$-symmetric quantum nonlinear $\sigma$ model, where the doping dependence is explicitly given by generalized fermionic susceptibilities which enter into the coupling constants of the $\sigma$ model and contain the fermionic band-structure that results from the spiral background. A stability condition of the field theory self-consistently determines the spiral wavevector as a function of the doping concentration. Furthermore, terms of topological nature like the $\theta$-vacuum term in (1+1)-dimensional nonlinear $\sigma$ models are obtained for the plane of the spiral.

PACS numbers: 11.15.Tk, 71.27.+a, 75.10.Jm
I. INTRODUCTION

The properties of doped magnetic systems have been intensively studied in the context of strongly correlated electronic systems like heavy-fermion compounds and cuprate superconductors. The coexistence of local magnetic moments and itinerant fermions in these materials has raised many questions concerning the interplay between magnetic and charge degrees of freedom close to a magnetic instability. A powerful tool for a quantitative investigation of the critical behaviour of given microscopic models is provided by renormalization group studies of the corresponding continuum theories.

The field-theoretical approach has led to a successful description of the two-dimensional (2D) Heisenberg quantum antiferromagnet, where even a quantitative comparison with experimental results for the undoped parent compounds of the cuprates was achieved [1]. Results obtained from studies of the appropriate continuum theory, the O(3) (2+1)-dimensional quantum nonlinear $\sigma$ model, through renormalization group analysis [1] and chiral perturbation theory [2], agree very well with experimental [3] and numerical [4] data.

More recently, field theories for frustrated 2D Heisenberg quantum antiferromagnets like the one on a triangular lattice [5–8] were derived. Since the starting point is the corresponding Néel state with a noncollinear order, the field theory is a (2+1)-dimensional SO(3) quantum nonlinear $\sigma$ model, as is generally expected for spiral states. Renormalization group analyses were performed for this model [7,9] as well as for the classical $\text{O}(3) \otimes \text{O}(2)$ nonlinear $\sigma$ model in $(2 + \epsilon)$ dimensions [10,11]. Furthermore, the scaling properties of the quantum phase transition from the ordered helical to the disordered phase were described [12], based on scaling arguments and on a scenario of deconfined spinons. An understanding of the critical properties of frustrated spin systems between two and four dimensions was also achieved recently [13]. Thus, a fairly large amount of theoretical predictions was obtained in recent years in the case of pure quantum spin systems.

However, the most interesting and relevant situation of strongly correlated systems, where magnetic as well as charge degrees of freedom interact, was until now investigated to a much lesser extent. Apart from numerous mean-field attempts to analyze those systems, to the best knowledge of the authors only few field-theories have been derived from microscopic models so far [14–17], besides phenomenological approaches [18], where fluctuations effects are duly taken into account.

The studies above were performed for holes in an antiferromagnetic background, such that no equivalent descriptions are available yet for the region of the phase diagram of strongly correlated systems, like some of the cuprates, in which incommensurate spin fluctuations appear. For doping concentrations in the superconducting regime, neutron scattering experiments have shown [19,20] that the Lanthanum compounds exhibit peaks in the magnetic scattering intensity at wave vectors which are shifted by $\pm \delta(\pi, 0)$ and $\pm \delta(0, \pi)$ from...
the antiferromagnetic point, where the shift δ increases with the doping. The correlation length of the incommensurate fluctuations has been found to be considerably larger than the lattice spacing, which makes a continuum approach reasonable. Our aim here is to establish a mathematically rigorous connection between a suitable, sufficiently general microscopic model and a continuum theory. Considering fluctuations around a helical spin configuration, we will develop a field theory that, in conjunction with the above-mentioned recent progress in renormalization group studies, leads to a quantitative description of the critical behaviour of an incommensurate spin system interacting with doped charge carriers.

Previous theoretical studies of microscopic models of strongly correlated fermions suggest that the competition of hopping and exchange effects may lead to the formation of a spiral phase in the spin subsystem. Mean-field calculations favour spiral against antiferromagnetic configurations in Hubbard models for small but non-zero doping [14,21]. Concentrating on the one-hole case, Shraiman and Siggia proposed spiral spin ordering based on a phenomenological continuum description of the t-J model [22]. They argued that the introduction of a low density of doped holes into a locally antiferromagnetic background may lead to an incommensurate, helical rotation of the staggered magnetization as a consequence of a coupling between the spin current of the holes and the magnetization current of the background. Such a coupling was in fact derived in a continuum action obtained directly from the spin-fermion model [17]. Later on, the phase diagram of the Shraiman-Siggia model was investigated within a \( \frac{1}{N} \) expansion [23]. A transition from the commensurate to an incommensurate magnetic phase was obtained with increasing doping, both in the ordered and in the quantum-disordered regime for suitable values of the Shraiman-Siggia coupling. However, apart from the fact that the starting model is a phenomenological one, it is not clear to us to which extent those results really apply to the case of a spiral phase, since the conclusions on the critical behavior are derived on the basis of an order-parameter in the manifold \( S^2 \) and not SO(3), as it should be in the non-collinear case. Finally, it should be mentioned that incommensurate spin configurations have further been shown to be induced by doping in numerical simulations of the one-band and the three-band Hubbard model away from half-filling [24,26].

The starting point of our work is the spin-fermion Hamiltonian [27], which describes mobile fermions interacting with a background of localized spins through an exchange term. It is a generalization of the Kondo lattice, since also a Heisenberg exchange interaction among the localized spins is included. We will concentrate on the two dimensional case that gives a realistic description of the cuprates. A central aspect that directs the choice towards the spin-fermion model is its being analytically tractable, an important advantage over other microscopic models for strongly correlated fermions. The study follows essentially the same steps as in the antiferromagnetic case [13,16], although the actual calculation (Sec. IV) greatly differs from that case. In a path-integral representation of the model, the fermionic
degrees of freedom appear bilinearly and are integrated out exactly. An action containing only the spin degrees of freedom is obtained in terms of a fermion determinant and a pure spin part. We consider incommensurate, short-ranged spiral configurations for the spins and discuss their parametrization for long-wavelength, low-energy fluctuations around the ordered spiral phase. The gradient expansion of the fermion determinant is carried out in energy and momentum space in such a way that the occurring infinite series can be summed to all orders of the coupling constant by using the constraint on the order parameter. We show that in the limit of long wavelengths and low energies of the spins, the continuum theory is not only given by an SO(3) quantum nonlinear $\sigma$ model, as assumed in previous phenomenological approaches, but additional terms appear. On the one hand, a term linear in derivatives is obtained, such that parity is explicitly broken. However, since this term is not positive definite, it should vanish in order to guarantee a stable spiral configuration. This requirement leads to two equations that determine the wavevector for the spiral as a function of doping and the parameters of the model. On the other hand, a term of topological character like the one obtained in the continuum limit of the one-dimensional antiferromagnetic Heisenberg model \cite{28} appears. However, in contrast to that case, the corresponding coupling constant is doping dependent, and hence, it can vary continuously. Therefore, our gradient expansion not only delivers the coupling constants of the SO(3) quantum nonlinear $\sigma$ model as functions of the microscopic parameters and generalized fermionic susceptibilities which contain the doping dependence, but shows important differences with phenomenological models obtained only on the basis of symmetry arguments. Finally, we would like to add that the present approach does not impose any restrictions on the energy and momentum scales of the fermions.

II. MICROSCOPIC MODEL AND PATH-INTEGRAL DESCRIPTION

We consider a spin-fermion Hamiltonian which describes spins localized on the vertices of a square lattice ("Cu-sites") interacting with fermions moving between sites situated on the bonds ("O-sites"). The problem remains nontrivial due to the coupling of the band fermions to the surrounding localized spins. The Hamiltonian is

$$H_{\text{sf}} = \sum_{<jj'>} t_{jj'} c_{j,\alpha}^\dagger c_{j',\alpha} + \sum_i \Big( \sum_{<jj',i>} J_{Kk}^{j,j',i} c_{j,\alpha}^\dagger \sigma_{\alpha\alpha'} c_{j',\alpha'} \Big) \cdot S_i$$

$$+ J_{H} \sum_{<ii'>} S_i \cdot S_{i'} .$$

Here $c_{j,\alpha}^\dagger$ and $c_{j,\alpha}$ are creation and annihilation operators, respectively, for holes with spin projection $\alpha = \uparrow, \downarrow$ on sites situated on the bonds of the square lattice, denoted by the index $j$. The index $i$ runs over the vertices of the square lattice. The kinetic term describes hopping
processes between the sites situated on the bonds, and it can contain a direct hopping between sites \( j \) and \( j' \) as well as an an effective hopping between sites \( j \) and \( j' \) mediated by a central site \( i \). The term proportional to \( J_K \) is a non-local, Kondo-like interaction between the localized spins \( S_i \) and the holes on the neighbouring sites. It includes spin-exchange processes with hopping in addition to pure exchange processes. The vector \( \sigma_{oo'} \) consists of the three Pauli matrices. The summation denoted by \( <jj',i> \) runs over all pairs of sites \( j, j' \) which are nearest neighbours to a given site \( i \). Finally, \( H_{sf} \) contains an antiferromagnetic Heisenberg superexchange interaction between nearest-neighbour spins \( S_i \).

The spin-fermion model can be obtained as the strong-coupling limit \([27]\) of the three-band Hubbard model \([29]\), which, according to numerical simulations \([25,30]\), consistently describes the CuO\(_2\) planes of the cuprates. Limiting cases of the spin-fermion model correspond to other models that are frequently discussed in the context of strongly correlated fermion systems. In the limit of zero doping, the spin-fermion model reduces to an antiferromagnetic \( S=1/2 \) Heisenberg model, which yields a quantitative description of the undoped cuprates \([1,2,4]\). For large \( J_K \), the doped holes on the O sites strongly bind to the central Cu ion to form a local singlet, so that for zero direct O-O hopping the low-energy dynamics of \( H_{sf} \) can be mapped onto the \( t-J \) model \([31]\). For finite O-O hopping, the spin-fermion Hamiltonian can be mapped onto a generalized \( t-J \) model containing second and third nearest-neighbour hopping and spin-flip hopping \([32]\). For vanishing Heisenberg interaction, \( H_{sf} \) is equivalent to a Kondo-lattice Hamiltonian with a nonlocal exchange between band fermions and localized spins.

In the following we discuss only briefly the path-integral representation of the model in order to set up the notation, since the same steps were already performed in the antiferromagnetic case \([15]\). We represent the partition function of the spin-fermion Hamiltonian as an imaginary-time path-integral. Employing a spin-\( 1/2 \) coherent state representation \([33]\) for the spin degrees of freedom and Grassmann variables for the hole degrees of freedom, we obtain for the partition function of \( H_{sf} \)

\[
Z_{sf} = \int_{S(0)=S(\beta)} D S \int_{c(0)=-c(\beta)} D c^* D c \exp \left( S_s + S_f \right),
\]

where \( \beta = 1/k_B T \). In Eq. (2), the pure spin part of the action is given by

\[
S_s = \int_0^{\beta} d\tau \left[ i \sum_i A \left( S_i(\tau)/S \right) \cdot \partial_\tau S_i(\tau) - J_H \sum_{<ii'>} S_i(\tau) \cdot S_{i'}(\tau) \right],
\]

where now \( S = S\Omega \) and \( \Omega = (\sin \theta_i(\tau) \cos \phi_i(\tau), \sin \theta_i(\tau) \sin \phi_i(\tau), \cos \theta_i(\tau)) \). As is well known, the first term in the action \( S_s \) is the Berry phase for the adiabatic transport of a quantum spin along a closed circuit and responsible for the correct quantization of the spins. The monopole potential \( A \) satisfies the constraint
\[ \epsilon^{abc} \frac{\partial A^b}{\partial Q^c} = \Omega^a \]  

(4)
on the unit sphere. The measure for the integration over the spin variables is given by

\[ \mathcal{D}S = \prod_{i, \tau} \frac{(2S+1)}{4\pi} \sin \theta_i(\tau) \, d\theta_i(\tau) \, d\phi_i(\tau). \]

In momentum and frequency space, and after diagonalizing the kinetic part of \( H_{sf} \), which yields two bands with a dispersion relation \( \epsilon(k, \lambda), \lambda = 1, 2 \), for which we refer to Ref. [15], we obtain

\[ S_f = \beta \sum_{\alpha, \alpha'} \sum_{k, k'} \sum_{\lambda, \lambda'} \epsilon_{\alpha k \lambda} \left[ \left( i\epsilon_n - \epsilon(k, \lambda) \right) \delta_{\alpha \alpha'} \delta_{kk'} \delta_{\lambda \lambda'} - 4J_K \sigma_{\alpha \alpha'} \cdot S_{k-k'} s^*(k, \lambda) s(k', \lambda') \right] \epsilon_{\alpha' k' \lambda'} , \]

(5)

where \( k = (\epsilon_n, k) \). Its zeroth component is a discrete Matsubara frequency defined by \( \epsilon_n = (2n - 1)\pi / \beta \) for fermionic fields, and by \( \omega_n = 2n\pi / \beta \) for bosonic fields. The form factors \( s(k, \lambda) \) contain information on the band structure of the free system and are given by\[ s(k, \lambda) = e_1(k, \lambda) \sin (k_x a/2) + e_2(k, \lambda) \sin (k_y a/2), \]
where \( e_{1,2}(k, \lambda) \) are the components of the eigenvectors [15] that diagonalize the kinetic part of \( H_{sf} \).

Since the action in Eq. (5) is bilinear in the Grassmann fields, the integration \( \mathcal{D}c^* \mathcal{D}c \) may be carried out within the path-integral to obtain

\[ Z_{sf} = \int_{S(0) = S(\beta)} \mathcal{D}S \exp \left( S_s + \text{Tr} \ln G^{-1} \right) , \]

(6)

where

\[ G^{-1} = G_0^{-1} - \Sigma \]

(7)

is the inverse propagator of the fermions in the presence of the dynamical spin field \( S_{k-k'} \). The free fermionic propagator is given by

\[ (G_0^{-1})_{\alpha k \lambda, \alpha' k' \lambda'} = \left( i\epsilon_n - \epsilon(k, \lambda) \right) \delta_{\alpha \alpha'} \delta_{kk'} \delta_{\lambda \lambda'} , \]

(8)

and \( \Sigma \) is the self-energy of the holes interacting with the Cu spins,

\[ (\Sigma)_{\alpha k \lambda, \alpha' k' \lambda'} = 4J_K \sigma_{\alpha \alpha'} S_{k-k'} s^†(k, \lambda) s(k', \lambda') . \]

(9)

The trace in Eq. (6) is to be taken over the indices \( k, \alpha \) and \( \lambda \). It is convenient to shift the form factors \( s(k, \lambda) \) from the self-energy \( \Sigma \) to the unperturbed fermionic propagator \( G_0 \). It can be checked that by writing the logarithm as a power series, \( \text{Tr} \ln(G^{-1} - \Sigma) = \text{Tr} \ln(G_0^{-1}) - \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}(G_0 \Sigma)^n \), and rearranging the terms in the matrix product, a redefinition of the free fermionic propagators
\begin{equation}
(\tilde{G}_0)_{\alpha k, \alpha' k'} = \sum_{\lambda=1}^{2} \frac{|s(k, \lambda)|^2}{(i\epsilon_n - \epsilon(k, \lambda))} \delta_{\alpha \alpha'} \delta_{kk'},
\end{equation}
and the self-energy
\begin{equation}
(\tilde{\Sigma})_{\alpha k, \alpha' k'} = 4J_K \sigma_{\alpha \alpha'} \cdot S_{k-k'}.
\end{equation}
yields the relation
\begin{equation}
\text{Tr}_{(a,k,\lambda)} \ln(G_0^{-1} - \Sigma) = \text{Tr}_{(a,k)} \ln\left(\sum_{\lambda=1}^{2} |s(k, \lambda)|^2 G_0^{-1} (3 - \lambda)\right) + \text{Tr}_{(a,k)} \ln(G_0^{-1} - \bar{\Sigma})).
\end{equation}
Since the first term on the right-hand side of Eq. (12) contains only the free propagator and no interaction contribution, it may be regarded as a normalization constant and will be ignored from now on. Note that the trace in the second term on the right-hand side of Eq. (12) no longer includes the band index \(\lambda\). Since in the following we will use the free fermionic propagator and the fermionic self-energy only in the forms (10) and (11), we will omit the bars to simplify the notation, and refer to Eqs. (10) and (11) simply as \(G_0\) and \(\Sigma\).

We further introduce the abbreviation
\begin{equation}
g_0(k) = \sum_{\lambda=1}^{2} \frac{|s(k, \lambda)|^2}{(i\epsilon_n - \epsilon(k, \lambda))}.
\end{equation}
Later on we will need the momentum-space symmetry properties of the function \(g_0(k) = g_0(\epsilon_n, k)\). Using the explicit expressions [15] for the energy eigenvalues and eigenvectors of the free fermion system, the following symmetry relations can be readily deduced:
\begin{align}
g_0(\epsilon_n, k) &= g_0(\epsilon_n, -k), \quad (14) \\
g_0(\epsilon_n, k_x, k_y) &= g_0(\epsilon_n, -k_x, k_y) = g_0(\epsilon_n, k_x, -k_y). \quad (15)
\end{align}
We have derived an action where the only degrees of freedom appearing explicitly are the spins, which are suitable for a continuum approximation. This continuum approximation does not affect the fermionic degrees of freedom which are taken into account with their full dispersion relation.

\section*{III. ORDER PARAMETER FOR THE SPIRAL}

For the spin configuration \(S_i\) along the lattice, we have to introduce an expression characterizing the spiral order expected in the classical ground state. We can then take into account long-wavelength, low-energy fluctuations around this ordered helical state. In the classical ground state, the spins lie in a plane in spin space. In order to derive a continuum
theory, we need to identify vector fields which are smooth in the long-wavelength, low-energy regime. In the case of a Heisenberg antiferromagnet on a triangular lattice, the constituent fields of the order parameter can be obtained from linear combinations of the physical spins within one magnetic cell [5,7], in analogy to the construction of the order parameter fields describing collinear antiferromagnet configurations [34]. In the case of an incommensurate helix however, there are infinitely many sublattices and a magnetic cell does not exist.

In order to construct the order parameter for the spiral, we consider long-wavelength fluctuations around a classical spiral configuration with wave-vector

\[ k_s = Q + \delta Q , \]

where

\[ Q = \left( \frac{\pi}{a}, \frac{\pi}{a} \right) \]

is the wave vector of the antiferromagnetic state, while

\[ \delta Q = (\delta Q_x, \delta Q_y) \]

is incommensurate with the lattice, i.e. \( m \delta Q \neq n Q \) for all \( m, n \) integer. We start with the vector product of neighbouring spins. We define first a vector \( n_{3i} = S_i \times S_{i+1} / N_i \), where \( N_i = S^2 \sin \phi_{i,i+1} \) and \( \phi_{i,i+1} \) is the angle enclosed between the spins \( S_i \) and \( S_{i+1} \).

This choice is well defined as long as the line joining the two spins is not perpendicular to \( k_s \). The vector \( n_3 \) reduces in the classical ground state to a vector parallel to the axis of the helix, which is independent of the lattice site. We now define a matrix \( R_i^{ab} = \cos(k_s \cdot r_i) \delta^{ab} + \left[1 - \cos(k_s \cdot r_i)\right] n_{3i}^a n_{3i}^b - \sin(k_s \cdot r_i) \epsilon^{abc} n_{3i}^c \) which performs rotations about \( n_{3i} \). Here the upper indices \( a, b, c = 1, 2, 3 \) denote components in spin space. We adopt the convention that summation over repeated indices is implied, although for clarity we will occasionally write the summation sign explicitly. Applying \( \hat{R}_i \) to the spin \( S_i \) rotates it back into a space-fixed axis, which provides the definition for a second vector being constant in the classical ground state: \( n_{1i} = \hat{R}_i S_i / S \). Defining a third vector \( n_{2i} = n_{3i} \times n_{1i} \), we have a complete set of orthonormal vectors which are constant in the classical ground state and fulfill

\[ n_b^a n_c^a = \delta_{bc} . \]

Using the above basis, we parametrize the incommensurate spiral configuration as

\[ S_{pq}(\tau) = S \left[ n_1(p, q, \tau) \cos(k_s \cdot r_{pq}) - n_2(p, q, \tau) \sin(k_s \cdot r_{pq}) \right] , \]

where now \( p, q \) (instead of \( i \)) denote the sites on the two-dimensional square lattice and \( r_{pq} = (p \cdot a, q \cdot a) \), with \( a \) the lattice constant. It is evident from Eq. (20) that in the
limit $\delta Q \to 0$, the antiferromagnetic ground state is reproduced, so that $\delta Q$ describes the deviation from perfect Néel order.

In order to evaluate the continuum limit of the action, we allowed the vectors $n_1$ and $n_2$ to be smooth functions of the lattice site and of the Euclidean time $\tau$. The constraint (19) must be satisfied at every given point in space-time. In a previous work [35], we have treated the special case of planar fluctuations, where the variation of $n_1(p, q, \tau)$ and $n_2(p, q, \tau)$ was confined to the plane which is defined by $n_1$ and $n_2$ in the classical ground state. The order parameter for the description of planar fluctuations around a spiral configuration is a two-component unit vector in the manifold $S_1$. The corresponding field theory was shown to be an $O(2)$ nonlinear $\sigma$ model. Here we extend this treatment to the general case of fully three-dimensional fluctuations. This generalization turns out to be highly non-trivial (see Sec. IV and V). In the present case, fluctuations of $n_1(p, q, \tau)$ and $n_2(p, q, \tau)$ imply fluctuations of $n_3(p, q, \tau)$, so that the excitations are described by an $SO(3)$ order parameter:

$$\hat{Q} = (n_1 n_2 n_3) \quad \text{or} \quad Q_{ab} = n^a_b. \quad (21)$$

In contrast to an antiferromagnetic configuration, where the ground state is invariant under rotations about the spin axis, in a noncollinear ground state global rotations about any axis lead to a new ground state. Thus in the spiral phase the global $O(3)$ rotational symmetry in spin space is completely broken and we expect three Goldstone modes.

In helical phases, however, there is still a local rotational symmetry, which was first identified in Ref. [36]: $S_{pq}(\tau)$ is invariant under rotations by an arbitrary local angle $\psi_{pq}(\tau)$ about the local axis $S_{pq}(\tau)$. This can be readily seen in a Schwinger boson representation of the spins $S = S \bar{\omega}^\alpha \sigma^{\alpha \beta} \omega^\beta$, where $\omega^\alpha$ is a doublet of complex scalar fields satisfying $\bar{\omega}^\alpha \omega^\alpha = 1$. In this representation, a local rotation about the local spin axis corresponds to the U(1) gauge transformation $\omega \to e^{i\psi/2} \omega$. However, under these gauge transformations the order parameter fields $n_i \ (i = 1, 2, 3)$ are not invariant. In the general case, in which $\delta Q$ is finite, an infinitesimal change in the angle $\psi_{pq}(\tau)$ between two sites will lead to a finite change in the fields $n_i$. If the fields $n_i$ are slowly varying in one choice of gauge, they will have rapid variations in other gauges. Thus, by focusing on continuous configurations, one explicitly breaks this local symmetry by fixing the gauge. Our prescription to determine the fields $n_i$, and therefore to fix the gauge is physically natural, since it is given by the actual configurations of the spin fields $S$.

IV. GRADIENT EXPANSION FOR THE DOPED SYSTEM

In order to parametrize the low-lying modes around the ground state of the classical spiral, we decompose the spin field into a helical and a uniform component so that
\[ S = S (n + aL) (1 + 2a n \cdot L + a^2 L^2)^{-1/2}. \]  

(22)

where we have introduced the abbreviation

\[ n = n_1 \cos(k_s \cdot r_{pq}) - n_2 \sin(k_s \cdot r_{pq}) \]  

(23)

and the \((p, q, \tau)\)-dependence of the fields is implicit. \( L \) is a slowly varying ferromagnetic field with \(|aL| \ll 1\), which corresponds to the net magnetization density. \( L \) is of the same order as a first-order derivative of \( n \). While \( n \) describes fluctuations around the spiral wave vector \( k_s \), \( L \) describes fluctuations around \( k = 0 \). Since the resulting action will be bilinear in the ferromagnetic fluctuation, \( L \) can be integrated out at the end of the calculation. The inverse square root factor in Eq. (22) puts \( S \) on the unit sphere.

Expanding the ansatz for the spin field (22) up to second order in \( a \) gives

\[ S = S \left\{ n + a \left[ L - (n \cdot L) n \right] - a^2 \left[ (n \cdot L) L + \frac{1}{2} L^2 n - \frac{3}{2} (n \cdot L)^2 n \right] \right\}. \]  

(24)

This expression will now be employed for the space-time-dependent spin field in the path-integral, Eq. (6). We will perform the gradient expansion of the pure spin part of the action, Eq. (3), in real space. It is technically advantageous to carry out the gradient expansion of the fermion determinant (see Eqs. (6)–(7)) in Fourier space, where it is approximately diagonal for long-wavelength spin fields. We therefore transform the expression Eq. (24) for \( S_{pq}(\tau) \) into \( k \)-space (where \( k = (\epsilon, k) \)) and insert it into the fermionic self-energy \( \Sigma \) (see Eq. (11)). This leads to an expansion of the fermionic self-energy in powers of \( a \):

\[ \Sigma = \Sigma^{(0)} + a \Sigma^{(1)} + a^2 \Sigma^{(2)}. \]  

(25)

Introducing the abbreviations:

\[ n_{\pm} = n_1 \pm i n_2, \]  

(26)

we obtain for the zeroth, first and second orders of the fermionic self-energy:

\[ (\Sigma^{(0)})_{ak,a'k'} = \frac{g}{2} \sigma^{a}_{a\alpha'} \left[ n_-^a(k-k'-k_s) + n_+^a(k-k'+k_s) \right], \]  

(27)

\[ (\Sigma^{(1)})_{ak,a'k'} = g \sigma^{a}_{a\alpha'} \left\{ L^a(k-k') - \frac{1}{4} \sum_{q_1,q_2} L^b(q_1) \times \right\} \]

\[ \sum_{r=\pm} n_r^b(q_1+r k_s) n_r^a(k-k'+r k_s - \sum_{i=1}^2 q_i) \]

\[ + 2 \sum_{d=1,2} n_{d}^b(q_2) n_{d}^a(k-k' - \sum_{i=1}^2 q_i) \right\}, \]  

(28)

\[ (\Sigma^{(2)})_{ak,a'k'} = g \sigma^{a}_{a\alpha'} \left\{ -\frac{1}{2} \sum_{q_1,q_2} L^a(q_1) L^b(q_2) \sum_{r=\pm} n_r^b(k-k'+r k_s - \sum_{i=1}^2 q_i) \right\}. \]
\[-\frac{1}{4} \sum_{q_1,q_2} L^b(q_1) L^b(q_2) \sum_{r=-,+} n^a_r (k-k'+rk_s - \sum_{i=1}^2 q_i) + \frac{3}{16} \sum_{q_1 \ldots q_4} L^b(q_1) L^c(q_2) \times \\
\left( \sum_{d=1}^2 n^b_d(q_3) n^c_d(q_4) \right) \sum_{r=-,+} n^a_r (k-k'+rk_s - \sum_{i=1}^4 q_i) + 2 \left( \sum_{d=1}^2 n^b_d(q_3) n^c_d(q_4) \right) \sum_{r=-,+} n^c_r (k-k'+rk_s - \sum_{i=1}^4 q_i) + \sum_{r=-,+} n^b_r(q_3+rk_s) n^c_r(q_4+rk_s) n^a_r(k-k'+rk_s - \sum_{i=1}^4 q_i) \right), \quad (29)\]

where we have defined the coupling constant
\[g = 4J_K S. \quad (30)\]

Note that \(k_s = (0, k_s)\). Inserting this expansion of the fermionic self-energy into the fermion determinant from Eq. (31), it may be written as:
\[\text{Tr } \ln(G_0^{-1} - \Sigma) = \text{Tr } \ln(\tilde{G}_0^{-1}) + \text{Tr } \ln(1 - a \tilde{G}_0 \Sigma^{(1)} - a^2 \tilde{G}_0 \Sigma^{(2)}), \quad (31)\]

where
\[\tilde{G}_0^{-1} = G_0^{-1} - \Sigma^{(0)} \quad (32)\]

The first term in Eq. (31) represents the helical contribution since the field \(L\) does not enter in it. The second term contains the contributions of the ferromagnetic field. The helical and the ferromagnetic contributions to the fermion determinant, Eq. (31), will be evaluated using different methods, which both rely crucially on the spin-field momenta being small.

\[\text{A. Spiral contribution to the fermion determinant}\]

As a first step, the logarithm in the first term on the right-hand side of Eq. (31) is expressed as a power series:
\[\text{Tr } \ln(G_0^{-1} - \Sigma) = \text{Tr } \ln(G_0^{-1}) - \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}(G_0 \Sigma^{(0)})^n. \quad (33)\]

The first term represents the free part and will enter the path integral as a multiplicative normalization constant. Our task is to take the trace over spin and momentum indices, and obtain a closed expression for general \(n\), so that the infinite series can be re-summed. It is useful to write down explicitly the trace over the matrix product in the above equation:
We abbreviate the product of propagators as
\[ \text{Tr} \left( G_0^{\Sigma(0)} \right)^n = \left( \frac{g}{2} \right)^n \sum_{\alpha_1 \ldots \alpha_n} g_0(k_1) \sigma_{\alpha_1 \alpha_2}^{\alpha_1} \left[ n_{a_1}^\alpha (k_1 - k_2 - k_s) + n_{a_2}^\alpha (k_1 - k_2 + k_s) \right] \]
\[ \quad \sum_{\alpha_1 \ldots \alpha_n} g_0(k_2) \sigma_{\alpha_2 \alpha_3}^{\alpha_2} \left[ n_{a_2}^\alpha (k_2 - k_3 - k_s) + n_{a_3}^\alpha (k_2 - k_3 + k_s) \right] \]
\[ \quad \vdots \]
\[ \quad g_0(k_n) \sigma_{\alpha_n \alpha_1}^{\alpha_n} \left[ n_{a_n}^\alpha (k_n - k_1 - k_s) + n_{a_1}^\alpha (k_n - k_1 + k_s) \right]. \quad (34) \]

Multiplying out the brackets leads to a sum
\[ \sum_{r_1 = -r_1^c} \ldots \sum_{r_n = -r_n^c} g_0(k_1) \sigma_{\alpha_1 \alpha_2}^{\alpha_1} n_{r_1}^{a_1} (k_1 - k_2 + r_1 k_s) \ldots g_0(k_n) \sigma_{\alpha_n \alpha_1}^{\alpha_n} n_{r_n}^{a_n} (k_n - k_1 + r_n k_s). \]
\[ (35) \]

Since we are taking the trace, the sum of all momentum transfers to the spin fields must be zero, so that \( k_{n+1} = \frac{1}{2} k_1 \). Since \( k_s = (0, k_s) \) (where \( k_s = Q + \delta Q \), see Eqs. (16)–(18)) is incommensurate, this can be fulfilled only if \( n \) is even and the number of \( r_i = (-) \) equals the number of \( r_i = (+) \). Consequently, the summation \( \sum_{r_1 \ldots r_n} \) over all configurations of the \( \{ r_i \} \) contains \( \binom{n}{n/2} \) terms, namely all permutations of \( n/2 \) \( n_- \)-fields with \( n/2 \) \( n_+ \)-fields. Since \( n \) is even, the trace in spin space over the string of Pauli matrices can be carried out using the trace reduction formula \[37\]:
\[ \text{Tr} (\sigma^{a_1} \sigma^{a_2} \ldots \sigma^{a_n}) = 2 \sum_{P} (-1)^P \delta^{a_1 a_2} \ldots \delta^{a_{n-1} a_n}, \]
\[ (36) \]

where \( P \) is the permutation
\[ P = \left( \begin{array}{cccc} 2 & 3 & 4 & \ldots & n \\ i_2 & i_3 & i_4 & \ldots & i_n \end{array} \right), \]
\[ (37) \]

and the sum \( \sum_{P} \) includes permutations between different index pairs only. In order to perform the gradient expansion of the expression \( \text{Tr}(G_0^{\Sigma(0)})^n \), the momenta in the arguments of the fields are redefined [35] in such a way that the free fermionic propagators \( g_0 \) appearing in Eq. (34) can be expanded in powers of the momentum transfer to the spin field. We obtain
\[ \text{Tr} \left( G_0^{\Sigma(0)} \right)^n = 2 \left( \frac{g}{2} \right)^n \sum_{q_1 \ldots q_{n-1}} \sum_{r_1 \ldots r_n} \sum_{P} (-1)^P \delta^{a_1 a_2} \ldots \delta^{a_{n-1} a_n} \]
\[ \sum_{k} g_0(k) g_0(k - q_1 + r_1 k_s) \ldots g_0(k - q_{n-1} + \sum_{i=1}^{n-1} r_i k_s) \times \]
\[ \quad n_{r_1}^{a_1}(q_1) n_{r_2}^{a_2}(q_2 - q_1) \ldots n_{r_n}^{a_n}(-q_{n-1}) \]
\[ (38) \]

We abbreviate the product of propagators as
\[ \Pi(q_1 \ldots q_{n-1}; r_1 \ldots r_n) = \sum_{k} g_0(k) g_0(k - q_1 + r_1 k_s) \ldots g_0(k - q_{n-1} + \sum_{i=1}^{n-1} r_i k_s). \]
\[ (39) \]
Note that the form of the function \( \Pi \) depends on the set of parameters \( \{ r_i \} \). The \( q \)'s are small, so we can expand the function \( \Pi \) around \( k \). It should be stressed that only the momenta exchanged with the spin fields are small but not their values for a given fermionic state. By Fourier transformation of the spin fields back into direct space and subsequent integration by parts, one obtains from an expansion up to \( \mathcal{O}(q^2) \) first and second order spatial and temporal derivatives of the spin fields. Defining

\[
\Gamma_i^\mu = \int d^2 x d\tau \sum_{l=1}^i \left[ n_{r_l}^{a_i}(x) \ldots \partial_\mu n_{r_l}^{a_i}(x) \ldots n_{r_l}^{a_i}(x) \right] n_{r_{l+1}}^{a_{l+1}}(x) \ldots n_{r_n}^{a_n}(x),
\]

\[
\Gamma_i^{\mu\nu} = \int d^2 x d\tau \sum_{l,m=1}^{i} \left[ n_{r_l}^{a_i}(x) \ldots \partial_\mu n_{r_l}^{a_i}(x) \ldots \partial_\nu n_{r_m}^{a_m}(x) \ldots n_{r_l}^{a_i}(x) \right] n_{r_{l+1}}^{a_{l+1}}(x) \ldots n_{r_n}^{a_n}(x),
\]

\[
\Gamma_{ij}^{\mu\nu} = \int d^2 x d\tau \left\{ \sum_{l,m=1}^{i} \left[ n_{r_l}^{a_i}(x) \ldots \partial_\mu n_{r_l}^{a_i}(x) \ldots \partial_\nu n_{r_m}^{a_m}(x) \ldots n_{r_l}^{a_i}(x) \right] n_{r_{l+1}}^{a_{l+1}}(x) \ldots n_{r_n}^{a_n}(x) \right\},
\]

we have

\[
\text{Tr} \left( G_0 \Sigma^{(0)} \right)^n = 2 \left( \frac{g}{2} \right)^n \frac{1}{a^2} \sum_{r_1 \ldots r_n} \sum_P (-1)^P \delta^{a_1 a_2} \ldots \delta^{a_{n-1} a_n} \left\{ \sum_{i=1}^{n-1} \left[ \frac{\partial \Pi}{\partial q_i^\mu} \right]_{q_i=0} \Gamma_i^\mu + \frac{1}{2} \sum_{i=1}^{n-1} (-1)^i \left[ \frac{\partial^2 \Pi}{\partial q_i^\mu \partial q_i^\nu} \right]_{q_i=0} \Gamma_i^{\mu\nu} + \right. \]

\[
\left. \frac{1}{2} \sum_{i,j=1}^{n-1} (-1)^i \left[ \frac{\partial^2 \Pi}{\partial q_i^\mu \partial q_j^\nu} \right]_{q_i=0} \Gamma_{ij}^{\mu\nu} \right\}. \tag{43}
\]

Here \( \partial_\mu \) is an abbreviation for \( \partial / \partial x^\mu \). Analogously to our notation for the 3-momentum, in which \( k = (\epsilon_n, k) \), we define \( x = (\tau, \mathbf{x}) \), so that the indices \( \mu \) and \( \nu \) in Eqs. (40)–(43) run over three values. In order to avoid confusion with the summation index \( i \), we use \( i \) to represent \( \sqrt{-1} \) in this section. The boundary terms resulting from the integration by parts, which we performed to arrive at Eq. (43), vanish because the field \( \mathbf{n}(x) \) has been required to be constant in infinity, which is a natural assumption in a low-temperature approach. As is evident from Eq. (43), the Kronecker deltas \( \delta^{a_i a_j} \) generate a pairwise contraction of the fields and their derivatives into inner products. Carrying out the sum \( \sum_P \), we sum over all possible pairwise contractions. In the derivation of Eq. (43), the local part containing the zeroth order term in the expansion of \( \Pi \) has been discarded since it contains no derivatives.
and is therefore constant due to the constraint, Eq. (44). Several types of inner products are encountered in Eq. (43):

\[
\begin{align*}
n_+^a n_+^a & = n_-^a n_-^a = 0, \\
(\partial_\mu n_+^a) n_+^a & = (\partial_\mu n_-^a) n_-^a = 0, \\
n_+^a n_-^a & = 2, \\
(\partial_\mu n_+^a) n_+^a & = - (\partial_\mu n_-^a) n_-^a = 2i (\partial_\mu n_+^a) n_-^a, \\
(\partial_\mu n_-^a) (\partial_\nu n_+^a) & = (\partial_\mu n_+^a) (\partial_\nu n_+^a) + (\partial_\mu n_-^a) (\partial_\nu n_-^a).
\end{align*}
\]

Pair contractions of the type \((\partial_\nu n^a_+) (\partial_\nu n^a_-)\) need not be considered since they imply a contraction of the type \(n^a_+ n^a_-\) (because the number of fields is even), which gives zero according to Eq. (44). From Eqs. (44)–(48), it is clear that one gets nonzero contributions only from those contractions \(\delta^{a_i a_j}\) for which \(r_i = - r_j\). Since there are \(\binom{n}{2}\) possibilities for such contractions producing inner products of \(\binom{n}{2}\) \(n_+\)-fields with \(\binom{n}{2}\) \(n_-\)-fields, the sum \(\sum_P^I\) in \(X_n^I\) has \(\binom{n}{2}\) terms.

We will now outline the combinatorial analysis which we have performed in order to permit an explicit evaluation of the sums appearing in Eqs. (40)–(43) to arbitrary order in \(n\). This analysis departs considerably from previous work in the antiferromagnetic case [Eq. (10)]. We denote the contribution of the first-order derivatives to \(\text{Tr}(G_0 \Sigma^{(0)})^n\) by

\[
X_n^I = 2 \left(\frac{g}{2}\right)^n \frac{1}{a_1 \cdots a_n} \sum_{r_1 \cdots r_n} (-1)^P \delta^{a_1 a_2} \cdots \delta^{a_{n-1} a_n} \sum_{i=1}^{n-1} i \frac{\partial \Pi}{\partial q_i} \Bigg|_{q_i=0} \Gamma_i^\mu,
\]

where \(\Gamma_i^\mu\) was defined in Eq. (11). We will show that the only nonvanishing contributions to the sum \(\sum_{r_1 \cdots r_n} X_n^I\) come from strictly alternating configurations of the set \(\{r_i\}\), i.e. those configurations which fulfill \(r_{i+1} = - r_i\).

Let us consider a nonalternating configuration of the \(\{r_i\}\), where at least two \(n_+\)-terms (and consequently two \(n_-\)-terms) are nearest neighbours, i.e. we assume \(r_i = r_{i+1}\) for some \(i\). One of the possible contractions connecting only \(n_+\)-fields with \(n_-\)-fields is:

\[
\delta^{a_i a_j} \delta^{a_{j+1} a_k}.
\]

Another contraction connecting only \(n_+\)-fields with \(n_-\)-fields by is obtained by one permutation:

\[
\delta^{a_{i+1} a_j} \delta^{a_j a_k}.
\]

Two sets of contractions differing by one permutation carry the sign \((-1)^P\), \((-1)^{P+1}\), respectively. Since the two terms (50) and (51) are identical except for their relative sign, they add up to zero in the sum \(\sum_P^I\). Since \(\binom{n}{2}\) is an even number, one can find for every
nonvanishing set of contractions another one differing only by a relative sign given a nonalternating configuration of the \( \{ r_i \} \). Thus all terms in the sum \( \sum_i' \) add to zero pairwise for a nonalternating configuration of the \( \{ r_i \} \). From the sum \( \sum_{r_1 \ldots r_n} \), we are left only with the two alternating configurations.

For a strictly alternating configuration of the \( \{ r_i \} \), where \( r_{i+1} = -r_i \forall i \), the total number of nonvanishing terms from the sum \( \sum_i' \) is obtained by the following consideration: If a pair-contraction \( \delta^{a_i a_j} \) with \( r_i = -r_j \) leaves a nonalternating sequence of the \( \{ r_i \} \), all terms will again add to zero pairwise. The only contractions that always leave an alternating sequence of the \( \{ r_i \} \) are contractions between nearest neighbours or between the first and last member of the remaining sequence. The number of these contractions is \( 2^{(\frac{n}{2}-1)} \), and they carry the same sign. Thus in Eq. (49), the sum \( \sum_i' (-1)^i \delta^{a_1 a_2} \ldots \delta^{a_{n-1} a_n} \) can be replaced by \( 2^{(\frac{n}{2}-1)} \) times the contribution of the zeroth permutation, \( \delta^{a_1 a_2} \delta^{a_3 a_4} \ldots \delta^{a_{n-1} a_n} \).

After the sums \( \sum_{r_1 \ldots r_n} \) and \( \sum_i' \) have been discussed, we still need to carry out the sums \( \sum_{i=1}^{n-1} \) and \( \sum_{i=1}^i \) (see Eqs. (10),(19)) in order to obtain a closed expression for \( X_n' \). First we consider the alternating configuration starting with \( r_1 = - \). For this configuration of the \( \{ r_i \} \) and the set of contractions \( \delta^{a_1 a_2} \delta^{a_3 a_4} \ldots \delta^{a_{n-1} a_n} \), every term with \( l \) odd gives \( (\partial n^a) n^a_n \), while every term with \( l \) even gives \( (\partial n^a) n^a_n \). There is always one contraction containing the derivative, for which we use Eq. (47), and \( (\frac{n}{2} - 1) \) contractions without derivative, for which we use Eq. (16). This leads to

\[
\sum_{i=1}^i \left[ n_{r_1}^{a_1} \ldots \partial_{\mu} n_{r_i}^{a_i} \ldots n_{r_n}^{a_n} \right] n_{r_{i+1}}^{a_{i+1}} \ldots n_{r_n}^{a_n} = \sum_{i=1}^i (-1)^{(l-1)} 2i (\partial_{\mu} n_1^a) n_2^a 2^{(\frac{n}{2}-1)} = \\
= 2^\frac{n}{2} i (\partial n_1^a) n_2^a \times \begin{cases} 
1 & \text{for } i \text{ odd}, \\
0 & \text{for } i \text{ even}. 
\end{cases}
\]  

(52)

For \( i \) odd, the first derivatives of the product of propagators \( \Pi \) (defined in Eq. (33)) for an alternating configuration starting with \( r_1 = - \) acquire the form

\[
\frac{\partial \Pi^{-+}}{\partial q_{\mu\text{odd}}} \bigg|_{q_i=0} = \sum_k [g_0(k)]^\frac{n}{2} [g_{0,-}(k)]^\frac{n}{2} \frac{\partial g_{0,-}(k-q)}{\partial q^\mu} \bigg|_{q=0} = \\
= \frac{2}{n} \sum_k [g_0(k)]^\frac{n}{2} \frac{\partial \left[ g_{0,-}(k-q) \right]^\frac{n}{2}}{\partial q^\mu} \bigg|_{q=0} .
\]

(53)

We define

\[ g_{0,\pm}(k) = g_0(k \pm k_s) . \]

(54)

Since \( \partial \Pi^{-+}/\partial q_{\mu\text{odd}} \bigg|_{q_i=0} \) has the same form for all \( i \) odd, the summation over \( i \) can be carried out: \( \sum_{i=1}^{n-1} \text{odd} 1^i = \frac{n}{2} \). Inserting these results into Eq. (19), and recalling that the sum \( \sum_i' \) over the permutations is replaced by \( 2^{(\frac{n}{2}-1)} \) times the contribution of the zeroth permutation, we obtain
\[ X^I_{n(+-)} = 2 \left( \frac{g}{2} \right)^n 2^{\frac{n-1}{2}} \frac{i}{a^2} \frac{n}{q_{odd}^{\alpha}} \frac{\partial \Pi^{++}}{\partial q_{odd}^{\alpha}} \bigg|_{q=0} \int d^2 x d\tau 2^{\frac{n}{2}} i (\partial_\mu n_1^\alpha) n_2^\alpha = -\frac{n}{2a^2} g^n \frac{\partial \Pi^{++}}{\partial q_{\mu}^{\alpha}} \bigg|_{q=0} \int d^2 x d\tau (\partial_\mu n_1^\alpha) n_2^\alpha. \quad (55) \]

For our fermion determinant, Eq. (33), we need \( -\sum_{n=1}^{\infty} \frac{1}{n} X^I_{n(+-)} \). We insert Eq. (53) into Eq. (54) and re-sum the power series in \( n \):

\[ -\sum_{n=2}^{\infty} \frac{1}{n} X^I_{n(-)} = \sum_{n=1}^{\infty} \frac{g^{2n}}{2n} \frac{1}{a^2} \sum_k \left[ g_0(k) \right]^n \frac{\partial}{\partial q^\mu} \left[ g_{0,-}(k-q) \right]^n \bigg|_{q=0} \int d^2 x d\tau (\partial_\mu n_1^\alpha) n_2^\alpha = -\frac{1}{2a^2} \frac{\partial}{\partial q^\mu} \sum_k \ln \left[ 1 - g^2 g_0(k) g_{0,-}(k-q) \right] \bigg|_{q=0} \int d^2 x d\tau (\partial_\mu n_1^\alpha) n_2^\alpha. \quad (56) \]

Adding also the contribution \( X^I_{n(+)} \) from the alternating sequence starting with \( r_1 = (+) \) and introducing

\[ \Phi^+_1(q) = \sum_k \ln(1 - g^2 g_0(k) g_{0,\pm}(k-q)), \quad (57) \]

we obtain

\[ -\sum_{n=2}^{\infty} \frac{1}{n} X^I_n = \frac{1}{2a^2} \frac{\partial}{\partial q^\mu} \left( \Phi^+_1(q) - \Phi^-_1(q) \right) \bigg|_{q=0} \int d^2 x d\tau (\partial_\mu n_1^\alpha) n_2^\alpha. \quad (58) \]

The sum \( \sum_k \) in our expressions for \( \Phi^+_1(q) \) denotes a sum \( \sum_{\epsilon, k} \). By shifting the summation index \( k \to -k \) in \( \Phi^+_1(q) \) and using the symmetry of the free propagator, Eq. (14), it can be shown that

\[ \frac{\partial \Phi^+_1(q)}{\partial q^0} \bigg|_{q=0} = \frac{\partial \Phi^-_1(q)}{\partial q^0} \bigg|_{q=0}, \]
\[ \frac{\partial \Phi^+_1(q)}{\partial q^{1,2}} \bigg|_{q=0} = -\frac{\partial \Phi^-_1(q)}{\partial q^{1,2}} \bigg|_{q=0}, \quad (59) \]

such that the time component of the derivative drops out and only the space components remain:

\[ -\sum_{n=2}^{\infty} \frac{1}{n} X^I_n = \sum_{\mu=1}^{2} \frac{1}{a^2} \frac{\partial \Phi^+_1(q)}{\partial q^\mu} \bigg|_{q=0} \int d^2 x d\tau n_1 \cdot \partial_\mu n_2. \quad (60) \]

For \( \delta Q = 0 \), i.e. a pure antiferromagnetic configuration, the first-order contributions vanish. We will discuss the relevance of this term for the resulting field theory at a later stage, after we have also obtained a contribution linear in the derivatives when we take the continuum limit of the Heisenberg part of the action.
We now turn to the second-order terms in the expression (43). The contribution of the quadratic derivatives is denoted by

\[ X_n^H = \left( \frac{g}{2} \right)^n \sum_{r_1, \ldots, r_n} \sum_P (-1)^P \delta^{a_1 a_2} \cdots \delta^{a_{n-1} a_n} \sum_{i=1}^{n-1} (-1) \frac{\partial^2 \Pi}{\partial q_i^\mu \partial q_i^\nu} \bigg|_{q_i=0} \Gamma_i^{\mu\nu} \]  

(see Eq. (11) for \( \Gamma_i^{\mu\nu} \)). We proceed similarly as in the case of the first-order derivatives, and investigate first whether for some special configurations of the \( r_i \) the terms in the sum over all contractions \( \sum_P \) add to zero. It will turn out that in the case of the quadratic derivatives not only the strictly alternating configurations of the \( r_i \) give a nonvanishing contribution, but also configurations where the alternation is broken once. Again, in a nonalternating sequence one can find two possible sets of pair-contractions, differing by one permutation, that connect only \((+-)\)-pairs and carry the signs \((-1)^P\) and \((-1)^{P+1}\), respectively. The corresponding terms will only add to zero if they are identical except for their signs. However, this is not always the case if we deal with the second-order-derivatives. It may occur that in one of the two sets the two derivatives are contracted,

\[ \left[ \ldots \partial_{\mu} n_{-i} \ldots \partial_{\nu} n_{+i} \ldots n_{-k} \ldots n_{+l} \ldots \right] \delta^{a_i a_j} \delta^{a_k a_l} . \]  

while in the set differing by one permutation the derivatives are contracted with other fields, i.e.

\[ \left[ \ldots \partial_{\mu} n_{-i} \ldots \partial_{\nu} n_{+i} \ldots n_{-k} \ldots n_{+l} \ldots \right] \delta^{a_i a_k} \delta^{a_j a_l} . \]

While a contraction of type (62) gives a term of the form \( [(\partial_{\mu} n_1 ) \cdot (\partial_{\nu} n_1 ) + (\partial_{\mu} n_2 ) \cdot (\partial_{\nu} n_2 )] \) (see Eq. (48)), the contraction (63) yields \( 4(n_1 \cdot \partial_{\mu} n_2 ) (n_1 \cdot \partial_{\nu} n_2 ) \) (see Eq. (17)). These terms are not identical but related through

\[ 2(n_1 \cdot \partial_{\mu} n_2 ) (n_1 \cdot \partial_{\nu} n_2 ) = (\partial_{\mu} n_1 ) \cdot (\partial_{\nu} n_1 ) + (\partial_{\mu} n_2 ) \cdot (\partial_{\nu} n_2 ) - (\partial_{\mu} n_3 ) \cdot (\partial_{\nu} n_3 ) . \]

Whether two terms carrying the signs \((-1)^P\) and \((-1)^{P+1}\) are identical depends on the position of the derivatives, which is denoted by \( l \) and \( m \) (see Eq. (11)) and varied in the sum \( \sum_{l,m=1}^i \). Thus we will now evaluate the sum \( \sum_P \) for all possible positions \( l \) and \( m \) of the derivatives and then compute the sum \( \sum_{l,m=1}^i \). Again, we first discuss the alternating configuration starting with \( r_1 = (--) \). The contribution of terms with \( l = m \) is

\[ T_{(l=m)}^{\mu\nu} = \sum_P (-1)^P \delta^{a_1 a_2} \cdots \delta^{a_{n-1} a_n} \int d^2 x d\tau \left[ n_{r_1}^{a_1} \cdots (\partial_{\mu} \partial_{\nu} n_{r_1}^{a_l}) \cdots n_{r_n}^{a_n} \right] \]

\[ = -2^{(n-2)} \sum_{b=1}^{2} (\partial_{\mu} n_b ) \cdot (\partial_{\nu} n_b ) . \]  

(65)

For terms with \( l \neq m \), where \( l, m \) are both even or both odd, we obtain \((z \in \mathbb{N} \setminus \{0\})\)
\[
T_{\mu \nu}^{(l=\pm m z)} = \sum_P' (-1)^P \delta^{a_1a_2} \ldots \delta^{a_{n-1}a_n} \int d^2x d\tau \left[ \eta^{a_1}_{\tau_1} \ldots (\partial_{\mu} \eta^{a_l}_{\tau_l}) \ldots (\partial_{\nu} \eta^{a_{n-1}}_{\tau_{n-1}}) \ldots \eta^{a_n}_{\tau_n} \right] \\
= -2^{(n-2)} 2 (\mathbf{n}_1 \cdot \partial_{\mu} \mathbf{n}_2)(\mathbf{n}_1 \cdot \partial_{\nu} \mathbf{n}_2).
\] (66)

Terms with \( l = m \pm 1 \) yield

\[
T_{\mu \nu}^{(l=\pm m \pm 1)} = \sum_P' (-1)^P \delta^{a_1a_2} \ldots \delta^{a_{n-1}a_n} \int d^2x d\tau \left[ \eta^{a_1}_{\tau_1} \ldots (\partial_{\mu} \eta^{a_l}_{\tau_l}) \ldots \eta^{a_{n-1}}_{\tau_{n-1}} \ldots \eta^{a_n}_{\tau_n} \right] \\
= 2^{(n-3)} 2 \sum_{b=1}^2 (\partial_{\mu} \mathbf{n}_b) \cdot (\partial_{\nu} \mathbf{n}_b) + 2^{(n-3)} 2 (\mathbf{n}_1 \cdot \partial_{\mu} \mathbf{n}_2)(\mathbf{n}_1 \cdot \partial_{\nu} \mathbf{n}_2).
\] (67)

Terms with \( l \neq m \pm 1 \), where \( l \) even, \( m \) odd or vice versa, contribute

\[
T_{\mu \nu}^{(l=\pm m z \pm 1)} = \sum_P' (-1)^P \delta^{a_1a_2} \ldots \delta^{a_{n-1}a_n} \int d^2x d\tau \left[ \eta^{a_1}_{\tau_1} \ldots (\partial_{\mu} \eta^{a_l}_{\tau_l}) \ldots \eta^{a_{n-1}}_{\tau_{n-1}} \ldots \eta^{a_n}_{\tau_n} \right] \\
= 2^{(n-2)} 2 (\mathbf{n}_1 \cdot \partial_{\mu} \mathbf{n}_2)(\mathbf{n}_1 \cdot \partial_{\nu} \mathbf{n}_2).
\] (68)

We now determine the number \( N \) of each of these four types of \( T_{\mu \nu} \)-terms in the sum \( \sum_{i,m=1}^n \), and denote the result of the summations \( \sum_{i,m=1}^n \sum_P' \) by \( T_i \), i.e. \( T_i = N_{(l=\text{even})} T_{\mu \nu}^{(l=\pm m z)} + N_{(l=\text{odd})} T_{\mu \nu}^{(l=\pm m \pm 1)} + N_{(l=\pm m z \pm 1)} T_{\mu \nu}^{(l=\pm m \pm 1)} \). Using Eq. (64), we obtain that \( T_i \) does not depend on explicitly on \( i \), namely,

\[
T_i(i=\text{odd}) = -2^{(n-2)} \sum_{b=1}^2 (\partial_{\mu} \mathbf{n}_b) \cdot (\partial_{\nu} \mathbf{n}_b),
\]

\[
T_i(i=\text{even}) = -2^{(n-2)} (\partial_{\mu} \mathbf{n}_3) \cdot (\partial_{\nu} \mathbf{n}_3).
\] (69)

For the alternating configuration starting with \( r_1 = (-) \) which we have discussed so far, the second derivative of the product of propagators from Eq. (69) becomes

\[
\left. \frac{\partial^2 \Pi^{++}}{\partial q_{\tau_1}^\mu \partial q_{\tau_1}^\nu} \right|_{q_i=0} = \sum_k g_0(k)^{\frac{n}{2}} g_{0,-}(k)^{\frac{n}{2}-1} \frac{\partial^2 g_{0,-}(k-q)}{\partial q^\mu \partial q^\nu} \bigg|_{q=0},
\]

\[
\left. \frac{\partial^2 \Pi^{++}}{\partial q_{\tau_1}^\mu \partial q_{\tau_1}^\nu} \right|_{q_i=0} = \sum_k g_0(k)^{\frac{n}{2}} g_{0,-}(k)^{\frac{n}{2}} \frac{\partial^2 g_{0,-}(k-q)}{\partial q^\mu \partial q^\nu} \bigg|_{q=0},
\] (70)

so that the sum \( \sum_{i=1}^{n-1} \) can be carried out. Adding the contribution of the alternating configuration starting with \( r_1 = (+) \), we obtain for the contribution of the two alternating configurations to the sum \( \sum_{r_1 \ldots r_n} \) in Eq. (71):

\[
X_{n(\text{alt})}^H = \left( \frac{g}{2} \right)^n 2^{(n-2)} \left[ \sum_k g_0(k)^{\frac{n}{2}} g_{0,-}(k)^{\frac{n}{2}} \frac{\partial^2 g_{0,-}(k-q)}{\partial q^\mu \partial q^\nu} \bigg|_{q=0} \right]
\]
For a nonalternating configuration, the terms \( T_{(l=m)}^{\mu \nu} \) and \( T_{(l=m \pm 1)}^{\mu \nu} \) vanish. The terms \( T_{(l=m \pm 1)}^{\mu \nu} \) and \( T_{(l=m \pm 2 \pm 1)}^{\mu \nu} \) can be nonvanishing when the contraction of the two derivatives is such that the remaining sequence is alternating, i.e. when the contraction of the derivatives ‘repairs’ the defects in the alternation. However, such terms can still add to zero in the sum \( \sum_{l,m=1}^{i} \). A term with fixed \( l, m \), where the contraction of the two derivatives restores the perfect alternation of the remaining sequence, adds to zero e.g. with the term \( \sum_{i=1}^{2} \), where the ‘repairing’ contraction differs by one permutation and thus carries the opposite the perfect alternation of the remaining sequence, occurs in \( \sum_{l,m=1}^{i} \). This happens precisely if the alternation of the configuration is broken only once. These configurations will be denoted by ‘one-kink configurations’ in the following.

We proceed with the discussion of the summations \( \sum_{i=1}^{n-1}, \sum_{l,m=1}^{i}, \sum_{p} \). We denote the site after which the kink occurs by \( \ell \). From the sum \( \sum_{i=1}^{n-1} \), \( \sum_{l,m=1}^{i} \) only the term with \( i = \ell \) contributes, and from the sum \( \sum_{l,m=1}^{\ell} \) only the term with \( l = 1, m = \ell \) (or vice versa) is nonvanishing. Further, for a one-kink configuration \( \ell \) is always even. The result of the summations \( \sum_{l,m=1}^{i} \sum_{p} \) is again independent of \( i \):

\[
T_{i}^{kink} = 2^{(n-2)} (\partial_\mu n_3) \cdot (\partial_\nu n_3). \tag{72}
\]

In order to carry out the sum over all possible one-kink configurations, we need the form of the function \( \partial^2 \Pi_{kink}^{\mu \nu} / \partial q_{\ell}^\mu \partial q_{\ell}^\nu \rvert_{q=0} \) for a general one-kink configuration of length \( n \). If it starts with \( r_1 = (-) \), we have

\[
\frac{\partial^2 \Pi_{kink}^{\mu \nu}}{\partial q_{\ell}^\mu \partial q_{\ell}^\nu} \bigg|_{q=0} = \sum_k g_{0,-}(k)^{\frac{\mu}{2}} g_{0,+}(k)^{\frac{\nu}{2} - \frac{1}{2}} \frac{\partial^2 g_0(k - q)}{\partial q^\mu \partial q^\nu} \bigg|_{q=0} . \tag{73}
\]

We add the configurations starting with \( r_1 = (-) \) and \( r_1 = (+) \) and perform the sum \( \sum_{\ell=2, \text{even}}^{n-2} \) over all possible one-kink configurations. We find for the contribution of the one-kink configurations to the sum \( \sum_{r_1 \cdots r_n} \) in Eq. (11):

\[
X_{n(kink)}^{II} = \left( \frac{g}{2} \right)^n \frac{2^{(n-1)}}{a^2} \left\{ \sum_k g_0(k)^{\frac{\mu}{2} - 1} \left[ g_{0,+}(k)^{\frac{\nu}{2} - 1} - g_{0,-}(k)^{\frac{\nu}{2} - 1} \right] \left[ g_{0,+}(k)^{-1} - g_{0,-}(k)^{-1} \right] \right\} \int d^2 x d\tau \left( \partial_\mu n_3 \right) \cdot (\partial_\nu n_3). \tag{74}
\]
The contribution of the mixed derivatives to $\text{Tr}(G_0\Sigma^{(0)})^n$ from Eq. (13) is

$$X_{n}^{ij} = \left( \frac{g}{2} \right)^n \frac{1}{a^2} \sum_{r_1 \ldots r_n} \sum' (-1)^P \delta^{n_1 a_2} \ldots \delta^{n_{n-1} a_n} \sum_{i,j=1}^{n-1} (-1) \frac{\partial^2 \Pi}{\partial q^\mu_r \partial q^\nu} \bigg|_{q_{i,j}=0} \Gamma_{ij}^{\mu\nu}, \quad (75)$$

where $\Gamma_{ij}^{\mu\nu}$ is given by Eq. (12). As we will show, nonvanishing contributions to the mixed derivatives originate from the strictly alternating configurations of the $\{r_i\}$ as well as from configurations where the alternation is broken up to two times. We will again study the case of the alternating configurations first. As can be seen from Eq. (42), $\Gamma_{ij}^{\mu\nu}$ falls into two parts, the sum $\sum_{l,m=1}^i \sum_{m=i+1}^j$ and the sum $\sum_{l=1}^i \sum_{m=i+1}^j$. The result of the summations $\sum_{l,m=1}^i \sum_{m=i+1}^j \sum' P$ is independent of the value that $j$ takes and its contribution is given by $T_i$ in Eq. (39). The terms occurring in the sum $\sum_{i=1}^l \sum_{m=i+1}^j$ are $T_{l,m=1}^{\mu\nu} T_{l,m=1}^{\mu\nu}$ and $T_{l,m=1}^{\mu\nu} T_{l,m=1}^{\mu\nu}$, whose contributions are given in Eqs. (60), (67), and (68).

We determine the number $M$ of each of these three types of terms in the sum $\sum_{l=1}^i \sum_{m=i+1}^j$ for every value of $i$ and $j$, and denote the result of the summations $\sum_{l,m=1}^i \sum_{m=i+1}^j \sum' P$ by $T_{ij}$, i.e. $T_{ij} = T_i + M_{l=m=1} T_{l,m=1}^{\mu\nu} + M_{l=m=1} T_{l,m=1}^{\mu\nu}$ + $M_{l=m=1} T_{l,m=1}^{\mu\nu} + M_{l=m=1} T_{l,m=1}^{\mu\nu}$. We find

$$T_{ij}(i = \text{odd}, j = \text{even}) = -2^{(n-3)}(\partial_\mu \mathbf{n}_3) \cdot (\partial_\nu \mathbf{n}_3),$$

$$T_{ij}(i = \text{even}, j = \text{odd}) = -2^{(n-3)}(\partial_\mu \mathbf{n}_3) \cdot (\partial_\nu \mathbf{n}_3),$$

$$T_{ij}(i = \text{even}, j = \text{even}) = -2^{(n-3)}(\partial_\mu \mathbf{n}_3) \cdot (\partial_\nu \mathbf{n}_3),$$

$$T_{ij}(i = \text{odd}, j = \text{odd}) = 2^{(n-3)}(\partial_\mu \mathbf{n}_3) \cdot (\partial_\nu \mathbf{n}_3) - 2^{(n-2)} \sum_{b=1}^2 (\partial_\mu \mathbf{n}_b) \cdot (\partial_\nu \mathbf{n}_b). \quad (76)$$

In order to perform the summation $\sum_{i,j=1, i \neq j}^{n-1}$, we determine the second derivatives of the product of propagators from Eq. (39) for an alternating configuration starting with $r_1 = (-)$,

$$\frac{\partial^2 \Pi}{\partial q^\mu_\text{odd} \partial q^\nu_\text{even}} \bigg|_{q_{i,j}=0} = \frac{\partial^2 \Pi}{\partial q^\mu_\text{even} \partial q^\nu_\text{odd}} \bigg|_{q_{i,j}=0} = \sum_k g_0(k) \frac{n}{2} g_{0_{-}}(k) \frac{n}{2} - 1$$

$$\frac{\partial g_0(k-q)}{\partial q^\mu} \bigg|_{q=0} \frac{\partial g_0_{-}(k-q)}{\partial q^\nu} \bigg|_{q=0},$$

$$\frac{\partial^2 \Pi}{\partial q^\mu_\text{even} \partial q^\nu_\text{even}} \bigg|_{q_{i,j}=0} = \sum_k g_0(k) \frac{n}{2} - 2 g_{0_{-}}(k) \frac{n}{2} - 2 \frac{\partial g_0(k-q)}{\partial q^\mu} \bigg|_{q=0} \frac{\partial g_0_{-}(k-q)}{\partial q^\nu} \bigg|_{q=0},$$

$$\frac{\partial^2 \Pi}{\partial q^\mu_\text{odd} \partial q^\nu_\text{odd}} \bigg|_{q_{i,j}=0} = \sum_k g_0(k) \frac{n}{2} g_{0_{-}}(k) \frac{n}{2} - 2 \frac{\partial g_0(k-q)}{\partial q^\mu} \bigg|_{q=0} \frac{\partial g_0_{-}(k-q)}{\partial q^\nu} \bigg|_{q=0}. \quad (77)$$

We add the alternating configuration starting with $r_1 = (+)$, and obtain from the summation $\sum_{i,j=1, i \neq j}^{n-1}$ the contribution of the alternating configurations to the sum $\sum_{r_1 \ldots r_n}$ in Eq. (73).
Next we discuss the contribution of the nonalternating configurations to the mixed derivatives. For nonalternating configurations the terms $T_{\ell=l}^{\mu\nu}$ and $T_{\ell=l+2}^{\mu\nu}$ vanish. As in the case of the quadratic derivatives, the terms $T_{\ell=l+1}^{\mu\nu}$ and $T_{\ell=l+2}^{\mu\nu}$ are nonzero if the contraction of $l$ and $m$ restores the alternation of the remaining sequence. For the mixed derivatives considered now, this is possible if the alternation of the configuration is broken not more than two times. We will first evaluate the case of the one-kink configurations.

The first sum in $\Gamma_{ij}^{\mu\nu}$ (see Eq. (42)), $\sum_{l,m=1}^{i} \sum_{n=1}^{m=i+1}$, has been discussed for one-kink configurations already, so that we just recall the result for $T_{i}^{\text{kink}}$, Eq. (72). In order to evaluate the sum $\sum_{l=1}^{i} \sum_{m=i+1}^{n}$, we first take $i < j$. The number of the site after which the kink occurs is again denoted by $\ell$. A contraction of the derivatives can restore the alternation of the remaining sequence only if $l = 1$ and $m = \ell$ or $l = 1$ and $m = (\ell + 1)$. The sum $\sum_{i,j=1}^{n-1} \sum_{i \neq j}$ only yields nonzero contributions for every $i$ with $i < j$ if $j = \ell$ (where only the term with $l = 1$ and $m = m_{\text{max}} = \ell$ contributes to $\sum_{l=1}^{i} \sum_{m=i+1}^{n} \sum_{r}$), and for every $j$ with $j > i$ if $i = \ell$ (where only the terms with $l = 1$ and $m = m_{\text{min}} = (\ell + 1)$ contributes to $\sum_{i=1}^{j} \sum_{m=i+1}^{n} \sum_{r}$). The summations $[\sum_{l,m=1}^{i} \sum_{n=1}^{m=i+1} \sum_{r}]$ can be calculated to give

$$T_{ij}^{\text{kink}}(i < j, j = \ell) = 2^{(n-3)}(\partial_{\mu} n_{3}) \cdot (\partial_{\nu} n_{3}),$$

$$T_{ij}^{\text{kink}}(i = \ell, j > i) = 2^{(n-3)}(\partial_{\mu} n_{3}) \cdot (\partial_{\nu} n_{3}).$$

At the present step of the evaluation, the contribution of the one-kink configurations starting with $r_{1} = (-)$ to the mixed derivatives takes the form:

$$X_{n(\text{kink})}^{IJ}(r_{1} = -) = \left(\frac{g}{2}\right)^{n} \frac{2^{(n-2)}}{a^{2}} \left\{ \sum_{k} g_{0}(k)^{\frac{n-2}{2}} g_{0,\ell}(k)^{\frac{n-2}{2}} \frac{\partial g_{0}(k-q)}{\partial q^{\mu}} \bigg|_{q=0} \frac{\partial g_{0}(k-q)}{\partial q^{\nu}} \bigg|_{q=0} \right. + \sum_{k} g_{0}(k)^{\frac{n-2}{2}} g_{0,\ell}(k)^{\frac{n-2}{2}} \frac{\partial g_{0}(k-q)}{\partial q^{\mu}} \bigg|_{q=0} \frac{\partial g_{0}(k-q)}{\partial q^{\nu}} \bigg|_{q=0} \right\} \times$$

$$\int d^{2}x d\tau \left[ \left( \frac{\partial_{\mu} n_{3}}{\partial_{\nu} n_{3}} \right) \right].$$

As to the second derivatives of the product of propagators from Eq. (39), the following cases are encountered:
mixed derivatives:

\[
\frac{\partial^2 \Pi_{kink}^{\mu}}{\partial q_{i\,even}^\mu \partial q_j^\nu} \bigg|_{q_i,j=0} = \frac{\partial^2 \Pi_{kink}^{\mu}}{\partial q_{i\,odd}^\mu \partial q_j^\nu} \bigg|_{q_i,j=0} = \sum_k g_0(k)^{\frac{n}{2} - 2} g_{0,-}(k)^{\frac{j}{2}} g_{0,+}(k)^{\frac{n}{2} - \frac{j}{2}} \\
\frac{\partial g_0(k-q)}{\partial q^\mu} \bigg|_{q=0} \frac{\partial g_0(k-q)}{\partial q^\nu} \bigg|_{q=0},
\]

\[
\frac{\partial^2 \Pi_{kink}^{\mu}}{\partial q_{i\,even}^\mu \partial q_j^\nu} \bigg|_{q_i,j=0} = \sum_k g_0(k)^{\frac{n}{2} - 1} g_{0,-}(k)^{\frac{j}{2}} g_{0,+}(k)^{\frac{n}{2} - \frac{j}{2} - 1} \\
\frac{\partial g_0(k-q)}{\partial q^\mu} \bigg|_{q=0} \frac{\partial g_0(k-q)}{\partial q^\nu} \bigg|_{q=0},
\]

\[
\frac{\partial^2 \Pi_{kink}^{\mu}}{\partial q_{i\,odd}^\mu \partial q_j^\nu} \bigg|_{q_i,j=0} = \sum_k g_0(k)^{\frac{n}{2} - 1} g_{0,-}(k)^{\frac{j}{2}} g_{0,+}(k)^{\frac{n}{2} - \frac{j}{2} - 1} \\
\frac{\partial g_0(k-q)}{\partial q^\mu} \bigg|_{q=0} \frac{\partial g_0(k-q)}{\partial q^\nu} \bigg|_{q=0},
\]

\[
(82)
\]

In the summations over \( i \) and \( j \) of Eq. (82), terms with \( (i \ even, \ j = \ell) \) or \( (i = \ell, \ j \ even) \) appear \( (\frac{n}{2} - 2) \) times, terms with \( (i \ odd, \ j = \ell) \) appear \( \frac{n}{2} \) times, and terms with \( (i = \ell, \ j \ odd) \) appear \( (\frac{n}{2} - \frac{j}{2}) \) times. Accounting for the cases \( i > j \) and adding the one-kink configurations starting with \( r_1 = (+) \), we finally perform the summation \( \sum_{i=2, \, even}^{n-2} \) and arrive at the result for the contribution of the one-kink configurations to the sum \( \sum_{r_1 \ldots r_n} \) in Eq. (73) for the mixed derivatives:

\[
X_{n(kink)}^{I,J} = (\frac{g}{2})^n \frac{2^{(n-2)}}{a^2} \sum_k \frac{\sum_{k} g_0(k)^{\frac{n}{2} - 2} \left[ g_{0,+}(k)^{\frac{n}{2} - 1} - g_{0,-}(k)^{\frac{n}{2} - 1} \right]}{[g_{0,-}(k) - g_{0,+}(k)]} \times \\
\left\{ (n - 4) g_{0,-}(k) g_{0,+}(k) \frac{\partial g_{0,-}(k-q)}{\partial q^\mu} \bigg|_{q=0} \\
+ \frac{n}{2} g_{0}(k) \frac{\partial [g_{0,-}(k-q) g_{0,+}(k-q)]}{\partial q^\nu} \bigg|_{q=0} \right\} \\
+ \sum_k g_{0}(k)^{\frac{n}{2} - 1} \left[ 1 - \frac{g_{0,-}(k)}{g_{0,+}(k)} \right]^{-2} \left\{ (\frac{n}{2} - 2) \left[ g_{0,+}(k)^{\frac{n}{2}} - g_{0,-}(k)^{\frac{n}{2}} \right] \\
- \frac{n}{2} \left[ g_{0,-}(k) g_{0,+}(k)^{\frac{n}{2} - 1} - g_{0,+}(k) g_{0,-}(k)^{\frac{n}{2} - 1} \right] \right\} \times \\
\frac{\partial g_{0}(k-q)}{\partial q^\mu} \bigg|_{q=0} \frac{\partial [g_{0,+}(k-q)^{-1} g_{0,-}(k-q)]}{\partial q^\nu} \bigg|_{q=0} \right\} \int dx (\partial_{\mu} n_3) \cdot (\partial_{\nu} n_3),
\]

\[
(83)
\]

What remains to be evaluated is the contribution of the two-kink configurations to the mixed derivatives Eq. (75). The first sum in \( \Pi_{ij}^{\mu \nu} \sum_{l,m=1}^{i,j} \) gives zero for configurations where the alternation is broken more than once. Thus we are left only with the sum \( \sum_{l=1}^{i} \sum_{m=i+1}^{j} \), where we first take \( i < j \). The number of the sites after which the first and second kink occur are denoted by \( \ell \) and \( I \), respectively. Note that \( \ell, I \) are either both even or both odd. The sum \( \sum_{i,j=1, i \neq j}^{n-1} \) only yields a nonzero contribution for \( i = \ell \land j = I \), where only the term with \( l = l_{max} = \ell \lor m = m_{max} = I \) contributes to \( \sum_{i=1}^{\ell} \sum_{m=i+1}^{j} \sum_{l,m=1}^{i,j} \). Thus the summations \( \left[ \sum_{i,m=1}^{\ell} + \sum_{i=1}^{j} \sum_{m=i+1}^{j} \sum_{l,m=1}^{i,j} \right] \) yield
\[ \mathcal{T}^{\text{two-kink}}_{ij} = -2^{n-3} (\partial_{\mu} n_3) \cdot (\partial_{\nu} n_3). \]  

Upon the determination of the second derivatives of the function \( \Pi \) for a general two-kink configuration, it turns out that after adding the configurations starting with \( r_1 = (-) \) and \( r_1 = (+) \), the form of \( \partial^2 \Pi^{\text{two-kink}}_{i,j} / \partial q^\mu_i \partial q^\nu_j \) depends only on the distance between the kinks. Introducing \( \delta = t - \xi \), we have

\[
\frac{\partial^2 \Pi^{\text{two-kink}}}{\partial q^\mu_i \partial q^\nu_j} \bigg|_{q_{i,j}=0} = \sum_k g_0(k)^{n/2-2} \left[ g_{0,-}(k)^{n/2} - \frac{g_{0,-}(k)^{n/2}}{g_{0,+}(k)^{n/2}} \right] \left[ g_{0,+}(k)^{n/2} - \frac{g_{0,+}(k)^{n/2}}{g_{0,-}(k)^{n/2}} \right]
\]

\[
\frac{\partial g_0(k-q)}{\partial q^\mu} \bigg|_{q=0} \quad \frac{\partial g_0(k-q)}{\partial q^\nu} \bigg|_{q=0},
\]

(85)

Now we carry out the sum over all possible two-kink configurations by summing over all possible distances \( \delta \) (\( \delta \) runs from 2 to \( n-2 \) over all even values) and weighting each distance \( \delta \) by a factor \( (n-\delta-1) \), which is the number of different two-kink configurations with the same inter-kink distance. We find the following contribution of the two-kink configurations to the sum \( \sum_{r_1,\ldots,r_n} \):

\[
X_{n(\text{two-kink})}^{IJ} = \left( \frac{g}{2} \right) \frac{2^{(n-2)}}{a^2} \left\{ \sum_k g_0(k)^{n/2-2} \left[ g_{0,+}(k)^{n/2-1} - g_{0,-}(k)^{n/2-1} \right] \left[ g_{0,-}(k)^{-1} - g_{0,+}(k)^{-1} \right] \right. \\
\left. \frac{\partial g_0(k-q)}{\partial q^\mu} \bigg|_{q=0} \quad \frac{\partial g_0(k-q)}{\partial q^\nu} \bigg|_{q=0} \right\} (n-2) \int d^2xd\tau \left( \partial_{\mu} n_3 \cdot \partial_{\nu} n_3 \right).
\]

(86)

In order to obtain a closed expression for the fermion determinant, we still have to re-sum the power series in \( n \) from Eq. (83) to infinite order, namely

\[
\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}(G_0 \Sigma^{(0)}_n) = \sum_{n=1}^{\infty} \frac{1}{n} \left[ X_{n(\text{alt})}^{II} + X_{n(\text{kink})}^{II} + X_{n(\text{alt})}^{IJ} + X_{n(\text{kink})}^{IJ} + X_{n(\text{two-kink})}^{IJ} \right],
\]

(87)

where the terms on the right-hand side were defined in Eqs. (74), (75), (78), (83), and (86). The summation is achieved in terms of the polylogarithmic functions \( \text{Li}_0, \text{Li}_1, \text{Li}_2 \), which have the power series expansions \( \text{Li}_0(z) = \sum_{n=1}^{\infty} z^n, \text{Li}_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = \ln(1/z) \), and \( \text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \). From the integral representation of \( \text{Li}_2(z) \) it follows, for example, that \( \frac{d}{dz} \text{Li}_2(z) = -\frac{\ln(1-z)}{z} \). This property will be needed when the derivatives of these functions, as shown in Eqs. (88)–(90) below, are to be performed for a later numerical evaluation. In addition, we use the relation \( \sum_{n=1}^{\infty} n z^n = \frac{z}{(1-z)^2} \). After some more algebra, we obtain for the contribution of the second-order derivatives to the helical part, Eq. (83), of the fermion determinant:

\[
-\sum_{n=2}^{\infty} \frac{1}{n} (X_n^{II} + X_n^{IJ}) = \chi^{\mu\nu} \int d^2xd\tau \sum_{a=1}^{2} \partial_{\mu} n_a \cdot \partial_{\nu} n_a + \bar{\chi}^{\mu\nu} \int d^2xd\tau \partial_{\mu} n_3 \cdot \partial_{\nu} n_3,
\]

(88)

23
where we have introduced the abbreviations

\[
\chi^{\mu \nu} = \frac{1}{8a^2} \frac{\partial^2}{\partial q^\mu \partial q^\nu} \left[ \Phi^-_1(q) + \Phi^+_1(q) \right]_{q=0}
\]

\[
\tilde{\chi}^{\mu \nu} = \frac{1}{8a^2} \frac{\partial^2}{\partial q^\mu \partial q^\nu} \left[ \Phi^-_2(q) + \Phi^+_2(q) - \Phi^-_3(q) - \Phi^+_3(q) \right]_{q=0} \\
+ \frac{1}{4a^2} \left[ (\Phi^-_4)^{\mu \nu} - (\Phi^+_4)^{\mu \nu} - (\Phi^-_5)^{\mu \nu} - (\Phi^+_5)^{\mu \nu} - (\Phi^-_6)^{\mu \nu} - (\Phi^+_6)^{\mu \nu} \right].
\]

Here \( \Phi^\pm_4(q) \) was defined in Eq. (57), and the explicit expressions for the functions \( \Phi^\pm_5(q) \), \( \Phi^\pm_6(q) \), \( (\Phi^+_7)^{\mu \nu} \), \( (\Phi^-_7)^{\mu \nu} \) are listed in the Appendix. As can be seen from Eq. (57) and Eqs. (A1)-(A5), the parameters of the initial model enter in a nonperturbative way into the functions \( \Phi \), which describe the influence of the doping.

**B. Ferromagnetic contribution to the fermion determinant**

We are left with the task of evaluating the part of the fermionic determinant which contains the ferromagnetic fluctuations \( L \), namely the second trace on the right-hand side of Eq. (21). Again we expand up to second order in powers of \( a \). Whereas in the spiral contribution \( \text{Tr} \ln(G_0^{-1} - \Sigma(0)) \) every order in the expansion of the logarithm had to be kept, the ferromagnetic contribution directly generates an expansion in powers of \( a \):

\[
\text{Tr} \ln[1 - a\tilde{G}_0 \Sigma^{(1)} - a^2\tilde{G}_0 \Sigma^{(2)}] = \text{Tr} \left[ -a\tilde{G}_0 \Sigma^{(1)} - a^2\tilde{G}_0 \Sigma^{(2)} - \frac{a^2}{2}\tilde{G}_0 \Sigma^{(1)} \tilde{G}_0 \Sigma^{(1)} \right].
\]

In order to evaluate this trace, we must determine the inverse of \( \tilde{G}_0^{-1} = G_0^{-1} - \Sigma(0) \). While \( G_0^{-1} \) is diagonal in spin and momentum indices (see Eq. (14)), \( \Sigma(0) \) splits into two parts depending on the wave vectors \( (k - k' - k_s) \) and \( (k - k' + k_s) \), respectively (see Eq. (27)). Since we consider long-wavelength spin fields, \( (k - k' \pm k_s) \) must be small. Thus, \( \Sigma(0) \) has matrix elements only on four side diagonals between \( k \) and \( k \pm k_s \). Hence \( \tilde{G}_0^{-1} \) has the form of a band matrix, where the number of matrix elements is determined by the number of \( k \) points and every \( k \) point splits into a \( 2 \times 2 \) spin space. Obviously, it is impossible to invert \( \tilde{G}_0^{-1} \) exactly. However, \( \tilde{G}_0 \) can be found perturbatively in powers of \( a \) by finding a matrix \( \tilde{G}_0 \) which, when multiplied on \( \tilde{G}_0^{-1} \), gives a diagonal matrix \( D \) plus off-diagonal terms of \( \mathcal{O}(a) \):

\[
\tilde{G}_0^{-1} \tilde{G}_0 = D + \mathcal{O}(a).
\]

This works because the square of the matrix \( \Sigma(0) \) is proportional to the unit matrix \( (\Sigma(0))_{k,k'}^2 = g^2 \delta_{aa} \delta_{kk'} \). Based on Eq. (27), we define \( (\Sigma(0))_{ak,ak'} = \frac{g}{2} \sigma^a n^a_\pm (k - k' \pm k_s) \), so that \( \Sigma(0) = \Sigma_+ + \Sigma_- \). Now we try as an ansatz for \( \tilde{G}_0 \):

\[
\tilde{G}_0 = P_1 + P_2 + \Sigma(0),
\]
multiply it onto $\tilde{G}_0^{-1}$ from the right and then choose $P_1$ and $P_2$ so that Eq. (92) can be fulfilled. An appropriate choice of $P_1$, $P_2$ should make terms which are manifestly non-diagonal in $k$-space either zero or small and of $O(a)$. Moreover, terms which were already diagonal in $k$-space should not lose that property through multiplication by $P_{1,2}$. This is accomplished by choosing

$$P_1 = \Sigma_+^{(0)} G_{0,-}^{-1} / g_0^2,$$
$$P_2 = \Sigma_-^{(0)} G_{0,+}^{-1} / g_0^2,$$  \quad (94)

where $G_{0,-}$ and $G_{0,+}$ are exactly diagonal in spin and momentum space. Their $k$-dependence is chosen to be

$$(G_{0,\pm}^{-1})_{ak,\alpha'k'} = g_0^{-1}(k) \delta_{\alpha\alpha'} \delta_{kk'}.$$  \quad (95)

Then from the multiplication

$$(G_0^{-1} - \Sigma_-^{(0)} - \Sigma_+^{(0)})(P_1 + P_2 + \Sigma_-^{(0)} + \Sigma_+^{(0)}) = D - F + K_- + K_+$$  \quad (96)

we obtain a purely diagonal contribution

$$(D)_{ak,\alpha'k'} = \left[ \frac{1}{2} g_0^{-1}(k) g_+(k) - g^2 \right] \delta_{\alpha\alpha'} \delta_{kk'},$$  \quad (97)

a contribution being approximately diagonal in momentum space but off-diagonal in spin space

$$(F)_{ak,\alpha'k'} = \frac{1}{2} g_0^{-1}(k) \sigma^a_{\alpha\alpha'} n^0_a (k - k') g_-(k'),$$  \quad (98)

and terms which are off-diagonal in spin and momentum space, but which are of $O(a)$:

$$K_\pm = G_0^{-1} \Sigma_\pm^{(0)} - \Sigma_\pm^{(0)} G_0^{-1}$$

$$= [g_0(k) - g_0(k' \mp k_s)] \sigma^a_{\alpha\alpha'} n^a_a (k - k' \pm k_s).$$  \quad (99)

We have used the abbreviation:

$$g_\pm(k) = g_0^{-1}(k) \pm g_0^{-1} (k).$$  \quad (100)

In order to diagonalize the remaining problem, we multiply Eq. (96) by a matrix $(\tilde{D} + \tilde{F})$ from the right, where

$$(\tilde{D})_{ak,\alpha'k'} = g_0^{-2}(k) \left[ \frac{1}{2} g_0^{-1}(k) g_+(k) - g^2 \right] \delta_{\alpha\alpha'} \delta_{kk'},$$  \quad (101)

$$(\tilde{F})_{ak,\alpha'k'} = \frac{1}{2} g_0^{-1}(k) \sigma^a_{\alpha\alpha'} n^0_a (k - k') g_0^{-1}(k').$$  \quad (102)
This leads to the desired expression Eq. (102). The matrix $\bar{G}_0$ which we have been looking for is thus given by

$$\bar{G}_0 = \left[ \Sigma^{-}(0) \Sigma^{+}(0) G^{-1}_{0,-}/g^2 + g^2 \Sigma^{-}(0) \Sigma^{+}(0) G^{-1}_{0,+}/g^2 + \Sigma^{-}(0) + \Sigma^{+}(0) \right] \left[ \bar{D} + \bar{F} \right], \quad (103)$$

and for the diagonal matrix $D$ in Eq. (102) we obtain

$$(D)_{\alpha k, \alpha' k'} = D  \bar{D} - F \bar{F} =$$

$$= \varphi^{-2}(k) \left[ g^4 - g^2 g^{-1}_{0,}(k) \varphi_{+}(k) + g^{-2}_{0,}(k) g^{-1}_{0,+}(k) g^{-1}_{0,-}(k) \right] \delta_{\alpha \alpha'} \delta_{kk'}. \quad (104)$$

In order to simplify the notation, we define

$$d(k) = g^4 - g^2 g^{-1}_{0,}(k) \varphi_{+}(k) + g^{-2}_{0,}(k) g^{-1}_{0,+}(k) g^{-1}_{0,-}(k). \quad (105)$$

The small off-diagonal terms are

$$aR = (D \bar{F} - F \bar{D}) + (K_{-} + K_{+})(\bar{D} + \bar{F}). \quad (106)$$

From Eq. (92), the inverse of $\bar{G}_0^{-1}$ to first order in $a$ is then found to be

$$\bar{G}_0 = \bar{G}_0 D^{-1} - a \bar{G}_0 D^{-1} R D^{-1}. \quad (107)$$

This has to be inserted into Eq. (101), together with the expressions Eqs. (28), (29) for the first and second orders of the fermionic self-energy. After inserting $\bar{G}_0$ from Eq. (103), $\text{Tr}[a \bar{G}_0 \Sigma^{(1)}]$ then becomes a sum of 240 terms, while $\text{Tr}[a^2 \bar{G}_0 \Sigma^{(2)}]$ contains 112 terms and $\text{Tr}[a^2 \bar{G}_0 \Sigma^{(1)} \bar{G}_0 \Sigma^{(1)}]$ becomes a sum of 1600 terms. (We classify different terms by different combinations of the order parameter fields. Every individual term still comprises the momentum- and spin-space summations arising from the matrix multiplications). These numbers grow considerably when the strings of Pauli matrices contained in these traces are evaluated, e.g. a string of six Pauli matrices leads to a sum of 15 different permutations of contractions. While strings of even numbers of Pauli matrices are carried out using the trace reduction formula Eq. (36), strings of odd numbers of Pauli matrices are evaluated by replacing two Pauli matrices according to the identity $\sum_{\alpha_2} \sigma_{\alpha_1 \alpha_2}^{a_1} \sigma_{\alpha_2 \alpha_3}^{a_2} = \delta^{a_1 a_2} \delta_{\alpha_1 \alpha_3} + i \varepsilon^{a_1 a_2 a_3} \sigma^{a_3}_{\alpha_1 \alpha_3}$. Obviously, a string of an odd number of Pauli matrices leads to inner products and vector products of the involved vectors. The traces are evaluated with an algebraic program which we developed using Mathematica [39]. It reduces the trace over the Pauli matrices, performs the contractions of the vectors, implements the constraint Eq. (19), and takes the trace in

\[\text{The program \texttt{TraceEval} is available via anonymous ftp from ftp.physik.uni-wuerzburg.de as /pub/dissertation/klee/TraceEval.m.}\]
momentum space by keeping only terms which are diagonal in the momentum indices. The result for the ferromagnetic contribution to the fermion determinant is

\[
S_{\text{ferro}} = -\int d^2x d\tau \left\{ \chi_1 L^2 + \chi_2 \left[ (L \cdot n_1)^2 + (L \cdot n_2)^2 \right] + \chi_3 (L \cdot n_3)^2 \\
+ \frac{\chi_4^\mu}{a} \left[ (L \cdot n_1)(n_1 \cdot \partial_\mu n_3) + (L \cdot n_2)(n_2 \cdot \partial_\mu n_3) \right] \\
- 2\frac{\chi_5^\mu}{a} L \cdot \partial_\mu n_3 + \frac{\chi_5^\mu}{a} (L \cdot n_3)(n_1 \cdot \partial_\mu n_2) \\
- \frac{\chi_6^\mu}{a} \left[ (L \cdot n_1)(n_2 \cdot \partial_\mu n_3) - (L \cdot n_2)(n_1 \cdot \partial_\mu n_3) \right] \\
+ \frac{\chi_7^\mu}{a} L \cdot \left( n_1 \times \partial_\mu n_1 + n_2 \times \partial_\mu n_2 \right) + \frac{\chi_8^\mu}{a} L \cdot \left( n_3 \times \partial_\mu n_3 \right) \right\},
\]

(108)

The quantities \(\chi_1 \ldots \chi_3, \chi_4^\mu \ldots \chi_8^\mu\) are generalized susceptibilities of the fermions in the presence of the spin fields. The explicit expressions are listed in the Appendix. The functions \(\chi\) consist of summations over \(\epsilon_n\) and \(k\) of polynomials of the inverse free fermionic propagator \(g_0^{-1}\) and its derivative. Shifting the summation index \(k \rightarrow -k\) and using the symmetry of the free propagator, Eq. (14), it can be shown that

\[
\chi_4^\mu = 0 \quad \text{for} \quad \mu = 0, \\
\chi_5^\mu = \chi_6^\mu = \chi_7^\mu = \chi_8^\mu = 0 \quad \text{for} \quad \mu = 1, 2,
\]

(109)

i.e. \(\chi_4^\mu\) is non-zero only for the spatial components \(\mu = 1, 2\), while \(\chi_5^\mu, \chi_6^\mu, \chi_7^\mu, \chi_8^\mu\) are non-zero only for the time-component \(\mu = 0\). Note that the susceptibilities are proportional to \(d^{-1}(k)\) or \(d^{-2}(k)\), where \(d(k)\) was defined in Eq. (105). The zeroes of \(d(k)\) determine the dispersion relations of the holes in the helical spin background. Since \(d(k)\) is fourth order in the free inverse propagators \(g_0^{-1}\), which contain contributions from two bands, \(d(k)\) is eighth order in \(i\epsilon_n\), and eight hole bands will result. This additional splitting in comparison to the antiferromagnetic case, in which four hole bands were obtained [15], results from the fact that the momentum transfer on the fermions interacting with the spin field is now \(Q \pm \delta Q\) instead of \(Q\) as in the antiferromagnetic case.

C. Contribution of the Heisenberg part

Finally we have to calculate the continuum limit of the pure spin part of the action, Eq. (3). The study of the continuum limit of this action, to which our model simplifies in the absence of doping, has in itself attracted much interest. In order to investigate the effect of frustration on the stability of the ordered state against quantum fluctuations, field-theoretic mappings have been performed for the Heisenberg antiferromagnet on a triangular lattice [3, 8], for frustrated Heisenberg antiferromagnets on a chain and on \(d\)-dimensional lattices.
The frustrating effect of either the lattice geometry or the competition between nearest-neighbour and higher-order interactions leads to noncollinear classical ground states, so that the order parameter is an element of SO(3). Thus our gradient expansion of the pure spin part, carried out in position space, is similar to the above-mentioned treatments. However, in these works the spins in the ground state have a periodicity, so that one can deal with a finite number of sublattices. This does not apply in the case of the incommensurate spiral.

It is useful to extract the antiferromagnetic modulation from \( n \) by writing \( n_{pq} = (-1)^{(p+q)} m_{pq} \), where we define

\[
m_{pq} = n_1 \cos(\delta Q \cdot r_{pq}) - n_2 \sin(\delta Q \cdot r_{pq})
\]

(110)

First, we consider the Berry phase. We inject Eq. (24) for \( S_{pq} \) and expand \( A(S_{pq}/S) \) and \( \partial_\tau S_{pq} \) to first order in \( a \). We exploit the gauge freedom that we have in the definition of \( A \) to choose it an even or an odd function of its argument, use the constraint satisfied by the monopole potential, Eq. (4), and the fact that total time derivatives drop out upon \( \tau \)-integration. Discarding terms which are of third or higher order in derivatives of \( m \), we obtain:

\[
S_{\text{Berry}} = \int_0^\beta d\tau i S \sum_{pq} \left[ (-1)^{(p+q)} A(m_{pq}) \cdot \partial_\tau m_{pq} + a L \cdot (m_{pq} \times \partial_\tau m_{pq}) \right].
\]

(111)

From the term involving the vector potential in Eq. (111), we first consider one chain of lattice sites by holding the index \( q \) fixed:

\[
S_A(q) = \int_0^\beta d\tau i S \sum_p (-1)^p A(m_{pq}) \cdot \partial_\tau m_{pq} = \int_0^\beta d\tau i S \sum_{p \text{ even}} \left[ A(m_{pq}) \cdot \partial_\tau m_{pq} - A(m_{p+1,q}) \cdot \partial_\tau m_{p+1,q} \right].
\]

(112)

Expanding our smooth order parameter fields according to \( n_{1,2}(p + 1, q) = n_{1,2}(p, q) + a \partial_x n_{1,2}(p, q) \), we find:

\[
m_{p+1,q} = m_{pq} \cos(\delta Q_x a) - \tilde{m}_{pq} \sin(\delta Q_x a) \\
+ a (\Delta_x m_{pq}) \cos(\delta Q_x a) - a (\Delta_x \tilde{m}_{pq}) \sin(\delta Q_x a),
\]

(113)

where we defined \( \tilde{m}_{pq} = n_1 \sin(\delta Q \cdot r_{pq}) + n_2 \cos(\delta Q \cdot r_{pq}) \). Note that in contrast to collinear spin configurations, the zeroth order contribution to \( m_{p+1,q} \) is not \( m_{pq} \), but a vector which we denote by \( m^0_{p+1,q} = m_{pq} \cos(\delta Q_x a) - \tilde{m}_{pq} \sin(\delta Q_x a) \). With expressions like \( \Delta_x m_{pq} \) we mean:

\[
\Delta_x m_{pq} = (\partial_x n_1) \cos(\delta Q \cdot r_{pq}) - (\partial_x n_2) \sin(\delta Q \cdot r_{pq}).
\]

(114)
After some algebra one obtains
\[ A(m_{p+1,q}) \cdot \partial_x m_{p+1,q} = A(m_{p+1,q}) \cdot \partial_x m_{p+1,q} \]
\[ -a m_{p+1,q}^0 \cdot (\Delta_x m_{p+1,q}^0 \times \Delta_y m_{p+1,q}^0). \]  \quad (115)

Now, the terms \( \int_0^\beta d\tau A(m_{pq}) \cdot \partial_\tau m_{pq} \) and \( \int_0^\beta d\tau A(m_{p+1,q}) \cdot \partial_\tau m_{p+1,q}^0 \) describe the area of the caps bounded by the paths of \( m_{pq} \) and \( m_{p+1,q}^0 \), respectively. Since \( m_{pq} \) and \( m_{p+1,q}^0 \) are connected by a space-time independent rotation about the axis \( n_3 \) by the angle \( (\delta Q, a) \), the solid angles spanned by their paths are equal. Thus the corresponding terms cancel when (113) is inserted into Eq. (112). After some tedious algebra, we get for the remaining term:
\[ m_{p+1,q}^0 \cdot (\Delta_x m_{p+1,q}^0 \times \Delta_y m_{p+1,q}^0) = -n_1 \cdot (\partial_\tau n_1 \times \partial_\tau n_1) \cos(\delta Q \cdot r_{pq} + \delta Q_x) \]
\[ + n_2 \cdot (\partial_\tau n_2 \times \partial_\tau n_2) \sin(\delta Q \cdot r_{pq} + \delta Q_x) \]  \quad (116)

Employing the relations \( \epsilon^{abc} n_3^a \partial_\tau n_3^b \partial_\tau n_3^c = \epsilon_{def} \partial_\tau n_3^e \partial_\tau n_3^f \), this result can be shown to agree with the one obtained for a spiral configuration in a frustrated Heisenberg chain within a different method [36]. Due to dimensional reasons, we must have: \( \delta Q = \delta \varphi / a \). We assume that in the continuum limit the angle \( \delta \varphi \) between neighbouring spins remains constant. Thus in the limit \( a \to 0 \) the terms containing \( \cos(\delta \varphi \cdot x / a) \), \( \sin(\delta \varphi \cdot x / a) \) oscillate very rapidly and drop out. Then the total Berry phase is given by the second term in Eq. (111), from which we keep only non-oscillatory contributions. Taking the continuum limit \( \lim_{a \to 0} \sum_{pq} a^2 = \int d^2 x \) leads to
\[ S_{\text{Berry}} = \int_0^\beta d\tau \int d^2 x \frac{iS}{2a} L \cdot (n_1 \times \partial_\tau n_1 + n_2 \times \partial_\tau n_2). \]  \quad (117)

We now turn to the real term in Eq. (3). We attribute to each site the quantity
\[ \frac{1}{2} J_H \left\{ \sum_{\delta = \pm 1} S_{pq} \cdot S_{p+\delta,q} + \sum_{\delta = \pm 1} S_{pq} \cdot S_{p,q+\delta} \right\} \]  \quad (118)
and then perform the sum over all sites. As before, we insert Eq. (24) for \( S \) and expand the fields at site \( (p+\delta, q) \) in terms of the corresponding fields at site \( (p, q) \) using Eq. (113). Again, we eliminate short-range oscillatory contributions. After some lengthy but straightforward algebra we obtain
\[ S_{\text{real}} = - \int_0^\beta d\tau \int d^2 x J_H S^2 \left\{ \sum_{\mu = 1} a^{-1} \sin(\delta Q_\mu a) n_1 \cdot \partial_\mu n_2 \right. \]
\[ + \sum_{\mu = 1} \frac{1}{4} \cos(\delta Q_\mu a) \left[ (\partial_\mu n_1)^2 + (\partial_\mu n_2)^2 \right] + \left[ 2 + \sum_{\mu = 1} \cos(\delta Q_\mu a) \right] L^2 \]
\[ + \left[ \frac{1}{2} \sum_{\mu = 1} \cos(\delta Q_\mu a) - \frac{1}{2} \sum_{\mu = 1} \cos^2(\delta Q_\mu a) \right] \left[ (L \cdot n_1)^2 + (L \cdot n_2)^2 \right] \} . \]  \quad (119)

Note that here the index \( \mu \) runs only over the space components, namely \( \mu = 1, 2 \).
V. DISCUSSION OF THE CONTINUUM THEORY

In order to express our continuum action in terms of the SO(3) order parameter \( \hat{Q} = (n_1, n_2, n_3) \), the ferromagnetic fluctuations \( L \) have to be integrated out. Since our resulting action is linear and quadratic in \( L \), this can be done by extremizing the action with respect to \( L \). Inserting Eqs. (108), (117), and (119) into the saddle-point equation \( \frac{\delta}{\delta L^a} (S_{\text{ferro}} + S_{\text{Berry}} + S_{\text{real}}) = 0 \), we obtain

\[
[\tilde{\chi}_1 \delta^{ab} + \tilde{\chi}_2 \left( n_1^a n_1^b + n_2^a n_2^b \right) + \chi_3 n_3^a n_3^b] L^b = v^a ,
\]

where \( v \) denotes the sum of all terms which were multiplied by \( L \) linearly in the action

\[
v^a = \frac{1}{2a} \left[ -2\chi_4 \mu \partial_\mu n_3^a + \chi_4^2 \sum_{i=1}^2 n_i^a \partial_i n_3^a - \chi_5 n_3^a \epsilon^{bcd} n_3^b n_2^c \partial_\mu n_2^d \\
+ \chi_6 \sum_{i,j=1}^2 n_i^a \epsilon^{bcd} n_2^c \partial_j n_3^d + \tilde{\chi}_7^\mu \epsilon^{abc} \sum_{i=1}^2 n_i^b \partial_\mu n_i^c + \chi_7^\mu \epsilon^{abc} n_3^b \partial_\mu n_3^c \right] ,
\]

and we have introduced the following abbreviations to incorporate fermionic and Heisenberg contributions:

\[
\begin{align*}
\tilde{\chi}_1 &= \chi_1 - J_R S^2 \left[ 2 + \sum_{\mu=1}^2 \cos(\delta Q_\mu a) \right] , \\
\tilde{\chi}_2 &= \chi_2 + J_R S^2 \left[ 2 + \frac{1}{2} \sum_{\mu=1}^2 \cos(\delta Q_\mu a) - \frac{1}{2} \sum_{\mu=1}^2 \cos^2(\delta Q_\mu a) \right] , \\
\tilde{\chi}_7^{\mu=0} &= \chi_7^{\mu=0} - \frac{i S a}{2} .
\end{align*}
\]

We solve the saddle-point equation for \( L \) by multiplying both sides of Eq. (120) by the inverse of the matrix \( [\tilde{\chi}_1 \delta^{ab} + \tilde{\chi}_2 \left( n_1^a n_1^b + n_2^a n_2^b \right) + \chi_3 n_3^a n_3^b] \) and obtain

\[
a L^a = - \zeta_1 \tilde{\chi}_7^a \epsilon^{abc} \left( n_1^b \partial_\mu n_1^c + n_2^b \partial_\mu n_2^c \right) - \zeta_1 \chi_8^a \epsilon^{abc} n_3^b \partial_\mu n_3^c \\
- n_1^a \left[ \zeta_2^a n_1^b \partial_\mu n_3^b - \zeta_3^a n_2^b \partial_\mu n_2^b \right] \\
- n_2^a \left[ \zeta_2^a n_2^b \partial_\mu n_3^b + \zeta_3^a n_1^b \partial_\mu n_1^b \right] \\
+ \zeta_4^a n_3^a n_1^b \partial_\mu n_2^b + 2 \zeta_1 \chi_8^a \partial_\mu n_3^c .
\]

The newly introduced abbreviations \( \zeta_1, \zeta_2^a, \zeta_3^a, \) and \( \zeta_4^a \) are defined in the Appendix. Recall that the upper index 0 on the generalized fermionic susceptibilities denotes their frequency component, which accompanies the time component of the field derivatives, while the index \( \alpha = 1, 2 \) runs over the wave-vector components of the susceptibilities, which accompany the spatial components of the field derivatives. In the above solution for \( L \), Eq. (123), we have applied the results of Eq. (109) to show explicitly how \( L \) is composed of first-order space and time derivatives of \( \hat{Q} = (n_1, n_2, n_3) \). Our expression for \( L \) must now be reinserted.
into Eqs. (108), (117), (119) for $S_{\text{ferro}}$, $S_{\text{Berry}}$, and $S_{\text{real}}$. In order to simplify the result, the following identities, derived from the constraint on the order parameter, Eq. (19), are needed:

\[
(n_1 \cdot \partial_\mu n_3)(n_1 \cdot \partial_\nu n_3) + (n_2 \cdot \partial_\mu n_3)(n_2 \cdot \partial_\nu n_3) = \partial_\mu n_3 \cdot \partial_\nu n_3,
\]

\[
(n_1 \cdot \partial_\mu n_3)n_2 \cdot \partial_\nu n_3) - (n_1 \cdot \partial_\nu n_3)(n_2 \cdot \partial_\mu n_3) = n_3 \cdot (\partial_\mu n_3 \times \partial_\nu n_3),
\]

\[
\partial_\nu n_3 \cdot (n_1 \times \partial_\mu n_1 + n_2 \times \partial_\mu n_2) = n_3 \cdot (\partial_\mu n_3 \times \partial_\nu n_3).
\] (124)

The total $L$-dependent part of the action can then be cast in the form

\[
S_L = -\tilde{\chi}^{\mu\nu} \int d^2x d\tau (\partial_\mu n_1 \cdot \partial_\nu n_1 + \partial_\mu n_2 \cdot \partial_\nu n_2)
\]

\[
- (\tilde{\chi}^{\mu\nu} - \tilde{\chi}^{\nu\mu}) \int d^2x d\tau \partial_\mu n_3 \cdot \partial_\nu n_3
\]

\[
- \tilde{\chi}^{\mu\nu} \int d^2x d\tau n_3 \cdot (\partial_\mu n_3 \times \partial_\nu n_3),
\] (125)

where the prefactors are given by

\[
\tilde{\chi}^{\mu\nu} = \frac{1}{2a^2} [\tilde{x}_1 + \chi_3]^{-1} \left[ \frac{1}{4} \chi_5^{\mu} \chi_5^{\nu} + \chi_7^{\mu} \chi_7^{\nu} - \chi_5^{\mu} \chi_7^{\nu} \right],
\] (126)

\[
\bar{\chi}^{\mu\nu} = \frac{1}{4a^2} [\tilde{x}_1 + \tilde{x}_2]^{-1} \left[ \chi_4^{\mu} \chi_4^{\nu} + \chi_6^{\mu} \chi_6^{\nu} + \chi_7^{\mu} \chi_7^{\nu} + \chi_8^{\mu} \chi_8^{\nu} + 2\chi_6^{\mu} \chi_7^{\nu} + 2\chi_7^{\mu} \chi_8^{\nu} \right],
\] (127)

\[
\bar{\chi}^{\mu\nu} = \frac{1}{2a^2} [\tilde{x}_1 + \tilde{x}_2]^{-1} \left[ \chi_6^{\mu} + \chi_7^{\mu} + \chi_8^{\mu} \right] \chi_4^{\nu}.
\] (128)

From Eqs. (64), (68) = (70) for the spiral contribution to the fermion determinant, Eq. (119) for the contribution of the Heisenberg part, and Eq. (125) for the contribution of the $L$-dependent part of the action, we find the total continuum theory to be given by an SO(3) quantum nonlinear $\sigma$ model, a term linear in the derivatives and a geometric term which is third order in the fields and contains first-order derivatives with respect to time and space,

\[
S = S_{\text{qnl}} + S_{\text{lin}} + S_{\text{geom}}.
\] (129)

The final form of our quantum nonlinear $\sigma$ model action is

\[
S_{\text{qnl}} = -\int d^2x d\tau \sum_{a=1}^3 p_a^{\mu\nu} \partial_\mu n_a(x) \cdot \partial_\nu n_a(x),
\] (130)

where $p_1^{\mu\nu} = p_2^{\mu\nu} \neq p_3^{\mu\nu}$. The coupling constants of the model are given as a function of the microscopic parameters and the generalized fermionic susceptibilities:

\[
p_1^{\mu\nu} = p_2^{\mu\nu} = \tilde{\chi}^{\mu\nu} - \chi^{\mu\nu} + \frac{J_H S^2}{4} \cos(\delta Q \mu a) \delta^{\mu\nu},
\] (131)

\[
p_3^{\mu\nu} = \tilde{\chi}^{\mu\nu} - \bar{\chi}^{\mu\nu} - \bar{\chi}^{\mu\nu}.
\] (132)
Equivalently to Eq. (130), the model can be written in SO(3) matrix form:

$$S_{\text{qu}l\text{erm}} = -\int d^2 x d\tau \text{Tr} \left[ \hat{P}^{\mu \nu} \partial_\mu \hat{Q}^T(x) \partial_\nu \hat{Q}(x) \right],$$

(133)

where the coefficient matrix is given by

$$\hat{P}^{\mu \nu} = \begin{pmatrix} p_1^{\mu \nu} & 0 & 0 \\ 0 & p_1^{\mu \nu} & 0 \\ 0 & 0 & p_3^{\mu \nu} \end{pmatrix}. \quad (134)$$

Among the nine fields $Q_{ab}(x) = n_a^b(x)$, taking into account the constraint Eq. (13), there are three independent fluctuating fields. Hence, our continuum action describes three massless modes, or spin waves, which result from the complete breaking of the O(3) rotation group by the ground state. Each spin wave excitation corresponds to infinitesimal rotations about one of the $n_a$’s. One Goldstone mode describes variations of the spin orientation within the plane defined by the classical ground state, and thus corresponds to infinitesimal rotations about the axis $n_3$, while the other two Goldstone modes describe out-of-plane fluctuations, and are related to rotations about $n_1$ and $n_2$. Eqs. (130) or (133) describe the linear spectrum and the interactions of these spin-wave excitations.

The coefficient matrix $\hat{P}$ in Eq. (134) is represented in internal space. Every matrix element of $\hat{P}^{\mu \nu}$, however, is still a matrix in the space-time indices $\mu$ and $\nu$. We will now consider the symmetry properties of the matrices $p_1^{\mu \nu}$ and $p_3^{\mu \nu}$. As regards $p_1^{\mu \nu}$, it can be seen from Eq. (131) that it is symmetric in $\mu$, $\nu$ when looking at the definitions of $\chi^{\mu \nu}$, Eq. (89), and $\bar{\chi}^{\mu \nu}$, Eq. (123), and taking into account Eq. (109), which implies that $\bar{\chi}^{\mu \nu}$ is nonzero only for $\mu = \nu = 0$. Inspection of the terms appearing in Eq. (132) for $p_3^{\mu \nu}$ (see Eqs. (104), (126) and (127)), shows that $p_3^{\mu \nu}$ contains also antisymmetric contributions. However, since we sum over $\mu$ and $\nu$ in Eq. (134) or (133), and the term $(\partial_\mu n_a(x) \cdot \partial_\nu n_a(x))$ is symmetric in $\mu$, $\nu$, the antisymmetric contributions cancel and we are left with a symmetric coefficient matrix.

Let us now consider the matrix elements $\hat{P}^{\mu \nu}$ which mix frequency and wave-vector indices (or, in the corresponding field derivatives, space and time indices). For the spiral contributions to the generalized fermionic susceptibilities, it can be shown by performing the derivatives and using the symmetry of the free propagator, Eq. (14), that for $\alpha = 1, 2$

$$\frac{\partial^2}{\partial q^a \partial q^a} \Phi_{1,2,3}^\alpha(q) \bigg|_{q=0} = -\frac{\partial^2}{\partial q^a \partial q^a} \Phi_{1,2,3}^{-\alpha}(q) \bigg|_{q=0}, \quad (135)$$

$$\left(\Phi_{4,5,6}^+\right)^{\alpha \alpha} = -\left(\Phi_{4,5,6}^-\right)^{\alpha \alpha}. \quad (136)$$

As a consequence, $\chi^{\alpha \alpha} = \bar{\chi}^{\alpha \alpha}$ and $\bar{\chi}^{\alpha \alpha} = \bar{\chi}^{\alpha \alpha}$ vanish, as can be seen from Eqs. (89) and (104). For the ferromagnetic contributions, we know that $\bar{\chi}^{\mu \nu} \neq 0$ only for $\mu = \nu = 0$ and that $\bar{\chi}^{\mu \nu}$ does also not mix frequency and wave-vector indices (see Eqs. (109) and (127)), so that

32
Therefore, the matrices $p_1^{\mu\nu}$ and $p_3^{\mu\nu}$ are of block-diagonal form (for clarity, we will use $\tau$ instead of 0 for the frequency component of the coefficients, and $\alpha = x, y$ instead of $\alpha = 1, 2$ for their wave-vector components),

$$
\hat{p}_a = \begin{pmatrix}
\hat{p}_a^{\tau \tau} & 0 & 0 \\
0 & \hat{p}_a^{xx} & \hat{p}_a^{xy} \\
0 & \hat{p}_a^{yx} & \hat{p}_a^{yy}
\end{pmatrix}, \quad a = 1, 3.
$$

The action, Eq. (130), can then be written as

$$
S_{qnl\sigma m} = -\int d^2x d\tau \text{Tr} \left[ \hat{P}^{\tau \tau} \partial_\tau \hat{Q}^T(x) \partial_\tau \hat{Q}(x) + \sum_{\alpha, \beta = x, y} \hat{P}^{\alpha \beta} \partial_\alpha \hat{Q}^T(x) \partial_\beta \hat{Q}(x) \right].
$$

This form, in which the coefficient of the time derivative term is a tensor in spin space and the coefficient of the space derivative term is a tensor both in spin space and in real space, was suggested as the most general case for systems with noncollinear spin orientation within a hydrodynamical theory [41]. For some special cases of the direction of the spiral wave vector, the matrices $\hat{p}_a$ become diagonal. If, as suggested by experiments [19] on cuprate superconductors and by theoretical studies [22] of models related to the spin-fermion model, the spiral wave vector lies along the (1,0) or (0,1) directions of the lattice, i.e. $\mathbf{k}_s = (\frac{\pi}{a} \pm \delta Q_x, \frac{\pi}{a})$ or $\mathbf{k}_s = (\frac{\pi}{a}, \frac{\pi}{a} \pm \delta Q_y)$, then, using the symmetry relation Eq. (15) it can be shown that

$$
\begin{align*}
\frac{\partial^2}{\partial q^x \partial q^y} \Phi_{1,2,3}^\pm(q) \bigg|_{q=0} &= 0, \\
(\Phi_{4,5,6}^\pm)^{xy} &= 0, \\
\chi^{xy} &= 0.
\end{align*}
$$

Thus, Eqs. (131) and (132) yield

$$
\hat{p}_a = \begin{pmatrix}
\hat{p}_a^{\tau \tau} & 0 & 0 \\
0 & \hat{p}_a^{xx} & \hat{p}_a^{xy} \\
0 & \hat{p}_a^{yx} & \hat{p}_a^{yy}
\end{pmatrix}.
$$

In the general case of Eq. (138), the symmetric $(2 \times 2)$ matrix in $x$ and $y$ can be diagonalized, which amounts to changing to a new basis which is a linear combination of the lattice basis vectors.

In order to make the properties of the spin waves resulting from our theory more transparent, we write the action in another equivalent form
\[ S_{\text{qnl}_{\text{sm}}} = - \int d^2xd\tau \text{Tr} [\hat{P}_{\text{diag}}^{\mu\nu} (\hat{Q}^\tau \partial_{\mu} \hat{Q})^2] \]  
(142)

(where we assume the coefficient matrix to be diagonalized). Now we represent the three degrees of freedom contained in the antisymmetric spin-space matrix \((\hat{Q}^{-1} \partial_{\mu} \hat{Q})\) according to \((\hat{Q}^{-1} \partial_{\mu} \hat{Q})^{12} = A_\mu, (\hat{Q}^{-1} \partial_{\mu} \hat{Q})^{13} = B_\mu, \) and \((\hat{Q}^{-1} \partial_{\mu} \hat{Q})^{23} = C_\mu, \) which leads to the action

\[
S_{\text{qnl}_{\text{sm}}} = - \int d^2xd\tau \left\{ 2p^{\tau\tau}_1 A_\tau^2 + (p^{\tau\tau}_1 + p^{\tau\tau}_3)(B_\tau^2 + C_\tau^2) + \sum_{\alpha=x,y} [2p^{\alpha\alpha}_1 A_\alpha^2 + (p^{\alpha\alpha}_1 + p^{\alpha\alpha}_3)(B_\alpha^2 + C_\alpha^2)] \right\}
\]  
(143)

The dispersion relations for the three spin-waves can now be read off:

\[
\omega_A^2 = \frac{p^{\tau\tau}_1}{p^{\tau\tau}_1} (k_A)_x^2 + \frac{p^{yy}_1}{p^{\tau\tau}_1} (k_A)_y^2,
\]

\[
\omega_{B,C}^2 = \frac{(p^{xx}_1 + p^{xx}_3)}{(p^{\tau\tau}_1 + p^{\tau\tau}_3)} (k_{B,C})_x^2 + \frac{(p^{yy}_1 + p^{yy}_3)}{(p^{\tau\tau}_1 + p^{\tau\tau}_3)} (k_{B,C})_y^2.
\]  
(144)

Thus, the spin-waves have different velocities which may depend on the direction of propagation, as was suggested by the hydrodynamical theory \[11\]. It is the fact that the theory is not Lorentz invariant, i.e. that the coupling-constant matrix \(\hat{P}_{\mu\nu}^{\tau\tau}\) for the time components is not proportional to that for the space components, which allows the three spin-waves to have different velocities. In our case, the two out-of-plane modes have the same spin-wave velocity, which differs from that of the in-plane mode. In the case in which the coupling-constant matrix is isotropic, i.e. \(\hat{P}^{xx} = \hat{P}^{yy}\), the spin-wave velocities become

\[
c_A = \sqrt{p^{xx}_1 / p^{\tau\tau}_1}
\]  
(145)

in the plane spanned by the classical ground state, and

\[
c_{B,C} = \sqrt{(p^{xx}_1 + p^{xx}_3) / (p^{\tau\tau}_1 + p^{\tau\tau}_3)}
\]  
(146)

out of this plane.

Now we briefly consider the internal-space symmetries of the action which we have obtained in Eq. \((130)\) or \((133)\). This action is invariant under global left \(O(3)\) rotations \(\hat{Q} \rightarrow \hat{U} \hat{Q}\), where \(\hat{U} \in O(3)\). This symmetry corresponds to the usual invariance under rotations of the basis vectors in spin space: \(n^a_c \rightarrow U^{ab} n^b_c\). In addition, the action is invariant under global right transformations \(\hat{Q} \rightarrow \hat{Q} \hat{V}\) if \([\hat{P}, \hat{V}] = 0\). The dimension of the group to which \(\hat{V}\) belongs depends on the values of the three coupling constants contained in \(\hat{P}\). In our case, in which two of the coupling constants are equal, \(\hat{V} \in O(2)\). This right transformation \(\hat{V}\) on \(\hat{Q}\) corresponds to a mixing of the basis vectors: \(n^a_c \rightarrow \sum_b n^a_b V_{bc}\). Since \(p_1 = p_2 \neq p_3\), the mixing occurs between the basis vectors \(n_1\) and \(n_2\). The right \(O(2)\)
invariance reflects the screw-axis-like symmetry of the spiral state, which is invariant under a rotation about the axis $\mathbf{n}_3$ by an angle enclosed between a pair of spins, followed by a lattice translation along the direction which connects the two spins. Thus, our action for the quantum nonlinear $\sigma$ model possesses an $O(3) \otimes O(2)$ symmetry as in previous work for frustrated quantum spin systems [3, 11].

In addition to the SO(3) quantum nonlinear $\sigma$ model, we find in our microscopic derivation the contribution (see Eqs. (60), (119))

$$S_{\text{lin}} = \sum_{\alpha=x,y} \left[ \frac{1}{a^2} \frac{\partial}{\partial q^\alpha} \Phi_1^- (q) \right]_{q=0} - \frac{J_H S^2}{a} \sin(\delta Q_\alpha a) \right) \int d^2 x d \tau \; \mathbf{n}_1 \cdot \partial_\alpha \mathbf{n}_2, \quad (147)$$

which is linear in the derivatives. The action $S_{\text{lin}}$ is of course invariant under rotations of the basis vectors, $n_c^a \to U^{ab} n_b^c$. Under the transformations $n_c^a \to \sum_b n_b^c V_{bc}$, however, $S_{\text{lin}}$ is invariant only if $\det V = +1$. Transformations with $\det V = -1$ change our right-handed set of basis vectors into a left-handed one. This means that $S_{\text{lin}}$ is not invariant under a change of helicity of the spiral.

Since the linear term is not positive definite, the weight of some field configurations in the path integral will tend to infinity, and hence this term leads to instabilities. To recover a stable ground state, we must ensure that the action is at a minimum. Thus, we impose the condition that the prefactors of the linear contributions must vanish:

$$\frac{1}{a} \frac{\partial}{\partial q^\alpha} \Phi_1^- (q) \bigg|_{q=0} \overset{!}{=} J_H S^2 \sin(\delta Q_\alpha a). \quad (148)$$

This condition yields two equations (one for $\alpha = x$ and one for $\alpha = y$) for the two spiral pitch parameters $\delta Q_x$ and $\delta Q_y$, whose values are then determined as a function of the microscopic parameters and the doping concentration. Thus, the wave vector of the spiral background is self-consistently determined from the stability argument. In order to check the consistency of the stability condition, Eq. (148), we consider again the special cases of the spiral wave vector mentioned above. For a spiral wave vector in the (1,0) direction, the condition in Eq. (148) is trivially fulfilled for $\alpha = y$ because both sides vanish (the symmetry relation, Eq. (13), implies that $\frac{\partial}{\partial q^y} \Phi_1^- (q)|_{q=0} = 0$ for this choice of $k_s$). If the spiral wave vector lies along the (0,1) direction, the terms on both sides of Eq. (148) vanish for $\alpha = x$. All other choices for $k_s$ lead to nonvanishing terms on both sides of Eq. (148) for $\alpha = x, y$.

Finally, our microscopic derivation gives rise to the following term which is third order in the fields and second order in the derivatives:

$$S_{\text{geom}} = -\hat{\chi}^{\mu\nu} \int d^2 x d \tau \; \mathbf{n}_3 \cdot (\partial_\mu \mathbf{n}_3 \times \partial_\nu \mathbf{n}_3). \quad (149)$$

From the definition $\hat{\chi}^{\mu\nu} = \frac{1}{2a^2} [\hat{\chi}_1 + \hat{\chi}_2]^{-1} [\chi_\mu^\nu + \hat{\chi}_7^\mu + \chi_8^\mu \chi_4^\nu]$ (see Eq. (128)), and using Eq. (109), it is seen that $S_{\text{geom}}$ is nonzero only for $\mu = \tau$ and $\nu = x, y$. For $\delta Q = 0$, 35
$\hat{\chi}^{\mu\nu}$ can be shown to be zero, so that $S_{\text{geom}}$ vanishes. Performing the first-order derivatives in Eqs. (A9), (A11)-(A13) and looking at Eq. (122), one finds that $\hat{\chi}^{\tau\alpha}$ is purely imaginary.

As is well-known, a term having the same structure as $S_{\text{geom}}$ appears in the continuum theory of one-dimensional quantum Heisenberg antiferromagnets with an order parameter in $S_2$. The Pontryagin index

$$Q = \frac{1}{4\pi} \int \text{d}x \text{d}\tau \ n \cdot (\partial_x n \times \partial_\tau n)$$ (150)

describes the winding number of the mapping $\pi_2(S_2) = \mathbb{Z}$. The long-wavelength action of the spin-$S$ antiferromagnetic Heisenberg chain contains the term $S_{AF}^{\text{AF}} = i2\pi S Q$, which has led Haldane [34] to conjecture that integer spin chains possess a gap in the excitation spectrum in contrast to half-integer ones. In a two-dimensional square lattice, however, the topological term is cancelled, since the summation of all rows of spins yields $\lim_{a\to0} \sum_q (-1)^q i2\pi S Q(q) = \int \text{d}y i\pi S \partial_y Q(y) = 0$ [42]. A similar situation arises in the triangular lattice, where the corresponding topological term does not contribute to the dynamics [43]. Using the symmetry properties of the coefficient $\hat{\chi}^{\mu\nu}$, we can write our geometric term as

$$S_{\text{geom}} = \sum_\alpha \hat{\chi}^{\tau\alpha} \int \text{d}x \text{d}y \text{d}\tau \ n_3 \cdot (\partial_x n_3 \times \partial_\tau n_3)$$

$$= 4\pi \hat{\chi}^{\tau x} \int \text{d}y \ Q(y) + 4\pi \hat{\chi}^{\tau y} \int \text{d}x \ Q(x).$$ (151)

While $Q$ is an integer, the coefficients in front of the topological terms $4\pi \hat{\chi}^{\tau\alpha}$ are not necessarily integer multiples of $\pi$ as in the undoped models, but are in general functions of the microscopic parameters and the doping. Therefore, the obtained field theory contains two $\theta$-vacuum terms [28] (one for each spatial direction) with continuously varying parameters, in contrast to the until now known field theories obtained for quantum spin systems in two dimensions.

It should be noted that $S_{\text{geom}}$ does not correspond to the topological invariant classifying the mapping $\pi_3(SO(3)) = \mathbb{Z}$, which is third order in derivatives and therefore is not contained in our expansion. It is given by [14]

$$\hat{Q} = 1/(24\pi^2) \int \text{d}^2x \text{d}\tau \varepsilon^{\lambda\mu\nu} \text{Tr} \left[ (\hat{Q}^T \partial_\lambda \hat{Q})(\hat{Q}^T \partial_\mu \hat{Q})(\hat{Q}^T \partial_\nu \hat{Q}) \right],$$ (152)

If we write $S_{\text{geom}}$ in terms of the $SO(3)$ matrix $\hat{Q}$, it becomes

$$S_{\text{geom}} = -f^{\mu\nu} \int \text{d}^2x \text{d}\tau \text{Tr} \left[ \hat{C}(\hat{Q}^T \partial_\mu \hat{Q})(\hat{Q}^T \partial_\nu \hat{Q}) \right],$$ (153)

where $C_{12} = -C_{21} = 1$ and $C_{ij} = 0$ otherwise, and we have introduced $f^{\mu\nu} = \frac{1}{2}(\hat{\chi}^{\mu\nu} - \hat{\chi}^{\nu\mu})$ which is completely antisymmetric in the momentum-space indices $\mu$ and $\nu$. A study up to $O(a^3)$ is left for the future.
Since the variation of $S_{\text{geom}}$ under an infinitesimal variation of the field vanishes, its value can only change in discrete steps and thus it is quantized. Field configurations falling in different homotopy sectors cannot be continuously deformed into one another. Non-trivial field configurations correspond in this case to instanton-like configurations for the plane of the spiral. Since until now the few studies carried out for the $\theta$-term considered only the one-dimensional case \[28\], such that an extension to $(2+1)$ dimensions is not straightforward, we restrict ourselves for the following discussion to the sector in which $S_{\text{geom}} = 0$, such that the results obtained previously in renormalization group analysis can be applied.

Renormalization group analyses of a $(2 + 1)$-dimensional SO(3) quantum nonlinear $\sigma$ model in the form which we have derived in Eq. (142) have been performed by Azaria et al. \[9\] and by Apel et al. \[7\]. Although previous microscopic derivations of the SO(3) quantum nonlinear $\sigma$ model from frustrated Heisenberg models resulted in an action with $p^{\tau\tau}_1 = p^{\tau\tau}_3$ and $p^{xx}_3 = p^{yy}_3 = 0$ \[5–8\], the renormalization group calculations were carried out for the model which we obtained in Eq. (142), because the aforementioned conditions are not stable under renormalization. Azaria et al. \[9\] considered the case in which the coupling constant matrix is isotropic in lattice space, i.e. $\hat{P}^{xx} = \hat{P}^{yy}$, so that the action contains four coupling constants, which correspond to $p^{\tau\tau}_1$, $p^{\tau\tau}_3$, $p^{xx}_1$, $p^{xx}_3$ in our case. Instead of using these four coupling constants, it is possible to work with two independent spin stiffnesses, $\rho_A \propto p^{xx}_1$ and $\rho_B \propto (p^{xx}_1 + p^{xx}_3)$, and two uniform spin susceptibilities, $\chi_A \propto p^{\tau\tau}_1$ and $\chi_B \propto (p^{\tau\tau}_1 + p^{\tau\tau}_3)$. (The spin-wave velocities are then given by $c_A = \sqrt{\rho_A/\chi_A}$ and $c_B = \sqrt{\rho_B/\chi_B}$.) In contrast to the case of the O(3) quantum nonlinear $\sigma$ model describing the long-wavelength properties of collinear spin configurations, here the spin-wave velocities $c_A$ and $c_B$ already renormalize at one-loop order. In fact, the spin-wave velocity in the O(3) quantum nonlinear $\sigma$ model at zero temperature is not renormalized at one-loop order \[4\] as a consequence of the Lorentz invariance of the theory. Such an invariance is absent in the present theory, as already pointed out before.

The one-loop renormalization group equations for the parameters $c_A$, $c_B$, $\rho_A$, $\rho_B$, $\beta$ admit a nontrivial fixed point for $T = 0$, $c^*_A = c^*_B$ and $\rho^*_A = \rho^*_B$. At this point, the theory is $O(3) \otimes O(3) \sim O(4)$ symmetric and Lorentz invariant. As in the case of the classical $O(3) \otimes O(2)$ nonlinear $\sigma$ model in $(2 + \epsilon)$ dimensions \[10,11\] (which contains two couplings in the isotropic case), the symmetry is dynamically enlarged at the fixed point. Connected to this fixed point in the five-dimensional parameter space is a critical surface separating an ordered phase, in our case a spiral one, from a phase disordered by quantum fluctuations. The scaling properties of this quantum transition have been described by Chubukov et al. \[12\]. The present work provides a way to determine quantitatively how a variation of doping will either drive the system from the helically ordered to the gapped spin-liquid phase, or in case the spin-wave velocities vanish, an instability to another state occurs.
VI. CONCLUSION

In this work, the long wavelength, low-energy sector of a spiral spin configuration in the microscopic spin-fermion model has been mapped onto an effective field theory. This mapping has been accomplished by a systematic gradient expansion of the spin action which was obtained by exactly integrating out the fermions from a coherent-state path-integral representation. The magnetically ordered spiral state completely breaks the O(3) rotation symmetry in spin space. This leads to an SO(3) order parameter. We have shown how the constituent fields of the order parameter are obtained from the physical spins, such that the ambiguity in describing the physical spin on $S_2$ with elements in SO(3) is avoided. The low-lying modes fluctuating around the classical spiral ground state were parametrized by a decomposition of the spin fields into a helical and a uniform component.

The gradient expansion of the fermionic determinant leads to an infinite series that can be summed to all orders of the microscopic coupling constants by a combinatorial method which exploits the constraint on the order parameter. We have carried out the gradient expansion up to second order, such that the relevant terms in the same order as those generally proposed in phenomenological approaches are contained. Such an expansion leads to an effective action containing first- and second-order space-time derivatives of the order parameter.

The first-order terms yield through a stability condition two equations which determine the two spiral pitch parameters as a function of the microscopic parameters and the doping. Thus, our theory is able to determine the wave vector of the spiral spin-background in a self-contained way.

The second-order terms, on the one hand, lead to an O(3)$\otimes$O(2) symmetric quantum nonlinear $\sigma$ model describing the long-wavelength spectrum of three spin waves. The coefficient matrix of the $\sigma$ model, which is responsible for the Lorentz noninvariance of our theory, determines the spin-wave velocities and spin-wave stiffnesses as a function of the microscopic parameters and the doping. It should be emphasized that the continuum approximation only refers to the spin dynamics, so that our method can deal with arbitrary dispersion relations for the fermions. Since the gradient expansion is nonperturbative in the coupling constants of the microscopic Hamiltonian, its results are also relevant for related models, like the Kondo lattice model ($J_H = 0$) and the $t$-$J$ model ($J_K \to \infty$).

On the other hand, our continuum theory yields geometric terms of the same form as the one obtained for the one dimensional antiferromagnetic Heisenberg model in the continuum limit. In our case however, the corresponding coefficient varies continuously as a function of doping. Whether these terms may close the gap in the quantum disordered phase for a given value of the coefficient (like at $\theta = \pi$ in nonlinear $\sigma$-models with a $\theta$ term in (1+1) dimensions), is a question left for further studies.
Our results show that by dealing with a given but rather general model for doped antiferromagnets, new terms appear that are in general not contained in phenomenological approaches, where only symmetry arguments are used. Furthermore, the influence of the doped charge carriers on the critical behaviour of the incommensurate spin system is obtained explicitly. The doping dependence is contained in the parameters of our SO(3) nonlinear \( \sigma \) model through generalized fermionic susceptibilities, which in addition yield the new dispersion relations of the holes in the presence of spiral spin fields. The numerical evaluation of the generalized fermionic susceptibilities is presently being carried out.

We would like to thank A. Angelucci for useful discussions. S.K. gratefully acknowledges support by the Studienstiftung des deutschen Volkes.

**APPENDIX A: GENERALIZED FERMIONIC SUSCEPTIBILITIES**

The quantities \( \Phi^\pm_2 \ldots \Phi^\pm_6 \) which appear in the prefactor of the \( (\partial_\mu \mathbf{n_3} \cdot \partial_\nu \mathbf{n_3}) \)-term of the action Eq. \((88)\) describing the spiral contribution to the fermion determinant are given by:

\[
\Phi^\pm_2(q) = \sum_k \text{Li}_2(g^2 g_0(k-q) g_0,+(k-q)),
\]

\[
\Phi^\pm_3(q) = \frac{1}{2} \sum_k \text{Li}_0(g^2 g_0(k) g_0,+(k)) \frac{g_0,+(k-q)}{g_0,+(k)},
\]

\[
(\Phi^\pm_4)_{\mu\nu} = \frac{1}{2} \sum_k \text{Li}_0(g^2 g_0(k) g_0,+(k)) \left[ \frac{g_0,+(k)}{g_0,+(k)} - 1 \right]^{-1} \left\{ \frac{\partial g_0(k-q)}{\partial q^\mu} \left| q=0 \right. \frac{\partial g_0,+(k-q)}{\partial q^\nu} \left| q=0 \right. \right. \\
\left. - \left. \frac{\partial g_0,+(k-q)}{\partial q^\mu} \left| q=0 \right. \frac{\partial g_0,+(k-q)}{\partial q^\nu} \left| q=0 \right. \right\},
\]

\[
(\Phi^\pm_5)_{\mu\nu} = \sum_k \text{Li}_1(g^2 g_0(k) g_0,+(k)) \left[ \frac{g_0,+(k)}{g_0,+(k)} - 1 \right]^{-1} \left\{ \frac{\partial g_0(k-q)}{\partial q^\mu} \left| q=0 \right. \frac{\partial g_0,+(k-q)}{\partial q^\nu} \left| q=0 \right. \right. \\
\left. - \left. \frac{\partial g_0,+(k-q)}{\partial q^\mu} \left| q=0 \right. \frac{\partial g_0,+(k-q)}{\partial q^\nu} \left| q=0 \right. \right\},
\]

\[
(\Phi^\pm_6)_{\mu\nu} = \frac{1}{8} \sum_k g^2 [\frac{\partial g_0(k-q)}{\partial q^\mu}] \left| q=0 \right. [\frac{\partial g_0,+(k-q)}{\partial q^\nu}] \left| q=0 \right. [1 - g^2 g_0(k) g_0,+(k)]^2,
\]

The generalized fermionic susceptibilities appearing in the ferromagnetic contribution to the fermion determinant (see Eq. \((108)\)) are defined as follows:

\[
\chi_1 = \sum_k g^2 \left\{ \left[ g^2 - \frac{1}{2} g_0^{-1}(k) g_+(k) \right] d^{-1}(k) \\
- \left[ g^2 - \frac{1}{2} g_0^{-1}(k) g_+(k) \right]^2 \left[ g^2 - g_0^{-1}(k) g_0,+(k) \right] d^{-2}(k) \\
- \frac{1}{4} g_0^{-2}(k) g_0,+(k) \left[ g^2 + g_0,+(k) g_0,+(k) \right] d^{-2}(k) \right\},
\]
\[ x_2 = -\frac{x_1}{2}, \quad \text{(A7)} \]
\[ x_3 = -\sum_k g^4 g_k^2 \left[ g^2 + g_0^{-2}(k) \right] d^{-2}(k), \quad \text{(A8)} \]
\[ \chi_4^\mu = \sum_k \frac{1}{4} g^3 g_k^2 \frac{\partial}{\partial q^\mu} \left[ g_0^{-2}(k-q) g_{0,+}^{-1}(k-q) g_{0,-}^{-1}(k-q) \right. \]
\[ \left. -g^2 g_0^{-1}(k-q) \rho+(k-q) \right] \bigg|_{q=0} d^{-2}(k), \quad \text{(A9)} \]
\[ \chi_5^\mu = \sum_k \frac{1}{6} g^3 g_k^2 \frac{\partial}{\partial q^\mu} \left[ g_0^{-3}(k-q) \right] \bigg|_{q=0} d^{-2}(k), \quad \text{(A10)} \]
\[ \chi_6^\mu = \sum_k \frac{1}{4} g^3 \frac{\partial}{\partial q^\mu} \left\{ -g^2 \left[ \frac{1}{2} g_0^{-1}(k) \rho_+^2(k-q) + \rho_-^2(k) g_0^{-1}(k-q) \right] \right. \]
\[ \left. +g_0^{-2}(k) \rho_-^2(k) \left[ g_{0,+}^{-1}(k) g_{0,-}^{-1}(k-q) - g_{0,-}^{-1}(k) g_{0,+}^{-1}(k-q) \right. \right. \]
\[ \left. \left. +\rho_-^{-1}(k) g_0^{-1}(k-q) \right] \right\} \bigg|_{q=0} d^{-2}(k), \quad \text{(A11)} \]
\[ \chi_7^\mu = \sum_k g^3 \left[ g^2 - \frac{1}{2} g_0^{-1}(k) \rho_+(k) \right]^2 \frac{\partial}{\partial q^\mu} \left[ g_0^{-1}(k-q) \right] \bigg|_{q=0} d^{-2}(k), \quad \text{(A12)} \]
\[ \chi_8^\mu = \sum_k \frac{1}{4} g^3 \frac{\partial}{\partial q^\mu} \left[ g_0^{-2}(k) \rho_-^2(k) \rho_+(k-q) \rho_-(k-q) \right. \]
\[ \left. +2g^2 \rho_-^2(k) g_0^{-1}(k-q) - g^2 g_0^{-1}(k) \rho_-^2(k-q) \right] \bigg|_{q=0} d^{-2}(k). \quad \text{(A13)} \]

In the expression Eq. (123), which is obtained for the ferromagnetic field \( \mathbf{L} \) from the saddle-point solution, the following abbreviations are used:

\[ \zeta_1 = \frac{\bar{\chi}_1 (\bar{\chi}_1 + 2\bar{\chi}_2 + \chi_3)}{2 (\bar{\chi}_1 + \bar{\chi}_2)^2 (\bar{\chi}_1 + \chi_3)}, \quad \text{(A14)} \]
\[ \zeta_2^\mu = \frac{\chi_4^\mu [(\bar{\chi}_1 - \bar{\chi}_2)(\bar{\chi}_1 + \chi_3) + 4\bar{\chi}_1 \bar{\chi}_2]}{2 (\bar{\chi}_1 + \bar{\chi}_2)^2 (\bar{\chi}_1 + \chi_3)}, \quad \text{(A15)} \]
\[ \zeta_3^\mu = \frac{\chi_6^\mu}{2 (\bar{\chi}_1 + \bar{\chi}_2)} + \frac{(\bar{\chi}_2^\mu + \chi_8^\mu) \bar{\chi}_2 (\chi_3 - \bar{\chi}_1)}{2 (\bar{\chi}_1 + \bar{\chi}_2)^2 (\bar{\chi}_1 + \chi_3)}, \quad \text{(A16)} \]
\[ \zeta_4^\mu = \frac{\bar{\chi}_7^\mu (\bar{\chi}_2^2 - \bar{\chi}_1 \chi_3)}{(\bar{\chi}_1 + \bar{\chi}_2)^2 (\bar{\chi}_1 + \chi_3)} - \frac{\chi_5^\mu}{2 (\bar{\chi}_1 + \chi_3)}. \quad \text{(A17)} \]
REFERENCES

[1] S. Chakravarty, B.I. Halperin, D.R. Nelson, Phys. Rev. Lett. 60 (1988) 1057; Phys. Rev. B 39 (1989) 2344.

[2] P. Hasenfratz, H. Leutwyler, Nucl. Phys. B 343 (1990) 241; P. Hasenfratz, F. Niedermayer, Phys. Lett. B 268 (1991) 231; P. Hasenfratz, F. Niedermayer, Z. Phys. B 92 (1993) 91.

[3] Y. Endoh et al., Phys. Rev. B 37 (1988) 7443.

[4] H.-Q. Ding, M.S. Makivić, Phys. Rev. Lett. 64 (1990) 1449.

[5] T. Dombre, N. Read, Phys. Rev. B 39 (1989) 6797.

[6] A. Angelucci, Int. J. Mod. Phys. B 4 (1990) 569.

[7] W. Apel, M. Wintel, H.U. Everts, Z. Phys. B 86 (1992) 139.

[8] P. Azaria, B. Delamotte, D. Mouhanna, Phys. Rev. Lett. 70 (1993) 2483.

[9] P. Azaria, B. Delamotte, D. Mouhanna, Phys. Rev. Lett. 68 (1992) 1762.

[10] P. Azaria, B. Delamotte, T. Jolicoeur, Phys. Rev. Lett. 64 (1990) 3175.

[11] P. Azaria, B. Delamotte, F. Delduc, T. Jolicoeur, Nucl. Phys. B 408 (1993) 485.

[12] A.V. Chubukov, T. Senthil, S. Sachdev, Phys. Rev. Lett. 72 (1994) 2089; Nucl. Phys. B 426 (1994) 601.

[13] P. Azaria, P. Lecheminant, and D. Mouhanna, Nucl. Phys. B 455 (1995) 648.

[14] H.J. Schulz, Phys. Rev. Lett. 65 (1990) 2462.

[15] A. Muramatsu und R. Zeyher, Nucl. Phys. B 346 (1990) 387.

[16] A. Muramatsu, Phys. Rev. Lett. 65 (1990) 2909.

[17] C. Küber, A. Muramatsu, Phys. Rev. B 47 (1993) 787; C. Küber, A. Muramatsu, Europhys. Lett. 30 (1995) 481.

[18] R. Shankar, Nucl. Phys. B 330 (1990) 433.

[19] G. Shirane et al., Phys. Rev. Lett. 63 (1989) 330; S.W. Cheong et al., Phys. Rev. Lett. 67 (1991) 1791; T.E. Mason et al., Phys. Rev. Lett. 68 (1992) 1414.

[20] T.R. Thurston, Phys. Rev. B 46 (1993) 9128.
[21] Z.Y. Weng, Phys. Rev. Lett. 66 (1991) 2156; R. Frésard, P. Wölfle, J. Phys.: Condens. Matter 4 (1992) 3625.

[22] B.I. Shraiman, E.D. Siggia, Phys. Rev. Lett. 61 (1988) 467; ibid. 62 (1989) 1564; B.I. Shraiman, E.D. Siggia, Phys. Rev. B 42 (1990) 2485.

[23] S. Sachdev, Phys. Rev. B 49 (1994) 6770.

[24] A. Moreo, D.J. Scalapino, R.L. Sugar, S.R. White, N.E. Bickers, Phys. Rev. B 41 (1989) 2313.

[25] G. Dopf, A. Muramatsu, W. Hanke, Europhys. Lett. 17 (1992) 559.

[26] F. F. Assaad, Phys. Rev. B 47 (1993) 7910.

[27] P. Prelovšek, Phys. Lett. A 126 (1988) 287; J. Zaanen, A.M. Olés, Phys. Rev. B 37 (1988) 9423; A. Muramatsu, R. Zeyher, D. Schmeltzer, Europhys. Lett. 7 (1988) 473.

[28] I. Affleck, Phys. Rev. Lett. 56 (1986) 408.

[29] V.J. Emery, Phys. Rev. Lett. 58 (1987) 2794.

[30] G. Dopf, A. Muramatsu, W. Hanke, Phys. Rev. Lett. 68 (1992) 353; G. Dopf, J. Wagner, P. Dieterich, A. Muramatsu, W. Hanke, Phys. Rev. Lett. 68 (1992) 2082.

[31] F.C. Zhang, T.M. Rice, Phys. Rev. B 37 (1988) 3759; F.C. Zhang, Phys. Rev. B 39 (1989) 7375.

[32] A.A. Aligia, M.E. Simon, C.D. Batista, Phys. Rev. B 49 (1994) 13061.

[33] J.R. Klauder, Phys. Rev. D 19 (1979) 2349.

[34] F.D.M. Haldane, Phys. Lett. 93 A (1983) 464; F.D.M. Haldane, Phys. Rev. Lett. 50 (1983) 1153.

[35] S. Klee, A. Muramatsu, Z. Phys. B 91 (1993) 407.

[36] A. Angelucci, Int. J. Mod. Phys. B 5 (1991) 659; Phys. Rev. B 44 (1991) 6849.

[37] M. Veltman, Nucl. Phys. B 319 (1989) 253.

[38] G. ’t Hooft, M. Veltman, Nucl. Phys. B 153 (1979) 365; L. Lewin, Polylogarithms and Associated Functions (North-Holland, New York, 1983).

[39] S. Wolfram, Mathematica (Addison Wesley, Redwood City, California, 1991).

[40] D. Allen, D. Senechal, Phys. Rev. B 51 (1995) 6394.

[41] B.I. Halperin, W.M. Saslow, Phys. Rev. B 16 (1977) 2154.
[42] T. Dombre and N. Read, Phys. Rev. B 38 (1988) 7181; F.D.M. Haldane, Phys. Rev. Lett. 61 (1988) 1029.

[43] M. Wintel, Dissertation, Univ. Hannover 1993.

[44] R. Jackiw, in Current Algebra and Anomalies, edited by S.B. Treiman et al. (World Scientific, Singapore, 1985).