ON THE MULTICANONICAL SYSTEMS OF QUASI-ELLIPTIC SURFACES

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ABSTRACT. We consider the multicanonical systems $|mK_S|$ of quasi-elliptic surfaces with Kodaira dimension 1 in characteristic 2. We show that for any $m \geq 6$ $|mK_S|$ gives the structure of quasi-elliptic fiber space, and 6 is the best possible number to give the structure for any such surfaces.

1. INTRODUCTION

Let $k$ be an algebraically closed field of characteristic $p \geq 0$, and let $S$ be a nonsingular complete algebraic surface with Kodaira dimension 1 defined over $k$. Then, $S$ has a structure of genus 1 fibration $\varphi : S \to B$. We denote by $K_S$ a canonical divisor of $S$ and we consider the multicanonical system $|mK_S|$. As is well known, the multicanonical system $|mK_S|$ gives the genus 1 fibration if $m$ is large enough. In Katsura and Ueno [5] and Katsura [3] (see also Iitaka [2]), we considered the following question:

**Question 1.1.** (1) Does there exist a positive integer $M$ such that if $m \geq M$, the multicanonical system $|mK_S|$ gives a structure of genus 1 fibration for any elliptic surface $S$ over $k$ with Kodaira dimension 1?

(2) What is the smallest $M$ which satisfies this property?

For this question, we have the following theorem.

**Theorem 1.2.** (1) For the complex analytic elliptic surfaces, $M = 86$ and 86 is best possible (cf. Iitaka [2]).

(2) For the algebraic elliptic surfaces, if the characteristic $p = 0$ or $p \geq 3$, then $M = 14$ and 14 is best possible (Katsura and Ueno [5] and Katsura [3]).

(3) For the algebraic elliptic surfaces, if the characteristic $p = 2$, then $M = 12$ and 12 is best possible (Katsura [3]).
If \( p = 2 \) or \( 3 \), there are two kinds of genus 1 fibrations, namely, the elliptic fibration and the quasi-elliptic fibration (cf. Bombieri and Mumford [1]). In these cases, we can also consider the same question for quasi-elliptic surfaces with Kodaira dimension 1. In characteristic 3, we already showed the following results (Katsura [4]).

**Theorem 1.3.** For the quasi-elliptic surfaces in characteristic 3, we have \( M = 5 \), and 5 is best possible.

Therefore, the remaining case of the question for the surfaces with Kodaira dimension 1 is the one in characteristic 2, and in this paper we show the following theorem. It finishes the answer to the question above for surfaces with Kodaira dimension 1 which S. Iitaka considered in the case of complex analytic elliptic surfaces in 1970 (cf. [2]).

**Theorem 1.4.** For the quasi-elliptic surfaces in characteristic 2, we have \( M = 6 \) and 6 is best possible.

In Section 2, we summarize basic facts on the theory of vector fields in positive characteristic and some results on quasi-elliptic surfaces. In Section 3, we give a criterion for a vector field that makes a singularity on the quotient of curve. In Section 4, we construct a quasi-elliptic surface over an elliptic curve with only one tame multiple fiber and examine the structure of its multicanonical system. In Section 5, we examine the multicanonical system of quasi-elliptic surfaces in characteristic 2 and show our main theorem.

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2. **Preliminaries**

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \), and let \( S \) be a nonsingular complete algebraic surface defined over \( k \). A non-zero rational vector field \( D \) on \( S \) is called \( p \)-closed if there exists a rational function \( f \) on \( S \) such that \( D^p = fD \).

We use a vector field to construct a quotient surface of \( S \). Let \( \{ U_i = \text{Spec} A_i \} \) be an affine open covering of \( S \) and we set \( A^D_i = \{ D(\alpha) = 0 | \alpha \in A_i \} \). Then, affine surfaces \( \{ U^D_i = \text{Spec} A^D_i \} \) glue together to define a normal quotient surface \( S^D \).

We now recall some results on vector fields by Rudakov and Shafarevich [8 Section 1]. Now, we assume that \( D \) is \( p \)-closed. Then, we know that the natural morphism \( \pi : S \to S^D \) is a purely inseparable morphism of degree \( p \). If the affine open covering \( \{ U_i \} \) of \( S \) is fine enough, then taking local coordinates \( x_i, y_i \) on \( U_i \), we see that there exist \( f_i, g_i \in A_i \) and a rational
function \( h_i \) such that the divisors defined by \( f_i = 0 \) and by \( g_i = 0 \) have no
common divisor and that the vector field \( D \) is expressed as

\[
D = h_i \left( f_i \frac{\partial}{\partial x_i} + g_i \frac{\partial}{\partial y_i} \right) \quad \text{on } U_i.
\]

Divisors \((h_i)\) on \( U_i \) give a global divisor \((D)\) on \( S \), and zero-cycles defined
by the ideal \((f_i, g_i)\) on \( U_i \) give a global zero cycle \( \langle D \rangle \) on \( S \). A point con-
tained in the support of \( \langle D \rangle \) is called an isolated singular point of \( D \). ([8,
Theorem 1, Corollary]). Rudakov and Shafarevich showed that \( S^D \) is non-
singular if and only if \( \langle D \rangle = 0 \). When \( S^D \) is nonsingular, they also showed
a canonical divisor formula

\[
(2.1) \quad K_S \sim \pi^* K_{S^D} + (p - 1)(D),
\]

where \( \sim \) means linear equivalence.

Now, we consider an irreducible curve \( C \) on \( S \) and we set \( C' = \pi(C) \).
Take an affine open set \( U_i \) above such that \( C \cap U_i \) is non-empty. The curve
\( C \) is said to be integral with respect to the vector field \( D \) if \( D \) is tangent to
\( C \) at a general point of \( C \cap U_i \). Rudakov-Shafarevich showed the following
proposition (cf. [8, Proposition 1]):

**Proposition 2.1.** (i) If \( C \) is integral, then \( C = \pi^{-1}(C') \) and \( C^2 = pC'^2 \).
(ii) If \( C \) is not integral, then \( pC = \pi^{-1}(C') \) and \( pC^2 = C'^2 \).

Now, let \( \varphi : S \longrightarrow B \) be a quasi-elliptic surface. We denote by \( g \) the
genus of the curve \( B \). As was shown in Katsura [4], we have \( Alb(S) \cong J(B) \), and \( \chi(O_S) \geq (1 - g)/3 \) (See also Lang [6] and Raynaud [7]). Here,
\( Alb(S) \) is the Albanese variety of \( S \) and \( J(B) \) is the Jacobian variety of \( B \).
As a corollary, we know that if \( g = 1 \), then \( \chi(O_S) \geq 0 \), and that if \( g = 0 \),
then \( \chi(O_S) \geq 1 \). We will freely use these inequalities in Section 5.

3. CUSPIDAL POINTS

From here on, let \( k \) be an algebraically closed field of characteristic 2,
if otherwise mentioned. Let \( S \) be a nonsingular complete algebraic surface
over \( k \), and let \( D \) be a non-zero 2-closed rational vector field on \( S \). Let \( U \)
be an affine open set of \( S \), and \( x, y \) be local coordinates of \( U \). Then, as in
Section 2, \( D \) is given by

\[
D = h(f \partial / \partial x + g \partial / \partial y),
\]

where \( f, g \) are regular functions on \( U \) such that \( f = 0 \) has no common
curves with \( g = 0 \), and where \( h \) is a rational function on \( S \).

**Lemma 3.1.** Under the assumption above, \( D(fg) = 0 \) holds.
Proof. We set $\alpha = hf$ and $\beta = hg$. Since there exists a rational function $\gamma$ such that $D^2 = \gamma D$, we have

\begin{align*}
\alpha \alpha_x + \beta \alpha_y &= \gamma \alpha, \\
\alpha \beta_x + \beta \beta_y &= \gamma \beta.
\end{align*}

Therefore, by direct calculation, we have $D(\alpha \beta) = 0$. Since $\alpha \beta = h^2 fg$, we conclude $D(fg) = 0$.

Corollary 3.2. $D(f/g) = 0$.

Proof. We have $D(f/g) = D(fg/g^2) = (1/g^2)D(fg) = 0$.

Definition 3.3. Let $D$ be a non-zero rational vector field on a nonsingular surface $S$, and $C$ be a nonsingular irreducible curve on $S$. Let $P$ be a point on $C$ which is not an isolated singular point of $D$. If $D$ is non-integral on $C$ and integral at a point $P$ on $C$, we call $P$ a cuspidal point of the vector field $D$.

Proposition 3.4. Under the notation in Definition 3.3 we consider the projection $\pi : S \to S^D$. Then, the image $\pi(P)$ of the cuspidal point $P$ is a singular point of the curve $\pi(C)$.

Proof. Let $O_P$ be the local ring of the cuspidal point $P$ and let $x, y$ be a system of parameters of $O_P$. Let $x = 0$ be a local equation of $C$ at the point $P$. By the definition of cuspidal points, there exist elements $\alpha, \beta, \gamma$ and $\delta$ of $O_P$ and a constant $c \in k$ such that $\beta \neq 0$ and $c \neq 0$, and such that $f = \alpha x + \beta y$ and $g = \gamma x + \delta y + c$. Since the situation is local, we may omit $h$ from $D$. By Corollary 3.2, we see that $D(x + (f/g)y) = 0$. Since $g(P) \neq 0$, $x + (f/g)y$ is contained in $O_P$. Considering the completion $\hat{O}_P$ of $O_P$, we have $\hat{O}_P \cong k[[x, y]]$. Since $k[[x, y]]^{D} \supset k[[y^2, x + (f/g)y]]$ and $\dim_k k[[x, y]]^{D}/k[[x^2, y^2]] = \dim_k k[[y^2, x + (f/g)y]]/k[[x^2, y^2]] = 2$, we have $k[[x, y]]^{D} = k[[y^2, x + (f/g)y]]$. Although by the general theory of the vector field the point $\pi(P)$ is a nonsingular point of $S^D$, this result also shows that $S^D$ is nonsingular at $\pi(P)$. We set $X = x^2$, $Y = y^2$ and $Z = x + (f/g)y$, and let $\tilde{f}, \tilde{g}$ be elements of $O_P$ whose coefficients are the squares of the ones of $f, g$, respectively. Let $S'$ be a surface defined by the equation

$$Z^2 = X + (\tilde{f}/\tilde{g})Y.$$ 

Since the degrees of the algebraic extensions $k(S)/k(S^{D})$ and $k(S)/k(S')$ of fields are 2 and $k(S^{D}) \supset k(S')$ holds, we have $k(S^{D}) = k(S')$, that is, $S'$ is birationally equivalent to $S^D$. Since $\tilde{g}(P) = c^2 \neq 0$, $S'$ is nonsingular at the point $(X, Y, Z) = (0, 0, 0)$. Therefore, by the Zariski main theorem the surface $S^D$ is isomorphic to $S'$ around $\pi(P)$. The curve $\pi(C)$ is defined
by $X = 0$ at the point $\pi(P)$. Therefore, the equation of the curve $\pi(C)$ at $\pi(P)$ on the plane $X = 0$ is given by

$$Z^2(\tilde{\delta}|_{X=0}Y + c^2) = \tilde{\beta}|_{X=0}Y^2.$$ 

Here, the notation of $\tilde{\beta}$ and $\tilde{\delta}$ are similar to $\tilde{f}$ and $\tilde{g}$. This equation for the curve $\pi(C)$ shows that $\pi(P)$ is a singular point of $\pi(C)$. □

4. A CONSTRUCTION OF A QUASI-ELLIPITIC SURFACE

Let $E$ be an elliptic curve and $\{U_0, U_\infty\}$ be an affine open covering and let $U_0$ (resp. $U_\infty$) be given by the equation

$$y^2 + y = x^3 \text{ (resp. } z^2 + z = w^3).$$

The change of coordinates is given by

$$y = 1/z, \quad x = w/z.$$ 

Let $\{V_0, V_\infty\}$ ($V_0 \cong V_\infty \cong \mathbb{A}^1$: an affine line) be affine open covering of the projective line $\mathbb{P}^1$ and $t$ (resp.$s$) be a coordinate of $V_0$ (resp. $V_\infty$). The change of coordinates is given by

$$t = 1/s.$$ 

We consider the algebraic surface $S = E \times \mathbb{P}^1$. Then, $\{U_i \times V_j \mid i = 0, \infty; j = 0, \infty\}$ gives an affine open covering of $S$. We have a projection

$$\psi : S \rightarrow E.$$ 

Let $C_\infty$ be the curve on $S$ defined by $s = 0$. We consider the following rational vector field $D$ on $U_0 \times V_0$.

$$(I) \quad D = y \frac{\partial}{\partial x} + (x^2 + x^2t + t^4) \frac{\partial}{\partial t}.$$ 

Then, $D$ gives a rational vector field on $S$ and on each affine chart it is concretely given as follows:

$$(II) \quad D = \frac{1}{z^2} \left\{ z^2 \frac{\partial}{\partial w} + (w^2 + w^2t + z^2t^4) \frac{\partial}{\partial t} \right\}$$

on $U_\infty \times V_0$

$$(III) \quad D = \frac{1}{s} \left\{ s^2 \frac{\partial}{\partial w} + (x^2s^4 + x^2s^3 + 1) \frac{\partial}{\partial s} \right\}$$

on $U_0 \times V_\infty$

$$(IV) \quad D = \frac{1}{z^2s^2} \left\{ z^2s^2 \frac{\partial}{\partial w} + (w^2s^4 + w^2s^3 + z^2) \frac{\partial}{\partial s} \right\}$$

on $U_\infty \times V_\infty$
Since $\frac{\partial y}{\partial x} = x^2$, we have $D^2 = x^2D$. Therefore, the rational vector field $D$ is 2-closed. The isolated singularities of $D$ on each affine chart are as follows.

On $U_0 \times V_0$ \quad $P : (x, y, t) = (0, 0, 0)$

On $U_\infty \times V_0$ \quad $Q_1 : (w, z, t) = (0, 0, 1)$

On $U_0 \times V_\infty$ \quad No isolated singular point

On $U_\infty \times V_\infty$ \quad $R : (w, z, s) = (0, 0, 0), Q_2 : (w, z, s) = (0, 0, 1)$.

On the surface $S$, $Q_1$ and $Q_2$ give the same point, and we denote it by $Q$. We set

$\psi(P) = P', \psi(Q) = \psi(R) = Q', \psi^{-1}(P') = F_0, \psi^{-1}(Q') = F_\infty$.

From here on, we use the same notation for the curve and the proper transform of the curve, if no confusion can occur. We blow-up at $P$, and denote the exceptional curve by $G_1$. Then, on the exceptional curve $G_1$ there exists one isolated singular point of the rational vector field $D$. We blow-up at the singular point, and denote the exceptional curve by $G_2$. Then, the vector field has no isolated singular point on $G_2$. Now, we blow-up at $Q$, and denote the exceptional curve by $E_1$. Then, the vector field has no isolated singular point on $E_1$. We again blow-up at $R$, and denote the exceptional curve by $E_2$. On the surface $\tilde{S}$ which we got by these blowing-ups the rational vector field $D$ has no isolated singularities. We have the morphism

$\tilde{\psi} : \tilde{S} \twoheadrightarrow E$

which is induced by $\psi$. Then, on $\tilde{S}$, by our construction we have the following lemma.

**Lemma 4.1.** On $\tilde{S}$, we have the following results.

1. $\tilde{\psi}^{-1}(P') = F_0 + G_1 + 2G_2, \tilde{\psi}^{-1}(Q') = F_\infty + E_1 + E_2$.

2. The curves $F_0, G_1$ and $F_\infty$ are integral with respect to the vector field $D$. The curves $G_2, E_1, E_2$ and $C_\infty$ are non-integral with respect to the vector field $D$.

3. $F_0^2 = -2, G_1^2 = -2, G_2^2 = -1, F_\infty^2 = -2, E_1^2 = -1, E_2^2 = -1$.

4. $(F_0, G_2) = (G_2, G_1) = 1, (F_0, G_1) = 0$.

5. $(F_\infty, E_1) = (F_1, E_2) = (C_\infty, E_2) = 1, (F_\infty, C_\infty) = (E_1, E_2) = (C_\infty, E_1) = 0$.

6. There is a cuspidal point of the vector field $D$ on $G_2$. There is also a cuspidal point of the vector field $D$ on $E_2$ where it intersects with $C_\infty$.

We consider the quotient surface $\tilde{S}^D$ of $\tilde{S}$ by $D$. We have the projection

$\pi : \tilde{S} \twoheadrightarrow \tilde{S}^D$
and a commutative diagram

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\pi} & \tilde{S}^D \\
\tilde{\psi} & \downarrow & \downarrow \psi' \\
E & \xrightarrow{F} & E^{(2)}
\end{array}
\]

Here, \( F \) is the Frobenius morphism and \( E^{(2)} \) is the Frobenius image. We set \( B = E^{(2)}, P'' = F(P') \) and \( Q'' = F(Q') \). For a curve \( C \) on \( \tilde{S} \), we denote the curve \( \pi(C) \) on \( \tilde{S}^D \) again by \( C \), if no confusion can occur. By Lemma 4.1 and Proposition 3.4, we have the following lemma.

**Lemma 4.2.** On \( \tilde{S}^D \), we have the following results.

1. \( \psi'^{-1}(P'') = 2F_0 + 2G_1 + 2G_2, \psi'^{-1}(Q'') = 2F_\infty + E_1 + E_2 \).
2. \( F_0^2 = -1, G_1^2 = -1, G_2^2 = -2, F_\infty^2 = -1, E_1^2 = -2, E_2^2 = -2, C_\infty^2 = -2 \).
3. \( (F_0, G_2) = (G_2, G_1) = 1, (F_0, G_1) = 0 \).
4. \( (F_\infty, E_1) = (F_1, E_2) = 1, (C_\infty, E_2) = 2, (F_\infty, C_\infty) = (E_1, E_2) = (C_\infty, E_1) = 0 \).
5. \( G_2 \) and \( E_2 \) are rational cuspidal curves.

First, we blow-down \( F_0, G_1 \) and \( F_\infty \), and then \( E_1 \) becomes an exceptional curve of the first kind and so we blow-down it:

\[
\eta : \tilde{S}^D \longrightarrow X.
\]

Then, we have a quasi-elliptic surface

\[
\varphi : X \longrightarrow B.
\]

The fiber \( \varphi^{-1}(P'') \) is the only one multiple fiber, and we have no other singular fiber.

Now, let’s calculate the canonical divisor \( K_X \). First, we have

\[
K_{\tilde{S}^D} \sim \eta^*K_X + F_0 + G_1 + E_1 + 2F_\infty.
\]

Therefore, we have

\[
\pi^*K_{\tilde{S}^D} \sim \pi^*\eta^*K_X + F_0 + G_1 + 2E_1 + 2F_\infty.
\]

On \( \tilde{S} \), by a direct calculation of \( D \) and \( K_\tilde{S} \), we have

\[
(D) = -2C_\infty - 4F_\infty + G_1 + 4G_2 - 3E_1 - 3E_2,
K_{\tilde{S}} \sim -2C_\infty + G_1 + 2G_2 + E_1 - E_2.
\]

Putting these data in the canonical bundle formula by Rudakov-Shafarevich:

\[
K_{\tilde{S}} \sim (D) + \pi^*K_{\tilde{S}^D},
\]

we have

\[
\pi^*\eta^*K_X \sim 2(F_\infty + E_1 + E_2) - (F_0 + G_1 + 2G_2).
\]
Therefore, we have
\[ \eta^*K_X \sim (2F_\infty + E_1 + E_2) - (F_0 + G_1 + G_2). \]

Hence, we have
\[ K_X \sim E_2 - G_2 \approx G_2, \]
where \( \approx \) means numerical equivalence. This means that there exists a divisor \( \mathcal{L} \) on \( B \) such that
\[ (4.1) \quad K_X \sim \varphi^*(\mathcal{L}) + G_2. \]

Therefore, the fiber \( \varphi^{-1}(P'') \) is a tame multiple fiber.

**Proposition 4.3.** The surface \( \varphi : X \rightarrow B \) which we constructed above is a quasi-elliptic surface with only one tame multiple fiber. It has no more singular fibers and \( \chi(\mathcal{O}_X) = 0 \) holds. The linear system \( |6K_X| \) gives the structure of the quasi-elliptic surface, and the linear system \( |5K_X| \) does not give the structure of the quasi-elliptic surface.

**Proof.** Take a general fiber \( G \). Then, we have \( G^2 = 0 \) and \( (K_X, G) = 0 \). Therefore, by the genus formula the virtual genus of \( G \) is 1. On the other hand, \( \tilde{\psi} : \tilde{S} \rightarrow E \) is a ruled surface. Therefore, \( G \) is not an elliptic curve. This means that \( \varphi : X \rightarrow B \) is a quasi-elliptic surface. By our construction, we have Betti numbers \( b_1(X) = 2 \) and \( b_2(X) = 2 \). Therefore, the Euler number \( c_2(X) = 1 - 2 + 2 - 2 + 1 = 0 \). Since \( K_X^2 = 0 \), we have \( \chi(\mathcal{O}_X) = 0 \) by Noether's formula. Since we have \( H^0(X, \mathcal{O}_X(6K_X)) \cong H^0(B, \mathcal{O}_B(3P'')) \) and the divisor \( 3P'' \) is very ample on \( B \), the linear system \( |6K_X| \) gives the structure of the quasi-elliptic surface. Since \( H^0(X, \mathcal{O}_X(5K_X)) \cong H^0(B, \mathcal{O}_B(2P'')) \) and the divisor \( 2P'' \) is not very ample on \( B \), the linear system \( |5K_X| \) does not give the structure of the quasi-elliptic surface. \( \square \)

**Remark 4.4.** In the above, we calculate the canonical divisor \( K_X \) by the construction of our quasi-elliptic surface. We give here one more proof for (4.1). On the quasi-elliptic surface \( \varphi : X \rightarrow B \), the cusp locus \( C_\infty \) is an elliptic curve and we have \( C_\infty^2 = -1 \) by considering the structure of blow-down. Therefore, by the genus formula, we have \( (K_X, C_\infty) = 1 \). On the other hand, by the canonical bundle formula for the quasi-elliptic surface \( X \), we have
\[ K_X \sim \varphi^*(\mathcal{L}) + aG_2 \]
with a line bundle \( \mathcal{L} \) on \( B \) and \( a = 0 \) or \( 1 \). Since \( 1 = (K_X, C_\infty) = 2\deg \mathcal{L} + a \), we conclude \( a = 1 \) and \( \deg \mathcal{L} = 0 \), which shows (4.1).
5. MULTICANONICAL SYSTEMS

Let \( \varphi : S \to B \) be a quasi-elliptic surface over an algebraically closed field \( k \) of characteristic \( p > 0 \). Such a surface exists only in characteristic \( p = 2 \) or \( 3 \). In this case, the multiplicity of a multiple fiber is equal to \( p \) (cf. Bombieri-Mumford [1]). We denote by \( pF_i \) \( (i = 1, \ldots, \lambda) \) the multiple fibers. Then, the canonical divisor formula is given by

\[
K_S \sim \varphi^*(K_B - f) + \sum_{i=1}^{\lambda} a_i F_i,
\]

where \( f \) is a divisor on \( B \) and \( -\deg f = \chi(O_S) + t \) with \( t = \text{length of the torsion part of } R^1\varphi_*O_S \), and \( 0 \leq a_i \leq p - 1 \). For details, see Bombieri-Mumford [1].

We denote by \( g \) the genus of the base curve \( B \). Then, we have the following theorem.

**Theorem 5.1.** Assume \( p = 2 \). Then, for any quasi-elliptic surface \( \varphi : S \to B \) with Kodaira dimension \( \kappa(S) = 1 \) over \( k \) and for any \( m \geq 6 \) \( |mK_S| \) gives the unique structure of quasi-elliptic surface, and 6 is the best possible number.

**Proof.** The method of the proof is similar to the one in Iitaka [2] and Katsura-Ueno [5] (see also Katsura [3] [4]). The Kodaira dimension of \( S \) is equal to 1 if and only if

\[
(*) \quad 2g - 2 + \chi(O_S) + t + \sum_{i=1}^{\lambda} (a_i/m_i) > 0.
\]

Therefore, we need to find the least integer \( m \) such that

\[
(**) \quad m(2g - 2 + \chi(O_S) + t) + \sum_{i=1}^{\lambda} [ma_i/m_i] \geq 2g + 1
\]

holds under the condition \( (*) \). Here, \( [r] \) means the integral part of a real number \( r \). We have the following 6 cases:

- Case (I) \( g \geq 2 \)
- Case (II-1) \( g = 1, \chi(O_S) + t \geq 1 \)
- Case (II-2) \( g = 1, \chi(O_S) = 0, t = 0 \)
- Case (III-1) \( g = 0, \chi(O_S) + t \geq 3 \)
- Case (III-2) \( g = 0, \chi(O_S) + t = 2 \)
- Case (III-3) \( g = 0, \chi(O_S) = 1, t = 0 \)

Case (I) We have \( 2g - 2 + \chi(O_S) \geq 5(g - 1)/3. \) Hence, if \( m \geq 3, (** \) holds.

Case (II-1) If \( m \geq 3, (** \) holds.
Case (II-2) All multiple fibers are tame in this case. If $m \geq 6$, (***) holds by $p = 2$. As we constructed in Section 4, there exists a quasi-elliptic surface with only one tame multiple fiber of type II and $\chi(O_S) = 0$ over an elliptic curve. Therefore, we need $m \geq 6$.

Case (III-1) (***) holds for $m \geq 1$.

Case (III-2) Since $\chi(O_S) \geq 1$, we have $t \leq 1$. Therefore, the number of wild fibers is less than or equal to 1. If there exists at least one tame multiple fiber then (***) holds for $m \geq 2$. If there exist no tame fibers and only one wild fiber, then by Katsura-Ueno [5] Lemma 2.4, this case is excluded in the case of $p = 2$.

Case (III-3) By $p = 2$, (***) holds for $m \geq 4$.

The result on the best possible number follows from the example in Section 4. □

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