Interval Graphs with Containment Restrictions

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Abstract

An interval graph is proper iff it has a representation in which no interval contains another. Fred Roberts [27] characterized the proper interval graphs as those containing no induced star \(K_{1,3}\). Proskurowski and Telle [26] have studied \(q\)-proper graphs, which are interval graphs having a representation in which no interval is properly contained in more than \(q\) other intervals. Like Roberts they found that their classes of graphs where characterized, each by a single minimal forbidden subgraph. This paper initiates the study of \(p\)-improper interval graphs where no interval contains more than \(p\) other intervals. This paper will focus on a special case of \(p\)-improper interval graphs for which the minimal forbidden subgraphs are readily described. Even in this case, it is apparent that a very wide variety of minimal forbidden subgraphs are possible.

1 Introduction

A finite, simple graph \(G = (V, E)\) is an interval graph iff there is an assignment \(\alpha : v \mapsto I_v\) of vertices \(v\) of \(G\) to intervals \(I_v\) on the real line such that \(vw \in E \iff I_v \cap I_w \neq \emptyset\). Interval graphs appear to have first been discussed by Hajos [15]. Now classical and well-known characterizations of interval graphs were given by Lekkerkerker and Boland [23] in 1962 and Gilmore and Hoffman [7] in 1964. Extensive investigations and generalizations have since followed [2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 26, 27, 28, 29]. An interval graph is proper iff it has a representation in which no interval contains another. Roberts [27] introduced proper interval graphs and characterized them as interval graphs containing no \(K_{1,3}\).
Proskurowski and Telle [26] generalized this to \(q\)-proper interval graphs, graphs having an interval representation in which no interval is properly contained in more than \(q\) others.

This paper will forbid containments in the opposite direction. A \(p\)-improper interval graph is one having an interval representation in which no interval contains more than \(p\) other intervals. The key difference between these generalizations is that Proskurowski and Telle [26] forbid supersets whereas here subsets are forbidden.

By a \(p\)-improper representation we mean an interval representation with no interval containing more than \(p\) other intervals. Obviously, if \(G\) has such a representation and \(H\) is a subgraph of \(G\), then deleting from a representation of \(G\) those intervals which correspond to vertices not in \(H\) yields a representation of \(H\). This hereditary property guarantees that the class \(I_p\) of \(p\)-improper interval graphs has a minimal forbidden subgraph characterization. The class of proper interval graphs (which coincides with the class of unit interval [14]) is thus the class \(I_0\).

The Lekkerkerker-Boland theorem [23] says that chordless cycles and asteroidal triples form a defining class of forbidden subgraphs for the class of interval graphs. Thus we will be interested in finding minimal forbidden subgraphs within the class of interval graphs. Let \(\mathcal{M}_p\) denote the set of minimal forbidden interval subgraphs (MFISG) for the class \(\mathcal{I}_p\) of \(p\)-improper interval graphs. The impropriety \(\text{imp}(G)\) of \(G\) is the smallest \(p\) such that \(G\) has a \(p\)-improper representation. Unlike the case of \(q\)-proper interval graphs which have an essentially unique MFISG for each \(q\), \(p\)-improper interval graphs show a great diversity of MFISGs, as we will see below. Fig. 3 shows a complete list of the MFISGs for the first class \(\mathcal{I}_1\) with \(p = 1\) [1]. These ten MFISGs show the breadth of possibilities right at the beginning. The star \(K_{1,p+3}\) is easily seen to be a MFISG for \(\mathcal{I}_p\). This is the easiest case. The next easiest case is the balanced case which includes three examples from Fig. 3. We will give a formal definition of balanced here and give a complete description of all MFISGs in this case.

2 Weight and Balance in Interval Graphs

Throughout this section \(G = (V, E)\) will denote a finite, connected, interval graph. First we establish the notation for the central ideas of the paper. Recall that a finite, simple graph \(G = (V, E)\) is an interval graph iff there is an assignment \(\alpha : v \to I_v\) of vertices \(v\) of \(G\) to intervals \(I_v\) on the real line such that \(vw \in E \iff I_v \cap I_w \neq \emptyset\). If a representation \(\alpha\) has been given, \(\ell_v\) and \(r_v\) will denote the left and right endpoints, resp., of the interval \(I_v\) representing \(v\). The support of a set \(W \subseteq V\) of vertices in a representation \(\alpha\) is the union of all intervals \(I_w\) where \(w \in W\). The impropriety \(\text{imp}_\alpha(z)\) of a vertex \(z\) of \(G\) with respect to the representation \(\alpha\)
Balanced, not critical  
Balanced, critical  
2-critical, not balanced  
3-critical, not balanced  
Balanced, multiple intervals with positive weight

Figure 1: Illustrations of Balance and Criticality

is the number of representing intervals which lie inside $I_z$ (not counting $I_z$ itself). The impropiety $\text{imp}(\alpha)$ of the representation $\alpha$ is the maximum of the impropieties $\text{imp}_v(z)$ over all vertices $z$ of $G$. The impropiety $\text{imp}(G)$ of $G$ is the minimum of $\text{imp}(\alpha)$ over all representations. A representation which minimizes the impropiety will be called a minimal representation. That is, a representation $\alpha$ is minimal iff $\text{imp}(\alpha) = \text{imp}(G)$.

For $z \in V$, a component of $G \setminus \{z\}$ will be called a local component at $z$ (or more simply, just a component at $z$). A local component is exterior iff it contains a vertex not adjacent to $z$.

**Lemma 2.1** A vertex $z$ in an interval graph can have at most two exterior (local) components.

*Proof.* If there are three exterior components $C_1, C_2, C_3$, choose vertices $a_1, a_2,$ and $a_3$ at distance two to $z$ with $a_i \in C_i$. Then $a_1, a_2,$ and $a_3$ form an asteroidal triple, which by [23] is forbidden in an interval graph. ■

A vertex $z$ of $G$ is type $k$ iff $z$ has exactly $k$ exterior components. By Lemma 2.1, $k$ can take on only three values: 0, 1, or 2.

We now introduce a quantity which provides a lower bound on — and sometimes an exact value for — the impropiety. Suppose $z$ has $n$ local components $C_1, C_2, C_3, \ldots, C_n$. The weight $\text{wt}(z)$ of $z$ is the sum of the $n - 2$ smallest orders of the non-exterior local components. The weight $\text{wt}(G)$ of $G$ is the maximum of the weights of its vertices. Note that the weight is defined in terms of the graph $G$ directly and does not depend on any particular representation. Impropiety, on the other hand, is defined in terms of representations of $G$. 

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Let us consider some examples of this somewhat confusing concept. Let $X_1$ and $X_2$ denote generic exterior components. Let $A, B, C, D, F$ denote local components with orders $A = 5, B = 5, C = 5, D = 4, F = 2$.

| Local components at $z$ are | Excluded Loc Comp | The counted orders are | Weight |
|-----------------------------|-------------------|-----------------------|--------|
| $X_1, X_2, A, B$            | $X_1, X_2$        | $5+5$                 | 10     |
| $X_1, X_2, C, F$            | $X_1, X_2$        | $5+2$                 | 7      |
| $X_1, A, B$                 | $X_1, A$          | 5                     | 5      |
| $X_1, C, F$                 | $X_1, C$          | 2                     | 2      |
| $A, B, C, D, F$             | $A, B$            | $5+4+2$               | 11     |
| $C, D, F$                   | $C, D$            | 2                     | 2      |

| Local components at $z$ are | Nr. of Comp. | $n - 2$ smallest Non-Exterior Local Comp. | Weight |
|-----------------------------|-------------|---------------------------------------|--------|
| $X_1, X_2, A, B, C, D, F$   | $n = 7$     | $A, B, C, D, F$                       | $5+5+5+4+2 = 21$ |
| $X_1, X_2, C, F$            | $n = 4$     | $C, F$                                | $5+2 = 7$ |
| $X_1, A, B, C$              | $n = 2$     | none                                  | 0      |
| $X_1, C, F$                 | $n = 3$     | $B, C$                                | $5+5 = 10$ |
| $A, B, C, D, F$             | $n = 3$     | $F$                                   | 2      |
| $C, D, F$                   | $n = 2$     | none                                  | 0      |
| $C, F$                      | $n = 1$     | none                                  | 0      |
| $D$                         |             |                                       | 0      |

**Theorem 2.2** If $z$ is any vertex of an interval graph $G$, the impropriety of $G$ is at least the weight of $z$.

**Proof.** Consider any interval representation $\alpha : v \rightarrow I_v$ of $G$. The supports of the local components are themselves disjoint intervals which lie left to right along the line. Say the local components in this ordering are $A_1, A_2, A_3, \ldots, A_n$. Then the components $A_2, A_3, \ldots, A_{n-1}$ must have supports entirely inside $I_z$. Thus each of these local components lies in the neighborhood of $z$. Hence if there are exterior components they must be $A_1$ or $A_n$, or both. In any case, the $n - 2$ components $A_2, A_3, \ldots, A_{n-1}$ are not exterior and thus the sum of their orders is at least $\text{wt}(z)$. Thus we have shown that in any representation, $I_z$ contains at least $\text{wt}(z)$ other intervals. Thus the impropriety of $G$ is at least $\text{wt}(z)$, as desired.

**Corollary 2.3** For any interval graph $G$, $\text{imp}(G) \geq \text{wt}(G)$.

$G$ is balanced iff $\text{wt}(G) = \text{imp}(G)$. If $G$ is balanced, a vertex $z$ such that $\text{wt}(z) = \text{imp}(G)$ is a basepoint of $G$. Equivalently, $z$ is a basepoint iff $G$ is
balanced and $z$ has maximum weight. Notice that a basepoint must have at least three local components since a vertex with only one or two local components has weight 0.

**Theorem 2.4** If $G$ is a connected, interval graph, then the vertices of positive weight induce a disjoint union of paths.

*Proof.* Suppose $\alpha$ is a representation of $G$ and suppose $I_v \subseteq I_w$. Then every neighbor of $v$ is also a neighbor of $w$. Hence in $G \setminus \{v\}$, all the neighbors of $v$ are still connected. Hence $v$ has only one local component, so $\text{wt}(v) = 0$.

Now suppose some vertex $v$ with $\text{wt}(v) > 0$ has three neighbors $a, b, c$ also with positive weight. Suppose $\alpha$ is a representation of $G$. We saw above that none of the intervals $I_a, I_b, I_c$ can be contained in $I_v$. Thus two of these intervals must exit $I_v$ on the same side. Say, $I_a$ and $I_b$ exit $I_v$ through the right end point $r_v$ of $I_v$. Without loss of generality, assume $\ell_a \leq \ell_b$. Since no interval of positive weight can contain another, $r_a \leq r_b$ is forced. Thus $I_a \subseteq I_v \cup I_b$. But this means that any neighbor of $a$ must be a neighbor of either $v$ or $b$. Since $v$ and $b$ are adjacent, it follows that $a$ has only one local component, and hence has $\text{wt}(a) = 0$, a contradiction.

\[\Box\]

### 3 $\mathbf{p}$-critical Interval Graphs

An interval graph $G$ is $\mathbf{p}$-critical with respect to impropriety iff $G$ has impropriety $p$ but every proper induced subgraph of $G$ has impropriety strictly less than $p$. Note that the concept of $p$-critical only makes sense for $p > 0$. Clearly, a $p + 1$-critical graph is a MFISG for the class $\mathcal{I}_p$ of $p$-improper interval graphs. The converse is not so clear. Fig. 2 gives an example where the impropriety changes drastically with the removal of a single vertex.
Figure 3: Minimal Forbidden Subgraphs for $\mathcal{I}_1$
**Theorem 3.1** Let \( z \) be a vertex of maximum weight in a balanced \( p \)-critical graph \( G \). If \( C \) is an exterior local component at \( z \), then \( C \) consists of exactly two vertices.

**Proof.** Let \( v \) be a vertex in \( C \) at distance 2 from \( z \), and let \( w \) be a common neighbor of \( v \) and \( z \). Let \( H \) be the graph obtained from \( G \) by deleting all vertices of \( C \) other than \( v \) and \( w \). The local components at \( z \) in \( H \) are the same as in \( G \) except that \( C \) is replaced by \( \{v, w\} \). Hence the \( n-2 \) smallest non-exterior local components at \( z \) in \( H \) are the same as in \( G \). Thus the weight of \( z \) in \( H \) is the same as the weight of \( z \) in \( G \). Since \( G \) is balanced and \( C \) contains vertices other than \( v \) and \( w \), then \( H \) is a proper induced subgraph of \( G \) and hence has a strictly smaller impropriety. Thus we have

\[
\text{wt}_H(z) \leq \text{imp}(H) < \text{imp}(G) = \text{wt}_G(z) = \text{wt}_H(z),
\]

a contradiction. Hence \( C \) must be just \( \{v, w\} \) as desired. \( \blacksquare \)

**Theorem 3.2** If \( G \) is balanced and \( p \)-critical, then \( G \) has exactly one basepoint.

**Proof.** Suppose \( y \) and \( z \) are distinct basepoints. Because \( G \) is connected, \( y \) must belong to some local component \( C \) of \( z \). This component must also contain all \( p \) of the vertices whose intervals are contained in \( I_y \). Since \( G \) is balanced and \( p \geq 1 \), any basepoint for \( G \) must have at least three local components and hence at least three neighbors. Thus since exterior components contain only 2 vertices by Lemma 3.1, \( C \) cannot be exterior. Dually, \( z \) is contained in a local component \( D \) at \( y \), which, dually, is not exterior. Since \( z \) has at least three local components, there is a local component \( A \) at \( z \) which is disjoint from \( C \). That is, \( z \) is adjacent to vertices not adjacent to \( y \). But that means, \( D \) is an exterior component at \( y \), a contradiction. \( \blacksquare \)

**Theorem 3.3** Suppose \( G \) is balanced and \( p \)-critical. Let \( z \) be the basepoint of \( G \).

**a)** If there is at most one exterior component at \( z \), then there are at least two local components at \( z \) which are cliques and have maximum order among the local components.

**b)** If there is no exterior component at \( z \), then there are at least three local components at \( z \) which are cliques and have maximum order among the local components.

**Proof.** Select a minimal representation \( \alpha \) of \( G \). As in the proof of Theorem 2.2 look at the supports of the local components. These are disjoint.
intervals, ordered from left to right. Call the leftmost and rightmost components the side components. The other components are inner components. By hypothesis, at most one local component can be exterior, so at least one of the side components is non-exterior. Call such a component A. For concreteness, suppose A is on the right side. The weight is determined by adding the orders of the non-side components. Since α is minimal and G is balanced, the impropriety equals the sum of the orders of the inner components. Hence A cannot contribute to the impropriety. Now consider v ∈ A. Since A is not exterior I_v ∩ I_z ≠ ∅. Thus ℓ_v ≤ r_z. Since A does not contribute to the impropriety, I_v is not contained in I_z. Since A is on the right side, this says r_z < r_v. Combining these inequalities, we find r_z ∈ I_v for all v ∈ A, so A is a clique.

If there are no exterior components, the above argument shows that both the right and left side components must be cliques.

Now let A and B the side components. If one of these is exterior, by symmetry it may be assumed to be B. Thus from the way that weight is defined and because α is a minimal representation, it follows that A is a component of maximum order. If there are no exterior components, then, by symmetry, A can be assumed to have order greater than or equal to B. Thus in either case, we can assume that A is local component of maximum order.

Suppose x ∈ A. Since G is p-critical, it follows that removing x will decrease the impropriety. That is, we need to find a representation of G \ {x} which has a lower impropriety. Any representation consists of the local components strung out in some order along I_z. Rearranging the inner components among themselves or changing the way they are represented will not decrease the number of intervals contained in I_z. Thus some inner component must trade places with one of the two side components. If exchanging an inner component for B has a helpful effect, this helpful effect would be present even if x is left in A. That is, this move could be used to give a representation for G with a smaller impropriety, contrary to the minimality of α. Thus the essential move is exchanging an inner component C for A \ {x}.

Suppose A has order m and C has order n. This exchange increases the number of intervals contained in I_z by m − 1 and decreases it by at most n. The inequality here arises if C is not a clique, so that some of its intervals must intersect I_z while avoiding other intervals from C. This would force some intervals arising from C to be wholly contained in I_z.

Now n ≤ m since A has maximum order. The decrease d in impropriety thus satisfies d = n − (m − 1) ≤ 1. Conversely, d ≥ 1 since G is p-critical. Thus n − (m − 1) = d = 1, so n = m. And this occurs iff all intervals in C can be moved out of I_z — that is, C is a clique.
Thus we have shown that there must be one side component $A$ that has maximum order and is a clique. Moreover, there must be an inner component $C$ that has maximum order and is a clique. If the type is 0, then $B$ exists and, as shown above, $B$ must be a clique. If it is not of maximum order, interchanging $B$ and $C$ would reduce the impropriety of the representation, contrary to the assumption that $\alpha$ is maximal.

4 Construction of Balanced Interval Graphs

Let $z$ denote an isolated vertex. Let $H := H_1, H_2, \ldots, H_n$ denote a sequence of interval graphs. Let $\text{BAL}_0(H)$ denote the join of $z$ with the disjoint union of the $H_i$. That is, $z$ is made adjacent to all vertices in all of the $H_i$. This is clearly an interval graph: represent $z$ by a long interval and draw representations of the $H_i$ inside smaller subintervals of this long interval. A pendant $P_3$ at $z$ is a path $xyz$ such that $y$ is adjacent only to $z$ and $x$ and $x$ is adjacent only to $y$. If in addition the maximum order of the $H_i$ is at least 2, $\text{BAL}_k(H)$ denotes $\text{BAL}_0(H)$ with $k \geq 1$ pendant $P_3$’s attached to $z$.

**Theorem 4.1** A graph $G$ is $p$-critical and balanced iff
  
  a) $G$ is isomorphic to $\text{BAL}_0(H)$ where three of the $H_i$ having maximum order are cliques;
  
  b) $G$ is isomorphic to $\text{BAL}_1(H)$ where two of the $H_i$ having maximum order are cliques;
  
  c) $G$ is isomorphic to $\text{BAL}_2(H)$ for interval graphs $H_i$.

**Proof.** If $G$ is $p$-critical and balanced, then by Theorems 3.1 and 3.3, $G$ has the form specified above. For the converse, suppose $G$ has the form specified above. It is convenient to assume that $H := H_1, H_2, H_3, \ldots, H_n$ is ordered so that $|H_i| \leq |H_{i+1}|$ and among the $H_i$ of maximum order, the cliques come last.

If $k = 2$, construct a representation $\alpha$ of $G = \text{BAL}_2(H)$ by putting the two pendant $P_3$’s at either ends of a long interval $I_z$ for $z$. Represent the $H_i$ inside smaller subintervals of $I_z$. The weight of $z$ in $G = \text{BAL}_2(H)$ is clearly $\Sigma := \sum_{i=1}^n |H_i|$. This is also the impropriety of $z$ in the representation $\alpha$. Thus $\Sigma = \text{wt}(z) \leq \text{wt}(G) \leq \text{imp}(G) \leq \text{imp}(\alpha) = \Sigma$. Therefore, $\text{wt}(G) = \text{imp}(G)$, so $\text{BAL}_2$-graphs are balanced.

To show $\text{BAL}_2$-graphs are critical, it suffices to show that if any interval from the representation $\alpha$ is removed, then the remaining intervals can be rearranged to reduce the impropriety. An inner interval contributes directly to the impropriety, so its removal reduces the impropriety. Thus consider a pendant $P_3$ $xyz$. If $y$ is removed, then $H_n$ can be moved to where $I_y$ was.
This decreases the impropriety by $|H_n|$. If $x$ is removed, then the interval $I_y$ for $y$ can be exchanged for $H_n$. This reduces the impropriety by $|H_n| - 1$. But $|H_n|$ is maximal, and by definition of BAL$_2$, there is a local component with at least two vertices. Thus $|H_n| - 1 > 0$, so the impropriety does go down.

If $k = 1$, put the pendant $P_3$ to the left of a long interval $I_z$ for $z$. Put small intervals for $H_n$, all containing the right endpoint of $I_z$. As before, represent the remaining $H_i$ in smaller intervals contained in $I_z$. The weight of $z$ in $G$ is $\sum_{i=1}^{n-1} |H_i|$. This is again $\text{imp}(\alpha)$. As in the case $k = 2$, this implies BAL$_1$-graphs are balanced.

In showing criticality, pendant $P_3$'s and inner intervals can be treated the same way as for $k = 2$. If a vertex is removed from $H_n$, then we can exchange $H_n$ for $H_{n-1}$ which is an interior clique of the same order as $H_n$ by hypothesis. This reduces the impropriety by 1.

If $k = 0$, $H_n$ and $H_{n-1}$ go on the ends. Removing an interior interval obviously reduces the impropriety as before. If an interval is removed from one of the end clique components, it can be exchanged for $H_{n-2}$. ■

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