CHEBYSHEV SERIES EXPANSION OF INVERSE POLYNOMIALS

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Abstract. An inverse polynomial has a Chebyshev series expansion
\[ \frac{1}{k} \sum_{j=0}^{k} b_j T_j(x) = \sum_{n=0}^{\infty} a_n T_n(x) \]
if the polynomial has no roots in \([-1, 1]\). If the inverse polynomial is decomposed into partial fractions, the \(a_n\) are linear combinations of simple functions of the polynomial roots. Also, if the first \(k\) of the coefficients \(a_n\) are known, the others become linear combinations of these with expansion coefficients derived recursively from the \(b_j\)'s. On a closely related theme, finding a polynomial with minimum relative error towards a given \(f(x)\) is approximately equivalent to finding the \(b_j\) in
\[ \frac{f(x)}{\sum_{k=0}^{k} b_j T_j(x)} = 1 + \sum_{k+1}^{\infty} a_n T_n(x), \]
and may be handled with a Newton method providing the Chebyshev expansion of \(f(x)\) is known.

1. Introduction and Scope

The Chebyshev polynomials \(T_n(x)\) are even or odd functions of \(x\) defined as
\[ T_0(x) = 1, \quad T_n(x) = \frac{n}{2} \sum_{m=0}^{[n/2]} (-1)^m \frac{(n-m-1)!}{m!(n-2m)!} (2x)^{n-2m}, \quad n = 1, 2, 3 \ldots \]
where the Gauss bracket \([.]\) denotes the largest integer not greater than the number it embraces. The reverse formula is
\[ x^n = 2^{1-n} \sum_{n-j \text{ even}}^{n} \left( \frac{n}{(n-j)/2} \right) T_j(x) \]
where the prime at the sum symbol means the first term (at \(j = 0\) and even \(n\)) is to be halved. The polynomials are orthogonal over the interval \([-1, 1]\) with weight function \(1/\sqrt{1-x^2}\)
\[ \int_{-1}^{1} T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \pi, & n = m = 0, \\ \pi/2, & n = m \neq 0, \\ 0, & n \neq m. \end{cases} \]

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The product rule is \[\frac{1}{2} (T_{|m-n|}(x) + T_{m+n}(x)).\]

The indefinite integral is \[\int T_n(x)\,dx = \begin{cases} T_1(x), & n = 0, \\ \frac{1}{4} T_2(x), & n = 1, \\ \frac{1}{n} \left( \frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1} \right), & n > 1, \end{cases}\]

which correlates to the derivative \[\frac{d}{dx} T_n(x) = 2n \sum_{l=0}^{n-1} T_l(x).\]

The expansion of an inverse polynomial of degree \(k\) in a power series is

\[\sum_{j=0}^{k} d_j x^j = \sum_{n=0}^{\infty} c_n x^n,\]

with recursively accessible [15, 0.313]

\[c_n = -\frac{1}{d_0} \sum_{n-j \leq k} d_{n-j} c_j, \quad c_0 = \frac{1}{d_0}, \quad n \geq 1.\]

The topic of this script is the equivalent arithmetic expansion of the inverse polynomial in a Chebyshev series,

\[\frac{1}{\sum_{j=0}^{k} d_j x^j} = \frac{1}{\sum_{j=0}^{\infty} b_j T_j(x)} = \sum_{n=0}^{\infty} a_n T_n(x),\]

i.e., computation of the coefficients

\[a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{T_n(x)}{\sum_{j=0}^{\infty} b_j T_j(x)} \frac{dx}{\sqrt{1-x^2}} \]

given the sets \(\{b_j\}\) or \(\{d_j\}\) that define the original function. Both sets are related via [39 (3), 31 (37)] and with [11] via

\[d_l = \frac{2^{l-1}}{l!} \sum_{j=0}^{k} (-)^{(j-l)/2} j \left(\frac{j+l}{2}\right)! b_j, \quad l = 0, \ldots, k.\]

The expansion (1.10) exists if the inverse polynomial is bound in the interval \([-1, 1]\), i.e., if \(\sum d_j x^j\) has no roots in \([-1, 1]\).

Characteristic generic methods of evaluating (1.10) are not reviewed here: (i) Fourier transform methods [7 (4.7), 6, 12, 10], (ii) sampling with Gauss-type quadratures [1 (25.4.38)], [25, 33, 20], which effectively means using an implicit intermediate interpolatory polynomial to represent \(1/\sum_{j=0}^{k} b_j T_j(x)\), (iii) approximation by truncation of (1.7), then insertion of (1.2), (iv) using the near-minimax properties of the Chebyshev series [23, 21].
Remark 1.1. The Fourier-Chebyshev series \[ T_n(z) - tT_{|n-m|}(z) \]
\[ \frac{1}{1 - 2tT_m(z) + t^2} = \sum_{k=0}^{\infty} T_{km+n}(z)t^k \]
provides special cases of polynomials with particularly simple expansions.

Remark 1.2. Insertion of \( n = 1 \) in (1.4) shows that the coefficients of
\[ f(x) = \sum_{n=0}^{\infty} f_n T_n(x) \]
and
\[ \frac{f(x)}{x} = \sum_{n=0}^{\infty} g_n T_n(x) \]
are related as
\[ f_0 = g_1, \quad 2f_{n-1} = g_{n-2} + g_n, \quad n \geq 2. \]

Chapter 2 explains how the \( a_n \) of (1.10) could be computed supposed the inverse polynomial has been decomposed into partial fractions. Chapter 3 provides a recursive algorithm to derive high-indexed \( a_n \) \((n \geq k)\) supposed the low-indexed \( a_n \) are given by other means. Chapter 4 touches on a (standard) integral-free method to compute approximate low-indexed \( a_n \), and Chapter 5 deals with a specific inverse problem—which is finding the \( b_j \) from partially known \( a_n \)—related to polynomial approximants with minimum relative error.

2. The Case of Known Partial Fractions

The straight way of computing the Chebyshev series uses the decomposition of \( \frac{1}{\sum d_j x^j} \) into partial fractions \[15, 2.102\], which reduces \[14\] to the calculation of the \( a_{n,s} \) in
\[ \frac{1}{(z-x)^s} \equiv \sum_{n=0}^{\infty} a_{n,s}(z) T_n(x), \]
where \( z \) is a root of the polynomial,
\[ \sum_{j=0}^{k} d_j z^j = 0. \]

Sign flips of \( z \) and \( x \) in \[24\] show that
\[ a_{n,s}(-z) = (-)^{n+s} a_{n,s}(z). \]
The case of \( s = 1 \) has been evaluated earlier \[17, (A.6)]\[27\] based on \[11\] (22.9.9)\[32\] (18),
\[ a_{n,1}(z) = \frac{2}{(z^2-1)^{1/2}} \frac{1}{w^n}, \quad w \equiv z + (z^2 - 1)^{1/2}, \quad z \notin [-1, 1]. \]
The branch cuts of \((z^2 - 1)^{1/2}\) must be chosen such that \(|w| > 1\).
Example 2.1.

\[(2.5) \frac{1}{1 + x^2} = i \frac{1}{2i - x} - i \frac{1}{2i-x} \]

consists of two terms,

\[(2.6) a_{n,1}(i) = -\frac{\sqrt{2}i^{1-n}}{(1 + \sqrt{2})^n}, \quad a_{n,1}(-i) = (-)^n \frac{\sqrt{2}i^{1-n}}{(1 + \sqrt{2})^n}, \]

which recombine with the two factors \(i/2\) and \(-i/2\) to \[28, (3.4.1a)]

\[(2.7) \frac{1}{1 + x^2} = \sqrt{2} \sum_{n=0,2,4,6,...} (-)^{n/2} \frac{T_n(x)}{(1 + \sqrt{2})^n} \]

Remark 2.2. The shifted Chebyshev polynomials \(T^*(x) \equiv T(2x-1)\) are orthogonal over \([0,1]\) with weight \(1/\sqrt{x(1-x)}\) \[1, (22.2.8)\]. From (2.1) we get

\[(2.8) \frac{1}{(z-x)^s} = 2^s \sum_{n=0}^{\infty} a_{n,s}(2z-1)T_n^*(x), \]

and (1.5) becomes

\[(2.9) \int T^*_n(x)dx = \left\{ \begin{array}{ll}
\frac{1}{4}T_n^*(x), & n = 0, \\
\frac{1}{4}T_n^*(x), & n = 1, \\
\frac{1}{4} \left( T_{n+1}^*(x) - T_{n-1}^*(x) \right), & n > 1.
\end{array} \right. \]

Example 2.3. An example of \(s = 1, z = -1\) in \[28\] is

\[(2.10) \frac{1}{1 + x} = -\frac{1}{-1-x} = -2 \sum_{n=0}^{\infty} a_{n,1}(-3)T_n^*(x), \]

where

\[(2.11) a_{n,1}(-3) = \frac{(-)^{n+1}}{\sqrt{2(3+2\sqrt{2})^n}} \]

according to \[28\].

Higher second indices \(s\) of the \(a_{n,s}\) are obtained from (2.1) by repeated derivation w.r.t. \(z\),

\[(2.12) (-)^s s! \frac{1}{(z-x)^{s+1}} = \sum_{n=0}^{\infty} \left( \frac{\partial}{\partial z} \right)^s a_{n,1}(z)T_n(x), \]

via \[15\ 0.432.1,\]

\[a_{0,s+1}(z) = \frac{2}{s!}(-)^s \left( \frac{\partial}{\partial z} \right)^s \frac{1}{(z^2-1)^{1/2}} \]

\[(2.13) = 2 \sum_{l=0}^{[s/2]} \frac{(-)^l}{l! (s-2l)!} \left( \frac{1}{2} \right)_{s-l} \frac{(2z)^{s-2l}}{(z^2-1)^{l+s-1}}, \quad s \geq 0, \]

with Pochhammer’s Symbol defined as \[1\ (6.1.22)\]

\[(2.14) (\alpha)_k \equiv \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1) = \Gamma(\alpha + k)/\Gamma(\alpha), \quad (\alpha)_0 = 1. \]
The formula
\[
(s - 1) \int \frac{dx}{(z - x)^s} = \frac{1}{(z - x)^{s-1}} + A_s
\]
\[
= (s - 1) \sum_{n=0}^{\infty} a_n, s \int T_n(x)dx = \sum_{n=0}^{\infty} a_{n,s-1} T_n(x) + A_s, \ s \geq 2
\]
in conjunction with the method quoted by Cody \cite{Cody} \cite[4.8]{Cody} \cite[25]{Cody} yields
\[
(2.15) \quad a_{n+1,s}(z) = a_{n-1,s}(z) - \frac{2n}{s-1} a_{n,s-1}(z), \ n \geq 1, \ s \geq 2.
\]
One needs \eqref{2.13} and
\[
a_{1,s+1}(z) = 2 \frac{\pi}{(s - 1)(z - x)^{s+1}} \frac{dx}{\sqrt{1-x^2}}
\]
\[
= -a_{0,s}(z) + za_{0,s+1}(z)
\]
to start the recurrence \eqref{2.15} and to obtain all coefficients in \eqref{2.1} for a particular \(z\). Closed form expressions for solving these recurrences in terms of Legendre Polynomials of \(z/\sqrt{z^2-1} = (w^2+1)/(w^2-1)\) have been given by Elliott \cite{Elliott}.

**Remark 2.4.** \eqref{2.16} may be generalized to
\[
\int_{-1}^{1} \frac{x^l}{(z-x)^n} \frac{dx}{\sqrt{1-x^2}} = \sum_{m=0}^{l} (-1)^m \binom{l}{m} z^{-m} a_{0,n-m}, \ l < n.
\]
and with \eqref{1.2} and \eqref{1.4} to
\[
\int_{-1}^{1} \frac{T_s(x)}{(z-x)^n} \frac{dx}{\sqrt{1-x^2}} = \frac{1}{2^{l}} \sum_{i=0}^{l} \binom{l}{i} \left[ a_{i-s,n} + a_{i+s,n} \right].
\]

**Example 2.5.** An example of degree \(k = 3\) is
\[
\frac{1}{(4-x)^2(5+x)} = \frac{78}{1} T_0(x) - \frac{23}{1} T_1(x) - \frac{1}{1} T_2(x) + \frac{1}{4} T_3(x)
\]
\[
= \frac{1}{9} \frac{1}{(4-x)^2} + \frac{1}{81} \frac{1}{(4-x)} - \frac{1}{81} \frac{1}{(-5-x)}.
\]
The root at \(z = 4\) yields
\[
a_{0,1}(4) = 2/\sqrt{15} \approx 0.5164
\]
from \eqref{2.21} and
\[
a_{0,2}(4) = 2 \cdot \frac{1}{2} \cdot \frac{2}{\sqrt{15}} \approx 0.1377
\]
from \eqref{2.22}. The root at \(z = -5\) yields
\[
a_{0,1}(-5) = 2/( - \sqrt{24}) \approx -0.4082
\]
from \eqref{2.23}. The combined total in \eqref{2.24} is
\[
a_0 = \frac{2}{\pi} \int_{-1}^{1} \frac{dx}{(4-x)^2(5+x)} \approx \frac{1}{9} \cdot 0.1377 + \frac{1}{81} \cdot 0.5164 - \frac{1}{81} (-0.4082) \approx 0.0267.
\]
Example 2.6. A case of $k = \infty$ is [15, 1.421.2]

\begin{equation}
(2.25) \quad \frac{\tanh(\pi x/2)}{x} = \frac{4}{\pi} \sum_{i=1}^{\infty} \frac{i}{2(2m-1)} \left[ \frac{1}{i(2m-1) - x} - \frac{1}{-i(2m-1) - x} \right].
\end{equation}

The roots at $z = \pm i(2m-1)$ yield

\begin{equation}
(2.26) \quad a_{0,1}(z) = 2/\left(\pm i \sqrt{4m^2 - 4m + 2}\right),
\end{equation}

and the combined total is

\begin{equation}
(2.27) \quad a_0 = \frac{2}{\pi} \int_{-1}^{1} \frac{\tanh(\pi x/2)}{x} \frac{dx}{\sqrt{1 - x^2}} = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1) \sqrt{m^2 - m + 1/2}} \approx 2.38.
\end{equation}

Remark 2.7. From (2.4)

\begin{equation}
(2.28) \quad \frac{\partial a_{n,1}(z)}{\partial z} = -a_{n,1} \left[ \frac{z}{z^2 - 1} + \frac{n}{(z^2 - 1)^{1/2}} \right],
\end{equation}

so the (linear) propagation of the absolute relative error in the root $z$ to the error in the coefficient $a_{n,1}$ is

\begin{equation}
(2.29) \quad \left| \frac{\Delta a_{n,1}}{a_{n,1}} \right| = \left| \frac{\Delta z}{z} \right| \cdot \left| \frac{z^2}{z^2 - 1} + \frac{n}{(z^2 - 1)^{1/2}} \right|.
\end{equation}

Remark 2.8. An associated factorization $\sum_{j=0}^{k} b_j T_j(x) \propto \prod_{m=1}^{l} (z_m - x)^{s_m}$, with $l$ different roots of multiplicities $s_m$, decomposes the square root of the polynomial into a $l$-fold product of series of the prototypical forms

\begin{equation}
(2.30) \quad \sqrt{z - x} = \sum_{n=0}^{\infty} q_n(z) T_n(x), \quad s_m = 1,
\end{equation}

\begin{equation}
(2.31) \quad z - x = z T_0(x) - T_1(x), \quad s_m = 2,
\end{equation}

where [15, 2.576.2]

\begin{equation}
(2.32) \quad q_0(z) = \frac{2}{\pi} \int_{0}^{\pi} dt \sqrt{z - \cos t} = \frac{4}{\pi} \sqrt{1 + z} E\left(\frac{2}{1 + z}\right)
\end{equation}

is related to Complete Elliptic Integrals of the Second Kind $E$ in the notation of [11, (17.3.4)]. The $q_n(z)$ with $n \geq 1$ follow recursively using [11, (17.1.4)]. In particular, one may expand $T_n(x)$ in terms of $P_n^{(0,-1/2)}(x)$ with [11, (1.4)] to obtain

\begin{equation}
(2.33) \quad q_n(1) = \frac{2^{5/2}}{\pi} \sum_{l=0}^{n} \frac{(-n)_l}{(3/2)_l (1/2)_l} (n)_l!, \quad n = 0, 1, 2, \ldots,
\end{equation}

for the Chebyshev coefficients of $\sqrt{1 - x}$. See [29] for an application.

3. Recurrence of Expansion Coefficients

The $T_n$ in [11, 10] may be decomposed into a unique product of a polynomial by the denominator plus a remainder of polynomial degree less than $k$. [The dependence on $x$ is omitted at all $T_n(x)$ for brevity.]

\begin{equation}
(3.1) \quad T_n = (d_0^{(n)} T_0 + d_1^{(n)} T_1 + \cdots + d_n^{(n)} T_{n-k})(b_0 T_0 + b_1 T_1 + \cdots + b_k T_k) + c_0^{(n)} T_0 + c_1^{(n)} T_1 + c_2^{(n)} T_2 + \cdots + c_k^{(n)} T_{k-1}.
\end{equation}
Expansion with \[14\] yields a system of linear equations for the vector of the unknowns \(a_j^{(n)}\) and \(c_j^{(n)}\):

\[
\begin{bmatrix}
1 & 0 & \ldots & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
\vdots \\
b_{k-1}
\end{bmatrix}
= \begin{bmatrix}
c_0^{(n)} \\
c_1^{(n)} \\
c_2^{(n)} \\
\vdots \\
c_{k-1}^{(n)}
\end{bmatrix}
\begin{bmatrix}
0 \\
\vdots \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

The \((n+1) \times (n+1)\) coefficient matrix \(A_{r,c}\) (row index \(r\) and column index \(c\) from 0 to \(n\)) is an upper triangular matrix. It hosts a \(k \times k\) unit matrix in the upper left corner, and is symmetric w.r.t. the minor diagonal that stretches from \(A_{0,k}\) to \(A_{n-k,n}\):

\[
A_{r,c} = \delta_{r,c}, \quad 0 \leq c \leq k - 1.
\]

\[
A_{r,k+c} = A_{c,k+r} = \begin{cases}
2b_0, & r = c = 0 \\
\frac{b_c}{b}, & r = 0, \quad 1 \leq c \leq k \\
\frac{1}{2}(b_{r-c} + b_{r+c}), & \text{otherwise}
\end{cases}
\]

This works with the auxiliary definition

\[
b_i = 0, \quad i > k \quad \text{or} \quad i < 0.
\]

Insertion of (3.3) into (1.10) yields

\[
a_n = 2d_0^{(n)} + \sum_{i=0}^{k-1} c_i^{(n)} a_i, \quad n \geq k,
\]

which means that entire sequence \(a_n\) can be generated recursively from its first \(k\) terms, if the \(d_0^{(n)}\) and \(c_1^{(n)}\) are generated at the same time via (3.2) or an equivalent method. Iterated full solution of (3.2) can be avoided through recursive generation.
of the set \( \{d_i^{(n+1)}, c_i^{(n+1)}\} \) from \( \{d_i^{(n)}, c_i^{(n)}\} \) and \( \{d_i^{(n-1)}, c_i^{(n-1)}\} \) as follows:

\[
\begin{align*}
(3.7) \quad d_0^{(n+1)} &= d_1^{(n)} + \frac{c_{k-1}}{b_k} - d_0^{(n-1)}, \\
(3.8) \quad d_1^{(n+1)} &= 2d_0^{(n)} + d_2^{(n)} - d_1^{(n-1)}, \\
(3.9) \quad d_j^{(n+1)} &= d_{j-1}^{(n)} + d_{j+1}^{(n)} - d_j^{(n-1)}, \quad j = 2, 3, \ldots, n-k+1. \\
(3.10) \quad \frac{c_0^{(n+1)}}{2} &= c_1^{(n)} - \frac{b_0 c_{k-1}}{b_k} - c_0^{(n-1)}, \\
(3.11) \quad c_j^{(n+1)} &= c_{j-1}^{(n)} + c_{j+1}^{(n)} - \frac{b_j c_{k-1}}{b_k} - c_j^{(n-1)}, \quad j = 1, 2, \ldots, k-1,
\end{align*}
\]

where the auxiliary definitions

\[
(3.12) \quad c_j^{(n)} = 0, \quad j \geq k, \quad \text{or} \quad j < 0,
\]

\[
(3.13) \quad d_j^{(n)} = 0, \quad j > n-k, \quad \text{or} \quad j < 0,
\]

are made to condense the notation.

**Proof.** Multiply (3.1) by \( 2T_1 \) and use (1.3) as

\[
2T_1 \sum_{j=0}^{n-k} d_j^{(n)} T_j = d_1^{(n)} T_0 + (2d_0^{(n)} + d_2^{(n)}) T_1
\]

\[
+ \sum_{j=2}^{n-k-1} (d_{j-1}^{(n)} + d_{j+1}^{(n)}) T_j + d_{n-k-1}^{(n)} T_{n-k} + d_{n-k}^{(n)} T_{n-k+1},
\]

\[
2T_1 \sum_{j=0}^{k-1} c_j^{(n)} T_j = c_1^{(n)} T_0 + \sum_{j=1}^{k-2} (c_{j-1}^{(n)} + c_{j+1}^{(n)}) T_j + c_{k-1}^{(n)} T_{k-1},
\]

Rewrite the last term in the previous equation

\[
c_{k-1}^{(n)} T_k = \frac{c_{k-1}}{b_k} \sum_{j=0}^{k} b_j T_j - \frac{c_{k-1}}{b_k} b_0 T_0 - \ldots - \frac{c_{k-1}}{b_k} b_{k-1} T_{k-1}.
\]

Construct

\[
2T_1 T_n = [(d_1^{(n)} + \frac{c_{k-1}}{b_k}) T_0 + (2d_0^{(n)} + d_2^{(n)}) T_1
\]

\[
+ \sum_{j=2}^{n-k-1} (d_{j-1}^{(n)} + d_{j+1}^{(n)}) T_j + d_{n-k-1}^{(n)} T_{n-k} + d_{n-k}^{(n)} T_{n-k+1}) \cdot \left[ \sum_{j=0}^{k} b_j T_j \right]
\]

\[
+ (c_1^{(n)} - \frac{c_{k-1}}{b_k} b_0) T_0 + \sum_{j=1}^{k-2} (c_{j-1}^{(n)} + c_{j+1}^{(n)} - \frac{c_{k-1}}{b_k} b_j) T_j + (c_{k-2}^{(n)} - \frac{c_{k-1}}{b_k} b_{k-1}) T_{k-1},
\]

and subtract \( T_{n-1} \) for identification of the \( d_j^{(n+1)} \) and \( c_j^{(n+1)} \),

\[
(3.17) \quad T_{n+1} = 2T_1 T_n - T_{n-1} = \left( \sum_{j=0}^{n-k+1} d_j^{(n+1)} T_j \right) \left( \sum_{j=0}^{k} b_j T_j \right) + \sum_{j=0}^{k-1} c_j^{(n+1)} T_j.
\]
Example 3.1. For (2.16), we obviously have

\[(3.18) \quad c^{(1)}_0 = c^{(1)}_2 = c^{(2)}_0 = c^{(2)}_1 = 0, \quad c^{(1)}_1 = c^{(2)}_2 = 1.\]

in (3.1). The formulas (3.7)–(3.11) predict at \( n = 2 \)

\[(3.19) \quad a^{(3)}_0 = \frac{1}{1/4}, \quad \frac{c^{(3)}_0}{2} = -\frac{78\frac{1}{2} \cdot 1}{1/4}, \quad c^{(3)}_1 = 1 - \frac{-23\frac{1}{2} \cdot 1}{1/4} - 1, \quad c^{(3)}_2 = -\frac{1\frac{1}{2} \cdot 1}{1/4}.\]

With these, (3.6) gives at \( n = 3 \)

\[(3.20) \quad a_3 = 8 + (-314) \cdot a_0 + 93 \cdot a_1 + 6 \cdot a_2\]

which is correct since

\[(3.21) \quad a_0 \approx 0.02671606, a_1 \approx 0.00412578, a_2 \approx 0.00087916, a_3 \approx 0.00013030.\]

The next step of the recursion is

\[(3.22) \quad a^{(4)}_4 = \frac{6}{1/4}, \quad \frac{c^{(4)}_0}{2} = 93 - \frac{78\frac{1}{2} \cdot 6}{1/4}, \quad c^{(4)}_1 = 2(-314) + 6 - \frac{-23\frac{1}{2} \cdot 6}{1/4}, \quad c^{(4)}_2 = 93 - \frac{1\frac{1}{2} \cdot 6}{1/4} - 1.\]

\[(3.23) \quad a_4 = 48 + (-1791) \cdot a_0 + (-64) \cdot a_1 + 128 \cdot a_2\]

which is also correct with

\[(3.24) \quad a_4 \approx 0.00002159.\]

4. Approximation by the Truncated Chebyshev Series

Approximations \( \hat{a}_n \) to the \( a_n \) of (1.9) may be calculated assuming that the \( a_n \) are negligible beyond some index \( N \):

\[(4.1) \quad \frac{1}{\sum_{j=0}^{k} b_j T_j(x)} \approx \sum_{n=0}^{N} \hat{a}_n T_n(x).\]

If this equation is multiplied by \( 2 \sum b_j T_j \), and we stay with (3.6) to keep the notation simple,

\[(4.2) \quad 2 \approx \sum_{n=0}^{N} \hat{a}_n \sum_{i=0}^{k+n} (b_{n-i} + b_{i+n} + b_{i-n}) T_i.\]

If the coefficients in front of \( T_0 \) to \( T_N \) are set equal on both sides, a system of linear equations for the \( \hat{a}_n \) ensues:

\[(4.3) \quad \begin{pmatrix}
  b_0 & b_1 & b_2 & b_3 & \ldots \\
  b_1 & 2b_0 + b_2 & b_1 + b_3 & b_2 + b_4 & \ldots \\
  b_2 & b_1 + b_3 & 2b_0 + b_4 & b_1 + b_5 & \ldots \\
  b_3 & b_2 + b_4 & b_1 + b_5 & 2b_0 + b_6 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix} \begin{pmatrix}
  \hat{a}_0 \\
  \hat{a}_1 \\
  \hat{a}_2 \\
  \hat{a}_3 \\
  \vdots \\
  \hat{a}_N \\
\end{pmatrix} = \begin{pmatrix}
  2 \\
  0 \\
  \vdots \\
  \vdots \\
  0 \\
\end{pmatrix}.\]
Example 4.3. The Chebyshev series of a neglected term via (3.6) to a linear combination of \( \hat{a} \) rank of the matrix and does therefore not improve on what is obtained from (4.3).

An idea of an improvement of this algorithm is: reduce the \( \hat{a} \) components (in)to the matrix—add the constant to the right hand side—in (4.8)

\[
\begin{align*}
(4.8) \quad & \hat{a}_0 = 0.02671606, \hat{a}_1 = 0.00412567, \hat{a}_2 = 0.00087845, \hat{a}_3 = 0.00012696. \\
(4.9) \quad & \hat{a}_0 = 0.02671606, \hat{a}_1 = 0.00412578, \hat{a}_2 = 0.00087914, \hat{a}_3 = 0.00013019, \\
& \hat{a}_4 = 0.00002111,
\end{align*}
\]

which is close to the exact results in (3.21) and (3.24). At \( N = 5 \), this improves further to

\[
\begin{align*}
(4.10) \quad & \hat{a}_0 = 0.02671606, \hat{a}_1 = 0.00412578, \hat{a}_2 = 0.00087916, \hat{a}_3 = 0.00013029, \\
& \hat{a}_4 = 0.00002158, \ldots
\end{align*}
\]

Remark 4.2. The matrix in (4.8) is the approximate, square upper left \((N+1) \times (N+1)\) submatrix of the “exact” solution. The approximate solution obtained could be considered as if the terms \( \sum_{n=N+1}^{N+k} \hat{a}_n (b_{n-l} + b_{n+l} + b_{l-k}) \) of (4.2) in the \( l \)th row of the system of linear equations had been neglected (as if the columns \( N+1 \) up to \( l+k \) had been chopped off). The neglected sum is nonzero only if \( l \geq N+1-k \). An idea of an improvement of this algorithm is: reduce the \( \hat{a}_n \) \((N+1 \leq n \leq l+k)\) in the neglected terms via (4.8) to a linear combination of \( \hat{a}_1, \ldots, k \), and re-introduce (add) these components (in)to the matrix—add the constant to the right hand side—in these rows \( l \geq N+1-k \). This update of the system of linear equations reduces the rank of the matrix and does therefore not improve on what is obtained from (4.8).

The algorithm may be extended to the division problem of finding the \( \hat{a}_n \) from given \( f_n \) in

\[
\begin{align*}
(4.11) \quad & f_n = \pi \sum_{s=0}^{\infty} \frac{(-)^s}{(2s+1)(s)!^2} \left( \frac{\pi}{4} \right)^{2s} = 2.552557924804531760415274, \\
(4.12) \quad & f_2 \approx -0.2852615691810360095702941, \\
(4.13) \quad & f_4 \approx 0.009118016006651802497767923, \\
(4.14) \quad & f_6 \approx -0.0001365875135419666724364765, \\
(4.15) \quad & f_8 \approx 0.000001184961857661690108290062,
\end{align*}
\]
CHEBYSHEV SERIES EXPANSION OF INVERSE POLYNOMIALS

\( f_n = \begin{cases} \frac{4(-)^{n/2} \sum_{s=n/2}^{\infty} J_{2s+1}(\pi/2)}{0}, & n \text{ even}, \\ \frac{\sum_{s=n/2}^{\infty} J_{2s}^{2s+1}(\pi/2)}{s}, & n \text{ odd}. \end{cases} \)

(Schonfelder lists \( 2f_{2n}/\pi \) for \( n \leq 16 \).) If we approximate \( f(x) \) by the polynomial \( \sum_{n=0}^{N} f_n T_n(x) \), calculation of the \( \hat{a}_n \) in

\[ \frac{f(x)}{\sum_{j=0}^{k} f_j T_j(x)} \approx \sum_{n=0}^{N} \hat{a}_n T_n(x) \]

via (4.9) at \( N = 8 \) predicts the relative error

\[ \sum_{n=0}^{N} \hat{a}_n T_n(x) - 1 \approx -6.74 \cdot 10^{-8} T_0(x) - 9.97 \cdot 10^{-7} T_2(x) \]

(4.17)

\[ -1.23 \cdot 10^{-5} T_4(x) - 1.09 \cdot 10^{-4} T_6(x) - 1.13 \cdot 10^{-5} T_8(x). \]

Remark 4.4. The functional relations (4.8) hold also for the shifted Chebyshev polynomials \( T^*(x) \):

\[ f(x) \approx \sum_{n=0}^{N} \hat{a}_n T^*_n(x); \quad f(x) \equiv \sum_{n=0}^{\infty} f_n T^*_n(x), \]

5. CHEBYSHEV APPROXIMATION FOR THE RELATIVE ERROR

The previous example of a truncated Chebyshev series had a maximum absolute error estimated at \( \sum_{n=0}^{8} |f_n| \approx 0.000138 \) if terms up to \( k = 4 \) were retained, and the maximum relative error of the same polynomial was estimated at \( \sum_{n=0}^{8} |\hat{a}_n| - 1 \approx 0.000134 \)—dominated by the \( \hat{a}_6 \) term in (4.11). To optimize the approximation of \( f(x) \) for the relative error in \([-1,1]\), one would rather like to find the \( k+1 \) coefficients \( b_j \) in (4.8) which force the relative error to be close to zero in the sense of

\[ \hat{a}_0 = 2, \quad \hat{a}_1 = \hat{a}_2 = \hat{a}_3 = \ldots = \hat{a}_k = 0. \]

As an inversion of the problem of Sec. 4, the matrix \( B \) in (4.9) is presumed unknown (up to some symmetry), and the first \( k+1 \) elements of the vector \( \hat{a}_c \) and all elements of \( f_r \) are known. The rationale is that removal of the ripples of \( T_1(x) \) to \( T_k(x) \) from the quotient expansion leaves a quotient with an appropriate number of “critical” points required by the alternating maximum theorem [7, 24, 36].

Remark 5.1. The case \( r = 0 \) in (4.9) in conjunction with (5.1) mandate

\[ b_0 = f_0/2. \]

Finding the constituents \( b_j \) of \( B \) that solve the bi-linear equation (4.9) may proceed with a vectorized first-order Newton method as follows:

- Chose a start solution \( b_j \), for example the obvious

\[ b_j = \begin{cases} f_0/2, & j = 0 \\ f_j, & j = 1, 2, \ldots, k \end{cases} \]

- Compute the \( \hat{a}_n \) \((n = 0, \ldots, N)\) from \( b_j \) by solving the linear system of equations (4.9).
• Compute an approximate \((N + 1) \times k\) Jacobi matrix

\[
J_{r,c} = \begin{pmatrix}
\frac{\partial a_0}{\partial b_1} & \frac{\partial a_0}{\partial b_2} & \cdots & \frac{\partial a_0}{\partial b_k} \\
\frac{\partial a_1}{\partial b_1} & \frac{\partial a_1}{\partial b_2} & \cdots & \frac{\partial a_1}{\partial b_k} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial a_N}{\partial b_1} & \frac{\partial a_N}{\partial b_2} & \cdots & \frac{\partial a_N}{\partial b_k}
\end{pmatrix}
\]

by partial derivation of the first \(N + 1\) equations of (5.3) w.r.t. the \(b_j\), i.e., by solving the \(k\) systems of \(N + 1\) linear equations

\[
\sum_{c=0}^{N} B_{r,c} J_{c,j} = -\left(\begin{array}{cccccc}
\hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \cdots & \hat{a}_{k-1} & \hat{a}_k \\
\hat{a}_0 + \hat{a}_2 & \hat{a}_1 + \hat{a}_3 & \hat{a}_2 + \hat{a}_4 & \cdots & \hat{a}_{k-1} + \hat{a}_{k+1} \\
\hat{a}_1 + \hat{a}_3 & \hat{a}_0 + \hat{a}_4 & \hat{a}_1 + \hat{a}_5 & \cdots & \hat{a}_{k-2} + \hat{a}_{k+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\hat{a}_{N-1} & \hat{a}_{N-2} & \hat{a}_{N-3} & \cdots & \hat{a}_{N-k-1} & \hat{a}_{N-k}
\end{array}\right)
\]

for \(r = 0, \ldots, N\) and \(j = 0, \ldots, k - 1\). The column \(\partial a_j / \partial b_0\) of the Jacobi matrix is not calculated, as \(b_0\) is assumed fixed according to (5.2).

• Compute the next iterated solution \(b_j + \Delta_j (j = 1, 2, \ldots, k)\) of the polynomial coefficients by solving the system of \(k\) linear equations

\[
\sum_{j=1}^{k} \frac{\partial \hat{a}_l}{\partial b_j} \Delta_j = -\hat{a}_l, \quad l = 1, \ldots, k
\]

for the first-order differences \(\Delta_j\). This equation is the first-order Taylor expansion of \(\hat{a}_l\) as a function of the \(b_j\) set to the target (5.3) for this update. The \(k \times k\) coefficient matrix \(\partial \hat{a}_l / \partial b_j\) is a square submatrix of the Jacobi matrix calculated in the previous step.

• Return to the second bullet for the next cycle until the \(\hat{a}_0\) to \(\hat{a}_k\) are sufficiently close to (5.3).

Remark 5.2. This algorithm involves only \(f_0\) to \(f_N\), but no higher order approximants to \(f(x)\). It therefore adapts a polynomial of degree \(k\) to a polynomial of degree \(N\).

Example 5.3. The error terms (4.17) for the polynomial \(\sum_{j=0}^{4} b_j T_j(x)\) change to

\[
\sum_{n=0}^{N} \hat{a}_n T_n(x) - 1 \approx 5.2 \cdot 10^{-12} T_0(x) + 4.7 \cdot 10^{-11} T_2(x) + 6.3 \cdot 10^{-12} T_4(x) - 1.08 \cdot 10^{-4} T_6(x) - 1.11 \cdot 10^{-5} T_8(x)
\]

after one Newton iteration, reducing the relative error to \(\sum_{n=0}^{8} |\hat{a}_n| - 1 \approx 0.000119\). During further iteration cycles the relative error stays about the same because it is dominated by \(\hat{a}_6 T_6(x)\) which is out of reach of the polynomial base with \(k = 4\).

Example 5.4. An IEEE “single” precision accuracy of \(f(x) = \sin(x^2) / x\) with a relative error smaller than \(2^{-24} \approx 6.0 \cdot 10^{-8}\) needs \(k = 8\). Truncation of the Chebyshev series for \(f(x)\) after \(k = 8\) yields an estimated maximum absolute error of \(\sum_{n=k+1}^{N} |f_n| \approx 6.7 \cdot 10^{-9}\) evaluated at \(N = 16\). The relative error of the same polynomial is also \(\sum_{n=0}^{N} |\hat{a}_n| - 1 \approx 6.7 \cdot 10^{-9}\). After four Newton iterations, this
value drops to $5.9 \cdot 10^{-9}$ with coefficients given in the following table—remaining very close to those cited after (4.10):

| $n$ | $b_n$ |
|-----|--------|
| 0   | 1.27627896240265880207637 |
| 2   | -0.2852615691810328617761446 |
| 4   | 0.9118016006289075331306166 $\cdot 10^{-2}$ |
| 6   | -0.1365874893444115901818408 $\cdot 10^{-3}$ |
| 8   | 0.1184206224108742454613850 $\cdot 10^{-5}$ |

**Example 5.5.** $g(x) = \cos(\pi x)$ has the expansion coefficients $[5, 23, 30]$ (5.8)

$$g_n = \begin{cases} 
2\left(-\frac{n}{2}\right) J_n(\pi/2), & n \text{ even}, \\
0, & n \text{ odd}.
\end{cases}$$

The approximation $g(x) \approx \sum_{n=0}^k b_n T_n(x)$ has an estimated maximum absolute error of $\sum_{n=k+1}^N |b_n| \approx 4.7 \cdot 10^{-8}$ for the polynomial of degree $k = 8$ evaluated at $N = 16$. Because $g(x)$ is zero at both ends of the interval $[-1, 1]$, the algorithm does not find polynomials $\sum_{j=0}^k b_j T_j(x)$ with a uniformly convergent Chebyshev expansion of the relative error—any $\hat{a}_n$ obtained depend strongly on $N$. We therefore “lift” both zeros by looking at $f(x) = \cos(\pi x)/(1 - x^2)$ instead, which has the expansion coefficients (5.9)

$$f_0 = \pi J_1(\pi/2),$$

$$f_2 = f_0 - 2g_0,$$

$$f_n = 2f_{n-2} - f_{n-4} - 4g_{n-2}, \quad n = 4, 6, 8, \ldots,$$

$$f_n = 0, \quad n \text{ odd}.$$ 

Truncation of the Chebyshev series for $f(x)$ after $k = 4$ yields an estimated maximum absolute error of $\sum_{n=k+1}^N |f_n| \approx 2.7 \cdot 10^{-5}$ evaluated at $N = 8$. The relative error of the same polynomial is $\sum_{n=0}^N |\hat{a}_n| - 1 \approx 3.3 \cdot 10^{-5}$. After four Newton iterations, this value drops to $3.1 \cdot 10^{-5}$ with coefficients given in the following table:

| $n$ | $b_n$ |
|-----|--------|
| 0   | 0.8903651967922106931461297 |
| 2   | -0.1072744347398521266520654 |
| 4   | 0.002332103968386755210894198 |

**Example 5.6.** The coefficients of the Chebyshev series of arcsin $x$ and $(\text{arcsin} x)/x$ (App. C) are slowly descending. The infinite slope of arcsin $x$ at $x = \pm 1$ renders both series inefficient, so we turn to $\frac{1}{2} \arcsin \frac{1}{\sqrt{2}}$ instead as configured in (D.10). Keeping terms up to $f_{36}$ yields an estimated maximum absolute error of $\sum_{n=38}^N |f_n| \approx 1.4 \cdot 10^{-17}$ evaluated at $N = 108$. The relative error of the same polynomial is $\approx 1.9 \cdot 10^{-17}$. After four Newton iterations, this value drops only slightly to $1.8 \cdot 10^{-17}$; obviously, there is not much room to improve the polynomial representation w.r.t. an optimized relative error in cases where the amplitude of the function is small over the $x$-interval.

**Example 5.7.** The expansion for $\exp(x)$ in $-1 \leq x \leq 1$ reads (9.6.19) (3.4.1e) (p. 69) (33)

$$\exp(x) = 2 \sum_{n=0}^{\infty} f_n(1) T_n(x).$$
Truncation after $k = 14$ yields an estimated maximum absolute error of $\sum_{n=k+1}^{N} |f_n| \approx 4.9 \cdot 10^{-17}$ evaluated at $N = 42$. The relative error of the same polynomial is $\sum_{n=0}^{\infty} |\tilde{a}_n| - 1 \approx 1.3 \cdot 10^{-16}$. After four Newton iterations, this value drops to $7.5 \cdot 10^{-17}$ with coefficients given in the following table, also listed as $f(x) \approx \sum_{n=0}^{N} d_n x^n$:

| $n$ | $b_n$         | $d_n$                  |
|-----|---------------|------------------------|
| 0   | 1.266065877520083355982446 | 1.00000000000000002107745526254 |
| 1   | 1.1303182079849700544135921 | 1.000000000000000635489461393241 |
| 2   | 0.271495339534076623657051 | 0.4999999999999792936666685291 |
| 3   | 0.433684984866380495257150-1 | 0.166666666666642610320391 |
| 4   | 0.547424044293732650276168-2 | 0.4166666666669875817272051-1 |
| 5   | 0.542926311919347503621352-3 | 0.8333333336026596258842-2 |
| 6   | 0.49773229542951466443872-4 | 0.138888888870286928616602-2 |
| 7   | 0.31984364624019950134121-5 | 0.198412697108614809924519-3 |
| 8   | 0.1992124806672795001043316-6 | 0.24801587802316103680909-4 |
| 9   | 0.11036717255163291577862-7 | 0.27557351523703425613644-5 |
| 10  | 0.550859607955188165792078-9 | 0.275572536928709036229172-6 |
| 11  | 0.249756604792065950497342-10 | 0.2504783672755758754944252-7 |
| 12  | 0.1093152481832518265611-11 | 0.208803415958673895181371-8 |
| 13  | 0.399067687421017034112272-12 | 0.16345812476748577123867-9 |
| 14  | 0.140023749972286678658850-14 | 0.11470745977297241385170-10 |

**Example 5.8.** The expansion for $J_0(\frac{\pi}{2} x)$ in $-1 \leq x \leq 1$ reads $\left[ \frac{15}{6.681.5} \right]$

\begin{equation}
J_0\left( \frac{\pi}{2} x \right) = 2 \sum_{n=0,2,4,\ldots}^{\infty} (-1)^n / n! \frac{\pi^2}{4} T_n(x).
\end{equation}

Truncation after $T_{16}(x)$ yields an estimated maximum absolute error of $\sum_{n=k+1}^{N} |f_n| \approx 7.3 \cdot 10^{-19}$ evaluated at $N = 48$. The relative error of the same polynomial is $\sum_{n=0}^{\infty} |\tilde{a}_n| - 1 \approx 1.6 \cdot 10^{-18}$. After four Newton iterations, this value drops to $1.3 \cdot 10^{-18}$ with coefficients given in the following table:

| $n$ | $b_n$         | $d_n$                  |
|-----|---------------|------------------------|
| 0   | 0.7252769164405135618043045 | 0.9999999999999991311745 |
| 2   | -0.26381081846140473471353 | -0.6168502750568034778603892 |
| 4   | 0.1072184541022420669250684-1 | 0.951260654628894620024320-10 |
| 6   | -0.1885687641235952967199171-3 | -0.6519837738512518004083602-2 |
| 8   | 0.1845983728936480887451460-5 | 0.251360212324872245916252-3 |
| 10  | -0.1150537141255094251800350-7 | -0.620906460960606041245435-5 |
| 12  | 0.4965028950145784947530764-10 | 0.1062698637612363296677914-6 |
| 14  | -0.1571252252452718608949964-11 | -0.1336990135568532922581048-10 |
| 16  | 0.3800986508122698831881511-15 | 0.124550725891645953230933-10 |

A set of $b_j$ in

\begin{equation}
R(x) \equiv \frac{f(x)}{\sum_{j=0}^{k} b_j T_j(x)} - 1
\end{equation}

found that way is also a starting point to calculate the solution with the minimax property of the relative error: This locates the local minima and maxima of $R(x)$, computes the mean of their absolute values, and iteratively adjusts the $b_j$ such
that the absolute values of the new alternating extrema equal that mean. The corrections $\Delta_j$ to the $b_j$ can be computed by expansion of (5.15) to first order in $\Delta_j$ keeping the abscissa of the extrema fixed, which ends up in a linear system of equations for the $\Delta_j$.

**Example 5.9.** An IEEE “double” precision accuracy of $f(x) = \sin(\pi x/2)/x$ with a relative error smaller than $2^{-53} \approx 1.1 \cdot 10^{-16}$ needs $k = 16$. Truncation of the Chebyshev series of Example 4.3 for $f(x)$ after $k = 16$ yields an estimated maximum absolute error of $\sum_{n=k+1}^{N} |f_n| \approx 4.1 \cdot 10^{-19}$ evaluated at $N = 32$. The relative error of the same polynomial is $\sum_{n=0}^{N} |\tilde{a}_n| - 1 \approx 3.8 \cdot 10^{-19}$. After four Newton iterations, this value drops to $3.5 \cdot 10^{-19}$ with coefficients $b_n$ given in the following table:

| $n$ | $b_n$ | $d_n$ |
|-----|-------|-------|
| 0   | 1.276278962402265880207637 | 1.570796326794896618868819 |
| 2   | -0.2852615691810300095702941 | -0.6459640975062461962319336 |
| 4   | 0.911801600665180249776792310^{-2} | 0.79692641655409762753310^{-1} |
| 6   | -0.136587513541966672436476510^{-3} | -0.468175413530346824088250610^{-2} |
| 8   | 0.118496185766169010828887210^{-5} | 0.160441184710011408803188110^{-3} |
| 10  | -0.670279160382744108170612110^{-8} | -0.359884301391732615952045610^{-5} |
| 12  | 0.266727859901790328386344310^{-10} | 0.569213565612242990194435710^{-7} |
| 14  | -0.787292200461570901859432510^{-13} | -0.668436943648410375793363310^{-9} |
| 16  | 0.179192909471828407211991610^{-15} | 0.58717932575728732475230710^{-11} |

The actual relative error of this approximation is shown in Fig. 1 as a continuous line, with a maximum of $2.9 \cdot 10^{-19}$. The dashed line with a relative error of $2.6 \cdot 10^{-19}$ in comparison results from further minimax optimization with coefficients shown in the next table:

![Figure 1](image-url)
Example 5.10. As an example for (4.18), consider \( \exp(x) = \sum_{n=0}^{\infty} f_n T_n^*(x) \) over \( 0 \leq x \leq 1 \) [1, (4.2.48)] [5, 19]. The \( f_n \) are represented via [1, (9.6.26)] through modified Bessel Functions \( I_n \),

\[
\begin{align*}
\begin{array}{ccc}
n & b_n & d_n \\
0 & 1.2762789624022658802075347 & 1.5707963267948966188314659 \\
2 & -0.2852615691810360095705230 & -0.6459640975062461915471363 \\
4 & 0.9118016006651802497528156 \cdot 10^{-2} & 0.7969262624616544421893744 \cdot 10^{-1} \\
6 & -0.1365875135419666726405733 \cdot 10^{-3} & -0.4681754135302704719117724 \cdot 10^{-2} \\
8 & 0.1184961857661699205427320 \cdot 10^{-5} & 0.1604411847070460989830944 \cdot 10^{-3} \\
10 & -0.6702791603827612608171959 \cdot 10^{-8} & -0.35988430076526555526627 \cdot 10^{-5} \\
12 & 0.26672785990198555592489579 \cdot 10^{-10} & 0.5692134921445583455723 \cdot 10^{-7} \\
14 & -0.7872921659616258733890169 \cdot 10^{-13} & -0.668432580312975131354658 \cdot 10^{-9} \\
16 & 0.1791589025538146793760922 \cdot 10^{-15} & 0.5870678918883399413795788 \cdot 10^{-11}
\end{array}
\end{align*}
\]

Truncation of the Chebyshev series for \( f(x) \) after \( k = 3 \) yields an estimated maximum absolute error of \( \sum_{n=k+1}^{N} |f_n| \approx 5.7 \cdot 10^{-4} \) evaluated at \( N = 9 \). The relative error of the same polynomial is \( \sum_{n=0}^{N} |a_n| - 1 \approx 5.1 \cdot 10^{-4} \). After four Newton iterations, this value drops to \( 4.0 \cdot 10^{-4} \) with coefficients given in the following table:

\[
\begin{align*}
\begin{array}{ccc}
n & b_n & f_n \\
0 & 3.506775308754180791443893 & 2 \sqrt{e} I_n(1/2); \\
1 & 0.8503916537808109665352350 & f_{n+1} = -4n f_n + f_{n-1}, \\
2 & 0.1052086936309369253029528 & \\
3 & 0.0087221047331556411612874 & \\
4 & 0.0005434368311501559635982758 & \\
5 & 0.00002711543491306869404046064 & \\
\end{array}
\end{align*}
\]

If we proceed to \( k = 12 \) at \( N = 36 \), the estimated maximum relative error becomes \( 6.1 \cdot 10^{-18} \) with the following coefficients:

\[
\begin{align*}
\begin{array}{ccc}
n & b_n & f_n \\
0 & 1.753387654377090395721946 & \\
1 & 0.850392561425088936327743 & \\
2 & 0.10519185208937847555014 & \\
3 & 0.00858708996902776771654559 & \\
\end{array}
\end{align*}
\]
### Chebyshev Series Expansion of Inverse Polynomials

| $n$ | $b_n$ | $d_n$ |
|-----|-------|-------|
| 0   | 1.7533876543770903957219464 | 1.0000000000000000060373678 |
| 1   | 0.85039165378081066532510   | 0.9999999999999788799411   |
| 2   | 0.105208693609369253029528   | 0.500000000000121648194572   |
| 3   | 0.8721047331556411161287410^{-2} | 0.1666666666639271874501180 |
| 4   | 0.54336831155015963598275810^{-3} | 0.8333331281515458169149710^{-2} |
| 5   | 0.27115439130686940405476510^{-4} | 0.138888986273893325816383910^{-2} |
| 6   | 0.1128132888208278896741610^{-5} | 0.19840982879736514642110310^{-3} |
| 7   | 0.4025582298707100276646710^{-7} | 0.248073462709246317680416410^{-4} |
| 8   | 0.125654412834225651702410^{-8} | 0.2827815155248445934907810^{-6} |
| 9   | 0.34880919365312561941432410^{-10} | 0.3419953309135672396023910^{-8} |

Equilibration of the local extrema with the following coefficients reduces this error to $5.0 \cdot 10^{-18}$:

| $n$ | $b_n$ | $d_n$ |
|-----|-------|-------|
| 0   | 1.7533876543770903961757996 | 1.0000000000000049913878 |
| 1   | 0.8503916537808109674449984 | 0.999999999999997887999411 |
| 2   | 0.105208693609369262175803 | 0.500000000000106363079379 |
| 3   | 0.8721047331556503519502710^{-2} | 0.1666666666642173677701902 |
| 4   | 0.5433683115015695765198677457910^{-3} | 0.8333331290713600667679710^{-2} |
| 5   | 0.27115439130694589690136010^{-4} | 0.13888898039056281871229210^{-2} |
| 6   | 0.11281328887305454637691810^{-5} | 0.19840996842631079252426310^{-3} |
| 7   | 0.40255822997040185490521810^{-7} | 0.248071259497116384524733510^{-4} |
| 8   | 0.125654412834225651702410^{-8} | 0.2827815155248445934907810^{-6} |
| 9   | 0.34880919365312561941432410^{-10} | 0.3419953309135672396023910^{-8} |

### 6. Summary

Besides some generic algorithms to compute the Chebyshev series of inverse polynomials, there are two specific aspects that facilitate this task: (i) the expansion coefficients can be derived from the partial fractions of the inverse polynomial. (ii) Expansion coefficients with indices larger than the polynomial degree are recursively linked to those of lower order. (iii) An algorithm has been presented which derives a polynomial of a given degree such that the first terms of the Chebyshev expansion of the relative error of a given function represented by this polynomial vanish.

### Appendix A. Chebyshev Series of $\ln(1 + x)$

The integral representation

\begin{equation}
\ln(1 + x) = \int \frac{dx}{1 + x}
\end{equation}

\section*{Appendix A. Chebyshev Series of $\ln(1 + x)$}

The integral representation

\begin{equation}
\ln(1 + x) = \int \frac{dx}{1 + x}
\end{equation}
and term-by-term integration of (2.10) on the r.h.s. with (2.9) yield that the Chebyshev coefficients of

\[(A.2)\]
\[f(x) = \ln(1 + x) \equiv \sum_{n=0}^{\infty} f_n T_n(x) \quad 0 \leq x \leq 1,\]

obey

\[(A.3)\]
\[2n f_n = a_{n+1,1}(-3) - a_{n-1,1}(-3), \quad n \geq 1,
\]

explicitly [13, p. 88] [14]

\[(A.4)\]
\[f_n = \frac{2(-)^{n+1}}{n(3 + 2\sqrt{2})^n}, \quad n \geq 1\]

from (2.11), as tabulated in [1] 4.1.45. The missing \(f_0\) is

\[(A.5)\]
\[f_0 = 2 \int_0^1 \frac{\ln(1 + x)}{\sqrt{x}(1-x)} dx = 2 \ln \frac{3 + 2\sqrt{2}}{4},\]

because insertion of \(x = 1\) in (A.2) yields

\[(A.6)\]
\[f_0 = 2 \left( f(1) - \sum_{n=1}^{\infty} f_n \right) \]

and \(f(1) = \ln 2\) and \(\sum_{n=1}^{\infty} f_n = 2 \ln(1 + \frac{1}{3+2\sqrt{2}})\) via [1] 4.1.24.

**Appendix B. Chebyshev Series of \(\arctan x\)**

Integrating (2.7) over \(x\) with

\[(B.1)\]
\[\int \frac{1}{1 + x^2} dx = \arctan x\]

and (1.5) we get [13] p. 89 [14]

\[(B.2)\]
\[\arctan x = 2 \sum_{j=1,3,5,7,\ldots} \frac{(-)^{[j/2]}}{j(1 + \sqrt{2})^j} T_j(x), \quad -1 \leq x \leq 1,\]

in particular at \(x = 1\)

\[(B.3)\]
\[\frac{\pi}{8} = \sum_{j=1,3,5,7,\ldots} \frac{(-)^{[j/2]}}{j(1 + \sqrt{2})^j}.\]

From (B.2) and (1.15), the coefficients of

\[(B.4)\]
\[\frac{\arctan x}{x} = \sum_{n=0,2,4,6,\ldots} g_n T_n(x)\]

as listed in [1] (4.4.50) [5] follow recursively, where \(g_0 = 2 \ln(1 + \sqrt{2})\) is obtained via [15] 4.531.12.
Appendix C. Chebyshev Series of arcsin \(x\)

The series of arcsin \(x\) \(= \sum_{n=1}^{\infty} g_n T_n(x)\) starts with \(g_1 = 4/\pi\). A combination of [1] (4.4.58), [14], [15] and [28] (3.4.1d) yields

\[
\sqrt{1-x^2} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{1-n^2} T_n(x)
\]

in this case, which can be unwound as \(g_n = 4/(\pi n^2)\). To find a formulation with controlled relative error, we would switch to \(h(x) = (\arcsin x)/x = \sum_{n=0}^{\infty} h_n T_n(x)\) to remove the zero in the spirit of example 5.5. With [1, (17.1.4)], the expansion coefficients are

\[
h_0 = 8\beta(2)/\pi, \quad h_{n+2} = -h_n + \frac{8}{\pi(n+1)^2},
\]

where \(\beta(2) \approx 0.915965594177219015054603515\) is Catalan’s constant [1, Tab 23.3].

Appendix D. Chebyshev Series of arcsin \((x/\sqrt{2})\)

The coefficients of

\[
\arcsin(x/\sqrt{2}) = \sum_{n=1,3,5,...} \infty k_n T_n(x)
\]

are found by partial integration of \(\int \arcsin((\cos \theta)/\sqrt{2}) \cos(n\theta) d\theta\)

\[
k_n = \frac{1}{n\pi} \left[ \int_0^\pi \cos[(n-1)\theta] \sqrt{2 - \cos^2 \theta} d\theta - \int_0^\pi \cos[(n+1)\theta] \sqrt{2 - \cos^2 \theta} d\theta \right], \quad n \text{ odd}
\]

where

\[
\int_0^\pi \frac{\cos(2m\theta)}{\sqrt{2 - \cos^2 \theta}} d\theta = G_{2m} = \begin{cases} \sqrt{2} F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2} \mid \frac{1}{2}\right) \approx 2.6220575542921198104648395899, & m = 0, \\ \sqrt{2} \left[ 3 F\left(\frac{1}{2},\frac{3}{2};\frac{5}{2} \mid \frac{1}{2}\right) - 4 E\left(\frac{1}{2}\right) \right] \approx 0.22577708482093539558499460534, & m = 1, \end{cases}
\]

are Complete Elliptic Integrals. To find a recurrence for these

\[
G_s = \int_{-1}^1 \frac{T_s(x)}{\sqrt{2-x^2}(1-x^2)} dx,
\]

we apply the method of [1] (17.1.4) to the quartic \(y^2 = (2-x^2)(1-x^2)\), with \(d(yT_s(x))/dx = y(dT_s(x)/dx) + T_s 1/2y (T_s(x) - 3T_1(x))\), insert [1.0] for the derivative on the r.h.s, replace the first \(y\) on the r.h.s. by \(y^2/y = (T_4/8 - T_2 + 7/8)/y\), expand all products with [1.3], and finally insert the upper limit \(x = 1\) where \(y(x)T_s(x) = 0\):

\[
G_{s+3} + G_{|s-3|} - 3(G_{s+1} + G_{|s-1|}) + \sum_{l=0}^{s-1} [G_{l+4} + G_{|l-4|} - 8(G_{l+2} + G_{|l-2|}) + 14G_l] = 0.
\]
Inserting \( s = 1, 3, 5 \) and 7, for example, yields
\[
\begin{align*}
(D.6) & \quad 3G_4 - 12G_2 + G_0 = 0, \\
(D.7) & \quad 5G_6 - 27G_4 + 15G_2 - G_0 = 0, \\
(D.8) & \quad 7G_8 - 41G_6 + 29G_4 - 3G_2 = 0, \\
(D.9) & \quad 9G_{10} - 55G_8 + 43G_6 - 5G_4 = 0,
\end{align*}
\]
and generates \( k_1 \) to \( k_9 \) in (D.24) from \( G_0 \) and \( G_2 \) shown in (D.23). A slowly converging series expansion is also known [1 806.01]. With (D.15) we find the coefficients 
\[
f_n = 2k_{n-1} - f_{n-2}
\]
for
\[
(D.10) \quad \frac{1}{x} \arcsin \frac{x}{\sqrt{2}} = \sum_{n=0,2,4,...}^{\infty} f_nT_n(x), \quad -1 \leq x \leq 1,
\]
starting at
\[
f_0 = \frac{4}{\pi} \int_0^1 \frac{1}{x} \arcsin \frac{x}{\sqrt{2}} \, dx = \sqrt{2} \sum_{l=0}^\infty \frac{[(2l-1)!!]^2}{2l(2l+1)(2l+1)!!^2}
\]
(D.11) \quad \frac{\pi}{2} - 4 \pi \sum_{q=1}^\infty \frac{1}{4q-1} [G_{4q-2} - G_{4q}] \approx 1.486664932871034689603296833.

The four coefficients \( \alpha_i \) that span (D.12)
\[
k_{n-1} = \sqrt{2} \left[ \alpha_1 \left( \frac{1}{\sqrt{2}} \right) + \alpha_2 E \left( \frac{1}{\sqrt{2}} \right) \right], \quad f_n = \sqrt{2} \left[ \alpha_3 K \left( \frac{1}{\sqrt{2}} \right) + \alpha_4 E \left( \frac{1}{\sqrt{2}} \right) \right] + (-)^n \sqrt{\frac{n}{2}} f_0,
\]
start as follows:
\[
\begin{array}{cccccc}
\hline
n & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\hline
2 & -2 & 4 & -4 & 8 \\
4 & -26/9 & 4 & -16/9 & 0 \\
6 & -638/75 & 292/25 & -3428/225 & 584/25 \\
8 & -22702/735 & 212/5 & -513088/11025 & 1536/25 \\
10 & -23722/189 & 4652/27 & -676346/33075 & 191128/675 \\
12 & -463174/847 & 2252/3 & -3558618544/4002075 & 822272/675 \\
14 & -162508858/65065 & 8691484/2535 & -2777152623884/676350675 & 643269592/114075 \\
\end{array}
\]
Because \( T_{2j}(x) = T_j(x^2) \), the following numbers coincide with [1 (4.451)] up to a factor \( \sqrt{2} \):
\[
\begin{array}{cccc}
\hline
n & f_n & \alpha_1 & \alpha_2 \\
\hline
2 & 0.3885303371652290716432228 \times 10^{-1} & 0.288421834475536563843289 \times 10^{-3} \\
4 & 0.2885441422084471126676825 \times 10^{-2} & 0.415877878052832866177270 \times 10^{-5} \\
8 & 0.322367192785279209254231 \times 10^{-4} & 0.755007844937152593425185 \times 10^{-7} \\
12 & 0.5496504525974164467345493 \times 10^{-6} & 0.15421803792814702156106 \times 10^{-8} \\
16 & 0.1067193805629843129424091 \times 10^{-7} & 0.33838563934277587100479 \times 10^{-10} \\
20 & 0.2268114598545151963877153 \times 10^{-9} & 0.77911392163624421446539 \times 10^{-12} \\
24 & 0.510893752437719224216916 \times 10^{-11} & 0.1856972621823422540637 \times 10^{-13} \\
28 & 0.119837859352895337866326 \times 10^{-12} & 0.4527928863282308147851 \times 10^{-15} \\
32 & 0.289618915439603461020997 \times 10^{-14} & 0.11341442569045599650971 \times 10^{-16} \\
36 & 0.7161678029265506176831289 \times 10^{-16} & 0.33838563934277587100479 \times 10^{-10} \\
\end{array}
\]
Appendix E. Chebyshev Series of $\psi(x + 2)$

An expansion of the Digamma function [34] is [11 (6.3.16)]

$$\psi(2 + x) = 1 - \gamma + x \sum_{k=2}^{\infty} \frac{1}{k(x + k)},$$

where $\gamma \approx 0.5772$ is Euler’s constant. Employing $a_{n,1}(-k)$ of [24],

$$\frac{1}{x + k} = -\frac{1}{-k - x} = -\frac{2}{\sqrt{k^2 - 1}} \sum_{n=0}^{\infty} \frac{(-)^n}{(k + \sqrt{k^2 - 1})^n} T_n(x), \quad -1 \leq x \leq 1,$$

$$\psi(2 + x) = 1 - \gamma + 2x \sum_{k=2}^{\infty} \frac{1}{k\sqrt{k^2 - 1}} \sum_{n=0}^{\infty} \frac{(-)^n}{(k + \sqrt{k^2 - 1})^n} T_n(x).$$

The auxiliary definition

$$K_n \equiv \sum_{k=2}^{\infty} \frac{1}{k\sqrt{k^2 - 1}(k + \sqrt{k^2 - 1})^n}, \quad n = 0, 1, 2, \ldots$$

turns [E.3] with the aid of [11.4] into

$$\psi(x + 2) = (1 - \gamma - K_1)T_0(x) - \sum_{n=1}^{\infty} (-)^n (K_{n-1} + K_{n+1})T_n(x), \quad -1 \leq x \leq 1,$$

where

$$K_n + K_{n+2} = 2 \sum_{k=2}^{\infty} \frac{1}{\sqrt{k^2 - 1}(k + \sqrt{k^2 - 1})^{n+1}}, \quad n = 0, 1, 2, \ldots$$

Alternatives to the slowly converging original series [E.3] at small $n$ are obtained in terms of the Riemann Zeta function $\zeta$ after reducing the fraction in [E.4] and/or [E.6] by $k - \sqrt{k^2 - 1}$,

$$K_0 = \sum_{k=2}^{\infty} \frac{1}{k^2} \left(1 - \frac{1}{k^2}\right)^{-1/2} = \sum_{l=0}^{\infty} (-)^l \left(-\frac{1}{l}\right) \zeta(2l + 2) - 1,$$

$$K_1 = \sum_{k=2}^{\infty} \frac{k - \sqrt{k^2 - 1}}{k\sqrt{k^2 - 1}} = \sum_{l=1}^{\infty} (-)^l \left(-\frac{1}{l}\right) \zeta(2l + 1) - 1,$$

$$K_0 + K_2 = 2 \sum_{l=1}^{\infty} (-)^l \left(-\frac{1}{l}\right) \zeta(2l) - 1,$$

$$K_1 + K_3 = 2 \sum_{l=2}^{\infty} (-)^l \left(-\frac{1}{l} + \frac{1}{l}\right) \zeta(2l - 1) - 1,$$

$$K_0 + K_n + K_{n+2} = \sum_{l=\lfloor(n+3)/2\rfloor}^{\infty} (-)^l \zeta(2l - n) - 1 \sum_{s=0}^{n+1} \binom{n+1}{s} \binom{n+1}{l/2}.$$
Linear combinations of these two equations are

$$(E.12) \quad \sum_{k=2}^{\infty} \frac{1}{k^2 k!} k + 1 + \sqrt{k^2 - 1} = \frac{1}{2},$$

$$(E.13) \quad \sum_{k=2}^{\infty} \frac{1}{k^2 k!} k + 1 - \sqrt{k^2 - 1} = 1.$$ 

Linear combinations of these two equations are

$$(E.14) \quad \sum_{k=2}^{\infty} \frac{(k-3)(k + \sqrt{k^2 - 1})}{k(k+1)(k+2)(k+3)} = 0,$$

$$(E.15) \quad \sum_{k=2}^{\infty} \frac{k + \sqrt{k^2 - 1}}{k(k+1)(k+2)(k+3)} = \frac{1}{8},$$

and these two can be combined to

$$(E.16) \quad \sum_{k=2}^{\infty} \frac{k + \sqrt{k^2 - 1}}{\sqrt{k^2 - 1}[(k+1)^2 - 1]} = 3.$$
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