MULTIPLIERS BETWEEN TWO OPERATOR SPACES

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Abstract. In a recent survey paper we introduced one-sided multipliers between two different operator spaces. Here we give some basic theory for these maps.

1. Introduction

In [3] we introduced the space \( M_l(X) \) (resp. \( A_l(X) \)) of left multipliers (resp. adjointable multipliers) on an operator space \( X \). See also [21]. This is an operator algebra (resp. \( C^* \)-algebra) whose elements are maps on \( X \); namely the maps satisfying variants of Theorems 6.1 or 8.1 (resp. 4.1) below (in the case that \( X = Y \)). Theory and applications of these maps may be found in a series of papers (e.g. [3, 6, 5, 2, 7]). A survey of this theory containing proofs and applications, may be found in [4]. In the latter paper we also began to consider one-sided multipliers between two different operator spaces \( X \) and \( Y \). In the present paper we give some basic theory of these maps.

To have any interesting notion of multipliers between two different spaces \( X \) and \( Y \) it does seem that there does need to be some relation between \( X \) and \( Y \), or between two \( C^* \)-algebras that are canonically associated with \( X \) and \( Y \). This is the approach taken in Sections 3 and 4, where we require the 'noncommutative Shilov boundaries' of \( X \) and \( Y \) to be a module over two equal, or comparable, \( C^* \)-algebras. We may then prove characterizations of the associated multipliers from \( X \) to \( Y \), which are satisfying except for the feature that they do depend on the particular Shilov boundaries chosen. For this reason we call these relative left multipliers.

The problem in the last paragraph also seems to be related to the issue of having a well defined ‘column sum’ of two operator spaces \( X \) and \( Y \). If \( X, Y \) are subspaces of an operator space \( V \) then this difficulty evaporates, we may simply define the column sum \( X \oplus V Y \) to be the algebraic sum \( X \oplus Y \) endow with an operator space structure by identifying it with a subspace of \( C_2(V) \) via the map

\[
(x, y) \mapsto \begin{bmatrix} x \\ y \end{bmatrix}, \quad x \in X, y \in Y.
\]

This is the approach taken in the final two sections of this paper, and in this case we can remove the ‘dependence on the Shilov boundary’ problem mentioned above.

2. Preliminaries

We reserve the symbols \( H, K \) for Hilbert spaces. We use standard notation for operator spaces (see e.g. [11, 16, 17]). In particular we write \( C_n(X) \) and \( R_n(X) \)

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for $M_{n,1}(X)$ and $M_{1,n}(X)$ respectively. We recall that the space $CB(X,Y)$ of completely bounded maps between two operator spaces is also an operator space. An injective operator space $X$ has the property that any completely contractive $T : Y \to X$ has a completely contractive extension $\tilde{T} : Z \to X$. Here $Z$ is any operator space with subspace $Y$. Alternatively, a subspace $X \subset B(K,H)$ is injective if and only if there is a completely contractive projection of $B(K,H)$ onto $X$.

We will need some basic theory of $C^*$-modules; see [14] or Section 3 of [4]. In fact we will need a few facts from the latter paper which we now repeat: Suppose that $W$ and $Z$ are right $C^*$-modules over $C^*$-algebras $B$ and $C$ respectively, where $B$ is a $C^*$-subalgebra of $C$. Then we may define an analogue of the $C^*$-module direct sum. Namely we define $W \oplus^C Z$ to be the algebraic sum $W \oplus Z$, endowed with matrix norms

$$\left\| \begin{bmatrix} w_{ij} & z_{ij} \end{bmatrix} \right\| = \left\| \sum_{k=1}^{n} \langle w_{k1}|w_{kj} \rangle + \langle z_{k1}|z_{kj} \rangle \right\|^{\frac{1}{2}}, \quad [w_{ij}] \in M_n(W), [z_{ij}] \in M_n(Z).$$

There are many ways to see that $W \oplus^C Z$ is an operator space: this is explained in [4] for example. In fact this will be obvious in most of the cases we are interested in the later sections.

The following is an extension of a result of Paschke (see [14, Theorem 2.8]).

**Theorem 2.1.** [14] Suppose that $W$ and $Z$ are right $C^*$-modules over $C^*$-algebras $B$ and $C$ respectively, where $B$ is a $C^*$-subalgebra of $C$. If $u : W \to Z$ is a $C^*$-linear map, then the following are equivalent:

(i) $u$ is a contractive $B$-module map;

(ii) $\langle u(w)|w(u(w)) \rangle \leq \langle w|w \rangle$, for all $w \in W$;

(iii) $\left\| \begin{bmatrix} u(w) \\ z \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} w \\ z \end{bmatrix} \right\|$, for all $w \in W, z \in Z$.

(iv) $\left\| \begin{bmatrix} u(w) \\ c \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} w \\ c \end{bmatrix} \right\|$, for all $w \in W, c \in C$.

Suppose that $Y, Z$ are $C^*$-modules over $B$, and that $J$ is a closed ideal of $B$ (or of the multiplier $C^*$-algebra $M(B)$ of $B$) containing the span of the ranges of the $B$-valued inner products on $Y$ and $Z$. We write $B_B(Y, Z)$ for the set of bounded $B$-module maps from $Y$ to $Z$. It clearly follows from Theorem 2.1 although it is also easy to prove directly using Cohen’s factorization theorem, that $Y$ and $Z$ are $C^*$-modules over $J$, and that

$$B_B(Y, Z) = B_J(Y, Z).$$

Thus there is not a truly essential dependence of $B_B(Y, Z)$ on $B$.

We write $\mathcal{B}(Y, Z)$ for the set of adjointable maps $T : Y \to Z$, that is those maps for which there is a map $S : Z \to Y$ such that

$$\langle Ty|z \rangle = \langle y|Sz \rangle, \quad y \in Y, z \in Z.$$

The space $\mathcal{K}(Y, Z)$, of so-called ‘compact’ adjointable maps, is the closure in $\mathcal{B}(Y, Z)$ of the span of the operators of the form $y \mapsto z|wy \rangle$ on $Y$, for $w \in Y, z \in Z$. The set $\mathcal{K}(Y) = \mathcal{K}(Y, Y)$ is a $C^*$-subalgebra of the $C^*$-algebra $\mathcal{B}(Y) = \mathcal{B}(Y, Y)$.

For any right $C^*$-module $Y$ over a $C^*$-algebra $B$ we may define the linking $C^*$-algebra $\mathcal{L}(Y)$ to be $\mathcal{K}(Y \oplus_c B)$. See e.g. [4, Section 3] for more details if needed.
Since $Y$ is a subspace of $\mathcal{L}(Y)$, it is consequently an operator space. We will always suppose that any $C^*$-module met in this paper has this operator space structure.

A $C^*$-module $W$ over a $W^*$-algebra $M$ such that $W$ has a predual Banach space will be called a $W^*$-module. By a result of Zettl (see [9] for a modern proof), this is exactly the (extremely important and well behaved) class of $C^*$-modules over $M$ which are self-dual. See [15] for the meaning of this term, and for basic theory of such modules.

Due to the work of Arveson and Hamana (see e.g. [11, 12]) it is known that every operator space $X$ has a noncommutative Shilov boundary. We will write this as $\partial X$ or $(\partial X, i)$, where $\partial X$ is a right $C^*$-module and $i : X \to \partial X$ is a linear complete isometry. We refer the reader to any one of [12, 3, 4], for example, for a better description of $\partial X$ and of its important universal property. We will not explicitly write down the latter property here but, loosely speaking, this universal property says that $\partial X$ is a smallest $C^*$-module containing $X$ completely isometrically. To be symmetrical, one should perhaps say 'M*-bimodule' here instead of $C^*$-module, but since we are emphasizing left multipliers we shall think of $\partial X$ as a right $C^*$-module over a $C^*$-algebra $B$ say. The $B$-valued inner product on $\partial X$ we shall write as $\langle \cdot | \cdot \rangle$; we refer to this as a (right) Shilov inner product. The ideal in $B$ densely spanned by the range of this inner product will be written as $\mathcal{F}(X)$, or $\mathcal{F}$ if $X$ is understood. By the afore-mentioned universal property one may see that the noncommutative Shilov boundary, the Shilov inner product, and the $C^*$-algebra $\mathcal{F}$, are essentially unique up to an appropriate isomorphism.

We review quickly Hamana’s method construction of a noncommutative Shilov boundary for $X$. We begin with an injective envelope $(I(X), i)$ of $X$. By this term, we mean that $I(X)$ is an injective operator space, that $i : X \to I(X)$ is a complete isometry, and that the identity map is the only completely contractive linear map from $I(X)$ to itself extending the identity map on $X$. This is called the rigidity property of the injective envelope. In fact $I(X)$ may be chosen to be a full right $C^*$-module over a $C^*$-algebra $\mathcal{D}$ (see e.g. [12]; also $\mathcal{D}$ is the algebra denoted $I(X)^*I(X)$ in [30, p.3]). We let $\partial X$ be the smallest closed subspace $E$ of $I(X)$ containing $i(X)$ for which $x, y, z \in E$ implies that $x\langle y | z \rangle \in E$. In this case the $C^*$-algebra $\mathcal{F}(X)$ in the last paragraph may be taken to be the closed span in $\mathcal{D}$ of the set $\{(y | z) : y, z \in \partial X\}$. Then $\partial X$ is a right $C^*$-module over $\mathcal{F}$.

We end this preliminaries section with a Lemma:

**Lemma 2.2.** Let $Z$ be a right $C^*$-module over a $C^*$-algebra $B$. Then there is an injective envelope $I(Z)$ of $Z$ which is a right $C^*$-module over a $C^*$-algebra $\mathcal{R}$, with the following properties: $\mathcal{R}$ contains $B$ as a $C^*$-subalgebra, and the module action $I(Z) \times \mathcal{R}$ restricted to $Z \times B$ is the original one.

**Proof.** We may suppose that $B$ is unital. Form the linking $C^*$-algebra $\mathcal{L}(Z)$ as in Section 2, and suppose that it is suitably represented nondegenerately as a $C^*$-subalgebra of $B(H \oplus K)$. Thus $I_B = I_K$. The following is a mild variant of the Hamana-Ruan construction of the injective envelope, and the reader may want to follow along with this construction in any of the sources [12, 15, 6, 16]. Consider the operator system

$$
\mathcal{S}_B(Z) = \left[ \begin{array}{cc}
\mathbb{C}I_H & Z \\
\overline{Z} & B
\end{array} \right] \subset B(H \oplus K).
$$
Let $\Phi$ be a \textit{minimal $S_B(Z)$ projection} on $B(H \oplus K)$, this is a completely positive idempotent map whose range is an injective envelope $I(S_B(Z))$ of $S_B(Z)$ (see the cited references). By a result of Choi and Effros [8], $I(S_B(Z))$ is a $C^*$-algebra with a new product $x \circ y = \Phi(xy)$. Also $I(S_B(Z))$ may be regarded as a $2 \times 2$ matrix algebra with respect to the canonical diagonal projections $p = I_H \oplus 0$ and $q = 0 \oplus I_K$. Let $\mathcal{R}$ be the 2-2-corner $qI(S_B(Z))q$, this is a $C^*$-subalgebra of $I(S_B(Z))$. By definition of the new product it is easy to see that $B$ is a $C^*$-subalgebra of $\mathcal{R}$. Let $E$ be the 1-2-corner $pI(S_B(Z))q$. Clearly $E$ is injective, and is also a right $C^*$-module over $\mathcal{R}$. In fact by definition of the new product it is easy to see that the right action of $\mathcal{R}$ on $E$ extends the action of $B$ on $Z$. By [6, Theorem 2.6], $E$ is injective in the category of operator $B$-modules. We wish to show that $E$ is an injective envelope of $Z$. To do this we first show that $Id_E$ is the only completely contractive $B$-module map $u: E \to E$ extending the identity map on $Z$. For by Suen’s variant on Paulsen’s lemma [20], such $u$ is the corner of a completely positive map $\Psi$ on the subspace $S_B(E)$ of $I(S_B(Z))$, such that $\Psi$ extends the identity map on $S_B(Z)$. Extend $\Psi$ further to a complete contraction from $I(S_B(Z))$ to itself. By the rigidity property of the injective envelope, this latter map and hence also $u$ must be the identity map.

The result is completed with an appeal to the fact from that the injective envelope of $Z$ is also the $B$-module injective envelope of $Z$ [6, Theorem 2.6]. The idea for this is as follows: by that result in [6] any injective envelope $I(Z)$ of $Z$ can be made into a $B$-module which is injective as an operator $B$-module. A routine diagram chase, using facts from the last paragraph, shows that $I(Z) \cong E$ completely isometrically and as $B$-modules.

\section{3. Relative left multipliers between two spaces}

In this section $X$ and $Y$ are operator spaces possessing noncommutative Shilov boundaries $(\partial X, i)$ and $(\partial Y, j)$ respectively, which will be fixed for the remainder of this section. We will assume also that $\partial X$ and $\partial Y$ are right $C^*$-modules over $C^*$-algebras $B$ and $C$ respectively, where $B$ is $C^*$-subalgebra of $C$.

The next result generalizes important facts about left multipliers on a single operator space. To explain the notation in this result: the inner products in (ii) are the (right) Shilov inner products on $Y$ and $X$ respectively, and the matrices there are indexed on rows by $i$, and on columns by $j$. The first norm in (iii) is just the norm in $M_{2n,n}(Y)$, the second is the norm on $M_n(\partial X \oplus C \partial Y)$ inherited from $M_n(\partial X \oplus C \partial Y)$. An explicit formula for this norm was given above Theorem 2.1.

Although the next result was stated in the survey [4], it was proved only in a special case.

\textbf{Theorem 3.1.} Let $X, Y, \partial X, \partial Y, B$ and $C$ be as above, where $B \subset C$, and let $T: X \to Y$ be a linear map. The following are equivalent:

(i) $T$ is the restriction to $X$ of a (necessarily unique) completely contractive right $B$-module map $S: \partial X \to \partial Y$.

(ii) $\|(T(x_1)T(x_2))\| \leq \|x_1\|\|x_2\|$ for all $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in X$.

(iii) For all $n \in \mathbb{N}$ and matrices $[x_{ij}] \in M_n(X)$, $[y_{ij}] \in M_n(Y)$ we have

$$\begin{bmatrix} T_{x_{ij}} & \cdot \\ \cdot & \cdot \end{bmatrix} \leq \begin{bmatrix} x_{ij} & \cdot \\ \cdot & \cdot \end{bmatrix}.$$ 

If $B = C$ then we may replace ‘completely contractive’ by ‘contractive’ in (i).
Proof. (i) ⇒ (ii) If \( B = C \) then we gave a simple proof of this implication in [4]. In the general case we sketch another argument. The point is that Paschke's proof of the implication (i) ⇒ (ii) in Theorem 2.1 has a matricial version that works here. To cite details, we will use the notation in Paschke’s proof (or in Section 10 of 4 where we reproduced Paschke’s proof). We need to replace \( h_n \) there by the matrix \( H = \left(\frac{1}{n!} I_m\right) \) by the row matrix \( v = [x_1, \ldots, x_n] H \), and the expression \( \langle x | x \rangle \) by \( \langle [x_i, x_j] \rangle \). One shows analogously to the proof in [4] that \( \|v\| \leq 1 \). Since \( T \) is completely contractive, \( T \) applied entrywise to \( v \) has norm \( 1 \); and then one proceeds along the earlier line. We leave the details as an exercise.

(ii) ⇒ (iii) This is easy using the formula above Theorem 2.1 (see e.g. 4).

(iii) ⇒ (i) Note that (iii) says that the map \( T : X \oplus Y \to C_2(Y) \) is completely contractive, when \( X \oplus Y \) is viewed as a subspace of \( \partial X \oplus^C \partial Y \). It follows from a tedious diagram chase that any injective envelope \( I(\partial Y) \) of \( \partial Y \) is an injective envelope \( I(Y) \) of \( Y \). Thus we may write \( I(Y) \) for the injective envelope of \( \partial Y \) in Lemma 2.2 this is a \( C^* \)-module over \( \mathcal{R} \), where \( C \) is a \( C^* \)-subalgebra of \( \mathcal{R} \). Then \( \partial X \oplus^C \partial Y \) is a subspace of \( \partial X \oplus^C \partial Y \). We may now follow the proof of the implication (iii) ⇒ (i) in [4], but with \( \mathcal{D} \) replaced by \( \mathcal{R} \), to extend \( T \) to a right \( \mathcal{R} \)-module map \( \tilde{T} : I(X) \to I(Y) \). Since \( \tilde{T} \) is also a right \( \mathcal{B} \)-module map we may conclude the proof as we did in [4].

For \( X, Y \) as above we define a relative left multiplier from \( X \) to \( Y \) to be a map \( T : X \to Y \) such that a positive scalar multiple of \( T \) satisfies the equivalent conditions of Theorem 3.1. We write \( \mathcal{M}_{rel}^l(X, Y) \) for the set of such relative left multipliers. Note that \( \mathcal{M}_{rel}^l(X, X) \) is simply the space \( \mathcal{M}_0(X) \) in the Introduction. To define an operator space structure on \( \mathcal{M}_{rel}^l(X, Y) \) we first observe that as in [3, p. 303] there is a canonical linear isomorphism

\[
\{ S \in CB_B(\partial X, \partial Y) : S(X) \subset Y \} \cong \mathcal{M}_{rel}^l(X, Y),
\]

given by \( S \mapsto S_{|X} \). Since \( CB(\partial X, \partial Y) \) is an operator space so is the set on the left side of the last displayed expression. We may therefore use the linear isomorphism above to give \( \mathcal{M}_{rel}^l(X, Y) \) an operator space structure. Note that Theorem 3.1 gives alternative descriptions of the unit ball of \( \mathcal{M}_{rel}^l(X, Y) \).

In [4] we gave some examples of relative left multipliers. We also proved the following result, which we shall not use in the present paper:

**Proposition 3.2.** If \( X, Y \) are as in Theorem 3.1 and if \( m, n \in \mathbb{N} \) then we have \( M_{m,n}(\mathcal{M}_{rel}^l(X, Y)) \cong \mathcal{M}_{rel}^l(C_n(X), C_m(Y)) \) completely isometrically.

4. Adjointable maps between two operator spaces

In this section we consider two operator spaces \( X, Y \) with fixed noncommutative Shilov boundaries \( \partial X \) and \( \partial Y \) which are right \( C^* \)-modules over the same \( C^* \)-algebra \( B \).

**Theorem 4.1.** Let \( X, Y \) be as above and suppose that \( T : X \to Y \). The following are equivalent:

(i) \( T \) is the restriction to \( X \) of an adjointable (in the usual \( C^* \)-module sense) \( B \)-module map \( R : \partial X \to \partial Y \) such that \( R(X) \subset Y \) and \( R^*(Y) \subset X \).

(ii) There exists a map \( S : Y \to X \) such that \( (T(x))y = \langle x | S(y) \rangle \) (these are the (right) Shilov inner products) for all \( x, y \in X \).
Moreover the set $\mathcal{A}_i(X,Y)$ consisting of maps $T$ satisfying condition (ii) above, is a closed subspace of $\mathcal{B}(X,Y)$ which is a $C^*$-bimodule over the algebras $\mathcal{A}_i(X)$ and $\mathcal{A}_i(Y)$. The module actions here on $\mathcal{A}_i(X,Y)$ are simply composition of operators. The $\mathcal{A}_i(X)$-valued inner product on $\mathcal{A}_i(X,Y)$ is $\langle T|R \rangle = SR$, for $T, R \in \mathcal{A}_i(X,Y)$ where $S$ is related to $T$ as in (ii) above.

Proof. We leave it to the reader to check that any $T \in \mathcal{A}_i(X,Y)$ is linear; that the map $S$ in (ii) is necessarily unique and linear; that $\mathcal{A}_i(X,Y)$ is an $\mathcal{A}_i(Y)$-$\mathcal{A}_i(X)$-bimodule; and that the $\mathcal{A}_i(X)$-valued inner product specified above does indeed take values in $\mathcal{A}_i(X)$. In fact the only nontrivial part of the proof that $\mathcal{A}_i(X,Y)$ is a right $C^*$-module consists in showing that for $T \in \mathcal{A}_i(X,Y)$, (a) $T^*T \geq 0$ in $\mathcal{A}_i(X)$, and (b) $\|T^*T\| = \|T\|^2$. Here $T^*$ denotes the map $S$ in (ii). In fact (a) follows from Theorem 4.10 (2) in [3], since

$$\langle T^*Tx|x \rangle = \langle Tx|Tx \rangle \geq 0, \quad x \in X.$$  

To prove (b) we first note that if $R \in \mathcal{A}_i(X)_+$, with $R = V^*V$ for a $V \in \mathcal{A}_i(X)$, then

$$\sup\{\|R_x\| : x \in \text{Ball}(X)\} = \sup\{\|V_xV_x\| : x \in \text{Ball}(X)\} = \|V\|^2 = \|R\|.$$ 

Setting $R = T^*T$ and using (a) we see that $\|T^*T\|$ equals

$$\sup\{\|T^*Tx|x\| : x \in \text{Ball}(X)\} = \sup\{\|Tx|Tx\| : x \in \text{Ball}(X)\} = \|T\|^2.$$ 

It follows that $\|T\| = \|T^*\|$ as in the Hilbert space case.

(i) $\Rightarrow$ (ii) This is obvious.

(ii) $\Rightarrow$ (i) Suppose that $T$ satisfies (ii). Then $T^*T \in \mathcal{A}_i(X)$ by the first part of the proof. For $x_1, \ldots, x_n \in X$ and $b_1, \ldots, b_n \in B$, we define $R(\sum_k x_kb_k) = \sum_k T(x_k)b_k$. To see that $R$ is well defined and bounded, set $u = \sum_k x_kb_k$, take $y_1, \ldots, y_m \in Y$ and $c_1, \ldots, c_m \in B$ and set $v = \sum_k ykc_k$. Then

$$\langle v| \sum_k T(x_k)b_k \rangle = \sum_{i,j} c_i^*(y_j|T(x_i))b_i = \sum_{i,j} c_i^*(T^*(y_j)|u) = \langle \sum_k T^*(y_k)c_k|u \rangle.$$ 

Setting $y_k = T(x_k)$ and $c_k = b_k$ we obtain from (2) and a Cauchy-Schwarz inequality:

$$\|\sum_k T(x_k)b_k\|^2 \leq \|\sum_k T^*T(x_k)b_k\| \|u\|.$$ 

Now the mapping $\sum_k x_kb_k \mapsto \sum_k T^*T(x_k)b_k$ is simply the unique $B$-module map on $\partial X$ extending $T^*T \in \mathcal{A}_i(X)$ (see [3] Theorem 4.10, in conjunction with the observation in equation (1)), and this extension has the same norm. Thus

$$\|\sum_k T(x_k)b_k\|^2 \leq \|T^*T\| \|u\|^2 = \|T\|^2 \|u\|^2.$$ 

Thus $R$ is bounded and well defined.

Since $R$ is bounded, it extends by density to a unique bounded $B$-module map $R : \partial X \to \partial Y$. Similarly $T^*$ extends to a bounded map $S : \partial Y \to \partial X$. We leave it as an exercise using (2), to check that $R$ is adjointable with adjoint $S$, and satisfies (i). 

As in the last proof we write $T^*$ for the unique $S$ related to $T$ in (ii) of the Theorem. We call such maps $T$ relatively adjointable. Strictly speaking we should write $\mathcal{A}_i^{rel}(X,Y)$ for what we wrote as $\mathcal{A}_i(X,Y)$ above, but for simplicity we use the shorter notation in this section. As was the case for $\mathcal{M}_i^{rel}(X,Y)$, the space
\(A_t(X, Y)\) is only defined relative to fixed noncommutative Shilov boundaries \(\partial X\) and \(\partial Y\). There are frameworks in which one may remove this ‘relative’ nature, as we shall see in the next sections.

**Remarks:** 1) It is easy to see that the set
\[
\{R \in \mathcal{B}_B(\partial X, \partial Y) : R(X) \subset Y, R^*(Y) \subset X\}
\]
is a right \(C^*\)-module over the \(C^*\)-algebra
\[
\{R \in \mathcal{B}(\partial X) : R(X) \subset X, R^*(X) \subset X\}.
\]
By basic properties of ‘ternary morphisms’ (see e.g. [12]), the restriction map from this \(C^*\)-module onto \(A_t(X, Y)\) is a completely isometric surjective ternary isomorphism.

2) If \(W\) is a third operator operator space whose noncommutative Shilov boundary \(\partial W\) is also a right \(C^*\)-module over the same algebra \(B\) as above, then ‘composition of operators’ is a well defined bilinear map \(A_t(X, Y) \times A_t(W, X) \to A_t(W, Y)\). Similar assertions hold for the \(M_t(\cdot, \cdot)\) spaces.

5. Multipliers relative to a superspace

In this section we consider a fairly general situation in which we can remove some of the relative nature of spaces \(M_t^{rel}(X, Y)\) and \(A_t^{rel}(X, Y)\) considered above.

**Definition 5.1.** Consider a pair \((X, Y)\) of closed subspaces of an operator space \(V\). Suppose that \(X\) has the property that there is a noncommutative Shilov boundary \((\partial V, i)\) of \(V\) such that the smallest closed \(\mathcal{F}(V)\)-submodule of \(\partial V\) containing \(i(X)\) is a noncommutative Shilov boundary of \(X\). Suppose that \(Y\) has the same property. Then we say that \((X, Y)\) is a \(\partial\)-compatible \(V\)-pair

By the universal property of the noncommutative Shilov boundary [12, 3], together with [12, Proposition 2.1 (iv)], and a routine diagram chase, it is easy to see that the notions above do not depend on the particular Shilov boundary of \(V\) considered above. We will not use this, but one may rephrase the statement “the smallest closed \(\mathcal{F}(V)\)-submodule of \(\partial V\) containing \(i(X)\)” as a combination of two statements: 1) the ‘subTRO’ \(Z\) of \(\partial V\) generated by \(i(X)\) (see e.g. [12, 4] for the definition of this) is a noncommutative Shilov boundary of \(X\), and 2) \(Z\) is a right \(\mathcal{F}(V)\)-submodule of \(\partial V\).

If \((X, Y)\) is a \(\partial\)-compatible \(V\)-pair, then we will henceforth in this section reserve the symbols \(\partial X\) and \(\partial Y\) for the particular noncommutative Shilov boundaries of \(X\) and \(Y\) respectively mentioned in the last paragraph; these are submodules of \(\partial V\). It is easy to see that \(\partial X\) and \(\partial Y\) are right \(C^*\)-modules over \(\mathcal{F}(V)\). Thus \(\partial X \oplus_c \partial Y\), the \(C^*\)-module sum, is a \(C^*\)-module over \(\mathcal{F}(V)\). We clearly have canonical complete isometric embeddings
\[
X \oplus_V Y \hookrightarrow \partial X \oplus_c \partial Y \hookrightarrow C_2(\partial V).
\]

The second matrix norm in (iii) below is the norm on \(M_n(X \oplus_V Y)\).

**Corollary 5.2.** Let \((X, Y)\) be a \(\partial\)-compatible \(V\)-pair, let \(\partial X, \partial Y, \mathcal{F}(V)\) be as above, and set \(C = \mathcal{F}(V)\). If \(T : X \to Y\) is a linear map then the following are equivalent:

(i) \(T\) is the restriction to \(X\) of a contractive right \(C\)-module map \(S : \partial X \to \partial Y\).

(ii) \([|T(x_i)|T(x_j)|] \leq [x_i|x_j]|\) for all \(m \in \mathbb{N}\) and \(x_1, \ldots, x_m \in X\).
For all \( n \in \mathbb{N} \) and matrices \( [x_{ij}] \in M_n(X) \), \( [y_{ij}] \in M_n(Y) \) we have

\[
\left\| \begin{bmatrix} Tx_{ij} \\ y_{ij} \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} x_{ij} \\ y_{ij} \end{bmatrix} \right\|.
\]

Proof. Follows from Theorem 5.1 with \( B = C = \mathcal{F}(V) \). \( \square \)

**Definition 5.3.** If \((X, Y)\) is a \( \partial \)-compatible \( V \)-pair, and if \( T : X \to Y \) is such that a positive scalar multiple of \( T \) satisfies the equivalent conditions of Corollary 5.2 then we call \( T \) a left \( V \)-multiplier from \( X \) to \( Y \). We write \( M^V_\partial(X, Y) \), or \( M(X, Y) \) when \( V \) is understood, for the set of such left \( V \)-multipliers. We write \( A^V_\partial(X, Y) \) for the \( C^* \)-bimodule in Theorem 5.4 (taking \( B = \mathcal{F}(V) \) there). The maps in \( A^V_\partial(X, Y) \) will be called left \( V \)-adjointable.

We will identify \( M^V_\partial(X, Y) \) with the operator space \( M^\partial_\partial(X, Y) \) from Section 3, where the noncommutative Shilov boundaries of \( X \) and \( Y \) are taken to be the ones mentioned at the start of the current Section.

**Proposition 5.4.** Let \( Y, Z \) be right \( C^* \)-modules over a \( C^* \)-algebra \( B \). Set \( V = Y \oplus_c Z \), and regard \( Y, Z \) as subspaces of \( V \). Then \((Y, Z)\) is a \( \partial \)-comparable \( V \)-pair, and \( M^V_\partial(Y, Z) \cong B_{\mathcal{F}}(Y, Z) \) and \( A^V_\partial(Y, Z) \cong \mathcal{B}(Y, Z) \).

Proof. Denote the closed span of the range of the canonical \( B \)-valued inner product on \( Y \oplus_c Z \) by \( \mathcal{F} \). In this case (see e.g. [3]) one can take \( \partial V = V \), viewed as a right \( C^* \)-module \( \mathcal{F} \). Then \( Y, Z \) are also \( C^* \)-modules over \( \mathcal{F} \), \((Y, Z)\) is a \( \partial \)-comparable \( V \)-pair, and \( M^V_\partial(Y, Z) \cong B_{\mathcal{F}}(Y, Z) \) and \( A^V_\partial(Y, Z) \cong \mathcal{B}_{\mathcal{F}}(Y, Z) \). We now may appeal to the principle in equation 10. \( \square \)

**Lemma 5.5.** If \((X, Y)\) is a \( \partial \)-compatible \( V \)-pair then \( \partial X \oplus_c \partial Y \) is a noncommutative Shilov boundary of \( X \oplus_V Y \).

Proof. First observe that \( \partial X \oplus_c \partial Y \) is a left operator \( \ell^\infty \)-submodule of \( C^*_p(\partial V) \). The canonical map \( X \oplus_V Y \to \partial X \oplus_c \partial Y \) is a complete isometry as noted above. Inside \( \mathcal{B}(\partial X \oplus_c \partial Y) \) there is a copy of \( \ell^\infty_2 \) (this is true for the sum of any two \( C^* \)-modules). We may follow the proof in [3] Theorem A.13]: one supposes that \( W \) is a ‘ternary ideal’ in \( \partial X \oplus_c \partial Y \) such that the canonical map \( X \oplus_V Y \to (\partial X \oplus_c \partial Y)/W \) is a complete isometry, and then one needs to show that \( W = \{0\} \). This is accomplished by letting \( W_1 = e_1 W \subset \partial X \), and \( W_2 = e_2 W \subset \partial Y \), where \( e_i \) is the ‘standard basis’ for \( \ell^\infty_2 \), and showing that the canonical maps \( X \to (\partial X)/W_1 \) and \( Y \to (\partial X)/W_2 \) are complete isometries. Since the reasoning is identical to that in [3] Theorem A.13] we omit the details. \( \square \)

We write \( \epsilon_X \) and \( P_X \) for the canonical inclusion and projection maps between \( X \) and \( X \oplus V Y \). Similarly for \( \epsilon_Y \) and \( P_Y \). These maps are restrictions of the canonical adjointable inclusion and projection maps between the \( C^* \)-module \( \partial X \oplus_c \partial Y \) and its summands. It is clear from the definitions in [3] that \( p = \epsilon_X \circ P_X \) is a left \( M \)-projection on \( X \oplus V Y \), onto the right \( M \)-summand \( X \oplus 0 \). Similarly \( q = \epsilon_Y \circ P_Y \) is the left \( M \)-projection onto \( 0 \oplus Y \).

For \( X, Y, V \) as above, \( X \oplus \{0\} \) and \( \{0\} \oplus Y \) are a \( \partial \)-compatible \( X \oplus_V Y \)-pair, as may be seen using Lemma 5.3. Thus it seems that in most situations we may assume without loss of generality that \( X, Y \) are complementary right \( M \)-summands in \( V \) (by ‘replacing’ \( V \) by \( X \oplus_V Y \), and using the observation in the last paragraph).
Conversely, if $(X, Y)$ is a $\partial$-compatible $V$-pair, and if also $X$ and $Y$ are ‘complementary’ right $M$-summands in $V$, then $V \cong X \oplus_Y Y$ completely isometrically. This is because, by one of the definitions of a right $M$-summand, the map

$$V \to C_2(V) : x + y \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$$

is a complete isometry, and its range is the space $X \oplus_Y Y$ defined in the Introduction.

The next observation we make is that since the operator algebra $M_l(X \oplus_Y Y)$ (resp. $C^*$-algebra $A_l(X \oplus_Y Y)$) contains the two canonical complementary projections $p, q$ mentioned a few paragraphs above, it splits as a $2 \times 2$ matrix algebra (resp. $C^*$-algebra). We first claim that the 1-1-corner is completely isometrically homomorphic to $M_l(X)$ (resp. $A_l(X)$). To see this consider the map $\theta : M_l(X) \to M_l(X \oplus_Y Y)$ taking $T$ to $\epsilon_X \circ T \circ P_X$. This may be viewed as the restriction to $M_l(X)$ of the map $R \to \epsilon_{\partial X} \circ R \circ P_{\partial X}$ from the space of bounded module maps on $X$, to the space of bounded module maps on $X \oplus_{\partial} Y$. Thus it is a well defined completely contractive homomorphism. From this argument, or directly, it is easy to see that $\theta$ is completely isometric. If $S = pS'p$ for a map $S' \in M_l(X \oplus_Y Y)$, then by the Lemma 5.6 and Theorem 5.4 $S'$ is the restriction to $X \oplus_Y Y$ of a bounded $F_l(V)$-module map $R'$ on $\partial X \oplus_{\partial} \partial Y$. Then $S$ is the restriction to $X \oplus_Y Y$ of $\epsilon_{\partial X} \circ P_{\partial X} \circ R' \circ \epsilon_{\partial X} \circ P_{\partial X}$. From this it is clear that $\theta(p_X \circ S' \circ \epsilon_X) = S$. Thus $\theta(M_l(X)) = pM_l(X \oplus_Y Y)p$. If $R \in B(X)$ then $\epsilon_{\partial X} \circ R \circ P_{\partial X} \in B(\partial X \oplus_{\partial} \partial Y)$. Thus it is easy to argue that $\theta$ induces a $*$-monomorphism from $A_l(X)$ onto $pA_l(X \oplus_Y Y)p$.

By identical reasoning we have completely isometries from $M_l(Y)$, $M_l(X, Y)$ and $M_l(Y, X)$ into the other three corners of $M_l(X \oplus_Y Y)$. Similar assertions hold for the $A_l(\cdot)$ spaces.

The following is the analogue of another important property of left $V$-multipliers on a single space [3]. It can be stated in many forms, but perhaps the following is the most concise:

**Proposition 5.6.** If $X, Y$ are complementary right $M$-summands of an operator space $V$ (see the discussion after Lemma 5.6), and if $(X, Y)$ is a $\partial$-compatible $V$-pair, then a linear map $T : X \to Y$ is a left $V$-multiplier if and only if there is a completely isometric linear embedding of $V$ into a $C^*$-algebra $A$, and an $a \in Ball(A)$ with $Tx = ax$ for all $x \in X$.

**Proof.** (⇒) Suppose that $T : X \to Y$ is a left $V$-multiplier. Since $M_l^V(X, Y)$ may be regarded as a corner of $M_l(X \oplus_Y Y)$, and since $X \oplus_Y Y \cong V$ by the discussion after Lemma 5.6, we may regard $T$ as a left multiplier $R$ of $V$. Thus by the ‘one-space variant’ of the result we are trying to prove, there is a completely isometric linear embedding $\sigma : V \to A$, and an $a \in Ball(A)$ with $\sigma(R(x)) = a\sigma(x)$, for all $x \in X$. However $R(x) = T(x)$.

(⇐) It is easy to show that the condition here implies Theorem 5.2 (iii). □

We recall from [2, 1] that $A_l(X)$ is a $W^*$-algebra if $X$ is a dual operator space. The following generalizes this important fact:

**Corollary 5.7.** If $(X, Y)$ is a $\partial$-compatible $V$-pair, where $V$ is a dual operator space and $X, Y$ are weak* closed subspaces of $V$, then $A_l^V(X, Y)$ is a $W^*$-module. Moreover, every $T \in A_l^V(X, Y)$ is automatically weak* continuous.
Theorem 6.1. Let $\mathcal{A}^V(X, Y)$ be a $W^*$-module it suffices to show that $\mathcal{A}^V(X, Y)$ is a dual space. However as we just saw, $\mathcal{A}^V(X, Y)$ is a ‘corner’ in $\mathcal{A}(X \oplus_V Y)$. Thus by the fact mentioned above the Corollary, it suffices to show that $X \oplus_V Y$ is a dual operator space. However this is clear since $C_2(V)$ is a dual operator space, and $X \oplus_V Y$ is easily seen to be weak$^*$ closed in $C_2(V)$.

The last assertion follows from the analogous fact for $\mathcal{A}(X \oplus_V Y)$ (see [5]), together with the fact that the canonical inclusion and projection maps between $X \oplus_V Y$ and its summands are weak$^*$ continuous in this case (which follows from basic operator space theory). □

The last result should be useful in the way that its ‘one-space predecessor’ was (see e.g. [4]). For example structural properties in a $W^*$-module (for example those considered in [13] or [19]) should have implications for the pair $X, Y$.

It is often useful that the adjointable maps on an operator space, or even on a Hilbert space, are characterizable as the span of the Hermitian (i.e. self-adjoint) ones. The following may be viewed as the ‘two-space’ analogue of this fact.

Corollary 5.8. Let $(X, Y)$ be a $\partial$-compatible V-pair, set $B = F(V)$, and let $T : X \to Y$ be a linear map. Then $T$ satisfies the equivalent conditions in Theorem 4.1 if and only if there is a map $S : Y \to X$ such that the map $(x, y) \mapsto (S(y), T(x))$ is a Hermitian in the Banach algebra $\mathcal{M}(X \oplus_V Y)$. In this case $T^* = S$.

Proof. We leave this as an exercise. The idea is very similar to the last proof and the discussion above it. That is, using the canonical inclusion and projection maps, we transfer the desired statement to a statement about maps between the $C^*$-modules $\partial X, \partial Y$ and $\partial X \oplus \partial Y$. □

6. Multipliers and the injective envelope

In this brief section we list some variants of results in the last section, but with the noncommutative Shilov boundary replaced by the injective envelope.

We consider an operator space $V$, and fix an injective envelope $(I(V), i)$ of $V$, which is a right $C^*$-module over a $C^*$-algebra $\mathcal{D} = \mathcal{D}(V)$ (see the paragraph before Lemma 2.3). We say that a subspace $X$ of $V$ is a (right) $\mathcal{D}$-subspace if there is a $\mathcal{D}$-submodule $W$ of $I(V)$ such that $(W, i)$ is an injective envelope of $X$. We call a pair $(X, Y)$ of $\mathcal{D}$-subspaces of $V$, an $I$-compatible V-pair. Clearly $W$ is a right $C^*$-module over $\mathcal{D}(V)$ too. We will write $W$ as $I(X)$, and $\mathcal{D}(X)$ for the $C^*$-subalgebra $W^*W$ of $\mathcal{D}(V)$. Similar notations hold for $Y$. In (i) below the notation $I_{11}(V)$ is used precisely in the sense of [4].

Theorem 6.1. Let $(X, Y)$ be an $I$-compatible V-pair. Suppose further that $\mathcal{D}(Y) \subset \mathcal{D}(X)$, where $\mathcal{D}(\cdot)$ is as defined above. If $T : X \to Y$ is a linear map, then the following are equivalent:

(i) There is an $a \in \text{Ball}(I_{11}(V))$ such that $i(Tx) = ai(x)$ for all $x \in X$.
(ii) $T \oplus \text{Id}_X : C_2(X) \to Y \oplus_V X$ is completely contractive.
(iii) $T \oplus \text{Id}_Y : X \oplus_V Y \to C_2(Y)$ is completely contractive.
(iv) There is a $C^*$-algebra $A$, a completely isometric embedding $V \hookrightarrow A$, and an $a \in \text{Ball}(A)$ such that $Tx = ax$ for all $x \in X$.
(v) $T$ is the restriction to $X$ of a contractive $\mathcal{D}(V)$-module map $S : I(X) \to I(Y)$. 

If further, \((X, Y)\) is a \(\partial\)-compatible \(V\)-pair, then the equivalent conditions above are also equivalent to conditions (i)–(iii) in Theorem \(6.2\).

We next claim that the discussion in the paragraphs between Lemma \(5.6\) and Proposition \(6.6\) above, is also valid for \(I\)-compatible \(V\)-pairs. To see this one needs the following result:

**Lemma 6.2.** If \((X, Y)\) is an \(I\)-compatible \(V\)-pair then \(I(X) \oplus_c I(Y)\) is an injective envelope for \(X \oplus_V Y\).

**Proof.** Since \(I(X)\) is a \(D(V)\)-submodule of \(I(V)\) it is a right \(M\)-ideal in \(I(V)\) by \([5]\) Theorem 6.6. Since \(I(X)\) is injective there is a contractive projection \(I(V)\) onto \(I(X)\). It follows from Theorem 3.10 (c) and Theorem 6.6 in \([5]\) that there exists a contractive \(D(V)\)-module map projection from \(I(V)\) onto \(I(X)\). Similarly for \(I(Y)\). It follows from Theorem 2.1 (iii) for example, that there is a contractive \(D(P)\)-module map projection from \(C_2(I(V))\) onto \(I(X) \oplus_c I(Y)\). Thus \(I(X) \oplus_c I(Y)\) is injective, since \(C_2(I(V))\) is injective. It suffices, by one of the equivalent definitions of the injective envelope \([12] [13]\), to show that if \(P\) is a completely contractive projection on \(I(X) \oplus_c I(Y)\) which restricts to the identity on \(X \oplus_V Y\), then \(P\) is the identity map. If \(\epsilon_X, P_X\) are as in the discussion below Lemma \(5.5\) then \(P_{I(X)} \circ P \circ \epsilon_{I(X)}\) is a complete contraction on \(I(X)\) which restricts to \(I\). By rigidly (see Section 2), \(P_{I(X)} \circ P \circ \epsilon_{I(X)} = I\). Similarly \(P_{I(Y)} \circ P \circ \epsilon_{I(Y)} = I\). Since \(P^2 = P\), by pure algebra we must conclude that \(P_{I(Y)} \circ P \circ \epsilon_{I(X)}\) and \(P_{I(X)} \circ P \circ \epsilon_{I(Y)}\) are zero. Thus \(P = I\).

Since (by the Lemma) the discussion in the paragraphs after Lemma \(5.6\) transfers to the present setting, one may check that the conclusions of Corollary \(5.7\) are true for \(I\)-compatible \(V\)-pairs too.

There is another characterization of left \(V\)-multipliers which is also analogous to the formulation of left multipliers in \([6]\). To state this characterization we suppose that \((X, Y)\) is an \(I\)-compatible \(V\)-pair. For simplicity we also suppose that \(D(Y) \subset D(X)\). We then have as above that \(I(X)\) and \(I(Y)\) are right \(C^\ast\)-modules over \(D(X)\), and hence also over the \(C^\ast\)-algebra multiplier algebra \(M(D(X))\). The latter \(C^\ast\)-algebra is injective too, by \([6]\) Corollary 1.8. Indeed \(M(D(X)) \cong I_{22}(X)\) in the language of \([13]\); and henceforth we shall just write \(I_{22}\) for \(M(D(X))\). We consider the ‘generalized linking \(C^\ast\)-algebra’ \(A = B_{I_{22}}(I(Y)) \oplus_c I(X) \oplus_c I_{22}\). With respect to the canonical diagonal projections corresponding to the identities of \(B(I(Y)), B(I(X))\) and \(I_{22}\) respectively, \(A\) may be written as a \(3 \times 3\) matrix \(C^\ast\)-algebra, whose \(k\)-\(\ell\)-corner we write as \(I_{k\ell}\), for \(k, \ell \in \{0, 1, 2\}\). Clearly \(I_{02} = I(Y)\) and \(I_{12} = I(X)\).

With a little more one can show that \(A\), and consequently also \(I_{ij}\), is injective. We will not use this here however. We write \(i\) and \(j\) for the canonical maps from \(Y\) and \(X\) into \(I_{02}\) and \(I_{12}\) respectively.

**Theorem 6.3.** Suppose that \((X, Y)\) is as in the first part of Theorem \(6.2\). Then a linear map \(T : X \to Y\) satisfies conditions (i)–(v) in that Theorem if and only if there exists an element \(a \in I_{01}\) such that \(i(Tx) = aj(x)\) for all \(x \in X\).

**Proof.** By \([6]\) Corollary 2.7 (iii) we have that

\[
I_{01} \cong B_{I_{22}}(I(X), I(Y)) = B_{I_{22}}(I(X), I(Y)).
\]

Hence

\[
I_{01} \cong B_{I_{22}}(I(X), I(Y)) = B_{D(X)}(I(X), I(Y)) = B_{D(V)}(I(X), I(Y)).
\]
using the principle in equation (1). The result is clear from this and Theorem 5.2 (v).

This result, and the matching part of the last theorem, may also be proved by a variation of the proof given in [16] of the analogous assertion for $M_l(X)$.

It should be interesting and useful to extend other known results about $M_l(X)$ and $A_l(X)$ (for example those in [7]) to the case of two spaces $X$ and $Y$.

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