On the Supremum of Certain Families of Stochastic Processes

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Summary

We consider a family of stochastic processes \( \{X_\epsilon^t, t \in T\} \) on a metric space \( T \), with a parameter \( \epsilon \downarrow 0 \). We study the conditions under which

\[
\lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{t \in T} |X_\epsilon^t| < \delta \right) = 1
\]

when one has an \textit{a priori} estimate on the modulus of continuity and the value at one point. We compare our problem to the celebrated Kolmogorov continuity criteria for stochastic processes, and finally give an application of our main result for stochastic integrals with respect to compound Poisson random measures with infinite intensity measures.

Key words: Compensated Poisson random measure, Generic chaining, Kolmogorov continuity criterion, Metric entropy, Suprema of stochastic processes

1 Introduction

Let \((T, d)\) be a metric space with finite diameter,

\[
D(T) = \sup \left\{ d(s, t) : s, t \in T \right\} < \infty.
\]

Let \(N(T, d, \delta)\) denote the covering number, \textit{i.e.}, for every \( \delta > 0 \), let \( N(T, d, \delta) \) denote the minimal number of closed \( d \)-balls of radius \( \delta \) required to cover \( T \). The supremum of a stochastic process \( X_t \) defined on \( T \), \( \sup_{t \in T} X_t \) can be quantified in terms of \( N(T, d, \delta) \) (see [Talagrand, 2005, Chapter 1] for instance) under various assumptions on the process \( X_t \).

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In this article we consider a family of stochastic processes \( X^\epsilon_t \) on \( T \), with a parameter \( \epsilon > 0 \). In certain applications in nonparametric statistics (see Section 4) it is of interest to study the limiting behaviour of the supremum, \( \lim_{\epsilon \to 0} \sup_{t \in T} X^\epsilon_t \) when one has an \textit{a priori} estimate of the form

\[
\mathbb{E}|X^\epsilon_t - X^\epsilon_s|^\beta \leq B_\epsilon \, d(s, t)^\gamma
\]

for some \( \beta, \gamma > 0 \) and \( B_\epsilon \to 0 \) as \( \epsilon \to 0 \). In particular, we would like to identify conditions under which, for every \( \delta > 0 \),

\[
\lim_{\epsilon \to 0} \mathbb{P}\left( \sup_{t \in T} |X^\epsilon_t| < \delta \right) = 1.
\]  

(1.1)

In our main result in Section 2, we find conditions in terms of the covering number \( N(T, d, \delta) \) that ensure (1.1) holds. Although our technique is based on well known chaining methods, our principle result appears to be new. In Section 3 we discuss briefly the optimality of our hypotheses and compare our theorem with the Kolmogorov criterion for continuity of stochastic processes. In Section 4 we present an application of our main theorem to random fields constructed from Lévy random measures.

2 Main Result

Let \((T, d)\) be a complete separable metric space and \((X^\epsilon_t)_{t \in T}\) a family of real-valued, centered, \( L_2 \) stochastic processes on \( T \), indexed by \( \epsilon > 0 \).

Let \( n_0 \) be the largest integer \( n \) such that \( N(T, d, 2^{-n}) = 1 \) (note \( n_0 < 0 \) is possible). For every \( n \geq n_0 \), fix a covering of \( T \) of cardinality \( N_n = N(T, d, 2^{-n}) \) by closed balls of radius \( 2^{-n} \). From this we can construct a partition \( \mathcal{A}_n \) of \( T \) of cardinality \( |\mathcal{A}_n| = N_n \) by Borel sets with diameter at most \( 2^{-n+1} \). For each \( n \geq n_0 \), fix a designated point in each element \( A \) of the partition \( \mathcal{A}_n \), and denote by \( T_n \) the collection of these points. Without loss of generality, let the designated point in the single element of partition \( \mathcal{A}_{n_0} = \{T\} \) be \( T_{n_0} = \{t_0\} \) for a point \( t_0 \in T \) to be specified in the statement of Theorem 1 below. For \( t \in T \) denote by \( \mathcal{A}_n(t) \) the partition element \( A \in \mathcal{A}_n \) that contains \( t \). For every \( t \) and every \( n \), let \( s_n(t) \) be the element of \( T_n \) in \( t \)'s partition element, so that \( t \in \mathcal{A}_n(s_n(t)) \). It is clear that \( d(t, s_n(t)) \leq 2^{-n+1} \) for every \( t \in T \) and \( n \geq n_0 \). By the triangle inequality

\[
d(s_n(t), s_{n-1}(t)) \leq d(s_n(t), t) + d(t, s_{n-1}(t)) \leq 2^{-n+1} + 2^{-n+2} = 6 \cdot 2^{-n}.
\]

Define the set

\[
H_n \equiv \left\{ (u, v) \in T_n \times T_{n-1} : d(u, v) \leq 6 \cdot 2^{-n} \right\}.
\]  

(2.2)

The following is our main result:

**Theorem 1.** Suppose that:
1. There exists a point $t_0 \in T$ such that
\[
\lim_{\epsilon \to 0} E(X_{t_0}^{\epsilon})^2 = 0.
\]
(2.3a)

2. There exist $\alpha, \beta > 0$ and positive numbers $\{B_\epsilon\}$ with $\lim_{\epsilon \to 0} B_\epsilon = 0$ such that for any $s, t \in T$
\[
E|X^{\epsilon}_t - X^{\epsilon}_s|^\beta \leq B_\epsilon d(s, t)^{1+\alpha}.
\]
(2.3b)

3. There exists a family of partitions $A_n$ of $T$ of sets of diameter no more than $2^{1-n}$ and a constant $\gamma < \alpha$ such that
\[
\sum_{n=1}^{\infty} |H_n| 2^{-(1+\gamma)n} < \infty
\]
(2.3c)
where $H_n$ is as defined in (2.2).

Then for any $\delta > 0$,
\[
\lim_{\epsilon \to 0} P\left(\sup_{t \in T} |X^{\epsilon}_t| < \delta\right) = 1.
\]
(2.4)

**Remark:** For each fixed $\epsilon > 0$, Equation (2.3b) guarantees the existence of a path-continuous version of $(X^{\epsilon}_t)$ (by Kolmogorov’s continuity criterion; see Durrett, 1996, p. 375. For more on this connection see Section 3). Since $H_n$ satisfies the bound
\[
|H_n| \leq |T_n| \cdot |T_{n-1}| \leq N^2(T, d, 2^{-n}),
\]
the monotonicity of $N(T, d, \delta)$ implies that the entropy condition (2.3c) holds whenever
\[
\int_0^{D(T)} a^\gamma N^2(T, d, a) \, da < \infty.
\]
Frequently in applications we have a bound of the form
\[
|H_n| \leq C \cdot |T_n| \leq C \cdot N(T, d, 2^{-n})
\]
for a universal constant $C$ and in this case (2.3c) holds if
\[
\int_0^{D(T)} a^\gamma N(T, d, a) \, da < \infty.
\]
(2.6)

For example, with $T = [0, 1]$ and $d(u, v) = |u - v|$, the dyadic partition
\[
A_n = \left\{ \left[i2^{1-n}, (i+1)2^{1-n}\right] : 0 \leq i < 2^{1-n} \right\}
\]
of $T$ into $N_n = 2^{n-1}$ $d$-balls of radius $2^{-n}$ for $n \geq n_0 = 1$ satisfies (2.3c) for $C = 5$. 3
Proof. Fix $\delta > 0$. First observe that
\[
\mathbb{P}\left( \sup_{t \in T} |X_t^\varepsilon| < \delta \right) \geq \mathbb{P}\left( \sup_{t \in T} |X_t^\varepsilon - X_{t_0}^\varepsilon| < \delta/2, |X_{t_0}^\varepsilon| < \delta/2 \right)
\geq \mathbb{P}\left( \sup_{t \in T} |X_t^\varepsilon - X_{t_0}^\varepsilon| < \delta/2 \right) - \mathbb{P}\left( |X_{t_0}^\varepsilon| \geq \delta/2 \right)
\]
and
\[
\mathbb{P}( |X_{t_0}^\varepsilon| \geq \delta/2) \leq 4\delta^{-2}\mathbb{E}|X_{t_0}^\varepsilon|^2 \to 0
\]
as $\varepsilon \to 0$ by equation (2.3a). Thus we only need to control $\sup_{t \in T} |X_t^\varepsilon - X_{t_0}^\varepsilon|$.

We employ the so-called generic chaining principle of Ledoux (1996) (see also Talagrand, 2005, or Xiao, 2009 for a refinement similar in spirit to our approach). The fundamental relation is the convergent telescoping sum
\[
X_t - X_{t_0} = \sum_{n>n_0} (X_{s_n(t)} - X_{s_{n-1}(t)})
\]
for every $t \in T$, where we note that $s_{n_0}(t) = t_0$ for every $t \in T$. Then,
\[
\sup_{t \in T} |X_t - X_{t_0}| \leq \sup_{t \in T} \sum_{n>n_0} |X_{s_n(t)} - X_{s_{n-1}(t)}| \leq \sum_{n>n_0} \max_{(u,v) \in H_n} |X_u - X_v|
\]
For $(u,v) \in T \times T$, let $\{w_n(u,v)\}_{n \geq n_0}$ be a sequence of non-negative real numbers such that $\sum_{n \geq n_0} w_n(u,v) = 1$. For any $\delta > 0$, by the triangle inequality
\[
\bigcap_{n>n_0} \bigcap_{(u,v) \in H_n} \left\{ |X_u - X_v| \leq w_n(u,v) \delta/2 \right\} \subset \left\{ \sup_{t \in T} |X_t - X_{t_0}| \leq \delta/2 \right\}.
\]
Therefore,
\[
\mathbb{P}\left( \sup_{t \in T} |X_t^\varepsilon - X_{t_0}^\varepsilon| > \delta/2 \right) \leq \mathbb{P}\left( \bigcup_{n>n_0} \bigcup_{(u,v) \in T_n} \left\{ |X_u^\varepsilon - X_v^\varepsilon| > w_n(u,v) \delta/2 \right\} \right)
\leq \sum_{n>n_0} \sum_{(u,v) \in H_n} \mathbb{P}( |X_u^\varepsilon - X_v^\varepsilon| > w_n(u,v) \delta/2 ).
\]
Next use Equations (2.3b) and (2.3c) to find optimal choices for $w_n(v)$ (the so-called “majorizing measure”, see Talagrand, 2005, Chapter 1). Set
\[
w_n(u,v) \equiv w_n \equiv (1 - 2^{-h})2^{-h(n-n_0)}, \quad h = (\alpha - \gamma)/\beta, \quad v \in T.
\]
Notice that $\sum_{n \geq n_0} w_n(u,v) = 1$. By Markov’s inequality and (2.3b), for $v \in T_n$,

$$
P(|X_u^\epsilon - X_v^\epsilon| \geq w_n \delta/2) \leq \left(\frac{w_n \delta/2}{\beta}\right)^\beta E|X_u^\epsilon - X_v^\epsilon|^\beta \leq \left(\frac{\delta/2}{\beta}\right)^\beta \left(1 - 2^{-h}\right)^{-\beta} 2^{\beta h(n-n_0)} B_\epsilon d(u,v)^{1+\alpha} \leq \left(\frac{\delta/2}{\beta}\right)^\beta \left(1 - 2^{-h}\right)^{-\beta} 2^{-\beta h n_0} 2^{\beta h n} B_\epsilon (6 \cdot 2^{-n})^{1+\alpha}.
$$

Putting all the estimates together,

$$
P\left(\sup_{t \in T} |X_t^\epsilon - X_0^\epsilon| \geq \delta/2\right) \leq \left(\frac{\delta/2}{\beta}\right)^\beta \left(1 - 2^{-h}\right)^{-\beta} 6^{1+\alpha} 2^{-\beta h n_0} B_\epsilon \left(\sum_{n > n_0} \sum_{(u,v) \in H_n} 2^{\beta h n} 2^{-(1+\gamma)n}\right) = CB_\epsilon \sum_{n > n_0} |H_n| 2^{-(1+\gamma)n}
$$

for a finite constant $C < \infty$. Since $B_\epsilon \to 0$ as $\epsilon \to 0$ and the sum converges by (2.3c),

$$
\lim_{\epsilon \to 0} P\left(\sup_{t \in T} |X_t^\epsilon - X_0^\epsilon| \geq \delta/2\right) = 0
$$

and the theorem is proved.

### 3 Near Optimality of Our Hypothesis

Kolmogorov’s continuity criterion asserts the existence of a path continuous version of any stochastic process $X_t$, $t \in [0,1]$ that satisfies

$$
E\left(|X_t - X_s|^\beta\right) \leq C |t-s|^{1+\alpha}
$$

for some fixed $\alpha, \beta > 0$, $C < \infty$ and all $0 \leq s, t \leq 1$ (cf. (2.3b)). Strict inequality $\alpha > 0$ is necessary, as illustrated by the well known counter example

$$
X_t = 1_{\{U \leq t\}}
$$

for $U \sim \text{Un}[0,1]$ which satisfies $E|X_t - X_s|^\beta \leq C|t-s|$ for all $\beta > 0$ and $C \geq 1$ but is almost surely discontinuous. In the spirit of this example, here we construct a stochastic process which shows that our hypothesis (2) in Theorem 1 is “very close” to optimal.

Let $U \sim \text{Un}[0,1]$, $0 < \epsilon < 1$ and $X_t^\epsilon = 1_{\{t < U \leq t+\epsilon\}}$, $0 \leq t \leq 1$. Then for any fixed $t \in [0,1],

$$
E(X_t^\epsilon)^2 = P(t < U \leq t + \epsilon) = \min(\epsilon, 1-t)
$$

(3.7)

Since

$$
E X_t^\epsilon X_s^\epsilon = \begin{cases} 
0 & \text{if } |t-s| > \epsilon \\
\epsilon - |t-s| & \text{if } |t-s| \leq \epsilon, \ min(s,t) \leq 1 - \epsilon \\
1 - \max(s,t) & \text{if } |t-s| \leq \epsilon, \ min(s,t) \geq 1 - \epsilon
\end{cases}
$$

Since
it follows that
\[ \mathbb{E}(X_t^\epsilon - X_s^\epsilon)^2 \leq 2 \min(\epsilon, |t - s|). \] (3.8)

By (3.7) \( X_t^\epsilon \) satisfies (2.3a) for any \( t_0 \in [0, 1] \), and by (3.8) we have bounds on \( \mathbb{E}[(X_t^\epsilon - X_s^\epsilon)^\beta] \) for \( \beta = 2 \) both of the form \( B|t - s| \) (for fixed \( B = 2 \)) and of the form \( B_\epsilon \to 0 \) (with \( B_\epsilon = 2\epsilon \)), but not quite a bound of the form required by (2.3b). The conclusion (2.4) of Theorem 1 fails for the process \( X_t^\epsilon \) since, for any \( \epsilon > 0 \), \( \sup_{0 \leq t \leq 1} X_t^\epsilon = 1 \) almost surely. We believe that the condition \( \alpha > 0 \) in equation (2.3b) cannot be relaxed and state this a conjecture.

**Conjecture 1.** Theorem 1 is not true if hypothesis (2) (equation (2.3b)) is replaced by
\[ \mathbb{E}|X_t^\epsilon - X_s^\epsilon|^\beta \leq B_\epsilon d(s, t). \]

### 4 Application: Compensated Poisson Random Measures

In this section we present an application of Theorem 1 to a stochastic process constructed from compensated Poisson random measures.

Let \( \Omega \) be a Polish space and \( \nu(du \, d\omega) \) be a positive sigma-finite measure on \((-1, 1) \times \Omega\) such that
\[
\nu((-a, a) \times \Omega) = \infty, \quad \forall a \in [0, 1]
\]
\[
\iint_{(-1,1) \times \Omega} u^2 \nu(du \, d\omega) < \infty.
\]

Let
\[ N(du \, d\omega) \sim \text{Po}(\nu) \]
be a Poisson random measure on \((-1, 1) \times \Omega\) which assigns independent \( \text{Po}(\nu(B_i)) \) distributions to disjoint Borel sets \( B_i \subset (-1, 1) \times \Omega \). Let
\[ \tilde{N}(du \, d\omega) \equiv N(du \, d\omega) - \nu(du \, d\omega) \]
denote the compensated Poisson measure with mean 0, an isometry from \( L_2((-1, 1) \times \Omega, \nu(du \, d\omega)) \) to the square-integrable zero-mean random variables (Sato, 1999, p. 38).

Let \( K(t, \omega) : [0, 1] \times \Omega \to \mathbb{R} \) be a Borel measurable function such that
\[
\iint_{(-1,1) \times \Omega} K^2(t, \omega) u^2 \nu(du \, d\omega) < \infty \tag{4.9}
\]
for all \( 0 \leq t \leq 1 \). For \( 0 < \epsilon \leq 1 \) define a stochastic process \( X_t^\epsilon \) by
\[
X_t^\epsilon \equiv \iint_{\{0 < |u| < \epsilon\} \times \Omega} K(t, \omega) u \tilde{N}(du \, d\omega), \quad t \in [0, 1]. \tag{4.10}
\]
For every $t \in [0, 1]$ the stochastic integral (4.10) is well defined by (4.9) (see Wolpert & Taqqu, 2005; Rajput & Rosiński, 1989). For $t \in [0, 1]$, we have:

$$E[X_t^\epsilon] = 0$$
$$E[(X_t^\epsilon)^2] = \int \int_{(-\epsilon, \epsilon) \times \Omega} K^2(t, \omega) \, u^2 \, \nu(du \, d\omega) < \infty$$
$$E[e^{i\zeta X_t^\epsilon}] = \exp \left\{ \int \int_{(-\epsilon, \epsilon) \times \Omega} \left[ e^{i\zeta K(t, \omega)} u - 1 - i\zeta K(t, \omega) u \right] \nu(du \, d\omega) \right\},$$

the Lévy-Khinchine formula for the characteristic function of an infinitely divisible random variable.

The stochastic process $\{X^\epsilon \equiv X_t^\epsilon, t \in [0, 1]\}$ is the discretization error arising from the approximation of certain stochastic integrals by finite sums (see Pillai & Wolpert, 2008; Wolpert et al., 2006). The limiting behaviour of the process $X^\epsilon$ as $\epsilon$ goes to zero (see Pillai & Wolpert, 2008, §3) is of particular interest; we would like to identify the conditions on the function $K$ under which

$$\lim_{\epsilon \to 0} P \left( \sup_{t \in [0,1]} |X_t^\epsilon| > \delta \right) = 0$$

(4.12)

for all $\delta > 0$, so the approximation error vanishes in the limit. Concentration equalities similar to (4.12) were studied by Reynaud-Bouret (2006) for finite intensity measures (i.e., $\nu((-1,1) \times \Omega) < \infty$) using methods that are not applicable to our infinite intensity case.

In the next proposition we apply Theorem 1 to identify conditions for the kernel $K(\cdot, \cdot)$ under which (4.12) holds.

**Proposition 1.** Let $K(t, \omega) : [0, 1] \times \Omega \to \mathbb{R}$ satisfy (4.9) and

$$|K(t, \omega) - K(s, \omega)|^2 \leq C(\omega) \, |t-s|^{1+\alpha}, \quad s, t \in [0, 1], \ \omega \in \Omega$$

(4.13)

for some $\alpha > 0$ and Borel measurable function $C : \Omega \to \mathbb{R}_+$ satisfying

$$\int \int_{(-1,1) \times \Omega} C(\omega) \, u^2 \, \nu(du \, d\omega) < \infty.$$  

(4.14)

Let $X^\epsilon$ be the stochastic process on $[0,1]$ given in (4.10). Then, for any $\delta > 0$,

$$\lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{t \in [0,1]} |X_t^\epsilon| > \delta \right) = 0.$$

**Proof.** For any $t_0 \in [0,1]$, by (4.9) and the dominated convergence theorem

$$\lim_{\epsilon \to 0} \mathbb{E} \left[ (X_{t_0}^\epsilon)^2 \right] = \lim_{\epsilon \to 0} \int_{|u| \leq \epsilon} K^2(t_0, \omega) \, u^2 \, \nu(du \, d\omega) = 0,$$

(4.15)
verifying hypothesis 1 (equation (2.3a)) of Theorem 1. For \( t, s \in [0,1] \), by (4.13) and by the isometric property of \( \tilde{N}(du \, d\omega) \),

\[
\mathbb{E} \left[ (X_t^\epsilon - X_s^\epsilon)^2 \right] = \int \int_{\{ |u| \leq \epsilon \} \times \Omega} |K(t, \omega) - K(s, \omega)|^2 \, u^2 \, \nu(du \, d\omega) \\
\leq B_\epsilon |t - s|^{1+\alpha}, \quad \text{where} \\
B_\epsilon \equiv \int \int_{\{ |u| \leq \epsilon \} \times \Omega} C(\omega) \, u^2 \, \nu(du \, d\omega) \to 0
\]

as \( \epsilon \to 0 \) by (4.14), so hypothesis 2 (equation (2.3b)) of Theorem 1 is satisfied with the Euclidean metric \( d(t, s) \equiv |t - s| \). For dyadic partitions of \([0,1]\), we have already shown that (2.5) holds. Since \( N([0,1], d, a) = \lceil \frac{1}{a} \rceil \leq \frac{2}{a} \) for all \( 0 < a < 1 \), for any \( \gamma > 0 \) (say, \( \gamma = \alpha/2 \)),

\[
\int_0^1 a^\gamma N([0,1], d, a) \, da \leq 2 \int_0^1 a^{\gamma-1} \, da = \frac{2}{\gamma} < \infty
\]

verifying Equation (2.6). Therefore by (4.15), (4.16), (4.17) and Theorem 1, it follows that for any \( \delta > 0 \),

\[
\lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{t \in [0,1]} |X_t^\epsilon| > \delta \right) = 0
\]

and we are done.

**Remark:** It is not known whether the conclusion of the above proposition still holds if (4.13) is weakened to

\[
|K(t, \omega) - K(s, \omega)|^2 \leq C(\omega) \, |t - s|, \quad s, t \in [0,1].
\]

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