I. INTRODUCTION

The possibility of a novel “relativity of locality” is being proposed [1, 2] as a candidate feature of quantum gravity. When locality is relative processes are still local in the coordinatization of spacetime by nearby observers, but they may appear to be nonlocal in the coordinatization of spacetime by distant observers. One can appreciate some of the main implications of this hypothesis by looking, for example, at a particle decay $a \rightarrow b + c$ decomposed into a disappearance event for particle $a$ and two particle-production events for particles $b$ and $c$. With absolute locality the 3 events coincide in the coordinatization of spacetime by any observer, but with relative locality only “nearby observers” (observers themselves “local to” the coincidence of events) describe the events as sharply coincident, whereas distant observers in general describe them as not exactly coincident. Of course, since so far all our observations are consistent with absolute locality, it must be assumed that the characteristic scale of the relativity of locality is very small, and the quantum-gravity intuition that motivates these studies provides [1, 2] a natural candidate for such a scale: the inverse of the Planck scale $M_p \sim 10^{28}\text{eV}$.

We shall here not dwell on the strength of the quantum gravity motivation for studies of relative-locality, for which readers find detailed arguments in Refs. [1, 2]. Our perspective is focused on investigating relative locality as a novel candidate relativistic feature, whose full understanding we expect to surely require as much dedicated effort as was needed for the Galilean relativity of rest and the Einsteinian relativity of simultaneity. For this purpose we return here to the simple perspective and formalization of the first studies [3–5] which noticed that some known deformations of relativistic symmetries, some of the ones studied within the “doubly-special relativity” research programme (see, e.g., Refs. [6–11]), could result in the property that coincidences of events established by nearby observers might not appear as coincidences of events in the coordinatization of spacetime by distant observers. The formalization adopted in Refs. [3,5] has the limitation of being confined to the description of free particles, which is evidently a severe limitation for what concerns the development of realistic physical theories with relative locality. This limitation was removed by the more powerful formulation of relative locality, including a description of interactions, advocated in Refs. [1, 2] (also see Refs. [12–16]). But for the characterization of certain relativistic issues, such as the ones which are here of our interest, the restriction to free particles is not an important limitation, and the simplicity of the formalism proves to be very advantageous.

Our objective here is to advance the understanding of relative locality by introducing a distinction between longitudinal relative locality and transverse relative locality: with longitudinal relative locality coincidences of
events established by nearby observers are described by distant observers as events that are non-coincident along the direction connecting the observer to the events, whereas with transverse relative locality the distant observer describes the events as non-coincident along a direction orthogonal to the direction connecting observer to events.

All relative-locality studies produced so far mainly focused on longitudinal relative locality. A brief mention of a feature of transverse relative locality is found in Ref. [3]: the analysis in Ref. [3] nearly exclusively focused on the case of coincidences of events established by nearby observers which would be described by distantly boosted observers as events that do not coincide along the direction connecting the distantly boosted observer to the events (a case of longitudinal relative locality), but Ref. [3] also mentioned briefly the possibility that for some distantly boosted observers the lack of coincidence could also be “transverse”, a lack of coincidence occurring in a direction orthogonal to the direction connecting the distantly boosted observer to the events. Another brief appearance of a manifestation of what we here label as transverse relative locality can be found implicitly in parts of Ref. [12], specifically the parts of Ref. [12] that concerned some analyses of the formalism of Refs. [1, 2], for interacting particles, providing evidence of “dual-gravity lensing”, intended as a manifestation of relative locality such that particles on parallel propagation according to some observers could be described by other observers as propagating along different directions. As discussed in some detail here below, this “dual-gravity lensing”\(^1\) is intimately connected with transverse relative locality.

We are here reporting a first dedicated study of transverse relative locality, whose humble objectives are focused on building a few elements of intuition on transverse relative locality, its possible dependence on different scales in some applications of interest, and its comparison to longitudinal relative locality. We shall in particular show that, at least within the confines of relativistic theories of free particles, all previously-obtained results for longitudinal relative locality have a clear (and no less significant) counterpart on the transverse-relative-locality side. Indeed in the next section we find that the doubly-special-relativity-based formalism that led, in Ref. [3], to the derivation of manifestations of longitudinal relative locality for some distantly boosted observers also produces transverse relative locality for some other distantly boosted observers, and the magnitude of the two classes of effects is comparable. Then in Section III we also expose some transverse-relative-locality coordinate artifacts that result from adopting spacetime-noncommutativity-inspired phase-space constructions, just like in phase-space constructions inspired by \(\kappa\)-Minkowski noncommutativity \[17–19\] it was recently established \[20\] that longitudinal-relative-locality coordinate artifacts are present. So the evidence we here provide suggests, however preliminarily, that longitudinal and transverse relative locality really need to be considered as equally meaningful aspects of relative locality.

We work throughout in 2+1 spacetime dimensions. Evidently transverse relative locality cannot be present in 1+1-dimensional theories, and on the other hand all the features of transverse relative locality we are here interested in are fully describable within a 2+1-dimensional setup (the generalization to a D+1-dimensional analysis is only modestly cumbersome, but adds nothing to the concepts and results here of interest).

II. TRANSVERSE RELATIVE LOCALITY FROM DISTANT Boosts

Our first task is to expose the presence of transverse relative locality and of the associated dual-gravity lensing within the phase-space setup introduced in Ref. [3], which was there used instead mainly to characterize longitudinal relative locality. So we follow Ref. [3] in introducing its three main ingredients: (I) ordinarily trivial Poisson brackets for the spacetime coordinates \(\{x_j, t\} = 0, \{x_j, x_k\} = 0\); (II) a completely standard description of the generators

\[\begin{align*}
\{x_j, t\} = 0, \{x_j, x_k\} = 0\end{align*}\]

\[\begin{align*}
\text{One can qualify this sort of effects as “dual-gravity lensing” in light of the thesis put forward in Refs. [1, 2] which characterizes relative locality as a manifestation of the, possibly curved, geometry of momentum space. The standard gravitational lensing is caused by spacetime curvature, and this relative-locality-induced “lensing” can be attributed, in light of Refs. [1, 2], to the “dual gravity” of momentum space.}\]
of space \((P_j)\) and time \((\Omega)\) translations
\[
\{\Omega, t\} = 1, \quad \{\Omega, x_i\} = 0, \\
\{P_t, t\} = 0, \quad \{P_t, x_j\} = -\delta_{ij}, \\
\{P_i, P_j\} = 0, \quad \{P_i, \Omega\} = 0 ;
\]
and of the generator of rotation
\[
\{R, x_i\} = \epsilon_{ij} x_j, \quad \{R, t\} = 0, \\
\{R, P_i\} = \epsilon_{ij} P_j, \quad \{R, \Omega\} = 0 ,
\]
(III) but an unconventional description, with deformation parameters \(\alpha, \beta, \gamma\), of the Poisson brackets between boost generators and generators of spacetime translations
\[
\{N_i, \Omega \} = P_i - \alpha \ell \Omega P_i \\
\{N_i, P_j\} = \Omega \delta_{ij} + \ell \left( (1 + \gamma - \alpha) \Omega^2 + \beta \vec{P}^2 \right) \delta_{ij} - \ell \left( \gamma + \beta - \frac{1}{2} \right) P_i P_j ,
\]
which in particular leads to \([3]\) the following one-parameter family of on-shell relations
\[
C_\ell = \Omega^2 - \vec{P}^2 + \ell (2 \gamma \Omega^3 + (1 - 2 \gamma) \Omega \vec{P}^2) ,
\]
\(\ell\) is easy to verify \([3]\) that all Jacobi identities are satisfied by these choices of Poisson brackets, and readers familiar with the doubly-special-relativity literature \([6-11]\) will recognize a rather standard DSR-type choice of boost generators.

For what concerns the derivation of the equations of motion (worldlines) in such a setup one can of course use \([3, 21, 22]\) the on-shell relation as Hamiltonian of evolution in an auxiliary worldline parameter \(\tau\). Evidently the momenta are conserved on the worldlines (since \(\{C_\ell, P_j\} = 0 = \{C_\ell, \Omega\}\)). And one finds
\[
\dot{t} \equiv \frac{\partial}{\partial \tau} = \{C_\ell, t\} = 2 \Omega + \ell (6 \gamma \Omega^2 + (1 - 2 \gamma) \vec{P}^2) ,
\]
\[
\dot{x}_i \equiv \frac{\partial}{\partial \tau} = \{C_\ell, x_i\} = 2 P_i (1 - \ell (1 - 2 \gamma) \Omega) ,
\]
from which in particular one obtains that for massless particles \((C_\ell = 0)\) the worldlines are governed by
\[
(x - x_0)_i = (1 - \ell |\vec{P}|) \frac{P_i}{|\vec{P}|} (t - t_0) .
\]
The fact that the speed of massless particles here depends on momentum\(^2\) is the main intriguing feature of this relativistic framework, and was the subject of several investigations (see, e.g., Refs. \([21, 23, 25]\)), including some which established the presence of longitudinal relative locality \([3, 5, 20]\).

\(^2\) The coefficients of the terms \(\ell \Omega^3\) and \(\ell \Omega \vec{P}^2\) in \(C_\ell\) were arranged in Ref. \([3]\) just so that this speed law for massless particles, \(1 - \ell |\vec{P}|\), would be produced. This is how from the more general two-parameter case \(C_\ell = \Omega^2 - \vec{P}^2 + \ell (\gamma' \Omega^3 + \gamma'' \Omega \vec{P}^2)\) one arrives at the one-parameter case considered here and in Ref. \([3]\): \(C_\ell = \Omega^2 - \vec{P}^2 + \ell (2 \gamma \Omega^3 + (1 - 2 \gamma) \Omega \vec{P}^2)\).
We shall here expose the presence in this framework of also transverse relative locality through a very explicit analysis. This analysis is centered on the properties of a coincidence of events, local to an observer Alice (i.e. occurring in the spacetime origin of Alice’s reference frame), with all events in the coincidence being events of emission of a massless particle, all propagating in the same direction (but with different momenta). The transverse relativity of locality will be evident once we establish how such a coincidence of events at Alice is described by a distant boosted observer, and specifically an observer who is at some distance from Alice along the direction of propagation of the particles and boosted in a direction orthogonal to the direction of propagation of the particles.

In preparation for this analysis let us first establish how a single massless particle emitted in Alice’s origin and propagating along its \( x_1 \) direction, and acting with it on the worldline from Alice’s origin just the right amount to detect in his origin a massless soft (\( \ell p \simeq 0 \)) particle emitted from Alice’s origin (so the translation is \( T_{a,a,0} \) with parameters \( a_1 = a, \ a_1 = a, \ a_2 = 0 \)). We therefore find that Alice’s worldline \( (8) \) is described by Bob as follows

\[
x_1^A(t^A) = (1 - \ell p)t^A, \quad x_2^A(t^A) = 0
\]

In order to obtain Bob’s description of this same worldline we must use the translation generators:

\[
T_{a,a,0} \triangleright t = t + a\{P_1, t\} - a\{\Omega, t\} = t - a
\]

\[
T_{a,a,0} \triangleright x_1 = x_1 + a\{P_1, x_1\} - a\{\Omega, x_1\} = x_1 - a
\]

\[
T_{a,a,0} \triangleright x_2 = x_2, \quad T_{a,a,0} \triangleright P_1 = P_1, \quad T_{a,a,0} \triangleright \Omega = \Omega,
\]

where, for definiteness and simplicity, we specialized to the case of an observer Bob whose spacetime origin is distant from Alice’s spacetime origin just the right amount to detect in his origin a massless soft (\( \ell p \simeq 0 \)) particle emitted from Alice’s origin (so the translation is \( T_{a,a,0} \) with parameters \( a_1 = a, \ a_1 = a, \ a_2 = 0 \)). We therefore find that Alice’s worldline \( (8) \) is described by Bob as follows

\[
x_1^B(t) = -a + (1 - \ell p)(t^B + a), \quad x_2^B(t) = 0
\]

We are now ready for the final step of our planned analysis of the massless particle of generic momentum \( p \) emitted in Alice’s origin along Alice’s \( x_1 \) axis, i.e. we can now perform the DSR-deformed boost along the \( x_2 \) direction to obtain the description of that particle according to observer Camilla. For that we can rely on the representation of the (3-parameter family of) boosts given in Eq. \( (10) \) which was already derived in Ref. \[3\]:

\[
\mathcal{N}_i = x_i\Omega - tP_i + \ell \left( at\Omega P_i + x_i (\beta P^2 + (1 + \gamma - \alpha)\Omega^2) - \left( \gamma + \beta - \frac{1}{2} \right) x_k P^k P_j \right).
\]

In particular this leads to the following action of the boosts on coordinates:

\[
\{\mathcal{N}_i, t\} = x_i + \ell (2(1 + \gamma - \alpha)x_i\Omega + atP_i)
\]

\[
\{\mathcal{N}_i, x_j\} = t\delta_{ij} - at\Omega\delta_{ij} + \ell \left( \gamma + \beta - \frac{1}{2} \right) (x_k P^k)\delta_{ij} + x_j P_i - 2\beta t x_i P_j.
\]

Specializing these formulae to the case of a boost purely in the \( x_2 \) direction, and acting with it on the worldline \( (9) \) we arrive at the sought Camilla description:

\[
x_1^C(t^C) = -a + (1 - \ell p)(t^C + a)
\]

\[
x_2^C(t^C) = -\xi_2 a + \xi_2 a (\alpha - \beta - \gamma + \frac{1}{2}) \ell p + \xi_2 \left( 1 - (\alpha - \beta - \gamma + \frac{1}{2}) \ell p \right) (t^C + a)
\]

where \( \xi_2 \) is the boost parameter for the transformation from Bob to Camilla, which is a pure boost along the \( x^2 \) direction.
Some indirect manifestations of transverse relative locality are already visible looking at this single worldline. In particular, by eliminating $t_C$ one obtains the projection of the worldline in the $x_1^C, x_2^C$ plane

$$x_2^C(x_1^C) = \xi_2 \left(1 - \left(\alpha - \beta - \gamma - \frac{1}{2}\right) \ell p\right) x_1^C + \ell \xi_2 a p$$

which has some remarkable properties.

We notice two main features that characterize this result with respect to the corresponding result that applies in the special-relativity limit ($\ell \to 0$):

(I) when distances of order $\ell \xi_2 a p$ are within the reach of available experimental sensitivities it will be appreciated that the worldline does not cross Camilla’s spatial origin, a feature we shall find convenient to label as “shift”;

(II) when $\ell \xi_2 p$ is within the reach of available angular resolutions (and $\alpha - \beta - \gamma - \frac{1}{2} \neq 0$) the angle in the $x_1, x_2$ plane by which Camilla sees the arrival of the particle is momentum dependent, which is the mentioned “dual-gravity lensing”.

None of this in itself provides a direct manifestation of relative locality, but as we shall see these two features of “shift” and “dual-gravity lensing” do play a role in the size of the transverse relative locality effects. At least within the framework we are here adopting, the effect of transverse relative locality could be described as composed of these two features, even in the very tangible sense that the magnitude of the transverse relative locality is obtained combining the magnitudes of the shift and of the dual-gravity lensing.

In order to examine the relative locality itself we must analyze contexts with distant coincidences of events, and the result we obtained above for a single massless particle emitted by Alice toward Bob and Camilla is all that we shall need in order to characterize such coincidences of events. Let us start by focusing on the case of 3 worldlines of that type; specifically 3 massless particles all emitted simultaneously in Alice’s origin toward Alice’s $x_1$ axis, but two of them are “soft” (i.e. their momenta, $p^{(1)}$ and $p^{(2)}$ are small enough that $\ell p^{(1)}$ and $\ell p^{(2)}$ can be neglected; $\ell p^{(1)} \simeq 0 \simeq \ell p^{(2)}$) while the third one has “hard” momentum $p^{(3)}$ and we keep track of terms with factors $\ell p^{(3)}$. So we have set up a coincidence of emission events established by the observer Alice, local to the coincidence, and with the work done above we can establish immediately how the relevant 3 worldlines are described by Camilla:

$$x_1^{(1)C}(t_C) = t_C, \quad x_2^{(1)C}(t_C) = \xi_2 t_C$$

(16)

$$x_1^{(2)C}(t_C) = t_C, \quad x_2^{(2)C}(t_C) = \xi_2 t_C$$

(17)

$$x_1^{(3)C}(t_C) = -a + (1 - \ell p^{(3)})(t_C + a)$$

$$x_2^{(3)C}(t_C) = -\xi_2 a + \xi_2 a \left(\left(\alpha - \beta - \gamma + \frac{1}{2}\right) \ell p^{(3)}\right) + \xi_2 (1 - (\alpha - \beta - \gamma + \frac{1}{2}) \ell p^{(3)}) (t_C + a)$$

(18)

3 Our choice of considering two soft massless particles and one hard massless particle is somewhat redundant but helps us keep the presentation clearer. With only one soft particle, plus the hard particle, one could already infer all the properties of the transverse relative locality which we are going to discuss (in fact in Fig. 1 only one soft and one hard particle are shown). By contemplating soft particles we have the luxury of seeing explicitly that coincidences of emission events of soft particles still behave with absolute locality, and this then renders more evident how the event of emission of a hard particle behaves anomalously (with relative locality). Moreover, there is a “relativist tradition” of viewing an event as a crossing of two worldlines, and from that perspective our 3 simultaneous emission events can be viewed as two independent crossing events: the crossing of two soft worldlines and the crossing of the hard worldline with one of the soft worldlines (it is of course irrelevant which one of the soft worldlines is taken into account for this). From this traditional relativist perspective one would describe the relativity of locality as the fact that for Alice the two crossings coincide whereas according to the coordinates of distantly boosted Camilla they do not coincide.
According to Camilla’s coordinatization the wordlines of the two soft particles (for which the ℓ-deformation is not felt) of course cross at \((-a, -ξ_2a, -a)\), just as they would in an ordinarily special-relativistic theory. But (see Fig. 1) the hard worldline does not go through \((-a, -ξ_2a, -a)\) and notably when \(t^C = -a\) the hard particle has coordinates \((-a, -ξ_2a(1 - (α - β - γ + \frac{1}{2})ξ_2ℓp(3)), -a)\), from which we establish a transverse relative locality of \(|Δx^C_s| = (α - β - γ + \frac{1}{2})ξ_2ℓp(3)\). Notice that (as also shown in Fig. 1) this amount of transverse relative locality can be described as a combination of the term we labeled “shift” and of a contribution proportional to the term we labeled “dual gravity lensing”.

Figure 1: 3D worldlines (top panels) and their 2D spatial projection (bottom panels) for a soft and a hard (respectively red and blue; violet when coincident) massless particles emitted simultaneously at Alice toward Bob and Camilla. Alice’s viewpoint is shown in the left panels. In Camilla’s coordinatization (right panels) the emissions are not coincident, and there is transverse relative locality, with contributions from “shift” an “dual-gravity lensing”.
So we have given a crisp characterization of how a distant coincidence of events, the crossings of the 3 worldlines $p^{(1)}, p^{(2)}, p^{(3)}$ at Alice, produces for a distantly boosted observer, Camilla, some transverse relative locality, which in general also involves “shift” and “dual-gravity lensing”.

It is interesting to check how the dual-gravity lensing depends on the momenta of the particles, and for this purpose it is useful to contemplate a fourth massless particle, with momentum $p^{(4)}$, again emitted at Alice’s origin simultaneously with the other 3 massless particles and emitted toward Alice’s $x_1$ direction. We take that $p^{(4)}$ is also “hard”, so that both terms with factors in $\ell p^{(4)}$ and in $\ell p^{(3)}$ should be taken into account. For this fourth worldline Camilla’s description evidently is

$$x^{(4)C}_1(t^C) = -a + (1 - \ell p^{(4)})(t^C + a)$$
$$x^{(4)C}_2(t^C) = -\xi_2a + \xi_2a \left( (\alpha - \beta - \gamma + \frac{1}{2}) \ell p^{(4)} \right) + \xi_2 \left( 1 - \left( \alpha - \beta - \gamma + \frac{1}{2} \right) \ell p^{(4)} \right) (t^C + a)$$  \hspace{1cm} (19)

Comparing these with $\ell p^{(4)}$ we see that in the $(x_1, x_2)$ plane the two hard wordlines reach Camilla from directions forming an angle

$$\theta = \arctan \left( \xi_2(1 - (\alpha - \beta - \gamma - \frac{1}{2} ) \ell p^{(4)}) \right) - \arctan \left( \xi_2(1 - (\alpha - \beta - \gamma - \frac{1}{2} ) \ell p^{(3)}) \right) \approx -\xi_2(\alpha - \beta - \gamma - \frac{1}{2} ) \ell (p^{(4)} - p^{(3)})$$

where notably the angle depends linearly on the difference of the momenta $p^{(4)} - p^{(3)}$.

Finally, let us contemplate a different situation, with only two such massless particles, of momenta $p^{(s)}$ and $p^{(h)}$ propagating again along Alice’s $x_1$ direction but emitted from Alice’s spatial origin with just the right difference of times of emission that they reach Bob’s spacetime origin simultaneously. Assuming $p^{(s)}$ is soft ($\ell p^{(s)} \approx 0$) and $p^{(h)}$ is “hard” ($\ell p^{(h)} \neq 0$) one has that the worldlines for these two particles are, according to Alice,

$$x^{(h)A}_1(t^A) = (1 - \ell p^{(h)})(t^A + \ell a p^{(h)}), \quad x^{(h)A}_2(t^A) = 0$$  \hspace{1cm} (20)
$$x^{(s)A}_1(t^A) = t^A, \quad x^{(s)A}_2(t^A) = 0$$  \hspace{1cm} (21)

Also for this situation we are interested in Camilla’s coordinatization. We have here a coincidence of (detection) events at Bob, so Camilla is purely boosted with respect to an observer who is local to a coincidence of events. Using again the results we derived above one easily finds that Camilla describes these two worldlines as follows

$$x^{(h)C}_1(t^C) = -a + (1 - \ell p^{(h)A})(t^C + a + \ell a p^{(h)A})$$
$$x^{(h)C}_2(t^C) = -\xi_2a \left( 1 - \left( \alpha - \gamma - \beta - \frac{1}{2} \right) \ell p^{(h)A} \right) + \xi_2 \left( 1 - \left( \alpha - \gamma - \beta + \frac{1}{2} \right) \ell p^{(h)A} \right) (t^C + a + \ell a p^{(h)A})$$
$$x^{(s)C}_1(t^C) = t^C, \quad x^{(s)C}_2(t^C) = \xi_2 t^C$$

This allows us to verify that, as expected $[1, 3, 20]$, the coincidence of events in Bob’s origin is also described as a coincidence of events by Camilla (both Camilla and Bob are nearby observers of a coincidence of events, so even in a relative-locality framework they should, and they do, agree on such coincidences of events).

There is however something noteworthy about directions of propagation and for which it is useful to characterize the two worldlines on the $(x_1, x_2)$ plane:

$$x^{(h)C}_2(x^{(h)C}_1) = \xi_2 \left( 1 - \left( \alpha - \beta - \gamma + \frac{1}{2} \right) \ell p^{(h)} \right) x^{(h)C}_1$$
$$x^{(s)C}_2(x^{(s)C}_1) = \xi_2 x^{(s)C}_1$$  \hspace{1cm} (22)

$$x^{(s)C}_2(x^{(s)C}_1) = \xi_2 x^{(s)C}_1$$  \hspace{1cm} (23)
From this we see that, while as expected no relative locality is seen by Camilla in such situations, the dual-gravity lensing survives (also see Fig. 2): the two worldlines emitted by Alice along parallel directions are detected by Camilla along directions forming an angle (if $\alpha - \beta - \gamma - \frac{1}{2} \neq 0$)

$$\theta = \arctan \left( \xi_2 (1 - (\alpha - \beta - \gamma - \frac{1}{2}) \ell p^{(h)}) \right) - \arctan (\xi_2 (\alpha - \beta - \gamma - \frac{1}{2}) \ell p^{(h)})$$

Figure 2: Alice emits the soft (red) and the hard (blue) massless particles from her spatial origin with a time-of-emission difference such that they reach Bob simultaneously. The simultaneous arrival of the two particles is also manifest in Camilla’s coordinates. But there still is some “dual-gravity lensing”: whereas according to Alice’s coordinatization the particles are on parallel trajectories, according to Camilla’s coordinatization the particles are not on parallel trajectories.

III. NONCOMMUTATIVITY-INSPIRED TRANSVERSE RELATIVE LOCALITY FROM PURE TRANSLATIONS

In the previous section we established for transverse relative locality results for distantly boosted observers in a theory of free particles, which are evidently as significant as the results for longitudinal relative locality established in [3, 4] in the same class of theories for free particles. We are, as announced, postponing more detailed studies of transverse relative locality for interacting theories, of the type introduced in Refs. [1, 2]. But there is another type of
manifestation of longitudinal relative locality for free-particle theories, which we can here reproduce in transverse-relative-locality version. These are the results for longitudinal relative locality for pure translations established in Ref. [20] as a peculiar class of coordinate artifacts present in a much-studied phase-space construction inspired by $\kappa$-Minkowski noncommutativity [17–19], in which one adopts "$\kappa$-Minkowski Poisson brackets" for the spacetime coordinates:

$$\{x_j, t\} = \frac{1}{\kappa} x_j \quad \text{with} \quad \{x_j, x_k\} = 0.$$ 

It is easy to verify that there is no transverse relative locality for pure translations in those $\kappa$-Minkowski-inspired phase-space constructions. However, in this section we want to show that some peculiar coordinate artifacts of transverse relative locality are found under pure translations within a closely related, still noncommutativity-inspired framework. For this purpose we introduce the following "$\rho$-Minkowski" Poisson brackets for spacetime coordinates

$$\{x_i, t\} = \rho \epsilon_{ij} x_j$$

$$\{x_i, x_j\} = 0$$

where $\rho$ is a parameter with dimension of length, and, consistently with the approach we already adopted in the previous section, we work at leading order in $\rho$.

For the description of space ($P_j$) and time ($\Omega$) translations, the requirement of enforcing the Jacobi identities\(^5\) leads us to

$$\{\Omega, t\} = 1, \quad \{\Omega, x_i\} = 0$$

$$\{P_i, t\} = 0, \quad \{P_i, x_j\} = - (\delta_{ij} + \rho \epsilon_{ij} \Omega)$$

$$\{P_i, P_j\} = 0, \quad \{P_i, \Omega\} = 0.$$  \quad (24)

The type of transverse-relative-locality coordinate artifacts we want to characterize in this section are due to the properties of these translation generators. And to see that these properties alone suffice to produce transverse-relative-locality coordinate artifacts we adopt for this section the standard on-shell relation

$$C = \Omega^2 - \vec{P}^2$$  \quad (25)

These ingredients are all that is required for the analysis in this section, but as a side remark let us observe that the Poisson brackets we introduced are covariant under classical spatial rotations. We have in fact that

$$\{x_i, t\} = \rho \epsilon_{ij} x_j, \quad \{x_i, x_j\} = 0$$

$$\Rightarrow \{x'_i, t'\} = \rho \epsilon_{ij} x'_j, \quad \{x'_i, x'_j\} = 0$$

if

$$t' = t, \quad x'_1 = x_1 \cos \theta + x_2 \sin \theta, \quad x'_2 = x_2 \cos \theta - x_1 \sin \theta.$$  

Moreover, postulating

$$\{R, x_i\} = \epsilon_{ij} x_j, \quad \{R, t\} = 0,$$

$$\{R, P_i\} = \epsilon_{ij} P_j, \quad \{R, \Omega\} = 0.$$  

\(^4\) Note that just like a 2+1D $\kappa$-Minkowski noncommutativity is linked to the algebra $\text{hom}(2)$ (euclidean-homotheties algebra) our $\rho$-Minkowski noncommutativity is linked to the algebra $\epsilon(2)$ (euclidean algebra).

\(^5\) In particular, it is easy to verify that instead the standard translations would not satisfy the Jacobi identities with $\rho$-Minkowski coordinates.
all Jacobi identities are satisfied.

Within this setup we shall expose transverse relative locality by analyzing a simultaneous emission of massless particles in the origin of an observer Alice, as described in the coordinatization of spacetime by a distant observer Bob.

In preparation for that let us first derive the worldlines that follow from the undeformed on-shell relation, when analyzed in terms of our $\rho$-deformed Poisson brackets. We have that

$$\dot{t} \equiv \frac{\partial t}{\partial \tau} = 2\Omega \{\Omega, t\} - 2P_k \{P_k, t\} = 2\Omega$$

$$\dot{x}_i \equiv \frac{\partial x_i}{\partial \tau} = 2\Omega \{\Omega, x_i\} - 2P_k \{P_k, x_i\} = 2P_k (\delta_{ik} + \rho \epsilon_{ik})$$

from which in particular one obtains that for massless particles ($\mathcal{C} = 0$) the worldlines are governed by

$$(x - x_0)_i = \left(\frac{P_i}{|P|} - \rho \epsilon_{ij} P_j\right) (t - t_0)$$

Notice that, as a result of the $\rho$-deformed Poisson brackets, the coordinate velocity depends on momenta in a peculiar way

$$v_i = \frac{P_i}{|P|} - \rho \epsilon_{ij} P_j$$

and in particular in order for the coordinate velocity to be directed along the $x_1$ direction actually the momentum must have a small $p_2$ component:

$$\vec{P} = (p, -\rho p^2), \Rightarrow \vec{v} = (1, 0)$$

Still it is easy to check that the speed of massless particles in this framework is always momentum independent:

$$|\vec{v}|^2 = v_i v_k \delta^{ik} = 1 - \rho \frac{1}{|P|} \delta^{ik} (\epsilon_{ij} P_j P_k + \epsilon_{kj} P_j P_i) = 1$$

We are now ready to consider the emission at Alice of two massless particles, a soft particle of momentum $p^{(s)}$ (with $\rho p^{(s)} \approx 0$) and a hard particle of momentum $p^{(h)}$ (such $\rho p^{(h)}$ cannot be neglected). And let us further restrict our focus to the case in which, according to Alice’s coordinates, the two massless particles are emitted toward the $x_1$ axis. In light of the results derived above it is evident that according to Alice the two worldlines are coincident:

$$x^{(s)A}_1(t^A) = t^A, \quad x^{(s)A}_2(t^A) = 0.$$  

$$x^{(h)A}_1(t^A) = t^A, \quad x^{(h)A}_2(t^A) = 0.$$  

We can now establish how the peculiar properties of our $\rho$-deformed translations affect the way in which a distant observer Bob describes these two worldlines. In general for our $\rho$-deformed translations we have

$$T_{a_1, a_2} \triangleright t = t - a_i \{\Omega, t\} + a_j \{P_j, t\} = t - a_t,$$

$$T_{a_1, a_2} \triangleright x_i = x_i - a_i \{\Omega, x_i\} + a_j \{P_j, x_i\} = x_i - a_i + \rho \Omega \epsilon_{ij} a_j.$$  

We are again interested in the description of the two worldlines given by an observer Bob, at rest with respect to Alice, and such that the soft massless particle reaches Bob in his spacetime origin. So the translation parameters
of our interest are $a_t = a$, $a_1 = a$, $a_2 = 0$. For this choice of translation parameters we find that the two worldlines are, according to Bob,

$$x_1^{(s)B}(t^B) = t^B, \quad x_2^{(s)B}(t^B) = 0. \quad (35)$$

$$x_1^{(h)B}(t^B) = t^B, \quad x_2^{(h)B}(t^B) = -\rho a^p(h). \quad (36)$$

As shown in Figure 3, these results confirm that with $\rho$-Minkowski coordinates one finds transverse relative locality which is completely analogous to the longitudinal relative locality found with $\kappa$-Minkowski coordinates in Ref. [20]. In particular we have seen that a coincident emission at Alice, is described as a pair of non-coincident emission events by distant observer Bob, and the lack of coincidence is seen by Bob in the direction orthogonal to the direction of the translation connecting Alice to Bob.

![Figure 3: In this case Alice (left panels) and Bob (right panels) use $\rho$-Minkowski coordinates. We show 3D worldlines (top panels) and their 2D spatial projection (bottom panels) for a soft and a hard (respectively red and blue; violet when coincident) massless particles, emitted simultaneously at Alice toward Bob. In Bob’s coordinates the two worldlines do not coincide and there is some transverse relative locality: the coincidence of emission events witnessed by nearby observer Alice, is not present in the coordinatization by distant observer Bob, and the difference between emission points in Bob coordinates is purely along the $x_2$ axis, orthogonal to the direction ($x_1$) of the translation transformation from Alice to Bob.]
IV. CLOSING REMARKS

We feel we have here accomplished the main task we had set for ourselves, by showing that transverse relative locality is an aspect of relative locality that cannot be dismissed, and actually may deserve as much attention as longitudinal relative locality. This was established here within the confines of relative-locality theories of free particles, but it is hard to imagine that for interacting relative-locality particles, which are described within the framework of Refs. [1, 2], the “balance of power” between longitudinal and transverse relative locality could be significantly shifted.

Our results on transverse relative locality and dual-gravity lensing in theories of free particles may also provide some guidance for future more detailed analyses of such features within the framework of Refs. [1, 2] for interacting particles. These should also take as starting point the preliminary results on dual-gravity lensing reported in parts of Ref. [12]. In particular, the feature of dual-gravity lensing exposed in Ref. [12] was proportional to the sum of the energies (momenta) of the two particles whose wordlines were experiencing lensing. It was already clear from Ref. [12] that this result of dependence on the sum of energies had only been checked within a very specific setup for the derivation, including definite choices among the many possible chains of interactions that could be considered in the interacting-particle framework. The fact that here, within the limitations of a theory of free particles, we found some dual-gravity lensing proportional to the difference of the energies (momenta) of the two particles whose wordlines experience lensing can provide encouragement for the search of other chains of interactions, in which the difference of energies governs the dual-gravity lensing.

The scopes of our analysis were still too limited for allowing speculations about phenomenology, but it is nonetheless noteworthy from that perspective that in Section II we found transverse-relative-locality effects of exactly the same magnitude of the effects of longitudinal relative locality previously found in Ref. [3], and those are (at least indirectly) testable [3, 20, 27], even if \( \ell \) is of the order of the Planck length.

The brief exploration, in Sec. III, of transverse relative locality when observers adopt our “\( \rho \)-Minkowski coordinates” must be viewed in exactly the same spirit as the analogous results obtained in Ref. [20] for observers adopting “\( \kappa \)-Minkowski coordinates”. In theories of classical particles these results may at best clarify possible confusion arising with the use of such non-standard coordinates (the confusion addressed in Ref. [20] being a particularly strong example, since it had obstructed progress in a relevant research area for more than a decade). But we feel such preliminary studies of classical theories with non-standard coordinates should have as ultimate goal the development of suitable quantum versions. And just like the use of “\( \kappa \)-Minkowski coordinates” in classical theories does prepare one’s intuition for studies of the “\( \kappa \)-Minkowski non-commutative spacetime” [18, 19], we expect that our preliminary observations on “\( \rho \)-Minkowski coordinates” might set the stage for intriguing studies of “\( \rho \)-Minkowski noncommutativity”.

Finally, we feel that the results we here reported should have some influence on future developments of the “doubly-special relativity” research programme [6–11]. Hundreds of papers have been devoted over the last decade to doubly-special-relativity results formulated exclusively in momentum space (see, e.g., Refs. [6–11] and references therein). Only recently some spacetime aspects of doubly-special relativity were satisfactorily analyzed in a handful of studies, which were however confined to essentially 1+1-dimensional analyses, and led to some of the first results on longitudinal relative locality [3–5]. We here reported, in Section II, an analysis in which the presence of more than one spatial dimension in a doubly-special-relativity framework plays a nontrivial role, and we feel this could now set the new standard for studies attempting to advance doubly-special-relativity research.

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