ON EQUATION FOR INITIAL VALUES IN THEORY OF THE SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract
We consider the properties of the second order nonlinear differential equations \( b'' = g(a, b, b') \) with the function \( g(a, b, b' = c) \) satisfying the following nonlinear partial differential equation
\[
\begin{align*}
g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2g_{bbcc} + 2cgg_{bc} + \\
+ y^2g_{cc} + (ga + cg_b)g_{ccc} - 4g_{abc} - 4cg_{bcb} - cgc_{gbc} \\
- 3gg_{bcc} - g_cg_{accc} + 4g_cg_{bccc} - 3g_bg_{ccc} + 6g_b = 0.
\end{align*}
\]
Any equation \( b'' = g(a, b, b') \) with this condition on function \( g(a, b, b') \) has the General Integral \( F(a, b, x, y) = 0 \) shared with General Integral of the second order ODE’s \( y'' = f(x, y, y') \) with condition \( \frac{\partial^4 f}{\partial y'^4} = 0 \) on function \( f(x, y, y') \) or
\[
y'' + a_1(x, y)y'^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0
\]
with some coefficients \( a_i(x, y) \).

1 Introduction
The relation between the equations in form
\[
y'' + a_1(x, y)y'^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0
\]
and
\[
b'' = g(a, b, b')
\]
with function \( g(a, b, b') \) satisfying the p.d.e
\[
\begin{align*}
g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2g_{bbcc} + 2cgg_{bc} + \\
+ g^2g_{cc} + (ga + cg_b)g_{ccc} - 4g_{abc} - 4cg_{bcb} - cgc_{gbc} -
\end{align*}
\]
$3gg_{bc} - g_3g_{a3} + 4g_3g_{bc} - 3g_3g_{cc} + 6g_{bb} = 0.$

from geometrical point of view was studied by E.Cartan [4].

In fact, according to the expressions on curvature of the space of linear elements $(x, y, y')$ connected with equation (1)

$$
\Omega^1_2 = a[\omega^2 \wedge \omega^1_1], \quad \Omega^0_1 = b[\omega^1 \wedge \omega^2], \quad \Omega^0_2 = h[\omega^1 \wedge \omega^2] + k[\omega^2 \wedge \omega^1_1].
$$

where:

$$
a = -\frac{1}{6} \frac{\partial^4 f}{\partial y'^4}, \quad h = \frac{\partial b}{\partial y'}, \quad k = -\frac{\partial^2 f}{\partial y'^2} - \frac{1}{6} \frac{\partial^3 f}{\partial y'^3}.
$$

and

$$
6b = f_{xxyy'} + 2y'f_{xyyy'} + 2f f_{xyyy'} + y'^2 f_{yyyy'} + 2y'f f_{yyyy'} \\
+ f^2 f_y f_{y'y'} + (f_x + y' f_y) f_{y'y'} - 4f_{xyy'} - 4y' f_{yy'} - y' f_y f_{xyy'} \\
- 3f f_{yy'} - f_y f_{xxyy'} + 4f_y f_{yy'} - 3f_y f_{yy'} + 6f_{yy}.
$$

two types of equations by a natural way are evolved: the first type from the condition $a = 0$ and second type from the condition $b = 0$.

The first condition $a = 0$ the equation in form (1) is determined and the second condition lead to the equations (2) where the function $g(a, b, b')$ satisfies the above p.d.e. (3).

From the elementary point of view the relation between both equations (1) and (2) is a result of the special properties of their General Integral

$$
F(x, y, a, b) = 0.
$$

So we have the following fundamental diagramm:

$$
\begin{align*}
F(x, y, a, b) &= 0 \\
y'' &= f(x, y, y') & b'' &= g(a, b, b') \\
\downarrow & \quad \leftrightarrow & \uparrow \\
M^3(x, y, y') & \quad \iff \quad N^3(a, b, b')
\end{align*}
$$

which is presented the General Integral $F(x, y, a, b) = 0$ (as some 3-dim orbifold) in form of the twice nontrivial fibre bundles on circles over corresponding surfaces:

$$
M^3(x, y, y') = U^2(x, y) \times S^1 \quad \text{and} \quad N^3(a, b, b') = V^2(a, b) \times S^1.
$$

2 An examples of solutions of dual equation

Let us consider the solutions of equation (3).

It has many types of reductions and the simplest of them are

$$
g = c^\alpha \omega[ac^{\alpha - 1}], \quad g = c^\alpha \omega[bc^{\alpha - 2}], \quad g = c^\alpha \omega[ac^{\alpha - 1}, bc^{\alpha - 2}], \quad g = a^{-\alpha} \omega[ca^{\alpha - 1}],
$$

$$
g = b^{1 - 2\alpha} \omega[cb^{\alpha - 1}], \quad g = a^{-1} \omega(c - b/a), \quad g = a^{-3} \omega[b/a, b - ac], \quad g = a^{3/\alpha - 2} \omega[b^\alpha/a^\beta, c^\alpha/a^\beta-a].
$$
For any type of reduction we can write corresponding equation (2) and then integrate it. As example for the function

$$g = a^{-\gamma}A(ca^{\gamma-1})$$

we get the equation

$$[A + (\gamma - 1)\xi^2A^1V + 3(\gamma - 2)[A + (\gamma - 1)\xi]A^{111} + (2 - \gamma)A^1A^{11} + (\gamma^2 - 5\gamma + 6)A^{11} = 0.$$ 

One solution of this equation is

$$A - (2 - \gamma)[\xi(1 + \xi^2) + (1 + \xi^2)^{3/2}] + (1 - \gamma)\xi$$

This solution is correspended to the equation

$$b'' = \frac{1}{a^2}[b'(1 + b'^2) + (1 + b'^2)^{3/2}]$$

with General Integral

$$F(x, y, a, b) = (y + b)^2 + a^2 - 2ax = 0$$

The dual equation has the form

$$y'' = -\frac{1}{2x}(y'^3 + y')$$

Remark that the first examples of solutions of equation (3) was obtained in [6-9].

**Proposition 1** Equation (3) can be represent in form

$$g_{ac} + gg_{cc} - g_c^2/2 + cg_{bc} - 2g_b = h(a, b, c),$$

$$h_{ac} + gh_{cc} - g_ch_c + ch_{bc} - 3h_b = 0.$$  (4)

From this is followed that exists the class of equations (2) with function g(a, b, c) satisfying the condition

$$g_{ac} + gg_{cc} - g_c^2/2 + cg_{bc} - 2g_b = 0.$$  (5)

which is more readily solved then equation (3).

Here we present some solutions of the equation (8) as function depending on two variables

$$g = g(a, c)$$

In case when $$g = g(a, c)$$ and $$h = 0$$ we have the equation

$$g_{ac} + gg_{cc} - \frac{1}{2}g_c^2 = 0.$$  

To integrate this equation we can transform its in more convenient form using variable $$g_c = f(a, c)$$. Then one obtains:

$$2f_cf_{ac} + (f^2 - 2f_a)f_{cc} = 0.$$  

After the Legendre-transformation we obtain the equation:

$$[(\xi\omega_\xi + \eta\omega_\eta - \omega)^2 - 2\xi]\omega_\xi\xi - 2\eta\omega_\xi\eta = 0.$$  

Using the new variable $$\xi\omega_\xi + \eta\omega_\eta - \omega = R$$ we have the new equation for $$R$$:

$$R_\xi - \frac{1}{2}R^2\omega_\xi\xi = 0.$$
and the following relations:

\[ \omega_\eta = \frac{\omega}{\eta} + \frac{R}{\eta} \xi + \frac{2\xi}{\eta R} - \frac{\xi A(\eta)}{\eta}, \]

\[ \omega_\xi = -\frac{2}{R} + A(\eta) \]

with arbitrary function \( A(\eta) \). From the conditions of compatibility is followed:

\[ 2\eta R_\eta + R_\xi (2\xi - R^2) + \eta A_\eta R^2 = 0. \]

Integrating this equation we can obtain general integral.

In the particular case: \( A = \frac{1}{\eta} \) we have:

\[ \frac{R^2}{R - 2\eta} = -\frac{\xi}{\eta} + \Phi\left(\frac{1}{\eta} - \frac{2}{R}\right). \]

At the condition \( A = 0 \) we obtain the equation:

\[ 2\eta R_\eta + (2\xi - R^2) R_\xi = 0, \]

which has the solution:

\[ R^2 = 2\xi + 2\eta \Phi(R), \]

were \( \Phi(R) \) is arbitrary function.

After choosing the function \( \Phi(R) \) we can find the function \( \omega \) and then using the inverse Legendre transformation the function \( g \) which is determined dual equation \( b'' = g(a, c) \).

**Remark 1** *The solutions of the equations of type*

\[ u_{xy} = uu_{xx} + \epsilon u_x^2 \]

*was constructed in [19]. In work of [20] was showed that they can be present in form*

\[ u = B'(y) + \int [A(z) - \epsilon y]^{(1-\epsilon)/\epsilon} dz, \]

\[ x = -B(y) + \int [A(z) - \epsilon y]^{1/\epsilon} dz. \]

*To integrate above equations we apply the parametric representation*

\[ g = A(a) + U(a, \tau), \quad c = B(a) + V(a, \tau). \tag{11} \]

*Using the formulas*

\[ g_c = \frac{g_\tau}{c_\tau}, \quad g_a = g_a + g_\tau \tau_a \]

*we get after the substitution in (10) the conditions*

\[ A(a) = \frac{dB}{da} \]

*and*

\[ U_{a\tau} - \left(\frac{V_a U_\tau}{V_\tau}\right)_\tau + U\left(\frac{U_a}{V_\tau}\right)_\tau - \frac{1}{2} \frac{U_a^2}{V_\tau} = 0. \]
So we get one equation for two functions $U(a, \tau)$ and $V(a, \tau)$. Any solution of this equation are determined the solution of equation (10) in form (11).

Let us consider the examples.

$$A = B = 0, \quad U = 2\tau - \frac{a\tau^2}{2}, \quad V = a\tau - 2\ln(\tau)$$

Using the representation

$$U = \tau \omega - \omega, \quad V = \omega$$

it is possible to obtain others solutions of this equation.

Equation

$$g_{ac} + gg_{cc} - \frac{1}{2}g_c^2 = B(a).$$

can be integrate in explicite form and solutions are

$$g = -H'(a) + \int \frac{dz}{[A(z) + \frac{1}{2}a]^3},$$

$$c = H(a) + \int \frac{dz}{[A(z) + \frac{1}{2}a]^2},$$

with arbitrary functions $H(a)$ and $A(z)$.

In fact, for $A(z) = z$ we have

$$g = -H'(a) + \int \frac{dz}{[z + \frac{1}{2}a]^3} = -H'(a) - \frac{1}{2} \frac{1}{[z + \frac{1}{2}a]^2},$$

and

$$c = H(a) + \int \frac{dz}{[z + \frac{1}{2}a]^2} = H(a) - \frac{1}{[z + \frac{1}{2}a]^3},$$

As result we get the solution

\textbf{Remark 2} In general case the equation

$$g_{ac} + gg_{cc} = 0,$$

is equivalent the equation

$$g_{ac} + gg_{cc} - \frac{1}{2}g_c^2 = B(a).$$

It can be integrate with help of Legender- transformation as in previous case.

Realy, we get

$$[(\xi \omega + \eta \omega - \omega)^2 - 2\xi + 2B(\omega_j)]\omega_{\xi \xi} - 2\eta \omega_{\xi \eta} = 0$$

and the relation

$$2R_{\xi} = [R^2 + 2B(\omega_j)\omega_{\xi \xi}.$$

It can be written in form

$$2 \frac{dR}{d\Omega} = R^2 + 2B(\Omega)$$

using the notation

$$\omega_{\xi} = \Omega.$$
Proposition 2 In case $h \neq 0$ and $g = g(a, c)$ the system (3) is equivalent the equation

$$\Theta_a \left( \frac{\Theta_a}{\Theta_c} \right)_{ccc} - \Theta_c \left( \frac{\Theta_a}{\Theta_c} \right)_{acc} = 1$$

(6)

where

$$g = -\frac{\Theta_a}{\Theta_c}, \; h_c = \frac{1}{\Theta_c}$$

To integrate this equation we use the presentation

$$c = \Omega(\Theta, a)$$

From the relations

$$1 = \Omega_{\Theta} \Theta_c, \; 0 = \Omega_{\Theta} \Theta_a + \Omega_c$$

we get

$$\Theta_c = \frac{1}{\Omega_{\Theta}}, \; \Theta_a = -\frac{\Omega_a}{\Omega_{\Theta}}$$

and

$$\frac{\Omega_a}{\Omega_{\Theta}} (\Omega_a)_{ccc} + \frac{1}{\Omega_{\Theta}} (\Omega_a)_{cca} = 1$$

Now we get

$$\Omega_{ac} = \frac{\Omega_{a\Theta}}{\Omega_{\Theta}} = (\ln \Omega_{\Theta})_a = K, \; \Omega_{acc} = \frac{K_{\Theta}}{\Omega_{\Theta}},$$

$$\Omega_{accc} = \frac{(K_{\Theta})_{\Theta}}{\Omega_{\Theta}} \frac{1}{\Omega_{\Theta}}, \; (\Omega_{acc})_a = \frac{K_{\Theta}}{\Omega_{\Theta}} - \frac{\Omega_a}{\Omega_{\Theta}} \frac{K_{\Theta}}{\Omega_{\Theta}}$$

As result the equation (6) take the form

$$\left[ \frac{(\ln \Omega_{\Theta})_{a\Theta}}{\Omega_{\Theta}} \right]_a = \Omega_{\Theta}$$

(7)

and can be integrate under the substitution

$$\Omega(\Theta, a) = \Lambda_a$$

So we get the equation

$$\Lambda_{\Theta} = \frac{1}{6} \Lambda_{\Theta}^3 + \alpha(\Theta) \Lambda_{\Theta}^2 + \beta(\Theta) \Lambda(\Theta) + \gamma(\Theta)$$

(8)

with arbitrary coefficients $\alpha, \beta, \gamma$.

This is Abel’s type of equation

$$y' = A(x)y^3 + B(x)y^2 + C(x)y + D(x)$$

It can be rewritten in form

$$y' = A(y - \phi)^3 + \theta(y - \phi)^2 + \lambda(y - \phi) + \phi'$$

or

$$z' = Az^3 + \theta z^2 + \lambda z$$
Let us consider the examples.

1. $\alpha = \beta = \gamma = 0$

The solution of equation (8) is

$$\Lambda = A(a) - 6\sqrt{B(a) - \frac{1}{3}\Theta}$$

and we get

$$c = A' - \frac{3B'}{\sqrt{B - \frac{1}{3}\Theta}}$$

or

$$\Theta = 3B - 27\frac{B'^2}{(c - A')^2}$$

This solution is corresponded to the equation

$$b'' = -\frac{\Theta_a}{\Theta_c} = -\frac{1}{18B'}b'^3 + \frac{A'}{6B'}b'^2 + \frac{B'' - A'^2}{6B'}b' + A'' + \frac{A'^3}{18B'} - \frac{A'B''}{B'}$$

cubical on the first derivatives $b'$ with arbitrary coefficients $A(a), B(a)$. This equation is equivalent to the equation

$$b'' = 0$$

under the point transformation.

The following example is the solution of equation (8) in form

$$g = b^{1-2\alpha} \omega[cb^{\alpha-1}]$$

Under this reduction one obtains the equation on the function $\omega(\xi = cb^{\alpha-1})$

$$\omega\omega'' - \frac{\omega'^2}{2} + (\alpha - 1)\xi^2\omega'' + (2 - 3\alpha)\xi\omega' + 2(2\alpha - 1)\omega = 0.$$  

To make the new variable $\theta = \omega + (\alpha - 1)\xi^2$ we obtain

$$\theta\theta'' - \frac{\theta'^2}{2} - \alpha\xi\theta' + 2\alpha\theta = 0.$$  

This equation has solution in parametrical form

$$\theta = \gamma\tau E(\tau), \quad \xi = \frac{\gamma E(\tau)}{\beta}$$

where

$$E(\tau) = \exp[-\int\frac{\tau d\tau}{(\tau - 1/2)^2 + \alpha/\beta^2 - 1/4}]$$  

(9)

where $\beta, \gamma$ are parameters and the explicit form of this integral depends on the value

$$\epsilon = \alpha/\beta^2 - 1/4.$$  

This solution is corresponed to the family of equations

$$b'' = b^{1-2\alpha}[\theta + (1 - \alpha)\xi^2]$$

and (1) forming dual pair.

The values of coefficients $a_i(x, y, \alpha, \beta, \gamma)$ in corresponding equation (1) can be change radically at the variation of parameters as it is showed the calculation of integral (10).
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