Decay of approximate solutions for the damped semilinear wave equation on a bounded 1d domain

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ABSTRACT

By relying on the probabilistic interpretation of the solution, we study the long time behavior for a semilinear hyperbolic system with space-dependent and nonlinear damping term. The analysis provides error estimates of a class of approximate solutions and exponential convergence in $L^\infty$ towards a stationary solution.

1. Introduction

In this paper we study the initial boundary value problem for the $2 \times 2$ system in one space dimension

$$\begin{cases}
\partial_t \rho + \partial_x J = 0, \\
\partial_t J + \partial_x \rho = -2k(x)g(J),
\end{cases}$$

(1.1)

where $x \in I = [0, 1]$ and $t \geq 0$, and

$$(\rho, J)(\cdot, 0) = (\rho_0, J_0)(\cdot), \quad J(0, t) = J(1, t) = J_b$$

(1.2)

for $(\rho_0, J_0) \in BV(I)$ and for a constant $J_b \in \mathbb{R}$. On the function $k = k(x)$ we assume that

$$k \geq 0, \quad \int_I k(x) \, dx > 0$$

(1.3)

while for $g = g(J)$ we require that

$$g \in C^1(\mathbb{R}), \quad g(0) = 0, \quad g'(J) > 0 \quad \forall J.$$ 

(1.4)

Problem (1.1)–(1.2) is related to the one-dimensional damped semilinear wave equation on a bounded interval: if $(\rho, J)(x, t)$ is a solution to (1.1), (1.2), then the function

$$u(x, t) = J_b t - \int_0^x \rho(y, t) \, dy$$

satisfies $u_x = -\rho, u_t = J$ and

$$\partial_{tt} u - \partial_{xx} u + 2k(x)g(\partial_t u) = 0.$$ 

(1.5)

The equation (1.5) has been considered in several papers, see for instance [8] for linear damping term and fixed ends as boundary conditions ($u = 0$ at both ends, corresponding to $J_b = 0$); the review paper [18] and the more recent one [1], where sharp energy decay rates for a large class of nonlinearly first-order damped systems.

This paper aims at studying the asymptotic properties of the solutions to (1.1)–(1.2), naturally described by the stationary solutions to (1.1):

$$\begin{cases}
\partial_x J = 0, \\
\partial_x \rho = -2k(x)g(J).
\end{cases}$$

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The initial and boundary conditions (1.2) lead to a stationary solution \( (\tilde{J}, \tilde{\rho}) \):

\[
\tilde{J}(x) = J_b, \quad \tilde{\rho}(x) = -2g(J_b) \int_0^x k(y) \, dy + C,
\]

(1.6)

the constant \( C \) being uniquely identified by the condition

\[
\int_0^1 \tilde{\rho}(x) \, dx = \int_0^1 \rho_0(x) \, dx,
\]

that results in

\[
C = \int_0^1 \rho_0(x) \, dx + 2g(J_b) \int_0^1 k(y)(1 - y) \, dy.
\]

(1.7)

For the system (1.1), a class of approximations of Well-Balanced type to the Cauchy problem was studied in [11, 12] and in the papers [2, 3, 4]. In these last papers, suitable \( L^1 \) error estimates are derived by means of stability analysis for hyperbolic systems of conservation laws, obtained through a suitable adaptation of the Bressan-Liu-Yang functional [5, 6].

The same approach to define approximate solutions is adopted in this paper, for the initial-boundary values problem (1.1)–(1.2). We remark that these approximate solutions can be regarded as wave-front tracking solutions [5], with a special choice of the approximate initial data, having discontinuities uniformly distributed on a grid.

The analysis performed in this paper is, however, very different from the one for the Cauchy problem. Indeed, the semilinear character of system (1.1) and the presence of the (reflecting) boundary conditions lead us to analyze the problem under an unusual perspective: it can be recasted as the time evolution of the solutions to a finite dimensional linear system, as follows,

\[
\sigma(t^n+ - B(t^n) \sigma(t^n-+)) = B(t^n)B(t^{n-1}) \cdots B(0+)\sigma(0+),
\]

(1.8)

where \( \sigma(t^n) \) denotes a vector of wave sizes appearing in the approximate solution to (1.1), (1.2) at time \( t^n \), while \( B(t^n) \) is a doubly stochastic matrix (that is, a nonnegative matrix for which the sum of all the elements by row is 1, as well as by column) that in general depends on time. We refer the reader to Section 4 for more details on the derivation of (1.8) and on the structure of \( B(t^n) \). The behavior of the vector \( \sigma \) is controlled by the spectral properties of the matrix \( B \); whenever \( g \) is nonlinear (that is, \( B \) is not constant in time), then the behavior of (1.8) is not trivial and may require advanced matrix analysis’ tools, such as the concept of Joint Spectral Radius ([16, 13]). In this paper we will use a different approach, based on a explicit representations of the matrix product in (1.8) and on the construction of a suitable contracting norm (see Section 5).

We introduce the main result of this paper. We will refer to \( (\rho_{\Delta x}, J_{\Delta x})(x, t) \) as the approximate solution for (1.1)–(1.2) defined according to the algorithm in Section 3 with \( N \in 2\mathbb{N}, \Delta x = 1/N \).

**Theorem 1.1.** Let \( g \) satisfy (1.4) and \( k \) satisfy

\[
0 < k_1 \leq k(x) \leq k_2,
\]

(1.9)

for some \( k_1, k_2 > 0 \). Given \( (\rho_0, J_0) \in BV(I) \) and \( J_b \in \mathbb{R} \), let \( (\tilde{\rho}, \tilde{J}) \) be the stationary solution as in (1.6)–(1.7). Define

\[
d_1 = k_1 \min_{J \in D_J} g'(J) > 0, \quad d_2 = k_2 \max_{J \in D_J} g'(J)
\]

(1.10)

where \( D_J \) is a closed bounded interval depending on the data, which is invariant for \( J \). Finally assume that

\[
e^{2d_2} - 2d_2 < e^{2d_1}.
\]

(1.11)
Then there exist constant values $\hat{C}_j > 0$, $j = 1, \ldots, 5$ that depend only on the coefficients of the equation and on the initial and boundary data, such that

\[
\|J_{\Delta x}(\cdot, t) - \tilde{J}\|_{\infty} \leq \hat{C}_1 \Delta x + \hat{C}_2 e^{-\hat{C}_3 t}
\]

\[
\|\rho_{\Delta x}(\cdot, t) - \tilde{\rho}(\cdot)\|_{\infty} \leq \hat{C}_4 \Delta x + \hat{C}_5 e^{-\hat{C}_3 t}
\]

(1.12)

where $\hat{C}_3$ is given by

\[
\hat{C}_3 = \frac{1}{2} |\log C(d_1, d_2)|, \quad C(d_1, d_2) = e^{-2d_1(e^{2d_2} - 2d_2)}.
\]

**Remark 1.2.**
(i) From (1.10), it is clear that $d_1 \leq d_2$ and that for every $d_1 > 0$ there exists a non-empty interval of values for $d_2$ for which (1.11) holds.

(ii) If $k(x) \equiv \bar{k} > 0$ and $g'(J) \equiv \bar{C} > 0$ are constant, then $d_1 = d_2 = d$ and then (1.11) is satisfied for every $d = \bar{k}\bar{C} > 0$. Moreover one has

\[
\hat{C}_3 = \frac{1}{2} |\log(1 - 2de^{-2d})| \sim d \quad \text{as } d \to 0.
\]

(iii) For (1.11) to hold, it is necessary that $d_1 > 0$ and hence that $g' > 0$ as in (1.4). In general, if $g'$ vanishes at some point, an exponential decay is no longer expected; see for instance [14].

The paper is organized as follows. In Section 2 we recall some preliminaries on Riemann problems and interaction estimates, that are required in the next sections. In Section 3 we describe the WB scheme and in Section 4 we introduce the evolution problem (1.8) mentioned above, focusing on the spectral properties of the matrix $B$.

Finally, in the long Section 5 we prove Theorem 1.1 whose proof is outlined at the beginning of the Section. The proof is based on a probabilistic interpretation of the solution [17, 10] and on the spectral properties of the evolution problem in (1.8). We use Birkhoff decomposition theorem for doubly stochastic matrixes and prove an exponential-type formula in Theorem 5.6. Thanks to these tools, we are able to prove that a norm of the iterated matrix in (1.8) decays in time on a suitable subspace. The proof of Theorem 1.1 is then summarized at the end of Section 5.

### 2. Preliminaries

In terms of the diagonal variables $f^\pm$, defined by

\[
\rho = f^+ + f^-, \quad J = f^+ - f^-
\]

the system (1.1) is rewritten as a discrete-velocity kinetic model

\[
\begin{aligned}
\partial_t f^- - \partial_x f^- = &\ k(x) g(f^+ - f^-), \\
\partial_t f^+ + \partial_x f^+ = &\ -k(x) g(f^+ - f^-).
\end{aligned}
\]

(2.2)

Now we recall some preliminary results from [2] dealing with Riemann problems and interaction estimates for system (2.2). Our approach is based on an alternative formulation of system (1.1) that is obtained by adding an equation for the antiderivative of $k$:

\[
a = a(x) = \int_0^x k(y) \, dy,
\]

(2.3)

which by (1.3) satisfies

\[
a \in AC(\mathbb{R}), \quad a_x = k \geq 0, \quad TV a = a(1) - a(0) = \|k\|_{L^1} > 0.
\]
This leads to consider the following non-conservative homogeneous $3 \times 3$ system

\[
\begin{align*}
\partial_t \rho + \partial_x J &= 0, \\
\partial_t J + \partial_x \rho + 2g(J)\partial_x a &= 0, \\
\partial_t a &= 0,
\end{align*}
\] (2.4)

which in diagonal variables (2.1) is written as

\[
\begin{align*}
\partial_t f^- - \partial_x f^- - g(f^+ - f^-)\partial_x a &= 0, \\
\partial_t f^+ + \partial_x f^+ + g(f^+ - f^-)\partial_x a &= 0, \\
\partial_t a &= 0.
\end{align*}
\] (2.5)

Notice that the non-conservative product $g(J)\partial_x a$, which in principle is ambiguous across the discontinuities of $a(x)$, is well-defined since $J$ is constant along stationary solutions.

Systems (2.4), (2.5) are introduced in order to be able to set up the WB algorithm: this procedure consists in localizing a source term of bounded extent into a countable collection of Dirac masses in order to integrate it inside a Riemann solver by means of an elementary wave, which is obviously linearly degenerate. The characteristic speed of system (2.5) are determined by corresponding right eigenvectors $(0, 1, 0)^t$, $(1, 0, 0)^t$ and $(-g, -g, 1)^t$. We call 0-wave curves those characteristic curves corresponding to the speed 0.

We denote either by $(\rho_\ell, J_\ell, a_\ell)$, $(\rho_r, J_r, a_r)$ or by $(f^-_\ell, f^+_\ell, a_\ell)$, $(f^-_r, f^+_r, a_r)$ the left and right states corresponding to Riemann data.

**Proposition 2.1.** [2] Assume (1.4) and consider the initial states

\[
U_\ell = (\rho_\ell, J_\ell, a_\ell) = (f^-_\ell, f^+_\ell, a_\ell), \quad U_r = (\rho_r, J_r, a_r) = (f^-_r, f^+_r, a_r).
\]

Let $m < M$, $a_\ell \leq a_r$ and set $\delta = a_r - a_\ell \geq 0$.

(i) The solution to the Riemann problem for system (2.5) and initial data $U_\ell, U_r$ is uniquely determined by

\[
U(x, t) = \begin{cases} 
U_\ell & x/t < -1 \\
U_* = (\rho_{*,\ell}, J_*, a_\ell) & -1 < x/t < 0 \\
U_{**} = (\rho_{*,r}, J_*, a_r) & 0 < x/t < 1 \\
U_r & x/t > 1
\end{cases}
\] (2.6)

with

\[
J_* + g(J_*)\delta = f^+_\ell - f^-_\ell, \quad \rho_{*,r} - \rho_{*,\ell} = -2g(J_*)\delta,
\] (2.7)

see Figure [1].

(ii) The rectangle $[m, M]^2$ is an invariant domain for the Riemann problem projected on the $(f^-, f^+)$-plane. This means that if $(f^-_\ell, f^+_\ell), (f^-_r, f^+_r) \in [m, M]^2$, then the solution $U(x, t)$ given in (2.6) satisfies $(f^-, f^+)(x, t) \in [m, M]^2$. This property is independent on $\delta \geq 0$.

(iii) For every pair $U_\ell, U_r$ with $(f^-_\ell, f^+_\ell), (f^-_r, f^+_r) \in [m, M]^2$, let $\sigma_{-1} = (J_* - J_\ell)$ and $\sigma_1 = (J_r - J_*)$. Hence

\[
||\sigma_1| - |f^+_\ell| - f^-_\ell|| \leq C_0 \delta, \quad ||\sigma_{-1}| - |f^-_r - f^-_\ell|| \leq C_0 \delta,
\] (2.8)

where $C_0 = \max\{g(M - m), -g(m - M)\}$. In particular $C_0$ is independent of $\delta$.

Since the introduction of $a(x)$ yields a nonlinearity, we need to study the interactions of waves in the solutions to (2.5). In the notation of Figure [1] the amplitude of waves is defined as

\[
\delta = a_r - a_\ell
\]
Figure 1. The solution to the Riemann problem in Proposition 2.1.

for a 0-wave and

\[ \sigma_{-1} = J_r - J_\ell = -(f_r^+ - f_\ell^-) = -(\rho_r - \rho_\ell), \]

\[ \sigma_1 = J_r - J_* = f_r^+ - f_*^+ = \rho_r - \rho_* . \]

In other words, if we denote by \( \Delta \phi \) the difference \( \phi_r - \phi_\ell \) for a certain quantity \( \phi \), the sizes \( \sigma_{\pm 1} \) are given by

\[ \sigma_{\pm 1} = \Delta J = \pm \Delta f^\pm = \pm \Delta \rho . \] (2.9)

In particular, we have

\[ \sigma_1 + \sigma_{-1} = (J_r - J_*) + (J_* - J_\ell) = J_r - J_\ell . \] (2.10)

The following proposition refines the statement of [2] Proposition 3.

**Proposition 2.2** (Multiple interactions). Assume that at a time \( t > 0 \) an interaction involving a \((+1)\)-wave, a 0-wave and a \((-1)\)-wave occurs, see Figure 2. Let \( \sigma_{-1}, \sigma_1 \) be the sizes of the incoming waves and \( \sigma_{+1}, \sigma_1^+ \) be the sizes of the outgoing ones. Let \( \delta = a_r - a_\ell \geq 0 \) be the size of the 0-wave that remains constant across the interaction and assume that

\[ (\sup g')\delta < 1 . \] (2.11)

Then, for some \( s \) it holds

\[ \left( \frac{\sigma_{+1}^-}{\sigma_1} \right) = \left( \frac{1 - c}{c} \frac{c}{1 - c} \right) \left( \frac{\sigma_{-1}}{\sigma_1} \right), \]

\[ c = \frac{g'(s)\delta}{g'(s)\delta + 1} , \] (2.12)

otherwise written as

\[ \sigma_{+1}^- = (1 - c)\sigma_{-1}^- + c\sigma_1^- , \]

\[ \sigma_1^+ = (1 - c)\sigma_1^- + c\sigma_{-1}^- . \] (2.13)

Moreover,

\[ |\sigma_{+1}^-| + |\sigma_1^+| \leq |\sigma_{-1}^-| + |\sigma_1^-| , \] (2.14)

\[ |\sigma_{+1}^- - \sigma_1^+| \leq |\sigma_{-1}^- - \sigma_1^-| \cdot \frac{1 - \delta(\inf g')}{1 + \delta(\inf g')}. \] (2.15)

**Proof.** Let \( J_{-1}^-, J_1^+ \) be the intermediate values of \( J \) before and after the interaction, respectively. By (2.7) these values satisfy

\[ J_1^+ + g(J_1^+)\delta = f_\ell^+ - f_r^- , \quad J_{-1}^- - g(J_{-1}^-)\delta = f_r^+ - f_\ell^- . \]

Since the quantity \( J_r - J_\ell \) remains constant across the interaction, we get

\[ J_r - J_\ell = (J_r - J_{-1}^-) + (J_1^+ - J_\ell) = (J_r - J_{-1}^-) + (J_{-1}^- - J_\ell) . \]
Then, by the definition of sizes \( (\sigma_{\pm 1} = \Delta J) \) we deduce the following identity
\[
\sigma_1^+ + \sigma_1^- = \sigma_1^+ + \sigma_1^- .
\] (2.16)
The same procedure can be applied to \( \rho_r - \rho_\ell \): by (2.7) and the fact that \( \sigma_{\pm 1} = \pm \Delta \rho \), we find
\[
\sigma_1^+ - \sigma_1^- - 2g(J^+_s)\delta = \sigma_1^- - \sigma_1^- - 2g(J^-_s)\delta ,
\] that can be rewritten as
\[
\sigma_1^+ - \sigma_1^- = \sigma_1^- - \sigma_1^- + 2 \left[ g(J^+_s) - g(J^-_s) \right] \delta = \sigma_1^- - \sigma_1^- + 2g(s) \left[ J^+_s - J^-_s \right] \delta \tag{2.17}
\]
for some \( s \in (\min\{J^+_s, J^-_s\}, \max\{J^+_s, J^-_s\}) \). Notice that
\[
J^+_s - J^-_s = (J^+_s - J_r) + (J_r - J^-_s) = -\sigma_1^+ + \sigma_1^- 
\]
and, replacing \( J_r \) with \( J_\ell \), one has
\[
J^+_s - J^-_s = \sigma_1^+ - \sigma_1^- .
\]
Since both equations are true, then one can combine them and write
\[
J^+_s - J^-_s = \frac{1}{2} \left( \sigma_1^- - \sigma_1^- + \sigma_1^- - \sigma_1^- \right) .
\]
By substitution into (2.17), we get
\[
\sigma_1^+ - \sigma_1^- = \sigma_1^- - \sigma_1^- + g'(s) \left( \sigma_1^- - \sigma_1^- + \sigma_1^- - \sigma_1^- \right) \delta ,
\]
which leads to
\[
(1 + \delta g'(s)) \left( \sigma_1^+ - \sigma_1^- \right) = (1 - \delta g'(s)) \left( \sigma_1^- - \sigma_1^- \right) .
\]
In conclusion, recalling (2.16), we have the following 2 \( \times 2 \) linear system
\[
\begin{align*}
\sigma_1^+ + \sigma_1^- &= \sigma_1^- + \sigma_1^- \\
\sigma_1^+ - \sigma_1^- &= \frac{1 - g'(s)\delta}{1 + g'(s)\delta} \left( \sigma_1^- - \sigma_1^- \right) = (1 - 2c) \left( \sigma_1^- - \sigma_1^- \right) .
\end{align*}
\] (2.18)
whose solution is given by (2.13), or equivalently by (2.12).

As for the second part of the proposition, the inequality (2.14) follows directly from (2.13). In order to prove (2.15), from assumption (2.11) and therefore from (2.18) we find
\[
|\sigma_{1-}^1 - \sigma_{1+}^1| \leq \frac{1 - \delta (\inf g')}{1 + \delta (\inf g')} |\sigma_{1-}^- - \sigma_{1+}^-| .
\]
This concludes the proof of Proposition 2.2.
Remark 2.3. As a consequence of (2.12), we can easily check that:
– the strength of the waves $|\sigma_1| + |\sigma_{-1}|$ remains constant across the interaction when $\sigma_{-1}\sigma_1^{-} \geq 0$ that is when the incoming waves have the same sign;
– on the other hand it decreases strictly whenever $\sigma_{-1}\sigma_1^{-} < 0$, leading therefore to a cancellation in terms of the wave strengths.

3. Approximate solutions

In this section we construct Well-Balanced approximate solutions for the initial boundary value problem associated to system (2.4) (or equivalently (2.5)) and initial-boundary condition (1.2). By the change of variable around the stationary solution

\[ v = \rho - \tilde{\rho}, \quad w = J - \tilde{J}, \]

the system (1.1)–(1.2) is rewritten as

\[
\begin{align*}
\partial_t v + \partial_x w &= 0 \\
\partial_t w + \partial_x v &= -2k(x)\tilde{g}(w; \tilde{J}) + \tilde{g}(w; \tilde{J}) = g(\tilde{J} + w) - g(\tilde{J})
\end{align*}
\]  

(3.2)

where $w \mapsto \tilde{g}(w; \tilde{J})$ has the same properties of $g$ in (1.4), and

\[ w(t, 0) = w(t, 1) = 0, \quad \int_I v_0 \, dx = 0. \]

Hence we can assume that

\[ J_b = 0, \quad \int_I \rho_0(x) \, dx = 0. \]

Let $D$ be the invariant domain in the $(f^-, f^+)$-variables of Proposition 2.1-(b), that is

\[ D = [\inf I f^-_0, \sup I f^-_0] \times [\inf I f^+_0, \sup I f^+_0], \]

and let

\[ D_J = [J_{\min}, J_{\max}] \]

(3.3)

denote the closed interval which is the projection of $D$ on the $J$-axis.

The construction proceeds as in the case of the Cauchy problem (see for instance [2] p.607) and is organized into the following steps. See Figure 4 for a picture of the scheme for $N = 4$.

**Step 1: approximation of initial data and of $k(x)$.** Let $N \in 2\mathbb{N}$ be a positive, even number and set

\[ \Delta x = 1/N, \quad x_j = j\Delta x, \quad j = 0, \ldots, N. \]

The interval $(0, 1)$ is then divided into $N$ cells of length $\Delta x$, with $x_0 = 0$ and $x_N = 1$. We approximate the initial data $f_0^\pm$ and $a(x)$ as

\[ (f_0^\pm)_{\Delta x}(x) = f_0^\pm(x_j^+), \quad a_{\Delta x}(x) = a(x_j), \quad x \in (x_j, x_{j+1}). \]

(3.4)

The size of the 0-wave at a point $0 < x_j < 1$ is given by

\[ \delta_j = \Delta(a_{\Delta x})(x_j) = a(x_j) - a(x_{j-1}) = \int_{x_{j-1}}^{x_j} k(x) \, dx. \]

(3.5)

Clearly, we have

\[ \sum_{j=1}^{N-1} \delta_j = \int_0^{1-\Delta x} k(x) \, dx \to \|k\|_{L^1} \text{ as } \Delta x = \frac{1}{N} \to 0. \]

(3.6)
Knowing that $k \in L^1(I)$ and using the absolute continuity of the Lebesgue integral, we can assume $\Delta x = 1/N$ to be sufficiently small so that

$$C_1 \cdot \delta_j < 1, \quad C_1 = \sup g'(J), \quad j = 1, \ldots, N - 1,$$

(3.7)

where the supremum is taken over the values of $J$ in the invariant set $D_J$. In this way the assumption (2.11) of Proposition 2.2 is satisfied.

For later use, recalling that $\int \rho_0 \, dx = 0$ and that $\rho = f^+ + f^-$, we easily deduce the following inequality:

$$\left| \int_I (f^+_0) \Delta x + (f^-_0) \Delta x \, dx \right| \leq \Delta x \text{TV} \rho_0.$$

(3.8)

**Step 2: solution at $t > 0$, small $t$.** At $t = 0$ each Riemann problem that arises at $0 < x_j < 1$ is solved using Proposition 2.1. Moreover, at $x = 0$ and $x = 1$ we have to deal with two boundary Riemann problems. For instance, at $x = 0, t = 0$ one has to solve the problem with $(f^-_0, f^+_0)(0^+)$ as initial data and $J_b = 0$ as boundary datum. The solution consists of a single (+1)-wave and the intermediate state $(f^+_0, f^-_0)$ between $x = 0$ and the (+1)-wave is uniquely determined by

$$f^+_0 = f^-_0, \quad f^+_0 - f^-_0 = 0 \quad \Rightarrow \quad f^+_0 = f^-_0.$$

The size of the outgoing wave is given by

$$\sigma_1 = \Delta J = (f^+_0 - f^-_0) = J_0(0^+).$$

(3.9)

**Step 3: solution at $t > 0$, general $t$.** At $t = t^n = n \Delta t$ with $n \geq 1$, multiple interactions of waves occur at $0 < x_j < 1$ and the newly generated Riemann problems are again solved as in Proposition 2.1.

At $x = 0$, let $\sigma^-_1$ be the size of a (-1)-wave that hits the boundary. Clearly, on the left of this wave the boundary condition $J_b = 0$ is satisfied. Being $J_r$ the value of $J$ on the right of the incoming wave, its size $\sigma^-_1$ satisfies

$$\sigma^-_1 = \Delta J = J_r.$$

The boundary Riemann problem is solved as before and a new (+1)-wave is issued at the point $x = 0, t = t^n$. Since the boundary condition is still satisfied after the interaction, the size of the new wave will be equal to

$$\sigma^+_1 = \Delta J = J_r = \sigma^-_1.$$

(3.10)

Hence the total variation does not change under reflection of waves at the boundaries. See Figure 3 for a picture of this interaction.

**Remark 3.1.** Below we summarize some basic properties of these approximations.
Invariant domains. Under the previous construction, the approximate solution attains its values in the invariant domain \( D \) for every \((x,t)\) as well as the component \( J \) is in \( D_J \).

Stationary solutions, stationary approximations. Recalling (1.6), let \( \tilde{J}(x) = J_b \in \mathbb{R} \) and \( \tilde{\rho}(x) = C - 2g(J_b)a(x) \) be a stationary solution, for some constant \( C \in \mathbb{R} \). In order to be stationary, the approximate initial data (3.4) must satisfy the boundary condition \( J = J_b \) and the following relation at \( x_j, j = 1, \ldots, N - 1 \):

\[
(f_0^\pm)_{\Delta x}(x_j+) - (f_0^\pm)_{\Delta x}(x_j-) = -g(J_b) [a_{\Delta x}(x_j+) - a_{\Delta x}(x_j-)].
\]

Since \( f^\pm = (\rho \pm J)/2 \), it is easy to check that the identity above is valid:

\[
(f_0^\pm)_{\Delta x}(x_j+) - (f_0^\pm)_{\Delta x}(x_j-) = f_0^\pm(x_j+) - f_0^\pm(x_{j-1}+) = \frac{1}{2} (\tilde{\rho}(x_j+) - \tilde{\rho}(x_{j-1}+)) = -g(J_b) (a(x_j+) - a(x_{j-1}+)) = -g(J_b) [a_{\Delta x}(x_j+) - a_{\Delta x}(x_j-)].
\]

Uniform bounds on \( TV(f^\pm) \). We define

\[
L^\pm(t) = \sum_{(\pm 1)-waves} |\Delta f^\pm|, \quad (3.11)
\]

\[
L_0(t) = \frac{1}{2} \left( \sum_{0-waves} |\Delta f^+| + |\Delta f^-| \right), \quad (3.12)
\]

that by (2.9) are related to \( \rho \) and \( J \) as

\[
L^\pm(t) = TV J(\cdot, t), \quad L^\pm(t) + L_0(t) = TV \rho(\cdot, t).
\]

As in the case of the Cauchy problem [2], we claim that \( L^\pm(t) \) is not increasing in time. Indeed, at time \( t \notin \Delta t\mathbb{N} \), the quantity \( L^\pm(t) \) remains constant, while at \( t \in \Delta t\mathbb{N} \) either it decreases by (2.14) for interactions inside the domain or it does not change for interactions at the boundary. Hence, we obtain that \( L^\pm(t) \leq L^\pm(0+) \). Moreover, using (2.8) and (3.9), we have

\[
L^\pm(t) \leq L^\pm(0+)
\]

\[
\leq TV f^+(\cdot, 0) + TV f^-(\cdot, 0) + |J_0(0+)| + |J_0(1-)| + 2C_0 TV a,
\]

\[
L_0(t) = \sum_j |g(J_*(x_j))| \Delta a(x_j) \leq C_0 TV a.
\]

In conclusion,

\[
TV f^+(\cdot, t) + TV f^-(\cdot, t) = L^\pm(t) + 2L_0(t)
\]

\[
\leq TV f^+(\cdot, 0) + TV f^-(\cdot, 0) + |J_0(0+)| + |J_0(1-)| + 4C_0 \|k\|_{L^1}.
\]

4. The iteration matrix

In this section we describe our strategy to study the long-time behavior of the approximate solutions. Let

\[
\sigma(t) = (\sigma_1, \ldots, \sigma_{2N}) \in \mathbb{R}^{2N}, \quad N \in 2\mathbb{N}
\]
be the vector of the sizes of the waves which are present in the solution at time \( t \), ordered according to increasing space position, and denote their location by

\[
y_1(t) < y_2(t) < \ldots < y_{2N}(t) \quad \forall t > 0, \ t \neq t^n, \ t \neq t^{n+1/2}.
\]

To study the evolution in time of the vector \( \boldsymbol{\sigma} \), we make iterative use of Proposition 2.2. An important role is played by the transition coefficients \( c = c_j^n \) that appear in (2.12) and correspond to a single interaction at time \( t^n \) and \( x = x_j \), that is:

\[
c_j^n = \frac{g'(s^n_j) \delta_j}{g'(s^n_j) \delta_j + 1} \geq 0, \quad s^n_j \in D_J, \quad j = 1, \ldots, N - 1, \ n \geq 1,
\]

where \( \delta_j \) is given in (3.5), \( D_J \) in (3.3) and \( s^n_j \) depends on the solution. We define

\[
\boldsymbol{c} = \boldsymbol{c}^n = (c_1^n, \ldots, c_{N-1}^n) \in \mathbb{R}^{N-1}.
\]

In the following we will often drop the index \( n \) when the time \( t = t^n \) is fixed and write \( c_j \) in place of \( c_j^n \), so that we denote \( \boldsymbol{c} = (c_1, \ldots, c_{N-1}) \).

We remark that the map

\[
D_J^{N-1} \ni (J_1, \ldots, J_{N-1}) \mapsto \boldsymbol{c} = \left( \frac{g'(J_1) \delta_1}{g'(J_1) \delta_1 + 1}, \ldots, \frac{g'(J_{N-1}) \delta_{N-1}}{g'(J_{N-1}) \delta_{N-1} + 1} \right)
\]

is continuous over the compact set \( D_J^{N-1} \subset \mathbb{R}^{N-1} \), hence its image is a compact set \( K \subset \mathbb{R}^{N-1} \), which is the set of all the possible values of the vectors \( \boldsymbol{c} \).

By the smallness of \( \delta_j \) (see (3.5) and (3.7)) we have that

\[
\inf g' \frac{\delta_j}{2} \leq c_j^n \leq \min\{C_1 \delta_j, 1/2\}, \quad j = 1, \ldots, N - 1.
\]

Let us give an estimate on the \( \ell_1 \)-norm of \( \boldsymbol{c}^n \), being \( \|\boldsymbol{c}^n\|_1 = \sum_{j=1}^{N-1} c_j^n \). Recalling (3.6) and (4.3), we immediately get

\[
\inf g' \frac{\int_0^{1-\Delta x} k(x) \ dx}{2} \leq \|\boldsymbol{c}^n\|_1 \leq C_1 \|k\|_{L^1}.
\]

In the next lemma we relate the iteration step to a suitable transition matrix \( B \).

**Lemma 4.1.** At time \( t^n = n \Delta t \) the vector \( \boldsymbol{\sigma} \) evolves according to

\[
\boldsymbol{\sigma}(t^n+) = B(\boldsymbol{c}) \boldsymbol{\sigma}(t^{n-1}+), \quad n \geq 1
\]
where \( B(c) \in \mathbb{R}^{2N \times 2N} \) is
\[
B(c) = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 1 - c_1 & \cdots & 0 & 0 & 0 \\
1 - c_1 & 0 & 0 & c_1 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & c_{N-1} & 0 & 0 & 1 - c_{N-1} \\
0 & 0 & 0 & \cdots & 1 - c_{N-1} & 0 & 0 & c_{N-1} \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0
\end{bmatrix}
\]
which is doubly stochastic. The following properties hold:

(i) The determinant of \( B \) is
\[
\det(B) = - (1 - 2c_1) \cdots (1 - 2c_{N-1}) .
\]

(ii) The eigenvalues \( \lambda_i \) of \( B \) satisfy \(|\lambda_i| \leq 1\) for all \( i = 1, \ldots, 2N \);

(iii) The values \( \lambda = \pm 1 \) are eigenvalues with corresponding (left and right) eigenvectors
\[
\lambda_- = -1 , \quad v_- = (1, -1, -1, 1, \ldots, 1, -1, -1, 1), \\
\lambda_+ = 1 , \quad e = (1, 1, \ldots, 1, 1).
\]

(iv) If
\[
c_j \cdot c_{j+1} > 0 \quad \text{for some} \; j,
\]
that is, if there are two consecutive coefficients that do not vanish, then the eigenvalues with maximum modulus are exactly two \((\lambda = \pm 1)\) and they are simple.

Proof. The construction is divided into three steps.

1. At time \( t = (n - \frac{1}{2})\Delta t, \; n \geq 1 \), each pair of components \( \sigma_{2i-1} \) and \( \sigma_{2i} \) are switched, \( i = 1, \ldots, N \). In matrix form, one has the permutation
\[
\sigma(t+) = B_1 \sigma(t-) , \quad B_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & c_{N-1} & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 - c_{N-1} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{bmatrix}
\]

2. At time \( t = n\Delta t \), by (4.12) we have
\[
\sigma(t+) = B_2 \sigma(t-) , \quad B_2(c) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 - c_1 & \cdots & 0 & 0 & 0 \\
0 & c_1 & 1 - c_1 & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & c_{N-1} & 1 - c_{N-1} & 0 \\
0 & 0 & 0 & \cdots & 1 - c_{N-1} & c_{N-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{bmatrix}
\]

3. Finally we write
\[
B(c) = B_2(c) B_1
\]
and obtain (4.6).
Proof of (i). By the Binet Theorem we have
\[
\det(B) = \det(B_2) \det(B_1)
\]
where
\[
\det(B_1) = 1, \quad \det(B_2) = (2c_1 - 1) \cdots (2c_{N-1} - 1).
\]
Since \((N - 1)\) is odd, we obtain (4.7).

Proof of (ii) and (iii). By Gershgorin Theorem all the eigenvalues of the matrix \(B\) are located in the circle of center 0 and radius 1 in the complex plane. Indeed, all the terms on the diagonal are 0 and \(\sum_{j=1, i \neq j}^{2N} |B_{ij}| = 1, \quad \forall i.\)

Hence (ii) follows. About (iii) it is immediate to check that
\[
Bv_\mp = -v_\mp, \quad v_i^t B = -v_i^t
\]
while \(Be = e\) and \(e^t B = e^t\) follow by the double stochastic character of \(B\).

Proof of (iv). It remains to prove that \(\lambda_\pm\) are the only eigenvalues of \(B\) with modulus 1, while all the other have modulus < 1.

We claim that \(B\) satisfies the hypotheses of Romanovsky Theorem, see [15, p. 541]. The latter result states that a nonnegative irreducible matrix \(A \in M_n(\mathbb{R})\) has exactly \(p \in \mathbb{N}\) eigenvalues with maximum modulus if, for any node of the corresponding directed graph, \(p\) is the greatest common divisor of the lengths of all the directed paths that both start and end at a same node.

See Figure 5 for a picture of the graph related to the matrix \(B = [B_{ij}]_{i,j=1,...,2N}\), where each node correspond to a row \(i\) and each directed arc \((i,j)\) corresponds to a non-zero element \(B_{ij}\). Remark that the graph of \(B\) can be deduced by noticing that the first row is represented by the arc \((1,2)\), the last row by the arc \((2N,2N-1)\) and that each \(2 \times 4\) submatrix occupying the block of rows \(2j,2j+1\) and columns \(2j-1,\ldots,2j+2\),
\[
\hat{B}_j = \begin{bmatrix}
c_j & 0 & 0 & 1 - c_j \\
1 - c_j & 0 & 0 & c_j
\end{bmatrix}
\]
j = 1, \ldots, N - 1,

corresponds to a squared subgraph made of the arcs \((2j,2j-1), (2j,2j+2), (2j+1,2j-1), (2j+1,2j+2)\). Notice that, if \(c_j = 0\), then only the upper arc \((2j,2j+2)\) and the lower one \((2j+1,2j-1)\) survive in the squared subgraph related to \(\hat{B}_j\). The whole graph is then obtained by juxtaposing the arcs \((1,2), (2N,2N-1)\) to the subgraphs representing \(\hat{B}_j\), for \(j = 1, \ldots, N - 1\).

First, notice that \(B\) is irreducible, which is equivalent to say that the graph is totally connected, namely that each node can be reached from any other node via a path made of arcs.
present in the graph: this holds true since one can always follow the circuit $(1, 2, 4, \ldots, 2j, 2j + 2, \ldots, 2N, 2N - 1, \ldots, 2j + 1, 2j - 1, \ldots, 3, 1)$ from any node in the graph. Secondly, the length of any path in the graph connecting a node to itself can be divided at most by 2, which means that in this case $p = 2$. Indeed, there is no way to obtain a path of odd length because there are no diagonal arcs. Moreover, by assumption there exists an index $j$ such that $c_j, c_{j+1}$ are not zero as in Figure 5.

Then, it is easy to see that there are at least two paths connecting the node 1 to itself of lengths $2j$ and $2j + 2$ and the greatest common divisor must be 2.

Now, by the Romanovsky Theorem we can conclude that $\lambda_{\pm}$ are the only two eigenvalues with modulus 1 and the proof of (iv) is complete.

**Remark 4.2.** Notice that in general $B_2$ depends on $t^n$, since the coefficients $c_j$ depend on $g'(J)$. However, the structure of the matrix $B$ (the coefficients which are $\neq 0$) does not change with $n$, in the sense that, for a fixed $j$, either $c_j^n \neq 0$ for every $n$ or $c_j^n = 0$ for every $n$.

It is well known that doubly stochastic matrices can be written as a convex combination of permutations by Birkhoff Theorem ([15 Theorem 8.7.2]). In the next proposition, for $c$ constant we give an explicit Birkhoff decomposition of the matrix $B(c)$.

**Proposition 4.3.** Let $c = c(1, \ldots, 1) \in \mathbb{R}^{N-1}$, for some constant $c \in [0, 1/2)$. Then the matrix $B$ can be decomposed as

$$B(c) = (1 - c)B(0) + cB_1.$$  \hfill (4.13)

**Proof.** Since $c$ is constant, then the matrix $B_2(c)$ in (4.11) can be written as

$$B_2(c) = (1 - c)B_2(0) + cI.$$  \hfill (4.14)

Recalling that $B(c) = B_2(c)B_1$ and substituting (4.14), we obtain (4.13). \hfill $\Box$

**Remark 4.4.** Assume that (1.9) holds, that is $0 < k_1 \leq k(x) \leq k_2$ for some positive $k_1, k_2$. Hence, see (3.5), $\delta_j$ is bounded as

$$\frac{k_1}{N} \leq \delta_j \leq \frac{k_2}{N}.$$  \hfill (4.15)

Let us define $d_1, d_2$ as in (1.10). By the monotonicity of the map $x \to \frac{x}{x+1}$, the bounds in (4.15) become:

$$\frac{d_1/N}{1 + d_1/N} \leq c_j^n \leq \frac{d_2/N}{1 + d_2/N}.$$  \hfill (4.15)

Hence

$$B(c^n) \leq \left(1 - \frac{d_1/N}{1 + d_1/N}\right)B(0) + \frac{d_2/N}{1 + d_2/N}B_1,$$

and after simple passages, it is rewritten as

$$B(c^n) \leq \left(1 + \frac{d_1}{N}\right)^{-1}\left[B(0) + \frac{d_2}{N}B_1\right].$$  \hfill (4.16)

Note that the inequality in (4.16) means entrywise inequality.
5. Long time behaviour of the approximate solutions

In this section we study the behaviour of $\sigma(t^n)$ as $n \to +\infty$ (i.e. as $t \to +\infty$) and as $N \to \infty$ ($\Delta x \to 0$). The main results are listed here below, each item corresponding to a subsection.

(1) Proposition 5.1 relates the $L^\infty$-norm of $J(\cdot, t^n)$, $\rho(\cdot, t^n)$ as $n \to \infty$ to the evolution of the $\ell_1$-norm of the operator $B_n$

$$B_n = \begin{bmatrix} B^{(n)} B^{(n-1)} \cdots B^{(2)} B^{(1)} \end{bmatrix}, \quad B^{(j)} = B(e^j), \quad n \in \mathbb{N}$$

(5.1)

on the eigenspace $E_\perp = \begin{bmatrix} e, v^- \end{bmatrix}^\perp$.

(2) Lemma 5.3 concerns a convenient decomposition of the vectors in $E_-$, along which a suitable cancellation occurs later on.

(3) In Theorem 5.6, the exponential formula $[B(0) + d N^{-1} B_1]^2N \in M_{2N}$ is estimated in terms of $d$ and $N$, the difficulty lying in the fact that the matrices $B(0)$ and $B_1$ do not commute. The proof relies on a detailed study of the expansion of the power whose coefficients are described by hypergeometric functions, and their sum is computed through modified Bessel functions.

Thanks to a careful expression of the first order in $1/N$, a cancellation property is identified (see Proposition 5.8). As a result, it is found that the $\|B_{2N}\|_1 < 1$ on $E_-$, where

$$\|A\|_1 = \max_j\sum_{i=1}^n |a_{ij}|, \quad A = (a_{ij}) \in M_n$$

is the maximum column sum matrix norm, which is induced by the $\ell_1$-norm on $\mathbb{R}^n$.

(4) Finally, in Subsection 5.4 we combine the previous results and prove Theorem 1.1 starting from the inequality (4.16) which is obtained by a Birkhoff decomposition of the generic matrix $B(c)$.

5.1. A first decomposition of the strength vector

We decompose the initial vector $\sigma(0+)$ as follows:

$$\sigma(0+) = \left(\frac{\sigma(0+) \cdot e}{2N}\right)e + \left(\frac{\sigma(0+) \cdot v^-}{2N}\right)v^- + \tilde{\sigma}(0+),$$

where $e$, $v^-$ are the eigenvectors defined at (4.8) and $\tilde{\sigma}(0+) \in E_-$. As a consequence of the boundary conditions $J(1-, t) = J(0+, t) = 0$, we get

$$\sigma(0+) \cdot e = \sum_{j=1}^{2N} \sigma^0_j = \sum \Delta J(x_j, 0) = J(1-, 0+)-J(0+, 0+) = 0.$$ 

Hence the decomposition of $\sigma(0+)$ reduces to

$$\sigma(0+) = \left(\frac{\sigma(0+) \cdot v^-}{2N}\right)v^- + \tilde{\sigma}(0+).$$

(5.3)

Consider the matrix $B_n$, defined at (5.1), obtained by iterating the step (4.5). By means of (4.8) and using again (4.8) for $v^-$, we get that

$$\sigma(t^n) = B_n \sigma(0+) = (-1)^n \left(\frac{\sigma(0+) \cdot v^-}{2N}\right)v^- + B_n \tilde{\sigma}(0+).$$

(5.4)

In the following proposition we employ (5.4) to obtain $L^\infty$-bounds on $J = J^{\Delta x}$, $\rho = \rho^{\Delta x}$. First, let us define the extended initial data $\bar{J}_0 : [0,1] \to \mathbb{R}$,

$$\bar{J}_0(x) = \begin{cases} J_0(x) & 0 < x < 1 \\ 0 & x = 0 \text{ or } 1. \end{cases}$$

(5.5)
It is clear that $TV \tilde{J}_0 = TV \{ \tilde{J}_0; [0, 1] \} = |J_0(0+)| + TV \{ J_0; (0, 1) \} + |J_0(1-)|$.

**Proposition 5.1.** For every $t \in (t^n, t^{n+1})$ one has

$$\| J(\cdot, t) \|_\infty \leq \frac{1}{2N} TV \tilde{J}_0 + \| B_n \tilde{\sigma}(0+) \|_{\ell^1}$$

(5.6)

$$\| \rho(\cdot, t) \|_\infty \leq \frac{2}{N} (1 + C_1 \| k \|_{L^1}) TV \tilde{J}_0 + 2 (1 + 2C_1 \| k \|_{L^1}) \| B_n \tilde{\sigma}(0+) \|_{\ell^1} + \frac{1}{N} TV \rho_0.$$  

(5.7)

**Proof.** We start by observing that the following inequality holds,

$$|\sigma(0+) \cdot v_-| \leq TV \tilde{J}_0.$$  

(5.8)

Indeed, by recalling the definition of $v_-$ in (4.8), we observe that

$$\sigma(0+) \cdot v_- = \sigma_1^0 + \sum_{j=1}^{N-1} (-1)^j (\sigma_{2j}^0 + \sigma_{2j+1}^0) + \sigma_{2N}^0.$$  

Recalling that $\sigma_{2j}^0, \sigma_{2j+1}^0$ are the two outgoing waves at $x_j = j \Delta x$ and time $t = 0$, then by (2.10) it holds

$$\sigma_{2j}^0 + \sigma_{2j+1}^0 = J(x_j+, 0) - J(x_j-, 0).$$  

Moreover, since the approximate solution satisfies the boundary conditions $J = 0$, for small $t$ we have

$$\sigma_1^0 = J(x_1-, 0) - J(0+, t) = J(x_1-, 0) = J(0+, 0), \quad \sigma_{2N}^0 = -J(1-, 0).$$  

Therefore,

$$\sigma(0+) \cdot v_- = J(x_1-, 0) + \sum_{j=1}^{N-1} (-1)^j (J(x_j+, 0) - J(x_j-, 0)) - J(x_{N-1}+, 0)$$  

(5.9)

and then, by recalling (3.4), we find that

$$|\sigma(0+) \cdot v_-| \leq |J_0(0+)| + TV \tilde{J}_0 + |J_0(1-)|$$

that gives (5.8).

**Proof of (5.6).** Let $y_\ell(t)$ denote the location of a $\pm 1$-wave at time $t$, for $\ell = 0, \ldots, 2N$. Observe that, for every $x \neq y_\ell$, the value of $J(x, t^n +)$ is expressed by a partial sum of the $\sigma^n_\ell$:

$$J(x, t^n +) = J(0+, t^n +) + \sum_{y_\ell < x} \Delta J(y_\ell, t^n +) = \sum_{y_\ell < x} \sigma^n_\ell = \sigma(t^n +) \cdot v$$

where

$$v = (v_1, \ldots, v_{2N}) \in \mathbb{R}^{2N}, \quad v_\ell = \begin{cases} 1 & \text{if } y_\ell < x \\ 0 & \text{if } y_\ell > x. \end{cases}$$  

(5.10)

By (5.4) we obtain

$$\sigma(t^n +) \cdot v = (-1)^n \frac{1}{2N} (\sigma(0+) \cdot v_-)(v_\cdot \cdot v) + B_n \tilde{\sigma}(0+) \cdot v.$$  

(5.11)

Recalling the definition of (4.8), observe that $v_- \cdot v \in \{ \pm 1, 0 \}$ and hence

$$|J(x, t^n +)| = |\sigma(t^n +) \cdot v|$$

$$\leq \frac{1}{2N} |\sigma(0+) \cdot v_-| + |B_n \tilde{\sigma}(0+) \cdot v|$$

$$\leq \frac{1}{2N} TV \tilde{J}_0 + \| B_n \tilde{\sigma}(0+) \|_{\ell^1}$$
where (5.8) is used and an $\ell_1 - \ell_\infty$ estimate is used for $B_n \tilde{\sigma}(0+) \cdot v$.

To complete the proof of (5.6), it remains to bound the values of $J$ at times $t \in (t^n + \Delta t/2, t^{n+1})$, since it may change due to the linear interaction of the waves. Recalling (3.10), we have

$$\sigma(t^{n+1} -) = B_1 \sigma(t^n +) = (-1)^n \frac{1}{2N} (\sigma(0+) \cdot v_-) B_1 v_- + B_1 B_n \tilde{\sigma}(0+)$$

with $B_1 v_- = -v_-$. By proceeding as before, we obtain

$$|J(x, t^{n+1})| = |\sigma(t^{n+1} -) \cdot v| \leq \frac{1}{2N} TV \tilde{J}_0 + \|B_1 B_n \tilde{\sigma}(0+)\|_{\ell_1}$$

where it is used that multiplication by $B_1$ leaves unaltered the $\ell_1$ norm (being a permutation matrix). Therefore, (5.6) is completely proved.

**Proof of (5.7).** For $x < x_j = j \Delta x$ and $x \neq y_{\ell}$, we have

$$\rho(x, t^n +) = \rho(0+, t^n +) + \sum_{y_{\ell} < x} \Delta \rho(y_{\ell}, t^n +) + \sum_{x_j < x} \Delta \rho(x_j, t^n +)$$

Recalling (3.8), we have

$$\left| \int_0^1 \rho(x, t^n +) \, dx \right| = \left| \int_0^1 \rho(x, 0) \, dx \right| \leq \Delta x TV \rho_0,$$

then

$$|\rho(0+, t^n +)| \leq \left| \int_0^1 [\rho(0+, t^n +) - \rho(x, t^n +)] \, dx \right| + \Delta x TV \rho_0$$

$$\leq \sup_x \left| \sum_{y_{\ell} < x} \Delta \rho(y_{\ell}, t^n +) \right| + \sup_x \left| \sum_{x_j < x} \Delta \rho(x_j, t^n +) \right| + \Delta x TV \rho_0$$

and hence

$$|\rho(x, t^n +)| \leq 2 \sup_x \left| \sum_{y_{\ell} < x} \Delta \rho(y_{\ell}, t^n +) \right| + 2 \sup_x \left| \sum_{x_j < x} \Delta \rho(x_j, t^n +) \right| + \Delta x TV \rho_0.$$

**• Estimate on (A).** Recalling that $\Delta \rho(y_{\ell}) = \pm \sigma_{\pm 1}$, we proceed similarly to (5.11):

$$\sum_{y_{\ell} < x} (\pm \sigma_{\pm 1}) = \sigma(t^n +) \cdot \tilde{v}$$

$$= (-1)^n \frac{1}{2N} (\sigma(0+) \cdot v_-) (v_- \cdot \tilde{v}) + B_n \tilde{\sigma}(0+) \cdot \tilde{v}$$

where $\tilde{v} = (v_1, \ldots, v_{2N}) \in \mathbb{R}^{2N}$,

$$v_{\ell} = \begin{cases} 1 & \text{if } y_{\ell} < x \text{ and } \ell \text{ odd} \\ -1 & \text{if } y_{\ell} < x \text{ and } \ell \text{ even} \\ 0 & \text{if } y_{\ell} > x. \end{cases}$$

Hence $|v_- \cdot \tilde{v}| \leq 2$ and then, by using (5.8), we get:

$$|(A)| = \left| \sum_{y_{\ell} < x} (\pm \sigma_{\pm 1}) \right| \leq \frac{1}{N} |\sigma(0+) \cdot v_-| + \|B_n \tilde{\sigma}(0+)\|_{\ell_1}$$

$$\leq \frac{1}{N} TV \tilde{J}_0 + \|B_n \tilde{\sigma}(0+)\|_{\ell_1}. $$
Estimate on (B). Recalling that $\Delta \rho(x_j) = -2g(J(x_j))\delta_j$, we have

$$
(B) = 2 \left| \sum_{x_j < x} g(J(x_j, t^n +))\delta_j \right| \leq 2C_1 \max_j |J(x_j, t^n +)| \cdot \left( \sum_{j=1}^{N-1} \delta_j \right) 
\leq 2C_1 \|k\|_{L^1} \left( \frac{1}{2N} \text{TV} \tilde{J}_0 + \|B_n\tilde{\sigma}(0+))\|_{\ell_1} \right).
$$

In conclusion, for every $x \in (0,1)$ we find that

$$
|\rho(x, t^n +)| \leq 2\Delta x (1 + C_1\|k\|_{L^1}) \text{TV} \tilde{J}_0 
+ 2 (1 + 2C_1\|k\|_{L^1}) \|B_n\tilde{\sigma}(0+))\|_{\ell_1} + \Delta x \text{TV} \rho_0
$$

which is (5.7) for $t \in (t^n, t^n + \Delta t/2)$. The estimate for $t \in (t^n + \Delta t/2, t^{n+1})$ is done similarly as the one for $J$.

Remark 5.2. (On the total variation of $J$). We remark that the total variation of $J_{\Delta x}$, being

$$
\text{TV} J_{\Delta x} (\cdot, t) = \|\sigma(t)\|_{\ell_1},
$$

does not necessarily vanish at $t \to \infty$. Indeed, from (5.4) it follows that

$$
\|\sigma(t^n+)\|_{\ell_1} \geq \frac{1}{2N} \|\sigma(0+) \cdot v_-\|_{\ell_1} \|\nu_{1, \Delta x} \|_{\ell_1} 
\geq |\sigma(0+) \cdot v_-| \|\nu_{1, \Delta x} \|_{\ell_1}
$$

where it is used that $\|v_-\|_{\ell_1} = 2N$ (see the definition of $v_-$ at (4.8)). By means of (5.9), and using the notation

$$
J_\ell = J(x_{\ell-1}+, 0) = J(x_\ell-, 0) = J_0(x_{\ell-1}+) \quad \ell = 1, \ldots, N
$$

we have

$$
|\sigma(0+) \cdot v_-| = \left| J_1 - J_N + \sum_{\ell=1}^{N-1} (-1)^\ell (J_{\ell+1} - J_\ell) \right|
\leq 2 \left| J_1 - J_N + \sum_{\ell=2}^{N-1} (-1)^{\ell-1} J_\ell \right|
\leq 2 \sum_{\ell=1}^{N/2} (J_{2\ell-1} - J_{2\ell})
$$

If the initial datum $J_0(x)$ is strictly monotone, then

$$
|\sigma(0+) \cdot v_-| = 2 |J_N - J_1| \to 2 |J_0(1-) - J_0(0+)| = 2 \text{TV} J_0 > 0, \quad N \to \infty.
$$

About the second term in the sum, when $c$ is constant in time we have $B_n = B(c)^n$ and

$$
\|B^n\tilde{\sigma}(0+)\|_{\ell_1} \to 0 \quad \text{as} \quad n \to +\infty
$$

since $\tilde{\sigma}(0+)$ belongs to the subspace $E_- = \langle e, v_- \rangle^\perp$ corresponding to the eigenvalues with modulus $< 1$. Therefore $\text{TV} J(\cdot, t)$ does not tend to zero as $t \to +\infty$ for $J_0$ strictly monotone, and the limit is uniformly positive as $\Delta x = 1/N \to 0$.

However, in (1.12), it will turn out that the $L^\infty$-norm of $J$ is of order $\Delta x$ for large $t$.

5.2. A refined decomposition of the strength vector

In this subsection we focus on the analysis of $\|B_n\tilde{\sigma}(0+)\|_{\ell_1}$. In particular we analyze the sequence $\{B_n\tilde{\sigma}\}_{n \in \mathbb{N}}$ whenever $\tilde{\sigma}$ belongs to the subspace $E_- = \langle e, v_- \rangle^\perp$. 

Let $N \in 2\mathbb{N}$ and consider $\tilde{\sigma} \in E_-$. By definition (1.8) of $e_\perp$, then $\tilde{\sigma}$ satisfies
\[
\begin{cases}
\tilde{\sigma}_1 + \tilde{\sigma}_2 + \cdots + \tilde{\sigma}_{2N} = 0,
\tilde{\sigma}_1 - \tilde{\sigma}_2 - \tilde{\sigma}_3 + \tilde{\sigma}_4 + \tilde{\sigma}_5 - \cdots + \tilde{\sigma}_{2N} = 0,
\end{cases}
\]
which is equivalent to
\[
\begin{cases}
\tilde{\sigma}_1 + \tilde{\sigma}_4 + \cdots + \tilde{\sigma}_{2N-3} + \tilde{\sigma}_{2N} = 0,
\tilde{\sigma}_2 + \tilde{\sigma}_3 + \cdots + \tilde{\sigma}_{2N-2} + \tilde{\sigma}_{2N-1} = 0.
\end{cases}
\]
We introduce the following subspaces in $\mathbb{R}^{2N}$, each of dimension $N - 1$:
\[
\begin{align*}
H_1' &= \{(x_1, \ldots, x_{2N}) \in \mathbb{R}^{2N} : x_1 + x_4 + \cdots + x_{2N-3} + x_{2N} = 0\}, \\
H_2' &= \{(x_1, \ldots, x_{2N}) \in \mathbb{R}^{2N} : x_2 + x_3 + \cdots + x_{2N-2} + x_{2N-1} = 0\}.
\end{align*}
\]
Hence we can write
\[
\tilde{\sigma} = \tilde{\sigma}' + \tilde{\sigma}'', \\
\tilde{\sigma}' \in H_1, \\
\tilde{\sigma}'' \in H_2.
\] (5.12)

Notice that, since $H_1$ and $H_2$ are complementary, we have
\[
\|\tilde{\sigma}\|_{\ell_1} = \|\tilde{\sigma}' + \tilde{\sigma}''\|_{\ell_1} = \|\tilde{\sigma}'\|_{\ell_1} + \|\tilde{\sigma}''\|_{\ell_1}.
\] (5.13)

For later use, we define the following sets of indices
\[
I' = \{1, 4, 5, 8, \ldots, 2N - 3, 2N\}, \\
I'' = \{2, 3, 6, 7, \ldots, 2N - 2, 2N - 1\}.
\] (5.14)

Let us define the vectors $v_{ij} \in \mathbb{R}^{2N}$ for $i, j$ either in $I'$ or in $I''$ as follows,
\[
(v_{ij})_i = 1, \\
(v_{ij})_j = -1, \\
(v_{ij})_k = 0 \quad \forall k \neq i, j.
\] (5.15)

Remark that $\tilde{\sigma}'$ and $\tilde{\sigma}''$ can be written as a linear combination of suitable $v_{ij}$’s, i.e. we can identify $\beta_{ij}', \beta_{ij}'' \in \mathbb{R}$ such that
\[
\tilde{\sigma}' = \sum_{i,j \in I'} \beta_{ij}' v_{ij}, \\
\tilde{\sigma}'' = \sum_{i,j \in I''} \beta_{ij}'' v_{ij}.
\] (5.16)

By the triangular inequality, one has that
\[
\|\tilde{\sigma}'\|_{\ell_1} \leq \sum_{ij} |\beta_{ij}'| \|v_{ij}\|_{\ell_1} = 2 \sum_{ij} |\beta_{ij}'|, \\
\|\tilde{\sigma}''\|_{\ell_1} \leq \sum_{ij} |\beta_{ij}''|.
\]

In the next Lemma we prove that, for a suitable choice of the decomposition, the sum above can be made an equality.

**Lemma 5.3.**
(i) There exists a choice of the vectors $v_{ij}$ such that (5.16) holds together with
\[
\|\tilde{\sigma}'\|_{\ell_1} = 2 \sum_{ij} |\beta_{ij}'|, \\
\|\tilde{\sigma}''\|_{\ell_1} = 2 \sum_{ij} |\beta_{ij}''|.
\] (5.17)

(ii) The following estimate holds,
\[
\|B_n \tilde{\sigma}\|_{\ell_1} \leq \sup_{i,j} \|B_n \frac{v_{ij}}{\|v_{ij}\|_{\ell_1}}\| \|\tilde{\sigma}\|_{\ell_1}, \quad \forall \tilde{\sigma} \in E_-.
\] (5.19)

**Proof.** We start with (i), it suffices to prove (5.17), since (5.18) is analogous.
First, we have to find a suitable linear decomposition of $\tilde{\sigma}'(0^+)$ in a basis of vectors of the form $v_{ij}$, with $i, j \in \mathcal{I}'$. By construction we have

$$\tilde{\sigma}' = (\tilde{\sigma}'_1, 0, 0, \tilde{\sigma}'_4, \tilde{\sigma}'_5, 0, \ldots, 0, \tilde{\sigma}'_{2N-3}, 0, 0, \tilde{\sigma}'_{2N})^t,$$

i.e. the components corresponding to indices in $\mathcal{I}''$ are zero. Therefore, we can simplify the notation and in place of $\tilde{\sigma}'$ consider

$$x = (x_1, x_2, \ldots, x_N) = (\tilde{\sigma}'_1, \tilde{\sigma}'_4, \ldots, \tilde{\sigma}'_{2N}) \in \mathbb{R}^N,$$

the vector obtained erasing from $\tilde{\sigma}'$ the zero components and satisfying $x_1 + x_2 + \cdots + x_N = 0$.

Below we describe an algorithm to decompose $x$ along a basis of $v_{ij}$’s, for $i, j \in \mathcal{I}'$.

**Step 1.** Let $x \neq 0$. Hence there exists a pair of indices $k_1, h_1 \in \{1, \ldots, N\}$ such that

$$x_{k_1} \cdot x_{h_1} < 0, \quad 0 < |x_{k_1}| = \min_{k=1, \ldots, N: x_k \neq 0} |x_k|.$$ 

In particular one has that $|x_{h_1}| \geq |x_{k_1}|$.

**Step 2.** Define the vector

$$x^{(1)} = x - x_{k_1}v_{k_1} \in \mathbb{R}^N,$$

and notice that it satisfies

$$(x^{(1)})_k = \begin{cases} 0 & k = k_1 \\ x_{h_1} + x_{k_1} & k = h_1 \\ x_k & k \neq k_1, h_1. \end{cases}$$

In particular,

$$|(x^{(1)})_{h_1}| = |x_{h_1}| - |x_{k_1}| \geq 0$$

and hence

$$\|x^{(1)}\|_{\ell_1} = \|x\|_{\ell_1} - 2|x_{k_1}| < \|x\|_{\ell_1}.$$

**Step 3.** We apply the same procedure to $x^{(1)}$, namely we choose suitable indexes $k_2, h_2 \in \{1, \ldots, N\}$ such that

$$(x^{(1)})_{k_2} \cdot (x^{(1)})_{h_2} < 0, \quad 0 < |(x^{(1)})_{k_2}| = \min_{k=1, \ldots, N: (x^{(1)})_{k} \neq 0} |(x^{(1)})_k|.$$ 

Notice that, since $(x^{(1)})_{k_1} = 0$, one has that $k_2, h_2$ are different from $k_1$. Moreover one has $|(x^{(1)})_{h_2}| \geq |(x^{(1)})_{k_2}|$.

As in Step 2, we define

$$x^{(2)} = x^{(1)} - (x^{(1)})_{k_2}v_{k_2} \in \mathbb{R}^N,$$

that is

$$(x^{(2)})_k = \begin{cases} 0 & k = k_2 \\ (x^{(1)})_{h_2} + (x^{(1)})_{k_2} & k = h_2 \\ (x^{(1)})_k & k \neq k_2, h_2. \end{cases}$$

Notice that

$$(x^{(2)})_k = 0 \quad \text{for} \ k = k_1, k_2$$

and that

$$|(x^{(2)})_{h_2}| = |(x^{(1)})_{h_2}| - |(x^{(1)})_{k_2}| \geq 0.$$
5.3. Linear damping

In this subsection we consider the special case when \( c \) is constant in space and time (which is the case if \( k \) is constant and \( g \) is linear) and hence \( B(c) \) does not depend on time. This means that the product of the matrices in (5.1) reduces to the \( n^{th} \) power of \( B(c) \). In particular we focus on the structure of the power for \( n = 2N \), since we can exploit the fact that the permutation \( B(0)^{2N} \) is the identity.

Assume that
\[
k(x) = \bar{k} > 0 \quad \forall x \in (0, 1), \quad g'(J) = \text{const.} = C_1
\]
and set
\[ d = \bar{k} C_1, \quad \gamma = \frac{d}{N}. \]  
(5.21)

By Proposition 4.3 and Birkhoff Theorem, the matrix \( B(c) \) can be written as
\[ B(c) = (1 - c)B(0) + cB_1 = (1 - c) \left[ B(0) + \frac{c}{1 - c} B_1 \right], \]
where \( c = c(1, 1, \ldots, 1) \in \mathbb{R}^{N-1} \) and
\[ c = \frac{\gamma}{\gamma + 1}, \quad \frac{c}{1 - c} = \gamma = \frac{d}{N}. \]

Hence
\[ B(c)^{2N} = (1 - c)^{2N} [B(0) + \gamma B_1]^{2N} \]  
(5.22)

It is clear that
\[ (1 - c)^{2N} = \left( 1 + \frac{d}{N} \right)^{-2N} \to e^{-2d} , \quad N \to \infty. \]

Let us focus on the second factor in (5.22), that is
\[ [B(0) + \gamma B_1]^{2N} = \sum_{k=0}^{2N} \gamma^k S_k(B(0), B_1), \]  
(5.23)

where each term \( S_k(B(0), B_1) \) is the sum of all products of 2\( N \) matrices which are either \( B_1 \) or \( B(0) \), and in which \( B_1 \) appears exactly \( k \) times, that is
\[ S_k(B(0), B_1) = \sum_{(\ell_1, \ldots, \ell_{k+1})} B(0)^{\ell_1} \cdot B_1 \cdot B(0)^{\ell_2} \cdot B_1 \cdots B(0)^{\ell_k} \cdot B_1 \cdot B(0)^{\ell_{k+1}} \]
\[ 0 \leq \ell_j \leq 2N - k, \quad \sum_{j=1}^{k+1} \ell_j = 2N - k. \]  
(5.24)

In what follows we use extensively the fact that \( B_1^2 = I_{2N} = B(0)^{2N} \) and the commutation property described in next proposition.

**PROPOSITION 5.4.** The following identity holds for any \( \ell \in \mathbb{N} \):
\[ B(0)^{\pm \ell} B_1 = B_1 B(0)^{\mp \ell}. \]  
(5.25)

**Proof.** Recalling (4.10)–(4.12), we have that \( B(0)^{-1} = (B_2(0)B_1)^{-1} = B_1 B_2(0) \). Then for every \( \ell \geq 0 \) we have
\[ B(0)^{-\ell} B_1 = (B_1 B_2(0) \cdots (B_1 B_2(0))) \cdot B_1 \]
\[ = B_1 \cdot (B_2(0) B_1) \cdots (B_2(0) B_1) \]
\[ = B_1 \cdot B(0)^{\ell}. \]

As for the identity for \( +\ell \), notice that
\[ B(0)^{\ell} B_1 = B(0)^{2N - (2N - \ell)} B_1 = B(0)^{2N} B(0)^{-2N - \ell} B_1 = B(0)^{-2N - \ell} B_1, \]
where we used that $B(0)^{2N} = I_{2N}$. Hence, by the first identity we get
\[ B(0)^{\ell} B_1 = B_1 \cdot B(0)^{2N-\ell} = B_1 \cdot B(0)^{-\ell}.\]

By means of \((5.25)\) and using that $B_1^2 = I_{2N}$, the generic term in the sum $S_k$ in \((5.24)\) can be conveniently rewritten. Indeed, one has $S_0 = S_{2N} = I_{2N}$. For $k = 1, \ldots, 2N - 1$, we have to distinguish the case of even/odd $k$.

- For $k$ even, we have
  \[ B(0)^{\ell_1} \cdot B_1 \cdot B(0)^{\ell_2} \cdot B_1 \cdots B(0)^{\ell_k} \cdot B_1 \cdot B(0)^{\ell_{k+1}} = B(0)^{\alpha-\beta}, \quad (5.26) \]
  where
  \[ \alpha = \sum_{j=1, j \text{ odd}}^{k+1} \ell_j, \quad \beta = \sum_{j=2, j \text{ even}}^{k+1} \ell_j = 2N - k - \alpha. \quad (5.27) \]
  Now let us count how many vectors $(\ell_1, \ldots, \ell_{k+1})$ lead, thanks to \((5.26)\), to the same matrix $B(0)^{\alpha-\beta} = B(0)^{2\alpha+k}$.

In the first sum of \((5.27)\) the indices are $k/2 + 1$, while in the second sum they are $k/2$. Hence, for a given $\alpha$, the number of the distinct vectors $(\ell_1, \ldots, \ell_{k+1})$ for which \((5.27)\) holds is
\[ \left( \frac{\alpha + \frac{k}{2}}{k/2} \right) \left( 2N - \alpha - \frac{1 - \frac{k}{2}}{k - 1} \right), \quad \alpha = 0, \ldots, 2N - k. \]

If we perform a change of variable $j = \alpha + k/2$, we get
\[ \left( \frac{j}{k/2} \right) \left( 2N - j - 1 \right), \quad j = \frac{k}{2}, \ldots, 2N - \frac{k}{2}, \]
and
\[ S_k(B(0), B_1) = \sum_{j=\frac{k}{2}}^{2N-\frac{k}{2}} \left( \frac{j}{k/2} \right) \left( 2N - j - 1 \right) B(0)^{2j}, \quad k = 2, 4, \ldots, 2N. \quad (5.28) \]

- For $k$ odd, we have
  \[ B(0)^{\ell_1} \cdot B_1 \cdot B(0)^{\ell_2} \cdot B_1 \cdots B(0)^{\ell_k} \cdot B_1 \cdot B(0)^{\ell_{k+1}} = B(0)^{\alpha-\beta} B_1 \]
  \[ = B(0)^{2\alpha+k} B_1 \]
  \[ = B(0)^{2\alpha+k-1} B_2(0), \]
where $\alpha, \beta = 2N - k - \alpha$ are given in \((5.27)\).

Here, the number of vectors $(\ell_1, \ldots, \ell_{k+1})$ for which \((5.24)\) are counted as follows. The indices $\ell_j$ are in total $(k + 1)/2$ for both sums, hence for a given $\alpha$ the number of terms is
\[ \left( \frac{\alpha + \frac{k-1}{2}}{k-1/2} \right) \left( 2N - \alpha - \frac{k-1}{2} - 1 \right), \quad \alpha = 0, \ldots, 2N - k. \]

\[ ^\dagger \text{Given } M \geq 0 \text{ and } a_j \geq 0 \text{ integers such that } \sum_{j=1}^{n} a_j = M, \text{ the number of distinct } (a_1, \ldots, a_n) \text{ is } \left( \begin{array}{c} M+n-1 \\ n-1 \end{array} \right) = \left( \begin{array}{c} M+n-1 \\ M \end{array} \right). \]
If we perform a change of variable $j = \alpha + \frac{k-1}{2}$, we get
$$
\left(\frac{j}{k-1}\right)\left(\frac{2N - j - 1}{k-1}\right), \quad j = \frac{k-1}{2}, \ldots, 2N - \frac{k+1}{2}.
$$

Hence,
$$
S_k(B(0), B_1) = \sum_{j=\frac{k-1}{2}}^{2N - \frac{k+1}{2}} \left(\frac{j}{k-1}\right)\left(\frac{2N - j - 1}{k-1}\right) B(0)^{2j} B_2(0) \quad k = 1, 3, \ldots, 2N - 1. \tag{5.29}
$$

The next proposition gives an explicit formula for the sum of the powers of $B(0)$.

**PROPOSITION 5.5.** Let $\hat{P}$ be the matrix defined by
$$
\hat{P} = \frac{1}{2} (ee^t + v_-v_-^t),
$$
which is the matrix composed by $N^2/4$ squared blocks as
$$
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}.
$$

Then, the following identity holds:
$$
\sum_{j=0}^{N-1} B(0)^{2j} = \sum_{j=1}^{N} B(0)^{2j} = \hat{P}. \tag{5.31}
$$

**Proof.** The first equality in (5.31) follows from the following identity:
$$
(I_{2N} - B(0)^2) \left(\sum_{j=0}^{N-1} B(0)^{2j}\right) = 0.
$$

Indeed,
$$
(I_{2N} - B(0)^2) \left(\sum_{j=0}^{N-1} B(0)^{2j}\right) = \left(\sum_{j=0}^{N-1} B(0)^{2j}\right) - \left(\sum_{j=1}^{N} B(0)^{2j}\right) = I_{2N} - B(0)^{2N} = 0.
$$

To prove the second identity in (5.31), observe that the matrix $B(0)^2$ contains the following two separated "cycles" of length $N$,
$$
1 \rightarrow 5 \rightarrow 9 \rightarrow \ldots \rightarrow 2N - 3 \rightarrow 2N - 4 \rightarrow \ldots \rightarrow 4 \rightarrow 1 \\
2 \rightarrow 3 \rightarrow 7 \rightarrow \ldots \rightarrow 2N - 1 \rightarrow 2N - 2 \rightarrow 2N - 6 \rightarrow \ldots \rightarrow 6 \rightarrow 2.
$$

In the first, second case the indexes are exactly the ones in $I'$, $I''$ respectively.

By summing all the permutations $B(0)^2$, $B(0)^{2N} = I_{2N}$ one obtains that every $i^{th}$ row, with $i \in I'$, has value $=1$ exactly at every index $i \in I'$ and value $=0$ otherwise. The same holds for every $i^{th}$ row with $i \in I''$. Hence (5.31) holds.

The next theorem provides an estimate on the components of $B(c)^{2N}$ in terms of $d$, $N$. 

**Theorem 5.6.** Let $N \in 2\mathbb{N}$. The following bound holds true:

$$
\left[ B(0) + \frac{d}{N} B_1 \right]^{2N} = I_{2N} + \frac{2d}{N} \tilde{P} + \sum_{j=0}^{2N-1} \zeta_{j,N} B(0)^{2j} B_2(0) + \sum_{j=1}^{2N-1} \eta_{j,N} B(0)^{2j}
$$

(5.32)

where

$$
0 \leq \sum_{j=0}^{2N-1} \zeta_{j,N} \leq \sinh(2d) - 2d + \frac{1}{N} f_0(d)
$$

(5.33)

and

$$
0 \leq \sum_{j=1}^{2N-1} \eta_{j,N} \leq \cosh(2d) - 1 + \frac{1}{N} f_1(d),
$$

(5.34)

where

$$
f_0(d) = \sum_{\ell=1}^{\infty} \frac{d^{2\ell+1}}{(\ell)!^2} = d[I_0(2d) - 1]
$$

(5.35)

$$
f_1(d) = \sum_{k=1}^{\infty} \frac{d^{2k}}{k! (k-1)!} = dI_1(2d),
$$

(5.36)

is a modified Bessel function of the first type.

**Proof.** From the identity (5.23) we have

$$
[B(0) + \gamma B_1]^{2N} = I_{2N} + \left( \sum_{k=1 \, \text{odd}}^{2N-1} \gamma^k S_k(B(0), B_1) \frac{m^2}{m! (m+\alpha)!} \right.\left. + \sum_{k=2 \, \text{even}}^{2N} \gamma^k S_k(B(0), B_1) \right)
$$

(5.37)

First, let us focus on the sum with $k$ odd in (5.37). By (5.29), we substitute the expression for $S_k$ and exchange the sum in $k$ and $j$ to get

$$
\sum_{k=1 \, \text{odd}}^{2N-1} \gamma^k S_k(B(0), B_1) = \sum_{j=0}^{2N-1} \tilde{\zeta}_{j,N} B(0)^{2j} B_2(0),
$$

(5.38)

where

$$
\tilde{\zeta}_{j,N} = \min\{2j+1, 4N-2j-1\} \sum_{k=1 \, \text{odd}}^{2N-1} \gamma^k \binom{j}{k-1} \binom{2N-j-1}{k-1}
$$

$$
= \min\{j, 2N-j-1\} \sum_{\ell=0}^{j} \gamma^{2\ell+1} \binom{j}{\ell} \binom{2N-j-1}{\ell - \ell}
$$

(5.39)

It is convenient to separate, in the expression of $\tilde{\zeta}_{j,N}$, the term with $\ell = 0$ and the sum for $\ell \geq 1$, since the former does not depend on $j, N$:

$$
\tilde{\zeta}_{j,N} = \gamma + \zeta_{j,N}, \quad \zeta_{j,N} = \sum_{\ell=1}^{\min\{j, 2N-j-1\}} \gamma^{2\ell+1} \binom{j}{\ell} \binom{2N-j-1}{\ell - \ell}
$$
Next we provide an estimate on the coefficients $\zeta_{j,N}$. Using the inequality
\[
\binom{n}{k} \leq \frac{n^k}{k!}, \quad 0 \leq k \leq n
\]
and the definition $\gamma = d/N$, we find that
\[
\zeta_{j,N} \leq \sum_{\ell=1}^{\infty} \frac{d^{2\ell+1} j^\ell (2N-j-1)^\ell}{(\ell!)^2 N^{\ell+1}}. \tag{5.40}
\]
Now we introduce another change of variable,
\[
x_j = -1 + \frac{j}{N}, \quad \frac{j}{N} = (1 + x_j), \quad j = 0, \ldots, 2N - 1. \tag{5.41}
\]
Thanks to the inequality (5.40) we get
\[
0 \leq \zeta_{j,N} \leq \sum_{\ell=1}^{\infty} \frac{d^{2\ell+1} (1 + x_j)^\ell}{(\ell!)^2} \left(1 - x_j - \frac{1}{N}\right)^\ell \leq \sum_{\ell=1}^{\infty} \frac{d^{2\ell+1} (1 - x_j^2)^\ell}{(\ell!)^2}.
\]
As a consequence, we deduce an estimate for the sum of the $\zeta_{j,N}$:
\[
0 \leq \sum_{j=0}^{2N-1} \zeta_{j,N} \leq \frac{1}{N} \sum_{j=0}^{2N-1} \sum_{\ell=1}^{\infty} \frac{d^{2\ell+1} (1 - x_j^2)^\ell}{(\ell!)^2} = \sum_{\ell=1}^{\infty} \frac{d^{2\ell+1} (1 - x_j^2)^\ell}{(\ell!)^2} \left\{ \frac{1}{N} \sum_{j=0}^{2N-1} (1 - x_j^2)^\ell \right\}
\]
where we used that $\Delta x = 1/N$. Using the definition (5.41) we notice that
\[
\frac{1}{N} \sum_{j=0}^{2N-1} (1 - x_j^2)^\ell \to \int_{-1}^{1} (1 - x^2)^\ell \, dx \quad \text{as} \quad N \to \infty, \quad \ell \geq 1;
\]
more precisely the following estimate holds,
\[
\sum_{j=0}^{2N-1} (1 - x_j^2)^\ell \Delta x = \left( \sum_{j=0}^{N-1} + \sum_{j=N+1}^{2N-1} \right) (1 - x_j^2)^\ell \Delta x + \Delta x
\leq \int_{-1}^{1} (1 - x^2)^\ell \, dx + \Delta x. \tag{5.42}
\]
Since $(1 + 2\ell)! = (1 + 2\ell)!! \cdot 2^\ell \cdot \ell!$, it is easy to check the following identities
\[
\int_{-1}^{1} (1 - x^2)^\ell \, dx = \frac{2^{\ell+1} \cdot \ell!}{(1 + 2\ell)!!} = \frac{2^{2\ell+1} \cdot (\ell!)^2}{(1 + 2\ell)!} \quad \ell \geq 1. \tag{5.43}
\]
By plugging the previous estimates into the sum of the $\zeta_{j,N}$ we get
\[
0 \leq \sum_{j=0}^{2N-1} \zeta_{j,N} \leq \sum_{\ell=1}^{\infty} \frac{d^{2\ell+1} 2^{2\ell+1} \cdot (\ell!)^2}{(\ell!)^2 \cdot (1 + 2\ell)!} + \Delta x \sum_{\ell=1}^{\infty} \frac{d^{2\ell+1}}{\ell!} \equiv f_0(d)
= \sum_{\ell=1}^{\infty} \frac{(2d)^{2\ell+1}}{(1 + 2\ell)!} + \Delta x f_0(d)
= \sinh(2d) - 2d + \Delta x f_0(d).
\]
Therefore (5.33) follows.

Analogously we treat the sum with \( k \) even in (5.37). By (5.28) we can exchange the sum in \( k \) and \( j \), hence we rewrite this term as

\[
\sum_{k=2}^{2N} \gamma^k S_k(B(0), B_1) = \sum_{j=1}^{2N-1} \eta_{j,N} B(0)^{2j},
\]

(5.44)

where we set

\[
\eta_{j,N} = \sum_{k=2}^{\min\{2j, 4N-2j\}} \gamma^k \left( \frac{j}{2} \right) \left( \frac{2N-j}{2} \right).
\]

Similarly to the estimate (5.40) for \( \zeta_{j,N} \) and using the change of variables (5.41), we find that

\[
\eta_{j,N} \leq \frac{1}{N} \sum_{h=1}^{\infty} \frac{d^{2h}}{h!(h-1)!} \left( 1 + x_j \right)^h \left( 1 - x_j - \frac{1}{N} \right)^{h-1}.
\]

The sum of the \( \eta_{j,N} \) can be estimated as follows,

\[
\sum_{j=1}^{2N-1} \eta_{j,N} \leq \sum_{h=1}^{\infty} \frac{d^{2h}}{h!(h-1)!} \left\{ \frac{1}{N} \sum_{j=1}^{2N-1} \left( 1 - x_j^2 \right)^{h-1} (1 + x_j) \right\}.
\]

By definition of the (5.41) and symmetry we have

\[
\sum_{j=1}^{2N-1} (1 - x_j^2)^{h-1} x_j = 0,
\]

while by (5.42) with \( \ell = h - 1 \) and by (5.43) we find that

\[
\frac{1}{N} \sum_{j=0}^{2N-1} (1 - x_j^2)^{h-1} \leq \int_{-1}^{1} (1 - x^2)^{h-1} dx + \frac{1}{N} = \frac{2^{2h-1} \cdot ((h-1)!)^2}{(2h-1)!} + \frac{1}{N}.
\]

Therefore

\[
\sum_{j=1}^{2N-1} \eta_{j,N} \leq \sum_{h=1}^{\infty} \frac{d^{2h}}{h!(h-1)!} \left( \frac{2^{2h-1} \cdot ((h-1)!)^2}{(2h-1)!} + \frac{1}{N} \right) \sum_{h=1}^{\infty} \frac{d^{2h}}{h!(h-1)!} = f_1(d).
\]
where

For every $d > 0$ and hence, from (5.38), we obtain:

\[
\text{that leads to (5.34).}
\]

Remark 5.7. For $a \in \mathbb{R}$ and $n \geq 0$, $n$ integer, we introduce the notation (rising Pochhammer symbol):

\[
(a)_n = \begin{cases} 1 & n = 0 \\ a(a+1) \cdots (a+n-1) & n \geq 1. \end{cases}
\]

(5.45)

With this notation we can write (1) as follows, (5.39) can be rewritten as follows, and it is clear that the above quantity vanishes for $\ell > \min\{j, 2N - j - 1\}$. Therefore the coefficients $\tilde{\zeta}_{j,N}$ is rewritten as

\[
\zeta_{j,N} = \sum_{\ell=1}^{\infty} \frac{\gamma^{2\ell+1}}{\ell!} \frac{(-j)_{\ell}(-2N+j+1)_{\ell}}{(1)_{\ell}}.
\]

(5.46)

The coefficients $\tilde{\zeta}_{j,N}$ in (5.38) can be rewritten in terms of the hypergeometric function,

\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n,
\]

where $a, b, c \in \mathbb{R}$.

In conclusion we have

\[
\tilde{\zeta}_{j,N} = \gamma \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad a, b, c \in \mathbb{R}
\]

and hence, from (5.38), we obtain:

\[
\sum_{k=1}^{2N-1} \gamma^k S_k = \gamma \sum_{j=0}^{2N-1} \frac{2F_1(-j, -2N + j + 1, 1; \gamma^2)B(0)^2jB_2(0)}{j!}
\]

Next, we want to prove a contractive estimate for $\|B(c)^{2N}v_{ij}\|_{\ell_1}$. We recall that here $c = c(1, \ldots, 1) \in \mathbb{R}^{N-1}$ with $c = d/N$ for some $d > 0$.

Proposition 5.8. For $N \in 2\mathbb{N}$, let $i, j$ be indices both either $\in I'$ or $\in I''$ (see (5.14)), For every $d > 0$ there is a constant $0 < C_N(d) < 1$ such that

\[
\|B(c)^{2N}v_{ij}\|_{\ell_1} < C_N(d)\|v_{ij}\|_{\ell_1},
\]

(5.47)

where

\[
C_N(d) \to (1 - 2de^{-2d}) < 1, \quad N \to \infty.
\]

(5.48)

Proof. Notice that

\[
B(c)^{2N}v_{ij} = B(c)^{2N}e_i - B(c)^{2N}e_j = B(c)^{2N}[i] - B(c)^{2N}[j],
\]

where $e_i, e_j$ are vectors of the canonical basis of $\mathbb{R}^{2N}$ and $B(c)^{2N}[i], B(c)^{2N}[j]$ denote the $i$-th and $j$-th column of the matrix $B(c)^{2N}$. Hence, $\|B(c)^{2N}v_{ij}\|_{\ell_1}$ corresponds to the distance between two columns of $B(c)^{2N}$ indicized by either $i, j \in I'$ or $\in I''$. 
Assume that $i, j \in \mathcal{T}'$, the other case being completely similar. We use the expression (5.22) for $B(c)^{2N}$ and Theorem 5.6 to get

$$\|B(c)^{2N}[i] - B(c)^{2N}[j]\|_{\ell_i} = \left(1 + \frac{d}{N}\right)^{-2N} \sum_{\ell=1}^{2N} |b_{\ell i} - b_{\ell j}|,$$

where $b_{\ell i}$ denotes the generic element of the matrix $B(c)^{2N}$ and where $b_{\ell i}, b_{\ell j} = 0$ if $\ell \notin \mathcal{T}'$.

A key observation is that, by applying formula (5.32) and recalling the definition (5.30) of $\hat{P}$, the contribution from the term $2dN\hat{P}$ is zero because

$$\hat{P}[i] - \hat{P}[j] = 0 \in \mathbb{R}^{2N}, \quad i, j \in \mathcal{T}'$$

(The same property holds if $i, j \in \mathcal{T}''$). Therefore

$$\sum_{\ell=1}^{2N} |b_{\ell i} - b_{\ell j}| \leq |b_{i i} - b_{i j}| + |b_{j i} - b_{j j}| + \sum_{\ell \neq i, j}^{2N} |b_{\ell i} - b_{\ell j}|$$

$$\leq 2 \left(1 + \sum_{j=0}^{2N-1} \zeta_{j,N} + \sum_{j=1}^{2N-1} \eta_{j,N}\right)$$

$$\leq 2 \left(\sinh(2d) - 2d + \frac{1}{N}f_0(d) + \cosh(2d) + \frac{1}{N}f_1(d)\right)$$

$$= \|v_{ij}\|_{\ell_i}\left[e^{2d} - 2d + \frac{1}{N}[f_0(d) + f_1(d)]\right].$$

By denoting

$$C_N(d) = \left(1 + \frac{d}{N}\right)^{-2N}\left[e^{2d} - 2d + \frac{1}{N}[f_0(d) + f_1(d)]\right],$$

we easily get that $C_N(d) \to (1 - 2de^{-2d})$ as $N \to \infty$, and this completes the proof of Prop. 5.8.

5.4. Nonlinear damping

In this subsection we prove Theorem 1.11.

Assume that (1.9) holds, that is $0 < k_1 \leq k(x) \leq k_2$ for some positive $k_1, k_2$ and recall the definition of $0 < d_1 \leq d_2$ given in (1.10). We study the behavior of

$$B_{2N} = \begin{bmatrix} B^{(2N)} & B^{(2N-1)} & \cdots & B^{(2)} \end{bmatrix}.$$ 

By the inequality (4.16) we have

$$B(c^n) \leq \left(1 + \frac{d_1}{N}\right)^{-1}\left[B(0) + \frac{d_2}{N}B_1\right], \quad \forall n,$$

and then

$$B_{2N} \leq \left(1 + \frac{d_1}{N}\right)^{-2N}\left[B(0) + \frac{d_2}{N}B_1\right]^{2N}. \quad (5.49)$$

**Proposition 5.9.** There exists a constant $C_N(d_1, d_2)$ such that as $N \to \infty$

$$C_N(d_1, d_2) \to e^{-2d_1}(e^{2d_2} - 2d_2) \equiv C(d_1, d_2) \quad (5.50)$$

and that for $i, j$ indices fixed either $\in \mathcal{T}'$ or $\in \mathcal{T}''$ it holds

$$\|B_{2N}v_{ij}\|_{\ell_i} \leq C_N(d_1, d_2)\|v_{ij}\|_{\ell_i}.$$
In particular, if \( d_1 \) and \( d_2 \) satisfy (1.11), then \( C_N(d_1, d_2) < 1 \) for \( N \) large enough.

**Proof.** From (5.49), one can estimate the term \( [\tilde{B}(0) + \frac{d_1}{N} B_1]^{2N} \) on the right hand side as in the proof of Theorem 5.6. Then as in the proof of Proposition 5.8, the conclusion follows easily with

\[
C_N(d_1, d_2) = \left( 1 + \frac{d_1}{N} \right)^{-2N} \left[ e^{2d_2} - 2d_2 + \frac{1}{N} [f_0(d_2) + f_1(d_2)] \right].
\]

\[\square\]

**Proof of Theorem 1.1.** To prove (1.12) in Theorem 1.1 we will employ the main results in this Section, namely Proposition 5.1, Lemma 5.3, Theorem 5.6 and Proposition 5.9. About the estimate for \( J \), we proceed as follows.

- We start from (5.6), that is

\[
\|J_{\Delta x}(\cdot, t)\|_\infty \leq \frac{1}{2N} TV \tilde{J}_0 + \|B_n \tilde{\sigma}(0+)\|_{\ell_1}.
\]

- Let \( n \in \mathbb{N}, 0 \leq h \in \mathbb{N} \) and \( 2Nh \leq n < 2N(h + 1) \), so that

\[
2h \leq \frac{n}{N} = n \Delta t = t^n < 2(h + 1), \quad h \geq 0.
\]

Since \( E_- \) is an invariant subspace for all \( B^{(j)} \), we have

\[
\tilde{\sigma}(t^n) = B_n \tilde{\sigma}(0+) \in E_- \quad \forall n.
\]

Hence by Proposition 5.9 and using that \( \|B^{(j)} v\|_{\ell_1} \leq \|v\|_{\ell_1} \) for all \( v \in \mathbb{R}^{2N} \), the following holds

\[
\|\tilde{\sigma}(t^n)\|_{\ell_1} = \|B_n \tilde{\sigma}(0+)\|_{\ell_1} \leq \|B_{2Nh} \tilde{\sigma}(0+)\|_{\ell_1} = \|B_{2N} \tilde{\sigma}(0+)\|_{\ell_1} \leq C_N \|\tilde{\sigma}(0+)\|_{\ell_1}.
\]

Let \( \delta > 0 \) satisfy \( [C - \delta, C + \delta] \subset (0, 1) \), and choose \( N \) be large enough so that \( C_N(d_1, d_2) \in [C - \delta, C + \delta] \). One can easily get

\[
|C_N(d_1, d_2) - C(d_1, d_2)| \leq \frac{1}{N} \left( 1 + \frac{d_1}{N} \right)^{-2N} \left[ f_0(d_2) + f_1(d_2) \right] + (e^{2d_2} - 2d_2) \cdot e^{-2d_1} \left( 1 + \frac{d_1}{N} \right)^2 - 1
\]

\[
\leq \frac{1}{N} \tilde{C}(d_1, d_2)
\]

for a suitable constant \( \tilde{C}(d_1, d_2) > 0 \). Therefore one has

\[
|C_N^h - C^h| \leq |C_N - C| \cdot h |\xi|^{h-1}, \quad \forall h \geq 1,
\]

for some \( \xi \in [C - \delta, C + \delta] \subset (0, 1) \). Since the quantity \( h |\xi|^{h-1} \) is uniformly bounded for \( h \geq 1 \) and \( \xi \in [C - \delta, C + \delta] \), then we deduce that for some \( \tilde{C}_0 > 0 \) one has

\[
\|B_n \tilde{\sigma}(0+)\|_{\ell_1} \leq \left( C^h + \frac{\tilde{C}_0}{N} \right) \|\tilde{\sigma}(0+)\|_{\ell_1}
\]

where \( n, N, h \) satisfy (5.51).
• From (5.3) we have that
\[
\tilde{\sigma}(0+) = \sigma(0+) - \frac{(\sigma(0+) \cdot v_-)}{2N} v_-, 
\]
and then
\[
\|\tilde{\sigma}(0+)\|_{\ell_1} \leq \|\sigma(0+)\|_{\ell_1} + \frac{\|\sigma(0+)\|_{\ell_1}}{2N} 2N = 2\|\sigma(0+)\|_{\ell_1}. 
\]
Moreover, using (2.8) and (3.9), we have
\[
\|\tilde{\sigma}(0+)\|_{\ell_1} \leq TV \rho_0 + TV \tilde{J}_0 + 2C_0 \|k\|_{L^1}, 
\]
where \(\tilde{J}_0\) is defined at (5.5). Therefore it holds, for \(h \leq \frac{n^2}{2} \leq (h + 1)\):
\[
\|E_h \tilde{\sigma}(0+)\|_{\ell_1} \leq 2 \left( C^h + \frac{\tilde{C}_0}{N} \right) \left( TV \rho_0 + TV \tilde{J}_0 + 2C_0 \|k\|_{L^1} \right). 
\]
Using the relation (5.51) for \(h, n\) and \(N\), we have
\[
C^h \leq C^h \leq C^h = \frac{1}{C} e^{-|\log C|((\frac{4}{N})^h)}. 
\]
In conclusion we get
\[
\|J_{\Delta x}(\cdot, t^n)\|_{\infty} \leq \frac{1}{2N} \left\{ TV \tilde{J}_0 + 4\tilde{C}_0 \left( TV \rho_0 + TV \tilde{J}_0 + 2C_0 \|k\|_{L^1} \right) \right\}
\]
\[
+ \frac{2}{C} e^{-|\log C|((\frac{4}{N})^h)} \left( TV \rho_0 + TV \tilde{J}_0 + 2C_0 \|k\|_{L^1} \right) 
\]
that leads to the first inequality in (1.12) for suitable constants \(\tilde{C}_j\) which are independent of \(\Delta x\) and \(t\). The constant \(\tilde{C}_3\) is given by
\[
\tilde{C}_3 = \frac{1}{2} \log C(d_1, d_2) \quad C(d_1, d_2) = e^{-2d_1(e^{2d_2} - 2d_2)}. 
\]
Starting from (5.7), the second inequality in (1.12), for the \(\rho\) variable, is obtained in a similar way.

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