SEMIGROUPS GENERATED BY ELLIPTIC OPERATORS IN NON-DIVERGENCE FORM ON $C_0(\Omega)$

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the uniform exterior cone condition. Let $A$ be a uniformly elliptic operator given by

$$A u = \sum_{i,j=1}^{n} a_{ij} \partial_{ij} u + \sum_{j=1}^{n} b_j \partial_j u + cu$$

where $a_{ij} \in C(\bar{\Omega})$ and $b_j, c \in L^\infty(\Omega), c \leq 0$.

We show that the realization $A_0$ of $A$ in

$$C_0(\Omega) := \{ u \in C(\bar{\Omega}) : u_{|\partial\Omega} = 0 \}$$

given by

$$D(A_0) := \{ u \in C_0(\Omega) \cap W^{2,n}_{loc}(\Omega) : Au \in C_0(\Omega) \}$$

$$A_0 u := Au$$

generates a bounded holomorphic $C_0$-semigroup on $C_0(\Omega)$. The result is in particular true if $\Omega$ is a Lipschitz domain. So far the best known result seems to be the case where $\Omega$ has $C^2$-boundary [Lun95, Section 3.1.5]. We also study the elliptic problem

$$-Au = f$$

$$u_{|\partial\Omega} = g .$$

0. Introduction

The aim of this paper is to study elliptic and parabolic problems for operators in non-divergence form with continuous second order coefficients and to prove the existence (and uniqueness) of solutions which are continuous up to the boundary of the domain. Throughout this paper $\Omega$ is a bounded open set in $\mathbb{R}^n, n \geq 2$, with boundary $\partial\Omega$. We consider the operator $A$ given by

$$A u := \sum_{i,j=1}^{n} a_{ij} \partial_{ij} u + \sum_{j=1}^{n} b_j \partial_j u + cu$$
with real-valued coefficients $a_{ij}, b_j, c$ satisfying

$$b_j \in L^\infty(\Omega), \ j = 1, \ldots, n, \ c \in L^\infty(\Omega), \ c \leq 0$$

$$a_{ij} \in C(\overline{\Omega}), \ a_{ij} = a_{ji},$$

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \Lambda|\xi|^2 \quad (x \in \overline{\Omega}, \xi \in \mathbb{R}^n)$$

where $\Lambda > 0$ is a fixed constant.

Our best results are obtained under the hypothesis that $\Omega$ satisfies the uniform exterior cone condition (and thus in particular if $\Omega$ has Lipschitz boundary). Then we show that for each $f \in L^n(\Omega), g \in C(\partial\Omega)$ there exists a unique $u \in C(\overline{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega)$ such that

$$\left\{ \begin{array}{ll}
-Au &= f \\
\nu_{|\partial\Omega} &= g
\end{array} \right.$$  

(Corollary 2.3). This result is proved with the help of Alexandrov’s maximum principle (which is responsible for the choice of $p = n$) and other standard results for elliptic second order differential operators (put together in the appendix). Our main concern is the parabolic problem

$$\begin{cases}
&u_t = Au \\
u(0, \cdot) &= u_0 \\
&u(t,x) = 0 \quad x \in \partial\Omega, \ t > 0
\end{cases} \quad (P)$$

with Dirichlet boundary conditions. Let $C_0(\Omega) := \{ v \in C(\overline{\Omega}) : v_{|\partial\Omega} = 0 \}$. Under the uniform exterior cone condition, we show that the realization $A_0$ of $A$ in $C_0(\Omega)$ given by

$$D(A_0) := \{ v \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : Av \in C_0(\Omega) \}$$

$$A_0u := Av$$

generates a bounded, holomorphic $C_0$-semigroup on $C_0(\Omega)$. This improves the known results, which are presented in the monographie of Lunardi [Lun95, Corollary 3.1.21] for $\Omega$ of class $C^2$ (and $b_j, c$ uniformly continuous).

If the second order coefficients are Lipschitz continuous, then the results mentioned so far hold if $\Omega$ is merely Wiener-regular. For elliptic operators in divergence form, this is proved in [GT98] Theorem 8.31 for the elliptic problem $(E)$ and in [AB99] Corollary 4.7 for the parabolic problem $(P)$. Concerning the elliptic problem $(E)$, and in particular the Dirichlet problem; i.e., the case $f = 0$ in $(E)$, there is earlier work by Krylov [Kry67, Theorem 4], who shows well-posedness of the Dirichlet problem if $\Omega$ is merely Wiener regular and the second order coefficients are Dini-continuous. Krylov also obtains the well-posedness of the Dirichlet problem for $a_{ij} \in C(\overline{\Omega})$ if $\Omega$ satisfies the uniform exterior cone condition [Kry67, Theorem 5]. He uses different (partially probabilistic) methods, though.

1. The Poisson problem

We consider the bounded open set $\Omega \subset \mathbb{R}^n$ and the elliptic operator $A$ from the Introduction. At first we consider the case where the second order conditions are
Lipschitz continuous. Then we merely need a very mild regularity condition on $\Omega$. We say that $\Omega$ is Wiener regular (or Dirichlet regular) if for each $g \in C(\partial \Omega)$ there exists a solution $u \in C^2(\Omega) \cap C(\Omega)$ of the Dirichlet problem
\[
\Delta u = 0 \\
u|_{\partial \Omega} = g.
\]
If $\Omega$ satisfies the exterior cone condition, then $\Omega$ is Dirichlet regular.

**Theorem 1.1.** Assume that the second order coefficients $a_{ij}$ are globally Lipschitz continuous. If $\Omega$ is Wiener-regular, then for each $f \in L^n(\Omega)$, there exists a unique $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$ such that
\[
-A u = f.
\]

The point is that for Lipschitz continuous $a_{ij}$ the operator $A$ may be written in divergence form. This is due to the following lemma.

**Lemma 1.2.** Let $h : \Omega \to \mathbb{R}$ be Lipschitz continuous. Then $h \in W^{1,\infty}(\Omega)$. In particular, $hu \in W^{1,2}(\Omega)$ for all $u \in W^{1,2}(\Omega)$ and $\partial_j(hu) = (\partial_j h)u + h\partial_j u$.

**Proof.** One can extend $h$ to a Lipschitz function on $\mathbb{R}^n$ (without increasing the Lipschitz constant, see [Min70]). Now the result follows from [Eva98, 5.8 Theorem 4]. □

**Proof of Theorem 1.1.** We assume that $\Omega$ is Dirichlet regular. Uniqueness follows from Aleksandrov’s maximum principle Theorem A.1. In order to solve the problem we replace $A$ by an operator in divergence form in the following way. Let $\tilde{b}_j := b_j - \sum_{i=1}^n \partial_i a_{ij}, j = 1, \ldots, n$. Then $\tilde{b}_j \in L^\infty(\Omega)$. Consider the elliptic operator $A_d$ in divergence form given by
\[
A_d u = \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j u) + \sum_{j=1}^n \tilde{b}_j \partial_j u + cu.
\]
a) Let $f \in L^q(\Omega)$ for $q > n$. By [GT98, Theorem 8.31] or [AB99, Corollary 4.6] there exists a unique $u \in C(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega)$ such that $-A_d u = f$ weakly, i.e.,
\[
\sum_{i,j=1}^d \int_{\Omega} a_{ij} \partial_j u \partial_i v - \sum_{j=1}^d \int_{\Omega} \tilde{b}_j \partial_j uv - \int_{\Omega} cuv = \int_{\Omega} fv
\]
for all $v \in D(\Omega)$ (the space of all test functions). We mention in passing that $u \in W^{1,2}_{\text{loc}}(\Omega)$ by [AB99, Lemma 4.2]. For our purposes, it is important that $u \in W^{2,2}_{\text{loc}}(\Omega)$ by Friedrich’s theorem [GT98, Theorem 8.8]. Here we use again that the $a_{ij}$ are uniformly Lipschitz continuous but do not need any further hypothesis on $b_j$ and $c$. It follows from Lemma 1.2 that $a_{ij} \partial_j u \in W^{1,2}_{\text{loc}}(\Omega)$ and $\partial_i(a_{ij} \partial_j u) = (\partial_i a_{ij}) \partial_j u + a_{ij} \partial_i \partial_j u$. Thus $A_d u = A u$. Now it follows from the interior Calderon-Zygmund estimate Theorem A.2 that $u \in W^{2,q}_{\text{loc}}(\Omega) \subset W^{2,n}_{\text{loc}}(\Omega)$. This settles the result if $f \in L^q(\Omega)$ for some $q > n$.

b) Let $f \in L^n(\Omega)$. Choose $f_k \in L^\infty(\Omega)$ such that $\lim_{k \to \infty} f_k = f$ in $L^n(\Omega)$. Let
$u_k \in W^{2,n}_{\text{loc}} \cap C_0(\Omega)$ such that $-\mathcal{A}u_k = f_k$ (use case a)). By Aleksandrov’s maximum principle Theorem [A.1] we have
\[ \|u_k - u\|_{L^\infty(\Omega)} \leq c\|f_k - f\|_{L^\infty(\Omega)}. \]
Thus $u_k$ converge uniformly to a function $u \in C_0(\Omega)$ as $k \to \infty$. By the Calderon-Zygmund estimate (Theorem [A.2]),
\[ \|u_k\|_{W^{2,n}(B_{\delta_0})} \leq c(\|u_k\|_{L^\infty(B_{2\delta_0})} + \|f_k\|_{L^\infty(B_{2\delta_0})}) \]
if $B_{\delta_0} \subset \Omega$, where the constant $c$ does not depend on $k$. Thus the sequence $(u_k)_{k \in \mathbb{N}}$ is bounded in $W^{2,n}(B_{\delta_0})$. It follows from reflexivity that $u \in W^{2,n}(B_{\delta_0})$ and $u_k \to u$ in $W^{2,n}(B_{\delta_0})$ as $k \to \infty$ after extraction of a subsequence. Consequently, $u \in W^{2,n}(\Omega) \cap C_0(\Omega)$. Since $-\mathcal{A}u_k = f_k$ for all $k \in \mathbb{N}$, it follows that $-\mathcal{A}u = f$.

Now we return to the general assumption $a_{ij} \in C(\bar{\Omega})$ and do no longer assume that the $a_{ij}$ are Lipschitz continuous. We need the following lemma which we prove for convenience.

**Lemma 1.3.**

a) There exist $\bar{a}_{ij} \in C^0(\mathbb{R}^n)$ such that $\bar{a}_{ij} = \bar{a}_{ji}, \bar{a}_{ij}(x) = a_{ij}(x)$ if $x \in \Omega$ and
\[ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \frac{\Lambda}{2} |\xi|^2 \]
for all $\xi \in \mathbb{R}^n, x \in \Omega$.

b) There exist $a_{ij}^k \in C^\infty(\bar{\Omega})$ such that $a_{ij}^k = a_{ij}^k, \sum_{i,j=1}^n a_{ij}^k(x)\xi_i\xi_j \geq \frac{\Lambda}{2} |\xi|^2$ and
\[ \lim_{k \to \infty} a_{ij}^k(x) = a_{ij}(x) \]
uniformly on $\bar{\Omega}$.

**Proof.**

a) Let $b_{ij} : \mathbb{R}^n \to \mathbb{R}$ be a bounded, continuous extension of $a_{ij}$ to $\mathbb{R}^n$. Replacing $b_{ij}$ by $\frac{b_{ij} + b_{ji}}{2}$, we may assume that $b_{ij} = b_{ji}$. Since the function $\varphi : \mathbb{R}^n \times S^1 \to \mathbb{R}$ given by $\varphi(x, \xi) := \sum_{i,j=1}^n b_{ij}(x)\xi_i\xi_j$ is continuous and $S^1 := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ is compact, the set $\Omega_1 := \{x \in \mathbb{R}^n : \varphi(x, \xi) > \frac{\Lambda}{4} \varphi_1(x) \}$ is open and contains $\bar{\Omega}$. Let $0 \leq \varphi_1, \varphi_2 \in C(\mathbb{R}^n)$ such that $\varphi_1(x) + \varphi_2(x) = 1$ for all $x \in \mathbb{R}^n$ and $\varphi_2(x) = 1$ for $x \in \mathbb{R}^n \setminus \Omega_1$. Then $\bar{a}_{ij} := \varphi_1 b_{ij} + \frac{\Lambda}{2} \varphi_2 \delta_{ij}$ fulfills the requirements.

b) Let $(g_k)_{k \in \mathbb{N}}$ be a mollifier satisfying $\text{supp} g_k \subset B_{1/k}(0)$. Then $a_{ij}^k = \bar{a}_{ij} * g_k \in C^\infty(\mathbb{R}^n)$ and $\lim_{k \to \infty} a_{ij}^k(x) = a_{ij}(x)$ uniformly in $x \in \bar{\Omega}$. If $\frac{1}{k} < \text{dist}(\partial\Omega_1, \Omega)$, then for $x \in \Omega, \xi \in \mathbb{R}^n$
\[ \sum_{i,j=1}^n a_{ij}^k(x)\xi_i\xi_j = \int_{|y| < 1/k} \sum_{i,j=1}^n \bar{a}_{ij}(x-y)\xi_i\xi_j g_k(y) \, dy \geq \frac{\Lambda}{2} \int_{|y| < 1/k} g_k(y) \, dy = \frac{\Lambda}{2}. \]

\[\Box\]
**Theorem 1.4.** Assume that $\Omega$ satisfies the uniform exterior cone condition. Then for all $f \in L^n(\Omega)$ there exists a unique $u \in C_0(\Omega) \cap W_{loc}^{2,n}(\Omega)$ such that $-Au = f$.

**Proof.** As for Theorem 1.1 we merely have to prove existence of a solution. We choose $a^k_{ij} \in C^\infty(\Omega)$ as in Lemma 1.3. Let $A_k$ be the elliptic operator with the second order coefficients $a_{ij}$ of $A$ replaced by $a^k_{ij}$. Let $f \in L^n(\Omega)$. By Theorem 1.1 for each $k \in \mathbb{N}$ there exists a unique $u_k \in W_{loc}^{2,n}(\Omega) \cap C_0(\Omega)$ such that $-A_k u_k = f$. By Hölder regularity (Theorem A.3) there exists a constant $c$ which does not depend on $k \in \mathbb{N}$ such that

$$
\|u_k\|_{C^n(\Omega)} \leq c(\|f\|_{L^n(\Omega)} + \|u_k\|_{L^n(\Omega)}).
$$

By Aleksandrov’s maximum principle $\|u_k\|_{L^{\infty}(\Omega)} \leq 2c_1 \|f\|_{L^n(\Omega)}$ for all $k \in \mathbb{N}$ and some constant $c_1$. Notice that the first order coefficients of $A_k$ are independent of $k \in \mathbb{N}$. Thus $(u_k)_{k \in \mathbb{N}}$ is bounded in $C^n(\Omega)$. By the Arcele-Ascoli theorem we may assume that $u_k$ converges uniformly to $u \in C_0(\Omega)$ as $k \to \infty$ (passing to a subsequence of necessary). Let $B_{2\rho} \subset \Omega$ where $B_{2\rho}$ is a ball of radius $2\rho$. Since the modulus of continuity of the $a^k_{ij}$ is bounded, by the interior Calderon-Zygmund estimate Theorem A.2

$$
\|u_k\|_{W^{2,n}(B_\rho)} \leq c_2 (\|u_k\|_{L^n(B_{2\rho})} + \|f\|_{L^n(B_{2\rho})})
$$

for all $k \in \mathbb{N}$ and some constant $c_2$. It follows from reflexivity that $u \in W^{2,n}(B_\rho)$ and $u_k \rightharpoonup u$ in $W^{2,n}(B_\rho)$ as $k \to \infty$ after extraction of a subsequence. Since $-A_k u_k = f$, it follows that $-Au = f$. In fact, since $u_k \rightharpoonup u$ weakly in $W^{2,n}(B_\rho)$, it follows that $\partial_{ij} u_k \rightharpoonup \partial_{ij} u$ in $L^n(B_\rho)$ as $k \to \infty$. Thus $\sup_k \|\partial_{ij} u_k\|_{L^n(B_\rho)} < \infty$. It follows that

$$(a^k_{ij} - a_{ij}) \partial_{ij} u_k \to 0 \text{ in } L^n(B_\rho) \text{ as } k \to \infty$$

and consequently $a^k_{ij} \partial_{ij} u_k \to a_{ij} \partial_{ij} u$ in $L^n(B_\rho)$. \hfill \Box

2. The Dirichlet problem

In this section we show the equivalence between well-posedness of the Poisson problem

$$(P) \quad -Au = f \quad u_{|_{\partial \Omega}} = 0$$

and the Dirichlet problem

$$(D) \quad Au = 0 \quad u_{|_{\partial \Omega}} = g$$

where $f \in L^n(\Omega)$ and $g \in C(\partial \Omega)$ are given. We consider the operator $A$ defined in the previous section and define its realization $A$ in $L^n(\Omega)$ (recall that $\Omega \subset \mathbb{R}^n$) by

$$D(A) \ := \ \{ u \in C_0(\Omega) \cap W_{loc}^{2,n}(\Omega) : Au \in L^n(\Omega) \}$$

$$Au \ := \ Au.$$. 
Thus the Poisson problem can be formulated in a more precise way by asking under which conditions $A$ is invertible (i.e. bijective from $D(A)$ to $L^n(\Omega)$ with bounded inverse $A^{-1} : L^n(\Omega) \to L^n(\Omega)$). Note that for $\mu > 0$, the operator $A - \mu := A - \mu I$ has the same form as $A$ (the order-0-coefficient $c$ being just replaced by $c - \mu$).

Proposition 2.1. The operator $A$ is closed and injective. Thus, $A$ is invertible whenever it is surjective. If $A - \mu$ is invertible for some $\mu \geq 0$, then it is so for all.

Proof. By the Aleksandrov maximum principle (Theorem A.1), there exists a constant $c_1 > 0$ such that
\begin{equation}
\|u\|_B \leq 2c_1\|\mu u - Au\|_{L^n(\Omega)}
\end{equation}
for all $u \in D(A), \mu \geq 0$. In order to show that $A$ is closed, let $u_k \in D(A)$ such that $u_k \rightharpoonup u$ in $L^n(\Omega)$ and $A u_k \to f$ in $L^n(\Omega)$. It follows from (2.1) that $u \in C_0(\Omega)$ and $\lim_{k} u_k = u$ in $C_0(\Omega)$. Let $B_{2\rho}$ be a ball of radius $2\rho$ such that $\overline{B}_{2\rho} \subset \Omega$. By the Calderon-Zygmund estimate (Theorem A.20)
\begin{equation}
\|u_k\|_{W^{2,n}(B_{2\rho})} \leq c_{0}(\|u_k\|_{L^n(B_{2\rho})} + \|Au_k\|_{L^n(B_{2\rho})})
\end{equation}
It follows that $(u_k)_{k \in \mathbb{N}}$ is bounded in $W^{2,n}(B_{2\rho})$. By passing to a subsequence we can assume that $u_k \to u$ in $W^{2,n}(B_{\rho})$. Consequently $Au_k \to Au$ in $L^n(\Omega)$. Thus $Au = f$ on $B_{\rho}$. Since the ball is arbitrary, it follows that $u \in D(A)$ and $Au = f$.

Now assume that $\mu_1 - A$ is invertible for some $\mu_1 \geq 0$. Let $\mu_2 \geq 0$. Define $B(t) = t(\mu_1 - A) + (1 - t)(\mu_2 - A)$. Since $(\mu_1 - A), (\mu_2 - A) \in \mathcal{L}(D(A), L^n(\Omega))$ where $D(A)$ is considered as a Banach space with respect to the graph norm $\|u\|_A := \|u\|_{L^n(\Omega)} + \|Au\|_{L^n(\Omega)}$, since by (2.1)
\begin{equation}
2c_1\|B(t)u\|_{L^n(\Omega)} \geq \|u\|_{C(\Omega)} \geq \frac{1}{|\Omega|^{1/n}}\|u\|_{L^n(\Omega)}
\end{equation}
for all $t \in [0,1]$ and since $B(1)$ is invertible, it follows from [GT98] Theorem 5.2 that $B(0)$ is also invertible.

We call a function $u$ on $\Omega$ $A$-harmonic if $u \in W_{loc}^{2,p}(\Omega)$ for some $p > 1$ and $Au = 0$. By [GT98] Theorem 9.16 each $A$-harmonic function $u$ is in $\bigcap_{q > 1} W_{loc}^{2,q}(\Omega)$.

Given $g \in C(\partial\Omega)$, the Dirichlet problem consists in finding an $A$-harmonic function $u \in C(\Omega)$ such that $u|_{\partial\Omega} = g$. We say that $\Omega$ is $A$-regular if for each $g \in C(\partial\Omega)$ there is a solution of the Dirichlet problem. Uniqueness follows from the maximum principle [GT98] Theorem 9.6]
\begin{equation}
-\|u^-\|_{L^\infty(\partial\Omega)} \leq u(x) \leq \|u^+\|_{L^\infty(\partial\Omega)}
\end{equation}
for all $x \in \Omega$, which holds for each $A$-harmonic function $u \in C(\Omega)$.

In particular,
\begin{equation}
\|u\|_{C(\bar{\Omega})} \leq \|u\|_{C(\partial\Omega)}.
\end{equation}

Theorem 2.2. The operator $A$ is invertible if and only if $\Omega$ is $A$-regular.
Proof. a) Assume that $A$ is invertible.

First step: Let $g \in C(\partial \Omega)$ be of the form $g = G|_{\partial \Omega}$ where $G \in C^2(\bar{\Omega})$. Then $\mathcal{A}G \in L^p(\Omega)$. Let $v = A^{-1}(\mathcal{A}G)$, then $u := G - v$ solves the Dirichlet problem for $g$.

Second step: Let $g \in C(\partial \Omega)$ be arbitrary. Extending $g$ continuously and mollifying we find $g_k \in C(\partial \Omega)$ of the kind considered in the first step such that $g = \lim_{k \to \infty} g_k$ in $C(\partial \Omega)$. Let $u_k \in C(\bar{\Omega})$ be $\mathcal{A}$-harmonic satisfying $u_k|_{\partial \Omega} = g_k$. By (2.3) $u := \lim_{k \to \infty} u_k$ exists in $C(\bar{\Omega})$. In particular, $u|_{\partial \Omega} = g$. Let $B_{2\rho} \subset \Omega$. Then by the Calderon-Zygmund estimate Theorem A.2

$$\|u_k\|_{W^{2,p}(B_{2\rho})} \leq C \|u_k\|_{L^p(B_{2\rho})} \leq c_p \epsilon \|u_k\|_{C(\bar{\Omega})}$$

(remember that $\mathcal{A}u_k = 0$). Thus $(u_k)_{k \in \mathbb{N}}$ is bounded in $W^{2,p}(B_{\rho})$. Passing to a subsequence, we can assume that $u_k \rightharpoonup u$ in $W^{2,p}(B_{\rho})$. This implies that $Au = 0$ in $B_{\rho}$. Since the ball is arbitrary, it follows that $u$ is $\mathcal{A}$-harmonic. Thus $u$ is a solution of the Dirichlet problem (D).

b) Conversely, assume that $\Omega$ is $\mathcal{A}$-regular. Let $f \in L^n(\Omega)$. We want to find $u \in D(\mathcal{A})$ such that $Au = f$. Let $B$ be a ball containing $\bar{\Omega}$ and extend $f$ by 0 to $B$. Then by Theorem 1.4 we find $v \in C_0(B) \cap W^{2,n}_{\text{loc}}(B)$ such that $\mathcal{A}v = f$. Here $\mathcal{A}$ is an extension of $\mathcal{A}$ to the ball $B$ according to Lemma 1.3. Let $g = v|_{\partial \Omega}$. Then by our assumption there exists an $\mathcal{A}$-harmonic function $w \in C(\bar{\Omega})$ such that $w|_{\partial \Omega} = g$. Let $u = v - w$. Then $u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega)$ and $\mathcal{A}u = \mathcal{A}v = f$; i.e. $u \in D(\mathcal{A})$ and $Au = f$. We have shown that $A$ is surjective, which implies invertibility by Proposition 2.1. \hfill \Box

Corollary 2.3. Assume that one of the following two conditions is satisfied:

a) $\Omega$ is Wiener regular and the coefficients $a_{ij}$ are globally Lipschitz continuous, or

b) $\Omega$ satisfies the exterior cone condition.

Then $\Omega$ is $\mathcal{A}$-regular. More generally, for all $f \in L^n(\Omega), g \in C(\partial \Omega)$ there exists a unique $u \in C(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega)$ satisfying

$$-\mathcal{A}u = f$$

$$u|_{\partial \Omega} = g.$$ 

Proof. Since $A$ is closed by Proposition 2.1 it follows from Theorem 1.1 (in the case a)) and from Theorem 1.4 (in the case b)) that $A$ is invertible. Thus $\Omega$ is $\mathcal{A}$ regular by Theorem 2.2. Let $f \in L^n(\Omega), g \in C(\partial \Omega)$. Since $\Omega$ is $\mathcal{A}$ regular, there exists an $\mathcal{A}$-harmonic function $u_1 \in C(\Omega)$ such that $u_1|_{\partial \Omega} = g$. Since $A$ is invertible, there exists a function $u_0 \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega)$ such that $-Au_0 = f$. Let $u := u_0 + u_1$. Then $u \in C(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega), u|_{\partial \Omega} = g$ and $-Au = f$. Uniqueness follows from Theorem 2.1. \hfill \Box

For the Laplacian $\mathcal{A} = \Delta$, $\Delta$-regularity is the usual regularity of $\Omega$ with respect to the classical Dirichlet problem, which is frequently called Wiener-regularity because of Wiener’s characterization via capacity [GT98 (2.37)]. It is a most interesting question how $\mathcal{A}$-regularity and $\Delta$-regularity are related. In general it is not true.
that \( \mathcal{A} \)-regularity implies Wiener regularity. In fact, K. Miller \cite{Mil70} gives an example of an elliptic operator \( \mathcal{A} \) with \( b_j = c = 0 \) such that the pointed unit disc \( \{ x \in \mathbb{R}^2 : 0 < |x| < 1 \} \) is \( \mathcal{A} \)-regular even though it is not \( \Delta \)-regular. The other implication seems to be open. The fact that the uniform exterior cone property (which is much stronger than \( \Delta \)-regularity) implies \( \mathcal{A} \)-regularity (Corollary \ref{cor:uniform_exterior_cone}) had been proved before by Krylov \cite[Theorem 5]{Kry67} with the help of probabilistic methods. If \( \Omega \) is merely \( \Delta \)-regular, then it seems not to be known whether \( \Omega \) is \( \mathcal{A} \)-regular. Known results concerning this question are based on further restrictive conditions on the coefficients \( a_{ij} \). In Theorem \ref{thm:globally_lipschitz} we gave a proof for globally Lipschitz continuous \( a_{ij} \). The best result seems to be \cite[Theorem 4]{Kry67} which goes in both directions: If the \( a_{ij} \) are Dini-continuous (in particular, if they are Hölder-continuous), then \( \Omega \) is \( \Delta \)-regular if and only if \( \Omega \) is \( \mathcal{A} \)-regular.

3. Generation results

An operator \( B \) on a complex Banach space \( X \) is said to generate a bounded holomorphic semigroup if \( (\lambda - B) \) is invertible for \( \Re \lambda > 0 \) and
\[
\sup_{\Re \lambda > 0} \| \lambda (\lambda - B)^{-1} \| < \infty.
\]
Then there exist \( \theta \in (0, \pi/2) \) and a holomorphic bounded function \( T : \Sigma_\theta \to \mathcal{L}(X) \) satisfying \( T(z_1 + z_2) = T(z_1)T(z_2) \) such that
\[
(3.1) \quad \lim_{n \to \infty} e^{tB_n} = T(t) \text{ in } \mathcal{L}(X)
\]
for all \( t > 0 \), where \( B_n = nB(n - B)^{-1} \in \mathcal{L}(X) \). Here \( \Sigma_\theta \) is the sector \( \Sigma_\theta := \{ re^{i\alpha} : r > 0, |\alpha| < \theta \} \).

If \( B \) is an operator on a real Banach space \( X \) we say that \( B \) generates a bounded holomorphic semigroup if its linear extension \( B_C \) to the complexification \( X_C \) of \( X \) generates a bounded holomorphic semigroup \( T_C \) on \( X_C \). In that case \( T_C(t)x \subseteq X \) (see \cite[Corollary 2.1.3]{Lun95}); in particular \( T_C(t) := T_C(t)|_X \in \mathcal{L}(X) \). We call \( T = (T(t))_{t>0} \) the semigroup generated by \( B \). It satisfies \( \lim_{t \to 0} T(t)x = x \) for all \( x \in X \) (i.e., it is a \( C_0 \)-semigroup) if and only if \( \overline{\text{D}(B)} = X \). We refer to \cite[Chapter 2]{Lun95} and \cite[Sec. 3.7]{ABHN01} for these facts and further information.

In this section we consider the parts \( A_c \) and \( A_0 \) of \( A \) in \( C(\hat{\Omega}) \) and \( C_0(\Omega) \) as follows:
\[
D(A_c) := \{ u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : Au \in C(\hat{\Omega}) \}
\]
\[
A_c u := Au
\]
\[
D(A_0) := \{ u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : Au \in C_0(\Omega) \}
\]
\[
A_0 u := Au.
\]

Thus \( A_c \) is the part of \( A \) in \( C(\hat{\Omega}) \) and \( A_0 \) the part of \( A_c \) in \( C_0(\Omega) \). Note that \( D(A_0) \subseteq D(A_c) \subseteq \bigcap_{q>1} W^{2,q}_{\text{loc}} \) by \cite[Lemma 9.16]{GT98}. The main result of this section is the following.
**Theorem 3.1.** Assume that $\Omega$ is $A$-regular. Then $A_c$ generates a bounded holomorphic semigroup $T$ on $C(\Omega)$. The operator $A_0$ generates a bounded holomorphic $C_0$-semigroup $T_0$ on $C_0(\Omega)$. Moreover, $T(t)C_0(\Omega) \subseteq C_0(\Omega)$ and

$$T_0(t) = T(t)|_{C_0(\Omega)}.$$ 

Recall that $\Omega$ is $A$-regular if one of the following conditions is satisfied:

(a) $\Omega$ satisfies the uniform exterior cone condition or

(b) $\Omega$ is Wiener regular and the coefficients $a_{ij}$ are Dini-continuous.

In particular, $\Omega$ is $A$-regular if

(a') $\Omega$ is a Lipschitz-domain or

(b') $\Omega$ is Wiener-regular and the $a_{ij}$ are Hölder continuous.

In the following complex maximum principle (Proposition 3.3) we extend $A$ to the complex space $W^{2,p}_{loc}(\Omega)$ without changing the notation. We first proof a lemma.

**Lemma 3.2.** Let $B \subseteq \Omega$ be a ball of center $x_0$ and let $u \in W^{2,p}(B), p > n$, be a complex-valued function such that $Au \in C(B)$. If $|u(x_0)| \geq |u(x)|$ for all $x \in B$, then

$$\text{Re} \left[ u(x_0)(Au)(x_0) \right] \leq 0.$$

**Proof.** We may assume that $x_0 = 0$. If the claim is wrong, then there exist $\varepsilon > 0$ and a ball $B_\varepsilon \subseteq B$ such that $\text{Re} \left[ u(x)(Au)(x) \right] \geq \varepsilon$ on $B_\varepsilon$.

Since $\partial_j|u|^2 = (\partial_j u)\bar{u} + u\partial_j \bar{u} = 2\text{Re} [\partial_j u\bar{u}]$, and $\partial_{i,j}(u\bar{u}) = (\partial_{i,j} u)\bar{u} + \partial_i u\partial_j \bar{u} + \partial_j u\partial_i \bar{u} + u\partial_{i,j} \bar{u}$, and since by ellipticity

$$\text{Re} \sum_{i,j} a_{ij} \partial_i u \partial_j \bar{u} \geq 0 \quad \text{Re} \sum_{i,j} a_{ij} \partial_j u \partial_i \bar{u} \geq 0,$$

it follows that

$$A|u|^2 \geq \text{Re} \sum_{i,j} a_{ij} (\partial_{i,j} u)\bar{u} + \text{Re} \sum_{i,j} a_{ij} u\partial_{i,j} \bar{u}$$

$$\geq \text{Re} (Au\bar{u}) \geq 2\varepsilon \quad \text{on} \quad B_\varepsilon.$$

Let $\psi(x) = |u|^2 - \tau |x|^2$, $\tau > 0$. Then $A|\psi|^2 \geq 2\varepsilon - c_1 \tau$ on $B_\varepsilon$ for all $\tau > 0$ and some $c_1 > 0$. Choosing $\tau > 0$ small enough, we have $A|\psi|^2 \geq \varepsilon$ on $B_\varepsilon$.

Since $\psi \in W^{2,p}(B_\varepsilon) \cap C(B_\varepsilon)$, by Aleksandrov’s maximum principle [GT98 Theorem 9.1], see Theorem [A.1], it follows that

$$|u(0)|^2 = |\psi(0)|^2 \leq \sup_{\partial B_\varepsilon(0)} \psi$$

$$= \sup_{\partial B_\varepsilon(0)} |u|^2 - \tau \theta^2$$

$$\leq |u(0)|^2 - \tau \theta^2 < |u(0)|^2,$$

a contradiction. \qed
Proposition 3.3. (complex maximum principle). Let \( u \in C(\Omega) \cap W^{2,0}(\Omega) \) such that \( \lambda u - Au = 0 \) where \( \Re \lambda > 0 \). If there exists \( x_0 \in \Omega \) such that \( |u(x)| \leq |u(x_0)| \) for all \( x \in \Omega \), then \( u \equiv 0 \). Consequently,

\[
\max_{\Omega} |u(x)| = \max_{\partial \Omega} |u(x)| .
\]

Proof. If \( |u(x)| \leq |u(x_0)| \) for all \( x \in \Omega \), then by Lemma 3.2, \( \Re \left( \frac{u(x_0)(Au)(x_0)}{u(x_0)} \right) \leq 0 \). Since \( \lambda u = Au \), it follows that

\[
\Re \lambda |u(x_0)|^2 = \Re \left( \frac{u(x_0)(Au)(x_0)}{u(x_0)} \right) \leq 0 .
\]

Hence \( u(x_0) = 0 \). \( \square \)

Next, recall that an operator \( B \) on a real Banach space \( X \) is called \( m\)-dissipative if \( \lambda - B \) is invertible and

\[
\lambda \|[\lambda - B]^{-1}\| \leq 1 \quad \text{for all} \quad \lambda > 0 .
\]

Now we show that the operator \( A_c \) is \( m\)-dissipative and that the resolvent is positive (i.e., maps non-negative functions to non-negative functions).

Proposition 3.4. Assume that \( \Omega \) is \( A \)-regular. Then \( A_c \) is \( m\)-dissipative and \( (\lambda - A_c)^{-1} \geq 0 \) for \( \lambda > 0 \).

Proof. Let \( \lambda > 0 \). Since by Theorem 2.2 the operator \( (\lambda - A) \) is bijective, also \( (\lambda - A_c) \) is bijective.

a) We show that \( (\lambda - A_c)^{-1} \geq 0 \). Let \( f \in C(\Omega), f \leq 0, u := (\lambda - A_c)^{-1} f \). Assume that \( u^* \neq 0 \). Since \( u \in C_0(\Omega) \), there exists \( x_0 \in \Omega \) such that \( u(x_0) = \max u > 0 \). Then by Lemma 3.2, \( Au(x_0) \leq 0 \). Since \( \lambda u - Au = f \), it follows that \( \lambda u(x_0) \leq f(x_0) \leq 0 \) a contradiction.

b) Let \( f \in C(\Omega), u = (\lambda - A_c)^{-1} f \). We show that \( \|\lambda u\|_{C(\Omega)} \leq \|f\|_{C(\Omega)} \). Assume first that \( f \geq 0, f \neq 0 \). Then \( u \geq 0 \) by a) and \( u \neq 0 \). Let \( x_0 \in \Omega \) such that \( u(x_0) = \|u\|_{C(\Omega)} \). Then \( (A_c u)(x_0) \leq 0 \) by Lemma 3.2. Hence \( \lambda u(x_0) \leq (\lambda - A_c)(x_0) = \lambda u(x_0) - \lambda (A_c u)(x_0) = f(x_0) \leq \|f\|_{C(\Omega)} \).

If \( f \in C(\Omega) \) is arbitrary, then by a) \( |(\lambda - A_c)^{-1} f| \leq (\lambda - A_c)^{-1} |f| \) and so \( \|\lambda (\lambda - A_c)^{-1} f\|_{C(\Omega)} \leq \|f\|_{C(\Omega)} \). \( \square \)

Now we consider the complex extension of \( A_c \) (still denoted by \( A_c \)) to the space of all complex-valued functions on \( \Omega \) which we still denote by \( C(\Omega) \). Our aim is to prove that for \( \Re \lambda > 0 \) the operator \( (\lambda - A_c)^{-1} \) is invertible and

\[
\| (\lambda - A_c)^{-1} \| \leq \frac{M}{|\lambda|} ,
\]

where \( M \) is a constant. For that, we extend the coefficients \( a_{ij} \) to uniformly continuous bounded real-valued functions on \( \mathbb{R}^n \) satisfying the strict ellipticity condition

\[
\Re \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \geq \frac{\Lambda}{2} |\xi|^2 .
\]
(ξ ∈ ℝ^n, x ∈ ℝ^n), keeping the same notation, see Lemma 3.5. We extend b_j, c to bounded measurable functions on ℝ^n such that c ≤ 0 (keeping the same notation). Now we define the operator B_∞ on L^∞(ℝ^n) by

\[
D(B_∞) := \{ u ∈ \bigcap_{p>1} W^{2,p}_0(ℝ^n) : u, Bu ∈ L^∞(ℝ^n) \}
\]

where

\[
B_∞u := Bu ,
\]

\[
B_∞u := \sum_{i,j=1}^d a_{ij} \partial_{ij} u + \sum_{j=1}^d b_j \partial_j u + cu \quad \text{for } u ∈ W^{2,p}_0(ℝ^n) .
\]

The operator B_∞ is sectorial. This is proved in [Lun95, Theorem 3.1.7] under the assumption that the coefficients b_j, c are uniformly continuous. We give a perturbation argument to deduce the general case from the case b_j = c = 0. The following lemma shows in particular that the domain of B_∞ is independent of b_j and c.

**Lemma 3.5.** One has D(B_∞) ⊂ W^{1,∞}(ℝ^n). Moreover, for each ε > 0 there exists c_ε ≥ 0 such that

\[
\|u\|_{W^{1,∞}(ℝ^n)} ≤ \varepsilon \|B_∞u\|_{L^{∞}(ℝ^n)} + c_ε \|u\|_{L^{∞}(ℝ^n)}
\]

for all u ∈ D(B_∞).

**Proof.** Consider an arbitrary ball B_1 in ℝ^n of radius 1 and the corresponding ball B_2 of radius 2. Let p > n. Since the injection of W^{2,p}(B_1) into C^{1}(B_1) is compact, for each ε > 0 there exists c_ε > 0 such that

\[
\|u\|_{C^{1}(B_1)} ≤ \varepsilon \|u\|_{W^{2,p}(B_1)} + c_ε \|u\|_{L^{∞}(B_1)} .
\]

By the Calderon-Zygmund estimate this implies that

\[
\begin{align*}
\|u\|_{C^{1}(B_1)} &≤ \varepsilon \|B_∞u\|_{L^{∞}(B_2)} + \|u\|_{L^{∞}(B_2)} \\
& \quad + c_ε \|u\|_{L^{∞}(B_1)} \\
& \leq \varepsilon \|B_∞u\|_{L^{∞}(ℝ^n)} + (\varepsilon c_1 + c_ε) \cdot \|u\|_{L^{∞}(ℝ^n)} .
\end{align*}
\]

Since \|u\|_{L^{∞}(ℝ^n)} = \sup_{B_1} \|u\|_{L^{∞}(B_1)}, where the supremum is taken over all balls of radius 1 in ℝ^n, the claim follows. □

**Theorem 3.6.** There exist M ≥ 0, ω ∈ ℝ such that (λ − B_∞) is invertible and

\[
\|λ(λ − B_∞)^{-1}\| ≤ M \quad (\text{Re} \lambda > ω) .
\]

**Proof.** Denote by B_0^{B_∞} the operator with the coefficients b_j, c replaced by 0. Lemma 3.5 implies that D(B_0^{B_∞}) = D(B_∞) and (applied to B_0^{B_∞}) that

\[
\| (B_∞ - B_0^{B_∞})u \|_{L^{∞}(ℝ^n)} ≤ \varepsilon \| B_0^{B_∞}u \|_{L^{∞}(ℝ^n)} + c_ε \|u\|_{L^{∞}(ℝ^n)}
\]

for all u ∈ D(B_0^{B_∞}), ε > 0 and some c_ε ≥ 0. Since B_0^{B_∞} is sectorial by [Lun95, Theorem 3.1.7] the claim follows from the usual holomorphic perturbation result [ABHN01, Theorem 3.7.23]. □
Now we use the maximum principle, Lemma 5.2, to carry over the sectorial estimate from \( \mathbb{R}^n \) to \( \Omega \). This is done in a very abstract framework by Lumer-Paquet [LP77], see [Are04, Section 2.5] for the Laplacian.

**Proof of Theorem 3.1.** Let \( \omega \) be the constant from Theorem 3.6 and let \( \text{Re}\lambda > \omega \), \( f \in C(\overline{\Omega}) \), \( u = (\lambda - A_c)^{-1}f \). Then

\[
  u \in C_0(\Omega) \cap \bigcap_{p>1} W^{2,p}_{\text{loc}}(\Omega) \quad \text{and} \quad \lambda u - Au = f.
\]

Extend \( f \) by 0 to \( \mathbb{R}^n \) and let \( v = (\lambda - B_\infty)^{-1}f \). Then \( \lambda v - Av = f \) on \( \Omega \) and \( \|\lambda v\|_{L^\infty(\Omega)} \leq M\|f\|_{C(\overline{\Omega})} \) by Theorem 3.6. Moreover, let \( w := v - u \in C(\overline{\Omega}) \cap \bigcap_{p \geq 1} W^{2,p}_{\text{loc}}(\Omega) \), \( \lambda w - Aw = 0 \) on \( \Omega \) and \( w(z) = v(z) \) for all \( z \in \partial \Omega \). Then by the complex maximum principle Proposition 3.3,

\[
  \|w\|_{C(\overline{\Omega})} = \max_{z \in \partial \Omega} |v(z)| \leq \frac{M}{|\lambda|}\|f\|_{C(\overline{\Omega})}.
\]

Consequently,

\[
  \|u\|_{C(\Omega)} = \|u - v + v\|_{C(\Omega)} \leq \|w\|_{C(\Omega)} + \|v\|_{C(\Omega)} \leq \frac{2M}{|\lambda|}\|f\|_{C(\overline{\Omega})}.
\]

This is the desired estimate which shows that \( A_c \) is sectorial. By [Lun95, Proposition 2.1.11] there exist a sector \( \Sigma : \omega := \{ \omega + re^{i\alpha} : r > 0, |\alpha| < \theta \} \) with \( \theta \in (\frac{\pi}{2}, \pi) \), \( \omega \geq 0 \), and a constant \( M_1 > 0 \) such that

\[
  (\lambda - A_c)^{-1} \quad \text{exists} \quad \text{for} \quad \lambda \in \Sigma + \omega \quad \text{and} \quad \|\lambda(\lambda - A_c)^{-1}\| \leq M_1.
\]

Thus there exists \( r > 0 \) such that \( (\lambda - A_c) \) is invertible and \( \|\lambda(\lambda - A_c)^{-1}\| \leq M \) whenever \( \text{Re}\lambda > 0 \) and \( |\lambda| > r \). Since \( A \) is invertible by Theorem 2.2 it follows that \( A_c \) is bijective. Since the resolvent set of \( A_c \) is nonempty, \( A_c \) is closed. Thus \( A_c \) is invertible. Since by Proposition 5.4 \( A_c \) is resolvent positive, it follows from [ABHN01, Proposition 3.11.2] that there exists \( \varepsilon > 0 \) such that \( (\lambda - A_c) \) is invertible whenever \( \text{Re}\lambda > -\varepsilon \). As a consequence,

\[
  \sup_{|\lambda| \leq r, \text{Re}\lambda > 0} \|\lambda(\lambda - A_c)^{-1}\| < \infty.
\]

Together with the previous estimates this implies that

\[
  \|\lambda(\lambda - A_c)^{-1}\| \leq M_2
\]

whenever \( \text{Re}\lambda > 0 \) for some constant \( M_2 \). Thus \( A_c \) generates a bounded holomorphic semigroup \( T \) on \( C(\overline{\Omega}) \). Since \( D(A_c) \subset C_0(\Omega) \) and \( D(\Omega) \subset D(A_c) \) it follows that \( D(A_c) = C_0(\Omega) \). The part of \( A_c \) in \( C_0(\Omega) \) is \( A_0 \). So it follows from [Lun95, Remark 2.1.5, Proposition 2.1.4] that \( A_0 \) generates a bounded, holomorphic \( C_0 \)-semigroup \( T_0 \) on \( C_0(\Omega) \) and \( T_0(t) = T(t)|_{C_0(\Omega)} \) on \( C_0(\Omega) \).

Finally we mention compactness and strict positivity.
Proposition 3.7. Assume that \( \Omega \) satisfies the uniform exterior cone condition. Then \((\lambda - A_c)^{-1}\) and \(T(t)\) are compact operators \((\lambda > 0, t > 0)\).

Proof. It follows from Theorem A.3 that \(D(A_c) \subset C^\alpha(\Omega)\). Since the embedding of \(C^\alpha(\Omega)\) into \(C(\bar{\Omega})\) is compact, it follows that the resolvent of \(A_c\) is compact. Since \(T\) is holomorphic, it follows that \(T(t)\) is compact for all \(t > 0\).

Proposition 3.8. Assume that \( \Omega \) is \( A \)-regular. Let \( t > 0, 0 \leq f \in C_0^0(\Omega), f \not\equiv 0 \).

Then \((T_0(t)f)(x) > 0\) for all \( x \in \Omega \).

Proof. a) We show that \( u := (\lambda - A_0)^{-1}f \) is strictly positive. Assume that \( u(x) \leq 0 \) for some \( x \in \Omega \). Let \( v = -u \). Then \( Av - \lambda v = f \geq 0 \). It follows from the maximum principle [GT98, Theorem 9.6] that \( v \) is constant. Since \( v \in C_0(\Omega) \), it follows that \( v \equiv 0 \). Hence also \( f \equiv 0 \).

b) It follows from a) that \( T_0 \) is a positive, irreducible \( C_0 \)-semigroup on \( C_0(\Omega) \).

Since the semigroup is holomorphic, the claim follows from [Na86, C-III. Theorem 3.2.(b)].

Appendix A. Results on elliptic partial differential equations

In this section, we collect some results on elliptic partial differential equations, which can be found in text books, for example [GT98]. We consider the elliptic operator \( A \) from the Introduction and assume that the ellipticity constant \( \Lambda > 0 \) is so small that \( \|a_{ij}\|_{L^\infty}, \|b_j\|_{L^\infty}, \|c\|_{L^\infty} \leq \frac{1}{\Lambda} \).

Theorem A.1 (Aleksandrov’s maximum principle, [GT98 Theorem 9.1]). Let \( f \in L^n(\Omega), u \in C(\bar{\Omega}) \cap W_{loc}^{2,n}(\Omega) \) such that

\[-Au \leq f.\]

Then

\[\sup_{\Omega} u \leq \sup_{\partial \Omega} u^+ + c_1\|f^+\|_{L^n(\Omega)}\]

where the constant \( c_1 \) depends merely on \( n, \text{diam} \Omega \) and \( \|b_j\|_{L^n(\Omega)}, j = 1 \ldots, n \).

Consequently, if \( u \in C_0(\Omega) \) and \(-Au = f\), then

\[\|u\|_{L^\infty(\Omega)} \leq 2c_1\|f\|_{L^n(\Omega)}\]

and \( u \leq 0 \) if \( f \leq 0 \).

Theorem A.2 (Interior Calderon-Zygmund estimate, [GT98 Theorem 9.11]). Let \( B_{2\varrho} \) be a ball of radius \( 2\varrho \) such that \( \overline{B_{2\varrho}} \subset \Omega \), and let \( u \in W^{2,p}(B_{2\varrho}) \), where \( 1 < p < \infty \). Then

\[\|u\|_{W^{2,p}(B_{\varrho})} \leq c_\varrho(\|Au\|_{L^p(B_{2\varrho})} + \|u\|_{L^p(B_{2\varrho})})\]

where \( B_{\varrho} \) is the ball of radius \( \varrho \) concentric with \( B_{2\varrho} \). The constant \( c \) merely depends on \( \Lambda, n, \varrho, p \) and the continuity moduli of the \( a_{ij} \).
Theorem A.3 (Hölder regularity, [GT98 Corollary 9.29]). Assume that $\Omega$ satisfies the uniform exterior cone condition. Let $u \in C_0(\Omega) \cap W^{2,n}_{loc}(\Omega)$ and $f \in L^q(\Omega)$ such that $-Au = f$. Then $u \in C^\alpha(\Omega)$ and

$$
\|u\|_{C^\alpha(\Omega)} \leq C(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)})
$$

where $\alpha > 0$ and $c > 0$ depend merely on $\Omega, \Lambda$ and $n$.

In [GT98 Corollary 9.29] it is supposed that $u \in W^{2,n}(\Omega)$. But an inspection of the proof and of the results preceding [GT98 Corollary 9.29] shows that $u \in W^{2,n}_{loc}(\Omega)$ suffices. The above Hölder regularity also holds for solutions of equations in divergence form when the right-hand side $f$ is in $L^q(\Omega)$ for some $q > \frac{n}{2}$, see [GT98 Theorem 8.29].

References

[AB99] Arendt, W., Bénilan, Ph.: Wiener regularity and heat semigroups on spaces of continuous functions. Progress in Nonlinear Differential Equations and Their Applications Vol. 35 Birkhäuser Basel 1999, 29–49.

[ABHN01] Arendt, W., Batty, C., Hieber, M., Neubrander, F.: Vector-valued Laplace Transforms and Cauchy Problems. Monographs in Mathematics. Birkhäuser, Basel, (2001) ISBN 3-7643-6549-8.

[ADN59] Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Communications on Pure and Applied Mathematics, 12 (1959), pp. 623-727.

[ADN64] Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II, Communications on Pure and Applied Mathematics, 17 (1964), pp. 35-92.

[Are04] Arendt, W. Semigroups and Evolution Equations: Functional Calculus, Regularity and Kernel Estimates. HANDBOOK OF DIFFERENTIAL EQUATIONS. Evolutionary Equations, Vol. 1. C.M. Dafermos and E. Feireisl eds., Elsevier (2004), 1–85.

[GT98] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order, Springer Verlag, 3.Auflage, Berlin (1998).

[Eva98] Evans, L.C.: Partial Differential Equations. American Math. Soc., Providence, R. I. 1998.

[Kry67] Krylov, N. V.: The first boundary value problem for elliptic equations of second order. Differential’nye Uravnenija 3 (1967), 315–326.

[LP77] Lumer, G., Paquet, L.: Semi-groupes holomorphes et équations d’évolution, CR Acad. Sc. Paris 284 Série A (1977), pp. 237–240.

[Lun95] Lunardi, A.: Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser Basel, (1995).

[Mil70] Miller, K.: Nonequivalence of regular boundary points for the Laplace and non-divergence equations, even with continuous coefficients. Ann. Scuola Norm. Sup. Pisa 3 24 (1970), 159–163.

[Min70] Minty, G. J.: On the extension of Lipschitz, Lipschitz-Hölder continuous, and monotone functions. Bull. Amer. Math. Soc. 76 (1970), 334–339.

[Nag86] Nagel, R. (ed.): One-parameter Semigroups of Positive Operators. Springer LN 1184, (1986) Berlin.
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