EFFECTIVE $l^2$ DECOUPLING FOR THE PARABOLA

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ABSTRACT. We make effective $l^2$ decoupling for the parabola $\{(\xi, \xi^2) : \xi \in [0,1]\}$ by following the argument of Bourgain and Demeter in [6]. This allows us to improve upon the bound of $O_\epsilon(\delta^{-\epsilon})$ on the decoupling constant. Our results are effective in that we make explicit all implied constants. Some technical features of the argument, such as the various weight functions, parabolic rescaling with weight $w_{B,E}$ and various integrality constraints, are clarified and made effective.

1. Introduction

In [4] and later with a more streamlined proof [6], Bourgain and Demeter prove that the decoupling constant associated to the paraboloid $\{(\xi_1, \xi_2, \ldots, \xi_{n-1}, \xi_1^2 + \cdots + \xi_{n-1}^2) : \xi_i \in [0,1]\}$ is $O_{n,\epsilon}(\delta^{-\epsilon})$ for $2 \leq p \leq 2^{(n+1)/n-1}$. In [3], Bourgain, Demeter, and Guth prove that the decoupling constant associated to the moment curve $\{(\xi, \xi^2, \ldots, \xi^n) : \xi \in [0,1]\}$ is $O_{n,\epsilon}(\delta^{-\epsilon})$ for $2 \leq p \leq n(n+1)$ which resolved Vinogradov’s mean value conjecture. Both the moment curve and the paraboloid are the same when $n = 2$. It is this case we study and make effective.

For each interval $J \subset [0,1]$ and $g : [0,1] \to \mathbb{C}$, let

$$(\mathcal{E}_J g)(x) := \int_J g(\xi) e(\xi x_1 + \xi^2 x_2) \, d\xi$$

where $e(z) = e^{2\pi i z}$. Note that $\mathcal{E}_{[0,1]} g$ is the extension operator for the parabola $\{(\xi, \xi^2) : \xi \in [0,1]\}$. For an integer $E \geq 1$ and a square $B = B(c_B, R) \subset \mathbb{R}^2$ centered at $c_B = (c_B1, c_B2)$ of side length $R$, let

$$w_{B,E}(x) := (1 + \frac{|x - c_B|}{R})^{-E}.$$
By making effective the arguments in [6], we have the following improvement over $D_{p,E}(\delta) \lesssim \delta^{-\varepsilon}$.

**Theorem 1.1.** Let $E \geq 100$ and $0 < \delta < 2^{-64E^{15}E}$ with $\delta \in \mathbb{N}^{-2}$.

(i) If $2 \leq p \leq 4$, then
$$D_{p,E}(\delta) \leq \exp(E^{6E}(\log \frac{1}{\delta})^{2/3}).$$

(ii) If $4 < p < 6$, then
$$D_{p,E}(\delta) \leq \exp(E^{6E}(\log \frac{1}{\delta})^{2/3} + \frac{1}{3} \log \log (\frac{E}{\delta}^2)).$$

(iii) If $p = 6$, then
$$D_{6,E}(\delta) \leq \exp(E^{6E} \frac{\log \frac{1}{\delta}}{\log \log \frac{1}{\delta}} \log \log \log \frac{1}{\delta}).$$

Using the trivial bound for $\delta > 2^{-64E^{15}E}$, one can obtain an upper bound on $D_{p,E}(\delta)$ that is valid for all $\delta \in \mathbb{N}^{-2}$.

Decoupling-type inequalities were first studied by Wolff in [23] in the context of local smoothing estimates. Bourgain in [2] was able to use induction on scales from [9] and multilinear restriction from [1] to partially resolve the $l^2$-decoupling conjecture for smooth compact hypersurfaces in the range $2 \leq p \leq \frac{2n}{n-1}$. Following the proof of the $l^2$-decoupling conjecture for smooth compact hypersurfaces by Bourgain and Demeter in [4] for the full range $2 \leq p \leq \frac{2(n+1)}{n-1}$, decoupling inequalities for various curves and surfaces have found many applications to PDE ([12, 13]) and analytic number theory ([3, 5, 7, 8, 10, 11, 14, 16]) most notably the proof of Vinogradov’s mean value theorem. This list is by no means exhaustive, for a more complete list see [20].

In the context of analytic number theory, Wooley developed over a series of papers [25, 26, 27, 29] the theory of efficient congruencing leading to a proof of Vinogradov’s mean value theorem in $n = 3$ ([28] with a simplified approach by Heath-Brown in [15]) and Vinogradov’s mean value theorem for translation-dilation invariant families via nested efficient congruencing [24]. See [20] for a comparison of efficient congruencing and decoupling. Wooley’s work on Vinogradov has been made effective in certain cases by Steiner [21] and this current paper can be viewed as an attempt to make certain cases of decoupling effective. See [16] and the MathOverflow question [19] for applications of an effective Bourgain-Demeter-Guth result. One key point is that it is important to work out the dependence on the dimension $n$.

In the proof of decoupling for the paraboloid or the moment curve in $n$ dimensions, one crucial input is a decoupling in $(n - 1)$ dimensions. This is most easily seen by the reliance on a Bourgain-Guth iteration to show the equivalence between linear and multilinear decoupling constants. In the case of the moment curve, this also makes an additional appearance in a step called lower dimensional decoupling (Lemma 8.2 of [8]) since various sections of the moment curve look lower dimensional at certain scales. Thus ultimately we are reduced to first studying explicit decoupling in $n = 2$
dimensions. Because of this reduction of dimension argument, the arguments of [6, 8] should give an upper bound on the decoupling constant that is worse than those stated in Theorem 1.1.

While the argument of this paper is similar to [6], we highlight some key features of this paper. One major feature of this paper is that we carefully work with the various weight functions that show up in the argument and obtain estimates with explicit constants. Section 2 develops all the estimates needed in this paper about the weight function $w_{B,E}$. The most crucial observation is that $w_{B(0,R),E} \ast w_{B(0,R'),E} \lesssim_{E} R'^2 w_{B(0,R),E}$ for $0 < R' \leq R$ (Lemma 2.1). The calculations in Section 2 can be easily generalized to $n$ dimensions. A careful study of the weight $w_{B,E}$ reveals that the decoupling constant with weight $w_{B,E}$ does not behave too well under parabolic rescaling, see Lemma 2.18, Remark 2.19, and the proof of Proposition 4.1. Essentially this is because $w_{B,E}$ weights all directions evenly and so it is well-adapted for squares and circles but not rectangles and ellipses. To accommodate this, we introduce a second weight

$$\tilde{w}_{B,E}(x) := w_{B,E}(x)(1 + \frac{|x_2 - cB^2|}{R})^{-E}$$

and let $\tilde{D}_{p,E}(\delta)$ be defined similarly as in (1) but with $w_{B,E}$ replaced with $\tilde{w}_{B,E}$. We will then need that $D_{p,E}(\delta) \sim_{E} \tilde{D}_{p,E}(\delta)$ which is the topic of Section 3. Once we have this, we then recover almost multiplicativity of $D_{p,E}(\delta)$ in Section 4 and other applications of parabolic rescaling. This also introduces some slight changes compared to [6], namely our multilinear decoupling constant in Section 5 is defined with weight $\tilde{w}_{B,E}$ rather than $w_{B,E}$ and in our iteration, $A_p$ uses weight $\tilde{w}_{B,E}$ rather than $w_{B,E}$. The ball inflation inequality of [6] is made effective in Section 6. We have chosen to keep track of the dependence on $E$ since estimates for the decoupling constant in higher dimensions for a specific $E$ may depend on an estimate for the decoupling constant at a lower dimension with a different $E$ (see for example, Theorems 5.1 and 8.4 of [6]).

Another key feature is that we do not ignore integrality constraints about partitioning intervals into an integer number of smaller intervals. Tracing all the integrality constraints on the parameters in the argument, the iteration in Sections 7 and 8 gives a good upper bound for the linear decoupling constant along a lacunary sequence of scales (Section 9). Using almost multiplicativity of the linear decoupling constant (Proposition 4.1) and the trivial bound, we can upgrade this to be a good upper bound on all scales. This is done in Section 10. Finally optimizing in Section 11 completes the proof of Theorem 1.1.

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2. Weight functions and consequences

2.1. The weights $w_B$ and $\tilde{w}_B$. As defined in Section 1, we recall that

$$w_B(x) := (1 + \frac{|x - c_B|}{R})^{-E}$$

and

$$\tilde{w}_B(x) := w_B(x)(1 + \frac{|x_2 - c_{B2}|}{R})^{-E}.$$ 

If $w$ is a weight function for $B$, let

$$\|f\|_{L^p_w} := \left(\frac{1}{|B|} \int_{\mathbb{R}^2} |f(x)|^p w(x) \, dx\right)^{1/p}.$$

We will make use of the following two inequalities that are immediate applications of Holder’s inequality: If $1/p = 1/q + 1/r$, then

$$\|fg\|_{L^p(w_{B,E})} \leq \|f\|_{L^q(w_{B,E})} \|g\|_{L^r(w_{B,E})}$$

and if $q > p$,

$$\|f\|_{L^p_w} \leq \|f\|_{L^q_w}.$$ 

(2)

The above two inequalities also hold with $w_{B,E}$ replaced with $\tilde{w}_{B,E}$. When $B$ is a square centered at the origin, $w_B$ and $\tilde{w}_B$ obey the following two important self-convolution estimates.

**Lemma 2.1.** Let $E \geq 10$. For $0 < R' \leq R$,

$$w_{B(0,R'),E} \ast w_{B(0,R'),E} \leq 4^E R'^2 w_{B(0,R),E}. \quad (3)$$

We also have

$$R^2 w_{B(0,R),E} \leq 3^E 1_{B(0,R)} \ast w_{B(0,R),E}. \quad (4)$$

The same inequalities with the same constants hold true when $w_{B(0,R),E}$ is replaced with $\tilde{w}_{B(0,R),E}$.

**Proof.** We first prove (3). We would like to give an upper bound for the expression

$$\frac{1}{R'^2} \int_{\mathbb{R}^2} (1 + \frac{|x - y|}{R'})^{-E} (1 + \frac{|y|}{R'})^{-E} (1 + \frac{|x|}{R'})^E \, dy$$

depending only on $E$. A change of variables in $y$ and rescaling $x$ shows that it suffices to give an upper bound for

$$\int_{\mathbb{R}^2} (1 + \frac{R'}{R} y)^{-E} (1 + |y|)^{-E} (1 + |x|)^E \, dy$$ 

(5)

depending only on $E$. If $|x| \leq 1$, then (3) is

$$\leq 2^E \int_{\mathbb{R}^2} (1 + |y|)^{-E} \, dy \leq 2^E.$$
If $|x| > 1$, then we split (5) into
\[
\left( \int_{|x - \frac{R'}{R}y| < \frac{|x|}{2}} + \int_{|x - \frac{R'}{R}y| > \frac{|x|}{2}} \right) (1 + |x - \frac{R'}{R}y|)^{-E} (1 + |y|)^{-E} (1 + |x|)^E dy.
\tag{6}
\]
In the case of the first integral in (6), $(R'/R)|y| \geq |x| - |x - (R'/R)y| \geq |x|/2$ and hence
\[
\int_{|x - \frac{R'}{R}y| < \frac{|x|}{2}} (1 + |x - \frac{R'}{R}y|)^{-E} (1 + |y|)^{-E} (1 + |x|)^E dy
\[
\leq \left( \frac{(1 + |x|)^E}{(1 + (R'/R)|x|/2)^E} \right) \int_{\mathbb{R}^2} (1 + |x - \frac{R'}{R}y|)^{-E} dy \leq (4R'/R)^E (R/R')^2 \leq 4^E.
\]
In the case of the second integral in (6),
\[
\int_{|x - \frac{R'}{R}y| > \frac{|x|}{2}} (1 + |x - \frac{R'}{R}y|)^{-E} (1 + |y|)^{-E} (1 + |x|)^E dy
\[
\leq \left( \frac{1 + |x|}{1 + |x|/2} \right)^E \int_{\mathbb{R}^2} (1 + |y|)^{-E} dy \leq 2^E.
\]
This then proves (3).

To prove (4) it suffices to give a lower bound for
\[
\frac{1}{R^2} \int_{B(0,R)} (1 + \frac{|x - y|}{R})^{-E} (1 + \frac{|x|}{R})^E dy
\]
which depends only on $E$. As before, rescaling $x$ and a change of variables in $y$ gives that it suffices to give a lower bound independent of $x$ for
\[
\int_{B(0,1)} \left( \frac{1 + |x|}{1 + |x - y|} \right)^E dy \geq \left( \frac{1 + |x|}{2 + |x|} \right)^E \geq 2^{-E}.
\]
Thus we have shown that $\frac{1}{R^2}(1_{B(0,R)} \ast w_{B(0,R),E}) \geq 2^{-E}w_{B(0,R),E}$ which shows (4).

We now prove the analogues for $w_{B(0,R),E}$. We first prove the analogue of (3). We would like to give an upper bound for the expression
\[
\frac{1}{R^2} \int_{\mathbb{R}^2} (1 + \frac{|x - y|}{R})^{-E} (1 + \frac{|x_2 - y_2|}{R})^{-E} \left( \frac{|y_2|}{R'} \right)^{-E} \times \left( 1 + \frac{|y_2|}{R'} \right)^{-E} \left( 1 + \frac{|x_2|}{R} \right)^E dy.
\]
A change of variables in $y$ and rescaling $x$ shows it suffices to bound
\[
\int_{\mathbb{R}^2} (1 + |x - \frac{R'}{R}y|)^{-E} (1 + |y|)^{-E} (1 + |x|)^E
\[
\times \left( 1 + \frac{|x_2 - \frac{R'}{R}y_2|}{R'} \right)^{-E} \left( 1 + \frac{|y_2|}{R'} \right)^{-E} \left( 1 + \frac{|x_2|}{R} \right)^E dy.
\tag{7}
\]
By the triangle inequality,
\[(1 + |x_2 - \frac{R'}{R} y_2|)^{-E} (1 + |y_2|)^{-E} (1 + |x_2|)^E \leq \left( \frac{1 + (R'/R)|y_2|}{1 + |y_2|} \right)^E \leq 1.\]

The upper bound for (7) then reduces to finding an upper bound for (5).

To prove the analogue of (4) for \(\tilde{w}_{B(0,R),E}\), it suffices to give a lower bound for
\[
\frac{1}{R^2} \int_{B(0,R)} (1 + \frac{|x - y|}{R})^{-E} (1 + \frac{|x_2 - y_2|}{R})^{-E} (1 + \frac{|x|}{R})^E (1 + \frac{|x_2|}{R})^E dy
\]
which depends only on \(E\). Once again, a change of variables in \(y\) and a rescaling in \(x\) show that it suffices to give a lower bound for
\[
\int_{B(0,1)} (1 + |x - y|)^{-E} (1 + |x|)^E (1 + |x_2 - y_2|)^{-E} (1 + |x_2|)^E dy.
\]

Since \(y \in B(0,1)\), the triangle inequality gives
\[
\frac{1 + |x_2|}{1 + |x_2 - y_2|} \geq \frac{1 + |x_2|}{3/2 + |x_2|} \geq \frac{2}{3}.
\]

Therefore (8) is bounded below by
\[
(2/3)^E \int_{B(0,1)} \left( \frac{1 + |x|}{1 + |x - y|} \right)^E dy \geq (2/3)^E \left( \frac{1 + |x|}{2 + |x|} \right)^E \geq 3^{-E}.
\]

This then proves the analogue of (4) for \(\tilde{w}_{B(0,R),E}\). This completes the proof of Lemma 2.1.

**Remark 2.2.** As a corollary of Lemma 2.1 and the observation that \(1_B \lesssim_E w_{B,E}\), we have \(w_{B(0,R),E} \ast w_{B(0,R),E} \sim_E R^2 w_{B(0,R),E}\). This is also true for \(\tilde{w}_{B(0,R),E}\).

**Remark 2.3.** Let \(I = [-R/2, R/2]\) and \(I' = [-R'/2, R'/2]\) with \(0 < R' \leq R\). For \(x \in \mathbb{R}\), let \(w_{I,E}(x) := (1 + \frac{|x|}{R})^{-E}\) and similarly define \(w_{I',E}\). The same proof as (3) gives that
\[
w_{I,E} \ast w_{I',E} \leq 4^E R' w_{I,E}.
\]

This estimate will be used extensively in the proof of Lemma 3.17.

Lemma 2.1 has an immediate corollary which serves as the continuous analogue of the localization lemma given in Lemma 4.1 of [6]. This will allow us to upgrade from unweighted to weighted estimates, see later in Proposition 2.11. The inequality below is from the proof of Theorem 5.1 in [6].

**Corollary 2.4.** For \(1 \leq p < \infty\) and \(E \geq 10\),
\[
\|f\|^p_{L^p(w_{B(0,R),E})} \leq 3^E \int_{\mathbb{R}^2} \|f\|^p_{L^p(B(y,R))} w_{B(0,R),E}(y) dy.
\]

This corollary is also true with \(w_{B(0,R),E}\) replaced with \(\tilde{w}_{B(0,R),E}\).
Proof. Lemma 2.1 implies that
\[ \int_{\mathbb{R}^2} \| f \|^p_{L^p(B(y,R))} w_{B(0,R),E}(y) \, dy = \int_{\mathbb{R}^2} |f(x)|^p \left( \frac{1}{R^2} 1_{B(0,R)} * w_{B(0,R),E} \right)(x) \, dx \geq 3^{-E} \| f \|^p_{L^p(w_{B(0,R),E})} \]
which completes the proof of Corollary 2.4. \( \square \)

We close this section by proving two lemmas about the interaction between \( \tilde{w}_B \) and rotations which will be used in the proof of Theorem 6.1.

Lemma 2.5. Let \( c_j \in [\delta/2, 1 - \delta/2] \),
\[ R_J = \frac{1}{\sqrt{1 + 4c_j^2}} \left( \frac{1}{2|c_j|} - \frac{2|c_j|}{1} \right), \]
and \( \theta_J \) be such that \( \cos \theta_J = 1/\sqrt{1 + 4c_j^2} \) and \( \sin \theta_J = 2|c_j|/\sqrt{1 + 4c_j^2} \). Suppose \( |a| \leq 2\delta^{-1} \), then
\[ \tilde{w}_{B(R_J(a,0)^\theta, \delta^{-1})}(s) \leq 16^E \tilde{w}_{B(0,\delta^{-1})}(s). \]

Proof. We want to give an upper bound for
\[ \frac{\delta^{-1} + |s|}{\delta^{-1} + |s - (\cos \theta_J, \sin \theta_J)a|} \left( \frac{\delta^{-1} + |s|}{\delta^{-1} + |s - (\cos \theta_J, \sin \theta_J)a|} \right)^E \] (9)
that only depends on \( E \). We first consider the first expression in (9). If \( |s| < 3\delta^{-1} \), then
\[ \frac{\delta^{-1} + |s|}{\delta^{-1} + |s - (\cos \theta_J, \sin \theta_J)a|} \leq 4. \]
If \( |s| \geq 3\delta^{-1} \), then
\[ \frac{\delta^{-1} + |s|}{\delta^{-1} + |s - (\cos \theta_J, \sin \theta_J)a|} = \left( \frac{\delta^{-1} + |s|}{\delta^{-1} + |s|} \right)^E \left( \frac{\delta^{-1} + |s|}{\delta^{-1} + |s|} \right)^E = \left( \frac{|s|}{|s - (\cos \theta_J, \sin \theta_J)a|} \right)^{-1}. \] (10)
Since \( |s| \geq 3\delta^{-1} \) and \( |a| \leq 2\delta^{-1} \),
\[ \frac{|s - (\cos \theta_J, \sin \theta_J)a|}{|s|} \geq 1 - \frac{|a|}{|s|} \geq \frac{1}{3}. \]
Therefore (10) is \( \leq 4 \) and so the first expression in (9) is \( \leq 4^E \). We next consider the second expression in (9). The proof is almost exactly the same. If \( |s_2| \leq 3\delta^{-1} \),
\[ \frac{\delta^{-1} + |s_2|}{\delta^{-1} + |s_2 - (\sin \theta_J)a|} \leq 4. \]
For \( |s_2| > 3\delta^{-1} \),
\[ \frac{\delta^{-1} + |s_2|}{\delta^{-1} + |s_2 - (\sin \theta_J)a|} = \left( \frac{\delta^{-1} + |s_2|}{\delta^{-1} + |s_2|} \right)^E \left( \frac{\delta^{-1} + |s_2|}{\delta^{-1} + |s_2|} \right)^E = \left( \frac{|s_2|}{|s_2 - (\sin \theta_J)a|} \right)^{-1}. \] (11)
Since $|s_2| > 3\delta^{-1}$ and $|a| \leq 2\delta^{-1}$,
$$\frac{|s_2 - (\sin \theta_j)a|}{|s_2|} \geq 1 - \frac{|a|}{|s_2|} \geq \frac{1}{3}.$$  
Therefore (11) is \(\leq 4\) and so the second expression in (9) is \(\leq 4E\). This completes the proof of Lemma 2.5. \(\square\)

**Lemma 2.6.** Let \(R_J\) be as in Lemma 2.5. Then

$$\left(1 + \frac{|(R_J^{-1}x_1)|}{\delta^{-1}}\right)^{-2E} \left(1 + \frac{|(R_J^{-1}x_2)|}{\delta^{-1}}\right)^{-2E} \leq \tilde{w}_{B(0,\delta^{-1}),E}. \tag{12}$$

**Proof.** Since \((1 + \delta|x|) \leq (1 + \delta|x_1|)(1 + \delta|x_2|)\), the left hand side of (12) is

$$\leq \left(1 + \frac{|R_J^{-1}x_1|}{\delta^{-1}}\right)^{-2E} = \left(1 + \frac{|x_1|}{\delta^{-1}}\right)^{-2E} \leq \tilde{w}_{B(0,\delta^{-1}),E}$$

where the equality is because \(R_J\) is a rotation. This completes the proof of Lemma 2.6. \(\square\)

2.2. **Explicit Schwartz functions.** In addition to our polynomial decaying weights \(w_B\) and \(\tilde{w}_B\), we will also need to construct an explicit Schwartz function weight. More specifically, in Corollary 2.9 we construct a nonnegative \(\eta\) in \(\mathbb{R}^2\) such that

$$1_{B(0,1)}(x) \leq \eta(x) \text{ and } \text{supp}(\tilde{\eta}) \subset B(0,1).$$

Such an \(\eta\) will be used in the proof of reverse Holder (Lemma 2.20), \(l^2L^2\) decoupling (Lemma 2.21), and will also allow us to reset the “\(E\) parameter” when we prove the equivalence of local decoupling constants in Section 3 (in particular, Lemma 3.8 and Proposition 3.11).

We also construct an explicit smoothed indicator function which is equal to 1 on \([-1,1]\) and vanishes outside \([-3,3]\). This will be used in the proof of ball inflation (Theorem 6.1) and the equivalence of local decoupling constants (Lemma 3.10).

Existence of such Schwartz functions is easy to justify, however our goal is to obtain explicit bounds and so not only will we need to construct such functions but also need to construct them in such a way as to make it easy to compute with. Both Schwartz functions rely on the following lemma which is a small modification of Theorem 1.3.5 of [17].

**Lemma 2.7.** Let \(a_0 \geq a_1 \geq \ldots\) be a positive sequence such that \(a := \sum_{i \geq 0} a_i < \infty\). For \(i \geq 0\), let

$$H_i(x) := \frac{1}{\tilde{a_i}}1_{[-a_i/2,a_i/2]}(x)$$

and let

$$u_k(x) := (H_0 * \ldots * H_k)(x).$$

Then for \(k \geq 2\), \(u_k \in C^{k-1}_c(\mathbb{R})\) is supported in \([-a/2,a/2]\) and converges (uniformly) to a function \(u \in C^{\infty}_c(\mathbb{R})\) as \(k \to \infty\) which is also supported in \([-a/2,a/2]\). Furthermore,

$$|u^{(j)}(x)| \leq \frac{2^j}{a_0a_1 \ldots a_j}$$
for $j \geq 0$ and
\[
\hat{u}(\xi) = \prod_{i=0}^{\infty} \text{sinc}(a_i \xi)
\]
where $\text{sinc}(x) = (\sin \pi x)/(\pi x)$.

**Proof.** The proof is the same as that in Theorem 1.3.5 of [17] except in this case we have
\[
u^{(j)}_k = \left( \prod_{i=0}^{j-1} \frac{1}{a_i} (\tau_{-a_i/2} - \tau_{a_i/2}) \right) (H_j \ast \cdots \ast H_k)
\]
for $j \leq k-1$ where $(f \ast f)(x) = f(x-a)$ and the product is a composition of operators.

For the claim about $\hat{u}$, note that $\hat{H}_i(\xi) = \text{sinc}(a_i \xi)$ which implies that $\hat{u}(\xi) = \prod_{i=0}^{k} \text{sinc}(a_i \xi)$. Since $u_k \to u$ uniformly as $k \to \infty$ and since $u_k$ and $u$ are both supported on $[-a/2, a/2]$, $\hat{u}_k \to \hat{u}$ uniformly as $k \to \infty$. This completes the proof of Lemma 2.7.

We use Lemma 2.7 to construct a function $\psi$ on $\mathbb{R}$ such that $\psi \geq 1_{[-1/2,1/2]}$ and $\text{supp}(\hat{\psi}) \subset [-1/2, 1/2]$.

**Lemma 2.8.** For $x \in \mathbb{R}$, let
\[
\psi(x) := 4 \left( \text{sinc}(\frac{x}{6}) \prod_{i=1}^{\infty} \text{sinc}(\frac{x}{6i^2}) \right)^2.
\]
Then $\psi \geq 1_{[-1/2,1/2]}$, $\text{supp}(\hat{\psi}) \subset [-1/2, 1/2]$, and for all $x \in \mathbb{R}$ and $E \geq 100$,
\[
|\psi(x)| \leq \frac{E^6}{(1 + |x|)^{2E}}.
\]

**Proof.** Let $u$ be as in Lemma 2.7 with $a_0 = 1$ and $a_i = 1/i^2$. Then
\[
\hat{u}(x) = \text{sinc}(x) \prod_{i=1}^{\infty} \text{sinc}(x/i^2)
\]
and $u$ is supported in $[-3/2, 3/2]$.

Observe that $\psi(x) = F(x)^2$ with $F(x) = 2 \hat{u}(x/6)$. Since $F$ is even, for $x \in [-1/2, 1/2]$, $F(x) \geq F(1/2) \geq 1$. As $\psi \geq 0$ for all $x \in \mathbb{R}$, $\psi \geq 1_{[-1/2,1/2]}$. From the support of $u$, the Fourier transform of $F$ is supported in $[-1/4, 1/4]$. Since $\hat{\psi} = \hat{F} \ast \hat{F}$, $\hat{\psi}$ is supported in $[-1/2, 1/2]$.

By the construction of $u$,
\[
|u^{(j)}(x)| \leq 2^j \prod_{k=0}^{j} a_k^{-1} = 2^j \prod_{k=1}^{j} k^2 \leq 2^j j^{2j}.
\]
The support of $u$ and integration by parts gives that for any $j \geq 0$ and $x \neq 0$,
\[
|\hat{u}(x)| \leq \frac{1}{(2\pi |x|)^j} \|u^{(j)}\|_{L^1(\mathbb{R})} \leq \frac{3j^{2j}}{\pi^j |x|^j}.
\]
Applying the above bound to \( j = E \) shows that for \( x \neq 0, |\hat{u}(x)| \leq E^2|x|^{-E} \). Then for \( |x| \geq 1 \),
\[
|\psi(x)| = 4|\hat{u}(x/6)|^2 \leq E^{5E}|x|^{-2E}
\]
Thus if \( |x| \geq 1 \), \((1 + |x|)^{2E}|\psi(x)| \leq E^{6E} \). If \( |x| \leq 1 \), then explicit computation gives that \((1 + |x|)^{2E}|\psi(x)| \leq 4E^{2E+1} \). This completes the proof of Lemma 2.8.

Since \( B(0,1) = [-1/2,1/2]^2 \) and \((1 + |x|)(1 + |x|) \leq (1 + |x_1|)(1 + |x_2|)^2 \), we immediately have the following corollary.

**Corollary 2.9.** Let \( \psi \) be as in Lemma 2.8. For \( x \in \mathbb{R}^2 \), let
\[
\eta(x) = \psi(x_1)\psi(x_2).
\]
Then \( \eta \geq 1_{B(0,1)}, \supp(\hat{\eta}) \subset B(0,1) \), and for all \( x \in \mathbb{R}^2 \) and \( E \geq 100 \),
\[
|\eta(x)| \leq \frac{E^{12E}}{(1 + |x_1|)^{2E}(1 + |x_2|)^{2E}}.
\]
For \( B = B(c_B,R) \), define
\[
\eta_B(x) := \eta\left(\frac{x - c_B}{R}\right).
\]
Then for all \( x \in \mathbb{R}^2 \) and arbitrary \( E \geq 100 \),
\[
\eta_B(x) \leq E^{12E}w_{B,E}(x) \leq E^{12E}w_B(x).
\]
We now construct our smoothed indicator function and estimate the size of the Fourier transform of its moments.

**Lemma 2.10.** Let \( u \) be as in Lemma 2.7 with \( a_0 := 1/3 \) and \( a_i := 1/(3i^2) \). Then
\[
\Psi(x) := (u * 1_{[-2,2]})(x)
\]
is a \( C_c^\infty(\mathbb{R}) \) function which is equal to 1 on \([-1,1]\) and vanishes outside \([-3,3]\). For \( k \geq 0, x \in \mathbb{R} \), and \( E \geq 100 \) we have
\[
|\int_{\mathbb{R}} t^k \Psi(t)e^{2\pi itx} \, dt| \leq \frac{6^k E^{5E}}{(1 + |x|)^{2E}}.
\]

**Proof.** From Lemma 2.7 \( u \) is supported in \([-1,1]\). Since \( u \geq 0 \), \( \|u\|_{L^1} = \hat{u}(0) = 1 \). Then
\[
\Psi(x) = \int_{[x-2,x+2] \cap [-1,1]} u(s) \, ds = \begin{cases} 
1 & \text{if } x \in [-1,1] \\
0 & \text{if } x \notin [-3,3].
\end{cases}
\]
To prove (13), we first prove that for \( k \geq 0 \),
\[
|\partial^{2E}(x^k \Psi(x))| \leq 6^{2E+k} E^{4E}
\]
where \( \partial^E = d^E/dx^E \). From Lemma 2.7 for \( j \geq 0 \), \( |u^{(j)}(x)| \leq 3(2^j) \prod_{i=1}^{j} 3i^2 = 3(6^j)(j!)^2 \). Thus for \( j \geq 0 \),
\[
|\Psi^{(j)}(x)| = |(u^{(j)} * 1_{[-2,2]})(x)| \leq 12(6^j)(j!)^2.
\]
First suppose $2E \leq k$. Then since $\Psi$ is supported on $[-3, 3]$, 
\[
|\partial^{2E}(x^k \Psi(x))| = |\sum_{j=0}^{2E} \binom{2E}{j} \partial^j(x^k)\Psi^{(2E-j)}(x)| 
\leq \sum_{j=0}^{2E} \binom{2E}{j} \frac{k!}{(k-j)!} 3^{k-j} 12(6^{2E-j})(2E-j)^2 
\leq 12(6^{2E+3k})(2E!) \sum_{j=0}^{2E} \binom{k}{j} \leq 12(6^{2E+k})(2E!)^2.
\]

Next suppose $k < 2E$. Then similarly, 
\[
|\partial^{2E}(x^k \Psi(x))| \leq \sum_{j=0}^{k} \binom{2E}{j} \frac{k!}{(k-j)!} 3^{k-j} 12(6^{2E-j})(2E-j)^2 \leq 12(6^{2E+k})(2E!)^2.
\]

Since $E \geq 100$, $12(2E!)^2 \leq E^{4E}$, and so when combined with the above implies 
\[
|\partial^{2E}(x^k \Psi(x))| \leq 6^{2E+k} E^{4E}
\]
which proves (14).

We now prove (13). Integration by parts and (14) gives that for $x \neq 0$, 
\[
|\int_{\mathbb{R}} t^k \Psi(t)e^{2\pi i tx} dt| \leq \frac{6}{(2\pi |x|)^{2E}} \|\partial^{2E}(t^k \Psi(t))\|_{L^\infty} \leq \frac{6^k E^{4E}}{|x|^{2E}}.
\]
Thus for $|x| \geq 1$, 
\[
(1 + |x|)^{2E} |\int_{\mathbb{R}} t^k \Psi(t)e^{2\pi i tx} dt| \leq 2^E 6^k E^{4E} \leq 6^k E^{5E}.
\]

Observe that 
\[
\int_{\mathbb{R}} |t^k \Psi(t)| dt \leq 3^k \|\Psi\|_{L^1} = 4(3^k)
\]
where the last equality we have used that $u \geq 0$ and $\|u\|_{L^1} = 1$. Then for $|x| < 1$, 
\[
(1 + |x|)^{2E} |\int_{\mathbb{R}} t^k \Psi(t)e^{2\pi i tx} dt| \leq 4^{E+1} 3^k.
\]

This completes the proof of Lemma 2.10 \hfill \square

2.3. Immediate applications. Corollary 2.4 allows us to upgrade from estimates in $L^p(B)$ and $L^p(\eta_B)$ to estimates in $L^p(w_B)$ and $L^p(\tilde{w}_B)$. We have the following proposition which contains all three different scenarios we will need to upgrade from an unweighted estimate to a weighted estimate.

**Proposition 2.11.** Let $I \subset [0, 1]$ and $\mathcal{P}$ be a disjoint partition of $I$.

(a) Suppose for some $2 \leq p < \infty$, we have 
\[
\|\mathcal{E}_j g\|_{L^p(B)} \leq C(\sum_{j \in \mathcal{P}} \|\mathcal{E}_j g\|_{L^p(w_B,E)}^2)^{1/2}
\]
The same results are also true with \( w \) replaced with \( \tilde{w} \).

Proof. We first prove (a). Since for \( a \in \mathbb{R}^2 \), \((E_J g)(x + a) = (E_J h)(x)\) where \( h(\xi) = g(\xi)e(a_1 \xi + a_2 \xi^2)\), a change of variables shows that it suffices to prove (15) in the case when \( B \) is centered at the origin. Corollary 2.4 implies that for all \( g : [0, 1] \to \mathbb{C} \) and all squares \( B \) of side length \( R \), we have

\[
\|E_J g\|_{L^p(w_{B,E})} \leq 12^E C \sum_{J \in \mathcal{P}} \|E_J g\|_{L^2(w_{B,E})} \frac{1}{2}
\]

(15)

for all \( g : [0, 1] \to \mathbb{C} \) and all squares \( B \) of side length \( R \).

(b) Suppose we have

\[
\|E_J g\|_{L^2(B)} \leq C \sum_{J \in \mathcal{P}} \|E_J g\|_{L^2(w_{B,E})} \frac{1}{2}
\]

for all \( g : [0, 1] \to \mathbb{C} \) and all squares \( B \) of side length \( R \). Then for each \( E \geq 100 \), we have

\[
\|E_J g\|_{L^2(w_{B,E})} \leq 12^{E/2} E^{12E} C \sum_{J \in \mathcal{P}} \|E_J g\|_{L^2(w_{B,E})} \frac{1}{2}
\]

(16)

for all \( g : [0, 1] \to \mathbb{C} \) and all squares \( B \) of side length \( R \).

(c) Suppose for some \( 1 \leq p < q < \infty \), we have

\[
\|E_J g\|_{L^q(B)} \leq C \|E_J g\|_{L^p(w_B^n)}
\]

for all \( g : [0, 1] \to \mathbb{C} \) and all squares \( B \) of side length \( R \). Then for each \( E \geq 100 \), we have

\[
\|E_J g\|_{L^q(w_{B,E})} \leq 12^E E^{12E} C \|E_J g\|_{L^p(w_{B,E})} \frac{1}{q}
\]

(17)

for all \( g : [0, 1] \to \mathbb{C} \) and all squares \( B \) of side length \( R \).

The same results are also true with \( w_{B,E} \) replaced with \( \tilde{w}_{B,E} \).

Proof. We first prove (a). Since for \( a \in \mathbb{R}^2 \), \((E_J g)(x + a) = (E_J h)(x)\) where \( h(\xi) = g(\xi)e(a_1 \xi + a_2 \xi^2)\), a change of variables shows that it suffices to prove (15) in the case when \( B \) is centered at the origin. Corollary 2.4 implies that

\[
\|E_J g\|_{L^p(w_{B,E})} \leq 3^E \int_{\mathbb{R}^2} \|E_J g\|_{L^p(w_{B,E})} \frac{1}{2}
\]

\[
\leq 3^E R^{-2} C^p \int_{\mathbb{R}^2} \left( \sum_{J \in \mathcal{P}} \|E_J g\|_{L^p(w_{B,E})} \right)^{p/2} \frac{1}{2}
\]

\[
= 3^E R^{-2} C^p \left( \sum_{J \in \mathcal{P}} \left( \int_{\mathbb{R}^2} \|E_J g\|_{L^p(w_{B,E})} \right)^{p/2} \right)^{2/p}.
\]

Since \( p \geq 2 \), we can interchange the \( L^p(w_{B,E}) \) and \( l^2_j \) norms and the above is

\[
\leq 3^E R^{-2} C^p \left( \sum_{J \in \mathcal{P}} \left( \int_{\mathbb{R}^2} \|E_J g\|_{L^p(w_{B,E})} \right)^{p/2} \right)^{2/p}.
\]

(18)

Since \( B \) is assumed to be centered at the origin,

\[
\int_{\mathbb{R}^2} \|E_J g\|_{L^p(w_{B,E})} \frac{1}{2}
\]

\[
= \|E_J g\|_{L^p(w_{B,E})} \leq 4^E R^2 \|E_J g\|_{L^p(w_{B,E})}.
\]
where the inequality is an application of Lemma 2.1. Inserting this into (18) gives that

$$\|E_Ig\|_{L^p(w_{B,E})}^p \leq 12^E C^p \left( \sum_{J \in P} \|E_Jg\|_{L^p(w_{B,E})}^2 \right)^{p/2}.$$

Taking $1/p$ powers of both sides completes the proof of (15).

We next prove (b). Once again it suffices to prove (16) in the case when $B$ is centered at the origin. Corollary 2.4 implies that

$$\|E_Ig\|_{L^2(w_B)}^2 \leq 3^E \int_{\mathbb{R}^2} \|E_Ig\|_{L^2(B(y,R))}^2 w_B(y) \, dy$$

$$= 3^E R^{-2} C^2 \sum_{J \in P} \int_{\mathbb{R}^2} \|E_Jg\|_{L^2(\eta_B^2 w_B)}^2 w_B(y) \, dy$$

$$= 3^E R^{-2} C^2 \sum_{J \in P} \|E_Jg\|_{L^2(\eta_B^2 w_B)}^2$$

(19)

By Corollary 2.9 and Lemma 2.1

$$\eta_B^2 w_B \leq E^{24E} w_{B,2E} \leq E^{24E} 4^E R^2 w_{B,E}$$

and hence (19) is

$$\leq E^{24E} 12^E C^2 \sum_{J \in P} \|E_Jg\|_{L^2(w_B)}^2.$$

Taking $1/2$ powers of both sides completes the proof of (16).

We finally prove (c). Again it suffices to prove (17) in the case when $B$ is centered at the origin. Corollary 2.4 implies that

$$\|E_Ig\|_{L^q(w_{B,E})}^q \leq 3^E \int_{\mathbb{R}^2} \|E_Ig\|_{L^q(\eta_B^2 w_B)}^q w_{B,E}(y) \, dy$$

$$\leq 3^E C^q R^{-2q/p} \int_{\mathbb{R}^2} \|E_Ig\|_{L^p(\eta_B^2 w_B)}^q w_{B,E}(y) \, dy$$

$$= 3^E C^q R^{-2q/p} \|E_Ig(s)\|_{L^p(\eta_B^2 w_B)}^q w_{B,E}(y) \, ds$$

(20)

Since $q > p$, we can interchange the norms and the above is

$$\leq 3^E C^q R^{-2q/p} \|E_Ig|_{L^q(B(y,R))}^q \|\eta_B^2 w_{B,E}\|_{L^p(L^p(x,w_{B,E}))}^{q/p}$$

$$= 3^E C^q R^{-2q/p} (\int_{\mathbb{R}^2} |E_Ig(s)|^p (\eta_B^2 w_{B,E})(s)^{p/q} ds)^{q/p}$$

Corollary 2.9 and Lemma 2.1 give that

$$\eta_B^2 w_{B,E} \leq E^{12E} w_{B,E} \leq E^{12E} 4^E R^2 w_{B,E}.$$

Inserting this into (21) shows that

$$\|E_Ig\|_{L^q(w_{B,E})}^q \leq 12^E E^{12E} C^q R^{-2q/p} \|E_Ig\|_{L^p(w_{B,E})}^q.$$

Changing $L^q$ and $L^p$ into $L^q_y$ and $L^p_y$, respectively, removes the factor of $R^{2-2q/p}$.

Taking $1/q$ powers of both sides then completes the proof of (17).
Since the same estimates hold for $\tilde{w}_{B,E}$ in Lemma 2.1 Corollary 2.4 and Corollary 2.9, the above proof also shows that the proposition also holds with every instance of $w_{B,E}$ replaced with $\tilde{w}_{B,E}$. This completes the proof of Proposition 2.11. □

Remark 2.12. Note that a change of variables as in the beginning of the proof of Proposition 2.11 shows that knowing
\[
\|E_{Ig}\|_{L^p(\Omega(0,R))} \leq C\left(\sum_{J \in P} \|E_{Jg}\|^2_{L^p(w_{B(0,R),E})}\right)^{1/2}
\]  
for all $g : [0, 1] \to \mathbb{C}$ implies that
\[
\|E_{Ig}\|_{L^p(B)} \leq C\left(\sum_{J \in P} \|E_{Jg}\|^2_{L^p(w_{B,E})}\right)^{1/2}
\]  
for all $g : [0, 1] \to \mathbb{C}$ and all squares $B$ of side length $R$. Therefore often to check the hypotheses of Proposition 2.11 we will just prove (21) instead.

Remark 2.13. Corollary 2.4 is not the only way to convert unweighted estimates to weighted estimates. Another approach is to prove an unweighted estimate where $B$ is replaced by $2^nB$ for all $n \geq 0$ and then use that $w_{B,E} = \sum_{n=0}^{\infty} 2^{-nE}1_{2^nB}$ to conclude the weighted estimate.

Proposition 2.14. Let $B$ be a square of side length $R$ and let $B$ be a disjoint partition of $B$ into squares $\Delta$ with side length $R' < R$. Then for $E \geq 10$,
\[
\sum_{\Delta \in B} w_{\Delta,E} \leq 48^E w_{B,E}.
\]  
This inequality remains true with $w_{\Delta,E}$ and $w_{B,E}$ replaced with $\tilde{w}_{\Delta,E}$ and $\tilde{w}_{B,E}$.

Proof. It suffices to prove the case when $B$ is centered at the origin. Since $B$ is a disjoint partition of $B$,
\[
\sum_{\Delta \in B} 1_{\Delta} \leq 1_B.
\]

Therefore
\[
\sum_{\Delta \in B} 1_{\Delta} \ast w_{B(0,R'),E} \leq 1_B \ast w_{B(0,R'),E}.
\]  
Lemma 2.1 gives that
\[
3^{-E} R'^2 \sum_{\Delta \in B} w_{\Delta,E} \leq \sum_{\Delta \in B} 1_{\Delta} \ast w_{B(0,R'),E}
\]  
and
\[
1_B \ast w_{B(0,R'),E} \leq 8^E R'^2 w_{B,E}
\]  
where here we have also used $1_B \leq 2^E w_{B,E}$. Rearranging then proves (22). Since $1_B \leq 4^E \tilde{w}_{B,E}$, the same proof then proves (22) with $w_{\Delta,E}$ and $w_{B,E}$ replaced with $\tilde{w}_{\Delta,E}$ and $\tilde{w}_{B,E}$, respectively. This completes the proof of Proposition 2.14. □

Remark 2.15. The only property we really need in Proposition 2.14 is that $\sum_{\Delta \in B} 1_{\Delta} \leq C1_B$ for some absolute constant $C$. In particular, the same proof will work with finitely overlapping covers and when $R/R' \notin \mathbb{N}$. 
Lemma 2.16. Let $E \geq 10$ and $S = \left( \frac{1}{11} \right)$ where $|a| \leq 2$. Then
\[ w_{B(0, R), E}(Sx) \leq 90^E w_{B(0, R), E}(x). \]

Proof. Since our weights are centered at the origin, rescaling $x$, it suffices to prove the case when $R = 1$. Since $|a| \leq 2$, $S^{-1}B(0, 1) \subset B(0, 3)$ and so $1_{B(0, 1)}(Sx) \leq 1_{B(0, 3)}(x)$ for all $x \in \mathbb{R}^2$. Therefore
\[ 1_{B(0, 1)}(x) \leq 1_{B(0, 3)}(S^{-1}x) \]
for all $x \in \mathbb{R}^2$. Convolving both sides by $w_{B(0, 1), E}$ and applying Lemma 2.1 gives that
\[ 3^{-E} w_{B(0, 1), E} \leq (1_{B(0, 3)} \circ S^{-1}) \ast w_{B(0, 1), E}. \]
Thus it remains to prove that
\[ (1_{B(0, 3)} \circ S^{-1}) \ast w_{B(0, 1), E} \leq 30^E w_{B(0, 1), E} \circ S^{-1}. \]
This is the same as showing that
\[ \int_{\mathbb{R}^2} 1_{B(0, 3)}(S^{-1}y)(1 + |x - y|)^{-E} \, dy \leq 30^E (1 + |S^{-1}x|)^{-E}. \tag{23} \]
If $x \in 2^5 S(B(0, 1))$, then $|S^{-1}x| \leq 2^4 \sqrt{2}$ and so
\[ \int_{\mathbb{R}^2} 1_{B(0, 3)}(S^{-1}y)(1 + |x - y|)^{-E} \, dy \leq 1 \leq 24^E (1 + |S^{-1}x|)^{-E} \]
which proves (23) in this case. Next let $x \in 2^{n+1} S(B(0, 1)) \setminus 2^n S(B(0, 1))$ for some $n \geq 5$. Then
\[ (1 + |S^{-1}x|)^{-E} \geq (1 + \sqrt{2} \cdot 2^n)^{-E} \geq (2 \cdot 2^n)^{-E}. \]
Thus in this case, to prove (23) it suffices to show that
\[ \int_{\mathbb{R}^2} 1_{B(0, 3)}(S^{-1}y)(1 + |x - y|)^{-E} \, dy \leq 15^E 2^{-nE}. \tag{24} \]
We have
\[ \int_{\mathbb{R}^2} 1_{B(0, 3)}(S^{-1}y)(1 + |x - y|)^{-E} \, dy = \int_{S(B(0, 3))} \frac{1}{(1 + |x - y|)^E} \, dy \]
\[ = \int_{x - S(B(0, 3))} \frac{1}{(1 + |y|)^E} \, dy. \tag{25} \]
For $y \in x - S(B(0, 3))$, write $y = Sa - Sb$ where $a \in B(0, 2^{n+1}) \setminus B(0, 2^n)$ and $b \in B(0, 3)$. Since $\|S^{-1}\| \leq 2 \|S^{-1}\|_{\max} \leq 4$,
\[ |y| = |S(a - b)| \geq \|S^{-1}\|^{-1} |a - b| \geq \frac{1}{4}(2^{n-1} - \frac{3}{2} \sqrt{2}) \geq \frac{1}{10} 2^n. \]
Therefore the right hand side of \((24)\) is bounded above by \(9(10^E)2^{-nE}\) which proves \((24)\) and hence \((23)\). This completes the proof of Lemma 2.16 \(\Box\)

**Remark 2.17.** The same proof also shows that \(w_{B(0,R),E}(S^t x) \leq 90^E w_{B(0,R),E}\) since the only two properties of \(S\) we used were \(S^{-1}B(0,1) \subset B(0,3)\) and \(\|S^{-1}\| \leq 4\). These properties are satisfied if we replace \(S\) with \(S^t\).

**Lemma 2.18.** For \(0 < \delta \leq \sigma < 1\) with \(\sigma^{-1/2} \in \mathbb{N}\), let
\[
T = \begin{pmatrix}
\sigma^{1/2} & 2a\sigma^{1/2} \\
0 & \sigma
\end{pmatrix}
\]
with \(0 \leq a \leq 1 - \sigma^{1/2}\) and \(B = B(0, \delta^{-1})\). Then \(T(B)\) is contained in a \(3\sigma^{1/2}\delta^{-1} \times \sigma\delta^{-1}\) rectangle centered at the origin. Let \(B\) denote the partition of this rectangle into \(3\sigma^{-1/2}\) many squares with side length \(\sigma\delta^{-1}\). Then for \(E \geq 100\),
\[
\sum_{\Delta \in B} \tilde{w}_{E} \leq 720^E w_{B,E} \circ T^{-1}.
\]

**Proof.** The proof is similar to what we did in Proposition 2.14 and Lemma 2.16. Since \(B\) is axis-parallel and centered at the origin, \(T(B)\) is a parallelogram centered at the origin with a base parallel to the \(x\)-axis and height \(\sigma\delta^{-1}\). The corners of \(B\) are given by \((\pm \delta^{-1/2}, \pm \delta^{-1/2})\) and hence the corners of \(T(B)\) are given by
\[
\left(\frac{1}{2}\sigma^{1/2}(1 + 2a)\delta^{-1}, \frac{1}{2}\sigma\delta^{-1}\right), \quad \left(-\frac{1}{2}\sigma^{1/2}(1 + 2a)\delta^{-1}, -\frac{1}{2}\sigma\delta^{-1}\right)
\]
\[
\left(\frac{1}{2}\sigma^{1/2}(1 - 2a)\delta^{-1}, -\frac{1}{2}\sigma\delta^{-1}\right), \quad \left(-\frac{1}{2}\sigma^{1/2}(1 - 2a)\delta^{-1}, \frac{1}{2}\sigma\delta^{-1}\right).
\]

Then \(T(B)\) is contained in a \(3\sigma^{1/2}\delta^{-1} \times \sigma\delta^{-1}\) rectangle centered at the origin.

Note that \(T(B) \subset \bigcup_{\Delta \in B} \Delta \subset 10T(B)\) (we actually have \(\bigcup_{\Delta \in B} \Delta \subset (3 + 2a)T(B)\), but this is not needed) and so
\[
\sum_{\Delta \in B} 1_{B(\varepsilon \Delta, \sigma \delta^{-1})} \leq 1_{B(0,10\delta^{-1})} \circ T^{-1}.
\]

Convolution with \(\tilde{w}_{B(0,\sigma \delta^{-1}),E}\) gives that
\[
(\sigma \delta^{-1})^2 \sum_{\Delta \in B} \tilde{w}_{\Delta,E} \leq 3^E \left(1_{B(0,10\delta^{-1})} \circ T^{-1}\right) * \tilde{w}_{B(0,\sigma \delta^{-1}),E}.
\]

Thus it suffices to show that
\[
(\sigma \delta^{-1})^{-2} \left(1_{B(0,10\delta^{-1})} \circ T^{-1}\right) * \tilde{w}_{B(0,\sigma \delta^{-1}),E} \leq 240^E w_{B(0,\delta^{-1}),E} \circ T^{-1}.
\]

That is,
\[
(\sigma \delta^{-1})^{-2} \int_{\mathbb{R}^2} 1_{B(0,10\delta^{-1})}(T^{-1} y)(1 + \frac{|x - y|}{\sigma \delta^{-1}})^{-E}(1 + \frac{|x_2 - y_2|}{\sigma \delta^{-1}})^{-E} dy
\]
\[
\leq 240^E (1 + \frac{|T^{-1} x|}{\delta^{-1}})^{-E}.
\]
Rescaling $x$ and $y$ (by setting $X = x/(\sigma \delta^{-1})$ and $Y = y/(\sigma \delta^{-1})$) shows it suffices to prove that
\[
\int_{\mathbb{R}^2} 1_{B(0,10)}(S^{-1}y)(1 + |x - y|)^{-E}(1 + |x_2 - y_2|)^{-E} \, dy \leq 240^E (1 + |S^{-1}x|)^{-E} \tag{27}
\]
for all $x \in \mathbb{R}^2$ where $S = \sigma^{-1}T = (\sigma^{-1/2} 2a \sigma^{-1/2})$. Suppose $x \in 2^n S(B)$. Then $|S^{-1}x| \leq 32\sqrt{2}$ and so
\[
\int_{\mathbb{R}^2} 1_{B(0,10)}(S^{-1}y)(1 + |x - y|)^{-E}(1 + |x_2 - y_2|)^{-E} \, dy \leq 1 \leq 50^E (1 + |S^{-1}x|)^{-E}.
\]
It then remains to prove (27) for $x \in 2^{n+1} S(B) \setminus 2^n S(B)$ for all $n \geq 6$.

Fix an $n \geq 6$. For $x \in 2^{n+1} S(B) \setminus 2^n S(B)$, $|S^{-1}x| \leq 2^{n+1/2}$ and so $(2^{n+1})^{-E} \leq (1 + |S^{-1}x|)^{-E}$. Therefore to prove (27) it is enough to prove
\[
\int_{10S(B(0,1))} \frac{1}{|x - y|^E(1 + |x_2 - y_2|)^E} \, dy \leq 120^E 2^{-nE}
\]
for all $x \in 2^{n+1} S(B(0,1)) \setminus 2^n S(B(0,1))$.

Fix an $x \in 2^{n+1} S(B(0,1)) \setminus 2^n S(B(0,1))$. First suppose $|x_2| \geq 2^{2n/E}$. If $y \in x - 10S(B(0,1))$, then $y = Sa - Sb$ for some $a \in B(0,2^{n+1}) \setminus B(0,2^n)$ and $b \in B(0,10)$. Since $\|S^{-1}\| \leq 2\|S^{-1}\|_{\text{max}} \leq 4$, we first have
\[
|y| = |S(a - b)| \geq \|S^{-1}\|^{-1}|a - b| \geq \frac{1}{4}|a - b| \geq \frac{1}{4} (2^{n-1} - 5\sqrt{2}) \geq \frac{1}{20} 2^n.
\]
Next, $y_2 = x_2 - (Sb)_2 = x_2 - b_2$ and $b_2 \in [-5,5]$ and so
\[
\frac{1 + |x_2|}{1 + |y_2|} = \frac{1 + |x_2|}{1 + |x_2 - b_2|} \leq 1 + |b_2| \leq 6.
\]
Therefore
\[
\int_{x-10S(B(0,1))} \frac{1}{|y|^E(1 + |y_2|)^E} \, dy \leq \left(\frac{6}{1 + |x_2|}\right)^E \int_{|y| > 2^n/20} \frac{1}{|y|^E} \, dy \leq 120^E \frac{2^{2n}}{(1 + |x_2|)^E 2^{-nE}}
\]
and since $|x_2| \geq 2^{2n/E}$, we have proven (28) in this case.

Next, suppose $|x_2| < 2^{2n/E}$. In this case, we claim that $y \in x - 10S(B(0,1))$ satisfies $|y| \geq 2^n \sigma^{-1/2}$ and so we can bound the integral trivially. By assumption, $|(S^{-1}x)_2| = |x_2| < 2^{2n/E}$. Since $S^{-1}x \in 2^{n+1} B(0,1) \setminus 2^n B(0,1)$, $|S^{-1}x| \geq 2^{n-1}$. Thus
\[
|(S^{-1}x)_1| \geq 2^{n-1} - 2^{2n/E}.
\]
Since $(S^{-1}x)_1 = \sigma^{1/2}x_1 - 2ax_2$, it follows that
\[
|x_1| \geq \sigma^{-1/2}(2^{n-1} - 3 \cdot 2^{2n/E}).
\]
As in the previous paragraph, write $y = x - Sb$ for some $b \in B(0,10)$. Then

$$|y| \geq |y_1| = |x_1| - \sigma^{-1/2}|b_1 + 2ab_2| \geq \sigma^{-1/2}(2^{n-1} - 3 \cdot 2^{2n/E} - 15) \geq \frac{1}{5} \sigma^{-1/2}2^n$$

where the last inequality we have used that $n \geq 6$ and $E \geq 100$. Thus in the case when $|x_2| < 2^{2n/E}$,

$$\int_{|x - 10S(B(0,1))| y|E(1 + |y_2|)^E}^{1} \, dy \leq (100\sigma^{-1/2})^5 E^{2/2 - nE} \leq 6^{E^{2 - nE}}$$

which proves (28) in this case. This completes the proof of Lemma 2.18.

\[\qed\]

Remark 2.19. The $\tilde{w}_{\Delta,E}$ on the left hand side of (26) was needed to make sure the $E$ on both sides stays the same which is needed when we iterate later (for example in Lemma 5.2). If the $\tilde{w}_{\Delta,E}$ is replaced with $w_{\Delta,E}$, then by the same method as the proof above, one can obtain $\sum_{\Delta \in E} w_{\Delta,E} \leq E w_{B,E - 2} \circ T^{-1}$. In this case, some loss in $E$ must occur since we can consider the analogue of (27) and (28) and let $a = 0$ and $x = (0, 2^{n-1})$.

2.4. Bernstein’s inequality. Another immediate application of Proposition 2.11 is Bernstein’s inequality (also called reverse Holder in [6]). This should be compared with (2) at the beginning of Section 2. Our proof of Lemma 2.20 is the same as that of Corollary 4.3 of [6] except we make effective all the implicit constants.

\[\text{Lemma 2.20.}\] Let $1 \leq p < q \leq \infty$, $E \geq 100$, $J \subset [0, 1]$ with $\ell(J) = 1/R$ and $B \subset \mathbb{R}^2$ a square with side length $R \geq 1$. If $q < \infty$, then

$$\|\mathcal{E}_J g\|_{L^q(\tilde{w}_{B,E})} \leq E^{23E} \|\mathcal{E}_J g\|_{L^p(\tilde{w}_{B,E,q})}. \quad (29)$$

If $q = \infty$, then

$$\sup_{x \in B} |(\mathcal{E}_J g)(x)| \leq E^{23E} \|\mathcal{E}_J g\|_{L^p(\tilde{w}_{B,E})}. \quad (30)$$

\[\text{Proof.}\] Let $\eta$ be as in Corollary 2.9. Since $\eta_B \geq 1_B$,

$$\|\mathcal{E}_J g\|_{L^q(B)} \leq \|\eta_B \mathcal{E}_J g\|_{L^q(\mathbb{R}^2)}.$$

Let $\theta(x) = \Psi(2x_1)\Psi(2x_2)$ where $\Psi$ is defined as in Lemma 2.10. Then $\theta = 1$ on $B(0,1)$ and vanishes outside $B(0,3)$. Since $\tilde{\eta}_B$ is supported on $B(0,1/R)$, the Fourier transform of $\eta_B \mathcal{E}_J g$ is supported in some square $S$ with side length $10/R$. Then we have the following self-replicating formula

$$\eta_B \mathcal{E}_J g = (\eta_B \mathcal{E}_J g) * \tilde{\theta}_S.$$  

Young’s inequality then gives

$$\|\eta_B \mathcal{E}_J g\|_{L^q(\mathbb{R}^2)} \leq \|\eta_B \mathcal{E}_J g\|_{L^p(\mathbb{R}^2)} \|\tilde{\theta}_S\|_{L^r(\mathbb{R}^2)} \|\mathcal{E}_J g\|_{L^p(\tilde{\eta}_B)}$$

where $1/q = 1/p + 1/r - 1$ (since $q > p$, we have $r > 1$ and $\tilde{\theta}_S \in L^r$). Since $\tilde{\theta}(\xi) = (1/4)\tilde{\Psi}(\xi/2)\tilde{\Psi}(\xi/2)$, $\|\tilde{\theta}\|_{L^r(\mathbb{R}^2)} = 4^{1/r-1}\|\tilde{\Psi}\|_{L^r(\mathbb{R})}^2$, applying Lemma 2.10 gives that

$$\|\tilde{\theta}_S\|_{L^r(\mathbb{R}^2)} = (10/R)^{2 - 2/r} \|\tilde{\theta}\|_{L^r(\mathbb{R}^2)} = 25^{1/r'} R^{-2/r'} \|\tilde{\Psi}\|_{L^r(\mathbb{R})}^2 \leq 25^{1/r'} R^{-2/r'} E^{10E}.$$
Therefore
\[ \|E_Jg\|_{L^q(B)} \leq 25^{1/r'} E^{10E} R^{-2/r'} \|E_Jg\|_{L^p(\tilde{\eta}_B)} \]  
for all squares \( B \subset \mathbb{R}^2 \) with side length \( R \). If \( q < \infty \), applying Proposition 2.11 and then using that \( q > p \geq 1 \) and \( E \geq 100 \) proves (29).

If \( q = \infty \), then (31) and Corollary 2.9 implies that
\[ \sup_{x \in B} |(E_Jg)(x)| \leq 25^{1/p} E^{2(2E - 2/p)} \|E_Jg\|_{L^p(\tilde{\eta}_B, E)}. \]
Since \( E \geq 100 \), (30) then follows. This completes the proof of Lemma 2.20. \( \square \)

2.5. \( l^2 L^2 \) decoupling. We now prove \( l^2 L^2 \) decoupling which will follow from almost orthogonality. This proof is the same as that of Proposition 6.1 of [6] except we once again make explicit all implicit constants.

**Lemma 2.21.** Let \( J \subset [0, 1] \) be an interval of length \( \geq 1/R \) such that \( |J| R \in \mathbb{N} \). Then for \( E \geq 100 \) and each square \( B \subset \mathbb{R}^2 \) with side length \( R \),
\[ \|E_Jg\|_{L^2(\tilde{\eta}_B, E)}^2 \leq E^{13E} \sum_{J' \in P_{1/R}(J)} \|E_{J'}g\|_{L^2(\tilde{\eta}_B, E)}^2. \]

**Proof.** Let \( \eta \) be as in Corollary 2.9. Since \( \eta_B^2 \geq 1_B \),
\[ \|E_Jg\|_{L^2(B)}^2 \leq \|E_Jg\|_{L^2(\tilde{\eta}_B^2)}^2 = \|E_B E_Jg\|_{L^2(\mathbb{R}^2)}^2 = \sum_{J' \in P_{1/R}(J)} \eta_{BJ'} g_{\tilde{\eta}_B}^2(\mathbb{R}^2). \]
Note that the Fourier transform of \( \eta_B E_Jg \) is supported in the \( 1/R \)-neighborhood of the piece of parabola above \( J' \). Therefore \( \eta_B E_Jg \) and \( \eta_B E_J'g \) have disjoint Fourier support if \( J' \) and \( J'' \) are separated by \( \geq 2 \) intervals. Applying this and Plancherel gives
\[
\sum_{J' \in P_{1/R}(J)} \eta_B E_Jg_{L^2(\mathbb{R}^2)}^2 \leq \sum_{J' \in P_{1/R}(J)} \sum_{J'' \in P_{1/R}(J)} \|E_B E_Jg\|_{L^2(\mathbb{R}^2)} \|E_B E_J'g\|_{L^2(\mathbb{R}^2)} \]
\[
\leq \left( \sum_{J' \in P_{1/R}(J)} \|E_B E_Jg\|_{L^2(\mathbb{R}^2)}^2 \right)^{1/2} \left( \sum_{J'' \in P_{1/R}(J)} \|E_B E_J'g\|_{L^2(\mathbb{R}^2)}^2 \right)^{1/2} \]
\[
\leq \sqrt{5} \left( \sum_{J' \in P_{1/R}(J)} \|E_B E_Jg\|_{L^2(\mathbb{R}^2)}^2 \right)^{1/2} \left( \sum_{J'' \in P_{1/R}(J)} \|E_B E_J'g\|_{L^2(\mathbb{R}^2)}^2 \right)^{1/2} \]
\[
\leq 5 \sum_{J' \in P_{1/R}(J)} \|E_Jg\|_{L^2(\tilde{\eta}_B^2)}^2. \]
Thus we have shown that
\[
\|EJg\|_{L^2(B)} \leq \sqrt{5} \left( \sum_{J \in P_{1/R}(J)} \|EJg\|_{L^2(n_p^2)}^2 \right)^{1/2}
\]
for all squares \(B \subset \mathbb{R}^2\) with side length \(R\). Applying Proposition 2.11 then completes the proof of Lemma 2.21.

\[\square\]

Remark 2.22. To modify the weights \(w_B\) and \(\tilde{w}_B\), the main properties the weights need to satisfy are Lemma 2.1 and Lemma 2.18. The other lemmas such as Lemmas 2.5, 2.6, and 2.16 are also desired, but these should be easy to satisfy.

3. Equivalence of Local Decoupling Constants

Recall that \(D_{p,E}(\delta)\) is defined similarly as \(D_{p,E}(\delta)\) except instead of \(w_{B,E}\) we use \(\tilde{w}_{B,E}\). The main goal of this section is to prove that

\[D_{p,E}(\delta) \sim_E \tilde{D}_{p,E}(\delta)\]  \hspace{1cm} (32)

for \(2 \leq p \leq 6\), \(E \geq 100\), and \(\delta \in \mathbb{N}^{-2}\). This is proven in Proposition 3.11. This equivalence is a consequence of a larger equivalence of a collection of local decoupling constants. This section is similar to Remark 5.2 of [4] and may be of independent interest since it shows that an array of slightly different local decoupling constants are essentially the same size. The restriction \(p \leq 6\) is very mild and can be removed with a bit more care (at the cost of introducing an implied constant that depends on \(p\)). However since \(2 \leq p \leq 6\) is precisely the range we need, we restrict to this range. The appearance of the weight \(\tilde{w}_B\) in parabolic rescaling (arising from Lemma 2.18) means that (32) will play an essential part of the argument (for example in Proposition 4.1, Lemma 5.2, and Lemma 8.11).

Let \(f_R\) denote the Fourier restriction of \(f\) to \(R\). For each \(J = [n_J \delta^{1/2}, (n_J + 1)\delta^{1/2}] \in P_{\delta^{1/2}}([0,1])\), let

\[\theta_J := \{(s, L_J(s) + t) : n_J \delta^{1/2} \leq s \leq (n_J + 1)\delta^{1/2}, -5\delta \leq t \leq 5\delta\}\]

where

\[L_J(s) := (2n_J + 1)\delta^{1/2}s - n_J(n_J + 1)\delta\]

and \(0 \leq n_J \leq \delta^{-1/2} - 1\). Here \(\theta_J\) is a parallelogram that has height \(10\delta\) and has base parallel to the straight line connecting \((n_J \delta^{1/2}, n_J^2\delta)\) and \(((n_J + 1)\delta^{1/2}, (n_J + 1)^2\delta)\). We note that for \(\xi \in \theta_J\),

\[|\xi_2 - L_J(\xi_1)| \leq 5\delta\]  \hspace{1cm} (33)

and

\[|L_J(\xi_1) - \xi_1^2| \leq \delta/4.\]  \hspace{1cm} (34)

Boundedness of the Hilbert transform implies that Fourier restriction to \(\theta_J\) is a bounded operator from \(L^p \to L^p\) with operator norm bounded independent of \(J\), we make this explicit with the following lemma.
**Lemma 3.1.** For each $J \in P_{3/2}([0,1])$ and $2 \leq p < \infty$, $\|f_{\theta_J}\|_p \leq C_p \|f\|_p$ with $C_p := (\frac{3}{2} + \frac{1}{2} \cot(\frac{\pi}{4p}))^4$.

**Proof.** Fix $J \in P_{3/2}([0,1])$. Let $R$ denote the operator defined by $\hat{R}f = \hat{f}1_{\theta_J}$. Let $S$ denote the operator defined by $\hat{S}f = \hat{f}1_{[0,\infty)}$. Each $\theta_J$ is the intersection of four half planes in $\mathbb{R}^2$. Since multiplier norms are unchanged after rotation and translation,

$$\|R\|_{p\to p} \leq \|S\|_{p\to p}^4.$$  

Note that here we have also used that the operator norm of Fourier restriction to a half plane is bounded above by $\|S\|_{p\to p}$ which follows from Fubini’s Theorem. If $H$ denotes the Hilbert transform, observe that $\hat{f}(\xi) + i\hat{H}f(\xi) = 2\hat{f}(\xi)1_{[0,\infty)}(\xi)$ almost everywhere. Since $2 \leq p < \infty$, $\|H\|_{p\to p} \leq \cot(\frac{\pi}{2p})$. Therefore

$$\|S\|_{p\to p} \leq \frac{1}{2} + \frac{1}{2} \cot(\frac{\pi}{2p}).$$

Inserting this into (35) then completes the proof of Lemma 3.1. \(\square\)

**Remark 3.2.** One can think of $\theta_J$ as a polygonal approximation of the set $\{(s, s^2 + t) : s \in J, |t| \leq \delta\}$. The reason why we use $\theta_J$ instead is because Fourier restriction to the aforementioned set is not bounded in $L^p$ for $p \neq 2$.

To prove (32), we introduce two more local decoupling constants and show that all four decoupling constants are equivalent.

**Definition 3.3.** Let $\delta \in \mathbb{N}^{-2}$, $2 \leq p < \infty$ and $E \geq 1$. Let $\eta$ be as in Corollary 2.9. Let $\overline{D}_p(\delta)$ be the smallest constant such that

$$\|\mathcal{E}_{[0,1]}g\|_{L^p(B)} \leq \overline{D}_p(\delta)(\sum_{J \in P_{3/2}([0,1])} \|\mathcal{E}_{\theta_J}g\|_{L^p(\eta B)}^2)^{1/2}$$

for all $g : [0,1] \to \mathbb{C}$ and all squares $B$ with side length $\delta^{-1}$. Let $\hat{D}_{p,E}(\delta)$ be the smallest constant such that

$$\|f\|_{L^p(B)} \leq \hat{D}_{p,E}(\delta)(\sum_{J \in P_{3/2}([0,1])} \|f_{\theta_J}\|_{L^p(w_{B,E})}^2)^{1/2}$$

for all $f$ Fourier supported in $\Theta = \bigcup_{J \in P_{3/2}([0,1])} \theta_J$ and all squares $B$ with side length $\delta^{-1}$.

From our definitions of $w_B$, $\tilde{w}_B$, and $\eta_B$, observe that

$$1_B \leq 2^E w_{B,E}, \quad 1_B \leq 4^E \tilde{w}_{B,E}, \quad 1_B \leq \eta_B.$$ 

Furthermore, note that by the triangle inequality followed by Cauchy-Schwarz, all four local decoupling constants we have defined are $\lesssim_{E,p} \delta^{-1/4}$. Taking a specific $g : [0,1] \to \mathbb{C}$ or a specific $f$ with Fourier support in $\Theta$ and using Proposition 2.11 shows that $D_{p,E}(\delta), \hat{D}_{p,E}(\delta)$, and $\hat{D}_{p,E}(\delta)$ are $\gtrsim_{E,p} 1$. We make this precise with $\hat{D}_{p,E}$ which is the only decoupling constant we need an explicit lower bound.
Remark 3.4. Another consequence of the equivalence of the four local decoupling constants is that $D_p(\delta) \lesssim_{E,p} 1$ but this is not immediate from the definition.

Lemma 3.5. For $p \geq 2$ and $E \geq 10$, $\hat{D}_{p,E}(\delta) \geq 12^{-E/p}$.

Proof. Let $\hat{D}_{p,E}'(\delta)$ be the smallest constant such that

$$\|f\|_{L^p(w_{B,E})} \leq \hat{D}_{p,E}'(\delta) \left( \sum_{J \in P_{\frac{1}{3},\frac{2}{3}}([0,1])} \|f_{\theta_J}\|_{L^p(w_{B,E})}^2 \right)^{1/2}$$

for all $f$ Fourier supported in $\Theta$ and all squares $B$ with side length $\delta^{-1}$. Proposition 2.11 implies that $\hat{D}_{p,E}'(\delta) \leq 12^{E/p} \hat{D}_{p,E}(\delta)$. From the definition,

$$\hat{D}_{p,E}(\delta) = \sup_{f,B} \left( \sum_{J \in P_{\frac{1}{3},\frac{2}{3}}([0,1])} \frac{\|f\|_{L^p(w_{B,E})}}{\|f_{\theta_J}\|_{L^p(w_{B,E})}} \right)^{1/2}$$

where the sup is taken over the $f$ and $B$ as mentioned at the beginning of this proof. Taking an $f$ with Fourier support on $\theta_{[0,\frac{1}{2}]}$ shows that $\hat{D}_{p,E}'(\delta) \geq 1$. Here we note that we needed that the numerator of the right hand side of (36) to be $L^p(w_{B,E})$ rather than $L^p(B)$. Therefore $\hat{D}_{p,E}(\delta) \geq 12^{-E/p}$ which completes the proof of Lemma 3.5.

Remark 3.6. The decoupling constants $D_{p,E}(\delta)$ and $\hat{D}_{p,E}(\delta)$ are useful because $w_B \ast w_B \sim E R^2 w_B$ and similarly for $\tilde{w}_B$. This allows us to use Proposition 2.11 to upgrade from unweighted to weighted estimates which is an important part of the argument. The same cannot be said with the Schwartz weight decoupling constant $\overline{D}_p(\delta)$ since we do not necessarily have $\eta_B \ast \eta_B \sim R^2 \eta_B$. This useful convolution property of the $w_B$ and $\tilde{w}_B$ makes $D_{p,E}(\delta)$ and $\hat{D}_{p,E}(\delta)$ ideal for iterative parts of the argument.

On the other hand, the decoupling constants $\overline{D}_p(\delta)$ and $\hat{D}_{p,E}(\delta)$ are more useful for Fourier type arguments since the Fourier transform of $w_B$ and $\tilde{w}_B$ are of sinc type and so do not work well with Fourier arguments. One corollary of the results proven in this section is that all four local decoupling constants are essentially equivalent so we can easily swap between them.

To prove (32) we will prove the chain of inequalities

$$D_{p,E}(\delta) \leq \tilde{D}_{p,E}(\delta) \leq E \overline{D}_p(\delta) \leq E \hat{D}_{p,G}(\delta) \leq E D_{p,E}(\delta)$$

for $2 \leq p \leq 6$ and some $G < E$ we will make explicit in our proof.

The first two inequalities follow from that $\eta_B \lesssim_E w_B \lesssim \tilde{w}_B$. The third inequality follows from boundedness of the Hilbert transform (Lemma 3.1) and the last inequality will follow from adapting the proof of Theorem 5.1 in [6] to our case and is the most technical.

Lemma 3.7. For $E \geq 100$ and $2 \leq p < \infty$,

$$D_{p,E}(\delta) \leq \tilde{D}_{p,E}(\delta) \leq E^{12E/p} \overline{D}_p(\delta).$$
The first inequality follows from the observation that \( \hat{w}_B \leq w_B \). The second inequality follows from Corollary 2.3, in particular, \( \eta_B \leq E^{12E} \hat{w}_{B,E} \). This completes the proof of Lemma 3.7. 

As mentioned above, the third inequality in (37) comes from boundedness of the Hilbert transform. In particular, we need the following lemma. Because \( \overline{D}_p \) does not depend on \( E \), this lemma allows us to “reset” the \( E \) parameter in \( D_{p,E} \). This is useful because going up in the \( E \) parameter of \( D_{p,E} \) is easy but going down is much harder.

**Lemma 3.8.** For \( \delta \in \mathbb{N}^- \), \( E \geq 1 \), and \( 2 \leq p < \infty \), we have

\[
\overline{D}_p(\delta) \leq (3C_p + 5 \cdot 12^{E/p}) \hat{D}_{p,E}(\delta)
\]

where \( C_p \) is as defined in Lemma 3.7.

**Proof.** We first assume that \( \delta \in \mathbb{N}^- \) and \( \delta \leq 1/36 \). Fix arbitrary \( g : [0, 1] \to \mathbb{C} \) and square \( B \) with side length \( \delta^{-1} \). We can write

\[
g = g_{1_{[0,1]}}^1 + g_{1_{[\delta^{1/2},1-\delta^{1/2}]}} \leq g_1 + g_2.
\]

Then

\[
\| \mathcal{E}_{[0,1]}g \|_{L^p(B)} \leq \| \mathcal{E}_{[0,1]}g_1 \|_{L^p(B)} + \| \mathcal{E}_{[0,1]}g_2 \|_{L^p(B)}.
\]

Using the support of \( g_1 \), the triangle inequality, \( 1_B \leq \eta_B \), and Lemma 3.5 we have

\[
\| \mathcal{E}_{[0,1]}g_1 \|_{L^p(B)} \leq \| \mathcal{E}_{[\delta^{1/2},1]}g \|_{L^p(B)} + \| \mathcal{E}_{[1-\delta^{1/2},1]}g \|_{L^p(B)} \leq 2 \cdot 12^{E/p} \hat{D}_{p,E}(\delta) \left( \sum_{J \in P_{1/2}([0,1])} \| \mathcal{E}_j g \|_{L^p(\eta_B)} \right)^{1/2}. \tag{38}
\]

Since \( g_2 \) is supported in \([\delta^{1/2}, 1 - \delta^{1/2}]\), the Fourier transform of \( \eta_B \mathcal{E}_{[0,1]}g_2 = \eta_B \mathcal{E}_{[\delta^{1/2},1-\delta^{1/2}]}g \) is supported in a \( \delta \)-neighborhood of this interval which is contained in \( \Theta \). Therefore

\[
\| \eta_B \mathcal{E}_{[0,1]}g_2 \|_{L^p(B)} \leq \hat{D}_{p,E}(\delta) \left( \sum_{J \in P_{1/2}([0,1])} \| \eta_B \mathcal{E}_{[0,1]}g_2 \|_{L^p(\eta_B)} \right)^{1/2}. \tag{39}
\]

Note that since \( g_2 = g_{1_{[\delta^{1/2},1-\delta^{1/2}]}} \), \( (\eta_B \mathcal{E}_{[0,1]}g_2)_{\theta_J} = (\eta_B \mathcal{E}_{[\delta^{1/2},1-\delta^{1/2}]}g)_{\theta_J} \)

\[
= \begin{cases} 
(\eta_B \mathcal{E}_{J}\theta_J)_{\theta_J} & \text{if } J = [0, \delta^{1/2}] \\
(\eta_B \mathcal{E}_{J}\theta_J + \eta_B \mathcal{E}_{J}g)_{\theta_J} & \text{if } J = [\delta^{1/2}, 2\delta^{1/2}] \\
(\eta_B \mathcal{E}_{J}\theta_J + \eta_B \mathcal{E}_{J}g + \eta_B \mathcal{E}_{J}g)_{\theta_J} & \text{if } J \in P_{\delta^{1/2}}([2\delta^{1/2}, 1 - 2\delta^{1/2}]) \\
(\eta_B \mathcal{E}_{J}\theta_J + \eta_B \mathcal{E}_{J}g)_{\theta_J} & \text{if } J = [1 - 2\delta^{1/2}, 1 - \delta^{1/2}] \\
(\eta_B \mathcal{E}_{J}\theta_J)_{\theta_J} & \text{if } J = [1 - \delta^{1/2}, 1].
\end{cases}
\]

where \( J_L \) and \( J_R \) denote the intervals to the left and right of \( J \). Lemma 3.1 gives that for \( J \in P_{\delta^{1/2}}([2\delta^{1/2}, 1 - 2\delta^{1/2}]), \)

\[
\left( \sum_{J' \in [J_L, J_R]} \| (\eta_B \mathcal{E}_{J}\theta_J)_{\theta_J} \|_{L^p(\eta_B)} \right)^{1/2} \leq \sum_{J' \in [J_L, J_R]} \| (\eta_B \mathcal{E}_{J}\theta_J)_{\theta_J} \|_{L^p(\eta_B)}.
\]
Similarly we have
\[
\|(\eta B\mathcal{E}[0,1]g_2)_{[0,\delta^{1/2}]}\|_{L^p(w_{B,E})} \leq C_p\|\mathcal{E}[\delta^{1/2},2\delta^{1/2}]g\|_{L^p(\eta B)}
\]
\[
\|(\eta B\mathcal{E}[0,1]g_2)_{[1-\delta^{1/2},2\delta^{1/2}]}\|_{L^p(w_{B,E})} \leq C_p\|\mathcal{E}[1-2\delta^{1/2},1-\delta^{1/2}]g\|_{L^p(\eta B)}
\]
\[
\|(\eta B\mathcal{E}[0,1]g_2)_{q [\delta^{1/2},2\delta^{1/2}]}\|_{L^p(w_{B,E})} \leq C_p(\|\mathcal{E}[\delta^{1/2},2\delta^{1/2}]g\|_{L^p(\eta B)} + \|\mathcal{E}[2\delta^{1/2},3\delta^{1/2}]g\|_{L^p(\eta B)})
\]
and
\[
\|(\eta B\mathcal{E}[0,1]g_2)_{[1-2\delta^{1/2},1-\delta^{1/2}]}\|_{L^p(w_{B,E})} \leq C_p(\|\mathcal{E}[1-3\delta^{1/2},1-2\delta^{1/2}]g\|_{L^p(\eta B)} + \|\mathcal{E}[1-2\delta^{1/2},1-\delta^{1/2}]g\|_{L^p(\eta B)})
\]
where here we have used that \(\delta \leq 1/36\). Applying Cauchy-Schwarz and using the above four inequalities gives that
\[
\sum_{J \in P_{3^{1/2}}([0,1])} \|(\eta B\mathcal{E}[0,1]g_J)\|_{L^p(w_{B,E})}^2 \leq 9C_p^2 \sum_{J \in P_{3^{1/2}}([0,1])} \|\mathcal{E}g\|_{L^p(\eta B)}^2
\]
Combining this with (39) and \(1_B \leq \eta B\) gives
\[
\|\mathcal{E}[0,1]g_2\|_{L^p(B)} \leq 3C_p\hat{D}_{p,E}(\delta)(\sum_{J \in P_{3^{1/2}}([0,1])} \|\mathcal{E}g\|_{L^p(\eta B)}^2)^{1/2}.
\]
Combining (38) and (40) proves that
\[
\overline{D}_p(\delta) \leq (3C_p + 2 \cdot 12^{E/p})\hat{D}_{p,E}(\delta)
\]
for all \(\delta \in \mathbb{N}^{-2}\) and \(\delta \leq 1/36\).

For \(\delta = 1, 1/4, 1/9, 1/16, \) and \(1/25\), we resort to the trivial bound. Proceeding as in the proof of (38) shows that for each such \(\delta = 1/i^2\), \(i = 1, 2, \ldots, 5\), we have
\[
\overline{D}_p(\delta) \leq 5 \cdot 12^{E/p}\hat{D}_{p,E}(\delta).
\]
Combining this with (41) then completes the proof of Lemma 3.8 \(\square\)

**Remark 3.9.** The reason why we split \(g\) up into \(g_1\) and \(g_2\) in proof above is because \(\eta B\mathcal{E}[0,1]g\) is Fourier supported in a set that is slightly bigger than \(\Theta\).

The last inequality in (37) is the most technical of the four inequalities. The proof is similar to that of Theorem 5.1 in [6] however our proof is more complicated since our definition of \(\hat{D}_{p,E}(\delta)\) uses Fourier restriction to the parallelogram \(\theta J_i\) (to take advantage of \(L^p\) boundedness) rather than Fourier restriction to a \(\delta\)-tube of a piece of parabola. We also want explicit constants and so we will need to spend some time to extract explicit constants from taking a large number of derivatives. We state our lemma below but due to the length of its proof, we defer the proof to the end of this section.

To simplify some constants, we also restrict to the range when \(2 \leq p \leq 6\) since this is the range we care about. The restriction that \(p \leq 6\) is only used once in the proof of Lemma 3.10 (in particular at the end of the proof of Lemma 3.16) and is a very mild assumption which can be removed with a bit more care.
Lemma 3.10. For $E \geq 10$ and $2 \leq p \leq 6$,
\[
\hat{D}_{p,E}(\delta) \leq E^{60E}D_{p,2E+7}(\delta).
\]

Since $w_{B,E_2} \leq w_{B,E_1}$ for $E_1 \leq E_2$, $D_{p,E_1}(\delta) \leq D_{p,E_2}(\delta)$ and so we can increase the $E$ parameter at no cost. Combining Lemmas 3.7, 3.10 proves the following result which shows (37) and hence (32).

Proposition 3.11. For $\delta \in \mathbb{N}^{-2}$, $E \geq 100$, and $2 \leq p \leq 6$, we have
\[
D_{p,E}(\delta) \leq \hat{D}_{p,E}(\delta) \leq E^{6E}\overline{D}_p(\delta) \leq E^{7E}\hat{D}_{p,G}(\delta) \leq E^{70E}D_{p,E}(\delta)
\]
where $G = [(E - 7)/2]$.

Proof. Fix arbitrary integer $E \geq 100$. Using Lemma 3.7 and that $2 \leq p \leq 6$, we have
\[
D_{p,E}(\delta) \leq \hat{D}_{p,E}(\delta) \leq E^{6E}\overline{D}_p(\delta).
\]

Now we use Lemma 3.8 to reset our $E$. Since $E \geq 100$, $G > 10$. From Lemmas 3.8 and 3.10
\[
E^{6E}\overline{D}_p(\delta) \leq E^{7E}\hat{D}_{p,G}(\delta) \leq E^{7E}G^{60G}D_{p,2G+7}(\delta)
\]
where in the first inequality we have used that $C_p \leq 32$ for $2 \leq p \leq 6$. Increasing $2G+7$ to $E$ bounds the above by $E^{70E}D_{p,E}(\delta)$. This completes the proof of Proposition 3.11. \(\square\)

3.1. Proof of Lemma 3.10. This proof is similar to the proof of Theorem 5.1 in [6]. Our goal is to show that if $f$ is Fourier supported on $\Theta = \bigcup_{J \in P_{3/2}([0,1])} \theta_J$, then
\[
\|f\|_{L^p(B)} \lesssim E^{-2} E_{p,2E+7}(\delta) \left( \sum_{J \in P_{3/2}([0,1])} \|f_{\theta_J}\|_{L^p(w_{B,E})}^p \right)^{1/2}
\]
for all squares $B$ with side length $\delta^{-1}$ and some implied constant that will be made explicit in our proof. It suffices to show that this is true in the case when $B$ is centered at the origin.

Since $f$ is Fourier supported on $\Theta$, for $x \in B$,
\[
f(x) = \sum_{J \in P_{3/2}([0,1])} \int_{\theta_J} \hat{f}(\xi) e(\xi \cdot x) \, d\xi
\]
\[
= \sum_{J \in P_{3/2}([0,1])} \int_{J \times [-5\delta,5\delta]} \hat{f}(s, L_J(s) + t) e(s x_1 + s^2 x_2) e((L_J(s) - s^2) x_2) e(t x_2) \, ds \, dt.
\]

Note that here both $t$ and $L_J(s) - s^2$ are of size $O(\delta)$ and $x_2$ is of size $O(\delta^{-1})$, so the contribution from $e((L_J(s) - s^2) x_2)$ and $e(t x_2)$ should be negligible. We make this rigorous. Since
\[
e(t x_2) = \sum_{j \geq 0} \frac{(2\pi)^j}{j!} \left( \frac{2ix_2}{\delta^{-1}} \right)^j \left( \frac{\delta^{-1}t}{2} \right)^j
\]
and
\[ e((L_J(s) - s^2)x_2) = \sum_{k \geq 0} \frac{(2\pi)^k}{k!} \left( \frac{2ix_2}{\delta^{-1}L_J(s) - s^2} \right)^k, \]
it follows that for \( x \in B \),
\[ \left| f(x) \right| \leq \sum_{j,k \geq 0} \frac{(2\pi)^k(2\pi)^j}{k!j!} \sum_{J \in \mathcal{P}_{\delta^1/2}(\{0,1\})} (E_J g_{j,k})(x) \]
where \( g_{j,k} : [0,1] \to \mathbb{C} \) is defined pointwise almost everywhere piecewise on each \( J \in \mathcal{P}_{\delta^1/2}(\{0,1\}) \) by
\[ g_{j,k}(s) = \left( \frac{\delta^{-1}(L_J(s) - s^2)}{2} \right)^k \int_{-5\delta}^{5\delta} \widehat{f}(s, L_J(s) + t) \left( \frac{\delta^{-1}t}{2} \right)^j \, dt \]
for \( s \in J \). Let \( F := 2E + 7 \). We then have
\[ \|f\|_{L^p(B)} \leq D_{p,F}(\delta) \sum_{j,k \geq 0} \frac{(2\pi)^k(2\pi)^j}{k!j!} \left( \sum_{J \in \mathcal{P}_{\delta^1/2}(\{0,1\})} \|E_J g_{j,k}\|_{L^p(w_{B,F})}^2 \right)^{1/2}. \quad (42) \]
It then remains to prove that
\[ \|E_J g_{j,k}\|_{L^p(w_{B,F})} \lesssim \exp(O(j) + O(k)) \|f_{\theta_j}\|_{L^p(w_{B,F})} \]
for some implied constants that will be made explicit in our proof. We first claim it suffices to only prove (43) when \( J = [0, \delta^{1/2}] \).

**Lemma 3.12.** Suppose we knew that
\[ \|E_{[0,\delta^{1/2}]} \left( \frac{\delta^{-1}(\delta^{1/2}s - s^2)}{2} \right)^k \int_{-5\delta}^{5\delta} \widehat{f}(s, \delta^{1/2}s + t) \left( \frac{\delta^{-1}t}{2} \right)^j \, dt \|_{L^p(w_{B,F})} \]
\[ \leq C \|f_{\theta_{[0,\delta^{1/2}]}}\|_{L^p(w_{B,F})}. \quad (44) \]
for some constant \( C \). Then
\[ \|E_{[n,\delta^{1/2},(n+1)\delta^{1/2}]} \left( \frac{\delta^{-1}(L_J(s) - s^2)}{2} \right)^k \int_{-5\delta}^{5\delta} \widehat{f}(s, L_J(s) + t) \left( \frac{\delta^{-1}t}{2} \right)^j \, dt \|_{L^p(w_{B,F})} \]
\[ \leq 90^{(E+F)/p} C \|f_{\theta_{[n,\delta^{1/2},(n+1)\delta^{1/2}]}}\|_{L^p(w_{B,F})}. \quad (45) \]

**Remark 3.13.** Here \( s \) is a dummy variable, so \( E_J g(s) \) means the extension operator applied to the function \( g(s) \) creating the function \( (E_J g)(x) \).

**Proof.** This proof is essentially a change of variables. The main idea is to translate \( \theta_{[n,\delta^{1/2},(n+1)\delta^{1/2}]} \) to the origin and apply a shear matrix to turn it into \( \theta_{[0,\delta^{1/2}]} \). Then apply (44) and finally undo the shear transformation. The weights \( w_B \) are preserved from (44) because of Lemma 2.16.
We have
\[
\left( E_{[nJ,\delta^{1/2},(nJ+1)\delta^{1/2}]}(\delta^{-1/2}(L_J(s) - s^2)\right) k \int_{-\delta}^{5\delta} \hat{f}(x, L_J(s) + t)(\frac{\delta^{-1/2} t}{2})^j dt \right) (x)
\]
\[
= \int_{[nJ,\delta^{1/2},(nJ+1)\delta^{1/2}]} \left( \frac{\delta^{-1/2}(L_J(s) - s^2)}{2} \right) k \int_{-\delta}^{5\delta} \hat{f}(x, L_J(s) + t)(\frac{\delta^{-1/2} t}{2})^j dt e(sx_1 + s^2x_2) ds.
\]

The change of variables \( u = s - nJ\delta^{1/2} \) and the observation that
\[
L_J(u + nJ\delta^{1/2}) - (u + nJ\delta^{1/2})^2 = \delta^{1/2}u - u^2
\]
gives that the above is equal in absolute value to
\[
\int_{[0,\delta^{1/2}]} \left( \frac{\delta^{-1/2}(L_J(u) - u^2)}{2} \right) k \int_{-\delta}^{5\delta} \hat{f}(u + nJ\delta^{1/2}, L_J(u + nJ\delta^{1/2}) + t)
\]
\[
\times \left( \frac{\delta^{-1/2} t}{2} \right)^j e(u(x_1 + 2nJ\delta^{1/2}x_2) + u^2x_2) du.
\]

Since \(|2nJ\delta^{1/2}| \leq 2\), after a change of variables and an application of Lemma 2.16, the right hand side of (15) is bounded above by
\[
90F^{2p} ||E_{[0,\delta^{1/2}]}(\frac{\delta^{-1/2}(L_J(s) - s^2)}{2})^k \times
\]
\[
\int_{-\delta}^{5\delta} \hat{f}(s + nJ\delta^{1/2}, L_J(s + nJ\delta^{1/2}) + t)(\frac{\delta^{-1/2} t}{2})^j dt\|_{L^p(w_B,F)} \] (46)

Observe that
\[
L_J(s + nJ\delta^{1/2}) = nJ^2\delta + (2nJ + 1)\delta^{1/2}s.
\]

Let
\[
g_J(x) := f(x)e^{-2\pi i x \cdot (nJ\delta^{1/2}, nJ^2\delta)}.
\]

Then
\[
\hat{f}(s + nJ\delta^{1/2}, L_J(s + nJ\delta^{1/2}) + t) = \hat{g}_J(s, (2nJ + 1)\delta^{1/2}s + t).
\]

This implies that
\[
E_{[0,\delta^{1/2}]}(\frac{\delta^{-1/2}(L_J(s) - s^2)}{2})^k \int_{-\delta}^{5\delta} \hat{f}(s + nJ\delta^{1/2}, L_J(s + nJ\delta^{1/2}) + t)(\frac{\delta^{-1/2} t}{2})^j dt
\]
\[
= \int_{0}^{\delta^{1/2}} \int_{-\delta}^{5\delta} \frac{\delta^{-1/2}(L_J(s) - s^2)}{2})^k \hat{g}_J(s, (2nJ + 1)\delta^{1/2}s + t)(\frac{\delta^{-1/2} t}{2})^j e(sx_1 + s^2x_2) dt ds
\]
which is equal to
\[
\int_{\theta_J - (nJ\delta^{1/2}, nJ^2\delta)} \frac{\delta^{-1/2}(L_J(s) - s^2)}{2})^k \times
\]
\[
\hat{g}_J(\xi)(\frac{\delta^{-1/2}(L_J(s) - s^2)}{2})^j e(sx_1 + s^2x_2) d\xi. \quad (47)
\]

Let
\[
T_J = \begin{pmatrix} 1 & 0 \\ -2nJ\delta^{1/2} & 1 \end{pmatrix}.
\]
Notice that $T_J$ sends $\theta_J - (n_J\delta^{1/2}, n_J^2\delta)$ to $\theta_{[0,\delta^{1/2}]}$. Letting $\mu = T_J\xi$ gives that (47) is equal to

$$\int_{\theta_{[0,\delta^{1/2}]}\setminus\theta_{[0,\delta^{1/2}]}} \frac{(\delta^{-1}(\delta^{1/2}\mu_1 - \mu_1^2))}{2} \psi_J(T_J^{-1}\mu)(\delta^{-1}(\mu_2 - \delta^{1/2}\mu_1)) \psi_J(T_J^{-1}\mu) e(\mu_1 x_1 + \mu_2 x_2) \mu \, d\mu$$

$$= \int_{\theta_{[0,\delta^{1/2}]}\setminus\theta_{[0,\delta^{1/2}]}} \frac{\psi_J(T_J^{-1}\mu)\psi_J(T_J^{-1}\mu)}{2} e(\mu_1 x_1 + \mu_2 x_2) \mu \, d\mu$$

$$= \int_0^{5\delta} \int_{-5\delta}^{5\delta} \frac{\psi_J(T_J^{-1}\mu)\psi_J(T_J^{-1}\mu)}{2} e(\mu_1 x_1 + \mu_2 x_2) \mu \, d\mu$$

Inserting the above into (46) and applying (44) shows that the left hand side of (45) is bounded by

$$90^{E/p}C \| (g_J \circ T_J^t)_{\theta_{[0,\delta^{1/2}]}\setminus\theta_{[0,\delta^{1/2}]}} \|_{L^p(w_{B,E})}.$$  

(48)

By Lemma 2.16 and the definitions of $T_J$ and $g_J$, we have

$$\| (g_J \circ T_J^t)_{\theta_{[0,\delta^{1/2}]}\setminus\theta_{[0,\delta^{1/2}]}} \|_{L^p(w_{B,E})}$$

$$= \int_{R^2} \left| \int_{R^2} \hat{g}_J(T_J^{-1}\xi) \xi e^{2\pi \xi \cdot x} \mu \, d\mu \right|^p_{w_{B,E}(x)} \, dx$$

$$= \int_{R^2} \left| \int_{R^2} \hat{g}_J(\mu) \xi e^{2\pi \xi \cdot \mu} \mu \, d\mu \right|^p_{w_{B,E}(T_J^{-t}x)} \, dx$$

$$= \int_{R^2} \left| \int_{R^2} \hat{f}(\mu + (n_J\delta^{1/2}, n_J^2\delta)) \xi \mu, (\mu + (n_J\delta^{1/2}, n_J^2\delta)) e^{2\pi \xi \cdot \mu} \mu \, d\mu \right|^p_{w_{B,E}(T_J^{-t}x)} \, dx$$

$$\leq 90^E \| f_{\theta_J} \|_{L^p(w_{B,E})}.$$  

(49)

Inserting this into (48) completes the proof of Lemma 3.12  

We now prove (43) when $J = [0,\delta^{1/2}]$, in other words we will prove (44). Corollary 2.4 implies that it is enough to show that

$$\int_{R^2} \| E_{[0,\delta^{1/2}]}^2 g_{j,k} \|_{L^p_{\psi_J}(B(y,\delta^{-1}))}^p w_{B,F}(y) \, dy$$

$$\leq E \exp(p(O(j) + O(k))) \| f_{\theta_{[0,\delta^{1/2}]}\setminus\theta_{[0,\delta^{1/2}]}} \|_{L^p(w_{B,E})}.$$  

(49)

We have

$$(E_{[0,\delta^{1/2}]}^2 g_{j,k})_x(x)$$

$$= \int_{\theta_{[0,\delta^{1/2}]}\setminus\theta_{[0,\delta^{1/2}]}} \hat{f}(\xi) \left( \frac{\delta^{-1}(\delta^{1/2}\xi_1 - \xi_1^2)}{2} \right) \left( \frac{\delta^{-1}(\xi_2 - \delta^{1/2}\xi_1)}{2} \right) e((\xi_1^2 - \xi_2)x_2)e(\xi \cdot x) \, d\xi.$$  

For $x \in B(y,\delta^{-1})$, since

$$e((\xi_1^2 - \xi_2)x_2) = e((\xi_1^2 - \xi_2)y_2)e((\xi_1^2 - \xi_2)(x_2 - y_2)),$$
Furthermore, for \( \xi \)

a Taylor expansion of \( e((\xi_1^2 - \xi_2)(x_2 - y_2)) \) gives that for \( x \in B(y, \delta^{-1}) \),

\[
|((\mathcal{E}_{[0,\delta^{1/2}]})g_{j,k})(x)| \leq \sum_{\ell \geq 0} \frac{(2\pi)^{\ell}}{\ell!} \left| \int_{[0,\delta^{1/2}]} \hat{f}(\xi)C_{j,k,\ell}(\xi)e((\xi_1^2 - \xi_2)y_2)e(\xi \cdot x) \, d\xi \right| \tag{50}
\]

where

\[
C_{j,k,\ell}(\xi) := \left( \frac{\delta^{-1}(\delta^{1/2}\xi_1 - \xi_1^2)}{2} \right)^{\ell} \left( \frac{\delta^{-1}(\xi_2 - \delta^{1/2}\xi_1)}{2} \right)^{\ell} \left( \frac{\delta^{-1}(\xi_1^2 - \xi_2)}{2} \right)^{\ell}.
\]

Let \( \Psi \) be as in Lemma 2.10 and so \( \Psi \in C_c^\infty(\mathbb{R}) \), \( \Psi = 1 \) on \([-1, 1]\) and vanishes outside \([-3, 3]\). For positive integer \( k \) and \( \lambda > 0 \), let

\[
M_{k,\lambda}(x) := x^k \Psi(x/\lambda).
\]

Because the integral on the right hand side of (50) is restricted to \( \theta_{[0,\delta^{1/2}]} \), we can insert some Schwartz cutoffs into \( C_{j,k,\ell} \). From (33) and (34), for \( \xi \in \theta_{[0,\delta^{1/2}]} \),

\[
\frac{\delta^{-1}}{2} |\delta^{1/2}\xi_1 - \xi_1^2| \leq \frac{1}{8}, \quad \frac{\delta^{-1}}{2} |\xi_2 - \delta^{1/2}\xi_1| \leq \frac{5}{2}, \quad \frac{\delta^{-1}}{2} |\xi_1^2 - \xi_2| \leq \frac{21}{8}.
\]

Furthermore, for \( \xi \in \theta_{[0,\delta^{1/2}]} \), \( |\xi_1| \leq \delta^{1/2} \) and \( |\xi_2| \leq 6\delta \). Let

\[
F(\xi) := \Psi(\delta^{-1/2}\xi_1)\Psi(\frac{\delta^{-1}\xi_2}{6}),
\]

\[
M_1(\xi_1) := M_{k,1/8}(\frac{\delta^{-1}(\delta^{1/2}\xi_1 - \xi_1^2)}{2}),
\]

\[
M_2(\xi_2) := M_{j,5/2}(\frac{\delta^{-1}(\xi_2 - \delta^{1/2}\xi_1)}{2}),
\]

\[
M_3(\xi) := M_{\ell,21/8}(\frac{\delta^{-1}(\xi_1^2 - \xi_2)}{2}),
\]

and

\[
\tilde{C}_{j,k,\ell}(\xi) := F(\xi)M_1(\xi_1)M_2(\xi_3)M_3(\xi).
\]

Thus we can replace the \( C_{j,k,\ell} \) on the right hand side of (50) with \( \tilde{C}_{j,k,\ell} \). It then remains to prove that

\[
\int_{\mathbb{R}^2} \left\| \int_{\theta_{[0,\delta^{1/2}]}} \hat{f}(\xi)\tilde{C}_{j,k,\ell}(\xi)e((\xi_1^2 - \xi_2)y_2)e(\xi \cdot x) \, d\xi \right\|_{L_p^p(B(y, \delta^{-1}))} \, w_{B,F}(y) \, dy \leq E \exp(p(O(j) + O(k) + O(\ell))) \int_{\theta_{[0,\delta^{1/2}]}} \left\| f_{[0,\delta^{1/2}]^2} \right\|_{L_p^p(w_{B,E})}^p.
\tag{52}
\]

For each fixed \( j, k, \ell, y \), let

\[
m(\xi) := e(\xi_1^2y_2)\tilde{C}_{j,k,\ell}(\xi) = e(\xi_1^2y_2)M_1(\xi_1)M_2(\xi_3)M_3(\xi)F(\xi).
\tag{53}
\]
Fix arbitrary $y \in \mathbb{R}^2$. Therefore
\[
\int_{\theta_{[0,\delta^{1/2}]}} \hat{f}(\xi)\hat{C}_{j,k,\ell}(\xi)e((\xi_1^2 - \xi_2)y_2)e(\xi \cdot x) \, d\xi
\]
\[
= \int_{\mathbb{R}^2} \hat{f}_{\theta_{[0,\delta^{1/2}]}}(\xi)m(\xi)e(\xi_1x_1 + \xi_2(x_2 - y_2)) \, d\xi
\]
\[
= (f_{\theta_{[0,\delta^{1/2}]}} \ast \tilde{m})(x_1, x_2 - y_2).
\]
This implies
\[
\left\| \int_{\theta_{[0,\delta^{1/2}]}} \hat{f}(\xi)\hat{C}_{j,k,\ell}(\xi)e((\xi_1^2 - \xi_2)y_2)e(\xi \cdot x) \, d\xi \right\|_p \leq \delta^2 \int_{\mathbb{R}^2} |f_{\theta_{[0,\delta^{1/2}]}} \ast \tilde{m}|^p(x)1_B(x_1 - y_1, x_2) \, dx.
\]
Holder’s inequality implies that
\[
|f_{\theta_{[0,\delta^{1/2}]}} \ast \tilde{m}|^p \leq (|f_{\theta_{[0,\delta^{1/2}]}}|^p \ast |\tilde{m}|)\|\tilde{m}\|_{L^1}^{p-1}.
\]
Note that the $L^1$ norm on the right hand side depends on $y$ since $\tilde{m}$ depends on $y$. To show (52), it is enough to show that for all $z \in \mathbb{R}^2$,
\[
\delta^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\tilde{m}|(x - z)1_B(x_1 - y_1, x_2) \|	ilde{m}\|_{L^1}^{p-1}w_{B,F}(y) \, dx \, dy \leq E \exp(p(O(j) + O(k) + O(\ell))w_{B,F}(z).
\]
We claim that for integers $a, b \geq 0$,
\[
\|\partial_{\xi_1}^a \partial_{\xi_2}^b m\|_{L^\infty} \leq C(a, b)(\delta^{-1/2} + \delta^{1/2}|y_2|)^a \delta^{-b}
\]
where
\[
C(a, b) = 12^5 40^a 3^3 15^3 3^{16} a^7 b^{20} (a + b)^4 (a + 1)^5 (b + 1)^3.
\]
The proof of (55) is deferred to the end of this section. The calculation is straightforward but rather tedious. With (55), integration by parts gives the following lemma.

Lemma 3.14. For $a, b \geq 0$, we have
\[
|\tilde{m}(x)| \leq 2^a 2^b 216 C(a, b)(\delta^{1/2}(1 + \frac{|x_1|}{\delta^{-1/2} + \delta^{1/2}|y_2|})^{-a})(\delta(1 + \frac{|x_2|}{\delta^{-1}})^{-b}).
\]

Proof. Note that for $|x| \leq 1, 1 \leq 2/(1 + |x|)$ and for $|x| \geq 1, 1/|x| \leq 2/(1 + |x|)$. There are four regions to consider.

First consider the case when $|x_1| > \delta^{-1/2} + \delta^{1/2}|y_2|$ and $|x_2| > \delta^{-1}$. Since $m$ is supported in a $6\delta^{1/2} \times 36\delta$ rectangle centered at the origin, integration by parts gives
that
\[
\left| \int_{\mathbb{R}^2} m(\xi) e^{2\pi i (x_1 \xi_1 + x_2 \xi_2)} \, d\xi \right| = \left| \int_{\mathbb{R}^2} m(\xi) \frac{1}{(2\pi|\xi_1| a)(2\pi|\xi_2| b)} \xi_1^a \xi_2^b \xi_1^c \xi_2^d e^{2\pi i (x_1 \xi_1 + x_2 \xi_2)} \, d\xi \right|
\]
\[
\leq \frac{216}{(2\pi |x_1| a)(2\pi |x_2| b)} C(a, b) (\delta^{1/2} + \delta^{1/2} |y_2|)^a \delta^{b/3}/2
\]
\[
\leq \frac{216C(a, b)}{(2\pi)^a(2\pi)^b} (\delta^{1/2}(\frac{|x_1|}{\delta^{-1/2} + \delta^{1/2} |y_2|})^{-a})(\delta(\frac{|x_2|}{\delta-1})^{-b})
\]
\[
\leq \frac{216C(a, b)}{\pi^a \pi^b} (\delta^{1/2}(1 + \frac{|x_1|}{\delta^{-1/2} + \delta^{1/2} |y_2|})^{-a})(\delta(1 + \frac{|x_2|}{\delta-1})^{-b}).
\]

Next consider the case when \( |x_1| \leq \delta^{-1/2} + \delta^{1/2} |y_2| \) and \( |x_2| \leq \delta^{-1} \). Then we just use the trivial bound in this case. We have
\[
\left| \int_{\mathbb{R}^2} m(\xi) e^{2\pi i (x_1 \xi_1 + x_2 \xi_2)} \, d\xi \right| \leq 216C(0, 0) \delta^{3/2}
\]
\[
\leq 2^a 2^b 216C(0, 0) (\delta^{1/2}(1 + \frac{|x_1|}{\delta^{-1/2} + \delta^{1/2} |y_2|})^{-a})(\delta(1 + \frac{|x_2|}{\delta-1})^{-b}).
\]

For the case when \( |x_1| \leq \delta^{-1/2} + \delta^{1/2} |y_2| \) and \( |x_2| > \delta^{-1} \) we integrate by parts in \( \xi_2 \). Thus
\[
\left| \int_{\mathbb{R}^2} m(\xi) e^{2\pi i (x_1 \xi_1 + x_2 \xi_2)} \, d\xi \right| \leq \frac{216}{(2\pi |x_2| b)} C(0, b) \delta^{-b} \delta^{3/2}
\]
\[
\leq \frac{2^a 216C(0, b)}{\pi^b} (\delta^{1/2}(1 + \frac{|x_1|}{\delta^{-1/2} + \delta^{1/2} |y_2|})^{-a})(\delta(1 + \frac{|x_2|}{\delta-1})^{-b}).
\]

Similarly, when \( |x_1| > \delta^{-1/2} + \delta^{1/2} |y_2| \) and \( |x_2| \leq \delta^{-1} \) we obtain
\[
\left| \int_{\mathbb{R}^2} m(\xi) e^{2\pi i (x_1 \xi_1 + x_2 \xi_2)} \, d\xi \right| \leq \frac{2^b 216C(a, 0)}{\pi^a} (\delta^{1/2}(1 + \frac{|x_1|}{\delta^{-1/2} + \delta^{1/2} |y_2|})^{-a})(\delta(1 + \frac{|x_2|}{\delta-1})^{-b}).
\]

Combining the estimates in the above four cases completes the proof of Lemma 3.14.

In particular, taking \( a, b = E \geq 10 \) in Lemma 3.14 gives the following corollary.

**Corollary 3.15.** For \( E \geq 10 \), let
\[
\phi_1(x_1) := \delta^{1/2}(1 + \frac{|x_1|}{\delta^{-1/2} + \delta^{1/2} |y_2|})^{-E}, \quad \phi_2(x_2) := \delta(1 + \frac{|x_2|}{\delta-1})^{-E}.
\]

Then
\[
|\tilde{m}(x)| \leq 15^k 3^{k^2} 16^k E^{30E} \phi_1(x_1) \phi_2(x_2).
\]

We now prove (5). The following lemma is the only place where \( p \leq 6 \) is used.
Lemma 3.16. For $2 \leq p \leq 6$,
\[
\| \tilde{m} \|_{L^1}^{p-1} \leq 15^j 3^k 16^\ell E^{30} (1 + \delta |y_2|)^5.
\]

Proof. From Corollary 3.15,
\[
\| \tilde{m} \|_{L^1} \leq 15^j 3^k 16^\ell E^{30} \int_R \phi_1(x_1) \, dx_1 \int_R \phi_2(x_2) \, dx_2.
\]

A change of variables gives that
\[
\int_R \phi_1(x_1) \, dx_1 = \delta^{1/2}(\delta^{-1/2} + \delta^{1/2}|y_2|) \int_R (1 + |x_1|)^{-E} \, dx_1 \leq 1 + \delta |y_2|
\]
and
\[
\int_R \phi_2(x_2) \, dx_2 = \int_R (1 + |x_2|)^{-E} \, dx_2 \leq 1.
\]
Therefore
\[
\| \tilde{m} \|_{L^1} \leq 15^j 3^k 16^\ell E^{30} (1 + \delta |y_2|).
\]

Raising both sides to the $p - 1$-power and then using that $p \leq 6$ completes the proof of the lemma. \qed

A change of variables gives
\[
\delta^2 \int_{\mathbb{R}^2} |\tilde{m}|(x - z)1_B(x_1 - y_1, x_2) \, dx = (|\tilde{m}| * \delta^2 1_B)(y_1 - z_1, -z_2)
\]
and so combining this with Lemma 3.16 shows that the left hand side of (54) is bounded above by
\[
15^j 3^k 16^\ell (p-1)^{30} E^{30}(p-1) \int_{\mathbb{R}^2} (|\tilde{m}| * \delta^2 1_B)(y_1 - z_1, -z_2)(1 + \delta |y_2|)^5 w_{B,F}(y) \, dy.
\]

Corollary 3.15 gives that
\[
(|\tilde{m}| * \delta^2 1_B)(x) \leq 15^j 3^k 16^\ell E^{30}(\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(x_1)(\phi_2 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(x_2).
\]
Since $1_{[-\delta^{-1}/2,\delta^{-1}/2]} \leq 2^E w_{[-\delta^{-1}/2,\delta^{-1}/2],E}$, Remark 2.3 shows
\[
(\phi_2 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(x_2) \leq 8^E \delta (1 + |x_2|/\delta^{-1})^{-E}.
\]
Therefore
\[
(|\tilde{m}| * \delta^2 1_B)(y_1 - z_1, -z_2) \leq 15^j 16^\ell E^{30} 8^E \delta (1 + |z_2|/\delta^{-1})^{-E} (\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1 - z_1).
\]

Thus (56) is bounded above by
\[
15^j 3^k 16^\ell 8^E E^{30} 8^E \times
\delta (1 + |z_2|/\delta^{-1})^{-E} \int_{\mathbb{R}^2} (\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1 - z_1)(1 + |y_2|/\delta^{-1})^5 w_{B,F}(y) \, dy.
\]

The following lemma will complete the proof of (54).
Lemma 3.17. Let \( E \geq 10 \) and \( F = 2E + 7 \), then
\[
\int_{\mathbb{R}^2} (\phi_1 \ast \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1 - z_1)(1 + \frac{|y_2|}{\delta^{-1}})^5 w_{B,F}(y) \ dy 
\leq 9 \cdot 128E\delta^{-1}(1 + \frac{|z_1|}{\delta^{-1}})^{-E}. \tag{58}
\]

Proof. We break the left hand side of (58) into the sum of integrals over the regions (recall that \( \delta \in \mathbb{N}^{-2} \)):
\[ I := \{ y : |y_2| \leq \delta^{-1} \} \]
\[ II := \bigcup_{1 \leq k < \delta^{-1/2}} \{ y : k\delta^{-1} < |y_2| \leq (k + 1)\delta^{-1} \} \]
\[ III := \bigcup_{k \geq 0} \{ y : 2^k\delta^{-3/2} < |y_2| \leq 2^{k+1}\delta^{-3/2} \}. \]
We also note that for \( a \geq 1 \),
\[
(1 + \frac{|x|}{a})^{-E} \leq a^E(1 + |x|)^{-E}. \tag{59}
\]
We first consider the integral over region I. When \(|y_2| \leq \delta^{-1}\),
\[
\phi_1(x_1) = \delta^{1/2}(1 + \frac{|x_1|}{\delta^{-1/2} + \delta^{1/2}|y_2|})^{-E} \leq \delta^{1/2}(1 + \frac{|x_1|}{2\delta^{-1/2}})^{-E} \leq 2^E \delta^{1/2}(1 + \frac{|x_1|}{\delta^{-1/2}})^{-E}.
\]
Therefore by Remark 2.3,
\[
(\phi_1 \ast \delta 1_{[-\delta^{-1/2},\delta^{-1/2}]})(y_1 - z_1) \leq 16E \delta(1 + \frac{|y_1 - z_1|}{\delta^{-1}})^{-E}
\]
and so
\[
\int_{I} (\phi_1 \ast \delta 1_{[-\delta^{-1/2},\delta^{-1/2}]})(y_1 - z_1)(1 + \frac{|y_2|}{\delta^{-1}})^5 w_{B,F}(y) \ dy 
\leq 16E \delta \int_{\mathbb{R}^2} (1 + \frac{|y_1 - z_1|}{\delta^{-1}})^{-E}(1 + \frac{|y_2|}{\delta^{-1}})^5(1 + \frac{|y_1|}{\delta^{-1}})^{-E}(1 + \frac{|y_2|}{\delta^{-1}})^{-E-7} \ dy.
\]
Applying Remark 2.3 in the \( y_1 \) variable bounds this by
\[
64E(1 + \frac{|z_1|}{\delta^{-1}})^{-E} \int_{\mathbb{R}} (1 + \frac{|y_2|}{\delta^{-1}})^{-E-2} \ dy_2 \leq 64E \delta^{-1}(1 + \frac{|z_1|}{\delta^{-1}})^{-E}. \tag{60}
\]
We next consider the integral over region II. For each \( 1 \leq k < \delta^{-1/2} \) and \( y \) such that \( k\delta^{-1} < |y_2| \leq (k + 1)\delta^{-1} \), we have
\[
\phi_1(x_1) = \delta^{1/2}(1 + \frac{|x_1|}{\delta^{-1/2} + \delta^{1/2}|y_2|})^{-E} \leq \delta^{1/2}(1 + \frac{|x_1|}{3k\delta^{-1/2}})^{-E} \leq 3^E \delta^{1/2}(1 + \frac{|x_1|}{k\delta^{-1/2}})^{-E}.
\]
Therefore by Remark 2.3,
\[
(\phi_1 \ast \delta 1_{[-\delta^{-1/2},\delta^{-1/2}]})(y_1 - z_1) \leq 24^E k\delta(1 + \frac{|y_1 - z_1|}{\delta^{-1}})^{-E}
\]
and so

\[ \int_{II} (\phi_1 * \delta 1_{[-\delta^{-1}/2, \delta^{-1}/2]})(y_1 - z_1)(1 + \frac{|y_2|}{\delta-1})^5 w_{B,F}(y) \, dy \]

\[ = \sum_{1 \leq k < \delta^{-1}/2} \int_{k\delta^{-1} < |y_2| \leq (k+1)\delta^{-1}} (\phi_1 * \delta 1_{[-\delta^{-1}/2, \delta^{-1}/2]})(y_1 - z_1)(1 + \frac{|y_2|}{\delta-1})^5 w_{B,F}(y) \, dy \]

\[ \leq 96E \sum_{1 \leq k < \delta^{-1}/2} k(1 + \frac{|z_1|}{\delta-1})^{-E} \int_{k\delta^{-1} < |y_2| \leq (k+1)\delta^{-1}} (1 + \frac{|y_2|}{\delta-1})^{-E-2} \, dy_2 \]

\[ \leq 96E \sum_{1 \leq k < \delta^{-1}/2} k(1 + \frac{|z_1|}{\delta-1})^{-E-2} k^{-E-2} \leq 4 \cdot 96E \delta^{-1}(1 + \frac{|z_1|}{\delta-1})^{-E} \tag{61} \]

where in the last inequality we have used that \( E \geq 10 \).

Finally we consider the integral over region III. For each \( k \geq 0 \) and \( y \) such that \( 2^k \delta^{-3/2} < |y_2| \leq 2^{k+1} \delta^{-3/2} \), we have

\[ \phi_1(x_1) = \delta^{1/2} (1 + \frac{|x_1|}{\delta^{-1/2} + \delta^{-1/2}|y_2|})^{-E} \leq \delta^{1/2} (1 + \frac{|x_1|}{4 \cdot 2^k \delta^{-1}})^{-E} \leq 4^E \delta^{1/2} (1 + \frac{|x_1|}{2^k \delta^{-1}})^{-E}. \]

Therefore by Remark 2.3

\[ (\phi_1 * \delta 1_{[-\delta^{-1}/2, \delta^{-1}/2]})(y_1 - z_1) \leq 32E \delta^{1/2} (1 + \frac{|y_1 - z_1|}{2^k \delta^{-1}})^{-E} \]

and so

\[ \int_{III} (\phi_1 * \delta 1_{[-\delta^{-1}/2, \delta^{-1}/2]})(y_1 - z_1)(1 + \frac{|y_2|}{\delta-1})^5 w_{B,F}(y) \, dy \]

\[ = \sum_{k \geq 0} \int_{2^k \delta^{-3/2} < |y_2| \leq 2^{k+1} \delta^{-3/2}} (\phi_1 * \delta 1_{[-\delta^{-1}/2, \delta^{-1}/2]})(y_1 - z_1)(1 + \frac{|y_2|}{\delta-1})^5 w_{B,F}(y) \, dy \]

\[ \leq 32E \delta^{3/2} \times \]

\[ \sum_{k \geq 0} \int_{R} (1 + \frac{|y_1 - z_1|}{2^k \delta^{-1}})^{-E} (1 + \frac{|y_1|}{\delta-1})^{-E} d y_1 \int_{2^k \delta^{-3/2} < |y_2| \leq 2^{k+1} \delta^{-3/2}} (1 + \frac{|y_2|}{\delta-1})^{-E-2} \, dy_2 \]

\[ \leq 128E \sum_{k \geq 0} \delta^{-1/2} (1 + \frac{|z_1|}{2^k \delta^{-1}})^{-E} 2^{k+1} \delta^{-3/2} (2^k \delta^{-1})^{-E-2} \]

\[ = 128E \sum_{k \geq 0} \delta^{-2+(E+2)/2} 2^{k+1-k(E+2)} (1 + \frac{|z_1|}{2^k \delta^{-1}})^{-E} \leq 4 \cdot 128E \delta (1 + \frac{|z_1|}{\delta^{-1}})^{-E} \]

where in the third inequality we have used (60). Summing this with (60) and (61) shows that the left hand side of (58) is bounded above by \( 9 \cdot 128E \delta (1 + |z_1|/\delta^{-1})^{-E} \) which completes the proof of Lemma 3.17.

Thus Lemma 3.17 shows that (57) is bounded above by

\[ 9 \cdot 15^{p \cdot 3^{kp}} 16^p E^{30E p^2 10^E} (1 + \frac{|z_1|}{\delta^{-1}})^{-E} (1 + \frac{|z_2|}{\delta^{-1}})^{-E} \leq 15^{p \cdot 3^{kp}} 16^p E^{40E p} w_{B,E}(z). \tag{62} \]
We now trace back all the implied constants to finish the proof of Lemma 3.10. From (62), the implied constants in (54) and (52) are both $15^{J_3p}16^p E^{40E_p}$. By (60) and (62), the left hand side of (49) is

\[
\left\| \mathcal{E}_{[0,\delta^{1/2}]} g_{j,k} \right\|_{L^p_y(B(y,\delta^{-1}))} \bigg\|_{L^p_y(w_{B,F})}^p \\
\leq \left( \sum_{\ell \geq 0} \frac{(2\pi)^\ell}{\ell!} \right) \int_{\theta_{[0,\delta^{1/2}]}} \hat{f}(\xi) \hat{C}_{j,k,\ell}(\xi) e((\xi_1^2 - \xi_2)y_2) e(\xi \cdot x) d\xi \bigg\|_{L^p_y(B(y,\delta^{-1}))} \bigg\|_{L^p_y(w_{B,F})}^p \\
\leq 15^{J_3p}3^{pE} e^{32\pi p} E^{40E_p} \| f_{\theta_j} \|_{L^p_y(w_{B,F})}^p.
\]

which gives the implied constant in (49). Using this, Lemma 3.12 and Lemma 2.4 we have

\[
\| \mathcal{E}_J g_{j,k} \|_{L^p_y(w_{B,F})} \leq \| f_{\theta_j} \|_{L^p_y(w_{B,F})} \times \begin{cases} 
3^{E/p} 15^{J_3k} E^{40E} e^{32\pi} & \text{if } J = [0, \delta^{1/2}] \\
90(E+F)/p 3^{E/p} 15^{J_3k} E^{40E} e^{32\pi} & \text{if } J \neq [0, \delta^{1/2}]
\end{cases}
\]

Inserting this estimate into (42) gives that

\[
\| f \|_{L^p(B)} \leq \text{Dec}(\delta, p, F) E^{54E} e^{32\pi} \left( \sum_{J \in P_{[\delta^{1/2}]}([0,1])} \| f_{\theta_j} \|_{L^p_y(w_{B,F})}^2 \right)^{1/2}.
\]

Since $E \geq 10$, $e^{36\pi} \leq 10^{50} \leq E^{5E}$ and this completes the proof of Lemma 3.10.

3.2. Proof of (55). Let $F, M_1, M_2, M_3$, and $m$ be as in (51) and (53). We will prove (55).

Lemma 3.18. Let $\lambda > 0$ and let

\[
M_{k,\lambda}(x) := x^k \Psi(x/\lambda)
\]

where $\Psi$ is as defined in Lemma 2.10. Then for integer $a \geq 0$,

\[
\| \partial^a M_{k,\lambda} \|_{L^\infty} \leq 12 \cdot 6^a 3^k (1 + \lambda^k (a!)^2).
\]

If $\lambda \geq 1$, this bound can be replaced with $12(6^a \lambda^k (a!)^2)$.

Proof. This proof is essentially the same as that of the beginning of the proof of Lemma 2.10. From the proof of Lemma 2.10 we have that $|\Psi^{(j)}(x)| \leq 12(6^j (j!)^2$ for all $j \geq 0$. Since $\Psi$ is supported in $[-3, 3]$, $\Psi(x/\lambda)$ is supported in $[-3\lambda, 3\lambda]$. 

If \( a = 0 \), then \( \|M_{k,\lambda}\|_{L^\infty} \leq 12(3\lambda)^k \) which proves (63) in this case. Now consider when \( a \geq 1 \). First suppose that \( a \leq k \), then

\[
|\partial^a(M_{k,\lambda}(x))| = \left| \sum_{j=0}^{a} \binom{a}{j} \partial^j(x^k)\Psi^{(a-j)}(x) \right|
\]

\[
\leq \sum_{j=0}^{a} \binom{a}{j} \frac{k!}{(k-j)!} (3\lambda)^{k-j} 12(6^{a-j})(a-j)!^2
\]

\[
\leq 12(6^a3^k)(a!)^2 \sum_{j=0}^{a} \binom{k}{j} \lambda^{k-j} \leq 12 \cdot 6^a3^k(1 + \lambda)^k(a!)^2.
\]

Next suppose that \( k < a \), then

\[
|\partial^a(M_{k,\lambda}(x))| \leq \sum_{j=0}^{k} \binom{a}{j} \frac{k!}{(k-j)!} (3\lambda)^{k-j} 12(6^{a-j})(a-j)!^2 \leq 12 \cdot 6^a3^k(1 + \lambda)^k(a!)^2.
\]

This completes the proof of Lemma 3.18. \( \square \)

Our goal is to obtain an estimate on \( \|\partial_{\xi_1}^a \partial_{\xi_2}^b m\|_{L^\infty} \) depending only on \( a, b, \delta \) and \( y_2 \) and where \( m \) is as defined in (53) and (54). Since we want exact constants, we will need to differentiate exactly each of the five functions that make up \( m(\xi) \). Note that since \( \Psi \) is supported in \([-3, 3]\), \( m \) is supported in a \( 6\delta^{1/2} \times 36\delta \) rectangle centered at the origin. In particular, for all \( \xi \in \text{supp}(m) \),

\[-3\delta^{1/2} \leq \xi_1 \leq 3\delta^{1/2}. \tag{64}\]

The bounds in Lemmas 3.20 and 3.21 are valid when we take no derivatives (either \( a = 0 \) or \( b = 0 \)) provided we use the convention that \( 0^0 = 1 \).

To compute \( \partial_{\xi_1}^a \partial_{\xi_2}^b m \), we will need to take arbitrarily many derivatives of a composition of functions. We will use the Faa di Bruno formula. We briefly recall all needed formulas (see [18] for a reference, note that Johnson defined \( B_{m,0} = 0 \) for \( m > 0 \) since the sum conditions would be vacuous). For \( m, k \geq 1 \), define the Bell polynomials

\[
B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) = \frac{1}{k!} \sum_{j_1 + \cdots + j_k = m} \binom{m}{j_1, \ldots, j_k} x_{j_1} \cdots x_{j_k}.
\]

Let

\[
Y_m(x_1, \ldots, x_m) := \sum_{k=1}^{m} B_{m,k}(x_1, \ldots, x_{m-k+1}). \tag{65}
\]

The Faa di Bruno formula states that

\[
\frac{d^m}{dt^m} g(f(t)) = \sum_{k=1}^{m} g^{(k)}(f(t)) B_{m,k}(f'(t), f''(t), \ldots, f^{(m-k+1)}(t)).
\]

Finally we will abuse notation slightly by writing \( Y_m(x, y, 0, \ldots, 0) \) as \( Y_m(x, y) \).
Lemma 3.19. Let \( m \geq 1 \) and \( x, y \neq 0 \) such that \(|x| \leq C|y|^{1/2}\) with \( C \geq 1 \). Then
\[
|Y_m(x, y)| \leq C^mm^m|y|^{m/2}.
\]

Proof. From [18, p. 220], \( Y_m(x, y) \) is equal to the determinant of the \( m \times m \) matrix
\[
\begin{pmatrix}
x & (m-1)y & 0 & \cdots & 0 \\
(m-2)y & (m-1)y & \cdots & 0 \\
-1 & x & (m-2)y & \cdots & 0 \\
0 & -1 & x & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & x+y \\
0 & 0 & 0 & \cdots & -1 \times x
\end{pmatrix}
\]
Cofactor expansion gives that \( Y_m(x, y) \) obeys the recurrence
\[
Y_m(x, y) = x^m + \sum_{j=1}^{\lfloor m/2 \rfloor} c_j x^{m-2j} y^j = y^m/2 \sum_{j=1}^{\lfloor m/2 \rfloor} c_j x^{m-2j} y^{m/2-j}
\]
and \( Y_m(1,1) = 1 + \sum_j c_j \leq m^m \). Thus \( Y_m(x,0) = x^m \) and
\[
|Y_m(x, y)| \leq |y|^{m/2}(C^m + \sum_{j=1}^{\lfloor m/2 \rfloor} c_j C^{m-2j}) \leq C^m m^m |y|^{m/2}.
\]
This completes the proof of Lemma 3.19.

Lemma 3.20. For \( a \geq 1 \) and \( \xi \in \supp(m) \),
\[
\|\hat{c}_{\xi} e^{2\pi iy_2 \xi_2^2}\|_{L^\infty} \leq (12\pi)^a a^a \times \begin{cases} \delta^{-a/2} & \text{if } |y_2| \leq \delta^{-1} \\ \delta^{a/2} |y_2|^a & \text{if } |y_2| > \delta^{-1}. \end{cases}
\]
In particular,
\[
\|\hat{c}_{\xi} e^{2\pi iy_2 \xi_2^2}\|_{L^\infty} \leq (12\pi)^a a^a (\delta^{-1/2} + \delta^{1/2} |y_2|)^a.
\]

Proof. If \( a = 0 \), then \( L^\infty \) norm is equal to 1 and the above formula still holds true. Now suppose \( a \geq 1 \). From Faa di Bruno’s formula,
\[
\hat{c}_{\xi} e^{2\pi iy_2 \xi_2^2} = \sum_{k=1}^a (2\pi i)^k e^{2\pi iy_2 \xi_2^2} B_{a,k}(2\xi_1 y_2, 2y_2, 0, \ldots, 0)
\]
and so,
\[
\|\hat{c}_{\xi} e^{2\pi iy_2 \xi_2^2}\|_{L^\infty} \leq (2\pi)^a Y_a(2|\xi_1||y_2|, 2|y_2|).
\]
Suppose \(|y_2| \leq \delta^{-1}\), then \( \delta^{1/2} |y_2| \leq |y_2|^{1/2} \) and so from (64),
\[
2|\xi_1||y_2| \leq 6|y_2|^{1/2}.
\]
Therefore Lemma 3.19 gives that
\[
Y_a(2|\xi_1||y_2|, 2|y_2|) \leq 6^a a^a |y_2|^{a/2} \leq 6^a a^a \delta^{-a/2}.
\]
Inserting this into (67) then finishes this case.

If \(|y_2| > \delta^{-1}\), then from (66),

\[
Y_a(2|\xi_1||y_2|, 2|y_2|) \leq Y_a(6\delta^{1/2}|y_2|, 2|y_2|) = 6^a\delta^{a/2}|y_2|^a(1 + \sum_{j=1}^{a/2} 18^{-j} c_j(\delta|y_2|)^{-j}).
\]

Since \(\delta|y_2| > 1\) and \(1 + \sum_j c_j \leq a^a\), the above is bounded by \(6^a a^a \delta^{a/2}|y_2|^a\) which completes the proof of Lemma 3.20. \(\square\)

Lemma 3.21. For integers \(a, b \geq 0\) and \(\xi \in \text{supp}(m),\)

\[
\| \partial_{\xi_1}^a M_1 \|_{L^\infty} \leq 12(21^a a^3 3^k) \delta^{-a/2} \tag{68}
\]

\[
\| \partial_{\xi_1}^a \partial_{\xi_2}^b M_2 \|_{L^\infty} \leq 12(6^a 3^b 15^j)(a + b)^2 \delta^{-b-a/2} \tag{69}
\]

\[
\| \partial_{\xi_1}^a \partial_{\xi_2}^b M_3 \|_{L^\infty} \leq 12(18^a 3^b 16^j)a^a(a + b)^2 \delta^{-b-a/2} \tag{70}
\]

\[
\| \partial_{\xi_1}^a \partial_{\xi_2}^b F \|_{L^\infty} \leq 12^2 6^a(a!)^2(b!)^2 \delta^{-b-a/2}. \tag{71}
\]

Proof. We first prove (68). If \(a = 0\), then from Lemma 3.18

\[
\| M_1 \|_{L^\infty} = \| M_{k,1/8} \|_{L^\infty} \leq 12 \cdot 3^k
\]

which proves (68) in this case. Next suppose \(a \geq 1\). We compute that

\[
\partial_{\xi_1}^a M_1 = \sum_{s=1}^{a} M_{k,1/8}^{(s)} \left( \frac{\delta^{-1}(\xi_1^2 - \xi_1^2)}{2} \right) B_{a,s}(\frac{1}{2} \delta^{-1/2} - \delta^{-1}, 0, \ldots, 0)
\]

and so applying Lemma 3.18 and (65) gives that

\[
\| \partial_{\xi_1}^a M_1 \|_{L^\infty} \leq 12(3^k 6^a(a!)^2 Y_a(\delta^{-1/2}|1/2 - \delta^{-1/2} \xi_1|, \delta^{-1}). \tag{72}
\]

Since

\[
\delta^{-1/2}|1/2 - \delta^{-1/2} \xi_1| \leq \frac{7}{2} (\delta^{-1})^{1/2},
\]

Lemma 3.19 implies that

\[
Y_a(\delta^{-1/2}|1/2 - \delta^{-1/2} \xi_1|, \delta^{-1}) \leq (7/2)^a a^a \delta^{-a/2}.
\]

Inserting this into (72) completes the proof of (68).

We now prove (69). We compute

\[
\partial_{\xi_1}^a \partial_{\xi_2}^b M_2 = \left( \frac{\delta^{-1}}{2} \right)^b \partial_{\xi_1}^a \partial_{\xi_2}^b \left( \frac{\delta^{-1} (\xi_2 - \delta^{1/2} \xi_1)}{2} \right)
\]

\[
= \left( \frac{\delta^{-1}}{2} \right)^b \left( \frac{\delta^{-1/2}}{2} \right)^a M_{j,5/2}^{(a+b)} \left( \frac{\delta^{-1} (\xi_2 - \delta^{1/2} \xi_1)}{2} \right).
\]

Applying Lemma 3.18 gives

\[
\| \partial_{\xi_1}^a \partial_{\xi_2}^b M_2 \|_{L^\infty} \leq 12(6^a 3^b 15^j)(a + b)^2 \delta^{-b-a/2}
\]

which proves (69).
Next we prove (70). If \( a = 0 \), then
\[
\partial_{\xi_2}^b M_3 = (- \frac{\delta^{-1}}{2})^b M_{\ell,21/8}(\frac{\delta^{-1}(\xi_1^2 - \xi_2)}{2})
\]
and so
\[
\| \partial_{\xi_2}^b M_3 \|_{L^\infty} \leq 12(3^b 16^c)(bt)^2 \delta^{-b}
\]
which proves (70) in this case. Now suppose \( a \geq 1 \). Faà di Bruno’s formula gives that

\[
\partial_{\xi_1}^a \partial_{\xi_2}^b M_3 = (- \frac{\delta^{-1}}{2})^b \sum_{s=1}^a M_{\ell,21/8}(\frac{\delta^{-1}(\xi_1^2 - \xi_2)}{2}) B_{a,s}(\delta^{-1}\xi_1, \delta^{-1}, 0, \ldots, 0).
\]

Applying Lemma 3.18 and (65) gives that

\[
\| \partial_{\xi_1}^a \partial_{\xi_2}^b M_3 \|_{L^\infty} \leq 12(6^a 3^b 16^c)(a+b)^2 \delta^{-b} Y_a(\delta^{-1}|\xi_1|, \delta^{-1})
\]

Since \( \delta^{-1}|\xi_1| \leq 3(\delta^{-1})^{1/2} \), it follows that

\[
Y_a(\delta^{-1}|\xi_1|, \delta^{-1}) \leq 3^a a^a \delta^{-a/2}.
\]

Inserting this into (73) completes the proof of (70).

Finally we prove (71). We compute

\[
\partial_{\xi_1}^a \partial_{\xi_2}^b F = \delta^{-a/2} \left( \frac{\delta^{-1}}{6} \right)^b \Psi^{(a)}(\delta^{-1/2}\xi_1) \Psi^{(b)}(\delta^{-1/2}\xi_2).
\]

Lemma 2.10 then implies that

\[
\| \partial_{\xi_1}^a \partial_{\xi_2}^b F \|_{L^\infty} \leq 12^2 6^a (a!)^2 (bt)^2 \delta^{-b-a/2}
\]

which proves (71). This completes the proof of Lemma 3.21.

We are now ready to prove (55).

**Lemma 3.22.** For \( a, b \geq 0 \),
\[
\| \partial_{\xi_1}^a \partial_{\xi_2}^b m \|_{L^\infty} \leq 12^2 5^a 3^b 15^c 3^b 16^c a^a b^b (a + b)!^4 (a + 1)^5 (b + 1)^3 (\delta^{-1/2} + \delta^{1/2}|y_2|) a \delta^{-b}.
\]

**Proof.** We compute

\[
\partial_{\xi_1}^a \partial_{\xi_2}^b m = \sum_{s_1+s_2+s_3=b \atop t_1+t_2+t_3+t_4+t_5=a \atop s_1, t_1 \geq 0} \left( \frac{b!}{s_1!s_2!s_3!} \right) \left( \frac{a!}{t_1!t_2!t_3!t_4!t_5!} \right) \times (\partial_{\xi_1}^{t_1} e(\xi_1^2 y_2)) (\partial_{\xi_1}^{t_2} M_1) (\partial_{\xi_1}^{t_3} \partial_{\xi_2}^{s_1} M_2) (\partial_{\xi_1}^{t_4} \partial_{\xi_2}^{s_2} M_3) (\partial_{\xi_1}^{t_5} \partial_{\xi_2}^{s_3} F).
\]

Applying crude bounds and Lemmas 3.20 and 3.21 gives that
\[
\| \partial_{\xi_1}^a \partial_{\xi_2}^b m \|_{L^\infty} \leq 12^2 5^a 3^b 15^c 3^b 16^c a^a b^b (1 + \delta|y_2|) a \times \sum_{s_1+s_2+s_3=b \atop t_1+t_2+t_3+t_4+t_5=a \atop s_1, t_1 \geq 0} t_1^{t_1} t_2^{t_2} t_3^{t_3} (t_3 + s_1)^2 (t_4 + s_2)^2 t_5! s_3!
\]
\[
\leq 12^2 5^a 3^b 15^c 3^b 16^c a^a b^b (a + b)!^4 (a + 1)^5 (b + 1)^3 \delta^{-b-a/2} (1 + \delta|y_2|)^a
\]
where in the first inequality we have used that
\[(\delta^{-1/2} + \delta^{1/2}|y_2|)^{t_1} = \delta^{-t_1/2}(1 + \delta|y_2|)^{t_1} \leq \delta^{-t_1/2}(1 + \delta|y_2|)^a\]
and we have removed a $t_2$! and $s_3$! using the multinomial coefficient. This completes the proof of Lemma 3.22 and the proof of (55).

\[\square\]

4. PARABOLIC RESCALING: AN APPLICATION

As an application of Lemma 2.18 and Proposition 3.11 we will prove that the decoupling constant is essentially multiplicative. This will play an important role in Section 10 when we upgrade knowledge about decoupling at a lacunary sequence of scales to knowledge about decoupling on all possible scales in $\mathbb{N}^{-2}$. The restriction that $p \leq 6$ is once again an artifact that only arises from our application of Proposition 3.11.

**Proposition 4.1.** Let $E \geq 100$ and $2 \leq p \leq 6$. For $0 < \delta < \sigma < 1$ with $\delta, \sigma, \delta/\sigma \in \mathbb{N}^{-2}$, we have

\[D_{p,E}(\delta) \leq E^{100E}D_{p,E}(\sigma)D_{p,E}(\delta/\sigma).\]

**Proof.** Fix an arbitrary $E \geq 100$ and $2 \leq p \leq 6$. We need to show that for all $g : [0,1] \to \mathbb{C}$ and all squares $B$ of side length $\delta^{-1}$, we have

\[\|\mathcal{E}_{[0,1]}g\|_{L^p(B)} \leq E^{100E}D_{p,E}(\sigma)D_{p,E}(\delta/\sigma)(\sum_{J \in P_{\delta/2}[0,1]} \|\mathcal{E}_J g\|_{L^p(w_{B,E})}^2)^{1/2}.\]

It suffices to assume that $B$ is centered at the origin.

Since $\delta/\sigma \in \mathbb{N}^{-2}$, we can partition $B$ into a collection of squares $\{\Sigma\}$ of side length $\sigma^{-1}$. Then

\[\|\mathcal{E}_{[0,1]}g\|_{L^p(\Sigma)} \leq D_{p,E}(\sigma)(\sum_{J \in P_{\delta/2}[0,1]} \|\mathcal{E}_J g\|_{L^p(w_{\Sigma,E})}^2)^{1/2}.\]

Raising both sides to the $p$th power on both sides and summing over all $\Sigma$, then using Minkowski’s inequality (since $p \geq 2$), and finally applying Proposition 2.14 gives that

\[\|\mathcal{E}_{[0,1]}g\|_{L^p(B)} \leq 48E^pD_{p,E}(\sigma)(\sum_{J \in P_{\delta/2}[0,1]} \|\mathcal{E}_J g\|_{L^p(w_{B,E})}^2)^{1/2}. \quad (74)\]

For each $J = [a, a + \sigma^{1/2}]$, we will first show that

\[\|\mathcal{E}_J g\|_{L^p(B)} \leq E D_{p,E}(\delta/\sigma)(\sum_{J \in P_{\delta/2}(J)} \|\mathcal{E}_J g\|_{L^p(w_{B,E})}^2)^{1/2}. \quad (75)\]

Afterwards we will apply Proposition 2.11 to (75) and then insert the result into (74) to finish.

Let $T$ be as in Lemma 2.18 $L(\xi) = (\xi - a)/\sigma^{1/2}$, $g_L = g \circ L^{-1}$. Then a change of variables gives that

\[\|\mathcal{E}_J g\|_{L^p(B)} = \sigma^{1/2} \frac{d}{d\xi} \|\mathcal{E}_{[0,1]}g_L\|_{L^p(T(B))}.\]
Let $\mathcal{B}$ be as in Lemma 2.18. Thus we cover $T(B)$ by a collection of squares $\mathcal{B} = \{\Delta\}$ of side length $\sigma / \delta$, use decoupling constant $\tilde{D}_{p,E}$ at scale $\delta / \sigma$ and undo change of variables. This gives

$$\sigma^{\frac{p}{2} - \frac{3}{2}} \| \mathcal{E}_{[0,1]} [g_L] \|_{L^p(T(B))}^p \leq \sigma^{\frac{p}{2} - \frac{3}{2}} \sum_{\Delta \in \mathcal{B}} \| \mathcal{E}_{[0,1]} [g_L] \|_{L^p(\Delta)}^p \leq \tilde{D}_{p,E} (\delta / \sigma)^p \sigma^{\frac{p}{2} - \frac{3}{2}} \sum_{\Delta \in \mathcal{B}} \sum_{J^p \in P_{(\delta / \sigma)^{1/2}}([0,1])} \| \mathcal{E}_{J^p} [g_L] \|_{L^p(\Delta \Delta, E)}^2 \left( \sum_{\Omega \in P_{\delta / \sigma}^3(J)} \| \mathcal{E}_{J^p} [g_L] \|_{L^p(\Omega \Omega \Omega \Omega, E)}^2 \right)^{p/2} \leq \tilde{D}_{p,E} (\delta / \sigma)^p (\sum_{J^p \in P_{\delta / \sigma}^3(J)} \| \mathcal{E}_{J^p} [g_L] \|_{L^p(\Omega \Omega \Omega \Omega, E)}^2 \left( \sum_{\Omega \in P_{\delta / \sigma}^3(J)} \| \mathcal{E}_{J^p} [g_L] \|_{L^p(\Omega \Omega \Omega \Omega, E)}^2 \right)^{p/2}$$

where the third inequality we have used Minkowski’s inequality and $p \geq 2$ and the last inequality we have used Lemma 2.18. Combining this with Proposition 3.11 gives that

$$\| \mathcal{E}_{J^p} [g_L] \|_{L^p(\Omega \Omega \Omega \Omega, E)} \leq E^{70E} 720^E D_{p,E} (\delta / \sigma)^{p/2} \left( \sum_{J^p \in P_{\delta / \sigma}^3(J)} \| \mathcal{E}_{J^p} [g_L] \|_{L^p(\Omega \Omega \Omega \Omega, E)} \right)^{1/2}.$$

Applying Proposition 2.11 gives that

$$\| \mathcal{E}_{J^p} [g_L] \|_{L^p(\Omega \Omega \Omega \Omega, E)} \leq E^{80E} D_{p,E} (\delta / \sigma)^{p/2} \left( \sum_{J^p \in P_{\delta / \sigma}^3(J)} \| \mathcal{E}_{J^p} [g_L] \|_{L^p(\Omega \Omega \Omega \Omega, E)} \right)^{1/2}.$$

Inserting this into (71) then completes the proof of Proposition 4.1.

**Remark 4.2.** Combining Propositions 3.11 and 4.1 we see that all four decoupling constants $D_{p,E}, \tilde{D}_{p,E}, \tilde{D}_p, \tilde{D}_{p,E}$ obey a similar multiplicative property.

5. **Bilinear Equivalence**

We now define the bilinear decoupling constant and show that it is essentially the same size as the linear decoupling constant. In [6], Bourgain and Demeter use a Bourgain-Guth type argument to do this. However in two dimensions, there is a simpler proof using Holder’s inequality and parabolic rescaling by Tao in [22]. It is this version we follow.

For each $m \in \mathbb{N}$, $E \geq 100$, let

$$\nu := 2^{-16 - 2m E^{10E}}.$$

For $\delta \in (0, 1)$ such that $\nu \delta^{-1/2} \in \mathbb{N}$, let $D_{p,E}(\delta, m)$ be the best constant such that

$$\| \text{geom} \|_{L^p(B)} \| J^p \|_{L^p(\Omega \Omega \Omega \Omega, E)} \leq D_{p,E}(\delta, m) \text{geom} \left( \sum_{J^p \in P_{\delta / \sigma}^3(I)} \| \mathcal{E}_{J^p} [g_L] \|_{L^p(\Omega \Omega \Omega \Omega, E)}^2 \right)^{1/2}$$

for all pairs of intervals $I_1, I_2 \subseteq P_{\nu}([0,1])$ which are at least $\nu$-separated, functions $g : [0,1] \to \mathbb{C}$, and squares $B$ of side length $\delta^{-1}$. Note that the right hand side uses the weight function $\tilde{w}_{B,E}$ rather than $w_{B,E}$.
We first give the trivial bound for the bilinear decoupling constant which is a useful bound at large scales.

**Lemma 5.1.** Let \( m, E, \nu \) be defined as above. If \( \nu \delta^{-1/2} \in \mathbb{N} \), then \( D_{p,E}(\delta, m) \leq 4E\nu^{1/2}\delta^{-1/4} \).

**Proof.** Holder’s inequality gives that
\[
\| \text{geom } \mathcal{E}_I, g \|_{L^p(B)} \leq \text{geom } \| \mathcal{E}_I, g \|_{L^p(B)}.
\]
The triangle inequality, Cauchy-Schwarz, and that \( 1_B \leq 4E\tilde{w}_{B,E} \) gives
\[
\| \mathcal{E}_I, g \|_{L^p(B)} = \| \sum_{J \in P_{\delta/2}(I)} \mathcal{E}_J, g \|_{L^p(B)} \leq 4E\nu^{1/2}\delta^{-1/4} \left( \sum_{J \in P_{\delta/2}(I)} \| \mathcal{E}_J, g \|_{L^p(B)}^2 \right)^{1/2}
\]
which completes the proof of Lemma 5.1. \( \square \)

**Lemma 5.2.** Let \( E \geq 100 \) and \( 2 \leq p \leq 6 \). If \( \delta^{1/2} \in 2^{-N} \) and \( \delta^{1/2} \nu^{-1} \in 2^{-N} \), then
\[
D_{p,E}(\delta) \leq E^{100E}(D_{p,E}(\delta) + \frac{1}{\nu}D_{p,E}(\delta, m)).
\]

**Proof.** This proof is essentially an application of parabolic rescaling. The restriction \( 2 \leq p \leq 6 \) comes only from the application of Proposition 3.11. Fix an arbitrary square \( B \) of side length \( \delta^{-1} \) and function \( g : [0, 1] \to \mathbb{C} \). It suffices to assume \( B \) is centered at the origin. Partition \([0, 1]\) into \( 1/\nu \) many intervals \( I_1, \ldots, I_{1/\nu} \) of length \( \nu \) (here we have used that \( \nu \in 2^{-N} \)). Then
\[
\| \mathcal{E}_{[0,1]}g \|_{L^p(B)} = \| \sum_{1 \leq i \leq 1/\nu} \mathcal{E}_{I_i}g \|_{L^p(B)} \leq \left( \sum_{1 \leq i, j \leq 1/\nu} \| \mathcal{E}_{I_i}g \|_{L^{p/2}(B)} \right)^{1/2} \leq \sqrt{2} \left( \sum_{1 \leq i, j \leq 1/\nu} \| \mathcal{E}_{I_i}g \|_{L^{p/2}(B)} \right)^{1/2}
\]
We first consider the off-diagonal terms. This will be controlled by the bilinear decoupling constant. Holder’s inequality gives that
\[
\left( \sum_{1 \leq i, j \leq 1/\nu} \| \mathcal{E}_{I_i}g \|_{L^{p/2}(B)} \right)^{p/2} \leq \nu^{-(p-2)} \sum_{1 \leq i, j \leq 1/\nu} (\| \mathcal{E}_{I_i}g \|_{L^{p/2}(B)} \|^2)
\]
and hence
\[
\int_B \left( \sum_{1 \leq i, j \leq 1/\nu} \| \mathcal{E}_{I_i}g \|_{L^{p/2}(B)} \right)^{p/2} dx \leq \nu^{-(p-2)} \sum_{1 \leq i, j \leq 1/\nu} \int_B (\| \mathcal{E}_{I_i}g \|_{L^{p/2}(B)} \) dx.
By bilinear decoupling, the above is bounded above by
\[ \nu^{-(p-2)} D_{p,E}(\delta, m)^p \sum_{1 \leq i,j \leq 1/\nu} \sum_{|i-j| > 1} \| \mathcal{E}_I g \|_{L^p(\hat{\omega}_{B,E})}^2 \| \mathcal{E}_J g \|_{L^p(\hat{\omega}_{B,E})}^2. \]

Note that here we have used that \( \nu / \delta^{1/2} \in 2^N \). Since \( \delta^{1/2} \) is dyadic and \( I_i \) and \( I_j \) are dyadic intervals, this is bounded above by
\[ \nu^{-p} D_{p,E}(\delta, m)^p \left( \sum_{J \in P_{3/2}(0,1)} \| \mathcal{E}_J g \|_{L^p(\hat{\omega}_{B,E})}^2 \right)^{p/2}. \]

Now we consider the diagonal contribution. The triangle inequality followed by Cauchy-Schwarz gives that
\[ \left\| \sum_{1 \leq i,j \leq 1/\nu} \| \mathcal{E}_I g \|_{L^p(\hat{\omega}_{B,E})} \| \mathcal{E}_J g \|_{L^p(\hat{\omega}_{B,E})} \right\|_{L^{p/2}(B)} \leq \sum_{1 \leq i,j \leq 1/\nu} \| \mathcal{E}_I g \|_{L^p(B)} \| \mathcal{E}_J g \|_{L^p(B)}. \tag{76} \]

Let \( I = a + [0, \nu] \) be an interval of length \( \nu \). Let \( L(\xi) = (\xi - a) / \nu, \ g_L := g \circ L^{-1} \), and \( T = (\nu^2 \xi) / \nu^2 \). A change of variables then gives that \( \| \mathcal{E}_I g \|_{L^p(B)} = \nu \| \mathcal{E}_I g \|_{L^p(T(B))} \) and therefore
\[ \| \mathcal{E}_I g \|_{L^p(B)} = \nu^{1-3/p} \| \mathcal{E}_I g \|_{L^p(T(B))}. \tag{77} \]

Note that \( T(B) \) is a parallelogram contained in a \( 3\nu \delta^{-1} \times \nu^2 \delta^{-1} \) rectangle. Covering \( T(B) \) by squares \( B = \{ \Delta \} \) of side length \( \nu^2 \delta^{-1} \) gives that
\[ \nu^{1-3/p} \| \mathcal{E}_I g \|_{L^p(T(B))} \leq \nu^{1-3/p} \left( \sum_{\Delta \in B} \| \mathcal{E}_I g \|_{L^p(\Delta)}^p \right)^{1/p}. \tag{78} \]

Applying the definition of the decoupling constant (and using that \( \nu \delta^{-1/2} \in 2^N \)), gives that for each square \( \Delta \),
\[ \| \mathcal{E}_{[0,1]} g \|_{L^p(\Delta)}^p \leq \tilde{D}_{p,E}(\delta / \nu^2)^p \left( \sum_{J \in P_{3/2}(0,1)} \| \mathcal{E}_J g \|_{L^p(\hat{\omega}_{\Delta,E})}^2 \right)^{p/2}. \]

Inserting this into (78) bounds the left hand side of (78) by
\[ \tilde{D}_{p,E}(\delta / \nu^2)^p \left( \sum_{\Delta \in B} \left( \sum_{J \in P_{3/2}(0,1)} \nu^{1-3/p} \| \mathcal{E}_J g \|_{L^p(\hat{\omega}_{\Delta,E})}^2 \right)^{p/2} \right)^{1/p}. \]

Applying the same change of variables as in (77) followed by Minkowski’s inequality (using that \( p \geq 2 \)) gives that the above is bounded by
\[ \tilde{D}_{p,E}(\delta / \nu^2)^p \left( \sum_{J \in P_{3/2}(I)} \| \mathcal{E}_J g \|_{L^p(\hat{\omega}_{[0,1]} \circ T)} \right)^{1/2} \leq 720^{E/p} \tilde{D}_{p,E}(\delta / \nu^2)^p \left( \sum_{J \in P_{3/2}(I)} \| \mathcal{E}_J g \|_{L^p(\hat{\omega}_{B,E})}^2 \right)^{1/2}. \]
By Proposition 5.1, $\tilde{D}_{p,E}(\delta) \leq E^{70E}D_{p,E}(\delta)$ and so the above gives that
$$\|\mathcal{E}_I g\|_{L^p(B)} \leq E^{75E}D_{p,E}(\delta/\nu^2)(\sum_{J \in P_{3/2}(I)} \|\mathcal{E}_J g\|_{L^p(w_{B,E})}^2)^{1/2}$$
for each interval $I$ of length $\nu$.

Using this for each interval that shows up on the right hand side of (76) gives an upper bound of
$$E^{150E}D_{p,E}(\delta/\nu^2)^2 \sum_{1 \leq i,j \leq 1/\nu, |i-j| \leq 1} \left( \sum_{J \in P_{3/2}(I_i)} \|\mathcal{E}_J g\|_{L^p(w_{B,E})}^2 + \sum_{J' \in P_{3/2}(I_j)} \|\mathcal{E}_{J'} g\|_{L^p(w_{B,E})}^2 \right) \leq 2 \cdot E^{150E}D_{p,E}(\delta/\nu^2)^2 \sum_{J \in P_{3/2}(I) \in [0,1]} \|\mathcal{E}_J g\|_{L^p(w_{B,E})}^2.$$

Therefore if $\delta^{1/2} \in 2^{-N}$ and $\delta^{1/2}\nu^{-1} \in 2^{-N}$, we have
$$D_{p,E}(\delta) \leq 2 \cdot E^{75E}D_{p,E}(\delta/\nu^2) + \left\| \frac{\sqrt{2}}{\nu} D_{p,E}(\delta, m) \right\|$$
which completes the proof of Lemma 5.2. $\square$

**Proposition 5.3.** Let $E \geq 100$ and $2 \leq p \leq 6$. Fix an arbitrary integer $m \geq 1$. Let $\delta^{1/2} \in 2^{-N}$ and $K$ be the largest positive integer such that $\delta^{1/2}\nu^{-K} \in 2^{-N}$. Then
$$D_{p,E}(\delta) \leq \delta^{100E\log_2 E \nu^{-1}} \max(1, \max_{i=0,1,\ldots,K-1} D_{p,E}(\delta \nu^{-2i}, m)).$$

**Proof.** Note that $\delta^{1/2} \in 2^{-N}$ and $\delta^{1/2} \nu^{-K} \in 2^{-N}$ imply that for $i = 0, 1, \ldots, K$, $\delta^{1/2}\nu^{-i} \in 2^{-N}$. In particular for each $i = 1, 2, \ldots, K$, both $\delta^{1/2}\nu^{-i+1}$ and $\delta^{1/2}\nu^{-i}$ are in $2^{-N}$ and hence
$$D_{p,E}(\delta \nu^{-2i+2}) \leq E^{100E}(D_{p,E}(\delta \nu^{-2i}) + \frac{1}{\nu} D_{p,E}(\delta \nu^{-2i+2}, m)).$$

Combining these $K$ inequalities then gives that
$$D_{p,E}(\delta) \leq E^{100EK}(D_{p,E}(\delta \nu^{-2K}) + 2\nu^{-1} \max_{i=0,1,\ldots,K-1} D_{p,E}(\delta \nu^{-2i}, m)). \quad (79)$$

To control $D_p((\delta \nu)^{-2K})$, we use the definition of $K$. In particular, since $\delta^{1/2} \in 2^{-N}$, $\delta^{1/2}\nu^{-(K+1)}$ is dyadic but $\geq 1$. Therefore $\delta^{1/2}\nu^{-(K-1)} \geq 1$ and so $\delta^{1/2}\nu^{-K} \geq \nu$. The trivial bound then gives that
$$D_{p,E}(\delta \nu^{-2K}) \leq 2^{E/p}(\delta \nu^{-2K})^{-1/4} \leq 2^E \nu^{-1/2}.$$

Since $\delta^{1/2}\nu^{-K} \leq 1$, $K \leq \log_2 \delta^{-1/2}$ and hence
$$E^{100EK} \leq \delta^{50E\log_2 E}.$$
Inserting the above two centered equations into (79) then completes the proof of Proposition 5.3.

6. Ball Inflation

We first discuss some basic geometry. Let \( \mathbb{P} := \{(\xi, \xi^2) : \xi \in [0,1]\} \) and \( \pi : \mathbb{P} \to [0,1] \) be the projection map which sends \( (\xi, \xi^2) \mapsto \xi \). Since \( I_1, I_2 \) are \( d \)-separated, for any \( P \in I_1, Q \in I_2 \), we have \(|P - Q| \geq d\). Observe that

\[
n(\pi^{-1}(P)) = \frac{(-2P,1)}{\sqrt{1 + 4P^2}}
\]

and similarly for \( Q \) (where here \( n(\pi^{-1}(P)) \) refers to the normal vector to the parabola at the point \( \pi^{-1}(P) \)). Let \( \theta \) be the angle between \( n_{\pi^{-1}(P)} \) and \( n_{\pi^{-1}(Q)} \). Then since \(|P - Q| \geq d\),

\[
\sin \theta = \frac{2|P - Q|}{\sqrt{(1 + 4P^2)(1 + 4Q^2)}} \geq \frac{2}{5}d.
\]

In the terminology of [6], \( I_1 \) and \( I_2 \) are \( 2d/5 \)-transverse.

We will now prove the following effective ball inflation inequality.

**Theorem 6.1.** Let \( p \geq 4, 0 < \delta < 1/10, E \geq 100, \) and \( 0 < d < 1/2 \). Let \( I_1, I_2 \subset [0,1] \) be two \( d \)-separated intervals of length \( \geq \delta \) such that \(|I_i|/\delta \in \mathbb{N}\). Let \( B \) be an arbitrary square in \( \mathbb{R}^2 \) with side length \( \delta^{-2} \) and let \( B \) be the unique partition of \( B \) into squares \( \Delta \) of side length \( \delta^{-1} \). Then for all \( g : [0,1] \to \mathbb{C} \), we have

\[
\frac{1}{|B|} \sum_{\Delta \in B} \text{geom} \left( \sum_{J \in P_3(I_i)} \|E_J g\|_{L_p^\gamma(\tilde{B}, B)}^2 \right)^{p/2} \leq E^{50\delta p} \sum_{J \in P_3(I_i)} \|E_J g\|_{L_p^\gamma(\tilde{B}_E B)}^2 \leq E^{50\delta p} \sum_{J \in P_3(I_i)} \|E_J g\|_{L_p^\gamma(\tilde{B}_E B)}^2.
\]

Furthermore, for \( p = 4 \), the estimate is true without the logarithm.

This inequality allows us to keep the frequency scale the same while increasing (inflating) the spatial scale and is a key step in the iteration. We will first prove a version of Theorem 6.1 where we additionally assume that all the \( \|E_J g\| \) are of comparable size (for each \( I_i \)). Then we remove this assumption by dyadic pigeonholing to obtain (80).

**Lemma 6.2.** Let \( p > 4 \) and everything else be as defined in Theorem 6.1. Furthermore, let \( F_1 \) be a collection of intervals in \( P_3(I_1) \) such that for each pair of intervals \( J, J' \in F_1 \), we have

\[
\frac{1}{2} < \frac{\|E_J g\|_{L_p^\gamma(\tilde{B}_E B)}^2}{\|E_J g\|_{L_p^\gamma(\tilde{B}_E B)}^2} \leq 2.
\]
Similarly define $\mathcal{F}_2$. Then for all $g : [0, 1] \to \mathbb{C}$ we have

$$\frac{1}{|B|} \sum_{\Delta \in B} \text{geom} \left( \sum_{J \in \mathcal{F}_i} \|\mathcal{E}_J g\|_{L^{p/2}(\hat{\omega}_{\Delta,E})}^{p/2} \right) \leq E^{30\epsilon P} d^{-1} \text{geom} \left( \sum_{J \in \mathcal{F}_i} \|\mathcal{E}_J g\|_{L^{p/2}(\hat{\omega}_{B,E})}^{p/2} \right)^{p/2}. \quad (82)$$

Proof. For each $J \in P_\delta(I_i)$ centered at $c_J$, cover $B$ by a set $\mathcal{T}_J$ of mutually parallel nonoverlapping boxes $P_J$ with dimension $\delta^{-1} \times \delta^{-2}$ with longer side pointing in the direction of the normal vector to $\mathbb{P}$ at $\pi^{-1}(c_J)$. Note that any $\delta^{-1} \times \delta^{-2}$ box outside $4B$ cannot cover $B$ itself. Thus we may assume that all the boxes in $\mathcal{T}_J$ are contained in $4B$. Finally, let $P_J(x)$ denote the box in $\mathcal{T}_J$ containing $x$ and let $2P_J$ be the $2\delta^{-1} \times 2\delta^{-2}$ box having the same center and orientation as $P_J$.

Since $p > 4$, Holder’s inequality yields that

$$(\sum_{J \in \mathcal{F}_i} \|\mathcal{E}_J g\|_{L^{p/2}(\hat{\omega}_{\Delta,E})}^p)^{p/2} \leq \left( \sum_{J \in \mathcal{F}_i} \|\mathcal{E}_J g\|_{L^{p/2}(\hat{\omega}_{\Delta,E})}^p \right)^2 |\mathcal{F}_i|^{p/2-2}. \quad (83)$$

Thus the left hand side of (82) is bounded above by

$$\left( \prod_{i=1}^2 |\mathcal{F}_i|^{p/4-1} \right) \frac{1}{|B|} \sum_{\Delta \in B} \prod_{i=1}^2 \left( \sum_{J \in \mathcal{F}_i} \|\mathcal{E}_J g\|_{L^{p/2}(\hat{\omega}_{\Delta,E})}^p \right)^{p/2-2}. \quad (84)$$

For $x \in 4B$, define

$$H_J(x) := \begin{cases} \sup_{y \in 2P_J(x)} \|\mathcal{E}_J g\|_{L^{p/2}(\hat{\omega}_{B(y,\delta^{-1},E})}^p & \text{if } x \in \bigcup_{J \in \mathcal{T}_J} P_J \\ 0 & \text{if } x \in 4B \setminus \bigcup_{J \in \mathcal{T}_J} P_J. \end{cases} \quad (85)$$

For each $x \in \Delta$, observe that $\Delta \subset 2P_J(x)$. Therefore for each $x \in \Delta$, $c_x \in 2P_J(x)$ and hence

$$\|\mathcal{E}_J g\|_{L^{p/2}(\hat{\omega}_{\Delta,E})}^p \leq H_J(x)$$

for all $x \in \Delta$. Thus

$$\frac{1}{|B|} \sum_{\Delta \in B} \prod_{i=1}^2 \left( \sum_{J \in \mathcal{F}_i} \|\mathcal{E}_J g\|_{L^{p/2}(\hat{\omega}_{\Delta,E})}^p \right)^{p/2-2} \leq \sum_{J_1 \in \mathcal{F}_1, J_2 \in \mathcal{F}_2} \frac{1}{|B|} \sum_{\Delta \in B} \|\mathcal{E}_{J_1} g\|_{L^{p/2}(\hat{\omega}_{\Delta,E})}^p \|\mathcal{E}_{J_2} g\|_{L^{p/2}(\hat{\omega}_{\Delta,E})}^p \frac{1}{|\Delta|} \int_{\Delta} dx$$

$$\leq \sum_{J_1 \in \mathcal{F}_1, J_2 \in \mathcal{F}_2} \frac{1}{|B|} \int_B H_{J_1}(x) H_{J_2}(x) dx \quad (86)$$

where the last inequality we have used (85). By how $H_J$ is defined, $H_J$ is constant on each $P_J \in \mathcal{T}_J$. That is, for each $x \in \bigcup_{J \in \mathcal{T}_J} P_J$,

$$H_J(x) = \sum_{J \in \mathcal{T}_J} c_J 1_{P_J}(x).$$
for some constants $c_{P_j} \geq 0$. Then
\[
\frac{1}{|B|} \int_B H_{J_1}(x) H_{J_2}(x) \, dx = \frac{1}{|B|} \sum_{P_{J_1} \in T_{J_1}} \sum_{P_{J_2} \in T_{J_2}} c_{P_{J_1}} c_{P_{J_2}} |(P_{J_1} \cap P_{J_2}) \cap B| \leq \frac{1}{|B|} \sum_{P_{J_1} \in T_{J_1}} \sum_{P_{J_2} \in T_{J_2}} c_{P_{J_1}} c_{P_{J_2}} |P_{J_1} \cap P_{J_2}|
\]
where the last inequality is because $c_{P_j} \geq 0$ for all $P_j$. Since $|P_j| = \delta^{-3}$ we also have
\[
\frac{1}{|B|} \int_{4B} H_{J}(x) \, dx = \frac{1}{|B|} \int_{4B} \sum_{P_{J_1} \in T_{J_1}} c_{P_{J_1}} 1_{P_{J_1}}(x) \, dx = \delta \sum_{P_{J_1} \in T_{J}} c_{P_{J}}.
\]
Recall that $J_1 \in \mathcal{F}_1 \subset P_0(I_1)$ and $J_2 \in \mathcal{F}_2 \subset P_0(I_2)$. Since $I_1$ and $I_2$ are $d$-separated, so are $J_1$ and $J_2$. Let $\angle J_1 J_2$ be the angle between the directions of $J_1$ and $J_2$. By geometry discussion at the beginning of this section, $\sin(\angle J_1 J_2) \geq 2d/5$. Therefore
\[
|P_{J_1} \cap P_{J_2}| \leq \frac{\delta^{-2}}{\sin(\angle J_1 J_2)} \leq \frac{\delta^{-2}}{2d/5}.
\]
Applying this gives
\[
\frac{1}{|B|} \sum_{P_{J_1} \in T_{J_1}} \sum_{P_{J_2} \in T_{J_2}} c_{P_{J_1}} c_{P_{J_2}} |P_{J_1} \cap P_{J_2}|
\leq \frac{3\delta^{-2} d^{-1}}{|B|} \prod_{i=1}^{2} \left( \frac{\delta^{-1}}{|B|} \right) \int_{4B} H_{J_i}(x) \, dx = \frac{3d^{-1}}{|B|^2} \prod_{i=1}^{2} \int_{4B} H_{J_i}(x) \, dx.
\]
Therefore (86) is bounded above by
\[
3d^{-1} \prod_{i=1}^{2} \left( \frac{1}{|B|} \right) \int_{4B} H_{J}(x) \, dx = 768d^{-1} \prod_{i=1}^{2} \left( \frac{1}{|4B|} \right) \int_{4B} H_{J}(x) \, dx.
\]
We now apply Lemma 6.3, proven later, to (87). This gives that an upper bound of
\[
E^{20E} d^{-1} \prod_{i=1}^{2} \left( \sum_{J \in \mathcal{F}_i} \|\mathcal{E}_J g\|_{L_{\#}^{p/2}(\tilde{w}_{B,E})}^{p/2} \right)
\]
where here we have also used that $E \geq 100$ and $p \geq 2$. Thus (83) is bounded above by
\[
E^{20E} d^{-1} \left( \prod_{i=1}^{2} |\mathcal{F}_i|^{p/4-1} \right) \prod_{i=1}^{2} \left( \sum_{J \in \mathcal{F}_i} \|\mathcal{E}_J g\|_{L_{\#}^{p/2}(\tilde{w}_{B,E})}^{p/2} \right).
\]
To obtain the right hand side of (82) we now use that intervals in \( \mathcal{F}_i \) satisfy (81). We have
\[
\left( \prod_{i=1}^{2} |\mathcal{F}_i|^{p/4-1} \right) \left( \prod_{i=1}^{2} \| \mathcal{E}_J g \|_{L_{p/2}^p(\tilde{w}_{B,E})}^{p/2} \right) \leq \prod_{i=1}^{2} |\mathcal{F}_i|^{p/4-1} \prod_{i=1}^{2} (|\mathcal{F}_i| \text{ max}_{J \in \mathcal{F}_i} \| \mathcal{E}_J g \|_{L_{p/2}^p(\tilde{w}_{B,E})}^{p/2}) \leq \left( \frac{2}{\prod_{i=1}^{2} (|\mathcal{F}_i| \text{ max}_{J \in \mathcal{F}_i} \| \mathcal{E}_J g \|_{L_{p/2}^p(\tilde{w}_{B,E})}^{p/2})^{1/2} \right)^{p/2} \leq \left( \prod_{i=1}^{2} \sum_{J \in \mathcal{F}_i} \| \mathcal{E}_J g \|_{L_{p/2}^p(\tilde{w}_{B,E})}^{2} \right)^{1/2} \leq 2^p \text{ geom}(\sum_{J \in \mathcal{F}_i} \| \mathcal{E}_J g \|_{L_{p/2}^p(\tilde{w}_{B,E})}^{2})^{p/2}
\]
where the second inequality is due to (81). Inserting this into (88) then completes the proof of Lemma 6.2.

\[\square\]

Lemma 6.3. Let \( H_J \) be as defined in (84). Then
\[
\frac{1}{|AB|} \int_{AB} H_J(x) \, dx \leq E^{8E_p} \| \mathcal{E}_J g \|_{L_{p/2}^p(\tilde{w}_{B,E})}^{p/2}.
\]

Proof. This is the inequality proven in (29) of [6] without explicit constants. We follow their proof, this time paying attention to the implied constants.

Fix arbitrary \( J \subset [0, 1] \) of length \( \delta \) and center \( c_j \). For \( x \in \bigcup_{J \in \mathcal{T}_J} P_J = \text{ supp } H_J \subset 4B \), fix arbitrary \( y \in 2P_J(x) \). Note that \( 2P_J(x) \) points is a rectangle of dimension \( 2\delta^{-1} \times 2\delta^{-2} \) with the longer side pointing in the direction of \((-2c_J, 1)\).

Let \( R_J \) and \( \theta_J \) be as in Lemma 2.5. Since \( c_J \in [\delta/2, 1 - \delta/2] \), both \( \cos \theta_J \) and \( \sin \theta_J \) are nonzero. Note that \( R_J \) is the rotation matrix such that \( R_J^{-1} \) applied to \( 2P_J(x) \) gives an axis parallel rectangle of dimension \( 2\delta^{-1} \times 2\delta^{-2} \) with the longer side pointing in the vertical direction. Since \( y \in 2P_J(x) \), we can write
\[
R_J^{-1}y = R_J^{-1}x + \tilde{y}
\]
where \( |\tilde{y}_1| \leq 2\delta^{-1} \) and \( |\tilde{y}_2| \leq 2\delta^{-2} \). We then have
\[
\| \mathcal{E}_J g \|_{L_{p/2}^p(\tilde{w}_{B(y, \delta^{-1})}, E)}^{p/2} = \int \| (\mathcal{E}_J g)(s + x + R_J(0, \tilde{y}), \tilde{y}) \|_{L_{p/2}^p(\tilde{w}_{B(y, \delta^{-1})}, E)}^{p/2} \, ds.
\]
Writing \( \tilde{y} = (\tilde{y}_1, 0)^T + (0, \tilde{y}_2)^T \) and a change of variables gives that the above is equal to
\[
\int \| (\mathcal{E}_J g)(s + x + R_J(0, \tilde{y}_2)^T) \|_{L_{p/2}^p(\tilde{w}_{B(R_J(\tilde{y}_1, 0)^T, \delta^{-1})}, E)}^{p/2} \, ds.
\]
Inserting Lemma 2.5 into (89) gives that
\[
\| \mathcal{E}_J g \|_{L_{p/2}^p(\tilde{w}_{B(y, \delta^{-1})}, E)}^{p/2} \leq 16E \int \| (\mathcal{E}_J g)(s + x + R_J(0, \tilde{y}_2)^T) \|_{L_{p/2}^p(\tilde{w}_{B(0, \delta^{-1})}, E)}^{p/2} \, ds.
\]
Observe that
\[ |(\mathcal{E}_J g)(s + x + R_J(0, \overline{y}_2)^T)| = \left| \int_{\mathbb{R}^2} \widehat{\mathcal{E}_J g}(\lambda) e(\lambda \cdot (s + x)) e(\lambda \cdot R_J(0, \overline{y}_2)^T) \, d\lambda \right|. \]
Since \( R_J \) is a rotation matrix, a change of variables gives that the above is equal to
\[ |\int_{\mathbb{R}^2} \widehat{\mathcal{E}_J g}(R_J \lambda) e(\lambda \cdot R_J^{-1}(s + x)) e(\lambda \cdot (0, \overline{y}_2)^T) \, d\lambda| \] (91)
Writing
\[ e(\lambda \cdot (0, \overline{y}_2)^T) = e((\lambda_2 - c_j^2) \overline{y}_2) e(c_j^2 \overline{y}_2) = e(c_j^2 \overline{y}_2) \sum_{k=0}^{\infty} \frac{(2\pi i)^k \overline{y}_2^k}{k!} (\lambda_2 - c_j^2)^k \]
and using that \(|\overline{y}_2| \leq 2\delta^{-2}\) shows that (91) is
\[ \leq \sum_{k=0}^{\infty} \frac{4 \pi^k}{k!} |\int_{\mathbb{R}^2} \widehat{\mathcal{E}_J g}(R_J \lambda) e(\lambda \cdot R_J^{-1}(s + x)) (\lambda_2 - c_j^2)^k \, d\lambda| \]
Applying the change of variables \( \eta = \lambda - \pi^{-1}(c_j) \) gives that the above is
\[ \leq \sum_{k=0}^{\infty} \frac{30^k}{k!} |\int_{\mathbb{R}^2} \widehat{\mathcal{E}_J g}(R_J(\eta + \pi^{-1}(c_j))) e(\eta \cdot R_J^{-1}(s + x)) \left( \frac{\eta_2}{2\delta} \right)^k \Psi \left( \frac{\eta_1}{2\delta} \right) \Psi \left( \frac{\eta_2}{2\delta} \right) \, d\eta| \] (92)
Note that \( \widehat{\mathcal{E}_J g}(R_J(\eta + \pi^{-1}(c_j))) \) is supported in a \( 4\delta \times 4\delta^2 \) box centered at the origin pointing in the horizontal direction. Thus we may insert the cutoff \( \Psi \) from Lemma 2.10 in (92). Then (92) becomes
\[ \sum_{k=0}^{\infty} \frac{30^k}{k!} |\int_{\mathbb{R}^2} \widehat{\mathcal{E}_J g}(R_J(\eta + \pi^{-1}(c_j))) e(\eta \cdot R_J^{-1}(s + x)) \left( \frac{\eta_2}{2\delta} \right)^k \Psi \left( \frac{\eta_1}{2\delta} \right) \Psi \left( \frac{\eta_2}{2\delta} \right) \, d\eta|. \]
Note that we are a bit wasteful since \( \Psi(\eta_1/(2\delta))\Psi(\eta_2/(2\delta)) \) is equal to 1 on \([-2\delta, 2\delta]^2\) rather than \([-2\delta, 2\delta] \times [-2\delta, 2\delta]^2\), but this will turn out to not matter.
Let \( \Phi_k(t) := t^k \Psi(t) \) and let
\[ (M_k f)(x) = \int_{\mathbb{R}^2} \hat{f}(R_J(\eta + \pi^{-1}(c_j))) e(\eta \cdot x) \Psi \left( \frac{\eta_1}{2\delta} \right) \Phi_k \left( \frac{\eta_2}{2\delta} \right) \, d\eta. \]
Thus we have shown that
\[ |(\mathcal{E}_J g)(s + x + R_J(0, \overline{y}_2)^T)| \leq \sum_{k=0}^{\infty} \frac{30^k}{k!} |(M_k \mathcal{E}_J g)(R_J^{-1}(s + x))| \]
and combining this with (91) gives that for \( x \in \bigcup_{P_J \in T_J} P_J \) and \( y \in 2P_J(x), \)
\[ \| \mathcal{E}_J g \|_{L^p_{\#}(\tilde{\omega}_{B(0, \delta^{-1}), E})}^{p/2} \leq 16 E^2 \int_{\mathbb{R}^2} \sum_{k=0}^{\infty} \frac{30^k}{k!} |(M_k \mathcal{E}_J g)(R_J^{-1}(s + x))|^{p/2} \tilde{\omega}_{B(0, \delta^{-1}), E}(s) \, ds. \]
Thus
\[
\frac{1}{|4B|} \int_{4B} H_J(x) \, dx
\]
\[
\leq 16^{E-1} \delta^6 \int_{4B} \int_{\mathbb{R}^2} \left( \sum_{k=0}^{\infty} \frac{30^k}{k!} |(M_k \mathcal{E}_J g)(R_{1,J}(s + x))|^p/2 \tilde{w}_{B(0,\delta^{-1})} E(s) \right) \, ds \, dx
\]
\[
= 16^{E-1} \delta^6 \int_{\mathbb{R}^2} \left( \sum_{k=0}^{\infty} \frac{30^k}{k!} |(M_k \mathcal{E}_J g)(u)|^p/2 \right) \left( \int_{4B} \tilde{w}_{B(\delta^{-1}),E}(R_J u) \, dx \right) \, du. \tag{93}
\]
As $1_{4B} \leq 4^E \tilde{w}_{4B,E} \leq 64^E \tilde{w}_{B,E}$ and since $B$ is centered at the origin,
\[
\int_{4B} \tilde{w}_{B(\delta^{-1}),E}(R_J u) \, dx = (1_{4B} \ast \tilde{w}_{B(0,\delta^{-1}),E})(R_J u)
\]
\[
\leq 64^E \tilde{w}_{B,E} \ast \tilde{w}_{B(0,\delta^{-1}),E}(R_J u) \leq 256^E \delta^{-2} \tilde{w}_{B,E}(R_J u).
\]
Thus it follows that (93) is bounded by
\[
2^{12E} \delta^4 \left( \sum_{k=0}^{\infty} \frac{30^k}{k!} \|M_k \mathcal{E}_J g \circ R_{1,J}^{-1}\|_{L^p/(\tilde{w}_{B,E})} \right)^{p/2}. \tag{94}
\]
Inserting an extra $e(R_J \pi^{-1}(c_J) \cdot s)$ and applying a change of variables gives
\[
|(M_k \mathcal{E}_J g)(R_{1,J} s)| = \left| \int_{\mathbb{R}^2} \mathcal{E}_J g(R_J(\eta + \pi^{-1}(c_J))) e(R_J \eta \cdot s) \Phi(\eta \delta_{2^\delta}) \Psi\left(\frac{\eta_1}{2^\delta}\right) \, d\eta \right|
\]
\[
= \left| \int_{\mathbb{R}^2} \mathcal{E}_J g(\gamma) e(\gamma \cdot s) \widehat{m}_k(\gamma) \, d\gamma \right|
\]
where
\[
\widehat{m}_k(\gamma) = \Psi\left(\frac{\gamma_1 \cos \theta_J + \gamma_2 \sin \theta_J - c_J}{2^\delta}\right) \Phi\left(\frac{\gamma_2 \cos \theta_J - \gamma_1 \sin \theta_J - c_J^2}{2^\delta}\right).
\]
Then $|M_k \mathcal{E}_J g \circ R_{1,J}^{-1}| = |\mathcal{E}_J g \ast m_k| \leq |\mathcal{E}_J g| \ast |m_k|$ and Holder’s inequality implies
\[
(|\mathcal{E}_J g| \ast |m_k|)^{p/2} \leq (|\mathcal{E}_J g|^{p/2} \ast |m_k|) \|m_k\|_{L^{p/2}}^{p/2 - 1}.
\]
Therefore
\[
\|M_k \mathcal{E}_J g \circ R_{1,J}^{-1}\|_{L^p/(\tilde{w}_{B,E})} \leq \|m_k\|_{L^{1}(\mathbb{R}^2)}^{1-2/p} \|\mathcal{E}_J g\|_{L^p/(\tilde{w}_{B,E} \ast m_k(-\cdot))} \tag{95}
\]
where here $|m_k|(-\cdot)$ is the function $|m_k|(-x)$. Since $\Phi$ and $\Psi$ are both Schwartz functions, our goal will be to use the rapid decay to show that $|m_k| \leq \tilde{w}_{B,E}$. A change of variables gives
\[
|m_k(x)| = \left| \int_{\mathbb{R}^2} \widehat{m}_k(\gamma) e^{2\pi i x \cdot \gamma} \, d\gamma \right|
\]
\[
= 4^2 \int_{\mathbb{R}} \Psi(w_1) e^{2\pi i (R_{1,J} x) \cdot 1(2 \delta w_1)} \, dw_1 \int_{\mathbb{R}} \Phi_k(w_2) e^{2\pi i (R_{1,J} x) \cdot 2(2 \delta w_2)} \, dw_2.
\]
Thus we have
\[ |\int_{\mathbb{R}} \Psi(w_1)e^{2\pi i (R_j^{-1}x_1)(2\delta w_1)} dw_1| \leq \frac{E_{5E}^E}{(1 + 2\delta |(R_j^{-1}x_1)|)^{2E}} \]
and
\[ |\int_{\mathbb{R}} \Phi_k(w_2)e^{2\pi i (R_j^{-1}x_2)(2\delta w_2)} dw_2| \leq \frac{6^k E_{5E}^E}{(1 + 2\delta |(R_j^{-1}x_2)|)^{2E}}. \]
Therefore
\[ |m_k(x)| \leq 4\delta^2 6^k E^{10E}(1 + \frac{|(R_j^{-1}x_1)|}{\delta-1})^{-2E}(1 + \frac{|(R_j^{-1}x_2)|}{\delta-1})^{-2E}. \tag{96} \]
Thus we have
\[ \|m_k\|_{L^2(\mathbb{R}^2)} \leq (6^k E^{11E})^{1-2/p}. \tag{97} \]
Applying Lemma 2.6 to (96) shows
\[ |m_k(x)| \leq 4(6^k E^{10E})\delta^2 \tilde{w}_{B(0,\delta^{-1}),E}(x). \]
Note that this inequality does not change if we replace \( x \) with \( -x \) on the left hand side since the right hand side is radial. Lemma 2.1 then implies that
\[ \tilde{w}_{B,E} \ast |m_k|(-\cdot) \leq 6^k E^{11E} \tilde{w}_{B,E} \]
and hence
\[ \|E_{J\ell}g\|_{L^{p/2}(\tilde{w}_{B,E} \ast m_k)(\cdot)} \leq (6^k E^{11E})^{2/p}\|E_{J\ell}g\|_{L^{p/2}(\tilde{w}_{B,E})}. \]
Combining this with (91), (95), and (97) shows that
\[ \frac{1}{|4B|} \int_{4B} H_j(x) \, dx \leq 2^{12E} E^{11E_{p/2}\delta^4} (\sum_{k=0}^{\infty} \frac{180^k}{k!} \|E_{J\ell}g\|_{L^{p/2}(\tilde{w}_{B,E})})^{p/2} \leq E^{8E_p} \|E_{J\ell}g\|_{L^{p/2}(\tilde{w}_{B,E})}^{p/2} \]
where in the last inequality we have used that \( E \geq 100 \) and \( p \geq 2 \). This completes the proof of Lemma 6.3.

\begin{proof}[Proof of Theorem 6.4] If \( p = 4 \), the proof of Lemma 6.2 (in particular (88)) implies that we can just take \( F_i = P_5(I_i) \) and discard the requirement in (81) since the only reason we dyadically decomposed and restricted to \( p > 4 \) was to match the \( L^{p/2} \) with the \( \ell^2 \) sum over \( \sum_{J \in F_i} \) in (82).

From now on we assume \( p > 4 \). For \( i = 1,2 \), let
\[ M_i := \max_{J \in P_5(I_i)} \|E_{J\ell}g\|_{L^{p/2}(\tilde{w}_{B,E})}. \]
For each \( i = 1,2 \), let \( F_{i,0} \) denote the set of intervals \( J' \in P_5(I_i) \) such that
\[ \|E_{J'\ell}g\|_{L^{p/2}(\tilde{w}_{B,E})} \leq \delta^3 M_i \]
and partition the remaining intervals in \( P_5(I_i) \) into \( \log_2(\delta^{-3}) \) many classes \( F_{i,k} \) (with \( k = 1,2,\ldots,\lceil \log_2(\delta^{-3}) \rceil \)) such that
\[ 2^{k-1} \delta^3 M_i < \|E_{J'\ell}g\|_{L^{p/2}(\tilde{w}_{B,E})} \leq 2^k \delta^3 M_i \]
for all $J' \in \mathcal{F}_{i,k}$. Note that $\mathcal{F}_{i,k}$ satisfies the hypothesis \((81)\) given in Lemma \(6.2\). For $1 \leq k, l \leq \lfloor \log_2(\delta^{-3}) \rfloor$, let

$$F_\Delta(k, l) := \left( \sum_{J \in \mathcal{F}_{i,k}} \|E_{J'}g\|_{L^p(w, \Delta, E)}^2 \right)^{p/4} \left( \sum_{J \in \mathcal{F}_{2,l}} \|E_{J'}g\|_{L^p(w, \Delta, E)}^2 \right)^{p/4}.$$  

Note that $F_\Delta(a, b) = F_\Delta(b, a)$.

The left hand side of \((80)\) is equal to

$$\frac{1}{|B|} \sum_{\Delta \in B} \left( \sum_{0 \leq k, l \leq \lfloor \log_2(\delta^{-3}) \rfloor} \sum_{J \in \mathcal{F}_{i,k}, J' \in \mathcal{F}_{2,l}} \|E_{J'}g\|_{L^p(w, \Delta, E)}^2 \|E_{J'}g\|_{L^p(w, \Delta, E)}^2 \right)^{p/4} \leq (\lfloor \log_2(\delta^{-3}) \rfloor + 1)^{p/2 - 1} \frac{1}{|B|} \sum_{\Delta \in B} \sum_{0 \leq k, l \leq \lfloor \log_2(\delta^{-3}) \rfloor} F_\Delta(k, l). \tag{98}$$

We then have

$$\frac{1}{|B|} \sum_{\Delta \in B} \sum_{k, l = 0}^{\lfloor \log_2(\delta^{-3}) \rfloor} F_\Delta(k, l) \leq \frac{1}{|B|} \sum_{\Delta \in B} F_\Delta(0, 0) + 2 \sum_{k = 1}^{\lfloor \log_2(\delta^{-3}) \rfloor} \frac{1}{|B|} \sum_{\Delta \in B} F_\Delta(0, k) + \sum_{k, l = 1}^{\lfloor \log_2(\delta^{-3}) \rfloor} \frac{1}{|B|} \sum_{\Delta \in B} F_\Delta(k, l). \tag{99}$$

We first consider the third sum on the right hand side of \((99)\). In this case, both families of intervals satisfy \((81)\) in Lemma \(6.2\). Thus applying Lemma \(6.2\) gives that

$$\sum_{k, l = 1}^{\lfloor \log_2(\delta^{-3}) \rfloor} \frac{1}{|B|} \sum_{\Delta \in B} F_\Delta(k, l) \leq \lfloor \log_2(\delta^{-3}) \rfloor^2 E^{30E_f} d^{-1} \text{geom} \left( \sum_{J \in \mathcal{F}_{i,0}} \|E_{J'}g\|_{L^p(w, \Delta, E)}^2 \right)^{p/2}. \tag{100}$$

The first two sums on the right hand side of \((99)\) are taken care of by trivial estimates. We consider the first sum. From Proposition \(2.13\), $w_{\Delta, E} \leq 48^E \tilde{w}_{B, E}$ (we can obtain a better constant using Lemma \(2.1\) and $1 \leq \Delta \leq 1_B$ but this is not needed). Therefore for $J' \in \mathcal{F}_{i,0}$,

$$\max_{\Delta \in B} \|E_{J'}g\|_{L^p(w, \Delta, E)}^2 \leq \delta^{-4/p} 48^{2E/p} \|E_{J'}g\|_{L^p(w, \Delta, E)}^2 \leq \delta^{3-4/p} 48^{2E/p} M_i. \tag{101}$$

Since $|\mathcal{F}_{i,0}| \leq |P_0(I_i)| \leq \delta^{-1}$,

$$\frac{1}{|B|} \sum_{\Delta \in B} F_\Delta(0, 0) \leq \left( |\mathcal{F}_{i,0}| \right) \delta^{12-16/p} 48^{2E/p} M_i^2 M_2^{p/4} \leq \delta^{5p/2 - 4} 48^{2E} \text{geom}(M_i^2)^{p/2}. \tag{102}$$

Since $p > 4$, $5p/2 - 4 > 6$ and so the union bound implies that \((102)\) is bounded by

$$48^{2E} \text{geom} \left( \sum_{J \in \mathcal{F}_{i,0}} \|E_{J'}g\|_{L^p(w, \Delta, E)}^2 \right)^{p/2}. \tag{103}$$
Finally we consider the second sum on the right hand side of (99). From the same proof as (101), for \( J' \in \mathcal{F}_{2,k} \) with \( k \neq 0 \) we have

\[
\max_{\Delta \in B} \| \mathcal{E}_J g \|_{L_\#^p(\tilde{\omega}_{\Delta,E})} \leq \delta^{-4/p} 48^{2E/p} M_2.
\]

Therefore by the same reasoning as in the previous paragraph we have

\[
\frac{1}{|B|} \sum_{\Delta \in B} F_\Delta(0, k) \leq (|\mathcal{F}_{1,0}| |\mathcal{F}_{2,k}| (\delta^{-4/p} 48^{2E/p} M_1)^2 (\delta^{-4/p} 48^{2E/p} M_2)^2) \frac{p}{4}
\]

\[
\leq \delta^{-4} 48^{2E} \text{geom}(M^2)^{p/2}.
\]

Since \( p > 4 \), we can discard the power of \( \delta \) and hence

\[
2 \sum_{k=1}^{[\log_2(\delta^{-3})]} \frac{1}{|B|} \sum_{\Delta \in B} F_\Delta(0, k) \leq 2 [\log_2(\delta^{-3})] 48^{2E} \text{geom} \left( \sum_{J \in P_3(I_1)} \| \mathcal{E}_J g \|_{L_\#^p(\tilde{\omega}_{B,E})} \right)^{p/2}.
\]

Combining this with (99), (100), and (103) shows that (98) (and hence the left hand side of (80)) is bounded above by

\[
(\cdots) \text{geom} \left( \sum_{J \in P_3(I_1)} \| \mathcal{E}_J g \|_{L_\#^p(\tilde{\omega}_{B,E})} \right)^{p/2}
\]

where \((\cdots)\) is equal to

\[
([\log_2(\delta^{-3})] + 1)^{\frac{p}{2} - 2} \left( [\log_2(\delta^{-3})]^2 E^{30E} d^{-1} + 2[\log_2(\delta^{-3})]^2 48^{2E} + 48^{2E} \right).
\]

Since \( \delta < 1/10 \) and \( E \geq 100 \), this is bounded above by \( E^{50E} d^{-1} (\log \frac{1}{\delta})^{p/2} \) which completes the proof of Theorem 6.1. \( \square \)

7. The iteration: Preliminaries

We now setup the iteration scheme as in [3] except this time we pay attention to various integrality constraints from previous sections. Let \( g : [0, 1] \rightarrow \mathbb{C}, t \geq 1, q \leq r, \) and \( I_1, I_2 \) two intervals in \([0, 1]\). Let \( B^r \) be a square in \( \mathbb{R}^2 \) with side length \( \delta^{-r} \). Define

\[
G_t(q, r) := \text{geom} \left( \sum_{J \in P_t(I)} \| \mathcal{E}_J g \|_{L_\#^p(\tilde{\omega}_{B^r,E})} \right)^{1/2}
\]

and

\[
A_p(q, r) = \left( \text{Avg}_{B^r \in P_{3-q}(B^r)} G_2(q, q)^p \right)^{1/p} := \left( \frac{1}{|P_{\delta^{-q}}(B^r)|} \sum_{B^r \in P_{3-q}(B^r)} G_2(q, q)^p \right)^{1/p}.
\]

Strictly speaking we should be writing \( G_t(q, B^r) \) instead of \( G_t(q, r) \) since this expression is different for different \( B^r \), however all that matters is keeping track of what our frequency and spatial scales are so for simplicity we will write \( r \) instead of \( B^r \).

Remark 7.1. Note that for \( G_t(q, r) \) and \( A_p(q, r) \) to be defined, we need \( |I_i| \delta^{-q} \in \mathbb{N} \) and \( \delta^{-r+q} \in \mathbb{N} \).
For a square $B^q$, note that $A_p(q, q) = G_2(q, q)$ for all $p$. In $A_p(q, r)$, increasing $q$ represents smaller frequency scales and increasing $r$ represents larger spatial scales.

We note that $G_t$ and $A_p$ here are essentially the same as $D_p$ and $A_p$, respectively in [6]. The only difference is that here we use the weight $\tilde{w}_B$ instead of $w_B$. This is because our bilinear decoupling constant is defined with weight $\tilde{w}_B$ rather than $w_B$.

Observe that $G_t$ and $A_p$ obey the following two basic properties. First the $t$ parameter in $G_t$ obeys Holder’s inequality.

**Lemma 7.2** (Holder’s inequality for $G_t$). For each square $B^r \subset \mathbb{R}^2$, if $(1 - \alpha)/p_1 + \alpha/p_2 = 1/t$, then

$$G_t(q, r) \leq G_{p_1}(q, r)^{1-\alpha} G_{p_2}(q, r)^{\alpha}.$$  

**Proof.** The factor $1/|B^r|$ in the definition of $G_t$ balances out by how $\alpha$ is defined and hence we may replace $L^p_q$, $L^p_{q1}$, and $L^p_{q2}$ with $L^1$, $L^{p1}$, and $L^{p2}$, respectively. Next, it suffices to prove that

$$\sum_{J \in P_{3q}(I)} \|\mathcal{E}_J g\|_{L^p(\tilde{w}_B)}^2 \leq \left( \sum_{J \in P_{3q}(I)} \|\mathcal{E}_J g\|_{L^{p1}(\tilde{w}_B)}^2 \right)^{1-\alpha} \left( \sum_{J \in P_{3q}(I)} \|\mathcal{E}_J g\|_{L^{p2}(\tilde{w}_B)}^2 \right)^{\alpha}.$$  

Applying Holder’s inequality gives that

$$\|\mathcal{E}_J g\|_{L^p}^2 \leq \left\| \mathcal{E}_J g \right\|_{L^{p_1}}^{2(1-\alpha)} \left\| \mathcal{E}_J g \right\|_{L^{p_2}}^{2\alpha} \leq \left\| \mathcal{E}_J g \right\|_{L^{p_1}}^{2(1-\alpha)} \left\| \mathcal{E}_J g \right\|_{L^{p_2}}^{2\alpha}$$

where here by $L^p$ we mean $L^p(\tilde{w}_B^r)$. This completes the proof Lemma 7.2. $\square$

Second, the averaging in the $r$ parameter in $A_p$ allows us to increase it.

**Lemma 7.3.** Fix arbitrary positive integers $r \leq s \leq t$ and suppose $\delta$ is such that $|I_t|^{r^{-1}} \in \mathbb{N}$, $\delta^{-r^{1+s}} \in \mathbb{N}$, and $\delta^{-t+s} \in \mathbb{N}$. Then for each square $B^t \subset \mathbb{R}^2$,

$$\text{Avg}_{B^t \in P_{3-s}(B^t)} A_p(r, s)^p = A_p(r, t)^p.$$  

**Proof.** Fix arbitrary square $B^t \subset \mathbb{R}^2$. Expanding the left hand side, we have

$$\text{Avg}_{B^t \in P_{3-s}(B^t)} A_p(r, s)^p = \text{Avg}_{B^t \in P_{3-s}(B^t)} \text{Avg}_{B^t \in P_{3-s}(B^t) \cap \text{geom}(B^t)} G_2(r, r)^p = \text{Avg}_{B^t \in P_{3-s}(B^t)} G_2(r, r)^p = A_p(r, t)^p.$$  

This completes the proof of Lemma 7.3. $\square$

Finally, we end this section with an outline of our strategy. As in Section 5 let $m \geq 1$, $E \geq 100$, $2 \leq p \leq 6$, and $\nu := 2^{-16} \cdot 2^m E^{10E}$. Let $I_1, I_2$ be two arbitrary intervals in $P_p([0, 1])$ which are at least $\nu$-separated.

**Lemma 7.4.** Suppose $\delta$ was such that $\delta^{-1/2m} \in 2^\mathbb{N}$ and $\nu \delta^{-1/2m} \in \mathbb{N}$. Then for each square $B^1$ of side length $\delta^{-1}$, we have

$$\|\text{geom} |\mathcal{E}_{I_1} g|\|_{L^p(B^1)} \leq E^{100E} \nu^{1/2} \delta^{-1/2m+1} A_p \left( \frac{1}{2^m}, 1 \right).$$
Proof. Note that since $\delta^{-1/2^m} \in 2\mathbb{N}$, $\delta^{-1+1/2^m} \in \mathbb{N}$ since $m \geq 1$. This proof is just an application of Holder, Minkowski, and Bernstein inequalities. We have

$$\| \text{geom} |E_i, g| \|^p_{L^p_B(B^1)} = \frac{1}{|B^1|} \int_{B^1} \text{geom} |E_i, g|^p = \frac{1}{|B^1|} \int_{B^1} \text{geom} \left( \sum_{J \in \mathcal{P}_{\delta^{-1/2^m}}(I_i)} |E_J g|^p \right)$$

$$\leq (\nu^{1/2} \delta^{-1/2^m+1})^p \frac{1}{|B^1|} \int_{B^1} \text{geom} \left( \sum_{J \in \mathcal{P}_{\delta^{-1/2^m}}(I_i)} |E_J g|^2 \right)^{p/2}$$

$$= (\nu^{1/2} \delta^{-1/2^m+1})^p \text{Avg}_{B^1/2^m \in \mathcal{P}_{\delta^{-1/2^m}}(B^1)} \| \text{geom} \left( \sum_{J \in \mathcal{P}_{\delta^{-1/2^m}}(I_i)} |E_J g|^2 \right)^{1/2} \|^p_{L^p_B(B^1/2^m)}.$$

Note that

$$\| \text{geom} \left( \sum_{J \in \mathcal{P}_{\delta^{-1/2^m}}(I_i)} |E_J g|^2 \right)^{1/2} \|^p_{L^p_B(B^1/2^m)} \leq \text{geom} \left( \sum_{J \in \mathcal{P}_{\delta^{-1/2^m}}(I_i)} |E_J g|^2 \right)^{1/2} \| \text{geom} \left( \sum_{J \in \mathcal{P}_{\delta^{-1/2^m}}(I_i)} |E_J g|^2 \right)^{p/2} \|_{L^p_B(B^1/2^m)}.$$ 

Since $p \geq 2$,

$$\left( \sum_{J \in \mathcal{P}_{\delta^{-1/2^m}}(I_i)} |E_J g|^2 \right)^{1/2} \|_{L^p_B(B^1/2^m)} \leq \left( \sum_{J \in \mathcal{P}_{\delta^{-1/2^m}}(I_i)} \| E_J g \|^2_{L^p_B(B^1/2^m)} \right)^{1/2} \|_{L^p_B(B^1/2^m)}.$$

Combining the above three centered equations gives that

$$\| \text{geom} (E_i, g) \|^p_{L^p_B(B^1)} \leq \nu^{1/2} \delta^{-1/2^m+1} \text{Avg}_{B^1/2^m \in \mathcal{P}_{\delta^{-1/2^m}}(B^1)} \| \text{geom} \left( \sum_{J \in \mathcal{P}_{\delta^{-1/2^m}}(I_i)} |E_J g|^2 \right)^{p/2} \|_{L^p_B(B^1/2^m)}^{1/p}.$$

Bernstein’s inequality (Lemma 2.20) and that $p \leq 6$, $E \geq 100$ gives that

$$\| E_J g \|^p_{L^p_B(B^1/2^m)} \leq 4^{pE/2} (pE/2)^{23pE/2} \| E_J g \|^2_{L^2_B \left( \mathcal{P}_{\delta^{-1/2^m}}(B^1), E \right)} \leq E^{100E} \| E_J g \|^2_{L^2_B \left( \mathcal{P}_{\delta^{-1/2^m}}(B^1), E \right)}.$$

Inserting this above gives that

$$\| \text{geom} (E_i, g) \|^p_{L^p_B(B^1)} \leq E^{100E} \nu^{1/2} \delta^{-1/2^m+1} \sum_{J \in \mathcal{P}_{\delta^{-1/2^m}}(I_i)} \| E_J g \|^2_{L^p_B(B^1/2^m)} \leq E^{100E} \nu^{1/2} \delta^{-1/2^m+1} A_p \left( \frac{1}{2^m}, 1 \right) \| \text{geom} (E_i, g) \|^p_{L^p_B(B^1)}$$

which completes the proof of Lemma 7.4.

Our target will be to prove an estimate of the form

$$A_p(2^{-m}, 1) \leq \delta_{\nu, E, m} G_p \left( \frac{1}{2}, 1 \right)$$

because then combining this with Lemma 7.4 gives an upper bound on the bilinear decoupling constant. Proposition 5.3 then allows us to control the linear decoupling constant. To prove (104), we will use ball inflation, $\ell^2 L^2$ decoupling to prove an estimate of the form $A_p(2^{-\ell}, 2^{-\ell+1}) \leq \nu, E A_p(2^{-\ell+1}, 2^{-\ell+1})$ for each $\ell = 2, 3, \ldots, m$. Then Lemma 7.3 allows us to patch all the estimates together.

The iteration is easier in the $2 \leq p \leq 4$ regime and so we will first do that case, then we will move on to the case when $4 < p < 6$. Finally, to control the decoupling constant at $p = 6$, we will apply Bernstein’s inequality and use the decoupling constant at $p'$ for some $p'$ suitably close to 6.
8. Control of the Bilinear Decoupling Constant

We now iterate to control the bilinear decoupling constant. We have two separate but similar cases. Our goal is to prove the following result.

**Proposition 8.1.** Fix integers \( m \geq 3 \) and \( E \geq 100 \). Let \( \nu := 2^{-162m}E^{10E} \) and suppose \( \delta \) is such that \( \delta^{-1/2m} \in 2^{\mathbb{N}} \) and \( \nu \delta^{-1/2m} \in \mathbb{N} \).

(a) If \( 2 \leq p \leq 4 \), then

\[
D_{p,E}(\delta, m) \leq \nu^{1/2}(E^{300E}\nu^{-1/4})^{m} \delta^{-\frac{1}{2m+1}}.
\]

(b) If \( 4 < p < 6 \), let \( a = \frac{p-4}{p-2} \), then

\[
D_{p,E}(\delta, m) \leq \nu^{1/2}(E^{300E}\nu^{-1/4}(\log \frac{1}{\delta})^{1/2})^{m} \delta^{-\frac{1}{2m+1}} D_{p,E}(\delta)^{1-(1-a)^{m-1}}.
\]

8.1. Case \( 2 \leq p \leq 4 \).

**Lemma 8.2.** Fix an integer \( 2 \leq \ell \leq m \). Suppose \( \delta^{-1/2m} \in 2^{\mathbb{N}} \) and \( \nu \delta^{-1/2m} \in \mathbb{N} \). Then for each square \( B^{2/2^\ell} \subset \mathbb{R}^2 \), we have

\[
A_{4}(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leq E^{100E}\nu^{-1/4}A_{4}(\frac{2}{2^\ell}, \frac{2}{2^\ell}).
\]

**Proof.** Fix an arbitrary square \( B^{2/2^\ell} \) of side length \( \delta^{-2/2^\ell} \). Note that our restrictions on \( \delta \) and \( \nu \) also imply that \( \nu \delta^{-2/2^\ell} \in \mathbb{N} \). We have

\[
A_{4}(\frac{1}{2^\ell}, \frac{2}{2^\ell})^{4} = \text{Avg}_{B^{1/2^\ell} \in \text{P}_{\delta^{-1/2^\ell}}(B^{2/2^\ell})} G_{2}(\frac{1}{2^\ell}, \frac{1}{2^\ell})^{4} \leq E^{200E}\nu^{-1}G_{2}(\frac{1}{2^\ell}, \frac{2}{2^\ell})^{4}
\]

where the inequality is by an application of Theorem 6.1. By \( l^2L^2 \) decoupling (Lemma 2.2.1), for each interval \( J \in \text{P}_{\delta^{-1/2^{\ell}}}(I_{i}) \), we have

\[
\|\mathcal{E}_{J}g\|^2_{L^2_{\#}(\tilde{\omega}_{B^{2/2^\ell}, E})} \leq E^{13E} \sum_{J \in \text{P}_{\delta^{-1/2^{\ell}}}(I_{i})} \|\mathcal{E}_{J^{'}}g\|^2_{L^2_{\#}(\tilde{\omega}_{B^{2/2^\ell}, E})}.
\]

Therefore

\[
\sum_{J \in \text{P}_{\delta^{-1/2^{\ell}}}(I_{i})} \|\mathcal{E}_{J}g\|^2_{L^2_{\#}(\tilde{\omega}_{B^{2/2^\ell}, E})} \leq E^{13E} \sum_{J \in \text{P}_{\delta^{-1/2^{\ell}}}(I_{i})} \sum_{J' \in \text{P}_{\delta^{-1/2^{\ell}}}(I_{i})} \sum_{J' \in \text{P}_{\delta^{-1/2^{\ell}}}(J)} \|\mathcal{E}_{J^{'}}g\|^2_{L^2_{\#}(\tilde{\omega}_{B^{2/2^\ell}, E})}.
\]

Since \( I_{i}, J \) and \( J' \) are all dyadic intervals, the above is equal to

\[
E^{13E} \sum_{J' \in \text{P}_{\delta^{-1/2^{\ell}}}(I_{i})} \|\mathcal{E}_{J^{'}}g\|^2_{L^2_{\#}(\tilde{\omega}_{B^{2/2^\ell}, E})}.
\]

Therefore

\[
G_{2}(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leq E^{13E/2}G_{2}(\frac{2}{2^\ell}, \frac{2}{2^\ell}) = E^{13E/2}A_{4}(\frac{2}{2^\ell}, \frac{2}{2^\ell}).
\]

Combining this with (105) completes the proof of Lemma 8.2. \( \square \)

Holder’s inequality allows us to change from \( A_{4} \) to \( A_{p} \) for \( 2 \leq p \leq 4 \) at no cost.
Corollary 8.3. Fix an integer $2 \leq \ell \leq m$. Suppose $\delta^{-1/2^\ell} \in 2^\mathbb{N}$ and $\nu \delta^{-1/2^\ell} \in \mathbb{N}$. Then for each square $B^{2/2^\ell} \subset \mathbb{R}^2$, we have

$$A_p\left(\frac{1}{2^\ell}, \frac{2}{2^\ell}\right) \leq E^{100E_p\nu^{-1/4}} A_p\left(\frac{1}{2^\ell}, \frac{2}{2^\ell}\right).$$

Proof. Applying Holder’s inequality to the definition of $A_p$ shows that for $2 \leq p \leq 4$, $A_p(q, r) \leq A_4(q, r)$. Lemma 8.2 and that

$$A_4\left(\frac{2}{2^\ell}, \frac{2}{2^\ell}\right) = G_2\left(\frac{2}{2^\ell}, \frac{2}{2^\ell}\right) = A_p\left(\frac{2}{2^\ell}, \frac{2}{2^\ell}\right)$$

then completes the proof of Corollary 8.3.

Now for each square $B^1$ with side length $\delta^{-1}$, we partition into squares of side length $\delta^{-2/2^\ell}$ and sum the previous corollary over all such squares. This yields the following result.

Lemma 8.4. Fix an integer $2 \leq \ell \leq m$. Suppose $\delta^{-1/2^\ell} \in 2^\mathbb{N}$ and $\nu \delta^{-1/2^\ell} \in \mathbb{N}$. Then for each square $B^1 \subset \mathbb{R}^2$, we have

$$A_p\left(\frac{1}{2^\ell}, 1\right) \leq E^{100E_p\nu^{-1/4}} A_p\left(\frac{1}{2^\ell}, 1\right).$$

Proof. Fix an arbitrary square $B^1$ of side length $\delta^{-1}$. Since $\delta^{-1/2^\ell} \in 2^\mathbb{N}$, we can dyadically partition $B^1$ into squares of side length $\delta^{-1/2^\ell}$. Lemma 7.3 and Corollary 8.3 then give that

$$A_p\left(\frac{1}{2^\ell}, 1\right)^p = \text{Avg}_{B^{2/2^\ell} \in P_{\delta^{-1/2^\ell}}(B^1)} A_p\left(\frac{1}{2^\ell}, \frac{2}{2^\ell}\right)^p$$

$$\leq E^{100E_p\nu^{-p/4}} \text{Avg}_{B^{2/2^\ell} \in P_{\delta^{-1/2^\ell}}(B^1)} A_p\left(\frac{2}{2^\ell}, \frac{2}{2^\ell}\right)^p = E^{100E_p\nu^{-p/4}} A_p\left(\frac{2}{2^\ell}, 1\right)^p.$$ 

This completes the proof of Lemma 8.4.

Now we combine the $m - 1$ inequalities together to obtain the following result.

Lemma 8.5. Suppose $\delta^{-1/2^m} \in 2^\mathbb{N}$ and $\nu \delta^{-1/2^m} \in \mathbb{N}$, then for each square $B^1 \subset \mathbb{R}^2$, we have

$$A_p\left(\frac{1}{2^m}, 1\right) \leq \left(E^{100E_p\nu^{-1/4}}\right)^{m-1} A_p\left(\frac{1}{2}, 1\right).$$

Proof. Since $\delta^{-1/2^m} \in 2^\mathbb{N}$, $\delta^{-1/2^\ell} \in 2^\mathbb{N}$ for $\ell = 1, 2, \ldots, m$. Since $\delta^{-1/2^m} \in 2^\mathbb{N}$ and $\nu \delta^{-1/2^m} \in \mathbb{N}$, it follows that $\nu \delta^{-1/2^{m-1}} \in \mathbb{N}$. Since $\delta^{-1/2^{m-1}} \in 2^\mathbb{N}$, we have that $\nu \delta^{-1/2^{m-2}} \in \mathbb{N}$. Continuing this shows that $\nu \delta^{-1/2^\ell} \in \mathbb{N}$ for $\ell = 1, 2, \ldots, m$. Iterating Lemma 8.4 a total of $m - 1$ times then completes the proof of Lemma 8.5.

We now finally relate $A_p(1/2, 1)$ to $G_p(1/2, 1)$ which will prove (104) in the case when $2 \leq p \leq 4$.

Lemma 8.6. If $\delta^{-1/2}, \nu \delta^{-1/2} \in \mathbb{N}$, then

$$A_p\left(\frac{1}{2}, 1\right) \leq 48^{E/p} G_p\left(\frac{1}{2}, 1\right).$$
Proof. Holder’s inequality \( \|f\|_p \leq \text{geom} \|f\|_q \) implies that
\[
G_2(\frac{1}{2}, \frac{1}{2}) \leq \text{geom} \left( \sum_{J \in P_{\delta-1/2}(L^1)} \|E_Jg\|_{L^p(\hat{w}_{B^{1/2},E})}^2 \right)^{1/2}.
\]
Since \( \|\text{geom} f_i\|_p \leq \|f_i\|_p \) and so
\[
A_p(\frac{1}{2}, 1) \leq \frac{1}{|P_{\delta-1/2}(B^1)|^{1/p}} \text{geom} \left( \sum_{B^1/2 \in P_{\delta-1/2}(B^1)} \sum_{J \in P_{\delta-1/2}(L^1)} \|E_Jg\|_{L^p(\hat{w}_{B^{1/2},E})}^2 \right)^{1/p^{2/3}}.
\]
Changing the \( L^p \) to \( L^2 \), interchanging the \( l^2 \) and \( L^p \) norms, and then applying Proposition \( 2.14 \) shows that this is \( \leq 48E/pG_p(1/2, 1) \) which completes the proof of Lemma \( 8.6 \).

Combining Lemmas \( 8.4 \) and \( 8.6 \) then proves \( 10.4 \) in the case when \( 2 \leq p \leq 4 \).

**Lemma 8.7.** Suppose \( \delta^{-1/2^m} \in 2^N \) and \( \nu\delta^{-1/2^m} \in N \), then for each square \( B^1 \subset \mathbb{R}^2 \), we have
\[
A_p(\frac{1}{2^m}, 1) \leq (E^{200E} \nu^{-1/4})^{m-1} G_p(\frac{1}{2}, 1).
\]

Combining Lemma \( 8.7 \) with Lemma \( 7.4 \) and applying the definition of the bilinear decoupling constant gives Proposition \( 8.1 \) in the case when \( 2 \leq p \leq 4 \).

**8.2 Case** \( 4 < p < 6 \). We now implement the iteration in the case when \( 4 < p < 6 \). This case is similar to the case when \( 2 \leq p \leq 4 \). For \( 4 < p < 6 \), \( a = \frac{p-1}{p-2} \) satisfies
\[
\frac{1}{p/2} = \frac{a}{p} + \frac{1-a}{2}.
\]
Note that \( 2(1-a) \) decreases monotonically to 1 as \( p \) increase to 6. The analogue of Lemma \( 8.2 \) and Corollary \( 8.3 \) is as follows.

**Lemma 8.8.** Fix an integer \( 2 \leq \ell \leq m \). Suppose \( \delta^{-1/2^\ell} \in 2^N \) and \( \nu\delta^{-1/2^\ell} \in N \). Then for each square \( B^{2^\ell} \subset \mathbb{R}^2 \), we have
\[
A_p(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leq E^{60E} \nu^{-1/4}(\log \frac{1}{\delta})^{1/2} A_p(\frac{2}{2^\ell}, \frac{2}{2^\ell})^{1-a} G_p(\frac{1}{2^\ell}, \frac{2}{2^\ell})^a.
\]

**Proof.** The proof is similar to that of Lemma \( 8.2 \). Since \( p \geq 4 \), in the definition of \( A_p \), we can increase the \( L^2(\hat{w}_{B^{1/2^\ell},E}) \) to \( L^2(\hat{w}_{B^{1/2^\ell},E}) \) using Holder’s inequality. Combining this with Theorem \( 6.1 \) gives that
\[
A_p(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leq E^{50E} \nu^{-1/4}(\log \frac{1}{\delta})^{1/2} G_p/2(1/2^\ell, 2/2^\ell).
\]
Holder’s inequality for \( G_2 \) (Lemma \( 7.2 \)) then shows that
\[
G_p(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leq G_p(\frac{1}{2^\ell}, \frac{2}{2^\ell})^a G_2(\frac{1}{2^\ell}, \frac{2}{2^\ell})^{1-a}.
\]
Proceeding as at the end of the proof of Lemma \( 8.2 \) gives that
\[
G_2(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leq E^{13E/2} A_p(\frac{2}{2^\ell}, \frac{2}{2^\ell})
\]
Putting the above three centered equations together then completes the proof of Lemma 8.8.

The analogue of Lemma 8.4 is as follows. The strategy of proof is essentially the same as that in Lemma 8.4 except this time we also need to deal with the $G_p(2^{-\ell}, 2^{-\ell+1})^\alpha$ term from Lemma 8.8.

**Lemma 8.9.** Fix an integer $2 \leq \ell \leq m$. Suppose $\delta^{-1/2^\ell} \in 2^N$ and $\nu\delta^{-1/2^\ell} \in \mathbb{N}$. Then for each square $B^1 \subset \mathbb{R}^2$, we have

$$A_p\left(\frac{1}{2^\ell}, 1\right) \leq E^{100E} \nu^{-1/4}(\log \frac{1}{\delta})^{1/2} A_p\left(\frac{1}{2^{\ell-1}}, 1\right)^{1-\alpha} G_p\left(\frac{1}{2^\ell}, 1\right)^\alpha.$$

**Proof.** Fix an arbitrary square $B^1$ of side length $\delta^{-1}$. Since $\delta^{-1/2^\ell} \in 2^N$, we can dyadically partition $B^1$ into squares of side length $\delta^{-1/2^\ell}$. Lemmas 7.3 and 8.8 and Holder’s inequality gives that

$$A_p\left(\frac{1}{2^\ell}, 1\right)^p = \operatorname{Av}_{B^{2/2^\ell} \subseteq P_{\delta^{-2/2^\ell}}(B^1)} A_p\left(\frac{1}{2^\ell}, \frac{2}{2^\ell}\right)^p \leq E^{60E} \nu^{-\frac{1}{2}} (\log \frac{1}{\delta})^{p/2} \left( \operatorname{Av}_{B^{2/2^\ell} \subseteq P_{\delta^{-2/2^\ell}}(B^1)} A_p\left(\frac{2}{2^\ell}, \frac{2}{2^\ell}\right)^p \right)^{1-\alpha} \left( \operatorname{Av}_{B^{2/2^\ell} \subseteq P_{\delta^{-2/2^\ell}}(B^1)} G_p\left(\frac{1}{2^\ell}, \frac{2}{2^\ell}\right)^p \right)^\alpha.$$

Lemma 7.3 gives that the first parenthetical term is equal to $A_p\left(\frac{2}{2^\ell}, 1\right)^p(1-\alpha)$. Thus the lemma is complete if we can show that

$$\operatorname{Av}_{B^{2/2^\ell} \subseteq P_{\delta^{-2/2^\ell}}(B^1)} G_p\left(\frac{1}{2^\ell}, \frac{2}{2^\ell}\right)^p \leq E^{40E} G_p\left(\frac{1}{2^\ell}, 1\right)^p.$$

Expanding definitions and interchanging geometric mean and the sum over $B^{2/2^\ell}$ gives that

$$\operatorname{Av}_{B^{2/2^\ell} \subseteq P_{\delta^{-2/2^\ell}}(B^1)} G_p\left(\frac{1}{2^\ell}, \frac{2}{2^\ell}\right)^p \leq \frac{1}{|B^1|} \operatorname{geom}( \sum_{B^{2/2^\ell} \subseteq P_{\delta^{-2/2^\ell}}(B^1)} \left( \sum_{J \in P_{\delta^{1/2^\ell}}(I_i)} \|E_{Jg}\|_{L^p(\tilde{w}_{B^{2/2^\ell}, E})}^2 \right)^{p/2} ).$$

Since $p \geq 2$, we can switch the $l^2$ and $l^p$ norms inside the geometric mean. Finally, apply Proposition 2.14 then proves that the above is $\leq 48E G_p\left(\frac{1}{2^\ell}, 1\right)^p$ which proves (106). This completes the proof of Lemma 8.9.

Combining the above $m-1$ inequalities in Lemma 8.9 gives the following result.

**Lemma 8.10.** Suppose $\delta^{-1/2^m} \in 2^N$ and $\nu\delta^{-1/2^m} \in \mathbb{N}$, then for each square $B^1 \subset \mathbb{R}^2$, we have

$$A_p\left(\frac{1}{2^m}, 1\right) \leq (E^{100E} \nu^{-1/4}(\log \frac{1}{\delta})^{1/2})^{m-1} A_p\left(\frac{1}{2^\ell}, 1\right)^{1-\alpha} \prod_{\ell=2}^m G_p\left(\frac{1}{2^\ell}, 1\right)^{\alpha(1-\alpha)^{m-\ell}}.$$
Lemma 8.11. Fix an integer $2 \leq \ell \leq m$. Suppose $\delta^{-1/2^\ell} \in 2^\mathbb{N}$ and $\nu \delta^{-1/2^\ell} \in \mathbb{N}$. Then

$$G_p\left(\frac{1}{2^\ell}, 1\right) \leq E^{100E} D_{p,E}(\delta) G_p\left(\frac{1}{2}, 1\right).$$

Proof. For each $J \in P_{\delta^{1/2^\ell}}(I_1)$, we have

$$\|\mathcal{E} g\|_{L^p(B^1)} = \|(g_{1,J})\|_{L^p(B^1)} \leq \tilde{D}_{p,E}(\delta) \left( \sum_{J' \in P_{\delta^{1/2}}([0,1])} \|\mathcal{E}_{J'}(g_{1,J})\|_{L^p(B^1)}^2 \right)^{1/2},$$

where the last equality is because both $\delta^{1/2^\ell}$ and $\delta^{1/2}$ are dyadic. Applying Propositions 2.11 and 3.11 then shows that

$$\|\mathcal{E} g\|_{L^p(B^1)} \leq 12E^{100} E^{70E} D_{p,E}(\delta) \left( \sum_{J' \in P_{\delta^{1/2}}(J)} \|\mathcal{E}_{J'} g\|_{L^p(B^1)}^2 \right)^{1/2}.$$

Combining this with the definition of $G_p(1/2^\ell, 1)$ then completes the proof of Lemma 8.11.

Combining Lemmas 8.6, 8.10, and 8.11 gives the following result.

Lemma 8.12. Suppose $\delta^{-1/2^m} \in 2^\mathbb{N}$ and $\nu \delta^{-1/2^m} \in \mathbb{N}$, then for each square $B^1 \subset \mathbb{R}^2$, we have

$$A_p\left(\frac{1}{2^m}, 1\right) \leq (E^{100E} \nu^{-1/4} (\log \frac{1}{\delta})^{1/2})^m D_{p,E}(\delta)^{1-(1-a)^m-1} G_p\left(\frac{1}{2}, 1\right).$$

This with Lemma 7.4 then proves Proposition 8.1 when $4 < p < 6$. Note that in this case we obtain a small improvement over the trivial bound of $D_{p,E}(\delta, m) \leq_{p,E} D_{p,E}(\delta)$ which is the key to obtaining control of the linear decoupling constant when $4 < p < 6$.

9. Decoupling at lacunary scales

Using Propositions 5.3 and 8.1 we bound the linear decoupling constant at a sequence of lacunary scales. The lacunary scales are because of the integrality conditions in Proposition 8.1. Our goal will be to prove the following result.

Proposition 9.1. Let $E \geq 100$, $m \geq 3$, $\nu := 2^{-16} 2^m E^{10E}$, and $\delta \in \{\nu^{-2^m n}\}_{n=1}^\infty$.

(a) If $2 \leq p \leq 4$, then

$$D_{p,E}(\delta) \leq 2^{m^2} E^{400E m} \nu^{-2^m} \delta^{1/m}. $$

(b) If $4 < p < 6$, then

$$D_{p,E}(\delta) \leq (2^{m^2} E^{400E m} \nu^{-2^m}) \frac{1}{|2/(p-2)|^{m-1}} \delta^{-2/(4(p-2))^{m-1}}.$$
(c) If \( p = 6 \), then for \( p' \in (4, 6) \), we have

\[
D_{6,E}(\delta) \leq E^{50E} (2m^2 E^{400E} m^{-2m}) \left( \frac{1}{(2/\nu - 2)m + 1} \right)^{1} \delta^{- \frac{2}{2(4/(\nu'-2))m+1} - 2\left( \frac{1}{\nu} - \frac{1}{2} \right)}.
\]

The proof of Proposition 9.1 actually shows that \( D_{p,E}(\delta) \leq E^{400E} m \nu^{-2m} \delta^{-1/2m} \) for \( 2 \leq p \leq 4 \), but the extra \( 2m^2 \) is harmless and will allow us to treat all three cases in essentially the same manner. Note that in Propositions 8.1 and 9.1, the bound when \( 2 \leq p \leq 4 \) is same as the bound for \( 4 < p < 6 \) except with \( p = 4 \) (and so \( a = 0 \)) and no \((\log \frac{1}{\delta})^{1/2} \). When we prove Proposition 9.1, we will only consider the more complicated case when \( 4 < p < 6 \) and \( p = 6 \).

9.1. Case \( 4 < p < 6 \). We first prove the following lemma.

**Lemma 9.2.** Let \( \nu = 2^{-16} 2^{10E} \), \( \delta^{1/2} \in 2^{-N} \), and \( a = \frac{p-4}{p-2} \). Let \( K \) be the largest integer such that \( \delta^{1/2} \nu^{-K} \in 2^{-N} \). Suppose \( \nu(\delta^{-2i})^{-1/2m} \in \mathbb{N} \) for all \( i = 0, 1, \ldots, K-1 \). Then

\[
D_{p,E}(\delta) \leq 2m^2 E^{400E} m \nu^{-2m} \delta^{-\frac{1}{2}} \max_{i=0, 1, \ldots, K-2^{m-1}-1} D_{p,E}(\delta \nu^{-2i})^{1-(1-a)^m-1}.
\]

**Proof.** Observe that

\[
\nu(\delta^{-2i})^{-1/2m} = (\delta^{-2(i+2^{m-1})})^{-1/2m}
\]

and so for \( i = 0, 1, \ldots, K-2^{m-1}-1 \), we have that \( \nu(\delta^{-2i})^{-1/2m} \in \mathbb{N} \).

For \( i = 0, 1, \ldots, K-2^{m-1}-1 \), we may apply Proposition 8.1 which gives that for such \( i \),

\[
D_{p,E}(\delta \nu^{-2i}, m) \leq (E^{300E} \nu^{-1/4} (\log \frac{1}{\delta})^{1/2})^{m} \delta^{-\frac{1}{2m+1}} D_{p,E}(\delta \nu^{-2i})^{1-(1-a)^m-1}.
\]

For \( i = K-2^{m-1}, \ldots, K-1 \), the trivial bound (Lemma 5.1) gives that

\[
D_{p,E}(\delta \nu^{-2i}, m) \leq 4E \nu^{1/2}(\delta \nu^{-2i})^{-1/4} \leq 4E \left( \delta^{-1/2} \nu^{-K} \right)^{1/2} \nu^{-\frac{1}{2}(2^{m-1}-1)}.
\]

By how \( K \) is defined, \( \delta^{1/2} \nu^{-K-1} \notin 2^{-N} \). Since \( \delta^{1/2} \) and \( \nu \) are dyadic numbers, we must then have \( \delta^{1/2} \nu^{-K-1} \notin 2 \mathbb{Z} \) and hence \( \delta^{1/2} \nu^{-K-1} \geq 1 \) which implies that \( \delta^{-1/2} \nu^{-K} \leq \nu^{-1} \).

Inserting this into (107) gives that for such \( i \),

\[
D_{p,E}(\delta \nu^{-2i}, m) \leq 4E \nu^{-2m/4}.
\]

Therefore Proposition 5.3 gives that

\[
D_{p,E}(\delta)
\leq \delta^{100E \log_{E} \nu^{-1}} \max (1, 4E \nu^{-2m/4}, \max_{i=0, 1, \ldots, K-2^{m-1}-1} D_{p,E}(\delta \nu^{-2i}, m))
\]

\[
\leq \delta^{100E \log_{E} \nu^{-1}} \max \left( 4E \nu^{-2m/4}, \right.
\]

\[
(E^{300E} \nu^{-1/4} (\log \frac{1}{\delta})^{1/2})^{m} \delta^{-\frac{1}{2m+1}} \max_{i=0, 1, \ldots, K-2^{m-1}-1} D_{p,E}(\delta \nu^{-2i})^{1-(1-a)^m-1}
\]

\[
\leq E^{300E} m \nu^{-2m} (\log \frac{1}{\delta})^{m/2} \delta^{-\frac{1}{2m+1}} + 100E \log_{E} \nu \max_{i=0, 1, \ldots, K-2^{m-1}-1} D_{p,E}(\delta \nu^{-2i})^{1-(1-a)^m-1}.
\]
where in the last inequality we have used that \( D_{p,E}(\delta) \geq 12^{-E/p} \) for all \( \delta \) which follows from the same proof as Lemma 3.5. Observe that \( \log \frac{1}{\delta} \leq \frac{1}{ae} \delta^{-a} \) for \( a > 0 \), and hence

\[
(\log \frac{1}{\delta})^{m/2} \leq 2^{m^2} E^{4Em} \delta^{-\frac{5}{2mEm^2}}.
\]

Furthermore, from our definition of \( \nu \), \( \delta_{100E\log \nu} \leq \delta^{-\frac{10}{2mEm^2}} \). Inserting this into the above completes the proof of Lemma 9.2.

Because of the generality of the statement of the previous lemma, we can upgrade the above result so that the same maximum appears on both left and right hand sides.

**Lemma 9.3.** Suppose \( \nu, \delta, K, \) and \( a \) are as in Lemma 9.2. The left hand side of the inequality in Lemma 9.2 can be replaced with \( \max_{i=0,1,...,K-2^{m-1}-1} D_{p,E}(\delta \nu^{-2i}) \).

**Proof.** Fix a \( j = 0,1,\ldots,K - 2^{m-1} - 1 \). Let \( K(j) := K - j \). Since \( K \) is the largest integer such that \( \delta^{1/2} \nu^{-K} \in 2^{-N} \), it follows that \( K(j) \) is the largest integer such that

\[
(\delta \nu^{-2j})^{1/2} \nu^{-K(j)} = \delta^{1/2} \nu^{-(K(j)+j)} \in 2^{-N}.
\]

We similarly also have \( (\delta \nu^{-2(i+j)})^{-1/2m} \in 2^N \) for \( i = 0,1,\ldots,K(j) - 1 \). Therefore Lemma 9.2 gives that

\[
D_{p,E}(\delta \nu^{-2j}) \leq 2^{m^2} E^{400Em} \nu^{-2m} \delta^{-\frac{5}{2m}} \max_{\ell=0,1,...,K-2^{m-1}-1-j} D_{p,E}(\delta \nu^{-2(j+\ell)})^{1-(1-a)^{m-1}}
\]

\[
\leq 2^{m^2} E^{400Em} \nu^{-2m} \delta^{-\frac{5}{2m}} \max_{\ell=0,1,...,K-2^{m-1}-1} D_{p,E}(\delta \nu^{-2\ell})^{1-(1-a)^{m-1}}.
\]

Since \( j \) on the left hand side of the above inequality is arbitrary and the right hand side is independent of \( j \), the above inequality is still true if we take the maximum over all \( j \) on the left hand side. This completes the proof of Lemma 9.3.

This gives the following corollary.

**Corollary 9.4.** Suppose \( \nu, \delta, K, \) and \( a \) are as in Lemma 9.2. Then

\[
\max_{\ell=0,1,...,K-2^{m-1}-1} D_{p,E}(\delta \nu^{-2\ell}) \leq (2^{m^2} E^{400Em} \nu^{-2m} \delta^{-\frac{5}{2m}})^{1-(1-a)^{m-1}}
\]

Taking \( \ell = 0 \) in Corollary 9.4 and observing that the choice of \( \delta \in \{\nu^{2m/n}\}_{n=1}^{\infty} \) satisfies the hypothesis of Lemma 9.2 completes the proof of Proposition 9.1 when \( 4 < p < 6 \). Indeed, with this choice of \( \delta \), \( K = 2^{m-1} n - 1 \) and so observe that

\[
(\delta \nu^{-2i})^{-1/2m} = (\nu^{-1})^{n-2i/2m}
\]

and for \( i = 0,1,\ldots,K - 1 \), we have \( n - 2i/2m \geq 0 \).
9.2. **Case** \( p = 6 \). At \( p = 6 \) the argument no longer gives a better than trivial estimate since here \( 2(1 - a) = 1 \). The advantage we have however is that we know a good bound on \( D_{p',E}(\delta) \) for all \( p' \) arbitrary close to 6. This combined with reverse Holder and Holder is enough to give a better than trivial bound at \( p = 6 \).

Let \( 4 < p' < 6 \) to be chosen later. The proof of Lemma 10.1 along with Corollary 2.9 and Proposition 2.11 imply that

\[
\|E_{[0,1]} g\|_{L^{p'}(B)} \leq 25^{(1/p' - 1/6)} E^{22E} \|E_{[0,1]} g\|_{L^{p'}(w_{B,E})} \leq E^{23E} D_{p',E}(\delta) \left( \sum_{J \in P_{6.6}([0,1])} \|E_{J} g\|_{L^{p'}(w_{B,E})}^2 \right)^{1/2}.
\]

Holder’s inequality to increase \( L^{p'} \) to \( L^6 \) then implies that

\[
D_{6,E}(\delta) \leq E^{50E} (\delta^{-2})^{1/p' - 1/6} D_{p',E}(\delta).
\]

Combining this with Proposition 9.1 for \( 4 < p' < 6 \) shows that under the hypothesis of Proposition 9.1 and arbitrary \( 4 < p' < 6 \), we have

\[
D_{6,E}(\delta) \leq E^{50E} \left( 2m^2 E^{400Em} \nu^{-2m} \right)^{1/(2(p' - 2)m)} \delta^{-2/(3(p' - 2)m + 1)} \delta^{-2/(3(p' - 2)m + 1)}
\]

Thus if we choose \( p' \) so that \( 1/p' - 1/6 \) is sufficiently small and then choose \( m \) sufficiently large, we once again can do better than the trivial bound of \( O_{E,p}(\delta^{-1/4}) \). This completes the proof of Proposition 9.1 when \( p = 6 \).

10. **Decoupling at all scales**

While Proposition 9.1 is for a lacunary sequence of scales, recall that the decoupling constant defined in (1) is for \( \delta \in \mathbb{N}^{-2} \). To upgrade Proposition 9.1 to all scales \( \delta \in \mathbb{N}^{-2} \) we use lacunarity and Proposition 4.1.

**Lemma 10.1.** Suppose \( \delta \in [\delta_1, \delta_2] \cap \mathbb{N}^{-2} \) and \( \delta_2/\delta_1 = c \). Then

\[
D_{p,E}(\delta) \leq E^{100E} 2^{E/p} c^{1/4} D_{p,E}(\delta_2).
\]

**Proof.** Using Proposition 4.1 and the trivial bound on decoupling we have

\[
D_{p,E}(\delta) \leq E^{100E} D_{p,E}(\delta_2) D_{p,E}(\delta/\delta_2)
\]

\[
\leq E^{100E} 2^{E/p} (\delta_2/\delta)^{1/4} D_{p,E}(\delta_2) \leq E^{100E} 2^{E/p} c^{1/4} D_{p,E}(\delta_2)
\]

which completes the proof of Lemma 10.1. \( \square \)

Combining this lemma with Proposition 9.1 gives the following result.

**Proposition 10.2.** Let \( E \geq 100 \), \( m \geq 3 \), and suppose \( \delta \in \mathbb{N}^{-2} \).

(a) If \( 2 \leq p \leq 4 \), then

\[
D_{p,E}(\delta) \leq 2^{4m} E^{15E} \delta^{-1/2m}.
\]

(b) If \( 4 < p < 6 \), then

\[
D_{p,E}(\delta) \leq (2^{4m} E^{15E} \delta^{-1/2m})^{1/(2(p-2)m+1)}.
\]
(c) If $p = 6$, then for $p' \in (4, 6)$ we have
\[ D_{p,E}(\delta) \leq (2^{4mE^{15E}}\delta^{-\frac{1}{2m}})^{\left\lfloor \frac{1}{2(p'-2)}m-1 \right\rfloor} \delta^{-2\left(\frac{1}{p} - \frac{1}{6}\right)} . \]

Proof. Recall that $\nu = 2^{-16}2^mE^{10E}$. The proof of all three parts is essentially the same, so we only concentrate on the $2 \leq p \leq 4$ case. If $\delta \in [\nu^{2m}, 1] \cap \mathbb{N}^{-2}$, the trivial bound gives that
\[ D_{p,E}(\delta) \leq 2^{Ep\nu^{-2m/4}} = 2^{E/p+4mE^{10E}}. \] (108)

From Lemma 10.1, if $\delta \in [\nu^{2m(n+1)}, \nu^{2m_n}] \cap \mathbb{N}^{-2}$ for some $n \geq 1$, then
\[ D_{p,E}(\delta) \leq E^{100E\nu^{2m/4}}D_{p,E}(\nu^{2m_n}). \]

Inserting the bound from Proposition 9.1 gives that the above is bounded by
\[ E^{100E\nu^{2m/4}}2^{m^2E^{100}E\nu^{-2m}}\delta^{-\frac{1}{2m}} \leq 2^{m^2E^{100}E\nu^{-2m}}\delta^{-\frac{1}{2m}}. \]

Using that $E \geq 100$ and the definition of $\nu$, we have
\[ 2^{m^2E^{100}E\nu^{-2m}} \leq 2^{100\cdot 4mE^{10E}} \leq 2^{4mE^{15E}}. \]

This then shows
\[ D_{p,E}(\delta) \leq 2^{4mE^{15E}}\delta^{-\frac{1}{2m}} \]
for all $\delta \in [\nu^{2m(n+1)}, \nu^{2m_n}]$, $n \geq 1$. Combining with (108) completes the proof of Proposition 10.2 when $2 \leq p \leq 4$. When $4 < p < 6$, $\frac{1}{2(p-2)} > 1$ and so we can repeat the same proof as above in the remaining two cases of the proposition. This completes the proof of Proposition 10.2. \qed

11. PROOF OF THEOREM 1.1

Since Proposition 10.2 is true for all $m \geq 3$ and $\delta \in \mathbb{N}^{-2}$, we now optimize the bound on $D_{p,E}(\delta)$ in $m$. This will give the proof of Theorem 1.1.

Proof of Theorem 1.1. We combine the cases of $2 \leq p \leq 4$ and $4 < p < 6$. Fix arbitrary $\delta \in \mathbb{N}^{-2}$ and $E \geq 100$. Let $m$ be the largest integer such that
\[ 2^{-m} \leq E^{5E}(\log_2 \delta^{-1})^{-1/3} < 2^{-m+1} . \] (109)

Since $\delta < 2^{-64E^{15E}}$, $m \geq 3$. Then
\[ 2^{4mE^{15E}}\delta^{-\frac{1}{2m}} \leq \exp\left(5(\log 2)^{1/3}E^{5E}(\log \frac{1}{\delta})^{2/3}\right) \leq \exp\left(5 \cdot E^{5E}(\log \frac{1}{\delta})^{2/3}\right) \]
(110) which finishes the case of Theorem 1.1 when $2 \leq p \leq 4$. For $4 < p < 6$, observe that
\[ \left(\frac{2}{p-2}\right)^{-(m-1)} = \exp\left(-(m-1) \log \frac{2}{p-2}\right) \leq 2\left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}(\log_2(\frac{2}{p-2}))}. \] (111)
Combining (110) and (111) then proves Theorem 1.1 in the case when $4 < p < 6$.
For the case when \( p = 6 \), choose \( m \) as in (109). Then for \( 4 < p' < 6 \),
\[
D_{6,E}(\delta) \leq \exp(10 \cdot E^{5E}(\log \frac{1}{\delta})^{\frac{2}{3} - \frac{1}{3} \log_2(\frac{2}{p'-2})})\delta^{-2(\frac{1}{p'} - \frac{1}{6})}
\leq \exp(E^{6E}(\log \frac{1}{\delta})^{\frac{1}{3} - \frac{1}{3} \log_2(\frac{p'-2}{2}) + (\frac{1}{p'} - \frac{1}{6})}).
\]
(112)

It thus remains to optimize
\[
(\log \frac{1}{\delta})^{-\frac{1}{3} \log_2(\frac{p'-2}{2}) + (\frac{1}{p'} - \frac{1}{6})}
\]
for \( 4 < p' < 6 \).

Let \( \lambda := \frac{1}{p'} - \frac{1}{6} \) and suppose we choose \( p' \) sufficiently close to 6 such that \( \lambda < 1/4 \). Then \( \frac{4}{p'-2} = \frac{1+6\lambda}{1-3\lambda} \) and
\[
\log \frac{4}{p'-2} \geq 8\lambda.
\]
Thus
\[
(\log \frac{1}{\delta})^{-\frac{1}{3} \log_2(\frac{p'-2}{2}) + (\frac{1}{p'} - \frac{1}{6})} \leq (\log \frac{1}{\delta})^{-3\lambda} + \lambda.
\]
Setting
\[
\lambda = \frac{\log(3 \log \frac{1}{\delta})}{3 \log \log \frac{1}{\delta}}
\]
gives that
\[
(\log \frac{1}{\delta})^{-3\lambda} + \lambda = \frac{1 + \log 3 + \log \log \log \frac{1}{\delta}}{3 \log \log \frac{1}{\delta}} \leq \frac{\log \log \log \frac{1}{\delta}}{\log \log \frac{1}{\delta}}
\]
(113)
where we have used that \( 1 + \log 3 \leq \log \log \log \frac{1}{\delta} \) for our range of \( \delta \). Note that for our range of \( \delta, \lambda < 1/4 \) since this is equivalent to \( 3 \log \log \frac{1}{\delta} < (\log \frac{1}{\delta})^{3/4} \) which is certainly satisfied if \( \delta^{-1} > 10^8 \). Inserting (113) into (112) then completes the proof of Theorem 1.1. □

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