A Universality Property of Gaussian Analytic Functions

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Abstract

We consider random analytic functions defined on the unit disk of the complex plane as power series such that the coefficients are i.i.d., complex valued random variables, with mean zero and unit variance. For the case of complex Gaussian coefficients, Peres and Virág showed that the zero set forms a determinantal point process with the Bergman kernel. We show that for general choices of random coefficients, the zero set is asymptotically given by the same distribution near the boundary of the disk, which expresses a universality property. The proof is elementary and general.

Keywords: Random analytic functions, Gaussian analytic functions.

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1 Main Result

Random analytic functions are a topic of classical interest [1, 12], which gained renewed interest, as a toy model for quantum chaos following work of Bogomolny, Bohigas and Leboeuf [5, 6]. A recent short paper about Gaussian analytic functions is entitled, “What is ... a Gaussian entire function,” [15].

Given a sequence of coefficients \( x = (x_0, x_1, x_2, \ldots) \), one may define the power series

\[
 f(x, z) = \sum_{n=0}^{\infty} x_n z^n .
\]

We consider random analytic functions defined by choosing a coefficient sequence \( X = (X_0, X_1, \ldots) \) where \( X_0, X_1, \ldots \) are i.i.d., complex valued random

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variables, with mean zero and unit variance, such that
\[ E[(\text{Re}X_i)^2] = E[(\text{Im}X_i)^2], \quad E\text{Re}[X_i]\text{Im}[X_i] = 0. \] (1.1)
By the Borel-Cantelli lemma, the radius of convergence of \( f(X, z) \) is 1, almost surely.

We wish to consider the case where the \( X_i \)'s are not necessarily Gaussian. But we first recall a beautiful result of Peres and Virág for Gaussian analytic functions [16].

**Theorem 1.1 (Peres and Virág, 2005)** Suppose that \( X_0, X_1, \ldots \) are i.i.d., and each \( X_i \) has density on \( \mathbb{C} \) given by \( \pi^{-1} \exp(-|z|^2) \). Then the zero set is a determinantal point process with Bergman kernel,
\[ K(z_1, z_2) = \frac{\pi}{(1 - z_1 \overline{z_2})^2}. \]
This kernel is invariant under the action of the symmetries of the hyperbolic plane modeled by the Poincaré disk. For each \( u \in \mathbb{C} \) with \( |u| < 1 \), the Möbius transformation is one such isometry
\[ \Phi(u; z) = \frac{z - u}{1 - \overline{u}z}. \]
This maps the open unit disk \( U(0, 1) = \{ z \in \mathbb{C} : |z| < 1 \} \) bijectively onto itself. Note that for a fixed \( z \in U(0, 1) \),
\[ |\Phi(u, z)| \uparrow 1 \quad \text{as} \quad |u| \uparrow 1. \]
In other words, taking \( |u| \uparrow 1 \) maps every point in the interior of the disk to a single point on the boundary. Our main result establishes a limit law at this boundary, for general random coefficients, not necessarily Gaussian.

**Theorem 1.2** Suppose that \( X_0, X_1, \ldots \) are i.i.d., complex valued random variables with mean zero and satisfying (1.1). Let \( Z(X) \) denote the random zero set \( \{ \xi \in U(0, 1) : f(X, \xi) = 0 \} \). Then, for any \( n \in \mathbb{N} \), and any \( n \) distinct points \( z_1, \ldots, z_n \in U(0, 1) \)
\[ \lim_{\epsilon \downarrow 0} \lim_{|u| \uparrow 1} \epsilon^{-2n} \mathbb{P} \left( \bigcap_{i=1}^{n} \{ U(z_i, \epsilon) \cap Z(X) \neq \emptyset \} \right) = \det \left( K(z_i, z_j) \right)_{i,j=1}^{n}, \]
where we write \( U(z_i, \epsilon) \) for the open ball \( \{ z \in \mathbb{C} : |z - z_i| < \epsilon \} \).

There have been many papers proving convergence of the first intensity measure of zeroes. In fact there are very precise and general results in this direction. See for example [9, 11, 17]. But this result addresses a slightly different issue because in principle it also gives correlations.

There is another important group of papers proving universality for Gaussian analytic functions, for the entire ensemble of correlations, by Bleher, Shiffman and Zelditch [2, 3, 4]. But these are in a different context.
In Section 2 we give the simple proof of this result. In Section 3 we describe the extensions to a related family of Gaussian analytic functions considered by Hough, Krishnapur, Peres and Virág in their recent monograph [10], which have interesting properties but whose zero sets are not determinantal.

2 Proof of the Main Result

A main step in proving Theorem 1.2 is the following elementary observation.

Lemma 2.1 Let \( Y = (Y_0, Y_1, \ldots) \) be i.i.d., complex Gaussians such that each \( Y_i \) has density equal to \( \pi^{-1} e^{-|y|^2} \) on \( \mathbb{C} \). Then for any \( n \in \mathbb{N} \), any \( z_1, \ldots, z_n \in U(0, 1) \) and any \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \), we have

\[
\text{Re} \left[ \sum_{i=1}^{n} \lambda_i \frac{f(X, \Phi(u, z_i))}{\Delta(u, z_i)} \right] \Rightarrow \text{Re} \left[ \sum_{i=1}^{n} \lambda_i f(Y, z_i) \right] \quad \text{as} \quad |u| \uparrow 1,
\]

where \( \Rightarrow \) denotes convergence in distribution, and

\[
\Delta(u, z) = \frac{1 - \pi z}{\sqrt{1 - |u|^2}}.
\]

We may prove this result using the following simple corollary of the Lindeberg-Feller central limit theorem.

Corollary 2.2 Suppose that \( X_0, X_1, \ldots \) are i.i.d., complex valued random variables with mean zero, satisfying condition (1.1). Suppose that \( \alpha_k(u) \in \mathbb{C} \) is defined for each \( u \in U(0, 1) \) and \( k \in \{0, 1, \ldots\} \), satisfying

(a) \( \frac{1}{2} \sum_{k=0}^{\infty} |\alpha_k(u)|^2 \to \sigma^2 > 0 \) as \( |u| \uparrow 1 \), and

(b) \( \sum_{k=0}^{\infty} |\alpha_k(u)|^p \to 0 \) as \( |u| \uparrow 1 \), for some \( p > 2 \).

Then \( \sum_{k=1}^{\infty} \text{Re}[\alpha_k(u)X_k] \) converges in distribution to \( \sigma \chi \) as \( |u| \uparrow 1 \), where \( \chi \) is a real standard normal random variable.

See, for example, [8] for the Lindeberg-Feller theorem. With this, we can prove the lemma.

Proof of Lemma 2.1 We can write

\[
\sum_{i=1}^{n} \lambda_i \frac{f(X, \Phi(u, z_i))}{\Delta(u, z_i)} = \sum_{k=0}^{\infty} \alpha_k(u)X_k,
\]

where

\[
\alpha_k(u) = \sum_{i=1}^{n} \lambda_i \frac{\Phi(u, z_i)^k}{\Delta(u, z_i)},
\]
since \( f(X, \Phi(u, z_i)) = \sum_{k=0}^{\infty} X_k \Phi(u, z_i)^k \). A simple calculation shows that
\[
\sum_{k=0}^{\infty} |\alpha_k(u)|^2 = \sum_{i,j=1}^{n} \lambda_i \lambda_j Q(u; z_i, z_j),
\]
where
\[
Q(u; z_i, z_j) = \frac{1}{\Delta(u, z_i) \Delta(u, z_j)} \sum_{k=0}^{\infty} \Phi(u, z_i)^k \Phi(u, z_j)^k.
\]
But an important property of the Möbius transform is that it leaves the covariance of this family of random analytic functions invariant, other than multiplying by the factors \( \Delta \). In fact, this is an important property of the Gaussian analytic functions studied by Peres and Virág, since it shows that their entire distributions are stationary, as the distribution of a Gaussian process is determined by the covariance. This is checked by summing the series to obtain
\[
Q(u; z_i, z_j) = \frac{1 - |u|^2}{(1 - uz_i)(1 - uz_j)} \cdot \frac{1}{1 - \Phi(u, z_i) \Phi(u, z_j)} = \frac{1}{1 - z_i z_j},
\]
which does not depend on \( u \). For the same reason it shows that
\[
\text{var} \left( \text{Re} \left[ \sum_{i=1}^{n} \lambda_i f(X, \Phi(u, z_i)) \right] \right) = \text{var} \left( \text{Re} \left[ \sum_{i=1}^{n} \lambda_i f(Y, z_i) \right] \right),
\]
for all \( u \). This takes care of condition (a) in Corollary 2.2.

To check condition (b) with \( p = 4 \), we note that
\[
\sum_{k=0}^{\infty} |\alpha_k(u)|^4 \leq \sqrt{n} \max_{i=1,\ldots,n} |\lambda_i|^4 \sum_{k=0}^{\infty} |\Phi(u, z_i)|^{4k} |\Delta(u, z_i)|^4.
\]
But we can sum the last series for each \( i \), to obtain
\[
\sum_{k=0}^{\infty} |\Phi(u, z_i)|^{4k} |\Delta(u, z_i)|^4 = \frac{(1 - |u|^2)^2}{1 - \nu z_i^2} \cdot \frac{1}{1 - \Phi(u, z_i)^4} = \frac{1 - |u|^2}{(1 - |z_i|^2)(1 - \nu z_i^2 + |z_i - u|^2)}.
\]
As long as all \( z_i \) are strictly inside the unit circle, this quantity converges to zero in the limit \( |u| \uparrow 1 \). This completes the proof of the lemma. \( \square \)

Lemma 2.1 implies that the random analytic function \( f(X, \Phi(u, z))/\Delta(u, z) \) converges to the random analytic function \( f(Y, z) \) in distribution, as \( |u| \uparrow 1 \), in the sense of finite dimensional marginals. But with this we may use the following lemma of Valko and Virág from their paper on random Schrödinger operators [19].

**Lemma 2.3 (Valko and Virág, 2010)** Let \( f_n(\omega, z) \) be a sequence of random analytic functions on a domain \( D \) (which is open, connected and simply connected) such that \( \mathbb{E}h(|f_n(z)|) < g(z) \) for some increasing unbounded function
and a locally bounded function $g$. Assume that $f_n(z) \Rightarrow f(z)$ in the sense of finite dimensional distributions. Then $f$ has a unique analytic version and $f_n \Rightarrow f$ in distribution with respect to local-uniform convergence.

Because of this result we see that $f(X, \Phi(u, z))/\Delta(u, z)$ converges in distribution to $f(Y, z)$ with respect to the local uniform convergence. But by Hurwitz’s theorem or Rouché’s theorem, this implies that the zero sets also converge in distribution, relative to the local weak topology on point processes. Since $\Delta(u, z)$ is finite and non-vanishing for $z \in U(0,1)$, the zero set is just the zero set of $f(X, \Phi(u, z))$. Combining this result with Peres and Virág’s Theorem 1.1 for the zero set of $f(Y, z)$ proves our theorem.

3 Discussion and Extensions

The proof presented here also may be extended to other families of Gaussian analytic functions. In a recent book by Hough, Krishnapur, Peres and Virág [10] several families of Gaussian analytic functions were studied, whose covariances are adapted to the classical symmetric spaces: the sphere, the plane and the hyperbolic plane. Some of the ensembles had been introduced before in [5, 6, 7, 13, 18].

In [10], there is presented a one-parameter family of Gaussian analytic functions, adapted to the Poincaré disk model of hyperbolic geometry, for all choices of Gaussian curvature $k < 0$, as well as a model for Gaussian analytic functions on the plane corresponding to $k = 0$. They also present Gaussian polynomials adapted to the sphere for quantized values of $k$: $k = 1/n$ for $n \in \mathbb{N}$. The special case considered by Peres and Virág in [16] corresponds to $k = -1$. Theorem 1.2 was just for the $k = -1$ model, but extends to all the models with $k \leq 0$, with analogous proofs mutatis mutandis. (See our original preprint [14] for full details.) For the spherical case, the diameter is finite, so no limit law is possible for fixed $k = 1/n$, $n \in \mathbb{N}$.

Finally, we end with a remark. One could consider the joint distribution of the two analytic functions $f(X, \Phi(u_1, z))$, $f(X, \Phi(u_2, z))$ in the limit that $u_1$ and $u_2$ both approach the boundary circle. A simple calculation shows that for any $z_1, z_2 \in U(0,1),

\text{cov}\left(\frac{f(X, \Phi(u_1, z_1))}{\Delta(u_1, z_1)}, \frac{f(X, \Phi(u_2, z_2))}{\Delta(u_2, z_2)}\right) = O\left(1 + \frac{|u_1 - u_2|}{\sqrt{(1 - |u_1|^2)(1 - |u_2|^2)}}\right)^{-1}

Since the processes are Gaussian, absence of correlations implies independence. Therefore, Lemma 2.3 may be used again to conclude that the zero sets of $f(X, \Phi(u_1, z))$ and $f(X, \Phi(u_2, z))$ are asymptotically independent if and only if $\Phi(u_1, u_2) \to \infty$. 
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