CANTOR UNIQUENESS AND MULTIPLICITY ALONG SUBSEQUENCES

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Abstract. We construct a sequence $c_l \to 0$ such that the trigonometric series $\sum c_l e^{ilx}$ converges to zero everywhere on a subsequence $n_k$. We show that any such series must satisfy that the $n_k$ are very sparse, and that the support of the related distribution is quite large.

1. Introduction

In 1870, Georg Cantor proved his famous uniqueness theorem for trigonometric series: if a series $\sum c_l e^{ilx}$ converges to zero for every $x \in [0,2\pi]$, then the $c_l$ are all zero [4]. The proof used important ideas from Riemann’s Habilitationsschrift, namely, that of taking the formal double integral $F(x) = \sum \frac{1}{2\pi} c_l e^{ilx}$ and examining the second Schwarz derivative of $F$. Cantor’s proof is now classic and may be found in many books, e.g. [16, §IX] or [3, §XIV]. A fascinating historical survey of these early steps in uniqueness theory, including why Riemann defined $F$ in the first place, may be found in [5]. (briefly, Riemann was writing necessary conditions for a function to be represented by a trigonometric series in terms of its double integral).

Cantor’s result may be extended in many directions, and probably the most famous one was the direction taken by Cantor himself, that of trying to see if the theorem still holds if the series is allowed not to converge at a certain set, which led Cantor to develop set theory, and led others to the beautiful theory of sets of uniqueness, see [9]. But in this paper we are interested in a different kind of extension: does the theorem hold when the series $\sum c_l e^{ilx}$ is required to converge only on a subsequence?

This problem was first tackled in 1950, when Kozlov constructed a non-trivial sequence $c_l$ and a series $n_k$ such that

$$\lim_{k \to \infty} \sum_{l=-n_k}^{n_k} c_l e^{ilx} = 0 \quad \forall x \in [0,2\pi].$$  

(1)

See [10] or [3, §XV.6]. A feature of Kozlov’s construction that was immediately apparent is that the coefficients $c_l$ are (at least for some $l$), very
large. Therefore it was natural to ask if it is possible to have (1) together with \( c_l \to 0 \). The problem was first mentioned in the survey of Talalyan [14] — this is problem 13 in §10 (note that there is a mistake in the English translation), and then repeated in [1] where the authors note, on page 1406, that the problem is “very hard”. In the same year, the survey of Ulyanov [15, page 20 of the English version] mentions the problem and conjectures that in fact, no such series exists. Skvortsov constructed a counterexample for the Walsh system [13], but not for the Fourier system.

1.1. Results. In this paper we answer this question in the positive. Here is the precise statement:

**Theorem 1.** There exist coefficients \( c_l \to 0 \), not all zero, and \( n_k \to \infty \) such that (1) holds.

The existence of such an example raises many new questions about the nature of the \( c_l \), of the distribution \( \sum c_le^{ilx} \), and of the numbers \( n_k \). We have two results which show some restrictions on these objects. The first states, roughly, that the \( n_k \) must increase at least doubly exponentially:

**Theorem 2.** Let \( c_l \to 0 \) and let \( n_k \) be such that (1) holds. Assume further that \( n_{k+1} = n_k^{1+o(1)} \). Then \( c_l \equiv 0 \).

Our second result is a lower bound on the dimension of the support of the distribution \( \sum c_le^{ilx} \). It is stated in terms of the upper Minkowski dimension (see, e.g., [6] where it is called the box counting dimension) which we denote by \( \text{dim}_{\text{Mink}} \).

**Theorem 3.** Let \( c_l \to 0 \) and let \( n_k \) be such that (1) holds. Let \( K \) be the support of the distribution \( \sum c_le^{ilx} \), and assume that

\[
\text{dim}_{\text{Mink}}(K) < \frac{1}{2}(\sqrt{17} - 3) \approx 0.561.
\]

Then \( c_l \equiv 0 \).

1.2. Comments and questions. An immediate question is the sharpness of the double exponential bound of theorem 2. The proof of theorem 1 which we will present is not quantitative, but it can be quantified with only a modicum of effort, giving:

There exists \( c_l \to 0 \) and \( n_k = \exp(\exp(O(k))) \) such that (1) holds.
(this quantitative version will not be proved in this paper). Thus in this setting the main problem remaining is the constant in the top. The reader might find it useful to think about the question as follows: suppose $n_{k+1} = n_k^\lambda$. For which value of $\lambda$ is it possible to construct a counterexample with this $n_k$?

But an even more interesting question is: what happens when the condition $c_l \to 0$ is removed from theorem 2? The answer is no longer doubly exponential, we are able to construct example with $n_k$ growing only slightly faster than exponentially. It seems interesting to determine the optimal rate of growth of $n_k$ in this case.

A variation of the problem where our upper and lower bound match more closely is the following: suppose that we require $c_l = 0$ for all $l < 0$ (often this is called an “analytic” version of the problem, because there is a naturally associated analytic function in the disk, $\sum c_l z^l$). In this case, the following can be proved. On the one hand, there is an example of a $|c_l| \leq \exp(Cl)$ and some $n_k$ such that

$$\lim_{k \to \infty} \sum_{l=0}^{n_k} c_l e^{ilx} = 0 \quad \forall x.$$ 

On the other hand, it is not possible to have such an example with $|c_l| \leq \exp(Cl/\log^2 l)$, or any other inverse of a non-quasianalytic sequence.

In a different direction, the condition $c_l \to 0$ can be improved: it is possible to require, in theorem 1, that the coefficients $c_l$ be inside $l^{2+\epsilon}$, for any $\epsilon > 0$. This, too, will not be shown in this paper, but the proof is a simple variation on the proof of theorem 1 below.

Another interesting question is the sharpness of the dimension bound in theorem 3. In our example the dimension of the support is 1 (even for the Hausdorff dimension, which is smaller than the upper Minkowski dimension). It would be very interesting to construct an example with dimension strictly smaller than 1. In the opposite direction, let us remark that in our example the support of the distribution $\sum c_l e^{ilx}$ has measure zero, but it is not difficult to modify the example so that the support would have positive measure. We find this interesting because this distribution is so inherently singular. The support must always be nowhere dense, see lemma 9 below.

1.3. Measures. It is interesting to note that the proofs of theorem 2 and 3 do not use the Riemann function in any way. In fact, the only element of classic uniqueness theory that appears in the proof is the localisation
principle. Thus the proof of theorem 2 is also a new proof of Cantor’s classic result. In the 150 years that passed since its original publication, the only other attempt we are aware of is [2], which gives a proof of Cantor’s theorem using one formal integration rather than two. To give the reader a taste of the ideas in the proofs of theorems 2 and 3, let us apply the same basic scheme to prove a simpler result: that no such construction is possible with $c_l$ being the Fourier coefficients of a measure.

**Proposition 4.** Let $\mu$ be a measure on $[0, 2\pi]$ with $\hat{\mu}(l) \to 0$ and let $n_k$ be a series such that

$$\lim_{k \to \infty} \sum_{l=-n_k}^{n_k} \hat{\mu}(l)e^{ilx} = 0 \quad \forall x \in [0, 2\pi].$$

Then $\mu = 0$.

**Proof.** Denote $S_n(x) = \sum_{l=-n}^{n} \hat{\mu}(l)e^{ilx}$. For every $x \in \text{supp} \mu$ there exists an $M(x)$ such that $|S_{n_k}(x)| \leq M(x)$ for all $k$ (certainly $M$ exists also for $x \notin \text{supp} \mu$ but we will not need it). By the Baire category theorem there is an interval $I$ and a value $M$ such that the set $\{x : M(x) \leq M\}$ is dense in $I \cap \text{supp} \mu$ (and $I \cap \text{supp} \mu \neq \emptyset$). By continuity, in fact $M(x) \leq M$ for all $x \in I \cap \text{supp} \mu$. Let $\varphi$ be a smooth function supported on $I$ (and $\varphi(x) \neq 0$ for all $x \in I$). We apply the localisation principle (see, e.g. [16, theorem IX.4.9]) and get that the series

$$\varphi(x) \sum \hat{\mu}(l)e^{ilx} \quad \text{and} \quad \sum \check{\varphi}\hat{\mu}(l)e^{ilx}$$

are uniformly equiconvergent. Hence $\varphi \mu$ satisfies the same property as $\mu$ i.e.

$$\lim_{k \to \infty} \sum_{l=-n_k}^{n_k} \check{\varphi}\hat{\mu}(l)e^{ilx} = 0 \quad \forall x \in [0, 2\pi]$$

and further, this convergence is bounded on $\text{supp} \varphi \mu$ (since $\text{supp} \varphi \mu = I \cap \text{supp} \mu$). If $\mu \neq 0$ then also $\varphi \mu \neq 0$.

The conclusion of the previous paragraph is that we could have, without loss of generality, assumed to start with that $S_{n_k}$ is bounded on $\text{supp} \mu$. Let us therefore make this assumption (so we do not have to carry around the notation $\varphi$). We now argue as follows:

$$\sum_{l=-n_k}^{n_k} |\hat{\mu}(l)|^2 = \sum_{l=-\infty}^{\infty} \overline{S_{n_k}(l)} \cdot \hat{\mu}(l) = \int \overline{S_{n_k}(x)} \, d\mu(x) \leq M||\mu||$$
where the second equality is due to Parseval and where $M$ is again the maximum of $|S_{nk}|$ on supp $\mu$. Since this holds for all $k$, we get that $\sum |\hat{\mu}(l)|^2 < \infty$, so $\mu$ is in fact an $L^2$ function. But this is clearly impossible, since the Fourier series of an $L^2$ function converges in measure to it. \hfill \Box

The crux of the proof is that $S_{nk}$ is small where $\mu$ is supported. The proofs of theorems 2 and 3 replace $\mu$ with a different partial sum, $S_s$ for some carefully chosen $s$ (roughly, for $s \approx n_k^{3/2}$) and show that $S_{nk}$ is small where $S_s$ is essentially supported. The details are below.

Let us remark that the only place where the condition $\hat{\mu}(l) \to 0$ was used in the proof of proposition 4 is in the application of the localisation principle. This can be circumvented, with a slightly more involved argument. See details in §4. Similarly theorems 2 and 3 may be generalised from $c_l \to 0$ to $c_l$ bounded, at the expense of a more involved use of the localisation principle.

2. Construction

It will be convenient to work in the interval $[0,1]$ and not carry around $\pi$-s, so define

$$e(x) = e^{2\pi ix}.$$ 

For an integrable function $f$ we define the usual Fourier partial sums,

$$S_n(f; x) = \sum_{l=-n}^{n} \hat{f}(l)e(ll).$$

In this paper “smooth” means $C^2$, but the proofs work equally well with higher smoothness (up to the quasianalytic threshold). We use $C$ and $c$ to denote arbitrary universal constant, whose value might change from line to line or even inside the same line. We use $C$ for constants which are large enough, and $c$ for constants which are small enough. We use $|| \cdot ||$ for the $L^2$ (or $l^2$) norm, other $L^p$ norms are denoted by $|| \cdot ||_p$ (except one place in the introduction where we used $||\mu||$ for the norm of the measure $\mu$). For a set $E \subset [0,1]$ we denote by $|E|$ the Lebesgue measure of $E$.

2.1. The localisation principle. Let us recall Riemann’s localisation principle: as formulated by Riemann, it states that the convergence of a trigonometric series at a point $x$ depends only on the behaviour of the Riemann function at a neighbourhood of $x$. See [16, §IX.4]. Rajchman found a formulation of the principle which does not use the Riemann function and has
a simple proof. It states that for any $c_l \to 0$ and any smooth function $\varphi$,

$$\varphi(x) \sum c_l e(lx) \text{ and } \sum (c \ast \widehat{\varphi})(l)e(lx) \text{ are equiconvergent}$$  \hspace{1cm} (2)

where $c \ast \widehat{\varphi}$ is a discrete convolution. See [16, theorem IX.4.9], or the proof of theorem 11 below, which follows Rajchman’s approach precisely. We will use Rajchman’s theorem both on and off the support of $\sum c_l e(lx)$ (denote this support by $K$). Off $K$, it has the following nice formulation: if $c_l \to 0$ then

$$\sum c_l e(lx) = 0 \quad \forall x \notin K. \hspace{1cm} (3)$$

and further, convergence is uniform on any closed interval disjoint from $K$.

To the best of our knowledge, this precise formulation first appeared in [7, Proposition 1, §V.3, page 54].

2.2. First estimates.

**Lemma 5.** For every $\epsilon > 0$ there exists a smooth function $u : [0,1] \to \mathbb{R}$ with $u(0) = u(1) = 0$, $u(x) \in [0,1]$, and $||\widehat{u} - 1||_{\infty} < \epsilon$.

When we say that $u$ is smooth we mean also when extended periodically (or when extended by 0, which is the same under the conditions above).

**Proof.** Take any standard construction of a smooth function satisfying $u(0) = u(1) = 0$, $u(x) \in [0,1]$ and $u(x) = 1$ for all $x \in \left[\frac{1}{2}\epsilon, 1 - \frac{1}{2}\epsilon\right]$. The condition on the Fourier coefficients then follows by $||\widehat{u} - 1||_{\infty} \leq ||u - 1||_{1}$. \hfill \Box

**Lemma 6.** For every $\epsilon > 0$ there exists a smooth function $h : [0,1] \to \mathbb{R}$ and an $n \in \mathbb{N}$ such that

1. $\widehat{h}(0) = 1$
2. $\text{supp } h \subset [0, \frac{1}{2}]$
3. For all $x \in [0, \frac{1}{2}]$, $|S_n(h; x)| < \epsilon$.

**Proof.** Let $P$ be an arbitrary trigonometric polynomial satisfying that $\widehat{P}(0) = 1$ and $|P(x)| < \epsilon$ for all $x \in [0, \frac{1}{2}]$. Let $n = \deg P$, let $m = 2n + 1$ and let $q$ be a smooth function supported on $[0, 1/2m]$ with $\widehat{q}(k) \neq 0$ for all $|k| \leq n$. Examine a function $h$ of the type

$$h(x) = \sum_{j=0}^{m-1} a_j q\left(x - \frac{j}{2m}\right).$$
Then $h$ is smooth, supported on $[0, \frac{1}{2}]$, and its Fourier coefficients are given by

$$\hat{h}(k) = \frac{\hat{P}(k)}{\hat{q}(k)} = \sum_{j=0}^{m-1} a_j e(-jk/2m).$$

The matrix $\{e(-jk/2m) : j \in \{0, \ldots, m-1\}, k \in \{-n, \ldots, n\}\}$ is a Vandermonde matrix hence invertible, so one may find $a_j$ such that $\sum a_j e(-jk/2m) = \hat{P}(k)/\hat{q}(k)$ for all $k \in \{-n, \ldots, n\}$. With these $a_j$ our $h$ satisfies $\hat{h}(k) = \hat{P}(k)$ for all $k$ such that $|k| \leq n$ so $S_n(h) = P$ which has the required properties.

**Remark.** The coefficients of the $h$ given by lemma 6 are typically large. The reason is the Vandermonde matrix applied. We need to invert the Vandermonde matrix and its inverse has a large norm, exponential in $n$ (the inverse of a Vandermonde matrix has an explicit formula). To counterbalance this last sentence a little, let us remark that $n$, the degree of the polynomial $P$ used during the proof can be taken to be logarithmic in $\epsilon$. This requires to choose a good $P$. For this purpose we apply the following theorem of Szegő: for every compact $K \subset \mathbb{C}$ there exists monic polynomials $Q_n$ with $\max_{x \in K} |Q_n(x)| = (\text{cap}(K) + o(1))^n$. See [12, corollary 5.5.5]. We apply Szegő’s theorem with $K = \{e(x) : x \in [0, \frac{1}{2}]\}$ and then define $P_n(x) = \text{Re}(e(-nx)Q_n(e(x)))$. We get that $\hat{P}_n(0) = 1$ and $\max_{x \in [0, \frac{1}{2}]} |P_n(x)| \leq (\text{cap}(K) + o(1))^n$. The capacity of $K$ can be calculated by writing explicitly a Riemann mapping between $\mathbb{C} \setminus K$ and $\{z : |z| > 1\}$ and is $1/\sqrt{2}$, and in particular smaller than 1 (see [12, theorem 5.2.3] for the connection to Riemann mappings). Hence it is enough to have $C \log \frac{1}{\epsilon}$ terms in the product to ensure that $P$ would satisfy $|P(x)| \leq \epsilon$ for all $x \in [0, \frac{1}{2}]$. With this $P$ the norm of $h$ would be polynomial in $\epsilon$.

2.3. **Reducing the coefficients.** In the next lemma we reduce the Fourier coefficients using a method inspired by the proof of the Menshov representation theorem (see, e.g., [11]). We separate the interval $[0, 1]$ into many small pieces and on each put a copy of the $h$ above, scaled differently. Unlike in typical applications of Menshov’s approach, we do not have each copy of $h$ sit in a distinct “spectral interval” but they are rather intertwined. The details are below. Still, like in other applications of Menshov’s technique, the resulting set is divided into many small intervals in a way that pushes the dimension up. This is why we are unable to construct an example supported on a set with dimension less than 1.
Lemma 7. For every $\epsilon > 0$ there exists a smooth function $f : [0, 1] \to \mathbb{R}$ and an $n \in \mathbb{N}$ with the following properties:

(i) $\hat{f}(0) = 1$.
(ii) For all $k \neq 0$, $|\hat{f}(k)| < \epsilon$.
(iii) For every $x \in \text{supp } f$, $|S_n(f; x)| < \epsilon$.

Proof. We may assume without loss of generality that $\epsilon < \frac{1}{2}$, and it is enough to replace requirement (i) by the weaker requirement $|\hat{f}(0) - 1| < \epsilon$ (and then normalise).

1. Let $h$ be the function given by lemma 6 with $\epsilon_{\text{lemma 6}} = \epsilon/4$, and denote $m = n_{\text{lemma 6}}$. In other words, $h$ satisfies

\[ \hat{h}(0) = 1 \quad \text{supp } h \subset [0, \frac{1}{2}] \quad |S_m(h; x)| < \frac{1}{4} \epsilon \quad \forall x \in [0, \frac{1}{2}]. \]

Let $a > 2||h||_1/\epsilon$ be some integer. Let $u$ be the function given by lemma 5 with $\epsilon_{\text{lemma 5}} = \epsilon/2$ i.e. $u$ is smooth on $[0, 1]$, $u(0) = u(1) = 0$ and $u$ satisfies $||\hat{u} - 1||_\infty < \frac{1}{2} \epsilon$.

Let $v(x) = u(xa)$ (extended to zero outside $[0, 1/a]$). Let $r$ be a large integer parameter to be fixed later, depending on all previously defined quantities ($\epsilon, h, m, a$ and $u$). Define

\[ f(x) = \sum_{j=0}^{a-1} v\left(x - \frac{j}{a}\right) h(x(r^3 + jr)). \]

The role of the quantities $r^3 + jr$ will become evident later.

Let us see that $f$ satisfies all required properties. It will be easier to consider trigonometric polynomials rather than smooth functions so define

\[ H := S_{\lfloor r/2 \rfloor - 1}(h) \quad V := S_{\lfloor r/2 \rfloor - 1}(v) \quad F(x) := \sum_{j=0}^{a-1} V\left(x - \frac{j}{a}\right) H(x(r^3 + jr)). \] (4)

The smoothness of $v$ and $h$ imply that $||\hat{v} - V||_1$ and $||\hat{h} - H||_1$ can be taken arbitrary small as $r \to \infty$. Since

\[ ||\hat{f} - F||_1 \leq \sum_{j=0}^{a-1} ||\hat{v} - V||_1 ||\hat{h}||_1 + ||\hat{V}||_1 ||\hat{h} - H||_1 \]
we may take \( r \) sufficiently large and get \( \| f - \hat{F} \|_1 < \frac{1}{2} \epsilon \) (but do not fix the value of \( r \) yet). Thus, with such an \( r \), we need only show

(i) \( \| \hat{F} - 1 \|_\infty < \frac{1}{2} \epsilon \)

(ii) For every \( x \in \text{supp} \, f \), \( | S_n(F; x) | < \frac{1}{2} \epsilon \) (note that we take \( x \) in \( \text{supp} \, f \) and not in \( \text{supp} \, F \)).

2. We start with the estimate of \( \hat{F} - 1 \). Examine one summand in the definition of \( F \), (4). Denoting \( G_j = V(x - j/a)H(x(r^3 + jr)) \) we have

\[
\hat{G}_j(l) = \begin{cases} 
\hat{V}(p)\hat{H}(q)e(-pj/a) & l = p + q(r^3 + jr), \ |p|, |q| < r/2 \\
0 & \text{otherwise}
\end{cases}
\] (5)

In particular, \( l \) and \( j \) determine \( p \) and \( q \) uniquely. An immediate corollary is:

\[
\| \hat{G}_j \|_\infty = \| \hat{V} \|_\infty \| \hat{H} \|_\infty \leq \| v \|_1 \| h \|_1 \leq \frac{\| h \|_1}{a} < \frac{\epsilon}{2}
\] (6)

where the last inequality is from the definition of \( a \). Assume now that \( r > a \).

Then we can extract another corollary from (5): that the different \( G_j \) have disjoint spectra, except at \( (-r/2, r/2) \). Hence

\[
| \hat{F}(l) | = \max_j | \hat{G}_j(l) | \leq \frac{\epsilon}{2} \quad \forall |l| \geq r/2.
\] (7)

Finally, for \( l \in (-r/2, r/2) \) we have that \( F \) “restricted spectrally to \((-r/2, r/2)\)” is simply \( \sum_j V(x - j/a) \) so its Fourier spectrum is simply that of \( u \) spread out. Since \( \| u \|_1 \| 1 \| \leq \frac{\epsilon}{2} \) we get also in this case \( |\hat{F} - 1(l)| < \frac{1}{2} \epsilon \). For those who prefer formulas, just note in (5) that if \( l \in (-r/2, r/2) \) then \( q = 0 \) and since \( \hat{H}(0) = 1 \) we get

\[
\hat{F}(l) = \sum_{j=0}^{a-1} \hat{V}(l)e(-lj/a) = \begin{cases} 
\frac{a}{a} \hat{V}(l) & l \equiv 0 \mod a \\
0 & \text{otherwise}
\end{cases}
\]

Recall that \( v(x) = u(xa) \) so for \( l \equiv 0 \mod a \) we have

\[
| V - 1(l) | \leq | v - 1(l) | = | u - 1(l/a) | < \frac{\epsilon}{a}. 
\]

With (7) we get \( \| \hat{F} - 1 \|_\infty < \frac{1}{2} \epsilon \), as needed.

3. Finally, we need to define \( n \) and see that \( S_n(F) \) is small on \( \text{supp} \, f \).

Assume \( r > m \) and define

\[
n = m(r^3 + r^2).
\]
This value of \( n \) has the property that
\[
\begin{align*}
  n &> m(r^3 + jr) + r/2 \\
  n &< (m + 1)(r^3 + jr) - r/2
\end{align*}
\]
for all \( j \in \{0, \ldots, a - 1\} \). We now see why it was important to choose the spacings of the arithmetic progressions to be \( r^3 + jr \): these spacings need to be different to have separation of the spectra of the different \( G_j \) (and they must be different by at least \( r \), because the spectra of the \( G_j \) are arranged in blocks of size \( r \)), but they need to be sufficiently close that even in the \( m \)th term it is still possible to “squeeze” an \( n \) between all the \( m \)th terms and all the \( m + 1 \)st terms. The \( r^3 \) in the spacings ensures that.

Using (5) gives that
\[
S_n(G_j; x) = S_m(H; x(r^3 + jr)) \cdot V\left(x - \frac{j}{a}\right).
\]
At this point it will be easier to compare to \( v \) rather than to \( V \), so write
\[
S_n(G_j; x) = S_m(H; x(r^3 + jr)) \cdot v\left(x - \frac{j}{a}\right) + E_j
\]
and note that for \( r \) sufficiently large \( E_j \) can be taken to be arbitrarily small. Take \( r \) so large as to have
\[
\left| S_n(F; x) - \sum_{j=0}^{a-1} S_m(H; x(r^3 + jr))v\left(x - \frac{j}{a}\right) \right| < \frac{1}{4} \epsilon. \tag{8}
\]
This is our last requirement from \( r \) and we may fix its value now.

For every \( x \in [0,1] \) there is at most one \( j_0 \) such that \( v(x - j_0/a) \neq 0 \), namely \( j_0 = \lfloor x/a \rfloor \). If \( x \in \text{supp} \ f \) then it must be the case that \( x(r^3 + j_0r) \in [0, \frac{1}{2}] \) mod 1. But in this case, by our definition,
\[
|S_m(H; x(r^3 + j_0r))| < \frac{1}{4} \epsilon.
\]
We get
\[
x \in \text{supp} \ f \implies \left| \sum_{j=0}^{a-1} S_m(H; x(r^3 + jr)) \cdot v\left(x - \frac{j}{a}\right) \right| < \frac{1}{4} \epsilon,
\]
and with (8) we get \( |S_n(F; x)| < \frac{1}{4} \epsilon \), as needed.

**Lemma 8.** Let \( f : [0,1] \to \mathbb{R} \) be smooth, \( \epsilon > 0 \) and \( N \in \mathbb{N} \). Then there exists a smooth function \( g : [0,1] \to \mathbb{R} \) satisfying
\[
\begin{align*}
  (i) \ &\text{supp} \ g \subseteq \text{supp} \ f, \\
  (ii) \ &\text{for all } n \in \mathbb{Z}, |\hat{g}(n) - \hat{f}(n)| < \epsilon
\end{align*}
\]
(iii) For some \( n > N \) we have

\[ |S_n(g; x)| < \epsilon \quad \forall x \in \text{supp} \, g. \]

Proof. Let \( h \) be the function from lemma 7 with \( \epsilon_{\text{lemma} \ 7} = \epsilon / 2 ||\hat{f}||_1 \). Denote by \( m \) the integer output of lemma 7 i.e. the number such that \( S_m(h; x) < \epsilon / 2 ||\hat{f}||_1 \) for all \( x \in \text{supp} \, h \). Let \( r \) be large enough so that

\[ \sum_{|k| \geq r/2} |\hat{f}(k)| < \epsilon / 2 ||\hat{h}||_1 \]

and such that \( r(m + 1/2) > N \) (let \( r \) be even). Denote

\[ g(x) := f(x)h(rx) \quad n := r(m + 1/2) \]

where \( h \) is extended periodically to \( \mathbb{R} \). Let us see that \( g \) and \( n \) satisfy the requirements of the lemma. The smoothness of \( g \) follows from those of \( f \) and \( h \). Condition (ii) follows because

\[ \hat{g}(k) - \hat{f}(k) = \sum_l (\hat{h} - 1(l)\hat{f}(k - lr)) \]

and because \( ||\hat{h} - 1||_{\infty} \leq \epsilon / 2 ||\hat{f}||_1 \). Finally, to see condition (iii) write

\[ F := S_{r/2}(f) \quad G(x) := F(x)h(rx) \]

and note that \( ||\hat{g} - G||_1 \leq ||\hat{f} - F||_1 ||\hat{h}||_1 < \frac{1}{2} \epsilon \). To estimate \( S_n(G) \), note that if \( x \in \text{supp} \, g \) then \( rx \in \text{supp} \, h \mod 1 \) and hence \( S_m(h; rx) < \epsilon / 2 ||\hat{f}||_1 \). But

\[ S_n(G; x) = F(x)S_m(h; rx) \]

but since \( |F(x)| \leq ||\hat{F}||_1 \leq ||\hat{f}||_1 \) we get

\[ |S_n(G; x)| \leq ||\hat{f}||_1 \frac{\epsilon}{2 ||\hat{f}||_1} = \frac{\epsilon}{2} \]

finishing the lemma.

\[ \square \]

2.4. Proof of theorem 1. The coefficients \( c_l \) will be constructed by inductively applying lemma 8. Define therefore \( f_1 = 1 \) and \( n_1 = 2 \), and for all \( k \geq 1 \) define \( f_{k+1} = g_{\text{lemma} \ 8} \) and \( n_{k+1} = n_{\text{lemma} \ 8} \) where lemma 8 is applied with \( f_{\text{lemma} \ 8} = f_k \), \( \epsilon_{\text{lemma} \ 8} = 2^{-k} / n_k \) and \( N_{\text{lemma} \ 8} = n_k + 1 \) (this last parameter merely ensures that the \( n_k \) are increasing). We now claim that \( \hat{f}_k(l) \) converges as \( k \to \infty \), and that the limit, \( c_l \), satisfies the requirements of the theorem.
The fact that \( \lim_{k \to \infty} \hat{f}_k(l) \) exists is clear, because \( \hat{f}_{k+1}(l) - \hat{f}_k(l) < 2^{-k}/n_k \). Denote
\[
c_l = \lim_{k \to \infty} \hat{f}_k(l).
\]
This also shows that \( c_l \to 0 \).

Denote now \( S_n = \sum_{l=-n}^n c_l e(lx) \). To see that \( S_n(x) \to 0 \) for all \( x \) we separate into \( x \in \cap \text{supp } f_k \) and the rest. Note that \( \cap \text{supp } f_k \) contains the support of the distribution \( \delta := \sum c_l e(lx) \). Indeed, if \( \varphi \) is a Schwartz test function supported outside \( \cap \text{supp } f_k \) then \( \text{supp } \varphi \cap \text{supp } f_k \) is a sequence of compact sets decreasing to the empty set (recall that \( \text{supp } f_{k+1} \subseteq \text{supp } f_k \)) so for some finite \( k_0 \) we already have \( \text{supp } \varphi \cap \text{supp } f_k = \emptyset \) for all \( k > k_0 \).

This of course implies that \( \langle \varphi, f_k \rangle = 0 \). Taking the limit \( k \to \infty \) we get \( \langle \varphi, \delta \rangle = 0 \) (we may take the limit since \( \| \hat{f}_k - \delta \|_\infty \to 0 \) while \( \hat{\varphi} \in l_1 \)). Since this holds for any \( \varphi \) supported outside \( \cap \text{supp } f_k \) we get \( \text{supp } \delta \subset \cap \text{supp } f_k \), as claimed.

Now, for \( x \not\in \cap \text{supp } f_k \) we use the localisation principle in the form (3) and get
\[
\lim_{n \to \infty} S_n(x) = 0 \quad \forall x \not\in \cap \text{supp } f_k
\]
i.e. outside the support it is not necessary to take a subsequence.

Finally, examine \( x \in \text{supp } f_k \). By definition
\[
S_n(f_k; x) < \frac{1}{2^n n_k}.
\]
For any \( j \geq k \), the condition \( |\hat{f}_{j+1}(k) - \hat{f}_j(k)| < 2^{-j}/n_j \leq 2^{-j}/n_k \) means that
\[
|S_{n_k}(f_{j+1}; x) - S_{n_k}(f_j; x)| < 3 \cdot 2^{-j}
\]
which we sum (also with (10)) to get
\[
|S_{n_k}(f_j; x)| < 7 \cdot 2^{-k} \quad \forall j \geq k
\]
and taking limit as \( j \to \infty \) gives
\[
|S_{n_k}(x)| < 7 \cdot 2^{-k} \quad \forall x \in \text{supp } f_k.
\]
We conclude
\[
\lim_{k \to \infty} S_{n_k}(x) = 0 \quad \forall x \in \cap \text{supp } f_k.
\]
With (9), the theorem is proved. \( \square \)
3. Proof of theorems 2 and 3

The following lemma summarises some properties of the support of the distribution.

**Lemma 9.** Let $c_l \to 0$ and $n_k \to \infty$ such that

$$\lim_{k \to \infty} S_{n_k}(x) = 0 \quad \forall x \quad S_n(x) = \sum_{l=-n}^{n} c_l \varphi(lx)$$

Let $K$ be the support of the distribution $\sum c_l \varphi(lx)$. Then

(i) $K = \{ x : \forall \epsilon > 0, S_{n_k} \text{ is unbounded in } (x - \epsilon, x + \epsilon) \}$.

(ii) $K$ is nowhere dense.

**Proof.** We start with clause (i). On the one hand, if $x \notin K$ then the localisation principle (3) tells us that $S_n \to 0$ uniformly in some neighbourhood of $x$. On the other hand, if $S_{n_k}$ is bounded in some neighbourhood $I$ of $x$ then for any smooth test function $\varphi$ supported on $I$ we have

$$\langle \varphi, \sum c_l \varphi(lx) \rangle = \sum_{l=-\infty}^{\infty} c_l \hat{\varphi}(l) = \lim_{k \to \infty} \sum_{l=-n_k}^{n_k} c_l \hat{\varphi}(l) = \lim_{k \to \infty} \int \varphi S_{n_k}$$

but the integral on the right-hand side tends to zero from the bounded convergence theorem. This shows (i).

To see clause (ii) apply the Baire category theorem to the function $N(x) = \sup_k |S_{n_k}(x)|$. We get, in every interval $I$, an open interval $J \subset I$ and an $M$ such that $N(x) \leq M$ on a dense subset of $J$. Continuity shows that in fact $N(x) \leq M$ on all of $J$ and hence $J \cap K = \emptyset$, as needed.

**Remark.** Without the condition $c_l \to 0$ it still holds that

$$K \subset \{ x : \forall \epsilon > 0, S_{n_k} \text{ is unbounded in } (x - \epsilon, x + \epsilon) \}$$

and that $K$ is nowhere dense. The proof is the same.

We will now make a few assumptions that will make the proof less cumbersome. First we assume that $c_{-l} = \overline{c_l}$ (or, equivalently, that the $S_n$ are real). It is straightforward to check that this assumption may be made without loss of generality in both theorems 2 and 3. Our next assumption is:

**Assumption.** In the next lemma we assume that $S_{n_k}$ is bounded on $K$, the support of the distribution $\sum c_l \varphi(lx)$. Further, whenever we write “$C$”, the constant is allowed to depend on $\sup\{|S_{n_k}(x)| : x \in K, k\}$. 
As in the proof of proposition 4, we will eventually remove this assumption by a simple localisation argument.

**Lemma 10.** Let $c_l, n_k, S_n$ and $K$ be as in the previous lemma. Let $r$ be in our sequence (i.e. $r = n_k$ for some $k$) and let $s > r$ not necessarily in the sequence. Then

$$||S_r||^2 \leq ||S_s|| \left( C + C \frac{||S_r||r \log^4 s}{s} \right). \quad (11)$$

The reader might find it useful to consider the case that $s > r^{3/2} \log^4 r$. In this case (11) takes on the particularly simple form $||S_s|| \geq c||S_r||^2$ (this uses the fact that $||S_r|| \leq C\sqrt{r}$ which follows because our coefficients $c_l$ are bounded).

It might be tempting to think that lemma 10 is a lemma on trigonometric polynomials, i.e. that it would have been possible to simply formulate it for $S_r$ being the Fourier partial sum of $S_s$. However, as the proof will show, we need to have the full distribution acting “in the background” restricting both what $S_r$ and $S_s$ may do.

**Proof.** Fix $r$ and $s > r$ as in the statement of the lemma. Further, we may assume $s/\log^4 s \geq r$ as otherwise (11) is trivial. Let now $I$ be a component of $K^c$ with $|I| > (2 \log^3 s)/s$. Let $\varphi_I$ be a function with the following properties:

(i) If $I = [a, b]$ then $\varphi_I$ restricted to $[a + (\log^3 s)/s, b - (\log^3 s)/s]$ is identically 1.

(ii) supp $\varphi_I \subset I$ (note that $I$ is open, so this inclusion must be strict).

(iii) $|\hat{\varphi}_I(l)| \leq C \exp \left( -C \sqrt{l \log^3 s}/s \right)$.

It is easy to see that such a $\varphi_I$ exists — take a standard construction of a $C^\infty$ function $\psi$ on $\mathbb{R}$ with $\psi|_{(-\infty, 0]} \equiv 0$, $\psi|_{[1, \infty)} \equiv 1$ and $||\psi^{(k)}||\infty \leq C(k!)^2$ (see e.g. [8, §V.2]), define $\varphi$ by mapping $\psi$ (restricted to an appropriate interval) linearly to each half of $I$ and estimate $\hat{\varphi}(l)$ by writing $|\hat{\varphi}(l)| \leq l^{-k} \cdot ||\varphi^{(k)}||\infty$ and optimising over $k$. We skip any further details.

Let

$$\varphi = \sum_I \varphi_I$$

where the sum is taken over all $I$ as above, i.e. $I$ is a component of $K^c$ with $|I| > (2 \log^3 s)/s$. Our lemma is based on the following decomposition

$$||S_r||^2 = \int S_s \cdot S_r = \int S_s \cdot S_r \cdot \varphi + \int S_s \cdot S_r \cdot (1 - \varphi).$$
To estimate the first summand, first note that
\[
|\hat{S_r \cdot \varphi_I(n)}| \leq \sum_{l=-r}^{r} |c_l \hat{\varphi_I}(n - l)| \leq C' \sum_{l=-r}^{r} \exp \left( -C \sqrt{(n - l) \log^3 s/s} \right) \\
\leq C' r \exp \left( -C \sqrt{(n \log^3 s/s)} \right).
\]

The inequality marked by (\star) is a simple exercise, but let us remark on it anyway. If \( n < 2s/\log^3 s \) then both sides of (\star) are \( \approx r \) and it holds. If \( n \geq 2s/\log^3 s \) then, because we assumed \( s/\log^4 s > r \), we get that \( |n - l| \geq \frac{1}{2} n \) and (\star) holds again.

Summing over \( I \) gives
\[
|\hat{S_r \cdot \varphi}(n)| \leq C r s \exp \left( -C \sqrt{(n \log^3 s/s)} \right).
\]

Next, because \( S_r \cdot \varphi \) is supported outside \( K \) we have
\[
\sum_{l=\pm \infty} c_l \hat{S_r \cdot \varphi}(l) = 0
\]
so
\[
\int S_s \cdot S_r \cdot \varphi = -\sum_{|l|>s} c_l \hat{S_r \cdot \varphi}(l)
\]
and then
\[
\left| \int S_s \cdot S_r \cdot \varphi \right| \leq \sum_{|l|>s} |c_l| \cdot |\hat{S_r \cdot \varphi}(l)| \leq C' \sum_{|l|>s} r s \exp \left( -C \sqrt{(l \log^3 s/s)} \right) \\
\leq C' \exp(-C \log^{3/2} s)
\]
which is negligible.

We move to the main term, \( \int S_r S_s (1 - \varphi) \), which we will estimate using Cauchy-Schwarz
\[
\left| \int S_s \cdot S_r \cdot (1 - \varphi) \right| \leq ||S_s|| \cdot ||S_r(1 - \varphi)||.
\]

Hence we need to estimate \( ||S_r(1 - \varphi)|| \). For this purpose, let us make the following definition. Let \( I \) be a component of \( K^c \) (not necessarily large, any component) and denote, for each such \( I \) and for each \( M \),
\[
A_{I,M} := \{ x \in I \cap E : |S_r^I| \in [M, 2M] \} \quad E := \text{supp}(1 - \varphi)
\]
We need a simple bound for the values of \( M \) that interest us, and we use that
\[
|S_r^I| \leq Cr^2 \quad \text{always (simply because the } c_l \text{ are bounded).}
\]
For any \( x \in I \cap E \) we may then estimate \( S_r \) itself by integrating \( S_r^I \) from the closest point of \( K \)
up to \( x \). We get

\[
|S_r(x)| \leq C + \sum_{M=1}^{C r^2} 2M A_{I,M}
\]  

where the word “scale” below the \( \Sigma \) means that \( M \) runs through powers of 2 (i.e., it is equivalent to \( \sum_{m=0}^{\lfloor \log_2 Cr^2 \rfloor} M = 2^m \)). Note that (13) uses our assumption that \( \max_{x \in K} |S_r(x)| \leq C \) for a constant \( C \) independent of \( r \) (and the additive constant \( C \) in (13) is the same \( C \)). Rewriting (13) as

\[
||S_r(1 - \varphi)|| \leq \sum_I 1_{I \cap E} \left( C + \sum_{M \text{ scale}} 2M A_{I,M} 1_{I \cap E} \right)
\]

gives

\[
||S_r(1 - \varphi)|| \leq C \left( \sum_I 1_{I \cap E} \right) + \sum_{M \text{ scale}} 2M \sqrt{\sum_I |I \cap E| A_{I,M}^2}.
\]

(14)

To estimate the sum notice that \( A_{I,M} \leq |I \cap E| \leq 2(\log s)^3/s \) so

\[
\sum_I |I \cap E| A_{I,M}^2 \leq \frac{4 \log^6 s}{s^2} \sum_M A_{I,M} \leq \frac{4 \log^6 s}{s^2} |\{ x : |S_r'(x)| \geq M \}|
\]

\[
\leq \frac{s^2}{4 \log^6 s} ||S_r'||^2 \frac{r^2}{M^2} \leq \frac{4 \log^6 s}{s^2} ||S_r'||^2 \frac{r^2}{M^2}
\]

where the inequality marked by (*) follows by Chebyshev’s inequality. The sum over scales in (14) has only \( C \log r \leq C \log s \) terms, so we get

\[
||S_r(1 - \varphi)|| \leq C \left( \sqrt{|E|} + \frac{||S_r|| r \log^4 s}{s} \right).
\]

or

\[
\left| \int S_s \cdot S_r \cdot (1 - \varphi) \right| \leq C ||S_s|| \left( \sqrt{|E|} + \frac{||S_r|| r \log^4 s}{s} \right)
\]

(15)

Recall that (12) showed that the other term in \( ||S_r||^2 \) is negligible, so we get the same kind of estimate for \( ||S_r||^2 \):

\[
||S_r||^2 \leq C ||S_s|| \left( \sqrt{|E|} + \frac{||S_r|| r \log^4 s}{s} \right).
\]

(16)

The lemma is finished since \( |E| \leq 1 \). Before putting the q.e.d. square, though, let us reformulate (16) in a way that will be useful in the proof of theorem 3. Recall that \( E = \text{supp}(1 - \varphi) \), that \( \varphi = \sum \varphi_I \) and that each \( \varphi_I \) is 1 except
in a \((\log^3 s)/s\) neighbourhood of \(K\). Hence \(E \subset K + [-(\log s)^3/s, (\log s)^3/s]\) (the sum here is the Minkowski sum of two sets) and (16) can be written as
\[
\|S_r\|^2 \leq C\|S_s\| \left( \sqrt{K + \left[ -\frac{\log^3 s}{s}, \frac{\log^3 s}{s} \right]} + \frac{\|S_r\|r \log^4 s}{s} \right).
\] (17)
Now we can put the little square.

**Proof of theorem 2.** Let \(K\) be the support of the distribution \(\sum c_l e(lx)\). We first claim that we can assume without loss of generality that \(S_{n_k}\) is bounded on \(K\). This uses the localisation principle exactly like we did in the proof of proposition 4, but let us do it in details nonetheless. Since \(S_{n_k}(x) \to 0\) everywhere \(\sup_k |S_{n_k}(x)|\) is finite everywhere. Applying the Baire category theorem to the function \(\sup_k |S_{n_k}(x)|\) on \(K\) we see that there is an open interval \(I\) such that \(S_{n_k}\) is bounded on a dense subset of \(K \cap I\), and \(K \cap I \neq \emptyset\). Continuity of \(S_{n_k}\) shows that they are in fact bounded on the whole of \(K \cap I\).

By the definition of support of a distribution, we can find a smooth test function \(\varphi\) supported on \(I\) such that \(\sum \widehat{\varphi}(l)c_l\) is not zero. Let \(d_l = c_l \ast \widehat{\varphi}\) (and hence \(d\) is not zero either). Then by the localisation principle (2), \(\sum_{-n_k}^{n_k} d_l e(lx)\) converges everywhere to zero and is bounded on \(K \cap I\), which contains the support of \(\sum d_l e(lx)\). Hence we can rename \(d_l\) to \(c_l\) and simply assume that \(S_{n_k}\) is bounded on \(K\).

We now construct a series \(r_i\) as follows: we take \(r_1 = n_1\) and for each \(i \geq 1\) let \(r_{i+1}\) be the first element of the series \(n_k\) which is larger than \(r_i^{7/4}\). Because \(n_{k+1} = r_k^{1+o(1)}\) we will have in fact that \(r_{i+1} = r_i^{7/4 + o(1)}\) and hence
\[
r_i = \exp((7/4 + o(1))^i).
\] (18)
We now apply lemma 10 with \(r_{\text{lemma }10} = r_i\) and \(s_{\text{lemma }10} = r_{i+1}\). We get
\[
\|S_{r_i}\|^2 \leq \|S_{r_{i+1}}\| \left( C + C \frac{\|S_{r_i}\|r_i \log^4 r_i}{r_i^{7/4 + o(1)}} \right) \leq C \|S_{r_{i+1}}\|
\]
where the second inequality follows from \(\|S_{r_i}\| \leq C \sqrt{r_i}\), because \(c_l\) are bounded. Denote the inverse of this last constant by \(\lambda\) for clarity (i.e. \(\|S_{r_{i+1}}\| \geq \lambda \|S_{r_i}\|^2\)). Iterating the inequality \(\|S_{r_{i+1}}\| \geq \lambda \|S_{r_i}\|^2\) starting from some \(i_0\) such that \(\|S_{r_{i_0}}\| > e/\lambda\) gives
\[
\|S_{r_i}\| \geq (\lambda \|S_{r_{i_0}}\|)^{2^{i-i_0}} > \exp(2^{i-i_0})
\]
Together with (18) we get
\[
\|S_{r_i}\| \geq \exp((\log r_i)^{1.2386 + o(1)})
\]
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(18)

Proof of theorem 3. Denote \( d = \dim_{\text{Mink}}(K) \) (recall that this is the upper Minkowski dimension). Assume by contradiction that \( c_l \equiv 0 \) and without loss of generality assume that \( c_0 = 1 \) (if \( c_0 = 0 \), shift the sequence \( c_l \) and note that the condition \( c_l \to 0 \) ensures that \( S_{n_k}(x) \to 0 \) even for the shifted sequence).

Fix \( s \in \mathbb{N} \) and let \( \varphi \) be as in the proof of lemma 10: let us remind the most important properties:

(i) \( \text{supp} \varphi \cap K = \emptyset \);
(ii) \( \text{supp}(1 - \varphi) \subset K + [- (\log^3 s)/s, (\log^3 s)/s] \); and
(iii) \( |\hat{\varphi}(l)| \leq Cs \exp \left( - C \sqrt{(l \log^3 s)/s} \right) \).

From this we can get a lower bound for \( ||S_s|| \). From \( \text{supp} \varphi \cap K = \emptyset \) we get
\[
\sum_{l=-\infty}^{\infty} c_l \hat{\varphi}(l) = 0
\]
so
\[
\int S_s \varphi = \sum_{l=-s}^{s} c_l \hat{\varphi}(l) = - \sum_{|l|>s} c_l \hat{\varphi}(l)
\]
giving
\[
\left| \int S_s \varphi \right| \leq \sum_{|l|>s} Cs \exp \left( - C \sqrt{(l \log^3 s)/s} \right) \leq C \exp(-c \log^{3/2} s).
\]

By assumption \( \int S_s = c_0 = 1 \) so for \( s \) sufficiently large
\[
\left| \int S_s (1 - \varphi) \right| = 1 - O(\exp(-c \log^{3/2} s)) > 1/2.
\]

Using Cauchy-Schwarz gives
\[
\frac{1}{2} < ||S_s|| \sqrt{|\text{supp}(1 - \varphi)|} \leq ||S_s|| \sqrt{K + [- \log^3 s/s, \log^3 s/s]} \leq ||S_s|| \cdot \sqrt{s^{d-1+o(1)}}
\]
where in the last inequality we covered \( K \) by intervals of size \( 1/s \) — no more than \( s^{d+o(1)} \) by the definition of upper Minkowski dimension — and inflated each one by \( (\log^3 s)/s \). We conclude that
\[
||S_s|| \geq s^{(1-d)/2-o(1)}.
\]
as \( s \to \infty \).
In the other direction, use (17) and get:

\[ \|S_r\|^2 \leq \|S_s\| \left( C s^{(d-1)/2 + o(1)} + C \frac{\|S_r\| r \log^4 s}{s} \right). \]

Choose \( s = (r \|S_r\|)^{2/(d+1)} \) (this makes the summands approximately equal) and get

\[ \frac{\|S_s\|}{\sqrt{s}} \geq \|S_r\|^2 \cdot (r \|S_r\|) \left( -\frac{d}{d+1} + o(1) \right) \]

\[ \geq r \left( -\frac{d}{d+1} + \frac{1}{2} \cdot \frac{d+2}{d+1} + o(1) \right) \]  

(20)

where the inequality marked by (\( * \)) follows from \( \|S_r\| \geq r^{(1-d)/2+o(1)} \), which is (19) with \( s \) replaced by \( r \). When \( d < \frac{1}{2}(\sqrt{17} - 3) \) the power of the \( r \) in (20) is positive. This means that \( \|S_s\|/\sqrt{s} \to \infty \), contradicting the boundedness of the coefficients \( c_i \). \( \square \)

4. LOCALISATION WITH BOUNDED COEFFICIENTS

Our last remark is that there is a version of the localisation principle suitable even when the coefficients of the series do not converge to zero, but are still bounded. Let us state it first

**Theorem 11.** Let \( c_l \) be bounded and \( n_k \) some sequence and let \( \varphi \) be a smooth function. Then there exists a subsequence \( m_k \) and two functions \( a \) and \( b \) such that

\[ \varphi(x) \sum_{l=-m_k}^{m_k} c_l e(lx) - \sum_{l=-m_k}^{m_k} (c * \hat{\varphi})(l)e(lx) + e^{im_kx}a(x) + e^{-im_kx}b(x) \]

converges to zero uniformly.

Further, \( a \) and \( b \) have some smoothness that depends on \( \varphi \) as follows:

\[ |\hat{a}(l)| \leq \sum_{|j|>|l|} |\hat{\varphi}(j)|. \]

and ditto for \( b \).

(recall that in the classic Rajchman formulation \( a \equiv b \equiv 0 \) and \( m_k \) can be taken to be \( n_k \), one does not need to take a subsequence).

**Proof.** Denote

\[ E_n(x) = \varphi(x) \sum_{l=-n}^{n} c_l e(lx) - \sum_{l=-n}^{n} (c * \hat{\varphi})(l)e(lx). \]
For $|j| > n$ only the first term appears in $\hat{E}_n(j)$ and we get
\[
\hat{E}_n(j) = \sum_{l=-\infty}^{\infty} c_{j-l} \hat{\varphi}(l) 1\{|j-l| \leq n\}
\]
and in particular $|\hat{E}_n(n+r)| \leq C \sum_{s \geq r} |\hat{\varphi}(s)|$, and similarly for $\hat{E}_n(-n-r)$. For $|l| \leq n$ the second term also appears, but since it is simply the sum without the restriction $|j-l| \leq n$ the difference takes the following simple form:
\[
\hat{E}_n(j) = -\sum_{l=-\infty}^{\infty} c_{j-l} \hat{\varphi}(l) 1\{|j-l| > n\}.
\]
Again we get $|\hat{E}_n(n-r)| \leq C \sum_{|s| \geq r} |\hat{\varphi}(s)|$ and similarly for $\hat{E}_n(-n+r)$.

These uniform bounds for $|\hat{E}_n(\pm n+r)|$ allow us to use compactness to take a subsequence $m_k$ of $n_k$ such that both $\hat{E}_{m_k}(m_k+r)$ and $\hat{E}_{m_k}(-m_k+r)$ converge for all $r$. Defining
\[
a(x) = -\sum_{r=-\infty}^{\infty} e(rx) \lim_{k \to \infty} \hat{E}_{m_k}(m_k+r)
\]
\[
b(x) = -\sum_{r=-\infty}^{\infty} e(rx) \lim_{k \to \infty} \hat{E}_{m_k}(-m_k+r)
\]
the theorem is proved. □

Theorem 11 can be used to strengthen both theorems 2 and 3 to hold for bounded coefficients rather than for coefficients tending to zero. But let us skip these applications and show only how to use it to strengthen proposition 4.

**Theorem 12.** Let $\mu$ be a measure and let $n_k$ be a series such that
\[
\lim_{k \to \infty} S_{n_k}(\mu; x) = 0 \quad \forall x.
\]
Then $\mu = 0$.

**Proof.** Let $K$ be the support of $\mu$ and let, as in the proof of proposition 4, $I$ be an interval such that $S_{n_k}(\mu)$ is bounded on $I$ and $I \cap K \neq \emptyset$. Let $\varphi$ be a smooth function supported on all of $I$. We use theorem 11 to find a subsequence $m_k$ of $n_k$ and an $a$ and a $b$ such that
\[
\varphi S_{m_k}(\mu) - S_{m_k}(\varphi \mu) + e^{im_kx} a + e^{-im_kx} b \to 0.
\]
This has two applications. First we conclude that $\varphi \mu \not\in L^2$. Indeed, if we had that $\varphi \mu \in L^2$ then we would get that $\varphi S_{m_k}(\mu) \to 0$ pointwise while
$S_{m_k}(\varphi \mu) \to \varphi \mu$ in measure, which can only hold if $\varphi \mu \equiv 0$ (also $a$ and $b$ need to be zero, but we do not need this fact). This contradicts our assumption that $I \cap K \neq \emptyset$ and that $\varphi$ is supported on all of $I$.

Our second conclusion from (21) is that $S_{m_k}(\varphi \mu)$ is bounded on $I \cap K$, which is the support of $\varphi \mu$. From here the proof continues as in the proof of proposition 4. □

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