ON BIRATIONAL SUPERRIGIDITY AND CONDITIONAL BIRATIONAL SUPERRIGIDITY OF CERTAIN FANO HYPERSURFACES

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Abstract. We prove birational superrigidity of hypersurfaces of degree \( N \) in \( \mathbb{P}^N \) with singular locus of dimension \( \delta \), under the assumption that \( N \geq 2\delta + 8 \) and it has only quadratic singularities of rank \( \geq N - \delta \). Combined with the results of I. A. Chel’tsov and T. de Fernex, this completes the list of birationally superrigid singular hypersurfaces with only ordinary double points except in dimension 4 and 6. Further we impose an additional condition on the base locus of a birational map to a Mori fiber space. Then we prove conditional birational superrigidity of certain smooth Fano hypersurfaces of index \( \geq 2 \), and birational superrigidity of smooth Fano complete intersections of index 1 in weak form.

Introduction

The notion of birational rigidity has its origin in the paper [IM] by V. A. Iskovskih and Ju. I. Manin, where they construct a counter-example to Lüroth problem, and nowadays it is re-defined as the property of certain Mori fiber spaces in the context of the minimal model program. Mori fiber spaces are considered to form, along with minimal models, fundamental classes in the birational classification of varieties, and compared to minimal models their birational geometry is rich in general. Then birational rigidity distinguishes Mori fiber spaces with simple birational structure, contrary to such tendency. In this paper we are mainly interested in birational rigidity of Fano complete intersections in a projective space, which are Mori fiber spaces in obvious way. We denote by \( X_{d_1,\ldots,d_k} \subset \mathbb{P}^N \) a type of complete intersections in \( \mathbb{P}^N \) defined by \( k \) hypersurfaces of degree \( d_1,\ldots,d_k \). It is Fano if \( \sum_{i=1}^{k} d_i \leq N \).

First consider a singular hypersurface \( X_N \subset \mathbb{P}^N \) with only ordinary double points. We recall that a Mori fiber space \( X \) is called birationally superrigid if any birational map to another Mori fiber space is isomorphism. Birational superrigidity is very strong condition: it implies \( X \) is non-rational and \( \text{Bir}(X) = \text{Aut}(X) \). When \( N = 4 \), the above hypersurface always has their birational automorphism induced by the projection away from a singular point, and thus the problem is whether it is birationally superrigid when \( N \geq 5 \). A. V. Pukhlikov affirmatively solved the problem in [Puk2], while its result is only valid for general members and no concrete examples are obtained. I. A. Chel’tsov affirmatively solved the problem for arbitrary members when \( N = 6 \) in [Ch2], so did T. de Fernex when \( N \geq 10 \) in [dF2]. The following theorem completes the case when \( N = 8 \) or 9. For a singular variety \( X \), let \( \delta(X) = \dim \text{Sing} X \).

**Theorem 0.1.** Let \( X = X_N \subset \mathbb{P}^N \) be a complex singular hypersurface. Assume that \( N \geq 2\delta(X) + 8 \) and \( X \) has only quadratic singularities of rank \( \geq N - \delta(X) \). Then \( X \) is birationally superrigid.

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Thus we can complete the list of birationally superrigid singular hypersurfaces with only ordinary double points except in dimension 4 and 6.

**Corollary.** For $N \geq 8$ and $N = 6$, every singular complex hypersurface $X = X_N \subset \mathbb{P}^N$ with only ordinary double points is birationally superrigid.

Next we consider a smooth hypersurface $X_d \subset \mathbb{P}^N$ with $d < N$. The recent result of A. V. Pukhlikov [Puk4] suggests a hypersurface of this type also has some birationally rigid property (see Section 3 for details). Since such a hypersurface is not birationally superrigid, we restrict the dimension of the undefined locus of a birational map to a Mori fiber space.

**Definition.** Let $X$ be a Fano manifold with Picard number one. For a positive integer $r \geq 2$, consider the following condition on $X$:

$(C_r)$ every birational map from $X$ to a Mori fiber space whose undefined locus has codimension at least $r$ is an isomorphism.

Further assume a generator $\mathcal{O}_X(1)$ of the Picard group is very ample and write $\mathcal{O}_X(-K_X) = \mathcal{O}_X(i_X)$. We say $X$ is conditionally birationally superrigid if $(C_{i_X+1})$ holds.

We only consider the case when $i_X \leq \dim X - 1$ since otherwise conditional birational superrigidity is trivial. When $i_X = 1$, conditional birational superrigidity is usual birational superrigidity. When $i_X \geq 2$, no $X$ satisfies $(C_{i_X})$ due to a general $(i_X - 1)$-dimensional linear system $\mathcal{L} \subset |\mathcal{O}_X(1)|$. We prove the following theorem.

**Theorem 0.2.** Every smooth complex hypersurface $X = X_N \subset \mathbb{P}^{N+1}$ $(6 \leq N \leq 10)$, $X_8 \subset \mathbb{P}^{10}$, $X_9 \subset \mathbb{P}^{11}$, $X_9 \subset \mathbb{P}^{12}$ is conditionally birationally superrigid.

Finally we consider a smooth complete intersection $X_{d_1, \ldots, d_k} \subset \mathbb{P}^{\sum_{i=1}^k d_i}$. A. V. Pukhlikov conjectured every smooth complete intersection of this type is birationally superrigid if its dimension is at least 5, and proved in [Puk3, Puk5] that it holds for general members of dimension at least 12 except three infinite series $X_{2, \ldots, 2}$, $X_{2, \ldots, 3}$ and $X_{2, \ldots, 4}$. We prove the following theorem.

**Theorem 0.3.** Every smooth complex complete intersection $X = X_{2, N} \subset \mathbb{P}^{N+2}$, $X_{3, N} \subset \mathbb{P}^{N+3}$, $X_{4, N} \subset \mathbb{P}^{N+4}$, $X_{2, 2, N} \subset \mathbb{P}^{N+4}$ satisfies $(C_3)$ for $N \geq 13, 18, 45, 81$ respectively.

Section 1 is devoted to review definitions and basic facts about singularities of pairs and Samuel multiplicities. In Section 2, we explain the common strategy to prove birational rigidity, so called the method of maximal singularities. Proofs of the main theorems are given in Sections 3, 4, 5. For the proofs, we essentially use inequalities giving lower-bounds of log-canonical thresholds, which were obtained by T. de Fernex, L. Ein, M. Mustață in a series of papers [dFEM1], [dFEM2], [dF1]. We provide in Appendix several straightforward generalizations of known results including these inequalities, which we use in the proofs of theorems. To prove theorem D in Appendix, we prove a claim which is missing in [dF2] (see Claim D.5.3). Further we need Proposition 5.1 as the key to prove Theorem 0.1.

**Notation and Convention.** Throughout this paper, we work over the field of complex numbers $\mathbb{C}$. Unless otherwise stated, a variety is assumed to be irreducible and reduced. For a scheme $S$ of finite type over $\mathbb{C}$, denote by $[S]$ its fundamental cycle. In Proposition 5.1 and Proposition A we use the following notations:

- for a variety $X$, denote by $X^{sm}$ the smooth locus of $X$;
• for pure-dimensional cycles \( \alpha_1, \alpha_2 \) on \( X \), we write \( \alpha_1 \sim \alpha_2 \) if \( \alpha_1 \) and \( \alpha_2 \) are rationally equivalent;
• for pure-dimensional cycles \( \beta, \gamma \) intersecting properly on \( X \) and an irreducible component \( T \) of the intersection, denote by \( \iota(T, \beta \cdot \gamma; X) \) the intersection multiplicity of \( T \) in \( \beta \cdot \gamma \) whenever the intersection product \( \beta \cdot \gamma \) is defined;
• for a cycle \( \delta \), we also denote its support by \( \delta \);
• for a projective variety \( Y \) embedded in some projective space \( \mathbb{P}^N \), denote by \( c_1(\mathcal{O}_Y(1)) \) the first Chern class of a hyperplane section, and by \( c_1(\mathcal{O}(1))^k \cap \eta \) its cup-product with a cycle \( \eta \);
• for projective varieties \( V, W \) and a point \( p \) in a projective space \( \mathbb{P}^N \),
\[
J(V, W) = \bigcup_{v \in V, w \in W} \langle v, w \rangle, \quad \text{Sec}(V) = J(V, V), \quad C(p, V) = J(p, V)
\]
and for a cycle \( Z = \sum_{i=1}^k n_i Z_i \) on \( \mathbb{P}^N \), as cycles,
\[
J(V, Z) = \sum_{i=1}^k n_i J(V, Z_i), \quad C(p, Z) = \sum_{i=1}^k n_i C(p, Z_i).
\]

For the definitions and basic properties of fundamental cycles, rational equivalence, intersection products, intersection multiplicities and Chern classes, we follow [Ful].

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1. Preliminaries

1.1. Singularities of pairs.

**Definition 1.1.** Let \( X \) be a variety. A prime divisor over \( X \) is a prime divisor on some smooth variety \( X' \) with a birational proper morphism \( f : X' \to X \). A center of a prime divisor over \( X \) is the image of its support on \( X \). A prime divisor over \( X \) is called exceptional if its center has codimension at least 2.

Assume \( X \) is normal with a \( \mathbb{Q} \)-Cartier canonical divisor \( K_X \). Fix a prime divisor \( E \) over \( X \) with \( X' \) and \( f \) as the above. Define
\[
k_E(X) := \text{ord}_E(K_{X'/X}),
\]
which is called the discrepancy of \( E \) with respect to \( X \). If \( \text{val}_E \) is the valuation of \( \mathbb{C}(X) \) defined by \( E \), \( k_E(X) \) only depends on \( \text{val}_E \) and is independent of the choice of \( f \). For a proper closed subscheme \( Z \) in \( X \), define
\[
\text{val}_E(Z) := \min \left\{ \text{val}_E(h) \mid h \in I_Z(U), \ U \subset X \text{ an affine open subset which intersects the intersection of } Z \\
\text{and the center of } E. \right\},
\]
For a cycle \( Z' = \sum c_i Z_i \) on \( X \) with real coefficients \( c_i \) and proper closed subschemes \( Z_i \), define
\[
a_E(X, Z') := k_E(X) + 1 - \sum c_i \text{val}_E(Z_i),
\]
which is called the log-discrepancy of \( E \) with respect to the pair \( (X, Z') \).

Fix a cycle \( Z' = \sum c_i Z_i \) on \( X \) as the above. The pair \( (X, Z') \) is called terminal if
\[
a_E(X, Z') > 1
\]
for every exceptional prime divisor $E$ over $X$, and called Kawamata log-terminal or klt for short if
\[ a_E(X, Z') > 0 \]
for every prime divisor $E$ over $X$. We say $(X, Z')$ is terminal (resp. klt) in codimension $r$ if the corresponding condition on the discrepancies holds for divisors over $X$ whose center has codimension at most $r$. If all $c_i \geq 0$, define
\[ \text{can}(X, Z') := \sup \{ c \in \mathbb{R} \mid (X, cZ') \text{ is terminal} \} , \]
which is called the canonical threshold of the pair $(X, Z')$.

1.2. Samuel multiplicities.

**Definition 1.2.** The Samuel multiplicity or multiplicity of a scheme $Z$ of finite type over $\mathbb{C}$ at a closed point $p \in Z$ is defined to be $e_p(Z) := e(m_{Z,p})$, the Samuel multiplicity of the maximal ideal $m_{Z,p} \subset O_{Z,p}$. For another definition using the Segre class and its agreement with the above definition, see [Ful, Section 4.3]. For an irreducible subvariety $S$ of $Z$, define
\[ e_S(Z) := \min \{ e_p(Z) \mid p \in S \} . \]
This is well-defined by the upper-semicontinuity of multiplicities [Ben]. We extend the definition of the multiplicity linearly to an arbitrary cycle where we use the convention $e_p(Z) = 0$ if $p \notin Z$.

If $Z$ is pure-dimensional,
\[ e_p(Z) = e_p([Z]) , \]
for every point $p \in Z$ by [Ful, Lemma 4.2]. Thus we identify the scheme $Z$ and the cycle $[Z]$ when we are concerned with its multiplicity (and also its degree).

Samuel multiplicities satisfies the following properties when we cut down a given pure-dimensional closed scheme by a hypersurface.

**Proposition 1.3.** Let $X$ be a positive, pure-dimensional closed subscheme in $\mathbb{P}^N$.

1. If $Y$ is a hypersurface in $\mathbb{P}^N$ which intersects $X$ properly,
\[ e_p(X \cap Y) \geq e_p(X) \cdot e_p(Y) \]
for every point $p \in X \cap Y$. The equality holds when $C_pX$ and $C_pY$ intersect properly, where $C_pX$ (resp. $C_pY$) is the tangent cone of $X$ (resp. $Y$) at $p$.

2. For an arbitrary hyperplane $H \subset (\mathbb{P}^N)^Y$, if $H \in \mathcal{H}$ is general,
\[ e_p(X \cap H) = e_p(X) \]
for every $p \in X \cap H$.

**Proof.** Since (2) follows from (1) in the same way as [dFEM2, Proposition 4.5], we only prove (1). Assume $\dim X \geq 2$. Take $p \in X \cap Y$. Let $\pi : \mathbb{P}^N \rightarrow \mathbb{P}^N$ be the blow-up of $\mathbb{P}^N$ at $p$ and $E$ be its exceptional divisor. Let $\widetilde{X}$ (resp. $\widetilde{Y}$, $(\widetilde{X} \cdot \widetilde{Y})$) be the strict transform of $X$ (resp. $Y$, $X \cdot Y$). We can write
\[ \pi^*Y = \widetilde{Y} + e_p(Y)E. \]
Since $\pi^*Y \cdot E = 0$ and $\dim X \cap Y \geq 1$,
\[ 0 = (\widetilde{X} \cdot \pi^*Y) \cdot E^{\dim X \cap Y} = \widetilde{X} \cdot \widetilde{Y} \cdot E^{\dim X \cap Y} + (-1)^{\dim X - 1} e_p(X) \cdot e_p(Y). \]
On the other hand, we can write
\[ \widetilde{X} \cdot \widetilde{Y} = (\widetilde{X} \cdot Y) + \alpha, \]
where $\alpha$ is a non-negative $(\dim X \cap Y)$-cycle and $\alpha = 0$ if $C_pX$ and $C_pY$ intersect properly. We have

$$\tilde{X} \cdot \tilde{Y} \cdot E_{\dim X \cap Y} = (-1)^{\dim X \cap Y - 1}e_p(X \cap Y) + (-1)^{\dim X \cap Y}a,$$

where $a = \deg \alpha$ as a cycle on $E = \mathbb{P}^{N-1}$. Therefore

$$e_p(X \cap Y) = e_p(X) \cdot e_p(Y) + \alpha.$$

In the case of $\dim X = 1$, using the same notation as above,

$$X \cdot Y = \tilde{X} \cdot \pi^*Y = \tilde{X} \cdot \tilde{Y} + e_p(X) \cdot e_p(Y).$$

Since the assertion is local, we may assume $p$ is the only intersection point of $X$ and $Y$. Then $X \cdot Y = X \cap Y = e_p(X \cap Y)$ by [Ful, Lemma 4.2]. The assertion follows since $\tilde{X} \cdot \tilde{Y} \geq 0$ and $\tilde{X} \cdot \tilde{Y} = 0$ if $C_pX$ and $C_pY$ intersect properly. The proof is done. \qed

2. The method of maximal singularities

In this section we explain the strategy to prove that a given Fano variety satisfies birationally rigid conditions.

**Definition 2.1.** A normal projective variety $X$ with $\mathbb{Q}$-factorial terminal singularities is called a *Mori fiber space* if it is endowed with an extremal Mori contraction of fiber type. If $X$ has Picard number one and is factorial, we say $X$ is a *Fano variety*.

Consider a Fano variety $X$. For a positive integer $r \geq 2$, we introduce the following condition on $X$:

- $(C_r)$: every birational map from $X$ to a Mori fiber space whose undefined locus has codimension at least $r$ is an isomorphism.

As we say in the introduction, we are mainly interested in Fano complete intersections in a projective space. Thus take a factorial terminal $X = X_{d_1, \ldots, d_k} \subset \mathbb{P}^N$ with $\sum_{i=1}^k d_i \leq N$. We follow the traditional way introduced by Iskovskih and Manin, so called the *method of maximal singularities*, where the Noether–Fano inequality plays an essential role.

**Theorem 2.2** (Noether–Fano Inequality [IM]). Let $X$ be a Fano variety. Let $Y$ be a Mori fiber space and let $\phi: X \dasharrow Y$ be a birational map. If $\phi$ is not an isomorphism, then there exists a natural number $\mu$ and a movable linear system $\mathcal{H} \subset | - \mu K_X|$ which determines $\phi$ such that $\text{can}(X, \text{Bs}(\mathcal{H})) < \frac{1}{\mu}$, where $\text{Bs}(\mathcal{H})$ is the base scheme of $\mathcal{H}$.

The non-canonical center of the pair $(X, \frac{1}{\mu} \text{Bs}(\mathcal{H}))$ is called *maximal singularities*. We have to exclude maximal singularities.

Here is the strategy: assume $X$ does not satisfy $(C_r)$. Then we have a linear system $\mathcal{H} \subset | - \mu K_X|$ in Theorem 2.2. If $r \geq 3$, we have an additional condition that $\text{Bs}(\mathcal{H})$ has at least codimension $r$ in $X$. As the first step, we have to bound the dimension of the non-terminal locus of the pair $(X, c \text{Bs}(\mathcal{H}))$ where $c = \text{can}(X, \text{Bs}(\mathcal{H}))$. We use Proposition A and B to do this. As the second step, we cut down the non-terminal center by hyperplane sections until it becomes a non-klt center. We use Inversion of Adjunction to do this. Then apply Theorem C or D which gives the lower-bound of log-canonical threshold of the restriction, and we get a contradiction combined with the inequality $c < \frac{1}{r}$.

Proposition A [A] and Theorem C [C] is proved in Appendix. All of those are straightforward generalizations of the fundamental results in [Puk1], [Kol], [TFEM2], [TF2].
3. Proof of Theorem 3.2 and Problem on Birational Geometry of
Fano Hypersurfaces of High Index

Proof. We only prove the first case since the other cases can be proved in the same
way. Take a smooth hypersurface $X = X_N \subset \mathbb{P}^{N+1}$ with $6 \leq N \leq 10$. Assume $X$
does not satisfy condition (C3). By the Noether–Fano inequality (Theorem 2.2),
there exists a natural number $\mu$ and a movable linear system $\mathcal{H} \subset |-\mu K_X|$ such
that $c = \text{can}(X, Bs(\mathcal{H})) < \frac{1}{\mu}$ and codim($Bs(\mathcal{H}), X) \geq 3$.

It is clear that the pair $(X, c Bs(\mathcal{H}))$ is terminal in codimension 2. Let $B_2 =
D_1 \cdot D_2$ be the complete intersection subscheme of $X$ defined by general members
$D_1, D_2$ of $\mathcal{H}$. Then since $B_2 \sim (2\mu)^3 \cdot c_1(\mathcal{O}_X(1))^3 \cap [X]$ and $c < \frac{1}{\mu}$, by Proposition [A]
\[ \dim \left\{ x \in B_2 \mid e_x(B_2) \geq \left( \frac{2}{\mu} \right)^2 \right\} \leq 1. \]
Therefore by Proposition [B] the non-terminal locus of the pair $(X, c Bs(\mathcal{H}))$ is at
most 1-dimensional.

Let $B = D_1' \cdot D_2' \cdot D_3'$ be the complete intersection subscheme of $X$ defined by
general members $D_1', D_2', D_3'$ of $\mathcal{H}$. Since $B \sim (2\mu)^3 \cdot c_1(\mathcal{O}(1))^3 \cap [X]$ and $c < \frac{1}{\mu}$, by Proposition [A]
\[ \dim \left\{ x \in B \mid e_x(B) \geq \left( \frac{2}{\mu} \right)^3 \right\} \leq 2. \]
After restricting to a suitable linear subspace $H = \mathbb{P}^{N-1} \subset \mathbb{P}^{N+1}$ of codimension 2,
by Inversion of Adjunction and Proposition 1.3 (2) we have a pair $(X_H, cB_H)$ with
$X_H$ smooth and a point $p \in B_3$ such that
\begin{enumerate}
  \item $p$ is the center of a prime divisor $E$ which satisfies $a_E(X_H, cB_H) \leq 0$,
  \item \[ \dim \left\{ x \in B_H \mid e_x(B) \geq \left( \frac{2}{\mu} \right)^3 \right\} \leq 0, \]
\end{enumerate}
where $X_H = X \cdot H$ and $B_H = B_3 \cdot H$. By (2), we have
\[ \dim \left\{ x \in B_H \mid e_x(B) \geq \frac{1}{2} \left( \frac{3}{\mu} \right)^3 \right\} \leq 0. \]
Apply Theorem [C] to the pair $(X_H, cB_H)$ and we have
\[ \begin{cases}
  \deg B_H \geq \left( \frac{3}{\mu} \right)^3 \cdot 3 & \text{if } N \geq 7, \\
  \deg B_H \geq \left( \frac{3}{\mu} \right)^3 \cdot (N-4) & \text{otherwise}.
\end{cases} \]
Since $\deg B_H = 8\mu^3 N$ and $c < \frac{1}{\mu}$,
\[ \begin{cases}
  N > 10 & \text{if } N \geq 7, \\
  6 > N & \text{otherwise},
\end{cases} \]
a contradiction. \qed

We formulate a variant of the problem suggested by T. de Fernex in [1F3, 8].
Consider a smooth hypersurface $X = X_N \subset \mathbb{P}^{N+m}$ with $m \geq 0$. Fix positive integers $k, d$ with $m \geq kd$. Then for a general $k$-dimensional linear system $\mathcal{L} \subset |\mathcal{O}_X(d)|$, an associated rational map $\Phi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}^k$ defines a birational Mori fiber space structure on $X$, whose fiber is $X_{N,d \ldots d} \subset \mathbb{P}^{N+m}$. When $d = 1$, $\Phi_{\mathcal{L}}$ is induced by a linear projection.

Definition 3.1. A birational Mori fiber space structure on $X$ defined as the above
for some $k, d$ is called standard.
When \( m = 0 \), \( X \) is birationally superrigid for \( N \geq 4 \) by \[ \text{(IP1)} \]. When \( m \geq 1 \), \( X \) in Theorem \[ \text{(4.2)} \] is conditionally birationally superrigid, i.e. \( X \) satisfies \( (C_{m+2}) \) while no \( X \) satisfies \( (C_{m+1}) \) due to standard Mori fibre space structures of type \( (k, d) = (m, 1) \). Observe there is a non-standard birational Mori fiber space structure on \( X \) when \( N \) is small. For example, when \( N = 3 \) and \( m \geq 1 \), \( X \) has a conic-bundle structure induced by the projection from a line on \( X \), and \( X \) is birational to a linear section of the Grassmannian \( G(2, 6) \) if \( m = 1 \) (see \[ \text{[Isk]}, \text{[Tak]} \]). Then the following problem arises.

**Problem 3.2.** Fix \( m \geq 1 \) and assume that \( N \) is sufficiently large. Then for every smooth hypersurface \( X = X_N \subset \mathbb{P}^{N+m} \), only birational Mori fiber space structures on \( X \) are standard ones?

Pukhlikov proved in \[ \text{(Puk4)} \] that the answer to Problem \[ 3.2 \] is affirmative when \( m = 1 \) for \( N \geq 14 \) if “every” in the statement is replaced by “general”.

### 4. Proof of Theorem 0.3

We prove the following theorem.

**Theorem 4.1.** Fix positive integers \( d_1, \ldots, d_k \geq 2 \). Let \( r \geq 2 \) be a positive integer. If \( \prod_{i=1}^k d_i < \frac{1}{r^2} \), every smooth complete intersection

\[
X = X_{d_1, \ldots, d_k, N} \subset \mathbb{P}^{N+k \sum_{i=1}^k d_i}
\]

satisfies \( (C_r) \) for sufficiently large \( N \).

“Sufficiently large \( N \)” in the statement means that \( N \) is larger or equal to some function which depends on \( k, d_1, \ldots, d_k \) and \( r \). Though we do not write it explicitly, the form of the function is clear from the proof.

**Proof.** Take a smooth complete intersection \( X = X_{d_1, \ldots, d_k, N} \subset \mathbb{P}^{N+k \sum_{i=1}^k d_i} \). If \( X \) does not satisfy \( (C_r) \), by the Noether–Fano inequality (Theorem \[ \text{(2.2)} \]), there exists a positive integer \( \mu \) and a movable linear system \( \mathcal{H} \subset |- \mu K_X| \) such that \( c = \text{can}(X, Bs(\mathcal{H})) \leq \frac{1}{r} \) and \( \text{codim}(Bs(\mathcal{H}), X) \geq r \).

Let \( D \in \mathcal{H} \). Since \( D \sim \mu \cdot c_k(O_X(1)) \cap [X] \) and \( c < \frac{1}{\mu} \), by Proposition \[ A \]

\[
\dim \left\{ x \in D \mid e_x(D) \geq \frac{1}{r} \right\} \leq k.
\]

Thus the non-terminal center of the pair \( (X, cBs(\mathcal{H})) \) has at most dimension \( k \) by Proposition \[ B \]

Let \( B = D_1 \cdot \ldots \cdot D_r \) be the complete intersection subscheme of \( X \) defined by general members \( D_1, \ldots, D_r \) of \( \mathcal{H} \). Since \( B \sim \mu^r \cdot c_k(O_X(1))^r \cap [X] \) and \( c < \frac{1}{\mu} \), by Proposition \[ A \]

\[
\dim \left\{ x \in B \mid e_x(B) \geq \frac{1}{c^r} \right\} \leq (k+1)r - 1.
\]

After restricting to a suitable linear subspace \( H = \mathbb{P}^{N-(k+1)r-1} \) of codimension \( (k+1)r - 1 \), by Inversion of Adjunction and Proposition \[ 4.3(2) \], we have the pair \( (X_H, cB_H) \) with \( X_H \) smooth and a point \( P \) in \( X_H \) such that

1. \( P \) is the center of a prime divisor \( E \) which satisfies \( a_E(X_H, cB_H) \leq 0 \),
2. \( \dim \left\{ x \in B_H \mid e_x(B_H) \geq \frac{1}{c^r} \right\} \leq 0 \),

where \( X_H = X \cdot H, B_H = B \cdot H \). Apply Theorem \[ D \] to the pair \( (X_H, cB_H) \) and we have

\[
\deg B_H \geq \frac{1}{r} \left( \frac{r}{c} \right)^r (\dim B_H + 1).
\]
Since $\deg B_H = \mu^r N \prod_{i=1}^{k} d_i$, $\dim B_H = N - k - r - (k + 1)r + 1$ and $c < \frac{1}{r}$,
\[ N \prod_{i=1}^{k} d_i > \frac{r^r}{r!} (N - k - r - (k + 1)r + 2). \]
If $N$ is sufficiently large, the above inequality contradicts to assumption. \qed

5. Proof of Theorem 0.1

First we prove a key proposition.

Proposition 5.1. Let $X \subset \mathbb{P}^N$ be a hypersurface with only ODPs and $N > 5$. Let $D_1, D_2$ be Cartier divisors of $X$ such that $D_1$ and $D_2$ intersect properly and $\alpha = D_1 \cdot D_2 \sim m \cdot c_1(O_X(1))^2 \cap [X]$. Then
\[ e_S(\alpha) \leq 2m \]
for every subvariety $S$ of $X$ with $\dim S \geq 2$.

Proof. By upper-semicontinuity of Samuel multiplicities, we may assume $\dim S = 2$ and $S$ is contained in the support of $\alpha$. Let $d = \deg X$.

For a general 2-tuple $(p_1, p_2) \in (\mathbb{P}^N \setminus X)^2$, set
\[ R_1 = X \cdot C(p_1, S) - S, \quad R_2 = X \cdot C(p_2, R_1) - R_1 \]
as cycles on $X$. By definition,
\[ \dim R_1 = \dim R_2 = \dim S = 2, \quad \deg R_2 = (d - 1) \cdot \deg R_1 = (d - 1)^2 \cdot \deg S. \]
By the following lemma, the intersection product $\alpha \cdot R_2$ on $X$ is a well-defined cycle.

Lemma 5.2.
\[ \dim \alpha \cap R_2 = 0. \]

We have
\[ m \deg R_2 = \alpha \cdot R_2 = \sum_{p \in \alpha \cap R_2} i(p, \alpha \cdot R_2; X) \geq \sum_{p \in S \cap R_2} i(p, \alpha \cdot R_2; X). \]
Let $D(q) = \{ x \in X^{nm} \mid q \in H_x \}$ for $q \in \mathbb{P}^N$, where $H_x$ is the embedded tangent space of $X$. Let $H(q) \subset \mathbb{P}^N$ be the hypersurface of degree $d - 1$ defining $D(q)$ on $X$, i.e. $D(q) = X \cdot H(q)$.

Lemma 5.3. The following holds.

1. For a general 2-tuple $(p_1, p_2) \in (\mathbb{P}^N \setminus X)^2$,
\[ S \cap R_2 = S \cap R_1 \cap R_2 = S \cap D(p_1) \cap D(p_2) = S \cap H(p_1) \cap H(p_2) \]
as a set. In particular, $S \cap R_2$ is not empty.

2. $i(p, \alpha \cdot R_2; X) \geq \frac{1}{2} e_p(\alpha) \cdot e_p(R_2)$ for every $p \in S \cap R_2$.

3. $e_p(R_2) \geq e_p(S)$ for every $p \in S \cap R_2$ which is an ODP of $X$.

We assume the above lemma for a moment and finish the proof of proposition. If we chose $(p_1, p_2)$ general enough, we can apply Lemma 5.3 (1), and for every $p \in S \cap R_2$ all the following conditions are satisfied:

a. $S$ is smooth at $p$ if $X$ is smooth at $p$,

b. $H(p_1), H(p_2)$ is smooth at $p$,

c. $C_p S, C_p H(p_1), C_p H(p_2)$ intersect properly at $p$,
by assumption, thus \( J \) by the same argument as in the proof of Claim A.6 in Appendix A where \( (p) \) is the equality

\[
S \cap (p) = \{ \text{Secality of } p \} \Rightarrow \text{check is the equality } S \cap (p) = \{ \text{Secality of } p \}.
\]

First we prove Lemma 5.2 and Lemma 5.3 (1). Observe for any surfaces \( T, U \) in \( \mathbb{P}^N \), \( \dim J(T, U) \leq \dim T + \dim U + 1 = 5 < N \) by assumption, thus \( J(T, U) \) is strictly contained in \( \mathbb{P}^N \). Take a general 2-tuple \( (p_1, p_2) \in (\mathbb{P}^N \setminus (p)) \). Then by Lemma 5.3 (2), (3), we have

\[
i(p, S \cdot H(p_1) \cdot H(p_2); \mathbb{P}^N) = \epsilon_p(S) \cdot \epsilon_p(H(p_1)) \cdot \epsilon_p(H(p_2))
\]

by [Ful, Corollary 12.4]. Then by Lemma [5.3](2), (3), we have

\[
i(p, \alpha \cdot R_2; X) \geq \frac{1}{2} \epsilon_S(\alpha) \cdot \epsilon_p(R_2)
\]

\[
\geq \frac{1}{2} \epsilon_S(\alpha) \cdot \epsilon_p(S)
\]

\[
= \frac{1}{2} \epsilon_S(\alpha) \cdot \epsilon_p(H(p_1)) \cdot \epsilon_p(H(p_2))
\]

\[
= \frac{1}{2} \epsilon_S(\alpha) \cdot i(p, S \cdot H(p_1) \cdot H(p_2); \mathbb{P}^N).
\]

Therefore

\[
\sum_{p \in S \cap R_2} i(p, \alpha \cdot R_2; X)
\]

\[
\geq \frac{1}{2} \epsilon_S(\alpha) \cdot \left( \sum_{p \in S \cap H(p_1) \cap H(p_2)} i(p, S \cdot H(p_1) \cdot H(p_2); \mathbb{P}^N) \right)
\]

\[
= \frac{1}{2} \epsilon_S(\alpha) \cdot (S \cdot H(p_1) \cdot H(p_2)) = \frac{1}{2} \epsilon_S(\alpha) \cdot (d - 1)^2 \cdot \deg S.
\]

The proof is done. \( \square \)

**Proof of Lemma [5.2] and [5.3].** First we prove Lemma [5.2] and Lemma [5.3] (1). Observe for any surfaces \( T, U \) in \( \mathbb{P}^N \), \( \dim J(T, U) \leq \dim T + \dim U + 1 = 5 < N \) by assumption, thus \( J(T, U) \) is strictly contained in \( \mathbb{P}^N \). Take a general 2-tuple \( (p_1, p_2) \in (\mathbb{P}^N \setminus (p)) \). Then \( S \cap R_1 = S \cap D(p_1) \). This follows from the generality of \( p_1 \) by the same argument as in the proof of [Puk1] Lemma 3], since \( \text{Sec}(S) \) is strictly contained in \( \mathbb{P}^N \) and \( X \) has only isolated singularities. It follows \( \alpha \cap S \cap R_1 = S \cap D(p_1) \) and

\[
\dim \alpha \cap S \cap R_1 = \dim S \cap D(p_1) = \dim S - 1 = 1.
\]

By the same argument as in the proof of Claim [A.6] in Appendix A,

\[
\dim \alpha \cap R_1 \setminus S \leq \dim \alpha + \dim S + 1 - N = \dim S - 2 = 0.
\]

Therefore

\[
\dim \alpha \cap R_1 = \dim \alpha \cap S \cap R_1 = 1.
\]

Since \( R_1 \) is irreducible by the generality of \( p_1 \), we can proceed in the same way after replacing \( S \) by \( R_1 \). We have \( R_1 \cap R_2 = R_1 \cap D(p_2) \) and

\[
\dim \alpha \cap R_2 = \dim \alpha \cap R_1 \cap R_2 = \dim \alpha \cap R_1 \cap D(p_2)
\]

\[
= \dim \alpha \cap R_1 - 1 = 0.
\]

Since \( S \cap R_1 \cap R_2 = S \cap D(p_1) \cap D(p_2) \) by the above argument, what remains to check is the equality \( S \cap R_2 = S \cap R_1 \cap R_2 \). Since \( J(S, R_1) \) is strictly contained in \( \mathbb{P}^N \) and \( S \cap R_2 \subset R_2 \) if \( p_2 \in \mathbb{P}^N \setminus J(S, R_1) \), the generality of \( p_2 \) implies the claim.

For Lemma [5.3] (2), if \( p \) is contained in the smooth locus of \( X \),

\[
i(p, \alpha \cdot R_2; X) \geq \epsilon_p(\alpha) \cdot \epsilon_p(R_2),
\]
Therefore

\[ \tilde{\alpha} \] where

Then we have

\[ O \]

\[ \tilde{\alpha} \]

Assume \( p \) blow-up at \( X \) by \([\text{Ful}, \text{Corollary 12.4}]\). Assume \( p_1 \) and let \( \tilde{X} \longrightarrow X \) be the blow-up at \( p \) with an exceptional divisor \( E \). Since the assertion is local, we may assume \( \tilde{X} \) is smooth. By projection formula, we have

\[ \alpha \cdot R_2 = D_1 \cdot D_2 \cdot R_2 = \pi_*(\pi^*D_1 \cdot \pi^*D_2 \cdot \tilde{R}_2), \]

where \( \tilde{R}_2 \) is the strict transform of \( R_2 \). Write

\[ \pi^*D_1 \cdot \pi^*D_2 = \tilde{\alpha} + m_\alpha E^2, \]

where \( \tilde{\alpha} \) is the strict transform of \( \alpha \). This is possible since \( \text{Pic} E \) is generated by \( O_E(-1) = N_E \tilde{X} \) as a consequence of Lefschetz theorem (See \([\text{Laz}, \text{Example 3.1.25}]\)). Then we have

\[ 0 = E^{\dim X - 2} \cdot \pi^*D_1 \cdot \pi^*D_2 = E^{\dim X - 2} \cdot \tilde{\alpha} + m_\alpha E^{\dim X} \]

\[ = (-1)^{\dim X - 3} e_p(\alpha) + (-1)^{\dim X - 1} 2m_\alpha. \]

Therefore \( m_\alpha = -\frac{1}{2} e_p(\alpha) \). Thus

\[ \pi^*D_1 \cdot \pi^*D_2 \cdot \tilde{R}_2 = \left( \tilde{\alpha} - \frac{1}{2} e_p(\alpha) E^2 \right) \cdot \tilde{R}_2 \]

\[ = \tilde{\alpha} \cdot \tilde{R}_2 + \frac{1}{2} e_p(\alpha) \cdot e_p(R_2), \]

where the second term is represented by a 0-cycle on \( E \). Since the contribution of the first term over \( p \) is non-negative, we obtain the desired inequality.

For Lemma \([13], (3)\), we know

\[ X \cdot C(p_1, S) = S + R_1, \]

\[ X \cdot C(p_2, R_1) = R_1 + R_2, \]

by definition of \( R_1 \) and \( R_2 \). Since

\[ e_p(X \cdot C(p_1, S)) \geq e_p(X) \cdot e_p(S) = 2e_p(S), \]

\[ e_p(X \cdot C(p_2, R_1)) \geq e_p(R_1), \]

by Proposition \([13], (1)\), comparing Samuel multiplicities of both sides of \( \boxminus \), we have

\[ e_p(R_2) \geq e_p(R_1) \geq e_p(S). \]

The proof is done.

Let \( X = X_N \subset \mathbb{P}^N \) be a singular hypersurface and assume that \( N \geq 2\delta(X) + 8 \) and \( X \) has only quadratic singularities of rank \( \geq N - \delta(X) \). Before beginning the proof of Theorem \([11]\), we briefly discuss the Picard group, factoriality and terminality of \( X \). \( \text{Pic} X \) is generated by the class of a hyperplane section by Lefschetz theorem (See \([\text{Laz}, \text{Example 3.1.25}]\)). \( X \) is factorial by the following theorem.

**Theorem 5.4 (Gro, Exp.XI).** Let \( A \) be a Noetherian local ring. If \( A \) is complete intersection and factorial in codimension \( \leq 3 \), then \( A \) is factorial.

To see \( X \) is terminal, first observe \( X \) is normal by \([\text{Mat}, \text{Theorem 23.8}]\) since \((R_1)\) and \((S_3)\) conditions are satisfied. By Inversion of Adjunction, we may assume \( X \) has only ODPs and \( \dim X \geq 7 \). Since the blow-up at singular points gives a resolution of \( X \) and

\[ a_E(X, \emptyset) = k_E(X) + 1 = \dim X - 1 > 1 \]

for each exceptional divisor \( E \) of the blow-up, \( X \) is terminal.
Proof of Theorem 1.14. Assume $X$ is not birationally superrigid. By the Noether–Fano inequality (Theorem 2.2), there exists a positive integer $\mu$ and a movable linear system $H \subset |-\mu K_X|$ such that $c = \text{can}(X, Bs(H)) < \frac{1}{\mu}$.

Let $D \in H$. Since $D \sim \mu \cdot c_1(\mathcal{O}_X(1)) \cap [X]$ and $c < \frac{1}{\mu}$, by Proposition A

$$\dim \left\{ x \in D \mid e_x(D) \geq \frac{1}{e} \right\} \leq \delta(X) + 1.$$ 
Therefore the non-terminal locus of the pair $(X, cBs(H))$ is at most $(\delta(X) + 1)$-dimensional by Proposition B and only the following cases are possible:

I. the non-terminal center is a subvariety $V$ of dimension at most $\delta(X) + 1$ such that $V \cap \text{Sing}(X) \neq \emptyset$ and $V \not\subset \text{Sing}(X)$,

II. the non-terminal center is a smooth point $P$ of $X$,

III. the non-terminal center is a subvariety $V$ of dimension at most $\delta(X)$ such that $V \subset \text{Sing}(X)$.

Let $B = D_1 \cdot D_2$ be the complete intersection scheme of $X$ defined by general members $D_1, D_2$ of $H$, then $B \sim \mu^2 \cdot c_1(\mathcal{O}_X(1)) \cap [X]$. Since the restriction of $X$ to a general linear subspace of codimension $\delta(X)$ has only ODPs, we can apply Proposition 5.1 to the restriction of $X$ and $B$. Since $c < \frac{1}{\mu}$,

$$\dim \left\{ x \in B \mid e_x(B) \geq \frac{2}{c^2} \right\} \leq \delta(X) + 1.$$ 

In the case of I and II, after restricting to a suitable linear subspace $H = \mathbb{P}^{N-\delta(X) - 2}$ of codimension $\delta(X) + 2$, by Inversion of Adjunction and Proposition 1.3 (2), we have the pair $(X_H, cB_H)$ with $X_H$ smooth and a point $Q$ in $X_H$ such that

1. $Q$ is the center of a prime divisor $E$ which satisfies $a_E(X_H, cB_H) \leq 0$,
2. $\dim \left\{ x \in B_H \mid e_x(B_H) \geq \frac{2}{c^2} \right\} \leq 0$,

where $X_H = X \cdot H, B_H = B \cdot H$. Apply Theorem 11 to the pair $(X_H, cB_H)$,

$$\deg B_H \geq \frac{2}{c^2} \cdot (\deg B_H + 1).$$
Since $\deg B_H = \mu^2 N, \dim B_H = N - \delta(X) - 5$ and $c < \frac{1}{\mu}$, we have

$$2\delta(X) + 8 > N,$$
a contradiction.

In the case of III, after restricting to a suitable linear subspace $H = \mathbb{P}^{N-\delta(X) - 1}$ of codimension $\delta(X) + 1$, by Inversion of Adjunction and Proposition 1.3 (2), we have the pair $(X_H, cB_H)$ with $X_H$ having only isolated singularities and a point $Q$ in $X_H$ such that

1. $Q$ is the center of a prime divisor $E$ which satisfies $a_E(X_H, cB_H) \leq 0$,
2. $\dim \left\{ x \in B_H \mid e_x(B_H) \geq \frac{2}{c^2} \right\} \leq 0$,
3. $Q$ is an ordinary double point of $X_H$,

where $X_H = X \cdot H, B_H = B \cdot H$. By [1F2] Proposition 2.3, (3) implies

$$m_{X_H,Q} \geq \text{Jac}_{X_H}$$
in $\mathcal{O}_{X_H,Q}$. Apply Theorem 11 to the pair $(X_H, cB_H)$ and we have

$$\deg B_H \geq \frac{2}{c^2} \cdot \dim B_H.$$ 
Since $\deg B_H = \mu^2 N, \dim B_H = N - \delta(X) - 4$ and $c < \frac{1}{\mu}$, we have

$$2\delta(X) + 8 > N,$$
a contradiction. The proof is done. \qed
We give a concrete example.

**Example 5.5.** Let
\[ X = \{X_0^2(X_1^2 + \cdots + X_8^2) + X_8^8 + \cdots + X_8^8 = 0\} \subset \mathbb{P}^8 = \text{Proj} \mathbb{C}[X_0, \cdots, X_8]. \]

\(X\) has 163969 ODPs and no other singular points. Thus \(X\) is non-rational and \(\text{Bir}(X) = \text{Aut}(X)\) by Theorem 0.1.

**Appendix**

A. BOUNDS FOR MULTIPLEITYS OF CYCLES ON COMPLETE INTERSECTIONS

**Proposition A.** Let \(X\) be a complete intersection variety in \(\mathbb{P}^N\) defined by \(k\) hypersurfaces with only isolated singularities and \(\alpha\) be an effective cycle of pure codimension \(r\) such that \(\alpha \sim m \cdot c_1(\mathcal{O}_X(1))^r \cap [X]\). Then \(e_S(\alpha) \leq m\) for every subvariety \(S\) of dimension at least \(kr\) which is disjoint from the singular points of \(X\).

**Remark.** Proposition A is proved when \(k = 1\) in [Puk1] and [IFEM2], and when \(r = 1\) in [Ch1]. The inequality in Proposition A is sharp when \(k = 1\) by the following example.

**Example A.1 (Zak, 1.12.).** Let \(X \subset \mathbb{P}^N\) be a smooth quadric hypersurface. Then its dual \(X^* \subset (\mathbb{P}^N)^*\) is also smooth quadric. Fix a integer \(n\) with \(1 \leq r \leq \left\lfloor \frac{N+1}{2} \right\rfloor\) and take a \((r-1)\)-dimensional linear subspace \(L\) of \((\mathbb{P}^N)^*\) contained in \(X^*\). Let \(L^* \subset \mathbb{P}^N\) be the dual of \(L\). Then \(L^*\) is a \((N-r)\)-dimensional linear subspace of \(\mathbb{P}^N\). Define
\[ \alpha := X \cdot L^* \sim c_1(\mathcal{O}_X(1))^r \cap [X]. \]

Since the singular locus of \(\alpha\) is a \((r-1)\)-dimensional linear subspace \(\bar{L}\) of \(\mathbb{P}^N\) at which \(L^*\) is tangent to \(X\), \(e_M(\alpha) \geq 2\) and \(e_S(\alpha) = 1\) for all subvarieties \(S\) of \(X\) with \(\dim S \geq r\).

**Proof of Proposition A.** By upper-semicontinuity of Samuel multiplicities, we may assume \(S\) is contained in the support of \(\alpha\) and \(\dim S = kr\). Let \(X_1, \cdots, X_k\) be hypersurfaces defining \(X\) with \(\deg X_i = d_i\). Fix a point \(p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N\) and set
\[ X'_i := C(p, X_i) \text{ for each } i = 1, \cdots, k, \quad X' := C(p, X) = \bigcap_{i=1}^k X'_i. \]

Then each \(X'_i\) is a hypersurface in \(\mathbb{P}^{N+1}\) with \(\deg X'_i = d_i\) and \(X'\) is a complete intersection in \(\mathbb{P}^{N+1}\) defined by \(X'_1, \cdots, X'_k\).

Take a pure-dimensional cycle \(T = \sum n_k T_k\) on \(X\) and a point \(q \in \mathbb{P}^{N+1} \setminus \left( \bigcup_{i=1}^k X'_i \right)\). Assume the support of \(T\) is disjoint from the singular points of \(X\) and \(\dim T \geq k-1\). We define \((\dim T - k + 1)\)-cycle \(R(q, T)\) on \(C = C(q, T)\) as follows:

First assume \(T\) is an irreducible subvariety. For each \(i = 1, \cdots, k\), let \(R^i(q, T)\) be the residual scheme to \(T\) in \(C \cap X'_i\). Then we can write scheme-theoretically
\[ C \cap X'_i = T \cup R^i(q, T). \]

\(R^i(q, T)\) is a Cartier divisor on \(C\) which satisfies
\[ R^i(q, T) \sim (d_i - 1) \cdot c_1(\mathcal{O}_C(1)) \cap [C]. \]

Define the intersection product
\[ R(q, T) := R^1(q, T) \cdot \cdots \cdot R^k(q, T) \sim \left( \prod_{i=1}^k (d_i - 1) \right) \cdot c_1(\mathcal{O}_C(1))^k \cap [C]. \]
If $T$ is a cycle, define $R(q, T) = \sum a_k R(q, T_k)$. Due to the following lemma, $R(q, T)$ is a well-defined cycle on $C$ for a general point $q \in \mathbb{P}^{N+1} \setminus \left( \bigcup_{i=1}^{k} X_i' \right)$, which satisfies

$$\dim R(q, T) = \dim T - (k-1), \quad \deg R(q, T) = \left( \prod_{i=1}^{k}(d_i - 1) \right) \cdot \deg T.$$  

**Lemma A.2.** $R^1(q, T), \ldots, R^k(q, T)$ intersect properly on $C$ for a general point $q \in \mathbb{P}^{N+1} \setminus \left( \bigcup_{i=1}^{k} X_i' \right)$ when $T$ is an irreducible subvariety.

Take a general $r$-tuple $(q_1, \ldots, q_r) \in \left( \mathbb{P}^{N+1} \setminus \left( \bigcup_{i=1}^{k} X_i' \right) \right)^r$. Define

$$R_0 := S, \quad R_j := (\pi_p)_* [R(q_j, R_{j-1})] \text{ for each } j = 1, \ldots, r,$$

where $\pi_p : \mathbb{P}^{N+1} \rightarrow \mathbb{P}^N$ is a linear projection from $p$. At each step of the definition, we can assume $R_j$ is disjoint from the singular points of $X$ by the generality of $(q_1, \ldots, q_r)$. For each $j = 1, \ldots, r$, $\pi_p$ is finite on each irreducible component of $R(q_j, R_{j-1})$ and therefore we have

$$\dim R_j = kr - (k-1)j, \quad \deg R_j = \left( \prod_{i=1}^{k}(d_i - 1)^j \right) \cdot \deg S.$$  

In particular,

$$\dim R_r = r, \quad \deg R_r = \left( \prod_{i=1}^{k}(d_i - 1)^r \right) \cdot \deg S.$$  

**Lemma A.3.** For a general $r$-tuple $(q_1, \ldots, q_r)$,

1. $\alpha$ and $R_r$ intersect properly on $X$, i.e. $\dim \alpha \cap R_r = 0$;
2. $S \cap R_r$ contains at least $\left( \prod_{i=1}^{k}(d_i - 1)^r \right) \cdot \deg S$ distinct points.

We assume the above lemma for a moment and finish the proof of the proposition. By Lemma A.3 (1), the intersection product $\alpha \cdot R_r$ on $X$ is a well-defined 0-dimensional cycle. The proof is done by the following inequality:

$$m \deg R_r = \alpha \cdot R_r = \sum_{t \in \alpha \cap R_r} i(t, \alpha \cdot R_r; X) \geq \sum_{t \in S \cap R_r} i(t, \alpha \cdot R_r; X) \geq \sum_{t \in S \cap R_r} e_t(\alpha) \cdot e_t(R_r) \geq \sum_{t \in S \cap R_r} e_s(\alpha) \cdot \left( \prod_{i=1}^{k}(d_i - 1)^r \right) \cdot \deg S,$$

where the second inequality follows from [Ful Corollary 12.4] and the last inequality follows from Lemma A.3 (2).

**Proof of Lemma A.2 and A.3** For a point $q \in \mathbb{P}^{N+1} \setminus \left( \bigcup_{i=1}^{k} X_i' \right)$, let

$$D^i(q) := \{ x \in (X')^{\ast m} | q \in H^i_{x} \} \subset X_i'$$

where $H^i_{x}$ is the embedded tangent space of $X_i'$ at $x$ for each $i = 1, \ldots, k$ and let

$$D(q) := \{ x \in (X')^{\ast m} | q \in H_{x} \} = \bigcap_{i=1}^{k} D^i(q, T) \subset X'$$

where $H_{x}$ is the embedded tangent space of $X'$ at $x$.

First we prove three claims.
Proof. It is enough to show for each secant variety \(\text{Sec}(T)\) of \(T\) that
\[
T \cap \left( \bigcap_{i=1}^{k} R^i(q, T) \right) = T \cap D(q).
\]

This follows from the same argument as in the proof of [Puk1, Lemma 3] since the secant variety \(\text{Sec}(T)\) of \(T\) is contained in \(\mathbb{P}^N\).

Claim A.5. Let \(T\) be a subvariety of \(X'\), and \(U\) be a subvariety of \(X\) which is disjoint from the singular points of \(X\). Then for a general point \(q \in \mathbb{P}^N \setminus \left( \bigcup_{i=1}^{k} X'_i \right)\),
\[
\dim T \cap U \cap \left( \bigcap_{i=1}^{q} R^i(q, U) \right) = \dim T \cap U - k \quad \text{if} \quad \dim T \cap U \geq k,
\]
\[
T \cap U \cap \left( \bigcap_{i=1}^{q} R^i(q, U) \right) = \emptyset \quad \text{otherwise}.
\]

Proof. Let \(I := \{ (x, p) \in (X')^{\infty} \times \mathbb{P}^{N+1} | p \in H\} \subset X' \times \mathbb{P}^{N+1}\). Take an irreducible component \(W\) of \(T \cap U\) and let \(I_W\) be an inverse image of \(W\) by the first projection of \(I\). Then the first projection \(\pi^W: I_W \to W\) defines \(\mathbb{P}^{N-k+1}\)-fibration structure of \(I_W\) over \(W\) since \(W\) is contained in the smooth locus of \(X'\) by assumption. Then \(I_W\) is irreducible and
\[
\dim I_W = \dim W + N - k + 1.
\]

On the other hand consider the second projection \(\pi^W_2: I_W \to \mathbb{P}^{N+1}\). \(\pi^W_2^{-1}(q) = W \cap D(q)\) for \(q \in \mathbb{P}^{N+1}\) by construction. Since \(\pi^W_2\) is surjective if and only if \(\dim W \geq k\), we have
\[
\begin{cases}
\dim W \cap D(q) = \dim W - k & \text{if} \ \dim W \geq k,
\dim W \cap D(q) = \emptyset & \text{otherwise}
\end{cases}
\]
for a general point \(q \in \mathbb{P}^N \setminus \left( \bigcup_{i=1}^{k} X'_i \right)\). Therefore
\[
\begin{cases}
\dim T \cap U \cap D(q) = \dim T \cap U - k & \text{if} \ \dim T \cap U \geq k,
\dim T \cap U \cap D(q) = \emptyset & \text{otherwise},
\end{cases}
\]
for a general point \(q \in \mathbb{P}^N \setminus \left( \bigcup_{i=1}^{k} X'_i \right)\). The proof is done by Claim A.4.

Claim A.6. Let \(T, U\) be subvarieties of \(X'\). Then for a general point \(q \in \mathbb{P}^{N+1} \setminus \left( \bigcup_{i=1}^{k} X'_i \right)\),
\[
\dim T \cap \left( \bigcap_{i=1}^{k} R^i(q, U) \right) \setminus U \leq \dim T + \dim U - N,
\]
where we use the convention \(\dim(\emptyset) = -\infty\).

Proof. If \(J(T, U) \not\subset \mathbb{P}^{N+1}\), take a point \(q\) not contained in \(J(T, U)\) and then \(T \cap \left( \bigcap_{i=1}^{k} R^i(q, U) \right) \subset U\).
If \(J(T, U) = \mathbb{P}^{N+1}\), let
\[
\mathcal{J} := \{ (t, u, q) \in (\mathbb{P}^{N+1} \times \mathbb{P}^{N+1} \setminus \Delta) \times \mathbb{P}^{N+1} | q \in \langle t, u \rangle \} \subset (\mathbb{P}^{N+1})^3
\]
where $\pi_1, \pi_2, \pi_3$ are projections. By the assumption, $\pi_3 : J_{T,U} := J \cap \pi_1^{-1}(T) \cap \pi_2^{-1}(U) \rightarrow \mathbb{P}^{N+1}$ is surjective. Since $\dim J_{T,U} = \dim T + \dim U + 1$, for a general point $q \in \mathbb{P}^{N+1}$,

$$\dim \pi_3^{-1}_{T,U}(q) = \dim T + \dim U - N.$$

On the other hand, for all $t \in T \cap \left( \bigcap_{i=1}^k R^i(q, U) \right) \setminus U$, there exists $u \in U$ such that $t, u, q$ are collinear, which implies $(t, u, q) \in \pi_3^{-1}_{T,U}(q)$. Thus the proof is done. □

We prove Lemma A.2. By Claim A.4, $T \cap \left( \bigcap_{i=1}^k R^i(q, U) \right) = T \cap D(q)$. By the same argument as in the proof of Claim A.5, $\dim T \cap D(q) = \dim T - k$ for a general point $q \in \mathbb{P}^{N+1} \setminus \left( \bigcup_{i=1}^k X_i^j \right)$. Therefore $\dim T \cap \left( \bigcap_{i=1}^k R^i(q, T) \right) = \dim T - k$. Since $T$ is a hyperplane section of $C = C(q, T)$ and $\bigcap_{i=1}^k R^i(q, T)$ is locally defined by $k$ elements in $C$, $\dim \left( \bigcap_{i=1}^k R^i(q, T) \right) = \dim T - k + 1$. The proof is done. We identify $\bigcap_{i=1}^k R^i(q, T)$ and $R(q, T)$ as a set in the following.

Next we prove Lemma A.3. For (1), we prove inductively

$$\dim \alpha \cap R_j = k(r - j) \text{ for } j = 1, \cdots, r.$$

As sets,

$$\alpha \cap R_j = \pi_p(C(p, \alpha) \cap R(q_j, R_{j-1}))$$

and

$$\alpha \cap R_{j-1} \cap R(q_j, R_{j-1}) = C(p, \alpha) \cap R_{j-1} \cap R(q_j, R_{j-1}).$$

Moreover observe that $R_{j-1} \cap R(q_j, R_{j-1})$ is the union of $Z \cap R(q_j, Z)$ where $Z$ runs all the irreducible component of $R_{j-1}$. This is because $Z_1 \cap R(q_j, Z_2) \subset Z_2$ for distinct irreducible components $Z_1, Z_2$ of $R_{j-1}$ by the generality of $q_j$ and the fact $J(Z_1, Z_2) \subset \mathbb{P}^N \not\subset \mathbb{P}^{N+1}$. Then by induction, Claim A.3 implies

$$\dim \alpha \cap R_{j-1} \cap R(q_j, R_{j-1}) = \dim \alpha \cap R_{j-1} - k = k(r - j).$$

Thus it is enough to show $\dim C(p, \alpha) \cap R(q_j, R_{j-1}) = \dim C(p, \alpha) \cap R_{j-1} \cap R(q_j, R_{j-1})$. This follows from the following inequality for each irreducible component $Z$ of $R_{j-1}$:

$$\dim C(p, \alpha) \cap R(q_j, Z) \setminus Z \leq \dim C(p, \alpha) + \dim Z - N = \dim R_{j-1} - k - r + 1 \leq \dim R_{j-1} - k - j + 1 = k(r - j) = \dim C(p, \alpha) \cap R_{j-1} \cap R(q_j, R_{j-1}),$$

where the first inequality is due to Claim A.6. We have

$$\dim \alpha \cap R_r = 0,$$

as desired.
For (2), observe that \( \dim S \cap R_r = 0 \) due to (1) since \( S \) is contained in the support of \( \alpha \) by assumption. We have the following inclusion by using Claim A.4:

\[
S \cap R_r \supseteq \bigcap_{j=0}^{r-1} R_j \cap R(q_r, R_{r-1})
\]

\[
\supseteq \bigcap_{j=0}^{r-1} R_j \cap D(q_r)
\]

\[
\supseteq \cdots
\]

\[
\supseteq S \cap \bigcap_{j=1}^{r} D(q_j).
\]

Let

\[
I_S := \left\{ (x, q_1, \ldots, q_r) \in S \times (\mathbb{P}^{N+1})^r \mid q_1, \ldots, q_r \in H_x \right\},
\]

where \( H_x \) is the embedded tangent space of \( X' \) at \( x \). Since \( S \) is contained in the smooth locus of \( X' \), the above set is well-defined and the first projection defines \( (\mathbb{P}^N - k + 1)^r \)-fibration structure over \( S \) on \( I_S \). Thus \( I_S \) is irreducible with \( \dim I_S = \dim S + (N - k + 1)r = (N + 1)r \), and the second projection \( \pi_2: I_S \to (\mathbb{P}^{N+1})^r \) is surjective and generically-finite. By the generic smoothness and the generality of \((q_1, \ldots, q_r)\), the number of the points of \( S \cap \bigcap_{j=1}^{r} D(q_j) \) is equal to

\[
S \cdot D(q_1) \cdot \cdots \cdot D(q_r) = \left( \prod_{i=1}^k (d_i - 1)^r \right) \cdot \deg S,
\]

where we use the fact that \( D(q_j) \sim \left( \prod_{i=1}^k (d_i - 1) \right) \cdot c_1(\mathcal{O}_{X'}(1))^k \) for each \( j = 1, \ldots, r \). \( \square \)

### B. Singularities and multiplicities of pairs

**Proposition B.** Let \( X \) be a smooth variety and let \( B \) be a closed subscheme of pure codimension \( r \). Let \( c \in \mathbb{Q}_{>0} \) and assume that \((X, cB)\) is terminal in codimension \( r \) and \( e_p(B) < \left( \frac{1}{2} \right)^r \) for all \( p \in B \). Then \((X, cB)\) is terminal.

**Remark.** Proposition B is well-known when \( r = 1 \). (See [Kol 3.14 Exercise] or [dF1, Proposition 8.8])

We explain by an example that the first condition on \( c \) in theorem cannot be omitted.

**Example B.1.** Let \( L \) be a linear subspace of \( \mathbb{P}^N \) of codimension 2. Consider the pair \((\mathbb{P}^N, L)\). Let \( \pi: \mathbb{P}^N \to \mathbb{P}^N \) be the blow-up at \( L \) with an exceptional divisor \( E \). Then

\[
a_E(\mathbb{P}^N, L) = k_E(\mathbb{P}^N) + 1 - \text{val}_E(L) = 1.
\]

Thus the pair \((\mathbb{P}^N, L)\) is not terminal in codimension 2. On the other hand,

\[
e_p(L) = 1 < 2^2
\]

for every \( p \in L \). For the proof, we use the following theorem by T. de Fernex, L. Ein and M. Mustață, which gives the relation between singularities and multiplicities of pairs.
Theorem B.2 ([FEXT1], Theorem 0.1). Let $X$ be a smooth variety of dimension $N$ and let $B$ be a closed subscheme whose support is a point $p$ in $X$. Let $c \in \mathbb{Q}_{>0}$ and assume that there exists a prime divisor $E$ over $X$ such that $a_E(X, cB) \leq 0$. Then

$$c_p(B) \geq \left(\frac{N}{c}\right)^N.$$  

Proof of Proposition 2. We may assume that $B$ is positive-dimensional. If Proposition is not true, there exists a prime divisor $E$ over $X$ with center $C$ such that $\text{codim}(C, X) > r$ and $a_E(X, cB) \leq 1$. Let $p \in C$ be a closed point, and let $Y \subset X$ be a general complete intersection subvariety containing $p$ with $\dim Y = \text{codim}(C, X)$. Let $X' \to X$ be a birational proper morphism where $X'$ is smooth and $E$ is a prime divisor on $X'$, then a strict transform $Y'$ of $Y$ in $X'$ intersects properly with $E$. If $E'$ is an irreducible component of $E|_{Y'}$, $a_{E'}(Y, cB|_Y) \leq 1$ and the center of $E'$ on $Y$ coincides with $p$. Since $\dim Y > r$, after restricted to a general complete intersection subvariety $Z$ of dimension $r$ through $p$, there exists a prime divisor $F$ over $Z$ with center $p$ such that $a_F(Z, cB|_Z) \leq 0$ by Inversion of adjunction. Since $\dim B|_Z = 0$, $c_p(B|_Z) \geq \left(\frac{r}{r}\right)^r$ by Theorem B.2. On the other hand, $c_p(B|_Z) < \left(\frac{r}{r}\right)^r$ by the assumption. A contradiction. □

Short preliminaries for Appendix C,D

Multiplier ideal sheaves.

Definition. Let $(X, Z)$ be a pair of a smooth variety $X$ and a $\mathbb{R}$-divisor $Z = \sum c_i Z_i$ on $X$. Assume that $c_i \geq 0$ for all $i$. For a log resolution $f : X' \to X$ of $(X, Z)$, the multiplier ideal of $(X, Z)$ is the ideal sheaf

$$\mathcal{J}(X, Z) := f_* \mathcal{O}_{X'}(K_{X'/X} - \left\lfloor \sum c_i f^{-1}(Z_i) \right\rfloor).$$

Theorem (Nadel Vanishing Theorem [Laz] Theorem 9.4.8). Let $X$ be a smooth variety, $D$ be a $\mathbb{R}$-divisor on $X$, and $L$ be a line bundle on $X$ such that $L - D$ is nef and big. Then

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, D)) = 0,$$

for all $i \geq 1$.

Restriction of divisorial valuations. Given a dominant morphism $f : X \to Y$ between two varieties, the valuation $\text{val}_E$ on $C(X)$ defined by a prime divisor $E$ over $X$ restricts to a divisorial valuation $\text{val}_E$ on $C(Y)$ where $p$ is a positive integer and $F$ is a prime divisor over $Y$ (see [FEXT2], Lemma 1.3). The center of $F$ coincides with the image of the center of $E$ by $f$.

C. Excluding maximal singularities : low-dimensional case

Theorem C. Let $X$ be a smooth projective variety in $\mathbb{P}^N$ and $B$ be a closed locally complete intersection subscheme of $X$ of pure codimension $r$. Assume there exists a point $p \in B$ and a positive real number $c > 0$ such that

1. $p$ is the center of a prime divisor $E$ which satisfies $a_E(X, cB) \leq 0$,
2. $\dim \{ x \in B \mid e_x(B) \geq \frac{r}{r} \cdot \left(\frac{r}{r}\right)^r \} \leq 0$.

Then

$$\begin{cases} 
\deg B \geq \left(\frac{r}{r}\right)^r \cdot 3 & \text{if } \dim B \geq 2, \\
\deg B \geq \left(\frac{r}{r}\right)^r \cdot (\dim B + 1) & \text{otherwise}.
\end{cases}$$
Remark: A hypersurface-case of Theorem [C] is the core of the proof of the birational superrigidity of low-dimensional hypersurfaces in [dFEM2]. For the proof of Theorem [C] we essentially follow [dFEM2]. To make the sufficient condition for the conclusion clear, we prove the theorem in more general setting.

For the proof, we use the following theorem proved by T. de Fernex, L. Ein and M. Mustaţă, which is necessary for estimating the change of log-canonical thresholds under linear projections.

**Theorem C.1** ([dFEM2], Theorem 1.1). Let $X, Y$ be smooth varieties and let $B$ be a closed locally complete intersection subscheme of $X$ of pure codimension $r$ with $\dim X - \dim Y = r - 1$. Let $f: X \to Y$ be a smooth proper surjective morphism such that $f|_B$ defines a finite morphism. Let $E$ be a prime divisor over $X$. Write $\text{val}_E|_Y = q \text{val}_E$ by some divisor $F$ and positive integer $q$. Then for any positive real number $c > 0$,

$$q a_F \left( Y, \left( \frac{c}{r} \right)^r f_*[B] \right) \leq a_E(X, cB).$$

**Proof of Theorem [C]** By Inversion of adjunction and Proposition [1.3] (2), the case when $\dim B > 2$ reduces to the case when $\dim B = 2$. Thus we assume $\dim B \leq 2$.

Consider the linear projection $\pi: \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{\dim B + 1}$ from a general $(N - \dim B - 2)$-dimensional linear subspace $\Lambda$ in $\mathbb{P}^N$. By the generality of $\Lambda$, the resolution of indeterminacy of $\pi|_X$ gives a morphism $\tilde{\pi}: \tilde{X} \to \mathbb{P}^{\dim B + 1}$ which is smooth and proper over a suitable open subset $U$ of $\mathbb{P}^{\dim B + 1}$ which contains the image of $p$, where $\tilde{X}$ is the strict transform of $X$. Since $B$ is disjoint from $\Lambda$ by the same reason, the strict transform $\tilde{B}$ of $B$ is isomorphic to $B$. Apply Theorem [C.1] to $\tilde{\pi}|_{\tilde{\pi}^{-1}(U)}$ and $(\tilde{\pi}^{-1}(U), c\tilde{B}|_{\tilde{\pi}^{-1}(U)})$, then the pair

$$\left( \mathbb{P}^{\dim B + 1}, \left( \frac{c}{r} \right)^r \pi_*[B] \right)$$

has a non-klt center at $q = \pi(p)$. Let $\Delta = \left( \frac{c}{r} \right)^r \pi_*[B]$.

On the other hand, by the generality of $\Lambda$, for each irreducible component $Z$ of $B$, $\pi$ is birational finite on $Z$. Moreover if $g = \pi|_B$, by [dFEM2] Proposition 4.6, 4.7,

$$\text{codim}(\Sigma \cup \Sigma', \mathbb{P}^{\dim B + 1}) \geq 3,$$

where

$$\Sigma = \left\{ q \in \pi(B) \mid e_\pi(\pi_*[B]) > \sum_{p \in \gamma^{-1}(q)} e_p(B) \right\}, \quad \Sigma' = \left\{ q \in \pi(B) \mid \#g^{-1}(q) \geq 3 \right\}.$$

Therefore $\dim \left\{ x \in \pi(B) \mid e_\pi(\pi_*[B]) \geq \left( \frac{c}{r} \right)^r \right\} \leq \max \{ \dim B + 1 - 3, 0 \} = 0$. Since $(\mathbb{P}^{\dim B + 1}, \Delta)$ is klt at every point $x$ with $e_\pi(\pi_*[B]) < \left( \frac{c}{r} \right)^r$, $(\mathbb{P}^{\dim B + 1}, \Delta)$ is klt away from $0$-dimensional locus. Thus the multiplier ideal sheaf $J(\mathbb{P}^{\dim B + 1}, \Delta)$ has non-empty $0$-dimensional co-support. Let $\Sigma$ be one of its irreducible components.

By Nadel vanishing theorem,

$$H^1(\mathcal{O}_{\mathbb{P}^{\dim B + 1}}(K_{\mathbb{P}^{\dim B + 1}} + [\deg \Delta + \epsilon]) \otimes J(\mathbb{P}^{\dim B + 1}, \Delta)) = 0$$

for all $\epsilon > 0$. Thus the homomorphism of cohomology induced by the restriction

$$H^0(\mathcal{O}_{\mathbb{P}^{\dim B + 1}}(K_{\mathbb{P}^{\dim B + 1}} + [\deg \Delta + \epsilon])) \to H^0(\Sigma, \mathcal{O}_\Sigma)$$

is surjective for all $\epsilon > 0$. Since the right hand side is non-zero, we have

$$-(\dim B + 2) + [\deg \Delta + \epsilon] \geq 0$$

for all $\epsilon > 0$. Thus we have

$$\deg \Delta \geq \dim B + 1.$$
Since $\deg \Delta = \left( \frac{c}{r} \right)^r \cdot \deg B$, the proof is done. \hfill \Box

D. Excluding maximal singularities: high-dimensional case

**Theorem D.** Let $X$ be a Gorenstein projective variety in $\mathbb{P}^N$ with only isolated singularities and $B$ be a closed subscheme of $X$ of pure-codimension $r$ which is arithmetically Cohen-Macaulay in $\mathbb{P}^N$. Assume there exists a point $P$ in $B$, a positive real number $c$ and a positive integer $\nu$ with $\dim B \geq \nu$ such that

1. $P$ is the center of a prime divisor $E$ which satisfies $a_E(X, cB) \leq 0$,
2. $\dim \{ x \in B \mid e_x(B) \geq \frac{1}{r} \cdot \left( \frac{c}{r} \right)^r \} \leq 0$,
3. $m_X^{-1} \subset \mathcal{J}ac_X$, where $m_{X,P}$ is the maximal ideal of $\mathcal{O}_{X,P}$ and $\mathcal{J}ac_X$ is the integral closure of the Jacobian ideal $\mathcal{J}ac_X$ in $\mathcal{O}_{X,P}$.

Then

$$\deg B \geq \frac{1}{r!} \cdot \left( \frac{c}{r} \right)^r \cdot (\dim B - \nu + 2).$$

**Remark.** A hypersurface-case of Theorem [D] is the core of the proof of birational superrigidity of high-dimensional hypersurfaces in [dF1] and [dF2]. For the proof of Theorem [D] we essentially follow [dF1] and [dF2]. By the same reason as in the case of Theorem [C] we prove the theorem in more general setting. We prove Claim D.5.6 which is missing in [dF2].

D.1 to D.3 are devoted to review basic facts about arc spaces and jet schemes. The proof of Theorem [D] is given in D.4 and D.5.

D.1. Arc spaces and maximal divisorial sets. Let $X$ be a variety.

**Definition D.1.1.** For $m \geq 0$, the $m$-th jet scheme $J^m X$ is characterized by

$$\text{Hom} \left( \text{Spec} A, J^m X \right) = \text{Hom} \left( \text{Spec} A[t]/(t^{m+1}), X \right)$$

for every $\mathbb{C}$-algebra $A$. The truncation maps $A[t]/(t^{m+1}) \to A[t]/(t^m)$ give rise to affine morphisms $J^m X \to J^{m-1} X$. The arc space $J^\infty X$ of $X$ is defined as the inverse limit of the inverse system $\{ J^m X \to J^{m-1} X \}_{m \geq 1}$. For $m \geq p \geq 0$ there are natural projections

$$\pi: J^\infty X \to X, \pi_m: J^\infty X \to J^m X, \text{ and } \pi_{m,p}: J^m X \to J^p X$$

given by truncation. For every subset $W \subset J^\infty X$ we denote $W_m = \pi_m(W) \subset J^m X$. For every $m \in \mathbb{N} \cup \{ \infty \}$, there is a trivial section $\iota_m: X \to J^m X$ that associates each point $P$ in $X$ with the constant jet or arc of $X$ at $P$. We denote $X_m = \iota_m(X)$ and $P_m = \iota_m(P)$. A morphism of schemes $f: X \to Y$ induces, by composition, natural maps $f_m: J^m X \to J^m Y$ that commute with the projections.

A $\mathbb{C}$-valued point of $J^\infty X$ is called an arc, and we call it a fat arc if it does not factor through a proper closed subset of $X$. For a fat arc $\gamma \in J^\infty X$, define

$$\text{ord}_\gamma(h) = \text{ord}_A(\gamma^*(h))$$

for every $h \in \mathbb{C}(X)^*$, where $\gamma^*: \mathbb{C}(X) \to \mathbb{C}((t))$ is the pull-back defined by $\gamma$.

**Definition D.1.2.** A cylinder of $J^\infty X$ is a subset $C \subset J^\infty X$ that is the inverse image of a constructible set on some finite level $J^p X$. If the image of $C$ in $X$ coincides with some point $P$, define

$$\mu_P(C) := \min \{ m \geq 0 \mid C_m \neq P_m \}.$$
Remark. For a closed irreducible cylinder $C$ of $J_\infty X$ whose image in $X$ is some point $P$, $\mu_P(C) = \text{val}_C(P)$ by the same argument as in [dF1, Lemma 4.4].

**Definition D.1.3.** For an arc $\gamma \in J_\infty X$ and a proper subscheme $Z \subset X$, define
\[
\text{ord}_\gamma(Z) = \text{ord}_\gamma^X(Z) := \max \{m \mid \gamma^*I_Z \subset t^m\mathbb{C}[\![t]\!], \} \in \mathbb{Z} \cup \{\infty\},
\]
where $I_Z \subset \mathcal{O}_{X,\pi(\gamma)}$ is the ideal locally defining $Z$ near $\pi(\gamma)$. In the case when $X$ is smooth, for every integer $q \geq 0$, define
\[
\text{Cont}^q(Z) := \{\gamma \in J_\infty X \mid \text{ord}_\gamma(Z) = q\}.
\]

For a prime divisor $E$ over $X$ and a positive integer $q$, fix a proper birational map $f : X' \to X$, where $X'$ is smooth and contains a smooth prime divisor $E$. We define
\[
W^q(E) := f_\infty(\text{Cont}^q(E)) \subset J_\infty X.
\]
$W^q(E)$ is called the maximal divisorial set of the divisorial valuation $q \text{val}_E$.

**Theorem D.1.4** ([ELM]). Assume $X$ is a smooth variety. The following holds.

1. For every prime divisor $E$ over $X$ and a positive integer $q$, $W^q(E)$ is a closed irreducible cylinder of codimension
\[
\text{codim}(W^q(E), J_\infty X) = q(k_E(X) + 1),
\]
and $\text{val}_{W^q(E)} = q \text{val}_E$.

2. For every closed irreducible cylinder $C$, $\text{val}_C = q \text{val}_E$ for some prime divisor $E$ and positive integer $q$, and $C$ is contained in $W^q(E)$.

**Theorem D.1.5** ([H], [CP1]). Assume $X$ is a singular variety. Then for a prime divisor $E$ over $X$ and a positive integer $q$, $W^q(E)$ is a closed irreducible quasi-cylinder, that is a cylinder away from the arc space of singular locus of $X$, which satisfies $\text{val}_{W^q(E)} = q \text{val}_E$.

---

**D.2. Higher order tangent spaces and principal tangent directions.** Let $X$ be a smooth variety and fix a closed point $P \in X$.

**Definition D.2.1.** Define
\[
T^{(m)}X := \pi_{m,m-1}(X_m) \quad (\text{resp. } T^{(m)}_P X := \pi_{m,m-1}^{-1}(P_{m-1}))
\]
and it is called the $m$-th order tangent bundle (resp. the $m$-th order tangent space over $P$).

$T^{(m)}X$ satisfies
\[
T^{(m)}X = \text{Hom}(\text{Spec} \mathbb{C}[t^m]/(t^{m+1}), X)
\]
and the embedding in $J_m X$ corresponds to the inclusion $\mathbb{C}[t^m]/(t^{m+1}) \subset \mathbb{C}[t]/(t^{m+1})$.

For all $m, n \geq 1$, let
\[
\psi_{m,n} : T^{(m)}X \to T^{(n)}X
\]
be a morphism corresponding to the the $\mathbb{C}$-algebra isomorphism $\mathbb{C}[t^m]/(t^{m+1}) \to \mathbb{C}[t^n]/(t^{n+1})$ mapping $a + bt^m$ to $a + bt^n$ where $a, b \in \mathbb{C}$. $\psi_{m,n}$ is an isomorphism of $\mathbb{A}^n$-bundles over $X$. We use the same symbol $\psi_{m,n}$ for its restriction to $T^{(m)}_P X$.

For a prime divisor $E$ over $X$ with center $P$ and a positive integer $q$, let $W = W^q(E) \subset J_\infty X$ be the maximal divisorial set of $q \text{val}_E$ and $\mu = \mu_P(W)$. If $F$ is the exceptional divisor of the blow-up of $X$ at $P$, the center of $E$ on $Bl_P X$, which we denote by $\hat{\Gamma}_E$, is contained in $F$. Since $F = \mathbb{P}_s(T_P X)$, we take the affine cone over $\hat{\Gamma}_E$ in $T_P X$ and denote it by $\hat{\Gamma}_E$.

**Proposition D.2.2** ([dF1], Proposition 4.6). $\psi_{\mu,1}(W_\mu) \subset \hat{\Gamma}_E$ is a dense constructible subset.
Definition D.2.3. The non-zero elements in $\psi_{N}(W_{\mu}) \subset T_{P}X$ are called the principal tangent vectors of $E$. The elements in $\Gamma_{E}$ given by homogeneous classes of non-zero elements in $\psi_{N}(W_{\mu})$ are called the principal tangent directions of $E$ at $P$.

D.3. $\mathbb{C}^{*}$-action on $J_{\infty}\mathbb{A}^{N}$ and homogeneous divisorial valuation. Fix a point $P \in \mathbb{A}^{N}$ as the origin. Corresponding to $\mathbb{C}^{*}$-action $z \mapsto z \cdot x$ on $\mathbb{A}^{N}$ along some linear coordinates $x = (x_{1}, \ldots, x_{N})$ centered at $P$, there is $\mathbb{C}^{*}$-action on $J_{\infty}\mathbb{A}^{N} = \mathbb{A}^{N(m+1)}$ which is compatible with each projection. Thus there is $\mathbb{C}^{*}$-action induced on $J_{\infty}\mathbb{A}^{N}$.

Definition D.3.1. For a prime divisor $E$ over $\mathbb{A}^{N}$ with center $P$, val$_{E}$ is called a homogeneous valuation if $W^{q}(E)$ is invariant under $\mathbb{C}^{*}$-action for some $q \geq 1$. This is equivalent to say val$_{E}(h(s \cdot x)) = val_{E}(h(x))$ for every $h(x) \in \mathbb{C}[x]$ and $s \in \mathbb{C}^{*}$, and thus the notion is independent of the choice of $q$.

D.4. Proof of Theorem [D]: the latter half. The proof consists of two parts: a linear projection and a restriction to a linear subspace. First we prove the latter half, where a restriction is carried out.

Proposition D.4.1. Fix the origin $P$ of $\mathbb{A}^{N}$. Let $D$ be a homogeneous hypersurface in $\mathbb{A}^{N}$ of degree $d$. Assume there exist a prime divisor $E$ over $\mathbb{A}^{N}$ with center $P$, a positive real number $c$ and a positive integer $\nu$ with $N - 1 \geq \nu$ such that

1. val$_{E}$ is a homogeneous valuation and $a_{E}(\mathbb{A}^{N}, cD + (\nu - 1)P) \leq 0$,
2. for some non-zero element $\xi \in \Gamma_{E}$, $e_{\xi}(D) < \frac{1}{c}$, where $L_{\xi}$ is a line through $P$ with the tangent direction $\xi$,
3. $e_{T}(D) < \frac{1}{c}$ for each irreducible component $T$ of $D$.

Then

$$d \geq \frac{1}{c}(N - \nu + 1).$$

For the proof, we use the special Inversion of Adjunction theorem proved by T. de Fernex.

Theorem D.4.2 ([HF1], Theorem 6.4). Let $E$ be a prime divisor over $X = \mathbb{A}^{N}$ with center a point $P \in X$. Assume val$_{E}$ is a homogeneous valuation with respect to a linear coordinate centered at $P$. Let $Y$ be a positive-dimensional linear subspace of $X$ of codimension $e$, which is tangent to some principal tangent vector of $E$. Then there exists a prime divisor $F$ over $Y$ with center $P$ and a positive integer $q$ such that for any positive real number $c$, any non-negative integer $m$ and any proper subscheme $Z$ not containing $Y$,

$$q a_{E}(Y, cZ|Y) + (m - e)P \leq a_{E}(X, cZ + mP).$$

Proof of Proposition D.4.1. Since the set of principal tangent vectors of $E$ forms a dense subset of $\Gamma_{E}$ by Proposition D.2.2, we may assume $\xi$ is a principal tangent vector by upper-semicontinuity of Samuel multiplicities. For a 2-dimensional linear subspace $k^{2} \subset k^{N}$ containing $L_{\xi}$,

$$(m_{k^{2}, P})^{N - \nu - 1} \not\subset J(k^{2}, cD|_{k^{2}}),$$

by assumption (1) and Theorem D.4.2. If $k^{2}$ is general, $e_{L}(D|_{k^{2}}) < \frac{1}{c}$ for every line $L$ in $k^{2}$ through $P$ by assumption (2), (3) and Proposition 1.3 (2). Since $D|_{k^{2}}$ is a union of line, this implies the pair $(k^{2}, cD|_{k^{2}})$ is klt away from $P$. Thus the multiplier ideal $J(k^{2}, cD|_{k^{2}})$ is co-supported at $P$. Let $q = J(k^{2}, cD|_{k^{2}})$ and $\Sigma$ be a scheme defined by $q$. Since $q$ is defined by homogeneous polynomials, by the above we need homogeneous elements of degree at least $N - \nu - 1$ to generate

$$\mathcal{O}_{\Sigma} = \mathbb{C}[x, y]/q.$$
Let $\Delta = cD|_{A^2}$ and $\overline{\Delta}$ be its projective closure in $\mathbb{P}^2$. By Nadel vanishing theorem,

$$H^1(\mathcal{O}_{\mathbb{P}^2}(K_{\mathbb{P}^2} + \lceil \deg \overline{\Delta} + \epsilon \rceil)) \otimes \mathcal{J}(\mathbb{P}^2, \overline{\Delta}) = 0$$

for all $\epsilon > 0$. Therefore the homomorphism of cohomology induced by restriction,

$$H^0(\mathcal{O}_{\mathbb{P}^2}(K_{\mathbb{P}^2} + \lceil \deg \overline{\Delta} + \epsilon \rceil)) \to H^0(\Sigma, \mathcal{O}_\Sigma),$$

is surjective for all $\epsilon > 0$. By the above argument, we have

$$-3 + \lceil \deg \overline{\Delta} + \epsilon \rceil \geq N - \nu - 1$$

for all $\epsilon > 0$. Thus we have

$$\deg \overline{\Delta} \geq N - \nu + 1.$$ 

Since $\deg \overline{\Delta} = cd$, the proof is done. \qed

D.5. **Proof of Theorem [1]: complete the proof.** In this section, we finish the
first half of the proof, where a linear projection is carried out, and complete the
proof of Theorem [1].

For the proof, we use the following theorem proved by T. de Fernex.

**Theorem D.5.1 ([1], Theorem 8.1).** Let $f : X \to Y$ be a composition of two
smooth proper surjective morphisms of smooth varieties $g : X \to U$ and $h : U \to
Y$, and let $r := \dim U - \dim Y + 1$. Let $B$ be a closed Cohen-Macaulay subscheme
of $X$ of pure dimension with $\dim Y = \dim B + 1$. Assume $f|_B$ defines a finite
morphism. Let $E$ be a prime divisor over $X$. Write $\text{val}_E|_{C(U)} = p \text{val}_E$ (resp.
$\text{val}_E|_{C(Y)} = q \text{val}_E$) for some prime divisor $F$ over $U$ (resp. $G$ over $Y$) and some
positive integer $p$ (resp. $q$). Then for any positive real number $c$,

$$qa_G \left( Y, r! \left( \frac{1}{r} \right)^r f_*[B] \right) \leq a_E(X \to U, cB),$$

where $a_E(X \to U, cB) = p(k_F(U) + 1) - c \text{val}_E(B)$.

**Proof of Theorem [1].** Take a general hyperplane $H \subset \mathbb{P}^N$ intersecting $B$ properly
and not containing $P$. Then we may assume

$$(\dagger) \quad \varepsilon_B(B|_H) < \frac{1}{r!} \left( \frac{r}{c} \right)^r$$

for every point $x \in B|_H$. Consider the complement $\mathbb{A}^N = \mathbb{P}^N \setminus H$ and $Y$ (resp. $Z$)
be the restriction of $X$ (resp. $B$) to $\mathbb{A}^N$. Take $P$ as the origin of $\mathbb{A}^N$.

Let $E_1 = E$. Then by assumption

$$a_{E_1}(Y, cZ) \leq 0.$$ 

Let $W = W^1(E_1) \subset J_\infty Y$ be the maximal divisorial set of $\text{val}_{E_1}$. $W$ is a cylinder
since $W$ is a quasi-cylinder by Theorem [1.3.5] and every quasi-cylinder in $J_\infty Y$ is a
cylinder due to the assumption that $Y$ has only isolated singularities. Fix a positive
integer $m$ such that $W$ is the inverse image of $W_m$ and $m > \text{val}_{E_1}(Z)$.

Let $W^0 \subset J_\infty \mathbb{A}^n$ be the flat degeneration of $W$ with respect to the $C^*$-action
$x \mapsto s \cdot x$ along some linear coordinates centered at $P$ and $s \to 0$. Define

$$\mu := \min \{ m' | (W^0)_{m'} \neq P_{m'} \},$$

and fix a non-zero element

$$\xi \in \psi_{h^{\mu}}((W^0)_\mu).$$

Let $n = \dim X$. Take a general linear projection $\tilde{\sigma} : \mathbb{A}^n \to \mathbb{A}^n$ and let $\sigma : Y \to \mathbb{A}^n$ be its restriction to $Y$. Then there exists a prime divisor $E_2$ over $\mathbb{A}^n$ and a
positive integer $q_2$ such that

$$\text{val}_{E_1}|_{C(\mathbb{A}^n)} = q_2 \text{val}_{E_2}. $$


Let \( W' = W'^{(1)}(E_2) \) be the maximal divisorial set of \( q_2 \) val\(_E\), and \((W')^0\) be its flat degeneration with respect to the \( \mathbb{C}^* \)-action with center \( Q = \sigma(P) \) compatible with those on \( \mathbb{A}^N \). Since \( \sigma \) is finite, \( \sigma_\infty(W) = W' \).

**Claim D.5.2.**

\[ \hat{\sigma}_m((W_m)^0) = (W^0)_m. \]

**Proof.** Since a general \((N - n - 1)\)-plane in \( \mathbb{P}^N \) does not intersect the embedded tangent cone of \( X \) at \( P \), the proof follows in the same way as [IF2] Lemma 4.1. \( \square \)

By Claim D.5.2,

\[ \mu = \mu_Q((W')^0). \]

Let

\[ \xi' = d\hat{\sigma}|_P(\xi) \in T_Q\mathbb{A}^n. \]

By the generality of \( \sigma \), we may assume \( \xi' \) is non-zero.

Take an irreducible component \( C \) of \((W')^0\) such that \( \psi_{\mu,1}(C_\mu) \) contains \( \xi' \). Then there exists a prime divisor \( E_3 \) and a positive integer \( q_3 \) such that

\[ \val_{E_3} = q_3 \val_{E_3} \text{ and } \codim(C, J_\infty\mathbb{A}^n) \geq \codim(W^0(E_3), J_\infty\mathbb{A}^n), \]

by Theorem D.1.4 (2). By construction the center of \( \psi_{\mu,1}(W^0(E_3)) \) in contained in \( \hat{\Gamma}_{E_3} \) by Proposition D.2.2.

**Claim D.5.3.** There exists a prime divisor \( E_4 \) over \( \mathbb{A}^N \) and a positive integer \( p \) such that

1. \( \val_{E_4} \mid_{(\mathbb{A}^N)} = p \val_{E_3}, \)
2. \( a_{E_4}\left(\mathbb{A}^N \xrightarrow{\sigma} \mathbb{A}^n, c\mathbb{Z}^0 + (\nu - 1)P\right) \leq 0, \)

where \( Z^0 \) is the flat degeneration of \( Z \).

**Proof.** We will give the proof later. \( \square \)

Let \( \tau: \mathbb{A}^n \to \mathbb{A}^{n-r+1} \) be a general linear projection and \( R = \tau(Q) \). Let \( \rho = \tau \circ \hat{\sigma} \) and \( D = \mathcal{F}|_{\mathbb{P}^n}^{\mathbb{Z}^0}|_{\mathbb{A}^{n-r+1}}, \) where \( \mathcal{F}: \mathbb{P}^n \to \mathbb{P}^{n-r+1} \) is induced by \( \rho \) and \( Z^0 \) is the projective closure of \( Z^0 \). By the generality of \( \rho, D \) is well-defined, homogeneous and \( \deg D = \deg Z^0 = \deg B \).

There exist a prime divisor \( E_5 \) and a positive integer \( q_5 \) such that

\[ \val_{E_5} \mid_{(\mathbb{A}^{n-r+1})} = q_5 \val_{E_5}. \]

Then \( \val_{E_5} \) is a homogeneous valuation. By Claim D.5.3,

\[ \val_{E_4} \mid_{(\mathbb{A}^{n-r+1})} = p q_5 \val_{E_5}. \]

Since \( Z^0 \) is Cohen-Macaulay by assumption, by Theorem D.5.1,

\[ pq_5 a_{E_3} \left(\mathbb{A}^{n-r+1}, r! \left(\frac{c}{r!}\right)^r D\right) \leq a_{E_4} \left(\mathbb{A}^N \xrightarrow{\sigma} \mathbb{A}^n, c\mathbb{Z}^0\right). \]

Since \( \val_{E_4}(P) \leq \val_{E_3}(\rho^{-1}(R)) = pq_5 \val_{E_5}(R) \), it follows by Claim D.5.3

\[ a_{E_4} \left(\mathbb{A}^{n-r+1}, r! \left(\frac{c}{r!}\right)^r D + (\nu - 1)R\right) \leq \frac{1}{pq_5} a_{E_4} \left(\mathbb{A}^N \xrightarrow{\sigma} \mathbb{A}^n, c\mathbb{Z}^0 + (\nu - 1)P\right) \leq 0. \]

Let

\[ \xi'' = d\tau|_Q(\xi') \in T_R\mathbb{A}^{n-r+1}. \]

By the generality of \( \tau \), we may assume \( \xi'' \) is non-zero. Since \( \xi' \in \hat{\Gamma}_{E_3}, \)

\[ \xi'' \in \hat{\Gamma}_{E_5}. \]
Claim D.5.4. For a general projection $\rho$, 
$$e_{L_{\xi''}(D)} < \frac{1}{r!} \left( \frac{T}{c} \right)^r,$$
where $L_{\xi''}$ is a line through $R$ with the tangent direction $\xi''$.

Proof. Due to (1), the claim follows in the same way as $\text{dF1}$ Lemma 9.3. □

By the generality of $\rho$, $\rho$ is birational finite on each irreducible component of $Z^0$. Thus by (1) and the fact that $Z^0$ is an affine cone over $B^r_H$, it also follows $e_T(D) < \frac{1}{r!} \left( \frac{T}{c} \right)^r$ for each irreducible component $T$ of $D$. Apply Proposition D.4.1 to the pair $(\mathbb{A}^n_{r+1}, r! \left( \frac{T}{c} \right)^r D + (\nu - 1)R)$ and $E_5$, and we have

$$\deg B \geq \frac{1}{r!} \left( \frac{T}{c} \right)^r (n - r - \nu + 2).$$

The proof is done. □

We prove Claim D.5.3 which is missing in $\text{dF2}$. The proof is the combination of those of $\text{dF1}$ Lemma 9.2 and $\text{dF2}$ Lemma 4.2.

Proof of Claim D.5.3. Take a resolution $\tilde{Y} \rightarrow Y$ (resp. $\tilde{\mathbb{A}}^n \rightarrow \mathbb{A}^n$) such that $E_1$ is a prime divisor on $\tilde{Y}$ (resp. $E_2$ is a prime divisor on $\tilde{\mathbb{A}}^n$) and which fits into the following commutative diagram:

$$\begin{array}{ccc}
\tilde{Y} & \xrightarrow{f} & Y \\
\downarrow g & & \downarrow \sigma \\
\tilde{\mathbb{A}}^n & \xrightarrow{} & \mathbb{A}^n
\end{array}$$

By the above,

$$K_{\tilde{Y}/Y} + f^*K_{Y/\tilde{\mathbb{A}}^n} = K_{\tilde{Y}/\tilde{\mathbb{A}}^n} + g^*K_{\mathbb{A}^n/\mathbb{A}^n}.$$ 

Comparing the order of $E_1$ on the both sides,

$$k_{E_1}(Y) + \val_{E_1}(K_{Y/\tilde{\mathbb{A}}^n}) = q_2 - 1 + q_2 k_{E_2}(\mathbb{A}^n).$$

Since $\val_{E_1}(K_{Y/\tilde{\mathbb{A}}^n}) = \val_{E_1}(\text{Jac}_X)$ for a general projection $\sigma$,

$$\begin{aligned}
(\dagger) & \quad k_{E_1}(Y) + 1 + \val_{E_1}(\text{Jac}_X) = q_2(k_{E_2}(\mathbb{A}^n) + 1).
\end{aligned}$$

Let $\tilde{W}$ be the inverse image of $W_m$ by the projection $\pi_m: J_{\infty, \tilde{\mathbb{A}}^n} \rightarrow J_m, \tilde{\mathbb{A}}^n$. Let $\tilde{W}^0$ be the flat degeneration of $W$. Then by the choice of $m$, $\tilde{W}^0 = \pi_m^{-1}((W_m)^0)$.

Since $((W')^0)_m = \tilde{\sigma}_m((W_m)^0)$ by Claim D.5.2 $\sigma_{\infty}((\tilde{W})^0) = (W')^0$. Let $C$ be an irreducible component of $((\tilde{W})^0) \cap \tilde{\sigma}_{\infty}^{-1}(C)$ such that $\sigma_{\infty}(C) = C$. Then there exists a prime divisor $E_4$ and a positive integer $a$ such that

$$a \val_{E_4} = \val_{\tilde{C}},$$

by Theorem D.1.4 (2). Then

$$a \val_{E_4|_{\tilde{C}(\tilde{\mathbb{A}}^n)}} = \val_{\tilde{C}|_{\tilde{C}(\tilde{\mathbb{A}}^n)}} = \val_{\tilde{C}} = q_3 \val_{E_3},$$

which implies $a$ divides $q_3$. Let $p = \frac{a}{q_3}$.

Since $m > \val_{E}(Z)$, $\val_{H}(Z) = \val_{\tilde{W}}(Z)$. Therefore by the semi-continuity

$$\val_{E_1}(Z) = \val_{H}(Z) = \val_{\tilde{W}}(Z) \leq \val_{\tilde{C}}(Z^0) = a \val_{E_4}(Z^0).$$

We also have

$$\val_{E_1}(P) = \val_{H}(P) = \val_{\tilde{W}}(P) \leq \val_{\tilde{C}}(P) = a \val_{E_4}(P).$$
On the other hand, by (†) and Theorem D.1.4 (1),
\[ q_3(k_{E_3}(\mathbb{A}^n) + 1) = \text{codim}(W^{\nu_3}(E_3), J_{\infty} \mathbb{A}^n) \leq \text{codim}(C, J_{\infty} \mathbb{A}^n) = \text{codim}(W^{\nu_3}(E_2), J_{\infty} \mathbb{A}^n) = q_2(k_{E_2}(\mathbb{A}^n) + 1) = k_{E_1}(Y) + 1 + \text{val}_{E_1}(\text{Jac}_X). \]

By assumption \( m_{X,P}^{-1} \subset \text{Jac}_X \),
\[ \text{val}_{E_1}(\text{Jac}_X) \leq (\nu - 1) \text{val}_{E_1}(P). \]

Thus
\[ a_{E_1}(\mathbb{A}^N \to \mathbb{A}^n, cZ^0 + (\nu - 1)P) = p(k_{E_3}(\mathbb{A}^n) + 1) - c \text{val}_{E_1}(Z^0) - (\nu - 1) \text{val}_{E_1}(P) \leq \frac{1}{a}(k_{E_1}(Y) + 1 + \text{val}_{E_1}(\text{Jac}_X) - c \text{val}_{E_1}(Z) - (\nu - 1) \text{val}_{E_1}(P)) \leq \frac{1}{a} a_{E_1}(Y, cZ) \leq 0. \]

The proof is done. \( \square \)

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