A posterior quantum dynamics for a continuous diffusion observation of a coherent channel

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We present the Belavkin filtering equation for the intense balanced heterodyne detection in a unitary model of an indirect observation. The measuring apparatus modelled by a Bose field is initially prepared in a coherent state and the observed process is a diffusion one. We prove that this filtering equation is relaxing: any initial square-integrable function tends asymptotically to a coherent state with an amplitude depending on the coupling constant and the initial state of the apparatus. The time-development of a squeezed coherent state is studied and compared with the previous results obtained for the measuring apparatus prepared initially in the vacuum state. © 2014 Optical Society of America

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1. Introduction

The theory of quantum measurements continuous in time is firmly based on quantum stochastic calculus (QSC) developed by Hudson and Parthasarathy [1, 2]. The time development of the posterior state conditioned by a trajectory of the results of a continuous measurement is given by the Belavkin filtering equation [3–6]. The measurement is taken on a Bose field interacting with the quantum system in question and enables one to perform its indirect observation. The Bose field can be treated as an approximation to the electromagnetic field. The filtering equation has the form of the Ito quantum stochastic differential equation and plays the rôle analogous to that of the Schrödinger equation for an unobserved quantum systems. The observed process has the properties of a diffusion or/and a counting one. The Belavkin filtering equation was obtained under the assumption that the Bose field modelling
the apparatus was initially prepared in the vacuum state. In [8–10] this assumption has been relaxed and the filtering equation has been derived for the Bose field prepared in a coherent state for the counting and diffusion observations.

The aim of this paper is twofold — we present the Belavkin equation for the diffusion observation with the apparatus prepared initially in a coherent state and we discuss the time development of a squeezed coherent state undergoing a balance heterodyne measurement [11]. In contrast to the result of [12], showing that for the apparatus initially prepared in the vacuum state a single and double heterodyne detection does not destroy the squeezed coherent state and drives the system asymptotically to the vacuum state, the state asymptotically relaxes to the coherent one with the amplitude independent of the initial state of the system. Next, we generalize this observation and prove that any square-integrable initial wave function relaxes to the coherent state with the given amplitude depending on the strength of the interaction between the system (single-mode field) and the apparatus and the initial state of the apparatus. Consequently, the considered filtering equation describes in fact a control of the quantum system by driving its state to the coherent one with the given amplitude.

Though the methods of QSC are neither widely used nor well-known by physicists, they deserve in our opinion more interest as the effective tools in modelling physical systems interacting with measuring devices. We refer the readers who not familiar with the theory of quantum measurements continuous in time to the books [13–15] or the recent papers [16–19]. More rigorous approach to the subject can be found in [20–23]. The experimental achievements in the area of continuous in time observations of quantum systems were reported, for instance, in [24–27].

The paper is organized as follows. In Section 2 we present the basic rules of quantum stochastic calculus. Section 3 is devoted to the presentation of a linear version of the Belavkin filtering equation for a balanced heterodyne detection of the diffusion type for the apparatus prepared initially in the coherent state. We choose the linear form of the filtering equation (for an unnormalized posterior wave function) instead of the nonlinear version derived in [12], because it is more convenient to deal with. The physical interpretation of the linear filtering equation one can find, for instance, in [22, 28]. We put forward here the approach of the generating map for the underlined continuous observation. In Section 4 we discuss the time development of a coherent and a squeezed coherent state evolving under a continuous diffusion observation of a coherent channel.

2. Quantum stochastic calculus

In this section we recall some basic rules of quantum stochastic calculus (QSC) in the boson Fock space [1, 2]. Denote by \( \mathcal{F} \) the (symmetrical) Fock space over the Hilbert space \( \mathcal{K} = \)
\( C^n \otimes L^2(\mathbb{R}_+) \) of all square integrable functions from \( \mathbb{R}_+ \) into \( C^n \). For any \( f \in \mathcal{K} \) one can define a coherent vector, \( e(f) \), by the formula
\[
e(f) = \exp \left( -\frac{1}{2} ||f||^2_k \right) (1, f, (2!)^{-1/2} f \otimes f, (3!)^{-1/2} f \otimes f \otimes f, \ldots) \, . \tag{1}
\]
In particular, \( e(0) = (1, 0, 0, \ldots) \in \mathcal{F} \) is the Fock vacuum. The annihilation, creation and number processes: \( B_j(t) \), \( B^\dagger_j(t) \) and \( A_{ij}(t) \) are defined on the dense in \( \mathcal{F} \) linear span of all coherent vectors as follows:
\[
B_j(t)e(f) = \int_0^t f_j(s) ds e(f) \, , \tag{2}
\]
\[
\langle e(g)|B^\dagger_j(t)e(f) \rangle = \int_0^t g_j(s) ds \langle e(g)|e(f) \rangle \, , \tag{3}
\]
\[
\langle e(g)|A_{ij}(t)e(f) \rangle = \int_0^t g_i(s)f_j(s) ds \langle e(g)|e(f) \rangle \, . \tag{4}
\]
These are the underlying processes for stochastic differential equations (QSDEs) of the Ito type:
\[
dM(t) = \sum_{j=1}^n \left( \sum_{i=1}^n F_{ji}(t) dA_{ji}(t) + E_j(t) dB_j(t) + D_j(t) dB^\dagger_j(t) \right) + C(t) dt \, . \tag{5}
\]
In (5) all the processes appearing at the Ito differentials are adapted processes on \( \mathcal{H} \otimes \mathcal{F} \), i.e. they depend on the processes up to \( t \) (present instant) and commute with the Ito differentials that “point to the future”. If \( M(t) \) is the process which satisfies an equation of the type (5), then the differential of the product \( M(t)M'(t) \) is given by the formula
\[
d(M(t)M'(t)) = dM(t)M'(t) + M(t)dM'(t) + dM(t)dM'(t) \, . \tag{6}
\]
The term \( dM(t)dM'(t) \) can be computed with the help of the multiplication table:
\[
dB_i(t) dB^\dagger_j(t) = \delta_{ij} dt \, , \quad dB_i(t) dA_{kj}(t) = \delta_{ik} dB_j(t) \, , \quad dA_{kj}(t) dB^\dagger_i(t) = \delta_{ji} dB^\dagger_k(t) \, , \quad dA_{ij}(t)dA_{kl}(t) = \delta_{jk} dA_{il}(t) \, , \tag{7}
\]
and all other products vanish.

3. Linear filtering equation for a balanced heterodyne scheme

Let us consider a harmonic oscillator (system \( S \)) interacting with an environment modelled by the Bose field in a coherent state \( e(f) \). We assume that the unitary evolution operator
\(U(t)\) of the compound system (system \(S\) plus one-dimensional Bose field) satisfies the QSDE \([20]\):
\[
dU(t) = \left[ \sqrt{\mu} a \, dB^\dagger(t) - \sqrt{\mu} \, a^\dagger \, dB(t) - \frac{\mu}{2} a^\dagger a \, dt - i \frac{\hbar}{\mu} H dt \right] U(t), \quad U(0) = I, \tag{8}
\]
where \(H = \hbar \omega (a^\dagger a + \frac{1}{2})\) is the Hamiltonian of \(S\), \(a\) is an annihilation operator, and \(\mu \in \mathbb{R}_+\) stands for a real coupling constant. Eq. \((8)\) is written in the interaction picture with respect to the free dynamics of the Bose field. The description of physical assumptions leading to this evolution can be found, for instance, in \([20, 29]\). In short, the coupling is linear in the field operators, the rotating-wave approximation (RWA) is made, the coupling constant are independent of frequency, and the spectrum of the reservoir is flat and broad. These assumptions are often made in quantum optics.

Though the Bose field disturbs the free evolution of \(S\), it also enables an indirect observation of \(S\) continuous in time. The input processes \(B(t), B^\dagger(t)\) refer to the field before its interaction with \(S\), whereas the output processes \(B^{\text{out}}(t) = U^\dagger(t)B(t)U(t), B^{\text{out}\dagger}(t) = U^\dagger(t)B^\dagger(t)U(t)\) describe the field after the interaction with \(S\).

In a balance heterodyne measurement depicted in Fig. 1 the output field, escaping from the cavity, is mixed with a strong laser field \(B_{lo}(t)\) (local oscillator). We assume that this auxiliary field, which does not interact with \(S\), is initially in a coherent state \(e(f_{lo})\) \([20, 30]\). In the paper we consider the filtering equation corresponding to the observation of the difference of photocurrents generated by the detectors monitoring the fields:
\[
B_1(t) = \frac{1}{\sqrt{2}} \left( B^{\text{out}}(t) + B_{lo}(t) \right), \quad B_2(t) = \frac{1}{\sqrt{2}} \left( B^{\text{out}}(t) - B_{lo}(t) \right). \tag{9}
\]
To derive the linear stochastic equation one can use, for example, the Belavkin method of generating functional \([3]\). The generating map, \(g(k, t)\), defined by \([3, 20]\)
\[
g(k, t) = g(k, t)[Z], \quad \langle \psi | g(k, t)[Z] | \psi \rangle = \langle \psi \otimes e(f) | G^{\text{out}}(k, t)Z_t \psi \otimes e(f) \rangle, \tag{10}
\]
where
\[
G^{\text{out}}(k, t) = \langle e(f_{lo}) | e(f_{lo}) \exp \left\{ \int_0^t \varepsilon k(t') \left( dA_{11}(t') - dA_{22}(t') \right) \right\} e(f_{lo}) \rangle, \tag{11}
\]
completely determines the observed process up to time \(t\). Here \(Z_t = U^\dagger(t)ZU(t)\) is the Heisenberg operator of \(S\), \(\psi\) stands for the initial state of \(S\), \(k\) is any integrable \(c\)-valued function, and \(\varepsilon^{-1} = |f_{lo}|\). In the limit \(\varepsilon \to 0\) of a very intense local oscillator field, the formula \((11)\) takes the form
\[
G^{\text{out}}(k, t) = \exp \left\{ \int_0^t k(t') \, dQ^{\text{out}}(t') \right\}, \tag{12}
\]

where the output process $Q^\text{out}$ reads

$$Q^\text{out}(t) = \int_0^t \left( e^{i\phi(t')} dB^\text{out\dagger}(t') + e^{-i\phi(t')} dB^\text{out}(t') \right),$$  

\[13\]

$\phi(t) = \arg f^\text{lo}_\nu(t)$. By using [5] and the Ito formula [6] one obtains

$$dQ^\text{out}(t) = e^{i\phi(t)} dB^\dagger(t) + e^{-i\phi(t)} dB(t) + \sqrt{\mu} \left( e^{i\phi(t)} a^\dagger + e^{-i\phi(t)} a_t \right) dt$$  

\[14\]

and $(dQ^\text{out}(t))^2 = dt$. An explicit expression for the generating map [10] one can find by solving the differential equation for $g(k, t)$. Using the method described in [3], one can check that the generating map $g(k, t)$ satisfies the equation

$$\frac{d}{dt} g(k, t)[Z] = g(k, t) \left[-(K + \sqrt{\mu} a^\dagger f(t) - \sqrt{\mu} \overline{a f(t)})^\dagger Z + \mu a^\dagger Z a \right.$$  

$$- Z(K + \sqrt{\mu} a^\dagger f(t) - \sqrt{\mu} \overline{a f(t)}) + \frac{1}{2} k^2(t) Z$$  

$$+ k(t)(\sqrt{\mu} a^\dagger + \overline{f(t)}) e^{i\phi(t)} Z + k(t)Z(\sqrt{\mu} a + f(t)) e^{-i\phi(t)} \right]$$  

\[15\]

with $g(k, 0)[Z] = Z$ and $K = \frac{1}{\hbar} H + \frac{\mu}{2} a^\dagger a$. The solution to the Eq. \[15\] can be written in the form

$$g(k, t)[Z] = \int_{\Omega^t} G(k, q^t) V^\dagger(q^t) Z V(q^t) \, d\nu(q^t),$$  

\[16\]

where $\nu$ is the probabilistic measure on the set $\Omega$ consisting of the continuous trajectories $q = [q(t)|t > 0]$ of the observed process $Q^\text{out}(t)$, restricted to the set $\Omega^t = \{q^t|q \in \Omega\}$ of the trajectories $q^t = [q(r)|r \leq t]$ up to $t$. The stochastic propagator $\hat{V}(t)(q^t) = V(q^t)$ satisfies the stochastic differential equation

$$d\hat{V}(t) = -\left( K + \sqrt{\mu} a^\dagger f(t) + \sqrt{\mu} a e^{-2i\phi(t)} f(t) \right) \hat{V}(t) \, dt + \sqrt{\mu} a e^{-i\phi(t)} \hat{V}(t) \, dQ(t), \quad \hat{V}(0) = I$$  

\[17\]

and

$$G(k, q^t) = G^\text{out}(k, t)(q) = \exp \left\{ \int_0^t k(t') \, dq(t') \right\}.$$  

\[18\]

Hence, the posterior unnormalized wave function $\hat{\psi}(t) = \hat{V}(t)\psi$ of the system $S$ satisfies the stochastic dissipative differential equation of the form

$$d\hat{\psi}(t) = -\left( K + \sqrt{\mu} \left( a^\dagger f(t) - a \overline{f(t)} \right) \right) \hat{\psi}(t) \, dt + \sqrt{\mu} a e^{-i\phi(t)} \hat{\psi}(t) \, dW(t), \quad \hat{\psi}(0) = \psi,$$  

\[19\]

where

$$dW(t) = dQ(t) - 2 \operatorname{Re} \left( e^{-i\phi(t)} f(t) \right) \, dt$$  

\[20\]
and \( W(t) \) is isomorphic to the standard Wiener process. Let us notice that the integral representation (30) of the generating map,

\[
\langle G^{\text{out}}(k, t) Z_t \rangle = \int \Omega G(k, q') \langle V(q') \psi | Z V(q') \psi \rangle \, d\nu(q'), \tag{21}
\]
gives for \( Z = I \) the mean value of the output process (12) as the generating function of the output probability measure

\[
d\zeta(q') = \langle V(q') \psi | V(q') \psi \rangle \, d\nu(q'). \tag{22}
\]

Therefore, the posterior wave function \( \psi(t) \) satisfying the filtering equation (19) is normalized to the probability density \( \langle V(q') \psi | V(q') \psi \rangle \) of the observed process with respect to the probability measure of the input process. From (21) we obtain the posterior mean value \( \langle Z \rangle(q') \) as

\[
\langle Z \rangle(q') = \langle \varphi(q') | Z \varphi(q') \rangle, \tag{23}
\]

where \( \varphi(t)(q) = \varphi(q') \) and \( \varphi(t) = \langle \hat{\psi}(t) | \hat{\psi}(t) \rangle^{-1/2} \hat{\psi}(t) \). For the normalized posterior wave function \( \hat{\varphi}(t) \) we get the nonlinear filtering equation

\[
d\hat{\varphi}(t) = -\left( \left( K + \sqrt{\mu} a^\dagger f(t) - \sqrt{\mu} a t f(t) + \mu \text{Re}^2(\langle a \rangle_t e^{-i\phi(t)})/2 \right) \hat{\varphi}(t) \, dt + \mu a e^{-i\phi(t)} \text{Re}(\langle a \rangle_t e^{-i\phi(t)}) \hat{\varphi}(t) \, dt + \left( \sqrt{\mu} a e^{-i\phi(t)} - \sqrt{\mu} a e^{-i\phi(t)} + \mu a e^{-i\phi(t)} \right) \hat{\varphi}(t) \, dt \times (dW(t) - 2\sqrt{\mu} \text{Re}(\langle a \rangle_t e^{-i\phi(t)} dt)), \tag{24}
\]

where \( \langle a \rangle_t = \langle \hat{\varphi}(t) | a \hat{\varphi}(t) \rangle \). And if the initial state of \( S \) is a mixed one, \( \hat{\rho}(0) = \rho \), then the posterior normalized density matrix \( \hat{\rho}(t) \) satisfies the nonlinear filtering equation of the form

\[
d\hat{\rho}(t) = \left( -\frac{i}{\hbar} [H, \hat{\rho}(t)] - \frac{\mu}{2} \left( a^\dagger a, \hat{\rho}(t) \right) + \left[ \sqrt{\mu} \, a \, t f(t), \hat{\rho}(t) \right] + \mu a \hat{\rho}(t) a^\dagger \right) \, dt + \left( \sqrt{\mu} a e^{-i\phi(t)} \hat{\rho}(t) + \hat{\rho}(t) \sqrt{\mu} a^\dagger e^{i\phi(t)} - 2 \sqrt{\mu} \text{Re}(\langle a \rangle_t e^{-i\phi(t)}) \right) \times (dW(t) - 2\sqrt{\mu} \text{Re}(\langle a \rangle_t e^{-i\phi(t)} dt)), \tag{25}
\]

where \( \langle a \rangle_t = \text{Tr}[\hat{\rho}(t)a] \). Eq. (25) is consistent with the result in [10].

4. Posterior evolution of a squeezed coherent state

We shall show that the coherent state survives the reduction of the state following the registered trajectory: the solution to Eq. (19) for the initial coherent state \( |\alpha_0\rangle \), \( \alpha_0 \in \mathbb{C} \), can be written in the form \( |\hat{\psi}(t)\rangle = l(t) |\alpha(t)\rangle \). Inserting the predicted solution into Eq. (19) and making use of the property

\[
a^\dagger |\alpha\rangle = \frac{\partial |\alpha\rangle}{\partial \alpha} + \frac{1}{2} \alpha |\alpha\rangle, \tag{26}
\]
on can write both sides of the equation in terms of linearly independent vectors $|\alpha\rangle$ and $\frac{\partial|\alpha\rangle}{\partial\alpha}$. Comparing the coefficients of the corresponding vectors one gets the consistent system of differential equations

$$d\alpha(t) = -\left(i\omega + \frac{\mu}{2}\right)\alpha(t)dt - \sqrt{\mu}f(t)dt,$$

(27)

$$\frac{dl(t)}{l(t)} = -\frac{i\omega}{2}dt - \frac{\mu}{2}|\alpha(t)|^2dt - \frac{3\sqrt{\mu}}{2}\alpha(t)f(t)dt$$

$$-\frac{\sqrt{\mu}}{2}\alpha(t)f(t)dt + \sqrt{\mu}\alpha(t)e^{-i\phi(t)}dW(t),$$

(28)

for the functions $\alpha(t)$ and $l(t)$ with the initial condition $\alpha(0) = \alpha_0$, $l(0) = 1$. To solve the stochastic equation for $l(t)$ one has to use the Ito rules and the formula

$$d\ln l(t) = \frac{1}{l(t)}dl - \frac{1}{2l^2(t)}(dl(t))^2.$$

(29)

Taking into account that $(dl(t))^2 = \mu \alpha^2(t)e^{-2i\phi(t)}dt$, we obtain the analytical solution of Eq. (19) for the initial coherent state in the form

$$|\hat{\psi}_\alpha(t)\rangle = \exp\left[-\frac{i\omega t}{2} - \sqrt{\mu}\int_0^t \left(\frac{\sqrt{\mu}}{2}|\alpha(t')|^2 + \alpha(t')f(t')\right)dt' - \frac{\sqrt{\mu}}{2}\alpha(t)f(t)dt'ight.$$

$$-\sqrt{\mu}\int_0^t \left(\text{Re}(\alpha(t')f(t')) + \frac{\sqrt{\mu}}{2}\alpha^2(t')e^{-2i\phi(t')}\right)dt'$$

$$+\sqrt{\mu}\int_0^t \alpha(t')e^{-i\phi(t')}dW(t')\right]\alpha(t)),$$

(30)

where the independent of the noise amplitude $\alpha(t)$ reads

$$\alpha(t) = \alpha_0 e^{-\left(i\omega + \frac{\mu}{2}\right)t} - \sqrt{\mu}\int_0^t e^{-\left(i\omega + \frac{\mu}{2}\right)(t-t')}f(t')dt'.$$

(31)

Since any initial state can be represented in the basis of the coherent states

$$|\psi(0)\rangle = \frac{1}{\pi} \int d^2\alpha_0 \langle \alpha_0 |\psi(0)\rangle |\alpha_0\rangle,$$

(32)

the linearity of Eq. (19) allows us to write the general solution as

$$|\hat{\psi}(t)\rangle = \frac{1}{\pi} \int d^2\alpha_0 \langle \alpha_0 |\psi(0)\rangle |\hat{\psi}_{\alpha_0}(t)\rangle.$$

(33)

The process of measurement changes the state of the system. When the system interacts with the external Bose field but the signal is not measured, the state in general becomes mixed. The nondemolition observation of the Belavkin type gives the opportunity to get
information about the system and allowing to retain some properties of the initial state, for example, its purity.

A coherent state is not the only one state invariant under the filtering equation (19). We shall prove that a squeezed coherent state is preserved under the considered observation as well. We will discuss the time development of the posterior wave function of $S$ for initial state of the form

$$|\hat{\psi}(0)\rangle = S(\xi_0)D(\alpha_0)|0\rangle = S(\xi_0)|\alpha_0\rangle = |\xi_0, \alpha_0\rangle,$$

where

$$D(\alpha_0) = \exp (\alpha_0 a^\dagger - \overline{\alpha}_0 a), \ \alpha_0 \in \mathbb{C}, \ (35)$$

and

$$S(\xi_0) = \exp \left( \frac{1}{2} \xi_0 a^2 - \frac{1}{2} \xi_0 (a^\dagger)^2 \right), \ \xi_0 = e^{i\theta_0} \overline{\alpha}_0 \in \mathbb{C}. \ (36)$$

The state is generated by displacing the vacuum state and then by squeezing. The amount of squeeze is described by $\rho_0 = |\xi_0|$ which is called the squeeze factor. Some details of description and detection of squeezed states of light one can find, for instance, in [11].

The method of computing a posterior dynamics for the initial squeezed coherent state by making a simple ansatz like in the previous case is cumbersome and laborious. Therefore, we shall use a more efficient method which allows to avoid arduous computation of a stochastic phase of posterior state.

We make use of the eigenvalue equation

$$S(\xi)aS^\dagger(\xi)|\xi, \alpha\rangle = \alpha |\xi, \alpha\rangle,$$

which can be readily found from the definition (34). The operator expansion theorem allows one to check that

$$S(\xi)aS^\dagger(\xi) = a \Gamma_1 + a^\dagger \Gamma_2,$$

where $\Gamma_1 = \cosh \varphi$, $\Gamma_2 = e^{i\theta} \sinh \varphi$. Let us notice that if the system $S$ remains in the squeezed coherent state at any time instant $t \geq 0$, then

$$S(\xi(t))aS^\dagger(\xi(t))\hat{\psi}(t) = \left[ a \Gamma_1(t) + a^\dagger \Gamma_2(t) \right] \hat{\psi}(t), \ (39)$$

and

$$S(\xi(t+dt))aS^\dagger(\xi(t+dt))\hat{\psi}(t+dt) = \left[ a \Gamma_1(t+dt) + a^\dagger \Gamma_2(t+dt) \right] \hat{\psi}(t+dt), \ (40)$$

where $|\hat{\psi}(t)\rangle = l(t)S(\xi(t))|\alpha(t)\rangle$, have to be fulfilled. Eqs. (39) and (40) can be reduced to a single condition of the form

$$\left[ a (\Gamma_1(t)+d\Gamma_1(t)) + a^\dagger (\Gamma_2(t)+d\Gamma_2(t)) - \alpha(t) - d\alpha(t) \right] d\hat{\psi}(t)
+ (a d\Gamma_1(t) + a^\dagger d\Gamma_2(t) - d\alpha(t)) \hat{\psi}(t) = 0. \ (41)$$
Finally inserting of the increment \(d\hat{\psi}(t)\) given by Eq. (19) into Eq. (11) we obtain the set of the two differential equations:

\[
\begin{align*}
\Gamma_2(t) \left[ -\Gamma_1(t) \left( i\omega + \frac{\mu}{2} \right) dt + \mu e^{-2i\phi(t)} \Gamma_2(t) dt + d\Gamma_1(t) \right] \\
-\Gamma_1(t) \left[ \Gamma_2(t) \left( i\omega + \frac{\mu}{2} \right) dt + d\Gamma_2(t) \right] &= 0,
\end{align*}
\]

(42)

\[
\begin{align*}
\alpha(t)\Gamma_1(t) \left[ -\Gamma_1(t) \left( i\omega + \frac{\mu}{2} \right) dt + \mu e^{-2i\phi(t)} \Gamma_2(t) dt + d\Gamma_1(t) \right] \\
-\alpha(t) \Gamma_2(t) \left[ \Gamma_2(t) \left( i\omega + \frac{\mu}{2} \right) dt + d\Gamma_2(t) \right] - \sqrt{\mu} \Gamma_1(t) f(t) dt \\
-\sqrt{\mu} \Gamma_2(t) f(t) dt - \sqrt{\mu} \Gamma_2(t) e^{-i\phi(t)} dW(t) - d\alpha(t) &= 0
\end{align*}
\]

(43)

with the initial condition: \(\Gamma(0) = \cosh \varrho_0\), \(\Gamma_2(0) = e^{i\varrho_0} \sinh \varrho_0\), \(\alpha(0) = \alpha_0\). The last step requires left-multiplying the Eq. (11) by \(S^\dagger(\xi(t))\) and use of the transformation

\[
S^\dagger(\xi) a S(\xi) = a \Gamma_1 - a^\dagger \Gamma_2,
\]

(44)

which one can easily get from (38). The Eqs. (42), (43) form a consistent set of equations and this completes the proof.

Eq. (42) can be rewritten in terms of the function \(\Gamma(t) = \Gamma_2(t)/\Gamma_1(t)\) as

\[
\begin{align*}
\frac{d}{dt} \Gamma(t) &= -2 \left( i\omega + \frac{\mu}{2} \right) \Gamma(t) + \mu e^{-2i\phi(t)} \Gamma^2(t).
\end{align*}
\]

(45)

The general solution to the Riccati differential equation (45) reads

\[
\Gamma(t) = \frac{\Gamma(0)e^{-(2i\omega+\mu)t}}{1 - \mu \Gamma(0) \int_0^t e^{-(2i\omega+\mu)t''-2i\phi(t'')} dt''}.
\]

(46)

In particular, for the phase \(\phi(t) = \pi/2 - \omega_0 t\), one gets

\[
\Gamma(t) = \frac{(2i\omega - 2i\omega_0 + \mu)\Gamma(0)}{e^{(2i\omega+\mu)t}[2i\omega - 2i\omega_0 + \mu(1 + \Gamma(0))] - \mu \Gamma(0)e^{2i\omega t}}.
\]

(47)

The integration of (43) yields the stochastic amplitude

\[
\begin{align*}
\alpha(t) &= \frac{1}{\sqrt{1 - |\Gamma(t)|^2}} \left[ \alpha_0 \sqrt{1 - |\Gamma(0)|^2} \\
&\times \exp \left( -i\omega t - \frac{\mu}{2} t + \mu \int_0^t e^{-2i\phi(t')} \Gamma(t') dt' \right) \\
&- \sqrt{\mu} \int_0^t \exp \left( - \left( i\omega + \frac{\mu}{2} \right) (t - s) + \mu \int_s^t e^{-2i\phi(t')} \Gamma(t') dt' \right) \\
&\times \left( f(s) ds + \Gamma(s) \overline{f(s)} ds + e^{-i\phi(s)} \Gamma(s) dW(s) \right) \right].
\end{align*}
\]

(48)
Therefore the posterior mean values of the optical quadratures $X = (a + a^\dagger)/2$ and $Y = (a - a^\dagger)/2i$ for the initial squeezed coherent state given as
\[
\langle X \rangle_t = \frac{\text{Re} \left(\alpha(t) - \bar{\alpha}(t)\Gamma(t)\right)}{\sqrt{1 - |\Gamma(t)|^2}},
\]
\[
\langle Y \rangle_t = \frac{\text{Im} \left(\alpha(t) - \bar{\alpha}(t)\Gamma(t)\right)}{\sqrt{1 - |\Gamma(t)|^2}}
\]
depend on the measurement noise, whereas the uncertainties of quadratures
\[
\Delta X(t) = \left(4\text{Re}\kappa(t)\right)^{-1/2},
\]
\[
\Delta Y(t) = |\kappa(t)| \left(4\text{Re}\kappa(t)\right)^{-1/2},
\]
where
\[
\kappa(t) = \frac{1 + \Gamma(t)}{1 - \Gamma(t)}
\]
remain deterministic. Moreover, the expressions for the mean value of optical quadratures include the parameter of the initial coherent state of Bose field, whereas the formulae for the uncertainties are exactly the same as for the case when the Bose field is initially in the vacuum $|0\rangle$.

The time dependence of the uncertainties $\Delta X$ and $\Delta Y$ has been illustrated by the parametric plots presented in Fig. 2. They show the dynamics of $\Delta X$ and $\Delta Y$ as functions of the dimensionless time $\tau = \omega t$ ($0 \leq \tau \leq 100$), for $\omega_0 = 0.05$, $\Gamma(0) = 0.8$ and three values of $\mu$. The uncertainties oscillate approaching the asymptotic values ($\Delta X = \Delta Y = 1/2$). The larger parameter $\mu$ the asymptotic values are reached faster.

In order to study the probability density of the output process $\hat{\psi}(t)$ one has to find the norm $||\hat{\psi}(t)|| = |l(t)|$. One can check that
\[
\frac{d||l(t)||^2}{||l(t)||^2} = \frac{2\sqrt{\mu} \text{Re} \left[\left(\alpha(t) - \bar{\alpha}(t)\Gamma(t)\right) e^{-i\phi(t)}\right]}{\sqrt{1 - |\Gamma(t)|^2}} dW(t).
\]
Hence
\[
||l(t)||^2 = \exp \left\{2\sqrt{\mu} \int_0^t \frac{\text{Re} \left[\left(\alpha(t') - \bar{\alpha}(t')\Gamma(t')\right) e^{-i\phi(t')}\right]}{\sqrt{1 - |\Gamma(t')|^2}} dW(t') \right. \\
-2\mu \int_0^t \left. \frac{\text{Re} \left[\left(\alpha(t') - \bar{\alpha}(t')\Gamma(t')\right) e^{-i\phi(t')}\right]}{1 - |\Gamma(t')|^2} dt' \right\}.
\]
Let us stress that the formula gives the probability of observed result $q^t \in \Omega^t$. 

10
We should remark that, due to \( \lim_{t \to \infty} \Gamma(t) = 0 \), the system asymptotically approaches the coherent state with the amplitude

\[
\alpha(t) = -\sqrt{\mu} \int_0^t e^{-(i\omega + \frac{\mu}{2})(t-t')} f(t') \, dt'.
\] (56)

Hence, any memory about the initial condition is lost after a transitional period. By the asymptotic posterior state for any initial state is the coherent state with the amplitude \( (56) \).

5. Final remarks

We have presented the derivation of the Belavkin filtering equation for a single-mode field in a cavity interacting with the Bose field (measuring apparatus) initially prepared in a coherent state. The considered balanced heterodyne observation is intense and is therefore considered as a diffusion one. In contrast to the case of the measuring apparatus prepared initially in the vacuum state, when the initial squeezed coherent state is preserved and driven to the vacuum asymptotic state, in the case studied in this paper the initial squeezed coherent state is asymptotically driven to the coherent one. This asymptotic state does not depend on the initial parameters of the initial state of the field in the cavity. Moreover, we have proved that any initial state described by a square integrable wave function relaxes to the coherent state, with the amplitude dependent on the coupling constant and the initial coherent state of the apparatus. The driving force from the field can control the system and drive its state to the coherent one with the given amplitude.

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Fig. 1. Balanced heterodyne detection

Figure captions
Fig. 2. The dependence of $\Delta X$ and $\Delta Y$ on the dimensionless time $\tau = \omega t$ is displayed for $\omega_0 = 0.05$, $\Gamma(0) = 0.8$ and for three values of $\mu$: 0.01 (a), 0.04 (b), and 0.08 (c).