Worldsheet CFTs for Flat Monodrofolds

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\textbf{Abstract}

We resolve a puzzle in the theory of strings propagating on locally flat spacetimes with nontrivial Wilson lines for stringy $\mathbb{Z}_N$ gauge symmetries. We find that strings probing such backgrounds are described by consistent worldsheet CFTs. The level mismatch in the twisted sectors is compensated by adjusting the quantization of momentum of strings winding around the Wilson line direction in units of $1/RN^2$ rather than $1/RN$, as might have been classically expected. We demonstrate in various examples how this improvement of the naive orbifold prescription leads to satisfaction of general physical principles such as level matching and closure of the OPE. Applying our techniques to construct a Wilson line for T-duality of a torus in the type II string ("T-fold"), we find a new 7D solution with $\mathcal{N} = 1$ SUSY where the moduli of the fiber torus are fixed. When the size of the base becomes small this simple monodrofold exhibits enhanced gauge symmetry and a self-T-duality on the $S^1$ base.

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1 Introduction

Worldsheet conformal field theories (CFTs) based on free fields represent the simplest type of weakly coupled string theory. The simplest of all are the cases in which the target space has no invariants which distinguish it from flat space. This set of examples includes Narain compactification, but is not limited to it. For instance, the Klein bottle can be thought of as an $S^1_{\text{fiber}}$ fibered over $S^1_{\text{base}}$, with an identification of the fiber under reflection as one traverses the base. Following particle physics terminology, we will refer to such parallel transport, which is locally trivial but nontrivial around noncontractable cycles in the base, as a Wilson line. In string theory, such constructions have been referred to as $T$-folds, monodrofolds, or when the fiber and base are tori, as twisted tori. [1–4, 7, 8, 11–13]

It is widely believed that any exact symmetry of string theory must be a gauge rather than a global symmetry, including discrete symmetries not embedded in any continuous group. The defining property of a gauge symmetry $g$ is that it is a local redundancy of description, which is to say, it should be possible to parallel transport an object around a closed loop and find that it comes back to itself up to a transformation by $g$. In the case that the loop is noncontractable, there are no local observables which can distinguish such a parallel transport law from the trivial one, and hence no possible nonzero contributions to a tree-level energy for such a configuration. $^1$

From this we conclude the following:

Given any string theory $\mathcal{W}$ in $\mathbb{R}^D$ with an unbroken internal symmetry $g$, there exists a Wilson line background of $\mathcal{W}$ on $\mathbb{R}^{D-1} \times S^1$ in which states are identified with themselves up to $g$ when parallel transported around the $S^1$.  

If the original string theory $\mathcal{W}$ has a weak coupling expansion, and we consider a symmetry $g$ which commutes with the limit $g_s \to 0$, then the resulting Wilson line theory should also have a weak coupling expansion characterized by a fundamental string worldsheet CFT.

When the “fiber” is a torus (namely, the internal worldsheet theory $\mathcal{I}$ of $\mathcal{W}$ is a free CFT), we can be even more precise about the properties such a solution should have. The $\alpha'$ expansion of the $\beta$-function equations is a derivative expansion in the $\alpha'$

Quantum effects such as Casimir forces can generate nonzero energies for Wilson line backgrounds, but we will focus in this paper on tree-level physics.
target space. There are no invariants in any patch of space which differentiate between the Wilson line background and flat space. It follows that any Wilson line background must be represented by a worldsheet CFT which locally in target space is a free CFT.

To learn more about the nature of this free CFT, observe that the Wilson line background is in some sense a quotient of a trivial background of $\mathcal{W}$ on $\mathbb{R}^{D-1} \times S'$, where $S'$ is a covering circle. Fix the size of the circle to be $2\pi R_{\text{base}}$, parametrized by the coordinate $X$. To the extent that string degrees of freedom are (approximately) local fields in $X$ space, we should be able to describe the Wilson line for a $\mathbb{Z}_N$ symmetry as a quotient of an $N$-times bigger covering circle with radius $R' = NR_{\text{base}}$ by a $\mathbb{Z}_N$ which acts as a shift $g_X : X \rightarrow X + 2\pi R_{\text{base}}$ by one $N^{th}$ the size of the covering circle, combined with the action of $g$ on all fields.

There is an argument—not quite precise—suggesting such a quotient should be accessible through the famous orbifold construction [18]. Certainly the projection onto gauge-invariant states is the correct one for strings which are approximately local fields in $X$ rather than extended objects. That is to say, starting from $I \times S'$ and imposing invariance under $g \times g_X$ projects onto states whose $X$-dependence is $\exp\{i(n+\alpha)X/R_{\text{base}}\}$ for some $n$ and whose phase under $g$ is $\exp\{-2\pi i\alpha\}$. So (approximately) local fields get parallel transported around the circle with a phase depending on their $g$-transformations. This is precisely the definition of a Wilson line.

In the twisted sectors, however, the naive orbifold construction of the Wilson line background can begin to break down. If $g$ acts asymmetrically on left- and right-moving degrees of freedom, sectors twisted by $g$ can have a level mismatch $L_0 - \tilde{L}_0$ which cannot in general be removed by acting with oscillators on the twisted ground states. We will give examples of the breakdown in the next section, but suffice it to say that the failure of level matching is a true, nonempty constraint on the orbifold construction: not every exact symmetry $g$ gives rise to a consistent, modular invariant theory through the orbifold construction.

Combining $g$ with $g_X$ does not cure the problem (see [3]). That is to say, orbifolding $I \times S'$ by $g \times g_X$ yields a modular invariant theory if and only if orbifolding by $g$ alone does so. This presents a paradox, because the argument for the existence of a Wilson line background is based on what should be an ironclad principle in string theory:

**Consistent boundary conditions for consistent string theories should yield new consistent string theories.**

We refer to this as the ‘consistent + consistent = consistent’ principle, or CCC principle.
We will see in this paper that the CCC principle is in fact vindicated on the string worldsheet. Namely, we will show that there always exists a consistent choice of momentum fractionation in the twisted sectors such that level matching, modular invariance and closure of the OPE can be maintained. That is, by allowing the momentum in the base to have certain values not necessarily satisfying $N p_{\text{base}} R_{\text{base}} \in \mathbb{Z}$, we will restore level matching and thereby save modular invariance. This works because the twisted sectors also carry winding in the $X$ direction, so the nonzero momentum and winding contribute to the level mismatch $\tilde{L}_0 - L_0$ by an amount $-p_{\text{base}} R_{\text{base}} w_{\text{base}}$ which precisely cancels the level mismatch due to asymmetric Casimir energies. This is in distinction to a standard $\mathbb{Z}_N$ orbifold construction, in which level mismatch is restricted to be in $\mathbb{Z}/N$ and can be canceled by adding oscillator energy. The important point is that both when the level mismatch is in $\mathbb{Z}/N$ and when it is not (so the standard orbifold prescription will apply), we can construct a Wilson line background for the symmetry, $g$. We will refer to this procedure as “wilsonization”, and distinguish the “tame” case which does not require momentum fractionation beyond $1/R_{\text{base}} N$ from the more general, “wild”, case.

It should also be noted that even when momenta are fractionated in units of $1/R N^2$ in the twisted sectors, in any given twisted sector, the base momenta differ only by multiples of $1/R N$. In this way, it is possible to reconcile the additional momentum fractionation with the closure of the OPE.

The presence of fractional momentum with $p_{\text{base}}$ not in $\mathbb{Z}/N R$ would seem to mean that the modes carrying such momentum are non-single-valued on the covering space. However this is not quite the right conclusion. These modes also carry winding, and so for large $R$ we should think of them as long winding strings with excitations on them carrying momentum along the direction of the string. In fact we shall see that the consistent treatment of the effective theory on the long string not only allows but demands the inclusion of sectors with $p_{\text{base}}$ not in $\mathbb{Z}/N R$.

We also have to wonder how this momentum fractionation can be reconciled with the naive expectation that the Wilson line theories should be accessible by orbifolding. As we will show, one can indeed reformulate our “wilsonization of $g$ over $S^1$” as an orbifoldization. What this requires is to let the $\mathbb{Z}_N$ action depend by a phase on the winding around the covering circle $S^1'$ (which is winding mod $N$ around the base circle $S^1$). In other words, the orbifold action on $I \times S^1'$ does not factorize between base and
fiber. The inclusion of these extra phases in the orbifold action is somewhat reminiscent of discrete torsion (by viewing winding around $S^1'$ as the twist quantum number of yet another $\mathbb{Z}_N$ orbifold). But it should be remembered that discrete torsion in orbifolds is usually a relative and not an absolute concept whereas in the present case we do not have a choice in picking these phases. In this sense, the wilsonization technique we will introduce in this paper can be viewed either as a refinement or as a generalization of the standard orbifold construction.

The outline of the paper is as follows. We will begin in the next section by studying some rather simple examples in the bosonic string, taken from both the “tame” and “wild” category. The first of those examples involves 32 free nonchiral fermions as the fiber theory and various discrete symmetries in $\text{Spin}(32)_L$ as the symmetry implemented by the Wilson line. The second example is a Wilson line for T-duality of a single circle at the self-dual radius. We will also study an example in which the fiber theory is not built on free fermions or bosons, but consists of an $SU(2)$ WZW model. This serves to illustrate that our methods are applicable also for fibering more general rational conformal field theories.

In section 3, we will then outline a general framework in which to understand wilsonization as a new worldsheet construction. This method is then the worldsheet realization of the CCC principle, and we hope that if the reader’s confidence in the CCC principle was ever unsettled, before the end of section 3 it will be completely restored.

Once the wilsonization technique is firmly established, we will then more leisurely present an application to the superstring in section 4. Specifically, we will construct a 7-dimensional background of the type II superstring with $\mathcal{N} = 1$ supersymmetry which is not equivalent to any geometric string theory. This example implements the transformation $\tau \to -\frac{1}{\tau}, \rho \to -\frac{1}{\rho}$ on a $T^2$ fiber as a Wilson line around an $S^1$ base. Sixteen supercharges are preserved by the Wilson line, which acts nontrivially only on the gravitini coming from left-moving spin fields on the string.

This example is interesting in itself, mainly because of its scarcity of massless fields. But we wish to reemphasize that the chief purpose of this paper is to show that the CCC principle is beyond any doubt satisfied in string theory and is useful for the construction of new, in general non-geometric, string backgrounds. In simple situations this can be verified explicitly at the worldsheet level.

We conclude the paper in section 5.
2 Wilson line theories in the bosonic string

Before laying out a general theory of string theory Wilson lines, let us study some examples to get the general idea. The bosonic string is the simplest arena in which to construct Wilson line solutions both of the wild and tame kind.

2.1 Chiral $\mathbb{Z}_2$ symmetries in a free fermion theory

Consider the bosonic string on $\mathbb{R}^{8,1} \times S^1 \times \mathcal{I}$, where $\mathcal{I}$ is an internal theory of $c = 16$. We will take the internal theory $\mathcal{I}$ to consist of an $SO(32)_L \times SO(32)_R$ current algebra $\mathcal{C}$ at level one, realized as 32 free fermions of each chirality. We will let the $(-1)^{F_W}$ projection on fermions be the maximally chiral one – that is, we restrict to states with left- and right-moving worldsheet fermion number separately even: $(-1)^{F_{RW}} = (-1)^{F_{LW}} = +1$, and allow right- and left-moving SO(32) spinor sectors independently.

Given our fiber theory $\mathcal{I}$, we wish to consider Wilson lines around $x_9$ for discrete symmetries $g$ of $\mathcal{I}$. For simplicity we will always consider $g$ to be $\mathbb{Z}_2$ in its action on fermions. (This will sometimes mean that it is $\mathbb{Z}_4$ in its action on SO(32) spinor states.)

Let us now analyze some of the simplest examples. We can classify the possible actions by the numbers $k_L, k_R$ of worldsheet fermions which are odd under the Wilson line symmetry $g$. To simplify the problem further we consider the case $k_R = 0$ where the action on all right-moving chiral operators is trivial.

The exact symmetry group of the theory is $Spin(32)_L/\mathbb{Z}_2 \times Spin(32)_R/\mathbb{Z}_2$.

2.1.1 The case $k_L = 0$

The simplest $\mathbb{Z}_2$ symmetry which is purely chiral is $-1$ to the total left-moving SO(32) spinor number. This is the quantum symmetry which acts with a $-1$ on sectors with periodic left-moving fermions. This corresponds to $k_L = 0$, because all the fermion fields $\tilde{\lambda}$ are even.

Let us try to construct the partition function by hand. The partition function for 32 free left-moving fermions is

$$\frac{1}{2}(I_0^0 + I_0^1)$$

(2.1)

A sector with periodic fermions is often called a 'Ramond' boundary condition for the fermions, but to avoid confusion of terminology we shall only use the term 'Ramond' when we study the superstring, to describe the boundary conditions of the worldsheet supercurrents.
in the SO(32) tensor sector and

$$\frac{1}{2} (I_0^0 + I_1^1)$$  \hspace{1cm} (2.2)$$

in the SO(32) spinor sector. Here, \(I_q^p\) is the path integral for fermions on the torus where the periodicity is \(\tilde{\lambda} \rightarrow (-1)^{q+1} \tilde{\lambda}\) on the spacelike cycle, with a factor of \((-1)^{pF_{LW}}\) inserted into the trace, which translates to a periodicity of \(\tilde{\lambda} \rightarrow (-1)^{p+1} \tilde{\lambda}\) on the timelike cycle. The numbers \(p, q\) are only meant to be taken mod 2.

**Partition functions for fermions**

In defining our partition functions it is useful to start by defining partition functions for complex fermions with various boundary conditions:

$$\tilde{F}_0^0(\bar{\tau}) \equiv \frac{1}{\eta(\bar{\tau})} \alpha_{00}(0, \bar{\tau})$$

$$\tilde{F}_0^1(\bar{\tau}) \equiv \frac{1}{\eta(\bar{\tau})} \alpha_{01}(0, \bar{\tau})$$

$$\tilde{F}_1^0(\bar{\tau}) \equiv \frac{1}{\eta(\bar{\tau})} \alpha_{10}(0, \bar{\tau})$$

$$\tilde{F}_1^1(\bar{\tau}) \equiv \frac{1}{\eta(\bar{\tau})} \alpha_{11}(0, \bar{\tau})$$  \hspace{1cm} (2.3)$$

The last of the four vanishes due to fermion zero modes. However we will treat this function as nonzero and demand modular invariance without using its vanishing, since the modular invariance of the partition function must hold even when the zero modes are lifted by the insertions of local operators. These functions have the modular transformation properties

$$\tilde{F}_0^0(\bar{\tau} + 1) = \exp\{-\frac{\pi i}{12}\} \tilde{F}_0^0(\bar{\tau})$$  \hspace{1cm} (2.4)$$

$$\tilde{F}_0^1(\bar{\tau} + 1) = \exp\{-\frac{\pi i}{12}\} \tilde{F}_0^1(\bar{\tau})$$  \hspace{1cm} (2.5)$$

$$\tilde{F}_1^0(\bar{\tau} + 1) = \exp\{+\frac{\pi i}{4} - \frac{\pi i}{12}\} \tilde{F}_1^0(\bar{\tau}) = \exp\{+\frac{\pi i}{4}\} \tilde{F}_1^0(\bar{\tau})$$  \hspace{1cm} (2.6)$$

$$\tilde{F}_1^1(\bar{\tau} + 1) = \exp\{+\frac{\pi i}{4} - \frac{\pi i}{12}\} \tilde{F}_1^1(\bar{\tau}) = \exp\{+\frac{\pi i}{4}\} \tilde{F}_1^1(\bar{\tau})$$  \hspace{1cm} (2.7)$$
\[
\tilde{F}_0^0(-\frac{1}{\tau}) = \tilde{F}_0^0(\bar{\tau}) \quad (2.8)
\]
\[
\tilde{F}_1^1(-\frac{1}{\tau}) = -i\tilde{F}_1^1(\bar{\tau}) \quad (2.11)
\]

We have assigned a phase of \(-i\) to the S-transformation of \(\tilde{F}_1^1\) in order to reproduce the S-transformation of the partition function for a complex left moving fermion with insertions. The phase can be computed with a single insertion of \(\tilde{\lambda}_8\tilde{\lambda}_9\), a weight \((1, 0)\) operator which lifts both fermion zero modes. Since the S-transformation is implemented by a 90 degree rotation, it acts with an extra phase of \(i\tilde{h} - \tilde{h}\) on an operator of weight \((\tilde{h}, h)\). This combines with a \(-\) sign from the \(\nu \to 0\) limit of the modular transformation of \(\frac{1}{\eta(\tau)}\alpha_{11}(n, \tau)\) to give an overall phase of \(-i\) in the modular S-transformation of \(\tilde{F}_1^1\).

**Fiber partition functions**

Explicit forms for \(I_{pq}^p\) are given by the theta functions

\[
I_0^0 = q^{-\frac{16}{\Delta}} \prod_{m=1}^{\infty} (1 + q^{m-\frac{1}{2}})^{32} \equiv \left( \frac{\alpha_{00}(0, \bar{\tau})}{\eta(\bar{\tau})} \right)^{16} \equiv \left( \tilde{F}_0^0(\tau) \right)^{16} \quad (2.12)
\]

\[
I_0^1 = q^{-\frac{16}{\Delta}} \prod_{m=1}^{\infty} (1 - q^{m-\frac{1}{2}})^{32} = \left( \frac{\alpha_{01}(0, \bar{\tau})}{\eta(\bar{\tau})} \right)^{16} \equiv \left( \tilde{F}_1^0(\tau) \right)^{16}
\]

\[
I_1^0 = q^\frac{16}{\Delta} \prod_{m=0}^{\infty} (1 + q^m)^{32} = \left( \frac{\alpha_{10}(0, \bar{\tau})}{\eta(\bar{\tau})} \right)^{16} \equiv \left( \tilde{F}_0^1(\tau) \right)^{16}
\]

\[
I_1^1 = q^{-\frac{16}{\Delta} + \frac{32}{\Delta}} \prod_{m=0}^{\infty} (1 - q^m)^{32} = 0 \equiv \left( \tilde{F}_1^1(\tau) \right)^{16}
\]

\(Z^{(R)}\) is the right-moving partition function, which is modular invariant up to the local gravitational anomaly contribution to the \(T\) transformation:

\[
Z^{(R)}(\tau + 1) = \exp\{-\frac{2\pi i}{3}\} Z^{(R)}(\tau) \quad (2.13)
\]

\[
Z^{(R)}(-\frac{1}{\tau}) = Z^{(R)}(\tau)
\]
The pieces $I^p_q$ have their 'classical' modular transformations, up to the contribution to the $T$ transformations:

$$I^p_q(\tau + 1) = \exp\left\{\frac{2\pi i}{3}\right\}I^{p+q+1}_q(\tau)$$

$$I^p_q(-\frac{1}{\tau}) = I^{-p}_q(\tau)$$

(2.14)

These can be computed from the modular transformations of the $\tilde F^p_q$, given above.

**Base partition functions**

Now we want to correlate the periodicity of states in the base with their SO(32) spinor number by combining the $I^p_q$'s with corresponding path integral sectors of the base. So we let $Y^a_b$ be the partition function for the $S^1$ base factor $X^9$ in the sector of the path integral where the $X^9$ winds through $2\pi a R_{\text{base}}$ around the timelike cycle of the worldsheet and $2\pi b R_{\text{base}}$ around the spacelike cycle. So in particular, the partition function in the sector with winding $b$ and momentum $n_{\text{base}} \equiv p_{\text{base}} R_{\text{base}}$ equal to $-\alpha \mod 1$ is given by

$$Y_b^{< -\alpha >} \equiv \sum_a \exp\{2\pi i a \alpha\} Y^a_b = |\eta(\tau)|^{-2} \sum_{n = -\alpha \mod 1} q^{\frac{1}{2} n_L^2} q^{\frac{1}{2} n_R^2}$$

(2.15)

where

$$p_L = \frac{n}{R_{\text{base}}} - R_{\text{base}} b \quad p_R = \frac{n}{R_{\text{base}}} + R_{\text{base}} b$$

(2.16)

are left- and right-moving momenta, respectively, and $q = e^{2\pi i \tau}$, as usual. For instance, $\alpha = \frac{1}{2}$ would correspond to the partition function over states with half-integral momentum on the base in units of $\frac{1}{R_{\text{base}}}$.

By inverting (2.15), one obtains the explicit expression

$$Y^a_b = |\eta(\tau)|^{-2} \int dn \ e^{2\pi i a \alpha} q^{\frac{1}{2} n_L^2} q^{\frac{1}{2} n_R^2} = |\eta(\tau)|^{-2} \frac{1}{\sqrt{\tau}} e^{(-\pi a^2 - \pi |\tau| b^2 - 2\pi i a b) R_{\text{base}}^2 / \tau}$$

(2.17)

for the $Y^a_b$'s. The functions $Y^a_b$ also have classical modular transformations:

$$Y^a_b(\tau + 1) = Y^{a+b}_b(\tau)$$

$$Y^a_b(-\frac{1}{\tau}) = Y^{-a}_b(\tau)$$

(2.18)
Full partition function

With these ingredients, the partition function in the untwisted (unwound) sector of the Wilson line theory can be written as

\[ Z = \frac{1}{2} Z^{(R)} \sum_a Y^a_0 (I^0_0 + I^1_0) + \frac{1}{2} \sum_a (-1)^a Y^a_0 (I^0_1 + I^1_1) \]  

(2.19)

For \( a \) even, this transforms unproblematically to itself under the modular \( T \) transformation \( \tau \to \tau + 1 \) and the modular \( S \) transformation \( \tau \to -1/\tau \). If we write the full partition function in the form \( \sum_{a,b} Y^a_b f^a_b \), then

\[ f^a_0 = \frac{1}{2} Z^{(R)} (I^0_0 + I^1_0 + (-1)^a I^0_1 + (-1)^a I^1_1) \]  

(2.20)

In order for modular invariance to be maintained, the functions \( f^a_0 \) must also transform classically under modular transformations. So

\[ f^0_b = \frac{1}{2} Z^{(R)} (I^0_0 + (-1)^b I^1_0 + I^0_1 + (-1)^b I^1_1) \]  

(2.21)

By applying modular transformations we find that

\[ f^1_0 = \frac{1}{2} Z^{(R)} (I^0_0 - I^1_0 + I^0_1 - I^1_1) \]  

(2.22)

and

\[ f^1_1 = \frac{1}{2} Z^{(R)} (-I^0_0 + I^1_0 + I^0_1 - I^1_1) \]  

(2.23)

One can check that \( f^a_b \) will depend on \( a \) and \( b \) only mod 2, so we define \( f^{a+2k}_{b+2l} = f^a_b \). Given the modular transformations of the \( I^p_q \), the symbols \( f \) then have classical transformation laws mod 2 as well. Fourier transforming back from \( a \) to \( \alpha = 0, \frac{1}{2} \), we then find the partition function in the odd winding sector is

\[ Z^{(R)} \left( Y^+_b I^-_1 + Y^-_b I^+_1 \right) \]  

(2.24)

where \( Y^\pm_b \equiv \sum_a (\pm)^a Y^a_b \) and \( I^\pm_b \) is defined similarly, with the sum on \( a \) running only from 0 to 1 and an overall factor of \( \frac{1}{2} \) in front.

In terms of states, then we have a theory containing states with even winding in which the SO(32) tensors have integer momentum and the SO(32) spinors have half integer momentum, and states with odd winding in which the SO(32) situation is the opposite. Furthermore, the winding mod 2 determines the \((-1)^{F_{ew}}\) projection. That is, in states with even winding the projection is positive (meaning positive-chirality
SO(32) spinors and even-rank SO(32) tensors) and in states with odd winding we have negative-chirality SO(32) spinors and odd-rank SO(32) tensors.

Note that it is important for level matching that the \((-1)^{F_L W}\) projection be reversed in sectors with odd winding in the base. Since the SO(32) tensors have half-integral momentum, there is a level mismatch in the base of \(1/2\) mod 1, and it must be cancelled by a level mismatch in the fiber of \(1/2\) as well, which comes from the odd number of \(\lambda\)-fermions in these states.

This example illustrates the general concept behind constructing stringy Wilson lines: we have some path integrals over internal degrees of freedom \(f^a_b\) with boundary conditions on the timelike and spacelike cycles labelled by \(a\) and \(b\), and we pair them with the \(Y^a_{ab}\). Defining the theory as a Wilson line means choosing the \(f^a_0\) to implement the boundary conditions for bulk fields. Modular invariance then determines the rest of the \(f^a_b\) by the requirement that their modular transformations must be classical, although the individual summands \(I^a_q\) of \(f^a_b\) need not have classical modular transformations. (We shall see that their transformations have anomalous phases in general.)

If we have made a consistent choice of boundary condition for local fields in the base, (i.e., a boundary condition which respects closure of the OPE on the worldsheet) then there should be a resulting set of \(f^a_b\) which transforms classically under the modular \(S\) and \(T\) transformations. This is the consequence of the CCC principle as discussed in the introduction.

Note, by the way, that some of the internal path integrals \(I^p_q\) may vanish (for instance \(I^0_0\) in this case), but we will not use any such vanishings. The reason is that when operators are inserted into the path integral, modular invariance must still hold. The structure of the partition function with insertions is exactly the same as the one without insertions, with the new \(I^p_q\)'s having the modular transformations we have assigned them here, but no longer vanishing.

2.1.2 The case \(k_L = 4\)

Next we consider the case \(k_L = 4\): a chiral discrete symmetry which inverts 4 of the \(\tilde{\lambda}\).

The untwisted sector is given by

\[
I^a_0 = \frac{1}{2} \left( (\hat{F}^a_0)^2(\hat{F}^0_0)^{14} + (\hat{F}^{a+1}_0)^2(\hat{F}^0_1)^{14} + (-1)^a(\hat{F}^a_0)^2(\hat{F}^0_1)^{14} + (-1)^a(\hat{F}^{a+1}_1)^2(\hat{F}^1_1)^{14} \right) \quad (2.25)
\]

These functions are periodic mod 2 in the upper index \(I^a_0 + 2 = I^a_0\). This expresses the fact that \(g\) is a \(\mathbb{Z}_2\) symmetry in the untwisted sector. The unprojected partition
functions in the twisted sectors are
\[ I^0_b = \frac{1}{2} \left( (\tilde{F}^0_b)^2 (\tilde{F}^0_0)^{14} + (-1)^b (\tilde{F}^1_b)^2 (\tilde{F}^1_1)^{14} + (\tilde{F}^0_{b+1})^2 (\tilde{F}^0_1)^{14} + (-1)^b (\tilde{F}^1_{b+1})^2 (\tilde{F}^1_1)^{14} \right) \] (2.26)

In the twisted NS+ ground state the eigenvalue of \( g \) is \(-1\), so
\[ I^{a,b} = \frac{1}{2} (-1)^{ab} \left( (\tilde{F}^a_b)^2 (\tilde{F}^0_0)^{14} + (-1)^b (\tilde{F}^{a+1}_b)^2 (\tilde{F}^1_1)^{14} \right) + (-1)^a (\tilde{F}^a_{b+1})^2 (\tilde{F}^0_1)^{14} + (-1)^{a+b} (\tilde{F}^{a+1}_{b+1})^2 (\tilde{F}^1_1)^{14} \] (2.27)

These functions transform under modular transformations as
\[ I^a_b(\tau + 1) = \exp\left\{ \frac{2\pi i}{3} \right\} \exp\left\{ \frac{\pi ib^2}{2} \right\} I^{a+b}_0(\tau) \] (2.28)

In fact \( g \) extends to twisted sectors as a \( \mathbb{Z}_2 \) symmetry as well, since fermions are odd or even under \( g \) in blocks of four, and a state with periodic fermions gets a phase of \( \pm i \) for each two periodic fermions odd under \( g \). This is reflected in the fact that \( I^a_b \) is periodic mod 2 in each index separately.

We can construct a unitary, modular invariant partition function by defining \( I^{a,ab}_b \equiv \exp\left\{ \frac{\pi iab}{2} \right\} I^a_b \). Up to the local gravitational anomaly, \( I^{a,ab}_b \) transforms classically under modular transformations:
\[ I^{a,ab}_b(\tau + 1) = \exp\left\{ \frac{2\pi i}{3} \right\} I^{a+b,ab}_0 \] (2.29)
\[ I^{a,ab}_b\left( \frac{-1}{\tau} \right) = I^{-b,a,ab}_0(\tau) \]
so the combination
\[ Z \equiv \sum_{ab} Y^a_b I^{a,ab}_b Z^{(R)} \] (2.30)

is modular invariant. It has an interpretation as a trace over states:
\[ Z = \sum_{a,b} (q\bar{q})^{-\frac{4}{3}} \text{tr}_{[\text{twisted by } g^b, w_{\text{base}}=b]} \left( g^a \cdot g^{a}_X \cdot q^{L_0} \bar{q}^{\bar{L}_0} \right) \exp\left\{ \frac{\pi ib^2}{2} \right\} \] (2.31)

which amounts to a projection onto states with \( g \cdot g_X = \exp\left\{ -\frac{\pi i w_{\text{base}}}{2} \right\} \). Since as we argued above \( g = \pm 1 \) for all states, twisted and untwisted, it follows that \( g_X \equiv \exp\left\{ 2\pi ip_{\text{base}} R_{\text{base}} \right\} \) must have eigenvalues \( \pm i \) when \( w_{\text{base}} \) is odd. In other words, \( p_{\text{base}} \) is fractionated in twisted sectors in units of \( \frac{1}{4R_{\text{base}}} \). This matches the value one would derive by demanding level matching in the sector \( w_{\text{base}} = 1 \).
2.1.3 Other values of \( k_L = 4m \)

In general\(^3\), we have

\[
I_b^a \equiv \frac{1}{2} (-1)^{mab} \left( (\tilde{F}^a_b)^{2m} (\tilde{F}^0_0)^{16-2m} + (-1)^{mb} (\tilde{F}^a_{b+1})^{2m} (\tilde{F}^1_0)^{16-2m} \right) + (-1)^{ma} (\tilde{F}^a_{b+1})^{2m} (\tilde{F}^0_0)^{16-2m} + (-1)^{m(a+b)} (\tilde{F}^{a+1}_b)^{2m} (\tilde{F}^1_0)^{16-2m} \] \tag{2.32}

\[
I_b^a (\tau + 1) = \exp\left\{ \frac{2\pi i}{3} \right\} \exp\left\{ \frac{\pi i mab}{2} \right\} I_a^{a+b}(\tau) \] \tag{2.33}

\[
I_b^a (-\frac{1}{\tau}) = (-1)^{mab} I_a^{-b}(\tau) \]

In fact \( g \) extends to twisted sectors as a \( \mathbb{Z}_2 \) symmetry as well, since fermions are odd or even under \( g \) in blocks of four, and a state with periodic fermions gets a phase of \( \pm i \) for each two periodic fermions odd under \( g \). This is reflected in the fact that \( I_b^a \) is periodic mod 2 in each index separately.

We can construct a unitary, modular invariant partition function by defining \( I_b^a \equiv \exp\left\{ \frac{\pi imab}{2} \right\} I_b^a \). Up to the local gravitational anomaly, \( I_b^a \) transforms classically under modular transformations:

\[
I_b^a (\tau + 1) = \exp\left\{ \frac{2\pi i}{3} \right\} I_b^{a+b} \] \tag{2.34}

\[
I_b^a (-\frac{1}{\tau}) = I_a^{-b}(\tau) \]

so the modular invariant combination is

\[
Z \equiv \sum_{ab} Y_a^b I_b^a Z^{(R)} = \sum_{ab} Y_a^b I_b^a Z^{(R)} i^{mab}. \tag{2.35}
\]

The partition function \( Z \) has an interpretation as a trace over states:

\[
Z = \sum_{a,b} (q \bar{q})^{-\frac{3}{2}} \ln_{\text{twisted by } g^b, \, w_{\text{base}} = b} \left( g^a \cdot g_X^a \cdot q^{L_0} \bar{q}^{\bar{L}_0} \right) \exp\left\{ \frac{\pi imab}{2} \right\} \] \tag{2.36}

which amounts to a projection onto states with \( g \cdot g_X = \exp\left\{ -\frac{\pi imw}{2} \right\} \). Since as we argued above \( g = \pm 1 \) for all states, twisted and untwisted, it follows that \( g_X \equiv \exp\{2\pi ip_{\text{base}} R_{\text{base}} \} \) must have eigenvalues \( \pm i^m \) when \( w_{\text{base}} \) is odd. \( p_{\text{base}} \) is fractionated in units of \( \frac{m}{4R_{\text{base}}} \mod \frac{1}{R_{\text{base}}} \). The Wilson line is tame when \( m \) is even and wild when \( m \) is odd.

\(^3\)Our \( k_L = 0 \) case in an earlier sub-subsection does not fall into this set – in that case we had an extra action of \( g \) on \( SO(32) \) spinor number in order to get a nontrivial action for \( g \) at all.
2.2 Wilson line for T-duality in the bosonic string

Now we turn to one of the more interesting and 'stringy' symmetries of the bosonic string compactified on a fiber circle of self-dual radius $\sqrt{\alpha'}$. As is well known, this theory has a duality symmetry which maps the theory to itself with an exchange of modes carrying momentum $n$ and winding $w$ on the fiber circle. We would like to compactify a second dimension on a circle to play the role of a base, and impose a Wilson line boundary condition such that all string states undergo a T-duality transformation when transported around the base.

Such a theory would be an example of a T-fold ([4]). Much work has been done on the construction of T-folds at the level of effective field theory in the base and also on some aspects of the worldsheet realizations ([1, 7, 3, 9, 2, 4, 10, 13, 16, 17]). However an explicitly modular invariant worldsheet description is lacking for even some of the simplest T-folds [8]. In this section and subsequent sections we will demonstrate how the worldsheet CFT construction of a string propagating on a T-fold or more general monodrofold ([3]) can be carried out in general.

In all that follows, $\mathcal{I}$ will be the theory of a circle $S^1$ compactified at the self-dual radius in the bosonic string. In this subsection, we give a direct discussion of the Wilson line theory, along the lines of the previous example. In the next subsection, we will identify several alternative derivations of the same result.

\textit{Action of T-duality symmetry in the fiber $\mathcal{I}$}

In order to make this construction, it is important to make one observation about the action of T-duality which appears to have gone unremarked in the literature. The technical basis for this observation is explained in detail in the Appendix.

The naive action

$$T_{\text{naive}} |n, w\rangle = |w, n\rangle$$

which simply reverses the momentum and winding of a state is not actually a good symmetry of the two-dimensional CFT, in the sense that the operator product expansion does not preserve it. For instance, it is possible to take the OPE of two states which have eigenvalue $+1$ under $T_{\text{naive}}$ and get a state on the right hand side of the OPE which has eigenvalue $-1$. The reason for this has to do with the zero-mode cocycle factors which appear in the definition of vertex operators carrying momentum and
winding in a toroidally compactified theory. There is still a true T-duality symmetry which switches momentum $n$ and winding $w$, with an extra phase depending on $n$ and $w$:

$$T_{\text{true}} |n, w\rangle = (-1)^{nw} |w, n\rangle$$  (2.38)

For any state with $n \neq w$ this phase can be absorbed into a redefinition of the phase of the state. However for states with $n = w$ the extra phase is meaningful. The naive T-duality operation always has positive eigenvalues for states with $n = w$, whereas the true T-duality operation has phase $+1$ when $n = w$ is even and $-1$ when $n = w$ is odd.

From here on, when we refer to T-duality we will always mean the true, conserved T-duality operation which switches $n$ with $w$ and acts with a minus sign on all left-moving oscillators $\tilde{\alpha}_n$ in the self-dual $c = 1$ circle CFT.

*Twisted sector*

Now we would like to make three more observations, relating to the properties of the twisted sectors. In a twisted sector, the left-moving current $\partial^-X$ is antiperiodic, and therefore all left-moving oscillators $\tilde{\alpha}_{n+\frac{1}{2}}$ are half-integrally moded.

In the twisted sector the left moving modes have a Casimir energy equal to $\frac{1}{16}$, and it is not even clear that we *should* expect the twisted sector to be level-matched. That is, there is no particular reason to expect an asymmetric orbifold by $\hat{T}$ to be consistent, so there is no *a priori* necessity of modular invariance which would demand level matching in the twisted sector. However we can forge ahead and explore the hypothesis that the orbifold by $\hat{T}$ may be modular invariant, seeing where this leads us.

The only possible source of fractional energy on the right would be a nontrivial quantization for the zero mode in the twisted sector. This is a natural possibility to consider, as the zero mode quantization condition $p_L - p_R \in 2\mathbb{Z}$ has no analog in the untwisted sector.

In order for the sector twisted by $\hat{T}$ to be level matched, the twisted ground state would have to have $p_R$ equal to $\pm \frac{1}{2}$ (mod 4), in order that the right-moving energy $\frac{1}{4}p_R^2$ should equal $\frac{1}{16}$. We will assume that this is the correct zero mode quantization in the twisted sectors.

Now to our second comment about the twisted sectors. In order for T-duality to act in a consistent way in the twisted sector, it must commute with T-even operators in $\mathcal{I}$ and anticommute with T-odd operators, in their action on twisted states. For this
to work out, there must be a nontrivial cocycle factor in the action of $T$ on the twisted sector which depends on the zero mode $p_R$, analogous to the cocycle phase $(-1)^{nw}$ in the untwisted states $I$. In the Appendix we work out a cocycle contribution to $T$ in the twisted sectors with the requisite properties.

Our third and last point is that for twisted states there is no mod 2 condition *per se* on the value of $p_R$. That is, if there exists a twisted state $T$ with some value of $p_R$, it is always possible to find an untwisted operator $O \in I$ with $p_R' = p_R + 1$, and the product $T \cdot O$ is a twisted state with the same eigenvalue of $T$ and the opposite value of $\exp\{\pi ip_R\}$.

Though it may seem surprising that there are 'fewer' constraints on $p_R$ mod 2 in the twisted sector than in the untwisted sector, there is a point of view from which it is less mysterious. The zero mode of $X^L$ is nondynamical, but it may nonetheless act as a discrete label which carries no energy and labels distinguishable but degenerate twisted sectors. These labels are quite familiar from geometric orbifold compactifications, where they simply label the fixed points of a geometric symmetry. In general they can be thought of as distinct eigenspaces for the zero modes of the orbifolded fields. In the twisted sectors, then, we can if we like consider states with $p_R = +\frac{1}{2}$ and those with $p_R = -\frac{1}{2}$ to be paired with distinct left-moving twisted sectors, distinguished by nondynamical labels of the left-moving zero modes. Then we can by fiat extend the quantum number $\exp\{\pi ip_L\}$ as a quantum number acting on twisted states, such that $\exp\{\pi ip_L\} = \exp\{\pi ip_R\}$ for all sectors, twisted and untwisted.

In the Appendix we construct the fiber partition functions of the asymmetric orbifold by $T$ in the bosonic string. We find that the correct $p_R$ projection in the twisted sectors is the vacuous one: for a $T$-even state of the left-moving degrees of freedom, both values $\pm\frac{1}{2}$ of $p_R$ are allowed. Likewise, for a $T$-odd state on the left, both values $\pm\frac{3}{2}$ are allowed.

The partition functions of the fiber theory are derived in the Appendix. In the $T$-even untwisted sector we have

$$I_0^+ \equiv \frac{1}{2} \left( I_0^0 + I_0^1 \right)$$

and in the $T$-odd untwisted sector we have

$$I_0^- \equiv \frac{1}{2} \left( I_0^0 - I_0^1 \right)$$
with

\[ I_0^0 = |\eta(\tau)|^{-2} \left( |\alpha_{00}(0,2\tau)|^2 + |\alpha_{10}(0,2\tau)|^2 \right) \]

(2.41)

\[ I_0^0 = \eta(\tau)^{-1} \frac{2\eta(\tau)}{\alpha_{10}(0,\tau)} \frac{1}{2} \alpha_{01}(0,2\tau) \]

where \( I_0^0 \) represents the partition function in the untwisted sector with an insertion of \( T^a \). That is, \( I_0^0 \) is simply the partition function for a \( S^1 \) CFT at the self-dual radius.

Similarly, in the twisted sector the partition functions are

\[ I_0^1 = \eta(\tau)^{-1} \left( \frac{\eta(\bar{\tau})}{\alpha_{10}(0,\bar{\tau})} \right)^{\frac{1}{2}} \alpha_{10}(0,\frac{\bar{\tau}}{2}) \]

(2.42)

\[ I_1^1 = \sqrt{2} \eta(\tau)^{-1} \left( \frac{\eta(\bar{\tau})}{\alpha_{00}(0,\bar{\tau})} \right)^{\frac{1}{2}} \alpha_{10}(-\frac{1}{4},\frac{\bar{\tau}}{2}) \]

and

\[ I_1^+ = \frac{1}{2} \left( I_0^0 + I_0^1 \right) \]

(2.43)

The modular transformations of these objects are simply the classical ones, \( I_0^a(\tau + 1) = I_0^{a+b}(\tau) \) and \( I_0^a(-\frac{1}{2}) = I_0^{-a-b} \), where \( I_0^a \) is defined to be periodic mod 2 in \( a \) and \( b \) separately.

This means in particular that it is simple to define a partition function for the asymmetric orbifold (as opposed to Wilson line) \( S^1/T \):

\[ Z_{\text{asym. orb.}} \equiv I_0^+ + I_1^+ \]

(2.44)

\[ = \frac{1}{2} \left( I_0^0 + I_0^1 + I_1^0 + I_1^1 \right) \]

which is unitary and modular invariant.

In order to define the Wilson line for \( T \) over a circle of radius \( R \), define \( Y_b^a \) to be the partition function in which the base coordinate \( X_{\text{base}} \) winds \( a \) times around the Euclidean timelike direction of the worldsheet, and \( b \) times around the spacelike direction of the worldsheet. This function is related to the unitary partition functions (2.15) for \( S^1 \) by

\[ Y_b^a = \int_0^1 d\alpha \exp\{2\pi i a \alpha\} Y_b^{<a>} \]

(2.45)
where \( Y_{b}^{<\alpha>} \) is the partition function for the base in the sector with winding \( w_{\text{base}} \) equal to \( b \), and momentum \( n_{\text{base}} \equiv R_{\text{base}} p_{\text{base}} \) equal to \( \frac{b}{2\pi} \mod 1 \). The functions \( Y_{b}^a \) also have classical modular properties

\[
Y_{b}^{a}(\tau + 1) = Y_{b}^{a+b}(\tau)
\]

(2.46)

and

\[
Y_{b}^{a}(-\frac{1}{\tau}) = Y_{b}^{a-b}(\tau)
\]

(2.47)

The partition function for the Wilson line is

\[
Z_{\text{Wilson line}} = \sum_{b} Y_{b}^{+} I_{b}^{+} + Y_{b}^{-} I_{b}^{-}
\]

(2.48)

where as above \( I_{b}^{\pm} \equiv \frac{1}{2} \left( I_{b}^{0} \pm I_{b}^{1} \right) \) and

\[
Y_{b}^{\pm} \equiv \sum_{a} (\pm 1)^{a} Y_{b}^{a}
\]

(2.49)

\[
\equiv Y_{b}^{<\frac{1}{4}(1\pm 1)>}
\]

With these definitions, we can rewrite our full partition function as

\[
Z_{\text{Wilson line}} = \frac{1}{2} \sum_{a,b} Y_{b}^{a} I_{b}^{a}
\]

(2.50)

Since both \( Y_{b}^{a} \) and \( I_{b}^{a} \) transform classically under modular transformations, the partition function \( Z_{\text{Wilson line}} \) is modular invariant.

In the sector of zero winding \( b = 0 \) the sum is

\[
Z_{w=0} \equiv \sum_{a} Y_{0}^{a} I_{0}^{a}
\]

(2.51)

which is precisely the sum over states in which the eigenvalue of \( T \) is equal to \( \exp\{\pi i p_{\text{base}} R_{\text{base}}\} \). The partition function is explicitly unitary, modular invariant and satisfies our definition of a Wilson line CFT. The Wilson line is self-evidently tame, a result of the classical, anomaly-free modular transformations of the internal partition functions \( I_{b}^{a} \).

Having constructed the Wilson line CFT for T-duality on a single \( S^1 \) of self-dual radius, it is straightforward to construct Wilson line theories in which parallel transport
around the base induces a simultaneous T-duality on $k$ self-dual $S^1$ factors at once. The internal partition functions are simply raised to the $k^{th}$ power:

$$Z \equiv \sum_{a,b} Y_a^a \left( I_b^a \right)^k$$

(2.52)

The Wilson line is unitary, modular invariant, and tame for all values of $k$.

Later we shall study a version of this Wilson line theory (with $k = 2$) in the type II superstring. We shall see that the type II Wilson line theory is still perfectly consistent, but no longer tame.

### 2.3 More on T-duality Wilson line and equivalence with other models

We will now reinterpret the discussion from the previous subsection in a somewhat more abstract language, starting from the chiral algebra of the circle at the self-dual radius $R_{\text{self-dual}} = 1$ and its $\mathbb{Z}_2$ orbifold [19].

**Asymmetric orbifold by T-duality**

In general, an orbifold model is defined by specifying the action of the symmetry group on the Hilbert space of the parent theory. For the free boson at the self-dual radius, this Hilbert space is obtained by the action of the $SU(2)_L \times SU(2)_R$ chiral algebra generated by

$$J_3 = \partial X_R, \quad J^\pm = e^{\pm 2iX_R}; \quad \tilde{J}_3 = \bar{\partial} X_L, \quad \tilde{J}^\pm = e^{\pm 2iX_L}$$

(2.53)

on the vacuum $|0\rangle$ and the non-trivial primary $e^{i(X_R+X_L)}|0\rangle$. In this language, the naive action of T-duality is as inversion of one of the two $U(1)$ current

$$T : \partial X_R \to \partial X_R \quad \tilde{\partial} X_L \to -\tilde{\partial} X_L$$

(2.54)

But as we have seen in the previous subsection, this is not the entire story. It is not hard to see that the “correct” action (2.38) which is required to preserve the OPE, must act on the full chiral algebra as

$$T : \quad J_3 \to J_3, \quad J^\pm \to -J^\pm; \quad \tilde{J}_3 \to -\tilde{J}_3, \quad \tilde{J}^\pm \to -\tilde{J}^\mp$$

(2.55)

This is the actual action of T-duality on the full chiral algebra of the free boson at the self-dual radius. Introducing $J_2 = \cos 2X_R$ and $J_1 = \sin 2X_R$, $\tilde{J}_2 = \cos 2X_L$, $\tilde{J}_1 = \sin 2X_L$, etc.
\[ \tilde{J}_1 = \sin 2X_L, \ T \ acts \ as \]
\[ J_3 \rightarrow J_3, \ J_2 \rightarrow -J_2, \ J_1 \rightarrow -J_1; \quad \tilde{J}_3 \rightarrow -\tilde{J}_3, \ J_2 \rightarrow -\tilde{J}_2, \ \tilde{J}_1 \rightarrow \tilde{J}_1 \]

(2.56) 

It is then not hard to see that (2.56) is equivalent by a chiral \( SU(2) \) rotation on the right to the standard, right-left symmetric reflection of the circle \( X \rightarrow -X \). As a consequence, the asymmetric orbifold by T-duality is equivalent, as an abstract conformal field theory, to the standard symmetric orbifold of the boson at the self-dual radius. Nevertheless, T-duality and the standard reflection are distinguished after breaking the \( SU(2)_R \times SU(2)_L \) symmetry by deforming away from the self-dual radius.

Given this (non-trivial) equivalence between the asymmetric orbifold and the standard orbifold, it is straightforward to explicitly compute the partition function and demonstrate modular invariance. One recovers the results from the previous subsection. We also find it instructive to describe the twisted sectors in the asymmetric language. While on the left, we have the standard twist fields (we will mostly follow the canonical conventions of [19]), one finds that the right-movers have a fractional momentum with respect to \( J_3 \), which is quantized in units of \( 1/2R_{\text{self\_dual}} \) instead of \( 1/R_{\text{self\_dual}} \), with the quantum depending on the twisted sector.

In hindsight, this is not surprising. Recall that the standard orbifold of \( S^1_{R_{\text{self\_dual}}} \) is also equivalent to the standard circle at twice the self-dual radius \( R_{\text{orbifold}} = 2R_{\text{self\_dual}} = 2 \), with an equivalence of chiral sectors given in table 1. The novelty is that besides viewing this theory as the boson at \( R_{\text{orbifold}} = 2 \), or as the orbifold of the boson at the self-dual radius, one can also view it as the asymmetric orbifold by T-duality. In this point of view, the left-movers live on \( S^1_{R_{\text{self\_dual}}} / \mathbb{Z}_2 \), and the right-movers live on \( S^1_{R_{\text{self\_dual}}} \), albeit with fractional momentum in the twisted sectors.\(^4\)

\(^4\) In this section, we use RCFT conventions. For a circle at radius \( R \) we have chiral momenta \( k_{L,R} = Rp_{L,R} \), which are related to integrally quantized momentum and winding by
\[ k_{R,L} = n \pm R^2 w \]

and the conformal weights are \( \Delta_{L,R} = k^2_{L,R}/4R^2 \). When \( R^2 = r/s \) is rational, the chiral algebra is extended by fields with \( (n,w) = (r,s) \) of dimension \( rs \), and there are \( 2rs \) primary fields labelled by \( sk_{L,R} \in \mathbb{Z} \mod 2rs \).
(chiral) boson sector labelled by momentum $k \mod 8$

| $k$ mod 8 | (chiral) orbifold sector |
|-----------|---------------------------|
| 0         | id                        |
| 4         | $J$                       |
| 2         | $(\frac{1}{4})_+$         |
| $-2$      | $(\frac{1}{4})_-$         |
| $\pm 1$   | $\sigma_{1,2}$            |
| $\pm 3$   | $\tau_{1,2}$              |

Table 1: Equivalence of chiral sectors between orbifold $S^1_{R_{\text{self-dual}}}/\mathbb{Z}_2$ and circle theory $S^1_{R_{\text{orbifold}}}$. Notations follow [19].

**Wilson line CFT**

We can now define the fibration of $S^1_{R_{\text{self-dual}}}$ over $S^1_L$ with a Wilson line for T-duality as the asymmetric orbifold of $S^1_{R_{\text{self-dual}}} \times S^1_{2R}/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts on the first factor as in (2.56) and on the base by a half-shift around the circle. We obtain table 2, in which the base sectors are labelled by momentum and winding, and the fiber sectors are labelled on the left by the chiral momentum and on the left by the orbifold sector (the latter two sets of labels are of course equivalent by table 1).

| sector          | base momentum, winding $(n^b, w^b) \mod (1, 2)$ | fiber $(k^f_R \mod 8, \text{orbifold sector})$ |
|-----------------|-----------------------------------------------|-----------------------------------------------|
| untwisted even  | $(0, 0)$                                      | $(0, \text{id}), (2, (1/4)_+), (4, J), (-2, (1/4)_-) $ |
| untwisted odd   | $(\frac{1}{2}, 0)$                            | $(0, J), (2, (1/4)_-), (4, 0), (-2, (1/4)_+) $ |
| twisted even    | $(0, 1)$                                      | $(1, \sigma_1), (3, \tau_1), (-3, \tau_2), (-1, \sigma_2) $ |
| twisted odd     | $(\frac{1}{2}, 1)$                            | $(1, \tau_2), (3, \sigma_2), (-3, \sigma_1), (-1, \tau_1) $ |

Table 2: Sectors of the CFT description of the Wilson line for T-duality.

It should be clear that since our orbifold action preserves a $U(1)^2$ chiral symmetry on both the right and on the left, this theory is equivalent to a particular point in the moduli space of a two-dimensional torus. The brute force method to determine this point is to identify the Narain lattice of charges with respect to the $U(1)^2_R \times U(1)^2_L$
current algebra. In notation explained in the last footnote, we find

\[
(k_f^R, k_f^L, k_b^R, k_b^L) \in (8, 0, 0, 0)\mathbb{Z} + (0, 1, 0, 1)\mathbb{Z} + (0, 2R^2, 0, -2R^2)\mathbb{Z} + (2, 0, 2, 0)\mathbb{Z}
\]

\[
+ (4, 1/2, 0, 1/2)\mathbb{Z} + (1, R^2, 1, -R^2)\mathbb{Z}
\]

\[
\cong (8, 0, 0, 0)\mathbb{Z} + (2, 0, 2, 0)\mathbb{Z} + (4, 1/2, 0, 1/2)\mathbb{Z} + (1, R^2, 1, -R^2)\mathbb{Z}
\] (2.57)

and the right/left metric

\[
4\Delta_{L,R} = \frac{(k_f^L)^2}{4} + \frac{(k_b^L)^2}{R^2}
\] (2.58)

The standard parametrization of the momentum lattice at a point in the moduli space of the torus is

\[
q_{L,R} = n - Bw \pm gw
\]

\[
4\Delta_{L,R} = g^{-1}(q_{L,R}, q_{L,R})
\] (2.59)

Here \(g\) is the metric and \(B\) the B-field on the torus, and \(n\) and \(w\) are now two-dimensional momentum and winding, respectively.

After a little bit of algebra, we find that the two lattices are isomorphic precisely if the moduli of the torus take the values (up to an \(SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})\) transformation)

\[
\rho = \frac{i}{R}, \quad \tau = \frac{1}{2} + \frac{i}{4R}
\] (2.60)

There is, of course, a more economical way to derive this result. Namely, we notice that T-duality is not only equivalent to the symmetric reflection, but also to half a shift around the circle at the self-dual radius. Therefore, our Wilson line CFT is an orbifold of two circles, one of radius \(2R_{\text{base}} = 2R\), and one of radius \(R_{\text{fiber}} = 1\), by the action

\[
(X^f, X^b) \rightarrow (X^f + 1/2, X^b + 1/2)
\] (2.61)

It is easy to see that the resulting torus has moduli given by (2.60).

Finally, we note that we can also view the Wilson CFT as the orbifold of \(S^1_{\text{self-dual}} \times S^1_{2R}\) by

\[
(X^f, X^b) \rightarrow (-X^f, X^b + 1/2)
\] (2.62)

which we recognize as the Klein bottle (as target space). This implies the amusing fact that the torus CFT at the point (2.60) in moduli space is equivalent to a Klein bottle CFT. The moduli space of the Klein bottle CFT is two-dimensional and it is not hard to see that the intersection locus of torus and Klein bottle moduli space is just the one-dimension here parametrized by \(R\).
2.4 Wilson lines in an SU(2) current algebra at level \( k \)

Going beyond free field theories, it is natural to test the set of ideas presented in this paper in the situation in which the fiber theory is described by a rational conformal field theory, such as a WZW model or a coset model. Such theories have a tendency to possess a host of discrete symmetries which can and have been used in a variety of different contexts. It is consequential to explore wilsonization of such symmetries.

We will here examine the simplest such example in detail, when the fiber theory is the SU(2) WZW model at level \( k \), and the symmetry which we want to wilsonize originates from chirally acting charge conjugation. The requisite representation theoretic data is most easily extracted from ref. [20], which treats \( \mathbb{Z}_2 \) orbifolds of WZW models in general and which will hence also be a useful input in constructing Wilson line theories for such symmetries.

The chiral algebra of the model of our interest is the SU(2) current algebra generated by \( J_3(z), J^\pm(z) \) on the left, and \( \bar{J}_3(\bar{z}), \bar{J}^\pm(\bar{z}) \) on the right. The finite number of primary fields are in correspondence with the finite number of integrable representations, \( \mathcal{H}_l \) of the \( \widehat{su}(2)_k \) affine Lie algebra and are labelled by twice the spin \( l = 0, 1, \ldots, k \). The modular invariant torus partition function is built from the chiral blocks \( \chi_l(\tau) = \text{tr}_{\mathcal{H}_l} q^{L_0 - c/24} \) and given by

\[
\mathbb{I}^0(\tau) = \sum_{l=0}^{k} \chi_l(\bar{\tau})\chi_l(\tau) \tag{2.63}
\]

The symmetry \( g \) we wish to wilsonize originates from the \( \mathbb{Z}_2 \) automorphism of the chiral algebra which acts on the generators in the chiral fashion \(^5\)

\[
g : \quad J_3 \rightarrow -J_3, \quad J^\pm \rightarrow J^\mp, \quad \bar{J}_3 \rightarrow \bar{J}_3, \quad \bar{J}^\pm \rightarrow \bar{J}^\mp \tag{2.64}
\]

and can be implemented on the highest weight modules as explained in [20]. For \( SU(2) \), charge conjugation is an inner automorphism, and (2.64) is in fact nothing but a chiral rotation by \( \pi \) around the \( y \)-axis. Since spinor representations pick up a minus sign under a \( 2\pi \) rotation, as in many examples in the previous subsections, the true symmetry of the OPE is a \( \mathbb{Z}_4 \) extension\(^6\) of \( \mathbb{Z}_2 \) charge conjugation by the \( \mathbb{Z}_2 \) center of \( SU(2) \). In other words, we are implementing \( g \) such that \( g^2 = (-1)^l \) on \( \mathcal{H}_l \).

\(^5\)In this section we act with our discrete chiral symmetry on the right rather than on the left.

\(^6\)which we will abusively continue to denote by \( g \).
The twisted partition functions $I^b_a$ of the fiber theory can be expressed in terms of twisted chiral blocks. We introduce

$$\chi_l^q(\tau) = \text{tr}_{H^l_q} q^{L_0 - c/24} \quad (2.65)$$

for the trace of $g$ in the untwisted sector as well as

$$\tilde{\chi}_l(\tau) = \text{tr}_{H^l_q} \tilde{q}^{L_0 - c/24} \quad (2.66)$$

for the twisted representations. Finally,

$$\tilde{\chi}_l^q(\tau) = \text{tr}_{H^l_q} \tilde{q}^{L_0 - c/24} \quad (2.67)$$

Explicit expressions for these characters are as follows [20]

$$\chi_l(\tau) = \chi_l[0,0](\tau,0) \quad \chi_l^q(\tau) = \chi_l[0,1/2](\tau,0)$$

$$\tilde{\chi}_l(\tau) = \chi_l[1/2,0](\tau,0) \quad \tilde{\chi}_l^q(\tau) = e^{-2\pi i k/8} \chi_l[1/2,1/2](\tau,0) \quad (2.68)$$

where

$$\chi_l[s_1,s_2](\tau,z) = \frac{\Theta_{l+1,k+2}[s_1,s_2](\tau,z) - \Theta_{l-1,k+2}[s_1,s_2](\tau,z)}{\Theta_{1,2}[s_1,s_2](\tau,z) - \Theta_{-1,2}[s_1,s_2](\tau,z)} \quad (2.69)$$

and

$$\Theta_{j,h}[s_1,s_2](\tau,z) = \sum_{n=-\infty}^{\infty} e^{2\pi i (j+hs_1+2nh)^2/4h} e^{2\pi i (z+s_2)(j+hs_1+2nh)/2} \quad (2.70)$$

is the twisted theta function associated with the root lattice of $\tilde{su}(2)$.

The modular transformation properties of the twisted characters are straightforward to work out. Some of the more important ones are

$$\chi_l(\tau + 1) = T_l \chi_l(\tau) \quad \chi_l^q(\tau + 1) = T_l \chi_l^q(\tau)$$

$$\tilde{\chi}_l(\tau + 1) = \zeta T_l \tilde{\chi}_l^q(\tau) \quad \tilde{\chi}_l(\tau + 2) = (-1)^l \zeta^2 T_l^2 \tilde{\chi}_l(\tau)$$

$$\chi_l(-1/\tau) = S_{\nu} \chi_{\nu}(\tau) \quad \chi_l^q(-1/\tau) = S_{\nu} \chi_{\nu}^q(\tau)$$

$$\tilde{\chi}_l(-1/\tau) = S_{\nu} \tilde{\chi}_{\nu}(\tau) \quad \tilde{\chi}_l^q(-1/\tau) = (-1)^l \zeta^2 S_{\nu} \tilde{\chi}_{\nu}^q(\tau) \quad (2.71)$$

where $T_l = e^{2\pi i [l(l+2)/4(k+2) - k/8(k+2)]}$ and $S_{\nu} = \sqrt{2/h \sin \left[ \pi (l + 1)(l' + 2)/(k + 2) \right]}$ are, respectively, modular T- and S-matrices of $\tilde{su}(2)$ at level $k$, and a sum over $l'$ is understood in all S-transformations. The phase $\zeta = e^{2\pi i k/16}$ is for general $k$ a 16-th root of unity and is the fundamental obstruction to the existence of the orbifold by $g$. 

25
In terms of these chiral blocks, the explicit expressions for the torus partition functions with $g^a$ twist in the space direction and $g^b$ twist in the time direction are

\begin{align*}
\mathbf{I}_0^0 &= \bar{\chi}_l \chi_l \\
\mathbf{I}_0^1 &= \bar{\chi}_l \bar{\chi}_l \\
\mathbf{I}_0^2 &= \bar{\chi}_l \chi_{k-l} \\
\mathbf{I}_0^3 &= \bar{\chi}_l \bar{\chi}_{k-l} \\
\mathbf{I}_1^0 &= \bar{\chi}_l \chi^g_l \\
\mathbf{I}_1^1 &= \bar{\chi}_l \bar{\chi}^g_l \\
\mathbf{I}_1^2 &= \bar{\chi}_l \chi^g_{k-l} \\
\mathbf{I}_1^3 &= \bar{\chi}_l \bar{\chi}^g_{k-l} \\
\mathbf{I}_2^0 &= \bar{\chi}_l \chi^g (\tau + 4) = \zeta^4 \mathbf{I}_1^0 (\tau) \\
\mathbf{I}_2^1 &= \bar{\chi}_l \bar{\chi}^g (\tau + 4) \\
\mathbf{I}_2^2 &= \bar{\chi}_l \chi^g_{k-l} (\tau + 4) \\
\mathbf{I}_2^3 &= \bar{\chi}_l \bar{\chi}^g_{k-l} (\tau + 4) \\
\mathbf{I}_3^0 &= \bar{\chi}_l \chi^g (\tau + 4) = \zeta^4 \mathbf{I}_0^0 (\tau)\\
\mathbf{I}_3^1 &= \bar{\chi}_l \bar{\chi}^g (\tau + 4) = \zeta^4 \mathbf{I}_1^1 (\tau) \\
\mathbf{I}_3^2 &= \bar{\chi}_l \chi^g_{k-l} (\tau + 4) = \zeta^4 \mathbf{I}_2^2 (\tau) \\
\mathbf{I}_3^3 &= \bar{\chi}_l \bar{\chi}^g_{k-l} (\tau + 4) = \zeta^4 \mathbf{I}_3^3 (\tau) \\
\end{align*}

(2.72)

It is straightforward to work out the modular transformation properties of these expressions. As expected, we cannot in general form an invariant expression by combining only the $I^b_a$’s. The fundamental obstruction is the level mismatch in the first twisted sector

\begin{equation}
\mathbf{I}_0^0 (\tau + 4) = \zeta^4 \mathbf{I}_1^0 (\tau) \\
(\zeta = e^{2\pi ik/16})
\end{equation}

(2.73)

The reason that for $\zeta^4 \neq 1$, we cannot project onto $g$ invariant states by summing over the T-orbit of $\mathbf{I}_0^0 (\tau)$ is that in these cases $g$ does not extend to a true symmetry of the OPE in the twisted sectors.

Let us clarify this statement. As far as the representation theory of the $\widehat{su}(2)$ affine Lie algebra is concerned, the $\mathbb{Z}_2$ automorphism $g$ is clearly realized in the untwisted (2.65) as well as in the twisted sectors, (2.66), (2.67). However, as can be seen for instance from the explicit fusion rules [20], $g$ can not be a symmetry of the full OPE. We have already seen that $g$ squares to $(-1)^l$ even in the untwisted sector, where it generates $G_c \cong \mathbb{Z}_4$. Including the twist, which is measured by the quantum symmetry group $G_q \cong \mathbb{Z}_4$, we expect that in general the OPE will have a symmetry group $\Gamma$ which is an extension

\begin{equation}
G_c \cong \mathbb{Z}_4 \to \Gamma \to \mathbb{Z}_4 \cong G_q ,
\end{equation}

(2.74)

depending on the value of $k \mod 4$. Specifically, we expect $\Gamma \cong \mathbb{Z}_{16}$ for $k$ odd, $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_8$ for $k = 2 \mod 4$ and $\Gamma \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ for $k = 0 \mod 4$. Only in the last case would an orbifold by $g$ make sense.

In any event, as is by now familiar, even if we cannot orbifold by $g$, we can still wilsonize it by fibering the sectors (2.72) over a free boson theory in the appropriate way. To this end, we will use again the path-integral sectors with fixed winding around space- and time-like worldsheet circles introduced in (2.15), (2.17). The partition

\footnote{Again, a sum over $l$ is implied and suppressed.}
function of the Wilson line theory is then given by

\[ Z = \sum_{a,b=0}^{3} \sum_{m,n \in \mathbb{Z}} \mathcal{I}_a Y^{4n+b, 4na-4mb+b\delta(a)} \]  
(2.75)

where the important phase corrections are given by

\[ \delta(0) = 0 \quad \delta(1) = 1 \quad \delta(2) = -2 \quad \delta(3) = -1 \]  
(2.76)

It is not hard to check that this expression is modular invariant, and has a \( q \)-expansion with positive integer coefficients. Specifically, we find that the projection in the \( a \)-th twisted sectors is onto states with momentum around the base \( p \in \frac{1}{4} \mathbb{Z} - \frac{ka}{16} \), and eigenvalue of \( g \) in the fiber theory given by

\[ \omega = e^{-2\pi i [p - km/4 + \delta(a)k/16]} \]  
(2.77)

where \( m \) is related to the winding \( w \) around the base circle (radius \( R \)) by \( w = 4m + a \).

This clearly shows that on the one hand, the projection in the fiber theory for unwound strings on the base is the correct one for a Wilson line theory implementing charge conjugation in the fiber around the base circle. On the other hand, the momentum in the twisted, wound sectors is generally fractionated in units of \( 1/16 = 1/4^2 \).

Only for \( k \) divisible by 4 does this Wilson line theory fall into the “tame” category.

### 3 General Theory of Wilson Line CFT

We have seen that Wilson line theories for discrete gauge symmetries in string theory exist in all the examples we examine. Two striking features are apparent from the set of examples we have considered so far. First, the momentum in the base is not always fractionated in the units we expect, namely \( \frac{1}{N_{\text{base}}} \) where \( N \) is the order of the symmetry group of the fiber theory \( \mathcal{I} \). Secondly, it is possible to Wilsonize a symmetry even when one cannot consistently orbifold by it.

#### 3.1 General properties of states and operators

It is now worth deriving some general properties of Wilson line CFTs in order to shed light on the relationship between these two observations. The goal of this section will be to formulate a criterion distinguishing those Wilson line CFT which can be represented straightforwardly as \( \mathbb{Z}_N \) orbifolds from those which cannot. We begin by
deriving some properties which hold for Wilson line CFT in general. To do this, let us formalize our definition of a Wilson line in string theory. We will take the base to be $S^1$, though the generalization to a $T^k$ base is self-evident.

**Def.** A Wilson line theory a CFT with operator algebra $\mathcal{W}$, whose data are a fiber CFT $\mathcal{I}$, a CFT base $S^1$ and an anomaly-free discrete symmetry $g$ of finite order $N$ acting on $\mathcal{I}$. The operator algebra of the base is a $U(1)_L \times U(1)_R$ current algebra $\mathcal{A}_{S^1}$ with a Narain lattice $\Lambda = \{ (p_L \equiv p_{\text{base}} - \frac{1}{\alpha'} R_{\text{base}} w_{\text{base}}, p_R \equiv p_{\text{base}} + \frac{1}{\alpha'} R_{\text{base}} w_{\text{base}}) \}$ which is neither necessarily even nor self dual. The Wilson line CFT $\mathcal{W}$ has the following defining properties:

- The subalgebra of $\mathcal{W}$ with $w_{\text{base}} = 0$ is the subalgebra of $\mathcal{A}_{S^1} \otimes \mathcal{I}$ such that $g$ and $g^{-1} \equiv \exp\{-2\pi i R_{\text{base}} p_{\text{base}}\}$ have the same eigenvalue.
- $\mathcal{W}$ has all the standard properties of a well-behaved CFT, in particular modular invariance, and single-valuedness and closure of the operator product expansion.

**Property 1:** For all Wilson line theories, the worldsheet boundary conditions on operators in $\mathcal{I}$ are a function of the winding $w_{\text{base}}$. In particular, operators in $\mathcal{I}$ have a periodicity $\mathcal{O} \rightarrow g^{w_{\text{base}}} \cdot \mathcal{O}$ around the spatial coordinate of the worldsheet in the sector with winding $w_{\text{base}}$.

**Proof:** Consider an operator $V_1$ with $w_{\text{base}} = 0$ and nonzero momentum $p_{\text{base}} \equiv n_{\text{base}}/R_{\text{base}}$. We would like to transport $V_1$ around the origin, where we have placed an operator with winding $w'_{\text{base}}$. When transported around $V_2$, the operator $V_1$ gets a phase $\exp\{2\pi i w'_{\text{base}} n_{\text{base}}\}$ from its base part. The phase contributed by the fiber part of the operators must cancel this phase. In particular, if the boundary condition at $z = 0$ is $\mathcal{O} \rightarrow g^m \cdot \mathcal{O}$ and the $g$-eigenvalue of $V_1$ is $\omega$, then $\omega^m$ must equal $\exp\{2\pi i n_{\text{base}} w\}$ for all $n_{\text{base}}$. By the defining property of a Wilson line CFT, $\omega = \exp\{w \pi i n_{\text{base}}\}$, so by taking $n_{\text{base}} = \frac{1}{N}$, we see that $m = w \mod N$. But the number $m$ is only meaningful mod $N$, since $g^N$ acts trivially on all operators in $\mathcal{I}$. So the boundary condition in the sector with winding $w_{\text{base}}$ is $\mathcal{O} \rightarrow g^{w_{\text{base}}} \cdot \mathcal{O}$, QED.
**Property 2**: For all Wilson line theories, the quantity \( n_{\text{base}} \equiv R_{\text{base}} p_{\text{base}} \) is completely determined mod 1 by \( w_{\text{base}} \) and by the eigenvalue of \( g \).

**Proof**: Consider two states \( V^{(1)} \) and \( V^{(2)} \) with the same winding number \( w_{\text{base}} \) and the same eigenvalue \( \omega \) of \( g \). Then \( V^\dagger_2 \) has winding number \( -w_{\text{base}} \) and \( g\)-eigenvalue \( \omega^* \). Then the OPE of \( V_1 \) with \( V^\dagger_2 \) contains states with zero \( w_{\text{base}} = 0 \) and \( g = 1 \), which by the defining property must have \( n_{\text{base}} \in \mathbb{Z} \). Therefore \( n_{\text{base}} \) values of \( V^{(1)} \) and \( V^{(2)} \) must differ by an integer, \( QED \).

*Wild versus tame Wilson lines*\(^8\)

At this point we wish to draw a sharp distinction between two different cases in which the symmetry \( g \) is of order \( N \) in its action on all local fields in the base.

In the first case, which we shall refer to as 'tame', the level mismatch in the fiber theory in the sector twisted by \( g \) is an integer multiple of \( \frac{1}{N} \). This means that the momentum in the base is fractionated in units of \( p_{\text{base}} = \frac{1}{NR_{\text{base}}} \). Two properties of tame Wilson lines follow directly from this.

**Property 3**: A tame Wilson line theory can be viewed as an orbifold of a product of the fiber with an \( N \)-fold cover of the base. The orbifold action is \( g \) on the fiber and a shift \( X_{\text{base}} \rightarrow X_{\text{base}} + 2\pi NR_{\text{base}} \) on the base. Each state, twisted or untwisted, carries a representation of \( g \) which may be nontrivial, but nonetheless satisfies the group relation \( g^N = 1 \) obeyed by \( g \).

**Proof**: We can always change the momentum of any state mod \( \frac{1}{NR_{\text{base}}} \) without changing the winding or twist, by taking its OPE with unwound states. By definition of a tame Wilson line all states, including winding sectors, carry momentum which vanishes mod \( \frac{1}{NR_{\text{base}}} \). It follows that every winding sector contains some states with \( p_{\text{base}} = 0 \). In particular, the sector with \( w_{\text{base}} = 1 \) contains a state with \( p_{\text{base}} = 0 \).

\(^8\)We are very grateful to A. Flournoy, B. Williams and B. Wecht for discussions relevant to this section.
Take the OPE of $N$ copies of that state, which produces a string with $w_{\text{base}} = N$ and $p_{\text{base}} = 0$. Call this vertex operator $V_1$, and call its $g$-eigenvalue $\omega^{(1)}$. Now, consider what happens when we circle $V_1$ around an operator $V_2$ in the twisted sector which has $p_{\text{base}} = \frac{n}{NR_{\text{base}}} (n \in \mathbb{Z})$. The phase one gets in circling $V_1$ once around $V_2$ is equal to

$$\exp\{2\pi i (n^{(1)} w^{(2)}(1) - n^{(2)} w^{(1)}(1))\} \cdot (\omega^{(1)})^{w_2} (\omega^{(2)})^{-w_1}, \quad (3.1)$$

with the second and third factors coming from the fact that the sector with winding $w_{\text{base}}$ is also twisted by $g^{w_{\text{base}}}$. For the two vertex operators we are considering, the first factor makes no contribution, since $n^{(1)} = 0$ and $n^{(2)} w^{(1)}$ is an integer. The third factor also makes no contribution since $w_1 = N$ and $\omega^{(2)}$ is an $N^{th}$ root of unity. So the phase is just $\omega^{(1)}$, since $w_2 = 1$. Now, for consistency of the OPE, the total phase must equal 1, so $\omega^{(1)} = 1$, proving that all operators with $w_{\text{base}} \in N\mathbb{Z}$ and $n_{\text{base}} \in \mathbb{Z}$ must have $\omega = 1$, QED.

**Corollary:** By closure of the OPE it follows that $\exp\{2\pi i n_{\text{base}}\} = \alpha^{w_{\text{base}}} \cdot \omega$, with $\alpha$ an $N^{th}$ root of unity. But since our symmetry $g$ was originally defined only in $\mathcal{I}$, we can choose to extend it to $\mathcal{I} \cdot \mathcal{A}^{S^1}$ in any way which respects the group relation $g^N = 1$. In particular if $\alpha$ is a root of unity we can redefine $g \rightarrow \alpha^{-w} \cdot g$, in which case our formula becomes $\exp\{2\pi i n_{\text{base}}\} = \omega$, for all states in the theory.

**Property 4:** The $R_{\text{base}} \rightarrow 0$ limit of a tame Wilson line is the product of an orbifold $\mathcal{I}/\{g\}$ of the fiber with the real line $\mathbb{R}$. The real line comes from the decompactification of the T-dual coordinate $\hat{X}_{\text{base}}$ to the base $X_{\text{base}}$.

**Proof:** The structure of the $R_{\text{base}} \rightarrow 0$ limit can be understood from consideration of the states which become light in the limit $R_{\text{base}} \rightarrow \infty$, which are exactly the states with $p_{\text{base}} = 0$. Then the states which survive in the limit $R_{\text{base}} \rightarrow 0$ must have $n_{\text{base}} = 0$, which implies $\omega = 1$. And for every winding state, there is a sector whose boundary conditions are $\mathcal{O} \rightarrow g^{w_{\text{base}}} \cdot \mathcal{O}$ around the spatial coordinate of the worldsheet.

So the theory at large $R_{\text{base}}$ describes a $g$-invariant set of states in $\mathcal{I}$ which form a
continuum with density \( \frac{dN}{dp_{\text{base}}} = \tilde{R}_{\text{base}} \), where

\[
\tilde{p}_{\text{base}} = \tilde{R}_{\text{base}}
\]

and a sector twisted by each element of \( G \), with the same spectral density \( N \). It follows that the \( R_{\text{base}} \to 0 \) limit of a tame Wilson line is simply the product of \( \mathbb{R}_{\tilde{\text{base}}} \) with the orbifold \( \mathcal{I}/G \), QED.

The wild case

In the more general case, the momentum \( p_{\text{base}} \) does not satisfy \( Nn_{\text{base}} \in \mathbb{Z} \) for all states. However we will now show that the denominator of \( n_{\text{base}} \) does obey a general bound.

**Property 5:** The momentum \( n_{\text{base}} \) always has a denominator dividing \( N^2 \). That is, \( N^2n_{\text{base}} \in \mathbb{Z} \) for any consistent Wilson line CFT.

*Proof:* It suffices to prove that the momentum in the sector with \( w_{\text{base}} = 1 \) satisfies \( N^2n_{\text{base}} \in \mathbb{Z} \). Let \( n \) be the value of \( n_{\text{base}} \) in any sector with \( w = 1 \). Then take the OPE of \( N \) copies of the state. The resulting piece has an untwisted fiber factor, so its contribution to the level mismatch is \( 0 \mod 1 \). Therefore the base contribution to the level mismatch is also \( 0 \mod 1 \). But the base contribution is \( N \cdot (Nn_{\text{base}}) \), which means the \( n_{\text{base}} \) we started with in the \( w_{\text{base}} = 1 \) sector must be an integer multiple of \( \frac{1}{N^2} \). QED.

It follows that the value of \( n_{\text{base}} \) in the sector with \( w_{\text{base}} = N \) is an integer multiple of \( \frac{1}{N^2} \), so there does exist a state \( V \) with winding \( N \) and \( p_{\text{base}} = 0 \). In general however, this state will not have \( \omega = 1 \).

**Property 6:** The value of \( p_{\text{base}} \mod \frac{1}{NR_{\text{base}}} \) is completely determined by the winding:

\[
n_{\text{base}} = \frac{\alpha w_{\text{base}}}{N^2} \pmod{\frac{1}{N}}
\]
where \( \alpha_1 \) is some integer characteristic of the theory.

**Proof**: For a state with \( w_{\text{base}} = 1 \) we have \( n_{\text{base}} = \frac{\alpha_1}{N^2} \) (mod 1) by property 5, so *a fortiori* we have \( n_{\text{base}} = \frac{\alpha b}{N^2} \) (mod \( \frac{1}{N} \)). The OPE of \( b \) copies of state yields a state \( V^{(1)} \) with \( w_{\text{base}} = b \) and \( n_{\text{base}} = \frac{\alpha b}{N^2} \) (mod \( \frac{1}{N} \)). If another state \( V^{(2)} \) also has \( w_{\text{base}} = b \), then \( V^{(1)} \dagger V^{(2)} \) has \( w_{\text{base}} = 0 \). From the defining property of a Wilson line CFT and the conservation of \( p_{\text{base}} \) in the OPE, it follows that \( V^{(2)} \) has \( n_{\text{base}} = \frac{\alpha_1 b}{N^2} \) (mod \( \frac{1}{N} \)), QED.

**Property 7**: In a general winding sector, the eigenvalue \( \omega \) of \( g \) is always an \((N^2)^{1/2}\)th root of unity.

*Proof*: Let the winding \( w_{\text{base}} \) of the state be \( b \) and its \( g \)-eigenvalue be \( \omega \). The OPE of \( N \) copies of the state has \( w_{\text{base}} = Nb \) and \( g \)-eigenvalue \( \omega^N \). Since \( w_{\text{base}} \in N\mathbb{Z} \), the fiber component of the state is untwisted, by property 1, which means it lives in \( \mathcal{I} \). The \( g \)-eigenvalues of all states in \( \mathcal{I} \) are \( N^{1/2} \)th roots of unity, which means \( (\omega^N)^N = \omega^{N^2} = 1 \). In other words, \( \omega \) is an \((N^2)^{1/2}\)th root of unity, QED.

**Property 8**: The eigenvalue \( \omega \) of \( g \) mod \( \exp\{2\pi i/N\} \) is completely determined by the winding:

\[
\omega = \exp\{2\pi i \beta_1 w_{\text{base}}/N^2\} \quad \text{(mod exp\{2\pi i/N\})} \quad (3.4)
\]

where \( \beta_1 \) is some integer characteristic of the theory.

*Proof*: For a state with \( w_{\text{base}} = 1 \), we have \( \omega = \exp\{2\pi i \beta_1/N^2\} \) with \( \beta_1 \in \mathbb{Z} \) by property 7. So *a fortiori* we have \( \omega = \exp\{2\pi i \beta_1/N^2\} \) (mod \( \exp\{2\pi i/N\} \)). Taking the OPE of \( b \) copies of such a state yields a state \( V^{(1)} \) with \( w_{\text{base}} = b \) and \( \omega = \exp\{2\pi i \beta_1 b/N^2\} \) (mod \( \exp\{2\pi i/N\} \)). If another state \( V^{(2)} \) also has \( w_{\text{base}} = b \), then \( V^{(1)} \dagger V^{(2)} \) has \( w_{\text{base}} = 0 \). From the defining property of a Wilson line CFT and the conservation of \( g \) in the OPE, it follows that \( V^{(2)} \) has \( g \)-eigenvalue \( \omega \) equal to
\[
\exp\left(\frac{2\pi i b}{N^2}\right) \pmod{\exp\{2\pi i/N\}}, \text{QED}.
\]

**Property 9**: The product \(\phi^{-1} \equiv \omega \cdot \exp\{2\pi in_{\text{base}}\}\), where \(\omega\) is the eigenvalue of \(g\), depends only on the value of \(w_{\text{base}}\).

**Proof**: Let \(V^{(1,2)}\) be two states with the same values of \(w_{\text{base}}\). Take the OPE of \(V^{(1)} \) with \(V^{(2)\dagger}\). The quantum numbers \(n_{\text{base}}\) and \(g\) are exactly conserved by the OPE, so the product \(\phi^{-1} \equiv \exp\{2\pi in_{\text{base}}\} \cdot \omega\) is conserved as well. The product \(V^{(1)} V^{(2)\dagger}\) has \(w_{\text{base}} = 0\), so by the defining property of a Wilson line CFT it has \(\exp\{2\pi in_{\text{base}}\} \cdot \omega = 1\). It follows that \(V^{(1)}\) and \(V^{(2)}\) share the same value of \(\phi^{-1} = \exp\{2\pi in_{\text{base}}\} \cdot \omega\), QED.

**Property 10**: Given a Wilson line CFT, there is a formula which applies to all physical states in the spectrum:

\[
\exp\{2\pi in_{\text{base}}\} \cdot \omega = \phi_1^{-w_{\text{base}}}
\]

where \(\phi_1\) is an \(\left(\frac{N^2}{2}\right)^{th}\) root of unity.

**Proof**: Given some state in the sector \(w_{\text{base}} = 1\) define \(\phi_1^{-1}\) to be the product \(\exp\{2\pi in_{\text{base}}\} \cdot \omega\). Both \(n_{\text{base}}\) and \(g\) are \(\left(\frac{N^2}{2}\right)^{th}\) roots of unity in the sector \(w_{\text{base}} = 1\), so \(\phi_1\) is an \(\left(\frac{N^2}{2}\right)^{th}\) root of unity as well. Taking the OPE of \(b\) copies of this state yields a state with \(w_{\text{base}} = b\) and \(\omega \cdot \exp\{2\pi in_{\text{base}}\} = \phi_1^{-b}\). By property 9, \(\omega \cdot \exp\{2\pi in_{\text{base}}\}\) depends only on \(b\), so \(\omega \cdot \exp\{2\pi in_{\text{base}}\} = \phi_1^{-b}\) holds for all states. The formula (3.5) follows, QED.

**Corollary**: The orbifold-like projection \(\omega = \exp\{-2\pi in_{\text{base}}\}\) cannot be extended in general from the \(w_{\text{base}} = 0\) sector to the winding sectors, even if \(g\) is extended to generate a \(\mathbb{Z}_{N^2}\) group when acting on twisted sectors. The action of \(g\) on untwisted sectors is already defined on operators in \(I\), which are the internal pieces of vertex...
operators with $w_{\text{base}} \in \mathbb{N}\mathbb{Z}$. And with that prior definition of $g$, the projection $\omega = \exp\{-2\pi im_{\text{base}}\}$ is not even obeyed in sectors with $w_{\text{base}} \in \mathbb{N}\mathbb{Z}$.

**Light states in wild Wilson theories in the $R_{\text{base}} \to 0$ limit**

What are the light states, in this case, as $R_{\text{base}} \to 0$? They are still the states with $p_{\text{base}} = 0$. But now the set of such states have fiber pieces which get a nontrivial phases under some elements $g$. This establishes that the $R_{\text{base}} \to 0$ limit does not contain a factor of $\mathcal{I}/G$.

To see what we do get in the limit $R_{\text{base}} \to \infty$, consider the character $\chi$ defined by the phase $\chi(g) = \omega$, with $\omega$ taken to be the eigenvalue of $g$ in the sector with $w_{\text{base}} = N$ and $p_{\text{base}} = 0$. Suppose $\chi$ generates the entire dual group $G^*$ of $G$. This means that there is no nontrivial element $h$ of $G$ such that $\chi(h) = 1$. Concretely, it means $\chi(g) = \exp\{2\pi im/N\}$ with $m$ relatively prime to $N$. In this case, all $N$ possible eigenvalues of $g$ occur for various states with winding $w_{\text{base}}$ in $\mathbb{N}\mathbb{Z}$ and $p_{\text{base}} = 0$. In such a case, the first state with $p_{\text{base}} = 0$ and $\omega = 1$ occurs for winding $w_{\text{base}} = N^2$. As $R_{\text{base}} \to 0$, these states form continua in a T-dual theory at large radius, with spectral density

$$
\frac{dN}{dp} = \tilde{R}_{\text{base}}
$$

(3.6)

for each sector. The twisted sectors have $n_{\text{base}} \notin \frac{1}{N}\mathbb{Z}$ so in particular they never have vanishing momentum, and thus they become infinitely heavy in the limit $R_{\text{base}} \to 0$.

The resulting theory, in the 'maximally wild' case $\gcd(m, N) = 1$, has the full complement of states in $\mathcal{I}$, with all possible eigenvalues of $g$, and no twisted sectors for small $R_{\text{base}}$. It follows that the limit $R_{\text{base}} \to 0$ of the 'maximally wild' Wilson line is simply $\mathbb{R}_{\text{base}} \times \mathcal{I}$. Intermediate cases can occur as well, in which the denominator of the momentum fractionation is an integer multiple of $N$ dividing $N^2$.

### 3.2 Effective field theory on the long winding string

The wild momentum fractionation in units of $\frac{1}{N^2R_{\text{base}}}$ may at first seem exotic and its introduction into the CFT ad hoc. One might be tempted to object that effective field theory would forbid, rather than mandate, the inclusion of states with $p_{\text{base}} \notin \frac{1}{N}\mathbb{Z}$. 

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The point is, this intuition is correct insofar as it refers to effective field theory in the bulk of the base in the limit \( R_{\text{base}} \gg \sqrt{\alpha'} \). Bulk states – states which are local fields in the base – do indeed have \( p_{\text{base}} \in \frac{1}{N} \mathbb{Z} \) with values mod 1 according to their \( g \)-eigenvalues, as dictated by the defining property of a Wilson line. But in the same limit one can analyze the effective theory on a long string winding the base, with attention to the spectrum of momenta \( p_0 \) for states of the long string.

What is the effective field theory on a long string with winding \( w_{\text{base}} = 1 \)? Since our backgrounds are represented by conventional worldsheet CFT with no fluxes or warping, the energies are given by

\[
E = \left( p_ip_i + p_{\text{base}}^2 + \frac{R_{\text{base}}^2 w_{\text{base}}^2}{\alpha'} \right) + \frac{2}{\alpha'} H_{\text{w.s.}} \quad (3.7)
\]

where \( p_i \) are the momenta in all noncompact spacelike, and \( m_{\perp}^2 \) is the mass-squared \( \frac{1}{\alpha'} H_{\text{w.s.}} \) contributed by the component of the worldsheet Hamiltonian \( H_{\text{w.s.}} \) other than \( X_{\text{base}} \) and \( X^0 \) (and their superpartners, in the heterotic or type II theory). So at long wavelengths in the c.m. frame \( p_i = 0 \) we have

\[
E \sim \frac{R_{\text{base}}}{\alpha'} + \frac{1}{R_{\text{base}}} H_{\text{w.s.}} \quad (3.8)
\]

Since \( H_{\text{w.s.}} \) is a Hamiltonian with respect to worldsheet time, it is dimensionless in the spacetime sense. For \( R_{\text{base}} \gg \sqrt{\alpha'} \), the theory on the long string is approximately conformal, with Hamiltonian \( \frac{1}{\alpha'} H_{\text{w.s.}} \) plus a constant. (The term \( p_{\text{base}}^2 \) is proportional to two powers of \( \sqrt{\alpha'}/R_{\text{base}} \) and is therefore subleading in the limit of interest.) The Hamiltonian of the long-wavelength effective CFT of the string wound on the base is therefore proportional to the Hamiltonian of the fundamental string CFT for the string degrees of freedom other than \( X^0, X_{\text{base}} \), with the worldsheet coordinates \((\sigma^1, \sigma^2)\) replaced by \( \frac{1}{R_{\text{WS}}} (X^0, X_{\text{base}}) \).

This much is common to winding string effective field theory in any compactification with a large \( S^1 \) direction. However the presence of the Wilson line means that the boundary conditions on the long string are ’twisted’ by the identification between the long string worldvolume degree of freedom and the base direction. Treating \( \sigma^1 \equiv X_{\text{base}}/R_{\text{base}} \) as an independent worldvolume coordinate and letting \( \mathcal{O} \) be any operator in \( \mathcal{I} \), the boundary condition for \( \mathcal{O} \) in the long string theory must be

\[
\mathcal{O}(y_1 + 2\pi) = g \cdot \mathcal{O}(y_1) \quad (3.9)
\]

in order for effective field theory on the long string to be consistent with effective field theory for the bulk modes in the base.
In other words, the long string Hilbert space is actually described by a 'twisted sector' of the internal theory $\mathcal{I}$. Such a sector would not be consistent as a state in a modular invariant worldsheet theory, but the long string effective theory need not satisfy modular invariance – it is impossible to perform a Dehn twist without changing the topology of the string in the target space, so there is no reason for modular invariance to be a good symmetry in this theory.

However the physical consequence of the failure of modular invariance is important. The sector twisted by $g$ will in generally fail to satisfy level matching, by an amount $\hat{L}_0 - L_0 = a \pmod{1}$. This quantity is a ground state value for the coordinate momentum $P_{\sigma^1}$ whose value can only be raised and lowered in units of $\frac{1}{N}$ by modes of local operators on the long string worldsheet. As we know from free field examples, the level mismatch in the sector twisted by $g$ will in general lie in $\frac{1}{N} \mathbb{Z} \mathbb{Z}$ rather than $\frac{1}{N} \mathbb{Z}$. Using the coordinate identification $\sigma^1 = \frac{1}{R_{\text{base}}} X_{\text{base}}$, we find that the level mismatch translates into a ground state value of the spacetime momentum in the base given by

$$p_{\text{base}} = \frac{\theta}{R_{\text{base}}}$$

(3.10)

where $\theta$ in general lies in $\frac{1}{N^2} \mathbb{Z}$ and not in $\frac{1}{N} \mathbb{Z}$, as we concluded from considerations of modular invariance in the fundamental string CFT.

So we find that the $\frac{1}{N^2}$ momentum fractionation in the base of the wild Wilson line string theory is not a mysterious worldsheet artifact but an inevitable consequence of the consistency of the effective dynamics of string theory at wavelengths long compared to the string scale.

### 3.3 Are flat monodrofolds orbifolds?

We should make a brief comment about the relation between the standard 'orbifold' construction and the construction of stringy Wilson lines described in this section. Are stringy Wilson lines just orbifolds of some kind? And if they are, what kind of orbifolds are they?

In terms of a projection on the sector with $w_{\text{base}} = 0$ is identical to the construction of the untwisted sector of an orbifold of $\mathcal{I} \times S^1$, where the radius of the $S^1$ is $2N\pi R_{\text{base}}$. The projection in the sector $w_{\text{base}} = 0$ is $g \cdot g_X = 1$, where $g$ acts on the fiber and $g_X$ acts by $X_{\text{base}} \rightarrow X_{\text{base}} + 2\pi R_{\text{base}}$.

Modular invariance also demands the inclusion of twisted sectors, where the sector with winding $w_{\text{base}}$ has $g$-covariant fields twisted by $g^{w_{\text{base}}}$. However the relation be-
tween the eigenvalue $\omega$ of $g$ and the momentum $R_{base}p_{base}$ mod 1 is not the same in the winding sectors as the naive orbifold projection would dictate. This can be fixed by including a winding-dependent phase in the orbifold projection in the winding sectors:

$$g \cdot gx = 1 \rightarrow g \cdot gx = \phi_{w_{base}}$$  

(3.11)

Broadly speaking this can be considered a type of orbifold projection. But one must use some caution, because the most straightforward application of the orbifold idea leads to incorrect conclusions. The orbifold projector is the generator of a group with structure $\mathbb{Z}_N$ on the covering space. That is, both $g^N$ and $g^N_X$ act separately as the identity all states on $S^1_{\text{cover}} \times \mathcal{I}$. However neither $g$ nor $g_X$ nor the product of the two is of order $N$ when extended to the twisted sector. In particular, if $g_X$ were of order $N$, all states would have momentum fractionated in units of $\frac{1}{NR_{base}}$. Likewise if $g$ were of order $N$ when extended to the twisted sector, the OPE of $N$ identical states would always be $g$-neutral. Neither is the case in general. So although one may refer loosely to Wilson lines of wild type as orbifolds of an $N$-fold cover, it is good to bear in mind that the simple terminology conceals important features of the spectrum.

The partition function which implements such a projection is of the form

$$Z \equiv \sum_{a,b \in \mathbb{Z}} Y^a_b I^a_b \phi_1^{ab}$$  

(3.12)

where $I^a_b$ is the trace over states of the internal theory (here, all degrees of freedom other than $X_{base}$) in the sector twisted by $g^b$, with $g^a$ inserted into the trace. The $I^a_b \equiv \phi_1^{ab} I^a_b$ are the related objects which transform classically under modular transformations. The phase $\phi_1$ is obtained by exponentiating the level mismatch

$$\phi_1 \equiv \exp\{2\pi i (\tilde{L}_0 - L_0)\} \big|_{\text{sector of } \mathcal{I} \text{ twisted by } g^1}$$  

(3.13)

evaluated in the sector of the internal theory twisted by $g^1$. All the examples we consider in the paper obey this classification.$^9$

4 A monodrofold of the type II superstring

The next most nontrivial case is a monodromy by a chiral action $g$ which acts on moduli as $\tau \rightarrow -\frac{1}{\tau}, \rho \rightarrow -\frac{1}{\rho}$. The fiber theory $\mathcal{I}$ is a square torus $X^{89}$, with each circle

$^9$Our $SU(2)$ current algebra examples at level $k$ use a different basis of internal partition functions, $I^a_b$, which are related to the $I^a_b$ by phases.
at the self-dual radius, and also worldsheet fermions $\psi^{8,9}, \tilde{\psi}^{8,9}$. The operation $g$ which sends $\tau \to -\frac{1}{\tau}$ and $\rho \to -\frac{1}{\rho}$ acts on left-moving worldsheet currents as $J_{8,9}^{\alpha} \to -J_{8,9}^{\alpha}$. By worldsheet supersymmetry it must also act on left-moving worldsheet fermions as $\tilde{\psi}^{8,9} \to -\tilde{\psi}^{8,9}$.

The discussion of the action of T-duality on the bosons $X^{8,9}$, as well as the corresponding partition functions and construction of the twisted sectors, is all precisely as it is for a single boson $X^9$. In particular, all the discussion of cocycles and zero-mode dependent phases is the same for two copies of a self-dual circle as it is for one.

In addition to acting with a $-\text{sign}$ on left-moving oscillators in the 8, 9 directions, the action of T-duality it also acts with a phase $(-1)^{n_8 w_8 + n_9 w_9}$, where $n_{8,9}$ and $w_{8,9}$ are the windings and momenta of the state on the torus. The argument is precisely as in the case of the bosonic string; the T-duality operation which is multiplicatively conserved in the OPE is the one with the extra cocycle factor in its action on momentum and winding. Since the symmetry acts as a 180 degree rotation on the left handed $89$ currents, it must act as a 90 degree phase rotation $\exp\{i\Gamma^8 \Gamma^9\} = \exp\{i \tilde{\psi}^8 \tilde{\psi}^9\}$ on left-handed R sectors. The symmetry is therefore $\mathbb{Z}_4$ rather than $\mathbb{Z}_2$ in its action on spacetime fields.

The partition functions $\tilde{F}_b^{a}(\bar{\tau}) \equiv \bar{q}^{-\frac{1}{2}} \text{tr}_{\beta,\tilde{\beta},\gamma,\tilde{\gamma}}[\text{twisted by } g^a] \left( q^{\bar{L}^0 a} g^a \right)$ for the left-moving fermions and superghosts are described by the following properties.

### 4.1 Worldsheet fermions in the type II superstring

In the type II superstring the transformation laws of the internal pieces are not as simple. In this case, it will be convenient to include all worldsheet fermions $\psi^\mu, \tilde{\psi}^\mu$ and superghosts $\beta, \tilde{\beta}, \gamma, \tilde{\gamma}$ in the partition function as well. Alternately we can omit the superghosts and the fermions $\psi^{0,1}, \tilde{\psi}^{0,1}$; the result is the same.
Each is a sum of four terms $\tilde{F}^a_b = \frac{1}{2} \omega^a|_b c|_d (\tilde{F}^c_d)^3 \tilde{F}^{a+c}_{b+d}$ with $\omega^a|_0 = r^{ab}$ and $|\omega^a|_d| = 1$, with the $\tilde{F}$ defined in (2.3). The right-hand indices denote boundary conditions for the supercurrent and the left-hand indices denote boundary conditions for $g$-covariant fields.

These phases satisfy $\omega^{a+2p|_c} = (-1)^{p(b+d)+q(a+c)} \omega^{a|_c}$. That is, shifting $a$ by two changes the sign of the untwisted Ramond sectors and twisted NS sectors in the path integral, and shifting $b$ by two changes the sign of the GSO and orbifold projections. In particular, the $\omega$’s, and hence the $\tilde{F}$, are periodic mod four in each index. All the $\tilde{F}$ are determined in this way by $\tilde{F}^a_b$ with $a$ and $b$ running from 0 to 1, which we write below.

The modular $T$-transformations of the $\tilde{F}$ are determined only by their lower indices: $\tilde{F}^a_b(\tau + 1) = \exp\left\{\frac{2\pi i}{3}\right\} \phi_b \tilde{F}^{a+b}_b(\tau)$ with $\phi_0 = 1, \phi_1 = \phi_3 = \sigma \exp\{-\pi i/4\}$, and $\phi_2 = -1$.

For the definition in terms of traces to make sense, we must define $g^2$ to act as $(-1)$ on left-moving Ramond sectors, which is implied by closure of the OPE: the product of two left-moving R sectors closes on operators with an odd number of fermions $\psi^{8,9}$, since the total number of $g$-odd complex fermions is 1. This feature of the OPE can be seen by bosonizing the fermion. Then $S$-invariance demands that we define the sector twisted by $g^2$ to be a sector with all GSO projections reversed in sign relative to the usual GSO.

The modular $S$-transformations are $\tilde{F}^a_b(-\frac{1}{\tau}) = i^{-ab} \tilde{F}^{a-b}_a(\tau)$.

The anomalous phases $\phi_b$ in the T-transformations encode the level mismatch mod 1 in the NS sector twisted by $g^b$, appropriately GSO projected. In the untwisted sector the level mismatch is 0; in the first and third twisted sectors it is $\frac{3}{8}$. In the second twisted sector the level mismatch is $\frac{1}{2}$; this is simply the reversal of the GSO projection. These values assume $\sigma = -1$. Taking $\sigma = +1$ simply changes the level mismatch to $-\frac{1}{8}$ instead of $+\frac{3}{8}$ in the first and third twisted sectors. Note that the level mismatch in the third twisted sector is always the same as that in the first twisted sector, rather than differing by $\frac{1}{2}$. That is because both the GSO and orbifold projections are opposite between the first and third twisted sectors, so the difference can be made up by acting with a zero mode $\tilde{\psi}^{8,9}_0$. 39
The four functions which determine the rest are

\[
\tilde{F}_0^0(\bar{\tau}) = \frac{1}{2} \left( \tilde{F}_{0|0}^0(\bar{\tau}) - \tilde{F}_{0|1}^0(\bar{\tau}) - \tilde{F}_{0|1}^0(\bar{\tau}) \mp \tilde{F}_{0|1}^0(\bar{\tau}) \right)
\]

\[
\tilde{F}_1^0(\bar{\tau}) = \frac{1}{2} \left( \tilde{F}_{1|0}^0(\bar{\tau}) - \tilde{F}_{1|1}^0(\bar{\tau}) - i\sigma\tilde{F}_{1|1}^0(\bar{\tau}) \right)
\]

\[
\tilde{F}_1^1(\bar{\tau}) = \frac{1}{2} \left( \tilde{F}_{1|0}^1(\bar{\tau}) + i\sigma\tilde{F}_{1|1}^1(\bar{\tau}) - \tilde{F}_{1|1}^0(\bar{\tau}) \right)
\]

\[
\tilde{F}_1^1(\bar{\tau}) = \frac{i}{2} \left( \tilde{F}_{1|0}^1(\bar{\tau}) - i\sigma\tilde{F}_{1|1}^1(\bar{\tau}) - \tilde{F}_{1|1}^0(\bar{\tau}) \right)
\]

(4.1)

4.2 The full type II theory

Now we combine all the elements of the type II worldsheet theory:

- The \(\tilde{F}_b^a\) functions of the left-moving superghosts and fermions.

- The partition functions \(Z_{\beta\gamma\psi}, Z_{\delta\epsilon\zeta}, Z_{X^{0-6}}\) for all the worldsheet fields which do not depend on \(a\) and \(b\), namely the bosonic ghosts, worldsheet fields \(X^{0-6}\), and right-moving superghosts and worldsheet fermions. (The latter are already defined as sums over right-moving spin structures.) These objects are inert under modular transformations (except for the gravitational anomaly \(\exp\{-\frac{2\pi i}{3}\}\) in the T-transformation \(\tau \to \tau + 1\) of \(Z_{\beta\gamma\psi}\) coming from the unbalanced central charge.)

- The partition functions \(I^{(X^8X^9)}_b^a \equiv (I^{(X^9)}_b^a)^2\) of the bosonic coordinates of the fiber torus. As we have seen, these path integrals have classical modular transformation properties. Under \(S\) and \(T\), the indices \(a\) and \(b\), which run from 0 to 1, simply transform as a doublet under \(SL(2, \mathbb{Z})\), with no anomalous phases in the \(I^{(9)}_b^a\).

- The path integrals \(Y_b^a\) for the bosonic base coordinate \(X^7\), where \(a\) and \(b\) define the windings of the two cycles of the worldsheet torus around the target space circle \(X^7\). These functions transform classically under \(SL(2, \mathbb{Z})\).

- The partition functions \(\tilde{F}_b^a\) of the left-moving superpartners of \(X^{8,9}\). These objects are periodic mod 4 in each index, and they transform with anomalous phases. That is, they define a projective representation of \(SL(2, \mathbb{Z}/4)\).
We can then combine $Z_{\beta\gamma\psi}$ with $\tilde{F}_b^{a}$ and and $I^{(XXX^\alpha)\beta}$ to get an object whose gravitational anomaly cancels, and whose anomalous transformations are entirely due to a failure of level matching. We will call this full object $I^a_b$, and we can equally well include the other factors $Z_{X\alpha\beta\gamma}$, $Z_{\alpha\beta\gamma\psi}$ with it. Its anomalous transformation properties come entirely from those of the $\tilde{F}_b^{a}$:

$$I^a_b(\tau + 1) = \phi_b I^{a+b}_b$$

$$I^a_b(-\frac{1}{\tau}) = \phi^{-2ab}_1 I^{-b}_a(\tau)$$

(4.2)

Despite forming a projective representation of $SL(2, \mathbb{Z}/4)$, we can use these objects to construct a true linear representation of $SL(2, \mathbb{Z})$ as follows. Define $I'_0^a$ as $I_0^a$. We can then define a set of functions $I'^a_b$ with no assumed periodicity on $a$ and $b$ by defining them to represent $PSL(2, \mathbb{Z})$ classically. That is, we define

$$I'^a_b(\tau) = I_0^a(\frac{A\tau + B}{C\tau + D})$$

(4.3)

if

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix}$$

(4.4)

The resulting objects $I'^a_b$ transform classically under $PSL(2, \mathbb{Z})$, with modular transformations acting on $a$ and $b$ as a doublet and no phases multiplying $I'^a_b$. This is automatically so, because every pair of integers is $PSL(2, \mathbb{Z})$ equivalent to a unique element of the form $(a,0)$, up to overall sign. Since $PSL(2, \mathbb{Z})$ acts on $\tau$, the set of all functions on the upper half plane must consistently represent $PSL(2, \mathbb{Z})$ in a linear way. It does not matter which $PSL(2, \mathbb{Z})$ transformation we use to transform $(a',0)$ to the form $(a,b)$ in order to define the function $I'^a_b$; the definition of the $I'$ will be independent of such choices.

It is clear from our earlier discussion that the $I'$ are equal to the corresponding $I$ up to phases. Specifically, $I'^a_b = \left( \phi_1 \right)^{ab} I^a_0$, with $\phi_1$ as defined above, $\phi_1 = \sigma \exp\left\{ \pi i \frac{a}{4} \right\}$.

The important point is that the $I'_0^a$ are equal to $I^a_0$ so the untwisted sector partition function is the same in terms of the $I'$ as in terms of the $I$.

**Interpretation**

We define the partition function of the full theory as $\sum_{a,b} I'^a_b \cdot Y_b^a$. Now consider the meaning of the sum for a fixed value in $b$. If we rewrite the $I'^a_b$ in terms of the original
traces $I_b^a$, the sum is

$$Z_b \equiv \sum_a Y_b^a (\phi_1)^a I_b^a$$

This is equal to

$$\sum_a \text{tr}_{[w=b, \text{ twisted by } g^b]} \phi_b^a \left( g_b^a g^a q^L q^\dagger^L \right)$$

Without the phases of $\phi_b^a$, this would be a projection onto states with winding $b$, twisted by $g^b$, satisfying $g \cdot g_x = 1$. That is, states for which the eigenvalue of $g$ was equal to the eigenvalue of the operator $g_x = \exp\{2\pi i R_{\text{base}} p_{\text{base}}\}$, which shifts $X^7$ by $2\pi R_{\text{base}}$. This would be just as in a conventional orbifold construction. The effect of the phase $\phi_b^a$ is to alter the orbifold projection to $g \cdot g_x = \phi_b^{-1} = \phi_1^{-b}$ in the sector with $w = b$. This is a direct derivation, starting from the imposition of modular invariance, of the anomalous momentum fractionation which we introduced $ad hoc$ earlier.

We add one final note about the phases $\phi_b$. The definition $I'_b^a = \left( \phi_1 \right)^a_{ab} I_b^a$ and the anomalous transformation $I_b^a(\tau + 1) = \phi_b I_b^{a+b}(\tau)$ implies $\phi_b = \left( \phi_1 \right)^b$. This reproduces the relation we derived in equation (3.5). In particular, the phase $\phi_b \equiv \left( \phi_1 \right)^b$ represents a multiplicatively conserved, winding-dependent modification of the projection in the sector $w_{\text{base}} = 0$.

### 4.3 Spectrum of light states

Let us use the requirement of level matching to determine the spectrum of states. The simplest way to proceed is to begin by considering states with no momentum or winding on either direction of the fiber.

Restrict attention to states which are in the NS sector on the right-hand side; the right moving R spectrum can be restored at the very end by taking the OPE with NS$_+/R_+$ states with no momentum, winding or phase under $g$.

The untwisted sector contains NS$_+/\text{NS}_+$ states with $(n_7, w_7) = (a, 0) \mod (1, 8)$ and eigenvalue $\exp\{\pi i \alpha\}$ under $g$, where $a = 0, \frac{1}{7}, \frac{2}{7}$. There are also R$_+/\text{NS}_+$ states of the same form with $\alpha = \frac{1}{7}, \frac{3}{7}$. It is clear, then, that the untwisted sector has a closed OPE and reproduces the expected boundary condition for local fields in the base $x^7$ in terms of their $g$ transformations.

Now consider the first twisted sector, in particular the NS/NS states. First, note that there is a single complex left-moving fermion which is periodic in this sector,

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\[\tilde{\psi}^8 + i\tilde{\psi}^9,\] so the ground state phase under \((-1)^{F_{LW}}\) is \(\pm i\) rather than \(\pm 1\). (If necessary this can be seen by bosonizing the complex fermion.) Next, we find that the level mismatch due to Casimir energy in this sector is \(\tilde{L}_0 - L_0 = \frac{1}{4}\) in the oscillator ground state. However as shown in the appendix, twisted sectors of the T-duality transformation should have the right-moving zero modes quantized in half-integer units. So there is an extra \(\frac{1}{16} + \frac{1}{16} = \frac{1}{8}\) contribution to \(L_0\) from the zero modes \(p^{8,9}_R\) which are each equal to \(\pm \frac{1}{2}\) in the lowest state.

Thus the total level mismatch of the lowest state in the twisted sector is \(\tilde{L}_0 - L_0 = +\frac{1}{8}\). This must be cancelled by a level mismatch contribution \(\Delta(\tilde{L}_0 - L_0) = -n_7w_7\). So we take \(n_7 = \frac{1}{8}\) in the first twisted sector, which has \(w_7 = 1\). The \(g\)-eigenvalue of the ground state is the same as its \((-1)^{F_{LW}}\) eigenvalue, because the fermion zero mode \(\tilde{\psi}^8 + i\tilde{\psi}^9_0\) gives the sole contribution to both. So the ground state has \(g = (-1)^{F_{LS}} = i\).

Now we can build the rest of the states by taking the OPE with the untwisted sector. In the sector with \(w = 1\), for instance, we have NS/NS states \((n_7, w_7) = (\frac{1}{8} + \alpha, 1)\) and \(g\)-eigenvalue \(\exp\{2\pi i\alpha\}\) with \(\alpha = \frac{1}{4}, \frac{3}{4}\). In the R/NS+ sector we get states with \((n_7, w_7) = (\frac{3}{8} + \alpha, 1)\) and \(g = \exp\{2\pi i\alpha\}\) where \(\alpha\) can be 0 or \(\frac{1}{2}\).

In the sector with \(w_7 = 2\) we get a particularly interesting set of NS/NS states. We can obtain them by taking the OPE of two identical R/NS+ states \((n_7, w_7) = (\pm \frac{3}{8}, \pm 1)\) mod (1, 8). The resulting sector is NS+/NS+ with \(g = +1\) and \((n_7, w_7) = (\mp \frac{1}{4}, \mp 2)\) mod (1, 8). This sector contains the \(W_{\pm}\)-bosons and off-diagonal adjoint scalars of an SU(2) gauge symmetry, of which the left-moving current \(\partial_- X^7\) is the generically unbroken generator. The states are of the form

\[\psi_{\mu}^{\mu} \mid p^{(L)}_7 = \pm \frac{1}{4R_7} \pm \frac{2R_7}{\alpha}, p^{(R)}_7 = \pm \frac{1}{4R_7} \mp \frac{2R_7}{\alpha} \otimes (\text{osc. vac.}) \otimes (\text{NS ghost vac.})\]
considered in this section implements the symmetry \( g \), which squares to \((-1)^F\). So it is altogether logical that the sector twisted by \( g^2 \) in the theory we consider here has a massless field content resembling that of the twisted sector in the Wilson line model of [11].

The rest of the behavior of our 7D theory parallels that of the Wilson line of [11], and we shall not re-derive the parallel results in detail. Instead, we list them briefly:

- The theory has sixteen supercharges in 7 dimensions. The massless gravitino comes from the right-moving spin fields. The left-moving spin fields give rise to fields which are antiperiodic around the \( X^7 \) circle, and therefore do not give rise to massless gravitini in 7 dimensions. However we can see that the left-moving gravitini do become light in the limit \( R_7 \to \infty \). This must be so, because when \( R_7 \) decompactifies, the information about the Wilson line is lost, so we must recover a theory on the moduli space of type II string theory compactified on a \( T^2 \), which has 32 supercharges in 8 dimensions.

- The massless field content at a generic radius is a gravitational multiplet and a \( \mathcal{N} = 1 \) abelian vector multiplet. In 7D the \( \mathcal{N} = 1 \) gravitational multiplet contains a single scalar and three abelian vector fields. The vector multiplet contains three scalars and one vector field. The four scalars come from the dilaton, the radius of the base, and the \( X^7 \) components of the two right-moving abelian vectors of the fiber torus. The three graviphotons are the abelian vectors corresponding to the right-moving worldsheet currents \( \partial^+ X^{7,8,9} \). The gauge field in the vector multiplet corresponds to the current \( \partial^- X^7 \) at a generic radius.

- The value of \( n_{\text{base}} \mod \frac{1}{4} \) is determined by the periodicity of the left-moving worldsheet supercurrent \( \tilde{G} \): in left-moving NS sectors \( n_{\text{base}} \) is equal to 0 mod \( \frac{1}{4} \) and in left-moving R sectors \( n_{\text{base}} \) is equal to \( \frac{1}{8} \mod \frac{1}{4} \). It follows immediately that there are no R/NS or R/R sectors which are massless as 7-dimensional fields. We therefore expect D-brane charges in 7D to be conserved at most modulo finite integers. This is a similar situation to that in the chiral Scherk-Schwarz compactification of ( [11]).

- The theory has a self-T-duality which takes \( R_7 \) to \( \alpha'/8R_7 \). Just as in ( [11]), this is a duality between type IIA and itself, or type IIB and itself, rather than exchanging the two type II theories.
At the self-dual radius the theory has an enhanced SU(2) gauge symmetry with a complete massless vector multiplet of SU(2).

5 Conclusions

In this paper we have demonstrated the consistency of Wilson line backgrounds for a range of stringy symmetries. The Wilson lines implement monodromies by symmetries \( g \) as parallel transport around a base. The symmetries \( g \) are symmetries of a compact internal 'fiber' string theory (in all cases we consider, a torus or some type of current algebra) at a point in moduli space where \( g \) is unbroken.

We contrast the class of models we consider to the case where one considers a Wilson line for a symmetry \( g \) which is spontaneously broken in the fiber theory. (See for example [7].)

In such a case, there must always be gradients of the fiber moduli over the base \( S^1 \), because the moduli \( \mathcal{M} \) must interpolate between \( \mathcal{M}_0 \) and \( \mathcal{M}' \equiv g \cdot \mathcal{M}_0 \) as one goes around the base. This means the theory must always have positive energy, which cannot be cancelled without positive curvature, orientifolds, or some other source of negative energy. The case where the gradient energies are cancelled with a positively curved two-dimensional base (with singular points) was considered in [1] and the case where the energy is cancelled by orientifolds was considered in [7]. However in neither case is the theory described by a free worldsheet CFT with an arbitrarily weak dilaton.

In our models, by contrast, the full moduli space is described by free, solvable CFT. This allows us to go beyond the large-base limit, or equivalently the adiabatic approximation for the dynamics of the fiber. Indeed, when the base to be smaller than string scale, interesting new phenomena appear, including enhanced gauge symmetry and self-T-duality.

To demonstrate the consistency of our theories, we calculated several partition functions, checking their modular invariance. We would like to emphasize particularly that the stringy Wilson line theories make sense even for symmetries \( g \) which would lead to anomalous, inconsistent theories if one were to attempt simply to orbifold by \( g \).

\(^{10}\)In special points of the moduli spaces in [1], the moduli of the fiber are fixed under the monodromy group, and in those cases there appears to be a free worldsheet CFT describing string propagation.
We draw special attention to a distinction between the ‘tame’ and ‘wild’ classes of Wilson line. The distinction can be expressed in terms of the denominator of $n_{\text{base}} \equiv p_{\text{base}} R_{\text{base}}$ and the order $N$ of the symmetry $g$. For tame Wilson lines, the momentum in the base is always an integer multiple of $\frac{1}{N R_{\text{base}}}$, in the twisted (winding) as well as untwisted sectors. Consequently these theories can be understood in the obvious way, as orbifolds of the $N$-fold cover of the base by a $2\pi R_{\text{base}}$ shift, combined with the action of $g$. For the wild case, the momentum $n_9$ in the winding sectors can have a denominator as large as $N^2$. We have seen that in order to interpret these wild Wilson lines as quotients of an $N$-fold cover, the orbifold action must contain additional phases which do not factorize between base and fiber.

The guiding principle in our constructions was the ‘CCC principle’: consistent boundary conditions for consistent string theories lead to consistent backgrounds. Though almost a tautology, the CCC principle is an extremely useful analytical tool when one wishes to find a description whose spacetime interpretation (in the effective theory after reduction on the fiber) is straightforward but whose worldsheet description may nonetheless be quite involved. The use of the CCC principle has allowed us to derive a detailed worldsheet descriptions of a number of new theories. As in the orbifold constructions of [5, 6], level matching and closure of the OPE turns out to be a sufficient condition for modular invariance in all cases.

Most interestingly, our type II model preserves sixteen supercharges in seven dimensions and its moduli space does not appear to be equivalent to the moduli space of any previously known compactification. In particular, it does not seem to be (perturbatively or non-perturbatively) connected to any of the moduli spaces on the list of, eg, ref. [21]. The simplest way to see this is to look at the rank of the gauge group.

The existence of highly supersymmetric, solvable nongeometric backgrounds is encouraging. It may be possible to fiber these examples further over a base to obtain lower-dimensional models, and/or to break some more SUSY by adding branes, fluxes and orientifolds to obtain interesting and controllable $\mathcal{N} = 1$ compactifications. As has been frequently pointed out [1, 12, 7, 9] nongeometric compactifications inevitably dominate the string landscape, even in ensembles preserving some low energy supersymmetry. This frontier is now open for exploration.

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A Cocycles and phases in the T-duality transformation

In this section of the Appendix we will develop a few technical components used in our description of Wilson lines for T-duality transformations on tori in the bosonic string and the superstring. First we will review the theory of the fiber circle $S^1$ at the self-dual radius in the bosonic string.

A.1 Local operators and cocycles in toroidal compactification

Consider the internal CFT $\mathcal{I}$, where $\mathcal{I}$ is a fiber circle $S^1$, described by the $c = 1$ CFT of a circle at the self-dual radius $\sqrt{\alpha'}$. Each state is characterized by a momentum $n_f$ and a winding number $w_f$, as well as a set of left-moving and right-moving occupation numbers $\tilde{N}^i_f$ and $N^i_f$. (When there is no danger of confusion we will drop the subscript and write $n, w, \tilde{N}^i$ and $N^i$.) It is often convenient to recombine the quantum numbers $n, w$ into the chiral quantum numbers $p_{R,L}$, which are related by

$$
\begin{align*}
  p_L &\equiv n - w \\
  p_R &\equiv n + w \\
  n &= \frac{1}{2}(p_L + p_R) \\
  w &= \frac{1}{2}(-p_L + p_R)
\end{align*}
$$

(A.1)

The conformal weights of the ground state of a sector with given momentum and winding are

$$
\begin{align*}
  \tilde{h}_{(n,w)} &= \frac{1}{4}(n - w)^2 = \frac{1}{4}p^2_L \\
  h_{(n,w)} &= \frac{1}{4}(n + w)^2 = \frac{1}{4}p^2_R
\end{align*}
$$

(A.2)

T-duality (for a review see [14]) is an unbroken symmetry of strings on a circle of radius $\sqrt{\alpha'}$ which inverts the left-moving part $X^L$ of the coordinate of the circle:

$$
X^L \rightarrow -X^L \quad X^R \rightarrow +X^R.
$$

(A.3)

This operation switches the quantum numbers $n$ and $w$, as well as acting with a sign $(-1)^{\tilde{N}} \equiv (-1)^{\Sigma_i \tilde{N}_i}$ depending on the number of left-moving oscillators excited in a given state. The symmetry we have just defined, which we shall call $T$, is cyclic of order 2 and is a symmetry of the conformal field theory.

We can decompose our space of states into two sectors, which we call $\mathcal{U}^\pm$, where the sign $\pm$ labels the eigenvalue of a sector under $T$. The $\mathcal{U}$ denotes the fact that all states in $\mathcal{I}$ have untwisted, periodic boundary conditions for $X^L$; their spin $\tilde{h} - h$ is equal to zero modulo the integers.
The discrete symmetry $X^L \to -X^L$ corresponds to an action on Hilbert space

$$\left| n, w, \tilde{N}_i, N^i \right> \to \Omega_{n,w} (-1)^{\sum \tilde{N}_i} \left| w, n, \tilde{N}_i, N^i \right> \quad (A.4)$$

which switches $n$ and $w$, and anticommutes with every left-moving oscillator $\tilde{\alpha}$. The only freedom is a possible phase $\Omega_{n,w}$ which depends on $n$ and $w$ and for consistency must satisfy $\Omega_{n,w} \Omega_{w,n} = 1$.

### A.2 Conserved and non-conserved T-duality operations

The simplest choice of phase would be $\Omega_{w,n} = 1$, and the corresponding naive T-duality operation on the single-string Hilbert space is a symmetry of the spectrum for all $w, n, \tilde{N}_i$. However it is not a symmetry of the CFT as a whole. In particular, this symmetry is not preserved by the operator product expansion. Therefore $\Omega_{w,n} = 1$ does not correspond to a symmetry which is conserved in string scattering interactions.

Let us see why the naive T-duality symmetry, which we will call $T_0$, is not preserved by the structure of the OPE. The origin of the nonconservation of $T_0$ is in the cocycle factor in the definition of vertex operators carrying momentum and winding. (For an introduction, see for instance [15]). The naive expressions

$$V^{(0)}_{(n,w)} = \exp\{i(n-w)X_L + i(n+w)X_R\} = \exp\{ip_L X_L + ip_R X_R\} \quad (A.5)$$

are not mutually local and are therefore cannot be good conformal fields corresponding to states via the state-operator correspondence.

To see that they are not local, we review the argument in [15]. The operator $X_L$ is not local with respect to itself, and similarly for $X_R$. The OPE of a chiral scalar with itself has a logarithmic singularity, which we can give a definite value by letting the branch cut run along the negative imaginary direction from the location of each operator. Then

$$V^{(0)}_{(n,w)}(\sigma_1 + i\epsilon)V^{(0)}_{(n',w')}(0) = \exp\{\pi i (nw' + wn')\} V^{(0)}_{(n,w)}(\sigma_1 - i\epsilon)V^{(0)}_{(n',w')}(0)$$

for $\sigma_1, \epsilon \in \mathbb{R}$ and $\epsilon \to 0^+$. Wick rotating, and defining the Lorentzian timelike direction as $i\sigma_2 = i\text{Im}z$, the discontinuity becomes a failure of the spacelike separated operators $V^{(0)}_{(n,w)}(\sigma_1, t)$ and $V^{(0)}_{(n',w')} (\sigma_1, t)$ to commute:

$$V^{(0)}_{(n,w)}(\sigma_1, t) \ V^{(0)}_{(n',w')} (\sigma_1, t) = \exp\{\pi i (nw' + wn')\} V^{(0)}_{(n',w')}(t) \ V^{(0)}_{(n,w)}(\sigma_1, t)$$

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To construct the appropriate local operators, multiply the naive expression for the vertex operator by the nonlocal expression \( \hat{C}_{(n,w)} \):

\[
V_{(n,w)} = \hat{C}_{(n,w)} \cdot V^{(0)}_{(n,w)}
\]

where \( \hat{C}_{(n,w)} \) is a nonlocal expression built from the zero modes of the field \( x_F \):

\[
\hat{C}_{(n,w)} \equiv \exp\{\pi i w \hat{n} - \pi i \frac{nw}{2}\}
\]

and \( V^{(0)}_{(n,w)} \) is the naive expression given above for the vertex operator.

Here, the symbol \( \hat{n} \) denotes the operator which acts on the Hilbert space as an infinitesimal translation, and \( n, w \) are the \( c \)-numbers which label the vertex operator \( V_{(n,w)} \). Concretely, \( \hat{C}_{(n,w)} \) is equal to \( i^{nw} \) if \( w \) is even and \( i^{nw} \cdot (-1) \hat{n} \) if \( w \) is odd.\(^{11}\)

This nonlocal operator \( \hat{C}_{(n,w)} \) cancels the nonlocality in the OPE of the naive vertex operators \( V_{(n,w)} \) with each other. There is some arbitrariness in the choice of \( \hat{C}_{(n,w)} \). Its only necessary properties are that it be made only from the operators \( \hat{p}_{L,R} \) and that it satisfy

\[
\hat{C}_{(n,w)} \cdot V^{(0)}_{(n',w')} = \gamma_{(n,w|n',w')} V^{(0)}_{(n',w')} \hat{C}_{(n,w)}
\]

with \( \gamma_{(n,w|n',w')} \) a phase satisfying

\[
\gamma_{(n,w|n',w')} \gamma_{(n',w'|n,w)} = \exp\{\pi i (n'w - w'n)\}
\]

The redefined vertex operators are good, mutually local conformal fields. With the effect of the cocycle taken into account, the OPE of two vertex operators is local:

\[
V_{(n,w)} \cdot V_{(n',w')} \sim \exp\{\frac{\pi i}{2} (w'n - n'w)\} (\Delta \bar{z})^{\frac{1}{2} p_{L'} p_{L}} (\Delta z)^{\frac{1}{2} p_{R'} p_{R}}.
\]

That is, if we define fractional powers of \( \Delta z \) with the branch cuts running from \( \Delta z = 0 \) to \( \Delta z = -i\infty \), then there is no branch cut in the overall OPE, as long as \( n \) and \( w \) are integers.

Now it is easy to check explicitly that the naive \( \mathbb{Z}_2 \) T-duality operation \( T_0 \) is not respected by the OPE, and therefore is not a symmetry of string interactions. Suppose \( \]

\(^{11}\)The phase \( i^{-nw} \) does not appear in the cocycle as defined in [15]. We include it so that the full vertex operator including the cocycle will obey the standard hermiticity condition \( V^+_{(n,w)} = V_{(-n,-w)} \). Our choice also makes the T-duality properties of our vertex operators simpler than those with the phase convention in [15].
we prepare states which are even under $T_0$: let
\[
\tilde{V}^+_{(n,w)} \equiv \frac{1}{\sqrt{2}} \left( V_{(n,w)} + V_{(w,n)} \right).
\] (A.10)

The vertex operators $\tilde{V}^+_{(n,w)}$ are even under $T_0$, by construction. If the symmetry operation $T_0$ were respected by the OPE, then the OPE of two $\tilde{V}^+$ operators would have to contain only vertex operators which are even under $T$. But we can see that this is not the case. Using the OPE we just derived, we have
\[
\tilde{V}^+_\ell(z_1, \bar{z}_1) \tilde{V}^+_{\ell'}(z_2, \bar{z}_2)
\sim \frac{1}{2} \bar{z}_{12}^{\frac{1}{2}k_Lk'_{L}} \bar{z}_{12}^{\frac{1}{2}k_Rk'_{R}} \left( i^{n'w-n'w} V_{(n+n',w+w')} + i^{n'n-w'n} V_{(w+w',n+n')} \right)
\] (A.11)

\[
+ \frac{1}{2} \bar{z}_{12}^{-\frac{1}{2}k_Lk'_{L}} \bar{z}_{12}^{-\frac{1}{2}k_Rk'_{R}} \left( i^{n'n-w'n} V_{(w+w',n+n')} + i^{n'w-n'w} V_{(w+n',n+w')} \right).
\]

The right hand side of the OPE contains the terms
\[
V_{(n+n',w+w')} + (-1)^{n'w-w'n} V_{(w+w',n+n')}
\] (A.12)
and
\[
V_{(n+w',w+n')} + (-1)^{w'n-n'n} V_{(w+n',n+w')},
\] (A.13)

which is even under $T_0$ only if $n'w - w'n$ and $n'n - w'w$ are both even, which is not the case in general.

Let us instead organize states according to their eigenvalues under the operation $T \equiv T_0 \cdot (-1)^{\hat{h}\hat{\bar{w}}} = (-1)^{\hat{h}\hat{\bar{w}}} \cdot T_0$. The operator $T$ is also a symmetry of the spectrum, but unlike $T_0$ we will now see it is also preserved by the OPE.

Define
\[
V^\pm_{(n,w)} \equiv \frac{1}{\sqrt{2}} (V_{(n,w)} \pm T \cdot V_{(w,n)})
\] (A.14)

The states $V^\pm_{(n,w)}$ have definite conformal weight $(\hat{h}_{(n,w)}, \hat{h}_{(w,n)})$ and definite eigenvalue $\pm$ under the redefined T-duality operation $T$. Note that $V_{(n,w)}$ is proportional but necessarily equal to $V^\pm_{(n,w)}$, and that some of the $V^\pm_{(n,w)}$ may vanish, when $w = n$.

Using our definition of $V^\pm_{(n,w)}$ and our OPE for the $V_{(n,w)}$, letting $\phi, \phi'$ take values in $\{ \pm 1 \}$ we find that the OPE of $V^\phi_{(n,w)}$ with $V^{\phi'}_{(n',w')}$ contains the primaries
\[
\alpha_1 V_{(n+n',w+w')} + \alpha_2 V_{(w+w',n+n')}
\] (A.15)
and
\[ \beta_1 V_{(n+w',w+n')} + \beta_2 V_{(w+n',n+w')} \] (A.16)
and their descendents, where
\[ \alpha_1 \equiv \frac{1}{2} i^{nw'-wn'} \]
\[ \alpha_2 = \frac{1}{2} i^{wn'-nw'}(-1)^{nw+n'w'} \phi \cdot \phi' \] (A.17)
\[ \beta_1 \equiv \frac{1}{2} \phi' (-1)^{n'+nw'} i^{nn'-ww'} \]
\[ \beta_2 \equiv \frac{1}{2} \phi (-1)^{nw} i^{ww'-nn'} \]

The overall normalization of the vertex operators on the RHS of the OPE is not of interest to us; therefore we only really care about the relative phase between \( \alpha_1 \) and \( \alpha_2 \), and similarly for \( \beta_{1,2} \). We have
\[ \alpha_2 = (-1)^{wn'-nw'+nn'+ww'} \cdot \phi \cdot \phi' \cdot \alpha_1 \] (A.18)
\[ = (-1)^{(n+n')(w+w')}(\phi \phi') \alpha_1 \]
and
\[ \beta_2 = (-1)^{n'w'+nw+ww'-nn'} \phi \phi' \beta_1 \]
\[ = (-1)^{(n+w')(n+n')}(\phi \phi') \beta_2 \] (A.19)

So the combinations which occur are precisely
\[ V_{(n+n',w+w')} + (\phi \phi')(-1)^{(n+n')(w+w')} V_{(w+w',n+n')} = \sqrt{2} V_{(n+n',w+w')} \] (A.20)
and
\[ V_{(n+w',w+n')} + (\phi \phi')(-1)^{(n+w')(w+n')} V_{(w+n',n+w')} = \sqrt{2} V_{(n+w',w+n')} \] (A.21)

The important point is that the OPE of two primaries with \( T \)-eigenvalues \( \phi \) and \( \phi' \) contains only primaries with \( T \)-eigenvalue \( \phi \phi' \); in other words, the \( \mathbb{Z}_2 \) symmetry group generated by \( T \) is preserved by the OPE.
A.3 Twisted sectors of the conserved T-duality

In order to define the Wilson line CFT, it is necessary to extend the action of $T$ from the theory of the $I$ circle by itself, to states with twisted boundary conditions on $X^L$ – that is, boundary conditions in which the field $X$ returns to itself up to a T-duality transformation. We need to extend $T$ in such a way that the OPE including twisted states will preserve $T$. Our ability to extend $T$ as an operation of order two is equivalent to the property that the Wilson line CFT in the bosonic string is tame.

In a twisted state $C_T$ the left-moving oscillators $\tilde{\alpha}$ are half-integrally moded with frequencies $a + \frac{1}{2}, a \in \mathbb{Z}, a \geq 0$. The left-moving zero mode $x_0^L$ is not present in this sector, so there is no $p_L$. Consequently $n = w$ in the twisted sector.

The twisted sectors have a Casimir momentum $\tilde{L}_0 - L_0 \in \mathbb{Z} + \frac{1}{16}$, which we cancel by an appropriate quantization of the value of $p_R = n = w$ in the twisted sector, as discussed in section 2:

$$p_R = \pm \frac{1}{2} \quad \left( \mod 1 \right) \quad (A.22)$$

The resulting twisted states satisfy level matching mod $\frac{1}{2}$ in the bosonic string, without the addition of a base direction $X_{\text{base}}$. The level mismatch of $\frac{1}{2}$ can be made up by acting with left-moving oscillators, which are half-integrally moded in the twisted sector.

In the next section of the Appendix we derive this quantization rule from an explicit modular transformation of the untwisted sector partition function with an insertion of $T$. For now, we simply assume this quantization rule for purposes of checking the consistency of operator product expansions involving two untwisted states and a twisted state.

An expression for the cocycle in the twisted sector

Our operator expression for untwisted vertex operators cannot be applied as an operator which acts on twisted states, since it contains an operator $\hat{p}_L$ which is not defined in the twisted sector. Therefore we need to find a definition of the untwisted operator $V_{(n,w)}$ which renders any two vertex operators local with respect to one another, yet can be defined as operators which act on twisted states. This is necessary if the OPE of a twisted and an untwisted state is to be given an unambiguous form.

Our expression for the cocycle in the twisted sector is something which can involve
the labels \(n, w \simeq k_L, k_R\) of the untwisted state, but can involve only the Hilbert space operator \(\hat{p}_R\) and not the operator \(\hat{p}_L\) because \(\hat{p}_L\) does not exist in the twisted sector.

We find that the candidate cocycle

\[
C^{(T)}_{(n,w)} \equiv \exp\left\{ -\frac{\pi i}{2} k_L \hat{p}_R + \frac{\pi i}{4} k_L k_R \right\}
\]  

satisfies the necessary condition (A.8). This means that the vertex operators

\[
V_{(n,w)} \equiv C^{(T)}_{(n,w)} V^{(0)}_{(n,w)}
\]

are mutually local with respect to one another in the presence of a twist field.

We also need an expression for \(T\) itself in the twisted sector which is consistent with multiplication by untwisted local operators. We would expect the consistent definition to be such that \(T = 1\) for the twisted states which are level matched mod 1, and \(T = -1\) for states with level mismatch equal to \(hh \mod 1\). A candidate \(T\) operator in the twisted sector is

\[
T^{(T)} = (-1)^{\left( \sum_{m=0}^{\infty} \hat{N}_m + \frac{1}{2} \right)} \exp\left\{ \frac{\pi i}{2} (\hat{p}_R^2 - \frac{1}{4}) \right\}
\]

As in the untwisted sector, the exponent is quadratic in the zero modes of \(X\).

It is not clear \textit{a priori} that the definition of \(T\) in the twisted sectors is consistent with the definition in the untwisted sectors. We can establish this by showing that the \(T\)-even and \(T\)-odd twisted states are separately modules over \(T\)-even untwisted vertex operators, defined with the use of our expression for the cocycle.

We can show this by explicit calculation. Let us summarize the action of \(T\)-even untwisted vertex operators \(U_{(pr=1)}^{(+)}\) with \(p_R\) equal to 1 \((\mod 4)\) on twisted states. We have:

\[
U_{(pr=1)}^{(+)} = \begin{pmatrix}
|p_R \equiv -\frac{1}{2} (\mod 4), N_L \equiv a (\mod 2)\
|p_R \equiv +\frac{1}{2} (\mod 4), N_L \equiv a (\mod 2)\
|p_R \equiv +\frac{3}{2} (\mod 4), N_L \equiv a (\mod 2)\
|p_R \equiv +\frac{5}{2} (\mod 4), N_L \equiv a (\mod 2)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
|p_R \equiv +\frac{1}{2} (\mod 4), N_L \equiv a + 1 (\mod 2)\
|p_R \equiv +\frac{3}{2} (\mod 4), N_L \equiv a + 1 (\mod 2)\
|p_R \equiv +\frac{5}{2} (\mod 4), N_L \equiv a (\mod 2)\
|p_R \equiv -\frac{1}{2} (\mod 4), N_L \equiv a + 1 (\mod 2)
\end{pmatrix}
\]
The other $T$-even untwisted vertex operators act as

$$U^{(+)}_{(p_R=2)} \begin{pmatrix} |p_R \equiv -\frac{1}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{3}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{1}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{5}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \end{pmatrix}$$ (A.27)

$$= \begin{pmatrix} |p_R \equiv \frac{3}{2} \pmod{4}, N_L \equiv a + 1 \pmod{2}\rangle \\ |p_R \equiv \frac{5}{2} \pmod{4}, N_L \equiv a + 1 \pmod{2}\rangle \\ |p_R \equiv -\frac{1}{2} \pmod{4}, N_L \equiv a + 1 \pmod{2}\rangle \\ |p_R \equiv \frac{1}{2} \pmod{4}, N_L \equiv a + 1 \pmod{2}\rangle \end{pmatrix}$$

$$U^{(+)}_{(p_R=3)} \begin{pmatrix} |p_R \equiv -\frac{1}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{3}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{1}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{5}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \end{pmatrix}$$ (A.28)

$$= \begin{pmatrix} |p_R \equiv \frac{5}{2} \pmod{4}, N_L \equiv a + 1 \pmod{2}\rangle \\ |p_R \equiv -\frac{1}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{1}{2} \pmod{4}, N_L \equiv a + 1 \pmod{2}\rangle \\ |p_R \equiv \frac{3}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \end{pmatrix}$$

$$U^{(+)}_{(p_R=0)} \begin{pmatrix} |p_R \equiv -\frac{1}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{3}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{1}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{5}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \end{pmatrix}$$ (A.29)

$$= \begin{pmatrix} |p_R \equiv -\frac{1}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{3}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{1}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \\ |p_R \equiv \frac{5}{2} \pmod{4}, N_L \equiv a \pmod{2}\rangle \end{pmatrix}$$
In particular, there exist two complementary sets of twisted states which are closed under the action of all $T$-even untwisted vertex operators $V^{(+)}$; namely the set of level matched states

$$\{ |p_R \equiv \pm \frac{1}{2} (\mod 4), N_L = \text{even} \} \oplus \{ |p_R \equiv \pm \frac{3}{2} (\mod 4), N_L = \text{odd} \}$$ (A.30)

and the set of un-level-matched states

$$\{ |p_R \equiv \pm \frac{1}{2} (\mod 4), N_L = \text{odd} \} \oplus \{ |p_R \equiv \pm \frac{3}{2} (\mod 4), N_L = \text{even} \}$$ (A.31)

Therefore we can define the orbifold projection in the twisted sector as the restriction to the first set. Level-matching follows immediately, as does closure of the untwisted-twisted vertex operator algebra. It remains to be shown is that two allowed twisted vertex operators close on an allowed untwisted vertex operators. To see this, we note the existence of an quantum number, namely $p_L \mod 1$, which commutes with T-duality. In order for the OPE to be potentially consistent, the value of $p_L \mod 1$ in the twisted sectors must be equal to 0 mod $\frac{1}{2}$. As a result, $\exp\{2\pi ip_L\}$ never changes under the operation $p_L \rightarrow -p_L$, meaning that the operator $\exp\{2\pi ip_L\}$, which measures $p_L \mod 1$ commutes with T-duality and therefore can be assigned a definite value in any twisted sector. This, despite the fact that $p_L$ is nondynamical in the twisted sectors and does not contribute to the energy or the Virasoro generators. To be consistent, then, the twisted sectors are paired with $p_R$ values in such a way that $p_R - p_L$ is always even, in the twisted as well as untwisted sectors. This guarantees the closure consistency of the twisted-twisted OPE onto untwisted sectors.

In the bosonic string, the orbifold by T-duality is the same as the unorbifoloded circle theory at twice the self-dual radius – equivalently, the interval theory at the special radius where it is equivalent to a circle theory. To see this, observe that the spectrum is left-right symmetric at low levels – note for instance the currents are $U(1)_L \times U(1)_R$ and nothing more. In fact the T-duality transformation, with the cocycle taken carefully into account, is a nonchiral symmetry, conjugate to a diagonal $\mathbb{Z}_2$ of the $SU(2)_L \times SU(2)_R$ symmetry.

When embedded as a symmetry of the superstring, however, the T-duality operation we describe here is chiral and therefore not conjugate to a diagonal $\mathbb{Z}_2$. 
B Fiber partition functions for T-duality Wilson lines

Here we work out the partition functions in twisted and untwisted sectors with insertions of T-duality transformations for worldsheet bosons and fermions.

B.1 Bosonic partition functions

In the untwisted sector, our projection again correlates the phase of $g$ with the momentum $n_7 \mod 1$ in the obvious way. Let us construct the partition function, enforcing modular invariance by hand. Here $Y^a_b$ is the path integral sector with $X^7$ winding $b$ times around the spacelike and $a$ times around the timelike cycle of the torus. Meanwhile we would like $I^p_q$ to represent something like the path integral with a '$g^p$ cut' along the timelike cycle and a '$g^q$ cut' along the spacelike cycle. Unfortunately, $g$ and its powers are really 'quantum' rather than classical operations, so it is far from clear what it should mean to define a path integral with cuts by $g^p$ and $g^q$ along cycles. However objects such as $I^p_q$ have been sufficiently useful in our study of Wilson line CFT for free fermion theories that we are tempted to try to define an analog here.

Here is how we do it. We think we know how the multiplicatively conserved $g$ charge is defined, at least in the untwisted sector. This gives a definition to $I^0_0$. Now, in order to modular transform, we want to define $I^0_p$ with a coefficient such that it is a sum of $q^h \bar{q}^{\bar{h}}$ with positive integer coefficients. From there we can continue.

So let us do this, first, for the simple T-duality Wilson line in the bosonic string. So how do we define $I^0_0$? Well let’s begin by factorizing $I^0_0$ which is just the partition function for the circle at the self-T-dual radius.

$$I^0_0 = b \, B^0_0 \, z^0_0$$ \hspace{1cm} (B.1)

where $b$ is the partition function for the right-moving degrees of freedom:

$$b \equiv q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)^{-1}$$ \hspace{1cm} (B.2)

and $B^0_0$ is the corresponding partition function for left-moving degrees of freedom

$$B^0_0 \equiv \bar{q}^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 - \bar{q}^n)^{-1}$$ \hspace{1cm} (B.3)

and $z^0_0$ is the zero mode partition function:

$$z^0_0 = \sum_{n,w} q^{\frac{1}{4}(n+w)^2} \bar{q}^{\frac{1}{4}(n-w)^2}$$ \hspace{1cm} (B.4)
Now let us define the functions with 'T-duality cuts' along the timelike direction. This just means inserting a $-1$ into the partition functions for each T-duality odd state. As discussed earlier, the one non-obvious aspect to this is that there is a factor of $(-1)^{nw}$ in the definition of the correct T-duality in the zero mode partition function. So

$$z_0^1 = \sum_n (-1)^{n^2} q^{n^2}$$

and of course

$$B_0^1 = \bar{q}^{-\frac{1}{2\pi}} \prod_{n=1}^\infty (1 + \bar{q}^n)^{-1}$$

Now, we will insert a phase into the definition of $z_0^0$ and of $B_0^0$ so that they are sums of powers of $q, \bar{q}$ with positive integer coefficients. We can perform the transformations by noting that these sums are equal to certain things, and using their known modular properties. For instance,

$$b(\tau) = \eta(\tau)^{-1}$$

$$z_0^0(\tau) = \alpha_{00}(0, 2\tau)\alpha_{00}(0, 2\tau) + \alpha_{10}(0, 2\bar{\tau})\alpha_{10}(0, 2\bar{\tau})$$

$$z_0^1(\tau) = \alpha_{01}(0, 2\tau)$$

$$B_0^0(\tau) = \eta(\bar{\tau})^{-1}$$

$$B_0^1(\tau) = \left( \frac{2\eta(\tau)}{\alpha_{10}(0, \tau)} \right)^{\frac{1}{2}}$$

and then define

$$I_p^p \equiv b \ z_p^p \ B_q^p$$

First let us verify the $S$ and $T$ transformations of $I_0^0$. We have

$$z_0^0 \rightarrow \frac{1}{2} \ | - i\tau| \left( \alpha_{00}(0, \tau/2)\alpha_{00}(0, \bar{\tau}/2) + \alpha_{01}(0, \tau/2)\alpha_{01}(0, \bar{\tau}/2) \right)$$

$$= | - i\tau| \left( \sum_{n_L, n_R} \frac{1}{2} (1 + (-1)^{n_L+n_R}) \bar{q}^{\frac{1}{4}n_L^2} q^{\frac{1}{4}n_R^2} \right)$$

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\( = | - i \tau | \left( \sum_{n,w} q^{\frac{1}{4}(n-w)^2} q^{\frac{1}{4}(n+w)^2} \right) \)

\( = | - i \tau | \left( \alpha_{00}(0, 2\tau)\alpha_{00}(0, 2\tau) + \alpha_{10}(0, 2\tau)\alpha_{10}(0, 2\tau) \right) \) \hspace{1cm} (B.10)

\( = | - i \tau | z_0^0 \)

so

\[ I_0^0(\tau) \equiv b(\tau) \ B_0^0(\tau) \ z_0^0(\tau) \] \hspace{1cm} (B.11)

transforms into itself under \( \tau \to -\frac{1}{\tau} \):

\[ I_0^0(-\frac{1}{\tau}) = I_0^0(\tau) \] \hspace{1cm} (B.12)

and the invariance under the T transformation is self-evident due to level matching.

As for \( I_1^0 \), the T transformation is clearly still fine. Under the S transformation the components transform as

\[ b(-\frac{1}{\tau}) = (-i\tau)^{-\frac{1}{2}} b(\tau) \]

\[ z_0^1(-\frac{1}{\tau}) = \frac{1}{\sqrt{2}} (-i\tau)^{\frac{1}{2}} \alpha_{10}(0, \frac{1}{2}\tau) = \frac{1}{\sqrt{2}} (-i\tau)^{\frac{1}{2}} z_0^0(\tau) \] \hspace{1cm} (B.13)

\[ B_0^1(-\frac{1}{\tau}) = \sqrt{2} \left( \frac{n(\tau)}{\alpha_{01}(0,\tau)} \right)^{\frac{1}{2}} = \sqrt{2} \ B_1^0(\tau) \]

with

\[ z_1^0(\tau) = \sum_n q^{\frac{n}{4}(n-\frac{1}{2})^2} = \alpha_{10}(0, \frac{1}{2}\tau) \]

\[ = 2 \sum_n q^{(n-\frac{1}{4})^2} \]

\[ B_1^0(\tau) = \left( \frac{n(\tau)}{\alpha_{01}(0,\tau)} \right)^{\frac{1}{2}} = q^{-\frac{1}{16}} \ q^{+\frac{1}{16}} \ \prod_{m=1}^{\infty} (1 - q^{16m})^{-1} \cdot (1 - q^{16m-\frac{1}{2}}) \]

The factor \( B_1^0 \) is the partition function for a set of half-odd-integrally moded left-moving bosonic oscillators, including the Casimir energy of \(+\frac{1}{16}\) from the antiperiodicity of the real boson. The factor \( z_1^0 \) describes a zero mode sum of a right-moving zero mode only, whose momenta are also half-odd-integers. We shall call this the ‘twisted sector’ of the T-duality projection. Indeed, it contains an infinite number of level matched
states, since the \(+\frac{1}{16}\) coming from the left-moving Casimir energy is balanced by the 
\(+\frac{1}{16}\) coming from the right-moving zero mode state with momentum \(\frac{1}{2}\). So we define

\[
I_1^0 \equiv b \ z_1^0 \ B_1^0
\]  

(B.15)

So far, the natural phases are all correct. In the last step we will have to be a bit 
careful about the phase.

Applying the T transformation, we have

\[
b \rightarrow \exp\{-\frac{\pi i}{12}\}b
\]

\[
z_1^0 \rightarrow \exp\{\frac{\pi i}{8}\}z_1^1
\]  

(B.16)

\[
B_1^0 \rightarrow \exp\{-2\pi i\left(-\frac{1}{24} + \frac{1}{16}\right)\}B_1^1
\]

where

\[
z_1^1 \equiv \sum_n \exp\{\frac{\pi i}{2}(n^2 - n)\}q^{\frac{1}{2}(n-\frac{1}{2})^2}
\]

\[
= 2 \sum_n (-1)^n q^{n(\frac{1}{2}-\frac{1}{2})^2}
\]

\[
= \sqrt{2} \sum_n \exp\{\pi i(\frac{1}{4} - \frac{n}{2})\} q^{\frac{1}{2}(n-\frac{1}{2})^2}
\]  

(B.17)

\[
B_1^1 = \bar{q}^{-\frac{\tau}{4}} q^{\frac{1}{2} \cdot \frac{1}{2}} \prod_{m=1}^{\infty} (1 + \bar{q}^{m-\frac{1}{2}})^{-1} = \left( \frac{\eta(\tau)}{\alpha_{00}(0,\tau)} \right)^{\frac{1}{2}}
\]

So letting \(I_1^1 \equiv b z_1^1 B_1^1\), we have \(I_1^a(\tau + 1) = I_1^{a+1}(\tau)\). Proving modular invariance now 
amounts to showing that \(I_1^1\) transforms to itself, with no phase, under \(\tau \rightarrow -\frac{1}{\tau}\). First, 
check the transformation of \(z_1^1\):

\[
z_1^1\left(-\frac{1}{\tau}\right) = \sqrt{2} \alpha_{10}(-\frac{1}{4}, -\frac{1}{2\tau})
\]

\[
= \sqrt{2} \alpha_{10}(\frac{1}{2\tau}, -\frac{1}{2\tau})
\]  

(B.18)

\[
= \sqrt{2} \exp\{\pi i(-\tau/2)^2/(2\tau)\} \left(-2i\tau\right)^{\frac{1}{2}} \alpha_{01}((-\frac{1}{2}), 2\tau)
\]
\[= 2 \exp\left\{ \frac{\pi i \tau}{8} \right\} (-i \tau)^{\frac{1}{2}} \alpha_{01}\left((-\frac{\tau}{2}), 2\tau\right)\]
\[= 2 (-i \tau)^{\frac{1}{2}} q^{\frac{1}{16}} \alpha_{01}\left((-\frac{\tau}{2}), 2\tau\right)\]
\[= 2 (-i \tau)^{\frac{1}{2}} \sum_n (-1)^n q^{\left(n-\frac{1}{2}\right)^2}\]
\[= (-i \tau)^{\frac{1}{2}} z_1^1(\tau)\]

Next, check that the transformation of \(B_1^1\) is trivial:
\[B_1^1(-\frac{1}{\tau}) = B_1^1(\tau)\]  
(B.20)

so
\[I_1^1(-\frac{1}{\tau}) = I_1^1(\tau)\]  
(B.21)

### B.2 Fermionic partition functions

Now we consider partition functions for left-moving worldsheet fermions in the type II string on a T-duality Wilson line background. T-duality acts on the fermions \(\tilde{\psi}^{8,9}\) with a \(-1\) sign, as dictated by worldsheet SUSY. The left-moving fermions therefore break up into a block of six \(\tilde{\psi}^{2-7}\) and a block of two \(\tilde{\psi}^{8,9}\) which transform differently under \(T\). We will define four partition functions

\[\tilde{F}_{a}^{b},\]  
(B.22)

with \(a\) and \(b\) defined mod two, corresponding to possible cuts by elements of the group element \(T\), along the two cycles of the torus. Each of these four partition functions can be decomposed further as a sum over four spin structures for left moving fermions. For instance, our definition of the untwisted sector means we can write

\[\tilde{F}_0^a \equiv \frac{1}{2} \left( \tilde{F}_{0|0}^{a|0} - \tilde{F}_{0|0}^{a|1} - (i \sigma)^a \tilde{F}_{0|1}^{a|0} + (i \sigma)^a \tilde{F}_{0|1}^{a|1} \right)\]  
(B.23)

where

\[\tilde{F}_{b|d}^{a|c} \equiv (\tilde{F}_d^c)^3 \tilde{F}_{b+d}^{a+c}\]  
(B.24)

and \(\sigma = \pm 1\) is a sign choice describing the \(g\)-projection \(g = \pm i\) in the left-moving Ramond sectors. The possibilities are \(\pm i\) rather than \(\pm 1\) because the product of two
identical spin fields for the $\tilde{\psi}^{8,9}$ fermions contains in its OPE only operators which are odd, rather than even, under $g$. This can be seen easily through bosonization.

The $\tilde{F}^a_b$, defined as traces of $\bar{q}^b\tilde{A}$ in the sector twisted by $b$, appropriately GSO projected and summed over R and NS states, sometimes have a nontrivial eigenvalue of $g$ even in the NS$_+$ sector. In the NS sector twisted by $g^b$, the ground state gets a phase $i^b$ under $g$ due to fermion zero modes. Therefore the contribution $\tilde{F}^a_{b|0}$ to $\tilde{F}^a_b$ has a coefficient $i^{ab}$. Combined with the requirement that the $\tilde{F}^a_b$ transform into one another under modular transformations up to a phase, this uniquely fixes the $\tilde{F}^a_b$. We have

$$\tilde{F}^a_b \equiv \frac{1}{2} \sum_{c,d} \omega^{a|c}_{b|d} (\tilde{F}^c_d)^3 \tilde{F}^{a+c}_{b+d}$$  \hspace{1cm} (B.25)$$

with $\omega^{a|0}_{b|0} = i^{ab}$ and $|\omega^{a|c}_{b|d}| = 1$. The values of $\tilde{F}^a_b$ with $a$ and $b$ running from 0 to 1 are

$$\tilde{F}^0_0(\tau) = \frac{1}{2} \left( \tilde{F}^{0|0}_{0|0}(\tau) - \tilde{F}^{0|1}_{0|0}(\tau) - \tilde{F}^{0|0}_{0|1}(\tau) + \tilde{F}^{0|1}_{0|1}(\tau) \right)$$

$$\tilde{F}^1_0(\tau) = \frac{1}{2} \left( \tilde{F}^{1|0}_{0|0}(\tau) - \tilde{F}^{1|1}_{0|0}(\tau) - i\tilde{F}^{1|0}_{0|1}(\tau) + i\sigma \tilde{F}^{1|1}_{0|1}(\tau) \right)$$

$$\tilde{F}^0_1(\tau) = \frac{1}{2} \left( \tilde{F}^{0|0}_{1|0}(\tau) + i\sigma \tilde{F}^{0|1}_{1|0}(\tau) - \tilde{F}^{0|0}_{1|1}(\tau) \pm i\sigma \tilde{F}^{0|1}_{1|1}(\tau) \right)$$

$$\tilde{F}^1_1(\tau) = \frac{i}{2} \left( \tilde{F}^{1|0}_{1|0} - i\sigma \tilde{F}^{1|1}_{1|0} - \sigma \tilde{F}^{1|0}_{1|1} \pm i\sigma \tilde{F}^{1|1}_{1|1} \right)$$  \hspace{1cm} (B.26)$$

The phases have a periodicity mod 2 which defines the rest:

$$\omega^{a+2p|c}_{b+2q|d} = (-1)^{p(b+d) + q(a+c)} \omega^{a|c}_{b|d}$$  \hspace{1cm} (B.27)$$

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