Communication in the Presence of a State-Aware Adversary

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Abstract—We study communication systems over the state-dependent channels in the presence of a malicious state-aware jamming adversary. The channel has a memoryless state with an underlying distribution. The adversary introduces a jamming signal into the channel. The message and the entire state sequence are known non-causally to both the encoder and the adversary. This state-aware adversary may choose an arbitrary jamming vector depending on the message and the state vector. Taking an Arbitrarily Varying Channel (AVC) approach, we consider two setups, namely, the discrete memoryless Gel’fand-Pinsker (GP) AVC and the additive white Gaussian Dirty Paper (DP) AVC. We determine the randomized coding capacity of both the AVCs under a maximum probability of error criterion. Similar to other randomized coding setups, we show that the capacity is the same even under the average probability of error criterion. Though the adversary can choose an arbitrary vector jamming strategy, we prove that the adversary cannot affect the rate any worse than when it employs a memoryless strategy which depends only on the instantaneous state. Thus, the AVC capacity characterization is given in terms of the capacity of the worst memoryless channels with state, induced by the adversary employing such memoryless jamming strategies. For the DP-AVC, it is further shown that among memoryless jamming strategies, none impact the communication more than a memoryless Gaussian jamming strategy which completely disregards the knowledge of the state. Thus, the capacity of the DP-AVC equals that of a standard AWGN channel with two independent sources of additive white Gaussian noise, i.e., the channel noise and the jamming noise.

Index Terms— Arbitrarily varying channels, state-aware adversary, refined Markov lemma, Gel’fand-Pinsker coding, dirty paper coding.

I. INTRODUCTION

We consider the problem of reliable communication over a state-dependent channel in the presence of a jamming adversary. In our generic problem setup depicted in Fig. 1 a message $M$ is to be communicated reliably over a channel with an independent and identically distributed (i.i.d.) state vector $S$ and an adversarial jamming signal $J$. The state is known non-causally to the encoder. The adversary too knows $M$ as well as state $S$ non-causally. The encoder and decoder share an unbounded amount of randomness, $\Theta$, pre-shared and unknown to the adversary. We consider both the discrete memoryless channel version and the additive white Gaussian version of the setup as elaborated later. Our aim is to determine the capacity of this communication system. An allied interest is to understand the behaviour of the adversary; specifically, its use of the knowledge of state $S$ in the design of its jamming strategy.

State-dependent channels, where the state is known non-causally at the transmitter, have been a subject of considerable interest since the seminal work of Gel’fand and Pinsker [1]. In their work, the capacity of the discrete memoryless channel version was established. Henceforth, we refer to this channel as the ‘Gel’fand-Pinsker (GP) channel’. Subsequently, using a coding scheme based on the technique in [1], called the dirty paper coding scheme, Costa [2] determined the capacity of the Gaussian version of this problem, i.e., the capacity of an AWGN channel with an additive white Gaussian state, where the state is known non-causally to the encoder. Interestingly, Costa showed that the effect of the additive state can be completely nullified. Hence, the capacity of this dirty paper channel was shown to be equal to that of a standard AWGN channel without state. Thus, an intelligent use of the state knowledge, even when available only at the encoder, enables the user to cancel its effect. In our setup, we additionally assume the presence of a state-aware adversary, i.e., an adversary with non-causal knowledge of the state. An intelligent adversary can use this knowledge to design a pernicious jamming strategy. We study the impact of such an adversary on reliable communication.

Our setup falls in the general framework of Arbitrarily Varying Channels (AVC), and the interest lies in determining the randomized coding capacity [3] of this setup. Note that many works on AVCs (for instance, see [3], [4]) refer to the adversary’s channel input $J$ as state. However, to avoid
confusion, in this work we use the word *state* to refer exclusively to the channel state $S$ and refer to $J$ as the jamming signal. Thus, while state $S$ is probabilistic, the jamming signal $J$ is adversarial. The adversary knows $M$ and $S$ prior to deciding the jamming vector $J$. So the process of coming up with $J$ by the adversary can be represented by a conditional distribution $Q_{J|M,S}$, unknown to the encoder and the decoder. Such stochastic vector jamming strategies clearly include deterministic jamming strategies which are functions of the message and the state vector. In addition, they capture possible randomization used by the adversary.

For each message, we take the maximum value of probability of error over all feasible stochastic jamming strategies $Q_{J|M,S}$. Further, our error criterion is the maximum (over messages) probability of error (see (1) for the expression) and we determine the randomized capacity under this error criterion. Note that the deterministic coding capacity problem in this setting is a hard problem and not addressed in this work. Hence, unless stated otherwise, the term *capacity* will hereafter refer to the randomized capacity. The adversary is aware of the state and the message, and furthermore, is assumed to also know the distribution of the randomized code. However, this randomized code is generated using randomness which is shared only between the encoder and decoder, and thus, its exact realization is unknown to the adversary. In particular, owing to the randomized encoding map the adversary does not know the transmitted codeword even though it knows the message. In this work, we consider two variants of the setup: the discrete memoryless Gel’fand-Pinsker A VC (GP-A VC) and the additive white Gaussian Dirty Paper A VC (DP-A VC), and determine their randomized capacity. As in many randomized coding setups (for instance, see [4], [5]), we show that the capacity is the same even under an average (over messages) probability of error criterion.

Subsequent to [9], where the A VC model was introduced, several works analysed different A VC models. In general, the capacity of an A VC communication system depends upon several factors, viz., possibility of randomization (unknown to the adversary) at the encoder/decoder, the probability of error criterion, assumptions on the adversary’s knowledge, etc. [3]. In the absence of state constraints, it is known that the deterministic coding capacity under average error criterion of the A VC exhibits a dichotomy - it is zero if the A VC is symmetrizable or is equal to the randomized coding capacity otherwise [10] (Theorem 1). An A VC $W_{Y|X,J}$, where $X \in \mathcal{X}$, $J \in \mathcal{J}$ and $Y \in \mathcal{Y}$, is said to be symmetrizable if for some conditional distribution $V_{J|X}$

\[
\begin{align*}
\sum_j W_{Y|X,J}(y|x,j)V_{J|X}(j|x') &= \sum_j W_{Y|X,J}(y|x',j)V_{J|X}(j|x), \text{ for every } x, x', y.
\end{align*}
\]

However, the deterministic coding capacity under the maximum error criterion, of which Shannon’s zero error capacity problem [8] is a special case, is not known in general [6], [7]. For a lucid exposition on AVCs and a survey of many useful results, see [6].

To provide context to our work, we review certain important results. In the standard point to point A VC setup under randomized coding, models of adversary ranging from the oblivious adversary (no knowledge of the codeword) to the codeword-aware adversary have been considered [4], [9], [11]. More generally, the myopic adversary which observes a noisy version of the codeword is analysed in [12], [13]. The Gaussian versions of these problems [5], [14], [15] have also been considered. An adversary with a causal view of the codeword [16] or a delayed view of the codeword [17], has also been studied. The capacity of an A VC version of the Gel’fand-Pinsker problem under deterministic coding is determined by Ahlswede [19]. Unlike our setup, this model has only an adversarial state (known to the encoder), but does not have an additional probabilistic state. The case where the decoder too is aware of the state is considered in [19]. A model similar to our DP-A VC, but with a *state-oblivious* adversary under deterministic coding, is analysed in [20]. The result under randomized coding also appears there without proof. Our models have a stronger, *state-aware*, adversary. Communication setups involving both jamming and secrecy have been studied in [15], [21], [22]. Achievability results for secret communication over the Gel’fand-Pinsker wiretap setup too have recently appeared [23].

Closely related to our problem are also problems on information hiding. Information hiding finds application in watermarking, fingerprinting, steganography, etc. (cf. [24], [25]). An information-theoretic approach to the problem of information hiding appears in [26], where information hiding under distortion-attack adversaries is studied. Further results on such *watermarking games* can be found in subsequent works like [27]–[29] and some of the references therein. However, there are important differences between these problems and our problem. In a generic watermarking game depicted in Fig. 2 (also see, for instance, Fig. 2), the aim is to reliably communicate a message $M$ over a channel controlled by an adversary, by embedding it into a covertext source (state $S$), like an image. The embedding process distorts $S$, and the resulting data $X$, called stegotext, is directly observed by the adversary. The adversary, who may or may not know $S$, is capable of distortion attacks, and hence, can further distort this text arbitrarily but within some overall distortion limit (adversary’s power constraint). In the watermarking game, unlike in our problem, the adversary knows the distorted covertext, and thus, can correlate with and cancel it, partially or fully, depending on its power. On the other hand, in our
setup the adversary knows the covertext (i.e., state $S$) but not $X$. Equivalently, this means that the adversary is state-aware but not aware of the transmitted vector. This difference has a major effect on the behaviour of the adversary as well as the capacity of the system. In the Gaussian analogue of the watermarking problem, considered for instance in [27], it is seen that a sufficiently strong adversary can force the capacity of this distortion attack channel to zero. On the contrary, it will be shown that for our setup, the capacity is always greater than zero for any finite value of adversary’s power.

Thus, our result subsumes the result for the state-oblivious adversary.

- As known in other randomized coding setups, we observe (see Remark 2 on page 5) that for both the GP-AVC and the DP-AVC, the capacity is identical under both the maximum and average probability of error criteria.

The proofs of Theorems 1 and 6 are given in Section IV. We discuss some implications of our work and make overall concluding remarks in Section V. The proofs of other auxiliary lemmas are given in the appendices.

II. NOTATION AND PROBLEM SETUP

A. Notation

We denote random variables by upper case letters (e.g. $X$), the values they take by lower case letters (e.g. $x$) and their alphabets by calligraphic letters (e.g. $\mathcal{X}$). We assume all discrete random variables to have alphabets of finite size, unless stated otherwise. The continuous random variables take values in the set of real numbers $\mathbb{R}$. Let $\mathbb{R}^+$ denote the set of non-negative real numbers. We use boldface notation to denote random vectors (e.g. $\mathbf{X}$) and their values (e.g. $\mathbf{x}$). Here the vectors are of length $n$ (e.g. $\mathbf{X} = (X_1, X_2, \ldots, X_n)$), where $n$ is the block length of operation. Let us also denote $\mathbf{X}^i = (X_1, X_2, \ldots, X_i)$ and $\mathbf{x}^i = (x_1, x_2, \ldots, x_i)$ as well as $\mathbf{X}^i_j = (X_i, X_{i+1}, \ldots, X_j)$ and $\mathbf{x}^i_j = (x_i, x_{i+1}, \ldots, x_j)$. We use the $l_\infty$ norm for discrete vectors and the $l_2$ norm for continuous vectors. We denote the former by $\|\cdot\|_\infty$ and the latter by $\|\cdot\|_2$, where we drop the subscript. For a set $\mathcal{X}$, let $\mathcal{P}(\mathcal{X})$ be the set of all probability distributions on $\mathcal{X}$. Similarly, let us write as $\mathcal{P}(\mathcal{X}|\mathcal{Y})$, the set of all conditional distributions of a random variable with alphabet $\mathcal{X}$ conditioned on another random variable with alphabet $\mathcal{Y}$. Let $X$ and $Y$ be two random variables. Then, we denote the distribution of $X$ by $P_X(\cdot)$, the joint distribution of $(X, Y)$ by $P_{X,Y}(\cdot, \cdot)$ and the conditional distribution of $X$ given $Y$ by $P_{X|Y}(\cdot|\cdot)$. Distributions corresponding to strategies adopted by the adversary are denoted by $Q$ instead of $P$ for clarity. In cases where the subscripts are clear from the context, we sometimes omit them to keep the notation simple. For an event $E$, let $\mathbb{P}(E)$ denote the probability of $E$. Functions will be denoted in lowercase letters (e.g. $f$). A Gaussian distribution with mean $\mu$ and variance $\sigma^2$ is denoted by $\mathcal{N}(\mu, \sigma^2)$. All logarithms are with base 2, and hence, all rates and capacities are expressed in bits.

B. The Gel’fand-Pinsker AVC (GP-AVC)

In the communication setup depicted in Fig. 1 there is an arbitrarily varying channel with input $X$, output $Y$, state $S$, and an input $J$ of an adversary. These random variables take values in the finite sets $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{S}$, and $\mathcal{J}$ respectively. The states in different channel uses are i.i.d. with distribution $P_S$. We assume without loss of generality that $P_S(s) > 0, \forall s \in \mathcal{S}$. The channel behaviour is given by the conditional distribution $W_{Y|X,S,J}$. A standard block-coding framework is considered where a message $M$ is communicated over $n$ channel uses. Let $X_i$, $Y_i$, $S_i$, and $J_i$ denote the symbols of the respective random variables associated with the $i$-th channel use. The
encoder as well as the adversary are assumed to know the state vector $S$ non-causally before deciding their input vectors $X$ and $J$ respectively. The encoder and the decoder share unlimited common randomness $\Theta$, unknown to the adversary. Thus, the transmitted vector $X$ is a function of $M$, $S$ and $\Theta$. Hence, we consider randomized coding. Similarly, the state-aware adversary chooses its own channel input $J$. Let the distribution used by the adversary be denoted by $Q_{J|M,S}$. Note that the adversary does not have knowledge of $\Theta$. For a given $x$, $s$ and $j$, the channel output $y$ is observed over the channel $W_{Y|X,S,J}$ with probability given by

$$P(Y = y|X = x, S = s, J = j) = \prod_{i=1}^{n} W_{Y_i|X_i,S_i,J_i}(y_i|x_i,s_i,j_i).$$

We call this channel the Gel’fand-Pinsker AVC (GP-AVC).

An $(n, R)$ deterministic code of block length $n$ and rate $R$ is a pair $(\psi, \phi)$ of mappings with encoder $\psi : \{1, 2, \ldots, 2^Rn\} \times S^n \rightarrow X^n$ and decoder $\phi : \mathbb{Y} \rightarrow \{0, 1, 2, \ldots, 2^Rn\}$, where an output of 0 indicates that the decoder declares an error. Here we have assumed $2^R$ to be an integer. The vector transmitted on the channel is given by $X = \psi(M, S)$.

An $(n, R)$ randomized code of block length $n$ and rate $R$ is a random variable ($\Theta$ in this case) which takes values in the set of $(n, R)$ deterministic codes. Let the pair $\Theta = (\Psi, \Phi)$ denote the encoder-decoder for the $(n, R)$ randomized code. In this case, the transmitted vector is given by $X = \Psi(M, S)$.

For this $(n, R)$ randomized code, the maximum probability of error is:

$$P_e^{(n)} = \max_m \max_{Q_{J|M}} \Pr(\Phi(Y) \neq m| M = m),$$

where the probability is over the state $S$, the adversary’s action $J$, the channel behavior and $\Theta = (\Psi, \Phi)$. The rate $R$ is achievable if for any $\epsilon > 0$, there exists an $(n, R)$ randomized code for some $n$ such that the corresponding $P_e^{(n)}$ is less than $\epsilon$. We define the capacity of the GP-AVC as the supremum of all achievable rates.

C. The Dirty Paper AVC (DP-AVC)

The communication channel depicted in Fig. 3 is a Gaussian arbitrarily varying channel with an additive white Gaussian state and an adaptive jamming interference. The encoder and decoder share an unbounded amount of common randomness $\Theta$, unknown to the adversary. Let us denote by $Y = (Y_1, Y_2, \ldots, Y_n)$, the signal received at the decoder. Then,

$$Y = X + S + J + Z,$$

where $X, S, J$ and $Z$ are the encoder’s input to the channel, the additive white Gaussian state, adversary’s channel input and the channel noise respectively. The components of $S$ are i.i.d. with $S_i \sim \mathcal{N}(0, \sigma_S^2)$ for $i = 1, 2, \ldots, n$. The components of $Z$ are i.i.d. with $Z_i \sim \mathcal{N}(0, \sigma_Z^2)$, $\forall i$. Similar to the GP-AVC, the state vector $S$ is known non-causally to both the encoder and the adversary, but it is not known to the decoder. Hence, the encoder’s output $X$ is a function of $M, S$ and $\Theta$. We call this channel the Dirty Paper AVC (DP-AVC). The encoder has a power constraint $P$, i.e. $\|X\|^2 \leq nP$. Similarly, the adversary’s power constraint is $\Lambda$, such that $\|J\|^2 \leq n\Lambda$. Let $\mathcal{J}(\Lambda) = \{j: \|j\|^2 \leq n\Lambda\}$ denote the set of feasible jamming signals.

An $(n, R, P)$ deterministic code of block length $n$, rate $R$ and average power $P$ is a pair $(\psi, \phi)$ of encoder map $\psi : \{1, 2, \ldots, 2^Pn\} \times \mathbb{S}^n \rightarrow \mathbb{X}^n$, such that $\|\psi(m, s)\|^2 \leq nP$, $\forall m, s$, and decoder map $\phi : \mathbb{Y} \rightarrow \{0, 1, 2, \ldots, 2^Pn\}$, where an output of 0 indicates that the decoder declares an error. Here we have assumed $2^P$ to be an integer. The transmitted vector is given by $X = \psi(M, S)$.

An $(n, R, P)$ randomized code is a random variable $(\Psi, \Phi)$ which forms the shared randomness $\Theta$ and takes values in the set of $(n, R, P)$ deterministic codes. Here the transmitted vector is given by $X = \Psi(M, S)$. For an $(n, R, P)$ randomized code with encoder-decoder pair $(\Psi, \Phi)$, the maximum probability of error is

$$P_e^{(n)} = \max_m \max_{Q_{J|M}, S} \Pr(\Phi(Y) \neq m| M = m),$$

where the probability is over the state $S$, the adversary’s action $J$, the channel behavior and $\Theta = (\Psi, \Phi)$. The rate $R$ is achievable if for every $\epsilon > 0$, there exists an $(n, R, P)$ randomized code for some $n$ such that $P_e^{(n)}$ is less than $\epsilon$. We define the capacity of the DP-AVC as the supremum of all achievable rates.

III. THE MAIN RESULTS

In this section, we present our main results. Theorem 1 characterizes the capacity of the GP-AVC while Theorem 6 determines the capacity of the DP-AVC.

Given a state distribution $P_S$ and for fixed distributions $P_{U,X|S}$ and $Q_{J|S}$, let $I(U;Y)$ and $I(U;S)$ denote respectively, the mutual information quantities evaluated with respect to the corresponding marginals $P_{U,Y}$ and $P_{U,S}$. In the following theorem, let $U$ denote the alphabet of $U$. 

![Fig. 3. The Dirty Paper AVC (DP-AVC) communication setup.](image-url)
Theorem 1. [GP-AVC Capacity] The capacity of the Gel’fand Pinsker AVC is\(^5\)

\[
C = \max_{P_{U|S}, \pi(\cdot)} \min_{Q_{JS}} (I(U;Y) - I(U;S)), \tag{3}
\]

where \(P_{U|S} \in \mathcal{P}(U|S)\), \(x : U \times S \rightarrow X\), \(Q_{JS} \in \mathcal{P}(J|S)\), and \(|U| \leq |X|^{S}|\).

The proof of this result is presented in Section [V].

Remark 2. Though the capacity is stated for the maximum probability of error criterion, the converse is proved for the average probability of error as defined in [6]. On the other hand, the achievability under maximum probability of error also implies the achievability under average probability of error. Thus, the GP-AVC capacity under the average probability of error criterion is the same as in (3).

This fact can also be seen directly from the definition itself. Clearly, capacity under average probability of error criterion cannot be smaller than that under maximum probability of error criterion. To see that the capacities must be the same, given any code with a certain average probability of error \(\bar{P}_{e}\) and hence whose maximal probability is \(P_{e}\), we can obtain a code whose probability of error under each message is \(P_{e}\) and hence whose maximal probability is \(P_{e}\). This can be done by simply using a part of the shared randomness \(\Theta\) to uniformly permute the messages. Specifically, a uniformly random permutation \(\Pi : \{1, \ldots, 2^{nR}\} \rightarrow \{1, \ldots, 2^{nR}\}\) is chosen using a part of \(\Theta\), and to send message \(m\), the permuted message \(\Pi_{m}\) is sent using the encoder which guarantees average probability of error \(P_{e}\). At the receiver the inverse map \(\Pi^{-1}\) is applied to the output of the decoder.

The above argument also shows how the adversary’s knowledge of \(M\) can be rendered essentially useless. Indeed, for an encoder-decoder pair which uses a random permutation as above, the optimal \(Q_{JM,S}\) must be such that it does not depend on \(M\), in other words, it must be of the form \(Q_{JS}\). Since the above random permutation can always be used without resulting in an increase in the maximal (and average) probability of error, it is clear that the capacity under a state-aware adversary who also knows the message \(M\) is the same as that under a state-aware adversary who does not know the message.

Remark 3. Every memoryless jamming strategy \(Q_{JS}\) induces some GP channel \(V_{Y|X,S}\). Thus, (3) can be expressed through the capacity of the worst memoryless channel that the adversary can induce through a memoryless strategy, i.e.,

\[
C = \max_{P_{U|S}, \pi(\cdot)} \min_{V_{Y|X,S}} (I(U;Y) - I(U;S)).
\]

Here

\[
V_{Y|X,S}(y|x,s) = \sum_{j} W_{Y|X,S,J}(y|x,s,j)Q_{JS}(j|s),
\]

where \(Q_{JS} \in \mathcal{P}(J|S)\).

\(^{5}\)The max-min exists as mutual information \(I(U;Y) - I(U;S)\) is a continuous function of these variables which take values over a compact set.

Remark 4. Recall that the standard GP channel capacity\(^7\) is given by

\[
C = \max_{P_{U|X}} \min_{P_{U|Y}} (I(U;Y) - I(U;S)) \tag{4}
\]

The standard argument for the reduction to (4) uses the fact that \((I(U;Y) - I(U;S))\) is a convex function of \(P_{X|U,S}\) for a fixed distribution \(P_{X|U,S}\) \([10]\). For the GP-AVC, though, such an approach fails as \(\min_{Q_{JS}} (I(U;Y) - I(U;S))\) is not necessarily a convex function of \(P_{X|U,S}\) for a fixed \(P_{U|S}\). However, in the proof of the converse of Theorem 1, we used a different approach to show that such a simplification is still possible for the GP-AVC.

Remark 5. Our bound on \(|U|\) in Theorem 1 follows from the set of Shannon strategies \([30]\, Remark 7.6\) at the encoder as there exist up to \(|X|^{S}|\) functions from \(S\) to \(X\). The details can be seen in the proof of the converse. In the standard GP channel, where the GP channel is fixed, a stronger bound of \(|U| \leq |X| \cdot |S|\) is known to hold using Support lemma \([31]\, Lemma 15.4\) (which uses Carathéodory’s theorem). However, we cannot use the Support lemma for the GP-AVC because the statistics of \(U\) depend upon the statistics of the output \(Y\) and the adversary can induce any of the infinitely many GP channels.

Our next result gives the capacity of the Dirty Paper AVC.

Theorem 6. [DP-AVC Capacity] The capacity\(^6\) of the Dirty Paper AVC is

\[
C = \frac{1}{2} \log \left( 1 + \frac{P}{\Lambda + \sigma^2} \right). \tag{5}
\]

The proof is given in Section IV. This result again implies that essentially a memoryless strategy is optimal for the adversary. Unlike in the case of the GP-AVC, here the adversary completely disregards the knowledge of the state. The adversary essentially inputs i.i.d. Gaussian jamming noise independent of the state. The effect of the additive random state \((S)\) is completely eliminated as in the standard dirty paper channel, and the capacity of the DP-AVC equals that of the dirty paper channel considered by Costa in \([2]\) where the noise variance is \((\Lambda + \sigma^2)\).

IV. PROOFS

A. Proof of Theorem 7. The Gel’fand-Pinsker AVC Capacity

In this section, we first discuss the converse for the Gel’fand-Pinsker AVC capacity theorem and then give a proof of achievability.

1) Converse: In the following, we prove the converse for an average probability of error criterion instead of the maximum probability of error criterion. For this stronger version of the converse, let the average probability of error be

\[
P_{e} = \frac{1}{2^{nR}} \sum_{m=1}^{2^{nR}} P_{e,m}. \tag{6}
\]

\(^{6}\)For the same reason as explained in Remark 2 the capacity is the same under both maximum error probability and average error probability criteria.
where

\[ P_{e,m}^{(n)} = \max_{Q_{j,M}=m} \mathbb{P}(\Phi(Y) \neq m| M = m). \]

To prove our converse, we will consider a specific memoryless (but not i.i.d.) jamming strategy, which depends on the randomized code (although as discussed in Section I, the actual realization of the encoding map is unknown to the adversary), and upper bound the rate of reliable communication possible under this strategy of the adversary.

Our proof starts along the lines of the standard Gel’fand-Pinsker converse [1]. Let us consider any sequence of codes given on top of the next page, from which it clearly follows that \( P_{U_i,X_i|S_i} \) depends on the randomized encoding map (in particular, on \( P_{X_i|M,\Theta} \)) as well as \( Q_{J_i|S_i} \), where \( k = 1, 2, \ldots, i - 1 \), but it does not depend on \( Q_{J_i|S_i} \). We now define \( Q_{J_i|S_i} \) inductively as follows. Given \( Q_{J_i|S_i} : k = 1, 2, \ldots, i - 1 \) and \( P_{U_i,X_i|S_i} \), let \( Q_{J_i|S_i} \) be the minimizer of \( (I(U_i; Y_i) - I(U_i; S_i)) \). Hence, from (7) we have

\[ nR \leq \max_{P_{U_i,X_i|S_i} Q_{J_i|S_i}} \min_{Q_{J_i|S_i}} (I(U_i; Y_i) - I(U_i; S_i)) + \epsilon_n, \]

for \( P_{U_i,X_i|S_i} \), \( i = 1, 2, \ldots, n \). Further, note that for \( i = 1, 2, \ldots, n \),

\[ \min_{Q_{J_i|S_i}} (I(U_i; Y_i) - I(U_i; S_i)) \leq \max_{P_{U_i,X_i|S_i} Q_{J_i|S_i}} \min_{Q_{J_i|S_i}} (I(U_i; Y_i) - I(U_i; S_i)). \]

Here the maximization is over all conditional distributions \( P_{U_i,X_i|S_i} \) with finite alphabet \( \mathcal{U} \) of \( U_i \). This inequality holds because the fixed \( P_{U_i,X_i|S_i} \) (induced by the code) on the RHS is such a distribution. Since the channel is memoryless, the RHS in (10) does not depend on \( i \), and thus we have

\[ R \leq \max_{P_{U_i,X_i|S_i} Q_{J_i|S_i}} \min_{Q_{J_i|S_i}} (I(U_i; Y_i) - I(U_i; S_i)) + \epsilon_n. \]

Since this holds for all \( n \) and \( \epsilon_n \rightarrow 0 \) as \( n \rightarrow \infty \), we have

\[ R \leq \max_{P_{U_i,X_i|S_i} Q_{J_i|S_i}} \min_{Q_{J_i|S_i}} (I(U_i; Y_i) - I(U_i; S_i)). \]

We now show that it is sufficient to perform the maximization in (11) over distributions \( P_{U_i|S} \) and functions \( x : \mathcal{U} \times S \rightarrow X \), i.e.,

\[ \max_{P_{U_i|S} Q_{J_i|S}} \min_{Q_{J_i|S}} (I(U_i; Y_i) - I(U_i; S_i)) = \max_{P_{U_i|S} \cdot x: \mathcal{U} \times \mathcal{S} \rightarrow \mathcal{X}} \min_{Q_{J_i|S}} (I(U_i; Y_i) - I(U_i; S_i)). \]

Let us fix the conditional distribution \( P_{U_i|S} \). We know from the functional representation lemma [30, pg. 626] that there exists a random variable \( W \) which is independent of \((U, S)\) such that \( X \) is a function of \((W, U, S)\). Let us define \( U' = (U, W) \) and denote its alphabet by \( \mathcal{U}' \), then we have \( P_{U_i|S}(u_i|w_i) = P_{U_i|S}(u_i|s_i)P_W(w) \). Let the function be denoted by \( x : \mathcal{U}' \times S \rightarrow X \). Note that \( U' \rightarrow (X, S) \rightarrow Y \) is a Markov chain. Then,

\[ I(U'; S) = I(U, W; S) = I(U; S) + I(W; S|U) = I(U; S), \]

where the last equality follows from \( W \perp (U, S) \). Further,
for any \( Q_{J|S} \in \mathcal{P}(J|S) \),
\[
I(U';Y) = I(U,W;Y) = I(U;Y) + I(W;Y|U) \\
\geq I(U;Y),
\]
and hence,
\[
\min_{Q_{J|S}} I(U';Y) \geq \min_{Q_{J|S}} I(U;Y). \quad (14)
\]

From (13) and (14), it then follows that
\[
\min_{Q_{J|S}} I(U;Y) - I(U;S) \leq \min_{Q_{J|S}} I(U';Y) - I(U';S).
\]

Here the LHS is evaluated under a conditional distribution \( P_{X|U,S} \) and the RHS under the corresponding \( P_{U'|S} \) and \( x : \mathcal{U}' \times \mathcal{S} \to \mathcal{X} \). Since the inequality holds for any \( P_{X|U,S} \), we have (12), and thus
\[
R \leq \max_{Q_{U'|S}} \min_{Q_{J|S}} (I(U;Y) - I(U;S)). \quad (15)
\]

For the bound on the cardinality of \( \mathcal{U} \), we use the Shannon strategy approach in a similar manner, for example, as in the context of channels with state with causal knowledge of the state at the encoder [10, Remark 7.6]. In particular, the maximization over functions \( x(u,s) \) in (15) can be equivalently viewed as a maximization over functions \( x_u : \mathcal{S} \to \mathcal{X} ; u \in \mathcal{U} \). Since there are exactly \(|\mathcal{X}|^{||S||}\) such distinct functions, without loss of generality, we can restrict \( \mathcal{U} \) to be of cardinality at most \(|\mathcal{X}|^{||S||}\). This completes the proof of the converse.

2) Achievability: To begin, let us introduce some useful notation. Given \( x, y \), the type of \( x \) will be denoted by \( T_x \), the joint type of \( (x,y) \) by \( T_{x,y} \) and the conditional type of \( x \) given \( y \) by \( T_{x|y} \). Here \( \forall (x,y) \) such that \( T_{x}(y) > 0 \),
\[
T_{x|y}(x,y) = \frac{T_{x,y}(x,y)}{T_{y}(y)}.\]

For any \( \epsilon \in (0,1) \), the set of \( \epsilon \)-typical sequences \( x \) for a distribution \( P_x \) is
\[
T^n_{\epsilon}(P_x) = \{ x : \|T_x - P_x\|_\infty \leq \epsilon \}, \quad (16)
\]
where \( \| \cdot \|_\infty \) is the \( l_\infty \) norm. For a joint distribution \( P_{X,Y} \) and \( x \in \mathcal{X}^n \), the set of conditionally \( \epsilon \)-typical sequences \( y \), conditioned on \( x \), is defined as
\[
T^n_{\epsilon}(P_{X,Y}|x) = \{ y : \|T_{x,y} - P_{X,Y}\|_\infty \leq \epsilon \}.
\]

We use randomized Gel’fand-Pinsker coding scheme [1], which involves an auxiliary random variable denoted by \( U \). We choose a rate \( R < C \), where \( C \) is as given in (3). Consider a conditional distribution \( P_{U|S} \) and a function \( x : \mathcal{U}' \times \mathcal{S} \to \mathcal{X} \) with \( X = x(U,S) \) such that
\[
R < \min_{Q_{J|S}} I(U;Y) - I(U;S).
\]

Note that here \( Q_{J|S} \) takes values from all conditional distributions in \( \mathcal{P}(J|S) \), and the encoder and the decoder clearly know this set.

**Code construction:**

- We generate a binned codebook \( C \) comprising \( 2^nR_C \) vectors \( U_{j,k} \), where \( j = 1,2,\ldots,2^nR \) and \( k = 1,2,\ldots,2^nR, \) \( R \geq 0 \) will be defined later. Here \( j \) indicates the bin index while \( k \) indicates the position within the bin. There are \( 2^nR \) bins with each bin containing \( 2^nR \) codewords. Every codeword \( U_{j,k} \) is chosen independently and uniformly at random from \( T^n_{\epsilon}(P_U) \) (the choice of \( \epsilon > 0 \) will be discussed later), where
\[
P_U(u) = \sum_s P_{U|S}(u,s)P_S(s), \forall u.
\]

The codebook is shared between the encoder and decoder as the shared randomness \( \Theta \).

**Encoding:**

- Given a message \( m \) and having observed the state \( S \), the encoder looks within the bin \( m \) for some \( U_{m,k} \) such that
\[
\|T_{U_{m,k},S} - P_{U,S}\|_\infty \leq \delta_1(\delta), \quad (17)
\]

for some \( \delta_1(\delta) > 0 \) (the choice of \( \delta_1(\delta) \) will be discussed later). Here \( P_{U,S} = P_{U|S}P_S \). The condition (17) implies that \( U_{m,k} \) and \( S \) are jointly typical according to \( P_{U,S} \). If no such \( U_{m,k} \) is found, then the encoder selects \( U_{1,1} \). If more than one \( U_{m,k} \) satisfying (17) exists, then the encoder chooses one uniformly at random from amongst them. Let \( U \) denote the chosen codeword.

- The encoder then generates \( X \), where \( X_i = x(U_i,S_i) \), \( i = 1,2,\ldots,n \) are independent, and transmits it over the channel.

**Decoding:**

- When \( Y \) is received at the decoder and given some fixed parameter \( \gamma(\delta) > 0 \) (the choice of \( \gamma(\delta) \) will be discussed later), the decoder determines the set
\[
L(y,\gamma(\delta)) = \{ u \in C : \exists Q_{J|S} \in \mathcal{P}(J|S) \}
\]
\[
\text{s.t. } \|T_{U,Y} - P_{U,Y}^{Q}\|_\infty \leq \gamma(\delta) \},
\]
where for $Q_{j|S} \in \mathcal{P}(\mathcal{J}|S)$,

$$P^{(Q)}_{U,Y}(u,y) = \sum_{x,s,j} P_S(s)P_{U|S}(u|s)1\{X=x(U,S)\}W_{Y|X,S,j}(y|x,s,j)Q_{j|S}(j|s), \forall (u,y).$$

Here the decoder lists all codewords $u \in C$ which are jointly typical with $y$ according to $P^{(Q)}_{U,Y}$, for some $Q_{j|S} \in \mathcal{P}(\mathcal{J}|S)$.

- If $L(y,\gamma(\delta))$ is not empty and all the bin indices of the codewords in it are identical, then the decoder outputs the common bin index $\hat{m}$. Otherwise, it declares an error by setting $\hat{m} = 0$.

**Probability of error analysis:**

A decoding error occurs if either the chosen codeword $U_{m,k}$ is not jointly typical with $Y$ or some other codeword $U_{m',k'}$, for some $m' \neq m$ and $k' \in \{1, 2, \ldots, 2^{nR}\}$, is jointly typical with $Y$. Here the typicality is according to $P^{(Q)}_{U,Y}$ for some $Q_{j|S} \in \mathcal{P}(\mathcal{J}|S)$. We show that the probability of this decoding error event is vanishing as $n \to \infty$. Let $\epsilon > 0$ be such that

$$R = \min_{Q_{j|S}} (I(U;Y) - I(U;S)) - \epsilon,$$

and

$$\hat{R} = I(U;S) + \epsilon/2.$$

Recall from earlier that $R_U = R + \hat{R}$, and hence,

$$R_U = \min_{Q_{j|S}} I(U;Y) - \epsilon/2.$$

Let $E = \{\hat{m} \neq M\}$ denote the decoding error event. Let the message sent be $M = m$ and let $U = U_{m,k}$ denote the chosen codeword. Then, we have

$$\mathbb{P}(E|M = m) = \mathbb{P}(U \not\in L(Y,\gamma(\delta))|M = m) + \mathbb{P}(\exists m', k': m' \neq m, U_{m',k'} \in L(Y,\gamma(\delta))|M = m).$$

From (1), we have

$$P_e^{(n)} = \max_m \min_{Q_{j|m}} \mathbb{P}(E|M = m),$$

and thus,

$$P_e^{(n)} \leq \max_m \min_{Q_{j|m}} \mathbb{P}(U \not\in L(Y,\gamma(\delta))|M = m) + \max_m \min_{Q_{j|m}} \mathbb{P}(\exists m', k': m' \neq m, U_{m',k'} \in L(Y,\gamma(\delta))|M = m).$$

(18)

We will show that for any $\epsilon > 0$, we can find a $\delta > 0$ such that both the terms go to zero as $n \to \infty$.

We now state some useful results which are required to bound the terms in the RHS of (18). Recall from (17) that $\delta_1(\delta)$ is the parameter which appears in the definition of the encoder. The following claim specifies this $\delta_1(\delta)$ parameter.

**Claim 7.** If $\hat{R} > I(U;S)$, then there exists $\delta_1: \mathbb{R}^+ \to \mathbb{R}^+$, where $\delta_1(\delta) \to 0$ as $\delta \to 0$, such that the probability that the encoder finds at least one $U_{m,k}$ such that $(U_{m,k}, S) \in T_{\delta_1}^{(n)}(P_{U,S})$ approaches 1 as $n \to \infty$.

This result follows from the use of the covering lemma, the proof of which is along the lines of the proof of [30, Lemma 3.3]. To bound the first term in (18), we will consider the conditional type $T_{j|S}$ of $j$ given $S$ and $j_i$. As $S$ is i.i.d. with $S_i \sim P_S$, $\forall i$, it follows that the pair $(S, J)$ will be jointly typical according to $P_S T_{j|S}$ with high probability. We now present a lemma which is a refined version of the Markov lemma in [30, Lemma 12.1]. This lemma will be used later (with $X \to J, Y \to S$, and $Z \to U$) to conclude that $(S, J, U)$ are jointly typical according to $P_S T_{j|S} P_{U|S}$ with high probability.

**Lemma 8 (Refined Markov Lemma).** Suppose $X \to Y \to Z$ is a Markov chain, i.e., $P_{X,Y,Z} = P_{X|Y} P_{Z|Y}$. Let $(x, y) \in T_{\delta_1}^{(n)}(P_{X,Y})$ and $Z \sim P_Z$ be such that

(a) for some $\epsilon > 0$,

$$\mathbb{P}(S \not\in T_{\delta_1}^{(n)}(P_{Y,Z})) \leq \epsilon,$$

(b) for every $z \in T_{\delta_1}^{(n)}(P_{Y,Z})$,

$$2^{-n(H(Z|Y) + g(\delta_0))} \leq P_Z(z) \leq 2^{-n(H(Z|Y) - g(\delta_0))},$$

for some $g: \mathbb{R}^+ \to \mathbb{R}^+$, where $g(\delta_0) \to 0$ as $\delta_0 \to 0$.

Then, there exists $\delta: \mathbb{R}^+ \to \mathbb{R}^+$, where $\delta(\delta_0) \to 0$ as $\delta_0 \to 0$, such that

$$\mathbb{P}(x \not\in T_{\delta_1}^{(n)}(P_{X,Y,Z})) \leq 2|X||Y||Z|e^{-nK} + \epsilon.$$

Here $K > 0$ and $K$ does not depend on $n$, $P_{X,Y}$, $P_Z$ or $(x, y)$ but does depend on $\delta_0$, $g$ and $P_{Z|Y}$. Further, the $\delta$ function does not depend on $(x, y)$, $P_{X,Y}$ or $P_Z$.

The proof of the lemma is presented in Appendix A.

**Remark 9.** (i) The Refined Markov lemma is a refinement of the Markov lemma [30]. Markov lemma gives the bound (see the proof in [30, Appendix 12A])

$$\mathbb{P}(x \not\in T_{\delta}^{(n)}(P_{X,Y,Z})) \leq 2(n + 1)2^{2n(g(\delta_0))}e^{-n(\delta - g(\delta_0))2^{-n(1+g(\delta_0))}},$$

where

$$P_{X,Y,Z}^{\min} := \min_{x,y,z: P_{X,Y,Z}(x,y,z) > 0} P(x,y,z),$$

and $\delta > 0$ is a constant. On the other hand, Lemma 8 gives a bound which does not depend on $P_{X,Y}$.

This refinement is crucial in our proof of achievability. Here the lemma will be used (in the proof of Claim 12 replacing $X \to J$, $Y \to S$ and $Z \to U$). Thus, we have the Markov chain $J \to S \to U$ with $P_{J,S,U} = P_{J,S} P_{U|S}$. For a given $J$ and $S$, we will take their joint type $T_{j,s}$ as the distribution $P_{j,s}$. Since $J$ is decided by the adversary based on their non-causal knowledge of $s$, the joint type $T_{j,s}$ can have non-zero components as small as $1/n$. This can be easily caused by the adversary by enforcing a pair of values $(j, s)$ only once in the length-$n$ pair of vectors. In such cases, the original Markov lemma does not guarantee any useful bound.

7 In fact, it will be be clear through the proof that even though the adversary can employ arbitrary vector jamming strategies of the form $Q_{j,M,S}$, its impact is completely captured through the conditional type $T_{j|S} \in \mathcal{P}(\mathcal{J}|S)$. See the proof of Lemma 10 for details.
on the probability $P((j, s, U) \not\in T_{\delta}^{n}(P_{U,S,J,U}))$. We believe that for similar reasons, our version of the Markov lemma may also be useful in achievable proofs in other systems with adversaries.

(ii) Another minor difference from the Markov lemma is that we use a slightly different notion of typicality \[16\] than the one used in \[30\]. This makes the analysis easier in the second part of the proof of Lemma \[10\]. However, the Refined Markov lemma can also be proved under the typicality notion used in \[30\] along the lines of our proof.

The following lemma bounds the two components of the probability of error in \[18\].

**Lemma 10.** Let the message be $M = m$. There exist $\gamma, \tilde{\gamma} : \mathbb{R}^{+} \to \mathbb{R}^{+}$, where $\gamma(\delta), \tilde{\gamma}(\delta) \to 0$ as $\delta \to 0$, such that for any jamming strategy $Q_{J|M,S}$,

(i) for $\epsilon_n$ independent of $m$, where $\epsilon_n \to 0$ as $n \to \infty$

\[
P(U \not\in L(Y, \gamma(\delta)) | M = m) \leq \epsilon_n,
\]

(ii) if $U' \sim \text{Unif}(T_{\delta}^{n}(P_{U}))$, independent of $(U, S, X, J, Y)$, then

\[
P(U' \in L(Y, \gamma(\delta)) | M = m) \leq 2^{-n \left(\min_{Q_{J}|\delta} I(U;Y) - \tilde{\gamma}(\delta)\right)}.
\]

Before we prove this lemma, we complete the proof of achievability. The proof of Lemma \[10\] follows immediately after and concludes this section. Claim \[7\] and Lemma \[8\] (Refined Markov lemma) are used in the proof of Lemma \[10\].

Coming back to the probability of error analysis, note that the first part of Lemma \[10\] implies that as $n \to \infty$, the first term in the RHS in \[18\] goes to zero. For the second RHS term, we have for any $Q_{J|M,S}$,

\[
\text{Pr}(\exists m', k' : m' \neq m, U_{m', k'} \in L(Y, \gamma(\delta)) | M = m) \leq \sum_{m' \neq m, k'} \text{Pr}(U_{m', k'} \in L(Y, \gamma(\delta)) | M = m) \leq 2^{nR_{U}} \text{Pr}(U_{m', k'} \in L(Y, \gamma(\delta)) | M = m) \leq 2^{nR_{U}} 2^{-n \left(\min_{Q_{J}|\delta} I(U;Y) - \tilde{\gamma}(\delta)\right))}.
\]

Here we get (a) using the union bound while (b) follows from the second part of Lemma \[10\]. Thus, by choosing a small enough $\delta$ such that

\[
R_{U} < \min_{Q_{J}|\delta} I(U;Y) - \tilde{\gamma}(\delta),
\]

it follows that the second term in the RHS of \[18\] can be made to go to 0 as $n \to \infty$. This implies that $P_{e}^{(n)} \to 0$ as $n \to 0$, and hence, concludes the proof of achievability.

Now, it only remains to prove Lemma \[10\]. For the proof of the first part of this lemma, we begin by stating a few useful claims. Recall our assumption that $P_{S}(s) > 0, \forall s$. In the following, when we write $s \in T_{\delta}^{n}(P_{S})$ we assume that $\delta_0$ is small enough and $n$ large enough such that $T_{s}(s) > 0, \forall s$. Hence, we may write

\[
T_{js}(j|s) = \frac{T_{s,j}(s,j)}{T_{s}(s)}, \forall (s,j).
\]

It will be seen through the following claims that the effect of the jamming input given the underlying adversarial strategy is completely captured through this conditional type $T_{js}$.

**Claim 11.** Let $(s,j)$ be a pair of vectors where $s \in T_{\delta}^{n}(P_{S})$. Then, $(s,j) \in T_{\delta}^{n}(P_{S}T_{js})$.

The proof is straightforward, and hence, omitted. Now, we note that under the event that the encoder succeeds in finding a typical $U$ codeword, $U \sim \text{Unif}(T_{\delta}^{n}(P_{U,S}|s))$ conditioned on $M = m$.

**Claim 12.** Let $(s,j) \in T_{\delta}^{n}(P_{S}T_{js})$. Then, there exists some $\delta_2(\delta_1) > 0$, where $\delta_2(\delta_1) \to 0$ as $\delta_1 \to 0$, such that if $U \sim \text{Unif}(T_{\delta}^{n}(P_{U,S}|s))$, where $P_{U,S} = P_{S}P_{U}|s$, then

\[
P(U \not\in T_{\delta}^{n}(P_{S}P_{U}|s) | M = m) \leq \epsilon_n,
\]

where $\delta_2$ and $\epsilon_n$ do not depend on $(s,j,m)$ and $\epsilon_n \to 0$ as $n \to \infty$.

**Proof:** We use Lemma \[8\] with $X \to J, Y \to S$ and $Z \to U$. Further, replace $\delta_0 \to \delta_1$ and $\delta(\delta_0) \to \delta_2(\delta_1)$ here. Next, the distribution $P_{S,J} = P_{S}T_{js}$ and $P_{U} = \text{Unif}(T_{\delta}^{n}(P_{U,S}|s))$. As $U \sim \text{Unif}(T_{\delta}^{n}(P_{U,S}|s))$, it follows that both the conditions of Lemma \[8\] are satisfied. In particular, the first condition is met with $\epsilon = 0$ as $U \in T_{\delta}^{n}(P_{U,S}|s)$, while the second condition is met as there exists some $g(\delta_0) > 0$, where $g(\delta_0) \to 0$ as $\delta_0 \to 0$, such that

\[
2^{n(H(U|S) - g(\delta_0))} \leq |T_{\delta}^{n}(P_{U,S}|s)| \leq 2^{n(H(U|S) + g(\delta_0))}.
\]

The claim now follows.

The following two claims follow from the conditional typicality lemma, where the proof of the latter is along the lines of the one which appears in \[30\] pg. 27.

**Claim 13.** Let $(u,s,j) \in T_{\delta}^{n}(P_{U,S,J})$, and let $X$ be generated from $(u,s)$ through the memoryless distribution $1_{(X = x(u,s))}$. Then there exists $\delta_3(\delta_2) > 0$, where $\delta_3(\delta_2) \to 0$ as $\delta_2 \to 0$, such that

\[
P((u,s,j) \not\in T_{\delta}^{n}(P_{U,S,J}1_{(X = x(u,s))}) | M = m) \leq \epsilon_n,
\]

where $\delta_3$ and $\epsilon_n$ do not depend on $(u,s,j,m)$, and $\epsilon_n \to 0$ as $n \to \infty$.

**Claim 14.** Let $(u,s,x,j) \in T_{\delta}^{n}(P_{U,S,X,J})$, and let $Y$ be generated from $(x,s,j)$ through the channel $W_{Y|X,S,J}$. Then there exists $\delta_4(\delta_3) > 0$, where $\delta_4(\delta_3) \to 0$ as $\delta_3 \to 0$, such that

\[
P((u,s,x,j,y) \not\in T_{\delta}^{n}(P_{U,S,X,J}W_{Y|X,S,J}) | M = m) \leq \epsilon_n,
\]

where $\delta_4$ and $\epsilon_n$ do not depend on $(u,s,x,j,m)$, and $\epsilon_n \to 0$ as $n \to \infty$.

To proceed with the proof of Lemma \[10\], let us define the following error event.

\[
E = \{U \not\in L(Y, \gamma(\delta))\}.
\]

The fact that these do not depend on $(s,j)$ is crucial, and it follows from our Refined Markov Lemma. They also do not depend on $m$, as the distribution of $U$ does not depend on $m$. 
From the definition of the decoder, it follows that this event $E$ occurs if there does not exist any $Q_{j|S} \in \mathcal{P}(\mathcal{F}|S)$, and correspondingly any resulting distribution $P_{U|Y}$, such that the chosen $U$ codeword and the received output $Y$ are jointly typical. In the following, we show that for the specific choice of $Q_{j|S} = T_{j|S}$, the correct codeword $U$ will satisfy the decoding criterion w.h.p. Toward this, we define some events:

$$E_1 = \{ S \notin T^{(n)}_{\delta_0}(P_S) \},$$
$$E_2 = \{ (S, J) \notin T^{(n)}_{\delta_0}(P_S T_{j|S}) \},$$
$$E_3 = \{ (S, U) \notin T^{(n)}_{\delta_1}(P_S P_U) \},$$
$$E_4 = \{ (S, J, U) \notin T^{(n)}_{\delta_2}(P_S T_{j|S} P_U) \},$$
$$E_5 = \{ (S, J, U, X) \notin T^{(n)}_{\delta_3}(P_S T_{j|S} P_U | S \mathbf{1}(X=x(U,S))) \},$$
$$E_6 = \{ (S, J, U, X, Y) \notin T^{(n)}_{\delta_4}(P_S T_{j|S} P_U | S \mathbf{1}(X=x(U,S))) W_Y | X, S, j, S \} \}.$$

Here $\delta_0(\delta) = \frac{\delta}{2}$ and $\delta_i$, $i = 1, 2, 3$ and $\gamma$ will be chosen such that as functions of $\delta$, they approach 0 as $\delta \to 0$. Using the union bound, we have

$$\mathbb{P}(E | M = m) = \sum_{(s,j) \in T^{(n)}_{\delta_0}(P_S T_{j|S})} \mathbb{P}(E_1 | M = m) + \mathbb{P}(E_2 | M = m) + \mathbb{P}(E_3 | M = m) + \mathbb{P}(E_4 | M = m) + \mathbb{P}(E_5 | M = m) + \mathbb{P}(E_6 | M = m).$$

Hence, we can conclude that $\mathbb{P}(E_3 | M = m) \to 0$ as $n \to \infty$.

For the fourth term, let $(s, j, u) \in T^{(n)}_{\delta_2}(P_S T_{j|S} P_U)$. Now conditioned on $(S, J, U, X, S, J, U, X, Y) \notin T^{(n)}_{\delta_4}(P_S T_{j|S} P_U | S \mathbf{1}(X=x(U,S))) W_Y | X, S, j, S \}$, we have

$$\mathbb{P}(E_6 | M = m) = \sum_{(s,j) \in T^{(n)}_{\delta_0}(P_S T_{j|S})} \mathbb{P}(E_1 | M = m) + \mathbb{P}(E_2 | M = m) + \mathbb{P}(E_3 | M = m) + \mathbb{P}(E_4 | M = m) + \mathbb{P}(E_5 | M = m) + \mathbb{P}(E_6 | M = m).$$

Hence, it follows that $\mathbb{P}(E_3 | M = m) \to 0$ as $n \to \infty$.

Similarly, for the final term, let

$$(s, j, u, x) \in T^{(n)}_{\delta_3}(P_S T_{j|S} P_U | S \mathbf{1}(X=x(U,S))) W_Y | X, S, J, U, X, S, J, U, X, Y).$$

Then, conditioned on $(S, J, U, X, S, J, U, X, Y) \notin T^{(n)}_{\delta_4}(P_S T_{j|S} P_U | S \mathbf{1}(X=x(U,S))) W_Y | X, S, J, U, X, S, J, U, X, Y)$.

From Claim [14] we know that there exists $\delta_4(\delta_3) > 0$, where $\delta_4(\delta_3) \to 0$ as $\delta_3 \to 0$, such that

$$\mathbb{P}((S, J, U, X, Y) \notin T^{(n)}_{\delta_4}(P_S T_{j|S} P_U | S \mathbf{1}(X=x(U,S))) W_Y | X, S, J, U, X, S, J, U, X, Y) \leq \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to \infty$ (here $\delta_4$ and $\epsilon_n$ do not depend on $(s, j, u, x, m)$).

We now assume $\gamma(\delta) = \delta_4(\delta_3(\delta_0(\delta_0(\delta))))$ in the definition of $E_6$. Then, by an argument similar to that
of the fourth term, it follows that
\[ \mathbb{P}(E_0 | E_1^c, E_2^c, E_3^c, E_4^c, M = m) \to 0 \]
as \( n \to \infty \).

As each term in the RHS of (19) is vanishing as \( n \to \infty \), we can conclude that \( \mathbb{P}(E | M = m) \to 0 \) as \( n \to \infty \). Thus, we have shown that, conditioned on \( M = m \), \( U \in L(Y, \gamma(\delta)) \) with probability approaching 1 as \( n \to \infty \). In particular, we have shown that the correct codeword satisfies the decoding condition w.r.t. \( Q_{JS} = T_{JS} \). This completes the proof of part (i) of the lemma.

We prove the second part using some well-known properties of types [31–33]. We begin by introducing some notation and useful quantities. Let \( H_{P(U,Y)}(U|Y) \) denote the conditional entropy of \( U \) given \( Y \) under the joint distribution \( P_{U,Y} \).

As discussed at the beginning of Section [1] to keep the notation simple, we drop the subscript in \( P_{U,Y} \) and denote this conditional entropy by \( H_P(U|Y) \) henceforth. Similarly, the mutual information between \( U \) and \( Y \) is denoted as \( I_P(U:Y) \).

Let \( \mathcal{T} \) denote the set of all types of length-\( n \) sequences \((u, y)\). For any type \( T_{U,Y} \in \mathcal{T} \), we define
\[
B_\delta(T_{U,Y}) = \{ \tau \in \mathcal{T} : \| \tau - T_{U,Y} \|_1 \leq \delta \}.
\]

By definition, if \( T_{u,y} \in B_\delta(T_{U,Y}) \), then \((u, y) \in T^n_u(P_{U,Y}) \). We know that if \((u, y) \in T^n_u(P_{U,Y}) \), then
\[
(\alpha) \text{ there exists } g(\delta) > 0, \text{ where } g(\delta) \to 0 \text{ as } \delta \to 0 \text{ and } g(\delta) \text{ does not depend on } P_{U,Y}, \text{ such that } H^n_P(U|Y) \leq 2^{-n(H_{P(U,Y)}(U|Y) + g(\delta))}.
\]

Thus, given \((u, y) \in T^n_u(P_{U,Y}) \) and for any \( \tau \in B_\delta(T_{U,Y}) \),
\[
|\{ \hat{u} : T_{u,y} = \tau \}| \leq |\{ \hat{u} : T_{u,y} \in B_\delta(T_{U,Y}) \}| = |\{ \hat{u} : (\hat{u}, y) \in T^n_u(P_{U,Y}) \}|
\]
\[
= \left( a \right) \{ \hat{u} : \hat{u} \in T^n_u(P_{U,Y}) \}
\]
\[
\leq 2^{n(H_{P(U,Y)}(U|Y) + g(\delta))}.
\]

where \( a \) follows from \( a \) above while \( b \) follows from \( b \).

Let
\[
P_{U,Y}^{(Q)}(u, y) = \sum_{x, s, j} P_S(s)P_{U|S}(u|s)\mathbb{1}_{\{X=x(u,s)\}}.
\]

be the joint distribution for \((U, Y)\) under the memoryless strategy \( Q_{JS} \in \mathcal{P}(J|S) \) of the adversary. Finally, let us denote
\[
Q^*_{JS} = \arg \min_{Q_{JS} \in \mathcal{P}(J|S)} I_P(Q_{JS})(U:Y).
\]

Note that the above minimum is achieved, and hence, at least one exists. If there are more than one minimizers, pick one arbitrarily from amongst them.

We now get a bound on the size of \( L(y, \gamma(\delta)) \).
\[
|L(y, \gamma(\delta))| = \left\{ u : T_{u,y} - P_{U,Y}^{(Q)} \right\}_\infty \leq \gamma(\delta),
\]
for some \( Q_{JS} \in \mathcal{P}(J|S) \).

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for some \( Q_{JS} \in \mathcal{P}(J|S) \).
Given a message $m$ and having observed the state $S$, the encoder looks within the bin $m$ for some $U_{m,k}$, $k \in 1,2,\ldots,2^{nR}$, such that
\[ |\langle U_{m,k} - \alpha S, S \rangle| \leq n\delta_1, \tag{23} \]
for some $\delta_1 > 0$ (the choice of $\delta_1$ will be discussed later in Lemma [19]). If no such $U_{m,k}$ is found, then the encoder chooses $U_{1,1}$. If more than one $U_{m,k}$ satisfying (23) exists, the encoder chooses one uniformly at random from amongst them. Let $U$ denote the chosen codeword.

**Decoding:**

- We employ the minimum angle decoder. When $\hat{y}$ is received at the decoder, its message estimate $\hat{m}$ is the solution of the following optimization problem.
\[ \hat{m} = \arg \max_{1 \leq j \leq 2^{nR}} \left( \max_{1 \leq k \leq 2^{nR}} \langle \hat{y}, \hat{u}, j, k \rangle \right). \]

Here the decoder finds the codeword $u \in C$ closest in angle to $y$.

- If no unique solution exists, the decoder declares an error by setting $\hat{m} = 0$.

**Probability of error analysis:**

Fix some $\epsilon_1, \epsilon > 0$, and let
\[ R = \frac{1}{2} \log \left( 1 + P/(\Lambda + \sigma^2) \right) - \epsilon. \]

Note that $R < C$ and $R$ approaches $C$ as $\epsilon_1, \epsilon \to 0$. Next, let $\hat{R} = \frac{1}{2} \log (P_U/P) + \epsilon/2$.

Recall that $R_U = R + \hat{R}$, and hence, we have
\[ R_U = \frac{1}{2} \log \left( \frac{(P_U + \sigma^2)P_U}{(\Lambda + \sigma^2)P} \right) - \epsilon/2. \tag{24} \]

Before we proceed, here is a brief outline of the analysis. Given any $\delta > 0$, we establish in Lemma [19] that irrespective of the adversary’s strategy, the inner product $\langle \hat{Y}, \hat{U} \rangle$ is at least $(\theta - \delta)$ (here $\theta$ is given in (26) w.h.p. for sufficiently large $n$. Now regardless of the strategy the adversary employs, a decoding error occurs only if either $\langle \hat{Y}, \hat{U} \rangle < (\theta - \delta)$ or some other codeword $U_{m',k'}$, for $m' \neq m$ and $k' \in \{1,2,\ldots,2^{nR}\}$, satisfies $\langle \hat{Y}, \hat{U}_{m',k'} \rangle \geq (\theta - \delta)$. Our aim will be to show that the probability of this decoding error event goes to zero as $n \to \infty$.

Let us denote the decoding error event by $\mathcal{E}$. Then, we have $\mathcal{E} = \{M \neq \hat{M}\}$. Let $M = m$ be the message sent. Given $\theta$ and for any $\delta > 0$, we then have
\[ \mathbb{P}(\mathcal{E} | M = m) \leq \mathbb{P} \left( \langle \hat{Y}, \hat{U} \rangle < \theta - \delta | M = m \right) + \mathbb{P} \left( \exists m', k' : m' \neq m, \langle \hat{Y}, \hat{U}_{m',k'} \rangle \geq \theta - \delta | M = m \right). \]

Using (24), it follows that
\[ P^e(m) = \max_m \max_{Q \in QJ_{m,S} : \mathcal{J}(\Lambda)} \mathbb{P}(\mathcal{E} | M = m). \]

Hence,
\[ P^e(m) \leq \max_m \max_{Q \in QJ_{m,S} : \mathcal{J}(\Lambda)} \mathbb{P} \left( \langle \hat{Y}, \hat{U} \rangle < \theta - \delta | M = m \right) + \mathbb{P} \left( \exists m', k' : m' \neq m, \langle \hat{Y}, \hat{U}_{m',k'} \rangle \geq \theta - \delta | M = m \right). \tag{25} \]

We will show that given any $\epsilon_1, \epsilon > 0$, we can find a $\delta > 0$ such that both the RHS terms above converge to $0$ as $n \to \infty$.

We now state some important lemmas which are needed to proceed with the probability of error analysis. We first state a main lemma. We use Lemmas 16, 17 and 18 towards proving this lemma which directly follows from [34, Lemma 2].

**Lemma 15.** Consider any $\mathbf{f}$ on the unit $n$-sphere and suppose an independent random vector $\mathbf{R}$ is uniformly distributed on this sphere. Then for any $\gamma$ satisfying $1/\sqrt{2\pi n} < \gamma < 1$, we have
\[ \mathbb{P}\{ \| \mathbf{f} - \mathbf{R} \| \geq \gamma \} \leq 2(n-1)^{1/2} \log(1-\gamma^2). \]

The above lemma is used in the proof of the next lemma, which guarantees encoding success with high probability.

**Lemma 16.** For any $\delta_1 > 0$ and message $M = m$, the probability that the encoder finds at least one $U_{m,k}$ satisfying (23) approaches 1 as $n \to \infty$.

The proof of this lemma appears in Appendix B. In the following lemma, we show that $U - \alpha S$ satisfies the encoder power constraint, and hence, $X = U - \alpha S$ with high probability.

**Lemma 17.** For any $\delta_2$ satisfying $0 < \delta_2 < \epsilon_1$ and message $M = m$,
\[ \mathbb{P} \left( \| U - \alpha S \|^2 - nP \| > n\delta_2 | M = m \right) \to 0, \]
as $n \to \infty$.

Refer Appendix B for the proof of this lemma. The following lemma captures the correlation that an adversary can induce with the chosen codeword through the choice of its jamming signal. We use Lemma 15 in the proof of this lemma as well.

**Lemma 18.** For any $\delta_3 > 0$ and message $M = m$, under any jamming strategy $Q_{J,M,S} : J \in \mathcal{J}(\Lambda)$,
\[ \mathbb{P} \left( \| J, U - \langle J, S \rangle S \|, S \| U \| > n\delta_3 | M = m \right) \to 0, \]
as $n \to \infty$.

The proof can be found in Appendix B. The following is the main lemma. We use Lemmas 16, 17 and 18 towards proving it. This lemma shows that given any $\delta > 0$, the inner product $\langle \hat{Y}, \hat{U} \rangle$ is at least $(\theta - \delta)$ with high probability irrespective of the adversary’s strategy $Q_{J,M,S} : J \in \mathcal{J}(\Lambda)$. Recall that $\delta_1$ is the parameter which appears in the definition of the encoder (see 23).
Lemma 19. There is a function $\delta_1 : \mathbb{R}^+ \to \mathbb{R}^+$, where $\delta_1(\delta) \to 0$ as $\delta \to 0$, such that for every message $M = m$, under any jamming strategy $Q_{J,M,S} : J \in \mathcal{J}(\Lambda)$ and for any $\delta > 0$, if the parameter $\delta_1$ in the definition of the encoder is chosen as $\delta_1(\delta)$, then

$$\mathbb{P}\left(\|\hat{Y}, \hat{U}\| < (\theta - \delta) \mid M = m\right) \to 0,$$

as $n \to \infty$, where

$$\theta = \sqrt{\frac{\alpha(P' + \alpha^2 \frac{\hat{Y}}{P_U})}{P_U}}.$$ (26)

The proof of this lemma is in Appendix B. Note that $\theta$ also depends on $\epsilon$. Coming back to the error analysis, note that Lemma 19 implies that the first RHS term in (25) can be made arbitrarily small by choosing a sufficiently large $n$, provided the encoder parameter $\delta_1$ is chosen suitably depending on $\delta$. Now, the second RHS term in (25) can be bounded using the union bound, and hence, for any $Q_{J,M,S} : J \in \mathcal{J}(\Lambda)$, we have,

$$\mathbb{P}\left(\exists m', k' : n' \neq m, \langle \hat{Y}, \hat{U}_{m',k'} \rangle \geq \theta - \delta \mid M = m\right) \leq \sum_{m' \neq m, k'} \mathbb{P}\left(\langle \hat{Y}, \hat{U}_{m',k'} \rangle \geq \theta - \delta \mid M = m\right).$$ (27)

For any $m' \neq m$ and $k'$, we have

$$\mathbb{P}\left(\langle \hat{Y}, \hat{U}_{m',k'} \rangle \geq \theta - \delta \mid M = m\right) \leq 2^{(n-1)\frac{1}{2} \log(1 - (\theta - \delta)^2)},$$ (28)

by Lemma 15 where we replace $(\hat{r}, \hat{r})$ by $(\hat{Y}, \hat{U}_{m',k'})$ and $\gamma$ by $(\theta - \delta)$. Using (28) in (27) and noting that the total number of codewords is $2^{nR_U}$, we can conclude that for any $Q_{J,M,S} : J \in \mathcal{J}(\Lambda)$,

$$\mathbb{P}\left(\exists m', k' : n' \neq m, \langle \hat{Y}, \hat{U}_{m',k'} \rangle \geq \theta - \delta \mid M = m\right) \leq 2^{nR_U} 2^{(n-1)\frac{1}{2} \log(1 - (\theta - \delta)^2)}.$$ (29)

We now give an alternate expression for $R_U$ in terms of $\theta$. Toward this, consider the following:

$$1 - \theta^2 \overset{(a)}{=} 1 - \frac{\alpha(P' + \alpha^2 \hat{Y})}{P_U} = \frac{P_U - \alpha^2 \hat{Y} - \alpha P'}{P_U} \overset{(b)}{=} \frac{P' - \alpha P'}{P_U} = \frac{(1 - \alpha)P'}{P_U} \overset{(c)}{=} \frac{(\Lambda + \sigma^2)P'}{(P' + \Lambda + \sigma^2)P_U},$$ (30)

where (26) gives (a), while (b) follows from noting that $P_U = P' + \alpha^2 \hat{Y}$. We get (c) as $\alpha = P'/(P' + \Lambda + \sigma^2)$. Recall from earlier in (24) our choice of $R_U$. Using (30), we observe that $R_U$ can be also expressed as

$$R_U = -\frac{1}{2} \log \left(1 - \theta^2\right) - \epsilon/2.$$

Now choosing a small enough $\delta > 0$ in (29) such that

$$R_U < -\frac{1}{2} \log \left(1 - (\theta - \delta)^2\right),$$ (31)

the RHS in (29), and hence, the second term in the RHS of (25), approaches $0$ as $n \to \infty$. Thus, $P_e^{(n)}$ goes to $0$ as $n \to \infty$, and this completes the proof of achievability.

2) Converse: We prove the converse for an average probability of error criterion instead of the maximum probability of error criterion. For this stronger version of the converse, we define the average probability of error (similarly as in (6)) by

$$P_e^{(n)} = \frac{1}{2nR} \sum_{m=1}^{2^{nR}} P_e^{(n,m)},$$ (32)

where

$$P_e^{(n,m)} = \max_{Q_{J,M,S} : J \in \mathcal{J}(\Lambda)} \mathbb{P}(\Phi(Y) \neq m | M = m).$$ (33)

Now let us consider any sequence of codes with rate $R$ and $P_e^{(n)} \to 0$ as $n \to \infty$. Even though the adversary can choose an arbitrary feasible vector jamming strategy $Q_{J,M,S} : J \in \mathcal{J}(\Lambda)$, we analyze the performance of the encoder-decoder pair under an i.i.d. Gaussian jamming strategy. For an arbitrarily small $\delta > 0$, let $\Lambda' = \Lambda - \delta$. We define $J'$ to be a vector of length $n$ generated i.i.d. with $J'_i \sim \mathcal{N}(0, \Lambda')$, $\forall i$. We emphasize that $J'$ is not a feasible jamming strategy as $\|J'\|$ can be greater than $\sqrt{n} \Lambda$. We also define a feasible jamming strategy $J$ whose distribution is the same as the conditional distribution of $J'$, conditioned on $J' \in \mathcal{J}(\Lambda)$. Let $\epsilon > 0$ here. Under the jamming strategy $J'$, let $P_e^{(n)}$ be the average probability of error achieved by the given sequence of randomized codes. Then,

$$P_e^{(n)} = \frac{1}{2nR} \sum_{i=1}^{2^nR} \mathbb{P}(\Phi(Y) \neq m | S + J' + Z) \neq i)$$

$$\leq \frac{1}{2nR} \sum_{i=1}^{2^nR} \mathbb{P}(\Phi(Y) \neq m | S + J' + Z) \neq i | J' \in \mathcal{J}(\Lambda)) \cdot \mathbb{P}(J' \in \mathcal{J}(\Lambda)) + \mathbb{P}(\|J'\|^2 > n\Lambda)$$

$$\leq \frac{1}{2nR} \sum_{i=1}^{2^nR} \mathbb{P}(\Phi(Y) \neq m | S + J' + Z) \neq i | J' \in \mathcal{J}(\Lambda)) + \epsilon$$

$$\leq \frac{1}{2nR} \sum_{i=1}^{2^nR} P^{(n)}_{e,i} + \epsilon$$

$$\leq P_e^{(n)} + \epsilon < 2\epsilon.$$

Note that there exists $\delta > 0$ such that (31) is satisfied. To see this, define $f(\delta) = -1/2 \log(1 - (\theta - \delta)^2)$. It can be easily verified that $f$ is a continuous and monotonically decreasing function of $\delta$. 
Here the probability is over the shared randomness, the channel, the state and adversary’s (i.i.d. Gaussian) action. As \( J' \) is i.i.d Gaussian with \( J'_n \sim N(0, \Lambda') \), \( \forall i \), we have \( P(\|J'\|^2 > n\Lambda) \to 0 \) as \( n \to \infty \). We choose \( n \) large enough such that \( P(\|J'\|^2 > n\Lambda) \leq \epsilon \), which gives (a). Then, (b) follows from (33) since \( J \) is a feasible jamming strategy, while (c) follows from (32). We now choose \( n \) large enough such that the probability \( P_{n}(^n) \) is less than \( \epsilon \), where \( \epsilon > 0 \). This gives us (d). Thus, we have shown that for any \( \epsilon > 0 \), under i.i.d. Gaussian (variance \( \Lambda' \)) jamming, the given sequence of randomized encoder-decoder pairs achieve \( P_{n}(^n) < 2\epsilon \) for large enough \( n \).

Under the jamming strategy \( J' \), the resulting channel is a dirty paper channel with noise variance \( \Lambda' + \sigma^2 \). Hence, the rate \( R \) must be smaller than the capacity of this channel, i.e.,
\[
C \leq \frac{1}{2} \log \left( 1 + \frac{P}{\Lambda' + \sigma^2} \right).
\]
Since this holds for any \( \Lambda' < \Lambda \), we have
\[
C \leq \frac{1}{2} \log \left( 1 + \frac{P}{\Lambda + \sigma^2} \right).
\]
This completes the proof of the converse.

V. DISCUSSION AND CONCLUSION

In this work, we analysed the performance of a communication system over a state-dependent channel in the presence of an adversary. Here both the encoder and the adversary were state-aware, i.e., they possessed non-causal knowledge of the state. The adversary induced an A VC through its jamming interference into the channel, where the interference could be designed using the non-causal knowledge of the state. We studied two versions, the discrete memoryless GP-A VC and the additive white Gaussian DP-A VC, and determined their randomized coding capacity under a maximum probability of error criterion. As in other randomized coding setups, we showed that the capacity for both our AVC setups was the same under the average probability of error criterion as well. Owing to the presence of shared randomness, it was seen that even with the non-causal knowledge of the state vector and the ability to use vector jamming strategies, the adversary could impact the communication rate no worse than by choosing memoryless strategies. Thus, the capacity of both the AVCs was characterized as that of the worst memoryless channel with state that the adversary could induce through some memoryless strategy. Furthermore, in the DP-AVC it was shown that the adversary, given its purpose, could do no better than to disregard the state knowledge entirely and introduce state-independent white Gaussian noise. Both deterministic coding capacity and the effect of limited shared randomness are natural next steps to this work. It would be interesting to know if, like for standard AVCs, O(log \( n \)) bits of randomness (in a block length of \( n \)) are sufficient to achieve randomized capacity. Finally, the results presented in this work could be similarly extended to state-dependent channels, where, in addition to the encoder and adversary, the decoder too is state-aware.

APPENDIX A

PROOF OF LEMMA [8]

The given distribution \( P_Z \) is ‘close’ to the uniform distribution over \( T_{n0}^n(\mathcal{P}_{X,Y|Z}(x,y)) \) due to the properties (a) and (b). Hence, in a two part proof, we first bound the probability \( P(\mathcal{Z} \notin T_{n0}^n(\mathcal{P}_{X,Y|Z}(x,y))) \) for \( \mathcal{Z} \sim \text{Unif}(T_{n0}^n(\mathcal{P}_{X,Y|Z}(x,y))) \). Then, in the second part, we appropriately modify this bound to obtain a bound on \( P(\mathcal{Z} \notin T_{n0}^n(\mathcal{P}_{X,Y|Z}(x,y))) \) under the given distribution \( P_Z \).

To prove the first part, we begin by assuming that \( \mathcal{Z} \sim \text{Unif}(T_{n0}^n(\mathcal{P}_{X,Y|Z}(x,y))) \). Then, as given on the next page, we can simplify \( P(\mathcal{Z} \notin T_{n0}^n(\mathcal{P}_{X,Y|Z}(x,y))) \) to (36), where (34) follows from the union bound, and (35) follows by relaxing the strict inequality. Since \( \mathcal{Z} \in T_{n0}^n(\mathcal{P}_{X,Y|Z}(x,y)) \) (with probability one), we have \( \forall (y,z) \in \mathcal{Y} \times \mathcal{Z} \)
\[
\left| \frac{N(y,z,y,Z)}{n} - P_Y(y)P_{Z|Y}(z|y) \right| < \delta_0.
\]
For every \( (y,z) \) such that \( P_{X,Y}(z|y) = 0 \), \( N(y,z,y,Z) \leq n\delta_0 \). This further implies that \( N(x,y,z|x,y,Z) \leq n\delta_0 \). By choosing \( \delta \) large enough such that \( \delta > \delta_0 \), we can guarantee that \( N(x,y,z|x,y,Z) < n\delta \), and hence, it follows that the probability of both the terms in the summation in (36) is zero.

For other values of \( (y,z) \), for which \( P_{X,Y}(z|y) > 0 \), we first note that \( P_{Z|Y}(z|y) \geq P_{Z|Y}^{\text{min}} \), where
\[
P_{Z|Y}^{\text{min}} := \min_{(y,z):P_{X,Y}(z|y)>0} P_{Z|Y}(z|y).
\]
We define \( \delta''_0 = \delta_0 + \sqrt{\delta_0} < 2\sqrt{\delta_0} \), and we assume that \( \delta > 3\sqrt{\delta_0} \). If \( P_Y(y) < \delta''_0 \), then \( P_{Y,Z}(y,z) < \delta''_0 \). This implies that
\[
N(y,z,y,Z) \leq n(\delta''_0 + \delta_0) < n(3\sqrt{\delta_0}) < n\delta.
\]
This again implies that
\[
N(x,y,z|x,y,Z) \leq n\delta,
\]
and thus, the probability of the first term in (36) is zero. Further, if \( P_Y(y) < \delta''_0 \), then
\[
P_Y(y)P_{X|Y}(x|y)P_{Z|Y}(z|y) \leq \delta''_0.
\]
This implies that
\[
-\delta + P_Y(y)P_{X|Y}(x|y)P_{Z|Y}(z|y) < 0,
\]
and hence, the probability of the second term in (36) is zero. We have, thus, shown that the probability terms in both the summations in the RHS of (36) are equal to zero. Based on the above observations, we now consider those \( (y,z) \in \mathcal{Y} \times \mathcal{Z} \) such that \( P_{Z|Y}(z|y) \geq P_{Z|Y}^{\text{min}} \) and \( P_Y(y) \geq \delta''_0 \).

We know that \( (x,y) \in T_{n0}^n(\mathcal{P}_{X,Y}) \). Hence,
\[
\left| \frac{N(x,y|x,y)}{n} - P_{X,Y}(x,y) \right| \leq \delta_0 \ \forall (x,y).
\]
We now make the following claim.
\[ \mathbb{P}(Z \not\in T^n(\mathcal{P}_{X,Y,Z|x,y})) = \mathbb{P}\left( \bigcup_{x,y,z} \left\{ \left| \frac{N(x, y, z|x, y, Z)}{n} - P_{X,Y}(x, y)P_{Z|Y}(z|y) \right| > \delta \right\} \right) \]
\[ \leq \sum_{x,y,z} \mathbb{P}\left( \frac{N(x, y, z|x, y, Z)}{n} - P_{X,Y}(x, y)P_{Z|Y}(z|y) < \delta \right) \]
\[ \leq \sum_{x,y,z} \mathbb{P}\left( \frac{N(x, y, z|x, y, Z)}{n} - P_{X,Y}(x, y)P_{Z|Y}(z|y) \geq \delta \right) \]
\[ = \sum_{x,y,z} \mathbb{P}\left( \frac{N(x, y, z|x, y, Z)}{n} \geq P_{X,Y}(x, y)P_{Z|Y}(z|y) + \delta \right) \]
\[ + \sum_{x,y,z} \mathbb{P}\left( \frac{N(x, y, z|x, y, Z)}{n} \leq P_{X,Y}(x, y)P_{Z|Y}(z|y) - \delta \right) \]

Claim 20. If \( z \in T^n_0(P_{Y,Z|Y}) \), then
\[ \left| \frac{N(y, z|y, z)}{N(y|y)} - P_{Z|Y}(z|y) \right| \leq \delta_0 - \forall (y, z), \quad (39) \]
where \( \delta_0 = 2\sqrt{\delta_0} \).

Proof: Since \((y, z) \in T^n_0(P_{Y,Z})\), we have \( \forall (y, z) \),
\[ N(y, z|y, z) - P_{Z|Y}(z|y) \leq \delta_0. \]
As \( P_Y(y) \geq \delta_0 \) and from \( 38 \), it follows that \( N(y|y)/n > 0 \).
Thus,
\[ \frac{N(y, z|y, z)}{N(y|y)} \leq \frac{P_{Z|Y}(z|y) + \delta_0}{N(y|y)}. \]
But, we know that
\[ \left| \frac{N(y|y)}{n} - P_Y(y) \right| \leq \delta_0 \quad \forall y. \]
Hence, it follows that
\[ \frac{N(y, z|y, z)}{N(y|y)} - P_{Z|Y}(z|y) \leq \frac{P_{Y,Z}(y, z) + \delta_0}{2\delta_0} - P_{Z|Y}(z|y) \]
\[ = \frac{\delta_0(1 + P_{Z|Y}(z|y))}{2\delta_0} - P_{Z|Y}(z|y) \]
\[ \leq \frac{P_{Y}(y) - \delta_0}{2\delta_0} \]
\[ \leq \delta_0 - \delta_0 \]
\[ \leq 2\sqrt{\delta_0}. \]

Here \((a)\) follows from \( P_Y(y) \geq \delta_0 \), and \((b)\) is true as \( \delta_0 = \delta_0 + \sqrt{\delta_0} \).
Similarly, it can be shown that
\[ \frac{N(y, z|y, z)}{N(y|y)} - P_{Z|Y}(z|y) \geq -2\sqrt{\delta_0}. \]
This completes the proof of the claim.

Continuing the analysis further, we consider a term inside the first sum in \( 36 \). We first recall that if \( N(x, y|x, y) < n\delta \), then \( N(x, y, z|x, y, Z) < n\delta \), and thus, the probability under consideration is zero. Hence, in the following, we assume w.l.o.g. that \( N(x, y|x, y) \geq n\delta \). We now get \( 42 \), as given on top of the next page, where
\[ t_1 = P_{\min Z|Y} \left( \left( \frac{\delta - \delta_0}{\delta_0} \right) - \frac{\delta_0}{P_{\min Z|Y}} \right), \]
and does not depend on \( n \). We choose \( \delta \) such that \( t_1 > 0 \). Recall that we have earlier required \( \delta > 3\sqrt{\delta_0} \) already. Note that \( 40 \) (given on the next page) follows from the upper bound for \( N(x, y|x, y)/n \) in \( 38 \), while \( 41 \) (given on the next page) follows as \( \forall (y, z) \) under consideration, \( P_{Z|Y}(z|y) \geq P_{\min Z|Y} \). The following claim now gives an exponentially decaying bound on the term appearing in \( 42 \).

Claim 21. If \( N(x, y|x, y) \geq n\delta \) and \( t_1 > 0 \),
\[ \mathbb{P}\left( \frac{N(x, y, z|x, y, Z)}{N(x, y|x, y)} - (P_{Z|Y}(z|y) + \delta_0) \geq t_1 \right) \leq e^{-2n\delta t_1^2}. \]

Proof: Let \( S(x, y|x, y) \) denote the indices of \( (x, y) \) with the value \( (x, y) \) and \( S(y|y) \) denote the indices of \( y \) with the value \( y \). We now consider a different but equivalent random experiment for generating \( Z \). First \( \tilde{Z} \) is chosen uniformly at random from \( T^n_0(P_{Y,Z|Y}) \), where \( P_{Y,Z} = P_Y P_{Z|Y} \), and then, for each \( y \), its components at \( S(y|y) \) are subjected to a permutation chosen uniformly at random from the set of all permutations of \( S(y|y) \). Since the set of sequences in \( T^n_0(P_{Y,Z|Y}) \) are invariant under such permutations, this two-step process results in the same final distribution of \( \tilde{Z} \), i.e., uniform over \( T^n_0(P_{Y,Z|Y}) \). From \( 39 \), \( N(y, z|y, \tilde{Z}) \) is bounded by
\[ N(y, z|y, \tilde{Z}) \leq N(y|y)(P_{Z|Y}(z|y) + \delta_0). \]

For a given \( S(y|y) \) and conditioned on \( N(y, z|y, \tilde{Z}) = k \), the number \( N(x, y, z|x, y, \tilde{Z}) \) can be considered as the number of positions in \( S(x, y|x, y) \) at which the letter \( z \) is assigned by the random permutation in the components in \( S(y|y) \). Thus, \( N(x, y, z|x, y, \tilde{Z}) \) is the number of times \( z \) is obtained when a total of \( |S(x, y|x, y)| = N(x, y|x, y) \) samples are drawn without replacement from a collection of \( |S(y|y)| \) components, of which \( k \) components have value \( z \). Now using
where the last step follows from (43) and

\[ P \left( \frac{N(x, y, z | x, y, Z)}{n} \right) \geq \delta + P_{X,Y}(x, y)P_{Z|Y}(z|y) \]

\[ = P \left( \frac{N(x, y, z | x, y, Z)}{N(x, y | x, y)} \right) \geq \delta + P_{X,Y}(x, y)P_{Z|Y}(z|y) \]

\[ \leq P \left( \frac{N(x, y, z | x, y, Z)}{N(x, y | x, y)} \right) \geq \frac{\delta + P_{X,Y}(x, y)P_{Z|Y}(z|y)}{\delta_0 + P_{X,Y}(x, y)} \]

\[ \leq \frac{N(x, y, z | x, y, Z)}{N(x, y | x, y)} \geq P_{Z|Y}(z|y) \frac{\delta + P_{X,Y}(x, y)P_{Z|Y}(z|y)}{\delta_0 + P_{X,Y}(x, y)} \]

\[ \leq P \left( \frac{N(x, y, z | x, y, Z)}{N(x, y | x, y)} \right) \geq P_{Z|Y}(z|y) \left( 1 + \frac{\delta - \delta_0}{1 + \delta_0} \right) \]

\[ \leq P \left( \frac{N(x, y, z | x, y, Z)}{N(x, y | x, y)} \right) - (P_{Z|Y}(z|y) + \delta_0') \geq P_{Z|Y}(z|y) \left( \frac{\delta - \delta_0}{1 + \delta_0} - \delta_0' \right) \]

Hence, for the rest of the analysis we assume that

\[ N(x, y | x, y) \geq (1/4) n \delta. \]

Note that this implies

\[ P_{X,Y}(x, y) - \delta_0 \geq \left( \frac{N(x, y | x, y)}{n} - \delta_0 \right) \]

\[ \geq \frac{\delta}{4} - 2\delta_0 \]

\[ \geq 0. \] (44)

Here, (a) follows from (38), and (b) follows by choosing \( \delta > 8\delta_0 \). We now get (47), given on top of the next page, where

\[ t_2 = P_{\min_{Z|Y}} \left( \left( \frac{\delta - \delta_0}{1 - \delta_0} - \frac{\delta_0'}{P_{\min_{Z|Y}}} \right) \right) \]

and does not depend on \( n \). Once again, we choose \( \delta \) so as to ensure that \( t_2 > 0 \). Observe that (45) (given on the next page) follows from the lower bound for \( N(x, y | x, y) / n \) in (38) as well as by choosing \( \delta > 8\delta_0 \) so that (44) is true. We get (46) (given on the next page) as we are analyzing for \( (y, z) \) for which \( P_{Z|Y}(z|y) \geq P_{\min_{Z|Y}} \).

\[ \text{Claim 22. If } N(x, y | x, y) \geq (1/4) n \delta \text{ and } t_2 > 0 \]

\[ \exists (x, y, z) \in Y \times Z \text{ such that } P_Y(y) \geq \delta_0' \]

\[ = \delta_0 + \sqrt{\delta_0} \]

and \( P_{Z|Y}(z|y) \geq P_{\min_{Z|Y}} \). Note that if \( N(x, y | x, y) \leq (1/4) n \delta \), then from (38),

\[ P_{X,Y}(x, y) \leq N(x, y | x, y) / n + \delta_0 \]

\[ \leq \frac{\delta}{4} + \delta_0. \]

Hence,

\[ P_{X,Y}(x, y)P_{Z|Y}(z|y) \leq \frac{\delta}{4} + \delta_0. \]

Then, the probability under consideration is zero if \( \delta_0 < 3\delta / 4. \)

Hoeffding’s inequality for sampling without replacement [37],

\[ P \left( \frac{N(x, y, z | x, y, \tilde{Z})}{n} > t_1 \right) \geq \frac{N(x, y | x, y)}{n} \]

\[ \leq e^{-2n \left( S(x, y | x, y) + t_1 \right)^2} \]

\[ \Rightarrow \left( \frac{N(x, y, z | x, y, \tilde{Z})}{n} > \frac{N(x, y | x, y)}{n} \right) \]

\[ \leq e^{-2n \left( S(x, y | x, y) + t_1 \right)^2} \]

\[ \Rightarrow \left( \frac{N(x, y, z | x, y, \tilde{Z})}{n} > \frac{N(x, y | x, y)}{n} \right) \]

\[ \geq e^{-2n \left( S(x, y | x, y) + t_1 \right)^2}, \]

where the last step follows from (43) and \( N(x, y | x, y) \geq n \delta. \)

This completes the proof of Claim 21.
By condition (a) of the Lemma, the perturbation in the distribution can increase the probability of any typical sequence by a factor of at most
\[
2^{-n(H(Z|Y)+\tilde{g}(\delta_0))}.
\]
By condition (b) of the Lemma, the perturbation in the distribution can increase the probability of any typical sequence by a factor of at most
\[
2^{-n(H(Z|Y)-\tilde{g}(\delta_0))/2} = 2^{n(h(\delta_0))}.
\]
Here \(h(\delta_0) > 0\) and \(h(\delta_0) \to 0\) as \(\delta_0 \to 0\).

Thus, the probability \(\mathbb{P}(Z \notin T_{\delta_0}^n(P_{X,Z}|Y))\) can now be bounded as follows. Given \(x\) and \(y\), let us define the set
\[
E = \{z : (x,y,z) \notin T_{\delta_0}^n(P_{X,Y,Z})\} = E_1 \cup E_2,
\]
where \(E_1 = E \cap T_{\delta_0}^n(P_{X,Y,Z})\) and \(E_2 = E \setminus E_1\). Then, using the union bound and (48), we have
\[
\mathbb{P}(E) \leq \mathbb{P}(E_1) + \mathbb{P}(E_2) \\
\leq \left(2 |\mathcal{X}||\mathcal{Y}|\mathcal{Z}| \epsilon + \frac{\epsilon}{2} \right)^2 2^{nh(\delta_0)} + \epsilon + \epsilon.
\]
We now choose a \(\delta(\delta_0)\) large enough such that \(K = \frac{1}{2}\delta^2 - h(\delta_0)ln 2 > 0\) as well as all the other conditions on \(\delta\) appearing in the proof are met. This completes the proof of Lemma 8.

**APPENDIX B**

**Proofs of Lemmas 16, 17, 18 and 19**

**A. Proof of Lemma 16**

Let \(M = m\) be the message and define the event (as a function of \(\delta_1 > 0\))
\[
E_0 = \left\{ \left| \|S\|^2 - n\sigma_S^2 \right| > n\delta_0(\delta_1) \right\},
\]
where \(0 < \delta_0(\delta_1) < \delta_1\) (the exact choice of \(\delta_0(\delta_1)\) will be discussed later in Claim 24), and \(\delta_0(\delta_1) \to 0\) as \(\delta_1 \to 0\). As \(S\) is an i.i.d. Gaussian vector, where \(S_i \sim N(0,\sigma_S^2), \forall i\), it follows that \(\mathbb{P}(E_0) \to 0\) as \(n \to \infty\) for given \(\delta_0 > 0\). Next, let us define
\[
\beta := \alpha \sqrt{\frac{\sigma_S^2}{P_U}}.
\]
Note that $\beta$ also depends on $\epsilon_1$ through the definition of $P_U$. We observe that
\[
\hat{R} > \frac{1}{2} \log \left( \frac{P_U}{P^{'}} \right) = \frac{1}{2} \log \left( \frac{1}{1 - \beta^2} \right),
\]
(51)
as $P_U = P' + \alpha^2 \sigma_2^2$ and by noting that $1 - \beta^2 = P' / P_U$ from [50]. Our aim is to show that for any $\delta_1 > 0$,
\[
\mathbb{P}(\hat{\beta} : |\langle U_{m,k} - \alpha \mathbf{S}, \mathbf{S} \rangle| \leq n\delta_1) \to 0,
\]
as $n \to \infty$. Note that
\[
\mathbb{P}(\hat{\beta} : |\langle U_{m,k} - \alpha \mathbf{S}, \mathbf{S} \rangle| \leq n\delta_1) \leq \mathbb{P}(E_0)
\]
\[
+ \int \mathbb{P}(\hat{\beta} : |\langle U_{m,k} - \alpha \mathbf{s}, \mathbf{s} \rangle| \leq n\delta_1|\mathbf{s} = \mathbf{s})dF_{\mathbf{S}}(\mathbf{s}),
\]
where $F_{\mathbf{S}}(\cdot)$ is the probability distribution function of $\mathbf{S}$. Recall from earlier that $\mathbb{P}(E_0) \to 0$ as $n \to \infty$. We now analyse the second term in the RHS of (52). Toward this, let us consider the following for any $\mathbf{s}$ satisfying $|||\mathbf{s}||^2 - n\sigma_2^2|| \leq n\delta_0(\delta_1)$ (i.e., $\mathbf{s} \in E^c_5$). Then,
\[
\mathbb{P}(\hat{\beta} : |\langle U_{m,k} - \alpha \mathbf{s}, \mathbf{s} \rangle| \leq n\delta_1|\mathbf{s} = \mathbf{s}) = \mathbb{P}(\langle U_{m,k} - \alpha \mathbf{s}, \mathbf{s} \rangle > n\delta_1, \forall k)
\]
\[
= \prod_{k=1}^{2n\delta_1} \mathbb{P}(\langle U_{m,k} - \alpha \mathbf{s}, \mathbf{s} \rangle > n\delta_1)
\]
\[
= \mathbb{P}(\langle U_{m,1} - \alpha \mathbf{s}, \mathbf{s} \rangle > n\delta_1)
\]
\[
= \mathbb{P}(\langle U_{m,1} - \alpha \mathbf{s}, \mathbf{s} \rangle \geq n\delta_1)
\]
\[
+ \mathbb{P}(\langle U_{m,1} - \alpha \mathbf{s}, \mathbf{s} \rangle < -n\delta_1)
\]
\[
= (1 - \mathbb{P}(\langle U_{m,1} - \alpha \mathbf{s}, \mathbf{s} \rangle \geq n\delta_1))2^{2n\delta_1}
\]
\[
(53)
\]
Here (a) follows as $U_{m,k}, \forall k$, are independently chosen, while (b) follows from the use of the union bound as well as relaxing the inequality in the second term.

To proceed, we require some additional results. We first state a lemma and then make a useful claim.

**Lemma 23.** Suppose $\hat{R}$ is chosen uniformly at random on the unit sphere surface. Then, for any unit vector $\hat{\mathbf{r}}$ and any $\gamma$ satisfying $0 < \gamma < 1$, we have
\[
\mathbb{P}(\langle \hat{\mathbf{r}}, \hat{\mathbf{R}} \rangle \geq \gamma) \geq 2^{-n\left(\frac{1}{2} \log \frac{1}{1 - \gamma^2} + f(n)\right)},
\]
where
\[
f(n) = \frac{1}{2n} \log \left( 2\pi n \gamma^2 (1 - \gamma^2) \right).
\]

There exists $n_0(\gamma)$ such that $f(n) \geq 0, \forall n \geq n_0(\gamma)$, and $\lim_{n \to \infty} f(n) = 0$.

**Proof:** The result directly follows from [38 eqn. (27)]. To see this, let $\angle(\hat{\mathbf{r}}, \hat{\mathbf{R}})$ denote the angle between the vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{R}}$. Then, from [38 eqn. (27)], we know that
\[
\mathbb{P}(\angle(\hat{\mathbf{r}}, \hat{\mathbf{R}}) \leq \theta) \geq \left( 1 - \frac{1}{n} \cos^2 \theta \right) \frac{1}{\sqrt{2\pi n}} \sin^{-1} \theta.
\]
Let us make the substitution $\gamma = \cos \theta$ in the above equation. Then,
\[
\mathbb{P}(\langle \hat{\mathbf{r}}, \hat{\mathbf{R}} \rangle \geq \gamma)
\]
\[
\geq \left( 1 - \frac{1}{n} \frac{1 - \gamma^2}{\gamma^2} \right) \frac{1}{\sqrt{2\pi n}} \sin^{-1} \frac{\sqrt{2\pi n}}{\gamma^2}
\]
\[
= 2^{-n\left(\frac{1}{2} \log \frac{1}{1 - \gamma^2} + f(n)\right)}.
\]
Thus, we have shown that
\[
\mathbb{P}(\langle \hat{\mathbf{r}}, \hat{\mathbf{R}} \rangle \geq \gamma) \geq 2^{-n\left(\frac{1}{2} \log \frac{1}{1 - \gamma^2} + f(n)\right)},
\]
where $f(n)$ is as given in the lemma. It is easily verified from the expression for $f(n)$ that there exists $n_0(\gamma)$ such that $f(n) \geq 0, \forall n \geq n_0(\gamma)$. This completes the proof of the lemma.

We now make the following claim. The previous lemma is used in the proof of this claim.

**Claim 24.** There exists $\delta_0(\delta_1)$, where $\delta_0(\delta_1) \to 0$ as $\delta_1 \to 0$ for $E_0$ as in [49]. Further, there exists $\delta_1(\delta_1) > 0$, where $\delta_1(\delta_1) \to 0$ as $\delta_1 \to 0$, such that for any $\mathbf{s} \in E^c_0$,

(i) there exists $n_0$, such that $f_1(n) \geq 0, \forall n \geq n_0$ and $\lim_{n \to \infty} f_1(n) = 0$, such that
\[
\mathbb{P}(\langle U_{m,1}, \mathbf{s} \rangle \geq \alpha||\mathbf{s}||^2 - n\delta_1)
\]
\[
\geq 2^{-n\left(\frac{1}{2} \log \frac{1}{1 - \gamma^2} + f_1(n)\right)},
\]
(ii) we have
\[
\mathbb{P}(\langle U_{m,1}, \mathbf{s} \rangle \geq \alpha||\mathbf{s}||^2 + n\delta_1) \leq 2^{-\left(\frac{n-1}{2} \log \frac{1}{1 - \gamma^2}\right)}.
\]

**Proof:** Consider any $\mathbf{s}$ satisfying $|||\mathbf{s}||^2 - n\sigma_2^2|| \leq n\delta_0(\delta_1)$ (where $\delta_0(\delta_1)$ is to be specified). We begin with the proof of part (i).
\[
\mathbb{P}(\langle U_{m,1}, \mathbf{s} \rangle \geq \alpha||\mathbf{s}||^2 - n\delta_1)
\]
\[
= \mathbb{P}(\langle \frac{U_{m,1}}{||U_{m,1}||}, \mathbf{s} \rangle \geq \frac{\alpha||\mathbf{s}||}{||U_{m,1}||} - \frac{n\delta_1}{||U_{m,1}||})
\]
\[
= \mathbb{P}(\langle \frac{U_{m,1}}{||U_{m,1}||}, \tilde{\mathbf{s}} \rangle \geq \frac{\alpha||\mathbf{s}||}{||U_{m,1}||} - \frac{n\delta_1}{||U_{m,1}||})
\]
\[
\geq \mathbb{P}(\langle \frac{U_{m,1}}{||U_{m,1}||}, \tilde{\mathbf{s}} \rangle \geq \frac{\alpha\sqrt{n(\sigma_2^2 + \delta_0)}}{\sqrt{nP_U}})
\]
\[
- \frac{n\delta_1}{\sqrt{n(\sigma_2^2 + \delta_0)\sqrt{nP_U}}}
\]
\[
(a)
\]
\[
(\beta)
\]
Coming back to the proof, it follows from (53) and Claim 24 that for any $s$ such that $||s||^2 - n \sigma_S^2 \leq \eta_0$,
\[
P(\exists k : |(U_{m,k} - \alpha s, s)| \leq \eta_1 |S = s| \leq \left(1 - 2^{-n\left(\frac{1}{2} \log \frac{1}{\tilde{\delta}_1} + f_1(n)\right)}\right) + 2^{-(n-1)\left(\frac{1}{2} \log \frac{1}{\tilde{\delta}_1}\right)} \leq \eta_0 \right).
\]  
(54)

Note that the upper bound does not depend on $s$. We use this fact to now simplify the RHS of (53) as follows.

\[
\int P(\exists k : |(U_{m,k} - \alpha s, s)| \leq \eta_1 |S = s| \, dF_S(s)
\]
\[
\leq \left[1 - 2^{-n\left(\frac{1}{2} \log \frac{1}{1 - (\tilde{\delta}_1)^2} + f_1(n)\right)}\right] + 2^{-(n-1)\left(\frac{1}{2} \log \frac{1}{1 - (\tilde{\delta}_1)^2}\right)} \leq \left[1 - 2 - 2^{-n\left(\frac{1}{2} \log \frac{1}{1 - (\tilde{\delta}_1)^2} + f_1(n)\right)}\right] + 2^{-(n-1)\left(\frac{1}{2} \log \frac{1}{1 - (\tilde{\delta}_1)^2}\right)} = \left[1 - 2 - 2^{-n\left(\frac{1}{2} \log \frac{1}{1 - (\tilde{\delta}_1)^2} + f_1(n)\right)}\right] + 2^{-(n-1)\left(\frac{1}{2} \log \frac{1}{1 - (\tilde{\delta}_1)^2}\right)} = \left[1 - 2 - 2^{-n\left(\frac{1}{2} \log \frac{1}{1 - (\tilde{\delta}_1)^2} + f_1(n)\right)}\right] + 2^{-(n-1)\left(\frac{1}{2} \log \frac{1}{1 - (\tilde{\delta}_1)^2}\right)}
\]

Here (54) gives (a), and we get (b) by defining

\[c(n) := \frac{1}{2} \log \frac{1 - (\beta - \tilde{\delta}_1)^2}{1 - (\beta + \tilde{\delta}_1)^2} - f_1(n) - \frac{1}{2n} \log \frac{1}{1 - (\beta + \tilde{\delta}_1)^2}.\]

We get (c) as follows. We choose $n$ large enough such that the exponent $c(n) > 0$ as well as $0 \leq f_1(n) \leq \eta_0$, for some $\eta_0 > 0$. The fact that such a choice of $n$ exists follows from part (i) of Claim 24 and since

\[\log \frac{1 - (\beta - \tilde{\delta}_1)^2}{1 - (\beta + \tilde{\delta}_1)^2} > 0.\]

We discuss the choice of $\eta$ later, but note that we can choose any $\eta > 0$. This gives us (c). Next, we define $\mu(n) := 1 -
2^{−n_c(n)} to get (d). We know that for any $x \in [0, 1]$ and any $k \geq 0$, $(1 − x)^k \leq e^{−kx}$. Now (e) follows from noting that $0 < \mu(n) < 1, \forall n$, implies $0 \leq \mu(n) \cdot 2^{−n_l} \leq 1$, for any $l \geq 0$.

Thus, given $δ₁ > 0$ (and hence, $δ₂(δ₁) > 0$) and from (51), it follows that we can choose an $η > 0$ small enough such that

$$\hat{R} \geq \left(\frac{1}{2} \log \frac{1}{1 - (\beta - δ₁)^2} + η\right).$$

This guarantees that the RHS in (55) goes to zero as $n \to ∞$. Using (55) in (52), it then follows that

$$\mathbb{P}(\mathbb{H}k : |⟨U_{m,k} − αS, S⟩| ≤ nδ₁) \to 0,$$

as $n \to ∞$. This concludes the proof.

B. Proof of Lemma 17

Given the message $M = m$, let us define the following events.

$$E₀ = \{||S||^2 - nσ₂^2| > nδ₀\},$$

$$E₁ = \{\exists k : |⟨U_{m,k} − αS, S⟩| ≤ nδ₁\}.$$

Here $δ₀, δ₁ > 0$ depend on $δ₂$, and will be chosen such that they approach 0 as $δ₂ \to 0$. Their choice will be specified later. Further, recall the proof of Lemma 16 where $δ₀$ is a function of $δ₁$. We use the same $δ₀$ function here, and hence, only need to specify $δ₁$. As $S$ is an i.i.d. Gaussian vector, where $S \sim \mathcal{N}(0, σ₂^2)$, $∀i$, $\mathbb{P}(E₀|M = m) \to 0$ as $n \to ∞$ for $δ₀ > 0$. From Lemma 16 it follows that for $δ₁ > 0$, $\mathbb{P}(E₁|M = m) \to 0$ as $n \to ∞$. Let $U$ denote the codeword chosen.

Let us define $E = E₀ ∪ E₁$. Conditioning on $E^c$ and noting that $U$ is chosen over the $n$-sphere with radius $\sqrt{nP_U}$, we have

$$\|U − αS\|^2 = \|U\|^2 + α^2\|S\|^2 − 2α ⟨U, S⟩ (a)$$

$$\geq \|U\|^2 + α^2\|S\|^2 − 2α(α\|S\|^2 + nδ₁)$$

$$= \|U\|^2 − α^2\|S\|^2 − n(2αδ₁)$$

$$\geq nP_U − nα^2(σ₂^2 + δ₀) − n(2αδ₁)$$

$$= n(P_U − α^2σ₂^2) − n(αδ₂ + 2αδ₁)$$

$$\geq nP' − nδ,$$

where $δ = (αδ₂ + 2αδ₁)$, and $δ \to 0$ as $δ₀, δ₁ \to 0$. Here (a) follows from Lemma 16 as conditioned on $E₁$, we have

$$α\|S\|^2 − nδ₁ ≤ ⟨U, S⟩ ≤ α\|S\|^2 + nδ₁.$$

We get (b) from noting that $P' = P_U − α^2σ₂^2$. Similarly, it can be shown that

$$\|U − αS\|^2 ≤ n(P' + δ).$$

We now ensure that $δ₀$ and $δ₁$ are chosen small enough such that

$$δ₀ + 2δ₁ < δ₂.$$  \hspace{1cm} (56)

As $α \leq 1$, this implies that $δ < δ₂$. Hence,

$$\mathbb{P}(\|U − αS\|^2 > nδ₂|M = m) \leq \mathbb{P}(E|M = m) \to 0$$

as $n \to ∞$. This completes the proof.

C. Proof of Lemma 18

Let $M = m$ be the message and let $U$ denote the chosen codeword. We resolve the components of $J$ and $U$ along directions parallel and orthogonal to $S$. We denote the latter components as $J^\perp$ and $U^\perp$ respectively.

$$J = ⟨J, S⟩ \hat{S} + J^\perp$$

$$U = ⟨U, S⟩ \hat{S} + U^\perp.$$

Note that $⟨J^\perp, \hat{S}⟩ = 0 = ⟨U^\perp, \hat{S}⟩$, and thus,

$$⟨J, U⟩ = ⟨J, \hat{S}⟩ \hat{S}, U⟩ + ⟨J^\perp, U^\perp⟩.$$ To prove this lemma, we need to show that for any $δ₃ > 0$,

$$\mathbb{P}(⟨J^\perp, U^\perp⟩ > nδ₃|M = m) \to 0$$

as $n \to ∞$, i.e., $J^\perp$ and $U^\perp$ are nearly orthogonal for large enough $n$.

To proceed, we introduce some notation. Let $S^n(0, r) = \{w \in \mathbb{R}^n : \|w\| = r\}$ be the surface of an $n$-sphere centered at the origin and with radius $r$. For any $w \in \mathbb{R}^n$, let $C^\perp(w)$ denote the $(n-1)$ subspace orthogonal to $w$. We now make the following claim.

Claim 25. Conditioned on $M = m$, $S = s$ and $⟨U, S⟩ = z$, the random vector $U$ is uniformly distributed over

$$\mathcal{B}_z(s) = \left\{z − \frac{s}{\|s\|^2} + v : v \in S^n(0, ρ_z(s)) \cap C^\perp(s) \right\},$$

where

$$ρ_z(s) = \sqrt{nP_U − \frac{σ^2}{\|s\|^2}}. \hspace{1cm} (58)$$

Proof: Given the symmetry of the codebook generation and the encoding, we know that the codeword vector $U$ is uniformly distributed over the set $S^n(0, \sqrt{nP_U})$. Now conditioned on message $M = m$, state $S = s$ and $⟨U, S⟩ = z$, it follows that the codeword vector $U$ is uniformly distributed over the set $\mathcal{B}_z(s)$.

i) To show $u \in \mathcal{B}_z(s) \Rightarrow u \in \mathcal{B}_z(s)$.

Let $u \in \mathcal{B}_z(s)$. Expressing $u$ through its two components, one in the direction parallel to $s$ and the other orthogonal to it, we get

$$u = ⟨u, s⟩ \frac{s}{\|s\|^2} + u^\perp.$$  \hspace{1cm}

Note here that $⟨u^\perp, s⟩ = 0$ and

$$\|u^\perp\| = \sqrt{nP_U − \frac{σ^2}{\|s\|^2}}.$$
Comparison with (57) completes the proof for the forward part.

ii) To show $u \in B_z(s) \Rightarrow u \in \tilde{B}_z(s)$.

Consider some vector $u \in B_z(s)$. Using (59), we can write

$$u = z \frac{s}{\|s\|^2} + v,$$

where

$$v \in S^n(0, \rho_z(s)) \bigcap C^\perp(s)$$

and $\rho_z(s)$ is as given in (58). It can be easily verified that $\|u\| = \sqrt{nP_U}$. Also, $\langle v, s \rangle = 0$, and hence, it can be immediately seen that $\langle u, s \rangle = z$. Thus, $u \in \tilde{B}_z(s)$.

This completes the proof of the claim.

The following claim, which is equivalent to the lemma as discussed earlier, completes the proof.

**Claim 26.** For any $\delta_3 > 0$,

$$P \left( \left| \langle \mathbf{j}^\perp, \mathbf{u}^\perp \rangle \right| > n\delta_3 \mid M = m \right) \to 0,$$

as $n \to \infty$.

**Proof:** We first prove the conditional version of this claim, where we condition on state $S = s$ and $\langle U, s \rangle = z$. From Claim 25 we know that

$$U = z \frac{s}{\|s\|^2} + V,$$

where

$$V \sim \text{Unif} \left( S^n(0, \sqrt{\rho_z(s)}) \bigcap C^\perp(s) \right)$$

with $\rho_z(s)$ as given in (58). Now for $\delta_3 > 0$, we have

$$P \left( \left| \frac{\langle \mathbf{j}^\perp, \mathbf{u}^\perp \rangle}{\sqrt{n}} \right| > \delta_3 \mid m = s, \langle U, s \rangle = z \right)$$

$$= P \left( \frac{1}{\sqrt{n}} \left| \frac{\langle \mathbf{j}^\perp, \mathbf{V} \rangle}{\|\mathbf{V}\|} \right| > \frac{\delta_3}{\sqrt{n}} \mid m = s, z \right)$$

$$\leq P \left( \frac{1}{\sqrt{n}} \left| \frac{\langle \mathbf{j}^\perp, \mathbf{V} \rangle}{\|\mathbf{V}\|} \right| > \frac{\delta_3}{\sqrt{n}} \mid m = s, z \right)$$

$$= P \left( \left| \langle \mathbf{j}^\perp, \mathbf{V} \rangle \right| > \frac{\delta_3}{\sqrt{n}} \mid m = s, z \right).$$

Here $(a)$ follows from noting that $\|\mathbf{j}^\perp\| \leq \|\mathbf{j}\| \leq \sqrt{n} \Lambda$ and $\|\mathbf{V}\| \leq \sqrt{n} P_U$.

Since the shared randomness $\Theta$ is unavailable to the adversary, conditioned on $M = m$, $S = s$ and $Z = z$, it follows that $\mathbf{j}^\perp$ and $\mathbf{V}$ are independent. Also, both $\mathbf{j}^\perp$ and $\mathbf{V}$ lie in the $(n-1)$ hyperplane orthogonal to $s$. Now using Lemma 15 with $\delta_3 = \delta_3 \sqrt{\Lambda} P_U > 0$, we have

$$P \left( \left| \langle \mathbf{j}^\perp, \mathbf{V} \rangle \rangle > \delta_3 \mid m = s, z \right)$$

$$\leq 2 \left( \frac{2(n-1)^{1/2} \log(1-\delta_3^2)}{\delta_3} \right) \forall m, s, z,$n

$$= 2 \left( \frac{2n-1}{2(n-1)^{1/2} \log(1-\delta_3^2)} \right) \forall m, s, z,$n

(60)

where

$$f(\delta_3) = \frac{1}{2} \log \left( \frac{1}{1-\delta_3^2} \right)$$

$$= \frac{1}{2} \log \left( \frac{P_U \Lambda}{P_U \Lambda - \delta_3^2} \right) > 0.$$

Since the upper bound in (60) tends to zero as $n \to \infty$, the conditional version of the claim follows. However, note that the bound in (60) does not depend on $m$, $s$ or $z$. Hence, the unconditioned version is also true, and the claim follows.

**D. Proof of Lemma 19**

Let $M = m$ be the message and let $U$ denote the chosen codeword. We know that

$$\langle Y, U \rangle = \langle U + (1-\alpha)S + J + Z, U \rangle$$

$$= \|U\|^2 + (1-\alpha) \langle S, U \rangle + \langle J, U \rangle + \langle Z, U \rangle$$

and

$$\|Y\|^2 = \langle U + (1-\alpha)S + J + Z, U + (1-\alpha)S + J + Z \rangle$$

$$= \|U\|^2 + (1-\alpha)^2 \|S\|^2 + \|J\|^2 + \|Z\|^2$$

$$+ 2(\langle U, Z \rangle + \langle J, Z \rangle + \langle J, U \rangle)$$

$$+ 2(1-\alpha)(\langle U, S \rangle + \langle J, S \rangle + \langle S, Z \rangle).$$

Let us define the following events:

$$E_0 = \{ \|S\|^2 - n\alpha^2 > n\delta_0 \},$$

$$E_1 = \{ \delta k : | \langle U_m,k - \alpha S, S \rangle | < n\delta_1 \},$$

$$E_2 = \{ \|U - \alpha S\|^2 - nP_U > n\delta_2 \},$$

$$E_3 = \{ \|J, U - \langle J, U \rangle \tilde{S}, \tilde{U} \rangle \| > n\delta_3 \},$$

$$E_4 = \{ \|U, Z \rangle | > n\delta_4 \},$$

$$E_5 = \{ \|S, Z \rangle | > n\delta_5 \},$$

$$E_6 = \{ \|J, Z \rangle | > n\delta_6 \},$$

$$E_7 = \{ \|Z\|^2 - n\alpha^2 > n\delta_7 \}.$$
\[ W = \frac{1}{n} \| \mathbf{J} \|^2. \]

Since \( \left| \langle \mathbf{j}, \mathbf{S} \rangle \right| \leq 1 \), we have \( V^2 \leq 1 \). It follows from \( \| \mathbf{J} \|^2 \leq n \Lambda \), that \( 0 \leq W \leq \Lambda \). Note that \( \mathbb{P}(E|M = m) \) approaches 0 for large enough \( n \) for \( \delta_i, i = 0, 1, \ldots, 7 \) as given above.

Recall that the codewords are those over the surface of an \( n \)-sphere of radius \( \sqrt{n \Lambda} \). Thus, from (61) and (62) as well as conditioned on the event \( E^c \),

\[
\langle \mathbf{Y}, \mathbf{U} \rangle \geq n \left( P_U + (1 - \alpha) \alpha \sigma_S^2 + V \alpha \sqrt{W \sigma_S^2 - \delta_a} \right), \quad (63)
\]

and

\[
\langle \mathbf{Y}, \mathbf{Y} \rangle \leq n \left( P_U + (1 - \alpha)^2 \sigma_S^2 + W + \alpha^2 + 2(1 - \alpha) \sigma_S^2 + 2V \alpha \sqrt{W \sigma_S^2 + 2(1 - \alpha)V \sqrt{W \sigma_S^2 + \delta_b}} \right). \quad (64)
\]

We know that

\[
\langle \hat{\mathbf{Y}}, \hat{\mathbf{U}} \rangle = \frac{\langle \mathbf{Y}, \mathbf{U} \rangle}{\langle \mathbf{Y}, \mathbf{Y} \rangle} \quad (65)
\]

Now substituting for \( \langle \mathbf{Y}, \mathbf{U} \rangle \) from (63) and \( \langle \mathbf{Y}, \mathbf{Y} \rangle \) from (64) in (65), and noting that \( P_U = P' + \alpha \sigma_S^2 \) and \( \alpha = P'/\left(P' + \Lambda + \sigma^2 \right) \), we get (66) (given on top of the next page), where \( \delta_a, \delta_b > 0 \) and \( \delta_a, \delta_b \to 0 \) as \( \delta_i \to 0, i = 0, 1, \ldots, 6 \). Hence, conditioned on \( E^c \), we have

\[
\langle \hat{\mathbf{Y}}, \hat{\mathbf{U}} \rangle \geq \frac{\sqrt{\alpha \left( P' + \alpha \sigma_S^2 + V \alpha \sqrt{W \sigma_S^2} \right)}}{P_U \left( P' + \alpha \sigma_S^2 + \alpha(W - \Lambda) + 2V \alpha \sqrt{W \sigma_S^2} \right)} - \tilde{\delta}, \quad (67)
\]

where \( \tilde{\delta} > 0 \) and \( \tilde{\delta} \to 0 \) as \( \delta_a, \delta_b \to 0 \). It can be verified that there exists a choice of \( \delta_i, i = 0, 1, \ldots, 7 \), as functions of \( \delta \), where \( \forall i, \delta_i \) approaches 0 as \( \delta \to 0 \), such that, firstly, \( \delta_0 \), \( \delta_1 \), and \( \delta_2 \) are such that \( \delta_2 < \epsilon_1 \) and they satisfy (66) as required in the proof of Lemma 17 earlier, and secondly, \( \delta \), which depends on \( \delta_i, \forall i \), is such that \( \delta < \delta \). Making this choice, conditioned on \( E^c \), it follows from (67) that

\[
\langle \hat{\mathbf{Y}}, \hat{\mathbf{U}} \rangle \geq \frac{\sqrt{\alpha \left( P' + \alpha \sigma_S^2 + V \alpha \sqrt{W \sigma_S^2} \right)}}{P_U \left( P' + \alpha \sigma_S^2 + \alpha(W - \Lambda) + 2V \alpha \sqrt{W \sigma_S^2} \right)} - \delta. \quad (68)
\]

We now make the following claim. The proof of this claim is discussed later.

Claim 27. If

\[
f(v, w) = \frac{\sqrt{\alpha \left( P' + \alpha \sigma_S^2 + v \alpha \sqrt{w \sigma_S^2} \right)}}{P_U \left( P' + \alpha \sigma_S^2 + \alpha(v - \Lambda) + 2v \alpha \sqrt{w \sigma_S^2} \right)} \quad (69)
\]

then for all \(-1 \leq v \leq 1 \) and \( 0 \leq w \leq \Lambda \),

\[
f(v, w) \geq \theta,
\]

where

\[
\theta = f(0, \Lambda)
\]

Using the above claim in (68), conditioned on \( E^c \), it follows that

\[
\langle \hat{\mathbf{Y}}, \hat{\mathbf{U}} \rangle \geq \theta - \delta.
\]

Thus, we can conclude that

\[
\mathbb{P}\left( \langle \hat{\mathbf{Y}}, \hat{\mathbf{U}} \rangle < \theta - \delta | M = m \right) \leq \mathbb{P}(E|M = m) \to 0
\]

as \( n \to \infty \). It only remains to prove Claim 27 above.

Proof of Claim 27. We show that for \(-1 \leq v \leq 1 \) and \( 0 \leq w \leq \Lambda \),

\[
f(v, w) \geq f(0, \Lambda). \quad (70)
\]

Let us first establish the simple fact that \( f(v, w) \geq 0 \). Consider the numerator term in (69),

\[
P' + \alpha \sigma_S^2 + v \alpha \sqrt{w \sigma_S^2}
\]

\[
= P' + \alpha \left( \sigma_S^2 + v \sqrt{w \sigma_S^2} \right)
\]

\[
= \frac{P'}{P' + \Lambda + \sigma^2} \left( P' + \Lambda + \sigma^2 + \sigma_S^2 + v \sqrt{w \sigma_S^2} \right)
\]

\[
= \frac{P'}{P' + \Lambda + \sigma^2} \left( P' + \Lambda + \sigma^2 + \sigma_S^2 + v \sqrt{w \sigma_S^2} \right)
\]

\[
\geq 0.
\]

Here (a) follows by substituting \( \alpha = P'/(P' + \Lambda + \sigma^2) \). Then, (b) follows since \( v \geq -1 \), while (c) follows from \( w \leq \Lambda \). Hence, we conclude that the numerator of (69) is non-negative, and \( f(v, w) \geq 0 \).

As \( f(v, \Lambda) \geq 0 \) for \(-1 \leq v \leq 1 \) and \( 0 \leq w \leq \Lambda \), to show (70), it is sufficient to prove

\[
(f(v, w))^2 \geq (f(0, \Lambda))^2, \quad (71)
\]

for \(-1 \leq v \leq 1 \) and \( 0 \leq w \leq \Lambda \). Hence, using (69) in (71), we want to show that

\[
\left( \frac{P' + \alpha \sigma_S^2 + v \alpha \sqrt{w \sigma_S^2}}{P_U \left( P' + \alpha \sigma_S^2 + \alpha(v - \Lambda) + 2v \alpha \sqrt{w \sigma_S^2} \right)} \right)^2
\]

\[
\geq \left( \frac{\sqrt{\alpha \left( P' + \alpha \sigma_S^2 + v \alpha \sqrt{w \sigma_S^2} \right)}}{P_U \left( P' + \alpha \sigma_S^2 + \alpha(v - \Lambda) + 2v \alpha \sqrt{w \sigma_S^2} \right)} \right)^2
\]

\[
\geq \left( \frac{\sqrt{\alpha \left( P' + \alpha \sigma_S^2 + v \alpha \sqrt{w \sigma_S^2} \right)}}{P_U \left( P' + \alpha \sigma_S^2 + \alpha(v - \Lambda) + 2v \alpha \sqrt{w \sigma_S^2} \right)} \right)^2
\]
\( (\hat{Y}, \hat{U}) \geq \sqrt[4]{\frac{(P_U + (1-\alpha)\alpha\sigma_S^2 + V\alpha\sqrt{W\sigma_S^2} - \delta_0)}{P_U (P_U + (1-\alpha)^2\sigma_S^2 + W + \sigma^2 + 2(1-\alpha)\alpha\sigma_S^2 + 2V\sqrt{W\sigma_S^2} + \delta_0)}} \)

\[
\geq \left( \frac{P' + \alpha\sigma_S^2 + \sqrt{\alpha}(P' + \alpha\sigma_S^2 + V\alpha\sqrt{W\sigma_S^2} - \delta_0)}{\sqrt{P_U'(P_U' + \alpha\sigma_S^2 + \alpha(W - \Lambda) + 2V\sqrt{W\sigma_S^2} + \alpha\delta_0)}} \right)^2
\]

Since \( w \leq \Lambda \), the RHS above is negative. However, \(-1 \leq v \leq 1\), and hence, \( v^2 \geq 0 \). Thus, (71) immediately follows and we conclude that \( f(v, w) \geq f(0, \Lambda) \), for \(-1 \leq v \leq 1\) and \( w, \Lambda \). This concludes the proof of the claim.

This completes the proof of Lemma [19].

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