Clifford Algebras, Multipartite Systems and Gauge Gravity Theory

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ABSTRACT: In this paper we present a multipartite formulation of gauge gravity theory based on the formalism of space-time algebra for gravitation developed by Lasenby and Doran (Lasenby, A. N., Doran, C. J. L, and Gull, S.F.: Gravity, gauge theories and geometric algebra. Phil. Trans. R. Soc. Lond. A, 582, 356:487 (1998)). We associate the gauge fields with description of fermionic and bosonic states using the generalized graded tensor product. Einstein’s equations are deduced from the graded projections and an algebraic Hopf-like structure naturally emerges from formalism. A connection with the theory of the quantum information is performed through the minimal left ideals and entangled qubits are derived. In addition applications to black holes physics and standard model are outlined.
1 Introduction

This work is dedicated to the memory of professor Waldyr Alves Rodrigues Jr., whose contributions in the field of mathematical physics were of great prominence, especially in the study of clifford algebras and their applications to physics [1].

The gauge gravity theory [2] is a gauge theory on gravitation based on geometric algebra formalism. Unlike the general relativity, it is formulated in a flat background spacetime with two basic principles related position-gauge invariance and rotation-gauge invariance so that gauge fields determine that relations between physical quantities are independent of the positions and orientations of the matter fields [2]. This formulation presents some advantages in relation to general relativity. Interesting connections with quantum theory are extensively explored through the ubiquitous character of clifford algebras although the issue of quantum gravitation is not explicitly discussed.

Motivated by this theory we propose an extension of these ideas in a multipartite context on which we believe to help in possible connections with the quantum gravity ([3], p. 175). In spite of the numerous proposals, the quantization of gravitational field remains one of major challenges of theoretical physics. The most promising proposals such as string theory [4] and loop quantum gravity [5] lack verifiable experimental predictions with the current technological apparatus. The verification of the existence of extra dimensions and elementary quantum granular structure of space time at the Planck scale are some examples. Recently, a new approach builds a bridge between quantum gravity and quantum information theory through the correspondence AdS / CFT (or gauge/gravity duality) [6–8]. In this context, Einstein equations are derived from entanglement constraints for small perturbations to AdS. In our proposal, we obtained the Einstein equations in a multipartite context following the prescription of gauge gravity theory and considering graded projection operators [9, 10] involving tensor products. For example, given $\nu$ multivectors $\Gamma_{\mu_1}, \Gamma_{\mu_2}, ..., \Gamma_{\mu_{\nu}}$, we define $\langle \sum_{\mu_1, \mu_2, ..., \mu_n} \Gamma_{\mu_1} \otimes \Gamma_{\mu_2} ... \otimes \Gamma_{\mu_n} \rangle_{1-r_2-...-r_s}$ as a sum in which the...
terms to be considered are only those of degree \( r_1 - r_2 - \ldots - r_s \) without taking into account the order in the tensorial product.

From another point of view, but still related to ideas of quantum information, an emerging gravity model \[11, 12\] has been proposed in which gravity is not a fundamental interaction; it is an emergent phenomenon, an entropic force arising from the statistical behavior of microscopic degrees of freedom in a holographic scenario. It is postulated that each region of space is associated with a tensor factor of the microscopic Hilbert space and the associated entanglement entropy satisfies an area law in according to the Bekenstein-Hawking formula \[12\]. In this perspective it is important to highlight the seminal proposal of Sakharov, called induced gravity - a model of free non-interacting fields where the magnitude of the gravitational interaction is dictated by the masses and equations of motion for the free particles \[13\].

In this work the Clifford algebras and associated Hopf algebras \[14–18\] constitute the underlying basic structures. An interesting approach to quantum gravity involving Hopf algebras was performed by Majid \[14, 16\]. This formulation based on Hopf algebra contains a symmetry between states and observables with ideas of non-commutative geometry \[14\]. Majid argues that both quantum theory and non-Euclidean geometry are needed for a self-dual picture which can be guaranteed with a bicrossproduct Hopf algebra structure. Compatibility conditions involving bicrossproduct structure result in second-order gravitational field equations and the solutions of these equations corresponds algebra of observables of a quantisation of a particle moving on a homogeneous spacetime as highlighted in \[14\]. Still in this reference toy models are proposed and it is emphasized the need of generalization of the notion of Hopf algebras in order to obtain more realistic models. A recent proposal of space-time quantization in which Clifford algebras and quantum groups (quasitriangular Hopf algebras) play a key role was given by \[19\]. There are several interesting papers that explore the relationship between Clifford and Hopf algebras \[20–25\]. Particularly, relations between Clifford-Hopf algebras and quantum deformations of the Poincaré algebra were derived by Rocha, Bernardini and Vaz \[20\]. In our formulation the relationship between Clifford and Hopf algebras arises from the need to use the primitive element \( \Gamma_\mu \hat{\otimes} 1 \) and the bicrossproduct \( 1 \hat{\otimes} \Gamma_\mu \) in order to guarantee the consistency of the commutation and anti-commutation relations. Here, we present the Hopf algebra associated to Clifford algebra through the generators of algebra and generalized graded tensor product, which is essential for obtaining the compatible co-algebra structure.

This paper is organized as follows: in section 2 we present the gauge gravity theory \[2\] with typical structures of Clifford and Hopf algebras in an multipartite gravity model resulting in einstein-like equations. Fields associated to bosonic and fermionic particles are deduced. Section 3 contains a connection with qubits and a deduction of entangled states in this formalism. In section 4 the Dirac equation is derived in the black holes background with the Schwarzschild solution following the reference \[26\]. In section 5 our formalism is applied to clifford algebra \( Cl(0,6) \) describing to the \( SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_B-R \) local gauge symmetries based on reference \[29\]. In the last section we present our conclusions and perspectives.
2 General formulation and Einstein field equations

In the gravity gauge theory, the basic structure is space-time algebra $Cl(1, 3)$ [2]:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}, \quad (2.1)$$

where $\eta_{\mu\nu}$ is the Minkowski with signature $(+,−−)$. Two gauge fields are introduced [2]. The first is a vector field $g^\mu(x)$ with transformation law:

$$g^\mu(x) \mapsto g^\mu(x)' = Rg^\mu(x)\tilde{R}, \quad (2.2)$$

where $R$ is a constant rotor. The second is the bivector-valued field whose law of transformation is given by

$$\Omega_\mu \mapsto \Omega'_\mu = R\Omega_\mu \tilde{R} - 2\partial_\mu R\tilde{R}, \quad (2.3)$$

The covariant derivative is defined by:

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{2} \Omega_\mu \psi, \quad (2.4)$$

The field gravitational equations are obtained from field strength through relation:

$$[D_\mu, D_\nu] \psi = \frac{1}{2} R_{\mu\nu} \psi, \quad (2.5)$$

where

$$R_{\mu\nu} = \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + \Omega_\mu \times \Omega_\nu \quad (2.6)$$

An additional field equation is given by:

$$D_\mu g_\nu - D_\nu g_\mu = 0, \quad (2.7)$$

One of the basic assumptions is that a quantum description of multipartite systems requires the structure of a tensor product associated with algebras analogously to the usual formulation of tensor products of Hilbert spaces in standard quantum mechanics. In order to implement such a structure preserving commutators and anticommutators we need to take into account primitive elements and the graded tensor product [16]. The start point is the generalized field strength:

$$[D_\mu \hat{\otimes} 1 \ldots \hat{\otimes} 1 + \ldots + 1 \hat{\otimes} 1 \ldots \hat{\otimes} D_\mu, D_\nu \hat{\otimes} 1 \ldots \hat{\otimes} 1 + \ldots + 1 \hat{\otimes} 1 \ldots \hat{\otimes} D_\nu] \Psi \quad (2.8)$$

where the $\hat{\otimes}$ is the generalized graded tensor product is given by

$$(a_1 \hat{\otimes} a_2 \hat{\otimes} a_3 \ldots \hat{\otimes} a_n)(b_1 \hat{\otimes} b_2 \hat{\otimes} b_3 \ldots \hat{\otimes} b_n) = (-1)^{\deg(a_2)\deg(b_1)\deg(a_3)\deg(b_2)\ldots\deg(a_n)\deg(b_{n-1})}(a_1 b_1 \otimes a_2 b_2 \otimes \ldots \otimes a_n b_n). \quad (2.9)$$
In compact form

\[ [\hat{D}^{(n)}_\mu, \hat{D}^{(n)}_\nu] \Psi = \frac{1}{2} \hat{R}^{(n)}_{\mu\nu} \Psi \]  

(2.10)

with the generalized covariant derivative

\[
\hat{D}^{(n)}_\mu = D_\mu \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes D_\mu
\]

\[
= (\partial_\mu + \Omega_{\mu} \times) \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes (\partial_\mu + \Omega_{\mu} \times)
\]

and

\[
\hat{R}^{(n)}_{\mu\nu} = (\partial_\mu \Omega_{\nu} - \partial_\nu \Omega_{\mu} + \Omega_{\mu} \times \Omega_{\nu}) \otimes 1 \otimes \ldots \otimes 1
\]

\[
+ \ldots + 1 \otimes 1 \otimes \ldots (\partial_\mu \Omega_{\nu} - \partial_\nu \Omega_{\mu} + \Omega_{\mu} \times \Omega_{\nu})
\]

where there are \( n \) terms. The Ricci tensor is given by

\[
\hat{R}^{(n)}_\nu = (g^{(n)}_{\nu\sigma} \hat{R}^{(n)}_{\mu\nu})_{v-s}
\]

(2.11)

where \( v - s \) is vector-scalar projection. Elements that have only these factors should be considered. Consequently

\[
\hat{R}^{(n)}_\nu = \langle (g^{\mu} \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes g^{\mu})(R_{\mu\nu} \otimes 1 \otimes \ldots \otimes 1
\]

\[
+ \ldots + 1 \otimes 1 \otimes \ldots \otimes R_{\mu\nu}) \rangle_{v-s}
\]

\[
= \langle g^{\mu} R_{\mu\nu} \otimes 1 \otimes \ldots \otimes 1 + g^{\mu} \otimes R_{\mu\nu} \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes g^{\mu} R_{\mu\nu} \rangle_{v-s}
\]

\[
= g^{\mu} R_{\mu\nu} \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes g^{\mu} R_{\mu\nu}
\]

\[
= R_{\nu} \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes R_{\nu}
\]

(2.12)

It is verify that law of transformation for \( \hat{R}^{(n)}_{\mu\nu} \) is given by:

\[
\hat{R}'^{(n)}_{\mu\nu} = \hat{R}^{(n)}_{\mu\nu} \hat{R}^{(n)} R^{(n)}
\]

(2.13)

where

\[
R^{(n)} = R \otimes R \otimes \ldots \otimes R
\]

(2.14)

and

\[
\hat{R}^{(n)} = \hat{R} \otimes \hat{R} \otimes \ldots \otimes \hat{R}
\]

(2.15)

The Ricci scalar can obtained as:

\[
\hat{R}^{(n)} = \langle g^{(n)}_{\nu} \hat{R}^{(n)}_\nu \rangle_{s-s}
\]

(2.16)

with

\[
\hat{R}^{(n)} = \langle (g^{\nu} \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes 1 \otimes \ldots \otimes g^{\nu})
\]

\[
\cdot (g^{\mu} R_{\mu\nu} \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes g^{\mu} R_{\mu\nu}) \rangle_{s-s}
\]

\[
= g^{\mu} R_{\mu\nu} \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes g^{\mu} R_{\mu\nu}
\]

\[
= R \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes R
\]

(2.17)
Therefore the Einstein equations are given by

\[
\begin{align*}
(R_\mu \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes R_\mu - \frac{1}{2} (R \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes R) \\
\cdot (g_\mu \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes g_\mu))_{v-s} = 8\pi G \langle (T_\mu \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots T_\mu) \rangle_{v-s}
\end{align*}
\]

or

\[
\frac{R^{(n)}_\mu}{2} - \frac{1}{2} R^{(n)} g^{(n)}_\mu = 8\pi \Omega_{\mu}
\]

The Einstein equation appears as an emergent equation of a multipartite scenario from multivector fields. Explicitly, in this formulation we can derive fields associated to bosonic states related to bivector-valued field $\Omega_v$ through linear combinations as follows

\[
\sigma_{12\ldots i\ldots j\ldots n}^{(n)} = \sum_{\mu_1, \mu_2} (\gamma_{\mu_1}^{\mu_2} \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes \gamma_{\mu_1}^{\mu_2})
\]

with $\gamma_{\mu_1}^{\mu_2}$, $\gamma_{\mu_2}^{\mu_1}$ $\neq$ $\gamma_{\mu_1}^{\mu_1}$, $\gamma_{\mu_2}^{\mu_2}$ or $\gamma_{\mu_1}^{\mu_2} = \gamma_{\mu_1}^{\mu_2}$ and $\gamma_{\mu_2}^{\mu_2} = \gamma_{\mu_2}^{\mu_2}$. Alternatively:

\[
\sigma_{12\ldots i\ldots j\ldots n}^{(n)} = \sum_{\mu_1, \mu_2} (\gamma_{\mu_1}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_1}^{1})
\]

\[
\cdot (\gamma_{\mu_2}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_2}^{1})
\]

\[
\cdot (\gamma_{\mu_1}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_1}^{1})
\]

\[
\cdot (\gamma_{\mu_2}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_2}^{1})
\]

\[
\cdot (\gamma_{\mu_1}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_1}^{1})
\]

\[
\cdot (\gamma_{\mu_2}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_2}^{1})
\]

\[
\cdot (\gamma_{\mu_1}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_1}^{1})
\]

\[
\cdot (\gamma_{\mu_2}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_2}^{1})
\]

\[
\cdot (\gamma_{\mu_1}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_1}^{1})
\]

\[
\cdot (\gamma_{\mu_2}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_2}^{1})
\]

\[
\cdot (\gamma_{\mu_1}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_1}^{1})
\]

\[
\cdot (\gamma_{\mu_2}^{1} \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \ldots \otimes \gamma_{\mu_2}^{1})
\]
with $\gamma^k_{\mu_1}, \gamma^k_{\mu_2} \neq \gamma^l_{\mu_1}, \gamma^l_{\mu_2}$ or $\gamma^k_{\mu_1} = \gamma^l_{\mu_1}$ and $\gamma^k_{\mu_2} = \gamma^l_{\mu_2}$. In order to ensure anticommutativity the number of factors must be even in the tensor product so that $n = 2(2m + 1)$, $m = 0, 1, 2, \ldots$. We define the subscript $(2m + 1)$ as $1_{(2m+1)} \equiv 1 \otimes 1 \otimes \ldots 1 \otimes 1$, $2m + 1$ times, i.e., $2m + 1$ factors. Analogously, $a_i \otimes a_i \otimes \ldots \otimes 1_{(2m+1)}$ represent tensor product of $a_i$, $2m + 1$ times, followed by the tensor product of $1, 2m + 1$ times, while $1_{(2m+1)} \otimes a_i \otimes \ldots \otimes a_i$ represent tensor product of $1, 2m + 1$ times, followed by the tensor product of $a_i, 2m + 1$ times. It is easy verify that

$$
\sigma_{12\ldots j\ldots n}^{(n)} = \sum_{\mu_1, \mu_2} (\gamma^1_{\mu_1} \gamma^1_{\mu_2} \otimes 1 \otimes \ldots 1 + \ldots + 1 \otimes 1 \otimes \ldots \gamma^1_{\mu_1} \gamma^1_{\mu_2})
\cdot (\gamma^2_{\mu_1} \gamma^2_{\mu_2} \otimes 1 \otimes \ldots 1 + \ldots + 1 \otimes 1 \otimes \ldots \gamma^2_{\mu_1} \gamma^2_{\mu_2})
\ldots
\cdot (\gamma^m_{\mu_1} \gamma^m_{\mu_2} \otimes 1 \otimes \ldots 1 + \ldots + 1 \otimes 1 \otimes \ldots \gamma^m_{\mu_1} \gamma^m_{\mu_2})
\ldots
\cdot (\gamma^n_{\mu_1} \gamma^n_{\mu_2} \otimes 1 \otimes \ldots 1 + \ldots + 1 \otimes 1 \otimes \ldots \gamma^n_{\mu_1} \gamma^n_{\mu_2})
= \sigma_{12\ldots j\ldots n}^{(n)}
(2.21)
$$

For the fields associated to fermionic states related to vector field $g^\mu(x)$ through linear combinations:

$$
g_{12\ldots j\ldots n}^{(n)} = \sum_{\mu} (\gamma^1_{\mu} \otimes 1 \otimes \ldots 1_{(2m+1)} + \ldots + 1 \otimes 1 \otimes \ldots \gamma^1_{\mu})
\cdot (\gamma^2_{\mu} \otimes 1 \otimes \ldots 1_{(2m+1)} + \ldots + 1 \otimes 1 \otimes \ldots \gamma^2_{\mu})
\ldots
\cdot (\gamma^m_{\mu} \otimes 1 \otimes \ldots 1_{(2m+1)} + \ldots + 1 \otimes 1 \otimes \ldots \gamma^m_{\mu})
\ldots
\cdot (\gamma^n_{\mu} \otimes 1 \otimes \ldots 1_{(2m+1)} + \ldots + 1 \otimes 1 \otimes \ldots \gamma^n_{\mu})
= g_{12\ldots j\ldots n}^{(n)}
(2.22)
$$

In order to ensure anti-commutativity the number of factors must be even in the tensor product. Analogously to bosonic case, it is easy verify that
The second field equation is given by:

\[
\langle \left( D_{\mu} \hat{\otimes} 1 \right) \left( g_{\nu} \hat{\otimes} 1 \right) - \left( D_{\nu} \hat{\otimes} 1 \right) \left( g_{\mu} \hat{\otimes} 1 \right) \rangle_{v-s} = 0
\]

where \( \langle \rangle_{v-s} \) is the vector-scalar projection. In a compact form:

\[
\langle D^{(n)}_{\mu} g^{(n)}_{\nu} \rangle_{v-s} = 0
\]  

We have an emergent Hopf algebra \((H, +, \cdot, \eta, \Delta, \epsilon, S, k)\) over \( k \) \([16, 17, 22]\) of this formalism. The coproduct is defined by

\[
\Delta(\gamma_i) = \frac{1}{\sqrt{2}} (\gamma_i \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_i)
\]

and

\[
\Delta(1) = 1 \hat{\otimes} 1
\]

so that

\[
\Delta(\{\gamma_i, \gamma_j\}) = \Delta(\gamma_i)\Delta(\gamma_j) + \Delta(\gamma_j)\Delta(\gamma_i) = \Delta(2\eta_{ij}) = 2\eta_{ij}(1 \hat{\otimes} 1)
\]

We also have the definitions:

\[
\epsilon(\gamma_i) = 0, \quad \epsilon(1) = 1;
\]

\[
S(\gamma_i) = -\gamma_i, \quad S(1) = 1;
\]

\[
\eta(1) = 1, \quad \eta(0) = 0
\]
1) Coalgebra:

\[(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta\]

\[(\epsilon \otimes \text{id}) \circ \Delta(c) = c = (\text{id} \otimes \epsilon) \circ \Delta(c),\]  

(2.29)

with \(c \in H\).

2) Bialgebra:

\[\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1) = 1 \otimes 1\]

\[\epsilon(gh) = \epsilon(g)\epsilon(h); \quad \epsilon(1) = 1\]  

(2.30)

3) Hopf algebra (together with 1 and 2):

\[(S \otimes \text{id}) \circ \Delta = (\text{id} \otimes S) \circ \Delta = \eta \circ \epsilon\]  

(2.31)

The elements of bosonic field can be obtained as \(\Delta(\gamma_i\gamma_j)\). Multipartite systems can be described making use of tensor product. This procedure can be generalized so that

\[\Delta^{(n)}(\gamma_i^{(2m+1)}) = \frac{1}{\sqrt{2}}(\gamma_i \hat{\otimes} \gamma_i \hat{\otimes} ... \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \gamma_i \hat{\otimes} ... \hat{\otimes} \gamma_i)\]

\[\Delta^{(n)}(1_{(2m+1)}) = 1 \hat{\otimes} 1 \hat{\otimes} ... \hat{\otimes} 1 \equiv 1_{2(2m+1)}\]  

(2.32)

with an even number of factors in the tensor product, where \(\gamma_i^{(2m+1)} \equiv \gamma_i \hat{\otimes} \gamma_i \hat{\otimes} ... \hat{\otimes} \gamma_i (2m+1)\) times. We have that

\[\Delta^{(n)}(\{\gamma_i^{(2m+1)}, \gamma_j^{(2m+1)}\}) = \Delta^{(n)}(\gamma_i^{(2m+1)})\Delta^{(n)}(\gamma_j^{(2m+1)})\]

\[+ \Delta^{(n)}(\gamma_j^{(2m+1)})\Delta^{(n)}(\gamma_i^{(2m+1)})\]

\[= \frac{1}{2}(\gamma_i \hat{\otimes} \gamma_i \hat{\otimes} ... \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \gamma_i \hat{\otimes} ... \hat{\otimes} \gamma_i) \cdot (\gamma_j \hat{\otimes} \gamma_j \hat{\otimes} ... \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_j \hat{\otimes} ... \hat{\otimes} \gamma_j)\]

\[+ \frac{1}{2}(\gamma_j \hat{\otimes} \gamma_j \hat{\otimes} ... \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \gamma_j \hat{\otimes} ... \hat{\otimes} \gamma_j) \cdot (\gamma_i \hat{\otimes} \gamma_i \hat{\otimes} ... \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \gamma_i \hat{\otimes} ... \hat{\otimes} \gamma_i)\]

\[= \Delta^{(n)}(2\eta_{ij}1_{(2m+1)})\]

\[= 2\eta_{ij}(1 \hat{\otimes} 1 \hat{\otimes} ... \hat{\otimes} 1)\]

\[= 1_{2(2m+1)}\]

It is important to point out that this realization was possible due to the definition of the generalized graded tensor product (2.9).
3 Qubits

In order to make a connection with quantum information, we consider the bipartite case with the algebra called $CI(2)(1, 3)$:

$$\{\Gamma_\mu, \Gamma_\nu\} = \{\gamma_\mu \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_\mu, \gamma_\nu \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_\nu\} = 2\eta_{\mu\nu}(1 \hat{\otimes} 1) \quad (3.1)$$

An primitive idempotent [30] associated with this algebra is given by

$$\hat{P} = \frac{1}{2} \left[ 1 \hat{\otimes} 1 + \frac{1}{2}(\gamma_3 \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_3)(\gamma_0 \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_0) \right]. \quad (3.2)$$

Let be the isomorphism $CI^+(1, 3) \cong CI(3, 0)$:

$$\sigma_\mu \in CI(3, 0) \leftrightarrow \gamma_\mu \gamma_0 \in CI^+(1, 3) \quad (3.3)$$

In the $CI(3, 0)$ algebra, we have the primitive

$$E = \frac{1}{2}(1 + \sigma_3) \quad (3.4)$$

Qubits can be identified as elements of minimal left ideals in these algebras [26–28]:

$$\begin{align*}
\gamma_3 \gamma_0 P &\leftrightarrow \sigma_3 E \leftrightarrow |0\rangle \\
\gamma_2 \gamma_1 P &\leftrightarrow i\sigma_3 E \leftrightarrow i|0\rangle \\
\gamma_1 \gamma_0 P &\leftrightarrow \sigma_1 E \leftrightarrow |1\rangle \\
\gamma_3 \gamma_2 P &\leftrightarrow i\sigma_1 E \leftrightarrow i|1\rangle
\end{align*}$$

where $\iota = \sigma_1 \sigma_2 \sigma_3$ and $P = \frac{1}{2}(1 + \gamma_3 \gamma_0)$. In the bipartite case, we have:

$$\begin{align*}
T^{(1)} &= (\gamma_3 \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_3)(\gamma_0 \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_0)(\gamma_2 \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_2)(\gamma_1 \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_1)\hat{P} \leftrightarrow |00\rangle \\
T^{(2)} &= (\gamma_1 \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_1)(\gamma_0 \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_0)(\gamma_3 \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_3)(\gamma_2 \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_2)\hat{P} \leftrightarrow |11\rangle
\end{align*}$$

so that

$$\Psi = \frac{1}{\sqrt{2}} \left( T^{(1)} + T^{(2)} \right) \hat{P} \quad (3.5)$$

corresponds to bipartite entangled state:

$$|\Psi\rangle = \frac{(1 + i)}{\sqrt{2}} (|00\rangle + |11\rangle) \quad (3.6)$$

Note that this is a possibility of building the fields so that commutativity is guaranteed. The bosonic field completely characterize the entangled qubits with the homomorphism $CI(2)(1, 3)/\hat{P} \cong CI(2)(1, 3)$ [28]. This procedure always can be generalized resulting in multipartite entangled states.
4 Blackholes background

According to the prescription derived from our formalism, we deduced an equation of Dirac in a multipartite context in a way analogous to the reference [26]. So let us consider the Schwarzschild metric in Cartesian coordinates [2]

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu - \frac{GM}{r}dt^2 - \frac{2}{r} \left( \frac{2GM}{r} \right)^{1/2} b_\mu dt dx^\mu,$$  \hspace{1cm} (4.1)

where $b_\mu = (0, x, y, z)$. In this case, we have the gauge fields [2, 26]

$$\begin{cases}
g_0 = \gamma_0 + \left( \frac{2GM}{r} \right)^{1/2} \frac{x^i}{r} \gamma_i \\
g_i = \gamma_i, \\
\Omega_0 = \frac{GM}{r} \sigma_r \\
\Omega_i = -\frac{1}{2r} \left( \frac{2GM}{r} \right)^{1/2} \left( 2\gamma_i \gamma_0 - 3\gamma_i \gamma_0 \cdot \sigma_r \sigma_r \right),
\end{cases}$$

where $\sigma_r = \frac{1}{r} x^i \gamma_i \gamma_0$ and $i = 1, 2, 3$. In our formulation the Dirac equation is given by

$$\langle (g^\mu \hat{\otimes} 1 \hat{\otimes} ... \hat{\otimes} 1 + ... + 1 \hat{\otimes} 1 \hat{\otimes} ... \hat{\otimes} D_\mu) \hat{\otimes} \Psi(1) \gamma_0 \hat{\otimes} 1 + ... + 1 \hat{\otimes} \Psi(1) \gamma_0 ) \rangle_{v-v} = 0$$

This equation can be rewritten as

$$\sum_\lambda \left[ (g^\mu D_\mu \Psi_\lambda^{(1)} I\sigma_3 \gamma_0 \hat{\otimes} 1 \hat{\otimes} ... \hat{\otimes} 1 + ... + 1 \hat{\otimes} 1 \hat{\otimes} (g^\mu D_\mu \Psi_\lambda^{(n)} I\sigma_3 - m \Psi_\lambda^{(n)} \gamma_0 ) \right] = 0$$

Therefore

$$D\Psi \equiv \sum_\lambda \left[ \left( \nabla \Psi_\lambda^{(1)} - \gamma_0 \left( \frac{2GM}{r} \right)^{1/2} \partial_r \Psi_\lambda^{(1)} - \frac{3}{4r} \left( \frac{2GM}{r} \right)^{1/2} \gamma_0 \Psi_\lambda^{(1)} \right) I\sigma_3 \\
- m \Psi_\lambda^{(1)} \gamma_0 \hat{\otimes} 1 \hat{\otimes} ... \hat{\otimes} 1 + ... + 1 \hat{\otimes} 1 \hat{\otimes} \left[ \nabla \Psi_\lambda^{(n)} \right] \\
- \gamma_0 \left( \frac{2GM}{r} \right)^{1/2} \partial_r \Psi_\lambda^{(n)} - \frac{3}{4r} \left( \frac{2GM}{r} \right)^{1/2} \gamma_0 \Psi_\lambda^{(n)} \right) I\sigma_3 - m \Psi_\lambda^{(n)} \gamma_0 \right] = 0$$  \hspace{1cm} (4.3)

This is Dirac equation in the blackholes background with our multipartite formalism. The existence of normalized bound state solutions associated to Dirac wave functions as well as non-hermiticity of the Hamiltonian associated to black hole singularity are carefully investigated issues in reference [26]. Here, we only intend to illustrate our formulation.
5 Standard model

In this section we will explore the formalism previously developed in the context of the standard model. Our formulation is applied to approach based on the framework of Clifford algebra $Cl(0,6)$ to chiral symmetry breaking and fermion mass hierarchies developed by Wei Lu [29]. The $SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-R}$ local gauge symmetries are characterized by this algebra. Thus, we consider the generators:

$$\Gamma_j \Gamma_k + \Gamma_k \Gamma_j = -2\delta_{jk}, \quad j,k = 1, 2, \ldots 6$$ \hspace{1cm} (5.1)

The generators of $Cl(1,3)$ subalgebra are given by [29]

$$\gamma_0 = \Gamma_1 \Gamma_2 \Gamma_3$$
$$\gamma_1 = \Gamma_4$$
$$\gamma_2 = \Gamma_5$$
$$\gamma_3 = \Gamma_6$$

The unit pseudoscalar is $i = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$. The $SU(3)_c$ strong interaction is given by $G^{(n)}_\mu = G^{k}_\mu T^{(n)}_k$, where the generators:

$$T_1 = \frac{1}{4}(\gamma_1 \Gamma_2 + \gamma_2 \Gamma_1) \hat{\otimes} 1 \ldots \hat{\otimes} 1 + \ldots + 1 \hat{\otimes} 1 \ldots \hat{\otimes} 1 \frac{1}{4}(\gamma_1 \Gamma_2 + \gamma_2 \Gamma_1)$$
$$T_2 = \frac{1}{4}(\Gamma_1 \Gamma_2 + \gamma_1 \gamma_2) \hat{\otimes} 1 \ldots \hat{\otimes} 1 + \ldots + 1 \hat{\otimes} 1 \ldots \hat{\otimes} 1 \frac{1}{4}(\gamma_1 \Gamma_2 + \gamma_1 \gamma_2)$$
$$T_3 = \frac{1}{4}(\Gamma_1 \Gamma_3 - \Gamma_2 \Gamma_2) \hat{\otimes} 1 \ldots \hat{\otimes} 1 + \ldots + 1 \hat{\otimes} 1 \ldots \hat{\otimes} 1 \frac{1}{4}(\Gamma_1 \Gamma_3 - \Gamma_2 \Gamma_2)$$
$$T_4 = \frac{1}{4}(\Gamma_1 \Gamma_3 + \Gamma_3 \Gamma_1) \hat{\otimes} 1 \ldots \hat{\otimes} 1 + \ldots + 1 \hat{\otimes} 1 \ldots \hat{\otimes} 1 \frac{1}{4}(\Gamma_1 \Gamma_3 + \Gamma_3 \Gamma_1)$$
$$T_5 = \frac{1}{4}(\Gamma_1 \Gamma_3 + \Gamma_3 \Gamma_1) \hat{\otimes} 1 \ldots \hat{\otimes} 1 + \ldots + 1 \hat{\otimes} 1 \ldots \hat{\otimes} 1 \frac{1}{4}(\Gamma_1 \Gamma_3 + \Gamma_3 \Gamma_1)$$
$$T_6 = \frac{1}{4}(\Gamma_2 \Gamma_3 + \Gamma_3 \Gamma_2) \hat{\otimes} 1 \ldots \hat{\otimes} 1 + \ldots + 1 \hat{\otimes} 1 \ldots \hat{\otimes} 1 \frac{1}{4}(\gamma_2 \Gamma_3 + \gamma_3 \Gamma_2)$$
$$T_7 = \frac{1}{4}(\Gamma_2 \Gamma_3 + \gamma_2 \gamma_3) \hat{\otimes} 1 \ldots \hat{\otimes} 1 + \ldots + 1 \hat{\otimes} 1 \ldots \hat{\otimes} 1 \frac{1}{4}(\Gamma_2 \Gamma_3 + \gamma_2 \gamma_3)$$
$$T_8 = \frac{1}{4\sqrt{3}}(\Gamma_1 \gamma_2 + \Gamma_2 \gamma_2 - 2\Gamma_3 \gamma_3) \hat{\otimes} 1 \ldots \hat{\otimes} 1$$
$$+ \ldots + 1 \hat{\otimes} 1 \ldots \hat{\otimes} 1 \frac{1}{4}(\Gamma_1 \gamma_2 + \Gamma_2 \gamma_2 - 2\Gamma_3 \gamma_3)$$ \hspace{1cm} (5.2)
or alternatively

\[
T_1 = \frac{1}{4} \left[ \gamma_1 \hat{\otimes} \gamma_1 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \gamma_1 \hat{\otimes} \ldots \hat{\otimes} \gamma_1 \right]
\cdot \left[ \Gamma_2 \hat{\otimes} \Gamma_2 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \Gamma_2 \hat{\otimes} \ldots \hat{\otimes} \Gamma_2 \right]
+ \frac{1}{4} \left[ \gamma_2 \hat{\otimes} \gamma_2 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \gamma_2 \hat{\otimes} \ldots \hat{\otimes} \gamma_2 \right]
\cdot \left[ \Gamma_1 \hat{\otimes} \Gamma_1 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \Gamma_1 \hat{\otimes} \ldots \hat{\otimes} \Gamma_1 \right]
\]

\[
T_2 = \frac{1}{4} \left[ \Gamma_1 \hat{\otimes} \Gamma_1 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \Gamma_1 \hat{\otimes} \ldots \hat{\otimes} \Gamma_1 \right]
\cdot \left[ \Gamma_2 \hat{\otimes} \Gamma_2 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \Gamma_2 \hat{\otimes} \ldots \hat{\otimes} \Gamma_2 \right]
+ \frac{1}{4} \left[ \gamma_1 \hat{\otimes} \gamma_1 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \gamma_1 \hat{\otimes} \ldots \hat{\otimes} \gamma_1 \right]
\cdot \left[ \gamma_2 \hat{\otimes} \gamma_2 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \gamma_2 \hat{\otimes} \ldots \hat{\otimes} \gamma_2 \right]
\]

\[
T_3 = \frac{1}{4} \left[ \Gamma_1 \hat{\otimes} \Gamma_1 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \Gamma_1 \hat{\otimes} \ldots \hat{\otimes} \Gamma_1 \right]
\cdot \left[ \gamma_1 \hat{\otimes} \gamma_1 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \gamma_1 \hat{\otimes} \ldots \hat{\otimes} \gamma_1 \right]
- \frac{1}{4} \left[ \Gamma_2 \hat{\otimes} \Gamma_2 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \Gamma_2 \hat{\otimes} \ldots \hat{\otimes} \Gamma_2 \right]
\cdot \left[ \gamma_2 \hat{\otimes} \gamma_2 \hat{\otimes} \ldots \hat{\otimes} 1_{(2m+1)} + 1_{(2m+1)} \hat{\otimes} \gamma_2 \hat{\otimes} \ldots \hat{\otimes} \gamma_2 \right]
\]
\[ T_4 = \frac{1}{4} \left[ \gamma_1 \otimes \gamma_1 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_1 \otimes \ldots \otimes \gamma_1 \right] \\
\otimes [\Gamma_3 \otimes \Gamma_3 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_3 \otimes \ldots \otimes \Gamma_3] \\
+ \frac{1}{4} \left[ \gamma_3 \otimes \gamma_3 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_3 \otimes \ldots \otimes \gamma_3 \right] \\
\otimes [\Gamma_1 \otimes \Gamma_1 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_1 \otimes \ldots \otimes \Gamma_1] \\
T_5 = \frac{1}{4} \left[ \Gamma_1 \otimes \Gamma_1 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_1 \otimes \ldots \otimes \Gamma_1 \right] \\
\otimes [\Gamma_3 \otimes \Gamma_3 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_3 \otimes \ldots \otimes \Gamma_3] \\
+ \frac{1}{4} \left[ \gamma_1 \otimes \gamma_1 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_1 \otimes \ldots \otimes \gamma_1 \right] \\
\otimes [\gamma_3 \otimes \gamma_3 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_3 \otimes \ldots \otimes \gamma_3] \\
T_6 = \frac{1}{4} \left[ \gamma_2 \otimes \gamma_2 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_2 \otimes \ldots \otimes \gamma_2 \right] \\
\otimes [\Gamma_3 \otimes \Gamma_3 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_3 \otimes \ldots \otimes \Gamma_3] \\
+ \frac{1}{4} \left[ \gamma_3 \otimes \gamma_3 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_3 \otimes \ldots \otimes \gamma_3 \right] \\
\otimes [\Gamma_2 \otimes \Gamma_2 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_2 \otimes \ldots \otimes \Gamma_2] \\
T_7 = \frac{1}{4} \left[ \Gamma_2 \otimes \Gamma_2 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_2 \otimes \ldots \otimes \Gamma_2 \right] \\
\otimes [\Gamma_3 \otimes \Gamma_3 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_3 \otimes \ldots \otimes \Gamma_3] \\
+ \frac{1}{4} \left[ \gamma_2 \otimes \gamma_2 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_2 \otimes \ldots \otimes \gamma_2 \right] \\
\otimes [\gamma_3 \otimes \gamma_3 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_3 \otimes \ldots \otimes \gamma_3] \\
T_8 = \frac{1}{4\sqrt{3}} \left[ \Gamma_1 \otimes \Gamma_1 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_1 \otimes \ldots \otimes \Gamma_1 \right] \\
\otimes [\gamma_1 \otimes \gamma_1 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_1 \otimes \ldots \otimes \gamma_1] \\
+ \frac{1}{4\sqrt{3}} \left[ \Gamma_2 \otimes \Gamma_2 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_2 \otimes \ldots \otimes \Gamma_2 \right] \\
\otimes [\gamma_2 \otimes \gamma_2 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_2 \otimes \ldots \otimes \gamma_2] \\
- \frac{2}{4\sqrt{3}} \left[ \Gamma_3 \otimes \Gamma_3 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_3 \otimes \ldots \otimes \Gamma_3 \right] \\
\otimes [\gamma_3 \otimes \gamma_3 \otimes \ldots \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_3 \otimes \ldots \otimes \gamma_3] \\
\] (5.3)

The SU(2) interaction is expressed by:

\[ W_{L\mu}^{(n)} = \left[ \frac{1}{2} (W_{L\mu}^L \Gamma_3 + W_{L\mu}^2 \Gamma_1 \Gamma_3 + W_{L\mu}^3 \Gamma_1 \Gamma_2) \right] \otimes 1_{\ldots} \otimes 1 \\
+ \ldots + 1_{\ldots} \otimes \left[ \frac{1}{2} (W_{L\mu}^1 \Gamma_2 \Gamma_3 + W_{L\mu}^2 \Gamma_1 \Gamma_3 + W_{L\mu}^3 \Gamma_1 \Gamma_2) \right] \\
\] (5.4)
or alternatively,

$$W^{(n)}_{L\mu} = \frac{1}{2} W_{L\mu}^{1} (\Gamma_{2} \otimes \Gamma_{2} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_{2} \otimes ... \otimes \Gamma_{2})$$

$$\cdot (\Gamma_{3} \otimes \Gamma_{3} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_{3} \otimes ... \otimes \Gamma_{3})$$

$$+ \frac{1}{2} W_{L\mu}^{2} (\Gamma_{1} \otimes \Gamma_{1} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_{1} \otimes ... \otimes \Gamma_{1})$$

$$\cdot (\Gamma_{3} \otimes \Gamma_{3} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_{3} \otimes ... \otimes \Gamma_{3})$$

$$+ \frac{1}{2} W_{L\mu}^{3} (\Gamma_{1} \otimes \Gamma_{1} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_{1} \otimes ... \otimes \Gamma_{1})$$

$$\cdot (\Gamma_{2} \otimes \Gamma_{2} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_{2} \otimes ... \otimes \Gamma_{2})$$

(5.5)

The $U(1)_{R}$ right handed weak interaction is given by

$$W^{(n)}_{R\mu} = \left[ \frac{1}{2} (W_{R\mu}^{3} \Gamma_{1}^{2} \Gamma_{2}) \right] \otimes 1 \otimes 1 + ... + 1 \otimes ... \otimes \left[ \frac{1}{2} (W_{R\mu}^{3} \Gamma_{1}^{2} \Gamma_{2}) \right]$$

(5.6)

or alternatively,

$$W^{(n)}_{R\mu} = \frac{1}{2} W_{R\mu}^{3} (\Gamma_{1} \otimes \Gamma_{1} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_{1} \otimes ... \otimes \Gamma_{1})$$

$$\cdot (\Gamma_{2} \otimes \Gamma_{2} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_{2} \otimes ... \otimes \Gamma_{2})$$

(5.7)

The $U(1)_{B-L}$ interaction $W_{BL\mu}$ can be written as:

$$W^{(n)}_{BL\mu} = \left[ \frac{1}{6} (W_{BL\mu}^{J} \gamma_{1} \Gamma_{1} + W_{BL\mu}^{J} \gamma_{2} \Gamma_{2} + W_{BL\mu}^{J} \gamma_{3} \Gamma_{3}) \right] \otimes 1 \otimes 1 + ...$$

$$+ 1 \otimes ... \otimes \left[ \frac{1}{6} (W_{BL\mu}^{J} \gamma_{1} \Gamma_{1} + W_{BL\mu}^{J} \gamma_{2} \Gamma_{2} + W_{BL\mu}^{J} \gamma_{3} \Gamma_{3}) \right].$$

(5.8)

Alternatively,

$$W^{(n)}_{BL\mu} = \frac{1}{6} W_{BL\mu}^{J} (\gamma_{1} \otimes \gamma_{1} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_{1} \otimes ... \otimes \gamma_{1})$$

$$\cdot (\Gamma_{1} \otimes \Gamma_{1} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_{1} \otimes ... \otimes \Gamma_{1})$$

$$+ \frac{1}{6} W_{BL\mu}^{J} (\gamma_{2} \otimes \gamma_{2} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_{2} \otimes ... \otimes \gamma_{2})$$

$$\cdot (\Gamma_{2} \otimes \Gamma_{2} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_{2} \otimes ... \otimes \Gamma_{2})$$

$$+ \frac{1}{6} W_{BL\mu}^{J} (\gamma_{3} \otimes \gamma_{3} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \gamma_{3} \otimes ... \otimes \gamma_{3})$$

$$\cdot (\Gamma_{3} \otimes \Gamma_{3} \otimes ... \otimes 1_{(2m+1)} + 1_{(2m+1)} \otimes \Gamma_{3} \otimes ... \otimes \Gamma_{3})$$

(5.9)

Several aspects related to the formulation based on reference [29] can be explored. For example, color projection operators, quark projection operators and lepton projection operators can be obtained in our multipartite scenario. However, our goal was to characterize the symmetries associated to the model by exploring two proposals of realization of the elements.
6 Conclusions

In conclusion, we deduce Einstein’s equations in a formulation based on geometric algebra using the gauge gravity [2] of a multipartite perspective. We interpret the gauge fields as fields associated to bosonic and fermionic states from which Einstein’s equations emerge. Such fields may be related with qubits and entangled states can be obtained from minimal left ideals using primitive idempotents. Therefore this point of view also suggests that gravity may appears as an emerging phenomenon. Using as key ingredients the generalized graded tensor product and the concept of primitive elements extensively explored in the context of Hopf algebras from Lie algebras, an underlying new algebraic structure of the Hopf type is deduced from the formalism so that the gauge fields as well as the qubits can be fully characterized. Applications for background black holes were performed by deducing a multipartite Dirac equation [26]. The symmetries of the \(SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-R}\) model [29] were also explored under two distinct ways in this multipartite formulation. As perspectives we intend to interpret which particles would be associated with the above mentioned fields through the spin. Another interesting question is the role of quantum entanglement in this formalism.

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