Boundary of the action of Thompson’s group $F$ on dyadic numbers

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Abstract

We prove that the Poisson boundary of a simple random walk on the Schreier graph of action $F \curvearrowright D$, where $D$ is the set of dyadic numbers in $[0,1]$, is non-trivial. This gives a new proof of the result of Kaimanovich: Thompson’s group $F$ doesn’t have Liouville property. In addition, we compute growth function of the Schreier graph of $F \curvearrowright D$.

1 Introduction

Let $G$ be a group equipped with a probability measure $\mu$. A right random walk on $(G, \mu)$ is defined as a Markov chain $Z$ with the state space $G$ and transitional probabilities $P(Z_{n+1} = g|Z_n = h) = \mu(h^{-1}g)$. Specifying initial measure $\theta$ (distribution of $Z_0$), we obtain a probability measure $P^\mu_\theta$ on the space of trajectories $(Z_i)_{i \geq 0} \in G^{\mathbb{Z}_+}$. Usually one takes $\theta = \delta_e$ - Dirac measure at the group identity. The Poisson boundary of the pair $(G, \mu)$ can be defined as the space of ergodic components of the time shift on the $(G^N, P^\mu_\delta)$ [7]. For more equivalent definitions of the boundary one can look at [6]. A pair $(G, \mu)$ is said to have Liouville property if the corresponding Poisson boundary is trivial, or, equivalently, when the space of bounded $\mu$-harmonic functions on $G$ is 1-dimensional, i.e. consists of constant functions. A group $G$ has Liouville property iff for every symmetric, finitely supported $\mu$ the pair $(G, \mu)$

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does. For a recent survey and results on Liouville property and Poisson boundaries see [2], [3] and [4].

In this note we prove that Richard Thompson’s group $F$ doesn’t have Liouville property. A survey on Thompson’s groups is presented in [1]. Here we only mention that question of amenability of $F$ is one of the major open problems now.

2 Main results

Consider a simple random walk on a locally finite graph $G = (V, E)$. Fix a starting point $x_0$. This enables trajectory space $V^\mathbb{Z}_+$ with a probability measure $P$. The notion of the boundary is easily adapted to this case: it is the space of ergodic components of the time shift on the $(V^\mathbb{Z}_+, P)$. We’ll use electrical networks formalism as it appears in [5]. Throughout the paper, $d(\cdot, \cdot)$ will denote standard graph distance. Let $B(x, n) = \{y \in V : d(x, y) \leq n\}$ - ball centered at $x$ of radius $n$. Define also $\partial B(x, n) = \{y \in V : d(x, y) = n\}$.

**Theorem 2.1.** Suppose a locally-finite graph $G$ is given. Fix any vertex $x_0$. Let $\Upsilon(x_0)$ be the set of geodesics starting at $x_0$. Define $gd(x, n) = \#\{\gamma = [x_0, ..., x_m] \in \Upsilon(x_0) : x \in \gamma$ and $d(x, x_m) = n\}$. Suppose there exist some real numbers $c, C > 0$ and $q > 1$ such that the following conditions are satisfied:

$$gd(n, x) \leq Cq^n \text{ for every } x \in X$$

$$cq^n \leq gd(x_0, n)$$

Then $\text{cap}(x_0) > 0$.

**Remark 2.2.** For example, it’s easy to see that conditions[1] are obviously satisfied for a regular $m$-tree, with $q = m$.

**Proof.** We have to show that if $f \in D_0(N)$, $f(x_0) = 1$ then it’s Dirichlet norm is bounded from below by some positive constant. Let $n$ be such that $\text{supp} f \subseteq B(x_0, n - 1)$. All resistances are equal to 1 in our case, so we may
write

\[ D(f) = \sum_{e \in E} (f(e^+) - f(e^-))^2 \geq \sum_{k=0}^{n-1} \sum_{x \in \partial B(x_0,k)}^{y \in \partial B(x_0,k+1)} (f(x) - f(y))^2 = \]

\[ \sum_{\gamma \in \Upsilon(x_0)}^{[x_0,\ldots,x_n]=\gamma} \sum_{k=0}^{n-1} \frac{(f(x_k) - f(x_{k+1}))^2}{\text{gd}(x_{k+1}, n - k - 1)} \geq \sum_{\gamma \in \Upsilon(x_0)}^{[x_0,\ldots,x_n]=\gamma} \frac{\left(\sum_{k=0}^{n-1} f(x_k) - f(x_{k+1})\right)^2}{\sum_{k=0}^{n-1} \text{gd}(x_{k+1}, n - k - 1)} \geq \frac{c(q-1)}{C} \]

For the first inequality, we cancel edges which connect vertices which connect vertices at the same distance from \(x_0\). For the second equality, we consider geodesics from \(x_0\) to points at the distance \(n\) and sum quantities \(\frac{(f(x_k) - f(x_{k+1}))^2}{\text{gd}(x_{k+1}, n - k - 1)}\) over them, getting \((f(x_k) - f(x_{k+1}))^2\) by definition of \(\text{gd}\). We use next Cauchy-Schwartz and finiteness of support of \(f\): \(f \equiv 0\) outside of \(B(x_0, n - 1)\). Thus we have

\[ D(f) \geq \frac{c(q-1)}{C} \]

for every finitely-supported \(f\), hence \(\text{cap}(x_0) > 0\) \(\square\)

This implies, by theorem (2.12) from [5], that simple random walk on \(G\) is transient. Now we are going to establish a theorem which connects transience of certain random walks to non-triviality of boundary. Following [6], we call subset \(A \subset G\) a trap, if \(\lim_n \mathbbm{1}(Z_n \in A)\) exists for almost all trajectories \(Z \in G^\mathbb{N}\). We call a graph transient if the simple random walk on it is transient.

**Theorem 2.3.** Let \(T\) be a tree with a root vertex \(v\) such that for each descendant \(v_1, \ldots, v_n\) of \(v\) \((n \geq 2)\) a subtree \(T_i\) rooted at \(v_i\) is transient. Then the boundary of simple random walk on \(T\) is nontrivial.

**Proof.** Take any \(T_i\). Almost surely, every trajectory hits \(v\) only finitely many times. The only way to move from \(T_i\) to \(T_j\) is to pass by \(v\). Therefore, for any \(i\), we’ll stay inside or outside of \(T_i\) from some moment. This means that \(T_i\) is a trap. Let’s prove that it is nontrivial, i.e. random walk will stay at \(T_i\)
with positive probability. If this is true for each \( i \), then every \( T_i \), \( 1 \leq i \leq n \), is a nontrivial trap, so boundary is indeed nontrivial. Consider the following set of trajectories of the simple random walk on \( T \):

\[
A = \{ Z : Z_1 = v_i, \forall k \geq 2 \ Z_k \neq v_i \}.
\]

In addition, consider the set of trajectories of the simple random walk on \( T_i \):

\[
A' = \{ Z' : Z'_0 = v_i, \forall i \geq 1 \ Z'_i \neq v_i \}.
\]

Collecting the following facts:
- simple random walk on \( T \) goes to \( v_i \) with probability \( 1/n \);
- probability of going from \( v_i \) not to \( v \) is \( \frac{\text{deg}(v_i) - 1}{\text{deg}(v_i)} \);
- \( (Z_{k+1})_{k \geq 0} \in A' \), and transition probabilities are the same for \( Z_{i+1} \) and \( Z'_i \) for \( i \geq 1 \)

we obtain

\[
\mathbb{P}(A) = \frac{1}{n} \frac{\text{deg}(v_i) - 1}{\text{deg}(v_i)} \mathbb{P}(A').
\]

This shows us that indeed \( \mathbb{P}(A) > 0 \), as \( \mathbb{P}(A') > 0 \) due to transience.

**Proposition 2.4.** Let \( H \) be a graph which is formed by adding a set of graphs \( G_v \) with pairwise disjoint sets of vertices to each vertex \( v \) in \( T \). Then the boundary of simple random walk on \( H \) is nontrivial.

**Proof.** Boundary of \( T \) is nontrivial, so we have non-constant bounded harmonic function \( h \) on \( T \). We can extend it to the whole \( H \) by setting

\[
\hat{h}(x) = \begin{cases} 
  h(x), & \text{if } x \in T \\
  h(v), & \text{if } x \in G_v 
\end{cases}
\]

This way we get non-constant bounded harmonic function \( \hat{h} \) on \( H \), so boundary is non-trivial.

Recall that Richard Thompson’s group \( F \) is defined as the group of all continuous piecewise linear transformations of \([0, 1]\), whose points of non-differentiability belong to the set of dyadic numbers and derivative, where it exists, is an integer power of 2. It is known to be 2-generated. Now we are ready to prove the main theorem.

**Theorem 2.5.** Thompson’s group \( F \) does not have Liouville property.
Figure 1: Schreier graph $\mathcal{H}$ of the action of $F$ on the orbit of $1/2$

Proof. First of all, we observe that if action $G \curvearrowright X$ is non-Liouville, i.e. there are bounded non-constant harmonic functions on the Schreier graph of this action, then $G$ itself is non-Liouville. Required harmonic function on $G$ is just a pullback of a harmonic function $h$ on $X$: $h'(g) = h(g.x)$. Obviously, $h'$ is harmonic if $h$ is. We consider the action of $F$ on the set of all dyadic numbers in $[0,1]$. We use presentation of its Schreier graph $\mathcal{H}$ constructed by D. Savchuk in [8]. It is illustrated on the Figure 1. We look at the tree $T$ rooted at 101 formed by grey vertices and white which are connected with the grey ones. We need to verify that $T$ satisfies condition [1]. Take $x_0$ to be the point $3/8 = 101$. For grey vertices $x$ we have $gd(n,x) = |\partial B(x_0,n)|$, and for white $y$ we have $gd(n,y) = |\partial B(x_0,n-1)|$. In fact, $|\partial B(x_0,n)|$ may be calculated explicitly ($T$ is a famous Fibonacci tree), and value $q = \frac{1+\sqrt{5}}{2}$
works. Hence, we can apply consequently 2.1, 2.3 and 2.4 to see that there are non-constant bounded harmonic functions on $\mathcal{H}$, so, by the remark in the beginning of the proof, on the Thompson’s group $F$.

**Remark 2.6.** The fact that simple random walk on the Thompson’s group $F$ has nontrivial boundary is first proven by Kaimanovich in [9].

### 3 Growth function of $\mathcal{H}$

In [10] different types of growth functions for groups are defined. We adapt these definitions to Schreier graphs of group actions. We’ll compute growth function of $\mathcal{H}$. Suppose we have a Schreier graph of action of a group $G$ on set $X$. Fix some starting point $p \in X$. A cone type of a vertex $x$ is defined as follows:

$$C(x) = \{ g \in G : \text{if } w \text{ is a geodesic from } p \text{ to } x, \text{ then } wg \text{ is a geodesic from } p \text{ to } g(p) \}.$$  

*Complete geodesic growth function* is defined as

$$L(z) = \sum_{g \in G : g \text{ is a geodesic for } g(p)} g z^{|g|}.$$  

*Geodesic growth function* is defined by sending all group elements to 1, namely

$$l(z) = \sum_{g \in G : g \text{ is a geodesic for } g(p)} z^{|g|}.$$  

*Orbit growth function* is defined as

$$\hat{l}(z) = \sum_{n=0}^{\infty} \# \{ x \in X : \exists g \in G - \text{geodesic : } |g| = n, g(p) = x \} z^n.$$  

Now consider $\mathcal{H}$. We are interested in geodesics starting at point $p = 1/2(100...)$. Then we have 5 cone types of vertices:

- Type 0: point 1 (which corresponds to 1/2);
- Type 1: black vertices;
- Type 2: grey vertices excluding 1;
- Type 3: white vertices on the tree;
Type 4: white vertices not on the tree.

Let’s write $\Lambda_i^n = \sum_{g \in C_i, |g| = n} g$, where $C_i$ is the i-th cone type. Then one gets recurrent relations:

$$\begin{align*}
\Lambda_0^n &= \Lambda_1^{n-1}a + \Lambda_3^{n-1}a^{-1} \\
\Lambda_1^n &= \Lambda_1^{n-1}a \\
\Lambda_2^n &= \Lambda_1^{n-1}a + \Lambda_2^{n-1}b + \Lambda_3^{n-1}a^{-1} \\
\Lambda_3^n &= \Lambda_2^{n-1}b + \Lambda_4^{n-1}(a^{-1} + b^{-1}) \\
\Lambda_4^n &= \Lambda_4^{n-1}(a^{-1} + b^{-1})
\end{align*}$$

(2)

Denote $L_i^n$ the number of geodesics of length $n$ starting from a vertex of type $i$, leading to different points, i.e. $L_i^n = \partial B(x_i, n)$ for $x_i$ being a vertex of type $i$. Then recurrent relations are:

$$\begin{align*}
L_0^n &= L_1^{n-1} + L_3^{n-1} \\
L_1^n &= L_1^{n-1} \\
L_2^n &= L_1^{n-1} + L_2^{n-1} + L_3^{n-1} \\
L_3^n &= L_2^{n-1} + L_4^{n-1} \\
L_4^n &= L_4^{n-1}
\end{align*}$$

(3)

Let $\Lambda^n = (\Lambda_0^n, \Lambda_1^n, \Lambda_2^n, \Lambda_3^n, \Lambda_4^n)^T$ and $L^n = (L_0^n, L_1^n, L_2^n, L_3^n, L_4^n)^T$. We compute

$$\tilde{L}(z) = \sum_{n=0}^{\infty} \Lambda^n z^n$$

- extended complete geodesic growth function and

$$\hat{L}(z) = \sum_{n=0}^{\infty} L^n z^n$$

- geodesic orbit growth function (if two geodesics lead to the same point, they are counted as one). By recurrent formulas, we have

$$\tilde{L}(z) = \sum_{n=0}^{\infty} A^n A_0 z^n = (I_5 - Az)^{-1} \tilde{\Lambda}_0 \quad \text{and} \quad \hat{L}(z) = (I_5 - Bz)^{-1} L_0,$$

where $\tilde{\Lambda}_0 = (e, e, e, e)^T$, $L_0 = (1, 1, 1, 1)^T$ and transition matrices $A, B, A$ are obtained from recurrent relations 2, 3. In particular, geodesic growth function is given by the first coordinate of vector-function

$$\tilde{l}(z) = (I_5 - Az)^{-1} \Lambda_0,$$

where $\Lambda_0 = L_0 = (1, 1, 1, 1)^T$. Performing calculations, one gets

$$\tilde{l}(z) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 - 2z & 1 - z & 1 - 3z + 2z^2 & 1 - 3z + 2z^2 & 1 - 2z \end{pmatrix}.$$
So, $\frac{1}{1-z}$ is the geodesic growth function for our Schreier graph. Also we get

$\tilde{L}(z) = \left( \frac{1+z}{1-z-z^2}, \frac{1}{1-z}, \frac{1+z}{1-2z+z^3}, \frac{1}{1-2z+z^3}, \frac{1}{1-z} \right)$.

Finally, we see that we’ve obtained $l(z) = \frac{1}{1-2z}$ and $\tilde{l}(z) = \frac{1+z}{1-z-z^2}$.

$\tilde{l}(z) = \frac{1+z}{1-z-z^2} = \frac{\varphi^2}{\sqrt{5}(1-\varphi z)} - \frac{\hat{\varphi}^2}{\sqrt{5}(1-\hat{\varphi} z)}$,

where $\varphi = \frac{\sqrt{5}+1}{2}$, $\hat{\varphi} = \frac{\sqrt{5}+1}{2}$. So, $L_n = |\partial B(p, n)| = \frac{\varphi^{n+2} - \hat{\varphi}^{n+2}}{\sqrt{5}}$. $|B(p, n)| = \sum_{k \leq n} L_k = \frac{\varphi^{n+3} - \varphi^3}{\sqrt{5}(\varphi-1)} - \frac{\hat{\varphi}^{n+3} - \hat{\varphi}^3}{\sqrt{5}(\hat{\varphi}-1)}$.

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