Landauer Conductance without Two Chemical Potentials

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Our approach avoids a problematic use of two chemical potentials in the same system.

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I. INTRODUCTION

Tunneling conductance through a potential barrier has been the subject of intense study since the original Landauer paper \cite{Landauer57} in 1957 (see Ref. \cite{Kamenev98} for reviews). It was recognized that the two–terminal measurements of the tunneling conductance, $g$, may be described by the Landauer formula

$$g = \frac{e^2}{\pi \hbar} \sum_i |t_i|^2 ,$$

where the $t_i$ are the eigenvalues of a barrier transmission matrix. The term “two–terminal” implies that the voltage drop is measured between the same two contacts, sufficiently far from the barrier, which are used to pass a current. Actually Landauer’s original expression \cite{Landauer57}, for the single channel case, had the different form

$$\tilde{g} = \frac{e^2}{\pi \hbar} \frac{|t|^2}{|r|^2} ,$$

where $r$ is the reflection coefficient of the barrier, obeying $|r|^2 + |t|^2 = 1$. Equation (2) is intuitively appealing since it leads to zero resistance for the fully transparent ($|t| = 1$) barrier. Only after the work of Thouless \cite{Thouless74} and Imry \cite{Imry79} was it realized that Eq. (2) implicitly relies on the ability to measure the voltage drop right across the tunneling barrier. One should thus assume the existence of another pair of contacts used solely as potential contacts, and the absence of any cross–talk between the left and right movers. As was first remarked by Böttiker, Imry, Landauer and Pinhas (BILP) \cite{Boettiker90}, there is no universal expression for the conductance of the barrier placed into a ballistic constriction.

While the single–channel four–terminal expression, Eq. (2), is generally agreed upon, the generalization to the multi–channel case has been a subject of controversy. In one dimension $n = 2mv_F/(\pi \hbar)$, where $v_F$ is the Fermi velocity, one recovers Eq. (2) for the conductance, in agreement with Thouless’s conclusion.

We present a theory of the four–terminal conductance for the multi–channel tunneling barrier, which is based on the self–consistent solution of Shrödinger, Poisson and continuity equations. We derive new results for the case of a barrier embedded in a long wire with and without disorder. We also recover known expressions for the conductance of the barrier placed into a ballistic constriction. The difference for the tunneling conductance, $g$, may be described by the Landauer formula

$$g = \frac{e^2}{\pi \hbar} \sum_i |t_i|^2 ,$$

where the $t_i$ are the eigenvalues of a barrier transmission matrix. The term “two–terminal” implies that the voltage drop is measured between the same two contacts, sufficiently far from the barrier, which are used to pass a current. Actually Landauer’s original expression \cite{Landauer57}, for the single channel case, had the different form

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We present a theory of the four–terminal conductance for the multi–channel tunneling barrier, which is based on the self–consistent solution of Shrödinger, Poisson and continuity equations. We derive new results for the case of a barrier embedded in a long wire with and without disorder. We also recover known expressions for the conductance of the barrier placed into a ballistic constriction. Our approach avoids a problematic use of two chemical potentials in the same system.
relaxation mechanisms, etc. \[1\]. BILP considered a particular case of scattering–free leads connected to two ideal thermal reservoirs. Each reservoir is taken to be in thermal equilibrium with different chemical potentials on the left and right of the barrier. The distribution functions of the left and right moving carriers are strictly imposed by the reservoirs from which they originated. Thus on each side of the barrier left and right moving electrons occupy Fermi hemispheres of different radii. The potential drop across the barrier is taken to be the difference of effective chemical potentials of the leads, which are determined from the postulated condition of having equal number of particles above and holes below them. For the case of no channel mixing, \( t_{ij} = t_i \delta_{ij} \) and \( r_{ij} = r_i \delta_{ij} \), BILP obtained

\[
\tilde{g}_{\text{BILP}} = \frac{e^2}{\pi \hbar} \sum_i v_i^{-1} \sum_i |t_i|^2, \quad (3)
\]

where \( i \) runs over the open channels with velocity \( v_i \). The same result was obtained earlier by Azbel \[2\], using a slightly different argumentation. For a single channel Eq. (3) reduces to the old Landauer result, Eq. (2), whereas for a very high barrier, \( |r_i| \approx 1 \), there is no difference between the four–terminal result, Eq. (3), and the two–terminal one, Eq. (4).

In this paper we develop a logically clean formulation of the four–terminal conductance which avoids the problematic introduction of different chemical potentials in various parts of the same system. Instead, we consider a pure Hamiltonian formulation of the problem with the external bias applied by a time dependent vector potential through a macroscopically large loop containing a barrier. We solve self-consistent equations for the electric field on the Hartree (or RPA) level to determine the voltage drop across the barrier. As a microscopic ingredient we need only an equilibrium linear conductance kernel, which is derived using the Kubo formula. Our formulation of the problem also allows us to tackle regimes not considered previously. As an example we consider a barrier embedded into a long multi–channel wire of uniform cross-section (a geometry different from that considered by BILP). If the wire length, \( L \), exceeds \( v_F/\omega \) (\( v_F \) is a Fermi velocity), the electron distribution function is replaced with a shifted (accelerated) Fermi sphere, rather than the BILP distribution. This results in a four–channel conductance which is different from the BILP expression. For example in the low barrier limit, \( |r_i| \ll 1 \), we find

\[
\tilde{g} = \frac{e^2}{\pi \hbar} \sum_i v_i^{-1} |r_i|^2, \quad (4)
\]

which is quite different from Eq. (3) in the same limit. For one thing, we do not expect discontinuities of conductance as function of the Fermi energy at the values where velocity of one of the channels is zero. In the opposite limit of a high barrier, \( |t_i| \ll 1 \), we obtain Eq. (4), in agreement with BILP and other approaches. Another regime we are able to treat is a barrier embedded in a disordered wire with \( L \gg l^d \), where \( l^d \) is an elastic mean free path. In this case, the momentum equilibration which takes place in the leads due to strong impurity scattering. We find that the same result, Eq. (4), holds for the case of disordered leads. That does not mean that the BILP expression, Eq. (3), is incorrect. We have recovered it, together with the conditions for its applicability, for a barrier embedded in a ballistic constriction, \( L < v_F/\omega, l^d \), between two wide reservoirs. Our procedure is quite different from that of BILP. In particular we do not require consideration of an unphysical, space dependent chemical potential — it may be defined using only standard equilibrium thermodynamics along with Poisson’s and linear response equations.

The outline of the paper is as follows: in Sec. \[II\] we formulate self–consistent equations for the induced electric field and charge density. In Sec. \[III\] we derive a microscopic expression for the non–local conductance in the case of the clean wire. We then solve the self–consistent field equations and obtain the four–terminal conductance in a number of regimes. Sec. \[IV\] deals with a barrier with disordered leads. In Sec. \[V\] we treat an adiabatic constriction geometry. Finally, in Sec. \[VI\] we discuss the results, their domains of validity and possible generalizations.

**II. SELF–CONSISTENT FIELD EQUATIONS**

Let us consider a multi–channel metallic wire of uniform cross-section, \( S \), along the \( z \)-direction. We assume that in a vicinity of \( z = 0 \) there is a potential barrier (or more generally some localized scattering region) across the wire. A weak uniform applied electric field, \( E^{\text{ext}}(\omega) \), with frequency \( \omega \) creates a current density, which in the linear response regime has the form

\[
j(r, \omega) = \int dr' g(r, r', \omega) \cdot E(r', \omega). \quad (5)
\]

Here the integration runs over the volume of the wire and \( E(r', \omega) \) is a total electric field at the point \( r' \). The non–local conductivity kernel, \( g(r, r', \omega) \), may be calculated using the Kubo formula.

We assume that, due to the quasi–one–dimensional geometry of the system, the electric field is practically independent of the transverse coordinates, \( E(r, \omega) \approx E(\zeta, \omega) \) and only the total current \( I(\zeta, \omega) = \int_S dS_j(r, \omega) \) is measurable experimentally. The total current, \( I(\zeta, \omega) \), and the electric field, \( E(\zeta, \omega) \), are linearly connected through the conductance kernel, \( g(\zeta, \zeta', \omega) \). Employing the Kubo
formula \(g(z, z', \omega)\) and disregarding the e–e interactions in the kernel \(g(z, z', \omega)\) (which is equivalent to the time-dependent Hartree approximation), one may express the non–local conductance as

\[
g(z, z', \omega) = \frac{e^2}{2im\hbar^2} \int d\epsilon \frac{f(\epsilon+) - f(\epsilon-)}{\omega} \times \sum_{ij} G^R_{ij}(\epsilon_0; z, z') \frac{\partial}{\partial z} G^A_{ij}(\epsilon_0; z', z),
\]

where \(A \nabla_z B \equiv A \nabla_z B - \nabla_z AB\), \(\epsilon_\pm = \epsilon \pm \omega/2\), \(f(\epsilon_\pm)\) is the Fermi function and \(G^{R(A)}(\epsilon)\) is the retarded (advanced) Green’s function of the system at energy \(\epsilon\) in the transverse channel basis,

\[
G_{ij}(\epsilon; z, z') = \int dS \int dS' \xi_i(x, y) G(\epsilon; r, r') \xi_j(x', y').
\]

Here \(\xi_i(x, y)\) is the eigenfunction of the \(i\)-th transverse channel with energy \(\epsilon_i\). It will sometimes be convenient to divide the conductance into two parts,

\[
g(z, z', \omega) = g_0(z - z', \omega) + g_b(z, z', \omega),
\]

where \(g_0(z - z', \omega)\) is the translationally invariant conductance of the wire without the barrier and \(g_b(z, z', \omega)\) is the contribution associated with the scattering by the barrier.

The linear response relation, Eq. (3), together with the continuity and Poisson’s equations, constitute a closed system of equations for three unknown quantities: the current \(I(z, \omega)\), the electric field \(E^{ind}(z, \omega)\) and the charge density \(\rho(z, \omega)\) all induced by the applied external field, \(E^{ext}(\omega)\),

\[
I(z, \omega) = \int dz' g(z, z', \omega)(E^{ext}(\omega) + E^{ind}(z', \omega)),
\]

\[
\frac{\partial}{\partial z} I(z, \omega) = i\omega \rho(z, \omega),
\]

\[
\frac{\partial}{\partial z} E^{ind}(z, \omega) = 4\pi \rho(z, \omega)/S.
\]

Eliminating \(I(z, \omega)\) and \(\rho(z, \omega)\), one obtains the following integral equation for \(E^{ind}(z, \omega)\),

\[
\frac{S}{4\pi} \frac{\partial}{\partial z} E^{ind}(z, \omega) - \frac{1}{i\omega} \int dz' \frac{\partial}{\partial z} g(z, z', \omega) E^{ind}(z', \omega) = \rho^{(0)}(z, \omega),
\]

where \(\rho^{(0)}(z, \omega)\) is the bare (unscreened) charge density produced by the external electric field near the barrier

\[
\rho^{(0)}(z, \omega) = \frac{1}{i\omega} \int dz' \partial_z g(z, z', \omega) E^{ext}(\omega).
\]

Since in the absence of the barrier no charge is induced in the leads, only the translationally non–invariant part of the conductance, \(g_b(z, z', \omega)\), contributes to \(\rho^{(0)}\). It is convenient to employ Fourier representation. Then, using Eq. (8), one may rewrite Eqs. (12), (13) in the following form

\[
\left[ \frac{i\omega S}{4\pi} - g_0(q, \omega) \right] E^{ind}(q, \omega) = \frac{1}{2\pi} \frac{d\delta}{dq} g_b(q, q', \omega) E^{ind}(q', \omega) = g_b(q, 0, \omega) E^{ext}(\omega),
\]

where e.g.

\[
g_b(q, q', \omega) = \int dz dz' e^{iqz} g_b(z, z', \omega) e^{-iq'z'}. \tag{15}
\]

Once Eq. (14) is solved, the induced voltage drop across the barrier is given by \(V = \int dz E^{ind}(z, \omega)\), where the limits of integration are taken to be much larger than any microscopic scale of the problem (see below), but still much smaller than the length, \(L\), of the ring which encloses the driving time–dependent magnetic flux. In the Fourier representation one obtains

\[
V = E^{ind}(q \to 0, \omega), \tag{16}
\]

where \(q \to 0\) in such a way that \(L^{-1} \ll q_0 \ll \omega/v_F\). The divergence-less part of the current, \(I_0\), originates from the convolution of \(g_0(z - z')\) and \(E^{ext}\):

\[
I_0 = g_0(q = 0, \omega) E^{ext}(\omega). \tag{17}
\]

All other terms in Eq. (14) describe currents localized near the barrier. As a result the four–terminal conductance, \(\tilde{g} = I_0/V\), may be expressed as

\[
\tilde{g} = \frac{g_0(0, \omega)}{E^{ext}(\omega)} \frac{E^{ext}(\omega)}{E^{ind}(q \to 0, \omega)} \tag{18}
\]

We shall proceed now by calculating the non–local conductance, \(g(z, z', \omega)\), and then solving Eq. (12) for the various cases of practical interest.

### III. BARRIER WITH LONG CLEAN LEADS

If there is no scattering of electrons in the leads, the Green’s functions outside the barrier region may be expressed as

\[
G_{ij}^{(0)}(\epsilon; z, z') = \frac{-im}{\sqrt{\rho_{ij}}} \left\{ \begin{array}{ll} t_{ij}(\epsilon) e^{ip_{ij}|z| + ip_{ij}|z'|}, & zz' < 0; \\ \delta_{ij} e^{ip_{ij}|z-z'|} + r_{ij}(\epsilon) e^{ip_{ij}|z|+ip_{ij}|z'|}, & zz' > 0, \end{array} \right. \tag{19}
\]

where \(p_{ij}^2/(2m) = m\epsilon_i^2/2 = \epsilon - \epsilon_i\), etc. The transmission and reflection matrices of the barrier, \(t_{ij}(\epsilon)\) and \(r_{ij}(\epsilon)\), obey
the unitarity condition
\[ \sum_j \left[ |t_{ij}(\epsilon)|^2 + |r_{ij}(\epsilon)|^2 \right] = 1. \] (20)

Substituting Eq. (19) into Eq. (1), one can perform the energy integration assuming that \( t_{ij}(\epsilon) \) and \( r_{ij}(\epsilon) \) are slowly varying functions of energy on the scales of both temperature and frequency, \( \omega \). As a result one obtains for the non–local conductance outside the barrier
\[ g(z, z', \omega) = \frac{e^2}{\pi h} \sum_{ij} \left\{ t_{ij}(t_{ij})^* e^{i\omega(|z|/v_i + |z'|/v_j)} + \delta_{ij} e^{i\omega(|z|/v_i - |z'|/v_j)} \right\}, \quad zz' < 0; \]
\[ g(z, z', \omega) = \frac{e^2}{\pi h} \sum_{ij} \left\{ r_{ij}(r_{ij})^* e^{i\omega(|z|/v_i + |z'|/v_j)} \right\}, \quad zz' > 0, \] (21)

where \( t_{ij} = t_{ij}(\epsilon_F) \) and \( \epsilon_F \) is the Fermi energy. Eq. (21) is the zeroth order term in the small parameter \( \omega/\epsilon_F \).

Note, however, that even for small \( \omega \) the combination \( \omega z/v_i \) is not necessarily small.

Setting \( \omega = 0 \) in Eq. (21) leads to
\[ g \equiv g(z, z', \omega = 0) = \frac{e^2}{\pi h} \text{Tr} tt^\dagger. \] (22)

The fact that at \( \omega = 0 \) the conductance kernel is coordinate independent is a manifestation of the continuity equation. As a result the total current is divergence-less and given by \( I_0 = g \int dz E(z) \). The integral on the r.h.s. is a total voltage drop across the entire wire (and not just across the barrier). Therefore, the quantity \( g \) defined by Eq. (22) represents the two–terminal conductance in agreement with the Landauer expression, Eq. (1). To analyze the four–terminal setup, one has to solve the self–consistent equation Eq. (12) and thus to keep frequency the dependence of Eq. (21).

For simplicity we restrict ourself to the case of no channel mixing inside the barrier region, i.e. \( t_{ij} = t_j \delta_{ij} \) and \( r_{ij} = r_j \delta_{ij} \), where, according to Eq. (21), \( |t_i|^2 + |r_i|^2 = 1 \). In this case Eq. (21) may be simplified further (see Ref. 8):
\[ g(z, z', \omega) = \frac{e^2}{\pi h} \sum_i \left[ e^{i\omega(|z|/v_i)} - |r_i|^2 e^{i\omega(|z'|/v_i)} \right]. \] (23)

The two terms on the right hand side correspond to \( g_0 \) and \( g_0 \) introduced in Eq. (8). In the Fourier representation these terms take the form
\[ g_0(q, \omega) = -\frac{e^2}{\pi h} \sum_i \frac{2v_i \omega}{v_i^2 q^2 - \omega^2}; \] (24)
\[ g_0(q', \omega) = -\frac{e^2}{\pi h} \sum_i |r_i|^2 \frac{2v_i \omega}{v_i^2 q^2 - \omega^2} + \frac{2v_i \omega}{v_i^2 q^2 - \omega^2}. \] (25)

These results may be derived directly in the momentum representation starting from the Kubo formula and employing the following expression for the Green’s functions
\[ G_j^R(\epsilon; p, p') = G_j^R(\epsilon; p) 2\delta_{p, p'} + \frac{iv_j G_j^R(\epsilon; p) G_j^R(\epsilon; p')}{2} \left\{ \frac{t_j - 1 + r_j + pp'}{t_j - 1 - r_j} \right\}. \] (26)

Here \( G_j^R(\epsilon; p) = (\epsilon - \epsilon_j - p^2/(2m) + i\eta)^{-1} \) is the retarded Green’s function of the \( j \)-th channel in the wire without a barrier (see Eq. (13)).

By introducing the dimensionless momentum \( l = q v_F/\omega \) and normalized induced electric field \( E(l) \equiv \omega E^\text{ind}(\omega/v_F, \omega)/(v_F E^\text{ext}(\omega)) \), Eq. (14) acquires the form
\[ \left[ \frac{2}{\tilde{\omega}_p^2} \frac{e^2}{\omega^2} - g_0(l) \right] \tilde{E}(l) = \int dl' \frac{g_b(l, l') \tilde{E}(l')}{2\pi}, \] (27)

where \( \tilde{\omega}_p^2 = 8e^2 v_F/(\hbar S) = \omega_p^2 3\pi \hbar^2/(k_F^2 S) \), with \( \omega_p^2 = 4\pi e^2 n/m \) being the plasma frequency. We have introduced the rescaled (frequency independent) conductance as
\[ g_0(l) \equiv -i \sum_i \frac{2\tilde{v}_i}{v_i^2 l^2 - 1}; \] (28)
\[ g_b(l, l') \equiv \sum_i |r_i|^2 \frac{2\tilde{v}_i}{v_i^2 l^2 - 1} \frac{2\tilde{v}_i}{v_i^2 (l')^2 - 1}, \] (29)

where \( \tilde{v}_i = v_i / v_F \). At small frequency, \( \omega \ll \tilde{\omega}_p \), one may neglect the first term on the l.h.s. of Eq. (27). The remaining equation is manifestly frequency independent. We shall, however, keep the frequency dependent term for the time being since it will allow us to discuss the characteristic length scales of the problem. We now proceed to solve Eq. (27) in some particular cases.

### A. Single–channel case

First we apply the formalism to the well–known single–channel case. In this case the integral equation (27) has a separable kernel and may be easily solved. We neglect for a moment the frequency dependent term. Substituting Eqs. (28) and (29) with \( \tilde{v}_1 = 1 \) into Eq. (27), one obtains
\[
\frac{\hat{E}(l)}{2i|v|^2} + \int dl' \frac{\hat{E}(l')}{(v')^2 - 1} = 1.
\]

(30)

The elementary solution of this equation is

\[
\hat{E}(l) = 2i\frac{|v|^2}{1 - |v|^2}.
\]

(31)

Employing Eq. (18), one immediately obtains the famous Landauer expression, Eq. (2). This agrees with the conclusions obtained by Thouless [3] by imposing strict charge neutrality outside the barrier. Indeed, the fact that \(E_{\text{ind}}(q) = \text{const}\) means that \(E_{\text{ind}}(z) \propto \delta(z)\). Thus, in this approximation there is no induced electric field and no induced charge density outside the barrier. To recover the shape of the charge distribution one must keep the frequency dependent term in Eq. (27). An elementary calculation gives

\[
\rho(z, \omega) \propto \text{sign}(z) \exp \left\{ -\frac{|z|\omega_p}{v_F} \sqrt{1 - \frac{\omega^2}{\omega_p^2}} \right\},
\]

(32)

showing that entire charge redistribution is confined to within the screening length, \(\kappa^{-1} = v_{F}/\omega_p\), near the barrier. Since we keep only the long wave–length components of the conductance kernel, Eq. (32) is valid only if the inequality \(\kappa < k_F\) is fulfilled (if not there are oscillations on the scale \((2k_F)^{-1}\)). Although the initial set of equations, (1)–(11), included the characteristic length \(v_{F}/\omega\), the expression (32) depends only on the much smaller scale, \(\kappa^{-1}\). This is a particular property of the single–channel case only. In the multi–channel case the length scale \(v_{F}/\omega\) does not drop out of the solution of the self–consistent equations. This demands separate treatments for wires whose lengths are larger or smaller than \(v_{F}/\omega\).

If one keeps the frequency dependent term in Eq. (14), the resulting four–terminal conductance is

\[
\tilde{g} = \frac{e^2}{\pi h} \left[ \frac{|v|^2}{|v|^2} \left( 1 - \frac{\omega^2}{\omega_p^2} \right) - i \frac{\omega}{\omega_p} \sqrt{1 - \frac{\omega^2}{\omega_p^2}} \right].
\]

(33)

At small frequency this expression describes the Landauer resistance in parallel with an effective capacitance

\[
C_0 \equiv \frac{S}{4\pi(2\kappa^{-1})},
\]

(34)

which is the classical result for a plane capacitor of the area \(S\) and spacing \(2\kappa^{-1}\). In our approximations the screening length is assumed to be much larger than both the wavelength and the barrier width. If this is not the case, one has to keep the next order in \(\omega/\epsilon_F\) as well as short wave–lengths in the expression for the conductance kernel, to obtain the effective capacitance.

**B. Equal velocity channels**

Another exactly solvable case, which will be used in Sec. [4] is that of \(N\) channels having the same velocity, \(\hat{v}_i = 1\); \(i = 1 \ldots N\). Calculations exactly parallel to the previous case lead to

\[
\tilde{g} = e^2 \frac{N}{\pi h} \sum_{i} |t_i|^2.
\]

(35)

This result is in agreement with BILP, Eq. (3). If only \(M \ll N\) channels are open, \(t_i = 0, |r_i| = 1\) for \(i = M + 1 \ldots N\), Eq. (35) simplifies to

\[
\tilde{g} = e^2 \frac{M}{\pi h} \sum_{i} |t_i|^2 + O \left( \frac{M}{N} \right).
\]

(36)

**C. Weakly reflecting channels**

The case of a weakly reflecting barrier, \(|r_i| \ll 1; \ i = 1 \ldots N\), may be considered by treating \(g_{b}(l, l')\) as a perturbation in Eq. (27). Employing obvious operator notations and omitting the frequency dependent term, one may write the formal solution of Eq. (27) as

\[
\tilde{E}(l) = - (\tilde{g}_0 + \tilde{g}_b)^{-1} \tilde{g}_b |\delta\rangle
\]

\[= - (\tilde{g}_0^{-1} \tilde{g}_b + \tilde{g}_0^{-1} \tilde{g}_b \tilde{g}_0^{-1} \tilde{g}_b |\delta\rangle),
\]

(37)

where \(|\delta\rangle = (1, 0, 0, \ldots)\), where the first entry refers to \(l = 0\) component. Since \(\tilde{g}_0\) is diagonal and may be easily inverted, all terms in the expansion, Eq. (37), may be written in quadratures. We restrict ourself to the leading term only. In this approximation one obtains

\[
\tilde{E}(l) = i \sum_{i} \frac{4\pi^2 |r_i|^2 (v_i^2 t_i^2 - 1)^{-1}}{\sum_{i} 2\epsilon_i (v_i^2 t_i^2 - 1)^{-1}}.
\]

(38)

Substitution of Eq. (35) into Eq. (18) leads to the result, announced in Sec. [4]

\[
\tilde{g} = e^2 \frac{N}{\pi h} \sum_{i} v_i^2 |r_i|^2.
\]

(39)

Note that, apart from the single–channel case, the approximate solution, Eq. (35), depends on the momentum, \(l\), and hence on the \(z\)–coordinate. This means that strict charge neutrality does not occur in general. Rather, the resulting charge density exhibits a spatial modulation on the scale \(v_F/\omega\).
D. Weakly transmitting channels

We turn now to the high barrier case, \(|t_i| \ll 1\); \(i = 1 \ldots N\). Employing \(|r_i|^2 = 1 - |t_i|^2\), we rearrange Eqs. (23), (24) as

\[
g(l, l') = g_1(l, l') + g_2(l, l') ,
\]

where

\[
g_1(l, l') = \sum_i \frac{2\bar{v}_i}{\bar{v}_i^2(l'^2 - 1)} \left[ -2\pi i \delta_{l,l'} + \frac{2\bar{v}_i}{\bar{v}_i^2(l'^2 - 1)} \right] ;
\]

\[
g_2(l, l') = -\sum_i |t_i|^2 \frac{2\bar{v}_i}{\bar{v}_i^2(l'^2 - 1)} \frac{2\bar{v}_i}{\bar{v}_i^2(l'^2 - 1)} .
\]

Here \(g_1\) describes two disconnected half-wires, and \(g_2\) – the perturbation due to the small tunneling transparency of the barrier. Since the conductance of the infinite barrier is zero, one expects that the operator \(\hat{g}_1\) has a zero eigenvalue. Indeed, it is easy to check that

\[
\hat{g}_1 |1\rangle = \int \frac{dl'}{2\pi} g_1(l, l') = 0 ,
\]

where \(|1\rangle\) is the abstract vector in the momentum space given by \((1, 1, 1 \ldots )\), whose entries refer to different values of \(l\). As a result, the operator \(\hat{g}_1\) is not invertible, which complicate the perturbation theory. To overcome this difficulty we pass to the basis of eigenstates of the operator \(\hat{g}_1\),

\[
\hat{g}_1 |a\rangle = \lambda_a |a\rangle ,
\]

where \(a = 1, 2, \ldots \) and \(\lambda_1 = 0\). In this basis Eq. (27) takes the form

\[
(g_1)_{aa'} E_{a'} = -|g_0|_1 ; \quad \lambda_a E_a + (g_1)_{aa'} E_{a'} = -|g_0|_a , \quad a \neq 1 ,
\]

where \(|g_0|_a = \langle a | g_0 | \delta\rangle\), and summation over repeated indices is assumed. Taking advantage of the smallness of \(\hat{g}_1 \propto |t_i|^2\), one may solve these equations iteratively. Neglecting first \(\hat{g}_1\) on the l.h.s. of Eq. (20), one obtains in the zeroth approximation

\[
E_a = -\frac{|g_0|_a}{\lambda_a} , \quad a \neq 1 ;
\]

\[
E_1 = -\frac{|g_0|_1}{(g_1)_{11}} + \sum_{a' \neq 1} \frac{(g_1)_{1a'}}{(g_1)_{11}} \frac{|g_0|_{a'}}{\lambda_a'} .
\]

Repeatedly substituting the solution back to Eq. (46), one may obtain the higher order terms. In the leading (minus first) order in \(\hat{g}_1\) the solution has the form

\[
E_a = -\frac{|g_0|_1}{(g_1)_{11}} \delta_{a,1} .
\]

To this order of accuracy on has to put \(|r_i|^2 = 1\) in the numerator and therefore \((g_0)_{1} = -g_0(0, \omega)\). By virtue of Eq. (13) \((\hat{g}_1)_{11} = (\tilde{g}_1)_{11}\). After substitution into Eq. (18) one obtains for the four-terminal conductance

\[
\tilde{g} = (\tilde{g})_{11} .
\]

This expression includes the exactly known \(|1\rangle\) eigenfunction only. A straightforward calculation of \((\tilde{g})_{11}\) gives the result, equivalent to the two–terminal expression,

\[
\tilde{g} = \frac{e^2}{\pi \hbar} \sum_i |t_i|^2 .
\]

This is the expected result, since for a high barrier, two and four–terminal measurements should yield the same result. Note, that to this order \(E(l) = \text{const}\), which is consistent with strict charge neutrality. To evaluate corrections to the leading order, Eq. (24), one needs an explicit form of the eigenfunctions of the \(\tilde{g}_1\) operator, Eq. (14), which is in general a hard problem.

IV. BARRIER WITH DISORDERED LEADS

We consider now the case, where the leads attached to the barrier contain weak elastic disorder. We restrict ourselves to uncorrelated short–range scatterers, with mean free path \(l^d\) and mean free time \(\tau\) related by \(l^d = v_F \tau\). The leads are assumed to be much longer than the mean free path: \(L \gg l^d\). As a result, the momentum distribution function far from the barrier is expected to be isotropic (spherical). If a current passes through the wire, the distribution function is a shifted Fermi–sphere. Our immediate goal is to calculate the disorder–averaged non–local conductance of such a system (leads with the barrier), substitute it to Eq. (14) and solve the latter. To this end one has to calculate the average product of two Green’s functions, Eq. (3). The average single particle Green’s function of the leads with the barrier is given by Eq. (20) where, in the disordered case [15],

\[
G^R_{jj}(\epsilon; p) = \frac{1}{\epsilon - \epsilon_j - p^2/(2m) + i/(2\tau)} .
\]

Employing the continuity relation, one may express the conductance kernel as

\[
g(q, q', \omega) = -\frac{2e^2 i \omega}{qq'} \left[ 2\pi \delta_{q,q'} \sum_i \frac{1}{\pi v_i} + i \omega \Pi(q, q', \omega) \right] .
\]

The two terms in the square brackets represent the static and dynamic parts of the compressibility. In the diffusive approximation \((k_F l^d >> 1)\) the second term is given by the sequence of the bubble diagrams [13], depicted in Fig.

6
Therefore the quantity \( \Pi(q, q', \omega) \) may be found as a solution of the following integral equation

\[
\Pi(q, q') = \eta(q, q') + \frac{1}{\tau \sum_i (\pi v_i)^{-1}} \int dq'' \eta(q, q'') \Pi(q'', q') ,
\]

(54)

Here \( \tau \sum_i (\pi v_i)^{-1} \) is the inverse scattering amplitude and \( \eta(q, q', \omega) \) is the single bubble (see Fig. [3]).

\[
\eta(q, q', \omega) = \frac{1}{2\pi} \sum_{p, p', \nu} G^{R}_{jj}(\nu_+; p_+, p'_+') G^{A}_{jj}(\nu_-; p'_-, p_-) ,
\]

(55)

where \( p_{\pm} = p \pm q/2 \), \( p'_{\pm} = p' \pm q'/2 \) and the average Green’s functions are given by Eqs. (26) and (52). Performing momentum summations, one obtains in the long wavelength limit \( q, q' \ll k_F, \omega \ll \epsilon_F \)

\[
\eta(q, q', \omega) = \sum_i \frac{1}{\pi v_i} \frac{1 - i \omega \tau}{(1 - i \omega \tau)^2 + (v_i q')^2} 2\pi \delta_{q, q'} + \frac{1}{2\pi} \sum_i \frac{|r_i|^2 4\pi v^2 q' q^4}{(1 - i \omega \tau)^2 + (v_i q')^2} .
\]

(56)

In the clean limit, \( \omega \tau \gg 1 \) the sequence is dominated by the single bubble resulting in \( \Pi = \eta \). Under these conditions Eqs. (53) and (54) lead to the previous result given by Eqs. (24), (25). In the following we concentrate on the opposite, diffusive limit, \( v_F q \tau \ll 1, \omega \tau \ll 1 \).

\[
\text{FIG. 3} \text{ Diagrammatic representation of the dynamic compressibility } \Pi(q, q', \omega), \text{ denoted by the shaded bubble; the quantity } \eta(q, q', \omega) \text{ is denoted by the empty bubble. The full lines represent electron Green's function, Eqs. (24), (25) and the full dot represents a point-like elastic scatterer with the amplitude } (\tau \nu)^{-1}.
\]

In the absence of the barrier, \( r_i = 0 \), and Eq. (54) may be easily solved, resulting in

\[
\Pi(q, q', \omega) = \nu \frac{2\pi \delta_{q, q'}}{D q^2 - i \omega} \equiv \Pi_0(q, \omega) 2\pi \delta_{q, q'} ,
\]

(57)

where the single particle density of states (per spin), \( \nu \), and the diffusion constant, \( D \), are defined as

\[
\nu = \sum_i \frac{1}{\pi v_i} ; \quad D = \frac{\tau \sum_i v_i}{\sum_i v_i} .
\]

(58)

Substituting Eq. (57) into Eq. (53), one obtains the standard diffusive expression for the conductance of a disordered wire

\[
g_0(q, \omega) = -\frac{e^2}{\pi h} \sum_i \frac{2v_i i \omega \tau}{D q^2 - i \omega} .
\]

(59)

Next we shall look for \( g_b(q, q', \omega) \) and then for solutions of the self-consistent equation (14) in some particular cases.

### A. Equal velocity channels

In the case of \( N \) channels with the same velocity, \( v_i = v, i = 1 \ldots N \), the integral equation (14) has a separable kernel and thus may be solved exactly. After simple algebra one obtains for the barrier-induced part of the conductance

\[
g_b(q, q', \omega) = \frac{-e^2/(\pi h) \sum_j |r_j|^2}{\sum_j (|t_j|^2 - |r_j|^2) \sqrt{i \omega \tau} / D q^2 - i \omega} 2\nu_i i \omega \tau \frac{2\nu_i i \omega \tau}{D q^2 - i \omega} .
\]

(60)

This expression is valid for \( \omega \tau \ll 1 \); however, we have retained the frequency dependent term in the denominator since the condition \( \sqrt{i \omega \tau} \ll |t_j|^2 / |r_j|^2 \) is not assumed. Substituting Eqs. (53), (54) into the self-consistent equation for the induced electric field, Eq. (14), and solving the latter, one obtains, for \( \omega \ll \omega_p \),

\[
E^{ind}(q) = 2\nu_i \frac{\sum_j |r_j|^2}{\sum_j |t_j|^2} E^{ext} .
\]

(61)

Finally, employing Eqs. (18) and (59), yields the four-terminal conductance

\[
\tilde{g} = \frac{e^2 N}{\pi h} \frac{\sum_j |r_j|^2}{\sum_j |t_j|^2} .
\]

(62)

in agreement with a clean case, cf. Eq. (15). Although for the sake of compactness we wrote all intermediate expressions for the small frequency limit, \( \omega \tau \ll 1 \), one may actually perform all the calculations and obtain the final result, Eq. (62), for any \( \omega \tau \) (the only limitation is \( \omega \ll \omega_p \)).
B. Weakly reflecting channels

One may formally solve the integral equation (54) employing perturbation theory with the small parameter, \(|r_i|^2 \ll 1\). The first order correction to \(\Pi(q, q', \omega)\) for \(\omega \tau \ll 1\) is

\[
\delta \Pi(q, q', \omega) = \frac{\tau^2}{2\pi} \sum_i |r_i|^2 \frac{2v_i q}{Dq^2 - i\omega Dq^2} g_i(q') \, ,
\]

Employing Eq. (53), one obtains in the same order

\[
g_i(q, q', \omega) = \frac{\tau^2}{\pi \hbar} \sum_i |r_i|^2 \frac{2v_i \omega \tau}{Dq^2 - i\omega Dq^2} \, .
\]

Note that Eqs. (53), (54) are almost exact analogs of Eqs. (24), (25), where the ballistic propagators are replaced by diffusive ones. The important difference, however, is that the validity of Eqs. (24), (25) is not restricted to small reflection coefficients, \(r_i\). On the other hand, in the diffusive problem an electron can bounce between the barrier and impurities, and thus the exact conductance contains higher powers of \(|r_i|^2\). Eqs. (53) and (54) are only the first two terms in the infinite series.

We now substitute Eqs. (53), (54) into Eq. (14) and solve the integral equation. Since the kernel, Eq. (53), is separable, its solution is elementary. The result for the induced electric field in the zero frequency limit is

\[
E^{ind}(q) = \frac{\tau}{\sum_i 2v_i |r_i|^2} E^{ext} \, .
\]

Employing Eq. (53) and (54), one obtains the same result for the four–terminals conductance, \(\tilde{g}\), as in the clean case – Eq. (6). This statement is actually valid for any value of \(\omega \tau\), provided \(|r_i|^2 \ll 1\) and \(\omega \ll \bar{\omega}_p\).

C. Weakly transmitting channels

For a completely reflecting barrier, \(|r_i| = 1\), one may easily check the following equality

\[
\int \frac{dq'}{2\pi} \left[ 2\pi \delta_{q,q'} - \frac{1-i\omega \tau}{\nu \tau} \eta_{|r_i|=1}(q, q', \omega) \right] \frac{1}{q'} = 0 \, .
\]

With its help and employing Eqs. (53), (54) and (56) with \(|r_i|^2 = 1\), one may prove that

\[
\int \frac{dq'}{2\pi} \eta_{|r_i|=1}(q, q', \omega) = 0
\]

for any \(q\) and \(\omega\). This identity simply reflects the fact that an electric field localized under a completely reflecting barrier cannot induce any current. Thus, to solve the self–consistent field equation for \(|t_i| \ll 1\) one has to invert an operator having one almost vanishing eigenvalue. Following the method described in section III D, one obtains for the the four–terminal conductance

\[
\tilde{g} = (\tilde{g})_{11} \, ,
\]

where \((\tilde{g})_{11} = (2\pi)^{-2} \int dq dq' g(q, q')\). Calculation of this matrix element is not as straightforward as in the clean case. To compute it one has first to solve the integral equation (54). Simple algebra reduces that problem to the inversion of the operator which differs by a small factor \((\sim |t_i|^2)\) from the one written in the square brackets on the left hand side of Eq. (53). According to Eq. (64) the latter has exactly zero eigenvalue, with the corresponding eigenvector \(\propto |q'|.\) One has to employ once again the method described in section III D to invert an operator with one small eigenvalue. As a result of this procedure one obtains Eq. (61), the same as in clean case.

V. BARRIER IN A BALLISTIC CONSTRICITION

We now consider a barrier embedded into a ballistic adiabatic constriction between the two wide reservoirs (see Fig. 11). We shall assume that the length of the constriction, \(L\), satisfies the conditions \(L < v_F/\omega\) and \(L < t_L^d\). This is the geometry considered by BILP [10] and others (see Ref. 3 and references therein). Since the cross–sectional area of the constriction, \(S(z)\), is a smooth function of \(z\), one may write the wave functions in the adiabatic approximation (10)

\[
\Psi(x, y, z) = \sum_i \xi_i(x, y; S(z)) \psi_i(z) \, ,
\]

where \(\xi_i(x, y; S(z))\) is a transverse wave function of the \(i\)-th channel with an eigenenergy \(\epsilon_i(S(z)) \equiv \epsilon_i(z)\). The longitudinal function, \(\psi_i(z)\), satisfies a one dimensional Shrödinger equation

\[
[-(2m)^{-1} \partial_z^2 + \epsilon_i(z) + V_B(z)] \psi_i(z) = \epsilon \psi_i(z) \, ,
\]

where \(V_B(z)\) is the localized tunneling barrier potential. Thus the problem is reduced to a one dimensional effective tunneling problem described by Eq. (71). (See also Fig. 3.) If the constriction is narrow enough it closes most of the channels, since in these channels \(\epsilon_i(z) > \epsilon_F\) for small enough \(z\). Hereafter we shall assume that this is the case. Even in the open channels there is a partial reflection of electrons due to the presence of the barrier, which is characterized by the barrier’s reflection coefficients, \(r_i\). For \(z \rightarrow z'\) outside the barrier region, by employing Eq. (3) and expressing the Green’s functions in the WKB approximation, one finds for the conductance kernel (cf. with Eq. (23) )
\[ g(z, z', \omega) = \frac{e^2}{\pi \hbar} \sum_i \left[ \exp \left\{ i \omega \left\{ \int_0^z \frac{dz'}{v_i(z')} \right\} \right\} - |r_i|^2 \exp \left\{ i \omega \left\{ \int_0^z \frac{dz'}{v_i(z')} + \int_z^{z'} \frac{dz}{v_i(z)} \right\} \right\} \right], \tag{71} \]

where

\[ v_i(z) = \sqrt{\frac{2(\epsilon_F - \epsilon_i(z))}{m}}, \tag{72} \]

and the summation runs over the open channels only. This expression will be used to find the four–terminal conductance of the barrier, measured by probing the voltage drop across it. (In very narrow energy intervals, when one of the channels is extremely near its transmission threshold it may not be tractable by WKB.)

Let us first imagine solving the problem on the large length scale, \(|z| \gg L\), where the entire constriction may be treated as a single localized scatterer. The results of previous sections are directly applicable to such a problem (see e.g. Sec. III B). One finds that at sufficiently low frequencies the induced electric field, \(E^{\text{ind}}\), is confined to the constriction region. A line integral over this field is a total voltage drop across the constriction

\[ \int dz E^{\text{ind}}(z, \omega) = V_0, \tag{73} \]

where the integral effectively runs over the region between the two points A and B (see Fig. 2a) located outside the constriction at a distance which is large compared with a bulk screening length (but still much smaller than \(L\)). Since only a few channels are open, the current is determined by Eq. (36) and given by

\[ I_0 \approx V_0 \frac{e^2}{\pi \hbar} \sum_i |t_i|^2. \tag{74} \]

There are corrections to this expression of the order of a ratio of number of open channels to a total number of channels \([13]\). Only the open channels contribute to the sum on the right hand side; the transmission coefficients of the open channels are those of the localized barrier.

To determine the voltage drop across the barrier itself one must determine the structure of \(E^{\text{ind}}(z, \omega)\) on a small length scale, which turns out to be the screening length, \(\kappa^{-1}(\ll L)\). The induced field \(E^{\text{ind}}\) is comprised by the two components, \(E^{\text{ind}} = E'^{\text{ind}} + E''^{\text{ind}}\), created by charge densities induced at spatially well separated locations. \(E'^{\text{ind}}\) is created by charges accumulated in the regions of the narrowing of the constriction. This field component is practically constant between points C an D in the constriction (see Fig. 2a). Its contribution to the voltage drop measured across the barrier is small in the parameter \(z_v/L \ll 1\), where \(2z_v\) is a distance between...
the voltage probes. The other component, $E_{\text{ind}}^{(v)}$, is induced by charges accumulated near the localized barrier. It is sharply peaked in the barrier region and is primarily responsible for the voltage measured across the barrier. The induced field, $E_{\text{ind}}(z, \omega)$, inside the constriction region is given by a solution of the self-consistent equation (74), with the conductance kernel Eq. (77). For low frequencies, $(\omega L/v_F) \ll 1$, one may expand the exponents in Eq. (74) in terms of this small parameter and obtain to leading order

$$
\frac{\hbar S(z)}{4e^2} \partial_z E_{\text{ind}}^{(v)}(z) - \left(\sum_i \frac{1}{v_i(z)}\right) \int dz' \text{sign}(z - z') E_{\text{ind}}^{(v)}(z') = - \left(\sum_i |r_i|^2 v_i(z)\right) \text{sign}(z) V_0, \tag{75}
$$

where Eq. (73) was employed on the right hand side. In the region where the effective potential is “flat” (see Fig. 2a) with coordinates $E \approx E_0$ and where $\sum v_i(z) \approx v_i = \text{const}$ and $S(z) \approx S = \text{const}$, the solution of Eq. (72) is

$$
E_{\text{ind}}^{(v)}(z) = \text{const} + V_0 \sum_i \frac{|r_i|^2 v_i^{-1}}{\sum v_i^{-1}} \exp\left[-|z|/2\kappa^{-1}\right], \tag{76}
$$

where the screening length is

$$
\kappa^{-1} = \sqrt{\frac{\hbar S}{8e^2 \sum v_i^{-1}}} = \frac{v_F}{\omega_p}. \tag{77}
$$

The constant on the right hand side of Eq. (76) represents $E_{\text{ind}}^{(v)}$, the component created by the distant charges of the constriction’s narrowing. Its value may be estimated from Eqs. (73) and (74) as

$$
E_{\text{ind}}^{(v)} \approx \text{const} \approx \frac{V_0}{L} \sum_i \frac{|t_i|^2 v_i^{-1}}{\sum v_i^{-1}}. \tag{78}
$$

The voltage drop across the barrier, measured by the probes located just outside the barrier at points E and F (see Fig. 2a) with coordinates $z = \pm z_0$, is given by $V = \int_{-z_0}^{z_0} dz E_{\text{ind}}^{(v)}$. Employing Eq. (78), one finds

$$
V = V_0 \left[\sum_i \frac{|t_i|^2 v_i^{-1}}{\sum v_i^{-1}} + O \left(\frac{z_0}{L}, e^{-z_0/\kappa}\right)\right]. \tag{79}
$$

Therefore in the asymptotic regime $\kappa^{-1} \ll z_0 \ll L$, one finds for the four–terminal conductance $\tilde{g} = I_0/V$,

$$
\tilde{g} = \frac{e^2}{\pi \hbar} \frac{\sum v_i^{-1} \sum_i |t_i|^2}{\sum v_i^{-1} |r_i|^2}. \tag{80}
$$

in agreement with BILP [10]. We therefore confirm their result for the barrier embedded into the constriction attached to the wide reservoirs if the inequalities $\kappa^{-1} \ll L < v_i/\omega$, $\delta \varepsilon$ are satisfied.

Eq. (80) predicts a sudden drop of the conductance if one (or several) $v_{i0} \to 0$. The physical reason for such behavior is an enhanced screening by the channel(s) with the small velocity and hence a large density of states. We stress that although a non–monotonic dependence of the conductance on the chemical potential may indeed take place, there is no actual discontinuity when a channel opens. The reason is that the above calculations resulting in Eq. (81) lose their validity if $v_{i0} \ll \hbar/(mL)$ [14] (or $v_{i0} \ll L\omega$). The two key approximations simultaneously go wrong in this case: (i) the long–wavelength limit for the non–local conductance, which neglects Friedel oscillations at $q = 2k_F$ and (ii) the WKB approximation, which neglects reflection by $\epsilon_{i0}(z)$, the effective barrier due to the constriction. If these two effects are properly taken into account the continuity of conductance is restored, however its value for $0 < v_{i0} < \hbar/(mL)$ depends both on the position of the probes and the specific shape of the constriction.

Also, at the beginning of this Section, we assumed that if a channel is “closed”, i.e. $v_i \to iv_i$, the corresponding current vanishes. In fact this requires negligible tunneling or $|v_{i0}| > \hbar/(mL)$. Thus Eq. (80) also fails just below an energy threshold.

Finally we address the frequency dependence of the four–terminal conductance. To this end we expand Eq. (71) to the next order in $\omega L/v_F$ and substitute into the self-consistent equation (72). This gives the frequency dependent correction to the induced electric field

$$
\delta E_{\text{ind}}^{(v)}(\omega, z) = i\omega \frac{\sum_i |t_i|^2 v_i^{-2}}{\sum v_i^{-1}} \exp\left[-|z|/2\kappa^{-1}\right] \int dz |z| E_{\text{ind}}^{(v)}(z), \tag{81}
$$

where $E_{\text{ind}}^{(v)}(z)$ is the zero frequency field given by Eqs. (76), (78). According to Eqs. (83), (84) the relation between the divergence–less current and the total voltage drop, Eq. (74), should be modified to include the classical capacitance of the constriction, $C_0$. The resulting four–terminal conductance acquires a capacity–like frequency dependent correction of the following form

$$
\delta g(\omega) = -i\omega (C_0 + C), \tag{82}
$$

where

$$
C = \frac{\sum_i |t_i|^2 v_i^{-2}}{\sum v_i^{-1}} \int dz |z| E_{\text{ind}}^{(v)}(z). \tag{83}
$$

The precise value of the last factor in this expression depends on $E_{\text{ind}}^{(v)}(z)$ on the scale $z \sim L$ and hence on the shape of the constriction. It may be estimated using Eqs. (70), (78). Provided that $(L\kappa)^{-1} < |t_i|^2$, which is
valid for all barriers except those with almost complete reflection, one finds for the additional capacitance

$$C = \gamma L \frac{e^2}{4\pi\hbar} \sum_i |t_i|^2 \frac{\sum_i |r_i|^2 v_i^{-2} \sum_i |t_i|^2 v_i^{-1}}{\left( \sum_i |r_i|^2 v_i^{-1} \right)^2}, \quad (83)$$

where $\gamma = O(1)$. In the opposite limit $|t_i|^2 \ll (L\kappa)^{-1}$ one obtains

$$C = \kappa^{-1} \frac{e^2}{\pi\hbar} \sum_i |t_i|^2 \frac{\sum_i |r_i|^2 v_i^{-2} \sum_i v_i^{-1}}{\left( \sum_i |r_i|^2 v_i^{-1} \right)^2}. \quad (84)$$

We stress the peculiar dependence of the additional capacitance on channel velocities and in particular a sharp rise at the point of a new channel opening. (As was mentioned above, there is no actual discontinuity, since Eqs. (83), (84) are not applicable once $v_0 < h/(mL)$.)

**VI. DISCUSSION OF THE RESULTS**

We have presented an approach to the Landauer four-terminal tunneling conductance, which does not assume the presence of two reservoirs which maintain different chemical potentials. Our theory is based on the solution of a self-consistent set of equations involving the non-local conductance, $g(z, z', \omega)$, the tunneling current, induced charge density and electric field.

There are three different regimes (see Fig. 3), defined by the length of the leads, $L$, the frequency, $\omega$, and the rate of elastic scattering, $\tau^{-1}$ (or the elastic mean free path, $l^e = v_F\tau$). The first “high” frequency regime denoted by I in Fig. 3 is defined by the conditions $\omega > \max\{\tau^{-1}, v_F/L\}$. We have treated it in section III. The second denoted by II is that of “disordered” leads. It is defined by the conditions $L > l^e; \omega < \tau^{-1}$ and was treated in section IV. Finally the third one denoted by III and defined by $L < l^e; \omega < v_F/L$ is the regime of “ballistic” motion. It was considered in section V.

![Fig. 3 Various physical regimes; area I corresponds to a “high” frequency regime, II to a “disordered leads” and III to a “ballistic constriction” geometry. Here $L$ is the length of the leads attached to the barrier, $\omega$ – the frequency of the driving field, $l^e$ and $\tau^{-1}$ are elastic mean free path and scattering rate correspondingly.](image)

A few remarks about the relationship of our work in the ballistic regime with a barrier, in the geometry of Fig. 2a, and the work of BILP [10]: First, the final result (our Eq. (80)) is the same in both treatments. However: in BILP the occupation of electronic states in the presence of a current is different in the regions to the left and to the right of the barrier. In our work the presence of a current does not modify the occupation of the electronic states (which extend from the left through the barrier to the right), but does modify their wave-functions – in accordance with standard Kubo transport theory. BILP’s chemical potential differences $\mu_1 - \mu_2$ and $\mu_A - \mu_B$ are respectively equal to the electrostatic potential differences $V_0$ and $V$ of Fig. 2c. We have not been able to derive the very simple picture of BILP from our considerations.

In the regimes I and II we have obtained results which are qualitatively different from ballistic regime III. The most striking difference occurs for the weakly reflecting barrier (cf. Eqs. (I) and (II)). In particular, for long and/or disordered leads the conductance does not exhibit discontinuities when the velocity of one of the channels vanishes. The physical reason for these differences lies in the difference of momentum distribution functions in the diffuse and high frequency regimes compared to that in the ballistic regime. In the opposite limit of weak transmission one finds, to leading order, the same result for all three regimes, which is identical with Landauer’s
two–terminal expression, Eq. (1). We believe that the predicted qualitatively different dependence on the channel velocities (and hence on the chemical potential) for diffusive and ballistic leads may be checked experimentally.

We also predict a peculiar dependence of the capacitance on the chemical potential, Eqs. (83), (84), which, we hope, may also be checked experimentally.

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[18] R. Landauer, J. Phys. 1, 8099 (1989);