Travelling Waves in Monostable and Bistable Stochastic Partial Differential Equations

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Abstract

In this review, we provide a concise summary of several important mathematical results for stochastic travelling waves generated by monostable and bistable reaction-diffusion stochastic partial differential equations (SPDEs). In particular, this survey is intended for readers new to the topic but who have some knowledge in any sub-field of differential equations. The aim is to bridge different backgrounds and to identify the most important common principles and techniques currently applied to the analysis of stochastic travelling wave problems. Monostable and bistable reaction terms are found in prototypical dissipative travelling wave problems, which have already guided the deterministic theory. Hence, we expect that these terms are also crucial in the stochastic setting to understand effects and to develop techniques. The survey also provides an outlook, suggests some open problems, and points out connections to results in physics as well as to other active research directions in SPDEs.

Keywords: travelling wave, reaction-diffusion equation, stochastic partial differential equation, monostable nonlinearity, bistable nonlinearity, stability, wave speed.

1 Introduction

We consider stochastic partial differential equations (SPDEs) of the form

\[ du = [\partial^2_x u + f(u)] \, dt + g(u) \, dW, \quad u(0, x) =: u_0(x), \]

where \( u = u(t, x) \in \mathbb{R}, x \in \mathbb{R}, t \in [0, \infty) \) and \( f \) is a given nonlinearity; the stochastic process \( W = W(t, x) \), the map \( g \) as well as the solution concept(s) for (1) will be specified precisely in Section 3. The SPDE (1) can also be written as

\[ \partial_t u = \partial^2_x u + f(u) + g(u)\xi, \quad \xi = \xi(t, x), \quad \partial_t W = \xi, \quad u(0, x) =: u_0(x). \]

We focus on the classical quadratic and cubic nonlinearities given by

\[ f(u) = f_2(u) = u(1 - u) \quad \text{and} \quad f_3(u) = u(1 - u)(u - a), \quad a \in (0, 1/2). \]

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For \( g(u) \equiv 0 \), the equation (1) becomes a partial differential equation (PDE) of reaction-diffusion type

\[
\partial_t v = \partial_x^2 v + f(v), \quad v(0, x) = v_0(x)
\]

where we use \( v = v(t, x) \) to emphasize that we work with a deterministic PDE. Particular cases of equation (3) have been studied intensively for almost a century. Observe that for \( f = f_2 \), there are two homogeneous steady states \( v_* = 0 \) and \( v_* = 1 \), while in the cubic case \( f = f_3 \) there is the additional steady state \( v_* = a \). Looking at perturbations via \( v = v_* + \varepsilon V \), one obtains to leading order in \( \varepsilon \), the linearized system

\[
\partial_t V = [\partial_x^2 + D_v f(v_*)]V, \quad V = V(t, x), \quad V(0, x) = V_0(x).
\]

Solving (4), e.g., via Fourier transform, one checks that for \( f = f_2 \) the state \( v_* \equiv 0 \) is unstable, while \( v_* \equiv 1 \) is linearly stable. Hence, one refers to \( f = f_2 \) as the monostable case. For \( f = f_3 \), the steady states \( v_* = 0 \) and \( v_* = 1 \) are linearly stable, while \( v_* = a \) is unstable; hence, this case is called bistable. The monostable PDE (3) is also referred to as Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP) equation [55, 93]. The bistable case is called Nagumo equation in (neuro-)biology [131], Allen-Cahn equation in materials science [3], \( \phi^4 \)-model in quantum field theory [70], Schlögl model in chemistry [159], and real Ginzburg-Landau equation [62] in the context of normal forms or amplitude/modulation equations [103, 161].

For the PDE (3) a very important class of non-steady solutions are travelling waves, i.e., solutions of the form

\[
v(t, x) = v(x - st) = v(\eta), \quad \eta := x - st,
\]

where \( s \in \mathbb{R} \) is the wave speed. If \( s > 0 \) (resp. \( s < 0 \)) then the wave is right-moving (resp. left-moving), while for \( s = 0 \) we have a standing wave. Existence and stability of travelling waves for the PDE (3) are well-studied. We recall certain parts of these results in Section 2. For the SPDE (1) a lot less is known rigorously about travelling waves. However, considerable insight has been gained already from the perspective of physical intuition, asymptotic approximations, direct numerical simulations, and complete proofs for certain particular cases/aspects. More generally, many open problems remain in theory of pattern formation for SPDEs. In fact, the area has been recognized for quite some time as one of the most challenging and fundamental research frontiers.

We briefly mention there are two important directions, which we do not detail here: (I) travelling waves in “random media”, or for random partial differential equations (RPDEs), i.e., when random coefficients are introduced to (3) as discussed e.g. in [181], and (II) travelling waves for discrete/microscopic versions of (1). We do not cover the RPDE case at all, yet we are going to comment on the relation to microscopic models at relevant places. The reason is that the form of \( g \) in (1) is often derived from microscopic considerations. We refer the reader for a detailed review from a physics perspective on discrete models to [137], for micro-macro model limits to [96], and to references therein.

Having covered the relevant deterministic PDE background for travelling waves in Section 2, we introduce some SPDE basics in Section 3. Then Section 4 contains an overview of current results on existence, speed, and stability of stochastic travelling waves for reaction-diffusion SPDEs.
with monostable ad bistable nonlinearities. We conclude in Section 5 with a brief summary, indicate connections to adjacent areas, and propose several directions for future work.

2 Deterministic Waves

First, let us remark that the existence and regularity theory of the PDE (3) is well-studied for a wide class of nonlinearities. For \( f = f_3 \), the highest-order polynomial term \(-v^3\) provides dissipativity \([76, 168]\) leading to global-in-time solutions in regular function spaces. Due to smoothing \([76, \text{Sec.3.3, Ex.3.7}]\), one can work with classical solutions for \( t > 0 \) if the initial condition is taken sufficiently regular, which we are going to assume from now on, say taking continuous and bounded data \( v_0 \in C^0(\mathbb{R}, \mathbb{R}) \). For \( f = f_2 \), an additional restriction of the initial condition to \( v_0 \geq 0 \) leads to the same conclusion of global-in-time existence and regularity. The condition \( v_0 \geq 0 \) is often natural for modelling purposes of the FKPP equation and also implies that \( v(t, x) \geq 0 \) for all \( t \geq 0 \) by using the maximum principle \([52]\).

There are three main classes of travelling waves we are going to consider: fronts, pulses, and wave trains. If \( v_{*,l} \) and \( v_{*,r} \) are steady states for (3), a travelling front from \( v_{*,l} \) to \( v_{*,r} \) is a solution such that

\[
\lim_{\eta \to -\infty} v(\eta) = v_{*,l} \quad \text{and} \quad \lim_{\eta \to +\infty} v(\eta) = v_{*,r}.
\]  
(6)

We also refer to \( v_{*,l} \) and \( v_{*,r} \) as (left and right) endstates of the wave. A travelling pulse to a single steady state (or endstate) \( v_* \) satisfies (6) with \( v_{*,l} = v_* = v_{*,r} \). A travelling wave train is a spatially periodic pattern \( v(\eta + \eta_0) = v(\eta) \) for some fixed \( \eta_0 > 0 \). Plugging in the travelling wave ansatz (5) into the PDE (3) and using the chain rule yields

\[
-s \frac{d^2 v}{d\eta^2} = \frac{d^2 v}{d\eta^2} + f(v),
\]  
(7)

which is a second-order ordinary differential equation (ODE). Re-writing this ODE via \( dv/d\eta =: w \), we get a planar first-order system

\[
\frac{dv}{d\eta} = \dot{v} = w, \quad \frac{dw}{d\eta} = \dot{w} = -sw - f(v).
\]  
(8)

The equilibrium points of (8) lie on the line \( \{ w = 0 \} \) with \( v_* \)-values \( v_* = 0, 1 \) and \( v_* = 0, a, 1 \) for the quadratic and cubic nonlinearities. The condition (8) is the defining property of a heteroclinic orbit in the system (8) from \((v_{l,*}, 0)\) to \((v_{r,*}, 0)\). Hence, travelling fronts correspond to heteroclinic orbits, travelling pulses to homoclinic orbits, and travelling wave trains to periodic orbits of the ODE (8).

For \( f = f_2 \), one checks that \((1, 0)\) is a saddle point, while \((0, 0)\) is a stable node for \( s \leq -2 \), an unstable node for \( s \geq 2 \), a spiral sink for \( s \in (-2, 0) \), a spiral source for \( s \in (0, 2) \), and a center for \( s = 0 \). Let us consider only the case \( s \geq 0 \) as the case \( s < 0 \) can be dealt with using the symmetry

\[
(s, t, v, w) \mapsto (-s, -t, v, -w)
\]  
(9)

of (8). Although it is possible for \( s \in [0, 2] \) to construct periodic, homoclinic \((s = 0)\) and heteroclinic \((s \in (0, 2))\) orbits for (8), we see, due to the complex eigenvalues near \((0, 0)\), that these orbits have negative \( v \)-values for certain \( \eta \). Hence, these solutions cannot be obtained for
the PDE if we adhere to the modelling constraint \( v(0, x) = v_0(x) \geq 0 \) in the monostable case. For \( s \geq 2 \), one may check using (8) that there exists for each fixed \( s \in [2, \infty) \) a unique heteroclinic orbit \( \gamma_s(\eta) \) with
\[
\lim_{\eta \to -\infty} \gamma_s(\eta) = (1, 0) \quad \text{and} \quad \lim_{\eta \to \infty} \gamma_s(\eta) = (0, 0).
\]
This family of orbits represents travelling front solutions in which the homogeneous state \( v_* = 1 \) of the original PDE invades the homogeneous state \( v_* = 0 \). The fronts are monotone functions of \( \eta \). One also refers to this travelling front scenario as propagation into an unstable stable case. One also refers to this travelling front scenario as a travelling front solution, which turns out to be a monotone function invading the PDE (3). Of course, we may ask, which of the family of solutions we may actually observe if we consider an initial condition for the PDE (3) with \( f = f_2 \) and \( v_0 \geq 0 \). This is related to the stability question to be reviewed below.

Before discussing stability, let us also consider existence for \( f = f_3 \). For \( a \in (0, 1/2) \), one finds that \((v_{r,s}, 0) = (1, 0)\) and \((v_{l,s}, 0) = (0, 0)\) are saddle points, while \((v_{m,s}, 0) = (a, 0)\) is locally stable for \( s > 0 \), unstable for \( s < 0 \), and a center for \( s = 0 \). For \( s = 0 \), one uses the Hamiltonian structure of (8) to show the existence of a homoclinic orbit representing a (standing) pulse solution. For \( s \neq 0 \), one can again restrict by the symmetry (9) the range of wave speeds. Let us now take \( s < 0 \). One can then prove (10) that there exists for each \( a \in (0, 1/2) \) a unique \( s_* = s_*(a) < 0 \), where (8) has a heteroclinic orbit \( \gamma_a(\eta) \) such that
\[
\lim_{\eta \to -\infty} \gamma_a(\eta) = (0, 0) \quad \text{and} \quad \lim_{\eta \to \infty} \gamma_a(\eta) = (1, 0).
\]
The heteroclinic orbit corresponds a front solution, which turns out to be a monotone function of \( \eta \) as a solution of the PDE. The wave speed \( s_*(a) \) can also be expressed via variational principles (12) (35) for quite general bistable scalar problems. In fact, for \( f = f_3 \) one may even write explicit formulas (51) (154) yielding
\[
\Phi(\eta) = \frac{1}{2} \left[ 1 + \tanh \left( \frac{\sqrt{2}}{2} \eta \right) \right], \quad s_*(a) = \sqrt{2} \left( a - \frac{1}{2} \right), \quad a \in (0, 1/2).
\]
Yet, it is rarely a good idea if one wants to generalize arguments to rely on these explicit formulas. Note that one can also consider the case \( a \in (1/2, 1) \) using a further symmetry, in which case the wave would be moving right instead of left. Clearly, although \( v_* = 0, 1 \) are locally stable for the PDE (3) for \( a \in (0, 1/2) \cup (1/2, 1) \), one stable state invades the other. This is easy to see understand using the gradient flow formulation
\[
\partial_t v = -\nabla_{L^2(\mathbb{R})} \mathcal{F}(u), \quad \mathcal{F}(u) := \int_{\mathbb{R}} -F(u) + \frac{1}{2} |\partial_x u|^2 \, dx
\]
where \( F' = f \) is an anti-derivative of \( f \). If \( a \in (0, 1/2) \), the state \( v_* = 1 \) invades \( v_* = 0 \) as it also is the unique global minimum of the potential \( \mathcal{F} \), while the situation is reversed for \( a \in (1/2, 1) \). The balanced potential case \( a = 1/2 \) is special and leads to metastability if the diffusion constant in front of the Laplacian is small (33) (103). We shall not cover this metastable case here but see Section (3) for further references.

The next natural question regarding the PDE (3) is to consider stability of travelling waves. Obviously, the non-trivial waves cannot be globally stable in any reasonable sense since we already have at least one locally stable homogeneous steady state. Yet, local stability of waves, potentially with quite large basins of attraction, is possible. The first step is to consider linear stability. Let
Φ = Φ(η) be a travelling wave, so that using the perturbation ansatz \( v(t, x) = Φ(η) + \varepsilon V(t, η) \) we get the linearized problem

\[
\partial_t V = \partial^2_η V + s\partial_η V + f'(Φ(η))V =: \mathcal{L}V.
\] (11)

We recall that the spectrum \( σ(\mathcal{L}) \) of a linear operator \( \mathcal{L} : \mathcal{X} → \mathcal{Y} \), where \( \mathcal{X}, \mathcal{Y} \) are suitable Banach or Hilbert spaces, consists of all \( λ ∈ \mathbb{C} \) such that \( (\lambda \text{Id} − \mathcal{L})^{-1} \) has no bounded inverse. We can decompose the spectrum \( σ(\mathcal{L}) = σ_{\text{pt}}(\mathcal{L}) \cup σ_{\text{ess}}(\mathcal{L}) \), where the essential spectrum \( σ_{\text{ess}}(\mathcal{L}) \) denotes all \( λ ∈ \mathbb{C} \) such that \( L : \mathcal{X} → \mathcal{Y} \) is not a Fredholm operator \(^2\). The point spectrum \( σ_{\text{pt}}(\mathcal{L}) = σ(\mathcal{L}) \setminus σ_{\text{ess}}(\mathcal{L}) \) consists of all eigenvalues \( λ \) of finite multiplicity solving the eigenvalue problem

\[
L V = λV, \quad λ ∈ \mathbb{C}, \ V ∈ \mathcal{X},
\] (12)

For our setting, one observes that \( \mathcal{L} \) is a special case of a Sturm-Liouville differential operator

\[
\mathcal{L} = \partial^2_η + a_1(η)\partial_η + a_0(η)
\] (13)

with \( a_1(η) ≡ s \) and \( a_0(η) = f'(Φ(η)) \), which are both coefficients which decay exponentially to asymptotic values if \( η → ±∞ \). Furthermore, we note that the travelling wave \( ∂_η Φ \) is always an eigenfunction with eigenvalue \( λ = 0 \) since we may use that a travelling wave solves the PDE and differentiate to obtain

\[
0 = \partial^2_η Φ + s\partial_η Φ + f(Φ) \quad ⇒ \quad 0 = \partial^2_η(∂_η Φ) + s\partial_η(∂_η Φ) + f'(Φ)∂_η Φ.
\] (14)

The direction associated to \( ∂_η Φ \) corresponds to the neutral direction induced by translation symmetry, i.e., if \( Φ(x − st) \) is a travelling wave then so is \( Φ(x − st + η_0) \) for \( η_0 ∈ \mathbb{R} \) fixed. The neutral mode does not contribute to the linear stability analysis for the PDE; it is also referred to as the Goldstone mode in the physics literature. In particular, a typical stability result is of the form

\[
\lim_{t → +∞} \| v(t, \cdot) − φ(\cdot − st + η_0) \| = 0
\] (15)

where \( η_0 > 0 \) is a constant, \( \| \cdot \| \) is norm on the (spatial) function space, and the initial condition \( v_0 \) of the solution \( v \) is taken from a certain class of data within a basin of attraction of the wave. So initial conditions just converge to some translate of the wave. The first standard setting is to consider \( \mathcal{X} = H^2(\mathbb{R}) \) and \( \mathcal{Y} = L^2(\mathbb{R}) \). In fact, the essential spectrum for our case can be inferred from the asymptotic operators linearized operators

\[
\mathcal{L}_± := \partial^2_η + s\partial_η + f'(Φ(±∞)).
\] (16)

To see this in a bit more detail, consider the linear problems

\[
\partial_t V = L_± V
\] (17)

and use the ansatz \( V(t, η) = e^{ikη − λt} \) (where \( i := \sqrt{-1} \)), which gives the dispersion relations

\[
d_± (ik, λ) := λ + (ik)^2 + cik + f'(Φ(±∞)) = 0.
\] (18)

\(^2\)There are several slightly different definitions of the essential spectrum of an operator, so one should check carefully, which definition each author uses.
A *dispersion relation* connects the temporal decay $\lambda$ to the spatial wave number $k$. The key objects enclosing the essential spectrum are the parabolic curves

$$\sigma_\pm := \{ \lambda \in \mathbb{C} | d_\pm(ik, \lambda) = 0, k \in \mathbb{R} \} = \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) = f'(\Phi(\pm \infty)) - (\text{Im}\lambda/s)^2 \}. \quad (19)$$

but this is non-trivial to prove [83, 152]. Although up to this point, the (linear) stability problem [152] can be set up in the same way for $f = f_2$ and $f = f_3$, the actual spectra turn out to differ substantially.

For $f = f_3$, one can prove that $\sigma_{\text{ess}}(\mathcal{L})$ is properly contained in the left-half of the complex plane for the pulse and front solutions if $s \leq 0$. Indeed, recall that $f = f_3$, we have $f'(\Phi(\pm \infty)) < 0$ as both endstates are stable. Therefore, the curves [19] both lie in the left half of the complex plane. So spectral stability is completely determined by the eigenvalue problem (12). For the travelling front and the travelling pulse, it is easy to check that the point spectrum consists of finitely many eigenvalues just using standard Sturm-Liouville theory [83]. The standing pulse

$$\text{Re}(\lambda) \leq 0$$

for the pulse in the bistable case, i.e., there exists a fixed constant $C_\ast = C_\ast(a) > 0$ such that if $\lambda \in \sigma(\mathcal{L}) \setminus \{ 0 \}$ then $\text{Re}(\lambda) < -C_\ast$. From this spectral gap, and upon factoring out the translation symmetry direction, one can establish *nonlinear orbital asymptotic stability* of the travelling pulse [83, 76].

For $f = f_2$, the stability question is very different. Clearly, the essential spectrum $\sigma_{\text{ess}}(\mathcal{L})$ now also contains parts in the right half of the complex plane as for the unstable endstate we have

$$\sigma_- = \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) = 1 - (\text{Im}\lambda/s)^2 \}. \quad (20)$$

However, one may use suitable exponentially weighted spaces $\mathcal{X}_w$ and $\mathcal{Y}_w$ to shift $\sigma_{\text{ess}}(\mathcal{L})$ for $\mathcal{L} : \mathcal{X}_w \rightarrow \mathcal{Y}_w$ to a half-plane $\{ \lambda \in \mathbb{C} : \text{Re}(\lambda) < -C \}$ for some $C > 0$. As an example, we can take $\mathcal{X}_w = H^2(\mathbb{R}; w)$ with norm

$$\| V \|_{H^2(\mathbb{R}; w)} = \| V(\cdot)(1 + e^{-c\cdot}) \|_{H^2(\mathbb{R})}, \quad V = V(x),$$

and we take $w(x) = e^{-cx}$ for some suitable constant $c > 0$. Similarly, one can define $\mathcal{Y}_w = L^2(\mathbb{R}; w)$. Hence, stability depends again on the eigenvalues problem $\mathcal{L}V = \lambda V$. One may prove that if the wave front $\Phi$ decays faster than $e^{-cx}$ as $x \to \infty$, then it is asymptotically stable (up to shifts) in the norm with weight $w(x) = e^{-cx}$. Furthermore, for $s > 0$ and $v_0 \geq 0$, only the travelling front with minimal speed $s = 2$ turns out to be locally stable for all initial data, which have at least exponential tails

$$| v(0, x) - \Phi(x + \eta_0) | = \mathcal{O}(e^{-x}) \quad \text{as } x \to \pm \infty \quad (21)$$

However, one may select other initial data to achieve different/faster speeds. If the initial data are even compactly supported, then one can prove that there is a left-moving and a right-moving front, each with minimal wave speed $|s| = 2$. In particular, the minimal wave speed is selected dynamically.

The front solution for the monostable case is also called a *pulled front* as its wave speed corresponds precisely to the linear spreading speed of small perturbations near the unstable steady state. More precisely, the speed of the front can be calculated from the linearization around the unstable state as follows: consider the general linearization around a steady state $v_\ast$ is given by [4], which can be written as

$$\partial_t V = \partial_x^2 V + Df(v_\ast)V =: \partial_x^2 V + a_\ast V. \quad (22)$$
Let $x_\kappa(t)$ be a level curve such that $V(t, x_\kappa(t)) = \kappa \in [0, 1]$ and define a linear spreading speed by

$$s_* := \lim_{t \to +\infty} \frac{dx_\kappa}{dt}.$$ 

Furthermore, applying the Fourier transform

$$\hat{V}(t, k) := \int_{\mathbb{R}} e^{ikx} V(t, x) \, dx$$

to (22) and substituting the ansatz $\hat{V}(t, k) = \hat{V}(0, k) \exp(-i\omega_*(k)t)$ into the resulting equation, yields the dispersion relation

$$\omega_*(k) = -i(k^2 - 1)$$

relating the (complex) frequency $\omega_*$ to the wave number $k$. A front is pulled [177] if its speed coincides with a linear spreading speed, which can just be calculated from the dispersion relation by the relations

$$s_* = \frac{d\omega_*}{dk}(k_*) \quad \text{and} \quad s_* = \frac{\text{Im}[\omega_*(k_*)]}{\text{Im}[k_*]},$$

where $k_* \in \mathbb{C}$ is a constant also called the linear spreading point. One easily checks that for the monostable case linearization at the unstable state we have $a_* = 1$ and $\omega_*(k) = i(1 - k^2)$. Therefore, the conditions (24) give $k_* = \pm i$ and $s_* = \pm 2$. Hence, we precisely recover the minimal wave speed for the monostable case, so the front is pulled. For the bistable case $f = f_3$, one can carry out the same linear calculation but finds that the linear spreading speed does not coincide with true wave speed. In this case, the front is called pushed.

Of course, we have just reviewed the relevant results we need here for travelling waves of the deterministic PDE (3) with monostable and bistable nonlinearities. For a lot more on deterministic travelling waves, we refer the reader to [9, 103, 152, 178] and references therein.

### 3 SPDE Background

To define the noise, let $\mathcal{H}$ be a Hilbert space and also be a suitable function spaces over $\mathbb{R}$, e.g., we may simply think of $\mathcal{H} = L^2(\mathbb{R})$ as the key example. Consider an operator $Q : \mathcal{H} \to \mathcal{H}$ and assume it has eigenfunctions $e_k = e_k(x)$ for $x \in \mathbb{R}$, and non-negative eigenvalues $\lambda_k$ such that $Qe_k = \lambda_k e_k$ for each $k \in \mathbb{N}$. We define an $\mathbb{R}$-valued $Q$-Wiener process as

$$W(t, x) := \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k(x) B_k(t), \quad W(t) := W(t, \cdot),$$

where $\{B_k(t)\}_{k=1}^{\infty}$ are independent identically distributed (iid) Brownian motions over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Using standard properties of Brownian motion, it is easy to see that we have zero mean $\mathbb{E}[W(t)] = 0$ and the correlation function $\mathbb{E}[W(t)W(s)] = \min(t, s)Q$ so $Q$ can be viewed as a covariance operator. Let $g : \mathbb{R} \to \mathbb{R}$ be a given function and now we may consider the SPDE (1) as an evolution equation on $\mathcal{H}$ for $u = u(t) = u(t, x)$ as follows

$$du = [\partial_x^2 u + f(u)] \, dt + g(u) \, dW, \quad u(0, x) =: u_0(x).$$

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In more generality, one can also use linear operator-valued maps instead of \( g \), i.e., one may take in the noise term \( G : \mathcal{H} \to \mathcal{L}(\mathcal{U}, \mathcal{H}) \) for some Hilbert space \( \mathcal{U} \) so that \((G(u)v)(x) = g(u(x))v(x)\) with \( v \in \mathcal{U} \) is a special case. However, we shall not need this more general viewpoint here. The SPDE (26) can be written equivalently as

\[
\partial_t u = \partial_x^2 u + f(u) + g(u)\xi, \quad \xi = \xi(t, x), \quad \partial_t W = \xi, \quad u(0, x) = u_0(x),
\]

where \( \xi \) is just a generalized stochastic process \( 8 \). Indeed, Brownian motion has only Hölder-regularity in time of \( 1/2 - \rho \) for any \( \rho > 0 \), so its time derivative is a generalized function/process. There are two common assumptions on \( Q \). If \( Q = \text{Id} \), then we have space-time white noise with \( E[\xi(t, x)\xi(s, y)] = \delta(t - s)\delta(x - y) \), where \( \delta \) is the usual Dirac-delta generalized function. If \( Q \) is a trace-class operator, i.e., \( \sum_{k=1}^{\infty} \lambda_k < +\infty \), then we have a spatially-correlated noise with \( E[\xi(t, x)\xi(s, y)] = \delta(t - s)C(x, y) \) for some spatial correlation function typically depending just on the difference between spatial locations \( C(x, y) = C(x - y) \). Spatially correlated noise has higher regularity than space-time white noise; see also \([144, 70]\) for more details. One possible solution concept to the SPDE (26) is to consider a mild solution

\[
u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))\, ds + \int_0^t S(t-s)g(u(s))\, dW(s),
\]

where \( S(t) = e^{t\partial_x^2} \) is the usual semigroup generated by the Laplacian on \( \mathcal{D}(\partial_x^2) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \) and one interpretation of the last integral in (28) is as an Itō integral \([144]\). As usual in the theory of stochastic differential equations, we could also consider the Stratonovich form of (26)

\[
u = [\partial_x^2 u + \tilde{f}(u)]\, dt + g(u) \circ dW, \quad \nu(0, x) = u_0(x), \tag{29}
\]

where the integral in an analogous mild solution formula (28) has to be interpreted as a Stratonovich integral if (29) is used. For trace-class noise one has the formal relation \( 61 \)

\[
\tilde{f}(u) = f(u) + C(0)g'(u)g(u),
\]

so that an Itō-Stratonovich correction/conversion term appears; see also \([16]\). The rigorous derivation of results such as (30) requires an infinite-dimensional Itō formula, i.e., an infinite-dimensional stochastic version of the chain rule. In general, it is not easy to prove rigorous Itō-type formulas for solutions of SPDEs; see the review in the introduction of \([143]\). Furthermore, correction terms obtained in (numerical) approximations of SPDEs pose similar technical issues \([71]\). It is important to point out that any Itō formula, as well as any Itō-Stratonovich correction such as (30), is expected to be “quadratic” in the noise term \( g \). Therefore, if \( g \) is multiplied by a small scalar \( \sigma > 0 \), then we obtain terms of order \( \sigma^2 \). It is hence natural to conjecture that in the small noise regime \( \mathcal{O}(\sigma^2) \)-terms only have a major impact on the dynamics if there is some form of instability in the problem; see also \([100] \) and the last part of Section 5.

Another frequently used solution concept for (26) are weak or variational solutions

\[
u(t) = \langle \zeta, u_0 \rangle + \int_0^t \langle \partial_x^2 \zeta, u(s) \rangle + \langle \zeta, f(u(s)) \rangle \, ds + \int_0^t \langle \zeta, g(u(s)) \rangle\, dW(s), \quad \forall \zeta \in \mathcal{D}(\partial_x^2),
\]

where \( \langle \cdot, \cdot \rangle \) is the inner product in \( \mathcal{H} \). Under reasonable conditions on \( f, g, \) and \( Q \), one can show that local-in-time mild solutions \([144]\) and weak solutions exist; under reasonable technical conditions these solutions even coincide \([145]\).
On a technical level, the \textit{uniqueness} or the \textit{weak uniqueness} of solutions is often needed to ensure the \textit{strong Markov property}, which - roughly speaking - means that the solution $u(t + \tau)$ for a stopping time $\tau$ is independent of $u(t)$. For general nonlinear non-Lipschitz reaction terms, such as $f = f_2$ and $f = f_3$, it is usually hard to prove uniqueness of solutions directly. Yet, first proving the existence of a random/pullback attractor \cite{39} and then cutting off the nonlinearity if $|u|$ is large to obtain a globally Lipschitz problem, provides the same work-around well-known for deterministic dissipative reaction-diffusion PDEs \cite{103, 148, 168}. Of course, one should be aware that for large noise, rare event fluctuations, and/or finite-time blow-up scenarios, the approach of cutting off the nonlinearity can influence the dynamics significantly.

There are other solution concepts such as strong, kinetic \cite{41}, martingale \cite{31}, pathwise mild \cite{104}, or renormalized \cite{70} solutions. Strong solutions rarely exist \cite{144} due to the roughness of the noise. The other classes of solutions for SPDEs are more complicated to define/construct. Furthermore, they are not immediately needed here to make sense of uniformly-parabolic scalar reaction-diffusion SPDEs on $\mathbb{R}$ and their travelling wave-type solutions; see Section \ref{5} for cases where other solution concepts enter the picture. Yet, what is often needed are \textit{comparison principles}. For example, if we assume that $f = f_2$ or $f = f_3$, and

$$ g(0) = 0, \quad g(1) = 0, \quad u_0(x) \in [0, 1], $$

then we intuitively expect that $u(x, t) \in [0, 1]$ for all $t > 0$ almost surely. An even stronger result would be a comparison theorem, i.e., one assumes that

$$ 0 \leq u_1(0, x) \leq u_2(0, x) \leq 1 $$

is true and then concludes that

$$ 0 \leq u_1(t, x) \leq u_2(t, x) \leq 1, \quad t > 0, $$

also holds almost surely. These types of invariant-region and comparison results indeed hold under various technical conditions; see \cite{10, 45, 68, 117, 124, 126, 95, 163, 169} and references therein. If $u(t, x) \in [0, 1]$ for $t > 0$ then it is easier to study the long-time asymptotic behaviour of travelling wave-type solutions as one already has a-priori boundedness and one can simply cut-off the nonlinearity outside of $u \in [0, 1]$ as necessary. However, for certain types of noises, such as additive noise given by $g \equiv \text{const.}$, simple comparison principles usually do not hold and large deviations of the solution from the region $u \in [0, 1]$ are are going to occur with positive probability. In this situation, it often makes more sense to focus on the behaviour of travelling wave-type solutions for times $t \in [0, T]$ for some fixed finite $T > 0$.

\section{Stochastic Waves}

It is already non-trivial, how to define a \textit{stochastic travelling wave}. Several related approaches exist, which we briefly review. Suppose we use a deterministic continuous initial condition $u_0$ resembling a front such that for some $K_0 > 0$ we have

$$ u_0(x) = \begin{cases} 
1 & \text{for } x < -K_0, \\
0 & \text{for } x > K_0, 
\end{cases} $$

and $u_0(x) \in [0, 1]$ for all $x \in \mathbb{R}$. Suppose we select the noise term $g$ so that an invariant-region/comparison principle holds so $u(x, t) \in [0, 1]$ for all $t \geq 0$. Furthermore, suppose we can
show that $u(-\infty, t) = 1$ and $u(+\infty, t) = 0$ for all $t \geq 0$, then the choice of the initial condition entails that the solution can only be different from the values of the endstates over a bounded set for $t > 0$. In this case, it is natural to consider level sets and define

$$a(t) := \sup\{z \in \mathbb{R} : u(x, t) = 1, \ x \leq z\}, \quad b(t) := \sup\{z \in \mathbb{R} : u(x, t) = 0, \ x \geq z\}. \quad (34)$$

These random variables measure the spread of left and right edges of a travelling wave front. Of course, the convention to look at endpoints of the support is somewhat arbitrary and we could also consider other level sets

$$c_\alpha(t) := \sup\{z \in \mathbb{R} : u(x, t) = \alpha, \ x \leq z\}, \quad \alpha \in (0, 1).$$

To obtain a well-defined fixed asymptotic wave speed, one often aims to show that at least one of following limits exists

$$\lim_{t \to +\infty} \frac{a(t)}{t}, \quad \lim_{t \to +\infty} \frac{b(t)}{t}, \quad \lim_{t \to +\infty} \frac{c_\alpha(t)}{t}. \quad (35)$$

We shall refer to concrete examples, how to use level sets and comparison principles in Section 4.1.

A related, somewhat weaker notion, is to define stochastic travelling waves via stationary laws [172]. Suppose we are interested in the case $W(t, x) = B_1(t)$, $f = f_2$ or $f = f_3$ and $g(0) = 0$, $g(1) = 0$, $u(t, x) \in [0, 1]$, (36)

where the relevant solutions are fronts connecting $u = 0$ to $u = 1$. Define the space

$$\mathcal{S} := \{\phi : \mathbb{R} \to [0, 1] : \phi(-\infty) = 1, \ \phi(+\infty) = 0, \ \phi \text{ decreasing and right-continuous}\}$$

with the $L^1_{\text{loc}}(\mathbb{R})$ topology. Then we can define a wave marker $C_\alpha$, similar to $c_\alpha$, for each $\phi \in \mathcal{S}$ and center the wave accordingly

$$C_\alpha(\phi) := \inf_{x \in \mathbb{R}} \{\phi(x) < \alpha\}, \quad \phi_\alpha(x) := \phi(C_\alpha(\phi) + x),$$

so that $\phi_\alpha$ is just $\phi$ re-centered at height $\alpha$. A stochastic travelling wave is a solution $u(t, x)$ with values in $\mathcal{D}$ for which the re-centered process $u_\alpha(t, x) = u(t, x + C_\alpha(u))$ is a stationary process in time. The law of the stochastic wave is then given by the law of $u_\alpha(0, x)$. On can actually prove under the assumptions [36] that if $u_0 \in \mathcal{S}$ almost surely, then $u(t) \in \mathcal{S}$ almost surely. Starting from the Heaviside function $u_0(x) = 1_{\{x < 0\}}$, assuming in addition for $f = f_3$ that $g(a) \neq 0$, one may use a stochastic ordering technique [172] to obtain stochastic travelling waves for monostable and bistable cases; see also [80] for an extension to $g(1) \neq 0$.

Another more general, and from a deterministic viewpoint potentially more natural, approach is to try to define the wave and its speed via a similar strategy as for PDEs, i.e., to go into a moving frame. Let

$$u_p(t, x) = u(t, x - p(t)). \quad (37)$$

For the classical PDE case, we just had $p(t) = st$, where $s$ is a fixed wave speed and $p(t)$ is the position of the wave. In the SPDE case, $p(t)$ is generically a stochastic process and it is sometimes assumed that

$$p(t) = \int_0^t \sigma(r) \, dr, \quad (38)$$
where $s = s(t)$ is another stochastic process. To see the reasoning behind this construction, beyond the obvious physical relation between speed and position, consider for simplicity the Stratonovich SPDE (29) so that we can use the standard chain rule applied to (37), which yields

$$du_p(t, x + p(t)) = (\partial_x u_p) \circ dp(t) + du.$$  

A time-dependent shift does not change the covariance so we just get

$$du_p = \left[ \partial^2_x u_p + \frac{dp}{dt} \partial_x u_p + \tilde{f}(u_p) \right] dt + g(u_p) \circ dW,$$

so we see that (38) is quite natural. In particular, $s(t)$ is a local/instantaneous wave speed. Of course, one can always consider an averaging procedure

$$\frac{1}{t - t_0} \int_{t_0}^t \mathbb{E}[s(r)] \, dr,$$

which is expected to be intimately related with time averages in (35). However, the key problem is that the SPDE (39) is not closed. There is currently no equation determining the unknown stochastic processes $s = s(t)$ or $p = p(t)$. The classical deterministic ODE/PDE arguments to determine the wave speed cannot be generalized directly. Of course, we can impose a constraint, e.g., by requiring

$$\min_{y \in \mathbb{R}} \|u(t) - u_{ref}(\cdot - y)\| = \min_{y \in \mathbb{R}} M(y),$$  

(40)

where there are two choices to be made. Firstly, the function $u_{ref}$ is a given reference solution usually taken as the family of deterministic waves $u_{ref} = \Phi_y = \Phi(\cdot - y)$ as discussed in [115]. Secondly, one has to select the type of (spatial) norm $\|\cdot\|$ to be used in (40). A common choice is just the $L^2(\mathbb{R})$-norm [115] or a weighted version $L^2(\mathbb{R}; \rho)$ with a weight function $\rho = \rho(x)$, which one could also adapt to the position of the wave $p(t)$; see [98]. In any case, let us agree from now on that we have chosen a Hilbert space so that $\|\cdot\| = \langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product. The minimization constraint then implicitly defines the position of the wave $p(t)$ by computing a minimizer $y$ for each $t \geq 0$ and setting $y := p(t)$. However, one first has to guarantee that a minimizer exists [81]. Although it does exist under reasonable conditions, it does not have to be unique. It is easy to imagine that there can be jumps for the global minimizer so $p(t)$ may have jumps. Of course, we can always aim to track a local minimizer. Differentiating $M(y)$ in $y$ yields a necessary critical point condition and so we get the constraint

$$\langle \Phi'_y, u(t) - \Phi_y \rangle = 0.$$  

(41)

If the second derivative of $M(y)$

$$\frac{d^2}{dy^2} M(y) = -2 \langle u(t), \Phi''_y \rangle,$$  

(42)

vanishes, the local minimizer becomes degenerate, so we have a criterion to test for potential jumps. Now, we have essentially given another implicit definition [115] [81] [98] of the wave speed for a stochastic travelling wave and we have chosen $p = p(t)$ as moving-frame coordinates for its measurement. There are obviously many variants of the construction (40), e.g., one could even
Consider the (pathwise-defined) RODE
\[ \text{consider (40), fix a deterministic wave } \Phi \text{ and subtract a deterministic speed in the position SODE (43).} \]

As an implicit input, even in the linear stability problem (11). The same effect precisely occurs we know that the deterministic stability problem always involves the shape of the travelling wave as an implicit input, even in the linear stability problem (11). The same effect precisely occurs in the position SODE (43).

The SODE (43) is rather involved and usually impossible to solve analytically, even if an explicit representation of the deterministic wave profile is available. Yet, this situation is completely expected from the classical analysis of deterministic travelling wave problems (see Section 2) since we are essentially trying to capture the deviation of the stochastic solution from a deterministic reference. Structurally, this amounts to studying an evolution equation for perturbations, and we are essentially trying to capture the deviation of the stochastic solution from a deterministic expected from the classical analysis of deterministic travelling wave problems (see Section 2) since we know that the deterministic stability problem always involves the shape of the travelling wave as an implicit input, even in the linear stability problem (11). The same effect precisely occurs in the position SODE (43). We highlight that one also finds further variants of the dynamic re-centering approach in the literature, which lead to slightly different evolution equations [98]. For example, one could consider (40), fix a deterministic wave \( \Phi \) and subtract a deterministic speed
\[ \min_{y \in \mathbb{R}} \| u(t) - \Phi(\cdot - st - y) \| =: \min_{y \in \mathbb{R}} M_s(y). \]

As another modification, suppose we want to dynamically minimize (44) by imposing dynamics along the negative gradient of \( M_s(y) \). Fixing a relaxation parameter \( r > 0 \) one then has to consider the (pathwise-defined) RODE
\[ \frac{dy^r}{dt} = -r \langle u(t) - \Phi(\cdot - st - y^r), \partial_x \Phi(\cdot - st - y^r) \rangle, \quad s^r(t) := \frac{dy^r}{dt} \]

where \( y^r = y^r(t) \) has an index \( r \) to indicate its dependence upon imposed relaxation parameter and \( s^r \) measures the deviation of the wave speed since it is the derivative of the position deviation \( y^r(t) = y^r \) from the deterministic profile. Now define \( p^r(t) := -st - y^r(t) \) and let
\[ u^r(t) := u(t) - \Phi(\cdot + p^r), \quad p^r = p^r(t), \]

so \( u^r = u^r(t) \) essentially measures the fluctuations of the SPDE around a dynamically phase-adapted deterministic wave. For additive noise \( g(u) \equiv \sigma \), the evolution equation of this deviation is then given by [98]
\[ du^r = \left[ \partial_x^2 u^r + f(u^r + \Phi(\cdot + p^r)) - f(\Phi(\cdot + p^r)) + \frac{dy^r}{dt} \partial_x \Phi(\cdot + p^r) \right] + \sigma \, dW. \]
Using Itô’s formula one can now also show that \( s^r(t) \) satisfies an SODE \cite{98, Lem. 3.2} since

\[
\frac{d\dot{y}}{dt} = -r(\dot{u}(t), \partial_x \Phi(\cdot - st - y^r))
\]

and we already have an equation for \( u^r \) given by \cite{16}. Yet, the evolution equation for \( s^r(t) \) is again fully coupled to the original SPDE. Hence, even if we aim to simplify the procedure by not looking at the position \( p \) and speed \( s \) but at the dynamically-adjusted deviation \( y^r \) of the position and deviation of the speed \( s^r(t) \), we still face a (set of) stochastic nonlinear evolution equation(s). To simplify the analysis, it is often assumed that one is interested in the case of small noise deviations from a deterministic profile

\[
0 < \sigma \ll 1, \quad u(0) = \Phi_0, \quad w = u - \Phi_0,
\]

so that \( w = w(t) \) is a stochastic process measuring the deviation from the deterministic solution as for \( \sigma = 0 \) we would have \( u(t, x) = \Phi(x - st) = \Phi_0(\eta) \). We shall encounter multiscale results based upon small-noise in Section 4.2.

### 4.1 Monostable Stochastic Waves

Here we collect results regarding different variants of the monostable FKPP-SPDE

\[
du = \left[ \partial_x^2 u + u(1 - u) \right] \, dt + g(u) \, dW, \quad u_0(x) = u(0, x).
\]  

(47)

The first question is whether we can prove that solutions resembling travelling waves can actually exist. Using a long-range voter model \cite{129} and/or the duality to particle systems \cite{44}, one natural setting to consider is

\[
g(u) = \sigma \sqrt{u(1 - u)}, \quad Q = \text{Id}, \quad \sigma > 0 \text{ sufficiently small.}
\]  

(48)

Suppose the initial condition is locally supported close to a front according to \cite{33}. It is then proven in \cite{127} for (47) with noise term (48) that defining \( a(t), b(t) \) according to \cite{34}, then we almost surely have that the limit

\[
\lim_{t \to +\infty} \frac{b(t)}{t} =: b_*(\sigma)
\]

exists and \( b_*(\sigma) \) is non-random; see also \cite{38, 135, 134}. In addition, the law of \( u_m(t, x + x) \) tends towards a stationary limit, and the interval \([a(t), b(t)]\) also has a non-degenerate limit providing a true front-type wave as \( t \to +\infty \). Then one may ask, how \( b_* \) depends upon \( \sigma \). Based on formal physical approximation arguments and numerical evidence, it has been found \cite{25, 26} (see also \cite{23, 86, 141}) that

\[
b_*(\sigma) = 2 - \frac{\pi^2}{\ln \sigma^2} + \text{higher-order terms} \quad \text{as } \sigma \to 0^+,
\]  

(49)

which is also known as the Brunet-Derrida conjecture. Note that the wave moves slower than the classical deterministic Fisher wave. The inverse-logarithmic correction in (49) is much larger than expected by a naive asymptotic expansion. Of course, it just results from the pushed nature of the front, i.e., from the interplay between small noise and instability \cite{100} near the unstable leading edge of the front. More precisely, if we are sufficiently close to the leading edge,
say \( u < \sigma^2 \), then \( u(1-u) < \sigma \sqrt{u(1-u)} \), so the noise term dominates the reaction term. The Brunet-Derrida conjecture \[26\] has been formed by relating the speed to the FKPP with a cut-off

\[ \partial_t v = \partial_x^2 v + v(1-v)1_{\{v \geq \sigma^2\}}, \tag{50} \]

which can be studied using elegant deterministic arguments \[48\,47\] based upon geometric singular perturbation theory \[101\]. The front for the cut-off system does behave rather “weakly pushed” \[139\]. However, even with the results for \[50\], one still has to rigorously connect the cut-off model \[50\] with the SPDE \[47\] or with an underlying particle system. It has been proven in \[125\] (see also \[14\]) that the small-noise expansion of the stochastic travelling front wave speed for \[47\]–\[48\] is indeed given by \[49\] with higher-order terms of order \( \mathcal{O}(\ln |\ln \sigma| |\ln \sigma|^{-3}) \); see also \[38\] for a lower bound. The same wave speed asymptotics also hold rigorously for the noise term \( g(u) = \sigma \sqrt{u} \), which can be derived from a long-range contact process \[129\]. Indeed, it has been proven \[171\] that travelling wave-type solutions also exist for \( g(u) = \sqrt{u}, \quad u_0(x) = \min(1, \max(-x, 0)), \quad Q = \text{Id}, \quad \sigma > 0 \) sufficiently small. \tag{51}

with a well-defined limit \( \lim_{t \to +\infty} b(t)/t \). In addition to looking at speeds, one can also define a diffusion coefficient of the front \[26\]

\[ D(\sigma) := \lim_{t \to +\infty} \frac{\mathbb{E}[b(t)^2] - \mathbb{E}[b(t)]^2}{t}, \]

yet its asymptotics for monostable equations seems to be more difficult to analyze rigorously \[29\,136\,149\]; we remark that the diffusion coefficient seems to have an interesting relation to coalescence times in the context of related particle models \[28\].

The next natural question is, what happens if we consider large(r) noise. Formal approximations and numerical simulations suggest that the wave speeds change substantially \[73\]. Even more drastically, if the noise is large enough, we may have propagation failure. For example, consider the suitably scaled monostable SPDE

\[ du = \left[ \frac{1}{2\sigma^2} \partial_x^2 u + \sigma^2 u(1-u) \right] dt + \sigma^2 u \, dW, \quad Q = \text{Id}, \quad u_0(x) \geq 0. \tag{52} \]

Then it is proven \[50\] that for small noise a wave exists. However, for any compact set \( \mathcal{K} \subset [0, \infty) \times \mathbb{R} \) there exists \( \sigma_0 > 0 \) sufficiently large and certain non-trivial sets of initial conditions such that for any \( \sigma \in [\sigma_0, +\infty) \) we have

\[ P\left( \sup_{(t,x) \in \mathcal{K}} u(t,x) > e^{-K_1 t \sigma^4} \right) \leq e^{-K_2 \sigma^2 \sigma_0^4} \tag{53} \]

for some constants \( K_{1,2} > 0 \). So the solution is exponentially small with high probability \[50\]; see also \[40\,49\]. In theoretical terms this implies \( u(t,x) \to 0 \) as \( \sigma \to +\infty \) and/or \( t \to +\infty \), while in practical terms it means that we can expect that a travelling wave-form initial condition may eventually be so close to zero that it is beyond any measurement or computer precision. This propagation failure is indeed observed in simulations for initial data with compact support such as approximate identities (“approximate \( \delta \)-distributions”) and strong noise \[50\,60\,99\] while in the deterministic case \( \sigma \equiv 0 \), we know there is one left-moving and one right-moving front for
such initial conditions. The dichotomy of propagation failure was made even more precise \[128\] for the SPDE
\[
du = \left[\partial_x^2 u + \theta u - u^2\right] + \sqrt{u} \, dW, \quad Q = \text{Id},
\]
for suitable non-negative compactly supported initial data. In this case, one can prove \[128\] that there exists a constant \(\theta_c\) independent of \(u_0\) such that
\[
P(u(t,0) \neq 0 \, \forall t > 0) \begin{cases} = 0 & \text{if } \theta < \theta_c, \\ > 0 & \text{if } \theta > \theta_c. \end{cases} \tag{54}
\]
So the process dies out if the local linear deterministic instability induced by the reaction term is not strong enough. In fact, the existence of travelling waves in the survival regime holds, i.e., we may replace the small-noise condition in (51) by \(\theta > \theta_c\); see \[171, 92, 91, 78\] for more details on limiting distributions and the role of initial conditions in this case. Of course, propagation failure effects such as (53) and (54) can also occur for various other noises \[79\].

We remark that there are strong indications that the statistical properties of a stochastic wave even contain early-warning signs to indicate closeness to propagation failure \[99\]; see also Section 5. In addition, there is growing evidence that suitably chosen noise can arrest/freeze pulled fronts or change their direction \[118\]. For viewpoints relating stochastic monostable dynamics to finite-size effects in autocatalytic reactions we refer to \[113, 116, 108\]. Another possible direction is to consider local modifications of the monostable reaction term to balance certain noise-induced effects \[138\] or to study the transition between pulled and pushed noisy waves \[16, 151\]. Now we are also going to transition and change from the monostable to the bistable case.

### 4.2 Bistable Stochastic Waves

It has been known for a quite a long time based on physical grounds and simulations that in the bistable case \(f = f_3\), front-like solutions exist, at least in the weak noise setting \[121\]. Next, one may ask similar questions regarding the wave speed of propagating fronts as for the monostable case. In \[7\] a noise term is derived from external fluctuations in the control parameter \(a\) to yield the model
\[
du = \partial_x^2 u + u(1-u)(u-a) \, dt + \sigma \tilde{g}(u) \, dW, \tag{55}
\]
where \(Q\) is of trace-class with correlation function \(C(x-y)\), the noise term is apparently to be interpreted in the Stratonovich form \[137\] in this context, and examples for \(\tilde{g}\) will be discussed below. A typical formal approximation approach \[7, 6\] for (55) is to reduce the problem to moments. One often finds a reference \[7, 6\] to “Novikov’s Theorem” \[133\] in the theoretical physics literature, which is used to derive approximating equations; see also \[137, 61, 153\] for some further references using this approach. Interestingly, the original reference by Novikov \[133\] frequently cited in this context of the stochastic travelling waves literature in the physical sciences does not contain any theorems but only proposes a formal approximation based on physical grounds for noise in the context of turbulence problems. The name “Novikov” and the transformation to a different stochastic process appearing in the theorem, may lead one intuitively to think of the so-called Novikov condition known to appear in the context of Girsanov’s Theorem \[146\]. Yet, the the Novikov condition is by A.A. Novikov \[132\] while “Novikov’s Theorem” is by E.A. Novikov \[133\]. In fact, it is probably better to use the convention Furutsu-Novikov
Theorem \[142, 94\] for the latter to avoid confusion. The Furutsu-Novikov Theorem states that
\[
\mathbb{E} \left[ \Xi(t) \mathcal{F}[\Xi(\cdot|t_0)] \right] = \int_{t_0}^t C(t, s) \mathbb{E} \left[ \frac{\delta \mathcal{F} \left[ \Xi(\cdot|t_0) \right]}{\delta \Xi(s)} \right] \mathrm{d}s, \tag{56}
\]
where \( \Xi \) is any zero-mean Gaussian process with given correlation function \( C(t, s) \), \( \mathcal{F} \left[ \Xi(\cdot|t_0) \right] \) is a functional of \( \xi \) over the time interval \([t_0, t]\) and \( \delta / \delta \xi(s) \) denotes the functional derivative with respect to \( \xi \) at \( s \). One idea to utilize this theorem for travelling waves is to notice that direct averaging of \[55\] will not produce a deterministic PDE to leading-order as \( \sigma \tilde{g}(u) \circ \partial_t W \) may not have zero mean. Hence, one tries to restore this zero mean property. The mean value of the noise can be calculated via the Furutsu-Novikov Theorem \[7, 6\]
\[
\sigma \mathbb{E}[\tilde{g}(u) \circ \partial_t W] = \sigma^2 C(0) \mathbb{E}[\tilde{g}'(u)\tilde{g}(u)], \tag{57}
\]
where we observe that the right-hand side of formula \[57\] is just the average of an Itô-Stratonovich correction term. Re-writing the SPDE \[55\] suggestively with \( \partial_t W =: \xi \) as
\[
\partial_t u = \partial^2_x u + f_3(u) + \sigma^2 C(0) \tilde{g}'(u)\tilde{g}(u) + \tilde{g}(u)\xi - \sigma^2 C(0) \tilde{g}'(u)\tilde{g}(u),
\]
\[
=: \partial^2_x u + f_3(u) + \sigma^2 C(0) \tilde{g}'(u)\tilde{g}(u) + R, \quad R = R(t, x, u) \tag{58}
\]
means that if we average now, the new noise term \( R \) should disappear due to zero mean. Furthermore, the noise does have a changed correlation function
\[
\mathbb{E}[R(t, x, u)R(s, y, u)] = \mathbb{E}[u(t, x)u(s, y)\xi(t, x)\xi(s, y)] + \mathcal{O}(\sigma), \quad \text{as } \sigma \searrow 0.
\]
Let us now illustrate, how to use \[58\] for the case
\[
\tilde{g}(u) = u(1 - u),
\]
to formally track suitable averages for front-like structures. Fix a sufficiently big interval \([-L, L]\) within we suspect the front-like structure connecting 0 to 1. Next, define
\[
\bar{m}_L(t) := \int_{-L}^L u(x, t) \mathrm{d}x
\]
which acts as an random-variable approximation of the front position. Then define the deviation of the position from the average as \( m_L(t) := \bar{m}_L(t) - \mathbb{E}[\bar{m}_L(t)] \); note that this defines a more complicated variant of the procedure described at the beginning of Section \[4\] where we focused on the deviation from the deterministic speed. Setting now \( u_{m_L}(t, x) := u(t, x + m_L(t)) \) in \[55\] one obtains
\[
\mathrm{d}u_{m_L} = \left[ \partial^2_x u_{m_L} + \frac{\mathrm{d}m_L}{\mathrm{d}t} \partial_x u_{m_L} + u_{m_L}(1 - u_{m_L})(u_{m_L} - a) \mathrm{d}t \right] + u_{m_L}(1 - u_{m_L}) \circ \mathrm{d}W. \tag{59}
\]
Then one can evidently average again by defining the average front shape \( u^0_{m_L} := \mathbb{E}[u_{m_L}] \). What is the evolution equation for \( u^0_{m_L} \)? Taking the average in \[59\] and using the Furutsu-Novikov Theorem gives an evolution equation for \( u^0_{m_L} \). Yet, the first moment is coupled to higher moments\[3\]
\footnote{I would like to thank Eulalia Nualart for pointing out the alternative attribution to Furutsu \[59\] to me, which I had not been previously aware of.}
in general, leading to the problem of moment closure [102]. However, let us decompose the dynamics into two parts
\[ u_{mL}(t, x) = u^0_{mL}(t, x) + u^c_{mL}(t, x). \]

If one postulates, say based upon numerical evidence, that the dynamics of the deviations \( u^c_{mL} \) decays quickly for any initial condition so that \( |u^c_{mL}| \ll 1 \), then keeping only lowest-order terms yields [7, 6]
\[ \partial_t u^0_{mL} = \partial^2_{x} u^0_{mL} + u^0_{mL}(1 - u^0_{mL})(c_0 u^0_{mL} - a_0), \quad c_0 = 1 - \sigma^2 C(0), a_0 = a + \frac{\sigma^2}{2} C(0). \quad (60) \]

The equation (60) can now be analyzed using PDE techniques. For example, we find that the wave speed changes depending upon varying \( C \). The preceding discussion did not a-priori rely on direct small noise expansions (or weak noise expansions) [140], which are another common tool, particularly for bistable equations [84, 121, 157, 158]. In the small-noise regime, one expects that if a travelling wave is deterministically locally stable, then the corresponding SPDE generates a similar wave-like profile with additional diffusive motion along the neutral (Goldstone) mode associated to the eigenvalue zero generated by translation invariance; see also Section 2. These results can be made rigorous for bistable systems and fronts under certain assumptions [98]. Suppose we start with the SODE (45) for the deviation of the position \( y^r(t) \) from the deterministic front \( \Phi \); recall that both quantities are dynamically adapted and the equations should be studied on a finite time interval, say \( t \in [0, T] \). Using these equations, and assuming that \( \sigma \) is small, one may prove there is a decomposition
\[ p^r(t) = -st - \sigma y^r_1(t) + o(\sigma), \quad \text{as } \sigma \to 0, \]
where \( s \) is the deterministic wave speed, \( y^r_1(t) = \int_0^t s^r_1(\tilde{t}) \, d\tilde{t} \) is the leading-order approximation of the position deviation. The leading-order approximation for the speed correction satisfies the SODE (62)
\[ ds^r_1 = -r s^r_1 \, dt - r \langle \partial_y \Phi(\cdot - st), dW \rangle, \quad (62) \]
where \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}, \rho(\cdot - st - \sigma y^r_1(t))))} \) is a weighted norm moving within the adapted reference frame and one considers the function \( \rho(y) = Z \exp(sy), \) where \( Z \) is constant computable from the deterministic wave profile \( \Phi \) as the normalization constant guaranteeing that
\[ \langle e^{\sigma t} \partial_y \Phi, \partial_y \Phi \rangle_{L^2(\mathbb{R})} = 1. \]

Indeed, one may check that the function \( e^{\sigma t} \partial_y \Phi \) is the eigenfunction to the neutral eigenvalue zero of the adjoint \( \mathcal{L}^* \) to the operator \( \mathcal{L} \) arising from the linearization around the deterministic travelling wave defined in [11]. Hence, we expect that in the weighted norm we just have that the wave speed correction \( s^r_1 \) arises from a projection argument onto the eigenspace spanned by the neutral (Goldstone) mode spanned by \( \partial_y \Phi \). This is indeed visible in the SODE (62) as there
is deterministic decay from the relaxation parameter and diffusive wandering projected onto the neutral mode as expected from physical intuition \[137\]. Furthermore, one may prove a leading order SPDE approximation for the fluctuations $u'(t)$

$$u'(t) = \sigma u_1'(t) + o(\sigma), \quad \text{as } \sigma \to 0,$$

where the SPDE for $u_1'(t) = u_1'(t, x)$ is \[98\]

$$du_1' = \left[\partial_x^2 u_1' + f_3' (\Phi(-st)) u_1' + s_1' \partial_x \Phi(-st)\right] dt + dW, \quad x \in \mathbb{R}. \quad (63)$$

The evolution equations (62) and (63) only hold up to a stopping time $\tau = \tau(\sigma)$ as discussed already at the beginning of Section 4 since one may encounter jumps of the processes involved. Yet, we can always ensure for a given fixed maximal time $T > 0$ that

$$\lim_{\sigma \to 0} \tau(\sigma) = T,$$

so we truly have a small-noise approximation in a precise sense on finite time intervals. Although the equations (62) and (63) are just the leading-order multiscale approximations one could in principle continue the expansion in $\sigma$ yielding more evolution equations for higher-order corrections.

The next natural question one can pose is, how can we analyze approximating equations, either without a small-noise or with a small-noise assumption? Since these evolution equations are often simpler, yet still impossible to solve explicitly, one aims for estimates \[166\]. For example, consider the SPDEs for the diffusion along the neutral mode (46) or its lead-order approximation (63) and the stopping times

$$\tau_K := \inf\{t \geq 0 : \|u'(t)\| > K\} \quad \text{and} \quad \tau_K^1 := \inf\{t \geq 0 : \|u_1'(t)\| > K\},$$

for some fixed spatial norm $\|\cdot\|$. These stopping times provide qualitative information about the times when fluctuations become larger than a given constant $K > 0$. In particular, a natural aim is to prove bounds on the distributions

$$P(\tau_K \leq \kappa) \quad \text{and} \quad P(\tau_K^1 \leq \kappa). \quad (64)$$

Alternatively, we could also look directly at the probabilities

$$P\left(\sup_{t \in [0,T]} \|u'(t)\| > \kappa\right) \quad \text{and} \quad P\left(\sup_{t \in [0,T]} \|u_1'(t)\| > \kappa\right), \quad (65)$$

for some $\kappa > 0$. Of course, similar remarks apply to probability bounds on stochastic corrections to the deterministic speed or position of the wave. It is then quite natural to expect that these probabilites can be estimated using arguments such as Markov/Chebyshev/Doob/Bernstein and related concentration inequalities for probabilities, or the Burkholder-Davis-Gundy inequality \[82\]. These inequalities often provide the natural way to convert the question to deterministic arguments about moment bounds. We may view estimates on the probabilities (64) and/or (65) of the fluctuations near a deterministic wave or related results on the speed/position as stability results, which have been studied already in quite some detail in the bistable setting \[74, 81, 98, 166, 167\].
To see that we indeed expect stability for small noise, we state at least one result in this direction in a very special case. Let $\Phi$ be the deterministic front for $f = f_3$ with speed $s$. Consider the SPDE

$$du = \left[ \partial_x^2 u + u(1-u)(u-a) \right] dt + \sigma \tilde{g}(u) dB, \quad u(0,x) = u_0(x),$$

(66)

where $B = B(t)$ is a standard real-valued Brownian motion and we impose that $g(\Phi) = -\sqrt{2} \partial_x \Phi$. Hence, the noise only acts “rigidly” on the stochastic wave as we have enforced an invariance along the neutral (Goldstone) mode for $g$. Upon starting with a well-prepared initial condition $u_0(x) = \Phi(x)$, one can actually ensure that we only see a stochastic wave with changed speed and fluctuating position but no additional fluctuations around the wave, so fluctuations estimates are trivial. Furthermore, (yet another) variant of the definition of position, say $p_*(t)$, is shown to satisfy

$$p_*(t) = \sqrt{1 - \sigma^2 st} - \sigma \sqrt{1 - \sigma^2} B(t).$$

(67)

Hence, this explicit (special-case) formula now easily yields stability type results as we only need to estimate the position and speed, which depend in a simple way on Brownian motion so well understood upper/lower bounds for Brownian motion can be applied. It is far more difficult to obtain general stability results but the bistable case, as illustrated by formula (67), is expected to be quite tame in the small noise regime; cf. formula (49) for the monostable case.

Lastly, we point out that one should always keep in mind that also for the bistable case, any theoretical result should be compared to microscopic modelling of the noise; see also [137].

5 Summary & Outlook

There are many topics closely connected to travelling waves for monostable and bistable SPDEs. We mention a few of these directions here. In fact, there is an even simpler SPDE, which can generate interface-like solutions given by

$$du = \partial_x^2 u \ dt + \sigma \sqrt{u(1-u)} \ dW, \quad Q = \text{Id}. \quad (68)$$

The interfaces of (68) behave like Brownian motion. The model (68) can be derived from a long-range voter model and is therefore microscopically related to the monostable FKPP equation, which has the same noise term.

Another topic related to the FKPP equation is its generalization to higher dimensions

$$du = [\Delta u + u(1-u)] \ dt + \sigma g(u) \ dW, \quad u = u(t,x), \ x \in \mathbb{R}^d, \quad (69)$$

for some $d \geq 2$. Upon using a certain multiplicative noise and a suitable initial condition, one can again obtain propagating front-like solutions invading the deterministically unstable state [122, 147]. Since the interface propagation has now a non-trivial spatial structure, e.g., a curve-like rough interface for $d = 2$, it is natural to try to connect its dynamics to the Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t h = \Delta h + (\nabla h)^2 + \partial_t W, \quad Q = \text{Id}, \ h = h(t,x), \ x \in \mathbb{R}^{d-1}. \quad (70)$$
The KPZ equation \[85\] is a normal-form type model or “universality class” for interface growth, where one can think of \(h\) as a height function of the interface for \(d = 2\). However, note that \((70)\) is not well-posed as written in the form \((70)\) since the regularity of the space-time white-noise \(\xi = \partial_t W\) does not allow one to define \((\nabla h)^2\) via a standard fixed-point argument to obtain the existence of solutions. Due to this regularity issue, the KPZ equation is an example of a singular SPDE. Under certain technical assumptions, it can be possible to renormalize a singular SPDE and analyze it within the framework of regularity structures [70, 69] or within paracontrolled distribution theory [65, 66]. Of course, if we view the monostable and bistable SPDEs in higher dimensions and/or with very irregular noise terms, they need renormalization as well, which has been noticed already in the context of numerical simulation; see [137] and references therein.

Instead of considering higher spatial dimensions \(d \geq 2\) for scalar equations, one may ask, what happens if \(d = 1\) but we consider systems of reaction-diffusion SPDEs with various nonlinearities? The theory for travelling waves in this context is even less developed. The typical results/effects for the one-component case are still key points for systems, e.g., front-like structures including propagation failure [130] or stability results [75] have been proven for two-component model problems. However, there are additional new phenomena possible if we consider systems for \(d \geq 2\) such as spiral-like structures [162, 43]. It is natural to conjecture that spiral-like waves can be found if we perturb the classical models for spiral waves such as the FitzHugh-Nagumo [56, 131] equation, the Barkley model [11], or the Oregonator system by noise [15, 17, 109, 155, 162].

Instead of generalizing to higher spatial dimensions, there are also first attempts to consider waves for other noise terms, such as Lévy noise [24]. Furthermore, one may replace the heat equation part \(\partial_t u = \Delta u\) by more general fractional derivative operators [24] derived from anomalous diffusion, or even observe anomalous diffusion from classical equations [150]. In fact, the analysis of travelling waves for deterministic PDE involving fractional operators is another recently emerging area for monostable [32, 42, 51] as well as bistable [2, 1, 36, 67, 179, 182] cases. One should even suspect that waves for fractional diffusion operators for PDEs and waves for SPDEs are deeply connected [150] since both underlying classes of differential equations are derived from very similar microscopic stochastic modelling principles.

Nonlocal fractional operators are just one class, where nonlocality has recently entered into focus. Another important recent class motivating research in stochastic travelling waves are stochastic neural field equations [20, 19, 53, 105]

\[
du = \left[ -\alpha u + \int_{\mathcal{I}} f(u(t, y)) w(\cdot, y) \, dy \right] \, dt + \sigma g(u) \, dW, \quad \alpha > 0, \quad u = u(t, x), \quad \mathcal{I} \subseteq \mathbb{R}, \quad (71)
\]

where \(w\) is a kernel modelling the connections of the neurons, and other variants of neural field equations place the nonlinearity \(f\) outside of the integral \(f(\int_{\mathcal{I}} \ldots)\). It has been proven that (71) has many analogies to classical local (S)PDEs [107, 110]. Furthermore, travelling waves have been studied, particularly in the bistable case, in quite some detail for stochastic neural fields, see e.g. [21, 22, 81, 59, 97, 111].

An important topic directly related to the bistable setting is the case \(a = 1/2\) for \(f = f_3\), so that the PDE has a standing wave. It is well-known that if we consider a weak diffusion

\[
\partial_t = \varepsilon^2 \partial_x^2 u + u(1 - u)(u - 1/2) \quad (72)
\]

then (72) develops quickly, for quite large sets of initial data, several sharp interfaces of width \(O(\varepsilon)\) between 0 and 1. These interfaces then move exponentially slowly on an (approximating)
invariant manifold at speed $O(\varepsilon^{-K/\varepsilon})$ for some constant $K > 0$; see e.g. \cite{33, 103}. Of course, one may then ask, how these interfaces form and move in the case, when (72) is perturbed by noise. This case has been studied in quite some detail showing that the interfaces still form and move \cite{57, 58, 112}. It is anticipated that an invariant manifold description still exists \cite{4, 5}, and that dynamics on this manifold is essentially Brownian motion \cite{18} under suitable conditions.

A very important future direction for research will be to connect waves/patterns for SPDEs more closely to applications \cite{61}. Stochastic wave-like structures in SPDEs have already appeared in an extremely diverse set of modelling contexts such as neuroscience \cite{156, 175, 176}, spin glasses \cite{30}, biological invasions \cite{165}, predator-prey systems \cite{87, 164}, directed polymers \cite{27}, evolutionary biology \cite{37, 72}, and epidemics \cite{114, 180}. Although small fluctuations in the modelling context are sometimes just neglected, this is generally a false hypothesis near instability, e.g., when we are close to propagation failure or when the deterministic PDE part undergoes a bifurcation. We remark that in this context the precise formulation of the SPDE via modelling will be crucial. The bifurcation (or phase/critical transition) aspect has been recognized early on as a key concept in SPDEs \cite{77} for steady-state-like patterns. A recently more detailed mathematical theory has begun to develop \cite{64, 106} to use critical slowing down in combination with stochastic perturbations as early-warning signs for transitions of steady-state-like stochastic dynamics. However, a similar idea also seems to have emerged early on in the context of numerical simulations of waves in noisy systems \cite{160}. Therefore, we conjecture that one may very efficiently compare data and SPDE models via fluctuation analysis via wave/pattern-forming instabilities.

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