GENERALISED MASS VARIATION FORMULA
FOR A STATIONARY AXISYMMETRIC STAR OR
BLACK HOLE WITH SURROUNDING ACCRETION DISC

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Abstract The general relativistic mass-energy variation formula for axisymmetric equilibrium states of a selfgravitating system is developed in the particular case for which the relevant matter consists of a perfectly conducting multiconstituent fluid (or superfluid) with stationary circular or convective motion.

1. Introduction

This communication presents an extension of the study\(^1\) of mass variation formulae for stationary axisymmetric systems – relativistic stars and black holes with surrounding material discs – that is contained in my contribution (chapter 6) to the Einstein Centenary Survey: General Relativity, ed S.W. Hawking and W. Israel (Cambridge U.P. 1979); all numbered references here refer to equations presented therein.

The present work starts from the general formula

\[
\delta M = \frac{1}{8\pi} \int \Omega^H \delta J^H - \frac{\kappa}{8\pi} \delta A + \frac{1}{8\pi} \int \left( G_{ab} k^a d\Sigma_b - \frac{1}{2} \int G^{cd} h_{cd} k^a d\Sigma_a + \Omega^H \delta J^H + \frac{\kappa}{8\pi} \delta A \right) \tag{1}
\]

(equation (6.342)\(^1\) ) for the variation of the total asymptotically defined mass \(M\) of the system between two neighbouring stationary axisymmetric equilibrium states, where \(\Omega^H, J^H, \kappa, A\) are the angular velocity, angular momentum, decay parameter (“surface gravity”) and surface area of the central black hole – if it is present – while \(G_{ab}\) is the Einstein tensor, \(h_{cd}\) the variation of the metric tensor, \(k^a\) the stationary Killing vector, and the integration is taken over a spacelike hypersurface \(\Sigma\) extending out to the asymptotically flat region with inner boundary on the black hole if it is present.

2. Electromagnetic and material contributions

We now introduce the energy momentum tensor \(T_{ab}\) and the corresponding momentum vector \(\Pi^a = -T^a_{\ b} k^b\) (decomposed into electromagnetic and purely material parts, labelled by suffices \(F\) and \(M\) ) as follows. When we substitute from the Einstein equations into the fundamental kinematic mass variation formula (1) we obtain

\[
\delta M - \Omega^H \delta J^H - \frac{\kappa}{8\pi} \delta A = \delta \int T_{\ b}^{\ a} k^b d\Sigma_a - \frac{1}{2} \int T_{\ c}^{\ b} h_{\ b}^{\ c} k^a d\Sigma_a \tag{2}
\]
\[ \delta M - \Omega^u \delta J_u - \frac{k}{8\pi} \delta A + \delta \int \Pi_F^a \{k\} \, d\Sigma_a + \frac{1}{2} \int T_{bc}^b h^c k^a \, d\Sigma_a \]

\[ = -\delta \int \Pi_M^a \{k\} \, d\Sigma_a - \frac{1}{2} \int T_{bc}^b h^c k^a \, d\Sigma_a \]

(3)

To evaluate the electromagnetic contribution we first use the analogue of (6.306) for the momentum flux of a stationary field, namely

\[ -\Pi^a_F = \Lambda_F k^a + (k^b A_b) j^a + \nabla_b (A_c k^c D^b F) \]

and then work out

\[ |g|^{-1/2} \delta |g|^{1/2} \Lambda_F k^a = \frac{1}{2} (T_{bc}^b h^c_b - D^bc \delta F_{bc}) k^a \]

\[ = (\frac{1}{2} T_{bc}^b h^c_b - j^b \delta A_b) k^a + 2 \nabla_b (D^{c[a} \delta A_{c]a}) \]

(5)

where the electromagnetic action density and displacement tensor are given – by (6.193) and (6.194) – as \( \Lambda_F = (1/16\pi) F^{ab} F_{ab} \) and \( D^{ab} = (1/4\pi) F^{ab} \).

With the aid of the Green theorem (and using the asymptotic flatness conditions to eliminate a surface contribution at infinity) we are thus able to obtain

\[ \delta \int_{\Sigma} \Pi^a_F d\Sigma_a - \frac{1}{2} \int_{\Sigma} T_{bc}^b h^c k^a \, d\Sigma_a = \int_{\Sigma} k^b A_b \delta (j^a \, d\Sigma_a) + \frac{1}{2} \int_{\Sigma} k^{[b} j^{a]} (\delta A_b) \, d\Sigma_a \]

\[ - \frac{1}{2} \oint_{S} \frac{1}{2} k^c A_c D^b F_{ab} \, dS_{ab} - \oint_{S} D^{c[b} (\delta A_{c]a}) k^a \, dS_{ab} \]

(6)

3. Perfectly conducting fluid

The result so far does not depend on the particular field equations satisfied by any matter that is present. In order to show the relationship with the first law of ordinary thermodynamics we shall now consider the explicit form of the right hand side of (3) in a particularly ideal case namely that of a perfectly conducting multiconstituent fluid medium. Previous descriptions of the first law by Bardeen 1973, Bardeen, Carter and Hawking 1973 have been restricted to the more specialised case of a non-conducting medium, electromagnetic effects being taken into account in the more extended treatment given by Carter 1973b.

The more general treatment presented here is based on the relativistic theory of a multiconstituent perfect fluid obtained as the natural extension of the two constituent case discussed by Carter 1976. According to this theory (which includes the Landau-London two-fluid model for superfluidity as a special case in the non-relativistic limit) the material equations of motion are derivable from a Lagrangian scalar \( \Lambda_M \) which is a covariant function of a set of conserved current vectors \( \vec{n}_X \) representing the fluxes of a set of independent constituents labelled by an index \( X \), one of the constituents being usually (except in zero-temperature models) the entropy. (Thus in a typical application \( X \) might take the values 1, II, III with \( \vec{n}_1 \) and \( \vec{n}_II \) representing fluxes of positively charged
ions and negatively charged electrons respectively, while \( \vec{n}_m \) is identified with an entropy flux vector \( \vec{s} \).

Under the influence of arbitrary infinitesimal variations of the fluxes \( \vec{n} \) and of the metric \( g \) the function \( \Lambda_m \) will have an infinitesimal variation of the form

\[
\delta\Lambda_m = \mu^x_a \delta n^a_X + \frac{1}{2} \mu^x_a n^b_X h^a_b. \tag{7}
\]

The covectors \( \mu^x \) so defined may appropriately be described as the chemical potential one-forms of the corresponding conjugate constituents. (In particular the covector conjugate to the entropy flux vector \( \vec{s} \) may appropriately be denoted by \( \Theta \) and interpreted as a temperature form.) Their constituent indices are written upstairs in view of the fact that they transform contravariantly with respect to the fluxes under a change of chemical basis in the vector space of constituents.

When the medium interacts with an electromagnetic field the total Lagrangian will be taken to have the simple additive form

\[
\Lambda = \Lambda_m + A_a j^a + \Lambda_F \tag{8}
\]

where \( \Lambda_F \) is the electromagnetic Lagrangian (given by (6.193)\(^1\)) and where the electromagnetic current \( j^a \) appearing in the interaction term is given by

\[
\vec{j} = e^X \vec{n}_X \tag{9}
\]

where \( e^X \) represents the mean electric charge per particle of the \( X \)th constituent. Like the chemical potentials, the charges behave as components of a contravariant vector in the chemical constituent space. (It would in fact be easy to work with a more general Lagrangian involving a more intimate relationship between the fluxes \( \vec{n}_X \) and the field \( \vec{E} \). The restriction to the simple additive form (8) – which means that polarisability effects are excluded – has been made merely to facilitate the interpretation of the formalism.)

Variation of the total Lagrangian now gives

\[
\delta\Lambda \equiv \pi^x_a \delta n^a_X + (j^a - j^a_F) \delta A_a - \nabla_c (D^{cb}_F \delta A_b)
+ \frac{1}{2} \left( n^a_X \pi^X_b - D^{ca}_F \nabla_b A_c + \nabla_c (D^{ca}_F A_b) + (j^a_F - j^a) A_b \right) h^b_a \tag{10}
\]

where we have used the abbreviation

\[
j^a_F \equiv \nabla_b D^{ab}_F \tag{11}
\]

in accordance with the scheme introduced in section 6.4\(^1\), and where we introduce the notation

\[
\pi^x = \mu^x + e^X \tag{12}
\]

for what will play the role of an effective four-momentum per particle for the corresponding constituent. If we required the volume integral of \( \Lambda \) to be invariant under arbitrary variations of the fields \( \vec{A} \) and \( \vec{n}_X \) we would obtain not only the correct equations of motion

\[
\vec{j}_F = \vec{j} \tag{13}
\]
for the electromagnetic field, but also the too restrictive requirement that the momenta \( \pi^X \) vanish altogether. To obtain reasonable equations of motion for the chemical constituents one must demand invariance not under arbitrary current variations but only under variations of the restricted class that arise from infinitesimal displacements of the world lines of the idealised particles, or directly from variations of \( g_{ab} \). Now it can easily be seen (e.g. using the formulae given by Carter 1973a) that an infinitesimal displacement \( \xi^X \) of the world lines of a conserved flux of particles (acting in conjunction with a variation \( h \) of the metric) induces a corresponding variation

\[
\delta n^a = -\frac{1}{2} n^a h^c_{\ c} - n^a \nabla_b \xi^b + n^b \nabla_b \xi^a - \xi^b \nabla_b n^a
\]  

(14)

in the corresponding current vector \( \vec{n} \).

The equations (7) to (13) were chemically covariant (i.e. unaffected by changes in the chemical basis of the space of constituents) in form although not necessarily in content (since the specific representation of the Lagrangian will in general depend on the chemical basis). However the choice of a particular set of chemical constituents implies the choice of a preferred basis in the vector space of constituents, so that the system of equations that results will not have even the appearance of chemical covariance. It is also to be remarked that although the identities derived in section 6.4.1 (which depended only on the covariance of the Lagrangian under general co-ordinate transformations of the space-time manifold as opposed to the chemical constituent vector space) will still be formally valid for the material lagrangian \( \Lambda_M \) – indeed (6.135) has been implicitly used in the derivation of the form (7) – and for the total Lagrangian \( \Lambda \), the quantities involved cannot be given the same direct interpretation (as representing energy-momentum etc. ...) for a restrained variational theory as in the free field theory for which they were originally intended. In the present theory the appropriate energy-momentum tensor cannot be obtained directly from the analogue of (6.138) but can be read out from the expression

\[
|g|^{-1/2} \delta (|g|^{1/2} \Lambda) \equiv T^{ab} h_{ab} + (j^a - j^a_\phi) \delta A_a - f^X A_a \xi^a + \nabla_a \left( 2\pi^X n^a \xi^b - A_b D^a \right)
\]  

(15)

where the \( \tilde{\xi}_X \) (which do not behave as components of a well behaved vector under chemical basis transformations) are a set of arbitrary displacement vector fields acting on the world lines of the corresponding fluxes, and where the fields \( f^X \) (to which the same remark applies) are to be interpreted as generalised 4-force densities acting on the particles of the corresponding constituents. One finds that the force covectors have the explicit form

\[
f^X \equiv \vec{n}_{i|X} \cdot \partial \vec{n}^X + \vec{n}^X \nabla \cdot \vec{n}_{i|X}
\]  

(16)

where the inclusion of the chemical indices within straight bars is introduced to denote suspension of the summation convention.

Each of the variational field equations

\[
f^X = 0
\]  

(16)

can be decomposed by first contracting with the corresponding flux \( \vec{n}^X \), to give first the conservation law

\[
\nabla \cdot \vec{n}_X = 0
\]
for each constituent, and then the canonical equation of motion

$$\vec{n}_{|X|} \cdot \partial \pi^X = 0$$  \hspace{1cm} (19)$$

which determines the acceleration of the corresponding world lines. This last equation may be decomposed into a material and an electromagnetic part in the form

$$\vec{n}_{|X|} \cdot \partial \mu^x = \vec{F} \cdot \vec{j}^x$$  \hspace{1cm} (20)$$

where

$$\vec{j}^x = e^x \vec{n}_{|X|}$$  \hspace{1cm} (21)$$

is the electric current density associated with the $X$th constituent. It is to be remarked that the condition (19) is precisely what is required for the flux of generalised vorticity

$$\vec{w}^x = \partial \pi^x$$  \hspace{1cm} (22)$$

to be conserved along the flow lines in the strong sense discussed in section 6.4.5\(^1\), meaning that

$$(\alpha \vec{n}_{|X|}) \cdot \nabla \pi^x = 0$$  \hspace{1cm} (23)$$

for an arbitrary scalar field $\alpha$, which implies that the flux $\vec{w}^x$ over any 2-surface, which by Stokes theorem is equal to the circulation

$$C^x = \oint_c \pi^x \cdot d\vec{x}$$  \hspace{1cm} (24)$$

around its boundary circuit $c$ say, is conserved in the sense of being unchanged when the circuit (or the original 2-surface enclosed by it) is dragged at an arbitrary rate along the flow lines.

Unlike the field equations themselves, the expression for the energy momentum that is read out from (15) has a chemically covariant form. After subtracting out the Maxwellian contribution in accordance with the decomposition (6.299)\(^1\) we are left with a material contribution to the energy-momentum given by

$$T^{a}_{M \ b} = n^a_X \mu^x_{\ a} - \Psi_M g^a_b$$  \hspace{1cm} (25)$$

where $\Psi_M$ is a thermodynamic potential density that is related to the material Lagrangian by

$$\Psi_M = n^a_X \mu^x_{\ a} - \Lambda_M \cdot$$  \hspace{1cm} (26)$$

Provided that the electromagnetic field equations (13) are satisfied, the expression (6.299)\(^1\) of the total metric energy momentum tensor as the sum of metric contributions $T^{ab}_F$ and $T^{ab}_M$ can be replaced by an expression in terms of canonical contributions in the form

$$T^{a}_{\ b} = T^{a}_{MC \ b} + T^{a}_{F \ b} + \nabla_c (D^c a A_b)$$  \hspace{1cm} (27)$$

where $T^{a}_{\ b}$ are components of the canonical electromagnetic energy momentum tensor introduced in (6.145)\(^1\), and where the corresponding canonical material energy momentum tensor is defined by

$$T^{a}_{MC \ b} = n^a_X \pi^x_{\ b} - \Psi_M g^a_b \cdot$$  \hspace{1cm} (28)$$
Subject to the field equations, the total energy-momentum tensor will of course be preserved, i.e.

\[ \nabla_b T^b_a = 0 \]  

(29)

and hence the separate parts will satisfy

\[ \nabla_b T^b_M_a = F_{abj}^b = -\nabla_b T^b_F_a , \]  

(30)

\[ \nabla_b T^b_{MC} a = j^b \nabla_a A_b = -\nabla_b T^b_F a . \]  

(31)

For any vector field \( \vec{\xi} \) we define associated canonical and metric energy-momentum flux vectors analogous to those introduced in section 6.4 \(^1\) by

\[ \vec{\mathcal{P}}\{\xi\} = \vec{\mathcal{P}}_{MC}\{\xi\} + \vec{\mathcal{P}}_F\{\xi\} , \quad \mathcal{P}^a_{MC}\{\xi\} = -\mathcal{T}^a_{MC b} \xi^b . \]  

(32)

\[ \vec{\Pi}\{\xi\} = \vec{\Pi}_{M}\{\xi\} + \vec{\Pi}_F\{\xi\} , \quad \Pi^a_{M}\{\xi\} = -T^a_{M b} \xi^b , \]  

(33)

the total being related by

\[ \mathcal{P}^a\{\xi\} = \Pi^a\{\xi\} + \nabla_b(D_{F}^{ba} A_c \xi^c) . \]  

(34)

It is obvious from (29) that the total metric energy-momentum fluxes will be conserved whenever \( \vec{\xi} \) is a Killing vector, and hence that the same applies to the canonical energy-momentum flux, i.e.

\[ \vec{\xi} \mathcal{L} \vec{g} = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \nabla \cdot \vec{\Pi}\{\xi\} = 0 \\ \nabla \cdot \vec{\mathcal{P}}\{\xi\} = 0 . \end{array} \right. \]  

(35)

(36)

However the canonical energy-momentum flux also has the property that is material part will be conserved separately whenever the fields are themselves invariant. Clearly the equation of motion (19) implies (by the Cartan formula (6.20) \(^1\)) that we shall have a generalised Bernoulli theorem to the effect that

\[ \vec{\xi} \mathcal{L} \vec{\pi}^X = 0 \quad \Rightarrow \quad \vec{n}_{|X|} \cdot \nabla(\vec{\xi} \cdot \vec{\pi}^X) = 0 , \]  

(37)

i.e. that \( \vec{\xi} \cdot \vec{\pi}^X \) is a constant of the motion, and hence, using (18) we see that

\[ \vec{\xi} \mathcal{L} \vec{\pi}^X = 0 , \quad \vec{\xi} \mathcal{L} \vec{\Psi}_M = 0 \quad \Rightarrow \quad \nabla \cdot \vec{\mathcal{P}}_{MC}\{\xi\} = 0 . \]  

(38)

This is to be compared with the corresponding result (6.157) \(^1\) for the electromagnetic component, which tells us that the same result would follow from purely electromagnetic invariance, i.e.

\[ \begin{array}{l} \vec{\xi} \mathcal{L} \vec{A} = 0 \\ \vec{\xi} \mathcal{L} \vec{g} = 0 \end{array} \quad \Rightarrow \quad \nabla \cdot \vec{\mathcal{P}}_{MC}\{\xi\} = 0 . \]  

(39)

It may be noticed that across any invariant hypersurface, i.e. one whose surface element satisfies

\[ \vec{\xi} \cdot d\Sigma = 0 \]  

(40)
the canonical material energy-momentum flux is interpretable as the total rate of transport of the Bernoulli quantity, i.e.

\[ \mathcal{P}_a \{ \xi \} \, d\Sigma_a = (\vec{\xi} \cdot \vec{\pi}^x) \, dN_x \]  

where \( dN_x \) denotes the number flux of the \( x \)-th constituent across \( d\Sigma \), i.e.

\[ dN_x = -\vec{n}_x \cdot d\Sigma. \]

### 4. Global energy and angular momentum

If we now consider the particular case of a stationary and/or axisymmetric system, in which there will be a conserved energy and/or angular momentum per particle given by

\[ \mathcal{E}^x = -\vec{k} \cdot \vec{\pi}^x \]

\[ \mathcal{J}^x = \vec{m} \cdot \vec{\pi}^x \]

then we shall have

\[ \mathcal{P}_a \{ k \} \, d\Sigma_a = -\mathcal{E}^x \, dN_x = \sum_x \vec{\mathcal{E}}_{CM}^x \cdot d\Sigma \]

\[ \mathcal{P}_a \{ m \} \, d\Sigma_a = \mathcal{J}^x \, dN_x = -\sum_x \vec{\mathcal{J}}_{CM}^x \cdot d\Sigma \]

across any respectively stationary or axisymmetric hypersurface, and in particular across a black hole horizon, where we define the fluxes

\[ \vec{\mathcal{E}}_{MC}^x = \mathcal{E}^x \vec{n}_{|x|} \]

\[ \vec{\mathcal{J}}_{MC}^x = \mathcal{J}^x \vec{n}_{|x|} \]

which evidently satisfy the separate conservation laws

\[ \nabla \cdot \vec{\mathcal{E}}_{MC}^x = 0 \]

\[ \nabla \cdot \vec{\mathcal{J}}_{MC}^x = 0. \]

In the case of angular momentum, the total exterior source contribution given by (6.322)\(^1\) may be expressed by

\[ J_{MC} = \sum_x \int_{\Sigma} \vec{\mathcal{J}}_{MC}^x \cdot d\Sigma \]

for any axisymmetric hypersurface \( \Sigma \), but the fact that \( \Sigma \) could not be invariant under \( \vec{k} \) prevents us from obtaining an equally simple formula for \( M_{MC} \).

We now proceed from (6) as in Section 6.6.1 using the strict stationary boundary conditions (6.314)\(^1\) to evaluate the surface integrals on the horizon in the form

\[ \delta \int_S \frac{1}{2} k^c A_c D_v^{ba} \, dS - \int_S D_v^c [b (\delta A_c) k^a] \, dS_{ab} = \Phi^u \delta Q_H + \Omega^u \delta J_{FH} \]

(52)
We next perform the analogous steps in the evaluation of the material contributions, using the formulae of the previous section which enable us to write firstly

\[ -\Pi^a_M \{ k \} = (\Lambda^a_M - n^b_X \mu^b_X )k^a + (k^b \mu^b_X )n^a_X \]  

and

\[ |g|^{-1/2} \delta (|g|^{1/2} \Lambda^a_M ) = \frac{1}{2} \left( T^b_M c h^b_b + n^b_X \mu^b_X h^c_c + \mu^b_X \delta n^b_X \right). \]  

Since we now only have source terms to deal with we proceed directly (i.e. without recourse to Green’s theorem) using (6.399) to obtain

\[ -\delta \int_\Sigma \Pi^a_M d\Sigma_a - \frac{1}{2} \int_\Sigma T^b_M c h^b_b k^a d\Sigma_a = \int_\Sigma k^b \mu^b_X \delta (n^a_X d\Sigma_a) + \frac{1}{2} \int_\Sigma k^b n^a_X (\delta \mu^b_X )d\Sigma_a \]  

in which the terms can be seen to be closely analogous to the volume integral terms in (6). Thus using the expression (12) for the total momenta we can combine (6) and (55) so as to obtain (with the aid of (52) and (6.320)) the final result

\[ \delta M - \Omega^u \delta J^1_B H - \Phi^u \delta Q^1_H - \frac{\kappa}{8\pi} \delta A = \int \bar{k} \cdot \bar{\pi} \delta (\bar{n} \cdot d\Sigma) + \int (\delta \bar{\pi} \cdot (\bar{k} \wedge \bar{n}) \cdot d\Sigma \]  

(in which \( J^1_B H = J_H + J^1_F H = J - J^1_{MC} \)).

5. Interpretation in the generic (convective) case

The first term on the right hand side of (56) can be interpreted as representing the static injection energy: it has the form

\[ \int \bar{k} \cdot \bar{\pi} \delta (\bar{n} \cdot d\Sigma) = \int \mathcal{E}^x \delta (dN_x) \]  

where \( dN_x \), as given by (42), is the flux of the \( X \)th constituent across \( d\Sigma \), and where \( \mathcal{E}^x \) is the canonical energy per particle, as given by (43), which satisfies the conservation law

\[ \bar{n}^x \cdot \nabla \mathcal{E}^x = 0 \]  

for each \( X \).

The form of the second term in (56) (which vanishes altogether in a static situation, i.e. one with all current vectors parallel to \( \bar{k} \)) may seem less familiar, but it is in fact a generalisation of the Faraday term \( I \delta F \) representing the energy needed to make a change \( \delta F \) in the magnetic flux \( F \) through a circuit carrying a current \( I \). The relationship can be made rather more apparent if we decompose each flux \( \bar{n}_X \) into a static part plus a spacelike part \( \bar{c}_X \) that is confined within the spacelike hypersurface \( \Sigma \), in the form

\[ \bar{n}_X = n_X^{(s)} \bar{k} + \bar{c}_X, \quad \bar{c}_X \cdot d\Sigma = 0 \]  

since we may then write

\[ \int (\delta \pi^x \cdot (\bar{k} - \bar{n}_X) \cdot d\Sigma = - \int (\bar{c}_X \delta \pi^x ) (\bar{k} \cdot d\Sigma). \]
We are always at liberty to decompose the hypersurface element $d\Sigma$ in the form
\[ d\Sigma = d\vec{x} \cdot dS \] (61)
where $d\vec{x}$ is an arbitrary infinitesimal displacement and $dS$ is a normal 2 surface element transverse to the displacement. If we take $d\vec{x}$ to be locally parallel to one of the fluxes, $\vec{c}_X$ say, we shall have
\[ -(\vec{c}_{X|}\delta \pi^X)(\vec{k} \cdot d\Sigma) = (d\vec{x} \cdot \delta \pi^X)dC_{|X|} \] (62)
where $dC_{|X|}$ represents the current of the $X$th constituent through the element $dS$ as defined by
\[ dC_X = \vec{c}_X \cdot (\vec{k} \cdot d\Sigma) \] (63)
Thus $dC_X$ represents the number of particles crossing the element $dS$ per unit of the ignorable (not proper) group time $t$. In general one can envisage that the flux lines in $\Sigma$ will have complicated ergodic properties. However if their global structure is such that they form sufficiently well behaved simple closed circuits then one can see from (62) that we shall have
\[ \delta M - \Omega^\mu \delta J_{BH} - \Phi^\mu \delta Q_{H} - \frac{\kappa}{8\pi} \delta A = \int \mathcal{E}^X \delta (dN_X) + \int (\delta C^X) dC_X \] (64)
where for each circuit the conserved circulation integral $C^X$, which is a generalisation of the Faraday flux, is as defined by (24).

6. The circular (non-convective) case

Despite the restriction imposed by the non-ergodicity hypothesis, the deceptively simple form (64) still represents quite a powerful generalisation compared not only with the original variation law of Bardeen et al (1973)\textsuperscript{2,3}, but even with the electromagnetic extension of Carter (1973b)\textsuperscript{4}, since these earlier versions applied only to flows with strictly (not just topologically) circular circuits. To show how these earlier versions are obtained in the appropriate, non-convective (and non-conducting) limit, we first introduce a vector $\vec{v}_X$ defined by
\[ \vec{v}_X = n_X^{(0)} \vec{v}_{|X|} \] (65)
and representing the mean displacement of the particles of the $X$th constituent per unit of the ignorable (not proper) group time $t$. (If there were any particles in the immediate neighbourhood of the black hole, their 3-velocities as thus defined would evidently have to tend towards the Damour\textsuperscript{7} velocity $\vec{v}_H$ – as defined in section 6.3\textsuperscript{1} – in the limit on the horizon.) We see that in terms of the corresponding unit flow vectors $\vec{u}_X$, defined by
\[ \vec{u}_X = n_X^{(0)} \vec{u}_{|X|}, \quad |\vec{u}_{|X|}|^2 = -1 \] (66)
the 3-velocity 4-vectors will be given by
\[ \vec{k} + \vec{v}_X = \tau_{|X|} \vec{u}_X \] (67)
where \( \hat{\tau}_X \) is the relevant time dilation (or “mean redshift”) factor. We can use these 3-velocities to regroup the terms on the right hand side of (56) so as to obtain the variation law

\[
\delta M - \Omega^\mu \delta J_{\mu H} - \Phi^\mu \delta Q_H - \frac{\kappa}{8\pi} \delta A = \int_\Sigma \pi^X_{(0)} \delta(dN_X) - \int_\Sigma \vec{\nu}_{X}^b \delta(\pi^X_{b} n_X^a) d\Sigma_a
\]

where

\[
\pi^X_{(0)} = -\left(\vec{k} + \vec{v}_{|X|}\right) \cdot \pi^X
\]

which is valid quite generally (regardless of the topological behaviour of the flow lines).

If we now impose the restriction that the flow lines be strictly circular, so that the 3-velocities take the form

\[
\vec{v}_X = \Omega_X \vec{m}
\]

where \( \Omega_X \) are the corresponding angular velocities then we shall have

\[
\delta M - \Omega^\mu \delta J_{\mu H} - \Phi^\mu \delta Q_H - \frac{\kappa}{8\pi} \delta A = \int_\Sigma \pi^X_{(0)} \delta(dN_X) + \int_\Sigma \Omega_X \delta(dJ_{MC}^X)
\]

with

\[
dJ_{MC}^X = \vec{J}_{MC}^X \cdot d\Sigma
\]

where the constituent angular momentum fluxes are given by (48). The quantity \( \pi^X_{(0)} \), which is a generalisation of what Bardeen has called the zero angular momentum injection energy, is given by

\[
\pi^X_{(0)} = \mathcal{E}^X - \Omega_{|X|} J^X = \mu^X_{(0)} + e^X \Phi_{|X|}(0)
\]

where the effective chemical potential \( \mu^X_{(0)} \) is related to the corresponding local rest frame chemical potential \( \mu^X \) by

\[
\mu^X_{(0)} = -\left(\vec{k} + \vec{v}_{|X|}\right) \cdot \mu^X = \hat{\tau}_{|X|} \mu^X
\]

with

\[
\mu^X = -\vec{u}_{|X|} \cdot \mu^X
\]

and where the corotating electric potential \( \Phi_{X(0)} \) is given by

\[
\Phi_{X(0)} = -\left(\vec{k} + \vec{v}_{|X|}\right) \cdot \vec{A} = \hat{\tau}_{|X|} \Phi_X
\]

with

\[
\Phi_X = -\vec{u}_{|X|} \cdot \vec{A}
\]
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