Non-zero $\theta_{13}$ in models for hierarchical neutrino mass spectrum

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Abstract

We introduce three right-handed Majorana neutrinos and combine the type-I seesaw and inert doublet mechanisms. The resultant (active) neutrino mass matrix is divided into rank = 1 and = 2 parts with different energy scales. The different energy scales are reduce to different mass scales in the hierarchical neutrino mass spectrum. We apply this scheme to both the inverted and normal hierarchy cases and find a correlation between the smallest mixing angle ($\theta_{13}$) and the lightest neutrino mass.

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I. INTRODUCTION

Thanks to neutrino oscillation experiments [1], we currently have convincing evidence that neutrinos have tiny masses and mix with each other through the Maki-Nakagawa-Sakata (MNS) leptonic mixing matrix. The recent global analysis of neutrino oscillation data yields the following best-fit values and 1σ errors [2]:

\[ \Delta m_{21}^2 = (7.59 \pm 0.20) \times 10^{-5} \text{eV}^2, \]

\[ \Delta m_{31}^2 = \begin{cases} 
- (2.36 \pm 0.11) \times 10^{-3} \text{eV}^2 & \text{for inverted hierarchy} \\
+ (2.46 \pm 0.12) \times 10^{-3} \text{eV}^2 & \text{for normal hierarchy} 
\end{cases}, \]

\[ \theta_{12} = (34.4 \pm 1.0)^\circ, \quad \theta_{23} = (42.8^{+4.7}_{-2.9})^\circ, \quad \theta_{13} = (5.6^{+3.0}_{-2.7})^\circ, \]

which indicate a bi-large mixing pattern and leave open three possibilities for the neutrino mass spectrum: the normal hierarchy \( m_3 \gg m_2 > m_1 \), inverted hierarchy \( m_2 > m_1 \gg m_3 \) and quasi-degenerate \( m_1 \simeq m_2 \simeq m_3 \) spectra. The individual neutrino masses as well as the correct mass spectrum remain unclear.

On the theoretical side, some extensions of the standard model (SM) to accommodate the tiny neutrino masses have been proposed. For instance, in the seesaw mechanisms [3–5], new heavy particles are introduced to generate neutrino masses suppressed by mass scales of the heavy particles, while such small neutrino masses can radiatively be induced from a loop diagram [6–8], too. Concerning the mixing, many constant-number-parametrizations (e.g., the democratic [9], bi-maximal [10] and tri-bimaximal [11] mixings) have been invented and a lot of efforts have been devoted to deriving them from a flavor symmetry. One of the most attractive features of these parametrizations is that they do not depend on the neutrino masses, so that no parameter tuning is required to obtain the desired mixing pattern. However, at the same time, it appears that this feature have made the mystery of the neutrino mass spectrum fade into the background. Theoretical studies on the mass spectrum seem subtle in comparison with those on the mixing: we still do not have any plausible model which can explain why only \( m_3 \) stands alone whereas \( m_1 \) and \( m_2 \) can be nearly degenerate in the hierarchical mass spectra, or why they are so degenerate in the quasi-degenerate spectrum.

In this Letter, we focus on the hierarchical mass spectra and explore a possibility that a mass generation mechanism for the lighter neutrino(s) is different from that for the...
TABLE I: The particle content and charge assignments in Scenario-A.

|       | $L_1$ | $N_S$ | $N_I$ | $H$ | $\eta$ |
|-------|-------|-------|-------|-----|--------|
| $SU(2)_L$ | 2     | 1     | 1     | 2   | 2      |
| $Z_2$   | +     | −     | +     | +   | −      |

heavier ones(one). Particularly, we consider the following two specific scenarios:

- Scenario-A
  
  The mass ordering is inverted. At the tree level, $m_{1,2}$ are non-zero and completely degenerate, while $m_3$ is vanishing. A small $m_3$ and mass splitting between $m_1$ and $m_2$ arise from radiative corrections.

- Scenario-B
  
  The mass ordering is normal. Only $m_3$ is non-zero at the tree level. $m_1$ and $m_2$ become non-zero after taking radiative corrections into account.

To this end, we combine the type-I seesaw [3] and inert doublet [8] mechanisms. The idea was originally proposed in Ref. [12] to simultaneously explain the relic abundance of dark matter, constrains from leptonic processes and the baryon asymmetry of the universe as well as the neutrino oscillation data. Here we take a closer look at the neutrino masses and try to find possible implications for the mixing angles; especially we are interested in correlations with the smallest mixing angle $\theta_{13}$. Similar studies are done in Refs. [13, 14] with a different particle content and/or setup.

This Letter is organized as follows. In Sec. II, we show a basic framework of our scheme and apply it to Scenario-A. We investigate Scenario-B in Sec. III and summarize our discussion in Sec. IV.

II. SCENARIO-A

A. basic framework

We extend the SM by introducing three right-handed Majorana neutrinos, $N$, and an inert $SU(2)_L$ doublet scalar, $\eta$, with a $Z_2$ symmetry. The particle content and charge
assignments are summarized in Table I. We require that all three right-handed neutrinos have super-heavy masses, say, $M = \mathcal{O}(10^{10-12})$ GeV, and that $\eta$ acquires a zero vacuum-expectation-value (VEV). The Lagrangian relevant to the following discussions is given by

$$L = Y_H \bar{L}_i \tilde{H} N_i + Y_\eta \bar{L}_i \tilde{\eta} N_S + \frac{1}{2} M_S N_S N_S + \frac{1}{2} M_I N_I N_I + h.c.$$, \hspace{1cm}(2)$$

$$V = \mu_1^2 H^\dagger H + \mu_2^2 \eta^\dagger \eta + \frac{\lambda_1}{2} (H^\dagger H)^2 + \frac{\lambda_2}{2} (\eta^\dagger \eta)^2 + \lambda_3 (H^\dagger H)(\eta^\dagger \eta) + \lambda_4 (H^\dagger \eta)(\eta^\dagger H) + \frac{\lambda_5}{2} [(H^\dagger \eta)^2 + h.c.]$$, \hspace{1cm}(3)$$

where $L_i$ stands for the left-handed $SU(2)_L$ doublet leptons and $H$ denotes the SM Higgs field with $\tilde{H} = i\sigma_2 H^\ast$. The subscript $i$ runs over 1 to 3 while $I$ is 1 or 2. Thus, $Y_H$ and $Y_\eta$ are $3 \times 2$ and $3 \times 1$ dimensional matrices, respectively. Notice that we have chosen the basis in which the charged lepton and right-handed neutrino mass matrices are diagonal, real and positive, and the real basis of $\lambda_5$ and $Y_\eta$.

By implementing the type-I seesaw mechanism, we obtain the following tree-level neutrino mass matrix:

$$M^0 = \frac{v^2}{M_1} \begin{pmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{pmatrix} + \frac{v^2}{M_2} \begin{pmatrix} D^2 & DE & DF \\ DE & E^2 & EF \\ DF & EF & F^2 \end{pmatrix}, \hspace{1cm}(4)$$

where $v = 174$ GeV is the VEV of the SM Higgs field and $A \cdots F$ are complex Yukawa couplings included in $Y_H$. Besides, we can induce a one-loop neutrino mass operator by exchanging $N_S$ and $\eta^0$ [8], and it results in

$$\delta M = \frac{v^2}{M_S} \begin{pmatrix} \alpha^2 & \alpha \beta & \alpha \gamma \\ \alpha \beta & \beta^2 & \beta \gamma \\ \alpha \gamma & \beta \gamma & \gamma^2 \end{pmatrix} \frac{\lambda_5}{8\pi^2} \left[ \ln \frac{M_S^2}{m_\eta^2} - 1 \right]$$, \hspace{1cm}(5)$$

where $\alpha \cdots \gamma$ are real Yukawa couplings included in $Y_\eta$. In Eq. (5), we have defined $m_\eta^2 \equiv \mu_2^2 + (\lambda_3 + \lambda_4)v^2$ and assumed $M_S^2 \gg m_\eta^2 \gg 2\lambda_5 v^2$ for simplicity. As one can see from Eqs. (4) and (5), the tree-level ($M^0$) and one-loop ($\delta M$) mass matrices are rank = 2 and 1, respectively, with different energy scales. Since $\delta M$ is suppressed with $\lambda_5/8\pi^2$ in comparison with $M^0$ in the case of $M_S \simeq M_I$, we conjecture that $M^0$ is responsible for the heavier-neutrino masses ($m_{1,2}$) and the lightest neutrino mass ($m_3$) originates in $\delta M$. Thus, this scheme suggests the inverted hierarchy spectrum.
B. neutrino masses and mixing

We apply the above scheme to Scenario-A and look at the neutrino mixing. Let us suppose that there exists a low-energy\(^1\) flavor symmetry which guarantees \(\theta_{13} = 0^\circ\) at the tree level. Hence, we consider the following tree-level mixing matrix:

\[
V^0 = \begin{pmatrix}
c_{12}^0 & s_{12}^0 & 0 \\
-s_{12}^0 c_{23}^0 & c_{12}^0 c_{23}^0 & s_{23}^0 \\
s_{12}^0 s_{23}^0 & -c_{12}^0 s_{23}^0 & c_{23}^0
\end{pmatrix},
\]

(6)

where \(c_{ij}^0 (s_{ij}^0) = \cos \theta_{ij}^0 (\sin \theta_{ij}^0)\). However, once we insist the degeneracy between \(m_1\) and \(m_2\) at the tree level, \(M^0\) in Eq. (4) may take the form of

\[
M^0 = V^0 \text{Diag}(m_0, m_0, 0) (V^0)^T = m_0 \begin{pmatrix}
1 & 0 & 0 \\
0 & (c_{23}^0)^2 & -s_{23}^0 c_{23}^0 \\
0 & -s_{23}^0 c_{23}^0 & s_{23}^0 c_{23}^0
\end{pmatrix}
\]

(7)

with a complex parameter \(m_0\), and this mass matrix is diagonalized by only \(\theta_{23}\). Thus, we start the discussion with Eqs. (7) and (6) with \(\theta_{13}^0 = 0^\circ\) at the tree level. Non-zero \(\theta_{12}, \theta_{13}, m_3\) and the mass splitting between \(m_1\) and \(m_2\) will arise after diagonalizing the full mass matrix \(M^\nu = M^0 + \delta M\) with the full mixing matrix

\[
V = V^0 \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23}^d & s_{23}^d \\
0 & -s_{23}^d c_{23}^d & c_{23}^d
\end{pmatrix} \begin{pmatrix}
c_{13} & 0 & s_{13} e^{-i\delta} \\
0 & 1 & 0 \\
-s_{13} e^{i\delta} & 0 & c_{13}
\end{pmatrix} \begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix} \Omega,
\]

(8)

where \(\theta_{23}^d = \theta_{23} - \theta_{23}^0\) and \(\Omega\) contains two Majorana CP-violating phases.

Prior to showing the results of numerical calculations, it may be useful to derive some approximate expressions of the mixing angles and masses. By taking the limit of \((s_{23}^d)^2 = 0\), we arrive at

\[
\tan 2\theta_{12} \simeq \frac{2\alpha(\beta c_{23} - \gamma s_{23}) c_{13} \delta m}{(c_{13}^2 - 1)m_0 + (\alpha c_{13})^2 \delta m - (\beta c_{23} - \gamma s_{23})^2 \delta m},
\]

(9)

\[
\tan 2\theta_{13} \simeq \frac{2\alpha(\beta s_{23} + \gamma c_{23}) \delta m}{|m_0 + \alpha^2 \delta m| e^{i\delta} - (\beta s_{23} + \gamma c_{23})^2 \delta m e^{-i\delta}|},
\]

(10)

\[
m_3 \simeq [(\beta s_{23} + \gamma c_{23}) c_{13}]^2 \delta m,
\]

(11)

\(^1\) We ignore corrections due to the RGE running effects.
FIG. 1: $\sin^2 \theta_{13}$ as a function of the lightest neutrino mass, $m_3$, (left panel) and $\sin^2 \theta_{23}$ (right panel) in Scenario-A. In the red (gray) region, $m_1$ ($m_2$) is slightly perturbed and decreased (increased) while corrections for $m_2$ ($m_1$) are negligibly small. The dotted and dashed lines display the 1σ upper bound of $\sin^2 \theta_{13}$ and best-fit values of $\sin^2 \theta_{13}$ and $\sin^2 \theta_{23}$, respectively.

where

$$\delta m = \frac{v^2}{M_S} \frac{\lambda_5}{8\pi^2} \left[ \ln \frac{M_S^2}{m_\eta^2} - 1 \right]$$

(12)

and $\alpha$, $\beta$ and $\gamma$ are real Yukawa couplings defined in Eq. (5). Notice that we have omitted some terms associated with $s_{13}$ in the expressions of $\tan 2 \theta_{12}$ and $m_3$. From the above expressions, one can see that $\theta_{12}$, $\theta_{13}$ and $m_3$ are not sensitive to the initial value of $\theta_{23}$ and find interesting correlations among them: e.g., when $\theta_{13}$ is non-zero (or zero), $m_3$ is also non-zero (or zero) since $\theta_{12} \neq 0$ restricts $\alpha$ to be non-zero. This correlation is not the result of the approximation we made. In Fig. 1 we numerically diagonalize the full neutrino mass matrix in the case of $\theta_{23}^0 = 45^\circ$ and plot $\sin^2 \theta_{13}$ as a function of the lightest neutrino mass, $m_3$, (left panel) with respect to the 1σ constraints of $\Delta m_{21}^2$, $\Delta m_{31}^2$, $\theta_{12}$ and $\theta_{23}$ given in Eq. (11). Since we are focusing on the hierarchical neutrino mass spectrum, we have fixed the absolute value of $m_0$ by the best-fit value of $\Delta m_{31}^2$, i.e., $|m_0| = \sqrt{2.36 \times 10^{-3}}$ eV, while varying its phase within 0 to 360°. As can be seen from the figure, there are two parameter regions in this model: in the red (gray) region, $m_1$ ($m_2$) is slightly perturbed and decreased (increased) while corrections for $m_2$ ($m_1$) are negligibly small. Nevertheless, the corrections for $m_1$ are sufficiently small in comparison.
FIG. 2: The rephasing-invariant Jarlskog parameter, $J_{CP}$, and the effective mass, $\langle m_{ee} \rangle$, of neutrinoless double beta decay. The legend of colored regions is the same as Fig. 1.

with $|m_0|$ and thus, $m_1$ can approximately be given by $m_1 \simeq |m_0|$. Therefore, the 1σ constraint of $\Delta m^2_{31}$ can be translated into an upper bound on $m_3$, which places an upper bound on $\theta_{13}$ and one can read off $\sin^2 \theta_{13} < 0.034$ ($\theta_{13} < 10.6^\circ$) from the red region. Interestingly, this upper bound is consistent with the recently reported T2K and MINOS results [15, 16], which indicate a relatively large $\theta_{13}$.

We also plot $\sin^2 \theta_{13}$ as a function of $\sin^2 \theta_{23}$ in the right panel. In the red regions, $\sin^2 \theta_{23}$ stays within $0.50 \pm 0.02$, while it can largely deviate from the initial value in the gray regions.

We remark that corrections to $\theta_{12}$ can in general be enhanced by the near degeneracy between $m_1$ and $m_2$ [17]. Therefore, we can always account for $\theta_{12} \simeq 34^\circ$ even starting from $0^\circ$.

C. CP violation

Since $\theta_{13}$ becomes non-zero after taking the radiative corrections into account and the model is described by a single CP-violating phase, it may be interesting to see a correlation
between the rephasing-invariant Jarlskog parameter:

\[ J_{CP} = \text{Im}[V_{e2}V_{\mu3}V_{e3}^{*}V_{\mu2}^{*}] \]  

(13)

and the effective mass of neutrinoless double beta decay (0\(\nu\)ββ):

\[ \langle m_{ee} \rangle = \left| V_{e1}^{2}m_{1} + V_{e2}^{2}m_{2} + V_{e3}^{2}m_{3} \right|. \]  

(14)

In Fig. 2 we plot \( \langle m_{ee} \rangle \) as a function of \( J_{CP} \) under the same conditions as Fig. 1. We find that the magnitude of \( \langle m_{ee} \rangle \) is around 0.046 \( \sim \) 0.049, which could be reachable in the near future experiments [18]. Moreover, \( J_{CP} \) is expected to be measured at long baseline neutrino oscillation experiments. Since \( J_{CP} \) and \( \langle m_{ee} \rangle \) are strongly correlated with each other in this model, these upcoming experiments may enable us to confirm or rule out the model.

III. SCENARIO-B

If we interchange the \( Z_{2} \) assignments of \( N_{S} \) and \( N_{I} \) in Table II in Sec. II-A becomes applicable to the normal hierarchy case\(^{2} \). In this case, \( N_{S} \) couples to the SM Higgs \((H)\) while \( N_{I} \) to the inert double \((\eta)\). Consequently, the tree-level and one-loop mass matrices turn out to be

\[ M^{0} = \frac{v^{2}}{M_{S}} \begin{pmatrix} \alpha^{2} & \alpha \beta & \alpha \gamma \\ \alpha \beta & \beta^{2} & \beta \gamma \\ \alpha \gamma & \beta \gamma & \gamma^{2} \end{pmatrix}, \]  

(15)

\[ \delta M = \delta m_{1} \begin{pmatrix} A^{2} & AB & AC \\ AB & B^{2} & BC \\ AC & BC & C^{2} \end{pmatrix} + \delta m_{2} \begin{pmatrix} D^{2} & DE & DF \\ DE & E^{2} & EF \\ DF & EF & F^{2} \end{pmatrix}, \]  

(16)

respectively, where the definitions of \( \delta m_{1} \) and \( \delta m_{2} \) are similar to that given in Eq. (12).

Let us apply this scheme to Scenario-B, namely, we presume that \( M^{0} \) is responsible for the heaviest neutrino mass \((m_{3})\) and \( \delta M \) for the lighter neutrino masses \((m_{1,2})\). Also,

\(^{2} \) Alternatively, one can simply assume \( \delta M \gg M^{0}. \)
we employ $V^0$ in Eq. (6) as the tree-level mixing matrix. As a result, $M^0$ may take the form of

$$M^0 = m_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & (s_{23}^0)^2 & s_{23}^0 c_{23}^0 \\ 0 & s_{23}^0 c_{23}^0 & (c_{23}^0)^2 \end{pmatrix}$$

and this mass matrix is again diagonalized by only $\theta_{23}$. The other neutrino masses and mixing angles are obtained after including $\delta M$ in Eq. (16). However, because $\delta M$ contains a lot of parameters, we cannot establish correlations among the neutrino masses and mixing angles. In order to do that, we simplify the mass matrix by imposing $C = B, F = -E, \theta_{23}^0 = 45^\circ$ and CP invariance. Then, the full neutrino mass matrix is given by

$$M'_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_0 \end{pmatrix} + \delta m_1 \begin{pmatrix} A^2 & 0 & \sqrt{2}AB \\ 0 & 0 & 0 \\ \sqrt{2}AB & 0 & 2B^2 \end{pmatrix} + \delta m_2 \begin{pmatrix} D^2 & \sqrt{2}DE & 0 \\ \sqrt{2}DE & 2E^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the diagonal basis of $M^0$. Roughly speaking, the second and third terms originate non-zero $\theta_{13}$ and $\theta_{12}$, respectively, and they are approximately expressed as

$$\tan 2\theta_{12} \simeq \frac{2\sqrt{2}DEc_{13} \delta m_2}{(2E^2 - D^2)\delta m_2 - A^2\delta m_1},$$

$$\tan 2\theta_{13} \simeq \frac{2\sqrt{2}AB\delta m_1}{(2B^2 - A^2)\delta m_1 - D^2\delta m_2 + m_0},$$

while corrections for $\theta_{23}$ are negligibly small. Moreover, $m_1$ and $m_2$ are given by

$$m_1 \simeq A^2c_{12}^2\delta m_1 + (Dc_{13}c_{12} - \sqrt{2}Es_{12})^2\delta m_2,$$

$$m_2 \simeq A^2s_{12}^2\delta m_1 + (Dc_{13}s_{12} + \sqrt{2}Ec_{12})^2\delta m_2.$$  

By requiring $m_0 = \sqrt{2.46 \times 10^{-3}}$ eV and 1$\sigma$ constraints of $\Delta m^2_{21}, \Delta m^2_{31}, \theta_{12}$ and $\theta_{23}$, we plot $\sin^2 \theta_{13}$ as a function of the lightest neutrino mass, $m_1$, (left panel) and $\sin^2 \theta_{23}$ (right panel) in Fig. 3. We note that corrections to $m_3$ are not negligible in this model, so that we have imposed $m_3 < \sqrt{(2.46 + 0.12) \times 10^{-3}}$ eV in order to keep the hierarchical mass spectrum. In this case, the 1$\sigma$ constraint of $\Delta m^2_{31}$ can be translated into an upper bound on $m_1$ and it leads to $\sin^2 \theta_{13} < 0.011$ ($\theta_{13} < 6.0^\circ$). Furthermore, $\theta_{23}$ remains almost maximal and this model indicates $\theta_{23} > 45^\circ$.

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3 A discrete flavor symmetry may realize these conditions. We show a simple realization based on the $D_4$ symmetry in Appendix.
IV. CONCLUSION

We have considered a combination of the type-I seesaw and inert doublet mechanisms with three right-handed Majorana neutrinos. The resultant (active) neutrino mass matrix is divided into rank = 1 and = 2 parts with different energy scales, and it suggests the hierarchical neutrino mass spectrum. We have applied this scheme to two scenarios in which both the lightest neutrino mass and a non-zero $\theta_{13}$ are radiatively induced via the inert doublet mechanism. We have found that the constraint of $\Delta m^2_{31}$ leads to an upper bound for the lightest neutrino mass, and it subsequently constraints the size of $\theta_{13}$. Given the $1\sigma$ constrains of Eq. (1), we have obtained $\sin^2 \theta_{13} < 0.034$ ($\theta_{13} < 10.6^\circ$) in Scenario-A. In Scenario-B, we have assumed a simple mass texture and gained $\sin^2 \theta_{13} < 0.011$ ($\theta_{13} < 6.0^\circ$).

As discussed in Refs. [12, 19], this kind of scheme possesses a great possibility for understanding other phenomena, such as the relic abundance of dark matter, some leptonic processes and the baryon asymmetry of the universe. Especially, since we have a unique CP-violating phase in Scenario-A, we may be able to directly relate the low-energy CP violation with leptogenesis. Further extensive studies including them could make a difference between our scheme and others. We shall study this issue elsewhere.
TABLE II: The particle content and charge assignments of the $D_4$ model.

|      | $L_1$ | $L_{D=2,3}$ | $N_S$ | $N_1$ | $N_2$ | $H$ | $\eta$ | $D$ | $S''$ | $S''''$ |
|------|-------|--------------|-------|-------|-------|-----|--------|-----|-------|---------|
| $D_4$ | $1'$   | 2            | 1     | $1''$ | $1''$ | 1   | 2      | 1'' | 1''   | $1''''$ |
| $Z_2'$ | +      | +            | -     | -     | -     | +   | -      | -   | -     | -       |

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Appendix A: $D_4$ flavor model

We show a simple realization of the mass matrix Eq. (18). In addition to the $Z_2$ symmetry, we introduce $D_4$-flavor and $Z_2'$-auxiliary symmetries with gauge singlet flavon fields $D$, $S''$ and $S''''$. The particle content and charged assignments are summarized in Table II and the tensor products of $D_4$ are given by

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} \otimes \begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = (x_1 y_1 + x_2 y_2) \oplus (x_1 y_1 - x_2 y_2)
\]

\[
2 \otimes 2 = 1 \oplus 1''
\]

\[
1' \otimes 1' = 1'' \otimes 1'' = 1''' \otimes 1''' = 1,
\]

\[
1' \otimes 1'' = 1''', \quad 1'' \otimes 1''' = 1', \quad 1' \otimes 1''' = 1''.
\]

Because of the symmetries, the Lagrangian of the neutrino sector is written as

\[
\mathcal{L} = \frac{\beta}{\Lambda} \mathcal{T}_D \bar{H} N_S D + \frac{A}{\Lambda} \mathcal{T}_{1\tilde{\eta}} N_1 S'' + \frac{B}{\Lambda} \mathcal{T}_{D\tilde{\eta}} N_1 D
\]
where we have written down only the leading terms and $\Lambda$ denotes a typical energy scale of the $D_4$ flavor symmetry. If we demand the VEV alignment: $\langle D \rangle \propto (1,1)$, the tree-level and one-loop neutrino mass matrices turn out to be

$$M^0 = \frac{\nu^2}{M_S} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta^2 & \beta^2 \\ 0 & \beta^2 & \beta^2 \end{pmatrix},$$  

(A5)

$$\delta M = \delta m_1 \begin{pmatrix} A^2 & AB & AB \\ AB & B^2 & B^2 \\ AB & B^2 & B^2 \end{pmatrix} + \delta m_2 \begin{pmatrix} D^2 & DE & -DE \\ DE & E^2 & -E^2 \\ -DE & -E^2 & E^2 \end{pmatrix},$$  

(A6)

respectively, where VEVs of the flavons and $\Lambda$ are included in the Yukawa couplings. $M^0$ can be diagonalized by the $45^\circ$ rotation in the 2-3 plane and then, we obtain the neutrino mass matrix given in Eq. (18). Furthermore, by adding extra Higgs doublets to the charged lepton sector, we can easily derive a diagonal charged lepton mass matrix [21].

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