COSMOLOGICAL SINGULARITIES*

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March 24, 2022

Abstract

An overview is provided of the singularity theorems in cosmological contexts at a level suitable for advanced graduate students. The necessary background from tensor and causal geometry to understand the theorems is supplied, the mathematical notion of a cosmology is described in some detail and issues related to the range of validity of general relativity are also discussed.

*To be published in the Springer LNP Proceedings of the First Aegean Summer School of Cosmology held on Samos, Greece, in September 21-29, 2001.
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1 Introduction

General relativity is the best theory we have for a dynamical description of spacetime, matter and gravitation. One might once hoped, with Einstein, that the evolution and structure of spacetime according to this theory would be free of singularities – places where the validity and predictions of the theory break down – and hence general relativity would represent the ‘final’ theory for the description of the physical world at the macroscopic level. However, general relativity assumes and implies the existence of spacetime and so the question of how such a spacetime structure can be created is logically outside the realm of this theory. This would not be a valid scientific question to ask if general relativity had an infinite range of validity or, equivalently, was a theory free from spacetime singularities for in that case it would have represented the final physical theory for macroscopic phenomena.

Unfortunately, the singularity theorems, first proven more than 30 years ago by Stephen Hawking, Robert Geroch and Roger Penrose, provide us with the bad news that indeed under certain conditions all generic spacetimes of general relativity will disappear in spacetime singularities either in the future or in the past. If this is true and the mathematical structure of general relativity implies the existence of spacetime singularities, then this theory cannot be the final one for the consistent description of the world. Its range of validity in such a case is finite and we must look for another theory that will answer the, now valid, question of how the spacetime of general relativity was created.

In such a new theory, the assumptions of the singularity theorems will lose their meaning in much the same way as that in which the assumptions of the Pythagorean theorem become vacuous in a curved space. This unknown, new, fully consistent framework has currently two offshoots: The first goes by the name of Quantum Theory of Gravity and asks for the complete connection of quantum mechanics and general relativity. It is hoped that in such a theory singularities are smoothed out in some way and a meaning of how the universe begun will emerge. The currently popular general approach to this problem is that quantum mechanics stays ‘untouched’ but general relativity is the one that needs modification. If this is assumed then there are four different ways in which modifications of general relativity can be accomplished:

1. Modify the action for gravity for example through the higher-order or scalar-tensor actions discussed below

2. Increase the number of spacetime dimensions in the ‘old Kaluza-Klein’ or in the new ‘Brane approach’ way
3. Introduce supersymmetry in the spacetime coordinates

4. Via String theory effective actions in which the spacetime worldlines are replaced by higher dimensional string worldsheets usually in combination with the three approaches above.

It is not our purpose here to discuss the different opinions which exist for these issues but only to state that there are at least two other, distinct from 1-4 above, quantum gravity approaches: One is the so-called Euclidean Quantum Gravity approach of Hawking and collaborators utilizing a complex-time approach through path integration, an offshoot of which is the theory of Quantum Cosmology (see for instance the collection of reprint papers in [1]). Last but not least, we mention an approach that Penrose has developed over the years and takes the point of view that it is not general relativity but quantum mechanics that needs modification and this modification will come as a consequence of our physical theory of spacetime – general relativity (see [2] for a popular account of this interesting set of ideas). It is also true that many important unsolved questions remain in this field.

The second, completely different (and inequivalent to the above) set of ideas in the search for the unknown, new theory is described by the String Theory/Noncommutative Geometry interface. In this framework, general relativity is not to be quantized but the problem is how classical spacetime, and general relativity, emerges from a completely new theory in which spacetime does not exist. What seem to exist instead at this level are some algebraic operators and certain abstract algebraic-geometric relations between them. It is not our purpose here to enter into the very interesting and, by and large, open problems of this highly complicated subject which is now in a state of intense development (cf. [3, 4]).

Of course an important issue not yet fully decided is connected with the nature of spacetime singularities in general relativity. All attempts up to now in this direction show that indeed singularities are typically connected with very complicated structures leading to the view discussed above. It may, however, be that spacetime singularities in general relativity eventually resemble the hydrodynamical shocks, but real progress in this direction seems almost impossible at present. We may therefore conclude that it is interesting (and legitimate!) to work on both aspects of the problem, namely, search for a new framework which will give meaning to spacetime itself or establish the true nature of generic spacetime singularities in general relativity.

Finalizing this speculative Introduction, we mention a last ingredient in our research path towards understanding these issues – cosmic censorship. This conjecture says that generic
spacetimes do not develop any singularities which are visible from infinity. That is, if this is true we may regard general relativistic singularities even if they exist in a most complex form as not so bad features after all and continue to believe in the ‘eternal power’ of general relativity. For a readable account of cosmic censorship, we refer the reader to [5].

It is therefore very basic to understand what is meant precisely by a singularity in general relativity and under what conditions singularities are expected to arise. We provide below an introduction to the singularity theorems of general relativity with special emphasis to those theorems that predict, under certain assumptions, the existence of spacetime singularities in the cosmological context. In the next Section we show what a cosmology is mathematically and how the results of the singularity theorems, usually assumed to hold only for general relativity, are in fact true for all theories which are conformally equivalent to general relativity provided the other assumptions of the theorems hold. More importantly, we show that violations of these theorems in such theories represent special cases and are not generic violations for these theories can all be regarded as containing special splitted forms of the energy-momentum tensors as compared to that of general relativity. Sections 3-7 provide the necessary background in spacetime geometry to understand the proof of the simplest singularity theorem and the statement of the most general result of this type given in Section 8. Finally, Section 9 gives (an introduction to) the physical interpretation of cosmological applications of the spacetime geometry techniques developed previously.

2 Cosmologies

The basic object of study in any mathematical approach to cosmological problems is that of a cosmology. There are three essential elements that go into a cosmology:

- A cosmological spacetime (CS)
- A theory of gravity (TG)
- A collection of matterfields (MF).

A cosmology is a particular way of combining these three basic elements into a meaningful whole:

\[
\text{Cosmology} = \text{CS} + \text{TG} + \text{MF}. \tag{2.1}
\]

Examples of cosmologies can be constructed by taking entries from the following table:

For instance we can consider the families:
Cosmologies

| Theories of gravity          | Cosmological spacetimes | Matterfields     |
|------------------------------|-------------------------|------------------|
| General Relativity           | Isotropic               | Vacuum           |
| Higher Derivative Gravity    | Homogeneous             | Fluids           |
| Scalar-Tensor theories       | Inhomogeneous           | Scalarfields     |
| String theories              | Generic                 | $n$-form fields  |

Table 1: Several members of each particular element of the three essential elements comprising a cosmology according Eq. (2.1).

- FRW/GR/vacuum
- Bianchi/ST/fluid
- Inhomogeneous/String/$n$-form
- Generic/GR/vacuum

and so on. In other Courses of this School you will study mathematical and physical properties of these and other cosmologies using exact (i.e., closed form) or special cosmological solutions. In this Course, however, we are interested in cosmological applications of certain geometric methods and results valid *independently* of specific choices of spacetimes, theories of gravity and matterfields, that is we develop (and then apply) *generic and global methods*.

The different theories of gravity appearing above are described by *actions* and their variations through the action principle give their associated field equations. For instance general relativity is described by the Einstein-Hilbert action and associated Einstein field equations:

$$ S = \int_{\mathcal{M}} (R + L_m) dv_g \Rightarrow R_{ab} - \frac{1}{2} g_{ab} R = T_{ab}. $$

Here $(\mathcal{M}, g)$ is the spacetime manifold with metric tensor $g_{ab}$, $dv_g$ is the invariant volume element of $\mathcal{M}$, $R_{ab}$ is the Ricci curvature tensor, $R$ the scalar curvature and $T_{ab}$ is the stress-energy tensor of the various matterfields appearing in the action defined through the *matter term* $L_m$ by

$$ T_{ab} = -\left( \frac{\partial L_m}{\partial g^{ab}} - g_{ab} L_m \right). $$

A general action for higher order gravity theories can be taken to be,

$$ S = \int_{\mathcal{M}} (f(R) + L_m) dv_g, $$

(2.4)
where \( f(R) \) is an arbitrary smooth function of the scalar curvature \( R \). Variation of this action leads to higher order field equations for the metric tensor describing the gravitational field. Next scalar-tensor theories are described by the general action,

\[
S = \int_{\mathcal{M}} (f(\phi, R) - g^{ab} \partial_a \phi \partial_b \phi + L_m) dv_g,
\]

where we see the general coupling of the scalar field \( \phi \) to the curvature. Furthermore string theories are multi-dimensional, scalar-tensor theories having additional scalar fields, formfields and supersymmetry.

Hence there is a large variety of seemingly different actions for the description of the gravitational field. A natural question arises as to which of these actions (if any) describes gravity correctly at all scales. In this connection, we quote a theorem proved in \([6, 7, 8]\) which clarifies the conformal structure of these (and other) gravity actions and shows in addition that all couplings of the scalar field to the kinetic term and to the curvature are essentially equivalent.

**Theorem 2.1 (Conformal equivalence)** *All higher order, scalar-tensor and string actions are conformally equivalent to general relativity with additional scalar fields which have particular (different in each case) self-interaction potentials.*

Hence all these actions are in fact *special cases* of the general relativity action (2.2) in the sense that any prediction made using any of the above variants of general relativity can also be made by going to the corresponding conformally related Einstein-matter system satisfied by the conformally equivalent metric tensor,

\[
\tilde{g}_{ab} = \Omega^2 g_{ab}, \quad \Omega^2 > 0.
\]

Here \( \Omega^2 \) is a positive function of the fields and takes a particular form when one conformally transforms a given action. For instance one takes \( \Omega^2 = f'(R) \) for higher order gravity whereas \( \Omega^2 = \phi \) for the simplest scalar-tensor theory, Brans-Dicke gravity. For these reasons we ‘restrict’ our attention to general relativity or, more accurately, the Einstein-Hilbert action (2.2).

It is indeed true that the conclusions of the singularity theorems when applied to cosmology can be changed when one modifies the action for gravity (for instance as in string theory) or considers ‘matter’ satisfying some ‘exotic’ energy condition etc. As a consequence of the above theorem on conformal equivalence, however, our viewpoint towards these changes is that they represent *special cases* and are not generic. In all such cases the matter lagrangian \( \tilde{L}_m \) in the so-called Einstein frame (and so the stress tensor) as a rule naturally *splits* into a term
describing the lagrangian of one or more scalar fields, $\tilde{L}_\phi$, and another term $\tilde{L}_M$ containing the remaining matter terms (fluids, formfields, vacuum, etc), i.e.,

$$\tilde{L}_m = \tilde{L}_\phi + \tilde{L}_M,$$

(2.7)
something that is not necessary in general. The singularity theorems, in particular, do not require such a splitting for the matter tensor among their assumptions. We therefore see that, as a consequence of the conformal equivalence theorem, all choices above (and others related to those) for possible ‘different’ gravity theories are actually included under the umbrella of the general Einstein-Hilbert action.

We return to our basic theme. There are two main areas in the study of global methods – *singularities and evolution*. We shall provide here an introduction to the theory of spacetime singularities and their applications in cosmology. The chapter by Choquet-Bruhat and York in this Volume, introduces the other main area in generic methods – global evolution.

A basic premise of this whole approach of applying spacetime differential geometry methods to cosmology and gravitation is that gravity is not like other physical fields. It is completely different because it shapes the spacetime on which it acts, while other fields act on a fixed background spacetime. In many respects we find ourselves in a similar situation as that of the ancient Greek geometers who, having constructed a fully consistent mathematical framework – euclidean geometry, proved many theorems corresponding to physical situations that gave rise to many interesting models of physical reality and eventually to the new framework we have today. Spacetime geometry and general relativity represent a radical departure from euclidean geometry that is here to stay irrespective of possible limitations (brought by the singularity theorems discussed below) that, one hopes, will lead in time to a new geometry and its physical interpretation.

The reader is referred to [9] for an introduction to current cosmological issues without the use of mathematics. Spacetime differential geometry, of which singularity theory forms an important component, has only comparatively recently (with a few important exceptions) stimulated the interest of mathematicians. It is really quite distinct from the more common Riemannian geometry, the latter now occurring in the study of submanifolds representing an instant of time in spacetime. Spacetime geometry has three branches, namely, the standard *tensor geometry* wherein curvature and geodesics are described, the more recent *causal geometry* where the various causality properties of spacetimes belong and *spin geometry* which is that branch concerned with spinor objects generalizing the usual tensor approach to geometry (or the equivalent exterior differential forms). Singularity theory, which we describe in these notes, can
be thought of as belonging to the interface of tensor and causal spacetime geometry. Helpful, general sources on spacetime geometry which are recommended for further reading are [10]-[19]. General references for cosmological applications of the material in these notes are [12], [16] and [18].

3 The spacetime metric

We assume some knowledge of manifolds, tensors and forms. An \(n\)-manifold \(\mathcal{M}\) is a topological space that locally looks like the Euclidean space \(\mathbb{R}^n\). This means that there is a homeomorphism (usually called a coordinate system or chart) of an open set of \(\mathcal{M}\) onto an open set of \(\mathbb{R}^n\). We assume that all manifolds in these notes are 4-dimensional, but actually all results will be valid in \(n \geq 2\) dimensions. Also our manifolds will be Hausdorff and connected.

Now let \(x \in \mathcal{M}\) and denote by \(T_x \mathcal{M}\) the tangent space of \(\mathcal{M}\) at \(x\), \(T_x \mathcal{M}^*\) the corresponding cotangent space, \(X(\mathcal{M})\) the set of all smooth vectorfields on \(\mathcal{M}\) and \(F(\mathcal{M})\) the set of all smooth, real–valued functions on \(\mathcal{M}\). For any non–negative integers \(r, s\), a tensor of type \((r,s)\) at \(x\) is an \(\mathbb{R}\)–multilinear map \(A : (T_x \mathcal{M}^*)^r \times (T_x \mathcal{M})^s \to \mathbb{R}\) and a smooth tensorfield of type \((r,s)\) on \(\mathcal{M}\) is an \(F(\mathcal{M})\)– multilinear map \((X(\mathcal{M}))^r \times (X(\mathcal{M}))^s \to F(\mathcal{M})\).

Thus a \((0,2)\) tensorfield can be identified with a (symmetric) bilinear form \(g(X,Y)\) on vectorfields of \(\mathcal{M}\). This is called nondegenerate provided \(g(X,Y) = 0\) for all \(Y \in X(\mathcal{M})\) implies \(X = 0\). A symmetric bilinear nondegenerate form \(g\) is called a scalar product.

At any \(x \in \mathcal{M}\) let \(e_1,\ldots,e_4\) be a basis of the tangent space at \(x\). The \(4 \times 4\) matrix \(g_{ab} = g(e_a,e_b)\) is called the matrix of the tensorfield \(g\) at \(x\) relative to the basis \(e_1,\ldots,e_4\). Then \(g\) is called Lorentzian if for every \(x \in \mathcal{M}\) there is a basis of the tangent space \(T_x \mathcal{M}\) relative to which the matrix of \(g\) at \(x\) has the form \(g_{ab} = \text{diag}(1,-1,-1,-1)\).

It is standard to write the scalar product of two vectorfields \(X^a, Y^b\) using the matrix \(g_{ab}\) in the form \(g(X,Y) = g_{ab}X^aY^b\). Since \(g\) is nondegenerate, \(g_{ab}\) is invertible and we denote the inverse matrix by \(g^{ab}\). Unless otherwise stated, we use the standard basis of vectorfields \(\partial_a\), denote by \(X^a\) the components of the vectorfield \(X\) with respect to that basis and call \(g_{ab}\) the components of the metric tensor \(g\) relative to the dual basis of one forms \(dx^a\) of \(\partial_a\), that is \(g = g_{ab}dx^adx^b\).

Sometimes it proves easier to deal with the function \(q\) defined pointwise on each tangent space by \(q(X) = g(X,X) = g(X^ae_a,X^be_b) = g_{ab}X^aX^b\) called the associated quadratic form of \(g\). Since the scalar product \(g\) is indefinite there may exist vectorfields \(X \neq 0\) with \(q(X) = 0\). \footnote{Such vectorfields are called null and exist only with indefinite scalar products. Two vectorfields \(X\) and \(Y\) are null if \(q(X) = q(Y) = 0\).}
Definition 3.1 (Spacetime) A spacetime is a pair \((\mathcal{M}, g)\) where \(\mathcal{M}\) is a manifold and \(g\) is a \((0,2)\) tensor field such that \(\mathcal{M}\) is: 4-dimensional, Hausdorff, connected, time-oriented and \(C^\infty\); \(g\) is globally defined, \(C^\infty\), nondegenerate and Lorentzian.

Time-orientability will be defined shortly. In a slight abuse of notation we often use \(\mathcal{M}\) to designate a spacetime. The simplest example of a spacetime is the Minkowski space \((\mathbb{R}^4, \eta_{ab})\) with \(\eta_{ab} = \text{diag}(1, -1, -1, -1)\). We note that,

Proposition 3.1 A spacetime is a paracompact manifold.

The geometric significance of the Lorentzian metric tensor in a spacetime \(\mathcal{M}\) is reflected in the nontrivial structure of the tangent spaces of \(\mathcal{M}\) at each point and derives from the following trichotomy.

Definition 3.2 (Vector character) Let \(\mathcal{M}\) be a spacetime and \(x \in \mathcal{M}\). We call a tangent vector \(v \in T_x \mathcal{M}\)

- spacelike if \(g_{ab}v^a v^b < 0\) (or, \(g(v, v) < 0\)) or \(v = 0\)
- null if \(g_{ab}v^a v^b = 0\) (or, \(g(v, v) = 0\)) and \(v \neq 0\)
- timelike if \(g_{ab}v^a v^b > 0\) (or, \(g(v, v) > 0\)).

We call \(v\) a causal vector if \(g_{ab}v^a v^b \geq 0\) (or, \(g(v, v) \geq 0\)).

The nullcone at \(x\) is the set of all null vectors of \(T_x \mathcal{M}\). The fact that the null cone is in fact a cone in the tangent space follows easily from the definition. From the definitions above also follows that timelike vectors are inside the null cone and spacelike ones are outside. The category into which a given tangent vector falls is called its causal character. The causal character of any vector \(v\) is the same as the causal character of the subspace \(\mathbb{R}v\) it generates.

A subspace \(\mathcal{W}\) of \(T_x \mathcal{M}\) is called nondegenerate if the restriction \(g|_{\mathcal{W}}\) is nondegenerate. A necessary and sufficient condition for the subspace \(\mathcal{W}\) of \(T_x \mathcal{M}\) to be nondegenerate is that \(T_x \mathcal{M} = \mathcal{W} \oplus \mathcal{W}^\perp\) where \(\mathcal{W}^\perp = \{v \in T_x \mathcal{M} : v \perp \mathcal{W}\}\), that is the tangent space at \(x\) is the direct sum of these two subspaces.

In our case (Lorentz scalar product) there will always be degenerate subspaces of \(T_x \mathcal{M}\), for example, the subspace spanned by a null vector. In fact there are three exclusive possibilities for \(\mathcal{W}\):

- orthogonal if \(g(X, Y) = 0\). Obviously a null vectorfield is orthogonal to itself.
\( g|_{\mathcal{W}} \) is positive definite; Then \( \mathcal{W} \) is an inner product space and \( \mathcal{W} \) in this case is said to be \textit{spacelike}.

\( g|_{\mathcal{W}} \) is nondegenerate; Then \( \mathcal{W} \) is \textit{timelike}.

\( g|_{\mathcal{W}} \) is degenerate; Then \( \mathcal{W} \) is \textit{null}.

The following lemma is used in the next proposition in an essential way.

**Lemma 3.2** If \( u \) is timelike then the subspace \( u^\perp \) is spacelike.

**Proof.** From the definition above it follows that since \( \mathbb{R}u \) is timelike \( g|_{\mathbb{R}u} \) is nondegenerate and so \( \mathbb{R}u \) is nondegenerate. Hence \( T_x\mathcal{M} = \mathbb{R}u \oplus u^\perp \) and \( u^\perp \) is nondegenerate. Therefore the index \( \text{ind} T_x\mathcal{M} = \text{ind} \mathbb{R}u + \text{ind} u^\perp \) i.e., \( \text{ind} u^\perp = 0 \) that is \( u^\perp \) is spacelike. □

**Proposition 3.3** The null cone disconnects the timelike vectors into two separate components.

**Proof.** Let \( \mathcal{T} \) be the set of all timelike vectors in \( T_x\mathcal{M} \). For any \( u \in \mathcal{T} \) define the \textit{future timecone} of \( u \),

\[
\mathcal{C}^+(u) = \{ v \in \mathcal{T} : g(u, v) > 0 \},
\]

(3.1)

and the \textit{past timecone} of \( u \),

\[
\mathcal{C}^-(u) = \{ v \in \mathcal{T} : g(u, v) < 0 \}.
\]

(3.2)

(Since \( u \) is timelike \( u \in \mathcal{C}^+(u) \) and \( -u \in \mathcal{C}^-(u) \).) Also obviously \( \mathcal{C}^+(u) \cap \mathcal{C}^-(u) = \emptyset \) and \( \mathcal{T} \subset \mathcal{C}^+(u) \cup \mathcal{C}^-(u) \) since, for instance, if \( v \) is timelike and does not belong to \( \mathcal{C}^+(u) \) then \( g(u, v) < 0 \) and so it belongs to \( \mathcal{C}^-(u) \). Finally the case where \( v \) is timelike and orthogonal to \( u \), \( g(u, v) = 0 \), is impossible since \( u^\perp \) is spacelike according to the preceding lemma. □

Therefore in each tangent space \( T_x\mathcal{M} \) of the spacetime there are two timecones \( \mathcal{C}^+ \) and \( \mathcal{C}^- \) which we can arbitrarily call the future timecone and the past timecone respectively. We call a vector \( u \in \mathcal{C}^+ \) \textit{future-pointing} and a \( v \in \mathcal{C}^- \) \textit{past-pointing}. There is no intrinsic way to distinguish a future timecone from a past timecone and to choose one of them is to \textit{time-orient} \( T_x\mathcal{M} \).

The existence of timecones discussed above raises a fundamental global question about a spacetime \( \mathcal{M} \): \textit{Can every tangent space in a spacetime be time-oriented in a suitable continuous way?}

**Definition 3.3 (Time orientability)** A spacetime \( \mathcal{M} \) (or the Lorentz metric \( g \) of \( \mathcal{M} \)) is called \textit{time-orientable} if it is possible to make a consistent continuous choice over all \( \mathcal{M} \) of one set of timelike vectors (say the future-pointing) at each point of \( \mathcal{M} \).
If such a choice has been made the spacetime is called *time-oriented*. In a time-oriented spacetime the null vectors at each point are also called future-pointing or past-pointing according as they are the limits of future-pointing or past-pointing timelike vectors.

**Definition 3.4 (Time function)** A (cosmic) **timefunction** on $\mathcal{M}$ is a map $t : \mathcal{M} \to T\mathcal{M} : x \mapsto C_x$ for each $x \in \mathcal{M}$. Such a function maps each point of $\mathcal{M}$ to a timecone at that point. A *timefunction* $t$ is smooth if for every $x \in \mathcal{M}$ there exists a neighborhood $\mathcal{U}$ of $x$ and a vectorfield $X$ on $\mathcal{U}$ such that for each $y \in \mathcal{U}$ we have $X_y \in C_y$.

A smooth timefunction on $\mathcal{M}$ is called a *time-orientation* on $\mathcal{M}$. To choose a specific time-orientation on $\mathcal{M}$ is to *time-orient* $\mathcal{M}$. Thus a spacetime is time-orientable if it admits a smooth timefunction (i.e., a time-orientation). Once we have chosen such a function we have time-oriented the spacetime. A basic criterion for time-orientability of a spacetime is provided by the following theorem.

**Theorem 3.4** Let $\mathcal{M}$ be a spacetime. Then the following two conditions are equivalent:

1. $\mathcal{M}$ is time-orientable
2. There exists a nowhere vanishing, smooth, timelike vectorfield $X$ on $\mathcal{M}$.

Directly related to the notion of time-orientability are the following two results which show respectively that not every smooth manifold can be made a spacetime but all spacetimes can essentially be regarded as time-oriented Lorentz manifolds.

**Theorem 3.5** For a smooth manifold $\mathcal{M}$ the following are equivalent:

1. There exists a Lorentz metric on $\mathcal{M}$
2. There exists a time-orientable Lorentz metric on $\mathcal{M}$
3. There exists a nowhere vanishing vectorfield on $\mathcal{M}$
4. Either $\mathcal{M}$ is noncompact, or $\mathcal{M}$ is compact and its Euler characteristic $\chi(\mathcal{M})$ is zero.

**Theorem 3.6** If a spacetime $\mathcal{M}$ is not time-orientable, there always exists a double-covering spacetime $\tilde{\mathcal{M}}$ which is time-orientable.

We now consider paths and curves. A *path* is a continuous map $\mu : I \to \mathcal{M}$ where $I$ is an open interval of $\mathbb{R}$. A *smooth path* is a path $\mu$ that is smooth and the differential $d\mu$ is nonzero for all values of the path parameter which we denote by $t \in I$. A (smooth) *curve* $\gamma$ is the image of a (smooth) path.
Definition 3.5 (Path character) A smooth path is called \textit{timelike (null)} if its tangent vector at every point is timelike (null). Such a path is called \textit{future-oriented} if its tangent vector is future pointing at every point of the path. We use the words \textit{causal path} for a timelike or a null path. A \textit{timelike curve} is the image of a timelike path. For a timelike curve we write $\gamma \subset \mathcal{M}$. Similarly, we speak of a \textit{future-oriented, smooth, causal curve}.

Notice that an arbitrary curve need not have a fixed causal character.

Next we define the notion of an \textit{endpoint} of a curve. A (smooth) \textit{curve segment} is a map $\gamma : [a, b] \to \mathcal{M}$, where $a = \inf I$ and $b = \sup I$ ($a, b$ can be $\mp\infty$ respectively), that has a continuous (smooth) extension to an open interval $J$ containing $[a, b]$. A point $x$ is an endpoint of a curve $\mu$ if $\mu$ enters and remains in every neighborhood of $x$. Notice that $x$ is not necessarily a point on $\mu$. The precise definition is as follows.

**Definition 3.6 (Endpoints)** An \textit{endpoint} of a path or a curve is a point $x \in \mathcal{M}$ such that, for all sequences $(a_i) \in I$ such that $a_i \to a$ we have that $\mu(a_i) \to x$, or if $a_i \to b$ we have $\mu(a_i) \to x$. If $\mu$ is causal and future-oriented then the first case defines a \textit{past endpoint} whereas the second a \textit{future endpoint}. Obviously a causal curve segment is causal at its endpoints. A timelike curve (or path) without a future (resp. past) endpoint must extend indefinitely into the future (resp. past) and is called \textit{future (resp. past) endless or inextendible}. A curve that is both future and past endless is called simply \textit{endless}.

An important function associated with any curve in spacetime is the \textit{length function} which we now define.

**Definition 3.7 (Length of curves)** Let $\mu$ be a smooth timelike curve in $\mathcal{M}$ with curve parameter $t \in [0, 1]$, endpoints $p = \mu(0)$ and $q = \mu(1)$ and let $V$ be its tangent vector field. We define the \textit{length of $\mu$} from $p$ to $q$ to be the function,

$$L(\mu) = \int_0^1 (g_{ab}V^aV^b)^{1/2} dt. \quad (3.3)$$

The length of a null curve is zero and for a spacelike curve we may take the definition above with a minus sign under the square root. We return to this function repeatedly below starting with our discussion of geodesics.

### 4 Derivatives

In this section we discuss collectively various useful derivative operators defined on the manifold $\mathcal{M}$:
Partial derivative: $\partial_\mu$ or $,\mu$

Differential of a map $\phi: \mathcal{M} \mapsto \mathcal{N}$: $\phi_*$

Exterior derivative: $d$

Lie derivative: $L_X$

Covariant derivative (connection or divergence): $\nabla$ or $;\mu$

Covariant derivative along a curve $\gamma(t)$: $\nabla_t$ or $\frac{D}{dt}$

Time derivative: $\dot{V}$

For simplicity, we discuss how these operators act on vector fields only, leaving their actions of higher rank tensor fields (or form fields) as an instructive exercise for the reader. We use both the index-free and index notations for vectors, tensors and forms invariably according to convenience.

We set as usual,

$$f, a \equiv \frac{\partial f}{\partial x^a} \cdot$$ (4.1)

Partial derivatives along vector fields coincide with the definition of the action of a vector field $X$ of $\mathcal{M}$ with coordinates $X^a$ on scalar fields (i.e., functions $f: \mathcal{M} \to \mathbb{R}$):

$$X(f) = X^a \partial_a f \equiv X^a f, a \cdot$$ (4.2)

Consider now a map $\phi: \mathcal{M} \to \mathcal{N}$ and a scalar field $f: \mathcal{N} \to \mathbb{R}$. We define the pull back of the scalar field $f$ to be a function on $\mathcal{M}$, $\phi^*f$, such that,

$$\phi^*f = f \circ \phi \cdot$$ (4.3)

Hence $\phi^*$ pulls back to $\mathcal{M}$ scalar fields defined on $\mathcal{N}$. For a vector field $X$ of $\mathcal{M}$ we define the derivative of $\phi$, $\phi_*$, to be

$$(\phi_*X)(f) = X(\phi^*f) \cdot$$ (4.4)

Thus the derivative of a map between two manifolds maps vector fields to vector fields in the way given above.
The exterior derivative operator $d$ maps $r$-formfields linearly to $(r+1)$-formfields. For example, the coordinate functions $x^a$ (0-formfields) are mapped to their differentials $dx^a$ (1-formfields). Consider now an $r$-formfield $A$ (that is a covariant tensorfield of rank $r$),

$$A = A^a \ldots d^{r-\text{times}} dx^a \ldots dx^d.$$  \hspace{1cm} (4.5)

Then the exterior derivative of $A$ is the $(r+1)$-formfield defined by,

$$dA = dA^a \ldots d^{r-\text{times}} dx^a \ldots dx^d.$$  \hspace{1cm} (4.6)

The chain rule for partial derivatives has an analogue expressed as the commutation of the exterior derivative and the pull back of a map, $\phi^*$, for an arbitrary $r$-formfield $A$:

$$d(\phi^* A) = \phi^* (dA).$$  \hspace{1cm} (4.7)

Next consider a fixed vectorfield $X$ and its flow, $\phi_t : M \rightarrow N$, $t \in \mathbb{R}$, that is a local 1-parameter group of diffeomorphisms that moves a point $p \in M$ a parameter distance $t$ along the integral curves of $X$ with the properties $\phi_{t+s} = \phi_t \circ \phi_s$, $\phi_{-t} = (\phi_t)^{-1}$ and $\phi_0 = \text{identity}$. Using the derivative $\phi_{t+s}$ of the flow we can carry any tensorfield $T$ of $M$ along the integral curves of the given vectorfield $X$ and observe how it evolves through the Lie derivative of the tensorfield $T$ with respect to the vectorfield $X$ defined by,

$$L_X T = \lim_{t \rightarrow 0} \frac{T - \phi_{t+s} T}{t}.$$  \hspace{1cm} (4.8)

(This is minus the Newton quotient.) One may easily show that $L_X$ is a linear map which preserves contractions and tensor type and satisfies a Leibniz rule. Further it follows from the definition that on scalar fields we have,

$$L_X f = X(f),$$  \hspace{1cm} (4.9)

and also for any vectorfield $Y$ one obtains (exercise),

$$(L_X Y)f = X(Yf) - Y(Xf) = [X,Y]f = -[Y,X]f.$$  \hspace{1cm} (4.10)

Thus two vectorfields $X,Y$ commute if the Lie derivative vanishes. In this case if one moves first along $X$ a parameter distance $t$ and then along $Y$ a distance $s$, one arrives at the same point as if he first goes along $Y$ a distance $s$ and then along $X$ a parameter distance $t$. This in turn means that the set of all points so visited forms a 2-dimensional immersed submanifold.
through the starting point\footnote{A $C^\infty$ map $\phi : \mathcal{M} \to \mathcal{N}$ is an immersion if $d\phi$ is one-to-one. An imbedding is an one-to-one immersion.}. From Eq. (4.7) it follows that the Lie derivative also commutes with the exterior derivative for any $r$-formfield $A$:

$$d(L_X A) = L_X (dA) .$$

The next derivative operator, the connection, is perhaps the most important of all. It satisfies:

**Theorem 4.1** In a spacetime $\mathcal{M}$ there is a unique torsion-free connection $\nabla$ under which the metric $g$ is covariantly constant (i.e., parallel).

Let $\partial_a$ be the standard coordinate vectorfields in the Minkowski space $\mathbb{M}^4$. As in the case of the Euclidean space $\mathbb{R}^n$, the covariant derivative $\nabla V W$ of a vectorfield $W \in X(\mathbb{M}^4)$ in the direction of a fixed $V \in X(\mathbb{M}^4)$ is given by

$$\nabla V W = V(W^a)\partial_a .$$

This definition however, does not extend to an arbitrary spacetime. We define a connection in spacetime by first giving a new definition of $\nabla V W$ valid in any given manifold $\mathcal{M}$ and then prove the above theorem in an equivalent (as we show) form which has been called the miracle of semi-Riemannian geometry.

**Definition 4.1** ($\nabla$) A connection on a manifold $\mathcal{M}$ is a map,

$$\nabla : X(\mathcal{M}) \times X(\mathcal{M}) \to X(\mathcal{M}) : (V, W) \mapsto \nabla V W,$$

such that for all $V, W, U, S \in X(\mathcal{M})$, the following properties hold:

(C1) $\nabla_{fV + gU} W = f\nabla_V W + g\nabla_U W$, for all $f, g \in F(\mathcal{M})$ (hence $\nabla$ is a tensor in the first argument $V$)

(C2) $\nabla_V (aW + bS) = a\nabla_V W + b\nabla_V S$, for all $a, b \in \mathbb{R}$

(C3) $\nabla_V (fW) = (Vf)W + f\nabla_V W$, for $f \in F(\mathcal{M})$.

The vectorfield $\nabla V W$ is called the covariant derivative of $W \in X(\mathcal{M})$ in the direction of $V \in X(\mathcal{M})$.

The components of a connection in a coordinate basis have a special significance.
Definition 4.2 (Christoffel symbols) Let \((x^a)\) be a coordinate system on \(U \subset \mathcal{M}\) and consider the vectorfield \(\nabla_{\partial_a} \partial_b\) which gives the rate of change of one coordinate vectorfield relative to another. Then the components of \(\nabla_{\partial_a} \partial_b\) in a coordinate basis \((x^a)\) are given by,
\[
\nabla_{\partial_a} \partial_b = \Gamma^c_{ab} \partial_c,
\]
and the functions \(\Gamma^c_{ab}\) are called the Christoffel symbols in the coordinate system \((x^a)\) on \(U\).

A property of the Christoffel symbols which is computationally advantageous is given in the following proposition.

Proposition 4.2 Let \(W = W^a \partial_a\) be a vectorfield on \(\mathcal{M}\) and \((x^a)\) a coordinate system on \(U \subset \mathcal{M}\). Then
\[
\nabla_{\partial_a} (W^b \partial_b) = \left( \partial_a W^c + \Gamma^c_{ab} W^b \right) \partial_c.
\]

Proof. Standard application of \((C3)\) with \(f = W^b\). Using \((C1)\) and \((4.14)\) we can compute the covariant derivative \(\nabla_V W\) for any \(V\).

It is not true that on an arbitrary manifold there exists a unique connection but in a spacetime the following theorem gives a uniqueness result for a connection that has three additional properties.

Theorem 4.3 On a spacetime \(\mathcal{M}\) there exists a unique connection called the Levi-Civita connection \(\nabla\) such that for all \(V, W, X \in X(\mathcal{M})\),

\[ (C4) \ [V, W] = \nabla_V W - \nabla_W V \quad \text{(torsion-free)} \]
\[ (C5) \ Xg(V, W) = g(\nabla_X V, W) + g(V, \nabla_X W) \quad \text{(Metric compatible)} \]
\[ (C6) \ 2g(\nabla_V W, X) = Vg(W, X) + Wg(X, V) - Xg(V, W) \]
\[ - g(V, [W, X]) + g(W, [X, V]) + g(X, [V, W]) \quad \text{(Koszul formula)}. \]

Condition \((C5)\) is equivalent to the metric \(g\) being covariantly constant i.e., \(\nabla_X g(V, W) = 0\). To see this, using the product rule we have,
\[
\nabla_X g(V, W) = (\nabla_X g)(V, W) + g(\nabla_X V, W) + g(V, \nabla_X W),
\]
and from \((C5)\),
\[
\nabla_X g(V, W) = (\nabla_X g)(V, W) + Xg(V, W),
\]
i.e., \((\nabla_X g)(V,W) = 0\) since on the left-hand side the covariant differential \(\nabla_X\) is a tensor derivation operating on the function \(g(V,W)\), which is just the last term on the right hand side of (4.16). Thus this is an equivalent version of the fundamental theorem 4.1.

**Proof of Theorem 4.3.** Let \(\nabla\) be a connection on \(\mathcal{M}\) satisfying properties (C1-C5). Applying (C5) on each of the first three terms on the right hand side of the Koszul formula, (C4) on each of the last three terms in the same formula and using the linearity and symmetry properties of \(g\) we find that most terms cancel leaving the term in the left hand side of (C6). Thus (C1-C5) imply the Koszul formula for \(\nabla\). If now \(\nabla\) is a second Levi-Civita connection on \(\mathcal{M}\) and we denote by \(F(V,W,X)\) the right hand side of the Koszul formula, then \(2g(\nabla V W, X) = F(V,W,X)\) and therefore \(g(\nabla V W, X) = g(\nabla V W, X)\) i.e., \(g(\nabla V W - \nabla V W, X) = 0\) for all \(X\). From the nondegeneracy of the metric \(g\) we conclude that the connection \(\nabla\) is unique. To show existence notice that the function \(X \mapsto F(V,W,X)\) is \(F(\mathcal{M})\)-linear and so a one-form. Therefore there exists a unique vectorfield, denote it by \(\nabla V W\), such that \(2g(\nabla V W, X) = F(V,W,X)\) for all \(X\). Hence the Koszul formula holds for \(\nabla V W\). It is a computational exercise to show that the Koszul formula implies properties (C1-C5). ■

Let now \(\mu : I \to \mathcal{M}\) be a smooth path on \(\mathcal{M}\) and denote by \(X(\mu)\) the set of all vectorfields along \(\mu\). A \(V \in X(\mu)\) takes each \(t \in I\) to a tangent vector in \(T_{\mu(t)}\mathcal{M}\).

**Definition 4.3 (\(\nabla_t\))** The **covariant derivative of \(V\) along \(\mu\)**, (also called the induced covariant derivative of \(V\)) is a map

\[
\nabla_t : X(\mu) \to X(\mu) : V \mapsto \nabla_t V
\]

with the following properties:

\(\text{IC1}\) \(\nabla_t(aV + bW) = a\nabla_t V + b\nabla_t W\), for \(a, b \in \mathbb{R}\)

\(\text{IC2}\) \(\nabla_t(fV) = f'V + f\nabla_t V\), for \(f \in F(I)\)

\(\text{IC3}\) \((\nabla_t W_\mu)(s) = \nabla_{\mu'(s)} W\), for all vectorfields \(W \in X(\mathcal{M})\) and \(s \in I\).

This definition is meaningful as we next show.

**Proposition 4.4** Let \(\mu : I \to \mathcal{M}\) be a smooth path on \(\mathcal{M}\). Then there exists a unique induced covariant derivative \(\nabla_t\) which has the properties (IC1)-(IC3) and also for every \(V, W \in X(\mu)\),

\[
\frac{d}{ds} g(V,W) = g(\nabla_t V, W) + g(V, \nabla_t W). \tag{4.17}
\]
Proof. Suppose that an induced covariant derivative $\nabla_t$ exists satisfying (IC1)-(IC3) and assume that the graph of $\mu$ lies in a single coordinate chart $x^a$. Then for $V \in X(\mu)$ with coordinates $V^a$ we have,

$$V(t) = V^a(t)\partial_a,$$

and by (IC1) and (IC2) it follows that on $\mu$,

$$\nabla_t V = \frac{dV^a}{dt}\partial_a + V^a\nabla_t \partial_a,$$

and since by (IC3) $\nabla_t \partial_a = \nabla_{\mu'(s)} \partial_a$ we deduce that,

$$\nabla_t V = \left(\frac{dV^a}{dt}\partial_a + V^a\nabla_{\mu'(s)} \partial_a\right),$$

which shows that $\nabla_t$ is solely determined by the unique $\nabla$. Hence uniqueness is proved. Using Eq. (4.20), it is not difficult to show that $\nabla_t V$ so defined satisfies properties (IC1)-(IC3) and Eq. (4.17) and therefore gives a single vectorfield in $X(\mu)$.  

In the special case for which the vectorfield $V$ along the path $\mu$ is the tangent vector of $\mu$, $V = \mu'$, the covariant derivative of $V$ along $\mu$ becomes the time derivative of $V$. To see this use (IC3) to write,

$$\nabla_t V = \nabla_{\dot{\mu}} V,$$

and calculate the vectorfield $\nabla_{\dot{\mu}} V$ in a coordinate basis to obtain,

$$\nabla_{\dot{\mu}} V = \nabla_{\dot{\mu}} V^a \partial_a = V^b \left[ (\partial_b V^a) \partial_a + \Gamma^a_{cb} V^c \partial_a \right],$$

that is

$$\nabla_{\dot{\mu}} V = V^b \nabla_b V^a \partial_a.$$

Using the index notation we thus have,

$$\dot{V}^a \equiv \nabla_t V^a = V^b \nabla_b V^a = V^b V^a,$$

This has the nice physical interpretation of being the acceleration of a flowline $\mu$ of a fluid and if $\mu$ is taken to mean the orbit of a particle moving on $\mathcal{M}$, $\dot{V}^a$ denotes the particle’s acceleration.

The time derivative of any tensorfield along $\mu$ is defined similarly (using the index notation) as,

$$\dot{T}^{a...d}_{e...g} = V^b \nabla_b T^{a...d}_{e...g},$$

where $V$ is the tangent vector to the path $\mu$.  

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5 Transport and geodesics

Definition 5.1 (Parallel transport) Let $\mu : I \to M$ be a smooth path on $M$ and $V \in X(\mu)$. The vectorfield $V$ is said to be parallelly transported along $\mu$ (or parallel) if,

$$\nabla_{\dot{\mu}} V = 0.$$ (5.1)

To see this condition expressed in coordinates set $\mu(t) = x^a(t)$ and take $V$ to be the tangent vectorfield, $V = \dot{x}^a$. Then as above $\nabla_{\dot{\mu}}$ becomes the time derivative of $V$ and using Eq. (4.23) we find,

$$\dot{V}^a + \Gamma^a_{bc}\dot{x}^b x^c = 0,$$ (5.2)
or,

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b \dot{x}^c = 0.$$ (5.3)

We see that the condition for parallel transport is equivalent to a nonlinear system of ODEs – the geodesic equations. By the fundamental existence and uniqueness theorem valid for such equations we deduce that for a path $\mu$ and points $p, q \in \mu$ one obtains a unique vector at $q$ by parallelly transporting a given vector at $p$ along $\mu$. Here by parallel transport along $\mu$ we mean a map,

$$P^b_a(\mu) : T_{\mu(a)}M \to T_{\mu(b)}M : v \mapsto V^b,$$ (5.4)

where the vectorfield $V$ is parallel and $V(a) = v$. It follows easily that $P^b_a$ is a linear isomorphism and from (4.17), if $V, W$ are parallel, it follows that $g(V, W)$ is constant. Hence taking two vectors at $p$ ($v, w \in T_p M$ with $V(a) = v$ and $W(a) = w$) we obtain,

$$g(P^b_a(v), P^b_a(w)) = g(V(b), W(b)) = g(V(a), W(a)) = g(v, w).$$ (5.5)

Therefore we have shown that the fundamental Theorem 4.1 takes still another equivalent form as follows:

Theorem 5.1 In a spacetime $M$ parallel transport along any curve preserves the scalar product defined by the metric $g$.

It is also true that in an arbitrary spacetime despite the situation in the Minkowski space $\mathbb{M}^4$ wherein one is using the same natural coordinates (distant parallelism), parallel transport depends on the particular path $\mu$ that we are using to move a vector placed at an initial point $p$. 

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to a final point $q$ via a parallel vectorfield. Equivalently, around a closed curve, the final vector $w$ obtained by parallelly transporting an initial vector $v$ need not be $v$, a phenomenon called holonomy. In this case all possible parallel vectorfields along $\mu$ are rotated through an angle called the holonomy angle.

As discussed above the induced covariant derivative operator $\nabla_t$ can be trivially applied to the tangent vectorfield $\mu'(s)$ of a path $\mu$ to give the vectorfield $\nabla_t \mu'$ called the acceleration of the path $\mu$. It is tempting sometimes to write for a vectorfield $V \in X(\mu)$, $\nabla_{\mu'} V = \nabla_t V$. In this notation, $\nabla_t \mu' = \nabla_{\mu'} \mu'$ which we sometimes abbreviate to $\mu''$. We shall see that this operation gives us important global information about the behaviour of a path.

**Definition 5.2 (Geodesics)** A geodesic in a spacetime $\mathcal{M}$ is a path $\mu : I \to \mathcal{M}$ such that for every $s \in I$,

$$\nabla_{\mu'} \mu' = 0,$$

that is $\mu$ has zero acceleration. Equivalently, a geodesic is a path such that its tangent vectorfield is parallel.

Thus geodesics satisfy (i.e., are solutions of) the geodesic equations. Geodesics have quite uniform behaviour. Every constant ($\mu' = 0$) curve is trivially a geodesic called a degenerate geodesic, but if for an $s_0 \in I$, $\mu'(s_0) \neq 0$ for a geodesic $\mu$ then, since geodesics by definition have constant speed, $\mu'(s) \neq 0$ for every $s \in I$. In this case $\mu$ is called a nondegenerate geodesic. In what follows all geodesics will be assumed nondegenerate. Thus a geodesic cannot slow down and stop.

Note also that the affine parameter $s$ of a geodesic $\mu$ is determined only up to transformations of the form $s \to as + b$ where $a, b$ are constants ($a$ corresponds to renormalizations of $\mu'$ of the form $\mu' \to (1/a)\mu'$ and $b$ to the freedom of choosing the initial point $\mu(0)$). All these affine parameters define the same geodesic.

**Definition 5.3 (Geodesic character)** A geodesic $\mu$ is called timelike, null, spacelike or causal if $\mu'$ is timelike, null, spacelike or causal respectively.

Notice that unlike an arbitrary curve a geodesic $\mu$ has necessarily one of the three causal characters. For if at an $s_0$ the tangent vector $\mu'$ is timelike ($g(\mu', \mu') > 0$) then, since by Theorem parallel transport preserves the scalar product defined by the metric $g$, $\mu'$ will always stay timelike.
Using the common abbreviation $x^a \circ \mu = x^a$ it follows from the existence and uniqueness theorem that the geodesic equations,

$$\ddot{x}^a + \Gamma_{bc}^a \dot{x}^b \dot{x}^c = 0,$$

yield for any $v \in T_x M$ a unique geodesic $\mu_v : I_v \to M$ that passes through $x$ i.e., $\mu_v(0) = x$ and has initial velocity $\mu'_v(0) = v$. We can therefore talk of a geodesic starting at $x$ with initial velocity $v$. We also deduce that the domain $I_v$ is the largest possible. Because of this $\mu_v$ is called maximal or inextendible. We therefore arrive at the following definition.

**Definition 5.4 (g-completeness)** A spacetime $M$ is called geodesically complete, if every maximal geodesic is defined on the entire real line. In this case we speak of complete geodesics.

For a different characterization and further properties of complete spacetimes we need to introduce some simple properties of the exponential map. Let $x \in T_x M$ and consider the subset $\Delta_x$ of $T_x M$ consisting of those $v \in T_x M$ such that the unique inextendible geodesic $\mu_v$ is defined at least on $[0,1]$ (that is $I_v \supset [0,1]$).

**Definition 5.5 (Exp map)** By the exponential map of $M$ at $x$ we mean a map,

$$\exp_x : \Delta_x \to M : v \mapsto \exp_x(v) = \mu_v(1).$$

If for every $x \in M$, $\Delta_x = T_x M$, then every inextendible geodesic of $M$ is defined on the whole real line that is $M$ is geodesically complete and so every inextendible geodesic extends to arbitrary parameter values.

The geometric meaning of the exponential map is obtained from the following result.

**Lemma 5.2** For any $x \in M$ the map $\exp_x$ carries radial lines through the origin of $T_x M$ to geodesics through $x$, that is $\exp_x(tv) = \mu_v(t)$.

**Proof.** Consider a fixed $t \in \mathbb{R}$ and a fixed $v \in T_x M$. Then the geodesic path $s \mapsto \mu_v(ts)$ has initial velocity $t \mu'_v(0)$, that is $tv$ which is obviously the initial velocity of $\mu_{tv}(s)$. Thus for all $s$ and $t$ we have $\mu_v(ts) = \mu_{tv}(s)$, and setting $s = 1$ we find,

$$\exp_x(tv) = \mu_{tv}(1) = \mu_v(t),$$

and this completes the proof of the Lemma. ■

It follows that the exponential map at a point on a spacetime collects together all geodesics passing through that point. Also if $Q$ is any neighborhood of the origin of $T_x M$, then $\exp(Q)$ is a neighborhood of $x$ diffeomorphic to $Q$ as the following Lemma shows.
Lemma 5.3 For every \( x \in M \) the exponential map at \( x \) carries a neighborhood \( Q \) of the origin \( 0 \in T_xM \) diffeomorphically onto a neighborhood \( N \) of \( x \) in \( M \).

Proof. This is easy using the inverse function theorem. It suffices to show that the differential map,

\[
d \exp_x : T_0(T_xM) \to T_xM,
\]

is the linear isomorphism \( v_0 \mapsto v \). If we set \( \rho(t) = tv \) and \( v_0 = \rho'(0) \) the claim follows from the fact that

\[
d \exp_x(v_0) = d \exp_x(\rho'(0)) = (\exp_x \circ \rho)'(0) = \mu'_x(0) = v.
\]

The proof is now complete.

In the following we assume that \( Q \) is starshaped about 0 in \( T_xM \), that is for each \( \lambda \in [0,1] \) the segments \( \lambda v \in Q \). In this case the neighborhood \( N \) in the above lemma is called a normal neighborhood of \( x \). When \( Q \) is an open disk of radius \( \varepsilon \) we speak of a normal neighborhood of radius \( \varepsilon \) and write \( N_{\varepsilon} \). We now give without proof the characteristic property of normal neighborhoods.

Proposition 5.4 If \( N \) is a normal neighborhood of \( x \in M \), then for each \( p \in N \) there exists a unique geodesic path \( \mu : [0,1] \to N \) joining \( x \) and \( p \) with initial velocity \( \mu'(0) = \exp^{-1}(x) \) in \( Q \). It follows that \( N \) uniquely describes \( Q \).

The part of a spacetime in a normal neighborhood of a point can be described in such a way that the length of a geodesic emanating from the point to one inside the normal neighborhood is given in a very simple form. To show this let \( e_{(a)} \) be a frame (orthonormal basis) of \( T_xM \) and \( N \) a normal neighborhood of \( x \). The normal coordinates \( x^a \) of a point \( p \in N \) are the coordinates of the tangent vector \( \exp_x^{-1}(p) \in Q \subset T_xM \) relative to the frame \( e_{(a)} \), that is \( \exp_x^{-1}(p) = x^a e_{(a)}[3] \)

Suppose now that a geodesic emanating from \( x \) to \( p \) has initial velocity \( \mu' = v \). The arc length of \( \mu \) is \( L(\mu) = \int_0^1 |\mu'(s)| ds \), where \( |\mu'(s)|^2 = g_{ab}(x^a \circ \mu)'(x^b \circ \mu)' \). We define the radius function of \( M \) at \( x \),

\[
r(p) = |\exp_x^{-1}(p)|,
\]

At \( x \) the metric \( g_{ab}(x) \) in terms of the normal coordinates takes the form \( g_{ab} = \text{diag}(1,-1,-1,-1) \) and thus \( \Gamma^a_{ab}(x) = 0 \). This also implies that covariant differentiation of any tensor at \( x \) is replaced, when normal coordinates at \( x \) are used, by common partial differentiation in terms of these normal coordinates.
which in normal coordinates is just \((x^1)^2 - (g_{ab}x^a x^b)^2\)^{1/2}. Since the exponential map sends lines to geodesics this definition uses implicitly the fact that the tangent vector to the geodesic \(\mu\) at \(p\) is \(\exp_{x}^{-1}(p)\). The radius function is thus smooth at all points except at \(x\). Since \(\mu'(s)\) is constant (arc length parametrization) we conclude that in these coordinates \(L(\mu) = r(p) = |v|\).

The radius function is positive, zero or negative according to whether the geodesic \(\mu\) from \(x\) to \(p\) is timelike, null or spacelike (that is \(\exp_{x}^{-1}(p)\) is!) respectively.

**Definition 5.6 (Simply convex)** Let \(\mathcal{M}\) be a spacetime. A set \(\mathcal{N} \subset \mathcal{M}\) is called **simply convex** if it is a normal neighborhood of each of its points. \(\mathcal{N}\) is called a **simple region** if it is open, simply convex and the closure \(\overline{\mathcal{N}}\) is a compact set contained in a simply convex, open set in \(\mathcal{M}\).

The entire manifold of Minkowski spacetime is simply convex. The characteristic property of a simply convex neighborhood \(\mathcal{N}\) is that there exists a unique geodesic lying entirely in \(\mathcal{N}\) connecting any pair of points \(p, q \in \mathcal{N}\) as it follows by applying Proposition 5.4. This nice local behaviour of geodesics in a normal neighborhood is in sharp contrast to what may happen globally in an arbitrary spacetime. In general there may exist points that can be connected by no geodesic or geodesics passing through a point may focus at some other point in a spacetime.

6 Conjugate points and geodesic congruences

The Riemann curvature tensor in this paper is determined by the equation,

\[
R(V,U)W = \nabla_{[V,U]}W - [\nabla_V, \nabla_U]W,
\]

for any three vectorfields \(V, U, W \in X(\mathcal{M})\). In this Section we discuss the problem of how to compare nearby geodesics. We start by introducing the notion of a 1-parameter family of geodesics. This is described by a map,

\[
x : [a, b] \times (-\varepsilon, \varepsilon) \to \mathcal{M} : (t, u) \mapsto x(t, u).
\]

We understand that for each constant value of the parameter \(u\) we have a geodesic parametrized by \(t\). Each such longitudinal geodesic has velocity \(x_t\) and acceleration \(x_{tt} = 0\). The transverse curves \(u \mapsto x(t, u)\) have velocity \(x_u\).

Consider now the variation vectorfield (or the vectorfield of geodesic variation) \(x_u(t,0) \equiv (d/du)_{u=0}x_u(t, u)\) on the longitudinal geodesic \(x(t, 0)\) (called the base or initial geodesic) which is a vectorfield along the geodesic. Since \(x_{tu} = x_{ut}\) we have,

\[
x_{utt} = x_{tut} = x_{ttu} + R(x_u, x_t)x_t = R(x_u, x_t)x_t,
\]

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i.e., the vectorfield \( x_u \), call it \( Y \), satisfies the linear second-order equation,

\[
x_{utt} = R(x_u, x_t)x_t.
\] (6.4)

This motivates the following more general notion.

**Definition 6.1 (Jacobi field)** Let \( \mu \) be a geodesic and \( Y \) a vectorfield along \( \mu \). We say that \( Y \) is a **Jacobi vectorfield** on \( \mu \) if \( Y \) satisfies the **Jacobi differential equation**, \( Y \)

\[
\nabla_{tt}Y = R(Y, \mu')\mu'.
\] (6.5)

This definition and the calculation above show that the variation vectorfield of the base geodesic in the 1-parameter family of geodesics is a Jacobi field. A way of interpreting the Jacobi equation that appears often in applications is to show that for a given vectorfield \( Y \) the values of both sides of the Jacobi equation are the same. Intuitively a Jacobi field connects points of the geodesic \( \mu \) with corresponding points of a neighboring geodesic \( \nu \).

The Jacobi equation is a linear differential equation and therefore the set of all Jacobi fields forms a real linear space. The dimension of this space is twice the dimension of \( T_x\mathcal{M} \) since any solution of the Jacobi equation is defined by specifying (arbitrarily) the value of the vector \( Y \) and that of the vector \( \nabla_t Y \) at any point on the geodesic.

**Definition 6.2 (Conjugate points)** We say that two points \( p = \mu(a) \) and \( q = \mu(b) \) on a geodesic \( \mu \) are **conjugate** along \( \mu \) provided there is a nonzero Jacobi field \( J \) on \( \mu \) such that \( J(a) = 0 \) and \( J(b) = 0 \).

Roughly speaking, a pair of conjugate points occurs when two neighboring geodesics meet at \( p \) and then refocus at \( q \). We can obtain a more general notion of conjugation if we replace one point in a pair of conjugate points by a submanifold of our spacetime. We restrict attention to the case of a **spacelike three-surface** \( \Sigma \) imbedded in the spacetime \( \mathcal{M} \). Thus \( \Sigma \) may be thought of as the three-dimensional graph of a \( C^2 \) function \( f \) defined locally by \( f = 0 \) and such that \( g^{ab}\nabla_a f \nabla_b f > 0 \) when \( f = 0 \). Consider then a congruence (see below) of timelike geodesics meeting \( \Sigma \) orthogonally. In this case we have the following definition.

**Definition 6.3 (Focal points)** A point \( p \) on a geodesic \( \gamma \) of a geodesic congruence orthogonal to \( \Sigma \) is called **conjugate to** \( \Sigma \) along \( \gamma \) (or a focal point) if there exists a Jacobi field along \( \gamma \) which is nonzero on \( \Sigma \) but vanishes at \( p \).

\(^{4}\)It is easily seen that the normal coordinate system defined by \( \exp_p \) becomes singular at \( q \).
We shall see below that geodesics are length-maximizing curves in a spacetime when they do not possess conjugate points. It is therefore basic to understand the precise conditions under which a pair of conjugate points will exist in a spacetime $\mathcal{M}$. Consider for this purpose a congruence of curves on spacetime. A congruence is a bunch of curves such that through each point of spacetime there passes precisely one curve from this bunch. A physically plausible way of visualizing this is to think of a congruence of curves as the flowlines of a fluid flow. Given a congruence of curves the tangent field is a well-defined vectorfield and conversely, it can be shown that every continuous vectorfield of $\mathcal{M}$ generates a congruence of curves. In the following we focus our attention to congruences of timelike geodesics. The theory of congruences of null geodesics, although analogous, is different and will not be treated here.

Consider a congruence of timelike geodesics and the cross-sectional area thought of as obtained by cutting the flowlines by a plane and taking the area enclosed by a small circle around the bunch. From the Jacobi equation through an argument which we omit it follows that, due to the tendency of geodesics to accelerate towards or away from each other, as the geodesic flowlines ‘move’ in spacetime this area can do three things: It can expand (or contract), it can get distorted or sheared, so that the circle enclosing it becomes an ellipse, or it can twist (i.e., rotate). How do we describe these three possible changes?

In the singularity theorems-related literature, the most common method to proceed is to form the covariant derivative $\nabla_b V^a$ of the tangent vectorfield $V^a (= g^{ab} V_b)$ to the flowlines of the geodesic congruence and eventually obtain a set of equations describing the evolution of the expansion and the other parameters defined above (the Raychaudhuri equations-see below). $\nabla_b V^a$ is the gradient of the fluid velocity vectorfield, and the aforementioned changes in the cross-sectional area during the evolution of the congruence are reflected in this derivative, for this measures the failure of the displacement vector between two neighboring geodesics to be parallely transported. To see this consider again the steps in the derivation of the Jacobi equation (6.5) and notice that the two coordinate vectorfields $\partial/\partial t$ and $\partial/\partial u$ (the second giving the displacement vector at each point) commute, that is if we call $V = \partial/\partial t$ and $Z = \partial/\partial u$ we find,

$$V^b \nabla_b Z^a = Z^b \nabla_b V^a,$$

which means that the failure of the RHS to be zero is equivalent to the failure of the connecting vector $Z$ is be parallely transported (i.e., LHS zero) along the geodesic flow. An intuitive way to say this is that an observer on the base geodesic will see nearby geodesics to be stretched and twisted.
Let $V$ be the unit tangent timelike vectorfield to the congruence ($g(V, V) = 1$) as above. This vectorfield clearly defines, at each point $q$ of the flow lines, a space $H_q$ orthogonal to it ($V_q \in T_q\mathcal{M}$ and $H_q \subset T_q\mathcal{M}$). Therefore we can take any vector $X_q \in T_q\mathcal{M}$ and project it in the direction orthogonal to $V_q$, that is in $H_q$. This can be done through the projection operator $h^a_b = g^{ac}h_{cb}$ where,

$$h_{ab} = g_{ab} + V_a V_b.$$  \hfill (6.7)

It can be shown that,

$$\nabla_b V_a = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab},$$  \hfill (6.8)

where we define the expansion $\theta$, the shear $\sigma_{ab}$ and the twist $\omega_{ab}$ of the geodesic congruence by

$$\theta = h^{ab}\nabla_b V_a,$$  \hfill (6.9)

$$\sigma_{ab} = \nabla_b(V_a) - \frac{1}{3}\theta h_{ab},$$  \hfill (6.10)

$$\omega_{ab} = \nabla_b [V_a].$$  \hfill (6.11)

To proceed further we understand that the basic quantities $\theta$, $\sigma_{ab}$ and $\omega_{ab}$ are functions of time and the question arises as to what are the equations that describe the evolution of these quantities. The resulting differential equations are the Raychaudhuri equations and play a central role in the proofs of the singularity theorems. We derive only the first such equation namely, the one that gives the evolution of the expansion $\theta$ as this is the most important for our purposes.

From our definition of the Riemann curvature tensor it follows that for any vectorfield $V$,

$$(\nabla_b \nabla_c - \nabla_c \nabla_b)V_a = -R_{bcd}^dV_d,$$  \hfill (6.12)

and so for our timelike $V$ we can calculate the time derivative of the tensorfield $\nabla_b V_a$,

$$V^c\nabla_c (\nabla_b V_a) = V^c\nabla_b(\nabla_c V_a) + R_{bca}^dV^cV_d$$  \hfill (6.13)

$$= \nabla_b(V^c\nabla_c V_a) - (\nabla_b V^c)(\nabla_c V_a) + R_{bca}^dV^cV_d$$

$$= - (\nabla_b V^c)(\nabla_c V_a) + R_{bca}^dV^cV_d.$$  

since the first term in the second line above contains the time derivative of the tangent vectorfield to the geodesic and is therefore zero. Tracing this equation with $h^{ab}$ and using the definitions $\theta$, $\sigma_{ab}$ and $\omega_{ab}$, we find,

$$V^c \nabla_c \theta \equiv \dot{\theta} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}^c V^c V^d.$$  \hfill (6.14)
This is the Raychaudhuri equation for the evolution of the expansion scalar $\theta$. Defining the scalar shear $\sigma$ and the scalar twist $\omega$ through the equations,

$$2\sigma^2 = \sigma_{ab} \sigma^{ab} > 0, \quad 2\omega^2 = \omega_{ab} \omega^{ab} > 0,$$

we can write the Raychaudhuri equation more compactly as,

$$\dot{\theta} = -\frac{1}{3} \theta^2 - 2\sigma^2 + 2\omega^2 - R_{ab} V^a V^b. \quad (6.16)$$

Suppose now that the congruence meets $\Sigma$ orthogonally (or that the curves in the congruence start from a point $p$). In this case the twist vanishes, $\omega_{ab} = 0$ (this follows from Frobenius theorem - see [16], page 435), and the Raychaudhuri equation (6.16) becomes,

$$\dot{\theta} = -\frac{1}{3} \theta^2 - 2\sigma^2 - R_{ab} V^a V^b. \quad (6.17)$$

This shows that provided $R_{ab} V^a V^b \geq 0$ all terms in the RHS are negative and so,

$$\dot{\theta} + \frac{1}{3} \theta^2 \leq 0, \quad (\theta^{-1})' \geq \frac{1}{3}, \quad (6.18)$$

or

$$\frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{1}{3} t, \quad (6.19)$$

where $\theta_0$ is the initial value of $\theta$. Therefore if we suppose that the congruence is initially converging, $\theta_0 < 0$, then within a time

$$t \leq \frac{3}{|\theta_0|}, \quad (6.20)$$

the expansion becomes infinite, $\theta \to -\infty$, that is there is a second conjugate point, say $q$, to $p$ or a focal point to $\Sigma$. In other words negative expansion implies refocusing or convergence of the geodesic congruence. This motivates the following definition.

**Definition 6.4 (TCC)** We say that a spacetime satisfies the timelike convergence condition if

$$R_{ab} V^a V^b \geq 0, \quad \text{for all timelike vectorfields } V^a. \quad (6.21)$$

If this holds for all null vectorfields $V^a$, then we call it the null convergence condition. By continuity the timelike implies the null convergence condition.

In the case when $R_{ab} V^a V^b = 0$ everywhere on $\gamma$, one can show that provided the tidal forces are not zero, $R_{abcd} V^a V^d \neq 0$, the shear term in the Raychaudhuri equation cannot vanish and therefore a similar argument as above establishes the existence of point conjugate to a point or a hypersurface. The above condition on the Riemann tensor is very important. We frame it in a definition which also includes the null case.
Definition 6.5 (GC) We say that a spacetime satisfies the **timelike generic condition** if

\[ R_{abcd} V^b V^c \neq 0. \tag{6.22} \]

We say it satisfies the **null generic condition** if

\[ V_{[a} R_{b]cd[e} V_f] V^c V^d \neq 0. \tag{6.23} \]

In such cases, we speak of a **generic vectorfield** \( V^a \).

When a vectorfield fails to satisfy a generic condition we call it **nongeneric**.

In the discussion above we have provided conclusive evidence (but not the proof which we omit) for the following result.

**Theorem 6.1** Let \((M, g)\) be a spacetime in which \( R_{ab} V^a V^b \geq 0 \) and also \( R_{abcd} V^a V^d \neq 0 \) for all timelike \( V \). Then every complete timelike geodesic possesses a pair of conjugate points. Also if \( \Sigma \) is a spacelike hypersurface with \( \theta_0 < 0 \) at some point \( q \in \Sigma \), then within a time \( t \leq 3/|C| \), \( C \) constant, there exists a point \( p \) conjugate to \( \Sigma \) along a geodesic \( \gamma \) orthogonal to \( \Sigma \) passing through \( q \) assuming that \( \gamma \) can be extended to these values.

Recall that an important property of geodesics is that they extremize the length function \( L(\gamma) \) among all possible curves connecting two points in spacetime. In fact, using the second variation of the length function we can show that this extremum is a maximum if and only if there are no conjugate points between the endpoints of the family.

**Theorem 6.2** Consider a 1-parameter family of smooth timelike curves connecting two points \( p \) and \( q \). Then the length function \( L(\gamma) \) has a maximum on a curve \( \gamma \) iff this curve satisfies the geodesic equations \( (5.3) \) with no points conjugate to \( p \) between \( p \) and \( q \).

In analogy with the point to point conjugation, the absence of conjugate points to the hypersurface \( \Sigma \) provides a necessary and sufficient condition for length maximization.

**Theorem 6.3** Consider a 1-parameter family of smooth timelike curves connecting a point \( p \) to \( \Sigma \). Then the length function \( L(\gamma) \) has a maximum on a curve \( \gamma \) iff this curve satisfies the geodesic equations \( (5.3) \) with no points conjugate to \( \Sigma \) between \( p \) and \( \Sigma \).

Before we proceed further we make the last definition of this Section. Let \( x \) be a 1-parameter family of geodesics through \( p \). The set of all points of \( M \) conjugate to \( p \) along geodesics from the family \( x \) is called a **caustic**. A caustic is, roughly speaking, the locus of points where consecutive geodesics intersect.
7 Causal geometry

We shall use the word \textit{trip} to indicate a curve that is piecewise a future–pointing, timelike geodesic and we understand that for any two points $x, y \in \mathcal{M}$ a trip from $x$ to $y$ is a trip with past endpoint $x$ and future endpoint $y$. Similarly we define a \textit{causal trip} to mean a curve that is piecewise a future pointing, causal (timelike or null) geodesic, possibly degenerate. Then two basic relations can be defined on $\mathcal{M}$.

\textbf{Definition 7.1 (Causality and chronology)} Let $\mathcal{M}$ be a spacetime and $x, y \in \mathcal{M}$. We say that $x$ \textit{chronologically precedes} $y$, $x \ll y$, if and only if there is a trip from $x$ to $y$.

$$x \ll y \iff \text{there is a trip from } x \text{ to } y.$$ (7.1)

We call the relation $\ll$ a \textit{chronology} relation on $\mathcal{M}$. We say that $x$ \textit{causally precedes} $y$, $x \prec y$, if and only if there is a causal trip from $x$ to $y$.

$$x \prec y \iff \text{there is a causal trip from } x \text{ to } y.$$ (7.2)

We call the relation $\prec$ a \textit{causality} relation on $\mathcal{M}$.

Evidently, $x \ll y$ implies $x \prec y$. Also, chronology and causality are transitive relations. Since degenerate causal geodesics are allowed it follows that $x \prec x$ is possible but in contrast $x \ll x$ means that there exists a closed trip with past and future endpoint $x$. A closed, nondegenerate causal trip connecting two distinct points $x, y$ is signified by $x \prec y$ and $y \prec x$.

The following definition gathers together the points $y$ of $\mathcal{M}$ that can be influenced by, or influence, a given point $x \in \mathcal{M}$. It is the fundamental definition of causal structure theory.

\textbf{Definition 7.2 (Futures and pasts)} Let $x$ be a fixed point in $\mathcal{M}$. The \textit{chronological future of} $x$ is the set

$$I^+(x) = \{ y \in \mathcal{M} : x \ll y \}.$$ (7.3)

The \textit{chronological past of} $x$ is the set

$$I^-(x) = \{ y \in \mathcal{M} : y \ll x \}.$$ (7.4)

The \textit{causal future of} $x$ is the set

$$J^+(x) = \{ y \in \mathcal{M} : x \prec y \}.$$ (7.5)
The causal past of $x$ is the set

$$J^-(x) = \{ y \in M : y < x \}. \tag{7.6}$$

For any given set $S \subset M$ we define its chronological future $I^+(S) = \cup_{x \in S} I^+(x)$ and similarly for the other ones.

The dual versions of any result are obvious and will always be assumed. We think of $I^+(x)$ as the set of all events in spacetime that can be influenced by what happens at $x$ and similarly for the other definitions. We call a spacetime chronological (resp. causal) if for every $x \in M$ we have $x \notin I^+(x)$ (resp. $x \notin J^+(x)$). Thus in a chronological (causal) spacetime there are no closed timelike (causal) curves. A spacetime is called distinguishing if for any two points $x, y \in M$ we have $x \neq y \Rightarrow I^\pm(x) \neq I^\pm(y)$. In a distinguishing spacetime points are distinguished by their chronological futures and pasts.

The following Proposition gives some simple properties of futures and pasts.

**Proposition 7.1** Let $x$ be any point in $M$ and $S \subset M$. Then the following are true:

1. $I^+(x)$ is open in $M$.
2. $I^+(S)$ is open in $M$.
3. $I^+(S) = I^+(\overline{S})$.
4. $I^+(S) = I^+(I^+(S)) \subset J^+(S) = J^+(J^+(S))$.

**Proof.** To show that $I^+(x)$ is open in $M$, take a $y \in I^+(x)$. Then there exists a trip $\gamma$ from $x$ to $y$. Consider a simple region $\mathcal{N}$ containing $y$ and choose a point $z$ in $\mathcal{N}$ which lies on the terminal segment of (the timelike, future oriented geodesic) $\gamma$. The initial velocity of the (timelike, future-oriented) geodesic $\mu = zy$ is $\mu'(0) = \exp_z^{-1}(y)$ and so it belongs to the open set $Q \subset \exp_z^{-1}(N)$ consisting of all timelike, future-pointing vectors of $\exp_z^{-1}(N)$. Since $\exp_z$ is a homeomorphism, the set $\exp_z(Q)$ is open in $M$ and obviously contains $y (= \exp_z(\mu'(0))$. By definition, $\exp_z(Q)$ contains all points that can be joined to $z$ by a timelike, future oriented geodesic and hence $\exp_z(Q) \subset I^+(z)$. From the transitivity property of $\ll$ we have that $I^+(z) \subset I^+(x)$ and the result follows.

The second claim is obvious since $I^+(S)$ is an arbitrary union of the open sets $I^+(x)$, $x \in M$.

The direction $I^+(S) \subset I^+(\overline{S})$ in the third claim is obvious. To prove the opposite inclusion take a $x \in \overline{S}$ and suppose that $y \in I^+(x)$. Then $x \in I^-(y)$ and if $x \in \partial S$ (otherwise the result is
obvious) then, since \( I^- (y) \) is open and every neighborhood of \( x \) contains a point in \( S \), it follows that there exists a neighborhood \( A \) of \( x \) in \( I^- (y) \) containing a point \( z \in S \). Hence \( z \in I^- (y) \) and so \( y \in I^+ (z) \), that is \( y \in I^+ (S) \).

Finally the last claim is shown as follows. Since \( x \ll y \) implies \( x \prec y \) we have \( \ldots \). Then we choose: The future endpoint of the connected component of \( \ldots \)

**Proof.** For the first suppose without loss of generality that \( x \ll y \), \( y \prec z \) and consider the trip \( \mu = xy \) and the causal trip \( \nu = yz \). Since \( \nu \) is compact there is a finite number of simple regions \( \mathcal{N}_1, \ldots, \mathcal{N}_r \) that cover \( \nu \). Set \( p_0 = y \in \mathcal{N}_{i_0} \) say. Then we choose: The future endpoint of the connected component of \( \nu \) in \( \mathcal{N}_{i_0} \) from \( p_0 \), call it \( p_1 \), and also a point in \( \mathcal{N}_{i_0} \), call it \( q_1 \), on the final segment of \( \mu \) different from \( p_0 \). Then \( q_1 p_1 \) is future-timelike. Then either \( p_1 = z \) in which case we are done, or \( p_1 \) is not in \( \mathcal{N}_{i_0} \). In the latter case suppose that \( p_1 \) is in some \( \mathcal{N}_{i_1} \). Then we choose: The future endpoint of the connected component of \( \nu \) in \( \mathcal{N}_{i_1} \) from \( p_1 \), call it \( p_2 \), and also a point in \( \mathcal{N}_{i_1} \), call it \( q_2 \), on the segment \( q_1 p_1 \) different from \( p_1 \). Then \( q_2 p_2 \) is future-timelike. Thus either \( p_2 = z \) in which case we are done, or we can repeat the argument.

The process will be terminated in a finite number of steps since there is a finite number of connected components of \( \nu \) in \( \mathcal{N}_i \).

For the second claim suppose that \( \mu \cup \nu \) is not a single (null) geodesic then this implies that there is a ‘joint’ at \( y \) meaning that the direction of \( \mu \) at \( y \) is different from the direction of \( \nu \) at \( y \). If we choose in a small neighborhood of \( y \), a point \( y_1 \in \mu \) and a point \( y_2 \in \nu \) then \( y_1 y_2 \) is a timelike geodesic and so we have \( x \ll y_1 \ll y_2 \prec y \), that is \( x \ll y \) by 1.
The last result is shown as follows. If the causal trip \( xy \) contains a timelike segment, then repeated application of (1) gives \( x \ll y \). If on the other hand all segments of \( xy \) are null, then by (2), \( x \ll y \) unless \( xy \) is a single null geodesic. This completes the proof of the last claim and that of the Proposition. ■

**Example 7.3** The Einstein cylinder provides a counterexample of the converse of (3) in Proposition 7.2. In this example, there exist two points \( x, y \) that can be joined by a null geodesic between and also \( x \ll y \). Also there is another pair of points \( x, z \) in which \( x \) and \( z \) can be joined by two null geodesics and \( x \not\ll z \).

For reasons that will be clear later we define a *causal curve* to be a curve \( \mu \) with the property that for all \( x, y \in \mu \) and for every open set \( A \) containing the portion of \( \mu \) from \( x \) to \( y \), there is a causal trip from \( x \) to \( y \) (or from \( y \) to \( x \)) contained entirely in \( A \).

Now fix a set \( S \subset \mathcal{M} \). We shall use the terminology *future set* for the chronological future \( I^+(S) \) and *past set* for the chronological past \( I^-(S) \) of \( S \). From Proposition 7.1 it follows that a set \( F \) is a future (respectively past) set if and only if \( F = I^+(F) \) (respectively \( F = I^-(F) \)). Future (and past) sets are obviously open sets, and if \( x \in F \) and \( x \ll y \) then also \( y \in F \). Examples of future sets include the chronological and causal futures of any set.

We call a set \( S \) *achronal* if there are no two points in \( S \) with a timelike separation, that is no two points are chronologically related: if \( x, y \in S \), then \( x \not\ll y \). Examples of achronal sets in Minkowski space include the future nullcone at the origin and the spacelike plane \( t = z \).

Consider next the boundary \( \partial \) of the future \( I^+ \) of a set \( S \) (that is the boundary of a future set) in the spacetime \( \mathcal{M} \).

**Definition 7.3 (Achronal boundary)** A set \( \mathcal{B} \subset \mathcal{M} \) is called an *achronal boundary* if it is the boundary of a future set, that is if

\[
\mathcal{B} = \partial I^+(S) .
\]  

(7.7)

Hence an achronal boundary \( \mathcal{B} \) is an achronal set for no two points in the boundary of a future set can be chronologically related (\( I^+(\mathcal{F}) \cap \partial \mathcal{F} = \emptyset \), where the closure of \( \mathcal{F} \) is defined as \( \overline{\mathcal{F}} = \mathcal{F} \cup \partial \mathcal{F} \)). It is also not difficult to establish that \( \mathcal{B} \) cannot be spacelike either (apart from possibly at \( S \) itself) and therefore it must be a null set. Achronal boundaries are topological (i.e., not necessarily smooth) 3-manifolds and are generated (i.e., made out) from null geodesics. The important properties of these null geodesic generators of achronal boundaries are summarized in the following result which we give without proof.
Theorem 7.4 Let $S \subset \mathcal{M}$ and $B = \partial I^+(S)$. If $x \in B \setminus \bar{S}$ is a future endpoint of a null geodesic $\mu \in B$ then $\mu$ is either past-endless or has a past endpoint on $\bar{S}$. Also every future extension of $\mu$ must leave $B$ and enter $I^+(S)$.

This basically says that every null geodesic generator has a future endpoint in the achronal boundary and if it intersects another generator it will have to leave the boundary and enter into the interior of the future. On the other hand, null geodesic generators are either past endless or can have past endpoints only on $S$.

Let $S$ be achronal. We define another achronal set the edge of $S$, $\text{edge}(S)$, which is the set of points $x \in S$ such that every neighborhood $Q$ of $x$ contains points $p \in I^-(x)$ and $q \in I^+(x)$, which can be joined by a trip in $Q$ that does not intersect $S$. If $\text{edge}(S) = \emptyset$, then $S$ is called edgeless. Every edgeless set must be closed.

There are three important constructions closely related to the notion of achronality, namely the domain of dependence, the Cauchy horizon and the Cauchy surface. We discuss each one of these in turn.

Let $S$ be an achronal, closed subset of $\mathcal{M}$.

Definition 7.4 (Domains of dependence) The domain of dependence (or Cauchy development) of $S$ is the set,

$$D(S) = \left\{ p \in \mathcal{M} : \text{every endless trip through } p \text{ meets } S \right\}.$$  \hfill (7.8)

The future domain of dependence of $S$ is the set,

$$D^+(S) = \left\{ p \in \mathcal{M} : \text{every past-endless trip through } p \text{ meets } S \right\}.$$  \hfill (7.9)

The past domain of dependence of $S$ is the set,

$$D^-(S) = \left\{ p \in \mathcal{M} : \text{every future-endless trip through } p \text{ meets } S \right\}.$$  \hfill (7.10)

It follows that $D(S) = D^+(S) \cup D^-(S)$ and that $S$ is contained in $D(S)$. Thus $D^+(S)$ is a closed set and denotes the region that can be predicted from knowledge of data on $S$. What are the future limits of that region? In other words, what is the set of those points $p \in D^+(S)$ such that events in $I^+(p)$ do not belong to $D^+(S)$? These questions lead to another achronal, closed set – the future boundary of the future domain of dependence: The future Cauchy horizon.

Definition 7.5 (Cauchy horizon) The future, past and total Cauchy horizon of $S$ is defined as respectively,

$$H^+(S) = \left\{ p \in \mathcal{M} : p \in D^+(S) \text{ but } I^+(p) \cap D^+(S) = \emptyset \right\}.$$  \hfill (7.11)
\[ H^{-}(S) = \left\{ p \in \mathcal{M} : p \in D^{-}(S) \text{ but } I^{-}(p) \cap D^{-}(S) = \emptyset \right\}, \tag{7.12} \]

\[ H(S) = H^{+}(S) \cup H^{-}(S). \tag{7.13} \]

or equivalently,

\[ H^{\pm}(S) = D^{\pm}(S) \setminus I^{\mp}(D^{\pm}(S)). \tag{7.14} \]

The future Cauchy horizon \( H^{+}(S) \) is another example of a achronal, closed set and it holds that \( H(S) = \partial D(S) \). Finally we have:

**Definition 7.6 (Cauchy surface)** A Cauchy surface for \( \mathcal{M} \) is an achronal set \( S \) such that,

\[ D(S) = \mathcal{M}. \tag{7.15} \]

To have a Cauchy surface \( S \) in a spacetime \( \mathcal{M} \) is a statement for both \( S \) and \( \mathcal{M} \). Intuitively speaking, if \( \mathcal{M} \) has a Cauchy surface then initial data on \( S \) determine the entire past and future evolution of \( \mathcal{M} \). The existence of a Cauchy surface is a global condition to impose on a spacetime, and it can happen that a surface \( S \) may appear to be a Cauchy surface for a spacetime \( \mathcal{M} \) during an early stage in the evolution, but later \( \mathcal{M} \) may develop in such a way so that no Cauchy surface can be admitted.

To proceed further we need to impose some global causality assumption on our spacetime. There is a number of such assumptions on the market the most important and also the most restrictive of all being the fundamental concept of global hyperbolicity, or \textit{hyperbolicité globale} if one wishes to be historically just!

**Definition 7.7 (Global hyperbolicity)** A spacetime \( \mathcal{M} \) is called \textit{globally hyperbolic} if it satisfies the following two conditions:

\textbf{Strong causality} It contains no closed or almost closed trips

\textbf{Compact diamond-shapes} For any two points \( p, q \), the intersection \( I^{+}(p) \cap I^{-}(q) \) has compact closure (or equivalently, the diamond-shaped sets \( J^{+}(p) \cap J^{-}(q) \) are compact)

The compactness of the sets \( J^{+}(p) \cap J^{-}(q) \) means that these diamond-shaped sets do not contain points at infinity or singular points, that is points that can be regarded as belonging to spacetime’s ‘edge’. This second condition requires intuitively speaking that in the region between any pair of points in spacetime there are no asymptotic regions or holes or singularities.
If on the other hand strong causality is violated in a spacetime this must be due to some *global* feature. In such a case, trips or causal trips starting near \( p \) will return to points near \( p \) without necessarily being closed. Thus global hyperbolicity is primarily a ‘cosmological’ condition. It can be shown that global hyperbolicity is equivalent to the existence of a Cauchy surface and is a stable property with respect to sufficiently small perturbations of the metric. We refer the interested reader to the excellent exposition of these basic causality properties given in [20].

We now give the last definition we need from causal structure theory.

**Definition 7.8 (Trapped)** A *future-trapped set* is an achronal, closed set \( S \subset \mathcal{M} \) for which the set,

\[
E^+(S) = J^+(S) \setminus I^+(S),
\]

(7.16)

*called the future horismos of \( S \),* is compact.

It follows that since \( S \subset E^+(S) \), any future-trapped set is itself compact. A very special example of a future-trapped set is an achronal, closed, spacelike hypersurface. The dual definition, a past-trapped set, is obvious.

### 8 Globalization and singularity theorems

The results of the Section 6 are local in character. They are concerned with conditions for the existence of length-maximizing curves in spacetime and obstructions to such curves. These conditions, we showed, are about the absence or existence of conjugate points to points or spacelike hypersurfaces in spacetime. However, the question of interest to us has to do with the structure of spacetime *globally*, that is on the whole, and it is unclear how the local results obtained so far concerning the behaviour of congruences of curves in spacetime can somehow be elevated to reveal information about the global behaviour of the spacetime itself. It is clear we need some method to *globalize* them. It is the purpose of the present Section to discuss this problem.

Consider the set \( \mathcal{K} \) of points in which \( \mathcal{M} \) is strongly causal (no closed or almost closed trips), a compact subset \( \mathcal{C} \) of \( \mathcal{K} \) and two closed subsets \( \mathcal{A} \) and \( \mathcal{B} \) of \( \mathcal{C} \). Denote by \( \mathcal{D} \) the set of all trips in \( \mathcal{K} \), by \( \mathcal{E} \) the set of all causal trips in \( \mathcal{K} \) and by \( \mathcal{F} \) the set of all causal curves in \( \mathcal{K} \). It is clear that \( \mathcal{D} \subset \mathcal{E} \subset \mathcal{F} \). We are interested in the sets of all causal curves in \( \mathcal{C} \) from a point of \( \mathcal{A} \) to a point of \( \mathcal{B} \) which we denote by \( C_\mathcal{C}(\mathcal{A}, \mathcal{B}) \) with \( C(\mathcal{A}, \mathcal{B}) \) meaning the set of all causal curves in \( \mathcal{K} \).

The basic idea can be summarized without proofs by the following *globalization procedure*:
GP1 Put a suitable topology on $C_C(A,B)$ so that $C_C(A,B)$ becomes a compact set.

GP2 Define a suitable length function $L : C_C(A,B) \to \mathbb{R}$ on this set and show it is upper semicontinuous.

GP3 Compactness on $C_C(A,B)$ implies that $L$ attains a maximum value on $C_C(A,B)$.

GP4 This maximum is always attained in globally hyperbolic spacetimes and can be arranged to be on a geodesic without conjugate points.

In this way all arguments of Section 6 can be globalized and we have extended the definition of the length of a smooth curve to that of a continuous curve.

We are now in a position to prove the simplest singularity theorem.

**Theorem 8.1** Let $(\mathcal{M}, g_{ab})$ be a spacetime such that the following conditions hold:

1. $(\mathcal{M}, g_{ab})$ is globally hyperbolic
2. $R_{ab} V^a V^b \geq 0$ for all timelike vectorfields $V^a$
3. There exists a smooth spacelike Cauchy surface $\Sigma$ such that the expansion $\theta$ of the past-directed geodesic congruence orthogonal to $\Sigma$ satisfies $\theta \leq C < 0$, where $C$ is a constant, everywhere on $\Sigma$.

Then no past-directed timelike curve from $\Sigma$ can have length greater than $3/|C|$, that is all past-directed timelike geodesics are incomplete.

**Proof.** The proof goes by reductio ad absurdum. Suppose there is a past-directed timelike curve $\mu$ and $p$ a point on $\mu$ lying beyond length $3/|C|$ from $\Sigma$. By (GP4), there exists a curve $\gamma$ of maximum length in $C(\Sigma, p)$ which obviously must have length greater than $3/|C|$ and $\gamma$ must be a geodesic with no conjugate points between $\Sigma$ and $p$. However, this contradicts Theorem 6.1 which predicts the existence of a conjugate point between $\Sigma$ and $p$. Therefore the curve $\mu$ cannot exist.

The following is (a corollary of) a general singularity theorem which combines both past and future singularities. Its proof can be found in pages 266-270 of [12].

**Theorem 8.2** $(\mathcal{M}, g_{ab})$ cannot be timelike and null geodesically complete if:

---

5By a technical argument which we omit, we can rule out the possibility that a continuous, nonsmooth curve exists which has length greater or equal to that of any geodesic. In essence one shows that a continuous, nonsmooth curve connecting any two points cannot be length maximizing, for if it fails to be a geodesic at a point we can deform it, in a convex normal neighborhood of that point, to obtain a curve of greater length.
1. $R_{ab}V^a V^b \geq 0$ for all timelike and null vectorfields $V^a$

2. The generic conditions hold for all timelike and null vectorfields

3. There are no closed timelike curves (the chronology condition)

4. $(\mathcal{M}, g_{ab})$ possesses at least one of the following: (a) A compact, achronal set without edge, (b) a closed trapped surface, (c) a point $p$ in $\mathcal{M}$ such that the expansion of every past (or every future) null geodesic congruence emanating from $p$ becomes negative along each geodesic of this congruence (i.e., the null geodesics from $p$ refocus).

9 Cosmological applications

The simplest singularity theorem, Theorem 8.1 (and in fact all such theorems about past-incomplete spacetimes), has profound implications in cosmology, for it predicts the existence of a cosmological singularity in the past a finite time ago in the form of timelike geodesic incompleteness for a universe which is globally hyperbolic and everywhere expanding at one instant of time. We now explain how this arises.

In order to apply the purely mathematical results about geodesic incompleteness discussed above to cosmology in a meaningful way, we have to connect somehow the geometry of the situation to the behaviour of matter in the real universe. As we discussed in the Introduction, we shall follow Einstein and postulate an interaction of spacetime geometry and the distribution of matter in the universe through the Einstein field equations,

$$R_{ab} - \frac{1}{2} g_{ab} R = T_{ab} \ .$$

When this is assumed, the purely geometric assumptions and results present in the singularity theorems above acquire immediate physical meanings. We comment on the physical ramifications of the singularity theorems here very briefly, referring those interested to the original papers [21]-[27].

Let us start with the convergence conditions (Assumption 2 of Theorem 8.1 or Assumption 2 of Theorem 8.2). If we write the Einstein equations in the equivalent form,

$$R_{ab} = T_{ab} - \frac{1}{2} g_{ab} T \ ,$$

the timelike convergence condition from Definition 6.4 yields restrictions on the matter content through the energy-momentum tensor.
Definition 9.1 (Energy conditions) The energy-momentum tensor $T_{ab}$ satisfies the **strong energy condition** if,

$$T_{ab}V^aV^b - \frac{1}{2} TV^aV_a \geq 0 \quad \text{for all timelike vectorfields } V^a. \quad (9.3)$$

It satisfies the **weak energy condition** if,

$$T_{ab}V^aV^b \geq 0 \quad \text{for all timelike vectorfields } V^a. \quad (9.4)$$

The weak energy condition is a weaker requirement on the energy-momentum tensor than the strong energy condition. By continuity, the former is also true for all null vectorfields.

We see that the singularity theorems place restrictions on the matterfields in the universe in the form of energy conditions on the energy-momentum tensor, independently of the detailed form of the matterfields. Also as we said in the Introduction, they do not assume any splitting in the form of the matter tensor. For example, since for an observer whose worldlines have tangent vectorfield $V^a$ we have,

$$T_{ab}V^aV^b = \text{energy density}, \quad (9.5)$$

it follows that the weak energy condition means that the energy density as measured by any observer is non-negative, which seems a very reasonable assumption on matter.

The strong energy condition is stricter than the weak energy condition but it is also reasonable on macroscopic scales since, through the Einstein equations, it is the corresponding inequality to the timelike convergence condition and so it means that ‘gravitation is always attractive’ in the sense that neighboring geodesics accelerate on the average towards each other. Only a positive cosmological constant $\Lambda > 0$ can induce a cosmic repulsion thus preventing gravity from being always attractive. Thus in all theories which in the Einstein frame (see Section 2) become like general relativity without a positive cosmological constant, and provided gravity remains attractive and the other conditions of the theorems hold, the singularity theorems apply.

The existence of closed timelike curves leads to severe difficulties of interpretation. For the simplest wave equation $u_{tt} - u_{xx} = 0$ on the $(x, t)$-torus, the only solution with $(t, x)$ identified with $(t+n, x+m\pi)$ is the $u = \text{const.}$ solution. So, it is not necessary for the singularity problem to consider closed timelike curves.

The generic conditions can fail along a geodesic only if we consider very special models (for more information, see [27], page 540). These conditions are therefore very general too.
We also see that the assumption of global hyperbolicity of Theorem 8.1 is absent from the general Theorem 8.2, and assumption 4 of this Theorem contains a much weaker version of assumption 3 of Theorem 8.1 namely, the universe is not assumed to be expanding everywhere. However, the results of this general singularity theorem are somewhat weaker as no information is given as to whether the singularity is in the future or past. One expects, however, that when a closed trapped surface is present the singularity is in the future and when the past nullcone starts reconverging the singularity is in the past.

We close this chapter with one final remark about the singularities discussed above. In general relativity singularities are described as timelike and null geodesic incompleteness and are shown to exist as a result of the singularity theorems proved above. These theorems use the Jacobi equation for the geodesic variation vectorfield, the timelike convergence (equivalently the energy) condition and some topological condition that comes as a result of a causality requirement. They do not use the Einstein equation (9.1). On the other hand, studies of the Cauchy and global existence problems for Eq. (9.1) (cf. [28, 29]) indicate that the general solution of Eq. (9.1) exists for an infinite proper time and lead us to suspect or expect that there would be no dynamical singularities for the Einstein field equations but these only appear when we impose some unphysical or nongeneric symmetry assumption (for example that the universe is described by a Friedmann or Bianchi metric). What is the relation between the apparent absence of any dynamical singularities of the Einstein equations and the general relativistic singularities in the geometric Hawking-Penrose sense of g-incompleteness via the Jacobi equation? We refer the reader interested in such questions to the very recent work [30].

The singularity theorems discussed in these notes are important results for cosmology and point to an aspect of cosmology which should not be forgotten namely, the application of rigorous mathematical techniques for the study of the universe is not a luxury. It may sometimes lead to profound changes of viewpoint for the approach and possible resolution of current and future cosmological issues. In many respects it reminds us of the well-known inscription in Plato’s Academy:

ΜΗΔΕΙΣ ΑΓΕΩΜΕΤΡΗΤΟΣ ΕΙΣΙΤΩ.

References

[1] G.W. Gibbons and S.W. Hawking (eds.): *Euclidean Quantum Gravity*, (World Scientific, 1993).
[2] R. Penrose: *The Emperor’s New Mind*, (Oxford University Press, 1989).

[3] D. Kastler (ed.): *Quantum Groups, Non-Commutative Geometry and Fundamental Physical Interactions*, (Nova Science Publishers, 1999); see also, D. Kastler: *Cyclic Cohomology within the Differential Envelope*, (Herman, Paris, 1988).

[4] S. Albeverio, J. Jost, S. Paycha and S. Scarlatti: *A Mathematical Introduction to String Theory*, (LNM225, Cambridge University Press, 1997).

[5] R. Penrose: *Singularities and Time-Asymmetry*. In: *General Relativity: An Einstein Centenary Survey*, S.W. Hawking and W. Israel (eds.), (Cambridge University Press, 1979).

[6] J.D. Barrow and S. Cotsakis: *Phys.Lett. B214* (1988) 515-518.

[7] K. Maeda: *Phys.Rev. D39* (1989) 3159.

[8] S. Cotsakis and J. Miritzis: *A Note on Wavemap-Tensor Cosmologies*, gr-qc/0107100.

[9] S. Cotsakis: *Current Trends in Mathematical Cosmology*, gr-qc/0107090.

[10] C.W. Misner, K.S. Thorne and J.A. Wheeler: *Gravitation*, (Freeman, 1973), Parts III, IV and VII.

[11] R. Penrose: *Techniques of Differential Topology in Relativity* (SIAM, 1972).

[12] S.W. Hawking and G.F.R. Ellis: *The Large Scale Structure of Space-Time* (Cambridge University Press, 1973).

[13] Y. Choquet-Bruhat, C. DeWitt-Morette and M. Dillard-Bleick: *Analysis, Manifolds and Physics: Basic Theory* (North Holland, 2nd ed., 1982).

[14] B. O’Neill: *Semi-Riemannian Geometry*, (Academic Press, 1983).

[15] R. Penrose, and W. Rindler: *Spinors and Space-Time, Volume I*, (Cambridge University Press, 1984).

[16] R.M. Wald: *General Relativity*, (Chicago University Press, 1984).

[17] J.K. Beem, P.E. Ehrlich and K.L. Easley: *Global Lorentzian Geometry*, (Dekker, 2nd ed., 1996).

[18] M. Kriele: *Spacetime*, (Springer, 1999).
[19] Y. Choquet-Bruhat, C. DeWitt-Morette: *Analysis, Manifolds and Physics: Applications* (North Holland, 2nd ed., 2000).

[20] R. Geroch: *J.Math.Phys.* **11** (1970) 437-49.

[21] R. Penrose: *Phys.Rev.Lett.* **14** (1965) 57-9.

[22] S.W. Hawking: *Phys.Rev.Lett.* **15** (1965) 689-90.

[23] S.W. Hawking: *Proc.Roy.Soc.Lond.* **A294** (1966) 511-21.

[24] S.W. Hawking: *Proc.Roy.Soc.Lond.* **A295** (1966) 490-93.

[25] R. Geroch: *Phys.Rev.Lett.* **17** (1966) 445-7.

[26] S.W. Hawking: *Proc.Roy.Soc.Lond.* **A300** (1967) 187-201.

[27] S.W. Hawking and R. Penrose: *Proc.Roy.Soc.Lond.* **A314** (1970) 529-48.

[28] D. Christodoulou and S. Klainerman: *The Global Nonlinear Stability of the Minkowski Space*, (Princeton University Press, 1993).

[29] Y. Choquet-Bruhat and V. Moncrief: *Future Global in Time Einsteinian Spacetimes with U(1) Isometry Group*, *Annales Henri Poincaré* (to appear).

[30] Y. Choquet-Bruhat and S. Cotsakis: *Global Hyperbolicity and Completeness*, *J.Geom.Phys.* (to appear), gr-qc/0201057.