TRANSFORMATIONS OF HARMONIC BUNDLES AND WILLMORE SURFACES

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Abstract. Willmore surfaces are the extremals of the Willmore functional (possibly under a constraint on the conformal structure). With the characterization of Willmore surfaces by the (possibly perturbed) harmonicity of the mean curvature sphere congruence [1, 5, 13, 19], a zero-curvature formulation follows [5]. Deformations on the level of harmonic maps prove to give rise to deformations on the level of surfaces, with the definition of a spectral deformation [5, 8] and of a Bäcklund transformation [9] of Willmore surfaces into new ones, with a Bianchi permutability between the two [9]. This text is dedicated to a self-contained account of the topic, from a conformally-invariant viewpoint, in Darboux’s light-cone model of the conformal n-sphere.

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1. INTRODUCTION

Among the classes of Riemannian submanifolds, there is that of Willmore surfaces, named after T. Willmore [23] (1965), although the topic was mentioned by Blaschke [1] (1929) and by Thomsen [21] (1923), as a variational problem of optimal geometric realization of a given compact surface in 3-space regarding the minimization of some natural energy.

Early in the nineteenth century, Germain [14, 15] studied elastic surfaces. On her pioneering analysis, she claimed that the elastic force of a thin plate is proportional to its mean curvature, \( H = (k_1 + k_2)/2 \), for \( k_1 \) and \( k_2 \) the maximum and minimum curvatures among all intersections of the surface with perpendicular planes, at each point. Since then, the mean curvature remains a key concept in the theory of elasticity.

In modern literature on the elasticity of membranes, a weighted sum of the total mean curvature, the total squared mean curvature and the total Gaussian curvature is considered the elastic energy of a membrane. By neglecting the total mean curvature, by physical considerations, and having in consideration the Gauss-Bonnet Theorem, T. Willmore defined the Willmore energy of a compact oriented surface \( \Sigma \), without boundary, isometrically immersed in \( \mathbb{R}^3 \) to be

\[
W = \int_{\Sigma} H^2 dA,
\]

averaging the mean curvature square over the surface.

From the perspective of energy extremals, the Willmore functional may be extended to isometric immersions \( \phi \) of compact oriented surfaces \( \Sigma \) in a general Riemannian manifold \( M \) of constant sectional curvature by means of half (or any other scale) of the total squared norm of

\[
\Pi_0 = \Pi - g_\phi \otimes \mathcal{H},
\]

the trace-free part of the second fundamental form \( \Pi \), for \( \mathcal{H} = \frac{1}{2} \text{tr}(\Pi) \), the mean curvature vector, and \( g_\phi \) the metric induced in \( \Sigma \) by \( \phi \). In fact, given \((X_i)_{i=1,2}\) a local orthonormal frame of \( T\Sigma \), the Gauss equation, relating the curvature tensors of \( \Sigma \) and \( M \), establishes, in particular,

\[
K - \hat{K} = (\Pi(X_1, X_1), \Pi(X_2, X_2) - (\Pi(X_1, X_2), \Pi(X_1, X_2),
\]

for \( K \) the Gaussian curvature of \( \Sigma \) and \( \hat{K} \) the sectional curvature of \( M \), and, therefore,

\[
|\Pi_0|^2 = \sum_{i,j} (\Pi_0(X_i, X_j), \Pi_0(X_i, X_j)) = 2(|\mathcal{H}|^2 - K + \hat{K}).
\]
Hence, for the particular case of surfaces in $\mathbb{R}^3$, the two functionals share critical points.

Willmore surfaces are the extremals of the Willmore functional. Constrained Willmore surfaces appear as the generalization of Willmore surfaces that arises when we consider extremals of the Willmore functional with respect to infinitesimally conformal variations, rather than with respect to all variations. The Euler-Lagrange equations include a Lagrange multiplier. Willmore surfaces are the constrained Willmore surfaces admitting the zero multiplier. The zero multiplier is not necessarily the only multiplier for a constrained Willmore surface with no constraint on the conformal structure, though. In fact, the uniqueness of multiplier characterizes non-isothermic constrained Willmore surfaces. Constant mean curvature surfaces in 3-dimensional space-forms are examples of isothermic constrained Willmore surfaces, as proven by J. Richter. A classical result by Thomson characterizes isothermic Willmore surfaces in 3-space as minimal surfaces in some 3-dimensional space-form.

It is well-known that the Levi-Civita connection is not a conformal invariant. Under a conformal change $g' = e^{2u}g$ of a metric $g$ on $\Sigma$, for some $u \in C^\infty(\Sigma, \mathbb{R})$, the Levi-Civita connections $\nabla$ and $\nabla'$ on $(\Sigma, g)$ and $(\Sigma, g')$, respectively, are related by

$$\nabla'_X Y = \nabla_X Y + (Xu)Y + (Yu)X - g(X, Y)(du)^*,$$

for $(du)^*$ the contravariant form of $du$ with respect to $g$, for all $X, Y \in \Gamma(T\Sigma)$. It follows that, under a conformal change of metric on a Riemannian manifold $M$, the second fundamental form of an isometric immersion $\phi : \Sigma \to M$ changes according to

$$(1.1) \quad \Pi'(X, Y) = \Pi(X, Y) - g_{\phi}(X, Y)\pi_{N_{\phi}}(\phi^*(du)^*),$$

for $\pi_{N_{\phi}}$ the orthogonal projection of the pull-back bundle $\phi^*TM$ onto the normal bundle $N_{\phi} = (d\phi(T\Sigma))^\perp$ and $\phi^*(du)^*$ the pull-back by $\phi$ of $(du)^*$; and, therefore,

$$\mathcal{H}' = e^{-2u\phi} \mathcal{H} - e^{-2u\phi} \pi_{N_{\phi}}(\phi^*(du)^*),$$

relating the respective mean curvature vectors. Hence, under a conformal change of the metric, the trace-free part of the second fundamental form remains invariant, so that its squared norm and the area element change in an inverse way, leaving the Willmore energy unchanged. In particular, this establishes the class of constrained Willmore surfaces as a conformally-invariant class.

Conformal invariance motivates us to move from Riemannian to conformal geometry. Our study is one of surfaces in $n$-dimensional space-forms from a conformally-invariant viewpoint. For this, we find a convenient setting in Darboux’s light-cone model of the conformal $n$-sphere, viewing the $n$-sphere not as the round sphere in the Euclidean space $\mathbb{R}^{n+1}$ but as the celestial sphere in the Lorentzian spacetime $\mathbb{R}^{n+1,1}$. 
A manifestly conformally-invariant formulation of the Willmore energy is presented, following the definition presented by Burstall, Ferus, Leschke, Pedit and Pinkall [7], in the quaternionic setting, for the particular case of $n = 4$.

A fundamental construction in conformal geometry of surfaces is the mean curvature sphere congruence, the bundle of 2-spheres tangent to the surface and sharing mean curvature vector with it at each point (although the mean curvature vector is not conformally-invariant, under a conformal change of the metric it changes in the same way for the surface and the osculating 2-sphere). From the early twentieth century, with the work of Blaschke [1], the family of mean curvature spheres has been known as the central sphere congruence. Nowadays, after Bryant’s paper [3], it goes as well by the name conformal Gauss map.

A key result by Blaschke [1] ($n = 3$) and, independently, Ejiri [13] and Rigoli [19] (general $n$) characterizes Willmore surfaces by the harmonicity of the central sphere congruence. The well-developed theory of harmonic maps, and, in particular, the integrable systems approach to these, then applies. The starting point is the fact that, for a map into a Grassmannian, harmonicity amounts to the flatness of a certain family of connections depending on a spectral parameter, according to Uhlenbeck [22]. A zero-curvature characterization of Willmore surfaces follows. This characterization generalizes to constrained Willmore surfaces, as established by Burstall and Calderbank [5].

The zero-curvature representation of the harmonic map equations allows one to deduce two kinds of symmetry: harmonic maps admit a spectral deformation [22], by exploiting a scaling freedom in the spectral parameter, and Bäcklund transformations, which arise by applying chosen gauge transformations to the family of flat connections, as studied by Terng and Uhlenbeck [20, 22]. Aiming to apply this theory to constrained Willmore surfaces, and in order to address the possibly non-harmonic central sphere congruences of constrained Willmore surfaces, the notion of perturbed harmonicity for a map into a Grassmannian is introduced [9]. It applies to the central sphere congruence and it provides a characterization of constrained Willmore surfaces.

A spectral deformation and Bäcklund transformations of perturbed harmonic maps into new ones are defined [9]. Some care is required to see that, when applied to the central sphere congruence of a constrained Willmore surface, each new map still is the central sphere congruence of a surface. Deformations on the level of perturbed harmonic maps prove [9] to give rise to deformations on the level of surfaces, with the definition of a spectral deformation and of Bäcklund transformations of constrained Willmore surfaces into new ones. This spectral deformation of constrained Willmore surfaces coincides, up to reparametrization, with the one presented by Burstall, Pedit
and Pinkall [8], in terms of the Schwarzian derivative and the Hopf differential, later defined by the action of a loop of flat connections, by Burstall and Calderbank [5].

The class of constrained Willmore surfaces is in this way established as a class of surfaces with strong links to the theory of integrable systems, admitting a spectral deformation and a Bäcklund transformation, with a Bianchi permutability between the two, as proven in [9]. All these transformations corresponding to the zero multiplier preserve the class of Willmore surfaces.

The isothermic surface condition is known [8] to be preserved under constrained Willmore spectral deformation. As for Bäcklund transformation of isothermic constrained Willmore surfaces, we believe it does not necessarily preserve the isothermic condition. In contrast, the constancy of the mean curvature of a surface in 3-dimensional space-form is preserved by both constrained Willmore spectral deformation, cf. [8], and constrained Willmore Bäcklund transformation, cf. [10], for special choices of parameters, with preservation of both the space-form and the mean curvature in the latter case. However, constant mean curvature surfaces are not conformally-invariant objects, requiring that we carry a distinguished space-form. This shall be the subject of a forthcoming paper.

2. Constrained Willmore surfaces and perturbed harmonicity

Consider \( \mathbb{C}^{n+2} = \Sigma \times (\mathbb{R}^{n+1,1})^C \) provided with the complex bilinear extension of the metric on \( \mathbb{R}^{n+1,1} \). In what follows, we shall make no explicit distinction between a bundle and its complexification, and move from real tensors to complex tensors by complex multilinear extension, with no need for further reference, preserving notation.

Throughout this text, we will consider the identification

\[
\wedge^2 \mathbb{R}^{n+1,1} \cong o(\mathbb{R}^{n+1,1})
\]

of the exterior power \( \wedge^2 \mathbb{R}^{n+1,1} \) with the orthogonal algebra \( o(\mathbb{R}^{n+1,1}) \) via

\[
u \wedge v(w) := (u, w)v - (v, w)u
\]

for \( u, v, w \in \mathbb{R}^{n+1,1} \). Given \( \mu, \eta \in \Omega^1(\Sigma \times o(\mathbb{R}^{n+1,1})) \), we use \( [\mu \wedge \eta] \) to denote the 2-form defined from the Lie Bracket \([ , ]\) in \( o(\mathbb{R}^{n+1,1}) \):

\[
[\mu \wedge \eta]_{(X,Y)} = [\mu_X, \eta_Y] - [\mu_Y, \eta_X],
\]

for all \( X, Y \in \Gamma(T\Sigma) \). We consider the bundle \( \text{End}(\mathbb{R}^{n+1,1}) \), and, more generally, any bundle of morphisms, provided with the metric defined by \( (\xi, \eta) := \text{tr}(\eta^t \xi) \) and we shall move from a connection on \( \mathbb{R}^{n+1,1} \) to a connection on \( \text{End}(\mathbb{R}^{n+1,1}) \) via \( \nabla \xi = \nabla \circ \xi - \xi \circ \nabla \), with preservation of notation. Note that, in the case of a metric connection \( \nabla \) on \( \mathbb{R}^{n+1,1} \), we have

\[
\nabla(u \wedge v) = \nabla u \wedge v + u \wedge \nabla v,
\]

for all \( u, v \in \Gamma(\mathbb{R}^{n+1,1}) \).
Our theory is local and, throughout this text, with no need for further reference, restriction to a suitable non-empty open set shall be underlying.

2.1. Real constrained Willmore surfaces.

2.1.1. Conformal geometry of the sphere. Our study is one of surfaces in the conformal \( n \)-sphere, with \( n \geq 3 \), in Darboux’s light-cone model \([12]\) of the latter. For this, contemplate the light-cone \( \mathcal{L} \) in the Lorentzian vector space \( \mathbb{R}^{n+1,1} \) and its projectivization \( \mathbb{P}(\mathcal{L}) \), provided with the conformal structure defined by a metric \( g_\sigma \) arising from a never-zero section \( \sigma \) of the tautological bundle \( \pi : \mathcal{L} \to \mathbb{P}(\mathcal{L}) \) via

\[
g_\sigma(X, Y) = (d\sigma(X), d\sigma(Y)).
\]

For \( v_\infty \in \mathbb{R}^{n+1,1}_\infty \), set

\[
S_{v_\infty} := \{ v \in \mathcal{L} : (v, v_\infty) = -1 \},
\]

an \( n \)-dimensional submanifold \( \mathbb{R}^{n+1,1} \). Given \( v \in S_{v_\infty} \),

\[
(2.1) \quad T_vS_{v_\infty} = \langle v, v_\infty \rangle^\perp.
\]

The fact that \( (v, v_\infty) \neq 0 \) establishes the non-degeneracy of the subspace \( \langle v, v_\infty \rangle \) of \( \mathbb{R}^{n+1,1} \), establishing a decomposition

\[
(2.2) \quad \mathbb{R}^{n+1,1} = \langle v, v_\infty \rangle \oplus T_vS_{v_\infty}.
\]

In its turn, the nullity of \( v \) establishes \( \langle v, v_\infty \rangle \) as a 2-dimensional space with a metric with signature \((1, 1)\), showing that \( S_{v_\infty} \) inherits from \( \mathbb{R}^{n+1,1} \) a positive definite metric. Furthermore: for \( v_\infty \) non-null, orthoprojection onto \( \langle v_\infty \rangle^\perp \) induces an isometry between \( S_{v_\infty} \) and \( \{ v \in \langle v_\infty \rangle^\perp : (v, v_\infty) = -1/(v_\infty, v_\infty) \} \), whereas, when \( v_\infty \) is null, for any choice of \( v_0 \in S_{v_\infty} \), orthoprojection onto \( \langle v_0, v_\infty \rangle^\perp \) restricts to an isometry of \( S_{v_\infty} \). We conclude that \( S_{v_\infty} \) inherits from \( \mathbb{R}^{n+1,1} \) a positive definite metric of (constant) sectional curvature \( -(v_\infty, v_\infty) \), defining a copy of the \( n \)-sphere, a copy of Euclidean \( n \)-space or two copies of hyperbolic \( n \)-space, according to the sign of \( (v_\infty, v_\infty) \).

By construction, the bundle projection \( \pi \) restricts to give a conformal diffeomorphism

\[
\pi_{|S_{v_\infty}} : S_{v_\infty} \to \mathbb{P}(\mathcal{L}) \setminus \mathbb{P}(\mathcal{L} \cap \langle v_\infty \rangle^\perp).
\]

In particular, choosing \( v_\infty \) time-like identifies \( \mathbb{P}(\mathcal{L}) \) with the conformal \( n \)-sphere,

\[
S^n \cong \mathbb{P}(\mathcal{L}).
\]

This model linearizes the conformal geometry of the sphere. For example, \( k \)-spheres in \( S^n \) are identified with \((k + 1, 1)\)-planes \( V \) in \( \mathbb{R}^{n+1,1} \) via \( V \mapsto \mathbb{P}(\mathcal{L} \cap V) \subset \mathbb{P}(\mathcal{L}) \).
2.1.2. Surfaces in the light-cone picture and central sphere congruence. For us, a map $\Lambda : \Sigma \to P(L)$ is the same as a null line subbundle of the trivial bundle $\mathbb{R}^{n+1,1} = \Sigma \times \mathbb{R}^{n+1,1}$, in the natural way. From this point of view, sections of $\Lambda$ are simply lifts of $\Lambda$ to maps $\Sigma \to \mathbb{R}^{n+1,1}$. Given such a $\Lambda$, we define

$$\Lambda^{(1)} := \langle \sigma, d\sigma(T\Sigma) \rangle,$$

for $\sigma$ a lift of $\Lambda$. For further reference, note that $\Lambda$ is an immersion if and only if the bundle $\Lambda^{(1)}$ has rank 3.

Let then $\Lambda : \Sigma \to P(L)$ be an immersion of an oriented surface $\Sigma$, which we provide with the conformal structure $C_\Lambda$ induced by $\Lambda$ and with $J$ the canonical complex structure (that is, $90^\circ$ rotation in the positive direction in the tangent spaces, a notion that is obviously invariant under conformal changes of the metric). Observe that every lift of $\sigma : \Sigma \to \mathbb{R}^{n+1,1}$ of $\Lambda$ is conformal: given $z$ a holomorphic chart of $\Sigma$, $(\sigma, \sigma_z) = 0$ (or, equivalently, $(\sigma_z, \sigma_{\bar{z}}) = 0$). Set

$$\Lambda^{1,0} := \Lambda \oplus d\sigma(T^{1,0}\Sigma), \quad \Lambda^{0,1} := \Lambda \oplus d\sigma(T^{0,1}\Sigma),$$

independently of the choice of a lift $\sigma$ of $\Lambda$, defining in this way two complex rank 2 subbundles of $\Lambda^{(1)}$, complex conjugate of each other,

$$\Lambda^{0,1} = \overline{\Lambda^{1,0}}.$$

The nullity and conformality of the lifts of $\Lambda$ establish the isotropy of (both) $\Lambda^{1,0}$ (and $\Lambda^{0,1}$), whilst the fact that $\Lambda$ is an immersion establishes that $\Lambda^{1,0}$ and $\Lambda^{0,1}$ intersect in $\Lambda$,

$$\Lambda^{1,0} \cap \Lambda^{0,1} = \Lambda.$$

Let $S : \Sigma \to G := \text{Gr}_{(3,1)}(\mathbb{R}^{n+1,1})$ be the central sphere congruence of $\Lambda$,

$$S = \Lambda^{(1)} \oplus \langle \triangle \sigma \rangle = \langle \sigma, \sigma_z, \sigma_{\bar{z}}, \sigma_{zz} \rangle,$$

for $\sigma$ any lift of $\Lambda$, $\triangle \sigma$ the Laplacian of $\sigma$ with respect to the metric $g_\sigma$ and $z$ a holomorphic chart of $\Sigma$. We use $\pi_S$ and $\pi_{S^\perp}$ to denote the orthogonal projections of $\mathbb{R}^{n+1,1}$ onto $S$ and $S^\perp$, respectively.

Given $z$ a holomorphic chart of $\Sigma$, let $g_z$ denote the metric induced in $\Sigma$ by $z$. Differentiation of $\langle \sigma, \sigma_z \rangle = 0$ gives $\langle \sigma, \sigma_{zz} \rangle = \langle \sigma_z, \sigma_{\bar{z}} \rangle$, which the conformality of $z$ proves to be never-zero. In many occasions, it will be useful to consider a special choice of lift of $\Lambda$, the normalized lift with respect to $z$, the section $\sigma_z^z : \Sigma \to L^+$ of $\Lambda$ (given a choice $L^+$ of one of the two connected components of $L$) defined by $g_{\sigma^z} = g_z$. For further reference, note that this condition establishes, in particular, that $(\sigma_{\bar{z}}, \sigma_{z\bar{z}})$ is constant,

$$(\sigma_{\bar{z}}, \sigma_{z\bar{z}}) = \frac{1}{2},$$

and, therefore,

$$\pi_S \sigma_{zz}^z \in \Gamma((\Lambda^{(1)})^\perp \cap S) = \Gamma \Lambda.$$

Consider the decomposition of the trivial flat connection $d$ on $\mathbb{R}^{n+1,1}$ as

$$d = D \oplus N.$$
for $\mathcal{D}$ the connection given by the sum of the connections $\nabla^S$ and $\nabla^{S^\perp}$ induced on $S$ and $S^\perp$, respectively, by $d$. Note that $\mathcal{D}$ is a metric connection and, therefore, $\mathcal{N}$ is skew-symmetric, 

$$\mathcal{N} \in \Omega^1(S \wedge S^\perp).$$

Note that, given $\xi \in \Gamma(S \wedge S^\perp)$, the transpose of $\xi|_S$ is $-\xi|_{S^\perp}$, and define a bundle isomorphism $S \wedge S^\perp \to \text{Hom}(S, S^\perp)$ by $\eta \mapsto \eta|_S$. Together with the canonical identification of $S^* \mathcal{T}G$ and $\text{Hom}(S, S^\perp)$, via $X \mapsto (\rho \mapsto \pi_{S^\perp}(d_X \rho))$, this gives an identification

$$(2.5) \quad S^* \mathcal{T}G \cong \text{Hom}(S, S^\perp) \cong S \wedge S^\perp,$$

of bundles provided with the canonical metrics and connections (for the connection $\mathcal{D}$ on $\mathbb{R}^{n+1,1}$, (see, for example, [10]), which we will consider throughout. Observe that, under the identification (2.5), we have

$$(2.6) \quad dS = \mathcal{N}.$$ 

We restrict our study to surfaces in $S^n$ which are not contained in any sub-sphere of $S^n$. This ensures, in particular, that, given $v_\infty \in \mathbb{R}^{n+1,1}$ non-zero, $\Lambda(\Sigma) \subset \mathbb{P}(\mathcal{L} \cap \langle v_\infty \rangle^\perp)$: if $v_\infty$ is space-like, $\mathbb{P}(\mathcal{L} \cap \langle v_\infty \rangle^\perp)$ is an hypersphere in $S^n$, whilst, in the case $v_\infty$ is time-like or light-like, this is always necessary the case. Hence, given $\sigma$ a lift of $\Lambda$, we have $(\sigma, v_\infty) \neq 0$ and we define a local immersion

$$\sigma_\infty := (\pi_{Sv_\infty})^{-1} \circ \Lambda = \frac{-1}{(\sigma, v_\infty)} \sigma : \Sigma \to Sv_\infty,$$

of $\Sigma$ into the space-form $Sv_\infty$, conformally diffeomorphic to the surface $\Lambda$. The normal bundle $N_\infty$ to $\sigma_\infty$ can be identified with the normal bundle $S^\perp$ to the central sphere congruence of $\Lambda$, as bundles provided with metrics and connections:

**Lemma 1.** [8] Let $\mathcal{H}_\infty$ denote the mean curvature vector of $\sigma_\infty$. Then

$$\xi \mapsto \xi + (\xi, \mathcal{H}_\infty)\sigma_\infty$$

defines an isomorphism

$$\mathcal{Q} : N_\infty \to S^\perp,$$

of bundles provided with a metric and a connection. Furthermore:

$$(2.7) \quad \mathcal{Q}(\mathcal{H}_\infty) = -\pi_{S^\perp}(v_\infty).$$

**Proof.** Let $g_\infty$ denote the metric induced in $\Sigma$ by $\sigma_\infty$ and $\nabla^{N_\infty}$ denote the connection induced in $N_\infty$ by the pull-back connection by $\sigma_\infty$ of the Levi-Civita connection on $(T\Sigma, g_\infty)$. According to (2.1) and (2.2), the pull-back bundle by $\sigma_\infty$ of $T\Sigma$ consists of the orthogonal complement in $\mathbb{R}^{n+1,1}$ of the non-degenerate bundle $\langle \sigma_\infty, v_\infty \rangle$,

$$\sigma_\infty^* T_{Sv_\infty}^* \cong \langle \sigma_\infty, v_\infty \rangle^\perp.$$ 

Let $\pi_{N_\infty}$ denote the orthogonal projection of

$$\mathbb{R}^{n+1,1} = d\sigma_\infty(T\Sigma) \oplus N_\infty \oplus \langle v_\infty, \sigma_\infty \rangle$$

onto $N_\infty$ and $\sigma_\infty^* T_{Sv_\infty}^*$. Then

$$\pi_{N_\infty} : \mathbb{R}^{n+1,1} \to N_\infty$$

is the orthogonal projection of $\mathbb{R}^{n+1,1}$ onto $N_\infty$. Let $\mathcal{D}_\infty$ denote the connection induced by the pull-back connection of the Levi-Civita connection on $(T\Sigma, g_\infty)$, via $X \mapsto (\rho \mapsto \pi_{N_\infty}(d_X \rho))$, and let $\mathcal{L}$ denote the tangent bundle of $\Lambda$. Then

$$\mathcal{D}_\mathcal{L} := \mathcal{D}_\infty.$$ 

In particular, the connection $\mathcal{D}_\mathcal{L}$ is a metric connection and, therefore, the connection $\mathcal{D}_\mathcal{L}$ is skew-symmetric, as required.
onto $N_\infty$. Since the metric in $S_{v_\infty}$ is the one inherited from $\mathbb{R}^{n+1,1}$, the second fundamental form $\Pi_\infty$ of $\sigma_\infty$ is simply given by

$$\Pi_\infty(X, Y) = \pi_{N_\infty}(d_X d_Y \sigma_\infty),$$

for $X, Y \in \Gamma(T \Sigma)$, so that, given $\xi \in \Gamma(N_\infty)$ and $(e_i)_i$ an orthonormal frame of $(T \Sigma, g_\infty)$, we have $(\xi, \sum_i d_{e_i} d_{e_i} \sigma_\infty) = 2(\xi, \mathcal{H}_\infty)$ and, therefore,

$$(\xi + (\xi, \mathcal{H}_\infty) \sigma_\infty, \sum_i d_{e_i} d_{e_i} \sigma_\infty) = 0.$$

Together with the fact that $N_\infty \subset \langle \sigma_\infty, v_\infty \rangle^\perp$, this shows that $\xi + (\xi, \mathcal{H}_\infty) \sigma_\infty$ is, in fact, a section of $S^\perp$.

Clearly, $Q$ is isometric, and, therefore, injective, as $N_\infty$ is non-degenerate. Now rank $N_\infty = n - 2 = \text{rank } S^\perp$ shows that $Q$ is an isometric isomorphism. Furthermore, given $\xi \in \Gamma(N_\infty)$,

$$\nabla S^\perp(Q(\xi)) = \pi_{S^\perp}(d\xi) + d(\xi, \mathcal{H}_\infty) \pi_{S^\perp}(\sigma_\infty) + (\xi, \mathcal{H}_\infty) \pi_{S^\perp}(d\sigma_\infty) = \pi_{S^\perp}(d\xi),$$

whilst

$$Q(\nabla N_\infty \xi) = \pi_{N_\infty}(d\xi) + (\pi_{N_\infty}(d\xi), \mathcal{H}_\infty) \sigma_\infty \in \Gamma(S^\perp).$$

To conclude that $Q$ preserves connections, we just need to verify that

$$d\xi - \pi_{N_\infty}(d\xi) \in \Gamma(S).$$

That is immediate: $d\xi$ is still a section of $\langle \sigma_\infty, v_\infty \rangle^\perp$,

$$(d\xi, \sigma_\infty) = (d\xi, \sigma_\infty) + (\xi, d\sigma_\infty) = 0 = (d\xi, v_\infty) + (\xi, dv_\infty) = (d\xi, v_\infty);$$

and, therefore, $d\xi - \pi_{N_\infty}(d\xi)$ is the orthogonal projection of $d\xi$ onto the tangent bundle to $\sigma_\infty$.

Finally, the fact that

$$(Q(\xi), \pi_{S^\perp}(v_\infty)) = (\xi, v_\infty) + (\xi, \mathcal{H}_\infty)(\sigma_\infty, v_\infty) = -(Q(\xi), Q(\mathcal{H}_\infty)),$$

for $\xi \in N_\infty$, establishes \((2.7)\) and completes the proof. \(\Box\)

2.1.3. The Willmore energy. Suppose, for the moment, that $\Sigma$ is compact. The Willmore energy $W(\Lambda)$ of $\Lambda$ is given by\(^1\)

$$W(\Lambda) = \int_\Sigma |\Pi_0|^2 dA,$$

for $\Pi_0$ the trace-free part of the second fundamental form of $\Lambda$ (calculated with respect to any representative metric on $S^n$ and independent of that choice).

Next we present a manifestly conformally-invariant formulation of the Willmore energy. It follows the definition presented in [7], in the quaternionic setting, for the particular case of $n = 4$. The intervention of the conformal structure will restrict to the Hodge $*$-operator, which is conformally-invariant on 1-forms over a surface.

\(^1\)In the literature, different scalings of the Willmore energy can be found. Our choice is justified by the classical scaling in the Dirichlet energy functional.
Given \( \mu, \eta \in \Omega^1(S^*T\Sigma) \), let \((\mu \wedge \eta)\) be the 2-form defined from the metric on \( S^*T\Sigma \):

\[
(\mu \wedge \eta)(X,Y) = \langle \mu X, \eta Y \rangle - \langle \mu Y, \eta X \rangle,
\]

for all \( X, Y \in \Gamma(T\Sigma) \). Note that

\[
(dS \wedge *dS) = -(dS \wedge dS) = (dS, dS) dA,
\]

\((dS \wedge *dS)\) is a conformally invariant way of writing \((dS, dS)\), for \( g \in \mathcal{C}_\Lambda \), with \( dA_g \) denoting the area element of \((\Sigma, g)\) and \((,)_g\) denoting the Hilbert-Schmidt metric on \( L((T\Sigma, g), S^*T\Sigma)\).

**Theorem 2.** \( W(\Lambda) = \frac{1}{2} \int \Sigma (dS \wedge *dS) \).

**Proof.** By (2.6), fixing a metric on \( \Sigma \),

\[
|dS|^2 = |N|^2.
\]

To prove the theorem, we fix \( v_\infty \in \mathbb{R}^{n+1} \) non-zero, provide \( \Sigma \) with the metric induced by \( \sigma_\infty \) and show that

\[
|N|^2 = 2 |\Pi_\infty|^2,
\]

for \( \Pi_\infty \) the trace-free part of the second fundamental form of \( \sigma_\infty \).

Fixing a local orthonormal frame \( \{X_i\}_i \) of \( T\Sigma \), we have

\[
|N|^2 = \sum_i \text{tr}(N_{X_i} N_{X_i}) = 2 \sum_i \text{tr}(N_{X_i}^\prime N_{X_i} |S|).
\]

Recall that if \( (e_i)_i \) and \( (\hat{e}_i)_i \) are dual basis of a vector space \( E \) provided with a metric \((,)_i\), then, given \( \mu \in \text{End}(E) \), \( \text{tr}(\mu) = \sum_i (\mu(e_i), \hat{e}_i) \). Let \( \hat{\sigma}_\infty \) be the section of \( S \) determined by the conditions \( (\hat{\sigma}_\infty, \hat{\sigma}_\infty) = 0 \), \( (\sigma_\infty, \hat{\sigma}_\infty) = -1 \) and \( (\hat{\sigma}_\infty, d\sigma_\infty) = 0 \). Then \( (\sigma_\infty, dX_1 \sigma_\infty, dX_2 \sigma_\infty, \hat{\sigma}_\infty) \) is a frame of \( S \) with dual \( (-\hat{\sigma}_\infty, dX_1 \sigma_\infty, dX_2 \sigma_\infty, -\sigma_\infty) \) and we conclude that

\[
|N|^2 = 2 \sum_{i,j} (\hat{N}_{X_i}(dX_j \sigma_\infty), N_{X_i}(dX_j \sigma_\infty)).
\]

Lemma 1 establishes \( \hat{N}_{X_i}(dX_j \sigma_\infty) = Q(\Pi_\infty(X_i, X_j)) \) and completes the proof.

**2.1.4. Willmore surfaces and harmonicity.** A conformal immersion is **Willmore** if it extremizes the Willmore functional and **constrained Willmore** if it extremizes the Willmore functional with respect to variations that infinitesimally preserve the conformal structure, that is, variations satisfying

\[
\frac{d}{dt}|_{t=0} (\delta_2, \delta_2) = 0,
\]

for the variation \((,)_t\) of the induced metric and \( z \) a holomorphic chart.

Theorem [2] makes it clear that

\[
W(\Lambda) = E(S, \mathcal{C}_\Lambda),
\]

the Willmore energy of \( \Lambda \) coincides with the Dirichlet energy of \( S \) with respect to any of the metrics in the conformal class \( \mathcal{C}_\Lambda \) (although the Levi-Civita connection is not conformally-invariant, the Dirichlet energy of a mapping of a surface is preserved under conformal changes of the metric.
Proof. Given a variation \( \Lambda_t \) of \( \Lambda \) and \( S_t \) the corresponding variation of \( S \) through central sphere congruences, the Dirichlet energy \( E(S_t, C_t) \) of \( S_t \) with respect to the conformal structure \( C_t \) induced in \( \Sigma \) by \( \Lambda_t \) is given by

\[
\frac{d}{dt}|_{t=0} E(S_t, C_t) = \frac{1}{2} \int_{\Sigma} ((d\dot{S} \wedge \ast dS) + (dS \wedge \ast d\dot{S})) + (dS \wedge \ast d\dot{S})
\]

abbreviating \( \frac{d}{dt}|_{t=0} \) by a dot. Let \( (J_t)_t \) be the corresponding variation of \( J \) through canonical complex structures. Differentiation at \( t = 0 \) of \( \ast dS_t = -(dS_t)J_t \) gives \( \ast dS = -(dS)\dot{J} \), whilst that of \( J^2_t = -I \) gives \( \dot{J}J = -JJ \) and, in particular, that \( J \) intertwines the eigenspaces of \( J \). The \( C_\Lambda \)-conformality of \( S_t \), \( (d_{X+ij}S_t, d_{X-ij}S_t) = 0 \), respectively, for \( X \in \Gamma(T\Sigma) \), establishes then \( (dS \wedge \ast dS) = 0 \) and, therefore,

\[
\frac{d}{dt}|_{t=0} E(S_t, C_t) = \frac{d}{dt}|_{t=0} E(S_t, C_0).
\]

It is now clear that if \( S : (\Sigma, C_\Lambda) \to \mathcal{G} \) is harmonic then \( \Lambda \) is Willmore.

Conversely, suppose \( \Lambda \) is Willmore, fix \( z \) a holomorphic chart of \( (\Sigma, C_\Lambda) \) and let us show that the tension field \( \tau_z \) of \( S : (\Sigma, g_z) \to \mathcal{G} \) vanishes. First of all, observe that

\[
4\nabla_{\delta_z} S_z = \tau_z = 4\nabla_{\delta_z} S_z
\]

to conclude that \( \Lambda^{(1)} \subset \ker \tau_z \) by [2,4],

\[
(\nabla_{\delta_z} S_z)\sigma^z = \nabla^S_{\delta_z} (\pi_{S\perp} \sigma^z_{S\perp}) - \pi_{S\perp} (\nabla^S_{\delta_z} \sigma^z_{S\perp}) = 0,
\]

and, similarly, \( (\nabla_{\delta_z} S_z)\sigma^z = 0 \) is immediate. It follows that \( \text{Im} \tau^1_z \subset \Lambda \). Fix \( \Lambda_t = (\sigma_t) \) a variation of \( \Lambda \) and let \( S_t \) be the corresponding variation of \( S \) through central sphere congruences. Then \( \tau^1_z(\pi_{S\perp} \dot{\sigma}) = \lambda \sigma_0 \), for some \( \lambda \in C^\infty(\Sigma, \mathbb{R}) \), and, therefore, \( \text{tr}(\tau^1_z \dot{\bar{S}}) = \lambda \) (note that \( \bar{S} \sigma = \pi_{S\perp} \dot{\sigma} \)). On the other hand, classically,

\[
0 = \frac{d}{dt}|_{t=0} E(S_t, C_\Lambda) = -\int_{\Sigma} (\dot{S}, \tau_z) dA_z = -\int_{\Sigma} \text{tr}(\tau^1_z \dot{\bar{S}}) dA_z,
\]
for $dA_z$ the area element of $(\Sigma, g_z)$. Now suppose $\tau_z$ is non-zero. Then so is $\tau_z^t \in \Gamma(\text{Hom}(S^t, S))$, so we can choose $\sigma_t$ such that $\lambda$ is positive, which leads to a contradiction and completes the proof. □

Having characterized Willmore surfaces by the harmonicity of the central sphere congruence, and recalling (2.6), we deduce the Willmore surface equation,

$$(2.8) \quad dD^* N = 0.$$  

More generally, we have a manifestly conformally-invariant characterization of constrained Willmore surfaces in space-forms, first established in [5] and reformulated in [5] as follows:

**Theorem 4.** [5] $\Lambda$ is a constrained Willmore surface if and only if there exists a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ with

$$(2.9) \quad dD q = 0$$

such that

$$(2.10) \quad dD^* N = 2 [q \wedge *N].$$

Such a form $q$ is said to be a [Lagrange] multiplier for $\Lambda$ and $\Lambda$ is said to be a $q$-constrained Willmore surface.

**Proof.** Calculus of variations techniques show that the variational Willmore energy relates to the variational surface by

$$\dot{W} = \int_{\Sigma} ((dD^* N) \wedge \dot{\Lambda}),$$

for some non-degenerate pairing $(\wedge)$. For general variations, $\dot{\Lambda}$ can be arbitrary, establishing the Willmore surface equation $\dot{W} = 0$ (2.8), whereas infinitesimally conformal variations are characterized by the normal variational being in the image of the conformal Killing operator $\overline{\partial}$, which, according to Weyl’s Lemma, consists of $(H^0 K)^{\perp}$, for $H^0 K$ the space of holomorphic quadratic differentials. The result follows by defining a multiplier $q$ from a quadratic differential $q^2 dz^2 \in (H^0 K)^{\perp}$ via $q_{\delta_z} \sigma_z = -\frac{1}{2} q^2 \sigma$, for $\sigma$ a lift of $\Lambda$ and $z$ a holomorphic chart of $\Sigma$. □

Willmore surfaces are the 0-constrained Willmore surfaces. The zero multiplier is not necessarily the only multiplier for a constrained Willmore surface with no constraint on the conformal structure, though. In fact, the uniqueness of multiplier characterizes non-isothermic constrained Willmore surfaces, as we shall see below in this text.

The characterization of constrained Willmore surfaces above motivates a natural extension to surfaces that are not necessarily compact.

Next we present a useful result, which establishes, in particular, that if $q$ is a multiplier for $\Lambda$, then $q^{1,0}$ takes values in $\Lambda \wedge \Lambda^{0,1}$. 
Lemma 5. Given \( q \in \Omega^1(\Lambda \wedge \Lambda^{(1)}) \) real,

\begin{enumerate}[i)]
\item if \( d^P q = 0 \) then \( q^{1,0} \in \Omega^{1,0}(\Lambda \wedge \Lambda^{0,1}) \) or, equivalently, \( q^{0,1} \in \Omega^{0,1}(\Lambda \wedge \Lambda^{1,0}) \);
\item \( d^P q = 0 \) if and only if \( d^P q^{1,0} = 0 \), or, equivalently, \( d^P q^{0,1} = 0 \);
\item \( d^P q = 0 \) if and only if \( d^P \ast q = 0 \).
\end{enumerate}

Proof. Fix \( z \) a holomorphic chart of \( \Sigma \). First of all, observe that a section \( \xi \) of \( \Lambda \wedge \Lambda^{(1)} \) is a section of \( \Lambda \wedge \Lambda^{1,0} \) if and only if \( \xi(\sigma_z^\ast) = 0 \). Suppose \( d^P q = 0 \). Then, in particular, \( d^P q (\delta_z, \delta_z) \sigma_z^\ast = 0 \), or, equivalently,

\[
\mathcal{D}_{\delta_z} (q \delta_z \sigma_z^\ast) - q \delta_z (\mathcal{D}_{\delta_z} \sigma_z^\ast) - \mathcal{D}_{\delta_z} (q \delta_z \sigma_z^\ast) + q \delta_z (\mathcal{D}_{\delta_z} \sigma_z^\ast) = 0,
\]

establishing

\begin{equation}
q \delta_z \sigma_z^\ast = q \delta_z \sigma_z^\ast.
\end{equation}

In its turn, \( d^P q (\delta_z, \delta_z) \sigma_z^\ast = 0 \) implies

\[
\mathcal{D}_{\delta_z} (q \delta_z \sigma_z^\ast) - \mathcal{D}_{\delta_z} (q \delta_z \sigma_z^\ast) + q \delta_z \sigma_z^\ast = 0,
\]

by (2.24). On the other hand, the skew-symmetry of \( q \) establishes \( (q \sigma_z^\ast, \sigma_z^\ast) = 0 \) and, therefore,

\begin{equation}
q \sigma_z^\ast = \mu \sigma_z + \eta \sigma_z^\ast,
\end{equation}

for some \( \mu, \eta \in \Omega^1(\mathbb{C}) \). Hence

\[
\mathcal{D}_{\delta_z} (q \delta_z \sigma_z^\ast) + \mu \delta_z \sigma_z^\ast = \mathcal{D}_{\delta_z} (q \delta_z \sigma_z^\ast) - \eta \delta_z \sigma_z^\ast.
\]

It is obvious that a section of \( \Lambda \wedge \Lambda^{(1)} \) transforms sections of \( \Lambda^{(1)} \) into sections of \( \Lambda \), so that, in particular, both \( q \delta_z \sigma_z^\ast \) and \( q \delta_z \sigma_z^\ast \) are sections of \( \Lambda \).

We conclude that \( \mathcal{D}_{\delta_z} (q \delta_z \sigma_z^\ast) + \mu \delta_z \sigma_z^\ast \) is a section of \( \Lambda^{1,0} \cap \Lambda^{0,1} = \Lambda \). Write \( q \delta_z \sigma_z^\ast = \lambda \sigma_z^\ast \), with \( \lambda \in \Gamma(\mathbb{C}) \). Then

\[
\lambda \sigma_z^\ast + (\lambda + \mu \delta_z) \sigma_z^\ast = \gamma \sigma_z^\ast,
\]

for some \( \gamma \in \Gamma(\mathbb{C}) \). In particular, \( \lambda = -\mu \delta_z \). Equation (2.12) establishes, on the other hand,

\[
q = -2 \mu \sigma_z^\ast \wedge \sigma_z^\ast - 2 \eta \sigma_z^\ast \wedge \sigma_z^\ast
\]

and, in particular, \( q \delta_z \sigma_z^\ast = \mu \delta_z \sigma_z^\ast \). Equation (2.11) completes the proof of \( i) \).

Next we prove \( ii) \). By (2.24),

\[
\mathcal{D}^{1,0} \Gamma(\Lambda^{1,0}) \subset \Omega^{1,0}(\Lambda^{1,0})
\]

or, equivalently,

\[
\mathcal{D}^{0,1} \Gamma(\Lambda^{0,1}) \subset \Omega^{0,1}(\Lambda^{0,1})
\]

and, therefore, following \( i) \), \( d^P q^{1,0} \in \Omega^2(\Lambda \wedge \Lambda^{0,1}) \) and \( d^P q^{0,1} \in \Omega^2(\Lambda \wedge \Lambda^{1,0}) \). Hence \( d^P q = 0 \) forces \( d^P q^{1,0} \) and \( d^P q^{0,1} \) to vanish separately. The reality of \( q \) completes the proof of \( ii) \).

As for \( iii) \), it is immediate from \( ii) \).
2.1.5. Constrained Willmore surfaces: a zero-curvature characterization. For a map into a Grassmannian, harmonicity amounts to the flatness of a family of connections, according to Uhlenbeck [22]. With the characterization of Willmore surfaces by the harmonicity of the central sphere congruence, a zero-curvature characterization of Wilmore surfaces follows. More generally, the constrained Willmore surface equations amount to the flatness of a certain family of connections too:

**Theorem 6.** $\Lambda$ is a constrained Willmore surface if and only if there exists a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ such that

$$d^\Lambda_q := D + \lambda^{-1} \mathcal{A}^{1,0} + \lambda \mathcal{A}^{0,1} + (\lambda^{-2} - 1)q^{1,0} + (\lambda^2 - 1)q^{0,1}$$

is flat for all $\lambda \in S^1$.

Before proceeding to the proof of the theorem, and for further reference, observe that, given $a, b \in \mathbb{R}^{n+1,1}$ and $T \in o(\mathbb{R}^{n+1,1})$,

$$[T, a \wedge b] = (Ta) \wedge b + a \wedge (Tb),$$

to conclude that

$$[\Lambda \wedge \Lambda^{(1)}, \Lambda \wedge \Lambda^{(1)}] \subset \Lambda \wedge \Lambda = \{0\}.$$  

Now we proceed to the proof of the theorem:

**Proof.** According to the decomposition

$$o(\mathbb{R}^{n+1,1}) = (\wedge^2 S \oplus \wedge^2 S^\perp) \oplus S \wedge S^\perp,$$

the flatness of $d$, characterized by

$$0 = RD + d^D\mathcal{N} + \frac{1}{2}[\mathcal{N} \wedge \mathcal{N}],$$

encodes two structure equations, namely,

$$RD + \frac{1}{2}[\mathcal{N} \wedge \mathcal{N}] = 0$$  
(2.15)

and

$$d^D\mathcal{N} = 0.$$  
(2.16)

Now suppose $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ is a real form. Given $\lambda \in S^1$, set

$$A^\lambda = d^\lambda_q - D \in \Omega^1(\text{End}(\mathbb{R}^{n+1,1})).$$

The curvature tensor of $d^\lambda_q$ is given by

$$R^{d^\lambda_q} = RD + d^DA^\lambda + \frac{1}{2}[A^\lambda \wedge A^\lambda].$$

Since there are no non-zero $(2, 0)$- or $(0, 2)$-forms over a surface, we have

$$\frac{1}{2}[A^\lambda \wedge A^\lambda] = [\mathcal{A}^{1,0} \wedge \mathcal{A}^{0,1} + (\lambda^{-1} - \lambda)([q^{1,0} \wedge \mathcal{A}^{0,1}] - [q^{0,1} \wedge \mathcal{A}^{1,0}]) + (2 - \lambda^{-2} - \lambda^2)[q^{1,0} \wedge q^{0,1}]$$

The associated family of flat connections presented in [9] corresponds to a different choice of orientation in $\Sigma$. 

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and equation (2.15) establishes then
\[ R^d\lambda = d^D A^\lambda + (\lambda^{-1} - \lambda)((q^{1,0} \wedge N^{0,1}) - [q^{0,1} \wedge N^{1,0}]) + \frac{1}{2} (2 - \lambda^{-2} - \lambda^2) [q \wedge q]. \]
But, according to (2.13), \([q \wedge q] = 0\). Hence
\[ R^d\lambda = d^D A^\lambda + (\lambda^{-1} - \lambda)((q^{1,0} \wedge N^{0,1}) - [q^{0,1} \wedge N^{1,0}]). \]
In its turn, equation (2.16) gives
\[ d^D N^{1,0} = i^2 d^D \star N = -d^D N^{0,1}. \]
We conclude that
\[ R^d\lambda = \frac{\lambda^{-1} - \lambda}{2} i (d^D \star N - 2[q \wedge \star N]) + (\lambda^{-2} - 1) d^D q^{1,0} + (\lambda^2 - 1) d^D q^{0,1}. \]
Yet again, according to the decomposition (2.14), it follows that \( R^d\lambda = 0 \) if and only if both
\[ \frac{\lambda^{-1} - \lambda}{2} i (d^D \star N - 2[q \wedge \star N]) = 0 \]
and
\[ (\lambda^{-2} - 1) d^D q^{1,0} + (\lambda^2 - 1) d^D q^{0,1} = 0 \]
hold. Organizing equations (2.17) and (2.18) by powers of \( \lambda \) completes the proof. \( \square \)

2.1.6. Constrained Willmore surfaces and the isothermic surface condition. Isothermic surfaces are classically defined by the existence of conformal curvature line coordinates. Equation (1.1) makes it clear that, although the second fundamental form is not conformally invariant, conformal curvature line coordinates are preserved under conformal changes of the metric and, therefore, so is the isothermic surface condition. The next result presents a manifestly conformally-invariant formulation of the isothermic surface condition, established in [6] and discussed also in [11].

Lemma 7. [6, 11] \( \Lambda \) is isothermic if and only if there exists a non-zero closed real 1-form \( \eta \in \Omega^1(\Lambda \wedge \Lambda^{(1)}) \). Under these conditions, we may say that \((\Lambda, \eta)\) is an isothermic surface. The form \( \eta \) is defined up to a real constant scale.

Remark 1. According to the decomposition (2.14), given \( \eta \in \Omega^1(\Lambda \wedge \Lambda^{(1)}) \),
\[ d\eta = d^D \eta + [N \wedge \eta] \]
vanishes if and only if \( d^D \eta = 0 = [N \wedge \eta] \).

Proposition 8. [9] A constrained Willmore surface has a unique multiplier if and only if it is not an isothermic surface. Furthermore:
1) if \( q_1 \neq q_2 \) are multipliers for \( \Lambda \), then \((\Lambda, \star(q_1 - q_2))\) is isothermic;
2) if \((\Lambda, \eta)\) is an isothermic \( q \)-constrained Willmore surface, then the set of multipliers to \( \Lambda \) is the affine space \( q + (\star \eta)_R \).
Proof. It is immediate, noting that \([N \wedge \eta] = [\ast \eta \wedge \ast N]\) and recalling Lemma 5 - iii).

A classical result by Thomsen [21] characterizes isothermic Willmore surfaces in 3-space as minimal surfaces in some 3-dimensional space-form. Constant mean curvature surfaces in 3-dimensional space-forms are examples of isothermic constrained Willmore surfaces, as proven by J. Richter [18]. However, isothermic constrained Willmore surfaces in 3-space are not necessarily constant mean curvature surfaces in some space-form, as established by an example, presented in [2], of a constrained Willmore cylinder that does not have constant mean curvature in any space-form.

2.2. Complexified constrained Willmore surfaces. The transformations of a constrained Willmore surface \(\Lambda\) we present below in this work are, in particular, pairs ((\(\Lambda^{1,0}\), \(\Lambda^{0,1}\))*) of transformations (\(\Lambda^{1,0}\))* and \((\Lambda^{0,1})^*\) of \(\Lambda^{1,0}\) and \(\Lambda^{0,1}\), respectively. The fact that \(\Lambda^{1,0}\) and \(\Lambda^{0,1}\) intersect in a rank 1 bundle will ensure that (\(\Lambda^{1,0}\))* and \((\Lambda^{0,1})^*\) have the same property. The isotropy of \(\Lambda^{1,0}\) and \(\Lambda^{0,1}\) will ensure that of (\(\Lambda^{1,0}\))* and \((\Lambda^{0,1})^*\) and, therefore, that of their intersection. The reality of the bundle \(\Lambda^{1,0} \cap \Lambda^{0,1}\) is preserved by the spectral deformation, but it is not clear that the same is necessarily true for B"acklund transformation. This motivates the definition of complexified surface.

Fix a conformal structure \(\mathcal{C}\) on \(\Sigma\) and consider the corresponding complex structure on \(\Sigma\). Let \(\hat{d}\) be a flat metric connection on \(\mathbb{C}^{n+2}\) and \(d\) denote the trivial flat connection. In what follows, omitting the reference to some specific connection shall be understood as an implicit reference to \(d\).

**Definition 1.** We define a complexified \(\hat{d}\)-surface to be a pair \((\Lambda^{1,0}, \Lambda^{0,1})\) of isotropic rank 2 subbundles of \(\mathbb{C}^{n+2}\) intersecting in a rank 1 bundle

\[\Lambda := \Lambda^{1,0} \cap \Lambda^{0,1}\]

such that

\[\hat{d}^{1,0} \Gamma \Lambda \subset \Omega^{1,0} \Lambda^{1,0}, \quad \hat{d}^{0,1} \Gamma \Lambda \subset \Omega^{0,1} \Lambda^{0,1} .\]

Obviously, given \(\Lambda\) a (real) surface in \(\mathbb{P}(\mathcal{L})\), \((\Lambda^{1,0}, \Lambda^{0,1})\) is a complexified surface with respect to \(\mathcal{C}_\Lambda\). Henceforth, we drop the term ”complexified” and use real surface when referring explicitly to a complexified surface \((\Lambda^{1,0}, \Lambda^{0,1})\) defining a real surface \(\Lambda\). Observe that \((\Lambda^{1,0}, \Lambda^{0,1})\) is a real surface if and only if \(\Lambda\) is a real bundle (recall that \(\Lambda\) is an immersion if and only if the bundle \(\Lambda^{1,0} + \Lambda^{0,1}\) has rank 3).

Observe, on the other hand, that, in the particular case of a real surface \((\Lambda^{1,0}, \Lambda^{0,1})\), the notation \(\Lambda^{1,0}\) and \(\Lambda^{0,1}\) is consistent with [23]. Indeed, the \(\mathcal{C}\)-isotropy of \(\Lambda^{1,0}\) characterizes the \(\mathcal{C}\)-conformality of the lifts of \(\Lambda\), or equivalently, the fact that \(\mathcal{C} = \mathcal{C}_\Lambda\).
2.2.1. Constrained Willmore surfaces and perturbed harmonic bundles. Theorem 6 motivates the definition of perturbed harmonicity for a bundle, which we present next and which will apply to the central sphere congruence to provide a characterization of constrained Willmore surfaces. In the particular case of a bundle of $(3,1)$-planes in $\mathbb{R}^{n+1,1}$, our notion of perturbed harmonicity coincides with the notion of 2-perturbed harmonicity introduced in [9].

Given $V$ a non-degenerate subbundle of $\mathbb{C}^{n+2}$, consider the decomposition $\hat{d} = D_V^\hat{d} + N_V^\hat{d}$ for $D_V^\hat{d}$ the metric connection on $\mathbb{C}^{n+2}$ given by the sum of the connections induced on $V$ and $V^\perp$ by $\hat{d}$.

**Definition 2.** A non-degenerate rank 4 subbundle $V$ of $\mathbb{C}^{n+2}$ is said to be a central sphere congruence of a $\hat{d}$-surface $(\Lambda^{1,0}, \Lambda^{0,1})$ if

$$\Lambda^{1,0} + \Lambda^{0,1} \subset V$$

and

$$(N_V^\hat{d})^{1,0}\Lambda^{0,1} = 0 = (N_V^\hat{d})^{0,1}\Lambda^{1,0}.$$ 

**Remark 2.** Let $(\Lambda^{1,0}, \Lambda^{0,1})$ be a surface, $\sigma \neq 0$ be a section of $\Lambda$ and $z$ be a holomorphic chart of $\Sigma$. Note that $\Lambda^{1,0} + \Lambda^{0,1} \subset V$ establishes

$$(N_V)|_{\Lambda} = 0$$

and then $N_V^{\Lambda^{1,0}}\Lambda^{0,1} = 0 = N_V^{\Lambda^{0,1}}\Lambda^{1,0}$ reads $\sigma_{zz} \in \Gamma V$. Hence, generically (if $\sigma \wedge \sigma_z \neq 0 \neq \sigma \wedge \sigma_{zz}$), $\Lambda^{1,0}, \Lambda^{0,1}$ and $V$ are all determined by $\Lambda$: $\Lambda^{1,0} = \langle \sigma, \sigma_z \rangle$, $\Lambda^{0,1} = \langle \sigma, \sigma_{zz} \rangle$ and

$$V = \langle \sigma, \sigma_z, \sigma_{zz} \rangle.$$ 

In particular, the complexification of the central sphere congruence of a real surface $\Lambda$ is the unique central sphere congruence of the corresponding surface $(\Lambda^{1,0}, \Lambda^{0,1})$.

For further reference:

**Lemma 9.** Suppose $V$ is a central sphere congruence of a $\hat{d}$-surface $(\Lambda^{1,0}, \Lambda^{0,1})$. Then

$$(D_V^\hat{d})^{1,0}\Gamma\Lambda^{1,0} \subset \Omega^{1,0}\Lambda^{1,0}, \quad (D_V^\hat{d})^{0,1}\Gamma\Lambda^{0,1} \subset \Omega^{0,1}\Lambda^{0,1}.$$ 

**Proof.** First of all, observe that, as $\text{rank} \Lambda^{1,0} = \frac{1}{2}\text{rank} V = \text{rank} \Lambda^{0,1}$, the isotropy of both $\Lambda^{1,0}$ and $\Lambda^{0,1}$ establishes their maximal isotropy in $V$. Write $\Lambda^{1,0} = \langle \sigma, \tau \rangle$, with $\sigma \in \Gamma\Lambda$. Since

$$(D_V^\hat{d})^{1,0}\sigma = \pi_V \circ \hat{d}^{1,0} \circ \pi_V \sigma \in \Omega^{1,0}\Lambda^{1,0},$$

the fact that $D_V^\hat{d}$ is a metric connection, together with the isotropy of $\Lambda^{1,0}$, shows that

$$((D_V^\hat{d})^{1,0}\tau, \sigma) = -\langle \tau, (D_V^\hat{d})^{1,0}\sigma \rangle = 0,$$

whereas

$$((D_V^\hat{d})^{1,0}\tau, \tau) = \frac{1}{2}d^{1,0}(\tau, \tau) = 0.$$
We conclude that \((D^\d_V)^{1,0}\tau \perp \Lambda^{1,0}\) and, therefore, that \((D^\d_V)^{1,0}\tau\) takes values in \(\Lambda^{1,0}\). A similar argument establishes \((D^\d_V)^{0,1}\tau\Lambda^{0,1}\subset \Omega^{0,1}\Lambda^{0,1}\). \(\square\)

**Definition 3.** A non-degenerate bundle \(V \subset \mathbb{C}^{n+2}\) is said to be \(\d\)-perturbed harmonic if there exists a 1-form \(q\) with values in \(\wedge^2 V \oplus \wedge^2 V^\perp\) such that, for each \(\lambda \in \mathbb{C}\setminus\{0\}\), the metric connection
\[
\tilde{\nabla}_V^\lambda q := \nabla^\lambda + \lambda^{-1}(\mathcal{N}_V^\lambda)^{1,0} + \lambda(\mathcal{N}_V^\lambda)^{0,1} + (\lambda^{-2} - 1)q^{1,0} + (\lambda^2 - 1)q^{0,1},
\]
on \(\mathbb{C}^{n+2}\), is flat. In this case, we say that \(V\) is \((q, \d)\)-perturbed harmonic or, in the case \(q = 0\), \(\d\)-harmonic. In the particular case of \(\d = d\), \(V\) a real bundle and \(q\) a real form, we say that \(V\) is a real \(q\)-perturbed harmonic bundle.

**Definition 4.** A \(\d\)-surface \((\Lambda^{1,0}, \Lambda^{0,1})\) is said to be \(\d\)-constrained Willmore if it admits a \((q, \d)\)-perturbed harmonic central sphere congruence with
\[(2.19)\]
\[q^{1,0} \in \Omega^{1,0}(\wedge^2 \Lambda^{1,0}), \quad q^{0,1} \in \Omega^{0,1}(\wedge^2 \Lambda^{1,0}).\]
If \(q = 0\), we say that \((\Lambda^{1,0}, \Lambda^{0,1})\) is \(\d\)-Willmore. In the particular case of \(\d = d\), \((\Lambda^{1,0}, \Lambda^{0,1})\) a real surface and \(q\) a real form, we say that \((\Lambda^{1,0}, \Lambda^{0,1})\) is a real \(q\)-constrained Willmore surface.

From Theorem 6 and Lemma 5, it follows that the real constrained Willmore surface condition is preserved under the correspondence
\[(\Lambda^{1,0}, \Lambda^{0,1}) \leftrightarrow \Lambda^{1,0} \cap \Lambda^{0,1}\]
for real surfaces \((\Lambda^{1,0}, \Lambda^{0,1})\), with preservation of multipliers:

**Theorem 10.** Suppose \((\Lambda^{1,0}, \Lambda^{0,1})\) is a real surface. Then \(\Lambda\) is constrained Willmore if and only if \(S\) is \(q\)-perturbed harmonic, for some real 1-form \(q\) with \(q^{1,0} \in \Omega^{1,0}(\wedge^2 \Lambda^{0,1})\).

3. **Transformations of perturbed harmonic bundles and constrained Willmore surfaces**

Fix a conformal structure \(\mathcal{C}\) on \(\Sigma\) and consider the corresponding complex structure on \(\Sigma\). Let \(\text{Ad}\) denote the adjoint representation of the orthogonal group on the orthogonal algebra. Note that, given \(T \in O(\mathbb{R}^{n+1,1})\) and \(u, v \in \mathbb{R}^{n+1,1}\),
\[\text{Ad}_T(u \wedge v) = Tu \wedge Tv.\]

Let \(V\) be a non-degenerate subbundle of \(\mathbb{C}^{n+2}\) and \(\pi_V\) and \(\pi_{V^\perp}\) denote the orthogonal projections of \(\mathbb{C}^{n+2}\) onto \(V\) and \(V^\perp\), respectively, and \(\rho\) denote reflection across \(V\),
\[\rho = \pi_V - \pi_{V^\perp}.\]
Let \((\Lambda^{1,0}, \Lambda^{0,1})\) be a surface admitting \(V\) as a central sphere congruence. As usual, we write \(\Lambda\) for \(\Lambda^{1,0} \cap \Lambda^{0,1}\). Suppose \(V\) is \(q\)-perturbed harmonic for some \(q \in \Omega^1(\wedge^2 V \oplus \wedge^2 V^\perp)\) satisfying conditions (2.19).
3.1. **Spectral deformation.** For each $\lambda \in \mathbb{C}\setminus\{0\}$, the flatness of the metric connection $d^\lambda q_V$ on $\mathbb{C}^{n+2}$ establishes the existence of an isometry

$$\phi^\lambda_V : (\mathbb{C}^{n+2}, d^\lambda q_V) \to (\mathbb{C}^{n+2}, d)$$

of bundles, preserving connections, defined on a simply connected component of $\Sigma$ and unique up to a Möbius transformation.

**Lemma 11.** Let $\hat{d}$ be a flat metric connection on $\mathbb{C}^{n+2}$ and

$$\phi : (\mathbb{C}^{n+2}, \hat{d}) \to (\mathbb{C}^{n+2}, d)$$

be an isometry of bundles, preserving connections. Then $V$ is $(q, \hat{d})$-perturbed harmonic, for some $q$, if and only if $\phi V$ is $\text{Ad}_\phi q$-perturbed harmonic.

**Proof.** It is immediate from the fact that

$$D_{\phi V} = \phi \circ D_V \circ \phi^{-1}, \quad N_{\phi V} = \phi \circ N_V \circ \phi^{-1}$$

and, therefore,

$$d^\lambda q_V = \phi \circ d_V^\lambda \circ \phi^{-1}.$$

\[\square\]

Set

$$q_\lambda := \lambda^{-2} q^{1,0} + \lambda^2 q^{0,1},$$

for $\lambda \in \mathbb{C}\setminus\{0\}$. The fact that $q$ takes values in $\wedge^2 V \oplus \wedge^2 V^\perp$ establishes, in particular,

$$D_V d^\lambda q_V = D_V + (\lambda^{-2} - 1)q^{1,0} + (\lambda^{0,1} - 1)q^{0,1},$$

whereas

$$N_V d^\lambda q_V = \lambda^{-1} N_V^{1,0} + \lambda N_V^{0,1},$$

and, therefore,

$$(d^\lambda q_V)^{\mu, q_\lambda} = d_V^{\lambda, q_{\lambda}},$$

for all $\lambda, \mu \in \mathbb{C}\setminus\{0\}$. From the flatness of $d^\lambda q_V$, for all $\lambda \in \mathbb{C}\setminus\{0\}$, we conclude that of $(d^\lambda q_V)^{\mu, q_\lambda}$, for all $\lambda, \mu \in \mathbb{C}\setminus\{0\}$ and, therefore, that $V$ is $d^\lambda q_V$-perturbed harmonic, for all $\lambda \in \mathbb{C}\setminus\{0\}$. We define then a spectral deformation of $V$ into new perturbed harmonic bundles by setting, for each $\lambda$ in $\mathbb{C}\{0\}$,

$$V^\lambda := \phi^\lambda q_V.$$

**Theorem 12.** \[9\] $V^\lambda q$ is $\text{Ad}_\phi q_{\lambda}$-perturbed harmonic, for each $\lambda \in \mathbb{C}\{0\}$.

A deformation on the level of constrained Willmore surfaces follows:

**Theorem 13.** \[9\] For each $\lambda \in \mathbb{C}\setminus\{0\}$, $(\phi^\lambda q V^{1,0}, \phi^\lambda q V^{0,1})$ is a $\text{Ad}_\phi q_{\lambda}$-constrained Willmore surface, admitting $V^\lambda$ as a central sphere congruence. Furthermore, if $(\Lambda^{1,0}, \Lambda^{0,1})$ is a real constrained Willmore surface, then so is

$$\Lambda^\lambda := \phi^\lambda q V,$$

for all $\lambda \in S^1$. 
Proof. By (2.19), together with the isotropy of $\Lambda_{i,j}$, for $i \neq j \in \{0, 1\}$, we have $q\Lambda = 0$. On the other hand, the centrality of $V$ with respect to $(\Lambda_{1,0}, \Lambda_{0,1})$ gives $N_V\Lambda = \pi_{V \perp} \circ d\Lambda = 0$. Hence
\[(3.2) \quad (d^\lambda q)_\Gamma \Lambda = d^\lambda \Gamma \Lambda\]
and we conclude that $(\phi^\lambda q \Lambda_{1,0}, \phi^\lambda q \Lambda_{0,1})$ is still a surface. Suppose, furthermore, that $(\Lambda_{1,0}, \Lambda_{0,1})$ is a real $q$-constrained Willmore surface. Given $\lambda \in S^1$, $d^\lambda q$ is real, so that we can choose $\phi^\lambda q$ to be real, in which case $\Lambda^\lambda q$ is a real surface. It is obvious, on the other hand, that as $V$ is a central sphere congruence of $(\Lambda_{1,0}, \Lambda_{0,1})$, $\phi^\lambda q V$ is a central sphere congruence of $(\phi^\lambda q \Lambda_{1,0}, \phi^\lambda q \Lambda_{0,1})$. Theorem 12 completes the proof. \(\square\)

Note that spectral deformation corresponding to the zero multiplier preserves the class of Willmore surfaces.

This spectral deformation of real constrained Willmore surfaces coincides, up to reparametrization, with the one presented in [8], in terms of the Schwarzian derivative and the Hopf differential (see [16, Section 6.4.1]).

An alternative perspective on this spectral deformation of perturbed harmonic bundles and constrained Willmore surfaces is that of a change of flat connection on $C^{n+2}$: if $V \subset (C^{n+2}, d)$ is perturbed harmonic, then so is $V \subset (C^{n+2}, d^\lambda q)$, as well as, if $(\Lambda_{1,0}, \Lambda_{0,1})$ is constrained Willmore [with respect to $d$] then $(\Lambda_{1,0}, \Lambda_{0,1})$ is still constrained Willmore with respect to $d^\lambda q$, for all $\lambda \in C \setminus \{0\}$. In the real case, this is the interpretation of loop group theory in [5].

3.2. Dressing action. We use a version of the dressing action theory of Terng and Uhlenbeck [20] to build transformations of $V$ into new perturbed harmonic bundles and thereafter transformations of $(\Lambda_{1,0}, \Lambda_{0,1})$ into new constrained Willmore surfaces. For that, we give conditions on a dressing $r(\lambda) \in \Gamma(O(C^{n+2}))$ such that the gauging $r(\lambda) \circ d^\lambda q \circ r(\lambda)^{-1}$ of $d^\lambda q$ by $r(\lambda)$ establishes the perturbed harmonicity of some bundle $\hat{V}$ from the perturbed harmonicity of $V$.

The $D_V$-parallelness of $V$ and $V^\perp$, together with the fact that $N_V$ intertwines $V$ and $V^\perp$, whereas $q$ preserves them, makes clear that
\[(3.3) \quad d^\lambda q = \rho \circ d^\lambda q \circ \rho^{-1},\]
for $\lambda \in C \setminus \{0\}$. Suppose we have $r(\lambda) \in \Gamma(O(C^{n+2}))$ such that $\lambda \mapsto r(\lambda)$ is rational in $\lambda$, $r$ is holomorphic and invertible at $\lambda = 0$ and $\lambda = \infty$ and twisted in the sense that
\[(3.4) \quad \rho r(\lambda) \rho^{-1} = r(-\lambda),\]
for \( \lambda \in \text{dom}(r) \). In particular, it follows that both \( r(0) \) and \( r(\infty) \) commute with \( \rho \), and, therefore, that

\[
(3.5) \quad r(0)|_V, r(\infty)|_V \in \Gamma(O(V)).
\]

Define \( \hat{q} \in \Omega^1(\wedge^2 V \oplus \wedge^2 V^\perp) \) by setting

\[
\hat{q}^{1,0} := \text{Ad}_{r(0)}q^{1,0}, \quad \hat{q}^{0,1} := \text{Ad}_{r(\infty)}q^{0,1}.
\]

Define a new family of metric connections on \( \mathbb{C}^{n+2} \) by setting

\[
d^\lambda \hat{q} := r(\lambda) \circ d^\lambda q \circ r(\lambda)^{-1}.
\]

Suppose that there exists a holomorphic extension of \( \lambda \mapsto d^\lambda \hat{q} \) to \( \lambda \in \mathbb{C} \setminus \{0\} \) through metric connections on \( \mathbb{C}^{n+2} \). We shall see later how to construct such \( r = r(\lambda) \), but assume, for the moment, that we have got one. In that case, we, crucially, verify next, the notation \( d^\lambda \hat{q} \) is not merely formal:

**Proposition 14.** [9]

\[
d^\lambda \hat{q} = D^\hat{q} + \lambda^{-1}(N^\hat{q})^{1,0} + \lambda(N^\hat{q})^{0,1} + (\lambda^{-2} - 1)\hat{q}^{1,0} + (\lambda^2 - 1)\hat{q}^{0,1},
\]

for the flat metric connection \( \hat{q} := d^\hat{q} = \lim_{\lambda \to 1} r(\lambda) \circ d^\lambda q \circ r(\lambda)^{-1} \) and \( \lambda \in \mathbb{C} \setminus \{0\} \).

**Proof.** The fact that \( r \) is holomorphic and invertible at \( \lambda = 0 \) and that \( (d^\lambda q)^{0,1} = D^{0,1} + \lambda N^{0,1} + (\lambda^2 - 1)q^{0,1} \) is holomorphic on \( \mathbb{C} \) establishes that the connection

\[
(d^\lambda q)^{0,1} = r(\lambda) \circ (d^\lambda q)^{0,1} \circ r(\lambda)^{-1},
\]

which admits a holomorphic extension to \( \lambda \in \mathbb{C} \setminus \{0\} \), admits, furthermore, a holomorphic extension to \( \lambda \in \mathbb{C} \). Thus, locally,

\[
(d^\lambda q)^{0,1} = A^{0,1}_0 + \sum_{i \geq 1} A^{0,1}_i,
\]

with \( A_0 \) connection and \( A_i \in \Omega^1(\rho(\mathbb{C}^{n+2})) \), for all \( i \). Considering then limits of

\[
\lambda^{-2} A^{0,1}_0 + \sum_{i \geq 1} \lambda^{i-2} A^{0,1}_i = r(\lambda) \circ (\lambda^{-2} D^{0,1} + \lambda^{-1} N^{0,1} + (1 - \lambda^{-2})q^{0,1}) \circ r(\lambda)^{-1},
\]

when \( \lambda \) goes to infinity, we get

\[
A^{0,1}_2 + \lim_{\lambda \to \infty} \sum_{i \geq 3} \lambda^{i-2} A^{0,1}_i = \text{Ad}_{r(\infty)} q^{0,1},
\]

which shows that \( A^{0,1}_i = 0 \), for all \( i \geq 3 \), and that \( A^{0,1}_2 = q^{0,1} \). Considering now limits of

\[
A^{0,1}_0 + \lambda A^{0,1}_1 + \lambda^2 q^{0,1} = r(\lambda) \circ (D^{0,1} + \lambda N^{0,1} + (\lambda^2 - 1)q^{0,1}) \circ r(\lambda)^{-1},
\]

when \( \lambda \) goes to 0, we conclude that

\[
A^{0,1}_0 = r(0) \circ (D^{0,1} - q^{0,1}) \circ r(0)^{-1}
\]
We conclude that
\[(d^\lambda \hat{q})^{0,1} = r(0) \circ (D_V^{0,1} - q^{0,1}) \circ r(0)^{-1} + \lambda A_1^{0,1} + \lambda^2 \hat{q}^{0,1}.\]

As for
\[(d^\lambda \hat{q})^{1,0} = r(\lambda) \circ (D_V^{1,0} + \lambda^{-1} A_1^{1,0} + (\lambda^{-2} - 1)q^{1,0}) \circ r(\lambda)^{-1},\]
which has a pole at \(\lambda = 0\), we have, for \(\lambda\) away from 0,
\[(3.6) \sum_i \lambda_i^{-1} A_i^{1,0} + A_0^{1,0} + \sum_i \lambda_i A_i^{1,0} = r(\lambda) \circ (D_V^{1,0} + \lambda^{-1} A_1^{1,0} + (\lambda^{-2} - 1)q^{1,0}) \circ r(\lambda)^{-1},\]
with \(A_i^{1,0} \in \Omega^1(o(\mathbb{C}^{n+2}))\), for all \(i \geq 1\). Considering limits of (3.6) when \(\lambda\) goes to infinity, shows that \(A_i^{1,0} = 0\), for all \(i \geq 1\), and that
\[A_0^{1,0} = r(\infty) \circ (D_V^{1,0} - q^{1,0}) \circ r(\infty)^{-1}.\]

Multiplying then both members of equation (3.6) by \(\lambda^2\) and considering limits when \(\lambda\) goes to 0, we conclude that \(A_{-2}^{1,0} = \hat{q}^{1,0}\) and that \(A_{-i}^{1,0} = 0\), for all \(i \geq 3\), and, ultimately, that
\[(d^\lambda \hat{q})^{1,0} = r(\infty) \circ (D_V^{1,0} - q^{1,0}) \circ r(\infty)^{-1} + \lambda^{-1} A_{-1}^{1,0} + \lambda^{-2} \hat{q}^{1,0}.\]

Thus
\[d^\lambda \hat{q} = r(0) \circ (D_V^{0,1} - q^{0,1} + q^{1,0}) \circ r(0)^{-1} + r(\infty) \circ (D_V^{1,0} - q^{1,0} + q^{0,1}) \circ r(\infty)^{-1} + \lambda^{-1} A_{-1}^{1,0} + \lambda A_1^{0,1} + (\lambda^{-2} - 1)q^{1,0} + (\lambda^2 - 1)\hat{q}^{0,1},\]
for \(\lambda \in \mathbb{C} \setminus \{0\}\), and, in particular,
\[d = r(0) \circ (D_V^{0,1} - q^{0,1} + q^{1,0}) \circ r(0)^{-1} + r(\infty) \circ (D_V^{1,0} - q^{1,0} + q^{0,1}) \circ r(\infty)^{-1} + A_{-1}^{1,0} + A_1^{0,1} + (\lambda^2 - 1)\hat{q}^{0,1},\]
for all \(\lambda \in \mathbb{C} \setminus \{0\}\) away from the poles of \(r\) and then, by continuity, on all of \(\mathbb{C} \setminus \{0\}\). The particular case of \(\lambda = 1\) gives \(\rho(A_{-1}^{1,0} + A_1^{0,1}) = -(A_{-1}^{1,0} + A_1^{0,1})\), showing that
\[A_{-1}^{1,0} + A_1^{0,1} \in \Omega^1(V \wedge V^\perp).\]

We conclude that
\[(3.7) r(0) \circ (D_V^{0,1} - q^{0,1} + q^{1,0}) \circ r(0)^{-1} + r(\infty) \circ (D_V^{1,0} - q^{1,0} + q^{0,1}) \circ r(\infty)^{-1} = D_V^\lambda\]
and
\[A_{-1}^{1,0} = (A_V^\lambda)^{1,0}, \quad A_1^{0,1} = (A_V^\lambda)^{0,1}.\]
completing the proof.

The flatness of $d^\lambda,q$ for all $\lambda \in \mathbb{C}\backslash\{0\}$ establishes that of $\hat{d}^\lambda,q$, for all non-zero $\lambda$ away from the poles of $r$ and then, by continuity, for all $\lambda \in \mathbb{C}\backslash\{0\}$. By Proposition \ref{prop:flatness}, we conclude that $V$ is $(\hat{q},\hat{d})$-perturbed harmonic. Suppose $1 \in \text{dom}(r)$. By Lemma \ref{lem:dom} it follows that:

**Theorem 15.** \cite{9} $r(1)^{-1}V$ is a $\text{Ad}_{r(1)^{-1}}\hat{q}$-perturbed harmonic bundle.

Note that this transformation preserves the harmonicity condition.

A transformation on the level of constrained Willmore surfaces follows, with some extra condition, as we shall see next. Set

$$\hat{\Lambda}^1,0 := r(\infty)\Lambda^1,0, \quad \hat{\Lambda}^0,1 := r(0)\Lambda^0,1$$

and

$$\hat{\Lambda} = \hat{\Lambda}^1,0 \cap \hat{\Lambda}^0,1.$$ 

Suppose, furthermore, that

$$\text{(3.8)} \quad \det r(0)|_V = \det r(\infty)|_V.$$ 

Then:

**Theorem 16.** \cite{9} $(r(1)^{-1}\hat{\Lambda}^{1,0},r(1)^{-1}\hat{\Lambda}^{0,1})$ is a $\text{Ad}_{r(1)^{-1}}\hat{q}$-constrained Willmore surface admitting $r(1)^{-1}V$ as a central sphere congruence.

**Proof.** First of all, note that, by \cite{219}, $\hat{q}^{i,j} \in \Omega^{i,j}(\wedge^2\hat{\Lambda}^{i,j})$ and, therefore,

$$\text{(Ad}_{r(1)^{-1}}\hat{q})^{i,j} \in \Omega^{i,j}(\wedge^2 r(1)^{-1}\hat{\Lambda}^{i,j}),$$

for $i \neq j \in \{0,1\}$. In the light of Theorem \ref{thm:15} we are left to verify that $(r(1)^{-1}\hat{\Lambda}^{1,0},r(1)^{-1}\hat{\Lambda}^{0,1})$ is a surface admitting $r(1)^{-1}V$ as a central sphere congruence.

The fact that $\Lambda^{1,0}$ and $\Lambda^{0,1}$ are rank 2 isotropic subbundles of $V$ ensures that so are $\hat{\Lambda}^{1,0}$ and $\hat{\Lambda}^{0,1}$, as $r(0)$ and $r(\infty)$ are orthogonal transformations and preserve $V$. To see that $\hat{\Lambda}$ is rank 1, we use some well-known facts about the Grassmannian $G_W$ of isotropic 2-planes in a complex 4-dimensional space $W$: it has two components, each an orbit of the special orthogonal group $SO(W)$, intertwined by the action of elements of $O(W)\backslash SO(W)$, and for which any element intersects any element of the other component in a line while distinct elements of the same component have trivial intersection. Since rank $\Lambda = 1$, $\Lambda_p^{1,0}$ and $\Lambda_p^{0,1}$ lie in different components of $G_{V_p}$ and the hypothesis \text{(3.8)} ensures that the same is true of $\hat{\Lambda}_p^{1,0}$ and $\hat{\Lambda}_p^{0,1}$, for all $p$.

We are left to verify that

$$\text{(3.9)} \quad \hat{d}^{1,0}\Gamma(\hat{\Lambda}) \subset \Omega^{1}(\hat{\Lambda}^{1,0}), \quad \hat{d}^{0,1}\Gamma(\hat{\Lambda}) \subset \Omega^{1}(\hat{\Lambda}^{0,1})$$

and that (recall equation \text{(3.1)})

$$\text{(3.10)} \quad (\Lambda^{\hat{d}})^{1,0}\hat{\Lambda}^{0,1} = 0 = (\Lambda^{\hat{d}})^{0,1}\hat{\Lambda}^{1,0}.$$
Equation (3.10) forces $N_\lambda^q \hat{\Lambda} = 0$, in which situation, (3.9) reads
\[(D^\lambda_V)^{1,0,1} \Gamma(\hat{\Lambda}) \subset \Omega^1(\hat{\Lambda}^{1,0}), \quad (D^\lambda_V)^{0,1} \Gamma(\hat{\Lambda}) \subset \Omega^1(\hat{\Lambda}^{0,1}),\]
which, in its turn, follows from
\[(3.11) \quad (D^\lambda_V)^{1,0} \Gamma(\hat{\Lambda}^{1,0}) \subset \Omega^1(\hat{\Lambda}^{1,0}), \quad (D^\lambda_V)^{0,1} \Gamma(\hat{\Lambda}^{0,1}) \subset \Omega^1(\hat{\Lambda}^{0,1}).\]
It is (3.10) and (3.11) that we shall establish.

First of all, note that, according to (3.7),
\[(D^\lambda_V)^{1,0} = r(\infty) \circ (D^\lambda_V)^{1,0} - q^{1,0} \circ r(\infty)^{-1} + \hat{q}^{1,0}\]
and
\[(D^\lambda_V)^{0,1} = r(0) \circ (D^\lambda_V)^{0,1} - q^{0,1} \circ r(0)^{-1} + \hat{q}^{0,1}.\]
Now $q^{1,0}$ takes values in $\Lambda \wedge \Lambda^{0,1}$, so $q^{1,0} \Lambda^{1,0} \subset \Lambda \subset \Lambda^{1,0}$, by the isotropy of $\Lambda^{1,0}$. On the other hand, since rank $\hat{\Lambda} = 1$, we have $\wedge^2 \Lambda^{0,1} = \Lambda \wedge \Lambda^{0,1}$ and, therefore, $\hat{q}^{1,0} \Lambda^{1,0} \subset \Lambda \subset \Lambda^{1,0}$. Together with Lemma 9, this establishes the $(1,0)$-part of (3.11). A similar argument establishes the $(0,1)$-part of it.

Finally, we establish (3.10). According to Proposition 14,
\[(N_V^q)^{1,0} = \lim_{\lambda \to 0} \lambda ((d^V_{\lambda q})^{1,0} - (d^\lambda_V)^{1,0} - (\lambda^{-2} - 1) \hat{q}^{1,0})\]
\[= \lim_{\lambda \to 0} \lambda ((d^V_{\lambda q})^{1,0} - \lambda^{-2} \hat{q}^{1,0})\]
\[= \lim_{\lambda \to 0} (r(\lambda) \circ (d^V_{\lambda q})^{1,0} \circ r(\lambda)^{-1} - \lambda^{-1} \text{Ad}_r(0) q^{1,0})\]
\[= \text{Ad}_r(0) N_V^{1,0} + \lim_{\lambda \to 0} \frac{1}{\lambda} (\text{Ad}_r(\lambda) - \text{Ad}_r(0)) q^{1,0}.\]
so that
\[(3.12) \quad (N_V^q)^{1,0} = \text{Ad}_r(0) N_V^{1,0} + \frac{d}{d\lambda|_{\lambda=0}} \text{Ad}_r(\lambda) q^{1,0};\]
and, similarly,
\[(N_V^q)^{0,1} = \lim_{\lambda \to \infty} \lambda^{-1} ((d^V_{\lambda q})^{0,1} - (d^\lambda_V)^{0,1} - (\lambda^{-2} - 1) q^{0,1})\]
\[= \text{Ad}_r(\infty) N_V^{0,1} + \lim_{\lambda \to \infty} (r(\lambda) \circ \lambda q^{0,1} \circ r(\lambda)^{-1} - \lambda \text{Ad}_r(\infty) q^{0,1})\]
and, therefore,
\[(3.13) \quad (N_V^q)^{0,1} = \text{Ad}_r(\infty) N_V^{0,1} + \frac{d}{d\lambda|_{\lambda=0}} \text{Ad}_r(\lambda^{-1}) q^{0,1}.\]
Furthermore, by (3.12),
\[(N_V^q)^{1,0} = \text{Ad}_r(0) (N_V^{1,0} + [r(0)^{-1} \frac{d}{d\lambda|_{\lambda=0}} r(\lambda) q^{1,0}]).\]
The centrality of $V$ with respect to $(\Lambda^{1,0}, \Lambda^{0,1})$ establishes, in particular, $N_V^{1,0} \Lambda^{0,1} = 0$, whilst the isotropy of $\Lambda^{0,1}$ ensures, in particular, that $q^{1,0} \Lambda^{0,1} = 0$. Hence
\[\text{Ad}_r(0) (N_V^{1,0} + r(0)^{-1} \frac{d}{d\lambda|_{\lambda=0}} r(\lambda) q^{1,0}) \hat{\Lambda}^{0,1} = 0.\]
On the other hand, differentiation of $r(\lambda)^{-1} = \rho r(-\lambda)^{-1} \rho$, derived from equation (3.4), gives

$$-r(\lambda)^{-1} \frac{d}{dk}|_{k=\lambda} r(k) r(\lambda)^{-1} = \rho r(-\lambda)^{-1} \frac{d}{dk}|_{k=-\lambda} r(k) r(-\lambda)^{-1} \rho,$$

or, equivalently,

$$\rho r(\lambda)^{-1} \frac{d}{dk}|_{k=\lambda} r(k) r(\lambda) = -r(-\lambda)^{-1} \frac{d}{dk}|_{k=-\lambda} r(k) r(-\lambda)^{-1} \rho r(\lambda) \rho,$$

and, therefore, yet again by equation (3.4),

$$\rho r(\lambda)^{-1} \frac{d}{dk}|_{k=\lambda} r(k) r(\lambda) = -r(0)^{-1} \frac{d}{d\lambda}|_{\lambda=0} r(\lambda).$$

Evaluation at $\lambda = 0$ shows then that

$$\rho r(0)^{-1} \frac{d}{d\lambda}|_{\lambda=0} r(\lambda) \rho = -r(0)^{-1} \frac{d}{d\lambda}|_{\lambda=0} r(\lambda).$$

Equivalently,

(3.14) $r(0)^{-1} \frac{d}{d\lambda}|_{\lambda=0} r(\lambda) \in \Gamma(V \wedge V^\perp).$

Since $qV^\perp = 0$, we conclude that

$$Ad_{r(0)}(q^{1,0} r(0)^{-1} \frac{d}{d\lambda}|_{\lambda=0} r(\lambda)) 0^{0,1} = 0$$

and, ultimately, that $(N^2 V_\perp^{1,0}) 0^{0,1} = 0$. A similar argument near $\lambda = \infty$ establishes $(N^2 V_\perp^{0,1}) 1^{1,0} = 0$, completing the proof. □

3.3. Bäcklund transformation. We now construct $r = r(\lambda)$ satisfying the hypothesis of the previous section. As the philosophy underlying the work of C.-L. Terng and K. Uhlenbeck [22] suggests, we consider linear fractional transformations. As we shall see, a two-step process will produce a desired $r$.

Given $\alpha \in \mathbb{C} \setminus \{-1, 0, 1\}$ and $L$ a null line subbundle of $\mathbb{C}^{n+2}$ such that, locally, $\rho L \cap L^\perp = \{0\}$, set

$$p_{\alpha,L}(\lambda) := I \begin{cases} \frac{\alpha+\lambda}{\alpha} & \text{on } L \\ \frac{\alpha-\lambda}{\alpha} & \text{on } (L \oplus \rho L)^\perp \\ \frac{\alpha+\lambda}{\alpha} & \text{on } \rho L \end{cases}$$

and

$$q_{\alpha,L}(\lambda) := I \begin{cases} \frac{\lambda-\alpha}{\lambda} & \text{on } L \\ \frac{\lambda+\alpha}{\lambda} & \text{on } (L \oplus \rho L)^\perp \\ \frac{\lambda+\alpha}{\lambda-\alpha} & \text{on } \rho L \end{cases},$$

for $\lambda \in \mathbb{C} \setminus \{\pm \alpha\}$, defining in this way maps

$$p_{\alpha,L}, q_{\alpha,L} : \mathbb{C} \setminus \{\pm \alpha\} \to \Gamma(O(\mathbb{C}^{n+2}))$$
that, clearly, extend holomorphically to \( \mathbb{P}^1 \setminus \{ \pm \alpha \} \), by setting

\[
p_{\alpha,L}(\infty) := I \begin{cases} 
-1 & \text{on } L \\
1 & \text{on } (L \oplus \rho L)^{\perp} \\
-1 & \text{on } \rho L
\end{cases}
\]

and

\[
q_{\alpha,L}(\infty) := I.
\]

Obviously, \( p_{\alpha,L}(\infty) \) and \( q_{\alpha,L}(\infty) \) do not depend on \( \alpha \). For further reference, note that, for all \( \lambda \in \mathbb{C} \setminus \{ \pm \alpha, 0 \} \), we have

\[(3.15) \quad p_{\alpha,L}(\lambda) = q_{\alpha^{-1},L}(\lambda^{-1}),\]

whilst

\[
p_{\alpha,L}(0) = q_{\alpha,L}(\infty), \quad p_{\alpha,L}(\infty) = q_{\alpha,L}(0).
\]

The isometry \( \rho = \rho^{-1} \) intertwines \( L \) and \( \rho L \) and, therefore, preserves \( (L \oplus \rho L)^{\perp} \), which makes clear that \( \rho \circ p_{\alpha,L}(\lambda) \) and \( p_{\alpha,L}(\lambda)^{-1} \circ \rho \) coincide in \( L, \rho L \) and \( (L \oplus \rho L)^{\perp} \) and, therefore,

\[(3.16) \quad \rho p_{\alpha,L}(\lambda) = p_{\alpha,L}(\lambda)^{-1} = p_{\alpha,L}(\lambda),\]

and, similarly,

\[(3.17) \quad \rho q_{\alpha,L}(\lambda) = q_{\alpha,L}(\lambda)^{-1} = q_{\alpha,L}(\lambda),\]

for all \( \lambda \) - both \( p_{\alpha,L} \) and \( q_{\alpha,L} \) are twisted in the sense of Section 3.2.

Since \( \rho L \) is not orthogonal to \( L \), \( \rho L \neq L \), so \( L \) is not a subbundle of \( V \) and, therefore, \( \text{rank } V \cap (L \oplus \rho L) = 1 \). We conclude that

\[
p_{\alpha,L}(\infty)|_V = q_{\alpha,L}(0)|_V = I \begin{cases} 
-1 & \text{on } V \cap (L \oplus \rho L) \\
1 & \text{on } V \cap (L \oplus \rho L)^{\perp}
\end{cases}
\]

has determinant \( -1 \neq 1 = \det p_{\alpha,L}(0)|_V = \det q_{\alpha,L}(\infty)|_V \), so we cannot take \( p_{\alpha,L} = r \) or \( q_{\alpha,L} = r \) in the analysis of Section 3.2. However, we will be able to take \( r = q_{\beta,L} p_{\alpha,L} \), for suitable \( \beta \) and \( \hat{L} \), as we shall see.

Now choose \( \alpha \in \mathbb{C} \setminus \{-1, 0, 1\} \) and \( L^\alpha \) a \( d^{\alpha,q}_V \)-parallel null line subbundle of \( \mathbb{C}^{n+2} \) such that, locally,

\[(3.18) \quad \rho L^\alpha \cap (L^\alpha)^{\perp} = \{0\}.
\]

Such a bundle \( L^\alpha \) can be obtained by \( d^{\alpha,q}_V \)-parallel transport of \( l^\alpha_p \in \mathbb{C}^{n+2} \), with \( l^\alpha_p \) null and non-orthogonal to \( \rho p^\alpha_p \), for some \( p \in \Sigma \).

**Lemma 17.** [9] There exists a holomorphic extension of

\[
\lambda \mapsto \tilde{d}_{p_{\alpha,L}}^{\lambda,q} := p_{\alpha,L^\alpha}(\lambda) \circ d^{\lambda,q}_V \circ p_{\alpha,L^\alpha}(\lambda)^{-1}
\]

to \( \lambda \in \mathbb{C} \setminus \{0\} \) through metric connections on \( \mathbb{C}^{n+2} \).
Proof. We prove holomorphicity at $\lambda = \alpha$. For $\lambda \in \mathbb{C}\{0, \alpha\}$, write

$$
d^{\lambda,q}_V = d^{\alpha,q}_{\alpha,L}(\lambda) + (\lambda - \alpha)A(\lambda),$$

with $\lambda \mapsto A(\lambda) \in \Omega^1(\mathbb{C}^{n+2})$ holomorphic. Decompose $d^{\alpha,q}_{\alpha,L} = D + \beta$ according to the decomposition

$$
\mathbb{C}^{n+2} = (L^\alpha \oplus \rho L^\alpha) \oplus (L^\alpha \oplus \rho L^\alpha)^\perp.
$$

The fact that $d^{\alpha,q}_{\alpha,L}$ is a metric connection establishes

$$
d^{\alpha,q}_{\alpha,L}(\rho L^\alpha) \subset \Omega^1(\rho L^\alpha)^\perp,$$

as well as

$$
d^{\alpha,q}_{\alpha,L}(\rho L^\alpha)^\perp \subset \Omega^1(\rho L^\alpha)^\perp,$$

in view of the $d^{\alpha,q}_{\alpha,L}$-parallelness of $L^\alpha$. By (3.18), we conclude that the 1-form $\beta \in \Omega^1((L^\alpha \oplus \rho L^\alpha) \wedge (L^\alpha \oplus \rho L^\alpha)^\perp)$ takes values in $L^\alpha \wedge (L^\alpha \oplus \rho L^\alpha)^\perp$. Hence

$$p_{\alpha,L^\alpha}(\lambda) \circ \beta \circ p_{\alpha,L^\alpha}(\lambda)^{-1} = \frac{\alpha - \lambda}{\alpha + \lambda} \beta.$$

On the other hand,

$$p_{\alpha,L^\alpha}(\lambda) \circ D \circ p_{\alpha,L^\alpha}(\lambda)^{-1} = D,$$

as $L^\alpha$, $\rho L^\alpha$, and $(L^\alpha \oplus \rho L^\alpha)^\perp$ are all $D$-parallel. Thus

$$d^{\lambda,q}_{p_{\alpha,L} L} = D + \frac{\alpha - \lambda}{\alpha + \lambda} \beta + (\lambda - \alpha) p_{\alpha,L^\alpha}(\lambda) A(\lambda) p_{\alpha,L^\alpha}(\lambda)^{-1}.$$

Lastly, note that, by the skew-symmetry of $A(\lambda)$, we have $A(\lambda)L^\alpha \subset (L^\alpha)^\perp$ and $A(\lambda)\rho L^\alpha \subset (\rho L^\alpha)^\perp$ and, therefore, by (3.18),

$$A(\lambda)L^\alpha \subset L^\alpha \oplus (L^\alpha \oplus \rho L^\alpha)^\perp$$

and

$$A(\lambda)\rho L^\alpha \subset \rho L^\alpha \oplus (L^\alpha \oplus \rho L^\alpha)^\perp.$$

We conclude that $(\lambda - \alpha) \Ad_{p_{\alpha,L^\alpha} L}(\lambda) A(\lambda)$ has at most a simple pole at $\lambda = -\alpha$ and, therefore, that $d^{\lambda,q}_{p_{\alpha,L} L}$ is holomorphic at $\lambda = \alpha$. Furthermore, the fact that $D$ is a metric connection establishes that so is $d^{\lambda,q}_{p_{\alpha,L} L}$, in view of the skew-symmetry of $A(\lambda)$ and of $\beta$.

Holomorphicity at $\lambda = -\alpha$ can either be proved in the same way, having in consideration that the $d^{\alpha,q}_{\alpha,L}$-parallelness of $L^\alpha$ establishes the $d^{\alpha,q}_{\alpha,L}$-parallelness of $\rho L^\alpha$, or by exploiting the symmetry $\lambda \mapsto -\lambda$.

Remark 3. 1) The same argument establishes the existence of a holomorphic extension of

$$\lambda \mapsto d^{\lambda,q}_{q_{\alpha,L^\alpha}} := q_{\alpha,L^\alpha}(\lambda) \circ d^{\lambda,q}_{V} \circ q_{\alpha,L^\alpha}(\lambda)^{-1}$$

to $\lambda \in \mathbb{C}\{0\}$ through metric connections on $\mathbb{C}^{n+2}$.

2) This argument uses nothing about the precise form of $d^{\lambda,q}_V$, only that it is holomorphic near $\lambda = \pm\alpha$. □
Now we can iterate the procedure starting with the connections \( d_{p,\alpha,\lambda}^{\lambda,q} \).
Choose \( \beta \neq \pm \alpha \) in \( \mathbb{C}\{\pm 1, 0, \pm 1\} \) and \( L^\beta \) a \( d_{\alpha}^{\beta,q} \)-parallel null line subbundle of \( \mathbb{C}^{n+2} \). The fact that
\[
p_{\alpha,\lambda,\alpha} : (\mathbb{C}^{n+2}, d_{\alpha}^{\lambda,q}) \to (\mathbb{C}^{n+2}, d_{\alpha,\lambda}^{\lambda,q})
\]
preserves connections establishes the \( d_{\alpha, L^{\alpha}}^{\beta,q} \)-parallelness of
\[
\hat{L}_{\alpha}^{\beta} := p_{\alpha, L^{\alpha}}(\beta)L^{\beta}.
\]
Choose \( L^{\beta} \) satisfying, furthermore, \( \rho \hat{L}_{\alpha}^{\beta} \cap (\hat{L}_{\alpha}^{\beta})^\perp = \{0\} \). Such a bundle \( L^{\beta} \) can be obtained by \( d_{\alpha}^{\beta,q} \)-parallel transport of \( l_{p}^{\alpha,L^{\alpha}} \), with \( l_{p}^{\alpha,L^{\alpha}} \) non-zero, non-orthogonal to \( \rho p_{p}^{\alpha,L^{\alpha}} \), for some \( p \in \Sigma \). Indeed, by (3.18),
\[
(\rho p_{p, L^{\alpha}}(\beta)l_{p}^{\alpha,L^{\alpha}}, p_{p, L^{\alpha}}(\beta)l_{p}^{\alpha,L^{\alpha}}) = (p_{\alpha, L^{\alpha}}(\beta)\rho p_{p}^{\alpha,L^{\alpha}}, p_{\alpha, L^{\alpha}}(\beta)l_{p}^{\alpha,L^{\alpha}}) = \frac{(\alpha - \beta)^2}{(\alpha + \beta)^2} (\rho p_{p}^{\alpha,L^{\alpha}}, l_{p}^{\alpha,L^{\alpha}}).
\]
It follows that
\[
\lambda \mapsto q_{\beta, L^{\alpha}}(\lambda) p_{\alpha, L^{\alpha}}(\lambda) \circ d_{\alpha}^{\lambda,q} \circ p_{\alpha, L^{\alpha}}(\lambda)^{-1} q_{\beta, L^{\alpha}}(\lambda)^{-1}
\]
admits a holomorphic extension to \( \lambda \in \mathbb{C}\{\pm 0\} \) through metric connections on \( \mathbb{C}^{n+2} \) and, furthermore, that
\[
r^{\ast} := q_{\beta, L^{\alpha}} p_{\alpha, L^{\alpha}}
\]
satisfies all the hypothesis of Section 3.2 on \( r \). Set
\[
q^{\ast} := \text{Ad}_{r^{\ast}(1)}^{-1}(\text{Ad}_{r^{\ast}(0)}q^{1,0} + \text{Ad}_{r^{\ast}(\infty)}q^{0,1}).
\]

**Definition 5.** The \( q^{\ast} \)-perturbed harmonic bundle
\[
V^{\ast} := r^{\ast}(1)^{-1} V
\]
is said to be the Bäcklund transform of \( V \) of parameters \( \alpha, \beta, L^{\alpha}, L^{\beta} \). The \( q^{\ast} \)-constrained Willmore surface
\[
(\Lambda^{1,0}, \Lambda^{0,1})^{\ast} := (r^{\ast}(1)^{-1} r^{\ast}(\infty)\Lambda^{1,0}, r^{\ast}(1)^{-1} r^{\ast}(0)\Lambda^{0,1})
\]
is said to be the Bäcklund transform of \( (\Lambda^{1,0}, \Lambda^{0,1}) \) of parameters \( \alpha, \beta, L^{\alpha}, L^{\beta} \).

Note that transformations corresponding to the zero multiplier preserve the class of Willmore surfaces.

For further reference, set
\[
((\Lambda)^{1,0}, (\Lambda)^{0,1}) := (r^{\ast}(1)^{-1} r^{\ast}(\infty)\Lambda^{1,0}, r^{\ast}(1)^{-1} r^{\ast}(0)\Lambda^{0,1}).
\]
3.3.1. Bianchi permutability. Next we establish a Bianchi permutability of type $p$ and type $q$ transformations, showing that starting the procedure above with the connections $d_{\alpha,\beta,\gamma}$ (when defined), instead of $d_{\alpha,\beta,\gamma}$, produces the same transforms. The underlying argument will play a crucial role when investigating the preservation of reality conditions by Bäcklund transformation, in the next section.

Suppose $\rho L_0 \cap (L^\beta)_{0} = \{0\}$ and set $\hat{L}_0^\alpha := q_{\beta,\gamma}(\alpha)L^\alpha$. Suppose, furthermore, that $\rho \hat{L}_0^\beta \cap (\hat{L}_0^\beta)_{0} = \{0\}$ (this is certainly the case for $L^\beta$ obtained by $d_{V}$-parallel transport of $l_p^\alpha \in L_p^\alpha$, with $l_p^\alpha$ non-zero, non-orthogonal to $r_p l_p^\alpha$, for some $p \in \Sigma$). Analogously to $r^*$, we verify that

$$\hat{r}^* := p_{\alpha,\beta,\gamma} q_{\beta,\gamma}$$

satisfies all the hypothesis of Section 3.2 on $r$. The next result, relating $\hat{r}^*$ to $r^*$, will be crucial in all that follows.

**Lemma 18.** [8]

(3.19) $$\hat{r}^* = K r^*,$$

for $K := q_{\beta,\gamma}(0) q_{\beta,\gamma}(0)$.

The proof of the lemma we present next will be based on the following result:

**Lemma 19.** [4] Let $\gamma(\lambda) = \lambda \pi L_1 + \pi L_0 + \lambda^{-1} \pi L_{-1}$ and $\hat{\gamma}(\lambda) = \lambda \hat{\pi} L_1 + \hat{\pi} L_0 + \lambda^{-1} \hat{\pi} L_{-1}$ be homomorphisms of $\mathbb{C}^{n+2}$ corresponding to decompositions

$$\mathbb{C}^{n+2} = L_1 \oplus L_0 \oplus L_{-1} = \hat{L}_1 \oplus \hat{L}_0 \oplus \hat{L}_{-1}$$

with $L_{-1}$ and $\hat{L}_{-1}$ null lines and $L_0 = (L_1 \oplus L_{-1})_{0} \perp (\hat{L}_1 \oplus \hat{L}_{-1})_{0}$. Suppose $\text{Ad} \gamma$ and $\text{Ad} \hat{\gamma}$ have simple poles. Suppose as well that $\xi$ is a map into $O(\mathbb{C}^{n+2})$ holomorphic near 0 such that $L_1 = \xi(0)L_1$. Then $\gamma \xi \hat{\gamma}^{-1}$ is holomorphic and invertible at 0.

Now we proceed to the proof of Lemma 18.

**Proof.** For simplicity, write $p_{\mu,\gamma}^{-1}$ and $q_{\mu,\gamma}^{-1}$ for $\lambda \mapsto p_{\mu,\gamma}(\lambda)^{-1}$ and, respectively, $\lambda \mapsto q_{\mu,\gamma}(\lambda)^{-1}$, in the case $p_{\mu,\gamma}$ and, respectively, $q_{\mu,\gamma}$ are defined. As $L^\alpha = q_{\beta,\gamma}(\alpha)^{-1} L_0^\alpha$, after an appropriate change of variable, we conclude, by Lemma 19, that $p_{\alpha,\beta,\gamma} q_{\beta,\gamma}^{-1} p_{\alpha,\beta,\gamma}^{-1}$ admits a holomorphic and invertible extension to $\mathbb{P}^1 \setminus \{\pm \beta, -\alpha\}$. On the other hand, in view of (3.19), the holomorphicity and invertibility of $p_{\alpha,\beta,\gamma} q_{\beta,\gamma}^{-1} p_{\alpha,\beta,\gamma}$ at the points $\alpha$ and $-\alpha$ are equivalent. Thus $p_{\alpha,\beta,\gamma} q_{\beta,\gamma}^{-1} p_{\alpha,\beta,\gamma}^{-1}$ admits a holomorphic and invertible extension to $\mathbb{P}^1 \setminus \{\pm \beta\}$, and so does, therefore, $(p_{\alpha,\beta,\gamma} q_{\beta,\gamma}^{-1} p_{\alpha,\beta,\gamma})^{-1} q_{\beta,\gamma}^{-1}$. A
similar argument shows that $p_{\alpha, L^\alpha} (q_{\beta, L^\beta}^{-1} p_{\alpha, L^\alpha} q_{\beta, L^\beta}^{-1})$ admits a holomorphic extension to $\mathbb{P}^1 \setminus \{ \pm \alpha \}$. But

$$p_{\alpha, L^\alpha} q_{\beta, L^\beta}^{-1} p_{\alpha, L^\alpha} q_{\beta, L^\beta}^{-1} = (p_{\alpha, L^\alpha} q_{\beta, L^\beta}^{-1} p_{\alpha, L^\alpha})^{-1} q_{\beta, L^\beta}^{-1}.$$

We conclude that $p_{\alpha, L^\alpha} q_{\beta, L^\beta}^{-1} p_{\alpha, L^\alpha} q_{\beta, L^\beta}^{-1}$ extends holomorphically to $\mathbb{P}^1$ and is, therefore, constant. Evaluating at $\lambda = 0$ gives

$$p_{\alpha, L^\alpha} q_{\beta, L^\beta}^{-1} p_{\alpha, L^\alpha} q_{\beta, L^\beta}^{-1} = q_{\beta, L^\beta}(0) q_{\beta, L^\beta}(0),$$

completing the proof.

According to (3.17), $\rho K \rho = K$, showing that $K$ preserves $V$ or, equivalently,

$$(3.20) \quad K V = V.$$

By (3.19), it follows that

$$r^*(1)^{-1} V = \tilde{r}^*(1)^{-1} V,$$

establishing a Bianchi permutability of type $p$ and type $q$ transformations of perturbed harmonic bundles, by means of the commutativity of the diagram in Figure 1 below.

![Figure 1](image-url)

**Figure 1.** A Bianchi permutability of type $p$ and type $q$ transformations of perturbed harmonic bundles.

Equation (3.19) makes clear, on the other hand, that

$$\tilde{r}^*(1)^{-1} \tilde{r}^*(\infty) \Lambda^{1,0} = r^*(1)^{-1} r^*(\infty) \Lambda^{1,0}$$

and

$$\tilde{r}^*(1)^{-1} \tilde{r}^*(0) \Lambda^{0,1} = r^*(1)^{-1} r^*(0) \Lambda^{0,1}.$$  

We conclude that, despite not coinciding, $r^*$ and $\tilde{r}^*$ produce the same transforms of perturbed harmonic bundles and constrained Willmore surfaces. As a final remark, note that, yet again by equation (3.19),

$$\hat{q}^* := \text{Ad}_{\tilde{r}^*(1)^{-1}} (\text{Ad}_{\tilde{r}^*(0)} q^{1,0} + \text{Ad}_{\tilde{r}^*(\infty)} q^{0,1}) = q^*.$$
3.3.2. Real Bäcklund transformation. As we verify next, Bäcklund transformation preserves reality conditions, for special choices of parameters.

Suppose $V$ is a real $q$-constrained harmonic bundle. Obviously, the reality of $V$ establishes that of $\rho$ and, therefore,
\begin{equation}
\tag{3.21}
\bar{p}_{\mu,L}(\lambda) = p_{\pi,\bar{L}}(\bar{\lambda}), \quad q_{\mu,L}(\lambda) = q_{\pi,\bar{L}}(\bar{\lambda}),
\end{equation}
for all $\mu, L$ and $\lambda \in \mathbb{C}\setminus\{\pm\mu\}$.

Lemma 20. Suppose $\alpha \in \mathbb{C}\setminus(S^1 \cup \{0\})$. Then we can choose $\beta = \bar{\alpha}^{-1}$ and $L^\beta = \bar{L}$ and both $r^*$ and $\bar{r}^*$ are defined.

Proof. The reality of $\rho$ makes it clear that the non-orthogonality of $L^\alpha$ and $\rho L^\alpha$ establishes that of $\bar{L}$ and $\rho \bar{L}$, as well as, together with (3.21) and (3.15), that, if
\begin{equation}
\rho p_{\alpha,L^\alpha}(\bar{\alpha}^{-1})\bar{L} \cap p_{\alpha,L^\alpha}(\bar{\alpha}^{-1})\bar{L} = \{0\},
\end{equation}
then
\begin{equation}
\rho q_{\bar{\alpha}^{-1},\bar{L}}(\alpha) L^\alpha \cap q_{\bar{\alpha}^{-1},\bar{L}}(\alpha) L^\alpha = \{0\}.
\end{equation}

On the other hand, the reality of $V$ establishes that of $D_V$ and $N_V$, so that, by the reality of $q$,
\begin{equation}
\bar{d}_{V}^{\alpha^{-1},q} = d_{V}^{\alpha,q}.
\end{equation}
Hence the $d_{V}^{\alpha,q}$-parallelness of $L^\alpha$ establishes the $d_{V}^{\alpha^{-1},q}$-parallelness of $\bar{L}$. Obviously, if $\alpha$ is non-unit, then $\alpha^{-1} \neq \pm\alpha$. We are left to verify that we can choose $L^\alpha$ a $d_{V}^{\alpha,q}$-parallel null line subbundle of $\mathbb{C}^{n+2}$ such that, locally, $\rho L^\alpha \cap L^\alpha = \{0\}$ and
\begin{equation}
\rho p_{\alpha,L^\alpha}(\bar{\alpha}^{-1})\bar{L} \cap p_{\alpha,L^\alpha}(\bar{\alpha}^{-1})\bar{L} = \{0\}.
\end{equation}

For this, let $v$ and $w$ be sections of $V$ and $V^\perp$, respectively, with $(v,v)$ never-zero, $(v,\bar{v}) = 0$ and $(w, w) = -(v, v)$. Define a null section of $\mathbb{C}^{n+2}$ by $l^\alpha := v + w$ and then $L^\alpha \subset \mathbb{C}^{n+2}$ by $d_{V}^{\alpha,q}$-parallel transport of $l^\alpha_p$, for some point $p \in \Sigma$.

Let us focus then on the particular case of Bäcklund transformation of parameters $\alpha, \beta, L^\alpha, L^\beta$ with
\begin{equation}
\alpha \in \mathbb{C}\setminus(S^1 \cup \{0\}), \quad \beta = \bar{\alpha}^{-1}, \quad L^\beta = \bar{L},
\end{equation}
which we refer to as Bäcklund transformation of parameters $\alpha, L^\alpha$. For this particular choice of parameters, we write $\bar{L}^\alpha$ and $\bar{L}_{\alpha}$ for $\bar{L}^\beta$ and $\bar{L}_{\alpha}$, respectively. Note that, by (3.15) and (3.21), $\bar{L}_{\alpha} = \bar{L}^\alpha$. On the other hand,
\begin{equation}
\bar{r}^*(1)^{-1} = p_{\alpha,L^\alpha}(1)^{-1} q_{\beta,\bar{L}_{\alpha}}(1)^{-1} = q_{\beta,L^\beta}(1)^{-1} p_{\alpha,L^\alpha}(1)^{-1},
\end{equation}
whilst, by (3.19),
\begin{equation}
r^*(1)^{-1} = (K^{-1}\bar{r}^*(1))^{-1} = q_{\beta,L^\beta}(1)^{-1} p_{\alpha,L^\alpha}(1)^{-1} K.
\end{equation}
Hence
\begin{equation}
(3.22) \quad r^*(1)^{-1} = r^*(1)^{-1}K^{-1}.
\end{equation}

By \((3.20)\), it follows that \(V^* = V\). Next we establish the reality of \(q^*\). Yet again by \((3.21)\),
\[
r^*(0) = q_{\alpha^{-1}, \tilde{L}_\alpha}(0) = p_{\alpha^{-1}, \tilde{L}_\alpha}(\infty)
\]
and, on the other hand, by \((3.19)\),
\[
r^*(\infty) = K^{-1}p_{\alpha, \tilde{L}_\alpha}(\infty),
\]
so that
\begin{equation}
(3.23) \quad r^*(0) = Kr^*(\infty).
\end{equation}

Together with \((3.22)\), this makes clear that \((q^*)_1^0, 0 = (q^*)_0^1, 1\), the reality of \(q\) establishes that of \(q^*\). We conclude that:

**Theorem 21.** \([9]\) If \(V\) is a real \(q\)-perturbed harmonic bundle, then the Bäcklund transform \(V^*\) of \(V\), of parameters \(\alpha, L^\alpha\), is a real \(q^*\)-perturbed harmonic bundle.

A real transformation on the level of constrained Willmore surface follows. Equation \((3.19)\) plays, yet again, a crucial role, by showing that
\[
r^*(1) = K^{-1}q_{\alpha^{-1}, \tilde{L}_\alpha}(1)p_{\alpha, \tilde{L}_\alpha}(1) = K^{-1}q_{\beta, \tilde{L}_\alpha}(1)p_{\alpha, \tilde{L}_\alpha}(1) = K^{-1}r^*(1),
\]
and, therefore,
\begin{equation}
(3.24) \quad r^*(1)^{-1} = r^*(1)^{-1}K.
\end{equation}
Suppose \((\Lambda^{1,0}, \Lambda^{0,1})\) is a real surface, so that, in particular, \(\Lambda^{1,0} = \Lambda^{0,1}\). By \((3.23)\) and \((3.24)\), it follows that \((\Lambda^*)^{1,0} = (\Lambda^*)^{0,1}\), establishing the reality of the bundle
\[
\Lambda^* := (\Lambda^*)^{1,0} \cap (\Lambda^*)^{0,1}.
\]

We conclude that:

**Theorem 22.** \([9]\) If \(\Lambda\) is a real \(q\)-constrained Willmore surface, then the Bäcklund transform \(\Lambda^*\) of \(\Lambda\), of parameters \(\alpha, L^\alpha\), is a real \(q^*\)-constrained Willmore surface.

3.4. **Spectral deformation versus Bäcklund transformation.** Bäcklund transformation and spectral deformation permute, as follows:

**Theorem 23.** \([9]\) Let \(\alpha, \beta, L^\alpha, L^\beta\) be Bäcklund transformation parameters to \(V\), \(\lambda \in \C\{0, \pm \alpha, \pm \beta\}\) and
\[
\phi^\lambda : (\C^{n+2}, d_{V^\lambda}) \to (\C^{n+2}, d)
\]
be an isometry of bundles preserving connections. The Bäcklund transform of parameters \(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, \phi^\lambda L^\alpha, \phi^\lambda L^\beta\) of the spectral deformation \(\phi^\lambda V\) of \(V\), of
transformations of harmonic bundles and willmore surfaces

parameter $\lambda$, corresponding to the multiplier $q$, coincides with the spectral deformation of parameter $\lambda$, corresponding to the multiplier $q^*$, of the Bäcklund transform of parameters $\alpha, \beta, L^\alpha, L^\beta$ of $V$. Furthermore, if

$$\phi_\lambda: (\mathbb{C}^{n+2}, d^\lambda_{\gamma^*}) \to (\mathbb{C}^{n+2}, d)$$

is an isometry preserving connections, then the diagram in Figure 2 commutes.

\[
\begin{array}{ccc}
\phi^\lambda_*(\Delta^*)^{1,0} & \phi^\lambda_*(\Delta^*)^{0,1} & \\
\lambda & \alpha, \beta, L^\alpha, L^\beta & \\
\alpha, \beta, L^\alpha, L^\beta & \phi^\lambda_1 & \\
\phi^\lambda_\Delta & \\
\end{array}
\]

Figure 2. A Bianchi permutability of spectral deformation and Bäcklund transformation of constrained Willmore surfaces.

**Proof.** It is trivial, noting that $\phi^\lambda_\gamma^*\phi^\lambda_{1,0}^{-1}\phi^\lambda_{1,1}(1): (\mathbb{C}^{n+2}, d^{\lambda, q^*}_{\gamma^*}) \to (\mathbb{C}^{n+2}, d)$ is an isometry of bundles preserving connections. □

For $\lambda \in \{\pm \alpha, \pm \beta\}$, it is not clear how the spectral deformation of parameter $\lambda$ relates to the Bäcklund transformation of parameters $\alpha, \beta, L^\alpha, L^\beta$.

3.5. Isothermic surfaces under constrained Willmore transformation. The isothermic surface condition is known to be preserved under constrained Willmore spectral deformation:

**Proposition 24.** [8] Constrained Willmore spectral deformation preserves the isothermic surface condition.

Next we derive it in our setting.

**Proof.** Suppose $(\Lambda, \eta)$ is an isothermic $q$-constrained Willmore surface, for some $\eta, q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$. Fix $\lambda \in S^1$ and $\phi_\Lambda^\gamma: (\mathbb{R}^{n+1,1}, d_\gamma^\lambda) \to (\mathbb{R}^{n+1,1}, d)$ an isometry preserving connections. Set

$$\eta_\lambda := \lambda^{-1}\eta^{1,0} + \lambda \eta^{0,1}.$$ 

To prove the theorem, we show that $(\phi^\lambda_\gamma \Lambda, \text{Ad}_{\phi^\lambda_\gamma} \eta_\lambda)$ is isothermic. Obviously, the reality of $\eta$ establishes that of $\text{Ad}_{\phi^\lambda_\gamma} \eta_\lambda$. Recall (3.2) to conclude that

$$((\phi^\lambda_\gamma \Lambda)^{(1)})^{(1)} = \phi^\lambda_\gamma \Lambda^{(1)}$$
and, therefore, that $\text{Ad}_{\phi_q^\lambda} \eta_\lambda$ takes values in $\phi_q^\lambda \Lambda \wedge (\phi_q^\lambda \Lambda)^{(1)}$. According to (2.13), we have
\[ [q^{1.0} \wedge \eta^{0.1}] = 0 = [q^{0.1} \wedge \eta^{1.0}] \]
and, therefore,
\[ d^{q_0^\lambda} \eta_\lambda = d^{D} \eta_\lambda + [(\lambda^{-1} N^{1.0} + \lambda N^{0.1} + (\lambda^{-2} - 1)q^{1.0} + (\lambda^2 - 1)q^{0.1}) \wedge \eta_\lambda] \]
\[ = d^{D} \eta_\lambda + [N \wedge \eta]. \]
According to the decomposition (2.14), we conclude that
\[ d(\text{Ad}_{\phi_q^\lambda} \eta_\lambda) = \phi_q^\lambda \circ d^{q_0^\lambda} \eta_\lambda \circ (\phi_q^\lambda)^{-1} \]
vanishes if and only if $d^{D} \eta_\lambda = 0 = [N \wedge \eta]$. Remark [1] and Lemma [5] complete the proof. □

As for Bäcklund transformation of isothermic constrained Willmore surfaces, we believe it does not necessarily preserve the isothermic condition. This shall be the subject of further work.

A very important subclass of isothermic constrained Willmore surfaces is the class of constant mean curvature surfaces in 3-dimensional space-forms. The constancy of the mean curvature of a surface in 3-dimensional space-form is preserved by both constrained Willmore spectral deformation, cf. [8], and constrained Willmore Bäcklund transformation, cf. [10], for special choices of parameters, with preservation of both the space-form and the mean curvature in the latter case. However, constant mean curvature surfaces are not conformally-invariant objects, requiring that we carry a distinguished space-form. This shall be the subject of a forthcoming paper. See [10] Section 8.2 and [17] for further details.

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