CATEGORICAL TORELLI THEOREM FOR HYPERSURFACES

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Abstract. Let \( X \subset \mathbb{P}^{n+1} \) be a smooth Fano hypersurface of dimension \( n \) and degree \( d \). The derived category of coherent sheaves on \( X \) contains an interesting subcategory called the Kuznetsov component \( A_X \). We show that this subcategory, together with a certain autoequivalence called the rotation functor, determines \( X \) uniquely if \( d > 3 \) or if \( d = 3 \) and \( n > 3 \). This generalizes a result by D. Huybrechts and J. Rennemo, who proved the same statement under the additional assumption that \( d \) divides \( n + 2 \).

1. Introduction

Reconstruction of an algebraic variety from some of its invariants is a classical endeavor that started with Torelli proving that a smooth projective curve is determined by its Jacobian as a polarized abelian variety. In 1997 A. Bondal and D. Orlov proved [BO01] that Fano varieties and varieties with ample canonical class can be reconstructed from their derived categories of coherent sheaves. In this note we are interested in some developments arising from Bondal–Orlov’s theorem.

To discuss our setting, we first need some notation. Let \( k \) be a field of characteristic zero, and let \( V \) be an \((n+2)\)-dimensional vector space. We work with an \( n \)-dimensional smooth Fano hypersurface \( X \subset \mathbb{P}(V) \) of degree \( d \). The bounded derived category of coherent sheaves on \( X \) admits a semiorthogonal decomposition:

\[
D^b_{\text{coh}}(X) = \langle A_X, \mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(n-d+1) \rangle,
\]

where the category \( A_X \) is called a Kuznetsov component or a residual category of \( D^b_{\text{coh}}(X) \). The decomposition (1.0.1) implies that, according to [Orl16, Prop. 3.8], the category \( D^b_{\text{coh}}(X) \), and hence by Bondal–Orlov’s theorem the variety \( X \) itself, can be reconstructed from the following pieces of data:

- The category \( \langle \mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(n-d+1) \rangle \) that depends only on \( n \) and \( d \), not on the choice of the hypersurface \( X \);
- The Kuznetsov component \( A_X \);
- A certain glueing data between the two categories above.

In general, it is impossible to determine \( X \) just from the Kuznetsov component \( A_X \), without the glueing data. An example with cubic fourfolds is described in [Per21]. In the paper [HR19] D. Huybrechts and J. Rennemo suggested a different approach to this problem. They used a particular autoequivalence \( \Phi_{A_X} \) of \( A_X \) called the rotation functor (see Definition 4.3). They proved the following reconstruction theorem:

1.1. Theorem ([HR19, Cor. 1.2]). A smooth Fano hypersurface \( X \subset \mathbb{P}(V) \) of degree \( d \) and dimension \( n \) with \( d | (n+2) \) is determined by the pair \( (A_X, \Phi_{A_X}) \) composed of the Kuznetsov component and the rotation functor (as a dg-category and a dg-endofunctor).
In this paper we generalize their approach to any Fano hypersurface, not necessarily satisfying the divisibility condition on $d$ and $n$:

1.2. Theorem ( = Theorem 5.3). A smooth Fano hypersurface $X \subset \mathbb{P}(V)$ of degree $d$ and dimension $n$ satisfying $d > 3$ or $(d = 3, n > 3)$ is determined by the pair $(\mathcal{A}_X, \Phi_{\mathcal{A}_X})$ composed of the Kuznetsov component and the rotation functor (as a dg-category and a dg-endofunctor).

Remark. The case of cubic threefolds, not included in this theorem, has been studied in [Ber+12]. For this case the category $\mathcal{A}_X$ alone suffices to determine $X$.

The strategy of the proof is similar to the one in [HR19]. The vector space of natural transformations from the identity functor of $\mathcal{A}_X$ to the rotation functor is isomorphic to $V^\vee$ (Lemma 4.4). The $d$'th power of the rotation functor is a shift-by-two functor on $\mathcal{A}_X$ (Theorem 4.6), and thus the $d$-fold composition of the natural transformations defines a morphism

$$S^dV^\vee \to \text{HH}^2(\mathcal{A}_X).$$

We show that the kernel of this morphism is exactly the $d$'th graded component of the Jacobian ideal of $X \subset \mathbb{P}(V)$, which determines the hypersurface $X$ uniquely. A subtle point of the argument is that we do not need an explicit computation of $\text{HH}^2(\mathcal{A}_X)$. Instead, we prove that the map (1.2.1) factors through $\text{HH}^2(X)$ and use an easy observation from Lemma 5.2 that shows that the restriction map $\text{HH}^2(X) \to \text{HH}^2(\mathcal{A}_X)$ is an injection on a large subspace of $\text{HH}^2(X)$.

There are many classes of varieties which admit semiorthogonal decompositions similar to the one in (1.0.1) in the sense that one of the components of the decomposition is the “interesting” one, and the others are very simple. In these situations one could investigate some refined version of the Bondal–Orlov’s theorem. For a review of known results along this direction, see [PS22]. We especially remark the Fano threefold case studied in [Jac+22], due to some similarities with our approach.

While preparing the work I learned of an upcoming paper by J. Rennemo [Ren22], who showed that if $d$ does not divide $n + 2$ and the pair $(d, n)$ is not of the form $(4, 4k)$, then the Kuznetsov component alone suffices to reconstruct $X$, with no dependency on the rotation functor. I believe that Theorem 1.2, though weaker outside of the case $(d = 4, n = 4k)$, is still of interest, in particular due to the uniform handling of all cases.

Structure of the paper. In Section 2 we recall the notion of the Jacobian ring of a hypersurface and its connection with Hochschild cohomology. In Section 3, following [Kuz04], we perform some computations related to the orthogonal to the structure sheaf in the derived category of a hypersurface. We use those results in Section 4 to study the $d$'th power of the rotation functor on the Kuznetsov component. Finally, in Section 5 we prove the main Theorem 1.2.

Notation. Let $K$ be a field of characteristic zero. In this paper all categories are assumed to be triangulated and $K$-linear, all pullbacks and pushforwards are assumed to be derived, and all varieties are assumed to be smooth.

Let $V$ be an $(n+2)$-dimensional vector space over $K$. We work with a smooth $n$-dimensional Fano hypersurface $X \subset \mathbb{P}(V)$ with $\deg(X) = d$, defined by an equation $f \in S^dV^\vee$. In particular, $d < n + 2$. 
We use the notation $\mathbb{P}$ for the projective space $\mathbb{P}(V)$. Since we mostly work with objects in the derived categories $D^b_{\text{coh}}(X)$ and $D^b_{\text{coh}}(X \times X)$, we sometimes omit the subscript $X$ on objects like the structure sheaf of the diagonal $\mathcal{O}_\Delta \in D^b_{\text{coh}}(X \times X)$ to avoid the symbol clutter in formulas when the risk of confusion is small.

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2. Hochschild cohomology and Jacobian ring

In [Don83] Ron Donagi proved a Hodge-theoretic Torelli theorem for a (very general) hypersurface satisfying some conditions on the degree and the dimension. The proof relied on the notion of the Jacobian ring of a hypersurface. We also need this notion for the categorical version of Torelli theorem.

2.1. Definition. The Jacobian ring of a hypersurface $X \subset \mathbb{P}(V)$ defined by an equation $f \in S^dV^\vee$ is the graded ring given by the quotient

$$J^\bullet(f) := S^dV^\vee / \langle \frac{\partial f}{\partial v} \rangle_{v \in V}. $$

The graded ideal generated by the partial derivatives of $f$ is called the Jacobian ideal of $X$.

What most interests us is a relation between the Jacobian ring and the Hochschild cohomology of the hypersurface. One aspect of this relation is demonstrated in the combination of Lemma 2.2 and Proposition 2.3. For general information on Hochschild cohomology see, for example, [Kuz09]. We only recall that Hochschild cohomology of a variety can be computed by the formula

$$\text{HH}^\bullet(X) = \text{Ext}^\bullet_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta),$$

and the Hochschild cohomology of an (admissible) subcategory $\mathcal{A}_X \subset D^b_{\text{coh}}(X)$ can be computed by the formula

$$\text{HH}^\bullet(\mathcal{A}_X) = \text{Ext}^\bullet_{X \times X}(P, P),$$

where the object $P \in D^b_{\text{coh}}(X \times X)$ is a Fourier–Mukai kernel of the (left or right) projection functor to $\mathcal{A}_X$.

2.2. Lemma. Consider the normal bundle short exact sequence:

$$0 \to T_X \to T_{\mathbb{P}|X} \to \mathcal{O}_X(d) \to 0.$$  

The connecting homomorphism in the long exact sequence of cohomology groups induces a morphism

$$S^dV^\vee \simeq H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(d)) \to H^0(X, \mathcal{O}_X(d)) \to H^1(X, T_X).$$

Then this map is surjective and it identifies $H^1(X, T_X)$ with the $d$’th graded component of the Jacobian ring $J^d(f)$.

Proof. The long exact sequence of cohomology groups of the normal short exact sequence contains the following fragment:

$$H^0(T_{\mathbb{P}|X}) \to H^0(\mathcal{O}_X(d)) \to H^1(T_X) \to H^1(T_{\mathbb{P}|X}).$$
Note that the tangent bundle of $\mathbb{P}$ fits into the Euler short exact sequence
\[0 \to \mathcal{O}_\mathbb{P} \to V \otimes \mathcal{O}_\mathbb{P}(1) \to T_\mathbb{P} \to 0.\]
Since $d < n + 2$ it is easy to compute that $H^1(X, T_\mathbb{P}|_X) = 0$ and $H^0(X, T_\mathbb{P}|_X) = V^\vee \otimes \mathcal{O}/(\text{id}_V)$. Thus the sequence (2.2.1) simplifies:
\[V^\vee \otimes \mathcal{O}/(\text{id}_V) \to S^dV^\vee/(f) \to H^1(T_X) \to 0.\]
Here the first map sends a decomposable tensor $\xi \otimes v \in V^\vee \otimes V$ to the element $\xi \cdot \frac{\partial f}{\partial v} \in S^d V^\vee$. The image of this map in $S^d V^\vee$ is thus the $d$’th component of the Jacobian ideal. Since $f$ lies in its own Jacobian ideal, we conclude that $H^1(X, T_X) \cong J_d(f)$. □

Recall the notion of the universal Atiyah class $At \in \text{Ext}^1(\mathcal{O}_\Delta, \Delta_* \Omega^1_X)$ (see, e.g., [KM09]). Abusing the notation, we denote the following composition also by At:
\[H^1(X, T_X) \cong \text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X) \xrightarrow{\Delta_*} \text{Ext}^1_{X \times X}(\Delta_* \Omega^1_X, \mathcal{O}_\Delta) \xrightarrow{-\circ At} \text{Ext}^2_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \text{HH}^2(X).\]
By the Hochschild–Kostant–Rosenberg theorem (e.g., [Swa96, Cor. 2.6]) this morphism is an injection.

Recall the (universal) linkage class $\epsilon_X$ for a hypersurface $X \subset \mathbb{P}(V)$ [KM09, Sec. 3]: the derived restriction of $\mathcal{O}_{\Delta'}$ to $X \times X \subset \mathbb{P} \times \mathbb{P}$ is a complex with two adjacent cohomology sheaves. Thus it fits into a distinguished triangle
\[(2.2.2) \quad \mathcal{O}_{\Delta'}|_{X \times X} \to \mathcal{O}_\Delta \xrightarrow{\epsilon_X} \mathcal{O}_\Delta(-d)[2].\]
The gluing morphism $\epsilon_X \in \text{Ext}^2_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta(-d))$ between the cohomology sheaves is called the universal linkage class.

2.3. Proposition. The composition with the universal linkage class defines a morphism:
\[S^d V^\vee \to H^0(X, \mathcal{O}_X(d)) \xrightarrow{\Delta_*} \text{Hom}_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta(d)) \xrightarrow{\epsilon_X \circ} \text{Ext}^2_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \text{HH}^2(X),\]
such that the following triangle commutes:
\[(2.3.1) \quad S^d V^\vee \xrightarrow{At} H^1(X, T_X) \xrightarrow{\text{HH}^2(X)} \]
where the horizontal arrow is the map from Lemma 2.2.

Proof. Let $\nu: \Omega^1 \to \mathcal{O}(-d)[1]$ be the extension class of the conormal exact sequence
\[0 \to \mathcal{O}(-d) \to \mathcal{O}^1_{\mathbb{P}}|_X \to \Omega^1_X \to 0.\]
By [KM09, Th. 3.2] the universal linkage class is equal to the composition
\[\mathcal{O}_\Delta \xrightarrow{At} \Delta_* \Omega^1[1] \xrightarrow{\Delta_* \nu} \mathcal{O}_\Delta(-d)[2]\]
of the universal Atiyah class and the pushforward of $\nu$ along the diagonal. Unwinding the definitions, we see that the diagram (2.3.1) commutes on an element $g \in S^d V^\vee$ if and only if
the diagram below commutes:

\[
\begin{array}{ccc}
\mathcal{O}\Delta & \xrightarrow{At} & \Delta_\ast \Omega^1[1] & \xrightarrow{\Delta_\ast \nu} & \mathcal{O}\Delta(\neg d)[2] \\
\downarrow g & & \downarrow g & & \\
\mathcal{O}\Delta(d) & \xrightarrow{At(d)} & \Delta_\ast \Omega^1(d)[1] & \xrightarrow{\Delta_\ast \nu(d)} & \mathcal{O}[2]
\end{array}
\]

This commutativity is clear since multiplication by \( g \) is a natural transformation. \( \square \)

3. The orthogonal to the structure sheaf

Since \( X \subset \mathbb{P}(V) \) is by assumption a Fano hypersurface, the structure sheaf \( \mathcal{O}_X \in D^b_{\text{coh}}(X) \) is an exceptional object. In this section we study some properties of the right-orthogonal subcategory \( \mathcal{O}_X^\perp \subset D^b_{\text{coh}}(X) \). The goal is to perform some explicit computations involving the projection functor to \( \mathcal{O}_X^\perp \). We will rely on them in Section 4 where we study the Kuznetsov component \( \mathcal{A}_X \subset D^b_{\text{coh}}(X) \). Nothing in this section is new, all results are already in [Kuz04].

We begin with a brief reminder on Fourier–Mukai transforms to fix the notation. For details, see [Huy06]. Given an object \( K \in D^b_{\text{coh}}(X \times X) \) we define the Fourier–Mukai functor \( \Phi_K : D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(X) \) by the formula \( \pi_2_*(\pi_1^*(\neg) \otimes K) \). For any \( d \in \mathbb{Z} \) the Fourier–Mukai transform \( \Phi_{\Delta(d)} \) is the functor given by the twist by \( \mathcal{O}(X) \). For any pair of objects \( F, G \in D^b_{\text{coh}}(X) \) the Fourier–Mukai transform along the exterior product \( F \boxtimes G \) is the functor

\[ \Phi_{F \boxtimes G}(\neg) := \text{R} \Gamma (\neg \otimes F) \otimes G. \]

The convolution of two kernels \( K_1, K_2 \in D^b_{\text{coh}}(X \times X) \) is defined by the formula

\[ K_1 \circ K_2 := \pi_{13_\ast}(\pi_{12}^*(K_1) \otimes \pi_{23}^*(K_2)), \]

and it satisfies \( \Phi_{K_1 \circ K_2} = \Phi_{K_1} \circ \Phi_{K_2} \). For line bundles of the form \( \mathcal{O}_X(a, b) := \mathcal{O}_X(a) \boxtimes \mathcal{O}_X(b) \) the convolution equals

\[ \mathcal{O}_X(a, b) \circ \mathcal{O}_X(a', b') \cong \text{R} \Gamma (\mathcal{O}_X(b + a')) \otimes \mathcal{O}_X(a, b'). \]

Now we return to the main object of this section.

3.1. Definition. We define the object \( Q_0 \in D^b_{\text{coh}}(X \times X) \) as the complex

\[ Q_0 := [\mathcal{O}_X \boxtimes \mathcal{O}_X \to \mathcal{O}_\Delta], \]

in degrees \(-1 \) and \( 0 \), representing the left projection functor to the subcategory \( \mathcal{O}_X^\perp \subset D^b_{\text{coh}}(X) \). We define the objects \( Q_i \), for \( i > 0 \) recursively:

\[ Q_i := Q_{i-1} \circ \mathcal{O}_\Delta(1) \circ Q_0, \]

where the symbol \( \circ \) denotes the convolution of Fourier–Mukai kernels in \( D^b_{\text{coh}}(X \times X) \).

Remark. The functor \( D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(X) \) represented by the object \( Q_1 \in D^b_{\text{coh}}(X \times X) \) is called a rotation functor in [Kuz04]. It can alternatively be described as a composition

\[ D^b_{\text{coh}}(X) \xrightarrow{Q_0} D^b_{\text{coh}}(X) \xrightarrow{- \otimes \mathcal{O}_X(1)} D^b_{\text{coh}}(X) \xrightarrow{Q_0} D^b_{\text{coh}}(X), \]

where the Fourier–Mukai transform along the object \( Q_0 \) is the left projection to the subcategory \( \mathcal{O}_X^\perp \).
3.2. Lemma. For any \( i \geq 0 \) there is a natural morphism \( \mathcal{O}_\Delta(i) \to Q_i \) which induces a map

\[ m_{Q_i} : S^iV^\vee \to \text{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta(i)) \to \text{Hom}(\mathcal{O}_\Delta, Q_i) \to \text{Hom}(Q_0, Q_i). \]

Furthermore, for \( i = 1 \) the map \( m_{Q_1} \) is an isomorphism of vector spaces.

Proof. By the definition of the left projection functor there is a morphism

\[ \mathcal{O}_\Delta \to Q_0. \]

We define the map \( \mathcal{O}_\Delta(i) \to Q_i \) by repeatedly using the map \( \mathcal{O}_\Delta \to Q_0 \) as follows:

\[ \xymatrix{ \mathcal{O}_\Delta(i) \ar[r] \ar[d] & \mathcal{O}_\Delta \circ \mathcal{O}_\Delta(1) \circ \mathcal{O}_\Delta \circ \cdots \circ \mathcal{O}_\Delta(1) \circ \mathcal{O}_\Delta \ar[d] \\ Q_i \ar@{=}[r] & Q_0 \circ \mathcal{O}_\Delta(1) \circ Q_0 \circ \cdots \circ \mathcal{O}_\Delta(1) \circ Q_0 } \]

The fact that \( m_{Q_i} \) is an isomorphism can be checked by a straightforward computation using the resolution

\[ Q_1 \cong [V^\vee \otimes \mathcal{O}_{X \times X} \to \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1) \to \mathcal{O}_\Delta(1)] \]

obtained by using Definition 3.1 and the formula (3.0.2).

3.3. Definition. We denote by \( B_\Delta \) the Beilinson’s resolution of the diagonal on \( \mathbb{P}(V) \times \mathbb{P}(V) \):

\[ B_\Delta := [\mathcal{O}_{\mathbb{P}(V)}(-n - 1) \otimes \Omega^{n+1}_{\mathbb{P}(V)}(n + 1) \to \cdots \to \mathcal{O}_{\mathbb{P}(V)}(-1) \otimes \Omega^1_{\mathbb{P}(V)}(1) \to \mathcal{O}_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathbb{P}(V)}]. \]

For any \( 0 \leq i \leq n + 1 \) we denote by \( s_{\geq -i}(B_\Delta) \) the stupid truncation of this resolution:

\[ s_{\geq -i}(B_\Delta) := [\mathcal{O}_{\mathbb{P}(V)}(-i) \otimes \Omega^i_{\mathbb{P}(V)}(i) \to \cdots \to \mathcal{O}_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathbb{P}(V)}]. \]

3.4. Theorem ([Kuz04]). For \( 0 \leq i < d \) the morphism \( \mathcal{O}_\Delta(i) \to Q_i \) from Lemma 3.2 fits into an exact triangle

\[ s_{\geq -i}(B_\Delta)|_{X \times X} \otimes (\mathcal{O}_X(i) \otimes \mathcal{O}_X) \xrightarrow{\psi_i} \mathcal{O}_\Delta(i) \to Q_i, \]

and the map \( \psi_i \) is a twist by \( \mathcal{O}_X(i) \otimes \mathcal{O}_X \) of the composition

\[ s_{\geq -i}(B_\Delta)|_{X \times X} \to B_\Delta|_{X \times X} \xrightarrow{\psi} (\mathcal{O}_{\Delta_{\mathbb{P}(V)}})|_{X \times X} \to \mathcal{O}_\Delta. \]

Proof. The statement is implicitly contained in the proof of [Kuz04, Lem. 4.2], and can be proved by an inductive computation. For the sake of clarity, we sketch the argument. The base case \( i = 0 \) is true by definition of \( Q_0 \). Suppose the statement holds for \( Q_i \) with \( i < d - 1 \) and we want to prove it for the object

\[ Q_{i+1} \cong Q_i \circ \mathcal{O}_\Delta(1) \circ [\mathcal{O}_X(0, 0) \to \mathcal{O}_\Delta] \cong \text{Cone}(Q_i \circ \mathcal{O}_X(0, 1) \to Q_i \circ \mathcal{O}_\Delta(1)). \]

This description as a cone, together with the formula for \( Q_i \) that we know by induction, shows that \( Q_{i+1} \) can be represented by the following complex:

\[
\begin{array}{c}
\mathcal{O}_X \otimes (H^0(\mathcal{O}_X(1) \otimes \Omega^i_P(i)|_X) \to \cdots \to \mathcal{O}_X \otimes (H^0(\mathcal{O}_X(i+1) \otimes \mathcal{O}_X) \to \mathcal{O}_X \otimes \mathcal{O}_X(i+1) \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{O}_X(1) \otimes \Omega^i_P(i)|_X \to \cdots \to \mathcal{O}_X(i+1) \otimes \mathcal{O}_X \to \mathcal{O}_\Delta(i+1)
\end{array}
\]
By induction hypothesis the differentials in the bottom row are given by the contraction with the restriction of the tautological section of \( O_{\mathbb{P}(V)}(-1) \otimes T_{\mathbb{P}(V)}(1) \), and the differentials in the top row are induced from that section. Note that \( H^0(\mathcal{O}_X(j)) \cong S^j V^\vee \) for any \( j < d \). Recall that on the projective space we have a resolution for \( \Omega^i + 1 \mathbb{P}(i + 1) \) given by a Koszul complex:
\[
0 \to \Omega^{i+1}_p(i + 1) \to V^\vee \otimes \Omega^i_p(i) \to \ldots \to \mathcal{O} \otimes S^{i+1} V^\vee \to \mathcal{O}_p(i + 1) \to 0.
\]
Thus, if \( i + 1 < d \), we recognize the upper row in the diagram (3.4.1) to be a complex quasiisomorphic to a single vector bundle \( \mathcal{O}_X \otimes \Omega^{i+1}_p(i + 1) \), put to the leftmost degree. This finishes the inductive argument.

\[\square\]

3.5. Theorem ([Kuz04]). There exists a morphism \( \varphi_{Q_d} : Q_d \to Q_0[2] \) such that the following diagram commutes:
\[
\begin{array}{cccc}
S^d V^\vee & \xrightarrow{m_{Q_d}} & \text{Hom}(Q_0, Q_d) \\
\downarrow & & \downarrow \\
\text{HH}^2(X) & \xrightarrow{\varphi_{Q_d} \circ -} & \text{Ext}^2(Q_0, Q_0)
\end{array}
\]

Here the left vertical map is from Proposition 2.3, the top horizontal map is from Lemma 3.2, and the bottom horizontal arrow comes from the identification of \( \text{Ext}^2(Q_0, Q_0) \) with \( \text{HH}^2(\mathcal{O}_X^\perp) \).

\[\text{Proof.}\] This is, again, implicitly contained in the proof of [Kuz04, Lem. 4.2]. To explain the commutativity of the diagram, we repeat the argument.

Denote by \( \widetilde{Q}_d \) the cone
\[
\text{(3.5.2)} \quad \widetilde{Q}_d := \text{Cone}(s_{\geq -d}(B_\Delta|_{X \times X}) \otimes \mathcal{O}(d, 0) \to \mathcal{O}_\Delta(d))
\]
as in Theorem 3.4. The diagram (3.4.1) for \( i = d - 1 \) shows that the difference between \( \widetilde{Q}_d \) and \( Q_d \) arises from the fact that \( H^0(X, \mathcal{O}_X(d)) \) is isomorphic not to \( S^d V^\vee \), but to the quotient \( S^d V^\vee / \langle f \rangle \), where \( f \) is the equation of the hypersurface \( X \subset \mathbb{P}(V) \). More precisely, there is a distinguished triangle
\[
\langle f \rangle \cdot \mathcal{O}_X \otimes \mathcal{O}_X[2] \to \widetilde{Q}_d \to Q_d.
\]

Note that the convolution \( \mathcal{O}_X \otimes \mathcal{O}_X \circ Q_0 \) is a zero object since the Fourier–Mukai transform along \( \mathcal{O}_X \otimes \mathcal{O}_X \) vanishes on \( \mathcal{O}_X^\perp \), and \( Q_0 \) is exactly the projector to \( \mathcal{O}_X^\perp \). Hence the convolution on the right with \( Q_0 \) produces an isomorphism
\[
\widetilde{Q}_d \circ Q_0 \to Q_d \circ Q_0 = Q_d.
\]

Consider now the commutative diagram of distinguished triangles arising from the stupid truncation of \( B_\Delta|_{X \times X} \cong \mathcal{O}_{\Delta_0}|_{X \times X} \):
\[
\text{(3.5.3)} \quad \begin{array}{cccc}
s_{\geq -d}(B_\Delta|_{X \times X}) \otimes \mathcal{O}(d, 0) & \xrightarrow{=} & \mathcal{O}_\Delta(d) & \xrightarrow{e_X} & \mathcal{O}_\Delta[2] \\
\downarrow & & \downarrow & & \\
B_\Delta|_{X \times X} \otimes \mathcal{O}(d, 0) & \to & \mathcal{O}_\Delta(d)
\end{array}
\]
Since $B$ is a resolution of the structure sheaf of the diagonal $\mathcal{O}_\Delta \in D^{b}_{\text{coh}}(\mathbb{P} \times \mathbb{P})$, the bottom horizontal triangle is the universal linkage class as defined in (2.2.2).

Using the rightmost vertical map, we define the morphism $Q_d \to Q_0[2]$ as the composition:

$$Q_d \simeq \tilde{Q}_d \circ Q_0 \to \mathcal{O}_\Delta[2] \circ Q_0 \simeq Q_0[2].$$

It only remains to show the commutativity of the diagram (3.5.1). Recall the natural morphism $\mathcal{O}_\Delta \to \tilde{Q}_d$ defined in Lemma 3.2. It is easy to see from the definition (3.5.2) that this morphism lifts to the map $\mathcal{O}_\Delta(d) \to \tilde{Q}_d$, and thus the map $m_{Q_d}$ factors through the map $S^dV^\vee \to \text{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta(d)) \to \text{Hom}(\mathcal{O}_\Delta, \tilde{Q}_d)$.

Since the rightmost square in the diagram (3.5.3) commutes, we additionally see that the map $m_{Q_d}$ factors through the composition with the universal linkage class:

$$S^dV^\vee \to \text{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta(d)) \xrightarrow{\ell_X o} \text{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta[2]),$$

and the commutativity of the diagram (3.5.1) follows from Proposition 2.3.

4. Kuznetsov components

We begin by discussing the basic properties of Kuznetsov components and their rotation functors.

4.1. Definition. The Kuznetsov component of the hypersurface $X \subset \mathbb{P}(V)$ of degree $d < n + 2$ is the category $\mathcal{A}_X$ defined as the left orthogonal to the exceptional sequence

$$\langle \mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(n - d + 1) \rangle$$

in the category $D^{b}_{\text{coh}}(X)$.

4.2. Definition. We define the object $P_0 \in D^{b}_{\text{coh}}(X \times X)$ to be the Fourier–Mukai kernel of the left projector to the subcategory $\mathcal{A}_X \subset D^{b}_{\text{coh}}(X)$. Recall the fundamental triangle of projector objects:

(4.2.1) $P'_0 \to \mathcal{O}_\Delta \to P_0$.

where $P'_0$ is the right projector to the subcategory $\langle \mathcal{O}_X, \ldots, \mathcal{O}_X(n - d + 1) \rangle$.

We define the objects $P_i$ for $i > 0$ recursively:

$$P_i := P_{i-1} \circ \mathcal{O}_\Delta(1) \circ P_0,$$

where the symbol $\circ$ denotes the convolution of Fourier–Mukai kernels in $D^{b}_{\text{coh}}(X \times X)$.

4.3. Definition. The rotation functor $\Phi_{\mathcal{A}_X}: \mathcal{A}_X \to \mathcal{A}_X$ of $\mathcal{A}_X$ is defined as the composition

$$\mathcal{A}_X \leftrightarrow D^{b}_{\text{coh}}(X) \xrightarrow{-\otimes \mathcal{O}_X(1)} D^{b}_{\text{coh}}(X) \to \mathcal{A}_X.$$

It can alternatively be described as the Fourier–Mukai transform along the kernel $P_1$.

The following result by Huybrechts and Rennemo computes the space of natural transformations from the identity functor on $\mathcal{A}_X$ to the rotation functor.
4.4. Lemma ([HR19, Lem. 3.1]). For any \( i \geq 0 \) there is a natural morphism \( \mathcal{O}_\Delta(i) \to P_i \), which induces a map

\[
m_{P_i} : S^i V^\vee \to \text{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta(i)) \to \text{Hom}(\mathcal{O}_\Delta, P_i) \to \text{Hom}(P_0, P_i).
\]

Furthermore, if \( d > 3 \) or if \( d = 3 \) and \( n > 3 \), the map \( m_{P_i} : V^\vee \to \text{Hom}(P_0, P_1) \) is an isomorphism of vector spaces.

For the proof of the main Theorem 1.2 we need some information about natural transformations from the identity functor of \( \mathcal{A}_X \) to the \( d \)'th power of the rotation functor. The following two results are sufficient for our purposes.

4.5. Lemma. There exists a natural morphism \( P_0 \to Q_0 \), which induces a map \( P_i \to Q_i \) for any \( i \geq 0 \). The precomposition with \( P_0 \) transforms this map into an isomorphism:

\[
P_i \approx P_i \circ P_0 \sim \to Q_i \circ P_0.
\]

Remark. Since \( P_0 \) is the projector to the subcategory \( \mathcal{A}_X \), the last claim of Lemma 4.5 essentially means that the Fourier–Mukai transforms \( D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(X) \) along the two kernels \( P_i \) and \( Q_i \) agree on the subcategory \( \mathcal{A}_X \subset D^b_{\text{coh}}(X) \).

Proof. Note that \( P_0 \) and \( Q_0 \) are projector functors to subcategories \( \mathcal{A}_X \) and \( \mathcal{O}_X^1 \), respectively. Since \( \mathcal{A}_X \subset \mathcal{O}_X^1 \), the statement is true for \( i = 0 \). Since \( P_i \) and \( Q_i \) are defined inductively in terms of \( P_0 \) and \( Q_0 \), it is enough to show that the convolution with the map \( P_0 \to Q_0 \) induces an isomorphism

\[
P_0 \circ \mathcal{O}_\Delta(1) \circ P_0 \sim \to Q_0 \circ \mathcal{O}_\Delta(1) \circ P_0.
\]

Since \( Q_0 \) and \( P_0 \) are projectors to the subcategories \( \mathcal{O}_X \) and \( \mathcal{A}_X := \langle \mathcal{O}_X, \ldots, \mathcal{O}_X(n-d+1) \rangle^\perp \), respectively, it is enough to show that the image of the Fourier–Mukai transform along the object \( \mathcal{O}_\Delta(1) \circ P_0 \) lies in the orthogonal to the exceptional sequence \( \langle \mathcal{O}_X(1), \ldots, \mathcal{O}_X(n-d+1) \rangle \). This is clear since \( P_0 \) is the projector to \( \mathcal{A}_X \). \( \square \)

4.6. Theorem ([Kuz04]). There is an isomorphism \( \varphi_{P_d} : P_d \simeq P_0[2] \) such that the following diagram commutes:

\[
\begin{array}{ccc}
S^d V^\vee & \xrightarrow{m_{P_d}} & \text{Hom}(P_0, P_d) \\
\downarrow & & \downarrow \varphi_{P_d} \circ - \\
\text{HH}^2(X) & \longrightarrow & \text{Ext}^2(P_0, P_0)
\end{array}
\]

Here the left vertical map is from Proposition 2.3, and the bottom horizontal arrow comes from the identification of \( \text{Ext}^2(P_0, P_0) \) with \( \text{HH}^2(\mathcal{A}_X) \).

Proof. This is proved in [Kuz04, Lem. 4.2]. We repeat the argument for the convenience of the reader. By Lemma 4.5 the convolution \( Q_d \circ P_0 \) is naturally isomorphic to \( P_d \). Recall the morphism \( \varphi_{Q_d} : Q_d \to Q_0[2] \) from Theorem 3.5. Let \( \varphi_{P_d} : P_d \to P_0[2] \) be the convolution \( \varphi_{Q_d} \circ P_0 \). Then by Theorem 3.5 the following diagram commutes.
The composition of the maps in the upper row is equal to $m_{P_d}$ by definition. Thus the commutativity of the diagram (4.6.1) is proved. It remains only to show that the map $\varphi_{P_d}$ is an isomorphism. To do this, consider the convolution of the diagram (3.5.3) used in the proof of Theorem 3.5 with the object $P_0$. Note that $Q_d \circ P_0 \simeq Q_d \circ P_0 \simeq P_d$. Thus the rightmost vertical map is exactly the morphism $\varphi_{P_d}: P_d \to P_0[2]$, and its cone is isomorphic to the object

\[(s_{\leq-d-1}(B_\Delta|_{X \times X}) \otimes \mathcal{O}(d,0)) \circ P_0[1].\]

To show that this cone is zero, by Definition 3.3 it is enough to check that the convolution

\[(\mathcal{O}_X(-k + d) \boxtimes \Omega^k_p(k)|_X) \circ P_0\]

vanishes for any $k$ satisfying $n + 1 \geq k \geq d + 1$. Since $P_0$ is the projector to the Kuznetsov component $A_X$, any object in the image of $P_0$ is right-orthogonal to $\mathcal{O}_X(-k + d)$ for $k \in [d+1; n+1]$, and hence the object (4.6.2) is zero. □

5. Rotation functors and Hochschild cohomology

Recall that we work with a Fano hypersurface $X \subset \mathbb{P}(V)$ of dimension $n$ and degree $d$. In particular, $d < n + 2$.

5.1. Lemma. If $d > 3$ or if $d = 3$ and $n \geq 3$, then the Hodge diamond of $X$ is not diagonal.

Proof. By Griffiths' theorem [Gri69, Th. 8.3] for any $0 \leq p \leq n$ there exists an isomorphism

\[H_{\text{prim}}^{p,n-p} \simeq J^v(f)\]

between the primitive part of the cohomology of $X$ and a particular graded component of the Jacobian ring, where $t_p = (n-p+1)d - (n+2)$. Since $X$ is smooth, the Jacobian ring is a finite-dimensional graded ring such that any graded component in degrees between 0 and $(d-2)(n+2)$ is nonzero (see, e.g., [Don83]). Thus it is enough to find some $p \neq n/2$ such that $t_p$ lies between 0 and $(d-2)(n+2)$.

If $n = 2m+1$ is an odd number, the condition $d \geq 3$ implies that we can take $p = m$, so we get that $H^{m,m+1}(X) \neq 0$. If $n = 2m$ is an even number, the condition $d \geq 3, n \geq 4$ implies that we can take $p = m - 1$, i.e., $H^{m-1,m+1}(X) \neq 0$. □

5.2. Lemma. Let $X \subset \mathbb{P}(V)$ be a hypersurface. Let $A_X \subset D_{\text{coh}}^b(X)$ be an admissible subcategory. Assume that the orthogonal subcategory $A^X_X \subset D_{\text{coh}}^b(X)$ has a full exceptional collection and the Hodge diamond of $X$ is not diagonal. Then the composition

\[H^1(X, T_X) \hookrightarrow \HH^2(X) \to \HH^2(A_X)\]

is an injection.

Remark. A similar idea in a different situation has been recently used in [Jac+22, Thm. 1.2].
Proof. Since $A_X$ is generated by an exceptional collection, by the additivity of Hochschild homology [Kuz09, Cor. 7.5] we have

\[ \text{HH}_i(X) \cong \text{HH}_i(A_X) \oplus \text{HH}_i(\text{pt})^\oplus k, \]

where $k$ is the length of the exceptional collection in $A_X$. In particular, for any $i \neq 0$ we have an equality $\text{HH}_i(X) = \text{HH}_i(A_X)$. There is an action of Hochschild cohomology on Hochschild homology, and this action is compatible with the decomposition (5.2.1) by construction (see, e.g., [AT14, Prop. 6.1] for the case where $A_X$ is isomorphic to $D^b_{\text{coh}}(S)$ for some smooth projective variety $S$; the proof works in general).

Let $\xi \in H^1(T_X)$ be a nonzero element. We want to show that its image in $\text{HH}^2(A_X)$ is nonzero. It is enough to show that the class of $\xi$ in $\text{HH}^2(X)$ acts on $\text{HH}_i(A_X)$ nontrivially. By (5.2.1) it is enough to show that $\xi$ acts nontrivially on the non-zero degree part of the Hochschild homology of $X$, i.e., to find a complementary class $\tilde{\xi} \in H^1(T_X)\otimes^{N-1}$ and some integer $a$ so that the action map

\[ \text{HH}_a(X) \xrightarrow{\xi \cdot -} \text{HH}_{a+2N}(X) \]

is nontrivial, $a \neq 0$ and $a + 2N \neq 0$.

Using the (Todd-twisted) Hochschild–Kostant–Rosenberg isomorphism [CRV12, Th. 1.4] the action of Hochschild cohomology on Hochschild homology can be reinterpreted in terms of the Hodge structure. Namely, under the isomorphisms

\[ \text{HH}_a(X) \cong \bigoplus_{i \geq 0} H^i(\Omega_X^{n+i}), \hspace{1cm} \text{HH}^2(X) \cong \bigoplus_{i \geq 0} H^i(\Lambda^{2-i}(T_X)) \]

the Hochschild-homological action of $H^1(T_X) \subset \text{HH}^2(X)$ on $\text{HH}_i(X)$ differs from the one induced from the contraction morphism $T_X \otimes \Omega_X \to O_X$ only by the multiplication with the Todd class. Since in (5.2.2) we have $a \neq 0$ and $a + 2N \neq 0$, we are only interested in what happens in the middle cohomology of the hypersurface, $H^n(X)$, and thus the twist by the Todd class does not matter for our purposes since it only changes the result by corrections in other cohomological degrees. Hence it is enough to find two integers, $p < q$, none of which is equal to $n/2$, and a complementary class $\tilde{\xi} \in H^1(T_X)\otimes^{q-p-1}$ such that the multiplication map

\[ H^{q,n-q}(X) \xrightarrow{\cdot \tilde{\xi}} H^{q-1,n-q+1}(X) \xrightarrow{\cdot \xi} H^{p,n-p}(X) \]

is nonzero.

By Lemma 5.1 the Hodge diamond of $X$ is not diagonal. By symmetry there are at least two integers $p < q$, none of which are equal to $n/2$, such that $H^{p,n-p}(X)$ and $H^{q,n-q}(X)$ are both nonzero. A refinement of Griffiths’ theorem (see, e.g., [Don83, Th. 2.2]) shows that not only those cohomology groups are isomorphic to the components of the Jacobian ring, but also the action of $H^1(X,T_X) \cong J^d(f)$ is given by the multiplication in the ring. The multiplication in the Jacobian ring is nondegenerate (see, e.g., [Don83, Th. 2.6]), and hence it is possible to choose a complementary class $\tilde{\xi}$ such that the multiplication (5.2.3) is nonzero. Thus the injectivity is proved.

We are now ready to prove the main theorem of this paper.
5.3. Theorem (= Theorem 1.2). Let \( X \subset \mathbb{P}(V) \) be a smooth \( n \)-dimensional hypersurface of degree \( d < n + 2 \) given by the equation \( f \in S^{d}V^{\vee} \). Assume that \( d \geq 4 \), or \( d \geq 3 \) and \( n > 3 \). Let \( A_{X} \subset D_{\text{coh}}^{b}(X) \) be the Kuznetsov component of \( X \) (Definition 4.1), and let \( \Phi_{A_{X}} \) be the rotation functor (Definition 4.3). Then the pair \((A_{X}, \Phi_{A_{X}})\), as a dg-category with a dg-endofunctor, determines \( X \) up to an isomorphism.

Proof. For any \( k \geq 0 \) the vector space of dg-natural transformations \( \text{id}_{A_{X}} \Rightarrow \Phi^{\circ k} \), i.e., the homotopy classes of maps in the dg-category of functors from \( A_{X} \) to itself, is naturally isomorphic to \( \text{Hom}_{X \times X}(P_{0}, P_{k}) \) [To¨e07]. By Lemma 4.4 the vector space \( \text{Hom}(P_{0}, P_{1}) \) is isomorphic to \( V^{\vee} \). The composition of the maps defines for any \( k \geq 0 \) a morphism

\[
(V^{\vee})^{\otimes k} \to \text{Hom}(P_{0}, P_{k}),
\]

and it factors through the map \( m_{P_{k}} : S^{k}V^{\vee} \to \text{Hom}(P_{0}, P_{k}) \) defined in Lemma 4.4 by construction. Thus for \( k = d \) we get a commutative diagram:

\[
\begin{array}{ccc}
S^{d}V^{\vee} & \longrightarrow & \text{dgNat}(\text{id}_{A_{X}}, \Phi_{A_{X}}^{d}) \\
& \searrow^{m_{P_{d}}} \swarrow_{\cong} & \\
\text{Hom}_{X \times X}(P_{0}, P_{d})
\end{array}
\]

By Theorem 4.6 the vector space \( \text{Hom}(P_{0}, P_{d}) \) is isomorphic to \( \text{Ext}^{2}(P_{0}, P_{0}) \), and the kernel of the diagonal arrow is equal to the kernel of the composition

\[
S^{d}V^{\vee} \to H^{1}(X, T_{X}) \hookrightarrow \text{HH}^{2}(X) \to \text{HH}^{2}(A_{X})
\]

By Lemma 5.2 the composition of the last two arrow is injective. By Proposition 2.3 the kernel of the first morphism is equal to the \( d \)'th component of the Jacobian ideal of \( f \).

Thus, up to an automorphism of \( V \), we reconstructed the \( d \)'th component of the Jacobian ideal of \( f \) as a subspace in \( S^{d}V^{\vee} \) from the pair \((A_{X}, \Phi)\). This subspace, in turn, recovers the hypersurface \( X \) up to an automorphism of \( V \) by Mather–Yau theorem [Don83, Prop. 1.1]. □

5.4. Corollary. Let \( X \subset \mathbb{P}(V) \) be an \( n \)-dimensional hypersurface of degree \( n + 1 \). Assume that \( n \geq 3 \). Then the Kuznetsov component \( A_{X} \subset D_{\text{coh}}^{b}(X) \), considered as a dg-category, determines \( X \) up to an isomorphism.

Proof. If \( d = n + 1 \), the endofunctor \( \Phi \) defined in the statement of Theorem 5.3 is, up to a shift, the inverse Serre functor of \( A_{X} \) [Kuz04, Lem. 4.1]. Thus it can be canonically recovered as a dg-endofunctor from the dg-structure on \( A_{X} \), and the result follows from Theorem 5.3. □

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