VOLUME GROWTH AND CURVATURE DECAY OF COMPLETE
POSITIVELY CURVED KÄHLER MANIFOLDS

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ABSTRACT. This paper constructs a class of complete Kähler metrics of positive holomorphic sectional curvature on $\mathbb{C}^n$ and finds that the constructed metrics satisfy the following properties:

As the geodesic distance $\rho \to \infty$, the volume of geodesic balls grows like $O(\rho^{\frac{2(\beta+1)}{\beta+2}})$ and the Riemannian scalar curvature decays like $O(\rho^{-\beta})$, where $\beta \geq 0$.

1. INTRODUCTION

We are concerned with the volume growth and curvature decay of complex $n$-dimensional ($n \geq 2$) complete noncompact Kähler manifolds (denoted by $M$) with positive holomorphic sectional curvature. For convenience, throughout this paper, $\rho = \rho(x_0, x)$ represents the geodesic distance from a fixed point $x_0$ to $x$ in $M$, $V(B(x_0, \rho))$ denotes the volume of the geodesic ball $B(x_0, \rho)$ centered at $x_0$ with radius $\rho$ and $R(x)$ the Riemannian scalar curvature at $x$. When $M$ is assumed to be of positive holomorphic bisectional curvature, it is known by the classical Bishop volume comparison theorem that $V(B(x_0, \rho)) \leq \omega(n)\rho^{2n}$ for any $x_0 \in M$, where $\omega(n)$ is the volume of the standard unit ball $B^{2n}$ in $\mathbb{R}^{2n}$. Also, if $V(B(x_0, \rho)) \geq C\rho^{2n}$ for some positive constant $C$, then the scalar curvature decays quadratically in the average sense. This is conjectured by Yau [8] and confirmed by Chen and Zhu [3]. On the other hand, it has been shown by Chen and Zhu [3] that

$$V(B(x_0, \rho)) \geq C\rho^n$$

for any $x_0 \in M$, where $C$ is a positive constant depending only on $x_0$ and $M$, and that the scalar curvature decays at least linearly in the average sense. In view of the above results, a complex $n$-dimensional Kähler manifold with positive holomorphic bisectional curvature is said to be of maximal volume growth if $V(B(x_0, \rho)) = O(\rho^{2n})$ and of minimal volume growth if $V(B(x_0, \rho)) = O(\rho^n)$.

There have been some examples of positively curved complete noncompact Kähler manifold with maximal (respectively, minimal) volume growth and quadratic (respectively, linear) curvature decay. In [3], Klembeck constructed a complete, rotationally symmetric Kähler metric $g$ of positive holomorphic sectional curvature on $\mathbb{C}^n$ with minimal volume growth and linear curvature decay. More precisely, the volume of the geodesic ball $B(0, \rho)$ with respect to the metric $g$ grows like $O(\rho^n)$ and the scalar curvature $R(x)$ of the metric $g$ decays like $O(\rho^{-1})$. Rotationally symmetric, complete gradient Kähler-Ricci solitons of positive Riemannian sectional curvature on $\mathbb{C}^n$ have been found by Cao ([1], [2]) and it has been known that such solitons can be divided into two branches: one is of maximal volume growth and quadratic curvature decay, and the other of minimal volume growth and linear curvature decay.

By the above analysis, it is natural to ask whether there exists a complete noncompact Kähler manifold of positive holomorphic sectional curvature satisfying

$$V(B(x_0, \rho)) = O(\rho^{n(1+\epsilon)})$$

and

$$R(x) = O(\rho^{-(1+\epsilon)})$$

for any fixed $\epsilon \in (0, 1)$.

To our best knowledge, examples of such manifolds seem to be elusive from the literature. In
this paper, we shall construct a class of complete Kähler metrics \( g \) of positive holomorphic sectional curvature on \( \mathbb{C}^n \) and find that the constructed metrics satisfy the following properties:

As the geodesic distance \( \rho \to \infty \), the volume of geodesic balls grows like \( O(\rho^{\frac{2(\beta+1)}{\beta+2}}) \) and the Riemannian scalar curvature decays like \( O(\rho^{-\frac{2(\beta+1)}{\beta+2}}) \), where \( \beta \geq 0 \).

Finally, we should mention the work of Ni et al. [7] for some results on the relation between volume growth and curvature decay of complete noncompact Kähler manifold with positive holomorphic bisectional curvature, and refer to the book of Greene and Wu [4] and Kobayashi and Nomizu [6] for background on differential geometry.

2. Method of Construction

In this section, we shall give the conditions under which there exists a complete rotationally symmetric Kähler metric on \( \mathbb{C}^n \) with positive holomorphic sectional curvature.

Recall that a complex \( n \)-dimensional complete Kähler manifold \( M \) is of positive holomorphic sectional curvature if \( \sum R_{ijkl}a_ia_ja_ka_l > 0 \) for all nonzero \( n \)-tuples \( (a_1, a_2, \cdots, a_n) \) of complex numbers, where \( R_{ijkl} \) is the components of the curvature tensor.

Let \( r^2 = \sum_{i=1}^n z_i \bar{z}_i \) on \( \mathbb{C}^n \). As in [5], we consider only rotationally symmetric metrics \( g_{ij} = \frac{\partial^2 f(r^2)}{\partial z_i \partial \bar{z}_j} \) on \( \mathbb{C}^n \), derived from a global potential function, where \( r \to f(r^2) \in C^\infty(\mathbb{R}) \). Clearly,

\[
g = f'(r^2) \left( \sum_{i=1}^n dz_i d\bar{z}_i \right) + f''(r^2) \left( \sum_{i=1}^n \bar{z}_i d\bar{z}_i \right) \left( \sum_{i=1}^n z_i dz_i \right).
\]

Thus \( g \) is a complete metric on \( \mathbb{C}^n \) if \( f \) satisfies the following two conditions:

(i) \( f'(r^2) > 0 \) and \( f'(r^2) + r^2 f''(r^2) > 0 \) for all \( r \),

(ii) \( \int_0^\infty \sqrt{f''(r^2) + r^2 f''(r^2)} dr \) diverges.

Since \( g \) is a rotationally symmetric Kähler metric, without loss of generality, we may restrict the computation of the curvature tensor \( R_{ijkl} \) of \( g \) to the complex line

\( L = \{ z_i = 0 | i > 1 \} \),

thus obtaining

\[
R_{ijkl} = - \left( \frac{\partial^2 g_{jk}}{\partial z_i \partial \bar{z}_m} - \sum_{p,q} g_{ip} \frac{\partial g_{jq}}{\partial z_l} \frac{\partial g_{km}}{\partial \bar{z}_q} \right)
= - f''(r^2) (\delta_{jk} \delta_{lm} + \delta_{jm} \delta_{lk})
- r^2 [f''(r^2) - \frac{f''(r^2)^2}{f'(r^2)}] (\delta_{jk} \delta_{lm} + \delta_{jm} \delta_{lk} + \delta_{jl} \delta_{km} + \delta_{jl} \delta_{km})
- \left[ r^4 f'''(r^2) - r^2 \frac{(f''(r^2)^2 + r^2 f'''(r^2))^2}{f'(r^2)^2} + 4 r^2 \frac{(f''(r^2)^2)}{f'(r^2)^2} \right] \delta_{jklm}. \tag{2.2}
\]

Let \( (a_1, a_2, \cdots, a_n) \) be any complex \( n \)-tuples of complex numbers. Then

\[
\sum R_{ijkl} a_ia_j a_k a_l = -(2A + 4B + C)|a_1|^4 - 4(A + B)|a_1|^2 \left( \sum_{j=2}^n |a_j|^2 \right) - 2A \left( \sum_{j=2}^n |a_j|^2 \right)^2
\]

where

\[
A = f''(r^2),
B = r^2 \left[ f'''(r^2) - \frac{(f''(r^2))^2}{f'(r^2)} \right],
\]
function in C from a function f. Therefore a complete metric of positive holomorphic sectional curvature on C^n can be generated from a function f ∈ C^∞(R) satisfying (i)-(v). In the next section, we shall introduce a class of functions in C^∞(R) satisfying (i)-(v).

3. Metrics of Positive Curvature

In this section, we shall show a family of complete Kähler metrics of positive holomorphic sectional curvature on C^n and discuss the curvature decay and volume growth of these metrics.

Let us consider the following function

\[ f(r^2) = \frac{1}{(\beta + 1)^{\alpha^3}} \int_0^{r^2} \left\{ \frac{[\alpha + \ln(1 + x)]^{\beta + 1} - \alpha^{\beta + 1}}{x} \right\} dx \in C^\infty(R) \]  

(3.1)

where \( \alpha > \beta \geq 0 \). Then

\[ f'(r^2) = \frac{1}{(\beta + 1)^{\alpha^3}} \left\{ \frac{[\alpha + \ln(1 + r^2)]^{\beta + 1}}{r^2} - \frac{\alpha^{\beta + 1}}{r^2} \right\}, \]

\[ f''(r^2) = -\frac{[\alpha + \ln(1 + r^2)]^{\beta + 1}}{(\beta + 1)\alpha^\beta r^4} + \frac{[\alpha + \ln(1 + r^2)]^{\beta}}{\alpha^\beta r^2(1 + r^2)} + \frac{\alpha}{(\beta + 1)r^4} \]

and

\[ f'(r^2) + r^2 f''(r^2) = \frac{[\alpha + \ln(1 + r^2)]^{\beta}}{\alpha^\beta(1 + r^2)}. \]

It follows that f satisfies (i), (ii) and (iii) (see Appendix A). Also,

\[ \frac{1}{4r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \ln(f'(r^2) + r^2 f''(r^2)) \right) \]

\[ = -\frac{\alpha(\alpha - \beta) + \beta r^2 + (2\alpha - \beta) \ln(1 + r^2) + \ln^2(1 + r^2)}{(1 + r^2)^2[\alpha + \ln(1 + r^2)]^2} \]

and

\[ f''(r^2) = \frac{r^2 f'''(r^2) - r^2 \left( \frac{f''(r^2)}{f'(r^2)} \right)^2}{f'(r^2)} \]

\[ = -\frac{\beta \alpha^{\beta + 1} e^{\gamma - \alpha} - \beta \alpha^{\beta + 1} - \gamma^{\beta + 2} + \gamma^{\beta + 1} e^{\gamma - \alpha} - \gamma^{\beta + 1} + \alpha^{\beta + 1} y}{y^{\beta r^2(1 + r^2)^2 [\alpha + \ln(1 + r^2)]^{\beta + 1} - \alpha^{\beta + 1}} \]  

(3.2)

which yield (iv) and (v) (see Appendix A). Here, y = \( \alpha + \ln(1 + r^2) \).

Therefore \( g_{ij} = \frac{\partial^2 f(r^2)}{\partial z_i \partial z_j} \) with f(r^2) defined by (3.1) is a class of complete Kähler metrics of positive sectional curvatures on C^n.

Now we turn to the computation of the volume growth and scalar curvature decay of C^n equipped with the metric \( g_{ij} = \frac{\partial^2 f(r^2)}{\partial z_i \partial z_j} \) with f(r^2) defined by (3.1). First, let us estimate the
volume growth of $C^n$. As before, our computation is restricted on $L$. Inserting (3.1) into (2.1), we have
\begin{align*}
g = \left\{ \frac{\alpha + \ln(1 + t^2)}{\alpha \beta (1 + r^2)} \right\} dz_1 d\bar{z}_1 + \left\{ \frac{\alpha + \ln(1 + t^2) + \beta + 1 - \alpha \beta + 1}{(\beta + 1)\alpha \beta t^2} \right\} \left( \sum_{i=2}^{n} dz_i d\bar{z}_i \right). \tag{3.3}
\end{align*}

And the volume form $\omega^n$ of (3.3) is given by
\begin{align*}
\omega^n = \left( \frac{\sqrt{-1}}{2} \right)^n \left\{ \frac{\alpha + \ln(1 + t^2)}{\alpha \beta (1 + t^2)} \right\} 
\times \left\{ \frac{\alpha + \ln(1 + t^2) + \beta + 1 - \alpha \beta + 1}{(\beta + 1)\alpha \beta t^2} \right\}^{n-1} \ dz_1 \wedge d\bar{z}_1 \otimes dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.
\end{align*}

Since the geodesic distance function $\rho$ from the origin of $C^n$ is given by
\begin{align*}
\rho = \int_0^r \sqrt{\frac{\alpha + \ln(1 + t^2)}{\alpha \beta (1 + t^2)}} \ dt
\end{align*}
and satisfies
\begin{align*}
\rho = O(\ln^{\frac{\beta+2}{\beta+1}} r) \ (r \to \infty), \tag{3.4}
\end{align*}
the volume growth $V(B(0, \rho))$ of geodesic ball $B(0, \rho)$ of $C^n$ equipped with (3.3) is
\begin{align*}
V(B(0, \rho)) = \int_{B_E(0,r)} \omega^n = \int_{S^{2n-1}(1)} \left[ \int_0^r \left( \frac{\sqrt{-1}}{2} \right)^{n+1} \left\{ \frac{\alpha + \ln(1 + t^2)}{\alpha \beta (1 + t^2)} \right\} 
\times \left\{ \frac{\alpha + \ln(1 + t^2) + \beta + 1 - \alpha \beta + 1}{(\beta + 1)\alpha \beta t^2} \right\}^{n-1} t^{2n-1} dt \right] d\theta
\end{align*}
\begin{align*}
&= O(\ln^{(\beta+1)n} r) \text{ as } r \to \infty \\
&= O(\rho^{\frac{2(\beta+1)n}{\beta+2}}) \text{ as } \rho \to \infty,
\end{align*}
where $B_E(0, r)$ is the Euclidean ball corresponding to the geodesic ball $B(0, \rho)$.

To determine the rate of curvature decay, without loss of generality, we may restrict the computation of the scalar curvature $R = \sum_{i,j} g^{ij} R_{ij}$ of $g$ to the complex line $L = \{ z_i = 0 | i > 1 \}$.

Then the Ric curvature, denoted by Ric, is as follows:
\begin{align*}
\text{Ric} &= -\sqrt{-1} \left\{ \beta \partial \bar{\partial} \ln \left[ \alpha + \ln(1 + r^2) \right] - \partial \bar{\partial} \ln(1 + r^2) \right\} \\
&= -\sqrt{-1} (n-1) \left\{ \partial \bar{\partial} \ln \left[ \frac{\beta}{\alpha + \ln(1 + r^2) - 1} \right] + (1 + r^2) \left( \sum_{i=2}^{n} dz_i \wedge d\bar{z}_i \right) \right\} \\
&= -\sqrt{-1} \left\{ \frac{\beta}{\alpha + \ln(1 + r^2) - 1} \right\} \frac{dz_1 \wedge d\bar{z}_1 + (1 + r^2) \left( \sum_{i=2}^{n} dz_i \wedge d\bar{z}_i \right)}{(1 + r^2)^2} \\
&+ \sqrt{-1} \frac{\beta r^2}{\alpha + \ln(1 + r^2)^2 (1 + r^2)^2} dz_1 \wedge d\bar{z}_1 \\
&= -\sqrt{-1} (n-1) \left\{ \frac{(\beta + 1) \alpha + \ln(1 + r^2)}{\alpha + \ln(1 + r^2)^2} \right\} \left( \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i \right) \\
&= -\sqrt{-1} (n-1) \left\{ \frac{(\beta + 1) \alpha + \ln(1 + r^2) - \beta - \alpha - \ln(1 + r^2)}{\alpha + \ln(1 + r^2)^2} \right\} \left( \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i \right) \\
&= -\sqrt{-1} (n-1) \left\{ \frac{(\beta + 1) \alpha + \ln(1 + r^2) - \beta - \alpha - \ln(1 + r^2)}{\alpha + \ln(1 + r^2)^2} \right\} \left( \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i \right) \\
&= -\sqrt{-1} (n-1) \left\{ \frac{(\beta + 1) \alpha + \ln(1 + r^2) - \beta - \alpha - \ln(1 + r^2)}{\alpha + \ln(1 + r^2)^2} \right\} \left( \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i \right)
\end{align*}
This implies that
\[
R_{11} = \left\{ \frac{\beta}{\alpha + \ln(1 + r^2)} - 1 \right\} \frac{1}{1 + r^2} + \frac{(n - 1)(\beta + 1)(\alpha + \ln(1 + r^2))^\beta 1}{(\alpha + \ln(1 + r^2))^{\beta+1} - \alpha^{\beta+1}} \frac{1}{1 + r^2} - \frac{(n - 1)(\beta + 1)(\alpha + \ln(1 + r^2))^{\beta-1}[\beta - \alpha - \ln(1 + r^2)] / r^2}{(\alpha + \ln(1 + r^2))^{\beta+1} - \alpha^{\beta+1}} \frac{1}{1 + r^2}
\]

and that for \( i \geq 2 \),
\[
R_{ii} = \left\{ \frac{\beta}{\alpha + \ln(1 + r^2)} - 1 \right\} \frac{1}{1 + r^2} + \frac{(n - 1)(\beta + 1)(\alpha + \ln(1 + r^2))^\beta 1}{(\alpha + \ln(1 + r^2))^{\beta+1} - \alpha^{\beta+1}} \frac{1}{1 + r^2}.
\]

Also, \( R_{ij} = 0 \) for \( i \neq j \). As a consequence, the scalar curvature \( R = \sum_{i=1}^{n} g_{ii} R_{ii} \) of \( C^n \) equipped with the rotationally symmetric metrics \( g_{ij} = \partial^2 f(r^2)/\partial z_i \partial z_j \) with \( f(r^2) \) defined by (3.1) decays like
\[
R = O \left( \frac{1}{\ln^{\beta+1}(r)} \right) \text{ as } r \to \infty.
\]

By (3.4), (3.5) can be written as
\[
R = O \left( \rho^{-2(\beta+1)/(\beta+2)} \right) \text{ as } \rho \to \infty.
\]

Appendix A.

Put \( G(x) = [\alpha + \ln(1 + x)]^{\beta+1} (1 + x) - (\beta + 1)x[\alpha + \ln(1 + x)]^\beta - \alpha^{\beta+1} (1 + x) \). Then
\[
f''(r^2) = \frac{G''(r^2)}{G(r^2)}. \text{ Since } \lim_{r^2 \to 0} f''(r^2) = -\frac{\alpha - \beta}{2\alpha} < 0 \text{ for } \alpha > \beta \geq 0, \text{ (iii) holds if } G(x) > 0 \text{ for all } x > 0. \text{ Since } G(0) = G'(0) = 0, \text{ it suffices to show that } G''(x) > 0 \text{ when } \alpha > \beta \geq 0. \text{ Indeed, if } \alpha > \beta \geq 0 \text{ and } x \geq 0, \text{ then}
\[
G''(x) = \frac{(2\alpha - \beta) + 2\alpha x}{(\alpha + \ln(1 + x))^{2-\beta}} + \frac{\alpha(\alpha - \beta) + 2\alpha^2 - \beta^2 + \beta x}{(\alpha + \ln(1 + x))^{2-\beta}} + \frac{\alpha^3}{(\alpha + \ln(1 + x))^{2-\beta}} + \frac{\ln^2(1 + x)}{(\alpha + \ln(1 + x))^{2-\beta}} > 0.
\]

To prove (v), let \( H(y) = \beta(\alpha + 1)^{\beta-1} e^{y - \alpha - \beta(\beta + 1)} - \beta \alpha^{\beta+1} - y^{\beta+2} + y^{\beta+1} e^{y - \alpha - \beta^{\beta+1} - \alpha^{\beta+1} y}. \) Since
\[
\lim_{r^2 \to 0} \left[ f''(r^2) + r^2 f'''(r^2) - r^2 \left( \frac{f''(r^2)}{f'(r^2)} \right)^2 \right] = -\frac{\alpha - \beta}{2\alpha} < 0,
\]
it is easily known from (5.2) that (v) holds if \( H(y) > 0 \) for all \( y > \alpha \). Since \( H'(\alpha) = H'(\alpha) = 0 \), it suffices to prove that
\[
H''(y) = \beta \alpha^{\beta+1} e^{y - \alpha} + y^{\beta}[y e^{y - \alpha} - \beta(\beta + 1)] + y^{\beta-1}[e^{\alpha - 1}\beta(\beta + 1) + 2y^{\beta}[e^{\alpha - 1}\beta(\beta + 1) + 1] > 0
\]
for all \( y > \alpha \) when \( \alpha > \beta \geq 0 \).
Let

$$I(y) = \beta \alpha^{\beta+1} e^{y-\alpha} + y^\beta [y e^{y-\alpha} - \beta (\beta + 1)].$$

Then we need to show that $I(y) > 0$ for all $y \geq \alpha$ when $\alpha > \beta \geq 0$.

Put $\alpha > \beta \geq 0$ and $I_n(y) = y I'_n(y)$ (n = 1, 2, 3, · · ·) and $I'_n(y) = I(y)$. Then

$$I_n(y) = -y^\beta \beta n^{n-1} (1 + \beta) + e^{y-\alpha} y^{n+\beta} + \alpha^{n+\beta} \beta P_{n-1}(y) + y^\beta Q_{n-1}(y)]$$

with $P_{n-1}(y) = 1 + \sum_{i=1}^{n-1} P_i^{n-1} y^i$ and $Q_{n-1}(y) = (1 + \beta)^n + \sum_{j=1}^{n-1} q_{n-1}^{n-1}(\beta) y^j$, where $P_i^{n-1}$ are positive integers, $q_{n-1}^{n-1}(\beta)$ are polynomials of degree j with respect to $\beta^j$ whose coefficients are positive integers, where $n = 1, 2, 3, \cdots, i = 1, 2, \cdots, n - 1, j = 1, 2, \cdots, n - 1$. It can be shown that $I_n(\alpha) > 0$ (n = 1, 2, 3, · · ·) and that

$$I_n(y) > -y^\beta \beta n^{n+1} (1 + \beta) + y^\beta (1 + \beta)^n > y^\beta (1 + \beta) [(1 + \beta)^n - \beta^n]$$

for all $y \geq \alpha > \beta \geq 0$. Hence there exists a positive integer $n_0$ which depends only on $\beta$, such that $I_n(y) > 0$ for all $y \geq \alpha > \beta \geq 0$ as $n \geq n_0$.

By the definition of $I_n(y)$, we know that $I'_{n_0-1}(y) > 0$ for all $y \geq \alpha$ when $\alpha > \beta \geq 0$. It follows that $I'_{n_0-1}(y) > 0$ for all $y \geq \alpha$ when $\alpha > \beta \geq 0$. Repeating the above analysis, we conclude that $I'_{n}(y) > 0$ and $I_n(y) > I_n(\alpha)$ (n = 0, 1, 2, 3, · · ·) for all $y \geq \alpha$ when $\alpha > \beta \geq 0$. In particular, $I(y) > 0$ for all $y \geq \alpha$ when $\alpha > \beta \geq 0$.

Acknowledgement

This paper is completed under the direction of Professor Zhu Xi-Ping. The authors would like to express their gratitude for his continuous guidance and much valuable advice. XF is also grateful for support from NSFC 10171114. ZJ is supported by NSFC 10271121 and sponsored by SRF for ROCS, SEM. The authors would also like to thank Dr Chen Binglong for his helpful comments on this paper. Finally, the authors would like to thank the referee of this paper for his helpful comments and suggestions.

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