ON CONDITIONS RELATING TO NONSOLVABILITY

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Abstract. Recent work of Kaplan and Levy refining a nonsolvability criterion proved by Thompson in his N-Groups paper prompts questions on whether certain conditions on groups are equivalent to nonsolvability.

In what follows, $G$ is a finite group with identity $1_G$ and $G^\# = G \setminus \{1_G\}$.

Thompson [4, Corollary 3] proved the following: A finite group $G$ is nonsolvable if and only if there are three elements $x$, $y$, and $z$ in $G^\#$, whose orders are coprime in pairs, such that $xyz = 1_G$.

How much tighter can one make this nonsolvability criterion? Can one always choose $x$, $y$, and $z$ to be elements of prime-power order, for distinct primes obviously? Call a group that satisfies this condition a 3PPO-group (for three prime-power orders). So is a group nonsolvable if and only if it is a 3PPO-group? Can one always choose $x$, $y$, and $z$ to be elements of prime order? Call a group that satisfies this condition a 3PO-group (for three prime orders).

In a recent paper [3], Kaplan and Levy show that $x$, $y$, and $z$ can be chosen so that $x$ has order a power of 2, $y$ has order a power of $p$ for an odd prime $p$, and $z$ has order coprime to $2p$. In other words, two of the three elements can be chosen to have order a power of a prime. In addition, they show that every nonabelian simple group is a 3PO-group.

In this short note, we show that not every nonsolvable group is a 3PO-group and we exhibit a condition equivalent to 3PPO.

Our first result below shows $SL(2, 5)$, the group of $2 \times 2$ matrices which entries in $GF(5)$ and determinant 1, is not a 3PO-group. Since $SL(2, 5)$ is a non-split extension of a central subgroup of order 2 by $A_5$, $SL(2, 5)$ has the smallest possible order of a nonsolvable group that is not simple and does not contain a simple group as a subgroup.

Theorem 1. In $SL(2, 5)$, there do not exist elements $x$, $y$, and $z$ in $SL(2, 5)$ of distinct prime orders with $xyz = e$.

Proof. In this proof, we use the character table of $2 \cdot A_5 \cong SL(2, 5)$ given on p. xxiv of [11] with its class labelings and its ordering of characters, which we label as $\chi_i$ with $1 \leq i \leq 9$. 

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Now the only possibility for three elements in $SL(2,5)$ to have distinct prime orders is for those orders to be 2, 3, and 5. The group $SL(2,5)$ has one element of order 2, namely $-I_2$, whose conjugacy class is labeled $1A_1$. In addition, $SL(2,5)$ has one conjugacy class of elements of order 3 labeled $3A_0$, and two conjugacy classes of elements of order 5 labeled $5A_0$ and $5B_0$. Now denote $-I_2$ by $g_2$, an element of the conjugacy class $3A_0$ by $g_3$, and elements of the conjugacy classes $5A_0$ and $5B_0$ by $g_5$ and $h_5$, respectively. Then

$$\sum_{k=1}^{9} \frac{1}{\chi_k(1_G)} \chi_k(g_2) \chi_k(g_3) \chi_k(g_5) = 1 + 0 + 0 + (-1) + 0 + b_5 + b_5^* + 1 + 0,$$

where the $k$th term on the right-hand side is $\frac{1}{\chi_k(1_G)} \chi_k(g_2) \chi_k(g_3) \chi_k(g_5)$.

This right-hand side simplifies to

$$b_5 + b_5^* + 1 = \frac{-1 + \sqrt{5}}{2} + \frac{-1 - \sqrt{5}}{2} + 1 = 0.$$

Similarly

$$\sum_{k=1}^{9} \frac{1}{\chi_k(1_G)} \chi_k(g_2) \chi_k(g_3) \chi_k(h_5) = 0.$$

By [2] Lemma 19.2, these two calculations show that there are no elements $x$, $y$, and $z$ of order 2, 3, and 5, respectively, in $SL(2,5)$ such that $xyz = 1_{SL(2,5)}$. \qed

We say that a group $G$ is a 3SS-group (for three Sylow subgroups) if and only if there are three Sylow subgroups $P_1$, $P_2$, and $P_3$ corresponding to three distinct primes $p_1$, $p_2$, and $p_3$ dividing $|G|$ such that $|P_1P_2P_3| < |P_1||P_2||P_3|$. (Here $P_1P_2P_3 = \{x_1x_2x_3 \mid x_i \in P_i, 1 \leq i \leq 3\}$.) Some time ago, Michael Ward and the present author tried unsuccessfully to prove that a group was nonsolvable if and only if it was a 3SS-group.

**Theorem 2.** A finite group $G$ is a 3PPO-group if and only if it is a 3SS-group.

**Proof.** Suppose that $G$ is a 3PPO-group. Then there are three distinct primes $p_1$, $p_2$, and $p_3$ dividing $|G|$, and three elements $x_1$, $x_2$, and $x_3$ in $G^\#$, such that $x_i$ is a $p_i$-element for $i = 1, 2, 3$ and $x_1x_2x_3 = 1_G$. If, for $i = 1, 2, 3$, $P_i$ is a Sylow $p_i$-subgroup containing $x_i$, then $|P_1P_2P_3| < |P_1||P_2||P_3|$, implying that $G$ is a 3SS-group.

Suppose that $G$ is a 3SS-group. Then there are three Sylow subgroups $P_1$, $P_2$, and $P_3$ corresponding to three distinct primes $p_1$, $p_2$, and $p_3$ dividing $|G|$ such that $|P_1P_2P_3| < |P_1||P_2||P_3|$. This implies that there are distinct triples $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ in $P_1 \times P_2 \times P_3$ such that $x_1x_2x_3 = y_1y_2y_3$, implying

$$(y_1^{-1}x_1)(x_2y_2^{-1})(y_2x_3y_3^{-1}y_2^{-1}) = 1_G.$$

Since the triples are distinct, there is an $i$ with $1 \leq i \leq 3$ such that $x_i \neq y_i$. From this it follows that for every $i$, $x_i \neq y_i$. Thus $y_1^{-1}x_1$, $x_2y_2^{-1}$, and $y_2x_3y_3^{-1}y_2^{-1}$ are non-trivial elements of prime-power order for three distinct primes, and this implies that $G$ is a 3PPO-group. \qed
To our knowledge, the question of whether the condition 3PPO is equivalent to nonsolvability remains open.

REFERENCES

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