Almost split sequences for polynomial \( G_rT \)-modules and polynomial parts of Auslander-Reiten components

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Abstract. In [8], Doty, Nakano and Peters defined infinitesimal Schur algebras, combining the approach via polynomial representations with the approach via \( G_rT \)-modules to representations of the algebraic group \( G = GL_n \). We study analogues of these algebras and their Auslander-Reiten theory for reductive algebraic groups \( G \) and Borel subgroups \( B \) by considering the categories of polynomial representations of \( G_rT \) and \( B_rT \) as full subcategories of \( \text{mod} \ G_rT \) and \( \text{mod} \ B_rT \), respectively. We show that every component \( \Theta \) of the stable Auslander-Reiten quiver \( \Gamma_\text{s}(G,T) \) of \( \text{mod} \ G_rT \) whose constituents have complexity 1 contains only finitely many polynomial modules. For \( G = GL_2, r = 1 \) and \( T \subseteq G \) the torus of diagonal matrices, we identify the polynomial part of the stable Auslander-Reiten quiver of \( G_rT \) and use this to determine the Auslander-Reiten quiver of the infinitesimal Schur algebras in this situation. For the Borel subgroup \( B \) of lower triangular matrices of \( GL_2 \), the category of \( B_rT \)-modules is related to representations of elementary abelian groups of rank \( r \). In this case, we can extend our results about modules of complexity 1 to modules of higher Frobenius kernels arising as outer tensor products.

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Introduction

Let $G = \text{GL}_n$ and $T \subseteq G$ the torus of diagonal matrices over an algebraically closed field of characteristic $p > 0$. In [5], Doty, Nakano and Peters investigated polynomial representations of the group scheme $G_r T$. These can be considered as modules over certain finite-dimensional algebras, the infinitesimal Schur algebras $S_d(G_r T)$. Using the general framework of [5], their definition can be extended to other group schemes $G$. In [16], the Auslander-Reiten theory of mod$G_r T$ was studied by considering $G_r T$-modules as $X(T)$-graded $G_r T$-modules, where $X(T) \cong \mathbb{Z}^n$ is the character group of $T$. We follow this approach and study, for a component $\Theta$ of the stable Auslander-Reiten quiver $\Gamma_s(G_r T)$ of mod$G_r T$, the polynomial part of $\Theta$, that is the intersection of $\Theta$ with the full subcategory of polynomial $G_r T$-modules. Here, $G \subseteq \text{GL}_n$ is either reductive or a Borel subgroup of a reductive group containing the center $Z(\text{GL}_n)$ of $\text{GL}_n$. The number of elements of the polynomial part of $\Theta$ is shown to be finite for components of complexity one. Since mod$G_r T$ is a Frobenius category and the category of polynomial representations is not, the polynomial part of $\Theta$ also contains natural types of $G_r T$-modules whose position in $\Theta$ one can determine, namely modules which are Ext-projective or have finite projective dimension in the full subcategory of polynomial $G_r T$-modules. Furthermore, the polynomial part of a component is part of the Auslander-Reiten quiver of an infinitesimal Schur algebra. We use this approach to fully determine the Auslander-Reiten quiver of the algebras $S_d(G_1 T)$ for $G = \text{GL}_2$ and $T \subseteq G$ the torus of diagonal matrices. It turns out that upon deleting projective-injective modules, the underlying directed graphs of all components in this situation are isomorphic to $\mathbb{Z}[A_{2s+1}]/(t^{2s+1})$ for some $s \in \mathbb{N}$. By reducing to the case $r = 1$, we are able to determine the polynomial part of all $G_r T$-components of type $\mathbb{Z}[A_{\infty}^r]$ whose restriction to $G_r$ has type $\mathbb{Z}[A_{12}]$ for higher $r$. This paper is organized as follows.

In the first section, we collect basic results about $\mathbb{Z}^n$-graded algebras and show how $G_r T$-modules can be considered as $X(T)$-graded $G_r$-modules.

In the second section, we consider a subcategory $\mathcal{T}$ of the category mod gr$\Lambda$ of $\mathbb{Z}^n$-graded modules over a $\mathbb{Z}^n$-graded algebra $\Lambda$, where $\mathcal{T}$ is closed with respect to some natural operations. We prove results about the Auslander-Reiten theory of $\mathcal{T}$ valid in this context. Here, we concentrate on the intersection of $\mathcal{T}$ with components of type $\mathbb{Z}[A_{\infty}]$. By [16] Proposition 8.2.2, Theorem 8.2.3, components of type $\mathbb{Z}[A_{\infty}]$ are the preeminent components of the stable Auslander-Reiten quiver of mod$G_r T$ for reductive groups $G$.

In Section 3, we define polynomial representations of $G_r T$ and infinitesimal Schur algebras and show how they fit into the general framework of Section 2. We then prove results about the position of special kinds of modules in the Auslander-Reiten quiver and give some criteria on modules in components to ensure that the polynomial part of these components is finite.

In Section 4, we turn to modules of complexity 1. In this context, we can combine results from [16] and the previous sections to show that the polynomial parts of their stable AR-components are finite. For $G = \text{GL}_n$ or $G \subseteq \text{GL}_n$ the Borel subgroup of lower triangular matrices, we show that these components contain a unique polynomial $G_r T$-module of...
maximal quasi-length, provided they contain a polynomial module whose quasi-length is large enough.

In Section 5, we completely determine the polynomial parts of components of mod $G_1 T$ for $G = \text{GL}_2$ and use this to determine the Auslander-Reiten quiver of the infinitesimal Schur algebras $S_d(G_1 T)$. For this, we rely heavily on the classification of indecomposable finite-dimensional $U_0(\mathfrak{sl}_2)$-modules obtained by Premet in [26]. By adapting Morita-equivalences between blocks of $G_{r-1} T$ and $G_r T$ to the setting of polynomial representations of $\text{GL}_n$, we are able to use our results in this case to obtain the aforementioned result for $G_r T$-components of type $\mathbb{Z}[A_\infty]$ whose restriction to $G_r$ has type $\mathbb{Z}[\tilde{A}_{12}]$.

In Section 5, we give further results on polynomial representations of $B_r T$, where $B$ is a Borel subgroup of a reductive group and $T \subseteq B$ a maximal torus. We show that in contrast to the case pertaining to $\text{GL}_n$ (cf. [7, Section 7]), the algebras $S_d(B_r T)$ are directed quasi-hereditary algebras. For $B \subseteq \text{GL}_2$ the Borel subgroup of lower triangular matrices, we can extend some results about modules of complexity 1 to modules which are outer tensor products of two modules, one of which has complexity 1. This case is of interest since representations of $B_r T$ can be viewed as graded modules over the truncated polynomial ring $k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)$, linking them to representations of elementary abelian $p$-groups of rank $r$.

For representations of algebraic groups and $G_r T$-modules, we refer the reader to [23]. Details about Auslander-Reiten theory and representations of associative algebras can be found in [1], [2].

1 $\mathbb{Z}^n$-graded algebras and $G_r T$-modules

In this section, we are going to establish basic notation and results for $\mathbb{Z}^n$-graded algebras and $G_r T$-modules.

Let $k$ be a field and $\Lambda$ be a finite-dimensional $k$-algebra. The algebra $\Lambda$ is called $\mathbb{Z}^n$-graded if there is a decomposition

$$\Lambda = \bigoplus_{i \in \mathbb{Z}^n} \Lambda_i$$

into $k$-subspaces such that $\Lambda_i \Lambda_j \subseteq \Lambda_{i+j}$ for all $i, j \in \mathbb{Z}^n$. Letting mod $\Lambda$ be the category of finite-dimensional $\Lambda$-modules, $M \in \text{mod} \Lambda$ is called $\mathbb{Z}^n$-graded if there is a decomposition

$$M = \bigoplus_{i \in \mathbb{Z}^n} M_i$$

such that $\Lambda_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}^n$. An element $m \in M \setminus \{0\}$ is called homogeneous of degree $i \in \mathbb{Z}^n$ if $m \in M_i$. In that case, we write $\deg(m) = i$ for the degree of $m$. A $\Lambda$-submodule $N \subseteq M$ is called homogeneous if

$$N = \bigoplus_{i \in \mathbb{Z}^n} N \cap M_i.$$
In that case, $N$ and the factor module $M/N$ have a natural structure as $\mathbb{Z}^n$-graded $\Lambda$-modules. If $M = \bigoplus_{i \in \mathbb{Z}^n} M_i, N = \bigoplus_{i \in \mathbb{Z}^n} N_i$, are $\mathbb{Z}^n$-graded modules, then a morphism of $\mathbb{Z}^n$-graded modules $M \to N$ is a $\Lambda$-linear map $f : M \to N$ such that $f(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}^n$. We denote the space of $\mathbb{Z}^n$-graded morphisms $M \to N$ by $\text{Hom}_{\text{gr}(\Lambda)}(M, N)$ and the category of finite dimensional $\mathbb{Z}^n$-graded $\Lambda$-modules by $\text{mod}_{\text{gr}(\Lambda)}$.

We denote by $F : \text{mod}_{\text{gr}(\Lambda)} \to \text{mod}(\Lambda)$ the forgetful functor. The modules in $F(\text{mod}_{\text{gr}(\Lambda)})$ are called gradable. For $i \in \mathbb{Z}^n$, there is a functor $[i] : \text{mod}_{\text{gr}(\Lambda)} \to \text{mod}(\Lambda)$, where $M[i]_{ij} := M_{j-i}$ for all $i \in \mathbb{Z}^n$ and all $M \in \text{mod}_{\text{gr}(\Lambda)}$ and the morphisms are left unchanged. Then $[i]$ is an auto-equivalence of $\text{mod}_{\text{gr}(\Lambda)}$ and we have $F \circ [i] = F$ for all $i \in \mathbb{Z}^n$.

The following results about $\mathbb{Z}^n$-graded algebras were first obtained in [19], [20] for $n = 1$ and it has been known for a long time that they hold for $n > 1$. Proofs for $n > 1$ can be found in [19].

**Proposition 1.1.** (1) A module $M \in \text{mod}_{\text{gr}(\Lambda)}$ is indecomposable iff $F(M)$ is indecomposable in $\text{mod}(\Lambda)$.

(2) If $M, N \in \text{mod}_{\text{gr}(\Lambda)}$ are indecomposable such that $F(M) \cong F(N)$, there is a unique $\lambda \in \mathbb{Z}^n$ such that $M \cong N[\lambda]$.

(3) If $(P_n)_{n \in \mathbb{N}}$ is a minimal projective resolution of $M$, then $(F(P_n))_{n \in \mathbb{N}}$ is a minimal projective resolution of $F(M)$.

**Theorem 1.2.** (1) The category $\text{mod}_{\text{gr}(\Lambda)}$ has almost split sequences.

(2) If $0 \to M \to E \to N \to 0$ is an almost split sequence in $\text{mod}_{\text{gr}(\Lambda)}$, then $0 \to F(M) \to F(E) \to F(N) \to 0$ is almost split in $\text{mod}(\Lambda)$.

For $M \in \text{mod}_{\text{gr}(\Lambda)}$, we define the support of $M$ as $\text{supp}(M) = \{ \lambda \in \mathbb{Z}^n \mid M_{\lambda} \neq 0 \}$. If $J \subseteq \mathbb{Z}^n$, we denote by $\text{mod}_{\text{gr}(\Lambda)}(J)$ the full subcategory of $\text{mod}_{\text{gr}(\Lambda)}$ whose objects are those $M \in \text{mod}_{\text{gr}(\Lambda)}$ such that $\text{supp}(M) \subseteq J$. The following result is useful for extending results about $\text{mod}(\Lambda)$ to $\text{mod}_{\text{gr}(\Lambda)}$.

**Proposition 1.3.** Let $J \subseteq \mathbb{Z}^n$ be finite. Then there is a finite-dimensional algebra $A$ and an equivalence of categories $\text{mod}_{\text{gr}(\Lambda)}(J) \to \text{mod}(A)$.

Now let $k$ be an algebraically closed field of characteristic $p > 0$. We want to apply results about $\mathbb{Z}^n$-graded algebras to the category $\text{mod}_k G, T$, where $G$ is a smooth connected algebraic group scheme over $k$, $T \subseteq G$ is a maximal torus and $G_r$ the $r$-th Frobenius kernel of $G$. The next proposition collects some results on this situation in a slightly more general context.

**Proposition 1.4.** Let $H$ be an affine group scheme, $G, T \subseteq H$ be closed subgroup schemes such that $T$ is a torus and $G$ is infinitesimal and normalized by $T$. Then the following statements hold:

(1) The category $\text{mod}_k G \rtimes T$ is equivalent to the category of $X(T)$-graded $G$-modules.
Recall that \((F\text{ and extensions and } T)\) VI.1.4,\(\text{Proof.}\) This can be proved as [14, 2.1].

(3) The forgetful functor induces a morphism of stable translation quivers \(\Gamma_s(GT) \to \Gamma_s(G)\).

\textbf{Proof.} This can be proved as [14, 2.1]. \(\square\)

\section{Almost split sequences in subcategories}

In this section, we want to establish basic results on almost split sequences and Auslander-Reiten theory in subcategories. This allows us to treat some results in the following sections in a uniform way. Let \(\Lambda\) be a finite-dimensional \(\mathbb{Z}^n\)-graded algebra and \(\mathcal{T}, \mathcal{F}\) be full subcategories of \(\text{mod gr } \Lambda\) closed with respect to finite direct sums. Recall that \((\mathcal{T}, \mathcal{F})\) is called a torsion pair if \(\mathcal{T} = \{V \in \text{mod gr } \Lambda \mid \text{Hom}_{\text{gr } \Lambda}(V, N) = 0\text{ for all } N \in \mathcal{F}\}\) and \(\mathcal{F} = \{V \in \text{mod gr } \Lambda \mid \text{Hom}_{\text{gr } \Lambda}(N, V) = 0\text{ for all } N \in \mathcal{T}\}\). In that case, \(\mathcal{T}\) is called a torsion class and \(\mathcal{F}\) is called a torsion-free class in \(\text{mod gr } \Lambda\). By [1 VI.1.4], \(\mathcal{T}\) is a torsion class if and only if \(\mathcal{T}\) is closed under images, finite direct sums and extensions and \(\mathcal{F}\) is a torsion-free class if and only if \(\mathcal{F}\) is closed under submodules, direct products and extensions. This is also equivalent to the existence of a subfunctor \(t\) of the identity functor \(\text{mod gr } \Lambda \to \text{mod gr } \Lambda\) called the torsion radical such that \(t \circ t = t\) and \(\mathcal{T} = \{M \mid tM = M\}, \mathcal{F} = \{M \mid tM = 0\}\). Additionally, we can define a functor \(u : \text{mod gr } \Lambda \to \text{mod gr } \Lambda\) mapping \(M \in \text{mod gr } \Lambda\) to \(M/tM\), the largest factor module of \(M\) belonging to \(\mathcal{F}\).

Since not all subcategories we consider in later sections are extension-closed, we formulate the results of this section for more general subcategories \(\mathcal{T}, \mathcal{F}\) whenever possible.

We will need the following lemma and a dual version which were proved for \(\text{mod } \Lambda\) in [22 Lemma 2, Lemma 3]. They can be translated to our setting using [13, Lemma 1.3]. We say that \(M \in \mathcal{T}\) is Ext-projective resp. Ext-injective in \(\mathcal{T}\) if \(\text{Ext}_{\text{mod gr } \Lambda}^1(M, -)|_{\mathcal{T}} = 0\) resp. \(\text{Ext}_{\text{mod gr } \Lambda}^1(-, M)|_{\mathcal{T}} = 0\).

\textbf{Lemma 2.1.} Let \((\mathcal{T}, \mathcal{F})\) be a torsion pair in \(\text{mod gr } \Lambda\) and \(V \in \mathcal{T}\) be indecomposable.

(1) The module \(V\) is Ext-projective in \(\mathcal{T}\) iff \(\tau_{\text{mod gr } \Lambda}(V) \in \mathcal{F}\).

(2) Suppose \(V\) is not Ext-projective. Then \(t(\tau_{\text{mod gr } \Lambda}(V))\) is indecomposable and if \(0 \to \tau_{\text{mod gr } \Lambda}(V) \xrightarrow{f} E \xrightarrow{g} V \to 0\) is the almost split sequence in \(\text{mod gr } \Lambda\) ending in \(V\), then the induced sequence \(0 \to t(\tau_{\text{mod gr } \Lambda}(V)) \to t(E) \to V \to 0\) is the almost split sequence in \(\mathcal{T}\) ending in \(V\).

\textbf{Proof.} (1) Let \(V \in \mathcal{T}\) be Ext-projective and let \(J = \text{supp}(V) \cup \text{supp}(\tau_{\text{mod gr } \Lambda}(V))\). Then \(J\) is finite and \((\mathcal{T} \cap \text{mod } \Lambda, \mathcal{F} \cap \text{mod } \Lambda)\) is a torsion pair in \(\text{mod } \Lambda\). As \(V\) is Ext-projective in \(\mathcal{T} \cap \text{mod } \Lambda\), a consecutive application of [13 and 22 Lemma 2] implies \(\tau_{\text{mod gr } \Lambda}(V) = \tau_{\text{mod } \Lambda}(V) \in \mathcal{F} \cap \text{mod } \Lambda\).
For the other direction, let \( \tau_{\text{mod gr } \Lambda}(V) \in \mathcal{F} \) and \( W \in \mathcal{T} \). Let \( J = \text{supp}(V) \cup \text{supp}(\tau_{\text{mod gr } \Lambda}(V)) \cup \text{supp}(W) \). Then all extensions of \( V \) by \( W \) belong to \( \text{mod}_J \text{gr } \Lambda \) and \( \tau_{\text{mod gr } \Lambda}(V) \in \mathcal{F} \cap \text{mod}_J \text{gr } \Lambda \). Thus, \cite{22} Lemma 2] yields \( \text{Ext}^1_{\text{mod gr } \Lambda}(V,W) = 0 \), so that \( V \) is \( \text{Ext} \)-projective in \( \mathcal{T} \).

(2) Let \( X \in \mathcal{T} \) and \( \alpha : X \to V \) not a split epimorphism. Set \( J = \text{supp}(\tau_{\text{mod gr } \Lambda}(V)) \cup \text{supp}(V) \cup \text{supp}(X) \). Then \( 0 \to t(\tau_{\text{mod gr } \Lambda}(V)) \to t(E) \to V \to 0 \) is the image of \( 0 \to \tau_{\text{mod gr } \Lambda}(V) \to E \to V \to 0 \) under the torsion radical of \( \mathcal{T} \cap \text{mod}_J \text{gr } \Lambda \), so that the sequence is almost split in \( \mathcal{T} \cap \text{mod}_J \text{gr } \Lambda \) by \cite{22} Lemma 2. As \( \alpha \) is not a split epimorphism in \( \mathcal{T} \cap \text{mod}_J \text{gr } \Lambda \), \( \alpha \) factors through \( t(g) \), so that \( t(g) \) is right almost split in \( \mathcal{T} \). Analogously, one shows that \( t(f) \) is left almost split in \( \mathcal{T} \), so that the sequence is almost split in \( \mathcal{T} \).

If \( \mathcal{T} \) is closed with respect to submodules and factor modules, we can define functors \( t, u : \text{mod gr } \Lambda \to \text{mod gr } \Lambda \) by letting \( t(V) \) be the largest submodule and \( u(V) \) be the largest factor module in \( \mathcal{T} \) of \( V \in \text{mod gr } \Lambda \). Then standard arguments show that \( t \) is right adjoint to the inclusion functor \( \mathcal{T} \to \text{mod gr } \Lambda \), so that \( t \) is left exact and maps injectives in \( \text{mod gr } \Lambda \) to injective objects in \( \mathcal{T} \), while \( u \) is left adjoint to the inclusion functor, so that \( u \) is right exact and maps projectives in \( \text{mod gr } \Lambda \) to projective objects in \( \mathcal{T} \). Thus, \( \mathcal{T} \) has enough projectives and enough injectives in this case and the notions of \( \text{Ext} \)-projective resp. \( \text{-injective and projective resp. injective} \) object in \( \mathcal{T} \) coincide. The notions of top, socle and radical in \( \text{mod gr } \Lambda \) and \( \mathcal{T} \) also coincide in this case and \( t \) resp. \( u \) maps the injective envelope resp. projective cover of \( V \in \text{mod gr } \Lambda \) to the injective envelope of \( t(V) \) resp. projective cover of \( u(V) \) in \( \mathcal{T} \). We call \( V \in \text{mod gr } \Lambda \) \( t \)-acyclic resp. \( u \)-acyclic if the highest right derived functors of \( t \) resp. left derived functors of \( u \) vanish on \( V \). If \( M \in \text{mod gr } \Lambda \) is \( t \)-acyclic, we can compute minimal injective resolutions and Heller shifts in \( \mathcal{T} \) from those in \( \text{mod gr } \Lambda \).

**Proposition 2.2.** Suppose that \( \mathcal{T} \) is closed with respect to submodules and factor modules. Let \( V \in \mathcal{T} \) be \( t \)-acyclic.

1. If \( 0 \to V \to I_0 \to I_1 \to \ldots \) is a minimal injective resolution of \( V \) in \( \text{mod gr } \Lambda \), then \( 0 \to V \to t(I_0) \to t(I_1) \to \ldots \) is a minimal injective resolution of \( V \) in \( \mathcal{T} \).

2. We have \( \Omega^{-i}_T(V) = t(\Omega^{-i}_{\text{mod gr } \Lambda}(V)) \) for all \( i \in \mathbb{N}_0 \).

**Proof.** This can be proved with the same arguments as \cite{7} Theorem 7.3], noting that the authors use positive superscripts instead.

We leave it to the reader to formulate a dual version about projective resolutions and positive Heller shifts for a \( u \)-acyclic \( V \in \mathcal{T} \). Recall that the injective resp. projective dimension of a module \( V \) is the length of a minimal injective resp. projective resolution of \( V \), i.e. the smallest natural number \( i \) such that \( \Omega^{-i+1}(V) = 0 \) resp. \( \Omega^{i+1}(V) = 0 \). We write \( id_T(V) \) resp. \( pd_T(V) \) for the injective resp. projective dimension of \( V \in \mathcal{T} \).
Corollary 2.3. Suppose $T$ is closed with respect to submodules and factor modules. Let $M \in T$ be $t$-acyclic such that $s = \text{id}_T(M) < \infty$. Then $T \cap \{\Omega_{\text{mod gr}\Lambda}^{-i}(M) \mid i \in \mathbb{N}_0\}$ is finite and all elements of this set have finite injective dimension in $T$.

Proof. By Corollary 2.2 (2), we have $t(\Omega_{\text{mod gr}\Lambda}^{-\ell}(M)) = 0$ for $\ell > s$, so that $\Omega_{\text{mod gr}\Lambda}^{\ell}(M) \notin T$ or $\Omega_{\text{mod gr}\Lambda}^{-\ell}(M) = 0$. Since minimal injective resolutions for $M$ in $T$ induce minimal injective resolutions for $t(M)$ in $T$, this also shows that all modules in $T \cap \{\Omega_{\text{mod gr}\Lambda}^{-i}(M) \mid i \in \mathbb{N}_0\}$ have finite injective dimension in $T$.

In the remainder of this section, let $\Theta$ be a regular component of type $\mathbb{Z}[A_\infty]$ of the Auslander-Reiten quiver $\Gamma(\text{mod gr}\Lambda)$ of $\text{mod gr}\Lambda$. By definition, $\Theta$ does not have any projective vertices and consequently, the arrows of $\Theta$ pointing downwards correspond to irreducible epimorphisms and the arrows pointing upwards correspond to irreducible monomorphisms. The next two results determine the position of Ext-projective modules in $T$ inside $\Theta$. Recall that $V \in \Theta$ is called quasi-simple if it belongs to the bottom layer of $\Theta$. In our context, this means that the middle term of the almost split sequence starting in $V$ is indecomposable. For each $M \in \Theta$, there is a unique quasi-simple module $N \in \Theta$ such that $N \subseteq M$, the quasi-socle of $M$. A path inside $\Theta$ is a sectional path if no vertex on the path is a $\tau_{\text{mod gr}\Lambda}$-shift of another vertex on the path. For a regular component of type $\mathbb{Z}[A_\infty]$, this is equivalent to all arrows being surjective or all arrows being injective. We denote by $ql(M)$ the quasi-length of $M \in \Theta$, that is the number of vertices on a sectional path from the quasi-socle of $M$ to $M$. Thus, if $M$ is quasi-simple, we have $ql(M) = 1$.

Proposition 2.4. Suppose $T$ is a torsion class and closed with respect to submodules. Let $M \in \Theta \cap T$ such that $M$ is Ext-projective in $T$ and $N$ be the quasi-socle of $M$. Then all modules on the sectional path from $N$ to $M$ are Ext-projective in $T$. If $ql(M) > 1$, then $N$ is simple.

Proof. By Proposition 2.1, $\tau_{\text{mod gr}\Lambda}(M)$ has no nontrivial $T$-submodules. If $M'$ is a predecessor of $M$ on the sectional path from $N$ to $M$, then $\tau_{\text{mod gr}\Lambda}(M')$ has no nontrivial $T$-submodules as it embeds into $\tau_{\text{mod gr}\Lambda}(M)$. By Proposition 2.1, $M'$ is Ext-projective in $T$. Now let $ql(M) > 1$. Then there is an almost split sequence in $T$ starting in $N$ such that the middle term of the sequence is Ext-projective in $T$. We use arguments dual to those of [2, V.3.3] to show that $N$ is simple. Let $B$ be the middle term and $C$ the right term of the almost split sequence. Without loss of generality, we may assume that $N$ is a submodule of $B$. Since the sequence is almost split, $N$ is a proper submodule of $B$. As $B$ is projective indecomposable in $T$, this implies $N \subseteq \text{Rad}(B)$. Thus, the sequence $0 \to \text{Top}(N) \to \text{Top}(B) \to \text{Top}(C) \to 0$ is not exact. Hence a dual version of [2, V.3.2] shows that $N$ is simple.

Corollary 2.5. Suppose $T$ is a torsion class and closed with respect to submodules. Let $M \in \Theta \cap T$ such that $M$ is Ext-projective in $T$. Suppose there is a duality $D : T \to T$ such that $D(S) \cong S$ for every simple module $S \in T$. Then $M$ is quasi-simple.
Proof. If \( ql(M) > 1 \), \( \text{2.4} \) provides a simple module \( S \in \mathcal{T} \) which is \( \text{Ext} \)-projective in \( \mathcal{T} \). Since \( D(S) \cong S \), the remarks preceding \( \text{2.4} \) yield that \( S \) is also \( \text{Ext} \)-injective in \( \mathcal{T} \), so the almost split sequence starting in \( S \) splits, a contradiction.

\[ \textbf{Lemma 2.6.} \quad \text{Let } \mathcal{T} \text{ be closed with respect to submodules and factor modules. Suppose the number of quasi-simple modules in } \Theta \cap \mathcal{T} \text{ is finite. Then } \Theta \cap \mathcal{T} \text{ is finite.} \]

\[ \text{Proof. Since } \mathcal{T} \text{ is closed with respect to taking submodules and factor modules, our assumption implies that the number of modules of any given quasi-length in } \Theta \cap \mathcal{T} \text{ is finite and that the quasi-length of modules in } \Theta \cap \mathcal{T} \text{ is bounded. Hence } \Theta \cap \mathcal{T} \text{ is finite.} \]

For \( M \in \Theta \) of quasi-length \( m \), we denote the quasi-socle of \( M \) by \( M(1) \) and the modules on the sectional path from \( M(1) \) to \( M \) by \( M(1), M(2), \ldots, M(m) = M \). We denote by \( \mathcal{W}(M) \) the wing of \( M \), namely the mesh-complete full subquiver of \( M \) containing the vertices \( \tau_{\text{mod gr } \Lambda}^s(M(s)) \) with \( 1 \leq s \leq m \) and \( 0 \leq r \leq m - s \), see \( \text{[28] 3.3} \) and the following figure.

\[ \text{Figure 1: The red vertices in this } \mathbb{Z}[A_{\infty}] \text{-component form the wing of the top red vertex } M. \]

\[ \textbf{Lemma 2.7.} \quad \text{Suppose } \mathcal{T} \text{ is closed with respect to submodules and factor modules and let } M \in \Theta \cap \mathcal{T} \text{. Then every module in the wing of } M \text{ belongs to } \mathcal{T}. \]

\[ \text{Proof. We show this by induction on the quasi-length } ql(M) \text{ of } M. \text{ If } ql(M) = 1, \text{ i.e. } M \text{ is quasi-simple, the statement is clear. Now let } m = ql(M) > 1. \text{ Since } \mathcal{T} \text{ is closed with respect to submodules and factor modules, } M(m - 1), \tau_{\text{mod gr } \Lambda}^{-1}(M(m - 1)) \in \mathcal{T}. \text{ As the remaining vertices in } \mathcal{W}(M) \text{ belong to } \mathcal{W}(M(m - 1)) \cup \mathcal{W}(\tau_{\text{mod gr } \Lambda}^{-1}(M(m - 1))), \text{ the result now follows by induction.} \]

The assumptions on \( M \) in the following lemma mean that two wings in \( \mathcal{T} \cap \Theta \) have adjacent quasi-simple modules. We use this to show that the modules between the wings also belong to \( \Theta \) and construct a module \( N \in \mathcal{T} \cap \Theta \) such that the wing of \( N \) contains the wing of \( M \) and \( \tau_{\text{mod gr } \Lambda}^*(M) \).
Lemma 2.8. Suppose $\mathcal{T}$ is a torsion class closed with respect to submodules. Let $s \in \mathbb{N}$ and $M \in \Theta \cap \mathcal{T}$ such that $2ql(M) > s$ and $\tau^s_{\text{mod gr } \Lambda}(M) \in \mathcal{T}$. Then $\tau^i_{\text{mod gr } \Lambda}(M) \in \mathcal{T}$ for $1 \leq i \leq s$ and there exists a module $N \in \Theta \cap \mathcal{T}$ such that $ql(N) = ql(M) + s$.

Proof. As $ql(M) = ql(\tau^s_{\text{mod gr } \Lambda}(M)) > \frac{s}{2}$, the wings of $\tau^s_{\text{mod gr } \Lambda}(M)$ and $M$ either intersect or the rightmost quasi-simple module $Y_1$ in the wing of $\tau^s_{\text{mod gr } \Lambda}(M)$ and the leftmost quasi-simple module $Y_2$ in the wing of $M$ satisfy $\tau_{\text{mod gr } \Lambda}(Y_2) = Y_1$. As $\mathcal{T}$ is extension-closed, this shows that $\tau^s_{\text{mod gr } \Lambda}(M) \in \mathcal{T}$ for $1 \leq i \leq s$.

Using again that $\mathcal{T}$ is extension-closed, we get that in the layer above a horizontal line $A, \tau_{\text{mod gr } \Lambda}(A), \ldots, \tau_{\text{mod gr } \Lambda}(A)$ of $(l + 1)$ modules in $\mathcal{T}$, there is a horizontal line of the same form of $l$ modules in $\mathcal{T}$. As the horizontal line in our situation has at least $(s + 1)$ modules in $\mathcal{T}$, the existence of $N$ now follows inductively.

Recall that a homogeneous tube is a component isomorphic to $\mathbb{Z}[A_\infty]/\langle \tau \rangle$. If $\Theta$ becomes a homogeneous tube upon forgetting the grading, we can determine when the almost split sequence in $\mathcal{T}$ ending in $N \in \mathcal{T} \cap \Theta$ is also almost split in $\text{mod gr } \Lambda$.

Proposition 2.9. Let $F : \text{mod gr } \Lambda \to \text{mod } \Lambda$ be the forgetful functor and $\mathcal{T}$ be a torsion class. Suppose that $F(\Theta)$ is a homogeneous tube and $N \in \Theta \cap \mathcal{T}$ is not Ext-projective in $\mathcal{T}$. Let

$$(0) \to V \to E \to N \to (0)$$

be the almost split sequence in $\mathcal{T}$ ending in $N$. Then the sequence is almost split in $\text{mod gr } \Lambda$ if and only if $l(V) = l(N)$ if and only if $\dim_k V = \dim_k N$.

Proof. By 2.1, $V$ is a submodule of the left term of the almost split sequence in $\text{mod gr } \Lambda$ ending in $V$. Since $F(\Theta)$ is a homogeneous tube, the left term of that sequence has the same length and dimension as $N$ by 1.2.

We say that $V,W \in \Theta$ belong to the same column if there are almost split sequences $\xi_1, \ldots, \xi_l$ such that $V$ is a non-projective direct summand of the middle term of $\xi_1$, $W$ is a non-projective direct summand of the middle term of $\xi_l$ and for each $1 \leq i \leq l - 1$, the middle terms of $\xi_i$ and $\xi_{i+1}$ have a common non-projective direct summand. This defines an equivalence relation on $\Theta$. We call the equivalence classes the columns of $\Theta$.

Proposition 2.10. Suppose there is a a duality $(-)^o : \text{mod gr } \Lambda \to \text{mod gr } \Lambda$. If $\Theta$ contains a module $V$ such that $V^o \cong V$, then $N^o \cong N$ for all $N$ in the column of $V$. The column of $V$ is a symmetry axis with respect to $(-)^o$.

Proof. As $(-)^o$ is a duality, $\Theta^o$ is a component of $\Gamma(\text{mod gr } \Lambda)$ and $\Theta \cap \Theta^o = \emptyset$ or $\Theta = \Theta^o$. Thus, $V \in \Theta \cap \Theta^o$ implies $\Theta = \Theta^o$. Since $(-)^o$ does not change the quasi-length of modules, any almost split sequence in $\Theta$ such that the middle term has a self-dual summand is mapped to itself by $(-)^o$ and every direct summand of a middle term of such a sequence is self-dual.
3 Representations of infinitesimal Schur algebras

In this section, we will introduce our main objects of study and show how they fit into the framework of the previous section. For algebraic groups, we follow the notation and terminology of [23]. For an introduction to algebraic monoids, we refer the reader to [27].

Let $k$ be an algebraically closed field of characteristic $p > 0$ and $\text{Mat}_n$ be the monoid scheme of $(n \times n)$-matrices over $k$. Let $d \in \mathbb{N}_0$ and denote by $A(n, d)$ the space generated by all homogeneous polynomials of degree $d$ in the coordinate ring $k[\text{Mat}_n] = k[X_{ij} \mid 1 \leq i, j \leq n]$ of $\text{Mat}_n$. Then $k[\text{Mat}_n] = \bigoplus_{d \geq 0} A(n, d)$ is a graded $k$-bialgebra with comultiplication $\Delta : k[\text{Mat}_n] \to k[\text{Mat}_n] \otimes_k k[\text{Mat}_n]$ and counit $\epsilon : k[\text{Mat}_n] \to k$ given by $\Delta(X_{ij}) = \sum_{l=1}^n X_{il} \otimes_k X_{lj}$, $\epsilon(X_{ij}) = \delta_{ij}$ and each $A(n, d)$ is a finite-dimensional subcoalgebra. As $\text{GL}_n$ is dense in $\text{Mat}_n$, the canonical map $k[\text{Mat}_n] \to k[\text{GL}_n]$ is injective, so that $k[\text{Mat}_n]$ and each $A(n, d)$ can be viewed as a subcoalgebra of $k[\text{GL}_n]$. Now let $G$ be a closed subgroup scheme of $\text{GL}_n$ and $\pi : k[\text{GL}_n] \to k[G]$ the canonical projection. Set $A(G) = \pi(k[\text{Mat}_n])$ and $A_d(G) = \pi(A(n, d))$ as well as $S_d(G) = A_d(G)^*$. Since $\pi$ is a homomorphism of Hopf algebras, $A(G)$ is a subbialgebra of $k[G]$ and each $A_d(G)$ is a subcoalgebra of $A(G)$. Thus, $S_d(G)$ is a finite-dimensional associative algebra in a natural way. Following [5], we say that a rational $G$-module $V$ is a polynomial $G$-module if the corresponding comodule map $V \to V \otimes_k k[G]$ factors through $V \otimes_k A(G)$. If the comodule map factors through $V \otimes_k A_d(G)$ for some $d \in \mathbb{N}_0$, we say that $V$ is homogeneous of degree $d$. Clearly, every $A_d(G)$-comodule is an $A(G)$-comodule and every $A(G)$-comodule is a $G$-module in a natural way. As $A(G)$ is a factor bialgebra of $k[\text{Mat}_n]$, it corresponds to a closed submonoid $M$ of $\text{Mat}_n$. Since $A(G) \subseteq k[G]$ is a subbialgebra, $G \subseteq M$ is a dense subscheme, so that $M$ is the closure of $G$ in $\text{Mat}_n$. Hence polynomial representations of $G$ can be regarded as rational representations of the algebraic monoid scheme $M = \overline{G}$. Note that all these notions depend on the given embedding $G \to \text{GL}_n$.

We are mainly interested in the algebras $S_d(G, T)$ and $S_d(B, T)$, where $G = \text{GL}_n$, $T \subseteq G$ is the maximal torus of diagonal matrices and $B \subseteq G$ is the Borel subgroup of upper triangular matrices. The algebra $S_d(G, T)$ is the infinitesimal Schur algebra introduced in [7] and $S_d(B, T)$ can be viewed as an infinitesimal analogue of the Borel-Schur algebra $S_d(B)$ introduced in [21]. Note that all results about $B, T$ in this paper can also be proved for the Borel subgroup of lower triangular matrices instead. If $D = \overline{T} \subseteq \text{Mat}_n$ is the monoid scheme of diagonal matrices, a comparison of coordinate rings shows $\overline{G_T} = M_T D$, where $M_T D = (F_{\text{Mat}_n}^r)^{-1}(D)$ with $F_{\text{Mat}_n}^r$ being the $r$-th iteration of the Frobenius homomorphism on $\text{Mat}_n$, see [7]. By the same token, we get for $L = \overline{B} \subseteq \text{Mat}_n$ the monoid scheme of lower triangular matrices that $\overline{B_T} = L_T D$, where $L_T D = (F_{L_T}^r)^{-1}(D)$ with $F_{L_T}^r$ the restriction of $F_{\text{Mat}_n}^r$ to $L$. We have $k[M_T D] = k[X_{ij} \mid 1 \leq i, j \leq n]/(X_{ij}^{p^n} \mid i \neq j)$ and $k[L_T D] = k[X_{ij} \mid 1 \leq i \leq j \leq n]/(X_{ij}^{p^n} \mid i \neq j)$.

From now on, we fix an embedding $G \subseteq \text{GL}_n$ of a smooth connected algebraic group scheme and let $T \subseteq G$ be a maximal torus. A routine verification shows
Lemma 3.1. The category of polynomial $G,T$-modules is closed with respect to submodules, factor modules and finite direct sums.

For $\mathcal{T} = \text{mod} \overline{G,T}$, we let $\mathcal{G}_{\mathcal{T}} = t, \mathcal{S}_{\mathcal{T}} = u$ be the functors defined above 2.2. If the center of $GL_n$ is contained in $G$, we can decompose every polynomial $G,T$-module into homogeneous constituents.

Proposition 3.2. Suppose that $Z(GL_n) \subseteq G$. Let $V \in \text{mod} \overline{G,T}$. Then there is a decomposition $V = \bigoplus_{d \in \mathbb{N}_0} V_d$ such that $V_d$ is an $S_d(G,T)$-comodule for all $d \in \mathbb{N}_0$.

Proof. This follows from [5, 1.3, 1.5].

From now on, we always assume $Z(GL_n) \subseteq G$. As any $S_d(G,T)$-comodule is also a $\overline{G,T}$-comodule, we get

Corollary 3.3. If $V \in \text{mod} \overline{G,T}$ is indecomposable, then $V = V_d$ for some $d \in \mathbb{N}_0$.

The two preceding results show that the blocks of mod $\overline{G,T}$ are just the blocks of the $S_d(G,T)$ for $d \in \mathbb{N}_0$. Thus, injective and projective indecomposables as well as almost split sequences in mod $S_d(G,T)$ coincide with those in mod $\overline{G,T}$. In particular, mod $\overline{G,T}$ has almost split sequences.

As $Z(GL_n) \subseteq G$ is a torus contained in the center of $G,T$, the $Z(GL_n)$-weight spaces of a $G,T$-module $V$ are $G,T$-submodules of $V$. This shows that $Z(GL_n)$ acts via a single character on indecomposable $G,T$-modules $V$ and that indecomposable modules affording different characters belong to different blocks. If $V$ is a module for $\overline{G,T}$, these submodules are just the homogeneous components of 3.2. We say that $V \in \text{mod} G,T$ is homogeneous of degree $d$ if $Z(GL_n)$ acts on $V$ via the character given by $d \in \mathbb{Z}$. If $V = V_d, W = W_{d'} \in \text{mod} G,T$ are homogeneous such that $d \neq d'$, then any exact sequence starting in $V$ and ending in $W$ splits as the middle term is the direct sum of its homogeneous constituents. As almost split sequences are non-split, we get

Proposition 3.4. Let $\Theta$ be a component of the stable Auslander-Reiten quiver of $\text{mod} G,T$. Then there is $d \in \mathbb{Z}$ such that all modules in $\Theta$ are homogeneous of degree $d$.

For $G \neq GL_n$ with maximal torus $T \subseteq G$, it is not known whether a $G,T$-module $V$ such that all $T$-weights of $V$ are polynomial is a polynomial $G,T$-module. However, for $G = GL_n$, this was proved by Jantzen in [25, Appendix]. We can also prove it for $B,T$, where $B \subseteq GL_n$ is the Borel subgroup of lower triangular matrices. In the remainder of this section, let $G = GL_n, T \subseteq GL_n$ the torus of diagonal matrices, $B \subseteq GL_n$ the Borel subgroup of lower triangular matrices, $U \subseteq B$ the subgroup of unipotent lower triangular matrices and $X \in \{ G,T, B,T \}$, $D \subseteq \text{Mat}_n$ the monoid scheme of diagonal matrices and $X(D)$ the character monoid of $D$. We identify $X(D)$ with a submonoid of $X(T)$ and call this submonoid the set of polynomial weights.

Theorem 3.5. If $V$ is an $X$-module such that all $T$-weights of $V$ are polynomial, then $V$ lifts to $\overline{X}$.
Proof. For $X = G_r T$, this is Jantzens result in [25] Appendix. Let $X = B_r T = U_r \times T$, $\rho : V \to V \otimes_l k[B_r T]$ be the comodule map and $v \in V$. Since all $T$-weights of $V$ are polynomial, $V|_T$ lifts to $D$, so there are $f_i \in k[D], v_i \in V$ such that

$$t.(v \otimes 1) = \sum_{i=1}^s v_i \otimes f_i(t)$$

for all $t \in T(A)$ and all commutative $k$-algebras $A$. Since $B_r T = U_r T = U_r \times T$, there are $g_{ij} \in k[U_r], v_{ij} \in V$ such that

$$(ut).(v \otimes 1) = u.(t.(v \otimes 1)) = \sum_{i=1}^s \sum_{j=1}^l v_{ij} \otimes g_{ij}(u)f_i(t)$$

for all $t \in T(A), u \in U_r(A)$ and all commutative $k$-algebras $A$. Applying this to $A = k[B_r T]$ and $id_A \in B_r T(A)$, we see that

$$\rho(v) = \sum_{i=1}^s \sum_{j=1}^l v_{ij} \otimes g_{ij} \otimes f_i \in V \otimes_k k[U_r] \otimes_k k[D] = V \otimes_k k[L_r D],$$

so that $V$ lifts to $L_r D$. \hfill $\Box$

Viewing $X$-modules as $\mathbb{Z}^n$-graded $G_r$ resp. $B_r$-modules, the theorem shows that $V \in \text{mod } \overline{X}$ if and only if $V \in \text{mod } X$ and $\text{supp}(V) \subseteq \mathbb{N}_0^n$.

Corollary 3.6. The category $\text{mod } \overline{X}$ is extension-closed in $\text{mod } X$.

Thus, $\text{mod } \overline{X}$ is a torsion class and a torsion-free class, so that all results of the previous section can be applied to $\text{mod } \overline{X}$ and we have $t = \overline{\delta}^X, u = G^X$ for the torsion radical $t$ and the functor $u(V) = V/t(V)$.

The transposition map $G_r T \to G_r T$ is an anti-automorphism inducing a duality $(-)^o : \text{mod } G_r T \to \text{mod } G_r T$ which restricts to a duality of $\text{mod } M_r D$ and $\text{mod } S_d(G_r T)$ for every $d \in \mathbb{N}_0$, see [8, 2.1]. Letting $B^{-} \subseteq \text{GL}_n$ be the Borel subgroup of lower triangular matrices and defining $L^{-}_r D$ as above, transposition is an anti-isomorphism $B_r T \to B^{-}_r T$ inducing anti-equivalences $\text{mod } B_r T \to \text{mod } B^{-}_r T$ and $\text{mod } L_r D \to \text{mod } L^{-}_r D$. Following [8], we call these dualities contravariant duality.

Remark. Analogues of this duality can be constructed for some other reductive subgroups $G \subseteq \text{GL}_n$ by using [27, 5.2] to extend the anti-automorphism of [23, II.1.16] to the closure of $G$ in $\text{Mat}_n$.

Applying the canonical isomorphism $(V^o)^o \cong V$ to the submodule $(G^X(V^o))^o$ of $(V^o)^o$, we get the following connection between $\overline{\delta}_X$ and $G^X$.

**Proposition 3.7.** There is an isomorphism $\overline{\delta}_X(V) \cong (G^X(V^o))^o$ natural in $V \in \text{mod } X$, where we abuse notation for $X = B_r T$ by letting $G^X(V^o)$ denote the largest $L^{-}_r D$-factor module of $V^o$. 

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In the remainder of this section, let $\Theta$ be a regular component of type $\mathbb{Z}[A_{\infty}]$ of the stable Auslander-Reiten quiver $\Gamma_s(G,T)$ of $G,T$. By \cite[II.9.6(13)]{23}, we have $\widetilde{L}_r(\lambda) \cong L_r(\lambda)$ for all $\lambda \in X(T)$ and the simple module $\widetilde{L}_r(\lambda)$. We can use this to determine the position of indecomposable Ext-projective modules $V \in \text{mod } M,D$ in $\Theta$.

**Proposition 3.8.** Let $V \in \Theta \cap \text{mod } M,D$ be Ext-projective in $\text{mod } M,D$. Then $V$ is quasi-simple and lies at the left side of a wing in $\Theta \cap \text{mod } M,D$.

**Proof.** This follows directly from \cite[2.1, 2.5 and 2.7]{2}.

Since $\text{mod } G,T$ is a Frobenius category, there are no non-projective $G,T$-modules of finite projective dimension. However, there are $M,D$-modules with finite projective dimension.

**Corollary 3.9.** Let $V \in \Theta \cap \text{mod } M,D$ and $P$ be the projective cover of $V$ in $\text{mod } G,T$. Suppose $P \in \text{mod } M,D$ and $\text{pd}_{M,D}(V) = 1$. Then $V$ is quasi-simple and $\tau_{G,T}(V) \notin \text{mod } M,D$.

**Proof.** We have $\tau_{G,T}(V) = \Omega^2_{G,T}(V)$ by \cite[7.2.3]{16}. Since $\text{pd}_{M,D}(V) = 1$ and $P \in M,D$, we have $\Omega^1_{G,T}(V) = \Omega^1_{M,D}(V)$ projective in $\text{mod } M,D$. In particular, the projective cover of $\Omega^1_{G,T}(V)$ in $\text{mod } G,T$ is not an $M,D$-module, so that $\text{pd}_{M,D}(V) = 3$. As $\Omega^2_{G,T}(V) \cong V$ by \cite[p. 338]{2}, \cite[3.8]{3} implies that $\Omega^2_{G,T}(V)$ is quasi-simple. Hence $V$ is also quasi-simple.

**Proposition 3.10.** Suppose $V \in \text{mod } M,D$ is indecomposable and $\mathcal{G}_{M,D}$-acyclic such that $\text{pd}_{M,D}(V) < \infty$. Then there are only finitely many $M,D$-modules in $\{\tau^i_{G,T}(V) \mid i \in \mathbb{N}_0\}$ and they all have finite projective dimension.

**Proof.** By \cite[7.2.3]{16}, we have $\tau_{G,T} \cong \Omega^2_{G,T}$. Setting $s = \text{pd}_{M,D}(V)$, a dual version of \cite[2.2]{2} shows that $\mathcal{G}_{M,D}(\tau^i_{G,T}(V)) = \mathcal{G}_{M,D}(\Omega^2_{G,T}(V)) = 0$ for $l > \frac{s}{2}$, so that $\tau^i_{G,T}(V) \notin \text{mod } M,D$. Since minimal projective resolutions for $V$ in $\text{mod } G,T$ induce minimal projective resolutions for $\mathcal{G}_{M,D}(V)$ in $\text{mod } M,D$, this also shows that all $M,D$-modules in $\{\tau^i_{G,T}(V) \mid i \in \mathbb{N}_0\}$ have finite projective dimension.

**Corollary 3.11.** Suppose $\Theta$ contains a $\mathcal{G}_{M,D}$-acyclic $M,D$-module $V$ which is quasi-simple such that $\text{pd}_{M,D}(V) < \infty$ and a self-dual module $N$. Then $\Theta$ contains only finitely many $M,D$-modules.

**Proof.** Since $X \in \text{mod } M,D$ iff $X^o \in \text{mod } M,D$ for all $X \in \text{mod } G,T$, this follows from \cite[3.10 and 2.10]{2}.

**Proposition 3.12.** Let $V \in \text{mod } M,D$ be $\mathcal{G}_{M,D}$-acyclic and indecomposable such that $\text{pd}_{M,D}(V) \leq 2n$ and $\tau^o_{G,T}(V) \in \text{mod } M,D$. Then $\tau^o_{G,T}(V)$ is projective in $\text{mod } M,D$. If moreover $V \in \Theta$, then $V$ is quasi-simple.
Proof. Let \( \ldots \to P_1 \to P_0 \to V \to 0 \) be a minimal projective resolution in \( \text{mod} \, G, T \). As \( V \) is \( \mathcal{G}_{M,D} \)-acyclic, a dual version of \( \mathbf{2.2} \) yields a minimal projective resolution \( \ldots \to \mathcal{G}_{M,D}(P_1) \to \mathcal{G}_{M,D}(P_0) \to V \to 0 \) in \( \text{mod} \, M, D \). Since \( \text{pd}_{M,D}(V) \leq 2n \), we get \( \mathcal{G}_{M,D}(P_{2n+1}) = 0 \). Now \( \mathcal{F}_{M,D}(\tau_{G,T}^{n+1}(V)) \cong \mathcal{F}_{M,D}(\mathcal{O}_{G,T}^{2n+2}(V)) \subseteq \mathcal{F}_{M,D}(P_{2n+1}) \cong \mathcal{G}_{M,D}(P_{2n+1})^0 = 0 \) by \( \mathbf{3.7} \) since \( P_{2n+1} \cong P_{2n+1} \). By \( \mathbf{2.1} \), \( \tau_{G,T}^n(V) \) is projective in \( \text{mod} \, M, D \). If \( V \in \Theta \), \( \mathbf{3.8} \) shows that \( V \) is quasi-simple. \( \square \)

In the next result, we adapt a standard Morita-equivalence between blocks of \( \text{mod} \, G, T \) to blocks of \( \text{mod} \, M, D \). Later we will also need that this equivalence is compatible with the restriction functor to \( (\text{SL}_n)_r \). We denote by \( St_s \cong \hat{Z}_s((p^n - 1)\rho) \) the \( s \)-th Steinberg module, where \( \rho \) is the half-sum of positive roots and we choose the root system of \( \text{GL}_n \) as in \( \mathbf{[23] \, 1.1.21} \). If \( V \in \text{mod} \, G, T \), we denote by \( V^{[s]} \) the module obtained by composing the module structure on \( V \) with the \( s \)-th iteration of the Frobenius morphism and by \( k_{1/2(p^n-1)(n-1)} \det_{|G,T}| \), the 1-dimensional module given by the character \( \frac{1}{2}(p^n - 1)(n - 1) \det_{|G,T}| \) of \( G, T \).

**Proposition 3.13.** Suppose \( p \geq 3 \). Let \( b \) be a block of \( M_{r-s}D \). The functor \( A : \text{mod} \, G_{r-s}T \to \text{mod} \, G, T, V \mapsto St_s \otimes_k V^{[s]} \otimes_k k_{1/2(p^n-1)(n-1)} \det_{|G,T}| \) commutes with the forgetful functors \( \text{mod} \, G_{r-s}T \to \text{mod} \, (\text{SL}_n)_{r-s} \), \( \text{mod} \, G, T \to \text{mod} \, (\text{SL}_n)_r \), and induces a Morita-equivalence from \( b \) to a block of \( M, D \).

**Proof.** We have \( \text{Ext}^1_{M,D}(V_1, V_2) \cong \text{Ext}^1_{G,T}(V_1, V_2) \) for all \( V_1, V_2 \in \text{mod} \, M, D \) by \( \mathbf{3.5} \), so there is a block \( b' \) of \( \text{mod} \, G_{r-s}T \) containing \( b \). By \( \mathbf{[23] \, II.10.5} \), \( V \mapsto St_s \otimes_k V^{[s]} \) induces a Morita-equivalence between \( b' \) and a block of \( G, T \). Since shifting with a character of \( G, T \) is an equivalence of categories, \( A \) also induces such an equivalence. We now show that \( V \in \text{mod} \, M_{r-s}D \) iff \( A(V) \in \text{mod} \, M, D \). Since \( St_s \cong \hat{Z}_s((p^n - 1)\rho) \), so that the set of weights of \( St_s \) is \( W \cong S_n \)-invariant by \( \mathbf{[23] \, II.9.16(1)} \) and \( -(n - 1) \) occurs as a coordinate of \( 2\rho \), we have that for every \( i \in \{1, \ldots, n\} \), there is a weight \( \lambda \) of \( St_s \) such that \( \lambda_i = -\frac{1}{2}(p^n - 1)(n - 1) \). Suppose there is a weight \( \mu \) of \( St_s \) with a smaller coordinate. Using the action of the Weyl group, we may assume that \( \mu_n < -\frac{1}{2}(p^n - 1)(n - 1) \). Then \( \mathbf{[23] \, II.9.2(6)} \) implies that \( (p^n - 1)\rho - \mu \) is a sum of positive roots. As \( (p^n - 1)\rho - (p^n - 1)\mu \) is a positive \( n \)-th coordinate, this is a contradiction. Thus, there is no weight \( \mu \) of \( St_s \) with \( \mu_n < -\frac{1}{2}(p^n - 1)(n - 1) \). As the grading on \( V^{[s]} \) arises by multiplying all degrees in the grading of \( V \) by \( p^n \), we get that \( V \) has a weight with a negative coordinate iff \( A(V) \) has a weight with a negative coordinate by the definition of the grading on a tensor product. Thus, \( V \in \text{mod} \, M_{r-s}D \) iff \( A(V) \in \text{mod} \, M, D \). Hence \( A \) maps the class of simple \( M_{r-s}D \)-modules of the \( G_{r-s}T \)-block containing \( b \) to the class of simple \( M, D \)-modules in the block of \( G, T \) containing \( A(b) \). As \( \text{Ext}^1_{M,D}(V_1, V_2) \cong \text{Ext}^1_{G,T}(V_1, V_2) \) for all \( V_1, V_2 \in \text{mod} \, M, D \), we see that \( V_1, V_2 \) belong to the same block of \( M_{r-s}D \) iff \( A(V_1), A(V_2) \) belong to the same block of \( M, D \), so that \( A \) induces a Morita-equivalence from \( b \) to a block of \( M, D \). Since \( \text{det} \) vanishes on \( \text{SL}_n \), the functor \( A \) commutes with the forgetful functors. \( \square \)
4 Modules of complexity one

In this section and the following sections, let \( p = \text{char}(k) \geq 3 \). Let \( G \subseteq \text{GL}_n \) be a reductive algebraic group over \( k \) with \( Z(\text{GL}_n) \subseteq G \), \( T \subseteq G \) a maximal torus, \( R \) the root system of \( G \) relative to \( T \) and \( B = U \rtimes T \subseteq G \) a Borel subgroup containing \( T \) with unipotent radical \( U \). Let \( R^+ \subseteq R \) be the set of positive roots determined by \( B \). Recall that the complexity \( cx(V) \) of a module \( V \) is the polynomial rate of growth of a minimal projective resolution of \( V \). Let \( X \in \{ G, T, B, T \} \) and \( \Theta \) be a component of \( \Gamma_s(X) \). By [13, Section 1] and [16, 8.1], the complexity of a module coincides with the dimension of its rank variety and we have \( cx_{G,T}(V_1) = cx_{G,T}(V_2) \) for all \( V_1, V_2 \in \Theta \), so that we may define \( cx_{G,T}(\Theta) = cx_{G,T}(V_1) \). As in the previous section, we denote by \( X \) the closure of \( X \) in \( \text{Mat}_n \). In this section, we will show that if \( cx_{G,T}(\Theta) = 1 \), then \( \Theta \cap \text{mod} X \) is finite.

**Lemma 4.1.** Let \( X \in \{ B, T, G, T \} \) and \( cx_X(\Theta) = 1 \).

1. If \( X = G, T \), there are \( \alpha \in R, 0 \leq s \leq r - 1 \) such that \( \tau_X^{p^s}(V) = V[p^s \alpha] \) for all \( V \in \Theta \).

2. If \( X = B, T \), there are \( \alpha \in R^+ \cup \{ 0 \}, 0 \leq s \leq r - 1 \) such that \( \tau_X^{p^s}(V) = V[p^s \alpha - 2p^s(p^s - 1)\rho] \) for all \( V \in \Theta \), where \( \rho = \frac{1}{2} \sum_{\beta \in R^+} \beta \) is the half-sum of positive roots.

**Proof.** (1) This follows directly from [16, 6.1.2, 7.2.3, 8.1.2], noting that the proof of [16, 8.1.2] only depends on the support variety and hence on the component of \( V \).

(2) Let \( X = B, T \). Then [16, 7.2.2] yields that \( \mathcal{N}(V) = V \otimes_k k_{\lambda_B|_{B,T}}[-p^s \lambda_B|_T] \) for the Nakayama functor \( \mathcal{N} \) of \( \text{mod} B, T \), where \( \lambda_B \) is the character of \( B \) given by \( \lambda_B = \det \circ \text{Ad} \). By the remarks preceding [16, 7.2.2], \( \lambda_B|_T = 2\rho \). As \( B, T \cong U_r \rtimes T \) with \( U_r \) unipotent, tensoring with \( k_{\lambda_B|_{B,T}} \) only changes the \( T \)-action on a module, so that \( V \otimes_k k_{\lambda_B|_{B,T}} \cong V[2\rho] \) and \( \mathcal{N}(V) = V[-2(p^s - 1)\rho] \). By [16, 6.1.2, 8.1.1], there exist \( \alpha \in R^+ \cup \{ 0 \} \) and \( s \in \mathbb{N}_0 \) such that \( \Omega^{p^s}_{B,T}(V) \cong V \otimes_k p^s \alpha \), so that \( \tau^{p^s}(V) \cong \mathcal{N}^{p^s}(V[p^s \alpha]) \cong V[p^s \alpha - 2p^s(p^s - 1)\rho] \).

**Lemma 4.2.** Let \( X \in \{ B, T, G, T \} \), \( V \) be an indecomposable \( X \)-module such that \( cx_X(V) = 1 \). Then there are only finitely many polynomial \( X \)-modules in the \( \tau_X \)-orbit of \( V \).

**Proof.** Let \( X = B, T \) and \( R^+ \subseteq R \) be the set of roots of \( B \). By [14, (2)], we have \( \tau^{p^s}_{B,T}(V) = V[p^s \alpha - 2p^s(p^s - 1)\rho] \) for some \( \alpha \in R^+ \cup \{ 0 \}, 0 \leq s \leq r - 1 \). It follows that

\[
\text{supp}(\tau^{ip^s}_{B,T}(V)) = \text{supp}(V) + ip^s \alpha - 2ip^s(p^s - 1)\rho
\]

for all \( i \in \mathbb{Z} \). Since \( p^s \alpha \neq 2p^s(p^s - 1)\rho \), we have \( \text{supp}(\tau^{ip^s}_{B,T}(V)) \neq \text{supp}(\tau^{jp^s}_{B,T}(V)) \) for \( i \neq j \). As \( Z(\text{GL}_n) \subseteq G \), the degree \( d \) weights of a maximal torus of \( \text{GL}_n \) containing \( T \) surject onto the degree \( d \) weights of \( T \). As all weights in the sets \( \text{supp}(\tau^{ip^s}_{B,T}(V)) \) have
degree $d$ and there are only finitely many polynomial weights of a given degree $d$ for $GL_n$, only finitely many of these sets consist entirely of polynomial weights. Thus, the $\tau_{B,T}^0$-orbit of $M$ contains only finitely many polynomial $B,T$-modules. As the $\tau_{B,T}$-orbit of $V$ is the union of the $\tau_{B,T}^i$-orbits of $V$, $\tau_{B,T}(V), \ldots, \tau_{B,T}^{p-1}(V)$, the result follows.

For $X = G_r,T$, apply \cite{1} (1), and use analogous arguments.

We denote by $F$ the forgetful functor $\text{mod } G_r,T \to \text{mod } G_r$ resp. $\text{mod } B,T \to \text{mod } U_r$.

**Proposition 4.3.** Let $X \in \{ B_r,T,G_r,T \}$ and $V$ be an indecomposable polynomial $X$-module such that $cx_X(V) = 1$ and $\Theta$ be the component of $\Gamma_s(X)$ containing $V$. Then $\Theta$ contains only finitely many polynomial $X$-modules.

**Proof.** We first show that $\Theta$ is either regular or has only finitely many $\tau_X$-orbits.

Let $F$ be the forgetful functor. If $X = G_r,T$, then $\Theta$ is regular; else there would be a projective indecomposable $G_r,T$-module $P$ such that $\text{Rad}(P) \in \Theta$ by \cite{2}. V.5.5, so that $cx_{G_r,T}(\text{Rad}(P)) = 1$ and the simple $G_r,T$-module $S = \Omega_{G_r,T}^{-1}(\text{Rad}(P))$ would also have complexity 1. But then $cx_{G_r,T}(F(S)) = 1$, a contradiction to \cite{17} Lemma 2.2]. If $X = B_r,T = U_r \rtimes T$ and $\Theta$ is a non-regular component of complexity 1, the above arguments show that there is a simple $B_r,T$-module $k_\lambda$ of complexity 1, so that $cx_{U_r}(F(k_\lambda)) = cx_{U_r}(k) = 1$. As the rank variety of $(U_r)_2$ is contained in that of $U_r$, we get $cx_{U_r}(k) \leq 1$ and \cite{18} Theorem 2.7 shows that $U_r$ has finite representation type. Thus, $F(\Theta)$ is finite, and \cite{3} 5.6 implies $F(\Theta) \cong \mathbb{Z}[A_p]((\tau))$ for some $l \in \mathbb{N}$, so that there are two distinct vertices in $F(\Theta)$ with only one successor and that all other vertices in $F(\Theta)$ have exactly two predecessors. Since $N \in \text{mod } B_r,T$ is projective iff $F(N) \in \text{mod } U_r$ is projective by \cite{1} it follows from \cite{2} that there are two distinct $\tau_{B,T}$-orbits in $\Theta$ such that every element of these orbits has exactly one predecessor and all other orbits consist of vertices with exactly two predecessors. Thus, $\Theta$ does not have tree class $A_\infty,A_\infty^\infty$ or $D_\infty$. Then \cite{17} 3.4 (which does not depend on the group being reductive) implies that $\Theta$ has only finitely many $\tau_{B,T}$-orbits.

Thus, $\Theta$ is either regular or has only finitely many $\tau_X$-orbits. In the second case, the result follows from \cite{4}. In the first case, \cite{3} 5.6 and \cite{16} 8.2.2 imply $\Theta \cong \mathbb{Z}[A_\infty]$. Letting $\Theta \cong \mathbb{Z}[A_\infty]$ and $N \in \Theta$ be quasi-simple, the set of quasi-simple modules in $\Theta$ is the $\tau_X$-orbit of $N$. Thus, \cite{4} shows that there are only finitely many quasi-simple polynomial $X$-modules in $\Theta$, so that the result follows from \cite{2}.6.

In the remainder of this section, let $G = GL_n$ with the notation of Section \cite{3}, $T \subseteq G$ the maximal torus of diagonal matrices and $B \subseteq G$ the Borel subgroup of upper triangular matrices.

**Proposition 4.4.** Let $F(\Theta) = \mathbb{Z}[A_\infty]/(\tau^{p-1})$ and $V \in \Theta \cap \text{mod } \overline{X}$ such that $ql(V) > \frac{3d-2}{2}$. Then the following statements hold:

1. For every $N \in \Theta \cap \text{mod } \overline{X}$ such that $ql(N) > \frac{d}{2}$, there is an undirected path from $V$ to $N$ in $\Theta \cap \text{mod } \overline{X}$.

2. $\Theta \cap \text{mod } \overline{X}$ contains a unique module of maximal quasi-length.
Proof. We first show:

(*) If \( \tau^p_X(N) \notin \text{mod} \overline{X} \), then \( \tau^j_X(N) \notin \text{mod} \overline{X} \) for all \( j > i \).

Let \( X = B_rT \). Since \( F(\Theta) \cong \mathbb{Z}[A_\infty]/\langle \tau^p \rangle \), we get that for each \( N \in \Theta \), there is \( \lambda_N \in X(T) \) such that \( \tau^p_X(N) = N[\lambda] \). As shifting with \( \lambda \) maps almost split sequences to almost sequences, we get \( \lambda_N = \lambda_{N'} \) for every \( N' \) which is a successor, predecessor or element of the \( \tau_X \)-orbit of \( N \). Since \( \Theta \) is a connected component, we inductively get that \( \lambda_N = \lambda_{N'} \) for all \( N, N' \in \Theta \). Thus, \( (\tau_X|_\Theta)^p = [\lambda] \) for some \( \lambda \in X(T) \). By \((1.1)\) (2), we also find suitable \( \alpha \in R^+ \cup \{0\}, s \in \mathbb{N}_0 \) such that \( (\tau_X|_\Theta)^p = [p' \alpha - 2p^s(p^r - 1)\rho] \) for all \( V \in \Theta \). As \( G = \text{GL}_n \) and we have chosen the roots as in \([23] \text{II.1.21}\), the first coordinate of \( p' \alpha - 2p^s(p^r - 1)\rho \) is negative and the last coordinate is positive, so that the same holds for \( \lambda \). We get that if \( N \in \text{mod} \overline{X} \cap \Theta \) and \( \tau^p_X(N) \notin \text{mod} \overline{X} \), then \( \tau^p_X(N) \notin \text{mod} \overline{X} \) for all \( j \in \mathbb{N} \), showing the statement for \( B_rT \). For \( X = G_rT \), use \((1.1)\) (1), and argue as above.

Analogously, one shows: If \( \tau^{-j}_X(N) \notin \text{mod} \overline{X} \), then \( \tau^{-j}_X(N) \notin \text{mod} \overline{X} \) for all \( j > i \).

(1) Since every module on the downward sectional path starting in \( N \) or \( V \) is also an \( \overline{X} \)-module, we may assume \( ql(N) = \frac{p+1}{2} \) and \( ql(V) = \frac{3p-1}{2} \). By following the downward sectional path starting in \( V \) and the upward sectional path from the quasi-socle of \( V \) to \( V \) for \( p' - 1 \) steps, we get a module \( N' \in \Theta \cap \text{mod} \overline{X} \) such that \( q(V') = q(V) - (p' - 1) = q(N) \). Then the modules \( \tau^i_{G_rT}(N') \) belong to the wing of \( V \) for \( 1 \leq i \leq p' - 1 \), so that they belong to \( \text{mod} \overline{X} \) by \((2.8)\). Thus, there are \( l \in \mathbb{N}_0 \) and \( 0 \leq i \leq p' - 1 \) such that \( N = \tau^{i+lp'}_X(N') \). By \((*)\), \( \tau^{lp'}_X(\tau^i_X(N')) \in \text{mod} \overline{X} \) implies \( \tau^k_X(\tau^i_X(N')) \in \text{mod} \overline{X} \) for \( 1 \leq k \leq l \). Since \( ql(N) = \frac{p+1}{2} \), \((2.8)\) shows that \( \tau^i_X(N') \in \text{mod} \overline{X} \) for \( 1 \leq j \leq i + lp' \). As \( \text{mod} \overline{X} \) is extension-closed, the middle terms of the almost split sequences defined by these modules also belong to \( \text{mod} \overline{X} \), so that we get an undirected path in \( \Theta \cap \text{mod} \overline{X} \) from \( N' \) to \( N \). As there is also such a path from \( V \) to \( N' \), the result follows.

(2) If \( N, N' \in \Theta \cap \text{mod} \overline{X} \) have maximal quasi-length and \( N \neq N' \), \((*)\) and \((2.8)\) imply the existence of a module with greater quasi-length in \( \Theta \cap \text{mod} \overline{X} \), a contradiction.

\[\square\]

**Corollary 4.5.** Let \( X \in \{B_rT, G_rT\} \). Suppose that \( F(\Theta) \) is a homogeneous tube and \( \Theta \cap \text{mod} \overline{X} \neq \emptyset \). Then there is a unique module \( N \in \Theta \cap \text{mod} \overline{X} \) such that \( \Theta \cap \text{mod} \overline{X} \) consists of the wing of \( N \).

**Proof.** Taking \( t = 0 \) in \((4.4)\), we see that \( \Theta \cap \text{mod} \overline{X} \) is connected. Since \( \text{mod} \overline{X} \) is extension-closed by \((3.5)\), we get that \( \Theta \cap \text{mod} \overline{X} \) is equal to the wing of a module \( V \in \Theta \cap \text{mod} \overline{X} \) with maximal quasi-length. \[\square\]

Note that for \( X = G_rT \) and \( r = 1 \), \( F(\Theta) \) always is a homogeneous tube by \((1.1)\).
5 The case $n = 2, r = 1$

In this section, let $G = \text{GL}_2$, $T \subseteq G$ be the torus of diagonal matrices. In [8], quiver and relations for the blocks of $S_d(G_1T)$ as well as the number of blocks were determined and it was shown that all blocks in this case are representation-finite. Our aim in this section is to determine the Auslander-Reiten quiver for the algebras $S_d(G_1T)$ by first considering the position of the relevant modules in the stable Auslander-Reiten quiver of $G_1T$. Since $G_1T = (\text{SL}_2)_1T$ and modules for $(\text{SL}_2)_1$ correspond to modules for the restricted enveloping algebra $U_0(\mathfrak{sl}_2)$, we have a restriction functor $F : \text{mod}G_1T \to \text{mod}U_0(\mathfrak{sl}_2)$ and the results of Section 1 apply to this functor. Contrary to the usual theory of $(\text{SL}_2)_1(T \cap \text{SL}_2)$-modules, we obtain a $\mathbb{Z}^2$-grading on $\mathfrak{sl}_2$ and $U_0(\mathfrak{sl}_2)$ given by $\deg(e) = (1, -1), \deg(f) = (-1, 1)$ and $\deg(h) = (0, 0)$ for the standard basis $e, f, h$ of $\mathfrak{sl}_2$. The indecomposable $U_0(\mathfrak{sl}_2)$-modules were classified by Premet in [26]. For $d \geq 0$, let $V(d)$ be the Weyl module of highest weight $d$ for the group scheme $\text{SL}_2$. Then $V(d)$ has a basis $v_0, \ldots, v_d$ such that for the standard basis $\{e, f, h\} \subseteq \mathfrak{sl}_2$, we have

$$e.v_i = (i + 1)v_{i+1}, f.v_i = (d - i + 1)v_{i-1}, h.v_i = (2i - d)v_i$$

For $d = sp + a$, where $s \geq 1$ and $0 \leq a \leq p - 2$, the space

$$W(d) := \bigoplus_{i=a+1}^{d} kv_i \subseteq V(d)$$

is a maximal $U_0(\mathfrak{sl}_2)$-submodule of $V(d)$. We record the classification of indecomposable $U_0(\mathfrak{sl}_2)$-modules in the following theorem:

**Theorem 5.1.** The following statements hold:

1. Let $C \subseteq \text{SL}_2$ be a complete set of coset representatives of $\text{SL}_2/B$, where $B \subseteq \text{SL}_2$ is the Borel subgroup of upper triangular matrices. Then any nonprojective indecomposable $U_0(\mathfrak{sl}_2)$-module is isomorphic to exactly one of the modules of the following list:

   (i) $V(d), V(d)^*$ for $d \geq p, d \neq -1 \mod p$;

   (ii) $V(r) =: L(r)$ for $0 \leq r \leq p - 1$;

   (iii) $g.W(d)$ for $g \in C$ and $d = sp + a$ with $s \geq 1$ and $0 \leq a \leq p - 2$.

In particular, the modules appearing in the list are pairwise nonisomorphic.

2. Up to isomorphism, every indecomposable $U_0(\mathfrak{sl}_2)$-module $N$ is uniquely determined by the triple $(\dim_k N, \text{Soc}_{\mathfrak{sl}_2}(N), V_{\mathfrak{sl}_2}(N))$, where $V_{\mathfrak{sl}_2}(N)$ is the $U_0(\mathfrak{sl}_2)$-rank variety of $N$.

The stable Auslander-Reiten quiver of $U_0(\mathfrak{sl}_2)$ is the disjoint union of $p - 1$ components of type $\mathbb{Z}[\hat{A}_{12}]$ and infinitely many homogeneous tubes. The modules $g.W(d)$ lie in homogeneous tubes and $gl(g.W(d)) = s$, where $d = sp + a$ ([13, 4.1.2]). We first determine all indecomposable homogeneous submodules and factor modules of $V(sp+a)$.
By definition (see for example \[15\] Section 4.1), the \(\mathfrak{sl}_2\)-structure on \(V(sp+a)\) comes from a twist of the dual of the \(d\)-th symmetric power of the natural \(\text{SL}_2\)-module \(L(1) = k^2\). This module is a \(\text{GL}_2\)-module in a natural way and the differential of the \(\text{GL}_2\)-action restricts to the \(\mathfrak{sl}_2\)-action on this module. Thus, we get an \(X(T)\)-grading given by \(\text{deg}(v_i) = (i, sp + a - i)\) compatible with the \(\mathfrak{sl}_2\)-action. Applying contravariant duality, we see that \(V(sp+a)^\circ\) has a basis with the same weights. Letting \(w_0 \in G\) be a representative of the nontrivial element of the Weyl group \(W\) of \(G\), the submodules \(W(sp+a)\) resp. \(w_0.W(sp+a)\) of \(V(sp+a)\) have bases \(v_{a+1}, \ldots, v_{sp+a}\) resp. \(v_0, \ldots, v_{sp-1}\), so they are homogeneous. Other modules of the form \(g.W(d)\) don't have \(T\)-invariant rank varieties and are thus not \(G_1T\)-modules. We have \(w_0.W(sp+a) = W(sp+a)^{\omega_0}\), the twist of \(W(sp+a)\) by \(w_0\). Since \(w_0, (t_1, t_2) = (t_2, t_1)\) for all \(t_1, t_2 \in T\), the grading on \(w_0.V\) is given by interchanging the coordinates of the grading of \(V\) for all \(V \in \text{mod} G_1T\). We denote the \(G_1T\)-structures defined above on \(V(sp+a), V(sp+a)^\circ, W(sp+a)\) and \(W(sp+a)^{\omega_0}\) by \(\hat{V}(sp+a), \hat{V}(sp+a)^\circ, \hat{W}(sp+a)\) and \(\hat{W}(sp+a)^{\omega_0}\), respectively. For \(0 \leq r \leq p - 1\), we also write \(\hat{L}(r) := \hat{V}(r)\). By the classification in \[5.1\] we have \(V(d) \cong V(d)\) for \(0 \leq d \leq p - 1\). This shows that \(F(\hat{V}(sp+a)^\circ) \cong V(sp+a)^\circ\).

**Lemma 5.2.** Let \(V\) be a nonprojective indecomposable \(G_1T\)-module. Then there are \(x, y \in \mathbb{Z}\) and a basis \(v_0, \ldots, v_l\) of \(V\) such that \(\text{deg}(v_i) = (x + i, y + i)\).

**Proof.** By \[5.1\] and its succeeding remarks, \(F(V)\) is isomorphic to either of \(V(sp+a) \cong F(\hat{V}(sp+a)), V(sp+a)^\circ \cong F(\hat{V}(sp+a)^\circ)\) with \(s \geq 0\) or \(W(sp+a) \cong F(\hat{W}(sp+a))\) or \(W(sp+a)^{\omega_0} \cong F(\hat{W}(sp+a)^{\omega_0})\) with \(s \geq 1\). Let \(F(V) \cong V(sp+a) \cong F(\hat{V}(sp+a))\). An application of \[1.1\] and \[14\] yields \(\lambda \in X(T)\) such that \(\lambda|_{T(sp+1)} = 1\) and \(V \cong \hat{V}(sp+a)[\lambda]\). By the remarks above, \(\hat{V}(sp+a)\) has a basis \(v_0, \ldots, v_{sp+a}\) such that \(\text{deg}(v_i) = (i, sp+a - i)\). In \(\hat{V}(sp+a)[\lambda]\), this basis has degrees \(\text{deg}(v_i) = (\lambda_1 + i, \lambda_2 + sp + a - i)\), so that \((x, y) = (\lambda_1, \lambda_2)\) yields the claim in this case. As \(\hat{V}(sp+a)^\circ\) has a basis with the same degrees, the preceding arguments also apply in the case \(F(V) \cong F(\hat{V}(sp+a)^\circ)\). In the remaining two cases, we apply the same arguments to the bases \(v_{a+1}, \ldots, v_{sp+a}\) resp. \(v_0, \ldots, v_{sp-1}\) of \(W(sp+a)\) resp. \(\hat{W}(sp+a)^{\omega_0}\), setting \((x, y) = (\lambda_1 + a + 1, \lambda_2)\) resp. \((x, y) = (\lambda_1, \lambda_2 + a + 1)\).

Using the lemma, we can now determine the \(M_1D\)-parts of the \(G_1T\)-components containing \(\hat{W}(sp+a), \hat{W}(sp+a)^{\omega_0}, \hat{V}(sp+a)\) and \(\hat{V}(sp+a)^\circ\).

**Proposition 5.3.** Let \(\Theta\) be the component of the stable Auslander-Reiten quiver of \(\text{mod} G_1T\) containing the module \(\hat{W}(sp+a)\) resp. \(\hat{W}(sp+a)^{\omega_0}\). Then \(\Theta\) is a \(\mathbb{Z}[A_{\infty}]\)-component such that \(\tau_{G_1T}|_{\Theta} = [(p, -p)]\) resp. \(\tau_{G_1T}|_{\Theta} = [(-p, p)]\) and \(\Theta \cap M_1D\) is the wing of \(\hat{W}(sp+a)\) resp. \(\hat{W}(sp+a)^{\omega_0}\). The \(i\)-th predecessor of \(\hat{W}(sp+a)\) resp. \(\hat{W}(sp+a)^{\omega_0}\) on the sectional path starting in its quasi-socle is \(\hat{W}((s - i)p + a)[(ip, 0)]\) resp. \(\hat{W}((s - i)p + a)^{\omega_0}[(0, ip)]\). The quasi-socle \(\hat{W}(p + a)((s - 1)p, 0])\) resp. \(\hat{W}(p + a)^{\omega_0}[(0, (s - 1)p)]\) is projective in \(\text{mod} M_1D\).

**Proof.** We show the result for \(\hat{W}(sp+a)\). For \(\hat{W}(sp+a)^{\omega_0}\), the result then follows since twisting with \(w_0\) is an equivalence of categories interchanging the two coordinates of the
Comparing the degrees of these basis elements with the degrees of the canonical basis of \(\hat{v}\) of the basis \(a\) and \(v\), so that \(\lambda, \mu, \nu \in X(T)\), where \(\tilde{W}(a) := 0\). Thus, \(\tau_{G_1T}(\hat{W}((s - 1)p + a))[\mu] \cong \hat{W}((s - 1)p + a)[\lambda]\). On the other hand, \(\text{[15]}\) yields \(\alpha \in R\) such that \(\tau_{G_1T}|_{\Theta} = [p\alpha]\). Hence \(\tau_{G_1T}|_{\Theta} = [p\alpha] = [\lambda - \mu]\) and \(\text{[15]}\) (2) yields \(p\alpha = \lambda - \mu\). In order to determine \(\lambda, \mu\), we determine all submodules and factor modules of \(\hat{W}(sp + a)\) isomorphic to a shift of \(\hat{W}((s - 1)p + a)\). Let \(A\) be such a submodule. In the canonical basis \(v_{a+1}, \ldots, v_{sp+a}\) of \(\hat{W}(sp + a)\), each \(v_i\) has degree \((i, sp+a-1)\), so that the subspace \(\hat{W}(sp + a)_{(i,sp+a-1)}\) is one-dimensional with basis \(v_i\). Hence any \(G_1T\)-submodule of \(\hat{W}(sp + a)\) is the span of a subset of the \(v_i\). As \(\hat{W}((s-1)p+a)\) is indecomposable and has dimension \((s-1)p, \text{[5.2]}\) provides a basis \(v'_0, \ldots, v'_{(s-1)p-1}\) of \(A\) and \(x, y \in Z\) such that \(\text{deg}(v'_i) = (x + i, y + sp - 1 - i)\). Comparing the degrees of these basis elements with the degrees of the \(v_i\) shows that there is \(0 \leq i \leq p\) such that \(A\) is spanned by \(v_{a+1}, \ldots, v_{(s-1)p+a+i}\). Suppose first that \(0 \leq i \leq p - a - 2\). Then \(e(v_{(s-1)p+a+i}) = ((s - 1)p + a + i + 1)v_{(s-1)p+a+i+1}\) yields \(v_{(s-1)p+a+i+1} \in A\), a contradiction. Thus, \(i \geq p - a - 1\). If \(p - a - 1 \leq i \leq p - 1\), then \(f.v_{a+i} = (sp + i)v_{a+i}\) yields \(v_{a+i} \in A\), a contradiction. As a result, we have \(i = p\) and \(A\) is spanned by \(v_{p+a+1}, \ldots, v_{sp+a}\). Comparing the degrees of the basis of \(A\) with the degrees of the canonical basis of \(\hat{W}((s - 1)p + a)\), we see that \(A \cong \hat{W}((s - 1)p + a)[(p, 0)]\), so that \(\lambda = (p, 0)\).

As \([\lambda - \mu] = [p\alpha]\) and \(\text{GL}_n\) has roots \((1, -1), (-1, 1)\), we have \(\mu = (2p, -p)\) or \(\mu = (0, 0)\). Since \(\hat{W}((s - 1)p + a)[\mu] \equiv \text{mod } M_1D\), \(\mu \neq (2p, -p)\); else the last base vector of the canonical basis of \(\hat{W}((s - 1)p + a)[\mu]\) would have a negative coordinate. Thus, \(\mu = (0, 0)\). As a result, we have \(\tau_{G_1T}|_{\Theta} = [(p, -p)]\). By considering the degrees of the canonical basis of \(\hat{W}(sp + a)\) again, we see that not all weights of \(\hat{W}(sp + a)[(p, -p)], \hat{W}(sp + a)[(-p, p)]\) are polynomial, so that \(\tau_{G_1T}(\hat{W}(sp + a)), \tau_{G_1T}^{-1}(\hat{W}(sp + a)) \notin \text{mod } M_1D\). Now \(\text{[15]}\) shows that \(\Theta \cap \text{mod } M_1D\) is equal to the wing of \(\hat{W}(sp + a)\). Now let \(s = 1\). The arguments above in particular yield an almost split sequence

\[
(0) \to \hat{W}((p + a)[(p, 0)]) \to \hat{W}(2p + a) \to \hat{W}(p + a)[(0, 0)] \to (0).
\]

Applying the shift functor \([(-p, 0)]\) to this sequence, we see that \(\tau_{G_1T}(\hat{W}(p + a)[(-p, p)]) = \hat{W}(p + a)\), so that \(\tau_{G_1T}|_{\Theta} = [(p, -p)]\) also holds in this case. Considering the degrees of the canonical basis for \(\hat{W}(p + a)[(-p, p)], \hat{W}(p + a)[(p, -p)]\), we see
that not all weights of these modules are polynomial, so that \( \Theta \cap \mod M_1 D \) is the wing of \( \widehat{W}(p + a) \).

For \( 1 \leq s \leq 2 \), the almost split sequences computed above now show that the leftmost quasi-simple module in the wing of \( \widehat{W}(sp + a) \) is \( \widehat{W}(p + a)(((s - 1)p, 0)] \). For \( s > 2 \), one shows by induction that the \( i \)-th predecessor of \( \widehat{W}(sp + a) \) on the sectional path starting in the quasi-socle of \( \widehat{W}(sp + a) \) is \( W((s - i)p + a][(i, 0)] \) by shifting the almost split sequence starting in \( \widehat{W}((s - i)p + a][(i, 0)] \) by \( [(i - 1)p, 0)] \).

The definition of the \( \mathfrak{sl}_2 \)-action on the basis vectors \( v_{a+1}, \ldots, v_{p+a} \) of \( W(p + a) \) shows that \( v_{a+1}, \ldots, v_{p-1} \) span a simple submodule of \( \widehat{W}(p + a) \) while each of the remaining basis vectors generates \( \widehat{W}(p + a) \) as an \( \mathfrak{sl}_2 \)-module, so that the socle of \( \widehat{W}(p + a) \) is spanned by \( v_{a+1}, \ldots, v_{p-1} \). As \( \deg(v_{a+1}) = (p - 1, a + 1) \), we get \( \mathfrak{F}_{M, D}(\soc(\widehat{W}(p + a)(sp, -p])) = 0 \), so that \( \mathfrak{F}_{M, D}(\widehat{W}(p + a)(sp, -p]) = 0 \) and \( W(p + a)(((s - 1)p, 0)] \) is projective in \( \mod M_1 D \).

For an illustration of the following proposition, see Figure 2 below.

**Proposition 5.4.** Let \( \Theta \) be the component of the stable Auslander-Reiten quiver of \( \mod G_1 T \) containing the module \( \widehat{V}(sp + a)[[(i, i)] \) for \( 0 \leq i \leq p - 1 \). Then \( \Theta \) is a \( \mathbb{Z}[A_{\infty}] \)-component such that \( \Theta = \Theta^o \). There is a column of simple module in \( \Theta \) which is a symmetry axis with respect to \( (-)^o \) and \( \Theta \cap \mod M_1 D \) consists of the modules on directed paths between \( \widehat{V}(sp + a)[[(i, i)] \) and \( \widehat{V}(sp + a)^o[[(i, i)] \). We also have \( \tau_{G_1 T}(\widehat{V}(sp + a)[[(i, i)] = \widehat{V}((s + 2)p + a)[[(i - p, i - p)] \).

**Proof.** By [15, Section 4.1], \( F(\Theta) \) is of type \( \mathbb{Z}[\hat{A}_{12}] \) and contains exactly one simple module \( L(a) \). By [14] \( F \) induces a morphism of stable translation quivers \( \Theta \rightarrow F(\Theta) \).

Since \( F(\Theta) \) has no quasi-simple modules and \( \Theta \) is isomorphic to either \( \mathbb{Z}[A_{\infty}], \mathbb{Z}[A_{\infty}^\circ] \) or \( \mathbb{Z}[D_{\infty}] \) by [14, 3.4], this shows that \( \Theta \) is of type \( \mathbb{Z}[A_{\infty}] \). We show that there is an almost split sequence

\[
\zeta : (0) \rightarrow \widehat{V}(sp + a) \rightarrow \widehat{V}((s - 1)p + a)[[(p, 0)] \oplus \widehat{V}((s - 1)p + a)[[(0, p)] \\
\rightarrow \widehat{V}((s - 2)p + a)[[(p, p)] \rightarrow (0)
\]

for \( s > 1 \) while the almost split sequence starting in \( \widehat{V}(p + a) \) is

\[
\zeta' : (0) \rightarrow \widehat{V}(p + a) \rightarrow \widehat{L}(a)[[(p, 0)] \oplus \widehat{L}(a)[[(0, p)] \oplus \widehat{P} \\
\rightarrow \widehat{V}(p + a)^o \rightarrow (0)
\]

with \( \widehat{P} \) projective indecomposable. By [2, V.5.5], there is an almost split sequence

\[
\xi : (0) \rightarrow \rad(P) \rightarrow \rad(P) / \soc(P) \oplus P \rightarrow P / \soc(P) \rightarrow (0)
\]

in \( \mod U_0(\mathfrak{sl}_2) \), where \( P \) is the projective cover of \( L(p - a - 2) \). By [11, IV.3.8.3] in conjunction with [17, 2.4], we have \( \rad(P) / \soc(P) \cong S \oplus S \) for some simple \( U_0(\mathfrak{sl}_2) \)-module \( S \) such that \( S \not\cong L(p - a - 2) \) and \( S, L(p - a - 2) \) are the only simple modules.
in the block of $S$. As $L(a)$ belongs to the block of $L(p - a - 2)$, we get $S \cong L(a)$. Hence $\text{Rad}(P)$ has dimension $p + a + 1$ and $\text{Soc}(\text{Rad}(P)) \cong L(p - a - 2)$, so that \((5.11)\) yields $\text{Rad}(P) \cong V(p + a)$. By \((1.2)\) in combination with \((1.3)\), the almost split sequence in mod $G_1T$ starting in $\hat{V}(p + a)$ is mapped to $\xi$ by $F$. Thus, the non-projective part of the middle term is a sum of two shifts of $\hat{L}(a)$ and is obtained by factoring out the socle of $\hat{V}(p + a)$. For the canonical basis $v_0, \ldots, v_{p + a}$ of $\hat{V}(p + a)$, the action of $\mathfrak{sl}_2$ on the basis vectors shows that the socle is spanned by $v(a)$, $v(b)$, and $v(c)$. Comparing the degrees of these vectors with the degrees in the canonical basis of $\hat{V}(p + a)$, we see that the factor module are spanned by the images of $v_0, \ldots, v_a$ and $v_p, \ldots, v_{p + a}$, respectively. Comparing the degrees of these vectors with the degrees in the canonical basis of $\hat{L}(a)$, we see that $\hat{V}(p + a)/\text{Soc}(\hat{V}(p + a)) \cong \hat{L}(a)[(p, 0)] \oplus \hat{L}(a)[(0, p)]$. As the middle term of the almost split sequence starting in $\hat{V}(p + a)$ is self-dual with respect to $(-)\circ$, the right term of this sequence is $\hat{V}(p + a)^{\circ}$, so that $\zeta$ is almost split. By applying the shift functors $[(0, p)], [(p, 0)]$ to $\zeta'$, we see that $\hat{V}(p + a)[(0, p)], \hat{V}(p + a)[(0, p)]$ are predecessors of $\hat{V}(a)[(p, p)]$ in $\Gamma_*(G_1T)$. As $\Theta$ is of type $\mathbb{Z}[A_2^\infty]$, this shows that the middle term of the almost split sequence ending in $\hat{V}(p + a)[(p, p)]$ has the claimed form. Inductively, to show that the almost split sequence for $s > 1$ has the claimed form, it suffices to show that if

$$
\xi' : (0) \to X_s \to \hat{V}((s - 1)p + a)[(p, 0)] \oplus \hat{V}((s - 1)p + a)[(0, p)]
$$

is almost split in mod $G_1T$, then $X_s \cong \hat{V}(sp + a)$. As above, $F(\xi')$ is almost split in mod $U_0(\mathfrak{sl}_2)$ and by exactness, we have $\text{dim}_k X_s = sp + a + 1$. Thus, \((5.11)\) yields that $F(X_s) \cong V(sp + a) \cong F(\hat{V}(sp + a))$ or $F(X_s) \cong V(sp + a)^\circ \cong F(\hat{V}(sp + a)^\circ)$, so that $X_s$ is isomorphic to a shift of $\hat{V}(sp + a)$ or $\hat{V}(sp + a)^\circ$ by \((1.1)\). We show that $\hat{V}(sp + a)$ does not have a submodule isomorphic to a shift of $\hat{V}((s - 1)p + a)^\circ$, so that $\hat{V}(sp + a)^\circ$ does not have a factor module isomorphic to a shift of $\hat{V}((s - 1)p + a)$ and the first alternative has to apply. Suppose $A$ is a submodule of $\hat{V}(sp + a)$ isomorphic to a shift of $\hat{V}((s - 1)p + a)^\circ$. Taking the canonical basis $v_0, \ldots, v_{sp + a}$ of $\hat{V}(sp + a)$ and comparing the degrees of this basis with the degrees of the basis of $A$ provided by \((5.2)\), we see that $A$ is spanned by $v_i, \ldots, v_{(s - 1)p + a + i}$ for some $0 \leq i \leq p$. Using the action of $U_0(\mathfrak{sl}_2)$ on the $v_j$, we get the following statements:

(1) If $v_j \in A$ for some $0 \leq j \leq a$, then $v_0, \ldots, v_a \in A$,

(2) if $v_j \in A$ for some $a + 1 \leq j \leq p - 1$, then $v_{a + 1}, \ldots, v_{p - 1} \in A$,

(3) if $v_j \in A$ for some $(s - 1)p + a + 1 \leq j \leq sp - 1$, then $v_{(s - 1)p + a + 1}, \ldots, v_{sp - 1} \in A$,

(4) if $v_j \in A$ for some $sp \leq j \leq sp + a$, then $v_{sp}, \ldots, v_{sp + a} \in A$.

Thus, $A$ is spanned by either $v_0, \ldots, v_{(s - 1)p + a}$ or by $v_p, \ldots, v_{sp + a}$. However, we have $e.v_{(s - 1)p + a} = ((s - 1)p + a + 1)v_{(s - 1)p + a + 1}$ and $f.v_p = ((s - 1)p + a + 1)v_{p + 1}$, so that in both cases, $A$ is not a submodule, a contradiction. Hence $X_s \cong \hat{V}(sp + a)[\lambda]$ for some
\( \lambda \in X(T) \). Since \( \xi' \) is almost split, \( \hat{V}(sp + a)[\lambda] \) has factor modules \( A_1 \cong \hat{V}((s - 1)p + a)[(p, 0)] \), \( A_2 \cong \hat{V}((s - 1)p + a)[(0, p)] \). Comparing degrees of bases and applying the arguments on submodules from above to factor modules and kernels of the canonical projections, we see that \( A_1 \) resp. \( A_2 \) are spanned by the images of the basis vectors \( v_p, \ldots, v_{sp + a} \) resp. \( v_0, \ldots, v_{(s-1)p + a} \) under the canonical projections and that \( \lambda = 0 \), so that \( X_\lambda \cong \hat{V}(sp + a) \) and \( \zeta \) is almost split. The statement about \( \tau_{G,T} \) now follows by shifting the sequence for \( s + 2 \) by \( [(i - p, i - p)] \). By applying the shift functor \( [(lp, l'p)] \) to the almost split sequences \( \zeta, \zeta' \), we see that for all \( l,l' \in \mathbb{Z} \), the element directly above resp. below \( \hat{V}(sp + a)[(lp, l'p)] \) in the column of \( \hat{V}(sp + a)[(lp, l'p)] \) is \( \hat{V}(sp + a)[(l + 1)p,(l' - 1)p] \) resp. \( \hat{V}(sp + a)[((l - 1)p,(l' + 1)p)] \). Thus, if \( 0 \leq j \leq s \) and we follow the upper resp. lower sectional path starting in \( \hat{V}(sp + a) \) for \( j \) steps, we arrive at the module \( \hat{V}((s - j)p + a)[(jp, 0)] \) resp. \( \hat{V}((s - j)p + a)[(0, jp)] \). The modules above \( \hat{V}((s - j)p + a)[(jp, 0)] \) have the form \( \hat{V}((s - j)p + a)[((j + 1)p, -lp)] \) while the modules below \( \hat{V}((s - j)p + a)[(0, jp)] \) have the form \( \hat{V}((s - j)p + a)[(-lp, j + lp)] \) for some \( l \in \mathbb{N} \). Thus, the degree of either \( v_0 \) or \( v_{(s-j)p+a} \) in their canonical bases has a coordinate which is a negative multiple of \( p \), so that these modules and their shifts by \( [(i,i)] \) are not polynomial \( G_1T \)-modules. All modules between \( \hat{V}((s - j)p + a)[(jp, 0)] \) and \( \hat{V}((s - j)p + a)[(0, jp)] \) in their column can be reached via directed paths starting in \( \hat{V}(sp + a) \). As all arrows on these paths correspond to irreducible epimorphisms, all modules obtained this way are polynomial \( G_1T \)-modules by 3.1. The arguments above show in particular that there is a column of simple modules in \( \Theta \) which is equal to \( \{ \hat{L}(a)[(sp + jp, -jp)] \mid j \in \mathbb{Z} \} \) and that a module to the left of this column belongs to \( \text{mod}\ M_1D \) iff it lies on a directed path from \( \hat{V}(sp + a) \) to this column. Since \( S^o \cong S \) for every simple \( G_1T \)-module \( S \) by [23, II.6.9(13)], we have \( \Theta^o = \Theta \) and the column of simple modules in \( \Theta \) is a symmetry axis with respect to \( (-)^o \). As \( V \in \text{mod}\ G_1T \) is a polynomial \( G_1T \)-module iff \( V^o \) is a polynomial \( G_1T \)-module, we get that any module to the right of the column of simple modules belongs to \( \text{mod}\ M_1D \) iff it lies on a directed path from the column of simple modules ending in \( \hat{V}(sp + a)^o \). As a result, \( \Theta \cap \text{mod}\ M_1D \) consists of all modules on directed paths from \( \hat{V}(sp + a) \) to \( \hat{V}(sp + a)^o \) for \( i = 0 \). For \( 1 \leq i \leq p - 1 \), the statement follows by applying the functor \( [(i,i)] \) to the component containing \( \hat{V}(sp + a) \), noting that the above arguments about negative coordinates also apply in this case.

By 1.4 the shifts of \( \hat{V}(sp + a) \) with a \( G_1T \)-structure are exactly those \( \hat{V}(sp + a)[\lambda] \) with \( \lambda \in X(T) \) such that \( \lambda|_{\text{SL}_2} = 0 \). These are the elements of \( pX(T) + \mathbb{Z}\det|_T \). The proof of 4.4 shows that all these modules belong to components of the type described in 5.4. As modules of the form \( V(sp + a) \) and their duals are the only \( U_0(\text{sl}_2) \)-modules contained in components of type \( \mathbb{Z}[\hat{A}_{12}] \) by 15. Section 4], we have found all of these modules for \( r = 1 \).

Since the restriction functor \( \text{mod}\ M_1D \rightarrow \text{mod}\ G_1T \) is fully faithful, almost split sequences in \( \text{mod}\ G_1T \) such that all constituents lift to \( M_1D \) are also almost split in \( \text{mod}\ M_1D \). In the following, we determine the almost split sequences in \( \text{mod}\ M_1D \) which are not almost split in \( \text{mod}\ G_1T \).
Figure 2: Component of $\Gamma_s(G_1T)$ containing the module $V = \hat{V}(2p+a)$. The blue
vertices are $M_1D$-modules and the blue line is the column of simple modules.

Lemma 5.5. There are almost split sequences

$$\xi_1 : (0) \to \hat{W}((s-l)p+a)[(0, lp)]$$
$$\to \hat{V}((s-l)p+a)[(0, lp)] \oplus \hat{W}((s-l-1)p+a)[(0, (l+1)p)]$$
$$\to \hat{V}((s-l-1)p+a)[(0, (l+1)p)] \to (0)$$

for $0 \leq l \leq s - 1$ and

$$\xi_2 : (0) \to \hat{V}((s-l)p-a-2)[(a + 1 + lp, a + 1)]^o$$
$$\to \hat{W}((s-l)p+a)[(lp, 0)]$$
$$\oplus \hat{V}((s-l+1)p-a-2)[(a + 1 + (l-1)p, a + 1)]^o$$
$$\to \hat{W}((s-l+1)p+a)[((l-1)p, 0)] \to (0)$$

for $s > 1$ and $1 \leq l \leq s - 1$ in mod $M_1D$, where $\hat{W}(a) = 0$.

Proof. We have almost split sequences

$$\xi_1^1 : (0) \to \hat{V}((s-l+1)p+a)[(-p, lp)]$$
$$\to \hat{V}((s-l)p+a)[(0, lp)] \oplus \hat{V}((s-l)p+a)[(-p, (l+1)p)]$$
$$\to \hat{V}((s-l-1)p+a)[(0, (l+1)p)] \to (0)$$

and

$$\xi_2^1 : (0) \to \hat{W}((s-l+1)p+a)[(lp, -p)]$$
$$\to \hat{W}((s-l)p+a)[(lp, 0)] \oplus \hat{W}((s-l+2)p+a)[((l-1)p, -p)]$$
$$\to \hat{W}((s-l+1)p+a)[((l-1)p, 0)] \to (0)$$

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λ = \{(v_{p+a+1}, \ldots, v_{(s-l+1)p+a}) | v_0, \ldots, v_{(s-l+1)p+a} is the canonical basis of \(\hat{V}'\))\}. As every weight space of \(\hat{V}'\) is one-dimensional and \(A\) is a homogeneous submodule, \(A\) is spanned by a subset of the \(v_i\).

Since the first coordinates of their weights are negative, we have to delete at least \(v_0, \ldots, v_{p-1}\), and since the \(U_0(\mathfrak{s}_2)\)-module spanned by each of the vectors \(v_p, \ldots, v_{p+a}\) contains \(v_{a+1}, \ldots, v_{p-1}\), we have to delete those as well, so that only the basis vectors spanning \(\langle v_{p+a+1}, \ldots, v_{(s-l+1)p+a} \rangle\) remain. As \(\langle v_{p+a+1}, \ldots, v_{(s-l+1)p+a} \rangle\) is stable under the \(\mathfrak{s}_2\)-action on \(\hat{V}'\), \(\mathfrak{s}_2\)-submodule which is isomorphic to \(L((-p, p))\) and spanned by homogeneous elements, it is a homogeneous submodule. Since all weights of the basis vectors occurring are polynomial, we get \(A = \langle v_{p+a+1}, \ldots, v_{(s-l+1)p+a} \rangle\). In particular, \(2.1\) shows that the right-hand term of \(\xi_1\) is not Ext-projective in mod \(M_1D\). Hence \(2.1\) implies that \(A\) is indecomposable, so that \(F(A)\) is indecomposable by \(1.1\). Since \(v_{p+a+1}, \ldots, v_{2p-1}\) span a simple \(\mathfrak{s}_2\)-submodule which is isomorphic to \(L((-p, p))\), \(L(-p, a - 2)\) occurs in the socle of \(F(A)\). As \(\dim_k A = (s-l)p\), \(5.1\) now yields \(F(A) \cong \hat{W}'((s-l)p+a)\) or \(F(A) \cong W((-s-l)p+a))w_0\). Using the action of \(f\) on \(A\), we see that the for \(0 \leq i \leq s-l-1\), the vectors \(v_{s+a+i+1}, \ldots, v_{2p+a+ip}\) span a \(p\)-dimensional \(U_0(\langle f \rangle)\) \(\cong k[f]/(f^p)\)-module which is generated by \(v_{2p+a+ip}\) and is hence projective indecomposable for \(U_0(\langle f \rangle)\).

As \(A|_{U_0(\langle f \rangle)}\) is the direct sum of these modules, \(A|_{U_0(\langle f \rangle)}\) is projective, so that \(f\) does not belong to the rank variety of \(F(A)\). Since \(f\) belongs to the rank variety of \(W((-s-l)p+a))w_0\), we get \(F(A) \cong \hat{W}'((s-l)p+a)\). Thus, \(A\) is isomorphic to a shift of \(\hat{W}'((s-l)p+a)\) and comparing the degrees in the canonical basis of \(\hat{W}'((s-l)p+a)\) to those of \(v_{p+a+1}, \ldots, v_{(s-l+1)p+a}\) yields \(A \cong \hat{W}'((s-l)p+a))\). This also shows that \(\xi_1\) is almost split in mod \(M_1D\).

We now compute \(B := \mathfrak{f}_{M,D}(\hat{W}'((s-l+1)p+a))((-p, -p))\). As above, \(B\) is spanned by a subset of the \(v_i\) as a vector space. Since their weights have a negative second coordinate, we have to delete at least the last \(p\) base vectors \(v_{(s-l)p+a+1}, \ldots, v_{(s-l+1)p+a}\). The \(U_0(\mathfrak{s}_2)\)-module spanned by each of the vectors \(v_{(s-l)p+1}, \ldots, v_{(s-l)p+a}\) contains the last \(p\) basis vectors, so they also have to be deleted. As the remaining vectors \(v_{a+1}, \ldots, v_{(s-l)p}\) span a homogeneous submodule of \(W((-s-l+1)p+a))((-p, -p))\), we get \(B = \langle v_{a+1}, \ldots, v_{(s-l)p} \rangle\). As above, \(B\) is indecomposable.

As \(\dim_k(B) = (s-l+1)p + a - 1\), \(5.1\) shows that \(F(B) \cong V((-s-l+1)p+a-2)\) or \(F(B) \cong V((-s-l+1)p+a-2)\). As \(v_{a+1}, \ldots, v_{p-1}\) is a simple \(\mathfrak{s}_2\)-submodule isomorphic to \(L((-s-l+1)p+a-2)\), we get \(F(B) \cong V((-s-l+1)p+a-2)\). Since \(L((-s-l+1)p+a-2)\) does not occur in the socle of \(V((-s-l+1)p+a-2)\). Now \(1.1\) yields \(B \cong \hat{V}'((s-l+1)p+a-2)\), for some \(\lambda \in X(T)\). Comparing the degrees of \(v_{a+1}, \ldots, v_{(s-l)p}\) with those of the canonical basis of \(\hat{V}'((s-l+1)p+a-2)\), yields \(\lambda = (a+1+lp, a+1)\). Thus, \(B \cong \hat{V}'((s-l+1)p+a-2)\). This also shows \(\mathfrak{f}_{M,D}(\hat{W}'((s-l+2)p+a))((-l+1)p, -p)) \cong \hat{V}'((-s-l)p+a-2)((a+1+l, a+1))\). Now \(2.1\) shows that \(\xi_2\) is almost split in mod \(M_1D\).
Lemma 5.6. There are almost split sequences

\[ \xi_1 : (0) \to \hat{W}((s-l)p + a)^{w_0}[(lp, 0)] \]
\[ \to \hat{V}((s-l)p + a)[(lp, 0)] \oplus \hat{W}((s-l-1)p + a)^{w_0}[(l+1)p, 0)] \]
\[ \to \hat{V}((s-l-1)p + a)[((l+1)p, 0)] \to (0) \]

for \( 0 \leq l \leq s - 1 \) and

\[ \xi_2 : (0) \to \hat{V}((s-l)p - a - 2)[(a+1, a+1 + lp)]^o \]
\[ \to \hat{W}(s+p+a)^{w_0}[(0, lp)] \]
\[ \oplus \hat{V}((s-l+1)p - a - 2)[(a+1, a+1 + (l-1)p)]^o \]
\[ \to \hat{W}((s-l+1)p + a)^{w_0}[(0, (l-1)p)] \to (0) \]

for \( s > 1 \) and \( 1 \leq l \leq s - 1 \) in \( \text{mod } M_1 D \), where \( \hat{W}(a) = 0 \).

Proof. Twisting with \( w_0 \) is an auto-equivalence of \( \text{mod } M_1 D \) mapping the almost split sequences from \( 5.5 \) to \( \xi_1, \xi_2 \). The result now follows. \( \square \)

Lemma 5.7. There is an almost split sequence

\[ (0) \to \hat{V}(sp - a - 2)[(a+1, a+1)]^o \]
\[ \to \hat{W}(sp + a) \oplus \hat{W}(sp + a)^{w_0} \]
\[ \to \hat{V}(sp + a) \to (0) \]

in \( \text{mod } M_1 D \) for \( s \geq 1 \).

Proof. By the proof of \( 5.4 \), there is an almost split sequence

\[ (0) \to \hat{V}((s+2)p + a)[-(p, p)] \]
\[ \to \hat{V}((s+1)p + a)[-(p, 0)] \oplus \hat{V}((s+1)p + a)[(0, -p)] \]
\[ \to \hat{V}(sp + a) \to (0) \]

in \( \text{mod } G_r T \). In the proof of \( 5.5 \) it was shown that \( \mathfrak{F}_{M_1, D}(\hat{V}((s+1)p + a)[-(p, 0)]) \cong \hat{W}(sp + a) \). Since \( \hat{V}((s+1)p + a)[-(p, 0)]^{w_0} \cong \hat{V}((s+1)p + a)[(0, -p)], \) this also shows \( \mathfrak{F}_{M_1, D}(\hat{V}((s+2)p + a)[(0, -p)]) \cong \hat{W}(sp + a)^{w_0} \). We compute \( C = \mathfrak{F}_{M_1, D}(\hat{V}((s+2)p + a)[-(p, p)]) \). Taking the canonical basis \( v_0, \ldots, v_{(s+2)p+a} \) of \( \hat{V}((s+2)p + a)[-(p, p)] \), the \( \mathfrak{sl}_2 \)-action on the base vectors shows that \( C \) is spanned by \( v_{p+a+1}, \ldots, v_{(s+1)p-1} \) as all other base vectors generate submodules which are not polynomial. Since \( \text{dim}_k(C) = sp - a - 1 \) and \( v_{p+a+1}, \ldots, v_{2p-1} \) span an \( \mathfrak{sl}_2 \)-module isomorphic to \( L(p-a-2), 5.1 \) yields \( F(C) \cong V(sp - a - 2)^* \cong F(\hat{V}(sp - a - 2)^o) \). An application of \( 1.1 \) and a comparison of degrees of the basis \( v_{p+a+1}, \ldots, v_{(s+1)p-1} \) of \( C \) and the canonical basis of \( \hat{V}(sp - a - 2)^o \) yields \( C \cong \hat{V}(sp - a - 2)[(a+1, a+1)]^o \). The claim now follows from \( 2.1 \). \( \square \)
By using the results about stable AR-components of mod $G_1 T$ and almost split sequences in mod $M_1 D$, we are able to determine the Auslander-Reiten quiver for the blocks of $S_{sp+a}(G_1 T)$ containing $\hat{V}(sp + a)$.

**Theorem 5.8.** The component of the AR-quiver of $S_{sp+a}(G_1 T)$ for $s \geq 1$ and $0 \leq a \leq p - 2$ containing $\hat{V}(sp + a)$ has the following form:

![AR-quiver diagram]

Here, the leftmost and the rightmost column are identified, the brown dots are projective-injective, the blue squares are the $M_1 D$-part of the $G_1 T$-component containing $\hat{V}(sp + a)$, the green squares are the $M_1 D$-part of the $G_1 T$-component containing $\hat{V}(sp - a - 2)\mod{(a + 1, a + 1)}$, the red squares on the left side are the $M_1 D$-part of the $G_1 T$-component containing $\hat{W}(sp + a)$, the orange squares on the left side are the $M_1 D$-part of the $G_1 T$-component containing $\hat{W}(sp + a)^{\omega_0}$ and the red and orange squares on the right side are the duals of those on the left side. All colored arrows represent morphisms which are irreducible in mod $G_1 T$, while the black arrows represent morphisms which are irreducible in mod $M_1 D$, but not in mod $G_1 T$. Thus, upon deleting projective-injective vertices, the underlying directed graph of the component is isomorphic to $\mathbb{Z}[A_{2s+1}]/\langle \tau^{2s+1} \rangle$.

**Proof.** For the left part of the quiver, one uses the shape of the $M, D$-part of the $G_1 T$-components containing $\hat{W}(sp + a), \hat{V}(sp + a), \hat{W}(sp + a)^{\omega_0}$ and $\hat{V}(sp - a - 2)^{\omega_0}[(a + 1, a + 1)]$ and the almost split sequences from \(5.5\), \(5.7\) and \(5.6\). The right part is now obtained from the left part by contravariant duality.

**Remark.** Using calculations similar to those in the proofs of this section, one can show that $\hat{W}(sp + a)^o \cong \hat{W}((s + 1)p - a - 2)^{\omega_0}[(a + 1, a + 1 - p)]$ and $(\hat{W}(sp + a)^{\omega_0})^o \cong \hat{W}((s + 1)p - a - 2)[(a + 1 - p, a + 1)]$ and use this to give a
different description of the modules in the right part of the component given above.

3.13 shows that components of this shape also occur for \( r > 1 \). We will show that all components of \( S_d(G_1 T) \) are shifts of a component of this shape.

For this, we first show that some shifts induce Morita-equivalences between blocks of different \( S_d(G_1 T) \).

**Proposition 5.9.** Let \( s \geq 0 \) and \( 0 \leq a \leq p - 2 \). Then for \( 1 \leq i \leq p - a - 2 \), the shift functor \([([i, i])]\) restricts to an equivalence of categories between the categories of finite dimensional modules belonging to the block of \( S_{sp+a}(G_1 T) \) containing \( \hat{V}(sp + a) \) and a block of \( S_{sp+a+2i}(G_1 T) \).

**Proof.** Since the block contains the simple module of highest weight \((sp + a, 0), p = (\frac{1}{2}, -\frac{1}{2})\), \( W = \{1, w_0\} \) and \( |T| = (1, 1) \), this follows from an application of [9, 4.2] to our situation. \(\square\)

By counting the number of shifts and comparing this to the number of non-semisimple blocks of \( S_{sp+a}(G_1 T)_1 \) determined in [8], we see that every non-semisimple block arises this way.

**Corollary 5.10.** Let \( b \) be a non-simple block of \( S_d(G_1 T) \). Then \( b \) is the shift of the block of \( S_d(G_1 T) \) containing \( \hat{V}(d') \) for some \( d' \leq d \) such that \( d' = sp + a \), \( 0 \leq a \leq p - 2 \).

**Proof.** Write \( d = sp + a \) with \( 0 \leq a \leq p - 1 \) and \( s > 1 \). We first consider the case \( a \neq p - 1 \).

By [5.9] the shifts by \([([i, i])]\) define a Morita-equivalence of the block of \( S_{sp+a-2i}(G_1 T) \) containing the module \( \hat{V}(sp + a - 2i) \) to a block of \( S_d(G_1 T) \) for \( 0 \leq i \leq \lfloor \frac{p}{2} \rfloor \). By the same token, the shifts by \([([a+i+1, a+i+1])]\) define a Morita-equivalence of the block of \( S_{(s-1)p+(p-a-2)-2i}(G_1 T) \) containing \( \hat{V}((s-1)p+(p-a-2)-2i) \) to a block of \( S_d(G_1 T) \) for \( 1 \leq i \leq \lfloor \frac{p-a-2}{2} \rfloor \). These blocks are pairwise distinct since the indecomposable modules of largest dimension belonging to them are \( \hat{V}(sp + a - 2i) \) with dimension \( sp + a - 2i + 1 \) resp. \( \hat{V}((s-1)p+(p-a-2)-2i) \) with dimension \( (s-1)p+(p-a-2)-2i+1 \). For \( a = p - 1 \), only the first kind of blocks occurs with \( 1 \leq i \leq \frac{p-1}{2} \). Thus, we have found \( \frac{p-1}{2} \) distinct blocks in both cases. By [8, 2.2], we have found all blocks not associated to a shift of a Steinberg module, hence all non-semisimple blocks. If now \( d = sp + a \) with \( 0 \leq a \leq p - 1 \), then \( S_d(G_1 T) \) is semisimple by [6, Theorem 3]. If \( d = sp + a \) for \( 0 \leq a \leq p - 1 \), then we get all blocks not associated to Steinberg modules as above, but only the first kind of blocks is non-semisimple since the second kind of blocks is a shift of a block of \( S_{d'}(G_1 T) \) with \( d' \leq p - 1 \). \(\square\)

Now let \( r, n \) be arbitrary again. By reducing to the case \( r = 1 \), we are able to determine the \( M_rD \)-parts of \( G_r T \)-components \( \Theta \) whose restrictions to \( G_r \) have type \( \mathbb{Z}[\tilde{A}_{12}] \). Since the restriction to \((\text{SL}_n)_r\) induces a morphism of stable translation quivers by [1, 4.3] so that the restriction of \( \Theta \) is a component of \( \Gamma_+(\text{SL}_n)_r \), [13, 4.1] implies that the restriction of \( \Theta \) to \((\text{SL}_n)_r \) is also of type \( \mathbb{Z}[\tilde{A}_{12}] \). Now [13, 4.2] implies \( n = 2 \) as \( \text{SL}_n \) is almost simple.
Corollary 5.11. Suppose that $F(\Theta)$ is of type $\mathbb{Z}[\tilde{A}_{12}]$ and $\Theta \cap \text{mod } M_r D \neq \emptyset$. Then $\Theta$ is of type $\mathbb{Z}[\tilde{A}_{\infty}]$, $\Theta^o = \Theta$ and there is a column of simple modules in $\Theta$ which is a symmetry axis with respect to $(-)^o$. There is an $M_r D$-module $V \in \Theta$ such that mod $M_r D \cap \Theta$ consists of all modules on directed paths from $V$ to $V^o$.

Proof. Note that taking the Morita-equivalence in 3.13 followed by the forgetful functor mod $G_r T \to \text{mod } (\text{SL}_2)_r$ is the same as taking the forgetful functor followed by the Morita-equivalence of [16, 5.1]. As the Morita-equivalence of 3.13 maps the $M_r D$-modules in a $G_r D$-block to the $M_r D$-modules in a $G_r T$-block and commutes with the forgetful functors, we can use the arguments of [16, 5.4] to reduce to the case $r = 1$. The result now follows from [5, 4] and its succeeding remarks. 

\hfill \Box

6 Polynomial representations of $B_r T$

In contrast to the ordinary Schur algebra, the algebra $S_d (G_r T)$ is not in general quasi-hereditary, see [21, Section 7]. However, as we show below, the algebras $S_d (B_r T)$ are directed, i.e. quasi-hereditary with simple standard modules. See [4] for the definition of a quasi-hereditary algebra.

Proposition 6.1. Let $G \subseteq \text{GL}_m$ be a reductive group, $T \subseteq G$ a maximal torus and $B$ a Borel subgroup of $G$ containing $T$. Then the algebra $S_d (B_r T)$ is quasi-hereditary and directed.

Proof. Let $\lambda$ be a polynomial weight for $T$ of degree $d$. By [23], II.9.5 and the remark before II.9.3, all weights $\mu$ of the projective cover $\tilde{Z}_r (\lambda)$ of $k_\lambda$ in mod $B_r T$ satisfy $\mu \leq \lambda$ and the weight space for the weight $\lambda$ is 1-dimensional. Since every module with top $k_\lambda$ is a factor module of $\tilde{Z}_r (\lambda)$, the ordering is adapted (see the remarks below [4, Lemma 1.2] for the definition of adapted ordering). Since $G_{B_r T}(\tilde{Z}_r (\lambda))$ is projective in mod $B_r T$ and has simple top $k_\lambda$, it is the projective cover of $k_\lambda$ in mod $B_r T$ and we get $G_{B_r T}(\tilde{Z}_r (\lambda)) \cong \Delta (\lambda)$ by definition, see [4, Section 1]. As the $\lambda$-weight space of $G_{B_r T}(\tilde{Z}_r (\lambda))$ is one-dimensional, $k_\lambda$ occurs only once as a composition factor of $G_{B_r T}(\tilde{Z}_r (\lambda))$, so that $G_{B_r T}(\tilde{Z}_r (\lambda))$ has endomorphism ring isomorphic to $k$ by [4, 1.3]. Now [4, Theorem 1] implies that $S_d (B_r T)$ is quasi-hereditary. Relative to the reverse ordering, $S_d (B_r T)$ obtains the structure of a quasi-hereditary algebra with simple standard modules by setting $\Delta (\lambda) = k_\lambda$, so that $S_d (B_r T)$ is directed. 

\hfill \Box

Now let $T \subseteq \text{GL}_2$ be the torus of diagonal matrices, $U \subseteq \text{GL}_2$ be the group of upper triangular unipotent matrices, $B = U \rtimes T \subseteq \text{GL}_2$ be the subgroup of upper triangular matrices and $L_r D = \overline{B_r T}$. It is well known that $U \cong G_a$, $k[U_r] \cong k[X]/(X^p)$ and $kU_r \cong k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)$, see for example [16, Section 2]. By considering the action of $T$ on $U_r$ and the induced actions on $k[U_r]$ and $kU_r$, we see that the $\mathbb{Z}^2 \cong X(T)$-grading on $k[U_r]$ is given by $\deg (x_i) = p^{i-1}(1, -1)$, where $x_i$ is the residue class of $X_i$ in $k[U_r]$. Then mod $B_r T = \text{mod } U_r \rtimes T$ is the category of $\mathbb{Z}^2$-graded $kU_r$-modules. In
this section, we consider all algebras $k[X_i, \ldots, X_j]/(X^p, \ldots X^p)$, $1 \leq i \leq j \leq r$, to be $\mathbb{Z}^2$-graded with these degrees. In this case, we can extend our results about modules of complexity one to tensor products of these modules with other indecomposable modules.

If $M, N$ as in the next result are indecomposable, general theory (\textit{\text{[3] L10E]}) in combination with \([1, 1]\) shows that $M \otimes k N$ is an indecomposable $B_r T$-module as $k$ is algebraically closed.

**Proposition 6.2.** Let $M \in \text{mod } B_{r-h} T$ be indecomposable such that $c x B_{r-h} T(M) = 1$ and let $N$ be a finite-dimensional non-projective indecomposable graded module over the algebra $k[X_{r-h+1}, \ldots, X_r]/(X^p, \ldots X^p)$. Suppose the component $\Theta \subset \Gamma_s(B_r T)$ containing $M \otimes_k N$ is regular. Then $\Theta \cong \mathbb{Z}[D_\infty]$ and the quasi-length of elements of $\Theta \cap \text{mod } L_r D$ is bounded. If $M \otimes_k N$ is quasi-simple, $\Theta \cap \text{mod } L_r D$ is finite.

**Proof.** As $\Theta$ is regular, we have $c x k[X_{r-h+1}, \ldots, X_r]/(X^p, \ldots X^p)(N) \geq 1$ and thus $c x B_r T(M \otimes_k N) = c x B_{r-h} T(M) + c x k[X_{r-h+1}, \ldots, X_r]/(X^p, \ldots X^p)(N) \geq 2$ (see for example \([24, 3.2.15]\)), so that $\Theta$ is not periodic. Now \([12] \text{ Theorem 1}\) implies $\Theta \cong \mathbb{Z}[D_\infty]$ as $k[X_1, \ldots, X_r]/(X^p, \ldots X^p)$ has wild representation type for $r > 1$ since $p \geq 3$.

We show that the $\tau_{B_r T}$-orbit of $M \otimes_k N$ contains only finitely many $L_r D$-modules. As $\text{mod } L_r D$ is closed with respect to submodules and factor modules, the result then follows from \([2, 6]\).

Let $P = (P_t)_{t \in \mathbb{N}_0}$ be a minimal projective resolution of $M$ as a graded $k[X_1, \ldots, X_r]/(X^p, \ldots X^p)$-module and $Q = (Q_t)_{t \in \mathbb{N}_0}$ be a minimal projective resolution of $N$ as a graded $k[X_{r-h+1}, \ldots, X_r]/(X^p, \ldots X^p)$-module. Then the complex $P \otimes_k Q$ is a minimal projective resolution of $M \otimes_k N$ as a $B_r T$-module, where

$$(P \otimes_k Q)_t = \bigoplus_{i+j=t} P_i \otimes_k P_j$$

and

$$d_{P \otimes_k Q, l} = \bigoplus_{i+j=l} d_{P_i} \otimes \text{id}_{Q_j} + (-1)^i \text{id}_{P_i} \otimes d_{Q_j}$$

for all $l > 0$. Thus, for all $i > 0$, we get $\Omega^{i}_{B_{r-h} T}(M) \otimes_k \Omega^{1}_{k[X_{r-h+1}, \ldots, X_r]/(X^p, \ldots X^p)}(N) \subseteq \Omega^{i}_{B_r T}(M \otimes_k N)$. Let $\lambda \in \mathbb{Z}^2$ such that $\Omega^{1}_{k[X_{r-h+1}, \ldots, X_r]/(X^p, \ldots X^p)}(N) \lambda \neq 0$. By definition of the grading on the tensor product of modules, we get $\text{supp}(\Omega^{i}_{B_{r-h} T}(M[\lambda])) \subseteq \text{supp}(\Omega^{i}_{B_r T}(M \otimes_k N))$ for all $i > 0$. Let $\alpha = (1, -1)$ be the positive root of $G$ relative to $B$. By the proof of \([4]\) the Nakayama functor $N$ of mod $B_r T$ is just the shift by $-(p^r - 1)\alpha$. For every $s \in \mathbb{N}_0$, we get

$$\text{supp}(\tau^{p^r}_{B_r T}(M \otimes_k N)) \supseteq \text{supp}(\Omega^{2p^r}_{B_{r-h} T}(M[\lambda])[-p^s(p^r - 1)\alpha])$$

By \([15] 8.1.1\), there is $s \in \mathbb{N}_0$ such that

$$\Omega^{2p^r}_{B_{r-h} T}(M[\lambda])[-p^s(p^r - 1)\alpha] = M[\lambda][-p^s(p^r - 1)\alpha + p^{-h}\alpha]$$

or

$$\Omega^{2p^r}_{B_{r-h} T}(M[\lambda])[-p^s(p^r - 1)\alpha] = M[\lambda][-p^s(p^r - 1)\alpha].$$
Since similar computations can also be applied to $\tau_{BrT}(M \otimes_k N) \cong N^i \circ \Omega^2_{BrT}(M \otimes_k N)$ for $1 \leq i \leq p^s - 1$, arguments analogous to those in the proof of 4.2 show that the $\tau_{BrT}$-orbit of $M \otimes_k N$ is finite.

Note that if $N$ in the situation of the previous result is projective, then $cx_{BrT}(M \otimes_k N) = 1$ and $\Theta \cap L_rD$ is finite by 4.3. If the component containing $M \otimes_k N$ is not regular, we have $cx_{U_r}(M \otimes_k N) = r$ and the Künneth-formula yields $h = r - 1$ and $cx_{k[X_2, \ldots, X_r]/(X_2^p, \ldots, X_r^p)}(N) = h$.

**Corollary 6.3.** Let $\lambda \in \mathbb{Z}^2$, $\Theta \subseteq \Gamma_s(BrT)$ be the component containing $k_\lambda$. Then $\Theta \cap \text{mod } L_rD$ is finite.

**Proof.** For $r = 1$, this follows from 4.3 since $cx_{BrT}(k_\lambda) = 1$. For $r \geq 2$, $cx_{BrT}(k_\lambda) = r > 1$, so that $k_\lambda$ is not periodic. Now [13, 5.6] shows $F(\Theta) \cong \mathbb{Z}[A_{\infty}]$, so that $k_\lambda$ is a quasi-simple module in a $\mathbb{Z}[A_{\infty}]$-component of $\Gamma_s(BrT)$. Suppose $\Theta$ is not regular. Then $F(\Theta)$ is not regular and the standard almost split sequence [2, V.5.5] shows that $\Omega_{U_r}(k) \in F(\Theta)$. As $\Omega_{U_r}$ is an auto-equivalence of the stable module category, this shows that $\Omega_{U_r}(F(\Theta)) = F(\Theta)$, so that $\Omega_{U_r}$ defines an automorphism of $F(\Theta)$. In particular, $k$ and $\Omega_{U_r}(k)$ have the same quasi-length, so that $\tau_{U_r} = \Omega^2_{U_r}$ implies $\Omega(k)^{2l+1}(k) = k$ for some $l \in \mathbb{Z}$. Thus, $k$ is periodic, a contradiction. Hence $\Theta$ is regular. Since $U_r$ acts trivially on $k_\lambda$, we have $k_\lambda \cong k_{\lambda BrT} \otimes_k k$, where we regard $k$ as the trivial module for $k[X_2, \ldots, X_r]/(X_2^p, \ldots, X_r^p)$. The result now follows from 6.2. \hfill $\square$

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