Out-of-equilibrium transport in the interacting resonant level model and comparison with the boundary sine-Gordon model

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We study the transport properties of the interacting resonant level model (IRLM) for arbitrary values of the interaction parameter $U$, and compare them with those of the Boundary sine-Gordon model (BSG), which are known analytically (Fendley, Ludwig and Saleur, 1995). While the IRLM and BSG models are expected not to be equivalent (except at two particular points: the free-fermion model and the self-dual point), we find that their $I$-$V$ curves are extraordinarily close, and can be distinguished only by a delicate analysis in the Infra Red (IR) regime. We also find that the shot noise in the two models is very similar for small enough values of $U$, but that marked differences appear for larger values of $U \gtrsim 5$. The numerical calculations of the IRLM are done using time-dependent Density Matrix renormalization Group (tDMRG) simulations. The shot noise is obtained by analyzing the second cumulant of the transferred charge, whose expectation value grows linearly with time, and extracting the corresponding rate from the simulations.

I. INTRODUCTION

The field of out-of-equilibrium quantum many body systems has developed rapidly in the last decade. From studies about the equilibration in isolated systems, to questions about transport properties, entanglement entropy, dissipation or the effect of disorder, the issues raised are very diverse, and often fundamental. However, despite the large amount of work - both analytical and numerical - devoted to this field, much fewer exact results are available than in the equilibrium case.

One of the simplest cases where exact out-of-equilibrium results might be achievable is the interacting resonant level model (IRLM) [1]. This model describes spinless fermions propagating on two semi-infinite leads and tunneling through a single site (“dot”) with some additional density-density interaction. From the point of view of mesoscopic physics, the IRLM is a simplified model to describe transport through a single-level quantum dot. From a theoretical point of view, it is a rich playground to investigate the out-of-equilibrium properties of an interacting system, where one can quantitatively compare the result of some powerful methods like integrability, bosonization or density matrix renormalization group (DMRG) [2] calculations. A typical question addressed in this context is to characterize the current-carrying steady state which appears in the long time limit when the system is initially prepared with different fermion densities in both leads.

This model is a free-fermion problem in the absence of density-density interaction, and is then easily solved. In presence of interactions the model is integrable [3], but, for a many-body problem of this type, the possibility to construct the exact eigenstates does not, in general, allow one to determine the steady states mentioned above. In other words, the existence of a Bethe-ansatz solution in equilibrium does not guarantee that the problem out-of-equilibrium can be solved. For such solution to exist, other difficulties have to be surmounted, involving in particular a careful handling of the initial state and the quenching procedure. What happens for the IRLM has been debated quite intensely. In [4], a new “open Bethe-ansatz” was proposed to calculate the steady current for the IRLM model when $U > 0$ and compare them with those of the Boundary Sine-Gordon (BSG) model, and the out-of-equilibrium solution of the latter [6, 7] could be imported to obtain the exact result for the $I$-$V$ curve exist for all values of $U \gtrsim 5$.

Interestingly, the non-interacting case $U = 0$ can also be mapped onto the the BSG model at the free-fermion point. Since formulas for the $I$-$V$ curve exist for all values of the interaction in the BSG case, it is tempting to ask whether they might after all correctly describe the results for the IRLM model when $U$ is varied. A quick analysis of the field theory for the IRLM (see below) shows that this cannot be expected, and that the two models coincide, in the scaling limit, for $U = 0$ and $U_{sd}$ only. Remarkably however, it turns out that the $I$-$V$ curves in both models can be matched to extremely good accuracy, for reasons that we do not fully understand. The difference reveals - especially when combined with evidence coming from studies of the shot-noise - the subtle yet profound differences between the two models.

The paper is organized as follows. After introducing
the model, some important observables and the simulation protocol in Sec. II, we discuss the similarities and differences between the IRLM and the BSG two models from a field theory point of view (Sec. III). We then report on extensive tDMRG simulations of the IRLM away from the self-dual point, focusing on two physical quantities: the steady current (Sec. IV) as well as the current fluctuations (or zero-frequency noise) (Sec. V).

II. DEFINITION OF THE MODEL

A. Model and quench protocol

The interacting resonant level model (IRLM) can be defined in terms of spinless fermions on a one-dimensional lattice:

\[ H_{\text{IRLM}} = H_L + H_R + H_d \]

\[ H_L = -J \sum_{r=-N/2}^{N/2-1} (c_r^+ c_{r+1} + \text{H.c}) \]

\[ H_R = -J' \sum_{r=1}^{N/2-1} (c_r^+ c_{r+1} + \text{H.c}) \]

\[ H_d = -J \sum_{r=\pm 1} \left( c_r^+ c_0 + \text{H.c} \right) + U \sum_{r=\pm 1} \left( c_r^+ c_r - \frac{1}{2} \right) \left( c_0^+ c_0 - \frac{1}{2} \right). \]

Here, \( H_L \) and \( H_R \) describe fermions hopping (or kinetic energy) in the left and the right “leads”, and \( H_d \) encodes the tunneling from the leads to the dot (level at \( r = 0 \)) and the density-density interaction (strength \( U \)) between the dot and the leads. In the following we will set the unit of energy to be the hopping amplitude \( J = 1 \) (bandwidth in the leads equal to \( W = 4J = 4 \)).

As previously done in several works [5, 8–11], the initial state is prepared with an inhomogeneous density, using a bias voltage \(+V/2\) in the left lead, and \(-V/2\) in the right one. Note that instead of using a step function, we use a smooth function interpolating between \(+V/2\) and \(-V/2\):

\[ H_{\text{bias}} = V/2 \sum_{r=-N/2}^{N/2} \tanh(r/w) c_r^+ c_r \]

where \( 2w \) is a smoothing width. In our numerics we typically use \( w = 10 \) lattice spacings. In addition, the initial state is prepared with the non-interacting Hamiltonian \( (U = 0) \) and \( J' = J = 1 \) (as in Ref. 8). This affects the initial state only close to the dot, and is not expected to change the steady state. This procedure reduces the Friedel-like oscillations in the initial density, and also reduces the left-right entanglement entropy (hence smaller matrices in the simulations).

For \( t > 0 \) the bias \( V \) is switched to zero, and the wave function evolves according to \( |\psi(t)\rangle = \exp(-iH_{\text{IRLM}}t)|\psi\rangle \). We are interested in the so-called scaling regime where the bandwidth is much larger than the other energies in the problem (\( 0 < J' \ll J \) and \( V \ll W \)). In this regime the model becomes equivalent to its continuum counterpart, modulo the replacement of \( U \) (lattice model) by \( U_c \) (in the continuum, see Sec. IV A).

B. Current

The (particle) current flowing through the dot is defined as

\[ I(t) = \frac{1}{2} \left[ I(-1, t) + I(0, t) \right] \]

where

\[ I(r, t) = 2J_{r,r+1} \text{Im} \langle \psi(t) | c_r^+ c_{r+1} | \psi(t) \rangle \]

is the expectation value of the current operator associated to the bond \( r, r+1 \). For the two bonds which connect the dot to the left and right leads the hopping amplitude is \( J_{r,r+1} = J' \) (and \( J_{r,r+1} = J \) in the leads). This current is expected to reach a steady value when \( t \) is large (but keeping \( tv_F \) smaller than the system size, where \( v_F = 2J = 2 \) is the Fermi velocity in the leads). This steady value is extracted from the numerical data by fitting \( I(t) \) to a constant plus damped oscillations (more details in Appendix C).

C. Charge fluctuations and shot noise

We will also consider the second cumulant \( C_2 \) of the charge in one lead. It is defined by

\[ C_2(t) = \langle \hat{Q}^2(t) \rangle - \langle \hat{Q}(t) \rangle^2, \]

where \( \hat{Q}(t) = \sum_{r=1}^{N/2} c_r^+ c_r(t) \) is the operator measuring the total charge in the right lead, at time \( t \) in the Heisenberg representation. A typical time evolution of

\[ \begin{array}{c}
\begin{array}{cccccccc}
V/2 & J & J' & J & J & J & J
\end{array}
\end{array} \]

\[ \begin{array}{cccccccc}
\text{Left lead} & U & \text{dot} & U & \text{Right lead}
\end{array} \]

FIG. 1: Schematics of the IRLM. The system is prepared at \( t = 0 \) in the ground state of the model with a chemical potential \( V/2 \) in the left lead, and \(-V/2\) in the right lead. For \( t > 0 \) the system then evolves with the bias switched to zero.
this cumulant is presented in Fig. 2. Since $C_2(t)$ grows linearly with time, a quantity of interest is the rate
\[
S = \frac{d}{dt} C_2(t),
\]
which goes to a constant in the steady regime. The long time limit of $S$ [34] is also a measure of the current noise, defined as the zero-frequency limit $\int_{-\infty}^{\infty} \langle \Delta \hat{I}(0) \Delta \hat{I}(\tau) \rangle d\tau$ of the current-current correlation function (see Sec. A).

This quantity will be studied in Sec. V, and its dependence on the bias will be compared with that of the BSG model.

### III. IRLM VERSUS BSG

#### A. A field-theoretic discussion

As discussed in [12], the IRLM admits two field theoretic formulations which are close to, but in general not identical with, the BSG model. Both formulations are obtained using bosonization. Their difference originates in the fact that one can first make linear combinations of the fermions in each lead then bosonize, or first bosonize, then make linear combinations of the resulting bosons. The first reformulation leads to an anisotropic Kondo Hamiltonian:

\[
H^{(1)}_{\text{IRLM}} = \sum_{\alpha=\pm} H_0(\phi_\alpha) + \frac{\gamma}{\sqrt{\pi}} \kappa_+ \left[ e^{i\beta \phi_+} S_+ + \text{H.c.} \right] \tag{10}
\]

where $H_0(\phi_\alpha) = \frac{1}{2} \int dx (\partial_x \phi_\alpha)^2$ is the free boson Hamiltonian. Here the two leads have been unfolded so that the bosons are chiral, with equal-time commutators $[\phi_a(x), \phi_b(x')] = d_{ab} \delta(x-x')$. We have $\phi_\pm = \frac{1}{\sqrt{2}} \left[ (\sqrt{\pi} - \alpha) \phi_+ \mp \alpha \phi_\pi \right]$, where $\alpha = \frac{U}{\gamma}$ and $\beta^2 = \frac{1}{2} (U - \pi)^2 + 2\pi; \kappa_+ = \eta \kappa_+$. The fields $\phi_\pm$ bosonize the even and odd combinations of physical fermions $\psi_\pm \equiv \frac{1}{\sqrt{2}} (\psi_1 \pm \psi_2); \psi_\pm = \eta \sqrt{\pi} e^{i\beta \phi_\pm}$. Meanwhile, $\eta \kappa_+$ are Klein factors. The self-dual case is $U_{sd} = \pi$ corresponding to $\alpha = \sqrt{\pi}$, while the non interacting case is $U = 0$. Finally, the amplitude $\gamma$ is related with amplitudes in the initial field theoretic formulation following [12]. How this amplitude is related with the tunneling term in the lattice Hamiltonian (the “bare” coupling constant) will be discussed below.

The second reformulation mixes somehow the Kondo and the BSG Hamiltonians, and reads

\[
H^{(2)}_{\text{IRLM}} = \sum_{\alpha=\pm} H_0(\phi_\alpha) + \frac{\gamma}{\sqrt{\pi}} [V_1(0) O_2(0) S_+ + \text{H.c.}] \tag{11}
\]

where $V_{\pm 1} = e^{\pm i\beta_1 \phi_1}, O_2 = \kappa_1 V_2 + \kappa_2 V_{-2}, V_{\pm 2} = e^{\pm i\pi/2} \phi_2, \kappa_\alpha = \kappa_\alpha$. We have set $\beta_1 = \sqrt{2\pi} - \alpha \sqrt{2}, \kappa_\alpha = \kappa_\alpha$. Meanwhile, $\gamma \kappa_2$ is related with amplitudes $V_1$ and the hamiltonian (11) where $V_{\pm 2} \rightarrow e^{\pm i\pi/4} V_{\pm 2}$.

Meanwhile, it is tempting to consider a BSG model

\[
H_{\text{BSG}} = H_0(\phi_+ \gamma \cos \phi_0(0) \tag{12}
\]

with Hamiltonian chosen in such a way that the tunneling term has the same dimension $g \equiv \frac{\beta^2}{\pi}$ as in the IRLM, and $\gamma' \propto \gamma$. The voltage can similarly be introduced by $e^{\pm i\pi/2} \phi_2 \rightarrow e^{\pm i(\phi_2 + V t)}$. The difference between (11) and (12) is obvious: the second Hamiltonian involves only two vertex operators, and does not contain any spin.

Note: from now on, we suppress mention of the coordinate in the exponentials of the fields at the origin. We do instead mention the time coordinate whenever we use a Heisenberg representation.

The current for the models (11) and (12) can be calculated perturbatively in the tunneling amplitudes $\gamma$ and $\gamma' \propto \gamma$. Since the dimensions of the perturbing operators are chosen to coincide, the corresponding Keldysh expansions are bound to be somewhat similar. At first order, only the two-point correlation functions are involved. For (11) we have the two-point function

\[
\langle \langle V_1(t) O_2(t) S_- + \text{H.c.} \rangle \rangle \langle V_1(t') O_2(t') S_- + \text{H.c.} \rangle
\]

but since $\langle V_2(t)V_2(t') \rangle = 0$ and $\langle V_2(t) V_{-2}(t') \rangle = \langle V_{-2}(t)V_2(t') \rangle$, as well as $\langle S_+ S_- \rangle = 0, \langle S_+ S_+ \rangle = \langle S_- S_- \rangle = 1$ (since we are dealing with spin 1/2), we
can write

\[ \langle (V_1(t)O_2(t)S_- + H.c) (V_1(t')O_2(t')S_- + H.c) \rangle = 4 \langle (V_1V_2(t)) (V_{-1}V_{-2}(t')) \rangle = 8 \cos \beta \phi_2(t) \cos \beta \phi_2(t') \] (13)

and thus the terms for the expansions for the IRLM and the BSG coincide after adjusting the relative scale of \( \gamma \) and \( \gamma' \). This coincidence breaks down beyond first order. While multiple correlators of \( O_2 \) have a similar structure to those of \( \cos \beta \phi_2 \), they are now multiplied by multiple correlators of \( (V_1S_- + H.c) \), which involve different kinds of terms - in particular, the spin operators \( S_\pm \) must alternate (since we are dealing with a spin 1/2), hence \( V_{\pm 1} \) also. This makes the Keldysh expansions of the current for the two Hamiltonians definitely different [13].

Rather than focus on the detailed difference between the perturbative terms in the ultra violet (UV) expansion, it is more satisfying physically to focus on the IR Hamiltonians. It is of course well known that the BSG and anisotropic Kondo fixed points are different, but this can be seen most clearly if we wonder how these fixed points (which, in our problem, both correspond to perfect transmission, with the impurity spin hybridized with the leads) are approached. This can be determined using general techniques of integrability. This time, it is more convenient to use the Hamiltonian (10). According to [14], the approach to the IR fixed point is given by an infinite series of operators \( O_{2n} \), which are well defined expressions in terms of the stress energy tensor, and whose coefficients are known exactly, and scale as \( \gamma^{-(2n-1)/2} \), where \( g \) is the dimension of the perturbation in the UV, \( g = \frac{\beta^2}{\pi} \). We have for instance

\[
O_2 = \frac{1}{2\pi} T \\
O_4 = \frac{1}{2\pi} T^2 : \\
O_6 = \frac{1}{2\pi} \left( T^3 : -\frac{c}{12} : T \partial^2 T : \right) \] (14)

while

\[ T = -2\pi : (\partial \phi_+)^2 : + i (1 - g) \sqrt{\frac{2\pi}{g}} \partial^2 \phi_+ \] (15)

and \( c = 1 - \frac{(1 - g)^2}{g} \). Here all the fields are defined exactly like before.

To understand the argument, it is now enough to consider the derivatives

\[
\partial \varphi_+ \sim (\psi_1^\dagger + \psi_2^\dagger) (\psi_1 + \psi_2) \sim \partial \phi_1 + \cos \sqrt{\pi} \phi_2 \\\n\partial \varphi_- \sim (\psi_1^\dagger - \psi_2^\dagger) (\psi_1 - \psi_2) \sim \partial \phi_1 - \cos \sqrt{\pi} \phi_2 \] (16)

where \( \sim \) means up to proportionality coefficients in all the terms on the right hand side. In the generic case, it follows that \( \partial \phi_+ \) is a linear combination of \( \partial \phi_1 \) and \( \cos \sqrt{\pi} \phi_2 \). Therefore, the stress tensor term \( T (15) \) is a combination involving, as far as charge transferring terms are concerned, \( \partial \phi_1 \cos \sqrt{\pi} \phi_2 \) and \( \partial \phi_2 \sin \sqrt{\pi} \phi_2 \) - the latter term coming from the \( \partial^2 \phi_+ \). Now - and this is the crucial point - when considering \( T^2 \) : and the products

\[ \partial \phi_1 (\text{resp.} 2) (z) \cos \sqrt{\pi} \phi_2 (z) \partial \phi_1 (\text{resp.} 2) (w) \cos \sqrt{\pi} \phi_2 (w) \] and using

\[ \partial \phi_1 (\text{resp.} 2) (z) \partial \phi_1 (\text{resp.} 2) (w) \sim \frac{1}{(z-w)^2} + \ldots \]

together with

\[ \cos \sqrt{\pi} \phi_2 (z) \cos \sqrt{\pi} \phi_2 (w) \sim \frac{1}{(z-w)^2} (1 + \ldots) + (z-w)^2 (\cos 2\sqrt{\pi} \phi_2 (w) + \ldots) \] (17)

we will generate in : \( T^2 \) : a term \( \cos 2\sqrt{\pi} \phi_2 \). Meanwhile, the bosonized expression of \( T \) itself involved a \( \partial \phi_1 \cos \sqrt{\pi} \phi_2 \) term, so we see we generate terms corresponding to different transfers of charge - that is, different integer multiples of \( \sqrt{\pi} \phi_2 \) in the exponentials. Meanwhile, all these terms come with the proper power of the UV coupling constant \( \gamma \), in this case:

\[
\partial \phi_1 \cos \sqrt{\pi} \phi_2, \text{ coupling } \gamma^{-1/(1-g)}, \text{ charge } e \\
\cos 2\sqrt{\pi} \phi_2, \text{ coupling } \gamma^{-3/(1-g)}, \text{ charge } 2e \] (18)

We see that the transfer of charge in the IR involves in general two terms at leading order, transporting respectively \( e \) and \( 2e \), with different scaling coefficients. The full counting statistics [15] at leading order for instance would be a function of \( \gamma^{-1/(1-g)} e^{-ix} \) and \( \gamma^{-3/(1-g)} e^{-2ix} \) (where \( \chi \) is the counting variable coupled to the charge. In particular, the leading term always corresponds to tunneling of electrons. This is in agreement with the fact that the anisotropic Kondo fixed point is a Fermi liquid.

Note that the argument breaks down if the amplitude of the leading term \( \gamma^{-1/(1-g)} e^{-ix} \) happens to vanish. This is precisely what happens in the self-dual case, since then \( \partial \phi_+ \) contains the term \( \cos \sqrt{\pi} \phi_2 \) only. In that case, the bosonized version of \( T \) does not contain a \( \cos \sqrt{\pi} \phi_2 \) term anymore, while the \( : T^2 : \) term still contains a \( \cos 2\sqrt{\pi} \phi_2 \) term as before. In fact, one can show that at all orders in the \( O_{2n} \), all that appears are \( \cos 2\sqrt{\pi} \phi_2 \) terms. Hence the transferred charges in this case are multiples of \( 2e \), not of \( e - a \) fact in agreement with the equivalence with the boundary sine-Gordon model at this point. Note that the result is not incompatible with the fixed point being Fermi liquid: what happens is simply that amplitudes in the mapping conspire to cancel the usual term describing tunneling of single electrons, and what is observed is tunneling of pairs instead.

The other case where the argument breaks down is the free-fermion case \( g = \frac{1}{2} \). In this case indeed, all quantities \( O_{2n} \) can be expressed solely as

\[ O_{2n} \sim \psi_+^\dagger \psi_{2n-1} \psi_+ \] (19)
and thus all involve only \( \cos \sqrt{\frac{8\pi}{3}} \phi \), corresponding to the transfer of charges \( e \), and an expansion for the FCS in terms of \( e^{-ix} \).

To summarize this technical section: the FCS in the IR involves in general \( \gamma^{-1/(1-g)} e^{-ix} \) and \( \gamma^{-3/(1-g)} e^{-2ix} \), corresponding at leading order to transfers of charge \( e \) and charge \( 2e \). It involves the first combination only when \( g = \frac{1}{2} \) and the second only when \( g = \frac{1}{4} \). The IR behavior is thus dominated by transfer of electrons for all values of \( U \) but \( U_{sd} \), where it then dominated by transfer of pairs of electrons.

This fact is the best way to state physically the difference between the IRLM and the BSG models. For the BSG model (12) the approach to the IR fixed point is up to operators that do not transfer charge, only by the operator \( \cos \) described, up to operators that do not transfer charge, \( \text{BSG model} \) (12) the approach to the IR fixed point is transfer of pairs of electrons.

Despite the foregoing long chain of arguments, it remains tempting to close one’s eyes and ask how well the BSG model (12) might reproduce the transport properties of the IRLM away from the two magic points \( U = 0 \) and \( U_{sd} \) to produce \( IV \) curves. The current for the BSG model - after some simple manipulations to allow for a slight difference in geometry - admits two series expansions [6, 7]

\[
I^\text{BSG} = \frac{V g}{2\pi} \sum_{n=1}^{\infty} a_n(g) \left( \frac{V}{T^\text{BSG}} \right)^{2n(g-1)}
\]

at large \( V \) (the UV regime) and

\[
I^\text{BSG} = \frac{V}{2\pi} - \frac{V}{2\pi g} \sum_{n=1}^{\infty} a_n(1/g) \left( \frac{V}{T^\text{BSG}} \right)^{2n(-1+1/g)}
\]

at small \( V \) (the IR regime). Here \( g = \frac{g^2}{8\pi} \),

\[
a_n(g) = (-1)^{n+1} \frac{\Gamma(3/2)\Gamma(n)}{\pi}\Gamma(n)(n+g-1+3/2)
\]

and \( T^\text{BSG} \propto \gamma^{1/1-g} \).

It will be convenient in what follows to use the scaling form

\[
I^\text{BSG} = T^\text{BSG} \vartheta(V/T^\text{BSG})
\]

where e.g. at small \( x \):

\[
\vartheta(x) = \frac{x}{2\pi} - \frac{x}{2\pi g} \sum_{n=1}^{\infty} a_n(1/g) x^{2n(-1+1/g)}.
\]

In the particular case \( g = 1/2 \) the series can easily be summed to give

\[
2\pi I^\text{BSG}(g = 1/2) = \frac{T^\text{BSG}}{2} \arctan \frac{2V}{T^\text{BSG}}
\]

This matches the well known result for the RLM [8] (IRLM at \( U = 0 \))

\[
2\pi I^\text{RLM}(U = 0) = 4t_B \arctan \frac{V}{4t_B}
\]

after the identification \( t_B = \frac{T^\text{BSG}}{8} \). Meanwhile, recall that if the perturbation in BSG is normalized precisely as \( 2\gamma^{\cos \frac{\theta_p}{\sqrt{2\pi}}} \) (so \( \gamma' = \sqrt{\frac{2}{\pi}} \gamma \)) we have the relation

\[
T^\text{BSG} = c^\text{BSG}\gamma^{1/1-g}
\]

with

\[
c^\text{BSG} = \frac{2}{g} \left( \sqrt{\frac{2\sin \pi g \Gamma(1-g)}{\sqrt{\pi}}} \right)^{1/1-g}.
\]

So when \( g = \frac{1}{2} \), \( T^\text{BSG} = 8\gamma^2 \), and \( t_B = \gamma^2 \).

We now propose to compare the measured \( I-V \) curves for the IRLM in the scaling limit with the analytical expressions for the BSG current. In order to do this we need to identify the coupling \( g = \frac{g^2}{8\pi} \) for the lattice IRLM: this can be done by studying the large \( V \) limit and comparing the algebraic decay of the current with the prediction from perturbation theory on (11) or (12). We are then left with the parameter \( T^\text{BSG} \) that we determine simply by a best fitting procedure. Note that, since the mapping on BSG is not supposed to work, there is no reason to use equation (28). We know by dimensional analysis that \( T^\text{BSG} \propto \gamma^{1/1-g} \), but it will be interesting to see what the dependency of the prefactor on \( g \) looks like, compared with (28).

IV. CURRENT: NUMERICS AND COMPARISON WITH THE BSG RESULTS

A. Power-law decay of the current at large bias, and associated exponent

In the scaling regime \( (J' \ll J \text{ and } V \ll W) \) the steady current of the IRLM vanishes as a power law

\[
I^\text{IRLM}(V) = \text{cst} \cdot V^{-b}
\]

in the large voltage limit [5, 8, 16], with an exponent given by

\[
b = \frac{1}{2} \frac{U_c}{\pi} (2 - \frac{U_c}{\pi}).
\]

The interaction constant \( U \) in the lattice model and its counter part \( U_c \) in a continuum limit have a simple relation (see the supplemental material of Ref. 17):
\[ U_c = \begin{cases} 
4 \arctan{U/2}, & U < 2 \\
4 \arctan{2/U}, & U > 2.
\end{cases} \]  

(31)

\( b \) reaches the maximum \( b_{\text{max}} = \frac{1}{2} \) at the self-dual point located at \( U = 2 \) (or equivalently \( g = \frac{1}{4} \) and \( U_c = \pi \)), and it is linear in \( U \) close to \( U = 0 \). As shown in Fig. 4, the exponent \( b(U) \) extracted from the numerics by fitting the current in the large bias regime is in good agreement with the analytical formula [Eqs. (30)-(31)]. It should be noted that for \( U < 0 \) the exponent \( b \) becomes negative, which means that the current keeps growing at large bias in presence of attractive interactions.

\[ U_c(U < 2) \quad \text{DMRG} \quad \text{U}_c(U > 2) \]

\[ \text{U}_c(U < 2) \quad \text{U}_c(U > 2) \]

FIG. 3: Interaction strength in continuum limit \( U_c \) versus interaction strength on the lattice \( U \). Red dots are constructed by inverting Eq. (30), where \( b \) is obtained from DMRG data; blue and green lines represent Eq. (31).

B. Comparison with the \( I-V \) curve of the BSG model

The current-voltage curve of the IRLM is known exactly in two cases: the noninteracting \( U = 0 \) case [8, 20–23] and self-dual point \( U = 2 \) [5]. At these two points, the IRLM maps exactly to the BSG. In this section we analyze to which extent the current \( I_{\text{BSG}} \) given by Eq. (20) could also describe the current of the IRLM away from the two cases above. In other words, we will attempt to describe the current of the IRLM in terms of the function \( \vartheta \) [Eq. (24)] defined in Sec. III for the BSG model.

1. Large bias, \( U, b \) and \( g \)

The exponent \( b \) of the IRLM is known exactly as a function of \( U \) Eqs. (30-31). In order to get the same large-bias exponent in the IRLM and in the BSG model, the parameter \( g \) of the BSG model has to become a function of \( U \) (or \( U_c \)):

\[ b = 1 - 2g \]  

(32)

with \( b \) given in Eq. (30).

2. Large bias and \( c_{\text{IRLM}} \)

Since the prefactor \( c_{\text{BSG}} \) appearing in the definition of the energy scale \( T_{\text{BSG}} \) a priori not a universal quantity, it is natural to redefine it for the IRLM. In other words, to compare the current in the IRLM with that of the BSG model, we introduce a scale

\[ T_B = c_{\text{IRLM}} \cdot J'/(1 - g), \]  

(33)

where \( J' \) appears with the same exponent as \( \gamma \) in \( T_{\text{BSG}} \) [Eq. (27)], but a different prefactor, \( c_{\text{IRLM}} \). The latter is adjusted numerically (fit) so that the analytical curve for a given \( g \) coincides with the DMRG data at large bias:

\[ J_{\text{IRLM}} \approx T_B \cdot \vartheta(V/T_B) \text{ when } V/T_B \gg 1 \]  

(34)

[withe \( \vartheta \) defined in Eq. (24)]. The result of this procedure is a function \( c_{\text{IRLM}}(b) \) (top panel of Fig. 5) or equivalently \( c_{\text{IRLM}}(U) \) (bottom panel of Fig. 5). For comparison, we also plotted \( c_{\text{BSG}} \) Eq. (28).

In case of the free-fermion problem, i.e. \( g = 1/2 \) and \( b = 0 \), the two models are equivalent and the theoretical value of crossover parameters is \( c_{\text{BSG}} = 8 = c_{\text{IRLM}} \), in a good agreement with DMRG data. Since the IRLM at \( U_c = \pi \) maps exactly onto the BSG model [5], we expect to have \( c_{\text{IRLM}} = c_{\text{BSG}} \) at this point too. The numerics give \( c_{\text{IRLM}} \approx 4.66 \) while the exact result is...
$c_{\text{BSG}} = 8 \cdot \Gamma(3/4)^{4/3}/\pi^{2/3} \approx 4.89$. This 5% discrepancy is presumably due to finite $J'$ effect, i.e. deviation from the scaling regime.

In Fig. 5 we also marked the value of $c_{\text{IRLM}}$ at the self-dual point which was found by Boulat et al. [5]. Their estimate for $T_B$ at this point is $2.7 c_0 (J')^{1/(1-g)} \approx 4.65$ (with $g = 1/4$ and $c_0 = 4\sqrt{\pi}/(17\Gamma^3(3/4))$). This value is in a good agreement with our data, but differs by about 5% from the exact value. Away from the free-fermion point and away from the self-dual point, the curves for $c_{\text{IRLM}}$ and $c_{\text{BSG}}$ are significantly different. $c_{\text{IRLM}}$ monotonically decreases with increasing $U$ but $c_{\text{BSG}}$ grows past the self-dual point at $U = 2$. From this point of view, the models are thus generically not equivalent, as discussed in Sec. III. As already mentioned, the prefactors $c_{\text{IRLM}}$ or $c_{\text{BSG}}$ are not expected to be universal quantities, so the fact that $c_{\text{IRLM}} \neq c_{\text{BSG}}$ is not surprising at all. We will now go further and investigate if the expression of Eq. (34) could also be used, at least approximately, for finite $V/T_B$.

3. Finite bias

Once the large-bias part of the current curve of the IRLM is adjusted to match that of the BSG (through $g$ and $c_{\text{IRLM}}$, as discussed above), we can see if the agreement persists at lower bias. The results are displayed in Figs. 6 and 7.

The remarkable and somewhat unexpected fact is that the BSG function Eqs. (30)-(31) is a very good approximation of the IRLM current, even when $V/T_B$ is of order 1. While the agreement is excellent at the self-dual point (as it should, and as already noted in Refs. [5, 8, 24]) the BSG function continues to describe well the IRLM current away from $U = 0$ and $U = 2$. In fact, for $U = 1$, 3 – 6, the deviation between BSG and IRLM is of the same order of magnitude as the numerical precision [35]!

As commented earlier, it is clear that one can expect a certain amount of similarity between the currents in the BSG and the IRLM. Thanks to our matching of the exponents and the $T_B$ scales numerical (Sec. IV B 2), the leading terms must agree by construction. On the other hand - as illustrated in Fig. 8 which shows (dotted or dashed lines) the leading term, or the sum of the first 2 or 10 terms in this expansion - it is clear that the leading term only is not enough to reproduce the IRLM data close to the maximum of the current. The agreement between the $J$-$V$ curves for the two models in this region also remains mysterious. What probably happens is that the first few terms in the UV expansion are very close to each other. We have not been able to check this, because of the difficulty in calculating the higher-order terms in the Keldysh expansion of the IRLM current (this calculation is easier in the BSG model, in part because of the underlying integrability). But this raises the question: could it be that the field theoretic arguments in section III are flawed and that the two models out-of-equilibrium are in the same universality class? To answer this question, we turn again to the IR properties.

C. Small bias and backscattered current

The low-bias expansion of the steady current in the IRLM has been computed up to order $O(V^6)$ by Freton and Boulat [17]. They computed the backscattered current $I_{\text{BS}} = V/(2\pi - J$, i.e. the difference between the current $I$ and the value of the current in absence of impurity.
FIG. 6: Rescaled $I$-$V$ curves for different values of interaction. $U = 1$ (top panel) and $U = 2$ (bottom). The symbol shapes encode $J'$. As expected the agreement between the IRLM numerics and the BSG is excellent for $U = 2$ (self-dual point), but it is also very good for $U = 1$, where the models are a priori not equivalent.

Their result reads [36]:

$$I_{BS} = \frac{X V^3}{48 g^2 T_B^2} \left[ 1 + \frac{3 \kappa_4 * 3 V^2 (X^2 - 10 X + 5)}{40 g^2 T_B^2} \right] + O(T_B^{-6}),$$

(35)

where $X = 4g - 1$ and

$$\kappa_{2n} = \frac{(g/\pi)^{n-1}}{(n - \frac{1}{2}) n! \Gamma \left( \frac{g}{2(1-g)} \right)} \frac{\Gamma \left( \frac{2n-1}{2(1-g)} \right)}{\Gamma \left( \frac{1}{2(1-g)} \right)}.$$  

(36)

This shows that for the IRLM $I_{BS}$ vanishes as $V^3$ at low bias. On the other hand, the backscattered current in the BSG model can be read of Eq. (21), and its leading term has an exponent which varies continuously with $g$: $I_{BS}^{bg}/I_{BS} \sim (V/T_{BS})^{2/g-1}$. So, for sufficiently low $V$ the numerical data for the IRLM should show a $V^3$ behavior and should depart from the BSG results if $U \neq 0$ and if $U \neq 2$ (at the self-dual point the coefficient of the $V^3$ and $V^5$ terms vanish and the leading term in $I_{BS}$ becomes $O(V^7)$ [17], in agreement with the BSG result at $g = 1/4$). All these features can be understood with an IR perturbative analysis similar to the one sketched in

FIG. 7: Same as Fig. 6 for $U = 4, 5$ and $U = 6$. The symbol shapes encode $J'$ (see the legend in Fig. 6). The full lines provide some comparison with the BSG solution, after the coupling $g$ and the crossover constant $c_{IRLM}$ have been adjusted (see text). The IRLM numerics and the BSG agree relatively well for $U = 4, 5$ and $U = 6$, although the models are a priori not equivalent.
section III. The data plotted in Fig. 9 illustrate the low-bias behaviors of \( I_{BS} \) at the self-dual point, and at a more generic value of \( U \). We see that, in fact, the currents in the BSG and IRLM have different analytical behaviors in this region - and that these behaviors are in agreement with the field theoretic analysis.

V. CHARGE FLUCTUATIONS AND CURRENT NOISE

A. Second charge cumulant rate

To investigate further the differences between IRLM and BSG models, we consider the current noise, as defined in Eq. (9). Results for the BSG model follow from more general calculations for the full counting statistics [15] of BSG [6, 7]:

\[
\langle BSG \rangle = \frac{V g}{\pi} \sum_{n=1}^{\infty} \frac{a_n(g)}{n} \left( \frac{V}{T_{BSG}} \right)^{2n(g-1)} \left( e^{i x_n/2} - 1 \right)
\]  

(37)
where $\chi$ is a “counting” parameter. This also can be expanded at the low bias $V$ regime as

$$F_{\text{BSG}}(\chi) = \frac{V}{2\pi} i\chi + \frac{V}{\pi} \sum_{n=1}^{\infty} \frac{a_n(1/g)}{n} \times \left( \frac{V}{T_{\text{BSG}}} \right)^{2n(-1+1/g)} \left( e^{-i\chi n/2g} - 1 \right).$$

From this function one can get the first charge cumulant, that is the mean current $I(V) = \partial F/\partial \chi$ [Eqs. (20), (21)]. One can also get the current noise:

$$S(V) = -\frac{\partial^2 F}{\partial \chi^2}.$$ (39)

Note that charges have been normalized so that tunneling at small coupling (large bias) is dominated by $\frac{\chi}{2g}$ charges. (This convention corresponds to one electron tunneling from one wire to the other in two steps). The expansion at low bias shows that, for BSG, the tunneling charge at large coupling is $\frac{\chi}{2g}$.

The current noise has already been investigated numerically using tDMRG for the RLM ($U = 0$) [20] as well as at the self-dual point [11, 25]. In Refs. 20, 25 $S$ was formulated in terms of the zero-frequency limit of the current-current correlations. In Ref. 11, using a modified time-evolution with an explicit counting field $\chi$, the cumulant generating function $F$ was estimated numerically and the noise was extracted as the coefficient of the $\chi^2$ term. More recently, a functional renormalization group approach [26] was used to compute the noise in the IRLM [27][37], specially in the regime of small $U$.

Instead, here we compute the current noise numerically using the relation between $S$ and the fluctuations of the charge, as described by Eq. (9). At any time, $C_2(t)$ is obtained by summing all the connected density-density correlations in the right lead:

$$C_2(t) = \sum_{r,r' \geq 1} G(r, r')$$ (40)

$$G(r, r') = \langle \psi(t)|c_r^t c_r c_{r'}^t c_{r'}|\psi(t)\rangle - \langle \psi(t)|c_r^t c_r|\psi(t)\rangle \langle \psi(t)|c_{r'}^t c_{r'}|\psi(t)\rangle.$$ (41)

$S$ is then obtained by extracting (fits) the coefficient of the linear growth of $C_2(t)$ with time.

The resulting $S$-$V$ curves are presented in Figs. 10-11 (see Fig. 13 for some raw data, without rescaling). For $U = 0$ (upper panel of Fig. 10) the data are in good agreement with the exact free-fermion result:

$$\frac{S(V)}{T_B} = \frac{1}{8\pi} \left( \arctan \left( \frac{2V}{T_B} \right) - \frac{2V/T_B}{1 + (2V/T_B)^2} \right)$$ (42)

Our data at $U = 0$ are also consistent with the results of Ref. 25.

At $U = 0$ as well as for $U \neq 0$ we observe a good collapse of the rescaled curves (obtained for different values of $J'$) onto a single master curve, as for rescaled current $I(V)$ or the entanglement entropy rate $\alpha(V)$ [8]. This indicates that, in the scaling regime, $S/T_B$ is a function of the rescaled voltage $V/T_B$.

![Graph showing rescaled charge cumulant rate $S/T_B$ versus $V/T_B$ for $U = 0$ (top), 1 (middle) and 2 (bottom). The symbol shape encodes $\nu$ (for details see the legend of Fig. 6). The full line for $U = 0$ corresponds to Eq. (42). For $U \neq 0$ the full lines correspond to Eqs. (37), (38) and (39).](image)

To analyze these data, we compare the shot noise $S/T_B$ of the IRLM with that of the BSG (full lines in Figs. 10-11). We stress that there is no new adjustable parameter, since for each $U$ and $J'$ we use the scale $T_B$ (and $c_{\text{IRLM}}$ determined from the analysis of the current. At $U = 2$
is approaches $e/2$ at large bias. It is however more difficult to extract the charge at small voltage in general. Since both $S$ and $I_{BS}$ become very small at low bias (almost perfect transmission), it is difficult to achieve a good numerical precision for these two quantities and for their ratio - the so-called backscattering Fano factor. The Fano factor is plotted in Fig. 12 for $U = 1$ (upper panel) and $U = 2$ (lower panel). Since errors (due to finite-time simulations) are the largest at low bias, one may discard the 3 or 4 lowest-bias data points. In that case, the Fano factor extrapolates to some value close to $\frac{e}{2g} = 2e$ at the exactly solvable point $g = \frac{1}{2} (U = 2)$. It should be noted that a very similar result has been obtained in Ref. [25]. The data for the other values of the interaction is unfortunately harder to analyze, but it is not incompatible with a charge $e$ for all other values of the interaction (as expected from the field theoretic discussion). At $U = 1$ for instance (top panel of Fig. 12), the low-bias limit of the Fano factor is indeed close to 1 (that is $e$) if we again allow ourselves to discard the three points at low $I_{BS}$, where we know - by comparison with the $U = 2$ case - that the error should be the largest.

**B. Charge of the carriers**

We checked numerically that we get the correct charge $e$ at large voltage, independently of the interaction. This was done by computing the ratio $S/I$, and checking that (bottom panel in Fig. 10) the data turn out to be in excellent agreement with the theoretical prediction for the BSG [Eqs. (38), (37) and (39)]. At $U = 1$, the agreement is still quite good, but for large values of $U$ [Fig. 11] the difference between the current noise of the IRLM and that of the BSG model become important.

**FIG. 11: Same as Fig. 10, for $U = 4, 5$ and 6.**

**FIG. 12: $S/I_{BS}$ versus rescaled bias $V/T_B$ for $U = 1$ and 2.** This ratio is also known as the backscattering Fano factor.
VI. SUMMARY AND CONCLUSIONS

Using time-dependent DMRG we have performed some numerical investigations of the $I$-$V$ curves of the IRLM, in a large range of voltage $V$ and interaction strength $U$. We focused on the scaling regime, where the rescaled steady current $I/T_B$ is a function of the rescaled bias $V/T_B$. The results were compared with the $I$-$V$ curves of a cousin model, the BSG model. Somewhat surprisingly, the BSG and the IRLM turn out to have very similar $I$-$V$ curves, even away from the two point where they are known to be equivalent (the free-fermion and self-dual points). It is only by doing a careful analysis in the limit of small bias (IR regime) that some differences between the two models could be observed: while the IRLM has a backscattered current which scales as $V^3$ [17], the exponent in the case of the BSG is a continuous function of the interaction strength. We also computed numerically the shot noise $S$ in the IRLM. The method we used is to extract $S$ from the rate at which the second cumulant of the transferred charge grows with time. The results show that, as for the current, the noise curves for values of $U$ less than 4 is striking, and, we feel, largely unexplained.

In conclusion, we have found that the transport properties of the IRLM and BSG model are different away from the non-interacting and the self-dual points. All predictions from field theory have been vindicated. In particular, we have found good agreement with the prediction that, for the IRLM, the tunneling charge in the IR is given by the electron charge which is a continuous function of the coupling (the dimension of the tunneling operator in the UV).

Nonetheless, the similarity of the $I$-$V$ curves for all values of $U$ and the noise curves for values of $U \leq 4$ is striking, and, we feel, largely unexplained.

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Appendix A: Second charge cumulant and zero-frequency current noise

To establish the (classical) relation between the rate of the second charge cumulant [Eq. (9)] and the current noise, one starts by writing the charge in right lead as an integral of the current

$$\dot{Q}(t) = \dot{Q}_0 + \int_0^t \dot{I}(\tau) d\tau. \quad (A1)$$

If we denote by $\text{var}[\dot{X}]$ the variance $\langle \dot{X}^2 \rangle - \langle \dot{X} \rangle^2$ of an operator $\dot{X}$, we have

$$C_2(t) = \text{var}[\dot{Q}(t)] = \text{var}[\dot{Q}_0 + \int_0^t \dot{I}(\tau) d\tau] = \text{var}[\dot{Q}(0) + \int_0^t \dot{I}(\tau) d\tau]. \quad (A2)$$

In the last equality we have used $\Delta \dot{I}(t) = \dot{I}(t) - \langle \dot{I}(t) \rangle$. Expanding Eq. (A2) we get

$$C_2(t) = \text{var}[\dot{Q}(0)] + \int_0^t \int_0^t \langle \dot{I}(\tau) \dot{I}(\tau') \rangle d\tau d\tau' + \int_0^t \langle \dot{I}(\tau) \dot{Q}(0) \rangle + \int_0^t \langle \dot{Q}(0) \dot{I}(\tau) \rangle. \quad (A3)$$

We then make the assumption that correlator $\langle \dot{Q}(0) \dot{I}(\tau) \rangle$ decays sufficiently quickly with $\tau$, such that the last line in the equation above is small compared to $t$ when $t \to \infty$. We further assume that $\langle \dot{I}(\tau) \dot{I}(\tau') \rangle$ decays sufficiently quickly with the time difference $|\tau - \tau'|$. In the limit $t \to \infty$, the double integral will be dominated by $\tau$ and $\tau'$ of the order of $O(t)$, and $|\tau - \tau'| \ll t$. At sufficient large times the system is in a (quasi) steady state and two-time correlations only depend on the time difference $|\tau - \tau'|$. It follows that the double integral can be approximated by $t \int_0^t \langle \dot{I}(0) \dot{I}(\tau) \rangle d\tau$, or by $t \int_{-\infty}^{\infty} \langle \dot{I}(0) \dot{I}(\tau) \rangle d\tau$. We finally get

$$C_2(t) \simeq t \int_{-\infty}^{\infty} \langle \dot{I}(0) \dot{I}(\tau) \rangle d\tau \quad (A4)$$

and

$$S \simeq \int_{-\infty}^{\infty} \langle \dot{I}(0) \dot{I}(\tau) \rangle d\tau. \quad (A5)$$

Appendix B: Details about the numerical simulations

The simulations are performed using a tDMRG algorithm [28, 29], implemented using the C++ iTensor library [30]. We approximate the evolution operator by a matrix-product operator (MPO) [31] with a 4-th order [8]
N size is tically Trotter scheme. The largest time for our numerics is typ-
tically \( \tau = 0.2 \), while the system size is \( N = 257 \) sites (128 sites in each lead).

The convergence of the data with respect to the maxi-
mum discarded weight \( \delta \) and Trotter time step \( \tau \) is illus-
trated in Fig. 14.

We are mostly interested in the scaling regime where \( J' \ll J \approx 1 \). However, if \( J' \) becomes very small the time to reach a (quasi-) steady state becomes very large, which is difficult to handle in the simulations. In practice we use \( J' \) from 0.08 up to 0.5 and \( V \lesssim 2 \) (to be compared with the bandwidth \( W = 4 \)). To check that the model is sufficiently close to the scaling regime, one verifies that rescaled quantities, like \( I/T_B \), do not depend too much on \( J' \) once they are plotted as a function of the rescaled bias \( V/T_B \).

Appendix C: Current oscillations

Since we are dealing with finite time simulations, the steady current is estimated by fitting \( I(t) \) to a constant plus damped oscillations:

\[
I(t) \approx I_0 + A \cos(\Omega t + \phi) e^{-t/\tau} + A_2 \cos(\Omega t + \phi_2) e^{-t/\tau^2}.
\]

For \( U \lesssim 2 \) the osci-
lations are well described by a frequency proportional to the bias, \( \Omega \approx V/2 \). On the other hand, a higher frequency appears for larger values of \( U \). This can be seen, for instance, in the bottom panel of Fig. 15. The dependence of \( \Omega_2 \) on \( U \) is displayed in Fig. 16; it grows linearly with \( U \) for large values of the interaction.

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Do not confuse with an entanglement entropy $S_{vN}$.

The latter precision can be estimated by looking at the approximate collapse of data obtained for different values of $J$ (but same $V/T_B$). The spread indicates to which extend the lattice model is close to the scaling limit.

For the Freton-Boulat expansion to match the exact formula at the free-Fermion point, we had to change the sign of the third term in their formula.

The functional renormalization group approach has also been employed to study other aspects of the out-of-equilibrium Physics of the IRLM, see for instance Refs. 18, 32, 33.
FIG. 14: Top: MPS bond dimension $M$ (link between the site $r = -1$ of the left lead and the dot at $r = 0$) as a function of time and for different values of the Trotter step $\tau$ and truncation parameter $\delta$ (see bottom panel for the legend). In most calculations the simulation is stopped when $M$ reaches 4000. Middle: von Neumann entanglement entropy $S_{vN}$ between the left and the right leads. Bottom: Current $2\pi I$ as a function of $t$. Parameters of the model: $N = 257, U = 6, J' = 0.3$ and $V = 0.6$. As far as the current or the entanglement entropy are concerned, all the simulations agree relatively well, apart from the less accurate one, with $\delta = 10^{-6}$ (crosses).
FIG. 15: Current $I(t)$ flowing through the dot for $U = -1.2$, 2, and 10. Parameters of the model: $J' = 0.08$, $V = 1.0$, $N = 257$. The current decays to a steady value, with oscillations dominated by two different frequencies. The first one is $\Omega = V/2$, while the second one, $\Omega_2$, may not have a simple relation in terms of the parameters (for $U = -1.2$ $\Omega_2 \simeq 2$, for $U = 2$ $\Omega_2 \simeq 2$, and for $U = 10$ $\Omega_2 \simeq 5.2$, amplitude also depends on parameters).
FIG. 16: Oscillation frequency $\Omega_2$ (see text) as a function of $U$. Parameters of the model: $J' = 0.08$, $V = 1.0$, $N = 257$. 