Generic properties of 2-step nilpotent Lie algebras and torsion-free groups
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To define the notion of a generic property of finite dimensional 2-step nilpotent Lie algebras we use standard correspondence between such Lie algebras and points of an appropriate algebraic variety, where a negligible set is one contained in a proper Zariski-closed subset. We compute the maximal dimension of an abelian subalgebra of a generic Lie algebra and give a sufficient condition for a generic Lie algebra to admit no surjective homomorphism onto a non-abelian Lie algebra of a given dimension. Also we consider analogous questions for finitely generated torsion free nilpotent groups of class 2.

1 Introduction and Results

1.1 Nilpotent Lie algebras of class 2

Consider a finite-dimensional 2-step nilpotent Lie algebra \( L \). Denote by \( S \) the commutator subalgebra of \( L \). Let \( z_1, \ldots, z_t \) be a basis of \( S \), and let \( V \subset L \) be such a subspace that \( L = V \oplus S \). Then the product of two elements \( x = \bar{x} + \bar{x} \) and \( y = \bar{y} + \bar{y} \) of \( L \) with \( \bar{x}, \bar{y} \in V \) and \( \bar{x}, \bar{y} \in S \) has the form

\[
[x, y] = \varphi_1(\bar{x}, \bar{y}) z_1 + \cdots + \varphi_t(\bar{x}, \bar{y}) z_t
\]

for some \( t \)-tuple of alternating bilinear forms \( \Phi = \Phi(L) = (\varphi_1, \ldots, \varphi_t) \) on \( V \).

On the other hand, given vector spaces \( S \) and \( V \), a basis \( z_1, \ldots, z_t \) of \( S \) and a \( t \)-tuple \( \Phi = (\varphi_1, \ldots, \varphi_t) \) of alternating bilinear forms on \( V \), one can define the product of two elements of \( L = L(\Phi) = V \oplus S \) by \( (1.1) \). Obviously, \( L(\Phi) \) is a 2-step nilpotent Lie algebra and \( S \) is a central subalgebra of \( L(\Phi) \).

Let \( z'_1, \ldots, z'_t \) be a new basis of \( S \) and let \( C = (c_{ij}) \) be the transformation matrix from old to new coordinates. Then \( (1.1) \) can be written in the following form:

\[
[x, y] = \sum_{i=1}^t \varphi_i(\bar{x}, \bar{y}) z_i = \sum_{i=1}^t \varphi_i(\bar{x}, \bar{y}) \sum_{j=1}^t c_{ij} z'_j = \sum_{j=1}^t (\sum_{i=1}^t \varphi_i(\bar{x}, \bar{y}) c_{ij}) z'_j = \sum_{i=1}^t \varphi'_i(\bar{x}, \bar{y}) z'_i,
\]

where

\[
\varphi'_i = \sum_{j=1}^t c_{ij} \varphi_j,
\]

That is \( C^\top \) is the transformation matrix to go from \( \Phi \) to \( \Phi' = (\varphi'_1, \ldots, \varphi'_t) \). This gives the following properties of a \( t \)-tuple \( \Phi \).
Proposition 1. The alternating bilinear forms of the \( t \)-tuple \( \Phi(L) \) corresponding to a Lie algebra \( L \) are linearly independent. And conversely, if the forms of a \( t \)-tuple \( \Phi \) are linearly independent and \( L(\Phi) = V \oplus S \) is a corresponding Lie algebra, then \( S = [L(\Phi), L(\Phi)] \).

Proof. Suppose, on the contrary, that for some Lie algebra \( L \) the forms of \( \Phi(L) \) are linearly dependent. Then there exists non-trivial linear combination \( \alpha_1 \varphi_1 + \ldots + \alpha_t \varphi_t = 0 \). It follows from \((1.3)\) that, putting \( \alpha_{ij} = \alpha_{ij} \), one can choose a basis \( z_1', \ldots, z_t' \) of \( S \) in such a way that \( \varphi'_1 \equiv 0 \). This contradicts with the fact that \( z'_t \) belongs to the commutator subalgebra \( S \).

Conversely, if the commutator subalgebra \([L(\Phi), L(\Phi)]\) is strictly less than \( S \), then there exists a basis \( z_1', \ldots, z_t' \) of \( S \) such that \( z_1, \ldots, z_k, \) where \( k < t \), generate \([L(\Phi), L(\Phi)]\). Then the corresponding bilinear form \( \varphi'_t \), which is a non-trivial linear combination of the forms of \( \Phi \), identically equals zero. \( \square \)

Proposition 2. If the linear spans of \( t \)-tuples \( \Phi \) and \( \Phi' \) coincide, then the corresponding Lie algebras \( L(\Phi) \) and \( L(\Phi') \) are isomorphic.

From now on we will assume that a ground field is infinite. It is appropriate to introduce some notation that will be used throughout the paper. We denote by \( \mathcal{N}_2^K(n, t) \) (or simply by \( \mathcal{N}_2(n, t) \), if it is not misleading) the set of all 2-step nilpotent Lie algebras \( L \) over a field \( K \) such that \( \dim L/[L, L] = n \) and \( \dim [L, L] = t \). Let \( B_n(K) \) be the set of all alternating bilinear forms on an \( n \)-dimensional space \( V \) over \( K \) and \( \mathbb{P}(B_n') \) be the projective space corresponding to the \( t \)-th direct power of \( B_n(K) \). Put \( M_{n,t} = \{ (\varphi_1 : \cdots : \varphi_t) \in \mathbb{P}(B_n') \mid \varphi_1, \ldots, \varphi_t \text{ are linearly independent} \} \). \( M_{n,t} \) is a Zariski-open subset of \( \mathbb{P}(B_n') \).

Definition 1. We say that \( \mathcal{A} \) is a generic property of \( \mathcal{N}_2(n, t) \) (a generic Lie algebra of \( \mathcal{N}_2(n, t) \) has a property \( \mathcal{A} \)) if the set of points of \( M_{n,t} \) corresponding to the algebras without the property \( \mathcal{A} \) is contained in some proper Zariski-closed subset.

Remark. Since \( M_{n,t} \) is irreducible, the dimension of any its closed subset is strictly less than the dimension of \( M_{n,t} \). (see. \([7]\))

The following theorem, which holds true for any infinite field, gives a simple example of a generic property.

Theorem 1. For a generic Lie algebra \( L \in \mathcal{N}_2(n, t) \) we have \( \dim Z(L) = 2 \) if \( t = 1 \) and \( n \) is odd, and \( Z(L) = [L, L] \) otherwise.

Here \( Z(L) \) is the center of a Lie algebra \( L \).

The next theorem allows to compute the maximal dimension of an abelian subalgebra of a generic Lie algebra.

Theorem 2. If a ground field \( K \) is algebraically closed and \( t > 1 \), then any Lie algebra of \( \mathcal{N}_2^K(n, t) \) contains a commutative subalgebra of dimension \( s = \left\lfloor \frac{2n+t^2+3}{t+2} \right\rfloor \). For any infinite ground field \( K \) a generic Lie algebra of \( \mathcal{N}_2^K(n, t) \) doesn’t have any commutative subalgebra of dimension \( s + 1 \).
This theorem immediately follows from [6]. However for the convenience of the reader we will give some details in Section 3.2. Also in this section we adduce an example of a Lie algebra from $\mathfrak{N}_2^3(4,3)$ without commutative subalgebras of dimension 5. Which shows that the condition for the ground field being algebraically closed can not be ignored.

Remark. The structure of Heisenberg Lie algebras, that is 2-step nilpotent Lie algebras with one-dimensional center, is well known. The dimension $m$ of such an algebra is always odd, and the dimension of a maximal commutative subalgebra is $\frac{m+1}{2}$.

The main result of the present paper is the following theorem, which holds true for an arbitrary infinite ground field.

Definition 2. We say that a nilpotent Lie algebra $L$ of class 2 has a property $S(n_0,t_0)$ $(1 \leq t_0 \leq \frac{n_0(n_0-1)}{2})$, if $L$ admits a surjective homomorphism onto a Lie algebra of $\mathfrak{N}_2(n_0,t_0)$.

Theorem 3. If the positive integers $n$, $n_0$, $t$ and $t_0$, where $n \geq n_0$, satisfy the inequality

\[ t < \frac{n(n-1)}{2} - \frac{n_0}{t_0} + \frac{n_0^2}{t_0} + t_0 - \frac{n_0(n_0-1)}{2}, \tag{1.4} \]

then a generic Lie algebra of $\mathfrak{N}_2(n,t)$ does not have the property $S(n_0,t_0)$. If $n$, $n_0$, $t$ and $t_0$, where $n \geq n_0$, satisfy the inequality

\[ t \geq \frac{n(n-1)}{2} - \frac{n_0(n_0-1)}{2} + t_0, \tag{1.5} \]

then the property $S(n_0,t_0)$ is true on $\mathfrak{N}_2(n,t)$.

The following statement is an immediate corollary of Theorem 3.

Corollary. A generic Lie algebra of $\mathfrak{N}_2(n,t)$ does not admit a surjective homomorphism onto a non-commutative Lie algebra of dimension $N < n$, if

\[ t < \frac{n^2}{2} - n \left( N - \frac{1}{2} \right) + \frac{N(N-1)}{2} + 1. \tag{1.6} \]

Remark. Obviously, if $n < N < n+t$, then any Lie algebra $L \in \mathfrak{N}_2(n,t)$ admits a surjective homomorphism onto a non-commutative Lie algebra of dimension $N$. It is enough to take a quotient group of $L$ by any central subalgebra of dimension $n+t-N < t$.

Actually, if a ground field is algebraically closed, then the set of point of $M_{n,t}$ corresponding to the algebras with property $S(n_0,t_0)$ forms a closed subset for any values of parameters (see Lemmas 2 and 3). It means that there are only two possibilities: either this subset coincides with $M_{n,t}$ and all the algebras have the property $S(n_0,t_0)$, or this subset is proper and almost all the algebras do not have this property. Moreover, if all the algebras of $\mathfrak{N}_2(n,t)$ have the property $S(n_0,t_0)$, then it is also true for any Lie algebra $L \in \mathfrak{N}_3(n,t+k)$, where $k > 0$, since for any $k$-dimensional subalgebra $H < [L,L]$ we have $L/H \in \mathfrak{N}_2(n,t)$. Thus, we get the following.
for some forms on a free abelian group formal products of the form (1.7). Define the product of two elements of that the subgroup \( G \) satisfies. From (1.9) it follows that \( G \) respectively. Then the elements of \( G/S \) group for example, \([1, \text{Section 8}]\), \( I \) of \( G \) putting \( C \) by \( S \) analogous way. Let \( \Phi \) be a finitely generated 2-step nilpotent group without torsion. Denote \( \Phi \) \( G \) is true on \( \Phi \). Conversely, let \( \Phi \) \( \Phi \) \( G \) \( \Phi \) \( I(G') \) also lies in the center. Hence \( S \) and the quotient group \( G/S \) are free abelian. Let \( b_1, \ldots, b_t \) and \( a_1, \ldots, a_n \) be bases of \( S \) and \( G \) modulo \( S \) respectively. Then the elements of \( G \) have the form \[ a_1^{k_1} \cdots a_n^{k_n} b_1^{l_1} \cdots b_t^{l_t}, \text{ with } k_i, l_j \in \mathbb{Z}, \text{ for } i = 1, \ldots, n, j = 1, \ldots, t. \] (1.7)

Since the commutator map on a nilpotent group of class 2 is bilinear, we get

\[
[x, y] = [a_1^{k_1} \cdots a_n^{k_n} b_1^{l_1} \cdots b_t^{l_t}, a_1^{p_1} \cdots a_n^{p_n} b_1^{q_1} \cdots b_t^{q_t}] = \prod_{i,j=1}^{n} [a_i, a_j]^{k_ip_j} \prod_{l=1}^{t} b_l^{\sum_{i,j=1}^{n} k_ip_j \varphi_l(a_i,a_j)} = \prod_{l=1}^{t} b_l^{\varphi_l(xS,yS)}. \tag{1.8}
\]

for some \( t \)-tuple of integer skew-symmetric bilinear forms \( \Phi(G) = \{\varphi_1, \ldots, \varphi_1\} \) on \( G/S \).

Conversely, let \( \Phi = \{\varphi_1, \ldots, \varphi_1\} \) be a \( t \)-tuple of integer skew-symmetric bilinear forms on a free abelian group \( \mathbb{Z}^n \) with a basis \( a_1, \ldots, a_n \). Consider the set \( G(\Phi) \) of formal products of the form (1.7). Define the product of two elements of \( G(\Phi) \) by

\[
a_1^{k_1} \cdots a_n^{k_n} b_1^{l_1} \cdots b_t^{l_t} \cdot a_1^{p_1} \cdots a_n^{p_n} b_1^{q_1} \cdots b_t^{q_t} = a_1^{k_1+p_1} \cdots a_n^{k_n+p_n} \prod_{k=1}^{t} b_k^{l_k+q_k+\sum_{i>j} k_ip_j \varphi_k(a_i,a_j)}. \tag{1.9}
\]

The product is well defined and it can be verified easily that the group axioms are satisfied. From (1.9) it follows that \( G(\Phi) \) is a nilpotent torsion-free group of class 2, that the subgroup \( S \) generated by \( b_1, \ldots, b_t \) is central and freely generated by \( b_1, \ldots, b_t \),  

\[ 1 \leq t < C(n, n_0, t_0), \]

a generic Lie algebra of \( \mathfrak{N}_2(n, t) \) doesn’t have the property \( S(n_0, t_0) \), and the property \( S(n_0, t_0) \) is true on \( \mathfrak{N}_2(n, t) \) if \[ C(n, n_0, t_0) \leq t < n(n-1)/2. \]

The relations (1.4) and (1.5) give upper and lower bounds for \( C(n, n_0, t_0) \). Computing \( C(n, n_0, t_0) \) precisely is subject to further investigation.

1.2 Finitely generated torsion-free nilpotent groups of class 2

Now let us introduce the notion of a generic property for nilpotent groups in an analogous way. Let \( G \) be a finitely generated 2-step nilpotent group without torsion. Denote by \( S = I(G') = \{x \in G \mid x^k \in G' \text{ for some } k\} \) the isolator of the commutator subgroup of \( G \). Since the center of a torsion-free nilpotent group coincides with its isolator (see, for example, \([1, \text{Section 8}]\), \( I(G') \) also lies in the center. Hence \( S \) and the quotient group \( G/S \) are free abelian. Let \( b_1, \ldots, b_t \) and \( a_1, \ldots, a_n \) be bases of \( S \) and \( G \) modulo \( S \) respectively. Then the elements of \( G \) have the form

\[ a_1^{k_1} \cdots a_n^{k_n} b_1^{l_1} \cdots b_t^{l_t}, \text{ with } k_i, l_j \in \mathbb{Z}, \text{ for } i = 1, \ldots, n, j = 1, \ldots, t. \] (1.7)
and the factor group \( G(\Phi)/S \) is abelian and freely generated by \( a_1S, \ldots, a_nS \). So we can naturally identify it with \( \mathbb{Z}^n \). Using (1.3), we get

\[
[a_i, a_j] = \prod_{l=1}^{t} b_l^{\varphi_l(a_i, a_j)} \quad \text{for } i, j = 1, \ldots, n.
\]

Further, since the commutator map is bilinear, we get that (1.8) holds.

It is easy to see that the \( t \)-tuple of bilinear forms corresponding to a new basis of \( S \) is defined by (1.3), where the transformation matrix \( C \) can be an arbitrary integer matrix with determinant 1. Therefore, as in the case of Lie algebras, we get the following property of \( \Phi(G) \).

**Proposition 1.** The alternating bilinear forms of the \( t \)-tuple \( \Phi(G) \) corresponding to a group \( G \) are linearly independent. And conversely, if the forms of a \( t \)-tuple \( \Phi \) are linearly independent and \( G(\Phi) \) is a corresponding 2-step nilpotent group, then \( S = I((G(\Phi))') \).

The proof is carried over from Proposition 1.

Let \( \mathcal{N}_2(n, t) \) be the set of all 2-step nilpotent torsion free groups \( G \) with \( \text{rk} G' = t \) and \( \text{rk} G/I(G') = n \). Denote by \( B_n(\mathbb{Z}) \) the set of all integer bilinear forms on \( \mathbb{Z}^n \). Given a basis \( a_1, \ldots, a_n \) of a vector space \( V \) over \( \mathbb{Q} \), we can identify the elements of \( B_n(\mathbb{Z}) \) with elements of \( B_n(\mathbb{Q}) \) represented in this basis by integer matrices.

**Definition 1.** We say that a property \( \mathcal{A} \) of \( \mathcal{N}_2(n, t) \) is true generically if the set of points of \( P_{n, t} \) corresponding to the groups without property \( \mathcal{A} \) is contained in some proper Zariski-closed subset of \( B_n(\mathbb{Q}) \).

A nilpotent group \( G \in \mathcal{N}_2(n, t) \) and the Lie algebra \( L = L(\Phi(G)) \in \mathcal{N}_2^Q(n, t) \), defined by the same tuple of the forms, have quite similar properties. In particular, \( G \) contains an abelian subgroup of rank \( s \) if and only if \( L \) has an \( s \)-dimensional commutative subalgebra. \( G \) admits a surjective homomorphism onto a group of \( \mathcal{N}_2(n_0, t_0) \) if and only if \( L \) has the property \( S(n_0, t_0) \). Also we have \( \text{rk} Z(G) = \dim Z(L) \) and \( \text{rk} G' = \dim[L, L] \). That is why the analogues of Theorems 1.3 holds true for the nilpotent groups. They do not need to be proved separately. This can be explained by the following arguments.

Denote by \( \sqrt{G} \) the Malcev completion of \( G \in \mathcal{N}_2(n, t) \), that is the smallest complete nilpotent torsion free group containing \( G \) (see, for example, [1] Section 8)). Notice that \( \sqrt{G} \) can be considered as the group of elements of the form (1.7) with multiplication (1.9), where \( (\varphi_1, \ldots, \varphi_t) = \Phi(G) \) and \( k_i, l_i, p_r, q_s \in \mathbb{Q} \).

On the other hand, it is well known (see [1]) that in every nilpotent Lie \( \mathbb{Q} \)-algebra \( L \) one can define multiplication ”\( \circ \)” by Campbell-Hausdorff formula in such a way that \( L \) is a torsion-free complete group, say \( L' \), of the same nilpotency class with respect to ”\( \circ \)”. (In the case of a 2-step nilpotent Lie algebra Campbell-Hausdorff formula has the
form $x \circ y = x + y + \frac{1}{2}[x, y]$.) The functor $\Gamma : L \to L^\circ, (f : L_1 \to L_2) \to (f : L_1^\circ \to L_2^\circ)$ provides an isomorphism from category of the nilpotent class $s$ Lie $\mathbb{Q}$ algebras to the category of the nilpotent class $s$ torsion-free complete groups. It is easy to see that $\sqrt{G} \simeq L(\Phi(G))^\circ$.

**Theorem 1.** For a generic group $G \in \mathcal{N}_2(n,t)$ we have $\text{rk}Z(G) = 2$ if $t = 1$ and $n$ is odd, and $Z(G) = I(G')$ otherwise.

Theorem 2. A generic group of $\mathcal{N}_2(n,t)$ doesn’t contain any abelian subgroup of rank $\left[\frac{2n+t^2+3t}{t+2}\right] + 1$.

Also this theorem follows from [5].

Theorem 3. Suppose that the positive integers $n$, $n_0$, $t$, and $t_0$ satisfy the inequality

$$t < \frac{n(n-1)}{2} - \frac{n_0}{t_0} + \frac{n_0^2}{t_0} + t_0 - \frac{n_0(n_0 - 1)}{2}.$$  

Then a generic group of $\mathcal{N}_2(n,t)$ does not admit a surjective homomorphism onto a group of $\mathcal{N}_2(n_0,t_0)$.

If $n$, $n_0$, $t$ and $t_0$, where $n \geq n_0$, satisfy the inequality

$$t \geq \frac{n(n-1)}{2} - \frac{n_0(n_0 - 1)}{2} + t_0,$$

then every group of $\mathcal{N}_2(n,t)$ admits a surjective homomorphism onto a group of $\mathcal{N}_2(n_0,t_0)$.

**Corollary.** A generic group of $\mathcal{N}_2(n,t)$ does not admit a surjective homomorphism on a non-abelian group of polycyclic rank $N < n$, if

$$t < \frac{n^2}{2} - n \left(N - \frac{1}{2}\right) + \frac{N(N-1)}{2} + 1.$$

2 Preliminaries on the algebraic varieties

We recall that $X \subset \mathbb{P}^n$ is a **closed in Zariski topology subset** if it consists of all points at which a finite number of homogeneous polynomials with coefficients in $K$ vanishes. This topology induces Zariski topology on any subset of the projective space $\mathbb{P}^n$. A closed subset of $\mathbb{P}^n$ is a **projective variety**, and an open subset of a projective variety is a **quasiprojective variety**. A nonempty set $X$ is called **irreducible** if it cannot be written as the union of two proper closed subsets.

Further, let $f : X \to \mathbb{P}^m$ be a map of a quasiprojective variety $X \subset \mathbb{P}^n$ to a projective space $\mathbb{P}^m$. This map is **regular** if for every point $x_0 \in X$ there exists a neighbourhood $U \ni x_0$ such that the map $f : U \to \mathbb{P}^m$ is given by an $(m+1)$-tuple
(F_0 : \ldots : F_m) of homogeneous polynomials of the same degree in the homogeneous coordinates of x \in \mathbb{P}^n, and F_i(x_0) \neq 0 for at least one i.

We will use the following properties of the \textit{dimension} of a quasiprojective variety.

- The dimension of \mathbb{P}^n is equal to n.
- If X is an irreducible variety and U \subset X is open, then \dim U = \dim X.
- The dimension of a reducible variety is the maximum of the dimension of its irreducible components.

For more details see [7]. We will need the following propositions which are also to be found in [7].

**Proposition 4.** If X = \bigcup U_\alpha with open sets U_\alpha, and Y \cap U_\alpha is closed in U_\alpha for each U_\alpha, then Y is closed in X.

**Proposition 5.** (see [4, Theorem 11.12]) Let f : X \to \mathbb{P}^n be a regular map of a quasiprojective variety X, and let Y be the closure of f(X). For each point p \in X, we denote by X_p = f^{-1}(f(p)) \subset X the fiber of f containing p, and by \dim_p X_p denote the local dimension of X_p at the point p, that is, the maximal dimension of an irreducible component of X_p, containing p. Then the set of all points p \in X such that \dim_p X_p \geq m is closed in X for any m. More over, if X_0 is an irreducible component of X and Y_0 \subset Y is the closure of the image f(X_0), then

\[
\dim X_0 = \dim Y_0 + \min_{p \in X_0} \dim_p X_p. \tag{2.1}
\]

**Proposition 6.** Let f : X \to Y be a regular map between projective varieties, with f(X) = Y. Suppose that Y is irreducible, and that all the fibers f^{-1}(y) for y \in Y are irreducible and of the same dimension. Then X is irreducible.

The following examples of projective varieties are important for our goals: a direct product of projective spaces, a Grassmannian and a Shubert cell.

Let \mathbb{P}^n, \mathbb{P}^m be projective spaces having homogeneous coordinates \((u_0 : \ldots : u_n)\) and \((v_0 : \ldots : v_m)\) respectively. Then the set \mathbb{P}^n \times \mathbb{P}^m of pairs \((x, y)\) with \(x \in \mathbb{P}^n\) and \(y \in \mathbb{P}^m\) is naturally embedded as a closed set into the projective space \(\mathbb{P}^{(n+1)(m+1)-1}\) with homogeneous coordinates \(w_{ij}\) by the rule \(w_{ij}(u_0 : \ldots : u_n; v_0 : \ldots : v_m) = u_i v_j\). Thus there is a topology on \(\mathbb{P}^n \times \mathbb{P}^m\), induced by the Zariski topology on \(\mathbb{P}^{(n+1)(m+1)-1}\).

**Proposition 7.** A subset \(X \subset \mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^N\) is a closed algebraic subvariety if and only if it is given by a system of equations

\[G_i(u_0 : \ldots : u_n; v_0 : \ldots : v_m) = 0 \quad (i = 1, \ldots, t)\]

homogeneous in each set of variables \(u_i\) and \(v_j\).
Proposition 8. Consider an $n$-dimensional vector space $V$ with a basis $\{e_1, \ldots, e_n\}$. Let $U$ be a $k$-dimensional subspace of $V$ with a basis $\{f_1, \ldots, f_k\}$. To $U$ we assign the point $P(U)$ of the projective space $\mathbb{P}(\Lambda^k V)$ by the rule

$$P(U) = f_1 \wedge \cdots \wedge f_k.$$ 

The point $P(U)$ has the following form in the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{i_1 < \cdots < i_k}$ of $\Lambda^k V$:

$$P(U) = \sum_{i_1 < \cdots < i_k} p_{i_1 \ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$ 

Then the homogeneous coordinates $p_{i_1 \ldots i_k}$ of $P(U)$ are called the Plücker coordinates of $U$, $P(U)$ is uniquely determined by $U$, and the following assertions hold.

(i) The subset of all points $p \in \mathbb{P}(\Lambda^k V)$ of the form $p = P(U)$ is closed in $\mathbb{P}(\Lambda^k V)$; this subset $G(k, n)$ (the Grassmannian or Grassmann variety) is defined by the relations

$$\sum_{r=1}^{k+1} (-1)^r p_{i_1 \ldots i_{k-1} j_r} p_{j_1 \ldots \hat{j_r} \ldots j_{k+1}} = 0 \quad (2.2)$$

for all sequences $i_1 \ldots i_{k-1}$ and $j_1 \ldots j_{k+1}$.

(ii) $\dim G(k, n) = k(n - k)$.

(iii) $G(k, n)$ is irreducible (see Section 2.2.7 of [8]).

(iv) Suppose, for example, that $p_{1 \ldots k} \neq 0$. If $p = (p_{i_1 \ldots i_k}) = P(U)$, then $U$ has a basis $\{f_1, \ldots, f_k\}$ such that

$$f_i = e_i + \sum_{r > k} a_{ir} e_r \quad \text{for } i = 1, \ldots, k,$$  

where

$$a_{ir} = (-1)^{k-i} \frac{p_{i_1 \ldots \hat{i} \ldots i_r \ldots i_k}}{p_{1 \ldots k}}. \quad (2.4)$$

(Here $\hat{i}$ means that the index $i$ is discarded.)

Proposition 9. (see [2] Section 14.7 or [3] Section 1.5) Consider an $n$-dimensional vector space $V$. For any increasing sequence

$$0 \subset V_1 \subset \cdots \subset V_k$$

of subspaces of $V$ put

$$W(V_1, \ldots, V_k) = \{p = P(U) \in G(k, n) \mid \dim(U \cap V_i) \geq i \text{ for } i = 1, \ldots, k\}.$$ 

Then all the subsets $W(V_1, \ldots, V_k)$, called Schubert cells, are closed in $G(k, V)$, and

$$\dim W(V_1, \ldots, V_k) = \sum_{i=1}^{k} (\dim V_i - i). \quad (2.5)$$
Corollary. Let $V$ be a vector space of dimension $n$, and let $s_0 = \max\{0, k + m - n\}$. Given an $m$-dimensional subspace $U \subset V$ ($m > 0$), put

$$G_s = G_s(U,V,k) = \{P(U') \in G(k,n) \mid \dim U \cap U' \geq s\}$$

(2.6)

for $s = s_0, s_0 + 1, \ldots, \min\{k,m\}$. Then the sets $G_s$ are closed in $G(k,n)$, and

$$\dim G_s = s(m - s) + (k - s)(n - k).$$

(2.7)

Proof. It is easy to check that the sets $G_s$ are Schubert cells. Indeed, choose a basis $\{e_1, \ldots, e_n\}$ of $V$ such that $e_1, \ldots, e_m$ span $U$. Then $G_s = W(V_1, \ldots, V_k)$ where

$$V_i = \langle e_1, \ldots, e_{m-s+i} \rangle \quad \text{for } i = 1, \ldots, s,$$

$$V_i = \langle e_1, \ldots, e_{n-k+i} \rangle \quad \text{for } i = s + 1, \ldots, k.$$  

(The conditions $\dim(U' \cap V_i) \geq i$ for $i > s$ are trivial, and for $i \leq s$ they follows from the condition $\dim(U' \cap V_s) = \dim(U' \cap U) \geq s$.) Using (2.5), we get

$$\dim G_s = \sum_{i=1}^{s}(m - s) + \sum_{i=s+1}^{k}(n - k) = s(m - s) + (k - s)(n - k). \quad \Box$$

3 Proofs

Lemma 1. Let $K$ be an infinite field. Denote by $\bar{K}$ its algebraic closure. Suppose that a generic Lie algebra of $N_2^K(n,t)$ has property $A$ and for any Lie algebra $L \in N_2^K(n,t)$ without property $A$ the Lie algebra $\bar{L} = L_K \otimes \bar{K} \in N_2^K(n,t)$ does not have property $A$ too. Then $A$ is also a generic property of $N_2^K(n,t)$.

Proof. The $t$-tuples $\Phi(L)$ and $\Phi(\bar{L})$, corresponding to the Lie algebras $L \in N_2^K(n,t)$ and $\bar{L} \in N_2^K(n,t)$ respectively, can be defined by the same $t$-tuple of matrices with coefficients in $K$. So, under the condition of the Lemma, the set of point of $M_{n,t}(K)$ corresponding to the algebras of $N_2^K(n,t)$ without property $A$ satisfies some finite system of homogeneous equations, say $(\ast)$, over $\bar{K}$. If the variables take values in $K$, $(\ast)$ is equivalent to some finite system of homogeneous equations over $K$. The last system is not zero on $M_{n,t}(K)$ identically, since $K$ is infinite and $(\ast)$ defines proper subset of $M_{n,t}(K)$. \Box

It follows from Lemma 1 that it is enough to prove Theorems 1, 2 just for the case of algebraically closed ground field $K$. So, from now on if not stated otherwise we will assume that $K$ is algebraically closed.

Propositions 1 and 2 show that we can define the correspondence between the elements of $N_2(n,t)$ and $G(t,B_n)$ by the rule $\varphi(L) = \varphi_1 \wedge \cdots \wedge \varphi_t$ for $L \in N_2(n,t)$ if $\Phi(L) = (\varphi_1, \ldots, \varphi_t)$, and vice versa, $L(\varphi) = L(\varphi_1, \ldots, \varphi_t)$ for $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_t \in G(t,B_n)$.
Lemma 2. \( \mathcal{A} \) is a generic property of \( \mathfrak{H}_2(n,t) \) if and only if the set of points of \( G(t,B_n) \) corresponding to the algebras without property \( \mathcal{A} \) belongs to some proper Zariski-closed subset of \( G(t,B_n) \).

Proof. Consider a regular map \( f : M_{n,t} \to G(t,B_n) \) such that \( f \) takes each \( t \)-tuple of alternating bilinear forms to the subspace spanned by it, that is,

\[
f(\varphi_1, \ldots, \varphi_t) = \varphi_1 \wedge \cdots \wedge \varphi_t.
\]

By definition, Lie algebras \( L(\Phi) \) and \( L(f(\Phi)) \) corresponding to the points \( \Phi \in M_{n,t} \) and \( f(\Phi) \) respectively are isomorphic.

Denote by \( M(\mathcal{A}) \subset M_{n,t} \) and \( G(\mathcal{A}) \subset G(t,B_n) \) the sets of points corresponding to the algebras without property \( \mathcal{A} \). Clearly, \( f^{-1}(G(\mathcal{A})) = M(\mathcal{A}) \). Suppose that \( G(\mathcal{A}) \) belongs to some proper closed subset \( X \subset G(t,B_n) \). Then \( M(\mathcal{A}) \) is contained in \( f^{-1}(X) \). Since \( f \) is surjective and continuous, \( f^{-1}(X) \) is a proper closed subset of \( M_{n,t} \).

Inversely, suppose that \( M(\mathcal{A}) \) belongs to some proper closed subset \( Y \subset M_{n,t} \). A fiber \( f^{-1}(\varphi) \) consists of all the bases of a given \( t \)-dimensional vector space considered up to proportionality. It can be parametrized by invertible matrices of size \( t \times t \). Hence, all the fibers \( f^{-1}(\varphi) \) for \( \varphi \in G(t,B_n) \) are isomorphic and of the same dimension \( t^2 - 1 \).

Obviously, \( L(\Phi) \) does not have property \( \mathcal{A} \) only if \( f^{-1}(f(\Phi)) \subset Y \). Hence, we can consider a set \( Y_0 = \{ \Phi \in Y \mid \dim_{\mathbb{F}}(f^{-1}(f(\Phi)) \cap Y) = t^2 - 1 \} \) instead of \( Y \). An application of Proposition \ref{prop} to the map \( f = f|_Y \), yields that \( Y_0 \) is also closed.

Denote by \( X_0 \) the closure of \( f(Y_0) \) in \( G(t,B_n) \). We have \( G(\mathcal{A}) \subset X_0 \). Let us show that \( X_0 \) is proper subset of \( G(t,B_n) \). Indeed, by \((\ref{eq:proof})\), for any irreducible component \( Y'_0 \subset Y_0 \) we have

\[
\dim Y'_0 = \dim X'_0 + t^2 - 1,
\]

where \( X'_0 \subset X_0 \) is a closure of the image \( f(Y'_0) \). Recalling that \( \dim M_{n,t} = \dim G(t,B_n) + t^2 - 1 \) and \( \dim Y'_0 < \dim M_{n,t} \), we obtain \( \dim X'_0 < \dim G(t,B_n) \). Consequently, \( \dim X_0 < \dim G(t,B_n) \). □

3.1 Proof of Theorem \[1\]

Consider a Lie algebra \( L \in \mathfrak{H}_2(n,t) \). Let \( S = [L,L] \), \( L = V \oplus S \) and \( \Phi(L) = (\varphi_1, \ldots, \varphi_n) \).

Fix a basis \( e_1, \ldots, e_n \) for \( V \) and identify forms \( \varphi_i \) with their matrices with respect to this basis. \( Z(L) \) is strictly greater than \( S \) if and only if there exists nonzero element \( c \in V \) such that

\[
\varphi_i(c, e_j) = 0 \quad i = 1, \ldots, t, \quad j = 1, \ldots, n. \tag{3.1}
\]

Denote by \( D \) the subset of \( H = \mathbb{P}(V) \times \mathbb{P}(B^t_n) \) consisting of pairs \( (c, \varphi) \) satisfying \((3.1)\). The system of equations \((3.1)\) is linear in each set of variables \( c \) and \( \varphi \). Therefore, by Proposition \ref{prop}, \( D \) is a projective variety.

Consider the projection \( \pi : D \to \mathbb{P}(B^t_n) \). Since \( \pi \) is a regular map, \( \pi(D) \) is closed in \( \mathbb{P}(B^t_n) \). Moreover, \( \pi(D) \cap M_{n,t} \) consists of \( t \)-tuples \( (\varphi_1, \ldots, \varphi_t) \) corresponding to the algebras with non-trivial center modulo the commutator ideal. Obviously, \( \pi(D) \cap M_{n,t} \) is a proper subset except the case when \( t = 1 \) and \( n \) is odd in which the only form \( \varphi_1 \)
defining a Lie algebra is always degenerate. In the last case the dimension of the center of a Lie algebra is greater than 2 only if the rank of $\varphi_1$ is strictly less than $n - 1$. This condition defines a proper closed subset in $M_{n,1}$. □

3.2 Proof of Theorem 2

Proof of Theorem 2. The Main Lemma of [6] states that if the positive integers $t \geq 2, k$ and $n$ satisfy the inequality

$$2n \geq t(k-1) + 2k; \quad (3.2)$$

then for any $t$-tuple $\{\varphi_1, \ldots, \varphi_t\} \in B_n^t(K)$ there exists a $k$-dimensional subspace that is simultaneously isotropic for all of the forms $\varphi_1, \ldots, \varphi_t$, i.e. on which all the forms are zero.

It follows easily from the proof of the Main Lemma that the set $\pi_2(S)$ (in the notation of Section 2.2 of [6]) consisting of all the $t$-tuples of forms with common $k$-dimensional isotropic subspace is a Zariski-closed subset of $\mathbb{P}(B_n^t)$. Furthermore, we have $\pi_2(S) = \mathbb{P}(B_n^t)$ if $t > 1$ and the inequality (3.2) holds, and $\pi_2(S)$ is a proper subset if (3.2) is not true.

Let $L(\Phi) = V \oplus S \in \mathfrak{N}_2(n,t)$ be a Lie algebra associated with a tuple $\Phi$. A subalgebra $H \leq L$ is commutative if and only if the subspace $H/(H \cap S)$ is isotropic for all of the forms of $\Phi$. Therefore each algebra of $\mathfrak{N}_2(n,t)$ contains a commutative subalgebra of dimension $s = k + t$, where $k$ is the maximal integer satisfying (3.2), that is $k = \left\lfloor \frac{2n + t}{t+2} \right\rfloor$. And a generic Lie algebra of $\mathfrak{N}_2(n,t)$ has no abelian subalgebras of dimension greater than $s$. □

If the ground field $K$ is not algebraically closed, then the maximal dimension of commutative subalgebras of a Lie algebra $L$ over $K$ may be strictly less than that of $\bar{L} = L_K \otimes \bar{K}$ over $\bar{K}$. And so the condition on the ground field to be closed in the first part of Theorem 2 is essential.

Example. Let $K$ be a subfield of the field of real numbers. There exists a Lie algebra $L \in \mathfrak{N}_2^K(4,3)$ without commutative subalgebras of dimension 5.

Proof. Consider a Lie algebra $L = V \oplus S$ defined by the following 3-tuple of matrices

$$\varphi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. $$

Elements $x, y \in V$ commute if and only if $\varphi_i(x, y) = 0$ for $i = 1, 2, 3$. Hence the set of all elements $y \in V$ commuting with a given element $x$ with coordinates $(x_1, x_2, x_3, x_4)$ is a solution space of the system of homogeneous linear equations with the matrix

$$M(x) = \begin{pmatrix} x_2 & -x_1 & x_4 & -x_3 \\ x_3 & -x_4 & -x_1 & x_2 \\ x_4 & x_3 & -x_2 & -x_1 \end{pmatrix}. $$

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The system has a solution not proportional to $x$ if and only if $M(x)$ is rank deficient. Let us compute $3 \times 3$ minors of $M(x)$:

$$A_i = (-1)^i x_i(x_1^2 + x_2^2 + x_3^2 + x_4^2) \quad \text{for } i = 1, 2, 3, 4.$$ 

Obviously, if all $A_i = 0$ and $x_i \in \mathbb{R}$, then $x_1 = x_2 = x_3 = x_4 = 0$. Consequently, $L$ has no commutative subalgebras of dimension greater than 1 modulo commutator ideal. □

### 3.3 Proof of Theorem 3

For any subspace $U$ of an $n$-dimensional vector space $V$, put

$$N_0(U) = \{ \varphi \in B_n \mid \varphi(x, u) = 0 \text{ for } x \in V \text{ and } u \in U \}.$$ 

Choose a basis $e_1, \ldots, e_n$ of $V$ such that $e_1, \ldots, e_k$ span $U$. Now an alternating bilinear form $\psi$ lies in $N_0(U)$ if and only if the matrix of $\psi$ is zero except lower right-hand $(n - k) \times (n - k)$ submatrix in this basis. Thus,

$$\dim N_0(U) = \frac{(n - k)(n - k - 1)}{2}. \quad (3.3)$$

**Lemma 3.** Consider a Lie algebra $L \in \mathcal{R}_2(n, t)$. Let $S = [L, L]$, $L = V \oplus S$ and $\Phi(L) = \langle \varphi_1, \ldots, \varphi_t \rangle$. $L$ has a property $S(n_0, t_0)$ if and only if there exists a subspace $U \subset V$ of dimension $n - n_0$ such that

$$\dim(N_0(U) \cap \langle \varphi_1, \ldots, \varphi_t \rangle) \geq t_0. \quad (3.4)$$

**Proof.** L has property $S(n_0, t_0)$ if and only if there exists an ideal $I \triangleleft L$ of dimension $(n + t - n_0 - t_0)$ such that $\dim(I \cap S) = t - t_0$. Indeed, in this case we have $L/I \in \mathcal{R}_2(n_0, t_0)$, since the image of a commutator ideal is the commutator ideal of the image. And so we can take $I$ as a kernel of a desired surjective map.

Assume that such $I$ exists. Choose a basis $z_1', \ldots, z_t'$ of $S$ such that $z_i + 1, \ldots, z_i' \in (S \cap I)$. Let $\langle \varphi_1', \ldots, \varphi_t' \rangle$ be a $t$-tuple of forms associated with $L$ in this basis. Denote by $U$ the projection of $I$ on $V$ along $S$. Clearly, $\dim U = n - n_0$. Since for any $x \in V$ and $u \in U$ we have $[x, u] = \sum_{i=1}^t \varphi_i'(x, u)z_i' \in \langle z_{t_0+1}, \ldots, z_t \rangle$, we get $\varphi_1', \ldots, \varphi_t' \in N_0(U)$. It follows from (1.3) that $\varphi_1', \ldots, \varphi_t'$ are linear combinations of $\varphi_1, \ldots, \varphi_t$ and, by Proposition 1, they are linearly independent. Consequently, (3.4) holds.

On the contrary, let $U$ be as in the lemma. Choose $t_0$ linearly independent forms $\varphi_1', \ldots, \varphi_{t_0}' \in (N_0(U) \cap \langle \varphi_1, \ldots, \varphi_t \rangle)$ and extend them to a basis $\varphi_1', \ldots, \varphi_{t_0}'$ of $\langle \varphi_1, \ldots, \varphi_t \rangle$. It follows from (1.2) and (1.3) that we can choose a basis $z_1', \ldots, z_t'$ of $S$ such that $\Phi(L) = \langle \varphi_1', \ldots, \varphi_t' \rangle$ in this basis. For any $x \in V$ and $u \in U$ we have $[x, u] = \sum_{i=t_0+1}^t \varphi_i'(x, u)z_i' \in \langle z_{t_0+1}, \ldots, z_t \rangle$. Hence, the linear span $I = \langle U, z_{t_0+1}, \ldots, z_t \rangle$ is a desired ideal. □

**Lemma 4.** Suppose that the positive integers $n$, $n_0$, $t$ and $t_0$ satisfy the inequality (1.5) and let $n \geq n_0$. Then any Lie algebra $L \in \mathcal{R}_2(n, t)$ has the property $S(n_0, t_0)$. 

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Proof. In the notation of Lemma 3 for any Lie algebra $L \in \mathfrak{h}_2(n,t)$ and for any subspace $U \subset V$ of dimension $n - n_0$ we have

$$\dim(N_0(U) \cap \langle \varphi_1, \ldots, \varphi_t \rangle) \geq \dim N_0(U) + \dim \langle \varphi_1, \ldots, \varphi_t \rangle - \dim B_n = \frac{n_0(n_0 - 1)}{2} + t - n(n - 1).$$

Combining this with inequality (1.5), we get (3.4). □

Obviously, if $n < n_0$ or $t < t_0$, there are no Lie algebras in $\mathfrak{h}_2(n,t)$ having the property $\mathcal{S}(n_0,t_0)$. And so in what follows to prove the theorem we fix integers $n$, $n_0$, $t$, $t_0$, $k$ such that $n = n_0 + k$, $t \geq t_0 \geq 1$, $n_0 \geq 2$, $k \geq 0$ and an $n$-dimensional vector space $V$. Denote by $D$ the following subset of the direct product $H = G(k,V) \times G(t,B_n) \subset \mathbb{P}(\Lambda^k V) \times \mathbb{P}(\Lambda^t B_n)$

$$D = \{ (p, \varphi) \in H \mid p = P(U), \varphi = P(\Omega), \dim(N_0(U) \cap \Omega) \geq t_0 \}.$$

Lemma 5. $D$ is a projective variety.

Proof. Fix a basis $e_1, \ldots, e_n$ of $V$ and a basis $\psi_1, \ldots, \psi_{n(n-1)}$ of $B_n$. Then the coordinates $(p_{i_1 \ldots i_k})$ and $(\varphi_{j_1 \ldots j_t})$ on $\mathbb{P}(\Lambda^k V)$ and $\mathbb{P}(\Lambda^t B_n)$ respectively (and, hence, on $H$) are naturally defined.

Let us prove that $D$ is closed in $H$. For this we consider the covering of the projective variety $H$ by the open sets

$$O_{i_1 \ldots i_k,j_1 \ldots j_t} = \{ (p, \varphi) \in H \mid p_{i_1 \ldots i_k} \neq 0 \text{ and } \varphi_{j_1 \ldots j_t} \neq 0 \}$$

and verify that $O_{i_1 \ldots i_k,j_1 \ldots j_t} \cap D$ is closed in $O_{i_1 \ldots i_k,j_1 \ldots j_t}$ for any sequences $i_1 \ldots i_k$ and $j_1 \ldots j_t$ with $i_1 < \cdots < i_k$, and $j_1 < \cdots < j_t$. Then, by Proposition 4, $D$ is closed in $H$. Hence, $D$ is also a projective variety.

Let us consider an arbitrary point $(p, \varphi) \in G(k,V) \times G(t,B_n)$. We can assume without loss of generality that $p_{i_1 \ldots i_k} \neq 0$ and $\varphi_{j_1 \ldots j_t} \neq 0$. Let $p = P(U)$ and $\varphi = P(\Omega)$. Then, according formula (2.3), $U$ has the basis

$$f_i = e_i + \sum_{r=k+1}^{n} a_{ir} e_r \quad \text{for } i = 1, \ldots, k,$$

where

$$a_{ir} = (-1)^{k-i} \frac{p_{1 \ldots i \hat{r} \ldots k}}{p_{1 \ldots k}};$$

and $\Omega$ has the basis

$$\varphi_i = \psi_i + \sum_{r=t+1}^{n(n-1)} b_{ir} \psi_r \quad \text{for } i = 1, \ldots, t,$$

where

$$b_{ir} = (-1)^{t-i} \frac{\varphi_{1 \ldots \hat{i} \ldots \hat{r} \ldots t}}{\varphi_{1 \ldots t}}.$$
It follows from (3.5) that the vectors \( f_1, \ldots, f_k, e_{k+1}, \ldots, e_n \) form a basis of \( V \). Denote by \( C \) the transformation matrix from the basis \( e_1, \ldots, e_n \) to this new one and by \( A_1, \ldots, A_{\frac{n(n-1)}{2}} \) the matrices of \( \psi_1, \ldots, \psi_{\frac{n(n-1)}{2}} \) in the initial basis. Then, by formula (3.7), matrices of \( \varphi_1, \ldots, \varphi_t \) have the form

\[
F_i = C^1(A_i + \sum_{r=1+1}^{\frac{n(n-1)}{2}} b_{ir}A_r)C \quad \text{for } i = 1, \ldots, t,
\]

in the new basis and, as we mentioned above, \( N_0(U) \) consists of the forms whose matrices are zero except lower right-hand \((n-k) \times (n-k)\) submatrix.

Let \( E_1, \ldots, E_{\frac{n(n-1)}{2}} \) be a basis of \( N_0(U) \). (These are matrices with constant coefficients.) The condition \( \dim(N_0(U) \cap \Omega) \geq t_0 \) holds if and only if the rank of the vector system \( \{F_1, \ldots, F_t, E_1, \ldots, E_{\frac{n(n-1)}{2}}\} \) is not greater than \( s = \frac{n_0(n_0-1)}{2} + t - t_0 \).

The last condition in turn is equivalent to the fact that all the minors of size \( s \) in the corresponding matrix are zero. That is, we have a system of polynomial equations in variables \( a_{ir} \) and \( b_{ir} \). Replacing this variables using (3.6) and (3.8) and multiplying both sides of the equations by the appropriate powers of \( p_{1\ldots k} \) and \( \varphi_{1\ldots t} \), we obtain the system of equations homogeneous separately in each set of variables \( p \) and \( \varphi \). According to Proposition 7, the set \( D \cap O_{1\ldots k, 1\ldots t} \), defined by the last system, is closed in \( O_{1\ldots k, 1\ldots t} \). □

Now consider the projections \( \pi_1 : D \to \mathbb{P}(\Lambda^k V) \) and \( \pi_2 : D \to \mathbb{P}(\Lambda^t B_n) \) such that \( \pi_1(p, \varphi) = p \), \( \pi_2(p, \varphi) = \varphi \). These are regular maps. Clearly \( \pi_1(D) = G(k, V) \).

**Lemma 6.** \( \varphi_1 \wedge \ldots \wedge \varphi_t \in G(t, B_n) \) belongs to \( \pi_2(D) \) if and only if the corresponding Lie algebra \( L(\varphi_1, \ldots, \varphi_t) \) has the property \( S(n_0, t_0) \). \( \pi_2(D) \) is closed in \( G(t, B_n) \).

**Proof.** The lemma follows from Lemma 3 and from the fact that the image of the projective variety \( D \) under the regular map \( \pi_2 \) is closed. □

**Lemma 7.** If the integers \( n, n_0, t \) and \( t_0 \) do not satisfy the relation (1.3), then for any point \( p \in G(k, V) \) the fiber \( \pi_1^{-1}(p) \) is an irreducible projective variety of dimension

\[
\dim \pi_1^{-1}(p) = t_0 \left( \frac{n_0(n_0-1)}{2} - t_0 \right) + (t-t_0) \left( \frac{n(n-1)}{2} - t \right).
\]  

(3.9)

**Proof.** For any point \( p = P(U) \in G(k, V) \) we have

\[
\pi_1^{-1}(p) = \{ P(\Omega) \in G(t, B_n) \mid \dim(N_0(U) \cap \Omega) \geq t_0 \}.
\]

That is, the fiber \( \pi_1^{-1}(p) \) is a set of type (2.6):

\[
\pi_1^{-1}(p) = G_{t_0}(N_0(U), B_n, t).
\]

Since (1.5) doesn’t holds, \( t_0 \) satisfies the condition of the Corollary of Proposition 9. We apply this corollary, taking in account that, by (3.3), \( \dim N_0(U) = \frac{n_0(n_0-1)}{2} \) and \( \dim B_n = \frac{n(n-1)}{2} \), and thus obtain (3.9). □
Lemma 8. \(D\) is an irreducible variety of dimension
\[
\dim D = t_0 \left( \frac{n_0(n_0 - 1)}{2} - t_0 \right) + (t - t_0) \left( \frac{n(n - 1)}{2} - t \right) + n_0(n - n_0). \tag{3.10}
\]

Proof. By the previous lemma, all the fibers \(\pi_1^{-1}(p)\) are irreducible and of the same dimension. The image \(\pi_1(D) = G(k, V)\) is also irreducible by Proposition 8 (iii). So, by Proposition 8, \(D\) is irreducible and we can use formula (??) to compute its dimension. Namely, \(\dim D = \dim \pi_1^{-1}(p) + \dim G(k, V)\). Using Lemma 7 and Proposition 8 (ii), we get (3.10). □

Proof of Theorem 3. To prove the first part of the theorem it is enough to show that under the condition (1.4) the inequality \(\dim \pi_2(D) < \dim G(t, B_n)\) holds and apply Lemmas 2 and 8. By (??), this inequality follows from
\[
\dim D < \dim G(t, B_n).
\]
Substituting for \(\dim D\) using Lemma 8 and recalling that \(\dim G(t, B_n) = t \left( \frac{n(n-1)}{2} - t \right)\) we obtain the relation equivalent to (1.4).

The second part of the theorem follows from Lemma 4. So, Theorem 3 is proved. □

Proof of the Corollary of Theorem 3. A Lie algebra \(L\) does not admit a surjective homomorphism on a non-abelian Lie algebra of dimension \(N\), if it does not have the property \(S(n_0, t_0)\) for any positive integers \(n_0, t_0\) such that \(n_0 + t_0 = N\) and \(t_0 \leq \frac{n_0(n_0 - 1)}{2}\). By Theorem 3, this condition holds for almost all the Lie algebras \(L \in \mathcal{N}_2(n, t)\) if for all of the stated values of \(n_0\) and \(t_0\) we have (1.4), that is,
\[
t < \frac{n^2}{2} + \frac{1}{2} n - 2n_0 - \frac{N(n - N)}{N - n_0} - \frac{n_0(n_0 - 1)}{2}.
\]
If \(N \leq n\), the right term of the inequality is decreasing in \(n_0\) function and, hence, it takes the minimum value at \(n_0 = N - 1\). □

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