Research Article

The Spectral Method for the Cahn-Hilliard Equation with Concentration-Dependent Mobility

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This paper is concerned with the numerical approximations of the Cahn-Hilliard-type equation with concentration-dependent mobility. Convergence analysis and error estimates are presented for the numerical solutions based on the spectral method for the space and the implicit Euler method for the time. Numerical experiments are carried out to illustrate the theoretical analysis.

1. Introduction

In this paper, we apply the spectral method to approximate the solutions of Cahn-Hilliard equation, which is a typical class of nonlinear fourth-order diffusion equations. Diffusion phenomena is widespread in the nature. Therefore, the study of the diffusion equation caught wide concern. Cahn-Hilliard equation was proposed by Cahn and Hilliard in 1958 as a mathematical model describing the diffusion phenomena of phase transition in thermodynamics. Later, such equations were suggested as mathematical models of physical problems in many fields such as competition and exclusion of biological groups [1], moving process of river basin [2], and diffusion of oil film over a solid surface [3]. Due to the important application in chemistry, material science, and other fields, there were many investigations on the Cahn-Hilliard equations, and abundant results are already brought about.

The systematic study of Cahn-Hilliard equations started from the 1980s. It was Elliott and Zheng [4] who first study the following so-called standard Cahn-Hilliard equation with constant mobility:

\[
\frac{\partial u}{\partial t} + \text{div}[k \nabla \Delta u - \nabla A(u)] = 0.
\]  (1.1)
Basing on global energy estimates, they proved the global existence and uniqueness of classical solution of the initial boundary problem. They also discussed the blow-up property of classical solutions. Since then, there were many remarkable studies on the Cahn-Hilliard equations, for example, the asymptotic behavior of solutions [5–8], perturbation of solutions [9, 10], stability of solutions [11, 12], and the properties of the solutions for the Cahn-Hilliard equations with dynamic boundary conditions [13–16]. In the mean time, a number of the numerical techniques for Cahn-Hilliard equations were produced and developed. These techniques include the finite element method [4, 17–24], the finite difference method [25–29], the spectral method, and the pseudospectral method [30–35]. The finite element method for the Cahn-Hilliard equation is well investigated by many researchers. For example, in [23, 36], (1.1) was discreted by conforming finite element method with an implicit time discretization. In [22], semidiscrete schemes which can define a Lyapunov functional and remain mass constant were used for a mixed formulation of the governing equation. In [37], a mixed finite element formulation with an implicit time discretization was presented for the Cahn-Hilliard equation (1.1) with Dirichlet boundary conditions. The conventional strategy to obtain numerical solutions by the finite difference method is to choose appropriate mesh size based on the linear stability analysis for different schemes. However, this conventional strategy does not work well for the Cahn-Hilliard equation due to the bad numerical stability. Therefore, an alternative strategy is proposed for general problems, for example, in [26, 38] the strategy was to design such that schemes inherit the energy dissipation property and the mass conservation by Furihata. In [27, 28], a conservative multigrid method was developed by Kim.

The advantage of the spectral method is the infinite order convergence; that is, if the exact solution of the Cahn-Hilliard equation is $C^k$ smooth, the approximate solution will be convergent to the exact solution with power for exponent $N^{-k}$, where $N$ is the number of the basis function. This method is superior to the finite element methods and finite difference methods, and a lot of practice and experiments convince the validity of the spectral method [39]. Many authors have studied the solution of the Cahn-Hilliard equation which has constant mobility by using spectral method. For example, in [33–35], Ye studied the solution of the Cahn-Hilliard equation by Fourier collocation spectral method and Legendre collocation spectral method under different boundary conditions. In [30], the author studied a class of the Cahn-Hilliard equation with pseudospectral method. However, the Cahn-Hilliard equation with varying mobility can depict the physical phenomena more accurately; therefore, there is practical meaning to study the numerical solution for the Cahn-Hilliard equation with varying mobility. Yin [40, 41] studied the Cahn-Hilliard equation with concentration-dependent mobility in one dimension and obtained the existence and uniqueness of the classical solution. Recently, Yin and Liu [42, 43] investigate the regularity for the solution in two dimensions. Some numerical techniques for the Cahn-Hilliard equation with concentration-dependent mobility are already studied with the finite element method [28] and with finite difference method [44].

In this paper, we consider an initial-boundary value problem for Cahn-Hilliard equation of the following form:

\[
\frac{\partial u}{\partial t} + D \left[ m(u) \left( D^3 u - D A(u) \right) \right] = 0, \quad (x, t) \in (0, 1) \times (0, T), \quad (1.2)
\]

\[
D u \bigg|_{x=0,1} = D^3 u \bigg|_{x=0,1} = 0, \quad t \in (0, T), \quad (1.3)
\]

\[
u(x, 0) = u_0, \quad x \in (0, 1), \quad (1.4)
\]
where $D = \partial / \partial x$ and
\[
A(s) = -s + \gamma_1 s^2 + \gamma_2 s^3, \quad \gamma_2 > 0.
\]

Here, $u(x,t)$ represents a relative concentration of one component in binary mixture. The function $m(u)$ is the mobility which depends on the unknown function $u$, which restricts diffusion of both components to the interfacial region only. Denote $Q_T = (0,1) \times (0,T)$. Throughout this paper, we assume that
\[
0 < m_0 \leq m(s) \leq M_0, \quad |m'(s)| \leq M_1, \quad \forall s \in \mathbb{R},
\]
where $m_0, M_0, \text{ and } M_1$ are positive constants. The existence and uniqueness of the classical solution of the problems (1.2)–(1.4) were proved by Yin [41].

In this paper, we will apply the spectral method to discretize the spatial variables of (1.2) to construct a semidiscrete system. We prove the existence and boundedness of the solutions of this semidiscrete system. Then, we apply implicit midpoint Euler scheme to discretize the time variable and obtain a full-discrete scheme, which inherits the energy dissipation property. The property of the mobility depending on the solution of (1.2) causes much troubles for the numerical analysis. Furthermore, with the aid of Nirenberg inequality we investigate the boundedness and convergence of the numerical solutions of the full-discrete equations. We also obtain the error estimation for the numerical solutions to the exact ones.

This paper is organized as follows. In Section 2, we study the spectral method for (1.2)–(1.4) and obtain the error estimate between the exact solution $u$ and the spectral approximate solution $u_N$. In Section 3, we use the implicit Euler method to discretize the time variable and obtain the error estimate between the exact solution $u$ and the full-discrete approximate solution $U_N$. Finally in Section 4, we present a numerical computation to illustrate the theoretical analysis.

2. Semidiscretization with Spectral Method

In this section, we apply the spectral method to discretize (1.2)–(1.4) and study the error estimate between the exact solution and the semidiscretization solution.

Denote by $\| \cdot \|_k$ and $| \cdot |_k$ the norm and seminorm of the Sobolev spaces $H^k(0,1)(k \in \mathbb{N})$, respectively. Let $(\cdot, \cdot)$ be the standard $L^2$ inner product over $(0,1)$. Define
\[
H^2_E(0,1) = \{ v \in H^2(0,1); Dv|_{x=0,1} = 0 \}.
\]

A function $u$ is said to be a weak solution of the problems (1.2)–(1.4), if $u \in L^\infty(0,T;H^2_E(0,1))$ and satisfies the following equations:
\[
\left( \frac{\partial u}{\partial t},v \right) + \left( D^2 u - A(u), D(m(u)Dv) \right) = 0, \quad \forall v \in H^2_E, \tag{2.2}
\]
\[
(u(\cdot,0),v) - (u_0,v) = 0, \quad \forall v \in H^2_E. \tag{2.3}
\]
For any integer $N > 0$, let $S_N = \text{span}\{\cos k\pi x, k = 0, 1, 2, \ldots, N\}$. Define a projection operator $P_N : H^2_E(I) \mapsto S_N$ by

$$
\int_0^1 (P_N)u(x)v(x)dx = \int_0^1 u(x)v(x)dx, \quad \forall v \in S_N.
$$

(2.4)

We collect some properties of this projection $P_N$ in the following lemma (see [39]).

**Lemma 2.1.** (i) $P_N$ commutes with the second derivation on $H^2_E(I)$, that is,

$$
P_N D^2v = D^2P_N v, \quad \forall v \in H^2_E(I).
$$

(2.5)

(ii) For any $0 \leq \mu \leq \sigma$, there exists a positive constant $C$ such that

$$
\|v - P_N v\|_\mu \leq CN^{\mu - \sigma}|v|_\sigma, \quad v \in H^\sigma(I).
$$

(2.6)

The following Nirenberg inequality is a key tool for our theoretical estimates.

**Lemma 2.2.** Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, $u \in W^{m,r}(\Omega)$, then we have

$$
\left\|D^j u\right\|_{L^p} \leq C_1 \|D^m u\|_{L^r}^j \|u\|_{L^{1+r}}^{1-j} + C_2 \|u\|_{L^r},
$$

(2.7)

where

$$
\frac{j}{m} \leq a < 1, \quad \frac{1}{p} = \frac{j}{n} + a \left(\frac{1}{r} - \frac{m}{n}\right) + (1 - a) \frac{1}{q}.
$$

(2.8)

By [41], we have the following.

**Lemma 2.3.** Assume that $m(s) \in C^{1+a}(\mathbb{R})$, $u_0 \in C^{4+a}(\bar{I})$, $D^i u_0(0) = D^i u_0(1) = 0$ ($i = 0, 1, 2, 3, 4$), $m(s) > 0$, then there exists a unique solution of the problems (1.2)–(1.4) such that

$$
u \in C^{1+a/4, 4+a}(\bar{Q}_T).
$$

(2.9)

The spectral approximation to (2.2) is to find an element $u_N(\cdot, t) \in S_N$ such that

$$
\left(\frac{\partial u_N}{\partial t}, v_N\right) + \left(D^2 u_N - P_N A(u_N), D(m(u_N)) Dv_N\right) = 0, \quad \forall v_N \in S_N,
$$

(2.10)

$$
(u_N(\cdot, 0), v_N) - (u_0, v_N) = 0, \quad \forall v_N \in S_N.
$$

(2.11)

Now we study the $L^\infty$ norm estimates of the function $u_N(\cdot, t)$ and $Du_N(\cdot, t)$ for $0 \leq t \leq T$. 
Theorem 2.4. Assume (1.6) and \( u_0 \in H_E^2 \). Then there is a unique solution of (2.10) and (2.11) such that

\[
\|u_N\|_{\infty} \leq C, \quad \|Du_N\| \leq C, \quad 0 \leq t \leq T,
\]

(2.12)

where \( C = C(u_0, T, \gamma_1, \gamma_2) \) is a positive constant.

Proof. From (2.11) it follows that \( u_N(\cdot, 0) = P_Nu_0(\cdot) \). The existence and uniqueness of the initial problem follow from the standard ODE theory. Now we study the estimate.

Define an energy function:

\[
F(t) = \langle H(u_N), 1 \rangle + \frac{1}{2}\|Du_N\|^2,
\]

(2.13)

where \( H(s) = \int_0^s A(t)dt = (1/4)\gamma_2 s^4 + (1/3)\gamma_1 s^3 - (1/2)s^2 \). Direct computation gives

\[
\frac{dF}{dt} = \left( A(u_N), \frac{\partial u_N}{\partial t} \right) + \left( Du_N, D\frac{\partial u_N}{\partial t} \right) = \left( P_NA(u_N), \frac{\partial u_N}{\partial t} \right) - \left( D^2u_N, \frac{\partial u_N}{\partial t} \right)
\]

\[
= \left( \frac{\partial u_N}{\partial t}, P_NA(u_N) - D^2u_N \right).
\]

(2.14)

Noticing that \( P_NA(u_N) - D^2u_N \in S_N \) and setting \( v_N = P_NA(u_N) - D^2u_N \) in (2.10), applying integration by part, we obtain

\[
\frac{dF}{dt} = \left( m(u_N) \left( D^3u_N - DPNA(u_N) \right), D \left( P_NA(u_N) - D^2u_N \right) \right)
\]

\[
= - \left( m(u_N) \left( D^3u_N - DPNA(u_N) \right), D^3u_N - DPNA(u_N) \right)
\]

\[
\leq -m_0 \left\| D \left( P_NA(u_N) - D^2u_N \right) \right\|^2 \leq 0.
\]

(2.15)

Hence,

\[
F(t) \leq F(0) = \int_0^1 H(P_Nu_0)ds + \frac{1}{2}\|DPNu_0\|^2, \quad \forall 0 \leq t \leq T.
\]

(2.16)

Applying Young inequality, we obtain

\[
w_N^2 \leq \varepsilon u_N^4 + C_{1\varepsilon}, \quad w_N^3 \leq \varepsilon u_N^4 + C_{2\varepsilon},
\]

(2.17)

where \( C_{1\varepsilon} \) and \( C_{2\varepsilon} \) are positive constants. Letting \( \varepsilon = 3|\gamma_2|/(8|\gamma_1| + 12) \), then for all \( 0 \leq t \leq T \),

\[
\int_0^1 H(u_N)dx \geq \int_0^1 \left( \frac{1}{4}|\gamma_2|u_N^4 - \frac{1}{3}|\gamma_1|u_N^3 - \frac{1}{2}u_N^2 \right)dx \geq \frac{|\gamma_2|}{8} \int_0^1 (u_N)^4dx - K_1,
\]

(2.18)
where $K_1$ is a positive constant depending only on $\gamma_1$ and $\gamma_2$. Therefore, we get

\[
\frac{1}{2} \|Du_N(\cdot,t)\|^2 + \frac{\gamma_2}{8} \int_0^1 (u_N(\cdot,t))^4 \, dx - K_1 \leq \frac{1}{2} \|Du_N(\cdot,t)\|^2 + \int_0^1 H(u_N(\cdot,t)) \, dx
\]

\[
= F(t) \leq F(0) = \int_0^1 H(P_Nu_0) \, dx + \|DP_Nu_0\|^2.
\]

Thus,

\[
\|Du_N(\cdot,t)\|^2 \leq 2K_1 + 2F(0) = 2K_1 + 2 \int_0^1 H(P_Nu_0) \, dx + \|DP_Nu_0\|^2 \triangleq C,
\]

\[
\int_0^1 (u_N(x,t))^4 \, dx \leq \frac{8}{\gamma_2} (K_1 + F(0)) = \frac{8}{\gamma_2} \left( K_1 + \int_0^1 H(P_Nu_0) \, dx + \frac{1}{2} \|DP_Nu_0\|^2 \right) \triangleq C,
\]

where $C = C(u_0, \gamma_1, \gamma_2)$ is a constant. By Hölder inequality, we obtain

\[
\|u_N\|^2 = \int_0^1 u_N^2 \, dx \leq \left( \int_0^1 u_N^4 \, dx \right)^{1/2} \left( \int_0^1 1^2 \, dx \right)^{1/2} = \left( \int_0^1 u_N^4 \, dx \right)^{1/2}.
\]

Therefore,

\[
\|u_N\| \leq C, \quad \|Du_N\| \leq C.
\]

From the embedding theorem it follows that

\[
\|u_N\|_\infty \leq C, \quad \forall 0 \leq t \leq T.
\]

**Theorem 2.5.** Assume (1.6) and let $u_N(\cdot,t)$ be the solution of (2.10) and (2.11). Then there is a positive constant $C = C(u_0, m, T, \gamma_1, \gamma_2) > 0$ such that

\[
\|Du_N\|_\infty \leq C, \quad \|D^2u_N\| \leq C.
\]

**Proof.** Setting $v_N = D^4u_N$ in (2.10) and integrating by parts, we get

\[
\frac{1}{2} \frac{d}{dt} \|D^2u_N\|^2 + \left( m(u_N) \left( D^4u_N - D^2P_NA(u_N) \right), D^4u_N \right) + \left( m'(u_N)Du_N \left( D^3u_N - DP_NA(u_N) \right), D^4u_N \right) = 0.
\]
Consequently,

\[
\frac{1}{2} \frac{d}{dt} \left\| D^2 u_N \right\|^2 + \left( m(u_N)D^4 u_N, D^4 u_N \right) = I_1 + I_2 + I_3, \tag{2.26}
\]

where

\[
I_1 = \left( m(u_N)D^2 P_N A(u_N), D^4 u_N \right),
\]

\[
I_2 = - \left( m'(u_N)Du_N D^3 u_N, D^4 u_N \right), \tag{2.27}
\]

\[
I_3 = \left( m'(u_N)Du_N D P_N A(u_N), D^4 u_N \right).
\]

Noticing (1.6), we have

\[
\frac{1}{2} \frac{d}{dt} \left\| D^2 u_N \right\|^2 + m_0 \left\| D^4 u_N \right\|^2 \leq \frac{1}{2} \frac{d}{dt} \left\| D^2 u_N \right\|^2 + \left( m(u_N)D^4 u_N, D^4 u_N \right). \tag{2.28}
\]

In terms of the Nirenberg inequality (2.7), there is a constant $C > 0$ such that

\[
\|Du_N\|_\infty \leq C \left( \|D^4 u_N\|^{3/8} \|u\|^{5/8} + \|u\| \right),
\]

\[
\|Du_N\|_\infty \leq C \left( \|D^2 u_N\|^{3/4} \|u\|^{1/4} + \|u\| \right), \tag{2.29}
\]

\[
\|D^3 u_N\|_\infty \leq C \left( \|D^4 u_N\|^{7/8} \|u\|^{1/8} + \|u\| \right).
\]

Noticing the definition of the function $A$ and the estimates in (2.12), we have

\[
\|A(u_N)\|_\infty \leq C, \quad \|A'(u_N)\|_\infty \leq C, \quad \|A''(u_N)\|_\infty \leq C, \tag{2.30}
\]
for some constant $C = C(u_0, T, \gamma_1, \gamma_2) > 0$. Applying Hölder inequality and Young inequality, for any $\varepsilon > 0$,

$$|I_1| \leq M_0 \left(\left\| A'(u_N) D^2 u_N \right\| + \left\| A''(u_N) (Du_N)^2 \right\| \right) \cdot \left\| D^4 u_N \right\|$$

$$\leq M_0 \left\| A'(u_N) \right\|_{\infty} \cdot \left\| D^2 u_N \right\| \left\| D^4 u_N \right\| + M_0 \left\| A''(u_N) \right\|_{\infty} \cdot \left\| Du_N \right\| \left\| Du_N \right\|_{\infty} \left\| D^4 u_N \right\|$$

$$\leq \varepsilon \left\| D^4 u_N \right\|^2 + C \left\| D^2 u_N \right\|^2 + \epsilon \left(\left\| D^4 u_N \right\|^{3/8} + 1 \right) \left\| D^4 u_N \right\|$$

$$\leq \varepsilon \left\| D^4 u_N \right\|^2 + C \left\| D^2 u_N \right\|^2 + C \left\| D^4 u_N \right\|^{11/8} + C \left\| D^4 u_N \right\|$$

$$\leq \varepsilon \left\| D^4 u_N \right\|^2 + C \left\| D^2 u_N \right\|^2 + \frac{\varepsilon}{2} \left\| D^4 u_N \right\|^2 + C + \frac{\varepsilon}{2} \left\| D^4 u_N \right\|^2 + C$$

$$\leq 2\varepsilon \left\| D^4 u_N \right\|^2 + C \left\| D^2 u_N \right\|^2 + C,$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2, \varepsilon) > 0$ is a positive constant. Similarly, we obtain

$$|I_2| \leq M_1 \left\| Du_N \right\| \left\| D^3 u_N \right\|_{\infty} \left\| D^4 u_N \right\|$$

$$\leq C \left(\left\| D^4 u_N \right\|^{7/8} + 1 \right) \left\| D^4 u_N \right\|$$

$$\leq C \left(\left\| D^4 u_N \right\|^{15/8} + \left\| D^4 u_N \right\| \right)$$

$$\leq \frac{\varepsilon}{2} \left\| D^4 u_N \right\|^2 + C + \frac{\varepsilon}{2} \left\| D^4 u_N \right\|^2 + C$$

$$\leq \varepsilon \left\| D^4 u_N \right\|^2 + C,$$

$$|I_3| \leq M_1 \left\| DP_A(u_N) \right\|_{\infty} \cdot \left\| Du_N \right\| \cdot \left\| D^4 u_N \right\|$$

$$\leq C \left\| DP_A(u_N) \right\| \left\| D^4 u_N \right\|$$

$$\leq C \left(\left\| D^2 P_A(u_N) \right\|^{3/4} \left\| P_N A(u_N) \right\|^{1/4} + \left\| P_N A(u_N) \right\| \right) \left\| D^4 u_N \right\|$$

$$\leq C \left(\left\| D^2 P_A(u_N) \right\|^{3/4} \left\| A(u_N) \right\|^{1/4} + \left\| A(u_N) \right\| \right) \left\| D^4 u_N \right\|$$

$$\leq C \left(\left\| D^2 P_A(u_N) \right\|^{3/4} \left\| A(u_N) \right\|_{\infty}^{1/4} + \left\| A(u_N) \right\|_{\infty} \right) \left\| D^4 u_N \right\|$$

$$\leq C \left(\left\| D^2 P_A(u_N) \right\|^{3/4} + 1 \right) \left\| D^4 u_N \right\|$$
\[
\begin{align*}
\leq & \ C\left(\|D^2 P_N A(u_N)\| + 1\right)\|D^4 u_N\| \\
\leq & \ C\left(\|D^2 A(u_N)\| + 1\right)\|D^4 u_N\| \\
\leq & \ C\left(\|A'(u_N)D^2 u_N\| + \|A''(u_N)(D u_N)^2\|\right) \cdot \|D^4 u_N\| \\
\leq & \ \epsilon \|D^4 u_N\|^2 + C.
\end{align*}
\] (2.32)

Hence,
\[
\frac{1}{2} \frac{d}{dt} \|D^2 u_N\|^2 + m_0 \|D^4 u_N\|^2 \leq 4\epsilon \|D^4 u_N\|^2 + C \|D^2 u_N\|^2 + C.
\] (2.33)

Taking \( \epsilon = m_0/8 \), we have
\[
\frac{1}{2} \frac{d}{dt} \|D^2 u_N\|^2 + \frac{m_0}{2} \|D^4 u_N\|^2 \leq C \|D^2 u_N\|^2 + C.
\] (2.34)

where \( C = C(u_0, m, T, \gamma_1, \gamma_2) > 0 \) is a positive constant. Therefore,
\[
\frac{1}{2} \frac{d}{dt} \|D^2 u_N\|^2 \leq C \|D^2 u_N\|^2 + C.
\] (2.35)

From Gronwall inequality it follows that
\[
\|D^2 u_N\|^2 \leq \exp^C \left(\|D^2 u_0\|^2 + 1\right) \leq \exp^C \left(\|D^2 u_0\|^2 + 1\right) \leq C,
\] (2.36)

where \( C = C(u_0, m, T, \gamma_1, \gamma_2) > 0 \) is a positive constant. According to the embedding theorem, we have
\[
\|D u_N\|_{\infty} \leq C, \quad 0 \leq t \leq T,
\] (2.37)

where \( C = C(u_0, m, T, \gamma_1, \gamma_2) > 0 \) is a positive constant.

Now, we study the error estimates between the exact solution \( u \) and the semidiscrete spectral approximation solution \( u_N \). Set the following decomposition:
\[
u - u_N = \eta + e, \quad \eta = u - P_N u, \quad e = P_N u - u_N.
\] (2.38)

From the inequality (2.6) it follows that
\[
\|\eta\| \leq C N^{-4}.
\] (2.39)

Hence, it remains to obtain the approximate bounds of \( e \).
Theorem 2.6. Assume that $u$ is the solution of (1.2)–(1.4), $u_N \in S_N$ is the solution of (2.10) and (2.11), and $m$ is smooth and satisfies (1.6), then there exists a constant $C = C(u_0, m, \gamma_1, \gamma_2)$ such that

$$
\|e\| \leq C \left( N^{-2} + \|e(0)\| \right).
$$

(2.40)

Before we prove this theorem, we study some useful approximation properties.

Lemma 2.7. For any $\varepsilon > 0$, we have

$$
-\left( D^2\eta + D^2e, D(m(u_N)De) \right) \leq -m_0 \left\| D^2e \right\|^2 + 3\varepsilon \left\| D^2e \right\|^2 + C \left( \|e\|^2 + N^{-4} \right),
$$

(2.41)

where $C = C(u_0, m, T, \gamma_1, \gamma_2, \varepsilon) > 0$ is a positive constant.

Proof. Direct computation gives

\[
-\left( D^2\eta + D^2e, D(m(u_N)De) \right) \\
= -\left( D^2e, m(u_N)D^2e \right) - \left( D^2\eta, m(u_N)D^2e \right) \\
- \left( D^2\eta, m'(u_N)Du_N De \right) - \left( D^2e, m'(u_N)Du_N De \right) \\
\leq -m_0 \left\| D^2e \right\|^2 + M_0 \left\| D^2e \right\| \left\| D^2\eta \right\|^2 \\
+ M_1 \left\| D^2\eta \right\| \left\| Du_N \right\| \|De\| + M_1 \left\| D^2e \right\| \left\| Du_N \right\| \|De\| \\
\leq -m_0 \left\| D^2e \right\|^2 + \varepsilon \left\| D^2e \right\|^2 + \frac{M_0^2 \left\| Du_N \right\|^2}{4\varepsilon} \left\| D^2\eta \right\|^2 \\
+ M_1 \left\| Du_N \right\| \left( \left\| De \right\|^2 + \left\| D^2\eta \right\|^2 \right) + \varepsilon \left\| D^2e \right\|^2 + \frac{M_1^2 \left\| Du_N \right\|^2}{4\varepsilon} \|De\|^2 \\
\leq -m_0 \left\| D^2e \right\|^2 + 2\varepsilon \left\| D^2e \right\|^2 + \left( \frac{M_0^2}{4\varepsilon} + M_1 \left\| Du_N \right\| \right) \left\| D^2\eta \right\|^2 \\
+ \left( M_1 \left\| Du_N \right\| + \frac{M_1^2 \left\| Du_N \right\|^2}{4\varepsilon} \right) \|De\|^2 \\
\leq -m_0 \left\| D^2e \right\|^2 + 2\varepsilon \left\| D^2e \right\|^2 + \left( \frac{M_0^2}{4\varepsilon} + M_1 \left\| Du_N \right\| \right) \left\| D^2\eta \right\|^2
\]
For any $\varepsilon > 0$, we have

\[
(A(u) - P_N A(u_N), D(m(u_N)\varepsilon)) \leq 3\varepsilon M^2 \|D^2 e\|^2 + C\left(\|e\|^2 + N^{-4}\right),
\]

(2.43)

where $C = C(u_0, m, T, \gamma_1, \gamma_2, \varepsilon) > 0$ is a positive constant.

**Lemma 2.8.** For any $\varepsilon > 0$, we have

\[
(A(u) - P_N A(u_N), D(m(u_N)\varepsilon)) \leq 3\varepsilon M^2 \|D^2 e\|^2 + C\left(\|e\|^2 + N^{-4}\right),
\]

(2.43)

where $C = C(u_0, m, T, \gamma_1, \gamma_2, \varepsilon) > 0$ is a positive constant.

**Proof.** Noticing that

\[
(A(u) - P_N A(u_N), D(m(u_N)\varepsilon))
\]

\[
= (A(u) - P_N A(u), D(m(u_N)\varepsilon)) + (P_N A(u) - P_N A(u_N), D(m(u_N)\varepsilon))
\]

\[
\leq \|A(u) - P_N A(u)\| \cdot \|D(m(u_N)\varepsilon)\| + \|P_N (A(u, u_N)(u - u_N))\| \cdot \|D(m(u_N)\varepsilon)\|
\]

\[
\leq \|A(u) - P_N A(u)\| \cdot \|D(m(u_N)\varepsilon)\| + \|A(u, u_N)(\eta + \varepsilon)\| \cdot \|D(m(u_N)\varepsilon)\|,
\]

(2.44)

where

\[
\bar{A}(s, \tau) = \gamma_2 \left(s^2 + s\tau + \tau^2\right) + \gamma_1 (s + \tau) - 1.
\]

(2.45)

From the boundedness of $u_N$ in Theorem 2.4 and the property of $u$ in Lemma 2.3, it follows that

\[
\|\bar{A}(u, u_N)\|_\infty \leq C.
\]

(2.46)
Lemma 2.9. Assume that

\[ \| A(u) - P_N A(u) \|^2 \leq C \left( \| u^3 - P_N u^3 \|^2 + \| u^2 - P_N u^2 \|^2 + \| u - P_N u \|^2 \right) \]

\[ \leq C N^{-8} \left( \| u^3 \|_4^4 + \| u^2 \|_4^4 + \| u \|_4^4 \right)^2 \leq C N^{-8}, \]

\[ \| A(u, u_N)(\eta + e) \|^2 \leq \| A(u, u_N) \|_\infty^2 \left( \| e \|^2 + \| \eta \|^2 \right) \leq C (N^{-8} + \| e \|^2), \]

\[ \| D(\mu_N) De \|^2 \leq 2 \| m(\mu_N) D^2 e \|^2 + 2 \| \mu'(\mu_N) Du_N De \|^2 \]

\[ \leq 2 M_2^2 \| D^2 e \|^2 + 2 M_2^2 \| Du_N \|^2 \| D e \|^2 \]

\[ \leq 2 M_2^2 \| D^2 e \|^2 + 3 M_2^4 \| D^2 e \|^2 + C \| e \|^2, \]

where \( C = C(u_0, m, T, \gamma_1, \gamma_2) > 0 \) is a positive constant. By Cauchy inequality, for any \( \varepsilon > 0 \), we have

\[ (A(u) - P_N A(u_N), D(\mu_N) De) \]

\[ \leq \| A(u) - P_N A(u) \| \cdot \| D(\mu_N) De \| + \| A(u, u_N)(\eta + e) \| \cdot \| D(\mu_N) De \| \]

\[ \leq \frac{\varepsilon}{2} \| D(\mu_N) De \|^2 + \frac{1}{2\varepsilon} \| A(u) - P_N A(u) \|^2 \leq \frac{\varepsilon}{2} \| D(\mu_N) De \|^2 + \frac{1}{2\varepsilon} \| A(u, u_N)(\eta + e) \|^2 \]

\[ \leq 3 \varepsilon M_2^2 \| D^2 e \|^2 + C \left( \| e \|^2 + N^{-8} \right), \]

(2.48)

where \( C = C(u_0, m, T, \gamma_1, \gamma_2, \varepsilon) > 0 \) is a positive constant.

Lemma 2.9. Assume that \( u \) is the solution of (1.2)–(1.4), there exists a positive constant \( C = C(u_0, m, \gamma_1, \gamma_2) \) such that

\[ (D^2 u - A(u), D(m(u) - m(u_N)) De) \leq \frac{m_0}{2} \| D^2 e \|^2 + C \left( \| e \|^2 + N^{-4} \right), \]

(2.49)

where \( C = C(u_0, m, T, \gamma_1, \gamma_2) > 0 \) is a positive constant.

Proof. By Lemma 2.3, we have

\[ \| D^3 u - DA(u) \|_\infty \leq \| D^3 u \|_\infty + \| A'(u) Du \|_\infty \leq C. \]

(2.50)
In the other hand, it follows that
\[
\|m(u) - m(u_N)\|^2 \leq M_1^2 \|u - u_N\|^2 \leq M_1^2 \left(\|e\|^2 + \|\eta\|^2\right).
\] (2.51)

By the Young inequality,
\[
\left(2.53 \quad \left(2.54 \quad \left(\begin{array}{c}
\sum^2 u - A(u), D(m(u) - m(u_N))De
\end{array}\right)
\right.\right.
\leq \left\|D^3 u - DA(u)\right\| \|m(u) - m(u_N)\| \cdot \|De\|
\leq C \|m(u) - m(u_N)\| \|De\|
\leq \epsilon \|De\|^2 + C\epsilon \left(\|e\|^2 + \|\eta\|^2\right)
\leq \epsilon \|D^2 e\|^2 + C\epsilon \left(\|e\|^2 + \|\eta\|^2\right).
\]

Choosing \(\epsilon = m_0/2\) in the previous inequality, we obtain (2.49).

\[\Box\]

**Proof of Theorem 2.6.** Setting \(v = e\) in (2.2), we obtain
\[
\left(\frac{\partial u}{\partial t}, e\right) + \left(\begin{array}{c}
2.53 \quad \left(2.54 \quad \left(\begin{array}{c}
D^2 u - A(u), D(m(u)De)
\end{array}\right)
\right.\right.
\right.\right.
\leq 0.
\] (2.53)

Setting \(v_N = e\) in (2.10), we get
\[
\left(\frac{\partial u_N}{\partial t}, e\right) + \left(\begin{array}{c}
2.53 \quad \left(2.54 \quad \left(\begin{array}{c}
D^2 u_N - P_N^A(u_N), D(m(u_N)De)
\end{array}\right)
\right.\right.
\right.\right.
\leq 0.
\] (2.54)

(2.53) minus (2.54) gives
\[
\left(\frac{\partial e}{\partial t}, e\right) = -\left(\begin{array}{c}
2.53 \quad \left(2.54 \quad \left(\begin{array}{c}
Du^2 - A(u), D(m(u) - m(u_N))De
\end{array}\right)
\right.\right.
\right.\right.
\left(\begin{array}{c}
2.53 \quad \left(2.54 \quad \left(\begin{array}{c}
D^2 u - D^2 u_N, D(m(u_N)De)
\end{array}\right)
\right.\right.
\right.\right.
\left(\begin{array}{c}
2.53 \quad \left(2.54 \quad \left(\begin{array}{c}
A(u) - P_N^A(u_N), D(m(u_N)De)
\end{array}\right)
\right.\right.
\right.\right.
\leq 0.
\] (2.55)

According to Lemmas 2.7, 2.8 and 2.9, we have
\[
\frac{1}{2} \frac{d}{dt} \|e\|^2 \leq -m_0 \left\|D^2 e\right\|^2 + 3\epsilon \left\|D^2 e\right\|^2 + C \left(\|e\|^2 + N^{-4}\right)
\leq 3\epsilon M_0^2 \left\|D^2 e\right\|^2 + C \left(N^{-8} + \|e\|^2\right)
\leq \left(-m_0 + \left(4 + 3M_0^2\right)\epsilon\right) \left\|D^2 e\right\|^2 + C \left(N^{-4} + \|e\|^2\right).
\] (2.56)
Set \( \varepsilon = m_0 / (4 + 3M_0^2) \), then there exists a positive constant \( C = C(u_0, m, \gamma_1, \gamma_2) \) such that

\[
\frac{d}{dt} \| e \|^2 \leq C \left( \| e \|^2 + N^{-4} \right).
\] (2.57)

By Gronwall inequality, we have

\[
\| e \| \leq \exp^{\| e_0 \| N^{-2}} \leq \exp^{\| e_0 \| N^{-2}} \leq C \left( \| e_0 \| + N^{-2} \right),
\] (2.58)

where \( C = C(u_0, m, T, \gamma_1, \gamma_2) > 0 \) is a constant.

Summing up the properties above, we obtain the following.

**Theorem 2.10.** Assume \( m(s) \) is sufficiently smooth and satisfies (1.6), \( u \) is the solution of (1.2)–(1.4), and \( u_N \) is the solution of (2.10) and (2.11). Then there exists a positive constant \( C = C(u_0, m, T, \gamma_1, \gamma_2) > 0 \) such that

\[
\| u - u_N \| \leq C \left( N^{-2} + \| u_0 - u_{0N} \| \right), \quad \forall t \in (0, T).
\] (2.59)

### 3. Full-Discretization Spectral Scheme

In this section, we apply implicit midpoint Euler scheme to discretize time variable and get a full-discrete form. Furthermore, we investigate the boundedness of numerical solution and the convergence of the numerical solutions of the full-discrete system. We also obtain the error estimates between the numerical solution and the exact ones.

Firstly, we introduce a partition of \([0, T]\). Let \( 0 = t_0 < t_1 < \cdots < t_{\Lambda} \), where \( t_j = jh \) and \( h = T/\Lambda \) is time-step size. Then the full-discretization spectral method for (1.2)–(1.4) reads: \( \forall v \in S_N \), find \( U^i_N \in S_N \) (\( i = 0, 1, 2, \ldots, \Lambda \)) such that

\[
\left( \frac{U_{N}^{i+1} - U_{N}^{i}}{h}, v \right) + \left( D^2 U_{N}^{i+1/2} - P_N \tilde{A} \left( U_{N}^{i+1}, U_{N}^{i} \right), D \left( m \left( U_{N}^{i+1/2} \right) Dv \right) \right) = 0,
\] (3.1)

\[
\left( U_{N}^{0}, v \right) - (u_0, v) = 0,
\] (3.2)

where \( U_{N}^{i+1/2} = (U_{N}^{i+1} + U_{N}^{i})/2 \) and

\[
\tilde{A} \left( \phi, \varphi \right) = \frac{Y_2}{4} \left( \phi^3 + \varphi^3 + \phi^2 \varphi + \phi \varphi^2 \right) + \frac{Y_1}{3} \left( \phi^2 + \phi \varphi + \varphi^2 \right) - \frac{1}{2} \left( \phi + \varphi \right).
\] (3.3)

The solution \( U_N^i \) has the following property.
Lemma 3.1. Assume that $U_j^i \in S_N$ ($j = 1, 2, \ldots, \Lambda$) is the solution of (3.1)-(3.2). Then there exists a constant $C = C(u_0, m, Y_1, Y_2) > 0$ such that

$$
\left\| U_j^i \right\|_\infty \leq C, \quad \left\| DU_j^i \right\| \leq C.
$$

(3.4)

Proof. Define a discrete energy function at time $t_j$ by

$$
F(j) = \frac{1}{2} \left\| DU_j^i \right\|^2 + \left( H(U_j^i), 1 \right).
$$

(3.5)

Notice that

$$
\frac{1}{h}(F(j + 1) - F(j)) = \left( DU_j^{i+1/2}, DU_j^{i+1} - DU_j^i \right) + \left( \tilde{A}(U_j^{i+1}, U_j^i), \frac{U_j^{i+1} - U_j^i}{h} \right)
$$

$$
= - \left( D^2 \tilde{U}_N^{i+1/2} - \tilde{A}(U_j^{i+1}, U_j^i), \frac{U_j^{i+1} - U_j^i}{h} \right).
$$

(3.6)

Setting $v = D^2 \tilde{U}_N^{i+1/2} - \tilde{A}(U_j^{i+1}, U_j^i) \in S_N$ in (3.1), we obtain

$$
\frac{1}{h}(F(j + 1) - F(j))
$$

$$
= - \left( m \left( \tilde{U}_N^{i+1/2} \right) \left( D^2 \tilde{U}_N^{i+1/2} - \tilde{A}(U_j^{i+1}, U_j^i) \right), D^3 \tilde{U}_N^{i+1/2} - \tilde{A}(U_j^{i+1}, U_j^i) \right)
$$

$$
\leq -m_0 \left\| D^3 \tilde{U}_N^{i+1/2} - \tilde{A}(U_j^{i+1}, U_j^i) \right\|^2 \leq 0,
$$

(3.7)

which implies

$$
F(j) \leq F(0) = \frac{1}{2} \left\| DU_0^i \right\|^2 + \left( H(U_0^i), 1 \right).
$$

(3.8)

By (2.18), we have

$$
\int_0^1 H(U_j^i) \, dx \geq \frac{12}{8} \int_0^1 (U_j^i)^4 \, dx - K_1.
$$

(3.9)
Then

\[
\frac{1}{2} \left\| DU_N^j \right\|^2 + \frac{K_1}{8} \int_0^1 \left( U_N^j \right)^4 dx - K_1 \leq \frac{1}{2} \left\| DU_N^j \right\|^2 + \int_0^1 H(U_N^j) dx = F(j) \\
\leq F(0) = \frac{1}{2} \left\| DU_N^0 \right\|^2 + \left( H(U_N^0), 1 \right).
\] (3.10)

So we obtain

\[
\left\| DU_N^j \right\|^2 \leq 2K_1 + \left\| DU_N^0 \right\|^2 + 2 \left( H(U_N^0), 1 \right) \equiv C_1, \\
\int_0^1 \left( U_N^j \right)^4 dx \leq \frac{8}{2} \left( K_1 + \frac{1}{2} \left\| DU_N^0 \right\|^2 + \left( H(U_N^0), 1 \right) \right) \equiv C_2,
\] (3.11)

where \( C_1 = C_1(u_0, m, \gamma_1, \gamma_2) \) and \( C_2 = C_2(u_0, m, \gamma_1, \gamma_2) \) are positive constants. By Holder inequality, we get

\[
\left\| DU_N^j \right\|^2 = \int_0^1 \left( U_N^j \right)^2 dx \leq \left( \int_0^1 \left( U_N^j \right)^4 dx \right)^{1/2} \leq C_2^{1/2}.
\] (3.12)

Therefore,

\[
\left\| U_N^j \right\| \leq C_2^{1/4}.
\] (3.13)

By the embedding theorem, we obtain

\[
\left\| U_N^j \right\|_\infty \leq C,
\] (3.14)

where \( C = C(u_0, m, \gamma_1, \gamma_2) > 0 \) is a constant.

\[ \square \]

**Lemma 3.2.** Assume that \( U_N^j \in S_N \) \((j = 1, 2, \ldots, \Lambda)\) is the solution of the full-discretization scheme (3.1)-(3.2), then there is a constant \( C = C(u_0, m, T, \gamma_1, \gamma_2) > 0 \) such that

\[
\left\| DU_N^j \right\|_\infty \leq C, \quad \left\| D^2 U_N^j \right\| \leq C.
\] (3.15)
Proof. Setting $v = 2D^4 \tilde{U}_N^{j+1/2} \in S_N$ in (3.1), we have

$$
\left( \frac{U_N^{j+1} - U_N^j}{h}, 2D^4 \tilde{U}_N^{j+1/2} \right) 
+ \left( D \left( m \left( \tilde{U}_N^{j+1/2} \right) D^3 \tilde{U}_N^{j+1/2} - D P_N \tilde{A} \left( U_N^{j+1}, U_N^j \right) \right) 2D^4 \tilde{U}_N^{j+1/2} \right) 
= \left( \frac{D^2 U_N^{j+1} - D^2 U_N^j}{h}, D^2 U_N^{j+1/2} + D^2 U_N^j \right) 
+ \left( m \left( \tilde{U}_N^{j+1/2} \right) D^4 \tilde{U}_N^{j+1/2}, 2D^4 \tilde{U}_N^{j+1/2} \right) 
+ \left( m' \left( \tilde{U}_N^{j+1/2} \right) D \tilde{U}_N^{j+1/2}, 2D \tilde{U}_N^{j+1/2} \right) 
- \left( m \left( \tilde{U}_N^{j+1/2} \right) D^2 P_N \tilde{A} \left( U_N^{j+1}, U_N^j \right), 2D^4 \tilde{U}_N^{j+1/2} \right) 
- \left( m' \left( \tilde{U}_N^{j+1/2} \right) D \tilde{U}_N^{j+1/2} D P_N \tilde{A} \left( U_N^{j+1}, U_N^j \right), 2D^4 \tilde{U}_N^{j+1/2} \right) = 0.
$$
(3.16)

Therefore,

$$
\frac{1}{h} \left( \| D^2 U_N^{j+1/2} \| - \| D^2 U_N^{j+1/2} \| \right) + 2 \left( m \left( \tilde{U}_N^{j+1/2} \right) D^4 \tilde{U}_N^{j+1/2}, 2D^4 \tilde{U}_N^{j+1/2} \right) 
= 2 \left( m' \left( \tilde{U}_N^{j+1/2} \right) D^4 \tilde{U}_N^{j+1/2} D \tilde{U}_N^{j+1/2}, 2D^4 \tilde{U}_N^{j+1/2} \right) 
+ 2 \left( m \left( \tilde{U}_N^{j+1/2} \right) D^2 P_N \tilde{A} \left( U_N^{j+1}, U_N^j \right), D^4 \tilde{U}_N^{j+1/2} \right) 
+ 2 \left( m' \left( \tilde{U}_N^{j+1/2} \right) D \tilde{U}_N^{j+1/2} D P_N \tilde{A} \left( U_N^{j+1}, U_N^j \right), D^4 \tilde{U}_N^{j+1/2} \right) 
= I_1' + I_2' + I_3'.
$$
(3.17)

By Nirenberg inequality (2.7), we have

$$
\| D U_N^j \|_\infty \leq C \left( \| D^4 U_N^j \|^{3/8} \| U_N^j \|^{5/8} + \| U_N^j \| \right), 
\| D U_N^j \|_\infty \leq C \left( \| D^2 U_N^j \|^{3/4} \| U_N^j \|^{1/4} + \| U_N^j \| \right), 
\| D^2 U_N^j \|_\infty \leq C \left( \| D^4 U_N^j \|^{7/8} \| U_N^j \|^{1/8} + \| U_N^j \| \right),
$$
(3.18)
According to (3.14), we obtain

\[
\begin{aligned}
\left\| \tilde{A}_1^i (u_{i+1}^j, u_i^j) \right\|_\infty & \leq C, \\
\left\| \tilde{A}_2^j (u_{i+1}^i, u_i^j) \right\|_\infty & \leq C, \\
\left\| \tilde{A}_{11}^i (u_{i+1}^i, u_i^j) \right\|_\infty & \leq C, \\
\left\| \tilde{A}_{12}^j (u_{i+1}^i, u_i^j) \right\|_\infty & \leq C, \\
\left\| \tilde{A}_{22}^i (u_{i+1}^i, u_i^j) \right\|_\infty & \leq C,
\end{aligned}
\]

(3.19)

where \( C = C(u_0, m, \gamma_1, \gamma_2) \) is a positive constant. By Young inequality, for any positive constant \( \varepsilon > 0 \), it follows that

\[
\begin{aligned}
|I_1^j| & \leq 2M_1 \left\| D^4 \tilde{U}_N^{i+1/2} \right\|_\infty \left\| D\tilde{U}_N^{i+1/2} \right\| \left\| D^4 \tilde{U}_N^{i+1/2} \right\| \\
& \leq C \left( \left\| D^4 \tilde{U}_N^{i+1/2} \right\|^{7/8} + 1 \right) \left\| D^4 \tilde{U}_N^{i+1/2} \right\| \\
& \leq C \left\| D^4 \tilde{U}_N^{i+1/2} \right\|^{15/8} + C \left\| D^4 \tilde{U}_N^{i+1/2} \right\| \\
& \leq \frac{\varepsilon}{2} \left\| D^4 \tilde{U}_N^{i+1/2} \right\|^2 + C \varepsilon + \frac{\varepsilon}{2} \left\| D^4 \tilde{U}_N^{i+1/2} \right\|^2 + C \varepsilon \\
& \leq \varepsilon \left\| D^4 \tilde{U}_N^{i+1/2} \right\|^2 + C \varepsilon, \\
|I_2^j| & \leq 2M_0 \left\| D^2 \tilde{A} (u_{i+1}^j, u_i^j) \right\| \left\| D^4 \tilde{U}_N^{i+1/2} \right\| \\
& \leq 2M_0 \left( A_{11}^u \left( D U_N^{i+1} \right)^2 + 2A_{12}^u D U_N^{i+1} D U_N^i + A_{22}^u \left( D U_N^i \right)^2 \\ + A_1^u D^2 U_N^{i+1} + A_2^u D^2 U_N^i \right) \left\| D^4 \tilde{U}_N^{i+1/2} \right\| \\
& \leq C \left( \left\| D^2 U_N^{i+1} \right\| + \left\| D^2 U_N^i \right\| + \left\| \left( D U_N^{i+1} \right)^2 \right\| + \left\| \left( D U_N^i \right)^2 \right\| \right) \left\| D^4 \tilde{U}_N^{i+1/2} \right\| \\
& \leq C \left( \left\| D^2 U_N^{i+1} \right\| + \left\| D^2 U_N^i \right\| + \left\| D U_N^{i+1} \right\| \left\| D U_N^{i+1} \right\|_\infty + \left\| D U_N^i \right\| \left\| D U_N^i \right\|_\infty \right) \left\| D^4 \tilde{U}_N^{i+1/2} \right\| \\
& \leq C \left( \left\| D^2 U_N^{i+1} \right\| + \left\| D^2 U_N^i \right\| + \left\| D U_N^{i+1} \right\|_\infty + \left\| D U_N^i \right\|_\infty \right) \left\| D^4 \tilde{U}_N^{i+1/2} \right\| \\
& \leq C \left( \left\| D^2 U_N^{i+1} \right\| + \left\| D^2 U_N^i \right\| + \left\| D^2 U_N^{i+1} \right\|^{3/4} + \left\| D^2 U_N^i \right\|^{3/4} + 1 \right) \left\| D^4 \tilde{U}_N^{i+1/2} \right\| \\
& \leq C \left( \left\| D^2 U_N^{i+1} \right\| + \left\| D^2 U_N^i \right\| + 1 \right) \left\| D^4 \tilde{U}_N^{i+1/2} \right\| \\
& \leq \varepsilon \left\| D^4 \tilde{U}_N^{i+1/2} \right\|^2 + C \varepsilon \left( \left\| D^2 U_N^{i+1} \right\|^2 + \left\| D^2 U_N^i \right\|^2 + 1 \right).
\end{aligned}
\]
\[ |I_3| \leq M_3 \left\| D\bar{U}_N^{j+1/2} \right\| \left\| D\bar{P}_N \bar{A}(U_{j+1}^N, U_N) \right\|_{\infty} \left\| D^4\bar{U}_N^{j+1/2} \right\| \\
\leq C \left( \left\| D^2 P_N \bar{A}(U_{j+1}^N, U_N) \right\|^{3/4} + 1 \right) \left\| D^4\bar{U}_N^{j+1/2} \right\| \\
\leq C \left( \left\| D^2 P_N \bar{A}(U_{j+1}^N, U_N) \right\| + 1 \right) \left\| D^4\bar{U}_N^{j+1/2} \right\| \\
\leq C \left( \left\| \bar{A}(U_{j+1}^N, U_N) \right\| \left\| D^4\bar{U}_N^{j+1/2} \right\| + \left\| D^4\bar{U}_N^{j+1/2} \right\| \right) \\
\leq \varepsilon \left\| D^4\bar{U}_N^{j+1/2} \right\|^2 + C_{\varepsilon} \left( \left\| D^2 U_N^{j+1} \right\|^2 + \left\| D^2 U_N^j \right\|^2 + 1 \right) + \varepsilon \left\| D^4\bar{U}_N^{j+1/2} \right\|^2 + C_{\varepsilon} \\
\leq 2\varepsilon \left\| D^4\bar{U}_N^{j+1/2} \right\|^2 + C_{\varepsilon} \left( \left\| D^2 U_N^{j+1} \right\|^2 + \left\| D^2 U_N^j \right\|^2 + 1 \right), \tag{3.20} \]

where \( C_{\varepsilon} = C_{\varepsilon}(u_0, m, \gamma_1, \gamma_2) > 0 \) is a constant. Therefore,

\[
\frac{1}{\tilde{h}} \left( \left\| D^2 U_N^{j+1} \right\|^2 - \left\| D^2 U_N^j \right\|^2 \right) + 2m_0 \left\| D^4 U_N^{j+1/2} \right\|^2 \\
\leq \frac{1}{\tilde{h}} \left( \left\| D^2 U_N^{j+1} \right\|^2 - \left\| D^2 U_N^j \right\|^2 \right) + 2 \left( \left\| D^4 \bar{U}_N^{j+1/2} \right\| + \left\| D^4\bar{U}_N^{j+1/2} \right\| \right) \tag{3.21} \\
\leq 4 \varepsilon \left\| D^2 U_N^{j+1/2} \right\|^2 + C_{\varepsilon} \left( \left\| D^2 U_N^{j+1} \right\|^2 + \left\| D^2 U_N^j \right\|^2 + 1 \right). \]

Setting \( \varepsilon = m_0/2 \), there is a positive constant \( C = C(u_0, m, \gamma_1, \gamma_2) \) such that

\[
\left\| D^2 U_N^{j+1} \right\|^2 \leq \left( 1 + \frac{2Ch}{1-C\tilde{h}} \right) \left\| D^2 U_N^j \right\|^2 + \frac{C}{1-C\tilde{h}} h. \tag{3.22} \]

Denoted by \( \tilde{C} = C/(1 - C\tilde{h}) \), if \( h \) is sufficiently small such that \( C\tilde{h} \leq 1/2 \), we have

\[
\left\| D^2 U_N^{j+1} \right\|^2 \leq \left( 1 + \tilde{C} h \right) \left\| D^2 U_N^j \right\|^2 + \frac{\tilde{C}}{2} h \\
\leq \left( 1 + \tilde{C} h \right)^{j+1} \left\| D^2 U_N^0 \right\|^2 + \frac{\tilde{C}}{2} h \sum_{i=1}^{j} \left( 1 + \tilde{C} h \right)^i \tag{3.23} \\
\leq \exp^{(j+1)\tilde{C} h} \left( \left\| D^2 U_N^0 \right\|^2 + \tilde{C} j h \right) \\
\leq \exp^{\tilde{C} \tilde{h}} \left( \left\| D^2 U_N^0 \right\|^2 + \tilde{C} \tilde{h} \right) \equiv C',
\]

where \( C' = C(u_0, m, T, \gamma_1, \gamma_2) > 0 \) is a positive constant. By the embedding theorem, the estimate (3.15) holds.

Next, we investigate the error estimates for the numerical solution \( U^i_N \) to the exact solution \( u(t_j) \). Our analysis is based on the error decomposition denoted by

\[
u(t_j) - U^i_N = \eta^i + e^i, \quad \eta^i = u(t_j) - P_N u(t_j), \quad e^i = P_N u(t_j) - U^i_N.\tag{3.24}
\]

The boundedness estimate of \( \eta^i \) follows from the inequality (2.6), that is, for any \( 0 \leq j \leq \Lambda \), there is a positive constant \( C = C(u_0, m, \gamma_1, \gamma_2) \) such that

\[
\| \eta^i \| \leq C N^{-4}.	ag{3.25}
\]

Hence, it remains to obtain the approximate bounds of \( e^i \). If no confusion occurs, we denote the average of the two instant errors \( e^i \) and \( e^{i+1} \) by \( \overline{e^{i+1/2}} \):

\[
\overline{e^{i+1/2}} = \frac{e^i + e^{i+1}}{2}, \quad \overline{\eta^{i+1/2}} = \frac{\eta^i + \eta^{i+1}}{2}.	ag{3.26}
\]

For later use, we give some estimates in the next lemmas.

**Lemma 3.3.** Assume that the solution of (1.2)–(1.4) is such that \( u_{tt} \in L^2(\Omega_T) \), then

\[
\| e^{i+1} \| \leq \| e^i \|^2 + 2h \left( u(t_{i+1/2}) - \frac{U^{i+1}_N - U^i_N}{h}, \overline{e^{i+1/2}} \right) + \frac{1}{320} h^4 \int_{t_j}^{t_{i+1}} \| u_{tt} \|^2 dt + h \| \overline{e^{i+1/2}} \|^2.	ag{3.27}
\]

**Proof.** Applying Taylor expansion about \( t_{i+1/2} \), we have

\[
u^i = u^{i+1/2} - \frac{h}{2} u_t^{i+1/2} + \frac{h^2}{8} u_{tt}^{i+1/2} - \frac{1}{2} \int_{t_j}^{t_{i+1/2}} (t - t_j)^2 u_{ttt} dt,
\]

\[
u^{i+1} = u^{i+1/2} + \frac{h}{2} u_t^{i+1/2} + \frac{h^2}{8} u_{tt}^{i+1/2} + \frac{1}{2} \int_{t_{i+1/2}}^{t_{i+1}} (t_{i+1} - t)^2 u_{ttt} dt.	ag{3.28}
\]

Then

\[
u_t^{i+1/2} - \frac{u^{i+1} - u^i}{h} = -\frac{1}{2h} \left( \int_{t_{i+1/2}}^{t_{i+1}} (t_{i+1} - t)^2 u_{ttt} dt + \int_{t_j}^{t_{i+1/2}} (t - t_j)^2 u_{ttt} dt \right).\tag{3.29}
\]
From Hölder inequality it follows that

$$
\left\| u_{j+1/2} - \frac{u^{j+1} - u^j}{h} \right\|^2 \leq \frac{1}{2h^2} \left( \left\| t_{j+1/2} (t_{j+1} - t)^2 u_{it} dt \right\|^2 + \left\| t_j (t - t_j)^2 u_{it} dt \right\|^2 \right)
$$

$$\leq \frac{h^3}{320} \int_{t_j}^{t_{j+1}} \| u_{itt} \|^2 dt. \quad (3.30)
$$

Noticing that for any $v \in S_N$, we have

$$
\begin{pmatrix}
  u_{j+1/2} - \frac{U_N^{j+1} - U_N^j}{h} \\
  u_{j+1/2} - \frac{u^{j+1} - u^j}{h}
\end{pmatrix}, v = \begin{pmatrix}
  u_{j+1/2} - \frac{U_N^{j+1} - U_N^j}{h} \\
  u^{j+1} - u^j
\end{pmatrix} + \frac{1}{h} \left( e^{j+1} - e^j, v \right). \quad (3.31)
$$

Taking $v = 2e^{j+1/2}$ in (3.31), we obtain

$$
\left\| e^{j+1} \right\|^2 = \| e^j \|^2 + 2h \left( u_{j+1/2}^i - \frac{U_N^{j+1} - U_N^j}{h}, e^{j+1/2} \right) - 2h \left( u_{j+1/2}^i - \frac{u^{j+1} - u^j}{\Delta t}, \bar{e}^{j+1/2} \right)
$$

$$\leq \| e^j \|^2 + 2h \left( u_{j+1/2}^i - \frac{U_N^{j+1} - U_N^j}{h}, e^{j+1/2} \right)
$$

$$+ \frac{1}{320} h^4 \int_{t_j}^{t_{j+1}} \| u_{itt} \|^2 dt + h \left\| e^{j+1/2} \right\|^2. \quad (3.32)
$$

Taking $v = \bar{e}^{j+1/2}$ in (2.2) and (3.1), respectively, we have

$$
\left( \frac{\partial u(t_{j+1/2})}{\partial t}, \bar{e}^{j+1/2} \right) + \left( D^2 u^{j+1/2} - A\left( u^{j+1/2} \right), D\left( m(u^{j+1/2}) D\bar{e}^{j+1/2} \right) \right) = 0, \quad (3.33)
$$

$$\left( \frac{U_N^{j+1} - U_N^j}{h}, \bar{e}^{j+1/2} \right) + \left( D^2 U_N^{j+1/2} - P_N \tilde{A}(U_N^{j+1}, U_N^j), D\left( m(U_N^{j+1/2}) D\bar{e}^{j+1/2} \right) \right) = 0. \quad (3.34)
$$
Comparing (3.33) and (3.34), we have

\[
\begin{aligned}
\left( u_i^{j+1/2} - \frac{U_N^{j+1} - U_N^j}{h} \right) \\
= - \left( D^2 u_i^{j+1/2} - \frac{D^2 U_N^{j+1} + D^2 U_N^j}{2} D\left( m\left( U_N^{j+1/2} \right) D\tilde{e}^{j+1/2} \right) \right) \\
+ \left( A\left( u_i^{j+1/2} \right) - P_N A\left( U_N^{j+1}, U_N^j \right), D\left( m\left( U_N^{j+1/2} \right) D\tilde{e}^{j+1/2} \right) \right) \\
+ \left( D^2 u_i^{j+1/2} - A\left( u_i^{j+1/2} \right), D\left( m\left( U_N^{j+1/2} \right) - m\left( u_i^{j+1/2} \right) \right) D\tilde{e}^{j+1/2} \right).
\end{aligned}
\]

(3.35)

Now we investigate the error estimates of the three items in the right-hand side of the previous equation.

**Lemma 3.4.** Assume that \( u \) is the solution of (1.2)–(1.4) such that \( Du_t \in L^2(Q_T) \), then there exists a positive constant \( C_1 = C_1(m, u_0, T, \gamma_1, \gamma_2) > 0 \) such that

\[
- \left( D^2 u_i^{j+1/2} - \frac{D^2 U_N^{j+1} + D^2 U_N^j}{2} D\left( m\left( U_N^{j+1/2} \right) D\tilde{e}^{j+1/2} \right) \right) \\
\leq - \frac{m_0}{2} \left\| D^2\tilde{e}^{j+1/2} \right\|^2 + C_1 \left( N^{-4} + \left\| e^j \right\|^2 + \left\| e^{j+1} \right\|^2 + h^3 \int_{t_j}^{t_{j+1}} \left\| D^2 u_t \right\|^2 dt \right).
\]

(3.36)

**Proof.** By Taylor expansion and Hölder inequality, we obtain

\[
\begin{aligned}
u^j = u_i^{j+1/2} - \frac{h}{2} u_i^{j+1/2} + \int_{t_j}^{t_{j+1/2}} (t - t_j) u_t dt, \\
u^{j+1} = u_i^{j+1/2} + \frac{h}{2} u_i^{j+1/2} + \int_{t_{j+1/2}}^{t_{j+1}} (t_{j+1} - t) u_t dt.
\end{aligned}
\]

(3.37)

Therefore,

\[
\frac{1}{2} (u^j + u^{j+1}) - u_i^{j+1/2} = \frac{1}{2} \left( \int_{t_j}^{t_{j+1/2}} (t - t_j) u_t dt + \int_{t_{j+1/2}}^{t_{j+1}} (t_{j+1} - t) u_t dt \right).
\]

(3.38)
By Hölder inequality, we have

\[
\left\| D^2 \left( u^{i+1/2} - \frac{1}{2} (u^i + u^{i+1}) \right) \right\|^2
\]
\[
= \frac{1}{4} \left( \left\| D^2 \left( \int_{t_j}^{t_{j+1/2}} (t-t_j)u_{tt}dt + \int_{t_{j+1/2}}^{t_{j+1}} (t_j+1-t)u_{tt}dt \right) \right\|^2 \right)
\]
\[
\leq \frac{1}{4} \left( \left\| \int_{t_j}^{t_{j+1/2}} (t-t_j)^2 dt \int_{t_j}^{t_{j+1/2}} (D^2 u_{tt})^2 dt + \int_{t_{j+1/2}}^{t_{j+1}} (t_{j+1}-t)^2 dt \int_{t_{j+1/2}}^{t_{j+1}} (D^2 u_{tt})^2 dt \right\|^2 \right)
\]
\[
= \frac{1}{4} \left( \left\| \left( \frac{h^3}{24} \int_{t_j}^{t_{j+1/2}} (D^2 u_{tt})^2 dt + \frac{h^3}{24} \int_{t_{j+1/2}}^{t_{j+1}} (D^2 u_{tt})^2 dt \right) \right\|^2 \right)
\]
\[
\leq \frac{1}{4} \cdot \frac{h^3}{24} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt.
\]

(3.39)

Direct computation gives

\[
\left\| D^2 \left( u^{i+1/2} - \frac{1}{2} (u^i + u^{i+1}) \right) \right\|^2 \leq \frac{h^3}{96} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt.
\]

(3.40)

Then

\[
- \left( D^2 \left( u^{i+1/2} - \frac{U_{j+1}^i + U_{j}^i}{2} \right), D \left( m \left( \overline{U}_{j+1/2} D \overline{e}_{j+1/2} \right) \right) \right)
\]
\[
= - \left( D^2 \left( u^{i+1/2} - \frac{u^{i+1} + u^i}{2} \right), D \left( m \left( \overline{U}_{j+1/2} D \overline{e}_{j+1/2} \right) \right) \right)
\]
\[
- \left( D^2 \left( \frac{u^{i+1} + u^i}{2} - \frac{U_{j+1}^i + U_{j}^i}{2} \right), D \left( m \left( \overline{U}_{j+1/2} D \overline{e}_{j+1/2} \right) \right) \right)
\]
\[
\leq \left\| D^2 \left( u^{i+1/2} - \frac{u^{i+1} + u^i}{2} \right) \right\| \cdot \left\| D \left( m \left( \overline{U}_{j+1/2} D \overline{e}_{j+1/2} \right) \right) \right\|
\]
\[
- \left( D^2 \left( \overline{e}_{j+1/2} + \overline{e}_{j+1/2} \right), D \left( m \left( \overline{U}_{j+1/2} D \overline{e}_{j+1/2} \right) \right) \right)
\]
\[
\frac{h^3}{96} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt \right)^{1/2} \cdot \left( \left\| m^{j+1/2} \right\| D^2 \bar{e}^{j+1/2} \right) + \left\| m^{j+1/2} \right\| D\bar{U}_{N}^{j+1/2} D\bar{e}^{j+1/2} \\
- \left( D^2 \bar{\eta}^{j+1/2}, m^{j+1/2} \right) \left( D^2 \bar{\eta}^{j+1/2}, D\bar{U}_{N}^{j+1/2} D\bar{e}^{j+1/2} \right) \\
- \left( D^2 \bar{e}^{j+1/2}, m^{j+1/2} \right) \left( D^2 \bar{e}^{j+1/2}, D\bar{U}_{N}^{j+1/2} D\bar{e}^{j+1/2} \right)
\]

\[\Delta \sigma_i = \sigma_1^i + \sigma_2^i + \sigma_3^i.\]

(3.41)

By Cauchy inequality, it follows that

\[
\left| \sigma_1^j \right| \leq \left\{ \frac{h^3}{96} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt \right\}^{1/2} \cdot \left( \left\| m^{j+1/2} \right\| D^2 \bar{e}^{j+1/2} \right) + \left\| m^{j+1/2} \right\| D\bar{U}_{N}^{j+1/2} D\bar{e}^{j+1/2} \\
+ \left\{ \frac{h^3}{96} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt \right\}^{1/2} \cdot \left( \left\| m^{j+1/2} \right\| D\bar{U}_{N}^{j+1/2} D\bar{e}^{j+1/2} \right) \\
\leq M_0 \left\{ \frac{h^3}{96} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt \right\}^{1/2} \left\| D\bar{\eta}^{j+1/2} \right\| \\
+ M_1 \left\| D\bar{U}_{N}^{j+1/2} \right\| \left\{ \frac{h^3}{96} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt \right\}^{1/2} \left\| D\bar{e}^{j+1/2} \right\| \\
\leq h^3 C_1e \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt + \varepsilon \left\| D\bar{\eta}^{j+1/2} \right\|^2 + \varepsilon \left\| D\bar{e}^{j+1/2} \right\|^2 \\
\leq h^3 C_1e \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt + \varepsilon \left\| D\bar{\eta}^{j+1/2} \right\|^2 + \varepsilon \left\| D\bar{e}^{j+1/2} \right\|^2 \\
\leq h^3 C_1e \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt + 2\varepsilon \left\| D\bar{\eta}^{j+1/2} \right\|^2, \\
\left| \sigma_2^j \right| \leq M_0 \left\| D\bar{\eta}^{j+1/2} \right\| \left\| D\bar{e}^{j+1/2} \right\| + M_1 \left\| D\bar{U}_{N}^{j+1/2} \right\| \left\| D\bar{\eta}^{j+1/2} \right\| \left\| D\bar{e}^{j+1/2} \right\| \\
\leq M_0^2 \frac{2\varepsilon}{2e} \left\| D\bar{\eta}^{j+1/2} \right\|^2 + \frac{\varepsilon}{2} \left\| D\bar{e}^{j+1/2} \right\|^2 + M_1^2 \frac{2\varepsilon}{2e} \left\| D\bar{U}_{N}^{j+1/2} \right\|^2 + \frac{\varepsilon}{2} \left\| D\bar{e}^{j+1/2} \right\|^2 \\
\leq C_2 \left\| D\bar{\eta}^{j+1/2} \right\|^2 + \frac{\varepsilon}{2} \left\| D\bar{e}^{j+1/2} \right\|^2 + \frac{\varepsilon}{2} \left\| D\bar{e}^{j+1/2} \right\|^2 \\
\leq C_2 \left\| D\bar{\eta}^{j+1/2} \right\|^2 + \frac{\varepsilon}{2} \left\| D\bar{e}^{j+1/2} \right\|^2 + \frac{\varepsilon}{2} \left\| D\bar{e}^{j+1/2} \right\|^2, \\
\left| \sigma_3^j \right| \leq \frac{\varepsilon}{2} \left\| D\bar{\eta}^{j+1/2} \right\|^2 + \frac{\varepsilon}{2} \left\| D\bar{e}^{j+1/2} \right\|^2 + \frac{\varepsilon}{2} \left\| D\bar{e}^{j+1/2} \right\|^2.
\]
Lemma 3.5. Assume that $u$ is the solution of (1.2)–(1.4) such that $u_{tt} \in L^2(Q_T)$ and $u_t \in L^\infty(Q_T)$. Then for any positive constant $\varepsilon > 0$, there exists a constant $C_2 = C_2(m, u_0, T, \gamma_1, \gamma_2) > 0$, such that

\[
(A(u_{i+1/2}^l) - P_N A(U_{N+1}^l, U_{N}^l), D\left(m\left(U_{N+1}^l\right)D\bar{e}^{j+1/2}\right)) \\
\leq \frac{m_0}{4} \left\| D^2 \bar{e}^{j+1/2} \right\| \leq C_2 \left\{ \left\| e^{j+1/2} \right\| + \left\| e^{j} \right\| + N^{-8} \right\} \\
+ h^3 \left( \int_{t_j}^{t_{j+1}} \left\| u_{tt} \right\|^2 dt + \left\| u_{i+1/2}^l \right\|^2 + \int_{t_j}^{t_{j+1}} \left\| u_t \right\|^2 dt \right).
\]

Proof. Firstly, we consider

\[
A(u_{i+1/2}^l) - \tilde{A}(u^{i+1}, u^l) = \gamma_2 \left( u^{i+1/2} \right)^3 - \frac{\gamma_2}{4} \left[ \left( u^{i+1} \right)^3 + \left( u^{i+1} \right)^2 u + u^{i+1} \left( u^l \right)^2 + \left( u^l \right)^3 \right] \\
+ \gamma_3 \left( u^{i+1/2} \right)^2 - \frac{\gamma_3}{3} \left[ \left( u^{i+1} \right)^2 + u^{i+1} u + \left( u^l \right)^2 \right] \\
- \left( u^{i+1/2} - \frac{1}{2} \left( u^{i+1} + u^l \right) \right) \triangleq \gamma_2 \rho^j_2 + \gamma_3 \rho^j_3 - \rho^j_3.
\]
Direct computation gives

\[
\|\rho_t^j\| = \frac{1}{2} \left\| \int_{t_j}^{t_j+1/2} (t-t_j) u_{tt} \, dt + \int_{t_j+1/2}^{t_{j+1}} (t_{j+1} - t) u_{tt} \, dt \right\| \leq \left( \frac{\Delta t^2}{96} \int_{t_j}^{t_{j+1}} \| u_{tt} \|^2 \, dt \right)^{1/2},
\]

\[
\|\rho_2^j\| = \left\| \frac{1}{6} \left[ \left( 2u^{j+1/2} \right)^2 - \left( u^{j+1} + u^j \right) \right] + \frac{1}{6} \left[ \left( 2u^{j+1/2} \right)^2 - \left( u^{j+1} - u^j \right)^2 \right] \right\|
\]

\[
\leq \frac{1}{6} \left\| \left( 2u^{j+1/2} - \left( u^{j+1} + u^j \right) \right) \left( 2u^{j+1/2} + \left( u^{j+1} + u^j \right) \right) \right\|
\]

\[
+ \frac{1}{6} \left\| \left( u^{j+1/2} - u^{j+1} \right) \left( u^{j+1/2} + u^j \right) \right\| \cdot \left\| 2u^{j+1/2} + \left( u^{j+1} + u^j \right) \right\|_{\infty}
\]

\[
\leq \frac{1}{6} \left\| \left( -\Delta t \frac{u_{j+1/2}^{j+1/2} - \int_{t_j}^{t_{j+1/2}} (t_{j+1} - t) u_{tt} \, dt \right) \right\| \left( u^{j+1/2} + u^j \right)
\]

\[
+ \left( \Delta t \frac{u_{j+1/2}^{j+1/2} - \int_{t_j}^{t_{j+1/2}} (t-t_j) u_{tt} \, dt \right) \right\| \left( u^{j+1/2} + u^j \right)
\]

\[
\leq C \Delta t^{3/2} \left\{ \left( \int_{t_j}^{t_{j+1/2}} \| u_{tt} \|^2 \, dt \right)^{1/2} + \frac{\Delta t^3}{12} \left\| u_{tt}^{j+1/2} \right\|_{\infty} \cdot \left( \int_{t_j}^{t_{j+1/2}} \| u_{tt} \|^2 \, dt \right)^{1/2},
\]

\[
\|\rho_3^j\| = \frac{1}{12} \left\| \left( \left( 2u^{j+1/2} \right)^3 - \left( u^{j+1} + u^j \right)^3 \right) + \frac{1}{6} \left[ \left( 2u^{j+1/2} \right)^3 - \left( u^{j+1} - u^j \right)^3 \right] \right\|
\]

\[
\leq \frac{1}{12} \left\| \left( 2u^{j+1/2} - \left( u^{j+1} + u^j \right) \right) \cdot 4 \left( u^{j+1/2} \right)^2 + 2 \left( u^{j+1} + u^j \right) u^{j+1/2} + \left( u^{j+1} + u^j \right)^2 \right\|
\]

\[
+ \frac{1}{6} \left\| \left( u^{j+1/2} - u^{j+1} \right) \left[ \left( u^{j+1/2} \right)^2 + u^{j+1} u^{j+1/2} + \left( u^{j+1} \right)^2 \right] \right\|
\]

\[
\leq C \Delta t^{3/2} \left\{ \left( \int_{t_j}^{t_{j+1/2}} \| u_{tt} \|^2 \, dt \right)^{1/2} + \left\| u_{tt}^{j+1/2} \right\|_{\infty} \cdot \left( \int_{t_j}^{t_{j+1/2}} \| u_{tt} \|^2 \, dt \right)^{1/2},
\}
\]

\[
(3.46)
\]

Then

\[
\left( A(u^{j+1/2}) - P_N \tilde{A}(U_N^{j+1}, U_N^{j}), D \left( m \left( \bar{U}_N^{j+1/2} \right) D \tilde{e}^{j+1/2} \right) \right)
\]

\[
\leq \frac{3}{4} \left\| A(u^{j+1/2}) - P_N A(u^{j+1/2}) \right\|^2 + \frac{\varepsilon}{3} \left\| D \left( m \left( \bar{U}_N^{j+1/2} \right) D \tilde{e}^{j+1/2} \right) \right\|^2
\]
Direct computation yields

\[
\|D\left(m\left(\bar{U}_N^{j+1/2}\right)\bar{D}^{j+1/2}\right)\| \leq \left(M_0^2 + M_1^2\right)\|D^2\bar{e}^{j+1/2}\|^2 + C\|\bar{e}^{j+1/2}\|^2,
\]

\[
\|A\left(u^{j+1/2}\right) - P_N A\left(u^{j+1/2}\right)\|^2 \leq C N^{-8},
\]

\[
\|A\left(u^{j+1/2}\right) - \bar{A}\left(u^{j+1}, u^j\right)\|^2 \leq C\left(\|\rho_1\|_2^2 + \|\rho_2\|_2^2 + \|\rho_3\|_2^2\right),
\]

\[
\|\bar{A}\left(u^{j+1}, u^j\right) - \bar{A}\left(U_{N+1}^{j+1}, U_N^j\right)\|^2 \leq \left(\|G_1\|_\infty^2 + \|G_2\|_\infty^2\right)\left(\|e^{j+1}\|_2^2 + \|e^j\|_2^2 + C N^{-8}\right),
\]

where

\[
G_1^j = \frac{\gamma_2}{8}\left(\left\|u^{j+1} + U_N^j\right\|^2 + \left\|U_N^{j+1} + U_N^j\right\|^2 + \left\|u^{j+1} + U_N^j\right\|^2\right) + \frac{\gamma_3}{3}\left(U_N^{j+1} + u^{j+1} + U_N^j\right) - \frac{1}{2},
\]

\[
G_2^j = \frac{\gamma_2}{8}\left(\left\|u^j + U_N^j\right\|^2 + \left\|u^j + u^{j+1}\right\|^2 + \left\|u^{j+1} + U_N^j\right\|^2\right) + \frac{\gamma_3}{3}\left(u^{j+1} + u^j + U_N^j\right) - \frac{1}{2}.
\]

Applying Lemma 2.3 and Theorem 3.7, we obtain that \(\|G_1\|_\infty \leq C(u_0, m, T, \gamma_1, \gamma_2)\) and \(\|G_2\|_\infty \leq C(u_0, m, T, \gamma_1, \gamma_2)\). Taking \(\varepsilon = m_0 / 4(M_0^2 + M_1^2)\) in (3.47), we have

\[
\left(\left\|A\left(u^{j+1/2}\right) - P_N \bar{A}\left(U_{N+1}^{j+1}, U_N^j\right), D\left(m^{j+1/2}D\bar{e}^{j+1/2}\right)\right\|\right)
\]

\[
\leq \frac{m_0}{4}\|D^2\bar{e}^{j+1/2}\|^2 + C_2\left\{\|e^{j+1}\|_2^2 + \|e^j\|_2^2 + N^{-8}
\]

\[
+ \Delta t^3\left(\int_{t_{j+1}}^{t_j} \|u_t\|^2 dt + \|u_t\|^2_\infty \int_{t_j}^{t_{j+1}} \|u_t\|^2 dt\right)\right\},
\]

where \(C_2 = C_2(u_0, m, T, \gamma_1, \gamma_2) > 0\) is a constant.
Lemma 3.6. Assume that $u$ is the solution of (1.2)–(1.4). Then there exists a positive constant $C = C(u_0, m, \gamma, \gamma_1) > 0$ such that

$$\left( D^2 u^{i+1/2} - A(u^{i+1/2}), D \left( m \left( \bar{U}_N^{i+1/2} \right) - m(u^{i+1/2}) \right) D\bar{e}^{i+1/2} \right) \leq \frac{m_0}{4} \left\| D^2 \bar{e}^{i+1/2} \right\|^2 + C \left( \left\| \bar{e}^{i+1/2} \right\|^2 + N^{-8} + h^3 \int_{t_j}^{t_{i+1}} \| u_t \| \, dt \right).$$

(3.51)

Proof. By (2.9), we have

$$\left\| D^3 u^{i+1/2} - DA (u^{i+1/2}) \right\|_\infty \leq \left\| D^3 u^{i+1/2} \right\|_\infty + \left\| A' \left( u^{i+1/2} \right) D u^{i+1/2} \right\|_\infty \leq \left\| D^3 u^{i+1/2} \right\|_\infty + \left\| A' \left( u^{i+1/2} \right) \right\|_\infty \left\| D u^{i+1/2} \right\|_\infty \leq C. \tag{3.52}$$

In the other hand,

$$\left\| m \left( \bar{U}_N^{i+1/2} \right) - m(u^{i+1/2}) \right\|^2 \leq M^2 \left\| \left( \bar{U}_N^{i+1/2} - u^{i+1/2} \right) \right\|^2 \leq C \left\| \bar{U}_N^{i+1/2} - \frac{u^i + u^{i+1}}{2} \right\|^2 + \left\| \frac{u^i + u^{i+1}}{2} - u^{i+1/2} \right\|^2 \leq C \left\| \bar{e}^{i+1/2} \right\|^2 + \left\| \eta^{i+1/2} \right\|^2 + \frac{h^3}{96} \int_{t_j}^{t_{i+1}} \| u_t \| \, dt. \tag{3.53}$$

By Young inequality, we obtain

$$\left( D^2 u^{i+1/2} - A(u^{i+1/2}), D \left( m \left( \bar{U}_N^{i+1/2} \right) - m(u^{i+1/2}) \right) D\bar{e}^{i+1/2} \right) \leq \left\| D^2 u^{i+1/2} - DA (u^{i+1/2}) \right\|_\infty \left\| m \left( \bar{U}_N^{i+1/2} \right) - m(u^{i+1/2}) \right\| \left\| D\bar{e}^{i+1/2} \right\| \leq \varepsilon \left\| D\bar{e}^{i+1/2} \right\|^2 + C \varepsilon \left\| m \left( \bar{U}_N^{i+1/2} \right) - m(u^{i+1/2}) \right\|^2 \leq \varepsilon \left\| D^2 \bar{e}^{i+1/2} \right\|^2 + \left\| \bar{e}^{i+1/2} \right\|^2 + \left\| \eta^{i+1/2} \right\|^2 \leq \varepsilon \left\| D^2 \bar{e}^{i+1/2} \right\|^2 + \left\| \bar{e}^{i+1/2} \right\|^2 + \left\| \eta^{i+1/2} \right\|^2 \leq \varepsilon \left\| D^2 \bar{e}^{i+1/2} \right\|^2 + \left\| \bar{e}^{i+1/2} \right\|^2 + \left\| \eta^{i+1/2} \right\|^2 + N^{-8} + h^3 \int_{t_j}^{t_{i+1}} \| u_t \| \, dt. \tag{3.54}$$

Choosing $\varepsilon = m_0/4$ in the previous inequality leads to (3.51).

Finally, we obtain the main theorem of this paper.
Theorem 3.7. Assume that \( u(x,t) \) is the solution of (1.2)–(1.4) and satisfies that

\[
\begin{align*}
D^4u &\in L^\infty(Q_T), \quad u_t \in L^\infty(Q_T), \\
D^2u_{tt} &\in L^2(Q_T), \quad u_{tt} \in L^2(Q_T).
\end{align*}
\]

\( \mathbf{U}_N^j \in S_N \) \( (j = 1, 2, \ldots, \Lambda) \) is the solution of the full-discretization (3.1) and (3.2). If \( h \) is sufficiently small, there exists a positive constant \( C \) such that, for any \( j = 0, 1, 2, \ldots, \Lambda, \)

\[
\begin{align*}
\left\| e^{j+1} \right\| &= \left\| P_N u(t_{j+1}) - \mathbf{U}_N^{j+1} \right\| \\
&\leq C \left( N^{-2} + \left\| e^j \right\| + B h^2 \right), \quad (3.56)
\end{align*}
\]

where \( B = \int_{t_j}^{t_{j+1}} \left( \left\| D^2u_{tt} \right\|^2 + \left\| u_t \right\|^2 + \left\| u_{tt} \right\|^2 + \max_{0 \leq s \leq j} \left\{ \left\| u_t^{j+1/2} \right\|_{\infty} \cdot \left\| u_t \right\|^2 \right\} \right) dt. \)

Proof. By (3.27), (3.36), (3.44), and (3.51), we have

\[
\begin{align*}
\left\| e^{j+1} \right\|^2 &\leq \left\| e^j \right\|^2 + \frac{h^4}{320} \int_{t_j}^{t_{j+1}} \left\| u_{tt} \right\|^2 dt + h \left\| e^{j+1/2} \right\|^2 \\
&\quad + 2h \left\{ -D^2u^{j+1/2} + D^2\mathbf{U}_N^{j+1} + D^2\mathbf{U}_N^j, D \left( m(\mathbf{U}_N^{j+1/2})D\bar{e}^{j+1/2} \right) \right\} \\
&\quad + \left( A(u^{j+1/2}) - \bar{A}(\mathbf{U}_N^{j+1}, \mathbf{U}_N^j), D \left( m(\mathbf{U}_N^{j+1/2})D\bar{e}^{j+1/2} \right) \right) \\
&\quad + \left( D^2u^{j+1/2} - A(u^{j+1/2}), D \left( m(\mathbf{U}_N^{j+1/2}) - m(u^{j+1/2}) \right)D\bar{e}^{j+1/2} \right) \} \quad (3.57)
\end{align*}
\]

\[
\begin{align*}
\leq \left\| e^j \right\|^2 + h C_1 \left( N^{-4} + \left\| e^{j+1} \right\|^2 + \left\| e^j \right\|^2 \right) \\
&\quad + C_2 h^4 \int_{t_j}^{t_{j+1}} \left( \left\| D^2u_{tt} \right\|^2 + \left\| u_t \right\|^2 + \left\| u_{tt} \right\|^2 + \left\| u_t^{j+1/2} \right\|^2 \left\| u_t \right\|^2 \right) dt,
\end{align*}
\]

where \( C_1 = C_1(m, u_0, T, \gamma_1, \gamma_2) > 0 \) and \( C_2 = C_2(m, u_0, T, \gamma_1, \gamma_2) > 0 \) are constants. For sufficiently small \( h \) such that \( C_1 h \leq 1/2, \) denoting \( \bar{C} = 2(C_1 + C_2), \) we obtain

\[
\begin{align*}
\left\| e^{j+1} \right\|^2 &\leq \left( 1 + \bar{C} h \right) \left\| e^j \right\|^2 + \bar{C} \left( h N^{-4} + h^4 B^j \right), \quad (3.58)
\end{align*}
\]

where

\[
B^j = \int_{t_j}^{t_{j+1}} \left( \left\| D^2u_{tt} \right\|^2 + \left\| u_t \right\|^2 + \left\| u_{tt} \right\|^2 + \left\| u_t^{j+1/2} \right\|^2 \left\| u_t \right\|^2 \right) dt. \quad (3.59)
\]
By the Gronwall inequality of the discrete form, we obtain

\[ \|e^{i+1}\|^2 \leq (1 + \tilde{C}h)^{i+1}\|e^0\|^2 + \tilde{C} \sum_{j=0}^{i} (1 + \tilde{C}h)^i \left( hN^{-4} + h^4B^j \right) \]

\[ \leq \exp\{\tilde{C}(j+1)h\} \left\{ \|e^0\|^2 + \tilde{C} \sum_{j=0}^{i} (hN^{-4} + h^4B^j) \right\} \]

\[ \leq \exp\{\tilde{C}(j+1)h\} \left\{ \|e^0\|^2 + \tilde{C} \left( jhN^{-4} + h^4 \sum_{l=0}^{j} B^l \right) \right\}. \] (3.60)

Direct computation gives

\[ \sum_{l=0}^{j} B^l \leq \int_{0}^{i+1} \left( \|D^2u_t\|^2 + \|u_t\|^2 + \|u_{tt}\|^2 + \max_{0\leq t\leq 1} \left\{ \|u_t^{l+1/2}\|_\infty \right\} \cdot \|u_t\|^2 \right) dt. \] (3.61)

Then we complete the conclusion (3.56).

Furthermore, we get the following theorem.

**Theorem 3.8.** Assume \( u \) is the solution of (1.2)--(1.4) and satisfies that

\[ D^4u \in L^\infty(Q_T), \quad u_t \in L^\infty(Q_T), \]

\[ D^2u_{tt} \in L^2(Q_T), \quad u_{ttt} \in L^2(Q_T). \] (3.62)

\( U_N^j \in S_N \ (j = 1, 2, \ldots, \Lambda) \) is the solution of the full-discretization scheme (3.1)-(3.2), and \( U^0 \) satisfies \( \|e^0\| = \|P_Nu^0 - U^0\| \leq CN^{-3}. \) If \( h \) is sufficiently small, there exists a constant \( C = C(u_0, m, T, \gamma_1, \gamma_2) > 0 \) such that

\[ \left\| u(x,t_j) - U_N^j \right\| \leq C \left( N^{-2} + h^2 \right), \quad j = 1, 2, \ldots, \Lambda. \] (3.63)

**4. Numerical Experiments**

In this section, we apply the spectral method described in (3.1) and (3.2) to carry out numerical computations to illustrate theoretical estimations in the previous section. Consider (2.2) with settings:

\[ m(s) = m_0 + s^2, \quad A(s) = s^3 - s, \] (4.1)
where \( m_0 > 0 \) is a constant. The full-discretization spectral method of (2.2) and (2.3) reads: find \( U^i_N = \sum_{j=0}^{N} \alpha_j \cos l \pi x \ (j = 1, 2, \ldots, \Lambda) \) such that

\[
\begin{align*}
\left( U^i_N - U^0_N, v \right) + \left( \frac{D^2 U^i_{N+1} + D^2 U^i_N}{2} - P_N \tilde{A} \left( U^i_{N+1}, U^i_N \right), v \right) ,
\end{align*}
\]

\[
D \left( m \left( \frac{D^2 U^i_{N+1} + D^2 U^i_N}{2} \right) Dv \right) = 0,
\]

\[
\left( U^0_N, v \right) - (u_0, v) = 0.
\]

In our computations we fix \( N = 32 \) and choose five different time-step sizes \( h_k \ (k = 1, 2, \ldots, 5) \). Let \( \Lambda_k \) be the integer with \( h_k \Lambda_k = T \). Since we have no exact solution of (2.2) and (2.3), we take \( N = 32 \) and \( h_0 = 0.15625 \times 10^{-4} \) to compute an approximating solution \( U^{\Lambda_0}_N \) with \( h_0 \Lambda_0 = T \) and regard this as an exact solution. We also choose five different time-step sizes \( h_k \ (k = 1, 2, \ldots, 5) \) with \( h_k \Lambda_k = T \) to obtain five approximating solutions \( U^{\Lambda_k}_N \ (k = 1, 2, \ldots, 5) \) and compute the error estimation. Define an error function:

\[
err(T, h_k) = \left( \int_0^1 \left( U^{\Lambda_0}_N - U^{\Lambda_k}_N \right)^2 dx \right)^{1/2}.
\]

This function characterizes the estimations with respect to time-step size.

### 4.1. Example 1

Take \( m_0 = 1 \) and \( T = 0.1 \). We also take two different initial functions \( u^0_1(x) = x^5(1 - x)^5 \) and \( u^0_2 = x^5(1 - x)^5 e^x \) to carry out numerical computations. Figure 1 shows the development of the solutions for time \( t \) from \( t = 0 \) to \( t = 0.1 \) with fixed step-size \( h_0 = 0.15625 \times 10^{-4} \).
Figure 2: The development of the solution of the full-discrete scheme with initial value \(u_0(x) = x^5(1 - x)^5\) (a) and \(u_0(x) = x^5(1 - x)^5 e^x\) (b).

Figure 3: The development of the solution of the full-discrete scheme with initial value \(u_0(x) = x^5(1 - x)^5\) (a) and \(u_0(x) = x^5(1 - x)^5 e^x\) (b).

Table 1: The error and the convergence order.

| \(h_k\)      | \(\text{err}_1(0.1, h_k)\) | Order_1 | \(\text{err}_2(0.1, h_k)\) | Order_2 |
|--------------|-----------------------------|---------|-----------------------------|---------|
| \(0.1 \times 10^{-2}\) | \(2.441 \times 10^{-6}\) | —       | \(2.114 \times 10^{-6}\) | —       |
| \(0.5 \times 10^{-3}\) | \(6.332 \times 10^{-7}\)  | 1.9466  | \(5.414 \times 10^{-7}\)  | 1.9651  |
| \(0.25 \times 10^{-3}\) | \(1.186 \times 10^{-7}\)  | 2.4159  | \(1.115 \times 10^{-7}\)  | 2.2799  |
| \(0.125 \times 10^{-3}\) | \(1.938 \times 10^{-8}\)  | 2.6142  | \(1.901 \times 10^{-8}\)  | 2.5515  |
| \(0.625 \times 10^{-4}\) | \(2.504 \times 10^{-9}\)  | 2.9518  | \(2.483 \times 10^{-9}\)  | 2.9368  |

Table 2: The error and the convergence order.

| \(h_k\)      | \(\text{err}_1(0.1, h_k)\) | Order_1 | \(\text{err}_2(0.1, h_k)\) | Order_2 |
|--------------|-----------------------------|---------|-----------------------------|---------|
| \(0.1 \times 10^{-2}\) | \(4.748 \times 10^{-8}\) | —       | \(8.864 \times 10^{-8}\) | —       |
| \(0.5 \times 10^{-3}\) | \(8.348 \times 10^{-9}\)  | 2.5077  | \(1.577 \times 10^{-8}\)  | 2.4907  |
| \(0.25 \times 10^{-3}\) | \(1.370 \times 10^{-9}\)  | 2.6069  | \(2.440 \times 10^{-9}\)  | 2.6922  |
| \(0.125 \times 10^{-3}\) | \(2.694 \times 10^{-10}\) | 2.3467  | \(4.457 \times 10^{-10}\) | 2.4525  |
| \(0.625 \times 10^{-4}\) | \(6.398 \times 10^{-11}\) | 2.0741  | \(1.059 \times 10^{-10}\) | 2.0736  |
We also choose five different time-step sizes $h_k$ to carry out numerical computations and apply the error function in (4.3) to illustrate the estimation and convergence order in time variable $t$, see Table 1.

### 4.2. Example 2

Take $m_0 = 0.05$ and $T = 0.1$. We also take two different initial functions $u_0^1(x) = x^5(1 - x)^5$ and $u_0^2 = x^5(1 - x)^5 e^x$ to carry out numerical computations. Figure 2 shows the development of the solutions for time $t$ from $t = 0$ to $t = 0.1$ with fixed step-sizes $h_0 = 0.15625 \times 10^{-4}$.

We also choose five different time-step sizes $h_k$ to carry out numerical computations and apply the error function in (4.3) to illustrate the estimation and convergence order in time variable $t$, see Table 2.

### 4.3. Example 3

Take $m_0 = 0.005$ and $T = 0.1$. We also take two different initial functions $u_0^1(x) = x^5(1 - x)^5$ and $u_0^2 = x^5(1 - x)^5 e^x$ to carry out numerical computations. Figure 3 shows the development of the solutions for time $t$ from $t = 0$ to $t = 0.1$ with fixed step-sizes $h_0 = 0.15625 \times 10^{-4}$.

We also choose five different time-step sizes $h_k$ to carry out numerical computations and apply the error function in (4.3) to illustrate the estimation and convergence order in time variable $t$, see Table 3.

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### References

[1] D. S. Cohen and J. D. Murray, “A generalized diffusion model for growth and dispersal in a population,” *Journal of Mathematical Biology*, vol. 12, no. 2, pp. 237–249, 1981.

[2] M. Hazewinkel, J. F. Kaashoek, and B. Leynse, “Pattern formation for a one-dimensional evolution equation based on Thom’s river basin model,” in *Disequilibrium and Self-Organisation (Klosterneuburg, 1983/Windsor, 1985)*, vol. 30 of *Mathematics and Its Applications*, pp. 23–46, Reidel, Dordrecht, The Netherlands, 1986.

[3] A. B. Tayler, *Mathematical Models in Applied Mechanics*, Oxford Applied Mathematics and Computing Science Series, The Clarendon Press Oxford University Press, New York, NY, USA, 1986.

[4] C. M. Elliott and S. Zheng, “On the Cahn-Hilliard equation,” *Archive for Rational Mechanics and Analysis*, vol. 96, no. 4, pp. 339–357, 1986.
Journal of Applied Mathematics

[5] X. Chen, “Global asymptotic limit of solutions of the Cahn-Hilliard equation,” *Journal of Differential Geometry*, vol. 44, no. 2, pp. 262–311, 1996.

[6] A. Novick-Cohen, “On Cahn-Hilliard type equations,” *Nonlinear Analysis A*, vol. 15, no. 9, pp. 797–814, 1990.

[7] S. Zheng, “Asymptotic behavior of solution to the Cahn-Hillard equation,” *Applicable Analysis*, vol. 23, no. 3, pp. 165–184, 1986.

[8] A. Voigt and K.-H. Hoffman, “Asymptotic behavior of solutions to the Cahn-Hilliard equation in spherically symmetric domains,” *Applicable Analysis*, vol. 81, no. 4, pp. 893–903, 2002.

[9] X. Chen and M. Kowalczyk, “Existence of equilibria for the Cahn-Hilliard equation via local minimizers of the perimeter,” *Communications in Partial Differential Equations*, vol. 21, no. 7-8, pp. 1207–1233, 1996.

[10] B. E. E. Stoth, “Convergence of the Cahn-Hilliard equation to the Mullins-Sekerka problem in spherical symmetry,” *Journal of Differential Equations*, vol. 125, no. 1, pp. 154–183, 1996.

[11] J. Brézis, A. Kupiainen, and J. Taskinen, “Stability of Cahn-Hilliard fronts,” *Communications on Pure and Applied Mathematics*, vol. 52, no. 7, pp. 839–871, 1999.

[12] E. A. Carlen, M. C. Carvalho, and E. Orlandi, “A simple proof of stability of fronts for the Cahn-Hilliard equation,” *Communications in Mathematical Physics*, vol. 224, no. 1, pp. 323–340, 2001.

[13] A. Miranville and S. Zelik, “Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions,” *Mathematical Methods in the Applied Sciences*, vol. 28, no. 6, pp. 709–735, 2005.

[14] J. Prüss, R. Racke, and S. Zheng, “Maximal regularity and asymptotic behavior of solutions for the Cahn-Hilliard equation with dynamic boundary conditions,” *Annali di Matematica Pura ed Applicata IV*, vol. 185, no. 4, pp. 627–648, 2006.

[15] R. Racke and S. Zheng, “The Cahn-Hilliard equation with dynamic boundary conditions,” *Advances in Differential Equations*, vol. 8, no. 1, pp. 83–110, 2003.

[16] H. Wu and S. Zheng, “Convergence to equilibrium for the Cahn-Hilliard equation with dynamic boundary conditions,” *Journal of Differential Equations*, vol. 204, no. 2, pp. 511–531, 2004.

[17] J. W. Barrett and J. F. Blowey, “An error bound for the finite element approximation of a model for phase separation of a multi-component alloy,” *IMA Journal of Numerical Analysis*, vol. 16, no. 2, pp. 257–287, 1996.

[18] J. W. Barrett and J. F. Blowey, “Finite element approximation of a model for phase separation of a multi-component alloy with non-smooth free energy,” *Numerische Mathematik*, vol. 77, no. 1, pp. 1–34, 1997.

[19] J. W. Barrett and J. F. Blowey, “Finite element approximation of a degenerate Allen-Cahn/Cahn-Hilliard system,” *SIAM Journal on Numerical Analysis*, vol. 39, no. 5, pp. 1598–1624, 2001/02.

[20] J. W. Barrett, J. F. Blowey, and H. Garcke, “Finite element approximation of the Cahn-Hilliard equation with degenerate mobility,” *SIAM Journal on Numerical Analysis*, vol. 37, no. 1, pp. 286–318, 1999.

[21] P. K. Chan and A. D. Rey, “A numerical method for the nonlinear Cahn-Hilliard equation with nonperiodic boundary conditions,” *Computational Materials Science*, vol. 3, no. 3, pp. 377–392, 1995.

[22] C. M. Elliott, D. A. French, and F. A. Milner, “A second order splitting method for the Cahn-Hilliard equation,” *Numerische Mathematik*, vol. 54, no. 5, pp. 575–590, 1989.

[23] C. M. Elliott and D. A. French, “A nonconforming finite-element method for the two-dimensional Cahn-Hilliard equation,” *SIAM Journal on Numerical Analysis*, vol. 26, no. 4, pp. 884–903, 1989.

[24] C. M. Elliott and S. Larsson, “Error estimates with smooth and nonsmooth data for a finite element method for the Cahn-Hilliard equation,” *Mathematics of Computation*, vol. 58, no. 198, p. 603–630, S33–S36, 1992.

[25] E. V. L. de Mello and O. T. da Silveira Filho, “Numerical study of the Cahn-Hilliard equation in one, two and three dimensions,” *Physica A*, vol. 347, no. 1–4, pp. 429–443, 2005.

[26] D. Furihata, “A stable and conservative finite difference scheme for the Cahn-Hilliard equation,” *Numerische Mathematik*, vol. 87, no. 4, pp. 675–699, 2001.

[27] J. Kim, “A numerical method for the Cahn-Hilliard equation with a variable mobility,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 12, no. 8, pp. 1560–1571, 2007.

[28] J. Kim, K. Kang, and J. Lowengrub, “Conservative multigrid methods for Cahn-Hilliard fluids,” *Journal of Computational Physics*, vol. 193, no. 2, pp. 511–543, 2004.

[29] Z. Z. Sun, “A second-order accurate linearized difference scheme for the two-dimensional Cahn-Hilliard equation,” *Mathematics of Computation*, vol. 64, no. 212, pp. 1463–1471, 1995.

[30] F. L. Bai, L. Yin, and Y. K. Zou, “A pseudo-spectral method for the Cahn-Hilliard equation,” *Journal of Jilin University*, vol. 41, no. 3, pp. 262–268, 2003.
[31] Y. He and Y. Liu, “Stability and convergence of the spectral Galerkin method for the Cahn-Hilliard equation,” *Numerical Methods for Partial Differential Equations*, vol. 24, no. 6, pp. 1485–1500, 2008.

[32] P. Howard, “Spectral analysis of planar transition fronts for the Cahn-Hilliard equation,” *Journal of Differential Equations*, vol. 245, no. 3, pp. 594–615, 2008.

[33] X. Ye, “The Fourier collocation method for the Cahn-Hilliard equation,” *Computers & Mathematics with Applications*, vol. 44, no. 1-2, pp. 213–229, 2002.

[34] X. Ye, “The Legendre collocation method for the Cahn-Hilliard equation,” *Journal of Computational and Applied Mathematics*, vol. 150, no. 1, pp. 87–108, 2003.

[35] X. Ye and X. Cheng, “The Fourier spectral method for the Cahn-Hilliard equation,” *Applied Mathematics and Computation*, vol. 171, no. 1, pp. 345–357, 2005.

[36] B. Nicolaenko, B. Scheurer, and R. Temam, “Some global dynamical properties of a class of pattern formation equations,” *Communications in Partial Differential Equations*, vol. 14, no. 2, pp. 245–297, 1989.

[37] Q. Du and R. A. Nicolaides, “Numerical analysis of a continuum model of phase transition,” *SIAM Journal on Numerical Analysis*, vol. 28, no. 5, pp. 1310–1322, 1991.

[38] D. Furihata, “Finite difference schemes for $\frac{\partial u}{\partial t} = (\frac{\partial}{\partial x})^3 \frac{\delta G}{\delta u}$ that inherit energy conservation or dissipation property,” *Journal of Computational Physics*, vol. 156, no. 1, pp. 181–205, 1999.

[39] C. Canuto and A. Quarteroni, “Error estimates for spectral and pseudospectral approximations of hyperbolic equations,” *SIAM Journal on Numerical Analysis*, vol. 19, no. 3, pp. 629–642, 1982.

[40] J. X. Yin, “On the existence of nonnegative continuous solutions of the Cahn-Hilliard equation,” *Journal of Differential Equations*, vol. 97, no. 2, pp. 310–327, 1992.

[41] J. X. Yin, “On the Cahn-Hilliard equation with nonlinear principal part,” *Journal of Partial Differential Equations*, vol. 7, no. 1, pp. 77–96, 1994.

[42] C. C. Liu, Y. W. Qi, and J. X. Yin, “Regularity of solutions of the Cahn-Hilliard equation with nonconstant mobility,” *Acta Mathematica Sinica*, vol. 22, no. 4, pp. 1139–1150, 2006.

[43] J. Yin and C. Liu, “Regularity of solutions of the Cahn-Hilliard equation with concentration dependent mobility,” *Nonlinear Analysis A*, vol. 45, no. 5, pp. 543–554, 2001.

[44] D. Kay and R. Welford, “A multigrid finite element solver for the Cahn-Hilliard equation,” *Journal of Computational Physics*, vol. 212, no. 1, pp. 288–304, 2006.