ASYMPTOTIC STABILITY FOR KÄHLER-RICCI SOLITONS

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Abstract. Let $X$ be a Fano manifold. We say that a hermitian metric $\phi$ on $-K_X$ with positive curvature $\omega_\phi$ is a Kähler-Ricci soliton if it satisfies the equation $\text{Ric}(\omega_\phi) - \omega_\phi = L_{V_{KS}} \omega_\phi$ for some holomorphic vector field $V_{KS}$. The candidate for a vector field $V_{KS}$ is uniquely determined by the holomorphic structure of $X$ up to conjugacy, hence depends only on the holomorphic structure of $X$. We introduce a sequence $\{V_k\}$ of holomorphic vector fields which approximates $V_{KS}$ and fits to the quantized settings. Moreover, we also discuss about the existence of the quantized Kähler-Ricci solitons attached to the sequence $\{V_k\}$.

1. Introduction

Let $X$ be an $n$-dimensional Fano manifold. Since the identity component $\text{Aut}_0(X)$ of the automorphism group is a linear algebraic group [Fuj78], we obtain a semidirect decomposition

$$\text{Aut}_0(X) = \text{Aut}_{r}(X) \rtimes R_u,$$

where $\text{Aut}_{r}(X) \subset \text{Aut}_0(X)$ is a reductive algebraic subgroup, which is the complexification of a maximal compact subgroup $K$, and $R_u$ is the unipotent radical. We often identify a holomorphic vector field $V$ such that $\text{Im}(V) \in k:= \text{Lie}(K)$ with its imaginary part $\xi_V := \text{Im}(V) \in k$. Let $PSH(X, -K_X)$ be the set of smooth hermitian metrics on the anti-canonical bundle $-K_X$ with positive curvature, where we regard $\phi$ as a psh weight, i.e., a psh function on $K_X \setminus \{0\}$-section satisfying the log-homogeneity property (for instance, see [BB10] for more detail). We say that a metric $\phi \in PSH(X, -K_X)$ is a Kähler-Ricci soliton if it satisfies the equation

$$\text{Ric}(\omega_\phi) - \omega_\phi = L_{V_{KS}} \omega_\phi$$

for some holomorphic vector field $V_{KS}$ (Kähler-Ricci soliton vector field). Let $T_C$ be the center of $K$ and put $t_C := \text{Lie}(T_C)$. Tian-Zhu showed that all Kähler-Ricci solitons are contained in the space of $K$-invariant smooth psh weights $PSH(X, -K_X)^K$ (cf. [TZ00]). They also showed that $\xi_{V_{KS}}$ is contained in $t_C$ and is uniquely determined as the minimizer of the following proper convex function on $k$:

$$\mathcal{F}(V) := \int_X e^{m_\phi(\xi_V)} MA(\phi),$$

where $\phi \in PSH(X, -K_X)^K$, $MA(\phi) := \frac{\omega_\phi^n}{c_1(X)^n}$, and $m_\phi$ is the moment map with respect to $\phi$ (cf. [TZ02]). The function $\mathcal{F}$ is a holomorphic invariant, i.e., independent

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of a choice of \( \phi \in PSH(X, -K_X)^K \) and its derivative at \( V \):

\[
Fut_V(W) := -\int_X m_\phi(\xi_W)e^{m_\phi(\xi_V)}MA(\phi)
\]

is called the modified Futaki invariant. By definition, the modified Futaki invariant with respect to \( V_{KS} \) vanishes on \( \mathcal{F} \).

In order to deal with the existence problem of canonical metrics, we study several functionals on the space of hermitian metrics and their asymptotics near the boundary. On the other hand, we also consider its analogue in the space of hermitian metrics on \( H^0(X, -K_X) \) (cf. [Don02], [Don09]). The main purpose of this theme is to construct balanced metrics and study their asymptotics as \( k \) tends to infinity. In the Kähler-Ricci soliton case, Berman-Nyström [BN14] showed that there exist a certain kind of balanced metrics called quantized Kähler-Ricci solitons and this sequence of metrics converges to a Kähler-Ricci soliton under some strong assumptions.

The purpose of this paper is to perturb the sequence of quantized Kähler-Ricci solitons and show their existence and convergence under weaker assumptions. First, we construct a sequence \( \{V_k\} \) of holomorphic vector fields which approximates \( V_{KS} \) and fits to the quantized settings. More concretely, this sequence is given as the following:

**Theorem 1.1.** Let \( X \) be a Fano manifold and \( K \) be a maximal compact subgroup of \( \text{Aut}_r(X) \). Then for sufficiently large \( k \), there exists a holomorphic vector field \( V_k \) such that its imaginary part is contained in \( \mathfrak{t}_C \) and the corresponding quantized modified Futaki invariant at level \( k \) vanishes on \( \mathfrak{t}_C \). The vector field \( V_k \) is characterized as the unique minimizer of the quantization of the function \( F|_{\mathfrak{t}_C} \) at level \( k \) and converges to \( V_{KS} \) as \( k \to \infty \) in the usual topology of the finite dimensional vector space \( \mathfrak{t}_C \).

Second, we introduce the quantized Kähler-Ricci solitons attached to this sequence and show that:

**Theorem 1.2.** Assume that \( (X, V_{KS}) \) is strongly analytically K-polystable (i.e., the corresponding modified Ding functional is coercive modulo \( \text{Aut}_0(X, V_{KS}) \)), then there exists a quantized Kähler-Ricci soliton attached to \( V_k \) if \( k \) is sufficiently large, which is unique modulo the action of \( \text{Aut}_0(X, V_{KS}) \) and as \( k \to \infty \), the corresponding Bergman metrics on \( X \) converge weakly, modulo automorphisms, to a Kähler-Ricci soliton on \( (X, V_{KS}) \).

As a corollary, we have the following:

**Corollary 1.3.** Assume that \( X \) is strongly analytically K-polystable (i.e., the Ding functional is coercive modulo \( \text{Aut}_r(X) \)), then there exists a quantized Kähler-Einstein metric attached to \( V_k \) if \( k \) is sufficiently large, which is unique modulo the action of \( \text{Aut}_r(X) \) and as \( k \to \infty \), the corresponding Bergman metrics on \( X \) converge weakly, modulo automorphisms, to a Kähler-Einstein metric on \( X \).

The crucial point is that in our results, we need not to assume that the vanishing of all the higher order (modified) Futaki invariants, which is, in the case of \( V_{KS} \equiv 0 \), an obstruction to the asymptotic Chow semi-stability (cf. [Fut04]).
The heart of this paper consists of mainly two ideas:

(1) While Berman-Nyström considered the torus $T_{KS}$ generated by the Kähler-Ricci soliton vector field $V_{KS}$ (see Lemma 2.1), we consider the center $T_C(\supset T_{KS})$ and the space of $T_C$-invariant hermitian metrics $PSH(X, -K_X)^{T_C}$. Actually, this setting seems to be natural since all of $\xi_{V_k}$ lie in its Lie algebra $t_C$ by Theorem 1.1.

(2) The condition $\text{Fut}_{V_{KS}} \equiv 0$ (resp. $\text{Fut}_{V_{k,k}} \equiv 0$) leads to the $\text{Aut}_0(X, V_{KS})$-invariance of $D_{g_{V_{KS}}}$ (resp. $D_{g_{V_{k,k}}}$). Hence the problem can be reduced to estimate the difference $|D_{g_{V_{KS}}} - D_{g_{V_{k,k}}}|$, which has linear growth along geodesics on the space of $T_C$-invariant hermitian metrics $\mathcal{H}_k^{T_C}$ on $H^0(X, -kK_X)$. On the other hand, the standard exhaustion function $J^{(k)}$ on $\mathcal{H}_k^{T_C}/\mathbb{R}$ has at least linear growth along geodesics (cf. [Don09]). Therefore $|D_{g_{V_{KS}}} - D_{g_{V_{k,k}}}|$ is bounded above by an affine function $\epsilon_k J^{(k)} + \epsilon'_k$ of $J^{(k)}$ for some positive numbers $\epsilon_k \to 0$ and $\epsilon'_k \to 0$. This leads to the coercivity of $D_{g_{V_{k,k}}}$ and therefore the existence of the quantized Kähler-Ricci soliton attached to $V_{k}$. 

Finally, we mention that our results are quite similar to those for extremal metrics proved in [Mab09].

2. Functionals on $PSH(X, -K_X)^T$

Let $X$ and $V_{KS}$ be as in Section 1.

**Lemma 2.1** ([BN14], Lemma 2.13). There is an algebraic torus $T_{KS} \subset \text{Aut}_r(X)$ which acts on $(X, -K_X)$ such that the imaginary part of $V_{KS}$ can be identified with an element $\xi_{V_{KS}} \in t_{KS} := \text{Lie}(T_{KS})$ and $PSH(X, -K_X)^{V_{KS}} = PSH(X, -K_X)^{T_{KS}}$.

Let $T \subset \text{Aut}_r(X)$ be an algebraic torus acting on $(X, -K_X)$. Now we recall several functionals on $PSH(X, -K_X)^T$ which play a central role in the study of Kähler-Ricci solitons. Let $\phi_0 \in PSH(X, -K_X)^T$ be a reference metric and $g$ be a positive continuous function on the moment polytope $P$ such that $g\nu^T$ is a probability measure on $P$, where $\nu^T$ denotes the Duistermaat-Heckman measure (cf. Proposition 3.4).

Following [BN14, Section 2.4 and 2.6], we define the $g$-Monge-Ampère energy by the formula

$$d\mathcal{E}_g|_{\phi}(\phi) = \int_X \phi g(m_\phi) MA(\phi), \quad \mathcal{E}_g(\phi_0) = 0$$

and define the $J_g$ functional by

$$J_g(\phi) = -\mathcal{E}_g(\phi) + \mathcal{L}_{\mu_0}(\phi), \quad \mathcal{L}_{\mu_0}(\phi) := \int_X (\phi - \phi_0) d\mu_0,$$

where $\mu_0 := MA(\phi_0)$. The functional $J$ defines an exhaustion function on $PSH(X, -K_X)^T/\mathbb{R}$ (cf. [BBGZ12, Lemma 3.3]).

Next set $g = g_V := C \exp(\langle \xi_V, \cdot \rangle)$ (where $C$ is a normalizing constant). We define the modified Ding functional by

$$D_{g_V}(\phi) = -\mathcal{E}_{g_V}(\phi) + \mathcal{L}(\phi), \quad \mathcal{L}(\phi) := -\log \int_X e^{-\phi}.$$
Then by [BN14, Lemma 3.4], we have
\[
\frac{d}{dt} D_g V(\exp(tW)\phi) = \text{Fut}_V(W).
\]
Moreover, critical points of $D_g V$ are Kähler-Ricci solitons with respect to $V$.

**Definition 2.2** ([BN14], Section 3.6). We say that a pair $(X, V_{KS})$ is strongly analytically K-polystable if the modified Ding functional $D_{g V_{KS}}$ is coercive modulo $\text{Aut}_0(X, V_{KS})$, i.e.,
\[
D_{g V_{KS}}(\phi) \geq \delta \inf_{F \in \text{Aut}_0(X, V_{KS})} J(F^* \phi) - C
\]
for some positive constants $\delta$ and $C$, where $\text{Aut}_0(X, V_{KS})$ be a subgroup of $\text{Aut}_0(X)$ consisting of elements which commute with the action generated by $V_{KS}$.

It is known that the coercivity of $D_{V_{KS}}$ leads to the existence of Kähler-Ricci solitons (cf. [BN14, Theorem 3.11]).

3. The properness of the functional $F_k$

Let $X$ be a Fano manifold. Let $T \subset \text{Aut}_r(X)$ be an algebraic torus of dimension $m \geq 1$ acting on $(X, -K_X)$ (where the action on $-K_X$ is defined by the canonical lift). For $\phi \in PSH(X, -K_X)^T$, we denote the moment map with respect to $\phi$ by
\[
m_\phi : X \to m_\phi(X) =: P \subset t^* \simeq \mathbb{R}^m,
\]
where we identify $t^* \simeq \mathbb{R}^m$ using a inner product on $t$. The image $P$ is a compact convex polytope of dimension $m$ and independent of a choice of $\phi \in PSH(X, -K_X)^T$. The polytope $P$ is characterized as the support of the Duistermaat-Heckman measure (see Proposition 3.4).

**Lemma 3.1.** Let $X$ and $T$ be as above. Then the polytope $P$ contains the origin in its interior $\text{int}(P)$.

**Proof.** Since the lift to $-K_X$ is canonical, we have an equation
\[
-\Delta m_\phi(\xi_V) + m_\phi(\xi_V) + V(\kappa_\phi) = 0
\]
for all $\xi_V \in t$, where $\kappa_\phi$ is the function defined by
\[
\text{Ric}(\omega_\phi) - \omega_\phi = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \kappa_\phi.
\]
Integrating by parts, we find that
\[
\int_X m_\phi(\xi_V)e^{\kappa_\phi} MA(\phi) = 0.
\]
Since $m \geq 1$, $m_\phi$ is not a constant. Thus the equation (3.1) implies that for any $\xi_V$, an inequality $m_\phi(\xi_V) > 0$ holds on some nonempty open subset of $X$. Now we assume that $0 \notin \text{int}(P)$, then we can choose an element $\xi \in \mathbb{R}^m$ so that $H_\xi \cap \text{int}(P) = \emptyset$, where $H_\xi$ a hyperplane which is orthogonal to $\xi$ and contains the origin. Then either of $m_\phi(\pm \xi_V)$ is semi-negative on $X$. This is a contradiction. \qed
We define the functions $F_k$ and $\text{Fut}_{V,k}$ as

\[
F_k(W) := k \text{Trace}(e^{W/k}) |_{H^0(X, -K_X)},
\]

\[
\text{Fut}_{V,k}(W) := - \frac{d}{dt} \bigg|_{t=0} F_k(V + tW).
\]

We set

\[N_k := \dim H^0(X, -K_X),\]

then these functions give the quantization of $F$ and $\text{Fut}_V$:

\textbf{Lemma 3.2} ([BN14], Proposition 4.7 or [Tak14], Proposition 2.8). Let $V$ be a holomorphic vector field generating a torus action and $W$ a holomorphic vector field generating a $\mathbb{C}^*$-action and commuting with $V$. Then we have identities

\[
F(V) = \lim_{k \to \infty} \frac{F_k(V)}{kN_k},
\]

\[
\text{Fut}_V(W) = \lim_{k \to \infty} \frac{1}{kN_k} \text{Fut}_{V,k}(W).
\]

If we apply the equivariant Riemann-Roch formula to $\text{Fut}_{V,k}(W)$, we have an expansion

\[
\text{Fut}_{V,k}(W) = \text{Fut}^{(0)}_V(W)k^{n+1} + \text{Fut}^{(1)}_V(W)k^n + \cdots,
\]

where $\text{Fut}^{(i)}_V(W)$ is the $i$-th order modified Futaki invariant introduced in [BN14, Section 4.4].

\textbf{Lemma 3.3}. The function $F_k|_{t_C}$ is a proper convex function if $k$ is sufficiently large.

\textit{Proof}. The convexity of $F_k|_{t_C}$ immediately follows from the convexity of exponential functions. In order to prove the properness, let $\{\xi_{W_j}\} \subset t_C \simeq \mathbb{R}^m$ be any sequence such that $|\xi_{W_j}| \to \infty$ as $j \to \infty$. We use the following proposition:

\textbf{Proposition 3.4} ([BN14], Proposition 4.1). Let $P_k := \{\lambda^{(k)}_i \subset \mathbb{Z}^m$ be the set of all weights for the action of the complexified torus $T^C$ on $H^0(X, -K_X)$, i.e., there is a decomposition

\[
H^0(X, -kK_X) = \bigoplus_{\lambda^{(k)}_i \in P_k} E_{\lambda^{(k)}_i}.
\]

Then the spectral measure:

\[
\nu_k := \frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{\lambda^{(k)}_i/k}
\]

supported on $P_k/k$ converges to the Duistermaat-Heckman measure $\nu^T := (m_\phi)_* MA(\phi)$ weakly as $k \to \infty$. In particular, $\nu^T$ does not depend on a choice of $\phi \in PSH(X, -K_X)^T$.

For $\epsilon > 0$, let $P_\epsilon$ be the interior compact convex polytope with faces parallel to those of $P$ separated by distance $\epsilon$. By Lemma 3.1, we can choose $\epsilon > 0$ so that
int($P_\epsilon$) contains the origin. Then Proposition 3.4 implies that there exists $k_0$ such that for all $k \geq k_0$ and $\xi_W \in \mathbb{R}^m$, there exists an eigenvalue $\lambda_i^{(k)}$ satisfying

$$\lambda_i^{(k)}/k \in P - P_\epsilon,$$

$$\cos(\text{angle}(\xi_W, \lambda_i^{(k)})) \geq 1 - \epsilon.$$ 

For each $\xi_{W_j}$, we choose the eigenvalue $\lambda_{j,i(j)}^{(k)}$ satisfying the above condition. Then we obtain

$$w_{j,i(j)}^{(k)} := \langle \lambda_{j,i(j)}^{(k)}, \xi_{W_j} \rangle \geq k|\xi_{W_j}| \cdot \inf_{\xi \in \partial P_\epsilon} |\xi| \cdot (1 - \epsilon) \to \infty$$

as $j \to \infty$. Hence we have

$$F_k(W_j) = k \sum_{i=1}^{N_k} \exp(w_{j,i}^{(k)}/k) \geq k \exp(w_{j,i(j)}^{(k)}/k) \to \infty$$

as $j \to \infty$. This completes the proof of Lemma 3.3.

Let $V_k$ be the unique minimizer of $F_k|_{\mathfrak{c}^k}$.

The proof of Theorem 1.1. By Lemma 3.2 and Lemma 3.3, we find that the minimizer $V_k$ converges to the Kähler-Ricci soliton vector field $V_{KS}$ as $k \to \infty$. Since $V_k \in \mathfrak{c}^k$ and $F_k$ is adjoint invariant (so is Trace in the defining equation (3.2) of $F_k$), the equation

$$(3.4) \quad \text{Fut}_{V_k,k}([V,W]) = 0$$

holds for all $V, W \in \mathfrak{c}^k$. Moreover, the formula $\mathfrak{c}^k = \mathfrak{c}^{k^c} \oplus [\mathfrak{c}^k, \mathfrak{c}^{k^c}]$ yields that $\text{Fut}_{V_k,k}$ vanishes on the entire space $\mathfrak{c}^k$. This completes the proof. □

4. The quantized Kähler-Ricci soliton attached to $\{V_k\}$

Let $X$ be a Fano manifold and $\mathcal{H}_k$ the space of hermitian metrics on $H^0(X, -K_X)$. We define two operators:

$$H\text{ilb}_k: PSH(X, -K_X)^T \to \mathcal{H}_k^{T^c},$$

$$FS_k: \mathcal{H}_k^{T^c} \to PSH(X, -K_X)^T$$

by the formula:

$$(4.1) \quad ||s||_{H\text{ilb}_k(\phi)}^2 := \int_X |s|^2 e^{-k\phi} \mu_\phi,$$

$$(4.2) \quad FS_k(H) := \frac{1}{k} \log \left( \frac{1}{N_k} \sum_{\iota=1}^{N_k} |s_\iota|^2 \right),$$

where $\mu_\phi$, a measure on $X$ defined by the density of the Hermitian metric $e^{-\phi} \in C^\infty(K_X + K_X)$ of $-K_X$ and $\{s_i\}$ is an $H$-orthonormal basis.
Definition 4.1. We say that a metric $H_k \in \mathcal{H}^{TC}_k$ is a quantized Kähler-Ricci soliton attached to $V$ if it satisfies the equation

$$\text{Hilb}_k \circ FS_k(H_k) = \exp(V)^* H_k.$$ 

Let $H_0 := \text{Hilb}_k(\phi_0) \in \mathcal{H}^{TC}_k$ ($\phi_0 \in \text{PSH}(X, -K_X)^{TC}$) be a reference metric. We define the quantization of the functional $\mathcal{E}_g$ as

$$\mathcal{E}_g^{(k)}(H) = \sum_{\lambda_i(k) \in \mathcal{P}_k} \log \det H_{|\lambda_i(k)}^{(k)}(H),$$ 

where we compute the determinant in reference to the metric $H_0$. Actually, using Proposition 3.4, we find that $\mathcal{E}_g^{(k)}(\text{Hilb}_k(\phi))$ converges to $\mathcal{E}_g(\phi)$ as $k \to \infty$ for any $\phi \in \text{PSH}(X, -K_X)^{TC}$ (cf. [BN14, Proposition 4.5]). We also have an isomorphism

$$\mathcal{H}_k \simeq GL(N_k, \mathbb{C})/U(N_k)$$ 

with respect to $H_0$, which implies that $\mathcal{H}_k$ is a Riemannian symmetric space and therefore geodesics are given as the decomposition of the exponential map and the projection $GL(N_k, \mathbb{C}) \to GL(N_k, \mathbb{C})/U(N_k)$. Let $s_i^{(k)}$ be an $H_0$-orthonormal and $H_0$-orthogonal basis. Then any geodesic can be represented by $H_t(s_i^{(k)}, s_i^{(k)}) = e^{-\mu_i^{(k)}t}H_0(s_i^{(k)}, s_i^{(k)})$ for some $\mu_i^{(k)}$. Thus we have

$$\frac{d}{dt} \mathcal{E}_g^{(k)}(H_t) = \frac{1}{kN_k} \sum_{i=1}^{N_k} g(\lambda_i^{(k)}/k, \mu_i^{(k)}).$$ 

Hence the functional $\mathcal{E}_g^{(k)}$ has linear growth along geodesics. We define the quantization of the functionals $J_g$ and $\mathcal{D}_g$ as follows:

$$J_g^{(k)}(H) := -\mathcal{E}_g^{(k)}(H) + \mathcal{L}_{\mu_0}(FS_k(H)), $$

$$\mathcal{D}_g^{(k)}(H) := -\mathcal{E}_g^{(k)}(H) + \mathcal{L}(FS_k(H)).$$

Then by [BN14, Proposition 4.7], we have

$$\mathcal{D}_g^{(k)}(\exp(tW)^* H_0) = \frac{\text{Fut}_{V,k}(W)}{kN_k}. $$

Moreover, critical points of $\mathcal{D}_g^{(k)}$ are quantized Kähler-Ricci solitons attached to $V$. 

Proof of the Theorem 1.2. This proof is mostly based on the original proof given by Berman-Nyström. The reader should refer to [BN14, Theorem 1.7].

The coercivity of $\mathcal{D}_{g^{KS}}$ implies that the equation

$$\mathcal{D}_{g^{KS}}(FS_k(H)) \geq \delta J(FS_k(F^*H)) - C$$

holds for some $F \in \text{Aut}_0(X, V^{KS})$, where we note that two operations $FS_k$ and $F^*$ are commutative. Then the LHS can be written as

$$\mathcal{D}_{g^{KS}}(FS_k(H)) = \mathcal{D}_{g^{KS}}(FS_k(F^*H)) \quad \text{(because Fut}_{V^{KS}} = 0) 
= J_{g^{KS}}(FS_k(F^*H)) + (\mathcal{L} - \mathcal{L}_{\mu_0})(FS_k(F^*H)).$$
On the other hand, since \( g_{V_{KS}} \) is bounded, we obtain
\[
\delta J(FS_k(F^*H)) - C \geq \delta' J_{g_{V_{KS}}}(FS_k(F^*H)) - C
\]
for sufficiently small \( \delta' > 0 \) depending only on \( g_{V_{KS}} \). Thus we obtain
\[
(4.3) \quad J_{g_{V_{KS}}}(FS_k(F^*H))(1 - \delta') + (\mathcal{L} - \mathcal{L}_{\mu_0})(FS_k(F^*H)) \geq -C.
\]
Now we use the following lemma, which compares the two functionals \( J_{g_{V_{KS}}} \circ FS_k \) and \( J_{g^k} \):

**Lemma 4.2** ([BN14], Lemma 4.10). There exists a sequence \( \delta_k \to 0 \) of positive numbers such that
\[
J_{g_{V_{KS}}}(FS_k(H)) \leq (1 + \delta_k)J_{g^k}(H) + \delta_k.
\]
Hence if we take \( k \) sufficiently large so that \((1 + \delta_k)(1 - \delta') \leq 1 - \frac{\delta'}{2} \) and \( \delta_k(1 - \delta') \leq C \) hold, we have
\[
(4.4) \quad J_{g_{V_{KS}}}(FS_k(F^*H))(1 - \delta') \leq J_{g^k}(F^*H) \left(1 - \frac{\delta'}{2}\right) + C.
\]
Thus we obtain
\[
D_{g_{V_{KS}}}^{(k)}(F^*H) = J_{g_{V_{KS}}}^{(k)}(F^*H) + (\mathcal{L} - \mathcal{L}_{\mu_0})(FS_k(F^*H))
\]
\[
\geq \frac{\delta'}{2} J_{g^k}(F^*H) - 2C \quad \text{(by (4.3) and (4.4))}
\]
\[
\geq \frac{\delta''}{2} J^{(k)}(F^*H) - 2C \quad \text{(because } g_{V_{KS}} \text{ is bounded)}.
\]
Now we consider the difference of the two modified Ding functionals:
\[
D_{g_{V_{KS}}}^{(k)} - D_{g^k}^{(k)} = -\mathcal{E}_{g_{V_{KS}}}^{(k)} + \mathcal{E}_{g^k}^{(k)},
\]
which has linear growth along geodesics explained above. On the other hand, the functional \( J^{(k)} \) is an exhaustion function on \( \mathcal{H}_{T_C}^k \) and has at least linear growth along geodesics. Hence we have an inequality
\[
(4.5) \quad -\epsilon_k J^{(k)} - \epsilon'_k \leq D_{g_{V_{KS}}}^{(k)} - D_{g^k}^{(k)} \leq \epsilon_k J^{(k)} + \epsilon'_k
\]
holds for some sequences of positive numbers \( \epsilon_k \to 0 \) and \( \epsilon'_k \to 0 \). Actually, put \( \epsilon_k := \text{sup}_P |g_{V_{KS}} - g_{V_k}| + 2^{-k} \) and consider a (non-trivial) geodesic \( H_t \) defined by eigenvalues \( (\mu_i^{(k)}) \). By the scaling invariance, we may assume that \( \sum_{i=1}^{N_k} ((g_{V_{KS}} - g_{V_k})(\lambda_i^{(k)}/k) - \epsilon_k)\mu_i^{(k)} = 0 \). Then \( (g_{V_{KS}} - g_{V_k})(\lambda_i^{(k)}/k) - \epsilon_k < 0 \) for each \( i \) and therefore \( \mu_{\max} := \max(\mu_i^{(k)}) \) is positive. Computing in the similar way as in [Don09, Proposition 3], we have
\[
(\epsilon_k J^{(k)} - D_{g_{V_{KS}}}^{(k)} + D_{g^k}^{(k)})(H_t) = \epsilon_k \mathcal{L}_{\mu_0}(FS_k(H_t)) \geq \epsilon_k \mu_{\max} t + (\text{const}),
\]
where we note that \( \epsilon_k \to 0 \) since \( V_k \to V_{KS} \) and \( g_{V_k} \to g_{V_{KS}} \) uniformly on \( P \). The assertion \( \epsilon'_k \to 0 \) follows from the fact that \( D_{g_{V_{KS}}}^{(k)}(H_0) \to D_{g_{V_{KS}}}^{(k)}(H_0) \) as \( k \to \infty \), which can be proved by the uniform convergence of density functions corresponding.
to Bergman measures with respect to $g_{V_{KS}}$ and $g_{V_k}$ (cf. [BN14, Proposition 4.4]). Hence we obtain
\[ D^{(k)}_{g_{V_k}}(H) = D^{(k)}_{g_{V_k}}(F^* H) \quad \text{(Because } \text{Fut}_{V_k,k} \equiv 0) \]
\[ \geq D^{(k)}_{g_{V_{KS}}} (F^* H) - \epsilon_k J^{(k)}(F^* H) - \epsilon'_k \]
\[ \geq \left( \frac{\delta''}{2} - \epsilon_k \right) J^{(k)}(F^* H) - 2C - \epsilon'_k. \]
Thus we have
\[ D^{(k)}_{g_{V_k}}(H) \geq \frac{\delta''}{3} \inf_{F \in \text{Aut}_0(X, V_{KS})} J^{(k)}(F^* H) - 3C \]
for sufficiently large $k$. Since $J^{(k)}$ is an exhaustion function on $H^{1,0}_{k}(X)/\mathbb{R}$, we find that there exists a unique quantized Kähler-Ricci soliton $H_k$ at level $k$ up to the action of $\text{Aut}_0(X, V_{KS})$ if $k$ is sufficiently large. We normalize $H_k$ so that the corresponding metric $\phi_k := FS_k(H_k)$ minimizes $J$ on the corresponding $\text{Aut}_0(X, V_{KS})$-orbit. Then the minimizing property of $H_k$ implies $D^{(k)}_{g_{V_k}}(H_k) \leq D^{(k)}_{g_{V_k}}(\text{Hilb}_k(\phi))$ for all $\phi \in PSH(X, -K_X)^{1,0}$. Thus letting $k \to \infty$, we obtain
\[ D^{(k)}_{g_{V_k}}(H_k) \leq D_{g_{V_{KS}}} (\phi) + \gamma_k \]
for all $\phi \in PSH(X, -K_X)^{1,0}$, where $\gamma_k = \gamma_k(\phi) \to 0$ is a sequence of constants depending on $\phi$. On the other hand, we have
\[ D_{g_{V_{KS}}} (\phi_k) \leq D^{(k)}_{g_{V_{KS}}} (H_k) + \delta_k J^{(k)}(H_k) + \delta_k \quad \text{(Lemma 4.2 and } g_{V_{KS}} \text{ is bounded)} \]
\[ \leq D^{(k)}_{g_{V_k}}(H_k) + \delta'_k J^{(k)}(H_k) + \delta'_k \quad \text{(by (4.5))}, \]
where $J^{(k)}(H_k)$ is bounded from above by (4.6) and (4.7). Thus we have
\[ \limsup_{k \to \infty} D_{g_{V_{KS}}} (\phi_k) \leq D_{g_{V_{KS}}} (\phi) \]
for all $\phi \in PSH(X, -K_X)^{1,0}$. This yields that $\phi_k$ is a minimizing sequence of the functional $D_{g_{V_{KS}}}$, and therefore $\phi_k$ converges to a Kähler-Ricci soliton weakly. This completes the proof.

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