Extended gravity theories from dynamical noncommutativity

Paolo Aschieri and Leonardo Castellani

Dipartimento di Scienze e Innovazione Tecnologica
INFN Gruppo collegato di Alessandria,
Università del Piemonte Orientale,
Viale T. Michel 11, 15121 Alessandria, Italy

Abstract

In this paper we couple noncommutative (NC) vielbein gravity to scalar fields. Noncommutativity is encoded in a $\star$-product between forms, given by an abelian twist (a twist with commuting vector fields). A geometric generalization of the Seiberg-Witten map for abelian twists yields an extended theory of gravity coupled to scalars, where all fields are ordinary (commutative) fields. The vectors defining the twist can be related to the scalar fields and their derivatives, and hence acquire dynamics. Higher derivative corrections to the classical Einstein-Hilbert and Klein-Gordon actions are organized in successive powers of the noncommutativity parameter $\theta^{AB}$. 

aschieri@to.infn.it
leonardo.castellani@mfn.unipmn.it
1 Introduction

In this paper we study the coupling of scalar fields to the noncommutative gravity theory constructed in [1] and further developed in [2, 3]. A noncommutative action is found, and generalizes the classical Einstein-Hilbert + Klein-Gordon actions. It is invariant under diffeomorphisms and noncommutative local Lorentz transformations. The noncommutativity is governed by an abelian twist, \( \mathcal{F} = e^{-\frac{i}{2} \theta^{AB} X_A \otimes X_B} \), and the corresponding \( \star \)-product between forms reads:

\[
\tau \star \tau' = \sum_{n=0}^{\infty} \left( \frac{i}{2} \right)^n \theta^{A_1 B_1} \cdots \theta^{A_n B_n} (\ell_{X_{A_1}} \cdots \ell_{X_{A_n}} \tau) \wedge (\ell_{X_{B_1}} \cdots \ell_{X_{B_n}} \tau')
\]

\[
= \tau \wedge \tau' + \frac{i}{2} \theta^{AB} (\ell_{X_A} \tau) \wedge (\ell_{X_B} \tau') + \frac{1}{2!} \left( \frac{i}{2} \right)^2 \theta^{A_1 B_1} \theta^{A_2 B_2} (\ell_{X_{A_1}} \ell_{X_{A_2}} \tau) \wedge (\ell_{X_{B_1}} \ell_{X_{B_2}} \tau') + \cdots
\]

where the mutually commuting vector fields \( X_A \) act on forms via the Lie derivatives \( \ell_{X_A} \). This product is associative, and the above formula holds also for \( \tau \) or \( \tau' \) being 0-forms (i.e. functions). A different study of a Klein-Gordon action in a curved background is presented in [5, 6], based on the metric formulation of twist noncommutative gravity [7], where noncommutative local Lorentz symmetry is absent.

Use of the geometric generalization [2] of the Seiberg-Witten map [8] between noncommutative and commutative local Lorentz symmetry allows to reinterpret the noncommutative vielbein gravity coupled to scalar fields as a theory with ordinary fields on commutative spacetime, invariant under diffeomorphisms and usual local Lorentz rotations. It is a particular higher derivative extension of Einstein gravity coupled to scalar fields.

The commuting vectors \( X_A \) present in the twist also enter the action, but they can be related to the scalar fields, so that the resulting theory contains only the vierbein, the spin connection and the scalars. Alternatively one can keep the vectors \( X_A \) as independent fields, and introduce a corresponding kinetic term coupled to the gravity action.

In the first scheme the particular extension of Einstein gravity depends on how the vectors \( X_A \) are related to the scalars. This relation is controlled by a function \( Z \) of the scalars, a sort of “potential” for noncommutativity. The \( X_A \) vector fields are given in terms of (derivatives of) \( Z \), and therefore are not anymore background spectators but acquire dynamics induced by the scalars. Some choices for \( Z \) are discussed.

The paper is organized as follows. In Section 2 we recall the geometric action for scalars coupled to gravity, and show how it can be recast in an index-free form, suitable for a noncommutative generalization. In Section 3 the noncommutative action is obtained, and its symmetries are discussed. In Section 4 we discuss the noncommutative field equations and how to find their solutions. We show that the nondynamical fields of the classical theory remain nondynamical in the NC theory. The geometric Seiberg-Witten map for abelian twists is recalled in Section 5, and applied in Section 6 to show that the first order correction (in \( \theta \)) of the action vanishes. In Section 7 we give the second order expansion of the scalar action, in a manifestly Lorentz gauge invariant form. In Section 8 we relate the \( X_A \) vector

---

\(^1\)when restricted to 0-forms, and if \( X_A = \delta^A_\mu \frac{\partial}{\partial x^\mu} \), the \( \star \)-product reduces to the well-known Moyal-Groenewold product [4].
fields to the scalars and discuss the potential \( Z \). An Appendix summarizes the \( D = 4 \) gamma matrix conventions.

2  Scalars coupled to gravity: classical action

2.1 Geometric first order action

The classical action for a scalar multiplet \( \phi^I \) coupled to gravity can be written in first order form as follows:

\[
S = \int \left( R^{ab} \wedge V^c \wedge V^d + \frac{1}{3} \varphi^I a d \phi^I \wedge V^b \wedge V^c \wedge V^d - \frac{1}{4!} (\varphi^I r + W(\phi^I)) V^a \wedge V^b \wedge V^c \wedge V^d \right) \varepsilon_{abcd} \tag{2.1} \]

The fundamental fields are:

i) the one-form spin connection \( \omega^{ab} \), entering the action through the Lorentz curvature

\[
R^{ab} \equiv d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \tag{2.2} \]

ii) a multiplet of \( N \) scalar fields (zero-forms) \( \phi^I \), \((I = 1, \ldots N)\)

iii) the zero-form auxiliary fields \( \varphi^I_a \);

iv) the vielbein one-form \( V^a \)

The variation of the action with respect to \( \varphi^I_a \) identifies the auxiliary field with the derivative of the scalar field:

\[
\varphi^I_a = \partial_a \phi^I \tag{2.3} \]

where \( \partial_a = V^\mu_a \partial_\mu \), \( V^\mu_a \) being the inverse matrix of \( V_a^\mu \), with \( V^a = V^a_\mu dx^\mu \). The field equation for \( \omega^{ab} \) gives the zero torsion condition:

\[
dV^a - \omega^{ab} \wedge V^b = 0 \tag{2.4} \]

which allows to express the spin connection in terms of derivatives of vielbeins and inverse vielbeins (second order formalism).

The variation with respect to the scalar fields \( \phi^I \) yields the Klein-Gordon equation in curved space:

\[
D_a \partial^a \phi^I + 12 \frac{\delta W}{\delta \phi^I} = 0 \tag{2.5} \]

where \( W \) is the scalar potential, depending only on \( \phi^I \), \( D_a = \text{Lorentz covariant derivative} \), and where repeated indices are summed with the flat Minkowski metric \( \eta_{ab} \).

Finally the variation of the vielbein yields the Einstein equations:

\[
R^{\alpha \beta} - \frac{1}{2} \delta^\alpha_\beta R = -\frac{1}{4} (\partial^a \phi^I \partial_a \phi^I - \frac{1}{2} \delta^a_\beta \partial^r \phi^I \partial_r \phi^I) - 3W \delta^\alpha_\beta \tag{2.6} \]

Note: the use of auxiliary fields \( \varphi^I_a \) in the action (2.1) allows to avoid the appearance of the Hodge star operation. cf. also the action for NC Yang-Mills coupled to gravity in
ref. [3]. This formulation is useful since the noncommutative generalization of the Hodge star operation in case of an arbitrary curved metric is presently an open question. The introduction of the auxiliary fields bypasses this problem, and still leads in the classical limit to usual Einstein gravity coupled to scalars.

2.2 Index-free action

The action (2.1) can be recast in index-free form as follows:

\[
S = \int Tr \left( i\gamma_5 (R \wedge V \wedge V - \frac{1}{4!} \varphi^I \varphi^I V \wedge V \wedge V + \frac{1}{3} \varphi^I D\Phi^I \wedge V \wedge V \wedge V) \right)
\]

where \( R \equiv \frac{1}{4} R^{ab} \gamma_{ab}, \Phi^I \equiv \phi^I, \varphi^I \equiv \varphi^I \gamma_a, V \equiv V^a \gamma_a, \) and the trace is taken on the spinor space. We take for simplicity \( W = 0. \) Use of the \( D = 4 \) gamma matrix identities

\[
Tr(\gamma_a \gamma_b \gamma_c \gamma_d) = -4i \varepsilon_{abcd}, \quad \gamma_{[a} \gamma_{b} \gamma_{c} \gamma_{d]} = -i \gamma_5 \varepsilon_{abcd}
\]

yields the action (2.1). The index-free definition of the Lorentz curvature is

\[
R = d\Omega - \Omega \wedge \Omega
\]

with \( \Omega \equiv \frac{1}{4} \omega^{ab} \gamma_{ab}. \) The definition (2.9) implies the Bianchi identities for \( R:\)

\[
DR \equiv dR - \Omega \wedge R + R \wedge \Omega = 0
\]

For notational economy, in the following we will occasionally omit the multiplet indices \( I \) in the scalar fields.

2.3 Symmetries

Apart from general coordinate invariance, obtained \textit{ab initio} through the use of Cartan calculus, the action (2.1) is invariant under local Lorentz transformations. In index-free form these transformations are

\[
\delta_\varepsilon \Phi = 0, \quad \delta_\varepsilon \varphi = 0 \quad (2.11)
\]

\[
\delta_\varepsilon V = -V \varepsilon + \varepsilon V \quad (2.12)
\]

\[
\delta_\varepsilon \Omega = d\varepsilon - \Omega \varepsilon + \varepsilon \Omega \quad \Rightarrow \quad \delta_\varepsilon R = -R \varepsilon + \varepsilon R \quad (2.13)
\]

with \( \varepsilon \equiv \frac{1}{4} \epsilon^{ab} \gamma_{ab}. \) Invariance of the action (2.1) under these transformations immediately follows from the cyclicity of the trace and the fact that \( \varepsilon \) commutes with \( \gamma_5. \)

3 Noncommutative action

The twisted noncommutative action is found by replacing in the index-free action (2.7) all products by twisted \( \star \)-products:

\[
S = \int Tr \left( i\gamma_5 (\hat{R} \wedge_\star \hat{V} \wedge_\star \hat{V} - \frac{1}{4!} \hat{\varphi} \star \hat{\varphi} \star \hat{V} \wedge_\star \hat{V} \wedge_\star \hat{V} \wedge_\star \hat{V} \wedge_\star \hat{V} + \frac{1}{6} (\hat{\varphi} \star D\hat{\Phi} + D\hat{\Phi} \star \hat{\varphi}) \wedge_\star \hat{V} \wedge_\star \hat{V} \wedge_\star \hat{V} \wedge_\star \hat{V}) \right)
\]

(3.1)
where the covariant exterior derivative is defined by:

\[ D\hat{\Phi} \equiv d\hat{\Phi} - \hat{\Omega} \star \hat{\Phi} + \hat{\Phi} \star \hat{\Omega} \quad (3.2) \]

and the curvature \( \hat{R}(\hat{\Omega}) \) is

\[ \hat{R}(\hat{\Omega}) = d\hat{\Omega} - \hat{\Omega} \wedge \hat{\Omega} \quad (3.3) \]

where \( \hat{\Omega} \) is the noncommutative spin connection matrix. This definition implies the Bianchi identity:

\[ D\hat{R}(\hat{\Omega}) \equiv d\hat{R}(\hat{\Omega}) - \hat{\Omega} \wedge \hat{\Omega} \star \hat{R}(\hat{\Omega}) + \hat{R}(\hat{\Omega}) \wedge \hat{\Omega} = 0 \quad (3.4) \]

The (associative) \( \star \)-exterior product between forms is defined by using Lie derivatives along a set of commuting vector fields \( X_A \) (see the formula given in the Introduction. For a summary on twist differential geometry see for ex. the Appendix of ref. [1]). The symmetrization in the third term of the noncommutative action (3.1) is necessary for the action to be real.

The NC fields have deformed transformation laws: to distinguish them from the ordinary fields transforming under the usual laws we denote them with a hat. In fact, the Seiberg-Witten map relates the hatted fields (the “noncommutative” fields) to the ordinary ones.

### 3.1 Noncommutative symmetries

The NC action (3.1) is invariant under general coordinate transformations (being the integral of a 4-form) and under the \( \star \)-gauge variations:

\[
\begin{align*}
\delta_{\hat{\epsilon}} \hat{\Omega} &= d\hat{\epsilon} - \hat{\Omega} \star \hat{\epsilon} + \hat{\epsilon} \star \hat{\Omega} \quad \Rightarrow \quad \delta_{\hat{\epsilon}} \hat{R}(\hat{\Omega}) = -\hat{R}(\hat{\Omega}) \star \hat{\epsilon} + \hat{\epsilon} \star \hat{R}(\hat{\Omega}) \\
\delta_{\hat{\epsilon}} \hat{V} &= -\hat{V} \star \hat{\epsilon} + \hat{\epsilon} \star \hat{V} \\
\delta_{\hat{\epsilon}} \hat{\varphi} &= -\hat{\varphi} \star \hat{\epsilon} + \hat{\epsilon} \star \hat{\varphi} \\
\delta_{\hat{\epsilon}} \hat{\Phi} &= -\hat{\Phi} \star \hat{\epsilon} + \hat{\epsilon} \star \hat{\Phi} \quad \Rightarrow \quad \delta_{\hat{\epsilon}}(D\hat{\Phi}) = -(D\hat{\Phi}) \star \hat{\epsilon} + \hat{\epsilon} \star (D\hat{\Phi})
\end{align*}
\]

(3.5)

with an arbitrary parameter \( \hat{\epsilon} \) commuting with \( \gamma_5 \).

The invariance of the noncommutative action under these transformations relies on the cyclicity of the integral (and of the trace) and on \( \hat{\epsilon} \) commuting with \( \gamma_5 \).

Because of noncommutativity, extra fields are entering in the expansions of \( \hat{\Omega}, \hat{V}, \hat{\Phi}, \hat{\varphi} \). Indeed now the \( \star \)-gauge variations of the fields (3.5) include also anticommutators of gamma matrices, due to the noncommutativity of the \( \star \)-product. Since for example the anticommutator \( \\{\gamma_{ab}, \gamma_{cd}\} \) contains \( 1 \) and \( \gamma_5 \), we see that the corresponding fields must be included in the expansion of \( \hat{\Omega} \). Similarly, \( \hat{V} \) must contain a \( \gamma_a \gamma_5 \) term due to \( \\{\gamma_{ab}, \gamma_c\} \), etc. The composition law for gauge parameters becomes:

\[
[\delta_{\hat{\epsilon}_1}, \delta_{\hat{\epsilon}_2}] = \delta_{\hat{\epsilon}_2 \star \hat{\epsilon}_1 - \hat{\epsilon}_1 \star \hat{\epsilon}_2}
\]

(3.6)

so that \( \epsilon \) must contain the 1 and \( \gamma_5 \) terms, since they appear in the composite parameter \( \hat{\epsilon}_2 \star \hat{\epsilon}_1 - \hat{\epsilon}_1 \star \hat{\epsilon}_2 \).
The \(SO(1,3)\) enveloping algebra generators \(\Gamma_{\alpha}\) are chosen to be \(\gamma_0\)-antihermitian:

\[
\Gamma_{\alpha} = \frac{1}{4} \gamma_{ab}, \ i1, \ \gamma_5 \tag{3.7}
\]

\[
(\Gamma_{\alpha})^\dagger = -\gamma_0 \Gamma_{\alpha} \gamma_0 \tag{3.8}
\]

The noncommutative fields are expanded as follows:

\[
\hat{\Omega} = \frac{1}{4} \hat{\omega}^{ab} \gamma_{ab} + i \hat{\omega} 1 + \hat{\omega} \gamma_5 \tag{3.9}
\]

\[
\hat{V} = \hat{V}^a \gamma_a + \hat{V}^a \gamma_5 \tag{3.10}
\]

\[
\hat{\Phi}^I = i \frac{1}{4} \hat{\phi}^{I ab} \gamma_{ab} + \hat{\phi}^I 1 + i \hat{\phi}^I \gamma_5 \tag{3.11}
\]

\[
\hat{\varphi}^I = \hat{\varphi}^{Ia} \gamma_a + \hat{\varphi}^{Ia} \gamma_5 \tag{3.12}
\]

Similarly, for the curvature and the gauge parameter the expansions are:

\[
\hat{R} = \frac{1}{4} \hat{R}^{ab} \gamma_{ab} + i \hat{r} + \hat{R} \gamma_5 \tag{3.13}
\]

\[
\hat{\epsilon} = \frac{1}{4} \hat{\epsilon}^{ab} \gamma_{ab} + i \hat{\epsilon} + \hat{\epsilon} \gamma_5 \tag{3.14}
\]

All the components along the \(SO(1,3)\) enveloping algebra generators are taken to be real, and therefore fields and curvatures satisfy the hermiticity properties:

\[
\hat{\Omega}^\dagger = -\gamma_0 \hat{\Omega} \gamma_0, \ \hat{V}^\dagger = \gamma_0 \hat{V} \gamma_0, \ \hat{\Phi}^I = \gamma_0 \hat{\Phi}_I, \ \hat{\varphi}^I = \gamma_0 \hat{\varphi}_I, \ \hat{R}^\dagger = -\gamma_0 \hat{R} \gamma_0 \tag{3.15}
\]

i.e. \(\hat{\Omega}\) and \(\hat{R}\) are \(\gamma_0\)-antihermitian, while \(\hat{V}\), \(\hat{\varphi}\) and \(\hat{\Phi}\) are \(\gamma_0\)-hermitian. Using these rules it is a quick matter to check that the noncommutative action (3.1) is real.

The NC action is also invariant under charge conjugation, defined on elementary fields as:

\[
\hat{V} \rightarrow \hat{V}^C = C \hat{V}^T C, \ \hat{\Omega} \rightarrow \hat{\Omega}^C = C \hat{\Omega}^T C, \ \hat{\Phi} \rightarrow \hat{\Phi}^C = -C \hat{\Phi}^T C, \ \hat{\varphi} \rightarrow \hat{\varphi}^C = C \hat{\varphi}^T C. \tag{3.16}
\]

Charge conjugation is extended linearly and anticommutatively on \(*\)-products (but not on matrix products) of fields, so that \((f * g)^C = g^C * f^C\), i.e.,

\[
(f * g)^C = f^C *_{-\theta} g^C \tag{3.17}
\]

where \(*_{-\theta}\) denotes the star product with opposite noncommutative parameters \(-\theta^{AB}\). This formula holds also for matrix valued fields.

The three addends in the action (3.1) are separately charge conjugation invariant. Charge conjugation symmetry of the second addend is easily verified:

\[
\left( \int Tr(i \gamma_5 \hat{\varphi} *_{-\theta} \hat{V} \wedge_\theta \hat{V} \wedge_\theta \hat{V}) \right)^C = \\
= \int Tr(i \gamma_5 \hat{\varphi}^{C} *_{-\theta} \hat{V}^{C} \wedge_{-\theta} \hat{V}^{C} \wedge_{-\theta} \hat{V}^{C} \wedge_{-\theta} \hat{V}^{C}) \tag{3.18}
\]
One proceeds similarly for the first addend, i.e. the pure NC gravity term (an explicit proof is in [2]). For the last term one uses the definition of charge conjugation on the scalar $\hat{\Phi}$ and the consequent property $(D\hat{\Phi})^C = -C(D\hat{\Phi})^T C$.

The noncommutative fields can be considered as dependent on $\theta$, since their $\star$-gauge transformed images are $\theta$ dependent. Indeed the $\star$-gauge transformations depend on $\theta$ (since the $\star$-product depends on $\theta$), so that the transformed fields necessarily depend on $\theta$. We can then expand the fields in power series of the noncommutativity parameter $\theta$, each coefficient of a given power of $\theta$ being a new field. These infinite degrees of freedom can be reduced by requiring the fields to satisfy the constraints

$$\hat{V}_C = \hat{V}_{-\theta}, \quad \hat{\Omega}^C = \hat{\Omega}_{-\theta}, \quad \hat{\Phi}^C = \hat{\Phi}_{-\theta}, \quad \hat{\varphi}^C = \hat{\varphi}_{-\theta}. \quad \text{(3.19)}$$

These conditions are compatible with the $\star$-gauge variations (3.15) provided that $\hat{\epsilon}^C = \hat{\epsilon}_{-\theta}$. They are equivalent to require that the component fields $\hat{\omega}^{ab}, \hat{V}^a, \hat{\phi}^I, \hat{\varphi}^{Ia}, \hat{\varepsilon}^{ab}$ are even in $\theta$ while the other components are odd in $\theta$.

We can further constrain these fields so that the noncommutative theory is reduced to a theory with the same degrees of freedom of the classical theory. This is done in Section 5 via the Seiberg-Witten map, that allows to express all noncommutative fields in terms of the commutative or classical fields $V^a, \omega^{ab}, \phi^I, \varphi^{Ia}$.

To conclude this section, we remark that conditions (3.19) imply that the NC action must be even in $\theta$. Indeed, because of (3.19) and (3.17), $S_\theta$ is mapped into $S_{-\theta}$ under charge conjugation. Invariance of the action under charge conjugation then implies invariance of the action under $\theta \to -\theta$. Finally $S_\theta = S_{-\theta}$ implies that all corrections to the classical action are even in $\theta$.

### 4 NC field equations and perturbative solutions

In this section $\star$-products, $\wedge_\star$ products and hats $\hat{\cdot}$ are omitted for notational brevity. The NC field equations, obtained by varying the NC action (3.1), are given by:

**Auxiliary field $\varphi$:**

$$Tr \left( \Gamma_{a,a5}([VVV\varphi] + 4\{VVV,D\Phi\}) \right) = 0 \quad \text{(4.1)}$$

**Spin connection $\Omega$:**

$$Tr \left( \Gamma_{ab,1,5}([T,V] + \frac{1}{6}([\Phi, VV\varphi] - [\Phi, \varphi VVV])) \right) = 0 \quad \text{(4.2)}$$
Scalar field $\Phi$:

$$Tr \left( \Gamma_{ab,15}(-\{VVV,D\varphi\} + [\varphi,TVV - VTV + VVT]) \right) = 0$$  \hfill (4.3)

Vielbein $V$:

$$Tr \left( \Gamma_{a,a5}(-\{V,R\} + \frac{1}{4!}(\{VVV,\varphi\varphi\} + \{V,V\varphi\varphi\}) - \frac{1}{6}(\{VV,\{\varphi,D\Phi\}\} - V\{\varphi,D\Phi\}V) \right) = 0$$  \hfill (4.4)

where $\Gamma_{a,a5}$ indicates $\gamma_{a}$ and $\gamma_{a}\gamma_{5}$ (thus there are two distinct equations) and likewise for $\Gamma_{ab,1,5}$ (three equations corresponding to $\gamma_{ab}$, 1 and $\gamma_{5}$). The noncommutative torsion two-form $T$ is defined by:

$$T \equiv T^a\gamma_a + \tilde{T}^a\gamma_a\gamma_5 \equiv DV \equiv dV - \Omega V - V\Omega$$  \hfill (4.5)

The NC field equations can be expanded in $\theta$, and have the general structure:

$$E_0(\phi) + E_1(\phi) + E_2(\phi) + \cdots = 0$$  \hfill (4.6)

where $\phi$ are the fields appearing in the action and $E_0$ is the classical field equation, $E_1$ is linear in $\theta$, $E_2$ is quadratic in $\theta$, etc. The solutions of these NC equations will in general depend on $\theta$:

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots$$  \hfill (4.7)

where $\phi_0$ is the classical field, $\phi_1$ is linear in $\theta$, $\phi_2$ is quadratic in $\theta$, etc. Substituting the expansion (4.7) into the NC equations (4.6), and requiring that the coefficients of all powers of $\theta$ vanish, we find

$$E_0(\phi_0) = 0$$

$$E_0(\phi_1, \phi_0) + E_1(\phi_0) = 0$$

$$E_0(\phi_2, \phi_1, \phi_0) + E_1(\phi_1, \phi_0) + E_2(\phi_0) = 0$$

$$\cdots$$  \hfill (4.8)

The zero-th order solution (the "classical solution") of the first equation can be substituted in the second equation, which then determines the first order correction $\phi_1$ in terms of $\phi_0$. Inserting $\phi_1$ into the third equation enables to find $\phi_2$ and so on: in this way the solution of the NC field equations can be constructed order by order.

Let us see how it works in our specific example of scalar fields coupled to NC gravity. We will not try here to solve the NC field equations for the dynamical fields $V$ and $\Phi$, but will show that the nondynamical fields of the classical theory, i.e. the spin connection $\Omega$ and the auxiliary field $\varphi$, remain nondynamical also in the NC context.

This can be understood as follows. The zero-th order field equations for $\varphi$ and $\Omega$ are just the classical ones given in (2.4) and (2.3), allowing to express the classical spin connection in terms of the classical vielbein and the classical auxiliary field as derivative of the classical scalar field. This is because the classical $\Omega$ and $\varphi$ appear algebraically in the classical field equations. For example $\varphi$ appears in the classical equation as $(\varphi_0)^I_{\alpha\beta\gamma\delta}V^\alpha_0V^\beta_0V^\gamma_0V^\delta_0$ (and is equated to $\partial_\alpha \Phi_I^{\beta\gamma\delta}V_0^\beta V_0^\gamma V_0^\delta dV_0^\alpha$). The higher order corrections $\Omega_i$, $\varphi_i$, $i \geq 1$ are determined by
the higher order equations in \([4.8]\), where they appear algebraically exactly as in the classical case: indeed the first order field equation contains \(\varphi_1\) as \((\varphi_1)_a^I \varepsilon_{bcde} V_0^b V_0^c V_0^d V_0^e\) and similarly for higher orders. Then all the higher order field equations can be solved algebraically in the same way for all the \(\varphi_i\). The same occurs for the higher order corrections of the spin connection that appears algebraically in the torsion field and hence algebraically in \([4.2]\).

Thus we can use the NC field equations to eliminate \(\varphi\) and \(\Omega\), i.e. the transition from first to second order formalism is possible also in the NC theory.

Finally, the same conclusion holds when use is made of the Seiberg-Witten map between noncommutative and commutative local Lorentz symmetry. In this case the fields are given from the start (off shell) a precise \(\theta\) dependence (in terms of the classical fields) dictated by the SW map. Substituting their expansion in the action, after expanding also the \(\star\) products, one obtains an extended action \(S_0 + S_1 + S_2 + \cdots\) expanded in powers of \(\theta\). Consider now the field equations: they involve only the classical fields and their higher derivatives, and can be expanded in powers of \(\theta\). We can look for perturbative solutions given by fields expanded in powers of \(\theta\). The reasoning is now identical to the one used in the previous paragraph. Since at zero-th order in \(\theta\) the auxiliary field \(\varphi\) and the spin connection \(\Omega\) are nondynamical, by the same argument the higher order corrections to these fields will also be nondynamical.

5 Geometric Seiberg-Witten map and fields at first order in \(\theta\) for a general abelian twist

As shown in ref. [2], the SW map can be recast in a coordinate-independent form, and generalized to a \(\star\)-product originating from an arbitrary abelian twist. We expand the noncommutative fields in addends of homogeneous degree in \(\theta\),

\[
\hat{\Omega} = \Omega + \Omega^1 + \Omega^2 + \ldots \Omega^n + \ldots \\
\hat{\epsilon} = \epsilon + \epsilon^1 + \epsilon^2 + \ldots \epsilon^n + \ldots \\
\hat{\phi} = \phi + \phi^1 + \phi^2 + \ldots \phi^n + \ldots
\]

(5.1) 
(5.2) 
(5.3)

where \(\Omega, \epsilon\) are the classical gauge potential and gauge parameter, and \(\phi\) is a classical scalar field, while the fields \(\Omega^n, \epsilon^n\) and \(\phi^n\) are homogeneous of order \(n\) in powers of the noncommutativity \(\theta\) and depend also from the derivatives of the gauge potential \(\Omega\), and, in case of the gauge parameter and the scalar field, also on the classical field \(\epsilon, \phi\) and their derivatives, respectively.

We recall the relevant formulae for the recursive relations determining the SW [2],

\[
\Omega^{n+1} = \frac{i}{4(n + 1)} \theta^{AB} \{\hat{\Omega}_A, \ell_B \hat{\Omega} + \hat{R}_B\}_\star^n
\]

(5.4)

\[
\epsilon^{n+1} = \frac{i}{4(n + 1)} \theta^{AB} \{\hat{\epsilon}_A, \ell_B \hat{\epsilon}\}_\star^n
\]

(5.5)

\[
R^{n+1} = \frac{i}{4(n + 1)} \theta^{AB} \left(\{\hat{\Omega}_A, (\ell_B + L_B)\hat{R}\}_\star^n - [\hat{R}_A, \hat{R}_B\}_\star^n\right)
\]

(5.6)
\[ \phi^{n+1} = \frac{i}{4(n+1)} \theta^{AB} \{ \tilde{\Omega}_A, (\ell_B + L_B) \phi \}^n, \quad \tilde{\delta}_\phi \phi = \tilde{\epsilon} \ast \phi - \phi \ast \tilde{\epsilon} \quad (5.7) \]

where \( \tilde{\Omega}_A, \tilde{R}_A \) are defined as the contraction \( i_A \) along the tangent vector \( X_A \) of the exterior forms \( \Omega, R \), i.e. \( \tilde{\Omega}_A \equiv i_A \tilde{\Omega}, \tilde{R}_A \equiv i_A \tilde{R} \). The apex \( n \) on the composite fields on the right hand side indicates that we are considering the term homogeneous of order \( \theta^n \) of the composite field. We have also introduced the covariant Lie derivative \( L_B \) along the tangent vector \( X_B \); it acts on \( \hat{\Omega} \) and \( \hat{\phi} \) as \( L_B \hat{\phi} = \ell_B \hat{\phi} - \hat{\Omega}_B \ast \hat{\phi} + \hat{\phi} \ast \hat{\Omega}_B \). In fact the covariant Lie derivative \( L_B \) can be written in the Cartan form:

\[ L_B = i_B D + D i_B \quad (5.8) \]

where \( D \) is the covariant derivative. The recursion formulae (5.4)-(5.7) relate the \( \theta^n+1 \) order to the \( \theta^n \) order of the NC fields.

For the fields in the index-free geometrical action (3.1) the above formulae at first order become:

\[ \varphi^1 = \frac{i}{4} \theta^{AB} \{ \Omega_A, (\ell_B + L_B) \varphi \} \quad (5.9) \]
\[ \Phi^1 = \frac{i}{4} \theta^{AB} \{ \Omega_A, (\ell_B + L_B) \Phi \} \quad (5.10) \]
\[ V^1 = \frac{i}{4} \theta^{AB} \{ \Omega_A, (\ell_B + L_B) V \} \quad (5.11) \]
\[ R^1 = \frac{i}{4} \theta^{AB} \{ \{ \Omega_A, (\ell_B + L_B) R \} - [R_A, R_B] \} \quad (5.12) \]

All these formulae are not \( SO(1,3) \)-gauge covariant, due to the presence of the “naked” connection \( \tilde{\Omega} \) and the non-covariant Lie derivative \( \ell_A \). However, when inserted in the NC action (3.1), the resulting action is gauge invariant order by order in \( \theta \). Indeed usual gauge variations induce the \( \ast \)-gauge variations under which the NC action is invariant. Therefore the NC action, re-expressed in terms of ordinary fields via the SW map, is invariant under usual gauge transformations. Since these do not involve \( \theta \), the expanded action is invariant under ordinary gauge variations order by order in \( \theta \). This will be explicitly checked in the next sections for the first and second order \( \theta \) correction of the NC action.

### 6 Action at first order in \( \theta \)

The first order correction of the Einstein term in (3.1) vanishes, as shown in ref. [2]. The remaining two terms contribute to the first order correction \( S^1 \) of the action:

\[ S^1 = S^1_{\varphi \varphi VVV} + S^1_{\phi D \phi VVV} \quad (6.1) \]

with

\[ S^1_{\varphi \varphi VVV} = -\frac{1}{4!} \int Tr \left( i \gamma_5 ((\varphi \ast \varphi)^1 V \wedge V \wedge V \wedge V + \varphi (V \wedge \ast V \wedge \ast V \wedge V)^1) \right) \]
 integrands in the action at first order are equal to minus the integrands.

\[ S_{\text{classical fields:}} \]

Note:

if they eventually have to vanish because of the particular matrix structure of the index-free covariant terms in the integrands. Thus expressions (6.2) and (6.6) are a useful check, even ensures gauge invariance of the result. This is confirmed by the appearance of only gauge-

via the geometrical Seiberg-Witten map, that, together with the cyclicity of the integral, ensures gauge invariance of the result. This is confirmed by the appearance of only gauge-

parity of the action discussed at the end of Section 3. To find nonzero contributions one has to compute the second order correction.

The expressions for the first order corrections (6.2) and (6.6) are obtained algebraically that does not contain the unit matrix, the only matrix with nonvanishing trace. Thus the curvature components

\[ R_{AB} \] are defined as:

\[ R_{AB} = i_B(R_A) = i_Bi_AR \] (6.3)

For the second term we need first to compute \((D\Phi)^1 \equiv d(\Phi)^1 - (\Omega \ast \Phi)^1 + (\Phi \ast \Omega)^1.\) We find:

\[ (D\Phi)^1 = \frac{i}{4} \theta^{AB}(\{\Omega_A, (l_B + L_B)D\Phi}\) - 2\{R_A, L_B\Phi\} \] (6.4)

Then

\[ S_{\varphi_DVVV}^1 = \frac{1}{6} \int Tr\left( i\gamma_5(\varphi^1 D\Phi + \varphi(D\Phi)^1 + (D\Phi)^1 \varphi + (D\Phi)\varphi^1 + \right. \]
\[ \left. + \frac{i}{2} \theta^{AB}(\ell_A\varphi\ell_B(D\Phi) + \ell_A(D\Phi)\ell_B\varphi) \wedge V \wedge V \wedge V + (\varphi D\Phi + D\Phi \varphi)(V \wedge V \wedge V) \right) \] (6.5)

Inserting the first order expressions for the fields yields

\[ S_{\varphi_DVVV}^1 = -\frac{1}{12} \theta^{AB} \int Tr\left( \gamma_5\left( (-\{\varphi, \{R_A, L_B\Phi\}\} + [L_A\varphi, L_B(D\Phi)] \right. \]
\[ \left. + \frac{1}{2}(R_{AB}; \varphi D\Phi + D\Phi \varphi) \wedge V \wedge V \wedge V \right. \]
\[ \left. + (\varphi D\Phi + D\Phi \varphi) \wedge (L_A V \wedge L_B(V \wedge V) + V \wedge L_A V \wedge L_B V) \right) \] (6.6)

Finally, inserting the expansions for the classical fields \( R \equiv \frac{1}{4} R^{ab}_{\gamma_{ab}}, \Phi^I \equiv \phi^I 1, \varphi^I \equiv \varphi^{Ia}_{\gamma a}, V \equiv V^a_{\gamma a}, \) and carrying out the trace on spinor space yields a vanishing result for both contributions \( S_{\varphi^1VVV}^1 \) and \( S_{\varphi_DVVV}^1.\) Indeed all terms have a gamma matrix content that does not contain the unit matrix, the only matrix with nonvanishing trace. Thus the first order (in \( \theta \)) correction to the classical action (2.1) vanishes, in agreement with the \( \theta \)-parity of the action discussed at the end of Section 3. To find nonzero contributions one has to compute the second order correction.

The expressions for the first order corrections (6.2) and (6.6) are obtained algebraically via the geometrical Seiberg-Witten map, that, together with the cyclicity of the integral, ensures gauge invariance of the result. This is confirmed by the appearance of only gauge-covariant terms in the integrands. Thus expressions (6.2) and (6.6) are a useful check, even if they eventually have to vanish because of the particular matrix structure of the index-free fields.

**Note:** one can also use the relations, due to the gamma matrix structure of the index-free classical fields:

\[ \Omega^T = C\Omega C, \ V^T = CV C, \ \Phi^T = -C\Phi C, \ \varphi^T = C\varphi C, \ R^T = CRC \] (6.7)

where \( C \) is the charge conjugation matrix (see the Appendix). It is easy to prove \( S_{\varphi^1VVVV}^1 = S_{\varphi_DVVV}^1 = 0,\) simply by checking that, before taking the spinor trace, the transpose of the integrands in the action at first order are equal to minus the integrands.
7 Action at second order in $\theta$

In this Section we expand the scalar action in (2.7) at second order in $\theta$ using the recursive formulae of the geometric SW map (3.4)-(5.7). In [9] we have developed a method that allows to write each term in the expansion of a generic noncommutative gauge theory action in explicit gauge invariant form. We apply this method to the scalar field action in (2.7) and obtain the second order corrections:

$$S^{2} = S_{\varphi \varphi VVV}^{2} + S_{\varphi \phi VVV}^{2}$$

(7.1)

with (we omit wedge products):

$$S_{\varphi \varphi VVV}^{2} = \frac{\theta^{AB} \theta^{CD}}{4!} \frac{1}{8} \int Tr i\gamma_{5} \left( \frac{1}{2} \{ R_{CD}, \frac{1}{2} \{ R_{AB}, \varphi \phi \} + 2 L_{A} \varphi L_{B} \varphi \} + \frac{1}{2} [ L_{C} R_{AB}, L_{D}(\varphi \phi)] ight.$$  

$$- \frac{1}{2} \{ \{ R_{AC}, R_{BD} \}, \varphi \phi \} + (L_{A} L_{C} \varphi)(L_{B} L_{D} \varphi) - \{ \{ R_{AC}, L_{B} \varphi \}, L_{D} \varphi \}\right) VVVV$$  

$$+ \{ \{ R_{CD}, \varphi \phi \} + 2 L_{C} \varphi L_{D} \varphi \} \left( \{ L_{A} V L_{B} V V \} + L_{A}(VV)L_{B}(VV) \right)$$  

$$+ \varphi \phi \left( - \{ \{ R_{AC}, L_{B} V \}, L_{D} V \}, V V \right) - \{ \{ R_{AC}, L_{B} V \}, V \}, L_{D} (VV) \right)$$  

$$+ [L_{C}(L_{A} V L_{B} V), L_{D}(VV)] + \{ \{ L_{A} L_{C} V \}(L_{B} L_{D} V), V V \} + 2 L_{A} V L_{B} V L_{C} V L_{D} V$$  

$$+ \{ \{ L_{A} L_{C} V, L_{B} V \}, L_{D} (VV) \} + L_{A} L_{C}(VV)L_{B} L_{D}(VV) \right)$$  

(7.2)

and

$$S_{\varphi \phi VVV}^{2} = -\frac{\theta^{AB} \theta^{CD}}{6 \cdot 8} \frac{1}{8} \int Tr i\gamma_{5} \left( \frac{1}{4} \{ R_{CD}, \{ R_{AB}, \{ \varphi, D \phi \} \} + 2 [ L_{A} \varphi, L_{B} D \phi ] - 2 \{ \varphi, \{ R_{A}, L_{B} \phi \} \} \right.$$  

$$+ \frac{1}{2} [ L_{C} R_{A B}, L_{D} \{ \varphi, D \phi \} ] - \frac{1}{2} \{ \{ R_{A C}, R_{B D} \}, \{ \varphi, D \phi \} \} - 2 [ L_{A} \phi, L_{B} \{ R_{C}, L_{D} \phi \} ]$$  

$$- [ \{ R_{C A}, L_{D} \phi \}, L_{B} D \phi ] \} - [ L_{A} \phi, \{ R_{C B}, L_{D} D \phi \} ] + \{ L_{C} L_{A} \phi, L_{D} (L_{B} D \phi) \}$$  

$$- \{ \varphi, \{ L_{C} R_{A}, L_{D} L_{B} \phi \} - \{ i_{A}(R_{C} R_{D}), L_{B} \phi \} - \{ i_{A}(R_{C}, L_{D} \phi) \} \} \}$$  

$$+ [\{ R_{C D}, \{ \varphi, D \phi \} \} + 2 [ L_{C} \varphi, L_{D} D \phi ] - 2 \{ \varphi, \{ R_{C}, L_{D} \phi \} \} \} (\{ L_{A} V \} L_{B}(VV) + V(L_{A} V)(L_{B} V))$$  

$$+ \{ \varphi, D \phi \}(L_{C} L_{A} V) L_{D} L_{B}(VV) + L_{A} V[L_{C} L_{B} V, L_{D} V] + V(L_{C} L_{A} V)(L_{D} L_{B} V)$$  

$$+(L_{C} V) L_{D}(L_{A} V L_{B} V) - \{ R_{C A}, L_{D} V \} L_{B}(VV)$$  

$$- L_{A} V \{ V, \{ R_{C B}, L_{D} V \} \} - V[\{ R_{C A}, L_{D} V \}, L_{B} V] \right)$$  

(7.3)

Note: by taking $\varphi = const.$ in $S_{\varphi \varphi VVV}^{2}$ one obtains the explicit second order correction to the cosmological term (simply by discarding all terms containing derivatives of $\varphi$).

8 Dynamical noncommutativity

The background commuting vector fields $X_{A}$, defining the twist, can become dynamical if we relate them to the scalar fields $\phi^{i}$. 

11
Consider a spacetime manifold $M$ that can be described by a single coordinate system, with coordinates $x^\mu$. In this case the vector fields $X_A$ (that are globally defined on $M$) can be written $X_A = X_\mu^A \partial_\mu$. They are then identified with the inverse of the matrix given by the derivatives of a “potential” $Z^A(\phi)$:

$$X_\mu^A \equiv (\partial_\mu Z^A)^{-1} \quad (8.1)$$

where the index $I$ labelling the scalars $\phi^I$ is now chosen to coincide with the index $A$ labelling the vector fields, and $A$ runs on $1,2,3,4$. This definition automatically ensures that the vector fields commute, i.e. that

$$[X_A, X_B] = 0 \iff X_\mu^A \partial_\mu X^\nu_B - X_\mu^B \partial_\mu X^\nu_A = 0 \quad (8.2)$$

For example, we can choose $Z^A = \phi^A$, as in ref. [10]. One has then to check whether there exist solutions of the scalar field equations and the Einstein equations for which $(\partial_\mu \phi^A)$ is invertible. The study of these solutions, and the analysis of their stability, is postponed to future work. If a solution exists with $\partial_\mu \phi^A = \delta^A_\mu \ (\phi^A = x^A + \text{const})$, it would correspond to $X_\mu^A = \delta_\mu^A$, i.e. to Moyal noncommutativity.

Another choice for $Z$ is:

$$Z^A = \frac{\phi^A}{\phi^2}, \quad \phi^2 \equiv \phi^B \phi^B \quad (8.3)$$

leading to

$$(X_\mu^A)^{-1} = \partial_\mu \left( \frac{\phi^A}{\phi^2} \right) = \frac{\partial_\mu \phi^B}{\phi^2} (\delta^{AB} - 2 \frac{\phi^A \phi^B}{\phi^2}) \quad (8.4)$$

or:

$$X_\mu^A = (\partial_\mu \phi^B)^{-1} \phi^2 (\delta^{AB} - 2 \frac{\phi^A \phi^B}{\phi^2}) \quad (8.5)$$

since $(\delta^{AB} - 2 \frac{\phi^A \phi^B}{\phi^2})$ is its own inverse. In this case a solution $\phi^A = x^A + \text{const}$ for $\text{const} = 0$ would correspond to

$$X_\mu^A = \delta_\mu^A x^2 - 2x^A x^\mu \quad (8.6)$$

so that spacetime becomes commutative close to the origin of coordinates.

### A Gamma matrices in $D = 4$

We summarize in this Appendix our gamma matrix conventions in $D = 4$.

\begin{align*}
\eta_{ab} &= (1, -1, -1, -1), \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad [\gamma_a, \gamma_b] = 2\gamma_{ab}, \quad (A.1) \\
\gamma_5 &\equiv i\gamma_0 \gamma_1 \gamma_2 \gamma_3, \quad \gamma_5 \gamma_5 = 1, \quad \varepsilon_{0123} = -\varepsilon^{0123} = 1, \quad (A.2) \\
\gamma_a^\dagger &= \gamma_0 \gamma_a \gamma_0, \quad \gamma_5^\dagger = \gamma_5 \quad (A.3) \\
\gamma_a^T &= -C \gamma_a C^{-1}, \quad \gamma_5^T = C \gamma_5 C^{-1}, \quad C^2 = -1, \quad C^\dagger = C^T = -C \quad (A.4)
\end{align*}
A.1 Useful identities

\begin{align}
\gamma_a \gamma_b &= \gamma_{ab} + \eta_{ab} \quad (A.5) \\
\gamma_{ab} &= \frac{i}{2} \varepsilon_{abcd} \gamma^{cd} \quad (A.6) \\
\gamma_{ab} \gamma_c &= \eta_{bc} \gamma_a - \eta_{ac} \gamma_b - i \varepsilon_{abcd} \gamma^d \quad (A.7) \\
\gamma_c \gamma_{ab} &= \eta_{ac} \gamma_b - \eta_{bc} \gamma_a - i \varepsilon_{abcd} \gamma^d \quad (A.8) \\
\gamma_{ab} \gamma_c &= \eta_{bc} \gamma_a + \eta_{ca} \gamma_b + i \varepsilon_{abcd} \gamma^d \quad (A.9) \\
\gamma^a \gamma_{cd} &= -i \varepsilon_{[a}^d [c] b] - 2 \delta_{cd} \quad (A.10) \\
\text{Tr}(\gamma_a \gamma_{bc} \gamma_d) &= 8 \delta_{bc} \quad (A.11) \\
\text{Tr}(\gamma_5 \gamma_a \gamma_{bc} \gamma_d) &= -4 i \varepsilon_{abcd} \quad (A.12) \\
\text{Tr}(\gamma^r_s \gamma_a \gamma_{bc} \gamma_d) &= 4(-2 \delta_{cd} \eta_{ab} + 2 \delta_{bd} \eta_{ac} - 3! \delta_{abc} \eta_{ed}) \quad (A.13) \\
\text{Tr}(\gamma_5 \gamma^r_s \gamma_a \gamma_{bc} \gamma_d) &= 4(-i \eta_{ab} \varepsilon^{rs}_{cd} + i \eta_{ac} \varepsilon^{rs}_{bd} + 2 i \varepsilon_{abc} \delta_{ed}) \quad (A.14)
\end{align}

where \( \delta_{cd} \equiv \frac{1}{2} (\delta_c^a \delta_d^b - \delta_c^b \delta_d^a) \), \( \delta_{abc}^{rs} \equiv \frac{1}{3!} (\delta_a^r \delta_b^s \delta_c^e + 5 \text{ terms}) \), and indices antisymmetrization in square brackets has total weight 1.

References

[1] P. Aschieri and L. Castellani, “Noncommutative D=4 gravity coupled to fermions,” JHEP 0906 (2009) 086 [arXiv:0902.3817 [hep-th]]; “Noncommutative supergravity in D=3 and D=4,” JHEP 0906 (2009) 087 [arXiv:0902.3823 [hep-th]].

[2] P. Aschieri and L. Castellani, “Noncommutative gravity coupled to fermions: second order expansion via Seiberg-Witten map,” JHEP 1207 (2012) 184 [arXiv:1111.4822 [hep-th]].

[3] P. Aschieri and L. Castellani, “Noncommutative gauge fields coupled to noncommutative gravity,” arXiv:1205.1911 [hep-th].

[4] J. E. Moyal, Quantum mechanics as a statistical theory, Proc. Camb. Phil. Soc. 45 (1949) 99 ; H. Groenewold, Physica 12 (1946) 405.

[5] T. Ohl and A. Schenkel, “Algebraic approach to quantum field theory on a class of noncommutative curved spacetimes,” Gen. Rel. Grav. 42 (2010) 2785 [arXiv:0912.2252 [hep-th]].

[6] A. Schenkel and C. F. Uhlemann, “Field Theory on Curved Noncommutative Spacetimes,” SIGMA 6 (2010) 061 [arXiv:1003.3190 [hep-th]].

[7] P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, “Noncommutative geometry and gravity,” Class. Quant. Grav. 23 (2006) 1883 [hep-th/0510059].
[8] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **9909**, 032 (1999) [hep-th/9908142].

[9] P. Aschieri, L. Castellani and M. Dimitrijevic, “Noncommutative gravity at second order via Seiberg-Witten map,” [arXiv:1207.4346 [hep-th]].

[10] P. Aschieri, L. Castellani and M. Dimitrijevic, “Dynamical noncommutativity and Noether theorem in twisted $\phi^4$ theory,” Lett. Math. Phys. **85** (2008) 39 [arXiv:0803.4325 [hep-th]].