**J-states and quantum channels between indefinite metric spaces**

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**Abstract**

In the present work, we introduce and study the concepts of state and quantum channel on spaces equipped with an indefinite metric. Exclusively, we will limit our analysis to the matricial framework. As it will be confirmed below, from our research it is noticed that, when passing to the spaces with indefinite metric, the use of the adjoint of a matrix with respect to the indefinite metric is required in the construction of states and quantum channels; which prevents us to consider the space of matrices of certain order \(M_n(\mathbb{C})\) as a \(C^*-\)algebra. In our case, this adjoint is defined through a \(J\)-metric, where the matrix \(J\) is a fundamental symmetry of \(M_n(\mathbb{C})\). In our paper, for quantum operators, we include the general setting in which, these operators map \(J_1\)-states into \(J_2\)-states, where \(J_2 \neq \pm J_1\) are two arbitrary fundamental symmetries. In the middle of this program, we carry out a study of the completely positive maps between two different positive matrices spaces by considering two different indefinite metrics on \(\mathbb{C}^n\).

**Keywords** Quantum channel · Fundamental symmetry matrix · Indefinite metric space

**Mathematics Subject Classification** Primary 81P16 · 81P45 · Secondary 15B48 · 46C20

1 **Introduction**

Quantum channels are the entities through which the encoded information of quantum systems is transmitted in the form of states. All these objects have their mathematical interpretation, in which the theory of completely positive maps stands out. Thus,
quantum states, completely positive maps and quantum channels are basic tools of the functional analysis underlying in the quantum information theory.

The aim of the present paper is to provide a more general setting for the theory of quantum information by means of tools arising from operators theory on indefinite metric spaces. In particular, we introduce the notion of $J$-state, completely $J$-positive map and quantum $J$-channel, where $J$ is a fundamental symmetry matrix of $M_n(\mathbb{C})$, that is, $J^* = J$ and $J^2 = J$. We consider the general situation in which the transmission of information is carried out between two different indefinite metric spaces with the same underlying vector space (see Sect. 4). The results of the article generalize those of the case $J = I_n$, where $I_n$ is the identity matrix of order $n$.

From a mathematical point of view, any of the problems that are studied for the usual quantum channels are susceptible of being transferred to the new situation considered in this article.

The paper is organized as follows. In Sect. 2, we introduce and study the notion of $J$-state which will derive in an analysis about the convenience or not of requiring or modifying the condition $Tr A = 1$ for a $J$-state. In Sect. 3, we introduce and study the new concepts of completely $J$-positive maps and quantum $J$-channels in which the Kraus type operators are revisited; this fact is characterized by the extensive use of the $J$-adjoint of a matrix. In Sect. 4, we also consider quantum channels which transform $J_1$-states in $J_2$-states belong to $M_n(\mathbb{C})$, here $J_2 \neq \pm J_1$ are two fundamental symmetries. Theses kinds of channels are called by us quantum $(J_1, J_2)$-channels.

1.1 Motivation, indefinite quantum mechanics, the $J$-Bloch ball

One of the achievements of quantum mechanics consisted in having incorporated statistical aspects into the physical theory of processes at the atomic level (in the opinion of the authors, it is not by chance that this idea has been suggested in parallel with the appearance and development of the analysis functional). On the other hand, but not least, also the mechanical aspects were revisited. We will then make a little summary of the physical and mathematical formalization of the new ideas for each of these two aspects.

In the mechanical aspects, to each dynamical variable (a physical notion) there corresponds a linear operator (a mathematical object). Then, the possible values of this dynamical variable are the eigenvalues of the operator. This approach is justified only by the complexity of the spectrum of the linear operators considered. The idea makes no sense, unless one proposes a rule for assigning a particular operator to a particular dynamic variable. It is commonly accepted that these linear operators act on a Hilbert space and that the values of a dynamical variable are reals. Hence, it is natural to assign self-adjoint (with respect to the definite inner product of the Hilbert space) linear operators to the dynamical variables, because these operators have all their eigenvalues on the real axis. However, this property is not exclusive to this class of operators. A self-adjoint operator with respect to an indefinite inner product on a Hilbert space has a discrete spectrum which is a symmetric set with respect to the real axis and therefore, in particular, all its eigenvalues could be on this axis. An important detail is that if the degree of negativity of the indefinite inner product is small, then
few eigenvalues will escape from the real axis. Thus, the self-adjoint linear operators with respect to an indefinite metric also turn out to be good candidates to be assigned to the dynamic variables.

In the general case, an operator assigned to a dynamic variable can have physical or significant eigenvalues, but also eigenvalues without any physical meaning, see [5]. Each one of these operators is named a generalized observable. On the other hand, it is known that some inconsistent quantum field theories can be made consistent by introducing an indefinite metric on the underlying Hilbert space [5].

For simplicity, we are going to consider that the operators assigned to dynamic variables belong to the vectorial space $M_n(\mathbb{C})$ composed of matrices of order $n$ over $\mathbb{C}$. By $D_n$, we denote the subspace of diagonal matrices. Then, from the statistical point of view, a state on $M_n(\mathbb{C})$ is a linear function $f : M_n(\mathbb{C}) \rightarrow \mathbb{C}$ such that $f(m) \geq 0$ for all positive matrix $m$ with respect to the usual inner product in $\mathbb{C}^n$ (the positive matrices are defined below) which satisfies $f(I_n) = 1$ where $I_n$ is the identity matrix of order $n$. A density matrix is an element $\rho \in M_n(\mathbb{C})$ such that $\rho$ is a positive matrix and $Tr \rho = \sum \rho_{ii} = 1$. It is possible to see that there is a bijective correspondence between states $f$ on $M_n(\mathbb{C})$ and density operators $\rho \in M_n(\mathbb{C})$, given by $f(m) = Tr(\rho m)$ for all $m \in M_n(\mathbb{C})$ (it is the expectation value of $m$, provided the system is in the state given by $\rho$). In other words, to each state can also be assigned a matrix (or a linear operator, in the general case). We suggest the reader consult the reference [35].

We recall the notion of probability distribution on finite sets. Let $X$ be a finite set. Then, a probability distribution on $X$ is a function $p : X \rightarrow [0, \infty)$ such that $\sum_x p(x) = 1$. Observe that really $p$ is a map $p : X \rightarrow [0, 1]$ such that $\sum_x p(x) = 1$. The case $X = \{1, \ldots, n\}$ is related with our exposition. In fact, there exists a bijective correspondence between states $f$ on $M_n(\mathbb{C})$ and probability distributions $p : \{1, \ldots, n\} \rightarrow [0, 1]$, such that $f(m) = \sum_i p(i) m_{ii}$ (see [35] for more details).

The first part of this article is devoted to extend some basic notions of quantum mechanics to the matrix space $M_n(\mathbb{C})$ when $\mathbb{C}^n$ is equipped with an indefinite metric. With this purpose, we widely use self-adjoint and positive matrices with respect to this indefinite metric. We must point out that from our approach, the classical notions and the known theory are recovered as a particular case.

Thus, early last century physicists’s need of computing probabilities of events concerning the subatomic world of elementary particles led to the arise of quantum probability (the interested reader can consult [30]). On the other hand, in the last century, there were some efforts to transfer quantum mechanics to spaces with indefinite metric which started with the work of P. M. Dirac, one of the founders of this type of spaces together with S.L. Sobolev in the former Soviet Union (Sobolev discovered the Pontryagin spaces $\Pi_\kappa$ of order 1, that is for $\kappa = 1$, in the case that, the indefinite metric is defined on a underlying vector spaces of infinite dimension). Some of these articles pointed out the possibility of using negative values for quantum probabilities (see [5], [20], [29], [34],). Below, we will try to approach our work in this context.

Let $p = (p_1, p_2)$ be a vector such that $p_1 > 0$, $p_2 < 0$ and $p_1 - p_2 = 1$. Then, we do the following observations:
1. It is clear that the vector \( p_1 - p_2 \) is a distribution of probability on the finite probability space \( \{1, 2\} \). Moreover, the vector \( s_p = (\sqrt{p_1}, \sqrt{-p_2}) \) is a usual state vector of \( \mathbb{C}^2 \). Note that

\[
p_1 - p_2 = \text{Tr} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} = \text{Tr} J P_J = 1,
\]

the matrix \( P_J \) will be called \( J \)-density matrix, in this case. Note that \( J P_J \) is a positive definite matrix for which \( \text{Tr} J P_J = 1 \).

2. Let \( A = (a_1, a_2) \) be an observable of phase space \( \mathbb{C}^2 \) which is the result of a measurement, thus \( A \in \mathbb{R}^2 \). The generalized probability distribution \( p \) is associated to \( A \) which is apparent in the usual metric of \( \mathbb{C}^2 \) but effective with respect to the indefinite metric induced by the matrix \( J \). Then, the \( J \)-expectation of \( A \) is, in this case

\[
E_J(A) = p_1 a_1 - p_2 a_2 = \text{Tr} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \text{Tr} J P_J A.
\]

Taking into account that the Bloch ball is a well known model of quantum mechanics, we will try to replicate this with respect to a certain indefinite metric on \( \mathbb{C}^2 \). Let us stay in the space of matrices of order 2 with complex entries which is denoted by \( M_2(\mathbb{C}) \). Any matrix \( S \in M_2(\mathbb{C}) \) is a linear combination of the identity matrix \( I_2 \) and the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

that is, \( S = \frac{1}{2} [z_0 I_2 + z_1 \sigma_1 + z_2 \sigma_2 + z_3 \sigma_3] \), where \((z_1, z_2, z_3, z_4) \in \mathbb{C}^4\). The crucial fact of this decomposition is that \( \text{Tr} S = z_0 \).

Denote by \( J \) the following signature matrix \( J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) which satisfies the following properties \( J^* = J \) and \( J^2 = I_2 \). Next, we will consider the indefinite metric space \((\mathbb{C}^2, [\cdot, \cdot])\) where \([x, y] = (Jx, y)\), being \((Jx, y)\) the usual inner product of \( Jx \) and \( y \) in \( \mathbb{C}^2 \). Then, it is easy to see that

\[
JS = \frac{1}{2} \left[ z_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - iz_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + iz_1 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + z_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right], \tag{3}
\]

thus \( \text{Tr} JS = z_3 \). Therefore, the next observations follow from (3):

- The matrix \( JS \) is self-adjoint if and only if \( z_3 = x_3 \) and \( z_0 = x_0 \) are real, moreover, \( z_2 = i x_2 \) and \( z_1 = -i x_1 \) being \( x_1, x_2 \) both real. In other words, every \( J \)-selfadjoint matrix \( S \) has the form

\[
S = \begin{pmatrix}
x_3 + x_0 & x_2 - i x_1 \\ x_2 + i x_1 & x_3 - x_0
\end{pmatrix}, \tag{4}
\]
Now, $JS$ is a positive definite matrix if and only if $\|(x_0, x_1, x_2)\|_{\mathbb{R}^3} \leq x_3$. This happens because the characteristic polynomial of $JS$ is, in our case, 
\[ \lambda^2 - x_3\lambda + \frac{x_1^2 - x_2^2 - x_0^2}{4} \], which will have two positive eigenvalues under the previous restriction.

One can ensure that if $z_3 = x_3 = 1$, then, $\text{Tr}JS = 1$ and we have $0 \leq JS$ provided that $\|(x_0, x_1, x_2)\|_{\mathbb{R}^3} \leq 1$. We call this set, the $J$-Bloch ball. The analysis for $J = I_2$ can be found in [39].

On the other hand, every $2 \times 2$ matrix $A$ such that $A = JA^*J = A^\sharp$ (this means that $A$ is $J$-selfadjoint) has the form
\[
A = \begin{pmatrix}
\frac{x_0 + x_3}{2} & \frac{y_2 + iy_1}{2} \\
\frac{y_2 - iy_1}{2} & -\frac{x_0 - y_3}{2}
\end{pmatrix},
\]
where the $y_i$ are real for $i = 1, 2, 3, 4$.

On the stage $(M_2(\mathbb{C}), J)$, we define the following notions:

- A $J$-observable is a matrix $A$ of type (5), such that, each $y_i$ is real for all $i$. Since $A$ is a $J$-selfadjoint matrix on $\Pi_1 = (\mathbb{C}^2, J)$, the spectrum of $A$ is symmetric with respect to the real axis. The more general result in this sense, can be found in [28], page 135. Then, the possible values of $A$ (its outcomes) are defined as the real part of the eigenvalues. Observe that in the classic quantum mechanics in which $J$ reduces to the identity matrix $I_2$, $A$ is a usual selfadjoint matrix, so the spectrum of $A$ is real, which is in accordance with the previously given definition.

- A quantum $J$-state is represented by a matrix of the form
\[
S = \begin{pmatrix}
\frac{1 + x_0}{2} & \frac{x_2 - ix_1}{2} \\
\frac{x_2 + ix_1}{2} & \frac{1 - x_0}{2}
\end{pmatrix},
\]
where $\|(x_0, x_1, x_2)\|_{\mathbb{R}^3} \leq 1$. Here, the variables $x_i$ denote real numbers for $i = 0, 1, 2$. The eigenvalues of $S$ are necessarily real (the proof that both eigenvalues are on the real axis follows from theorem 3.27 page 111 of [7]). Even more, we can calculate exactly these eigenvalues
\[
p_1(S) = \frac{x_0 + 1}{2}, \quad p_2(S) = \frac{x_0 - 1}{2} = -\frac{1 - x_0}{2},
\]
and taking into account that $|x_0| < 1$, we can assume that $0 < x_0 < 1$ corresponding to case (1). In fact, it is clear that $p_1 - p_2 = 1$. We say that $p(S) = (p_1(S), p_2(S))$ is the generalized probability distribution corresponding to $S$.

Now, we proceed as in [30]. With this purpose, we return to the $J$-observable matrices (5). Let us recall that $x_0, x_1, x_2, x_3 \in \mathbb{R}$. Suppose next $y_1^2 + y_2^2 < y_3^2$ further $y_1 \neq 0$ or $y_2 \neq 0$. Then, the eigenvalues of $A$, that is, its possible values, are real and
they have the following form (the calculations that follow were done with MATLAB)

\[
\lambda_1 = \frac{y_0 + \sqrt{y_3^2 - y_1^2 - y_1^2}}{2}, \quad \lambda_2 = \frac{y_0 - \sqrt{y_3^2 - y_1^2 - y_1^2}}{2},
\]

and its corresponding eigenvectors are

\[
E_1 = \left(\frac{-y_3 + \sqrt{y_3^2 - y_1^2 - y_1^2}}{y_2 - iy_1}, 1\right), \quad E_2 = \left(\frac{y_3 - \sqrt{y_3^2 - y_1^2 - y_1^2}}{y_2 - iy_1}, 1\right),
\]

which are two linearly independent vectors of \(\mathbb{C}^2\) because

\[|E_1 E_2| = \frac{2\sqrt{y_3^2 - y_1^2 - y_1^2}}{y_2 - iy_1} \neq 0,\]

it shows that \(\mathbb{C}^2 = \langle E_1 \rangle \oplus \langle E_2 \rangle\), where \(\langle E_k \rangle\) is the subspace spanned by \(E_k\) for \(k = 1, 2\). On the other hand, since \(\lambda_1 \neq \lambda_2\) we have \([E_1, E_2] = \langle JE_1, E_2 \rangle = 0\), which implies that even more \(\mathbb{C}^2 = \langle E_1 \rangle[+]\langle E_2 \rangle\). Here, the symbol \([+]\) means \(J\)-orthogonal direct sum. Define \(V_k = \frac{E_k}{||E_k||}\) for \(k = 1, 2\). Then, any vector \(V \in \mathbb{C}^2\) can be written in the form \(V = [., V_1]V_1 + [., V_2]V_2\). It shows that

\[
A V = \lambda_1 [V, V_1]V_1 + \lambda_2 [V, V_2]V_2 = \lambda_1 \Pi_1 V + \lambda_2 \Pi_2 V,
\]

for any \(V \in \mathbb{C}^2\), where we have denoted \(\Pi_k = [., V_k]V_k = \langle J., V_k \rangle V_k\) which is a \(J\)-orthogonal projector for each \(k\), that is, \(\Pi_k^2 = J\Pi_k^*J = \Pi_k\) and \(\Pi_k \Pi_k = 0\).

**Definition 1** Let us denote \(H_k = \Pi_k \mathbb{C}^2\). The probability of observing a system in a subspace \(H_k\) is defined as \(\text{Tr}(S \Pi_k)\). The expectation of the \(J\)-observable \(A\) relative to \(S\) is \(E(A) = \text{Tr}SA = \lambda_1 \text{Tr}SP_1 + \lambda_2 \text{Tr}SP_2\). It means that a single measurement of \(A\) in state \(S\) produces a values \(\lambda_k\) with probability \(\text{Tr}(S \Pi_k)\). We call the pair \((\mathbb{C}^2, S)\) a quantum probability \(J\)-space.

**Remark 2** Consider now a \(J\)-unitary matrix \(V\) belongs to \(M_2(\mathbb{C})\), that is, a matrix for which \(VV^* = V^*V = I_2\). We recall again that \(V^* = JV^*J\). This condition is equivalent to the following \(V^*J V = VJ V^* = J\). The \(J\)-unitary matrices of \(M_2(\mathbb{C})\) have the form

\[
V = \left(\begin{array}{cc}
\alpha & \beta \\
-\overline{\beta} & \overline{\alpha}
\end{array}\right), \quad |\alpha|^2 - |\beta|^2 = 1.
\]

It is easy to show that if \(A\) is a \(J\)-observable and \(S\) is a \(J\)-state, then \(VAV^*\) (respectively \(V^*AV\)) is a \(J\)-observable and \(VS V^*\) (respectively \(V^*SV\)) is a \(J\)-state.
Hence, it is convenient (and in occasions necessary. For example, when measurements cannot be made) to study the dynamic of the following matrix functions:

\[ A(t) = V_1(t)AV_1^*(t), \quad S(t) = V_2(t)SV_2^*(t), \]  

where \( V_1(t) \) and \( V_2(t) \) are families of \( J \)-unitary matrices. It is well known that (11) is equivalent to the equations of Lax type

\[ \frac{dA}{dt} = [B_1(t), A(t)], \quad \frac{dS}{dt} = [B_2(t), S(t)], \]  

where \( B_1(t) = -V_1(t)\frac{dV_1^*(t)}{dt} \) and \( B_2(t) = -V_2(t)\frac{dV_2^*(t)}{dt} \) are both skew \( J \)-selfadjoint matrices, that is, \( B_k^*(t) = -B_k(t) \) for \( k = 1, 2 \). We can take, as usual, \( V_1(t) = V_2(t) = e^{-itM} \) where in our context \( M \) is a \( J \)-observable matrix, that is, \( M^* = M \) of the form (5).

To finish this subsection, we wish to note that the theory of linear operators on spaces with indefinite metric is not restricted to the habitual theory of linear operators on Hilbert spaces. This is revealed in the differences that appear, for instance, in the spectral properties of the operators acting on these spaces. We show an example: it is well known that a selfadjoint operator has all its eigenvalues on the real axis, while some conjugated pairs of eigenvalues of a \( J \)-selfadjoint operator can escape from the real axis. The reader may consult [7], [11] and [28] for more details.

## 2 Overview on \( J \)-indefinite linear algebra. The notion of \( J \)-state

In all quantum system, a state describes the current condition of that system. For instance, the states are relevant to study any important quantum information experiment. In this section, we introduce a generalization of this notion from the point of view of the theory of indefinite metric spaces. Here, these new introduced states are called \( J \)-states and they live in space of \( J \)-positive matrices where \( J \) is a fundamental symmetry matrix of \( Mn(\mathbb{C}) \). Briefly, we expose the bases of our indefinite quantum proposal presented in this section:

- Our quantum system is described by \( \mathbb{C}^n \) equipped with one or two indefinite metrics of the following form \( \langle ., . \rangle = \langle J ., . \rangle_{\mathbb{C}^n} \), where \( J \) is a fundamental symmetry matrix (see below for details) and \( \langle ., . \rangle_{\mathbb{C}^n} \) is the usual inner product in \( \mathbb{C}^n \). This system shall be called indefinite quantum system.

- In our approach, the quantum states will be introduced following an analogous procedure to that of operator theory in spaces with indefinite metric (see [7] and [19]). Specifically, in what follows, a \( J \)-state will be a matrix \( B \) such that \( JB \) is a usual state or density matrix in quantum mechanics.

The objective of this section is to introduce and study the notion of quantum state in indefinite metric spaces. According to the opinion of the authors, the relevance of the present study is given by its usefulness in the security of the transmission.
of information through quantum \( J \)-channels. In this sense, we believe that send the information by a \( J \)-channel will provide bigger chances to encrypt the information.

Next, we review some aspects of indefinite linear algebra. We say that an indefinite inner product (or metric) is given in \( \mathbb{C}^n \) if additionally to the usual inner product \( \langle \cdot , \cdot \rangle_{\mathbb{C}^n} \) we have a function \([\cdot , \cdot ]\) from \( \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C} \), which satisfies the following axioms:

1. \([\alpha x_1 + \beta x_2 , y] = \alpha [x_1 , y] + \beta [x_2 , y], \ \forall x_1 , x_2 , y \in \mathbb{C}^n, \ \forall \alpha , \beta \in \mathbb{C},\)
2. \([x , y] = \overline{[y , x]}, \ \forall x , y \in \mathbb{C}^n,\)
3. if \([x , y] = 0, \ \forall y \in \mathbb{C}^n \) then \( x = 0.\)

One can check that all invertible \( n \times n \) selfadjoint matrix \( H \) induces an indefinite metric, through the formula

\[ [x , y] = [x , y]_H = \langle Hx , y \rangle_{\mathbb{C}^n}, \quad \text{for all } x , y \in \mathbb{C}^n. \]

In this paper, we will only concentrate in a special case of this type of indefinite metrics, when \( H = J \in M_n(\mathbb{C}) \) is a fundamental symmetry matrix, which means that \( J^* = J \) and \( J^2 = I_n. \) One can easily exhibit the structure of every fundamental symmetry \( J. \) Define \( P_+ = \frac{I_n + J}{2}, \) then \( P_+^* = P_+ \) and \( P_+^2 = P_+ \), that is \( P_+ \) is an orthogonal projection matrix. Put \( P_− = I_n − P_+ = \frac{I_n − J}{2}. \) It shows that \( J = P_+ − P_− \) with \( P_+ + P_− = I_n. \) Conversely, suppose that \( P , Q \) are two orthogonal projection matrices, such that, \( P + Q = I_n \) then \( J = P − Q \) is a fundamental symmetry.

We recall that the matrix \( M^2 \) is the \( J \)-adjoint of a matrix \( M \) if by definition \([Mx , y] = [x , M^2 y]\) for all \( x , y \in \mathbb{C}^n \) where \([x , y] = \langle Jx , y \rangle_{\mathbb{C}^n}. \) The \( J \)-adjoint \( M^2 \) of \( M \) is unique, even more \( M^2 = JM^*J \) for all \( M \in M_n(\mathbb{C}) \), where \( M^* \) stands the usual adjoint of \( M. \) It is easy to see that \((M^*)^2 = (M^2)^*.\)

A matrix \( A \) is said to be \( J \)-positive, if \( 0 \leq [Ax , x] \) for all \( x \in \mathbb{C}^n, \) in which case we have \( A^2 = A. \) Next, the set of all \( J \)-positive matrices is denoted by \( M_n^+(\mathbb{C})(J). \) Notice that if \( A \) is a \( J \)-positive matrix, then, it implies that \( JA \) is a positive matrix in the usual sense. An important property of the \( J \)-adjoint of matrices is the following:

\((AB)^2 = B^2A^2\) which is easy of to prove. Indeed, \((AB)^2 = J(AB)^*J = JB^*A^*J = JB^*J^2A^*J = B^2A^2.\) For more on the indefinite metric spaces, see [7] and [19].

Suppose that the matrix \( A \) is \( J \)-positive, which as was mentioned before, it is equivalent to the fact that \( JA \) is a positive matrix (so self-adjoint and of trace class). Then,

\[ JA = \sum_{i=1}^{n} \lambda_i \langle \cdot , e_i \rangle_{\mathbb{C}^n} e_i = \sum_{i=1}^{n} \lambda_i e_i \otimes e_i, \quad Tr \ JA = \sum_{i=1}^{n} \lambda_i, \quad (13) \]

where some of the \( 0 \leq \lambda_i \) could be zero, and \( \{e_i\} \) is an orthonormal basis of \( \mathbb{C}^n. \) That is,

\[ I_n = \sum_{i=1}^{n} \langle \cdot , e_i \rangle_{\mathbb{C}^n} e_i, \quad \langle e_i , e_j \rangle_{\mathbb{C}^n} = \delta_{ij}, \quad (14) \]
and, therefore, from (13) it follows that

$$A = \sum_{i=1}^{n} \lambda_i \langle \cdot, e_i \rangle_C J e_i = \sum_{i=1}^{n} \lambda_i (J e_i) \otimes e_i.$$  \hspace{1cm} (15)

Thus, all $J$-positive matrix $A$ can be written in the form (15), where $\{\lambda_i\} \subset \mathbb{R}_+$ and $\{e_i\}$ constitutes an orthonormal basis. From (13) and (14) it follows that the first equation of (13) represents the spectral decomposition of $B = J A$. Since $B^* = B$, then we obtain

$$\text{Tr} B^2 = \text{Tr} J B^* J = \text{Tr} J B J = \text{Tr} B = \sum_{i=1}^{n} \lambda_i.$$ \hspace{1cm} (16)

**Proposition 3** A matrix $A$ is $J$-positive if and only if $A^2 = A$ and there exists $B$ such that $A = B^2 J B$.

**Proof** Suppose first that $A$ is $J$-positive, then $J A$ is a positive matrix and so self-adjoint. Thus, $J A = (J A)^* = A^* J$ which implies that $A = A^2$. On the other hand, it is well known that there is a matrix $B$ such that $J A = B^* B$, that is, $A = J B^* J^2 B = B^2 J B$. The other implication is proved similarly. $\square$

An interesting and simple class of maps related with a fundamental symmetry $J$ is one constituted by linear $J$-positive functionals.

**Remark 4** Let $\Theta$ be a linear $J$-positive functional on $M_n(\mathbb{C})$, that is, $\Theta(A) \geq 0$ for all matrix $A$ which is $J$-positive. Then, one can see that $\Theta$ is necessarily bounded, even more $\|\Theta\| = \Theta(J)$. This result is a consequence of the fact that $\hat{\Theta}(A) = \Theta(J A)$ is a positive linear functional and that $X \rightarrow J X$ is a bijective map. Moreover, $\Theta(A^2) = \Theta(A)$ for any $A \in M_n(\mathbb{C})$, it implies that if $A$ is a $J$-selfadjoint matrix then $\Theta(A)$ is real. In this case, $\langle A, B \rangle_\Theta = \Theta(A J B^2)$ constitutes a pre-inner product on $M_n(\mathbb{C})$ (really an indefinite metric). Observe that from (15) it follows that, in particular, $\Theta(A) = \text{Tr} A J = \text{Tr} J A$ is a linear $J$-positive functional.

### 2.1 The notion of $J$-state in $M_n(\mathbb{C})$ provided with an indefinite metric

In quantum mechanics, a state is a mathematical object, for example, a unit vector, a positive linear functional, or a density matrix which associates a probability distribution to the set of possible values of each measurement in a system. The knowledge of the evolution rules of the system together with the state represent everything that can be predicted about the behavior of the system.

Now, we present a notion of quantum state in spaces with an indefinite metric and immediately after, we study some of its properties. In this point, we recall that a usual quantum state of $M_n(\mathbb{C})$ is a matrix $S$ which is positive and moreover it satisfies that $\text{Tr} S = 1$. The geometry of the space of all quantum states can be found in [10].

We directly propose
Definition 5 A matrix $B$ will be called a quantum $J$-state once checked that $J B$ is a quantum state.

We have

Lemma 6 The matrix $B$ is a quantum $J$-state if and only if $B$ is a $J$-positive matrix and $\text{Tr} B J = 1$.

Proof Suppose that $B$ is a quantum $J$-state, then $A = J B$ is a quantum state. Hence, $A$ is a positive matrix and $\text{Tr} A = 1$. First, it implies that $B = J A$ is a $J$-positive matrix and second $\text{Tr} B J = \text{Tr} J A J = \text{Tr} A = 1$. Conversely, assume that $B$ is $J$-positive and $\text{Tr} B J = 1$ then $A = J B$ is positive and also $\text{Tr} A = \text{Tr} J A J = \text{Tr} B J = 1$, that is, $A$ is a quantum state. □

Example 1 Observe that the simplest quantum $J$-state is $\Pi_1 = \langle \cdot, e \rangle$ where $\|e\| = 1$. It is called a pure quantum $J$-state. One can see that $\Pi_1^2 = \Pi$ and $\Pi J \Pi = \Pi$. The set of all pure quantum $J$-states is denoted by $\mathcal{P}_J(\mathbb{C})$. A quantum $J$-state that is not pure is called mixed quantum $J$-state.

Theorem 7 Let $\mathcal{S}_J(\mathbb{C}^n)$ be the set of all quantum $J$-states. Then, $\mathcal{S}_J(\mathbb{C}^n)$ is a convex set.

Proof Suppose that $B_1, B_2 \in \mathcal{S}_J(\mathbb{C}^n)$ and $\beta \in (0, 1)$. From the previous lemma, we must see that $B = \beta B_1 + (1 - \beta) B_2$ is $J$-positive and also $\text{Tr} B J = 1$. Note that both $J B_1$ and $J B_2$ are positive matrices, it shows that $J B$ is a positive matrix, in other words, $B$ is a $J$-positive. On the other hand, $\text{Tr} B J = \beta \text{Tr} (B_1 J) + (1 - \beta) \text{Tr} (B_2 J) = 1$. □

Proposition 8 A quantum $J$-state is pure, if and only if it is not a convex combination of elements of $\mathcal{S}_J(\mathbb{C}^n)$.

Proof A pure quantum $J$-states $\Pi$ can not be a convex combination of elements in $\mathcal{S}_J(\mathbb{C}^n)$, otherwise, $J \Pi$ should be a convex combination of two ordinary quantum states which is impossible. Conversely, if a quantum $J$-states $A$ is not a convex combination of elements of $\mathcal{S}_J(\mathbb{C}^n)$, then, it shall be of the form $\langle \cdot, e \rangle \otimes e$, because in the opposite case

$$A = \sum_{i=1}^{l} \lambda_i \langle \cdot, e_i \rangle \otimes e_i = \sum_{i=1}^{l} \lambda_i (J e_i) \otimes e_i,$$

with $2 \leq l, \sum \lambda_i = 1$ and $\{e_i\}$ is an orthonormal set. Hence,

$$A = \lambda_1 \langle \cdot, e_1 \rangle \otimes e_1 + (1 - \lambda_1) \sum_{i=2}^{l} \frac{\lambda_i}{(1 - \lambda_1)} \langle \cdot, e_i \rangle \otimes e_i,$$

which is a contradiction. It shows that the pure quantum $J$-states are the extreme points of convex set $\mathcal{S}_J(\mathbb{C}^n)$. □

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Let $M \in M_n(\mathbb{C})$ be a fixed matrix. Then, there exist two orthonormal systems $\{f_i\}$ and $\{g_i\}$ such that

$$Mx = \sum_{i=1}^{v(M)} s_i(M) \langle x, f_i \rangle \mathbb{C}^n g_i, \quad \forall x \in \mathbb{C}^n,$$

where the $s_i$ for $i = 1, \ldots, v(M)$ are the singular values of $M$, that is, the nonzero eigenvalues of $(M^*M)^{\frac{1}{2}}$. This is the so-called singular value decomposition of $M$. In the case, when $M^* = M$ then the $s_i(M)$ for $i = 1, \ldots, v(M)$ are the eigenvalues of $M$ and $f_i = g_i$ for all $i = 1, \ldots, v(M)$. Note that (15) constitutes the Schmidt decomposition for a $J$-positive matrix $A$. Similarly, from (15) it follows that $AJ$ is a positive operator because

$$AJ = \sum_{i=1}^{n} \lambda_i \langle \cdot, Je_i \rangle \mathbb{C}^n Je_i = \sum_{i=1}^{n} \lambda_i (Je_i) \otimes Je_i,$$

moreover, $\text{Tr} AJ = \text{Tr} JA = \sum_{i=1}^{n} \lambda_i$. Finally, observe that $\langle Je_i, Je_j \rangle \mathbb{C}^n = \delta_{ij}$.

For $1 \leq p < \infty$ one defines the following norms

$$\|M\|_p = \left( \sum_{i=1}^{v(M)} (s_i(M))^p \right)^{\frac{1}{p}},$$

and $\|M\|_\infty = \max_i s_i(M) = \|M\|$. Observe that if $A$ is positive then $\|A\|_1 = \sum_{i=1}^{v(M)} s_i(M) = \text{Tr} A$. Denote by $S_p$ the Banach space $(M_n(\mathbb{C}), \|\cdot\|_p)$. From now on, the norm $\|\cdot\|_1$ is called the trace-norm. On the other hand, as it was seen before if $A$ is a $J$-positive matrix, then $\|JA\|_1 = \|AJ\|_1$.

Next, we recall some facts related with these spaces $S_p$:

- $\|ASB\|_1 \leq \|A\| \|S\|_1 \|B\|$, \quad $\forall A, S, B \in M_n$,
- For any fixed matrix $T$, the function $F_T(\cdot) : S \mapsto F_T(S) = \text{Tr}(ST)$ defines a continuous linear functional and

$$|F_T(S)| \leq \|S\|_p \|T\|_q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. In the case $p = 1$, this reduces to $|F_T(S)| \leq \|S\|_1 \|T\|$. Observe that if the matrix $T$ is a quantum effect which means that $0 \leq T \leq I_n$ then $0 \leq \|T\| \leq 1$ and so $0 \leq F_T(S) \leq 1$ for all quantum state $S$. Hence, the value $P_S(T) = F_T(S) = \text{Tr} ST$ can be considered as the probability that the effect $T$ emerges in the quantum state $S$, giving rise to a new quantum state.

**Definition 9** A quantum $J$-effect is a matrix $E$ such that $JE$ is a usual quantum effect in the explained above sense.
We shall denote by $\mathcal{E}_J(\mathbb{C}^n)$ the set of all $J$-effects which turns out to be a convex set. In fact, if $E_1$ and $E_2$ are $J$-effects then $0 \leq J E_1 \leq I_n$ and $0 \leq J E_1 \leq I_n$ hence for all $\beta \in (0, 1)$ we have $0 \leq \beta J E_1 + (1 - \beta) J E_2$ and moreover $\beta (J E_1 x, x)_{\mathbb{C}^n} + (1 - \beta) (J E_2 x, x)_{\mathbb{C}^n} \leq \|x\|^2$ for all $x \in \mathbb{C}^n$ which means that $0 \leq \beta J E_1 + (1 - \beta) J E_2 \leq I_n$. Thus, $\beta E_1 + (1 - \beta) E_2$ is a quantum $J$-effect.

It is useful to introduce $J$-state automorphisms on $\mathcal{S}_J(\mathbb{C}^n)$.

**Definition 10** A function $s : \mathcal{S}_J(\mathbb{C}^n) \rightarrow \mathcal{S}_J(\mathbb{C}^n)$ is a $J$-state automorphism if

1. The function $s$ is a bijection,
2. $s(\beta S_1 + (1 - \beta) S_2) = \beta s(S_1) + (1 - \beta) s(S_2)$ for all $S_1, S_2 \in \mathcal{S}_J(\mathbb{C}^n)$ and all $\beta \in (0, 1)$.

The case in which $J = I_n$, that is, $J$ is the identity matrix can be consulted in [13].

It is clear that the set $\text{Aut}_J = \text{Aut}_J(\mathcal{S}(\mathbb{C}^n))$ of all $J$-state automorphisms is a group with respect to the composition of functions. We recall that $\mathcal{P}_J(\mathbb{C}^n) = \{\langle \cdot, e \rangle_{\mathbb{C}^n} J e | \|e\|_{\mathbb{C}^n} = 1\}$ is the set of all pure quantum $J$-states.

We already know that $S \in \mathcal{S}_J(\mathbb{C}^n)$, if and only if $S$ is a $J$-positive matrix and $\text{Tr} SJ = 1$.

**Remark 11** From lemma 3, we know that if a matrix $A$ is $J$-positive then it is $J$-selfadjoint, that is, $A^\sharp = A$. Hence, being $B = C - D$ where $C$ and $D$ are $J$-positive, we have $B^\sharp = B$. In fact, $B^\sharp = C^\sharp - D^\sharp = C - D$.

**Proposition 12** Let $s \in \text{Aut}_J$, then

- $s$ is the restriction of a unique linear operator $\tilde{s}$ on the real vector space $M_n(\mathbb{C})^{sa}(J) = \{A | A^\sharp = A\}$ such that $\text{Tr} (\tilde{s}(T) J) = \text{Tr} T J$ for all $T \in M_n(\mathbb{C})^{sa}(J)$. Moreover, $\tilde{s}$ is a bijection of $M_n(\mathbb{C})^{sa}(J)$ on $M_n(\mathbb{C})^{sa}(J)$.
- $s(\mathcal{P}_J) \subset \mathcal{P}_J$.

**Proof** First, we will extend $s$ to the set $M_n^+(\mathbb{C})(J)$ of all $J$-positive matrices. We put

$$\tilde{s}(T) = \|T J\|_1 s \left( \frac{T}{\|T J\|_1} \right), \quad \forall T \neq O_n \in M_n^+(\mathbb{C})(J),$$

and $\tilde{s}(O_n) = O_n$. It is convenient to indicate that the image of a $J$-positive matrix when applying $\tilde{s}$ is, by definition, a $J$-positive matrix.

Now, when $0 \leq \lambda$ and $T$ is a $J$-positive matrix, then $\lambda T$ is also $J$-positive. In this case, we obtain $\tilde{s}(\lambda T) = \lambda \tilde{s}(T)$ (the positive homogeneity of $\tilde{s}$). Indeed,

$$\tilde{s}(\lambda T) = \|\lambda T J\|_1 s \left( \frac{\lambda T}{\|\lambda T J\|_1} \right) = \lambda \tilde{s} \left( \frac{T}{\|T J\|_1} \right).$$

Suppose now that $T_1$ and $T_2$ are $J$-positive, then we can write the sum $T_1 + T_2$ in the following form

$$T_1 + T_2 = (\|T_1 J\|_1 + \|T_2 J\|_1) \left( \frac{\|T_1 J\|_1}{\|T_1 J\|_1 + \|T_2 J\|_1} T_1 + \frac{\|T_2 J\|_1}{\|T_1 J\|_1 + \|T_2 J\|_1} T_2 \right).$$
Let us pay attention to the following details of the previous equality

- Observe that \( \hat{T} = \left( \frac{||T_1 J||}{||T_1 J||_1 + ||T_2 J||_1} T_1 + \frac{||T_2 J||}{||T_1 J||_1 + ||T_2 J||_1} T_2 \right) \) \( \in \mathcal{S}_J(C^n) \), because \( \hat{T} \) is a convex combination of two matrices of \( \mathcal{S}_J(C^n) \).

- \( T_1 + T_2 = \lambda \hat{T} \) where \( \hat{T} \) is \( J \)-positive and \( 0 \leq \lambda = (||T_1 J||_1 + ||T_2 J||_1) \).

Hence, the property 2. of \( s \) and the positive homogeneity of \( \tilde{s} \) imply that

\[
\tilde{s}(T_1 + T_2) = \tilde{s}(T_1) + \tilde{s}(T_2).
\]

Let us extend \( s \) to \( M_n(C)_{sa}(J) \). For this purpose, consider a \( T \in M_n(C)_{sa}(J) \) arbitrary, then \( T^s = T \), that is, \( JT^s J = T \) which implies that \( JT = (JT)^s \). Hence, \( JT = A_+ - A_- \) where \( A_+ \) and \( A_- \) are usual positive matrices. It shows that \( T = T_+ - T_- \) where both matrices \( T_+ \) and \( T_- \) are \( J \)-positive. Then, one defines \( \tilde{s}(T) = \tilde{s}(T_+) - \tilde{s}(T_-) \). Now, taking into account that \( \tilde{s}(T_+) \) and \( \tilde{s}(T_-) \) are \( J \)-positive matrices from remark 11, it follows that \( \tilde{s}(T) \in M_n(C)_{sa}(J) \).

It is not hard to see that \( \tilde{s} \) defined in this form is linear. Indeed, let \( T = \kappa_1 T_1 + \kappa_2 T_2 \), where \( \kappa_1, \kappa_2 \in \mathbb{R} \) and \( T_1, T_2 \in M_n(C)_{sa}(J) \), then

\[
T = \kappa_1 T_1 + \kappa_2 T_2 = \kappa_1[(T_1)_+ - (T_1)_-] + \kappa_2[(T_2)_+ - (T_2)_-],
\]

where each of the matrices \( (T_1)_\pm, (T_2)_\pm \) is \( J \)-positive. On the other hand, without loss of generality, we can assume that \( 0 \leq \kappa_1, \kappa_2 \). In fact, otherwise, if for instance \( \kappa_i < 0 \) for some \( i \) we have

\[
\kappa_i[(T_i)_+ - (T_i)_-] = -\kappa_i[(T_i)_- - (T_i)_+] = \tilde{\kappa}[(H_i)_+ - (H_i)_-],
\]

where \( 0 < \tilde{\kappa} \) and the \( (H_i)_\pm \) are \( J \)-positive matrices. Thus,

\[
T = [\kappa_1(T_1)_+ + \kappa_2(T_2)_+] - [\kappa_1(T_1)_- + \kappa_2(T_2)_-],
\]

note that each \( \kappa_i(T_i)_\pm \) is a \( J \)-positive matrix for \( i = 1, 2 \). Taking into account the way in which \( \tilde{s} \) has been defined, we obtain

\[
\tilde{s}(\kappa_1 T_1 + \kappa_2 T_2) = \tilde{s}(\kappa_1(T_1)_+ + \kappa_2(T_2)_+) - \tilde{s}(\kappa_1(T_1)_- + \kappa_2(T_2)_-)
= (\tilde{s}(\kappa_1(T_1)_+) + \kappa_2 \tilde{s}(T_2)_+) - \tilde{s}(\kappa_1(T_1)_-) - \kappa_2 \tilde{s}(T_2)_-
= (\kappa_1 \tilde{s}((T_1)_+) + \kappa_2 \tilde{s}((T_2)_+)) - (\kappa_1 \tilde{s}((T_1)_-) + \kappa_2 \tilde{s}((T_2)_-))
= \kappa_1 (\tilde{s}((T_1)_+) - \tilde{s}((T_1)_-)) + \kappa_2 (\tilde{s}((T_2)_+) - \tilde{s}((T_2)_-))
= \kappa_1 (\tilde{s}((T_1)_+) + (T_1)_-)) + \kappa_2 (\tilde{s}((T_2)_+) + (T_2)_-))
= \kappa_1 \tilde{s}(T_1) + \kappa_2 \tilde{s}(T_2).
\]

On the other hand, suppose that \( T = T_1 - T_2 \) with both \( T_1, T_2 \) \( J \)-positive. Then \( T_+ + T_2 = T_1 + T_- \) from which follows that \( \tilde{s} \) is well defined (because \( \tilde{s}(T_+) - \tilde{s}(T_-) = \tilde{s}(T_1) - \tilde{s}(T_2) \)). A main fact in our construction is that \( \tilde{s} \) maps \( M_n^+(C)(J) \) into \( M_n^+(C)(J) \).
Now, let $T$ be an arbitrary $J$-selfadjoint matrix, that is, $T^\natural = T$. Then $T = T_+ - T_-$ so

$$\tilde{s}(T) = \|T_+ J\|_1 s\left( \frac{T_+}{\|T_+ J\|_1} \right) - \|T_- J\|_1 \tilde{s}\left( \frac{T_-}{\|T_- J\|_1} \right), \quad (19)$$

observe that $\frac{T_+}{\|T_+ J\|_1}$ and $\frac{T_-}{\|T_- J\|_1}$ are quantum $J$-states, hence $s(\frac{T_+}{\|T_+ J\|_1})$ and $s(\frac{T_-}{\|T_- J\|_1})$ are quantum $J$-states and

$$Tr\left( s\left( \frac{T_+}{\|T_+ J\|_1} \right) J \right) = 1 = Tr\left( s\left( \frac{T_-}{\|T_- J\|_1} \right) J \right), \quad (20)$$

therefore, combining (19) and (20), we find that

$$Tr(\tilde{s}(T) J) = \|T_+ J\|_1 - \|T_- J\|_1 = Tr(T_+ J) - Tr(T_- J) = Tr TJ.$$

Suppose now that $\tilde{s}$ is another linear operator which extends $s$ such that $\tilde{s}(M^+_n(\mathbb{C})(J)) \subset M^+_n(\mathbb{C})(J)$. Then, for any Matrix $T$ for which $T^\natural = T$, we obtain (using the linearity of $\tilde{s}$)

$$\tilde{s}(T) = \tilde{s}(T_+ - T_-) = \tilde{s}(T_+) - \tilde{s}(T_-) = \|T_+ J\|_1 s\left( \frac{T_+}{\|T_+ J\|_1} \right) - \|T_- J\|_1 \tilde{s}\left( \frac{T_-}{\|T_- J\|_1} \right)$$

It shows that $\tilde{s}$ is unique. Notice that if $T_1 \neq O_n$ and $T_2 \neq O_n$ are $J$-positive and $\tilde{s}(T_1) = \tilde{s}(T_2)$, then it implies that $T_1 = T_2$. In fact, if we suppose that

$$\tilde{s}(T_1) = \|T_1 J\|_1 s\left( \frac{T_1}{\|T_1 J\|_1} \right) = \|T_2 J\|_1 \tilde{s}\left( \frac{T_2}{\|T_2 J\|_1} \right) = \tilde{s}(T_2),$$

then

$$\|T_1 J\|_1 = Tr(\tilde{s}(T_1) J) = Tr(\tilde{s}(T_2) J) = \|T_2 J\|_1,$$

and since $s$ is a bijection, it follows that $T_1 = T_2$. Next, suppose that $T_1, T_2 \in M_n(\mathbb{C})^{sa}(J)$ such that $\tilde{s}(T_1) = \tilde{s}(T_2)$. Then,

$$\tilde{s}(T_1) = \tilde{s}(T_1^1 - T_1^-) = \tilde{s}(T_1^1) - \tilde{s}(T_1^-) = \tilde{s}(T_1^2) - \tilde{s}(T_1^-) = \tilde{s}(T_1^2 - T_1^-) = \tilde{s}(T_2),$$

it shows that

$$\tilde{s}(T_1^1 + T_1^-) = \tilde{s}(T_2^1) + \tilde{s}(T_2^-) = \tilde{s}(T_2^2) + \tilde{s}(T_2^-) = \tilde{s}(T_2^2 + T_2^-),$$

and from this, we obtain $T_1^1 + T_1^- = T_2^2 + T_2^-$ because both sides are $J$-positive matrices, thus $T_1 = T_2$. This tells us that $\tilde{s} : M_n(\mathbb{C})^{sa}(J) \rightarrow M_n(\mathbb{C})^{sa}(J)$ is an injective map.

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Now, we shall show that it is also surjective. Indeed, let $T \in M_n(\mathbb{C})^{sa}(J)$, that is, $T^\sharp = T$ then $T = T_+ - T_-$ and
\[
T = \|T_+J\|_1 \left( \frac{T_+}{\|T_+J\|_1} \right) - \|T_-J\|_1 \left( \frac{T_-}{\|T_-J\|_1} \right) = \|T_+J\|_1 s(S_1) - \|T_-J\|_1 s(S_2),
\]
for some $S_1, S_2 \in \mathcal{G}(\mathbb{C}^n)$, thus using the linearity of $\mathcal{F}$ we have
\[
T = \|T_+J\|_1 \mathcal{F}(S_1) - \|T_-J\|_1 \mathcal{F}(S_2) = \mathcal{F}(\|T_+J\|_1 S_1 - \|T_-J\|_1 S_2).
\]

Now, we turn to prove that $s(\mathcal{P}_J) \subset \mathcal{P}_J$. It is easy to see that $s^{-1}(\beta T_1 + (1 - \beta)T_2) = \beta s^{-1}(T_1) + (1 - \beta)s^{-1}(T_2)$ for all $T_1, T_2 \in \mathcal{G}_J(\mathbb{C}^n)$ and $\beta \in (0, 1)$. Suppose that $\Pi \in \mathcal{P}_J$ and $s(\Pi) = \beta T_1 + (1 - \beta)T_2$ then $\Pi = \beta s^{-1}(T_1) + (1 - \beta)s^{-1}(T_2)$ so from remark 8, it follows that $\Pi = s^{-1}(T_1) = s^{-1}(T_2)$, that is $s(\Pi) = T_1 = T_2$, therefore making use of remark 8 again, we may simply obtain $s(\Pi) \in \mathcal{P}_J$. It makes the proof to be concluded.

Remark 13 Observe that if $s \in \text{Aut}_J$ and $s(\Pi) = \Pi$ for all $\Pi \in \mathcal{P}_J$ then $s$ is the identity.

Next, we will construct a concrete $J$-state automorphism of $\mathcal{G}_J(\mathbb{C}^n)$.

Example 2 Define $s^J_V(A) = V^\sharp AV$ such that $V^\sharp JV = J = JV V^\sharp$ (in short, this means that $V^*V = VV^* = I_n$) then $s^J_V \in \text{Aut}_J$. In fact, if $A \in \mathcal{G}_J(\mathbb{C}^n)$ we have $[s^J_V(A)x, x] = [V^\sharp AVx, x] = [AVx, Vx] \succeq 0$ for all $x \in \mathbb{C}^n$ because $A$ is a $J$-positive matrix. It shows that $s^J_V(A)$ is a $J$-positive matrix. On the other hand,
\[
\text{Tr } s^J_V(A)J = \text{Tr } V^\sharp AVJ = \text{Tr } V^\sharp (AJ)(JVJ) = \text{Tr } AJ = 1,
\]
here, we have used that $V^\sharp (JVJ) = (JVJ)V^\sharp = J^2 = I_n$. From this follows that $s^J_V(A) \in \mathcal{G}_J(\mathbb{C}^n)$ and hence $s^J_V(\mathcal{G}_J(\mathbb{C}^n)) \subset \mathcal{G}_J(\mathbb{C}^n)$. Let us assume that $s^J_V(A_1) = s^J_V(A_2)$ where $A_1, A_2 \in \mathcal{G}_J(\mathbb{C}^n)$, then since $V^\sharp JV = J = JV V^\sharp$, it is easy to conclude that $A_1 = A_2$. In other words, $s^J_V$ is injective. Let $B \in \mathcal{G}_J(\mathbb{C}^n)$ be arbitrary and define $A = (JVJ)B(JV^\sharp J)$. Then, one can see that $A \in \mathcal{G}_J(\mathbb{C}^n)$ and $s^J_V(A) = B$. In the next section, this type of maps will be studied in detail.

This last example shows that the notion of quantum $J$-state as it was introduced in this section is basically related to the usual group of unitary matrices. Hence, it suggests to call to this as a quantum $J$-state of unitary origin.

Remark 14 There is another possible geometrical notion of quantum state in a space with an indefinite metric which is related with a $J$-unitary matrix.

Definition 15 We say that a Matrix $A$ is a quantum $J$-state of $J$-unitary origin if $A$ is $J$-positive and $\text{Tr } A = 1$. 
Observe that if $A$ is a quantum $J$-state of $J$-unitary origin, then the same happens with $V^\dagger AV$ whenever $V^\dagger V = VV^\dagger = I_n$. On the other hand, the set of all quantum $J$-state of $J$-unitary origin is a convex set. A pure quantum $J$-state of $J$-unitary origin is one of the following form $\Gamma_1 = \{ e \}$, $e = \langle J_\cdot, e \rangle C_n e = \Pi^*$ (see example 1).

We would like to indicate that the concept of quantum $J$-state of $J$-unitary origin leads to a symmetric theory to which was previously developed with the definition of $J$-state of unitary origin. We show this fact with the following table (for a matrix $M$ the notation $0 \leq J M$ means that $M$ is a $J$-positive matrix),

| $J$-state of unit. orig. $B$ | then if $A = JB$ | $J$-state of $J$-unit. orig. $B$ | then if $A = JB$ |
|-----------------------------|-----------------|-----------------------------|-----------------|
| $0 \leq J B$, $Tr BJ = 1$  | $0 \leq A$, $Tr A = 1$ | $0 \leq J B$, $Tr B = 1$  | $0 \leq A$, $Tr AJ = 1$ |

### 3 Completely $J$-positive type maps on $M_n(\mathbb{C})$ and quantum $J$-channel

In describing the physical evolution (reversible or not) of a part of a larger system, some requirements must be followed [39]. Consider an evolution which, in the Schrödinger representation, is described by a map $T : B(H) \rightarrow B(H')$ where $H$ and $H'$ are Hilbert spaces, while $B(H)$ and $B(H')$ denote the algebras of bounded linear operators on $H$ and $H'$ respectively. Then, when describing a physically meaningful evolution $T$ should satisfy the following three conditions:

- **Linearity.** This property is related to locality, i.e., the fact that a spatially localized action does not instantaneously affect distant regions. In general, this is an intrinsic quantum mechanical requirement. Linearity means

  $$T(\lambda A + C) = \lambda T(A) + T(C), \quad \forall A, C \in B(H), \text{ and } \forall \lambda \in \mathbb{C}.$$  

- **Preservation of the trace in the following sense, $Tr T(A) = Tr A$ for all $A \in B(H)$, in particular $T$ has to map density operators (density matrices) onto density operators (density matrices).**

- **Complete positivity.** The first two requirements imply the positivity of $T$, that is, $T(A^* A) \geq 0$ for any $A \in B(H)$. Positivity alone is, however, not sufficient. Suppose that $H$ is part of a bipartite system so that the evolution of the larger system is described by $T \otimes id$. In other words, the additional system merely plays the role of a spectator as the evolution on this part is the trivial one. Then, a requirement stronger than positivity must be imposed on $T \otimes id$. The relevant condition is the complete positivity of $T$ which means positivity of $T \otimes I_n$ for all $n \in \mathbb{N}$ where $I_n$ is the identity matrix of order $n$.

In this section, we study maps that satisfy properties similar to the first two requirements above, while the third requirement is studied from the perspective of the positivity of the elements of $B(H)$ with respect to an indefinite metric defined on $H$. Below, we consider only the case in which $H$ is a finite-dimensional Hilbert spaces.

The study of positive maps on $C^*$-algebras began long before the boom of the quantum theories of computation and information; which were suggested mainly by
Paul Benioff, Richard Feynman and Yuri Manin in the 80s of the last century. These maps were introduced around 1950 by R. V. Kadison in the papers [25], [26]. Later in 1955 Stinespring introduced completely positive maps and proved his important dilation theorem [36], simultaneously. The relationship between completely positive maps and the theory of dilation was extensively formalized by Arveson [2], [3] and [4]. At that time, the topic was not popular among mathematicians and practically it was barely known to researches from other fields, however remarkable progress was made. The situation changed in the 1990s when the importance of completely positive maps in quantum information theory was evidenced. It can be said that, currently, the subject is consolidated and the interest of the scientific community about this is constantly growing.

We want to mention that the study of certain types of completely positive maps on groups, using representation theory on Hilbert spaces equipped with an indefinite metric defined by means of fundamental symmetries, it was first carried out by J. Heo in [24]. There are excellent texts on completely positive maps among which we have only selected a few of them: [10], [31] and [37].

On the other hand, theoretically quantum channels or quantum operators are the fundamental objects through which information is transmitted. They constitute completely positive maps that preserve the matrix trace. A detailed discussion of the theory of quantum channels in the finite dimensional case is presented in [38]. In the class of quantum channels, we must mention some of them that have particular characteristics and perform important functions within the theory of quantum information. For example, random unitary quantum channels which have the form

$$\Phi(A) = \sum_{s=1}^{l} \mu_s U_s^* A U_s,$$

where each $U_s$ is a unitary matrix and $\mu_s$ are positive weights, such that $\sum_{s=1}^{l} \mu_s = 1$ [15].

This class of quantum channels is very important, since the action of such channels can be considered as the random application of one of the unitary transformations $U_s$, with respective probabilities $\mu_s$. Moreover, because these have particular properties. For example, random unitary channels have been used to disprove the additivity of minimum output entropy, see [22]. We refer to some papers, where this kind of quantum channels has been studied, for example, [6], [22] and [23].

Other important quantum channels are the quantum Gaussian channels which have certain behavior with respect to the so-called characteristic function on trace-class matrices, these have a main role in quantum communication theory because they determine the attenuation and the noise affecting any electromagnetic signal, in the quantum regime. In this regard, the reader can consult e.g [16] and [17].

In summary, the purpose of this section is to study completely positive maps and quantum channels between indefinite metric spaces. Throughout this section the fundamental symmetry $J \in M_n$ is fixed.
3.1 Kraus $J$-maps and completely $J$-positive maps

In the space $M_n(\mathbb{C})$ of $n \times n$ matrices, we consider the following kind of map

$$
\Phi(A) = \sum_{s=1}^{\nu} V_s^\dagger A V_s, \quad \forall A \in M_n(\mathbb{C}),
$$

(21)

here $M^\dagger$ denotes the $J$-adjoint for an arbitrary matrix $M$ and $(V_1, \cdots, V_\nu) \in (M_n(\mathbb{C}))^\nu$ is a fixed matrix vector. These linear maps are called Kraus $J$-maps by us, and the number $\nu$ is named the Kraus index for the corresponding $\Phi$. In addition, observe that they transform $J$-positive $n \times n$ matrices into $J$-positive matrices of the same order. Indeed, from (21), it follows that for all $x \in \mathbb{C}^n$ and any $A \in M_n^+(\mathbb{C})(J)$

$$
[\Phi(A)x, x] = \left[ \left( \sum_{s=1}^{\nu} V_s^\dagger A V_s \right) x, x \right] = \sum_{s=1}^{\nu} [V_s^\dagger A V_s x, x] = \sum_{s=1}^{\nu} [AV_s x, V_s x] \geq 0.
$$

We say that $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ linear is $J$-positive if $\Phi(M_n^+(\mathbb{C})(J)) \subset M_n^+(\mathbb{C})(J)$. Thus, the map $\Phi$ defined by (21) is $J$-positive. Let us assume that $\Phi(\cdot)$ is a Kraus $J$-map then $\Phi_M(\cdot) = M^\dagger \Phi(\cdot) M$ is also a Kraus $J$-map for all matrix $M$ of order $n$, and the Kraus index of $\Phi$ and $\Phi_M$ are equal. The simplest Kraus $J$-map is the identity map, that is $\Phi(A) = A$, because $I_n^\dagger = I_n$ and so $\Phi(A) = I_n$.

The following lemma will be very important for our goals. It shows the general form of a $J$-positive map.

Lemma 16 Let $\Psi$ be a $J$-positive map on $M_n(\mathbb{C})$, then there is $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ positive, such that, $\Psi(\cdot) = J \Phi(J \cdot)$.

Proof We directly define $\Phi(\cdot) = J \Psi(J \cdot)$ which is evidently a linear map because $\Psi$ is linear by definition. Let $N \in M_n^+(\mathbb{C})$ be arbitrary then $JN$ is a $J$-positive matrix. Hence, taking into account that $\Psi$ is a $J$-positive map, we shall have $\Psi(JN) = JL$ for some $L \in M_n^+(\mathbb{C})$. It shows that $\Phi(N) = L$ and so $\Phi$ is a positive map. It is now easy to see that $\Psi(\cdot) = J \Phi(J \cdot)$.

Let $\Psi^{jp}$ be a $J$-positive map. Then, the positive map which was described in the lemma 16 and that it was put in correspondence with $\Psi^{jp}$ will be denoted by $\Phi^{jp}_{\Psi^{jp}}$. We refer to $\Phi^{jp}_{\Psi^{jp}}$ as the positive map associated to $\Psi^{jp}$. Below, such a correspondence shall also be indicated in the following way $\Psi^{jp} \xrightarrow{\Psi^{jp}} p \Phi$.

Theorem 17 Suppose that $\Phi(\cdot)$ is a usual completely positive map on $M_n(\mathbb{C})$ then $\Psi(\cdot) = J \Phi(J \cdot)$ is a Kraus $J$-map on $M_n(\mathbb{C})$. Conversely, if $\Psi(\cdot)$ is a Kraus $J$-map on $M_n(\mathbb{C})$ then $\Phi(\cdot) = \Psi(J \cdot) J$ is a completely positive map on $M_n(\mathbb{C})$.

Proof Since $\Phi$ is a completely positive map then it can be represented in the following form

$$
\Phi(A) = \sum_{s=1}^{\nu} V_s^* A V_s, \quad \forall A \in M_n(\mathbb{C}),
$$

(22)
for some \( \nu \in \mathbb{Z}_+ \), and some matrices \( V_s \in M_n(\mathbb{C}) \) for \( s = 1, \ldots, \nu \). This result can be found in [38], page 82. Hence, from (22) it follows that

\[
\Psi(A) = J \Phi(JA) = \sum_{s=1}^{\nu} J V_s^* J AV_s = \sum_{s=1}^{\nu} V_s^* AV_s, \quad \forall A \in M_n(\mathbb{C}).
\]

Next, we prove the converse. Suppose that \( \Psi \) is a Kraus \( J \)-map on \( M_n(\mathbb{C}) \), that is

\[
\Psi(A) = \sum_{s=1}^{\nu} V_s^* AV_s, \quad \forall A \in M_n(\mathbb{C}),
\]

where \( \nu \in \mathbb{Z}_+ \) and the \( V_s \in M_n(\mathbb{C}) \) for \( s = 1, \ldots, \nu \) depend of \( \Psi \), then

\[
\Phi(A) = \Psi(JA)J = \sum_{s=1}^{\nu} J V_s^* AV_s J = \sum_{s=1}^{\nu} (V_s J)^* A(V_s J), \quad \forall A \in M_n(\mathbb{C}),
\]

then, from the result in [38] page 82 previously referred, we conclude that \( \Phi \) is a completely positive map. \( \square \)

From now on, each \( kn \times kn \) matrix will be written in block form

\[
C = \begin{pmatrix}
C_{11} & \cdots & C_{1k} \\
\vdots & \ddots & \vdots \\
C_{k1} & \cdots & C_{kk}
\end{pmatrix},
\]

here each block \( C_{ij}, i, j = 1, \ldots, k \) is a complex \( n \times n \) matrix and \( k = 1, 2, \ldots \). In particular, for later use, we denote by \( J_k \) the following block \( kn \times kn \) diagonal matrix

\[
J_k = \begin{pmatrix}
J & O_n & \cdots & O_n \\
O_n & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & O_n \\
O_n & \cdots & O_n & J
\end{pmatrix},
\]

where \( k = 1, 2, \ldots \). It is clear that \( J_k \) is a fundamental symmetry on \( \mathbb{C}^{kn} \) for all \( k \geq 1 \), that is, \( J_k^* = J_k \) and \( J_k^2 = I_{kn} \); finally observe that \( J_1 = J \). On the other hand, recall that the \( kn \times kn \) block matrix \( J_k \) induces the following indefinite metric on \( \mathbb{C}^{kn} \)

\[
[x, y]_{J_k} = \langle J_k x, y \rangle_{\mathbb{C}^{kn}},
\]

for all \( k = 1, 2, \ldots \).
Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given. Then, it induces for each $k \in \mathbb{N}$ a map $\Phi^k : M_{kn}(\mathbb{C}) \to M_{kn}(\mathbb{C})$ which is defined in the following form

$$
\Phi^k(\mathcal{C}) = \begin{pmatrix}
\Phi(C_{11}) \cdots \Phi(C_{1k}) \\
\vdots & \ddots & \vdots \\
\Phi(C_{k1}) \cdots \Phi(C_{kk})
\end{pmatrix},
$$

where, clearly $\Phi^1 = \Phi$.

**Definition 18** We say that $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a completely $J$-positive map if, for all $k \geq 1$, the map $\Phi^k$ is $\mathcal{J}_k$-positive. In other words, $\Phi$ is completely $J$-positive if, for all $k = 1, 2, \ldots$,

$$
\Phi^k(M_{kn}^+(\mathbb{C})(\mathcal{J}_k)) \subset M_{kn}^+(\mathbb{C})(\mathcal{J}_k).
$$

**Lemma 19** Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a fixed map and $\Psi(\cdot) = J\Phi(J\cdot)$, then $\Psi^k(\cdot) = \mathcal{J}_k\Phi^k(\mathcal{J}_k\cdot)$ for all $k \geq 1$. On the other hand, if we define $\Theta(\cdot) = \Phi(J\cdot)J$, one has $\Theta^k(\cdot) = \Phi^k(J\mathcal{J}_k\cdot)\mathcal{J}_k$.

**Proof** If we choose an arbitrary block matrix $\mathcal{C} \in M_{kn}(\mathbb{C})$ then, we obtain

$$
\Psi^k(\mathcal{C}) = \begin{pmatrix}
\Psi(C_{11}) \cdots \Psi(C_{1k}) \\
\vdots & \ddots & \vdots \\
\Psi(C_{k1}) \cdots \Psi(C_{kk})
\end{pmatrix} = \begin{pmatrix}
J\Phi(JC_{11}) \cdots J\Phi(JC_{1k}) \\
\vdots & \ddots & \vdots \\
J\Phi(JC_{k1}) \cdots J\Phi(JC_{kk})
\end{pmatrix} = \mathcal{J}_k\Phi^k(\mathcal{J}_k\mathcal{C}).
$$

The proof that $\Theta^k(\cdot) = \Phi^k(J\mathcal{J}_k\cdot)\mathcal{J}_k$ is very similar; therefore, it will be omitted. □

The particular selection of $\Psi$ in the next proposition is justified by virtue of the lemma 16. We have

**Theorem 20** Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a completely positive map, that is, for all $k = 1, 2, \ldots$, $\Phi^k : M_{kn}(\mathbb{C}) \to M_{kn}(\mathbb{C})$ is a habitual positive map. Then, $\Psi(\cdot) = J\Phi(J\cdot)$ is a completely $J$-positive map.

**Proof** Since $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a completely positive map it admits a representation of the form

$$
\Phi(A) = \sum_{i=1}^{\nu} V_i^*AV_i, \quad \forall A \in M_n(\mathbb{C}).
$$

We must prove that $\Psi^k$ is $\mathcal{J}_k$-positive for all $k \in \mathbb{Z}_+$. From lemma 19, we have $\Psi^k(\cdot) = \mathcal{J}_k\Phi^k(\mathcal{J}_k\cdot)$ for all $k \in \mathbb{Z}_+$. Now, let $k$ be fixed but arbitrary and let $\mathcal{C}$ be any
matrix of order $kn$ which is $\mathcal{J}_k$-positive, that is, $0 \preceq [Cx, x]_{\mathcal{J}_k}$ for all $x \in \mathbb{C}^{kn}$. Then, for every $x^T = (x_1, \ldots, x_k)$, $y^T = (y_1, \ldots, y_k) \in \mathbb{C}^{kn}$ we obtain

$$[\Psi^k(C)x, y]_{\mathcal{J}_k} = [\mathcal{J}_k \Phi^k(\mathcal{J}_k C) x, y]_{\mathcal{J}_k} = (\mathcal{J}_k \Phi^k(\mathcal{J}_k C) x, y)_{\mathbb{C}^{kn}} = (\Phi^k(\mathcal{J}_k C)x, y)_{\mathbb{C}^{kn}}$$

$$= \left( \begin{array}{c} \Phi(JC_{11}) \cdots \Phi(JC_{1k}) \\ \vdots \cdots \cdots \\ \Phi(JC_{k1}) \cdots \Phi(JC_{kk}) \end{array} \right) x, y = \left( \begin{array}{c} \sum_{i=1}^\nu V_i^* JC_{11} V_i \cdots \sum_{i=1}^\nu V_i^* JC_{1k} V_i \\ \vdots \cdots \cdots \\ \sum_{i=1}^\nu V_i^* JC_{k1} V_i \cdots \sum_{i=1}^\nu V_i^* JC_{kk} V_i \end{array} \right) x, y$$

$$= \sum_{i=1}^\nu \left( \begin{array}{c} V_i^* JC_{11} V_i \cdots V_i^* JC_{1k} V_i \\ \vdots \cdots \cdots \\ V_i^* JC_{k1} V_i \cdots V_i^* JC_{kk} V_i \end{array} \right) x, y = \sum_{i=1}^\nu \left( \begin{array}{c} \sum_{i=1}^\nu V_i^* JC_{11} V_i \cdots \sum_{i=1}^\nu V_i^* JC_{1k} V_i \\ \vdots \cdots \cdots \\ \sum_{i=1}^\nu V_i^* JC_{k1} V_i \cdots \sum_{i=1}^\nu V_i^* JC_{kk} V_i \end{array} \right) x, y$$

$$= \sum_{i=1}^\nu \left( \begin{array}{c} JC_{11} \cdots JC_{1k} \\ JC_{k1} \cdots JC_{kk} \end{array} \right) \left( \begin{array}{c} V_i x_1 \\ \vdots \\ V_i x_k \end{array} \right), \left( \begin{array}{c} V_i y_1 \\ \vdots \\ V_i y_k \end{array} \right) = \sum_{i=1}^\nu \left( \begin{array}{c} C_{11} \cdots C_{1k} \\ C_{k1} \cdots C_{kk} \end{array} \right) \left( \begin{array}{c} V_i x_1 \\ \vdots \\ V_i x_k \end{array} \right), \left( \begin{array}{c} V_i y_1 \\ \vdots \\ V_i y_k \end{array} \right) = \sum_{i=1}^\nu \left[ \begin{array}{c} C_{11} \cdots C_{1k} \\ C_{k1} \cdots C_{kk} \end{array} \right] \left( \begin{array}{c} V_i x_1 \\ \vdots \\ V_i x_k \end{array} \right), \left( \begin{array}{c} V_i y_1 \\ \vdots \\ V_i y_k \end{array} \right) \right]_{\mathcal{J}_k}$$

now, taking into account that $C$ is a $\mathcal{J}_k$-positive matrix, we conclude that $0 \preceq [\Psi^k(C)z, z]_{\mathcal{J}_k}$ for all $z \in \mathbb{C}^{kn}$. $\square$

Next, we summarize some remarks. Observe that from the proof of the previous theorem, it follows that if $\Phi(A) = \sum_{i=1}^\nu V_i^* AV_i$ for any $A \in M_n(\mathbb{C})$, being $\Phi$ completely positive and we define the map $\Psi(\cdot) = J \Phi(J \cdot)$, then for all $k \in \mathbb{Z}_+$, one obtains

$$\Psi^k(C) = \mathcal{J}_k \Phi^k(\mathcal{J}_k C) = \sum_{i=1}^\nu \mathcal{J}_k^i C V_i, \quad \forall C \in M_{kn}(\mathbb{C}), \quad (23)$$

where for $i = 1, \ldots, \nu$

$$\mathcal{J}_k^i = \begin{pmatrix} V_i & O_n & \cdots & O_n \\ O_n & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & O_n \\ O_n & \cdots & O_n & V_i \end{pmatrix}, \quad \mathcal{J}_k^\nu = \begin{pmatrix} V_i^\nu & O_n & \cdots & O_n \\ O_n & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & O_n \\ O_n & \cdots & O_n & V_i^\nu \end{pmatrix}$$

are matrices of order $kn$ and for each $i$ the matrix $\mathcal{J}_k^i$ denotes the $\mathcal{J}_k$-adjoint of $V_i$ with respect to the indefinite metric $[\cdot, \cdot]_{\mathcal{J}_k} = \langle \mathcal{J}_k \cdot, \cdot \rangle_{\mathbb{C}^{kn}}$. On the other hand, for all
\( k \in \mathbb{Z}_+, \) (23) implies

\[
\Psi^k(\mathcal{J}_kC)\mathcal{J}_k = \sum_{i=1}^{\nu} \mathcal{J}_kC_i^*C\mathcal{V}_i\mathcal{J}_k = \sum_{i=1}^{\nu} \mathcal{D}_i^*\mathcal{D}_i = \mathcal{J}_k\Phi^k(C)\mathcal{J}_k,
\]

in which \( \mathcal{D}_i = \mathcal{V}_i\mathcal{J}_k \) for \( i = 1, \cdots, \nu. \) It shows that for all \( k \in \mathbb{Z}_+ \)

\[
\Phi^k(\cdot) = \mathcal{J}_k\Psi^k(\mathcal{J}_k \cdot) .
\] (25)

**Proposition 21** Each Kraus J-map

\[
\Psi(A) = \sum_{i=1}^{\nu} W_i^*AW_i, \quad \forall A \in M_n(\mathbb{C}),
\]

is a completely J-positive map.

**Proof** Define \( \Phi(A) = J\Psi(JA) \) for all \( A \in M_n(\mathbb{C}). \) It is clear that \( \Psi \xrightarrow{jp} \Phi \) in the sense of lemma 16. Then,

\[
\Phi(A) = \sum_{i=1}^{\nu} W_i^*AW_i, \quad \forall A \in M_n(\mathbb{C}),
\]

which implies that \( \Phi \) is a completely positive map (see [38], page 82). Now, the proposition follows from theorem 20. \( \square \)

### 3.2 Admissible Kraus J-positive maps and Quantum J-channels

We recall that an ordinary quantum operator (or ordinary quantum channel) is a completely positive matrix map \( \Phi \) which is trace-preserving, that is, \( \text{Tr} \Phi(A) = \text{Tr} A. \) Clearly, a quantum operator maps states into states, where by a state, we mean a positive matrix whose trace is 1. It is well known that \( \Phi \) is a quantum channel, if and only if (see [38] page 89)

\[
\Phi(A) = \sum_{i=1}^{\nu} V_i^*AV_i, \quad \forall A \in M_n(\mathbb{C}), \quad (26)
\]

and

\[
\sum_{i=1}^{\nu} V_i V_i^* = I_n .
\] (27)
From (26) and (27) follow that being $\Phi$ a quantum operator and $U \in M_n(\mathbb{C})$ a unitary matrix, that is, $U^*U = UU^* = I_n$, then $\Theta(\cdot) = U^*\Phi(\cdot)U$ is a quantum channel.

Suppose that $\Phi$ is a quantum operator, then we already know that $\Psi(\cdot) = J\Phi(J\cdot)$ is a completely $J$-positive map, and from (26) we obtain

$$\Psi(A) = \sum_{i=1}^{\nu} V_i^\sharp AV_i, \quad \forall A \in M_n(\mathbb{C}),$$  

(28)

and (27) implies

$$\sum_{i=1}^{\nu} V_i J V_i^\sharp = J.$$  

(29)

Condition (27) is equivalent to (29). However, in our exposition, we will maintain (29) to achieve a more related format to the new situation. In other words, we would only like to use the involution $\sharp$ in the involved equations.

A map $\Psi$ satisfying (28) and (29) is called an admissible Kraus $J$-positive map.

Let $U$ be a $J$-unitary $n \times n$ matrix, that is, $UU^\sharp = U^\sharp U = I_n$ and assume that $\Psi$ is an admissible Kraus $J$-positive map, that is, we can find $\nu \in \mathbb{Z}_+$ and a matrix vector $(V_1, \cdots, V_{\nu}) \in (M_n(\mathbb{C}))^\nu$ which depend on $\Psi$ such that (28)-(29) hold then $\Theta(\cdot) = U^\sharp \Psi(\cdot)U$ is also an admissible Kraus $J$-positive map. To prove this fact, it is enough to note that

$$(AB)^\sharp = J(AB)^*J = JB^*A^*J = JB^*J^2A^*J = B^\sharp A^\sharp.$$  

Definition 22 The map $\Psi$, is said to be a quantum $J$-operator (or quantum $J$-channel) if it is a completely $J$-positive map such that $J\Psi(J\cdot)$ is trace-preserving (it implies that $\Psi(J\cdot)J$ is also trace-preserving).

Now, we will discuss the relation between admissible Kraus $J$-positive maps and quantum $J$-channels.

Theorem 23 Assume that $\Psi$ is an admissible Kraus $J$-positive map, then it is a quantum $J$-channel.

Proof Suppose that $\Psi$ is an admissible Kraus $J$-positive map, then there is $\nu$ and there exists a matrix vector $(V_1, \cdots, V_{\nu}) \in (M_n(\mathbb{C}))^\nu$ such that (29) holds and $\Psi$ admits the representation (28). Define $\Phi(\cdot) = J\Psi(J\cdot)$, so from (28) and (29), we can recover (26) and (27) which imply that $\Phi$ is a quantum channel. Hence, theorem 20 shows that $\Psi$ is a completely $J$-positive map. On the other hand, $J\Psi(J\cdot) = \Phi(\cdot)$ is trace-preserving. □

Proposition 24 Assume that $\Psi$ is a completely $J$-positive map and $\Phi$ such that $\Psi \xrightarrow{\text{jp}} p \Phi$ then $\Phi$ is a usual completely positive map. Even more, suppose that $\Psi$ is a quantum $J$-operator, then this $\Phi$ will be an ordinary quantum channel.
This proposition can be considered a reciprocal one, in regard to proposition 20.

**Proof** From lemma 19, one knows that $\Phi^k(\cdot) = J_k \Psi^k(\mathcal{J}_k \cdot)$ for all $k \in \mathbb{Z}_+$ where by hypothesis each $\Psi^k$ is $\mathcal{J}_k$-positive. It implies that $\Psi^k(\mathcal{J}_k N) \in M_{kn}^+(\mathbb{C})(\mathcal{J}_k)$ for all $N \in M_{kn}^+(\mathbb{C})$ so $\Psi^k(\mathcal{J}_k N) = \mathcal{J}_k L$ for some $L \in M_{kn}^+(\mathbb{C})$. It follows that $\Phi^k$ is positive on $M_{kn}^+(\mathbb{C})$. Indeed, $\Phi^k(N) = L$, in other words, $\Phi^k(M_{kn}^+(\mathbb{C})) \subset M_{kn}^+(\mathbb{C})$. On the other hand, being $\Psi$ a quantum $J$-operator, by definition, it is a completely $J$-positive, so from the first part of our proof it follows that $\Phi$ is a completely positive map. Finally, since $J \Psi(J \cdot)$ is preserving trace then $\Phi(\cdot) = J \Psi(J \cdot)$ is a quantum operator. The proposition has been proved. \[\Box\]

**Proposition 25** Each completely $J$-positive map $\Psi$ is a Kraus $J$-map. Even more, from (23) it follows that each $\Psi^k$ is a Kraus $\mathcal{J}_k$-map. Finally, every quantum $J$-channel is an admissible Kraus $J$-positive map.

**Proof** Let $\Phi$ be the completely positive map such that $\Psi \xrightarrow{jp} \Phi$ (we are taking into account the previous proposition), then

$$\Phi(A) = \sum_{i=1}^{v} V_i^* A V_i, \quad \forall A \in M_n(\mathbb{C}),$$

and since $\Phi(\cdot) = J \Psi(J \cdot)$, the statement follows. If, on the contrary, we suppose something stronger, let’s say, that $\Psi$ is a quantum $J$-channel, then this $\Phi$ is now a quantum operator, so the matrix vector $(V_1, \cdots, V_v)$ satisfies the condition

$$\sum_{i=1}^{v} V_i V_i^* = I_n,$$

from which we easily obtain (29). It concludes the proof. \[\Box\]

**Corollary 26** The map $\Psi$ is completely $J$-positive if and only if it is a Kraus $J$-map.

**Proof** It is a consequence of propositions 21 and 25. \[\Box\]

The following conclusive corollary represents a summary of the results of this section.

**Corollary 27** $\Psi$ is a quantum $J$-operator, if and only if $\Psi$ is an admissible Kraus $J$-positive map.

**Proof** The statement follows from theorem 23 and proposition 25. \[\Box\]

Now, we relate the two sections by means of the following proposition

**Proposition 28** A quantum $J$-operator maps quantum $J$-states of unitary origin into quantum $J$-states of unitary origin.
Proof We recall that a quantum $J$-state of unitary origin $B$ satisfies the following two properties: $B$ is $J$-positive and $\text{Tr} BJ = 1$. Let $\Psi$ be an arbitrary quantum $J$-channel. Then, there is a matrix vector $(V_1, \ldots, V_v)$ such that, in particular, for any quantum $J$-state $B$ of unitary origin

$$
\Psi(B) = \sum_{i=1}^v V_i^\flat B V_i.
$$

(30)

Hence, it shows that $\Psi(B)$ is a $J$-positive matrix for all quantum $J$-state $B$ of unitary origin. Note also that from (30), it follows

$$
\Psi(B)J = \sum_{i=1}^v (JV_i^*J)(BJ)(JViJ) = \sum_{i=1}^v Z_i^*(BJ)Z_i,
$$

where $Z_i = JV_iJ$ for $i = 1, \ldots, v$. Define $\Phi(M) = \sum_{i=1}^v Z_i^*MZ_i$ for all $M \in M_n(\mathbb{C})$. Obviously, it is a completely positive map. We claim that actually $\Phi$ is a quantum channel. Indeed, since $\Psi$ is quantum $J$-operator, we know that

$$
\sum_{i=1}^v V_i JV_i^\flat = \sum_{i=1}^v V_i J(JV_i^*J) = J,
$$

thus

$$
\sum_{i=1}^v (JV_iJ)(JV_i^*J) = \sum_{i=1}^v Z_i Z_i^* = I_n,
$$

so $\Phi$ is a quantum channel. Note also that $\Psi(B)J = \Phi(BJ)$, it implies that $\text{Tr} \Psi(B)J = \text{Tr} \Phi(BJ) = \text{Tr} BJ = 1$. Then, $\Psi(B)$ is a quantum $J$-state of unitary origin. $\square$

Theorem 29 Let $\Psi$ be a completely $J$-positive map, that is, there is a matrix vector $(V_1, \ldots, V_l) \in M_n(\mathbb{C})$ such that

$$
\Psi(A) = \sum_{i=1}^l V_i^\flat AV_i, \quad \forall A \in M_n(\mathbb{C}).
$$

Then, $\Psi$ is trace preserving, if and only if $\sum_{i=1}^l V_i V_i^\flat = I_n$.

Proof The proof is similar to the case $J = I_n$, that is, when $A^\sharp = A^*$ for every $A \in M_n(\mathbb{C})$. This is based on two well known facts: first, the trace is invariant under cyclic permutations and, second, $S_2$ is a Hilbert space with respect to the inner product $\langle A, B \rangle = \text{Tr} A^*B$. $\square$
Remark 30 Observe that the completely $J$-positive maps, which are trace preserving, transform quantum $J$-states of $J$-unitary origin into quantum $J$-states of $J$-unitary origin.

4 A generalization of the previous section

In this part of our work, results of the previous section are generalized. Moreover, the fact between the working conditions, which leads to an essential change, is that all maps $\Psi$ considered below transform $J_1$-positive matrices into $J_2$-positive matrices, where $J_2 \neq \pm J_1$ are both fundamental symmetries of $M_n(\mathbb{C})$. Next, we generally keep the same notation of the previous section.

Let $J_1$ and $J_2$ be two different matrices belonging to $M_n(\mathbb{C})$ which are fundamental symmetries. Thus, they introduce two different structures of indefinite metric space on $\mathbb{C}^n$, denoted by $(\mathbb{C}^n, [\cdot, \cdot]_1)$ and $(\mathbb{C}^n, [\cdot, \cdot]_2)$ respectively, where $[\cdot, \cdot]_1 = \langle J_1 \cdot, \cdot \rangle_{\mathbb{C}^n}$ and $[\cdot, \cdot]_2 = \langle J_2 \cdot, \cdot \rangle_{\mathbb{C}^n}$. Suppose that $A$ and $B$ are two matrices such that

$$[Ax, y]_1 = [x, By]_2, \quad \forall x, y \in \mathbb{C}^n,$$

then, we say that $B$ is the generalized indefinite adjoint of $A$ and it is denoted by $A^b$.

Clearly, $A^b = J_2 A^* J_1$.

We shall make a few of remarks about operation $\cdot^b$.

Theorem 31 The following statements are true

1. $(A_1 + A_2)^b = A_1^b + A_2^b$, for all $A_1, A_2 \in M_n(\mathbb{C})$.
2. $(\lambda A)^b = \overline{\lambda} A^b$, for any $\lambda \in \mathbb{C}$ and all $A \in M_n(\mathbb{C})$.
3. $(A^b)^b = A$ for all $A \in M_n(\mathbb{C})$ if and only if $J_2 = \pm J_1$.

Proof The first two statements are obvious. We have $(A^b)^b = J_2 J_1 A J_2 J_1$ thus $(A^b)^b = A$, if and only if $J_2 J_1 = \pm I_n$ so, from the uniqueness of the inverse in an associative algebra, the latter could be true, if and only if $J_2 = \pm J_1$. \ \Box

From the previous theorem it follows that the operation $\cdot^b$ is not an involution if $J_2 \neq \pm J_1$ (which is in correspondence with our initial assumption). Now, we introduce the concept of completely positive map in indefinite metric.

Definition 32 We say that $\Lambda : (\mathbb{C}^n, [\cdot, \cdot]_1) \rightarrow (\mathbb{C}^n, [\cdot, \cdot]_2)$ is positive in indefinite metric if it is a linear map. Moreover, $\Lambda$ maps $J_1$-positive matrices into $J_2$-positive matrices, that is, $\Lambda(M_+^+(\mathbb{C})J_1)) \subset M_+^+(\mathbb{C})(J_2)$. It is said to be completely positive in indefinite metric, if $\Lambda^k$ is positive in indefinite metric for all $k \in \mathbb{N}$. Here, for an arbitrary linear map $\Theta$ and for all $k \in \mathbb{N}$ the $k$-th block map $\Theta^k$ is defined in a similar way it was done in the previous section.

For a fixed $k$, that the matrix $\Lambda^k$ is positive in indefinite metric means for us the following $\Lambda^k(M_+^{kn}(\mathbb{C})(J_1^k)) \subset M_+^{kn}(\mathbb{C})(J_2^k)$ with respect to the indefinite metrics $[\cdot, \cdot]_i^k = \langle J_i^k \cdot, \cdot \rangle_{\mathbb{C}^{kn}}$ for $i = 1, 2$; where, as in the last section, $J_i^k$ is the following
block diagonal fundamental symmetry matrix of order $kn \times kn$

\[
\mathcal{J}^k_i = \begin{pmatrix}
J_i & O_n & \cdots & O_n \\
O_n & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & O_n \\
O_n & \cdots & O_n & J_i
\end{pmatrix},
\]

being $i = 1, 2$ and $k = 1, 2, \ldots$.

We have

**Lemma 33** The map $\Lambda : (\mathbb{C}^n, \langle \cdot, \cdot \rangle_1) \rightarrow (\mathbb{C}^n, \langle \cdot, \cdot \rangle_2)$ is positive in indefinite metric, if and only if there is a regular positive map $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ such that $\Lambda(\cdot) = J_2 \Phi (J_1 \cdot)$.

**Proof** Let $\Phi$ be a usual positive map of $M_n(\mathbb{C})$ into $M_n(\mathbb{C})$ and define $\Lambda(\cdot) = J_2 \Phi (J_1 \cdot)$, then $\Lambda$ is clearly linear. We know that $B$ is a $J_1$-positive matrix, if and only if $J_1 B$ is a positive matrix, hence for all $x \in \mathbb{C}^n$ and any matrix $B$, which is $J_1$-positive

\[
\langle \Lambda(B)x, x \rangle_2 = \langle J_2\Lambda(B)x, x \rangle_{\mathbb{C}^n} = \langle \Phi(J_1B)x, x \rangle_{\mathbb{C}^n} \geq 0,
\]

it shows that for all $B \in M^+_n(\mathbb{C})(J_1)$ one has $\Lambda(B) \in M^+_n(\mathbb{C})(J_2)$, that is, $\Lambda$ is a positive map in indefinite metric. On the other hand, if $\Lambda$ is a positive map in indefinite metric one can prove that $\Phi(\cdot) = J_2 \Lambda(J_1 \cdot)$ is a positive map (the proof is similar to the one of lemma 16) and this leads us to the other implication. \qed

If $\Psi$ is a positive map in indefinite metric, notation $\Psi \Rightarrow \Phi$ indicates that $\Phi$ is the positive map associated to $\Psi$ through the lemma 33, that is, $\Lambda(\cdot) = J_2 \Phi (J_1 \cdot)$. The proofs of the following facts are similar to those when $J_2 = J_1$, hence they are omitted. Suppose that $\Psi \Rightarrow \Phi$ where $\Phi$ is a completely positive map, then

1. $\Psi$ admits the following representation

\[
\Psi(A) = \sum_{i=1}^{\nu} V_i^b A V_i, \quad \forall A \in M_n(\mathbb{C}). \tag{32}
\]

2. $\Psi$ is completely positive in indefinite metric. This is because one has $\Psi^k(\cdot) = J_2^k \Phi^k (J_1^k \cdot)$ for all $k \in \mathbb{N}$.

3. Suppose that, additionally, $\Phi$ is trace preserving, that is, $\Phi$ is a usual quantum channel, then $Tr J_2 \Psi(J_1 A) = Tr A$.

**Remark 34** As a consequence of the previous observations, it follows that every linear map of the form (32) is completely positive in indefinite metric.

Next, we develop some results of Stinespring type in indefinite metric spaces. Suppose that we have a linear operator $\pi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ such that $\pi(MN) =$
\[ \pi(M) \pi(N) = (\pi(M))^* \pi(N) \] for all \( M, N \in M_n(\mathbb{C}) \). In this case, we say that \( \pi \) is a representation of \( M_n(\mathbb{C}) \) in itself. Observe that \( \pi \) transforms definite positive matrices into definite positive matrices. It follows from the following fact \( \pi(M^* M) = \pi(M^*) \pi(M) = (\pi(M))^* \pi(M) \).

Probably, the following result belongs to the folklore of the subject, however, its proof has been included to guarantee this paper to be more comprehensive. We have

**Lemma 35** It turns out that \( \pi^k : M_{kn}(\mathbb{C}) \to M_{kn}(\mathbb{C}) \) is a positive map for all \( k \in \mathbb{N} \).

**Proof** It is enough to prove that \( \pi^k(M^*) = (\pi^k(M))^* \) and \( \pi^k(MN) = \pi^k(M) \pi^k(N) \). For two arbitrary block matrices \( M, N \in M_{kn}(\mathbb{C}) \), one obtains

\[
(\pi^k(MN))_{ij} = \pi(\sum_{s=1}^{k} M_{is} N_{sj}) = \sum_{s=1}^{k} \pi(M_{is}) \pi(N_{sj})
\]

if \( 1 \leq i, j \leq k \). Thus, \( \pi^k(MN) = \pi^k(M) \pi^k(N) \). On the other hand, for \( 1 \leq i, j \leq k \)

\[
(\pi^k(M^*))_{ij} = \pi(M^*_{ji}) = (\pi(M_{ji}))^* = ((\pi^k(M))^*)_ij,
\]

it shows that \( \pi^k(M^*) = (\pi^k(M))^* \). The lemma is proved \( \square \)

**Remark 36** Observe that we can obtain a similar result assuming that \( \pi \) satisfies the following conditions \( \pi(MN) = \pi(M) \pi(N) \) and \( \pi(M^*) = (\pi(M))^* \) for all \( M, N \in M_n(\mathbb{C}) \). In which case \( \pi \) is said to be also a representation of \( M_n(\mathbb{C}) \) in itself.

**Example 3** Suppose that \( U \) is a unitary matrix then \( \pi_1(A) = UA U^* \) and \( \pi_2(A) = U A^* U^* \) are representations of \( M_n(\mathbb{C}) \) in itself.

We may arrive to the following interesting result in indefinite metric spaces

**Theorem 37** Let \( \pi \) be a representation of \( M_n(\mathbb{C}) \) in itself and \( V \in M_n(\mathbb{C}) \), then \( \Psi(\cdot) = J_2 V^* \pi(J_1 \cdot) V \) is completely positive in indefinite metric.

**Proof** First of all, observe that for any \( k \in \mathbb{N} \) and all block matrix \( C \) of order \( kn \times kn \), we have

\[
\Psi^k(C) = J_2^k V_k^* \pi^k(J_1^k C) V_k,
\]

where, as above,

\[
V_k = \begin{pmatrix} V & O_n & \cdots & O_n \\ O_n & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & O_n \\ O_n & \cdots & O_n & V \end{pmatrix}, \quad V_k^* = \begin{pmatrix} V^* & O_n & \cdots & O_n \\ O_n & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & O_n \\ O_n & \cdots & O_n & V^* \end{pmatrix}.
\]
are block diagonal matrices of order $kn \times kn$. We claim that $\Psi^k(\cdot) : (\mathbb{C}^{kn}, [, \cdot, \cdot]_1^k) \rightarrow (\mathbb{C}^{kn}, [, \cdot, \cdot]_2^k)$ is positive in indefinite metric for all $1 \leq k$. In fact, let $k$ be a fixed positive integer but arbitrary. Then, from the lemma 35 it follows that $\Phi_k(\cdot) = \mathcal{V}_k^* \pi^k(\cdot) \mathcal{V}_k$ is a usual positive map on $M_{kn}(\mathbb{C})$. On the other hand, $\Psi^k(\cdot) = \mathcal{J}_2^k \Phi_k(\mathcal{J}_1^k)$, which implies that $\Psi^k$ is positive in indefinite metric for all $1 \leq k$ due to lemma 33. Thus, the map $\Psi$ is completely positive in indefinite metric.

As above, suppose that $J_1$ and $J_2$ are fundamental symmetries. For $i = 1, 2$, define $A^i = J_iA^*J_j$ for all $A \in M_n(\mathbb{C})$. Then, as it was already mentioned the operations $\pi_i$ are involutions on $M_n(\mathbb{C})$.

**Definition 38** We say that $\pi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a representation in indefinite metric of $M_n(\mathbb{C})$ in itself, if it is a linear map, such that $\pi(A J_1 B) = \pi(A) J_2 \pi(B)$ and $\pi(A^i_1) = (\pi(A))^i_2$.

We give an example

**Example 4** Suppose that $\pi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a linear map such that

$$\pi(AB) = \pi(A)\pi(B), \quad \pi(A^*) = (\pi(A))^*, \quad \pi(J_1) = J_2, \quad \forall A, B \in M_n(\mathbb{C}),$$

then $\pi$ is a representation in indefinite metric of $M_n(\mathbb{C})$ in itself. In fact, we have $\pi(A J_1 B) = \pi(A)\pi(J_1)\pi(B) = \pi(A) J_2 \pi(B)$ for all $A, B \in M_n(\mathbb{C})$. On the other hand, $\pi(A^i_1) = \pi(J_1 A^*J_1) = \pi(J_1)\pi(A^*)\pi(J_1) = J_2(\pi(A))^*J_2 = (\pi(A))^i_2$ for any $A \in M_n(\mathbb{C})$.

**Remark 39** Observe that if $\pi$ is a representation in indefinite metric of $M_n(\mathbb{C})$ in itself, then it is positive in indefinite metric. Indeed, according to the previous definition, we have $\pi(B^i_1 J_1 B) = \pi(B^i_1) J_2 \pi(B) = (\pi(B))^i_2 J_2 \pi(B)$ for all $B \in M_n(\mathbb{C})$. On the other hand, if $A = A^i_1$, then, $\pi(A) = \pi(A^i_1) = (\pi(A))^i_2$. It implies that $\pi$ maps $J_1$-positive matrices into $J_2$-positive matrices.

We have

**Theorem 40** Any representation $\pi$ in indefinite metric of $M_n(\mathbb{C})$ in itself is a completely positive map in indefinite metric.

**Proof** We should prove that for each $k \in \mathbb{Z}_+$ the map $\pi^k$ satisfies the following properties $\pi^k(C \mathcal{J}_1^k D) = \pi^k(C) \pi^k D = \pi(C \mathcal{J}_2^k \pi^k(D)$ and $\pi^k(C^2(\mathcal{J}_1^k)) = (\pi^k(C))^2(\mathcal{J}_2^k)$ for all $C, D \in M_{kn}(\mathbb{C})$ where $C^2(\mathcal{J}_1^k) = \mathcal{J}_1^k C^* \mathcal{J}_1^k$ for $i = 1, 2$. Consider a fixed $k$ but arbitrary, if $1 \leq i, j \leq k$ then, for two block matrices $\mathcal{M}, \mathcal{N} \in M_{kn}(\mathbb{C})$ we have

$$\begin{align*}
(\pi^k(\mathcal{M} \mathcal{J}_i^k \mathcal{N}))(ij) &= \pi(\sum_{s=1}^{k} M_{is} J_1 N_{sj}) = \sum_{s=1}^{k} \pi(M_{is} J_1 N_{sj}) = \sum_{s=1}^{k} \pi(M_{is}) J_2 \pi(N_{sj}) \\
&= \sum_{s=1}^{k} (\pi^k(\mathcal{M}))_{is} J_2 (\pi^k(\mathcal{N}))_{sj} = (\pi^k(\mathcal{M}) \mathcal{J}_2^k \pi^k(\mathcal{N}))(ij),
\end{align*}$$

$\square$ Springer
which implies that $\pi^k(\mathcal{M} J^k_1 \mathcal{N}) = \pi^k(\mathcal{M}) J^k_2 \pi^k(\mathcal{N})$. Moreover,

$$(\pi^k(\mathcal{M} \natural(\mathcal{J}^k_1)))_{ij} = (\pi^k(\mathcal{J}^k_1 \mathcal{M}^* \mathcal{J}^k_1))_{ij} = \pi(J_1 M^*_i J_1) = \pi(M^*_i) = (\pi(M))_{ij}^* = J_2(\pi(M))_{ij} = J_2(\pi(\mathcal{M}))_{ij} = (J_2(\pi(\mathcal{M}^* \mathcal{J}^k_2)))_{ij},$$

thus $\pi^k(\mathcal{M} \natural(\mathcal{J}^k_1)) = (\pi^k(\mathcal{M})) \natural(\mathcal{J}^k_2)$ for all $\mathcal{M} \in M_n(\mathbb{C})$. The theorem follows from remark 39. 

**Definition 41** We say that a linear map $\Psi$ is a quantum $(J_1, J_2)$-channel if $\Psi$ can be represented in the form (32) and moreover it is trace preserving.

**Theorem 42** A linear map $\Psi$ is a quantum $(J_1, J_2)$-channel, if and only if it has the form (32) and the matrix vector $(V_1, \cdots, V_\nu)$ which arises from this representation satisfies the property

$$\sum_{i=1}^{\nu} V_i^* V_i = I_n.$$

**5 Conclusions**

To conclude this paper, two open problems are being proposed, in which the authors of the present article are currently making progress:

1. The study of the dynamics of linear operators is an important and popular topic. For example, in this sense, the reader can browse a classic reference, like book [9]. In particular, from this point of view, completely positive maps, have not been exempt from analysis. In this direction, they were object of interest in many works. For example, [33], [12] and references therein. Specifically, about 15 years ago, the study of quantum channels fixed points was subjected to strong research. On the other hand, it is well known that the multiplicative properties of quantum channels play a central role in order to obtain these results. We suggest to study the dynamics of the quantum channels defined on spaces with an indefinite metric.

2. Inspired by the first problem, we consider that it would be interesting and feasible to reveal the relationship between quantum $J$-channels and the Jordan algebras (for the usual case $J = I_n$, we suggest to see the book [37]).

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