Perfect colorings of hypergraphs

A. A. Taranenko*

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Abstract

Perfect colorings (equitable partitions) of graphs are extensively studied, while the same concept for hypergraphs attracts much less attention. The aim of this paper is to develop basic notions and properties of perfect colorings for hypergraphs. Firstly, we introduce a multidimensional matrix equation for perfect colorings of hypergraphs and compare this definition with a standard approach based on the incidence graph. Next, we show that the eigenvalues of the parameter matrix of a perfect coloring are eigenvalues of the multidimensional adjacency matrix of a hypergraph. We consider coverings of hypergraphs as a special case of perfect colorings and prove a theorem on the existence of a common covering of two hypergraphs. As an example, we show that a \( k \)-transversal in a hypergraph corresponds to a perfect coloring and calculate its parameters. At last, we find all perfect 2-colorings of the Fano’s plane hypergraph and compute some eigenvalues of this hypergraph.

Keywords: perfect coloring; eigenvalues of hypergraphs; coverings of hypergraphs; multidimensional adjacency matrix.

MSC2020: 05C15; 05C50; 05C65.

1 Introduction

A coloring of vertices of a graph is called perfect if a color of a vertex uniquely defines colors of adjacent vertices.

Perfect colorings of graphs have been studied for quite a long time and arose in the literature under different names. One of the most known of them is “equitable partition” that was introduced by Delsarte [8]. In book [7] objects equivalent to perfect colorings are named as divisors of graphs. An algebraic definition of perfect colorings firstly appeared in book [11]. At last, in [12] one finds some information on graph coverings that can be considered as perfect colorings of a special type.

Graph perfect colorings have plenty of applications in coding theory and the theory of combinatorial designs. There are many results on enumeration or characterization of perfect colorings in certain families of graphs (see, e.g., survey papers [2,3]).

Meanwhile, the specifics of perfect colorings in hypergraphs are hardly studied. As it was noted in “Handbook of combinatorics” [13 Ch. 31.2], a perfect coloring of a hypergraph is equivalent to a certain perfect coloring of its incidence graph. In [19], it was shown that some combinatorial designs correspond to perfect colorings of hypergraphs. In particular, every transversal in a regular uniform hypergraph is a perfect 2-coloring.

Reduction of perfect colorings of hypergraphs to colorings of graphs implies immediately some of their properties. On the other hand, this approach uses redundant parameters of a coloring of hyperedges that can be deduced from a coloring of vertices.

*Sobolev Institute of Mathematics, Novosibirsk, Russia; taa@math.nsc.ru
The main aim of this paper is to propose and develop another perspective for perfect colorings of hypergraphs based on their multidimensional adjacency matrices.

The paper is structured as follows.

Section 2 is preliminary. It provides all definitions and auxiliary results on hypergraphs, their adjacency and incidence matrices, multidimensional matrices and their products, and eigenvalues of multidimensional matrices and hypergraphs. In this section we also briefly review the main results on perfect colorings in graphs that will be later generalized for hypergraphs.

Section 3 is the main part of the paper. Firstly, we consider in detail the definition of a hypergraph perfect coloring as a perfect coloring of its incidence graph. Although the concept of the hypergraph perfect coloring was introduced earlier, it is a debatable question how to define their parameters so that they still have the most nice properties of the graph case. Here we pay special attention to this question because it was omitted in previous works. Also we deduce some simple corollaries from the approach based on the incidence graph that were not given before. In particular, we get necessary and sufficient conditions for a pair of matrices to be parameters of a perfect coloring of some hypergraph (Theorem 3) and establish the analogue of the Weisfeiler–Leman–Vizing theorem on the refinement of perfect coloring (Theorem 4).

In the second part of Section 3 we propose a new definition of a hypergraph perfect coloring in terms of its multidimensional adjacency matrix. In Theorem 5 we show that for uniform hypergraphs this definition is equivalent to the previous one and introduce the multidimensional parameter matrix of a perfect coloring. We also prove that the multidimensional parameter matrix can be symmetrized (Theorem 6) and connect it with the 2-dimensional parameter matrices of the coloring (Corollary 1). The major asset of the multidimensional parameter matrix is that all its eigenvalues are also the eigenvalues of the adjacency matrix of a hypergraph (Theorem 7). We obtain it as a corollary from a more general statement for multidimensional matrices (Theorem 7). These results allow one to find several eigenvalues of a multidimensional matrix (or hypergraph) on the basis of eigenvalues of certain smaller matrices (hypergraphs).

In Section 4, we consider coverings of hypergraphs as a special case of perfect colorings. This allows us to easily get the following properties of coverings: characterization of hypergraphs that cover a given one (Theorem 9), preservation of parameters of a perfect coloring in coverings (Theorem 10), and inclusion of the spectrum of hypergraphs into the spectrum of their coverings (Theorem 10). The last result was recently obtained in [23] by another method. At the end of the section we generalize Leighton’s theorem [15] on common coverings from graphs to hypergraphs (Theorem 12 and Corollary 4).

At last, in Section 5 we consider several examples of hypergraph perfect colorings. We find the parameter matrices of a hypergraph transversal as a perfect 2-coloring and calculate their eigenvalues (Theorem 13). Moreover, we show that multidimensional parameters may distinguish hypergraphs with no transversals, while the 2-dimensional approach does not. Finally, we compute the parameters of perfect 2-colorings of regular 3-uniform hypergraphs and find all perfect 2-colorings of the Fano’s plane hypergraph and some of its eigenvalues.

## 2 Definitions and preliminaries

A hypergraph $\mathcal{G}$ is a pair of sets $X = X(\mathcal{G})$ and $E = E(\mathcal{G})$, $|X| = n$, $|E| = m$. $X$ is called a set of vertices and $E$ is a set of hyperedges, each $e \in E$ is some (nonempty) subset of $X$. A vertex $x \in X$ is said to be incident to a hyperedge $e$ if $x \in e$. Vertices $x$ and $y$ are adjacent if there is $e \in E$ incident to both of them. If $E$ is a multiset, then $\mathcal{G}$ is said to be a multihypergraph.

A hypergraph is connected if there is an interchanging sequence of incident vertices and hyperedges that connects a vertex $x$ with any other vertex $y$. Such a sequence is called a Berge path from $x$ to $y$.

A hypergraph $\mathcal{G}$ is said to be $d$-uniform if every hyperedge of $\mathcal{G}$ consists of $d$ vertices. Simple
graphs are exactly 2-uniform hypergraphs. A $d$-uniform hypergraph is \textit{complete} if its hyperedges are all $d$-element subsets of the vertex set.

A \textit{degree} of a vertex $x$ is the number of hyperedges containing $x$. A hypergraph $G$ is \textit{$r$-regular} if every vertex of $G$ has degree $r$.

A hypergraph $G$ is said to be \textit{$d$-partite} if there is a partition of its vertex set into $d$ disjoint subsets (parts) such that each hyperedge of $G$ contains no more than one vertex from each part.

A \textit{$k$-transversal} in a hypergraph $G = (X, E)$ is a set of vertices $Y \subseteq X$ such that every hyperedge of $G$ contains exactly $k$ vertices from $Y$. A \textit{$k$-factor} in $G$ is a set of hyperedges $U \subseteq E$ such that every vertex of $G$ is incident to exactly $k$ hyperedges from $U$. 1-factors are said to be \textit{perfect matchings} and 1-transversals are known as \textit{transversals}.

Transversals and factors are dual in the following sense. For every hypergraph $G$ one can consider a dual hypergraph $G^*$ obtained from $G$ by interchanging sets of vertices and hyperedges and preserving the incidence relations between them. Then every $k$-transversal in $G$ is a $k$-factor in $G^*$ and vice versa.

Every hypergraph $G = (X, E)$ corresponds to the \textit{incidence graph} $G'$ that is a bipartite graph with parts $X$ and $E$: $x \in X$ and $e \in E$ are adjacent in $G'$ if and only if they are incident in $G$. Incidence graph is also known as the Levi graph or the bipartite representation of a hypergraph.

### 2.1 Matrices and eigenvalues of hypergraphs

Let $A = \{a_{i,j}\}, i = 1, \ldots, n, j = 1, \ldots, m$ be a matrix of size $n \times m$. If $n = m$ then $A$ is a matrix of order $n$.

We will say that a matrix $A$ of size $n \times m$ is a \textit{block matrix} with blocks $B_{i,j}$ of sizes $n_i \times m_j$, $i = 1, \ldots, k$, $j = 1, \ldots, l$, $\sum_{i=1}^{k} n_i = n$, $\sum_{j=1}^{l} m_j = m$, if (after, possibly, appropriate permutations of rows and columns) the matrix $A$ can be presented in a form

$$
A = \begin{pmatrix}
B_{1,1} & \cdots & B_{1,l} \\
\vdots & \ddots & \vdots \\
B_{k,1} & \cdots & B_{k,l}\end{pmatrix}.
$$

For shortness, we write $A = \{B_{i,j}\}$.

Let $[n] = \{1, \ldots, n\}$. A \textit{$d$-dimensional matrix} $A$ of order $n$ is an array of entries $(a_\alpha)$, $a_\alpha \in \mathbb{R}$, indexed by tuples $\alpha \in [n]^d$, $\alpha = (\alpha_1, \ldots, \alpha_d)$. Vectors can be considered as 1-dimensional matrices, and usual matrices have 2 dimensions.

Let $A$ be a $d$-dimensional matrix of order $n$. A \textit{hyperplane} $\Gamma$ of \textit{direction} $i$ in $A$ is a $(d - 1)$-dimensional submatrix obtained by fixing a value of index component $\alpha_i$ and letting the other $d - 1$ components vary.

A matrix $A$ is said to be \textit{symmetric} if for every permutation $\sigma \in S_d$ and for every index $\alpha \in [n]^d$ we have $a_\alpha = a_{\sigma(\alpha)}$, where $\sigma(\alpha) = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(d)})$. In other words, the matrix $A$ does not change under permutations of directions of hyperplanes.

The \textit{$d$-dimensional identity matrix} $I$ is the matrix with entries $i_\alpha = 1$ if $\alpha_1 = \cdots = \alpha_d$ and $i_\alpha = 0$ otherwise.

Let $G = (X, E)$ be hypergraph on $n$ vertices with $m$ hyperedges. The \textit{incidence matrix} $B$ of the hypergraph $G$ is a matrix of size $n \times m$ such that $b_{x,e} = 1$ if a vertex $x$ is incident to a hyperedge $e$ and $b_{x,e} = 0$ otherwise.

If $G$ is a uniform hypergraph, then we can define its multidimensional adjacency matrix. The \textit{adjacency matrix} $A$ of a $d$-uniform hypergraph $G = (X, E)$ on $n$ vertices is a $d$-dimensional matrix of order $n$ in which entries $a_\alpha$ with index $\alpha = (x_1, \ldots, x_d) \in E$ are equal to $(d - 1)!^{-1}$ and all other entries of $A$ are 0. Such scale of entries of the matrix $A$ is taken for the sake of compactness of future expressions. By definition, the adjacency matrix $A$ is symmetric.
If \( G \) is a graph, then an \textit{eigenvalue} of \( G \) is an eigenvalue of its 2-dimensional adjacency matrix. For hypergraphs, there are several ways to define eigenvalues, the most popular approaches use the following:

1. maximization of some multilinear form generated by a hypergraph;
2. eigenvalues of the 2-dimensional signless Laplacian matrix \( BB^T \);
3. eigenvalues of the multidimensional adjacency matrix \( A \).

In this paper we will utilize the last of them. For a brief survey on other definitions of hypergraph eigenvalues see, e.g., the introduction of [5]. To define eigenvalues of multidimensional matrices, we use the following operation.

Let \( A \) be a \( d \)-dimensional matrix of order \( n \) and \( B \) be a \( t \)-dimensional matrix of the same order. Define the \textit{product} \( A \circ B \) to be the \((d - 1)(t - 1) + 1\)-dimensional matrix \( C \) of order \( n \) with entries

\[
c_{i_1, \beta_2, \ldots, \beta_d} = \sum_{i_2=1}^{n} \cdots \sum_{i_d=1}^{n} a_{i_1, i_2, \ldots, i_d} \cdot b_{i_2, \beta_2} \cdots b_{i_d, \beta_d},
\]

where indices \( \beta_2, \ldots, \beta_d \in [n]^{\delta - 1}, i \in [n] \).

The following properties of the product \( \circ \) can be derived directly from the definition or found in [21].

\textbf{Proposition 1.} Let \( A, B, \) and \( C \) be multidimensional matrices of appropriate sizes.

1. If dimensions of matrices \( A \) and \( B \) do not exceed 2, then \( \circ \) is a standard matrix (dot) product: \( A \circ B = AB \).
2. The product of multidimensional matrices is associative: \((A \circ B) \circ C = A \circ (B \circ C) \).
3. If \( \lambda \in \mathbb{C} \) and \( A \) is a \( d \)-dimensional matrix, then \( A \circ (\lambda B) = \lambda^{d-1} (A \circ B) \).

Let \( A \) be a \( d \)-dimensional matrix of order \( n \). We will say that \( \lambda \) is an \textit{eigenvalue} of \( A \) if there is a vector \( x = (x_1, \ldots, x_n) \) such that \( A \circ x = \lambda I \circ x \), where \( I \) is a \( d \)-dimensional identity matrix of order \( n \) and \( I \circ x \) is a vector \((x_1^{d-1}, \ldots, x_n^{d-1})\). The vector \( x \) is called an \textit{eigenvector} corresponding to the eigenvalue \( \lambda \), a pair \((\lambda, x)\) is said to be an \textit{eigenpair} for \( A \).

If \( G \) is a \( d \)-uniform hypergraph with the \( d \)-dimensional adjacency \( A \), then the eigenvalues of the matrix \( A \) are said to be the \textit{eigenvalues} of \( G \), the eigenvectors of \( A \) are the \textit{eigenvectors} of \( G \).

The set \( V(\lambda) \subset \mathbb{C}^n \) of all eigenvectors for an eigenvalue \( \lambda \) is a complex algebraic variety, and the \textit{geometric multiplicity} of the eigenvalue \( \lambda \) is the dimension of \( V(\lambda) \).

The present definition of eigenvalues and eigenvectors for tensors was proposed in 2005 by Lim [17] and was studied for symmetric matrices by Qi [20]. They used the theory of determinants of multidimensional matrices and resultants of multilinear systems developed in book [10] by Gelfand, Kapranov, and Zelevinsky. In particular, they proved that all eigenvalues of a multidimensional matrix \( A \) are roots of its characteristic polynomial \( \varphi_A \).

\textbf{Theorem 1} [20]. For a \( d \)-dimensional matrix \( A \) of order \( n \), there is a characteristic polynomial \( \varphi_A \) of degree \( n(d - 1)^{n-1} \) such that a number \( \lambda \in \mathbb{C} \) is an eigenvalue of \( A \) if and only if \( \lambda \) is a root of \( \varphi_A \).

The \textit{algebraic multiplicity} of an eigenvalue \( \lambda \) is said be its multiplicity as a root of the characteristic polynomial. It is well known that for 2-dimensional matrices geometric and algebraic multiplicities coincide. Unfortunately, in the multidimensional case it is not true. For more information on relations between these multiplicities see [14].
The study of eigenvalues of multidimensional matrices and hypergraphs grows now extensively, so we mention here only several papers. In [6] Cooper and Dutle proved some properties of hypergraph spectra and calculate eigenvalues of certain classes of hypergraphs. In [21] it was introduced several products of hypergraphs and multidimensional matrices and then considered the eigenvalues of the result. Algebraic properties of eigenvalues and determinants of multidimensional matrices with respect to various matrix operations were studied in [22]. At last, for some survey on the spectral theory of nonnegative matrices see [4].

2.2 Perfect colorings and coverings of graphs

Let us review some basic definitions and facts on perfect colorings of graphs that we aim to generalize for hypergraphs. Most of them can be found in [24].

A coloring of a graph $G = (X, E)$ in $k$ colors is a surjective function $f : X \to \{1, \ldots, k\}$. To every $k$-coloring $f$, we can put into a correspondence a rectangular color matrix $P$ of size $|X| \times k$ with entries $p_{x,i} = 1$ if $f(x) = i$ and $p_{x,i} = 0$ otherwise. Note that each row of $P$ contains exactly one nonzero entry. A coloring of a graph parts its vertex set $X$ into disjoint color classes $\{C_1, \ldots, C_k\}$, where $C_i = \{x : f(x) = i\}$. In what follows, we usually define a coloring with the help of the color matrix $P$, but in some cases we also refer to the coloring function $f$ and the color classes $\{C_1, \ldots, C_k\}$.

If $G$ is a bipartite graph and $f$ is a coloring of $G$, we will say that $f$ is a bipartite coloring if there are no color classes of $f$ that intersect both parts of $G$.

A $k$-coloring $f$ of a graph $G$ is called perfect if there exist integer $s_{i,j}$, $i, j \in \{1, \ldots, k\}$, such that every vertex of color $i$ in $f$ is adjacent to exactly $s_{i,j}$ vertices of color $j$. The matrix $S = (s_{i,j})$ of order $k$ is called the parameter matrix of the perfect coloring.

It is not hard to see that if $A$ is the adjacency matrix of a graph $G$, then $P$ is a perfect coloring with the parameter matrix $S$ if and only if $AP = PS$. Moreover, if $y$ is an eigenvector of the parameter matrix $S$ with an eigenvalue $\lambda$, then $Py$ is the eigenvector of $A$ with the same eigenvalue $\lambda$.

Given a coloring $P$, let us denote by $N$ the matrix $P^TP$. It can be checked that $N$ is a diagonal matrix with entries $n_{i,i}$ equal to the number of vertices of color $i$ and that the matrix $NS = P^TPS$ is symmetric.

We will say that a coloring $f$ of a graph $G$ with color classes $\{C_1, \ldots, C_k\}$ is a refinement of a coloring $g$ with color classes $\{D_1, \ldots, D_l\}$ if each color class $D_i$ is a union of some classes $C_j$. For a given graph $G$, its colorings (and perfect colorings) compose a partially ordered set with respect to the refinement.

For every coloring of a graph, there is the unique coarsest refinement that will be a perfect coloring. The method of finding such refinements is known as Weisfeiler–Leman algorithm [25].

**Theorem 2** ([25]). Given a graph $G$, there is a perfect coloring $f$ such that every other perfect coloring of $G$ is a refinement of $f$. Moreover, for every coloring $g$ of $G$ there is a refinement $h$ of $g$ such that $h$ is a perfect coloring, and any other perfect coloring that refines $g$ is a refinement of $h$.

A graph $G = (X, E)$ covers a graph $H = (Y, U)$ if there exists a surjective function $\varphi : X \to Y$ such that for each $x \in X$ the equality $\{\varphi(x')|(x, x') \in E\} = \{y|(y, \varphi(x)) \in U\}$ holds. A covering $\varphi$ of a graph $H$ by $G$ is a $k$-covering if for every $y \in Y$ there are exactly $k$ vertices $x \in X$ such that $\varphi(x) = y$. It is well known that every covering is a $k$-covering for some $k \in \mathbb{N}$.

It is not hard to see that a graph $G$ covers a graph $H$ if and only if there is a perfect coloring of $G$ with the parameter matrix equal to the adjacency matrix of $H$. It implies, for example, the following simple properties of coverings:

- If a graph $G$ covers a graph $H$ and a graph $H$ covers a graph $F$, then $G$ covers $F$. 

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• If a graph $G$ covers a graph $H$ and $\lambda$ is an eigenvalue of $H$, then $\lambda$ is an eigenvalue of $G$.

In [1, 15], it was proved that if graphs $H_1$ and $H_2$ have perfect colorings with the same parameter matrices, then there exists a graph $G$ that covers both $H_1$ and $H_2$.

For more information of graph coverings and their applications see [12].

3 Perfect colorings of hypergraphs

There are two equivalent ways to define perfect colorings in uniform hypergraphs. The first of them uses a perfect coloring of the incidence graph, and the second one is based on an equation for the multidimensional adjacency matrix of a hypergraph. The first approach is more general and can be applied to non-uniform hypergraphs. Although it appeared in [13] and [19], the parameters of such colorings were not discussed before. The approach based on multidimensional matrices is completely new. In this section we also study the interrelations between these approaches.

3.1 Colorings of the incidence graph

Let $G(X, E)$ be a hypergraph. We will say that a surjective function $f : X \rightarrow \{1, \ldots, k\}$ is a coloring of $G$ into $k$ colors (or $k$-coloring). In other words, $f$ defines a partition of the set $X$ in $k$ color classes. Given a coloring $f$ and a hyperedge $e$, let the color range $f(e)$ be the multiset of colors of all incident vertices: $f(e) = \{f(x)|x \in e\}$.

Let $G$ be the incidence graph of the hypergraph $G$. To each coloring $f$ of $G$, we associate an induced coloring $g$ of $G$ such that for every $x \in X$ we have $g(x) = f(x)$ and for every $e \in E$ we define $g(e) = f(e)$. Here the color ranges $f(e)$ of hyperedges are considered as colors of the coloring $g$. Note that every induced coloring of $G$ is bipartite, but not every bipartite coloring of $G$ is induced by some coloring of $G$.

A coloring $f$ of a hypergraph $G$ is said to be perfect if each two vertices $x$ and $y$ of the same color have the same set of color ranges of incident hyperedges. Directly from the definitions, it follows that $f$ is a perfect coloring of $G$ if and only if the induced bipartite coloring $g$ of $G$ is perfect.

We will say that a rectangular $(0,1)$-matrix $P$ is a color matrix if every row of $P$ contains exactly one 1. To every $k$-coloring $f$ of a hypergraph $G(X, E)$ on $n$ vertices, we associate the color matrix $P$ of size $n \times k$ such that $p_{x,i} = 1$ if $f(x) = i$ and $p_{x,i} = 0$ otherwise.

Assume that a hypergraph $G(X, E)$ has $n$ vertices, $m$ hyperedges, and the incidence matrix $B$. Let $f$ be a perfect $k$-coloring of $G$ such that the number of different color ranges $\gamma$ of hyperedges is equal to $l$. Then the induced coloring $g$ of the incidence graph $G$ gives us the following matrix equation:

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} 0 & P \\ R & 0 \end{pmatrix} = \begin{pmatrix} 0 & P \\ R & 0 \end{pmatrix} \begin{pmatrix} 0 & W \\ V & 0 \end{pmatrix}$$

or, equivalently,

$$BR = PV; \quad B^TP = RW,$$

(1)

where

• $P$ and $R$ are vertex and hyperedge color matrices of sizes $n \times k$ and $m \times l$, respectively; the coloring $R$ is induced by the coloring $P$.

• $V$ and $W$ are matrices of sizes $k \times l$ and $l \times k$, respectively. An entry $v_{i,\gamma}$ of $V$ is equal to the number of hyperedges of color range $\gamma$ in $G$ incident to a vertex of color $i$, entry $w_{\gamma,i}$ of $W$ is equal to the number of vertices of color $i$ in $G$ contained in a hyperedge of color range $\gamma$.
We will say that the pair \((V, W)\) is the *incidence parameters* of the perfect coloring \(f\): the matrix \(V\) is said to be the *XE-parameter matrix* (describing incidence of vertices \(x\) to hyperedges \(e\)), and \(W\) is the *EX-parameter matrix* (describing incidence of hyperedges \(e\) to vertices \(x\)).

**Example 1.** Let \(G\) be a 3-uniform hypergraph with the vertex set \(X = \{x_1, x_2, x_3, x_4, x_5, x_6\}\) and the hyperedge set \(E = \{e_1, e_2, e_3, e_4\}\), where \(e_1 = \{x_1, x_2, x_3\}\), \(e_2 = \{x_1, x_4, x_5\}\), \(e_3 = \{x_2, x_4, x_6\}\), \(e_4 = \{x_3, x_5, x_6\}\).

Let \(f\) be a coloring of \(X\) into two colors such that \(f(x_1) = f(x_2) = f(x_3) = 1\) and \(f(x_4) = f(x_5) = f(x_6) = 2\). Then the color ranges of hyperedges are \(f(e_1) = \{1, 1, 1\}\) and \(f(e_2) = f(e_3) = f(e_4) = \{1, 2, 2\}\). Note that each vertex of color 1 is incident to one hyperedge of color range \(\{1, 1, 1\}\) and one hyperedge of color range \(\{1, 2, 2\}\), and each vertex of color 2 is incident to two hyperedges of color range \(\{1, 2, 2\}\). Therefore, \(f\) is a perfect coloring of \(G\).

The incidence graph \(G\) of \(G\) with the induced perfect coloring \(g\) is given at Figure 1.

![Figure 1: The perfect coloring \(g\) of the incidence graph \(G\) for the hypergraph \(G\)](image)

The incidence matrix of the hypergraph \(G\) is

\[
B = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}.
\]

By the definition, the vertex color matrix \(P\) and the hyperedge color matrix \(R\) for the perfect coloring \(f\) are

\[
P = \begin{pmatrix}
1 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 1 \\
0 & 1
\end{pmatrix}; \quad R = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix},
\]

and the incidence parameters \((V, W)\) are

\[
V = \begin{pmatrix}
1 & 1 \\
0 & 2
\end{pmatrix}; \quad W = \begin{pmatrix}
3 & 0 \\
1 & 2
\end{pmatrix}.
\]

It can be checked directly that \(BR = PV\) and \(B^T P = RW\).

Let us derive several simple observations from the given definition of a perfect coloring of hypergraphs.
First of all, if vertices $x$ and $y$ of a hypergraph $G$ have different degrees, then $f(x) \neq f(y)$ for every perfect coloring $f$ of $G$ because the same is true for perfect colorings of graphs. The sum of entries in the $i$-th row of the $XE$-parameter matrix $V$ is equal to the degree of vertices of color $i$, and the sum of entries in the $\gamma$-th row of the $EX$-parameter matrix $W$ is equal to the size of a hyperedge of the color range $\gamma$.

Next, if $P$ is a coloring of vertices and $R$ is a coloring of hyperedges, then $N = P^T P$ and $M = R^T R$ are diagonal matrices. Entries $n_{i,i}$ of $N$ are equal to the numbers of vertices colored by color $i$ in $P$, and entries $m_{j,j}$ of $M$ are the numbers of hyperedges colored by color $j$ in $R$.

We also prove the following results on the structure of the incidence matrix $B$ and a relation between the incidence parameters $(V, W)$ and matrices $N$ and $M$.

**Theorem 3.** 1. Let $G$ be a hypergraph with the incidence matrix $B$, $P$ be a coloring of $G$, $R$ be an induced coloring of hyperedges, $N = P^T P$ and $M = R^T R$ be the diagonal matrices with entries $n_i$ and $m_j$ at the diagonal. If a coloring $P$ is perfect for $G$ with incidence parameters $(V, W)$, then $NV = W^T M$ and $B$ is a block matrix $\{A_{i,j}\}$, where $A_{i,j}$ are $(0, 1)$-matrices of sizes $n_i \times m_j$ with row sums $v_{i,j}$ and column sums $w_{j,i}$.

2. Let $V$ and $W$ be integer nonnegative matrices and $N$ and $M$ be integer diagonal matrices with $n_i > 0$ and $m_j > 0$ such that $NV = W^T M$. Then there exist a hypergraph $G$ and its perfect coloring $P$ having incidence parameters $(V, W)$.

**Proof.** (1). Assume that $P$ is a perfect coloring of $G$. Recall that equations (1) for perfect colorings give

$$BR = PV; \quad B^T P = RW.$$  

Multiplying the first equation by $P^T$ and the second equation by $R^T$, we have

$$P^T BR = P^T PV = NV; \quad R^T B^T P = R^T RW = MW.$$  

Note that the left-hand side of one equation is the transposition of the other, so $NV = W^T M$. On the other hand, these equations mean that matrices $P$ and $R$ define a partition of the incidence matrix $B$ into blocks $A_{i,j}$, where $A_{i,j}$ has sizes $n_i \times m_j$, row sums $v_{i,j}$, and column sums $w_{j,i}$.

(2). Equality $NV = W^T M$ means that $n_i v_{i,j} = w_{j,i} m_j$ for all $i$ and $j$. Choose a nonnegative integer $t$ so that for all $i$ and $j$ there exist $(0, 1)$-matrices $A_{i,j}$ of sizes $tn_i \times tm_j$ such that each row of $A_{i,j}$ contains exactly $v_{i,j}$ ones and each column contains exactly $w_{j,i}$ ones.

Let a rectangular matrix $B$ is given by the block matrix $\{A_{i,j}\}$. We treat $B$ as the incidence matrix of a hypergraph $G$. It can be verified directly that a $k$-coloring of $G$

$$P = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 0 \\ 0 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 1 \end{pmatrix},$$

where the $i$-th column contains exactly $tn_i$ ones, is a perfect coloring with the incidence parameters $(V, W)$.

At last, we define refinements of perfect colorings in hypergraphs similar to graphs. We will say that a coloring $f$ of a hypergraph $G$ is a refinement of a coloring $g$ if the coloring of the incidence graph of $G$ induced by $f$ is a refinement of the coloring induced by $g$. The set of perfect colorings of a hypergraph with respect to the refinement forms a partially ordered set. **Theorem 2** implies the following.
Theorem 4. Given a hypergraph $G = (X, E)$, there is a perfect coloring $f$ of $G$ such that any other perfect coloring is a refinement of $f$. Moreover, for every coloring $g$ of $G$ there is a refinement $h$ of $g$ such that $h$ is a perfect coloring and any other perfect coloring that refines $g$ is a refinement of $h$.

Proof. To construct the coloring $f$, it is sufficient to apply Theorem 2 to the trivial bipartite coloring of the incidence graph $G$, in which the part $X$ is colored by one color, and the part $E$ is colored by another color.

The second statement of the theorem is obtained by the application of Theorem 2 to the coloring of the incidence graph induced by the coloring $g$.

We will say that the coloring $f$ from Theorem 4 is the minimal perfect coloring of the hypergraph $G$.

3.2 The multidimensional matrix equation for perfect colorings

Let $G(X, E)$ be a $d$-uniform hypergraph on $n$ vertices, $A$ be the $d$-dimensional adjacency matrix of $G$, and $P$ be a coloring of $G$ into $k$ colors (color matrix of vertices). We will say that $P$ is perfect if there exists a $d$-dimensional matrix $S$ of order $k$ such that

$$A \circ P = P \circ S,$$

where the product $\circ$ of multidimensional matrices is defined in Section 2. The matrix $S$ is called the parameter matrix of the perfect coloring $P$.

Let us show that this definition of perfect colorings in case of uniform hypergraphs is equivalent to the previous one. Moreover, we express entries of the parameter matrix $S$ by the incidence parameters ($V, W$).

Theorem 5. Let $G$ be $d$-uniform hypergraph with the $d$-dimensional adjacency matrix $A$. Then a coloring $P$ of $G$ into $k$ colors is a perfect if and only if $A \circ P = P \circ S$. Moreover, the entries $s_\gamma$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d)$, of $S$ are

$$s_\gamma = v_{\gamma_1, \gamma} \cdot \binom{d - 1}{d_1, \ldots, d_k},$$

where $v_{\gamma_1, \gamma}$ is the number of hyperedges of color range $\gamma$ incident to a vertex of color $\gamma_1$, every color $l$ appears in the multiset $\{\gamma_2, \ldots, \gamma_d\}$ exactly $d_l$ times, and $\binom{d - 1}{d_1, \ldots, d_k}$ is the multinomial coefficient.

Proof. $\Leftarrow$ Suppose that $f$ is a $k$-coloring of the hypergraph $G$ with a color matrix $P$ such that

$$A \circ P = P \circ S.$$

Let us consider entries of the matrix $C = A \circ P$. Using the definition of the multidimensional matrix product, for a given vertex $x$ and colors $j_1, \ldots, j_{d-1}$ we have

$$c_{x,j_1,\ldots,j_{d-1}} = \sum_{x_1,\ldots,x_{d-1}=1}^{n} a_{x,x_1,\ldots,x_{d-1}} p_{x_1,j_1} \cdots p_{x_{d-1},j_{d-1}}. \quad (2)$$

Note that an entry $a_{x,j_1,\ldots,j_{d-1}}$ of $A$ is equal to $\frac{1}{(d-1)!}$ if and only if $x, x_1, \ldots, x_{d-1}$ are different vertices of $G$ and constitute a hyperedge, and the entries $p_{x_i,j_i} = 1$ if and only if $f(x_i) = j_i$. Let $t_{x,j_1,\ldots,j_{d-1}}$ be the number of hyperedges $(x, x_1, \ldots, x_{d-1})$ such that $\{f(x_1), \ldots, f(x_{d-1})\}$ and $\{j_1, \ldots, j_{d-1}\}$ coincide as multisets. We also assume that each color $l$ appears in these
multisets exactly \( d_l \) times, \( \sum_{l=1}^{k} d_l = d - 1 \). Then equation (2) is equivalent to \( c_{x,j_1,...,j_{d-1}} = t_{x,j_1,...,j_{d-1}}(d_{d_1,...,d_k})^{-1} \), because every ordering of vertices \( x_1,\ldots,x_{d-1} \) that does not change the sequence of colors \( f(x_1),\ldots,f(x_{d-1}) \) gives a summand \( \frac{1}{(d-1)!} \) in counting of \( c_{x,j_1,...,j_{d-1}} \).

On the other hand, using \( C = P \circ S \) and again the definition of \( \circ \)-operation, we have

\[
c_{x,j_1,...,j_{d-1}} = \sum_{i=1}^{k} p_{x,i} s_{i,j_1,...,j_{d-1}}.
\]

Since \( p_{x,i} = 1 \) if and only if \( f(x) = i \), we have that if a vertex \( x \) has color \( i \), then \( c_{x,j_1,...,j_{d-1}} = s_{i,j_1,...,j_{d-1}} \) and, consequently, \( s_{i,j_1,...,j_{d-1}} = t_{x,j_1,...,j_{d-1}}(d_{d_1,...,d_k})^{-1} \). Therefore, the color ranges \( \{i,j_1,\ldots,j_{d-1}\} \) of hyperedges incident to a vertex \( x \) of color \( i \) are uniquely defined by entries of \( S \), and so the coloring \( P \) is perfect. Replacing \( v_{i,(i,j_1,...,j_{d-1})} = t_{x,j_1,...,j_{d-1}} \) for all vertices \( x \) of color \( i \), we have the statement of the theorem.

\( \Rightarrow \): Assume that a coloring \( P \) is perfect. Repeating the calculation of entries of the matrix \( A \circ P \) and using the fact that the values \( t_{x,j_1,...,j_{d-1}} \) are uniquely defined by the color of vertex \( x \), we see that there exists the required \( d \)-dimensional matrix \( S \) satisfying \( A \circ P = P \circ S \).

\[\Box\]

Note that a perfect coloring \( P \) defines only the factor \( v_{\gamma_1,\gamma} \) in the entries \( s_\gamma \) of the matrix \( S \), while the factor \( (d_{d_1,...,d_k})^{-1} \) in \( s_\gamma \) depends only on the index \( \gamma \) and it is the same for all \( d \)-dimensional parameter matrices \( S \).

**Corollary 1.** Let \( G \) be a \( d \)-uniform hypergraph, \( P \) be a perfect coloring of \( G \), \( n_l \) be the number of vertices of color \( l \), and \( m_\gamma \) be the number of hyperedges of a color range \( \gamma \). Suppose that \( W = (w_{\gamma_1}) \) is the EX-parameter matrix and \( S \) is the multidimensional parameter matrix of the perfect coloring \( P \). Then entries \( s_\gamma \) of \( S \) are

\[
s_\gamma = d \cdot \frac{m_\gamma}{n_{\gamma_1}} \cdot \left( \frac{d}{w_{\gamma_1},\ldots,w_{\gamma_k}} \right)^{-1}.
\]

**Proof.** By Theorem 5, the entries \( s_\gamma \) of the parameter matrix \( S \) are equal to \( v_{\gamma_1,\gamma} \cdot (d_{d_1,...,d_k})^{-1} \), where \( v_{\gamma_1,\gamma} \) are entries of the XE-parameter matrix \( V \) and each color \( l \) appears in \( \{\gamma_1,\ldots,\gamma_d\} \) exactly \( d_l \) times. By the definition of incidence parameters, \( \gamma = \{\gamma_1,\ldots,\gamma_d\} \) contains the \( l \)-th color exactly \( w_{\gamma_1,l} \) times, so \( d_{\gamma_1} = w_{\gamma_1,\gamma_1} - 1 \) and \( d_{\gamma} = w_{\gamma_1,\gamma} \) for all other \( \gamma_i \).

By Theorem 3(1), \( n_{\gamma_1} v_{\gamma_1,\gamma} = w_{\gamma_1,\gamma_1} m_\gamma \). Consequently,

\[
s_\gamma = \frac{d_{\gamma_1}! \cdots d_k! \cdot v_{\gamma_1,\gamma} \cdot w_{\gamma_1,\gamma_1}! \cdots w_{\gamma_k,\gamma_k}!}{(d-1)!} = \frac{d \cdot m_\gamma \cdot n_{\gamma_1}}{n_{\gamma_1}} \cdot \left( \frac{d}{w_{\gamma_1,1},\ldots,w_{\gamma_k,\gamma}} \right)^{-1}.
\]

\[\Box\]

**Example 2.** Let \( G \) be the 3-uniform hypergraph on 6 vertices from Example 1 and \( P \) be its perfect coloring:

\[
P = \begin{pmatrix}
1 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 1 \\
0 & 1
\end{pmatrix}
\]
Recall that the incidence parameters \((V, W)\) for this coloring are
\[
V = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}; \quad W = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}.
\]

The adjacency matrix \(A\) of the hypergraph \(G\) is the following 3-dimensional matrix of order 6:
\[
A = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

It can be checked directly that \(A \circ P = P \circ S\) for the 3-dimensional parameter matrix
\[
S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\]
where \(s_{1,1,1} = v_{1,1}/1 = 1, s_{1,2,2} = v_{1,2}/1 = 1, s_{2,2,1} = v_{2,1}/2 = 1\). Note that \(s_{1,1,2} = s_{1,2,1} = s_{2,1,1} = s_{2,2,2} = 0\) because there are no hyperedges of color ranges \(\{1,1,2\}\) and \(\{2,2,2\}\) in the perfect coloring \(P\).

Using the obtained results, let us prove several properties of the parameter matrix \(S\).

**Proposition 2.** Let \(P\) be a perfect coloring in a \(d\)-uniform hypergraph \(G\) with the parameter matrix \(S\). Then the sum of entries of \(S\) along the \(i\)-th hyperplane of the first direction is equal to the degree of vertices of color \(i\).

**Proof.** By Theorem 5, the entries \(s_{\gamma}\) of the parameter matrix \(S\) are equal to \(v_{\gamma_{1},\gamma} \cdot (d_{1} \ldots d_{k})^{-1}\), where \(v_{\gamma_{1},\gamma}\) is the number of hyperedges of color range \(\gamma\) incident to a vertex of color \(\gamma_{1}\). Note that every hyperedge with color range \(\gamma\) gives exactly \((d_{1} \ldots d_{k})^{-1}\) nonzero entries \(s_{\gamma}\) in the \(\gamma_{1}\)-th hyperplane of the first direction for the parameter matrix \(S\). So the total contribution of such hyperedges to this sum is \(v_{\gamma_{1},\gamma}\), and the total sum of entries along this hyperplane is the degree of a vertex of color \(\gamma_{1}\). \(\square\)

**Theorem 6.** Let \(G\) be a \(d\)-uniform hypergraph. Assume that \(P\) is a perfect coloring of \(G\) with the parameter matrix \(S\) and \(N = P^{T} P\). Then the matrix \(H = N \circ S\) is symmetric.

**Proof.** Recall that \(N = P^{T} P\) is a diagonal matrix whose \(i\)-th diagonal entry is the number \(n_{i}\) of vertices of color \(i\) in the coloring \(P\).

By Corollary 1, an entry \(s_{\gamma}\) of the matrix \(S\) is equal to \(d \cdot \frac{m_{\gamma}}{m_{\gamma_{1}}} \cdot (w_{\gamma_{1},1}, \ldots, w_{\gamma_{1,k}})^{-1}\), where \(w_{\gamma_{1}}\) is the number of appearances of symbol \(l\) in \(\gamma\) and \(m_{\gamma}\) is the number of hyperedges of \(G\) having the color range \(\gamma\).

Using the definition of the product of multidimensional matrices, we see that \(H = N \circ S\) is a \(d\)-dimensional matrix with entries
\[
h_{\beta} = \sum_{j=1}^{k} n_{\beta_{1},j} \cdot s_{j,\beta_{2},\ldots,\beta_{d}} = d \cdot m_{\beta} \cdot \left(\frac{d}{w_{\beta,1}, \ldots, w_{\beta,k}}\right)^{-1}.
\]
Since for every permutation $\sigma \in S_d$ it holds $h_\beta = h_{\sigma(\beta)}$, we have that the matrix $H$ is symmetric.

It is well known that for graphs the spectrum of the parameter matrix of a perfect coloring is contained in the spectrum of the adjacency matrix. A similar fact (regarding geometric multiplicities) is true for hypergraphs. Moreover, this property holds not only for perfect colorings of hypergraphs, but for general matrices satisfying the same equation. To prove this statement, we need the following auxiliary result.

**Proposition 3.** Let $P$ be a color matrix of size $n \times k$ and $I$ be the $d$-dimensional identity matrix. Then

$$I \circ P = P \circ I.$$

**Proof.** The proof is based on the definitions of $\circ$-operation and matrices $I$ and $P$.

Let us show that $(\alpha_1, \alpha_2, \ldots, \alpha_d)$-th entries of matrices in left-hand and right-hand sides of the equation $I \circ P = P \circ I$ are equal to 1 if $\alpha_2 = \cdots = \alpha_d$ and $p_{\alpha_1, \alpha_2} = 1$ and they are 0 otherwise.

Indeed, an $(\alpha_1, \alpha_2, \ldots, \alpha_d)$-th entry of the matrix $I \circ P$ is

$$\sum_{\beta_2, \ldots, \beta_d=1}^n i_{\alpha_1, \beta_2, \ldots, \beta_d} \cdot p_{\beta_2, \alpha_2} \cdots p_{\beta_d, \alpha_d}.$$

Note that $i_{\alpha_1, \beta_2, \ldots, \beta_d} = 1$ if and only if $\alpha_1 = \beta_2 = \cdots = \beta_d$. Otherwise, $i_{\alpha_1, \beta_2, \ldots, \beta_d} = 0$. By the definition of a color matrix, there is a unique $j$ such that $p_{\alpha_1, \alpha_2} = 1$.

On the other hand, an $(\alpha_1, \alpha_2, \ldots, \alpha_d)$-th entry of the matrix $P \circ I$ is

$$\sum_{j=1}^k p_{\alpha_1, \alpha_2} \cdot i_{j, \alpha_2, \ldots, \alpha_d}.$$

Again, there is a unique $j$ such that $p_{\alpha_1, \alpha_2} = 1$ and $i_{j, \alpha_2, \ldots, \alpha_d} = 1$ if and only if $j = \alpha_2 = \cdots = \alpha_d$. 

**Theorem 7.** Let $A$ be a $d$-dimensional matrix of order $n$, $P$ be a color matrix of size $n \times k$, and $B$ be a $d$-dimensional matrix of order $k$ such that $A \circ P = P \circ B$. If $(\lambda, x)$ is an eigenpair for the matrix $B$, then $(\lambda, P \circ x)$ is an eigenpair for $A$.

**Proof.** By the definition, $(\lambda, x)$ is an eigenpair for the matrix $B$ if and only if $B \circ x = \lambda (I \circ x)$. Using Proposition 3 and the equality $P \circ I = I \circ P$ from Proposition 3 we conclude that

$$A \circ (P \circ x) = A \circ (P \circ x) = (A \circ P) \circ x = (P \circ B) \circ x = P \circ (B \circ x) = P \circ (I \circ x) = \lambda (I \circ x) \circ x = \lambda (I \circ P) \circ x = \lambda I \circ (P \circ x) = \lambda I \circ (P \circ x).$$

This theorem easily implies that all eigenvalues of the parameter matrix of a perfect coloring are also eigenvalues of the adjacency matrix of a hypergraph.

**Theorem 8.** Let $G$ be a $d$-uniform hypergraph with the adjacency matrix $A$ and $P$ be a perfect coloring with the $d$-dimensional parameter matrix $S$. If $(\lambda, x)$ is an eigenpair for the matrix $S$, then $(\lambda, P \circ x)$ is an eigenpair for the matrix $A$.

**Proof.** It is sufficient to note that $A \circ P = P \circ S$ and apply Theorem 7.

**Corollary 2.** Assume that $P$ is a perfect coloring of a $d$-uniform hypergraph $G$ into $k$ colors with the parameter matrix $S$ and $\lambda$ is an eigenvalue of $S$. Then there is an eigenvector $y$ of $G$ for eigenvalue $\lambda$ such that components of $y$ attain at most $k$ different values.

**Proof.** Let $x$ be an eigenvector of $S$ for the eigenvalue $\lambda$. Set $y = P \circ x$ and repeat the proof of Theorem 7.
4 Coverings of hypergraphs

We will say that a hypergraph $G$ covers a hypergraph $H$ if there is a map (covering) $\varphi : X(G) \to X(H)$ such that for each hyperedge $e \in E(G)$ the set $\{\varphi(x) | x \in e\}$ is a hyperedge of $H$ and $\varphi$ saves a collection of incident hyperedges for each vertex.

Note that a covering preserves degrees of vertices and sizes of hyperedges, so every covering of a $d$-uniform $r$-regular hypergraph is also $d$-uniform and $r$-regular.

Every covering $\varphi$ of a hypergraph $H$ by a hypergraph $G$ may be considered as a coloring of $G$ by vertices of $H$. Moreover, $\varphi$ colors the hyperedges of $G$ by colors corresponding to hyperedges of $H$. It gives us the following statement.

**Proposition 4.** A map $\varphi : X(G) \to X(H)$ is a covering of a hypergraph $H$ by a hypergraph $G$ if and only if $\varphi$ is a perfect coloring of $G$ with the parameter matrix $S$ equal to the adjacency matrix $A_H$ of $H$. The incidence parameters of this coloring are $(C, C^T)$, where $C$ is the incidence matrix of $H$.

An equivalent definition of coverings of hypergraphs was proposed in [16]. In particular, in [16, Theorem 7] it was stated that hypergraph coverings are equivalent to coverings of their incidence graphs. It is also straightforward from our notions and Proposition 4.

**Proposition 5.** A coloring $\varphi$ is a covering of a hypergraph $H$ by a hypergraph $G$ if and only if the induced bipartite perfect coloring $\overline{\varphi}$ of the incidence graph $G$ is a covering of the incidence graph $H$.

We will say that a covering of a hypergraph $H$ by $G$ is a $k$-covering if for every $y \in X(H)$ there are exactly $k$ vertices $x \in X(G)$ such that $\varphi(x) = y$. Proposition 5 implies that every covering of a hypergraph is a $k$-covering.

**Proposition 6.** For every covering $\varphi$ of a hypergraph $H = (Y, U)$ by $G = (X, E)$ there is $k \in \mathbb{N}$ such that $\varphi$ is $k$-covering. In particular, we have $|X| = k|Y|$ and $|E| = k|U|$.

At last, we note that $k$-coverings of hypergraphs can be nicely described in terms of incidence matrices.

**Theorem 9.** Let $G$ and $H$ be hypergraphs with incidence matrices $B$ and $C$, respectively. If there is a $k$-covering of $H$ by $G$, then the matrix $B$ is a block matrix $\{D_{i,j}\}$, where $D_{i,j}$ is a permutation matrix of order $k$ if $c_{i,j} = 1$ and $D_{i,j}$ is the zero matrix of order $k$ otherwise.

**Proof.** By Proposition 4 we can consider a $k$-covering $\varphi$ as a perfect coloring of $G$ with the incidence parameters $(C, C^T)$. In particular, we have that the size of each color class in $\varphi$ is $k$. To prove the theorem, it only remains to apply Theorem 3(1) to the coloring $\varphi$. \hfill $\square$

**Example 3.** Let $G$ be the 3-uniform hypergraph with the vertex set $X = \{x_1, x_2, \ldots, x_8\}$ and eight hyperedges $e_1 = \{x_3, x_6, x_7\}$, $e_2 = \{x_4, x_5, x_8\}$, $e_3 = \{x_1, x_6, x_7\}$, $e_4 = \{x_2, x_5, x_8\}$, $e_5 = \{x_1, x_4, x_8\}$, $e_6 = \{x_2, x_3, x_7\}$, $e_7 = \{x_2, x_3, x_5\}$, $e_8 = \{x_1, x_4, x_6\}$, and $H$ be the 3-uniform hypergraph with the vertex set $X' = \{x'_1, x'_2, x'_3, x'_4\}$ and four hyperedges $e'_1 = \{x'_2, x'_3, x'_4\}$, $e'_2 = \{x'_1, x'_3, x'_4\}$, $e'_3 = \{x'_1, x'_2, x'_4\}$, $e'_4 = \{x'_1, x'_2, x'_3\}$.

Then the map $\varphi : X \to X'$ such that $\varphi(x_1) = \varphi(x_2) = x'_1$, $\varphi(x_3) = \varphi(x_4) = x'_2$, $\varphi(x_5) = \varphi(x_6) = x'_3$, $\varphi(x_7) = \varphi(x_8) = x'_4$ is a 2-covering of the hypergraph $H$ by the hypergraph $G$. In particular, $\varphi(e_1) = \varphi(e_2) = e'_1$, $\varphi(e_3) = \varphi(e_4) = e'_2$, $\varphi(e_5) = \varphi(e_6) = e'_3$, and $\varphi(e_7) = \varphi(e_8) = e'_4$. 
The incidence matrices $B$ and $C$ of the hypergraphs $G$ and $H$, respectively, are

\[
B = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 
\end{pmatrix}; \quad C = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 
\end{pmatrix}.
\]

It can be checked that they satisfy the condition of Theorem 9.

The correspondence between coverings and colorings allows us to establish several properties of coverings of hypergraphs. First of all, let us show that if a uniform hypergraph $G$ covers a hypergraph $H$, then eigenvalues of $H$ are also eigenvalues of $G$. This result was recently obtained in [23, Theorem 4.1].

**Theorem 10.** Assume that a $d$-uniform hypergraph $G$ with the adjacency matrix $A_G$ covers a hypergraph $H$ with the adjacency matrix $A_H$. Then every eigenvalue of $A_H$ is an eigenvalue of $A_G$ (regarding geometric multiplicities).

**Proof.** By the definition of coverings, there is a perfect coloring $P$ of $G$ with the parameter matrix $A_H$. By Theorem 8 every eigenvector of the matrix $A_H$ for an eigenvalue $\lambda$ gives an eigenvector of the matrix $A_G$ for the same eigenvalue. \qed

Next, let us prove that a covering preserves the set of perfect colorings of hypergraphs.

**Theorem 11.** Assume that a $d$-uniform hypergraph $G$ with the adjacency matrix $A_G$ covers a hypergraph $H$ with the adjacency matrix $A_H$. Then for every perfect coloring of $H$ with the parameter matrix $S$ there is a perfect coloring of $G$ with the same parameter matrix $S$.

**Proof.** Let $P$ be a perfect coloring of $H$ with the parameter matrix $S$:

$$A_H \circ P = P \circ S.$$ 

Since $G$ covers $H$, there is a color matrix $R$ such that

$$A_G \circ R = R \circ A_H.$$ 

Then, using properties of the product of multidimensional matrices from Proposition 1 we have

$$A_G \circ (RP) = (A_G \circ R) \circ P = (R \circ A_H) \circ P = R \circ (A_H \circ P) = R \circ (P \circ S) = RP \circ S.$$ 

Consequently, $RP$ is a perfect coloring of $G$ with the parameter matrix $S$. \qed

At last, we establish a result on the existence of common coverings for hypergraphs with several corollaries. They can be obtained by an accurate combination of the theorem on the existence of common coverings for graphs (see [15]) and the correspondence between coverings of hypergraphs and their incidence graphs (Proposition 3). Meanwhile, here we provide a direct proof of this result, reformulating the ideas from [15] in terms of incidence matrices. For future applications, we state this result for multihypergraphs.

**Theorem 12.** If connected (multi)hypergraphs $H$ and $H'$ have perfect colorings with the same incidence parameters $(V, W)$, then there is a hypergraph $G$ that covers both $H$ and $H'$. 

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Proof. Assume that $C$ and $C'$ are the incidence matrices of $\mathcal{H}$ and $\mathcal{H}'$ and $\varphi$ and $\varphi'$ are their perfect colorings with incidence parameters $(V, W)$, respectively. Let $n_i$ and $m_j$ ($n'_i$ and $m'_j$), $i = 1, \ldots, k$, $j = 1, \ldots, l$ be the number of vertices of color $i$ and the number of hyperedges of color $j$ in the induced bipartite perfect coloring of $\mathcal{H}$ (of $\mathcal{H}'$). By Theorem $3(1)$, there are numbers $\pi_{i,j}$ and $\pi'_{i,j}$ such that

\[
\pi_{i,j} = n_i v_{i,j} = w_{j,i} m_j, \quad \pi'_{i,j} = n'_i v_{i,j} = w_{j,i} m'_j.
\]

Moreover, by Theorem $3(1)$, the incidence matrix $C$ is the block matrix $\{A_{i,j}\}$ such that $A_{i,j}$ is a $(0, 1)$-matrix of size $n_i \times m_j$ with row sums $v_{i,j}$ and column sums $w_{j,i}$. In particular, the block $A_{i,j}$ contains $\pi_{i,j}$ ones. Let us index all nonzero entries $a_{x,e}$ of $A_{i,j}$ by pairs $(\mu, \nu)$, $\mu \in \{0, \ldots, w_{j,i} - 1\}$, $\nu \in \{0, \ldots, v_{i,j} - 1\}$, so that $\mu(x)$ is the position of $a_{x,e}$ in the $\nu$-th column of $A_{i,j}$ with respect to other nonzero entries (from up to down), and $\nu(e)$ is the position of this entry in the $x$-th row of $A_{i,j}$ (from left to right).

Similarly, the incidence matrix $C'$ is a block matrix $\{A'_{i,j}\}$ with blocks $A'_{i,j}$ of sizes $n'_i \times m'_j$, row sums $v_{i,j}$, and column sums $w_{j,i}$. We also label all nonzero entries of $\{A'_{i,j}\}$ by pairs $(\mu', \nu')$ in the same way.

Let $\Pi$ denote the least common multiple of the numbers $\pi_{i,j}$: $\Pi = \text{lcm}(\pi_{i,j})$, and let $t_i = \Pi/n_i$ and $s_j = \Pi/m_j$. Note that all $t_i$ and $s_j$ are integer.

We construct the incidence matrix $B$ of a covering (multi)hypergraph $\mathcal{G}$ as a block matrix $\{D_{i,j}\}$, $i = 1, \ldots, k$, $j = 1, \ldots, l$. The size of each block $D_{i,j}$ will be $t_i n_i n'_i \times s_j m_j m'_j$.

Let $t$ be the size of blocks $D_{i,j}$ remaining the same, if we use $t'_i$ and $s'_j$ instead $t_i$ and $s_j$. It is shown in the following claim.

Claim: For all $i = 1, \ldots, k$, $j = 1, \ldots, l$, we have that $t_i = t'_i$, and $s_j = s'_j$, where $t_i = \Pi/n_i$, $t'_i = \Pi'/n'_i$, $s_j = \Pi/m_j$, $s'_j = \Pi/m'_j$, $\Pi = \text{lcm}(\pi_{i,j})$, $\Pi' = \text{lcm}(\pi'_{i,j})$.

Proof of claim. Thanks to symmetry, it is sufficient to prove the equalities for all $t_i$ and $t'_i$ Without loss of generality, we show that $t_1 = t'_1$, i.e., $\frac{\text{lcm}(\pi_{1,j})}{n_1} = \frac{\text{lcm}(\pi'_{1,j})}{n'_1}$. Since $\mathcal{H}$ is a connected (multi)hypergraph, there are Berge paths from a vertex of color 1 to any vertex of color $c \neq 1$. Using relations $n_i v_{i,j} = w_{j,i} m_j$ from Theorem $3(1)$ for colors of vertices $i$ and colors of hyperedges $j$ along this path, we conclude that there are rational numbers $r_{1,e}$ depending only $V$ and $W$ such that $n_c = r_{1,e} n_1$ for every $c \geq 1$. Using these equalities, we conclude that there is a rational number $q$ depending only on $V$ and $W$ such that $\Pi = \text{lcm}(\pi_{1,j}) = q n_1$.

Acting similarly, in case of the $\mathcal{H}'$ we get the same $q$ such that $\Pi' = \text{lcm}(\pi'_{1,j}) = q n'_1$. Consequently, $t_1 = t'_1$.

Let us describe the structure of blocks $D_{i,j}$. Given $i$ and $j$, each $D_{i,j}$ is a block matrix $\{F_{a,b}\}$ with blocks of sizes $t_i \times s_j$, each index $a$ has a form $(x, x')$, $x \in X(\mathcal{H})$, $x' \in X(\mathcal{H}')$, and an index $b$ has a form $(e, e')$, $e \in E(\mathcal{H})$, $e' \in E(\mathcal{H}')$. Each block $F = F_{a,b}$, where $a = (x, x')$, $b = (e, e')$, is a $(0, 1)$-matrix defined as follows.

- Assume that for entries of $A_{i,j}$ and $A'_{i,j}$ it holds $a_{x,e} = 1$ and $a'_{x',e'} = 1$. If $\sigma \equiv \nu(e) - \nu'(e') \mod v_{j,i}$ and $\sigma \equiv \mu(x) - \mu'(x') \mod w_{j,i}$, then the $\sigma$-th row and the $\delta$-th column of the block $F_{a,b}$ contain exactly one 1, otherwise they are filled by 0.

- If we have $a_{x,e} = 0$ or $a'_{x',e'} = 0$, then the block $F_{a,b}$ is the zero matrix.

Note that the nonzero blocks $F_{a,b}$ contain exactly $t_i v_{i,j} = \frac{\Pi}{n_i v_{i,j}}$ nonzero rows that coincides with the number $s_j w_{j,i} = \frac{\Pi}{m_j w_{j,i}}$ of nonzero columns. Since both of these numbers are integers, the blocks $F_{a,b}$ are well defined.

It only remains to show that the constructed $(0, 1)$-matrix $B = \{D_{i,j}\}$ (with $D_{i,j} = \{F_{a,b}\}$) is the incidence matrix of a (multi)hypergraph $\mathcal{G}$ covering both $\mathcal{H}$ and $\mathcal{H}'$. Without loss of generality, let us prove that $\mathcal{G}$ covers $\mathcal{H}$. 15
For every \((x, e)\)-entry of the incidence matrix \(C\) of \(\mathcal{H}\), consider a submatrix (block) \(J_{x,e}\) of \(B\) composed of blocks \(F_{\alpha,\beta}\) such that \(\alpha = (x, x')\), \(\beta = (e, e')\), \((x', e')\) runs over all entries of \(C'\). It can be checked that \(J_{x,e}\) is a square matrix of order \(II'\).

Let \(x\) has color \(i\) and \(e\) has color \(j\) in the perfect coloring \(\varphi\) of \(\mathcal{H}\). Then \(J_{x,e}\) is a submatrix contained in the block \(D_{i,j}\) of \(B\).

The definition of blocks \(F_{\alpha,\beta}\) implies that if \((x, e)\)-entry of \(C\) is 0, then the corresponding block \(J_{x,e}\) is the zero matrix.

Suppose that \((x, e)\)-entry of \(C\) is equal to 1. Then every row of the submatrix \(J_{x,e}\) contains exactly one 1, because, (by the construction of \(F_{\alpha,\beta}\)) there is a unique \((x', e')\)-entry of \(C'\) that gives the block \(F_{\alpha,\beta}\) having the unity entry exactly at this row. For similar reasons, every column of \(J_{x,e}\) contains exactly one 1. Consequently, \(J_{x,e}\) is a permutation matrix.

By Theorem \([5]\), it means that the constructed (multi)hypergraph \(\mathcal{G}\) covers \(\mathcal{H}\). If \(\mathcal{G}\) has multiple hyperedges (i.e. \(B\) contains identical columns), then one can cover \(\mathcal{G}\) by some hypergraph \(\mathcal{G}'\) that also covers \(\mathcal{H}\) and \(\mathcal{H}'\).

\[\Box\]

**Remark 1.** In definition of matrices \(F_{\alpha,\beta}\) instead relations \(\sigma \equiv \nu(e) - \nu'(e') \mod v_{i,j}\) and \(\delta \equiv \mu(x) - \mu'(x') \mod w_{j,i}\) we can use any other latin squares of orders \(v_{i,j}\) and \(w_{j,i}\), whose entries we treat as triples \((\nu(e), \nu'(e'), \sigma)\) and \((\mu(x), \mu'(x'), \delta)\), respectively.

Let us obtain several corollaries of Theorem \([12]\). Firstly, we reformulate it in terms of multidimensional matrices.

**Corollary 3.** Let \(\mathcal{H}\) and \(\mathcal{H}'\) be d-dimensional adjacency matrices of d-uniform hypergraphs \(\mathcal{H}\) and \(\mathcal{H}'\). If there are color matrices \(P\) and \(P'\) and a d-dimensional matrix \(S\) such that \(A \circ P = P \circ S\) and \(A' \circ P' = P' \circ S\), then there are color matrices \(R\) and \(R'\) and a d-dimensional matrix \(L\) such that \(L \circ R = R \circ A\) and \(L \circ R' = R' \circ A'\).

**Proof.** The statement directly follows from the existence of a common covering (Theorem \([12]\), an interpretation of a covering as a perfect coloring (Proposition \([4]\)), and the definition of perfect colorings by the means of multidimensional adjacency matrices (Theorem \([5]\)).

**Corollary 4.** Let \(\mathcal{H}\) and \(\mathcal{H}'\) be d-uniform connected hypergraphs. There are perfect colorings of \(\mathcal{H}\) and \(\mathcal{H}'\) with the same parameter matrix \(S\) if and only if there exists a hypergraph \(\mathcal{G}\) that covers both \(\mathcal{H}\) and \(\mathcal{H}'\).

**Proof.** Necessity is proved in Corollary \([3]\) and in Theorem \([12]\).

Let us prove sufficiency. Suppose that a hypergraph \(\mathcal{G}\) covers hypergraphs \(\mathcal{H}\) and \(\mathcal{H}'\) with adjacency matrices \(A\) and \(A'\), respectively. Assume that there are no perfect colorings of \(\mathcal{H}\) and \(\mathcal{H}'\) with the same parameter matrix \(S\). In particular, the minimal perfect colorings \(P\) and \(P'\) of these hypergraphs (that exist by Theorem \([3]\)) have different parameter matrices \(S\) and \(S'\):

\[A \circ P = P \circ S; \quad A' \circ P' = P' \circ S'.\]

By Theorem \([11]\) the hypergraph \(\mathcal{G}\) has perfect colorings with the parameter matrices \(S\) and \(S'\). Using again Theorem \([4]\) find the minimal perfect coloring of the hypergraph \(\mathcal{G}\). It has the parameter matrix \(T\) such that \(T\) is different from at least one of the matrices \(S\) and \(S'\). Without loss of generality, assume that \(T\) is not equal to \(S\). Then the perfect coloring of \(\mathcal{G}\) with the parameter matrix \(S\) is a refinement of the minimal perfect coloring of \(\mathcal{G}\). Equivalently, there exists a color matrix \(R\) such that \(S \circ R = R \circ T\). Using Proposition \([1]\) we get

\[A \circ (P \circ R) = A \circ (P \circ R) = (A \circ P) \circ R = (P \circ R) \circ R = P \circ (S \circ R) = P \circ (R \circ T) = (PR) \circ T.\]

Therefore, \(PR\) is a perfect coloring of \(\mathcal{H}\) in which some color classes are a union of color classes of \(P\): a contradiction with the minimality of the perfect coloring \(P\).

\[\Box\]
We also prove that every regular uniform hypergraph can be covered by a multipartite hypergraph that can be partitioned into perfect matchings.

**Corollary 5.** For every \( d \)-uniform \( r \)-regular hypergraph \( \mathcal{H} \) there is a hypergraph \( \mathcal{G} \) such that \( \mathcal{G} \) covers \( \mathcal{H} \) and \( \mathcal{G} \) is \( d \)-partite hypergraph that can be partitioned into \( r \) perfect matchings.

**Proof.** Consider an auxiliary multihypergraph \( \mathcal{F} \) with the incidence matrix \( B \) of size \( d \times r \) whose all entries equal to 1. It consists of \( d \) vertices \( x_1, \ldots, x_d \) and a hyperedge \( e = (x_1, \ldots, x_d) \) of multiplicity \( r \). By the definition, each hyperedge of \( \mathcal{F} \) is a perfect matching and \( \mathcal{F} \) is a \( d \)-partite hypergraph (each vertex is a part).

Colorings of all vertices into one color are perfect colorings of \( \mathcal{H} \) and \( \mathcal{F} \) that also induce monochrome colorings of hyperedges of these hypergraphs. The incidence parameter matrices \( W \) and \( V \) of these colorings are the same, have size \( 1 \times 1 \) and, by the definition, \( W = (d) \) and \( V = (r) \). By Theorem 12, there is a hypergraph \( \mathcal{G} \) covering both \( \mathcal{H} \) and \( \mathcal{F} \). Since the covering preserves the incidence relations between vertices and hyperedges, we see that every vertex of \( \mathcal{F} \) corresponds to a part of \( \mathcal{G} \) and every hyperedge gives a perfect matching in \( \mathcal{G} \). So \( \mathcal{G} \) is a \( d \)-partite hypergraph that can be partitioned into \( r \) perfect matchings.

### 5 Examples of perfect colorings of hypergraphs

#### 5.1 Transversals and perfect matchings

A \( k \)-transversal in uniform regular hypergraphs gives us the simplest nontrivial example of perfect colorings: the vertices of a hypergraph are colored into two colors (transversal and non-transversal), while all hyperedges have the same color. So if \( \mathcal{G} \) is a \( d \)-uniform \( r \)-regular hypergraph, then a \( k \)-transversal is a perfect coloring with the incidence parameters

\[
V = \begin{pmatrix} r \\ r \end{pmatrix}; \quad W = \begin{pmatrix} k & d-k \end{pmatrix}.
\]

By Theorem 5, the parameter matrix \( S \) of a \( k \)-transversal is the \( d \)-dimensional matrix of order 2 with entries

\[
s_{\gamma} = \begin{cases} r^{(d-k-1)} & \text{if } \gamma_1 = 1 \text{ and } \#\{\gamma_i : \gamma_i = 1\} = k; \\ r^{(d-k-1)} & \text{if } \gamma_1 = 2 \text{ and } \#\{\gamma_i : \gamma_i = 1\} = k; \\ 0 & \text{otherwise}. \end{cases}
\]

For example, the parameter matrices of \( k \)-transversals in \( d \)-uniform \( r \)-regular hypergraphs for some small \( d \) and \( k \) are the following:

\[
d = 2, \ k = 1 : \quad S = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix};
\]

\[
d = 3, \ k = 1 : \quad S = \begin{pmatrix} 0 & r/2 & 0 \\ r/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};
\]

\[
d = 4, \ k = 1 : \quad S = \begin{pmatrix} 0 & r/3 & 0 & 0 \\ r/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};
\]

\[
d = 4, \ k = 2 : \quad S = \begin{pmatrix} 0 & 0 & 0 & r/3 \\ 0 & r/3 & r/3 & 0 \\ 0 & r/3 & r/3 & 0 \\ r/3 & 0 & 0 & 0 \end{pmatrix}.
\]

Let us find eigenvalues of transversals as perfect colorings of hypergraphs.
Theorem 13. Suppose that $S$ is the parameter matrix of the perfect coloring corresponding to a $k$-transversal in a $d$-uniform $r$-regular hypergraph $G$. Then the eigenvalues of $S$ are $\lambda_0 = 0$ and $\lambda_j = r\xi^j$, where $\xi$ is a $d$-th primitive root of unity. In particular, the matrix $S$ has $\frac{d}{\gcd(k,d)}$ different nonzero eigenvalues.

Proof. Knowing the entries of the matrix $S$ from Theorem 5, we find that the equation $S \circ x = \lambda I \circ x$ on eigenvalues $\lambda$ is equivalent to the following system:

$$
\begin{align*}
-\lambda x_1^{d-1} + r x_1^{k-1} x_2^{d-k} &= 0; \\
rx_1^{k} x_2^{d-k-1} - \lambda x_2^{d-1} &= 0.
\end{align*}
$$

From here it is easy to see that $\lambda_0 = 0$ is an eigenvalue with the corresponding eigenvectors $(x_1, 0)$ and $(0, x_2)$.

Assume that $x = (x_1, x_2)$ is an eigenvector with both nonzero components: $x_1 = tx_2$, $t \neq 0$. Then the above system of equations can be rewritten as

$$
\begin{align*}
-\lambda t^{d-1} + rt^{k-1} &= 0; \\
rt^{k} - \lambda &= 0.
\end{align*}
$$

Consequently, $\lambda = rt^k$ and $t^d = 1$, so $t$ is a $d$-th root of unity. So all nonzero eigenvalues of the matrix $S$ are $\lambda = r\xi^j$, $j = 1, \ldots, d$, where $\xi$ is a $d$-th primitive root of unity.

We believe that all nonzero eigenvalues of $S$ have the same algebraic multiplicity. Moreover, we propose the following conjecture on the characteristic polynomial of $S$.

Conjecture 1. Let $S$ be the parameter matrix of a $k$-transversal in a $d$-uniform $r$-regular hypergraph $G$. Then the characteristic polynomial of $S$ is

$$
\varphi(\lambda) = \lambda^{d-2} \prod_{i=j}^d (\lambda - \xi^{jk}r),
$$

where $\xi$ is a $d$-th primitive root of unity.

Combining Theorem 13 with Theorem 8, we obtain that every $d$-uniform $r$-regular hypergraph with a $k$-transversal should have values $r\xi^j$ in its spectrum. It gives a spectral condition on existence of $k$-transversals that cannot be reduced to the spectrum of the induced coloring of the incidence graph.

Indeed, consider complete 3-uniform hypergraphs on $n \geq 4$ vertices. It is obvious that they do not contain transversals.

Using computational results for [6] (available at [9]), one can check that complete 3-uniform hypergraphs on $n$ vertices, $4 \leq n \leq 8$, do not have eigenvalues $re^{\pm \frac{2\pi i}{3}}$. It indicates that they do not contain a transversal.

On the other hand, the parameter matrix of the induced perfect coloring of the incidence graph of a $d$-uniform $r$-regular hypergraph is

$$
\begin{pmatrix}
0 & W \\
V & 0
\end{pmatrix} = \begin{pmatrix}
0 & 1 & d-1 \\
r & 0 & 0 \\
r & 0 & 0
\end{pmatrix}.
$$

The eigenvalues of this matrix are $\lambda_1 = 0$ and $\lambda_{2,3} = \pm \sqrt{dr}$, where the latter values are contained in the eigenspectrum of every bipartite biregular graph with degrees of parts $r$ and $d$. Eigenvalue 0 belongs to the spectrum of many of bipartite biregular graphs, for example, the incidence graph of the complete 3-uniform hypergraph on 5 vertices.
5.2 Perfect 2-colorings of 3-uniform hypergraphs

Let us completely describe the smallest case of hypergraph perfect colorings, i.e., perfect colorings \( P \) of \( r \)-regular 3-uniform hypergraphs in 2 colors.

Assume that \( n_1 \) and \( n_2 \) are the numbers of vertices of first and second colors and let \( \chi = n_1/n_2 \leq 1 \). In a general case, the incidence parameters \((V, W)\) of the coloring \( P \) are

\[
V = \begin{pmatrix}
v_{1,1} & v_{1,2} & v_{1,3} & 0 \\
0 & v_{2,2} & v_{2,3} & v_{2,4}
\end{pmatrix}; \quad W = \begin{pmatrix}
3 & 0 \\
2 & 1 \\
1 & 2 \\
0 & 3
\end{pmatrix},
\]

where \( v_{1,1} + v_{1,2} + v_{1,3} = v_{2,2} + v_{2,3} + v_{2,4} = r \). If some color range of hyperedges is absent in the coloring \( P \), then the matrices \( W \) and \( V \) lack the corresponding row and column.

From the equality \( NV = W^T M \) (see Theorem 3(1)), we deduce that

\[
n_1v_{1,2} = 2n_2v_{2,2}; \quad n_2v_{2,3} = 2n_1v_{1,3},
\]

and, consequently,

\[
\frac{v_{2,2}}{v_{1,2}} = \frac{\chi}{2}; \quad \frac{v_{2,3}}{v_{1,3}} = 2\chi.
\]

The parameter matrix of a perfect coloring \( P \) is a 3-dimensional matrix of order 2:

\[
\mathbb{S} = \begin{pmatrix}
s_{1,1,1} & s_{1,1,2} & s_{2,1,1} & s_{2,1,2} \\
s_{1,2,1} & s_{1,2,2} & s_{2,2,1} & s_{2,2,2}
\end{pmatrix}.
\]

With the help of Theorem 5, we find entries of the parameter matrix \( \mathbb{S} \):

\[
s_{1,1,1} = v_{1,1}; \quad s_{1,1,2} = s_{1,2,1} = v_{1,2}/2; \quad s_{1,2,2} = v_{1,3};
\]

\[
s_{2,1,1} = v_{2,1}; \quad s_{2,1,2} = s_{2,2,1} = v_{2,3}/2; \quad s_{2,2,2} = v_{2,4}.
\]

Using additionally equalities (3), we reduce the number of essential parameters of the matrix \( \mathbb{S} \) and denote them by \( a, b, c \) and \( d \):

\[
\mathbb{S} = \begin{pmatrix}
a & b & \chi b & \chi c \\
b & c & \chi c & d
\end{pmatrix}.
\]

To calculate the characteristic polynomial \( \varphi(\lambda) \) of the matrix \( \mathbb{S} \), we use a formula from [18] for resultants \( R_{2,2} \):

\[
\varphi(\lambda) = \lambda^4 - 2\lambda^3(a + d) + \lambda^2(d^2 + 4ad + a^2 - 6\chi bc) + \lambda((a + d)(6\chi bc - 2ad) - 4\lambda^2c^3 - \chi b^3) + a^2d^2 - 3\lambda^2b^2c^2 - 6\chi abcd + 4\lambda^2ac^3 + \chi b^3d.
\]

5.3 Fano’s plane hypergraph

Here we illustrate that perfect colorings can be used for finding eigenvalues of some hypergraphs and multidimensional matrices.

Consider a hypergraph \( \mathcal{F} \) corresponding to the Fano’s plane: \( \mathcal{F} \) is a 3-uniform hypergraph on 7 vertices with the following 7 hyperedges:

\[
(x_1, x_2, x_3), \ (x_1, x_4, x_5), \ (x_1, x_6, x_7), \ (x_2, x_4, x_6), \ (x_2, x_5, x_7), \ (x_3, x_4, x_7), \ (x_3, x_5, x_6).
\]

Due to the symmetry of the hypergraph \( \mathcal{F} \), it is not hard to list all its 2-colorings and find that only two of them are perfect.

1. Color one vertex of \( \mathcal{F} \) into the first color, and all other six vertices into the second color (Figure 2):
The incidence parameters are
\[ V = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}; \quad W = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}. \]

Using the help of Theorem 5, we find that the multidimensional parameter matrix of this perfect coloring is
\[ S_1 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1/2 \\ 1/2 & 2 \end{pmatrix}. \]

To calculate the characteristic polynomial \( \varphi_1(\lambda) \) of the matrix \( S_1 \), we use a formula from [18] or the result for 2-colorings of 3-uniform hypergraphs from the previous section:
\[ \varphi_1(\lambda) = \lambda(\lambda - 3)(\lambda^2 - \lambda + 1). \]

So the parameter matrix \( S_1 \) has eigenvalues 0, 3, and \( \frac{1}{2}(1 \pm i\sqrt{3}) \).

2. Another perfect coloring of the hypergraph \( F \) is presented at Figure 3.

The incidence parameters of this coloring are
\[ V = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}; \quad W = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}. \]

Theorem 5 gives the following multidimensional parameter matrix:
\[ S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3/2 \\ 3/2 & 0 \end{pmatrix}. \]

Using the result for 2-colorings of 3-uniform hypergraphs from previous section, we find the characteristic polynomial for the matrix \( S_2 \):
\[ \varphi_2(\lambda) = (\lambda - 3)(\lambda - 1)(\lambda^2 + 2\lambda + 6). \]

So the parameter matrix \( S_2 \) has eigenvalues 1, 3, and \( -1 \pm i\sqrt{5} \).

Thus, the hypergraph \( F \) of the Fano’s plane has at least 7 different eigenvalues:
\[ \lambda_0 = 0, \quad \lambda_1 = 1, \quad \lambda_2 = 3, \quad \lambda_{3,4} = \frac{1}{2}(1 \pm i\sqrt{3}), \quad \lambda_{5,6} = -1 \pm i\sqrt{5}. \]
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