Sullivan completions

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Abstract

The Sullivan construction associates to each path connected space or connected simplicial set, \( X \), a special cdga, its minimal model \((\wedge V, d)\), and to each such cdga \( \wedge W \) its geometric realisation \( \langle \wedge W \rangle \). The composite of these constructions is the Sullivan completion, \( X_\mathbb{Q} \), of \( X \). In this paper we give a survey of the main properties of Sullivan completions, and include explicit examples.

1 Introduction

The fundamental contributions of Quillen \cite{Quillen} and Sullivan \cite{Sullivan} in 1970 were initial landmarks in formalizing the rationalization of a path connected space. In fact, Quillen completely solved the problem for a simply connected space, identifying the rationalization as the realization of a differential graded Lie algebra. Sullivan then formalized his approach via his minimal models, \cite{Sullivan}.

Intuitively, the homology and homotopy groups of the rationalization of a space should be the tensor product of the original groups with \( \mathbb{Q} \), and this is the case with Quillen’s construction. Since this is not possible for non-simply connected spaces, \( X \), one looks instead for useful ”completions”, inspired in particular by the classical completion \( B \to \hat{B} \) of an augmented algebra \( B \xrightarrow{\varepsilon} \mathbb{Q} \):

\[
\hat{B} := \lim_{\leftarrow} B/I^k,
\]

\( I^k \) denoting the \( k \)th power of the augmentation ideal \( I := \ker \varepsilon \).

For path connected spaces, or connected simplicial sets, the \( R \)-completions \( R_\infty(X) \) of Bousfield-Kan \cite{BousfieldKan}, defined for any commutative ring \( R \), are prime examples, and it follows from \((\text{\cite{BousfieldKan} (4.3), p.137})\) that up to homotopy these coincide with Quillen’s construction for simply connected spaces when \( R = \mathbb{Q} \). On the other hand, there is a similarity to the constructions of \( \mathbb{Q}_\infty(X) \) and of Sullivan completions \( X_\mathbb{Q} \). For instance, Sullivan completions can be described as a functor \( \ell_\mathbb{Q} : X \to X_\mathbb{Q} \), characterized by a universal property analogous to that of \( \mathbb{Q}_\infty(X) \) established in \((\text{\cite{BousfieldKan} III (6.2)})\). The Bousfield-Kan completion of \( X \) is the inverse limit

\[
X \to R_\infty(X) = \lim_{\leftarrow} R_k(X)
\]
of a tower of fibrations, and inducing an isomorphism $\lim_{\rightarrow k} H^*(R_k(X);R) \xrightarrow{\cong} H^*(X;R)$. Analogously, the Sullivan completion of $X$ is the inverse limit

$$\ell : X \to X_\mathbb{Q} = \lim_{\alpha} X_\alpha$$

of a specific inverse system inducing an isomorphism $\lim_{\rightarrow \alpha} H^*(X_\alpha;\mathbb{Q}) \xrightarrow{\cong} H^*(X;\mathbb{Q})$. Here the $X_\alpha$ are nilpotent spaces defined via the Sullivan model of $X$, and for which $\oplus_{i \geq 2} \pi_i(X)$ and each $\pi_1(X)^n/\pi_1(X)^{n+1}$ is a finite dimensional rational vector space. Here $\pi_1(X)^n$ denotes the $n^{th}$ term of the lower central series of $\pi_1(X)$.

Finally, as Bousfield has shown ([3]), the completions $\mathbb{Q}_\infty(X)$ and $X_\mathbb{Q}$ are directly related: the map $X \to X_\mathbb{Q}$ factors up to homotopy as $X \to \mathbb{Q}_\infty(X) \to X_\mathbb{Q}$. Moreover, if in the terminology of [5] $X$ is $\mathbb{Q}$-good, then $\mathbb{Q}_\infty(X)_\mathbb{Q} \to X_\mathbb{Q}$ is a homotopy equivalence. Furthermore, ([4, Theorem 12.2]) if $H^*(X;\mathbb{Q})$ is a graded vector space of finite type then $\mathbb{Q}_\infty(X) \to X_\mathbb{Q}$ is a homotopy equivalence. Finally we note that for a map $f : X \to Y$ between path connected spaces the following conditions are equivalent: (i) $H^*(f;\mathbb{Q})$ is an isomorphism, (ii) $\mathbb{Q}_\infty(f) : \mathbb{Q}_\infty(X) \to \mathbb{Q}_\infty(Y)$ is a homotopy equivalence, and (iii) $f_\mathbb{Q} : X_\mathbb{Q} \to Y_\mathbb{Q}$ is a homotopy equivalence.

Here we shall focus on the properties of Sullivan completions, $X_\mathbb{Q}$. These are constructed as the simplicial realization of a minimal Sullivan model $(\land V,d)$ for $X$. Because $H(\land V,d) \cong H^*(X;\mathbb{Q})$ this model provides an often computable approach to the cohomology of $X$. On the other hand, there is a natural bijection

$$\pi_*(X_\mathbb{Q}) \cong \text{Hom}(V,\mathbb{Q})$$

which provides access to the structure and properties of $\pi_*(X_\mathbb{Q})$.

Some Sullivan completions are very simple. For instance $S^1_\mathbb{Q}$ is the rationalization of $S^1$, constructed by a telescope process. But in general, Sullivan completions are more complicated and with quite mysterious cohomology. For example, Ivanov and Mikhailov have recently proved ([11]) that $H_2((S^1 \lor S^1)_\mathbb{Q};\mathbb{Q})$ is uncountable. Moreover it is unclear whether $H_{\geq 1}(X_\mathbb{Q};\mathbb{Z})$ is a rational vector space, though this is true for the $X_\alpha$.

For simplicity we adopt the following

Conventions

(i) By "space" we mean either a CW complex or a simplicial set.

(ii) For a space $X$ we write $H(X) := H^*(X;\mathbb{Q})$.

(iii) We write $(\_\_\_\_\_\_\_\_\_\_)^\vee := \text{Hom}(\_\_\_\_\_\_\_\_\_,\mathbb{Q})$.

(iv) Where there is no ambiguity we suppress the differentials from the notation for a complex and write $A$ instead of $(A,d)$.

Finally, our thanks to Pete Bousfield, [3], for a number of helpful suggestions, including the observation that $X \to X_\mathbb{Q}$ factors through $\mathbb{Q}_\infty(X)$. 2
2 Basic constructions

We briefly review the basic facts and notation from Sullivan’s theory. A \( \Lambda \)-algebra is a commutative differential graded algebra (cdga) of the form \( (\wedge V, d) \), where \( V = V^0 \) is a graded vector space and \( \wedge V \) is the free graded commutative algebra generated by \( V \). Here the differential is required to satisfy the \textit{Sullivan condition}: \( V = \cup_{n \geq 0} V(n) \), where

\[
V(0) = V \cap \ker d \quad \text{and} \quad V(n + 1) = V \cap d^{-1}(\wedge V(n)).
\]

(1)

Following our convention, when there is no ambiguity we suppress the differential from the notation and write

\[\wedge V = (\wedge V, d).\]

The cylinder object for a cdga \( A \) is the cdga \( A \otimes \wedge(t, dt) \) in which \( \deg t = 0 \), together with the morphisms \( \varepsilon_0, \varepsilon_1 : A \otimes \wedge(t, dt) \to A \) sending \( t \mapsto 0, 1 \). Then two morphisms \( \varphi_0, \varphi_1 : \wedge V \to A \) are \textit{homotopic} if some morphism \( \varphi : \wedge V \to A \otimes \wedge(t, dt) \) satisfies \( \varepsilon_i \circ \varphi = \varphi_i \). This is denoted by \( \varphi_0 \sim \varphi_1 \). In particular, a quasi-isomorphism between \( \Lambda \)-algebras satisfying \( H^0 = Q \) is a homotopy equivalence.

Now \( \wedge V = \oplus_{p \geq 0} \wedge^p V \), where \( \wedge^p V \) denotes the linear span of the monomials in \( V \) of length \( p \); \( p \) is called the \textit{wedge degree}. In particular, a \( \Lambda \)-algebra is \textit{minimal} if \( d : V \to \wedge^2 V \) and \textit{quadratic} if \( d : V \to \wedge^2 V \). Thus a minimal \( \Lambda \)-algebra \( (\wedge V, d) \) determines the associated quadratic \( \Lambda \)-algebra \( (\wedge V, d_1) \) defined by: \( d_1 v \) is the component of \( dv \) in \( \wedge^2 V \).

A Sullivan algebra is a \( \Lambda \)-algebra \( (\wedge V, d) \) with \( V = V^1 \). A (minimal) Sullivan model for a cga \( A \) is a quasi-isomorphism \( \wedge V \xrightarrow{\cong} A \) from a (minimal) Sullivan algebra, and if \( H^0(A) = Q \) then \( A \) has a minimal Sullivan model. Moreover, given Sullivan models for \( A \) and \( B \), a morphism \( \sigma : A \to B \) determines a unique homotopy class of morphisms, \( \psi_\sigma \), for which

\[
\begin{array}{ccl}
\wedge V & \xrightarrow{\cong} & A \\
\wedge W & \xrightarrow{\cong} & B \\
\xrightarrow{\psi_\sigma} & & \xrightarrow{\sigma}
\end{array}
\]

is homotopy commutative; \( \psi_\sigma \) is a \textit{Sullivan representative} for \( \sigma \). Finally \( \Lambda \)-algebras can be reduced to minimal Sullivan algebras: if \( \wedge W \) is a \( \Lambda \)-algebra and \( H^0(\wedge W) = Q \) then \( \wedge W \cong (\wedge V, d) \otimes (U \oplus dU) \), where \( (\wedge V, d) \) is a minimal Sullivan algebra and \( d : U \xrightarrow{\cong} dU \). Thus the inclusion \( \wedge V \to \wedge W \) is a homotopy equivalence, as is the surjection \( \wedge W \to \wedge V \).

Sullivan algebras link topological spaces and simplicial sets with their Sullivan completions via two pairs of adjoint functors

\[
Sing : \text{Top} \rightsquigarrow \text{Simp} \quad \text{and} \quad | : \text{Simp} \rightsquigarrow \text{Top},
\]

and

\[
A_{PL} = \text{Simp}(-, A_{PL}^\ast) : \text{Simp} \rightsquigarrow \text{Cdga} \quad \text{and} \quad \langle \cdot \rangle = \text{Cdga}(-, A_{PL}^\ast) : \text{Cdga} \rightsquigarrow \text{Simp}.
\]

(Here \( \text{Simp} \) is the category of simplicial sets and \( Sing \) is the simplicial set of singular simplices on \( X \), and \( A_{PL}^\ast \) is the simplicial cdga with \( (A_{PL})_n \) the rational cdga generated by the coordinate functions on \( \Delta^n \).) For simplicity we write \( A_{PL}(X) := A_{PL}(Sing X) \) for topological space, \( X \).
For any $\Lambda$-algebra $\wedge V$, $id_{\langle \wedge V \rangle}$ is adjoint to a morphism

$$m_{\Lambda V} : \wedge V \to A_{PL}(\wedge V).$$

Moreover, the functor $\langle \rangle$ associates to each morphism of $\Lambda$-algebras, $\varphi : \wedge V \to \wedge W$, a morphism of simplicial sets $\langle \wedge W \rangle \to \langle \wedge V \rangle$, and

$$A_{PL}(\varphi) \circ m_{\wedge V} = m_{\wedge W} \circ \varphi.$$ 

It is immediate from the definition that $\langle \rangle$ has the following properties

(i) $\langle \wedge V \otimes \wedge W \rangle = \langle \wedge V \rangle \times \langle \wedge W \rangle$

(ii) If $\varphi_0 \sim \varphi_1 : \wedge V \to \wedge W$ then $\langle \varphi_0 \rangle$ and $\langle \varphi_1 \rangle$ are homotopic.

(iii) $\langle \rangle$ converts direct limits to inverse limits and quasi-isomorphisms to homotopy equivalences. 

This yields

From the adjoint functors above it follows that, for a simplicial set $X$ and a $\Lambda$-algebra $\wedge V$, adjoint to any morphism $\varphi : \wedge V \to A_{PL}(X)$ is a simplicial map

$$\bar{\varphi} : X \to \langle \wedge V \rangle.$$ 

On the other hand, suppose $\varphi : \wedge V \to A_{PL}(X)$ is any cdga morphism from a Sullivan algebra. Then $\varphi$ and $m_{\wedge V}$ are connected via the commutative diagram

$$\begin{array}{c}
A_{PL}(X) \\
\varphi \\
\wedge V
\end{array} \leftarrow \begin{array}{c}
A_{PL}(\bar{\varphi}) \\
m_{\wedge V}
\end{array} \rightarrow \begin{array}{c}
A_{PL}(\langle \wedge V \rangle)
\end{array}$$ 

**Definition.**

(i) If $A$ is a cdga then $\langle A \rangle$ is its simplicial realization.

(ii) A (minimal) Sullivan model for a connected space $X$, is a quasi-isomorphism from a (minimal) Sullivan algebra,

$$\varphi : \wedge V \cong \longrightarrow A_{PL}(X).$$ 

(iii) For such a minimal Sullivan model,

$$\bar{\varphi} : X \longrightarrow X_Q := \langle \wedge V \rangle$$

is the Sullivan completion of $X$.

(iv) A Sullivan representative for a map $f : X \to Y$ is a Sullivan representative for $A_{PL}(f)$. Its simplicial realization, $f_Q : X_Q \to Y_Q$, satisfies $\bar{\varphi}_Y \circ f \sim f_Q \circ \bar{\varphi}_X$. 

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Example. If each \( \dim V_i < \infty \) then \( H(\wedge V) \) is a graded vector space of finite type, and by (\( \text{S} \) Theorem 5.4) \( m_{\wedge V} \) is a quasi-isomorphism. Therefore, in this case
\[
H(\wedge V) = H((\wedge V); \mathbb{Q}) \quad \text{and} \quad (\wedge V) = (\langle \wedge V \rangle)_{\mathbb{Q}}.
\] (4)

Next, recall (\( \text{S} \) Theorem 3.1) that any morphism \( \varphi : \wedge V \to \wedge W \) of Sullivan algebras factors as
\[
\wedge V \to \wedge V \otimes \wedge Z \cong \wedge W, \quad v \mapsto v \otimes 1,
\]
in which \( Z = \bigcup_{k \geq 0} Z(k) \) and \( d : 1 \otimes Z(k+1) \to \wedge V \otimes \wedge Z(k) \). The inclusion \( \wedge V \to \wedge V \otimes \wedge Z \) is called a \( \Lambda \)-extension, and \( \wedge V \otimes \wedge Z = \wedge (V \oplus Z) \) and the quotient \( (\wedge Z, \overline{d}) := Q \otimes_{\wedge V} (\wedge V \otimes \wedge Z) \) are \( \Lambda \)-extensions. By (\( \text{S} \) Proposition 17.9) the sequence
\[
(\langle \wedge V \rangle) \leftarrow (\langle \wedge V \otimes \wedge Z \rangle) \leftarrow (\langle \wedge Z \rangle)
\]
is a Serre fibration and \( (\langle \wedge V \otimes \wedge Z \rangle) \) is homotopy equivalent to \( (\langle \wedge W \rangle) \).

In particular, when \( \varphi \) is a Sullivan representative for \( f : X \to Y \) then \( (\langle \wedge Z \rangle) \) is homotopy equivalent to the homotopy fibre of \( F_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}} \). However it is important to note that \( \langle \wedge Z \rangle \) may not be the Sullivan completion of a connected space.

Finally, any simplicial map \( \sigma : X \to \langle \wedge V \rangle \) is the adjoint of a morphism \( \varphi : \wedge V \to A_{PL}(X) \). If \( X \) is connected and \( \psi : \wedge W \to A_{PL}(X) \) is a Sullivan model, then \( \varphi \) lifts up to homotopy through \( \psi \) to give a morphism \( \chi : \wedge V \to \wedge W \). It is immediate from the definition that the diagram
\[
\begin{array}{ccc}
X_{\mathbb{Q}} = (\langle \wedge W \rangle) & \xrightarrow{\langle \chi \rangle} & (\langle \wedge V \rangle) \\
\downarrow{\psi} & & \downarrow{\sigma} \\
X & & \langle \wedge V \rangle
\end{array}
\]
(5)
homotopy commutes.

3 Homotopy groups, examples

Let \( \wedge V \) be a minimal Sullivan algebra. Then the adjoint functors in \( \S \), together with the fact that \( H(S^n) \) and \( A_{PL}(S^n) \) have isomorphic minimal Sullivan models, yield natural set bijections (\( \text{S} \) Theorem 1.3),
\[
\pi_n(\langle \wedge V \rangle) \cong (V^n)^\vee, \quad n \geq 1.
\] (6)

These are isomorphisms of abelian groups when \( n \geq 2 \) (\( \text{S} \) Theorem 1.4), and so identify \( \pi_{\geq 2}(\langle \wedge V \rangle) \) as a rational vector space. The group multiplication in \( \pi_1(\langle \wedge V \rangle) \) also has an explicit description in terms of \( \wedge V \) (cf \( \S \)).

Then, if \( \varphi : \wedge V \to \wedge W \) is a morphism of minimal Sullivan algebras, a linear map \( \varphi_0 : V \to W \) is defined by \( \varphi_0 v \) is the component of \( \varphi v \) in \( W \). It is immediate from the construction of the bijections (\( \text{S} \)) that they identify
\[
\pi_*(\varphi) = \varphi_0^\vee.
\] (7)

In particular there follows
Proposition 1. Suppose $\varphi : \wedge V \to \wedge W$ is a morphism of $\Lambda$-algebras for which $H^0(\wedge V) = Q = H^0(\wedge W)$. Then the following conditions are equivalent

(i) $H(\varphi)$ is an isomorphism

(ii) $\langle \varphi \rangle$ is a homotopy equivalence.

proof: Because these $\Lambda$-algebras are the tensor product of an acyclic $\Lambda$-algebra and a minimal Sullivan algebra, it is sufficient to consider the case both are minimal. In this case if $H(\varphi)$ is an isomorphism $\varphi$ must be an isomorphism and $\langle \varphi \rangle$ is a homeomorphism.

In the other direction, suppose $\langle \varphi \rangle$ is a homotopy equivalence. Then it follows from (7) that $\varphi_0$ is an isomorphism and hence $\varphi$ is an isomorphism. □

Example. Note that $(S^1 \vee S^2)_Q \neq S^1_Q \vee S^2_Q$. Indeed let $\wedge V$ be the minimal Sullivan model of $S^1 \vee S^2$. Then $\dim V^2 = \infty$ and so $\pi_2((S^1 \vee S^2)_Q)$ is an uncountable vector space. On the other hand, since the universal cover of $S^1_Q \vee S^2_Q$ is a wedge of countably many rational spheres $S^2_Q$, $\pi_2(S^1_Q \vee S^2_Q)$ is a countable vector space.

Example. For a path connected space $X$ with minimal Sullivan model $(\wedge V, d)$, by [8, Theorem 9.2], $\text{cat}(\wedge V) \leq \text{cat} X$, where $\text{cat}(\wedge V)$ is the Lusternik-Schnirelmann category of $\wedge V$ defined in [8] §9.1. Since the natural map $m_{\wedge V} : \wedge V \to A_{PL}(\wedge V)$ has a homotopy section ([8, Theorem 1.13]), we have also

$$\text{cat}(\wedge V) \leq \text{cat}(\wedge V).$$

$\Lambda$-algebras, $\wedge V$, are equipped with an additional structure: the family, $J_V = \{V_\alpha\}$, of finite dimensional subspaces $V_\alpha \subset V$ for which $\wedge V_\alpha$ is preserved by $d$. For simplicity, we write

$$J_V = \{\alpha\}.$$

It follows from the Sullivan condition that $J_V$ is closed under arbitrary intersections and finite sums. In particular, $J_V$ is a directed set under inclusion, and

$$V = \lim_{\alpha} V_\alpha. \quad (8)$$

Because $(\quad)$ and $(\quad)^\vee$ convert direct limits to inverse limits, this gives

$$\langle \wedge V \rangle = \lim_{\alpha} \langle \wedge V_\alpha \rangle \quad \text{and} \quad \pi_*(\wedge V) = \lim_{\alpha} \pi_*(\wedge V_\alpha). \quad (9)$$

If $\wedge V$ is a minimal Sullivan model of $X$, we write $X_\alpha = \langle \wedge V_\alpha \rangle$ and then

$$X_Q = \lim_{\alpha} X_\alpha, \quad H(X) = \lim_{\alpha} H(X_\alpha), \quad \text{and} \quad \pi_*(X_Q) = \lim_{\alpha} \pi_*(X_\alpha).$$

Moreover, by (11), $X_\alpha = (X_\alpha)_Q$.

Example. A minimal Sullivan algebra $\wedge V$ has a countable basis if and only if $H(\wedge V)$ has a countable basis, and this condition holds if and only if in the filtration (11) each $V[n] := V^{\leq n}(n)$ is finite dimensional. In this case $\{V[n]\}$ is cofinal with $J_V$. In particular,
if $\wedge V$ is the minimal Sullivan model of a space $X$ then this condition holds if and only if each $H^n(X)$ is finite dimensional.

Next, suppose $\varphi : \wedge W \to A_{PL}(BG)$ is the minimal Sullivan model of the classifying space of a discrete group $G$. Then $\varphi$ restricts to a morphism

$$\varphi_1 : \wedge W^1 \to A_{PL}(BG).$$

**Definition.** The group homomorphism

$$\pi_1(\varphi_1) : G \to \pi_1(\langle \wedge W^1 \rangle)$$

is the **Sullivan completion** of $G$, and we write

$$G_\mathbb{Q} := \pi_1(\langle \wedge W^1 \rangle).$$

If $\wedge V$ and $\wedge W$ are respectively the minimal Sullivan models of $X$ and $B\pi_1(X)$ then ([8, diagram 7.3]) the Sullivan algebras $\wedge V^1$ and $\wedge W^1$ coincide. It follows that

$$\pi_1(X_\mathbb{Q}) = [\pi_1(X)]_\mathbb{Q}. \quad (10)$$

**Example.** Let $X$ be the wedge of two circles, $X = S^1 \vee S^1$. Then $X_\mathbb{Q} = K(G,1)$ where $G$ is the Sullivan completion of a free group on two generators. Since $S^1$ is a retract of $X$, $S^1_\mathbb{Q}$ is a retract of $X_\mathbb{Q}$ and by [6, Example 4.10], $\text{cat}(X_\mathbb{Q}) \geq 2$.

A minimal Sullivan algebra also provides an expression for the **Hurewicz homomorphism**, $\text{hur} : \pi_*(\wedge V) \to H_*(\langle \wedge V \rangle; \mathbb{Q})$.

In fact, division by $\wedge^{\geq 2} V$ induces a linear map $H^{\geq 1}(\wedge V) \xrightarrow{\xi} V$. Dualizing gives a linear map $\xi^\vee : \pi_*(\wedge V) \to (H(\wedge V))^\vee$. If $\dim V < \infty$ then the inclusion $H_*(\langle \wedge V \rangle; \mathbb{Q}) \to (H^*(\wedge V))^\vee$ is an isomorphism. In this case, by [8, Proposition 1.19], $\xi$ is the classical Hurewicz homomorphism.

For general minimal Sullivan algebras note that dualizing $H(m_{\wedge V})$ yields the linear map

$$j : H_*(\langle \wedge V \rangle; \mathbb{Q}) \to H(\langle \wedge V \rangle)^\vee \to H(\wedge V)^\vee = \lim_{\alpha} H(\wedge V_\alpha)^\vee = \lim_{\alpha} H_*(\langle \wedge V_\alpha \rangle; \mathbb{Q}).$$

Here we have the commutative diagram

$$\begin{array}{cccc}
\pi_*(\wedge V) & \xrightarrow{\cong} & \lim_{\alpha \in \mathcal{I}_V} \pi_*(\wedge V_\alpha) & \\
\downarrow \text{hur} & & \downarrow \text{lim}_{\alpha \in \mathcal{I}_V} \text{hur}_\alpha & \\
H_*(\langle \wedge V \rangle; \mathbb{Q}) & \xrightarrow{j} & \lim_{\alpha \in \mathcal{I}_V} H_*(\langle \wedge V_\alpha \rangle; \mathbb{Q}). & \\
\end{array}$$

Finally, the homology $H_*(X_\mathbb{Q}; \mathbb{Z})$ remains mysterious, as it is even unknown whether or not is is a rational vector space. We do have, as pointed to us by J. Rosenberg, the following.
Proposition 2. If in a minimal Sullivan algebra, $\wedge V$, $\dim V^1 < \infty$, then $H_{\geq 1}(\langle \wedge V \rangle; \mathbb{Z})$ is a rational vector space.

proof: Denote $\pi_1(\wedge V)$ by $G$. According to [7, Proposition 17.9], $\langle \wedge V \rangle$ decomposes as a (Serre) fibration

$$\langle \wedge V^2 \rangle \to \langle \wedge V \rangle \to \langle \wedge V^1 \rangle.$$ 

The Serre spectral sequence for this fibration then converges to $H_*(\langle \wedge V \rangle; \mathbb{Z})$ from the group homology

$$\text{Tor}_{p}^{\mathbb{Z}[G]}(\mathbb{Z}, H_q(\langle \wedge V^2 \rangle; \mathbb{Z})).$$

It is therefore sufficient to show that $\text{Tor}_{\geq 1}^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})$ and $H_{\geq 1}(\langle \wedge V^2 \rangle; \mathbb{Z})$ are rational vector spaces.

The second assertion is ([10, Chap II, Proposition 1.1]), since $\pi_*(\langle \wedge V^2 \rangle)$ is a rational vector space. For the first assertion, note that $\langle \wedge V^1 \rangle$ is an Eilenberg-MacLane space $K(G, 1)$. Since $G$ is the direct limit of finitely generated nilpotent groups $G_\lambda$, it is sufficient to show that the image of $H_{\geq 1}(K(G_\lambda, 1); \mathbb{Z})$ in $H_{\geq 1}(K(G, 1); \mathbb{Z})$ is contained in an abelian subgroup which is itself a rational vector space.

But since $\dim V^1 < \infty$, $\langle \wedge V^1 \rangle = \langle \wedge (v_1, \ldots, v_r) \rangle$. Since $\langle \wedge v_i \rangle = S^1_i$ and $H_{\geq 1}(S^1_i, \mathbb{Z}) = \mathbb{Q}$, it follows by induction via a Serre spectral sequence argument that $H_{\geq 1}(\langle \wedge (v_1, \ldots, v_r) \rangle; \mathbb{Z})$ is a rational vector space. \qed

Finally, even in the simply connected case, the homology of a Sullivan completion can be enormous, as the following Proposition shows.

Proposition 3. If $(\wedge V, d)$ is a minimal Sullivan algebra in which $H_{\geq 1}(\wedge V) = H^2(\wedge V)$ is countably infinite, then $H_3(\langle \wedge V, d \rangle)$ is uncountable.

Corollary. If $X$ is a countable wedge of 2-spheres then $H_3(X; \mathbb{Q})$ is uncountable.

proof of Proposition 3. $H^2(\wedge V)$ is the union of an increasing sequence of finite dimensional vector spaces $A(n)$. Let $(\wedge V(n), d)$ be the minimal model of the cdga $(\mathbb{Q} \oplus A(n), 0)$. Denote by $L$ the homotopy Lie algebra of $(\wedge V, d)$ and by $L(n)$ the homotopy Lie algebras for the $(\wedge V(n), d)$.

Since $A(n)$ is finite dimensional, by [7, Example 2, §18]) $L(n)$ is the free graded Lie algebra $\mathbb{L}(L(n)_1)$. On the other hand, $L = \varprojlim_n L(n)$. Thus the natural map

$$\varphi : \mathbb{L}(L_1) \to L$$

is injective and an isomorphism in degree 1.

Now decompose a basis of $V^2$ into two infinite sequences, $x_k, k \geq 1$ and $y_k, k \geq 1$. There are then elements $z_{ij}, t_{ij}$ and $t'_{ij}$ in $V^3$ with

$$dz_{ij} = x_iy_j, \ dt_{ij} = x_ix_j, \ dt'_{ij} = y_iy_j.$$ 

Then let $W$ be the linear span of the $z_{ij}$. The dual elements $f_i \in W^\vee$ defined by $f_i(z_{jj}) = \delta_{ij}$ are the basis of a countable subspace of $W^\vee$, fix a direct summand $Q \subset W^\vee$ of this subspace. Then choose elements $u_\alpha \in L_2$ so that the $su_\alpha$ vanish on the $t_{ij}$, the $t'_{ij}$ and on the $z_{ij}$, $i \neq j$, and so that when restricted to $W$ the $su_\alpha$ are a basis of $Q$. 

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Thus the $su_\alpha$ are an uncountable set, and no linear combination of the $su_\alpha$ can vanish on all but finitely many $z_i$. In particular the $u_\alpha$ represent linearly independent elements in $L_2/L^2(L_1)$

Next set $X = \langle \wedge V \rangle$, and let $S$ be a wedge of rational spheres $S^2_\mathbb{Q}$ admitting a map $f : S \to X$ that induces an isomorphism on $\pi_2$. Then $\pi_\leq 2(X, S) = 0$, and

$$\pi_3(S) \to \pi_3(X) \to \pi_3(X, S) \to 0$$

is exact. Denote by $\mathfrak{su}_\alpha$ the image of $su_\alpha$ in $\pi_3(X, S)$. Since the image of $\pi_3(S)$ in $\pi_3(X)$ is contained in $sL^2(L_1)$, the $\mathfrak{su}_\alpha$ are linearly independent.

Next, denote by $C$ the homotopy cofiber of $f$, $C = X \cup_S CS$, where $CS$ is the cone on $S$. Then

$$\pi_r(CS, S) \cong \pi_{r-1}(S).$$

Thus the pairs $(X, S)$ and $(CS, S)$ are both 2-connected. It follows from the Blakers-Massey theorem ([1]) that the natural map

$$\pi_3(X, S) \to \pi_3(C, CS)$$

is an isomorphism. From the commutative diagram

$$\begin{array}{ccc}
\pi_3(X) & \xrightarrow{g} & \pi_3(C) \\
\downarrow & & \downarrow \\
\pi_3(X, S) & \cong & \pi_3(C, CS)
\end{array}$$

it follows that the $g(\mathfrak{su}_\alpha)$ are linearly independent.

Now remark that $C$ is 2-connected, and that the Hurewicz theorem yields a commutative diagram

$$\begin{array}{ccc}
\pi_3(X) & \xrightarrow{g} & \pi_3(C) \\
\downarrow & & \downarrow \\
H_3(X) & \cong & H_3(C)
\end{array}$$

It follows that the images under the Hurewicz map of the elements $su_\alpha$ are linearly independent in $H_3(X; \mathbb{Q})$, and so $H_3(X; \mathbb{Q})$ is uncountable. $\square$

4 The homotopy Lie algebra, $L_V$, and the completions of $UL_V$ and $H_*(\Omega X; \mathbb{Q})$

For each minimal Sullivan algebra $\wedge V$, the associated quadratic differential, $d_1$, determines the homotopy Lie algebra,

$$L_V := \{(L_V)_p\}_{p \geq 0}.$$

$L_V$ is defined by a (degree 1) suspension isomorphism $s : (L_V)_p \cong (V^{p+1})^\wedge$ and

$$\langle v, s[x, y] \rangle = (-1)^{deg y+1} \langle d_1 v, sx, sy \rangle, \quad v \in V.$$
Moreover, the family $I = \{V_\alpha\}$ endows $L_V$ with additional structure. Denote by $L_\alpha$ the homotopy Lie algebra of $\wedge V_\alpha$. Since $\dim V_\alpha < \infty$, $L_\alpha$ is a nilpotent graded Lie algebra. The duals of the inclusions $V_\alpha \to V$ then desuspend to Lie algebra surjections $\rho_\alpha : L_V \to L_\alpha$, which define an isomorphism

$$L_V \xrightarrow{\cong} \lim_{\alpha} L_\alpha.$$ 

On the other hand, recall that the classical completion of the universal enveloping algebra, $UL$, of a graded Lie algebra $L$ is the inverse limit

$$\hat{UL} = \lim_{\leftarrow n} UL/I^n,$$ 

$I^n$ denoting the $n^{th}$ power of the augmentation ideal $I$ generated by $L$. It turns out that if $L_V$ is the homotopy Lie algebra of a minimal Sullivan algebra, $\wedge V$, then a more useful notion is the Sullivan completion, $UL_V$, given by

$$UL_V = \lim_{\leftarrow \alpha \in I_V} \hat{UL}_\alpha.$$ 

In general, the natural map $\hat{UL}_V \to UL_V$ may not be an isomorphism, but it is an isomorphism when $L_V$ is finitely generated.

**Remark.** Let $x_i$ be a graded basis for $UL_\alpha$; then $\hat{UL}_\alpha = \prod_i \mathbb{Q} \cdot x_i$. It follows that $UL_V = \prod_j \mathbb{Q} \cdot y_j$, where $y_j$ is a graded basis for $UL$.

The isomorphism $L_V = \lim_{\leftarrow \alpha} L_\alpha$ also identifies the product structure in $\pi_1(\wedge V)$. Since $\dim L_\alpha < \infty$, bijections

$$\exp_\alpha : (L_\alpha)_0 \to G_\alpha \subset \hat{UL}_\alpha$$

onto the group $G_\alpha$ of units of $\hat{UL}_\alpha$ are given by $x \mapsto \sum_{n \geq 0} x^n / n!$ ([17, Chapter 4], [8, Chapter 2]). The inverse bijection $\log_\alpha : G_\alpha \to (V^1_\alpha)\vee$ is then given by the standard power series. Moreover, by ([8, Theorem 2.4]) the composites

$$\pi_1(\wedge V) \to (V^1_\alpha)\vee \to (L_\alpha)_0 \to G_\alpha$$

are isomorphisms $\pi_1(V_\alpha) \xrightarrow{\cong} G_\alpha$ of groups. Thus passing to inverse limits yields a group isomorphism,

$$\exp : \pi_1(\wedge V) \xrightarrow{\cong} \lim_{\alpha \in I_V} G_\alpha \subset \overline{UL}_V,$$

whose inverse bijection is $\log = \lim_{\alpha \in I_V} \log_\alpha$.

In particular, for $x \in (L_\alpha)_0$, $\exp(px) = (\exp x)^p$, $p \in \mathbb{N}$, it follows that $a \mapsto a^p$ is a bijection in each $G_\alpha$. Therefore

$$x \mapsto x^p$$

is a bijection in $\pi_1(\wedge V)$; i.e., $\pi_1(\wedge V)$ is uniquely divisible.

Finally, for any minimal Sullivan algebra $\wedge V$ the Whitehead product

$$[\cdot, \cdot]_W : \pi_p(\wedge V) \times \pi_q(\wedge V) \to \pi_{p+q-1}(\wedge V)$$

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may be computed directly from $L_V$ as follows

$$[sx, sy]_W = \begin{cases} 
(1)^{\deg x} s[x, y], & x, y \in (L_V)_{\geq 1} \\
 Ad(\exp x)y - sy, & x \in (L_V)_0, y \in (L_V)_0 \\
 s(\log[\exp x, \exp y]), & x, y \in (L_V)_0.
\end{cases}$$

(11)

This follows from [8, Chapter 2 and Theorem 4.2] for each $\wedge V$, and then by an inverse limit argument.

**Remark:** The family of finite dimensional subspaces $W \subset V$ for which $\wedge W$ is preserved by $d_1$ is cofinal with $I_V$. It follows that $UL_V$, and the Whitehead products including the group structure in $\pi_1(\wedge V)$ depend only on the associated quadratic Sullivan algebra $(\wedge V, d_1)$.

**Example.** Let $(\wedge V, d)$ be the minimal Sullivan model of $X = S^1 \vee S^3$. Then $V^1 = Qv$ and $V^3$ has a basis $y_0, \ldots, y_n, \ldots$ satisfying $dy_0 = 0$ and for $n \geq 1$, $dy_n = y_{n-1}v$. Choose $x \in \pi_1(X)$ with $\langle v, x \rangle = 1$. We identify $\pi_3(\wedge V, d) = (V^3)^\vee$ with the space of series $Q[[t]]$ by associating to a linear map $f$ the series $g(t) = \sum \langle y_n, f \rangle t^n$. Using this identification, by [8, Theorem 4.6], the action of $\pi_1(\wedge V)$ on $\pi_3(\wedge V)$ is given as the product of two series:

$$(\alpha \cdot g)(t) = \exp(t \log \alpha) \cdot g(t) \quad \alpha \in \pi_1(\wedge V^1).$$

Next, in any minimal Sullivan algebra, $\wedge V$, each $\alpha \in I_V$ yields the map

$$f_\alpha : (\wedge V) \to (\wedge V_\alpha),$$

and if $\alpha \leq \beta$ then $f_\alpha = f_{\alpha \beta} \circ f_\beta$ where $f_{\alpha \beta} : (\wedge V_\beta) \to (\wedge V_\alpha)$ is induced by the inclusion. This defines a map

$$H_*(\Omega(\wedge V); Q) \to \lim_{\alpha} H_*(\Omega(\wedge V_\alpha); Q).$$

On the other hand, completing $H_*(\Omega(\wedge V_\alpha); Q)$ with respect to the augmentation ideal yields natural isomorphisms ([9, Proposition 3.3])

$$\hat{H}(\Omega(\wedge V_\alpha); Q) \cong \hat{UL}_\alpha.$$

Thus we obtain the morphism

$$H_*(\Omega(\wedge V); Q) \to UL_V := \lim_{\alpha} UL_\alpha,$$

which identifies $UL_V$ as a completion of $H_*(\Omega(\wedge V); Q)$. We denote this by

$$\overline{H_*(\Omega(\wedge V); Q)} := UL_V.$$

Moreover, the isomorphisms $s : L_V \xrightarrow{\cong} \pi_*(\wedge V)$ and $s : \pi_*\Omega(\wedge V) \xrightarrow{\cong} \pi_*\wedge V$ define an isomorphism

$$L_V \xrightarrow{\cong} \pi_*\Omega(\wedge V).$$
In view of (11), this converts the Lie bracket in \( L_V \) to the Samelson bracket in \( \pi_0 \Omega(\wedge V) \), in degrees \( \geq 1 \), and thus extends the Samelson bracket to all of \( \pi_0(\Omega(\wedge V)) \).

**Remark.** If \( \wedge V \) is the minimal Sullivan model of a path connected space \( X \), then a homological invariant of \( X_\mathbb{Q} = \langle \wedge V \rangle \) provides a lower bound for the Lusternik-Schnirelmann category, \( \text{cat} X \) (19),

\[
\text{Ext}^p_{UL}(\mathbb{Q}, H_*(\Omega X_\mathbb{Q})) \neq 0, \quad \text{some } p \leq \text{cat } X.
\]

5 The Sullivan topology

Fix a minimal Sullivan algebra, \( \wedge V \), and recall that \( I_V = \{ \alpha \} \) is the index set for the finite dimensional subspaces \( V_\alpha \subset V \) for which \( \wedge V_\alpha \) is preserved by \( d \). Then by (6) we may identify

\[
\pi_0 \langle \wedge V, d \rangle = V^\vee \quad \text{and} \quad \pi_0 \langle \wedge V_\alpha \rangle = V_\alpha^\vee.
\]

Thus the inclusions \( V_\alpha \to V \) dualize to surjections

\[
\rho_\alpha : \pi_0 \langle \wedge V \rangle \to \pi_0 \langle \wedge V_\alpha \rangle.
\]

Moreover, since \( I_V \) is closed under arbitrary intersections and finite sums, the set of subspaces \( \{ \ker \rho_\alpha \} \) is closed under finite intersections and arbitrary sums. Thus (similar to an observation of Lefschetz (13)) it follows that sets of the form

\[
\mathcal{O} = \bigcup_{x_\alpha \in L_V, \alpha \in I_V} \ker \rho_\alpha + x_\alpha
\]

are the open sets of a topology in \( \pi_0 \langle \wedge V \rangle \). Desuspension then transfers this to a topology in \( L_V \).

**Definition.** The topology just defined is the **Sullivan topology** in \( L_V \).

**Remark.** The Sullivan topology may be identified with that introduced by Lefschetz. In fact every finite dimensional subspace \( S \subset V \) induces a surjection \( \rho_S : V^\vee \to S^\vee \). Moreover, each \( S \) is a subspace of some \( V_\alpha \), so that \( \{ \ker \rho_\alpha \} \) is cofinal with \( \{ \ker \rho_S \} \). But the topology in \( V^\vee \) determined by \( \{ \ker \rho_S \} \) is the topology introduced by Lefschetz.

The next Proposition in particular exhibits \( (\wedge V, d_1) \) as the continuous dual of the classical *Cartan-Chevalley-Eilenberg* differential graded coalgebra \( (\wedge sL_V, \partial) \), defined by \( \partial(sx \wedge sy) = (-1)^{1 + \deg x} s[x, y] \).

**Proposition 4.** Let \( (\wedge V, d) \) be a minimal Sullivan algebra. With the notation above:

(i) The Whitehead products in \( \pi_0 \langle \wedge V \rangle \) are continuous in the Sullivan topology. In particular, \( \pi_1 \langle \wedge V \rangle \) is a topological group.

(ii) The Sullivan topology in \( L_V \) coincides with that determined by the associated quadratic Sullivan algebra \( (\wedge V, d_1) \).

(iii) The Lie bracket \([ , ] : L_V \times L_V \to L_V\) is continuous in the Sullivan topology.
(iv) \( \wedge^p V \) may be identified with the continuous alternating \( p \)-linear maps

\[
s_L V \times \cdots \times s_L V \to \mathbb{Q},
\]

where \( \mathbb{Q} \) has the discrete topology, and the differential, \( d \), is continuous.

**proof:** (i) The product in \( \pi_1(\wedge^1) = (V^1)^{\vee} \) is given by \( x \cdot y = \log(\exp x \cdot \exp y) \), where \( \exp x \), \( \exp y \) are elements in the group of units in \( UL_V \). Since \( U_\rho : U L_V \to U L_\alpha \) commutes with \( \exp \) and \( \log \) it follows that if \( \hat{x} \in \ker \rho_\alpha + x \) and \( \hat{y} \in \ker \rho_\alpha + y \) then \( \hat{x} \cdot \hat{y} \in \ker \rho_\alpha + x \cdot y \). Thus multiplication is continuous. Similarly inversion is also continuous. The same argument shows that \( \pi_1(\wedge^1) \) acts continuously in \( L_V \). Thus (i) follows from (III).

(ii) This follows from the fact that \( \rho_\beta : L_V \to L_\beta \) are the surjections determined by \( (\wedge V, d_1) \) then \( \{ \ker \rho_\beta \} \) is cofinal with \( \{ \ker \rho_\alpha \} \).

(iii) This follows because if \( [x, y] \in \ker \rho_\alpha + z \) then \( [x + \ker \rho_\alpha, y + \ker \rho_\alpha] \subset \ker \rho_\alpha + z \).

(iv) If \( \Phi \in \wedge^p V \) then \( \Phi \in \wedge^p V_\gamma \), some \( \gamma \). Now \( (V_\gamma, s \ker \rho_\gamma) = 0 \). Thus, if \( x_1, \ldots, x_p \in L_V \), then \( (\Phi, s(x_1 + \ker \rho_\gamma), \ldots, s(x_p + \ker \rho_\gamma)) = (\Phi, sx_1, \ldots, sx_p) \). This shows that \( \Phi \) acts continuously.

On the other hand, suppose \( \Phi : s L_V \times \cdots \times s L_V \to \mathbb{Q} \) is any continuous alternating multilinear function. Then for some \( \gamma, \Phi \) vanishes on \( s \ker \rho_\gamma \times \cdots \times s \ker \rho_\gamma \). But this implies that \( \Phi \) is in the image of \( \wedge^p V_\gamma \) and hence in \( \wedge^p V \).

**Corollary.** Suppose a graded Lie algebra, \( E \), is the homotopy Lie algebra of two quadratic Sullivan algebras \( (\wedge V, d_1) \) and \( (\wedge W, d_1) \). If the induced Sullivan topologies in \( E \) coincide, then \( (\wedge V, d_1) \cong (\wedge W, d_1) \).

However, the following remains an open problem:

**Problem.** Can a graded Lie algebra \( E \) be the homotopy Lie algebra of two non-isomorphic quadratic Sullivan algebras?

Note that with additional data, it is possible to construct a quadratic Sullivan algebra from a graded Lie algebra \( E \), and with a certain finiteness hypothesis, Proposition 5 below then shows that this Sullivan algebra is unique.

First suppose a graded Lie algebra \( E = E_{\geq 0} \) is equipped with a family of surjections \( \rho_\alpha : E \to \hat{E}_\alpha \) onto nilpotent and finite dimensional graded Lie algebras. Then the dual of the Cartan-Chevalley-Eilenberg constructions are quadratic Sullivan algebras \( \wedge s E_{\alpha}^\vee \). If also \( E = \lim_{\alpha} E_\alpha \) and \( \{ \ker \rho_\alpha \} \) is closed under finite intersections and arbitrary sums then \( \{ s E_{\alpha}^\vee \} \) is a directed set and

\[
\wedge W := \lim_{\alpha} \wedge s E_{\alpha}^\vee
\]

is a quadratic Sullivan algebra with homotopy Lie algebra \( E \). Moreover, \( \{ s E_{\alpha}^\vee \} \) is cofinal with \( J_W \), so that the topology in \( E \) determined by \( \{ \ker \rho_\alpha \} \) is the Sullivan topology.

Finally, recall that the lower central series for a graded Lie algebra \( E = E_{\geq 0} \) is the descending sequence of ideals \( E^n \) in which \( E^n \) is the linear span of iterated commutators \([x_1, [x_2, \ldots, [x_n] \ldots]]\) of length \( n \). By definition, \( E \) is pronilpotent if \( E \xrightarrow{\sim} \lim_{n} E/E^n \).

**Proposition 5.** Suppose \( E \) is a pronilpotent Lie algebra and \( E/E^2 \) is a graded vector space of finite type. Then \( E \) is the homotopy Lie algebra of a unique quadratic Sullivan algebra, \( \wedge W \).
proof: Since the Lie bracket defines linear surjections \( E/E^2 \otimes E^n/E^{n+1} \to E^{m+1}/E^{m+2} \) it follows that each \( E/E^n \) is a graded vector space of finite type. Since \( E/E^n \) is also nilpotent, the dual of the Cartan-Chevalley-Eilenberg construction is a quadratic Sullivan algebra \( \wedge(s(E/E^n))^\vee \). It follows that \( \wedge W = \lim_{\to \alpha} \wedge(s(E/E^n))^\vee \) is also a quadratic Sullivan algebra and, since \( E \) is pronilpotent, \( W^\vee = \lim_{\to \alpha} sE/E^n = sE \). It is immediate that the corresponding Lie bracket in \( E \) is the original Lie bracket.

On the other hand, suppose \( E \) is also the homotopy Lie algebra of another quadratic Sullivan algebra, \( ^\wedge V \). Because \( E/E^n \) has finite type, there is a subspace \( V(n) \subset V \) such that \( \langle V(n), sE^n \rangle = 0 \) and the induced pairing \( V(n) \times sE/E^n \to \mathbb{Q} \) is non degenerate.

These conditions imply that \( ^\wedge V(n) \) is a sub quadratic Sullivan algebra with homotopy Lie algebra \( E/E^n \). In particular, \( ^\wedge V(n) \) is the dual of the Cartan-Chevalley-Eilenberg complex. But it is immediate that \( ^\wedge V = \lim_{\to \alpha} ^\wedge V(n) \), and so \( ^\wedge V \cong ^\wedge W \). □

6 Lower central series

In this section we fix a minimal Sullivan algebra, \( ^\wedge V \), with associated quadratic differential, \( d_1 \).

Here, and subsequently, we shall rely on the following property [2]:

Suppose \( 0 \to A, \to B, \to C, \to 0 \) are exact sequences of morphisms of inverse systems of vector spaces. Then if each \( \dim C_\alpha < \infty \),

\[
0 \to \lim_{\to \alpha} A_\alpha \to \lim_{\to \alpha} B_\alpha \to \lim_{\to \alpha} C_\alpha \to 0 \quad (12)
\]

is also exact.

The classical lower central series of a graded Lie algebra \( L \) is the sequence of ideals \( L = L^1 \supset L^2 \supset \cdots \supset L^r \supset \cdots \) in which \( L^r \) is the linear span of iterated commutators of length \( r \), \( [x_1, \ldots, [x_{r-1}, x_r]] \). Again, a more natural role in Sullivan’s theory is played by the ideals

\[
L_V^{(r)} := \lim_{\to \alpha} L_\alpha^r,
\]

when \( L_V \) is the homotopy Lie algebra of \( ^\wedge V \). The sequence

\[ L_V = L_V^{(1)} \supset \cdots \supset L_V^{(r)} \supset \cdots \]

is the **Sullivan lower central series**. It satisfies

\[ L_V^r \subset L_V^{(r)}, \quad [L_V^{(r)}, L_V^{(s)}] \subset L_V^{(r+s)}, \quad \text{and} \quad \cap_r L_V^{(r)} = 0. \]

Further, since each \( L_\alpha \) is finite dimensional,

\[ L_V^{(r)} / L_V^{(s)} = \lim_{\to \alpha} L_\alpha^r / L_\alpha^s, \quad \text{and} \]

\[
L_V = \lim_{\to \alpha} L_\alpha = \lim_{\to \alpha} \lim_{r} L_\alpha^r = \lim_{r} \lim_{\to \alpha} L_\alpha^r = \lim_{r} L_V / L_V^{(r)}. \quad (13)
\]
This exhibits $L_V$ as complete with respect to the Sullivan lower central series. By contrast the map $L_V \to \lim_{r} L_V/L_V^r$ may not always be an isomorphism, although we have the

**Example.** If $H^1(\Lambda V)$ and each $V^n, n \geq 2, are finite dimensional then

$$L_V^{(r)} = L_V^r, \quad r \geq 1.$$  

Moreover if $\Phi \in \Lambda^{d+1}V$ is a $d$-cycle then the component $\Phi_1$ of $\Phi$ in $V$ is a $d_1$-cycle. It follows that the Hurewicz map, $\xi^\vee : sL_V \to H(\Lambda V)^\vee$, described in §3 vanishes on $sL_V^2$. Hence $sL_V^2$ vanishes on $H(\Lambda V_\alpha)^\vee$. Since $H(\Lambda V)^\vee = \lim_{\alpha} H(\Lambda V_\alpha)^\vee$ it follows that $sL_V^{(2)}$ vanishes on $H(\Lambda V)^\vee$, so that the Hurewicz map factors to give

$$s(L_V/L_V^{(2)}) \to H(\Lambda V)^\vee.$$  

(14)

Analogously, the lower central series for a group $G$ is the sequence of normal subgroups $G = G^{(1)} \supset \cdots \supset G^{(r)} \supset \cdots$, where $G^{(r)}$ is the subgroup generated by the elements $aba^{-1}b^{-1}$, $a \in G$, $b \in G^r$. Analogous to the construction above, when $G = \pi_1(\Lambda V)$, we set

$$G^{(r)} := \lim_{\alpha \in \Lambda V} G^{r}_\alpha,$$

where $G^{r}_\alpha = \pi_1(\Lambda V_\alpha)$. This defines the *Sullivan lower central series*,

$$G = G^{(1)} \supset \cdots \supset G^{(r)} \supset \cdots$$  

for $G$. As in the Lie algebra case, it is immediate that

$$G^r \subset G^{(r)}, \quad [G^{(r)}, G^{(s)}] \subset G^{(r+s)}, \text{ and } \cap_r G^{(r)} = \{e\}.$$  

Moreover ([8, Theorem 2.4 and Corollary 2.4]), the exponential map restricts to bijections $(L_\alpha)^0_0 \xrightarrow{\cong} G^{r}_\alpha$, which then factor to yield linear isomorphisms $(L_\alpha)_0^0/(L_\alpha)_0^{r+1} \xrightarrow{\cong} G^{r}_\alpha/G^{r+1}_\alpha$. In particular, these are finite dimensional vector spaces. Passing to inverse limits yields linear isomorphisms

$$(L_V)_0^0/(L_V)_0^{r+1} \xrightarrow{\cong} G^{r}/G^{r+1}, \quad r \geq 1.$$  

It follows from this, and induction on $s \geq r$ that

$$G^{(r)}/G^{(s)} \xrightarrow{\cong} \lim_{\alpha \in \Lambda V} G^{r}_\alpha/G^{s}_\alpha$$

is an isomorphism of groups, and that when $s = r + 1$ these are isomorphisms of rational vector spaces.

Finally, filtering by the normal subgroups $G^{(r)}$ produces an associated graded group

$$\text{gr} G = \oplus_r G^{(r)}/G^{(r+1)}.$$  

The properties above for the filtration imply that $G$ satisfies condition N of Lazard [12], and hence the commutator $[a, b] = aba^{-1}b^{-1}$ makes $\text{gr} G$ into a graded Lie algebra. On the
other hand, the filtration $L^{(r)}_V$ makes $(L_V)_0$ into a filtered Lie algebra, and it is straightforward from the properties above that the associated graded Lie algebra $gr(L_V)_0$ is the Lazard Lie algebra, $gr G$:

$$gr G = gr(L_V)_0.$$ 

**Example.** The space $H_1(⟨∧V⟩)$. 

For each $α ∈ I_V$ the isomorphism $H_1(⟨∧V_α⟩; Z) = G_α/G^2_α ≅ L_α/[L_α, L_α]$ identifies $H_1(⟨∧V_α⟩; Z)$ as the rational vector space $L_α/[L_α, L_α]$. Passing to inverse limits yields the diagram

$$
\begin{array}{ccc}
G/G^2 & \cong & H_1(⟨∧V⟩; Z) \\
\downarrow & & \downarrow \\
G/G^{(2)} & \cong & \mathop{\lim}\limits_{\leftarrow} H_1(⟨∧V_α⟩; Z).
\end{array}
$$

Thus if $G^2 = G^{(2)}$ (equivalently if $L^2 = L^{(2)}$) then this exhibits $H_1(⟨∧V⟩; Z)$ as a rational vector space.

Now define the **Sullivan completion** of the group ring $Q[G]$ by

$$
\hat{Q}[G] := \mathop{\lim}\limits_{\leftarrow} \hat{Q}[G_α]
$$

where $\hat{Q}[G_α]$ is the completion of $Q[G_α]$ with respect to its augmentation ideal. According to ([15], [9, Proposition 3.2]), there is a natural isomorphism

$$
\hat{Q}[G_α] ≅ (UL_α)_0.
$$

Taking inverse limits gives the isomorphism,

$$
\hat{Q}[G] = \mathop{\lim}\limits_{\leftarrow} \hat{Q}[G_α] ≅ \mathop{\lim}\limits_{\leftarrow} (UL_α)_0 = (UL)_0.
$$

Next observe that filtration (1) of $V$ determined by $d_1$ and denoted $\{V_n\}$ is given by

$$
V_0 = V ∩ ker d_1 \quad \text{and} \quad V_{n+1} = V ∩ d_1^{-1}(∧^2 V_n), \quad n ≥ 0.
$$

Restricting $sL_V$ to $V_n$ gives linear maps $sL_V → V_n^\vee$.

**Proposition 6.** The linear maps $sL_V → V_n^\vee$ factor over the surjections $sL_V → s(L_V/L^{(n+2)}_V)$ to give degree 1 isomorphisms

$$
L_V/L^{(n+2)}_V \cong V_n^\vee, \quad n ≥ 0.
$$

**Corollary.** If $H(⟨∧V, d_1⟩)$ has finite type then $dim L_V/L^{(n+1)}_V < ∞$, $n ≥ 0$.

**proof of Proposition 6**. We establish the equivalent assertions,

$$
L^{(n+2)}_V \cong (V/V(n))^\vee, \quad n ≥ 0. \quad (15)
$$
For this, let $I_{V,1}$ be the set of finite dimensional subspaces $V_{\alpha} \subset V$ such that $\wedge V_{\alpha}$ is preserved by $d_1$. Then let $J_V$ be the set of subspaces $W \subset V$ for which $\wedge W$ is preserved by $d$ and $\dim W \cap \ker d < \infty$. Note that for any two subspaces $S(1)$ and $S(2)$ of a graded vector space $S$, that $\wedge S(1) \cap \wedge S(2) = \wedge (S(1) \cap S(2))$. \[
abla V_{\alpha} \subset V \text{ such that } \wedge V_{\alpha} \text{ is preserved by } d_1 \text{. Then let } J_V \text{ be the set of subspaces } W \subset V \text{ for which } \wedge W \text{ is preserved by } d \text{ and } \dim W \cap \ker d < \infty. \text{ Note that for any two subspaces } S(1) \text{ and } S(2) \text{ of a graded vector space } S, \text{ that } \wedge S(1) \cap \wedge S(2) = \wedge (S(1) \cap S(2)) \text{. In particular, if } V_{\alpha} \in I_{V,1} \text{ and } W \in J_V \text{ then it follows by induction that } (V_{\alpha})_n \cap W = (V_{\alpha} \cap W)_n. \text{ Hence} \]

\[
L^{(n+2)}_{V_{\alpha}} = \lim_{W \in J_V} \lim_{W_{\alpha} \in I_{W,1}} L^{n+2}_{W_{\alpha}} = \lim_{W \in J_V} L^{n+2}_W. \]

On the other hand, for $W \in J_V$ the subspaces $W_k$ are finite dimensional and cofinal in $I_{W,1}$. Therefore \[
L^{(n+2)}_W = \lim_k L^{n+2}_{W_k}. \]

Since $(W(k)_n = W_n$ for $k \geq n$ it follows in this case, exactly as in the proof of Theorem 2.1, p.50 in [8], that \[
L^{n+2}_{W_k} \xrightarrow{\xi} (W_k / W_n)^{\vee}. \text{ This gives} \]

\[
L^{(n+2)}_W = \lim_k (W_k / W_n)^{\vee} = (W / W_n)^{\vee} = (W / W \cap V_n)^{\vee}. \text{ This gives} \]

Formula (15) follows. \[\square\]

The Hurewicz map (14) for $(\wedge V, d_1)$ may be regarded as a linear map of degree 1, \[\xi^{\vee} : L_V \rightarrow H(\wedge V, d_1)^{\vee}. \]

**Corollary.** The Hurewicz map for $(\wedge V, d_1)$ factors as \[
L_V \rightarrow L_V / L_V^{(2)} \xrightarrow{\xi} V_0^{\vee} \subset H^2(\wedge V, d_1)^{\vee}. \]

In particular, in degree 1 it translates to \[
G \rightarrow G / G^{(2)} \xrightarrow{\xi} (H^1(\wedge V)^{\vee}, \text{ where } G = \pi_1(\wedge V) \]

**proof:** Because $\xi$ is induced by the surjection $\wedge^1 V \rightarrow V$ with kernel $\wedge^2 V$, and because $(\wedge V, d_1)$ is quadratic, it follows that $\xi$ factors as $H^1(\wedge V, d_1) \xrightarrow{\xi} V_0^{\vee} \rightarrow \wedge V$. This gives the first assertion. The second follows because $H^1(\wedge V, d) = H^1(\wedge V, d_1)$. \[\square\]

**Remark.** A second application of the proof of Theorem 2.1 in [8] also gives \[
L^{(n)}_V = L^n_V \quad n \geq 1, \]

if $\dim V \cap \ker d_1 < \infty$.

## 7 The holonomy representation of a $\Lambda$-extension

Now recall from §2 that a $\Lambda$-extension of a Sullivan algebra $\wedge V$ is sequence of cdga morphisms of the form \[
\begin{align*}
\wedge V & \xrightarrow{\lambda} \wedge V \otimes \wedge Z \xrightarrow{\rho} \wedge Z, \\
\lambda v & = v \otimes 1, 
\end{align*}
\]

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in which \((\wedge Z, \overline{d}) = \mathbb{Q} \otimes_{\wedge V} (\wedge V \otimes \wedge Z)\).

Now suppose that \(\wedge V\) is a minimal Sullivan algebra. Then the differential in \(\wedge V \otimes \wedge Z\) satisfies
\[
d(1 \otimes \Phi) = 1 \otimes \overline{d} \Phi + \sum_i v_i \otimes \theta_i \Phi + \Omega,
\]
where \(\overline{d}\) is the differential in \(\wedge Z\), \(v_i\) is a basis of \(V\) and \(\Omega \in \wedge^{\geq 2} V \otimes \wedge Z\). Here the \(\theta_i\) are derivations in \((\wedge Z, \overline{d})\), and setting
\[
\overline{\theta}(x) = -(v_i, sx) H(\theta_i), \quad x \in L_V
\]
defines the holonomy representation of \(L_V\) in \(H(\wedge Z)\).

Next, suppose \(\wedge V \otimes \wedge Z\) is a \(\Lambda\)-extension of a minimal Sullivan algebra, and set \(W = V \oplus Z\). Then \(W\) is the union of the finite dimensional subspaces \(W_\alpha = V_\alpha \oplus Z_\alpha\) for which \(\wedge V_\alpha \otimes \wedge Z_\alpha\) is preserved by \(d\). Under inclusion, these form a directed set \(J\). Moreover \(\{V_\alpha, \alpha \in J\}, \{W_\alpha, \alpha \in J\}\), and \(\{Z_\alpha, \alpha \in J\}\) are respectively cofinal with \(J_V, J_W,\) and \(J_Z\).

In particular, if \(\beta \geq \alpha \in J\) then the \(\Lambda\)-extension \(\wedge V_\beta \otimes \wedge Z_\alpha\) yields a holonomy representation of \(L_{V_\beta}\) in \(H(\wedge Z_\alpha)\); The defining condition for \(\Lambda\)-extensions implies that this representation extends to a representation of \(UL_{V_\beta}\). Passing to inverse limits then gives a representation of \(UL_V\) in \(H(\wedge Z_\alpha)\), and passing to direct limits gives a representation of \(UL_V\) in \(H(\wedge Z)\).

**Definition.** This extension of the holonomy representation of \(L_V\) to \(UL_V\) is the holonomy representation of \(UL_V\) in \(H(\wedge Z)\).

**Remark.** If \(A \otimes M \rightarrow M\) is a representation of a graded algebra in a graded vector space \(M\) then we define \(M^\vee \otimes A \rightarrow M^\vee\) by
\[
\langle m, f \cdot a \rangle = (-1)^{\deg a} \langle a \cdot m, f \rangle.
\]
This is a right representation of \(A\): the dual right representation. In particular the holonomy representation dualizes to a right representation of \(UL_V\) in \(H(\wedge Z)^\vee\).

In the special case that \(\wedge W = \wedge V \otimes \wedge Z\) is itself a minimal Sullivan algebra, the homotopy Lie algebras form a short exact sequence
\[
0 \leftarrow L_V \leftarrow L_W \leftarrow L_Z \leftarrow 0
\]
identifying \(L_Z\) as an ideal in \(L_W\). In particular, for \(\beta \geq \alpha \in J\), \(L_{Z_\alpha}\) is a finite dimensional ideal in \(L_{W_\beta}\), and the right adjoint representation of \(L_{W_\beta}\) in \(L_{Z_\alpha}\) is nilpotent. As above, these representations extend to a right representation of \(UL_W\) in \(L_{Z_\alpha}\), and passing to inverse limits defines a representation of \(UL_W\) in \(L_Z\), extending the right adjoint representation of \(L_W\) dual to the adjoint representation.

**Definition.** This representation, denoted \(\text{ad}_R\), is the right adjoint representation of \(UL_W\) in \(L_Z\).

Again recall from (14) that the Hurewicz map \(\xi^\vee\) induces a linear map \(L_Z/L_Z^{(2)} \rightarrow H(\wedge Z)^\vee\) of degree 1.
Proposition 7. Suppose $\Lambda W = \Lambda V \otimes \Lambda Z$ decomposes a minimal Sullivan algebra as a $\Lambda$-extension of a minimal Sullivan algebra, $\Lambda V$. Then

(i) Each $L^r_Z$ is a sub $UL_W$-module.

(ii) The surjection $L_W \to L_V$ extends to a surjection $\pi : UL_W \to UL_V$.

(iii) The quotient right representations of $UL_W$ in $L^r_Z / L^{r+1}_Z$ factor over $\pi$ to yield right representations of $UL_V$.

(iv) The Hurewicz map $\xi^\vee : L_Z / L^{(2)}_Z \to H^{\geq 1}(\Lambda Z)^\vee$ is a morphism of $UL_V$-modules.

Proof. (i) This is immediate, because if $\beta \geq \alpha$ then each $L^r_Z(\alpha)$ is a $UL_W(\beta)$-module.

(ii) This is immediate because $\pi$ is the inverse limit of the surjections, $\pi_{r,\alpha} : UL_{W,\alpha} / I_{r,\alpha} \to UL_{V,\alpha} / J_{r,\alpha}$, between finite dimensional vector spaces, where $I_{r,\alpha}$ and $J_{r,\alpha}$ respectively denote the augmentation ideals in $UL_{W,\alpha}$ and $UL_{V,\alpha}$.

(iii) This follows because in each $\Lambda V, \otimes \Lambda Z$, the adjoint representation of $L_Z(\alpha)$ is zero in $L^r_Z(\alpha)$.

(iv) Recall that division by $\Lambda^{\geq 2}Z$ is the linear map $\Lambda^{\geq 1}Z \to Z$ which induces $\xi : H^{\geq 1}(\Lambda Z) \to Z$. Because $\Lambda V \otimes \Lambda Z$ is itself a minimal Sullivan algebra, the derivations $\theta_i$ of (16) preserve $\Lambda^{\geq 1}Z$. Define $\hat{\theta}_i : Z \to Z$ by requiring

$$\xi \circ \theta_i = \hat{\theta}_i \circ \xi.$$ 

Then for $x \in L_W$, $y \in L_Z$ and $z \in Z$ we have, where $v_i$ is a basis of $V$,

$$\langle d_1(1 \otimes z), sx, sy \rangle = \sum \langle v_i \otimes \hat{\theta}_iz, sx, sy \rangle = \sum (-1)^{(\deg x+1)(\deg y+1)} \langle v_i, sx \rangle \langle \hat{\theta}_i z, sy \rangle,$$

where $\pi$ is the image of $x$ in $L_V$.

Now suppose $z = \xi \Phi$ for some $\mathcal{F}$-cycle $\Phi \in \Lambda Z$. Then $\hat{\theta}_i z = \xi \theta_i \Phi$. It follows that

$$\langle d_1(1 \otimes z), sx, sy \rangle = -(-1)^{(\deg y+1)(\deg x+1)} \langle \xi \theta_i \Phi, sy \rangle.$$ 

On the other hand, by definition

$$\langle d_1(1 \otimes z), sx, sy \rangle = (-1)^{\deg y+1} \langle z, [x, y] \rangle = (-1)^{\deg y+1} \langle z, ad x(y) \rangle.$$ 

A straightforward computation now gives that

$$\xi^\vee \circ ad_R(\pi) = \mathcal{F}^\vee(\pi) \circ \xi^\vee.$$ 

Finally (iv) follows from (17) applied to the sub $\Lambda$-extension of the form $\Lambda V, \otimes \Lambda Z$, in which $\dim V, \otimes Z, < \infty$, together with a standard limit argument.

Example. Acyclic Closures

Suppose $(\Lambda V \otimes \Lambda U, d)$ is the acyclic closure of a minimal Sullivan algebra, $\Lambda V$. This is the $\Lambda$-extension of $\Lambda V$ satisfying $\varepsilon : \Lambda V \otimes \Lambda U \to \mathbb{Q}$, where $\varepsilon$ is the augmentation sending
\[ V, U \to 0. \] It has the important property that \( d : U \to \wedge^+ V \otimes \wedge U, \) so that the quotient differential in \( \wedge U \) is zero. Thus in this case the holonomy representation is a representation of \( UL_V \) in \( \wedge U. \) This then dualizes to a right representation of \( UL_V \) in \( (\wedge U)^\vee. \)

Now let \( \varepsilon_U : \wedge U \to \mathbb{Q} \) be the augmentation vanishing on \( \wedge^+ U. \) Then we have

**Proposition 8.** An isomorphism of right \( UL_V \)-modules

\[ UL_V \congto (\wedge U)^\vee \]

is given by \( a \mapsto \varepsilon_U \cdot a. \)

**proof:** Suppose \( V_\alpha \subset V \) is a finite dimensional subspace for which \( dV_\alpha \subset \wedge V_\alpha. \) Then the inclusion \( V_\alpha \hookrightarrow V \) extends to an inclusion of acyclic closures \( \wedge V_\alpha \otimes \wedge U_\alpha \hookrightarrow \wedge V \otimes \wedge U \) which maps \( U_\alpha \) to a subspace of \( U. \) Now because \( \dim V_\alpha < \infty, \) ([8, Theorem 6.1]) asserts that

\[ UL_V \alpha \congto (\wedge U_\alpha)^\vee. \]

Passing to inverse limits gives the isomorphism of the Proposition, because \( \wedge U = \varprojlim_\alpha \wedge U_\alpha \) and so \( (\wedge U)^\vee = \varprojlim_\alpha (\wedge U_\alpha)^\vee. \) □

# 8 The Sullivan rationalization, \( \ell_\mathbb{Q} \)

In this section all spaces are connected and based CW complexes or simplicial sets. All maps and homotopies preserve base points, and "homotopy" is denoted "\( \sim \)." In particular the augmentation \( \wedge V \to \mathbb{Q} \) in a Sullivan algebra defines a base point in \( \langle \wedge V \rangle \) and the maps \( \tilde{\varphi} : X \to \langle \wedge V \rangle \) of §1 are base point preserving with respect to any base point of \( X. \)

Here we define homotopy localization, exhibit Sullivan’s construction as an example, and examine its properties from that perspective. Homotopy localizations are defined via the category \( \mathcal{C} \) described next, when there is no ambiguity, we may use the same symbol to denote a map and its homotopy class.

- The objects in \( \mathcal{C} \) are the families \( (X) := \{X_\alpha\} \) of connected based spaces, together with the assignment for each pair \( X_\alpha, X_\beta \in (X) \) of a single homotopy class of homotopy equivalences,

\[ \omega_{\alpha,\beta} : X_\beta \congto X_\alpha. \]

These are required to satisfy \( \omega_{\alpha,\beta} \circ \omega_{\beta,\gamma} \sim \omega_{\alpha,\gamma} \) and \( \omega_{\sigma,\sigma} \sim \text{id}_{X_\sigma}. \)

- A morphism \( (g) : (X) \to (Y) \) in \( \mathcal{C} \) is the assignment for each pair \( X_\alpha \in (X), Y_\beta \in (Y) \) of a homotopy class of maps,

\[ g_{\alpha,\beta} : X_\beta \to Y_\alpha, \]

satisfying \( g_{\alpha,\beta} \circ \omega_{\beta,\gamma} \sim \omega_{\alpha,\delta} \circ g_{\delta,\gamma} \) for all \( \alpha, \beta, \gamma, \delta. \)

**Remark.** Given objects \( (X) \) and \( (Y) \) in \( \mathcal{C}, \) a single map \( g_{\beta,\alpha} : X_\alpha \to Y_\beta \) extends uniquely to a morphism \( (g) \) in \( \mathcal{C}. \) The map \( g_{\beta,\alpha} \) is called a representative for \( (g). \) A morphism \( (g) \) is a homotopy equivalence if some \( g_{\beta,\alpha} \) (equivalently all \( g_{\beta,\alpha} \)) is a homotopy equivalence.
Moreover, the maps $\omega_{\alpha,\beta}$ provide a consistent identification of the spaces $X_\alpha \in (X)$ as a single homotopy type, and the maps $g_{\alpha,\beta}$ provide a consistent identification of $(g)$ as the homotopy class of a map $(X) \to (Y)$.

In particular, if $(X) \in \mathcal{C}$ then the homotopy equivalences $\omega_{\alpha,\beta}$ identify the groups $\pi_i(X_\alpha)$ with a single group $\pi_i(X)$ and the graded algebras $H(X_\alpha)$ with a single graded algebra $H(X)$. This then defines functors

$$\pi_i : \mathcal{C} \to \mathcal{G} \quad \text{and} \quad H : \mathcal{C} \to \mathcal{A}$$

to the category of groups and graded algebras. In particular, the various constructions of the classifying space of a group $G$ define a natural transformation $B : \mathcal{G} \to \mathcal{C}$ from the category of groups, characterized by $\pi_1 \circ B = \text{id}$, and $\pi_i \circ B = 0$, $i \geq 2$.

On the other hand, the homotopy category of spaces and homotopy classes of maps is a subcategory of $\mathcal{C}$ via a canonical functor, $h : \mathcal{H} \to \mathcal{C}$, defined as follows:

- $hX$ contains the single space $X$,
- $\mathcal{C}(hX, hY)$ is the set of homotopy classes of maps from $X$ to $Y$.

**Definition.** A homotopy localization is a functor $\ell : \mathcal{H} \to \mathcal{C}$ together with a natural transformation $t : h \to \ell$. Thus if $tX = \{X_\sigma, (\omega_{\sigma,\tau})\}$ then $t(X)$ is a family of homotopy classes of maps $f_\sigma : X \to X_\sigma$ satisfying $\omega_{\sigma,\tau} \circ f_\tau \sim f_\sigma$.

For the definition of Sullivan rationalization it is convenient to extend the definition of Sullivan models to quasi-isomorphisms $\wedge W \to A_{PL}(X)$ from $\Lambda$-algebras. Because our spaces are connected such a Sullivan model always decomposes as a tensor product $(\wedge V, d) \otimes (\wedge (U \oplus dU))$ in which $\wedge V$ is a minimal Sullivan algebra and $d : U \xrightarrow{\cong} dU$. In particular, this implies that the inclusion $\wedge V \to \wedge W$ induces a homotopy equivalence $\langle \wedge V \rangle \leftarrow \langle \wedge W \rangle$.

**Definition.** The Sullivan rationalization is the natural transformation $t_Q : h \to \ell_Q$, defined next.

- $\ell_Q(X) = \{\langle \wedge V_\sigma \rangle\}$, indexed by the Sullivan models $\varphi_\sigma : \wedge V_\sigma \xrightarrow{\cong} A_{PL}(X)$, together with the homotopy equivalences $\langle \varphi_{\tau,\sigma} \rangle : \langle \wedge V_\sigma \rangle \to \langle \wedge V_\tau \rangle$ determined by the unique class of morphisms $\varphi_{\tau,\sigma} : \wedge V_\sigma \xrightarrow{\cong} \wedge V_\tau$ satisfying $\varphi_\tau \circ \varphi_{\tau,\sigma} \sim \varphi_\sigma$.
- If $f : X \to Y$ then $\ell_Q f = \{\langle \psi_{\tau,\sigma} \rangle : \langle \wedge V_\tau \rangle \to \langle \wedge W_\tau \rangle\}$, where the $\psi_{\tau,\sigma}$ are the Sullivan representatives of $f$.
- The natural transformation $t_Q : h \to \ell_Q$ assigns for each $X$ the homotopy classes of the maps $\varphi_\sigma : X \to \langle \wedge V_\sigma \rangle$ defined in §1.
Remark. If $\ell_Q(X) = \ell_Q(Y)$ then $X$ and $Y$ have the same Sullivan models. Moreover, if $f : X \to Y$, the single homotopy class determined by $\ell_Q(f)$ is a homotopy equivalence if and only if a Sullivan representative $\varphi : \wedge V \to \wedge W$ of $f$ is a quasi-isomorphism.

Our principal objective in this section (Proposition 8) is to characterize Sullivan rationalization, without reference to Sullivan models, in terms of

- its behaviour on elementary spaces, defined next, and
- its behaviour with respect to inverse systems.

Definition. An elementary space is a nilpotent space, $X$, such that each $\pi_1(X)^r/\pi_1(X)^{r+1}$ is a finite dimensional rational vector space, and $\oplus_{i \geq 2} \pi_i(X)$ is also a finite dimensional rational vector space.

As for inverse systems, we consider only those whose index set is directed set satisfying

For each $\gamma \in S$ there is an $n$ such that if $\gamma_1 < \cdots < \gamma_r = \gamma$, then $r \leq n$. \hspace{1cm} (18)

For a directed set $S$ satisfying (18), denote by $n(\gamma)$, $\gamma \in S$, the least $n$ for which (18) holds. Then $S = \cup_0 S(n)$, where

$S(n) = \{ \gamma \in S \mid n(\gamma) \leq n \}$.

Where there is no ambiguity we denote $\lim \leftarrow$ and $\lim \rightarrow$ by $\lim \leftarrow$ and $\lim \rightarrow$.

Proposition 9. Sullivan rationalization satisfies

(i) If $X$ is an elementary space, then $t_Q(X) : X \to \ell_Q(X)$ is a homotopy equivalence.

(ii) If $f : X \to Y$ represents $t_Q(X) : X \to \ell_QX$ then there is a map $g = \{ g_\sigma \} : Y \to \lim \leftarrow Y_\sigma$ in which the $Y_\sigma$ are elementary spaces and the composite

$$\lim \rightarrow H(Y_\sigma) \to H(Y) \to H(X)$$

is an isomorphism.

(iii) If a map $g = \{ g_\sigma \} : X \to \lim \leftarrow X_\sigma$ induces an isomorphism

$$H(X) \leftarrow \lim \rightarrow H(X_\sigma)$$

then

$$\pi_*(\ell_QX) \rightarrow \lim \leftarrow \pi_*(\ell_QX_\sigma).$$

(iv) If $\hat{t} : h \to \hat{t}$ is any homotopy localization satisfying conditions (i)-(iii), then for any space $X$, $\ell_Q(X)$ and $\hat{t}(X)$ have the same homotopy type.

Lemma 1. The following conditions on a connected space $X$ are equivalent

(i) $X$ is an elementary space.
(ii) The minimal Sullivan model of $X$ has the form $\wedge V$ with $\text{dim } V^i < \infty$, and the map $X \to \langle \wedge V \rangle$ is a homotopy equivalence.

**proof:**

(i) $\Rightarrow$ (ii) Denote $\pi_1(X)$ by $G$ and suppose $G^{n+1} = \{0\}$. If $G$ is abelian then $BG = S^1_Q \times \cdots \times S^1_Q = \langle \wedge V \rangle$, where $\wedge V \to A_{PL}(BG)$ is the minimal Sullivan model. In general ([8, Theorem 5.1]) gives a commutative diagram

$$
\begin{array}{c}
A_{PL}(BG/G^2) \\
\wedge W
\end{array} \longrightarrow 
\begin{array}{c}
A_{PL}(BG) \\
\wedge W \otimes \wedge Z
\end{array} \longrightarrow 
\begin{array}{c}
A_{PL}(BG^2) \\
\wedge Z
\end{array}
$$

in which the vertical arrows are Sullivan models. Apply $\langle \rangle$ to get a map of fibrations

$$
\begin{array}{c}
BG/G^2 \\
\langle \wedge W \rangle
\end{array} \longrightarrow 
\begin{array}{c}
BG \\
\langle \wedge W \otimes \wedge Z \rangle
\end{array} \longrightarrow 
\begin{array}{c}
BG^2 \\
\langle \wedge Z \rangle
\end{array}
$$

By induction on $n$, and the case $n = 1$, the left and right arrows are homotopy equivalences. Thus so is the central arrow.

Finally, the same argument applied to the fibration

$$
BG \leftarrow \hat{X} \leftarrow \bar{X}
$$

where $\hat{X} \simeq X$ and $\bar{X}$ is the universal cover, shows that if $\wedge V$ is the minimal Sullivan model of $X$ then $X \to \langle \wedge V \rangle$ is a homotopy equivalence.

(ii) $\Rightarrow$ (i) If (ii) holds then $\pi_i(X) = (V^i)^\vee$ and so (i) follows at once from [8, Chap. 2]. $\square$

**Lemma 2.** If $g_{\sigma, \tau} : X_\sigma \leftarrow X_\tau$, $\sigma \leq \tau \in S$, defines an inverse system of maps, then there is a family of commutative diagrams

$$
\begin{array}{c}
A_{PL}(X_\sigma) \\
\wedge V_\sigma
\end{array} \xrightarrow{A_{PL}(g_{\sigma, \tau})} 
\begin{array}{c}
A_{PL}(X_\tau) \\
\wedge V_\tau
\end{array}
$$

in which

(i) the $\psi_{\sigma, \tau}$ form an inductive system of morphisms of $\Lambda$-algebras;

(ii) each $\psi_{\sigma, \tau}$ restricts to an inclusion $V_\sigma \to V_\tau$;

(iii) each $\varphi_\sigma$ is a quasi-isomorphism.
proof: The construction is by induction. If \( \tau \in S(1) \) we let \( \varphi_\tau : \wedge V_\tau \to A_{PL}(Y_\tau) \) be any Sullivan model. Suppose the construction is accomplished for \( \tau \in S(n) \) and let \( \tau \in S(n+1)/S(n) \). Then set \( V_\tau = \lim_{\sigma \prec \tau} \wedge V_\sigma \), so that

\[
\lim_{\sigma < \tau} \varphi_\sigma : \wedge V_\tau \to \lim_{\sigma < \tau} A_{PL}(V_\sigma)
\]

is a quasi-isomorphism from a \( \Lambda \)-algebra satisfying \( H^0(\wedge V_\tau) = \mathbb{Q} \).

Because \( Y_\tau \) is connected it follows from ([8, Theorem 3.1]) that the composite

\[
\wedge V_\tau \to \lim_{\sigma < \tau} A_{PL}(Y_\sigma) \to A_{PL}(Y_\tau)
\]

extends to a quasi-isomorphism \( \wedge V_\tau \otimes \wedge Z \xrightarrow{\sim} A_{PL}(X_\tau) \) from a \( \Lambda \)-algebra. This is the desired quasi-isomorphism \( \varphi_\tau : \wedge V_\tau \to A_{PL}(Y_\tau) \), and this closes the induction. \qed

**Lemma 3.** Suppose \( \wedge V = \lim_{\sigma} \wedge V_\sigma \) is the direct limit of an inductive system of \( \Lambda \)-algebras in which the morphisms map \( V_\sigma \to V_\tau \). If each \( H^0(\wedge V_\sigma) = \mathbb{Q} \) then \( \pi_* \langle \wedge V_\sigma \rangle \xrightarrow{\sim} \lim_{\sigma} \pi_* \langle \wedge V_\sigma \rangle \).

**proof:** Define differentials \( d_0 \) in \( V \) and \( V_\sigma \) by setting \( d_0v \) to be the component of \( dv \) in \( V(0 \in V_\sigma) \). Then in the minimal models \( \wedge Z \) and \( \wedge Z_\sigma \) for \( \wedge V \) and \( \wedge V_\sigma \) we have isomorphisms which identify

\[
Z_\sigma \cong H(V_\sigma, d_0) \quad \text{and} \quad Z \cong H(V, d_0) = \lim_{\sigma} H(V_\sigma, d_0).
\]

These isomorphisms identify \( \pi_* \langle \wedge V_\sigma \rangle = H(V_\sigma, d_0)^\vee \) and \( \pi_* \langle \wedge V \rangle = H(V, d_0)^\vee \). Since dualizing converts direct limits to inverse limits, it follows that

\[
\pi_* \langle \wedge V \rangle = \left[ \lim_{\sigma} H(V_\sigma, d_0) \right]^\vee = \lim_{\sigma} \pi_* \langle \wedge V_\sigma \rangle.
\]

\qed

**proof of Proposition 2**

(i) This is immediate from Lemma 1.

(ii) We may assume \( f = \bar{\varphi} : X \to \langle \wedge V \rangle \), where \( \varphi : \wedge V \to A_{PL}(X) \) is a Sullivan model. This identifies \( H(f) : H(Y) \to H(X) \) with \( H(\bar{\varphi}) \). Now as described in §3, \( \wedge V = \lim_{\sigma} \wedge V_\sigma \), where the \( \wedge V_\sigma \) are sub Sullivan algebras and \( \dim V_\sigma < \infty \). Thus by Lemma 1, each \( Y_\sigma = \langle \wedge V_\sigma \rangle \) is a Sullivan space. Moreover, since \( \langle \rangle \) converts direct limits to inverse limits, \( Y = \lim_{\sigma} Y_\sigma \). Finally, by ([8, Theorem 5.4]) adjoint to \( \text{id}_{\wedge V_\sigma} \) is the quasi-isomorphism \( \varphi_\sigma : \wedge V_\sigma \xrightarrow{\sim} A_{PL}(\langle \wedge V_\sigma \rangle) \). Since by the definition \( \varphi = \lim_{\sigma} \varphi_\sigma \) it follows from (3) that the composite

\[
\lim_{\sigma} H(Y_\sigma) \to H(Y) \to H(X)
\]

is the isomorphism \( H(\bar{\varphi}) \).

(iii) Lemma 2 provides an inductive system of quasi-isomorphisms \( \varphi_\sigma : \wedge V_\sigma \to A_{PL}(X_\sigma) \) in which

\[
\varphi : \wedge V = \lim_{\sigma} \wedge V_\sigma \xrightarrow{\sim} \lim_{\sigma} A_{PL}(X_\sigma) \xrightarrow{\sim} A_{PL}(X)
\]

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exhibits \( \tilde{\varphi} : X \to \langle \land V \rangle \) as a representative of \( tQ(X) \). Moreover, the quasi-isomorphisms \( \varphi_{\sigma} \) in Lemma 2 identify \( \pi_*(\ell Q X_{\sigma}) \) with \( \pi_*(\land V_{\sigma}) \) and \( \pi_*(\ell Q X) \) with \( \pi_*(\land V) \). This identifies \( \pi_*(\ell Q X) \to \lim \pi_*(\ell Q X_{\sigma}) \) with \( \pi_*(\land V) \to \lim \pi_*(\land V_{\sigma}) \). This is an isomorphism by Lemma 3.

(iv) Suppose \( \hat{f} : h \to \hat{l} \) is a second functor and natural transformation satisfying (i)-(iii). Then, for a space \( X \), let

\[
\begin{array}{c}
X \xrightarrow{f=\hat{l}(X)} Y = \hat{X} \xrightarrow{\{g_{\sigma}\}} \lim_{\sigma} Y_{\sigma}
\end{array}
\]

be the maps provided by (ii). Then Lemma 2 provides a morphism

\[
\psi : \land V = \lim_{\sigma} \land V_{\sigma} \xrightarrow{\lim_{\sigma} \varphi_{\sigma}} \lim_{\sigma} A_{PL}(Y_{\sigma}) \xrightarrow{\sim} A_{PL}(Y)
\]

from a \( \Lambda \)-algebra \( \land V \). Moreover, by hypothesis,

\[
A_{PL}(f) \circ \psi : \land V \xrightarrow{\sim} A_{PL}(X)
\]

is a Sullivan model.

On the other hand, we have the commutative diagram,

\[
\begin{array}{c}
\pi_*(Y) \xrightarrow{\{\pi_*(g_{\sigma})\}} \lim_{\sigma} \pi_*(Y_{\sigma})
\end{array}
\]

\[
\begin{array}{c}
\pi_*\langle \land V \rangle \xrightarrow{\{\pi_*\tilde{\varphi}_{\sigma}\}} \lim_{\sigma} \pi_*\langle \land V_{\sigma} \rangle
\end{array}
\]

Now \( Y = \hat{X} \) and, since each \( Y_{\sigma} \) is an elementary space, by (i) there is a natural identification \( \pi_*(Y_{\sigma}) = \pi_*(\hat{Y}_{\sigma}) \). Thus by (iii), the upper horizontal arrow is an isomorphism.

On the other hand, since each \( Y_{\sigma} \) is an elementary space, Lemma 1 asserts that each \( \tilde{\varphi}_{\sigma} \) is a homotopy equivalence. Thus the right hand vertical arrow is an isomorphism. Finally, Lemma 3 asserts that the lower horizontal arrow is an isomorphism. Therefore, \( \pi_*(\tilde{\psi}) \) is an isomorphism and \( \tilde{\psi} \) is a homotopy equivalence. Since \( A_{PL}(f) \circ \psi : \land V \xrightarrow{\sim} A_{PL}(X) \) it follows that \( \langle \land V \rangle \in \ell Q(X) \) and

\[
\tilde{\psi} : \ell Q X \xrightarrow{\sim} \hat{X}.
\]

Recall now from [5] that the Bousfield-Kan completion \( Q_\infty(X) \) can be expressed as the inverse limit

\[
X \xrightarrow{\lim_{\kappa}} Q_k(X) = Q_\infty(X)
\]

of a tower of fibrations in which \( \lim_{\kappa} H(Q_k(X)) \xrightarrow{\sim} H(X) = 2^k \). Moreover, Proposition [9] is analogous to [3, Tower Lemma 6.2]. Here we provide a proof of the following result of Bousfield:

**Proposition 10.** For any connected space, the map \( X \to \ell Q(X) \) factors up to homotopy as

\[
X \to Q_\infty(X) \to \ell Q(X).
\]
Proof: Apply Lemma 2 to $X \rightarrow Q_\infty(X) = \lim_k Q_k(X)$ to obtain a quasi-isomorphism,

$$\varphi: \wedge V = \lim_k \wedge V_k \rightarrow A_{PL}(Q_\infty(X)) \rightarrow A_{PL}(X),$$

which then gives the map $Q_\infty(X) \rightarrow \langle \wedge V \rangle = \ell_Q(X).$ \hfill \Box

Corollary. If $H(Q_\infty(X)) \xrightarrow{\sim} H(X)$ - ($X$ is $\mathbb{Q}$-good in the terminology of [5]) - then $Q_\infty(X) \mathbb{Q} \xrightarrow{\sim} X.$

Finally, Proposition[9] provides a proof of a major theorem of Bousfield and Gugenheim:

**Proposition 11.** ([4, Theorem 12.2]) If $X$ is a connected space and $H(X)$ is a graded vector space of finite type, then $Q_\infty(X) \rightarrow X$ is a homotopy equivalence.

**proof:** As observed in the third example in §3, in a minimal Sullivan model $\wedge V$ for $X$, $V$ admits an increasing filtration $V[0] \subset \cdots \subset V[n] \subset \cdots$ in which $d: V[n+1] \rightarrow \wedge V[n]$ and each $V[n]$ is finite dimensional. Set $X[n] = \langle \wedge V[n] \rangle$. Then $X$ is the inverse limit of the tower of fibrations

$$X[0] \leftarrow \cdots \leftarrow X[n] \leftarrow$$

But this tower satisfies conditions of ([5, Chapter 5]) which identify the inverse limit as $Q_\infty(X).$ \hfill \Box

**Remark.** Suppose that for some $n$, card $H_n(X; \mathbb{Q})$ is an infinite cardinal $k = \text{card} \oplus_i H_i(X)$. Then Bousfield has shown [3] that card $Q_\infty(X) \leq 2^k$. On the other hand, if $\wedge V$ is a minimal Sullivan model of $X$, then card $V = \text{card} \wedge V = \text{card} H(X) = 2^k$. Then

$$\text{card} \pi_*(X) = \text{card} V^v = 2^{2k} > \text{card} Q_\infty(X).$$

In particular, the hypothesis of finite type in the result of Bousfield and Gugenheim is necessary.

**References**

[1] A. Blakers and M. Massey, *The homotopy groups of a triad, III*, Annals of Math. 58 (1953), 409-417.

[2] N. Bourbaki, *Eléments de Mathématique, Théorie des Ensembles, tome 3*, 1970, Hermann.

[3] A.K. Bousfield, Private communication.

[4] A.K. Bousfield and V.K. Gugenheim, *On PL de Rham Theory and Rational Homotopy Type*, Mem. Amer. Math. Soc. 179 (1976)

[5] A.K. Bousfield et D. Kan, *Homotopy limits, Completions and Localizations*, Lecture Notes in Math. 304, Springer-Verlag, 1972

[6] O. Cornea, G. Lupton, J. Oprea and D. Tanré, *Lusternik-Schnirelman Category*, Math. Surveys and Monographs 103 (2003), Amer. Math. Soc.
[7] Y. Félix, S. Halperin and J.-C. Thomas, *Rational Homotopy Theory*, Graduate texts in Mathematics 205, Springer-Verlag, 2001

[8] Y. Félix, S. Halperin and J.-C. Thomas, *Rational homotopy Theory II*, World Scientific, 2015

[9] Y. Félix and S. Halperin, *The depth and LS category of a topological space*, Math. Scand. 123 (2018), 220-238

[10] P. Hilton, G. Mislin and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, North-Holland Mathematics Studies, 1975.

[11] S. Ivanov and R. Mikhailov, *A finite \( \mathbb{Q} \)-bad space*, arXiv, 2017

[12] M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Annales Scientifiques de l'Ecole Normale Supérieure 71.2 (1954), 101-190.

[13] S. Lefschetz, *Algebraic Topology*, American Mathematical Society, 1942

[14] A. Mikhalev, V. Shipilrain and A. Zolotykh, *Subalgebras of free algebras*, Proc. Amer. Math. Soc. 124 (1996), 1977-1984.

[15] P.F. Pickel, *Rational cohomology of nilpotent groups and Lie algebras*, Communications in Algebra 6 (1978), 409-419

[16] D. Quillen, *Rational Homotopy Theory*, Ann. of Math. 90 (1969), 205-295.

[17] J.P. Serre, *Lie algebras and Lie groups*, Benjamin Inc. 1965.

[18] A.I. Sirsov, *On free Lie rings*, Mat. Sb. 45 (1958), 113-122.

[19] D. Sullivan, *Geometric Topology Part I, Localization, Periodicity, and Galois Symmetry*, MIT Press (1970)

[20] D. Sullivan, *Infinitesimal computations in topology*, Publ. IHES 47 (1977), 269-331.

[21] E. Witt, *Die Unterringe der freien Lieschen Ringe*, Math. Zeit. 64 (1956), 195-216.

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