Fixed point theorems on partial randomness

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Abstract

In this talk, we show the following fixed point theorem on partial randomness, from the point of view of algorithmic randomness.

Theorem [fixed point theorem on partial randomness]

For every \( T \in (0, 1) \), if \( \Omega(T) \) is a computable real number, then

(i) \( T \) is right-computable and not left-computable,
(ii) \( T \) is weakly Chaitin \( T \)-random and \( T \)-compressible,
(iii) \( \lim_{n \to \infty} \frac{H(T \mid n)}{n} = T. \Rightarrow \text{The compression rate of } T \text{ equals to } T. \)

After that, we introduce variants of this theorem, and investigate their properties and relation.
Preliminaries: Program-size Complexity

- \( \{0, 1\}^* := \{\lambda, 0, 1, 00, 01, 10, 11, 000, \ldots \} \). The set of finite binary strings.
- For any \( s \in \{0, 1\}^* \), \( |s| \) denotes the \textit{length} of \( s \).
- Let \( V \subset \{0, 1\}^* \). We say \( V \) is \textit{prefix-free} if for any distinct \( s \) and \( t \in V \), \( s \) is not a prefix of \( t \).

For example \( \{0, 10\} \): prefix-free \( \{0, 01\} \): not prefix-free

Let \( U \) be a universal \textit{self-delimiting} Turing machine.

\( \text{Dom } U \), i.e., the domain of definition of \( U \), is a prefix-free set.

**Definition** The \textit{program-size complexity} (or \textit{Kolmogorov complexity}) \( H(s) \) of \( s \in \{0, 1\}^* \) is defined by

\[
H(s) := \min \left\{ |p| \mid p \in \{0, 1\}^* \& U(p) = s \right\}.
\]

\( H(s) \): The length of the shortest input for the universal self-delimiting Turing machine \( U \) to output \( s \). \( \Rightarrow H(s) \): The degree of randomness of \( s \).
**Preliminaries: Randomness of Real Number**

**Definition** For any $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}^+$, we denote by $\alpha \upharpoonright n$ the first $n$ bits of the base-two expansion of $\alpha - \lfloor \alpha \rfloor$, i.e., the fractional part of $\alpha$.

**Definition** [weak Chaitin randomness, Chaitin 1975]
We say $\alpha \in \mathbb{R}$ is **weakly Chaitin random** if $n \leq H(\alpha \upharpoonright n) + O(1)$, i.e., any prefix of the base-two expansion of $\alpha$ cannot be compressed by $H$.

This notion is equivalent to Martin-Löf randomness (Schnorr).

**Definition** [Chaitin’s halting probability $\Omega$, Chaitin 1975]

$$\Omega := \sum_{p \in \text{Dom } U} 2^{-|p|}.$$ 

**Theorem** [Chaitin 1975] $\Omega$ is weakly Chaitin random.
Preliminaries: Partial Randomness of Real Number

The partial randomness (degree of randomness) of a real number can be characterized by a real number.

**Definition** [weak Chaitin $T$-randomness, Tadaki 2002] Let $T \in [0, 1]$. We say $\alpha \in \mathbb{R}$ is **weakly Chaitin $T$-random** if $Tn \leq H(\alpha \upharpoonright n) + O(1)$.

In the case of $T = 1$, the weak Chaitin $T$-randomness results in the weak Chaitin randomness.

**Definition** [$T$-compressibility] Let $T \in [0, 1]$. We say $\alpha \in \mathbb{R}$ is **$T$-compressible** if $H(\alpha \upharpoonright n) \leq Tn + o(n)$, which is equivalent to $\limsup_{n \to \infty} \frac{H(\alpha \upharpoonright n)}{n} \leq T$.

**Remark** If $\alpha \in \mathbb{R}$ is weakly Chaitin $T$-random and $T$-compressible, then

$$\lim_{n \to \infty} \frac{H(\alpha \upharpoonright n)}{n} = T.$$

The **compression rate** of $\alpha$ by program-size complexity equals to $T$. (The converse does not necessarily hold.)
Definition [generalization of Chaitin's $\Omega$, Tadaki 1999]

$$\Omega(T) := \sum_{p \in \text{Dom } U} 2^{-\frac{|p|}{T}} \quad (T > 0).$$

$\Omega(1) = \Omega$. 

Theorem [Tadaki 1999] Let $T \in \mathbb{R}$.

(i) If $0 < T \leq 1$ and $T$ is computable, then $\Omega(T)$ is weakly Chaitin $T$-random and $T$-compressible. $\Rightarrow$ The compression rate of $\Omega(T)$ equals to $T$.

(ii) If $1 < T$, then $\Omega(T)$ diverges to $\infty$.

Here, $T$ is called computable if the mapping $\mathbb{N}^+ \ni n \mapsto T \upharpoonright n$ is a total recursive function.
**Fixed Point Theorem on Partial Randomness**

**Theorem** [fixed point theorem on partial randomness, Tadaki, CiE 2008]

For every $T \in (0, 1)$, if $\Omega(T)$ is a computable real number, then

(i) $T$ is right-computable and not left-computable,

(ii) $T$ is weakly Chaitin $T$-random and $T$-compressible,

(iii) $\lim_{n \to \infty} \frac{H(T \mid n)}{n} = T$. $\Rightarrow$ The compression rate of $T$ equals to $T$ itself.

Here, a real $\alpha$ is called **right-computable** if the set $\{ r \in \mathbb{Q} \mid \alpha < r \}$ is r.e., and $\alpha$ is called **left-computable** if $-\alpha$ is right-computable.
Proof of Fixed Point Theorem

Theorem [fixed point theorem on partial randomness,] [posted again]
For every $T \in (0, 1)$, if $\Omega(T)$ is a computable real number, then
(i) $T$ is right-computable and not left-computable,
(ii) $T$ is weakly Chaitin $T$-random and $T$-compressible,
(iii) $\lim_{n \to \infty} \frac{H(T | n)}{n} = T$. □

Lemma [upper bound I] For every $T \in (0, 1)$, if $\Omega(T)$ is right-computable then $T$ is also right-computable. □

Lemma [upper bound II] For every $T \in (0, 1)$, if $\Omega(T)$ is left-computable and $T$ is right-computable, then $T$ is $T$-compressible. □

Lemma [lower bound] For every $T \in (0, 1)$, if $\Omega(T)$ is right-computable then $T$ is weakly Chaitin $T$-random. □
Proofs of the three lemmas

**Lemma [upper bound I]** For every $T \in (0, 1)$, if $\Omega(T)$ is right-computable then $T$ is also right-computable.

**Proof** For each $k \in \mathbb{N}^+$ and $x \in (0, 1)$, let $\omega_k(x) = \sum_{i=1}^{k} 2^{-|p_i|/x}$, where $p_1, p_2, p_3, \ldots$ is a particular recursive enumeration of $\text{Dom } U$.

Then we see that, for every $r \in \mathbb{Q} \cap (0, 1)$, $T < r$ if and only if there exists $k \in \mathbb{N}^+$ such that $\Omega(T) < \omega_k(r)$. This is because $\Omega(x)$ is an increasing function of $x \in (0, 1]$ and $\lim_{k \to \infty} \omega_k(r) = \Omega(r)$.

Since $\Omega(T)$ is right-computable, the set $\{ r \in \mathbb{Q} \cap (0, 1) \mid \exists k \in \mathbb{N}^+ \, \Omega(T) < \omega_k(r) \}$ is r.e. and therefore the set $\{ r \in \mathbb{Q} \cap (0, 1) \mid T < r \}$ is also r.e. □

**Lemma [upper bound II]** For every $T \in (0, 1)$, if $\Omega(T)$ is left-computable and $T$ is right-computable, then $T$ is $T$-compressible.

**Proof** Omitted. □
**Lemma** [lower bound] For every $T \in (0, 1)$, if $\Omega(T')$ is right-computable then $T$ is weakly Chaitin $T$-random.

**Proof** The following procedure calculates a partial recursive function $\Psi : \{0, 1\}^* \to \{0, 1\}^*$ such that $Tn - Tc < H(\Psi(T | n))$. The lemma follows from $H(\Psi(T | n)) \leq H(T | n) + O(1)$. Let $\omega_k(x) = \sum_{i=1}^{k} 2^{-|p_i|/x}$.

**Procedure:** Given $T | n$, one can effectively find $k_0$ which satisfies

$$\Omega(T') < \omega_{k_0}(0.(T | n) + 2^{-n}).$$

This is possible because $\Omega(x)$ is an increasing function of $x$, $\lim_{k \to \infty} \omega_k(r) = \Omega(r)$ for every $r \in \mathbb{Q} \cap (0, 1)$, and $\Omega(T)$ is right-computable. It follows that

$$\sum_{i=k_0+1}^{\infty} 2^{-\frac{|p_i|}{T}} = \Omega(T) - \omega_{k_0}(T) < \omega_{k_0}(0.(T | n) + 2^{-n}) - \omega_{k_0}(T) < 2^{c-n}.$$

Hence, for every $i > k_0$, $2^{-\frac{|p_i|}{T}} < 2^{c-n}$ and therefore $Tn - Tc < |p_i|$. Thus, by calculating the set $\{ U(p_i) \mid i \leq k_0 \}$ and picking any one finite binary string $s$ which is not in this set, one can then obtain $s \in \{0, 1\}^*$ such that $Tn - Tc < H(s)$.
Remark on the sufficient condition in the fixed Point Theorem

**Theorem** [fixed point theorem on partial randomness] [posted again]
For every $T \in (0, 1)$, if $\Omega(T)$ is computable then $T$ is weakly Chaitin $T$-random and $T$-compressible.

Note that $\Omega(x)$ is a strictly increasing continuous function of $x \in (0, 1)$, and the set of all computable real numbers is dense in $\mathbb{R}$. Thus,

**Theorem** The set $\{T \in (0, 1) \mid \Omega(T) \text{ is computable}\}$ is dense in $(0, 1)$.

**Corollary** [density of the fixed points]
The set $\{T \in (0, 1) \mid T \text{ is weakly Chaitin } T\text{-random and } T\text{-compressible}\}$ is dense in $(0, 1)$.

At this point, the following question would arise naturally:

**Question** Is this sufficient condition, i.e., the computability of $\Omega(T)$, also necessary for $T$ to be a fixed point?

**Answer** Completely not !! (as we can see through the following argument)
Thermodynamic Quantities in AIT: Definition

The thermodynamic quantities in AIT (algorithmic information theory) is introduced in the following manner.

Definition Let \( q_1, q_2, q_3, \ldots \) be an arbitrary enumeration of Dom \( U \).

Note that the results of this talk are independent of the choice of \( \{q_i\} \).

Definition [thermodynamic quantities in AIT, Tadaki, CiE 2008] Let \( T > 0 \).

(i) Partition Function: \( Z(T) := \lim_{k \to \infty} Z_k(T) \), where \( Z_k(T) = \sum_{i=1}^{k} 2^{-\frac{|q_i|}{T}} \).

(ii) Free Energy: \( F(T) := \lim_{k \to \infty} F_k(T) \), where \( F_k(T) = -T \log_2 Z_k(T) \).

(iii) Energy: \( E(T) := \lim_{k \to \infty} E_k(T) \), where \( E_k(T) = \frac{1}{Z_k(T)} \sum_{i=1}^{k} |q_i| 2^{-\frac{|q_i|}{T}} \).

(iii) Entropy: \( S(T) := \lim_{k \to \infty} S_k(T) \), where \( S_k(T) = \frac{E_k(T) - F_k(T)}{T} \).

Remark (i) \( Z(T) = \Omega(T) \). (ii) The real \( T \) corresponds to “temperature”.
Thermodynamic Quantities in AIT: Properties

The thermodynamic quantities $F(T)$, $E(T)$, and $S(T)$ has the almost same randomness properties as $\Omega(T)$, i.e., $Z(T)$.

Theorem [free energy $F(T)$] Let $T \in \mathbb{R}$.

(i) If $0 < T \leq 1$ and $T$ is computable, then $F(T)$ converges to a real number which is weakly Chaitin $T$-random and $T$-compressible. (same as for $\Omega(T)$)

(ii) If $1 < T$, then $F(T)$ diverges to $-\infty$.

\[ \square \]

Definition We say $\alpha \in \mathbb{R}$ is Chaitin $T$-random if $\lim_{n \to \infty} H(\alpha | n) - Tn = \infty$.

Theorem [energy $E(T)$] Let $T \in \mathbb{R}$.

(i) If $0 < T < 1$ and $T$ is computable, then $E(T)$ converges to a real number which is Chaitin $T$-random and $T$-compressible.

(ii) If $1 \leq T$, then $E(T)$ diverges to $\infty$.

\[ \square \]

Theorem [entropy $S(T)$] Let $T \in \mathbb{R}$.

(i) If $0 < T < 1$ and $T$ is computable, then $S(T)$ converges to a real number which is Chaitin $T$-random and $T$-compressible.

(ii) If $1 \leq T$, then $S(T)$ diverges to $\infty$.

\[ \square \]
Thermodynamic Quantities in AIT: **Fixed Point Theorems**

In the fixed point theorem, $\Omega(T)$ can be replaced by each of the thermodynamic quantities $F(T)$, $E(T)$, and $S(T)$.

**Theorem** [fixed point theorem by the free energy $F(T)$]
For every $T \in (0, 1)$, if $F(T)$ is computable, then

(i) $T$ is right-computable and not left-computable,
(ii) $T$ is weakly Chaitin $T$-random and $T$-compressible.

This theorem has the exactly same form as for $\Omega(T)$.

**Theorem** [fixed point theorem by the energy $E(T)$]
For every $T \in (0, 1)$, if $E(T)$ is computable, then

(i) $T$ is right-computable and not left-computable,
(ii) $T$ is Chaitin $T$-random and $T$-compressible.

**Theorem** [fixed point theorem by the entropy $S(T)$]
For every $T \in (0, 1)$, if $S(T)$ is computable, then

(i) $T$ is right-computable and not left-computable,
(ii) $T$ is Chaitin $T$-random and $T$-compressible.
Proof of the fixed point theorem by free energy $F(T)$

**Theorem** [general form of fixed point theorem] Let $f: (0, 1) \rightarrow \mathbb{R}$. Suppose that $f$ is a strictly increasing function and there is $g: (0, 1) \times \mathbb{N}^+ \rightarrow \mathbb{R}$ which satisfies the following conditions:

(i) $\forall T \in (0, 1) \lim_{k \to \infty} g(T, k) = f(T)$.

(ii) The mapping $\mathbb{Q} \times (\mathbb{Q} \cap (0, 1)) \ni (r, k) \mapsto g(r, k)$ is computable.

(iii) $\forall T \in (0, 1) \exists k_0 \in \mathbb{N}^+ \exists a, b \in \mathbb{N} \forall k \geq k_0$

$$2^{-\frac{|p_{k+1}|}{T}} - a \leq g(T, k + 1) - g(T, k) \leq 2^{-\frac{|p_{k+1}|}{T}} + b.$$ 

(iv) $\forall T \in (0, 1) \exists t \in (T, 1) \exists k_0 \in \mathbb{N}^+ \exists c, d \in \mathbb{N} \forall k \geq k_0 \forall x \in (T, t)$

$$2^{-c}(x - T) \leq g(x, k) - g(T, k) \leq 2^d(x - T).$$

(v) $\forall t_1, t_2 \in (0, 1)$ with $t_1 < t_2 \exists k_0 \in \mathbb{N}^+ \forall k \geq k_0 \forall x \in [t_1, t_2] g(x, k) \leq f(x)$.

(vi) $\forall k \in \mathbb{N}^+ \forall T \in (0, 1) \lim_{x \to T+0} g(x, k) = g(T, k)$.

Then, for every $T \in (0, 1)$, if $f(T)$ is computable, then $T$ is weakly Chaitin $T$-random and $T$-compressible.
Proof of the fixed point theorem by free energy $F(T)$

**Theorem** [fixed point theorem by free energy $F(T)$] [posted again]

For every $T \in (0, 1)$, if $F(T)$ is a computable real number, then $T$ is weakly Chaitin $T$-random and $T$-compressible.

**A portion of the proof:**

Using the mean value theorem and the lemma below,

$$S_k(T)(x - T) \leq F_k(T) - F_k(x) \leq S_k(t)(x - T)$$

for every $k \in \mathbb{N}^+$ and every $T, x, t \in (0, 1)$ with $T < x < t$. On the other hand, for every $T \in (0, 1)$, there exists $k_0 \in \mathbb{N}^+$ such that, for every $k \geq k_0$,

$$0 < S_{k_0}(T) \leq S_k(T) \leq S(T).$$

**Lemma** [thermodynamic relation] Let $T \in (0, 1)$ and $k \in \mathbb{N}^+$.

(i) $F'_k(T) = -S_k(T)$, $E'_k(T) = C_k(T)$, and $S'_k(T) = C_k(T)/T$.

(ii) $F'(T) = -S(T)$, $E'(T) = C(T)$, and $S'(T) = C(T)/T$.

(iii) $S_k(T), C_k(T) \geq 0$ and $S(T), C(T) > 0$. 


Relation between the sufficient conditions of FPTs I

**Theorem** There does not exist $T \in (0, 1)$ such that both $\Omega(T)$ and $F(T)$ are computable.

**Proof** Contrarily, assume that both $\Omega(T)$ and $F(T)$ are computable for some $T \in (0, 1)$. Since the statistical mechanical relation $F(T) = -T \log_2 \Omega(T)$ holds,

$$T = -\frac{F(T)}{\log_2 \Omega(T)}.$$

Thus, $T$ is computable, and therefore $\Omega(T)$ is weakly Chaitin $T$-random, i.e., $Tn \leq H((\Omega(T)) \upharpoonright n) + O(1)$. However, this is impossible, since $\Omega(T)$ is computable and therefore $H((\Omega(T)) \upharpoonright n) \leq 2 \log_2 n + O(1)$. Thus we have a contradiction. 

\[
\{ T \in (0,1) \mid \Omega(T) \text{ is computable} \} \cap \{ T \in (0,1) \mid F(T) \text{ is computable} \} = \emptyset.
\]

In particular, this shows that the computability of $\Omega(T)$ is not a necessary condition for $T$ to be a fixed point in the fixed point theorem by $\Omega(T)$. 

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Relation between the sufficient conditions of FPTs II

**Theorem**  There does not exist $T \in (0, 1)$ such that all of $\Omega(T)$, $E(T)$, and $S(T)$ are computable.

**Proof**  Use the statistical mechanical relation

$$S(T) = \frac{E(T)}{T} + \log_2 \Omega(T).$$

**Theorem**  There does not exist $T \in (0, 1)$ such that all of $F(T)$, $E(T)$, and $S(T)$ are computable.

**Proof**  Use the thermodynamic relation

$$S(T) = \frac{E(T) - F(T)}{T}.$$
Some other property of the sufficient condition in FPTs

Using the fixed point theorem by $\Omega(T)$, some property of the computability of $\Omega(T)$ is derived.

Let $T \in (0, 1)$ and $a \in (0, 1]$. Assume that $a$ is computable.

$\Omega(aT)$ is computable $\Rightarrow \lim_{n \to \infty} \frac{H((aT) \upharpoonright n)}{n} = aT$ $\Rightarrow \lim_{n \to \infty} \frac{H(T \upharpoonright n)}{n} = aT.$

by FPT by $H((aT) \upharpoonright n) = H(T \upharpoonright n) + O(1)$

**Theorem** $S_a \cap S_b = \emptyset$ for any distinct computable real numbers $a, b \in (0, 1]$, where $S_a = \{ T \in (0, 1) \mid \Omega(aT) \text{ is computable} \}.$

**Example** For every $T \in (0, 1)$, if $\Omega(T)$ is computable, then for each integer $n \geq 2$, $\Omega(T/n)$ is not computable. Namely,

for every $T \in (0, 1)$, if the sum $\sum_{p \in \text{Dom } U} 2^{-|p|/T}$ is computable, then its power sum $\sum_{p \in \text{Dom } U} \left(2^{-|p|/T}\right)^n$ is not computable for every integer power $n \geq 2.$
Summary

In this talk, we introduced and showed the following fixed point theorem on partial randomness, from the point of view of algorithmic randomness.

**Theorem** [fixed point theorem on partial randomness]

For every $T \in (0, 1)$, if $\Omega(T')$ is a computable real number, then

(i) $T$ is right-computable and not left-computable,

(ii) $T$ is weakly Chaitin $T$-random and $T$-compressible,

(iii) $\lim_{n \to \infty} \frac{H(T \upharpoonright n)}{n} = T$.

After that, we introduced several variants of this theorem, and investigate their properties and relation. In particular, we showed that the sufficient condition for $T$ to be a fixed point is not a necessary condition in the fixed point theorems.