CROKE-KLEINER ADMISSIBLE GROUPS: PROPERTY (QT) AND QUASICONVEXITY

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Abstract. Croke-Kleiner admissible groups firstly introduced by Croke-Kleiner in [CK02] belong to a particular class of graph of groups which generalize fundamental groups of 3–dimensional graph manifolds. In this paper, we show that if $G$ is a Croke-Kleiner admissible group, acting geometrically on a CAT(0) space $X$, then a finitely generated subgroup of $G$ has finite height if and only if it is strongly quasi-convex. We also show that if $G \acts X$ is a flip CKA action then $G$ is quasi-isometric embedded into a finite product of quasi-trees. With further assumption on the vertex groups of the flip CKA action $G \acts X$, we show that $G$ satisfies property (QT) that is introduced by Bestvina-Bromberg-Fujiwara in [BBF19].

1. Introduction

In [CK02], Croke and Kleiner study a particular class of graph of groups which they call admissible groups and generalize fundamental groups of 3–dimensional graph manifolds and torus complexes (see [CK00]). If $G$ is an admissible group that acts geometrically on a Hadamard space $X$ then the action $G \acts X$ is called Croke-Kleiner admissibe (see Definition 2.1) termed by Guilbault-Mooney [GM14]. The CKA action is modeling on the JSJ structure of graph manifolds where the Seifert fibration is replaced by the following central extension of a general hyperbolic group:

\[ 1 \to \mathbb{Z}(G_v) = \mathbb{Z} \to G_v \to H_v \to 1 \]

However, CKA groups can encompass much more general class of groups and can actually serve as one of simplest algebraic means to produce interesting groups from any finite number of hyperbolic groups.

Let $\mathcal{G}$ be a finite graph with $n$ vertices, each of which are associated with a hyperbolic group $H_i$. We then pick up an independent set of primitive loxodromic elements in $H_i$ which crossed with $\mathbb{Z}$ are the edge groups $\mathbb{Z}^2$. We identify $\mathbb{Z}^2$ in adjacent $H_i \times \mathbb{Z}$’s by flipping $\mathbb{Z}$ and loxodromic elements as did in flip graph manifolds by Kapovich and Leeb [KL98]. These are motivating examples of flip CKA groups and actions, for the precise definition of flip CKA actions, we refer the reader to Section 4.2.

The class of CKA actions has manifested a variety of interesting features in CAT(0) groups. For instance, the equivariant visual boundaries of admissible actions are completely determined in [CK02]. Meanwhile, the non-homeomorphic visual boundaries of torus complexes were constructed in [CK00] and have sparked an intensive research on boundaries of CAT(0) spaces. So far, the most of research on CKA groups is centered around the boundary problem (see [GM14], [Gre16]). In the rest of Introduction, we shall explain our results on the coarse geometry of Croke-Kleiner admissible groups and their subgroups.

1.1. Proper actions on finite products of quasi-trees. A quasi-tree is a geodesic metric space quasi-isometric to a tree. Recently, Bestvina, Bromberg and Fujiwara [BBF19] introduced a (QT) property for a finitely generated group: $G$ acts properly on a finite product of quasi-trees so that the orbital map from $G$ with word metrics is a quasi-isometric embedding. This is a stronger property of the finite asymptotic dimension by recalling that a quasi-isometric embedding implies finite asdim of $G$. It is known that Coxeter groups have property (QT) (see [DJ99]), and thus every

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right-angled Artin group has property (QT) (see Induction 2.2 in [BBF19]). Furthermore, the fundamental group of a compact special cube complex is undistorted in RAAGs (see [HW08]) and then has property (QT). As a consequence, many 3-manifold groups have property (QT), among which we wish to mention chargeless (including flip) graph manifolds [HP15] and finite volume hyperbolic 3-manifolds [Wis20]. In [BBF19], residually finite hyperbolic groups and mapping class groups are proven to have property (QT). It is natural to ask which other groups have property (QT) rather than these groups above.

The main result of this paper adds flip CKA actions into the list of groups which have property (QT). The notion of an omnipotent group is introduced by Wise in [Wis00] and has found many applications in subgroup separability. We refer the reader to Definition 5.8 for its definition and note here that free groups [Wis00], surfaces groups [Baj07], and the more general class of virtually special hyperbolic groups [Wis20] are omnipotent.

**Theorem 1.1.** Let \( G \curvearrowright X \) be a flip admissible action where for every vertex group the central extension \( (1) \) has omnipotent hyperbolic quotient group. Then \( G \) acts properly on a finite product of quasi-trees so that the orbital map is a quasi-isometric embedding.

**Remark 1.2.** It is an open problem whether every hyperbolic group is residually finite. In [Wis00, Remark 3.4], Wise noted that if every hyperbolic group is residually finite, then any hyperbolic group is omnipotent.

**Remark 1.3.** As a corollary, Theorem 1.1 gives another proof that flip graph manifold groups have property (QT). This was indeed one of motivations of this study (without noticing [HP15]).

In [HS13], Hume-Sisto prove that the universal cover of any flip graph manifold is quasi-isometrically embedded in the product of three metric trees. However, it does not follow from their proof that the fundamental group of a flip graph manifold has property (QT).

We now give an outline of the proof of Theorem 1.1 and explain some intermediate results, which we believe are of independent interest.

**Proposition 1.4.** Let \( G \curvearrowright X \) be a flip CKA action. Then there exists a quasi-isometric embedding from \( X \) to a product \( X_1 \times X_2 \) of two hyperbolic spaces.

If \( G_v = H_v \times Z(G_v) \) for every vertex \( v \in T^0 \) and \( G_e = Z(G_{e-}) \times Z(G_{e+}) \) for every edge \( e \in T^1 \), then there exists a subgroup \( \hat{G} < G \) of finite index at most 2 such that the above Q.I. embedding is \( \hat{G} \)-equivariant.

Let us describe briefly the construction of \( X \in \{ X_1, X_2 \} \). By Bass-Serre theory, \( G \) acts on the Bass-Serre tree \( T \) with vertex groups \( G_v \) and edge groups \( G_e \). Let \( V \) be one of the two sets of vertices in \( T \) with pairwise even distance. Note that \( G_v \) is the central extension of a hyperbolic group \( H_v \) by \( \mathbb{Z} \), so acts geometrically on a metric product \( Y_v = Y_v \times \mathbb{R} \) where \( H_v \) acts geometrically on \( Y_v \) and \( Z(G_v) \) acts by translation on \( \mathbb{R} \) lines. Roughly, the space \( X \) is obtained by isometric gluing of the boundary lines of \( Y_v \)'s over vertices \( v \) in the link of every \( w \in T^0 - V \). In proving Proposition 1.4, the main tool is the construction of a class of quasi-geodesic paths called *special paths* between any two points in \( X \). See Section 3 for the details and related discussion after Theorem 1.6 below.

To endow an action on \( X \), we pass to an index at most 2 subgroup \( \hat{G} \) preserving \( V \) and the stabilizer in \( \hat{G} \) of \( v \in V \) is \( G_v \) by Lemma 4.6. Under the assumptions on \( G_v \) and \( G_e \)'s, \( \hat{G} \) acts by isometry on \( X \) and the Q.I. embedding is \( \hat{G} \)-equivariant.

To prove Theorem 1.1, we exploit the strategy as [BBF19] to produce a proper action on products of quasi-trees. By Lemma 4.15, we first produce enough quasi-lines

\[
\bar{A} = \bigcup_{v \in V} A_v
\]

for the hyperbolic space \( X \) where \( A_v \) is a \( H_v \)-finite set of quasi-lines in \( Y_v \) so that the so-called distance formula follows in Proposition 4.18. On the other hand, the “crowd” quasi-lines in \( \bar{A} \) may
This is done in the following two steps:

Geodesic in quasi-convex finitely generated group \( G \), crucially the omnipotence, with details given in index subgroups \( K \) sparse. This follows the same argument in [BBF19]. Secondly, we need to reassemble those finite from the boundary lines. See Proposition 5.5.

\( X \) in Proposition 5.7 so that \( Y \) from different pieces that each \( A \) is quasi-isometric embedded into the product of \( \mathcal{X} \) with a quasi-tree from the boundary lines. See Proposition 5.5.

The goal is then to find a finite index subgroup \( \bar{G} < G \) so that each orbit in \( \mathbb{A} \) is sparse. This is done in the following two steps:

By residual finiteness of \( H_v \), we first find a finite index subgroup \( K_v < H_v \) whose orbit in \( \mathbb{A}_v \) is sparse. This follows the same argument in [BBF19]. Secondly, we need to reassemble those finite index subgroups \( K_v \) as a finite index group \( \bar{G} \) so that the orbit in \( \mathbb{A} \) is sparse. This step uses crucially the omnipotence, with details given in §5.4. The projection axioms thus fulfilled for each \( \bar{G} \)-orbit produce a finite product of actions on quasi-trees, and finally, the distance formula finishes the proof of Theorem 1.1.

1.2. Strongly quasi-convex subgroups. The height of a finitely generated subgroup \( H \) in a finitely generated group \( G \) is the maximal \( n \in \mathbb{N} \) such that there are distinct cosets \( g_1 H, \ldots, g_n H \in G/H \) such that the intersection \( g_1 H g_1^{-1} \cap \cdots \cap g_n H g_n^{-1} \) is infinite. The subgroup \( H \) is called strongly quasi-convex in \( G \) if for any \( L \geq 1, C \geq 0 \) there exists \( R = R(L, C) \) such that every \((L, C)\)-quasigeodesic in \( G \) with endpoints in \( H \) is contained in the \( R \)-neighborhood of \( H \). We note that strong quasiconvexity does not depend on the choice of finite generating set of the ambient group and it agrees with quasiconvexity when the ambient group is hyperbolic. In [GMRS98], the authors prove that quasi-convex subgroups in hyperbolic groups have finite height. It is a long-standing question asked by Swarup that whether or not the converse is true (see Question 1.8 in [Bes]). Tran in [Tra19] generalizes the result of [GMRS98] by showing that strongly quasi-convex subgroups in any finitely generated group have finite height. It is natural to ask whether or not the converse is true in this setting (i.e., finite height implies strong quasiconvexity). If the converse is true, then we could characterize strongly quasi-convex subgroup of a finitely generated group purely in terms of group theoretic notions.

In [NTY], the authors prove that having finite height and strong quasiconvexity are equivalent for all finitely generated 3–manifold groups except the only ones containing the Sol command in its sphere-disk decomposition, and the graph manifold case was an essential case treated there. More precisely, Theorem 1.7 in [NTY] states that finitely generated subgroups of the fundamental group of a graph manifold are strongly quasi-convex if and only if they have finite height. The second main result of this paper is to generalize this result to Croke-Kleiner admissible action \( G \acts X \).

Theorem 1.6. Let \( G \acts X \) be a CKA action. Let \( K \) be a nontrivial, finitely generated infinite index subgroup of \( G \). Then the following are equivalent.

1. \( K \) is strongly quasi-convex.
2. \( K \) has finite height in \( G \).
3. \( K \) is virtually free and every infinite order elements are Morse.
4. Let \( G \acts T \) be the action of \( G \) on the associated Bass-Serre tree. \( K \) is virtually free and the action of \( K \) on the tree \( T \) induces a quasi-isometric embedding of \( K \) into \( T \).
We prove Theorem 1.6 by showing that (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) \(\Rightarrow\) (1) and (3) \(\iff\) (4). Similarly as in [NTY], the heart part of Theorem 1.6 is the implication (3) \(\Rightarrow\) (1). We briefly review ideas in the proof of Theorem 1.7 in [NTY]. Suppose that \(K\) is a finitely generated finite height subgroup of \(\pi_1(M)\) where \(M\) is a graph manifold. Let \(M_K \rightarrow M\) be the covering space of \(M\) corresponding to \(K\). The authors in [NTY] prove that \(K\) is strongly quasi-convex in \(\pi_1(M)\) by using Sisto’s notion of path system \(\mathcal{PS}(\tilde{M})\) in the universal cover \(\tilde{M}\) of \(M\), and prove that the preimage of the Scott core of \(M_K\) in \(\tilde{M}\) is \(\mathcal{PS}(\tilde{M})\)–contracting in the sense of Sisto. In this paper, the strategy of the proof of Theorem 1.6 is similar to the proof of Theorem 1.7 in [NTY] where we still use Sisto’s path system in \(X\) but details are different. Sisto’s construction of special paths are carried out only in flip graph manifolds. Our construction of \((X, \mathcal{PS}(X))\) relies on the work of Croke-Kleiner [CK02] and applies to any admissible space \(X\) (so any nonpostively curved graph manifold). We then construct a subspace \(C_K \subset X\) on which \(K\) acts geometrically and show that \(C_K\) is contracting in \(X\) with respect to the path system \((X, \mathcal{PS}(X))\). As a consequence, \(K\) is strongly quasi-convex in \(G\).

To conclude the introduction, we list a few questions and problems.

Quasi-isometric classification of graph manifolds has been studied by Kapovich-Leeb [KL98] and a complete quasi-isometric classification for fundamental groups of graph manifolds is given by Behrstock-Neumann [BN08]. Kapovich-Leeb prove that for any graph manifold \(M\), there exists a flip graph manifold \(N\) such that their fundamental groups are quasi-isometric. We would like to know that whether or not such a result holds for admissible groups.

Question 1.7. Let \(G\) be an admissible group such that each vertex group is the central extension of an omnipotent hyperbolic CAT(0) group by \(\mathbb{Z}\). Does there exist flip CKA action \(G' \acts X\) so that \(G\) and \(G'\) are quasi-isometric?

Question 1.8 (Quasi-isometry rigidity). Let \(G \acts X\) be a flip CKA action, and \(Q\) be a finitely generated group which is quasi-isometric to \(G\). Does there exist a finite index subgroup \(Q' < Q\) such that \(Q'\) is a flip CKA group?

With a positive answer to the above questions, we hope one can try to follow the strategy described in [BN08] to attack the following.

Problem 1.9. Under the assumption of Theorem 1.1, give a quasi-isometric classification of admissible actions.

In [Liu13], Liu showed that the fundamental group of a non-positively curved graph manifold \(M\) is virtually special (the case \(\partial M \neq \emptyset\) was also obtained independently by PrzytyckiWise [PW14]). Thus, it is natural to ask the following.

Question 1.10. Let \(G \acts X\) be a CKA action where vertex groups are the central extension of a virtually special hyperbolic group by the integer group. Is \(G\) virtually special?

As above, a positive answer to the question (with virtual compact specialness) would give an other proof of Theorem 1.1 under the same assumption.

Overview. In Section 2, we review some concepts and results about Croke-Kleiner admissible groups. In Section 3, we construct special paths in admissible spaces and give some results that will be used in the later sections. The proof of Theorem 1.5 and Proposition 1.4 is given in Section 4. We prove Theorem 1.1 and Theorem 1.6 in Section 5 and Section 6 respectively.

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2. Preliminary

Admissible groups firstly introduced in [CK02]. This is a particular class of graph of groups that includes fundamental groups of 3–dimensional graph manifolds (i.e, compact 3–manifolds are obtained by gluing some circle bundles). In this section, we review admissible groups and their properties that will used throughout in this paper.

**Definition 2.1.** A graph of group \( G \) is admissible if

1. \( G \) is a finite graph with at least one edge.
2. Each vertex group \( G_v \) has center \( Z(G_v) \cong \mathbb{Z} \), \( H_v : G_v / Z(G_v) \) is a non-elementary hyperbolic group, and every edge subgroup \( G_e \) is isomorphic to \( \mathbb{Z}^2 \).
3. Let \( e_1 \) and \( e_2 \) be distinct directed edges entering a vertex \( v \), and for \( i = 1, 2 \), let \( K_i \subset G_v \) be the image of the edge homomorphism \( G_{e_i} \rightarrow G_v \). Then for every \( g \in G_v \), \( gK_1g^{-1} \) is not commensurable with \( K_i \), and for every \( g \in G_v - K_i \), \( gK_i g^{-1} \) is not commensurable with \( K_i \).
4. For every edge group \( G_e \), if \( \alpha_i : G_e \rightarrow G_v \) are the edge monomorphism, then the subgroup generated by \( \alpha_1^{-1}(Z(G_{e_1})) \) and \( \alpha_2^{-1}(Z(G_{e_2})) \) has finite index in \( G_e \).

A group \( G \) is admissible if it is the fundamental group of an admissible graph of groups.

**Definition 2.2.** We say that the action \( G \curvearrowright X \) is a Croke-Kleiner admissible (CKA) if \( G \) is an admissible group, and \( X \) is a Hadamard space, and the action is geometrically (i.e, properly and cocompactly by isometries)

Examples of admissible actions:

1. Let \( M \) be a noneometric graph manifold that admits a nonpositively curve metric. Lift this metric to the universal cover \( \tilde{M} \) of \( M \), and we denote this metric by \( d \). Then the action \( \pi_1(M) \curvearrowright (\tilde{M}, d) \) is a CKA action.
2. Let \( T \) be the torus complexes constructed in [CK00]. Then \( \pi_1(T) \curvearrowright \tilde{T} \) is a CKA action.
3. Let \( H_1 \) and \( H_2 \) be two torsion-free hyperbolic groups such that they act geometrically on \( CAT(0) \) spaces \( X_1 \) and \( X_2 \) respectively. Let \( G_i = H_i \times \mathbb{Z} \) (with \( i = 1, 2 \)), then \( G_i \) acts geometrically on the \( CAT(0) \) space \( Y_i = X_i \times \mathbb{R} \). A primitive hyperbolic element in \( H_i \) gives a totally geodesic torus \( T_i \) in the quotient space \( Y_i / G_i \). Choose a basis on each torus \( T_i \). Let \( f : T_1 \rightarrow T_2 \) be a flip map. Let \( M \) be the space obtained by gluing \( Y_1 \) to \( Y_2 \) along the homomorphism \( f \). We note that there exists a metric on \( M \) such that with respect to this metric, \( M \) is a locally \( CAT(0) \) space. Then \( G \curvearrowright \tilde{M} \) is a CKA action.

Let \( G \curvearrowright X \) be an admissible action, and let \( G \curvearrowright T \) be the action of \( G \) on the associated Bass-Serre tree. Let \( T^0 = \text{Vertex}(T) \) and \( T^1 = \text{Edge}(T) \) be the vertex and edge sets of \( T \). For each \( \sigma \in T^0 \cup T^1 \), we let \( G_{\sigma} \leq G \) be the stabilizer of \( \sigma \). For each vertex \( v \in T^0 \), let \( Y_v := \text{Minset}(Z(G_v)) := \cap_{g \in Z(G_v)} \text{Minset}(g) \) and for every edge \( e \in E \) we let \( Y_e := \text{Minset}(Z(G_e)) := \cap_{g \in Z(G_e)} \text{Minset}(g) \). We note that the assignments \( v \rightarrow Y_v \) and \( e \rightarrow Y_e \) are \( G \)-equivariant with respect to the natural \( G \) actions.

The following lemma is well-known.

**Lemma 2.3.** If \( H = \mathbb{Z}^k \) for some \( k \geq 1 \) then \( \text{Minset}(H) = \cap_{h \in H} \text{Minset}(h) \) splits isometrically as a metric product \( C \times \mathbb{R}^k \) so that \( H \) acts trivially on \( C \) and as a translation lattice on \( \mathbb{R}^k \). Moreover, \( Z(H,G) \) acts cocompactly on \( C \times \mathbb{R}^k \).

As a corollary, we have

1. \( G_v \) acts cocompactly on \( Y_v = \overline{Y}_v \times \mathbb{R} \) and \( Z(G_v) \) acts by translation on the \( \mathbb{R} \)-factor and trivially on \( \overline{Y}_v \) where \( \overline{Y}_v \) is a Hadamard space.
2. \( G_e = \mathbb{Z}^2 \) acts cocompactly on \( Y_e = \overline{Y}_e \times \mathbb{R}^2 \subset Y_v \) where \( \overline{Y}_e \) is a compact Hadamard space.
3. if \( \langle t_1 \rangle = Z(G_{v_1}), \langle t_2 \rangle = Z(G_{v_2}) \) then \( \langle t_1, t_2 \rangle \) generates a finite index subgroup of \( G_e \).
We summarize results in Section 3.2 of [CK02] that will be used in this paper.

**Lemma 2.4.** Let $G \curvearrowright X$ be an CKA action. Then there exists a constant $D > 0$ such that the following holds.

1. $\cup_{v \in T_0} N_D(Y_v) = \cup_{e \in T^1} N_D(Y_e) = X$. We define $X_v := N_D(Y_v)$ and $X_e := N_D(Y_e)$ for all $v \in T_0$, $e \in T^1$.

2. If $\sigma, \sigma' \in T^0 \cup T^1$ and $X_\sigma \cap X_{\sigma'} = \emptyset$ then $d_T(\sigma, \sigma') < D$.

**Strips in admissible spaces:** (see Section 4.2 in [CK02]). We first choose, in a $G$–equivariant way, a plane $F_e \subset Y_e$ for each edge $e \in T^1$. Then for every pair of adjacent edges $e_1$, $e_2$, we choose, again equivariantly, a minimal geodesic from $F_{e_1}$ to $F_{e_2}$; by the convexity of $Y_v = \overline{Y}_v \times \mathbb{R}$, $v := e_1 \cap e_2$, this geodesic determines a Euclidean strip $S_{e_1 e_2} := \gamma_{e_1 e_2} \times \mathbb{R}$ (possibly of width zero) for some geodesic segment $\gamma_{e_1 e_2} \subset \overline{Y}_v$. Note that $S_{e_1 e_2} \cap F_{e_1}$ is an axis of $Z(G_v)$. Hence if $e_1, e_2, e \in E$, $e_1 \cap e = v_i \in V$ are distinct vertices, then the angle between the geodesics $S_{e_1 e} \cap F_e$ and $S_{e_2 e} \cap F_e$ is bounded away from zero.

**Remark 2.5.** (1) We note that it is possible that $\gamma_{e_1 e_2}$ is just a point. The lines $S_{e_1, e_2} \cap F_{e_1}$ and $S_{e_1, e_2} \cap F_{e_2}$ are axes of $Z(G_v)$.

(2) There exists a uniform constant such that for any edge $e$, the Hausdorff distance between two spaces $F_e$ and $X_e$ is no more than this constant.

**Remark 2.6.** There exists a $G$–equivariant coarse $L$–Lipschitz map $\rho$: $X \to T^0$ such that $x \in X_{\rho(x)}$ for all $x \in X$. The map $\rho$ is called *indexed map*. We refer the reader to Section 3.3 in [CK02] for existence of such a map $\rho$.

**Definition 2.7 (Templates, [CK02]).** A *template* is a connected Hadamard space $\mathcal{T}$ obtained from disjoint collection of Euclidean planes $\{W\}_{W \in Wall_\mathcal{T}}$ (called *walls*) and directed Euclidean strips $\{S\}_{S \in Strip_\mathcal{T}}$ (a direction for a strip $S$ is a direction for its $R$–factor $S \simeq I \times \mathbb{R}$) by isometric gluing subject to the following conditions.

1. The boundary geodesics of each strip $S \in Strip_\mathcal{T}$, which we will refer to as *singular geodesics*, are glued isometrically to distinct walls in $Wall_\mathcal{T}$.

2. Each wall $W \in Wall_\mathcal{T}$ is glued to at most two strips, and the gluing lines are not parallel.

**Notations:** We use the notion $a \lesssim_K b$ if the exists $C = C(K)$ such that $a \leq Cb + C$, and we use the notion $a \simeq_K b$ if $a \lesssim_K b$ and $b \lesssim_K a$. Also, when we write $a \asymp_K b$ we mean that $a/C \leq b \leq Ca$.

Denote by $Len^1(\gamma)$ and $Len(\gamma)$ the $L^1$–length and $L^2$–length of a path $\gamma$ in a metric product space $A \times B$. These two lengths are equal for a path if it is parallel to a factor; in general, they are bilipschitz.

### 3. Special paths in CKA action $G \curvearrowright X$

Let $G \curvearrowright X$ be a CKA action. In this section, we are going to define *special paths* (see Definition 3.6) in $X$ that will be used on the latter sections. Roughly speaking, each special path in $X$ is a concatenation of geodesics in consecutive pieces $Y_i$’s of $X$ and they are uniform quasi-geodesics in the sense that there exists a constant $\mu = \mu(X)$ such that every special path is $(\mu, \mu)$–quasi-geodesic.

We first introduce the class of *special paths* in a template which shall be mapped to *special paths* in $X$ up to a finite Hausdorff distance.

#### 3.1. Special paths in a template.

**Definition 3.1.** Let $\mathcal{T}$ be the template given by Definition 2.7. A (connected) path $\gamma$ in $\mathcal{T}$ is called *special path* if $\gamma$ is a concatenation $\gamma_0 \gamma_1 \cdots \gamma_n$ of geodesics $\gamma_i$ such that each $\gamma_i$ lies on the strip $S_i$ adjacent to $W_i$ and $W_{i+1}$.
Remark 3.2. By the construction of the template, the endpoints of \( \gamma_i \) \((1 \leq i < n)\) must be the intersection points of singular geodesics on walls \( W_i, W_{i+1} \).

We use Lemma 3.3 in the proof of Proposition 3.8.

**Lemma 3.3.** Assume that the angles between the singular geodesics on walls are between \( \beta \) and \( \pi - \beta \) for a universal constant \( \beta \in (0, \pi) \). There exists a constant \( \mu \geq 1 \) such that any special path is a \((\mu, 0)\)-quasi-geodesic.

**Proof.** Let \( \gamma \) be a special path with endpoints \( x, y \). We are going to prove that \( \text{Len}(\gamma) \leq \mu d(x, y) \) for a constant \( \mu \geq 1 \). Since any subpath of a special path is special, this proves the conclusion.

Let \( \alpha \) be the unique \( \text{CAT}(0) \) geodesic between \( x \) and \( y \). By the construction of the template, if \( \alpha \) does not pass through the intersection point \( z_W \) of the singular geodesics on a wall \( W \), then it passes through a point \( x_W \) on one singular geodesic \( L_- \) and then a point \( y_W \) on the other singular geodesic \( L_+ \). Recall that the angle between the singular geodesics on walls are uniformly between \( \beta \) and \( \pi - \beta \). There exists a constant \( \mu \geq 1 \) depending on \( \beta \) only such that \( d(x_W, z_W) + d(y_W, z_W) \leq \mu d(x_W, y_W) \). We thus replace \([x_W, y_W]\) by \([x_W, z][z, y_W]\) for every possible triangle \( \Delta(x_W, y_W, z_W) \) on each wall \( W \). The resulted path then connects consecutively the points \( z_W \) on the walls \( W \) in the order of their intersection with \( \alpha \), so it is the special path \( \gamma \) from \( x \) to \( y \) satisfying the following inequality

\[
\text{Len}(\gamma) \leq \mu d(x, y)
\]

Thus, we proved that \( \gamma \) is a \((\mu, 0)\)-quasi-geodesic. \( \square \)

We are going to define a template associated to a geodesic in the Bass-Serre tree as the following.

**Definition 3.4** (Standard template associated to a geodesic \( \gamma \subset T \)). Let \( \gamma \) be a geodesic segment in the Bass-Serre tree \( T \). We begin with a collection of walls \( W_e \) and an isometry \( \phi_e : W_e \to F_e \) for each edge \( e \subset \gamma \). For every pair \( e, e' \) of adjacent edges of \( \gamma \), we let \( \hat{S}_{e,e'} \) be a strip which is isometric to \( S_{e,e'} \) if the width of \( S_{e,e'} \) is at least 1, and isometric to \([0,1] \times \mathbb{R}\) otherwise; we let \( \phi_{e,e'} : \hat{S}_{e,e'} \to S_{e,e'} \) be an affine map which respects product structure \((\phi_{e,e'} \) is an isometry if the width of \( S_{e,e'} \) is greater than or equal to 1 and compresses the interval otherwise). We construct \( T_\gamma \) by gluing the strips and walls so that the maps \( \phi_e \) and \( \phi_{e,e'} \) descend to continuous maps on the quotient, we denote the map from \( T_\gamma \to X \) by \( \hat{\psi}_\gamma \).

The following lemma is cited from Lemma 4.5 and Proposition 4.6 in [CK02].

**Lemma 3.5.**

1. There exists \( \beta = \beta(X) > 0 \) such that the following holds. For any geodesic segment \( \gamma \in T \), the angle function \( \alpha_\gamma : \text{Wall}_{\gamma}^o \to (0, \pi) \) satisfies \( 0 < \beta \leq \alpha_\gamma \leq \pi - \beta < \pi \).
2. There are constants \( L, A > 0 \) such that the following holds. Let \( \gamma \) be a geodesic segment in \( T \), and let \( \psi_\gamma : T_\gamma \to X \) be the map given by Definition 3.4. Then \( \psi_\gamma \) is a \((L, A)\)-quasi-isometric embedding. Moreover, for any \( x, y \in [\bigcup_{e \subset \gamma} X_e] \cup [\bigcup_{e, e' \subset \gamma} S_{e,e'}] \), there exists a continuous map \( \alpha : [x, y] \to T \) such that \( d(\psi_\gamma \circ \alpha, id[x,y]) \leq L \).

**3.2. Special paths in the admissible space \( X \)**. In this subsection, we are going to define special paths in \( X \).

Recall that we choose a \( G \)-equivariant family of Euclidean planes \( \{F_e : F_e \subset Y_e\}_{e \in \mathcal{T}} \). For every pair of planes \((F_e, F_{e'})\) so that \( v = e \cap e' \), a minimal geodesic between \( F_e, F_{e'} \) in \( Y_v \) determines a strip \( S_{e,e'} = \gamma_{e,e'} \times \mathbb{R} \) for some geodesic \( \gamma_{e,e'} \subset Y_v \). It is possible that \( \gamma_{e,e'} \) is trivial so the width of the strip is zero. Let \( x \in X_v \) and \( e \) an edge with an endpoint \( v \). The minimal geodesic from \( x \) to \( F_e \) (possibly not belong to \( Y_v \)) also define a strip \( S_{xe} = \gamma_{xe} \times \mathbb{R} \) where the geodesic \( \gamma_{xe} \subset Y_v \) is the projection to \( Y_v \) of the intersection of this minimal geodesic with \( Y_v \). Thus, \( x \) is possibly not in the strip \( S_{xe} \) but within its \( D \)-neighborhood by Lemma 2.4.

**Definition 3.6** (Special paths in \( X \)). Let \( \rho : X \to T^0 \) be the indexed map given by Remark 2.6. Let \( x \) and \( y \) be two points in \( X \). If \( \rho(x) = \rho(y) \) then we define a special path in \( X \) connecting \( x \) to \( y \).
is the geodesic \([x, y]\). Otherwise, let \(e_1 \cdots e_n\) be the geodesic edge path connecting \(\rho(x)\) to \(\rho(y)\) and let \(p_i \in F_{e_i}\) be the intersection point of the strips \(S_{e_{i-1}e_i}\) and \(S_{e_ie_{i+1}}\), where \(e_0 := x\) and \(e_{n+1} := y\). The special path connecting \(x\) to \(y\) is the concatenation of the geodesics
\[
[x, p_1][p_1, p_2] \cdots [p_n, y].
\]

**Remark 3.7.** By definition, the special path except the \([x, p_1]\) and \([p_n, y]\) depends only on the geodesic \(e_1 \cdots e_n\) in \(T\) and the choice of planes \(F_e\).

In this section, we are going to prove the following proposition.

**Proposition 3.8.** There exists a constant \(\mu > 0\) such that every special path \(\gamma\) in \(X\) is a \((\mu, \mu)\)-quasi-geodesic.

To get into the proof of Proposition 3.8, we need several lemmas (see Lemma 3.9 and Lemma 3.10).

**Lemma 3.9.** [CK02, Lemma 3.17] There exists a constant \(C > 0\) with the following property. Let \([x, y]\) be a geodesic in \(X\) with \(\rho(x) \neq \rho(y)\) and \(e_1 \cdots e_n\) be the geodesic edge path connecting \(\rho(x)\) to \(\rho(y)\). Then there exists a sequence of points \(z_i \in [x, y] \cap N_C(F_{e_i})\) such that \(d(x, z_i) \leq d(x, z_j)\) for any \(i \leq j\).

Let \([x, y]\) be a geodesic in \(Y_v = \overline{Y}_v \times \mathbb{R}\) with \(y \in F_e\). We apply a minimizing horizontal slide of the endpoint \(y \in F_e\) to obtain a point \(z \in F_e\) so that \([y, z]\) is parallel to \(\overline{Y}_v\) and the projection of \([x, z]\) on \(\overline{Y}_v\) is orthogonal to \(F_e \cap \overline{Y}_v\).

**Lemma 3.10.** Let \(x \in X_{v_0}, y \in X_{v_n}\) where \(v_0, v_n\) are the endpoints of a geodesic \(e_1 \cdots e_n\) in \(T\). Then there exists a universal constant \(C > 0\) depending on \(X\) such that for each \(1 \leq i \leq n\), we have
\[
d(x, p_i) + d(p_i, y) \leq Cd(x, y) + C
\]
where \(p_i = S_{e_{i-1}e_i} \cap S_{e_ie_{i+1}}\) and \(e_0 := x\) and \(e_{n+1} := y\).

**Proof.** We use the notion \(a \asymp b\) if there exists \(K = K(X)\) such that \(a/K \leq b \leq Ka\).

Let \(D\) be the constant given by Lemma 2.4 and satisfying Lemma 3.9 such that \(X_v = N_D(Y_v)\).

Without loss of generality, we can assume \(x \in Y_{v_0}, y \in Y_{v_n}\).

Denote \(e = e_i\) and \(e^− = e_i−1, e^+ = e_{i+1}\) in this proof. By Lemma 3.9, there exists a point \(q \in F_e\) such that \(d(q, [x, y]) \leq D\). For ease of computation, we will consider the mixed length \(\|x − q\|_1 + d(q, y)\) of the path \([x, q]q[y]\) which satisfies
\[
\|x − q\|_1 + d(q, y) \asymp d(x, q) + d(q, y) \leq d(x, y) + 2D
\]
where \(\| \cdot \|_1\) is the \(L^1\)-metric on the metric product \(Y_v = \overline{Y}_v \times \mathbb{R}\).
Note that the Euclidean plane $F_e \subset Y_v \cap Y_{v'}$ for $e = v\overline{w}$ contains two non-parallel lines $l^+_{e} := S_{v,-e} \cap F_{e}$ and $l^-_{e} := S_{e,+} \cap F_{e}$. So we can apply a minimizing horizontal slide of the endpoint $q$ of $[x, q]$ in $Y_v$ to a point $z$ on $l^-_{e}$. On the one hand, since the line $l^-_{e}$ on $F_e$ is $\mathbb{R}$-factor of $Y_v$, this slide decreases the $L_1$-distance $||x-p||_1$ by $d(q, z) + C$ for a constant $C$ depending on hyperbolicity constant of $Y_v$. On the other hand, by the triangle inequality, this slide increases $d(q, y)$ by at most $d(q, z)$. Hence, we obtain

$$\left(||x-q||_1 + d(q, y)\right) - \left(||x-z||_1 + d(z, y)\right) \leq C$$

Similarly, by a minimizing horizontal slide of the endpoint $z$ of $[z, y]$ in $Y_{v'}$ to $p$,

$$\left(||x-p||_1 + d(p, y)\right) - \left(||x-z||_1 + d(z, y)\right) \leq C$$

yielding

$$\left(||x-q||_1 + d(q, y)\right) - \left(||x-p||_1 + d(p, y)\right) \leq 2C$$

Together with (2) this completes the proof of the lemma. □

**Proof of Proposition 3.8.** Let $\gamma$ be the special path from $x$ to $y$ for $x, y \in X$ so that $\rho(x) \neq \rho(y)$; otherwise it is a geodesic, and thus there is nothing to do. Let $e_1 \cdots e_n$ be the geodesic in $T$ from $\rho(x)$ to $\rho(y)$. With notations as above (see Definition 3.6),

$$\gamma = [x,p_1][p_1,p_2] \cdots [p_{n-1},p_n][p_n,y]$$

By Lemma 3.10, there exists a constant $C \geq 1$ such that

$$d(x, p_1) + d(p_1, p_n) + d(p_n, y) \leq Cd(x, p_1) + Cd(p_1, y) + C$$

$$\leq C^2d(x, y) + C^2 + C$$

Denoting $\alpha = [p_1,p_2] \cdots [p_{n-1},p_n]$, it remains to give a linear bound on $\ell(\alpha)$ in terms of $d(p_1, p_n)$.

By Lemma 3.5, there exists a $K$-template $(T, f, \phi)$ for the $e_1 \cdots e_n$ such that $\phi$ is a $(L, A)$-quasi-isometric map from the template $T$ to the union of the planes $\{F_{e_i} : 1 \leq i \leq n\}$ with the strips $\{S_{e_i : 1 \leq i \leq n}\}$. Moreover, $\phi$ sends walls and strips of $T$ to the $K$–neighborhood of planes $F_{e_i}$ and strips $S_{e_i}$ of $T$ accordingly. Hence, $\phi$ maps the intersection point on $W_{e_i}$ of the singular geodesics of two strips in $T$ to a finite $K$–neighborhood of $p_i$ ($1 \leq i \leq n$). Since the map $\phi$ is affine on strips and isometric on walls of $T$, we conclude that there exists a special path $\overline{\alpha}$ in $T$ such that $\phi(\overline{\alpha})$ is sent to a finite neighborhood of the special path $\alpha$. Lemma 3.3 then implies that $\overline{\alpha}$ is a $(C, C_1)$–quasi-geodesic for some $C_1 > 1$ so $\alpha$ is a $(\mu, \mu)$–quasi-geodesic for some $\mu$ depending on $L, A, K, C_1$. The proof is complete. □

4. **Quasi-isometric embedding of admissible groups into product of trees**

A quasi-tree is a geodesic metric space quasi-isometric to a tree. In this section, we are going to prove Theorem 1.5 that states if $G \curvearrowright X$ is a flip CKA action (see Definition 4.1) then $G$ is quasi-isometric embedded into a finite product of quasi-trees. The strategy is that we first show that the space $X$ is quasi-isometric embedded into product of two hyperbolic spaces $X_1, X_2$ (see Subsection 4.2). We then show that each hyperbolic space $X_i$ is quasi-isometric embedded into a finite product of quasi-trees (see Subsection 4.3).

4.1. **Flip CKA actions and constructions of two hyperbolic spaces.** Let $G \curvearrowright X$ be a CKA action. Recall that each $Y_v$ decomposes as a metric product of a hyperbolic Hadamard space $Y_v$ with the real line $\mathbb{R}$ such that $Y_v$ admits a geometric action of $H_v$. Recall that we choose a $G$–equivariant family of Euclidean planes $\{F_e : F_e \subset Y_v\}_{e \in T^1}$.

**Definition 4.1** (Flip CKA action). If for each edge $e := [v, w] \in T^1$, the boundary line $\ell = \overline{Y_v \cap F_e}$ is parallel to the $\mathbb{R}$–line in $Y_w = \overline{Y_w \times \mathbb{R}}$, then the CKA action is called flip in sense of Kapovich-Leeb.
Let $L_v$ be the set of boundary lines of $Y_w$ which are intersections of $Y_w$ with $F_e$ for all edges $e$ issuing from $v$. Thus, there is a canonical one-to-one correspondence between $L_v$ and the link of $v$ denoted by $Lk(v)$.

**Definition 4.2.** A flat link is the countable union of (closed) flat strips of width 1 glued along a common boundary line called the binding line.

**Construction of hyperbolic spaces $X_1$ and $X_2$:** We first partition the vertex set $T^0$ of the Bass-Serre tree into two disjoint class of vertices $V_1$ and $V_2$ such that if $v$ and $v'$ are in $V_i$ then $d_T(v, v')$ is even.

Given $V \in \{V_1, V_2\}$, we shall build a geodesic (non-proper) hyperbolic space $X$ by gluing $Y_v$ for all $v \in V$ along the boundary lines via flat links.

Consider the set of vertices in $V$ such that their pairwise distance in $T$ equals 2. Equivalently, it is the union of the links of every vertex $w \in T^0 - V$. For any $v_1 \neq v_2 \in Lk(w)$, the edges $e_1 = [v_1, w]$ and $e_2 = [v_2, w]$ determine two corresponding boundary lines $\ell_1 \in L(v_1)$ and $\ell_2 \in L(v_2)$ which are the intersections of $Y_{v_i}$ with $F_{e_i}$ for $i = 1, 2$ respectively. There exists a canonical identification between $\ell_1$ and $\ell_2$ so that their $R$-coordinates equal in the metric product $Y_w = Y_v \times R$.

Note the link $Lk(w)$ determines a flat link $Fl(w)$ so that the flat strips are one-to-one correspondence with $Lk(w)$. In equivalent terms, it is a metric product $Lk(w) \times R$, where $R$ is parallel to the binding line.

For each $w \in T^0 - V$, the set of hyperbolic spaces $Y_v$ where $v \in Lk(w)$ are glued to the flat links $Fl(w)$ along the boundary lines of flat strips and of hyperbolic spaces with the identification indicated above. Therefore, we obtain a metric space $X$ from the union of $\{Y_v : v \in V\}$ and flat links $\{Fl(w), w \in T^0 - V\}$.

**Remark 4.3.** By construction, $Y_v$ and $Y_{v'}$ are disjoint in $X$ for any two vertices $v, v' \in V$ with $d_T(v, v') > 2$. Endowed with induced length metric, $X$ is a hyperbolic geodesic space but not proper since each $Y_v$ is glued via flat links with infinitely many $Y_{v'}$'s where $d_T(v', v) = 2$.

**Definition 4.4.** Let $g$ be an element in $G$. The translation length of $g$ is defined to be $|g| := \inf_{x \in T} d(x, gx)$. Let $\text{Axis}(g) = \{x \in T | d(x, gx) = |g|\}$. If $\text{Axis}(g) \neq \emptyset$ and $|g| > 0$ then $g$ is called elliptic. If $\text{Axis}(g) \neq \emptyset$ and $|g| > 0$, it is called loxodromic (or hyperbolic).

**Remark 4.5.** We note that $\text{Axis}(g) \neq \emptyset$ for any $g \in G$. If $g$ is loxodromic, $\text{Axis}(g)$ is isometric to $R$, and $g$ acts on $\text{Axis}(g)$ as translation by $|g|$.

**Lemma 4.6.** There exists a subgroup $\hat{G}$ of index at most 2 in $G$ so that $\hat{G}$ preserves $V_1$ and $V_2$ respectively and $G_v \subset \hat{G}$ for any $v \in T^0$.

**Proof.** Observe first that if $d_T(go, o) = 0 \pmod{2}$ for some $o \in T^0$ and $g \in G$, then $d_T(gv, v) = 0 \pmod{2}$ holds for any $v \in T^0$. Indeed, if $g$ is elliptic and thus rotates about a point $o$, the geodesic $[ov, v]$ for any $v$ is contained in the union $[o, v] \cup [o, ov]$ and thus has even length. Otherwise, $g$ must be a hyperbolic element and leaves invariant a geodesic $\gamma$ acted upon by translation. By a similar reasoning, if $g$ moves the points on $\gamma$ with even distance, then $d_T(gv, v) = 0 \pmod{2}$ for any $v \in T^0$.

Consider now the set $\hat{G}$ of elements $g \in G$ such that $d_T(gv, v) = 0 \pmod{2}$ for any $v \in T^0$. Using the tree $T$ again, if $g, h \in \hat{G}$, then $d_T(gv, hv) = 0 \pmod{2}$ for any $v \in T^0$. Thus, $G$ is a group of finite index 2. □

Let $\hat{G}$ be the subgroup of $G$ given by the lemma. By Bass-Serre theory, it admits a finite graph of groups where the underlying graph $\hat{G} = T/\hat{G}$ is bipartite with vertex sets $V = V/\hat{G}$ and $W = W/\hat{G}$ where $W := T^0 - V$, and the vertex groups are isomorphic to those of $G$.

**Lemma 4.7.** The space $X$ is a $\delta$–hyperbolic Hadamard space where $\delta > 0$ only depends on the hyperbolicity constants of $Y_v$ ($v \in V$).
If for every $v \in T^0$, $G_v = H_v \times Z(G_v)$ and for each edge $e := [v, w] \in T^1$, $G_e = Z(G_v) \times Z(G_w)$ then the subgroup $\hat{G} < G$ given by Lemma 4.6 acts on $X$ with the following properties:

1. for each $v \in V$, the stabilizer of $\overline{Y}_v$ is isomorphic to $G_v$ and $H_v$ acts geometrically on $\overline{Y}_v$, and
2. for each $w \in W$, the flat link $\text{Fl}(w)$ admits an isometric group action of $G_w$ so that $G_w$ acts by translation on the line parallel to the binding line and on the set of flat strips by the action on the line $\text{Lk}(w)$.

Proof. On one hand, $G_v$ acts on the boundary line $\ell_e = F_e \cap Y_v$ through $G_v \rightrightarrows G_v/Z(G_v) = Z(G_w)$. On the other hand, $Z(G_w)$ acts on the boundary line of the flat strip corresponding to the edge $e$. Since these two actions are compatible with gluing of $Y_v$’s where $v \in \text{Lk}(w)$, we can extend the actions on $\overline{Y}_v$’s and flat links $\text{Fl}(w)$’s to get the desired action of $\hat{G}$ on $X$.

4.2. Q.I. embedding into the product of two hyperbolic spaces.

**Proposition 4.8.** Let $G \ltimes X$ be a flip CKA action and $\hat{G}$ the subgroup in $G$ of index at most 2 given by Lemma 4.6. Let $X_i$ ($i = 1, 2$) be the hyperbolic space constructed in Section 4.1 with respect to $V_i$. Then there exists a quasi-isometric embedding map $\phi$ from $X$ to $X_1 \times X_2$.

If for every $v \in T^0$, $G_v = H_v \times Z(G_v)$ and for each edge $e := [v, w] \in T^1$, $G_e = Z(G_v) \times Z(G_w)$ then the above map $\phi$ can be made $\hat{G}$-equivariant.

**Proof.** Let $\rho : X \to T^0$ be the indexed map given by Remark 2.6. Choose a vertex $v \in T^0$ and a point $x_0 \in Y_v$ such that $\rho(x_0) = v$. Note that $\rho$ is a $G$–equivariant, hence $\rho(g(x_0)) = g\rho(x_0) = gv$. Without loss of generality, we can assume that $v \in V_1$. Let $\hat{X} = \hat{G}(x_0)$ be the orbit of $x_0$ in $X$.

For any $x \in \hat{X} = \hat{G}(x_0)$ then $x = g(x_0)$ for some $g \in \hat{G}$. We remark here that in general it is possible that $g(x_0)$ belong to several $Y_w$’s for some $w \in T^0$. However, recall that we have the given indexed map $\rho : X \to T^0$. This indexed function will tell us exactly which space we should project $g(x_0)$ into, i.e. $g(x_0)$ should project to $\overline{Y}_\rho(g(x_0)) = \overline{Y}_gv$.

We recall that $\hat{G}$ has finite index in $G$ and it preserves $V_1$ and $V_2$.

**Step 1:** Construct a quasi-isometric embedding map $\phi : \hat{X} \to X_1 \times X_2$.

We are going to define the map $\phi = \phi_1 \times \phi_2 : \hat{X} \to X_1 \times X_2$ where $\phi_i : \hat{X} \to X_i$. We first define a map $\phi_1 : \hat{X} \to X_1$.

For any $x \in \hat{X}$ then $x = g(x_0)$ for some $g \in \hat{G}$, and thus $g(x_0) \in Y_{gv}$. Since we assume that $v \in V_1$ and $\hat{G}$ preserves $V_1$, it follows that $gv \in V_1$. We define $\phi_1(x) := \pi_{\overline{Y}_gv}(x)$ where $\pi_{\overline{Y}_gv}$ is the projection of $Y_{gv} = \overline{Y}_gv \times \mathbb{R}$ to the factor $\overline{Y}_v$. We define $\phi_2(x)$ to be the point on the binding line of the flat link $\text{Fl}(v)$ so that its $\mathbb{R}$–coordinate is the same as that of $x$ in the metric product $Y_{gv} = \overline{Y}_gv \times \mathbb{R}$.

**Step 2:** Verifying $\phi$ is a quasi-isometric embedding.

We are now going to show that $\phi = \phi_1 \times \phi_2 : \hat{X} \to X_1 \times X_2$ is a quasi-isometric embedding. Before getting into the proof, we clarify here an observation that will be used later on.

Observation: By the tree-like construction of $X$, any geodesic $\alpha$ in $X$ with endpoints $\alpha_- \in \overline{Y}_{v_1}$, $\alpha_+ \in \overline{Y}_{v_2}$ crosses $\overline{Y}_v$ for alternating vertices $v \in [v_1, v_2]$ in their order appearing in the interval, where $[v_1, v_2]$ is the geodesic in the tree $T$. Using the convexity of boundary lines in a hyperbolic CAT(0) space, we see that the intersection $\alpha \cap \overline{Y}_v$ connects two boundary lines $\ell, \ell'$ in $\overline{Y}_v$ so that the projection $\pi_{\ell}(\ell')$ is uniformly close to $\alpha$. We can then construct a quasi-geodesic $\beta$ in $X$ with the same endpoints as $\alpha$ so that $\beta \cap \overline{Y}_v$ connects $\pi_{\ell}(\ell')$ to $\pi_{\ell'}(\ell)$.

**Claim:** There exists a constant $C \ge 2$ such that $d(x, y)/C - C \le d(\phi(x), \phi(y)) \le Cd(x, y) + D$ for any $x, y \in \hat{X}$.

Indeed, let $g \in \hat{G}$ and $g' \in \hat{G}$ be two elements such that $x = g(x_0)$ and $y = g'(x_0)$. Note that $x \in Y_{gv}$ and $y \in Y_{g'v}$. We consider the following cases.
Case 1: $gv = g'v$. In this case, it is easy to see the claim holds since

$$\text{Len}^1[x, y] = \text{Len}^1(\phi([x, y])) + \text{Len}^1(\phi_2([x, y]))$$

Case 2: $gv \neq g'v$. Note that they both belong to $V_1$, so $d_T(gv, g'v)$ is even. Let $2n = d_T(gv, g'v)$. We write

$$[gv, g'v] = e_1 \cdots e_{2n}$$

as the edge paths in $T$. Let $v_{i-1}$ be the initial vertex of $e_i$ with $i = 1, \ldots, 2n$ and $v_{2n}$ be the terminal vertex of $e_{2n}$. We note that $v_0, v_2, \ldots, v_{2n} \in V_1$ and $v_1, v_3, \ldots, v_{2n-1} \in V_2$

With notations in in Definition 3.6, the special path $\gamma$ between $x, y$ decomposes as the concatenation of geodesics:

$$\gamma = [x, p_1][p_1, p_2] \cdots [p_{2n}, y].$$

Denote $(x_1, x_2) = (\phi_1(x), \phi_2(x)) \in \mathcal{X}_1 \times \mathcal{X}_2$ and $(y_1, y_2) = (\phi_1(y), \phi_2(y)) \in \mathcal{X}_1 \times \mathcal{X}_2$. By the above observation, we connect $x_1$ to $y_1$ by a quasi-geodesic $\beta_1$ in $\mathcal{X}_1$ so that whenever $\beta_1$ passes through $\mathcal{Y}_{v_2}, \mathcal{Y}_{v_4}, \ldots, \mathcal{Y}_{v_{2n-2}}$, it is orthogonal to the boundary lines. In this way, we can write $\beta_1$ as the concatenation of geodesic segments $\beta_1^0, \beta_1^1, \beta_1^2, \ldots, \beta_1^{2i}$, where $\beta_1^{2i}$ are maximal segments contained in the flat links. The first $\beta_1^0$ and last $\beta_1^{2i}$ may have overlap with boundary lines, and the other $\beta_1^{2i}$ are orthogonal to the boundary lines of $\mathcal{Y}_{v_{2i}}$.

Similarly, let $\beta_2$ be a quasi-geodesic from $x_2$ to $y_2$ in $\mathcal{X}_2$ as the concatenation of geodesic segments $\beta_2^0, \beta_2^1, \beta_2^2, \ldots, \beta_2^{2i}$.

We relabel $x$ by $p_0$ and relabel $y$ by $p_{2n+1}$. For each vertex $v_i$, let $\pi_{Y_i}$ and $\pi_{\mathbb{R}_i}$ denote the projections of $Y_{v_i} = \mathcal{Y}_{v_i} \times \mathbb{R}$ to the factor $\mathcal{Y}_i$ and $\mathbb{R}$ respectively.

By the construction of $\phi_1$ and $\phi_2$ we note that there exists a constant $A = A(A')$ such that

$$\text{Len}_{\mathcal{X}_1}(\beta_1) \sim_A \sum_{i=0}^{n} \text{Len}_{\mathcal{X}}(\pi_{\mathcal{Y}_{2i}}[p_{2i}, p_{2i+1}]) + \sum_{i=0}^{n-1} \text{Len}_{\mathcal{X}}(\pi_{\mathcal{R}_{2i+1}}[p_{2i+1}, p_{2i+2}])$$

and

$$\text{Len}_{\mathcal{X}_2}(\beta_2) \sim_A \sum_{i=0}^{n} \text{Len}_{\mathcal{X}}(\pi_{\mathcal{R}_{2i}}[p_{2i}, p_{2i+1}]) + \sum_{i=0}^{n-1} \text{Len}_{\mathcal{X}}(\pi_{\mathcal{Y}_{2i+1}}[p_{2i+1}, p_{2i+2}])$$

Summing over two equations above, we obtain

$$d(x, y) \sim_B (\text{Len}_{\mathcal{X}_1}(\beta_1) + \text{Len}_{\mathcal{X}_2}(\beta_2))$$

for some constant $B = B(A)$.

Since $\beta_i$ is a $(\kappa, \kappa)$-quasi-geodesic connecting two points $\phi_i(x)$ and $\phi_i(y)$ (for some uniform constant $\kappa$ that does not depend on $x, y$), we have that $\ell(\beta_i) \sim_\kappa d(x_i, y_i)$ with $i = 1, 2$. This fact together with formula (3) and the fact $d(\phi(x), \phi(y)) \geq \sqrt{2} d(x_1, y_1) + d(x_2, y_2)$ give a constant $c = C(B, \kappa)$ such that $d(x, y) \sim_C d(\phi(x), \phi(y))$. The claim is verified. Therefore, $\phi$ is a quasi-isometric embedding.

4.3. Q.I. embedding into a finite product of trees. In Section 4.2 we have shown that $X$ is quasi-isometric embedded into a product of two hyperbolic spaces $\mathcal{X}_1$ and $\mathcal{X}_2$. In order to prove Theorem 1.5, the next step is to show that each hyperbolic space $\mathcal{X}_1$ is quasi-isometric embedded into a finite product of quasi-trees.

We shall make use of the work of Bestvina-Bromberg-Fujiwara [BBF15] on a quasi-tree of spaces. Their theory applies to any collection of spaces $\mathcal{Y}$ equipped with a family of projection maps

$$\{\pi_Y : \mathcal{Y} - \{Y\} \times \mathcal{Y} - \{Y\} \to Y\}_{Y \in \mathcal{Y}}$$

satisfying the so-called projection axioms with projection constant $\xi \geq 0$. The precise formulation of projection axioms is irrelevant here. We only mention that their results applies to a collection of quasi-lines with bounded projection property in a (not necessarily proper) hyperbolic space, where the projection maps are shortest point projections. (See Proposition 4.9.)
Fix $K > 0$. In [BBF15], a quasi-tree of spaces $C_K(\mathcal{Y})$ is constructed for given $(\mathcal{Y}, \pi_K)$ satisfying projection axioms with constant $\xi$. Again, we do not use the precise construction but recall their principal result in our setting:

If $K > 4\xi$ and $\mathcal{Y}$ is a collection of uniform quasi-lines, then $C_K(\mathcal{Y})$ is an unbounded quasi-tree. If $\mathcal{Y}$ admits a group action of $G$ so that $\pi_{g\mathcal{Y}} = g\pi_{\mathcal{Y}}$ for any $g \in G$ and $Y \in \mathcal{Y}$, then $G$ acts on $C_K(\mathcal{Y})$.

The following results summarize what we need from [BBF15] in the present paper. Let $\pi_\gamma$ be the shortest projection map in $Y$ and $d_\gamma(x, y) = \text{diam}(\pi_\gamma(\{x, y\}))$ for $x, y \in Y$. Set $[t]_K = t$ if $t \geq K$ otherwise $[t]_K = 0$.

**Proposition 4.9.** [BBF19, Proposition 2.4] Let $\mathcal{A}$ be a collection of quasi-lines in a $\delta$–hyperbolic space $Y$. If there is $\theta > 0$ such that $\text{diam}(\pi_\beta(\alpha)) \leq \theta$ for all $\alpha \neq \beta \in \mathcal{A}$, then $(\mathcal{A}, \pi_\gamma)$ satisfies the projection axioms with projection constant $\xi$ depending on $\theta$, and for any $x, y \in Y$,

$$\frac{1}{4} \sum_{\gamma \in \mathcal{A}} [d_\gamma(x, y)]_K \leq d_{C_K(\mathcal{A})}(x, y) \leq 2 \sum_{\gamma \in \mathcal{A}} [d_\gamma(x, y)]_K + 3K$$

for all $K \geq 4\xi$.

**Remark 4.10.** By [BBF15], the projection constant $\xi$ only depends on the value of $\theta$.

As a corollary, the distance formula still works when the points $x, y$ are perturbed up to bounded error.

**Corollary 4.11.** Under the assumption of Theorem 4.9, if $d(x, x'), d(y, y') \leq R$ for some $R > 0$, then exists $K_0 = K_0(R, \xi, \delta)$ such that

$$\frac{1}{8} \sum_{\gamma \in \mathcal{A}} [d_\gamma(x, y)]_K \leq d_{C_K(\mathcal{A})}(x, y) \leq 4 \sum_{\gamma \in \mathcal{A}} [d_\gamma(x', y')]_K + 3K$$

for all $K \geq 2K_0$.

**Proof.** If $d(x, x'), d(y, y') \leq R$ then there exists a constant $K_0 = K_0(R, \xi, \delta)$ such that $|d_\gamma(x, y) - d_\gamma(x', y')| \leq K_0$ for any $\gamma \in \mathcal{A}$. Assuming $d_\gamma(x, y) > K \geq 2K_0$ then $d_\gamma(x', y') \geq K_0$ we see that

$$\frac{1}{2}[d_\gamma(x', y')]_K \leq [d_\gamma(x, y)]_K \leq 2[d_\gamma(x', y')]_K$$

yielding the desired formula. \(\square\)

**Definition 4.12** (Acylindrical action). [Bow08][Osi16] Let $G$ be a group acting by isometries on a metric space $(X, d)$. The action of $G$ on $X$ is called acylindrical if for any $r \geq 0$, there exist constants $R, N \geq 0$ such that for any pair $a, b \in X$ with $d(a, b) \geq R$ then we have

$$\#\{g \in G \mid d(ga, a) \leq r \text{ and } d(gb, b) \leq r\} \leq N$$

The following property in hyperbolic groups is probably known to experts, but is referred to a more general result [Yan19, Lemma 2.14] since we could not locate a precise statement as follows. A group is called non-elementary if it is neither finite nor virtually cyclic.

**Lemma 4.13.** Let $H$ be a non-elementary group admitting a co-bounded and acylindrical action on a $\delta$–hyperbolic space $(\mathcal{Y}, d)$. Fix a basepoint $o$. Then there exist a set $F \subset H$ of three loxodromic elements and $\lambda, c > 0$ with the following property.

For any $h \in H$ there exists $f \in F$ so that $h^f$ is a loxodromic element and the bi-infinite path

$$\gamma = \bigcup_{i \in \mathbb{Z}} [(hf)^i o, (hf)^{i+1} o]$$

called axis below is a $(\lambda, c)$–quasi-geodesic.
Sketch of the proof. This follows from the result [Yan19, Lemma 2.14] which applies to any isometric action of $H$ on a metric space with a set $F$ of three pairwise independent contracting elements (loxodromic elements in hyperbolic spaces). If $X$ denotes the set of $G$–translated quasi-axis of all elements in $F$, the pairwise independence condition is equivalent (defined) to be the bounded projection property of $X$. Thus, the existence of such $F$ is clear in a proper action of a non-elementary group. For acylindrical actions, this is also well-known, see [BBF19, Proposition 3.4], recalled in Proposition 5.11 below. 

**Convention 4.14.** When speaking of quasi-lines in hyperbolic spaces with actions satisfying Lemma 4.13 we always mean $(\lambda, c)$–quasi-geodesics where $\lambda, c > 0$ depend on $F$ and $\delta$.

**Lemma 4.15.** Let $H$ be a non-elementary group admitting a co-bounded and acylindrical action on a $\delta$–hyperbolic space $(\overline{Y}, d)$. Assume that $\mathbb{L}$ is a $H$–finite collection of quasi-lines. Then for any sufficiently large $K > 0$, there exist a $H$–finite collection of quasi-lines $\mathbb{L} \subset \mathbb{A}$ in $\overline{Y}$ and a constant $N = N(K, \delta, \mathbb{A}) > 0$, such that for any $x, y \in \overline{Y}$, the following holds

$$\frac{1}{N} \sum_{\gamma \in \mathbb{A}} [d_\gamma(x, y)]_K \leq d(x, y) \leq 2 \sum_{\gamma \in \mathbb{A}} [d_\gamma(x, y)]_K + 2K.$$

**Proof.** Fixing a point $o \in \overline{Y}$, the co-bounded action of $H$ on $(\overline{Y}, d)$ gives a constant $R > 0$ such that $N_R(Ho) = \overline{Y}$. By hyperbolicity, if $\gamma$ is a $(\lambda, c)$–quasi-geodesic, then there exists a constant $C = C(\lambda, c, R) > 0$ such that $\text{diam}([x, y] \cap N_R(\gamma)) > C$ implies

$$|d_\gamma(x, y) - \text{diam}([x, y] \cap N_R(\gamma))| \leq C.$$

Fix $K > 2C$ and denote $\tilde{K} = K + 2C$. Let $S = \{h \in H : |d(o, ho) - \tilde{K}| \leq 2R\}$ and consider the set $\tilde{S}$ of loxodromic elements $hf$ where $h \in S$ and $f \in F$ is provided by Lemma 4.13. Note that $\sharp \tilde{S} = \sharp S$. Let $\mathbb{A}$ be the set of all $H$–translated axis of $hf \in \tilde{S}$. It is possible that $\sharp \mathbb{A}/H \leq \sharp \tilde{S}$ since two elements in $\tilde{S}$ may be conjugate.

Assume that $d(x, y) > \tilde{K}$. Consider a geodesic $\alpha$ from $x$ to $y$ and choose points $x_i$ on $\alpha$ for $0 \leq i \leq n + 1$ such that $d(x_i, x_{i+1}) = \tilde{K}$ for $0 \leq i \leq n - 1$ and $d(x_n, x_{n+1}) \leq \tilde{K}$ where $x_0 = x, x_{n+1} = y$. Since $N_R(Ho) = \overline{Y}$, there exists $h_i \in H$ so that $d(x_i, h_i o) \leq R$. It implies that $\tilde{K} - 2R \leq d(o, h_i^{-1} h_{i+1} o) \leq \tilde{K} + 2R$, and thus we have $h_i^{-1} h_{i+1} \in S$ for $0 \leq i \leq n - 1$. Noting that $[h_i o, h_{i+1} o]$ is contained in a $H$-translated axis of some loxodromic element in $\tilde{S}$, we thus obtain $n$ axis $\gamma_0, \cdots, \gamma_{n-1} \in \mathbb{A}$ (with possible multiplicities: $\gamma_i = \gamma_j$ for $i \neq j$) satisfying $\text{diam}(N_R(\gamma_i) \cap \alpha) \geq \tilde{K}$ so that

$$\alpha - [x_n, x_{n+1}] \subset \bigcup_{0 \leq i < n} N_R(\gamma_i) \cap \alpha$$

which yields

$$\text{Len}(\alpha) \leq \sum_{0 \leq i < n} \text{diam}(N_R(\gamma_i) \cap \alpha) + \tilde{K}$$

where the constant $\tilde{K}$ bounds the length of the last segment $[x_n, x_{n+1}]$.

By the equation (4), $d_\gamma(x, y) \geq \text{diam}(N_R(\gamma_i) \cap \alpha) - 2C \geq \tilde{K} > 2C$ and then $\text{diam}(N_R(\gamma_i) \cap \alpha) \leq d_\gamma(x, y) + C \leq 2d_\gamma(x, y)$. Thus, we obtain

$$\text{Len}(\alpha) \leq \sum_{0 \leq i < n} 2[d_\gamma(x, y)]_K + \tilde{K},$$

implying the upper bound where $\gamma \in \mathbb{A}$. Of course, the upper bound holds as well after adjoining $\mathbb{L}$ into $\mathbb{A}$.

The remainder of the proof is to prove the lower bound. By assumption, the union of $\mathbb{L}$ and $\mathbb{A}$ is $H$–finite, so is locally finite. In particular, any ball of radius $(2R + \tilde{K})$ intersects at most $D = D(\mathbb{A}, \mathbb{L}, K, R)$ quasi-lines in $\mathbb{A} \cup \mathbb{L}$. 

CROKE-KLEINER ADMISSIBLE GROUPS: PROPERTY (QT) AND QUASICONVEXITY 14
We look at the set $B$ of quasi-lines $\gamma \in A \cup L$ satisfying $\text{diam}(N_R(\gamma) \cap \alpha) \geq \tilde{K}$. By the local finiteness, those can be divided into at most $D$ sub-collections of quasi-lines in which $N_R(\gamma) \cap \alpha$ is disjoint with $N_R(\gamma') \cap \alpha$ for any two $\gamma \neq \gamma'$. Thus,

$$D \cdot \text{Len}(\alpha) \geq \sum_{\gamma \in B} \text{diam}(N_R(\gamma) \cap \alpha) \geq \frac{1}{2} \sum_{\gamma \in B} [d_\gamma(x,y)]_K$$

where $\text{diam}(N_R(\gamma) \cap \alpha) \geq d_\gamma(x,y) - C \geq \frac{1}{2} d_\gamma(x,y)$ follows from Eq. (4).

To conclude the proof, we now consider $\gamma \in (A \cup L) - B$ for which $\text{diam}(N_R(\gamma) \cap \alpha) < \tilde{K}$. Note that the set of axes $\gamma_0, \ldots, \gamma_{n-1}$ obtained as above is contained into $B$. The $R$–neighborhood of the union $\gamma_0 \cup \cdots \cup \gamma_{n-1}$ covers $\alpha$ except the last segment $[x_n, x_{n+1}]$. Hence, there must exist but at most $D$ quasi-lines $\gamma' \in B$ such that $\gamma \cap N_{2R+K'}(\gamma') \neq \emptyset$. If $d_\gamma(x,y) \geq K$, then $\text{diam}(N_R(\gamma) \cap \alpha) + C < \tilde{K} + C$ by (4). Consequently,

$$\sum_{\gamma \in (A \cup L) - B} [d_\gamma(x,y)]_K \leq D(\tilde{K} + C) \cdot \sum_{\gamma' \in B} \text{diam}(N_R(\gamma') \cap \alpha) \leq 2D(\tilde{K} + C) \cdot \sum_{\gamma' \in B} [d_\gamma'(x,y)]_K.$$

Finally, we have that

$$\text{Len}(\alpha) \geq \frac{1}{N} \sum_{\gamma \in A \cup L} [d_\gamma(x,y)]_K$$

for some constant $N := N(D, \tilde{K} + C)$. The proof is complete by renaming $A := A \cup L$. \hfill $\Box$

Remark 4.16. The statement of Lemma 4.15 is a re-package of Proposition 3.3 and Theorem 3.5 in [BBF19], but $H$ is allowed to have torsion.

The reminder of this section is devoted to the proof of Theorem 1.5. We start by explaining the choice of the constants and the collection of quasi-lines $A$ in $X_1$ that will be used in the rest of this subsection.

The constants $D$ and $\theta$ and $\xi = \xi(\theta)$: Let $X_1$ and $X_2$ be two $\delta$–hyperbolic spaces given by Lemma 4.7 where $\delta > 0$ depends on the hyperbolicity constants of $Y_v$.

Note that each $Y_v$ for $v \in V_1$ are isometrically embedded into $X_1$ and thus $\delta$–hyperbolic. We follow the Convention 4.14 on the quasi-lines which are $(\lambda, c)$–quasi-geodesics in $Y_v$ and $X_1$.

By the $\delta$–hyperbolicity of $X_1$, there exist constants $D, \theta > 0$ depending on $\delta$ (and also $\lambda, c$) such that if any $(\lambda, c)$–quasi-lines $\alpha \neq \beta$ have a distance at least $D$ then $\text{diam}(\pi_\beta(\alpha)) \leq \theta$.

We then obtain the projection constant $\xi = \xi(\theta)$ by Proposition 4.9.

The collection of quasi-lines $A$ in $X_1$: Fix any sufficiently large number $K > \max\{4\xi, \theta, 2\}$ depending on $L_v$, where $L_v$ is the collection of boundary lines of $Y_v$. By Lemma 4.15, there exist a locally finite collection of quasi-lines $L_v \subset A_v \in Y_v$ and a constant $N = N(K, A_v, \delta) > 0$ such that

$$\frac{1}{N} \sum_{\gamma \in A_v} [d_\gamma(x,y)]_K \leq d_{\text{max}}(x,y) \leq 2 \sum_{\gamma \in A_v} [d_\gamma(x,y)]_K + 2K$$

for any $x, y \in Y_v$. Since there are only finitely many $\tilde{G}$–orbits of $(H_v, Y_v)$ we assume furthermore $A_w = gA_v$ if $w = gv$ for $g \in \tilde{G}$. Then $A := \cup_{v \in V_1} A_v$ is a locally finite collection of quasi-lines in $X_1$, preserved by the group $\tilde{G}$.

We use the following lemma in the proof of Proposition 4.18 that gives us a distance formula for $X_1$.

Lemma 4.17. There exists a constant $L > 0$ depending only on $K$ with the following properties.

1. For any $\ell \neq \gamma \in A$, we have $\text{diam}(\pi_\gamma(\ell)) \leq L$.
2. For any $v \in V_1$ and $x, y \in Y_v$, there are at most $L$ quasi-lines $\gamma$ in $A - A_v$ such that $L \geq d_\gamma(x,y) \geq K$.\hfill $\Box$
Proof. Since there are only finitely many $\mathcal{Y}_w$’s up to isometry, and $\mathcal{A}_w = g\mathcal{A}_w$ if $w = gv$, the union $\mathcal{A}$ of quasi-lines containing $\cup_{w \in \mathcal{V}_1} \mathcal{Y}_w$ is uniformly locally finite: any ball of a fixed radius in $\mathcal{X}_1$ intersects a uniform number of quasi-lines depending only on the radius. By the hyperbolicity of $\mathcal{X}_1$, the local finiteness implies the bounded projection property, so gives the desired constant $L$ in the assertion (1).

By the construction of $\mathcal{X}_1$, the shortest projection of a point $x \in \mathcal{Y}_v$ to $\gamma \in \mathcal{A}_w$ for $w \neq v$ has to pass through a boundary line $\ell \in L_w$ of $\mathcal{Y}_w$, so is contained in the projection of $\ell$ to $\gamma$. By the assertion (1) we have $d_\gamma(x,y) \leq \text{diam}(\pi_\gamma(\ell)) \leq L$. If $d_\gamma(x,y) \geq K \geq \theta$ for $x, y \in \mathcal{Y}_v$ and $\gamma \in \mathcal{A}_w$ with $w \neq v$, then $d(\gamma, \ell) \leq D$ by the above defining property of $D$ and $\theta$. By local finiteness, there are at most $L = L(D)$ quasi-lines $\ell$ with this property, proving the assertion (2). \hfill \Box

\begin{proposition}[Distance formula for $\mathcal{X}_1$]
For any $x,y \in \mathcal{X}_1$, there exists a constant $\mu = \mu(L, K) > 0$ such that

$$
\frac{1}{\mu} \sum_{\gamma \in \mathcal{A}} [d_\gamma(x,y)]_K + d_T(\rho(x), \rho(y)) - L^2 \leq d_{\mathcal{X}_1}(x,y) \leq \mu \sum_{\gamma \in \mathcal{A}} [d_\gamma(x,y)]_K + 4K \cdot d_T(\rho(x), \rho(y)).
$$

\end{proposition}

\begin{proof}
Since the 1–neighborhood of the union $\cup_{v \in \mathcal{V}_1} \mathcal{Y}_v$ is $\mathcal{X}_1$, assume for simplicity $x \in \mathcal{Y}_{v_1}$ and $y \in \mathcal{Y}_{v_n}$ where $v_1 = \rho(x), v_n = \rho(y) \in \mathcal{V}_1$. Thus, $d_T(v_1, v_n) = 2n - 1$. By the construction of $\mathcal{X}_1$, a geodesic $[x, y]$ travels through $\mathcal{Y}_{v_i}$ and then flat links $L(\ell_i)$, where $v_i \in \mathcal{V}_1$ and $w_i \in \mathcal{V}_2$ appear alternatively on $[v_1, v_n] \subset T$. Let us denote the exit point on the boundary line $\ell_i$ of $\mathcal{Y}_{v_i}$ and entry point on $\ell_{i+1}$ of $\mathcal{Y}_{v_{i+1}}$ by $y_i$ and $x_{i+1}$ respectively for $1 \leq i \leq n$ where $x_1 := x$ and $y_n := y$ by convention. Thus,

$$
d_{\mathcal{X}_1}(x,y) = \sum_{1 \leq i \leq n-1} d_{\mathcal{X}_1}(y_i, x_{i+1}) = \sum_{1 \leq i \leq n} d_{\mathcal{Y}_{v_i}}(x_i, y_i).
$$

Therefore, we shall derive (6) from (7) which requires to apply the formula (5) for $d_{\mathcal{Y}_{v_i}}(x_i, y_i)$. To that end, we need the following estimates. Recall that $\asymp_{L,K}$ means the equality holds up to a multiplicative constant depending on $L, K$.

\begin{claim}
(1) If there is $\gamma \in \mathcal{A}_{v_i}$ such that $d_\gamma(x_i, y_i) \geq K$ then

$$
[d_\gamma(x_i, y_i)]_K \asymp_{L,K} [d_\gamma(x,y)]_K
$$

(2) $d_{\mathcal{X}_1}(y_i, x_{i+1}) \geq 2$. If $d_{\mathcal{X}_1}(y_i, x_{i+1}) > K + 2$, then

$$
[d_\gamma(y_i, x_{i+1})]_K \asymp_{L,K} [d_\gamma(x,y)]_K, \quad [d_{\ell_{i+1}}(y_i, x_{i+1})]_K \asymp_{L,K} [d_{\ell_{i+1}}(x,y)]_K
$$

\end{claim}

\begin{proof}
If there is $\gamma \in \mathcal{A}_{v_i}$ such that $d_\gamma(x_i, y_i) \geq K$ we then have

$$
|d_\gamma(x,y) - d_\gamma(x_i, y_i)| \leq \text{diam}(\pi_\gamma(\ell_i)) + \text{diam}(\pi_\gamma(\ell_i)) \leq 2L,
$$

where Lemma 4.17 is applied, after taking the cutoff function $[.]_K$,

$$
|[d_\gamma(x,y)] - [d_\gamma(x_i, y_i)]_K| \leq 2L + K.
$$

This in turn implies (8).

Recall that $[y_i, x_{i+1}]$ is contained in the union of two flat strips with width 1 in a flat link, and is from one boundary line $\ell_i$ to the other $\ell_{i+1}$. Thus, $d_{\mathcal{X}_1}(y_i, x_{i+1}) \geq 2$. If $d_{\mathcal{X}_1}(y_i, x_{i+1}) > K + 2$ is assumed, then $d_{\ell_i}(y_i, x_{i+1}) > K$ and $d_{\ell_{i+1}}(y_i, x_{i+1}) > K$. The assertion (2) follows similarly as above. \hfill \Box
\end{proof}
Recalling $K \geq 2$, the assertion (2) of the Claim 1 implies a constant $\mu_1 = \mu_1(L, K) > 1$ such that

$$d_T(\rho(x), \rho(y)) \leq \sum_{1 \leq i \leq n-1} d_{\mathcal{A}_i}(y_i, x_{i+1}) \leq \mu_1 \sum_{\ell \in \mathcal{A}} [d_{\ell(x,y)}]_K + 2Kd_T(\rho(x), \rho(y)).$$

Using (8), we now replace $[d_\gamma(x,y)]_K$ by $[d_\gamma(x,y)]_K$ in the formula (5) for $d_{\mathcal{T}_v}(x_i, y_i)$. Hence, there exists a constant $\mu_2 = \mu_2(K, L) > 1$ so that

$$\frac{1}{\mu_2} \sum_{\gamma \in \mathcal{A}_v} [d_\gamma(x,y)]_K \leq d_{\mathcal{T}_v}(x_i, y_i) \leq \mu_2 \sum_{\gamma \in \mathcal{A}_v} [d_\gamma(x,y)]_K + 2K.$$

Noting $d_T(\rho(x), \rho(y)) = 2n - 1$, we deduce from Eq. (7) and (9) that

$$d_{\mathcal{A}_1}(x, y) \leq (\mu_1 + \mu_2) \sum_{1 \leq i \leq n} \sum_{\gamma \in \mathcal{A}_v} [d_\gamma(x,y)]_K + 4K \cdot d_T(\rho(x), \rho(y))$$

so the upper bound in (6) follows by setting $\mu := \mu_1 + \mu_2$.

We now derive the lower bound from those of Eq. (7) and (9):

$$d_{\mathcal{A}_1}(x, y) \geq \sum_{1 \leq i \leq n} d_{\mathcal{T}_v}(x_i, y_i) + dt_T(\rho(x), \rho(y)) \geq \frac{1}{\mu_2} \sum_{1 \leq i \leq n} \sum_{\gamma \in \mathcal{A}_v} [d_\gamma(x,y)]_K + dt_T(\rho(x), \rho(y))$$

By the Claim 1, there are at most $L$ quasi-lines $\gamma \in \bigcup \{\mathcal{A}_v : v \in \mathcal{V}_1 - [\rho(x), \rho(y)]^0\}$ satisfying $L \geq [d_\gamma(x,y)]_K > 0$. Hence, the following holds

$$d_{\mathcal{A}_1}(x, y) \geq \frac{1}{\mu_2} \sum_{v \in \mathcal{V}_1} \sum_{\gamma \in \mathcal{A}_v} [d_\gamma(x,y)]_K + dt_T(\rho(x), \rho(y)) - L^2$$

completing the proof of the lower bound.

\square

**Lemma 4.19.** The collection $\mathcal{A}$ can be written as a union (possibly non-disjoint) $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$ with the following properties for each $\mathcal{A}_i$:

1. for any two quasi-lines $\alpha \neq \beta \in \mathcal{A}_i$ we have $d(\alpha, \beta) \geq D$,
2. the $(D + R)$–neighborhood of the union $\cup_{\gamma \in \mathcal{A}_i} \gamma$ contains $\mathcal{A}_i$,
3. for any $K > 4\xi$ the quasi-tree of quasi-lines $(\mathcal{C}_K(\mathcal{A}_i), d_{\mathcal{A}_i})$ is a quasi-tree.

**Proof.** Since $H_v$ acts geometrically on $Y_v$ for $v \in \mathcal{V}_1$ and $\mathcal{V}_1$ is $G$–finite, there exists a constant $R > 0$ such that the $R$–neighborhood of the union $\cup_{\gamma \in \mathcal{A}_i} \gamma$ contains $Y_v$ for each $v \in \mathcal{V}_1$. Since $\mathcal{A}$ is locally finite and $G$–invariant, the $D$–neighborhood of any quasi-line in $\mathcal{A}$ intersects $n$ quasi-lines from $\mathcal{A}$ for some $n = n(D) \geq 1$.

We can now write $\mathcal{A}$ as the (possibly non-disjoint) union $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$ with the following two properties for each $\mathcal{A}_i$:

1. for any two quasi-lines $\alpha \neq \beta \in \mathcal{A}_i$ we have $d(\alpha, \beta) \geq D$,
2. the $(D + R)$–neighborhood of the union $\cup_{\gamma \in \mathcal{A}_i} \gamma$ contains $\mathcal{A}_i$.

Indeed, by definition of $R$, any ball of radius $R$ intersects a quasi-line so for each $\alpha \in \mathcal{A}_i$, there exists $\beta \in \mathcal{A}$ such that $D \leq d(\alpha, \beta) \leq D + 2R$. Starting from a quasi-line $\gamma_1$, we inductively choose the quasi-lines which intersect the $(D + R)$–neighborhood of the already chosen ones, and by the axiom of choice, a collection $\mathcal{A}_1$ of quasi-lines containing $\gamma_1$ is obtained so that the properties (1) and (2) are true. The other collections $\mathcal{A}_i$ for $n \geq i \geq 1$ is obtained similarly from the other $n - 1$ quasi-lines intersecting the $D$–neighborhood of $\gamma_1$. The property (2) guarantees $\mathcal{A} \subseteq \cup_{1 \leq i \leq n} \mathcal{A}_i$ from the choice of $R$. We do allow $\mathcal{A}_i \cap \mathcal{A}_j \not= \emptyset$, but any $\gamma \in \mathcal{A}$ would appear at most once in $\mathcal{A}_i$.

By the defining property of $D$, the collection $\mathcal{A}_i$ of quasi-lines in the hyperbolic space $\mathcal{X}_1$ satisfies $\text{diam}(\pi_\beta(\alpha)) \leq \theta$ for all $\alpha \neq \beta \in \mathcal{A}_i$. By Proposition 4.9, $(\mathcal{A}_i, \pi_\gamma)$ satisfies projection axioms with projection constant $\xi = \xi(\theta)$. For given $K > 4\xi$, the quasi-tree of quasi-lines $(\mathcal{C}_K(\mathcal{A}_i), d_{\mathcal{A}_i})$ is a quasi-tree by [BBF15].

\square
Proof of Theorem 1.5. Let $\mathcal{X}_i$ ($i = 1, 2$) be the hyperbolic space constructed in Section 4.1 with respect to $\mathcal{V}_i$. By Proposition 4.8, the admissible group $G$ admits a quasi-isometric embedding into $\mathcal{X}_1 \times \mathcal{X}_2$. Thus, to complete the proof of Theorem 1.5, we only need to show that each hyperbolic space $\mathcal{X}_i$ is quasi-isometric embedded into a finite product of quasi-trees. We give the proof for $\mathcal{X}_1$ and the proof for $\mathcal{X}_2$ is symmetric.

Let $\rho : X \to T^0$ be the indexed map given by Remark 2.6. Let $\mathbb{A}_1, \ldots, \mathbb{A}_n$ be the collection of quasi-lines given by Lemma 4.19.

Let $\bar{X}_1 := \cup_{v \in \mathcal{V}_1} \bar{V}_v$. Since the 1–neighborhood of $\bar{X}_1$ is $\mathcal{X}_1$, it suffices define a quasi-isometric embedding map from $\bar{X}_1$ to a finite product of quasi-trees.

We now define a map

$$\Phi : \bar{X}_1 \to T \times \prod_{1 \leq i \leq n} C_K(\mathbb{A}_i),$$

where $T$ is the Bass-Serre tree of $G$.

Let $x \in \bar{X}_1 = \cup_{v \in \mathcal{V}_1} \bar{V}_v$ and assume $x \in \bar{V}_v$. By the property (2) of Lemma 4.19, we choose a point $\Phi_i(x) \in \cup_{A \in \mathbb{A}_i} \gamma$ for $1 \leq i \leq n$ such that $d(x, \Phi_i(x)) \leq R + D$. Denote $\bar{R} = R + D$. Let $\Phi(x) = (\rho(x), \Phi_1(x), \cdots, \Phi_n(x))$.

We now verify that $\Phi$ is a quasi-isometric embedding. Since $d(x, (\Phi_i(x))) \leq \bar{R}$ and $d(y, (\Phi_i(y))) \leq \bar{R}$, let $K_0 = K_0(\bar{R}, \xi, \delta)$ be given by Corollary 4.11 so that for $K > K_0$, the following distance formula holds

$$d_{C_1}(\Phi_i(x), \Phi_i(y)) \sim_K \sum_{\gamma \in \mathbb{A}_i} [d_\gamma(x, y)]_K$$

for each $1 \leq i \leq n$. Therefore,

$$d(\Phi(x), \Phi(y)) = d_T(\rho(x), \rho(y)) + \sum_{i=1}^n d_{C_i}(\Phi_i(x), \Phi_i(y))$$

$$\sim_K d_T(\rho(x), \rho(y)) + \sum_{i=1}^n \sum_{\gamma \in \mathbb{A}_i} [d_\gamma(x, y)]_K.$$ 

Recall that $\mathbb{A} = \cup_{1 \leq i \leq n} \mathbb{A}_i$ is a possibly non-disjoint union, but any quasi-line $\gamma$ with $[d_\gamma(x, y)]_K > 0$ in the above sum appears at most once in each $\mathbb{A}_i$. We thus obtain

$$d(\Phi(x), \Phi(y)) \sim_{K,n} d_T(\rho(x), \rho(y)) + \sum_{\gamma \in \mathbb{A}} [d_\gamma(x, y)]_K$$

which together with distance formula (6) for $\mathcal{X}_1$ concludes the proof of Theorem.

5. Proper action on a finite product of quasi-trees

In this section, under a stronger assumption on vertex groups as stated in Theorem 1.1, we shall promote the quasi-isometric embedding to be an orbital map of an action of the admissible group on a (different) finite product of quasi-trees.

By [BBF19, Induction 2.2], if $H < G$ has finite index and acts on a finite product of quasi-trees, then so does $G$. We are thus free to pass to finite index subgroups in the proof.

Recall that $T^0 = \mathcal{V}_1 \cup \mathcal{V}_2$ where $\mathcal{V}_i$ consists of vertices in $T$ with pairwise even distances, and $X$ is the hyperbolic space constructed from $\mathcal{V} \in \{\mathcal{V}_1, \mathcal{V}_2\}$. By Lemma 4.6, let $\hat{G} < G$ be the subgroup of index at most 2 preserving $\mathcal{V}_1$ and $\mathcal{V}_2$.

5.1. Construct cone-off spaces: preparation. In this preparatory step, we first introduce another hyperbolic space $\mathcal{X}$ which is the cone-off of the previous hyperbolic space $X$ over boundary lines of flat links. We then embed $\mathcal{X}$ into a product of $\mathcal{X}$ and a quasi-tree built from the set of binding lines from the flat links.

Definition 5.1 (Hyperbolic cones). [BH99, Part I, Ch. 5] For a line $\ell$ and a constant $r > 0$, a hyperbolic $r$–cone denoted by $cone_r(\ell)$ is the quotient space of $\ell \times [0, r]$ by collapsing $\ell \times 0$ as a point called apex. A metric is endowed on $cone_r(\ell)$ so that it is isometric to the metric completion
of the universal covering of a closed disk of radius $r$ in the real hyperbolic plane $\mathbb{H}^2$ punctured at the center.

A hyperbolic multicone of radius $r$ is the countable wedge of hyperbolic $r$–cones with apex identified.

If $\xi$ is an isometry on $\ell$ then $\xi$ extends to a natural isometric action on the hyperbolic cone $cone_\ell(\ell)$ which rotates around the apex and sends the radicals to radicals.

Similar to the flat links, the link $Lk(w)$ for $w \in T^0$ determines a hyperbolic multicone of radius $r$ denoted by $Cones_\ell(w)$ so that the set of hyperbolic cones is bijective to the set of vertices adjacent to $w$. And $G_w = H_w \times Z(G_w)$ acts on $Cones_\ell(w)$ so that the center of $G_w$ rotates each hyperbolic cone around the apex and $G_w/Z(G_w)$ permutes the set of hyperbolic cones by the action of $G_w$ on $Lk(w)$.

Construction of the cone-off space $\hat{X}$. Let $V \in \{V_1, V_2\}$ and $r > 0$. Let $\hat{X}$ be the disjoint union of $\{\overline{Y}_v : v \in V\}$ and hyperbolic multicones $(Cones_\ell(w), w \in T^0 - V)$ glued by isometry along the boundary lines of $\bigcup_{v \in Lk(w)} \overline{Y}_v$ and those of hyperbolic multicones $Cones(w)$.

Remark 5.2. We note that $\hat{X}$ is obtained from $X$ by replacing each flat links by hyperbolic multicones. However, the identification in $\hat{X}$ between boundary lines of $\overline{Y}_v$ and multicones $Cones(w)$ is only required to be isometric, while the $\mathbb{R}$–coordinates of the boundary lines in constructing $X$ have to be matched up.

We now give an alternative way to construct the cone-off space $\hat{X}$, which shall be convenient in the sequel.

Let $\hat{Y}_v$ be the disjoint union of $\overline{Y}_v$ and hyperbolic cones $cone_\ell(\ell)$ glued along boundary lines $\ell \in \mathbb{L}_v$. If $E(\ell)$ denotes the stabilizer in $H_v$ of the boundary $\ell \in \mathbb{L}_v$, then $E(\ell)$ is virtually cyclic and almost malnormal. By [Bow12], $H_v$ is hyperbolic relative to $\{E(\ell) : \ell \in \mathbb{L}_v\}$ and the action on the coned-off Cayley graph of $H_v$ is acylindrical (see [Bow08, Lemma 3.3], [Osi16, Prop. 5.2]). Since the coned-off Cayley graph is quasi-isometric to $\hat{Y}_v$, the action of $H_v$ on $\hat{Y}_v$ is co-bounded and acylindrical.

Alternatively, the cone-off space $\hat{X}$ could be obtained from the disjoint union of $\{\hat{Y}_v\}_{v \in V}$ by identifying the apex of hyperbolic cones from the same link $Lk(w)$ where $w \in T^0 - V$.

Since $\mathbb{L}_v$ has the bounded intersection property, by [DGO17, Corollary 5.39], for a sufficiently large constant $r$, the space $\hat{Y}_v$ is a hyperbolic space with constant depending only on the original one. Thus, the space $\hat{X}$ is a hyperbolic space.

By Lemma 4.6, a subgroup $G$ of index at least 2 in $G$ leaves invariant $V_1$ and $V_2$. The following lemma is proved similarly as Lemma 4.7.

Lemma 5.3. Fix a sufficiently large $r > 0$. The space $\hat{X}$ is a $\delta$–hyperbolic space where $\delta > 0$ only depends on the hyperbolicity constants of $\hat{Y}_v$ ($v \in V$).

If $G_v = H_v \times Z(G_v)$ for every $v \in T^0$ and $G_e = Z(G_v) \times Z(G_w)$ for every edge $e = [v, w] \in T^1$, then a subgroup $\hat{G}$ of index at most 2 in $G$ acts on $\hat{X}$ with the following properties:

1. for each $v \in V$, the stabilizer of $\hat{Y}_v$ is isomorphic to $G_v$ and $H_v$ acts coboundedly on $\hat{Y}_v$, and
2. for each $w \in T^0 - V$, the stabilizer of the apex of $Cones_\ell(w)$ is isomorphic to $G_w$ so that $H_w$ acts on the set of hyperbolic cones by the action on the link $Lk(w)$ and $Z(G_w)$ on acts by rotation on each hyperbolic cone.

Let $\mathbb{L}$ be the set of all binding lines in the flat links $Fl(w)$ over $w \in T^0 - V$. Note that $\mathbb{L}$ is disjoint with the union $\bigcup_{v \in V} \mathbb{L}_v$, although the binding lines in flat links $Fl(w)$ are parallel to the boundary lines of $(the$ flat strips and) $\overline{Y}_v$ for $v \in Lk(w)$.

We now relate the metric geometry of $\hat{X}$ and $X$. Let $\pi_\ell$ denote the shortest projection to $\ell$ in $X$ and $d_\ell(x, y)$ the diameter of the projection of the points $x, y$ to $\ell$. 
Lemma 5.4. There exists $K_0 > 0$ such that for any two points $x, y \in \mathcal{X}$ and $K > K_0$, we have
\begin{equation}
    d_\mathcal{X}(x, y) \sim_K d_\hat{\mathcal{X}}(x, y) + \sum_{\ell \in \mathbb{L}} [d_\gamma(x, y)]_K.
\end{equation}

Proof. Let $\gamma$ be a $\hat{\mathcal{X}}$-geodesic with endpoints $x, y \in \mathcal{X}$. Let us consider the generic case that $\rho(x) \neq \rho(y)$; the case $\rho(x) = \rho(y)$ is much easier and left to the reader. Then $\gamma$ can be written as the union of geodesics in $\hat{\mathcal{X}}$'s with endpoints at the apex. If $c$ is the maximal subpath of $\gamma$ contained in some hyperbolic multicones $\text{Cones}_v(w)$ passing through the apex, we replace $c$ by an $\hat{\mathcal{X}}$-geodesic with the same endpoints $c_-, c_+$ of $c$: it is the geodesic in the corresponding flat links with same endpoints, whose binding line is denoted by $\ell_c$. This replacement becomes non-unique when different $c$'s have overlap (in the subspace $\hat{\mathcal{Y}}_v \subset \hat{\mathcal{Y}}_v$). However, the bounded intersection $\ell$ gives a uniform upper bound on the overlap. Let $K$ be any constant sufficiently bigger than this bound. We then number those subpaths $c$ of $\gamma$ with $d_\gamma(c, \gamma(c)) > K$ in a fixed order (eg. from left to right): $c_1, \ldots, c_n$. Up to bounded modifications, we obtain a well-defined notion of lifted path $\hat{\gamma}$ with same endpoints of $\gamma$. By construction, we have
\begin{equation}
    \text{Len}_\mathcal{X}(\hat{\gamma}) \sim_K d_\hat{\mathcal{X}}(x, y) + \sum_{i=1}^n [d_\gamma(x, y)]_K.
\end{equation}

Using the local finiteness and bounded intersection $\mathbb{L}$, for each $\ell_i$ with $[d_\gamma(x, y)]_K > 0$, there are only finitely many $\ell \in \mathbb{L}$ such that $[d_\gamma(x, y)]_K > 0$. Hence, by worsening the multiplicative constant, we have
\begin{equation}
    \text{Len}_\mathcal{X}(\hat{\gamma}) \sim_K d_\hat{\mathcal{X}}(x, y) + \sum_{\ell \in \mathbb{L}} [d_\gamma(x, y)]_K.
\end{equation}

The proof is then concluded by the well-known fact that $\hat{\gamma}$ is a quasi-geodesic. One proof proceeds as follows: it is an efficient semi-polygonal path in the sense of Bowditch in [6, Section 7]. This result follows as a consequence of [Bow12, Lemma 7.3].

Proposition 5.5. For any $K > K_0$, there exists a $G$-equivariant quasi-isometric embedding from $\mathcal{X}$ to the product $\hat{\mathcal{X}} \times \mathcal{C}_K(\mathbb{L})$.

Proof. We first define the map $\Phi = \Phi_1 \times \Phi_2 : \mathcal{X} \to \hat{\mathcal{X}} \times \mathcal{C}_K(\mathbb{L})$. If $x \in \hat{\mathcal{Y}}_v$, define $\Phi_1(x) = x$ in $\hat{\mathcal{X}}$. Choosing a (not unique) closest quasi-line $\ell$ in $\mathbb{L}_v$ to $x$ we define $\Phi_2(x) = \pi_\ell(x)$ and extend by $G$-equivariance $\Phi_2(gx) = g\pi_\ell(x)$ for all $g \in G$. If $x$ lies in the flat links $Fl(w)$ define $\Phi_1(x)$ to be the apex of $\text{Cones}_v(w)$ and $\Phi_2(x) = \pi_\ell(x)$ where $\ell$ is the binding line of $Fl(w)$.

The quasi-isometric embedding follows from the distance formula (10) in Lemma 5.4 and the formula in Proposition 4.9, which says that $\sum_{\ell \in \mathbb{L}} [d_\gamma(x, y)]_K$ is bi-Lipschitz up to bounded error to the distance from $x$ to $y$ in $\mathcal{C}_K(\mathbb{L})$.

5.2. Construct the collection of quasi-lines in $\hat{\mathcal{X}}$. Recall that $\hat{\mathcal{X}}$ is the hyperbolic cone-off space constructed from $\mathcal{V} \in \{\mathcal{V}_1, \mathcal{V}_2\}$. In this subsection, after Proposition 5.5 we are working in the cone-off space $\hat{\mathcal{X}}$ endowed with length metric $\hat{d}$. In particular, all quasi-lines are understood with this metric, and boundary lines $\mathbb{L}_v$ of $\hat{\mathcal{Y}}_v$ are of bounded diameter so not quasi-lines anymore in $\hat{\mathcal{X}}$.

Observe that for every $v \in \mathcal{V} \in \{\mathcal{V}_1, \mathcal{V}_2\}$, the cone-off space $\hat{\mathcal{Y}}_v$ admits a co-bounded and acylindrical action of $H_v$ by [Bow08], [Osi16]. Thus, when talking about quasi-lines, we follow the Convention 4.14: quasi-lines are $(\lambda, c)$-quasi-geodesics in $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}_v$'s (isometrically embedded into the former), where $\lambda, c > 0$ are given by Lemma 4.13 applied to those actions of $H_v$ on $\hat{\mathcal{Y}}_v$.

If $\gamma$ is a quasi-line in $\hat{\mathcal{X}}$, denote by $d_\gamma(x, y)$ the $\hat{d}$-diameter of the shortest projection of $x, y \in \hat{\mathcal{X}}$ to $\gamma$ in $\hat{\mathcal{X}}$. By Lemma 5.3, $\hat{\mathcal{X}}$ is $\delta$-hyperbolic for a constant $\delta > 0$. The coning-off construction is crucial to obtain the uniform constant $\theta$ below that will be used later on.
Lemma 5.6. There exists a constant $\theta > 0$ depending on $\delta$ (and also $\lambda, c$) with the following property: for any $((\lambda, c)-)$quasi-lines $\alpha$ in $\hat{Y}_v$ and $\beta$ in $\hat{Y}_v'$ with $v \neq v' \in \mathcal{V}$ we have $\text{diam}_{\hat{X}_1}(\pi_\beta(\alpha)) \leq \theta$.

Proof. By the construction of $\hat{X}$, any geodesic from $\alpha$ to $\beta$ has to pass through the apex between $\hat{Y}_v$ and $\hat{Y}_v'$, and thus the shortest projection $\pi_\beta(\alpha)$ is contained in the projection of the apex to $\beta$. By hyperbolicity, there exists a constant $\theta$ depending only on $\lambda, c, \delta$ such that the diameter of the projection of any point to every quasi-line is bounded above by $\theta$. The conclusion then follows. \qed

The goal of this subsection is to introduce a collection $\mathcal{A}$ of quasi-lines so that a distance formula holds for $\hat{X}$.

We first apply Lemma 4.15 to the cone-off space $\hat{Y}_v$, which admits a cobounded and acylindrical action of $H_v$. For any sufficiently large number $K > \max\{4\xi, \theta\}$, there exist a locally finite collection of quasi-lines $\mathcal{A}_v$ in $\hat{Y}_v$ and a constant $N = N(\mathcal{A}_v, K, \delta)$ such that

$$\frac{1}{N} \sum_{\gamma \in \mathcal{A}_v} [d_\gamma(x, y)]_K \leq d_{\hat{Y}_v}(x, y) \leq 2 \sum_{\gamma \in \mathcal{A}_v} [d_\gamma(x, y)]_K + 2K$$

for any $x, y \in \hat{Y}_v$.

Recall that $\hat{G}$ preserves $\mathcal{V}_1$ and $\mathcal{V}_2$, and by Lemma 5.3, there are only finitely many $\hat{G}$–orbits in $\{\hat{Y}_v : v \in \mathcal{V}\}$, so we can assume furthermore $\mathcal{A}_w = g\mathcal{A}_v$ if $w = gv$ for $g \in \hat{G}$.

Recall that $r$ is the radius of the multicones in constructing $\hat{X}_i$ for $i = 1, 2$.

Proposition 5.7. The $\mathcal{A} = \bigcup_{v \in \mathcal{V}} \mathcal{A}_v$ is a $\hat{G}$–invariant collection of quasi-lines in $\hat{X}$ such that for any $x, y \in \hat{X}$,

$$\frac{1}{N} \sum_{\gamma \in \mathcal{A}} [d_\gamma(x, y)]_K + r \cdot d_T(\rho(x), \rho(y)) \leq d_{\hat{X}_1}(x, y) \leq 2 \sum_{\gamma \in \mathcal{A}} [d_\gamma(x, y)]_K + 2K \cdot d_T(\rho(x), \rho(y)).$$

Proof. The proof proceeds similarly as that of Proposition 4.18, so only the differences are spelled out. Assume that $x, y \in \hat{X}$ are not in any hyperbolic multicones. We can then write the geodesic $[x, y]$ as the following union

$$[x, y] = \bigcup_{v \in \mathcal{V} \cap [\rho(x), \rho(y)]} \bigcup_{w \in \mathcal{T}_v \cap [\rho(x), \rho(y)]} c_w$$

where $\gamma_v$ is a geodesic in the cone-off space $\hat{Y}_v$ whose endpoints are on boundary lines of $Y_v \subset \hat{Y}_v$ where $v \in \mathcal{V} \cap [\rho(x), \rho(y)]$ and $c_w$ penetrates the apex and is of length $2r$ in the hyperbolic $r$–cones.

By the formula (11), we sum up the lengths of geodesics $\gamma_v$ in $\hat{Y}_v$ yielding the upper bound in (12).

By the choice of $K > \theta$ and Lemma 5.6, we have $[d_\gamma(x, y)]_K = 0$ for any quasi-line $\gamma$ in $\mathcal{V} - [\rho(x), \rho(y)]^0$. The lower bound of (12) is obtained as well by summing up distances in the formula (11), by taking into account that $[x, y]$ goes through $d_T(\rho(x), \rho(y))/2$ hyperbolic cones with radius $r$ so the term $r \cdot d_T(\rho(x), \rho(y))$ is added. \qed

5.3. Reassembling finite index vertex groups. By Bass-Serre theory, the finite index subgroup $\hat{G} < G$ from Lemma 4.6 acts on the Bass-Serre tree of $G$ and can be represented as a finite graph $\mathcal{G} = T/\hat{G}$ of groups where the vertex subgroups are isomorphic to those of $G$.

Let $e$ be an oriented edge in $\mathcal{G}$ from $e_-$ to $e_+$ (it is possible that $e_- = e_+$ because $e$ could be a loop) and $\bar{e}$ be the oriented edge with reversed orientation. A collection of finite index subgroups $\{G'_v, G''_v : v \in \mathcal{G}^0, e \in \mathcal{G}^1\}$ is called compatible if whenever $v = e_-$, we have

$$G_v \cap G'_v = G'_v \cap G_v.$$
We shall make use of [DK18, Theorem 7.51] to obtain a finite index subgroup in \( \hat{G} \) from a compatible collection of finite index subgroups. For this purpose, we assume the quotient \( H_v \) of each vertex group \( G_v \) for \( v \in \mathcal{G}^0 \) is omnipotent in the sense of Wise.

In a group two elements are independent if they do not have conjugate powers. (see Definition 3.2 in [Wis00])

**Definition 5.8.** A group \( H \) is omnipotent if for any set of pairwise independent elements \( \{h_1, \cdots, h_r\} \) \((r \geq 1)\) there is a number \( p \geq 1 \) such that for every choice of positive natural numbers \( \{n_1, \cdots, n_r\} \) there is a finite quotient \( H \to \hat{H} \) such that \( \hat{h}_i \) has order \( n_i p \) for each \( i \).

**Remark 5.9.** If \( H \) is hyperbolic, two loxodromic elements \( h, h' \) are usually called independent if the collection of \( H \)–translated quasi-axis of \( h, h' \) has the bounded projection property. When \( H \) is torsion-free, it is equivalent to the notion of independence in the above sense.

Let \( g \) be a loxodromic element in a hyperbolic group and \( E(g) \) be the maximal elementary group containing \( \langle g \rangle \). By [BH99, Ch. II Theorem 6.12], \( G_v \) contains a subgroup \( K_v \) intersecting trivially with \( Z(G_v) \) so that the direct product \( K_v \times Z(G_v) \) is a finite index subgroup. Thus, the image of \( K_v \) in \( G_v/Z(G_v) \) is of finite index in \( H_v \) and \( K_v \) acts geometrically on hyperbolic spaces \( \mathcal{Y}_v \).

The following result will be used in the next subsection to obtain desired finite index subgroups.

**Lemma 5.10.** Let \( \{K_v < K_v : v \in \mathcal{G}^0\} \) be a collection of finite index subgroups. Then there exists a compatible collection of finite index subgroups \( \{G'_v, G'_v : v \in \mathcal{G}^0, e \in \mathcal{G}^1\} \) such that \( G'_v \subset \hat{K}_v \times Z(G_v) \) for each \( v \in \mathcal{G}^0 \), where \( \hat{K}_v \) is of finite index in \( \hat{K}_v \).

**Proof.** Let \( e \) be an oriented edge in \( \mathcal{G} \) from \( e_- \) to \( e_+ \) (it is possible that \( e_- = e_+ \)) and \( \pi \) be the oriented edge with reversed orientation.

If \( v = e_- \), then the abelian group \( K_v \cap G_v \) is a cyclic group contained in a maximal elementary \( E(b_e) \) in \( K_v \) where \( b_e \) is a primitive loxodromic element. Similarly, for \( w = e_+ \), let \( b_\pi \in K_w \) be a primitive loxodromic element in \( E(b_\pi) \) containing \( K_w \cap G_v \). Then \( b_e \) and \( b_\pi \) preserve two lines respectively which are orthogonal in the Euclidean plane \( F_e \) and thus generate an abelian group \( \hat{G}_e := \langle b_e, b_\pi \rangle \) of rank 2 so that \( G_e \subset \hat{G}_e \) is of finite index.

Let \( \{b_{e_1}, \ldots, b_{e_r}\} \) be the set of primitive loxodromic elements in \( K_v \) in correspondence with the collection of all oriented edges \( e_1, \ldots, e_r \) in \( \mathcal{G}^1 \) such that \( (e_i)_- = v \).

By assumption, \( \hat{K}_v \) is a finite index subgroup of \( K_v \). Note that a finite index subgroup of an omnipotent group is omnipotent, so \( \hat{K}_v \) are omnipotent. By the finite index of \( G_v \) in \( \hat{G}_v \), there exists a set of powers of \( b_{e_i} \)'s in \( \hat{K}_v \cap G_{e_i} \) denoted by \( \{h_{e_1}, \ldots, h_{e_r}\} \) and for which we apply the omnipotence of \( \hat{K}_v \), let \( p_v \) be the constant given by Definition 5.8. Let

\[
s = \prod_{v \in \mathcal{G}^0} p_v
\]

Define \( n_i = \frac{s}{p_v} \) with \( i \in \{1, \ldots r\} \). By the omnipotence of \( \hat{K}_v \) there is a finite index subgroup \( \hat{K}_v \) of \( \hat{K}_v \) such that \( \hat{h}_{e_i}^{p_v n_i} = h_{e_i}^s \in \hat{K}_v \).

For each vertex \( v \in \mathcal{G} \) and for each edge \( e \) in \( \mathcal{G} \), we define

\[
G'_v := \hat{K}_v \times \langle h_{e_i}^s \rangle
\]

and

\[
G'_e := \langle h_{e_i}^s \rangle \times \langle h_{e}^s \rangle = s\mathbb{Z} \times s\mathbb{Z}
\]

To conclude the proof, it remains to note the collection \( \{G'_v, G'_e : v \in \mathcal{G}^0, e \in \mathcal{G}^1\} \) is compatible. It is obvious that \( G'_v \subset G'_v \), so \( G'_v \leq G'_v \cap G_{e_i} \). Conversely, \( G'_v \cap G_{e_i} \subset G'_v \cap G_{e_i} \subset (\hat{K}_v \cap \langle \hat{h}_{e_i} \rangle) \times \langle h_{e_i}^s \rangle \subset G'_v \cap G_{e_i} \). \( \square \)
5.4. Partition $\mathcal{A}$ into sub-collections with good projection constants: completion of the proof. Recall that $K_v$ is a subgroup of $G_v$ so that $K_v \times Z(G_v)$ is of finite index in $G_v$ and $K_v \cap Z(G_v) = \varnothing$. Thus (the image of) $K_v$ is of finite index in $H_v$ and acts co-boundedly and acylindrically on $\mathbb{A}_v$. Since $H_v$ is omnipotent and then residually finite, without loss of generality we can assume that $K_v$ is torsion-free.

Let $\mathcal{V} \in \{\mathcal{V}_1, \mathcal{V}_2\}$. Let us recall the data we have now:

(1) For every $v \in \mathcal{V}$, $\mathbb{A}_v$ is a locally finite $K_v$-invariant collection of quasi-lines in $\hat{Y}_v$ so that the distance formula (11) holds for $\hat{Y}_v$. (Lemma 4.15)

(2) Let $\mathbb{A} = \cup_{v \in \mathcal{V}} \mathbb{A}_v$ be the $G$-invariant collection of quasi-lines so that the formula (12) holds. (Proposition 5.7)

The first step is passing to a further index subgroup $\check{K}_v$ of $K_v$ so that $\mathbb{A}_v$ is partitioned into $\check{K}_v$-invariant sub-collections with projection constants $\xi$. It follows closely the argument in [BBF19] which is presented below for completeness.

Proposition 5.11. [BBF19, Proposition 3.4] Assume that a hyperbolic group $H$ acts acylindrically on $Y$. Let $\mathbb{A}$ be the set of all $H$-translated axis of a loxodromic element $g \in H$. Then there exists a constant $\theta > 0$ depending on $\mathbb{A}$ such that for any $\gamma \in \mathbb{A}$

$$\{h \in G : \text{diam}(\pi_\gamma(h\gamma)) \geq \theta\}$$

is a finite union of double $E(g)$-cosets.

In particular, there are only finitely many distinct pairs $(\gamma, \gamma') \in \mathbb{A} \times \mathbb{A}$ satisfying $\text{diam}(\pi_\gamma(\gamma')) > \theta$ up to the action of $H$.

The constants $\theta$ and $\xi$: The constant $\theta > 0$ is chosen so that it satisfies Proposition 5.11 and Lemma 5.6 simultaneously. Then $\xi = \xi(\theta)$ is given by Proposition 4.9.

Lemma 5.12. Let $\mathbb{A}_v$ be a $K_v$-finite collection of quasi-lines obtained as above by Lemma 4.15. Then there exists a finite index subgroup $\check{K}_v < K_v$ such that any two quasi-lines in the same $\check{K}_v$-orbit has $\theta$-bounded projection.

Proof. By construction, the quasi-lines in $\mathbb{A}_v$ are quasi-axis of loxodromic elements whose maximal elementary group is virtually cyclic. Recalling that $K_v$ is torsion-free, the maximal elementary group is cyclic and thus $E(g)$ is the centralizer $C(g) := \{h \in K_v : hg = gh\}$ of $g$. By [BBF19, Lemma 2.1], since $K_v$ is residually finite, then the centralizer of any element $g \in K_v$ is separable, i.e. the intersection of all finite index subgroups containing $C(g)$.

Proposition 5.11 implies that $E := \{h \in K_v : \text{diam}(\pi_\gamma(h\gamma)) \geq \theta\}$ consists of finite double $C(g)$-cosets. Since $C(g)$ is separable, we use the remark after Lemma 2.1 in [BBF19] to get a finite index $\check{K}_v < K_v$ such that $E \cap \check{K}_v = \varnothing$. The proof is complete. \(\square\)

The next step is re-grouping appropriately the collections of quasi-lines $\cup_{v \in \mathcal{V}} \mathbb{A}_v$ in Lemma 5.12.

By [DK18, Theorem 7.51], the compatible collection of finite index subgroups from Lemma 5.10 determines a finite index group $G_0 < \hat{G} < G$ such that $G_0 \cap G_v = G'_v, G_0 \cap G_e = G'_e$ and $G'_v = \check{K}_v \times Z(G_v) < \hat{K}_v \times Z(G_v)$ for every vertex $v$ and edge $e$.

By Bass-Serre theory, $G_0$ acts on the Bass-Serre tree $T$ of $G$ with finitely many vertex orbits. To be precise, let $\{v_0, \ldots, v_m\}$ be the full set of vertex representatives.

Since for each $1 \leq i \leq m$, $\check{K}_{v_i} \subset \hat{K}_{v_i}$ is of finite index, Lemma 5.12 implies that $\mathbb{A}_{v_i}$ consists of finitely many $\check{K}_v$-orbits, say $\check{K}^j_i (1 \leq j \leq l_i)$,

$$\mathbb{A}_{v_i} = \cup_{j=1}^{l_i} \check{K}^j_i,$$

each of which satisfies projection axioms with projection constant $\xi$.

Recall that $G_0 \subset \hat{G}$ acts on $\mathcal{X}_1$ and $\mathcal{X}_2$. We now set $\mathbb{A}_{ij} := \cup_{g \in G_0} g \check{K}^j_i$ so we have

$$\mathbb{A} = \cup_{i=1}^m \cup_{j=1}^{l_i} \mathbb{A}_{ij}.$$
We summarize the above discussion as the following.

**Proposition 5.13.** For each \( X \in \{X_1, X_2\} \), there exists a finite partition \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cdots \cup \mathcal{A}_n \) where \( n = \sum_{i=1}^n l_i \) such that for each \( 1 \leq i \leq n \), \( \mathcal{A}_i \) is \( G_0 \)-invariant and satisfies projection axioms with projection constant \( \xi \).

We are now ready to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 4.8, \( \hat{G} \) acts on the product \( X_1 \times X_2 \) so that the orbital map is quasi-isometrically embedded. Furthermore, there exists a \( \hat{G} \)-equivariant quasi-isometric embedding of each \( X_i \) \( (i = 1, 2) \) into the product of the cone-off space \( \hat{X}_i \) and a quasi-tree by Proposition 5.5. Therefore, it suffices to establish a \( G_0 \)-equivariant quasi-isometric embedding of \( \hat{X}_i \) into a finite product of quasi-trees.

By construction, each \( \hat{A}^j_i \) \( (1 \leq j \leq l_i) \) is \( \hat{K}_{0i} \)-invariant and \( \hat{K}_{0i} \) acts co-boundedly on \( Y_{vi} \), so there exists some \( R \) independent of \( i, j \) so that the union of quasi-lines in \( \hat{A}^j_i \) is \( R \)-cobounded in \( Y_{vi} \).

Let \( x \in X \) and \( x \in \hat{Y}_{vi} \), we choose a point \( \Phi_i(x) \in \bigcup_{\gamma \in \hat{A}^j_i} \gamma \) for \( 1 \leq i \leq n \) such that \( d(x, \Phi_i(x)) \leq R \). By \( G_0 \)-equivariance we define \( \Phi_i(gx) = g\Phi_i(x) \) for any \( g \in G_0 \).

By Proposition 5.7, the formula (12) holds for any \( x, y \in X \). Note the sum

\[
\sum_{\gamma \in \hat{A}^j_i} [d_{\gamma}(x, y)]_K = \sum_{i=1}^n \sum_{\gamma \in \hat{A}^j_i} [d_{\gamma}(x, y)]_K
\]

For each \( \mathcal{A}_i \), let \( \mathcal{C}_K(\mathcal{A}_i) \) be the quasi-tree of quasi-lines and by Proposition 4.9

\[
\sum_{\gamma \in \hat{A}^j_i} [d_{\gamma}(x, y)]_K \sim d_{\mathcal{C}_i}(\Phi_i(x), \Phi_i(y))
\]

Hence the formula (12) implies

\[
\hat{X} \to T \times \prod_{i=1}^n \mathcal{C}_K(\mathcal{A}_i)
\]

is a \( G_0 \)-equivariant quasi-isometric embedding. The proof of the Theorem is thus completed. \( \square \)

6. **Finite height subgroups in a CKA action \( G \curvearrowright X \)**

In this section, we are going to prove Theorem 1.6 that basically says having finite height and strongly quasiconvexity are equivalent to each other in the context of CKA actions, and both properties can be characterized in term of their group elements. The heart of the proof of this theorem belongs to the implication \((3) \Rightarrow (1)\) where we use Sisto’s notion of path systems \((\text{Sis18})\).

We first review some concepts finite height subgroups, strongly quasi-convex subgroups as well as some terminology in \((\text{Sis18})\).

**Definition 6.1.** Let \( G \) be a group and \( H \) a subgroup of \( G \). We say that conjugates \( g_1Hg_1^{-1}, \cdots, g_kHg_k^{-1} \) are **essentially distinct** if the cosets \( g_1H, \cdots, g_kH \) are distinct. We call \( H \) has **height at most \( n \)** in \( G \) if the intersection of any \( (n+1) \) essentially distinct conjugates is finite. The least \( n \) for which this is satisfied is called the **height of \( H \)** in \( G \).

**Definition 6.2** \((\text{Strongly quasiconvex, (Tra19)})\). A subset \( Y \) of a geodesic space \( X \) is called **strongly quasiconvex** if for every \( K \geq 1, C \geq 0 \) there is some \( M = M(K, C) \) such that every \( (K, C) \)-quasigeodesic with endpoints on \( Y \) is contained in the \( M \)-neighborhood of \( Y \).

Let \( G \) be a finitely generated group and \( H \) a subgroup of \( G \). We say \( H \) is strongly quasiconvex in \( G \) if \( H \) is a strongly quasi-convex subset in the Cayley graph \( \Gamma(G, S) \) for some (any) finite generating set \( S \). A group element \( g \) in \( G \) is **Morse** if \( g \) is of infinite order and the cyclic subgroup generated by \( g \) is strongly quasiconvex.
Remark 6.3. The strong quasiconvexity of a subgroup does not depend on the choice of finite generating sets, and this notion is equivalent to quasiconvexity in the setting of hyperbolic groups. It is shown in [Tra19] (see Theorem 1.2) that strongly quasi-convex subgroups of a finitely generated group are finitely generated and have finite height.

The following proposition is cited from Proposition 2.3 and Proposition 2.6 in [NTY].

Proposition 6.4. (1) Let $G$ be a group such that the centralizer $Z(G)$ of $G$ is infinite. Let $H$ be a finite height infinite subgroup of $G$. Then $H$ must have finite index in $G$.

(2) Assume a group $G$ is decomposed as a finite graph $T$ of groups that satisfies the following.

(a) For each vertex $v$ of $T$ each finite height subgroup of vertex group $G_v$ must be finite or have finite index in $G_v$.

(b) Each edge group is infinite.

Then, if $H$ is a finite height subgroup of $G$ of infinite index, then $gHg^{-1} \cap G_v$ is finite for each vertex group $G_v$ and each group element $g$. In particular, if $H$ is torsion free, then $H$ is a free group.

Definition 6.5 (Path system, [Sis18]). Let $X$ be a metric space. A path system $PS(X)$ in $X$ is a collection of $(c,c)$–quasi-geodesic for some $c \geq 1$ such that any subpath of a path in $PS(X)$ is in $PS(X)$, and all pairs of points in $X$ can be connected by a path in $PS(X)$.

Definition 6.6 ($PS$–contracting, [Sis18]). Let $X$ be a metric space and let $PS(X)$ be a path system in $X$. A subset $A$ of $X$ is called $PS(X)$–contracting if there exists $C > 0$ and a map $\pi: X \to A$ such that

1. For any $x \in A$, then $d(x, \pi(x)) \leq C$.

2. For any $x, y \in X$ such that $d(\pi(x), \pi(y)) \geq C$ then for any path $\gamma$ in $PS(X)$ connecting $x$ to $y$ then $d(\pi(x), \gamma) \leq C$ and $d(\pi(y), \gamma) \leq C$.

The map $\pi$ will be called $PS(X)$–projection on $A$ with constant $C$.

Lemma 6.7. [Sis18, Lemma 2.8] Let $A$ be a $PS(X)$–contracting subset of a metric space $X$, then $A$ is strongly quasi-convex.

Theorem 6.8. Let $G \acts X$ be a CKA action. Let $PS(X)$ be the collection of all special paths defined in Definition 3.6. Then $(X, PS(X))$ is a path system.

Proof. The proof follows from Proposition 3.8. □

For the rest of this section, we fix a CKA action $G \acts X$ and $G \acts T$ the action of $G$ on the associated Bass-Serre tree. We also fix the path system $(X, PS(X))$ in Theorem 6.8.

To get into the proof of Theorem 1.6, we need several lemmas. The following lemma tells us that finite height subgroups in the CKA action $G \acts X$ are virtually free.

Lemma 6.9. Let $K \leq G$ be a nontrivial finitely generated infinite index subgroup of $G$. Suppose that $K$ has finite height in $G$ then $K$ is virtually free.

Proof. Suppose that $K$ has finite height in $G$. Since the centralizer $Z(G_v)$ each each vertex group is isomorphic to $Z$, it follows from Proposition 6.4 that for any $g \in G$ and $v \in T^0$, the intersection $K \cap gG_vg^{-1}$ is finite. Thus, $K$ acts properly on the tree $T$ and the stabilizer in $K$ of each vertex in $T$ is finite. It follows from [DK18, Theorem 7.51] that $K$ is virtually free. □

Remark 6.10. Let $K \leq G$ be a nontrivial finitely generated infinite index subgroup of $G$. Suppose that $K$ is a free group of finite rank and every nontrivial element in $K$ is not conjugate into any vertex group. Then there exists a subspace $C_K$ of $X$ such that $K$ acts geometrically on $C_K$ with respect to the induced length metric on $C_K$. The subspace $C_K$ is constructed as the following.
Fix a vertex $v$ in $T$, and fix a point $x_0$ in $Y_v$ such that $\rho(x_0) = v$. Let $\{g_1, g_2, \ldots, g_n\}$ be a generating set of $K$. For each $i \in \{1, 2, \ldots, n\}$, let $g_{a+i} = g_i^{-1}$. Let $\gamma_j$ be the geodesic in $X$ connecting $x_0$ to $g_j(x_0)$ with $j \in \{1, 2, \ldots, 2n\}$. Let $C_K$ be the union of segment $g(\gamma_j)$ where $g$ varies elements of $K$ and $j \in \{1, \ldots, 2n\}$.

The following lemma is well-known (see Lemma 2.9 in [CK02] or Lemma 4.5 in [GM14] for proofs).

**Lemma 6.11.** Let $X$ be a $\delta$–hyperbolic Hadamard space. Let $\gamma_1$ and $\gamma_2$ be two geodesic lines of $X$ such that $\partial_\infty \gamma_1 \cap \partial_\infty \gamma_2 = \emptyset$. Let $\eta$ be a minimal geodesic segment between $\gamma_1$ to $\gamma_2$. Then any geodesic segment running from $\gamma_1$ to $\gamma_2$ will pass within distance $D = D(\gamma_1, \gamma_2)$ of both endpoints of $\eta$. Moreover, when $d(\gamma_1, \gamma_2) > 4\delta$ then we may take $D = 2\delta$.

Lemma 6.12 and Lemma 6.13 below are used in the proof of Proposition 6.15.

**Lemma 6.12.** Let $e$ and $e'$ be two consecutive edges in $T$ with a common vertex $v$. Let $A$ be a subset of $Y_v$ such that $A \cap F_e \neq \emptyset$ and $A \cap F_{e'} \neq \emptyset$ and $\text{diam}(A) \leq \mu$ for some $\mu > 0$. Then there exists $r = r(\mu, e, e') > 0$ such that the following holds. Let $p$ be a point in the line $\ell := \overline{Y_v} \cap F_e$ and $q$ be a point in the line $\ell' := \overline{Y_v} \cap F_{e'}$ such that the geodesic $[p, q]$ is a shortest path joining two lines $\ell$ to $\ell'$. For any $x \in F_e \cap A$ and $y \in F_{e'} \cap A$, let $u$ and $v$ be the projections of $x$ and $y$ into the lines $\ell$ and $\ell'$ respectively. Then $d(u, p) \leq r$ and $d(v, q) \leq r$.

**Proof.** We recall that $Y_v = \overline{Y_v} \times \mathbb{R}$ and $H_v$ acts properly and cocompactly on $\overline{Y_v}$. Since $H_v$ is a nonelementary hyperbolic group, it follows that $\overline{Y_v}$ is a $\delta_v$–hyperbolic space for some $\delta_v \geq 0$.

Let $D = D(\ell, \ell') > 0$ be the constant given by Lemma 6.11. Let $r = 4D + \mu$. Since $\partial_\infty \ell \cap \partial_\infty \ell' = \emptyset$ and $u \in \ell$, $v \in \ell'$, it follows from Lemma 6.11 that there exist $p', q' \in [u, v] \cap A$ such that $d(p, p') \leq D$ and $d(q, q') \leq D$. By the triangle inequality, we have $d(u, p) + d(p, q) + d(q, v) \leq 4D + d(u, v)$.

Since $u$ and $v$ are projection points of $x$ and $y$ into the factor $\overline{Y_v}$ of $Y_v = \overline{Y_v} \times \mathbb{R}$ respectively, it follows that $d(u, v) \leq d(x, y)$. Since $x, y \in A$ and $\text{diam}(A) \leq \mu$, it follows that $d(x, y) \leq \mu$. Hence $d(u, v) \leq d(x, y) \leq \mu$. Thus, $d(u, p) \leq 4D + d(u, v) \leq 4D + \mu = r$ and $d(v, q) \leq 4D + d(u, v) \leq 4D + \mu = r$.

**Lemma 6.13.** Let $K \leq G$ be a finitely generated, finite height subgroup of $G$ of infinite index. Suppose that $K$ is a free group of finite rank, and let $C_K$ be the subspace of $X$ given by Remark 6.10. Then there exists a constant $R > 0$ such that if $\gamma$ is a special path in $X$ (see Definition 3.6) connecting two points in $C_K$ then $\gamma \subset N_R(C_K)$.

**Proof.** By the construction of $C_K$, we note that there exists a constant $\mu > 0$ such that $\text{diam}(C_K \cap X_v) < \mu$ for any vertex $v \in V(T)$. For any consecutive edges $e$ and $e'$ in $T$ with a common vertex $v$, let $r_v = r(\mu, e, e')$ be the constant given by Lemma 6.12. Since there are only finitely many $r_v$ up to the action of $G$, we let $r$ be the maximum of these numbers.

Recall that we choose a $G$–equivariant family of Euclidean planes $\{F_e : F_e \subset Y_v\}_{e \in T_1}$. Let $\{S_{e' e''} \cap F_{e'} \cap F_{e''} \}$ be the collection of strips in $X$ given by Section 2. For any three consecutive edges $e, e', e''$ in the tree $T$, two lines $S_{e' e''} \cap F_{e'}$ and $S_{e' e''} \cap F_{e''}$ in the plane $F_{e'}$ determine an angle in $(0, \pi)$. However, there are only finitely many angles shown up. We denote these angles by $\theta_1, \ldots, \theta_k$.

Let $D$ be the constant given by Lemma 2.4 such that $X_v = N_D(Y_v)$ for every vertex $v \in T^0$. Let

$$\xi = 2\mu + r + \max \left\{ \frac{2\mu + r}{\sin(\theta_j)} + \frac{2\mu + r}{\sin(\pi - \theta_j)} \mid j \in \{1, \ldots, k\} \text{ and } \theta_j \neq \pi/2 \right\}$$

and

$$R = 2r + \mu + 2\xi + D$$

Let $x$ and $y$ be the initial and terminal points of $\gamma$. We note that $x \in X_{\rho(x)}$ and $y \in X_{\rho(y)}$. We consider the following cases:
Proof of the claim. Let \( O_1 \) be the projection of \( O \) into the line \( F_{e_1} \cap Y_{v_0} \). Let \( V_1 \) be the projection of \( O_1 \) into the line \( F_{e_1} \cap \overline{Y}_{v_1} \). By Lemma 6.12, we have
\[
|d(V_1, S_{e_1 e_2} \cap F_{e_1} \cap \overline{Y}_{v_1})| \leq r
\]
(we note that \( S_{e_1 e_2} \cap F_{e_1} \cap \overline{Y}_{v_1} = \gamma_{e_1 e_2}(0) \)). Since \( O_1 \) and \( p_0 = x \) belong to \( X_{v_0} \cap C_K \) and \( diam(X_{v_0} \cap C_K) \leq \mu \), it follows that \( d(O_1, p_0) \leq \mu \). Let \( \overline{p}_0 \) be the projection of \( p_0 \) into the factor \( Y_{v_0} \) of \( Y_{v_0} = \overline{Y}_{v_0} \). We have that \( d(O_1, \overline{p}_0) \leq d(O_1, p_0) \). Since \( d(\overline{p}_0, S_{e_1 e_2} \cap F_{e_1} \cap \overline{Y}_{v_0}) \) is the minimal distance from \( p_0 \) to the line \( F_{e_1} \cap \overline{Y}_{v_0} \) and \( O_1 \in F_{e_1} \cap \overline{Y}_{v_0} \), we have that \( d(p_0, S_{e_1 e_2} \cap F_{e_1} \cap \overline{Y}_{v_0}) \leq d(\overline{p}_0, O_1) \leq \mu \). Using the triangle inequality for three points \( \overline{p}_0, O_1, \) and \( S_{e_1 e_2} \cap F_{e_1} \cap \overline{Y}_{v_0} \), we have
\[
|d(O_1, S_{e_1 e_2} \cap F_{e_1} \cap \overline{Y}_{v_0})| \leq 2\mu
\]
(14)

Let \( A \) be the projection of \( O_1 \) into the line \( F_{e_1} \cap S_{e_1 e_2} \). Using formula (13), we have
\[
|d(O_1, A)| = |d(V_1, S_{e_1 e_2} \cap F_{e_1} \cap \overline{Y}_{v_1})| \leq r
\]
(15)

Let \( T \) be the projection of \( O_1 \) into the line \( S_{e_1 e_2} \cap F_{e_1} \). Using formula (14), we have \( |d(O_1, T)| = |d(V_1, S_{e_1 e_2} \cap F_{e_1} \cap \overline{Y}_{v_1})| \leq 2\mu \). Thus, we have \( |d(A, T)| = |d(A, O_1) + d(O_1, T)| \leq r + 2\mu \). An easy application of Rule of Sines to the triangle \( \Delta(T, p_1, A) \) together with the fact \( d(A, T) \leq 2\mu + r \) give us that \( d(p_1, A) < \xi \) and \( d(p_1, T) < \xi \). Combining these inequalities with formula (15), we obtain that \( d(O_1, p_1) \leq d(O_1, A) + d(A, p_1) < r + \xi \). The claim is verified.

Using the facts \( d(O_1, p_0) \leq \mu \) and \( d(O_1, p_1) < r + \xi \) we have \( d(p_0, p_1) \leq d(p_0, O_1) + d(O_1, p_1) \leq \mu + r + \xi < R \). Since \( p_0 = x \in C_K \), it follows that \( [p_0, p_1] \subset N_R(C_K) \).

Proof of the case \( j = 1 \):

Since \( C_K \cap F_{e_2} \neq \emptyset \), we choose a point \( O_2 \in C_K \cap F_{e_2} \). Since \( O_1, O_2 \) belong to \( C_K \) and \( diam(C_K \cap Y_{v_1}) \leq \mu \), we have \( d(O_1, O_2) \leq \mu \). By a similar argument as in the proof of the claim of the case \( i = 0 \), we can show that \( d(O_2, p_2) < r + \xi \). Thus, \( d(p_1, p_2) \leq d(p_1, O_1) + d(O_1, O_2) + d(O_2, p_2) < (r + \xi) + \mu + (r + \xi) = 2r + \mu + 2\xi \). Since \( O_1, O_2 \in C_K \), it is easy to see that \( [p_1, p_2] \subset N_{3r+\mu+3\xi}(Y) \subset N_R(Y) \).

We recall that an infinite order element \( g \) in a finitely generated group is Morse if the cyclic subgroup generated by \( g \) is strongly quasi-convex.
Lemma 6.14. If an infinite order element \( g \) in \( G \) is Morse, then it is not conjugate into any vertex group of \( G \).

Proof. Since \( g \) is Morse, it follows that the infinite cyclic subgroup \( \langle g \rangle \) generated by \( g \) is strongly quasi-convex in \( G \). We would like to show that \( g \) is not conjugate into any vertex group. Indeed, by way of contradiction, we assume that \( g \in xG_ex^{-1} \) for some \( x \in G \) and for some vertex group \( G_v \). Hence, the cyclic subgroup generated by \( h = x^{-1}gx \) is strongly quasi-convex in \( G \). Since \( G_v \) is undistorted in \( G \) (as \( G_v \) acts geometrically on \( Y_v \) and \( Y_v \) is undistorted in \( X \)), it follows from Proposition 4.11 in [Tra19] that \( \langle h \rangle \) is strongly quasi-convex in \( G_v \). By Theorem 1.2 in [Tra19], \( \langle h \rangle \) has finite height in \( G_v \). Since the centralizer \( Z(G_v) \) of \( G_v \) is isomorphic to \( Z \), it follows from Proposition 6.4 that \( \langle h \rangle \) has finite index in \( G_v \). This contradicts to the fact that \( G_v \) is not virtually cyclic group. Therefore \( g \) is not conjugate into any vertex group of \( G \).

\( \square \)

Proposition 6.15. Let \( K \) be a free subgroup of \( G \) of infinite index with a finite generating set \( \{g_1, g_2, \ldots, g_n\} \). Suppose that all nontrivial elements in \( K \) are Morse in \( G \). Let \( C_K \) be the subspace of \( X \) given by Remark 6.10 with respect to the generating set \( \{g_1, g_2, \ldots, g_n\} \). Then \( C_K \) is contracting in \( (X, \mathcal{PS}(X)) \). As a consequence, \( K \) is strongly quasi-convex in \( G \).

Proof. Since \( K \) is a free subgroup of \( G \) and all nontrivial elements in \( K \) are Morse in \( G \), it follows from Lemma 6.14 that every nontrivial element in \( K \) is not conjugate into any vertex group \( G_v \). Hence, \( K \) acts freely on the Bass-Serre tree \( T \). To show that \( C_K \) (we note that \( K(x_0) \subset C_K \)) is contracting in \( (X, \mathcal{PS}(X)) \), we need to define a \( \mathcal{PS}(X) \)-projection \( \pi: X \to C_K \) satisfying conditions (1) and (2) in Definition 6.6.

Step 1: Constructing \( \mathcal{PS}(X) \)-projection \( \pi: X \to C_K \) on \( C_K \).

Let \( \rho: X \to T^0 \) be the indexed map given by Remark 2.6. Let \( T' \) be the minimal subtree of \( T \) that contains the orbit \( K(v) \). Let \( R: T \to T' \) be the nearest point projection from \( T \) to \( T' \). Let \( x \) be any point in \( X \). If \( \rho(x) \in K(v) \) then we define \( \pi(x) \) to be \( g(x_0) \) where \( g \in K \) such that \( g(v) = R(\rho(x)) \). If \( \rho(x) \notin K(v) \) then there is an unique geodesic path \( e_1 \cdots e_m \) in the tree \( T' \) such that the following holds.

1. \( R(\rho(x)) \) is a vertex in the path \( e_1 \cdots e_m \).
2. The initial vertex of \( e_1 \) is \( g(v) \) for some \( g \in K \) and the terminal vertex of \( e_m \) is \( gg_j(v) \) for some \( j \in \{1, 2, \ldots, 2n\} \) (here we extend \( \{g_1, \ldots, g_n\} \) to \( \{g_1, \ldots, g_{2n}\} \) where \( g_{n+i} = g_i \) with \( i = 1, \ldots, n \)).

There exists uniquely a point \( y \) in the segment \( g(\gamma_j) \) such that

\[
d(y, g(x_0)) = \text{Len}(\gamma_j) \left( \frac{d_T(R(\rho(x)), g(v))}{m} \right)
\]

We define \( \pi(x) := y \).

Step 2: Verifying the condition (1) in Definition 6.6. Let \( \delta = \max \{ \text{Len}(\gamma_i) \mid i \in \{1, \ldots, n\} \} \). Let \( L \) be the constant given by Remark 2.6. Let \( \lambda = \max \{ d_T(v, g_i(v)) \mid i \in \{1, \ldots, n\} \} \). Let \( \mu > 0 \) be a constant such that \( \text{diam}(C_K \cap Y_u) \leq \mu \) for any vertex \( u \in T^0 \). Let \( R \) be the constant given by Lemma 6.13. By the definition of \( C_K \), there exists a constant \( \epsilon \geq 1 \) such that for any \( g \) and \( g' \) in \( K \) then

\[
d(g(x_0), g'(x_0)) / \epsilon - \epsilon \leq d_T(g(v), g'(v)) \leq \epsilon d(g(x_0), g'(x_0)) + \epsilon
\]

Claim 1: Let \( C \) be a constant such that \( 2\delta + \epsilon(\epsilon \delta + 2L \delta + 2L) + 5\epsilon \lambda + \mu + R < C \). Then \( d(x, \pi(x)) \leq C \) for any \( x \in C_K \). Indeed, since \( C_K \subset N_{\delta}(K(x_0)) \), there exists \( k \in K \) such that \( d(x, k(x_0)) \leq \delta \). By the definition of \( \pi \), there exists an element \( g \in K \) and \( j \in \{1, \ldots, 2n\} \) such that \( \pi(x) \in g(\gamma_j) \) and \( R(\rho(x)) \in [g(v), gg_j(v)] \). It follows that \( d(\pi(x), gg_j(x_0)) \leq \delta \) and

\[
d_T(R(\rho(x)), gg_j(v)) \leq d_T(g(v), gg_j(v)) \leq \epsilon \text{Len}(\gamma_j) + \epsilon \leq \epsilon \delta + \epsilon
\]
We also have
\[d_T(\rho(x), k(v)) = d(\rho(x), \rho(k(x_0)))\]
\[\leq Ld(x, k(x_0)) + L \leq L\delta + L\]

Since $\mathcal{R}: T \to T'$ is the nearest point projection, we obtain that $d_T(\rho(x), \mathcal{R}(\rho(x))) \leq d_T(\rho(x), k(v)) \leq L\delta + L$. Putting the above inequalities together with formula (16), we have
\[d(x, \pi(x)) \leq d(x, k(x_0)) + d(k(x_0), \pi(x)) \leq \delta + d(k(x_0), \pi(x))\]
\[\leq \delta + d(k(x_0), gg_j(x_0)) + d(gg_j(x_0), \pi(x)) \leq \delta + d(k(x_0), gg_j(x_0)) + \delta\]
\[\leq 2\delta + \epsilon + \epsilon (d_T(gg_j(v), \mathcal{R}(\rho(x))) + d_T(\mathcal{R}(\rho(x)), \rho(x)) + d_T(\rho(x), k(v)))\]
\[\leq 2\delta + \epsilon (\delta + 2L\delta + 2L)\]

Now, we will choose a constant $C > 0$ such that $2\delta + \epsilon (\delta + 2L\delta + 2L) < C$. Claim 1 is confirmed.

Step 3: Verifying condition (2) in Definition 6.6.

Claim 2: Let $C$ be the constant given by Claim 1. Then the projection $\pi: X \to C_K$ satisfies condition (2) in Definition 6.6 with respect to this constant $C$.

Let $x$ and $y$ be two points in $X$ such that $d(\pi(x), \pi(y)) \geq C$. Let $\gamma$ be a special path in $X$ connecting $x$ to $y$. We would like to show that $d(\pi(x), \gamma) \leq C$ and $d(\pi(y), \gamma) \leq C$. We recall that $X_u = N_D(Y_u)$ for any vertex $u \in T^d$. Thus we assume, without loss of generality that $x \in Y_\rho(x)$ and $y \in Y_\rho(y)$. We also further assume that $\mathcal{R}(\rho(x)) \notin K(v)$ and $\mathcal{R}(\rho(y)) \notin K(v)$ (since the proof for the cases $\mathcal{R}(\rho(x)) \in K(v)$ or $\mathcal{R}(\rho(y)) \in K(v)$ is similar). There exists $g, g' \in K$ and $i, j \in \{1, \ldots, 2n\}$ such that $\mathcal{R}(\rho(x)) \in [g(v), gg_j(v)]$ and $\mathcal{R}(\rho(y)) \in [g'(v), g'g_i(v)]$. Since $\pi(x) \in [g(x_0), gg_j(x_0)]$ and $\pi(y) \in [g'(x_0), g'g_i(x_0)]$, we have that
\[d(\pi(x), gg_j(x_0)) \leq \delta \quad \text{and} \quad d(\pi(y), g'g_i(x_0)) \leq \delta\]

Let $\kappa$ be the number of elements in $[gg_j(v), g'g_i(v)] \cap K(v)$. We claim that $\kappa > 20$. Indeed, using formula (16) we have $d(gg_j(x_0), g'g_i(x_0)) \leq \epsilon d_T(gg_j(v), g'g_i(v)) + \epsilon \leq \epsilon \lambda \kappa + \epsilon$. Thus,
\[d(\pi(x), \pi(y)) \leq d(\pi(x), gg_j(x_0)) + d(gg_j(x_0), g'g_i(x_0)) + d(g'g_i(x_0), \pi(y))\]
\[\leq \delta + d(gg_j(x_0), g'g_i(x_0)) + \delta \leq 2\delta + \epsilon \lambda \kappa + \epsilon\]

Hence, $2\delta + \epsilon + 20\epsilon \lambda < C \leq d(\pi(x), \pi(y)) \leq 2\delta + \epsilon \lambda \kappa + \epsilon$. It implies that $20 < \kappa$.

Choose $\sigma \in \{g, gg_j\}$ and $\tau \in \{g', g'g_i\}$ such that $\sigma(v)$ and $\tau(v)$ lie in the geodesic $[\mathcal{R}(\rho(x)), \mathcal{R}(\rho(y))]$. Since $\mathcal{R}(\rho(x)) \in [g(v), gg_j(v)]$, $\mathcal{R}(\rho(y)) \in [g'(v), g'g_i(v)]$, and the number of elements in $[gg_j(v), g'g_i(v)] \cap K(v)$ is $\kappa$. It follows that the number of elements in $[\mathcal{R}(\rho(x)), \mathcal{R}(\rho(y))] \cap K(v)$ is $\geq \kappa - 3 > 16$. We thus can choose $\sigma$ and $\tau$ in $K$ such that $\mathcal{R}(\rho(x)), \mathcal{R}(\rho(y)) \in [\mathcal{R}(\rho(x)), \mathcal{R}(\rho(y))]$ and the number of elements of $K(v)$ in $[\sigma(v), \mathcal{R}(\rho(x))] \setminus \{\sigma(v)\}$ is 2 as well as the number of elements of $K(v)$ in $[\tau(v), \mathcal{R}(\rho(y))] \setminus \{\tau(v)\}$ is 2.

Let $\alpha$ be a special path in $X$ connecting $\sigma(x_0) \in Y_{\sigma(v)}$ to $\tau(x_0) \in Y_{\tau(v)}$. We remark here that there exists a subpath $\gamma'$ of $\gamma$ and there exists a subpath $\alpha'$ of $\alpha$ such that $\gamma'$ and $\alpha'$ connect some point in $Y_{\sigma(v)}$ to some point in $Y_{\tau(v)}$. By Remark 3.7, we have that $\alpha' \cap X_u = \gamma' \cap X_u$ for any vertex $u$ in the geodesic $[\mathcal{R}(\rho(x)), \mathcal{R}(\rho(y))]$.

Let $R$ be the constant given by Lemma 6.13. It follows from Lemma 6.13 that $\alpha \cap Y_{\sigma(v)} \subset N_R(Y \cap Y_{\sigma(v)})$. Choose $z \in \alpha \cap Y_{\sigma(v)}$ and $z' \in C_K \cap X_{\sigma(v)}$ such that $d(z, z') \leq R$. Since $\sigma(v) \in \{g(v), gg_j(v)\}$ and the number of elements of $K(v) \cap [\sigma(v), \mathcal{R}(\rho(x))]$ is 4, it follows that the number of elements of $K(v) \cap [gg_j(v), \mathcal{R}(\rho(x))]$ is no more than 5. Using this fact together with formula (16) to obtain
\[d(gg_j(x_0), \mathcal{R}(x_0)) \leq \epsilon d(gg_j(v), \mathcal{R}(v)) + \epsilon \leq 5\epsilon \lambda + \epsilon\]
We also have \( d(z', \sigma(x_0)) \leq \mu \) because both \( z' \) and \( \sigma(x_0) \) belong to \( C_K \cap Y_{\pi(x)} \). Hence,

\[
d(\pi(x), \gamma) \leq d(\pi(x), z) \leq d(\pi(x), z') + d(z', z) \leq d(\pi(x), z') + R \\
\leq d(\pi(x), gg_j(x_0)) + d(gg_j(x_0), \sigma(x_0)) + d(\sigma(x_0), z') + R \\
\leq \delta + 5\epsilon \lambda + \mu + R < C
\]

Similarly, we can show that \( d(\pi(y), \gamma) < C \). Thus Claim 2 is established.

Combining Step 1, Step 2, and Step 3 together, we conclude that \( C_K \) is contracting in \( (X, \mathcal{PS}(X)) \). By Lemma 6.7, \( C_K \) is strongly quasi-convex in \( X \). Since \( K \) acts geometrically on \( C_K \subset X \), \( G \) acts geometrically on \( X \), and \( C_K \) is strongly quasi-convex in \( X \), we conclude that \( K \) is strongly quasi-convex in \( G \). The proposition is proved. \( \square \)

Proposition 6.15 has the following corollary that characterize Morse elements and contracting element in the admissible group \( G \).

**Corollary 6.16.**

1. An infinite order element \( g \) in \( G \) is Morse if and only if \( g \) is not conjugate into any vertex group \( G_v \).

2. An element of \( G \) is contracting with respect to \( (X, \mathcal{PS}(X)) \) if and only if its acts hyperbolically on the Bass-Serre tree \( T \).

**Proof.** (1). By Lemma 6.14, if \( g \) is Morse in \( G \) then it is not conjugate into vertex group of \( G \). Conversely, if \( g \) is not conjugate into any vertex group of \( G \) then \( g \) acts hyperbolically on the Bass-Serre tree \( T \). If \( g \in G \) acts hyperbolically on the Bass-Serre tree \( T \). Let \( K \) be the infinite cyclic subgroup generated by \( g \). Since \( g \in G \) acts hyperbolically on the Bass-Serre tree \( T \), it is not conjugate into any vertex subgroup. Fix a point \( x_0 \in X \), by Proposition 6.15 the orbit space \( K(x_0) \) is contracting in \( (X, \mathcal{PS}(X)) \) (because \( C_K \) is contracting in \( (X, \mathcal{PS}(X)) \)). Thus \( g \) is a contracting element with respect to \( (X, \mathcal{PS}(X)) \). By Lemma 2.8 in [Sis18], \( g \) must be Morse.

(2). If \( g \in G \) acts hyperbolically on the Bass-Serre tree \( T \). Let \( K \) be the infinite cyclic subgroup generated by \( g \). Since \( g \in G \) acts hyperbolically on the Bass-Serre tree \( T \), it is not conjugate into any vertex subgroup. Fix a point \( x_0 \in X \), by Proposition 6.15 the orbit space \( K \cdot x_0 \) is contracting in \( (X, \mathcal{PS}(X)) \) (because \( C_K \) is contracting in \( (X, \mathcal{PS}(X)) \)). Thus \( g \) is a contracting element with respect to \( (X, \mathcal{PS}(X)) \).

Conversely, if \( g \in G \) is contracting with respect to \( (X, \mathcal{PS}(X)) \), then \( g \) is Morse. By the assertion (1), \( g \) is not conjugate into any vertex group. Thus it acts hyperbolically on the Bass-Serre tree \( T \). \( \square \)

We now ready to prove Theorem 1.6.

**Proof of Theorem 1.6.** We are going to prove the following implications: \( (1) \Rightarrow (2) \), \( (2) \Rightarrow (3) \), \( (3) \Rightarrow (1) \), and \( (3) \iff (4) \).

\( (1) \Rightarrow (2) \): The implication just follows from Theorem 1.2 in [Tra19].

\( (2) \Rightarrow (3) \): By Lemma 6.9, \( K \) has a finite index subgroup \( K' \) such that \( K' \) is a free group of finite rank. Let \( x \) be an infinite order element in \( K' \). By way of contradiction, suppose that \( x \) is not Morse in \( G \). By Corollary 6.16, \( x \) is conjugate into a vertex group of \( G \). In other words, \( x \in gG_v g^{-1} \) for some vertex group \( G_v \) and for some \( g \in G \). Hence \( x \in K' \cap gG_v g^{-1} \subset K \cap gG_v g^{-1} \) that is finite by Proposition 6.4. So, \( x \) has finite order that contradicts to our assumption that \( x \) is an infinite order element. Since any infinite order element in \( K \) has a power that belongs to \( K' \), the implication \( (2) \Rightarrow (3) \) is verified.

\( (3) \Rightarrow (1) \): Let \( K' \) be a finite index subgroup of \( K \) such that \( K' \) is free. It follows from Proposition 6.15 that \( K' \) is strongly quasi-convex in \( G \). Since \( K' \) is a finite index subgroup of \( K \), it follows that \( K \) is also strongly quasi-convex in \( G \).
(3) ⇒ (4): Let $K'$ be a finite index subgroup of $K$ such that $K'$ is free. Let $T' \subset T$ be the minimal subtree of $T$ that contains $K'(v)$. Since the map $K' \to T'$ given by $k \to k(v)$ is quasi-isometric and the inclusion map $T' \to T$ is also a quasi-isometric embedding, it follows that the composition $K' \to T' \to T$ is a quasi-isometric embedding. Since $K'$ is a finite index subgroup of $K$, it follows that the map $K \to T$ given by $k \to k(v)$ is a quasi-isometric embedding.

(4) ⇒ (3): By way of contradiction, suppose that there exists an infinite order element $g \in K$ such that $g$ is not Morse in $G$. It follows from Corollary 6.16 that $g$ is conjugate to a vertex group, hence $g$ fixes a vertex $v$ of $T$. By our assumption, there is a vertex $w$ in $T$ such that the map $K \to T$ given by $k \to k(w)$ is a quasi-isometric embedding. It implies that the map $K \to T$ given by $k \to k(v)$ is also a $\lambda$–quasi-isometric embedding for some $\lambda > 0$. Choose an integer $n$ large enough such that $|g^n| > \lambda^2$. We have $1/\lambda |g^n| - \lambda \leq d(g(v), g^n(v)) = d(v, v) = 0$. Hence $|g^n| < \lambda^2$. This contradicts to our choice of $n$. □

References

[Baj07] Jitendra Bajpai. Omnipotence of surface groups. Master’s thesis, McGill University, 2007.

[BBF15] Mladen Bestvina, Ken Bromberg, and Koji Fujiwara. Constructing group actions on quasi-trees and applications to mapping class groups. Publ. Math. Inst. Hautes Études Sci., 122:1–64, 2015.

[BBF19] Mladen Bestvina, Kenneth Bromberg, and Koji Fujiwara. Proper actions on finite products of quasi-trees. arXiv e-prints, page arXiv:1905.10813, May 2019.

[Bes] Mladen Bestvina. Questions in geometric group theory. M. Bestvinas home page, 2004.

[BH99] Martin R. Bridson and André Haefliger. Metric spaces of non-positive curvature, volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999.

[BN08] Jason A. Behrstock and Walter D. Neumann. Quasi-isometric classification of graph manifold groups. Duke Math. J., 141(2):217–240, 2008.

[Bow08] Brian H. Bowditch. Tight geodesics in the curve complex. Invent. Math., 171(2):281–300, 2008.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. Internat. J. Algebra Comput., 22(3):1250016, 66, 2012.

[CK00] Christopher B. Croke and Bruce Kleiner. Spaces with nonpositive curvature and their ideal boundaries. Topology, 39(3):549–556, 2000.

[CK02] C. B. Croke and B. Kleiner. The geodesic flow of a nonpositively curved graph manifold. Geom. Funct. Anal., 12(3):479–545, 2002.

[DGO17] F. Dahmani, V. Guirardel, and D. Osin. Hyperbolically embedded subgroups and rotating families in hyperbolic spaces. Mem. Amer. Math. Soc., 245(1156):v+152, 2017.

[DJ99] A. Dranishnikov and T. Januszkiewicz. Every Coxeter group acts amenably on a compact space. In Proceedings of the 1999 Topology and Dynamics Conference (Salt Lake City, UT), volume 24, pages 135–141, 1999.

[DK18] Cornelia Drutu and Michael Kapovich. Geometric group theory, volume 63 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2018. With an appendix by Bogdan Nica.

[GM14] Craig R. Guilbault and Christopher P. Mooney. Boundaries of Croke–Kleiner-admissible groups and equivariant cell-like equivalence. J. Topol., 7(3):849–868, 2014.

[GMRS98] Rita Gitik, Mahan Mitra, Eliyahu Rips, and Michah Sageev. Widths of subgroups. Trans. Amer. Math. Soc., 350(1):321–329, 1998.

[Gre16] Sebastian G節sing. Virtual boundaries of Hadamard spaces with admissible actions of higher rank. Math. Z., 284(1-2):1–22, 2016.

[HP15] Mark F. Hagen and Piotr Przytycki. Cocompactly cubulated graph manifolds. Israel J. Math., 207(1):377–394, 2015.

[HS13] David Hume and Alessandro Sisto. Embedding universal covers of graph manifolds in products of trees. Proc. Amer. Math. Soc., 141(10):3337–3340, 2013.

[HW08] Frédéric Haglund and Daniel T. Wise. Special cube complexes. Geom. Funct. Anal., 17(5):1551–1620, 2008.

[KL98] M. Kapovich and B. Leeb. 3-manifold groups and nonpositive curvature. Geom. Funct. Anal., 8(5):841–852, 1998.

[Liu13] Yi Liu. Virtual cubulation of nonpositively curved graph manifolds. J. Topol., 6(4):793–822, 2013.

[NTY] Hoang Thanh Nguyen, Hung Cong Tran, and Wenyuan Yang. Quasiconvexity in 3-manifold groups. arXiv:1911.07807. To appear in Mathematische Annalen.
D. Osin. Acylindrically hyperbolic groups. *Trans. Amer. Math. Soc.*, 368(2):851–888, 2016.

Piotr Przytycki and Daniel T. Wise. Graph manifolds with boundary are virtually special. *J. Topol.*, 7(2):419–435, 2014.

Alessandro Sisto. Contracting elements and random walks. *J. Reine Angew. Math.*, 742:79–114, 2018.

Hung Cong Tran. On strongly quasiconvex subgroups. *Geom. Topol.*, 23(3):1173–1235, 2019.

Daniel T. Wise. Subgroup separability of graphs of free groups with cyclic edge groups. *Q. J. Math.*, 51(1):107–129, 2000.

Daniel T. Wise. *The Structure of Groups with a Quasiconvex Hierarchy*, volume AMS-209. Annals of Mathematics Studies, 2020.

Wen-yuan Yang. Statistically convex-cocompact actions of groups with contracting elements. *Int. Math. Res. Not. IMRN*, (23):7259–7323, 2019.

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