Phase diagram for two-layer ReLU neural networks at infinite-width limit

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Summary

How neural network behaves during the training over different choices of hyperparameters is an important question in the study of neural networks. However, except for specific examples with particular choices of hyperparameters, e.g., neural tangent kernel (NTK), mean-field model, this question is largely unanswered. In this work, inspired by the phase diagram in statistical mechanics, we draw the phase diagram for the two-layer ReLU neural network at the infinite-width limit for a complete characterization of its dynamical regimes and their dependence on hyperparameters. Through both experimental and theoretical approaches, we identify three regimes in the phase diagram, i.e., linear regime, critical regime and condensed regime, based on the relative change of input weights as the width approaches infinity, which tends to 0, \( O(1) \) and \( +\infty \), respectively. In the linear regime, NN training dynamics is approximately linear similar to a random feature model with an exponential loss decay. In the condensed regime, we demonstrate through experiments that active neurons are condensed at several discrete orientations. The critical regime serves as the boundary between above two regimes, which exhibits an intermediate nonlinear behavior with the mean-field model as a typical example. Overall, our phase diagram for the two-layer ReLU NN serves as a map for the future studies and is a first step towards a more systematical investigation of the training behavior and the implicit regularization of NNs of different structures.

1 Introduction

It has been widely observed that, given training data, neural networks (NNs) may exhibit distinctive dynamical behaviors during the training, depending on the choices of hyperparameters. As an example, we consider a two-layer NN with \( m \) hidden neurons

\[
 f^\alpha_\theta(x) = \frac{1}{\alpha} \sum_{k=1}^{m} a_k \sigma(w^T_k x),
\]

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where \( x \in \mathbb{R}^d \), \( \alpha \) is the scaling factor, \( \theta = \text{vec}(\{a_k, w_k\}_{k=1}^m) \) is the set of parameters initialized by \( a_k^0 \sim N(0, \beta^2_1) \), \( w_k^0 \sim N(0, \beta^2_2 I_d) \). The bias term \( b_k \) can be incorporated by expanding \( x \) and \( w_k \) to \( (x^\top, 1)^\top \) and \( (w_k^\top, b_k)^\top \). At the infinite-width limit \( m \to \infty \), given \( \beta_1, \beta_2 \sim O(1) \), for \( \alpha \sim \sqrt{m} \), the gradient flow of NN can be approximated by a linear dynamics of neural tangent kernel (NTK) \([1, 2, 3]\), whereas for \( \alpha \sim m \), gradient flow of NN exhibits highly nonlinear mean-field dynamics \([4, 5, 6, 7]\). The current situation of NN study is similar to an early era of statistical mechanics, when we observe different states of a matter at several discrete conditions without the guidance of a unified phase diagram.

In this work, we present the first phase diagram for the two-layer neural networks with rectified linear units (ReLU NN). To this end, two difficulties need to be overcome. The first difficulty is that one can not identify sharply distinctive regimes/states required for a phase diagram with finite neurons. This situation is similar to the analysis in statistical mechanics, e.g., Ising model, where phase transition can not happen with finite particles. Therefore, in analogy to the thermodynamic limit, we take the infinite-width limit \( m \to \infty \) as our starting point and successfully identify three dynamical regimes of NNs, i.e., linear regime, critical regime, and condensed regime. In the linear regime, \( \theta_w \) almost does not change and NN training dynamics can be linearized around the initialization similar to an NTK or a random feature model. In the condensed regime, the relative change of \( \theta_w \) tends to infinity and is condensed at several discrete directions in the feature space. In the critical regime, which serves as the boundary between above two regimes, relative change of \( \theta_w \) is \( O(1) \) with the mean-field model as an example. The second difficulty is the identification of phase diagram coordinates. For the vanilla gradient flow training dynamics of NN in Eq. (1), there are three hyperparameters \( \alpha, \beta_1 \) and \( \beta_2 \), which in general are functions of \( m \). However, through appropriate rescaling and normalization of the gradient flow dynamics, which accounts for the dynamical similarity up to a time scaling, we arrive at two independent coordinates

\[
\gamma = \lim_{m \to \infty} -\frac{\log \beta_1 \beta_2 / \alpha}{\log m}, \quad \gamma' = \lim_{m \to \infty} -\frac{\log \beta_1 / \beta_2}{\log m}.
\]

The resulting phase diagram is shown in Fig. 1. Examples studied in previous literature are also marked, for example, Ref. [8] studied NNs with settings represented by the red dashed line.

This phase diagram is obtained through experimental and theoretical approaches. We first present an intuitive scaling analysis to provide a rationale for the boundary that separates the linear regime and the condensed regime. Then, we experimentally demonstrate the transition across this boundary in the phase diagram for an 1-d dataset. Finally, we establish a rigorous theory for general datasets.

Our work is a first step towards a systematical effort in drawing the phase diagrams for NNs of different structures. With the guidance of these phase diagrams, detailed experimental and theoretical works can be done to further characterize the dynamical behavior and the corresponding implicit regularization effect at each of the identified regime.

2 Related works

The study of regimes in the literature usually revolves around the choice of scaling factor \( \alpha \) in specific power-law relations to the width \( m \). For example, the NTK scaling \( \alpha \sim \sqrt{m} \) \([1, 2, 3]\) and the mean-field scaling \( \alpha \sim m \) \([4, 5, 6, 7]\) has been studied extensively. In [9], the authors identify the lazy training behavior for \( \lim_{m \to \infty} m / \alpha = \infty \), by which NN parameters stay close to
Figure 1: Phase diagram of two-layer ReLU NNs at infinite-width limit. The marked examples are studied in existing literature (see Table 1 for details.)
initialization during the training. In [10], for two-layer ReLU network, lazy and active regimes and their corresponding regularization effect are studied for 1-d problems. Their analysis uses different quantities for regime separation, which cannot serve as coordinates for a phase diagram. All above works do not account for the effect of specific power-law scaling of initialization over different layers used in practice.

In [8], for two-layer NNs with $\alpha = 1$, $\beta_2 \sim O(1)$, the authors study the effect of $\beta$ ($\sim \beta_1$) in relation to $m$. Specifically, they prove that NN training dynamics can be linearized for $\beta = o(m^{-1/6})$ as $m \to \infty$, which constitutes a line in Fig. 1. In [11], they further study such cases in the under-parameterized and mildly over-parameterized settings and experimentally identified the quenching-activation behavior for finite $m$, which phenomenologically is closely related to the condensed regime we identified at $m \to \infty$.

Another work related to the condensed regime is [12]. The authors study the two-layer ReLU NNs and prove that, as the initialization of parameters goes to zero, a quantization effect emerges, that is, the weight vectors tend to concentrate at a small number of orientations determined by the input data at an early stage of training. However, the limit of $m \to \infty$ is not considered in their work.

3 Rescaling and the normalized model

Identification of the coordinates is important for drawing the phase diagram. Unlike in some thermodynamic systems where temperature and pressure are natural choices, for NNs, it is not obvious which quantities of hyperparameters are keys to the regime separation. However, there are some guiding principles for finding the coordinates of a phase diagram at $m \to \infty$:

(i) They should be effectively independent.

(ii) Given a specific coordinate in the phase diagram, the learning dynamics of all the corresponding NNs statistically should be similar up to a time scaling.

(iii) They should well differentiate dynamical differences except for the time scaling.

Guided by above principles, in this section, we perform the following rescaling procedure for a fair comparison between different choices of hyperparameters and obtain a normalized model with two independent quantities irrespective of the time scaling of the gradient flow dynamics. We start with the original model (1)

$$f^\theta_\alpha(x) = \frac{1}{\alpha} \sum_{k=1}^{m} a_k \sigma(\mathbf{w}_k^\top \mathbf{x}),$$

(3)
defined on a given sample set $S = \{(\mathbf{x}_i, y_i)\}_{i=1}^{n}$ where $\mathbf{x}_i \in \mathbb{R}^d$, $i \in [n]$, network width $m$ and a scaling parameter $1/\alpha$ and $\sigma = \text{ReLU}$. The parameters are initialized by

$$a^0_k \sim N(0, \beta_1^2), \quad \mathbf{w}_k^0 \sim N(0, \beta_2^2 \mathbf{I}_d),$$

(4)

where $a_k$ and $\mathbf{w}_k$ are separated into to different scales $\beta_1$ and $\beta_2$. The empirical risk is

$$R_S(\theta) = \frac{1}{2m} \sum_{i=1}^{n} (f^\theta_\alpha(\mathbf{x}_i) - y_i)^2.$$
Then the training dynamics based on gradient descent (GD) at the continuous limit obeys the following gradient flow of $\theta$,

$$\frac{d\theta}{dt} = -\nabla_\theta R_S(\theta).$$  \hspace{1cm} (6)

More precisely, $\theta = \text{vec}((q_k)_{k=1}^m)$ with $q_k = (a_k, w_k^\top)^\top$, $k \in [m]$ solves

$$\frac{da_k}{dt} = -\frac{1}{n} \sum_{i=1}^n \frac{1}{\alpha} \sigma(w_k^\top x_i) \left( \frac{1}{\alpha} \sum_{k=1}^m a_k \sigma(w_k^\top x_i) - y_i \right)$$

$$\frac{dw_k}{dt} = -\frac{1}{n} \sum_{i=1}^n \frac{1}{\alpha} a_k \sigma'(w_k^\top x_i) x_i \left( \frac{1}{\alpha} \sum_{k=1}^m a_k \sigma(w_k^\top x_i) - y_i \right).$$

Let

$$\bar{a}_k = \beta_1^{-1} a_k, \quad \bar{w}_k = \beta_2^{-1} w_k, \quad \bar{t} = \frac{1}{\beta_1 \beta_2} t,$$

then

$$\frac{d\bar{a}_k}{dt} = \frac{\beta_2}{\beta_1} \left( \frac{1}{n} \sum_{i=1}^n \frac{\beta_1 \beta_2}{\alpha} \sigma(\bar{w}_k^\top x_i) \right) \left( \frac{\beta_1 \beta_2}{\alpha} \sum_{k=1}^m \bar{a}_k \sigma(\bar{w}_k^\top x_i) - y_i \right),$$

$$\frac{d\bar{w}_k}{dt} = \frac{\beta_1}{\beta_2} \left( \frac{1}{n} \sum_{i=1}^n \frac{\beta_1 \beta_2}{\alpha} \bar{a}_k \sigma'(\bar{w}_k^\top x_i) x_i \right) \left( \frac{\beta_1 \beta_2}{\alpha} \sum_{k=1}^m \bar{a}_k \sigma(\bar{w}_k^\top x_i) - y_i \right).$$

We introduce two scaling parameters

$$\kappa := \frac{\beta_1 \beta_2}{\alpha}, \quad \kappa' := \frac{\beta_1}{\beta_2},$$

where $\kappa$ and $\kappa'$ are called the energetic scaling parameter and the dynamical scaling parameter, respectively. Then the above dynamics can be written as

$$\frac{d\bar{a}_k}{dt} = -\frac{1}{\kappa} \left( \frac{1}{n} \sum_{i=1}^n \kappa \sigma(\bar{w}_k^\top x_i) \right) \left( \kappa \sum_{k=1}^m \bar{a}_k \sigma(\bar{w}_k^\top x_i) - y_i \right),$$

$$\frac{d\bar{w}_k}{dt} = -\kappa' \left( \frac{1}{n} \sum_{i=1}^n \kappa \bar{a}_k \sigma'(\bar{w}_k^\top x_i) x_i \right) \left( \kappa \sum_{k=1}^m \bar{a}_k \sigma(\bar{w}_k^\top x_i) - y_i \right).$$

The above rescaled dynamics can be treated as a weighted gradient flow of NN scaled by $\kappa$ equipped with the empirical risk

$$f_\theta^\kappa(x) = \kappa \sum_{k=1}^m \bar{a}_k \sigma(\bar{w}_k^\top x),$$

$$R_{S,\kappa}(\theta) = \frac{1}{2n} \sum_{i=1}^n (f_\theta^\kappa(x_i) - y_i)^2,$$

with the following initialization

$$\bar{a}_j^0 \sim N(0, 1), \quad \bar{w}_j^0 \sim N(0, I_d).$$  \hspace{1cm} (11)
where we can see they are of standard normal distributions. The weighted GD dynamics then can be written simply as
\[
\frac{dq_j}{dt} = -M_{\kappa'} \nabla q_k R_{S,\kappa}(\bar{\theta}),
\]
where the mobility matrix
\[
M_{\kappa'} = \left( \frac{1}{\kappa'} \kappa' I_d \right).
\]
In the following discussion throughout this paper, we will refer to this rescaled model \((\ref{eq:rescaled_model})\) as normalized model and drop superscript \(\kappa\) and all the “bar”s of \(a_k, w_k, t\) for simplicity. Note that \(\kappa\) and \(\kappa'\) do not follow principle (ii) and (iii) above at infinite-width limit. They are in general functions of \(m\), which attains 0, \(O(1)\), +\(\infty\) at \(m \to \infty\). For example, \(\kappa = 0\) and \(\kappa' = 1\) for both the NTK and mean-field model, however, they are known to have distinctive training behaviors. To account for such dynamical difference under different widely considered power-law scalings of \(\alpha, \beta_1\) and \(\beta_2\) shown in Table 1, we arrive at
\[
\gamma = \lim_{m \to \infty} -\frac{\log \kappa}{\log m}, \quad \gamma' = \lim_{m \to \infty} -\frac{\log \kappa'}{\log m},
\]
which meets all above principles as demonstrated later by theory and experiments.

**Remark 1.** We remark that the above rescaling technique can be viewed in analogy to the nondimensionalization in physics, which is the partial or full removal of physical dimensions from an equation involving physical quantities by a suitable substitution of variables. In more general point of view, nondimensionalization can also recover characteristic properties of a system, which in our case recovers the different behaviors of training dynamics for different regimes.

More specifically, we can view, in the original model \((\ref{eq:original_model})\), \(q_k = (a_k, w_k^\top)^\top, k \in [m]\) as the generalized coordinates which have the unit of “length” denoted as [L]. Then in the two-layer NN \((\ref{eq:two_layer_model})\), \(\alpha\) should have the unit of “volume” as a normalization factor depending on \(m\) to avoid blowing up of the model. Particularly, if \(\sigma\) is ReLU then we can think \(\alpha\)’s unit is [L]^2 (unit of area on a plane).

Finally, following above analysis, \(\kappa = \frac{\beta_1 \beta_2}{\alpha}\) and \(\kappa' = \frac{\beta_1}{\beta_2}\) are two nondimensional parameters (without unit) so as for \(\gamma\) and \(\gamma'\), which are suitable to serve as the coordinations of our phase diagram.

**Remark 2.** Here we list some commonly-used initialization methods and/or related works with their scaling parameters as shown in Table 4.

### 3.1 Typical cases over the phase diagram

With \(\gamma\) and \(\gamma'\) as coordinates, in this subsection, we illustrate through experiments the behavior of a diversity of typical cases over the phase diagram using a simple 1-d problem of 4 training points, which allows easy visualization.

The first row in Fig. 2 shows typical learning results over different \(\gamma'\)’s, from a relatively jagged interpolation (NTK scaling) to a smooth cubic-spline-like interpolation (mean-field scaling) and further to a linear spline interpolation. To probe into details of their parameter space representation, we notice for the ReLU activation that the parameter pair \((a_k, w_k)\) of each neuron can be separated into a unit orientation feature \(\hat{w} = w/\|w\|_2\) and an amplitude \(A = |a|\|w\|_2\) indicating
Table 1: Initialization methods with their scaling parameters

| Name [related works] | $\alpha$ | $\beta_1$ | $\beta_2$ | $\gamma$ | $\gamma'$ |
|----------------------|---------|---------|---------|---------|---------|
| LeCun [13]           | 1       | $\sqrt{\frac{m}{\pi}}$ | $\sqrt{\frac{\beta}{\pi}}$ | $\frac{\pi}{2}$ | $\frac{3}{2}$ |
| He [14]              | 1       | $\sqrt{\frac{2}{m}}$ | $\sqrt{\frac{2}{\pi}}$ | $\frac{4}{2\pi}$ | $\frac{3}{2}$ |
| Xavier [15]          | 1       | $\sqrt{\frac{2}{m+1}}$ | $\sqrt{\frac{2}{m+\pi}}$ | $\sqrt{\frac{m+1}{(m+1)(m+\pi)}}$ | $\frac{1}{2}$ |
| NTK [1]              | $\frac{1}{\sqrt{m}}$ | 1       | 1       | $\frac{1}{m}$ | 0       |
| Mean-field [4, 7, 5] | $m$     | 1       | 1       | $\frac{1}{\beta}$ | 1       |
| E et al. [8]         | $\beta$ | 1       | $\beta'$ | $\beta$ | $\frac{1}{\log_2 m}$ | $\frac{1}{\log_2 m}$ |

4 Phase diagram

In this section, with $\gamma$ and $\gamma'$ as coordinates, we characterize at $m \to \infty$ the dynamical regimes of NNs and identify their boundaries in the phase diagram through experimental and theoretical approaches. How to characterize and classify different types of training behaviors of NNs is an important open question. Currently, a behavior of NN dynamics, by which gradient flow of the NN can be effectively linearized around initialization during the training, has been extensively studied both empirically and theoretically [1, 16, 2, 8]. We refer to the regime with this behavior as the linear regime. As shown in Fig. 1, many works have proved that a specific point or line in the phase diagram belong to the linear regime. However, its exact range in the phase diagram remains unclear. On the other hand, NN training dynamics can also be highly nonlinear at $m \to \infty$ as widely studied for the mean-field model as a point shown in the phase diagram [4, 7, 5]. However, whether there are other points in the phase diagram that has similar training behavior is not well understood. In addition, it is not clear if there are other regimes in the phase diagram that are nonlinear but behaves distinctively comparing to the mean-field model. In the following, we will address these problems and draw the phase diagram.

4.1 Regime identification and separation

The linear regime refers to the set of coordinates with which the gradient flow of $f_{\theta}$ at any $t$ is well approximated by gradient flow of its linearized model, i.e.,

$$f_{\theta}^{\text{lin}} = \nabla_{\theta} f_{\theta(0)} \cdot (\theta(t) - \theta(0)).$$  \hspace{1cm} (15)

Note that, the zeroth order term $f_{\theta(0)}$ does not appear because, without loss of generality, it is always offset to 0 by the ASI trick to eliminate the extra generalization error induced by a random
Figure 2: Learning four data points by two-layer ReLU NNs with different $\gamma$'s are shown in the first row. The corresponding scatter plots of initial (cyan) and final (red) $\{(A_k, \hat{w}_k)\}_{k=1}^m$ are shown in the second row. $\gamma' = 0$ ($\beta_1 = \beta_2 = 1$), hidden neuron number $m = 1000$.

initial function as studied in [3]. In general, this linear behavior only happens when $\theta(t)$ always stays within a small neighbourhood of $\theta(0)$ such that the first order Taylor expansion is a good approximation. For a two-layer NN, because its output layer is always linear w.r.t. output weights, this requirement of small neighbourhood is reduced to the one for the input weights, that is, $\theta_w(t)$ always stays within a neighbourhood of $\theta_w(0)$. Since the size of this neighbourhood of good linear approximation scales with $\|\theta_w(0)\|_2$, therefore we use the following relative distance as an indicator of how far $\theta_w(t)$ deviates from $\theta_w(0)$ during the training

$$RD(\theta_w(t)) = \frac{\|\theta_w(t) - \theta_w(0)\|_2}{\|\theta_w(0)\|_2}. \quad (16)$$

Specifically, we focus on quantity $\sup_{t \in [0, +\infty)} RD(\theta_w(t))$, which is the maximum deviation of $\theta_w(t)$ from initialization during the training. As $m \to \infty$, if $\sup_{t \in [0, +\infty)} RD(\theta_w(t)) \to 0$, then the NN training dynamics falls into the linear regime. Otherwise, if it approaches $O(1)$ or $+\infty$, then NN training dynamics is nonlinear. Note that, for the latter case, in which $\theta_w$ deviates infinitely far away from initialization, a very strong nonlinear dynamical behavior of condensation in feature space can be observed as illustrated in Fig. 2f. We refer to the regime of $\sup_{t \in [0, +\infty)} RD(\theta_w(t)) \to +\infty$ the condensed regime, which is justified latter in Sec. 4.2 by detailed experiments. For $\sup_{t \in [0, +\infty)} RD(\theta_w(t)) \to O(1)$, NNs exhibit an intermediate level of nonlinear behavior. We refer to this regime as the critical regime.

In the following, we will separate exactly the linear regime and the condensed regime in the phase diagram through experimental and theoretical approaches. We first present an intuitive
scaling analysis to provide a rationale for the boundary that separates these two regimes in the phase diagram. Then, we experimentally demonstrate the validity of this boundary in regime separation in the phase diagram for an 1-d dataset. Finally, we establish a rigorous theory which proves the transition across this boundary for two-layer ReLU NNs at \( m \to \infty \) for general datasets.

4.1.1 Intuitive scaling analysis

Before we jump into a detailed analysis, through an intuitive scaling analysis, we first illustrate the separation between the linear regime and the condensed regime. The capability, i.e., the magnitude of target function that can be fitted, of the two-layer ReLU NN around initialization can be roughly estimated as

\[
C = m\beta_1 \beta_2 / \alpha = mk.
\]

Without loss of generality, the target function is always \( O(1) \). Therefore, a necessary condition for the linear regime is that NN has the capability of fitting the target in the vicinity of initialization, i.e., \( C \gtrsim O(1) \). Therefore,

\[
\kappa \gtrsim 1/m,
\]

yielding \( \gamma \leq 1 \) at \( m \to \infty \). We further notice that, the output layer is always linear. Therefore, even when the output weight \( \theta_o \) changes significantly, the dynamics can still be linearized if the input layer weight \( \theta_w \) stays in the vicinity of its initialization. As indicated by the dynamics Eq. (12), this is possible when (i) \( \kappa' \ll 1 \) at initialization and (ii) the scale of \( a \), say quantified by expectation \( E(|a|) \), satisfies \( E(|a|) \ll \beta_2 \) throughout the training. In this case, at the end of the training,

\[
C = m\beta_2 E(|a|) / \alpha \ll m\beta_2^2 / \alpha = mk / \kappa'.
\]

(17)

Because \( C \gtrsim O(1) \), we got

\[
1/\kappa' \gg 1/mk,
\]

(18)

which yields the condition \( \gamma' > \gamma - 1 \) for \( \gamma' > 0 \) at \( m \to \infty \).

In contrary, if \( \gamma' < \gamma - 1 \) and \( \gamma > 1 \), i.e., \( mk \ll 1 \) and \( mk / \kappa' \ll 1 \) as \( m \to \infty \), then the NN has no capability in fitting a \( O(1) \) target when \( \theta_w \) stays at the vicinity of its initialization. The capability of NN must undergo a magnificent increase to be able to fit the data, which is a feature of the condensed regime.

Above scaling analysis provides an intuitive argument about the separation of linear and condensed regimes by the boundary \( \gamma = 1 \) for \( \gamma' \leq 0 \) and \( \gamma' = \gamma - 1 \) for \( \gamma' > 0 \) in the phase diagram. To further demonstrate the criticality of this boundary, we sort to the following experimental studies for a specific case.

4.1.2 Experimental demonstration

To experimentally distinguish the linear and nonlinear regimes, we need to estimate \( \sup_{t \in [0, +\infty)} \text{RD}(\theta_w(t)) \), which empirically can be approximated by \( \text{RD}(\theta_w^*) \) (\( \theta_w^* := \theta_w(\infty) \)) without loss of generality. Next, because we can never run experiments at \( m \to \infty \), we alternatively quantify the growth of \( \text{RD}(\theta_w^*) \) as \( m \to \infty \). By Fig. 3 (a-c), they approximately have a power-law relation. Therefore we define

\[
S_w = \lim_{m \to \infty} \frac{\log \text{RD}(\theta_w^*)}{\log m},
\]

(19)
Figure 3: Growth of $\text{RD}(\theta^*_w)$ w.r.t. $m \to \infty$ with $\gamma' = 0$. For (a-c), the plot is $\text{RD}(\theta^*_w)$ vs. $m$ of NNs with 1000, 5000, 10000, 20000, 40000 hidden neurons indicated by five blue dots, respectively. The gray line is a linear fit with slope indicated. For (d), the plot is $S_w$ vs. $\gamma$ for $\gamma' = 0$. Each line is for a pair of $\beta_1$ and $\beta_2$: Blue: $\beta_1 = 1$, $\beta_2 = 1$; Orange: $\beta_1 = m^{-1/2}$, $\beta_2 = m^{-1/2}$; Blue: $\beta_1 = m^{-1}$, $\beta_2 = m^{-1}$.

Figure 4: $S_w$ estimated on NNs of 1000, 5000, 10000, 20000, 40000 neurons over $\gamma$ (ordinate) and $\gamma'$ (abscissa). The stars are zero points obtained by the linear interpolation over different $\gamma$ for each fixed $\gamma'$. Dashed lines are auxiliary lines indicating the theoretically obtained boundary. which is empirically obtained by estimating the slope in the log-log plot like in Fig. 3. As shown in Fig. 3(d), NNs with the same pair of $\gamma$ and $\gamma'$, but different $\alpha$, $\beta_1$, and $\beta_2$, have very similar $S_w$, which validates the effectiveness of the normalized model. In the following experiments, we only show result of one combination of $\alpha$, $\beta_1$, and $\beta_2$ for a pair of $\gamma$ and $\gamma'$.

Then, we visualize the phase diagram by experimentally scanning $S_w$ over the phase space. The result for the same 1-d problem as in Fig. 2 is presented in Fig. 4. In the red zone, where $S_w$ is less than zero, $\text{RD}(\theta^*_w) \to 0$ as $m \to \infty$, indicating a linear regime. In contrast, in the blue zone, where $S_w$ is greater than zero, $\text{RD}(\theta^*_w) \to \infty$ as $m \to \infty$, indicating a highly nonlinear behavior. Their boundary are experimentally identified through interpolation indicated by stars in Fig. 4, where $\text{RD}(\theta^*_w) \sim O(1)$. They are close to the boundary identified through the scaling analysis indicated by the auxiliary lines, justifying its criticality.
4.1.3 Theoretical results for general two-layer ReLU NNs

The intuitive scaling analysis and the experimental demonstration result in a consistent boundary to separate the linear and condensed regimes. A question naturally arises—is there a theory that makes the intuitive scaling analysis rigorous and generalizes above empirical phase diagram for an 1-d example to general high-dimensional data for two-layer ReLU NNs. In the following, we address this question by providing two theorems in informal statements, which proves the criticality of \( \lim_{m \to +\infty} \sup_{t \in [0, +\infty)} \text{RD}(\theta_w(t)) \) at above identified boundary in the phase diagram. Their rigorous statements can be found in Section 5.

**Theorem 1*.** (Informal statement of Theorem 1) If \( \gamma < 1 \) or \( \gamma' > \gamma - 1 \), then with a high probability over the choice of \( \theta^0 \), we have

\[
\lim_{m \to +\infty} \sup_{t \in [0, +\infty)} \text{RD}(\theta_w(t)) = 0.
\]

(20)

**Theorem 2*.** (Informal statement of Theorem 2) If \( \gamma > 1 \) and \( \gamma' < \gamma - 1 \), then with a high probability over the choice of \( \theta^0 \), we have

\[
\lim_{m \to +\infty} \sup_{t \in [0, +\infty)} \text{RD}(\theta_w(t)) = +\infty.
\]

(21)

**Remark 3.** \( \lim_{m \to +\infty} \sup_{t \in [0, +\infty)} \text{RD}(\theta_w(t)) \) is like an order parameter in the analysis of phase transition in statistical mechanics, which is key to the regime separation and exhibits discontinuity at the boundary.

In Theorem 1*, focusing on the linear regime, the negligible relative change of \( w \) is essentially proved by showing the kernel of the training dynamics undergoes no significant change during the whole dynamics. However, the kernel of the training dynamics might be out of control for the condensed regime. This difficulty makes the result of Theorem 2* nontrivial. Instead of studying the kernel, more detailed information of the dynamics should be used. Indeed, we establish a neural-wise estimate, \( |a_k(t)| \lesssim \frac{1}{\kappa'} \|w_k(t)\|_2 \), which holds for any \( \kappa, \kappa' \) and any \( t \geq 0 \). We believe that this estimate can be extended to other network structures and general activation functions for the regimes of nonlinear dynamics.

Above two theorems complete the phase diagram of two-layer ReLU NN with distinctive dynamical regimes separated based on \( \lim_{m \to +\infty} \sup_{t \in [0, +\infty)} \text{RD}(\theta_w(t)) \). The behavior of NN in the linear regime, e.g., exponential decay of loss, implicit regularization in terms of a RKHS norm, and etc., is very well studied. However, the critical and condense regimes is largely not understood. In the following, we make a further step to unravel a signature nonlinear behavior—condensation as \( m \to \infty \) through experiments, which sheds light on future theoretical study.

4.2 Critical and condensed regimes

By Fig. 2(d-f) in previous section, it can be observed that the condensation of NN representation in feature space \( \{(A_k, \hat{w}_k)\}_{k=1}^m \) comparing to initialization is a distinctive feature for the nonlinear training dynamics of NNs. Specifically, we care about this condensation at the \( m \to \infty \) limit when relative change of \( \theta_w \) approaches \( +\infty \). Therefore, using the same 1-d data as in Fig. 2, we scan the
learned distribution of \((A_k, \hat{w}_k)\) pair for \(m = 10^3, 10^4, 10^6\) over the phase diagram to experimentally find out the limiting behavior. The result is shown in Fig. 5. It is easy to observe that, right to the boundary indicated by blue boxes, the condensation becomes stronger as \(m \to \infty\), implying a \(\delta\)-like condensation behavior at the limit. This conforms with our intuition that the farther away \(\theta_w\) deviates from initialization, the stronger nonlinearity of NNs exhibited here in the form of condensation. Therefore, as introduced before, we refer to this regime as the condensed regime. In the critical regime as the boundary between the linear and the condensed regimes, the level of condensation is almost fixed as \(m \to \infty\), which resembles a mean-field behavior. Indeed, the well-studied mean-field model is one point in the critical regime shown in the phase diagram Fig. 1.

In general, the mechanism of condensation as well as its implicit regularization effect is not well understood, which remain as important open questions for the future research.

5 Theoretical characterization of the linear regime

We illustrate above how our phase diagram Fig. 1 is obtained through experimental and theoretical approaches. To obtain a more detailed understanding of general properties of these regimes, we present our theoretical results in detail in this section, which follows a rigorous description of our notations and definitions in the beginning. The proofs can be found in the appendix.

To start with, let us consider a two layer neural network

\[
f_{\theta}(x) := \frac{1}{\kappa} f^\kappa_{\theta}(x) = \sum_{k=1}^{m} a_k \sigma(\hat{w}_k^T x), \quad (22)
\]

with the activation function \(\sigma(z) = \text{ReLU}(z) = \max(z, 0)\). Denote the dataset

\[
S = \{(x_i, y_i)\}_{i=1}^{n}, \quad (23)
\]

where \(x_i\)'s are i.i.d. sampled from the (unknown) distribution \(D\) over \(\Omega = [0, 1]^d\) with \((x_i)_d = 1\) and \(y_i = f(x_i) \in [0, 1]\) for all \(i \in [n]\).

We denote \(e_i = \kappa f_{\theta}(x_i) - y_i = \kappa f_{\theta}(x_i) - f(x_i), i \in [n]\) and \(e = (e_1, e_2, \ldots, e_n)^T\). Then the empirical risk can be written as

\[
R_S(\theta) := R_{S,\kappa}(\theta) = \frac{1}{2n} \sum_{i=1}^{n} (\kappa f_{\theta}(x_i) - y_i)^2 = \frac{1}{2n} e^T e. \quad (24)
\]

Its gradient descent (GD) dynamics is

\[
\dot{\theta} = -M_{\kappa} \nabla_{\theta} R_{S}(\theta), \quad (25)
\]

with a more explicit form for \(a_k\) and \(w_k\) respectively

\[
\begin{align*}
\dot{a}_k &= -\frac{1}{\kappa'} \nabla_{a_k} R_{S}(\theta) = -\frac{\kappa}{\kappa' n} \sum_{i=1}^{n} e_i \sigma(\hat{w}_k^T x_i), \\
\dot{w}_k &= -\kappa' \nabla_{w_k} R_{S}(\theta) = -\frac{\kappa \kappa'}{n} \sum_{i=1}^{n} e_i a_i \sigma'(\hat{w}_k^T x_i x_i). 
\end{align*} \quad (26)
\]
Figure 5: Condensation map. Each color in each box is a scatter of \(\{ (A_k, \hat{w}_k) \}_{k=1}^m \) for a NN with corresponding \( \gamma \) and \( \gamma' \). The hidden neuron numbers are: \( m = 10^3 \) (blue), \( 10^4 \) (red), \( 10^6 \) (yellow). The abscissa coordinate is \( \gamma \) and the ordinate one is \( \gamma' \).
Here $\kappa, \kappa'$ are scaling parameters proposed in Section 3. The parameters are initialized as
\begin{align}
    a_k^0 := a_k(0) & \sim N(0, 1), \\
    w_k^0 := w_k(0) & \sim N(0, I_d), \\
    \theta^0 := \theta(0) & = \text{vec}(\{a_k^0, w_k^0\}_{k=1}^m).
\end{align}

The kernels $k^{[a]}$ and $k^{[w]}$ of the GD dynamics are
\begin{align}
    k^{[a]}(x, x') = E_w \sigma(w^T x) \sigma(w^T x'), \\
    k^{[w]}(x, x') = E_{(a, w)} a^2 \sigma'(w^T x) \sigma'(w^T x') x \cdot x'.
\end{align}

The Gram matrices $K^{[a]}$ and $K^{[w]}$ of an infinite width two-layer network are
\begin{align}
    K^{[a]}_{ij} = k^{[a]}(x_i, x_j), & \quad K^{[a]} = (K^{[a]}_{ij})_{n \times n}, \\
    K^{[w]}_{ij} = k^{[w]}(x_i, x_j), & \quad K^{[w]} = (K^{[w]}_{ij})_{n \times n}.
\end{align}

The Gram matrices $G^{[a]}$, $G^{[w]}$, and $G$ of a finite width two-layer network have the following expressions
\begin{align}
    G^{[a]}_{ij}(\theta) &= \frac{1}{\kappa m} \sum_{k=1}^m \nabla a_k \kappa f_\theta(x_i) \cdot \nabla a_k \kappa f_\theta(x_j) = \frac{\kappa^2}{\kappa m} \sum_{k=1}^m \sigma(w_k^T x_i) \sigma(w_k^T x_j), \\
    G^{[w]}_{ij}(\theta) &= \frac{\kappa'}{m} \sum_{k=1}^m \nabla w_k \kappa f_\theta(x_i) \cdot \nabla w_k \kappa f_\theta(x_j) = \frac{\kappa^2 \kappa'}{m} \sum_{k=1}^m \sigma^2(w_k^T x_i) \sigma'(w_k^T x_j) x_i \cdot x_j, \\
    G &= G^{[a]} + G^{[w]}.
\end{align}

**Assumption 1.** Suppose that the Gram matrices are strictly positive definite. In other words,
\begin{equation}
    \lambda := \min \{\lambda_a, \lambda_w\} > 0,
\end{equation}
where
\begin{equation}
    \lambda_a := \lambda_{\text{min}} \left( K^{[a]} \right), \quad \lambda_w := \lambda_{\text{min}} \left( K^{[w]} \right).
\end{equation}

**Assumption 2.** Suppose that the following limits exist
\begin{equation}
    \gamma := \lim_{m \to \infty} - \frac{\log \kappa}{\log m}, \quad \gamma' := \lim_{m \to \infty} - \frac{\log \kappa'}{\log m}.
\end{equation}

**Remark 4.** We expect that
\begin{equation}
    G^{[a]}(\theta^0) \approx \frac{\kappa^2}{\kappa'} K^{[a]}, \quad G^{[w]}(\theta^0) \approx \kappa \kappa' K^{[w]},
\end{equation}
and these will be rigorously achieved in the following proofs. We also remark that $\lambda \leq d$, which will be used in the following proofs.

**Remark 5.** When $\gamma \leq \frac{1}{2}$, we consider NNs with non-zero initial parameters and zero initial output, which can be achieved in NNs by applying the AntiSymmetrical Initialization (ASI) trick 3.
Our main results are as follows.

**Theorem 1** (linear regime). Given $\delta \in (0, 1)$ and the sample set $S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega$ with $x_i$'s drawn i.i.d. from some unknown distribution $D$. Suppose that Assumption 1 and Assumption 2 hold. ASI is used when $\gamma \leq \frac{1}{2}$. Suppose that $\gamma < 1$ or $\gamma' > \gamma - 1$ and the dynamics $(24)-(29)$ is considered. Then for sufficiently large $m$, with probability at least $1 - \delta$ over the choice of $\theta^0$, we have

(a) (changes of $\theta$ and $\theta_w$)

$$\sup_{t \in [0, +\infty)} \|\theta_w(t) - \theta_w^0\|_2 \leq \sup_{t \in [0, +\infty)} \|\theta(t) - \theta^0\|_2 \lesssim \frac{1}{\sqrt{m\kappa}} \log m. \quad (37)$$

(b) (linear convergence rate)

$$R_S(\theta(t)) \leq \exp \left( - \frac{2m\kappa^2\lambda t}{n} \right) R_S(\theta^0). \quad (38)$$

Moreover, for sufficiently large $m$, with probability at least $1 - \delta - 2\exp \left( - \frac{C_0 m(d+1)}{\kappa^2} \right)$ over the choice of $\theta^0$, we have

(c) (relative change of $\theta$)

$$\sup_{t \in [0, +\infty)} \frac{\|\theta(t) - \theta^0\|_2}{\|\theta^0\|_2} \lesssim \frac{1}{m\kappa} \log m. \quad (39)$$

In particular, if $\gamma < 1$, $\sup_{t \in [0, +\infty)} \frac{\|\theta(t) - \theta^0\|_2}{\|\theta^0\|_2} < 1$.

(d) (relative change of $\theta_w$)

$$\sup_{t \in [0, +\infty)} RD(\theta_w(t)) = \sup_{t \in [0, +\infty)} \frac{\|\theta_w(t) - \theta_w^0\|_2}{\|\theta_w^0\|_2} \lesssim \begin{cases} \frac{1}{m\kappa} \log m, & \gamma < 1, \\ \frac{\kappa'}{m\kappa} \log m, & \gamma' > \gamma - 1. \end{cases} \quad (40)$$

In particular, if either $\gamma < 1$ or $\gamma' > \gamma - 1$, $\sup_{t \in [0, +\infty)} RD(\theta_w(t)) = \sup_{t \in [0, +\infty)} \frac{\|\theta_w(t) - \theta_w^0\|_2}{\|\theta_w^0\|_2} << 1$.

**Remark 6.** In the regions $\gamma < 1$ or $\gamma' > \gamma - 1$, Theorem [7] shows that with a high probability over the initialization, the relative changes of $w_k$'s are negligible. This implies that the features change only slightly during the whole gradient descent dynamics. Therefore, in this regime and with large width $m$, one can expect the training result to be close to that of some proper linear regression model. Note that the relative change of $\theta$ is negligible only in the sub-region $\gamma < 1$. For $\gamma \geq 1$ and $\gamma' > \gamma - 1$, the relative changes of $a_k$'s can be fairly large, which may lead to unbounded relative change of $\theta$. The relative changes of $\theta$ and $a_k$'s are also empirically validated in Appendix D.

In order to obtain the theorem that characterize the condensed regime, we need further assumption as follows,
Assumption 3. We assume that, without loss of generality,

\[
\max_{i \in [n]} y_i \geq \frac{1}{2},
\]

and that the neural network can be well-trained to the empirical risk less than \(O\left(\frac{1}{n}\right)\). More quantitively, we require that there exists a \(T^* > 0\) such that

\[
R_S(\theta(T^*)) \leq \frac{1}{32n}.
\]

Then we can get the following theorem

**Theorem 2** (condensed regime). The sample set \(S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega\) with \(x_i\)'s drawn i.i.d. from some unknown distribution \(D\). Suppose that Assumption 1, Assumption 2, and Assumption 3 hold. Suppose that \(\gamma > 1\) and \(\gamma' < \gamma - 1\) and the dynamics (26)–(29) is considered. Then for sufficiently large \(m\), with probability at least \(1 - 2 \exp\left(-\frac{C_0(n, d+1)}{4^{C_0^2}}\right)\) over the choice of \(\theta_0\), we have

\[
\sup_{t \in [0, +\infty)} R_D(\theta_w(t)) = \sup_{t \in (0, +\infty)} \frac{\|\theta_w(t) - \theta_0\|_2}{\|\theta_0\|_2} \gg 1.
\]

6 Conclusions and discussion

In this paper, we characterized the linear, critical, and condensed regimes with distinctive features and draw the phase diagram for the two-layer ReLU NN at the infinite-width limit. We experimentally demonstrate and theoretically prove the transition across the boundary (critical regime) in the phase diagram. Through experiments, we further identify the condensation as the signature behavior in the condensed regime of very strong nonlinearity.

A phase diagram serves as a map that guides the future research. In our phase diagram for two-layer ReLU NNs, the linear regimes is very well understood both theoretically and experimentally. However, the critical and condensed regimes are still largely not understood from both experimental and theoretical perspectives. The following problems for these regimes requires further studies: (i) whether the dynamics always converges to a global minimizer; (ii) what is the convergence rate; (iii) what is the mechanism of condensation; (iv) how to characterize the implicit regularization of condensation.

Our phase diagram is obtained specifically for the ReLU activation, however, our methodology and thus obtained regime characterization can be naturally extended to more general activations, which is an immediate next step of this work. In addition, how to characterize the effect of other hyperparameters, e.g., choice of optimization method, learning rate, regularization techniques, and etc., to the NN training dynamics requires future studies. Other important future problems include drawing the phase diagram for NNs of three or more layers or for convolutional networks.

In analogy to statistical mechanics, a clean regime separation may be only possible at the infinite width limit, which is not realistic in practice. Nevertheless, rich insight about a finite size system often can be derived from the analysis at the limit, which is usually much easier. Therefore, we believe it is an important task to systematically draw such phase diagrams for NNs of different structures through a combination of theoretical and experimental approaches as demonstrated in this work. These phase diagrams can be continuously refined and provides a clear pathway to open the black box of deep learning.
7 Acknowledgement

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A Technical lemmas

This section collects some technical lemmas and propositions. For convenience, we define the two quantities
\[ \alpha(t) := \max_{k \in [m], s \in [0, t]} |a_k(s)|, \quad \omega(t) := \max_{k \in [m], s \in [0, t]} \|w_k(s)\|_\infty. \] (44)

Lemma 1 (bounds of initial parameters). Given \( \delta \in (0, 1) \) and the sample set \( S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega \) with \( x_i \)'s drawn i.i.d. from some unknown distribution \( D \). Suppose that Assumption 1 holds. We have with probability at least \( 1 - \delta \) over the choice of \( \theta^0 \)

\[ \max_{k \in [m]} \{ |a_k^0|, \|w_k^0\|_\infty \} \leq \sqrt{2 \log \frac{2m(d + 1)}{\delta}}, \] (45)

Proof. If \( X \sim N(0, 1) \), then \( \mathbb{P}(|X| > \varepsilon) \leq 2e^{-\frac{1}{2}\varepsilon^2} \) for all \( \varepsilon > 0 \). Since \( a_k^0 \sim N(0, 1), (w_k^0)_\alpha \sim N(0, 1) \) for \( k = 1, 2, \ldots, m, \alpha = 1, \ldots, d \) and they are all independent, by setting

\[ \varepsilon = \sqrt{2 \log \frac{2m(d + 1)}{\delta}}, \]
one can obtain
\[
P \left( \max_{k \in [m]} \{ |a_k^0|, \|w_k^0\|_\infty \} > \varepsilon \right) = P \left( \max_{k \in [m], \alpha \in [d]} \{ |a_k^0|, |(a_k^0)_\alpha| \} > \varepsilon \right)
\]
\[
= P \left( \bigcup_{k=1}^m (|a_k^0| > \varepsilon) \cup \bigcup_{\alpha=1}^d (|(w_k^0)_\alpha| > \varepsilon) \right)
\]
\[
\leq \sum_{k=1}^m P (|a_k^0| > \varepsilon) + \sum_{k=1}^m \sum_{\alpha=1}^d P (|(w_k^0)_\alpha| > \varepsilon)
\]
\[
\leq 2m e^{-\frac{1}{2} \varepsilon^2} + 2md e^{-\frac{3}{2} \varepsilon^2}
\]
\[
= 2m(d+1)e^{-\frac{1}{2} \varepsilon^2} = \delta.
\]

\[\text{Lemma 2 (bound of initial empirical risk). Given } \delta \in (0, 1) \text{ and the sample set } S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega \text{ with } x_i \text{'s drawn i.i.d. from some unknown distribution } D. \text{ Suppose that Assumption 1 holds. We have with probability at least } 1 - \delta \text{ over the choice of } \theta^0,
\]
\[R_S(\theta^0) \leq \frac{1}{2} \left[ \frac{1}{m} m \sum_{k=1}^m a_k^0 \sigma(w_k^0 \cdot x) - \mathbb{E}_{(\alpha, w)} a\sigma(w^T x) \right] \leq \frac{4m(d+1)}{\delta} \left( 2 + 3\sqrt{2 \log(8/\delta)} \right) \kappa \sqrt{m}.
\]

**Proof.** Let
\[G = \{g_x(a, w) \mid g_x(a, w) := a\sigma(w^T x), x \in \Omega\}.
\]
Lemma 1 implies that with probability at least \(1 - \delta/2\) over the choice of \(\theta^0\), we have
\[|g_x(a_k^0, w_k^0)| \leq d|a_k^0||w_k^0| \leq 2d \log \frac{4m(d+1)}{\delta}
\]

Then
\[
\frac{1}{m} \sup_{x \in \Omega} |f_{\theta^0}(x)| = \sup_{x \in \Omega} \left| \frac{1}{m} m \sum_{k=1}^m a_k^0 \sigma(w_k^0 \cdot x) - \mathbb{E}_{(\alpha, w)} a\sigma(w^T x) \right|
\]
\[
\leq 2 \text{Rad}_{\theta^0}(G) + 6d \left( \log \frac{4m(d+1)}{\delta} \right) \sqrt{2 \log(8/\delta)} \sqrt{d}/\sqrt{m}.
\]

The Rademacher complexity can be estimated by
\[
\text{Rad}_{\theta^0}(G) = \frac{1}{m} \mathbb{E}_{\tau} \left[ \sup_{x \in \Omega} m \sum_{k=1}^m \tau_k a_k^0 \sigma(w_k^0 \cdot x) \right]
\]
\[
\leq \frac{1}{m} \sqrt{2 \log \frac{4m(d+1)}{\delta}} \mathbb{E}_{\tau} \left[ \sup_{x \in \Omega} m \sum_{k=1}^m \tau_k w_k^0 \cdot x \right]
\]
\[
\leq \sqrt{2 \log \frac{4m(d+1)}{\delta}} \sqrt{2d \log \frac{4m(d+1)}{\delta}} \sqrt{d} \sqrt{m}
\]
\[
= \frac{2d \log \frac{4m(d+1)}{\delta}}{\sqrt{m}}.
\]
Therefore
\[ \sup_{x \in \Omega} |f_{\theta}(x)| \leq 2d \left( \log \frac{4m(d + 1)}{\delta} \right) \left( 2 + 3\sqrt{2\log(8/\delta)\sqrt{m}} \right), \]
and
\[ R_S(\theta^0) \leq \frac{1}{2n} \sum_{i=1}^{n} (1 + \kappa |f_{\theta}(x_i)|)^2 \leq \frac{1}{2} \left[ 1 + 2d \left( \log \frac{4m(d + 1)}{\delta} \right) \left( 2 + 3\sqrt{2\log(8/\delta)\kappa\sqrt{m}} \right) \right]^2. \]

**Remark 7.** If \( \gamma > \frac{1}{2} \), then \( \kappa = o \left( \frac{1}{\sqrt{\log m}} \right) \) and \( R_S(\theta^0) = O(1) \). One can use ASI trick [3] to guarantee \( R_S(\theta^0) \leq \frac{1}{2} \) for any \( \kappa \).

Next we introduce the sub-exponential norm of a random variable and the sub-exponential Bernstein’s inequality.

**Definition 1** (sub-exponential norm [17]). The sub-exponential norm of a random variable \( X \) is defined as
\[ \|X\|_{\psi_1} := \inf\{s > 0 \mid E_X[e^{s|X|/s}] \leq 2 \}. \] (48)
In particular, we denote the sub-exponential norm of a \( \chi^2(d) \) random variable \( X \) by \( C_{\psi,d} := \|X\|_{\psi_1} \).
Here the \( \chi^2 \) distribution with \( d \) degrees of freedom has the probability density function
\[ f_X(z) = \frac{1}{2^{d/2}\Gamma(d/2)}z^{d/2-1}e^{-z/2}. \]

**Remark 8.** Note that
\[ E_{X \sim \chi^2(d)} e^{X/s} = \int_0^{+\infty} \frac{1}{2^{d/2}\Gamma(d/2)}z^{d/2-1} e^{-z/s} \, dz = +\infty, \]
and
\[ \lim_{s \to +\infty} E_{X \sim \chi^2(d)} e^{X/s} = \lim_{s \to +\infty} \int_0^{+\infty} \frac{1}{2^{d/2}\Gamma(d/2)}z^{d/2-1} e^{-z/s} \, dz = 1 \leq 2. \]
These imply that \( 2 \leq C_{\psi,d} < +\infty \).

**Lemma 3.** Suppose that \( w \sim N(0, I_d) \), \( a \sim N(0, 1) \) and given \( x_i, x_j \in \Omega \). Then we have
(i) if \( X := \sigma(w^T x_i)\sigma(x \cdot x_j) \), then \( \|X\|_{\psi_1} \leq dC_{\psi,d} \).
(ii) if \( X := a^2 \sigma'(w^T x_i)\sigma'(w^T x_j)x_i \cdot x_j \), then \( \|X\|_{\psi_1} \leq dC_{\psi,d} \).

**Proof.** Let \( Z := \|w\|^2_2 = \chi^2(d) \).
(i) \( |X| \leq d\|w\|^2_2 = dZ \) and
\[ \|X\|_{\psi_1} = \inf\{s > 0 \mid E_X\exp(|X|/s) \leq 2 \} \]
\[ = \inf\{s > 0 \mid E_w\exp(|\sigma(w^T x_i)\sigma(w^T x_j)|/s) \leq 2 \} \]
\[ \leq \inf\{s > 0 \mid E_w\exp(d\|w\|^2_2/s) \leq 2 \} \]
\[ = \inf\{s > 0 \mid E_{Z}\exp(d|Z|/s) \leq 2 \} \]
\[ = d\inf\{s > 0 \mid E_Z\exp(|Z|/s) \leq 2 \} \]
\[ = d\|\chi^2(d)\|_{\psi_1} \]
\[ \leq dC_{\psi,d}. \]
Assumption 1 holds. If $m_S$ set $\theta$ of $-\theta$, therefore with probability at least $1$ we have

$$\mathbb{P}\left(\left|\frac{1}{m} \sum_{k=1}^{m} X_k - \mu \right| \geq s\right) \leq 2 \exp\left(-C_0 m \min\left(s^2, \frac{s}{\|X_1\|_\psi, 1} \right)\right), \quad (49)$$

where $C_0$ is an absolute constant.

**Proposition 1** (norm of initial parameters). Given $\delta \in (0, 1)$ and the sample set $S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega$ with $x_i$’s drawn i.i.d. from some unknown distribution $\mathcal{D}$. Suppose that Assumption 1 holds. We have with probability at least $1 - 2 \exp\left(-\frac{C_0 m(d+1)\delta}{4C_\psi, 1}\right)$ over the choice of $\theta^0$

$$\sqrt{\frac{md}{2}} \leq \|\theta^0\|_2 \leq \sqrt{\frac{3md}{2}}, \quad (50)$$

$$\sqrt{\frac{md}{2}} \leq \|\theta^0_{\theta, \psi}\|_2 \leq \sqrt{\frac{3md}{2}}. \quad (51)$$

**Proof.** Let $X_1, \ldots, X_m$ be the squares of the entries of $\theta^0$, which are drawn i.i.d. from $\chi^2(1)$. The latter is sub-exponential and $\mathbb{E}X_k = 1$. Then by Theorem 3

$$\mathbb{P}\left(\frac{1}{m(d+1)} \sum_{k=1}^{m(d+1)} X_k - 1 \geq s\right) \leq 2 \exp\left(-C_0 m(d+1) \min\left(s^2, \frac{s}{C_\psi, 1}\right)\right).$$

Note that $C_\psi, 1 \leq 1$. Setting $s = \frac{3}{2}$, we have

$$\mathbb{P}\left(\frac{1}{2} \leq \frac{1}{m(d+1)} \sum_{k=1}^{m(d+1)} X_k \leq \frac{3}{2}\right) \leq 2 \exp\left(-C_0 m(d+1) \min\left(\frac{1}{4C_\psi, 1}, \frac{1}{2C_\psi, 1}\right)\right)$$

$$= 2 \exp\left(-\frac{C_0 m(d+1)}{4C_\psi, 1}\right).$$

Therefore, with probability at least $1 - 2 \exp\left(-\frac{C_0 m(d+1)\delta}{4C_\psi, 1}\right)$ over the choice of $\theta^0$,

$$\frac{1}{2} \leq \frac{1}{m(d+1)} \sum_{k=1}^{m(d+1)} X_k = \frac{1}{m(d+1)} \|\theta^0\|_2^2 \leq \frac{3}{2}.$$

In other words, (50) holds. The proof of (51) is similar. \qed

**Lemma 4** (minimal eigenvalue of Gram matrix $G$ at initial). Given $\delta \in (0, 1)$ and the sample set $S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega$ with $x_i$’s drawn i.i.d. from some unknown distribution $\mathcal{D}$. Suppose that Assumption 1 holds. If $m \geq \frac{16n^2d^2C_\psi, 4}{C_0 \delta} \log \frac{4n^2}{\delta}$, then with probability at least $1 - \delta$ over the choice of $\theta^0$, we have

$$\lambda_{\min}(G(\theta^0)) \geq 3 \frac{1}{4} \left(\frac{1}{\kappa} \lambda_n + \kappa' \lambda_w\right).$$

(52)
Proof. For any \( \varepsilon > 0 \), we define

\[
\Omega^{[a]}_{ij} := \left\{ \theta^0 \mid \frac{\kappa'}{\kappa^2} G^{[a]}_{ij}(\theta^0) - K^{[a]}_{ij} \leq \frac{\varepsilon}{n} \right\},
\]

\[
\Omega^{[w]}_{ij} := \left\{ \theta^0 \mid \frac{1}{\kappa^2 \kappa'} G^{[w]}_{ij}(\theta^0) - K^{[w]}_{ij} \leq \frac{\varepsilon}{n} \right\}.
\]

By Theorem and Lemma, if \( \frac{\varepsilon}{mC_{\psi,d}} \leq 1 \), then

\[
P(\Omega^{[a]}_{ij}) \geq 1 - 2 \exp\left( - \frac{mC_{\psi,d}^2}{n^2 d^2 C_{\psi,d}^2} \right),
\]

\[
P(\Omega^{[w]}_{ij}) \geq 1 - 2 \exp\left( - \frac{mC_{\psi,d}^2}{n^2 d^2 C_{\psi,d}^2} \right),
\]

so with probability at least \( \left[ 1 - 2 \exp\left( - \frac{mC_{\psi,d}^2}{n^2 d^2 C_{\psi,d}^2} \right) \right]^{2n^2} \geq 1 - 4n^2 \exp\left( - \frac{mC_{\psi,d}^2}{n^2 d^2 C_{\psi,d}^2} \right) \) over the choice of \( \theta^0 \), we have

\[
\left\| \frac{\kappa'}{\kappa^2} G^{[a]}(\theta^0) - K^{[a]} \right\|_F \leq \varepsilon,
\]

\[
\left\| \frac{1}{\kappa^2 \kappa'} G^{[w]}(\theta^0) - K^{[w]} \right\|_F \leq \varepsilon,
\]

Hence by taking \( \varepsilon = \lambda/4 \), that is, \( \delta = 4n^2 \exp\left( - \frac{mC_{\psi,d}^2}{16n^2 d^2 C_{\psi,d}^2} \right) \)

\[
\lambda_{\min} \left( G(\theta^0) \right) \geq \lambda_{\min} \left( G^{[a]}(\theta^0) \right) + \lambda_{\min} \left( G^{[w]}(\theta^0) \right)
\]

\[
\geq \frac{\kappa^2}{\kappa'} \lambda_a - \frac{\kappa^2}{\kappa'} \left\| \frac{\kappa'}{\kappa^2} G^{[a]}(\theta^0) - K^{[a]} \right\|_F
\]

\[
+ \frac{\kappa^2 \kappa'}{\kappa^2} \lambda_w - \frac{\kappa^2 \kappa'}{\kappa^2} \left\| \frac{1}{\kappa^2 \kappa'} G^{[w]}(\theta^0) - K^{[w]} \right\|_F
\]

\[
\geq \frac{\kappa^2}{\kappa'} (\lambda_a - \varepsilon) + \kappa^2 \kappa' (\lambda_w - \varepsilon)
\]

\[
\geq \frac{3}{4} \frac{\kappa^2}{\kappa'} \left( \frac{1}{\kappa^2} \lambda_a + \kappa' \lambda_w \right).
\]

\[
\square
\]

In the following we denote

\[
t^* = \inf \{ t \mid \theta(t) \notin \mathcal{N}(\theta^0) \},
\]

where

\[
\mathcal{N}(\theta^0) := \left\{ \theta \mid \| G(\theta) - G(\theta^0) \|_F \leq \frac{1}{4} \kappa^2 \left( \frac{1}{\kappa'} \lambda_a + \kappa' \lambda_w \right) \right\}.
\]

Then we have the following lemma.
Lemma 5 (local in time exponential decay of $R_S$, $\theta$-lazy training). Given $\delta \in (0, 1)$ and the sample set $S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega$ with $x_i$’s drawn i.i.d. from some unknown distribution $D$. Suppose that Assumption 1 holds. If $m \geq \frac{16n^2d^2\psi_0 C^2_{\theta}}{\lambda^2\kappa^2} \log \frac{4n^2}{\delta}$, then with probability at least $1 - \delta$ over the choice of $\theta^0$, we have for any $t \in [0, t^*)$

$$R_S(\theta(t)) \leq \exp \left( - \frac{m\kappa^2}{n} \left( \frac{1}{\kappa'} \lambda_a + \kappa' \lambda_w \right) \right) R_S(\theta^0).$$

(55)

Proof. Lemma 4 implies that for any $\delta \in (0, 1)$, with probability at least $1 - \delta$ over the choice of $\theta_0$ and for any $\theta \in N(\theta^0)$, we have

$$\lambda_{\min} (G(\theta)) \geq \lambda_{\min} (G(\theta^0)) - \|G(\theta) - G(\theta^0)\|_F \geq \frac{3}{4} \kappa^2 \left( \frac{1}{\kappa'} \lambda_a + \kappa' \lambda_w \right) - \frac{1}{4} \kappa^2 \left( \frac{1}{\kappa'} \lambda_a + \kappa' \lambda_w \right) = \frac{1}{2} \kappa^2 \left( \frac{1}{\kappa'} \lambda_a + \kappa' \lambda_w \right).$$

Note that

$$G_{ij} = G^a_{ij} + G^w_{ij} = \frac{\kappa^2}{\kappa' m} \sum_{k=1}^m \nabla_{a_k} f_\theta(x_i) \cdot \nabla_{a_k} f_\theta(x_j) + \frac{\kappa^2}{\kappa' m} \sum_{k=1}^m \nabla_{w_k} f_\theta(x_i) \cdot \nabla_{w_k} f_\theta(x_j),$$

and

$$\nabla_{a_k} R_S(\theta) = \frac{1}{n} \sum_{i=1}^n e_i \nabla_{a_k} f_\theta(x_i),$$

$$\nabla_{w_k} R_S(\theta) = \frac{1}{n} \sum_{i=1}^n e_i \nabla_{w_k} f_\theta(x_i).$$

Thus

$$\frac{m}{n^2} e^T G e = \frac{1}{\kappa'} \sum_{k=1}^m \nabla_{a_k} R_S(\theta) \cdot \nabla_{a_k} R_S(\theta) + \kappa' \sum_{k=1}^m \nabla_{w_k} R_S(\theta) \cdot \nabla_{w_k} R_S(\theta).$$

Then finally we get

$$\frac{d}{dt} R_S(\theta(t)) = - \left( \frac{1}{\kappa'} \sum_{k=1}^m \nabla_{a_k} R_S(\theta) \cdot \nabla_{a_k} R_S(\theta) + \kappa' \sum_{k=1}^m \nabla_{w_k} R_S(\theta) \cdot \nabla_{w_k} R_S(\theta) \right),$$

$$= - \frac{m}{n^2} e^T G e,$$

$$\leq - \frac{2m}{n} \lambda_{\min} (G(\theta(t))) R_S(\theta(t)) \leq - \frac{m\kappa^2}{n} \left( \frac{1}{\kappa'} \lambda_a + \kappa' \lambda_w \right) R_S(\theta(t)),$$

and an integration yields the result. \(\square\)
that Assumption 1 holds. If we obtain sample set $\{x_i, y_i\}_{i=1}^n \subset \Omega$ with $x_i$’s drawn i.i.d. from some unknown distribution $D$. Suppose that Assumption 1 holds. If $m \geq \max \left\{ \frac{16n^2 d^2 c^2 \log \frac{8n^2}{\delta}}{\lambda^2 c_0^2}, \frac{4 \sqrt{2dn} \sqrt{R_S(\theta^0)}}{\kappa(\lambda_\alpha/\kappa + \kappa' \lambda_w)} \right\}$, then with probability at least $1 - \delta$ over the choice of $\theta^0$, for any $t \in [0, t^*)$ and any $k \in [m]$,

$$
\max_{k \in [m]} |a_k(t) - a_k(0)| \leq 2 \max \left\{ \frac{1}{\kappa'}, 1 \right\} \sqrt{2 \log \frac{4m(d+1)}{\delta}} \frac{\sqrt{2dn} \sqrt{R_S(\theta^0)}}{m \kappa (\lambda_\alpha/\kappa + \kappa' \lambda_w)}.
$$

(56)

$$
\max_{k \in [m]} \|w_k(t) - w_k(0)\|_{\infty} \leq 2 \max \{ \kappa', 1 \} \sqrt{2 \log \frac{4m(d+1)}{\delta}} \frac{\sqrt{2dn} \sqrt{R_S(\theta^0)}}{m \kappa (\lambda_\alpha/\kappa + \kappa' \lambda_w)}.
$$

(57)

and

$$
\max_{k \in [m]} \{ |a_k(0)|, \|w_k(0)\|_{\infty} \} \leq \sqrt{2 \log \frac{4m(d+1)}{\delta}}.
$$

(58)

**Proof.** Since $\alpha(t) = \max_{k \in [m], s \in [0, t]} |a_k(s)|$, $\omega(t) = \max_{k \in [m], s \in [0, t]} \|w_k(s)\|_{\infty}$, we obtain

$$
|\nabla_{a_k} R_S|^2 \leq \left| \frac{1}{n} \sum_{i=1}^n e_i \kappa \sigma(w_k^T x_i) \right|^2 \leq 2 \|w_k\|_2^2 \kappa^2 R_S(\theta) \leq 2d^2 (\omega(t))^2 \kappa^2 R_S(\theta),
$$

$$
\|\nabla_{w_k} R_S\|^2 \leq \left| \frac{1}{n} \sum_{i=1}^n e_i \kappa a_k \sigma'(w_k^T x_i) x_i \right|^2 \leq 2 |a_k|^2 \kappa^2 R_S(\theta) \leq 2(\alpha(t))^2 \kappa^2 R_S(\theta).
$$

By Lemma 5, we have if $m \geq \frac{16n^2 d^2 c^2 \log \frac{8n^2}{\delta}}{\lambda^2 c_0^2}$, then with probability at least $1 - \delta/2$ over the choice of $\theta^0$,

$$
|a_k(t) - a_k(0)| \leq \frac{1}{\kappa'} \int_0^t |\nabla_{a_k} R_S(\theta(s))| \, ds
\leq \frac{\sqrt{2dn}}{\kappa'} \int_0^t \omega(s) \sqrt{R_S(\theta(s))} \, ds
\leq \frac{\sqrt{2dn} \omega(t)}{\kappa'} \int_0^t \sqrt{R_S(\theta^0)} \exp \left( -\frac{m \kappa^2}{2n} \left( \frac{1}{\kappa'} \lambda_\alpha + \kappa' \lambda_w \right) s \right) \, ds
\leq \frac{2 \sqrt{2dn} \sqrt{R_S(\theta^0)}}{m \kappa \kappa' (\lambda_\alpha^{(0)}/\kappa + \kappa' \lambda_w)} \omega(t)
= \frac{p}{\kappa'} \omega(t),
$$

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where \( p := \frac{2 \sqrt{2d} \sqrt{R_S(\theta^0)}}{m \kappa (\lambda_a / \kappa' + \kappa' \lambda_w)} \). On the other hand,
\[
\| w_k(t) - w_k(0) \|_\infty \leq \kappa' \int_0^t \| \nabla w_k R_S(\theta(s)) \|_\infty \, ds \\
\leq \sqrt{2} \kappa' \int_0^t \alpha(s) \sqrt{R_S(\theta(s))} \, ds \\
\leq \sqrt{2} \kappa' \alpha(t) \int_0^t \sqrt{R_S(\theta^0)} \exp \left( -\frac{m \kappa^2}{2n} \left( \frac{1}{\kappa'} \lambda_a + \kappa' \lambda_w \right) s \right) \, ds \\
\leq \frac{2 \sqrt{2} n \sqrt{R_S(\theta^0)} \kappa'}{m \kappa (\lambda_a / \kappa' + \kappa' \lambda_w)} \alpha(t) \\
\leq p \kappa' \alpha(t).
\]

Thus
\[
\alpha(t) \leq \alpha(0) + p \omega(t) \frac{1}{\kappa'}, \\
\omega(t) \leq \omega(0) + p \alpha(t) \kappa'.
\]

By Lemma 1, we have with probability at least \( 1 - \delta/2 \) over the choice of \( \theta^0 \),
\[
\max_{k \in [m]} \{ |a_k(0)|, \| w_k(0) \|_\infty \} \leq \sqrt{\frac{2 \log 4m(d + 1)}{\delta}}.
\]

If
\[
m \geq \frac{4 \sqrt{2} n \sqrt{R_S(\theta^0)}}{\kappa (\lambda_a / \kappa' + \kappa' \lambda_w)},
\]
then we have
\[
p = \frac{2 \sqrt{2} n \sqrt{R_S(\theta^0)}}{m \kappa (\lambda_a / \kappa' + \kappa' \lambda_w)} \leq \frac{1}{2}.
\]

Thus
\[
\alpha(t) \leq \alpha(0) + \frac{p \omega(t)}{\kappa'} + p^2 \alpha(t), \\
\alpha(t) \leq \frac{4}{3} \alpha(0) + \frac{2}{3} \kappa' \omega(t).
\]

Therefore
\[
\alpha(t) \leq 2 \max \{ 1, \frac{1}{\kappa'} \} \sqrt{\frac{2 \log 4m(d + 1)}{\delta}}.
\]

Similarly, one can obtain the estimate of \( \omega(t) \) as
\[
\omega(t) \leq 2 \max \{ 1, \kappa' \} \sqrt{\frac{2 \log 4m(d + 1)}{\delta}}.
\]

Finally we have for any \( t \in [0, t^*] \) with probability at least \( 1 - \delta \) over the choice of \( \theta^0 \),
\[
\max_{k \in [m]} | a_k(t) - a_k(0) | \leq 2 \max \left\{ 1, \frac{1}{\kappa'} \right\} \sqrt{\frac{2 \log 4m(d + 1)}{\delta}} \, p, \\
\max_{k \in [m]} \| w_k(t) - w_k(0) \|_\infty \leq 2 \max \{ \kappa', 1 \} \sqrt{\frac{2 \log 4m(d + 1)}{\delta}} \, p,
\]

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which completes the proof.

To show our main results with \( \gamma' > \gamma - 1 \), we further define

\[
  t^*_a = \inf \{ t \mid \theta(t) \in \mathcal{N}_a(\theta^0) \}, \quad t^*_w = \inf \{ t \mid \theta(0) \in \mathcal{N}_w(\theta^0) \},
\]

where

\[
  \mathcal{N}_a(\theta^0) := \left\{ \theta \mid \|G^{[a]}(\theta) - G^{[a]}(\theta^0)\|_F \leq \frac{1}{4} \kappa^2 \kappa' \lambda_a \right\},
\]

and

\[
  \mathcal{N}_w(\theta^0) := \left\{ \theta \mid \|G^{[w]}(\theta) - G^{[w]}(\theta^0)\|_F \leq \frac{1}{4} \kappa^2 \kappa' \lambda_w \right\}.
\]

**Lemma 6** (minimal eigenvalue of Gram matrix \( G^{[a]} \) at initial). Given \( \delta \in (0, 1) \) and the sample set \( S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega \) with \( x_i \)'s drawn i.i.d. from some unknown distribution \( D \). Suppose that Assumption 1 holds. If

\[
  m \geq \frac{16n^2d^2C^2_\psi,d}{C_0 \kappa^2} \log \frac{2n^2}{\delta},
\]

then we have with probability at least \( 1 - \delta \) over the choice of \( \theta^0 \),

\[
  \lambda_{\min} \left( G^{[a]}(\theta^0) \right) \geq \frac{3 \kappa^2}{4} \kappa' \lambda_a.
\]

**Proof.** For any \( \varepsilon > 0 \) define

\[
  \Omega^{[a]}_{ij} := \left\{ \theta^0 \mid \left| \frac{\kappa'}{\kappa^2} G^{[a]}_{ij}(\theta^0) - K_{ij}^{[a]} \right| \leq \frac{\varepsilon}{n} \right\}.
\]

By Theorem 3 and Remark 4 if \( \frac{\varepsilon}{nC_\psi,d} \leq 1 \) then

\[
  P(\Omega^{[a]}_{ij}) \geq 1 - 2 \exp \left( - \frac{mC_0 \varepsilon^2}{n^2d^2C^2_\psi,d} \right),
\]

with probability at least

\[
  \left[ 1 - 2 \exp \left( - \frac{mC_0 \varepsilon^2}{n^2d^2C^2_\psi,d} \right) \right] n^2 \geq 1 - 2n^2 \exp \left( - \frac{mC_0 \varepsilon^2}{n^2d^2C^2_\psi,d} \right)
\]

over the choice of \( \theta^0 \), we have

\[
  \left\| \frac{\kappa'}{\kappa^2} G^{[a]}(\theta^0) - K^{[a]} \right\|_F \leq \varepsilon.
\]

Taking \( \varepsilon = \lambda_a / 4 \), i.e., \( \delta = 2n^2 \exp \left( - \frac{mC_0 \lambda_a^2}{16n^2d^2C^2_\psi,d} \right) \), we obtain the estimate

\[
  \lambda_{\min}(G^{[a]}(\theta^0)) \geq \frac{\kappa^2}{\kappa'} \lambda_a - \frac{\kappa^2}{\kappa'} \left\| \frac{\kappa'}{\kappa^2} G^{[a]}(\theta^0) - K^{[a]} \right\|_F
\]

\[
  \geq \frac{\kappa^2}{\kappa'} (\lambda_a - \varepsilon)
\]

\[
  \geq \frac{3 \kappa^2}{4} \kappa' \lambda_a.
\]

\[\square\]
Proposition 3 (local in time exponential decay of $R_S$, $w$-lazy training). Given $\delta \in (0, 1)$ and the sample set $S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega$ with $x_i$'s drawn i.i.d. from some unknown distribution $D$. Suppose that Assumption 1 holds. If $m \geq \frac{16n^2d^2C^2_s}{c_0\lambda^2_{\text{c}}} \log \frac{2m^2}{\delta}$, then with probability at least $1 - \delta$ over the choice of $\theta^0$, for $t \in [0, t_a^\ast)$,

$$R_S(\theta(t)) \leq \exp\left( -\frac{m\kappa_\ast^2}{\kappa^\ast n} \lambda_\ast t \right) R_S(\theta^0).$$

Proof. By Lemma 6 for any $\delta \in (0, 1)$ with probability $1 - \delta$ over the choice of $\theta^0$ and for any $\theta \in \mathcal{N}_\ast(\theta^0)$,

$$\lambda_{\min}(G^{[a]}(\theta)) \geq \lambda_{\min}(G^{[a]}(\theta^0)) - \|G(\theta) - G(\theta^0)\|_F$$

$$\geq \frac{3\kappa^2}{4\kappa^\ast\lambda_\ast} - \frac{1\kappa^2}{4\kappa^\ast\lambda_\ast}$$

$$\geq \frac{1\kappa^2}{2\kappa^\ast\lambda_\ast}.$$ 

Therefore

$$\frac{d}{dt} R_S(\theta(t)) = -\frac{m}{n^2} e^T G e$$

$$\leq -\frac{m}{n^2} e^T G^{[a]} e$$

$$\leq -\frac{2m}{n} \lambda_{\min}\left(G^{[a]}(\theta(t))\right) R_S(\theta(t))$$

$$\leq -\frac{m\kappa^2}{\kappa^\ast n} \lambda_\ast R_S(\theta(t)).$$

This leads to the linear convergence rate. □

Proposition 4 (bounds on the change of parameters, $w$-lazy training). Given $\delta \in (0, 1)$ and the sample set $S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega$ with $x_i$'s drawn i.i.d. from some unknown distribution $D$. Suppose that Assumption 1 holds. If $m \geq \frac{16n^2d^2C^2_s}{c_0\lambda^2_{\text{c}}} \log \frac{4m^2}{\delta}$ and $\max \frac{4\sqrt{23n} \sqrt{R_S(\theta^0)}}{\lambda_\ast} \kappa^\ast \leq 1$, then with probability at least $1 - \delta$ over the choice of $\theta^0$ and for any $\theta \in [0, t_a^\ast)$, $k \in [m]$,

$$\max_{k \in [m]} |a_k(t) - a_k(0)| \leq 2\frac{1}{\kappa^\ast} \sqrt{2 \log \frac{4m(d+1)}{\delta}} p_a,$$  

(65)

$$\max_{k \in [m]} \|w_k(t) - w_k(0)\|_\infty \leq 2\sqrt{2 \log \frac{4m(d+1)}{\delta}} p_a,$$  

(66)

and

$$\max_{k \in [m]} \{|a_k(0)|, \|w_k(0)\|_\infty\} \leq \sqrt{2 \log \frac{4m(d+1)}{\delta}},$$ 

(67)

where $p_a = \frac{2\sqrt{23n} \sqrt{R_S(\theta^0)}}{m\kappa^\ast\lambda_\ast\kappa^\ast}.$

Proof. Since

$$\alpha(t) = \max_{k \in [m], s \in [0,t]} |a_k(s)|, \quad \omega(t) = \max_{k \in [m], s \in [0,t]} \|w_k(s)\|_\infty,$$ 

(27)
then
\[ |\nabla_{\alpha_k} R_S|^2 = \left| \frac{1}{n} \sum_{i=1}^{n} e_i \kappa a \sigma (w_k^T x_i) \right|^2 \leq 2 \| w_k \|^2 \kappa^2 R_S(\theta) \leq 2 \| w_k \|^2 \kappa^2 R_S(\theta), \]
\[ \| \nabla_{w_k} R_S \|^2 = \left| \frac{1}{n} \sum_{i=1}^{n} e_i \kappa a \sigma (w_k^T x_i) x_i \right|^2 \leq 2 \| a_k \|^2 \kappa^2 R_S(\theta) \leq 2 \| a_k \|^2 \kappa^2 R_S(\theta). \]

By Lemma 5, we have if
\[ m \geq \frac{16 \kappa^2 \ln C_2}{\lambda^2 C_0} \log \frac{8n^2}{\delta}, \]
then with probability at least \( 1 - \delta/2 \) over the choice of \( \theta^0 \),
\[ |a_k(t) - a_k(0)| \leq \frac{1}{\kappa'} \int_0^t \| \nabla_{a_k} R_S(\theta(s)) \| ds \]
\[ \leq \frac{\sqrt{2}\kappa}{\kappa'} \int_0^t \omega(s) \sqrt{R_S(\theta(s))} ds \]
\[ \leq \frac{\sqrt{2}\kappa}{\kappa'} \omega(t) \int_0^t \sqrt{R_S(\theta^0)} \exp \left( -\frac{m\kappa^2}{2n} \left( \frac{1}{\kappa'} \lambda a + \kappa' \lambda w \right) s \right) ds \]
\[ \leq \frac{2 \sqrt{2} \kappa \omega(t)}{m \kappa' (\lambda S / \kappa' + \kappa' \lambda w)} \]
\[ = \frac{p_a}{\kappa'} \omega(t), \]
where \( p_a := \frac{2 \sqrt{2} \kappa \omega(t)}{m \kappa' (\lambda S / \kappa' + \kappa' \lambda w)} \).

On the other hand,
\[ \| w_k(t) - w_k(0) \|_\infty \leq \kappa' \int_0^t \| \nabla_{w_k} R_S(\theta(s)) \| ds \]
\[ \leq \sqrt{2} \kappa \int_0^t \alpha(s) \sqrt{R_S(\theta(s))} ds \]
\[ \leq \sqrt{2} \kappa \alpha(t) \int_0^t \sqrt{R_S(\theta^0)} \exp \left( -\frac{m\kappa^2}{2n} \left( \frac{1}{\kappa'} \lambda a + \kappa' \lambda w \right) s \right) ds \]
\[ \leq \frac{2 \sqrt{2} \kappa \alpha(t)}{m \kappa (\lambda S / \kappa' + \kappa' \lambda w)} \alpha(t) \]
\[ \leq p_a \kappa' \alpha(t). \]

Thus
\[ \alpha(t) \leq \alpha(0) + p_a \omega(t) \frac{1}{\kappa'}, \]
\[ \omega(t) \leq \omega(0) + p_a \alpha(t) \kappa'. \]

By Lemma 3, we have with probability at least \( 1 - \delta/2 \) over the choice of \( \theta^0 \),
\[ \max_{k \in [m]} \{ |a_k(0)|, \| w_k(0) \|_\infty \} \leq \sqrt{\frac{2 \log \frac{4m(d+1)}{\delta}}{\delta}}. \]
If
\[ m \geq \frac{4\sqrt{2}dn \sqrt{R_S(\theta^0)}}{\kappa (\lambda_a/\kappa' + \kappa' \lambda_w)}, \]
then
\[ p_a = \frac{2\sqrt{2}dn \sqrt{R_S(\theta^0)}}{m \kappa \lambda_a/\kappa'} \leq \frac{1}{2}. \]

Thus
\[ \alpha(t) \leq \alpha(0) + \frac{p_a}{\kappa'} \omega(0) + p_a^2 \alpha(t), \]
\[ \alpha(t) \leq \frac{4}{3} \alpha(0) + \frac{2}{3} \frac{1}{\kappa'} \omega(0), \]

Therefore
\[ \alpha(t) \leq 2 \frac{1}{\kappa'} \sqrt{2 \log \frac{4m(d+1)}{\delta}}. \]

Similarly, one can obtain the estimate of \( \omega(t) \) as
\[ \omega(t) \leq 2 \frac{1}{\kappa'} \sqrt{2 \log \frac{4m(d+1)}{\delta}}. \]

Finally, with probability at least \( 1 - \delta \) over the choice of \( \theta^0 \) and for any \( t \in [0, t^*_a) \), we have
\[ \max_{k \in [m]} |a_k(t) - a_k(0)| \leq 2 \frac{1}{\kappa'} \sqrt{2 \log \frac{4m(d+1)}{\delta}} p_a, \]
\[ \max_{k \in [m]} \|w_k(t) - w_k(0)\|_\infty \leq 2 \frac{1}{\kappa'} \sqrt{2 \log \frac{4m(d+1)}{\delta}} p_a, \]
which completes the proof. \( \square \)

**B Proof of Theorem 1**

We further divide the linear regime into two parts: \( \gamma < 1 \) where the training dynamics is \( \theta \)-lazy and \( \gamma' > \gamma - 1 \) where the training dynamics is \( w \)-lazy. Theorem 1 is hence covered by Proposition 5 and Proposition 6 whose proofs are given in this section.

**Proposition 5** (\( \theta \)-lazy training). Given \( \delta \in (0, 1) \) and the sample set \( S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega \) with \( x_i \)'s drawn i.i.d. from some unknown distribution \( D \). Suppose that Assumption 1 and Assumption 2 hold. ASI is used if \( \gamma \leq \frac{1}{2} \). Suppose that \( \gamma < 1 \) and the dynamics (26)–(29) is considered. Then for sufficiently large \( m \), with probability at least \( 1 - \delta \) over the choice of \( \theta^0 \), we have

(a) \[ \sup_{t \in [0, +\infty)} \|\theta(t) - \theta^0\|_2 \leq \frac{1}{\sqrt{\kappa}} \log m. \]

(b) \[ R_S(\theta(t)) \leq \exp \left( -\frac{2m \kappa^2 \lambda}{n} \right) R_S(\theta^0). \]

Moreover, we have with probability at least \( 1 - \delta - 2 \exp \left( -\frac{C_0 m(d+1)}{4C_1^2 \psi^2} \right) \).

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(c) \( \sup_{t \in [0, +\infty)} \frac{\|\theta(t) - \theta^0\|_2}{\|\theta^0\|_2} \leq \frac{1}{mK} \log m. \)

Proof. Let \( t \in [0, t^*), \) \( p = \frac{2\sqrt{2dn\sqrt{R_S(\theta^0)}}}{mK(\lambda_a/\kappa' + \kappa'\lambda_w)} \) and \( \xi = \sqrt{2 \log \frac{8m(d+1)}{\delta}}. \)

(a) From Proposition 2 we have with probability at least \( 1 - \delta/2 \) over the choice of \( \theta^0, \)

\[
\sup_{t \in [0, t^*]} \|\theta(t) - \theta^0\|_2 \leq \left[ (m+1) \left( 2 \sqrt{2 \log \frac{8m(d+1)}{\delta}} \right)^2 \right]^{1/2} = \max \left\{ \kappa', \frac{1}{\kappa'} \right\} \sqrt{m(d+1)} \sqrt{2 \log \frac{8m(d+1)}{\delta}} \frac{\kappa^2}{mK(\lambda_a/\kappa' + \kappa'\lambda_w)} \leq \max \left\{ \kappa', \frac{1}{\kappa'} \right\} \frac{4d + 1dn \sqrt{\log \frac{8m(d+1)}{\delta}} \sqrt{R_S(\theta^0)}}{\sqrt{mK} \lambda} \leq \frac{1}{\sqrt{mK}} \log m,
\]

where we use the fact
\[
\max \left\{ \kappa', \frac{1}{\kappa'} \right\} \leq \max \left\{ \kappa', \frac{1}{\kappa'} \frac{1}{\lambda_a/\kappa' \lambda_w} \right\} \leq \frac{1}{\lambda}.
\]

(b) The linear convergence rate is essentially proved by Lemma 3 with \( t^* = +\infty. \) We divide the proof into the following three steps. In particular, \( t^* = +\infty \) is proved in the step (iii).

(i) Let \( g_{ij}^{[a]}(w) := \sigma(w^T x_i)\sigma(w^T x_j), \)

then
\[
\left| G_{ij}^{[a]}(\theta(t)) - G_{ij}^{[a]}(\theta(0)) \right| \leq \frac{K^2}{mK} \sum_{k=1}^{m} \left| g_{ij}^{[a]}(w_k(t)) - g_{ij}^{[a]}(w_k(0)) \right|.
\]

By mean value theorem, for some \( c \in (0, 1), \)
\[
\left| g_{ij}^{[a]}(w_k(t)) - g_{ij}^{[a]}(w_k(0)) \right| \leq \|\nabla g_{ij}^{[a]}(cw_k(t) + (1 - c)w_k(0))\|_\infty \|w_k(t) - w_k(0)\|_1,
\]

where
\[
\nabla g_{ij}^{[a]}(w) = \sigma'(w \cdot \xi_i)\sigma(w^T x_j) x_i + \sigma(w^T x_i)\sigma'(w^T x_j) x_j,
\]

and
\[
\|\nabla g_{ij}^{[a]}(w)\|_\infty \leq 2\|w\|_1.
\]

From Proposition 2 we have with probability at least \( 1 - \delta/2 \) over the choice of \( \theta^0, \)
\[
\|w_k(t) - w_k(0)\|_\infty \leq po(t)\kappa' \leq 2\max\{\kappa', 1\} \xi_p,
\]
\[
\|w_k(t) - w_k(0)\|_1 \leq 2d \max\{\kappa', 1\} \xi_p.
\]
Thus
\[ \|cw_k(t) + (1 - c)w_k(0)\|_1 \leq d (\|w_k(0)\|_\infty + \|w_k(t) - w_k(0)\|_\infty) \leq d (\xi + 2 \max\{\kappa', 1\} \xi p) \leq 2d\xi \max\{\kappa', 1\}. \]

Then
\[ |G_{ij}^{[a]}(\theta(t)) - G_{ij}^{[a]}(\theta(0))| \leq 8d^2 \kappa^2 \xi^2 \max\left\{\kappa', \frac{1}{\kappa'}\right\} p, \]
and
\[ \|G^{[a]}(\theta(t)) - G^{[a]}(\theta(0))\|_F \leq 8d^2 n \kappa^2 \max\left\{ \kappa', \frac{1}{\kappa'} \right\} \left(2 \log \frac{8m(d + 1)}{\delta} \right) \frac{2\sqrt{2}dnR_S(\theta)}{m(\lambda_a/\kappa' + \kappa' \lambda_w)} \]
\[ \leq \kappa \max\left\{ \kappa', \frac{1}{\kappa'} \right\} \frac{32\sqrt{2}d^3n^2 \left(\log \frac{8m(d + 1)}{\delta} \right) \sqrt{R_S(\theta)}}{m(\lambda_a/\kappa' + \kappa' \lambda_w)}. \]

If we choose
\[ m\kappa \geq \frac{128\sqrt{2}d^3n^2 \left(\log \frac{8m(d + 1)}{\delta} \right) \sqrt{R_S(\theta)}}{\lambda^2}, \] (68)
then noticing that
\[ \frac{1}{\lambda^2} \geq \frac{1}{4 \left(\frac{1}{27} \lambda_a^3 \lambda_w\right)^{1/2}} \]
\geq \frac{1}{\left(\lambda_a/\kappa' + \kappa' \lambda_w\right)^2 \kappa'}
= \frac{1}{\left(\lambda_a/\sqrt{\kappa'} + (\kappa')^{3/2} \lambda_w\right)^2},
\[ \frac{1}{\lambda^2} \geq \frac{1}{4 \left(\frac{1}{27} \lambda_a^3 \lambda_w\right)^{1/2}} \]
\geq \frac{\kappa'}{\left(\lambda_a/\kappa' + \kappa' \lambda_w\right)^2}
= \frac{1}{\left(\lambda_a/(\kappa')^{3/2} + \sqrt{\kappa'} \lambda_w\right)^2},
we have
\[ m\kappa \geq \max\left\{ \kappa', \frac{1}{\kappa'} \right\} \frac{256\sqrt{2}d^3n^2 \left(\log \frac{8m(d + 1)}{\delta} \right) \sqrt{R_S(\theta)}}{(\lambda_a/\kappa' + \kappa' \lambda_w)^2}. \]

Therefore
\[ \|G^{[a]}(\theta(t)) - G^{[a]}(\theta(0))\|_F \leq \frac{1}{8\kappa^2} \left(\frac{1}{\kappa'} \lambda_a + \kappa' \lambda_w\right). \] (69)
(ii) Define

\[ D_{i,k} = \{ \omega_k(0) \mid \| w_k(t) - w_k(0) \|_\infty \leq 2\xi \max\{\kappa', 1\} p, \] 
\[ \sigma'(w_k(t^*) \cdot x_i) \neq \sigma'(w_k(0) \cdot x_i) \} \].

If \( |w_k(0) \cdot x_i| > 4d \max\{\kappa', 1\} \xi p \), then

\[ |w_k(t) \cdot x_i - w_k(0) \cdot x_i| \leq \| x_i \|_1 \| w_k(t) - w_k(0) \|_\infty \leq 2d \sqrt{\frac{2 \log 8m(d+1)}{\delta}} p, \]

thus \( w_k(t) \cdot x_i \) and \( w_k(0) \cdot x_i \) have the same sign which means \( D_{i,k} \) is empty. Recall that \( x_i \in [0, 1]^d \) with \( (x_i)_d = 1 \), then \( \| x_i \|_2 \geq 1 \). Let \( \hat{x}_i = \frac{x_i}{\| x_i \|_2} \) then \( |w_k(0) \cdot x_i| \geq |w_k(0) \cdot \hat{x}_i| \) and

\[ P(D_{i,k}) \leq P(\{w_k(0) \cdot x_i| \leq 4d \max\{\kappa', 1\} \xi p \}) \]
\[ \leq P(\{w_k(0) \cdot \hat{x}_i| \leq 4d \max\{\kappa', 1\} \xi p \}) \]
\[ = P(\{w_k(0) \cdot (1, 0, 0, \ldots, 0)^T \} \leq 4d \max\{\kappa', 1\} \xi p) \]
\[ = \frac{2}{\sqrt{2\pi}} \int_0^{4d \max\{\kappa', 1\} \xi p} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \]
\[ \leq \frac{8}{\sqrt{2\pi}} d \max\{\kappa', 1\} \xi p \]
\[ \leq 4d \max\{\kappa', 1\} \xi p. \]

Then

\[ |C_{i,j}^{\omega}(\theta(t)) - C_{i,j}^{\omega}(\theta(0))| \leq \frac{\kappa^2 \kappa'}{m} \sum_{k=1}^{m} \left| a_k^2(t^*) \sigma'(w_k(t) \cdot x_i) \sigma'(w_k(t) \cdot x_j) \right| \]
\[ - a_k^2(0) \sigma'(w_k(0) \cdot x_i) \sigma'(w_k(0) \cdot x_j) \right| \]
\[ \leq \frac{\kappa^2 \kappa'}{m} \sum_{k=1}^{m} \left[ a_k^2(t)|D_{k,i,j}| + |a_k^2(t) - a_k^2(0)| \right], \]

where

\[ D_{k,i,j} := \sigma'(w_k(t) \cdot x_i) \sigma'(w_k(t) \cdot x_j) - \sigma'(w_k(0) \cdot x_i) \sigma'(w_k(0) \cdot x_j). \]

Thus

\[ E|D_{k,i,j}| \leq P(D_{k,i} \cup D_{k,j}) \leq 8d \max\{\kappa', 1\} \xi p. \]

At the same time

\[ |a_k^2(t) - a_k^2(0)| \leq |a_k(t) - a_k(0)|^2 + 2|a_k(0)||a_k(t) - a_k(0)| \]
\[ \leq \left( 2 \max\left\{ \frac{1}{\kappa', 1} \right\} \xi p \right)^2 + 2\xi \left( 2 \max\left\{ \frac{1}{\kappa', 1} \right\} \xi p \right) \]
\[ \leq 6\xi^2 \max\left\{ \frac{1}{\kappa'^2}, 1 \right\} p, \]

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so
\[ a_k^2(t) \leq |a_k^2(t) - a_k^2(0)| + a_k^2(0) \leq \left( 2 \max \left\{ \frac{1}{\kappa'}, 1 \right\} \xi p \right)^2 + 2\xi \left( 2 \max \left\{ \frac{1}{\kappa'}, 1 \right\} \xi p \right) + \xi^2 \]
\[ \leq 4 \max \left\{ \frac{1}{\kappa'^2}, 1 \right\} \xi^2. \]

Then
\[ \mathbb{E} \sum_{i,j=1}^{n} \left| G_{ij}^{[w]}(\theta(t)) - G_{ij}^{[w]}(\theta(0)) \right| \leq \sum_{i,j=1}^{n} \frac{\kappa^2 \kappa'd}{m} \sum_{k=1}^{m} \left( 4 \max \left\{ \frac{1}{\kappa'^2}, 1 \right\} \xi^2 \mathbb{E}|D_{k,i,j}| + 6 \max \left\{ \frac{1}{\kappa'^2}, 1 \right\} \xi^2 p \right) \]
\[ \leq \sum_{i,j=1}^{n} \frac{\kappa^2 \kappa'd}{m} \sum_{k=1}^{m} \left( 4 \max \left\{ \frac{1}{\kappa'^2}, 1 \right\} \xi^2 n \max \{\kappa', 1\} \xi p + 6 \max \left\{ \frac{1}{\kappa'^2}, 1 \right\} \xi^2 p \right) \]
\[ \leq \kappa^2 \kappa'd n^2 \left( 32 d \xi \max \{\kappa', \frac{1}{\kappa'^2}\} + 6 \max \left\{ \frac{1}{\kappa'^2}, 1 \right\} \right) \xi^2 p \]
\[ \leq 40 \kappa^2 d^2 n^2 \left( 2 \log \frac{8m(d+1)}{\delta} \right)^{3/2} \max \{\kappa'^2, \frac{1}{\kappa'^2}\} p. \]

By Markov’s inequality, with probability at least \( 1 - \delta/2 \) over the choice of \( \theta^0 \), we have
\[ \|G^{[w]}(\theta(t)) - G^{[w]}(\theta(0))\|_{\mathcal{F}} \leq \sum_{i,j=1}^{n} \left| G_{ij}^{[w]}(\theta(t)) - G_{ij}^{[w]}(\theta(0)) \right| \]
\[ \leq \max \{\kappa'^2, \frac{1}{\kappa'}\} \frac{40 \kappa^2 d^2 n^2 \left( 2 \log \frac{8m(d+1)}{\delta} \right)^{3/2} p}{\delta/2} \]
\[ \leq \max \{\kappa'^2, \frac{1}{\kappa'}\} \frac{80 \kappa^2 d^2 n^2 2 \sqrt{2}}{\delta} \left( \log \frac{8m(d+1)}{\delta} \right)^{3/2} \frac{2 \sqrt{2} d n \sqrt{R_S(\theta^0)}}{m (\lambda_a / \kappa' + \kappa' \lambda_w)} \]
\[ \leq \kappa \max \{\kappa'^2, \frac{1}{\kappa'}\} \frac{640 d^3 n^3 \left( \log \frac{8m(d+1)}{\delta} \right)^{3/2} \sqrt{R_S(\theta^0)} \delta^{-1}}{m (\lambda_a / \kappa' + \kappa' \lambda_w)}. \]

If
\[ m \geq \frac{5120 \delta^{-1} d^3 n^3 \left( \log \frac{8m(d+1)}{\delta} \right)^{3/2} \sqrt{R_S(\theta^0)}}{\lambda^2}, \]
then noticing that
\[ \frac{1}{\lambda^2} \geq \frac{1}{4 \left( \frac{1}{27} (\lambda_a)^3 \lambda_w \right)^{1/4}} \]
\[ \geq \frac{1}{(\lambda_a / \kappa' + \kappa' \lambda_w)^2 \kappa'} \]
\[ = \frac{1}{\left( \lambda_a / \sqrt{\kappa'} + (\kappa')^{3/2} \lambda_w \right)^2}, \]

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and
\[ \frac{1}{\lambda^2} \geq \left( 4 \left( \frac{1}{27} \lambda_a (\lambda_w) \right)^{1/4} \right)^2 \]
\[ \geq \frac{\kappa'}{(\lambda_a / \kappa' + \kappa' \lambda_w)^2} \]
\[ = \frac{1}{(\lambda_a / (\kappa')^{3/2} + \sqrt{\kappa'} \lambda_w)^2}, \]
we have
\[ \|G^w(\theta(t)) - G^w(\theta(0))\|_F \leq \frac{1}{8} \kappa'^2 \left( \frac{1}{\kappa'} \lambda_a + \kappa' \lambda_w \right). \] (70)

(iii) For \( t \in [0, t^*), \)
\[ R_S(\theta(t)) \leq \exp \left( -\frac{m \kappa^2}{n} \left( \frac{1}{\kappa'} \lambda_a + \kappa' \lambda_w \right) t \right) R_S(\theta^0) \leq \exp \left( -\frac{2m \kappa^2 \lambda}{n} \right) R_S(\theta^0). \]

Suppose that \( t^* < +\infty \) then one can take the limit \( t \to t^* \) in (69) and (70). This will lead to a contradiction with the definition of \( t^* \). Therefore \( t^* = +\infty \).

(c) By Proposition 1, we have with probability at least \( 1 - \frac{2 \exp \left( -\frac{C_0 m (d+1)}{4 C_{\psi,d}} \right)}{2} \) over the choice of \( \theta^0 \),
\[ \|\theta^0\| \geq \sqrt{\frac{m(d+1)}{2}}, \]
Therefore, with probability at least \( 1 - \delta - \frac{2 \exp \left( -\frac{C_0 m (d+1)}{4 C_{\psi,d}} \right)}{2} \) over the choice of \( \theta^0 \), we have
\[ \sup_{t \in [0, +\infty)} \frac{\|\theta(t) - \theta^0\|_2}{\|\theta^0\|_2} \leq \sqrt{\frac{2}{m(d+1)}} \sup_{t \in [0, +\infty)} \|\theta(t) - \theta^0\|_2 \]
\[ \leq \sqrt{\frac{2}{m(d+1)}} \frac{4 \sqrt{d + 1} \sqrt{d} \log \frac{8m(d+1)}{\delta}}{\sqrt{m \kappa \lambda}} R_S(\theta^0) \]
\[ \leq \frac{1}{m \kappa} \frac{4 \sqrt{dn} \sqrt{d} \log \frac{8m(d+1)}{\delta}}{\lambda} R_S(\theta^0) \]
\[ \leq \frac{1}{m \kappa} \log m. \]

\[ \square \]

Remark 9. The proof indicates more quantitative conditions on \( m \) and \( \kappa \) for Proposition 5 to hold:
\[ m \geq \frac{16n^2 d^2 C^2_{\psi,d}}{\lambda^2 C_0} \log \frac{16n^2}{\delta}, \] (71)
and

\[ m\kappa \geq \max \left\{ \frac{2\sqrt{2dnR_S(\theta^0)}}{\lambda}, \frac{128\sqrt{2d^3n^2} \left( \log \frac{8m(d+1)}{\delta} \sqrt{R_S(\theta^0)} \right)}{\lambda^2}, \frac{512\delta^{-1}d^3n^3 \left( \log \frac{8m(d+1)}{\delta} \right)^{3/2}}{\lambda^2} \right\}. \]  

(72)

**Proposition 6** (\(w\)-lazy training). Given \(\delta \in (0, 1)\) and the sample set \(S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega\) with \(x_i\)’s drawn i.i.d. from some unknown distribution \(D\). Suppose that Assumption 1 and Assumption 2 hold. Suppose that \(\gamma' > \gamma - 1, \gamma' > 0\), and the dynamics (26)–(29) is considered. Then for sufficiently large \(m\), with probability at least \(1 - \delta\) over the choice of \(\theta^0\), we have

(a)

\[ \sup_{t \in [0, +\infty)} \| \theta_w(t) - \theta^0_w \|_2 \leq \sup_{t \in [0, +\infty)} \| \theta(t) - \theta^0 \|_2 \lesssim \frac{1}{\sqrt{m\kappa}} \log m. \]

(b) \(R_S(\theta(t)) \leq \exp \left( -\frac{m\kappa^2\lambda_t}{k'\kappa n} \right) R_S(\theta^0).\)

Moreover we have with probability at least \(1 - \delta - 2\exp \left( -\frac{C \kappa m(d+1)}{4Cw_1} \right)\).

(c)

\[ \sup_{t \in [0, +\infty)} \frac{\| \theta(t) - \theta^0 \|_2}{\| \theta^0 \|_2} \lesssim \frac{1}{m\kappa} \log m, \quad \text{(not} \ll 1), \]

\[ \sup_{t \in [0, +\infty)} \frac{\| \theta_w(t) - \theta^0_w \|_2}{\| \theta^0_w \|_2} \lesssim \frac{k'}{m\kappa} \log m, \quad \text{((not} 1). \]

**Proof.** Let \(t \in [0, t_*]\), \(p_a = \frac{2\sqrt{2dnR_S(\theta^0)}}{m\kappa \lambda_a / k'}\) and \(\xi = \sqrt{2\log \frac{8m(d+1)}{\delta}}\).

(a) From Proposition 2 we have with probability at least \(1 - \delta/2\) over the choice of \(\theta^0\)

\[ \sup_{t \in [0, t_*]} \| \theta_w(t) - \theta^0_w \|_2 \leq \sup_{t \in [0, t_*]} \| \theta(t) - \theta^0 \|_2 \]

\[ \leq \left[ \frac{m}{k'^2} + md \right] \left( 2\sqrt{2\log \frac{8m(d+1)}{\delta} p_a} \right)^{2^{\frac{1}{2}}} \]

\[ = \sqrt{m(d+1)} \frac{1}{k'} \sqrt{2\log \frac{8m(d+1)}{\delta}} \sqrt{2dnR_S(\theta^0)} \frac{m\kappa \lambda_a / k'}{\lambda_w} \]

\[ \leq \max \left\{ k', \frac{1}{k'} \right\} \frac{4\sqrt{d+1}d \log \log \frac{8m(d+1)}{\delta} R_S(\theta^0)}{\sqrt{m\kappa} \lambda_a / k' \lambda_w} \]

\[ \leq \frac{8\sqrt{d+1}d \log \log \frac{8m(d+1)}{\delta} R_S(\theta^0)}{\sqrt{m\kappa} \lambda_w}. \]

(b) We divide this proof into the following two steps.
(i) Let
\[ G^{[a]}_{ij}(w) := \sigma(w^\top x_i)\sigma(w^\top x_j), \]
then
\[ |G^{[a]}_{ij}(\theta(t)) - G^{[a]}_{ij}(\theta(0))| \leq \frac{\kappa^2}{m\lambda_\alpha} \sum_{k=1}^m |g^{[a]}_{ij}(w_k(t)) - g^{[a]}_{ij}(w_k(0))|. \]
By the mean value theorem, for some \( c \in (0, 1) \),
\[ |g^{[a]}_{ij}(w_k(t)) - g^{[a]}_{ij}(w_k(0))| \leq \|\nabla g^{[a]}_{ij}(cw_k(t) + (1-c)w_k(0))\|_{\infty} \|w_k(t) - w_k(0)\|_1, \]
where
\[ \nabla g^{[a]}_{ij}(w) = \sigma'(w \cdot \xi_i)\sigma(w^\top x_j)x_i + \sigma(w^\top x_i)\sigma'(w^\top x_j)x_j, \]
and
\[ \|\nabla g^{[a]}_{ij}(w)\|_{\infty} \leq \|w\|_1. \]
From Proposition 2, we have with probability at least \( 1 - \delta/2 \) over the choice of \( \theta^0 \),
\[ \|w_k(t) - w_k(0)\|_{\infty} \leq p_a \alpha(t) \kappa' \leq 2\xi_p, \]
\[ \|w_k(t) - w_k(0)\|_1 \leq 2d\xi_p. \]
Thus
\[ \|cw_k(t) + (1-c)w_k(0)\|_1 \leq d (\|w_k(0)\|_{\infty} + \|w_k(t) - w_k(0)\|_{\infty}) \]
\[ \leq d (\xi + 2\xi_p) \]
\[ \leq 2d\xi. \]
Then
\[ |G^{[a]}_{ij}(\theta(t)) - G^{[a]}_{ij}(\theta(0))| \leq 8d^2\frac{\kappa^2}{\kappa'} \xi^2 p_a, \]
and
\[ \|G^{[a]}(\theta(t)) - G^{[a]}(\theta(0))\|_F \leq 16d^2n \left( \log \frac{8m(d+1)}{\delta} \right) \frac{\kappa^2}{\kappa'} p_a \]
\[ \leq \frac{32\sqrt{2}d^3n^2 \left( \log \frac{8m(d+1)}{\delta} \right) \sqrt{R_S(\theta^0)}\kappa}{m\lambda_\alpha}. \]
If
\[ \frac{m\kappa}{\kappa'} \geq \frac{256\sqrt{2}d^3n^2 \left( \log \frac{8m(d+1)}{\delta} \right) \sqrt{R_S(\theta^0)}}{\lambda_\alpha^2}, \]
then we have
\[ \|G^{[a]}(\theta(t)) - G^{[a]}(\theta(0))\|_F \leq \frac{1}{8} \frac{\kappa^2}{\kappa'} \lambda_\alpha. \] (73)
(ii) For \( t \in [0, \tau^*_a) \) by Lemma 5
\[ R_S(\theta(t)) \leq \exp \left( -\frac{m\kappa^2\lambda_\alpha t}{n\kappa'} \right) R_S(\theta^0). \]
Suppose that \( \tau^*_a < +\infty \) then one can take the limit \( t \to \tau^*_a \) in (73). This will lead to a contradiction with the definition of \( \tau^*_a \). Therefore \( \tau^*_a = +\infty \).
(c) By Proposition 1, we have with probability at least $1 - 2\exp(-\frac{C_0 m (d+1)}{4c_0^2})$ over the choice of $\theta^0$, $\|\theta^0\|_2^2 \geq \frac{d + 1}{2} m$.

So with probability at least $1 - \delta - 2\exp\left(-\frac{C_0 m (d+1)}{4c_0^2}\right)$ over the choice of $\theta^0$, we have

$$
\sup_{t \in [0, +\infty)} \frac{\|\theta(t) - \theta^0\|_2}{\|\theta^0\|_2} \leq \sqrt{\frac{2}{m(d+1)} \sup_{t \in [0, +\infty)} \|\theta(t) - \theta^0\|_2}
$$

$$
\leq \sqrt{\frac{2}{m(d+1)} 8\sqrt{d + 1} d n \log \frac{8m(d+1)}{\delta} R_S(\theta^0)}
$$

$$
\leq \frac{1}{m\kappa} \sqrt{\frac{8\sqrt{d} d n \log \frac{8m(d+1)}{\delta}}{\lambda}} R_S(\theta^0)
$$

$$
\leq \frac{1}{m\kappa} \log m.
$$

Similarly, by Proposition 1, we have with probability at least $1 - 2\exp(-\frac{C_0 m (d+1)}{4c_0^2})$ over the choice of $\theta^0$, $\|\theta^0_w\|_2^2 \geq \frac{d}{2} m$.

So with probability at least $1 - \delta - 2\exp\left(-\frac{C_0 m (d+1)}{4c_0^2}\right)$ over the choice of $\theta^0$, we have

$$
\sup_{t \in [0, +\infty)} \frac{\|\theta_w(t) - \theta^0_w\|_2}{\|\theta^0_w\|_2} \leq \sqrt{\frac{2}{dm} \sup_{t \in [0, +\infty)} \|\theta_w(t) - \theta^0_w\|_2}
$$

$$
\leq \sqrt{\frac{2}{dm} \frac{\kappa'}{\sqrt{m\kappa}} 8\sqrt{d} d n \log \frac{8m(d+1)}{\delta} R_S(\theta^0)}
$$

$$
\leq \frac{\kappa'}{m\kappa} \log m.
$$

We remark that in fact we can prove a similar result about the change of parameter $a_k$'s. We state this result as follows without proof.

Proposition 7 (a-lazy training). Given $\delta \in (0, 1)$ and the sample set $S = \{(x_i, y_i)\}_{i=1}^n \subset \Omega$ with $x_i$'s drawn i.i.d. from some unknown distribution $D$. Suppose that Assumption 1 and Assumption 2 hold. Suppose that $\gamma' < \gamma - 1$, $\gamma' < 0$, and the dynamics (26)-(29) is considered. Then for sufficiently large $m$, with probability at least $1 - \delta$ over the choice of $\theta^0$, we have

(a) $\sup_{t \in [0, +\infty)} \|\theta(t) - \theta^0\|_2 \leq \sup_{t \in [0, +\infty)} \|\theta(t) - \theta^0\|_2 \leq \frac{1}{\sqrt{m\kappa}} \log m.$
(b) $R_S(\theta(t)) \leq \exp \left( -\frac{m^2 \kappa' \Delta \theta t}{n} \right) R_S(\theta^0)$.

Moreover we have with probability at least $1 - \delta - 2 \exp \left( -\frac{C_0 (m+1)}{4 \psi} \right)$ over the choice of $\theta^0$, we have

(c) \[
\sup_{t \in [0, \infty)} \frac{\|\theta(t) - \theta^0\|_2}{\|\theta^0\|_2} \lesssim \frac{1}{mk} \log m, \quad (\text{not } \ll 1),
\]

\[
\sup_{t \in [0, \infty)} \frac{\|\theta_\alpha(t) - \theta^0\|_2}{\|\theta_\alpha\|_2} \lesssim \frac{1}{mk\kappa'} \log m, \quad (\ll 1).
\]

C Proof of Theorem 2

In order to characterize the condensed regime, we need a crucial proposition that ravel a natural relation between $a_k(t)$ and $w_k(t)$ during the GD training dynamics.

**Proposition 8.** Consider the GD training dynamics \[26 - 29\], then we have

\[|a_k(t)| \leq \sqrt{\frac{1}{\kappa'}} |a^0_k| \|w_k(t)\|_2, \quad (74)\]

which holds for any $t \geq 0$ and $k \in [m]$.

**Proof.** Multiplying equations in \[26\] by $\kappa' a_k$ and $\frac{w_k}{\kappa'}$ respectively, we obtain

\[
\begin{aligned}
\kappa' \dot{a}_k &= -\frac{\kappa}{n} \sum_{i=1}^n e_i a_k \sigma(w_k^\top x_i), \\
\frac{1}{\kappa'} \dot{w}_k &= -\frac{\kappa}{n} \sum_{i=1}^n e_i a_k \sigma'(w_k^\top x_i) w_k^\top x_i.
\end{aligned}
\]

Notice that for ReLU activation $\sigma(z) = z \sigma'(z)$, $z \in \mathbb{R}$, by comparing the right hand side of \[75\], one can obtain

\[
\kappa'^2 \frac{d}{dt} |a_k|^2 = \frac{d}{dt} \|w_k\|^2_2.
\]

Integrating this from 0 to $t$ leads to

\[
\kappa'^2 \left( |a_k(t)|^2 - |a^0_k|^2 \right) = \|w_k(t)\|^2_2 - \|w_k^0\|^2_2,
\]

which then can be written as

\[
|a_k(t)|^2 = \frac{1}{\kappa'^2} \left( \|w_k(t)\|^2_2 - \|w_k^0\|^2_2 \right) + |a^0_k|^2. \quad (76)
\]

Finally we have

\[
|a_k(t)| \leq \sqrt{\frac{1}{\kappa'^2} \|w_k(t)\|^2_2 + |a^0_k|^2} \leq \sqrt{\frac{2}{\kappa'}} |a^0_k| \|w_k(t)\|_2.
\]

\[\square\]
Proof of Theorem 3. By Assumption 3, there exits a $T^* > 0$ such that

$$R_S(\theta(T^*)) \leq \frac{1}{32n}. $$

Without loss of generality, we assume $f(x_1) \geq \frac{1}{2}$. Therefore

$$\frac{1}{2n} e_1(T^*)^2 \leq \frac{1}{2n} e(T^*)^\top e(T^*) = R_S(\theta(T^*)) \leq \frac{1}{32n}, $$

which means

$$|e_1(T^*)| \leq \frac{1}{4}.$$

Recalling the definition that $e_1 = \kappa f_\theta(x_1) - f(x_1)$, we have

$$\kappa \sum_{k=1}^{m} a_k(T^*) \sigma(w_k(T^\top)\mathbf{x}_1) \geq f(x_1) - \frac{1}{4} \geq \frac{1}{4}.$$

So

$$\frac{1}{4\kappa} \leq \sum_{k=1}^{m} a_k(T^*) \sigma(w_k(T^\top)\mathbf{x}_1)$$

$$\leq \sqrt{d} \sum_{k=1}^{m} |a_k(T^*)||w_k(T^*)|_2$$

$$\leq \frac{\sqrt{2d}}{\kappa'} \sum_{k=1}^{m} |a_k^0||w_k(T^*)|_2^2$$

$$\leq \frac{\sqrt{2d}}{\kappa'} \max_{k \in [m]} |a_k^0||\theta w(T^*)|_2,$$

where we have used Proposition 8 and equivalently

$$||\theta w(T^*)||_2 \geq \kappa' \sqrt{\frac{1}{\kappa' \frac{1}{4} \sqrt{2d} \max_{k \in [m]} |a_k^0|}}.$$

Finally, by Proposition 1, we have with probability at least $1 - 2 \exp(-\frac{C_{\text{sim}}(d+1)}{4C_{\text{sim}}})$ over the choice of $\theta^0$,

$$||\theta^0_w|| \leq \sqrt{\frac{3}{8}dm},$$

therefore

$$\sup_{t \in [0, +\infty)} \frac{||\theta_w(t) - \theta^0_w||_2}{||\theta^0_w||_2} \geq \frac{||\theta_w(T^*) - \theta^0_w||_2}{||\theta^0||_2}$$

$$\geq \sqrt{\frac{\kappa'}{\frac{1}{8} \sqrt{2d} \max_{k \in [m]} |a_k^0|}} - 1$$

$$\geq \sqrt{\frac{\kappa'}{\kappa m \log m}} - 1.$$
Figure 6: \( S_\theta \) in (a) and \( S_w \) in (b) estimated on NNs of 1000, 5000, 10000, 20000, 40000 neurons over \( \gamma \) (ordinate) and \( \gamma' \) (abscissa). The stars are zero points obtained by the linear interpolation over different \( \gamma \) for each fixed \( \gamma' \). Dashed lines are auxiliary lines.

If \( \gamma' < \gamma - 1 \), then

\[
\frac{\kappa'}{\kappa m \log m} \gg 1,
\]

which completes the proof.

**Remark 10.** If \( \gamma > 1 \) and \( \gamma' < 1 - \gamma \), then Theorem 2 can hold without taking Assumption 3.

Actually, for any \( \delta \in (0, 1) \), Proposition 7 guarantees the Assumption 3 with probability at least \( 1 - \delta \) over the choice of \( \theta_0 \), when \( m \) is sufficiently large. Therefore, under Assumptions 1 and 2, if \( m \) is sufficiently large, then we have with probability at least \( 1 - \delta \) over the choice of \( \theta_0 \), the relative change

\[
\sup_{t \in [0, +\infty)} \text{RD}(\theta_w(t)) \gg 1.
\]

### D Relative deviation of parameters

For completion, we can also similarly define the slope of the relative deviation for \( \theta \) and \( a \) denoted by \( S_\theta \) and \( S_a \), respectively. As shown in Fig.6(a), the boundary for the \( \theta \) is \( \gamma = 1 \), regardless of \( \gamma' \), that is, all parameters are close to their initialization after training. For output weight \( a \), as shown in Fig.6(b), the boundary consists of two rays, one is \( \gamma = 1 \) and \( \gamma' \geq 0 \), the other is \( \gamma + \gamma' = 1 \) and \( \gamma' \leq 0 \). This verifies that, in the area between \( \gamma = 1 \) and \( \gamma - \gamma' = 1 \) of \( \gamma' \leq 0 \), the change of scatter plot from a Gaussian initialization is induced by the change of \( a \).