Exact Solutions of Berkovits’ String Field Theory

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Abstract

The equation of motion for Berkovits’ WZW-like open (super)string field theory is shown to be integrable in the sense that it can be written as the compatibility condition (“zero-curvature condition”) of some linear equations. Employing a generalization of solution-generating techniques (the splitting and the dressing methods), we demonstrate how to construct nonperturbative classical configurations of both $N=1$ superstring and $N=2$ fermionic string field theories. With and without $u(n)$ Chan-Paton factors, various solutions of the string field equation are presented explicitly.

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1 Introduction

String field theory opens new possibilities of exploring nonperturbative structures in string theory. Starting from Witten’s bosonic string field theory action \([1]\) there have been several attempts to construct a generalization to the supersymmetric case \([2, 3, 4]\) (for a general discussion of the subject see \([3, 5]\)). The crucial novelty in the superstring case is the appearance of the so-called picture degeneracy. Each physical state is now represented by an infinite number of BRST-invariant vertex operators in the covariant RNS formalism \([6]\). This enforces the choice of a definite picture number for each vertex operator; for open superstring fields, one usually chooses all Ramond (R) string fields to carry \(-\frac{1}{2}\) and all Neveu-Schwarz (NS) string fields to carry \(-1\) units of picture charge. As the total picture number must equal \(-2\) for open superstrings and Witten’s bosonic Chern-Simons-like action contains a quadratic and a cubic term in the string fields, this entails the introduction of picture raising and lowering operators (carrying picture number +1 and \(-1\), respectively). The resulting action \([2]\), however, is not gauge-invariant \([7, 8]\) due to contact term divergences occurring when two picture raising operators collide. Although one can choose other pictures for the string fields \([3, 4]\), there is no choice for which the action is cubic and gauge-invariant \([9]\).

As an alternative, Berkovits proposed a (nonpolynomial) WZW-like action \([10]\) for the NS sector which contains in its Taylor expansion a kinetic term and a cubic interaction similar to Witten’s superstring field theory action:

\[
S = \frac{1}{2g^2} \text{tr} \left\{ (e^{-\Phi^+} G^+ e^\Phi)(e^{-\Phi^-} \tilde{G}^+ e^\Phi) - \int_0^1 dt (e^{-\hat{\Phi}} \partial_t e^\hat{\Phi}) \{ e^{-\hat{\Phi}} G^+ e^\hat{\Phi}, e^{-\hat{\Phi}} \tilde{G}^+ e^\hat{\Phi} \} \right\}.
\]

(1.1)

Here \(e^\Phi = I + \Phi + \frac{1}{2} \Phi \ast \Phi + \ldots\) is defined via Witten’s midpoint gluing prescription (\(I\) denotes the identity string field), \(\Phi\) is a NS string field carrying \(u(n)\) Chan-Paton labels\(^1\) with an extension \(\hat{\Phi}(t)\) interpolating between \(\hat{\Phi}(t = 0) = 0\) and \(\hat{\Phi}(t = 1) = \Phi\). The BRST-like currents \(G^+\) and \(\tilde{G}^+\) are the two superpartners of the energy-momentum tensor in a twisted small superconformal \(N = 4\) algebra with positive \(U(1)\)-charge. The action of \(G^+\) and \(\tilde{G}^+\) on any string field is defined in conformal field theory language as taking the contour integral, e.g.

\[
(G^+ e^\Phi)(z) = \oint \frac{dw}{2\pi i} G^+(w) e^\Phi(z), \quad (\tilde{G}^+ e^\Phi)(z) = \oint \frac{dw}{2\pi i} \tilde{G}^+(w) e^\Phi(z),
\]

(1.2)

with the integration contour running around \(z\).

The action (1.1) is invariant under the transformation

\[
\delta e^\Phi = \Lambda e^\Phi + e^\Phi \tilde{\Lambda} \quad \text{with} \quad G^+ \Lambda = 0, \quad \tilde{G}^+ \tilde{\Lambda} = 0,
\]

(1.3)

and arguments based on this gauge invariance suggest that beyond reproducing the correct four-point tree amplitude all \(N\)-point tree amplitudes are correctly reproduced by (1.1). The corresponding equation of motion reads

\[
\tilde{G}^+(e^{-\Phi} G^+ e^\Phi) = 0,
\]

(1.4)

where contour integrations are implied again. This is the equation we set out to solve.

Solutions to the string field equation (1.4) describe classical string fields, which convey information about nonperturbative string configurations. There has been renewed interest in such solutions \(^1\)All states in the string field \(\Phi\) are taken to be GSO(+). For including GSO(−) states (e.g. for the study of tachyon condensation) one has to add internal Chan-Paton labels to \(\Phi\) and also to \(G^+\) and \(\tilde{G}^+\) (for a review on this subject see \([11, 12]\)).
for open string field theories, initiated by a series of conjectures due to Sen (see [13] for a review and a list of references). The latter have been tested in various ways, in particular within the above-described nonpolynomial superstring field theory: Using the level-truncation scheme, numerical checks were performed e.g. in [14, 15, 16, 17], and the predicted kink solutions describing lower-dimensional D-branes were found [18]. On the analytic side, a background-independent version of Berkovits’ string field theory was proposed in [19, 20], and the ideas of vacuum string field theory and computations of the silver state of bosonic string field theory were transferred to the superstring case [21, 22, 23]. More recently, there have been some attempts [24, 25, 23] to solve the string field equation (1.4) but, to our knowledge, no general method for finding explicit solutions has been presented so far.

In the present paper we show that the WZW-like string field equation (1.4) is integrable in the sense that it can be written as the compatibility condition of some linear equations with an extra “spectral parameter” \( \lambda \). This puts us in the position to parametrize solutions of (1.4) by solutions of linear equations on extended string fields (depending on the parameter \( \lambda \)) and to construct classes of explicit solutions via various solution-generating techniques. This discussion will be independent of any implementation of \( G^+ \) and \( \tilde{G}^+ \) in terms of matter multiplets and is therefore valid for \( N = 1 \) superstrings as well as for \( N = 2 \) strings.

We discuss two related approaches to generating solutions of the WZW-like string field equation (1.4). First, considering the splitting method and using the simplest Atiyah-Ward ansatz for the matrix-valued string field of the associated Riemann-Hilbert problem, we reduce eq. (1.4) to the linear equations \( \tilde{G}^+ G^+ \rho_k = 0 \) with \( k = 0, \pm 1 \). Here \( \rho_0 \) and \( \rho_{\pm 1} \) are some string fields parametrizing the field \( e^\phi \). However, our discussion of the splitting approach is restricted to the \( n = 2 \) case, i.e. \( u(2) \) Chan-Paton factors, or a certain embedding of \( u(2) \) into \( u(2 + l) \). Second, we consider the (related) dressing approach which overcomes this drawback. With this method, new solutions are constructed from an old one by successive application of simple (dressing) transformations. Using them, we write down an ansatz which reduces the nonlinear equation of motion (1.4) to a system of linear equations. Solutions of the latter describe nonperturbative field configurations obeying (1.4). Finally, we present some explicit solutions of the WZW-like string field equation.

The paper is organized as follows: In the next section we describe the superconformal algebras needed for the formulation of the action (1.1) and give their implementation in terms of massless \( N = 1 \) and \( N = 2 \) matter multiplets for later use. In section 3 we discuss the reality properties of some operators in the superconformal algebra. This issue is crucial for the treatment of the linear equations. We prove the integrability of the WZW-like string field equation in section 4. The construction of solutions via solving a Riemann-Hilbert problem and via dressing a seed solution is subject to sections 5 and 6, respectively. Section 7 presents some explicit solutions. Finally section 8 concludes with a brief summary and open problems.

2 Definitions and conventions

Small \( N = 4 \) superconformal algebra. To begin with, we consider an \( N = 2 \) superconformal algebra generated by an energy-momentum tensor \( T \), two spin 3/2 superpartners \( G^\pm \) and a \( U(1) \) current \( J \). It can be embedded into a small \( N = 4 \) superconformal algebra with two additional superpartners \( \tilde{G}^\pm \) and two spin 1 operators \( J^{++} \) and \( J^{--} \) supplementing \( J \) to an \( SU(2) \) (or \( SU(1, 1) \))
current algebra \[\text{3}\]. To this end \[\text{1}\], \(J\) can be “bosonized” as \(J = \partial H\), and we define
\[
J^{++} := e^H, \quad J^{--} := e^{-H},
\]
(2.1)
where \(H(z)\) has the OPE
\[
H(z)H(0) \sim \frac{c}{3} \log z
\]
(2.2)
with a central charge \(c\). Then \(\tilde{G}^{\pm}\) can be defined by
\[
\tilde{G}^{-}(z) := \oint \frac{dw}{2\pi i} J^{--}(w)G^{+}(z) = [J^{--}_0, G^{+}(z)],
\]
(2.3)
\[
\tilde{G}^{+}(z) := \oint \frac{dw}{2\pi i} J^{++}(w)G^{-}(z) = [J^{++}_0, G^{-}(z)],
\]
(2.4)
so that \((G^{+}, \tilde{G}^{-})\) and \((\tilde{G}^{+}, G^{-})\) transform as doublets under \(SU(2)\) (or \(SU(1,1)\)). We will see below, however, that there is a certain freedom to embed the \(N=2\) superconformal algebra into a small \(N=4\) superconformal algebra, parametrized by \(SU(2)\) (or \(SU(1,1)\)). So, the embedding given above corresponds to a special choice (see section \(\text{3}\) for more details).

This small \(N=4\) algebra in general has a nonvanishing central charge. In topological string theories \[\text{2}\], it is removed by “twisting” \(T\) by the \(U(1)\) current \(J\), i.e. \(T \rightarrow T + \frac{1}{2} \partial J\), so that the resulting algebra has vanishing central charge. This will be shown explicitly in the subsequent paragraphs.

**Realization in terms of \(N=1\) matter multiplets.** For the construction of an anomaly-free \(N=1\) superstring theory, ten massless matter multiplets are needed. A massless \(N=1\) matter multiplet \((X, \psi)\) consists of real bosons \(X\) (the ten string coordinates) and \(so(1,1)\) Majorana spinors \(\psi\) (their superpartners, each splitting up into a left- and a right-handed Majorana-Weyl spinor). Due to the reparametrization invariance of the related supersymmetric sigma model its covariant quantization entails the introduction of world-sheet (anti)ghosts \(b\) and \(c\) and their superpartners \(\beta\) and \(\gamma\). The superghosts are bosonized in the usual way \[\text{3}\], thereby bringing about the anticommuting fields \(\eta\) and \(\xi\). The realization of the previously mentioned \(N=2\) superconformal algebra in terms of these multiplets is given by \[\text{2}\]
\[
T = T_{N=1} + \frac{1}{2} \partial (bc + \xi \eta),
\]
\[
G^- = b, \quad G^+ = j_{BRST} + \partial^2 c + \partial (c \xi \eta),
\]
\[
J = cb + \eta \xi,
\]
(2.5)
where the energy-momentum tensor \(T_{N=1}\) is the usual one with central charge \(c = 0\) and \(j_{BRST}\) is the BRST current. These generators make up an \(N=2\) superconformal algebra with \(c = 6\), i.e. with the same central charge as the critical \(N=2\) superstring. A straightforward method for calculating scattering amplitudes would be to introduce an \(N=2\) superghost system with \(c_{\text{gh}} = -6\) compensating the positive central charge \[\text{2}\]. However, there is a more elegant method \[\text{2}\]: One can embed the \(N=2\) algebra into a small \(N=4\) algebra (as described above) and afterwards twist

\[\text{3}\]The case of an \(SU(2)\) current algebra applies to \(N=1\) strings, and the case of an \(SU(1,1)\) current algebra corresponds to the \(N=2\) string.
by the $U(1)$ current $J$. Then,
\[
\tilde{G}^- = [Q, b\xi] = -b\mathcal{X} + \xi T_{N=1}, \quad \tilde{G}^+ = \eta,
\]
\[
J^- = b\xi, \quad J^++ = c\eta, \tag{2.6}
\]
where $\mathcal{X}$ denotes the picture raising operator and $Q$ is the BRST-operator of the original $N=1$ string theory. These generators together make up a small $N=4$ superconformal algebra with $c = 6$. The twist $T \rightarrow T + \frac{1}{2}\partial J$ in this case amounts to removing the term $\frac{1}{2}\partial(bc + \xi\eta)$ from $T$, thereby reproducing the original $T_{N=1}$ with $c = 0$. It shifts the weight of each conformal field by $-1/2$ of its $U(1)$ charge — in particular, $G^+$ and $\tilde{G}^+$ after twisting become fermionic spin 1 generators which subsequently serve as BRST-like currents. Their zero modes are exactly $Q$ and $\eta_0$, respectively.

**Realization in terms of $N=2$ matter multiplets.** For later use, we also discuss the representation in terms of $N=2$ multiplets \cite{25, 26, 29, 30}: From the world-sheet point of view, critical $N=2$ strings in flat Kleinian space $\mathbb{R}^{2,2} \cong \mathbb{C}^{1,1}$ are described by a theory of $N=2$ supergravity on a $(1+1)$-dimensional (pseudo)Riemann surface, coupled to two chiral $N=2$ massless matter multiplets $(Z, \psi)$. The components of these multiplets are complex scalars (the four string coordinates) and $so(1,1)$ Dirac spinors (their four NSR partners). The latter can be chosen to have unit charge $\pm 1$ under the $U(1)$ current $J$ of the $N=2$ superconformal algebra; we denote them by $\psi^\pm$ (restricting ourselves to the chiral half). One has the following realization of the $N=2$ super Virasoro algebra in terms of free fields:
\[
T = -\frac{1}{4}\eta_{a\bar{a}} \left( 2\partial Z^a \partial \bar{Z}^{\bar{a}} - \psi^{+a} \partial \psi^{-\bar{a}} - \psi^{-\bar{a}} \partial \psi^{+a} \right),
\]
\[
G^+ = \eta_{a\bar{a}} \psi^{+a} \partial \bar{Z}^{\bar{a}}, \quad G^- = \eta_{a\bar{a}} \psi^{-\bar{a}} \partial Z^a,
\]
\[
J = \frac{1}{2}\eta_{a\bar{a}} \psi^{-\bar{a}} \psi^{+a}.
\]
Here $a \in \{0, 1\}$, $\bar{a} \in \{\bar{0}, \bar{1}\}$, and the metric ($\eta_{a\bar{a}}$) with nonvanishing components $\eta_{1\bar{1}} = -\eta_{0\bar{0}} = 1$ defines an $SU(1,1)$-invariant scalar product. Using the operator product expansions
\[
Z^a(z, \bar{z}) \bar{Z}^{\bar{a}}(w, \bar{w}) \sim -2\eta^{a\bar{a}} \ln |z-w|^2 \quad \text{and} \quad \psi^{+a}(z)\psi^{-\bar{a}}(w) \sim -\frac{2\eta^{a\bar{a}}}{z-w},
\]
\[
\tag{2.8}
\]
it is easy to check that these currents implement the $N=2$ superconformal algebra with central charge $c = 6$. This central charge is usually compensated by that of the $N=2$ (super)ghost Virasoro algebra. Equivalently, the $N=2$ super Virasoro algebra can be embedded into a small $N=4$ superconformal algebra \cite{24} which after twisting has vanishing central charge. Note that in this approach, contrary to the $N=1$ case, we do not need to introduce reparametrization ghosts. The $N=4$ extension is achieved by adding the currents
\[
J^{++} = \frac{1}{4}\varepsilon_{ab} \psi^{+a} \psi^{+b}, \quad J^{-} = \frac{1}{4}\varepsilon_{\bar{a}\bar{b}} \psi^{-\bar{a}} \psi^{-\bar{b}},
\]
\[
\tilde{G}^+ = -\varepsilon_{ab} \psi^{+a} \partial Z^b, \quad \tilde{G}^- = -\varepsilon_{\bar{a}\bar{b}} \psi^{-\bar{a}} \partial \bar{Z}^{\bar{b}},
\]
\[
\text{where we choose the convention that } \varepsilon_{01} = \varepsilon_{\bar{0}\bar{1}} = -\varepsilon^{01} = -\varepsilon^{\bar{0}\bar{1}} = 1. \quad \text{The twist } T \rightarrow T + \frac{1}{2}\partial J \text{ shifts the weights by } -1/2 \text{ of their fields’ } U(1) \text{ charge, and again the supercurrents } G^+ \text{ and } \tilde{G}^+ \text{ become fermionic spin 1 currents (cf. the } N=1 \text{ case).}
\]

**String fields.** In the rest of the paper, we will consider string fields as functionals of $X^\mu(\sigma, \tau)$, $\psi^\mu(\sigma, \tau)$, $b(\sigma, \tau)$, $c(\sigma, \tau)$, $\beta(\sigma, \tau)$, and $\gamma(\sigma, \tau)$ for $N=1$ and of $Z^a(\sigma, \tau)$, $\bar{Z}^{\bar{a}}(\sigma, \tau)$, $\psi^{+a}(\sigma, \tau)$, and
ψ^{-\bar{a}}(\sigma, \tau) for \( N = 2 \). We will often suppress the \( \tau \)-dependence of the world-sheet fields and only indicate their \( \sigma \)-dependence when necessary. As an abbreviation for this list of world-sheet fields, we use \( X \) and \( \psi \) and thus denote a string field \( \Phi \) by \( \Phi[X, \psi] \) (in the \( N = 2 \) case, one may think of \( X \) as the real and imaginary parts of \( Z \), similarly with \( \psi \)). Throughout this paper, all string fields are understood to be multiplied via Witten’s star product. Equivalently, one can consider the string fields as operators (see e.g. [31] and references therein).

3 Reality properties

Real structures on \( sl(2, \mathbb{C}) \). We would like to introduce a real structure on the Lie algebra \( sl(2, \mathbb{C}) \). A real structure \( \sigma \) on a vector space \( V \) is by definition an antilinear involution \( \sigma : V \to V \), i.e. \( \sigma(\zeta X + Y) = \zeta \sigma(X) + \sigma(Y) \) for \( \zeta \in \mathbb{C}, X, Y \in V \) and \( \sigma^2 = \text{id}_V \) (here we choose \( V = sl(2, \mathbb{C}) \)). There are three choices for a real structure on \( sl(2, \mathbb{C}) \). Acting onto the defining representation of \( sl(2, \mathbb{C}) \), we can write them as

\[
\sigma_\varepsilon \begin{pmatrix} a & b \\ c & -a \end{pmatrix} := \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{pmatrix} \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \tag{3.1}
\]

\[
\sigma_0 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} := \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{pmatrix} \tag{3.2}
\]

for \( \varepsilon = \pm 1 \) and \( a, b, c \in \mathbb{C} \). We denote the space of fixed points by \( V_\mathbb{R} \), i.e. \( V_\mathbb{R} := \{ X \in V : \sigma(X) = X \} \). For the real structures (3.1), it is straightforward to check that \( V_\mathbb{R} \cong su(2) \) for \( \varepsilon = -1 \) and \( V_\mathbb{R} \cong su(1,1) \) for \( \varepsilon = 1 \), whereas, for the real structure \( \sigma_0 \), we obtain \( V_\mathbb{R} \cong sl(2, \mathbb{R}) \). Real linear combinations of vectors within \( V_\mathbb{R} \) are again contained in \( V_\mathbb{R} \), so they make up real linear subspaces in \( sl(2, \mathbb{C}) \). Because of the form of eqs. (3.1) and (3.2) it is clear that \( \sigma \) preserves the Lie algebra structure, i.e. the \( V_\mathbb{R} \) are real Lie subalgebras. In the case of \( \sigma_\varepsilon \), complex conjugation can be “undone” by a conjugation with the matrix \( \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \) on an element of this linear subspace. Let us already here note that this conjugation matrix is contained within the group \( SU(2) \) for \( \varepsilon = -1 \), but is outside \( SU(1,1) \) for \( \varepsilon = 1 \). This means [34, 35] that conjugation by \( \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \) is an inner automorphism on \( su(2) \) and an outer automorphism on \( su(1,1) \).

Bar operation. As stated above, an \( N = 4 \) superconformal algebra contains a subalgebra a current algebra generated by \( J, J^{++}, \text{ and } J^{--} \). Its horizontal Lie algebra, when complexified, is \( sl(2, \mathbb{C}) \). On \( sl(2, \mathbb{C}) \), let us introduce a real structure \( \sigma_\varepsilon \) from (3.1), where we choose \( \varepsilon = -1 \) for \( N = 1 \) and \( \varepsilon = 1 \) for \( N = 2 \) strings. The action of complex conjugation on the \( N = 4 \) superconformal generators, as determined in [26, 28], looks as follows:

\[
J^* = -J, \quad (J^{++})^* = J^{--}, \quad (G^+)^* = G^-, \quad (\tilde{G}^+)^* = \tilde{G}^- \tag{3.3}
\]

Now, the sign flip of \( J \) in (3.3) necessitates an additional rotation to reestablish the original twist \( T \to T + \frac{1}{2} \partial J \) of the \( N = 4 \) superconformal algebra. As mentioned in the previous paragraph, this can be accomplished by an inner automorphism in the case of \( N = 1 \) strings and by an outer automorphism of the current Lie algebra in the case of \( N = 2 \) strings. These automorphisms consist in conjugating an element of the defining representation with the matrix \( \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \). In the following, we want to scrutinize how the rotation reestablishing the twist acts onto the other elements of the \( N = 4 \) superconformal algebra.

\footnote{In the following, we will be mainly interested in \( \sigma_\varepsilon \). The case of eq. (3.2) has been used in [12] and [33].}
As explained in [28], the four spin $\frac{3}{2}$ superpartners of the energy-momentum tensor transform under the current group (therefore the current indices $\pm$) as well as under a group\footnote{For a discussion of its existence, see [28].} of additional automorphisms $SU(2)$ (or $SU(1,1)$). Explicitly,

$$
(G^{\tilde{\alpha}\alpha}) = \begin{pmatrix} G^+ & \tilde{G}^+ \\ \varepsilon \tilde{G}^- & G^- \end{pmatrix} \quad (3.4)
$$

transforms under current group transformations by left-multiplication (note that the columns of this matrix form doublets under the current algebra, cf. (2.3) and (2.4)). Under additional automorphisms it transforms by right-multiplication. Obviously, the rotation reestablishing the original twist acts onto the complex conjugate matrix as

$$
\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} G^- & \tilde{G}^- \\ \varepsilon \tilde{G}^+ & G^+ \end{pmatrix} = \begin{pmatrix} \varepsilon \tilde{G}^+ & G^+ \\ \varepsilon G^- & \varepsilon \tilde{G}^- \end{pmatrix} \quad (3.5)
$$

Having established the action of this rotation on the $N=4$ superconformal algebra, we denote the combined operation of complex conjugation and “twist-restoring” transformation by a bar, i.e.

$$
\begin{align*}
G^+ & = \varepsilon \tilde{G}^+ , & \tilde{G}^+ & = G^+ , & \tilde{G}^+ & = G^- , & G^- & = \varepsilon \tilde{G}^- .
\end{align*} \quad (3.6)
$$

On elements of the complexified current algebra, the bar operation acts by (3.1), in particular it acts trivially onto the real $su(2)$ or $su(1,1)$ subalgebra.

**Additional automorphisms.** The aforementioned additional automorphisms of the $N=4$ superconformal algebra act on the “$G$-matrix” from the right [28]. We take them to be elements of $SL(2,\mathbb{C})$ and therefore define

$$
\begin{pmatrix} G^+ (v) & \tilde{G}^+ (v) \\ \varepsilon \tilde{G}^- (v) & G^- (v) \end{pmatrix} := \begin{pmatrix} G^+ & \tilde{G}^+ \\ \varepsilon \tilde{G}^- & G^- \end{pmatrix} \begin{pmatrix} v_3 & v_1 \\ v_4 & v_2 \end{pmatrix} \quad (3.7)
$$

for $v_i \in \mathbb{C}$ with $v_2 v_3 - v_1 v_4 = 1$. The action of the additional automorphisms should be compatible with the above-defined bar operation (3.6), i.e.

$$
\begin{align*}
\overline{G^+ (v)} & = \varepsilon \tilde{G}^+ (v) , & \overline{G^+ (v)} & = G^+ (v) , & \overline{\tilde{G}^+ (v)} & = G^- (v) , & \overline{G^- (v)} & = \varepsilon \tilde{G}^- (v) .
\end{align*} \quad (3.8)
$$

These four equations all lead to the same requirements $v_3 = \tilde{v}_2 =: u_1$ and $v_4 = \varepsilon \tilde{v}_1 =: u_2$, which in effect restricts the group of additional automorphisms from $SL(2,\mathbb{C})$ to $SU(2)$ for $\varepsilon = -1$ and to $SU(1,1)$ for $\varepsilon = 1$:

$$
\begin{pmatrix} G^+ (u) & \tilde{G}^+ (u) \\ \varepsilon \tilde{G}^- (u) & G^- (u) \end{pmatrix} := \begin{pmatrix} G^+ & \tilde{G}^+ \\ \varepsilon \tilde{G}^- & G^- \end{pmatrix} \begin{pmatrix} u_1 & \varepsilon \tilde{u}_2 \\ u_2 & \tilde{u}_1 \end{pmatrix} \quad (3.9)
$$

with $|u_1|^2 - \varepsilon |u_2|^2 = 1$. Note that

$$
\begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 & \varepsilon \tilde{u}_2 \\ u_2 & \tilde{u}_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} = \begin{pmatrix} \tilde{u}_1 & \varepsilon u_2 \\ \tilde{u}_2 & u_1 \end{pmatrix} . \quad (3.10)
$$

Obviously, eq. (3.8) entails the introduction of the same real structure on the group of additional automorphisms as the one chosen for the current group. Since we are only interested in the
ratio of the prefactors of $G^+$ and $\tilde{G}^+$, and of $G^-$ and $\tilde{G}^-$, respectively, we define for later use the combinations

$$
\tilde{G}^+(\lambda) := \tilde{G}^+ + \lambda G^+ = \frac{1}{\bar{u}_1} \tilde{G}^+(u),
$$

$$
\tilde{G}^-(\lambda) := \tilde{G}^- + \bar{\lambda} G^- = \frac{1}{u_1} \tilde{G}^-(u),
$$

$$
G^+(\lambda) := G^+ + \varepsilon \bar{\lambda} G^+ = \frac{1}{u_1} G^+(u),
$$

$$
G^-(\lambda) := G^- + \varepsilon \lambda G^- = \frac{1}{u_1} G^-(u).
$$

(3.11)

Here, $(\bar{u}_1 : \varepsilon \bar{u}_2)$ can be regarded as homogeneous coordinates on the sphere $S^2 \cong \mathbb{C}P^1$, and $\lambda \equiv \varepsilon \bar{u}_2/\bar{u}_1$ is a local coordinate for $\bar{u}_1 \neq 0$.

Let us reconsider the action of the bar operation on the matrix $(3.9)$. Remember that the bar operation consists of a complex conjugation and a subsequent twist-restoring rotation as in $(3.7)$. Acting on $(3.9)$, we have

$$
\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} G^- & \tilde{G}^- \\ \varepsilon \tilde{G}^+ & G^+ \end{pmatrix} \begin{pmatrix} \bar{u}_1 & \varepsilon \bar{u}_2 \\ \bar{u}_2 & u_1 \end{pmatrix} = \begin{pmatrix} \varepsilon \tilde{G}^+ & G^+ \\ \varepsilon G^- & \tilde{G}^- \end{pmatrix} \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 & \varepsilon \bar{u}_2 \\ u_2 & \bar{u}_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} G^+ & \tilde{G}^+ \\ \varepsilon \tilde{G}^- & G^- \end{pmatrix} \begin{pmatrix} \bar{u}_2 & u_1 \\ \bar{u}_1 & u_2 \end{pmatrix}.
$$

(3.12)

For the first equality, we have used $(3.3)$ and $(3.10)$. An additional right-multiplication by $(\begin{smallmatrix} 0 & 1 \\ \varepsilon & 0 \end{smallmatrix})$ transforms the “u-matrix” on the right hand side back to its original form (cf. $(3.14)$), thereby taking $\varepsilon \bar{\lambda}^{-1}$ to $\lambda$. This mediates a map between operators defined for $|\lambda| < \infty$ and operators defined for $|\lambda| > 0$.

The action of the bar operation on $\tilde{G}^+(\varepsilon \bar{\lambda}^{-1})$ etc. can be determined from the fact that $\varepsilon \bar{\lambda}^{-1} = u_1/u_2$ and therefore

$$
\begin{pmatrix} G^+(\varepsilon \bar{\lambda}^{-1}) & \tilde{G}^+(\varepsilon \bar{\lambda}^{-1}) \\ \varepsilon \tilde{G}^+(\varepsilon \bar{\lambda}^{-1}) & G^-(\varepsilon \bar{\lambda}^{-1}) \end{pmatrix} = \begin{pmatrix} G^+ & \tilde{G}^+ \\ \varepsilon \tilde{G}^- & G^- \end{pmatrix} \begin{pmatrix} \bar{u}_2 & u_1 \\ \bar{u}_1 & u_2 \end{pmatrix} \begin{pmatrix} u_2^{-1} & 0 \\ 0 & u_2^{-1} \end{pmatrix}.
$$

(3.13)

Multiplying the complex conjugate matrices from the left by a twist-restoring rotation, we obtain

$$
\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} G^- & \tilde{G}^- \\ \varepsilon \tilde{G}^+ & G^+ \end{pmatrix} \begin{pmatrix} u_2 & \bar{u}_1 \\ \bar{u}_2 & u_1 \end{pmatrix} \begin{pmatrix} u_2^{-1} & 0 \\ 0 & \bar{u}_2^{-1} \end{pmatrix} = \begin{pmatrix} \bar{\lambda}^{-1} G^+(\lambda) & \lambda^{-1} \tilde{G}^+(\lambda) \\ \varepsilon \bar{\lambda}^{-1} \tilde{G}^-(\lambda) & \varepsilon \bar{\lambda}^{-1} G^-(\lambda) \end{pmatrix}.
$$

(3.14)

So, “barring” accompanied by the transformation $\lambda \mapsto \varepsilon \bar{\lambda}^{-1}$ maps $\tilde{G}^+(\lambda)$ (defined for $|\lambda| < \infty$) to $\frac{1}{\lambda} \tilde{G}^+(\lambda)$ (defined for $|\lambda| > 0$),

$$
\tilde{G}^+(\varepsilon \bar{\lambda}^{-1}) = \frac{1}{\lambda} \tilde{G}^+(\lambda).
$$

(3.15)

This result will be needed in section 4.

For the selection of an $N=2$ superconformal subalgebra within the $N=4$ algebra, there is obviously the freedom to choose a linear combination of the $J$’s as the $U(1)$ current. All choices are equivalent through current $SU(2)$- (or $SU(1,1)$-) rotations (acting on the matrices in $(3.3)$ from
the left). In addition, there is the freedom to choose which linear combination of the positively charged \( G \)'s will be called \( G^+ \) \[36\]. This freedom is parametrized by another \( SU(2) \) (or \( SU(1,1) \)), cf. \[34\]. Since in our case only the ratio of the prefactors of \( G^+ \) and \( \tilde{G}^+ \) is important, we arrive at generators \( \{3.11\} \) parametrized by \( \lambda \in \mathbb{C}P^1 \). So, we obtain a one-parameter family of \( N=2 \) superconformal algebras embedded into a small \( N=4 \) algebra \[26, 28\].

4 Integrability of Berkovits’ string field theory

In this section we show that the equation of motion \( (1.4) \) of Berkovits’ string field theory can be obtained as the compatibility condition of some linear equations. In other words, solutions of these linear equations exist iff eq. \( (1.4) \) is satisfied. For \( N=2 \) strings the integrability of Berkovits’ string field theory was shown in \[33\], and here we extend this analysis to the \( N=1 \) case. The payoff for considering such (integrable) models is the availability of powerful techniques for the construction of solutions to the equation(s) of motion.

Sphere \( \mathbb{C}P^1 \). Let us consider the Riemann sphere \( S^2 \cong \mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\} \) and cover it by two coordinate patches

\[
\mathbb{C}P^1 = U_+ \cup U_-, \quad U_+ := \{ \lambda \in \mathbb{C} : |\lambda| < 1 + \epsilon \}, \quad U_- := \{ \lambda \in \mathbb{C} \cup \{\infty\} : |\lambda| > (1 + \epsilon)^{-1} \}
\]

for some \( \epsilon > 0 \) with the overlap

\[
U_+ \cap U_- \supset S^1 = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}.
\]

We will consider \( \lambda \in U_+ \) and \( \tilde{\lambda} \in U_- \) as local complex coordinates on \( \mathbb{C}P^1 \) with \( \tilde{\lambda} = \frac{1}{\lambda} \) in \( U_+ \cap U_- \). Recall that \( \lambda \in \mathbb{C}P^1 \) was introduced in the previous section as a parameter for a family of \( N=2 \) subalgebras in the twisted small \( N=4 \) superconformal algebra with fermionic currents \( \{3.11\} \).

Linear system. Taking the string field \( \Phi \) from \( \{1.1\} \) and operators \( \tilde{G}^+(\lambda) \) from \( \{3.11\} \), we introduce the following equation:

\[
(\tilde{G}^+ + \lambda G^+ + \lambda A)\Psi = 0,
\]

where \( A := e^{-\Phi}G^+e^\Phi \) and \( \Psi \) is a matrix-valued string field depending not only on \( X \) and \( \psi \) but also (meromorphically) on the auxiliary parameter \( \lambda \in \mathbb{C}P^1 \). As in eq. \( \{1.2\} \) the action of \( G^+ \) and \( \tilde{G}^+ \) on \( \Psi \) implies a contour integral of the (fermionic) current around \( \Psi \). Note that \( G^+ \) and \( \tilde{G}^+ \) are Grassmann-odd and therefore \( (G^+)^2 = (\tilde{G}^+)^2 = G^+\tilde{G}^+ + \tilde{G}^+G^+ = 0 \).

If \( e^\Phi \) is given, then \( \{1.3\} \) is an equation for the field \( \Psi \). Solutions \( \Psi \) of this linear equation exist if the term in brackets squares to zero, i.e.

\[
(\tilde{G}^+ + \lambda G^+ + \lambda A)^2 = 0 \quad \Leftrightarrow \quad \lambda^2(G^+ + A)^2 + \lambda\tilde{G}^+A = 0
\]

for any \( \lambda \). Here we have used the Grassmann nature of \( G^+ \) and \( \tilde{G}^+ \). So, we obtain two equations:

\[
G^+A + A^2 = 0, \quad \tilde{G}^+A = 0.
\]

The first of these two equations is trivial since \( G^+ \) acts as a derivation on the algebra of string fields. The second equation coincides with the equation of motion \( \{1.4\} \).
Chiral string fields. As a special case of eq. (4.3) one can consider the equations

\[
\begin{align*}
\left( \tilde{G}^+ + \lambda G^+ + \lambda A \right) \Psi_+ &= 0, \\
\left( \frac{1}{\lambda} \tilde{G}^+ + G^+ + A \right) \Psi_- &= 0,
\end{align*}
\]

(4.7) (4.8)

where \( \Psi_+ \) and \( \Psi_- \) are invertible matrix-valued string fields depending holomorphically on \( \lambda \) and \( \frac{1}{\lambda} \), respectively.

Considering \( \lambda \to 0 \) in (4.7), we see that \( \tilde{G}^+ \Psi_+(\lambda = 0) = 0 \), and one may choose \( \Psi_+(\lambda = 0) = I \).

Analogously, taking \( \lambda \to \infty \) in (4.8), we obtain

\[
A = e^{-\Phi} G^+ e^{\Phi} = \Psi_-^{-1}(\lambda = \infty)\tilde{G}^+ \Psi_-^{-1}(\lambda = \infty),
\]

(4.9)

and one may choose \( \Psi_-(\lambda = \infty) = e^{-\Phi} \) as a solution thereof. From this we derive the asymptotic behavior of our fields:

\[
\begin{align*}
\Psi_+ &= \mathcal{I} + O(\lambda) \quad \text{for } \lambda \to 0, \\
\Psi_- &= e^{-\Phi} + O(\lambda^{-1}) \quad \text{for } \lambda \to \infty.
\end{align*}
\]

(4.10) (4.11)

We see that \( \Psi_- \) may be considered as an \( \lambda \)-augmented solution of eq. (4.4), and all information about \( e^\Phi \) is contained in \( \Psi_\pm \).

Suppose that we find solutions \( \Psi_+ \) and \( \Psi_- \) of eqs. (4.7) and (4.8) for a given \( \Phi \). Then one can introduce the matrix-valued string field

\[
\Upsilon_{+\mp} := \Psi_\mp^{-1} \Psi_\pm
\]

(4.12)

defined for \( \lambda \in U_+ \cap U_- \). From eqs. (4.7) and (4.8) it follows that

\[
\tilde{G}^+(\lambda) \Upsilon_{+\mp} \equiv (\tilde{G}^+ + \lambda G^+) \Upsilon_{+\mp} = 0.
\]

(4.13)

String fields annihilated by the operator \( \tilde{G}^+(\lambda) \) will be called chiral.

Reality properties. W. r. t. the bar operation from section 3, the string field \( \Phi \) is real, i.e.

\[
\Phi[X(\pi - \sigma), \psi(\pi - \sigma)] = \Phi[X(\sigma), \psi(\sigma)].
\]

(4.14)

To see the behavior of the extended string field under the bar operation we scrutinize eqs. (4.7) and (4.8). We already saw that \( \tilde{G}^+(\varepsilon \lambda^{-1}) = \frac{1}{\lambda} \tilde{G}^+ + G^+ \), and by definition, \( \mathcal{I} = (\tilde{G}^+ e^\Phi) e^{-\Phi} = \varepsilon (\tilde{G}^+ e^\Phi) e^{-\Phi} \) under \( \sigma \mapsto \pi - \sigma \). Then mapping \( \lambda \mapsto \varepsilon \lambda^{-1} \) and \( \sigma \mapsto \pi - \sigma \) in (4.7) and conjugating, we obtain

\[
\left( \frac{1}{\lambda} \tilde{G}^+ + G^+ \right) \Psi_+^{-1} - \frac{1}{\lambda}(\tilde{G}^+ e^\Phi) e^{-\Phi} \Psi_+^{-1} = 0.
\]

(4.15)

This coincides with eq. (4.8) if we set

\[
(\Psi_+)^{-1} \left[ X(\pi - \sigma), \psi(\pi - \sigma), \frac{\varepsilon}{\lambda} \right] = e^\Phi \Psi_- \left[ X(\sigma), \psi(\sigma), \lambda \right].
\]

(4.16)

---

6The world-sheet parity transformation \( \sigma \mapsto \pi - \sigma \) is accompanied by a transposition of Chan-Paton matrices. Note that hermitian generators are used for the \( u(n) \) Chan-Paton algebra. Therefore, \( e^\Phi \) does not necessarily take values in \( U(n) \).
Then from (1.13), it follows that
\[
\Upsilon_{+-} = \Psi_- e^\Phi \Psi_-
\]
is real, i.e. \( \Upsilon_{+-}[X(\pi - \sigma), \psi(\pi - \sigma), \varepsilon \lambda^{-1}] = \Upsilon_{+-}[X(\sigma), \psi(\sigma), \lambda] \).

**Gauge freedom.** Recall that gauge transformations of the fields \( e^\Phi \) and \( A \) have the form
\[
e^\Phi \mapsto e^{\Phi'} = Be^{\Phi}C \quad \text{with} \quad G^+B = 0, \quad \bar{G}^+C = 0,
A \mapsto A' = C^{-1}AC + C^{-1}G^+C. \tag{4.18}
\]
Under \( C \)-transformations the fields \( \Psi_\pm \) transform as
\[
\Psi_\pm \mapsto \Psi_\pm' = C^{-1} \Psi_\pm. \tag{4.19}
\]
It is easy to see that the chiral string field \( \Upsilon_{+-} \) is invariant under the transformations (4.19). On the other hand, the field \( A \) will remain unchanged after the transformations
\[
\Psi_+ \mapsto \Psi_+ h_+, \quad \Psi_- \mapsto \Psi_- h_- \tag{4.20}
\]
where \( h_+ \) and \( h_- \) are chiral string fields depending holomorphically on \( \lambda \) and \( \bar{\lambda} \), respectively. In the special case when \( h_+ \) and \( h_- \) are independent of \( \lambda \) the transformations (4.20) with \( h_+ = \mathcal{I}, h_- =: B^{-1} \) induce the \( B \)-transformations \( e^\Phi \mapsto Be^\Phi \) in (4.18). In general, eqs. (4.20) induce the transformations
\[
\Upsilon_{+-} \mapsto h_+^{-1} \Upsilon_{+-} h_-
\]
on the space of solutions to eq. (1.13), and any two solutions differing by a transformation (4.21) are considered to be equivalent.

**Splitting.** Up to now, we discussed how to find \( \Upsilon_{+-} \) for a given \( A = e^{-\Phi}G^+e^\Phi \). Consider now the converse situation. Suppose we found a string field \( \Upsilon_{+-} \) which depends analytically on \( \lambda \in S^1 \) and satisfies the linear equation (4.13). Being real-analytic \( \Upsilon_{+-} \) can be extended to a string field depending holomorphically on \( \lambda \in U_+ \cap U_- \). Then we can formulate an operator version of the Riemann-Hilbert problem: Split \( \Upsilon_{+-} = \Psi_+^1 \Psi_- \) into matrix-valued string fields \( \Psi_\pm \) depending on \( \lambda \in U_+ \cap U_- \) such that \( \Psi_+ \) can be extended to a regular (i.e. holomorphic in \( \lambda \) and invertible) matrix-valued function on \( U_+ \) and that \( \Psi_- \) can be extended to a regular matrix-valued function on \( U_- \). From eq. (1.13) it then follows that
\[
\Psi_+(\bar{G}^+ + \lambda G^+)\Psi_+^{-1} = \Psi_-(\bar{G}^+ + \lambda G^+)\Psi_-^{-1} = \tilde{A} + \lambda A, \tag{4.22}
\]
where \( \tilde{A} \) and \( A \) are some \( \lambda \)-independent string fields. The last equality follows from expanding \( \Psi_+ \) and \( \Psi_- \) into power series in \( \lambda \) and \( \lambda^{-1} \), respectively. If we now choose a function \( \Psi_+ (\lambda) \) such that \( \Psi_+ (\lambda = 0) = \mathcal{I} \) then \( \tilde{A} = 0 \) and
\[
A = \Psi_- (\lambda = \infty)G^+\Psi_-^{-1} (\lambda = \infty) \quad \Rightarrow \quad \Psi_- (\lambda = \infty) =: e^{-\Phi}. \tag{4.23}
\]
In the general case, we have
\[
e^{-\Phi} = \Psi_+^{-1} (\lambda = 0) \Psi_- (\lambda = \infty). \tag{4.24}
\]
\(^7\)This can always be achieved by redefining \( \Psi_+ (\lambda) \mapsto \Psi_+^{-1} (\lambda = 0) \Psi_+ (\lambda), \Psi_- (\lambda^{-1}) \mapsto \Psi_-^{-1} (\lambda = 0) \Psi_- (\lambda^{-1}) \).
So, starting from $\Upsilon_{+-}$ we have constructed a solution $e^{\Phi}$ of eq. (1.4).

Suppose that we know a splitting for a given $\Upsilon_{+-}$ and have determined a correspondence $\Upsilon_{+-} \leftrightarrow e^{\Phi}$. Then for any matrix-valued chiral string field $\tilde{\Upsilon}_{+-}$ from a small enough neighborhood of $\Upsilon_{+-}$ (i.e. $\tilde{\Upsilon}_{+-}$ is close to $\Upsilon_{+-}$ in some norm) there exists a splitting $\tilde{\Upsilon}_{+-} = \tilde{\Psi}^{-1} \tilde{\Psi}$ due to general deformation theory arguments. Namely, there are no obstructions to a deformation of a trivial holomorphic vector bundle $E$ over $\mathbb{C}P^1$ since its infinitesimal deformations are parametrized by the group $H^1(\mathbb{C}P^1, E)$. This cohomology group is trivial because of $H^1(\mathbb{C}P^1, \mathcal{O}) = 0$ where $\mathcal{O}$ is the sheaf of holomorphic functions on $\mathbb{C}P^1$. But from the correspondence $\tilde{\Upsilon}_{+-} \leftrightarrow e^{\Phi}$ it follows that any solution $e^{\Phi}$ from an open neighborhood (in the solution space) of a given solution $e^{\Phi}$ can be obtained from a “free” chiral string field $\tilde{\Upsilon}_{+-}$. In this sense, Berkovits’ string field theory is an integrable theory; in other words, it is completely solvable.

To sum up, we have described a one-to-one correspondence between the gauge equivalence classes of solutions to the nonlinear equation of motion (1.4) and equivalence classes of solutions (chiral string fields) to the auxiliary linear equation (4.13). The next step is to show how this correspondence helps to solve (1.4).

5 Exact solutions by the splitting approach

Atiyah-Ward ansatz. As described in the previous section, solutions of the string field equations (1.4) can be obtained by splitting a given matrix-valued chiral string functional $\Upsilon_{+-}$. In general, splitting is a difficult problem but for a large class of special cases it can be achieved. The well known cases (for $n = 2$) are described e.g. by the infinite hierarchy of Atiyah-Ward ansätze [37] generating instantons in four-dimensional $SU(2)$ Yang-Mills theory. These ansätze are easily generalized [38] to the case of noncommutative instantons, the first examples of which were given in [39]. Here, we consider the first Atiyah-Ward ansatz from the above-mentioned hierarchy [37] and discuss its generalization to the string field case.

We start from the $2 \times 2$ matrix

$$\Upsilon_{+-} = \begin{pmatrix} \rho & \lambda^{-1} \mathcal{I} \\ \varepsilon \lambda \mathcal{I}^{-1} & 0 \end{pmatrix}, \tag{5.1}$$

where $\rho$ is a real and chiral string field, i.e.

$$\mathcal{P} \left[ X, \psi, \varepsilon \lambda \right] = \rho [X, \psi, \lambda] \tag{5.2}$$

and

$$(\tilde{G}^+ + \lambda G^+) \rho = 0. \tag{5.3}$$

We assume that $\rho$ depends on $\lambda \in S^1$ analytically and therefore can be extended holomorphically in $\lambda$ to an open neighborhood $U_+ \cap U_-$ of $S^1$ in $\mathbb{C}P^1$. From (5.2) and (5.3) it follows that the matrix $\Upsilon_{+-}$ in (5.1) is chiral and real.

Splitting. We now expand $\rho$ into a Laurent series in $\lambda$,

$$\rho = \sum_{k=-\infty}^{\infty} \lambda^k \rho_k = \rho_- + \rho_0 + \rho_+, \quad \rho_- = \sum_{k<0} \lambda^k \rho_k, \quad \rho_+ = \sum_{k>0} \lambda^k \rho_k, \tag{5.4}$$
and obtain from (5.3) for

$$\rho_k = \oint \frac{d\lambda}{2\pi i} \lambda^{-k-1} \rho$$

(5.5)

the recursion relations

$$\tilde{G}^+ \rho_{k+1} = -G^+ \rho_k.$$  

(5.6)

Using (5.4), one easily checks that

$$\Upsilon_{+-} = \hat{\Psi}^{-1}_+ \hat{\Psi}^-$$

(5.7)

where

$$\hat{\Psi}^- = \rho_0^{-1/2} \left( \rho_0 + \rho_1 + \lambda \rho_1 + \lambda^{-1} \mathcal{I} \right), \quad \hat{\Psi}^+ = \rho_0^{-1/2} \left( \rho_0 + \rho_1 + \lambda^{-1} \mathcal{I} \right).$$  

(5.8)

However, the asymptotic value of $\hat{\Psi}_+$,

$$\hat{\Psi}_+(\lambda = 0) = \rho_0^{-1/2} \left( \mathcal{I} - \varepsilon \lambda^{-1} \rho_1 \right) \bigg|_{\lambda = 0} = \rho_0^{-1/2} \left( \mathcal{I} - \varepsilon \rho_1 \right) \neq 1_{2\mathcal{I}},$$

(5.9)

shows that this splitting corresponds to a more general gauge than the one used in eq. (1.4) (see [32, 33] for a discussion of this gauge in the case of $N = 2$ strings).

To obtain the asymptotic behavior (4.10) one may exploit the “gauge freedom” contained in (5.7) and introduce the fields

$$\Psi_+ := \tilde{\Psi}^{-1}_+(\lambda = 0) \tilde{\Psi}_+ \quad \text{and} \quad \Psi_- := \tilde{\Psi}^{-1}_+(\lambda = 0) \tilde{\Psi}_-$$

(5.10)

which by definition have the right asymptotic behavior. These functionals yield the same chiral string field since

$$\Upsilon_{+-} = \hat{\Psi}^{-1}_+ \hat{\Psi}^- = \tilde{\Psi}^{-1}_+ \tilde{\Psi}_+(\lambda = 0) \tilde{\Psi}^{-1}_+(\lambda = 0) \tilde{\Psi}_- = \Psi^{-1}_+ \Psi_. $$

(5.11)

**Explicit solutions.** Now, from

$$\Psi_- = \begin{pmatrix} \rho_0 + \rho_1 + \lambda \rho_1 & -\lambda^{-1} \rho_1 \\ -\varepsilon \lambda \rho_0^{-1} & -\varepsilon \rho_0^{-1} \end{pmatrix}$$

(5.12)

we can determine a solution of (1.4) with the help of (4.11),

$$e^{-\Phi} = \Psi_-(\lambda = \infty) = \begin{pmatrix} \rho_0 - \rho_1 \rho_0^{-1} \rho_1 & -\rho_1 \rho_0^{-1} \\ -\varepsilon \rho_0^{-1} & -\varepsilon \rho_0^{-1} \end{pmatrix}. $$

(5.13)

A direct calculation shows that this satisfies eq. (1.4) iff $\rho_0$, $\rho_1$ and $\rho_-$ satisfy the linear recursion relations (5.6) for $k = -1, 0$. Moreover, substituting (5.13) into (1.4) yields

$$\tilde{G}^+ G^+ \rho_0 = 0$$

(5.14)

which is the analogue of the Laplace equation in the case of instantons in four-dimensional Euclidean space [38, 39]. Note that (5.14) is just one of an infinite set of equations,

$$\tilde{G}^+ G^+ \rho_k = 0 \quad \forall k \in \mathbb{Z},$$

(5.15)
which can easily be obtained from the recursion relations (5.6). So, the ansatz (5.1) for \( \Upsilon^+ - \Upsilon^- \) and its splitting reduce the nonlinear string field theory equation (1.4) to the linear equations (5.6) which are equivalent to the chirality equation (5.3).

Finally, notice that in the case of \( N = 1 \) strings, one may take \( |\rho_0\rangle = \xi_0|V\rangle \) where \( |V\rangle \) is a state in the “small” Hilbert space \( \mathfrak{h} \). Then eq. (5.14) reduces to

\[
Q_{\rho_0}(\xi_0|V\rangle) = Q(-\xi_0\eta_0|V\rangle + |V\rangle) = 0,
\]

i.e.

\[
Q|V\rangle = 0 \quad \text{and} \quad \eta_0|V\rangle = 0.
\]

This fits in nicely with the discussion in [40].

6 Exact solutions via dressing of a seed solution

**Extended solutions.** In the previous section we discussed solutions \( \Psi_+ \) and \( \Psi_- \) of the linear system which are holomorphic in \( \lambda \) and \( 1/\lambda \), respectively. Now we are interested in those solutions \( \Psi \) of eq. (4.3) which are holomorphic in open neighborhoods of both \( \lambda = 0 \) and \( \lambda = \infty \) and therefore have poles at finite points \( \lambda = \mu_k, k = 1, \ldots, m \). Again we see from (4.3) that \( \Psi(\lambda = \infty) \) coincides with \( e^{-\Phi} \) up to a gauge transformation and we fix the gauge by putting

\[
\Psi^{-1}(\lambda = \infty) = e^{\Phi}.
\]

The string field \( \Psi[X, \psi, \lambda] \) will be called the extended solution corresponding to \( e^{\Phi} \). Recall that \( e^{\Phi} \) is a solution of (1.4) where \( \Phi \) carries \( u(n) \) Chan-Paton labels.

The reality properties of extended solutions are derived in very much the same way as the reality properties of \( \Psi_+ \) and \( \Psi_- \). Namely, one can easily show that if \( \Psi[X, \psi, \lambda] \) satisfies eq. (4.3) then \( e^{-\Phi}\Psi^{-1}[X, \psi, \varepsilon\lambda^{-1}] \) satisfies the same equation and therefore

\[
\Psi[X(\pi - \sigma, \tau), \psi(\pi - \sigma, \tau), \frac{\varepsilon}{\lambda}] = \left[ \Psi[X(\sigma, \tau), \psi(\sigma, \tau), \lambda] \right]^{-1} e^{-\Phi},
\]

or equivalently,

\[
\Psi[X(\sigma, \tau), \psi(\sigma, \tau), \lambda] \Psi[X(\pi - \sigma, \tau), \psi(\pi - \sigma, \tau), \frac{\varepsilon}{\lambda}] = e^{-\Phi}.
\]

Using (6.3), one can rewrite eq. (4.3) in the form

\[
\left( \frac{1}{\lambda} \hat{G}^+ + G^+ \right) \Psi[X, \psi, \lambda] \Psi[X, \psi, \frac{\varepsilon}{\lambda}] = -A e^{-\Phi}.
\]

Notice that \( \Psi \) satisfies the same equation as \( \Psi_- \), and therefore \( \Xi := \Psi^{-1}\Psi_- \) is annihilated by the operator \( \hat{G}^+(\lambda) \). Thus,

\[
\Psi_- = \Xi \Rightarrow \Psi_+^{-1} = \Xi e^{\Phi} = \Xi \Psi e^{\Phi}
\]

for some matrix-valued chiral string field \( \Xi \). Moreover, from (6.2) and (6.5) we see that

\[
\Upsilon_{++} = \Psi_+^{-1}\Psi_- = \Xi \Psi e^{\Phi} \Xi = \Xi \Xi.
\]
This establishes a connection with the discussion in section 5.

**Dressing.** The dressing method is a recursive procedure generating a new extended solution from an old one. A solution \( e^\Phi \) of the equation of motion (1.4) is obtained from the extended solution via (6.1). Namely, let us suppose that we have constructed an extended seed solution \( \Psi_0 \) by solving the linear equation (4.3) for a given (seed) solution \( e^\Phi_0 \) of eq. (1.4). Then one can look for a new extended solution in the form

\[
\Psi_1 = \chi_1 \Psi_0 \quad \text{with} \quad \chi_1 = \mathcal{I} + \frac{\lambda \alpha_1}{\lambda - \mu_1} P_1,
\]

(6.7)

where \( \alpha_1 \) and \( \mu_1 \) are complex constants and the matrix-valued string field \( P_1[X,\psi] \) is independent of \( \lambda \). The transformation \( \Psi_0 \mapsto \Psi_1 \) is called dressing. Below, we will show explicitly how one can determine \( \Psi_1 \) by exploiting the pole structure (in \( \lambda \)) of eq. (6.4) together with (6.7). An \( m \)-fold repetition of this procedure yields as the new extended solution

\[
\Psi_m = \prod_{j=1}^{m} \left( \mathcal{I} + \frac{\lambda \alpha_j}{\lambda - \mu_j} P_j \right) \Psi_0.
\]

(6.8)

We will choose below the vacuum seed solution \( \Phi_0 = 0, \Psi_0 = \mathcal{I} \).

**First-order pole ansatz for \( \Psi \).** Choose the complex constants \( \mu_j \) in (6.8) such that they are mutually different. Then using a decomposition into partial fractions, one can rewrite the multiplicative ansatz (6.8) in the additive form

\[
\Psi_m = \left( \mathcal{I} + \lambda \sum_{q=1}^{m} \frac{R_q}{\lambda - \mu_q} \right) \Psi_0,
\]

(6.9)

where the matrix-valued string fields \( R_q[X,\psi] \) are some combinations of (products of) \( P_j \). As already mentioned we now choose the vacuum \( \Phi_0 = 0, \Psi_0 = \mathcal{I} \) and consider \( R_q \) of the form \[11, 12, 43, 44\]

\[
R_q = -\sum_{p=1}^{m} \mu_q T_p \Gamma^{pq} T_q,
\]

(6.10)

where \( T_p[X,\psi] \) are taken to be the \( n \times r \) matrices for some \( r \geq 1 \) and \( \Gamma^{pq}[X,\psi] \) are \( r \times r \) matrices for which an explicit expression is going to be determined below.

From (6.9) and (6.10) it follows that

\[
\Psi = \mathcal{I} - \lambda \sum_{p,q=1}^{m} \mu_q T_p \Gamma^{pq} T_q \frac{\lambda - \mu_q}{\lambda - \mu_p},
\]

(6.11)

\[
\overline{\Psi} = \mathcal{I} + \sum_{k,\ell=1}^{m} T_{k} \Gamma^{k\ell} T_{\ell} \frac{\lambda - \epsilon}{\lambda - \epsilon / \bar{\mu}_{\ell}}.
\]

(6.12)

Here we omitted the index \( m \) in \( \Psi_m \) and \( \overline{\Psi}_m \). In accordance with (6.3) we have to choose \( \Gamma^{pq} \) in
such a form that $\Psi \Psi$ will be independent of $\lambda$. A splitting into partial fractions yields

$$
\Psi \Psi = I + \sum_{k,\ell} \frac{\varepsilon T_k \Gamma^{k\ell} T_k}{\mu_k - \varepsilon/\mu_\ell} - \sum_{p,q} \frac{\mu_q T_p \Gamma^{pq} T_q}{\lambda - \mu_q} \left(\frac{\varepsilon}{\mu_q} + \mu_q\right) \frac{T_p \Gamma^{pq} T_q T_k \Gamma^{k\ell} T_k}{(\lambda - \mu_q)(\lambda - \varepsilon/\mu_\ell)}
$$

$$
= I + \sum_{k,\ell} \frac{\varepsilon T_k \Gamma^{k\ell} T_k}{\mu_k - \varepsilon/\mu_\ell} - \sum_{p,q} \frac{\mu_q T_p \Gamma^{pq} T_q}{\lambda - \mu_q} \left(\frac{\varepsilon}{\mu_q} + \mu_q\right) \frac{T_p \Gamma^{pq} T_q T_k \Gamma^{k\ell} T_k}{(\lambda - \mu_q)(\lambda - \varepsilon/\mu_\ell)}
$$

$$
- \sum_{p,q,k,\ell} \frac{\varepsilon \mu_q^2 \mu_\ell}{\mu_k - \varepsilon} \left(\frac{1}{\lambda - \mu_q} - \frac{1}{\lambda - \varepsilon/\mu_\ell}\right) T_p \Gamma^{pq} T_q T_k \Gamma^{k\ell} T_k.
$$

(6.13)

This motivates us to define

$$
\tilde{\Gamma}_{q\ell} := -\varepsilon \frac{T_q T_\ell}{\mu_k - \varepsilon},
$$

(6.14)

and, as the matrix $\Gamma = (\Gamma^{pq})$ has not yet been specified, to take it to be inverse to $\tilde{\Gamma} = (\tilde{\Gamma}_{q\ell})$,

$$
\sum_{q=1}^m \Gamma^{pq} \tilde{\Gamma}_{q\ell} = \delta_{p\ell} I.
$$

(6.15)

Upon insertion of eqs. (6.14) and (6.15) into (6.13) nearly all terms cancel each other and we are left with

$$
\Psi \Psi = I - \sum_{p,q} \frac{\mu_q T_p \Gamma^{pq} T_q}{\mu_k - \varepsilon} = e^{-\Phi}.
$$

(6.16)

This expression is independent of $\lambda$ and, therefore, we can identify it with $e^{-\Phi}$ as in (6.3). We see that for the above choice of the $\Gamma$-matrices the reality condition is satisfied, and the solution $e^{\Phi}$ of eq. (1.4) is parametrized by the matrix-valued string fields $T_k$, $k = 1, \ldots, m$. Note that (6.16) coincides with $\Psi \Psi |_{\lambda=\infty} = \Psi |_{\lambda=\infty}$.

**Pole structure.** We are now going to exploit eq. (6.4) in combination with the ansatz (6.11). First, it is easy to show that

$$
\Psi |_{\lambda = \frac{\varepsilon}{\mu_k}} T_k = \left( I + \sum_{p,q} \frac{\varepsilon \mu_q T_p \Gamma^{pq} T_q}{\mu_k - \varepsilon} \right) T_k = T_k - \sum_{p,q} T_p \Gamma^{pq} \tilde{\Gamma}_{q\ell} = 0
$$

(6.17)

and

$$
T_k \Psi |_{\lambda = \mu_k} = T_k \left( I + \sum_{p,q} \frac{\mu_q T_q \Gamma^{pq} T_p}{\mu_k - \varepsilon} \right) = T_k - \sum_{p,q} \tilde{\Gamma}_{q\ell} \Gamma^{pq} T_p = 0.
$$

(6.18)

Second, note that the right hand side of eq. (6.4) is independent of $\lambda$ and therefore the poles on the left hand side have to be removable. Putting to zero the corresponding residue at $\lambda = \frac{\mu_k}{\mu_k}$ we obtain, due to (6.17),

$$
\Psi |_{\lambda = \frac{\varepsilon}{\mu_k}} \left\{ (\varepsilon \tilde{\mu}_k \tilde{G}^+ + G^+) T_k \right\} \sum_{\ell} \Gamma^{k\ell} T_\ell = 0.
$$

(6.19)
Obviously, a sufficient condition for a solution is
\[
\left( \tilde{G}^+ + \frac{\varepsilon}{\bar{\mu}_k} G^+ \right) T_k = T_k Z_k ,
\] (6.20)
with an arbitrary operator \( Z_k \) having the same Grassmann content as the operator \( \tilde{G}^+ \left( \frac{\varepsilon}{\bar{\mu}_k} \right) \). In the same way, the residue at \( \lambda = \mu_k \) should vanish,
\[
\left( \sum_p \mu_k T_p \Gamma^p k \right) \left\{ \left( \frac{1}{\mu_k} \tilde{G}^+ + G^+ \right) T_k \right\} \Psi|_{\lambda=\mu_k} = 0
\] \Rightarrow (\tilde{G}^+ + \mu_k G^+) T_k = Z'_k T_k ,
\] (6.21)
with another Grassmann-odd operator \( Z'_k \). Comparing eqs. (6.20) and (6.21), we learn that
\[
Z_k T_k = \varepsilon Z'_k T_k \Rightarrow Z'_k = \varepsilon Z_k .
\] (6.22)
In other words, eqs. (6.21) are not independent but follow from eq. (1.20) by conjugation. For every collection \( \{ T_k, k = 1, \ldots, m \} \) of solutions to eqs. (6.20) we can determine a solution to eq. (1.4) from eqs. (6.14)–(6.16).

**Projectors.** Now let us consider the simplest case \( m = 1 \). Then, eq. (6.11) simplifies to
\[
\Psi = I + \frac{\lambda \varepsilon (1 - \varepsilon |\mu|^2)}{\lambda - \mu} P ,
\] (6.23)
where \( P := T \bar{T} T^{-1} \bar{T} \) is a hermitian projector, \( P^2 = P = \bar{P} \), parametrized by an \( n \times r \) matrix \( T \). In the abelian \((n = 1)\) case \( r \) is the rank of the projector \( P \) in the Hilbert space \( \mathcal{H} \) of string field theory. In the nonabelian \((n > 1)\) case \( r \leq n \) can be identified with the rank of the projector in the \( u(n) \) factor of the \( u(n) \otimes \mathcal{H} \) Hilbert space.

From the extended solution \( \Psi \) we obtain the solution
\[
e^{-\Phi} = \Psi|_{\lambda=\infty} = I - (1 - \varepsilon |\mu|^2) P
\] (6.24)
of the equation of motion (1.4). Thus, the simplest solutions are parametrized by projectors in the string field theory Hilbert space.

To conclude this section, we summarize the main idea of the dressing approach as follows: One has to extend the string field theory Hilbert space \( u(n) \otimes \mathcal{H} \) to \( u(n) \otimes \mathcal{H} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \), there solve the equations on the extended string field \( \Phi[X, \psi, \lambda] \) such that \( \Psi^{-1}[X, \psi, \lambda] = e^{\Phi[X, \psi, \lambda]} \), and then project back onto \( u(n) \otimes \mathcal{H} \). In this way, one obtains a solution \( e^{\Phi} = \Psi^{-1}(\lambda = \infty) \) of the initial equation of motion (1.4), where the extended solution \( \Psi \) is parametrized by \( T_k[X, \psi, \lambda = \varepsilon \bar{\mu}_k^{-1}] \) with \( k = 1, \ldots, m \).

**7 Solutions of the linear equations**

**\( \tilde{G}^+(\lambda) \)-exact solutions.** In the previous section we have shown that in the dressing approach solving the nonlinear string field equation (1.4) reduces to solving the linear equations (6.21). Solutions \( T_k, k = 1, \ldots, m \), of these equations parametrize solutions \( e^{\Phi} \) of eq. (1.4) (cf. (6.16)). Obviously, for obtaining some examples of solutions it is sufficient to find solutions for \( Z_k = 0 \),
\[
\tilde{G}^+(\varepsilon \bar{\mu}_k^{-1}) T_k \equiv \left( \tilde{G}^+ + \frac{\varepsilon}{\bar{\mu}_k} G^+ \right) T_k = 0 .
\] (7.1)
Here, we present two classes of solutions to these equations.

Recall that \( (\bar{G}^+(\lambda))^2 = 0 \) and, therefore,

\[
T_k = \bar{G}^+ (\varepsilon \bar{\mu}_k^{-1}) W_k
\]  

(7.2)
is a solution of eq. (7.1) for any string field \( W_k \in \text{Mat}(n \times r, \mathbb{C}) \otimes \mathcal{H} \). These solutions are in general nontrivial because they are not annihilated by \( G^+ \) and \( \bar{G}^+ \) separately.

This discussion is valid for both \( N = 1 \) strings (\( \varepsilon = -1 \)) and \( N = 2 \) strings (\( \varepsilon = 1 \)). Substituting (7.2) into (6.16), we get explicit solutions \( e^\Phi \). In the \( N = 1 \) case, other obvious solutions are all BRST-closed vertex operators \( T_k \) in the small Hilbert space of [6] as they satisfy \([Q, T_k] = 0 \) and \([\eta_0, T_k] = 0 \) separately.

For the case of \( N = 2 \) strings we will discuss two classes of explicit solutions of (6.20) (for both \( Z_k = 0 \) and \( Z_k \neq 0 \)) which in general do not have the form (7.2).

**N = 2 string solutions for \( Z_k = 0 \).** We already presented the realization of \( G^+ \) and \( \bar{G}^+ \) in terms of the constituents of an \( N = 2 \) matter multiplet in eqs. (2.7) and (2.9). Using this realization, one can factorize the “\( G \)-matrix” in (3.4) according to

\[
\begin{pmatrix}
G^+ & \bar{G}^+
\end{pmatrix}
\begin{pmatrix}
G^- & G^-
\end{pmatrix}
= \begin{pmatrix}
\psi^{+1} & -\psi^{+0} \\
-\psi^{-0} & \psi^{-1}
\end{pmatrix}
\begin{pmatrix}
\partial \bar{Z}^1 & \partial Z^0 \\
\partial Z^0 & \partial \bar{Z}^1
\end{pmatrix}
\]

(7.3)

As in section 3 this matrix transforms under current \( SU(1,1) \)-rotations acting from the left (note that the world-sheet fermions are charged under the current group) and under the additional \( SU(1,1) \)-rotations as in (3.9) acting from the right. The latter transform (7.3) to

\[
\begin{pmatrix}
G^+(u) & \bar{G}^+(u) \\
G^-(u) & G^-(u)
\end{pmatrix}
= \begin{pmatrix}
\psi^{+1} & -\psi^{+0} \\
-\psi^{-0} & \psi^{-1}
\end{pmatrix}
\begin{pmatrix}
\partial \bar{Z}^1 & \partial Z^0 \\
\partial Z^0 & \partial \bar{Z}^1
\end{pmatrix}
\begin{pmatrix}
u_1 & \bar{u}_2 \\
\bar{u}_1 & u_2
\end{pmatrix}
\]

(7.4)

By right-multiplication with \( \frac{u_1^{-1}}{0 \ a^{-1}_i} \) as in (3.11) we can express everything in terms of \( \lambda \),

\[
\begin{pmatrix}
G^+(\lambda) & \bar{G}^+(\lambda) \\
G^-(\lambda) & G^-(\lambda)
\end{pmatrix}
= \begin{pmatrix}
\psi^{+1} & -\psi^{+0} \\
-\psi^{-0} & \psi^{-1}
\end{pmatrix}
\begin{pmatrix}
\partial \bar{Z}^1(\lambda) & \partial Z^0(\lambda) \\
\partial Z^0(\lambda) & \partial \bar{Z}^1(\lambda)
\end{pmatrix}
\]

(7.5)

where the coordinates

\[
Z^0(z, \bar{z}, \lambda) := Z^0(z, \bar{z}) + \lambda \bar{Z}^1(z, \bar{z}) \quad \text{and} \quad Z^1(z, \bar{z}, \lambda) := Z^1(z, \bar{z}) + \lambda \bar{Z}^0(z, \bar{z})
\]

(7.6)
define a new complex structure on the target space \( \mathbb{C}^{1,1} \). From eq. (2.8) we derive that \( Z^a(z, \bar{z}, \lambda) \) are null coordinates:

\[
Z^a(z, \bar{z}, \lambda) \bar{Z}^b(w, \bar{w}, \lambda) \sim 0.
\]

(7.7)

The derivation of the string field algebra in (3.20) can be written entirely in terms of \( Z^a(z, \bar{z}, \lambda) \),

\[
\bar{G}^+(z, \lambda) = \bar{G}^+(z) + \lambda G^+(z) = -\varepsilon_{ab} \psi^{+a}(z) \partial Z^b(z, \lambda),
\]

(7.8)

and from (7.7) it follows that

\[
\oint \frac{dw}{2\pi i} \bar{G}^+(w, \lambda) Z^a(z, \bar{z}, \lambda) = 0
\]

(7.9)
with the integration contour running around \( z \). This equation implies that every analytic functional \( T_k \) of the new spacetime coordinates \( Z^0(z, \bar{z}, \mu_k^{-1}) \) and \( Z^1(z, \bar{z}, \mu_k^{-1}) \) solves (1.4). Indeed, it can be easily checked that for any integer \( p, q \) with the integration contour running around \( z \), we immediately derive

\[
\mathcal{G}^+ \left( w, \frac{1}{\mu_k} \right) : (Z^0)^p(Z^1)^q(z, \bar{z}, \mu_k^{-1}) = 0. \tag{7.10}
\]

The functional \( T_k \) may also depend on arbitrary derivatives \( \partial^\ell Z^a(z, \mu_k^{-1}) \) (note that, for \( \ell = 1 \), \( \partial Z^a(z, \mu_k^{-1}) \) is \( \tilde{G}^+ (\mu_k^{-1}) \)-exact). Due to (2.8), it may furthermore depend on \( \psi^{+a}(z) \) or its derivatives. Given some analytic functionals \( T_k[Z^a(z, \bar{z}, \mu_k^{-1}), \partial^\ell Z^a(z, \mu_k^{-1}), \psi^{+a}(z), \partial^p \psi^{+a}(z)] \) with values in \( \text{Mat}(n \times r, \mathbb{C}) \) for \( k = 1, \ldots, m \), we can determine a solution of (1.4) with the help of eq. (6.16). Note that we do not claim to have found all solutions.

**N = 2 string solutions for \( Z_k \neq 0 \).** We restrict ourselves to the abelian case \( n = 1 \). In addition to the coordinates \( Z^a(z, \bar{z}, \mu_k^{-1}) \) from above, we introduce vertex operators

\[
Y^0(z, \bar{z}, \mu_k^{-1}) := \frac{1}{2} \left( Z^1(z, \bar{z}) - \frac{1}{\mu_k} Z^0(z, \bar{z}) \right) \quad \text{and} \quad Y^1(z, \bar{z}, \mu_k^{-1}) := \frac{1}{2} \left( Z^0(z, \bar{z}) - \frac{1}{\mu_k} Z^1(z, \bar{z}) \right). \tag{7.11}
\]

They satisfy the following OPE with \( Z^a(z, \bar{z}, \mu_k^{-1}) \):

\[
Z^a(z, \bar{z}, \mu_k^{-1}) Y^b(w, \bar{w}, \mu_k^{-1}) \sim -2 \varepsilon^{ab} \ln |z - w|^2. \tag{7.12}
\]

From this, we immediately derive

\[
\mathcal{G}^+ \left( w, \frac{1}{\mu_k} \right) : e^{\alpha_a^k Y^a}(z, \bar{z}, \mu_k^{-1}) = -2 : e^{\alpha_a^k Y^a}(z, \bar{z}, \mu_k^{-1}) \alpha_k^b \psi^{+b}(z), \tag{7.13}
\]

where \( \alpha_a^k \) are complex constants. We see that for \( k = 1, \ldots, m \),

\[
T_k := e^{\alpha_a^k Y^a}(z, \bar{z}, \mu_k^{-1}) : \tag{7.14}
\]

satisfy eq. (6.20) and therefore produce a solution of (1.4) via (6.16).

**8 Conclusions**

In this paper we have demonstrated that the equation of motion for Berkovits’ WZW-like string field theory is integrable. Two approaches to generating solutions of this equation were discussed and adapted to string field theory: the splitting technique and the dressing method. In essence, both procedures reduce the nonpolynomial equation of motion to some linear equations. The solutions of these linear equations give us nonperturbative solutions of the original equation of motion. Our discussion was kept general enough to apply to the case of \( N = 1 \) superstrings as well as to the case of \( N = 2 \) strings.

In order to demonstrate the power of our methods we explicitly constructed some solutions to the linear equations via the dressing approach. For \( N = 1 \) superstrings, a quite general class of solutions was presented; for \( N = 2 \) strings, the same and additional classes of solutions were found. Following the recipe given in section 3, one can easily translate all these to classical configurations of Berkovits’ (super)string field theory.

A lot of work remains to be done: In order to establish which among our solutions represent soliton-like objects within the theory, one has to evaluate their energy. It would be interesting to
find criteria on the $T_k$ for the solution to be a soliton, an instanton, or a monopole. In the case of $N=2$ strings explicit solitonic solutions to the corresponding field theory equations have been constructed earlier \cite{44,46,47}; it is plausible that they can be promoted to the string level. An examination of the fluctuations around these nonperturbative solutions should determine what kind of object they represent in string theory. If some of these solutions turn out to describe D-branes, perhaps another check of Sen’s conjecture on the relation between the tension of D-branes and the string field theory action is feasible. Due to our choosing the simplest ansätze for the splitting and the dressing methods, we have obtained not the broadest classes of field configurations. However, nothing prevents one from employing more general ansätze and thereby creating more general solutions for $N=1$ superstring field theory.

It is also desirable to elucidate the geometrical interpretation of all these classical configurations. To this end, we hope that further work will shed some light on the physical meaning of the projector $P$, which was central to the ansatz for the simplest solution to the linear equations (cf. section 6). Another direction for future investigation could be the transfer of our analysis to cubic superstring field theory. The knowledge of nonperturbative solutions should help to understand the relation of Berkovits’ nonpolynomial with Witten’s cubic superstring field theory. In all instances, it would be exciting to make contact with the current discussion of supersliver states.

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