On the conjectural Leibniz cohomology for groups

Simon Covez

Abstract

The goal of this paper is to present results which are consistent with conjectures about Leibniz (co)homology for discrete groups, due to J. L. Loday in 2003. We show that rack cohomology has properties very close to the properties expected for the conjectural Leibniz cohomology. In particular, we prove the existence of a graded dendriform algebra structure on rack cohomology, and we construct a graded associative algebra morphism $H^\bullet(-) \to H R^\bullet(-)$ from group cohomology to rack cohomology which is injective for $\bullet = 1$.

Mathematics subject classification (2000): 17A32, 20J06

Key words: conjectural Leibniz K-theory, Leibniz algebra, Dendriform algebra, Zinbiel algebra, rack cohomology, group cohomology.

Introduction

Problems and results. The starting point of this article are conjectures by J.-L. Loday in [Lod03]. There J.-L. Loday conjectures the existence of a Leibniz homology $HL_\bullet(-)$ for groups and some properties it should satisfy. Here we focus on two properties

1. $HL_\bullet(G)$ is a graded Zinbiel coalgebra (and, a fortiori, a graded cocommutative coalgebra),

2. There exists a morphism of graded cocommutative coalgebras from $HL_\bullet(G)$ to the usual group homology $H_\bullet(G)$.

Here we work in the cohomological context and we show two results which are consistent with these conjectures (Theorem 4.5 and Theorem 5.8). We prove that there exists a cohomology theory $HR^\bullet$, defined for groups, and satisfying :

1. $HR^\bullet(G, A)$ is a graded dendriform algebra (and, a fortiori, a graded associative algebra),

2. There exists a non zero morphism of graded associative algebras from $H^\bullet(G, A)$ to $HR^\bullet(G, A)$ which is injective for $\bullet = 1$.

where $A$ is an associative algebra considered as a trivial $G$-module. To understand what is $HR^\bullet(-)$ and where it comes from, let us explain some background information about these conjectures.

Algebraic K-theory and additive K-theory. The starting point is a program of J.-L. Loday exposed in [Lod03] which intends to solve two problems arising in algebraic $K$-theory using the knowledge of the solutions of these problems in the additive $K$-theory context. These two problems are :
1. Have a small presentation of the algebraic $K$-groups of a field, or, at least, a small presentation of a chain complex whose homology would give these $K$-groups. (By small, J.-L. Loday means no matrices in the presentation).

2. Determine the obstructions to the periodicity.

Rationally, the algebraic $K$-theory $K_\bullet(A)\mathbb{Q}$ of a unitary ring $A$ can be defined using the group homology of $GL(A)$, as the primitive part of the graded connected cocommutative Hopf algebra $H_\bullet(GL(A),\mathbb{Q})$, hence, using the Milnor-Moore theorem, we have the following isomorphism of graded Hopf algebras

$$H_\bullet(GL(A),\mathbb{Q}) \simeq S(K_\bullet(A)\mathbb{Q}).$$

Using this relation, the additive $K$-theory can be seen as the "tangent" version of algebraic $K$-theory. Let $K$ be a field of characteristic zero and $A$ be a unitary $K$-algebra. The additive $K$-theory $K^+_\bullet(A)$ is defined similarly to algebraic $K$-theory, replacing the group $GL(A)$ by the Lie algebra $gl(A)$, and group homology by Lie algebra homology. Therefore $K^+_\bullet(A)$ is defined as the primitive part of the graded connected commutative and cocommutative Hopf algebra $H_\bullet(gl(A),K)$, and by the Milnor-Moore theorem we have the following graded isomorphism

$$H_\bullet(gl(A),K) \simeq S(K^+_\bullet(A)).$$

In the additive $K$-theory context, the solutions of our problems are given by :

1. The cyclic homology $HC_\bullet$ by the Loday-Quillen-Tsygan Theorem (cf. [LQ84, FT87])

Theorem 0.1 (Loday-Quillen-Tsygan). Let $K$ be a field containing $\mathbb{Q}$ and $A$ be a unital associative $K$-algebra. Then there is a natural isomorphism :

$$K^+_n(A) \simeq HC_{n-1}(A), \ n \geq 1.$$

2. The Hochschild homology $HH_\bullet$ by the Connes’ periodicity exact sequence

$$\cdots \rightarrow HH_n(A) \rightarrow HC_n(A) \rightarrow HC_{n-2}(A) \rightarrow HH_{n-1}(A) \rightarrow \cdots .$$

The following theorem is the starting point of this program. In the same way as the Loday-Quillen-Tsygan Theorem links $HC_\bullet(A)$ and $gl(A)$, the Loday-Cuvier Theorem (cf. [Lod98, Cuv94]) gives a link between $HH_\bullet(A)$ and $gl(A)$.

Theorem 0.2 (Loday-Cuvier). For any associative unital algebra $A$ over a characteristic zero field $K$ there is an isomorphism of graded modules :

$$HL_\bullet(gl(A),K) \simeq T(HH_{\bullet-1}(A)).$$

In this theorem $HL_\bullet(\cdot)$ is the Leibniz homology, the homology theory naturally associated to Leibniz algebras (cf. [Lod93, Lod97, Lod98]). Leibniz algebras are a non-commutative version of Lie algebras and their homology gives new invariants for Lie algebras. The existence of $HL_\bullet(\cdot)$ and its following properties (cf. [Lod95]) can explain the reason of the conjectures :

1. $HL_\bullet(g)$ is a graded Zinbiel algebra,

2. There exists a non zero morphism of graded commutative algebra from $H_\bullet(g)$ to $HL_\bullet(g)$ which is an isomorphism for $\bullet = 1$.

Lie algebra homology being the "tangent" version of group homology, Leibniz algebra homology should be the "tangent" version of the expected homology $HR_\bullet(\cdot)$. This hypothesis led J.-L. Loday to the formulation of the coquecigrue problem (cf. [Lod93, Lod93, Lod12]).
The coquecigrue problem. There are (at least) two ways to construct a Lie algebra from a group. One is to consider the graded abelian group \( \bigoplus_n \frac{G^{(n)}}{G^{(n+1)}} \) associated to the descending central series \( \{G^{(n)} = [G, G^{(n-1)}] \}_{n \in \mathbb{N}} \). This object is provided with a Lie algebra structure where the bracket is induced by the commutator in \( G \), the Jacobi identity being a consequence of the so-called Phillip Hall relation.

Another way is to consider the tangent space at the neutral element of a Lie group. The bracket is induced by the conjugation in the group, and the Leibniz identity by the self-distributivity of the conjugation, that is, the relation:

\[
x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z) \quad \text{where} \quad x \triangleright y = xyx^{-1}.
\]

The coquecigrue problem is to find a generalization of one of these constructions for Leibniz algebra. Therefore the problem is to know if there exists a mathematical object, dubbed coquecigrue, generalizing groups, such that a Leibniz algebra is naturally associated to it (by one of these constructions).

In [Cov10] a local solution to the coquecigrue problem has been given for the second construction using racks, especially the links between (Lie) rack cohomology \( HR^\ast(-) \) and Leibniz algebra cohomology \( HL^\ast(-) \). Originally, racks were defined by knot theorists to construct invariants for knots (cf. [CS03] and references therein), and M. K. Kinyon in [Kin07] is the first to state a relation between Lie racks and Leibniz algebras.

Since a group is a rack it is natural to suppose that rack cohomology is our conjectural Leibniz cohomology \( HR^\ast(-) \), and so, to study the existence of an algebraic product structure on it and its link with usual group cohomology.

The plan for this article is the following:

Section 1: Dendriform and Zinbiel algebras. This section is based on [LFCG01]. We recall the definitions of dendriform and Zinbiel algebras and give some examples. We recall also the relations between these algebras and others types of algebras such as associative, commutative, Lie, Leibniz and diassociative algebras.

Section 2: Racks. Racks are sets provided with a product which encapsulates some properties of the conjugation in a group (especially the self-distributivity). This section is based on [FR92] for the general theory of racks and on [EG03] for the definition of the rack cohomology \( HR^\ast(-) \). We recall some basic definitions about racks and their cohomology.

Section 3: The category of trunks. Trunks have been introduced by R. Fenn, C. Rourke and B. Sanderson in [FRS95] in order to define a classifying space for racks. Trunks are objects loosely analogous to categories. Categories can be seen as graphs with a fixed set of triangle
(the graph of the composition) satisfying the associativity condition.

Like categories, trunks are graphs with a fixed set of cubes

and corner trunks are trunks where the set of cubes satisfies the bidistributivity condition.

In the same way as a group can be seen as a category with one object and all its arrows invertible, to any rack $X$ we can associate a corner trunk $X_{\text{Tr}}$ (Example 3.4). Moreover, one may construct a (pre)simplicial set from a category (the nerve), in the same way there is a precubical set $N(T)$ constructed from a trunk $T$, that is, a contravariant functor from the category $\square$ to the category $\text{Set}$, by $N(T)(-) := \text{Hom}_{\text{Trunk}}(-, T)$ (cf. [6]). In the case where the trunk is the trunk canonically associated to a rack, the description of the precubical nerve is easy to compute (Theorem 3.10), and these result allows us to give a cubical description of the rack cohomology (Corollary 3.12). Then we use this description to define a dendriform structure (formulas (10) and (11)). Our interest for this cubical definition is that the proof that the differential graded dendriform structure on $HR^\bullet(-)$ is well defined becomes a succession of combinatorial lemmas.
Section 4: A graded dendriform algebra structure on $HR^\bullet(X,A)$. This section and the following are the heart of this paper. We use the cubical description of rack cohomology using trunks described in Section 3 to define a graded dendriform algebra structure on the cochain complex $\{CR^n(X,A), d^n\}_{n \in \mathbb{N}}$ computing rack cohomology of a rack $X$ with coefficients in an associative algebra (considered as a trivial $X$-module) (Theorem 4.2). Then we show that this structure is compatible with the differential (Theorem 4.3).

Dendriform and Zinbiel algebras are closely related to shuffles as we can see examples 1.1 and 1.3. Then, in order to define such an algebraic structure on our cochain complex, we have to link shuffles and some sets of trunk maps. The link is provided by the map $\rho$ defined on $\rho : \text{Sh}_{p_1,p_2} \to \text{Hom}_{\text{Set}}(N(X_{\text{Tr}})(\square_{p_1+p_2}), N(X_{\text{Tr}})(\square_{p_1}) \times N(X_{\text{Tr}})(\square_{p_2}))$, which satisfies good properties with respect to the composition $\circ$ and the concatenation product $\star$ (2) defined on $S_n$ (Lemma 4.1).

Section 5: A graded associative algebra morphism from $H^\bullet(G,A)$ to $HR^\bullet(\text{Conf}(G), A)$. In this section we recall the definitions of group cohomology $H^\bullet(-)$ and of the cup product defined on it. Viewing a group $G$ as a category $G_{\text{Cat}}$ and using the simplicial nerve $B(G_{\text{Cat}})$, we recall the simplicial definition of group cohomology (with trivial coefficient) and of its cup product. Now, on the one hand we have a cohomology theory based on simplicial relations and on the other hand a cohomology theory based on cubical relations. Therefore, in order to construct the expected morphism from $H^\bullet(G,A)$ to $HR^\bullet(\text{Conf}(G), A)$, we have to find a relation between simplices and cubes. The construction of this morphism (Theorem 5.6) is based on the decomposition of the $n$-cubes into $n!$ $n$-simplex, that is, the existence of functors $\sigma : \Delta_n \to \square_n$ for all $\sigma \in S_n$. We can represent geometrically such a functor in the following manner ($n = 3, \sigma = (13)$)

We finish this section by proving that this morphism is compatible with the graded associative structures on $H^\bullet(-)$ and $HR^\bullet(-)$ (Theorem 5.8).

1 Dendriform and Zinbiel algebras

Shuffles. For $n \in \mathbb{N}$, we denote by $S_n$ the group of permutations of $\{1, \ldots, n\}$. For $p_1, p_2 \in \mathbb{N}$, a $(p_1, p_2)$-shuffle is an element $\sigma$ of $S_{p_1+p_2}$ satisfying $\sigma(1) < \cdots < \sigma(p_1)$ and $\sigma(p_1 + 1) < \cdots < \sigma(p_1 + p_2)$.

Remark that $\sigma$ satisfies $\sigma(p_1) = p_1 + p_2$ or $\sigma(p_1 + p_2) = p_1 + p_2$.
Denote by $\text{Sh}_{p_1,p_2}$ the set of $(p_1, p_2)$-shuffles, $\text{Sh}^p_{p_1,p_2}$ the subset of elements $\sigma \in \text{Sh}_{p_1,p_2}$ satisfying $\sigma(p_1) = p_1 + p_2$ and $\text{Sh}^{p_1+p_2}_{p_1,p_2}$ the subset of elements $\sigma \in \text{Sh}_{p_1,p_2}$ satisfying $\sigma(p_1 + p_2) = p_1 + p_2$.

Identifying a permutation $\sigma \in S_n$ with the $n$-tuple $(\sigma(1), \ldots, \sigma(n))$, the set $\text{Sh}_{p_1,p_2}$ corresponds to the set of $n$-tuples $(a_1, \ldots, a_{p_1+p_2})$ satisfying $a_1 < \cdots < a_{p_1}$ and $a_{p_1+1} < \cdots < a_{p_1+p_2}$.

The cardinal of $\text{Sh}_{p_1,p_2}$ (resp. $\text{Sh}^p_{p_1,p_2}$; $\text{Sh}^{p_1+p_2}_{p_1,p_2}$) is the binomial coefficient $\binom{p_1 + p_2}{p_1}$ (resp. $\binom{p_1 + p_2 - 1}{p_1 - 1}$).

In the same way, given $p_1, p_2, p_3 \in \mathbb{N}$, the set of $(p_1, p_2, p_3)$-shuffles is defined as the subset of elements $\sigma \in \text{Sh}^{p_1+p_2+p_3}_{p_1,p_2,p_3}$.

\[ \sigma(1) < \cdots < \sigma(p_1), \sigma(p_1 + 1) < \cdots < \sigma(p_1 + p_2) \quad \text{and} \quad \sigma(p_1 + p_2 + 1) < \cdots < \sigma(p_1 + p_2 + p_3). \]

For $p_1, p_2, p_3 \in \mathbb{N}$, there are bijections

\[ \text{Sh}^{p_1+p_2+p_3}_{p_1,p_2,p_3} \xrightarrow{\alpha} \text{Sh}^{p_1+p_2}_{p_1,p_2} \quad \text{and} \quad \text{Sh}^{p_1+p_2+p_3}_{p_1,p_2,p_3} \xrightarrow{\beta} \text{Sh}^{p_1+p_2}_{p_1,p_2} \]

given by $\alpha(\sigma, \gamma) = \sigma \circ (1_{p_1} \star \gamma)$ and $\beta(\sigma, \gamma) = \sigma \circ (\gamma \star 1_{p_3})$ where $\star : S_p \times S_q \to S_{p+q}$ is the map defined by

\[ (\sigma \star \gamma)(k) := \begin{cases} \sigma(k) & \text{if } 1 \leq k \leq p, \\ p + \gamma(k - p) & \text{if } p + 1 \leq k \leq p + q. \end{cases} \]

In subset notation, the formulas for $\alpha$ and $\beta$ are

\[ \alpha((a_1, \ldots, a_{p_1+p_2+p_3}), (b_1, \ldots, b_{p_2+p_3})) = (a_1, \ldots, a_{p_1}, a_{p_1+b_1}, \ldots, a_{p_1+b_{p_2+p_3}}) \]

and

\[ \beta((a_1, \ldots, a_{p_1+p_2+p_3}), (b_1, \ldots, b_{p_1+p_2})) = (a_{b_1}, \ldots, a_{b_{p_1+p_2}}, a_{p_1+b_1+p_2}, \ldots, a_{p_1+b_{p_2}}). \]

As we will see in the examples 1.1 and 1.3, the notion of a shuffle is closely related to the notion of a dendriform algebra.

**Dendriform algebras.** A (graded) dendriform algebra is a (graded) vector space $D$ provided with two (graded) linear maps,

\[ \prec, \prec : D \otimes D \to D, \]

which satisfy for all $x, y, z \in D$ the following relations

\[ x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z \]

\[ (x \succ y) \prec z = x \succ (y \prec z) \]

\[ (x \prec y) \prec z = x \prec (y \prec z) + (x \succ y) \]

\[ \text{Example 1.1. } T(V) \text{ and the free dendriform algebra } F(V) \]

- Let $V$ be a vector space. We define a dendriform algebra structure on $T(V)$ by setting

\[ v_1 \ldots v_p \succ v_{p+1} \cdots v_{p+q} := \sum_{\sigma \in \text{Sh}^{p+q}_{p,q}} v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(p+q-1)} v_{p+q}, \]

and

\[ v_1 \ldots v_p \prec v_{p+1} \cdots v_{p+q} := \sum_{\sigma \in \text{Sh}^{p+q}_{p,q}} v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(p+q-1)} v_p. \]
• Let $V$ be a vector space. J.-L. Loday has defined a structure of dendriform algebra on $F(V) := \bigoplus_{n \geq 0} \mathbb{K}[Y_n] \otimes V^\otimes n$ where $Y_n$ is the set of planar binary trees with $n + 1$ leaves (cf. [LFCG01]). This algebra is the free dendriform algebra associated to $V$.

• The next example of a graded dendriform algebra uses the well known Pascal formula.

Let $A$ be an associative algebra and consider the graded $\mathbb{K}$-module $\bigoplus_{n \geq 0} A_n$ where $A_n = A$ for all $n \in \mathbb{N}$. We define the structure of a graded dendriform algebra on it by setting for $a \in A_p$ and $b \in A_q$

$$a \succ b := \left(\frac{p + q - 1}{p}\right)ab,$$

and

$$a \prec b := \left(\frac{p + q - 1}{p - 1}\right)ab.$$

**Relation with associative algebras.** A (graded) dendriform algebra is a particular example of a (graded) associative algebra. Indeed, let $(D, \succ, \prec)$ be a (graded) dendriform algebra, we define a new product $\ast$ on $D$ by setting

$$x \ast y := x \succ y + x \prec y.$$

**Proposition 1.2.** The product $\ast$ is associative.

Hence there exists a functor from the category of dendriform algebras to the category of associative algebras

$$\text{Dend} \to \text{As}$$

The product $\ast$ is not necessarily commutative, but it becomes commutative if $x \succ y = y \prec x$. This condition leads us to the notion of a commutative dendriform algebra, which is also called Zinbiel algebra.

**Zinbiel algebras.** A Zinbiel algebra is a vector space $D$ provided with a linear map

$$\succ: D \otimes D \to D,$$

which satisfies for all $x, y, z \in D$ the following relation

$$x \succ (y \succ z) = (x \succ y) \succ z + (y \succ x) \succ z.$$

Remark that the variables do not stay in the same order, thus in the graded case we have to be careful. A graded Zinbiel algebra is a graded vector space $D$ provided with a graded linear map of degree 0 $\succ: D \otimes D \to D$ which satisfies for all $x, y, z \in D$ the relation

$$x \succ (y \succ z) = (x \succ y) \succ z + (-1)^{pq}(y \succ x) \succ z,$$

where $p$ is the degree of $x$ and $q$ is the degree of $y$.

**Example 1.3.** $T(V), HL^*(g, A)$

• Let $V$ be a vector space. There is a Zinbiel algebra structure defined on $T(V)$ by setting

$$v_1 \ldots v_p \succ v_{p+1} \ldots v_{p+q} := \sum_{\sigma \in S_{p+q}^{p+q}} v_{\sigma^{-1}(1)} \ldots v_{\sigma^{-1}(p+q-1)} v_{p+q}.$$
Let $g$ be a Leibniz algebra and $A$ a commutative algebra. In [Lod95] J.-L. Loday defines a graded Zinbiel product on $HL^*(g,A)$, the Leibniz cohomology of $g$ with values in the trivial $g$-representation $A$.

Relation with dendriform algebras and commutative algebras. As stated above, a Zinbiel algebra is kind of a commutative dendriform algebra. Indeed, let $(D, >)$ be a Zinbiel algebra, and define a second product $\prec$ by the formula $x < y := y \succ x$ for all $x, y \in D$.

**Proposition 1.4.** $(D, >, \prec)$ is a dendriform algebra, and $(D, *)$ is commutative. Conversely, a dendriform algebra $(D, >, \prec)$ where $x > y = y \prec x$ for all $x, y \in D$, is a Zinbiel algebra.

Hence, there exist functors from the category of Zinbiel algebras to the category of commutative algebras, and to the category of dendriform algebras. All these functors fit into the following diagram

\[ \begin{array}{ccc}
Dend & \Rightarrow & Zinb \\
\downarrow & & \downarrow \\
Com & \Rightarrow & As \\
\end{array} \]

**Remark 1.5.** This diagram fits into a (beautiful) butterfly diagram (cf. [LFCG01])

\[ \begin{array}{ccc}
Dend & \Rightarrow & Dialg & \Rightarrow & Zinb \\
\downarrow & & \downarrow & & \downarrow \\
Com & \Rightarrow & As & \Rightarrow & Leib \\
\end{array} \]

\[ \begin{array}{ccc}
\Rightarrow & & \Rightarrow \\
S& & S \\
\end{array} \]

where Dialg is the category of dialgebras, Leib is the category of Leibniz algebras, and Lie is the category of Lie algebras. We can remark that each operad on the left (Zinb, Dend, Com, As) is Koszul dual to the operad on the right (Leib, Dialg, Lie, As) which is symmetric to it with respect to the vertical line passing through As.

2 Racks

Racks are sets equipped with a product which encapsulates some properties of the conjugation in a group (especially the self-distributivity). This algebraic structure has been introduced in knot theory to define invariant for knots and links (cf. [CS03] or [FR92]). We recall some basic definitions about racks and its cohomology theory, and the relations between the category of racks and the category of groups.

**Shelves.** A shelf is a set $X$ provided with a binary product $\triangleright : X \times X \rightarrow X$ satisfying $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$. A pointed shelf is the data of a shelf $(X, \triangleright)$ provided with an element $1 \in X$, called the unit, satisfying $1 \triangleright x = x$ and $x \triangleright 1 = 1$ for all $x \in X$.

**Notation:** Let $(X, \triangleright)$ be a shelf. For all $x \in X$ denote by $c_x$ the map from $X$ to $X$ defined by $c_x(y) = x \triangleright y$. Because in a shelf the product $\triangleright$ is non associative, we have to be careful with
the parenthesis. In the sequel the expression \((c_{x_1} \circ \cdots \circ c_{x_n}) (x_n)\) (bracketing from the right) is denoted by \(x_1 \triangleright \cdots \triangleright x_n\) or \(\prod_{1 \leq i \leq n} x_i\).

Let \(X\) and \(X'\) be two shelves. A morphism of shelves from \(X\) to \(X'\) is a map \(f : X \to X'\) which preserves products. If the shelves are pointed then a morphism of pointed shelves is a morphism of shelves which preserves units.

The category of shelves (resp. pointed shelves) is denoted by \(\text{Shelf}\) (resp. \(\text{pShelf}\)).

**Racks.** A (pointed) rack is a (pointed) shelf \((X, \triangleright)\) where for all \(x \in X\) the map \(c_x : X \to X\) is a bijection. Let \(X\) and \(X'\) be two (pointed) racks. A morphism of (pointed) racks from \(X\) to \(X'\) is a morphism of (pointed) shelves \(f : X \to X'\).

The category of racks (resp. pointed racks) is denoted by \(\text{Rack}\) (resp. \(\text{pRack}\)).

**Example 2.1.** Groups and augmented racks

- Let \(G\) be a group. Define a rack structure on the set \(G\) by \(x \triangleright y = xyx^{-1}\). This rack is denoted \(\text{Conj}(G)\). It is pointed by \(e\), the unit of \(G\).

- Let \(G\) be a group, \(X\) be a \(G\)-set and \(f : X \to G\) a \(G\)-map (where \(G\) acts on itself by conjugation). Define a rack structure on \(X\) by \(x \triangleright y = f(x) \cdot y\). If there exists a fixed point \(1 \in X\) such that \(f(1) = e\), then this rack is pointed by \(1\).

**Example 2.1** defines a functor \(\text{Conj} : \text{Grp} \to \text{Rack}\) from the category of groups to the category of racks. There is a functor \(\text{As} : \text{Rack} \to \text{Grp}\) defined by

\[
\text{As}(X) := \mathcal{F}(X)/\langle x \triangleright y = xyx^{-1} \rangle,
\]

\[
\text{As}(f) := \overline{\mathcal{F}(f)},
\]

where \(\mathcal{F}(X)\) is the free group generated by the set \(X\), and \(\overline{\mathcal{F}(f)}\) is the map induced by \(\mathcal{F}(f)\) by passing to quotients.

**Proposition 2.2.** The functor \(\text{As}\) is left adjoint to the functor \(\text{Conj}\).

**Rack modules.** Let \(X\) be a rack. A left \(X\)-module is an abelian group \(A\) provided with a right linear map

\[
\cdot : X \times A \to A; (x, a) \mapsto x \cdot a
\]

satisfying \(x \cdot (y \cdot a) = (x \triangleright y) \cdot (x \cdot a)\) for all \(a \in A\) and \(x, y \in X\). If the rack is pointed then we demand the additional axiom : \(1 \cdot a = a\), \(\forall a \in A\). Let \(A\) and \(B\) be two left \(X\)-modules. A map \(f : A \to B\) is a morphism of left \(X\)-modules if \(f(x \cdot a) = x \cdot f(a)\) for all \((a, x) \in A \times X\). We denote by \(X - \text{Mod}\) the category of left \(X\)-modules.

**Example 2.3.** Let \(G\) be a group and \(A\) be a left \(G\)-module, then \(A\) is a left \(\text{Conj}(G)\)-module.

**Remark 2.4.** Let \(X\) be a rack and \(A\) be a left \(X\)-module. There is an equivalence of categories between \(X - \text{Mod}\) and \(\text{As}(X) - \text{Mod}\).
Rack cohomology. Let $X$ be a rack and $A$ be a left $X$-module. We define a cochain complex \( CR^n(X,A) \) by

\[
CR^n(X,A) := \text{Hom}_{\text{Set}}(X^n, A),
\]

\[
d_R^{n+1} f := \sum_{i=1}^{n+1} (-1)^i (d_{i,0}^{n+1} f - d_{i,1}^{n+1} f),
\]

where

\[
d_{i,\epsilon}^{n+1} f(x_1, \ldots, x_{n+1}) := \begin{cases} f(x_1, \ldots, x_i, x_i \triangleright x_{i+1}, \ldots, x_{n+1}) & \text{if } \epsilon = 0, \\ (x_1 \triangleright \ldots \triangleright x_i) \cdot f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) & \text{if } \epsilon = 1. \end{cases}
\]

The fact that $d_R$ is a differential comes from the cubical identities satisfied by the family of maps \( \{d_{i,\epsilon}^n\} \), that is, for all $1 \leq i < j \leq n + 1$ and $\epsilon, \omega \in \{0,1\}$, we have the identities:

\[
d_{i,\epsilon}^n \circ d_{j-1,\omega}^n = d_{i,\omega}^{n+1} \circ d_{i,\epsilon}^n.
\]

The cohomology associated to this cochain complex is called the rack cohomology of $X$ with coefficients in $A$ and denoted $HR^\bullet(X,A)$.

Remark 2.5. In this paper we will consider the rack cohomology with trivial coefficients, that is when $A$ is a trivial left $X$-module. In this case we have $d_R^{n+1} f = d_{n+1,1}^{n+1} f$ for all $f \in CR^n(X,A)$.

Remark 2.6. If $X$ is a pointed rack, then a subcomplex $CR^\bullet_p(X,A) = \{CR_p^n(X,A), d_R^n\}_{n \geq 1}$ is defined by

\[
CR_p^n(X,A) := \{ f \in \text{Hom}_{\text{Set}}(X^n, A) \mid f(x_1, \ldots, x_n) = 0 \}.
\]

The cohomology associated to this cochain complex is called the pointed rack cohomology of $X$ with coefficients in $A$ and denoted $HR^\bullet_p(X,A)$.

Remark 2.7. In the definition of \( \{CR^n(X,A), d_R^n\}_{n \in \mathbb{N}} \) (resp. \( \{CR_p^n(X,A), d_R^n\}_{n \in \mathbb{N}} \)) we don’t use the second axiom of a rack, that is the bijectivity of $e_x$ for all $x \in X$. Hence this cochain complex is well defined for a shelf (resp. pointed shelf). It corresponds to the cochain complex associated to the multi-shelf $\langle X, \triangleright, \triangleleft \rangle$, where $x \triangleleft y = x$, defined in [PS11].

3 The category of trunks

Trunks are the fundamental tools in order to solve our problems combinatorially. This section is based on [FRS95]. Note that our definitions are slightly different from theirs for we are working in the left rack context instead of the right rack context. The two theories are obviously equivalent.

3.1 Definitions and examples

Categories. Following [ML98], let us recall the definition of a (small) category as a graph provided with extra properties. Let $\mathcal{G} = (V, E, s, t)$ be a directed graph with $V$ the set of vertices, $E$ the set of edges, $s : E \to V$ the source map and $t : E \to V$ the target map. We define a subset of $E \times E \times E$ by

\[
E \times_V E \times_V E := \{(a, b, c) \in E \times E \times E \mid t(a) = s(b), t(b) = s(c)\},
\]
and subsets of $E \times E$ by
\[
E \times V E := \{ (a, b) \in E \times E \mid t(a) = s(b) \},
\]
\[
V \times V E := \{ (s(a), a) \in V \times E \mid a \in E \},
\]
\[
E \times V V := \{ (a, t(a)) \in E \times V \mid a \in E \}.
\]

A category is a directed graph $C = (V, E, s, t)$ provided with a composition map $c : E \times V E \to E$ and an identity map $i : V \to E$ satisfying the following commutative diagrams.

\[
\begin{array}{ccc}
E \times V E \times V E & \xrightarrow{1 \times 1} & E \times V E \\
1 \times c & \downarrow & c \\
E \times V E & \xrightarrow{c} & E \\
\end{array}
\]

\[
\begin{array}{ccc}
V \times V E \times V E & \xrightarrow{1 \times i} & E \times V V \\
\pi_2 \circ c & \downarrow & \pi_1 \\
E \times V E & \xrightarrow{c} & E \\
\end{array}
\]

The left diagram states the associativity of the composition $c$ and the right diagram states that each $i(x)$ is a left and right unit for the composition.

**Example 3.1.** Simplexes, cubes and groups.

- Let $n \in \mathbb{N}$, and let us consider the directed graph $\Delta_n = \{ V(\Delta_n), E(\Delta_n), s, t \}$ where $V(\Delta_n) := \{0, \ldots, n\}$, $E(\Delta_n) := \{ (i, j) \in V(\Delta_n) \times V(\Delta_n) \mid i \leq j \}$, $s(i, j) = i$ and $t(i, j) = j$.
Define a composition map $c$ by $c((i, j), (j, k)) = (i, k)$ and an identity map $i$ by $i(j) = (j, j)$. In this way, the standard simplex $\Delta_n$ gives rise to a category denoted $\Delta_n$.

- Let $n \in \mathbb{N}$, let us consider the directed graph $\square_n = \{ V(\square_n), E(\square_n), s, t \}$ where $V(\square_n) := \{ A \subseteq \{1, \ldots, n\}\}$, $E(\square_n) := \{ (A, B) \in V(\square_n) \times V(\square_n) \mid A \subseteq B \}$, $s(A, B) = A$ and $t(A, B) = B$. Define a composition map $c$ by $c((A, B), (B, C)) = (A, C)$ and an identity map $i$ by $i(A) = (A, A)$. In this way, the standard cube $\square_n$ gives rise to a category denoted $\square_n$.

- Let $G$ be a group. Consider the directed graph $G = (\{\ast\}, G, s, t)$ with $s = t : G \to \{\ast\}$, the canonical map from $G$ to the terminal object $\{\ast\}$ in $\textbf{Set}$. Define a composition map $c : G \times G \to G$ by $c(a, b) = \mu(a, b)$ (the product of $a$ and $b$ in $G$) and an identity map $i : \{\ast\} \to G$ by $i(\ast) = e$ (the unit in $G$). In this way, a group $G$ give rise to a category.

**Functors.** Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A functor from $\mathcal{C}$ to $\mathcal{C}'$ is a graph map $F : \mathcal{C} \to \mathcal{C}'$ which preserves composition and units, i.e $F$ satisfies the following commutative diagrams.

\[
\begin{array}{ccc}
E \times V E & \xrightarrow{F \times F} & E \\
F \times c \downarrow & & F \downarrow \\
E' \times V E' & \xrightarrow{c'} & E'
\end{array}
\]

\[
\begin{array}{ccc}
V \times V E \times V E & \xrightarrow{1 \times i} & E \times V V \\
\pi_2 \circ c \downarrow & & \pi_1 \\
E \times V E & \xrightarrow{c} & E \\
\end{array}
\]

**Example 3.2.** Faces of simplexes, faces of cubes and permutations.

- Let $n \in \mathbb{N}$. For all $0 \leq i \leq n$ define a functor $\partial^n_i$ from the category $\Delta_{n-1}$ to the category $\Delta_n$ by

\[
\partial^n_i(k) := \begin{cases} 
  k & \text{if } 0 \leq k \leq i - 1, \\
  k + 1 & \text{if } i \leq k \leq n - 1.
\end{cases}
\]

Given two vertices in $\Delta_{n-1}$ there is exactly one edge between them, hence there is only one way to define $\partial^n_i$ on the set of edges in order to have a graph map.
\begin{itemize}
  \item Let \( n \in \mathbb{N} \). For all \((i, \epsilon) \in \{1, \ldots, n\} \times \{0, 1\}\) define a functor \( \partial^n_{i, \epsilon} \) from the category \( \square_{n-1} \) to the category \( \square_n \) by
    \[
    \partial^n_{i, \epsilon}(A) = \begin{cases} 
    A_{<i} \amalg t_{+1}(A_{\geq i}) & \text{if } \epsilon = 0, \\
    A_{<i} \amalg t_{+1}(A_{\geq i}) \amalg \{i\} & \text{if } \epsilon = 1. 
    \end{cases}
    \]
    where \( A_{<i} = \{a \in A \mid a < i\} \), \( A_{\geq i} = \{a \in A \mid a \geq i\} \) and \( t_{+1}(B) = \{b + 1 \mid b \in B\} \). Given two vertices in \( \square_{n-1} \) there is at most one edges between them, hence there is only one way to define \( \partial^n_{i, \epsilon} \) on the set of edges in order to have a graph map.
  \item Let \( n \in \mathbb{N} \). For all \( \sigma \in S_n \) define a functor \( \sigma \) from the category \( \Delta_n \) to the category \( \square_n \) by
    \[
    \sigma(0) := \emptyset \quad \text{and} \quad \sigma(k) := \{\sigma(1), \ldots, \sigma(k)\} \quad \forall k \geq 1.
    \]
\end{itemize}

As explained in the introduction, in order to define trunks we want to replace the associativity relation for the composition in a category by the self-distributivity relation.

**Trunks.** Let \( G = (V, E, s, t) \) be a directed graph. We denote by \( S(G) \) the subset of \( E \times V \times E \times V \) of elements \((a, b, c, d)\) satisfying \( s(a) = s(c), t(a) = s(b), t(c) = s(d) \) and \( t(b) = t(d) \). A trunk is a directed graph \( T = (V, E, s, t) \) provided with a subset \( \Gamma \) of \( S(T) \). An element in \( \Gamma \) is called a preferred square.

A pointed trunk is a trunk \((T = (V, E, s, t), \Gamma)\) together with an identity map \( i : V \to E \) satisfying \((a, i(t(a)), i(s(a)), a) \in \Gamma \) and \((i(s(a)), a, a, i(t(a))) \in \Gamma \) for all \( a \in E \).

**Example 3.3.** Categories and cubes.

\begin{itemize}
  \item Let \( C = (V, E, s, t) \) be a category. We define a trunk \( \text{Tr}(C) \) by the pair \((\mathcal{C}, \Gamma)\) with \( \Gamma \) the set of commutative diagrams in \( \mathcal{C} \)
    \[
    \Gamma = \{(a, b, c, d) \in S(E) \mid c(a, b) = c(c, d)\}.
    \]
    The identity map \( i \) in the category \( \mathcal{C} \) provides \( \text{Tr}(\mathcal{C}) \) with a pointed trunk structure.
  \item Let \( n \in \mathbb{N} \). Consider the graph \( \square_n \) with set of vertices \( V(\square_n) = \{A \subseteq \{1, \ldots, n\}\} \) and set of edges \( E(\square_n) = \{(A, A \amalg \{k\}) \mid k \notin A\} \). Take as set of preferred square \( \Gamma_{\square_n} \) the set equals to
    \[
    \{(\{A, A \amalg \{k\}\}, A \amalg \{k\}, A \amalg \{l\}, (A \amalg \{l\}, A \amalg \{k, l\}) \mid k < l\}.
    \]
    Then, in this way, the standard cube \( \square_n \) give rise to a trunk denoted \( \square_n \).
\end{itemize}

**Corner trunks.** A corner trunk is a trunk \((T, \Gamma)\) where \( \Gamma \) is the graph of a map \( c : E \times V \to E \times V \) satisfying the following commutative diagram
\[
\begin{array}{ccc}
E \times V & \xleftarrow{1 \times c} & E \times V \\
\downarrow{c \times 1} & & \downarrow{1 \times c} \\
E \times V & \xrightarrow{1 \times c} & E \times V \\
\end{array}
\]

The map \( c \) is called the composition map. The relation satisfied by \( c \) can be decomposed into three relations, called bidistributivity relations, according to the projections onto the three components in \( E \times V \times V \). To describe these relations, let us denote by \( \triangleright \) (resp. \( \triangleleft \)) the composition \( \text{pr}_1 \circ c \) (resp. \( \text{pr}_2 \circ c \)). This defines maps \( \triangleright \) and \( \triangleleft \) from \( E \times V \) to \( E \), and the relations are :
Example 3.4. Racks and shelves.

Let \( X, \triangleright \) be a rack or a shelf. Consider the directed graph \( T = (\{\star\}, X, s, t) \) with \( s = t : X \to \{\star\} \), the unique map from \( X \) to the terminal object in \( \text{Set} \). Define a composition map \( c : X^2 \to X^2 \) by \( c(a, b) = (a \triangleright b, a) \). The set of preferred square \( \Gamma_X \) is equal to \( S(X) = X^4 \). In this case, because the map \( \langle \rangle \) is the first projection, the second and third bidistributivity relations are trivialities and the first is equivalent to the self-distributivity of \( \triangleright \). Remark that \( T \) is a pointed corner trunk if and only if \( X \) is a pointed rack/shelf.

Consider the graph \( N = (\{\star\}, N, s, t) \) with \( s = t : N \to \{\star\} \), the unique map from \( N \) to the terminal object in \( \text{Set} \). Define a composition map \( c : N^2 \to N^2 \) by \( c(i, j) = (\max(i, j), \min(i, j)) \). The set of preferred square \( \Gamma_N \) equals to \( S(N) = N^4 \). Then, in this way, the set of natural numbers \( N \) give rise to a corner trunk.

Trunk maps. Let \((T, \Gamma)\) and \((T', \Gamma')\) be two trunks. A trunk map from \( T \) to \( T' \) is a graph map \( F : T \to T' \) mapping \( \Gamma \) to \( \Gamma' \). If the trunks are pointed then \( F \) is a pointed trunk map if \( F \) preserves the units. In case the trunks \( T \) and \( T' \) are corner trunks, \( F : T \to T' \) is a trunk map if and only if \( F \) preserves the composition map \( c \), or equivalently, if and only if \( F \) preserves the products \( \triangleright \) and \( \langle \rangle \).

Example 3.5. Functors, faces of a cube and shuffles.

Let \( F \) be a functor from a category \( \mathcal{C} \) to a category \( \mathcal{C}' \). Because \( F \) maps a commutative diagram in \( \mathcal{C} \) to a commutative diagram in \( \mathcal{C}' \), it induces a pointed trunk map \( \text{Tr}(F) \) from the pointed trunk \( \text{Tr}(\mathcal{C}) \) to \( \text{Tr}(\mathcal{C}') \) (cf. Example 3.3).

Let \( n \in \mathbb{N} \) and consider the graph map \( \partial^n_{\triangle_0} : \square_{n-1} \to \square_n \) defined in Example 3.2.

Given \( 1 \leq k < l \leq n-1 \), we have \( \partial^n_{\triangle_0}(\{k\}) < \partial^n_{\triangle_0}(\{l\}) \). Moreover for all \( A, B \) such that \( A \cap B = \emptyset \) we have \( \partial^n_{\triangle_0}(A \amalg B) = \partial^n_{\triangle_0}(A) \amalg \partial^n_{\triangle_0}(B) \). Thus \( \partial^n_{\triangle_0} \) maps a preferred square in \( \square_{n-1} \) to a preferred square in \( \square_n \). Therefore, the maps \( \partial^n_{\triangle_0} \) are trunk maps.

Let \( p_1, p_2 \in \mathbb{N} \) and \( \sigma \in \text{Sh}_{p_1, p_2} \). The permutation \( \sigma \) induces a graph map \( \sigma : \square_{p_1+p_2} \to \square_{p_1+p_2} \) on the vertices by

\[
\sigma(\{a_1, \ldots, a_k\}) := \{\sigma(a_1), \ldots, \sigma(a_k)\}.
\]

Define two graph maps \( \sigma \circ i_{p_1} : \square_{p_1} \to \square_{p_1+p_2} \) and \( \sigma \circ i_{p_2} : \square_{p_2} \to \square_{p_1+p_2} \) where

\[
i_{p_1} := \partial^{p_1}_{p_1+p_2, 0} \circ \cdots \circ \partial^{p_1+1}_{p_1+1, 0} \\
i_{p_2} := \partial^{p_2}_{1, 1} \circ \cdots \circ \partial^{p_2+1}_{1, 1}.
\]
We have

\[(\sigma \circ i_p)(A) = \{\sigma(a_1), \ldots, \sigma(a_k)\},\]
\[(\sigma \circ i_p)(A) = \{\sigma(1), \ldots, \sigma(p_1), \sigma(p_1 + a_1), \ldots, \sigma(p_1 + a_k)\} .\]

Let \(1 \leq k < l \leq p_1, \sigma \in \text{Sh}_{p_1,p_2}\) implies \(\sigma(k) < \sigma(l)\). Moreover for all \(A, B\) such that \(A \cap B = \emptyset\) we have \((\sigma \circ i_{p_2})(A \sqcup B) = (\sigma \circ i_{p_2})(A) \sqcup (\sigma \circ i_{p_2})(B)\), thus \((\sigma \circ i_{p_2})(A \sqcup \{k\}) = (\sigma \circ i_{p_2})(A) \sqcup \{\sigma(k)\}\) and \((\sigma \circ i_{p_2})(A \sqcup \{l\}) = (\sigma \circ i_{p_2})(A) \sqcup \{\sigma(l)\}\) with \(\sigma(k) < \sigma(l)\). Thus \(\sigma \circ i_{p_2}\) maps a preferred square in \(\Box_{p_1}\) to a preferred square in \(\Box_{p_1 + p_2}\) and \(\sigma \circ i_{p_2} : \Box_{p_1} \rightarrow \Box_{p_1 + p_2}\) is a trunk map.

In the same way, let \(1 \leq k < l \leq p_2, \sigma \in \text{Sh}_{p_1,p_2}\) implies \(\sigma(p_1 + k) < \sigma(p_1 + l)\). Moreover for all \(A, B\) such that \(A \cap B = \emptyset\) we have \((\sigma \circ i_{p_2})(A \sqcup B) = (\sigma \circ i_{p_2})(A) \sqcup (\sigma \circ i_{p_2})(B)\), thus \((\sigma \circ i_{p_2})(A \sqcup \{k\}) = (\sigma \circ i_{p_2})(A) \sqcup \{\sigma(p_1 + k)\}\) and \((\sigma \circ i_{p_2})(A \sqcup \{l\}) = (\sigma \circ i_{p_2})(A) \sqcup \{\sigma(p_1 + l)\}\) with \(\sigma(p_1 + k) < \sigma(p_1 + l)\). Thus \(\sigma \circ i_{p_2}\) maps a preferred square in \(\Box_{p_2}\) to a preferred square in \(\Box_{p_1 + p_2}\) and \(\sigma \circ i_{p_2} : \Box_{p_2} \rightarrow \Box_{p_1 + p_2}\) is a trunk map.

Using these maps, we will define the graded dendriform algebra product on rack/shelf cohomology.

The category of (pointed) trunks is the category with the (pointed) trunks as vertices and the (pointed) trunk maps as edges. This category is denoted by \(\text{Trunk}\) (resp. \(p\text{Trunk}\)).

**Relations between categories and trunks.** In Example 3.3 we have seen that a category \(\mathcal{C}\) gives rise to a trunk \(\text{Tr}(\mathcal{C})\). Moreover we have seen in Example 3.5 that a functor \(F\) from a category \(\mathcal{C}\) to a category \(\mathcal{C}'\) determines a trunk map \(\text{Tr}(F)\) from \(\text{Tr}(\mathcal{C})\) to \(\text{Tr}(\mathcal{C}')\). Hence a functor \(\text{Tr} : \text{Cat} \rightarrow \text{Trunk}\), from the category of categories \(\text{Cat}\) to the category of trunks \(\text{Trunk}\) defined by

\[
\begin{align*}
\text{Cat} & \xrightarrow{\text{Tr}} \text{Trunk} \\
\mathcal{C} & \mapsto \text{Tr}(\mathcal{C}) \\
F & \mapsto \text{Tr}(F)
\end{align*}
\]

Let \((\mathcal{T}, \Gamma)\) be a trunk. Consider the free category \(\mathcal{F}(\mathcal{T})\) generated by the graph \(\mathcal{T}\). For all pairs of objects \((x, y)\) in \(\mathcal{F}(\mathcal{T})\) we define a binary relation \(R_{x,y}\) on \(\text{Hom}_{\mathcal{F}(\mathcal{T})}(x, y)\) by \((a_1, \ldots, a_n)R_{x,y}(b_1, \ldots, b_m)\) if and only the two following points are satisfied:

1. \(n = m\),
2. For all \(1 \leq k \leq n\) such that \(a_k \neq b_k\) then \((a_k, a_{k+1}, b_k, b_{k+1}) \in \Gamma\) or \((a_{k-1}, a_k, b_{k-1}, b_k) \in \Gamma\).

We define a category \(\text{Cat}(\mathcal{T})\) by taking the quotient of \(\mathcal{F}(\mathcal{T})\) by the binary relation \(R\) (cf. [ML98] for the definition of a quotient category). Then the category \(\text{Cat}(\mathcal{T})\) has the same vertices as \(\mathcal{T}\) and the set of edges is the set of strings of composable edges in \(\mathcal{T}\) where two strings of edges are identified if they have the same length and are equal "up to preferred squares in \(\Gamma\)". Hence all commutative diagrams in \(\text{Cat}(\mathcal{T})\) comes from preferred squares in \((\mathcal{T}, \Gamma)\).

\[
x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \cdots \xrightarrow{a_n} x_{n+1} \sim x_0 \xrightarrow{b_0} x_1 \xrightarrow{b_1} \cdots \xrightarrow{b_n} x_{n+1}
\]
if and only if

\[
\begin{array}{ccc}
  x_0 & \overset{a_0}{\rightarrow} & x_1 \\
  b_1 & \downarrow & b_2 \\
  x_2 & \overset{a_1}{\rightarrow} & x_3 \\
  y_2 & \downarrow & y_3 \\
  x_4 & \overset{a_2}{\rightarrow} & x_5 \\
  \ldots & \downarrow & \ldots \\
  x_{n-2} & \overset{a_{n-3}}{\rightarrow} & x_{n-1} \\
  b_{n-2} & \downarrow & b_{n-1} \\
  x_{n-1} & \overset{a_{n-2}}{\rightarrow} & x_n \\
  y_{n-2} & \downarrow & y_{n-1} \\
  x_n & \overset{a_n}{\rightarrow} & x_{n+1}
\end{array}
\]

Let \( F \) be a trunk map from \((\mathcal{T}, \Gamma)\) to \((\mathcal{T}', \Gamma')\). As \( F \) is a graph map, \( \mathcal{F}(F) \) is a functor from \( \mathcal{F}(\mathcal{T}) \) to \( \mathcal{F}(\mathcal{T}') \). As \( F \) maps preferred squares in \( \Gamma \) to preferred squares in \( \Gamma' \), \( \mathcal{Cat}(F) \) is a well defined functor from \( \mathcal{Cat}(\mathcal{T}) \) to \( \mathcal{Cat}(\mathcal{T}') \).

Hence, there is a functor \( \mathcal{Cat} : \mathcal{Trunk} \to \mathcal{Cat} \), from the category of trunks to the category of (small) categories, defined by

\[
\mathcal{Trunk} \xrightarrow{\mathcal{Cat}} \mathcal{Cat} \\
\mathcal{T} \mapsto \mathcal{Cat}(\mathcal{T}) \\
F \mapsto \mathcal{Cat}(F)
\]

and \( \mathcal{Cat} \) is left adjoint to \( \mathcal{Trunk} \).

\[
\mathcal{Cat} \dashv \mathcal{Tr}.
\]  

(3)

**Remark 3.6.** For all \( n \in \mathbb{N} \) we can consider \( \square_n \) as a category (Example 3.1) or as a trunk (Example 3.3). We have

\[
\mathcal{Cat}(\square_n) = \square_n,
\]  

(4)

where on the left \( \square_n \) is seen as a trunk and on the right as a category, and

\[
\mathcal{Tr}(\square_n) \hookrightarrow \square_n,
\]

where on the left \( \square_n \) is seen as a category and on the right as a trunk.

### 3.2 Nerve of a trunk

**The category \( \square \).** Objects of the **the cubical category** \( \square \) are the graphs \( \square_n \) defined in Example 3.1. Given \( n \in \mathbb{N} \) vertices of \( \square_n \) has been defined as the subsets of \( \{1, \ldots, n\} \). The bijection between the set of subsets of \( \{1, \ldots, n\} \) and the set \( \{0, 1\}^n \), given by \( A \mapsto (\epsilon_1, \ldots, \epsilon_n) \) with \( \epsilon_k = 1 \) if \( k \in A \) and \( 0 \) if not, allows us to describe \( \square_n \) in a second way. In the first case, we will say that \( \square_n \) is described in subset notation, and in the second case, in coordinate notation.

For all \( m, n \in \mathbb{N} \), define \( \text{Hom}_{\square}(\square_m, \square_n) \) as the subset of \( \text{Hom}_{\text{Graph}}(\square_m, \square_n) \) generated by the face maps \( \partial_{i, \epsilon}^k \) defined in Example 3.5. In other words a morphism \( \text{Hom}_{\square}(\square_m, \square_n) \) is a graph map from \( \square_m \) to \( \square_n \) which is a composition of face maps. Using the description of \( \square_n \) in terms of coordinates, \( \partial_{i, \epsilon}^k \) is the graph map defined by

\[
\partial_{i, \epsilon}^k(\epsilon_1, \ldots, \epsilon_{n-1}) = (\epsilon_1, \ldots, \epsilon_{i-1}, \epsilon, \epsilon, \epsilon, \ldots, \epsilon_{n-1}).
\]

The next proposition states that face maps satisfy relations called **cubical relations**. This allows us to write a morphism in \( \text{Hom}_{\square}(\square_m, \square_n) \) in a canonical way.
Proposition 3.7 (Cubical relations). For all $1 \leq i < j \leq k + 1$, and $\epsilon, \omega \in \{0, 1\}$ we have the following relation:

$$\partial_{i,\epsilon}^{k+1} \circ \partial_{j-1,\omega}^{k} = \partial_{j,\omega}^{k+1} \circ \partial_{i,\epsilon}^{k}.$$  \hspace{1cm} (5)

Corollary 3.8. Let $m < n \in \mathbb{N}$. Each element $f$ in $\text{Hom}(\square_m, \square_n)$ can be written uniquely as

$$f = \partial_{m-n, \epsilon}^{n} \circ \cdots \circ \partial_{i_{1}, \epsilon_{1}}^{m+1},$$

with $i_1 < i_2 < \cdots < i_{n-m}$.

Cubical object in a category. Let $\mathcal{C}$ be a category. A precubical object in $\mathcal{C}$ is a functor $N : \square^{op} \to \mathcal{C}$. Dually a precocubical object in $\mathcal{C}$ is a functor $N : \square \to \mathcal{C}$.

Let $K$ be a commutative ring and $\mathcal{C}$ be the category $\text{Mod}_K$ of modules over $K$. From a precocubical $K$-module $M$ (that is a precocubical object in $\text{Mod}_K$), define a cochain complex $M^\bullet = \{M^n, d^n\}$ by

$$M^n := M(\square_n), d^n := \sum_{i=1}^{n} (-1)^i (M(\partial^n_{i,0}) - M(\partial^n_{i,1})).$$

The graded map $d^n$ is a differential thanks to the cubical relations (Proposition 3.7).

Remark 3.9. Let $A$ be a $K$-module. From a precubical set $S : \square^{op} \to \text{Set}$ we can always define a precocubical $K$-module by postcomposed $S$ by the functor $\text{Hom}_{\text{Set}}(-, A)$.

In the following paragraph we present an example of precubical set associated to a trunk.

Nerve of a trunk. We have seen in Example 3.3 that $\square_n$ has a trunk structure and that $\partial^n_{i,\epsilon}$ is a trunk map. Let $T$ be a trunk, the nerve of $T$ is the precubical set $N(T) : \square^{op} \to \text{Set}$ defined by

$$N(T)(\square_n) := \text{Hom}_{\text{Trunk}}(\square_n, T),$$

$$N(T)(\partial^n_{i,\epsilon}) := \text{Hom}_{\text{Trunk}}(\square_n, T)(\partial^n_{i,\epsilon}) = (\partial^n_{i,\epsilon})^*.$$  \hspace{1cm} (6)

In the case of the trunk associated to a rack, the nerve is easy to compute.

Theorem 3.10. Let $X$ be a rack. There is a bijection $N(X)(\square_n) \cong X^n$, and under this bijection we have

$$N(X)(\partial^n_{i,\epsilon})(x_1, \ldots, x_n) = \begin{cases} (x_1, \ldots, x_{i-1}, x_i \triangleright x_{i+1}, \ldots, x_1 \triangleright x_n) & \text{if } \epsilon = 0, \\
(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \text{if } \epsilon = 1. \end{cases}$$

Lemma 3.11. If $X$ is a rack, $F \in \text{Hom}_{\text{Trunk}}(\square_n, X)$ and $A \to A \amalg \{b\}$ is an edge in $\square_n$, then

$$F(A \to A \amalg \{b\}) = \prod_{1 \leq x \leq b} F([x-1] \to [x]).$$  \hspace{1cm} (7)

where $[x] := \{1, \ldots, x\}$. 

16
Proof. (of Lemma 3.11) First recall that in the trunk associated to a rack, we have \( a \triangleleft b = a \). Suppose \( A = \{a_1, \ldots, a_p\} \) with \( a_1 < \cdots < a_p \). We can suppose that \( b > a_p \). Indeed, if this is not the case, then

\[
F(A \to A \parallel \{b\}) = F((A \setminus \{a_p\}) \to (A \setminus \{a_p\}) \parallel \{b\}) \leq F((A \setminus \{a_p\}) \parallel \{b\}) \to A \parallel \{b\})
\]

because

\[
\begin{array}{c}
A \\
\downarrow \\
A \setminus \{a_p\} \longrightarrow (A \setminus \{a_p\}) \parallel \{b\}
\end{array}
\]

is a preferred square in \( \square_n \). Continuing this reduction until there is no element in \( A \) bigger than \( b \) shows that we may suppose that \( b > a_p \).

Now let us show by induction on the cardinal of \( \{1, \ldots, n\} \setminus A \) the expected equality (7).

- **Initialization**: Suppose that the cardinal of \( \{1, \ldots, n\} \setminus A \) is equal to 1. Because \( b > a_p \), necessarily \( A = \{1, \ldots, n - 1\} \) and \( b = n \), so the equality (7) is true.

- **Induction**: Suppose the equality true for rank \( k \). Let \( b_1 = \min\{1 \leq x \leq b \mid x \notin A\} \). If \( b_1 = b \), then \( A = \{1, \ldots, b - 1\} \) and we have the expected equality. If \( b_1 < b \) then the following equality holds by induction hypothesis

\[
F(A \parallel \{b_1\} \to A \parallel \{b_1, b\}) = \prod_{x \notin A \parallel \{b_1\}} F([x - 1] \to [x]).
\]

Because \( b_1 < b \), the square

\[
\begin{array}{c}
A \parallel \{b\} \\
\downarrow \\
A \parallel \{b_1, b\}
\end{array}
\]

is preferred in \( \square_n \), therefore

\[
F(A \to A \parallel \{b\}) = F(A \to A \parallel \{b_1\}) \triangleright F(A \parallel \{b_1\} \to A \parallel \{b_1, b\}).
\]

Now using the same reduction as in the beginning, we have

\[
F(A \to A \parallel \{b_1\}) = F(A_{<b_1} \to A_{<b_1} \parallel \{b_1\}),
\]

and by definition of \( b_1 \), we set \( A_{<b_1} = \{1, \ldots, b_1 - 1\} \). Thus the expected equality is true.

\[\square\]

Proof. (Theorem 3.10) First, let \( \eta_n \) the set map defined from \( N(X)(\square_n) \to X^n \) by

\[\eta_n : F \mapsto (x_1, \ldots, x_n) \quad \text{(8)}\]

with \( x_{k+1} = F([k] \to [k + 1]) \) for all \( 0 \leq k \leq n - 1 \). By Lemma 3.11 this map is a bijection.

Let \( (x_1, \ldots, x_n) \in X^n \). Let \( F \) be the trunk map in \( \text{Hom_{Trunk}}(\square_n, X) \) corresponding to this \( n \)-tuple by the previous bijection. By definition, \( N(X)(\partial^n_{\text{Trunk}})(x_1, \ldots, x_n) = (y_1, \ldots, y_{n-1}) \) with

\[
y_k = (F \circ \partial^n_{\text{Trunk}})([k - 1] \to [k]).
\]
• $\epsilon = 1$ : We have

$$\partial^n_{i,1}([k-1] \to [k]) = \begin{cases} [k-1] \coprod \{i\} \to [k] \coprod \{i\} & \text{if } k < i, \\ [k] \to [k+1] & \text{if } k \geq i. \end{cases}$$

Then

$$y_k = \begin{cases} F([k-1] \coprod \{i\} \to [k] \coprod \{i\}) = F([k-1] \to [k]) = x_k & \text{if } k < i, \\ F([k] \to [k+1]) = x_{k+1} & \text{if } k \geq i. \end{cases}$$

• $\epsilon = 0$ : We have

$$\partial^n_{i,0}([k-1] \to [k]) = \begin{cases} [k-1] \to [k] & \text{if } k < i, \\ [k] \setminus \{i\} \to [k+1] \setminus \{i\} & \text{if } k \geq i. \end{cases}$$

The square

$$\begin{array}{ccc}
[k+1] \setminus \{i\} & \to & [k+1] \\
\uparrow & & \uparrow \\
[k] \setminus \{i\} & \to & [k]
\end{array}$$

is preferred in $\square_n$, so

$$F([k] \setminus \{i\} \to [k+1] \setminus \{i\}) = F([k] \setminus \{i\} \to [k]) \triangleright F([k] \to [k+1]).$$

By Lemma 3.11 this is equal to

$$F([i-1] \to [i]) \triangleright F([k] \to [k+1]),$$

and finally

$$y_k = \begin{cases} F([k-1] \to [k]) = x_k & \text{if } k < i, \\ x_i \triangleright x_{k+1} & \text{if } k \geq i. \end{cases}$$

Let $X$ be a rack and $A$ be an abelian group. By Remark 3.9, the composition $\text{Hom}_{\text{Set}}(-, A) \circ N(X)$ defines a precocubical abelian group. The cochain complex associated to this precocubical abelian group is denoted by $\{C^n(N(X), A), d^n\}_{n \in \mathbb{N}}$.

**Corollary 3.12.** Let $X$ be a rack and $A$ be an abelian group (considered as a trivial $X$-module). There is an isomorphism of cochain complexes

$$\{CR^n(X, A), d^n_R\}_{n \in \mathbb{N}} \cong \{C^n(N(X), A), d^n\}_{n \in \mathbb{N}}.$$

## 4 A graded dendriform algebra structure on $HR^\bullet(X, A)$

Let $X$ be a rack and $A$ be an associative algebra considered as a trivial $X$-module. The goal of this section is to define a graded dendriform algebra structure on $HR^\bullet(X, A)$, the rack cohomology of $X$ with coefficients in $A$. First we define two graded dendriform products on $\{C^n(N(X), A), d^n\}_{n \in \mathbb{N}}$, the graded module associated to the nerve of the rack $X$. Then we show that these products are compatible with the differential, and so define a graded dendriform algebra structure on $HR^\bullet(X, A)$. 


Products. Let $\sigma \in \text{Sh}_{p_1,p_2}$. In Example 3.5 we deduced from $\sigma$ two trunk maps $\sigma \circ i_{p_1} : \Box_{p_1} \to \Box_{p_1+p_2}$ and $\sigma \circ i_{p_2} : \Box_{p_2} \to \Box_{p_1+p_2}$. Let $\rho_\sigma$ denote the (set theoretical) map from $N(X)(\Box_{p_1+p_2})$ to $N(X)(\Box_{p_1}) \times N(X)(\Box_{p_2})$ defined by

$$\rho_\sigma := ((\sigma \circ i_{p_1})^*, (\sigma \circ i_{p_2})^*).$$  \hspace{1cm} (9)

We define two graded products $\succ$ and $\prec$ on $\{C^n(N(X), A), d^n\}_{n \in \mathbb{N}}$ by

$$f_1 \succ f_2 := \sum_{\sigma \in \text{Sh}_{p_1+p_2}^{p_1+p_2}} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ \rho_\sigma,$$  \hspace{1cm} (10)

and

$$f_1 \prec f_2 := \sum_{\sigma \in \text{Sh}_{p_1+p_2}^{p_1+p_2}} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ \rho_\sigma.$$  \hspace{1cm} (11)

Dendriform structure. In this paragraph we prove that $\{C^n(N(X), A), \succ, \prec\}_{n \in \mathbb{N}}$ is a graded dendriform algebra. For this we use the bijections

$$\text{Sh}_{p_1,p_2+p_3} \times \text{Sh}_{p_2,p_3} \xrightarrow{\alpha} \text{Sh}_{p_1,p_2+p_3} \simeq \text{Sh}_{p_1+p_2,p_3} \times \text{Sh}_{p_1,p_2}.$$  \hspace{1cm} (12)

described in (1).

First, similarly to what we did in Example 3.5, let us define from a shuffle $\sigma \in \text{Sh}_{p_1,p_2,p_3}$ three trunk maps $\sigma \circ i_{p_j} : \Box_{p_j} \to \Box_{p_1+p_2+p_3}$ where

$$i_{p_1} := \partial_{p_1+p_2+p_3} \circ \cdots \circ \partial_{p_1+1,0},$$

$$i_{p_2} := \partial_{p_1+p_2+p_3} \circ \cdots \circ \partial_{p_1+p_2+1,0} \circ \partial_{p_1+p_2+1,1} \circ \cdots \circ \partial_{1,1},$$

$$i_{p_3} := \partial_{p_1+p_2+p_3} \circ \cdots \circ \partial_{1,1}.$$

If $A = \{a_1, \ldots, a_p\} \subseteq \Box_{p_j}$, then

$$(\sigma \circ i_{p_j})(A) = \{\sigma(a_1), \ldots, \sigma(a_p)\},$$

$$(\sigma \circ i_{p_2})(A) = \{\sigma(1), \ldots, \sigma(p_1), \sigma(p_1 + a_1), \ldots, \sigma(p_1 + a_p)\},$$

$$(\sigma \circ i_{p_3})(A) = \{\sigma(1), \ldots, \sigma(p_1 + p_2), \sigma(p_1 + p_2 + a_1), \ldots, \sigma(p_1 + p_2 + a_p)\}.$$  \hspace{1cm} (13)

Let $\rho_\sigma$ the set map from $N(X)(\Box_{p_1+p_2+p_3})$ to $N(X)(\Box_{p_1}) \times N(X)(\Box_{p_2}) \times N(X)(\Box_{p_3})$ defined by

$$\rho_\sigma := ((\sigma \circ i_{p_1})^*, (\sigma \circ i_{p_2})^*, (\sigma \circ i_{p_3})^*).$$

Lemma 4.1. Let $\sigma \in \text{Sh}_{p_1+p_2+p_3}$ (resp. $\sigma \in \text{Sh}_{p_1,p_2+p_3}$) and $\gamma \in \text{Sh}_{p_2,p_3}$ (resp. $\gamma \in \text{Sh}_{p_1,p_2}$). The following equality holds

$$\rho_{\sigma(1\gamma)} = (1 \times \rho_\gamma) \circ \rho_\sigma \text{ resp. } \rho_{\sigma(\gamma 1)} = (\rho_\gamma \times 1) \circ \rho_\sigma.$$  \hspace{1cm} (14)

Proof. Let $\sigma \in \text{Sh}_{p_1+p_2+p_3}$ and $\gamma \in \text{Sh}_{p_2,p_3}$. Let $F \in N(T)(\Box_{p_1+p_2+p_3})$, by definition

$$\rho_{\sigma(1\gamma)}(F) = (F \circ (\sigma \circ (1 \gamma) \circ i_{p_1}), F \circ (\sigma \circ (1 \gamma) \circ i_{p_2}), F \circ (\sigma \circ (1 \gamma) \circ i_{p_3})).$$

and

$$((1 \times \rho_\gamma) \circ \rho_\sigma)(F) = (F \circ \sigma \circ i_{p_1}, F \circ \sigma \circ i_{p_2+p_3} \circ \gamma \circ i_{p_2}, F \circ \sigma_{p_2+p_3} \circ \gamma_{p_3}).$$

19
Let $A = \{a_1, \ldots, a_p\} \subseteq \square_{p1}$.

$$(\sigma \circ (1 \star \gamma) \circ i_{p1})(A) = \{(\sigma \circ (1 \star \gamma))(a_1), \ldots, (\sigma \circ (1 \star \gamma))(a_p)\},$$

$$= \{\sigma(a_1), \ldots, \sigma(a_p)\},$$

$$= (\sigma \circ i_{p1})(A).$$

Let $A = \{a_1, \ldots, a_p\} \subseteq \square_{p2}$.

$$(\sigma \circ (1 \star \gamma) \circ i_{p2})(A) = \{(\sigma \circ (1 \star \gamma))(1), \ldots, (\sigma \circ (1 \star \gamma))(p_1), (\sigma \circ (1 \star \gamma))(p_1 + a_1), \ldots, (\sigma \circ (1 \star \gamma))(p_1 + a_p)\},$$

$$= \{\sigma(1), \ldots, \sigma(p_1), \sigma(p_1 + \gamma(a_1)), \ldots, \sigma(p_1 + \gamma(a_p))\},$$

$$= (\sigma \circ i_{p2+p_2} \circ \gamma \circ i_{p2})(A).$$

Let $A = \{a_1, \ldots, a_p\} \subseteq \square_{p3}$.

$$(\sigma \circ (1 \star \gamma) \circ i_{p3})(A) = \{(\sigma \circ (1 \star \gamma))(1), \ldots, (\sigma \circ (1 \star \gamma))(p_1 + p_2), (\sigma \circ (1 \star \gamma))(p_1 + p_2 + a_1), \ldots, (\sigma \circ (1 \star \gamma))(p_1 + p_2 + a_p)\},$$

$$= \{\sigma(1), \ldots, \sigma(p_1), \sigma(p_1 + \gamma(1)), \ldots, \sigma(p_1 + \gamma(p_2)), \sigma(p_1 + \gamma(p_2 + a_1)), \ldots, \sigma(p_1 + \gamma(p_2 + a_p))\},$$

$$= (\sigma \circ i_{p2+p_2+p_3} \circ \gamma \circ i_{p3})(A).$$

Thus $\rho_{\sigma \circ (1 \star \gamma)} = (1 \times \rho_1) \circ \rho_2$. The proof for the other equality is similar. \qed

**Theorem 4.2.** Let $X$ be a rack and $A$ be an associative algebra. The graded module $\{C^n(N(X), A), >, <\}_{n \in \mathbb{N}}$ provided with the products $>$ and $<$ is a graded dendriform algebra.

**Proof.** Let $p_1, p_2, p_3 \in \mathbb{N}$ and $f_1 \in C^{p_1}(N(X), A), f_2 \in C^{p_2}(N(X), A), f_3 \in C^{p_3}(N(X), A)$. We have to prove the three equalities

1. $f_1 > (f_2 > f_3) = (f_1 > f_2 + f_1 \times f_2) > f_3$.
2. $f_1 > (f_2 < f_3) = (f_1 > f_2) < f_3$.
3. $(f_1 < f_2) < f_3 = f_1 < (f_2 < f_3 + f_2 > f_3)$.

Consider the bijection from $\text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_2+p_2} \times \text{Sh}_{p_1+p_2}$ to $\text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_1+p_2}$

$$(\sigma, \gamma) \mapsto (\sigma', \gamma') = (\beta^{-1} \circ \alpha)(\sigma, \gamma).$$

In particular $\epsilon(\sigma)\epsilon(\gamma) = \epsilon(\sigma')\epsilon(\gamma')$.

1. Let $(\sigma, \gamma) \in \text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_2+p_2+p_3}$, necessarily $(\sigma', \gamma') \in \text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_1+p_2}$. Then, because of cardinality, the map $\beta^{-1} \circ \alpha$ induces a bijection

$$\text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_2+p_2+p_3} \simeq \text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_1+p_2}.$$

Applying the change of variable $\beta^{-1} \circ \alpha$ to the left term in the equality 1. and using the Lemma 4.1, we find the right term in 1..

2. To show this equality, use the bijection $\beta^{-1} \circ \alpha$ restricted to $\text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_2+p_2+p_3}$. It gives a change of variables

$$\text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_2+p_2} \simeq \text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_1+p_2}.$$

3. To show this equality, use the bijection $\beta^{-1} \circ \alpha$ restricted to $\text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_1+p_2+p_3}$. It gives a change of variables

$$\text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_2+p_2} \simeq \text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_1+p_2+p_3} \times \text{Sh}_{p_1+p_2}.$$

\qed
Compatibility of the differential with the products. Let $X$ be a rack and $A$ be an associative algebra considered as a trivial $X$-module. By Theorem 1.2 there is a graded dendriform algebra structure on $C^*(N(X), A)$. In this paragraph we prove that this dendriform structure is compatible with the differential, that is, $\{C^n(N(X), A), d^n, \succ, \prec\}_{n \in \mathbb{N}}$ is a differential graded dendriform algebra.

Theorem 4.3. Let $X$ be a rack and $A$ be an associative algebra. The cochain complex $\{C^n(N(X), A), d^n\}_{n \in \mathbb{N}}$ provided with the products $\succ$ and $\prec$ is a differential graded dendriform algebra.

Proof. We have to prove that

\[(d^{p_1+1}f_1) \succ f_2 + (-1)^{p_1}f_1 \succ (d^{p_2+1}f_2) = d^{p_1+p_2+1}(f_1 \succ f_2)\]  

(12)

and

\[(d^{p_1+1}f_1) \prec f_2 + (-1)^{p_1}f_1 \prec (d^{p_2+1}f_2) = d^{p_1+p_2+1}(f_1 \prec f_2)\]  

(13)

for all $f_1 \in C^{p_1}(N(X), A)$ and $f_2 \in C^{p_2}(N(X), A)$.

To prove these two equalities, let us introduce the map $\phi$ from $S_{n+1} \times \{1, \ldots, n+1\}$ to $S_n$ defined by

\[\phi(\sigma, i) := (n+1 \ldots \sigma(i)) \circ \sigma \circ (i \ldots n+1)\]

Two important properties of $\phi$ are the following equalities, satisfied for all $\sigma \in S_{n+1}, i \in \{1, \ldots, p_1 + p_2 + 1\}$ and $\epsilon \in \{0, 1\}$:

\[\partial_{\sigma(i), \epsilon}^n \circ \phi(\sigma, i) = \sigma \circ \partial_{\epsilon, \epsilon}^n,\]  

(14)

\[(-1)^{p(i)}\epsilon(\phi(\sigma, i)) = (-1)^{i}\epsilon(\sigma).\]  

(15)

Remark 4.4. The equality (14) takes place in the category $\text{Gph}$.

In the sequel, the restriction to suitable subsets of the map $\psi : S_{n+1} \times \{1, \ldots, n+1\} \to S_n \times \{1, \ldots, n+1\}$ defined by $\psi(\sigma, i) = (\phi(\sigma, i), \sigma(i))$ will provide the right change of variables needed in order to prove (12) and (13).

Compatibility with $\succ$: Let $f_1 \in C^{p_1}(N(X), A)$ and $f_2 \in C^{p_2}(N(X), A)$. The left term in (12) is equal to the sum of four terms (a), (b), (c) and (d) with

\[\text{(a)} = \sum_{\sigma \in \text{Sh}_{p_1+p_2+1}} \sum_{i=1}^{p_1} (-1)^i \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\sigma \circ i_{p_1+1} \circ \partial_{0,0}^{i+1})(\sigma \circ i_{p_2})^*),\]

\[\text{(b)} = \sum_{\sigma \in \text{Sh}_{p_1+p_2+1}} \sum_{i=1}^{p_2} (-1)^{i+p_1} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\sigma \circ i_{p_1})^*, (\sigma \circ i_{p_2+1} \circ \partial_{0,0}^{i+1})^*),\]

\[\text{(c)} = \sum_{\sigma \in \text{Sh}_{p_1+p_2+1}} \sum_{i=1}^{p_1} (-1)^{i+1} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\sigma \circ i_{p_1+1} \circ \partial_{i,1}^{i+1})(\sigma \circ i_{p_2})^*),\]

\[\text{(d)} = \sum_{\sigma \in \text{Sh}_{p_1+p_2+1}} \sum_{i=1}^{p_2} (-1)^{i+p_1+1} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\sigma \circ i_{p_1})^*, (\sigma \circ i_{p_2+1} \circ \partial_{i,1}^{i+1})^*).\]
The right term in (12) is equal to the sum of two terms (A) and (B), with

\[
\begin{align*}
(A) &= \sum_{\sigma \in \mathbb{S}_p^{1+p_2}} \sum_{i=1}^{p_1+p_2} (-1)^i \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \sigma \circ i_p)_*) \circ ((\partial^{p_1+p_2+1}_{i_1} \circ \sigma \circ i_p)_*),
\end{align*}
\]

\[
(B) = \sum_{\sigma \in \mathbb{S}_p^{1+p_2}} \sum_{i=1}^{p_1+p_2} (-1)^{i+1} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \sigma \circ i_p)_*) \circ ((\partial^{p_1+p_2+1}_{i_1} \circ \sigma \circ i_p)_*).}
\]

In order to prove (12), we are going to show that

\[(a) + (c) + (d) = (B) \quad \text{and} \quad (b) = (A).
\]

• (b) = (A) : This equality is proved using the change of variables

\[
\psi \circ t_{p_1} : \mathbb{S}_p^{1+p_2+1} \times \{1, \ldots, p_2\} \to \mathbb{S}_p^{1+p_2} \times \{1, \ldots, p_1 + p_2\}
\]

with \(t_{p_1}(i) = p_1 + i\). Using this change of variables, we have to prove for all \(\sigma \in \mathbb{S}_p^{1+p_2+1}\) and \(i \in \{1, \ldots, p_2\}\):

1. \(((\sigma \circ i_{p_1})_*; (\sigma \circ i_{p_1})_*; 0; 0; 0; 0) ; (\partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \phi(\sigma, p_1 + i) \circ i_{p_1})_*; (\partial^{p_1+p_2+1}_{\sigma(i_1)} \circ \phi(\sigma, p_1 + i) \circ i_{p_1})_*; (\partial^{p_1+p_2+1}_{\sigma(i_2)} \circ \phi(\sigma, p_1 + i) \circ i_{p_1})_*; (\partial^{p_1+p_2+1}_{\sigma(i_3)} \circ \phi(\sigma, p_1 + i) \circ i_{p_1})_*; (\partial^{p_1+p_2+1}_{\sigma(i_4)} \circ \phi(\sigma, p_1 + i) \circ i_{p_1})_*; (\partial^{p_1+p_2+1}_{\sigma(i_5)} \circ \phi(\sigma, p_1 + i) \circ i_{p_1})_*; (\partial^{p_1+p_2+1}_{\sigma(i_6)} \circ \phi(\sigma, p_1 + i) \circ i_{p_1})_*) \]

2. \((-1)^{p_1+1} \epsilon(\sigma) \phi(\sigma, p_1 + i) \circ i_{p_1}) = (-1)^{p_1+1} \epsilon(\sigma).
\]

1. Let \(\sigma \in \mathbb{S}_p^{1+p_2+1}\) and \(i \in \{1, \ldots, p_2\}\). We want to prove for all \(F \in \text{Hom}_{\text{Trunk}}(\square_{p_1+p_2+1}, X)\)

\[
F \circ \sigma \circ i_{p_1} = F \circ \partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \phi(\sigma, p_1 + i) \circ i_{p_1},
\]

\[
F \circ \sigma \circ i_{p_2+1} \circ \partial^{p_2+1}_{i_0} = F \circ \partial^{p_1+p_2+1}_{\sigma(i_1)} \circ \phi(\sigma, p_1 + i) \circ i_{p_2}.
\]

By Theorem 3.10 this is equivalent to

\[
\begin{cases}
(F \circ \sigma \circ i_{p_1})([k-1] \to [k]) = (F \circ \partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \phi(\sigma, p_1 + i) \circ i_{p_1})([k-1] \to [k]), & 1 \leq k \leq p_1, \\
(F \circ \sigma \circ i_{p_2+1} \circ \partial^{p_2+1}_{i_0})([k-1] \to [k]) = (F \circ \partial^{p_1+p_2+1}_{\sigma(i_1)} \circ \phi(\sigma, p_1 + i) \circ i_{p_2})([k-1] \to [k]), & 1 \leq k \leq p_2.
\end{cases}
\]

Let \(0 \leq k \leq p_1\) and \((\sigma \circ i_{p_1})([k_1]) = \sigma([k])\) and \((\sigma \circ i_{p_1})([k_2]) = \sigma([k])\) \(\sigma \circ i_{p_1})([k]) = \sigma([k])\) the first equality is true. Let \(0 \leq k \leq p_2\),

\[
(\sigma \circ i_{p_2+1} \circ \partial^{p_2+1}_{i_0})([k]) = \begin{cases}
(\sigma \circ i_{p_2+1})([k]) & \text{if } k < i, \\
(\sigma \circ i_{p_2+1})([k+1] \setminus \{i\}) & \text{if } k \geq i,
\end{cases}
\]

and

\[
\partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \phi(\sigma, p_1 + i) \circ i_{p_2})([k]) = \partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \phi(\sigma, p_1 + i) \circ i_{p_2})([k+1])
\]

\[
\begin{cases}
(\sigma \circ i_{p_1+p_2+1})([k]) & \text{if } k < i, \\
(\sigma \circ i_{p_1+p_2+1})([k+1] \setminus \{i\}) & \text{if } k \geq i,
\end{cases}
\]

and

\[
\partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \phi(\sigma, p_1 + i) \circ i_{p_2})([k]) = \partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \phi(\sigma, p_1 + i) \circ i_{p_2})([k+1])
\]

\[
\begin{cases}
(\sigma \circ i_{p_1+p_2+1})([k]) & \text{if } k < i, \\
(\sigma \circ i_{p_1+p_2+1})([k+1] \setminus \{i\}) & \text{if } k \geq i,
\end{cases}
\]

and

\[
\partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \phi(\sigma, p_1 + i) \circ i_{p_2})([k]) = \partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \phi(\sigma, p_1 + i) \circ i_{p_2})([k+1])
\]

\[
\begin{cases}
(\sigma \circ i_{p_1+p_2+1})([k]) & \text{if } k < i, \\
(\sigma \circ i_{p_1+p_2+1})([k+1] \setminus \{i\}) & \text{if } k \geq i,
\end{cases}
\]

and

\[
\partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \phi(\sigma, p_1 + i) \circ i_{p_2})([k]) = \partial^{p_1+p_2+1}_{\sigma(i_0)} \circ \phi(\sigma, p_1 + i) \circ i_{p_2})([k+1])
\]

\[
\begin{cases}
(\sigma \circ i_{p_1+p_2+1})([k]) & \text{if } k < i, \\
(\sigma \circ i_{p_1+p_2+1})([k+1] \setminus \{i\}) & \text{if } k \geq i,
\end{cases}
\]
so the second equality is true.

2. Deduced from [15].

(a) + (c) + (d) = (B) : To prove this equality, we need to decompose (d) into two sums. Let us denote by \( I^2 \) and \( I^c \) the subsets of \( \text{Sh}_{p_1 + p_2 + 1} \times \{1, \ldots, p_2\} \) defined by

\[
I^2 := \{(\sigma, i) \in \text{Sh}_{p_1 + p_2 + 1} \times \{1, \ldots, p_2\} \mid \sigma(1 + i) \geq p_1 + 1 \}
\]

\[
= \{(\sigma, i) \in \text{Sh}_{p_1 + p_2 + 1} \times \{1, \ldots, p_2\} \mid \sigma(1 + i) = p_1 + 1 \}
\]

and

\[
I^c := (\text{Sh}_{p_1 + p_2 + 1} \times \{1, \ldots, p_2\}) \setminus I^2.
\]

Then we are going to prove that

\[
(a) + \sum_{I^c} = 0 \quad \text{and} \quad (c) + \sum_{I^c} = (B).
\]

(a) + \( \sum_{I^c} = 0 \): To prove this equality, we use the change of variables

\[
\theta : \text{Sh}_{p_1 + p_2 + 1} \times \{1, \ldots, p_1\} \to I^c
\]

defined by \( \theta(\sigma, i) = (\rho(\sigma, i), \sigma(i) + i) \) with \( \rho(\sigma, i) = \sigma \circ (i \ldots p_1 + 1) + \sigma(i) - i \). Using this change of variables, we have to prove for all \( \sigma \in \text{Sh}_{p_1 + p_2 + 1} \) and \( i \in \{1, \ldots, p_1\} \):

1. \((\sigma \circ i_{p_1 + 1} \circ \partial_{i,0}^{p_1 + 1} \sigma(\sigma \circ i_{p_1})^*) = ((\rho(\sigma, i) \circ i_{p_1})^*, (\rho(\sigma, i) \circ i_{p_2 + 1} \circ \partial_{i,0}^{p_2 + 1} \sigma(\sigma \circ i_{p_1} + (\sigma \circ i_{p_1}))^*)

\]

2. \((-1)^{p_1 + 1 + \sigma(i)} \epsilon(\rho(\sigma, i)) = (-1)^i \epsilon(\sigma).

1. Let \( \sigma \in \text{Sh}_{p_1 + p_2 + 1} \) and \( i \in \{1, \ldots, p_1\} \). Using Theorem 3.10, we are going to prove, for all \( F \in \text{Hom}_{\text{Trunk}}([p_1 + p_2 + 1, X] \), the following equalities:

\[
\begin{cases}
(F \circ \sigma \circ i_{p_1 + 1} \circ \partial_{i,0}^{p_1 + 1} \sigma([k - 1] \to [k])) = (F \circ \rho(\sigma, i) \circ i_{p_1})([k - 1] \to [k]) & \text{if } 1 \leq k \leq p_1, \\
(F \circ \sigma \circ i_{p_2})([k - 1] \to [k]) = (F \circ \rho(\sigma, i) \circ i_{p_2 + 1} \circ \partial_{i,0}^{p_2 + 1} \sigma([k - 1] \to [k]) & \text{if } 1 \leq k \leq p_2.
\end{cases}
\]

Let \( 0 \leq k \leq p_1 \),

\[
(\sigma \circ i_{p_1 + 1} \circ \partial_{i,0}^{p_1 + 1} \sigma([k]) = \begin{cases}
\sigma([k]) & \text{if } k < i, \\
\sigma([k + 1] \setminus \{i\}) & \text{if } k \geq i.
\end{cases}
\]

and

\[
(\rho(\sigma, i) \circ i_{p_1})([k]) = (\sigma \circ (i \ldots p_1 + 1 + \sigma(i) - i))([k]).
\]

\[
= \begin{cases}
\sigma([k]) & \text{if } k < i, \\
\sigma([k + 1] \setminus \{i\}) & \text{if } k \geq i.
\end{cases}
\]

(remark that \( \sigma(i) - i \geq 0 \) for \( i \in \{1, \ldots, p_1\} \), so the first equality is true.

Let \( 0 \leq k \leq p_2 \), we have \((\sigma \circ i_{p_2})([k]) = \sigma([p_1 + 1 + k])\) and

\[
(\rho(\sigma, i) \circ i_{p_2 + 1} \circ \partial_{i,0}^{p_2 + 1} \sigma([k])) = \begin{cases}
(\rho(\sigma, i) \circ i_{p_2 + 1})([k] \cup [1 + \sigma(i) - i]) & \text{if } k < 1 + \sigma(i) - i, \\
(\rho(\sigma, i) \circ i_{p_2 + 1})([k + 1]) & \text{if } k \geq 1 + \sigma(i) - i, 
\end{cases}
\]

\[
= \begin{cases}
\rho(\sigma, i)\{[p_1 + k] \cup [p_1 + 1 + \sigma(i) - i]) & \text{if } k < 1 + \sigma(i) - i, \\
\rho(\sigma, i)\{[p_1 + k + 1]) & \text{if } k \geq 1 + \sigma(i) - i, 
\end{cases}
\]

\[
\sigma([p_1 + k + 1]),
\]

23
so the second equality is true.

2. Clear by definition of $\rho(\sigma,i)$.

(c) $\sum_{I^c} = (B)$ : To prove this equality, we need to decompose (B) into two sums, one equals $\sum_{I^c}$ and the other equals (c). Let us denote by $J^c$ and $J^<\subseteq$ the subsets of $\text{Sh}_{p_1+p_2}^{p_1+p_2} \times \{p_1 + 1, \ldots, p_1 + p_2\}$ defined by

$$J^c := \{(\sigma, i) \in \text{Sh}_{p_1+p_2}^{p_1+p_2} \times \{p_1 + 1, \ldots, p_1 + p_2\} \mid \sigma(i) \geq i\}$$

and

$$J^< := (\text{Sh}_{p_1+p_2}^{p_1+p_2} \times \{p_1 + 1, \ldots, p_1 + p_2\}) \setminus J^c.$$

Then we are going to prove that

$$\sum_{I^c} = \sum_{J^c} \text{ and } (c) = \sum_{J^<}.$$

$\sum_{I^c} = \sum_{J^c}$ : This equality is proved using the change of variables

$$\psi \circ t_{p_1} : I^c \rightarrow J^c.$$

In this case we have $(\psi \circ t_{p_1})(\sigma, i) = (\sigma, p_1 + i)$. Using this change of variables, we have to prove for all $\sigma \in \text{Sh}_{p_1+p_2+1}$ and $i \in \{1, \ldots, p_2\}$ :

$$((\sigma \circ i_{p_1}), (\sigma \circ i_{p_1+1} \circ \partial_i^{p_2+1})*) = ((\partial_i^{p_1+p_2+1} \circ \sigma \circ i_{p_1}), (\partial_i^{p_1+p_2+1} \circ \sigma \circ i_{p_2})*).$$

Let $\sigma \in \text{Sh}_{p_1+p_2+1}$ and $i \in \{1, \ldots, p_2\}$. Using Theorem [3.10] we are going to prove, for all $F \in \text{Hom}_{\text{Trunk}(\square_{p_1+p_2+1}, X)}$, the following equalities :

$$\left\{ \begin{array}{ll}
(F \circ \sigma \circ i_{p_1})([k - 1]) \rightarrow [k] &= (F \circ \partial_{i_{p_1+1}} \circ \sigma \circ i_{p_1})([k - 1] \rightarrow [k]) \quad \text{if } 1 \leq k \leq p_1, \\
(F \circ \sigma \circ i_{p_2+1} \circ \partial_i^{p_2+1})([k - 1] \rightarrow [k]) &= (F \circ \partial_{i_{p_1+1}} \circ \sigma \circ i_{p_2})([k - 1] \rightarrow [k]) \quad \text{if } 1 \leq k \leq p_2.
\end{array} \right.$$

Let $0 \leq k \leq p_1$, $(\sigma \circ i_{p_1})([k]) = \sigma([k])$ and $(\partial_{i_{p_1+1}} \circ \sigma \circ i_{p_1})([k]) = (\sigma \circ i_{p_1+1})([k]) \sqcup (\sigma \circ i_{p_1})([k])$. Hence the first equality is equivalent to

$$F(\sigma([k - 1])) \rightarrow \sigma([k]) = F((\sigma([k - 1]) \sqcup (p_1 + i)) \rightarrow (\sigma([k]) \sqcup (p_1 + i))).$$

Because $\sigma \in I^c$, necessarily $\sigma(k) < p_1 + i$, so the square

$$\begin{array}{ccc}
\sigma([k - 1]) \sqcup (p_1 + i) & \rightarrow & \sigma([k]) \\
\downarrow & & \downarrow \\
\sigma([k - 1]) & \rightarrow & \sigma([k])
\end{array}$$

is a preferred square in $\square_{p_1+p_2+1}$, and the expected equality si proved.

Let $0 \leq k \leq p_2$, we have

$$(\sigma \circ i_{p_2+1} \circ \partial_i^{p_2+1})([k]) = \left\{ \begin{array}{ll}
\sigma([p_1 + k]) \sqcup (p_1 + i) & \text{if } k < i, \\
\sigma([p_1 + k + 1]) & \text{if } k \geq i.
\end{array} \right.$$
Using this change of variables, we have to prove for all 
so the second equality is true.

(c) = \sum_{J \subset \varepsilon}: The following equality is proved using the change of variables

Using this change of variables, we have to prove for all \( \sigma \in \text{Sh}_{p_1 + p_2} \) and \( i \in \{1, \ldots, p_1\} \):

1. \( ((\sigma \circ i_{p_1 + 1} \circ \partial_{i_1}^{p_1 + 1})^*, (\sigma \circ i_{p_2})^*) = ((\partial_{\sigma(i_1)1}^{p_1 + p_2 +1} \circ \phi(\sigma, i) \circ i_{p_1})^*, (\partial_{\sigma(i)1}^{p_1 + p_2 +1} \circ \phi(\sigma, i) \circ i_{p_2})^*), \)

so the first equality is true.

2. (-1)^{i+1} \epsilon(\sigma) = (-1)^{\sigma(i)+1} \epsilon(\phi(\sigma, i)).

1. Let \( \sigma \in \text{Sh}_{p_1 + p_2} \) and \( i \in \{1, \ldots, p_1\} \). Using Theorem 3.10 we are going to prove, for all \( F \in \text{Hom}_{\text{Trunk}}(\mathbb{C}_{p_1 + p_2}, X) \), the following equalities:

\[
\begin{cases}
(F \circ (\sigma \circ i_{p_1 + 1} \circ \partial_{i_1}^{p_1 + 1})((k - 1) \rightarrow [k]) = (F \circ (\partial_{\sigma(i_1)1}^{p_1 + p_2 +1} \circ \phi(\sigma, i) \circ i_{p_1})((k - 1) \rightarrow [k]) & \text{if } 1 \leq k \leq p_1, \\
(F \circ (\sigma \circ i_{p_2}))([k - 1] \rightarrow [k]) = (F \circ (\partial_{\sigma(i_1)1}^{p_1 + p_2 +1} \circ \phi(\sigma, i) \circ i_{p_2})(([k - 1] \rightarrow [k]) & \text{if } 1 \leq k \leq p_2.
\end{cases}
\]

2. Deduced from \([15]\).

Compatibility with \( \prec \): The proof is essentially the same as before. Let \( f_1 \in C^{p_1}(N(X), A) \) and \( f_2 \in C^{p_2}(N(X), A) \). The left term in \([13]\) is equal to the sum of four terms (a), (b), (c) and (d)
with

(a) = \sum_{\sigma \in \text{Sh}_{p_1+1, p_2}} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (-1)^{i+j} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\sigma \circ i_{p_1+1} \circ \partial_{i,0}^{p_1+1})^*, (\sigma \circ i_{p_2})^*),

(b) = \sum_{\sigma \in \text{Sh}_{p_1+1, p_2}} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (-1)^{i+j} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\sigma \circ i_{p_1} \circ \partial_{i,0}^{p_1+1})^*, (\sigma \circ i_{p_2+1} \circ \partial_{i,0}^{p_2+1})^*),

(c) = \sum_{\sigma \in \text{Sh}_{p_1+1, p_2}} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (-1)^{i+j} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\sigma \circ i_{p_1+1} \circ \partial_{i,1}^{p_1+1})^*, (\sigma \circ i_{p_2})^*),

(d) = \sum_{\sigma \in \text{Sh}_{p_1+1, p_2}} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (-1)^{i+j} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\sigma \circ i_{p_1} \circ \partial_{i,1}^{p_1+1})^*, (\sigma \circ i_{p_2+1} \circ \partial_{i,1}^{p_2+1})^*).

The right term in (13) is equal to the sum of two terms (A) and (B), with

(A) = \sum_{\sigma \in \text{Sh}_{p_1+1, p_2}} \sum_{i=1}^{p_1+p_2} \sum_{j=1}^{\max(p_1+1, p_2)} (-1)^{i+j} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\partial_{i,0}^{p_1+p_2+1} \circ \sigma \circ i_{p_1+1})^*, (\partial_{i,0}^{p_2+1} \circ \sigma \circ i_{p_2+1})^*),

(B) = \sum_{\sigma \in \text{Sh}_{p_1+1, p_2}} \sum_{i=1}^{p_1+p_2} \sum_{j=1}^{\max(p_1+1, p_2)} (-1)^{i+j} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ ((\partial_{i,1}^{p_1+p_2+1} \circ \sigma \circ i_{p_1+1})^*, (\partial_{i,1}^{p_2+1} \circ \sigma \circ i_{p_2+1})^*).

In order to prove (13), we are going to show that

(a) + (b) + (d) = (A) and (c) = (B).

• (c) = (B) : The following equality is proved using the change of variables

\psi : \text{Sh}_{p_1+1, p_2+1} \times \{1, \ldots, p_1\} \rightarrow \text{Sh}_{p_1, p_2} \times \{1, \ldots, p_1, p_2\}.

• (a) + (b) + (d) = (A) : To prove this equality, we need to decompose (a) into two sums. Let us denote by \( I^2 \) and \( I^< \) the subsets of \( \text{Sh}_{p_1+1, p_2+1} \times \{1, \ldots, p_1\} \) defined by

\[ I^2 := \{(\sigma, i) \in \text{Sh}_{p_1+1, p_2+1} \times \{1, \ldots, p_1\} \mid \sigma(i) \geq p_2 + i\}, \]

\[ I^< := \{(\sigma, i) \in \text{Sh}_{p_1+1, p_2+1} \times \{1, \ldots, p_1\} \mid \sigma(i) = p_2 + i\}, \]

and

\[ I^< := (\text{Sh}_{p_1+1, p_2+1} \times \{1, \ldots, p_1\}) \setminus I^<. \]

Then we are going to show that

(d) + \sum_{I^<} = 0 and (b) + \sum_{I^>} = (A).

(d) + \sum_{I^<} = 0 : The following equality is proved using the change of variables

\theta : \text{Sh}_{p_1} \times \{1, \ldots, p_2\} \rightarrow I^<.
defined by \( \theta(\sigma, i) = (\rho(\sigma, i), \sigma(p_1 + i) - i + 1) \) with \( \rho(\sigma, i) = \sigma \circ (p_1 + i) \).

(b) \( \sum_{I \geq} \) : To prove this equality, we need to decompose (A) into two sums, one equals to \( \sum_{I \geq} \) and the other equals to (b). Let us denote by \( J^\geq \) and \( J^< \) the subsets of \( \text{Sh}^n_{p_1,p_2} \times \{1,\ldots,p_1\} \) defined by

\[
J^\geq := \{ (\sigma, i) \in \text{Sh}^n_{p_1,p_2} \times \{p_2 + 1,\ldots,p_1 + p_2\} \mid \sigma(i - p_2) \geq i \}
\]

and

\[
J^< := (\text{Sh}^n_{p_1,p_2} \times \{1,\ldots,p_1\}) \setminus J^\geq .
\]

Then we are going to show that

\[
\sum_{I \geq} = \sum_{I <} \quad \text{and} \quad (b) = \sum_{J <} .
\]

\( \sum_{I \geq} = \sum_{I <} : \) The following equality is proved using the change of variables

\[
\psi : I^\geq \to J^\geq .
\]

(b) = \( \sum_{J <} : \) The following equality is proved using the change of variables

\[
\psi \circ t_{p_1} : \text{Sh}^n_{p_1,p_2+1} \times \{1,\ldots,p_2\} \to J^< .
\]

\[ \square \]

**Explicit formulas for \( \succ \) and \( \prec \).** Using the cochain complex isomorphism \( \eta^* \) between \( \{CR^n(X,A), d^n_R\}_{n \in \mathbb{N}} \) and \( \{C^n(N(X),A), d^n\}_{n \in \mathbb{N}} \) (Corollary 3.12), the differential graded dendriform algebra structure on \( \{CR^n(X,A), d^n_R\}_{n \in \mathbb{N}} \) is defined by the formulas:

\[
f_1 \succ f_2 := \sum_{\sigma \in \text{Sh}^n_{p_1,p_2} \times \{p_2 + 1,\ldots,p_1 + p_2\}} \epsilon(\sigma) \mu_\sigma \circ (f_1 \times f_2) \circ (\eta_{p_1} \times \eta_{p_2}) \circ \rho_\sigma \circ \eta^{-1}_{p_1+p_2},
\]

and

\[
f_1 \prec f_2 := \sum_{\sigma \in \text{Sh}^n_{p_1,p_2} \times \{p_2 + 1,\ldots,p_1 + p_2\}} \epsilon(\sigma) \mu_\sigma \circ (f_1 \times f_2) \circ (\eta_{p_1} \times \eta_{p_2}) \circ \rho_\sigma \circ \eta^{-1}_{p_1+p_2}.
\]

Then to have formulas for \( \succ \) and \( \prec \) on \( \{CR^n(X,A), d^n_R\} \) we have to compute

\[
(\eta_{p_1} \times \eta_{p_2}) \circ \rho_\sigma \circ \eta^{-1}_{p_1+p_2} : X^{p_1+p_2} \to X^{p_1} \times X^{p_2}.
\]

Let \((x_1,\ldots,x_{p_1+p_2}) \in X^{p_1+p_2}, \) by [8] and [9]

\[
((\eta_{p_1} \times \eta_{p_2}) \circ \rho_\sigma \circ \eta^{-1}_{p_1+p_2})(x_1,\ldots,x_{p_1+p_2}) = ((y_1,\ldots,y_{p_1}), (z_1,\ldots,z_{p_2})),
\]

where

\[
y_k = (F \circ \sigma \circ i_{p_1})([k - 1] \to [k]) \quad \text{and} \quad z_k = (F \circ \sigma \circ i_{p_2})([k - 1] \to [k]).
\]

27
Thus, using Lemma (3.11)

\[ y_k = (F \circ \sigma)([k - 1] \to [k]), \]
\[ = F(\sigma([k - 1]) \to \sigma([k])), \]
\[ = F(\sigma([k - 1]) \to \sigma([k - 1]) \{\sigma(k)\}), \]
\[ = \prod_{1 \leq x \leq \sigma(\sigma(k))} F([x - 1] \to [x]), \]
\[ = x_{i_1} \triangleright \cdots \triangleright x_{i_j} \triangleright x_{\sigma(k)}, \]

with \( x_{i_1} < \cdots < x_{i_j} < \sigma(k) \) and \( i_i \in \{\sigma(p_1 + 1), \ldots, \sigma(p_1 + p_2)\} \). In the same way

\[ z_k = (F \circ \sigma)([p_1 + k - 1] \to [p_1 + k]), \]
\[ = F(\sigma([p_1 + k - 1]) \to \sigma([p_1 + k])), \]
\[ = F(\sigma([p_1 + k - 1]) \to \sigma([p_1 + k - 1]) \{\sigma(p_1 + k)\}), \]
\[ = \prod_{1 \leq x \leq \sigma(\sigma(p_1 + k))} F([x - 1] \to [x]), \]
\[ = x_{\sigma(p_1 + k)}. \]

Finally we have proved the following theorem.

**Theorem 4.5.** Let \( X \) be a rack and \( A \) be an associative algebra. The cochain complex \( \{CR^n(X, A), d_R^n\}_{n \in \mathbb{N}} \) provided with the products \( \triangleright \) and \( \prec \) defined by

\[
(f_1 \triangleright f_2)(x_1, \ldots, x_{p_1+p_2}) = \sum_{\sigma \in \mathbb{Sh}_{p_1+p_2}^+} \epsilon(\sigma) f_1(y_1, \ldots, y_{p_1}) f_2(z_1, \ldots, z_{p_2}),
\]
\[
(f_1 \prec f_2)(x_1, \ldots, x_{p_1+p_2}) = \sum_{\sigma \in \mathbb{Sh}_{p_1+p_2}^-} \epsilon(\sigma) f_1(y_1, \ldots, y_{p_1}) f_2(z_1, \ldots, z_{p_2}),
\]

where

\[
\begin{cases}
  y_k = x_{i_1} \triangleright \cdots \triangleright x_{i_j} \triangleright x_{\sigma(k)}, \\
  z_k = x_{\sigma(p_1+k)},
\end{cases}
\]

with \( x_{i_1} < \cdots < x_{i_j} < \sigma(k) \) and \( i_i \in \{\sigma(p_1+1), \ldots, \sigma(p_1+p_2)\} \), is a differential graded dendriform algebra. In particular, \( HR^\bullet(X, A) \) is provided with a graded dendriform algebra structure.

**Remark 4.6.** If the rack is pointed, then these formulas are well defined on the subcomplex \( \{CR^n_p(X, A), d_R^n\}_{n \in \mathbb{N}} \). Notice that these formulas hold also in the case of a shelf (resp. pointed shelf).

### 5 A graded associative algebra morphism from \( H^\bullet(G, A) \) to \( HR^\bullet(\text{Conj}(G), A) \)

Let \( G \) be group and \( A \) be an associative algebra over \( \mathbb{Z} \) considered as a trivial \( G \)-module. The cochain complex \( \{C^n(G, A), d_G^n, \cup\} \) calculating the group cohomology with coefficients in \( A \) is provided with a differential graded associative algebra structure given by the cup product \( \cup \).
Moreover, considering the rack $\text{Conj}(G)$ associated to $G$ (Example 2.1), we have shown (Theorem 4.5) that the cochain complex $\{CR^n(\text{Conj}(G), A), d^n_R, \cup\}$, with $\star = \succ + \prec$, is a differential graded associative algebra. In this section, we define a differential graded associative algebra morphism from $\{C^n(G, A), d^n_G, \cup\}$ to $\{CR^n(\text{Conj}(G), A), d^n_R, \star\}$.

**Group cohomology.** Let $G$ be a group and $A$ be a left $G$-module. The cochain complex $\{C^n(G, A), d^n_G\}_{n \in \mathbb{N}}$ is defined by

$$C^n(G, A) := \text{Hom}_{\text{Set}}(G^n, A),$$

$$d^n_G := \sum_{i=0}^{n+1} (-1)^i d^i_{n+1} f,$$

where

$$d^i_{n+1} f(x_1, \ldots, x_{n+1}) := \begin{cases} x_1 \cdot f(x_2, \ldots, x_{n+1}) & \text{if } i = 0, \\ f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) & \text{if } 1 \leq i \leq n, \\ f(x_1, \ldots, x_n) & \text{if } i = n + 1. \end{cases}$$

The family of maps $\{d^n_i\}$ satisfies the simplicial identities, that is, for all $0 \leq i < j \leq n + 1$, we have the identities:

$$d^i_{n+1} \circ d^j_{n} = d^j_{n+1} \circ d^i_{n},$$

so $d_G$ is a differential.

The cohomology associated to this cochain complex is called the *group cohomology of G with coefficients in A* and denoted $H^\bullet(G, A)$.

**Cup product on group cohomology.** Let $G$ be a group and $A$ be an associative algebra with product denoted by $\mu_A$. Let $p_1, p_2 \in \mathbb{N}$, let $\rho$ denote the set map from $G^{p_1 + p_2}$ to $G^{p_1} \times G^{p_2}$ defined by

$$\rho(x_1, \ldots, x_{p_1 + p_2}) := ((x_1, \ldots, x_{p_1}), (x_{p_1 + 1}, \ldots, x_{p_1 + p_2})).$$

A graded product $\cup$, called *cup product*, is defined on $\{C^n(G, A), d^n_G\}_{n \in \mathbb{N}}$ by the formula

$$f_1 \cup f_2 := \mu_A \circ (f_1 \times f_2) \circ \rho.$$

**Theorem 5.1.** Let $G$ be a group and $A$ be an associative algebra (considered as a trivial $G$-module). The cochain complex $\{C^n(G, A), d^n_G\}_{n \in \mathbb{N}}$ provided with the cup product $\cup$ is a differential graded associative algebra structure.

**Equivalent definitions of the group cohomology and the cup product.** Let $G_{\text{Cat}}$ denote the category canonically associated to a group $G$ (Example 3.1). Let $B(G)$ denote its presimplicial nerve of $G_{\text{Cat}}$, that is, the functor from $\Delta^{\text{op}}$ to $\text{Set}$ defined by

$$B(G)(\Delta_n) := \text{Hom}_{\text{Cat}}(\Delta_n, G_{\text{Cat}}),$$

$$B(G)(\partial^n_\lambda) := \text{Hom}_{\text{Cat}}(-, G_{\text{Cat}}) (\partial^n_\lambda) = (\partial^n_\lambda)^*.$$

**Proposition 5.2.** Let $G$ be a group. There is a bijection $B(G)(\Delta_n) \overset{\lambda}{\to} G^n$, and under this bijection we have

$$(\partial^n_\lambda)^*(x_1, \ldots, x_n) = \begin{cases} (x_2, \ldots, x_{n+1}) & \text{if } i = 0, \\ (x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) & \text{if } 1 \leq i \leq n, \\ (x_1, \ldots, x_n) & \text{if } i = n + 1. \end{cases}$$
Corollary 5.3. Let $G$ be a group and $A$ be an abelian group (considered as a trivial left $G$-module). There is an isomorphism of cochain complexes

$$\{C(G,A), d^n_G\}_{n \in \mathbb{N}} \cong \{C^n_G(A), d^n_G\}_{n \in \mathbb{N}}.$$ 

Let $p_1, p_2 \in \mathbb{N}$. We define two functors $j_{p_1} : \Delta_{p_1} \to \Delta_{p_1+p_2}$ and $j_{p_2} : \Delta_{p_2} \to \Delta_{p_1+p_2}$ by $j_{p_1}(k) = k$ and $j_{p_2}(k) = p_1 + k$. Let $\rho$ denote the set map from $B(G)(\Delta_{p_1+p_2})$ to $B(G)(\Delta_{p_1}) \times B(G)(\Delta_{p_2})$ defined by

$$\rho := (j_{p_1}, j_{p_2}).$$

A graded product $\cup$ on $\{C^n_G(A), d^n_G\}_{n \in \mathbb{N}}$ is defined by

$$f_1 \cup f_2 := \mu_A \circ (f_1 \times f_2) \circ \rho.$$

Proposition 5.4. Let $G$ be a group and $A$ be an associative algebra (considered as a trivial $G$-module). The cochain complex $\{C^n(G,A), d^n_G\}_{n \in \mathbb{N}}$ provided with the product $\cup$ is a differential graded associative algebra.

Proposition 5.5. Let $G$ be a group and $A$ be an associative algebra (considered as a trivial $G$-module). The cochain complex isomorphism between $\{C^n(G,A), d^n_G\}_{n \in \mathbb{N}}$ and $\{C^n_G(A), d^n_G\}_{n \in \mathbb{N}}$ given in Corollary 5.3 is a differential graded associative algebra isomorphism.

Cubical cohomology of groups. We have seen in Example 3.1 and 3.2 that $\Box_n$ has a category structure and that $\partial^n_{e_a}$ is a functor. Hence, there is a precubical set $N(C) : \Box^{op} \to \text{Set}$ associated to any category $C$, called the cubical nerve of $C$, and defined by

$$N(C)(\Box_n) := \text{Hom}_{\text{Cat}}(\Box_n, C),$$

$$N(C)(\partial^n_{e_a}) := \text{Hom}_{\text{Cat}}(-, C)(\partial^n_{e_a}) = (\partial^n_{e_a})^*.$$

Let $G$ be a group and $A$ be a left $G$-module. The composition $\text{Hom}_{\text{Set}}(-, A) \circ B(G)$ defines a precubical abelian group. Denote by $\{C^n_G(A), d^n_G\}_{n \in \mathbb{N}}$ the cochain complex associated to this precubical abelian group. The cohomology associated to this cochain complex is called cubical cohomology of $G$ with coefficient in $A$ and denoted $H^n_G(G,A)$.

The morphism $H^\bullet(G,A) \xrightarrow{\mathfrak{S}^\bullet} C^\bullet(\text{Conj}(G), A)$. Define a graded abelian group morphism $S^\bullet$ as the following composition:

$$\begin{array}{c}
C^\bullet(G,A) \xrightarrow{x} \xrightarrow{\mathfrak{S}^\bullet} \xrightarrow{(\eta^*)^{-1}} CR^\bullet(\text{Conj}(G), A) \\
C^\bullet_G(A) \xrightarrow{\mathfrak{S}^\bullet} C^\bullet_G(A) \xrightarrow{\mathfrak{T}^\bullet} C^\bullet(N(\text{Tr}(G_{\text{Cat}})), A) \xrightarrow{\mathfrak{I}^\bullet} C^\bullet(N(\text{Conj}(G)), A)
\end{array}$$

where

- $\mathfrak{I}^n = (\text{Hom}_{\text{Set}}(-, A) \circ \text{Hom}_{\text{Trunk}}(\Box_n, -))(\text{inc})$ with $\text{inc} \in \text{Hom}_{\text{Trunk}}(\text{Conj}(G), \text{Tr}(G_{\text{Cat}}))$ is the inclusion of $\text{Conj}(G)$ into $\text{Tr}(G_{\text{Cat}})$.
• $T^n = \text{Hom}_{\text{Set}}(-, A)(\theta(\Box_n, G_{\text{Cat}}))$ with $\theta(\Box_n, G_{\text{Cat}})$ the bijection induced by the adjunction $\text{Cat} \vdash \text{Tr}$ (cf. [3]) and the equality $\text{Cat}(\Box_n) = \Box_n$ (cf. [4]).

• $\Sigma^n = \sum_{\sigma \in S_n} \epsilon(\sigma) s_\sigma$ with $s_\sigma = (\text{Hom}_{\text{Set}}(-, A) \circ \text{Hom}_{\text{Cat}}(-, G_{\text{Cat}}))(\sigma)$ where $\sigma \in \text{Hom}_{\text{Cat}}(\Delta_n, \Box_n)$ is the functor defined in Example [3.2].

**Theorem 5.6.** Let $G$ be a group and $A$ be an abelian group (considered as a trivial $G$-module). The graded linear map $S^\bullet$ is a chain complex morphism.

**Proof.** The map $I^\bullet$ and $T^\bullet$ are induced by cubical set morphisms, hence they are chain complex morphisms. It remains to prove that $\Sigma^\bullet$ is a chain complex morphism.

We want to prove that $d_{n+1}^\Sigma(\Sigma^n(f)) = \Sigma^{n+1}(d_{\Delta}^{n+1}(f))$. The left hand term of this equation is equal to the sum of two terms (A) and (B) with

\[
\begin{align*}
(A) &= \sum_{\sigma \in S_n} \sum_{i=1}^{n+1} (-1)^{i+1} \epsilon(\sigma) f \circ (\partial_{i,1}^{n+1} \circ \sigma)^*, \\
(B) &= \sum_{\sigma \in S_n} \sum_{i=1}^{n+1} (-1)^i \epsilon(\sigma) f \circ (\partial_{i,0}^{n+1} \circ \sigma)^*.
\end{align*}
\]

The right hand term is equal to sum of three terms (a), (b) and (c) with

\[
\begin{align*}
(a) &= \sum_{\sigma \in S_{n+1}} \epsilon(\sigma) f \circ (\sigma \circ \partial_{0}^{n+1}), \\
(b) &= \sum_{\sigma \in S_{n+1}} (-1)^{n+1} \epsilon(\sigma) f \circ (\sigma \circ \partial_{n+1}^{n+1}), \\
(c) &= \sum_{\sigma \in S_{n+1}} \sum_{i=1}^{n} (-1)^{i} \epsilon(\sigma) f \circ (\sigma \circ \partial_{i}^{n+1}).
\end{align*}
\]

In order to prove the wanted equality we are going to show that

\[
(a) = (A), \quad (b) = (B) \quad \text{and} \quad (c) = 0.
\]

• (a) = (A): This equality is proved using the change of variables

\[
\nu : S_n \times \{1, \ldots, n+1\} \rightarrow S_{n+1}
\]

defined by $\nu(\sigma, i) := (\{i \cdot - 1 \ldots 1\} \circ (1 \ast \sigma)$. Using this change of variables, we have to prove for all $\sigma \in S_n$ and $i \in \{1, \ldots, n+1\}$:

1. $(\partial_{i,1}^{n+1} \circ \sigma)^* = (\nu(\sigma, i) \circ \partial_{0}^{n+1})^*$,

2. $(-1)^{i+1} \epsilon(\sigma) = \epsilon(\nu(\sigma, i))$.

1. Let $\sigma \in S_n$ and $i \in \{1, \ldots, n+1\}$. We have $(\partial_{i,1}^{n+1} \circ \sigma)(k) = \sigma([k])_{<i} \Pi t_{\ast+1}(\sigma([k])_{\geq i}) \Pi \{i\}$ and

\[
(\nu(\sigma, i) \circ \partial_{0}^{n+1})(k) = \nu(\sigma, i)(k+1),
\]

\[
= (\{i \cdot 1 \ldots 1\} \circ (1 \ast \sigma)(1), \ldots, 1 + \sigma(k)),
\]

\[
= [i] \Pi t_{\ast-1}(1 + \sigma([k]))_{<i} \Pi t_{\ast+1}(\sigma([k])_{\geq i}),
\]

\[
= \sigma([k])_{<i} \Pi t_{\ast+1}(\sigma([k])_{\geq i}) \Pi \{i\}.
\]

31
2. Clear.

• (b) = (B) : This equality is proved using the change of variables

\[ \xi : S_n \times \{1, \ldots, n + 1\} \rightarrow S_{n+1} \]

defined by \( \xi(\sigma, i) := (ii + 1 \ldots n + 1) \circ (\sigma \ast 1) \). Using this change of variables, we have to prove for all \( \sigma \in S_n \) and \( i \in \{1, \ldots, n + 1\} \):

1. \( (\partial_{\iota,0}^{n+1} \circ \sigma)^* = (\xi(\sigma, i) \circ \partial_{\iota,1}^{n+1})^* \),

2. \( (-1)^i \epsilon(\sigma) = (-1)^{n+1} \epsilon(\xi(\sigma, i)) \).

1. Let \( \sigma \in S_n \) and \( i \in \{1, \ldots, n + 1\} \). We have \( (\partial_{\iota,0}^{n+1} \circ \sigma)(k) = \sigma([k]) \llcorner \llcorner t_1(\sigma([k]) \geq 1) \) and

\[
(\xi(\sigma, i) \circ \partial_{\iota,1}^{n+1})(k) = \xi(\sigma, i)(k),
\]

\[
= (ii + 1 \ldots n + 1)((\sigma(1), \ldots, \sigma(k)),
\]

\[
= \sigma([k]) \llcorner \llcorner t_1(\sigma([k]) \geq 1).
\]

2. Clear.

• (c) = 0 : This equality is proved using the change of variables

\[ \kappa : \{\sigma \in S_{n+1} | \epsilon(\sigma) = 1\} \times \{1, \ldots, n\} \rightarrow \{\sigma \in S_{n+1} | \epsilon(\sigma) = -1\} \times \{1, \ldots, n\} \]

defined by \( \kappa(\sigma, i) := ((\sigma(i) \sigma(i + 1)) \circ \sigma, i) \). Using this change of variables, we have to prove for all \( \sigma \in S_{n+1}, \epsilon(\sigma) = 1 \) and \( i \in \{1, \ldots, n\} \):

1. \( (\sigma \circ \partial_{\iota,0}^{n+1})^* = (\kappa(\sigma, i) \circ \partial_{\iota,1}^{n+1})^* \),

2. \( (-1)^i \epsilon(\sigma) + (-1)^i \epsilon(\kappa(\sigma, i)) \).

1. Let \( \sigma \in S_{n+1} \) and \( i \in \{1, \ldots, n\} \). We have

\[
(\kappa(\sigma, i) \circ \partial_{\iota,1}^{n+1})(k) = \begin{cases} 
\kappa(\sigma, i)(k) & \text{if } k < i, \\
\kappa(\sigma, i)(k + 1) & \text{if } k \geq i.
\end{cases}
\]

\[
= \begin{cases} 
\sigma([k]) & \text{if } k < i, \\
\sigma([k + 1]) & \text{if } k \geq i.
\end{cases}
\]

\( = (\sigma \circ \partial_{\iota,1}^{n+1})(k) \).

2. Clear.

\[ \square \]

**Explicit formula for \( S^\bullet \).** By definition, for all \( f \in C^n(G, A) \)

\[
S^n(f) = \sum_{\sigma \in S_n} \epsilon(\sigma) f \circ \lambda \circ \sigma^* \circ \theta \circ \text{inc}_* \circ \eta^{-1}.
\]

Then to have an explicit formula for \( S^\bullet \), we have to compute

\[ \lambda \circ \sigma^* \circ \theta \circ \text{inc}_* \circ \eta^{-1} : \text{Conj}(G)^n \rightarrow G^n \]

Let \( (x_1, \ldots, x_n) \in \text{Conj}(G)^n \),

\[ (\lambda \circ \sigma^* \circ \theta \circ \text{inc}_* \circ \eta^{-1})(x_1, \ldots, x_n) = (y_1, \ldots, y_n), \]

32
where \( y_k = (F \circ \sigma)(k-1 \to k) \) for all \( 1 \leq k \leq n \) with \( F = (\eta^{-1})(x_1, \ldots, x_n) \). Thus using Lemma 3.11 we find

\[
y_k = (F \circ \sigma)(k-1 \to k),
\]
\[
= F(\sigma([k-1]) \to \sigma([k])),
\]
\[
= \prod_{1 \leq i \leq \sigma(k)} F([x-1] \to [x]),
\]
\[
= x_1 \triangleright \cdots \triangleright x_i \triangleright \ldots \triangleright x_\sigma(k),
\]
with \( x_i < \cdots < x_j < \sigma(k) \) and \( i \in \{\sigma(k+1), \ldots, \sigma(n)\} \).

**Theorem 5.7.** Let \( G \) be a group and \( A \) be an abelian group (considered as a trivial \( G \)-module). The graded abelian group morphism \( S^\bullet \) from \( C^\bullet(G, A) \) to \( CR^\bullet(\text{Conj}(G), A) \) defined by

\[
S^n(f)(x_1, \ldots, x_n) := \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) f(y_1, \ldots, y_n)
\]

with \( y_k = x_1 \triangleright \cdots \triangleright x_i \triangleright \ldots \triangleright x_\sigma(k) \) where \( x_i < \cdots < x_j < \sigma(k) \) and \( i \in \{\sigma(k+1), \ldots, \sigma(n)\} \), is a cochain complex morphism.

**S\(^\bullet\) is a differential graded associative algebra morphism.** Let \( G \) be a group and \( A \) be an associative algebra (considered as a trivial \( G \)-module). There are two differential graded associative algebras :

- \( \{C^n(G, A), d^n_G, \cup\}_{n \in \mathbb{N}}, \)
- \( \{CR^n(\text{Conj}(G), A), d^n_R, \star = \triangleright + \lt\}_{n \in \mathbb{N}}. \)

and a cochain complex morphism \( \{S^n\}_{n \in \mathbb{N}} \) between these two cochain complexes is defined in \([16]\). The following theorem states that \( \{S^n\}_{n \in \mathbb{N}} \) respects their associative structures.

**Theorem 5.8.** Let \( G \) be a group and \( A \) be an associative algebra (considered as a trivial \( G \)-module). The cochain complex morphism

\[
\{S^n\}_{n \in \mathbb{N}} : \{C^n(G, A), d^n_G\}_{n \in \mathbb{N}} \longrightarrow \{CR^n(\text{Conj}(G), A), d^n_R\}_{n \in \mathbb{N}}
\]

is a differential graded associative algebra morphism. Hence it induces a graded associative algebra morphism \( \{[S^n]\}_{n \in \mathbb{N}} \) from \( \{H^n(G, A), \cup\}_{n \in \mathbb{N}} \) to \( \{HR^n(\text{Conj}(G), A), \star\}_{n \in \mathbb{N}}. \)

**Proof.** By construction \( \lambda^\bullet \) and \( \eta^\bullet \) are graded associative algebra morphisms, hence we have to prove that \( I^\bullet \circ T^\bullet \circ \Sigma^\bullet \) is a graded associative algebra morphism.

Let \( f_1 \in C_{\Delta}^{p_1}(G, A) \) and \( f_2 \in C_{\Delta}^{p_2}(G, A) \), on one hand

\[
(I^{p_1+p_2} \circ T^{p_1+p_2} \circ \Sigma^{p_1+p_2})(f_1 \cup_{\Delta} f_2) := \sum_{\sigma \in \mathfrak{S}_{p_1+p_2}} \epsilon(\sigma) \mu_A \circ (f_1 \times f_2) \circ \rho \circ \sigma^* \circ \theta \circ \text{inc}_*,
\]

and on the other hand

\[
(I^{p_1} \circ T^{p_1} \circ \Sigma^{p_1})(f_1) \star (I^{p_2} \circ T^{p_2} \circ \Sigma^{p_2})(f_2) := \sum_{\gamma \in \text{Sh}_{p_1+p_2}} \sum_{\alpha \in \mathfrak{S}_{p_1}} \sum_{\beta \in \mathfrak{S}_{p_2}} \epsilon(\gamma) \epsilon(\alpha) \epsilon(\beta) \mu_A \circ (f_1 \times f_2) \circ ((\alpha^* \circ \theta \circ \text{inc}_*) \times ((\beta^* \circ \theta \circ \text{inc}_*) \circ \rho_\sigma.
\]

33
To show the equality of these two sums we are going to use the change of variables

$$\varphi : Sh_{p_1,p_2} \times S_{p_1} \times S_{p_2} \rightarrow S_{p_1+p_2}$$

defined by $$\varphi(\gamma, \alpha, \beta) = \gamma \circ (\alpha * \beta)$$. Using this change of variables we have to prove for all $$\gamma \in Sh_{p_1,p_2}, \alpha \in S_{p_1}$$ and $$\beta \in S_{p_2}$$:

1. $$\rho \circ \sigma^* \circ \theta \circ \inc_*=((\alpha^* \circ \theta \circ \inc_*) \times (\beta^* \circ \theta \circ \inc_*)) \circ \rho_{\sigma},$$

2. $$\epsilon(\gamma)\epsilon(\alpha)\epsilon(\beta) = \epsilon(\gamma \circ (\alpha * \beta)).$$

1. Let $$\gamma \in Sh_{p_1,p_2}, \alpha \in S_{p_1}$$ and $$\beta \in S_{p_2}$$. We have to prove for all $$F \in N(\mathrm{Conj}(G))(\square_{p_1+p_2})$$ the following equalities:

$$\left\{ \begin{array}{ll}
(F \circ \sigma \circ (\alpha * \beta) \circ j_{p_1})(k-1 \rightarrow k) = (F \circ \sigma \circ i_{p_1} \circ \alpha)(k-1 \rightarrow k) & \text{if } 1 \leq k \leq p_1, \\
(F \circ \sigma \circ (\alpha * \beta) \circ j_{p_2})(k-1 \rightarrow k) = (F \circ \sigma \circ i_{p_2} \circ \beta)(k-1 \rightarrow k) & \text{if } 1 \leq k \leq p_2.
\end{array} \right.$$  

Let $$1 \leq k \leq p_1$$,

$$(\sigma \circ (\alpha * \beta) \circ j_{p_1})(k) = (\sigma \circ (\alpha * \beta))(k) = (\sigma \circ (\alpha * \beta))(\lfloor k \rfloor) = \sigma(\alpha(\lfloor k \rfloor))$$

and

$$(\sigma \circ i_{p_1} \circ \alpha)(k) = (\sigma \circ i_{p_1})(\alpha(\lfloor k \rfloor)) = \sigma(\alpha(\lfloor k \rfloor)).$$

Let $$1 \leq k \leq p_2$$,

$$(\sigma \circ (\alpha * \beta) \circ j_{p_2})(k) = (\sigma \circ (\alpha * \beta))(p_1 + k) = (\sigma \circ (\alpha * \beta))(\lfloor p_1 + k \rfloor) = \sigma(t_{p_1} \beta(\lfloor k \rfloor))$$

and

$$(\sigma \circ i_{p_2} \circ \beta)(k) = (\sigma \circ i_{p_2})(\beta(\lfloor k \rfloor)) = \sigma(t_{p_2} \beta(\lfloor k \rfloor)).$$

2. Clear. \hfill \Box

References

[Cov10] S. Covez. The local integration of Leibniz algebras. ArXiv e-prints, November 2010. http://arxiv.org/abs/1011.4112.

[CS03] J. Scott Carter and Masahico Saito. Quandle homology theory and cocycle knot invariants. In Topology and geometry of manifolds (Athens, GA, 2001), volume 71 of Proc. Sympos. Pure Math., pages 249–268. Amer. Math. Soc., Providence, RI, 2003.

[Cuv94] C. Cuvier. Algèbres de Leibnitz: définitions, propriétés. Ann. Sci. École Norm. Sup. (4), 27(1):1–45, 1994.

[EG03] P. Etingof and M. Graña. On rack cohomology. J. Pure Appl. Algebra, 177(1):49–59, 2003.

[FR92] Roger Fenn and Colin Rourke. Racks and links in codimension two. J. Knot Theory Ramifications, 1(4):343–406, 1992.

[FRS95] Roger Fenn, Colin Rourke, and Brian Sanderson. Trunks and classifying spaces. Appl. Categ. Structures, 3(4):321–356, 1995.
[FT87] B. L. Feigin and B. L. Tsygan. Additive $K$-theory. In $K$-theory, arithmetic and geometry (Moscow, 1984–1986), volume 1289 of Lecture Notes in Math., pages 67–209. Springer, Berlin, 1987.

[Kin07] Michael K. Kinyon. Leibniz algebras, Lie racks, and digroups. J. Lie Theory, 17(1):99–114, 2007.

[LFCG01] J.-L. Loday, A. Frabetti, F. Chapoton, and F. Goichot. Dialgebras and related operads, volume 1763 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2001.

[Lod93] Jean-Louis Loday. Une version non commutative des algèbres de Lie: les algèbres de Leibniz. In R.C.P. 25, Vol. 44 (French) (Strasbourg, 1992), volume 1993/41 of Prépubl. Inst. Rech. Math. Av., pages 127–151. Univ. Louis Pasteur, Strasbourg, 1993.

[Lod95] Jean-Louis Loday. Cup-product for Leibniz cohomology and dual Leibniz algebras. Math. Scand., 77(2):189–196, 1995.

[Lod97] Jean-Louis Loday. Overview on Leibniz algebras, dialgebras and their homology. In Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), volume 17 of Fields Inst. Commun., pages 91–102. Amer. Math. Soc., Providence, RI, 1997.

[Lod98] Jean-Louis Loday. Cyclic homology, volume 301 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 1998. Appendix E by Maria O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[Lod03] Jean-Louis Loday. Algebraic $K$-theory and the conjectural Leibniz $K$-theory. $K$-Theory, 30(2):105–127, 2003. Special issue in honor of Hyman Bass on his seventieth birthday. Part II.

[Lod12] Jean-Louis Loday. Some problems in operad theory. In Operads and universal algebras (Tianjin, China, July 2010), volume 9 of Proc. Int. Conf. in Nankai Series in Pure, Applied Mathematics and Theoretical Physics, pages 139–146. World Scientific, 2012.

[LQ84] Jean-Louis Loday and Daniel Quillen. Cyclic homology and the Lie algebra homology of matrices. Comment. Math. Helv., 59(4):569–591, 1984.

[ML98] Saunders Mac Lane. Categories for the working mathematician, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1998.

[PS11] J. H. Przytycki and A. S. Sikora. Distributive Products and Their Homology. ArXiv e-prints, may 2011. http://arxiv.org/abs/1105.3700.