THE INTEGRATED PERIODOGRAM OF A DEPENDENT EXTREMAL EVENT SEQUENCE

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ABSTRACT. We investigate the asymptotic properties of the integrated periodogram calculated from a sequence of indicator functions of dependent extremal events. An event in Euclidean space is extreme if it occurs far away from the origin. We use a regular variation condition on the underlying stationary sequence to make these notions precise. Our main result is a functional central limit theorem for the integrated periodogram of the indicator functions of dependent extremal events. The limiting process is a continuous Gaussian process whose covariance structure is in general unfamiliar, but in the iid case a Brownian bridge appears. In the general case, we propose a stationary bootstrap procedure for approximating the distribution of the limiting process. The developed theory can be used to construct classical goodness-of-fit tests such as the Grenander-Rosenblatt and Cramér-von Mises tests which are based only on the extremes in the sample. We apply the test statistics to simulated and real-life data.

1. Introduction

1.1. Regularly varying sequences. We consider a strictly stationary $\mathbb{R}^d$-valued sequence $(X_t)$ for some $d \geq 1$ with a generic element $X$ and assume that its finite-dimensional distributions are regularly varying. This means that for every $h \geq 1$, there exists a non-null Radon measure $\mu_h$ on the Borel $\sigma$-field $\mathcal{B}^{dh} = \mathbb{R}^{dh} \setminus \{0\}$, $\mathbb{R} = (-\infty, \infty)$, such that

$$P(x^{-1}(X_1, \ldots, X_h) \in \cdot) \xrightarrow{v} \mu_h(\cdot),$$

where $\xrightarrow{v}$ denotes vague convergence in $\mathcal{B}^{dh}$; cf. Resnick [25, 26], Kallenberg [20]. The limiting measure $\mu_h$ necessarily has the property $\mu_h(t \cdot) = t^{-\alpha} \mu_h(\cdot)$, $t > 0$, for some $\alpha \geq 0$, the index of regular variation. In what follows, we assume that $\alpha > 0$. Relation (1.1) is equivalent to the sequential definition

$$nP(x^{-n}(X_1, \ldots, X_h) \in \cdot) \xrightarrow{v} \mu_h(\cdot), \quad n \to \infty,$$

where $(a_n)$ is chosen such that $P(|X| > a_n) \sim n^{-1}$ as $n \to \infty$. We will say that the sequence $(X_t)$ and any of the vectors $(X_1, \ldots, X_h)$, $h \geq 1$, are regularly varying with index $\alpha$.

Examples of regularly varying strictly stationary sequences are linear and stochastic volatility processes with iid regularly varying noise, GARCH processes, infinite variance stable processes and max-stable processes with Fréchet marginals. These examples are discussed e.g. in Davis et al. [7] [10], Mikosch and Zhao [21].

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1.2. The extremogram. Consider a $\mu_1$-continuity Borel set $D_0 = A \subset \mathbb{R}^d$ bounded away from zero and such that $\mu_1(A) > 0$. Then the sets $D_h = A \times \mathbb{R}^{d(h-1)} \times A$ are bounded away from zero as well and are continuity sets with respect to the corresponding limiting measures $\mu_{h+1}$, $h \geq 1$. We conclude from (1.2) that the limits
\begin{equation}
\gamma_A(h) = \lim_{n \to \infty} n P(a_n^{-1} X_0 \in A, a_n^{-1} X_h \in A) = \mu_{h+1}(D_h), \quad h \geq 0,
\end{equation}
exist. For $t \in \mathbb{Z}$, it is not difficult to see that
\[ n \text{cov}(I_{\{a_n^{-1} X_t \in A\}}, I_{\{a_n^{-1} X_{t+h} \in A\}}) \sim n E I_{\{a_n^{-1} X_t \in A, a_n^{-1} X_{t+h} \in A\}} = n P(a_n^{-1} X_0 \in A, a_n^{-1} X_h \in A) \to \gamma_A(h), \quad n \to \infty. \]
Hence $\gamma_A$ constitutes the covariance function of a stationary process. We refer to $\gamma_A$ as the extremogram relative to the set $A$. We will also consider the standardized extremogram given as the limiting sequence
\[ \rho_A(h) = \lim_{n \to \infty} P(a_n^{-1} X_h \in A \mid a_n^{-1} X_0 \in A) = \frac{\mu_{h+1}(D_h)}{\mu_1(D_0)}, \quad h \geq 0. \]
The quantities $\rho_A(h)$ have an intuitive interpretation as limiting conditional probabilities. Moreover, $\rho_A$ is the autocorrelation function of a stationary process. The quantities $\rho_A(h)$ are generalizations of the upper tail dependence coefficient of a two-dimensional vector $(Y_1, Y_2)$ with identical marginals given as the limit $\lim_{x \to \infty} P(Y_2 > x \mid Y_1 > x)$.

The extremogram was introduced in Davis and Mikosch [7] as a measure of serial extremal dependence in a strictly stationary sequence. There and in Davis et al. [10][11] various aspects of the estimation of the extremogram were discussed, including asymptotic theory and the use of the stationary bootstrap for the construction of confidence bands.

1.3. The sample extremogram. Natural estimators of the extremograms $\gamma_A$ and $\rho_A$ are given by their respective sample analogs
\[ \tilde{\gamma}_A(h) = \frac{m}{n} \sum_{t=1}^{n-h} \tilde{I}_t \tilde{I}_{t+h} \quad \text{and} \quad \tilde{\rho}_A(h) = \frac{\tilde{\gamma}_A(h)}{\gamma_A(0)} , \quad h \geq 0. \]
Here $m = m_n$ is any integer sequence satisfying the conditions $m_n \to \infty$ and $m_n/n = o(1)$ and
\[ I_t = I_{\{a_n^{-1} X_t \in A\}} , \quad \tilde{I}_t = I_t - p_0 , \quad \text{and} \quad p_0 = E I_t = P(a_n^{-1} X \in A), t \in \mathbb{Z}. \]
It is shown in Davis and Mikosch [7] that the conditions $m_n \to \infty$ and $m_n/n = o(1)$ are needed for the validity of the asymptotic properties $E \tilde{\gamma}_A(h) \to \gamma_A(h)$ and $\text{var}(\tilde{\gamma}_A(h)) \to 0$ as $n \to \infty$. Moreover, under a mixing condition, the finite-dimensional distributions of $\tilde{\gamma}_A$ and $\tilde{\rho}_A$ satisfy a central limit theorem with normalization $(n/m)^{1/2}$; cf. Lemma 11 below.

1.4. Spectral density and periodogram. Since $\gamma_A$ and $\rho_A$ are the autocovariance and autocorrelation functions of a stationary process, respectively, it is possible to enter the corresponding frequency domain. If $\gamma_A$ is square summable one can define the spectral densities
\[ h_A(\lambda) = \sum_{h \in \mathbb{Z}} \gamma_A(h) e^{-i h \lambda} \quad \text{and} \quad f_A(\lambda) = \sum_{h \in \mathbb{Z}} \rho_A(h) e^{-i h \lambda}, \quad \lambda \in [0, \pi] = \Pi. \]
A natural estimator of the spectral density is the periodogram. Since the sample autocovariances $\tilde{\gamma}_A(h)$ are derived from the triangular array of the stationary sequences $(\tilde{I}_t)$, an analog of the classical
periodogram for $h_A$ is given by

$$I_{n,A}(\lambda) = \frac{m}{n} \left| \sum_{t=1}^{n} \tilde{I}_t e^{-i t \lambda} \right|^2 = \tilde{\gamma}(0) + 2 \sum_{h=1}^{n-1} \tilde{\gamma}(h) \cos(h \lambda), \quad \lambda \in \Pi,$$

and the periodogram for the standardized spectral density $f_A$ is obtained as the scaled periodogram $I_{n,A}/\tilde{\gamma}(0)$. Mikosch and Zhao [21] showed under mixing conditions that the extremal periodogram ordnates $I_{n,A}(\lambda)$ share various of the classical properties of the periodogram ordnates for a stationary sequence (cf. Brockwell and Davis [3]): consistency in the mean, convergence in distribution to independent exponential random variables with expectation $h_A(\lambda)$ at distinct fixed frequencies $\lambda_j \in (0, \pi)$ and at distinct Fourier frequencies $\omega_n(j) = 2\pi j/n \in (0, \pi)$ provided these frequencies converge to a limit $\lambda_j \in (0, \pi)$ as $n \to \infty$. The latter property ensures that weighted versions of the periodogram $I_{n,A}$ at fixed frequencies $\lambda \in (0, \pi)$ converge in mean square to $h_A(\lambda)$.

For practical purposes, one will mostly work with the periodogram at the Fourier frequencies $\omega_n(j) \in (0, \pi)$. Then

$$I_{n,A}(\omega_n(j)) = \frac{m}{n} \sum_{t=1}^{n} \tilde{I}_t e^{-i t \omega_n(j)} \left| \right|^2,$$

i.e., centering of the indicator functions $I_t$ is not needed. However, for proving asymptotic theory it will be convenient to work with the extremal periodogram $I_{n,A}$ based on the centered quantities $\tilde{I}_t$, $t = 1, \ldots, n$.

1.5. The integrated periodogram. The integrated periodogram of a stationary sequence has a long history in time series analysis, starting with classical work of Grenander and Rosenblatt [14], and was extensively used in the monographs Hannan [15], Priestley [24], Brockwell and Davis [3], to name a few references. Dahlhaus [4] discovered a close relationship of the integrated periodogram, considered as a process indexed by functions, and empirical process theory. Under entropy conditions, he proved uniform convergence results over suitable classes of index functions; see also the survey paper Dahlhaus and Polonik [5]. These papers gave some general theoretical background for various periodogram based techniques such as Whittle estimation of the parameters of a FARIMA process and goodness of fit tests for linear processes as mentioned in Grenander and Rosenblatt [14] and Priestley [24].

In this paper, we will consider the integrated periodogram

$$J_{n,A}(g) = \int_{\Pi} I_{n,A}(\lambda) g(\lambda) d\lambda = c_0(g) \tilde{\gamma}(0) + 2 \sum_{h=1}^{n-1} c_h(g) \tilde{\gamma}(h),$$

and its standardized version

$$J_{n,A}^o(g) = \frac{1}{\tilde{\gamma}(0)} \int_{\Pi} I_{n,A}(\lambda) g(\lambda) d\lambda = c_0(g) + 2 \sum_{h=1}^{n-1} c_h(g) \tilde{\rho}(h),$$

where $g$ is non-negative and square integrable with respect to Lebesgue measure on $\Pi$ (we write $g \in L^2(\Pi)$) with corresponding Fourier coefficients

$$c_h(g) = \int_{\Pi} \cos(h \lambda) g(\lambda) d\lambda, \quad h \in \mathbb{Z}.$$
We will understand $J_{n,A}(g)$ and $J_{n,A}^\circ(g)$ as natural estimators of

\begin{equation}
J_A(g) = \int_\Pi h_A(\lambda) g(\lambda) d\lambda = c_0(g) \gamma_A(0) + 2 \sum_{h=1}^\infty c_h(g) \gamma_A(h),
\end{equation}

respectively. The latter identities hold if $\sum_{h=0}^\infty \gamma_A(h) < \infty$, a condition we assume throughout this paper; see also Remark 5 below.

The main results of this paper (see Section 3) are functional central limit theorems for the integrated periodogram $J_{n,A}$ with $g = h I_{[0,1]}$ for a sufficiently smooth function $h$ on $\Pi$. The limit processes are Gaussian whose covariance structure strongly depends on the limit measures ($\mu_h$). The rate of convergence in these results is typically slower than $\sqrt{n}$. However, in the case of an iid sequence, the limiting process is a Brownian bridge and the convergence rates are much faster than in the case of a dependent sequence. These results differ from classical theory for the periodogram of a stationary sequence $(X_t)$ (see e.g. Dahlhaus [4], Klüppelberg and Mikosch [19]), where the limiting process is completely determined by the covariance structure of $(X_t)$. The methods of proof combine classical techniques of weak convergence and strong mixing (e.g. Billingsley [1]) with extreme value theory for dependent sequences (e.g. Davis and Mikosch [7]). The proofs are rather technical due to the fact that the sequences of indicator functions $(I_t)$ have triangular structure:

As in classical time series analysis, the functional central limit theory for the integrated periodogram can be used to construct asymptotic goodness-of-fit tests such as the Grenander-Rosenblatt and Cramér-von Mises tests. In contrast to their classical counterparts, these tests are based only on the extremal part of the underlying sample, i.e., we test whether the extremes of the sample are in agreement with the null hypothesis about a given type of time series model. Such tests may be useful, for example, for distinguishing between a GARCH and a stochastic volatility model fitted to a return time series. The aforementioned two types of models may have similar autocorrelation structure for the data, their absolute values and squares, so their spectral properties are very similar as well, while their extremograms are rather distinct: the extremogram $\gamma_A$ relative to the set $A = (1, \infty)$ decays exponentially fast for GARCH and for the simple stochastic volatility model $\gamma_A$ vanishes at all positive lags; see Davis and Mikosch [7].

The paper is organized as follows. We start in Section 2 with some moment calculations and we also introduce the relevant mixing conditions and central limit theory for the sample extremogram. In Section 2.4 we provide a result about the mean square consistency of the integrated periodogram; the proof is given in Section 3. The main results (Theorems 15 and 17) are functional central limit theorems for the integrated periodogram. They are given in Section 3; the corresponding proofs are provided in Sections 6 and 7. The covariance structure of the limiting Gaussian processes in Theorem 15 is rather complicated. Therefore in Section 4 we supplement the asymptotic theory by consistency results for the stationary bootstrap applied to the integrated periodogram of extremal events in a strictly stationary sequence. The corresponding proofs are given in Section 8. In Section 4.4 we indicate how the integrated periodogram works for simulated and real-life data.

2. Preliminaries

2.1. Some moment calculations. Recall the notation and conditions of Section 1. We write

$$p_0 = P(a_m^{-1} X_0 \in A) \quad \text{and} \quad p_h = P(a_m^{-1} X_0, a_m^{-1} X_h \in A), \quad h \geq 1,$$

where
where as above, $m_n \to \infty$ and $m_n/n = o(1)$ as $n \to \infty$. For integers $s, t, u, v \geq 0$, we set

$$
\Gamma(s, t, u, v) = E\overline{I_s}I_t\overline{I_u}I_v,
$$

$$
\Gamma(s, t, u) = E\overline{I_s}I_t\overline{I_u},
$$

$$
\Gamma(s, t) = E\overline{I_s}I_t = p_{|s-t|} - p_0^2.
$$

We will often have to calculate variances and covariances of the sample extremogram $\tilde{\gamma}_A$. We provide some of these formulas for further use.

**Lemma 1.** Let $(X_t)$ be a strictly stationary sequence. Then, for $1 \leq h \leq n - 1$,

$$(n/m)^2 E\tilde{\gamma}_A^2(h) = (n-h)E(\overline{I_0}\overline{I_h})^2 + 2 \sum_{t=1}^{n-h-1} (n-h-t)\Gamma(0,h,t,t+h)$$

and for $1 \leq h < h + u \leq n - 1$,

$$(n/m)^2 E\tilde{\gamma}_A(h+u)
= (n-h-u)\Gamma(0,h,0,h+u)
+ \sum_{t=1}^{n-h-u-1} (n-h-u-t)\Gamma(0,h,t+t+h+u)
+ \sum_{t=1}^{n-h-1} \min(n-h-u,n-h-t)\Gamma(0,h+u,t,t+h).$$

### 2.2. Mixing conditions.

The following two mixing conditions were introduced in Davis and Mikosch [7] for a strongly mixing $\mathbb{R}^d$-valued sequence $(X_t)$ with rate function $(\xi_h)$.

**Condition (M).** There exist integer sequences $m = m_n \to \infty$ and $r_n \to \infty$ such that $m_n/n \to 0$, $r_n/m_n \to 0$ and

$$
\lim_{n \to \infty} m_n \sum_{h=r_n} \xi_h = 0.
$$

Moreover, an anti-clustering condition holds:

$$
\lim_{n \to \infty} \limsup_{k \to \infty} \sum_{h=k}^{r_n} P(|X_h| > \epsilon a_m \mid |X_0| > \epsilon a_m) = 0, \quad \epsilon > 0.
$$

**Condition (M1).** Assume (M) and that the sequences $(m_n)$, $(r_n)$, $k_n = [n/m_n]$ from (M) also satisfy the growth conditions $k_n \xi_{r_n} \to 0$, and $m_n = o(n^{1/3})$.

**Remark 2.** The condition $m_n = o(n^{1/3})$ in (M1) can be replaced by $\frac{m_n r_n^3}{n} \to 0$ and $\frac{m_n^4}{n} \sum_{j=r_n}^{m_n} \xi_j \to 0$ which is often much weaker.

Condition (2.1) is easily satisfied if the mixing rate $(\xi_h)$ is geometric, i.e., exponentially decaying to zero. Under mild conditions, the popular classes of ARMA, max-stable, GARCH and stochastic volatility processes are strongly mixing with geometric rate; cf. Davis et al. [7] [10] [11] [21] for discussions of these examples. Condition (2.2) is similar to (2.8) in Davis and Hsing [6]. It serves the purpose of establishing the convergence of a sequence of point processes to a limiting cluster point process. This condition is much weaker than the anti-clustering condition $D'(ea_n)$ of Leadbetter; cf. Section 5.3.2 in Embrechts et al. [15].

The mixing rate $(\xi_h)$ in conditions (M) and (M1) is useful for finding bounds on the moments $\Gamma(s, t, u, v)$ introduced above. In what follows, $c$ will denote any (possibly different) constants whose value is not of interest.
Lemma 3. Let \((X_i)\) be a strongly mixing sequence with mixing rate \((\xi_h)\). Then for integers \(h, l, u \geq 1\) and for some constants \(c > 0\) which do not depend on \(n\),

\[
\begin{align*}
&\Gamma(0, h, h + l, h + l + u) \leq c \min(\xi_h, \xi_u), \\
&\Gamma(0, h, h + l, h + l + u) - (p_h - p_0^2)(p_u - p_0^2) \leq c \xi_l, \\
&\Gamma(0, h, h + l) \leq c \min(\xi_h, \xi_l), \\
&\Gamma(0, h) \leq \xi_h.
\end{align*}
\]

The proof of Lemma 3 follows by a direct application of Theorem 17.2.1 in Ibragimov and Lin-

2.3. Central limit theory for the sample extremogram. In this section we recall a central

Lemma 4. Assume that \((X_i)\) is an \(\mathbb{R}^d\)-valued strictly stationary regularly varying sequence with

\[
\begin{align*}
&\gamma_A(i) \overset{P}{\to} \gamma_A(h), \\
&(n/m)^{1/2}(\gamma_A(i) - E\gamma_A(i)) \underset{i=0, \ldots, h}{\overset{d}{\to}} (Z_i)_{i=0, \ldots, h},
\end{align*}
\]

where \((Z_i)_{i=0, \ldots, h}\) is Gaussian with mean zero and covariance matrix \(\Sigma_h = (\sigma_{ij})_{i,j=0, \ldots, h}\) given by

\[
\sigma_{ij} = \gamma_A(i, j) + \sum_{l=1}^{\infty} [\gamma_A(i, l, l + j) + \gamma_A(j, l, l + i)], \quad i, j = 0, \ldots, h,
\]

and for \(u, s, t \geq 0\),

\[
\gamma_A(u, s, t) = \lim_{n \to \infty} n P(a_n^{-1}X_0 \in A, a_n^{-1}X_u \in A, a_n^{-1}X_s \in A, a_n^{-1}X_t \in A),
\]

with the convention that \(\gamma_A(u, t) = \gamma_A(u, u, t)\). Moreover, we have for \(h \geq 1\)

\[
\begin{align*}
&\tilde{\rho}_A(h) \overset{P}{\to} \rho_A(h), \\
&(n/m)^{1/2}(\tilde{\rho}_A(i) - p_i^A/p_0^A) \underset{i=1, \ldots, h}{\overset{d}{\to}} \frac{1}{\gamma_A(0)}(Z_i - \rho_A(i)Z_0)_{i=1, \ldots, h}.
\end{align*}
\]

Proof. The proof of (2.7) was given in Section 3 of Davis and Mikosch [7]. There we can also find the

Proof. The proof of (2.3) in a more general context. Here we will calculate the covariance matrix \(\Sigma_h\) explicitly.

The expressions for \(\sigma_{ii}, i \geq 0\), were derived in Davis and Mikosch [7] for \(i = 0\) and \(i \geq 1\) in Theorem

3.1 and Lemma 5.2, respectively. We notice that \(\gamma_A(i, l, l + j) \leq \gamma_A(l)\) and therefore the infinite series in \(\sigma_{ij}\) are finite.
The summability condition on $\gamma_A$ also exist for infinite variance stable and max-stable processes with Fréchet specification of the process.

For $i \neq j$, similar calculations as for Lemma 1 yield for $k \geq 1$ and $r_n/m_n \to 0$,

$$
\frac{m}{n} \text{cov}\left(\sum_{t=1}^{n} \tilde{I}_t \tilde{I}_{t+i}, \sum_{s=1}^{n} \tilde{I}_s \tilde{I}_{s+j}\right)
= m \Gamma(0,0,i,j) + m \sum_{t=1}^{n} \left[(1 - l/n)\Gamma(0,i,l,l) + \Gamma(0,j,l,l+i) - (p_i - p_0^2)(p_j - p_0^2)\right]
= m \Gamma(0,0,i,j) + m \sum_{t=k+1}^{n} \left[(1 - l/n)\Gamma(0,i,l,l) + \Gamma(0,j,l,l+i) - (p_i - p_0^2)(p_j - p_0^2)\right]
= Q_1 + Q_2 + Q_3 + Q_4.
$$

By regular variation, for fixed $k \geq 1$ as $n \to \infty$,

$$
Q_1 + Q_2 \to \gamma_A(i,j) + \sum_{i=1}^{k} \left[\gamma_A(i,l,l) + \gamma_A(j,l,l+i)\right],
$$

and the right-hand side converges to $\sigma_{i,j}$ as $k \to \infty$. By (2.2), we have

$$
\lim_{k \to \infty} \limsup_{n \to \infty} |Q_3| = 0.
$$

Using (2.4) and (2.1), we also have $|Q_4| \leq cn_n \sum_{l=r_n+1}^{\infty} \xi_l \to 0$ as $n \to \infty$. This proves (2.7) and (2.8). Relations (2.9) and (2.10) follow by a continuous mapping argument, observing that for $1 \leq i \leq h$,

$$
\left(\frac{n}{m}\right)^{1/2} \left(\rho_A(i) - p_i/p_0\right) = \left(\frac{n}{m}\right)^{1/2} \frac{\overline{\gamma}_A(i)}{\overline{\gamma}_A(0)} - \frac{E\overline{\gamma}_A(i)}{\overline{\gamma}_A(0)}\frac{(n/m)^{1/2}(\overline{\gamma}_A(0) - E\overline{\gamma}_A(0))}{\overline{\gamma}_A(0)E\overline{\gamma}_A(0)} + o_P(1)
\xrightarrow{d} \frac{1}{\overline{\gamma}_A(0)} \left(Z_i - \rho_A(i)Z_0\right).
$$

Remark 5. The summability condition on $\gamma_A$ which we assume in the previous lemma and throughout this paper is satisfied for a large variety of regularly varying time series models; see the calculation of $\gamma_A$ in Davis et al. [7, 10, 11]. For example, finite order ARMA models with regularly varying iid noise and GARCH models have exponentially decaying extremogram, and the simple stochastic volatility model with log-normal volatility process has vanishing extremogram at all positive lags. Formulas for $\gamma_A$ also exist for infinite variance stable and max-stable processes with Fréchet marginals. Also for these processes the summability condition on $\gamma_A$ may hold, depending on the specification of the process.

Recall that a strictly stationary process $(X_t)$ is $\eta$-dependent for some integer $\eta \geq 0$ if $(X_t)_{t<0}$ and $(X_t)_{t>\eta}$ are independent. For such a process we observe that $\sigma_{hh} = 0$ for $h > \eta$ and hence (2.5) collapses into $(n/m)^{0.5}\gamma_A(h) \xrightarrow{P} 0$ for $h > \eta$. In particular, for an iid sequence $(X_t)$, $Z_h = 0$ a.s. for $h \geq 1$, while $(n/m)^{0.5}\gamma_A(0) \xrightarrow{d} Z_0$ and $Z_0$ is $N(0, \gamma_A(0))$ distributed.

In these cases, the rate of convergence in (2.8) can be improved.
Lemma 6. Assume that \((X_t)\) is an \(\mathbb{R}^d\)-valued \(\eta\)-dependent regularly varying strictly stationary sequence with index \(\alpha > 0\) for some \(\eta \geq 0\), and the Borel set \(A\) satisfies the conditions of Section 1.2. Additionally, assume that for \(j \geq i > \eta\) and \(1 \leq t \leq \eta - (j - i)\), the following limits exist:

\[
\gamma_A(t, i, t + j) = \lim_{m \to \infty} m^2 P(a_m^{-1}X_0 \in A, a_m^{-1}X_i \in A, a_m^{-1}X_t \in A, a_m^{-1}X_{t+j} \in A).
\]

(2.11)

Then for \(h \geq 1\), \(n^{0.5} \gamma_A(\eta, i + \eta))_{i=1,\ldots,h} \overset{d}{\to} (Z_i)_{i=1,\ldots,h}\), where \((Z_i)_{i=1,\ldots,h}\) is Gaussian \(N(0, \Sigma_h)\) whose covariance matrix \(\Sigma_h\) is given by

\[
\sigma_{ij} = \gamma_A(0) \gamma_A(j - i) + \sum_{t=1}^{\eta-j-i} [\gamma_A(t, i, t + j) + \gamma_A(t, j, t + i)],
\]

\(1 \leq i \leq j\).

Remark 7. Condition (2.11) is an additional asymptotic independence condition. Indeed, regular variation of \((X_t)\) only implies that the limits

\[
\lim_{m \to \infty} mP(a_m^{-1}X_0 \in A, a_m^{-1}X_i \in A, a_m^{-1}X_t \in A, a_m^{-1}X_{t+j} \in A)
\]

exist and are finite. Then (2.11) implies that the latter limits must be zero. In Example 8 we consider some simple cases when (2.11) is satisfied.

Remark 8. Assume \(j - i > \eta\). Then, by \(\eta\)-dependence, \(\gamma_A(j - i) = 0\) and the index set in (2.12) is empty. Hence \(\sigma_{ij} = 0\) for \(j - i > \eta\). In particular, if \((X_t)\) is iid, \(\sigma_{ij} = 0\) for \(i \neq j\) and \(\sigma_{ii} = \gamma_A^2(0)\).

Proof. We start by calculating the asymptotic covariances. Assume \(j \geq i > \eta\). Then, using the independence of \(I_0\) and \((I_t, I_t I_{t+j}, I_t I_{t+i})\) for \(t > \eta\) and of \(I_{t+j}\) and \(I_0 I_t I_i\) for \(t \leq \eta\) and \(t \geq \eta - (j - i)\), we obtain

\[
\text{cov}(n^{0.5} \gamma_A(i), n^{0.5} \gamma_A(j)) = mn^2 \sum_{t=1}^{\eta} [E \bar{I}_0 \bar{I}_t \bar{I}_j + E \bar{I}_0 \bar{I}_j \bar{I}_{t+i}] + o(1)
\]

\[
= \gamma_A(0) \gamma_A(j - i) + mn^2 \sum_{t=1}^{\eta-j-i} [E \bar{I}_0 \bar{I}_t \bar{I}_j \bar{I}_{t+i}] + o(1)
\]

\[
\to \gamma_A(0) \gamma_A(j - i) + \sum_{t=1}^{\eta-j-i} [\gamma_A(t, i, t + j) + \gamma_A(t, j, t + i)], \quad n \to \infty.
\]

In the last step we used (2.11). This completes the calculation of \(\Sigma_h\). Furthermore, we observe that for \(h \geq 1\),

\[
n^{0.5}(\gamma_A(i))_{i=\eta+1,\ldots,\eta+h} = (m/n^{0.5}) \sum_{t=1}^{\eta} (\bar{I}_t \bar{I}_{t+i})_{i=\eta+1,\ldots,\eta+h} + oP(1).
\]

(2.13)

The vector sequence \((\bar{I}_t \bar{I}_{t+i})_{i=\eta+1,\ldots,\eta+h}, t = 1, 2, \ldots\) is strictly stationary and \((h + \eta)\)-dependent. Now an application of the central limit theorem for strongly mixing triangular arrays in Rio [27] and the Cramér-Wold device to (2.13) conclude the proof.

The following examples fulfill the conditions of Lemma 6.
Example 9. An iid regularly varying sequence \((X_t)\) is 0-dependent, and thus \((2.11)\) holds. Its limiting covariance matrix \(\sum_h\) is a diagonal matrix with entries \(\gamma^2_A(0) = (\mu_1(A))^2\) on the main diagonal.

We consider the stochastic volatility model \(X_t = \sigma_t V_t\) where \((\sigma_t)\) is independent of \((V_t)\), \((\sigma_t)\) is a positive \(\eta\)-dependent strictly stationary sequence and \((V_t)\) is a regularly varying iid sequence with index \(\alpha > 0\); see Davis and Mikosch [8]. Assume that \(E\sigma^{\alpha+\varepsilon} < \infty\) for some \(\varepsilon > 0\). In this case, \((X_t)\) is \(\eta\)-dependent, strictly stationary and regularly varying with index \(\alpha\). We will show that \((2.11)\) holds with \(\varphi_A(u,s,t) = 0\) for \(0 < u < s < t\). Since \(A\) is bounded away from zero, there exists a \(\delta > 0\) such that

\[
\varphi_A(u,s,t) \leq \limsup_{m \to \infty} m^2 P(\sigma_m^{-1} \min(|X_0|, |X_u|, |X_s|, |X_t|) > \delta)
\]

assuming the asymptotic independence condition \((AI)\) and that the Borel set \(A \in \mathbb{R}^d\) possesses a finite density \(P(\sigma_0 V_0 > a_m) = P(\sigma_a > a_m)\) by virtue of Breiman’s lemma; see [2].

In the iid case, the limiting quantities \(Z_h, h \geq 1\), in Lemma 4 vanish. The same observation can be made in the case of a strictly stationary sequence with asymptotic (extremal) independence in the following sense:

**Condition (AI):** Assume there exist sequences \(m = m_n \to \infty\) and \(r_n \to \infty\) such that \(m = o(n)\) and \(r_n = o(m)\) as \(n \to \infty\) and the following conditions are satisfied for any Borel set \(A \subset \mathbb{R}^d\) bounded away from zero and the axes such that \(\mu_1(\partial A) = 0\):

1. \(\lim_{n \to \infty} m^2 p_h \) exists and is finite for \(h \geq 1\),
2. \(\lim_{n \to \infty} m^2 \sup_{1 \leq i < j \leq r_n} P(a_m^{-1} X_0 \in A, a_m^{-1} X_i \in A, a_m^{-1} X_j \in A) = 0\),
3. \(\lim_{n \to \infty} m^2 \sup_{1 \leq i < j < i \leq r_n} P(a_m^{-1} X_0 \in A, a_m^{-1} X_i \in A, a_m^{-1} X_j \in A, a_m^{-1} X_t \in A) = 0\).

Example 10. We consider the stochastic volatility model from Example 9 but we drop the condition of \(\eta\)-dependence. Conditions (AI.2) and (AI.3) are verified in the same way as in Example 9. We also observe that for some constant \(c > 0\),

\[
m^2 p_h \sim c \frac{P(a_m^{-1} X_0 \in A, a_m^{-1} X_h \in A)}{P(\min(V_1, V_2) > a_m)} = c \frac{P(a_m^{-1} \text{diag}(\sigma_0, \sigma_h)(V_1, V_2)' \in A \times A)}{P(\min(V_1, V_2) > a_m)}.
\]

An application of a Breiman-type result for regularly varying vectors on cones due to Janssen and Drees [18] ensures the existence and finiteness of the limits \(\lim_{m \to \infty} m^2 p_h\) for \(h \geq 1\). This is (AI.1).

**Lemma 11.** Assume that \((X_t)\) is an \(\mathbb{R}^d\)-valued strongly mixing strictly stationary regularly varying sequence with index \(\alpha > 0\) and that the Borel set \(A\) satisfies the conditions of Section 7.2. We also assume the asymptotic independence condition (AI) and the mixing condition

\[
(2.14) \quad \lim_{n \to \infty} n m^2 \sum_{h=r_n}^n \xi_h = 0.
\]
Then

\[ n^{0.5}(\bar{\gamma}_A(i) - \bar{\gamma}_A(i)) \xrightarrow{d} \mathcal{N}(0, \gamma_i^2), \quad i \geq 1, \tag{2.15} \]

where \((Z_i)_{i=1, \ldots, h}\) are independent Gaussian with mean zero and variances

\[ \text{var}(Z_i) = \lim_{m \to \infty} m^2 p_i, \quad i \geq 1. \]

**Proof.** We will apply the central limit theorem in Rio [27] for strongly mixing triangular arrays to the left-hand side in (2.15). For this reason, we have to calculate the asymptotic covariance matrix of the left-hand vector. We observe that for fixed \(j > i \geq 1\), in view of the mixing condition (2.14) as \(n \to \infty\),

\[
\text{cov}(n^{0.5}\bar{\gamma}_A(i), n^{0.5}\bar{\gamma}_A(j)) = m^2 \text{cov}(\tilde{I}_0 I_j, \tilde{I}_0 I_i) + m^2 \sum_{t=1}^{n} \left[ \text{cov}(\tilde{I}_0 \tilde{I}_i, \tilde{I}_t \tilde{I}_{t+j}) + \text{cov}(\tilde{I}_0 \tilde{I}_j, \tilde{I}_t \tilde{I}_{t+i}) \right] + o(1) \tag{2.16}
\]

Condition (AI) implies that \(m^2 \text{cov}(\tilde{I}_0 I_j, \tilde{I}_0 I_i) \to 0\) as \(m \to \infty\). The same argument also implies that the first \(j\) summands in (2.16) vanish as \(n \to \infty\). Therefore it suffices to consider

\[ m^2 \sum_{t=j+1}^{n} \left[ \text{cov}(\tilde{I}_0 \tilde{I}_i, \tilde{I}_t \tilde{I}_{t+j}) + \text{cov}(\tilde{I}_0 \tilde{I}_j, \tilde{I}_t \tilde{I}_{t+i}) \right]. \]

In the latter sum, the indices \(0, i, t, t+j\) are distinct and the same observation applies to \(0, j, t, t+i\). Direct calculation with condition (AI) shows that this sum is asymptotically negligible. This implies that the covariance matrix of the limiting vector is diagonal. The calculation of the asymptotic variances is similar by observing that as \(n \to \infty\),

\[
\text{var}(n^{0.5}\bar{\gamma}_A(h)) = m^2 \text{var}(\tilde{I}_0 I_h) + 2 m^2 \sum_{t=1}^{n} \text{cov}(\tilde{I}_0 \tilde{I}_i, \tilde{I}_t \tilde{I}_{t+i}) + o(1) = m^2 p_h + o(1).
\]

**Remark 12.** Although \(\bar{\gamma}_A(h) \xrightarrow{d} 0, h \geq 1\), it is in general not possible to avoid centering in (2.15). However, under (A1.1), \(n^{0.5} \bar{\gamma}_A(h) \to 0\) if \(n/m^2 = o(1)\) as \(n \to \infty\), and the latter condition can even be weakened if \(m^2(p_h - p^2_0) \to 0\) as \(m \to \infty\).

2.4. Mean square consistency of the integrated periodogram. Recall the definitions of \(J_{n,A}(g)\) and \(J_A(g)\) for \(g \in L^2_+(\Pi)\) from [1.2] and [1.3], respectively. The following elementary result deals with the convergence of the first and second moments of \(J_{n,A}(g)\) for a given function \(g\).

**Lemma 13.** Consider an \(\mathbb{R}^d\)-valued strictly stationary regularly varying sequence \((X_t)\) with index \(\alpha > 0\). Assume that the Borel set \(A \subset \mathbb{R}_0^d\) satisfies the conditions of Section 1.2 and \(\gamma_A(l) < \infty\) and (M) holds. Then the following asymptotic relations hold for \(g \in L^2_+(\Pi)\).

1. \(EJ_{n,A}(g) \to J_A(g)\) as \(n \to \infty\).
2. If in addition, \( m \log^2 n/n = O(1) \) as \( n \to \infty \), and there exists a constant \( c > 0 \) such that
\[
|c_n(g)| \leq c/h, \quad h \geq 1,
\]
then \( E(J_{n,A}(g) - J_A(g))^2 \to 0 \) and \( J_{n,A}(g) \overset{p}{\to} J_A(g) \) as \( n \to \infty \).

The proof of the lemma is given in Section 5.

**Remark 14.** Condition (2.17) holds under mild smoothness conditions on \( g \), e.g. if \( g \) is Lipschitz or has bounded variation on \( \Pi \); see Theorem 4.7 on p. 46 and Theorem 4.12 on p. 47 in Zygmund [30].

3. **Functional central limit theorem for the integrated periodogram**

Recall the definition of the spectral density \( h_A \) from Section 1.2. In this section, we assume that the weight function \( g \) is a non-negative continuous function. Abusing notation, we define the empirical spectral distribution function with weight function \( g \) by
\[
J_{n,A}(x) = J_{n,A}(gI_{[0,x]}) = \int_0^x I_{n,A}(\lambda) g(\lambda) \, d\lambda, \quad x \in \Pi.
\]

Under the conditions of Lemma 13, again abusing notation, we have
\[
J_{n,A}(x) \overset{p}{\to} J_A(x) = J_A(gI_{[0,x]}) = \int_0^x h_A(\lambda) g(\lambda) \, d\lambda, \quad x \in \Pi.
\]

In view of the monotonicity and continuity of the functions \( J_{n,A} \) and \( J_A \) we also have
\[
\sup_{x \in \Pi} |J_{n,A}(x) - J_A(x)| \overset{P}{\to} 0.
\]

Our next goal is to complement this consistency result by a functional central limit theorem of the type \( (n/m)^{0.5} (J_{n,A} - J_A) \overset{d}{\to} G \), in \( C(\Pi) \), the space of continuous functions on \( \Pi \) equipped with the uniform topology, for a suitable Gaussian limit process \( G \).

However, this result is unlikely to hold in general, due to asymptotic bias problems. It is mentioned in Davis and Mikosch [7] in relation with the central limit theorem for the sample extremogram (see Lemma 2 above) that the pre-asymptotic centerings \( E\tilde{\gamma}_A(i) = ((n-i)/n)m(p_i - p_0^A) \) can in general not be replaced by their limits \( \gamma_A(i) \) due to the failure of the relation \( (n/m)^{0.5}m(p_i - p_0^A) \to 0 \) as \( n \to \infty \). Therefore we will equip the empirical spectral distribution function \( J_{n,A} \) with the pre-asymptotic centering \( EJ_{n,A} \). It follows from Lemma 13 that under (M), \( EJ_{n,A}(x) \to J_A(x) \) for every \( x \in \Pi \), and again using monotonicity of \( EJ_{n,A} \) and \( J_A \), we have \( \sup_{x \in \Pi} |EJ_{n,A}(x) - J_A(x)| \to 0 \).

We observe that
\[
J_{n,A}(x) = \psi_0(x) \tilde{\gamma}_A(0) + 2 \sum_{h=1}^{n-1} \psi_h(x) \tilde{\gamma}_A(h),
\]
\[
J_{n,A}^o(x) = \psi_0(x) + 2 \sum_{h=1}^{n-1} \psi_h(x) \tilde{\rho}_A(h),
\]
where \( \psi_h(x) = \int_0^x \cos(h\lambda) g(\lambda) \, d\lambda, \quad x \in \Pi \). We also consider a Riemann sum approximation of the coefficients \( \psi_h(x) \) at the Fourier frequencies \( \omega_n(i) = 2i\pi/n \in \Pi \) given by
\[
\tilde{\psi}_h(x) = \frac{2\pi}{n} \sum_{i=1}^n g(\omega_n(i)) \cos(h\omega_n(i)), \quad x \in \Pi,
\]
Theorem 15. Assume that \( \alpha > 0 \) and the Borel set \( A \subset \mathbb{R}^d \) is bounded away from zero, \( \mu_1(\partial A) = 0 \) and \( \mu_1(A) > 0 \). Let \( g \) be a non-negative \( H^\beta \)-H"older continuous function with \( \beta \in (3/4, 1] \). If the conditions (M1) and \( \sum_{i=1}^\infty \gamma_A(l) < \infty \) hold then in \( \mathbb{C}(\Pi) \),

\[
(n/m)^{0.5} (J_{n,A} - EJ_{n,A}) \overset{d}{\to} G, \ n \to \infty,
\]

(3.3)

\[
(n/m)^{0.5} (\tilde{J}_{n,A} - E\tilde{J}_{n,A}) \overset{d}{\to} G, \ n \to \infty,
\]

(3.4)

where the limit process is given by the infinite series

\[
G = \psi_0 Z_0 + 2 \sum_{h=1}^\infty \psi_h Z_h,
\]

(3.5)

which converges in distribution in \( \mathbb{C}(\Pi) \). \((Z_h)\) is a mean zero Gaussian sequence such that \((Z_0, \ldots, Z_h)\) has the covariance matrix \((\Sigma_h)\), \(h \geq 0\), given in Lemma [4]. Moreover, the following limit relations hold

\[
(n/m)^{0.5} (J^\circ_{n,A} - EJ^\circ_{n,A}/(mp_0)) \overset{d}{\to} G^\circ, \ n \to \infty,
\]

(3.6)

\[
(n/m)^{0.5} (\tilde{J}^\circ_{n,A} - E\tilde{J}^\circ_{n,A}/(mp_0)) \overset{d}{\to} G^\circ, \ n \to \infty,
\]

(3.7)

where the limit process is given by the infinite series

\[
G^\circ = \frac{2}{\gamma_A(0)} \sum_{h=1}^\infty \psi_h (Z_h - \rho_A(h)Z_0).
\]

The proof of this result is given in Section [6].

Remark 16. For practical purposes, the discretized version \( \tilde{J}_{n,A} \) will be preferred to \( J_{n,A} \) since it does not involve the calculation of integrals. Moreover, since \( \sum_{t=1}^n e^{i \omega_n(j)t} = 0 \) for \( \omega_n(j) \in (0, \pi) \), centering of the indicators \( I_t \) with the unknown parameter \( p_0 \) in the periodogram ordinates \( I_{n,A}(\omega_n(j)) = (m/n)|\sum_{t=1}^n I_t e^{i \omega_n(j)t}|^2 \) is not needed.

For an \( \eta \)-dependent sequence \((X_t)\), we know that \( Z_h = 0 \) a.s. for \( h > \eta \). Then we conclude from Theorem [13] and Lemma [3] that the limit process \( G \) collapses into \( G = \psi_0 Z_0 + 2 \sum_{h=1}^\infty \psi_h Z_h \). However, taking into account Lemma [8] a more sophisticated result with a different convergence rate can be derived. The corresponding result for \( J^\circ_{n,A} \) is similar and therefore omitted.

Theorem 17. Assume that \((X_t)\) is an \( \mathbb{R}^d \)-valued strongly mixing strictly stationary \( \eta \)-dependent regularly varying sequence with index \( \alpha > 0 \) for some \( \eta \geq 0 \) and the Borel set \( A \subset \mathbb{R}^d \) is bounded away from zero, \( \mu_1(\partial A) = 0 \) and \( \mu_1(A) > 0 \). Also assume that the limits in (2.11) exist. Let \( g \) be a
non-negative $\beta$-Hölder continuous function with $\beta \in (3/4, 1]$. Then the relations
\[
\sqrt{n} \left( J_{n,A} - \psi_0 \tilde{\gamma}_A(0) - 2 \sum_{h=1}^{n} \psi_h \tilde{\gamma}_A(h) \right) \xrightarrow{d} \overline{G},
\]
\[
\sqrt{n} \left( \hat{J}_{n,A} - \hat{\psi}_0 \tilde{\gamma}_A(0) - 2 \sum_{h=1}^{n} \hat{\psi}_h \tilde{\gamma}_A(h) \right) \xrightarrow{d} \overline{G},
\]
hold in $\mathcal{C}(\Pi)$, where the limit process is given by the a.s. converging infinite series
\[
\overline{G} = 2 \sum_{h=1}^{\infty} \psi_{\eta+h} Z_h,
\]
and $(Z_h)$ is a mean zero Gaussian sequence such that $(Z_1, \ldots, Z_h)$ has covariance matrix $\Sigma_h$, $h \geq 1$, given in Lemma 6.

The proof is given in Section 7.

**Example 18.** Assume that $(X_t)$ is an iid regularly varying sequence with index $\alpha > 0$. Then $(Z_h)$ is an iid mean zero Gaussian sequence with $\text{var}(Z) = \gamma_A^2(0) = (\mu_1(A))^2$. If we choose the function $g \equiv 1$ we obtain
\[
\psi_h(x) = \int_0^x \cos(h \lambda) d\lambda = \frac{\sin(hx)}{h}, \quad h \geq 0, \quad x \in \Pi,
\]
and
\[
\overline{G}(x) = 2 \sum_{h=1}^{\infty} \frac{\sin(hx)}{h} Z_h, \quad x \in \Pi.
\]
We notice that $\overline{G}$ is a series representation of a Brownian bridge; see Hida [10].

In the case of asymptotic (extremal) independence a result similar to Theorem 17 holds.

**Theorem 19.** Assume that $(X_t)$ is an $\mathbb{R}^d$-valued strictly stationary regularly varying sequence with index $\alpha > 0$ and the Borel set $A \subset \mathbb{R}^d_0$ is bounded away from zero and the axes, $\mu_1(\partial A) = 0$ and $\mu_1(A) > 0$. Also assume the mixing condition (2.14) and the asymptotic independence condition (AI). Let $g$ be a non-negative $\beta$-Hölder continuous function with $\beta \in (3/4, 1]$. Then the relations
\[
\sqrt{n} \left( J_{n,A} - EJ_{n,A} - \psi_0 \tilde{\gamma}_A(0) - E\tilde{\gamma}_A(0) \right) \xrightarrow{d} \hat{G},
\]
\[
\sqrt{n} \left( \hat{J}_{n,A} - E\hat{J}_{n,A} - \hat{\psi}_0 \tilde{\gamma}_A(0) - E\tilde{\gamma}_A(0) \right) \xrightarrow{d} \hat{G},
\]
hold in $\mathcal{C}(\Pi)$, where the limit process is given by the a.s. converging infinite series
\[
\hat{G} = 2 \sum_{h=1}^{\infty} \psi_h Z_h,
\]
and $(Z_h)$ is a sequence of independent mean zero Gaussian variables with variances $\text{var}(Z_h) = \lim_{m \to \infty} m^2 \rho_h$, $h \geq 1$.

The proof is based on Lemma 11 and tightness arguments which are similar to the proofs of Theorem 15 and 17. We omit further details. In view of Remark 12, centering in Theorem 19 can be avoided if $n/m^2 = o(1)$ as $n \to \infty$.

As in classical limit theory for the empirical spectral distribution (see Grenander and Rosenblatt [14], Dahlhaus [4]), an application of the continuous mapping theorem to Theorems 15 and 17 yields limit theory for functionals of the integrated periodogram. These functionals can be used for testing the integrated periodogram of a dependent extremal event sequence.
the goodness of fit of the spectral density of the time series model underlying the data, under the
null hypothesis that the model is correct. From Theorem 15 we get the following limit results for
the corresponding test statistics.

- **Grenander-Rosenblatt test:**
  \[
  (n/m)^{0.5} \sup_{x \in \Pi} \left| J_{n,A}(x) - E J_{n,A}(x) \right| \xrightarrow{d} \sup_{x \in \Pi} |G(x)|.
  \]  
  (3.8)

- **\(\omega^2\)- or Cramér-von Mises test:**
  \[
  (n/m) \int_{\Pi} \left( J_{n,A}(x) - E J_{n,A}(x) \right)^2 dx \xrightarrow{d} \int_{\Pi} G^2(x) dx.
  \]
  If \((X_t)\) is an \(\eta\)-dependent sequence satisfying the conditions of Theorem 17 the corresponding limit results read as follows:

  - **Grenander-Rosenblatt test:**
    \[
    \sqrt{n} \sup_{x \in \Pi} \left| J_{n,A}(x) - \psi_0(x) \tau_A(0) - 2 \sum_{h=1}^{\eta} \psi_h(x) \tau_A(h) \right| \xrightarrow{d} \sup_{x \in \Pi} |\overline{G}(x)|.
    \]  
    (3.9)

  - **\(\omega^2\)-statistic or Cramér-von Mises test:**
    \[
    n \int_{\Pi} \left( J_{n,A}(x) - \psi_0(x) \tau_A(0) - 2 \sum_{h=1}^{\eta} \psi_h(x) \tau_A(h) \right)^2 dx \xrightarrow{d} \int_{\Pi} \overline{G}^2(x) dx.
    \]  
    (3.10)

In Figures 1 and 2 we show the estimated densities of the test statistics in (3.9) and (3.10) for
\(n = 2,000\) and \(n = 10,000\), for different thresholds \(a_m\) and \(g \equiv 1\). We compare the estimated densities
with their corresponding limits. The samples are iid \(t\)-distributed with \(\alpha = 3\) degrees of freedom.
We mention that the density of \(\sup_{x \in \Pi} |\overline{G}(x)|\) is given by \(4\pi^{-2} \sum_{j=1}^{\infty} (-1)^{j+1} x \exp \left( -j^2 x^2 / \pi^2 \right)\),
\(x > 0\); see Shorack and Wellner [28]. We use the identity in law \(\int_{\Pi} \overline{G}^2(x) dx = \sum_{j=1}^{\infty} (2/j^2)N_j^2\)
for an iid standard normal sequence \((N_j)\) (see [28]) for the simulation of the limiting density of the
\(\omega^2\)-statistic.

Not surprisingly, these graphs show that one needs rather large sample sizes to make the tests
reliable. The Grenander-Rosenblatt statistic shows a better overall behavior in comparison with the
\(\omega^2\)-statistic. The distribution of the former statistic is close to its limit for a variety of thresholds
like \(p_0 = 0.1, 0.05\) and even for \(p_0 = 0.03\). In contrast, the \(\omega^2\)-statistic is rather sensitive to the
choice of threshold and sample size; the best overall approximation is achieved for \(n = 10,000\) and
\(p_0 = 0.05\). For applications, one would need to focus on the quality of the approximation of high/low
quantiles of the test statistics by the limiting quantiles. This task is not addressed in this paper.

4. **The bootstrapped integrated periodogram**

With a few exceptions, the limit processes \(G\) and \(\overline{G}\) in Theorem 15 and 17 have an unfamiliar
dependence structure and then it is impossible to give confidence bands for the test statistics
mentioned in the previous section. One faces a similar problem when dealing with the sample extremograms whose asymptotic covariance matrix is a complicated function of the measures \(\mu_h\) in (1.2). Davis et al. [10] proposed to apply the stationary bootstrap for constructing confidence bands for the sample extremogram. The stationary bootstrap can also be used for the integrated periodogram, as we will show below.
Figure 1. Density of the left-hand side in (3.9) with \( \eta = 0 \) (dotted line) and its limit \( \sup_{x \in \Pi} |G(x)| \) (solid line) for \( g \equiv 1 \). We choose the set \( A = (1, \infty) \), different thresholds \( a_m \) with \( p_0 = P(X > a_m) \) and different sample sizes \( n \). The underlying sequence \((X_t)\) is iid \( t \)-distributed with \( \alpha = 3 \) degrees of freedom. The sample sizes are chosen as \( n = 2,000 \) in the first row and \( n = 10,000 \) in the second row. The thresholds \( a_m \) are chosen such that \( p_0 = 0.1 \) in the first column, \( p_0 = 0.05 \) in the second column and \( p_0 = 0.03 \) in the third column.

4.1. Stationary bootstrap. The stationary bootstrap was introduced by Politis and Romano [23] as an alternative block bootstrap method. First, we describe this procedure for a strictly stationary sequence \((Y_t)\). Given a sample \( Y_1, \ldots, Y_n \), consider the bootstrapped sequence

\[
Y_{K_1}, \ldots, Y_{K_1 + L_1 - 1}, \ldots, Y_{K_N}, \ldots, Y_{K_N + L_N - 1}, \ldots,
\]

(4.1)

where \((Y_t), (K_i), (L_i)\) are independent sequences, \((K_i)\) is an iid sequence of random variables uniformly distributed on \( \{1, \ldots, n\} \), \((L_i)\) is an iid sequence of geometrically distributed random variables with distribution \( P(L_1 = i) = \theta (1 - \theta)^{i-1}, i = 1, 2, \ldots, \) for some \( \theta = \theta_n \in (0, 1) \) such that \( \theta_n \to 0 \) as \( n \to \infty \) and \( N = N_n = \inf \{ i \geq 1 : \sum_{j=1}^{i} L_j \geq n \} \). If any element \( Y_t \) in (4.1) has an index \( t > n \), we replace it by \( Y_{t \mod n} \). As a matter of fact, \((Y_t)_{t \geq 1}\) constitutes a strictly stationary sequence. The stationary bootstrap sample is now chosen as the block of the first \( n \) elements in (4.1). In what follows, we write \((Y_t^{*})_{t \geq 1}\) for the bootstrap sequence (4.1), indicating that this sequence is nothing but the original \( Y \)-sequence sampled at the random indices \((K_1, \ldots, K_1 + L_1 - 1, K_2, \ldots, K_2 + L_2 - 1, \ldots)\) with the convention that indices larger than \( n \) are taken modulo \( n \).

In what follows, the probability measure generated by the bootstrap procedure is denoted by \( P^\ast \), i.e., \( P^\ast(\cdot) = P(\cdot | (X_t)) \). The corresponding expected value, variance and covariance are denoted by \( E^\ast, \text{var}^\ast \) and \( \text{cov}^\ast \).
Figure 2. Density of the left-hand side in (3.10) with \( \eta = 0 \) (dotted line) and its limit \( \int_{x \in \Pi} G^2(x) \, dx \) (solid line) for \( g \equiv 1 \). We choose the same setting as in Figure 2.

4.2. The bootstrapped sample extremogram. Davis et al. [10] applied the stationary bootstrap to the sequence of lagged vectors

\[ I_t(h) = (I^2_t, I_t I_{t+1}, \ldots, I_t I_{t+h}), \quad t = 1, 2, \ldots, \]

for fixed \( h \geq 0 \) and showed consistency of the bootstrapped sample extremogram. In particular, they showed the following result which we cite for further reference. A close inspection of the proof in [10] shows that the results remain true if in \( I_t(h) \) we replace the quantities \( I_s \) by \( \tilde{I}_s \), \( s = t, \ldots, t+h \).

We denote the corresponding vector by \( \tilde{I}_t(h) \). Consider the stationary bootstrap sequence \( (\tilde{I}_t(h)) \) and write

\[ \tilde{\gamma}_A(i) = \frac{m}{n} \sum_{t=1}^{n-i} \tilde{I}_t \tilde{I}_{t+i}, \quad i = 0, \ldots, h. \]

**Theorem 20.** Consider an \( \mathbb{R}^d \)-valued strictly stationary regularly varying sequence \( (X_t) \) with index \( \alpha > 0 \) and assume the following conditions:

1. The mixing condition (M1) and in addition \( \sum_{h=1}^{\infty} h \xi_h < \infty \).
2. The growth conditions \( \theta = \theta_n \to 0 \) and \( n\theta^2/m \to \infty \).
3. The set \( A \) is bounded away from zero, \( \mu_A(\partial A) = 0 \) and \( \mu_1(A) > 0 \).

Then the following bootstrap consistency results hold for \( h \geq 0 \):

\[ E^* \left( \tilde{\gamma}_A(h) \right) \xrightarrow{P} \gamma_A(h) \quad \text{and} \quad \text{var}^* \left( (n/m)^{0.5} \tilde{\gamma}_A(h) \right) \xrightarrow{P} \sigma_{hh}, \]

where the covariance matrix \( \Sigma_h = (\sigma_{ij}) \) is given in Lemma 4. Moreover, writing \( d \) for any metric describing weak convergence in Euclidean space and \( (Z_i)_{i=0,\ldots,h} \) for an \( N(0, \Sigma_h) \) Gaussian vector, we also have

\[ d \left( (n/m)^{1/2} (\tilde{\gamma}_A(i) - \tilde{\gamma}_A(i))_{i=0,\ldots,h}, (Z_i)_{i=0,\ldots,h} \right) \xrightarrow{P} 0, \quad n \to \infty. \]
In what follows, we will write \( d \) for any metric describing weak convergence in any space of interest.

4.3. The bootstrapped integrated periodogram. Bootstrapping the sequence \((I_t(h))\) has the advantage that we preserve the neighbors \(I_t^∗ + i\) of \(I_t^∗\) from the original sequence \((I_s)\). However, this method depends on the lag \(h\) and creates problems if the number of lags increases with the sample size \(n\). In what follows, we will apply the stationary bootstrap directly to \((I_t)\). Then we have to re-define the bootstrap sample extremogram at any lag \(h < n\). Write

\[ I_n = n^{-1} \sum_{t=1}^{n} I_t \quad \text{and} \quad \tilde{I}_t = I_t - I_n, \quad t \in \mathbb{Z}, \]

and define the corresponding bootstrap sample extremogram

\[ \tilde{\gamma}_A^∗(h) = \frac{n^{-1}}{n} \sum_{t=1}^{n-h} \tilde{I}_t \tilde{I}_{(t+h)}^*, \quad h = 0, \ldots, n - 1, \]

and the bootstrap periodogram

\[ I_{n,A}^*(\lambda) = \frac{n}{m} \left| \sum_{t=1}^{n} \tilde{I}_t e^{-it\lambda} \right|^2, \quad \lambda \in \Pi. \]

Note the crucial difference: in general, \(I_t \cdot I_{(t+h)}^* \neq I_t \cdot I_{(t+h)}^* + h\), but, as we will see in Lemma 25, the quantities \(\tilde{\gamma}_A^*(h)\) and \(\tilde{\gamma}_A^*(h)\) are asymptotically close for fixed \(h \geq 0\).

In what follows, we focus on the bootstrap for the continuous version \(J_{n,A}^\ast\) of the integrated periodogram for a given smooth weight function \(g\); bootstrap consistency can also be shown for the discretized version \(\hat{J}_{n,A}^\ast\); we omit further details. In the definition of \(J_{n,A}^\ast\) in (3.1), we simply replace \((I_t)\) by \((\tilde{I}_t)\), resulting in its bootstrap version

\[ J_{n,A}^*(\lambda) = \int_0^\lambda I_{n,A}^*(x) g(x) dx = \psi_0 \tilde{\gamma}_A^*(0) + 2 \sum_{h=1}^{n-1} \psi_h \tilde{\gamma}_A^*(h), \quad \lambda \in \Pi. \]

Now we can formulate a bootstrap analog of Theorem 15 which shows the consistency of the stationary bootstrap procedure.

**Theorem 21.** Assume the conditions of Theorem 15 and 20. Then

\[ d\left((n/m)^{1/2}(J_{n,A}^* - EJ_{n,A}^*) \cdot G\right) \overset{L_2}{\to} 0, \quad n \to \infty, \]

where the Gaussian process \(G\) is defined in Theorem 16 and \(d\) is any metric which describes weak convergence in \(C(\Pi)\).

**Remark 22.** Recall that, in general, it is not possible to replace the centering \(EJ_{n,A}^\ast\) of \(J_{n,A}^\ast\) in the functional central limit theorem of Theorem 15 by its limit \(\int_0^\lambda h_A(\lambda) g(\lambda) d\lambda\). A similar remark applies to Theorem 21. Although \(\sup_{\lambda \in \Pi} |E^\ast J_{n,A}^*(\lambda) - J_{n,A}(\lambda)| \overset{L_2}{\to} 0\), under the conditions of Theorem 21, it is in general not possible to replace the centering \(E^\ast J_{n,A}^\ast\) by \(J_{n,A}^\ast\); see Lemma 28. Thus, Theorem 21 does not yield bootstrap consistency in a textbook sense but it rather provides a simulation technique for the limit process \(G\). In turn, the simulation of this process makes it possible to provide confidence bands for the goodness of fit test statistics considered above. We will apply this simulation procedure in Section 4.4.
4.4. A simulation study. We focus on the Grenander-Rosenblatt statistic (GRS) on the left-hand side of (3.8) for different time series models, distinct thresholds and sample sizes. Under the null hypothesis of a particular time series model, one can simulate the quantiles of the GRS from the theoretical model. In this study we also follow a different approach. First, we determine the expected value function \( EJ_{n,A} \) and the threshold \( a_m \) such that \( p_0 = P(X > a_m) = 1/m \) by simulation from the theoretical model and then we use the stationary bootstrap to calculate the asymptotic quantiles of the GRS. This distribution is be obtained by repeated simulation of \( (n/m)^{0.5} \sup_{x \in \Pi} |J_{n,A}^*(x) - E^* J_{n,A}^*(x)| \); Theorem 21 provides a justification for this approach. In the cases when the expected value function \( EJ_{n,A} \) can be replaced by its limit, i.e., when the bias of \( J_{n,A} \) is negligible, this approach has the advantage that the test is non-parametric. An example are models satisfying the asymptotic independence condition (AI) and \( n/m^2 \to 0 \) as \( n \to \infty \); see Theorem 19 and the remark following it. Of course, for an iid sequence or \( \eta \)-dependent sequence one can also use the quantiles of the limit distribution of the GRS which are known or can be simulated; see (3.3) and (3.11).

In what follows, we apply the Grenander-Rosenblatt test (GRT) to various univariate (real-life or simulated) time series \( X_t, t = 1, \ldots, n \) for different sample sizes \( n \) and thresholds \( a_m \). We always choose \( A = (1, \infty) \) and \( g \equiv 1 \). Whenever we apply the stationary bootstrap we choose the geometric parameter \( \theta = 1/50 \). Density plots and simulated quantiles are derived from 4,000 independent repetitions, also in the bootstrap case.

In Figure 4 we illustrate how the stationary bootstrap works for different thresholds \( a_m \) and sample size \( n = 2,000 \). We show the density of the normalized GRS on the left-hand side of (3.8) and its bootstrap approximation. We choose regularly varying ARMA(1,1) and GARCH(1,1) models. The densities of the GRS and its bootstrap approximation are close to each other. We take this fact as justification of using the bootstrap quantiles of the GRS in the test. While the densities in the ARMA case do not seem too sensitive to the choice of the high threshold \( a_m \), the shape of the densities change for the GARCH model when switching from \( p_0 = 0.10 \) to \( p_0 = 0.05 \), while they look similar for \( p_0 = 0.05 \) and \( p_0 = 0.01 \).

In Figure 4 we show sample paths of the normalized and centered integrated periodogram \( (n/m)^{0.5} |J_{n,A} - E J_{n,A}| \) with \( p_0 = 0.05 \) for samples of size \( n = 2,000 \) from ARMA(1,1) and GARCH(1,1) models together with 95%-quantiles of the GRS both under the correct and under an incorrect null hypothesis. Due to the need of centering with \( EJ_{n,A} \) these sample paths are affected both by the sample and the model. Indeed, if the model is chosen incorrectly we will typically subtract the incorrect centering and calculate an incorrect threshold \( a_m \). When using both the bootstrap-based or true 95%-quantiles of the GRS, the model is not rejected if the sample is in agreement with the null hypothesis. However, if the sample comes from a model whose parameters slightly deviate from the parameters of the null hypothesis the incorrect expected value \( EJ_{n,A} \) and wrong threshold \( a_m \) change the sample path of the integrated periodogram in such a way that the bootstrap-based GRT rejects the null hypothesis while it does not reject the null if one uses the quantiles based on the null hypothesis. It is advantageous to show both 95%-quantiles: they deviate rather significantly, indicating that we chose an incorrect null model.

In Figure 5 we consider a stochastic volatility model \( X_t = \sigma_t Z_t \), where \( (\sigma_t) \) is a log-normal stationary process independent of the iid \( t \)-distributed sequence \( (Z_t) \). The \( \alpha \) degrees of freedom of the \( t \)-distribution coincide with the index \( \alpha \) of regular variation of \( (X_t) \). The extremogram of this sequence vanishes at all positive lags. This fact is in agreement with the extremogram of an iid sequence but is in contrast to a GARCH(1,1) process. Choosing \( p_0 = 0.05 \), we apply the GRS to a stochastic volatility sample of size \( n = 2,000 \) under the incorrect null hypothesis of a GARCH(1,1) process.

\footnote{Throughout, to exploit the power of the Fast Fourier Transform, we use the Riemann sum approximations to the integrated periodograms. We do not indicate this fact in the notation.}
model with tail index close to the chosen $\alpha$. The test clearly rejects the null hypothesis. We also run a GRT for the stochastic volatility sample under the null hypothesis of an iid $t$-distributed sequence with $\alpha$ degrees of freedom. We use the approximation of the distribution of the GRS by the distribution of the supremum of a Brownian bridge; see Example 18. Also in this case, the null is clearly rejected.

In Figure 6 we deal with a time series $(X_t)$ of 1,560 1-minute log-returns of Goldman Sachs stock from the period November 7-10, 2011. It has estimated tail index $\alpha \approx 3$. Using standard software, we fitted a GARCH(1,1) model such that $\sigma_t^2 = 0.019 + 0.1X_{t-1}^2 + 0.87\sigma_{t-1}^2$. Hill and QQ plots of the residuals of this model indicate that the noise is well fitted by a $t$-distribution with (approximately) 4 degrees of freedom. The theoretical index of regular variation of this GARCH(1,1) model is $\alpha = 3.13$; see Table 2 in [9]. We test the null hypothesis of a GARCH(1,1) model with the aforementioned parameters. This hypothesis is rejected. On the other hand, the GRT passes under the hypothesis of an iid sequence, where we choose $a_m$ as the 95% empirical quantile. This means that the extremes of this data set are more in agreement with an iid than with a GARCH structure. This is perhaps not surprising in view of a high frequency data return series while GARCH seems more suitable for fitting low frequency returns.

A GARCH(1,1) model is often considered to give a good fit to daily log-returns of stock prices and foreign exchange (FX) rates. For example, such a judgement may be based on tests for zero autocorrelation of the residuals, their absolute values and squares. We did not find evidence of GARCH behavior in the extremes of three 5-year time series of daily Euro-USD FX rate log-returns: from 2002 to 2006 (before the financial crisis), from 2006 to 2010 (including the financial crisis), from 2009 to 2013 (after the financial crisis); see Figure 7. We choose different thresholds $a_m$. When $p_0 = 0.05$ the null hypothesis of an iid sequence is accepted for 2002-2006 and 2009-2013, but not for 2006-2010. The null hypothesis of a fitted GARCH process with $\sigma_t^2 = 2.37 \times 10^{-7} + 0.1X_{t-1}^2 + 0.87\sigma_{t-1}^2$ and iid $t$-distributed noise with 4 degrees of freedom is also rejected by the GRT for 2006-2010. For this latter period, the stationarity assumption may be doubted. We repeat the GRTs for $p_0 = 0.02$ in the periods 2002-2006 and 2009-2013. In the latter case the iid null hypothesis is still not rejected while it is rejected in the former case. The abrupt change of the behavior of the GRT may be due to the sample size (roughly 1,280 for each time series). For $p_0 = 0.02$ one would use only 2% of the data for the calculation of the GRT.

Our simulation study points at some of the problems one has to face when using goodness of fit tests based on the extremes of a time series. A major problem is the choice of the threshold $a_m$. A data driven choice would be preferable but we do not have a theoretical answer to the problem. We propose to use graphical methods to compare the shapes of the extremogram and the integrated periodogram for different thresholds and to choose a sufficiently high threshold where the shapes stabilize. A message from the simulations is that the sample size $n$ should not be too small. For example, the GRTs in Figure 7 with $n \approx 1,280$ give rather distinct answers when switching from $p_0 = 0.05$ to $p_0 = 0.02$. The sample extremogram and the integrated periodogram render meaningless for too high thresholds because most indicator functions of extreme events will be zero. The simulation study indicates that it is useful to exploit the true quantiles of the GRS (obtained by simulation from a model under the null hypothesis) as well as corresponding bootstrap-based quantile of the GRS. In particular, when the null hypothesis is incorrect the two 95% quantiles (say) will typically differ, pointing at the incorrect null hypothesis. We do not address the problem
Figure 3. Density of the normalized GRS (solid line) and its bootstrap approximation. The sample size is $n = 2,000$, and the thresholds $a_m$ are chosen such that $p_0 = P(X > a_m) = 0.10, 0.05, 0.03$ corresponding to the first, second and third column. Top: The sample is drawn from the ARMA$(1,1)$ process $X_t = 0.8X_{t-1} + 0.1Z_{t-1} + Z_t$, where $(Z_t)$ is iid $t$-distributed with $\alpha = 3$ degrees of freedom. Bottom: The sample is drawn from the GARCH$(1,1)$ process $X_t = \sigma_t Z_t$, where $\sigma^2_t = 0.1 + 0.1X^2_{t-1} + 0.84\sigma^2_{t-1}$ and $(Z_t)$ is iid $t$-distributed with 4 degrees of freedom. In this case, the index of regular variation for $(X_t)$ is $\alpha = 3.49$; see Table 2 in Davis and Mikosch [9].

of goodness of fit tests in the case when the null hypothesis depends on estimated parameters; the asymptotic theory does not change under mild conditions on the convergence rates of the estimators.

5. Proof of Lemma 13

Part 1. Recall the series representations of $J_{n,A}(g)$ and $J_A(g)$ from (1.4) and (1.5), respectively. Then for every fixed $k \geq 1$, large $n$,

$$J_{n,A}(g) - J_A(g) = \left( c_0(g)[\tilde{\gamma}_A(0) - \gamma_A(0)] + 2 \sum_{h=1}^{k} c_h(g) [\tilde{\gamma}_A(h) - \gamma_A(h)] \right)$$

$$+ 2 \sum_{h=k+1}^{n-1} c_h(g) [\tilde{\gamma}_A(h) - \gamma_A(h)] - 2 \sum_{h=n}^{\infty} c_h(g) \gamma_A(h)$$

$$= I_1(k) + I_2(k) - I_3.$$
Figure 4. Paths of the integrated periodogram \((n/m)^{0.5}|J_{n,A} - EJ_{n,A}|\) with \(p_0 = 0.05\) for samples of size \(n = 2,000\). **Top:** We work under the null hypothesis of the ARMA(1,1) model \(X_t = 0.8X_{t-1} + 0.3Z_{t-1} + Z_t\), where \((Z_t)\) is iid \(t\)-distributed with \(\alpha = 3\) degrees of freedom. **Left:** The sample is drawn from the null model. The lower and upper dotted lines \(y = 11.9\) and \(y = 12.9\) correspond to the bootstrap-based and true 95%-quantiles of the GRS, respectively. The null hypothesis would be accepted. **Right:** The sample is drawn from the ARMA(1,1) process \(X_t = 0.8X_{t-1} + 0.1Z_{t-1} + Z_t\) with the same distribution for \((Z_t)\). The lower dotted line \(y = 6.84\) is the bootstrap-based 95%-quantile of the GRS. Based on it, the test would reject the null. However, it would accept the null if one chose the 95%-quantile of the null model. **Bottom:** We work under the null hypothesis of the GARCH(1,1) process \(X_t = \sigma_tZ_t\), where \(\sigma_t^2 = 10^{-7} + 0.1X_{t-1}^2 + 0.81\sigma_{t-1}^2\) and \((Z_t)\) is iid \(t\)-distributed with 4 degrees of freedom. **Left:** The sample is chosen from the null model. The lower and upper dotted lines \(y = 6.4\) and \(y = 8\) correspond to the bootstrap-based and true 95%-quantiles of the GRS, respectively. The null would be accepted for both quantiles. **Right:** The sample is drawn from a GARCH(1,1) process with \(\sigma_t^2 = 10^{-7} + 0.1X_{t-1}^2 + 0.84\sigma_{t-1}^2\) and the same distribution of \((Z_t)\). The lower dotted line \(y = 7.4\) is the bootstrap-based 95%-quantile of the GRS. The null would be rejected in this case while it would be accepted if one used the 95%-quantile \(y = 8\) based on the null hypothesis.
Then $I_3 \to 0$ as $n \to \infty$ since $(\gamma_A(h))$ is summable and $EI_1(k)$ converges to zero as $n \to \infty$ due to regular variation, for every $k$. In view of (2.1) in (M),

$$
|E \sum_{h=r_n+1}^{n-1} \tilde{\gamma}_A(h)c_h(g)| = \left| \frac{m}{n} \sum_{h=r_n+1}^{n-1} (n-h) c(h) (p_h - p_0^2) \right| \\
\leq cm \sum_{h=r_n+1}^{\infty} \xi_h \to 0, \quad n \to \infty,
$$

and (2.2) in (M) implies

$$
\lim_{k \to \infty} \limsup_{n \to \infty} \left| E \sum_{h=k+1}^{r_n} \tilde{\gamma}_A(h)c_h(g) \right| \leq c \lim_{k \to \infty} \limsup_{n \to \infty} m \sum_{h=k+1}^{r_n} p_h = 0.
$$

Since $\lim_{k \to \infty} \sum_{h=k+1}^{\infty} \gamma_A(h) = 0$, we have $\lim_{k \to \infty} \limsup_{n \to \infty} |EI_2(k)| = 0$. This proves Part 1.

**Part 2.** It follows from Theorem 3.1 in Davis and Mikosch [7] that $\tilde{\gamma}_A(h) \overset{L^2}{\to} \gamma_A(h)$, $h \geq 1$. Hence $I_1(k) \overset{L^2}{\to} 0$ as $n \to \infty$ for fixed $k \geq 1$. It remains to show that $\lim_{k \to \infty} \limsup_{n \to \infty} \var(I_2(k)) = 0$. 

![Figure 5. The sample of size $n = 2,000$ is drawn from a stochastic volatility process $X_t = \sigma_t Z_t$ with log-volatility $\log \sigma_t = 0.9 \log \sigma_{t-1} + \epsilon_t$ for an iid standard normal sequence $(\epsilon_t)$. $Z_t$ is $t$-distributed with 3.6 degrees of freedom. Left: Sample path of $(n/m)^{0.5} |J_n,A - EJ_n,A|$ with $p_0 = 0.05$. The lower and upper dotted lines $y = 7.8$ and $y = 10.2$ correspond to the true and bootstrap-based 95%-quantiles of the GRS under the null hypothesis of a GARCH(1,1) process $\tilde{X}_t = \tilde{\sigma}_t \tilde{Z}_t$ with $\tilde{\sigma}_t^2 = 6.23 \times 10^{-3} + 0.1 \tilde{X}_{t-1}^2 + 0.8 \tilde{\sigma}_{t-1}^2$ and iid $t$-distributed ($\tilde{Z}_t$) with 4 degrees of freedom. This process has tail index 3.68; see Table 1 in [9]. The test clearly rejects the null hypothesis. Right: Sample path of the integrated periodogram absolute value $n^{0.5} |J_n,A - \psi_0 \tilde{\gamma}_A(0)|$. The dotted line is the 95%-quantile of the distribution of the supremum of the absolute values of a Brownian bridge. The test clearly rejects the null hypothesis that $(X_t)$ is iid.]}
We have

\[ I_2(k) = 2 \left( \sum_{h=k+1}^{r_n} + \sum_{h=r_n+1}^{n-1} \right) c_h(g) [\tilde{\gamma}_A(h) - \gamma_A(h)] = 2I_{21}(k) + 2I_{22}. \]

In view of Lemma \( \Pi \) we get the bound

\[
\text{var}(I_{21}(k)) \leq \frac{m^2}{n} \sum_{h=k+1}^{r_n} \sum_{l=0}^{r_n-h} |c_h(g)c_{h+l}(g)| \times \\
\left( |\Gamma(0, h, 0, h + l)| + \sum_{t=1}^{n-h-l} |\Gamma(0, h, t, t + h + l)| + \sum_{t=1}^{n-h} |\Gamma(0, h + l, t, t + h)| \right) \\
= Q_1 + Q_2 + Q_3.
\]

Since \(|c_h(g)| \leq c/h\) (see (2.17)),

\[
|Q_1| \leq \frac{c m^2}{n} \sum_{h=k+1}^{r_n} |c_h(g)| \sum_{s=h}^{r_n} |c_s(g)| p_s = \frac{c m^2}{n} \sum_{s=k+1}^{r_n} |c_s(g)| p_s \sum_{h=0}^{r_n} |c_h(g)| \\
\leq \frac{c m^2}{n} \sum_{s=k+1}^{r_n} p_s s^{-1} \log s,
\]

Figure 6. GRTs for 1,560 Goldman Sachs 1-minute log-returns. Left: The integrated periodogram \((n/m)^{0.5} |J_{n,A} - EJ_{n,A}| \) with \( p_0 = 0.05 \) under the null hypothesis that the data are generated by the GARCH(1, 1) model \( \sigma^2_t = 0.019 + 0.1X^2_{t-1} + 0.87\sigma^2_{t-1} \) with iid \( t \)-distributed noise with 4 degrees of freedom. The lower and upper dotted lines \( y = 7.6 \) and \( y = 12.3 \) represent the true and bootstrap-based 95%-quantiles of the GRS under the null hypothesis. The hypothesis of GARCH(1, 1) is clearly rejected. Right: The integrated periodogram \((n/m)^{0.5} |J_{n,A} - \psi_0 \gamma_A(0)| \) with \( p_0 = 0.05 \) under the null hypothesis of an iid sequence. The dotted line represents the asymptotic 95%-quantile based on the approximation of the GRS by the supremum of the absolute values of a Brownian bridge. The null hypothesis is not rejected.
Figure 7. GRTs for daily Euro-USD FX rate log-returns 2002-2006 (top, \( n = 1,280 \)), 2006-2010 (middle, \( n = 1,279 \)), 2009-2013 (bottom, \( n = 1,281 \)). The graphs show the integrated periodograms \( n^{0.5}|J_{n,A} - \psi_0 J_A(0)| \) under the null hypothesis of an iid sequence and \( (n/m)^{0.5}|J_{n,A} - E J_{n,A}| \) in the case of a fitted GARCH model. Under the iid hypothesis, the dotted lines represent the 95%-quantile obtained from the limiting supremum of the absolute values of a Brownian bridge. Under the GARCH hypothesis, the dotted line represents the bootstrap-based 95%-quantile of the GRS. Top: FX rate log returns 2002-2006 (\( n = 1,280 \)). We test under the iid null hypothesis. For \( p_0 = 0.05 \) (left), the null is not rejected. This is in contrast to the case \( p_0 = 0.02 \) (right) which leads to a clear rejection. The qualitative difference may be due to the relatively small sample size which renders the test statistics meaningless. Middle: FX rate log returns 2006-2010 (\( n = 1,279 \)). Left. The iid null hypothesis with \( p_0 = 0.05 \) is rejected. Right. A GARCH(1,1) model with \( \sigma_t^2 = 2.37 \times 10^{-7} + 0.1 X_{t-1}^2 + 0.8 \sigma_{t-1}^2 \) and iid \( t \)-distributed noise with 4 degrees of freedom is fitted to the data. The null hypothesis of this GARCH is clearly rejected. Bottom: FX rate log returns 2009-2013 (\( n = 1,281 \)). The iid null hypothesis with \( p_0 = 0.05 \) (left) and \( p_0 = 0.02 \) (right) is not rejected.
and the right-hand side converges to 0 by first letting \( n \to \infty \) and then \( k \to \infty \), using (2.4). Since the structures of \( Q_2 \) and \( Q_3 \) are similar we restrict ourselves to showing \( Q_2 \to 0 \) as \( n \to \infty, k \to \infty \).

We observe that
\[
Q_2 \leq \frac{c^2}{n} \sum_{h=k+1}^{r_n} \frac{1}{h^2} \left( \sum_{t=1}^{2r_n} - \sum_{t=2r_n+1}^{r_n} \right) |\Gamma(0, h, t, t + s)|
\]
\[
\leq \frac{c}{n} \sum_{h=k+1}^{r_n} \log^2 m \sum_{h=k+1}^{r_n} p_h + c \frac{\log^2 m}{n} \sum_{h=r_n+1}^{r_n} \xi_h + cn^{-1} \left( \sum_{h=k+1}^{r_n} p_h/h \right)^2.
\]

In the last step, we used (2.3). The right-hand side vanishes as \( n \to \infty \) and \( k \to \infty \). Finally, we conclude that \( \lim_{k \to \infty} \limsup_{n \to \infty} \var(I_{21}(k)) = 0 \).

Now we turn to bounding \( \var(I_{22}) \). In view of Lemma 1 we have
\[
\var(I_{22}) \leq \frac{m^2}{n} \sum_{h=r_n+1}^{r_n} \sum_{s=h}^{r_n} |c_h| \var(|E_{\tilde{I}_0} \tilde{I}_s|)
\]
\[
+ \sum_{t=1}^{n-s} |\Gamma(0, s, t, t + h)| = Q_4 + Q_5 + Q_6.
\]

We have by (2.17),
\[
Q_4 \leq \frac{c m^2}{n} \sum_{h=r_n+1}^{r_n} \sum_{s=h}^{r_n} |c_h| \var(|E_{\tilde{I}_0} \tilde{I}_s|)
\]
\[
\leq \frac{c m^2}{n} \sum_{h=r_n+1}^{r_n} \log^2 m \sum_{s=h}^{r_n} [(p_s - p_0^2) + p_0^2]
\]
\[
\leq c \frac{m}{nr_n} \sum_{h=r_n+1}^{r_n} \xi_h + \frac{(p_0 m)^2}{r_n} = o(1), \quad n \to \infty.
\]

The terms \( Q_5 \) and \( Q_6 \) can be treated in a similar way; we focus on \( Q_5 \). By (2.17),
\[
Q_5 \leq \frac{c m^2}{n} \sum_{h=r_n+1}^{r_n} \sum_{s=h}^{r_n} (hs)^{-1} \sum_{t=1}^{r_n} |\Gamma(0, h, t, t + s)|
\]
\[
+ \frac{c m^2}{n} \sum_{h=r_n+1}^{r_n} \sum_{s=h+1}^{r_n} \sum_{t=r_n+1}^{r_n} (hs)^{-1} |\Gamma(0, h, t, t + s)|
\]
\[
+ \frac{c m^2}{n} \sum_{h=r_n+1}^{r_n} \sum_{s=h}^{r_n} \sum_{t=1}^{r_n} (hs)^{-1} |\Gamma(0, h, t, t + s)|
\]
\[
= Q_{51} + Q_{52} + Q_{53},
\]

and
\[
Q_{51} \leq \frac{c m^2}{n} \sum_{h=r_n+1}^{r_n} \sum_{s=h}^{r_n} (hs)^{-1} \sum_{t=1}^{r_n} [(p_h - p_0^2) + p_0^2]
\]
\[
\leq c \frac{m}{nr_n} \sum_{h=r_n+1}^{r_n} \xi_h + (mp_0)^2 \frac{r_n}{n} \to 0, \quad n \to \infty.
\]
Next we consider $Q_{52}$ and $Q_{53}$. By (2.3), we have

$$Q_{52} \leq c \frac{2m^2}{n} \sum_{h=r_n+1}^{n-1} \sum_{s=h}^{n-1} \xi_t \leq c \frac{m \log^2 n}{n} m \sum_{t=r_n+1}^{\infty} \xi_t.$$ 

The right-hand side converges to zero by using the assumption $m \log^2 n/n = O(1)$ and the condition (2.4). Similarly, using (2.3), we obtain

$$Q_{53} \leq c \frac{m}{n} m \sum_{h=r_n+1}^{\infty} \xi_h.$$ 

We conclude that $\text{var}(I_{22}) \to 0$ as $n \to \infty$.

We proved above that $E((J_{n,A} - J_A(g))^2) \to 0$, hence $J_{n,A}(g) \overset{P}{\to} J_A(g)$, combined with (2.4), yields $J_{n,A}(g) \overset{P}{\to} J_A(g)$.

6. PROOF OF THEOREM 15

We start by proving (3.3). An application of the continuous mapping theorem in $\mathbb{C}(\Pi)$ and Lemma 11 yield in $\mathbb{C}(\Pi)$ for every $k \geq 1,$

$$\left(\frac{m}{n}\right)^{0.5} \psi_0(\tilde{\gamma}_A(0) - E\tilde{\gamma}_A(0)) + 2 \sum_{h=1}^{k} \psi_h(\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \overset{d}{\to} \psi_0 \xi_0 + 2 \sum_{h=1}^{k} \psi_h \xi_h.$$ 

Here $(\xi_h)$ is a mean zero Gaussian process with covariance structure specified in Lemma 11. In view of Theorem 2 in Dehling et al. [12], relation (3.3) will follow if we can prove the following result.

**Lemma 23.** Assume that the conditions of Theorem 15 hold. Then for any $\varepsilon > 0$,

$$\lim_{k \to \infty} \limsup_{n \to \infty} P\left(\frac{(n/m)^{0.5}}{\sup_{\lambda \in \Pi} |n^{-1} \sum_{h=k+1}^{n-1} \psi_h(\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h))| > \varepsilon}\right) = 0.$$ 

**Proof of Lemma 23.** We borrow the techniques of the proof of Theorem 3.2 in Klüppelberg and Mikosch [19]. Without loss of generality we assume that $k = 2^a - 1$ and $n = 2^{a+1}$ where $a < b$ are integers; if $k$ or $n$ do not have this representation we have to modify the proof slightly but we omit details. For integer $q > 0$ and some constant $\kappa > 0$ to be chosen later, let $\varepsilon_q = 2^{-2q/\kappa}$. We have for $\varepsilon > 0$,

$$Q = P\left(\frac{(n/m)^{0.5}}{\sup_{\lambda \in \Pi} |n^{-1} \sum_{h=k+1}^{n-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \psi_h(\lambda)| > \varepsilon}\right)$$

$$\leq P\left(\frac{(n/m)^{0.5}}{\sup_{q=a} \sum_{q=a}^{b} 2^{q+1-1} \sum_{h=2^q}^{2^{q+1}-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \psi_h(\lambda)| > \varepsilon}\right)$$

$$\leq P\left(\frac{(n/m)^{0.5}}{\sup_{q=a} \sum_{q=a}^{b} 2^{q+1-1} \sum_{h=2^q}^{2^{q+1}-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \psi_h(\lambda)| > \varepsilon_q}\right)$$

$$\leq \sum_{q=a}^{b} P\left(\frac{(n/m)^{0.5}}{\sup_{\lambda \in \Pi} |n^{-1} \sum_{h=2^q}^{2^{q+1}-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \psi_h(\lambda)| > \varepsilon_q}\right) = \sum_{q=a}^{b} Q_q.$$ 

In the last steps we used that $P(\sum_{q=a}^{b} \varepsilon_q > \varepsilon)$ vanishes for fixed $\varepsilon$ and sufficiently large $a$. Next we will bound the expressions $Q_q$. Write $J_{q,v} = \{(v-1)2^q + 1, \ldots, v2^q\}$ and for $j \in J_{q,v}$ and
\[\lambda \in [0, 2^{-2q}\pi],\]
\[Y_{qj}(\lambda) = (n/m)^{0.5} \sum_{h=2^q}^{2^{q+1}-1} (\gamma_A(h) - E\gamma_A(h)) \psi_h(\lambda + (j-1)\pi 2^{-2q}).\]

Then
\[Q_q = P\left((n/m)^{0.5} \max_{v=1,\ldots,2^q} \max_{j' \in J_{q,v}, \lambda \in [(j-1)\pi 2^{-2q}+1, j\pi 2^{-2q}+1]} \left| \sum_{h=2^q}^{2^{q+1}-1} (\gamma_A(h) - E\gamma_A(h)) \psi_h(\lambda) \right| > \varepsilon_q \right) \leq \sum_{v=1}^{2^q} P\left((n/m)^{0.5} \max_{j' \in J_{q,v}, \lambda \in [0, 2^{-2q+1}\pi]} |Y_{qj}(\lambda)| > \varepsilon_q \right) = \sum_{q,v} Q_{qv}.\]

We will bound each of the terms \(Q_{qv}\) by twice applying the maximal inequality of Theorem 10.2 in Billingsley \(\Pi\). For this reason we have to control the variance of the increments of the process \(Y_{qj}\) both as a function of \(\lambda\) and \(j\). In particular, we will derive the following bound
\[(6.1) \quad \frac{n}{m} E\left(\sum_{h=2^q}^{2^{q+1}-1} (\gamma_A(h) - E\gamma_A(h)) d_h(\omega, \lambda, j, j')\right)^2 \leq c |j - j'|^2 |\lambda - \omega|^{2\beta} K_{k,n},\]
where \(\beta\) is the Hölder coefficient of the function \(g\),
\[K_{k,n} \leq c \left[ m \sum_{h=r_n+1}^{\infty} \xi_h + m \sum_{h=k+1}^{r_n} p_h + r_n/m \right] \]
and for \(j < j'\) in \(J_{q,v}, h \in \{2^q, \ldots, 2^{q+1} - 1\}\) and \(\omega < \lambda\) in \([0, 2^{-2q+1}\pi]\),
\[(6.2) \quad d_h(\omega, \lambda, j, j') = \left( \psi_h(\lambda + (j' - 1)\pi 2^{-2q+1}) - \psi_h(\lambda + (j - 1)\pi 2^{-2q+1}) \right) \]
\[\quad - \left( \psi_h(\omega + (j' - 1)\pi 2^{-2q+1}) - \psi_h(\omega + (j - 1)\pi 2^{-2q+1}) \right) \]
\[\quad = \int_{\omega+(j-1)\pi 2^{-2q+1}}^{\omega+(j'+1)\pi 2^{-2q+1}} g(x) \cos(hx) dx - \int_{\omega+(j-1)\pi 2^{-2q+1}}^{\omega+(j'+1)\pi 2^{-2q+1}} g(x) \cos(hx) dx \]
\[\quad = \int_{(j-1)\pi 2^{-2q+1}}^{(j'+1)\pi 2^{-2q+1}} \left( g(x + \lambda) \left[ \cos(h(x + \lambda)) - \cos(h(x + \omega)) \right] \right. \]
\[\left. \quad - g(x + \lambda) - g(x + \omega) \right) \cos(h(x + \omega)) dx.\]

Since \(g\) is \(\beta\)-Hölder continuous we have
\[\left| \int_{(j-1)\pi 2^{-2q+1}}^{(j'+1)\pi 2^{-2q+1}} [g(x + \lambda) - g(x + \omega)] \cos(h(x + x)) dx \right| \leq c(\lambda - \omega)\beta(j' - j)2^{-2q}.\]

Similarly,
\[\left| \int_{(j-1)\pi 2^{-2q}}^{(j'+1)\pi 2^{-2q}} g(x + \lambda) \left[ \cos(h(\lambda + x)) - \cos(h(\omega + x)) \right] dx \right| \]
\[\quad = \left| \int_{(j-1)\pi 2^{-2q}}^{(j'+1)\pi 2^{-2q}} g(x + \lambda) (2 \sin(h(\lambda - \omega)/2) \sin(h(\lambda + \omega + 2x)/2)) dx \right| \]
\[\leq c h(\lambda - \omega)(j' - j)2^{-2q} \leq c(\lambda - \omega)(j' - j)2^{-q}.\]
The last two inequalities yield for a constant $c$ only depending on $g$,

\begin{equation}
|d_h(\omega, \lambda, j, j')| \leq c|\lambda - \omega|^{\beta} |j' - j| 2^{-q}.
\end{equation}

Using this bound, we have

\begin{equation}
\frac{n}{m} E \left( \sum_{h=2^q}^{2^{q+1}-1} (\gamma_A(h) - E\gamma_A(h)) d_h(\omega, \lambda, j, j') \right)^2 \leq c |j - j'|^2 |\lambda - \omega|^{2\beta} 2^{-2q} \frac{n}{m} \sum_{h=2^q}^{2^{q+1}-1} \sum_{s=h}^{2^{q+1}-1} |\text{cov}(\gamma_A(h), \gamma_A(s))|.
\end{equation}

In what follows, it will be convenient to write $\sum_{h,l}^{(q)} = \sum_{h=2^q}^{2^{q+1}-1} \sum_{l=0}^{2^{q+1}-h-1}$. In view of Lemma \[\text{I}\] we can bound the last term in (6.4) as follows:

\begin{align*}
\frac{n}{m} \sum_{h,l}^{(q)} |\text{cov}(\gamma_A(h), \gamma_A(h + l))| \\
= \frac{m}{n} \sum_{h,l}^{(q)} \left( (n - h - l) \Gamma(0, h, 0, h + l) + \sum_{t=1}^{n-h-l-1} (n - h - l - t) \Gamma(0, h, t, t + h + l) \\
+ \sum_{t=1}^{n-h-l-1} \min(n - h - l, n - h - t) \Gamma(0, h + l, t, t + h) \\
- (n - h)(n - h - l)(p_h - p_0)(p_{h+l} - p_0) \right) \\
\leq m \sum_{h,l}^{(q)} \left[ \Gamma(0, h, 0, h + l) + \sum_{t=1}^{h+r_n} \Gamma(0, t, h, t + h + l) + \sum_{t=1}^{h+l+n} \Gamma(0, h + l, t, t + h) \\
+ \frac{1}{n} \sum_{t=h+r_n+1}^{n-h-l-1} (n - t - h - l) \Gamma(0, h, t, t + h + l) \\
+ \sum_{t=h+l+r_n+1}^{n-h-1} (n - t - h) \Gamma(0, h + l, t, t + h) \\
- (n - h)(n - h - l)(p_h - p_0)(p_{h+l} - p_0) \right] \\
= W_1 + W_2 + W_3 + W_4.
\end{align*}

We will treat two cases of interest for the sums $\sum_{h,l}^{(q)}$, when $2^{q+1} - 1 \leq r_n$ and $2^q > r_n$. If $2^q \leq r_n < 2^{q+1} - 1$ the sums $\sum_{h,l}^{(q)}$ can be split into two sums corresponding to $h \leq r_n$ and $h > r_n$ and these can be treated in a similar fashion.

We start by studying the case $2^{q+1} - 1 \leq r_n$. Then $r_n \geq 2^{q+1} - 1 \geq h \geq 2^q > k$ and consequently $2^{q+1} - h - 1 \leq 2^q$. Thus, $W_1 \leq c 2^q m \sum_{h=k+1}^{r_n} p_h$. The terms $W_2, W_3$ have a similar structure and can be treated in the same way; we focus on $W_2$. Then we get the following bound from Lemma \[\text{I}\]

\begin{equation}
W_2 \leq c 2^q \left[ m \sum_{h=k+1}^{r_n} p_h + m \sum_{h=r_n+1}^{2r_n} \xi_h + (r_n/m) \right].
\end{equation}
In view of \([2.4]\), we also have

\[
W_4 \leq m \sum_{n=1}^{m} \sum_{h=2}^{m} \sum_{l=0}^{h} \left( \sum_{t=0}^{n-1} (n-t-h-l) \right) \left( \Gamma(0, h, t, t + h + l) - (p_{h} - p_{0}^2)(p_{h+l} - p_{0}^2) \right)
\]

\[
+ \sum_{n=1}^{m} \sum_{l=0}^{h} (n-t-h-l) \left( \Gamma(0, h, t, t + h + l) - (p_{h} - p_{0}^2)(p_{h+l} - p_{0}^2) \right)
\]

\[
+ c^2 m \left( |p_{h} - p_{0}^2| (p_{h+l} - p_{0}^2) \right)
\]

\[
\leq c 2^{q} m \sum_{n=1}^{m} \xi_h + c 2^{q} \frac{m}{m} \left( m \sum_{n=1}^{m} \xi_h \right)^2.
\]

Next we assume that \(2^q > r_n\). By \([2.3]\) and \([2.4]\),

\[
W_1 \leq m \sum_{n=2}^{r_n} \sum_{l=0}^{h} \left( \sum_{t=0}^{n-1} (n-t-h-l) \right) \left( \Gamma(0, h, t, t + h + l) - (p_{h} - p_{0}^2)(p_{h+l} - p_{0}^2) \right)
\]

\[
+ m \sum_{n=2}^{r_n} \sum_{l=1}^{h} \left( \sum_{t=0}^{n-1} (n-t-h-l) \right) \left( \Gamma(0, h, t, t + h + l) - (p_{h} - p_{0}^2)(p_{h+l} - p_{0}^2) \right)
\]

\[
\leq c 2^{q} m \sum_{h=1}^{r_n} \xi_h + c 2^{q} \frac{r_n}{m} (mp_0)^2.
\]

We again focus on \(W_2\); \(W_3\) can be treated in a similar way.

\[
W_2 \leq m \sum_{n=2}^{r_n} \sum_{l=0}^{h} \left( \sum_{t=0}^{n-1} (n-t-h-l) \right) \left( \Gamma(0, h, t, t + h + l) - (p_{h} - p_{0}^2)(p_{h+l} - p_{0}^2) \right)
\]

\[
\leq c 2^{q} m \sum_{h=1}^{r_n} \xi_h + c 2^{q} \frac{r_n}{m} (mp_0)^2
\]

To obtain the bounds for \(W_4\) we use \([2.4]\):

\[
W_4 \leq m \sum_{n=2}^{r_n} \sum_{l=0}^{h} \left( \sum_{t=0}^{n-1} (n-t-h-l) \right) \left( \Gamma(0, h, t, t + h + l) - (p_{h} - p_{0}^2)(p_{h+l} - p_{0}^2) \right)
\]

\[
+ m \sum_{n=1}^{m} \sum_{l=0}^{h} (n-t-h-l) \left( \Gamma(0, h, t, t + h + l) - (p_{h} - p_{0}^2)(p_{h+l} - p_{0}^2) \right)
\]

\[
+ c 2^{q} \left( |p_{h} - p_{0}^2| (p_{h+l} - p_{0}^2) \right)
\]

\[
\leq c 2^{q} m \sum_{h=1}^{r_n} \xi_h + c 2^{q} \frac{r_n}{m} \left( m \sum_{h=1}^{r_n} \xi_h \right)^2.
\]

Collecting the bounds for \(W_i\), \(i \leq 4\), and using \([6.4]\), we finally proved \([6.1]\).

Using this bound, we can apply the maximal inequality of Theorem 10.2 in Billingsley \([1]\) with respect to the variable \(\lambda \leq 2^{-2q} \pi\) and for fixed \(j, j'\):

\[
P\left( \max_{0 \leq \lambda \leq 2^{-2q} \pi} |Y_j(\lambda) - Y_{j'}(\lambda)| > \varepsilon \right) \leq c \varepsilon^{-2} (2^{-2q} \pi)^{2\beta} (j - j')^2 K_{k,n}
\]

\[
\leq c 2^{q} (1 - \beta + \kappa^{-1}) (j - j')^2 K_{k,n}.
\]
Another application of this maximal inequality to \( \max_{0 \leq \lambda \leq 2^{-2q\pi}} |Y_j(\lambda)| \) with respect to the variable \( j \in J_{q,v} \) yields

\[
Q_{qv} = P\left( \max_{j \in \{(v-1)2^{q+1},\ldots,v2^q\}} \max_{0 \leq \lambda \leq 2^{-2q\pi}} |Y_j(\lambda)| > \varepsilon_q \right) \leq c 2^{4q(2^{-1} - \beta + \kappa - 1)} K_{k,n}.
\]

Then we also have

\[
Q_q \leq \sum_{v=1}^{2^q} Q_{qv} \leq c 2^{4q(3/4 - \beta + \kappa - 1)} K_{k,n}.
\]

The right-hand side converges to zero as \( q \to \infty \) provided \( \beta \in (3/4, 1] \) and \( \kappa \) is chosen sufficiently large. Therefore we conclude for every \( \varepsilon > 0 \),

\[
Q \leq b \sum_{q=a}^{b} Q_q \leq c K_{k,n} \sum_{q=a}^{\infty} 2^{4q(3/4 - \beta + \kappa - 1)}.
\]

The right-hand side converges to zero by first letting \( n \to \infty \) and then \( k \to \infty \). This concludes the proof of (6.3).

Next we turn to the proof of (6.3). It will follow from (6.3) once we prove the following lemma.

**Lemma 24.** Assume that the conditions of Theorem 15 hold. Then for any \( \varepsilon > 0 \), as \( n \to \infty \),

\[
P\left( (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \left( \hat{J}_{n,A}(\lambda) - E\hat{J}_{n,A}(\lambda) \right) - \left( J_{n,A}(\lambda) - EJ_{n,A}(\lambda) \right) \right| > \varepsilon \right) \to 0.
\]

**Proof of Lemma 24.** For any fixed \( k \geq 1 \) we have

\[
P\left( (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \left( \hat{J}_{n,A}(\lambda) - E\hat{J}_{n,A}(\lambda) \right) - \left( J_{n,A}(\lambda) - EJ_{n,A}(\lambda) \right) \right| > \varepsilon \right)
\leq P\left( (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=0}^{k} (\hat{\gamma}_A(h) - E\hat{\gamma}_A(h))(\psi_h(\lambda) - \hat{\psi}_h(\lambda)) \right| > \varepsilon/3 \right)
+ P\left( (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n} (\hat{\gamma}_A(h) - E\hat{\gamma}_A(h))\psi_h(\lambda) \right| > \varepsilon/3 \right)
+ P\left( (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n} (\hat{\gamma}_A(h) - E\hat{\gamma}_A(h))\hat{\psi}_h(\lambda) \right| > \varepsilon/3 \right)
= V_1 + V_2 + V_3.
\]

An application of Chebyshev’s and Hölder’s inequalities yields,

\[
V_1 \leq 9 \varepsilon^{-2} \frac{n}{m} E \sup_{\lambda \in \Pi} \left| \sum_{h=0}^{k} (\hat{\gamma}_A(h) - E\hat{\gamma}_A(h))(\psi_h(\lambda) - \hat{\psi}_h(\lambda)) \right|^2
\leq c \frac{n}{m} E \sup_{\lambda \in \Pi} \sum_{h=0}^{k} (\hat{\gamma}_A(h) - E\hat{\gamma}_A(h))^2 \left| \psi_h(\lambda) - \hat{\psi}_h(\lambda) \right| \sum_{s=0}^{k} \left| \psi_s(\lambda) - \hat{\psi}_s(\lambda) \right|
\leq c k \frac{n}{m} \sum_{h=0}^{k} \text{var}(\hat{\gamma}_A(h)) \sup_{x \in \Pi} \left| \psi_h(x) - \hat{\psi}_h(x) \right|.
\]

Next we will study \( \sup_{\lambda \in \Pi} |\psi_h(\lambda) - \hat{\psi}_h(\lambda)| \). Trivially, for \( x \in \Pi \),

\[
\left| \int_{\omega_n(x_n)}^{x} \cos(h(\lambda)) g(\lambda) \, d\lambda \right| \leq c/n,
\]
where the constant $c$ only depends on $g$. We also have for the frequencies $x \in \Pi$,

\[
\begin{align*}
|\hat{\psi}_h(\omega_n(x_n)) - \hat{\psi}_h(\omega_n(x_n))| &= \left| \sum_{i=1}^{x_n} \left( \int_{\omega_n(i-1)}^{\omega_n(i)} \cos(h\lambda) g(\lambda) d\lambda - \omega_n(1) \cos(h\omega_n(i)) g(\omega_n(i)) \right) \right| \\
&\leq \sum_{i=1}^{x_n} \left| \int_{\omega_n(i-1)}^{\omega_n(i)} \cos(h\lambda) (g(\lambda) - g(\omega_n(i))) d\lambda \right| \\
&\quad + \sum_{i=1}^{x_n} g(\omega_n(i)) \left( \frac{\sin(h\omega_n(i)) - \sin(h\omega_n(i-1))}{h} - \omega_n(1) \cos(h\omega_n(i)) \right) \right|.
\end{align*}
\]

(6.6)

Since $g$ is $\beta$-Hölder continuous there exists a constant $c > 0$ such that

\[ |g(\lambda) - g(\omega_n(i))| \leq cn^{-\beta}, \quad \lambda \in [\omega_n(i-1), \omega_n(i)]. \]

Therefore the term in $|c|n^{-\beta}$. A Taylor expansion as $z \to 0$ yields $\sin(z) = z - z^3/3! + o(z^3)$. Then we have for $h \leq n$,

\[
\begin{align*}
\left| \frac{\sin(h\omega_n(i)) - \sin(h\omega_n(i-1))}{h} - \omega_n(1) \cos(h\omega_n(i)) \right| &= 2h^{-1} \sin(h\omega_n(0.5)) \cos(h\theta(i + 0.5)) - \omega_n(1) \cos(h\omega_n(i)) \\
&= 2h^{-1} \sin(h\omega_n(0.5)) - \omega_n(0.5) \cos(h\theta(i + 0.5)) - \omega_n(0.5) \cos(h\omega_n(i)) \\
&\quad + \omega_n(1) (\cos(h\omega_n(i + 0.5)) - \cos(h\omega_n(i))) \\
&\leq c h\omega_n(1)^3 + \omega_n(1) \left| 2 \sin(h\omega_n(0.25)) \sin(h\omega_n(i + 0.25)) \right| \leq c (h^3 n^{-3} + h n^{-2}).
\end{align*}
\]

(6.7)

Consequently, we have the bound $c(k/n)(1 + k^2/n)$ for (6.7) uniformly for $x \in \Pi$ and $h \leq k$, Thus, uniformly for $h \leq k$,

\[
\sup_{x \in \Pi} |\hat{\psi}_h(x) - \hat{\psi}_h(x)| \leq c \left[ n^{-\beta} + (k/n)(1 + k^2/n) \right].
\]

As we have shown in Lemma 31 $(n/m) \sum_{h=0}^{k} \var{\hat{\gamma}_\lambda(h)} \leq c k$; see also Davis and Mikosch 7. Lemma 5.2. Thus, as $n \to \infty$,

\[
V_1 \leq c k^2 n^{-\beta} + (k^3/n)(1 + k^2/n) \to 0.
\]

It follows from Lemma 23 that $\lim_{k \to \infty} \limsup_{n \to \infty} V_2 = 0$. We adapt the proof of Lemma 23 for the case $V_3$. Abusing notation, consider

\[
d_h(\omega, \lambda, j, j') = (\hat{\psi}_h(\lambda + (j' - 1)\pi 2^{-2q+1}) - \hat{\psi}_h(\lambda + (j - 1)\pi 2^{-2q+1}))
\]

\[
- (\hat{\psi}_h(\omega + (j' - 1)\pi 2^{-2q+1}) - \hat{\psi}_h(\omega + (j - 1)\pi 2^{-2q+1})).
\]

Recall that we assume $n = 2^b$ for some integer $b$ and $x_n = [nx/(2\pi)]$. Therefore for $\lambda \in \Pi$ and integer $j$,

\[
(\lambda + (j - 1)\pi 2^{-2q+1})_n = [n\lambda/(2\pi) + (j - 1)2^{-2q+b}]
\]

\[
= [n\lambda/(2\pi)] + (j - 1)2^{-2q+b}.
\]

\[
\lambda_n + (j - 1)2^{-2q+b}.
\]
Thus we can write

\[ d_h(\omega, \lambda, j, j') = \frac{2\pi}{n} \sum_{i=\lambda_n+(j'-1)2^{b-2q}}^{\lambda_n+j-1} g(\omega_n(i)) \cos(h\omega_n(i)) \]

\[ - \frac{2\pi}{n} \sum_{i=\omega_n+(j-1)2^{b-2q}}^{\omega_n+j-1} g(\omega_n(i)) \cos(h\omega_n(i)) \]

\[ = \frac{2\pi}{n} \sum_{i=(j-1)2^{b-2q}}^{(j'-1)2^{b-2q}} \left( g(\omega_n(\lambda_n + i)) \cos(h\omega_n(\lambda_n + i)) - \cos(h\omega_n(\omega_n + i)) \right) \cos(h\omega_n(\omega_n + i)) = T_1 + T_2. \]

Calculation yields

\[ |T_1| \leq c |\omega_n(\lambda_n) - \omega_n(\omega_n)| |j' - j| 2^{-2q} |2^q \leq c |(\lambda_n - \omega_n)/n| |j' - j| 2^{-2q} |2^q, \]

\[ |T_2| \leq c |\omega_n(\lambda_n) - \omega_n(\omega_n)| |j' - j| 2^{-2q} \leq c |(\lambda_n - \omega_n)/n| |j' - j| 2^{-2q}. \]

Combining these bounds, we have,

\[ |d_h(\omega, \lambda, j, j')| \leq c |(\lambda_n - \omega_n)/n| |j' - j| 2^{-2q} |2^q. \]

In the remaining argument we can follow the proof of Lemma 23, the only difference is that we have to replace the supremum over \( \lambda, \omega \in [0, j 2^{-2q+1}] \) by the corresponding quantities \( \lambda_n/n, \omega_n/n \in [0, j 2^{-2q+1}] \). This proves \( \lim_{k \to \infty} \limsup_{n \to \infty} V_3 = 0 \) and concludes the proof of the lemma.

The proofs of (3.6) and (3.7) are completely analogous. Instead of the relations (2.8) one has to use (2.10).

## 7. Proof of Theorem 17

We adapt the proof of Theorem 15. We need to prove that

\[ n \sum_{h,l}^{(g)} |\text{cov}(\tilde{\gamma}_A(h), \tilde{\gamma}_A(h + l))| \leq c 2^q. \]
Lemma 25. Under the conditions and with the notation of Theorem 20, for extremogram \( \hat{\gamma}_A(h) \),

Proof. We start by observing (see Lemma 27) that for the central limit theory of Lemma 6.

In the above calculation, we use the facts that for \( \hat{\gamma}_A(h) \) defined in the proof of Lemma 23. Here \( h > \eta \).

In the remaining argument we can follow the proof of Theorem 15; instead of Lemma 4 we use the central limit theory of Lemma 6.

\[ \sum_{h=2}^{q} | \Gamma(0, 0, h, h) | + m^2 \sum_{h=2}^{q} \sum_{l=1}^{\eta} | \Gamma(0, 0, h, h + l) | \]

\[ + m^2 \sum_{h=2}^{q} \sum_{l=1}^{\eta} | \Gamma(0, h, h + l) + m^2 \sum_{h=2}^{q} \sum_{l=1}^{\eta} | \Gamma(0, h + l, h + l) | \]

\[ \leq c^2 \]

In the above calculation, we use the facts that for \( s \leq t \leq u \), \( \Gamma(s, t, u, v) = 0 \) where \( t - s > \eta \) or \( v - u > \eta \).

In the remaining argument we can follow the proof of Theorem 15 instead of Lemma 4.

8. Proof of Theorem 21

We will mimic the proof of Theorem 15. We start by proving a result for the bootstrapped sample extremogram \( \hat{\gamma}_A^* \) analogous to Theorem 20.

Lemma 25. Under the conditions and with the notation of Theorem 20 for \( h \geq 0 \),

\[ d \left( \left( n/m \right)^{0.5} \left( \hat{\gamma}_A^*(i) - E^* \hat{\gamma}_A^*(i) \right) \right)_{i=0, \ldots, h} \xrightarrow{P} 0, \quad n \to \infty. \]

Proof. We start by observing (see Lemma 27) that for \( h \geq 0 \)

\[ E^* \hat{\gamma}_A^*(h) = \frac{m}{n} (n - h) E^* \hat{I}_1, \hat{I}_{1+h} \]

\[ = (1 - h/n) \left[ \hat{\gamma}_A(h) + \frac{m}{n} \sum_{t=n-h+1}^{n} \hat{I}_t \hat{I}_{t+h} \right], \]

\[ E^* \hat{\gamma}_A^*(h) = \frac{m}{n} (n - h) E^* \hat{I}_1, \hat{I}_{(1+h)} \]

\[ = (1 - h/n) (1 - \theta)^h \left[ \hat{\gamma}_A(h) + \frac{m}{n} \sum_{t=n-h+1}^{n} \hat{I}_t \hat{I}_{t+h} \right], \]

where we interpret indices larger than \( n \) modulo \( n \), and therefore

\[ (n/m)^{0.5} \left[ (1 - \theta)^h E^* \hat{\gamma}_A^*(h) - E^* \hat{\gamma}_A^*(h) \right] = O_P(m^{-1}) \xrightarrow{P} 0, \]

(8.1)
In terms with normalization structure of the covariances in Lemma 27. The expressions for the covariances in Lemma 27 contain 

\[ Q \]

omit the details for 

\[ P \]

Markov’s inequality ensures that it suffices to prove that

\[ \frac{n}{m} \operatorname{var}^* ((1 - \theta)^h \gamma^*_A(h) - \gamma^*_A(h)) \] \( \overset{P}{\rightarrow} 0 \), \( n \to \infty \).

We observe that

\[ \frac{n}{m} \operatorname{var}^* ((1 - \theta)^h \gamma^*_A(h) - \gamma^*_A(h)) \]

\[ = m \left( 1 - \frac{h}{n} \right) \operatorname{var}^* (\tilde{I}_i \tilde{I}_{(1+h)^*} - (1 - \theta)^h \tilde{I}_i \tilde{I}_{1+h} ) \]

\[ = m \left( 1 - \frac{h}{n} \right) \operatorname{var}^* (\tilde{I}_i \tilde{I}_{(1+h)^*} - (1 - \theta)^h \tilde{I}_i \tilde{I}_{1+h} ) \]

\[ + 2m \sum_{s=1}^{n-h-1} \left( 1 - \frac{h + s}{n} \right) \left[ \operatorname{cov}^* (\tilde{I}_i \tilde{I}_{(1+h)^*} , \tilde{I}_{(1+s+h)^*} ) + (1 - \theta)^h \gamma^*_A(h) - \gamma^*_A(h) \right] \]

\[ = m \left( 1 - \frac{h}{n} \right) \operatorname{var}^* (\tilde{I}_i \tilde{I}_{(1+h)^*} - (1 - \theta)^h \tilde{I}_i \tilde{I}_{1+h} ) \]

\[ + 2m \sum_{s=1}^{n-h-1} \left( 1 - \frac{h + s}{n} \right) \left[ \operatorname{cov}^* (\tilde{I}_i \tilde{I}_{(1+h)^*} , \tilde{I}_{(1+s+h)^*} ) \right] \]

\[ + (1 - \theta)^2 \gamma^*_A(h) - \gamma^*_A(h) \]

\[ = Q_1 + Q_2. \]

We will show that the right-hand side converges to zero in \( P \)-probability, where we focus on \( Q_2 \) and omit the details for \( Q_1 \). We start by looking at the summands in \( Q_2 \) for fixed \( s \leq h \), using the structure of the covariances in Lemma 27. The expressions for the covariances in Lemma 27 contain terms with normalization \( n^{-2} \). For example, by (S.7) a corresponding term in \( Q_2 \) is of the order

\[ m \left( n^{-1} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \right)^2 = m^{-1} \left( \frac{m^2}{n} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \right)^2 = O_P(m^{-1}), \]

since \( \frac{m}{n} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \overset{P}{\to} \gamma_A(h) \); see Lemma 3. In the latter sums, the \( \tilde{I}_i \)'s can be exchanged by the \( I_i \)'s or the \( \tilde{I}_i \)'s. Therefore all other terms in \( Q_2 \) with normalization \( m n^{-2} \) converge to zero in \( P \)-probability. Another appeal to Lemma 27 shows that it remains to consider those expressions in \( Q_2 \) that are normalized by \( m n^{-1} \) again for fixed \( s \leq h \). From (S.9) and (S.10) we see that, on one hand, we have to deal with the differences

\[ (1 - \theta)^{s+h} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+s+h} \tilde{I}_{i+s+h} - (1 - \theta)^{s+2h} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \tilde{I}_{i+s+h}, \]

but both sums are consistent estimators of \( \lim_{n \to \infty} m P (a_{m}^{-1} X_0 \in A, a_{m}^{-1} X_s \in A, a_{m}^{-1} X_h \in A, a_{m}^{-1} X_{s+h} \in A) \) (see [2], Theorem 3.1). Therefore (S.2) converges to zero in \( P \)-probability. On the other hand, in view of (S.7) and (S.8) we have to deal with the differences, for \( s \leq h \),

\[ (1 - \theta)^{s+2h} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+s+h} \tilde{I}_{i+s+h} - (1 - \theta)^{2h} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \tilde{I}_{i+s+h}, \]

which again converge to zero in \( P \)-probability. These arguments finish the proof for \( s \leq h \).
An inspection of the covariances in Lemma 27 shows that for \( s > h \) all expressions with normalization \( n^{-2} \) do not depend on \( s \). The corresponding aggregated terms in \( Q_2 \) are then given by

\[
2m \sum_{s=h+1}^{n-h-1} \left( 1 - \frac{h+s}{n} \right) \left[ (1 - \theta)^{s+h} \left( n^{-1} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \right)^2 + (1 - \theta)^{s+h} \left( n^{-1} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \right) \right] + \frac{1}{n} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \right) \sum_{s=h+1}^{n-h-1} \left( 1 - \frac{h+s}{n} \right) (1 - \theta)^{s+h}
\]

\[
-2m^{-1} \left( \frac{m}{n} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} - \frac{m}{n} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \right) \left( \frac{m}{n} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \right) \sum_{s=h+1}^{n-h-1} \left( 1 - \frac{h+s}{n} \right) (1 - \theta)^{s+2h}
\]

\[
O_p(1/(\theta \sqrt{mn})) = o_P(1)
\]

In the last step we used \( \text{[8.1]} \) and the assumption \( n\theta^2/m \to \infty \). Finally, we deal with the remaining terms in \( Q_2 \). In view of Lemma 27 they are given by

\[
2m \sum_{s=h+1}^{n-h-1} \left( 1 - \frac{h+s}{n} \right) \left[ (1 - \theta)^{s+h} \left( n^{-1} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \tilde{I}_{i+s+h} \right) - (1 - \theta)^{s+2h} n^{-1} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \tilde{I}_{i+s+h} \right]
\]

\[
= \sum_{s=h+1}^{n-h-1} \left( 1 - \frac{h+s}{n} \right) \left( \frac{m}{n} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \tilde{I}_{i+s+h} \right) \frac{m}{n} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \tilde{I}_{i+s+h} \right)
\]

Using the assumption \( n\theta^2/m \to \infty \), we have

\[
E|J_0| \leq c m \sum_{s=h+1}^{n-h-1} \left( 1 - \theta \right)^{s+h} E|\tilde{I}_0 \tilde{I}_h - \tilde{I}_0 \tilde{I}_h|
\]

\[
\leq c m E|\tilde{I}_0 - \tilde{I}_n| \sum_{s=h+1}^{n-h-1} \left( 1 - \theta \right)^{s+h} \leq c (m/n)^{0.5} \theta^{-1} = o(1)
\]

This finishes the proof of the lemma. \( \square \)

We conclude from Lemma 25 that for any \( k \geq 1 \), as \( n \to \infty \),

\[
d \left( \frac{n}{m} \right)^{0.5} \left( \psi_0 (\gamma^*_A(0) - E^* \gamma^*_A(0)) + 2 \sum_{h=1}^{k} \psi_h (\gamma^*_A(h) - E^* \gamma^*_A(h)) \right),
\]

\[
\psi_0 Z_0 + 2 \sum_{h=1}^{k} \psi_h Z_h \overset{p}{\to} 0.
\]
where the dependence structure of \((Z_h)\) is defined in Lemma 4.

The proof of the theorem is finished by the following result which parallels Lemma 23.

**Lemma 26.** Assume the conditions of Theorem 21. Then the following relation holds for \(\delta > 0\)

\[
\lim_{k \to \infty} \lim_{n \to \infty} \sup_{\lambda \in \Pi} P \left( \sup_{h=k+1}^{n-1} \psi_h(\lambda) \left( \bar{\gamma}_\Lambda^*(h) - E^* \bar{\gamma}_\Lambda^*(h) \right) > \delta \right) = 0 .
\]

(8.3)

**Proof.** We follow the lines of the proof of Lemma 23 and use the same notation. We again assume without loss of generality that \(k = 2^a - 1\) and \(n = 2^{b+1} \) for integers \(a < b\), \(a\) chosen sufficiently large, and we write \(\varepsilon_q = 2^{-2q/\kappa}\) for \(\kappa > 0\) to be chosen later. Then, for large \(a\) depending on \(\varepsilon > 0\), the steps of the proof lead to the inequality (cf. (6.5))

\[
Q^* = P^* \left( (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n-1} (\bar{\gamma}_\Lambda^*(h) - E^* \bar{\gamma}_\Lambda^*(h)) \psi_h(\lambda) \right| > \varepsilon \right)
\]

\[
\leq c \sum_{q=a}^{b} 2^{4q(0.75 - \beta + \kappa^{-1})} K_q,
\]

where \(\beta \in (3/4, 1]\) is the H"older coefficient of the function \(g\), the number \(\kappa > 0\) can be chosen arbitrarily large and

\[
K_q = \frac{n}{m} \sum_{h=2^q}^{2^{q+1} - 1} \sum_{s=h}^{2^{q+1} - 1} |\text{cov}^*(\bar{\gamma}_\Lambda^*(h), \bar{\gamma}_\Lambda^*(s))| .
\]

By the Cauchy-Schwarz inequality, for \(s, h \in [2^q, 2^{q+1})\) and \(h \leq s\),

\[
(n/m)^2 |\text{cov}^*(\bar{\gamma}_\Lambda^*(h), \bar{\gamma}_\Lambda^*(s))|^2 \leq (n/m) \text{var}^*(\bar{\gamma}_\Lambda^*(h)) (n/m) \text{var}^*(\bar{\gamma}_\Lambda^*(s)) .
\]

We will show that

\[
(8.4) \quad (n/m) \text{var}^*(\bar{\gamma}_\Lambda^*(h)) \leq c
\]

for some constant \(c\), uniformly for \(k \leq h \leq n\) and \(n\). Then

\[
EQ^* \leq c \sum_{q=a}^{b} 2^{4q(3/4 - \beta + \kappa^{-1})} \leq c \sum_{q=a}^{\infty} 2^{4q(3/4 - \beta + \kappa^{-1})} .
\]

The right-hand side converges since \(\beta \in (3/4, 1]\) and \(\kappa\) can be chosen arbitrarily large. Moreover, the right-hand side converges to zero as \(k \to \infty\).
Thus it remains to show (8.4). In view of Lemma 27 we have

\[(n/m)\text{var}(\tilde{\theta}_h^*) = \frac{(m/n)(n-h)\text{var}(\tilde{I}_1, \tilde{I}_{(1+h)})}{2} \]

\[+ 2 \sum_{t=1}^{n-h-1} (n-h-t) \text{E} \text{cov}(\tilde{I}_1 \tilde{I}_{(1+h)}, \tilde{I}_{(1+t)} \tilde{I}_{(1+t+h)}) \]

\[= \left[ m(1-h/n)(1-\theta)^{2h} \left( E(\tilde{I}_1 \tilde{I}_{(1+h)})^2 - E \left( n^{-1} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{(1+h)} \right)^2 \right) \right] \]

\[+ 2m \sum_{t=1}^{n-h-1} \left( 1 - (h+t)/n \right) \left[ n^{-1} \sum_{i=1}^{n} E(\tilde{I}_1 \tilde{I}_{(1+h)} \tilde{I}_{(1+t)} \tilde{I}_{(1+t+h)}) \right] (1-\theta)^{t+h} \]

\[+ 2m \sum_{t=1}^{n-h-1} \left( 1 - (h+t)/n \right) E \left( n^{-1} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{(1+t)} \right)^2 \left( (1-\theta)^{2t} - (1-\theta)^{t+h} \right) \]

\[- 2m \sum_{t=1}^{n-h} \left( 1 - (h+t)/n \right) E \left( n^{-1} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{(1+h)} \right)^2 (1-\theta)^{t+h} \]

\[\leq m E(\tilde{I}_1 \tilde{I}_{(1+h)})^2 + 2m \sum_{t=1}^{n-h-1} \left( 1 - (h+t)/n \right) \left( n^{-1} \sum_{i=1}^{n} E(\tilde{I}_1 \tilde{I}_{(1+h)} \tilde{I}_{(1+t)} \tilde{I}_{(1+t+h)}) \right) (1-\theta)^{t+h} \]

\[+ 2m \sum_{t=1}^{n-h-1} \left( 1 - (h+t)/n \right) E \left( n^{-1} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{(1+t)} \right)^2 (1-\theta)^{2t} \]

\[= V_1 + V_2 + V_3. \]

We observe that, for some constant \(c_0 > 0\),

\[V_1 \leq m E(\tilde{I}_1 \tilde{I}_{(1+h)})^2 \leq cm \left[ E \text{I}_1 \text{I}_{1+h} + (E \text{I}_n)^2 \right] \leq cm \eta_0 \leq c_0. \]

For \(V_2\), we observe that for \(i \leq n\),

\[mn^{-1} \left| E \left[ \tilde{I}_1 \tilde{I}_{(1+h)} \tilde{I}_{(1+t)} \tilde{I}_{(1+t+h)} - \tilde{I}_1 \tilde{I}_{(1+h)} \tilde{I}_{(1+t)} \tilde{I}_{(1+t+h)} \right] \right| \]

\[\leq cm \eta^{-1} \left| E \left( \text{I}_1 \text{I}_n - \eta_0 \right) \right| = O \left( \sqrt{m/n} \eta^{-1} \right) = o(1), \]

by virtue of the condition \(n \theta^2/m \to \infty\). Therefore, for showing that \(|V_2| \leq c\) uniformly for \(h, n\), it suffices to show that \(|V_2| \leq c\), where \(V_2\) is obtained from \(V_2\) by replacing the \(\tilde{I}_i\)’s by the corresponding \(\text{I}_i\)’s. Taking into account \(E \tilde{I} \tilde{I}_{1+t} = p_t - p_0^2\) and the Cauchy-Schwarz inequality, we have for a fixed integer \(M > 0\),

\[|\tilde{V}_2| \leq c m \sum_{t=1}^{n-h-1} \left| n^{-1} \sum_{i=1}^{n} E \tilde{I}_1 \tilde{I}_{(1+h)} \tilde{I}_{(1+t)} \tilde{I}_{(1+t+h)} \right| \]

\[= c m \sum_{t=1}^{n-h-1} \left| E \tilde{I}_1 \tilde{I}_{(1+h)} \tilde{I}_{(1+t)} \tilde{I}_{(1+t+h)} \right| \]

\[\leq (m \eta_0)M + cm \sum_{t=M+1}^{\infty} (p_t + p_0^2) + cm \sum_{t=r_n+1}^{\infty} \xi_t \leq c, \]
in view of condition (M) and regular variation. A similar argument as for $V_2$ shows that one may replace the $\tilde{I}_i$’s in $V_3$ by the corresponding $\tilde{I}'_i$’s. We denote the resulting quantity by $\tilde{V}_3$. Then we have

$$
\tilde{V}_3 \leq m \sum_{t=1}^{n} (1 - \theta)^t E \left( n^{-1} \sum_{i=1}^{n-t} \tilde{I}_i \tilde{I}_{i+t} + n^{-1} \sum_{i=n-t+1}^{n} \tilde{I}_i \tilde{I}_{i+t-n} \right)^2,
$$

$$
\leq cm \sum_{t=1}^{n} (1 - \theta)^t E \left( n^{-1} \sum_{i=1}^{n-t} \tilde{I}_i \tilde{I}_{i+t} \right)^2 + cm \sum_{t=1}^{n} (1 - \theta)^t E \left( n^{-1} \sum_{i=n-t+1}^{n} \tilde{I}_i \tilde{I}_{i+t-n} \right)^2,
$$

$$
= \tilde{V}_{31} + \tilde{V}_{32}.
$$

We will only deal with $\tilde{V}_{31}$, the other term can be bounded in a similar way. We observe that for fixed $M > 1$, using condition (M),

$$
\tilde{V}_{31} \leq c \frac{m}{n} \sum_{t=1}^{n} (1 - \theta)^t \left( E(\tilde{I}_1 \tilde{I}_{1+t})^2 + 2 \sum_{s=1}^{n-t-1} |E\tilde{I}_s \tilde{I}_{s+t} \tilde{I}_{s+t+s}| \right),
$$

$$
\leq o(1) + \frac{cm}{n} \sum_{t=1}^{n} (1 - \theta)^t \sum_{s=1}^{n-t-1} |E\tilde{I}_s \tilde{I}_{s+t} \tilde{I}_{s+t+s}|,
$$

$$
\leq o(1) + \frac{cm}{n} \sum_{t=1}^{n} (1 - \theta)^t \sum_{s=M+1}^{n-t-1} (p_s + p_0^2) + c \frac{m}{n} \sum_{t=1}^{n} (1 - \theta)^t \sum_{r_n+1 \leq s \leq n-t-1, s \geq t} \xi_s,
$$

$$
+ c \frac{m}{n} \sum_{t=1}^{n} (1 - \theta)^t \sum_{r_n+1 \leq s \leq n-t-1, s > t} \left( |E\tilde{I}_s \tilde{I}_{s+t} \tilde{I}_{s+t+s} - (p_t - p_0^2)| + (p_t - p_0^2)^2 \right).
$$

In view of condition (M), the first two terms on the right-hand side are negligible as $n \to \infty$. The third term is bounded by

$$
c \frac{m}{n} \sum_{t=1}^{n} (1 - \theta)^t \sum_{r_n+1 \leq s \leq n-t-1, s > t} \xi_{s-t} + cm \sum_{t=1}^{n} (1 - \theta)^t (p_t - p_0^2)^2.
$$

Multiple use of (M) again shows that the right-hand side is negligible. This proves \textbf{(8.4)}. □ □
Lemma 27. Under the conditions of Theorem 27, the following relations hold for $s, h \geq 0$.

(8.5) \[ E^* \hat{I}_s = 0, \]

(8.6) \[ E^* \hat{I}_s \hat{I}_t = (1 - \theta)^h n^{-1} \sum_{i=1}^{n} \hat{I}_i \hat{I}_{i+h}, \quad E^* \hat{I}_s \hat{I}_{t+s} = n^{-1} \sum_{i=1}^{n} \hat{I}_i \hat{I}_{i+s}, \]

(8.7) \[ \text{cov}^*(\hat{I}_s \hat{I}_{t+h}, \hat{I}_{(1+s)} \hat{I}_{(1+s)+h}) = (1 - \theta)^{h+s} \left( n^{-1} \sum_{i=1}^{n} \hat{I}_i \hat{I}_{i+s} \hat{I}_{i+h} \hat{I}_{i+s+h} - \left( n^{-1} \sum_{i=1}^{n} \hat{I}_i \hat{I}_{i+h} \right) \left( n^{-1} \sum_{i=1}^{n} \hat{I}_i \hat{I}_{i+h} \right) \right), \]

(8.8) \[ \text{cov}^*(\hat{I}_s \hat{I}_{t+h}, \hat{I}_{(1+s)} \hat{I}_{(1+s)+h}) = (1 - \theta)^{s} \left( n^{-1} \sum_{i=1}^{n} \hat{I}_i \hat{I}_{i+s} \hat{I}_{i+h} \hat{I}_{i+s+h} - \left( n^{-1} \sum_{i=1}^{n} \hat{I}_i \hat{I}_{i+h} \right) \left( n^{-1} \sum_{i=1}^{n} \hat{I}_i \hat{I}_{i+h} \right) \right), \]

(8.9) \[ \text{cov}^*(\hat{I}_s \hat{I}_{t+h}, \hat{I}_{(1+s)} \hat{I}_{(1+s)+h}) = (1 - \theta)^{s+1} \left( n^{-1} \sum_{i=1}^{n} \hat{I}_i \hat{I}_{i+s} \hat{I}_{i+h} \hat{I}_{i+s+h} - \left( n^{-1} \sum_{i=1}^{n} \hat{I}_i \hat{I}_{i+h} \right) \left( n^{-1} \sum_{i=1}^{n} \hat{I}_i \hat{I}_{i+h} \right) \right) \]

Proof. Relations (8.5) and (8.6) follow from the defining properties of the stationary bootstrap; see Politis and Romano [29].

We will only show that (8.10) holds; (8.7)–(8.9) can be proved in a similar (and even simpler) way but we omit further details. First assume $s < h$. Recall $L_1$ from the construction of the stationary bootstrap scheme. Consider the following decomposition

\[ E^*[\hat{I}_s \hat{I}_{(1+h)} \hat{I}_{(1+s)} \hat{I}_{(1+s)+h}] = E^*[\hat{I}_s \hat{I}_{(1+h)} \hat{I}_{(1+s)} \hat{I}_{(1+s)+h} | L_1 \leq s] P(L_1 \leq s) + E^*[\hat{I}_s \hat{I}_{(1+h)} \hat{I}_{(1+s)} \hat{I}_{(1+s)+h} | s < L_1 \leq h] P(s < L_1 \leq h) + E^*[\hat{I}_s \hat{I}_{(1+h)} \hat{I}_{(1+s)} \hat{I}_{(1+s)+h} | h < L_1 \leq s+h] P(h < L_1 \leq s + h) + E^*[\hat{I}_s \hat{I}_{(1+h)} \hat{I}_{(1+s)} \hat{I}_{(1+s)+h} | L_1 > s + h] P(L_1 > s + h) \]

We start with $Q_1$. For $L_1 \leq s < h$, \( \hat{I}_s \) is independent of \( \hat{I}_{(1+h)} \), \( \hat{I}_{(1+s)} \), \( \hat{I}_{(1+s)+h} \), given \( (X_t) \), but \( \hat{I}_s \hat{I}_{(1+h)} = 0 \) by (8.3) and therefore $Q_1 = 0$. Similarly, for $h < L_1 \leq s + h$, \( \hat{I}_{(1+s)+h} \) is independent of \( \hat{I}_s \), \( \hat{I}_{(1+h)} \), \( \hat{I}_{(1+s)} \), given \( (X_t) \), and since \( \hat{I}_s \hat{I}_{(1+s)+h} = 0 \), $Q_3 = 0$. Each of the values $i = 1, \ldots, n$ has the same chance to be chosen by the bootstrap, i.e., \( \hat{P}_i(\hat{I}_i = \hat{I}_i) = n^{-1} \) for $i = 1, \ldots, n$. Thus, for $L_1 > s + h$ and the chosen $i$, the natural ordering \( (1^*, (1 + h)^*, (1 + s)^*, (1 + s + h)^*) = (i, i + h, i + s, i + s + h) \) is preserved and
Therefore
\[ Q_4 = n^{-1} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \tilde{I}_{i+s+h} P(L_1 > s + h) \]
\[ = n^{-1} \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \tilde{I}_{i+s+h} (1 - \theta)^{s+h}. \]

By a similar argument, (8.6) and using stationarity, we have
\[ Q \]
\[ \text{show that} \]
\[ h \]
\[ \text{the case} \]
\[ \text{Lemma 28.} \]
\[ (8.10). \]
\[ □ □ \]

We proceed with the case \( s > h \). Then we have the corresponding decomposition
\[ E^*[\tilde{I}_1 \tilde{I}_{1+h} \tilde{I}_{1+s+h} \tilde{I}_{1+s+h+h}] \]
\[ = E^*[\tilde{I}_1 \tilde{I}_{1+h} \tilde{I}_{1+s+h} \tilde{I}_{1+s+h+h} \mid L_1 \leq h] P(L_1 \leq h) \]
\[ + E^*[\tilde{I}_1 \tilde{I}_{1+h} \tilde{I}_{1+s+h} \tilde{I}_{1+s+h+h} \mid h < L_1 \leq s] P(h < L_1 \leq s) \]
\[ + E^*[\tilde{I}_1 \tilde{I}_{1+h} \tilde{I}_{1+s+h} \tilde{I}_{1+s+h+h} \mid s < L_1 \leq s + h] P(s < L_1 \leq s + h) \]
\[ + E^*[\tilde{I}_1 \tilde{I}_{1+h} \tilde{I}_{1+s+h} \tilde{I}_{1+s+h+h} \mid L_1 > s + h] P(L_1 > s + h) \]
\[ = Q'_1 + Q'_2 + Q'_3 + Q'_4. \]

We observe that the left-hand side is symmetric in \( h, s \) and therefore the same arguments as above show that \( Q'_1 = Q'_3 = 0, Q'_4 = Q'_4' \) and
\[ Q'_2 = E^*[\tilde{I}_1 \tilde{I}_{1+h} \mid h < L_1 \leq s] E^*[\tilde{I}_{1+s+h} \mid L_1 > s + h] P(h < L_1 \leq s) \]
\[ = n^{-2} \left( \sum_{i=1}^{n} \tilde{I}_i \tilde{I}_{i+h} \right)^2 (1 - \theta)^h \left( (1 - \theta)^s - (1 - \theta)^h \right). \]

The case \( h = s \) can be considered as a degenerate case, where \( Q'_2 = 0 \). This completes the proof of (8.10). \( □ □ \)

We conclude with a short discussion of the bias problem of the bootstrapped integrated periodogram mentioned in Remark 22.

\[ \text{Lemma 28. Assume the conditions of Theorem 22 and the additional condition} \]
\[ \sup_{x \in \Pi} |\psi_h(x)| \leq c/h \text{ for } h \geq 1 \text{ and a constant } c. \text{ Then the following relation holds as } n \to \infty, \]
\[ \sup_{\lambda \in \Pi} \left| \psi_{\lambda}(\lambda) \left( E^* \tilde{\gamma}_A(0) - \tilde{\gamma}_A(0) \right) + 2 \sum_{h=1}^{n-1} \psi_{\lambda}(\lambda) \left( E^* \tilde{\gamma}_A(h) - (1 - \theta)^h \tilde{\gamma}_A(h) \right) \right| \]
\[ \xrightarrow{P} 0. \]

\[ (8.11) \]

\[ \text{Proof. We observe that for } h \geq 0, \]
\[ E^* \tilde{\gamma}_A(h) - (1 - \theta)^h \tilde{\gamma}_A(h) = (1 - \theta)^h \left( \tilde{\gamma}_A(h) + \tilde{\gamma}_A(n - h) - \tilde{\gamma}_A(h) \right) \]
\[ = (1 - \theta)^h \left[ \tilde{\gamma}_A(n - h) - m(p_0 - T_n)^2 \right]. \]

\[ (8.12) \]

For fixed \( h \) we have \( (n/m)^{0.5} n (p_0 - T_n)^2 \xrightarrow{P} 0 \) as \( n \to \infty \) and
\[ (n/m)^{0.5} E|\tilde{\gamma}_A(n - h)| \leq c (m/n)^{0.5} hp_0 \to 0, \quad n \to \infty. \]
Therefore it suffices to show that
\[
\lim_{k \to \infty} \limsup_{n \to \infty} P \left( \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n-1} \psi_h(\lambda) \left( E^* \gamma_A^*(h) - (1 - \theta)^h \gamma_A(h) \right) \right| > \delta \right), \quad \delta > 0.
\]
Keeping in mind (8.12), we have
\[
(n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n-1} \psi_h(\lambda) (1 - \theta)^h \gamma_A(n - h) \right|
\leq (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=1}^{n-k-1} \psi_{n-h}(\lambda) (1 - \theta)^{n-h} [\gamma_A(h) - m(1 - h/n)(p_h - p_0^2)] \right|
+ (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=1}^{n-k-1} \psi_{n-h}(\lambda) (1 - \theta)^{n-h} m(1 - h/n)(p_h - p_0^2) \right|
= I_1 + I_2.
\]
Under the assumption \( \sup_{c \in \Pi} |\psi_c(x)| \leq c/h \) uniformly for \( h \geq 1 \), we have for small \( \varepsilon > 0 \),
\[
I_2 \leq (m/n)^{0.5} c \sum_{h=1}^{\infty} \xi_h \to 0, \quad n \to \infty.
\]
Now we can adapt the proof of Lemma 23 to prove that
\[
\lim_{k \to \infty} \limsup_{n \to \infty} P(I_1 > \delta) = 0, \quad \delta > 0.
\]
This proves (8.11). \( \square \)

However, under the assumptions of Theorem 15 it is in general not possible to replace the quantities \((1 - \theta)^h \gamma_A(h) \) in (8.11) by \( \gamma_A(h) \), i.e., in general we do not have the relation \((n/m)^{0.5} (E^* J_{n,A}^* - J_{n,A}) \) \( \to 0 \). Indeed, taking into account (8.11) and assuming \( \eta \)-dependence for \( (X_t) \), we have \( E \gamma_A(h) = 0 \) for \( h > \eta \) and
\[
(n/m)^{0.5} (E^* J_{n,A}^* - J_{n,A})
= 2(n/m)^{0.5} \sum_{h=1}^{n-1} \psi_h(\lambda) [(1 - \theta)^h - 1] \gamma_A(h) + o_p(1)
= 2(n/m)^{0.5} \sum_{h=1}^{n-1} \psi_h(\lambda) [(1 - \theta)^h - 1] (\gamma_A(h) - E \gamma_A(h))
+ 2(n/m)^{0.5} \sum_{h=1}^{n} \psi_h(\lambda) [(1 - \theta)^h - 1] (1 - h/n)(p_h - p_0^2) + o_p(1).
\]
An argument similar to the proof of Theorem 15 shows that the first term on the right-hand side is stochastically bounded, while the second term may diverge (for example, if \( \gamma_A(\eta) > 0 \) and \( \psi_\eta \neq 0 \)) since it is of the order \( \theta(n/m)^{0.5} \) which converges to infinity in view of the assumption \( \theta^2 n/m \to \infty \) which is vital for the proof of the consistency of the stationary bootstrap.
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