Infinitesimal Bishop–Gromov condition for Alexandrov spaces

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Abstract.

We prove the infinitesimal version of Bishop–Gromov volume comparison condition for Alexandrov spaces.

§1. Introduction

We first present the definition of the infinitesimal Bishop–Gromov volume comparison condition for Alexandrov spaces.

For a real number $\kappa$, we set

$$s_\kappa(r) := \begin{cases} 
\sin(\sqrt{\kappa}r)/\sqrt{\kappa} & \text{if } \kappa > 0, \\
 r & \text{if } \kappa = 0, \\
\sinh(\sqrt{|\kappa|}r)/\sqrt{|\kappa|} & \text{if } \kappa < 0.
\end{cases}$$

The function $s_\kappa$ is the solution of the Jacobi equation $s''_\kappa(r) + \kappa s'_\kappa(r) = 0$ with initial condition $s_\kappa(0) = 0, s'_\kappa(0) = 1$.

Let $M$ be an Alexandrov space and set $r_p(x) := d(p, x)$ for $p, x \in M$, where $d$ is the distance function. For $p \in M$ and $0 < t \leq 1$, we define a subset $W_{p,t} \subset M$ and a map $\Phi_{p,t} : W_{p,t} \rightarrow M$ as follows. We first set $\Phi_{p,t}(p) := p \in W_{p,t}$. A point $x (\neq p)$ belongs to $W_{p,t}$ if and only if there exists $y \in M$ such that $x \in py$ and $r_p(x) : r_p(y) = t : 1$, where $py$ is a minimal geodesic from $p$ to $y$. Since a geodesic does not branch on an Alexandrov space, for a given point $x \in W_{p,t}$ such a point $y$ is unique and we set $\Phi_{p,t}(x) := y$. The triangle comparison condition implies the...
local Lipschitz continuity of the map $\Phi_{p,t} : W_{p,t} \to M$. We call $\Phi_{p,t}$ the \textit{radial expansion map}.

Let $\mu$ be a positive Radon measure with full support in $M$, and $n \geq 1$ a real number.

**Infinitesimal Bishop–Gromov Condition $BG(\kappa, n)$ for $\mu$:**

For any $p \in M$ and $t \in (0, 1]$, we have

$$d(\Phi_{p,t*}\mu)(x) \geq \frac{ts_\kappa(tr_p(x))^{n-1}}{s_\kappa(r_p(x))^{n-1}} d\mu(x)$$

for any $x \in M$ such that $r_p(x) < \pi/\sqrt{\kappa}$ if $\kappa > 0$, where $\Phi_{p,t*}\mu$ is the push-forward by $\Phi_{p,t}$ of $\mu$.

For an $n$-dimensional complete Riemannian manifold, the Riemannian volume measure satisfies $BG(\kappa, n)$ if and only if the Ricci curvature satisfies $\text{Ric} \geq (n-1)\kappa$ (see Theorem 3.2 of [10] for the ‘only if’ part). We see some studies on similar (or same) conditions to $BG(\kappa, n)$ in [2, 18, 6, 7, 15, 10, 21] etc. $BG(\kappa, n)$ is sometimes called the Measure Contraction Property and is weaker than the curvature-dimension (or lower $n$-Ricci curvature) condition, $CD((n-1)\kappa, n)$, introduced by Sturm [19, 20] and Lott–Villani [9] in terms of mass transportation. For a measure on an Alexandrov space, $BG(\kappa, n)$ is equivalent to the $((n-1)\kappa, n)$-MCP introduced by Ohta [10]. In our paper [5, 8], we prove a splitting theorem under $BG(0, N)$. For a survey of geometric analysis on Alexandrov spaces, we refer to [17].

The purpose of this paper is to prove the following

**Theorem 1.1.** Let $M$ be an $n$-dimensional Alexandrov space of curvature $\geq \kappa$. Then, the $n$-dimensional Hausdorff measure $H^n$ on $M$ satisfies the infinitesimal Bishop–Gromov condition $BG(\kappa, n)$.

Note that we claimed this theorem in Lemma 6.1 of [6], but the proof in [6] is insufficient. The theorem also completes the proof of Proposition 2.8 of [10].

For the proof of the theorem, we have the delicate problem that the topological boundary of the domain $W_{p,t}$ of the radial expansion $\Phi_{p,t}$ is not necessarily of $H^n$-measure zero. In fact, we have an example of an Alexandrov space such that the cut-locus at a point is dense (see Remark 2.2), in which case the boundary of $W_{p,t}$ has positive $H^n$-measure. This never happens for Riemannian manifolds. To solve this problem, we need some delicate discussion using the approximate differential of $\Phi_{p,t}$.

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§2. Preliminaries

2.1. Alexandrov spaces

In this paper, we mean by an Alexandrov space a complete locally compact geodesic space of curvature bounded below locally and of finite Hausdorff dimension. We refer to [1, 12, 4] for the basics for the geometry and analysis on Alexandrov spaces. Let $M$ be an Alexandrov space of Hausdorff dimension $n$. Then, $n$ coincides with the covering dimension of $M$ which is a nonnegative integer. Take any point $p \in M$ and fix it. Denote by $\Sigma_pM$ the space of directions at $p$, and by $K_pM$ the tangent cone at $p$. $\Sigma_pM$ is an $(n-1)$-dimensional compact Alexandrov space of curvature $\geq 1$ and $K_pM$ an $n$-dimensional Alexandrov space of curvature $\geq 0$.

**Definition 2.1** (Singular Point, $\delta$-Singular Point). A point $p \in M$ is called a singular point of $M$ if $\Sigma_pM$ is not isometric to the unit sphere $S^{n-1}$. For $\delta > 0$, we say that a point $p \in M$ is $\delta$-singular if $\mathcal{H}^{n-1}(\Sigma_pM) \leq \text{vol}(S^{n-1}) - \delta$. Let us denote the set of singular points of $M$ by $S_M$ and the set of $\delta$-singular points of $M$ by $S_\delta$.

We have $S_M = \bigcup_{\delta > 0} S_\delta$. Since the map $M \ni p \mapsto \mathcal{H}^n(\Sigma_pM)$ is lower semi-continuous, the set $S_\delta$ of $\delta$-singular points in $M$ is a closed set.

**Lemma 2.1** ([14]). Let $\gamma$ be a minimal geodesic joining two points $p$ and $q$ in $M$. Then, the space of directions, $\Sigma_xM$, at all interior points of $\gamma$, $x \in \gamma \setminus \{p,q\}$, are isometric to each other. In particular, any minimal geodesic joining two non-singular (resp. non-$\delta$-singular) points is contained in the set of non-singular (resp. non-$\delta$-singular) points (for any $\delta > 0$).

The following shows the existence of differentiable and Riemannian structure on $M$.

**Theorem 2.1.** For an $n$-dimensional Alexandrov space $M$, we have the following:

1. There exists a number $\delta_n > 0$ depending only on $n$ such that $M^* := M \setminus S_{\delta_n}$ is a manifold ([1]) and has a natural $C^\infty$ differentiable structure ([4]).

2. The Hausdorff dimension of $S_M$ is $\leq n - 1$ ([1, 12]).

3. We have a unique continuous Riemannian metric $g$ on $M \setminus S_M \subset M^*$ such that the distance function induced from $g$ coincides with the original one of $M$ ([12]). The tangent space at each point in $M \setminus S_M$ is isometrically identified with the tangent cone ([12]). The volume measure on $M^*$ induced from $g$ coincides with the $n$-dimensional Hausdorff measure $\mathcal{H}^n$ ([12]).
Remark 2.1. In [4] we construct a $C^\infty$ structure only on $M \setminus B(S_{\delta_n}, \epsilon)$, where $B(A, \epsilon)$ denotes the $\epsilon$-neighborhood of $A$. However this is independent of $\epsilon$ and extends to $M^*$. The $C^\infty$ structure is a refinement of the structures of [12, 11, 13] and is compatible with the DC structure of [13].

Note that the metric $g$ is defined only on $M^* \setminus S_M$ and does not continuously extend to any other point of $M$.

Definition 2.2 (Cut-locus). Let $p \in M$ be a point. We say that a point $x \in M$ is a cut point of $p$ if no minimal geodesic from $p$ contains $x$ as an interior point. Here we agree that $p$ is not a cut point of $p$. The set of cut points of $p$ is called the cut-locus of $p$ and denoted by $\text{Cut}_p$.

Note that $\text{Cut}_p$ is not necessarily a closed set. For the $W_{p,t}$ defined in §1, it follows that $\bigcup_{0 < t < 1} W_{p,t} = X \setminus \text{Cut}_p$. The cut-locus $\text{Cut}_p$ is a Borel subset and satisfies $\mathcal{H}^m(\text{Cut}_p) = 0$ (Proposition 3.1 of [12]).

Remark 2.2. There is an example of a 2-dimensional Alexandrov space $M$ such that $S_M$ is dense in $M$ (see [12]). For such an example, $\text{Cut}_p$ for any $p \in M$ is also dense in $M$.

2.2. Approximate differential

Definition 2.3 (Density; cf. 2.9.12 in [3]). Let $X$ be a metric space with a Borel measure $\mu$. A subset $A \subset X$ has density zero at a point $x \in X$ if

$$\lim_{r \to 0} \frac{\mu(B(x, r) \cap A)}{\mu(B(x, r))} = 0.$$ 

Definition 2.4 (Approximate Differential; cf. 3.1.2 in [3]). Let $A \subset \mathbb{R}^m$ be a subset and $f : A \to \mathbb{R}^n$ a map. A linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ is called the approximate differential of $f$ at a point $x \in A$ if the approximate limit of

$$\frac{|f(y) - f(x) - L(y - x)|}{|y - x|}$$

is equal to zero as $y \to x$, i.e., for any $\delta > 0$, the set

$$\left\{ \ y \in A \setminus \{x\} \ \bigg| \ \frac{|f(y) - f(x) - L(y - x)|}{|y - x|} \geq \delta \right\}$$

has density zero at $x$, where we consider the Lebesgue (or equivalently $m$-dimensional Hausdorff) measure on $\mathbb{R}^m$ to measure the density. We say that $f$ is approximately differentiable at a point $x \in A$ if the approximate differential of $f$ at $x$ exists. Denote by ‘$\text{ap df}_x$’ the approximate
differential of $f$ at $x$. It is unique at each approximate differentiable point.

Let $M$ and $N$ be two differentiable manifolds and let $A \subset M$. We give a map $f : A \to N$ and a point $x \in A$. Take two charts $(U, \varphi)$ and $(V, \psi)$ around $x$ and $f(x)$ respectively. The map $f$ is said to be \textit{approximately differentiable at $x$} if $\psi \circ f \circ \varphi^{-1}$ is approximately differentiable at $\varphi(x)$. If $f$ is approximately differentiable at $x$, then the \textit{approximate differential} $\text{ap} d_{f}x$ of $f$ at $x$ is defined by

$$\text{ap} d_{f}x := (d\psi_{f(x)})^{-1} \circ \text{ap} d(\psi \circ f \circ \varphi^{-1})\varphi(x) \circ d\varphi_{x} : T_{x}M \to T_{f(x)}N.$$  

The approximate differentiability of $f$ at $x$ and $\text{ap} d_{f}x$ are both independent of $(U, \varphi)$ and $(V, \psi)$.

\section{Proof of Theorem 1.1}

Let $M$ be an Alexandrov space of curvature $\geq \kappa$. We first investigate the exponential map on $M$. Denote by $o_{p}$ the vertex of the tangent cone $K_{p}M$ at a point $p \in M$. We denote by $U_{p} \subset K_{p}M$ the \textit{inside of the tangential cut-locus of $p$}, i.e., $v \in U_{p}$ if and only if there is a minimal geodesic $\gamma : [0, a] \to M$ from $p$ with $a > 1$ such that $\gamma'(0) = v$, where $\gamma'(t)$ denotes the element of $K_{\gamma(t)}M$ tangent to $\gamma|_{[t, t+\epsilon]}$, $\epsilon > 0$, and whose distance from $o_{\gamma(t)} \in K_{\gamma(t)}M$ is equal to the speed of parameter of $\gamma$. Note that $U_{p}$ is not necessarily an open set. Since the exponential map $\exp_{p}|U_{p} : U_{p} \to M \setminus \text{Cut}_{p}$ is a homeomorphism and since $W_{p, t} \cap \overline{B}(p, r)$ is compact for any $0 < t \leq 1$ and $r > 0$, the set

$$U_{p} = \bigcup_{0 < t \leq 1, \ r > 0} (\exp_{p}|U_{p})^{-1}(W_{p, t} \cap \overline{B}(p, r))$$

is a Borel subset of $K_{p}M$.

Denote by $\Theta(t|a, b, \ldots)$ a function of $t, a, b, \ldots$ such that $\Theta(t|a, b, \ldots) \to 0$ as $t \to 0$ for any fixed $a, b, \ldots$. We use $\Theta(t|a, b, \ldots)$ as Landau symbols.

\textbf{Lemma 3.1.} For any $p \in M$, $r > 0$, and for any $\mathcal{H}^{n}$-measurable subset $A \subset B(o_{p}, r) \subset K_{p}M$, we have

1. $|\mathcal{H}^{n}(\exp_{p}(A \cap U_{p})) - \mathcal{H}^{n}(A)| \leq \Theta(r|p, n) r^{n},$

2. $\mathcal{H}^{n}(B(o_{p}, r) \setminus U_{p}) \leq \Theta(r|p, n) r^{n}.$

Note that $\Theta(r|p, n)$ here is independent of $A$. 
Proof. Let \( p \in M \) and \( r > 0 \). By the triangle comparison condition, \( \exp_p : U_p \cap B(o_p, r) \rightarrow M \) is Lipschitz continuous with Lipschitz constant \( 1 + \Theta(r|p|) \). Therefore, for any \( \mathcal{H}^n \)-measurable \( A \subset B(o_p, r) \),
\[
\mathcal{H}^n(A) \geq (1 - \Theta(r|p|, n)) \mathcal{H}^n(\exp_p(A \cap U_p)),
\]
\[
\mathcal{H}^n(B(o_p, r) \setminus A) \geq (1 - \Theta(r|p|, n)) \mathcal{H}^n(B(p, r) \setminus \exp_p(A \cap U_p)).
\]
According to Lemma 3.2 of [16], we have
\[
\lim_{\rho \to 0} \frac{\mathcal{H}^n(B(p, \rho))}{\rho^n} = \frac{\mathcal{H}^n(B(o_p, 1))}{r^n}.
\]
Combining those three formulas we have the lemma. Q.E.D.

Let \( p \in M \) and \( 0 < t \leq 1 \). We restrict the domain of the radial expansion map \( \Phi_{p,t} : W_{p,t} \rightarrow M \) to the subset
\[
W'_{p,t} := W_{p,t} \setminus (\Phi_{p,t}^{-1}(\text{Cut}_p) \cup S_{\delta_n}),
\]
where \( S_{\delta_n} \) is as in Theorem 2.1.

**Lemma 3.2.** We have \( \Phi_{p,t}(W'_{p,t}) = M \setminus (\text{Cut}_p \cup S_{\delta_n}) \) and the map \( \Phi_{p,t}|_{W'_{p,t}} : W'_{p,t} \rightarrow M \setminus (\text{Cut}_p \cup S_{\delta_n}) \) is bijective. In particular, the sets \( W'_{p,t} \) and \( \Phi_{p,t}(W'_{p,t}) \) are both contained in the \( C^\infty \) manifold \( M^* = M \setminus S_{\delta_n} \) without boundary.

**Proof.** Let us first prove \( \Phi_{p,t}(W'_{p,t}) \subset M \setminus (\text{Cut}_p \cup S_{\delta_n}) \). It is clear that \( \Phi_{p,t}(W'_{p,t}) \subset M \setminus \text{Cut}_p \). To prove \( \Phi_{p,t}(W'_{p,t}) \subset M \setminus S_{\delta_n} \), we take any point \( x \in W'_{p,t} \). Since \( \Phi_{p,t}(x) \) is not a cut point of \( p \) and by Lemma 2.1, \( \Phi_{p,t}(x) \) is not \( \delta_n \)-singular. Therefore, \( \Phi_{p,t}(W'_{p,t}) \subset M \setminus (\text{Cut}_p \cup S_{\delta_n}) \).

Let us next prove \( \Phi_{p,t}(W'_{p,t}) \supset M \setminus (\text{Cut}_p \cup S_{\delta_n}) \). Take any point \( y \in M \setminus (\text{Cut}_p \cup S_{\delta_n}) \) and join \( p \) to \( y \) by a minimal geodesic \( \gamma : [0, 1] \rightarrow M \). Then, \( \Phi_{p,t}(\gamma(t)) = y \). Since \( y \not\in \text{Cut}_p \), the geodesic \( \gamma \) is unique and so \( \Phi_{p,t}|_{W'_{p,t}} \) is injective. By Lemma 2.1, \( \gamma(t) = (\Phi_{p,t}|_{W'_{p,t}})^{-1}(y) \) is not \( \delta_n \)-singular and belongs to \( W'_{p,t} \). This completes the proof. Q.E.D.

By the local Lipschitz continuity of \( \Phi_{p,t} \) and by 3.1.8 of [3], \( \Phi_{p,t}|_{W'_{p,t}} \) is approximately differentiable \( \mathcal{H}^n \)-a.e. on \( W'_{p,t} \). The following lemma is essential for the proof of Theorem 1.1.

**Lemma 3.3.** Let \( p \in M \) and \( 0 < t < 1 \). Then, the approximate Jacobian determinant of \( \Phi_{p,t}|_{W'_{p,t}} \) satisfies that
\[
|\det ap d(\Phi_{p,t}|_{W'_{p,t}})x| \leq \frac{s_\kappa(r_p(x)/t)^{n-1}}{t^s r_p(x)^{n-1}}
\]
for any approximately differentiable point \( x \in W'_{p,t} \setminus S_M \) of \( \Phi_{p,t}|_{W'_{p,t}} \).
Proof. Let \( x \in W_{p,t}' \setminus S_M \) be an approximately differentiable point of \( \Phi_{p,t}|_{W_{p,t}'} \) and let \( \epsilon > 0 \) be a small number. Note that \( K_x M \) and \( K_{\Phi_{p,t}(x)} M \) are both isometric to \( \mathbb{R}^n \) and identified with the tangent spaces. We take two charts \((U, \varphi)\) and \((V, \psi)\) of \( M \setminus S_{\delta_n} \) around \( x \) and \( \Phi_{p,t}(x) \) respectively such that \( \frac{|\varphi(y) - \varphi(z)|}{d(y, z) - 1} < \epsilon \) for any different \( y, z \in U \) and \( \psi \) satisfies the same inequality on \( V \). In particular, every eigenvalue of the differentials \( d\varphi_x : K_x M \to \mathbb{R}^n \) and \( d\psi_{\Phi_{p,t}(x)} : K_{\Phi_{p,t}(x)} M \to \mathbb{R}^n \) is between \( 1 - \epsilon \) and \( 1 + \epsilon \). Put

\[
\tilde{\Phi} := \psi \circ \Phi_{p,t}|_{W_{p,t}' \cap U} \circ \varphi^{-1} : \varphi(W_{p,t}' \cap U) \to \psi(V),
\]

\[
\bar{x} := \varphi(x), \quad L := ap d\tilde{\Phi}_x : \mathbb{R}^n \to \mathbb{R}^n.
\]

For simplicity we set \( D := ap d(\Phi_{p,t}|_{W_{p,t}'}) : K_x M \to K_{\Phi_{p,t}(x)} M \). Then,

\[
D = (d\psi_{\Phi_{p,t}(x)})^{-1} \circ L \circ d\varphi_x.
\]

By the definition of the approximate differential, for any \( r > 0 \) with \( B(x, r) \subset U \), the set of \( \bar{y} \in B(\bar{x}, r) \) satisfying

\[
| \tilde{\Phi}(\bar{y}) - \tilde{\Phi}(\bar{x}) - L(\bar{y} - \bar{x}) | \geq \epsilon | \bar{x} - \bar{y} |
\]

has \( \mathcal{H}^n \)-measure \( \leq \Theta(r|\tilde{\Phi}, \bar{x}) \mathcal{H}^n(B(\bar{x}, r)) \), where \( B(\bar{x}, r) \) is a Euclidean metric ball. Take any \( u \in \Sigma_x M \) and fix it. Let \( r > 0 \) be any number. We set

\[
C(u, r, \epsilon) := \{ v \in B(o_x, r) \setminus \{o_x\} \subset K_x M | \langle u, v \rangle < \epsilon \}.
\]

It follows from Lemma 3.1(1) that

\[
\mathcal{H}^n(\varphi(\exp_x(C(u, r/2, \epsilon) \cap U_x))) \geq (1 - \epsilon)^n \mathcal{H}^n(\exp_x(C(u, r/2, \epsilon) \cap U_x)) \geq (1 - \epsilon)^n (\mathcal{H}^n(C(u, 1/2, \epsilon)) - \Theta(r|x, n)) r^n.
\]

Since \( \mathcal{H}^n(C(u, 1/2, \epsilon)) \) is positive, we have

\[
\lim_{r \to 0} \frac{\mathcal{H}^n(\varphi(\exp_x(C(u, r/2, \epsilon) \cap U_x)))}{\mathcal{H}^n(B(\bar{x}, r))} > 0.
\]

Note that \( \varphi(\exp_x(C(u, r/2, \epsilon) \cap U_x)) \) is contained in \( B(\bar{x}, r) \) because \( \epsilon \) is small enough. Therefore, supposing \( r \ll \epsilon \), there is a point \( \bar{y} \in B(\bar{x}, r) \) such that

\[
\bar{y} \in \varphi(\exp_x(C(u, r/2, \epsilon) \cap U_x)),
\]

\[
| \tilde{\Phi}(\bar{y}) - \tilde{\Phi}(\bar{x}) - L(\bar{y} - \bar{x}) | < \epsilon d(\bar{x}, \bar{y}).
\]
Setting \( y := \varphi^{-1}(\tilde{y}) \) and \( v_{xy} := (\exp_x | u_x |)^{-1}(y) \), we have \( \angle(u, v_{xy}) < \epsilon \). For simplicity we write \( a \leq (1 + \Theta(\epsilon|p, t, x|)) b + \Theta(\epsilon|p, t, x|) \) by \( a \lesssim b \). Note that since \( r \ll \epsilon \), all \( \Theta(r \cdots) \) become \( \Theta(\epsilon \cdots) \). Since \( |v_{xy}| = d(x, y) \) and \( |d\varphi_x(v_{xy}) - (\tilde{y} - \tilde{x})| \leq \Theta(\epsilon|x|) d(x, y) \) (cf. Lemma 3.6(2) of [12]), we have

\[
|D(u)| \lesssim |D(v_{xy}/v_{xy})| \lesssim \frac{|L(\tilde{y} - \tilde{x})|}{d(x, y)} \lesssim \frac{|\Phi(\tilde{y}) - \Phi(\tilde{x})|}{d(x, y)} \lesssim \frac{d(\Phi_{p,t}(x), \Phi_{p,t}(y))}{d(x, y)}.
\]

We are going to estimate the last formula. Denote by \( M^2(\kappa) \) a complete simply connected 2-dimensional space form of curvature \( \kappa \). We take three points \( \tilde{p}, \tilde{x}, \tilde{y} \in M^2(\kappa) \) such that \( d(\tilde{p}, \tilde{x}) = d(p, x), \ d(\tilde{p}, \tilde{y}) = d(p, y), \) and \( d(\tilde{x}, \tilde{y}) = d(x, y) \). The triangle comparison condition tells that

\[
d(\Phi_{p,t}(x), \Phi_{p,t}(y)) \leq d(\Phi_{\tilde{p},t}(\tilde{x}), \Phi_{\tilde{p},t}(\tilde{y})),
\]

where \( \Phi_{\tilde{p},t} \) is the radial expansion on \( M^2(\kappa) \). Since \( d(\tilde{x}, \tilde{y}) = d(x, y) < r \ll \epsilon \), we have

\[
\frac{d(\Phi_{\tilde{p},t}(\tilde{x}), \Phi_{\tilde{p},t}(\tilde{y}))}{d(\tilde{x}, \tilde{y})} \lesssim |d(\Phi_{\tilde{p},t}(\tilde{x}, v_{xy}/v_{xy})|).
\]

Let \( \hat{\gamma} \) be the minimal geodesic from \( \tilde{p} \) passing through \( \tilde{x} \). We denote by \( \hat{\theta} \) the angle between \( v_{xy} \) and \( \hat{\gamma}'(t_{\hat{x}}) \), where \( t_{\hat{x}} \) is taken in such a way that \( \hat{\gamma}(t_{\hat{x}}) = \tilde{x} \). Set

\[
\lambda(\xi) := \sqrt{\frac{1}{t^2} \cos^2 \xi \frac{s_\kappa(r_p(x)/t)^2}{s_\kappa(r_p(x))^2} \sin^2 \xi}, \quad \xi \in \mathbb{R}.
\]

A calculation using Jacobi fields yields \( |d(\Phi_{\tilde{p},t}(\tilde{x}, v_{xy}/v_{xy})| = \lambda(\hat{\theta}) \). Combining the above estimates, we have

\[
|D(u)| \lesssim \lambda(\hat{\theta}).
\]

Let \( \gamma \) be the minimal geodesic from \( p \) passing through \( x \) and let \( t_x \) be a number such that \( \gamma(t_x) = x \). Denote by \( \theta \) the angle between \( v_{xy} \) and \( \gamma'(t_x) \) and by \( \theta_u \) the angle between \( u \) and \( \gamma'(t_x) \). It follows from \( \angle(u, v_{xy}) < \epsilon \) that \( |\theta - \theta_u| < \epsilon \). By 5.6 of [1] we have \( |\theta - \hat{\theta}| \leq \Theta(r|p, t, x|) \leq \Theta(\epsilon|p, t, x|). \) Therefore we have \( |D(u)| \lesssim \lambda(\theta_u) \). Taking the limit as \( \epsilon \to 0 \) yields that

\[
|D(u)| \leq \lambda(\theta_u)
\]

for any \( u \in \Sigma_x M \), which together with Hadamard's inequality implies

\[
|\det D| \leq \lambda(0) \lambda(\pi/2)^{n-1} = \frac{s_\kappa(r_p(x)/t)^{n-1}}{t s_\kappa(r_p(x))^{n-1}}.
\]
This completes the proof of Lemma 3.3. Q.E.D.

Proof of Theorem 1.1. For the proof, it suffices to prove that

\[ \int_{W_{p,t}} f \circ \Phi_{p,t}(x) \, dH^n(x) \geq \int_M f(y) \frac{t s_\kappa(t r_p(y))^{n-1}}{s_\kappa(r_p(y))^{n-1}} \, dH^n(y) \]

for any $H^n$-measurable function $f : M \to [0, +\infty)$ with compact support. Since $\Phi_{p,t}|_{W_{p,t}} : W_{p,t} \to M \setminus (\text{Cut}_p \cup S_{\delta_n})$ is bijective, the area formula (cf. 3.2.20 of [3]) implies that

\[ \int_{W_{p,t}} F \circ \Phi_{p,t}(x) \, |\det d(\Phi_{p,t}|_{W_{p,t}})_x| \, dH^n(x) \]

\[ = \int_{M \setminus (\text{Cut}_p \cup S_{\delta_n})} F(y) \, dH^n(y) \]

for any $H^n$-measurable function $F : M \to [0, +\infty)$ with compact support. We set

\[ F(y) := f(y) \frac{t s_\kappa(t r_p(y))^{n-1}}{s_\kappa(r_p(y))^{n-1}}, \quad y \in M \setminus \text{Cut}_p, \]

in (3.2). Then, since $H^n(\text{Cut}_p) = H^n(S_{\delta_n}) = 0$ and by Lemma 3.3, we obtain (3.1). This completes the proof of the theorem. Q.E.D.

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