Robust inference for general framework of projection structures

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Abstract: We develop a general framework of projection structures and study the problem of inference on the unknown parameter within this framework by using empirical Bayes and penalization methods. The main inference problem is the uncertainty quantification, but on the way we solve the estimation and posterior contraction problems as well (and a weak version of the structure recovery problem). The approach is local in that the quality of the inference procedures is measured by the local quantity, the oracle rate, which is the best trade-off between the approximation error by a projection structure and the complexity of that approximating projection structure. The approach is also robust in that the stochastic part of the general framework is assumed to satisfy only certain mild condition, the errors may be non-iid with unknown distribution. We introduce the excessive bias restriction (EBR) under which we establish the local (oracle) confidence optimality of the constructed confidence ball.

As the proposed general framework unifies a very broad class of high-dimensional models interesting and important on their own right, the obtained general results deliver a whole avenue of results (many new ones and some known in the literature) for particular models and structures as consequences, including white noise model and density estimation with smoothness structure, linear regression and dictionary learning with sparsity structures, biclustering and stochastic block models with clustering structure, covariance matrix estimation with banding and sparsity structures, and many others. Various adaptive minimax results over various scales follow also from our local results.

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Contents

1 Introduction ...................................................... 3
   1.1 Uncertainty quantification problem ................................ 4
   1.2 The scope .................................................. 5
2 Preliminaries .................................................... 8
   2.1 Notation ..................................................... 8
   2.2 Slicing into structural layers, complexity of a layer ............... 9
   2.3 Conditions ................................................... 11
   2.4 Multivariate normal prior ..................................... 13
   2.5 Empirical Bayes posterior ..................................... 15
3 Main results ................................................................. 16
  3.1 Oracle rate ............................................................ 16
  3.2 Estimation and contraction results with oracle rate .......... 17
  3.3 Confidence ball under EBR ........................................ 19
  3.4 Confidence ball of $N^{1/4}$-radius without EBR ............... 21
4 Technical lemmas ...................................................... 23
5 Proofs of the theorems ................................................. 25
6 Applications ............................................................ 32
  6.1 Signal+noise model with smoothness structure ............... 34
     6.1.1 Sobolev ellipsoids ............................................ 35
     6.1.2 Sobolev hyperrectangles .................................... 36
     6.1.3 Analytic and tail classes .................................... 36
  6.2 Signal+noise model under wavelet basis ....................... 36
     6.2.1 Besov scale .................................................. 37
  6.3 Signal+noise model with (multi-level) sparsity structure ... 38
     6.3.1 Minimax results for nearly black vectors $\ell_0$ and weak $\ell_q$-balls ... 39
     6.3.2 Multi-level sparsity (clustering) structure ................ 40
  6.4 Function on a large graph with smoothness structure ....... 42
     6.4.1 Minimax results for a Laplacian graph ................... 43
  6.5 Density estimation with smoothness structure ............... 44
  6.6 Biclustering model ............................................... 46
     6.6.1 Minimax results for the biclustering model .............. 49
     6.6.2 Stochastic block model ..................................... 50
     6.6.3 Minimax results for the stochastic block model ......... 50
     6.6.4 Graphon classes ............................................. 51
  6.7 Linear regression with various structures .................... 52
     6.7.1 Linear regression with sparsity structure, inference on $\beta$ ... 53
     6.7.2 Linear regression with group sparsity ..................... 59
     6.7.3 Linear regression with group clustering ................... 60
     6.7.4 Linear regression with mixture structure ................. 61
  6.8 Aggregation ......................................................... 64
  6.9 Isotonic, unimodal and convex regressions .................... 66
     6.9.1 Minimax results for isotonic, unimodal and convex regressions ... 67
     6.9.2 Log factor and universality of the results ............... 68
     6.9.3 No EBR-like condition for shape-restricted structures .... 69
  6.10 Dictionary learning ............................................... 70
     6.10.1 Minimax results for sparse dictionary learning ........ 73
  6.11 Mean matrix with submatrix sparsity .......................... 73
     6.11.1 Minimax results for $F(k_1, k_2, n_1, n_2)$ ............. 74
  6.12 Covariance matrix with banding or sparsity structure ....... 74
     6.12.1 Banding structure ......................................... 75
     6.12.2 Minimax results for the scale $\{g_{\beta, \beta > 0}\}$ ....... 78
1. Introduction

Suppose we observe a random element \((Y, X) \in (\mathcal{Y} \times \mathcal{X})\):

\[
Y \sim \mathbb{P}_\theta = \mathbb{P}_{\theta, X}, \quad \theta \in \Theta \subseteq \mathcal{Y}, \quad \text{such that} \quad \mathbb{E}_\theta Y = \theta(X) = \theta,
\]

where \(\mathbb{P}_\theta\) is the probability measure of \(Y\) (\(\mathbb{E}_\theta\) is the corresponding expectation) depending on an unknown high-dimensional parameter of interest \(\theta\). By default, \(\Theta = \mathcal{Y} = \mathbb{R}^N\), \(\mathcal{X} = \mathbb{R}^{d_X}\) for “big” \(N, d_X \in \mathbb{N}\) (with the usual norm \(||\cdot|||\)), unless stated otherwise. In some particular models (see Section 6), \(\mathcal{Y}, \Theta \subseteq \mathbb{R}^\infty\) can be infinite dimensional and \(\Theta\) can be a proper subset of \(\mathcal{Y}\), e.g., \(\mathcal{Y} = \mathbb{R}^\infty\) and \(\Theta = \ell_2\). Those particular models can also be reduced to the high-dimensional case by assuming that any \(\theta \in \Theta \subseteq \mathbb{R}^\infty\) can be arbitrarily well approximated by \(\bar{\theta} \in \mathbb{R}^N\) for sufficiently large \(N \in \mathbb{N}\).

Useful inference in high-dimensional models is not possible without some (approximate) structure on the parameter of interest, the basic idea is to reduce the “effective” dimensionality of the high-dimensional \(\theta\). The most popular structural assumptions are smoothness, sparsity and clustering. These structures and many others can be represented via appropriate families of linear spaces \(L_I \subseteq \mathcal{Y}, I \in I\). Precisely, we introduce a finite (or countable) family \(I\) of possible structures \(I\) and an associated family of linear subspaces \(\{L_I, I \in I\}\) of \(\mathcal{Y}\), which express these structures. This in turn determines the family of corresponding projection operators \(\{P_I, I \in I\}\) onto linear subspaces \(\{L_I, I \in I\}\). The true \(\theta\) is “approximately structured” according to the family \(I\) if \(||\theta - P_I \theta||^2 = \min_{I \in I} ||\theta - P_I \theta||^2\) is close to zero. If \(||\theta - P_I \theta||^2 = 0\), the true \(\theta\) happens to be exactly structured, i.e., \(\theta \in L_I\) and \(I^*\) has the meaning of the “true structure” of the true \(\theta\).

Let \(\sigma \xi = Y - \mathbb{E}_\theta Y\) (any \(Y\) is its expectation plus zero mean “noise”), \(\sigma > 0\) be the “noise intensity”, then

\[
Y = \theta(X) + \sigma \xi = \theta + \sigma \xi \quad \text{with} \quad \mathbb{E}_\theta \xi = 0. \tag{1.1}
\]

The (known) parameter \(\sigma\) is introduced to accommodate certain asymptotic regimes in particular models, \(\sigma \rightarrow 0\) reflects an information increase. How the underlying measure \(\mathbb{P}_{\theta, X}\) of \(Y\) (and \(\theta = \theta(X) = \mathbb{E}_\theta Y\)) depends on \(X\) will only be exploited when constructing specific families of structures \(\mathcal{I} = \mathcal{I}(X)\) (e.g., in the linear regression model from Section 6.7). We skip the dependence of structures on \(X\) in further notation.

The goal is to make inference on the parameter \(\theta\) based on the data \((Y, X)\): recovery of \(\theta\) (estimation and posterior contraction), structure recovery, and uncertainty quantification by constructing an optimal confidence set. We pursue local inference in the sense that no structure on \(\theta\) is a priori imposed, rather we aim to extract as much structure (according to the family of structures \(\mathcal{I}\), once \(\mathcal{I}\) is chosen) as there is in the underlying \(\theta\). We will make
this notion precise later. We also pursue robust inference in the sense that the distribution of $\xi$ is unknown and can depend on $\theta$ (often we suppress this dependence in notation), the coordinates $\xi_i$’s of $\xi$ do not have to be iid, even not independent. The distribution of $\xi$ is assumed to satisfy only certain mild condition; see Condition (A1) in Section 2.

We derive non-asymptotic results, which imply asymptotic assertions if needed. Possible asymptotic regimes are: high-dimensional setup $N \to \infty$ (the leading case in the literature for high-dimensional models $Y = \mathbb{R}^N$), decreasing noise level $\sigma \to 0$, or their combination, e.g., $\sigma = N^{-1/2}$ and $N \to \infty$.

For inference on $\theta$, we exploit the empirical Bayes approach and make connection with the penalization method. Since any Bayesian approach always delivers also a posterior $\pi(\theta|Y)$ (in the posteriors, we will use the variable $\vartheta$ to distinguish it from the true parameter $\theta$), an accompanying problem of interest is the contraction of the resulting (empirical Bayes) posterior to the “true” structured $\theta$ from the frequentist perspective of the “true” measure $\mathbb{P}_\vartheta$, which is the distribution of data $Y$ from (1.1). The quality of posterior is characterized by the the posterior contraction rate. In this paper we allow this to be a local quantity, i.e., depending on the true $\theta$, while usually in the literature on Bayesian nonparametrics it is a global quantity related to the minimax estimation rates over certain classes. Despite the rapidly growing number of papers about particular high-dimensional and nonparametric models and structures, there are very few approaches in both frequentist and Bayesian literature that can deal with general classes of high-dimensional and nonparametric models: general posterior contraction rate results are studied in [29, 58, 28, 30], general frameworks for estimation in [28, 37]. We should especially highlight the paper [28] which provided us with important insights for the present study (although our approach is very different). However, all estimation (and posterior contraction) results do not reveal how far the optimal estimator (posterior) is from the “true” $\theta$. It is of great importance to quantify this uncertainty, which can be cast into the problem of constructing confidence sets for $\theta$.

### 1.1. Uncertainty quantification problem

One of the prime goals in this paper is to construct confidence sets with optimal properties for model (1.1). The size of a confidence set is measured by the smallest radius of a ball containing this set, hence it suffices to consider confidence balls. For the usual norm $\|\cdot\|$ in $\Theta$, a random ball in $\Theta$ is $B(\hat{\theta}, \hat{r}) = \{\theta \in \Theta : \|\hat{\theta} - \theta\| \leq \hat{r}\}$, where the center $\hat{\theta} = \hat{\theta}(Y) : \mathcal{Y} \mapsto \Theta$ and radius $\hat{r} = \hat{r}(Y) : \mathcal{Y} \mapsto \mathbb{R}_+ = [0, +\infty]$ are measurable functions of the data $Y$. Let us introduce the optimality framework for uncertainty quantification. The goal is to construct such a confidence ball $B(\hat{\theta}, C\hat{r})$ that for any $\alpha_1, \alpha_2 \in (0, 1]$ and some functional $R(\theta) = R_{\sigma,N}(\theta)$, $R : \Theta \mapsto \mathbb{R}_+$, there exist $C, c > 0$ such that

\[
\sup_{\theta \in \Theta_0} \mathbb{P}_\vartheta(\theta \notin B(\hat{\theta}, C\hat{r})) \leq \alpha_1, \quad \sup_{\theta \in \Theta_1} \mathbb{P}_\vartheta(\hat{r} \geq cR(\theta)) \leq \alpha_2,
\]

(1.2)

for some $\Theta_0, \Theta_1 \subseteq \Theta$. The function $R(\theta)$, called the radial rate, is a benchmark for the effective radius of the confidence ball $B(\theta, C\hat{r})$. The first expression in (1.2) is called coverage
relation and the second size relation. Notice that our approach is local (and hence genuinely adaptive) as the radial rate $R(\theta)$ is a function of the “true” parameter $\theta$. The minimax adaptive version of (1.2) for a scale $\{\Theta_s, s \in S\}$ (indexed by a structural parameter $s \in S$, e.g., smoothness or sparsity) would be obtained by taking $\Theta_0 = \Theta_1 = \Theta_s$ and the global radial rate $R(\theta) = r(\Theta_s)$, for all $\theta \in \Theta_s, s \in S$, where $r^2(\Theta_s) = \inf_{\tilde{\theta}} \sup_{\theta \in \Theta_s} \mathbb{E}_\theta ||\tilde{\theta} - \theta||^2$ is the minimax estimation rate over the sets $\Theta_s$.

Coming back to our local framework (1.2), it is desirable to find the smallest $R(\theta)$ and the biggest $\Theta_0, \Theta_1$, for which (1.2) holds. These are contrary requirements, so we have to trade them off against each other. There are different ways of doing this, leading to different optimality frameworks. A traditional optimality framework commonly pursued in the literature (in earlier papers on the topic) was to insist on $\Theta_0 = \Theta$ in (1.2), i.e., considered confidence sets (called “honest” in some papers) must satisfy the coverage property uniformly over the entire space. Then one tries to find a “honest” confidence set with the fastest radial rate $R(\theta)$ and the biggest set $\Theta_1$ (preferably $\Theta = \mathbb{R}^N$). However, it turned out that pursuing such an optimality framework often leads to discarding many good procedures and optimality of uninteresting ones. Many “good” confidence sets are not “honest”, therefore cannot be optimal and effectively excluded from the consideration. Besides, the results of [39, 3] (formulated for the high-dimensional setting $\Theta = \mathbb{R}^N$) say basically that the radial rate $R(\theta)$ cannot be of a faster order than $\sigma N^{1/4}$ for every $\theta$ and is at least of the order $\sigma N^{1/2}$ for some $\theta$. This means that, in the situations when the targeted optimal size $r(\theta)$ (local oracle or global minimax $r(\Theta_s)$) can be of a smaller order than $\sigma N^{1/4}$ for some $\theta$’s (which is typically the case, e.g., for smoothness and sparsity structures), this optimal size cannot be attained in the size relation uniformly over $\Theta$ and necessarily $R(\theta) \gg r(\theta)$ for some $\theta \in \Theta'$. Thus, insisting on $\Theta_0 = \Theta$ implies that either the radial rate $R(\theta)$ or the set $\Theta_1$ in the size relation has to be sacrificed: $\Theta_1 = \Theta$ but $R(\theta) \gg r(\theta)$ for $\theta \in \Theta'$, or $R(\theta) = r(\theta)$ but $\Theta_1 = \Theta \setminus \Theta'$.

Another, seemingly more reasonable approach to optimality developed recently in the literature is to sacrifice in the set $\Theta_0 = \Theta \setminus \Theta_{\text{dec}}$ by removing a preferably small portion of “deceptive parameters” $\Theta_{\text{dec}}$ from $\Theta$ in the coverage property, so that the size property would then hold with $R(\theta) = r(\theta)$ uniformly over $\Theta_1 = \Theta$. To summarize, there are situations when the overall uniform coverage and optimal size properties cannot hold together and it is necessary to sacrifice at least one of these. This is the core of the so called deceptiveness issue in the uncertainty quantification problem. This deceptiveness phenomenon is well understood only for smoothness and sparsity structures; see [4, 8] and further references therein. Typically, global radial rates for specific smoothness (or sparsity) structures are studied in the literature. Local results, delivering also (global) adaptive minimax results for smoothness and sparsity structures, are obtained in [4, 8].

1.2. The scope

The keywords summarizing the main novel features of our approach in this paper are general, robust, local, refined, and EBR-scale.
The approach is general because we develop a general framework of projection structures and study inference problems within this framework by using empirical Bayes and penalization methods. The main inference problem is the uncertainty quantification, but on the way we solve the estimation problem, posterior contraction problem and (a weak version of) structure recovery problem as well. As the proposed general framework unifies a broad class of models with various structures, interesting and important on their own right (including graphical/network models), the general framework results deliver a whole avenue of results (many new ones and some known in the literature) for particular models and structures as consequences. There are numerous examples of models and structures falling into our general framework. In Section 6 of this paper, we consider the following models and structures: 1) signal+noise model with smoothness structure (Sobolev ellipsoids and hyperrectangles, analytic and tail classes); 2) signal+noise model under wavelet basis (Besov balls); 3) signal+noise model with (multi-level) sparsity structure (multi-level sparsity is considered for the first time); 4) noisy function on a large graph (Laplacian graph) with smoothness structure; 5) density estimation with smoothness structure; 6) biclustering model (also for stochastic block model and graphon classes); 7) linear regression with sparsity structure, with group sparsity, with group clustering, and with mixture structure; 8) aggregation in nonparametric regression; 9) isotonic, unimodal and convex regressions; 10) dictionary learning; 11) mean matrix with submatrix sparsity; 12) covariance matrix with banding and sparsity structures. Almost all the results on uncertainty quantification problem are new, many known results are improved (obtained local rates improve upon global ones from the literature), some new structures (like multi-level sparsity) are studied for the first time. We emphasize that the scope of our approach extends further than just these specific cases. In fact, the results are readily obtained for any specific model and structure falling into the proposed general framework.

The approach is robust in that the model is allowed to be misspecified. Namely, we introduce a family of normal priors, propose an empirical Bayes procedure, and use the normal likelihood, whereas the true model in (1.1) does not have to be normal. In fact, the distribution of $\xi$ in (1.1) is not known, the $\xi_i$’s are not necessarily iid (and not even independent), but only satisfying certain mild exchangeable exponential moment condition; see Condition (A1) in Section 2.

The approach is local in that the quality of the inference procedures is measured by the local quantity, the oracle rate $r(\theta)$ (the best rate over a certain family of local rates) which is the best trade-off between the approximation error by a projection structure and the complexity of that approximating projection structure. In a way, it measures the amount of projection structure for each $\theta$: the smaller $r(\theta)$, the more structured $\theta$. To the best of our knowledge, there is no general framework on uncertainty quantification, neither in global nor in local settings. This paper attempts to fill this gap by developing the novel local approach, namely, the radial rate $R(\theta)$ in (1.2) is allowed to be a function of $\theta$, preferably coinciding with the oracle rate $r(\theta)$. The proposed local approach is more powerful than global in that we do not need to impose any specific projection structure, because the local approach automatically exploits the “effective” projection structure of each underlying $\theta$. 


and our local results imply a whole panorama of global minimax adaptive results over various scales at once, see examples in Section 6.

We construct a confidence ball by using the empirical Bayes posterior quantities. Since we want the size of our confidence sets to be of the oracle rate order, this comes with the price that the coverage property can hold uniformly only over some set of parameters satisfying the so-called excessive bias restriction (EBR) $\Theta_0 = \Theta_{eb} \subseteq \Theta$. The main result consists in establishing the optimality (1.2) of the constructed confidence ball for the optimality framework $\Theta_0 = \Theta_{eb}$, $\Theta_1 = \Theta$ and the local radial rate $r(\theta)$. In addition, we also treat the optimality framework with $\Theta_0 = \Theta_1 = \Theta$ in (1.2) by constructing an alternative confidence ball such that its radius is of the order $\sigma N^{1/4} + r(\theta)$. Thus, insisting on the overall uniformity in the coverage and size relations leads to the extra term $\sigma N^{1/4}$ in the expression for the effective radial rate $R(\theta) = r(\theta) + \sigma N^{1/4}$. This fact has also been observed by [45] for the case of linear regression with two sparsity classes. Interestingly, this alternative construction of confidence ball is more preferable for some particular models and structures, e.g., biclustering model (stochastic block model), dictionary learning; see alternative construction of confidence ball is more preferable for some particular models and structures, e.g., biclustering model (stochastic block model), dictionary learning; see Section 6. The point is that, for those models and structures, the extra term $\sigma N^{1/4}$ does not increase the order of the radial rate because $\sigma N^{1/4} \leq cr(\theta)$ for the “majority” of $\theta$’s, precisely, for all $\theta \in \Theta \setminus \tilde{\Theta}$, with some “thin” set $\tilde{\Theta}$. The set $\tilde{\Theta}$ can be informally described as a set of “highly structured” parameters. This means that, modulo the set $\tilde{\Theta}$ of “highly structured” parameters, there is no deceptiveness issue for those cases. (Speaking informally, these models and structures are already “too difficult” for the term $\sigma N^{1/4}$ to spoil the radial rate.)

The approach is refined, namely, we derive the local posterior contraction result for the resulting empirical Bayes posterior $\hat{\pi}(\theta|Y)$ in the refined non-asymptotic exponential probability bound formulation: $\sup_{\theta \in \Theta} E_{\theta} \hat{\pi}(|\theta - \theta|^2 \geq M_0 \sigma^2(\theta) + M \sigma^2|Y| \leq H_0 e^{-m_0 M}$ for some fixed $M_0, H_0, m_0 > 0$ and arbitrary $M \geq 0$, uniformly in $\theta \in \Theta$. Besides, we obtain the local estimation result for the empirical Bayes posterior mean estimator, also as a non-asymptotic exponential probability bound. This refined formulation provides a rather sharp characterization of the quality of the posterior and the estimator, (finer than, e.g., traditional oracle inequalities in expectation or asymptotic claims for posterior contraction), allowing subtle analysis for various asymptotic regimes. These results, besides being ingredients for the uncertainty quantification problem, are of interest and importance on its own as they establish the local (oracle) optimality of the empirical Bayes posterior and estimator in this refined formulation. As we have mentioned already, the local results imply in turn the corresponding global (minimax adaptive) results, also in the refined formulation.

Finally, recall that in order to have the targeted optimal radial rate $r(\theta)$ in the size relation of the optimality framework (1.2), the overall uniformity in the coverage property of (1.2) must be sacrificed: $\Theta_0 = \Theta_{eb} \subseteq \Theta$, where $\Theta_{eb}$ is a set of parameters satisfying the so-called excessive bias restriction (EBR). This set is expected to be of the same type for all models and structures coming from the general framework (1.1): $\Theta_{eb} = \Theta_{eb}(t) = \{\theta \in \Theta : b(\theta) \leq tV(\theta)\}$, where $b(\theta)$ and $V(\theta)$ are the approximation and complexity (or “bias” and “variance”) parts of the squared oracle rate $r^2(\theta) = b(\theta) + V(\theta)$ for the
corresponding particular model and structure. It turns out that the EBR leads to a new
EBR-scale \( \{ \Theta_{eb}(t), t \geq 0 \} \), which gives a slicing of the entire space: \( \Theta = \bigcup_{t \geq 0} \Theta_{eb}(t) \). This
slicing is very suitable for uncertainty quantification and provides a new perspective at the
deceptiveness issue: basically, each parameter \( \theta \) is deceptive (or non deceptive) to some
extent. It is the parameter \( t \) that measures the deceptiveness in \( \Theta_{eb}(t) \) and affects the size
of the confidence ball needed to provide a guaranteed high coverage uniformly over \( \Theta_{eb}(t) \).

The paper is organized as follows. In Section 2 we introduce the notation, the prior,
describe the empirical Bayes procedure in detail, make a link with the penalization method,
and provide some conditions. Section 3, where we also introduce the EBR, contains the
main results of the paper. The proofs of the lemmas and theorems are given in Sections 4
and 5 respectively. In Section 6, we demonstrate how the main general results specify to
the above mentioned examples of models and structures in local and minimax settings.

2. Preliminaries

First we introduce some notation and notions, then introduce some conditions and a mixture
normal prior. Next, by applying the empirical Bayes approach to the normal likelihood
(recall that the true model does not have to be normal), we derive an empirical Bayes posterior
which we will use in the construction of the estimator and the confidence ball.

At first reading, one may want to skip this section and go ahead to Section 3 (one will
only need to consult some definitions from Section 2) which contains the main results of
the paper.

2.1. Notation

For \( n \in \mathbb{N} \), denote \([n] = \{1, 2, \ldots, n\} \) and \([n]_0 = \{0\} \cup [n] \); for a Hilbert space \( \mathcal{Y} \), \((y, z)\)
denotes the scalar product between \( y, z \in \mathcal{Y} \), \( \mathbb{R}_+ = [0, +\infty] \), \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \). For an \((n_1 \times n_2)\)-matrix
\( x = (x_{ij}) \in \mathbb{R}^{n_1 \times n_2} \), we will interchangeably use the same notation \( x \) to denote the
vector \( x = \text{vec}[(x_{ij})] = (x_{11}, x_{12}, \ldots, x_{n_1n_2})^T \). Conversely, for any \( x \in \mathbb{R}^{n_1n_2} \) we can use
matricized indexing \( x = (x_{11}, x_{12}, \ldots, x_{n_1n_2})^T \). Most of the time the vector notation will be used,
and it should be clear from the context which notation is meant in each expression.

For two nonnegative sequences \((a_l)\) and \((b_l)\), \( a_l \lesssim b_l \) means \( a_l \leq cb_l \) for all \( l \) (its range
should be clear from the context) with some absolute \( c > 0 \), and \( a_l \asymp b_l \) means that \( a_l \lesssim b_l \)
and \( b_l \lesssim a_l \).

For a set \( S \), \(|S|\) denotes its cardinality. We will often denote matrices and operators
by upright capital letters, the identity matrix is denoted by \( I \), \( 1_E = 1\{E\} \) stands for the
indicator function of the event \( E \). As usual, \( N(\mu, \Sigma) \) is the multivariate normal distribution
with mean \( \mu \) and covariance matrix \( \Sigma \), its density at point \( x \) is denoted by \( \varphi(x, \mu, \Sigma) \).
The dimensions of matrices and normal distributions should be clear from the context. Let
\( P_I = I - P_I \) be the projection operator onto the orthogonal complement \( \mathbb{L}_I^\perp \) of \( \mathbb{L}_I \). We use
both notation \( P_I \) and \( P_{\mathbb{L}_I} \) \((P_I^\perp \) and \( P_{\mathbb{L}_I}^\perp \) to denote the projection operator onto the linear
subspace \( \mathbb{L}_I \) (onto the orthogonal complement \( \mathbb{L}_I^\perp \) of \( \mathbb{L}_I \)).
The symbol \( \triangleq \) will refer to equality by definition, for \( a, b \in \mathbb{R} \), \( (a \lor b) = \max\{a, b\} \), \( (a \land b) = \min\{a, b\} \), \( \lfloor a \rfloor = \max\{m \in \mathbb{Z} : m \leq a\} \). Throughout we assume the conventions: \( |\emptyset| = 0 \), \( \sum_{I \in \emptyset} a_I = 0 \) for any \( a_I \in \mathbb{R} \) and \( 0 \log (a/0) = 0 \) (hence \((a/0)^0 = 1\)) for any \( a > 0 \).

2.2. Slicing into structural layers, complexity of a layer

As we mentioned in the introduction, useful inference is not possible without some structure on parameter \( \theta \). The structure on each \( \theta \in \Theta \) is represented by the slicing \( \Theta \subseteq \bigcup_{I \in \mathcal{I}} \mathbb{L}_I \). Indeed, then for any \( \theta \in \Theta \) there exists an \( I \in \mathcal{I} \) such that \( \theta \in \mathbb{L}_I \), this \( I \) (and \( \mathbb{L}_I \)) has the meaning of the structure of that \( \theta \). Structure \( I \) of \( \theta \) is always determined via the corresponding linear space \( \mathbb{L}_I \ni \theta \), so by saying “\( I \) is the structure of \( \theta \)” we mean that \( \theta \in \mathbb{L}_I \) (or can be well approximated by \( P_I \theta \)).

**Remark 1.** There may be \( \mathbb{L}_I = \mathbb{L}_{I'} \) for different \( I, I' \in \mathcal{I} \), in other words, the family of structures \( \mathcal{I} \) can have redundancy. Without loss of generality, we could assume that the family \( \mathcal{I} \) is “cleaned up” in the sense that each subspace \( \mathbb{L} \in \mathcal{L}_\mathcal{I} \) is represented in \( \mathcal{I} \) by only one (arbitrary) element \( J = J(\mathbb{L}) \) from the set \( \{I \in \mathcal{I} : \mathbb{L}_I = \mathbb{L}\} \). Mathematically, this means that the resulting “cleaned up” family \( \mathcal{I} \) of structures consists of equivalence classes on the original collection of all structures with the equivalence relation: \( I_1 \sim I_2 \) if and only if \( \mathbb{L}_{I_1} = \mathbb{L}_{I_2} \), so that \( |\mathcal{I}| = |\mathcal{L}_\mathcal{I}| \) in this case.

However, in general \( |\mathcal{L}_\mathcal{I}| \leq |\mathcal{I}| \) and this redundancy can be beneficial in some practical situations when searching (or optimizing in an inference procedure) over a possibly redundant family of structures \( \mathcal{I} \) can be described and realized easier than over the “cleaned up” version of it. The only price for this redundancy is a bigger sum (because of more terms) in Condition (A2), resulting in a bigger constant \( C_\nu \) in Condition (A2) for a redundant \( \mathcal{I} \). In many situations this is a mild price, see for example Sections 6.7.3 and 6.7.4.

Clearly, \( \theta \) can have many structures, and we would like to distinguish the “simplest” structure of \( \theta \). For that, we introduce a measure of structure complexity below. Suppose the following condition is fulfilled for the vector \( \xi \) from (1.1): for some \( \alpha > 0 \) and \( (d_I)_{I \in \mathcal{I}} \) with \( d_I \geq 0 \),

\[
\mathbb{E}_\theta \exp \left\{ \alpha \|P_I \xi\|^2 \right\} \leq \exp(d_I) \quad \text{for all } I \in \mathcal{I} , \quad \theta \in \Theta. \tag{A0}
\]

**Remark 2.** It is desirable to have the bound (A0) in the tightest possible form, by determining the smallest sequence \( (d_I)_{I \in \mathcal{I}} \) for which (A0) holds with a given \( \alpha > 0 \). (A0) is useful only when all the \( d_I \)’s are finite. (Notice that (A0) always holds for any \( \alpha > 0 \), if the \( d_I \)’s are allowed to be infinite.) Thus, instead of (A0), we could equivalently assume \( \sup_{\theta \in \Theta} \mathbb{E}_\theta \exp \left\{ \alpha \|P_I \xi\|^2 \right\} < \infty \) for all \( I \in \mathcal{I} \). Then the smallest \( d_I \)’s for which (A0) holds are

\[
d_I = \sup_{\theta \in \Theta} \log \left( \mathbb{E}_\theta \exp \left\{ \alpha \|P_I \xi\|^2 \right\} \right), \quad I \in \mathcal{I}. \tag{2.1}
\]
The quantity $d_I$ can be seen as a \textit{statistical dimension} of structure $I$, reflecting in a way the \textit{complexity} of the structure $I$ (space $\mathbb{L}_I$): the bigger $d_I$, the more complex the structure $I$. Notice that if the distribution of $\xi$ does not depend on $\theta$, then there is no $\sup_{\theta \in \Theta}$ in (2.1). Typically, in such cases $d_I \propto \dim(\mathbb{L}_I)$. The bound (A0) holds, for example, for standard normal $\xi$ with $\alpha = 0.43$ and $d_I = \dim(\mathbb{L}_I)$; see Remark 5 below.

A subfamily $\mathcal{J} \subseteq \mathcal{I}$ of structures is called \textit{structural layer} (or just layer) in $\mathcal{I}$. By $\mathcal{L}_J = \{\mathbb{L}_I, I \in J\}$ we denote the corresponding layer in the family of all linear subspaces $\mathcal{L}_I = \{\mathbb{L}_I, I \in \mathcal{I}\}$. Now we characterize the \textit{complexity of a layer $\mathcal{J} \subseteq \mathcal{I}$}. A version of this key notion (in a different context) is present in [28]. Let us give a simple (but important) heuristics for controlling the maximal projected error $\max_{I \in J} \|P_I \xi\|^2$ over a layer $J$. Denote $d_J = \max_{I \in J} d_I$. Using (A0) and Jensen’s inequality, we derive

$$\exp \left\{ \alpha \mathbb{E}_\theta \max_{I \in J} \|P_I \xi\|^2 \right\} = \exp \left\{ \alpha \mathbb{E}_\theta \max_{L_I \in \mathcal{L}_J} \|P_I \xi\|^2 \right\} \leq \mathbb{E}_\theta \exp \left\{ \alpha \max_{L_I \in \mathcal{L}_J} \|P_I \xi\|^2 \right\} \leq \sum_{L_I \in \mathcal{L}_J} \mathbb{E}_\theta e^{\alpha \|P_I \xi\|^2} \leq e^{d_J + \log |\mathcal{L}_J|}.$$  

Hence, under (A0) we control the maximal projected error $\max_{I \in J} \|P_I \xi\|^2$ up to the order of $d_J + \log |\mathcal{L}_J|$. It is this sum that characterizes the complexity of the layer $J$ (layer $\mathcal{L}_J$). Define the complexity of the layer $J$ (layer $\mathcal{L}_J$) as

$$c(J) \triangleq d_J + \log |\mathcal{L}_J| \leq d_J + \log |J|, \quad \text{where} \quad d_J = \max_{I \in J} d_I.$$  

The second equality holds because in general $|\mathcal{L}_J| \leq |J|$ in view of Remark 1.

Next, introduce a surjective function $s : \mathcal{I} \mapsto \mathcal{S}$, for some set $\mathcal{S}$, called the \textit{structural slicing mapping}. This function slices the family $\mathcal{I}$ in layers

$$\mathcal{I}_s = \{I \in \mathcal{I} : s(I) = s\}, \quad s \in \mathcal{S},$$

i.e., $\mathcal{I} = \cup_{s \in \mathcal{S}} \mathcal{I}_s$. $\mathcal{S}$ marks the collection of all layers $\mathcal{I}_s$. Clearly, the structure $I$ belongs to the layer $\mathcal{I}_{s(I)}$, and any slicing of $\mathcal{I}$ can be realized by appropriate function $s(I), \ I \in \mathcal{I}$. This also leads to the corresponding slicing of the parameter space $\Theta \subseteq \cup_{s \in \mathcal{S}} \mathbb{L}_{\mathcal{I}_s}$. The quantity $s(I)$ typically describes some features of the space $\mathbb{L}_I$. For example, $s(I)$ can be the dimension (or some function of it) of $\mathbb{L}_I$.

According to (2.3) and Remark 1, the complexity of the layer $\mathcal{I}_s$ is then

$$c(s) = d_s + \log |\mathcal{L}_s| \leq d_s + \log |\mathcal{I}_s|, \quad s \in \mathcal{S},$$

where, slightly abusing notation, we adopted the following notional conventions for brevity: $c(s) = c(\mathcal{I}_s)$, $d_s = d_{\mathcal{I}_s} = \max_{J \in \mathcal{I}_s} d_I$, $\mathcal{L}_s = \mathcal{L}_{\mathcal{I}_s} = \{\mathbb{L}_I, I \in \mathcal{I}_s\}$, with $c(J)$ defined by (2.3).
2.3. Conditions

Recall that we have introduced a structural slicing mapping \( s : \mathcal{I} \mapsto \mathcal{S} \), which slices the family of structures \( \mathcal{I} \) and the parameter space \( \Theta \) in structural layers: \( \mathcal{I} = \bigcup_{s \in \mathcal{S}} \mathcal{I}_s \) and \( \Theta \subseteq \bigcup_{s \in \mathcal{S}} \mathcal{I}_s \), \( s \in \mathcal{S} \), where \( \mathcal{I}_s = \{ I \in \mathcal{I} : s(I) = s \} \) and \( \mathcal{L}_s = \{ L_I : s(I) = s \} \).

In previous section we proposed condition (A0) which led to the notion (2.3) of layer complexity \( c(J) \). We can relax condition (A0) by imposing it on the layers \( \mathcal{I}_s \) only (instead of all \( I \in \mathcal{I} \)). Precisely, throughout the rest of the paper we impose the following so called exchangeable exponential moment condition on the random vector \( \xi \) from (1.1).

**Condition (A1).** For some structural slicing mapping \( s : \mathcal{I} \mapsto \mathcal{S} \), sequence \((d_s)_{s \in \mathcal{S}}\) and \( \alpha > 0 \),

\[
\mathbb{E}_\theta \exp \left\{ \alpha \| P_I \xi \|^2 \right\} \leq \exp \{ d_s(I) \}, \quad I \in \mathcal{I}, \quad \theta \in \Theta.
\]

Without loss of generality, assume \( \alpha \in (0, 1] \) and \( d_s(I) \gtrsim \dim(\mathcal{L}_I), \ I \in \mathcal{I} \).

**Remark 3.** Note that there is no need to assume \( d_s(I) \gtrsim \dim(\mathcal{L}_I), \ I \in \mathcal{I} \), as all the results below hold true (with small adjustments) for any \((d_s)_{s \in \mathcal{S}}\) for which (A1) is fulfilled. The only place where this is used is the relation (4.4), and this can easily be fixed by modifying the empirical Bayes posterior (2.17) for \( I \) (use \( d_s(I) \) instead of \( \dim(\mathcal{L}_I) \) in (2.17)).

**Remark 4.** In view of Remark 2, (A1) is equivalent to the following condition: for \( \alpha > 0 \),

\[
\sup_{\theta \in \Theta} \mathbb{E}_\theta \exp \left\{ \alpha \| P_I \xi \|^2 \right\} < \infty, \quad I \in \mathcal{I}.
\]

Then the smallest \( d_s \)'s for which (A1) is fulfilled are \( d_s = \max_{I \in \mathcal{I}_s} d_I \), where \( d_I \) is defined by (2.1). Referring to Remark 2, \( d_s \) can be interpreted as statistical dimension of the layer \( \mathcal{I}_s \), which is the first term of its total complexity (2.4).

**Remark 5.** Condition (A1) holds for high-dimensional independent normal \( \xi \)'s with \( d_s(I) = \dim(\mathcal{L}_I) \), irrespective of the linear spaces \( \mathcal{L}_I, \ I \in \mathcal{I} \). In fact, in this case even stronger Condition (A0) holds with \( d_I = \dim(\mathcal{L}_I) \). Indeed, if \( \xi_i \overset{\text{ind}}{\sim} N(0, 1), \ |P_I \xi|^2 \sim \chi^2_{\dim(\mathcal{L}_I)} \), the chi-squared distribution with \( d_I = \dim(\mathcal{L}_I) \) degrees of freedom. Hence, for any \( t < \frac{1}{2} \) we have that \( \mathbb{E} \exp \{ t \| P_I \xi \|^2 \} = (1 - 2t)^{-d_I/2} \). Since \( (1 - 2t)^{-d_I/2} \leq e^{d_I} \) for any \( t \leq (1 - e^{-2})/2 \approx 0.432. \) By taking \( t = 0.4 \), we derive \( \mathbb{E} e^{0.4 |P_I \xi|^2} \leq e^{d_I} \). Hence, Condition (A0) is fulfilled with \( \alpha = 0.4 \) and \( d_I = \dim(\mathcal{L}_I) \).

Importantly, Condition (A1) allows quite some flexibility, which is crucial when treating concrete models and significantly broadens the range of models falling into our general framework. First, the distribution of \( \xi \) may depend on \( \theta \) (note however that in many important models from Section 6, the distribution of \( \xi \) does not depend on \( \theta \)). Besides, the coordinates \( \xi_i \)'s of \( \xi \) do not have to be iid and may even be non-independent. For example, for the “signal+noise” model with the sparsity structure, it was shown in [8] that Condition (A1) is fulfilled for the \( \xi_i \)'s generated according an autoregressive model.
In case of “signal+noise” model with the sparsity structure, one can relate (cf. [8]) Condition (A1) to the so called sub-gaussianity condition on \( \xi = (\xi_i, i \in [N]) \). The vector \( \xi \) is called sub-gaussian with parameter \( \rho > 0 \) if
\[
P(|x, \xi| > t) \leq e^{-\rho t^2} \quad \text{for all } t \geq 0, \quad v \in \mathbb{R}^N \text{ such that } \|v\| = 1. \tag{2.5}
\]

In this case, in [8] we showed that the sub-gaussianity condition is equivalent to Condition (A1) for independent \( \xi \)'s, for dependent \( \xi \)'s, the sub-gaussianity condition (2.5) and Condition (A1) are close, but in general incomparable. For example, if \( \xi_i = \xi_0, \ i \in [n] \), for some bounded random variable \( \xi_0 \) (say, uniform on \([-1, 1]\)), then a version of Condition (A1) (for the sparsity structure) trivially holds whereas the sub-gaussianity condition is not fulfilled.

It is desirable to use a slicing \( s : \mathcal{I} \mapsto \mathcal{S} \) that is parsimonious in the sense that the maximum \( d_s = \max_{I \in \mathcal{I}} d_I \) degenerates, i.e., \( d_I = d_J \) for all \( I, J \in \mathcal{I}_s \), so \( d_s(I) = d_I \). In other words, \( d_I = h(s(I)) \), \( I \in \mathcal{I} \), for some function \( h : \mathcal{S} \mapsto \mathbb{N}_0 \). Since typically \( d_I \asymp \dim(\mathbb{L}_I) \), in many examples of Section 6 we will take a slicing \( s \) such that \( \dim(\mathbb{L}_I) = h(s(I)) \), \( I \in \mathcal{I} \), for some function \( h \).

The complexity \( c(s) \) of the layer \( \mathcal{I}_s \) is defined in the previous section by (2.4). Instead, from now on we will work with a majorant of the layer complexity, some function \( \rho : \mathcal{S} \mapsto \mathbb{R}_+ \) such that
\[
\rho(s) \geq d_s + \log |\mathcal{I}_s| \geq c(s), \quad s \in \mathcal{S}. \tag{2.6}
\]

**Remark 6.** We can use an up-to-a-constant majorant \( \rho(s) \gtrsim c(s) \) instead of (2.6) by adjusting \( \alpha \in (0, 1] \) in (A1), but without loss of generality we stick to (2.6) for the sake of a clean mathematical exposition.

The reasoning in (2.2) gives a heuristic explanation why later on at least a multiple of the complexity \( c(s) \) must be present in the prior on \( I \) (and as penalty term in the penalization method). This quantity will also enter the local (oracle) rate. It is therefore desirable to use the smallest possible majorant \( \rho(s) \) in (2.6) in order to derive stronger results. In this light, the best majorant in (2.6) is the complexity itself \( \rho(s) = d_s + \log |\mathcal{I}_s| = c(s) \) for the “cleaned up” family of structures \( \mathcal{I} \) (see Remark 1). On the other hand, in the end any majorant \( \rho(s) \gtrsim c(s) \) will do the job. The reason to allow an arbitrary majorant \( \rho(s) \) is that \( d_s \) and \( |\mathcal{I}_s| \) (or \( |\mathcal{L}_s| \)) may be difficult to compute, whereas some closed form upper bounds can be derived. Of course, this comes at the price of a bigger resulting local rate because this majorant will then enter the local rate.

In the proof of Theorem 1 below, we will need a bound for \( \mathbb{E}_\theta \|P_I \xi\|^4 \) for each \( I \in \mathcal{I} \). Condition (A1) ensures such a bound. Indeed, since \( x^2 \leq e^{2x} \) for all \( x \geq 0 \), by the Hölder inequality and (A1), we obtain for any \( t \in (0, 1/2] \) and \( I \in \mathcal{I} \),
\[
\mathbb{E}_\theta \|P_I \xi\|^4 \leq \frac{\mathbb{E}_\theta e^{2\alpha \|P_I \xi\|^2}}{(t\alpha)^2} \leq \frac{(\mathbb{E}_\theta e^{\alpha \|P_I \xi\|^2})^{2t}}{(t\alpha)^2} \leq \frac{e^{2t\rho(s(I))}}{(t\alpha)^2}. \tag{2.7}
\]
In case $\xi_i \overset{\text{ind}}{\sim} N(0, 1)$, Condition (A0) is fulfilled with $d_I = \dim(L_I)$, so that (A1) holds also with $d_s = \max_{I \in \mathcal{I}} d_I$. As $\|P_I \xi\|^2 \sim \chi^2_{d_I}$, instead of (2.7), a better bound can be used in this case: $[\mathbb{E}\|P_I \xi\|^2]^{1/2} = (d_I^2 + 2d_I)^{1/2} / 2 < d_I + 1 \leq \rho(s(I)) + 1$.

We finish this section by introducing the conditions on the family of structures $\mathcal{I}$ and the majorant $\rho(s)$ from (2.6), which we will need in the theorems.

**Condition (A2).** For some $\nu, C_\nu > 0$, $\sum_{I \in \mathcal{I}} e^{-\nu \rho(s(I))} \leq C_\nu$. \hfill (A2)

**Remark 7.** Notice that in view of (2.6), for $\nu \geq 1$ we obtain the bound

$$\sum_{I \in \mathcal{I}} e^{-\nu \rho(s(I))} = \sum_{s \in \mathcal{S}} \sum_{I \in L_s} e^{-\nu \rho(s(I))} \leq \sum_{s \in \mathcal{S}} e^{-\nu d_s -(\nu - 1) \log |L_s|} \leq \sum_{s \in \mathcal{S}} e^{-\nu d_s}.$$  

So, (A2) is satisfied if $\sum_{s \in \mathcal{S}} e^{-\nu d_s} \leq C_\nu$ for $\nu \geq 1$. Informally, this condition means that the majorant $\rho$ is large enough to match the massiveness (reflected by the cardinalities $|L_s|$, $s \in \mathcal{S}$) and the complexity (reflected by the sequence $(d_s)_{s \in \mathcal{S}}$) of the family of structures $\mathcal{I}$.

**Condition (A3).** For any $I_0, I_1 \in \mathcal{I}$ there exists $I' = I'(I_0, I_1) \in \mathcal{I}$ such that

$$(L_{I_0} \cup L_{I_1}) \subseteq L_{I'} \text{ and } \rho(s(I')) \leq \rho(s(I_0)) + \rho(s(I_1)).$$ \hfill (A3)

**Remark 8.** Typically, Condition (A3) is fulfilled with $I' = I'(I_0, I_1) \in \mathcal{I}$ such that $L_{I'} = L_I + L_{I_1}$. This is the case for almost all examples in Section 6.

Let us formulate a slightly stronger version of Condition (A3) called Condition (A3'): for any $I_0, I_1 \in \mathcal{I}$ there exist $I' = I'(I_0, I_1) \in \mathcal{I}$ and $I'' = I''(I_0, I_1) \in \mathcal{I}$ such that $L_{I'} = L_{I_0} + L_{I_1}$, $L_{I''} = L_{I_0} \cap L_{I_1}$, $\rho(s(I')) \leq \rho(s(I_0)) + \rho(s(I_1))$ and $\rho(s(I'')) \leq \rho(s(I_0))$.

The constants $\alpha \in (0, 1]$ and $\nu > 0$ will be fixed throughout and we omit the dependence on these constants in all further notation.

### 2.4. Multivariate normal prior

In this section we construct an (empirical) Bayesian procedure and make connection with the penalization method for making inference on $\theta$. Recall that the true parameter $\theta$ is assumed to be well approximated by its structured version $P_I \theta$ (e.g., it itself can be structured $\theta = P_I \theta$, for some “true” structure $I^* \in \mathcal{I}$. The true structure $I^*$ is unknown, so at a later stage we will put a prior on the family of structures $\mathcal{I}$. For now, given a structure $I$, consider the model $Y = P_I \theta + \sigma \xi$, approximating the original model (1.1), where $P_I$ is the projection operator onto space $L_I$, and put first an “unstructured” prior $\Pi$ on the “unstructured” $\theta \in \Theta$ as follows: $\theta \sim \Pi = N(\mu, \kappa \sigma^2 I)$, where $\kappa = e - 1$ and the parameter $\mu \in \mathcal{Y}$ is to be chosen by the empirical Bayes method later. The “unstructured” prior $\Pi$ on $\theta$ leads to the “structured” prior $\pi_I$ on the “structured” $\theta^I \equiv P_I \theta$:

$$\pi_I(\theta) = N(P_I \mu, \kappa \sigma^2 P_I), \quad I \in \mathcal{I}, \quad \kappa = e - 1.$$ \hfill (2.8)
In this way, we constructed the conditional prior on $\theta$ given $I$: $\theta|I \sim \pi_I(\theta)$. The rather specific choice of $\kappa = e^{-1}$ is made for the sake of clean mathematical exposition in later calculations, many other choices are actually possible.

The next very important step in the Bayesian analysis below is that we use the normal likelihood $\ell(\theta, Y) = \bigotimes_i N(\theta_i, \sigma^2)$, whereas the “true” model $Y \sim P_\theta$ is not assumed to be normal, but only satisfying Condition (A1). Formally applying Bayesian approach to this prior and the normal likelihood $\ell(\theta, Y)$ delivers the marginal distribution $Y \sim P_{Y,I} = N(P_I\mu, I + \kappa P_I)$ and the following posterior distribution on $\theta$:

$$\pi_I(\vartheta|Y) = N\left(\frac{1}{\kappa+1}P_I\mu + \frac{\kappa}{\kappa+1}P_I Y, \frac{\kappa \sigma^2}{\kappa+1} P_I\right).$$ (2.9)

Note that in general the covariance matrix in (2.8) is not invertible, but the Bayes formula for the conjugate normal-normal model still holds with the Moore-Penrose inverse $P_I^-$ of $P_I$ instead of the usual inverse (recall that $P^-=P$ for any projection operator $P$).

Let us now put a prior on $I$:

$$\lambda_I = c_\nu e^{-\nu\rho(s(I))}, \quad I \in I,$$ (2.10)

where $c_\nu$ is the normalizing constant (i.e., $\sum_{I \in I} \lambda_I = 1$), $\rho(s)$ satisfies (2.6), the mapping $s(I)$ is from Condition (A1), the parameter $\nu$ satisfies the relation

$$\nu \geq \bar{\nu} = (32\nu + 10 + \alpha)/(4\alpha),$$ (2.11)

$\alpha$ and $\nu$ are from Conditions (A1) and (A2), respectively.

Combining (2.8) and (2.10) gives the mixture prior on $\theta$: $\pi = \sum_{I \in I} \lambda_I \pi_I$. This leads to the marginal distribution of $Y$: $P_Y = \sum_{I \in I} \lambda_I P_{Y,I}$, $P_{Y,I} = N(P_I\mu, \sigma^2(I + \kappa P_I))$, where the density of the distribution $P_{Y,I} = N(P_I\mu, \sigma^2(I + \kappa P_I))$ is

$$\varphi(\vartheta, P_I\mu, \sigma^2(I + \kappa P_I)) = \frac{e^{-(y-P_I\mu)^T(I-\frac{\kappa}{\kappa+1}P_I)(y-P_I\mu)/(2\sigma^2)}}{(2\pi\sigma^2)^n/2(1+\kappa)^{\dim(L_I)/2}},$$ (2.12)

because $(I + \kappa P)^{-1} = I - \frac{\kappa}{\kappa+1}P$, $\det(I + \kappa P) = (1 + \kappa)^{\rank(P)}$ for any projection operator $P$, and $\rank(P_I) = \dim(L_I)$. The posterior of $\theta$ becomes

$$\pi(\vartheta|Y) = \pi_\nu(\vartheta|Y) = \sum_{I \in I} \pi(\vartheta, I|Y) = \sum_{I \in I} \pi(\vartheta|Y, I) \pi(I|Y),$$ (2.13)

where $\pi(\vartheta|Y, I) = \pi_I(\vartheta|Y)$ is defined by (2.9) and the posterior for $I$ is

$$\pi(I|Y) = \frac{\lambda_I P_{Y,I}}{\sum_{J \in I} \lambda_J P_{Y,J}}.$$ (2.14)
2.5. Empirical Bayes posterior

The parameter $\mu$ is yet to be chosen in the prior. We apply the empirical Bayes approach. The marginal likelihood $P_Y$ is readily maximized with respect to $\mu$: $\text{argmin}_{\mu} \{ (Y-P_IY)^T(I-\frac{\kappa}{\kappa+1}P_J)(Y-P_IY) \} = Y$. Substituting $Y$ instead of $\mu$ in the expressions (2.9), (2.13) and (2.14) yields the empirical Bayes posterior

$$\tilde{\pi}(\vartheta|Y) = \tilde{\pi}_{\vartheta}(\vartheta|Y) = \sum_{I \in \mathcal{I}} \tilde{\pi}(\vartheta|Y,I)\tilde{\pi}(I|Y),$$  \hspace{1cm} (2.15)

called empirical Bayes model averaging (EBMA) posterior, where the EBMA posterior for $\theta$ given $I$ is

$$\tilde{\pi}(\vartheta|Y,I) = \tilde{\pi}_I(\vartheta|Y) = N(P_IY, \frac{\kappa \sigma^2}{\kappa+1} P_I)$$  \hspace{1cm} (2.16)

and the empirical Bayes posterior for $I$ is

$$\tilde{\pi}(I|Y) = \tilde{\pi}_I = \frac{\lambda_I \exp\{-\frac{1}{2\sigma^2}[\|I-P_IY\|^2 + \sigma^2\text{dim}(L_I)]\}}{\sum_{J \in \mathcal{I}} \lambda_J \exp\{-\frac{1}{2\sigma^2}[\|I-P_JY\|^2 + \sigma^2\text{dim}(L_J)]\}}.$$  \hspace{1cm} (2.17)

When deriving (2.17), we used (2.12), $\kappa = e-1$ and the fact that $(I-P)(I-\frac{\kappa}{\kappa+1}P)(I-P) = (I-P)$ for any projection operator $P$. Let $\tilde{E}$ and $\tilde{E}_I$ be the expectations with respect to the EBMA measures $\tilde{\pi}(\vartheta|Y)$ and $\tilde{\pi}_I(\vartheta|Y)$, respectively. Then $\tilde{E}_I(\vartheta|Y) = P_IY, I \in \mathcal{I}$. Introduce the **EBMA posterior mean estimator**

$$\hat{\theta} = \tilde{E}(\vartheta|Y) = \sum_{I \in \mathcal{I}} \tilde{E}_I(\vartheta|Y)\tilde{\pi}(I|Y) = \sum_{I \in \mathcal{I}} (P_IY)\tilde{\pi}(I|Y).$$  \hspace{1cm} (2.18)

Consider yet alternative empirical Bayes posterior. First derive an empirical Bayes structure selector $\hat{I}$ by maximizing $\tilde{\pi}(I|Y)$ over $I \in \mathcal{I}$. This boils down to

$$\hat{I} = \text{argmax}_{I \in \mathcal{I}} \tilde{\pi}(I|Y) = \text{argmin}_{I \in \mathcal{I}} \{ \|Y - P_IY\|^2 + \sigma^2\text{pen}(I) \},$$  \hspace{1cm} (2.19)

which is essentially the **penalization method** with the penalty $\text{pen}(I) = 2\lambda \rho(s(I)) + \text{dim}(L_I)$. Plugging in this into $\tilde{\pi}_I(\vartheta|Y)$ defined by (2.16) gives the corresponding empirical Bayes posterior (called empirical Bayes model selection (EBMS) posterior), yielding also the EBMS mean estimator for $\theta$:

$$\tilde{\pi}(\vartheta|Y) = \tilde{\pi}_I(\vartheta|Y) = N(P_IY, \frac{\kappa \sigma^2}{\kappa+1} P_I), \quad \hat{\theta} = \tilde{E}(\vartheta|Y) = P_IY,$$  \hspace{1cm} (2.20)

where $\tilde{E}$ denotes the expectation with respect to the EBMS measure $\tilde{\pi}(\vartheta|Y)$. Notice that, like (2.15), $\tilde{\pi}(\vartheta|Y)$ defined by (2.20) can also be seen formally as mixture

$$\tilde{\pi}(\vartheta|Y) = \tilde{\pi}_I(\vartheta|Y) = \sum_{I \in \mathcal{I}} \tilde{\pi}_I(\vartheta|Y)\tilde{\pi}(I|Y), \quad \tilde{\pi}(I|Y) = 1\{I = \hat{I}\},$$  \hspace{1cm} (2.21)

where the mixing distribution $\tilde{\pi}(I|Y) = 1\{I = \hat{I}\}$, the empirical Bayes posterior for $I$, is degenerate at $\hat{I}$.
Remark 9. In a way, the EBMA posterior \( \hat{\pi}(\theta|Y) \) defined by (2.15) is “more Bayesian” than the EBMS posterior \( \hat{\pi}(\theta|Y) \) defined by (2.20). Note however that, while the penalization method gives only an estimator, the EBMS method does also yield a posterior.

From now on, by \( \hat{\pi}(\theta|Y) \) we denote either \( \hat{\pi}(\theta|Y) \) defined by (2.15) or \( \hat{\pi}(\theta|Y) \) defined by (2.20), and \( \theta \) will stand either for \( \hat{\theta} \) defined by (2.18) or for \( \theta \) defined by (2.20). In case \( \hat{\pi}(I|Y) = \hat{\pi}(I|Y) = 1\{I = \hat{I}\} \), the meaning of \( \hat{\pi}(I \in G|Y) \) for any \( G \subseteq \mathcal{I} \) is as follows: \( \hat{\pi}(I \in G|Y) = \hat{\pi}(I \in G|Y) = 1\{\hat{I} \in G\} \) so that \( E_\theta \hat{\pi}(I \in G|Y) = P_\theta(\hat{I} \in G) \).

3. Main results

In this section we present the main results of the paper.

3.1. Oracle rate

For \( I \in \mathcal{I} \), consider the projection estimator \( P_I Y \) for estimating \( \theta \). By Condition (A1) and Jensen’s inequality, we obtain the following upper bound for the estimator \( P_I Y \): for some \( C > 0 \),

\[
E_\theta \| \theta - P_I Y \|^2 = \| \theta - P_I \theta \|^2 + \sigma^2 E_\theta \| P_I \xi \|^2 \leq \| \theta - P_I \theta \|^2 + C \sigma^2 d_s(I).
\]

Ideally, we would like to mimic the local rate for the best (oracle) choice of the projection structure \( \min_{I \in \mathcal{I}} (\| \theta - P_I \theta \|^2 + \sigma^2 d_s(I)) \), uniformly in \( \theta \in \Theta \). However, as is shown for some particular models, this is impossible unless we use a majorant \( \rho(s(I)) \geq d_s(I) + \log |\mathcal{I}_s(I)| \) instead of just \( d_s(I) \). The layer complexity term \( \log |\mathcal{I}_s(I)| \) is the “price” for not knowing the structure. This motivates the following definition. Introduce the family of local rates

\[
r^2(I, \theta) = \| \theta - P_I \theta \|^2 + \sigma^2 \rho(s(I)), \quad I \in \mathcal{I},
\]

for some \( \rho(s) \) satisfying (2.6) and the slicing mapping \( s(I) \) from Condition (A1). For each \( \theta \) there exists the best structure \( I_o = I_o(\theta) = I_o(\theta, \sigma^2) \) (if not unique, take any minimizer) corresponding to the fastest local rate

\[
r^2(\theta) = \min_{I \in \mathcal{I}} r^2(I, \theta) = r^2(I_o, \theta) = \| \theta - P_{I_o} \theta \|^2 + \sigma^2 \rho(s(I_o)), \quad (3.1)
\]

representing the optimal trade-off between the approximation term \( \| \theta - P_{I_o} \theta \|^2 \) and the complexity term \( \rho(s(I_o)) \) satisfying (2.6). We call \( I_o \) by oracle structure (or just oracle) and the quantity \( r^2(\theta) \) by oracle rate.

Remark 10. Often we will have that \( d_s(I) = \dim(\mathbb{L}_I) = d_I \) and \( \rho(s(I)) = d_I + \log |\mathcal{I}_s(I)| \). This is the case in many particular models and structures that we consider in Section 6. If \( I^* \in \mathcal{I} \) is the true structure, i.e., \( \theta \in \mathbb{L}_{I^*}, s^* = s(I^*) \) and \( \mathcal{I}_{s^*} = \{I^*\} \), then, by the oracle definition (3.1) and the facts that \( \| \theta - P_{I^*} \theta \|^2 = 0 \) and \( \mathbb{L}_{I^*} \subseteq \mathbb{R}^N \), we have

\[
r^2(\theta) \leq r^2(I^*, \theta) = \sigma^2 \rho(s(I^*)) = \sigma^2 d_{s(I^*)} = \sigma^2 \dim(\mathbb{L}_{I^*}) \leq N \sigma^2. \quad (3.2)
\]
If such a true structure does not exist, we can assume without loss of generality that there is an \( I \in \mathcal{I} \) such that \( \mathbb{L}_I = \mathbb{R}^N, \bar{s} = s(I) \) and \( \mathcal{I}_s = \{I\} \). This would lead again to the bound (3.2): \( r^2(\theta) \leq r^2(\bar{I}, \theta) = \sigma^2 \dim(\mathbb{L}_I) = N\sigma^2 \). This is of course not surprising as the oracle performance should not be worse than that of the simplistic procedure \( \hat{\theta} = \bar{Y} \).

**Remark 11.** Suppose we have two different family of structures \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \), with corresponding (different) families of linear spaces \( \mathbb{L}_I \) and (different) majorants. We say that the family \( \mathcal{I}_1 \) **covers** the family \( \mathcal{I}_2 \) if for any \( I \in \mathcal{I}_2 \) there exists \( I' = I'(I) \in \mathcal{I}_1 \) such that \( r^2(I', \theta) \leq r^2(I, \theta) \) for all \( \theta \in \Theta \) (up-to-a-constant relation will do as well). If \( \mathcal{I}_1 \) covers \( \mathcal{I}_2 \), there is no point in considering the family \( \mathcal{I}_2 \), one should use the family \( \mathcal{I}_1 \). The family \( \mathcal{I}_1 \) and the family \( \mathcal{I}_2 \) covered by \( \mathcal{I}_1 \) could be of very different natures. But sometimes \( \mathcal{I}_1 \) can be a subfamily of \( \mathcal{I}_2 \). This happens when a chunk of structures in \( \mathcal{I}_2 \) can be dominated by just one structure. Then we can remove those structures without any harm, obtaining a new adjusted family \( \mathcal{I}_1 \). The complexity term in the majorant gets adjusted for some \( I \in \mathcal{I}_1 \), leading to an **elbow effect** in the rate and improving the resulting oracle rate. We will see how this elbow effect is exhibited in several examples of Section 6.

### 3.2. Estimation and contraction results with oracle rate

Recall the quantities: the empirical Bayes posterior \( \hat{\pi}(\theta|Y) \), which is either EBMA posterior \( \hat{\pi}(\theta|Y) \) defined by (2.15) or EBMS posterior \( \hat{\pi}(\theta|Y) \) defined by (2.20); the empirical Bayes posterior mean \( \hat{\theta} \), which is either \( \hat{\theta} \) defined by (2.18) or \( \hat{\theta} \) defined by (2.20); and the oracle rate \( r(\theta) \) defined by (3.1). The following theorem establishes that the empirical Bayes posterior \( \hat{\pi}(\theta|Y) \) contracts (from the frequentist \( \mathbb{P}_\theta \)-perspective) to \( \theta \) with the oracle rate \( r(\theta) \), and the empirical Bayes posterior mean \( \hat{\theta} \) converges to \( \theta \) with the oracle rate \( r(\theta) \), uniformly over the entire parameter space.

**Theorem 1.** Let Conditions (A1) and (A2) be fulfilled. Then there exist constants \( M_0, M_1, H_0, H_1, m_0, m_1 > 0 \) such that for any \( \theta \in \Theta \) and any \( M \geq 0 \),

\[
\mathbb{E}_\theta \hat{\pi}(\|\theta - \theta\|^2 \geq M_0 r^2(\theta) + M \sigma^2 | Y) \leq H_0 e^{-m_0 M}, \tag{3.3}
\]

\[
\mathbb{P}_\theta(\|\hat{\theta} - \theta\|^2 \geq M_1 r^2(\theta) + M \sigma^2) \leq H_1 e^{-m_1 M}. \tag{3.4}
\]

The constants in the theorem depend only on \( \alpha \) and some also on \( \kappa \), the exact expressions can be found in the proof.

**Remark 12.** Notice that already claim (3.3) of Theorem 1 contains an oracle bound for the estimator \( \hat{\theta} \). Indeed, by Jensen’s inequality, we get the oracle inequality in expectation:

\[
\mathbb{E}_\theta \|\hat{\theta} - \theta\|^2 \leq \mathbb{E}_\theta \hat{\pi}(\|\theta - \theta\|^2 | Y) \leq M_0 r^2(\theta) + H_0 \int_0^{+\infty} e^{-m_0 u/\sigma^2} du \\
= M_0 r^2(\theta) + \frac{H_0 \alpha^2}{m_0}. \tag{3.5}
\]

Similarly we can show that also (3.4) implies (3.5). This means that claim (3.4) is actually stronger than (3.5) and therefore requires a separate proof.
Remark 13. The non-asymptotic exponential probability bounds in the both claims of the theorem provide a very refined characterization of the quality of the posterior $\hat{\pi}(\theta | Y)$ and estimator $\hat{\theta}$, finer than, e.g., the traditional oracle inequalities in expectation like (3.5) (since (3.5) follows from (3.4), see Remark 12). This refined formulation allows for subtle analysis in various asymptotic regimes ($N \to \infty$, $\sigma \to 0$, or their combination) as we can let $M$ depend in any way on $N$, $\sigma$, or both.

Now we give several technical definitions which we will need in the claims. For the constants $\alpha$ from Condition (A1) and $\varkappa$ from (2.10), define

$$\tilde{\tau} = \tilde{\tau}(\varkappa, \alpha) \triangleq 3(1 + \varkappa \alpha + \alpha / 2) / \alpha. \tag{3.6}\$$

Next, for some $\delta \in (0, 1)$, fix some $\tau_0 = \tau_0(\delta)$ such that $\tau_0 > \frac{1 + \delta}{1 - \delta} \tilde{\tau}$, where $\tilde{\tau}$ is defined by (3.6). For example, take $\delta = 0.1$ and $\tau_0 = \frac{1 + \delta}{1 - \delta} + 0.1$. For this $\tau_0$ and any $\theta \in \Theta$, define

$$I_\ast = I_\ast(\theta) = I_\ast(\theta, \delta) \triangleq I_{\theta, \delta}^0(\theta) = I_0(\theta, \tau_0 \sigma^2), \tag{3.7}\$$

where $I_0(\theta, \sigma^2)$ is defined by (3.1). We call the quantity $I_\ast^0 = I_{\theta}^0(\theta) = I_0(\theta, \tau \sigma^2)$, for $\tau \geq 0$, by $\tau$-oracle, which is just the oracle defined by (3.1) with $\sigma^2$ substituted by $\tau \sigma^2$. Notice that $\rho(s(I_{\theta}^0)) \geq \rho(s(I_{\tau}^0))$ for $\tau_1 \leq \tau_2$. All $\tau$-oracle rates are related to the oracle rate by the trivial relations: $r^2(\theta) \leq r^2(I_{\theta}^\ast, \theta) \leq \tau r^2(\theta)$ for $\tau \geq 1$, and $r^2(I_{\theta}^\ast, \theta) \leq r^2(\theta) \leq \tau^{-1} r^2(I_{\theta}^\ast, \theta)$ for $0 < \tau < 1$.

When proving the above theorem, as byproduct we also obtain a result about the frequentist behavior of the structure selector $\hat{I}$ and the empirical Bayes posterior for $I$, saying basically that $\hat{I}$ and $\hat{\pi}(I | Y)$ “live” on a set that is, in a sense, almost as good as the oracle $I_0$.

Theorem 2. Let Conditions (A1) and (A2) be fulfilled, $\nu, C_\nu$ be from Condition (A2). The following relations hold for any $\theta \in \Theta$ and $M \geq 0$.

(i) Let $c_1, c_2, c_3$ be the constants defined in Lemma 2. Then

$$\mathbb{E}_\theta \hat{\pi}(I \in \mathcal{I} : r^2(I, \theta) \geq c_3 r^2(\theta) + M \sigma^2 | Y) \leq C_\nu e^{-c_2 M}. \tag{3.8}\$$

(ii) Let $\varkappa \geq \alpha^{-1} \nu$ (implied by (2.11)) and Condition (A3) be fulfilled. Then there exists $m_1' > 0$ such that

$$\mathbb{E}_\theta \hat{\pi}(I \in \mathcal{I} : \rho(s(I)) \leq \delta \rho(s(I_\ast)) - M | Y) \leq C_\nu e^{-m_1' M}, \tag{3.8}\$$

where $I_\ast = I_\ast(\theta, \delta)$ is defined by (3.7).

(iii) Let $\varkappa \geq \frac{2\nu + 2\alpha + 3}{2\alpha}$ (implied by (2.11)) and Condition (A3') be fulfilled (given in Remark 8). Then there exists $M_0' > 0$ such that

$$\mathbb{E}_\theta \hat{\pi}(I \in \mathcal{I} : \rho(s(I)) \geq M_0' \rho(s(I_0)) + M | Y) \leq C_\nu e^{-M / 2}. \tag{3.8}\$$
We can interpret the above theorem as structure recovery, but in a somewhat weak sense. Namely, Theorem 2 says basically that the empirical Bayes posterior \(\hat{\pi}(\theta|Y)\) and the structure selector \(\hat{I}\) “live” in the set of structures resembling the oracle structure \(I_o\), in the sense that the rates and complexities for the structures from this set are in a proximity of the oracle rate and complexity, respectively. Recall that in general the oracle structure is not the same as the true structure. Notice that this weak structure recovery is always possible as there are no conditions on \(\theta\).

### 3.3. Confidence ball under EBR

Theorem 1 establishes strong local optimal properties of the empirical Bayes posterior \(\hat{\pi}(\theta|Y)\) and mean \(\hat{\theta}\), but this is not enough to solve the uncertainty quantification problem yet. As a first candidate for confidence ball, let us construct a credible ball by using the EBMS posterior \(\tilde{\pi}(\theta|Y)\) defined by (2.20). As \(\tilde{\pi}(\theta|Y) = N\left(\hat{\theta}, \frac{\sigma^2}{\alpha+1}P_l\right)\), denoting by \(\chi^2_{\alpha} \) the (1 - \(\alpha\))-quantile of \(\chi^2\)-distribution, we have

\[
\tilde{\pi}\left(\|\theta - \hat{\theta}\|^2 \leq \sigma^2\chi^2_{\text{dim}(L_o),\alpha}|Y\right) \geq \tilde{\pi}\left(\|\theta - \hat{\theta}\|^2 \leq \frac{\sigma^2}{\alpha+1}\chi^2_{\text{dim}(L_o),\alpha}|Y\right) = 1 - \alpha.
\]

Since \(\chi^2_{\text{dim}(L_o),\alpha}\) is bounded by a constant multiple of \(\text{dim}(\mathbb{L})\), for simplicity the latter can replace the former to obtain a credible ball. Then \(B(\hat{\theta}, M\sigma[\text{dim}(\mathbb{L})]^{1/2})\) is a credible ball for \(\theta\), which can be guaranteed to have a given level of credibility by choosing a sufficiently large \(M\). However, \(B(\hat{\theta}, M\sigma[\text{dim}(\mathbb{L})]^{1/2})\) cannot have a guaranteed coverage, since otherwise in some particular models (cf. [8]) this would mean that the estimator \(\hat{\theta}\) would converge to \(\theta\) uniformly in \(\theta \in \Theta\) at the smaller oracle rate with \(\hat{\rho}(s(I_o)) = \text{dim}(L_o)\) instead of \(\rho(s(I_o)) = d_{s(I_o)} + \log |I_{s(I_o)}|\). But \(\log |I_{s(I_o)}|\) can be the dominating term in \(\rho(s(I_o))\), e.g., for sparsity structures (see [8] and Section 6). This would contradict the lower bounds from the literature. Basically, the posterior \(\tilde{\pi}(\theta|Y)\) is well concentrated (in fact, “too concentrated”), but not around the truth, rather around its mean \(\hat{\theta}\) which in general is away from the truth by the distance of a bigger order than the concentration rate. Hence, to obtain coverage, the radius of confidence balls must be at least of the order \(\rho(s(I_o)) = d_{s(I_o)} + \log |I_{s(I_o)}|\). Of course, the oracle structure \(I_o\) is not known, but we have the structure selector \(\hat{I}\) defined by (2.19), which is in a way close to the oracle structure, according to Theorem 2.

The above heuristics suggests to use \(\rho(s(\hat{I}))\) as a proxy for \(\rho(s(I_o))\). According to Theorem 2, \(\hat{I}\) lives in the “complexity shell” \(\delta \sigma^2 \rho(s(I_o)) - M\sigma^2 \leq \sigma^2 \rho(s(\hat{I})) \leq c_3 \sigma^2(\hat{\theta}) + M\sigma^2\) with a large probability. So, if we want the size of confidence ball to be not of a bigger order than oracle rate, it seems reasonable to use the following data dependent radius

\[
r^2 = \hat{r}^2(Y) = \sigma^2 + \sigma^2 \rho(s(\hat{I})). \tag{3.9}
\]

We will show that the size property holds for the radial rate equal to the oracle rate, uniformly over \(\theta \in \Theta\). But then there is an inevitable problem with coverage: the coverage property does not hold uniformly. Indeed, the complexity shell can be too wide if
\( \sigma^2 \rho(s(I_\ast)) \ll r^2(\theta) \). If this happens (for deceptive \( \theta \)'s), then the coverage property of a ball with radius of order \( \hat{r} \) cannot be guaranteed because its radius can be of a smaller order than the oracle rate \( r^2(\theta) \). This problem will not occur for those \( \theta \)'s (called non-deceptive) for which the approximation term of the oracle rate is within a multiple of its complexity term. This discussion motivates introducing the following condition.

**Condition EBR.** We say that \( \theta \in \Theta \) satisfies the excessive bias restriction (EBR) condition with structural parameter \( t \geq 0 \) if \( \theta \in \Theta_{eb}(t) \), where the corresponding set (called the EBR class) is

\[
\Theta_{eb}(t) = \Theta_{eb}(t, \tau_0) = \{ \theta \in \Theta : \| \theta - P_t \theta \|^2 \leq t \sigma^2 (1 + \rho(s(I_\ast))) \}, \tag{3.10}
\]

where the \( \tau_0 \)-oracle structure \( I_\ast = I_0(\theta, \tau_0 \sigma^2) \) is defined by (3.7). The condition EBR essentially requires that the approximation term of the \( \tau_0 \)-oracle rate \( r^2(I_\ast, \theta) \) is dominated by a multiple of its complexity term (additional \( \sigma^2 \) is needed to handle the case \( \rho(s(I_\ast)) = 0 \)). Clearly, \( \Theta_{eb}(t_1) \subseteq \Theta_{eb}(t_2) \) for \( t_1 \leq t_2 \).

Now we use the center \( \hat{\theta} \) and the radius \( \hat{r} \) to construct a confidence ball for \( \theta \). The following theorem describes the coverage and size properties of the confidence ball based on \( \hat{\theta} \) and \( \hat{r} \).

**Theorem 3.** Let Conditions (A1), (A2) and (A3) be fulfilled, \( \Theta_{eb}(t) \) be defined by (3.10). Then there exist constants \( M_2, M_3, H_2, H_3, m_2, m_3 > 0 \) such that for any \( t, M \geq 0 \), with \( \hat{R}_M = \hat{R}_M^2(M_2) = (t + 1)M_2 \hat{r}^2 + (t + 2)M \sigma^2 \),

\[
\sup_{\theta \in \Theta_{eb}(t)} \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, \hat{R}_M)) \leq H_2 e^{-m_2 M},
\]

\[
\mathbb{P}_\theta(\hat{r}^2 \geq M_3 \rho^2(\theta) + (M + 1)\sigma^2) \leq H_3 e^{-m_3 M}.
\]

The size (second) relation holds uniformly in \( \theta \in \Theta \) without Condition (A3).

Moreover, if, instead of Condition (A3), stronger Condition (A3') is fulfilled, then a stronger version of the size relation holds: \( \mathbb{P}_\theta(\hat{r}^2 \geq M'_3 \rho(s(I_0))\sigma^2 + (M + 1)\sigma^2) \leq C_v e^{-M/2} \), where the constants \( M'_3 \) and \( C_v \) are from Theorem 2.

**Remark 14.** Recall that \( I_\ast \) from (3.10) is actually the \( \tau_0 \)-oracle. It may be desirable to impose an EBR condition in terms of the “standard” oracle \( I_0 \) rather than the \( \tau_0 \)-oracle. By rewriting the original model (1.1) as \( Y_0^{-1/2} = \theta_0^{-1/2} + \sigma \xi_0^{-1/2} \), it is not difficult to see that we can construct a confidence ball with the radius \( \sqrt{\hat{r}_0} \hat{R}_M \) satisfying the coverage property as above, but now uniformly over \( \Theta_{eb}(t, 1) \).

**Remark 15.** When proving the coverage relation of Theorem 3, we actually established the following uniform local assertion: there exist constants \( M_2, M_3, H_2, m_2 > 0 \) such that for any \( \theta \in \Theta \) and any \( M \geq 0 \),

\[
\mathbb{P}_\theta(\theta \notin B(\hat{\theta}, [(b(\theta) + 1)M_2 \hat{r}^2 + (b(\theta) + 2)M \sigma^2]^{1/2})) \leq H_1 e^{-m_1 M} + C_v e^{-m_2 M} \leq H_2 e^{-m_2 M}, \tag{3.11}
\]
where the constants $H_1, m_1$ are defined in Theorem 1, $C_1$ is from Condition (A2), and the quantity $b(\theta)$ (called excessive bias ratio) is defined by

$$b(\theta) = b(\theta, \tau_0) = \frac{\|\theta - P_1 \theta\|^2}{\sigma^2 + \sigma^2 \rho(s(I^*)]}.$$  

(3.12)

Although the newly formulated coverage relation (3.11) is now uniform over the entire space $\theta \in \Theta$, the main (and unavoidable) problem is its dependence on $b(\theta)$. That is why we introduced the EBR condition which essentially provides control over the quantity $b(\theta)$: indeed, $\Theta_{eb}(t) = \\{\theta \in \Theta : b(\theta) \leq t\}$.

**Remark 16.** The smaller constant $\tau_0$ (involved in the definition of the EBR condition) is, the less restrictive the EBR condition is, the limiting case $\tau_0 \downarrow 0$ corresponds basically to no condition. We treat a general situation, with only Condition (A1) assumed for $\xi$, so that we have a lower bound for $\tau_0$ in terms of $\alpha$ which is possibly too conservative for each specific distribution of $\xi$. However, even for any specific distribution of $\xi$, the value of the constant $\tau_0 > 0$ in the EBR condition is always bounded away from zero (further from zero for “bad” $\xi$’s).

**Remark 17.** The EBR leads to the new EBR-scale $\{\Theta_{eb}(t), t \geq 0\}$ which gives a slicing of the entire space $\Theta = \cup_{t \geq 0} \Theta_{eb}(t)$ This slicing is very suitable for uncertainty quantification and provides a new perspective at the deceptiveness issue (discussed in the Introduction): basically, each parameter $\theta$ is deceptive (or non deceptive) to some extent. It is the parameter $t$ that measures the deceptiveness in $\Theta_{eb}(t)$ and affects the size of the confidence ball needed to provide a guaranteed high coverage uniformly over $\Theta_{eb}(t)$.

### 3.4. Confidence ball of $N^{1/4}$-radius without EBR

Suppose we want to construct a confidence ball of a full coverage uniformly over the whole space $\Theta$. Recall however that for “signal+noise” models, in view of the negative results of [39, 18, 3, 45] mentioned in the Introduction, no data dependent ball can have uniform coverage and adaptive size simultaneously. When insisting on the uniform coverage, one must have an additional term of the order $\sigma N^{1/4}$ in the radial radius. Let us give a heuristics behind this. An idea is to mimic the quantity $\|\theta - \hat{\theta}\|^2$ by $\hat{R}^2 = \|Y - \hat{\theta}\|^2$. Clearly, there is a lot of bias in $\hat{R}^2$, the biggest part of which is due to the term $\sigma^2 \|\xi\|^2$ contained in $\hat{R}$. To de-bias for that part, we need to subtract its expectation $\sigma^2 \mathbb{E}\|\xi\|^2$. However, even the de-biased version of $\hat{R}^2$ can only be controlled up to a margin of the order $\sigma^2 \sqrt{N}$. That is why a term of the order $\sigma N^{1/4}$ is necessary in the radius of the confidence ball to provide coverage uniformly over the whole space $\Theta$.

To handle some technical issues, we impose the following condition.

**CONDITION (A4).** Besides $Y$ given by (1.1), we also observe $Y' = \theta + \sigma \xi'$ independent of
are to be interpreted as the coverage and size relations in the optimality framework (1.2) with \( \Theta_0 = \Theta_1 = \Theta \) and the effective radial rate \( R(\theta) = \sqrt{g_M(\theta, N)} \approx r(\theta) + \sigma N^{1/4} \) (for now disregarding the constants and
the inflating factor \( M \) as we consider only the order of the radial rate). Since the both sets \( \Theta_0 = \Theta_1 = \Theta = \mathbb{R}^N \) are the biggest possible, the deceptiveness phenomenon manifests itself only in the effective radial rate \( R(\theta) \), which can be of a bigger order than the oracle rate \( r(\theta) \) for \( \theta \in \tilde{\Theta} \), where (for some \( c > 0 \))

\[
\tilde{\Theta} = \tilde{\Theta}(c) = \{ \theta \in \Theta : r^2(\theta) \leq c\sigma N^{1/2} \}.
\]

Equivalently, this can be seen as the optimality framework (1.2) with \( \Theta_0 = \Theta = \mathbb{R}^N \), \( \Theta_1 = \Theta \setminus \tilde{\Theta} = \mathbb{R}^N \setminus \tilde{\Theta} \) and the effective radial rate \( R(\theta) \asymp r(\theta) + \sigma N^{1/4} \lesssim C \rho(\theta) \) is of the oracle rate order for \( \theta \in \Theta_1 \). Now the deceptiveness phenomenon manifests itself in the fact that \( \Theta_1 = \Theta \setminus \tilde{\Theta} \), not the whole \( \Theta \).

In fact, the massiveness of the set \( \tilde{\Theta} \) measures how much the deceptiveness phenomenon is present in particular models and structures. Loosely speaking, models and structures, where “good” estimation \( R(\theta) \lesssim \sigma N^{1/2} \) is possible for “many” \( \theta \)'s (\( \tilde{\Theta} \) is massive), suffer more from the deceptiveness phenomenon. For example, these are all models with sparsity structures in Section 6.7, as the set \( \tilde{\Theta} \) is a substantial part of \( \Theta = \mathbb{R}^N \) in those cases.

On the other hand, the deceptiveness phenomenon becomes effectively marginal for some “uninformative” particular models and structures, e.g., biclustering model (stochastic block model), dictionary learning (see Section 6), because in those cases the set \( \tilde{\Theta} \) is a very “thin” subset of \( \mathbb{R}^N \) and can informally be described as set of highly structured parameters. In these cases the extra term \( \sigma N^{1/4} \) in the radial rate \( R(\theta) \) does not increase its order as \( \sigma N^{1/4} \lesssim r(\theta) \) for the “majority” of \( \theta \)'s: \( \theta \in \Theta_1 = \Theta \setminus \tilde{\Theta} \). This means that, modulo the set \( \tilde{\Theta} \) of highly structured parameters, there is no deceptiveness issue for those cases. Indeed, there is no payment in terms of removing deceptive parameters from the parameter space \( \Theta \) in the coverage relation and the size relation holds uniformly over \( \Theta_1 = \Theta \setminus \tilde{\Theta} \) which is “almost” the whole space \( \Theta \).

4. Technical lemmas

We provide a couple of technical lemmas used in the proofs of the main results. Recall that \( \hat{\pi}(I|Y) \) is either \( \hat{\pi}(I|Y) \) defined by (2.17) or \( \hat{\pi}(I|Y) = 1\{I = \hat{I}\} \) defined by (2.21). In the latter case \( \mathbb{E}_\theta \hat{\pi}(I|Y) = \mathbb{P}_\theta(\hat{I} = I) \). In what follows, denote \( \hat{p}_I = \hat{\pi}(I|Y) \) for brevity.

**Lemma 1.** Let Condition (A1) be fulfilled. Then for any \( \theta \in \Theta \) and \( I, I_0 \in \mathcal{I} \)

\[
\mathbb{E}_\theta \hat{p}_I \leq \left( \frac{N}{N_0} \right)^h \exp \left\{ -\frac{1}{\sigma^2}(A_h \|P_I^\perp \theta\|^2 - B_h \|P_{I_0}^\perp \theta\|^2) + C_h \rho(s(I)) + D_h \rho(s(I_0)) \right\},
\]

where \( h = \frac{\alpha}{3} \) and the constants \( A_h = \frac{\alpha}{16} \), \( B_h = \frac{3\alpha}{16} \) and \( C_h = \frac{5}{8} \), \( D_h = \frac{3+\alpha}{8} \).

If \( \mathbb{I}_I \subseteq \mathbb{I}_{I_0} \), then

\[
\mathbb{E}_\theta \hat{p}_I \leq \left( \frac{N}{N_0} \right)^{\alpha} \exp \left\{ -A_\alpha \sigma^{-2}(\|P_I^\perp \theta\|^2 - \|P_{I_0}^\perp \theta\|^2) + D_\alpha \rho(s(I_0)) \right\},
\]

where the constants \( A_\alpha = \frac{\alpha}{3} \) and \( D_\alpha = 1 + \frac{\alpha}{2} \).
If $\mathbb{L}_{I_0} \subseteq \mathbb{L}_I$, then

$$E_\theta \hat{P}_I \leq \left( \frac{\lambda_I}{\lambda_{I_0}} \right)^\alpha \exp \left\{ B_\alpha \sigma^2 (\|P_I \theta\|^2 - \|P_{I_0} \theta\|^2) + C_\alpha \rho(s(I)) + D_\alpha \rho(s(I_0)) \right\},$$

where the constants $B_\alpha = \alpha, C_\alpha = 1$ and $D_\alpha = \frac{\alpha}{2}$.

**Proof.** Recall that $P_I$ is the projection onto $\mathbb{L}_I$. Since $P_I - P_{I_0} = P_{I_0}^\perp - P_I^\perp$, the bound

$$Y^T (P_I - P_{I_0}) Y = \theta^T (P_I - P_{I_0}) \theta + 2 \sigma^2 \xi^T (P_I - P_{I_0}) \xi$$

$$\leq -\|P_I^\perp \theta\|^2 + \|P_{I_0}^\perp \theta\|^2 + 2 \sigma^2 \xi^T (P_I - P_{I_0}) \xi + \sigma^2 \|P_I \xi\|^2 - \sigma^2 \|P_{I_0} \xi\|^2$$

holds for any $I, I_0 \in \mathcal{I}$. Using the relations $P_I - P_{I_0} = (P_I - P_{I_0})P_{L_I + L_{I_0}}, \|P_{L_I + L_{I_0}} x\|^2 \leq \|P_I x\|^2 + \|P_{I_0} x\|^2, x \in \mathcal{Y}$, and the inequality $2ab \leq a^2/4 + 4b^2$ (for any $a, b \in \mathbb{R}$), we derive

$$2 \sigma^2 \xi^T (P_I - P_{I_0}) \xi \leq \frac{1}{2} \|(P_I - P_{I_0}) \theta\|^2 + 4 \sigma^2 \|P_{L_I + L_{I_0}} \xi\|^2$$

The last bound and (4.1) imply that

$$Y^T (P_I - P_{I_0}) Y \leq -\frac{1}{2} \|P_I^\perp \theta\|^2 + \frac{3}{2} \|P_{I_0}^\perp \theta\|^2 + 5 \sigma^2 \|P_I \xi\|^2 + 3 \sigma^2 \|P_{I_0} \xi\|^2.$$

(4.2)

In case $\hat{P}_I = \hat{\pi}(I|Y) = \hat{\pi}(I|Y) = 1\{ \hat{I} = I \}$, (2.17), the definition (2.19) of $\hat{I}$ and the Markov inequality imply that, for any $I, I_0 \in \mathcal{I}$ and any $h \geq 0$,

$$E_\theta \hat{P}_I = P_\theta (\hat{I} = I) \leq P_\theta (\hat{\pi}(I|Y) \geq 1) \leq E_\theta \left[ \hat{\pi}(I|Y) \right]^h.$$

(4.3)

In case $\hat{P}_I = \hat{\pi}(I|Y) = \hat{\pi}(I|Y)$, (2.17) implies (4.3) for any $I, I_0 \in \mathcal{I}, h \in [0,1]$. Combining (4.2) and (4.3), we derive for any $I, I_0 \in \mathcal{I}$ and any $h \in [0,1],$

$$E_\theta \hat{P}_I \leq E_\theta \left[ \frac{\lambda_I \exp \left\{ -\frac{1}{2 \sigma^2} \|Y - P_I Y\|^2 + \sigma^2 \dim(L_I) \right\} }{\lambda_{I_0} \exp \left\{ -\frac{1}{2 \sigma^2} \|Y - P_{I_0} Y\|^2 + \sigma^2 \dim(L_{I_0}) \right\} } \right]^h$$

$$= \left( \frac{\lambda_I}{\lambda_{I_0}} \right)^h E_\theta \exp \left\{ \frac{h}{2 \sigma^2} (Y^T (P_I - P_{I_0}) Y + \sigma^2 \dim(L_{I_0}) - \sigma^2 \dim(L_I)) \right\}$$

$$\leq \left( \frac{\lambda_I}{\lambda_{I_0}} \right)^h e^{-\frac{h}{2 \sigma^2} \|P_I^\perp \theta\|^2 + \frac{3h}{4 \sigma^2} \|P_{I_0}^\perp \theta\|^2 + \frac{h}{2 \sigma^2} \rho(s(I))} E_\theta \left[ \frac{\lambda_I}{\lambda_{I_0}} \right]^h (5 \|P_I \xi\|^2 + 3 \|P_{I_0} \xi\|^2).$$

(4.4)

The lemma follows for $h = \frac{\alpha}{2}$ from the last display and the relation

$$E_\theta \exp \left\{ \frac{3h}{8} \|P_I \xi\|^2 + \frac{3h}{8} \|P_{I_0} \xi\|^2 \right\} \leq \left[ E_\theta e^{\alpha \|P_I \xi\|^2} \right]^\frac{3}{8} \left[ E_\theta e^{\alpha \|P_{I_0} \xi\|^2} \right]^\frac{3}{8}$$

$$\leq \exp \left\{ \frac{3}{8} \rho(s(I)) + \frac{3}{8} \rho(s(I_0)) \right\},$$
which is in turn obtained by using the Hölder inequality and Condition (A1).

In case $\mathbb{L}_I \subseteq \mathbb{L}_{I_0}$, take $h = \alpha$ in (4.4) and, instead of (4.2) use $Y^T(P_I - P_{I_0})Y = -\|P_{L_I \cap \mathbb{L}_{I_0}} Y\|^2 \leq -\frac{2}{3}\|P_{L_I \cap \mathbb{L}_{I_0}} \theta\|^2 + 2\sigma^2\|P_{L_{I_0} \cap \mathbb{L}_{I_0}} \xi\|^2 \leq \frac{2}{3}(\|P_I \theta\|^2 - \|P_{I_0} \theta\|^2) + 2\sigma^2\|P_{I_0} \xi\|^2 = -\frac{2}{3}\|P_{I_0} \theta\|^2 + \frac{2}{3}\|P_{I_0} \theta\|^2 + 2\sigma^2\|P_{I_0} \xi\|^2 \geq 2a^2/3 - 2b^2$ and $P_{I_0} - P_I = P_{L_{I_0} \cap \mathbb{L}_{I_0}}$.

In case $\mathbb{L}_{I_0} \subseteq \mathbb{L}_I$, take $h = \alpha$ in (4.4) and, instead of (4.2) use $Y^T(P_I - P_{I_0})Y = \|P_{L_{I_0} \cap \mathbb{L}_I} Y\|^2 \leq 2\|P_{L_{I_0} \cap \mathbb{L}_I} \theta\|^2 + 2\sigma^2\|P_{L_{I_0} \cap \mathbb{L}_I} \xi\|^2 \leq 2(\|P_I \theta\|^2 - \|P_{I_0} \theta\|^2) + 2\sigma^2\|P_I \xi\|^2 = -2\|P_I \theta\|^2 + 2\|P_{I_0} \theta\|^2 + 2\sigma^2\|P_I \xi\|^2 \geq (a + b)^2 \leq 2a^2 + 2b^2$ and $P_I - P_{I_0} = P_{L_{I_0} \cap \mathbb{L}_I}$.

Note that above lemma holds for any $I_0 \in \mathcal{I}$. By taking $I_0 = I_0$ defined by (3.1), we derive the next lemma.

**Lemma 2.** Let Condition (A1) be fulfilled. Then there exist positive constants $c_1 = c_1(\alpha) > 2\nu, c_2$ and $c_3 = c_3(\alpha)$ such that for any $\theta \in \Theta$

$$\mathbb{E}_\theta \hat{p}_I \leq \exp \left\{ -c_1\rho(s(I)) - c_2\sigma^{-2}[r^2(I, \theta) - c_3r^2(\theta)] \right\}.$$

**Proof.** With constants $h, A_h, B_h, C_h, D_h$ defined in Lemma 1, define the constant $c_1 = c_1(\alpha) = \nu + \frac{5}{8} - \frac{5}{8} > 2\nu$ as $\alpha > \nu$ by (2.11). The definition (2.10) of $\lambda_I$ entails that

$$(\lambda_I / \lambda_{I_0})^h = \exp \left\{ h\nu\rho(s(I_0)) - (c_1 + C_h + A_h)\rho(s(I)) \right\}.$$ Combining the last relation with Lemma 1 (for $I_0 = I_0$), we derive that

$$\mathbb{E}_\theta \hat{p}_I \leq \exp \left\{ -c_1\rho(s(I)) - c_2\sigma^{-2}[r^2(I, \theta) - \max\{B_h, D_h + h\nu\}r^2(\theta)] \right\} \leq \exp \left\{ -c_1\rho(s(I)) - c_2\sigma^{-2}[r^2(I, \theta) - c_3r^2(\theta)] \right\},$$

which completes the proof with the constants $c_1 = c_1(\alpha) = \frac{5}{8} - \frac{5}{8} + \frac{10}{8} > 2\nu$ and $c_3 = c_3(\alpha) = A_h^{-1}\max\{B_h, D_h + h\nu\} = \frac{16}{10}\max\left\{\frac{3\alpha}{16}, \frac{3\alpha}{8} + \frac{\alpha\nu}{4}\right\} = \frac{6}{\alpha} + 2 + 4\nu$ because $\alpha > \nu > 1$ by (2.11).

### 5. Proofs of the theorems

Here we gather the proofs of all the theorems. By $C_1, C_2$ etc., we denote constants which are different in different proofs.

**Proof of Theorem 1.** Recall the constants $c_1, c_2, c_3$ defined in the proof of Lemma 2 and the notation $\hat{p}_I = \hat{\pi}(I|Y)$. For any $\theta \in \Theta, M \geq 0$ and some constant $M_0$ to be chosen later, denote $\Delta_M = \Delta_M(\theta) = M_0r^2(\theta) + Ma^2$. Next, introduce the set $\mathcal{O}_M = \mathcal{O}_M(\theta) = \{I \in \mathcal{I} : r^2(I, \theta) \leq c_3r^2(\theta) + C_1M_0a^2\}$ and the events $A_M(I) = \{\alpha\|P_I \xi\|^2 \leq (\nu + 1)\rho(s(I)) + C_2M\}$, $I \in \mathcal{I}$, where constants $C_1, C_2 > 0$ are to be chosen later. We have

$$\hat{\pi}(\|\theta - \theta\|^2 \geq \Delta_M|Y) = \sum_{I \in \mathcal{I}} \hat{p}_I(\|\theta - \theta\|^2 \geq \Delta_M|Y)\hat{p}_I$$
\[
\leq \sum_{I \in \mathcal{I}} \hat{p}_I A_{\delta_M}(I) + \sum_{I \in \mathcal{O}_M^c} \hat{p}_I + \sum_{I \in \mathcal{O}_M} \hat{\pi}_I \left( \|\theta - \theta\|^2 \geq \Delta_M |Y| \right) \hat{p}_I A_{\delta_M}(I) \\
= T_1 + T_2 + T_3. \tag{5.1}
\]

Now we need to bound the quantities \(\mathbb{E}_\theta T_1, \mathbb{E}_\theta T_2\) and \(\mathbb{E}_\theta T_3\).

By using the Markov inequality and Condition (A1), we have
\[
\mathbb{P}_\theta (A_M^\delta (I)) = \mathbb{P}_\theta \left( e^{\alpha \|P_1 \xi\|^2} > e^{(\nu + 1)\rho(s(I)) + C_2 M} \right) \leq e^{-\nu \rho(s(I)) - C_2 M}.
\]

The last relation and Condition (A2) yield the bound for \(\mathbb{E}_\theta T_1\):
\[
\mathbb{E}_\theta T_1 \leq \sum_{I \in \mathcal{I}} \mathbb{P}_\theta (A_M^\delta (I)) \leq \sum_{I \in \mathcal{I}} \exp \left\{ -\nu \rho(s(I)) - C_2 M \right\} \leq C_\nu e^{-C_2 M}. \tag{5.2}
\]

If \(I \in \mathcal{O}_M^c\), then \(r^2(I, \theta) > c_3 r^2(\theta) + C_1 M \sigma^2\). Using this, Lemma 2 and the fact that \(\sum_{I \in \mathcal{I}} e^{-c_1 \rho(s(I))} \leq C_\nu\) (in view of Condition (A2) and because \(c_1 > 2\nu\)), we bound \(\mathbb{E}_\theta T_2\) as follows:
\[
\mathbb{E}_\theta T_2 = \sum_{I \in \mathcal{O}_M^c} \mathbb{E}_\theta \hat{p}_I \leq \sum_{I \in \mathcal{O}_M^c} \exp \left\{ -c_1 \rho(s(I)) - c_2 \sigma^2 \left[ r^2(I, \theta) - c_3 r^2(\theta) \right] \right\}
\leq \sum_{I \in \mathcal{I}} \exp \left\{ -c_1 \rho(s(I)) - c_2 C_1 M \right\} \leq C_\nu \exp \left\{ -c_2 C_1 M \right\}. \tag{5.3}
\]

It remains to establish the last bound for \(\mathbb{E}_\theta T_3\). For \(I \in \mathcal{O}_M\), we have that
\[
A_M(I) \leq \left\{ \|\theta - P_I \theta\|^2 + \sigma^2 \|P_I \xi\|^2 \leq c_3 r^2(\theta) + \frac{\nu + 1}{\alpha} \sigma^2 \rho(s(I)) + (C_1 + C_2 \alpha) M \sigma^2 \right\}.
\]

Recall that, in view of (2.16), \(\hat{\pi}_I(\theta | Y) = N(P_I Y, (1 - e^{-1}) \sigma^2 P_I I)\). Let \(\mathbb{P}_Z\) be the measure of \(Z = (Z_1, \ldots, Z_N)\), with \(Z_i \overset{\text{ind}}{\sim} N(0, 1)\). In Remark 5, we established \(\mathbb{E} e^{0.4 \|P_I Z\|^2} \leq e^{\theta_I},\) which implies \(\mathbb{P}_Z (\|P_I Z\|^2 \geq \frac{5}{2} d_I + M) \leq e^{-2M/5}, I \in \mathcal{I}\). Using this, the last display and the fact that \(\frac{r^2(\theta)}{\sigma^2} \geq c_3^{-1}(\rho(s(I)) - C_1 M)\) for \(I \in \mathcal{O}_M\), we obtain that, for any \(I \in \mathcal{O}_M\),
\[
\hat{\pi}_I (\|\theta - \theta\|^2 \geq \Delta_M |Y|) 1_{A_M(I)} \\
= \mathbb{P}_Z (\|P_I Y + (1 - e^{-1}) \sigma^2 P_I Z - \theta\|^2 \geq \Delta_M) 1_{A_M(I)} \\
\leq \mathbb{P}_Z (2 \sigma^2 \|P_I Z\|^2 + 2 \|P_I Y - \theta\|^2 \geq M_0 r^2(\theta) + M \sigma^2) 1_{A_M(I)} \\
= \mathbb{P}_Z (\sigma^2 \|P_I Z\|^2 + \|\theta - P_I \theta\|^2 + \sigma^2 \|P_I \xi\|^2 \geq \frac{M_0}{2} r^2(\theta) + \frac{M \sigma^2}{2}) 1_{A_M(I)} \\
\leq \mathbb{P}_Z (\|P_I Z\|^2 \geq (\frac{M_0}{2c_3} - c_3) \frac{r^2(\theta)}{\sigma^2} - \frac{\nu + 1}{\alpha} \rho(s(I)) + \frac{M}{2} - (C_1 + C_2 \alpha) M) \\
\leq \mathbb{P}_Z (\|P_I Z\|^2 \geq \frac{5}{2 \alpha} \rho(s(I)) + \frac{M}{2}) \leq e^{-M/10},
\]
where we have chosen \( M_0 = \frac{c_3(2(\nu+7\alpha+2))}{\alpha}, \) \( C_1 = \frac{\alpha}{4(2\nu+\alpha+2)} \) and \( C_2 = \frac{8}{\alpha} \) (so that \( \frac{M}{2C_2} = \frac{\nu+\alpha+1}{\alpha} = \frac{5}{2}, \) \( C_2^\alpha = \frac{1}{3}, \) \( M_0C_1^\alpha = \frac{1}{3} \)). Thus we have derived

\[
\mathbb{E}_\theta T_3 = \mathbb{E}_\theta \sum_{I \in \mathcal{O}_M} 1_{A_M(I)} \hat{\pi}_I (\|\theta - \theta\|^2 \geq \Delta_M | Y) \hat{\mu}_I \leq \mathbb{E}_\theta \sum_{I \in \mathcal{I}} \exp\left(-\frac{M}{\hat{\mu}_I}\right) \hat{\mu}_I \leq e^{-M/10}.
\]

This completes the proof of the first assertion since, in view of (5.1), (5.2), (5.3) and the last display, we established the claim (3.3): \( \mathbb{E}_\theta \hat{\pi}(\|\theta - \theta\|^2 \geq M_0 r^2(\theta) + M \sigma^2 | Y) \leq \mathbb{E}_\theta (T_1 + T_2 + T_3) \leq H_0 e^{-m_0M}, \) with the constants \( M_0 = \frac{c_3(2(\nu+7\alpha+2))}{\alpha}, H_0 = 1 + 2C_\nu \) and \( m_0 = \min\{C_2, c_2C_1, 1/10\}. \)

The proof of the assertion (3.4) proceeds along similar lines. Introduce the set \( J_M = J_M(\theta) = \{ I \in \mathcal{I} : r^2(I, \theta) \leq 2c_3 r^2(\theta) + C_3 M \sigma^2 \} \) and the events \( B_M(I) = \{ \alpha \| P_I \xi \|^2 \leq 2(\nu+1)\rho(s(I)) + C_4 M \}, \) \( I \in \mathcal{I}, \) where constants \( C_3, C_4 > 0 \) are to be chosen later.

If \( M \in [0,1], \) the claim (ii) holds for \( H_1 = e^m. \) Let \( M \geq 1. \) Denote for brevity \( R_I^2 = R_I^2(\theta, Y) = \|\theta - P_I Y\|^2 = \|\theta - P_I \theta\|^2 + \sigma^2 \| P_I \xi \|^2, \Delta_M' = \Delta_M'(\theta) = M_{1} r^2(\theta) + M \sigma^2 \) and \( \hat{\mu}_I = \hat{\pi}(I | Y), \) where \( M_{1} > 0 \) is to be chosen later. Applying the Cauchy-Schwarz inequality, we have

\[
\mathbb{P}_\theta(\|\hat{\mu} - \mu\|^2 \geq \Delta_M') \leq \mathbb{P}_\theta \left( \sum_{I \in \mathcal{I}} R_I^2 \hat{\mu}_I \geq \Delta_M' \right)
\]

\[
\leq \mathbb{P}_\theta \left( \sum_{I \in \mathcal{J}_M} R_I^2 \hat{\mu}_I (1_{B_M(I)} + 1_{B_M'(I)}) + \sum_{I \in \mathcal{J}_M} R_I^2 \hat{\mu}_I \geq \Delta_M' \right)
\]

\[
\leq \mathbb{P}_\theta \left( \sum_{I \in \mathcal{J}_M} R_I^2 \hat{\mu}_I 1_{B_M(I)} \geq \frac{\Delta_M'}{3} \right) + \mathbb{P}_\theta \left( \sum_{I \in \mathcal{J}_M} R_I^2 \hat{\mu}_I 1_{B_M'(I)} \geq \frac{\Delta_M'}{3} \right)
\]

\[
+ \mathbb{P}_\theta \left( \sum_{I \in \mathcal{J}_M} R_I^2 \hat{\mu}_I \geq \frac{\Delta_M'}{3} \right) = T_1 + T_2 + T_3.
\]

Let us evaluate \( T_1. \) For any \( I \in \mathcal{J}_M, \) under \( B_M(I), \) we have that \( R_I^2 = \|\theta - P_I \theta\|^2 + \sigma^2 \| P_I \xi \|^2 \leq \|\theta - P_I \theta\|^2 + \frac{2(\nu+1)}{\alpha} \sigma^2 \rho(s(I)) + \frac{C_4}{\alpha} M \sigma^2 \leq \frac{2(\nu+1)}{\alpha} r^2(I, \theta) + \frac{C_4}{\alpha} M \sigma^2 \leq \frac{4c_3(\nu+1)}{\alpha} r^2(\theta) + \frac{2C_3(\nu+1)+C_4}{\alpha} M \sigma^2. \) Using this, we derive

\[
T_1 = \mathbb{P}_\theta \left( \sum_{I \in \mathcal{J}_M} R_I^2 \hat{\mu}_I 1_{B_M(I)} \geq \frac{\Delta_M'}{3} \right)
\]

\[
\leq \mathbb{P}_\theta \left( \frac{4c_3(\nu+1)}{\alpha} r^2(\theta) + \frac{2C_3(\nu+1)+C_4}{\alpha} M \sigma^2 \geq \frac{\Delta_M'}{3} \right) = 0,
\]

as \( \frac{4c_3(\nu+1)}{\alpha} = \frac{M_1}{3} \) and \( \frac{2C_3(\nu+1)+C_4}{\alpha} < \frac{1}{3} \) because we choose \( M_1 = \frac{12c_3(\nu+1)}{\alpha}, C_3 = \frac{\alpha}{12(\nu+1)} \) and \( C_4 = \frac{\alpha}{\nu}. \)

Next, we evaluate \( T_2. \) By Condition (A1) and the Markov inequality,

\[
\mathbb{P}_\theta(B_M'(I)) = \mathbb{P}_\theta(\alpha \| P_I \xi \|^2 > (2\nu + 2) \rho(s(I)) + C_4 M) \leq e^{-(2\nu+2)\rho(s(I)) - C_4 M}.
\]
It follows from (2.7) with \( t = \frac{1}{2} \) that \( \left[ \mathbb{E}_\theta \| P_I \xi \|^4 \right]^{1/2} \leq \frac{2}{\alpha} \exp \{ \rho(s(I))/2 \} \) for any \( I \in \mathcal{I} \). By Condition (A2), \( \sum_{I \in \mathcal{I}} \exp \{ -\nu \rho(s(I)) \} \leq C_\nu \). Besides, for any \( I \in \mathcal{I}_M \), \( \| \theta - P_I \theta \|^2 / \Delta'_M \leq (2c_3 \nu^2(\theta) + C_3 M \sigma^2)/(M_1 r^2(\theta) + M \sigma^2) \leq \frac{2c_3}{M} + C_3 \) and \( \Delta'_M \geq M \sigma^2 \geq \sigma^2 \) (as \( M \geq 1 \)). Collecting all the derived relations for evaluating \( T_2 \) and using the Markov and Cauchy-Schwarz inequalities, we obtain

\[
T_2 = \mathbb{P}_\theta \left( \sum_{I \in \mathcal{I}_M} R_I^2 \hat{p}_I 1_{B_M^c(I)} \geq \Delta'_M / 3 \right)
\leq \frac{\mathbb{E}_\theta \sum_{I \in \mathcal{I}_M} \left( \| \theta - P_I \theta \|^2 + \sigma^2 \| P_I \xi \|^2 \right) \hat{p}_I 1_{B_M^c(I)}}{\Delta'_M / 3}
\leq \frac{\sum_{I \in \mathcal{I}_M} \| \theta - P_I \theta \|^2 \mathbb{P}_\theta (B_M^c(I))}{\Delta'_M / 3} + \frac{\sigma^2 \sum_{I \in \mathcal{I}_M} \left[ \mathbb{E}_\theta \| P_I \xi \|^4 \right]^{1/2} \left[ \mathbb{E}_\theta (B_M^c(I)) \right]^{1/2}}{\Delta'_M / 3}
\leq 3 \left( \frac{2c_3}{M} + C_3 \right) e^{-C_4 M} \sum_{I \in \mathcal{I}_M} e^{-(2\nu + 1)\rho(s(I))} + \frac{6}{\alpha} e^{-C_4 M / 2} \sum_{I \in \mathcal{I}_M} e^{-\nu \rho(s(I))}
\leq 3C_\nu \left( \frac{2c_3}{M} + C_3 \right) e^{-C_4 M} + \frac{6C_3}{\alpha} e^{-C_4 M / 2}.
\]

(5.6)

It remains to bound \( T_3 \). Applying first the Markov inequality and then the Cauchy-Schwarz inequality, we have

\[
T_3 = \mathbb{P}_\theta \left( \sum_{I \in \mathcal{I}_M^c} R_I^2 \hat{p}_I \geq \Delta'_M / 3 \right) \leq \sum_{I \in \mathcal{I}_M^c} \| \theta - P_I \theta \|^2 \mathbb{E}_\theta \hat{p}_I
\leq \frac{\sigma^2 \sum_{I \in \mathcal{I}_M^c} \left[ \mathbb{E}_\theta \| P_I \xi \|^4 \right]^{1/2} \left[ \mathbb{E}_\theta (B_M^c(I)) \right]^{1/2}}{\Delta'_M / 3}
\leq \frac{\sum_{I \in \mathcal{I}_M^c} \| \theta - P_I \theta \|^2 \mathbb{P}_\theta (B_M^c(I))}{\Delta'_M / 3} = T_{31} + T_{32}.
\]

(5.7)

For each \( I \in \mathcal{I}_M^c \), we have \( c_3 \nu^2(\theta) \leq \frac{1}{2} \nu^2(I, \theta) - \frac{C_2}{2} M \sigma^2 \), yielding the bound

\[
\frac{c_2}{2\sigma^2} \left( r^2(I, \theta) - c_3 \nu^2(\theta) \right) \geq \frac{c_2}{4\sigma^2} r^2(I, \theta) + \frac{c_2 C_3}{4} M.
\]

The last relation and Lemma 2 entail that, for each \( I \in \mathcal{I}_M^c \),

\[
\left[ \mathbb{E}_\theta \hat{p}_I \right]^{1/2} \leq \exp \left\{ -\frac{c_1}{2} \rho(s(I)) - \frac{c_2}{4\sigma^2} r^2(I, \theta) - \frac{c_2 C_3}{4} M \right\}.
\]

(5.8)

Since \( M \geq 1 \), \( \Delta'_M \geq M \sigma^2 \geq \sigma^2 \). Using this, the relation (5.8), the facts that \( \max_{x \geq 0} \{ x e^{-cx} \} \leq (ce)^{-1} \) for any \( c > 0 \) and \( \sum_{I \in \mathcal{I}} e^{-c_1 \rho(s(I))} \leq C_\nu \) (in view of Condition (A2) as \( c_1 > 2\nu \)), we bound the term \( T_{31} \) as follows:

\[
T_{31} = \sum_{I \in \mathcal{I}_M^c} \| \theta - P_I \theta \|^2 \mathbb{E}_\theta \hat{p}_I
\leq 3 \sum_{I \in \mathcal{I}_M^c} \frac{r^2(I, \theta)}{\sigma^2} \exp \left\{ -c_1 \rho(s(I)) - \frac{c_2}{2\sigma^2} r^2(I, \theta_0) - \frac{c_2 C_3}{2} M \right\}
\]
With all these together with (5.8), we obtain

\[
\bar{T}_{32} = \frac{\sigma^2 \sum_{I \in J^* M} (E \|P_I \xi\|^4)^{1/2} [E \theta P_I]^{1/2}}{\Delta^M/3}
\]

\[
\leq \frac{3e^{-c_2 C_3 M/4}}{\alpha t_0} \sum_{I \in J^* M} \exp \left\{ - \frac{c_1}{2} \rho(s(I)) + \frac{c_2}{4} \rho(s(I)) - \frac{c_3}{4} r^2(I, \theta) \right\}
\]

\[
\leq \frac{3e^{-c_2 C_3 M/4}}{\alpha t_0} \sum_{I \in I} \exp \left\{ - \frac{c_2}{2} \rho(s(I)) \right\} \leq \frac{3C_{\nu}}{\alpha t_0} e^{-c_2 C_3 M/4}.
\]

Combining (5.4), (5.5), (5.6), (5.7), (5.9) and the last relation finishes the proof of claim (3.4) with the constants \( M_1 = \frac{12c_3(\nu+1)}{\alpha}, H_1 = \max\{C_{\nu}\left[3(\frac{c_2}{M_1} + C_3) + \frac{6}{\alpha} + \frac{6}{c_2} + \frac{3}{\alpha \min\{1/2, c_2/4\}}\right], e^{m_1}\} \) and \( m_1 = \min\{\frac{C_{\nu}}{2}, \frac{c_2}{4}\} \).

**Proof of Theorem 2.** First we prove (i). Denote \( G_1 = G_1(\theta, M) = \{I \in I : r^2(I, \theta) \geq c_3 r^2(\theta) + M \sigma^2\} \), where the constants \( c_1 > 2\nu, c_2, c_3 \) are defined in Lemma 2. Applying Lemma 2 and Condition (A2), we obtain

\[
E_{\theta} \hat{\pi}(I \in G_1 | Y) = \sum_{I \in G_1} E_{\theta} \hat{\pi}(I | Y) \leq e^{-c_2 M} \sum_{I \in I} e^{-c_1 \rho(s(I))} \leq C_{\nu} e^{-c_2 M},
\]

which completes the proof of (i).

Now we prove (ii). By Condition (A3), for any \( I, I_1 \in I \) there exists \( I' = I'(I, I_1) \in I \) such that \( (L_{I} \cup L_{I_1}) \subseteq L_{I'} \). Fix \( I_1 \in I \) and define \( G_2(M, I_1) = \{I \in I : \theta^T[P_{I'} - P_I] \theta \geq \tau \rho(s(I')) + M \sigma^2\} \), where \( \tau \) is defined by (3.6).

As \( L_I \subseteq L_{I'} \), by using (2.10) and applying Lemma 1 with \( h = \alpha \) and \( I_0 = I' \), we obtain that, for each \( I \in G_2(M, I_1) \),

\[
E_{\theta} \hat{\rho}_I \leq \left( \frac{\lambda}{\alpha t_0} \right)^\alpha \exp \left\{ - \frac{\alpha t_0}{3\sigma^2} \left[ \theta^T(P_{I'} - P_I) \theta \right] + \left( 1 + \frac{\sigma}{2} \right) \rho(s(I')) \right\}
\]

\[
= \exp \left\{ - \alpha \rho(s(I)) - \frac{\alpha t_0}{3\sigma^2} \left[ \theta^T(P_{I'} - P_I) \theta \right] + \left( 1 + \frac{\sigma}{2} + \alpha \rho(s(I')) \right) \right\}
\]

\[
\leq \exp \left\{ - \alpha \rho(s(I)) - \left[ \frac{\alpha t_0}{3} - (1 + \frac{\sigma}{2} + \alpha \rho(s(I')) \right] \right\}
\]

\[
= e^{-\alpha \rho(s(I))} e^{-\frac{\alpha t_0}{3} M}.
\]

Since \( \alpha \geq \alpha^{-1} \nu \), by Condition (A2) we have that \( \sum_{I \in I} e^{-\alpha \rho(s(I))} \leq C_{\nu} \). This relation and the last display imply that, with \( m_0' = \alpha/3 \),

\[
E_{\theta} \hat{\pi}(I \in G_2(M, I_1) | Y) = \sum_{I \in G_2(M, I_1)} E_{\theta} \hat{\pi}_I \leq C_{\nu} \exp \{ - m_0' M \}. \tag{5.10}
\]
Now take $I_1 = I_s$ defined by (3.7). By Condition (A3) there exists $I'(I, I_s) \in \mathcal{I}$ such that $(L_I \cup L_{I_s}) \subseteq L'_{I'}$ and $\rho(s(I')) \leq \rho(s(I)) + \rho(s(I_s))$. If $\rho(s(I)) \leq \delta \rho(s(I_s)) = M$, then $\rho(s(I')) \leq \rho(s(I)) + \rho(s(I_s)) \leq (1 + \delta)\rho(s(I_s)) - M$. Hence, $\rho(s(I_s)) \geq \frac{1}{1+\delta} \rho(s(I')) + \frac{M}{1+\delta}$ and $P_{I'} \geq P_{I_s}$, which, together with the definition of the $\tau$-oracle, imply
\[
\theta^T[P_{I'} - P_I] \theta \geq \theta^T[P_{I_s} - P_I] \theta \geq \tau_0 \sigma^2 [\rho(s(I_s)) - \rho(s(I))]
\geq \tau_0 \sigma^2 (1 - \delta) \rho(s(I_s)) + \tau_0 M \sigma^2 \geq \tau' \sigma^2 \rho(s(I')) + \tau_0 M \sigma^2,
\]
where $\tau' \triangleq \frac{1 - \delta}{1 + \delta} \tau_0 > \bar{\tau}$ by the condition of the theorem. It follows that \{$I \in \mathcal{I} : \rho(s(I)) \leq \delta \rho(s(I_s)) - M$\} \(\subseteq \mathcal{G}_2(\tau_0 M, I_s)\). Thus, we obtain
\[
\mathbb{E}_\theta \hat{\pi}(I \in \mathcal{I} : \rho(s(I)) \leq \delta \rho(s(I_s)) - M | Y) \leq \mathbb{E}_\theta \hat{\pi}(\mathcal{G}_2(\tau_0 M, I_s) | Y).
\]
The last relation and (5.10) imply claim (ii) with $m'_I = \tau_0 m'_0 = \tau_0 \alpha / 3$.

Finally, we prove (iii). Condition (A3) implies that $L_{I'} = L_{I_o} + \mathbb{I}_I \oplus (L_I \cap L_{I_o}^\perp)$. If the inequality $\sigma^2 \rho(s(I)) < \|P_{L_I \cap L_{I_o}^\perp} \theta\|^2$ would hold, then
\[
r^2(I', \theta) = \|\theta - P_{I_o} \theta\|^2 + \sigma^2 \rho(s(I'))
\leq \|\theta - (P_{I_o} + P_{L_I \cap L_{I_o}^\perp}) \theta\|^2 + \sigma^2 (\rho(s(I)) + \rho(s(I_s)))
\leq \|P_{L_I \cap L_{I_o}^\perp} \theta\|^2 + \|\theta - (P_{I_o} + P_{L_I \cap L_{I_o}^\perp}) \theta\|^2 + \sigma^2 \rho(s(I))
= \|\theta - P_{I_o} \theta\|^2 + \sigma^2 \rho(s(I)) = r^2(\theta),
\]
which contradicts the definition of the oracle. Hence, $\|P_{L_I \cap L_{I_o}^\perp} \theta\|^2 \leq \sigma^2 \rho(s(I))$.

Take $I_0 \in \mathcal{I}$ such that $L_{I_o} = L_I \cap L_{I_o}$. Using $\alpha \geq \frac{2\alpha + 2\alpha + 3}{2\alpha}$, the fact that $\theta^T(P_{L_I} - P_{I_o}) \theta = \|P_{L_I \cap L_{I_o}^\perp} \theta\|^2 \leq \sigma^2 \rho(s(I))$ and Lemma 1 (in case $L_{I_o} \subseteq L_I$) with $h = \alpha$, we obtain for each $I \in \mathcal{G}_0 = \{I \in \mathcal{I} : \rho(s(I)) \geq M'_0 \rho(s(I_o)) + M \}$ with $M'_0 = 2(\alpha \alpha + \frac{\alpha}{2})$,
\[
\mathbb{E}_\theta \hat{\pi}(I \in \mathcal{I} : \rho(s(I)) \geq M'_0 \rho(s(I_o)) + M | Y)
\leq \sum_{I \in \mathcal{G}_0} \mathbb{E}_\theta \hat{\pi}(I \in \mathcal{G}_0 | Y) \leq C_0 e^{-M/2}.
\]

**Proof of Theorem 3.** We first establish the coverage property. The constants $M_1$, $H_1$ and $m_1$ are defined in Theorem 1. Take $M_2 = \frac{\delta}{3}$ where $\delta \in (0, 1)$ is from (3.7). From (3.1), it
follows that \( r^2(\theta) \leq r^2(I, \theta) = (b(\theta) + 1)\sigma^2\rho(s(I)) + b(\theta)\sigma^2 \leq (b(\theta) + 1)\sigma^2(\rho(s(I)) + 1) \), where \( b(\theta) \) is given by (3.12). Combining this with the claim (3.4) from Theorem 1, the claim (ii) from Theorem 2 and the definition (3.9) of \( \hat{r} \) yields the coverage property:

\[
\mathbb{P}_\theta(\theta \notin B(\hat{\theta}, (b(\theta) + 1)M_2\hat{r}^2 + (b(\theta) + 2)M\sigma^2)^{1/2})) \\
\leq \mathbb{P}_\theta(||\hat{\theta} - \theta||^2 > (b(\theta) + 1)M_2\hat{r}^2 + (b(\theta) + 2)M\sigma^2, \hat{r}^2 \geq \delta\sigma^2\rho(s(I)) + \sigma^2 - \frac{M\sigma^2}{M_2}) \\
+ \mathbb{P}_\theta(\hat{r}^2 < \delta\sigma^2\rho(s(I)) + \sigma^2 - \frac{M\sigma^2}{M_2}) \\
\leq \mathbb{P}_\theta(||\hat{\theta} - \theta||^2 > M_1r^2(\theta) + M\sigma^2) + \mathbb{P}_\theta(\rho(s(\hat{I})) < \delta\rho(s(I)) - \frac{M}{M_2}) \\
\leq H_1e^{-m_1M} + C_\nu e^{-m_2M} \leq H_2e^{-m_2M},
\]

where \( m_1' = m_1'/M_2, H_2 = H_1 + C_\nu, m_2 = m_1 \wedge m_1'; m_1' \) is defined in Theorem 2. Since \( b(\theta) \leq t \) for all \( \theta \in \Theta_{\Theta}(t) \), the coverage relation follows.

Let us show the size property. For \( M \geq 0 \), introduce the set \( \mathcal{G}(M) = \mathcal{G}(M, \theta) = \{I \in \mathcal{I} : \sigma^2\rho(s(I)) \geq c_3r^2(\theta) + M\sigma^2\} \), where \( c_3 \) is defined in Lemma 2. Then for all \( I \in \mathcal{G}(M) \),

\[
r^2(I, \theta) - c_3r^2(\theta) \geq \sigma^2\rho(s(I)) - c_3r^2(\theta) \geq M\sigma^2.
\]

Remind the notation \( \hat{\pi}_I = \hat{\pi}(I | Y) = 1\{I = \hat{I}\} \) defined by (2.21). From Lemma 2 and the last relation, it follows that for all \( I \in \mathcal{G}(M) \)

\[
\mathbb{E}_\theta \hat{\pi}_I \leq \exp \{ -c_1\rho(s(I)) - c_2\sigma^2[r^2(I, \theta) - c_3r^2(\theta)] \} \leq e^{-c_1\rho(s(I)) - c_2M}.
\]

The last display implies that, for any \( \theta \in \Theta \),

\[
\mathbb{P}_\theta(\hat{r}^2 \geq c_3r^2(\theta) + (M + 1)\sigma^2) = \mathbb{P}_\theta(\sigma^2\rho(s(\hat{I})) \geq c_3r^2(\theta) + M\sigma^2) \\
\leq \sum_{I \in \mathcal{G}(M)} \mathbb{E}_\theta \hat{\pi}_I \leq e^{c_2M} \sum_{I \in \mathcal{I}} e^{-c_1\rho(s(I))} \leq H_3e^{-c_2M},
\]

because \( \sum_{I \in \mathcal{I}} e^{-c_1\rho(s(I))} \leq C_\nu \) in view of Condition (A2) as \( c_1 > 2\nu \). The size relation follows with \( M_3 = c_3, H_3 = C_\nu \) and \( m_3 = c_2 \).

If, instead of Condition (A3), stronger Condition (A3') is fulfilled, then the stronger version of the size relation follows immediately from property (iii) of Theorem 2: \( \mathbb{P}_\theta(\hat{r}^2 \geq M_0'\sigma^2\rho(s(I_0)) + (M + 1)\sigma^2) \leq C_\nu e^{-M/2} \), where the constants \( M_0' \) and \( C_\nu \) are defined in Theorem 2.

**Proof of Theorem 4.** Since \( Y' = P_{I^*} + \xi' \), we rewrite (3.13) as

\[
\mathcal{R}_M^2 = (\|Y' - \hat{\theta}\|^2 - 2\sigma^2V(Y') + 2\sigma G_M\sqrt{N})_+ \\
= (\|\theta - \hat{\theta}\|^2 + 2\sigma^2(\|\xi'\|^2 - V(Y')) + 2\sigma(\xi', (\theta - \hat{\theta})) + 2\sigma G_M\sqrt{N})_+.
\]

Introduce the events \( D_M = D_M(\theta) = \{||\hat{\theta} - \theta|| \geq M_1r^2(\theta) + M\sigma^2\} \) and \( E_M = E_M(\theta) = \{2(\xi', (\theta - \hat{\theta})) \geq \sqrt{M(M_1r^2(\theta) + M\sigma^2)}\} \). According to Condition (A4), \( \hat{\theta} \) and \( \hat{I} \) are based
on $Y$ and independent of $\xi'$. Using this fact, the first relation from (A4) and Theorem 1, we obtain that

$$\mathbb{P}_\theta(E_M) = \mathbb{E}_\theta \mathbb{P}_\theta(E_M \cap D_{M}^c | Y) + \mathbb{P}_\theta(E_M \cap D_M)$$

$$\leq \mathbb{E}_\theta \left[ \psi_1 \left( \frac{(M1r^2(\theta) + M\sigma^2)}{4||\theta - \hat{\theta}||^2} \right) 1_{D_{M}^c} \right] + \mathbb{P}_\theta(D_M) \leq \psi_1(M/4) + H_1 e^{-m_1 M}. \quad (5.12)$$

Since, by (3.2), $r^2(\theta) \leq \sigma^2 N$, the event $E_M^c$ implies that $2\sigma \langle \xi', (\theta - \hat{\theta}) \rangle > -\sigma \sqrt{M(M_1 \sigma^2 N + M\sigma^2)} \geq -\sigma^2 G_M \sqrt{N}$. Combining this with (5.11), (5.12) and the second relation from (A4) yields the coverage property:

$$\mathbb{P}_\theta(\theta \notin B(\hat{\theta}, \tilde{R}_M)) = \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, \tilde{R}_M), E_M^c) + \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, \tilde{R}_M), E_M)$$

$$\leq \mathbb{P}_\theta(||\theta - \hat{\theta}||^2 \geq \tilde{R}_M^2, E_M^c) + \mathbb{P}_\theta(E_M)$$

$$\leq \mathbb{P}_\theta(0 \geq \sigma^2 (||\xi'||^2 - V(Y')) + \sigma^2 G_M \sqrt{N}) + \mathbb{P}_\theta(E_M)$$

$$\leq \mathbb{P}_\theta(||\xi'||^2 - V(Y') \leq -M \sqrt{N}) + \psi_1(M/4) + H_1 e^{-m_1 M}$$

$$\leq \psi_2(M) + \psi_1(M/4) + H_1 e^{-m_1 M}.$$ 

Let us show the size property. By (5.12), $\mathbb{P}_\theta(2\sigma \langle \xi', (\theta - \hat{\theta}) \rangle \geq \sigma^2 G_M \sqrt{N}) \leq \mathbb{P}_\theta(2\langle \xi', (\theta - \hat{\theta}) \rangle > \sqrt{M(M_1 r^2(\theta) + M\sigma^2)}) \leq \mathbb{P}_\theta(E_M) \leq \psi_1(M/4) + H_1 e^{-m_1 M}$. This, Theorem 1 and (5.11) imply

$$\mathbb{P}_\theta(\hat{R}_M^2 \geq g_M(\theta, N)) \leq \mathbb{P}_\theta(||\theta - \hat{\theta}||^2 \geq M_1 r^2(\theta) + M\sigma^2)$$

$$+ \mathbb{P}_\theta(\sigma^2 (||\xi'||^2 - V(Y')) \geq \sigma^2 G_M \sqrt{N} + \mathbb{P}_\theta(2\langle \xi', (\theta - \hat{\theta}) \rangle \geq \sigma^2 G_M \sqrt{N})$$

$$\leq H_1 e^{-m_1 M} + \psi_2(M) + \psi_1(M/4) + H_1 e^{-m_1 M}. \quad \square$$

6. Applications

In this section we introduce a number of particular models and structures which fall in the studied general framework and for which local and adaptive (global) minimax results (for all the three problems: estimation, posterior contraction rate and confidence sets) can be derived as consequences of our local results for the general framework.

For all the examples (of particular models with particular structures) considered below, we specify the structures $T$, the structural slicing mapping $s : T \mapsto S$ and the layer complexity function $\rho(s)$. We will further verify Conditions (A1), (A2), (A3) and (A4), whenever appropriate. In view of Remarks 5 and 18, Conditions (A1) and (A4) hold with $d_{\rho(I)} = \dim(L_I)$ in all particular models with $\xi_i \overset{\text{ind}}{\sim} N(0,1)$. Therefore, we will not verify Conditions (A1) and (A4) for models with $\xi_i \overset{\text{ind}}{\sim} N(0,1)$.

We keep the same notation for all the quantities involved as for the general framework, with the understanding that these are specialized for the particular models and structures, and some constants must be adjusted. First we summarize the results of Theorems 1, 2, 3 and 4 by the following two corollaries.
Corollary 1. Let Conditions (A1) and (A2) be fulfilled. Then for any $M \geq 0$

\[
\sup_{\theta \in \Theta} \mathbb{E}_\theta \hat{\pi}(\|\hat{\theta} - \theta\|^2 \geq M_0 r^2(\theta) + M \sigma^2 |Y) \leq H_0 e^{-m_0 M},
\]

(i)

\[
\sup_{\theta \in \Theta} \mathbb{P}_\theta (\|\hat{\theta} - \theta\|^2 \geq M_1 r^2(\theta) + M \sigma^2 ) \leq H_1 e^{-m_1 M},
\]

(ii)

\[
\sup_{\theta \in \Theta} \mathbb{E}_\theta \hat{\pi}(I \in \mathcal{I} : r^2(I, \theta) \geq c_3 r^2(\theta) + M \sigma^2 |Y) \leq C(r e^{-c_2 M},
\]

(iii)

\[
\sup_{\theta \in \Theta} \mathbb{P}_\theta (r^2 \geq M_3 r^2(\theta) + (M + 1) \sigma^2 ) \leq H_3 e^{-m_3 M}.
\]

(iv)

If in addition Condition (A3) is fulfilled, then for any $M, t \geq 0$

\[
\sup_{\theta \in \Theta, \psi(t)} \mathbb{P}_\theta (\theta \notin B(\hat{\theta}, \hat{R}_M)) \leq H_2 e^{-m_2 M}.
\]

(v)

If in addition Condition (A4) is fulfilled, then for any $M \geq 0$,

\[
\sup_{\theta \in \Theta} \mathbb{P}_\theta (\theta \notin B(\hat{\theta}, \tilde{R}_M)) \leq \psi_1(M/4) + \psi_2(M) + H_1 e^{-m_1 M},
\]

(vi)

\[
\sup_{\theta \in \Theta} \mathbb{P}_\theta (\tilde{R}_M^2 \geq M^2(\theta, N)) \leq \psi_1(M/4) + \psi_2(M) + 2H_1 e^{-m_1 M}.
\]

(vii)

Remark 19. The properties (ii) and (iii) of Theorem 2 can also be included in Corollary 1, but we omit it, because these properties are only auxiliary results used for proving the size relations of Theorem 3. If additionally Condition (A3') is assumed for the property (iv), then the stronger uniform version of (iv) holds: $\sup_{\theta \in \Theta} \mathbb{P}_\theta (r^2 \geq M_1 \rho(s(I_n)) \sigma^2 + (M + 1) \sigma^2 ) \leq C(r e^{-M/2}$. Claim (v) of Corollary 1 can be formulated for the local version of coverage relation of Theorem 3 in terms of $b(\theta)$ (given by (3.12)) if needed.

Consider scales of classes $\{\Theta_{\beta}, \beta \in \mathcal{B}\}$, where $\beta \in \mathcal{B}$ is the structural parameter, for instance, $\beta$ could measure the amount of smoothness or sparsity of $\theta \in \Theta_{\beta}$. The above local results imply adaptive (global) minimax results for estimation and posterior contraction rate problems over all scales $\{\Theta_{\beta}, \beta \in \mathcal{B}\}$ at once, whose minimax rate

\[
r^2(\Theta_{\beta}) \triangleq \inf_{\hat{\theta}} \sup_{\theta \in \Theta_{\beta}} \mathbb{E}_\theta \|\hat{\theta} - \theta\|^2
\]

is bounded from below by a multiple of the local rate, namely

\[
r^2(\Theta_{\beta}) \geq c r^2(\theta) \quad \text{for all } \theta \in \Theta_{\beta}, \beta \in \mathcal{B}.
\]

(6.1)

Remark 20. Typically, (6.1) is established by comparing the oracle rate with the rate for some appropriately chosen structure $I^* = I^*(\theta)$. The reasoning goes usually as follows: first show that $\sup_{\theta \in \Theta_{\beta}} \|\theta - P_{I^*} \theta\|^2 \lesssim r^2(\Theta_{\beta})$ and $\sigma^2 \rho(s(I^*)) \lesssim r^2(\Theta_{\beta})$, then argue $r^2(\theta) \leq r^2(I^*, \theta) = \|\theta - P_{I^*} \theta\|^2 + \sigma^2 \rho(s(I^*)) \lesssim r^2(\Theta_{\beta})$ uniformly in $\theta \in \Theta_{\beta}$. Often $I^*$ is the so-called “true structure”, i.e., $\theta \in \mathbb{L}_{I^*}$, then $r^2(I^*, \theta) = \sigma^2 \rho(s(I^*)) \lesssim r^2(\Theta_{\beta})$. 
If (6.1) holds, we say that the oracle \( r^2(\theta) \) covers the scale \( \{ \Theta_\beta, \beta \in B \} \). Under (6.1), the adaptive (with respect to the structural parameter \( \beta \in B \)) minimax result follows immediately from Theorem 1: \( \sup_{\theta \in \Theta_\beta} \mathbb{P}(\| \hat{\theta} - \theta \|^2 \geq \frac{M_1}{C} r^2(\Theta_\beta) + M\sigma^2) \leq H_1 e^{-m_1 M} \).

Moreover, Theorems 1 and 3 imply the minimax versions of the posterior contraction result, the estimation result and the size relation in the uncertainty quantification problem, which are summarized by the following corollary.

**Corollary 2.** Let (6.1) and Conditions (A1) and (A2) be fulfilled. Then for any \( M \geq 0 \),

\[
\sup_{\theta \in \Theta_\beta} \mathbb{E}_\theta \hat{\pi}(\| \theta - \theta \|^2 \geq M_0 C^{-1} r^2(\Theta_\beta) + M\sigma^2) \leq H_0 e^{-m_0 M},
\]

\[
\sup_{\theta \in \Theta_\beta} \mathbb{P}(\| \hat{\theta} - \theta \|^2 \geq M_1 C^{-1} r^2(\Theta_\beta) + M\sigma^2) \leq H_1 e^{-m_1 M},
\]

\[
\sup_{\theta \in \Theta_\beta} \mathbb{P}(\hat{r}^2 \geq M_3 C^{-1} r^2(\Theta_\beta) + M\sigma^2) \leq H_3 e^{-m_3 M}.
\]

In case the radius of confidence ball is of the order \( r(\theta) + \sigma N^{1/4} \), we assume that the conditions of Theorem 4 instead of Theorem 3 are fulfilled and the third claim of Corollary 2 is replaced as follows:

\[
\sup_{\theta \in \Theta_\beta} \mathbb{P}(\hat{R}_M^2 \geq g_M'(\theta, N)) \leq \psi_1(M/4) + \psi_2(M) + 2H_1 e^{-m_1 M},
\]

where \( g_M'(\theta, N) = M_1 C^{-1} r^2(\Theta_\beta) + M\sigma^2 + 4\sigma^2 G_M \sqrt{N} \). We do not specialize Theorem 2 and the coverage relation of Theorems 3 and 4 for the scale \( \{ \Theta_\beta, \beta \in B \} \), because it does not make much sense to specialize these claims for any scale. Theorem 2 holds uniformly in \( \theta \in \Theta \), hence uniformly over any \( \Theta_\beta \). The coverage relation in Theorem 3 holds uniformly over the EBR class \( \Theta_{eb} \), so it will certainly hold uniformly over the intersection \( \Theta_{eb} \cap \Theta_\beta \). Similarly, the coverage relation in Theorem 4 will certainly hold uniformly over \( \Theta_\beta \).

Below we verify the required conditions to obtain Corollaries 1 and 2 for concrete models and structures. For brevity sake, for some examples and some claims of Corollaries 1 and 2, we will not present all the computations for verifying the required conditions, since these computations can be done similarly to the previously considered cases.

### 6.1. Signal+noise model with smoothness structure

Assume that the data \( Y = (Y_i)_{i \in N} \) come from the model

\[
Y_i = \theta_i + \frac{1}{\sqrt{n}} \xi_i, \quad i \in N,
\]

where \( \theta = (\theta_i)_{i \in N} \in \Theta = \ell_2 \) is an unknown parameter and \( \xi_i \overset{\text{iid}}{\sim} N(0, 1) \). A local approach for this model, delivering also the adaptive minimax results for many smoothness structures simultaneously, is considered by [1, 28] for posterior contraction rates and by [4] for uncertainty quantification problem.
Admittedly, this is an infinite dimensional model as compared with the default high-dimensional general framework (1.1), but in this case all the results go through with one minor adjustment: all the sums over $I \in \mathcal{I}$ become countable infinite instead of finite. Alternatively, we could consider a finite dimensional model approximating the original infinite dimensional model with arbitrary accuracy.

In this case, the smoothness structure is modeled by the linear spaces

$$\mathbb{L}_I = \{x \in \ell_2 : x_i = 0 \text{ for all } i \geq I + 1\}, \quad I, I_0 \in \mathbb{N}_0.$$  \hspace{1cm} (6.2)

We have $\|\theta - P_I \theta\|^2 = \sum_{i=I+1}^\infty \theta_i^2$, $d_I = \dim(\mathbb{L}_I) = I$, the structural slicing mapping is taken to be $s(I) = I$, so that $\mathcal{S} = \mathcal{I} = \mathbb{N}_0$ and $\mathcal{I}_{s(I)} = \{I\}$. Hence $\log |\mathcal{I}_s| = 0$ for all $s \in S$, and we thus take the majorant $\rho(s(I)) = d_s(I) + \log |\mathcal{I}_{s(I)}| = d_I = I$. The oracle rate is

$$r^2(\theta) = \min_{I \in \mathbb{N}_0} \left( \sum_{i \geq I+1} \theta_i^2 + \frac{L_0}{n} \right) = \sum_{i \geq I_0+1} \theta_i^2 + \frac{L_0}{n}.$$

Recall that, in view of Remarks 5 and 18, Conditions (A1) and (A4) hold with $d_{s(I)} = \dim(\mathbb{L}_I)$. Condition (A2) is fulfilled since, in view of Remark 7, $\sum_{i \in I} e^{-\nu s(I)} = \nu \sum_{s \in S} e^{-\nu s} = \frac{c_\nu}{\nu - 1} = C_\nu$ for any $\nu > 0$. Finally, Condition (A3) is also fulfilled. Indeed, for any $I_0, I_1 \in \mathcal{I}$ define $I'(I_0, I_1) = I_0 \cup I_1$, then ($\mathbb{L}_{I_0} \cup \mathbb{L}_{I_1} \subseteq \mathbb{L}_{I'}$) and $\rho(s(I')) = I_0 \cup I_1 \leq I_0 + I_1 = \rho(s(I_0)) + \rho(s(I_1))$.

As a consequence of our general results, we obtain the local results of Corollary 1 for this case with the local rate $r^2(\theta)$ defined above. In turn, by virtue of Corollary 2 the local results will imply global minimax adaptive results at once over all scales $\{\Theta_\beta, \beta \in \mathcal{B}\}$ covered by the oracle rate $r^2(\theta)$ (i.e., for which (6.1) holds). Below we present a couple of examples of scales $\{\Theta_\beta, \beta \in \mathcal{B}\}$ covered by the oracle rate $r^2(\theta)$.

6.1.1. Sobolev ellipsoids

For $\beta, Q > 0$, introduce the Sobolev ellipsoids

$$\Theta_\beta = \Theta_\beta(Q) = \{\theta \in \ell_2 : \sum_{i \in \mathbb{N}} i^{2\beta} \theta_i^2 \leq Q\}. \hspace{1cm} (6.3)$$

It is well known that the corresponding minimax rate is $r^2(\Theta_\beta) = n^{-2\beta/(2\beta + 1)}$; see, for example, [7] or [46]. The adaptive minimax results for Sobolev ellipsoids were considered by [1, 53] (see further references therein) for posterior contraction rates, and by [4, 52, 54] (see further references therein) for constructing optimal confidence balls. By taking $I_0 = \lceil n^{1/(2\beta + 1)} \rceil$, we obtain (6.1):

$$\sup_{\theta \in \Theta_\beta(Q)} r^2(\theta) = \sup_{\theta \in \Theta_\beta(Q)} \left( \sum_{i=I_0+1}^\infty \theta_i^2 + \frac{L_0}{n} \right) \leq \sup_{\theta \in \Theta_\beta(Q)} \sum_{i=I_0+1}^\infty i^{2\beta} \theta_i^2 + \frac{L_0}{n} \leq \frac{L_0}{n} + \frac{Q}{I_0^{2\beta}} \leq n^{-2\beta/(2\beta + 1)} \times r^2(\Theta_\beta).$$

Corollary 2 follows for this case with the minimax rate $r^2(\Theta_\beta)$ defined above.
6.1.2. Sobolev hyperrectangles

Consider the so called hyperrectangles in $\ell_2$:

$$\Theta_{\beta} = \Theta_{\beta}(Q) = \{ \theta \in \ell_2 : |\theta_i| \leq \sqrt{Q_i^{-\beta}} \}, \quad \beta > 1/2.$$

It is not difficult to show that the corresponding minimax rate is $r_2^2(\Theta_{\beta}) = n^{-2(2\beta-1)/(2\beta)}$.

The adaptive minimax results for Sobolev hyperrectangles were considered by [1, 4] for posterior contraction rates, and by [4, 52, 54] (see further references therein) for constructing optimal confidence balls. By taking $I_0 = \lfloor n^{1/(2\beta)} \rfloor$, we obtain (6.1):

$$\sup_{\theta \in \Theta_{\beta}(Q)} r_2^2(\theta) = \sup_{\theta \in \Theta_{\beta}(Q)} \sum_{i=I_0+1}^{\infty} \frac{Q_i - \beta}{n} \leq \frac{I_0}{n} + \frac{Q}{(2\beta-1)n^{2\beta-1}} \lesssim n^{-2(2\beta-1)/(2\beta)} \asymp r_2^2(\Theta_{\beta}).$$

Corollary 2 follows for this case with the minimax rate $r_2^2(\Theta_{\beta})$ defined above.

6.1.3. Analytic and tail classes

Similarly, we can derive the adaptive minimax results for two more scales of exponential ellipsoids (or analytic classes) and tail classes. Exponential ellipsoids are defined as follows:

$$\Theta_{\beta} = \Theta_{\beta}(Q) = \{ \theta \in \ell_2 : \sum_{k \in \mathbb{N}} e^{2\beta k} \theta_k^2 \leq Q \}, \quad \beta > 0.$$

For the analytic scale, the relation (6.1) is $\sup_{\theta \in \Theta_{\beta}} r_2^2(\theta) \lesssim r_2^2(\Theta_{\beta}) \asymp \frac{\log n}{n}$.

The tail classes are

$$\Theta_{\beta} = \Theta_{\beta}(Q) = \{ \theta \in \ell_2 : \sum_{k=m}^{\infty} \theta_k^2 \leq Q m^{-\beta}, m \in \mathbb{N} \}, \quad \beta > 0.$$

In this case, the relation (6.1) is $\sup_{\theta \in \Theta_{\beta}} r_2^2(\theta) \lesssim r_2^2(\Theta_{\beta}) \asymp n^{-\beta/(\beta+1)}$.

Corollary 2 follows for the both scales with the corresponding minimax rates $r_2^2(\Theta_{\beta})$ defined above.

6.2. Signal+noise model under wavelet basis

We adopt the notation and conventions from [32]. Consider the observations

$$Y_{jk} = \theta_{jk} + \frac{1}{\sqrt{n}} \xi_{jk}, \quad \xi_{jk} \sim \text{N}(0,1), \quad (jk) \in \mathcal{K} = \{(jk) : j \in \mathbb{N}_0, k \in [2^j]\}.$$ 

This model is obtained as the result of the orthogonal wavelet transform of an additive regression function observed in Gaussian noise with $\sigma^2 = n^{-1}$, or just as a sequence version (with respect to some wavelet basis) of the continuous white noise model. We could also
consider a high dimensional “projected” (see (9.57) in [32]) variant, where \( j \in \lbrack J_0 \rbrack \) with \( 2^{J_0+1} = n \). For details and many interesting connections and relations to the function estimation theory we refer to the very comprehensive and insightful account [32] on this topic.

The smoothness structure of \( \theta = (\theta_{j k}, (j k) \in K) \) is modeled by the linear spaces

\[
\mathbb{L}_I = \{(x_{jk}, (jk) \in K) : x_{jk} = 0 \ \forall j \in \lbrack j_0 \rbrack_0, k \in I_j^c \ \text{and} \ \forall j > j_0, k \in \lbrack 2^j \rbrack \},
\]

where \( I = (j_0, I_0, \ldots, I_{J_0}) \) with \( I_j \subseteq \lbrack 2^j \rbrack, j \in \lbrack J_0 \rbrack_0 \). The structural slicing mapping is \( s(I) = (j_0, |I_0|, \ldots, |I_{j_0}|) \) and \( d_I = \dim(\mathbb{L}_I) = \sum_{j_0 = 0}^{J_0} |I_m| \). Compute \( |\mathcal{I}_{s(I)}| = \prod_{k = 0}^{J_0} (|I_k|_k^2) \), hence \( \log |\mathcal{I}_{s(I)}| = \sum_{k = 0}^{J_0} \log (2k) \leq \sum_{k = 0}^{J_0} |I_k| \log (\frac{2k}{|I_k|}) \). Since \( d(s(I)) + \log |\mathcal{I}_{s(I)}| = d_I + \log |\mathcal{I}_{s(I)}| \leq 2 \sum_{k = 0}^{J_0} |I_k| \log (\frac{2k}{|I_k|}) \), we take the majorant \( \rho(s(I)) = 2 \sum_{k = 0}^{J_0} |I_k| \log (\frac{2k}{|I_k|}) \).

Conditions (A1) and (A4) hold with \( d(s(I)) = \dim(\mathbb{L}_I) \) in view of Remarks 5 and 18. Condition (A2) is also fulfilled, since, according to Remark 7, for any \( \nu > 2 \)

\[
\sum_{I \in \mathcal{I}} e^{-\nu \rho(s(I))} \leq \sum_{s \in S} e^{-(\nu - 1) \rho(s(I))} \leq \sum_{j_0 = 0}^{\infty} \sum_{k_0 = 1}^{2^0} \cdots \sum_{k_m = 1}^{2^j} e^{-(\nu - 1)(k_0 + \cdots + k_m)} \leq \sum_{j_0 = 0}^{\infty} \left( \frac{e^{\nu - 1} - 1}{\nu - 1} \right)^{j_0 + 1} \leq \frac{1}{e^{\nu - 1} - 2} = C_\nu.
\]

Finally, for any \( I^0, I^1 \in \mathcal{I} \) define \( j_0'' = \min\{j_0^0, j_1^1\} \), \( j_0' = \max\{j_0^0, j_1^1\} \) and \( I'(I^0, I^1) \in \mathcal{I} \) such that

\[
I'(I^0, I^1) = (I_0^0 \cup I_1^0) \cup (I_0^1 \cup I_1^1) \cup (I_0^0 \cup I_1^1)_{j_0^0 + 1} \cup (I_0^1 \cup I_1^0)_{j_0^1 + 1} \cup (I_0^0 \cup I_1^1)_{j_0^0 + 1} \cup (I_0^1 \cup I_1^0)_{j_0^1 + 1}.
\]

Then \( (\mathbb{L}_{I^0} \cup \mathbb{L}_{I^1}) \subseteq \mathbb{L}_{I'} \) and

\[
\sum_{m = 0}^{j_0^0} |I_m^0| \log (\frac{2^m}{|I_m^0|}) \leq \sum_{m = 0}^{j_0^1} |I_m^1| \log (\frac{2^m}{|I_m^1|}) \leq \sum_{m = 0}^{j_0^1} |I_m^1| \log (\frac{2^m}{|I_m^1|}) + \sum_{m = 0}^{j_0^0} |I_m^0| \log (\frac{2^m}{|I_m^0|}),
\]

which entails Condition (A3).

As a consequence of our general results, we obtain Corollary 1 for this case with the local rate \( r^2(\theta) = \min_{I \in \mathcal{I}} \left\{ \|\theta - P_I \theta\|_2 + \frac{1}{2} \rho(s(I)) \right\} \). Below we present the example of Besov scale, for which the global minimax adaptive results follow from the local results. We should mention that there are of course more scales covered by the oracle rate \( r^2(\theta) \), the reader is invited to make computations for other interesting scales. Besides, the results can be extended to non-normal, not independent \( \xi_{j k} \)'s, but only satisfying Condition (A1).

### 6.2.1. Besov scale

Assume that the true signal \( \theta \) belongs to a Besov ball

\[
\Theta^\beta_{p,q}(Q) = \left\{ \theta : \sum_{j = \theta}^{\infty} 2^a q_j \left( \sum_{k = 1}^{2^j} \theta_{j k}^p \right)^{q/p} \leq Q^q \right\}, \quad a = \beta + \frac{1}{2} - \frac{1}{p},
\]
for some $p, q, Q > 0$ and $\beta \geq 1/p$. The minimax rate over $\Theta_{p,q}^\beta(Q)$ is known to be $r^2(\Theta_{p,q}^\beta(Q)) \asymp n^{-\frac{\beta}{2p+1}}$. The adaptive minimax results for the scale of the class $\Theta_{p,q}^\beta(Q)$ were considered by [51, 31, 28] and many others for posterior contraction rates, and [16] for constructing optimal confidence balls.

Let $j_* = \lfloor \log_2 n \rfloor$. Define $\mathcal{I}_s = \{I \in \mathcal{I} : j_0(I) = j_*\}$ and note that $\mathcal{I}_s \subset \mathcal{I}$. Hence, for any $\theta \in \Theta_{p,q}^\beta(Q)$,

$$r^2(\theta) \leq \min_{I \in \mathcal{I}_s} \{ ||\theta - P_I\theta||^2 + \frac{1}{n} \rho(s(I)) \} \leq \sum_{j=0}^{j_*} \sum_{k \in I_{s,j}} \theta_{j,k}^2 + \sum_{j=0}^{j_*} \frac{|I_{s,j}|}{n} \log(e^{2^j/k}) \leq C_2 n^{-\frac{2\beta}{2p+1}} + C_3 n^{-1} \lesssim n^{-\frac{\beta}{2p+1}} \asymp r^2(\Theta_{p,q}^\beta(Q)),$$

where $\theta_{j,k}^2$ denotes the $l$-th largest value among $\{\theta_{j,k}^2, j \in [2^k]\}$. The third inequality of the last display follows from Theorem 12.1 in [32] under the assumption $\beta \geq 1/p$. We thus established the relation (6.1) for the Besov scale, and Corollary 2 follows with the minimax rate $r^2(\Theta_{p,q}^\beta(Q))$ defined above.

### 6.3. Signal+noise model with (multi-level) sparsity structure

Assume that the data $Y = (Y_i)_{i \in [n]}$ come from the model

$$Y_i = \theta_i + \sigma \xi_i, \ i \in [n],$$

(6.4)

where $\theta = (\theta_i)_{i \in [n]} \in \Theta = \mathbb{R}^n$ is an unknown parameter and $\xi_i \overset{\text{ind}}{\sim} N(0, 1)$. The high-dimensional vector $\theta$ is assumed to be sparse. A local approach for this model, delivering also the adaptive minimax results for various sparsity structures simultaneously, is considered in [8, 28] for posterior contraction rates and by [8] for uncertainty quantification problem.

The classical sparsity structure is modeled by the linear spaces

$$\mathbb{L}_I = \{ x \in \mathbb{R}^n : x_i = 0, i \in I^c \}, \ I \in \mathcal{I} = \{ J : J \subseteq [n]\}.$$ 

In this case, $d_I = \dim(\mathbb{L}_I) = |I|$, $\|\theta - P_I\theta\|^2 = \sum_{i \in I} \theta_i^2$, the structural slicing mapping is defined to be $s(I) = |I| \in S \triangleq [n]_0$. Compute $|I_s(I)| = \binom{n}{|I|}$, hence $\log |I_s(I)| = \log \binom{n}{|I|} \leq |I| \log \left( \frac{\binom{n}{|I|}}{|I|} \right)$. Since $d_{s(I)} + \log |I_{s(I)}| = d_I + \log |I_s(I)| \leq |I| + |I| \log \left( \frac{\binom{n}{|I|}}{|I|} \right)$, we take the majorant $\rho(s(I)) = 2|I| \log \left( \frac{\binom{n}{|I|}}{|I|} \right)$. 

Conditions (A1) and (A4) hold with $d_{s(I)} = \dim(\mathbb{L}_I)$ in view of Remarks 5 and 18. Condition (A2) is fulfilled, since, according to Remark 7, for any $\nu > 1$

$$\sum_{I \in \mathcal{I}} e^{-\nu \rho(s(I))} \leq \sum_{s \in \mathcal{S}} e^{-(\nu - 1)\rho(s(I))} \leq \sum_{s=0}^{n} \left( \frac{en}{s} \right)^{-(\nu - 1)s} \leq \frac{1}{1 - e^{1-\nu}} = C_\nu.$$  

Finally, for any $I_0, I_1 \in \mathcal{I}$ define $I' = I_0 \cup I_1$. Then $(\mathbb{L}_I \cup \mathbb{L}_{I_I}) \subseteq \mathbb{L}_{I'} = \mathbb{L}_0 + \mathbb{L}_{I_1}$ and $|I'| \log \left( \frac{en}{|I'|} \right) \leq |I_0| \log \left( \frac{en}{|I_0|} \right) + |I_1| \log \left( \frac{en}{|I_1|} \right)$, which entails Condition (A3).

**Remark 21.** We can take a slightly better majorant, $\rho'(s(I)) = \max\{|I|, \log \left( \frac{en}{|I|} \right) \}$.

As a consequence of our general results, we obtain Corollary 1 with the local rate $r^2(\theta) = \min_{I \in \mathcal{I}} \left\{ \|\theta - P_I \theta\|^2 + \sigma^2 \rho(s(I)) \right\}$. In view of Remark 21, the results hold also with the local rate $r^2(\theta) = \min_{I \in \mathcal{I}} \left\{ \|\theta - P_I \theta\|^2 + \sigma^2 \rho'(s(I)) \right\}$. As $\rho'(s) \leq \rho(s)$ for all $s \in \mathcal{S}$, the local rate with $\rho'(s)$ is smaller than the rate with $\rho(s)$ implying a stronger version of Corollary 1. However, the quantity $\rho(s)$ is easier to compute and we will thus use the majorant $\rho(s)$.

Below we present a couple of examples of scales $\{\Theta_\beta, \beta \in \mathcal{B}\}$, for which the global minimax adaptive results follow from the local results. Recall that the results can be extended to non-normal and not necessarily independent $\xi_i$’s, but only satisfying Condition (A1). For example, as demonstrated in [8], $\xi_i$’s originating from a certain AR(1)-model also satisfy Condition (A1).

### 6.3.1. Minimax results for nearly black vectors $\ell_0$ and weak $\ell_q$-balls

**Nearly black vectors $\ell_0$.** By $I^*(\theta)$ and $s(\theta)$, we denote respectively the active index set and the sparsity of $\theta \in \mathbb{R}^n$. For $p \in [n]$, introduce the sparsity class (also called nearly black vectors)

$$\ell_0[p] = \{ \theta \in \mathbb{R}^n : \|\theta\|_0 = |I^*(\theta)| \leq p \}, \quad I^*(\theta) = \{ i \in [n] : \theta_i \neq 0 \}.$$  

(6.5)

The minimax estimation rate over the class of nearly black vectors $\ell_0[p]$ with the sparsity parameter $p$ is known to be $r^2(\ell_0[p]) \asymp \sigma^2 p \log \left( \frac{2n}{p} \right)$ (usually in the literature $p = p_n = o(n)$ as $n \to \infty$, but we do not impose this restriction); see [24]. The adaptive minimax results for nearly black vectors were considered in [8, 22, 43, 56] and many others for posterior contraction rates, and in [8, 57] for constructing optimal confidence balls.

By the definition (3.1) of the oracle rate $r^2(\theta)$, we have that $r^2(\theta) \leq r^2(I^*(\theta), \theta)$. Then we obtain trivially that

$$\sup_{\theta \in \ell_0[p]} r^2(\theta) \leq \sup_{\theta \in \ell_0[p]} r^2(I^*(\theta), \theta) \leq \sigma^2 p \log \left( \frac{en}{p} \right) \leq r^2(\ell_0[p]).$$

We thus established the relation (6.1) for the scale $\{\ell_0[p], 0 \leq p \leq n\}$, and Corollary 2 follows with the minimax rate $r^2(\ell_0[p])$ defined above.
Weak $\ell_q$-balls. For $q \in (0, 2)$, the weak $\ell_q$-ball of sparsity $p_n$ is defined by

$$m_q[p_n] = \{ \theta \in \mathbb{R}^n : \theta_{[i]}^2 \leq (p_n/n)^2(n/i)^{2/q}, \, i \in [n] \},$$

where $p_n = o(n \log n)$ as $n \to \infty$, $\theta_{[1]}^2 \geq \ldots \geq \theta_{[n]}^2$ are the ordered $\theta_1^2, \ldots, \theta_n^2$. This class can be thought of as Sobolev hyperrectangle for ordered (with unknown locations) coordinates: $m_q[p_n] = \mathcal{H}(\beta, \delta_n) = \{ \theta \in \mathbb{R}^n : |\theta_{[i]}| \leq \delta_n i^{-\beta} \}$, with $\delta_n = p_n n^{-1+1/q}$ and $\beta = 1/q > 1/2$.

Denote $j = O_\theta(i)$ if $\theta_{[i]} = \theta_{[j]}$, with the convention that in the case $\theta_{i_1} = \ldots = \theta_{i_k}$ for $i_1 < \ldots < i_k$, we let $O_\theta(i_l) = O_\theta(i_l-1) + 1$, $l = 1, \ldots, k-1$. The minimax estimation rate over this class is $r^2(m_q[p_n]) = n(p_n/n)^q[\sigma^2 \log(n p_n)]^{1-q/2}$ when $n^{2/q}(p_n/n)^2 \geq \sigma^2 \log n$, and $r^2(m_q[p_n]) = n^{2/q}(p_n/n)^2 + \sigma^2$ when $n^{2/q}(p_n/n)^2 < \sigma^2 \log n$, as $n \to \infty$; see [25, 14].

The adaptive minimax results for the scale of weak $\ell_q$-balls were considered in [8, 22] for posterior contraction rates and in [8] for constructing optimal confidence balls. We take $I_0(\theta) = \{ i \in [n] : O_\theta(i) \leq p_n^* \}$, with $p_n^* = \max(\frac{p_n}{n \log n})^{-q/2}$ in the case $n^{2/q}(p_n/n)^2 \geq \sigma^2 \log n$, to derive (6.1):

$$r^2(\theta) \leq \sup_{\theta \in m_q[p_n]} r^2(I_0(\theta), \theta) \leq \sigma^2 p_n^* \log(n p_n) + n^{2/q}(p_n/n)^2 \sum_{i > p_n^*} i^{-2/q} \approx C_1 \sigma^2 p_n^* \log(n p_n) + C_2 n^{2/q}(p_n/n)^2 p_n^* \log(n p_n) \leq n(p_n/n)^q[\sigma^2 \log(n p_n)]^{1-q/2} \approx r^2(m_q[p_n]).$$

The case $n^{2/q}(p_n/n)^2 < \sigma^2 \log n$ is treated similarly by taking $p_n^* = 0$. Corollary 2 follows for this case with the minimax rate $r^2(m_q[p_n])$ defined above.

6.3.2. Multi-level sparsity (clustering) structure

Consider the same model (6.4), but now with the so-called multi-level sparsity structure, an extension of the traditional sparsity structure. In the usual one-level sparsity structure we have just one known sparsity level, which is by default zero. The first attempt to study a version of such structure has been undertaken in [9], here we propose a systematic approach to this from the general perspective of the linear spaces for the first time. To the best of our knowledge, this structure has never been systematically studied in the literature.

First we extend the classical sparsity structure by allowing the sparsity level to be an unknown constant, not necessarily zero. This extended unknown level sparsity structure is described by the linear spaces:

$$\mathbb{L}_I = \{ x \in \mathbb{R}^n : x_i = x_j, \forall i, j \in I \}, \quad I \in \mathcal{I} = \{ J : J \subseteq [n] \}.$$

Then $d_I = \dim(\mathbb{L}_I) = (|I| + 1) \wedge n$, $\| \theta - P_I \theta \|^2 = \sum_{i \in I^c} (\theta_i - \bar{\theta}_I)^2$ (where $\bar{\theta}_I = |I^c|^{-1} \sum_{i \in I^c} \theta_i$), and the structural slicing mapping $s(I) = |I| \in \mathcal{S} \triangleq [n]_{\mathbb{R}}$. Compute $|\mathcal{I}_s| = \binom{n}{2}$, hence $d_{s(I)} + \log |\mathcal{I}_s | = d_I + \log \binom{n}{2} \leq (|I| + 1) \wedge n + |I| \log(n |I|)$. The majorant is $\rho(s(I)) = (|I| + 1) \wedge n + |I| \log(n |I|)$. 

E. Belitser and N. Nurushev/Inference for projection structures 40
Next, we extend the one-level sparsity structure to the multi-level sparsity structure (with unknown sparsity levels) by introducing the following linear spaces: for a partition \( I = (I_i, i \in [m]) \) of the set \([n]\) into \( m + 1 \) parts,

\[
L_I = \{ x \in \mathbb{R}^n : x_j = x_{j'}, \forall j, j' \in I_i, i \in [m] \}, \quad I \in \mathcal{I},
\]

where \( \mathcal{I} = \mathcal{I}_m \) is the family of all partitions of \([n]\) into \( m + 1 \) parts (some possibly empty), and \( m = 2, \ldots, n - 1 \). This can also be seen as clustering structure, where the partition \( I \) determines clustering of the coordinates of \( \theta \) into \( m + 1 \) groups. In this case, compute

\[
\|\theta - P_I \theta\|^2 = \sum_{k=1}^m \sum_{i \in I_k} (\theta_i - \bar{\theta}_{I_k})^2
\]

with the group averages \( \bar{\theta}_{I_k} = \frac{1}{|I_k|} \sum_{i \in I_k} \theta_i \), the structural slicing mapping is taken to be \( s(I) = (|I_i|, i \in [m]) \in S \), where \( S = S(n, m + 1) = \{ (n_i, i \in [m]) : n_i \in [n], \sum_{i \in [m]} n_i = n \} \) is the family of the so called weak compositions of \( n \) into \( m + 1 \) parts. It is well known that \( |S| = \binom{n+m}{m} \). Further we have \( d_{s(I)} = d_I = \dim(L_I) = (|I_i| + m) \land n \) and \( |I_{s(I)}| = \binom{n}{|I_i|, \ldots, |I_m|} \) is the multinomial coefficient.

To ensure Condition (A2), we have to compensate for the number \(|S|\) by adding the term \( \log |S| = \log \binom{n+m}{m} \) in the complexity majorant \( \rho(s) \). Hence, we take the majorant \( \rho(s(I)) = (|I_i| + m) \land n + \log \binom{n}{|I_i|, \ldots, |I_m|} + \log \binom{n+m}{m} \).

Conditions (A1) and (A4) hold with \( d_{s(I)} = \dim(L_I) \) in view of Remarks 5 and 18. Condition (A2) is fulfilled, since for any \( \nu \geq 1 \)

\[
\sum_{I \in \mathcal{I}} e^{-\nu \rho(s(I))} = \sum_{S \in S} \sum_{I \in \mathcal{I}_S} e^{-\nu \rho(s(I))} \leq \sum_{S \in S} e^{-\nu \log |S|} \leq 1.
\]

Unfortunately, we were unable to establish Condition (A3) for this structure, which is needed for the uncertainty quantification results under the EBR condition. What we can claim are the relations (i)-(iv) and (vi)-(vii) of Corollary 1 with the local rate

\[
r^2(\theta) = \min_{I \in \mathcal{I}} \left\{ \|\theta - P_I \theta\|^2 + \sigma^2 \rho(s(I)) \right\}
\]

\[
= \min_{I \in \mathcal{I}} \left\{ \sum_{k=1}^m \sum_{i \in I_k} (\theta_i - \bar{\theta}_{I_k})^2 + \sigma^2 \left[ (|I_0| + m) \land n + \log \binom{n}{|I_0|, \ldots, |I_m|} + \log \binom{n+m}{m} \right] \right\}.
\]

For \( m = 1 \) we get the classical one-level local sparsity results which also imply the global minimax results over sparsity scales, as is considered in the previous paragraph. For \( m \geq 2 \), the obtained local results (i)-(iv) and (vi)-(vii) of Corollary 1 are new to the best of our knowledge. The most problematic term is \( \log \binom{n}{|I_0|, \ldots, |I_m|} \), this term is of a smaller order than \( n \) if \( |I_0| \) and any \( m - 1 \) values among \( |I_1|, \ldots, |I_m| \) (e.g., \( |I_0|, |I_1|, \ldots, |I_{m-1}| \)) are themselves of the smaller order than \( n \).

**Remark 22.** It is an open problem to establish Condition (A3). This is important in the uncertainty quantification problem, namely, the coverage relation (v) from Corollary 1 relies on this condition.
If we are to verify Condition (A3), for any \( I, I' \in \mathcal{I} \) we would define

\[
I'' = I''(I, I') = \left( I_0 \cup I'_0, (I_i \cap I'_i, i, i' \in [m]) \right).
\]

Clearly, \( L_{I''} \subseteq L_{I'} \subseteq L_I \subseteq L_{I'} + L_{I''} \) and \( \max\{ s(I), s(I') \} \leq s(I'') \leq s(I) + s(I') \), implying \( \rho(s(I'')) \leq \rho(s(I)) + \rho(s(I')) \), and seems that Condition (A3) is fulfilled. However, the problem is that the resulting \( I'' \) may in general not lie in \( \mathcal{I} \) but rather in \( \mathcal{I}_{m^2} \). An idea to fix this would be to let the number \( m \) of parts in partitions \( I \in \mathcal{I} \) free (any integer from 0 to \( n \)). But then the problem will emerge in another place: there are too many choices as the family \( \mathcal{S} \) of all compositions of \( n \) becomes \( |\mathcal{S}| = 2^{n-1} \). Then we will have to put the term \( \log |\mathcal{S}| \propto n \) in the complexity majorant \( \rho(s(I)) \) to meet Condition (A2), which makes the local rate \( r^2(\theta) \geq n \sigma^2 \) trivially large and therefore uninteresting.

**Remark 23.** We should mention that the global minimax multi level sparsity results are not going to be useful, at least if we try to extend one-level sparsity scales to multi-level sparsity scales in the usual way. Indeed, even if we assume sparsity in the sense that \( |I_0| \leq s \) for some small \( s = s_n \ll n \), i.e., \( \theta \in \Theta_s = \cup_{I \in \mathcal{I} : |I_0| \leq s} \mathcal{I}_I \), the minimax rate over \( \Theta_s \) will presumably be

\[
r^2(\Theta_s) \asymp \sigma^2 \max_{I \in \mathcal{I} : |I_0| \leq s} \rho(s(I)) \asymp \sigma^2 \max_{I \in \mathcal{I} : |I_0| \leq s} \log \left( \frac{n}{|I_0|} \right) \geq n \sigma^2,
\]

and this is not a useful result. This means that the multilevel counterpart \( \Theta_s \) for the traditional one-level sparsity class \( \ell_0[s] \) is too “massive” in the minimax sense.

One can propose other scales \( \{\Theta_\beta, \beta \in B\} \) with more structure for which at least minimax consistency would hold, i.e., \( r^2(\Theta_\beta) \ll \sigma^2 n \). For example, consider \( \Theta_{s,m} = \cup\{ L_I, I \in \mathcal{I}, i \in [m] : |I_j| \leq s, j \in [m] \setminus \{i\} \} \), with \( s = s_n \) and \( m = m_n \) such that \( (s_n + 1)m_n \log n \ll n \). Then we can show that for any \( \theta \in \Theta_{s,m} \)

\[
r^2(\theta) \leq \sigma^2 \left[ (|I_0| + m_n) + \log \left( \frac{n}{|I_0|} \right) \right] \leq \sigma^2 (s_n(m_n + 1) + m_n) \log n.
\]

### 6.4. Function on a large graph with smoothness structure

We adopt the notation and conventions from [33]. Let \( G \) be a connected, simple (i.e., no loops, multiple edges or weights), undirected graph with \( n \) vertices labelled as \( 1, \ldots, n \). A function \( f \) on the (vertices of the) graph is a mapping \( f : [n] \mapsto \mathbb{R} \). We will write \( f \) both for the function and for the associated vector of function values \( (f(1), f(2), \ldots, f(n)) \) in \( \mathbb{R}^n \).

We assume that \( Y_1, \ldots, Y_n \) are the observations at the vertices of the graph, satisfying

\[
Y_i = f(i) + \frac{1}{\sqrt{n}} \xi_i, \quad i \in [n],
\]

where \( f = (f(i))_{i \in [n]} \in \mathbb{R}^n \) is the vector of values of unknown function on the graph \( G \) and \( \xi_i \overset{\text{ind}}{\sim} \mathcal{N}(0, 1) \). In this case, \( \theta = f \in \Theta \triangleq \mathbb{R}^n \). To the best of our knowledge, there are no local results on estimation, posterior contraction rate and uncertainty quantification problems for this model.
The smoothness structure of function $f$ is described by the linear spaces

$$L_I = \{ x \in \mathbb{R}^n : x_i = 0 \text{ for all } i = I + 1 \ldots n \}, \quad I \in \mathcal{I} = [n]_0.$$  

In this case, $\| f - P_I f \|^2 = \sum_{i=I+1}^n f^2(i)$, the structural slicing mapping $s(I) = I$, so that $\mathcal{S} = \mathcal{I} = [n]_0$ and $\mathcal{I}_s = \mathcal{I}_I = \{ I \}$. Hence $\log |\mathcal{I}_s| = 0$. Further, in view of Remark 5, Condition (A1) is fulfilled with $\alpha = 0.4$, $d_{s(I)} = d_I = \dim(L_I) = I$, and we arrive at the majorant $\rho(s(I)) = \rho(I) = d_I = I$. The oracle rate is

$$r^2(f) = \min_{I \in [n]} \left( \sum_{i=I+1}^n f^2(i) + \sigma^2 I \right) = \sum_{i=I_0+1}^n f^2(i) + \sigma^2 I_0.$$

Further, Condition (A4) holds in view of Remark 18. Condition (A2) is fulfilled since, according to Remark 7, for any $\nu > 0$,

$$\sum_{I \in \mathcal{I}} e^{-\nu \rho(I)} = \sum_{s \in \mathcal{S}} e^{-\nu s} = \frac{e^{\nu}}{e^{\nu} - 1} = C_\nu.$$

Condition (A3) is also fulfilled. Indeed, for any $I_0, I_1 \in \mathcal{I}$ define $I'(I_0, I_1) = I_0 \cup I_1$, then $(L_{I_0} \cup L_{I_1}) \subseteq L_{I'}$ and $\rho(s(I')) = I_0 \cup I_1$. As consequence of our general results, we obtain the local results of Corollary 1 for this case with the local rate $r^2(f)$ defined above. In turn, by virtue of Corollary 2 the local results will imply global minimax adaptive results over all scales $\{ \Theta_{\beta}, \beta \in \mathcal{B} \}$ at once, covered by the oracle rate $r^2(f)$ (i.e., for which (6.1) holds). Below we present the Laplacian scale $\{ H^\beta, \beta > 0 \}$ and show that it is covered by the oracle rate $r^2(f)$.

### 6.4.1. Minimax results for a Laplacian graph

One common approach to learn functions on graphs is Laplacian regularisation; see, for example, [11, 33]. The graph Laplacian is defined as $L = D - A$, where $A$ is the adjacency matrix of the graph and $D$ is the diagonal matrix with the degrees of the vertices on the diagonal. When viewed as a linear operator, the Laplacian acts on a function $f$ as

$$L f(i) = \sum_{j \sim i} (f(i) - f(j)),$$

where we write $i \sim j$ if vertices $i$ and $j$ are connected by an edge. Denote the Laplacian eigenvalues, ordered by magnitude, by $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. As in [33], we assume without loss of generality that there exist $i_0 \in \mathbb{N}$, $C_1 > 0$ such that for all $n$ large enough and $r \geq 1$,

$$\lambda_i \geq C_1 \left( \frac{i}{n} \right)^{2/r}, \quad i > i_0,$$

and $f \in H^\beta = H^\beta(Q) = \{ f : \sum_{i=1}^n (1 + n^{2/\beta} \lambda_i^\beta) f^2(i) \leq Q \}$, with smoothness $\beta > 0$. The minimax estimation rate over the class $H^\beta$ is $r^2(H^\beta) = \inf_{\tilde{f}} \sup_{f \in H^\beta} \mathbb{E}_{\tilde{f}} \| \tilde{f} - f \|^2 \asymp n^{-\frac{2\beta}{2\beta + r}}$; see [34].
By taking \( I_0 = \lfloor n^{r/(2\beta + r)} \rfloor \), we establish (6.1) in this case:

\[
\sup_{f \in H^\beta} r^2(f) = \sup_{f \in H^\beta} \frac{1}{n} \sum_{i=I_0+1}^{n} f^2(i) + \frac{L^2}{n} \leq \sup_{f \in H^\beta} \frac{1}{n} \sum_{i=I_0+1}^{n} f^2(i) + \frac{L^2}{n} 
\leq \frac{Q}{1+n^{2\beta/r} \lambda^2} + \frac{L^2}{n} \lesssim n^{-\frac{2\beta}{2\beta + r}} \times r^2(H^\beta).
\]

Hence, Corollary 2 follows for this case with the minimax rate \( r^2(H^\beta) \) defined above. As compared to the Theorems 3.2 and 3.3 in [33], there is no restriction on the range of the smoothness \( \beta \) in Corollary 2, and we do not have any extra logarithmic factor in the rate.

### 6.5. Density estimation with smoothness structure

We observe \( X_1, \ldots, X_n \sim f \), where \( f \) is a density on \([0,1]\). Let \( \{\varphi_i, i \in \mathbb{N}\} \) be an orthonormal basis in \( L_2[0,1] \). For simplicity consider a bounded basis \( \{\varphi_i, i \in \mathbb{N}\} \), implying \( \sup_{x \in [0,1]} |\varphi_i(x)| \leq c_\varphi \) for some \( c_\varphi > 0 \) (e.g., for the trigonometric basis \( c_\varphi = \sqrt{2} \)).

We can expand the density function \( f \) in a Fourier series
\[
f(x) = \sum_{i=1}^{\infty} \theta_i \varphi_i(x), \quad x \in [0,1],
\]
where \( f(x) \) is a density on \([0,1]\). Let \( \{\varphi_i, i \in \mathbb{N}\} \) be an orthonormal basis in \( L_2[0,1] \). For simplicity consider a bounded basis \( \{\varphi_i, i \in \mathbb{N}\} \), implying \( \sup_{x \in [0,1]} |\varphi_i(x)| \leq c_\varphi \) for some \( c_\varphi > 0 \) (e.g., for the trigonometric basis \( c_\varphi = \sqrt{2} \)). We can expand the density function \( f \) in a Fourier series
\[
f(x) = \sum_{i=1}^{\infty} \theta_i \varphi_i(x), \quad x \in [0,1],
\]
where \( \theta = (\theta_i)_{i \in \mathbb{N}} \) is an unknown high-dimensional parameter of interest with \( \theta_i = \mathbb{E} f Y_i = \int_0^1 \varphi_i(x) f(x) dx \), \( Y_i = \frac{1}{n} \sum_{i=1}^{n} \varphi_i(X_i) \), and \( \sigma_n \xi_i = Y_i - \theta_i \). Since \( |Y_i| = \frac{1}{n} \sum_{i=1}^{n} \varphi_i(X_i) \leq c_\varphi \), we have \( \sigma_n |\xi_i| \leq |Y_i| + |\theta_i| \leq 2c_\varphi \) and \( \text{Var}(\sigma_n \xi_i) \leq \frac{c_\varphi^2}{n} \). The parameter \( \sigma_n \) will be chosen later, for now it is any sequence \( \sigma_n \in [0,1] \).

Notice that we reduced the original density estimation problem to a finite dimensional version of the model from Section 6.1, however the errors \( \xi_i \)'s are now not iid normals, which complicates the study of the present model. Consider the same smoothness structure (6.2) as in Section 6.1, with the difference that we restrict the family of structures \( I \in \mathcal{I} = [n]_0 \). The oracle rate becomes

\[
r^2(\theta) = \min_{I \in [n]_0} \left( \sum_{i \geq I+1} \theta_i^2 + \sigma_n^2 I \right) = \sum_{i \geq I_0+1} \theta_i^2 + \sigma_n^2 I_0.
\]

Conditions (A2) and (A3) are met in the same way as for the signal+noise model from Section 6.1. However, in order to derive at least the local estimation and posterior contraction results, we also need Condition (A1). This condition is now not immediate since the errors \( \xi_i \)'s are non-normal and dependent in the model (6.7) (actually, the \( \xi_i \)'s are asymptotically normal, but we are not going to rely on this). We apply the following strategy: introduce certain event and establish that the probability of this event is exponentially small (in \( n \));
next, under this event establish Condition (A1); finally, combine these two facts to derive the local estimation and posterior contraction results.

The following proposition is a direct consequence of McDiarmid’s inequality; see, for instance Theorem 6.2 in [15].

**Proposition 1.** For any $t > 0$ and $i \in [n],$

$$\mathbb{P}(\sigma_n|\xi_i| \geq t) \leq 2 \exp \left\{-c_\varphi^{-2}t^2/2\right\}.$$  

The relation $\mathbb{P}(\max_{i \in [n]}|\xi_i| \geq t) \leq \sum_{i \in [n]} \mathbb{P}(|\xi_i| \geq t)$ and Proposition 1 imply that, for the event $E = \{\max_{i \in [n]}|\xi_i| \leq \sqrt{2c_\varphi}\},$

$$\mathbb{P}(E^c) = \mathbb{P}(\max_{i \in [n]}|\xi_i| > \sqrt{2c_\varphi}) \leq 2 \exp\{-n\sigma_\varphi^2 + \log n\}. \quad (6.8)$$

Now, by using (6.8), we ensure Condition (A1) under the event $E = \{\max_{i \in [n]}|\xi_i| \leq \sqrt{2c_\varphi}\}$ with $\alpha = 1 \wedge 1/(2c_\varphi^2).$ Exactly, for any $I \in [n]_0,$

$$\mathbb{E}\exp\left\{\alpha\|P_I\xi\|^2\right\}1_E = \mathbb{E}\exp\left\{\alpha \sum_{i=1}^I \xi_i^2\right\}1\{\max_{i \in [n]}|\xi_i| \leq \sqrt{2c_\varphi}\} \leq \exp\{\alpha 2c_\varphi^2 I\} = e^I = \exp\{d_{s(I)}\}. \quad (6.9)$$

We have thus verified the conditional version of Condition (A1) (under event $E$) and Conditions (A2) and (A3) for the model (6.7). This means that we can derive results on estimation, posterior contraction and uncertainty quantification for the density $f$ in terms of the model (6.7). These are the counterparts of claims (i)-(v) of Corollary 1 summarized by Theorem 5 below. To the best of our knowledge, local results on uncertainty quantification for the density are new. In the below theorem, we keep the same notation for all the quantities involved as in the general framework, with the understanding that these are specialized for the model (6.7) with the smoothness structure and the oracle rate $r^2(\theta)$.

**Theorem 5.** Let the constants $M_0, M_1, M_3, H_0, H_1, H_2, H_3, m_0, m_1, m_2, m_3, c_2, c_3, C_\nu$ be defined in Theorems 1-3 and (6.8). Then for any $M \geq 0,$

$$\sup_{\theta \in \ell_2} \mathbb{E}_\theta \hat{\sigma}(\|\theta - \vartheta\|^2 \geq M_0r^2(\theta) + M\sigma_\varphi^2|Y) \leq 2e^{-n\sigma_\varphi^2 + \log n} + H_0e^{-m_0M},$$

$$\sup_{\theta \in \ell_2} \mathbb{P}_\theta(||\theta - \vartheta||^2 \geq M_1r^2(\theta) + M\sigma_\varphi^2) \leq 2e^{-n\sigma_\varphi^2 + \log n} + H_1e^{-m_1M},$$

$$\sup_{\theta \in \ell_2} \mathbb{E}_\theta \hat{\sigma}(I : r^2(I, \theta) \geq c_3r^2(\theta) + M\sigma_\varphi^2|Y) \leq 2e^{-n\sigma_\varphi^2 + \log n} + C_\nu e^{-c_2M},$$

$$\sup_{\theta \in \ell_2} \mathbb{P}_\theta(r^2 \geq M_3r^2(\theta) + (M + 1)\sigma_\varphi^2) \leq 2e^{-n\sigma_\varphi^2 + \log n} + H_3e^{-m_3M},$$

$$\sup_{\theta \in \ell_2 \cap \Theta_{eb}} \mathbb{P}_\theta(\theta \notin B(\hat{\theta}, \hat{R}_M)) \leq 2e^{-n\sigma_\varphi^2 + \log n} + H_2e^{-m_2M}. $$
Let us outline the idea of the proof (which is omitted) of the first claim of the above theorem; the same reasoning applies to the remaining claims. The expectation of the empirical Bayes posterior probability $E_{\theta} \Pi = E_{\theta} \hat{\pi} \left( \| \theta - \theta \|^2 \geq M_0 r^2(\theta) + M_2^2 |Y\right)$ is bounded by the sum of two terms $E_{\theta} \Pi \leq P_{\theta}(E^c) + E_{\theta} \Pi |E$. The first term is evaluated by using (6.8) (obtaining the bound $2e^{-n\sigma_n^2 + \log n}$); the second term is evaluated exactly in the same way as in the proof Theorem 1, because Condition (A1) is fulfilled under the event $E$ according to (6.9). Counterparts of assertions (ii) and (iii) of Theorem 2 can also be formulated and proved in the same way. Notice that the results that rely on Condition (A4) are not claimed as we are unable to verify this condition at the moment.

As to the choice of $\sigma_n^2$ in the oracle rate, clearly, we would want it to be as small as possible. On the other hand, we want the claims of the theorem to be non-void, which is ensured only if $\sigma_n^2 n \geq C \log n$, or $\sigma_n^2 \geq C \log n$, for sufficiently large $C > 0$. In the sequel we take therefore $\sigma_n^2 = \frac{C \log n}{n}$. An extra log factor thus appeared which will also enter the minimax rates in the global results. We conjecture that one can get rid of that factor by using more accurate concentration inequalities when establishing Condition (A1).

As usually, the local results of Theorem 5 will imply global minimax adaptive results simultaneously over all scales $\{\Theta_\beta, \beta \in B\}$ covered by the oracle rate $r^2(\theta)$ (i.e., for which (6.1) holds). Hence, the same adaptive minimax results for the same scales as in Section 6.1 follow, up to a log factor as we have $\sigma_n^2 \asymp \frac{\log n}{n}$ in the model (6.7) instead of $n^{-1}$ in the model from Section 6.1. The reader is invited to formulate a number of local and adaptive minimax results for this case. We should mention that it seems possible to extend the results to other structures (e.g., sparsity) and scales (e.g., Besov scales).

6.6. Biclustering model

Suppose we observe a matrix $Y = (Y_{ij}) \in \mathbb{R}^{n_1 \times n_2}$:

$$Y_{ij} = \theta_{ij} + \sigma \xi_{ij}, \quad i = 1, \ldots, n_1, \quad j = 1, \ldots, n_2,$$

where $\theta = (\theta_{ij}) \in \mathbb{R}^{n_1 \times n_2}$ is an unknown high-dimensional parameter of interest with biclustering structure, $\sigma > 0$ is the known noise intensity, $\xi = (\xi_{ij}) \in \mathbb{R}^{n_1 \times n_2}$ is a random matrix with $E_{\theta} \xi_{ij} = 0$. These model and structure were studied at length in [10], here we demonstrate that the results obtained in [10] also follow from our general framework results.

The essence of biclustering structure is to reduce dimensionality of a large matrix of parameters by simultaneous grouping of the rows and columns. For example, if the rows of $\theta$ correspond to objects and the columns to features, a biclustering structure means that only a few features are relevant for identifying a few groups of similar objects. Biclustering structure means that the rows and columns of the matrix $\theta = (\theta_{ij}) \in \mathbb{R}^{n_1 \times n_2}$ are split into $k_1$ and $k_2$ clusters, respectively, and the values $\theta_{ij}$ are the same for $i, j$ from the same clusters. Let us give the mathematical formalization of this idea.

For $(k_1, k_2) \in [n_1] \times [n_2]$, consider a mapping $z = (z_1, z_2) : [n_1] \times [n_2] \mapsto [k_1] \times [k_2]$, where $z_1 : [n_1] \mapsto [k_1]$ and $z_2 : [n_2] \mapsto [k_2]$. Each mapping $z \in [k_1]^{[n_1]} \times [k_2]^{[n_2]}$ determines
Assume that \( I = I(z) \) of the rows and columns of any matrix \((M_{ij}) \in \mathbb{R}^{n_1 \times n_2}\) into \(k_1 \times k_2\) blocks:

\[
[n_1] \times [n_2] = z^{-1}([k_1] \times [k_2]) = z_1^{-1}([k_1]) \times z_2^{-1}([k_2]) = \cup_{(I_1^i, I_2^j) \in I}(I_1^i, I_2^j),
\]

where \( I_1^i = z_1^{-1}(i) \) and \( I_2^j = z_2^{-1}(j) \). The biclustering structure is nothing else but just this partition \( I = I(z) = (I_1^i, I_2^j) \), where \( I_1^i = I_1^i(z_1) = (I_1^i : i \in [k_1]) \) is the row partition and \( I_2^j = I_2^j(z_2) = (I_2^j : j \in [k_2]) \) is the column partition. So, the collection of all mappings \( \mathcal{Z} = \mathcal{Z}(n_1, n_2) = \{(z_1, z_2) \in [k_1]^{n_1} \times [k_2]^{n_2}, (k_1, k_2) \in [n_1] \times [n_2]\} \) yields the collection of all biclustering structures (which are all biclustered partitions of \([n_1] \times [n_2]\)):

\[
\mathcal{I} = \mathcal{I}(n_1, n_2) = \{(I(z), z \in [k_1]^{n_1} \times [k_2]^{n_2}, (k_1, k_2) \in [n_1] \times [n_2]\}.
\]

A biclustering structure \( I \in \mathcal{I} \) in terms of parameter \( \theta \) is expressed by imposing \( \theta \in \mathbb{L}_I \subseteq \mathbb{R}^{n_1 n_2} \), where the linear subspace \( \mathbb{L}_I \) is defined as

\[
\mathbb{L}_I = \{x \in \mathbb{R}^{n_1 n_2} : x_{ij} = x_{i'j'}, \forall (i,j), (i',j') \in (I_1, I_2), \forall (I_1, I_2) \in I\}, \quad (6.10)
\]

Assume that \( \mathcal{I} \) is “cleaned up” in the sense that \( \mathbb{L}_I \neq \mathbb{L}_{I'} \) for all \( I \neq I' \) (see Remark 1).

The structural slicing mapping \( s : \mathcal{I} \mapsto \mathcal{S} \) is defined as \( s(I) = (s_1(I), s_2(I)) \in [n_1] \times [n_2] \triangleq \mathcal{S} \), where \((s_1(I), s_2(I)) \) denotes the numbers of nonempty row and column blocks in the structure \( I \in \mathcal{I} \). Then \( d_{s(I)} = d_I = \text{dim}(\mathbb{L}_I) = s_1(I)s_2(I) \).

Let us propose a reasonable majorant \( \rho(s) \) for the layer complexity \( d_s + \log |\mathcal{I}_s| = s_1 s_2 + \log |\mathcal{I}_s| \). Clearly, \( |\mathcal{I}_s| \leq N(n_1, s_1)N(n_2, s_2) \), where \( N(n,k) \) is the number of ways to put \( n \) different objects into \( k \) different boxes so that each box contains at least one object. Notice that \( S(n,k) = N(n,k)/k! = \frac{1}{n!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \) is a Stirling number of the second kind. To have a simple closed form expression for a majorant of the complexity, instead of \( N(n_1, s_1)N(n_2, s_2) \) we can use its upper bound \( s_1^{n_1} s_2^{n_2} \) (all the partitions of \([n_1] \times [n_2]\) into \(s_1 \times s_2\) blocks, some of which are possibly empty). However, the bound \( |\mathcal{I}_s| \leq s_1^{n_1} s_2^{n_2} \) becomes too crude for some \( s \in \mathcal{S} \). In particular, this bound is too crude for the cases (i) \((s_1, s_2) \in \mathcal{S}_1 = \{(s_1, s_2) \in [n_1] \times [n_2] : s_1 < n_1, s_2 = n_2\} \), (ii) \((s_1, s_2) \in \mathcal{S}_2 = \{(s_1, s_2) \in [n_1] \times [n_2] : s_1 = n_1, s_2 < n_2\} \), and (iii) \((s_1, s_2) \in \mathcal{S}_3 = \{(n_1, n_2)\} \).

Indeed, let \( \text{id}_m : [m] \mapsto [m] \) with \( \text{id}_m(s) = s, s \in [m] \), the identity mapping of \([m]\).

Then it is easy to see that \( \mathbb{L}_{I(z_1, z_2)} = \mathbb{L}_{I(z_1, \text{id}_{z_2})} \) for all \( z_2 \in [n_2]^{n_2} \) and all \( z_1 \in [n_1]^{n_1}, s_1 \in [n_1] \).

Similarly, \( \mathbb{L}_{I(z_1, z_2)} = \mathbb{L}_{I(\text{id}_{z_1}, z_2)} \) for all \( z_1 \in [n_1]^{n_1}, z_2 \in [n_2]^{n_2} \). Hence, \( |\mathcal{I}_s| \leq |\mathcal{S}_1| \leq s_1^{n_1} \) for \((s_1, s_2) \in \mathcal{S}_1, |\mathcal{I}_s| \leq s_2^{n_2} \) for \((s_1, s_2) \in \mathcal{S}_2, |\mathcal{I}_s| \leq 1 \) for \((s_1, s_2) \in \mathcal{S}_3 \). Thus, we improve the bound \( d_s + \log |\mathcal{I}_s| \leq s_1 s_2 + \log(s_1^{n_1} s_2^{n_2}) \) by proposing the following majorant \( \rho(s) \) for the complexity of the layer \( \mathcal{I}_s \):

\[
\rho(s) \triangleq \begin{cases} 
  s_1 s_2 + n_1 \log s_1 + n_2 \log s_2, & s_1 < n_1, s_2 < n_2, \\
  s_1 n_2 + n_1 \log s_1, & s_1 < n_1, s_2 = n_2, \\
  n_1 s_2 + n_2 \log s_2, & s_1 = n_1, s_2 < n_2, \\
  n_1 n_2, & s_1 = n_1, s_2 = n_2.
\end{cases} \quad (6.11)
\]
This is an example of the so called elbow effect mentioned in Remark 11.

In case $\xi_i \sim \text{N}(0,1)$, Conditions (A1) and (A4) hold with $d_{e(I)} = \dim(L_I)$ in view of Remarks 5 and 18. Let us show that Condition (A1) is also fulfilled in case $Y_{ij} \sim \text{Bernoulli}(\theta_{ij})$, which is typically used for modeling indirect network graphs. Indeed, we have $\xi_{ij} \in \{1 - \theta_{ij}, -\theta_{ij}\} \subseteq [-1,1]$ and $\mathbb{E}_\theta \xi_{ij} = 0$, $(i, j) \in [n_1] \times [n_2]$. Note that in this case the error distribution depends on $\theta$. We represent the projection $P_I = BB^T$, where $B$ is the $(n_1 n_2 \times k_1 k_2)$-matrix whose columns $(b_{11}, b_{12}, (I_1, I_2) \in I)$ form an orthonormal basis of $L_I$. Then $\|P_I \xi\|^2 = \|B^T \xi\|^2 = \|\eta\|^2$, with $\eta = (\eta_{I_1 I_2}, (I_1, I_2) \in I)$, $\eta_{I_1 I_2} = b_{I_1 I_2}^T \xi$. We choose the following orthogonal basis of $L_I$: $b_{I_1 I_2} = ((|I_1||I_2|)^{-1/2}1\{(i, j) \in (I_1, I_2)\}, (i, j) \in [n_1] \times [n_2]), (I_1, I_2) \in I$, so that $\eta_{I_1 I_2} = \frac{1}{\sqrt{|I_1||I_2|}}\sum_{(i, j) \in (I_1, I_2)} \xi_{ij}$. Hoeffding’s inequality implies that for any $t \geq 0$

$$\mathbb{P}_\theta(\|\eta_{I_1 I_2}\| \geq t) \leq 2e^{-t^2/2}, \text{ for all } (I_1, I_2) \in I.$$ 

Using this, we obtain for any $0 < b < 1/2$

$$\mathbb{E}_\theta e^{b \eta_{I_1 I_2}^2} = 1 + \int_1^\infty \mathbb{P}_\theta(e^{b \eta_{I_1 I_2}^2} \geq t) dt \leq 1 + 2\int_1^\infty e^{-(\log t)/(2b)} dt = 1 + \frac{4b}{1 - 2b}.$$ 

By taking $b_0 = \frac{e - 1}{2(1 + e)}$, we derive

$$\mathbb{E}_\theta \exp\{b_0 \|P_I \xi\|^2\} = \mathbb{E}_\theta \exp\{b_0 \|\eta\|^2\} \leq \left(1 + \frac{4b_0}{1 - 2b_0}\right)|I_1||I_2| = e^{|I_1||I_2|} = e^{d_{e(I)}},$$

which is Condition (A1) with the constant $\alpha = b_0 = \frac{e - 1}{2(1 + e)}$. Of course, the above argument applies (with minor adjustments) to any independent zero mean bounded errors $\xi'_{ij} \in [-c, c]$ for some $c > 0$.

Let us verify Condition (A2): for any $\nu \geq 1$,

$$\sum_{I \in \mathcal{I}} e^{-\nu p(s(I))} \leq \sum_{(s_1, s_2) \in [n_1] \times [n_2]} e^{-\nu s_1 s_2} = (e^\nu + e^{-\nu} - 2)^{-1} = C_\nu.$$ 

Thus, the properties (i)-(iv) of Corollary 1 follow for the biclustering model with the $\xi_i'$ that are independent and either normal or binomial, in fact, for any $\xi$ satisfying Condition (A1).

One can also check Condition (A3), so that the coverage property (v) of Corollary 1 holds under EBR as well. However, the peculiarity of the biclustering structure is that the size and coverage claims (vi)-(vii) for the confidence ball $B(\hat{\theta}, \hat{R}_M)$ are stronger and more useful in this case than the corresponding claims (iv)-(v) for the confidence ball $B(\bar{\theta}, \bar{R}_M)$.

Indeed, the coverage property (v) holds uniformly only under the EBR, whereas the coverage property (vii) is uniform over the entire space $\Theta = \mathbb{R}^{n_1 \times n_2}$. So, basically the deceptiveness issue is not present in the coverage property (vii) for the confidence ball $B(\bar{\theta}, \bar{R}_M)$, it appears only marginally in the size relation (vi) of Corollary 1. Indeed, the
size \( \tilde{R}_M \) of the ball is of the oracle rate order uniformly in \( \theta \in \Theta \setminus \tilde{\Theta} = \mathbb{R}^{n_1 \times n_2} \setminus \tilde{\Theta} \), where \( \tilde{\Theta} \) is defined by (3.14). By the definition of \( \tilde{\Theta} \), \( r^2(\theta) \geq c \sigma^2 \sqrt{n_1 n_2} \) for \( \theta \in \Theta \setminus \tilde{\Theta} \). For the biclustering model, we can take \( c = \log 2 \) and \( \tilde{\Theta} \) can be written as \( \tilde{\Theta} = \{ \theta \in \mathbb{R}^{n_1 \times n_2} : \min \{ s_{o1}(\theta), s_{o2}(\theta) \} = 1 \} \) with \( (s_{o1}(\theta), s_{o2}(\theta)) = s(I_o(\theta)) \), where the oracle \( I_o(\theta) \) is defined by (3.1). Hence, for the biclustering model, \( \Theta \) is indeed a “thin” subset of \( \mathbb{R}^{n_1 \times n_2} \) consisting of highly structured parameters, whose oracle number of either row or block columns is 1.

As we have already discussed at the end of Section 3.4, this means that, modulo highly structured parameters, there is no deceptiveness phenomenon in the biclustering model.

Consider an example of scale \( \{ \Theta_\beta, \beta \in \mathcal{B} \} \) covered by the local rate \( r^2(\theta) \).

### 6.6.1. Minimax results for the biclustering model.

In [26], classes \( \Theta_{k_1 k_2}^{\text{asym}} \) are introduced (and classes \( \Theta_{k_1 k_2}(M) \) from [27]). In our notation, \( \Theta_{s_1 s_2}^{\text{asym}} = \bigcup_{I \in \mathcal{I}} \Theta_I \), where \( s = (s_1, s_2) \in [n_1] \times [n_2] \triangleq \mathcal{S} \), \( \Theta_I \triangleq \mathcal{L}_I \cap \mathcal{L}_I' \) and \( \mathcal{L}_I \) is defined by (6.10). So, the family of classes \( \Theta_{s_1 s_2}^{\text{asym}} \) is nothing else but the scale \( \{ \Theta_s, s \in \mathcal{S} \} \).

The minimax rate \( r^2(\Theta_s) \triangleq s_1 s_2 + n_1 \log s_1 + n_2 \log s_2 \) over \( \Theta_s \) is derived in [26], under the assumption \( \log s_1 \ll \log s_2 \). It is easy to see that the oracle rate \( r^2(\theta) \) covers the scale \( \{ \Theta_s, s \in \mathcal{S} \} \) in the sense of (6.1). Indeed, if \( \theta \in \Theta_s \), then \( \theta \in \mathcal{L}_{I'} \) for some \( I' \in \mathcal{I}_s \), so that \( P_{I'} \theta = \theta \) and hence

\[
r^2(\theta) \leq r^2(I', \theta) = \rho(s(I')) = \rho(s) \leq r^2(\Theta_s), \quad \theta \in \Theta_s.
\]

(6.12)

Corollary 2 follows for this case with the minimax rate \( r^2(\Theta_s) \) defined above.

**Remark 24.** From (6.12), we have that \( r^2(\theta) \leq s_1 s_2 + n_1 \log s_1 + n_2 \log s_2 \) for each \( \theta \in \Theta_s \).

Next, for any \( I \in \mathcal{I}_s \) with \( s = (s_1, s_2) \) and any \( \mathcal{L}_I \), there exist \( I' = I'(I) \) and \( \mathcal{L}_{I'} \) such that \( \mathcal{L}_I \subseteq \mathcal{L}_{I'} \) where \( I' \in \mathcal{I}_s' \) with \( s' = (s_1, n_2) \). Then for any \( \theta \in \Theta_s \), \( \theta \in \mathcal{L}_I \subseteq \mathcal{L}_{I'}(I) \) for some \( I' \in \mathcal{I}_s' \) with \( s' = (s_1, n_2) \), implying \( P_{I'} \theta = \theta \). In view of (6.11), we obtain that \( r^2(\theta) \leq r^2(I', \theta) = \rho(s(I')) = \rho(s') = \rho(2 s' + n_1 \log s_1) \) for all \( \theta \in \Theta_s \). Similarly, we derive that \( r^2(\theta) \leq n_1 s_2 + n_2 \log s_2 \) and \( r^2(\theta) \leq n_1 n_2 \) for all \( \theta \in \Theta_s \). Thus, instead of (6.12), we established the following stronger bound for any \( \theta \in \Theta_s \)

\[
r^2(\theta) \leq \min \{ r^2(\Theta_s), s_1 s_2 + n_1 \log s_1, n_1 s_2 + n_2 \log s_2, n_1 n_2 \} \triangleq \tilde{r}^2(\Theta_s).
\]

Notice that for some \( s \in \mathcal{S} \), the quantity \( \tilde{r}^2(\Theta_s) \) can be less than the minimax rate \( r^2(\Theta_s) = s_1 s_2 + n_1 \log s_1 + n_2 \log s_2 \). Recall however that the minimax rate \( r^2(\Theta_s) \) is claimed in [26] only under the assumption \( \log s_1 \ll \log s_2 \), and, in this case, indeed \( r^2(\Theta_s) \succ \tilde{r}^2(\Theta_s) \). In general, the minimax rate over \( \Theta_s \) for arbitrary \( s \in \mathcal{S} \) cannot be bigger than \( \tilde{r}^2(\Theta_s) \), we conjecture that it is \( \tilde{r}^2(\Theta_s) \) for all \( s \in \mathcal{S} \).

**Remark 25.** In view of Remark 18, Condition (A4) is always fulfilled whenever \( \xi_i \overset{\text{ind}}{\sim} N(0,1) \). However, for the biclustering model (and the stochastic block model, described below), a more appropriate distribution for the observations is binomial, i.e., \( Y_{ij} \overset{\text{ind}}{\sim} \)
Bernoulli($\theta_{ij}$). This case is important in relation to network modeling. Unfortunately, we were unable to establish Condition (A4) for the binomial case (hence, unable to establish claims (vi)–(vii) of Corollary 1). Verification of Condition (A4) for the binomial observations essentially boils down to the problem of estimating the functional $F(\theta) = \sum_{i,j} \theta_{ij}^2$ with the rate $\sqrt{N} = \sqrt{n_1n_2}$. It is not known to us whether it is possible to construct such an estimator (possibly exploiting the biclustering structure of $\theta$), which is an interesting and challenging problem on its own.

6.6.2. Stochastic block model

Here we briefly discuss a particular case of biclustering model, the stochastic block model (SBM) which is used in the literature on networks to model undirected network graphs. Oracle estimation and posterior contraction rate results for stochastic block model were recently derived in [28, 36]. Precisely, to get the SBM from the biclustering model, we assume additionally $s_1 = s_2 = s$, $n_1 = n_2 = n$, $z_1 = z_2 = z$. For a mapping $z \in [s]^{[n]}$, the pertinent row partition in the SBM is $I = I(z) = (z^{-1}(i), i \in [s])$, which is the same as the column partition.

In the binomial case $Y_{ij} \overset{\text{ind}}{\sim} \text{Bernoulli}(\theta_{ij})$, the observations $Y_{ij}$ can be associated with network data. In this case $Y_{ij}$ stands for the presence or absence of an edge between vertices $i$ and $j$ in the network interpretation. To model undirected network graphs, some conditions (called network conditions) are then additionally assumed: the “no self-loop” condition $Y_{ii} = \theta_{ii} = 0$ and symmetry condition $Y_{ij} = Y_{ji}$ and $\theta_{ij} = \theta_{ji}$. Denote by $\Theta_{\text{net}}$ the parameters $\theta \in \mathbb{R}^{n_1 \times n_2}$ satisfying these additional network conditions.

All the quantities, conditions and claims specialize to the SBM by setting $s_1 = s_2 = s$, $n_1 = n_2 = n$, $z_1 = z_2 = z$ in all the above formulas for the biclustering model. The linear subspaces $L_I$ defined by (6.10) will get adjusted since $z_1 = z_2$, the family $\mathcal{I}_s$ can be associated with the collection of all possible partitions of $[n]$ into $s$ blocks, parametrized by mappings $z \in [s]^{[n]}$, $|\mathcal{I}_s| \leq s^n$, $s \in \mathcal{S} \triangleq [n]$. The structural slicing mapping $s(I)$ is the number of blocks in the partition $I$. Notice that under additional network conditions $cs^2(I) \leq \dim(L_I) \leq s^2(I)$, so that we can use $s^2(I)$ (instead of the true $d_I = \dim(L_I)$) in the complexity part of the local rate as it is still of the same order, although some constants can be improved because of this extra network structure. We have $d_{s(I)} = d_I = \dim(L_I) \leq s^2(I)$, $\log |\mathcal{I}_s| \leq n \log s$, and we take $\rho(s(I)) = s^2(I) + n \log s(I)$. Conditions (A1)–(A4) are fulfilled in the same way as for the biclustering model, leading to Corollary 1. As to the binomial case, see Remark 25.

Consider a couple of examples of scales $\{\Theta_{\beta}, \beta \in \mathcal{B}\}$ covered by the local rate $r^2(\theta)$.

6.6.3. Minimax results for the stochastic block model

In [26] (cf. [36]), classes $\Theta_k$ were introduced. In our notation, $\Theta_s = \cup_{I \in \mathcal{I}_s} \Theta_I$, where $s = k$, $\Theta_I = L_I \cap \Theta_{\text{net}} \cap [0, 1]^{n^2}$, $I \in \mathcal{I}_s$, $s \in \mathcal{S}$. So, we have the scale $\{\Theta_s, s \in \mathcal{S}\}$ and the adaptive
minimax results over this scale follow from the local results given by Corollary 1. Indeed, as is shown in [26], the minimax rate over \( \Theta_s \) in the SBM is \( r^2(\Theta_s) = \inf_{\beta} \sup_{\theta \in \Theta_s} \mathbb{E}_\theta \| \hat{\theta} - \theta \|^2 \propto s^2 + n \log s = \rho(s) \). On the other hand, for each \( \theta \in \Theta_s \) there exists \( I \in I_s \) such that \( \theta \in \mathbb{L}_I \). Hence, \( P_I \theta = \theta \) and \( r^2(\theta) \leq r^2(I, \theta) = \rho(s) \times r^2(\Theta_s) \). This implies Corollary 2 for this scale.

**Remark 26.** As to the deceptiveness phenomenon in the SBM, for the confidence ball \( B(\theta, \hat{R}_M) \) we again have the coverage property uniformly over the whole scale \( \{ \Theta_s, s \in \mathcal{S} \} \), whereas the size property with the optimal radial rate holds over all classes \( \{ \Theta_s, s = 2, \ldots, n \} \), but one: \( \Theta_1 \). Indeed, the class \( \Theta_1 \) consists of highly structured parameters \( \theta \in \mathbb{R}^{n^2} \), whose coordinates are all equal. The case \( \theta \in \Theta_1 \) reduces to just one-dimensional signal+noise model with \( N = n^2 \) observations. Since the effective radial rate \( g_M(\theta, N) \) for the confidence ball \( B(\theta, \hat{R}_M) \) is always at least of the order \( n \sigma^2 \gg \sigma^2 = r^2(\Theta_1) \), we could not attain the optimal rate \( r^2(\Theta_1) \) in the size relation only for the highly structured parameters \( \theta \in \Theta_1 \).

### 6.6.4. Graphon classes

It is also possible to derive the global minimax results for the function class of graphons as consequence of our local results. We use the same notation as in [26]. Consider a random graph with adjacency matrix \( \{ Y_{ij} \} \in \{0, 1\}^{n \times n} \). Assume again the network conditions: \( Y_{ii} = \theta_{ii} = 0, Y_{ij} = Y_{ji}, \theta_{ij} = \theta_{ji} \). For any \( i > j, Y_{ij} \) is sampled as follows:

\[
(\xi_1, \ldots, \xi_n) \sim P_\xi, \quad Y_{ij}|(\xi_i, \xi_j) \overset{\text{ind}}{\sim} \text{Bernoulli}(\theta_{ij}), \quad \theta_{ij} = f(\xi_i, \xi_j).
\]

The function \( f \) on \([0, 1]^2\), which is assumed to be symmetric, is called *graphon*. Introduce the derivative operator \( \nabla_{jk} f(x, y) = \frac{\partial^{j+k}}{\partial x^j \partial y^k} f(x, y) \), with the convention \( \nabla_{00} f(x, y) = f(x, y) \).

For \( \beta > 0 \), the Hölder norm is defined by

\[
\|f\|_{H^\beta} = \max_{j+k \leq \beta} \sup_{x,y} |\nabla_{jk} f(x, y)| + \max_{j+k = \beta} \sup_{(x, y) \neq (x', y')} \frac{|\nabla_{jk} f(x, y) - \nabla_{jk} f(x', y')|}{\| (x-x', y-y') \|^\beta - (|j|-|k|)}.
\]

For \( \beta, Q > 0 \), the Hölder graphon class is

\[
\mathcal{F}_\beta = \mathcal{F}_\beta(Q) = \{ f : \|f\|_{H^\beta} \leq Q, f(x, y) = f(y, x), 0 \leq f(x, y) \leq 1 \text{ for } x \geq y \}.
\]

Recall that \( \log_\theta = f(\xi_i, \xi_j) \). Slightly abusing notation, we will write \( \theta \in \mathcal{F}_\beta(Q) \) if \( f \in \mathcal{F}_\beta(Q) \).

The next proposition is Lemma 2.1 from [26], which we give here (in our notation) for completeness. The proof can be found in [26].

**Proposition 2.** For any \( \theta \in \mathcal{F}_\beta(Q), s_0 \in \mathcal{S} \), there exists a partition \( I_0 = I_0(\theta, s_0) \in I_{s_0} \) such that, for some universal constant \( \tilde{C}_1 > 0 \),

\[
\| \theta - P_{I_0} \theta \|^2 \leq \tilde{C}_1 Q^2 n^2 s_0^{-2 \min(\beta,1)}.
\]
Corollary 2 follows for the scale \( \{F_{\beta}, \beta > 0\} \) with the minimax rate \( r^2(F_{\beta}) \asymp n^{2/(\beta+1)} + n \log n \). The second claim of Corollary 2 recovers the same minimax estimation rate as in [26] and [36].

6.7. Linear regression with various structures

Let us consider a general setting for linear regression:

\[
Y = X\beta + \sigma \xi, 
\]

(6.13)

where \( X = \text{diag}(X^1, \ldots, X^m) \in \mathbb{R}^{m \times np} \) is a block diagonal matrix, whose blocks \( X^1, \ldots, X^m \in \mathbb{R}^{n \times p} \) are design matrices, \( \sigma > 0 \) is the known noise intensity, \( \beta = (\beta^1, \ldots, \beta^m) \in \mathbb{R}^{np} \) is a concatenation of \( m \) unknown \( p \)-dimensional vectors \( \beta^1, \ldots, \beta^m \in \mathbb{R}^p \), \( Y = (Y^1, \ldots, Y^m) \in \mathbb{R}^{mn} \) is a concatenation of observed vectors \( Y^1, \ldots, Y^m \in \mathbb{R}^n \), \( \xi = (\xi_i, i \in [mn]), \xi_i \overset{\text{ind}}{\sim} N(0, 1) \).

Wherever appropriate, by \( M_I \) we denote the submatrix of \( M \) with columns \( (M_i, i \in I) \), \( x_I \) is the \( |I| \)-dimensional subvector of \( x \in \mathbb{R}^p \) with coordinates \( i \in I, \|\beta\|_0 \) denotes the number of non-zero elements of \( \beta \), i.e., the cardinality of the support \( \text{I}^*(\beta) = \text{supp}(\beta) = \{i : \beta_i \neq 0\} \) of \( \beta \). Under \( \xi_i \overset{\text{ind}}{\sim} N(0, 1) \), Conditions (A1) and (A4) will hold with \( d_{s(I)} = \text{dim}(\mathbb{L}_I) \) for all cases of this section in view of Remarks 5 and 18.

We should emphasize that in this section we denote by \( \beta \) the vector of unknown parameters in (6.13), notation commonly used in the literature. This is not to be confused with the structural parameter \( \beta \) for indexing the scales of classes \( \{\Theta_{\beta}, \beta \in \mathcal{B}\} \) which we use in other sections, in this section we will instead use the notation \( \{\Theta_\gamma, \gamma \in \Gamma\} \) for scales.

Many particular linear models can be put in (6.13) by choosing appropriately \( m, p \) and \( X \). The case \( m = 1, p = n, X = I \) is already considered in Section 6.1 for the smoothness structure and in Section 6.3 for the sparsity structure. In the following subsections we consider several other specific models and structures in detail.

Remark 27. The true model does not have to be exactly of the form (6.13), it can be \( Y = \theta + \sigma \xi \), where \( \theta \neq X\beta \). In that case, (6.13) is an approximating model of the true model and all the local results hold with \( \theta \) substituted everywhere instead of \( X\beta \). The global minimax results will have to be modified by including the approximation term \( \sup_{\theta \in \Theta_\gamma} E_{\theta} \|\theta - P_L \theta\|^2 \) in the minimax rate \( r^2(\Theta_\gamma) \).
6.7.1. Linear regression with sparsity structure, inference on $\beta$

Consider the classical linear regression model, i.e., for $m = 1$ in (6.13). In the high-dimensional setting, typically $p \gg n$. So, to be able to make sensible inference, one needs to exploit a structure on $\beta$. The vector $\beta = (\beta_1, \ldots, \beta_p)^T \in \mathbb{R}^p$ with a structure $I \subseteq [p]$ is assumed to be sparse in the sense that $\beta_i = 0$ for $i \in I^c$, that is, the predictors $(X_i : i \in I^c)$ of design matrix $X = (X_1, \ldots, X_p)$ are irrelevant from the perspective of structure $I$. Denote $r = r(X) = \text{rank}(X)$.

A local approach for this model, delivering also the adaptive minimax results for many sparsity structures simultaneously, is considered in [28, 6] for the posterior contraction rates. In [6] also the uncertainty quantification is treated. Note that “signal+noise” model is a special case of linear regression with $X = I$ and $p = n$, and it has been considered in several previous sections.

In this model, the sparsity structure is expressed by the linear spaces

$$L_I = \{X_I x_I \in \mathbb{R}^n : x_I \in \mathbb{R}^{\lvert I \rvert}\} = \{X x \in \mathbb{R}^n : x \in \mathbb{R}^p, x_i = 0 \text{ for } i \notin I\}, \quad \text{(6.14)}$$

$I \in \mathcal{I}$, where the family of structures is $\mathcal{I} = \mathcal{I}_1 \cup \{I_r\}$ with $\mathcal{I}_1 = \{I \subseteq [p] : 2\lvert I \rvert \log(ep/\lvert I \rvert) \leq r\}$ (recall that $r = \text{rank}(X)$) and $I_r = (i_1, \ldots, i_r) \subseteq [p]$ such that $(X_i)_{i \in I_r}$ are $r$ linearly independent columns of $X$. Then $\lvert I \rvert \leq 2^p$, the structural slicing mapping is taken to be $s(I) = \lvert I \rvert \in \mathcal{S} \triangleq [r]_0$. Further, we have $\theta = X\beta$, $d_I = \dim(L_I) \leq \min\{\lvert I \rvert, r\} \leq \min\{\lvert I \rvert, n, p\}$ for $I \in \mathcal{I}_1$ and $d_{I_r} = \dim(L_{I_r}) = r$. Clearly, $\lvert \mathcal{I}_{s(I)} \rvert \leq \log \binom{p}{\lvert I \rvert} \leq \lvert I \rvert \log \binom{p}{\lvert I \rvert}$ for $I \in \mathcal{I}_1$ and $\log |\mathcal{I}_{s(I_r)}| = 0$. Since $d_{s(I)} + \log |\mathcal{I}_{s(I)}| \leq \lvert I \rvert + \lvert I \rvert \log \binom{p}{\lvert I \rvert} \leq 2\lvert I \rvert \log \binom{p}{\lvert I \rvert}$ for $I \in \mathcal{I}_1$ and $d_{s(I_r)} + \log |\mathcal{I}_{s(I_r)}| = d_{I_r} = r$, we take the majorant

$$\rho(s(I)) = 2\lvert I \rvert \log \binom{p}{\lvert I \rvert} 1\{I \in \mathcal{I}_1\} + r 1\{I = I_r\}, \quad I \in \mathcal{I}. \quad \text{(6.15)}$$

Notice that we could use a smaller majorant $\rho'(s(I)) = d_I + \log \binom{p}{\lvert I \rvert}$ for $I \in \mathcal{I}_1$ (the best choice), but this majorant is not practical to use.

**Remark 28.** In the majorant $\rho(s(I))$ defined above, we see the elbow effect mentioned in Remark 11, this elbow effect will enter the rate as well. Let us explain how this elbow effect has emerged in this model.

Notice that we could consider the more natural full family of structures $\mathcal{I} = \{J : \lvert J \rvert \leq [p]\}$, so that $|I| = 2^p$, with the same structural slicing mapping $s(I) = \lvert I \rvert \in \mathcal{S} \triangleq [p]_0$, but defined on the family $\mathcal{I}$. As before, $d_I = \dim(L_I) \leq \min\{\lvert I \rvert, r\}$ and $|\mathcal{I}_{s(I)}| = \binom{p}{\lvert I \rvert}$. Since $d_I + \log |\mathcal{I}_{s(I)}| \leq \lvert I \rvert + \lvert I \rvert \log \binom{p}{\lvert I \rvert} \leq 2\lvert I \rvert \log \binom{p}{\lvert I \rvert}$, the majorant would be $\rho(s(I)) = 2\lvert I \rvert \log \binom{p}{\lvert I \rvert}$, $I \in \mathcal{I}$. The idea of the family $\mathcal{I}$ is that, even though $\mathcal{I} \subseteq \mathcal{I}$, the family $\mathcal{I}$ still covers $\mathcal{I}$ in the sense of Remark 11. Indeed, $r^2(I, \beta) = \lVert X\beta - P_I X\beta \rVert^2 + \sigma^2 \rho(s(I)) = \lVert X\beta - P_I X\beta \rVert^2 + \sigma^2 \rho(s(I)) = r^2(I, \beta)$ for $I \in \mathcal{I}_1$, and $r^2(I_r, \beta) = \sigma^2 r \leq \sigma^2 2\lvert I \rvert \log \binom{p}{\lvert I \rvert} \leq r^2(I, \beta)$ for all $I \in \mathcal{I} \setminus \mathcal{I}_1$, as $P_I X\beta = X\beta$ for any $\beta \in \mathbb{R}^p$.

Here we considered an important case when a seemingly right (full) family $\mathcal{I}$ of structures can be reduced to a subfamily $\mathcal{I} \subset \mathcal{I}$ that has a reduced complexity but still covers the
original family $\mathcal{I}$ in the sense of Remark 11, thus improving the resulting oracle rate. This is a typical situation exhibiting the “elbow effect” in the complexity term of the rate; below there are a couple of such example (Sections 6.7.3, 6.7.4 and 6.10).

**Remark 29.** We could further reduce the family of structures to $\mathcal{I}' = \{I_r\} \cup \mathcal{I}'_1$, with $\mathcal{I}'_1 = \{I \subseteq \mathcal{I}_1 : \text{the columns } (X_i, i \in I) \text{ are linearly independent}\}$ (with the same structural slicing mapping $s(I) = |I|$), so that $\mathcal{I}' \subseteq \mathcal{I}$. In this case, we have $d_I = \dim(\mathbb{L}_I) = |I|$, $|I| \leq r \leq \min\{n, p\}$ for each $I \in \mathcal{I}'$, the layer is $\mathcal{I}_s(I) = \{J \subseteq \mathcal{I}_1 : \dim(\mathbb{L}_J) = \dim(\mathbb{L}_I)\}$ for $I \in \mathcal{I}_1'$ and $\mathcal{I}_s(I_r) = \{I_r\}$. Then we can take the majorant $\tilde{\rho}(s(I)) = (|I| + \log |\mathcal{I}_s(I)|)1\{I \in \mathcal{I}_1\} + r1\{I = I_r\}$. When implementing the Bayesian or penalization procedure, the majorant $\rho(s(I)) = 2|I|\log(ep/|I|)1\{I \in \mathcal{I}_1\} + r1\{I = I_r\}$ is more practical to use also for the family $\mathcal{I}'$. But then $\mathcal{I}$ also covers $\mathcal{I}'$, thus boiling down to the same resulting oracle rate. Therefore, as soon as we use the same majorant, it does not matter which family of structures, $\mathcal{I}$ or $\mathcal{I}'$, we take. A slightly bigger constant in Condition (A2) will be for the family $\mathcal{I}$ as there are more terms in the sum. We will use the family $\mathcal{I}$.

Condition (A2) is fulfilled, since, according to Remark 7, for any $\nu > 1$

$$\sum_{I \in \mathcal{I}} e^{-\nu \rho(s(I))} \leq \sum_{s \in S} e^{-(\nu - 1)\rho(s(I))} \leq \sum_{s=0}^{p} e^{-(\nu - 1)s} \leq \frac{1}{1 - e^{1-\nu}} = C_\nu.$$

As to Condition (A3), for any $I_0, I_1 \in \mathcal{I}$, take $I' = I_r$ if either $I_0 = I_r$ or $I_1 = I_r$ or $2|I_0 \cup I_1| \log(ep/|I_0 \cup I_1|) > r$; otherwise take $I' = I_0 \cup I_1$. Since $(\mathbb{L}_{I_0} \cup \mathbb{L}_{I_1}) \subseteq L_{I'} = \mathbb{L}_{I_0} + \mathbb{L}_{I_1}$ and $\rho(s(I')) \leq \rho(s(I_0)) + \rho(s(I_1))$, Condition (A3) is also fulfilled.

As a consequence of our general results, we obtain Corollary 1 with the local (prediction) rate $r^2(\beta) = \min_{I \in \mathcal{I}} r^2(I, \beta) = \min_{I \in \mathcal{I}} \{|[X\beta - P_I X\beta]|^2 + \sigma^2 \rho(s(I))\}$, where the majorant $\rho(s(I))$ is defined by (6.15). In particular,

$$r^2(\beta) \leq r^2(I^*(\beta), \beta) \land r^2(I_r, \beta) = \sigma^2 [\rho(s(I^*(\beta)) \land \rho(s(I_r))] \lesssim \sigma^2 ([|I^*(\beta)| \log(ep/|\mathbb{L}_{I^*(\beta)}|) \land r].$$

Notice that the claims (i)-(vii) of Corollary 1 deliver finer and stronger versions of the corresponding results from [6]; besides, we can drop the normality and independence assumptions and impose only Condition (A1) instead.

Next, by virtue of Corollary 2 the local results imply global minimax adaptive results at once over all scales $\{\Theta_\gamma, \gamma \in \Gamma\}$ covered by the oracle rate $r^2(\beta)$ (i.e., for which (6.1) holds). Below we present a couple of scales $\{\Theta_\gamma, \gamma \in \Gamma\}$ covered by the oracle rate $r^2(\beta)$.

**Nearly black vectors.** For $s \in [p]$, introduce

$$\ell_0[s] = \{\beta \in \mathbb{R}^p : \|\beta\|_0 = |I^*(\beta)| \leq s\}, \quad \text{where } I^*(\beta) = \{i \in [p] : \beta_i \neq 0\},$$

the set of vectors with at most $s$ nonzero elements. Under certain conditions on the parameters $s, p, n$ and the design matrix $X$ (at least, $s \log(ep/s) \lesssim r = \text{rank}(X)$ has
to hold), the minimax prediction estimation rate over \( \ell_0[s] \) is known to be \( r^2(\ell_0[s]) = \inf_{\beta} \sup_{\beta \in \ell_0[s]} \mathbb{E}_\beta \| X\beta - X\beta \|^2 \approx \sigma^2 s \log(ep/s) \); see \([17, 48]\). The adaptive minimax results for \( \ell_0 \)-balls were considered by \([21, 28, 42]\) for posterior contraction rates and by \([45]\) for uncertainty quantification problem.

If \( \beta \in \ell_0[s] \), then \( X\beta \in \mathbb{L}_{I^*(\beta)} \) for \( I^*(\beta) \in \mathbb{I}_s \) such that \( \mathbb{P}_{I^*(\beta)} X\beta = X\beta \) and \( |I^*(\beta)| \leq s \), and hence \( r^2(I^*(\beta), \beta) \leq \sigma^2 |I^*(\beta)| \log(ep/|I^*(\beta)|) \leq \sigma^2 s \log(ep/s) \). By the definition (3.1) of the oracle rate \( r^2(\beta) \), we have that \( r^2(\beta) \leq r^2(I^*(\beta), \beta) \wedge r^2(I_r, \beta) \). Then we obtain trivially that

\[
\sup_{\beta \in \ell_0[s]} r^2(\beta) \leq \sup_{\beta \in \ell_0[s]} \left[ r^2(I^*(\beta), \beta) \wedge r^2(I_r, \beta) \right] \leq \sigma^2 \left[ r \wedge (s \log(ep/s)) \right] \lesssim \sigma^2 \left( \log(ep/s) \right).
\]

We thus established the relation (6.1) for the scale \( \ell_0[s] \), and Corollary 2 follows with the minimax rate \( r^2(\ell_0[s]) \) defined above.

**Weak \( \ell_q \)-balls.** For \( q \in (0, 1] \), the weak \( \ell_q \)-ball is defined by

\[
\ell_q[R] = \{ \beta \in \mathbb{R}^p : \beta_{[i]}^2 \leq R^2 i^{-2/q}, i \in [p] \}, \quad R^2 \geq \sigma^2 \log p,
\]

where \( \beta_{[1]}^2 \geq \ldots \geq \beta_{[p]}^2 \) are the ordered \( \beta_1^2, \ldots, \beta_p^2 \). We assume that there exists a constant \( L > 0 \) such that \( \max_{i \in [p]} \| X_i \|^2 \leq nL^2 \). The minimax prediction estimation rate over \( \ell_q[R] \) in \( \ell_2 \)-prediction norm is known to be \( r^2(\ell_q[R]) = \inf_{\beta} \sup_{\beta \in \ell_q[R]} \mathbb{E}_\beta \| X\beta - X\beta \|^2 = R^4 n^{q/2} \sigma^2 \left[ \log \left( 1 + \frac{p \sigma^q}{n^{q/2} R^4} \right) \right]^{-q/2} \) when \( R^2 \geq \sigma^2 \log p \); see \([55]\) (cf. \([25, 14]\)). The adaptive minimax results for weak \( \ell_q \)-balls in \( \ell_2 \)-prediction norm were considered by \([28]\) for posterior contraction rates.

Define \( j = O_{\beta}(i) \) if \( \beta_{[j]}^2 = \beta_{[i]}^2 \), with the convention that in the case \( \beta_{[1]}^2 = \ldots = \beta_{[k]}^2 \) for \( i_1 < \ldots < i_k \) we set \( O_{\beta}(i_{k+1}) = O_{\beta}(i_1) + 1, l = 1, \ldots, k - 1 \). Let \( I^* = I^*(\beta) = \{ i \in [p] : O_{\beta}(i) \leq R^* \} \) with \( R^* = e(\frac{\sigma^2}{\sigma^2})^{n^{q/2} \log \left( \frac{\sigma^q}{n^{q/2} R^2} \right)} \) and \( \beta^* = \beta^*(\beta) = (\beta_{[i]}')_{i \in [p]} : \beta_{[i]}' = \beta_{[i]} - \beta_{[i]} = 0 \) for \( j \notin I^* \).

There exists \( I^* \in \mathbb{I}_s \) such that \( X\beta^* \in \mathbb{L}_{I^*} \) and \( \| X\beta - P_{I^*} X\beta \|^2 = \| X\beta - X\beta^* \|^2 \). By using this and the fact that \( \max_{i \in [p]} \| X_i \|^2 \leq L^2 n \), we derive (6.1):

\[
\sup_{\beta \in \ell_q[R]} r^2(\beta) \leq \sup_{\beta \in \ell_q[R]} \left[ r^2(I^*, \beta) \right] \leq \sigma^2 R^2 \log \left( \frac{\sigma^q}{n^{q/2} R^2} \right) + \sup_{\beta \in \ell_q[R]} \| X\beta - X\beta^* \|^2
\]

\[
\lesssim \sigma^2 R^2 \log \left( \frac{\sigma^q}{n^{q/2} R^2} \right) + L^2 n R^2 (R^*)^{1-2/q}
\]

\[
\lesssim R^4 n^{q/2} \sigma^2 \left[ \log \left( 1 + \frac{p \sigma^q}{n^{q/2} R^2} \right) \right]^{1-2/q} \times r^2(\ell_q[R]).
\]

Corollary 2 follows for this case with the minimax rate \( r^2(\ell_q[R]) \) defined above.

**Model selection.** Besides inference on \( \theta = X\beta \), several interesting corollaries were established in \([6]\) and they follow from our results exactly in the same way, we provide them here for completeness.
The first corollary concerns a bound on the size of the selected model. Similar to [21] and [42], the following assertion shows that the models with substantially higher size than the true one are unlikely according to the posterior \( \hat{\pi}(I|Y) \) (which is in essence the penalization method in case \( \hat{\pi}(I|Y) = \hat{\pi}(I|Y) \)).

**Proposition 3.** Under the conditions of Corollary 1, for sufficiently large \( M'_0 \)

\[
\sup_{\beta \in \mathbb{R}^p} \mathbb{E}_\beta \hat{\pi}(I : |I| > C_0\|\beta\|_0|Y) \leq C_0 \exp \left\{ - c_2 \left( \frac{M'_0}{2} - c_3 \right) \|\beta\|_0 \log \left( \frac{en}{\|\beta\|_0} \right) \right\},
\]

where \( C_0 = \max\{M'_0, M''_0/e\} \).

**Proof.** Note that for any \( M'_0 > 2c_3, |I| \geq M'_0\|\beta\|_0 \) implies that

\[
r^2(I, \beta) \geq \sigma^2 |I| \log \left( \frac{en}{|I|} \right) \geq M'_0\sigma^2\|\beta\|_0 \log \left( \frac{en}{M'_0\|\beta\|_0} \right) \geq \frac{M'_0}{2}\sigma^2\|\beta\|_0 \log \left( \frac{en}{\|\beta\|_0} \right),
\]

provided \( \|\beta\|_0 < en/M'_0^2 \). Since \( r^2(\beta) \leq r^2(I^*(\beta), \beta) \leq \sigma^2\|\beta\|_0 \log (en/\|\beta\|_0) \), the above display implies that \( r^2(I, \beta) \geq c_2 r^2(\beta) + M''_0\sigma^2 \), where \( M''_0 = (M'_0/2 - c_3)\|\beta\|_0 \log (en/\|\beta\|_0) \). By (i) of Theorem 2 (\( M''_0 \) corresponds to \( M \) from Theorem 2), the assertion holds for any \( |I| \geq M'_0\|\beta\|_0 \) whenever \( \|\beta\|_0 < en/M'_0^2 \). If \( \|\beta\|_0 \geq en/M'_0^2 \), the result trivially holds for any \( |I| \geq M''^2_0\|\beta\|_0/e \). Hence, the choice \( C_0 = \max\{M'_0, M''^2_0/e\} \) ensures the result for any \( \beta \in \mathbb{R}^p \).

The above claim, being non-asymptotic and uniform in \( \beta \in \mathbb{R}^p \), can be specialized to certain situations. In particular, it leads to an interesting conclusion under the asymptotic setting \( p = p_n \to \infty \) and \( \|\beta\|_0 \leq s_n \to (p_n) \) as \( n \to \infty \). Then the probability bound goes to 0 as \( n \to \infty \), uniformly in \( \beta \in \ell_0[s_n] \). Further, when \( s_n = o(p_n) \), the constant \( C_0 \) can be chosen smaller, which makes the conclusion of the claim stronger.

**Inference on \( \beta \) under the compatibility condition.** The next several corollaries concern inference on \( \beta \) rather than on \( \theta \). Besides optimal prediction, it is of interest to infer on the parameter \( \beta \) itself. Because the dimension \( p \) may be (and generally is) larger than \( n \), the correspondence between \( X\beta \) and \( \beta \) is not unique, and hence additional conditions are necessary even in the noiseless situation. As is commonly adopted in the literature (see, e.g., [28]), we will need to assume a condition lower bounding the norm of \( X\beta \) by a positive multiple of a norm on \( \beta \) for sparse vectors which is in turn a condition on the design matrix \( X \).

There is yet another issue: recall that inference in the general framework is on \( \theta \) and is based on the posterior \( \hat{\pi}(\vartheta|Y) \) for \( \theta = X\beta \), not for \( \beta \). In order to infer on \( \beta \), we need to construct a prior \( \Pi \) on \( \beta \) that leads to an (empirical Bayes) posterior \( \hat{\Pi}(\vartheta|Y) \) such that \( \vartheta = X\beta \sim \hat{\pi}(\vartheta|Y) \). This is not difficult: indeed, recall the construction (2.8) of the conditional prior \( \pi_I(\vartheta|Y) \) on \( \theta \). Since in this case the unstructured \( \hat{\theta} = X\beta \) and the conditional prior \( \pi_I(\vartheta|Y) \) was formally constructed as prior on \( \theta^I = P_I\theta = P_IX\beta = X_I\beta \), we can derive the
corresponding conditional prior \( \Pi_I(b|Y) \) for \( \beta = (X_I^T X_I)^{-1} \theta^I \) because \( \theta^I = X_I \beta \) is invertible with respect to \( \beta \) for any \( I \in \mathcal{I} \). The corresponding conditional prior on \( \beta \) becomes

\[
\beta|I \sim \Pi_I(b|Y) = N((X_I^T X_I)^{-1}X_I \mu, \kappa \sigma^2 (X_I^T X_I)^{-1}) \otimes \delta_{\theta^I|\mu},
\]

which means that subvector \( \beta_I \) with coordinates in \( I \) is normally distributed with the above parameters, and the remaining coordinates \( I^c \) of \( \beta \) are set to zero. From this point on, we can apply the (empirical) Bayesian approach in the same way as for \( \theta \). We thus construct the (empirical Bayes) posteriors on \( \beta \): \( \tilde{\Pi}(b|Y), \Pi(b|Y), \Pi(b|Y) \); and the estimators \( \tilde{\beta} = \sum_{I \in \mathcal{I}} \tilde{\beta}_I \pi(I|Y) \) and \( \hat{\beta} = \hat{\beta}_I \), where \( \tilde{\mathcal{I}} \) is defined by (2.19) and \( \hat{\beta}_I = (X_I^T X_I)^{-1}X_I Y \) is just the ordinary least squares estimator of \( \beta \) based on the design matrix \( X_I \) of full column rank as \( I \in \mathcal{I} \). Similarly, we can define \( \Pi(\beta|Y) \) as being either \( \Pi(\beta|Y) \) or \( \Pi(\beta|Y) \), and \( \hat{\beta} \) as being either \( \tilde{\beta} \) or \( \hat{\beta} \). The details of Bayesian construction for \( \beta \) can be found in [6]. For us what only matters is the fact that if \( \beta \sim \tilde{\Pi}(b|Y) \) then \( \theta = X \beta \sim \pi(\theta|Y) \).

Introduce some additional notation. Recall that \( \| \beta \|_0 \) denotes the number of non-zero elements of \( \beta \). Further let \( \| \beta \|_1 = \sum_{j=1}^p |\beta_j| \) be the \( \ell_1 \)-norm of \( \beta \) and \( \| X \|_{\text{max}} = \max_{k=1, \ldots, p} \| X_k \| \) (notice that if the design matrix \( X \) is normalized so that \( \| X_k \|^2 = n, k \in [p] \), then \( \| X \|_{\text{max}} = \sqrt{n} \)). For \( l \in \mathbb{N} \), let

\[
\phi_1(l) = \inf \left\{ \frac{\sqrt{l} \| X \beta \|_1}{\| X \|_{\text{max}} \| \beta \|_1} : \| \beta \|_0 \leq l, \operatorname{supp}(\beta) \in \mathcal{I} \right\},
\]

\[
\phi_2(l) = \inf \left\{ \frac{\| X \beta \|_1}{\| X \|_{\text{max}} \| \beta \|_1} : \| \beta \|_0 \leq l, \operatorname{supp}(\beta) \in \mathcal{I} \right\}.
\]

Because \( \| \beta \|_1 \leq \sqrt{\| \beta \|_0 \| \beta \|_0} \), it follows that \( \phi_1(l) \geq \phi_2(l) \). Positivity of \( \phi_1 \) at an argument \( l \) is called the compatibility condition, and is stronger if \( \phi_1(l) \) is larger. If any of \( \phi_1 \) or \( \phi_2 \) is zero at its argument, then the corresponding result below becomes trivial but remains valid.

The following claims say basically that, under the compatibility condition, the (empirical Bayes) posterior \( \Pi(b|Y) \) on \( \beta \) contracts around the truth with the optimal rate.

**Proposition 4.** Under the conditions of Corollary 1, for sufficiently large \( M_0 \) and any \( M \geq 0 \)

\[
\begin{align*}
\mathbb{E}_\beta \hat{\Pi} \left( \| b - \beta \|_1 \geq \sqrt{(C_0 + 1) \| \beta \|_0 (M_0 \sigma^2 + M \sigma^2)} \right) Y & \leq H_0 e^{-m_0 M} + C_0 \exp \left\{ - c_2 (M_0^2 / 2 - c_3) \| \beta \|_0 \log \left( \frac{\exp(\| \beta \|_0)}{\| \beta \|_0} \right) \right\}, \\
\mathbb{E}_\beta \hat{\Pi} \left( \| b - \beta \|_1 \geq \frac{\sqrt{M_0 \sigma^2 + M \sigma^2}}{\| X \|_{\text{max}} \| \beta \|_1} \right) Y & \leq H_0 e^{-m_0 M} + C_0 \exp \left\{ - c_2 (M_0^2 / 2 - c_3) \| \beta \|_0 \log \left( \frac{\exp(\| \beta \|_0)}{\| \beta \|_0} \right) \right\}, 
\end{align*}
\]

uniformly in \( \beta \in \mathbb{R}^p \), where \( C_0 = \max\{M_0, M_0^2 / e\} \).
Proof. By the definition of compatibility coefficient, on models \( I \) with \(|I| \leq C_0 \| \beta \|_0 \), the quantity \( \| b - \beta \|_1 \) is bounded by

\[
\sqrt{(C_0 + 1) \| \beta \|_0 \| X(b - \beta) \| / \| \max \phi_1((C_0 + 1) \| \beta \|_0) \|},
\]

since the cardinality of \( \text{supp}(b - \beta) \) is at most \( (C_0 + 1) \| \beta \|_0 \). By Theorem 1, the \( E_\beta \)-expectation of the posterior probability of \( \| X(b - \beta) \| = \| \theta - \theta \| > \sqrt{M_0 r^2(\theta) + M \nu^2} \) is bounded by \( H_0 e^{-m_0 M} \), while by Proposition 3, the event \( \{ I : |I| \geq C_0 \| \beta \|_0 \} \) has probability bounded by

\[
C_\nu \exp \left\{ -c_2(M'_0/2 - c_3) \| \beta \|_0 \log(\frac{eP_{\beta \| \beta \|_0}}{\| \beta \|_0}) \right\}.
\]

The first assertion follows, the proof of the second claim is similar. \( \square \)

Notice that the above result implies the Corollary 5.4 in [28] and obtains optimal estimation rates for both \( \ell_1 \) and \( \ell_2 \) loss functions. Moreover, the dependence on the quantities \( \phi_1(l) \) and \( \phi_2(l) \) are optimal; cf. [48]. Next, we also obtain the optimal estimation result for both \( \ell_1 \)- and \( \ell_2 \)-norms.

**Proposition 5.** Under the conditions of Corollary 1, for sufficiently large \( M'_0 \) and any \( M \geq 0 \)

\[
P_\beta(\| \hat{\beta} - \beta \|_1 \geq \sqrt{(C_0 + 1) \| \beta \|_0 \| M_1 r^2(\beta) + M \nu^2 \| / \| \max \phi_1((C_0 + 1) \| \beta \|_0) \|})
\leq H_1 e^{-m_1 M} + C_\nu \exp \left\{ -c_2(M'_0/2 - c_3) \| \beta \|_0 \log(\frac{eP_{\beta \| \beta \|_0}}{\| \beta \|_0}) \right\},
\]

uniformly in \( \beta \in \mathbb{R}^p \), where \( C_0 = \max\{M'_0, M_0^2/e\} \).

**Proof.** Consider the case \( \hat{\theta} = \tilde{\theta} = X\hat{\beta} \), where \( \hat{\theta} \) is defined by (2.20). Denote for brevity \( \Delta = \sqrt{(C_0 + 1) \| \beta \|_0 \| M_1 r^2(\beta) + M \nu^2 \| / \| \max \phi_1((C_0 + 1) \| \beta \|_0) \|} \) and introduce the event \( E_M = \{ |I| \leq C_0 \| \beta \|_0 \} \), where \( I \) is defined by (2.19). By the definition of compatibility coefficient, in case \( |I| \leq C_0 \| \beta \|_0 \), \( \| \hat{\beta} - \beta \|_1 \) is bounded by \( \sqrt{(C_0 + 1) \| \beta \|_0 \| X(\hat{\beta} - \beta) \| / \| \max \phi_1((C_0 + 1) \| \beta \|_0) \|} \), since the cardinality of \( \text{supp}(\hat{\beta} - \beta) \) is at most \( (C_0 + 1) \| \beta \|_0 \). By Theorem 1, \( \| X(\hat{\beta} - \beta) \| > \sqrt{M_1 r^2(\beta) + M \nu^2} \) has probability bounded by \( H_1 e^{-m_1 M} \). Using this and Proposition 3, we have

\[
P_\beta(\| \hat{\beta} - \beta \|_1 \geq \Delta) = P_\beta(\| \hat{\beta} - \beta \|_1 \geq \Delta, E_M) + P_\beta(\| \hat{\beta} - \beta \|_1 \geq \Delta, E^c_M)
\leq P_\beta(\| X\hat{\beta} - X\beta \|_2 \geq M_1 r^2(\beta) + M \nu^2) + P_\beta(E^c_M)
\leq H_1 e^{-m_1 M} + P_\beta(E^c_M)
\leq H_1 e^{-m_1 M} + E_\beta \tilde{\pi}(I : |I| > C_0 \| \beta \|_0 Y)
\leq H_1 e^{-m_1 M} + C_\nu \exp \left\{ -c_2(M'_0/2 - c_3) \| \beta \|_0 \log(\frac{eP_{\beta \| \beta \|_0}}{\| \beta \|_0}) \right\}
\]

The proof of the second claim for the case \( \hat{\theta} = \tilde{\theta} = X\hat{\beta} \) and the proofs of the both claims for the case \( \hat{\theta} = \tilde{\theta} = X\hat{\beta} \) are similar and therefore omitted. \( \square \)
6.7.2. Linear regression with group sparsity

Assume that the unknown regression vectors \( \beta^1, \ldots, \beta^m \in \mathbb{R}^p \) in (6.13) share the same support. Note that the model considered in Section 6.7.1 is a special case of linear regression with group sparsity with \( m = 1 \). Local results for linear regression with group sparsity were derived in [40], and posterior contraction rate results in [28]. The group sparsity structure is modeled by the linear spaces

\[
\mathbb{L}_I = \{ \text{vec}(X^j x^j_1, \ldots, X^m x^m_I) \in \mathbb{R}^{nm} : x^j_I \in \mathbb{R}^{|I|}, j \in [m] \}, \quad I \in \mathcal{I},
\]

where \( \mathcal{I} = \mathcal{I}_1 \cup \{ \mathcal{I}_p \} \), with \( \mathcal{I}_p = [p] \), \( \mathcal{I}_1 = \{ I \subseteq [p] : m|I| + |I| \log(ep/|I|) \leq r \} \), \( r = \sum_{i=1}^m \text{rank}(X^i) \). Clearly, \( |\mathcal{I}| \leq 2^p \) and \( d_I = \dim(\mathbb{L}_I) \leq m|I| \) for \( I \in \mathcal{I}_1 \) and \( d_{I_p} = r \).

In this case, \( \theta = X\beta \) with \( \beta = (\beta^1, \ldots, \beta^m) \in \mathbb{R}^{mp} \), the structural slicing mapping is \( s(I) = |I| \in \mathcal{S} := [p]_0 \). Further, we have \( |\mathcal{L}_{s(I)}| = (|I|) \) for \( I \in \mathcal{I}_1 \) and \( |\mathcal{L}_{s(I_p)}| = 1 \), hence \( \log |\mathcal{L}_{s(I)}| = \log (|I|) \leq |I| \log(ep/|I|) \) for \( I \in \mathcal{I}_1 \) and \( \log |\mathcal{L}_{s(I_p)}| = 0 \). Since \( d_{s(I)} + \log |\mathcal{L}_{s(I)}| \leq m|I| + |I| \log(ep/|I|) \) for \( I \in \mathcal{I}_1 \) and \( d_{I_p} + \log |\mathcal{L}_{s(I_p)}| = r \), we take the majorant

\[
\rho(s(I)) = (m|I| + |I| \log(ep/|I|))1\{I \in \mathcal{I}_1\} + r1\{I = I_p\}, \quad I \in \mathcal{I}.
\]

Notice the elbow effect in the majorant that emerges here for the same reason as in Section 6.7.1.

Conditions (A2) and (A3) are fulfilled in the same way as for the model in Section 6.7.1. As consequence of our general results, we obtain Corollary 1 for this case with the local rate

\[
r^2(\beta) = \min_{I \in \mathcal{I}} r^2(I, \beta) = \min_{I \in \mathcal{I}} \{\| (I - P_I)X\beta \|^2 + \sigma^2 \rho(s(I)) \}.
\]

**Remark 30.** We can redefine the structural slicing mapping as \( s(I) = \dim(\mathbb{L}_I) \), and the bound \( \log |\mathcal{S}| = \log (|I|) \leq s \log(ep/s) \) would still be valid. Notice further that we can slightly improve the above oracle rate by using the exact quantity \( d_{s(I)} = d_I = \dim(\mathbb{L}_I) \) instead of its upper bound \( m|I| \) in the expression for the complexity \( \rho(s(I)) \), which would make the oracle rate \( r^2(\beta) \) slightly smaller.

One can formulate the minimax results for the appropriate scales. For example, introduce the scale of classes

\[
\ell_0^m[s] = \{ \text{vec}(\beta^1, \ldots, \beta^m) \in \mathbb{R}^{pm} : \beta^i, \beta^j \in \mathbb{R}^{|I^i|^t} \} \leq (|I^i|, |I^j|) \leq s \forall i, j \in [m] \},
\]

where \( \beta^i = \{ i \in [p] : \beta_i \neq 0 \} \). The minimax rate over this class is established in [40] (under some conditions):

\[
r^2(\ell_0^m[s]) = \inf_{\beta} \sup_{\beta \in \ell_0^m[s]} \mathbb{E}_\beta ||x_{\beta} - X\beta||^2 \leq \sigma^2[m s + s \log(ep/s)].
\]

Then we can easily show that the oracle rate implies this global rate since

\[
r^2(\beta) \leq r^2(I^*(\beta), \beta) \land r^2(I_p, \beta) \leq \sigma^2[m|I^*(\beta)| + |I^*(\beta)| \log(\frac{ep}{|I^*(\beta)|})] \land r
\]

\[
\leq \sigma^2[ms + s \log(ep/s)] \times r^2(\ell_0^m[s]) \quad \text{for all} \quad \beta \in \ell_0^m[s].
\]
6.7.3. Linear regression with group clustering

Assume now a clustering structure shared by \( m \) unknown regression vectors \( \beta^1, \ldots, \beta^m \in \mathbb{R}^p \). That is, there is some mapping \( z : [m] \mapsto [k] \) such that \( \beta^i = \beta^{z(i)}, j \in [m] \). Let the design matrix \( X = \text{diag}[X^1, \ldots, X^m] \) in (6.13) be such that \( X^1 = \ldots = X^m = \bar{X} \), with \( \det(\bar{X}^T \bar{X}) > 0 \). Full column rankness of the \((n \times p)\)-matrix \( \bar{X} \) implies \( p \leq n \). Each mapping \( z \in [k]^{[m]} \) determines (uniquely) the pertinent partition \( I = I(z) = (I_i, i \in [k]) \) of the vectors \( \beta^1, \ldots, \beta^m \) into \( k \) groups \( I_i = I_i(z) = z^{-1}(i) \subseteq [m], i \in [k], \) such that \( \bigcup_{i \in [k]} I_i = [m] = z^{-1}([k]) \). Thus, the collection of all mappings \( \mathcal{Z} = \mathcal{Z}(m) = \{ z \in [k]^{[m]}, k \in [m] \} \) yields the collection of all clustering partitions of \([m]\): \( \mathcal{I} = \mathcal{I}(m) = \{ I(z), z \in [k]^{[m]}, k \in [m] \} \). Some local posterior contraction rate results for this model are claimed in [28], where this model is called by multi-task learning. We will call this model rather by linear regression with group clustering. To the best of our knowledge, there are no adaptive minimax results on estimation and uncertainty quantification problems for this model.

In this model, the structures \( I \) are going to be certain partitions from \( \mathcal{I} \). Let \( \bar{I} = ([1], \ldots, [m]) \) be the finest partition of \([m]\) into \( m \) one-point clusters and the structural slicing mapping \( s(I) \) be the number of blocks in the partition \( I \), so that \( \mathcal{S} = [m] \). The group clustering structure is modeled by the following linear spaces

\[
\mathbb{L}_I = \{ \text{vec}(\bar{X} x^1, \ldots, \bar{X} x^m) \in \mathbb{R}^{nm} : x^j \in \mathbb{R}^p, j \in [m], \text{ such that } x^j = x^{j'} \forall j, j' \in I_i, I_i \in I, i \in [s(I)] \},
\]

where \( I \in \mathcal{I} \triangleq \mathcal{I}_1 \cup \{ \bar{I} \} \) with \( \mathcal{I}_1 = \{ I \in \bar{I} : p s(I) + m \log s(I) \leq p m \} \). In this case, \( \theta = X \beta, \delta_I = \dim(\mathbb{L}_I) = p s(I) \) and \( |\mathcal{Z}(s(I))| = N(m, s(I)) \) for \( I \in \mathcal{I}_1 \), where \( N(m, s) \) is the number of ways to put \( m \) different objects into \( s \) different boxes so that each box contains at least one object. Then \( |\mathcal{Z}(s(I))| \leq \log s^m(I) = m \log s(I) \) for \( I \in \mathcal{I}_1 \). Besides, we have \( d_s = \dim(\mathbb{L}_I) = p m \) and \( |\mathcal{Z}(s(I))| = 1 \). Since \( d_s + \log |\mathcal{Z}(s(I))| \leq p s(I) + m \log s(I) \) for \( I \in \mathcal{I}_1 \) and \( d_s + \log |\mathcal{Z}(s(I))| = p m \), we take the majorant

\[
\rho(s(I)) = (p s(I) + m \log s(I))1\{I \in \mathcal{I}_1\} + p m 1\{I = \bar{I}\}.
\]

Remark 31. As before, we have an elbow effect, again for the same reason. The idea of the elbow in the majorant should be clear now: there is no point (although possible) to model the structures \( I \in \bar{I} \setminus \mathcal{I} \), because all these structures are dominated by the structure \( \bar{I} \in \mathcal{I} \). Indeed, for each \( I \in \bar{I} \setminus \mathcal{I} \),

\[
\frac{1}{2} r^2(I, \beta) = \| (I - P_I) X \beta \|^2 + \sigma^2 \rho(s(I)) = \| (I - P_I) X \beta \|^2 + \sigma^2 (p s(I) + m \log s(I)) \geq \sigma^2 p m = \| (I - P_I) X \beta \|^2 + \sigma^2 p m = r^2(\bar{I}, \beta),
\]

because \( P_I X \beta = X \beta \).

Condition (A2) is fulfilled, since, according to Remark 7, for any \( \nu \geq 1 \)

\[
\sum_{I \in \mathcal{I}} e^{-\nu \rho(s(I))} \leq \sum_{I \in \mathcal{I}_1} e^{-\nu \rho(s(I))} + e^{-\nu p m} \leq \sum_{s \in [m]} e^{-\nu p m} + e^{-\nu p m} \leq (e^{\nu p} - 1)^{-1} + 1 = C_\nu.
\]
Thus, this structure redundancy \( \bar{\mathcal{L}} \) under some mild condition. Namely, we could allow redundancy by associating the same space \( \mathbb{L}_I \) to each \( I \in \bar{\mathcal{L}} \setminus \mathcal{I} \). The majorant becomes \( \tilde{\rho}(s(I)) = (ps(I) + m \log s(I)) \mathbb{1}_{\{I \in \mathcal{I}_1\}} + pm \mathbb{1}_{\{I \in \bar{\mathcal{L}} \setminus \mathcal{I}_1\}} \), defined now for all \( I \in \bar{\mathcal{L}} \). Then, if \( p \gtrsim \log m \), Condition (A2) is fulfilled for sufficiently large \( \nu \):

$$
\sum_{I \in \bar{\mathcal{L}}} e^{-\nu \rho(s(I))} \leq \sum_{I \in \mathcal{I}_1} e^{-\nu \rho(s(I))} + \sum_{I \in \bar{\mathcal{L}} \setminus \mathcal{I}_1} e^{-\nu \rho(s(I))} \\
\leq \sum_{s \in [m]} e^{-\nu ps} + \sum_{s \in [m]} s^m e^{-\nu pm} \leq (e^{\nu p} - 1)^{-1} + C = C_\nu.
$$

Thus, this structure redundancy \( \bar{\mathcal{L}} \setminus \mathcal{I}_1 \) does not affect the final local rate, only constant \( C_\nu \) becomes slightly larger (and the condition \( p \gtrsim \log m \) has to hold).

Condition (A3) is also fulfilled. Indeed, for any \( I^0, I^1 \in \mathcal{I} \) define the partition refinement \( I' = I'(I^0, I^1) = I^0 \vee I^1 = (I_i \cap J_j, I_i \in I^0, J_j \in I^1) \).

Clearly, \( \mathbb{L}_{I^0} \cup \mathbb{L}_{I^1} \subseteq \mathbb{L}_{I'} \subseteq \mathbb{L}_{I^0} + \mathbb{L}_{I^1} \) and \( \max\{s(I^0), s(I^1)\} \leq s(I') \leq s(I^0) + s(I^1) \), implying \( \rho(s(I')) \leq \rho(s(I^0)) + \rho(s(I^1)) \), which entails Condition (A3).

As consequence of our general results, we obtain the local results of Corollary 1 for these model and structure with the local rate

$$
r^2(\beta) = \min_{I \in \bar{\mathcal{I}}} \{ ||(I - \mathbb{P}_I)X\beta||^2 + \sigma^2 \rho(s(I)) \}.
$$

In turn, by virtue of Corollary 2, the local results will imply global minimax adaptive results at once over all scales \( \{\Theta_\gamma, \gamma \in \Gamma\} \) covered by the oracle rate \( r^2(\beta) \) (i.e., for which (6.1) holds). For example, let \( \Theta_{GC}(s) = \bigcup_{I \in \mathcal{I}, s(I) \leq s} \mathbb{L}_I \). To the best of our knowledge, there are no minimax results over \( \Theta_{GC}(s) \). We conjecture that the minimax rate over \( \Theta_{GC}(s) \) is

$$
r^2(\Theta_{GC}(s)) \triangleq \inf_{\beta} \sup_{\beta, X, \beta \in \Theta_{GC}(s)} \mathbb{E}_\beta \|X\beta - \mathbb{X}\beta\|^2 \asymp \sigma^2 \min\{ps + m \log s, pm\}.
$$

It is not difficult to show that the local rate \( r^2(\beta) \) covers this scale. Indeed, for each \( \theta = X\beta \in \Theta_{GS}(s) \) there exists \( I_\ast = I_\ast(\theta) \in \bar{\mathcal{I}} \) such that \( \theta = X\beta \in \mathbb{L}_{I_\ast} \) and \( s(I_\ast) \leq s \). If \( ps + m \log s \leq pm \), then \( I_\ast \in \mathcal{I}_1 \). Hence, \( r^2(\beta) \leq r^2(I_\ast, \beta) = \sigma^2 \rho(s(I_\ast)) = \sigma^2 (ps(I_\ast) + m \log s(I_\ast)) = \sigma^2 (ps + m \log s) \) because \( \mathbb{P}_{I_\ast}X\beta = X\beta \) and \( s(I_\ast) \leq s \). If \( ps + m \log s > pm \), then \( r^2(\beta) \leq r^2(I, \beta) = \sigma^2 \rho(s(I)) = \sigma^2 pm \) because \( \mathbb{P}_I X\beta = X\beta \).

Summarizing, \( r^2(\beta) \leq \sigma^2 \min\{ps + m \log s, pm\} \). We thus established the relation (6.1) for this scale, and Corollary 2 follows with the minimax rate \( r^2(\Theta_{GS}(s)) \) defined above.

6.7.4. Linear regression with mixture structure

Consider the regression model (6.13) with \( p \in [n] \) such that \( X^1 = \ldots = X^m = \bar{X}, \bar{X} = (\bar{X}_{ij}) \in \{0,1\}^{m \times p} \), and \( \sum_{j \in [p]} \bar{X}_{ij} = 1 \) for all \( i \in [n] \), i.e., each row of the matrix \( \bar{X} \) has
n − 1 zeros and only one entry equals to 1. Recently, some estimation results for this model were derived in [37]. To the best of our knowledge, there are no local results on posterior contraction rate and uncertainty quantification problems for mixture model.

In this case, \( \theta = X\beta \) and \( \dim(\beta) = p \in [n] \) is now not fixed but rather a varying ingredient of the structure. Another ingredient of the structure are the locations \( I_i \) of 1’s in the \( i \)th \( p \)-dimensional row of the matrix \( \bar{X} \), \( i \in [n] \). Putting these together, we encode the whole structure as \( \mathcal{I} = \{ [\bar{I}, i \in [n]] \} \) where \( \bar{I} \in [p], p \in [n] \). Thus, the full family of all structures is

\[
\mathcal{I} = \{ [\bar{I}, i \in [n]] : I_i \in [p], p \in [n] \}.
\]

Let \( X_I = \text{diag}\{\bar{X}_1, \ldots, \bar{X}_{\bar{I}}\} \), \( \bar{X}_I = (\bar{X}_{ij}) \) be the \( (n \times p(I)) \)-matrix corresponding to the structure \( I \in \mathcal{I} \), that is, \( \bar{X}_{iI} = 1 \) for \( i \in [n] \) and all the other entries of this matrix are zeros. By \( p(I) \) we denote the first ingredient of the structure \( I \), the number of columns in the matrix \( \bar{X}_I \). The structural slicing mapping is \( s(I) = r(I) \), where \( r(I) = \text{rank}(\bar{X}_I) \), the number of linearly independent columns in the matrix \( \bar{X}_I \). So, \( S = [n] \) and notice that \( r(I) \leq p(I) \).

The structures in this model are modeled by the linear spaces

\[
\mathbb{L}_I = \{ \text{vec}(\bar{X}_Ix^1, \ldots, \bar{X}_Ix^m) \in \mathbb{R}^{nm} : x^j \in \mathbb{R}^{p(I)}, j \in [m] \},
\]

where \( I \in \mathcal{I} \triangleq \mathcal{I}_1 \cup \{ \bar{I} \} \) with \( \mathcal{I}_1 = \{ I \in \mathcal{I} : mr(I) + n \log p(I) \leq nm \} \) and \( \bar{I} = [n, [n]] \) (so that \( \bar{X}_I = 1 \) is the \( n \)-dimensional identity matrix). In this case, \( \theta = X\beta \), \( d_I = \dim(\mathbb{L}_I) = m \text{rank}(\bar{X}_I) = mr(I) \leq mp(I) \) and \( |\mathcal{I}_{s(I)}| \leq p^n(I) \) for \( I \in \mathcal{I}_1 \), because \( p^n(I) \) is the number of possibilities to choose locations of 1’s in the \( n p(I) \)-dimensional rows of the design matrix \( \bar{X}_I \). Further, \( d_I = \dim(\mathbb{L}_I) = nm \) (as \( \bar{X}_I = 1 \)) and \( |\mathcal{I}_{s(I)}| = 1 \). Since \( d_{s(I)} + \log |\mathcal{I}_{s(I)}| = mr(I) + n \log p(I) \) for \( I \in \mathcal{I}_1 \) and \( d_{s(I)} + \log |\mathcal{I}_{s(I)}| = nm \), we take the majorant

\[
p(s(I)) = (mr(I) + n \log p(I))1\{I \in \mathcal{I}_1\} + nm1\{I = \bar{I}\}. \tag{6.19}
\]

The reason for considering the restricted family of structures \( \mathcal{I} \) instead of the full family \( \mathcal{I} \) in this model is the same as for the model from Section 6.7.3 and is explained in Remark 31.

Condition (A2) is fulfilled since, according to Remark 7, for any \( \nu \geq 1 \)

\[
\sum_{I \in \mathcal{I}} e^{-\nu p(s(I))} \leq \sum_{s \in S} e^{-\nu d_s} \leq \sum_{s \in [n]} e^{-\nu ms} + e^{-\nu nm} \leq (e^{\nu m} - 1)^{-1} + 1 = C_\nu.
\]

Condition (A3) can also be verified, which would ensure the coverage property (v) of Corollary 1 under EBR as well. However, there is no point in verifying Condition (A3) because for this linear regression model with mixture structure we have the same peculiar situation as for the biclustering model from Section 6.6: the size and coverage claims (vi)-(vii) for the confidence ball \( B(\hat{\theta}, \hat{R}_M) \) are stronger and more useful than the corresponding claims (iv)-(v) for the confidence ball \( B(\hat{\theta}, \hat{R}_M) \). Let us demonstrate that the linear regression with mixture structure does not suffer from the deceptiveness phenomenon, modulo the so-called highly structured parameters.
Indeed, as consequence of our general results, we obtain the local results (i)-(iv) and (vi)-(vii) of Corollary 1 for this case with the local rate $r^2(\beta) = \min_{I \in \mathcal{I}} \{ \| (I - P_I) X \beta \|^2 + \sigma^2 \rho(s(I)) \}$, with $P_I$ as projection onto $\mathbb{L}_I$ defined above and the majorant $\rho(s(I))$ defined by (6.19). The coverage property (v) for the confidence ball $B(\hat{\theta}, \hat{R}_M)$ can be shown to hold also, but uniformly only under the EBR, whereas the coverage property (vii) for the confidence ball $B(\hat{\theta}, \hat{R}_M)$ is uniform over the entire space $\Theta = \mathbb{R}^{n \times m}$. The size $\hat{R}_M$ is of the oracle rate order (as the radius $\hat{R}_M$ uniformly in $\theta \in \Theta \setminus \tilde{\Theta} = \mathbb{R}^{n \times m} \setminus \tilde{\Theta}$ where $\tilde{\Theta}$ is defined by (3.14). Since in this model the total number of observations is $N = nm$, it is easy to see that $\tilde{\Theta} \subseteq \{ \theta \in \mathbb{R}^{n \times m} : p(I_o(\theta)) = 1 \}$ (i.e., $X_{I_o} = 1_n$, where $1_n$ is the $n$-dimensional column of 1’s) where the oracle structure $I_o(\theta)$ is defined by (3.1). Clearly, the $m$-dimensional $\tilde{\Theta}$ is a “thin” subset of $\mathbb{R}^{n \times m}$ consisting of highly structured parameters $\theta$ whose oracle number of columns in the design matrix $X_{I_o}$ is $p(I_o(\theta)) = 1$. As we have already discussed at the end of Section 3.4, this means that, modulo these highly structured parameters, there is no deceptiveness phenomenon in this model.

Remark 33. Notice that our local results for the linear regression model with mixture structure actually improve upon the results of [37] as we have $mr(I) \leq mp(I)$ instead of $mp(I)$ (as in [37]) in the expression of the local rate $r^2(\beta)$. This means that this oracle rate $r^2(\beta)$ defined above is smaller than the one from [37]. Notice that the below global minimax results over the considered class cannot be improved as the worst case of both local rates is the same.

Finally, by virtue of Corollary 2 the local results will imply global minimax adaptive results at once over all scales $\{ \Theta_\gamma, \gamma \in \Gamma \}$ covered by the oracle rate $r^2(\beta)$ (i.e., for which (6.1) holds). Below we present one such scale, covered by the oracle rate $r^2(\beta)$.

Minimax results for mixture model. Define the class

$$\Theta_M(p) = \cup_{I \in \mathcal{I}, \rho(I) \leq p} \mathbb{L}_I.$$ 

As is shown in [37], the minimax rate over $\Theta_M(p)$ is

$$r^2(\Theta_M(p)) \triangleq \inf_{\beta} \sup_{\beta : X\beta \in \Theta_M(p)} \mathbb{E}_\beta \| X\hat{\beta} - X\beta \|^2 \geq \sigma^2 \min \{ mp + n \log p, nm \}. $$

For each $\theta = X\beta \in \Theta_M(p)$ there exists $I_* = I_*(\theta) \in \mathcal{I}$ such that $\theta \in \mathbb{L}_{I_*}$ and $p(I_*) \leq p$. If $mp + n \log p \leq nm$, then $r(I_*) \leq p(I_*), p(I_*) \leq p$, so that $r^2(\beta) \leq r^2(I_*, \beta) = \sigma^2 \rho(s(I_*)) \leq \sigma^2 (mp + n \log p)$ because $P_{I_*} X\beta = X\beta$. If $mp + n \log p > nm$, then $r^2(\beta) \leq r^2(I, \beta) = \sigma^2 \rho(s(I)) = \sigma^2 nm$ because $P_I X\beta = X\beta$.

Piecing these together, we obtain that $r^2(\beta) \leq \sigma^2 \min \{ mp + n \log p, nm \}$ for all $\beta$ such that $\theta = X\beta \in \Theta_M(p)$. We thus established the relation (6.1) for this scale, and Corollary 2 follows with the minimax rate $r^2(\Theta_M(p))$ defined above.
6.8. Aggregation

Aggregation in nonparametric regression has been considered by [44, 17, 50, 55] and many others. We consider the regression model with a fixed design. The observation model is as follows:

$$Y_i = f(x_i) + \sigma \xi_i, \ i \in [n],$$

(6.20)

where $x_i \in \mathcal{X}$ are nonrandom, $\mathcal{X}$ is an arbitrary set, $f : \mathcal{X} \to \mathbb{R}$ is an unknown function, and $\xi_i \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$. We use the notation $\|f\|^2 = \sum_{i \in [n]} f^2(x_i)$.

Assume we are given a collection of functions $\{f_1, \ldots, f_p\}$, called dictionary. For $\beta \in \mathbb{R}^p$, let $f_\beta = \sum_{j=1}^p \beta_j f_j$. By choosing a rich dictionary $\{f_1, \ldots, f_p\}$ and an appropriate $\beta \in \mathcal{B} \subseteq \mathbb{R}^p$, one can expect $f_\beta$ to be close to $f$ under some assumptions. For a certain choice of $\mathcal{B}$, the so called aggregation problem consists basically in determining the “best” $\beta \in \mathcal{B}$ on the basis of the data $Y$ such that $(\sum_{j=1}^p \hat{\beta}_j f_j(x_i), i \in [n])$ well estimates the true $(f(x_i), i \in [n])$.

Introduce the sets $\mathcal{B}$ studied in the literature: the sets $\mathcal{B}(MS), \mathcal{B}(C), \mathcal{B}(L), \mathcal{B}(La), \mathcal{B}(Ca)$ are defined as in [50]. Precisely, let $B_1(1) = \{\beta \in \mathbb{R}^p : \|\beta\|_1 = \sum_{j=1}^p \beta_j \leq 1\}$ and $B_0(s) = \ell_0[s] = \{\beta \in \mathbb{R}^p : \|\beta\|_0 \leq s\}$ for $s \in [p]$. Next, define $\mathcal{B}(MS) = B_0(1)$, $\mathcal{B}(C)$ is a closed convex subset of $B_1(1)$, $\mathcal{B}(L) = B_0(p)$, $\mathcal{B}(La) = B_0(s) \cap B_1(1)$. Thus $\mathcal{B} \in \{\mathcal{B}(MS), \mathcal{B}(C), \mathcal{B}(L), \mathcal{B}(La), \mathcal{B}(Ca)\}$.

First recall the main estimation results from [50] (lower bounds are also established in that paper). The so called exponential screening estimator $f_{\beta ES}$ is proposed in [50]. Under the assumptions $\max_{j \in [p]} \|f_j\| \leq \sqrt{n}$, $p \geq 2$, $n \geq 1$, $s \in [p]$, the following oracle estimation result is derived in [50]: for some constant $C > 0$,

$$\mathbb{E} f \|f_{\beta ES} - f\|^2 \leq \inf_{\beta \in \mathcal{B}} \|f_\beta - f\|^2 + C \sigma^2 \psi_{n,p}(\mathcal{B}),$$

(6.21)

where $\psi_{n,p}(\mathcal{B}) = \min \{\psi^*_{n,p}(\mathcal{B}), r\}$ is the optimal rate of aggregation for the corresponding classes $\mathcal{B} \subseteq \{\mathcal{B}(MS), \mathcal{B}(C), \mathcal{B}(L), \mathcal{B}(La), \mathcal{B}(Ca)\}$, $r = \text{rank}(X)$, $\psi^*_{n,p}(\mathcal{B})$ is defined as follows:

$$\psi^*_{n,p}(\mathcal{B}) = \begin{cases} \log p, & \mathcal{B} = \mathcal{B}(MS), \\ \sqrt{n \log \left(1 + \frac{ep\sigma}{\sqrt{n}}\right)}, & \mathcal{B} = \mathcal{B}(C), \\ r, & \mathcal{B} = \mathcal{B}(L), \\ s \log(1 + ep/s), & \mathcal{B} = \mathcal{B}(La), \\ \min \left\{ \sqrt{n \log \left(1 + \frac{ep\sigma}{\sqrt{n}}\right)}, s \log(1 + ep/s) \right\}, & \mathcal{B} = \mathcal{B}(Ca). \end{cases}$$

An advantageous feature of the above results is its universality: the aggregation is attained over the five classes simultaneously. This result follows from Lemma 8.2 and Theorem 3.1 of [50]. Lemma 8.2 is fulfilled as soon as Theorem 3.1 holds and $\max_{j \in [p]} \|f_j\| \leq \sqrt{n}$. The result of Theorem 3.1 from [50] in our notation reads as follows: for any $p, n \geq 1$

$$\mathbb{E} f \|f_{\beta ES} - f\|^2 \leq \min_{\beta \in \mathbb{R}^p} \left\{ \|f - f_\beta\|^2 + \sigma^2 \left[ r \wedge (9 |I^*(\beta)| \log(1 + \frac{ep}{|I^*(\beta)| \log n}) \right) \right\}.$$
A very useful fact concluded in [50] is that, under the condition \( \max_{j \in [p]} \|f_j\| \leq \sqrt{n} \), any estimator satisfying (6.22) (possibly with different constants in the right hand side) leads to the universal oracle inequality (6.21).

Let us demonstrate that we can derive the same type of estimation results as in [50], again as consequences of our general approach for particular choice of sparsity structures. In fact, we improve upon certain aspects and also provide the results on uncertainty quantification, again as consequence of our general framework results.

The aggregation problem considered here for the model (6.20) can be associated with the standard linear regression problem (6.13) (with \( m = 1 \)) studied in Section 6.7.1. Indeed, let \( \theta = (f(x_i), i \in [n]) \) and notice that the vector \( f_\beta = (f_\beta(x_i), i \in [n]) \) can be represented as \( X\beta \), where \( \beta \in \mathbb{R}^p \) is the unknown high-dimensional parameter and the design \((n \times p)\)-matrix \( X \) has the entries \( X_{ij} = f_j(x_i), (i,j) \in [n] \times [p] \). In doing so, we arrive to the general setting \( Y = \theta + \sigma \xi \), but now we take the same family of structures \( \mathcal{I} \) and the corresponding family of linear spaces given by (6.14), as in Section 6.7.1. The conditions are already verified in Section 6.7.1. Then, according to Remark 27, the general results imply Corollary 1 with the oracle rate

\[
r^2(\theta) = \min_{I \in \mathcal{I}} r^2(I, \theta) = \min_{I \in \mathcal{I}} \{\|\theta - P_I \theta\|^2 + \sigma^2 \rho(s(I))\},
\]

where the majorant \( \rho(s(I)) \) is defined by (6.15).

Recall the full family of structures \( \mathcal{I} = \{J : J \subseteq [p]\} \). Since \( \|\theta - P_I \theta\|^2 = \min_{I \in \mathcal{I}} \|\theta - P_I \theta\|^2 \), it is easy to see that

\[
r^2(\theta) = \left[ \min_{I \in \mathcal{I}} r^2(I, \theta) \right] \land r^2(I, \theta) = \min_{I \in \mathcal{I}} \{\|\theta - P_I \theta\|^2 + \sigma^2 [\rho(s(I)) \land r]\}
= \min_{\beta \in \mathbb{R}^p} \{\|f_\beta - f\|^2 + \sigma^2 [r \land (2[I^* (\beta)] \log(\frac{\rho}{I^* (\beta)})^2)]\}. \tag{6.23}
\]

In particular, property (ii) of Corollary 1 entails that for some \( C_0, C_1 > 0 \)

\[
\mathbb{E}_f \|f - \hat{\theta}\|^2 \leq C_0 r^2(\theta) + C_1 \sigma^2,
\]

where \( r^2(\theta) \) is defined by (6.23), which is in fact property (6.22) for our estimator \( \hat{\theta} \). As is mentioned above, [50] established that, under the additional assumption \( \max_{j \in [p]} \|f_j\| \leq \sqrt{n} \), this in turn leads to the universality property (6.21), now for the estimator \( \hat{\theta} \); for some \( C_0, C_2 > 0 \),

\[
\mathbb{E}_f \|f - \hat{\theta}\|^2 \leq C_0 \inf_{\beta \in \mathbb{B}} \|f_\beta - f\|^2 + C_2 \sigma^2 \psi_{n,p}(\mathbb{B}).
\]

We should mention that the constants in the universality property for our estimator \( \hat{\theta} \) may be worse than those for the estimator \( \tilde{f}_{\beta \in \mathbb{B}} \). On the other hand, notice that the claim (ii) of Corollary 1, being a uniform exponential inequality in probability, is itself finer and stronger versions of the corresponding oracle result in expectation (like (6.21)). Moreover,
we additionally obtain claims (i) and (iv)-(vii) of Corollary 1 for the posterior concentration rate and uncertainty quantification. Global results over appropriate scales can also be derived as consequences of Corollary 2. Besides, we can drop the normality and independence assumptions and impose only Condition (A1) instead. One can readily formulate these results.

6.9. Isotonic, unimodal and convex regressions

Assume that the observations are \( Y = (Y_i)_{i \in [n]} \) according to the model

\[
Y_i = \theta_i + \sigma \xi_i, \quad i \in [n],
\]

where \( \xi_i \overset{\text{ind}}{\sim} N(0,1) \) and \( \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n \) is the unknown high-dimensional parameter, possibly belonging to one of the three classes:

\[
\begin{align*}
\mathcal{S}^1 &= \{ \theta \in \mathbb{R}^n : \theta_i \leq \theta_{i+1}, i = 1, \ldots, n-1 \}, \quad n \geq 2; \\
\mathcal{U}^m &= \{ \theta \in \mathbb{R}^n : \theta_1 \geq \cdots \geq \theta_m \leq \theta_{m+1} \leq \cdots \leq \theta_n \}, \quad m \in [n], \quad n \geq 2; \\
\mathcal{C} &= \{ \theta \in \mathbb{R}^n : 2\theta_i \leq \theta_{i+1} + \theta_{i-1}, i = 2, \ldots, n-1 \}, \quad n \geq 3.
\end{align*}
\]

The isotonic, unimodal and convex regression problems concern the classes \( \mathcal{S}^1, \mathcal{U}^m \) and \( \mathcal{C} \), respectively. Recently, oracle estimation results for these problems were derived by [23, 12, 13]. To the best of our knowledge, there are no local results on posterior contraction rate and uncertainty quantification problems for these structures.

First, to model parameters from \( \mathcal{S}^1 \) and \( \mathcal{U}^m \), introduce the linear spaces

\[
\mathbb{L}_I = \{ x \in \mathbb{R}^n : x_i = x_{i+1}, \quad i \notin I \}, \quad I \subseteq \mathcal{I} = [n-1],
\]

where \( d_I = \dim(\mathbb{L}_I) = |I| + 1 \). The structural slicing mapping is \( s(I) = |I|, \) so that \( \mathcal{S} = [n-1] \). Compute \( |\mathcal{I}_{s(I)}| = \binom{n-1}{|I|} \), hence \( \log |\mathcal{I}_{s(I)}| \leq |I| \log(n-|I|). \) Since \( d_{s(I)} + |\mathcal{I}_{s(I)}| \leq |I| + 1 + \log(n-|I|) \leq 1 + 2|I| \log(n/|I|) \), we take the majorant \( \rho(s(I)) = 1 + 2|I| \log(n/|I|) \).

Next, the parameters from \( \mathcal{C} \) are modeled by the linear spaces

\[
\mathbb{L}'_I = \{ x \in \mathbb{R}^n : 2x_i = x_{i+1} + x_{i-1}, \quad i \notin I \}, \quad I \subseteq \mathcal{I}' = \{2, \ldots, n-1\},
\]

where \( d'_I = \dim(\mathbb{L}_I) \leq (2|I| \lor 1) \land n \leq 2|I| + 1 \). The structural slicing mapping in this case is \( s'(I) = |I|, \) so that \( \mathcal{S}' = \{2, \ldots, n-1\}. \) Compute \( |\mathcal{I}'_{s'(I)}| = \binom{n-2}{|I|-1} \), hence \( \log |\mathcal{I}'_{s'(I)}| \leq |I| \log(n/|I|). \) Since \( d_{s'(I)} + |\mathcal{I}'_{s'(I)}| \leq 2|I| + 1 + |I| \log(n/|I|) \leq 1 + 3|I| \log(n/|I|) \), we take the majorant \( \rho'(s'(I)) = 1 + 3|I| \log(n/|I|) \).

We introduced two different families of structures with two corresponding (different) families of linear spaces, but the majorants in the both cases can be chosen the same (up to a multiplicative constant). Conditions (A1)–(A4) for the both cases are fulfilled in the same way as for the model considered in Section 6.3, we omit the argument and computations.
that are very much along the same lines as in Section 6.3. As consequence of our general results, we obtain the local results of Corollary 1 for the both cases with the local rate \( r^2(\theta) = \min_{I \in \mathcal{I}} \{ \| \theta - P_I \theta \|^2 + \sigma^2 \rho(s(I)) \} \) and \( r'^2(\theta) = \min_{I \in \mathcal{I}} \{ \| \theta - P_I \theta \|^2 + \sigma^2 \rho(s'(I)) \} \). In turn, by virtue of Corollary 2 the local results will imply global minimax adaptive results at once over all scales \( \{ \Theta_\beta, \beta \in \mathcal{B} \} \) covered by the oracle rate \( r^2(\theta) \) and \( r'^2(\theta) \) (i.e., for which (6.1) holds). Below we present a couple of examples of such scales \( \{ \Theta_\beta, \beta \in \mathcal{B} \} \).

6.9.1. Minimax results for isotonic, unimodal and convex regressions

Following [12], for \( \theta \in \mathbb{R}^n \), denote the number of relations \( \theta_i \neq \theta_{i+1} \) for \( i \in [n-1] \) by \( k(\theta) - 1 \) (number of jumps of \( \theta \)), and for \( \theta \in \mathcal{C} \), the number of inequalities \( 2 \theta_i \leq \theta_{i+1} + \theta_{i-1} \) that are strict for \( i = 2, \ldots, n-1 \) by \( q(\theta) - 1 \). Let \( \mathcal{U} = \bigcup_{m=1}^n \mathcal{U}^m \), where \( \mathcal{U}^m \) is defined by (6.25). Define the classes of monotone and unimodal parameters with at most \( k \) jumps and the class of piecewise linear convex parameters with at most \( q \) linear pieces as follows: for \( k, q \geq 1 \),

\[
S^+_k = \{ \theta \in S^1 : k(\theta) \leq k \}, \quad U_k = \{ \theta \in \mathcal{U} : k(\theta) \leq k \}, \quad C_q = \{ \theta \in \mathcal{C} : q(\theta) \leq q \},
\]

where \( S^+ \) and \( C \) are defined by (6.24) and (6.26). Define \( \Theta^+_k = \bigcup_{I \in \mathcal{I}, |I|+1 \leq k} \mathcal{L}_I \) and notice that for each \( \theta \in S^+_k \) (or \( \theta \in U_k \)) there exists \( I_* \in \mathcal{I} \) such that \( \theta \in \mathcal{L}_{I_*} \) and \( k(\theta) = |I_*| + 1 \), implying that \( S^+_k \subseteq \Theta^+_k \) (and \( U_k \subseteq \Theta^+_k \)). Similarly, we define \( \Theta_q = \bigcup_{I \in \mathcal{I}, |I|+1 \leq q} \mathcal{L}_I \) and derive that \( C_q \subseteq \Theta_q \).

As is shown in [12], the minimax rates over \( S^+_k \) and \( C_q \), with \( k, q \geq 1 \), are \( r^2(S^+_k) \triangleq \inf_{\theta \in S^+_k} \sup_{\| \theta - \hat{\theta} \|^2} \rho \sigma^2 k \) and \( r^2(C_q) \asymp \sigma^2 q \), respectively. Due to the fact that \( S^+ \subseteq \mathcal{U} \), we also have \( r^2(U_k) \asymp \sigma^2 k \). Now, for each \( \theta \in S^+_k \) (or \( \theta \in U_k \)) there exists \( I_* \in \mathcal{I} \) such that \( \theta \in \mathcal{L}_{I_*} \) (so that \( P_{I_*} \theta = \theta \)) and \( |I_*| + 1 = k(\theta) \leq k \). Hence, \( r^2(\theta) \leq r^2(I_*, \theta) = \sigma^2 (1 + 2|I_*| \log(\frac{|I_*|}{k}))) \leq \sigma^2 k \log(\frac{|I_*|}{k}) \) for all \( \theta \in S^+_k \) and \( \theta \in U_k \). Similarly, we show that \( r^2(\theta) \asymp \sigma^2 q \log(\frac{m}{q}) \) for all \( \theta \in C_q \). We thus established the relation (6.1) for the classes \( S^+_k \), \( U_k \) and \( C_q \), which imply the minimax results (up to a logarithmic factor) of Corollary 2 for all these three classes.

Finally introduce the classes of (shape-restricted) monotone, unimodal and convex parameters \( \theta \) with bounded total variation: \( S^0(V) = \{ \theta \in S^1 : V(\theta) \leq V \}, \mathcal{U}(V) = \{ \theta \in \mathcal{U} : V(\theta) \leq V \}, C(V) = \{ \theta \in \mathcal{C} : V(\theta) \leq V \} \), where \( V(\theta) = \max_{i,j}(\theta_i - \theta_j) \) (notice that \( V(\theta) = \theta_n - \theta_1 \) for \( \theta \in S^+ \)). It is known that the minimax rates over \( S^0(V), \mathcal{U}(V) \) and \( C(V) \) are respectively \( r^2(S^0(V)) \asymp \max\{n^{1/3}(\sigma^2 V)^{2/3}, \sigma^2 \}, r^2(\mathcal{U}(V)) \asymp \max\{n^{1/3}(\sigma^2 V)^{2/3}, \sigma^2 \} \) and \( r^2(C(V)) \asymp n^{1/5}(\sigma^4 V)^{2/5} \) if \( V \geq \sigma \). To derive the Corollary 2 for these classes, we need the next proposition, where claim (i) is Lemma 2 from [12] and claim (ii) is Lemma 4.1 from [12]. We give these claims here (in our notation) for completeness, the proofs can be found in the mentioned references.

**Proposition 6.** Let \( k, q \in [n] \). Then the following properties hold.
(i) For any \( \theta \in \mathcal{S}_1 \) (or \( \theta \in \mathcal{U} \)) there exists a \( \theta^* = \theta^*(\theta) \in \Theta_k^\top \) such that

\[
\|\theta - \theta^*\|^2 \leq C_1 \frac{nV^2(\theta)}{k^2}
\]

for some absolute constant \( C_1 > 0 \).

(ii) For any \( \theta \in \mathcal{C} \) there exists a \( \theta^* = \theta^*(\theta) \in \Theta_q \) such that \( \|\theta - \theta^*\|^2 \leq C_2 \frac{nV^2(\theta)}{\sigma^2} \) for some absolute constant \( C_2 > 0 \).

According to Proposition 6, \( \theta^* \in \Theta_k^\top \), then there exists an \( I_* \in \mathcal{I} \) such that \( \theta^* \in \mathbb{L}_{I_*} \) and

\[
\|\theta - P_{I_*} \theta\|^2 \leq \|\theta - \theta_*\|^2 \leq C_1 \frac{nV^2(\theta)}{k^2}.
\]

It follows therefore that for any \( k \in [n] \) and any \( \theta \in \mathcal{S}_1 \) (or \( \theta \in \mathcal{U} \)),

\[
r^2(\theta) \leq r^2(I_*, \theta) \lesssim \frac{nV^2(\theta)}{k^2} + \sigma^2 k \log \left( \frac{en}{k} \right).
\]

Let \( \theta \in \mathcal{S}_1 \) (or \( \theta \in \mathcal{U} \)) and let us take \( k = k_* = \lceil (\frac{nV^2(\theta)}{\sigma^2 \log(en)})^{1/3} \rceil + 1 \). Now, if \( k_* = 1 \), we have \( nV^2(\theta) \leq \sigma^2 \log(en) \). If \( k_* > 1 \), then by definition of \( k_* \),

\[
\frac{nV^2(\theta)}{k_*^2} \leq n^{1/3} (\sigma^2 V(\theta) \log(en))^{2/3}.
\]

We conclude that if \( V(\theta) \leq V \), then for all \( \theta \in \mathcal{S}_1(V) \) and \( \theta \in \mathcal{U}(V) \),

\[
r^2(\theta) \lesssim \max \{ n^{1/3} (\sigma^2 V \log(en))^{2/3}, \sigma^2 \log(en) \}.
\]

Thus, we established the relation (6.1) and Corollary 2 follows with the minimax rate \( \max \{ n^{1/3} (\sigma^2 V)^{2/3}, \sigma^2 \} \) (up to a logarithmic factor) for the both classes \( \mathcal{S}_1(V) \) and \( \mathcal{U}(V) \) simultaneously.

Similarly, we establish that for all \( \theta \in \mathcal{C}(V) \)

\[
r^2(\theta) \lesssim \max \{ n^{1/5} (\sigma^4 V^{2/5} \log(en))^{4/5}, \sigma^2 \log(en) \}.
\]

This means that we also established the relation (6.1) and hence also Corollary 2 with the minimax rate (up to a logarithmic factor) for the class \( \mathcal{C}(V) \).

Below we present some further remarks and extensions for isotonic, unimodal and convex regressions.

6.9.2. Log factor and universality of the results

It should be recognized that we attain the minimax rates for the classes \( \mathcal{S}_k^\top, \mathcal{U}_k, \mathcal{C}_q, \mathcal{S}_1(V), \mathcal{U}(V) \) and \( \mathcal{C}(V) \) only up to a logarithmic factor. On the other hand, we obtain the optimal rates over the bigger scales \( \{ \Theta_k^\top, k \in [n] \} \) and \( \{ \Theta_k, k \in [n] \} \). Moreover, as consequence of our general results we have also solved the uncertainty quantification problem and the problem of structure recovery (in a weak sense). Our constants in the estimation results may be worse than those from the above mentioned references, but on the other hand we do not require that the vector \( \xi \) is normal and its coordinates are independent, only mild Condition (A1) is to be fulfilled.
Interestingly, this extra log factor in the local rate can also be seen as “price” for certain universality in the results. Indeed, recall that the results for the family of structures $I$ with corresponding linear spaces $L_I, I \in I$, cover the scale $\{\Theta^I_k, k \in [n]\}$. This in turn implies the minimax results for the scales $\{S^I_k, k \in [n]\}$ and $\{\mathcal{U}_k, k \in [n]\}$ (adaptively with respect to $k \in [n]$) and over the global shape-restricted classes $S^I(V)$ and $\mathcal{U}(V)$ of monotone and unimodal parameters, simultaneously for all the mentioned scales. Thus, at the log factor price, one approach handles several structures at once.

Actually our approach allows to extend the universality property even further. Indeed, let us unite the two structures $\bar{I} = I \cup I'$ and the corresponding families of the linear spaces $\{L_I, I \in \bar{I}\}$ and consider the resulting procedure. This procedure makes sense because the majorants for the both families are of the same order, so we only need to adjust a multiplicative constant in front of the majorant $\rho(s(I)) = 1 + 2|I|\log(\frac{|I|}{n})$ that will now handle the both families of structures. In doing so, we get the local result with the oracle rate over the both families at the price of a bigger multiple of the majorant. This means that the resulting procedure will mimic the oracle structure over the union of the two families, i.e., the resulting oracle rate will cover both scales $\{\Theta^I_k, k \in [n]\}$ and $\{\Theta^I_k, k \in [n]\}$ simultaneously. This in turn implies the minimax results for the scales $\{S^I_k, k \in [n]\}, \{\mathcal{U}_k, k \in [n]\}$ and $\{\mathcal{C}_q, q \in [n]\}$ (adaptively with respect to $k, q \in [n]$) and over the global shape-restricted classes $S^I(V), \mathcal{U}(V)$ and $\mathcal{C}(V)$ of monotone, unimodal and convex parameters, simultaneously for all the mentioned scales.

### 6.9.3. No EBR-like condition for shape-restricted structures

The last important aspect to discuss for this model and the considered shape-restricted structures is one peculiar phenomenon recently discovered by some researchers in related settings: for certain shape-restricted classes, the uniform coverage and optimal size properties in the uncertainty quantification problem can be derived without imposing any EBR-like condition. For example, one can construct a confidence ball for monotone $\theta$’s with a high coverage and a radius of the optimal order $n^{1/3}(\sigma^2 V)^{2/3}$, uniformly over monotone $\theta \in S^I(V)$ and without any EBR-like condition.

Let us show that we can also achieve this (up to a logarithmic factor) by using our approach. We consider only the family of structures $I$ (and the corresponding family of linear spaces) for modeling monotone and unimodal $\theta$’s, similar argument can be given for the family of structures $I'$. To ensure the EBR-condition, we simply restrict the family of structures $I$ to the subfamily $I_1 = \{I \in I : |I| \geq C(n/\sigma^2)^{1/3}\}$ for some sufficiently large $C > 0$. Then the results go through in the same way as before with the difference that the oracle rate is now $r^2(\theta) = \min_{\bar{I} \in I_1} \{||\theta - P_{\bar{I}}\theta||^2 + \sigma^2 \rho(s(I))\}$, with respect to the family $I_1$, rather than $I$. Since for any $I \in I_1, |I| \geq C(n/\sigma^2)^{1/3}$, this and Proposition 6 imply that for any $I \in I_1$ and any $\theta \in S^I(V)$ (or $\bar{\theta} \in \mathcal{U}(V)$) there exists a $\bar{\theta} \in \Theta^I_{\bar{I}}$ such that

$$||\theta - P_{\bar{I}}\theta||^2 \leq ||\theta - \bar{\theta}||^2 \leq C_1 \frac{n \nu^2}{|I|} \lesssim n^{1/3} \sigma^{4/3} \lesssim \sigma^2 |I| \lesssim \sigma^2 \rho(s(I)),$$
which ensures the EBR condition (3.10). Thus, the EBR condition is fulfilled automatically for the family of structures \( \mathcal{I}_1 \). At the same time, the oracle rate \( r^2(\theta) \) covers the both scales \( \mathcal{S}^\dagger(V) \) and \( \mathcal{U}(V) \). Indeed, by taking \( I_\ast \in \mathcal{I}_1 \) such that \(|I_\ast| = \lfloor C(n/\sigma^2)^{1/3} \rfloor + 1 \), we obtain that uniformly over \( \theta \in \mathcal{S}^\dagger(V) \cup \mathcal{U}(V) \)

\[
 r^2(\theta) \leq r^2(I_\ast, \theta) \leq C_1 n^{1/3} \rho^2(s(I_\ast)) + n^{1/3} \sigma^{4/3} \log(en),
\]

which is the minimax rate (up to a logarithmic factor) over the both classes \( \mathcal{S}^\dagger(V) \) and \( \mathcal{U}(V) \) simultaneously.

### 6.10. Dictionary learning

Dictionary learning can be considered as a linear regression problem when the design matrix and (sparse) vector of regressors are both unknown. The data \( Y = (Y_i)_{i \in [mn]} \) are observed according to the model:

\[
 Y = \bar{D}r + \sigma \xi,
\]

where \( \xi = (\xi_i, i \in [mn]), \xi_i \sim \text{N}(0,1) \), \( \bar{D} = \text{diag}\{D, \ldots, D\} \in \mathbb{R}^{mn \times mp} \) is an \( m \) block diagonal matrix with \( p \in \mathbb{N} \), whose block \( D = (D_1, \ldots, D_p) \in \mathbb{R}^{n \times p} \) is an unknown dictionary matrix, \( p \leq n \) without loss of generality, \( \sigma > 0 \) is the known noise intensity, \( r = (r^1, \ldots, r^m) \in \mathbb{R}^{mp} \) is a concatenation of unknown representations \( r^1, \ldots, r^m \in \mathbb{R}^p \) such that each entry \( r^j_i \) of each \( r^j \) comes from a (known) finite set of numbers: \( r^j_i \in \mathcal{R}_K = \{r_1, \ldots, r_K\} \) (for instance, \( \mathcal{R}_3 = \{-1, 0, 1\} \)), for some \( r_k \in \mathbb{R}, k \in [K] \). Recently, posterior contraction rate and oracle estimation results for this model were derived by [28] and [37], respectively. To the best of our knowledge, there are no local results on uncertainty quantification problem for dictionary learning.

In this model, we have \( \theta = \bar{D}r \). The structure \( I \) consists of two parts: \( m \) sparsity patterns \( I^m = (I_1, \ldots, I_m) \subseteq [p]^m \) (\( I_j \) determines which columns are taken in the \( j \)-th diagonal block \( D \) of \( \bar{D} \)) and \( m \) sparse versions of representation vectors \( R^m = (r^1_1, \ldots, r^m_1) \) according to the sparsity patterns \( I^m \), where \( r^j_i = (r^j_i, i \in I_j) \) with \( r^j_i \in \mathcal{R}_K, i \in I_j, j \in [m] \). We encode the structure \( I \) as \( I = (I^m, R^m) \), and the whole family of structures is

\[
 \tilde{\mathcal{I}} = \{(I^m, R^m) : r^j_i \in \mathcal{R}_K, i \in I_j, j \in [m]; I_k \subseteq [p], k \in [m]\}.
\]

The structural slicing mapping is defined as \( s(I) = (|I_k|, k \in [m]) \in \mathcal{S} = [p]_0^m \). Further, introduce the subfamily \( \mathcal{I}_1 \) of \( \tilde{\mathcal{I}} \):

\[
 \mathcal{I}_1 = \{I \in \tilde{\mathcal{I}} : np + l_K(I) \leq nm\},
\]

where the quantity \( l_K(I) \) is defined as

\[
 l_K(I) \triangleq \sum_{j \in [m]} |I_j| \log(\frac{\rho}{|I_j|}) + (\log K) \sum_{j \in [m]} |I_j|.
\]
This quantity has the meaning of the log of the cardinality of the structural layer \( \mathcal{I}_s(I) \) and its motivation to appear here will become clear later.

The structures in this model are modeled by the linear spaces

\[
\mathbb{L}_I = \{ \text{vec}(D_{I,1}r^1_I, \ldots, D_{I,m}r^m_I) \in \mathbb{R}^{nm} : D_{I,k} \in \mathbb{R}^{n \times |I_k|}, k \in [m] \},
\]

where \( I \in \mathcal{I} \triangleq \mathcal{I}_1 \cup \{ \bar{I} \} \) and \( \bar{I} \) is one special structure (the finest possible) such that \( s(\bar{I}) = (p, \ldots, p) \) (\( m \)-dimensional vector of \( p \)'s) and the associated linear space is \( \mathbb{L}_I = \{ \text{vec}(x^1, \ldots, x^m) \in \mathbb{R}^{nm} : x^j \in \mathbb{R}^n, j \in [m] \}. \) If some \( I_j = \emptyset \), then the corresponding column \( D_{I,j}r^j_I \) is the zero column.

In this case, \( \theta = \bar{D} \in \mathbb{R}^{n \times m} \) (recall that whenever appropriate we treat \( \theta \) as vector: \( \theta \in \mathbb{R}^{nm} \), \( d_I = \dim(\mathbb{L}_I) = n \mid \bigcup_{k \in [m]} I_k \mid \leq np \) for \( I \in \mathcal{I}_1 \) and \( d_I = \dim(\mathbb{L}_I) = nm \). The layer \( \mathcal{I}_s(I) \) consists of all the structures \( I \) which have the same \( s(I) = (|I_k|, k \in [m]) \). Clearly, \( |\mathcal{I}_s(I)\| = 1 \) because there is only one structure \( \bar{I} \) in the layer \( \mathcal{I}_s(I) \). To count the number of structures in \( \mathcal{I}_s(I) \) for \( I \in \mathcal{I}_1 \), notice that there are \( \prod_{j \in [m]} \binom{p}{|I_j|} \) possible choices of the sparsity patterns \( I^m \) and there are \( K \sum_{k \in [m]} |I_k| \) possible choices of sparse representation vectors \( R_{I^m} \), yielding the cardinality \( |\mathcal{I}_s(I)\| = \prod_{j \in [m]} \binom{p}{|I_j|} \times K \sum_{k \in [m]} |I_k| \). Hence,

\[
\log |\mathcal{I}_s(I)| \leq \sum_{j \in [m]} |I_j| \log(\frac{op}{|I_j|}) + (\log K) \sum_{j \in [m]} |I_j| = l_K(I) \quad \text{for} \quad I \in \mathcal{I}_1,
\]

where \( l_K(I) \) is introduced by (6.27). The last relation explains the origin of the quantity \( l_K(I) \). Since \( d_s(I) + \log |\mathcal{I}_s(I)| \leq np + l_K(I) \) for \( I \in \mathcal{I}_1 \) and \( d_s(I) + \log |\mathcal{I}_s(I)| = nm \), we take the majorant

\[
\rho(s(I)) = (np + l_K(I))1\{I \in \mathcal{I}_1\} + nm1\{I = \bar{I}\}, \tag{6.28}
\]

As for some previous models and structures, we have an elbow effect expressed basically by the quantity \( l_K(I) \) in the majorant and there is no need to consider the structures \( I \in \mathcal{I} \setminus \mathcal{I}_1 \), because these structures are dominated by the structure \( \bar{I} \), the same reasoning applies as in Remark 31.

Conditions (A1) and (A4) hold with \( d_s(I) = \dim(\mathbb{L}_I) \) in view of Remarks 5 and 18. Denote \( S_1 = \{ s(I) : I \in \mathcal{I}_1 \} \). Condition (A2) is fulfilled, since, according to Remark 7, for a sufficiently large \( \nu > 1 \)

\[
\sum_{I \in \mathcal{I}} e^{-\nu \rho(s(I))} \leq \sum_{I \in \mathcal{I}_1} e^{-\nu \rho(s(I))} + e^{-\nu nm} \leq e^{-\nu np} \sum_{s \in S_1} e^{-(\nu - 1)l_K(I)} + e^{-\nu nm}
\]

\[
\leq e^{-\nu np} \sum_{|I_1| = 0}^{p} \ldots \sum_{|I_m| = 0}^{p} e^{-(\nu - 1)l_K(I)} + e^{-\nu nm}
\]

\[
\leq e^{-\nu np} \left( \sum_{l = 0}^{p} e^{-(\nu - 1)l} \right)^m + 1 \leq \frac{e^{-\nu np}}{1 - e^{-\nu p}} + 1 \leq C_\nu,
\]

under the assumption that \( m \lesssim np \).
Remark 34. Notice the emerging condition $m \lesssim np$. This is not completely surprising: $m$ should not be too big in order not to have too many structures in the layers. Alternatively, instead of imposing this condition, we can make the majorant slightly bigger by setting $np \lor m$ instead of just $np$ in (6.28). Yet another fix would be to remove those structures $I$ from $\mathcal{I}_{1}$ for which $\sum_{j \in [m]} |I_{j}| < m$. One can show that in this case the above sum will be uniformly bounded.

As for the previous model (linear regression with mixture structure), there is no point in verifying Condition (A3) because the size and coverage claims (vi)-(vii) for the confidence ball $B(\theta, \hat{R}_{M})$ are stronger and more useful for this model and this structure than the corresponding claims (iv)-(v) for the confidence ball $B(\theta, \hat{R}_{M})$. Let us demonstrate that this model in essence does not suffer from the deceptiveness phenomenon, modulo the so-called highly structured parameters.

Indeed, as consequence of our general results, we obtain the local results (i)-(iv) and (vi)-(vii) of Corollary 1 for this case with the local rate $r^{2}(\theta) = \min_{I \in \mathcal{I}} \left\{ \| (1 - P_{I}) \theta \|^{2} + \sigma^{2} \tilde{\rho}(s(I)) \right\}$, where $\theta = \hat{\theta} r$, with majorant $\tilde{\rho}(s(I))$ defined above and $P_{I}$, the projection onto $\mathbb{L}_{I}$ defined above. The coverage property (v) for the confidence ball $B(\theta, \hat{R}_{M})$ can be shown to hold also, but uniformly only under the EBR, whereas the coverage property (vii) for the confidence ball $B(\theta, \hat{R}_{M})$ is uniform over the entire space $\theta = \hat{\theta} r \in \Theta = \mathbb{R}^{n \times m}$. The size $\hat{R}_{M}$ is of the oracle rate order (as the radius $\hat{R}_{M}$) uniformly in $\theta \in \Theta \setminus \hat{\Theta} = \mathbb{R}^{n \times m} \setminus \hat{\Theta}$, where $\hat{\Theta}$ is defined by (3.14). In this model the total number of observations is $N = nm$ and $\hat{\Theta} = \{ \theta \in \mathbb{R}^{n \times m} : \sigma^{-2} \| (1 - P_{I_{0}(\theta)}) \theta \|^{2} + np + l_{K}(I_{0}(\theta)) \lesssim \sqrt{N} = \sqrt{nm} \}$, where $I_{0}(\theta)$ is the oracle structure defined by (3.1). Clearly, $\hat{\Theta}$ is a “thin” subset of $\mathbb{R}^{n \times m}$ consisting of highly structured parameters $\theta$, in this case ultra-sparse parameters as their oracle structure must be very sparse: $l_{K}(I_{0}(\theta)) \lesssim C \sqrt{nm} - np$. Actually, $\hat{\Theta} = \emptyset$ if $m \lesssim p^{2}n$ which is a very mild assumption on the dimensions $n, p, m$ only. To summarize, under the assumption $m \lesssim p^{2}n$, in the dictionary learning model there is no deceptiveness issue at all.

Remark 35. Notice that we actually established stronger local results: the local rate $r^{2}(\theta) = \min_{I \in \mathcal{I}} \left\{ \| (1 - P_{I}) \theta \|^{2} + \sigma^{2} \tilde{\rho}(s(I)) \right\}$ is with a smaller majorant $\tilde{\rho}(s(I)) = \min \{ n | \cup_{i \in [m]} I_{i}| + \sum_{i \in [m]} |I_{i}| \log(\frac{\hat{\mu}_{I_{i}}}{\sigma^{2}}) + (\log K) \sum_{i \in [m]} |I_{i}|, nm \}$, under the assumption $m \lesssim np$. If we want to avoid the assumption $m \lesssim np$, then we should put $(n | \cup_{i \in [m]} I_{i}) \lor m$ instead of $n | \cup_{i \in [m]} I_{i}$ in the expression of the majorant $\tilde{\rho}(s(I))$.

Finally, by virtue of Corollary 2 the local results will imply global minimax adaptive results at once over all scales $\{ \Theta_{\beta}, \beta \in \mathcal{B} \}$ covered by the oracle rate $r^{2}(\theta)$ (i.e., for which (6.1) holds). Below we present one example of scale $\{ \Theta_{\beta}, \beta \in \mathcal{B} \}$ covered by the oracle rate $r^{2}(\theta)$.
6.10.1. Minimax results for sparse dictionary learning

Define the sparsity class for the dictionary learning model: for $\bar{s} \in [p]_0$, $\Theta_{SDL}(\bar{s}) = \cup \{ \mathbb{L}_I : I \in \mathcal{I}, |I| \leq \bar{s}, i \in [m] \}$. As is shown in [37], the minimax rate over $\Theta_{SDL}(\bar{s})$ is

$$r^2(\Theta_{SDL}(\bar{s})) \asymp \sigma^2 \min \{ np + m\bar{s}\log\left(\frac{np}{\bar{s}}\right), nm \}.$$  

For each $\theta = \bar{D}r \in \Theta_{SDL}(\bar{s})$ there exists $I_* \in \mathcal{I}$ such that $\theta \in \mathbb{L}_{I_*}$, hence $P_{I_*} \theta = \theta$ and $r^2(I_*(\theta), \theta) = \sigma^2 \rho(s(I_*))$. Further, since $|I_{si}(\theta)| \leq \bar{s}$, $i \in [m]$, we have $l_K(I_*(\theta)) = \sum_{i \in [m]} |I_{si}(\theta)| \log\left(\frac{ep}{l_{si}(\theta)}\right) + (\log K) \sum_{i \in [m]} |I_{si}(\theta)| \leq (1 + \log K)m\bar{s}\log\left(\frac{np}{\bar{s}}\right)$. Therefore, if $np + (1 + \log K)m\bar{s}\log\left(\frac{np}{\bar{s}}\right) \leq nm$, then $np + l_K(I_*(\theta)) \leq np + (1 + \log K)m\bar{s}\log\left(\frac{np}{\bar{s}}\right) \leq nm$, hence $I_*(\theta) \in \mathcal{I}_1$ and $r^2(\theta) \leq r^2(I_*(\theta), \theta) = \sigma^2 \rho(s(I_*(\theta))) = \sigma^2 (np + l_K(I_*(\theta))) \leq \sigma^2 (np + (1 + \log K)m\bar{s}\log\left(\frac{np}{\bar{s}}\right))$ in this case. Besides, recall that $P_\theta \theta = \theta$, so that $r^2(\theta) \leq r^2(I, \theta) = \sigma^2 \rho(s(I)) = \sigma^2 nm$. Piecing these together, we obtain that

$$r^2(\theta) \lesssim \sigma^2 \min \{ np + m\bar{s}\log\left(\frac{np}{\bar{s}}\right), nm \} \asymp r^2(\Theta_{SDL}(\bar{s})) \text{ for all } \theta \in \Theta_{SDL}(\bar{s}).$$

We thus established the relation (6.1) for this scale, and Corollary 2 follows with the minimax rate $r^2(\Theta_{SDL}(\bar{s}))$ defined above.

6.11. Mean matrix with submatrix sparsity

Suppose we observe a matrix $Y = (Y_{ij}) \in \mathbb{R}^{n_1 \times n_2}$:

$$Y_{ij} = \theta_{ij} + \sigma \xi_{ij}, \quad i \in [n_1], \quad j \in [n_2],$$

where $\sigma > 0$ is the known noise intensity, $\xi_{ij} \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$, $\theta = (\theta_{ij}) \in \mathbb{R}^{n_1 \times n_2}$ is an unknown high-dimensional parameter of interest with at most $k_1$ nonzero rows and $k_2$ nonzero columns, which are not necessarily consecutive. To the best of our knowledge, there are no local results on estimation, posterior contraction rate and uncertainty quantification problems for this model.

The submatrix sparsity structure is modeled by the linear subspaces

$$\mathbb{L}_I = \{ \text{vec}(x) \in \mathbb{R}^{n_1n_2} : x_{ij} = 0 \ \forall (i, j) \in \left( (I_1^c \times [n_2]) \cup ([n_1] \times I_2^c) \right) \},$$

where $I = (I_1, I_2) \in \mathcal{I} = \{ (I'_1, I'_2) : I'_1 \subseteq [n_1], I'_2 \subseteq [n_2] \}$ and $d_I = \dim(\mathbb{L}_I) = |I_1||I_2|$. The structural slicing mapping is $s(I) = (|I_1|, |I_2|)$, so that $\mathcal{S} = ([n_1][0], [n_2][0])$. Compute $|\mathcal{I}_{s(I)}| = \prod_{i=1}^2 \binom{n_i}{|I_i|}$, hence

$$\log |\mathcal{I}_{s(I)}| = \log \binom{n_1}{|I_1|} + \log \binom{n_2}{|I_2|} \leq \sum_{i=1}^2 |I_i| \log \left( \frac{e n_i}{|I_i|} \right).$$

Since $d_{s(I)} = d_I = |I_1||I_2|$ and $d_{s(I)} + \log |\mathcal{I}_{s(I)}| \leq d_I + \sum_{i=1}^2 |I_i| \log \left( \frac{en_i}{|I_i|} \right)$, we take the majorant $\rho(s(I)) = |I_1||I_2| + |I_1| \log \left( \frac{en_1}{|I_1|} \right) + |I_2| \log \left( \frac{en_2}{|I_2|} \right)$.  

Conditions (A1) and (A4) hold with \( d_{s(I)} = \text{dim}(\mathbb{L}_I) \) in view of Remarks 5 and 18. Condition (A2) is fulfilled, since, according to Remark 7, for any \( \nu > 1 \)
\[
\sum_{I \in \mathcal{I}} e^{-\nu \rho(s(I))} \leq \sum_{|I_1| = 0}^{n_1} \left( \frac{en_1}{|I_1|} \right)^{-(\nu-1)|I_1|} \sum_{|I_2| = 0}^{n_2} \left( \frac{en_2}{|I_2|} \right)^{-(\nu-1)|I_2|} \leq \frac{1}{1-e^{-\nu}} = C_\nu.
\]
For any \( I^0, I^1 \in \mathcal{I} \) define \( I' = I'(I^0, I^1) = (I^0 \cup I^1, I^0 \cup I^1) \). Then \( (\mathbb{L}_{I^0} \cup \mathbb{L}_{I^1}) \subseteq \mathbb{L}_{I'} \) and \( \rho(s(I')) \leq \rho(s(I^0)) + \rho(s(I^1)), \) which entails Condition (A3).

As consequence of our general results, we obtain the local results of Corollary 1 for this case with the local rate \( r^2(\theta) = \min_{I \in \mathcal{I}} \{ \| \theta - P_I \theta \|^2 + \sigma^2 \rho(s(I)) \} \). In turn, by virtue of Corollary 2 the local results will imply global minimax adaptive results at once over all scales \( \{ \Theta_\beta, \beta \in \mathcal{B} \} \) covered by the oracle rate \( r^2(\theta) \) (i.e., for which (6.1) holds). Below we present the example of scales \( \{ \Theta_\beta, \beta \in \mathcal{B} \} \) covered by the oracle rate \( r^2(\theta) \).

### 6.11.1. Minimax results for \( \mathcal{F}(k_1, k_2, n_1, n_2) \)

Let \( \mathcal{F}(k_1, k_2, n_1, n_2) \) be the collection of matrices \( \theta = (\theta_{ij}) \in \mathbb{R}^{n_1 \times n_2} \) with at most \( k_1 \) nonzero rows and \( k_2 \) nonzero columns, which are not necessarily consecutive. Classes \( \mathcal{F}(k_1, k_2, n_1, n_2) \) were introduced in [41]. In our notation, \( \mathcal{F}(k_1, k_2, n_1, n_2) = \bigcup_{I \in \mathcal{I} : |I_1| \leq k_1, |I_2| \leq k_2} \mathbb{L}_I \). As is shown in [41], the minimax rate over \( \mathcal{F}(k_1, k_2, n_1, n_2) \) is
\[
r^2(\mathcal{F}(k_1, k_2, n_1, n_2)) \asymp \sigma^2(k_1 k_2 + k_1 \log(\frac{en_1}{k_1}) + k_2 \log(\frac{en_2}{k_2})).
\]

On the other hand, for each \( \theta \in \mathcal{F}(k_1, k_2, n_1, n_2) \) there exists \( I_* \in \mathcal{I} \) such that \( \theta \in \mathbb{L}_{I_*} \) and \( |I_1| \leq k_1 \) and \( |I_2| \leq k_2 \). Hence, \( P_{I_*} \theta = \theta \) and
\[
r^2(\theta) \leq r^2(I_*, \theta) = \sigma^2 \rho(s(I_*)) = \sigma^2(|I_1||I_2| + |I_1| \log(\frac{en_1}{|I_1|}) + |I_2| \log(\frac{en_2}{|I_2|})) \leq \sigma^2(k_1 k_2 + k_1 \log(\frac{en_1}{k_1}) + k_2 \log(\frac{en_2}{k_2}))) \asymp r^2(\mathcal{F}(k_1, k_2, n_1, n_2)).
\]

We thus established the relation (6.1) for this scale, and Corollary 2 follows with the minimax rate \( r^2(\mathcal{F}(k_1, k_2, n_1, n_2)) \) defined above.

### 6.12. Covariance matrix with banding or sparsity structure

Suppose we observe \( n \) iid \( p \)-dimensional vectors \( X_1, \ldots, X_n, X_i = (X_{i1}, \ldots, X_{ip})^T, i \in [n], \) with \( \mathbb{E}X_i = 0, \mathbb{E}(X_{ij})^4 \leq C_X, (i, j) \in [p] \times [p], \) and unknown covariance matrix \( \mathbb{E}(X_i X_i^T) = \Sigma, i \in [n] \). Without loss of generality, we set \( C_X = 1 \) for the rest of this section. Let \( \mathcal{C} \subseteq \mathbb{R}^{p \times p} \) denote the set of all \( p \)-dimensional covariance matrices. Assume that for some (known and independent of \( p \)) \( \varepsilon_0 > 0 \),
\[
\Sigma \in \mathcal{C}_{\varepsilon_0} = \{ M \in \mathcal{C} : \varepsilon_0 \leq \lambda_{\text{min}}(M) \leq \lambda_{\text{max}}(M) \leq \varepsilon_0^{-1} \}.\]
Here, \( \lambda_{\text{max}}(M) \) and \( \lambda_{\text{min}}(M) \) are the maximum and minimum eigenvalues of \( M \). We assume that \( X_i^{\text{ind}} \sim N(0_n, \Sigma) \), where \( 0_n \) is the \( n \)-dimensional vector of zeros. The normality assumption is not important to us, this only plays a role in that we can use certain auxiliary result below (Proposition 7) which is available only for the normal case.

We are interested in recovering the covariance matrix \( \Sigma = \{\Sigma_{ij}\}_{1 \leq i,j \leq p} \) which is assumed to have the banding or sparsity structure, to be specified later. The maximum likelihood estimator of \( \Sigma \) is \( \tilde{\Sigma} = \frac{1}{n} \sum_{l=1}^{n} (X_l - \bar{X})(X_l - \bar{X})^T = \frac{1}{n} \sum_{l=1}^{n} X_l X_l^T - \bar{X} \bar{X}^T \), where \( \bar{X} = \frac{1}{n} \sum_{l=1}^{n} X_l \). Since \( \bar{X} \bar{X}^T \) is a higher order term (see Remark 1 in [19]), we shall ignore this term and focus on the dominating term \( \frac{1}{n} \sum_{l=1}^{n} X_l X_l^T \) for estimating the covariance matrix \( \Sigma \).

Let \( Y = (Y_{ij})_{i,j \in [p]} = \frac{1}{n} \sum_{l=1}^{n} X_l X_l^T \), \( Y = \text{vec}([Y_{ij}]) = (Y_{11}, Y_{12}, \ldots, Y_{pp})^T \). We obtained the following model:

\[
Y_{ij} = \Sigma_{ij} + \sigma_n \xi_{ij}, \quad i, j \in [p], \tag{6.29}
\]

where \( \sigma_n \xi_{ij} = Y_{ij} - \mathbb{E}Y_{ij} = Y_{ij} - \Sigma_{ij} \), so that \( \mathbb{E} \xi_{ij} = 0 \) and

\[
\sigma_n^2 \text{Var}(\xi_{ij}) = \frac{1}{n} \text{Var}(X_i X_j^T) = \frac{1}{n} \mathbb{E}(X_i X_j^T)^2 \leq \frac{1}{n} \mathbb{E}(X_i^4) \frac{1}{n} \mathbb{E}(X_j^4) = \frac{1}{n}.
\]

The parameter \( \sigma_n \) will be chosen later, for now it is any sequence \( \sigma_n \in [0,1] \). We thus have a particular case of general framework model (1.1), where the parameter of interest is now denoted by \( \Sigma \) instead of \( \theta \). Recall that we work with the usual norm of vectorized version of the parameter \( \Sigma = (\Sigma_{ij})_{i,j \in [p]} \), that is, if \( \Sigma \) is seen as matrix, then \( ||\Sigma|| \) means its Frobenius norm. We denote the probability measure of \( Y \) from the model (6.29) by \( \mathbb{P}_\Sigma \), and the corresponding expectation \( \mathbb{E}_\Sigma \).

### 6.12.1. Banding structure

Assume that the covariance matrix \( \Sigma = (\Sigma_{ij})_{i,j \in [p]} \) has a banding structure, i.e., \( \Sigma_{ij} = 0 \) for all \( i, j \in [p] \) such that \( |i - j| > I \) for some \( I \in [p] \). To model this structure, define the linear spaces

\[
L_I = \{ \text{vec}(x) \in \mathbb{R}^{p^2} : x_{ij} = x_{ji} \forall i, j \in [p] ; x_{ij} = 0 \text{ if } |i - j| > I \}, \quad I \in \mathcal{I} = [p].
\]

Then \( ||\Sigma - P_I \Sigma||^2 = \sum_{i,j > I} \Sigma_{ij}^2 \) is the structural slicing mapping \( s(I) = I, \mathcal{S} = [p-1]_0, \log |\mathcal{S}| = 0, d_s(I) = p + I(p-I+1)/2 \), leading to the dominant \( \rho(s(I)) = d_I = p + I(p-I+1)/2 \).

Condition (A2) is fulfilled, since \( \sum_{I \in \mathcal{I}} e^{-\nu \rho(s(I))} \leq \sum_{s \in \mathcal{S}} e^{-\nu s} \leq \frac{e^\nu}{e^{-1}} = C_\nu \) for any \( \nu > 0 \). However, in order to derive at least the local estimation and posterior contraction results, we also need Condition (A1). This condition is now not easy to check since the errors \( \xi_{ij} \)’s are dependent in the model (6.29). We apply the following strategy (in the same spirit as in Section 6.5): introduce certain event and establish that the probability of this
event is exponentially small (in \( n \)); next, under this event establish Condition (A1); finally, combine these two facts to derive the local estimation and posterior contraction results.

The following proposition (formulated in our notation) is Lemma 12 from Appendix of supplementary material [38] and is given here for completeness, its proof can be found in [38].

**Proposition 7.** Let \( \nu_{ij} = \max\{ (\Sigma_{ii} \Sigma_{jj})^{1/2} - \Sigma_{ij}, (\Sigma_{ii} \Sigma_{jj})^{1/2} + \Sigma_{ij} \} \), \( i, j \in [p] \). Then for any \( t \in [0, \nu_{ij}/2] \)
\[
\mathbb{P}(\sigma_n | \xi_{ij} | \geq t) \leq 4\exp\{-\frac{3n^2}{16\nu_{ij}^2}\}.
\]

The relation \( \mathbb{P}(\max_{i,j \in [p]} | \xi_{ij} | \geq t) \leq \sum_{i,j \in [p]} \mathbb{P}(| \xi_{ij} | \geq t) \) and Proposition 7 imply that, for the event \( E = \{ \max_{i,j \in [p]} | \xi_{ij} | \leq t_0 \} \) with \( t_0 = \frac{\min_{i,j \in [p]} \nu_{ij}}{\sqrt{n}} \),
\[
\mathbb{P}(E^c) = \mathbb{P}(\max_{i,j \in [p]} | \xi_{ij} | \geq t_0) \leq H' \exp\{-c_1 n\sigma_n^2 + c_2 \log p\}, \tag{6.30}
\]
where \( \nu_{ij} \) is defined in Proposition 7, \( H' = 4, 0 < c_1 = \frac{3\varepsilon_0^2}{320} \leq \frac{3}{80} \min_{i,j \in [p]} \nu_{ij}^2 \) (because \( \varepsilon_0 \leq \min_{i,j \in [p]} \nu_{ij} \leq \max_{i,j \in [p]} \nu_{ij} \leq 2\varepsilon_0^{-1} \)) and \( c_2 = 2 \). For (6.30) to be useful, we need to assume the asymptotic relation \( \log p = o(n\sigma_n^2) \) as \( n \to \infty \).

By the assumptions on \( \Sigma \), we have that \( \min_{i,j \in [p]} \nu_{ij} \leq 2\varepsilon_0^{-1} \), so that \( t_0^2 = \min_{i,j \in [p]} \nu_{ij}^2/5 \leq \frac{4}{5\varepsilon_0^2} \). Using this and (6.30), we ensure Condition (A1) under the event \( E = \{ \max_{i,j \in [p]} | \xi_{ij} | \leq t_0 \} \) with \( \alpha = 1 \wedge (5\varepsilon_0^2)/4 \). Exactly,
\[
\mathbb{E} \exp\{\alpha|P_I \xi|^2\}1\{E\} = \mathbb{E} \exp\{\alpha \sum_{i,j \in [p]} \xi_{ij}^2\}1\{\max_{i,j \in [p]} | \xi_{ij} | < t_0\}
\leq \exp\{\alpha t_0^2 (p + (p - (I + 1)/2))\} \leq \exp\{d_{s(I)}\}. \tag{6.31}
\]
Condition (A3) holds as well. Indeed, for any \( I_0, I_1 \in \mathcal{I} \) take \( I' = I_0 \vee I_1 \) and verify that \( (\mathbb{L}_{I_0} \cup \mathbb{L}_{I_1}) \subseteq \mathbb{L}_{I'} = \mathbb{L}_{I_0} + \mathbb{L}_{I_1} \) and \( \rho(s(I')) \leq \rho(s(I_0)) + \rho(s(I_1)) \).

The oracle rate is in this case \( r^2(\Sigma) = \min_{I \in \mathcal{I}} r^2(I, \Sigma) \), where
\[
r^2(I, \Sigma) = \| \Sigma - P_I \Sigma \|^2 + \sigma_n^2 \rho(s(I)) = \sum_{|i-j| > I} \Sigma_{ij}^2 + \sigma_n^2 (p + (p - (I + 1)/2)),
\]
and the EBR-set \( \Theta_{eb} = \Theta_{eb}(t) \) is given by (3.10), but now in terms of the bias and variance parts of the oracle rate \( r^2(\Sigma) \).

We have thus verified the conditional version of Condition (A1) (under the event \( E \)) and Conditions (A2) and (A3) for the model (6.29) with the banding structure. This means that we can derive results on estimation, posterior contraction and uncertainty quantification for this model. These are the counterparts of claims (i)-(v) of Corollary 1 summarized by Theorem 6 below. To the best of our knowledge, there are no local results on estimation, posterior contraction rate and uncertainty quantification problems for this model.
A couple of conventions concerning notation in Theorem 6: as compared to the general framework notation, in the model (6.29), the parameter of interest is denoted by \( \Sigma \) instead of \( \theta \) and the corresponding estimator becomes \( \hat{\Sigma} \) instead of \( \hat{\theta} \); in the posteriors for \( \Sigma \) we use the variable \( \Sigma \) to distinguish it from the “true” \( \Sigma \in \mathcal{C}_0 \). We keep the same notation for all other quantities involved as in the general framework (like \( \hat{r}, \hat{R}_M, B(\Sigma, \hat{R}_M) \)), with the understanding that these are specialized for the model (6.29) with the banding structure and the oracle rate \( r^2(\Sigma) \).

**Theorem 6.** Let the constants \( M_0, M_1, M_3, H_0, H_1, H_2, H_3, m_0, m_1, m_2, m_3, c_2, c_3, C_\nu, H', \tilde{c}_1, \tilde{c}_2 \) be defined in Theorems 1-3 and (6.30). Then for any \( M \geq 0 \),

\[
\sup_{\Sigma \in \mathcal{C}_0} \mathbb{E}_\Sigma \hat{\pi} \left( \| \Sigma - \Sigma \|^2 \geq M_0 r^2(\Sigma) + M \sigma_n^2 | Y \right) \leq H' e^{-\tilde{c}_1 n \sigma_n^2 + \tilde{c}_2 \log p + H_0 e^{-m_0 M}},
\]

\[
\sup_{\Sigma \in \mathcal{C}_0} \mathbb{P}_\Sigma \left( \| \Sigma - \Sigma \|^2 \geq M_1 r^2(\Sigma) + M \sigma_n^2 \right) \leq H' e^{-\tilde{c}_1 n \sigma_n^2 + \tilde{c}_2 \log p + H_1 e^{-m_1 M}},
\]

\[
\sup_{\Sigma \in \mathcal{C}_0} \mathbb{E}_\Sigma \hat{\pi} \left( I : r^2(\Sigma, \Sigma) \geq c_3 r^2(\Sigma) + M \sigma_n^2 | Y \right) \leq H' e^{-\tilde{c}_1 n \sigma_n^2 + \tilde{c}_2 \log p + C_\nu e^{-c_2 M}},
\]

\[
\sup_{\Sigma \in \mathcal{C}_0} \mathbb{P}_\Sigma \left( \hat{r}^2 \geq M_3 r^2(\Sigma) + (M + 1) \sigma_n^2 \right) \leq H' e^{-\tilde{c}_1 n \sigma_n^2 + \tilde{c}_2 \log p + H_3 e^{-m_3 M}},
\]

\[
\sup_{\Sigma \in \mathcal{C}_0} \mathbb{P}_\Sigma \left( \Sigma \notin B(\hat{\Sigma}, \hat{R}_M) \right) \leq H' e^{-\tilde{c}_1 n \sigma_n^2 + \tilde{c}_2 \log p + H_2 e^{-m_2 M}}.
\]

Let us outline the idea of the proof (which is omitted) of the first claim of the above theorem; the same reasoning applies to the remaining claims. The expectation of the empirical Bayes posterior probability \( \mathbb{E}_\Sigma \pi = \mathbb{E}_\Sigma \hat{\pi} \left( \| \Sigma - \Sigma \|^2 \geq M_0 r^2(\Sigma) + M \sigma_n^2 | Y \right) \) is bounded by the sum of two terms \( \mathbb{E}_\Sigma \pi \leq \mathbb{P}_\Sigma (\Sigma' \in \mathcal{E}) + \mathbb{E}_\Sigma \pi I_{\mathcal{E}} \). The first term is evaluated by using (6.8) (obtaining the bound \( H' e^{-\tilde{c}_1 n \sigma_n^2 + \tilde{c}_2 \log p} \)); the second term is evaluated exactly in the same way as in the proof Theorem 1 because Condition (A1) is fulfilled under the event \( \mathcal{E} \) according to (6.31). Counterparts of assertions (ii) and (iii) of Theorem 2 can also be formulated and proved in the same way.

As to the choice of \( \sigma_n^2 \), this quantity is in the oracle rate, so that we would want it to be as small as possible. On the other hand, we want the claims of the theorem to be non-void, which is ensured only if \( \sigma_n^2 \log p \), or \( \sigma_n^2 \geq \frac{C \log p}{n} \), for sufficiently large \( C > 0 \). In the sequel we take therefore \( \sigma_n^2 = \frac{C \log p}{n} \). An extra log factor thus appeared which will also enter the minimax rates in the global results. We conjecture that one can get rid of that factor by using more accurate concentration inequalities when establishing Condition (A1).

As usually, the local results of Theorem 6 will imply global minimax adaptive results at once over all scales \( \{ \Theta_\beta, \beta \in \mathcal{B} \} \) covered by the oracle rate \( r^2(\Sigma) \) (i.e., for which (6.1) holds). Below we present the example of scales \( \{ \Theta_\beta, \beta \in \mathcal{B} \} \) covered by the oracle rate \( r^2(\Sigma) \).
6.12.2. Minimax results for the scale $\{G_\beta, \beta > 0\}$

For $\beta, L, \varepsilon_0 > 0$, define

$$G_\beta = G_\beta(L, \varepsilon_0^{-1}) = \{\Sigma \in C_{\varepsilon_0} : |\Sigma_{ij}| \leq L|i - j|^{-(\beta+1)} \text{ for } i \neq j\}.$$ 

The rate $r^2(G_\beta) = \min \{pn^{-\frac{2\beta+1}{2(\beta+1)}}, p^2n^{-1}\}$ is minimax over the class $G_\beta$ under the Frobenius norm; see [19]. If $(\frac{n}{\log p})^{\frac{1}{2(\beta+1)}} \leq p$, taking $I_* = \lceil (n/\log p)^{\frac{1}{2(\beta+1)}} \rceil$ and recalling $\sigma_n^2 = \frac{C \log p}{n}$, we derive that, uniformly in $\Sigma \in G_\beta$,

$$r^2(\Sigma) \leq r^2(I_*, \Sigma) = \sum_{|i-j|>I_*} \Sigma_{ij}^2 + \sigma_n^2(p + I_* (p - (I_* + 1)/2)) \lesssim pI_*^{-(2\beta+1)} + \frac{pL \log p}{n} \lesssim p(\frac{n}{\log p})^{-\frac{2\beta+1}{2(\beta+1)}}.$$

If $(\frac{n}{\log p})^{\frac{1}{2(\beta+1)}} > p$, we take $I_* = p$ to derive $\sup_{\Sigma \in G_\beta} r^2(\Sigma) \lesssim p^2(\frac{n}{\log p})^{-1}$.

To summarize, we established that

$$\sup_{\Sigma \in G_\beta} r^2(\Sigma) \lesssim \min \{p(\frac{n}{\log p})^{-\frac{2\beta+1}{2(\beta+1)}}, p^2(\frac{n}{\log p})^{-1}\} = \tilde{r}^2(G_\beta),$$

where $\tilde{r}^2(G_\beta)$ is the minimax rate (up to a logarithmic factor) for the class $G_\beta$. Then the last relation and Theorem 6 imply the global minimax results for the scale $\{G_\beta, \beta > 0\}$. These results will look as the ones from Theorem 6 with the difference that the class $G_\beta$ stands instead $C_{\varepsilon_0}$ and the rate $\tilde{r}^2(G_\beta)$ stands instead of $r^2(\Sigma)$. For the results to be most useful, we take $\sigma_n^2 = \frac{C \log p}{n}$ with sufficiently large $C > 0$ and $M = M_n \to \infty$ as $n \to \infty$ such that $M_n \sigma_n^2 \leq \tilde{r}^2(G_\beta)$.

Remark 36. The obtained local and global results on uncertainty quantification for the covariance matrix with a banding structure are new to the best of our knowledge. Notice however that we derived only the uncertainty quantification results based on the EBR condition, whereas counterparts of claims (vi)-(vii) of Corollary 1 could not be established because we were unable to verify Condition (A4). The point is that the set $\hat{\Theta}$ of highly structured parameters defined by (3.14) is empty in this case: as $N = p^2$,

$$r^2(\Sigma) \geq \sigma_n^2 \rho(s(I)) \gtrsim \sigma_n^2 p = \sigma_n^2 N^{1/2}.$$

This means that the uncertainty quantification claims based on Condition (A4) would be more valuable for this model because they are free of the deceptiveness phenomenon. Indeed, if we would have established Condition (A4), then the confidence ball $B(\hat{\Sigma}, \hat{R}_M)$ would have been of asymptotically full coverage and of the optimal oracle size, uniformly over $C_{\varepsilon_0}$ (because $\hat{\Theta}$ turns out to be empty in this case). It is an open problem to verify Condition (A4) for the model (6.29), the main issue is to find an appropriate statistics $V(Y')$ for which the second relation of Condition (A4) is fulfilled.
6.12.3. Sparsity structure

Here we briefly discuss the case of sparsity structure for the model (6.29).

Denote by $\Sigma_{-,i}$ the $i$-th column of $\Sigma$ with $\Sigma_{ii}$ removed. Let $p \geq 2$. For any $i \in [p]$ the vector $\Sigma_{-,i} \in \mathbb{R}^{p-1}$ is assumed to be sparse so that $\Sigma_{ki} = 0$, $k \notin I_i$ (or $\Sigma_{ik} = 0$), where $I_i \subseteq [p] \setminus \{i\}$. To model this sparsity structure, introduce the linear spaces

$$\mathbb{L}_I = \{ \text{vec}(x) \in \mathbb{R}^p : x_{ki} = 0, \ k \notin I_i, \ i \in [p] \},$$

where the structure is $I = (I_1, \ldots, I_p) \in \mathcal{I} \triangleq \{(J_1, \ldots, J_p) : J_i \subseteq [p] \setminus \{i\}, i \in [p]\}$. Then $||\Sigma - P_I \Sigma||^2 = \sum_{i \in [p]} \sum_{j \notin I_i} \Sigma_{ji}^2$, $\dim(\mathbb{L}_I) = p + \sum_{i=1}^p |I_i|$. The structural slicing mapping is $s(I) = (|I_1|, \ldots, |I_p|) \in \otimes_{i \in [p]} [p-1]_0 = \mathcal{S}$, $\log |I_{s(I)}| = \sum_{i \in [p]} \log (p-1) \leq \sum_{i \in [p]} |I_i| \log \left(\frac{(p-1)}{|I_i|}\right)$, $d_{s(I)} = p + \sum_{i \in [p]} |I_i|$. Thus, we take the majorant $\rho(s(I)) = p + \sum_{i \in [p]} |I_i| \log \left(\frac{(p-1)}{|I_i|}\right)$.

Condition (A2) is fulfilled, since, according to Remark 7,

$$\sum_{I \in \mathcal{I}} e^{-\nu \rho(s(I))} \leq e^{-\nu p} \sum_{s_1=0}^{p-1} e^{-(\nu-1)s_1} \cdots \sum_{s_p=0}^{p-1} e^{-(\nu-1)s_p} \leq \frac{e^{-\nu p}}{(1-e^{-\nu})^p} = C_\nu,$$

for sufficiently large $\nu > 1$.

Next, along the same lines as in Section 6.12.1, we can verify the conditional version of Condition (A1) (under event $E$) and Conditions (A2) and (A3) for the model (6.29) with the sparsity structure. This means that we can derive results on estimation, posterior contraction and uncertainty quantification for the model (6.29), now with the sparsity structure. We can readily formulate a theorem containing the local results for this structure: it will take the form of Theorem 6 with the oracle rate $r^2(\Sigma) = \min_{I \in \mathcal{I}} \left\{ ||\Sigma - P_I \Sigma||^2 + \sigma_n^2 \rho(s(I)) \right\}$, where $\sigma_n^2 = C \log p$ with sufficiently large $C > 0$. To the best of our knowledge, there are no local results on estimation, posterior contraction rate and uncertainty quantification problems for this model. Also for this sparsity structure we have the same issue (described in Remark 36) with Condition (A4) as for the banding structure.

Finally, consider one scale covered by the oracle rate for the sparsity structure.

6.12.4. Minimax results for weak $\ell_q$-balls

Recall the weak $\ell_q$ ball of radius $c$ in $\mathbb{R}^m$ containing elements with fast decaying ordered magnitudes of components,

$$B_q^m(c) = \{ \zeta \in \mathbb{R}^m : |(\zeta|_{(k)})^q \leq c k^{-1}, \ k \in [m] \},$$

where $|(\zeta|_{(k)})$ denotes the $k$th largest element in magnitude of the vector $\zeta$. For $0 \leq q < 1$, define the class $\mathcal{G}_q(c_n,p)$ of covariance matrices by

$$\mathcal{G}_q(c_n,p) = \{ \Sigma \in \mathcal{C}_0 : \Sigma_{-,j} \in B_q^{p-1}(c_n,p), j \in [p] \},$$
that is, each column \( \Sigma_{-j} \) of \( \Sigma \in \mathcal{G}_q(c_{n,p}) \) must be in a weak \( \ell_q \) ball, \( j \in [p] \). The minimax estimation rate over this class is 
\[
\sup_{\Sigma \in \mathcal{G}_q(c_{n,p})} r^2(\Sigma) \leq \sup_{\Sigma \in \mathcal{G}_q(c_{n,p})} r^2(I^*, \Sigma)
\]
\[
\leq \mathcal{N} \sum_{i \in [p]} \sum_{j \notin I^*_i} \Sigma^2_{ji} + \sigma_n^2 \big( p + pp^* \log \left( \frac{e(p-1)}{p^*} \right) \big)
\]
\[
\leq \mathcal{N} \sum_{j > p^*} j^{-2/q} + \sigma_n^2 \left( \frac{pc_{n,p}(\log p)^{1-q/2} + \sigma_n^2}{p^*} \right) \log p.
\]

This relation and the local results imply the global minimax results (up to the logarithmic factor \( \log p \)) on estimation, posterior contraction and uncertainty quantification for the model (79) for the scale \( \mathcal{G}_q(c_{n,p}) \).

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