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Koch-Medina, Pablo; Moreno-Bromberg, Santiago; Ravanelli, Claudia; Sikic, Mario

Abstract: We study capital management and investment decisions of a value-maximizing insurance firm with a broad ownership base in a discrete-time setting. We highlight that the valuation measure used to determine the value of the cash flows to shareholders should reflect two economically sound requirements: market-consistency and indifference to idiosyncratic risk. We provide a rigorous construction of this economic valuation measure and use it to derive the optimal capital-management and investment strategies that realize the economic value of the firm. Our objective is to shed light on the controversial question of whether insurers should invest in liquidly-traded risky assets. Decomposing firm value into net tangible value, default option value, and franchise value, we find that whether to take investment risk is optimal or not essentially depends on the tradeoff between the impact of investment risk on the owner’s option to default and on the firm’s franchise value. A variety of numerical examples illustrate how changes in the regulatory and financial environment can result in materially different optimal investment strategies.

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Revisiting optimal investment strategies of value-maximizing insurance firms

Pablo Koch-Medina a,b,*, Santiago Moreno-Bromberg a, Claudia Ravanelli a, Mario Šikić a

a Center for Finance and Insurance, University of Zurich, Switzerland
b Swiss Finance Institute, Switzerland

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A B S T R A C T
We study capital management and investment decisions of a value-maximizing insurance firm with a broad ownership base in a discrete-time setting. We highlight that the valuation measure used to determine the value of the cash flows to shareholders should reflect two economically sound requirements: market-consistency and indifference to idiosyncratic risk. We provide a rigorous construction of this economic valuation measure and use it to derive the optimal capital-management and investment strategies that realize the economic value of the firm. Our objective is to shed light on the controversial question of whether insurers should invest in liquidly-traded risky assets. Decomposing firm value into net tangible value, default option value, and franchise value, we find that whether to take investment risk is optimal or not essentially depends on the tradeoff between the impact of investment risk on the owner’s option to default and on the firm’s franchise value. A variety of numerical examples illustrate how changes in the regulatory and financial environment can result in materially different optimal investment strategies.

0. Introduction

This paper revisits questions related to the capital and investment strategies of publicly-traded, limited-liability insurance firms with a diffuse shareholder base. By capital strategy we mean how much dividends the firm should pay and how much capital it should raise and by investment strategy how much investment risk it should take. Of these, the most controversial, both amongst academics and practitioners, is the question of whether insurance firms should take investment risk at all. In reality, most insurance firms do engage, to varying degrees, in risk taking on the asset side. Indeed, some practitioners view insurance firms as little more than investment funds that are leveraged through the sale of insurance strategies. In Warren Buffet’s letters to shareholders in the annual reports of Berkshire Hathaway, the attractiveness of the insurance business is typically explained as follows: “... insurers receive premiums upfront and pay claims later. In extreme cases, such as claims arising from exposure to asbestos, payments can stretch over many decades. This collect-now, pay-later model leaves P/C companies holding large sums – money we call “float” – that will eventually go to others. Meanwhile, insurers get to invest this float for their own benefit.”

Regardless of what insurers actually do, academic research has tried to provide a normative answer to what insurers should do. In this respect, Froot (2007) states that “financial intermediaries should shed all liquid risks in which they have no ability to outperform and devote their entire risk budgets toward an optimally diversified portfolio in exposures where they have an edge.” On the other extreme, some authors, e.g., Højgaard and Taksar (2004), conclude that investment in the risky asset is always optimal. In this paper, we present a more nuanced picture, so it is interesting to understand what is driving these absolute and antipodal answers in Froot (2007) and Højgaard and Taksar (2004). Both of these papers seek, as we do, to identify the optimal investment strategy of a value-maximizing insurance firm, where firm value

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corresponds to the present value of cash flows to shareholders. Thus, the specification of cash flows to shareholders and of the present value rule will ultimately determine which investment strategies are optimal. As we will argue below, the model in Froot (2007) suffers from a misspecification of cash flows to shareholders because it ignores the option to default, and the model in Højgaard and Taksar (2004) from a misspecification of the present value rule because it values cash flows using expectations with respect to the “physical” probability measure.

The economic valuation measure

The present-value rule advocated in this paper is in line with a general shift in perspective witnessed over the past decades which favors economic over traditional accounting approaches. The rationale for our advocacy relies on the following argument. In the absence of financial frictions, a diffuse shareholder base is typically associated with shareholders being indifferent to idiosyncratic risk because it can be diversified; see e.g. Mayers and Smith (1982). For a company that is not exposed to financial-market risk, this results in a present-value rule in which the physical probability is the valuation measure. However, for financial firms or, more generally, for firms whose cash flows to shareholders depend on financial-market prices, care needs to be taken when selecting the appropriate valuation measure. Indeed, the valuation measure must be simultaneously consistent with financial-market prices and with the firm’s indifference toward idiosyncratic risk: it needs to reproduce market prices when applied to the payoffs of traded assets and coincide with expected cash flows discounted at the risk-free rate when applied to cash flows that contain only diversifiable risk. Since our framework is intrinsically one of incomplete markets, there are infinitely many valuation measures that are consistent with market prices. One of our contributions is to show that only one of them is consistent with indifference to idiosyncratic risk. We call it the economic valuation measure. The valuation of cash flows to shareholders then entails taking expected values with respect to this special measure and discounting them at the risk-free rate. Indeed, the economic valuation measure is the only valuation measure that should be used when determining the present value of cash flows to shareholders of any publicly-traded firm with a diffuse shareholder base and, as a useful special test case, of cash flows of an investment fund investing in liquidly-traded assets that has a diffuse ownership base. We emphasize that the assumption of a diffuse shareholder base is critical to be able to postulate that all shareholders will agree on this single valuation measure. With several owners but without a diffuse ownership base, the owners may not be able to agree on which measure to use.

The insistence on market consistency is not without justification. The choice of the wrong valuation measure does not only produce wrong firm values but results in wrong decision making. For instance, using the physical probability as the valuation measure makes the market seem to be undervaluing the risky asset due to its higher expected return. This creates a clear bias toward a risky investment strategy which is easily seen to be problematic. Consider a fund investing in a risk-free and a risky asset, both liquidly traded, and assume that its managers value it using the physical probability. As the risky return is higher than the risk-free rate, the fund’s value would be highest when fully invested in the risky asset. This contradicts the fact that the value of the fund should always be equal to the market price of the assets it holds. It also explains the bias of such a fund toward risky investments.

Capital and investment strategies

We use the economic valuation framework described above to provide insight into the capital and investment management decisions of value-maximizing insurance firms in a discrete-time, dynamic framework. The controls at the manager’s disposal are the decision to either liquidate the firm or continue operations, the amount of dividends to pay, the amount of capital to raise, and the amount of investment risk to take. In particular, premia are not a control variable. In fact, in order to focus on the capital and investment decisions of the firm, we assume that, in each period, it sells the same portfolio of insurance strategies. Our model incorporates several well-documented frictions that are typical for the environment in which insurance companies operate: (i) carry costs of capital, i.e. deadweight costs associated with holding capital within the firm, which may include double-taxation, as well as agency and financial-distress costs; (ii) cost of raising new capital, which we assume to be fixed, and (iii) minimum regulatory capital requirements that the firm must satisfy in order to operate. It is well known that these financial frictions generate costs at the corporate level which ultimately affect the cost of taking risk and thus the choice of optimal strategies.

Each choice of capital and investment strategies generates a stream of cash flows to shareholders, obtained by netting dividend payments and capital injections at each date. The manager’s task is to select an optimal strategy, i.e. a strategy that maximizes added value. We establish the existence of deterministic, stationary optimal capital and investment strategies by applying the dynamic-programming principle. As a result, firm value is described by a value function, depending exclusively on the current level of equity capital. The focus on added value implies that raising capital is meaningful only if doing so adds value, i.e. only if the value of the firm increases by more than the amount raised, and that paying out dividends makes sense only if keeping these funds within the firm does not add value. These simple observations allow us to provide a comprehensive description of the capital strategies of the firm. First, since holding capital is costly, the firm never recapitalizes unless capital falls below the regulatory minimum. Second, there exists a liquidation barrier, such that the firm is liquidated whenever capital falls below it and recapitalized whenever capital lies between it and the regulatory minimum. Third, there exists an upper dividend barrier such that any capital in excess of it is paid out as dividends. The upper dividend barrier is also the capital level at which added value for the firm is maximal. As a consequence, whenever the firm is recapitalized, it is recapitalized to this level. Fourth, if raising capital is costless, the
upper dividend barrier coincides with the regulatory minimum. Finally, at capital levels between the regulatory minimum and the upper dividend barrier we may encounter either intermediate dividend payments, so-called “dividend bands”, or no dividend payments at all. We provide examples exhibiting each of these behaviors in Section 5.

What about the investment strategy? If the value function were concave, the firm would never take investment risk and if it were convex, the firm would seek maximum exposure to investment risk. However, as we show in Section 4, the value function is typically neither concave nor convex. As a consequence, we cannot hope to have a universal answer as to the optimal amount of investment risk to take (see Section 5). To illustrate the critical mechanism driving the optimal investment strategy, we identify three different components of firm value: net tangible capital, the value of the default option, and franchise value. We show that risky investments have no impact on net tangible value, and affect default option value and franchise value in opposite directions. Indeed, by increasing the likelihood and size of a potential default, risky investments increase the default option value. On the other hand, risky investments typically increase the likelihood of liquidation, which leads to a loss of franchise value, or the likelihood of recapitalization, which is costly. Both of these effects have a negative impact on firm value. Interestingly, it is also possible that, at low capital levels, risky investments increase the likelihood of the firm reaching capital levels at which franchise value is significantly higher. In this case, given that no recapitalization costs are incurred to reach more value-adding capital levels, risky investments may be seen as a substitute for costly capital raising. Our analysis shows that the optimal investment strategy will depend on how all these tradeoffs resolve.

Our work shows that if we account for the limited liability when specifying cash flows to shareholders and use the right valuation measure, value-maximizing insurance firms will exhibit widely different optimal behaviors depending on the environment in which they operate. The range of optimal behaviors is illustrated by the examples in Section 5, which show how investment risk impacts the two key value components: default option value and franchise value.

**Embedding in the literature**

To date, most of the literature on optimal investment strategies for insurers does not consider a value-maximizing insurer but mainly focuses on an insurance firm that either maximizes the utility of surplus, minimizes the probability of ruin, or takes decisions within a mean–variance framework. By contrast, our approach is based on an insurer that maximizes value, defined as the market-consistent value of the stream of dividends net of capital injections. The first suggestion that an insurer should aim to maximize the present value of dividend streams rather than minimize the probability of ruin, thus far the approach in the actuarial literature, can be found in De Finetti (1957). De Finetti’s work spawned a vast body of literature on dividend-distribution problems. To the best of our knowledge all of this literature uses a present value rule based on the physical probability. This is not an issue for the bulk of it because it does not consider investments in a risky asset. It is, however, an issue as soon as the firms considered are exposed to financial risk. Unfortunately, the only two papers dealing with investments in a risky asset we know of, Højgaard and Taksar (2004) and Azcue and Muler (2010), fail to adopt a market-consistent approach to valuation. Both these papers work in a continuous-time setting, allow investments in financial markets of Black–Scholes type, and rely on the dynamic-programming principle. Under the realistic assumption that the risk-free rate of return is strictly smaller than the expected rate of return of the risky asset, the optimal investment strategies in Højgaard and Taksar (2004) and in Azcue and Muler (2010) always involve either partial or full investment in the risky asset. In particular, in Højgaard and Taksar (2004) investment in the risky asset is always optimal even though the value function is concave. This bias toward the risky asset is mainly a consequence of their valuation not being market consistent. Indeed, their valuation rule consists in taking the expected cash flows under the physical probability and discounting them at a constant dividend discount rate that exceeds the risk-free rate. Since the discount rate does not depend on the investment strategy, this valuation rule cannot be market consistent.

The series of papers (Froot et al., 1993), Froot and Stein (1998), and Froot (2007) provides another relevant point of comparison in the existing literature. In particular, Froot (2007) derives the optimal financial strategies of an insurer under a market-consistent valuation. While their model accounts for deadweight cost of capital, it does not contemplate the possibility of firm default, as we do. In this setting, Froot (2007) concludes that insurers should not take liquid investment risk. The gist of the argument is that taking liquid investment risk does not create value per se and that, by taking investment risk, insurers may end up having less capital with which to exploit value-creating opportunities in the future. The result seems plausible as long as the limited-liability option of the firm has no value. In the more realistic case where there is a possibility of default, this logic is less compelling because risky investments can increase the value of the default option.

Finally, our work is related to, but different from, an important strand of the literature that focuses on the “market-consistent” or “fair” valuation of insurance liabilities. This literature aims to extend well-known premium principles used to value insurance contracts in a market-consistent fashion, i.e. in a way that reproduces market prices for hedgeable contracts. Early papers in this area are (Babbel, 1999), Babbel and Merrill (1999), Hancock et al. (2001a) and Hancock et al. (2001b). These papers focus on advancing the use of replication for the hedgeable part of liabilities as well as on describing the role of financial frictions in the valuation of insurance contracts. More recently, a more systematic study of market-consistent valuation of insurance contracts has been undertaken (see, for instance, Pelsser and Stadje (2014), Dhaene et al. (2017), Barigou et al. (2019), Delong et al. (2019a), Delong et al. (2019b) and the references cited therein). The main

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12. This is a simple consequence of Jensen’s inequality and the market consistency of the valuation rule. As pointed out in the section embedding our work in the literature, if the valuation rule is not market consistent, it may be optimal to invest in the risky asset even when the value function is concave.

13. Net tangible capital corresponds to the value of current assets less the default-free value of current insurance liabilities, i.e. excluding liabilities arising from future business.

14. This decomposition of value is similar to the one described but not formalized in Babbel (1999).

15. See e.g. Pun and Wong (2015), Peng and Wang (2016), Gu et al. (2018), Sun et al. (2019), and Zhang and Chen (2019).

16. See e.g. Paulsen and Gjessing (1997), Paulsen (1998), Hipp and Plum (2000), Gaier et al. (2003), Bai and Guo (2008), Azcue and Muler (2009), Thonhauser (2013), Badouzi and Fernández (2013), and Zhang et al. (2016).

17. See e.g. Shen and Zeng (2015), Zhao et al. (2016), Zeng et al. (2016), Li et al. (2016), and Bi and Cai (2019).

18. See e.g. Schmidli (2008), Albrecher and Thonhauser (2009), and Avanzi (2009).

19. See Footnote 12.

20. Froot et al. (1993), Froot and Stein (1998), and Froot (2007) explicitly mention that they consider a default-free firm. However, in their analytical model the possibility of default cannot be ruled out because it assumes normal returns.
difference with our work is that we are not concerned with the valuation of insurance contracts but rather with the valuation of the insurance firm, i.e. of the cash flows to the firm's shareholders.

In fact, for our particular objective, we do not need to postulate anything about how insurance premia are set but take them as given. Although our objective is different, some of the valuation measures put forward for valuing insurance contracts coincide with the economic valuation measure advocated in this paper.\footnote{The construction of the economic valuation measure first appeared in Koch-Medina et al. (2018), an earlier version of this paper.}

In particular, Artzner et al. (2020) use the concept of "insurance arbitrage" to justify the valuation of insurance liabilities using essentially the same probability measure that we propose.\footnote{Note, however, that we work with an infinite time horizon, whereas the literature on market-consistent valuation of insurance contracts (naturally) focuses on a finite time horizon. As we explain in the Online Appendix, dealing with an infinite time horizon requires more delicate measure theoretic arguments.}

1. The model

We work in a discrete-time, infinite-horizon setting with dates indexed by $n \in \mathbb{N}$.\footnote{We use the standard notation $\mathbb{N} = \{0, 1, 2, \ldots \}$.} For every $n \in \mathbb{N} \setminus \{0\}$, the period starting at date $n - 1$ and ending at date $n$ is denoted by $[n-1, n]$. We consider a limited-liability insurance firm with a diffuse shareholder base contributing the firm's equity capital. To be allowed to operate, the regulator requires the firm to have a minimum amount of equity capital. At the beginning of each period, the firm decides whether to stay in business or to liquidate. If it decides to liquidate, any remaining equity capital is returned to shareholders as a dividend and the firm ceases to exist. If it decides to continue operating, the firm may pay dividends but needs to make sure that minimum capital requirements are met. This may possibly require recapitalization. The firm then sells a standardized insurance portfolio of one-period insurance policies in exchange for a premium and decides how to invest collected premiums and capital in assets that are traded in a frictionless and arbitrage-free market. At the end of the period, investment returns, insurance losses are realized and insurance claims are settled. Thus, to steer the firm, the manager has four controls at his disposal: (i) the decision to continue operations or to liquidate the firm, possibly defaulting if assets do not suffice to cover claims; (ii) the amount of dividends to pay (if any) to shareholders; (iii) the amount of new capital (if any) to raise from existing owners;\footnote{To avoid having to deal with dilution effects, we only allow existing shareholders to inject capital. This is akin to a rights issue where incumbent shareholders purchase all newly-issued equity pro rata to their existing share holdings.} and (iv) the amount of investment risk (if any) to (in)directly invest in assets that are traded in a frictionless and arbitrage-free market. All payoffs, prices and values are silently understood to be in a common currency.

1.1. The financial market, insurance losses and the valuation measure

We proceed to describe the underlying probabilistic model. A more detailed and technically rigorous description can be found in the Online Appendix. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ described below represents uncertainty in the economy with $\mathbb{P}$ being the 'physical’ probability measure. If $\mathcal{G}$ is a sub-$\sigma$-algebra of $\mathcal{F}$ and $X : \Omega \to \mathbb{R}$ is a random variable, $\mathbb{E}_\mathbb{P}[X]$ denotes the expectation of $X$ with respect to $\mathbb{P}$ and $\mathbb{E}_\mathbb{P}[X | \mathcal{G}]$ the conditional expectation of $X$ with respect to $\mathbb{P}$ and $\mathcal{G}$. Finally, for any $p \geq 0$, $L^p(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of $p$-integrable random variables and $\| \cdot \|_{p, \mathbb{P}}$ the corresponding norm. Stochastic processes are denoted by $X = (X_t, t \geq 0)$. Inequalities and equalities between random variables are always in the "$\mathbb{P}$-almost sure” sense.

The financial market

Even though the insurer takes all decisions, including investment decisions, at discrete dates, we assume that financial markets trade continuously. From a modeling perspective, this is consistent with the fact that insurance firms take decisions, such as rebalancing their investment portfolio, at a significantly lower frequency than trading takes place in financial markets. We assume a Black–Scholes market in which a risk-free money market account and a risky security, e.g. an index, are traded. The money-market account pays a deterministic, instantaneous interest rate $r > 0$, i.e. one unit of currency follows the process $B = (B_t, t \geq 0)$, where $B_t = e^{rt}$, $t \geq 0$.

The risky security has initial price $s_0 > 0$ and its price process $S = (S_t, t \geq 0)$ follows a geometric Brownian motion with drift $\mu > r$ and volatility $\sigma > 0$, i.e.

$$s_t = s_0 \exp \left\{ \sigma W_t + \left( \mu - \frac{1}{2} \sigma^2 \right) t \right\},$$

where $W = (W_t, t \geq 0)$ is a standard $\mathbb{P}$-Brownian motion.

The flow of information in the financial market is described by the market filtration $\mathbb{F}^W = \{ \mathcal{F}^W_t, t \geq 0 \}$.\footnote{The market filtration is slightly larger than the raw filtration generated by $W$. The raw filtration is the smallest filtration $\mathbb{F}^W$ such that $W$ is $\mathbb{F}^W$-adapted. See the Online Appendix for further details.} We denote by $\mathcal{F}^W$ the smallest $\sigma$-algebra containing all the $\sigma$-algebras in $\mathbb{F}^W$, i.e. $\mathcal{F}^W = \sigma(\bigcup_{t \geq 0} \mathcal{F}^W_t)$.

For every finite maturity $T > 0$, the Black–Scholes model is arbitrage-free and complete.\footnote{Completeness implies that every cash flow $X \in L^2(\Omega, \mathcal{F}^W, \mathbb{P})$ maturing at date $T$ has a unique, well-defined market price at every date $t \leq T$. In the Online Appendix we show that there exists a unique probability measure $\mathbb{P}^*$ defined on $\mathcal{F}^W$,\footnote{The standard theorems in mathematical finance only imply the existence of $\mathbb{P}^*$ and the existence of the corresponding norm. Stochastic processes are described further below in this section.} such that $\mathcal{F}^W$ is equivalent to $\mathbb{P}^*$,\footnote{In our setting we work with an infinite time horizon and thus the standard existence theorems only apply to $\mathbb{P}^*$ restricted to $\mathcal{F}_T$ for each fixed maturity $T > 0$, for which we need to establish the existence of a single probability measure that works for all maturities. A rigorous construction of $\mathbb{P}^*$ is provided in the Online Appendix.} the pricing measure, such that the market price at date $t \leq T$ of any cash flow $X \in L^2(\Omega, \mathcal{F}^W, \mathbb{P})$ maturing at an arbitrary date $T$ is given by

$$\pi_{t,T}(X) = \mathbb{E}_{\mathbb{P}^*}\left[ e^{-(T-t)\rho} \mathbb{E}_{\mathbb{P}}[X | \mathcal{F}_T] \right] \in L^1(\Omega, \mathcal{F}^W, \mathbb{P}).$$

(1)

For every $T > 0$, the probability measure $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ when restricted to $\mathcal{F}^W_T$ but not on $\mathcal{F}^W$.\footnote{The lack of equivalence of the market valuation measure and the physical probability on $(\Omega, \mathcal{F}^W)$ results from the infinite time horizon, but is of no consequence in our setting because we only consider cash flows that have fixed, albeit arbitrarily long, maturities. Hence, we only need to establish equivalence on $(\Omega, \mathcal{F}^W_T)$ for every $T > 0$. Note also that, while $\mathcal{F}^W_T$-measurability has a clear economic interpretation (cash flows that are $\mathcal{F}^W_T$ measurable are fully known as soon as uncertainty resolves at date $T$), we attach no economic interpretation to $\mathcal{F}^W$-measurability.}
Insurance losses and the insurer’s filtration

The standardized portfolio of one-period strategies sold by the insurer is characterized by a fixed aggregate premium \( p \) and an i.i.d. sequence \( L = (L_n, n \in \mathbb{N} \setminus \{0\}) \) of nonzero, nonnegative random variables, where \( L_n \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \) represents aggregate losses over the period \([n-1, n]\). We assume that, for every \( n \in \mathbb{N} \setminus \{0\} \), \( L_n \) is independent of the \( \sigma \)-algebra \( \mathcal{F}_W \), i.e. independent of \( \mathcal{F}_W \). The random variables
\[
R_n := E[\pi L_n] - L_n, \quad n \in \mathbb{N} \setminus \{0\},
\]
represent the insurance risk. \( R = (R_n, n \in \mathbb{N} \setminus \{0\}) \) is an i.i.d. sequence of \( \mathbb{P} \)-square-integrable, centered random variables that are also independent of \( \mathcal{F}_W \). The insurer has at his disposal the information contained in the market filtration but, as time elapses, he also learns about realized insurance losses. This means that he has access to the information contained in a larger filtration \( \mathcal{F} = (\mathcal{F}_t, t \geq 0) \), that we call the insurer’s filtration. We assume that \( \mathcal{F} \) is the smallest filtration such that \( \mathcal{F}_t \) contains \( \mathcal{F}_W \) and information of the insurance losses \( \sigma(R_n; n \leq t) \). We set \( \mathcal{F} := \sigma(\bigcup_{t \geq 0} \mathcal{F}_t) \).

The extended financial market

The insurance firm has an exposure both to financial market risk and to insurance risk. Thus, cash flows to shareholders are, in general, adapted to the filtration \( \mathcal{F} \), which describes the financial market and insurance loss information available to the insurer, rather than \( \mathcal{F}_W \), which only captures the financial market information. Therefore, we need to extend the Black–Scholes economy described above, keeping the same traded assets but replacing the flow of information \( \mathcal{F}_W \) by the insurer’s filtration \( \mathcal{F} \). After passing from \( \mathcal{F}_W \) to \( \mathcal{F} \), the market remains arbitrage free for every finite maturity but loses completeness. As a result, a replicable cash flow \( X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \) maturing at date \( T > 0 \) has a well-defined market price \( \pi_T(X) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \) at every date \( t \) prior to maturity.\(^{30}\) Incompleteness with respect to the enlarged filtration \( \mathcal{F} \) leads to an infinite number of market-consistent valuation measures, i.e. of probability measures \( \mathbb{Q} \) defined on \( \mathcal{F} \) such that \( \pi_T(X) = \mathbb{E}_{\mathbb{Q}}[e^{-\int_T^T r \, dr} X | \mathcal{F}_T] \) for every replicable cash flow \( X \) maturing at date \( T > 0 \). To choose one of these measures in order to value cash flows to shareholders, i.e. the cash flow stream of dividends net of the cost of capital injections, we need to require more than just market consistency.

The economic valuation measure

With a diffuse shareholder base and in the absence of financial market risk, a firm manager is typically assumed to behave in a risk-neutral manner because shareholders are able to diversify all risk.\(^{31}\) In this case, the present value of cash flows to shareholders is new.\(^{32}\) The projection formula (3) implies that the economic value of a general cash flow \( X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \) with maturity \( T \) can be computed as follows: i) decompose \( X \) into a collection of \( H_k = X_k + E_k \), where \( E_k = \mathbb{E}_\mathbb{P}[X | \mathcal{F}_k] \) is its replicable, or hedgeable, component and \( E_k = X - \mathbb{E}_\mathbb{P}[X | \mathcal{F}_k] \) is an error term that is uncorrelated with the financial market and, hence, has zero economic value in our model; ii) compute the unique market price \( \pi_{0,T}(H_k) \) of the replicable component of \( X \), which yields the economic value of \( X \) at date 0.

Economic value of discrete-time cash flow streams

Cash flows to shareholders are \( \mathbb{P} \)-adapted sequences \( X = (X_n, n \in \mathbb{N}) \), where \( X_n \) denotes the cash flow due at date \( n \in \mathbb{N} \).

\(^{30}\) In the new market, replicating strategies are allowed to be \( (\mathcal{F}_t, t \in [0, T]) \)-predictable. For instance, if at date \( n \) the market goes up, one may choose a different strategy depending on whether the insurance has to settle a small or a large loss. This genuinely enlarges the set of self-financing trading strategies and results in a larger set of replicable cash flows. In particular, because \( \mathcal{F} \) is an enlargement of \( \mathcal{F}_W \), all cash flows in \( L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \) remain replicable in the new economy and have the same price as in the original Black–Scholes economy. For this reason we continue to use the same symbol for the market price of a cash flow.

\(^{31}\) Indifference to diversifiable risk, does not imply indifference to possible side effects of taking diversifiable risk. Indeed, taking risk may generate costs due to financial frictions or benefits if part of the risk is externalized. Thus, even a risk neutral firm may be led to behave in a manner that appears to be risk averse or a risk seeking depending on whether costs outweigh benefits or the other way around.

\(^{32}\) To our knowledge, the explicit construction of the economic valuation measure is new. It can be shown that, in the finite-time horizon case, the probability measure \( \mathbb{Q} \) coincides with the so-called variance optimal measure. See, e.g. Delbaen and Schachermayer (1996), Schweizer (1996), and the references therein.

\(^{33}\) We emphasize that the economic justification for using the economic valuation measure critically depends on the fact that the insurance firm has a diffuse shareholder base.
If $X$ is bounded in $L^1(\Omega, \mathcal{F}, Q^*)$, then its economic value at date $0$ can be defined in a natural way as the sum of the economic values of the individual components

$$
\Pi_0^e(X) := X_0 + \sum_{n \geq 1} \frac{1}{(1+r)^n} E^Q[X_n],
$$

where $r := e^\delta - 1$ denotes the one-period risk-free rate of return. In our model, the strategy to invest an amount in the money market account at a date $n \geq 0$ and liquidate the position at date $n + 1$ mimics a risk-free instrument for the period $[n, n+1]$ that earns the risk-free return $r$. We refer to this instrument as the (one-period) risk-free asset.

2. Admissible strategies and capital dynamics

The manager’s objective of maximizing the economic value of the firm can be analyzed using dynamic-programming techniques. At any date $n \geq 0$, the firm may be either in a state $m \in \mathbb{R}$ representing the firm’s capital if the firm is still active or in the state $\star$ if the firm has been liquidated. Hence, the state space is $\mathcal{X} := \mathbb{R} \cup \{\star\}$. The “cemetery state” $\star$ is assumed to be an absorbing state, i.e. once the firm reaches that state it remains there forever. A (financial) strategy $\pi$ is an $\mathbb{F}$-adapted process $\mathcal{S} = (\mathcal{X}, \mathcal{F}, \lambda, \delta)$, where, for every $n \in \mathbb{N}$: (i) $\delta_n \in [0, 1]$ with 0 indicating liquidation (which sends the firm into state $\star$) and 1 continuation; (ii) $z_{n+1} = 0$ is the amount of dividends to be paid; (iii) $\kappa_n = 0$ is the amount of capital to be raised, and (iv) $\lambda_n \in [0, 1]$ is the proportion of capital to be invested in the risky asset. The random vector $(\delta_n, z_n, \kappa_n)$ corresponds to financing operations at date $n$ and the random variable $\lambda_n$ to the investment strategy at date $n$.

The case in which the initial state of the firm is $\star$ is devoid of interest since the firm then remains inactive for ever. In this case, by convention, the only possible strategy $\mathcal{S}$ is $\delta_0 = 0, z_0 = 0, \kappa_0 = 0$ and $\lambda_n = 0$ for every $n \geq 0$. Assume now that the initial state of the firm is $m \in \mathbb{R}$ and that the manager has chosen strategy $\mathcal{S}$. Set $M_{m}^{n, \mathcal{S}} = m$ and, for $n \geq 1$, denote by $M_{m}^{n, \mathcal{S}}$ the state of the firm at the end of the period $[n-1, n]$ under $\mathcal{S}$. The liquidation time is the stopping time

$$
\tau^{n, \mathcal{S}} := \inf\{n \geq 0 \mid \delta_n = 0\}
$$

representing the date at which the company is liquidated according to strategy $\mathcal{S}$. If not liquidated, to be able to operate, the firm must satisfy minimum regulatory capital requirements given by $M_{\text{reg}} \geq 0$. Hence, for every $n \geq 0$, any strategy must satisfy

$$
M_{\text{reg}} \leq M_{m}^{n, \mathcal{S}} - z_n + \kappa_n, \quad \text{on } \{\tau^{n, \mathcal{S}} > n\}. \tag{5}
$$

When the firm is liquidated, the manager is only allowed to return any remaining capital to shareholders, i.e. we have

$$
\delta_n = 0, \quad z_n = \max(M_{m}^{n, \mathcal{S}}, 0), \quad \kappa_n = 0, \quad \lambda_n = 0, \quad \text{on } \{\tau^{n, \mathcal{S}} = n\}. \tag{6}
$$

where setting $\lambda_n = 0$ is just a useful convention since, in this case, there is nothing to invest. Moreover, since the cemetery state $\star$ is absorbing, we must have

$$
\delta_n = 0, \quad z_n = 0, \quad \kappa_n = 0, \quad \lambda_n = 0 \quad \text{on } \{\tau^{m, \mathcal{S}} < n\}, \tag{7}
$$

where setting $\lambda_n = 0$ is, again, just a convention.

For the remainder of the paper, when the initial state of the firm is $m \in \mathbb{R}$, any strategy is always assumed to satisfy Conditions (5), (6), and (7).

2.1. End-of-period capital dynamics

Let $m \in \mathbb{R}$ be the initial state of the firm and assume that $M_{m}^{n, \mathcal{S}} \neq \star$. If $\delta_n = 1$, i.e. if the manager decides to continue operations during the period $[n, n+1]$, the amount $z_n \geq 0$ of dividends to pay and the amount $\kappa_n \geq 0$ of capital to raise must be chosen. The firm collects the premium $p$ when it writes new business. We decompose the premium as

$$
p = l + q,
$$

where $l = E^Q[M_n]/(1+r)$ is the (actuarially) fair premium and $q \geq 0$ is the margin (over the fair value). After selling insurance, the capital in the firm increases to $M_{m}^{n, \mathcal{S}} - z_n + \kappa_n + p$. However, holding capital is costly due to the presence of carry costs, which are accounted for by scaling down the amount of capital by a fixed factor $\gamma \in (0, 1)$. Thus, capital available for investments amounts to $\gamma (M_n - z_n + \kappa_n + p)$. We refer to $1 - \gamma$ as the carry cost of capital. We assume that the manager always matches liabilities when investing the fair premium, i.e. $l$ is invested in the money-market account. On the other hand, the manager has the choice of investing $\gamma (M_n - z_n + \kappa_n + q)$ in a mix of the risk-free and the risky assets, with $\lambda_n \in [0, 1]$ being the portion invested in the risky asset.

Given that $(1+r) = E^Q[L_{n+1}]$, the total return resulting from the investment of $l$ in the money-market account after netting claims $L_{n+1}$ is precisely $R_{n+1}$. Hence, the firms capital at the end of the period $[n, n+1]$ is given by

$$
\gamma (M_{m}^{n, \mathcal{S}} - z_n + \kappa_n + q)(1 + r + \lambda_n R_{n+1}) + R_{n+1},
$$

where

$$
\rho_{n+1} := \frac{S_{n+1} - S_n}{S_n} - r \tag{8}
$$

is the one-period excess return on the risky asset, which satisfies $E^Q[\rho_{n+1} | \mathcal{F}_n] = 0$ for $n \geq 0$. In summary we have that end-of-period capital at date $n+1$ is given by

$$
M_{n+1}^{m, \mathcal{S}} = \begin{cases} 
\gamma (M_{m}^{n, \mathcal{S}} - z_n + \kappa_n + q)(1 + r + \lambda_n R_{n+1}) + R_{n+1}, & \text{on } \{\tau^{n, \mathcal{S}} > n\}; \\
\star, & \text{on } \{\tau^{n, \mathcal{S}} \leq n\}. 
\end{cases} \tag{9}
$$

34 This means that the components of $X$ belong to $L^1(\Omega, \mathcal{F}, Q^*)$ and there exists a constant $B > 0$ such that $|X_n|_{L^1} \leq B$ for every $n \geq 0$. It is worth pointing out that, by Theorem 1.1, for every $n \in \mathbb{N}$, we have that $X_n \in L^1(\Omega, \mathcal{F}^n)$ whenever $X_n \in L^2(\Omega, \mathcal{F}^n)$. 35 Our main reference for dynamic programming is (Hernández-Lerma and Lasserre, 1999) where what we call a strategy is called a deterministic policy. 36 We allow for the possibility that $M_{\text{reg}} = 0$ in which case the insurer can operate with zero capital, i.e. without putting own capital at risk. Moreover, assuming that $M_{\text{reg}}$ is fixed, is a justifiable approximation because the insurer can only underwrite a fixed standardized book of business. See, however, Section 6, where we discuss possible extensions. 37 Recall that we assume that the firm can raise capital only from existing shareholders. 38 Since the insurer sells a fixed standardized portfolio for a fixed premium, the fair premium and the margin are fixed. For our purposes, it is not necessary to specify how the insurer prices insurance contracts. 39 Carry costs include agency costs and double-taxation costs; see e.g. Flannery (1994), Froot et al. (1993), Froot and Stein (1998), Froot (2007), Goosbee (1998) and Hann et al. (2013). In the literature, carry costs are also referred to as deadweight or frictional costs. 40 Since $\lambda_n \in [0, 1]$ the insurer can only take long positions in the money-market account and the risky asset. This amounts to disallowing leverage.
2.2. Cash flows to shareholders and admissible strategies

Raising capital is assumed to incur a fixed cost, e.g. legal fees, that is independent of the amount $\kappa > 0$ being raised. Hence, for the firm to receive the amount $\kappa \geq 0$ of capital, shareholders have to inject $\kappa + C(\kappa)$ where $C(\kappa) := C \cdot (1_{\kappa > 0})$. Under strategy $S$, the cash flow to shareholders at date $n \geq 0$ is

$$CF_n := \begin{cases} z_n - \kappa_n - C(\kappa_n), & \text{on } \{r^{m.S} \geq n\} ; \\ 0, & \text{on } \{r^{m.S} < n\}. \end{cases}$$

The cash flow stream $CF^S := (CF_0^S, CF_1^S, \ldots)$ is referred to as cash flows to shareholders under the strategy $S$.

We now argue that the firm’s manager can focus on strategies satisfying Properties (S1)–(S3) specified below. In the Online Appendix, we provide a formal proof that there is no loss in generality in doing so. Here we are content to explain why restricting the attention to these strategies makes economic sense.

Assuming the firm’s initial state is $m \in \mathbb{R}$, the first defining property of an admissible strategy is that the firm should never simultaneously raise capital and pay dividends, i.e. we require that

$$\Pi(z_n, \kappa_n = 0) = 1 \quad \text{for every } n \geq 0. \quad (S1)$$

Economically, this is clear because, when raising capital is costless, there is no discernible benefit to raising capital and using it to pay dividends or paying dividends only to have raise capital to be able to operate. In contrast, when raising capital is costly, this procedure is clearly disadvantageous.

The second defining property is that the firm should not raise capital unless regulatory requirements are not met:

$$\Pi(M^{m.S}_n \geq M_{reg}, \kappa_n > 0) = 0 \quad \text{for every } n \geq 0. \quad (S2)$$

The rationale for this property is that, unless it is absolutely necessary to be able to meet regulatory requirements, raising any amount $K$ of capital can always be delayed to the next period and, by doing so, the firm can avoid incurring the carry costs $(1 - \gamma)K$ and save the interest, $rC$, on the cost of raising it.

The final defining property is that, at any date, the firm should avoid holding too much capital above the regulatory minimum. More precisely,

$$M^{m.S}_n - z_n + \kappa_n \leq \frac{C}{(1 - \gamma)(1 + r)} + M_{reg}, \quad \text{on } \{r^{m.S} > n\}. \quad (S3)$$

This property is reasonable because dividends should be paid at the latest when the cost of carrying the excess above the regulatory minimum, $(1 - \gamma)(M^{m,S}_n - z_n + \kappa_n - M_{reg})$ (incurred at the start of the period) exceeds the discounted cost of raising capital at the end of the period $C/(1 + r)$.

Strategies satisfying Properties (S1)–(S3) are said to be admissible for the initial state $m \in \mathbb{R}$. The set of admissible strategies for $m$ is denoted by $S(m)$. Since $S = (0, 0, 0, 0)$ is the only possible strategy when $m = \bullet$, it makes sense to call $S = (0, 0, 0, 0)$ admissible for $\bullet$ and to set $S(\bullet) = \{(0, 0, 0, 0)\}$. From now on we restrict the manager’s choices to admissible strategies. For convenience, we will write statements such as “for $S \in S(m)$” instead of the more cumbersome “for $m \in \mathbb{R}$ and $S \in S(m)$”.

The following proposition establishes that admissible strategies are bounded in $L^1(\Omega, F, \mathbb{Q}^m)$ and, thus, admit an economic value. The proof can be found in the Online Appendix.

**Proposition 2.1.** There exists a constant $B > 0$ such that, for every $S \in S(m)$ and $n \in \mathbb{N} \setminus \{0\}$,

$$||z_n - \kappa_n - C(\kappa_n)||_{L^\infty} \leq B. \quad (10)$$

In particular, for every strategy $S \in S(m)$, the cash flow stream $CF^S$ received by shareholders is bounded in $L^1(\Omega, F, \mathbb{Q}^m)$.

2.3. Decision functions and stationary strategies

Given the stationary nature of our model, i.e. that the parameters of our model are time independent, it is particularly interesting to search for strategies that result from applying at each date a decision rule prescribing a “one-step strategy” that depends only on the state of the firm at that date. For any $m \in \mathcal{X}$, the set $S_0(m)$ of all admissible, one-step strategies at date 0 consists of all tuples $(z, \kappa, \lambda, \delta) = s$ for some strategy $S \in S(m)$. Given any one-step strategy $s \in S_0(m)$ we set

$$M^{m,s}_n = \begin{cases} y((m - z + \kappa + q)(1 + r + \lambda\rho_1) + R_1, \quad \text{if } \delta = 1 \quad ; \\ \bullet, \quad \text{if } \delta = 0. \end{cases} \quad (11)$$

Thus, the random variable $M^{m,s}_n$ represents the capital of the firm at the end of period $[0, 1]$ provided the manager implements $s$ at date 0.

A decision function is a measurable mapping

$$D(\cdot) = (z(\cdot), \kappa(\cdot), \lambda(\cdot), \delta(\cdot)) : \mathcal{X} \rightarrow \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1] \times [0, 1]$$

such that $D(m) \in S_0(m)$ for every $m \in \mathcal{X}$. Given a decision function $D$ we can define the process $(M^{m,D}_n, n \geq 0)$ by setting

$$M^{m,D}_n = m$$

for $n < 1$ and

$$M^{m,D}_n = y(M^{m,D}_{n-1} - z(M^{m,D}_{n-1}) + \kappa(M^{m,D}_{n-1}) + q(1 + r + \lambda(M^{m,D}_{n-1})\rho_1) + R_{n-1},$$

for $n \geq 1$. The associated strategy $S^D$ is defined by setting, for every $n \geq 0$,

$$S^D_n = (z_n, \kappa_n, \lambda_n, \delta_n) = D(M^{m,D}_n) \in S_0(M^{m,D}_n).$$

It is clear that $S^D$ is admissible, i.e. $S \in S(m)$. Strategies that can be generated in this way are said to be stationary.

3. Firm value, added value, and optimal strategies

Whenever the firm’s manager adopts a specific strategy $S \in S(m)$, the corresponding cash flows to shareholders admit an economic value by Proposition 2.1. The manager’s task is to select a strategy with the highest possible economic value. This section is devoted to providing precise definitions of firm value and firm added value and establishing the existence of “optimal” strategies that are stationary. The proofs of the results in this section are found in Appendix A1.1.

3.1. Firm value and added value

The value of a strategy $S \in S(m)$ is denoted by $V^S(m)$ and corresponds to the economic value of cash flows to shareholders $CF^S$, i.e.

$$V^S(m) := \Pi^S_0(CF^S) = E_{\mathbb{Q}^m} \left[ \sum_{n=0}^{\infty} \frac{1}{(1 + r)^n} (z_n - \kappa_n - C(\kappa_n)) \right].$$

43 Since the constant $B$ does not depend on $m \in \mathbb{R}$, Inequality (10) cannot be true for $n = 0$. Indeed, if $m > 0$, we may choose a strategy satisfying $z_0 = m$ making the left-hand side arbitrarily large as $m$ tends to infinity.

44 In Appendix A we show that $\mathcal{X}$ can be naturally viewed as a metric space. Measurability should be understood as being with respect to the corresponding Borel $\sigma$-algebra.
Clearly, if \( S \in \mathcal{S}(m) \) and \( \delta_0 = 0 \), then \( V^S(m) = \max\{m, 0\} \). We set \( V^S(\ast) = 0 \).

To introduce the notion of “added value” it is useful to separate the impact of financing operations and the “productive” use of capital. To this effect, consider a strategy \( S \in \mathcal{S}(m) \) for which \( \delta_0 = 1 \). It is useful to bear in mind that the capital that is being put to productive use is not \( m \) but \( m - z_0^* + k_0^* \). Denote by \( S^+ \) the admissible strategy in \( \mathcal{S}(m - z_0^* + k_0^*) \) defined by

\[
\begin{align*}
  z_0^* &= 0, \quad k_0^* = 0, \quad \lambda_0^* = \lambda_0, \quad \delta_0^* = 1; \\
  z_n^+ &= z_n, \quad k_n^+ = k_n, \quad \lambda_n^+ = \lambda_n, \quad \delta_n^+ = \delta_n, \quad n \geq 1.
\end{align*}
\]

We clearly have

\[
V^S(m) = z_0^* - C(k_0) + \max\{m, 0\} + \frac{\text{cost of raising capital within the firm}}{\text{value due to capital put to productive use within the firm}}.
\]

The expression \( z_0^* - C(k_0) \) represents the cash flow to shareholders resulting from financing operations at date \( 0 \). It is natural to define the added value of \( S \in \mathcal{S}(m) \) with \( \delta_0 = 1 \) as

\[
\mathcal{A}V^S(m) := V^S(m - z_0^* + k_0^*) - (m - z_0^* + k_0^*)
\]

To obtain the total cost of reaching the capital level \( m - z_0^* + k_0^* \), we need to take the cost of raising capital, \( C(k_0^*) \), and add \( \max\{m, 0\} \), which represents the amount of outstanding liabilities the firm needs to settle before being allowed to continue operations. Added value of \( S \in \mathcal{S}(m) \) with \( \delta_0 = 0 \) is set at \( 0 \). It is immediate to see that, for every \( S \in \mathcal{S}(m) \), we have

\[
\mathcal{A}V^S(m) = \left\{ \begin{array}{ll} V^S(m) - \max\{m, 0\}, & \text{if } \delta_0 = 1; \\
0, & \text{if } \delta_0 = 0. \end{array} \right.
\]

Finally, we point out that the value of \( S \) depends only on added value after financing operations and on whether capital had to be raised, i.e.

\[
V^*(m) = m - C(k_0) + \mathcal{A}V^S(m - z_0^* + k_0^*). \tag{16}
\]

Firm value and firm added value are defined assuming the firm makes optimal use of its resources and will be seen to correspond to the value, respectively added value, of an “optimal” admissible strategy.

Firm value given initial capital \( m \in \mathcal{X} \) is defined as

\[
V(m) := \sup_{S \in \mathcal{S}(m)} V^S(m). \tag{17}
\]

The map \( V : \mathcal{X} \to [0, \infty) \) is called the value function. Firm added value given initial capital \( m \in \mathcal{X} \) is defined as

\[
\mathcal{A}V(m) := \sup_{S \in \mathcal{S}(m)} \mathcal{A}V^S(m) = V(m) - \max\{m, 0\}. \tag{18}
\]

The map \( \mathcal{A}V : \mathcal{X} \to [0, \infty) \) is called the added-value function. We always have

\[
V(m) = \max\{m, 0\} + \mathcal{A}V(m), \quad \text{for every } m \in \mathbb{R}. \tag{19}
\]

The following result collects some important monotonicity properties of the value and added-value functions.

**Proposition 3.1.** The value function \( V : \mathcal{X} \to \mathbb{R} \) satisfies the following properties:

(i) \( V(m) \geq \max\{m, 0\} \) for every \( m \in \mathbb{R} \).

(ii) \( V(m_2) - V(m_1) \geq m_2 - m_1 \) for every \( m_2 \geq m_1 \) with \( V(m_1) > 0 \).

(iii) The set \( \{m \in \mathbb{R} : V(m) > 0\} \) is an interval containing \((0, \infty)\) and \( V \) is strictly increasing on the interval \([M_\gamma, \infty)\), where \( M_\gamma := \inf\{m \in \mathbb{R} : V(m) > 0\} \in (-\infty, 0] \).

(iv) The added-value function \( \mathcal{A}V : \mathcal{X} \to \mathbb{R} \) takes only nonnegative values and is increasing, but not strictly increasing, on \( \mathbb{R} \).

**Remark 3.2.** At any capital level \( m \), both the value function \( V(m) \) and the added-value function \( \mathcal{A}V(m) \) are decreasing in the recapitalization cost \( C \), decreasing in the carry cost of capital \( 1 - \gamma \), and increasing in the margin \( q \).

3.2. Existence of stationary optimal strategies

A key question is whether, for every \( m \in \mathcal{X} \), there exists an optimal strategy, i.e. an admissible strategy \( S^* \in \mathcal{S}(m) \) such that

\[
V(m) = V^S(m),
\]

or, equivalently,

\[
\mathcal{A}V(m) = \mathcal{A}V^S(m).
\]

A stationary optimal strategy is an optimal strategy \( S^* \in \mathcal{S}(m) \) such that

\[
S^* = S^0
\]

for some decision function \( D^* \). In such a case, we say that the decision function \( D^* \) is optimal. Given the stationary nature of our model, it is reasonable to expect that stationary optimal strategies exist; a decision that is optimal at a particular date should also be optimal at any other date as long as capital before financing operations is the same. Our next result establishes the existence of an optimal stationary strategy.

**Theorem 3.3.** The value function satisfies the Dynamic Programming Equation

\[
V(m) = \sup_{s \in \delta_0(m)} \left( z - C(s) + \frac{1}{1 + r} E_{q^0} \left[ V(M_1^{m_s}) \right] \right), \quad \text{for every } m \in \mathcal{X},
\]

where

\[
M_1^{m_s} = \begin{cases} y(m - z + \kappa + q)(1 + r + \lambda \rho_1) + R_1, & \text{if } \delta = 1; \\
\ast, & \text{if } \delta = 0. \end{cases}
\]

The value and added-value functions are both continuous from the right, i.e.

\[
\lim_{m \downarrow m_\gamma} V(m) = V(m_\gamma) \quad \text{and} \quad \lim_{m \downarrow m_\gamma} \mathcal{A}V(m) = \mathcal{A}V(m_\gamma) \quad \text{for every } m_\gamma \in \mathbb{R}.
\]

There exists an optimal decision function \( D^*(\cdot) = (z^*(\cdot), \kappa^*(\cdot), \lambda^*(\cdot), \delta^*(\cdot)) \). In particular,

\[
V(m) = z^*(m) - C^*(m) + \frac{1}{1 + r} E_{q^0} \left[ V(M_1^{m_s}) \right],
\]

for all \( m \in \mathcal{X} \).

3.3. Capital management

We can now provide a comprehensive picture of the capital management decisions of the firm, i.e. of its optimal liquidation, dividend and capital-raising strategies.
The dividend principle

The following Dividend Principle for optimal strategies is intuitive: as long as a unit of capital adds extra value when left in the firm, the manager should not pay out dividends. On the other hand, if an additional unit of capital does not add value, then it should be returned to shareholders. To formalize this, it is useful to introduce notation for the intervals on which AV is constant. For every \( m \in \mathbb{R} \), we consider the interval \( J(m) := \{ n \in \mathbb{R} \mid \text{AV}(m) \geq \text{AV}(n) \} \). The interval \( J(m) \) may be degenerate and have \( m \) as its only element. We note that there is at most a countable number of \( J(m) \)'s that are not degenerate. Because the function \( \text{AV} \) is right continuous, if \( J(m) \) is non degenerate, it must be closed to the left and open to the right unless \( \text{AV} \) is continuous at its right end.

Our next result shows that we can always choose the optimal decision function in Theorem 3.3 to be aligned with the Dividend Principle as formalized in Property (23).

**Proposition 3.4 (Dividend Principle).** The decision function \( D^*(\cdot) \) in Theorem 3.3 can always be chosen in such a way that, for every \( m \in \mathbb{R} \),

\[
z^*(m) = \begin{cases} m - d(m), & \text{if } \text{AV}(m) > 0; \\ \max(m, 0), & \text{if } \text{AV}(m) = 0, \end{cases}
\]

where \( d(m) := \inf J(m) \).

The nondegenerate intervals \( J(m) \) are called dividend bands. Whenever capital belongs to a band, the firm pays dividends down to the band's left end. Proposition 3.4 says that we can always choose an optimal decision function whose dividend strategies are of band type. If there is only one band, then the dividend strategy is of barrier type. For the remainder of this paper we assume that \( D^*(\cdot) \) is an optimal decision function satisfying the above Dividend Principle.

Property (23) implies that the firm is liquidated if and only if \( \text{AV}(m) = 0 \), in which case \( V(m) = \max(m, 0) \). To make the problem studied in this paper worthwhile, we make the following Nontriviality Assumption:

\[
\text{AV}(M_{\text{reg}}) > 0.
\]

Liquidation and upper-dividend barriers

There are two critical thresholds associated with the added-value function. The first one is the upper-dividend barrier

\[
\overline{M} := \inf \{ m ; \text{AV} = \text{constant on } [m, \infty) \}.
\]

By the Dividend Principle (23), when capital levels are above \( \overline{M} \), the firm pays dividends down to \( \overline{M} \) because any capital in excess of \( \overline{M} \) does not add value. Moreover, the monotonicity of the added-value function immediately implies that added value is maximal at \( \overline{M} \), i.e. \( \text{AV}(\overline{M}) \geq \text{AV}(m) \) for every \( m \in \mathbb{R} \).

**Proposition 3.5.** The upper-dividend barrier satisfies:

\[
M_{\text{reg}} \leq \overline{M} \leq \frac{C}{(1 - \gamma)(1 + r)} + M_{\text{reg}}.
\]

In particular, \( \overline{M} = M_{\text{reg}} \) whenever \( C = 0 \). Moreover, for every \( m \geq \overline{M} \), the firm pays dividends down to \( \overline{M} \) and continues operations, i.e. \( \delta^*(m) = 1, z^*(m) = m - \overline{M} \) and \( \kappa^*(m) = 0 \).

An implication of the above result is that costly capital raising should be the only reason for an insurance firm to choose to operate above the regulatory minimum. As illustrated by the examples in Section 5, no general statement about the investment strategy can be made when raising capital is costly. These examples also show that, when \( M > M_{\text{reg}} \), the region between \( M_{\text{reg}} \) and \( M \) may contain multiple dividend bands.

The second critical threshold is the liquidation barrier

\[
\underline{M} := \sup \{ m ; \text{AV} \equiv 0 \text{ on } (-\infty, m) \}.
\]

The next proposition shows that the firm is liquidated whenever capital is below \( \underline{M} \). It also shows that, when the firm recapitalizes, it always does so up to \( \overline{M} \), i.e. up to the level at which added value is greatest. Indeed, recapitalizing to any other level would result in less added value at the same cost. Hence, for the recapitalization option to have any value, i.e. for \( \underline{M} < M_{\text{reg}} \) to hold, the cost of raising capital must be lower than added value at the upper-dividend barrier.

**Proposition 3.6.** The liquidation barrier satisfies:

\[
\underline{M} = \begin{cases} M_{\text{reg}}, & \text{if } \text{AV}(\overline{M}) \leq C; \\ C - \text{AV}(\overline{M}), & \text{if } \text{AV}(\overline{M}) > C. \end{cases}
\]

If \( \text{AV}(\overline{M}) \leq C \), then the firm is liquidated if and only if \( m < M_{\text{reg}} \). If \( \text{AV}(\overline{M}) > C \), e.g. if \( C = 0 \), then \( \underline{M} < 0 \). In this case, the firm is liquidated if and only if \( m < \underline{M} \), whereas, if \( m \in (\underline{M}, M_{\text{reg}}) \), then

\[
(i) \ k^*(m) = \overline{M} - m; \quad (ii) \ V(m) = m + \text{AV}(\overline{M}) - C, \quad \text{and} \quad (iii) \ \text{AV}(m) = \min(m, 0) + \text{AV}(\overline{M}) - C.
\]

In particular, \( V \) and \( \text{AV} \) are affine on \( [\underline{M}, M_{\text{reg}}) \) with \( \text{AV} \equiv \text{AV}(\overline{M}) - C \) on \([0, M_{\text{reg}}) \).

The firm is liquidated when capital is at \( \underline{M} \) if \( M < M_{\text{reg}} \), but not if \( M = M_{\text{reg}} \). This is because recapitalizing adds zero value in the first case and continuing operations always adds value in the second case by the Nontriviality Assumption (24). Observe also that the recapitalization option is always valuable if raising capital is costless. The next result establishes that, ceteris paribus, there exists a maximal cost of raising capital above which the recapitalization option is worthless.

**Proposition 3.7.** Consider firms that are identical with the exception of their recapitalization costs. Denote by \( \text{AV}_C \) the added-value function of the firm with recapitalization cost \( C \geq 0 \) and by \( \overline{M}_C \) and \( M_C \) the corresponding upper-dividend and liquidation barriers, respectively. There exists \( \overline{C} > 0 \) such that \( M_C < M_{\text{reg}} \) if and only if \( C < \overline{C} \). Moreover, \( \overline{C} \leq \text{AV}_0(\overline{M}_0) \).

3.4. Decomposition of value and added value

To identify the drivers of firm value, we decompose it into components attributable to financing operations, business of the current period, the limited liability option of shareholders, and the ability to write profitable new business in the future. Assume \( m \) is a capital level at which the firm continues to operate, i.e. \( \delta^*(m) = 1 \), and set

\[
M_* = M_*^{\text{ST}} = \gamma(m - z^*(m) + k^*(m) + q)(1 + r + \lambda^*(m) \rho_1) + R_1.
\]

By (22),

\[
V(m) = \text{FO}(m) + \frac{1}{1 + r} \text{E}_{\text{AV}}[V(M_*^{\text{ST}})],
\]

where the impact of financing operations \( \text{FO}(m) \) is defined as

\[
\text{FO}(m) := z^*(m) - k^*(m) - C(k^*(m)),
\]

This is precisely the case when \( \text{AV}(m) > \text{AV}(n) \) for every \( n < m \).
and reflects how dividend payments and recapitalization affect firm value. Since \( V(M^*_1) = \max[M^*_1, 0] + \mathcal{A}(M^*_1) \), we can write

\[
V(M^*_1) = \frac{M^*_1}{1+r} + \max[-M^*_1, 0] + \mathcal{A}(M^*_1)
\]

Net tangible capital corresponds to the value of internal funds available at the end of the period assuming no default is possible. It includes the profit margin from business from the current period and is given by

\[
\text{NTC}(m) := \frac{1}{1+r} E_q^r [M^*_1] = \gamma(m - z^*(m) + k^*(m) + q), \tag{28}
\]

where we have used Expression (26) and the fact that \( E_q^r [\rho_1] = E_q^r [R_1] = 0 \). The firm’s default option value reflects the limited liability of shareholders. Its value is

\[
D(m) := \frac{1}{1+r} E_q^r [\max[-M^*_1, 0]]. \tag{29}
\]

The firm’s franchise value reflects its ability to collect economic rents, i.e. to add value, from future operations and is given by

\[
FV(m) := \frac{1}{1+r} E_q^r [\mathcal{A}(M^*_1)]. \tag{30}
\]

With this terminology we immediately obtain the following decomposition of firm value.

**Proposition 3.8.** Assume that starting capital \( m \) is such that \( \delta^*(m) = 1 \). Then,

\[
V(m) = FQ(m) + \text{NTC}(m) + DQ(m) + FV(m).
\]

To obtain the added-value counterpart of the above decomposition, we first define economic profit at the start of the period to be

\[
\mathcal{B}(m) := q - (1 - \gamma) m - z^*(m) + k^*(m) + q - C(k^*(m)) - \max[-m, 0].
\]

Economic profit at the start of the period corresponds to the margin \( q \) adjusted for costs incurred, which include: the carry cost of capital \((1 - \gamma) m - z^*(m) + k^*(m) + q\); the cost of raising capital \( C(k^*(m))\); and the cost of not defaulting on claims \( \max[-m, 0] \). The following decomposition of added value is just a restatement of Proposition 3.8 in terms of added value.

**Corollary 3.9.** Assume that starting capital \( m \) is such that \( \delta^*(m) = 1 \). Then,

\[
\mathcal{A}(m) = V(m) - \max[m, 0] = \mathcal{B}(m) + DQ(m) + FV(m). \tag{31}
\]

We can further split franchise value to highlight the value of the recapitalization option. Noting that the set \( \{M_1 < M^*_1 < M_{\text{rec}}\} \) corresponds to those states in which the firm can continue to operate but only after recapitalizing, it makes sense to define the firm’s franchise value due to recapitalization as

\[
\mathcal{F}V(m) := \frac{1}{1+r} E_q^r [\mathcal{A}(M^*_1) \cdot \mathbb{1}_{\{M_1 < M^*_1 < M_{\text{rec}}\}}]
\]

\[
= \frac{1}{1+r} \left( \mathcal{A}(M_1) - C q [M_1 \leq M^*_1 < M_{\text{rec}}] \right)
\]

\[
+ \frac{1}{1+r} E_q^r [M^*_1 \cdot \mathbb{1}_{\{M_1 < M_{\text{rec}}\}}], \tag{32}
\]

where the equality follows from Proposition 3.6. The second term after the equality sign reflects the cost of settling outstanding liabilities when recapitalizing from negative capital levels. On the other hand, \( \{M^*_1 \geq M_{\text{rec}}\} \) corresponds to the states in which the firm can continue operating business without having to recapitalize so that franchise value from internal resources is

\[
\mathcal{F}V(m) := \frac{1}{1+r} E_q^r [\mathcal{A}(M^*_1) \cdot \mathbb{1}_{\{M^*_1 \geq M_{\text{rec}}\}}]. \tag{33}
\]

Note that

\[
FV(m) = \mathcal{F}V(m) + \mathcal{F}V(m).
\]

3.5. Capital buffers and policyholder protection

Assume initial capital \( m \) is such that \( \delta^*(m) = 1 \). Net tangible capital \( \text{NTC}(m) \), defined by Expression (28), corresponds to the amount of “physical” capital available at date 0 to absorb unexpected losses from the insurance portfolio. However, policyholders may effectively enjoy more protection than that provided by net tangible capital. Indeed, whenever \( M < 0 \) and \( \text{NTC} \leq m < 0 \), instead of defaulting, the firm chooses to recapitalize. This means that policyholders can view \( \max[-M, 0] \) as an additional layer of protection over and above net tangible capital. For this reason we refer to

\[
\text{EBB}(m) := \text{NTC}(m) + \max[-M, 0]
\]

as the effective capital buffer. **Proposition 3.6** characterizes the situations in which \( \text{EBB}(m) \) is strictly larger than \( \text{NTC}(m) \). Note also that \( \text{EBB}(m) = \text{NTC}(m) \) whenever \( M = M_{\text{rec}} \).

If \( M = M_{\text{rec}} \), then the firm always operates at the regulatory minimum and \( \text{EBB}(m) \) does not depend on the initial capital level \( m \). On the other hand, if \( M > M_{\text{rec}} \), then the effective capital buffer will depend on \( m \). Indeed, \( \text{NTC}(m) \) is maximal whenever capital after financing operations equals \( M \) and minimal whenever capital after financing operations equals \( M_{\text{rec}} \).

We highlight that whether or not the effective capital buffer is higher than what regulatory minimum requires is driven purely by the manager’s desire to maximize value and not by any concern for policyholder safety. Examples in Section 5 illustrate that in the relation \( M \leq \text{NTC} \leq M \) the inequalities can be strict or not, in all possible combinations depending on the environment in which the insurer operates.

4. Convexity of the value function and investment strategies

In this section we focus on optimal investment strategies. For this reason, we restrict our attention to situations where the firm is not liquidated. Moreover, we need only consider capital levels at or above \( M_{\text{rec}} \); since the amount of capital available for investments is always determined after financial operations. For all \( m \geq M_{\text{rec}} \) and \( \lambda \in [0, 1] \) we set

\[
M_{1}^{m, \lambda} := \gamma(m - z^*(m) + q)(1 + r + \lambda \rho_1) + R_1,
\]

which corresponds to end-of-period capital if the manager adopts the investment strategy \( \lambda \). Denoting by \( \text{ess inf}(X) \) the essential infimum of the random variable \( X \), we have that

\[
P(M_{1}^{m, \lambda} \geq \text{ess inf}(R_1)) = 1
\]

holds for every \( m \geq M_{\text{rec}} \) and \( \lambda \in [0, 1] \). Hence, for all practical purposes, properties of the value function \( V \) matter only on the interval \( (\text{ess inf}(R_1), \infty) \). If the value function is affine on \( (\text{ess inf}(R_1), \infty) \), then the firm is indifferent between taking investment risk or not. By contrast, if the value function is convex but not affine on this interval, then the firm seeks maximal investment risk exposure. The situations in which the value function exhibits these two behaviors are characterized in Proposition 4.2, if \( R_1 = 0 \), and in Proposition 4.4, if \( R_1 \neq 0 \).

4.1. No insurance losses

In the limiting case where there are no insurance losses, i.e. \( R_1 = 0 \), the firm receives the margin \( q \) without assuming any obligation. This, however, requires holding at least the minimum regulatory capital and incurring the corresponding carry cost.
Therefore, operating the firm is only worthwhile if the effective, per-period margin \( \gamma q \) exceeds the corresponding carry cost of capital \( (1 - \gamma)M_{\text{reg}} \), or, equivalently, if the per-period upfront profit \( \gamma q - (1 - \gamma)M_{\text{reg}} \) is strictly positive.

**Lemma 4.1.** If \( R_1 \equiv 0 \), then the Nontriviality Assumption (24) holds if and only if
\[
\gamma q - (1 - \gamma)M_{\text{reg}} > 0.
\]

The manager's task is to capture the benefits of receiving \( q \) while incurring the smallest possible cost. This translates into holding the least possible amount of capital, i.e. \( \bar{M} = M_{\text{reg}} \), and, if raising capital is costly, avoiding investment strategies that could make it necessary to recapitalize.

**Proposition 4.2.** If \( R_1 \equiv 0 \) or, equivalently, \( \text{ess inf}(R_1) = 0 \), then \( \bar{M} = M_{\text{reg}} \) and
\[
A(V)(M_{\text{reg}}) = \frac{1 + r}{r} (\gamma q - (1 - \gamma)M_{\text{reg}}). \tag{34}
\]
Moreover, the recapitalization option is valuable, i.e. \( M < M_{\text{reg}} \), if and only if
\[
\frac{1 + r}{r} (\gamma q - (1 - \gamma)M_{\text{reg}}) > C. \tag{35}
\]
Finally,

(i) If \( C = 0 \) or \( M_{\text{reg}} = 0 \), then the value function is affine on \( [0, \infty) \). In this case, the set of optimal investment strategies is the entire interval \( [0, 1] \).

(ii) If \( C > 0 \) and \( M_{\text{reg}} > 0 \), then the value function is neither convex nor concave on \( [0, \infty) \). In this case, the set of optimal investment strategies is the closed interval \( [0, \bar{\lambda}] \) with
\[
\bar{\lambda} := 1 - \frac{1}{1 + r} \frac{M_{\text{reg}}}{\gamma(M_{\text{reg}} + q)} \in (0, 1]. \tag{36}
\]

**Proposition 4.2** shows that, if the firm pursues a risk-free investment strategy, end-of-period capital never falls below \( M_{\text{reg}} \) and the firm receives the amount \( \gamma q - (1 - \gamma)M_{\text{reg}} \) perpetually without ever incurring recapitalization costs. Indeed, Formula (34) is just the present value of a perpetuity entitling shareholders to receive \( \gamma q - (1 - \gamma)M_{\text{reg}} \) at the beginning of every period. This applies to any investment strategy under which end-of-period capital cannot fall under \( M_{\text{reg}} \). If \( M_{\text{reg}} = 0 \), this is true for all investment strategies. If \( M_{\text{reg}} > 0 \), this is true only if the firm takes a sufficiently small amount of investment risk. This is precisely what the condition \( \lambda < \bar{\lambda} \) ensures.

**Proposition 4.2** can be used to derive an upper bound for the value of the firm in the general case when insurances losses are not zero. The value of an insurance firm exposed to insurance losses is clearly bounded from above by the value of a firm that obtains the amount \( I + q \) in every period and carries no insurance risk.\(^{46}\) Hence, when \( R_1 \not\equiv 0 \), we have for every \( m \geq 0 \):
\[
A(V)(m) \leq \frac{1 + r}{r} (I + \gamma q - (1 - \gamma)M_{\text{reg}}). \tag{37}
\]
This proves the following necessary condition for the Nontriviality Assumption (24) to be satisfied when insurance losses are nonzero.

**Corollary 4.3.** Assume \( R_1 \not\equiv 0 \). If the Nontriviality Assumption (24) is satisfied, then \( I + \gamma q - (1 - \gamma)M_{\text{reg}} > 0 \).

It follows that, for a fixed \( 0 < \gamma < 1 \), the regulatory capital requirement \( M_{\text{reg}} \) cannot be arbitrarily large.

46 This is the case because the new firm receives exactly the same premium as the original firm, but has no liabilities.

47 The proof of this fact relies on the conditional version of Jensen's inequality and on the market consistency of the valuation rule.
5. Examples

Optimal strategies depend on the financial and the regulatory conditions in which the insurer operates. When there is no insurance risk ($R = 0$) or recapitalization is costless ($C = 0$), these strategies can be fully characterized (see Proposition 4.2 and Proposition 4.4). When recapitalization is costly and insurance risk is nontrivial, a whole range of financial and investment strategies can occur. In particular, whether or not investing in the risky asset is optimal cannot be established without a narrower model specification, as the following examples illustrate.\footnote{The numerical examples are computed using the “value iteration method”, which relies on the approach used to prove Theorem 3.3, i.e. on applying the contraction principle to find a fixed point of the operator $\tau$, defined by
\[ \tau(v)(m) = \sup_{x \in \mathcal{D}(m)} \left( x - C(x) + \frac{1}{1 + r} E_{\tau}[v(M^x_M)] \right), \] in an appropriate function space.}

That being said, our numerical examples suggest that the firm will tend to seek investment risks in regions where the value function is locally convex and to avoid it in regions in which it is locally concave.

How to read the figures

It is helpful to bear two things in mind when reading the figures. First, as per our convention, whenever capital before financing operations $m$ is smaller or equal to $M$, we have set $\lambda^*(m) = 0$, although, strictly speaking, the firm has no capital to invest because it is liquidated. Second, whenever capital before financing operations $m$ is such that the firm continues to operate, we have $\lambda^*(m) = \lambda^*(m - z^*(m) + \kappa^*(m))$ because the amount relevant for investments is capital after financing operations. In particular, whenever $M < M_{\text{reg}}$ and $m \in [M, M_{\text{reg}}]$, then $\lambda^*(m) = \lambda^*(M)$ and whenever $m \geq M_{\text{reg}}$, $\lambda^*(m) = \lambda^*(M)$. Similarly, if $M = M_{\text{reg}}$, then the firm always operates at the regulatory minimum and the investment strategy for all capital levels $m$ at which the firm continues to operate is $\lambda^*(M_{\text{reg}})$.\footnote{This corresponds to an instantaneous interest rate of $r = 4.87\%$ for the money-market account.}

5.1. Financial and investment strategies in different financial environments

We start introducing a base case that we use as a benchmark. In the base case, it is optimal to hold capital above the regulatory minimum ($M > M_{\text{reg}}$) and the recapitalization option is valuable ($M < M_{\text{reg}}$). Moreover, depending on the level of capital, various levels of exposure to investment risk are optimal. We illustrate how these optimal strategies change with the cost of recapitalization $C$, the insurance margin level $q$, the carry cost of capital $1 - \gamma$, and the minimum regulatory capital requirement $M_{\text{reg}}$.

Throughout this section we consider a one-period risk-free rate $\rho = 5\%$, a volatility of the risky asset $\sigma = 10\%$, and a loss distribution constructed as follows: the densities of the $\mathcal{N}(0, 0.01)$ and $\mathcal{N}(3, 0.5)$ distributions\footnote{$\mathcal{N}(\mu, \sigma)$ denotes the normal distribution with mean $\mu$ and standard deviation $\sigma$.} are truncated outside the interval $[0, 6]$, added up, and normalized to obtain a new density with support $[0, 6]$. The resulting loss distribution is a smooth approximation of a Bernoulli random variable taking the values 0 and 3 with equal probabilities.\footnote{This loss distribution allows for a richer range of optimal strategies than, for instance, the more commonly used exponential distribution. In effect, Loeffen and Renault (2010) prove that when the tail of the Lévy measure is log-convex (as is the case with exponentially-distributed losses), all optimal dividend strategies are of barrier type.}

5.1.1. Base case

For the base case we specify the following parameters: $C = 0.1$, $q = 10\%$, $\gamma = 0.005$ and $M_{\text{reg}} = 0.25$. Fig. 1 superposes the graphs of the added-value function $A$, the dividend strategy (left vertical axis), and the investment strategy (right vertical axis); all regarded as functions of initial capital $C$. For our choice of $C$, $M = -2.4 < M_{\text{reg}}$, which implies that the recapitalization option is valuable. The optimal dividend strategy is of barrier type with $M = 3.6$. The optimal investment strategy is of bang–bang type, i.e. the firm chooses either full exposure...
to investment risk ($\lambda^* = 1$) or no investment risk at all ($\lambda^* = 0$). Interestingly, risky investment is optimal precisely at capital levels where the added value function is strictly convex.

5.1.2. Changing the cost of recapitalization

We now increase the recapitalization cost from $C = 0.1$ in the base case to $C = 2.5$. Fig. 2 shows that $\bar{M} = M_{reg}$ so that recapitalization is no longer valuable. The prohibitive recapitalization cost makes it optimal to increase the capital buffer above $M_{reg}$. Indeed, the upper-dividend barrier is now $\bar{M} = 7.4$, which is significantly higher than the 3.6 of the base case. The dividend-distribution strategy is richer than in the base case. The firm dividends out any capital in excess of $\bar{M}$, does not pay dividends in the interval $[1.3, \bar{M})$, and pays dividends down to $M_{reg}$ when capital belongs to the dividend-distribution band $[M_{reg}, 1.3]$. This pattern is a consequence of the shape of the loss distribution and the fact that the firm never recapitalizes: at levels of capital at which falling below $M_{reg}$ and having to liquidate is likely, there would be no benefit to compensate the carry cost of holding capital above the regulatory minimum. By contrast, at higher levels of capital, it does make sense to keep the excess within the firm because it helps avoid liquidation. The investment strategy is no longer of bang–bang type. Indeed, it is optimal to take full investment risk when capital lies in the interval $(1.47, 1.62)$, to only take partial investment risk when capital lies in the interval $(2.75, 3.1)$, and not to take any investment risk otherwise. As in the base case, investment risk is taken precisely when capital belongs to intervals in which the added value function is strictly convex.

5.1.3. Changing the margin

We now decrease the margin from $q = 10\%l = 0.142$ in the base case to $q = 5\%l = 0.071$. Fig. 3 shows that after decreasing $q$ it is no longer attractive to hold capital above the regulatory minimum, i.e. $\bar{M} = M_{reg}$, resulting in a dividend strategy of barrier type. This is because, with a lower margin, there is less added value to protect. The reduction in added value also makes recapitalization less attractive, as can be seen from the increase of the liquidation barrier from $M = -2.4$ in the base case to $M = -1.4$. As long as the firm operates, capital after financial operations remains always at the same level, with the amount $\gamma(M_{reg} + q)$ available for investment. As a result, the optimal investment strategy does not depend on $m$ and, in this case, requires taking full investment risk, i.e. $\lambda^* = 0$. The reason is that, by taking investment risk, the firm stands to gain more from the increase in value of the default option than what it can potentially lose in terms of added value.

Remark 5.1. If, together with decreasing $q$ to 0.071, we increase the cost of raising capital to $C = 2.5$, recapitalization becomes too costly, that is $\bar{M} = M_{reg}$. In such a setting the firm operates with capital at the regulatory minimum $M_{reg}$ and is liquidated as soon as capital falls below $M_{reg}$.

5.1.4. Changing the carry cost of capital

Next we consider the effect of increasing the carry cost of capital from $1 - \gamma = 0.005$ in the base case to $1 - \gamma = 0.03$. Fig. 4 shows that, with this particular increase in $1 - \gamma$, holding capital becomes so costly that it is no longer optimal to keep any capital above the regulatory minimum, i.e. $\bar{M} = M_{reg}$, implying a dividend strategy of barrier type. As in the preceding example, at capital levels where the firm is not liquidated, capital after financial operations remains at the same level and the optimal investment strategy does not depend on $m$. Here, it is also optimal to take full investment risk, i.e. $\lambda^* = 1$. Clearly, the higher carry cost of capital, reduces the amount of value the firm can add, leading to an increase of the liquidation barrier from $-2.4$ in the base case to $-1.83$.

5.1.5. Changing the minimal regulatory capital requirement

Finally, we increase the minimal regulatory capital requirement from $M_{reg} = 0.25$ in the base case to $M_{reg} = 3$. Having to incur carry costs on a higher level of capital reduces the firm’s potential to add value and results in an increase of the liquidation barrier from $-2.4$ in the base case to $-2.2$ as shown in Fig. 5. Capital above $M_{reg}$ is held to avoid having to incur recapitalization.
costs. Since, at any capital level, an increase in $M_{\text{reg}}$ makes the likelihood of having to recapitalize higher, the upper-dividend barrier $\overline{M}$ also increases from 3.6 in the base case to 6. However, due to the presence of carry costs, $\overline{M}$ does not increase by the same amount as $M_{\text{reg}}$. It is optimal to take full investment risk at capital levels between $M_{\text{reg}}$ and 4. As capital increases from 4 to 4.2, the proportion of investments in the risky asset gradually decreases from 1 down to 0.

5.2. Risky investments can also support franchise value

The previous section shows that increasing the value of the default option can justify pursuing risky investment strategies; see, for instance, Cases 5.1.3 and 5.1.4. In particular, Proposition 5.4 establishes that, when recapitalization is costless and there are states in which it is worthwhile defaulting, taking maximal investment risk to increase the value of the default option is the only optimal choice. However, increasing the value of the default option is not the only possible reason for investing in the risky asset. To illustrate this, consider a firm with the following features by increasing the margin from $\varrho = 10\%$ to $\varrho = 0.142$, in the base case, to $\varrho = 20\%$ = 0.284.\footnote{In this case, the liquidation barrier is $\overline{M} = -5.48$ and the worst possible insurance loss is $\text{ess inf}[\tilde{R}] = -4.572$. Hence, for an operating firm, capital can never fall below $\overline{M}$, i.e. the firm is never liquidated.} We now apply the decomposition of firm value in Proposition 3.8 at a capital level $m \in [M_{\text{reg}}, \overline{M}]$. First observe that

\begin{equation}
F_0(m) = 0, \quad \text{and} \quad D_0(m) = -\frac{1}{1 + r}E_{\varrho}(\overline{M}^*|M_{\text{reg}} < \overline{M}^*),
\end{equation}

where we have used that $Q^*(M_{\text{reg}}^* < \overline{M}) = 0$. It follows that

\begin{equation}
V(m) = NTC(m) + IFV(m) + D_0(m) + RF(m)
\end{equation}

\begin{equation}
= NTC(m) + IFV(m) - \frac{1}{1 + r}E_{\varrho}(\overline{M}^*|M_{\text{reg}} < \overline{M}^*)
\end{equation}

\begin{equation}
\quad + \frac{1}{1 + r}(\overline{M}^* - C)Q^*(\overline{M} \leq M_{\text{reg}}^* < \overline{M})
\end{equation}

\begin{equation}
\quad + \frac{1}{1 + r}E_{\varrho}(\overline{M}^*|M_{\text{reg}} < \overline{M}^*)
\end{equation}

\begin{equation}
\quad \text{value of outstanding liabilities when capital is negative}
\end{equation}

\begin{equation}
= NTC(m) + IFV(m) + \frac{1}{1 + r}(\overline{M}^* - C)Q^*(\overline{M} \leq M_{\text{reg}}^* < \overline{M}).
\end{equation}

Observe that any increase in the value of the default option is offset by the value of outstanding liabilities in those instances where the firm could have defaulted but chooses to recapitalize. This is particularly interesting because it shows that any gain from investing in the risky asset cannot be attributed to an increase in the default option value but rather to a boost in franchise value. Fig. 6 shows this is what occurs at levels of capital between $M_{\text{reg}} = 0.25$ and 1.4 where taking maximal investment risk is optimal.

6. Potential extensions

In this section we comment on three of our main assumptions: independence of insurance and financial market risk, regulatory capital requirements, and the cost of raising capital.

**Independence of insurance and financial market risk.** Independence allows the explicit construction of the underlying probability space as a product space. However, the general approach would also work under the weaker requirement that the additional information that the insurer possesses does not introduce arbitrage opportunities for the insurer.\footnote{Although $2^*$ is measured in nominal terms and $\lambda^*$ is a ratio, in this figure both quantities are measured along the same vertical axis for illustration purposes.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Decreasing the margin; $C = 0.1$, $M_{\text{reg}} = 0.25$, $1 - \varrho = 0.005$ and $\varrho = 5\%$ = 0.071 (\varrho = 10\% = 0.142 in the base case). The liquidation barrier is $\overline{M} = -1.38$. The upper-dividend barrier is $\overline{M} = M_{\text{reg}}$. The optimal dividend strategy is of barrier type and no capital in excess of $M_{\text{reg}}$ is kept within the firm. The optimal investment strategy prescribes maximal investment risk.}
\end{figure}
Fig. 4. Increasing the carry cost of capital: $C = 0.1$, $M_{\text{reg}} = 0.25$, $1 - \gamma = 0.03$ ($1 - \gamma = 0.005$ in the base case), and $q = 0.142$. The liquidation barrier is $\overline{M} = -1.83$. The upper-dividend barrier is $M = M_{\text{reg}}$. The optimal dividend strategy is of barrier type and no capital in excess of $M_{\text{reg}}$ is kept within the firm. The optimal investment strategy prescribes maximal investment risk.

Fig. 5. Increasing the minimal capital requirement: $C = 0.1$, $M_{\text{reg}} = 3$ ($M_{\text{reg}} = 0.25$ in the base case), $1 - \gamma = 0.005$ and $q = 0.142$. The liquidation barrier is $\overline{M} = -2.2$. The upper-dividend barrier is $M = 6$. The optimal dividend strategy is of barrier type: the firm pays any capital in excess of 6 as dividends and pays no dividends for $m \in (M_{\text{reg}}, 6)$. The optimal investment strategy prescribes maximal investment risk when $m \in (M_{\text{reg}}, 4)$, partial investment risk when $m \in (4, 4.2)$ (gradually dropping from maximal to no investment risk), and no investment risk when $m \geq 4.2$. The region of strict convexity of the added-value function is highlighted.

Turning to regulatory capital requirements and the cost of raising capital, none of the changes discussed below have a qualitative impact either on the monotonicity properties of the value and added-value functions, or on the basic structure of dynamic programming problem. In particular, the results on the regularity properties of the value and added-value functions and the existence of optimal stationary strategies remain valid. We thus comment only on their implications on the optimal recapitalization and investment behavior of the firm.

**Regulatory capital requirements**: If we replace fixed capital requirements by a “risk-sensitive capital requirement”, such as the VaR-based capital requirement in Solvency II, the amount of required capital $M_{\text{reg}}(\lambda)$ will be an increasing function of $\lambda \in [0, 1]$, the proportion of financial risk the firm takes. Under such a
In the solvency regime, there would still exist an upper dividend barrier $\overline{M}$ because, from a given level of capital on, the costs of holding additional capital outweighs the benefits of holding it. The monotonicity of the value added function implies that, also in this case, added value is highest at $\overline{M}$. To understand the firm's recapitalization behavior, we recall that in the fixed capital requirement case, the rationale for not recapitalizing before falling below the minimum regulatory requirement was to avoid incurring the carry cost of capital as long as it is not strictly necessary. This rationale remains valid under a risk-sensitive solvency regime so that the firm will never recapitalize unless it falls below $M_{\text{reg}}(0)$. As the cost of recapitalization does not depend on how much capital the firm raises, the firm will always recapitalize up to $\overline{M}$, where added value is highest.

To understand what happens with the optimal investment strategy, it is useful to benchmark the risk-sensitive model against two models with fixed capital requirements at $M_{\text{reg}}^-(\lambda) := M_{\text{reg}}(0)$ and $M_{\text{reg}}^+(\lambda) := M_{\text{reg}}(1)$. In particular, we have:

$$M_{\text{reg}}^-(\lambda) \leq M_{\text{reg}}(\lambda) \leq M_{\text{reg}}^+(\lambda) \quad \text{for every } \lambda \in [0, 1].$$

As a result, investing in the risky asset in an environment with risk-sensitive capital requirements is more capital intensive (i.e. costlier) than doing so under the fixed capital requirement $M_{\text{reg}}^-$ and less capital intensive (i.e. less costly) than under the fixed capital requirement $M_{\text{reg}}^+$. This implies that, whenever the optimal policy under the constraint $M_{\text{reg}}$ is risk free, the optimal investment strategy under a variable capital requirement will also be risk free. Similarly, whenever the optimal investment strategy under the constraint $M_{\text{reg}}^-$ requires to invest in the risky asset, the optimal strategy under a variable capital requirement will also involve investing in the risky asset.

**Cost of raising capital:** Assume that the cost of raising capital has an additional component that is proportional to the amount of capital raised, i.e.

$$C(\kappa) = c_1 1_{\kappa > 0} + c_2 \kappa, \quad \kappa \geq 0,$$

with $c_1, c_2 > 0$. In this case, the same arguments as in the fixed cost model imply that there is an upper dividend barrier $\overline{M} = M_{\text{reg}}$ so that capital in excess of it is always paid back as dividends.

Moreover, also here, the monotonicity of the value added function implies that added value is highest at $\overline{M}$. The incentive to avoid the cost of holding capital when not strictly necessary, will also lead the firm not to recapitalize unless capital falls below the regulatory minimum. The firm will always recapitalize up to the level where the difference between the added value at that level and the cost of raising the necessary capital is maximized. As a result, when the firm chooses to recapitalize, the level it recapitalizes to will not necessarily be the upper dividend barrier.

Adding a proportional component to the cost of raising capital will make taking financial market risk costlier because it increases the likelihood and size of costly recapitalization. Hence, we would expect to see the same firm take less investment risk under this more general cost structure than under a fixed cost of raising capital. The extent of this effect will depend on the magnitude of $c_2$.

**7. Conclusions**

We have studied a dynamic model for a value-maximizing insurance firm that takes decisions on its capital and investment strategies, including the possibility of liquidation. In a first step we developed a rigorous economic valuation framework for insurers with a broad ownership base. Because in an insurance firm cash flows to shareholders generally depend on financial-market returns, it is necessary to use a market-consistent valuation measure to assess their value. Using discounted expected cash flows with respect to the “objective” probability measure, as often done in the literature, would mean second guessing market prices and creating a bias toward risky investments due to their higher expected returns. This would result in an incorrect firm value and suboptimal investment strategies. On the other hand, due to the firm’s broad ownership base, the manager should be indifferent to idiosyncratic risk and act as a risk-neutral investor whenever valuing cash flows that do not depend on financial-market returns. We proved the existence of a unique economic valuation measure that is both market consistent and captures indifference toward idiosyncratic risk. Using this measure, the economic value of the firm is the NPV of cash flows to shareholders under an
optimal strategy. We used dynamic-programming techniques to prove the existence of stationary optimal strategies.

To analyze investment strategies, we derived a decomposition of firm value and added value that isolates the four sources of value: the impact of financing operations, net tangible capital, default option value, and franchise value. The investment strategy only impacts the default option value and the franchise value and, typically, in opposite directions. Hence, whether or not to take investment risk is the result of how this trade-off resolves. We provide a full description of optimal investment strategies when either recapitalization is costless or there is no insurance risk. When recapitalization is costly and insurance risk nontrivial, we find that optimal investment strategies can cover the full range from risk-free to maximally risky investments depending on the particular constellation of the financial and regulatory environment within which the firm operates. Interestingly, investment in risky assets may not only be driven by the desire to increase firm value by boosting the option to default. Indeed, it may also serve as a surrogate for recapitalization by helping the firm reach capital levels at which franchise value is greater.

Appendix A

This Appendix provides proofs for the results in the paper with the exception of the existence and uniqueness of the economic valuation measure, which are relegated to the Online Appendix. In the Online Appendix we also substantiate the claim that the firm manager can focus exclusively on admissible strategies.

A.1. Proofs of results in Section 3

Proof of Proposition 3.1. (i) If \( m \in \mathbb{R} \) and \( S = (\max[m, 0], 0, 0, 0) \), then \( S \) is always admissible and \( 0 \leq \max[m, 0] \leq V^S(m) \leq V(m) \).

(ii) Let \( S \in S(m_1) \) be arbitrary and \( m_2 \geq m_1 \). Consider the strategy \( \tilde{S} \in S(m_2) \) which, after date 0, is identical to \( S \) and, at date 0, is given by \( \tilde{z}_0 = \max[m_2 - m_1 + z_0 - \kappa_0, 0] \), \( \tilde{\kappa}_0 = \max[m_1 - m_2 + \kappa_0 - z_0, 0] \), \( \kappa_0 = \lambda_0 \) and \( \delta_0 = \delta_1 \). In particular, \( m_2 - \tilde{z}_0 + \kappa_0 = m_2 - z_0 + \kappa_0 \). Writing \( V^S(m_1) \) and \( V^S(m_2) \) as in (14) and noting that \( \tilde{S} = S^* \), we get

\[
V^S(m_1) = z_0 - \kappa_0 - C(\kappa_0) + V^{S^*}(m_1 - z_0 + \kappa_0) \\
= m_1 - m_2 + C(\kappa_0) - C(\kappa_0) \\
+ \tilde{z}_0 - \kappa_0 - C(\kappa_0) + V^{S^*}(m_2 - \tilde{z}_0 + \kappa_0) \\
\leq m_1 - m_2 + V^{S^*}(m_2) \\
\leq m_1 - m_2 + V(m_2) \tag{A.1}
\]

The first inequality holds since \( m_2 \geq m_1 \) implies \( \tilde{\kappa}_0 \leq \kappa_0 \), thus \( C(\kappa_0) - C(\kappa_0) \leq 0 \). Taking the supremum over all \( S \in S(m_1) \) yields the result.

(iii) This follows directly from (i) and (ii).

(iv) This follows from (i)-(iii) since, for every \( m \in \mathbb{R} \), we have \( V^A(m) = V(m) - \max[m, 0] \) by the definition of \( V^A \) through Expression (18). \( \square \)

Proof of Theorem 3.3. Recall that \( \mathcal{X} = \mathbb{R} \cup \{ \ast \} \), equipped with the disjoint union topology, is a separable, completely metrizable space. Define, for every \( a > 0 \), the weight function

\[
w_a(x) := \begin{cases} 
1 + ([x]_a)_+, & x \in \mathcal{X} \setminus \{ \ast \} \\
1, & x = \ast.
\end{cases}
\]

We search for a solution to the dynamic-programming equation (20) within the space \( B_{w_a}(\mathcal{X}) \) of Borel-measurable functions

\[f : \mathcal{X} \to \mathbb{R} \quad \text{satisfying} \quad \|f\|_a < \infty, \quad \text{where} \quad \|f\|_a := \sup_{x \in \mathcal{X}} |f(x)| w_a(x).\]

For brevity we introduce the impact of financing operations

\[c(m, s) := z - \kappa - C(\kappa) \quad m \in \mathcal{X}, \ s \in S_0(m).\]

Step 1. We verify that our model satisfies Assumptions 8.5.1-8.5.3 of Section 8.5 in Hernández-Lerma and Lasserre (1999).

We start with the three parts of Assumption 8.5.1 in Hernández-Lerma and Lasserre (1999):

(i) We first establish that the map \( m \mapsto S_0(m) \) is compact valued and upper semicontinuous. By definition, \( S_0(m) \) is compact for all \( m \in \mathcal{X} \). Theorem 17.20 in Aliprantis and Border (2006) guarantees that upper semicontinuity of \( m \mapsto S_0(m) \) is equivalent to sequential closedness of its graph. To see that this is the case, consider a convergent sequence \( (m_n, s_n), n \geq 0 \) with elements in the graph of \( S_0 \) and denote its limit by \( (m, s) \). There are two possibilities:

(a) If \( m = \ast \) (equivalently \( \delta = 0 \), for \( n \) large enough, we must have \( m_n = \ast \) and \( s_n = (0, 0, 0, 0) \). We conclude that \( (s_n, n \geq 0) \) converges to \((0, 0, 0, 0) \in S_0(\ast)\). 

(b) If \( m \in \mathbb{R} \) and \( \delta = 1 \), then, for every \( n \in \mathbb{N} \), \( \delta_n = 1 \) and \( 0 \leq z_n, 0 \leq \kappa_n, \lambda_n \in [0, 1] \), and

\[
M_{\text{reg}} \leq m_n - z_n + \kappa_n \leq M_{\text{reg}} + \frac{C}{(1 - \gamma)(1 + r)}. 
\]

It follows that these conditions also hold for \( m, z, \kappa, \lambda \). Moreover, since \( z_n \kappa_n = 0 \) and \( z_n \kappa_n \to z \kappa \), we have \( z \kappa = 0 \). Hence, \( s = (z, \kappa, \lambda, 1) \in S_0(m) \).

(ii) Next we must show that, for every \( m \in \mathcal{X} \), the map \( s \mapsto c(m, P) \) is upper semicontinuous. This follows directly from the lower semicontinuity of the map \( m \mapsto C(\kappa) = C \cdot \mathbb{I}_{(0, \infty)}(\kappa) \).

(iii) Finally, we prove that, for all \( m \in \mathcal{X} \), the map \( s \mapsto E_{Q^m}[w_0(M_1^{m,s})] \) is continuous on \( S_0(m) \). In the case \( m = \ast \), we have \( s \mapsto E_{Q^m}[w_0(M_1^{m,s})] \equiv 1 \), which is clearly continuous. Otherwise, consider \( s \in S_0(m) \) and note that

\[w_0(M_1^{m,s}) = |M_1^{m,s}| + 1 \leq |s - Q^m| + 1 \leq \gamma |s - Q^m| + \gamma^r |Q^m| + |r_1| + 1 \leq \gamma \frac{C}{(1 - \gamma)(1 + r)}(M_{\text{reg}} + |Q^m|) + |r_1| + 1. \tag{A.2}\]

Given that the right-hand side of the above expression is integrable, the dominated convergence theorem implies the continuity of the map \( s \mapsto E_{Q^m}[w_0(M_1^{m,s})] \).

To establish Assumption (8.5.2) in Hernández-Lerma and Lasserre (1999) we show that there exist \( a > 0, \tau \geq 1, \beta \in [1, 1 + r) \) such that, for all \( m \in \mathcal{X} \),

\[
\sup_{s \in S_0(m)} |c(m, s)| \leq \tau w_a(m), \quad \text{and} \quad \sup_{s \in S_0(m)} E_{Q^m}[w_0(M_1^{m,s})] \leq \beta w_a(m) \tag{A.3}, \tag{A.4}
\]

If \( m = \ast \), then \( S_0(m) = \{(0, 0, 0, 0)\} \). In this case, both conditions are satisfied with \( \tau = \beta = \gamma \) for any choice of \( a > 0 \) and \( \beta \geq 1 \). For \( m \in \mathbb{R} \), consider an arbitrary \( s \in S_0(m) \). Regardless of the value of \( \delta \), by definition of \( S_0(m) \)

\[|c(m, s)| \leq |m| + C + M_{\text{reg}} + C \frac{1}{(1 - \gamma)(1 + r)} = |m| + \tilde{C} \tag{A.5}. \]

Since the function \( f(m) := |m| + \tilde{C}/w_a(m) \) is bounded, there exists \( \tau \geq 1 \) such that \( f(m) \leq \tau \) for all \( m \). This implies that \( |c(m, s)| \leq \tau w_a(m) \) holds for all \( s \in S_0(m) \) and, therefore, that Condition (A.3) is satisfied. If \( s \in S_0(m) \) is such that \( \delta = 0 \) then
It follows that \( w_2(M_1^m) = 1 \leq w_2(m) \leq \beta w_2(m) \) for any choice of \( \beta \in [1, 1 + r] \) and, hence, that Condition (A.4) is satisfied. On the other hand, if \( \delta = 1 \), then

\[
|M_1^{\max}| = |\gamma (m - \kappa + \rho) + R_1| \\
\leq \gamma \left( \frac{1}{1 - \gamma} \right) (1 + \rho) + |R_1| =: Y. 
\]

The right-hand side is \( \mathcal{Q}^\star \)-integrable, hence there exist \( a \geq 1 \) such that \( E_{\mathcal{Q}^\star}\{\max[Y - a, 0]\} \leq \frac{\gamma}{2} \). As a result,

\[
E_{\mathcal{Q}^\star}\{w_2(M_1^m)\} = 1 + E_{\mathcal{Q}^\star}\{\max[1, -a, 0]\} \\
\leq E_{\mathcal{Q}^\star}\{\max[Y - a, 0]\} < 1 + \frac{\gamma}{2} =: \beta. 
\]

Together with the fact that \( w_2 \geq 1 \), this implies that \( E_{\mathcal{Q}^\star}\{w_2(M_1^m)\} \leq \beta w_2(m) \) for all \( m \in S_0(m) \), which yields Condition (A.4).

Finally, we verify Assumption 8.5.3 in Hernández-Lerma and Lasserre (1999) by proving that the map \( s \mapsto E_{\mathcal{Q}^\star}\{u(M_1^m)\} \) is continuous for every \( m \in \mathcal{X} \) and every bounded, continuous function \( u: \mathcal{X} \to \mathcal{R} \). If \( m = \star \), then the only admissible strategy \( s \) is \( (0, 0, 0) \), and continuity of \( s \mapsto E_{\mathcal{Q}^\star}\{u(M_1^m)\} \) is trivial. For \( m \in \mathcal{R} \) and \( s \in S_0(m) \), the map \( \omega \mapsto u(M_1^m(\omega)) \) is measurable. Moreover, for every \( \omega \in \Omega \), the map \( s \mapsto u(M_1^m(\omega)) \) is continuous. Consider a sequence \( (s_n, n \in \mathbb{N}) \subset S_0(m) \) converging to some \( s \), which must belong to \( S_0(m) \) by compactness. By the assumptions on \( u \), the sequence \( (u(M_1^m(n)), n \in \mathbb{N}) \) is uniformly bounded. The result now follows by direct application of the dominated convergence theorem.

**Step 2.** Given that our problem satisfies Assumptions 8.5.1-8.5.3 in Hernández-Lerma and Lasserre (1999), we immediately have the following:

- The value function \( V \) is the unique solution in \( B_\mathcal{C}(\mathcal{X}) \) to the fixed point problem

\[
V(m) = \sup_{s \in S_0(m)} \left( z - \kappa - C(s) + \frac{1}{1 + r} E_{\mathcal{Q}^\star}\{V(M_1^m)\} \right). \tag{A.6}
\]

- From Expression (8.5.3) in Hernández-Lerma and Lasserre (1999), we have that \( V \) (thus \( \mathcal{A} \)) is upper semicontinuous and, as \( V \) and \( \mathcal{A} \) are nondecreasing, this is equivalent to continuity from the right.

- There exists \( \mathcal{A} \) (a Borel) measurable selection

\[
D^\star(\cdot) = (z^\star(\cdot), \kappa^\star(\cdot), \lambda^\star(\cdot), \delta^\star(\cdot)): \mathcal{X} \to \mathcal{R}_+ \times \mathcal{R}_+ \times [0, 1] \times [0, 1] \]

of \( S_0 \), i.e. \( s^\star(m) \in S_0(m) \) for all \( m \in \mathcal{X} \), that attains the maximum in (A.6), i.e.

\[
V(m) = z^\star(m) - \kappa^\star(m) - C(\kappa^\star(m)) + \frac{1}{1 + r} E_{\mathcal{Q}^\star}\{V(M_1^m)\}. \tag{A.7}
\]

It follows that \( D^\star \) is an optimal decision function. \( \square \)

**Proof of Proposition 3.5.** We know from the defining Property (53) of admissible strategies that, if capital is above \( C \), dividends should be paid out so that, by the Dividend Principle (23), \( \mathcal{A} \) is constant on \( \{1 - \gamma\} + M_{\text{reg}} \), and

\[
\overline{M} := \inf\{r : \mathcal{A} \text{ is constant on } [r, \infty)\} \leq \frac{C}{1 - \gamma + M_{\text{reg}}}. 
\]

Invoking again the Dividend Principle (23), we conclude that whenever \( m > \overline{M} \) dividends should be paid out down to \( \overline{M} \). \( \square \)

**Proof of Proposition 3.6.** From the Dividend Principle (23) it follows that the firm is always liquidated if capital \( m < M \). Given that \( \mathcal{A}(m) = V(m) - \max\{0, m\} \), it is easy to see that

\[
\mathcal{A}(m) = \mathcal{A}(m - z^\star(m) + \kappa^\star(m)) - C(\kappa^\star(m)) + \min\{m, 0\}, \quad \text{for } m \in \mathcal{R}. \tag{A.8}
\]

Assume that \( \mathcal{A}(\overline{M}) \leq C \). Then, since \( \mathcal{A}(m - z^\star(m) + \kappa^\star(m)) \leq \mathcal{A}(m) \), Expression (A.8) yields that \( \mathcal{A}(m) = 0 \) for \( m < M_{\text{reg}} \). This implies that \( \overline{M} = M_{\text{reg}} \). Moreover, the Nontriviality Assumption, equation (24), ensures that the firm is not liquidated at \( M_{\text{reg}} \).

Assume now that \( \mathcal{A}(\overline{M}) > C \). Since firm added value attains its maximum at \( \overline{M} \), Expression (A.8) tells us that, for \( m < M_{\text{reg}} \), added value is strictly positive if and only if \( m > C - \mathcal{A}(\overline{M}) \) in which case the firm recapitalizes to \( \overline{M} \) as it is easily seen from Expression (A.8). It follows that \( \overline{M} = \mathcal{A}(\overline{M}) - C < 0 \) and that the firm is liquidated if and only if \( m \leq \overline{M} \). It also follows, that, when \( m < M_{\text{reg}} \), the firm recapitalizes up to \( \overline{M} \), and proves (i).

If \( m \in (M_{\text{reg}}, \overline{M}) \), then we see from Expression (A.8) that

\[
\mathcal{A}(m) = \mathcal{A}(\overline{M}) - C + \min\{m, 0\}, \quad \text{which implies that } V(m) = \max\{m, 0\} + \mathcal{A}(m) = m + \mathcal{A}(M) - C.
\]

proving (ii). Item (iii) now follows since \( \mathcal{A}(m) = V(m) - \max\{m, 0\} \).

The proof of Proposition 3.7 requires the following lemma establishing that the value function depends continuously on the recapitalization costs.

**Lemma A.1.** Denote by \( V_\mathcal{C} \) the value function of the insurance firm with recapitalization cost \( C \geq 0 \). Then \( |V_\mathcal{C}(m) - V_\mathcal{C}(m)| \leq \frac{1}{2} |C_1 - C_2| \), for every \( m \in \mathcal{R} \) and \( C_1, C_2 \geq 0 \). The same is true if we replace \( V_\mathcal{C} \) by \( V/C \).

**Proof.** Fix \( m \in \mathcal{R} \) and denote by \( S^\mathcal{C}(z^\star, \kappa^\star, \lambda^\star, \delta^\star) \) an optimal strategy when refinancing costs are \( C \geq 0 \). Assume first that \( 0 \leq C_1 \leq C_2 \).

\[
V_\mathcal{C}(m) \leq V_\mathcal{C}(m) = E_{\mathcal{Q}^\star}\left[ \sum_{n=0}^{\infty} \frac{\delta_n}{1 + r^n} (z_n^\mathcal{C}_{C_1} - \kappa_n^\mathcal{C}_{C_1} - C_1 1_{e_n^0}) \right] \\
\leq E_{\mathcal{Q}^\star}\left[ \sum_{n=0}^{\infty} \frac{\delta_n}{1 + r^n} (z_n^\mathcal{C}_C - \kappa_n^\mathcal{C} - C_2 1_{e_n^0}) \right] + \left[ \frac{1}{r} |C_1 - C_2| \right] \\
\leq V_\mathcal{C}(m) + \left[ \frac{1}{r} |C_1 - C_2| \right], \tag{A.9}
\]

where we have used that, when \( C_1 \leq C_2 \), the strategy \( S^\mathcal{C}_C \) is also admissible for the firm with recapitalization costs \( C_2 \). The above estimate is equivalent to \( |V_\mathcal{C}(m) - V_\mathcal{C}(m)| \leq \frac{1}{2} |C_1 - C_2| \). If \( C_2 \leq C_1 \) then we exchange the roles of \( C_1 \) and \( C_2 \) to obtain the same estimate. \( \square \)

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**Proof of Proposition 3.7.** We use the notation of Lemma A.1 and denote by $M$ the liquidation barrier for the firm with recapitalization cost $C \geq 0$. By Proposition 3.6, $M_{reg} < M_{reg}$ if and only if $A'(C) > C$. By Remark 3.2 we have $A'(M_{reg}) \leq A'(M_{reg})$. Hence, if $C \leq A'(M_{reg})$ we automatically have $C \geq A'(C)$ and, hence, $M = M_{reg}$. It is clear that the range of recapitalization costs at which the firm recapitalizes is an interval: if a firm has an incentive to recapitalize, then any firm that is an identical but has lower recapitalization costs will have an even greater incentive to do so. By Proposition 3.6, $M < 0$ if $C = 0$. Hence, this interval is not empty. We show that it is right open. Indeed, Lemma A.1 implies that if $A'(C) > C$ holds for some $C$, we will have $A'(M) > C$ for any $C \geq 0$ that is sufficiently close to $C$. □

**A.2. Proofs of results in Section 4**

**Proof of Lemma 4.1.** We start with a preliminary remark. As there are no insurance losses, the value of the default option is zero and the decomposition of added value of Corollary 3.9 yields

$$ A'(M) = \gamma q - (1 - \gamma) M + \frac{1}{1+r} E_q R[A'(M_{reg})]. \quad (A.10) $$

Using that the added-value function attains its maximum at $M$ and that $M_{reg} \leq M$, Expression (A.10) implies that $A'(M) \leq \gamma q - (1 - \gamma) M_{reg} + \frac{1}{1+r} A'(M_{reg})$. It follows that

$$ A'(M_{reg}) \leq A'(M) \leq \frac{1 + r}{1 + r} (\gamma q - (1 - \gamma) M_{reg}) \quad (A.11) $$

If the Nontriviality Assumption (24) holds, i.e. $A'(M_{reg}) > 0$, then Inequality (A.11) immediately implies that $\gamma q - (1 - \gamma) M_{reg} > 0$ must hold. On the other hand, assume now that $\gamma q - (1 - \gamma) M_{reg} > 0$ is satisfied. We define the strategy $S = S(M_{reg})$ recursively. For the first step set $z_0 = 0$, $k_0 = 0$, $\lambda_0 = 0$ and $\delta_0 = 1$ so that $M_1 = M_{reg}^1 = \gamma (M_{reg} + q)(1 + r)$. Then, if $S_{n-1}$ has already been defined, set

$$ z_n = \max(M_n - M_{reg}, 0), \quad k_0 = 0, \quad \lambda_n = 0, \quad \delta_n = 1, $$

so that $M_{n+1} = \gamma (M_n - z_n + q)(1 + r) = \gamma (M_n + q)(1 + r)$. For every $n \in \mathbb{N}$, we have

$$ M_n - M_{reg} = (1 + r) \left[ \gamma q - \frac{1}{1 + r} \gamma M_{reg} \right]. $$

Since $\gamma q - (1 - \gamma) M_{reg} > 0$, this implies that $M_n > M_{reg}$ for all $n \in \mathbb{N}$ and, hence, that $S$ indeed belongs to $S(M_{reg})$. Firm value at $M_{reg}$ given this strategy is easily computed to be

$$ V^s(M_{reg}) = E_q \left[ \sum_{n=1}^{\infty} \frac{M_n - M_{reg}}{(1 + r)^n} \right] = 1 + r \left[ \gamma q - (1 - \gamma) M_{reg} \right] + M_{reg}. $$

Assumption $\gamma q - (1 - \gamma) M_{reg} > 0$ now implies that $0 < A'(M_{reg}) \leq A'(M_{reg})$, i.e. the Nontriviality Assumption holds.

Because we need it in the proof of Proposition 4.2, we point out that we have actually proved that $M = M_{reg}$ and that the strategy $S$ is optimal. Indeed, Inequality (A.11) and Expression yield

$$ A'(M_{reg}) \leq A'(M_{reg}) \leq A'(M) \leq \frac{1 + r}{1 + r} (\gamma q - (1 - \gamma) M_{reg}) = A'(M_{reg}) \quad (A.12) $$

implying equality everywhere. Hence, $A'(M_{reg}) = \frac{1 + r}{1 + r} \gamma q - (1 - \gamma) M_{reg} + M_{reg}$. □

**Proof of Proposition 4.2.** In the preceding proof we showed that $M = M_{reg}$ and that paying dividends down to $M_{reg}$ and pursuing a risk-free investment strategy is always optimal. We also provided the explicit Expression (34) for $A'(M_{reg})$, from which Condition (35) immediately follows. Below we will use that, for every $M \geq M_{reg}$, we always have

$$ V(m) = m + A'(M_{reg}). \quad (A.12) $$

If $M_{reg} > 0$, then, for every $m \in [0, M_{reg})$, we have that

$$ V(m) = \begin{cases} m + A'(M_{reg}) - C, & \text{if } M < M_{reg} ; \\ m, & \text{if } M = M_{reg}. \end{cases} \quad (A.13) $$

(i) Note that if $C = 0$, then $M < M_{reg}$. From identities (A.12) and (A.13) we immediately obtain that, whenever $M_{reg} = 0$ or $C = 0$, $V$ is affine on $[0, \infty)$. We now prove that, in both cases, all investment strategies are optimal. To this effect, we use that, in the absence of insurance losses, admissible strategies at $M_{reg}$ are completely determined by the investment strategy $0 \leq \lambda \leq 1$. For any investment strategy $0 \leq \lambda \leq 1$ set $M_{\lambda} = \gamma (M_{reg} + q)(1 + r + \lambda \rho_1) = M_{\lambda} + A'(M_{reg})$ for every $\lambda \in [0, 1]$. Since the value function is affine on $[0, \infty)$ and $M_{\lambda} \geq 0$ almost surely for every $\lambda \in [0, 1]$, we get $E_q[V(M_{\lambda})] = E_q[V(M_{\lambda}^1)]$. The dynamic programming equation (22), implies $\lambda$ is optimal.

(ii) If $C > 0$ and $M_{reg} > 0$, then we immediately see from Identities (A.12) and (A.13) that $V$ is not affine on $[0, \infty)$. Moreover, for $m \in [0, M_{reg})$ either $V(m) = m$ (when $M = M_{reg}$) or $V(m) = m + A'(M_{reg}) - C$ (when $M < M_{reg}$). In particular,

$$ V(M_{\lambda}) = M_{\lambda} + A'(M_{reg}) \quad (A.14) $$

$$ V(M_{\lambda}) < M_{\lambda} + A'(M_{reg}) \quad (A.15) $$

It is immediate to see that $M_{\lambda} \geq M_{reg}$ almost surely if and only if $\lambda \in [0, \bar{\lambda}]$ with

$$ \bar{\lambda} := 1 - \frac{1}{1 + r} \frac{M_{reg}}{\gamma M_{reg} + q} \in (0, 1). $$

Hence, for $\lambda \in [0, \bar{\lambda}]$, Identity (A.14) implies optimality, as we argued in part (i). For $\lambda \in (\bar{\lambda}, 1]$ we have $V(M_{\lambda}^1) < M_{\lambda}^1 + A'(M_{reg})$, so $E_q[V(M_{\lambda}^1)] < E_q[M_{\lambda}^1 + A'(M_{reg})] = M_{\lambda}^1 + A'(M_{reg}) = E_q[V(M_{\lambda}^1)]$, by Expression (A.15). As a consequence, no $\lambda \in [\bar{\lambda}, 1]$ can be optimal. □

**Proof of Proposition 4.4.** If $C = 0$, Propositions 3.5 and 3.6 imply that $-\infty < M < M_{reg} = \bar{M}$ as well as that $V(m) = \max(m - M, 0)$. Hence, $V$ is always convex. Moreover, it is affine on $[\text{ess inf}(R_1), \infty)$ if and only if $M \leq \text{ess inf}(R_1)$.

Whenever the firm is not liquidated, it operates with capital equal to $M_{reg}$. As a result optimal strategies are fully determined by the investment strategy. Set $M_{\lambda}^1 = \gamma (M_{reg} + q)(1 + r + \lambda \rho_1) + R_1$ for every $\lambda \in [0, 1]$. Then, $\lambda^{*} \in [0, 1]$ is optimal if and only if

$$ f(\lambda^{*}) = \max_{\lambda \in [0, 1]} f(\lambda) $$

where $f(\lambda) := E_q[V(M_{\lambda}^1)]$ for $\lambda \in [0, 1]$. Note that, as $\lambda \mapsto V(M_{\lambda}(\omega))$ is convex for every fixed $\omega$, the function $f$ is also convex.

(i) If $M \leq \text{ess inf}(R_1)$, then we already know that $V$ is affine on $[\text{ess inf}(R_1), \infty)$ and the firm is never liquidated. It follows that

$$ f(\lambda) = E_q[V(M_{\lambda}^1)] = E_q[M_{\lambda}^1 - M] = E_q[M_{\lambda}^1] - M $$

and, hence, $f$ is constant on $[0, 1]$ and every $\lambda \in [0, 1]$ is optimal.

(ii) If $M > \text{ess inf}(R_1)$, then we already know that $V$ is convex but not affine. Moreover, we know that $\text{ess inf}(R_1) < M$ implies that $0 < P(M_{\lambda}^1 < M) < 1$. In particular, the probability of
recapitalization and the probability of default are both strictly positive whenever taking maximal investment risk applying. Applying the conditional version of Jensen's inequality, we obtain

\[
E_{Q}(V(M^1_t)) - R_t \geq V(E_{Q}[M^1_t]) = \mathbb{V}(M^0_t). \tag{A.16}
\]

Here, we have used that \(\rho_Q \) and \( R_t \) are \( Q \)-independent. The above inequality cannot be an almost sure equality. To see this, set \( E = [M^0 \leq M] \) and observe that \( E \in \sigma(R_t) \). Assume first that \( P(E) > 0 \). Because \( \rho \) is unbounded above, we have that \( [M^1_t > M] \cap E \) has strictly positive probability. This means that \( V(M^1_t) \) is positive and non-zero and, as a result, \( E_{Q}[V(M^1_t)|E] \) is also positive and nonzero. Since \( V(M^1_t) \equiv 0 \) on \( E \), inequality (A.16) is not an almost sure equality. Assume now that \( P(E) = 0 \). Then, recalling that \( P(M^1_t < M) > 0 \), we have

\[
E_{Q}[V(M^1_t)|E] = E_{Q}[V(M^1_t + \mathbb{A}(\text{reg}))|E]\mathbb{E}[M^0|E] \geq \mathbb{E}(M^1_t) = \mathbb{V}(M^0_t).
\]

This concludes the proof that inequality (A.16) is not an almost sure equality. It follows that

\[
f(1) = E_{Q}[\text{ess inf}(V(M^1_t)|E_1)] > E_{Q}[\text{ess inf}(V(M^1_t))] = f(0).
\]

Since \( f \), as a convex function, cannot first rise and then fall, this implies that \( \lambda^* = 1 \) is a global maximum for \( f \).

**Proof of Proposition 4.5.** Assume that \( C > 0 \). If \( M<0 \leq \text{reg} \), then \( V(m) = m + \mathbb{A}(M) \) – \( C \) on \([M, \text{reg}]\) and \( V(m) = m + \mathbb{A}(M) \) for \( M \geq \text{reg} \). Hence, \( V \) is affine on \([M, \text{reg}]\) and \((\text{reg}, \infty)\) with slope 1 on bounded intervals. Both convexity and concavity would force \( V \) to be affine on \([\text{reg}, \infty)\), which is not possible since \( C > 0 \). Hence, \( V \) is neither concave nor convex on \([\text{ess inf}(R), \infty)\). On the other hand, if \( M = \text{reg} > 0 \), then \( V(m) = m + (0, \text{reg}) \) and \( V(m) = m + \mathbb{A}(M) \) for \( M \geq \text{reg} \). The same argument as above shows that \( V \) can be neither concave nor convex on \([0, \infty)\) and, hence, on \([\text{ess inf}(R), \infty)\). Finally, if \( M = M<0 \), then \( V(m) = 0 \) for \( m \in (-\infty, 0) \) but \( V(0) > 0 \). On the Nontriviality Assumption (24) Hence, \( V \) has a discontinuity at 0 and can be neither concave nor convex on \([\text{ess inf}(R), \infty)\). \( \square \)

**Appendix B. Supplementary material**

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.insmatheco.2021.03.013.

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