Observability of ‘Cascade Mixing’ in 
\[ B^o \rightarrow J/\psi \; K^o \]

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Abstract

In high statistics observations of \( B^o \rightarrow J/\psi K^o \) originating from the process \( \Upsilon(4S) \rightarrow B^o \bar{B}^o \) it should be possible to observe ‘cascade mixing’, where one mixing particle, the \( B^o \), turns into another, the \( K^o \). This is possible despite the difficulty that the length of the beam crossing region makes a precise definition of the primary vertex impossible. This difficulty is circumvented by using an ‘away side’ tag to specify the initial time. We review the formalism for describing such processes, and first apply it to simple \( B^o \) mixing, noting it gives a transparent description for CP and T asymmetries. In particular we show that three different asymmetries of the CP and T type, with neglect of direct CP violation, are given by the same expression.

For “cascade mixing” we present predictions for processes of the type \( B_i \rightarrow K_j \) via \( J/\psi \), where in the limit of no direct CP violation each state \( i \) or \( j \) is determined by a simple tag. There are 16 such simple measurable processes, involving 10 functions of the two time intervals involved. The coefficients of the functions are different for each of the processes and are given in terms of the mass splitting and the CP, T violating parameter of the \( B^o \) mass matrix \( m_2 \). The results presented here are just consequences of the quantum mechanics of particle mixing and do not involve any particular model of CP violation.

1 Introduction

Some years ago the idea of “double” or “cascade” mixing, where one mixing system turns into another via a decay process was introduced [1], [2], [3]. Such processes would exhibit amusing quantum mechanical interference effects and could also provide information on certain properties of the interfering systems.

The most discussed process of this type involves decays of \( B^o \) mesons to a \( J/\psi K^o \) state. The first mixing system would be the \( B^o \) and the second the \( K^o \), while the decay through the \( J/\psi \) provides a kind of “regeneration” or “filter”
for the initial state of the $K^o$. This can provide a new and interesting tool in the manipulation of mixing systems, analogous to the passing of a $K^o$ beam through a material of variable thickness, but where additionally the quantum numbers of the decay, such as for the p-wave $J/\psi$, play a role in determining the evolution. Thus in addition to $J/\psi$, one may consider processes mediated by other particles, with different predictions for the behavior of the $K^o$ oscillations.

Since the original proposals considerable time has elapsed and much data has been accumulated and is promised for $B^o$ processes. It would therefore appear appropriate to reconsider the possibility of studying “cascade mixing” experimentally.

However, there would appear to be a difficulty. Study of the process involves the determination of two time differences: $\tau(1,0)$, the proper time for the interval between the $B^o$ creation and the decay to $J/\psi K^o$; and $\tau(2,1)$, the proper time for the interval between this decay and the final $K^o$ decay. The difficulty is that primary vertex of the process, seemingly necessary to determine $\tau(1,0)$, is not well determined in space by the apparatus. For example at Belle, where one studies $e^+e^- \rightarrow \Upsilon(4S) \rightarrow B^o \bar{B}^o$, the length of the beam crossing region, is on the order of several cm. Such distances are much larger than $(\Delta m_{B^o})^{-1} \sim 0.5 \times 10^{-12}s \sim 1.5 \times 10^{-2}cm$ relevant for $B^o$ oscillations, making it seem unrealistic to observe any oscillatory effects in $\tau(1,0)$; and it might be feared that the rapid oscillations in $\tau(1,0)$ will wash out any interference effects at all.

1.1 ‘Away side’ tag

Nevertheless, in the $\Upsilon(4S) \rightarrow B^o \bar{B}^o$ process there is a way around this difficulty. Namely, one may use the method of the “away side tag” to determine the initial time. In this method, one uses the p-wave nature of the $\Upsilon(4S) \rightarrow B^o \bar{B}^o$ decay to say that if one $B^o$ is observed to be in a given state, then the other member of the pair must be in the opposite, orthogonal state. For example, if one meson is observed to be in the $B^o$ state then the other one –at the same time– is in the $\bar{B}^o$ state. For our present purposes ‘at the same time’ is the important point. This implies that a measurement of the ‘away side’ specifies not only the state of the meson under consideration, but also the time when it was in this state—without having to know the original vertex. The possibility of such measurements to sufficient accuracy has been demonstrated by the Belle and Babar groups in their studies of CP violation.

In the following we shall refer to the $B^o$ opposite to the initial ‘away side’ tag and its subsequent development, as the ‘same side’, since this is the system we wish to study.

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1 Although ‘at the same time’ sounds like a frame-dependent, non-covariant specification, we have explained elsewhere that the procedure can be put in covariant language and that the $\Upsilon$ rest frame is indeed the correct frame for the procedure.
2 Formalism

We use the formalism introduced in ref [2], which we briefly review here. One operates in a two-dimensional vector space spanned by $B^o$ and $\bar{B}^o$ before the $J/\psi$ decay or by $K^o$ and $\bar{K}^o$ after the decay. All quantities are either ‘spinors’ in this space, or 2 x 2 matrices. Examples of ‘spinors’ are states like $B_1$ or $K_s$ mesons. The propagation in time of these states, or the conversion of a $B^o$ to a $K^o$ state via the decay, are given by matrices. We shall also use density matrices $\rho$ to describe the initial or final states of the two-state system.

Either for the ordinary one-time mixing, or for the ”cascade mixing” with two times, the expressions needed will have the form

$$\text{Tr}[\rho(b)M\rho(a)M^\dagger].$$

This expression represents the probability amplitude squared to begin with a state $a$ of the two-state system and to end with a state $b$ of the two-state system. For ordinary $B^o$ mixing $a$ and $b$ are different (or perhaps the same ) $B^o$ states and $M$ will depend only on one time difference. For ‘cascade mixing’ $a$ is a $B^o$ state and $b$ is a $K^o$ state, and the expression depends on the two time intervals. To convert Eq 1 into an experimental rate for a given final channel $\alpha$, it is necessary to multiply it by a rate constant $\Gamma(b \to \alpha)$ giving the decay rate for $b$ into that channel. We discuss the relation between the spinor representing $b$ and the channel $\alpha$ in the next section.

$M$ is a matrix describing the evolution in the two-state space. For ‘cascade mixing’ it is a product of factors describing the evolution of the system as a $B^o$, its transition to a $K^o$ and finally its evolution as a $K^o$. Since there no external disturbances (“decoherence”) the entire process may be simply regarded as a coherent evolution in the generalized two-state space. The time evolution factors are governed by the mass matrices of the $B^o$ and $K^o$, and there is a ‘flip’ amplitude $A(1)$ at the time 1 giving the transition from the $B^o$ to $K^o$. We will not be concerned with absolute normalizations, so only the structure of these factors in the two-state space will be of interest and multiplicative constants ignored. We finally normalize the whole expression to some particular process.

The $\rho(a)$ and $\rho(b)$ are density matrices characterizing the initial and final states. They arise from the expression $\rho = vv^\dagger$, where $v$ is the ‘spinor’ of the two-state system describing the initial or final state. The definition of these states will be discussed in the next section.

Finally, we note that by taking the hermitian conjugate and using the permutation property of the trace together with $\rho = \rho^\dagger$, one can show that Eq 1 is always real. It is also positive (or zero) since it is the absolute value squared of a certain quantity, namely $v^\dagger(b)Mv(a)$. Hermiticity of $M$ is not assumed.

2.1 Definition of ‘Particle’

It is important to recognize, as had been stressed in ref [9] that a particular decay channel can be used to define some ‘particle’ in the two-state system.
Any decay channel $\alpha$ is described by two complex numbers $\alpha, \alpha'$, giving the amplitude from say a $B^o$ or a $\bar{B}^o$ into the given channel. It is then possible to find two $B^o$ states, one not going into the channel $\alpha$ and one that does go. The one that does not decay can be constructed as $\alpha' |B^o\rangle - \alpha |\bar{B}^o\rangle$, since it will be seen that the decay amplitude for this state $\sim (\alpha\alpha' - \alpha'\alpha) = 0$. On the other hand the orthogonal state $\alpha^* |B^o\rangle + \alpha'^* |\bar{B}^o\rangle$ goes into the channel $\alpha$. We have thus obtained two orthogonal states, one going and the other not going into the given channel. Following ref [9] one may call these two 'particles' $B^\alpha$ and $B^\alpha_\perp$. In this way there is a pair of $B^o$ states defined by every decay channel. The same of course applies to states in $K^o$ decay.

In 'measurement theory' language, if one thinks of the decay as a 'measurement' of the state of the 'spinor', then that state which does decay (like $B^\alpha$) is the eigenstate for the 'measurement'. Naturally, if there is some conserved quantum number such as CP, various decay channels carrying this quantum number can in fact define the same state of the two-state system.

Evidently, if in Eq 1 one uses for $b$ that state which does decay into $\alpha$, then multiplying by the rate constant $\Gamma(b \rightarrow \alpha)$ gives the experimental rate. Thus experimental rates at different times for a given channel may be compared by simply using Eq 1 with the 'eigenspinor' for the decay. To compare different final channels in an absolute manner, knowledge of the different 'eigenspinors' and partial $\Gamma$’s is necessary.

Finally, a significant point about $B^\alpha_\perp$ is that, for a p-wave pair as in $\Upsilon(4S)$ decay, if one side is a $B^\alpha$, then the other side is necessarily a $B^\alpha_\perp$. This just follows from the linear quantum mechanics of a two-state system with Bose-Einstein statistics and do not involve any symmetry properties such as CP. Indeed, without further assumptions or information there is no definite relation between the decays of $|B_\alpha\rangle$ and those of $|B_{\alpha\perp}\rangle$, except as said, that $|B_{\alpha\perp}\rangle$ does not go into the channel $\alpha$.

2.2 Density Matrices

We shall use density matrices in the following form:

$$\rho(d) = \frac{1}{2}(1 + d \cdot \sigma)$$

where the $\sigma$ are the three pauli matrices. One has $Tr\rho = 1$, reflecting the normalization of the state to one. We use standard notation where $\sigma_3$ corresponds to the flavor direction, $\sigma_3 |B^o\rangle = + |B^o\rangle$, $\sigma_3 |\bar{B}^o\rangle = - |\bar{B}^o\rangle$, while $\sigma_2$ is the pure imaginary and anti-symmetric matrix. Some definitions and relations are given in the Appendix.

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2 The complex conjugates appearing here explain why our basic expression Eq[1] has a somewhat different ordering of the factors than the expression used in ref [2] (Eq 3). In that paper we used an approach based on amplitudes and not state vectors, so that the corresponding state vector for a given channel would have the complex conjugations. Here to avoid possible misunderstandings we use the conventional language, where a decay channel is characterized by a state vector, a certain $B^o$ or $K^o$ state, and so Eq[1] has the conventional form with the initial state on the right and the final state on the left.
Furthermore we take $d$ to be a unit vector, $d^2 = 1$. This gives the properties

$$\rho^2(d) = \rho(d) \quad \quad \rho(d)\rho(-d) = 0 \quad (3)$$

The first of these properties reflects the fact that we will always have to do with 'pure' states in the following \[10\]. The density matrix for the state $|B_{\alpha\perp}\rangle$ has the $d$ opposite to that for $|B_{\alpha}\rangle$. This is the meaning of the second relation in Eq.3.

We now consider some definite states of interest and list the properties of the associated density matrix in Table\[1\]. The first entries, for $B^o$ and $\bar{B}^o$ could be determined by a flavor tag, as in leptonic decays of the type $B^o \to l^+...$. When the density matrix refers to an initial state determined by an 'away side' tag in $\Upsilon(4S)$ decay, then this initial state has the opposite $d$ to the tag.

Next we can consider tags involving $J/\psi K^o$ and $J/\psi \bar{K}^o$. In principle this decay amplitude has four basic possibilities according to whether $B^o$ or $\bar{B}^o$ decays and whether the kaon is $K^o$ or $\bar{K}^o$. However in the Standard Model particle to antiparticle processes like $B^o \to J/\psi K^o$ and $\bar{B}^o \to J/\psi \bar{K}^o$ involve higher order weak transition and are expected to be very small compared to the other two. We thus neglect these, leaving the $B^o \to J/\psi K^o$ and $\bar{B}^o \to J/\psi \bar{K}^o$ amplitudes.

In the tags involving $K^o$’s we shall neglect CP violation in the $K^o$ system, so for example a $K_s$ refers to the detection of $\pi\pi$. Thus our discussion in these cases can be inaccurate at the $10^{-3}$ level.

We now come to the next lines of Table\[1\] involving $J/\psi K_s$ and $J/\psi K_l$. Without any particular assumptions these channels may be taken as defining two ‘particles’ $B^o$ and $\bar{B}^o$ and their orthogonal states, as discussed in section\[2.1\].

In principle the amplitudes for these channels could be found by taking $\pm$ combinations of the $K^o,\bar{K}^o$ amplitudes. However, these amplitudes are not completely known. Although the $K^o$ and $\bar{K}^o$ amplitudes refer to CPT conjugates, it is not permissible to use CPT for a single channel in a many-channel situation. Hence it is not possible to say more about the $J/\psi K_s$ or $J/\psi K_l$ channels without further information or assumptions.

### 2.3 Neglect of Direct CP Violation

Direct CP Violation is expected to be small for $B^o \to J/\psi K^o$ and experimentally it is below the few percent level \[11\]. If we permit ourselves to neglect it in the following and further neglect all CP violation in the $K^o$ system ($10^{-3}$ level), a great simplification ensues. All CP violation arises through the mixing in the $B^o$ time evolution. Then the states defined by various tags may simply be given their naive CP assignments – since no time evolution is involved.

Thus a $K_s$, identified by $\pi\pi$ decay, is approximately the CP even $K_1$. The tag $J/\psi K_s$, involving the parity odd $l=1$, identifies the parent $B^o$ as the particle we may call the CP odd $B_2$.

Then the $B^o$ Table\[1\] is simply a $B_1$ and $B^o_\perp = B^o$ is the CP odd $B_2$. We show these designations in the last lines of the table. These identifications are
Table 1: Values of $d$ for some decay channels under different assumptions. As explained in the text, a ‘particle’, that is, a certain linear combination in the two-state system, may be defined by a decay channel. In $\Upsilon(4S) \to B^o \bar{B}^o$ when determining the initial state by an ‘away side’ tag, one has $-d$ for the ‘same side’.

certainly approximate, but probably good to the percent level. As we shall see below certain simple relations in the simple one-time $B^o$ mixing problem follow from this assumption, and so larger violations of these relations can be taken as a suggestion of direct CP violation.

In using the naive CP assignments to make an ‘away side’ tag of definite CP it is of course not necessary that the tag be $J/\psi \ K^o$, but it must also be one where direct CP violation is small.

3 Simple $B^o$ Mixing

We first consider the time evolution within the $B^o$ system only. This will help in establishing the method and showing the relevant parameters. This is of course a much studied subject [11], and we will mostly reproduce known results in the present language. In the notation of [2], the time evolution of the $B^o$ system is given by $S(1,0) = e^{-iM_B \tau(1,0)}$, where $M_B$ is the complex, not necessarily hermitian, mass matrix. For the purposes of this section with only one time, we can call the time variable simply $\tau$ and our basic expression Eq (1) becomes

$$Tr[\rho(b)e^{-iM_B \tau}\rho(a)(e^{-iM_B \tau})^\dagger] \quad (4)$$

A great simplification, as compared with the $K^o$ system, ensues here in that the non-hermitian part of $M_B$, called $\frac{1}{2}\Gamma_B$, may be taken as proportional to the identity matrix. That is, to a good approximation [12] one has $\Delta \Gamma_B \approx 0$. With this approximation $\Gamma$ factors out of the exponential, leaving a unitary matrix $U'$.

$$S = e^{-\frac{1}{2} \Gamma_B \tau} U' \quad \Delta \Gamma_B \approx 0 \,, \quad (5)$$
and $U'$ gives simple unitary ‘rotations’ in the $B^\circ$ system:

$$U' = e^{-iM_B^H\tau},$$

(6)

where $M_B^H$ is the hermitian part of the mass matrix $M_B^H = \frac{1}{2}(M_B + M_B^\dagger)$.

The fact that the time evolution is essentially given by a unitary matrix allows for simplification of the basic expression Eq. 4. Expanding the two $\rho$ gives four terms. The ‘1’ term gives simply $\frac{1}{2}$. The linear in $d_b \cdot \sigma$ terms give zero using $U'U'^\dagger = 1$ and $Tr\sigma = 0$, leaving

$$Tr[\rho(b)S\rho(a)S^\dagger] = e^{-\Gamma_B\tau}\left(\frac{1}{2} + \frac{1}{4}Tr[(d_b \cdot \sigma)(d_a \cdot \sigma)(U'U'^\dagger)]\right)$$

(7)

$$\Delta\Gamma_B \approx 0,$$

With the assumption of CPT invariance for the mass matrix the diagonal elements of $M$ are equal, so that

$$M_B^H = m_B^{av}I + m_B = m_B^{av} + m_1\sigma_1 + m_2\sigma_2,$$  \hspace{1cm} CPT good

(8)

with $m_B^{av}$ the average mass of the $B^\circ$. The traceless part of the hermitian mass matrix

$$m_B = m_1\sigma_1 + m_2\sigma_2$$

(9)

will play the most important role in the following. For the mass splitting $\Delta m_B$ we have

$$\frac{1}{2}\Delta m_B = \sqrt{m_1^2 + m_2^2},$$

(10)

and finally

$$S = e^{-(im_B^{av} + \frac{1}{2}\Gamma_B)\tau}U \quad U = e^{-im_B\tau} = e^{-i(m_1\sigma_1 + m_2\sigma_2)\tau}$$

(11)

and Eq 7 becomes

$$Tr[\rho(b)S\rho(a)S^\dagger] = e^{-\Gamma_B\tau}\left(\frac{1}{2} + \frac{1}{4}Tr[(d_b \cdot \sigma)U(d_a \cdot \sigma)U'^\dagger]\right)$$

(12)

CPT good, \hspace{1cm} $\Delta\Gamma_B \approx 0$.

The evaluation of the traces in various expressions may be simplified by using the absence of the $\sigma_3$ term in $U$ and the anticommutation of the $\sigma$ to give Eq. 11 of the appendix

$$\sigma_3U(\tau) = U(\tau)\sigma_3. \hspace{1cm} (13)$$

Also, note $U'^\dagger(\tau) = U(-\tau)$, independently of CPT.
4 CP and T Asymmetries in Simple Mixing

For the further discussion one needs the values of $m_1$ and $m_2$ individually. The mass splitting $\Delta m_B$ gives $(m_1^2 + m_2^2)$ via Eq 10 and is known \cite{4} to be, $\Delta m_B = 0.51 \times 10^{-12}s = 3.3 \times 10^{-10}MeV$. As explained next, the measurements for CP or T asymmetries give $m_2$, so then both m’s are determined.

We thus proceed to describe CP and T tests in simple $B^0$ mixing in our formalism, with the assumptions of neglecting both direct CP violation and a possible CPT violation. With the neglect of direct CP violation, both CP and T violation effects will arise from the $m_2$ term in $S$. As should be expected from the CPT theorem the violation of one symmetry will imply the violation of the other, via $m_2 \neq 0$.

4.1 CP Asymmetry

Here one has compared \cite{7,8}, as a function of time, $B^0$ and $\bar{B}^0$ going to a common, presumably CP self-conjugate, final state, namely $J/\psi K_s$. Identifying $J/\psi K_s$ with a decay of $B_2$ as explained earlier, one defines the asymmetry

$$A = \frac{\text{Rate}(\bar{B}^0 \to B_2) - \text{Rate}(B^0 \to B_2)}{\text{Rate}(\bar{B}^0 \to B_2) + \text{Rate}(B^0 \to B_2)}$$

for different time intervals.

Using Eq 12 with the $d$ from the first, second, and last lines of Table 1, one has

$$A = \frac{\frac{2}{7}Tr[\sigma_1 U \sigma_3 U^\dagger]}{\frac{1}{7} + \frac{1}{7}} = \frac{1}{2}Tr[\sigma_1 U \sigma_3 U^\dagger].$$

Using Eq 61 to write $\frac{1}{2}Tr[\sigma_1 U(\tau)\sigma_3 U^\dagger(\tau)] = \frac{1}{2}Tr[-i\sigma_2 U(-\tau)U^\dagger(\tau)] = \frac{1}{2}Tr[-i\sigma_2 U(-2\tau)]$, one has

$$A = -\frac{1}{2}Tr[i\sigma_2 U(-2\tau)]$$

We now use Eq 59

$$A = -\frac{1}{2}Tr[i\sigma_2 U(-2\tau)] = -\frac{m_2}{\Delta m_B} \sin(\Delta m_B \tau) = -\frac{m_2}{\sqrt{m_1^2 + m_2^2}} \sin(\Delta m_B \tau).$$

As was to be expected, with neglect of direct CP violation the result is proportional to $m_2$. The formula is of course in agreement with standard results \cite{11} with neglect of direct CP violation. We note that this result, and in particular the fact that $m_2/\sqrt{m_1^2 + m_2^2}$ is less than or equal to one, follows essentially from the quantum mechanics of mixing and is independent of any definite model of CP violation.

Thus the parameters $m_1$ and $m_2$ needed for our description are given by $\Delta m_B$ and the coefficient of $sin(\Delta m_B \tau)$ in the asymmetry Eq 13. Experimentally, our $\frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ is usually referred to in the context of the CKM model as $sin2\beta$, and has \cite{4} the value 0.68, suggesting–in an interesting coincidence–that $m_1$ and $m_2$ are about equal.
4.2 CP Asymmetry Relations

Because of the simple structure (within our approximations) of the formulas it is easy to establish relations between various other asymmetries like Eq 14. Let us write $A(\bar{B}_0, B^o; B_2)$ for the asymmetry of Eq 14.

First we can consider replacing the $K_s$ in the $J/\psi K_s$ final state with a $K_l$ so that we have a $B_1$ as the final state. According to the last line of Table 1 this means we should replace $\sigma_1$ in Eq 15 with $-\sigma_1$. Thus

$$A(\bar{B}_0, B^o; B_1) \approx -A(\bar{B}_0, B^o; B_2) \quad (18)$$

Next, we can consider a reverse type of procedure where starting with a CP eigenstate, $B_1$ or $B_2$, we go to two different but conjugate states, $B^o$ or $\bar{B}_0$. The initial $B_1$ or $B_2$ must of course be established by an ‘away side’ tag. In an obvious notation

$$A(B_2; \bar{B}_0, B^o) = \frac{\text{Rate}(B_2 \rightarrow \bar{B}_0) - \text{Rate}(B_2 \rightarrow B^o)}{\text{Rate}(B_2 \rightarrow \bar{B}_0) + \text{Rate}(B_2 \rightarrow B^o)} \quad (19)$$

This amounts to exchanging $\sigma_1$ and $\sigma_3$ in Eq 15. Looking at Eq 17 we see that $i\sigma_2$ will be replaced by $-i\sigma_2$. Thus

$$A(B_2; \bar{B}_0, B^o) \approx -A(\bar{B}_0, B^o; B_2) \quad (20)$$

For $B_1$, we should, according to Table 1, just change the sign:

$$A(B_1; \bar{B}_0, B^o) \approx -A(B_2; \bar{B}_0, B^o) \approx +A(\bar{B}_0, B^o; B_2) \quad (21)$$

All these relations are expected to hold as a function of time. The approximations made are $\Delta \Gamma_B \approx 0$ and neglect of direct CP violation in the $B^o$ system. Neglect of CP violation in the $K^o$ system is also implied, insofar that identification of a $B^o$ state involves the assignment of a definite CP to a $K^o$.

Because of the approximations, particularly that of neglecting direct CP violation in the $B^o$ system, the equalities may only be good to about the percent level. Alternatively, a breakdown of the relations may be used to look for violations of the assumptions.

5 T Asymmetry

An interesting test showing manifest T violation in the $K^o$ system was carried out by the LEAR group in the 90’s [13] and recently analogous tests have been discussed and presented [14] for the $B^o$ system. In these tests a certain time evolution ‘forwards’ and ‘backwards’ is compared, and a difference in the two rates is a manifest violation of T. One thus defines the asymmetry

$$A(B^a \rightarrow B^b) = \frac{\text{Rate}(B^a \rightarrow B^b) - \text{Rate}(B^b \rightarrow B^a)}{\text{Rate}(B^a \rightarrow B^b) + \text{Rate}(B^b \rightarrow B^a)} \quad (22)$$

According to Eq 17 the numerator here is
A notable consequence of this expression is that if the $B^b$ state is the orthogonal state to the $B^a$ state, so that $d_b = -d_a$, then $A$ is zero

$$A(B^a \rightarrow B^b) = 0$$

(24)

For LEAR the comparison was between $K^o \rightarrow \bar{K}^o$ and $\bar{K}^o \rightarrow K^o$, involving in fact orthogonal states. Thus the nonzero result found there is due to the significant $\Delta \Gamma \neq 0$ in the $K^o$ system. In the $B^o$ case however, the analogous asymmetry $A(B^o \rightarrow \bar{B}^o)$ should be essentially zero since $\Delta \Gamma_{B} = 0$ holds to high accuracy [12]. These points are in agreement with ref [15] where it was pointed out that a nonzero $\Delta \Gamma$ is needed for tests of this type.

### 5.1 T Test Formulas

For tests of the type Eq (22) it is thus necessary to chose channels where $B^a, B^b$ are not orthogonal states, and discussion [15] has centered around $B^a = B^o, B^b = B_2$, with $B_2$ identified via the $J/\psi K_s$ tag. In this case Eq (22) becomes

$$A(B^o \rightarrow B_2) = \frac{\text{Rate}(B^o \rightarrow B_2) - \text{Rate}(B_2 \rightarrow B^o)}{\text{Rate}(B^o \rightarrow B_2) + \text{Rate}(B_2 \rightarrow B^o)},$$

(25)

where with the identification of the $J/\psi K_s$ tag with $B_2$ we neglect direct CP violation in the $B^o$ system and CP violation in the $K^o$ system. Employing Eq (7) with $d_a = (0, 0, 1)$ and $d_b = (-1, 0, 0)$

$$A(B^o \rightarrow B_2) = \frac{1}{4} \left( \frac{-Tr \left[ \sigma_1 U \sigma_3 U^\dagger \right] + Tr \left[ \sigma_3 U \sigma_1 U^\dagger \right]}{1 + Tr \left[ \sigma_1 U \sigma_3 U^\dagger \right] + Tr \left[ \sigma_3 U \sigma_1 U^\dagger \right]} \right),$$

(26)

we see we have to do with the expressions $Tr \left[ \sigma_3 U \sigma_1 U^\dagger \right]$ and $Tr \left[ \sigma_1 U \sigma_3 U^\dagger \right]$. These quantities are equal and of opposite sign as follows by using Eq (61) and $U(-\tau) = U^\dagger(\tau)$. Then

$$A(B^o \rightarrow B_2) = \frac{1}{2} Tr \left[ \sigma_3 U \sigma_1 U^\dagger \right] = \frac{1}{2} Tr \left[ i \sigma_2 U(-2\tau) \right].$$

(27)

But this is the same as Eq (16) up to the sign. Therefore

$$A(B^o \rightarrow B_2) \approx -A(B^o, B^o; B_2),$$

(28)

which was evaluated in Eq (17). It should be remarked, however, that the CP test of Eq (16) and the T test here are not the same quantities. The CP asymmetry need not vanish at $t = 0$ when direct CP violation is not neglected [11], while here the vanishing is an identity, following from $Tr[\rho(a)\rho(b)] = Tr[\rho(b)\rho(a)]$. 

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As with the CP asymmetries of Eq 18 and Eq 21, different variants of $A(B^o \to B_2)$ are simply related [16]. Replacing $B^o$ by $\bar{B}^o$ leads to a minus sign due to $d \to -d$, and similarly for changing $B_1$ to $B_2$:

$$A(\bar{B}^o \to B_2) \approx A(B^o \to B_1) \approx -A(\bar{B}^o \to B_1) \approx -A(B^o \to B_2) \tag{29}$$

All of these are experimentally distinct possibilities, with different combinations of ‘away side’ and ‘same side’ tags and, within the approximations, are all given by Eq 17. That the $T$ asymmetries are equal or opposite to the CP asymmetries is not very surprising since, given the assumptions, both originate from the same $m_2$ term in the $B^o$ time evolution.

The approximations used for Eq 28 and Eq 29 are, as before, $\Delta \Gamma_B \approx 0$, CPT, neglect of CP violation in the kaon system, and neglect of direct CP violation in the $B^o$ system. We thus should expect the relations to hold at least to the percent level. Violation of the various equalities would indicate a breakdown of the assumptions, most likely that of neglecting direct CP violation in the $B^o$ system, and violation above the percent level could be suggestive of new physics.

### 6 Cascade Mixing

Having established the formalism and some parameters. We come finally to “cascade mixing” where we study the behavior in the two times $\tau(1,0)$ and $\tau(2,1)$. As explained earlier, in $\Upsilon(4s) \to B^o\bar{B}^o$ time $0$ and the starting $B^o$ state can be determined by the ‘away side’ tag. Time ‘1’ is the time of the ‘same side’ decay to $J/\psi K^o$ and time ‘2’ is the time of the final $K^o$ decay. In Eq 1, $\rho(a)$ is the density matrix for the initial state of the $B^o$, $\rho(b)$ that for the final state of the $K^o$ and $\mathcal{M}$ is

$$S(2,1)A(1)S(1,0), \tag{30}$$

$S(1,0)$ giving the propagation in time from 0 to 1 of the two-state system, $A(1)$ the transition within the system induced by the decay to $J/\psi K^o$, and $S(2,1)$ gives the propagation in time from 1 to 2. $S(1,0)$ was established above with the parameters as discussed in section 4.1.

Concerning $S(2,1)$ the neglect of CP violation for the $K^o$ (and of course good CPT) allows us to write the mass matrix as $M_K^{\nu} + m_K$, where $m_K$ is a matrix proportional to $\sigma_1$, so that

$$S(2,1) = e^{-iM_K^{\nu}\tau(2,1)}e^{-im_K\tau(2,1)}, \tag{31}$$

with the first term a scalar and the second a matrix operator. In contrast to the $B^o$ situation, $\Delta \Gamma$ here, although approximately proportional to $\sigma_1$, is significantly different from zero and must be retained. The matrix $m_K$ thus represents the complex $K^o$ masses, with the hermitian part of $m_K$ representing $\frac{1}{2}$ the mass splitting of $K_l$, $K_s$ and the antihermite part half of the lifetime difference $\Delta \Gamma_K = \frac{1}{2}(\Gamma_s - \Gamma_l)$ (see Appendix). Because of the retention of the
matrix $\Delta \Gamma_K \sim \sigma_1$, we now longer have the evolution in terms of a unitary matrix as in Eq 6, with $UU^\dagger = 1$.

Instead we have

$$
\mathcal{M}\mathcal{M}^\dagger = e^{-\Gamma_K^\sigma \tau(2,1)}e^{-\Gamma_B \tau(1,0)} \times \\
(e^{-im_K \tau(2,1)} A(1)e^{-im_B \tau(1,0)}e^{+im_B \tau(1,0)} A^\dagger(1)e^{+im_K \tau(2,1)})
= e^{-\Gamma_K^\sigma \tau(2,1)}e^{-\Gamma_B \tau(1,0)} \times (e^{-im_K \tau(2,1)}e^{+im_K \tau(2,1)})
= e^{-\Gamma_K^\sigma \tau(2,1)}e^{-\Gamma_B \tau(1,0)} \times e^{-\Delta \Gamma_K \sigma_1 \tau(2,1)},
$$

where in the next-to-last step we used $m_B = m_B^\dagger$. We also took $A(1)A^\dagger(1) = 1$ as will be used next in Eq 34.

We now proceed to find the analog of the simple Eq 7. The lifetime prefactor now becomes $e^{-\Gamma_B \tau(1,0)}e^{-\Gamma_K \tau(2,1)}$. So that Eq 1 now is

$$
\text{Tr}\left[\rho(b)e^{-im_K \tau(2,1)} A(1)e^{-im_B \tau(1,0)} \rho(a)e^{+im_B \tau(1,0)} A^\dagger(1)e^{+im_K \tau(2,1)}\right]
$$

with $m_B = m_1 \sigma_1 + m_2 \sigma_2$.

Turning now to $A(1)$, we use the neglect of particle - antiparticle transitions to set the amplitudes for $B^o \rightarrow J/\psi \ K^o$ and $B^o \rightarrow J/\psi \ K^\circ$ to zero. This implies that $A(1)$ is a diagonal operator in our two-state flavor basis, leaving a linear combination of $I$ and $\sigma_3$ as possibilities. However, if we continue with the approximation of neglecting direct CP violation for the $B^o$ and all CP violation for the $K^o$, which permits us to make naive CP identifications, then only $B^1 \rightarrow K^2$ and $B^2 \rightarrow K^3$ amplitudes are allowed. This then excludes $I$ as a component of $A(1)$ and we may set

$$
A(1) \sim \sigma_3.
$$

### 6.1 Evaluation of ‘Cascade’ Terms

There are in principle 16 different simple experimental observables for Eq 33. For $\rho(a)$ the initial $B^o$ could be tagged as a $B^o$, $\bar{B}^o$, $B_1$, or $B_2$, while for $\rho(b)$ the final $K^o$ can be the analogous $K^o$, $\bar{K}^o$, $K_1$, or $K_2$. We proceed by considering the four terms resulting from the product of the two $\rho = \frac{1}{2}(1 + d \cdot \sigma)$

#### 6.1.1 ‘1’ term

The ‘1’ term simply leads to the product evaluated in Eq 32, hence this contribution is, taking the trace,

$$
\frac{1}{4} \text{Tr}[e^{-\Delta \Gamma_K \sigma_1 \tau(2,1)}] = \frac{1}{4}(e^{-\Delta \Gamma_K \tau(2,1)} + e^{+\Delta \Gamma_K \tau(2,1)}) = \frac{1}{2} \cosh(\Delta \Gamma_K \tau(2,1))
$$

(35)
6.1.2 linear in d terms

Unlike the discussion for Eq 7, these linear terms do not vanish since for the $K^0$ one deals with a non-unitary evolution with a nonzero $\Delta \Gamma$. It will be seen that all terms in this section are proportional to $\sinh \Delta \Gamma_K$ and so would vanish in the limit $\Delta \Gamma_K = 0$.

There are two terms, for $d_a \cdot \sigma$ and for $d_b \cdot \sigma$. For $B^0, B_1$ etc we then only have the two cases, up to a sign, $d_3 \sigma_3$ and $d_1 \sigma_1$, for each $d$.

Linear in $d_b$:

$d_b$: With only the ‘1’ term from $\rho(a)$, Eq 33 simplifies using $A^2(1) = 1$, so we are left with $Tr[(d_b \cdot \sigma)e^{-i \Delta \Gamma_K \sigma_1 \tau(2,1)}]$. A $\sigma_3$ term gives zero so there is only a contribution from a $\sigma_1$ term:

- Case $d_3 \sigma_3$: $Tr = 0$
- Case $d_1 \sigma_1$: Using Eq 60 we obtain
  \[- \frac{1}{2} \sinh(\Delta \Gamma_K \tau(2,1))\] (36)

Linear in $d_a$:

Now turning to the terms proportional to $d_a$, with only the ‘1’ term from $\rho(b)$, we need to find

\[\frac{1}{4}Tr[e^{-im_B \tau(1,0)}(d_a \cdot \sigma)e^{+im_B \tau(1,0)}e^{-i \Delta \Gamma_K \sigma_1 \tau(2,1)}] = \frac{1}{4}Tr[e^{-im_B \tau(1,0)}(d_a \cdot \sigma)e^{+im_B \tau(1,0)}e^{+\Delta \Gamma_K \sigma_1 \tau(2,1)}],\] (37)

where in the last line we have used Eq 61 to pass through the $\sigma_3$. At this point we need Eq 60 which we insert in the last line of Eq 37, the $\cosh$ term vanishes by the unitarity of the $m_B$ expression, leaving

\[\frac{1}{4}Tr[e^{-im_B \tau(1,0)}(d_a \cdot \sigma)e^{+im_B \tau(1,0)} \sigma_1 \sinh(\Delta \Gamma_K \tau(2,1))\] (38)

to be evaluated.

There are now the two cases $(d_a \cdot \sigma) = \sigma_3$ or $\sigma_1$, corresponding to a flavor or CP eigenstate for the initial $B^0$.

- Case $d_3 \sigma_3$: Here Eq 61 can be used again to pass through the $\sigma_3$, giving for the trace $\frac{1}{4}Tr[e^{-2im_B \tau(1,0)}i \sigma_2]$. This can be evaluated using Eq 59, and we finally have

\[\frac{1}{2} \sinh(\Delta \Gamma_K \tau(2,1)) \frac{m_2}{\sqrt{m_1^2 + m_2^2}} \sin(\Delta m_B \tau(1,0)),\] (39)

so that this term is proportional to the CP and T violating parameter $m_2$. 

13
Case \(d_1\sigma_1\): Here in Eq 43 we have to do with the expression \(\sigma_1 e^{+im_B\tau(1,0)}\sigma_1\). Using Eq 62, Eq 38 becomes

\[
\frac{1}{4} Tr \left[ e^{-im_B\tau(1,0)} e^{+im_B\tau(1,0)} \right] \sinh(\Delta \Gamma_K \tau(2,1)) = \\
\frac{1}{2} \sinh(\Delta \Gamma_K \tau(2,1)) \left( \cos^2 \left( \frac{1}{2} \Delta m_B \tau(1,0) \right) + \left( \frac{m_1^2 - m_2^2}{m_1^2 + m_2^2} \right) \sin^2 \left( \frac{1}{2} \Delta m_B \tau(1,0) \right) \right).
\]

(40)

A somewhat simpler form results if we add and subtract 1 so that Eq 40 is

\[
\frac{1}{2} \sinh(\Delta \Gamma_K \tau(2,1)) \left( 1 + \left( \frac{-2m_2^2}{m_1^2 + m_2^2} \right) \sin^2 \left( \frac{1}{2} \Delta m_B \tau(1,0) \right) \right),
\]

(41)

In using the above results, it should be kept in mind that the sign in front is relevant; thus \(d = (1, 0, 0)\) can correspond to a \(B_1\) while \(d = (-1, 0, 0)\) can correspond to a \(B_2\) and so on.

6.1.3 \(d_a, d_b\) Term

We now come to the last and most complicated term, where the trace in Eq 63 is

\[
\frac{1}{4} Tr \left[ (d_b \cdot \sigma) e^{-im_K\tau(2,1)} \sigma_3 e^{-im_B\tau(1,0)} (d_a \cdot \sigma) e^{+im_B\tau(1,0)} \sigma_3 e^{+im_K\tau(2,1)} \right]
\]

(42)

Since we consider flavor or CP tags, we have 4 possibilities here: \((\sigma_3, \sigma_3), (\sigma_3, \sigma_1), (\sigma_1, \sigma_3)\) and \((\sigma_1, \sigma_1)\).

Term \((\sigma_3, \sigma_3)\) By repeated use of Eq 61 Eq 42 can be reduced to

\[
\frac{1}{4} Tr \left[ e^{+im_K\tau(2,1)} e^{-im_B\tau(1,0)} e^{-im_B\tau(1,0)} e^{+im_K\tau(2,1)} \right] = \\
\frac{1}{4} Tr \left[ e^{-i2m_B\tau(1,0)} e^{+i(m_1 + m_2)\tau(2,1)} \right] = \frac{1}{4} Tr \left[ e^{-i2m_B\tau(1,0)} e^{+i\Delta m_K \sigma_1 \tau(2,1)} \right]
\]

(43)

Expanding \(e^{+i\Delta m_K \sigma_1 \tau(2,1)} = \cos(\Delta m_K \tau(2,1)) + i\sigma_1 \sin(\Delta m_K \tau(2,1))\) and using Eq 17 gives finally

\[
\frac{1}{4} \cos(\Delta m_B \tau(1,0)) \cos(\Delta m_K \tau(2,1)) + \frac{1}{2} \frac{m_1}{\sqrt{m_1^2 + m_2^2}} \sin(\Delta m \tau(1,0)) \sin(\Delta m_K \tau(2,1))
\]

(44)

Through the \(\sin \sin\) term this expression is sensitive to the relative sign of \(\Delta m_B, \Delta m_K\); with \(m_2 = 0\) it would be simply \(\frac{1}{4} \cos(\Delta m_B \tau(1,0) - \Delta m_K \tau(2,1))\), as was found in ref 2 with CP conservation.

Term \((\sigma_1, \sigma_1)\) Using Eq 61 Eq 42 is now reduced to

\[
-\frac{1}{4} Tr \left[ e^{-im_K\tau(2,1)} \sigma_2 e^{-im_B\tau(1,0)} \sigma_2 e^{-im_B\tau(1,0)} e^{+im_K\tau(2,1)} \right]
\]

(45)
We expand the $\Delta \Gamma$ term according to Eq 60. The coefficient of the $cosh$ was evaluated in Eq 40 so we have

$$m_1 \expanding the two exponentials via Eq 59 one finds the traces of $\sigma_i$ We now use Eq 62, which are zero, so Eq 47 is the only contribution to this term.

$$m_2 \expanding the $\Delta \Gamma$ term according to Eq 60 The coefficient of the $cosh$ term was evaluated in Eq 40 so we have

$$- \frac{1}{4} \cosh(\Delta \Gamma_K \tau(2,1)) \left( \cos^2 \left( \frac{1}{2} \Delta m_B \tau(1,0) \right) + \frac{m_1^2 - m_2^2}{m_1^2 + m_2^2} \sin^2 \left( \frac{1}{2} \Delta m_B \tau(1,0) \right) \right) =$$

$$- \frac{1}{4} \cosh(\Delta \Gamma_K \tau(2,1)) \left( 1 + \frac{-2m_2^2}{m_1^2 + m_2^2} \sin^2 \left( \frac{1}{2} \Delta m_B \tau(1,0) \right) \right).$$ (47)

The coefficient of the $sinh$ involves $Tr[(m - \tilde{m})\sigma_1]$ and $Tr[\tilde{m}n\sigma_1]$, both of which are zero, so Eq 47 is the only contribution to this term.

**Term** $(\sigma_1, \sigma_3)$

Here we need

$$\frac{1}{4} \text{Tr}\left[ \sigma_1 e^{-imK\tau(2,1)} \sigma_3 e^{-imB\tau(1,0)} \sigma_3 e^{+imB\tau(1,0)} \sigma_1 e^{+imK\tau(2,1)} \right] =$$

$$- \frac{1}{4} \text{Tr}\left[ i\sigma_2 e^{-i2mB\tau(1,0)} e^{-\Delta \Gamma_K \sigma_1 \tau(2,1)} \right].$$ (48)

Expanding the two exponentials via Eq 59 one finds the traces of $\sigma_2$, $\sigma_2\sigma_1$, $\sigma_2\sigma_2\sigma_1$ or $\sigma_2\sigma_1\sigma_1$, all of which are zero. The only nonzero factor is that of $m_2\sigma_2\sigma_2$, so this term contributes

$$- \frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}} \cosh \Delta \Gamma_K \tau(2,1) \sin(\Delta m_B \tau(1,0)).$$ (49)

**Term** $(\sigma_3, \sigma_1)$

Here we need

$$\frac{1}{4} \text{Tr}\left[ \sigma_3 e^{-imK\tau(2,1)} \sigma_3 e^{-imB\tau(1,0)} \sigma_1 e^{+imB\tau(1,0)} \sigma_3 e^{+imK\tau(2,1)} \right] =$$

$$\frac{1}{4} \text{Tr}\left[ e^{-imB\tau(1,0)} \sigma_1 e^{+imB\tau(1,0)} \sigma_3 e^{+i(m_K + m_2^1)\tau(2,1)} \right] =$$

$$- \frac{1}{4} \text{Tr}\left[ e^{-imB\tau(1,0)} i\sigma_2 e^{-imB\tau(1,0)} e^{+i(m_K + m_2^1)\tau(2,1)} \right].$$ (50)

We expand the $m_K$ exponential, the $sin$ term giving

$$- \frac{1}{4} \sin(\Delta m_K \tau(2,1)) \text{Tr}\left[ i\sigma_1 e^{-imB\tau(1,0)} i\sigma_2 e^{-imB\tau(1,0)} \right] =$$

$$\frac{1}{4} \sin(\Delta m_K \tau(2,1)) \text{Tr}\left[ i\sigma_3 e^{+i\tilde{m} B \tau(1,0)} e^{-imB\tau(1,0)} \right] =$$

$$\frac{1}{4} \sin(\Delta m_K \tau(2,1)) \frac{\sin^2 \left( \frac{1}{2} \Delta m_B \tau(1,0) \right)}{m_1^2 + m_2^2} \text{Tr}\left[ i\sigma_3 \tilde{m} B m_B \right] =$$

$$\frac{m_1 m_2}{m_1^2 + m_2^2} \sin(\Delta m_K \tau(2,1)) \sin^2 \left( \frac{1}{2} \Delta m_B \tau(1,0) \right).$$ (51)
while the cos term is

$$\frac{1}{2} \cos(\Delta m_K \tau(2, 1)) Tr \left[ e^{-i m_B \tau(1, 0)} \sigma_2 e^{-i m_B \tau(1, 0)} \right] =$$

$$\frac{1}{2} \cos(\Delta m_K \tau(2, 1)) Tr \left[ i \sigma_2 e^{i m_B \tau(1, 0)} e^{-i m_B \tau(1, 0)} \right] =$$

$$\frac{1}{2} \cos(\Delta m_K \tau(2, 1)) \frac{\sin(\Delta m_B \tau(1, 0)) \cos(\frac{1}{2} \Delta m_B \tau(1, 0))}{\sqrt{m_1^2 + m_2^2}} Tr \left[ \sigma_2 (m_B - \bar{m}_B) \right] =$$

$$\frac{1}{2} m_2 \sqrt{m_1^2 + m_2^2} \cos(\Delta m_K \tau(2, 1)) \sin(\Delta m_B \tau(1, 0)). \quad (52)$$

### 6.2 Tabulation

We now have completed the evaluation of the terms that will appear in the cascade mixing of all 16 combinations of initial and final tags. We present these in tabular form. Table 2 lists the functions that occur. They are normalized so that at $\tau(2, 1) = \tau(1, 0) = 0$, they are equal to either zero or one. A given process, beginning with a $B^o$ and ending with a $K^o$, called 'a' and 'b' respectively, will be given by sums of these functions with different coefficients. The coefficients for each process are given in Table 3.

If desired, the factors $\sin^2 \frac{1}{2} \Delta m_B \tau(1, 0))$ can be expanded using half angle identities to give full angle expressions; thus combinations of the functions I and J can be made to exhibit expressions of the type $\sin(\Delta m_B \tau(1, 0) + \Delta m_K \tau(2, 1))$, where the relative sign of the $\Delta m$ enters, as was noted in ref [2]. Similarly E and F when combined in the limit $m_2 = 0$ give $\cos(\Delta m_B \tau(1, 0) - \Delta m_K \tau(2, 1))$.

One may also multiply the $\sinh \Delta m_K \tau(2, 1)$ or $\cosh \Delta m_K \tau(2, 1)$ factors by the prefactor $e^{-\Gamma_K^o \tau(2, 1)}$ to exhibit the contributions of the two $K^o$ lifetime eigenstates. While $\Delta m_K$ occurs in the functions of Table 2, the coefficients in Table 3 depend only on the parameters of the $B^o$ mass matrix. This is because with the assumption of neglect of CP violation in the $K^o$ system, there is essentially only one parameter, the coefficient of $\sigma_1$, and one has simply $e^{-i \Delta m_K \sigma_1 \tau} = \cos(\Delta m_K \tau) - i \sigma_1 \sin(\Delta m_K \tau)$.

| Name | Function | reference |
|------|----------|-----------|
| A    | $\cosh \Delta m_K \tau(2, 1)$ | Eq[35] Eq[47] |
| B    | $\sinh \Delta m_K \tau(2, 1)$ | Eq[36] Eq[41] |
| C    | $\sinh \Delta m_K \tau(2, 1) \sin \Delta m_B \tau(1, 0)$ | Eq[39] |
| D    | $\sinh \Delta m_K \tau(2, 1) \sin^2 \frac{1}{2} \Delta m_B \tau(1, 0)$ | Eq[41] |
| E    | $\cos \Delta m_K \tau(2, 1) \cos \Delta m_B \tau(1, 0)$ | Eq[44] |
| F    | $\sin \Delta m_K \tau(2, 1) \sin \Delta m_B \tau(1, 0)$ | Eq[44] |
| G    | $\cosh \Delta m_K \tau(2, 1) \sin^2 \frac{1}{2} \Delta m_B \tau(1, 0)$ | Eq[47] |
| H    | $\cosh \Delta m_K \tau(2, 1) \sin \Delta m_B \tau(1, 0)$ | Eq[49] |
| I    | $\cos \Delta m_K \tau(2, 1) \sin \Delta m_B \tau(1, 0)$ | Eq[52] |
| J    | $\sin \Delta m_K \tau(2, 1) \sin^2 \frac{1}{2} \Delta m_B \tau(1, 0)$ | Eq[51] |

Table 2: Functions employed with their designations and reference in the text.
| a  | b  | d₀   | d₀  |
|----|----|------|------|
| B₀ | K₀ | (0,0,1) | (0,0,1) |
|    |    | ½ | 0 | $\frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | $\frac{1}{2} \frac{m_1}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 | 0 | 0 |
| B₀ | K₀ | (0,0,1) | (0,0,-1) |
|    |    | ½ | 0 | $-\frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | $-\frac{1}{2} \frac{m_1}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 | 0 | 0 |
| B₁ | K₀ | (0,0,1) | (1,0,0) |
|    |    | ½ | ½ | 0 | $-\frac{1}{2} \frac{m_2}{m_1^2 + m_2^2}$ | 0 | 0 | 0 | 0 | $\frac{1}{2} \frac{m_2}{m_1^2 + m_2^2}$ | $-\frac{m_1 m_2}{m_1^2 + m_2^2}$ |
| B₀ | K₀ | (0,0,1) | (1,0,0) |
|    |    | ½ | ½ | 0 | $-\frac{1}{2} \frac{m_2}{m_1^2 + m_2^2}$ | 0 | 0 | 0 | 0 | $-\frac{1}{2} \frac{m_2}{m_1^2 + m_2^2}$ | $\frac{m_1 m_2}{m_1^2 + m_2^2}$ |
| B₀ | K₀ | (0,0,1) | (0,0,1) |
|    |    | ½ | ½ | 0 | $\frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | $-\frac{1}{2} \frac{m_1}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 | 0 | 0 |
| B₀ | K₀ | (0,0,1) | (0,0,-1) |
|    |    | ½ | ½ | 0 | $\frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | $-\frac{1}{2} \frac{m_1}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 | 0 | 0 |
| B₀ | K₁ | (1,0,0) | (0,0,1) |
|    |    | ½ | $-\frac{1}{2}$ | $\frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 | 0 | 0 | $\frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 |
| B₀ | K₁ | (1,0,0) | (0,0,-1) |
|    |    | ½ | $-\frac{1}{2}$ | $-\frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 | 0 | 0 | $-\frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 |
| B₀ | K₁ | (1,0,0) | (1,0,0) |
|    |    | ½ | $-\frac{1}{2} + \frac{1}{2}$ | $-\frac{1}{2} + \frac{1}{2}$ | 0 | $-\frac{m_2}{m_1^2 + m_2^2}$ | 0 | 0 | $\frac{m_2}{m_1^2 + m_2^2}$ | 0 | 0 |
| B₀ | K₂ | (-1,0,0) | (0,0,1) |
|    |    | ½ | $-\frac{1}{2} + \frac{1}{2}$ | $-\frac{1}{2} - \frac{1}{2}$ | 0 | 0 | 0 | 0 | $-\frac{1}{2} \frac{m_2}{m_1^2 + m_2^2}$ | 0 | 0 |
| B₀ | K₂ | (-1,0,0) | (0,0,-1) |
|    |    | ½ | $-\frac{1}{2} + \frac{1}{2}$ | $-\frac{1}{2} - \frac{1}{2}$ | 0 | 0 | 0 | 0 | $-\frac{1}{2} \frac{m_2}{m_1^2 + m_2^2}$ | 0 | 0 |
| B₁ | K₁ | (1,0,0) | (0,0,1) |
|    |    | ½ | $-\frac{1}{2}$ | $-\frac{1}{2} + \frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 | 0 | 0 | $\frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 |
| B₁ | K₁ | (1,0,0) | (1,0,0) |
|    |    | ½ | $-\frac{1}{2} + \frac{1}{2}$ | $-\frac{1}{2} - \frac{1}{2}$ | 0 | $-\frac{m_2}{m_1^2 + m_2^2}$ | 0 | 0 | $\frac{m_2}{m_1^2 + m_2^2}$ | 0 | 0 |
| B₂ | K₁ | (1,0,0) | (1,0,0) |
|    |    | ½ | $-\frac{1}{2} + \frac{1}{2}$ | $-\frac{1}{2} - \frac{1}{2}$ | 0 | $-\frac{m_2}{m_1^2 + m_2^2}$ | 0 | 0 | $\frac{m_2}{m_1^2 + m_2^2}$ | 0 | 0 |
| B₀ | K₂ | (-1,0,0) | (0,0,1) |
|    |    | ½ | $-\frac{1}{2}$ | $-\frac{1}{2} + \frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 | 0 | 0 | $\frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 |
| B₂ | K₂ | (-1,0,0) | (0,0,-1) |
|    |    | ½ | $-\frac{1}{2}$ | $-\frac{1}{2} + \frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 | 0 | 0 | $\frac{1}{2} \frac{m_2}{\sqrt{m_1^2 + m_2^2}}$ | 0 | 0 |

Table 3: Coefficients of functions arising in $B^o \rightarrow (J/ψ \rightarrow K^o)$, for 16 possible combinations of initial $B^o$ and final $K^o$. To find the relative rates for a given process as function of $\tau(1,0)$ and $\tau(2,1)$, use the functions given in Table 2 summed with the coefficients given here; and multiply by the prefactor $e^{-F_{B^o} \tau(1,0)} e^{-F_{K^o} \tau(2,1)}$. The normalization is such that the value for $B^o \rightarrow K^o$ at $\tau(1,0) = \tau(2,1) = 0$ is 1. Values of the $d$ are given as $(d_1, d_2, d_3)$. Where a coefficient arises from two different equations in the text, we exhibit the contributions individually. The $B^o$ mass splitting is $2\sqrt{m_1^2 + m_2^2}$ and $m_2$ gives the CP and T violation in the $B^o$ mass matrix.
7 Conclusions

We have explained how using an ‘away side tag’ can make it possible to observe ‘cascade mixing’ in $\Upsilon(4S) \to B^o \bar{B}^o$ without precision knowledge on the location of the primary vertex.

Confirmation of our predictions, as summarized in Table 3, would verify the validity of this pretty extension of the physics of particle mixing and also provide an additional approach to the parameters of the $B^o$ mass matrix, particularly the CP violating parameter, here called $m_2$.

Since, as explained in section 4.1, it appears that $m_1$ and $m_2$ are of about the same size, the coefficients in Table 3 should have substantial values.

The role of the first flight time as a ‘variable regenerator’ depending on $\tau(1,0)$, can be nicely exhibited, data permitting, by exhibiting the different oscillation patterns arising for the $K^o$ according to the value of $\tau(1,0)$. By setting $\Delta m_B \tau(1,0)$ in the vicinity of $0, \pi, 2\pi$...for example, various of the functions in Table 2 can be made to disappear or to reverse sign, giving distinctly different patterns in $\tau(2,1)$.

Those processes involving the function F can give information about the relative sign of $\Delta m_B, \Delta m_K$ (ref 1), while the coefficients of function J involve the relative sign of $m_1, m_2$.

Our approximations include $\Delta \Gamma_B \approx 0$, good CPT, neglect of CP violation in the $K^o$ system, neglect of particle-antiparticle processes, and neglect of direct CP violation. The latter is probably the most significant, and enters in two ways. One is that it allows the use of specific decay channels to identify states of the $K^o$ or $B^o$ systems with their naive CP values, as discussed in section 2.3.

Secondly it allows for the simple form of the transition amplitude $A(1)$ in Eq 34. Because of these assumptions, the predictions can only be taken to be good to the percent level.

Many extensions and generalizations using the method can be envisioned. These could include replacing the $J/\psi$ which induces the conversion from $B^o$ to $K^o$, with another particle or particle state, for example one where the CP is not ‘flipped’, or one where there is direct CP violation, in which case the transition operator Eq 33 would contain a mixture of terms. One may consider introducing the $D^o$ (most amusing would be a ‘three-time cascade’ $B^o \to D^o \to K^o$ which could be treated by the methods here) and undoubtedly many other possibilities.

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9 Appendix

We list some of the definitions and $\sigma$ relations used in the text.

9.1 Mass Matrices

The mass matrices used, after removal of the average mass and average $\Gamma$ are

\[ m_B = m_1 \sigma_1 + m_2 \sigma_2 \quad m_K = \frac{1}{2}(\Delta m_K - i\Delta \Gamma_K)\sigma_1, \] (53)

where the terms proportional to the I matrix, namely $-i\frac{1}{2}\Gamma_B$, $-i\frac{1}{2}\Gamma_{K}^{av}$, as well as the average real masses have been taken out. These definitions for $m_B$ and $m_K$ represent our approximations of neglecting $\Delta \Gamma_B$ and neglecting CP violation in the $K^o$ system. In terms of the hermitian and antihermitian parts for the $K^o$

\[ (m_K + m_K^\dagger) = \Delta m_K \sigma_1 \quad -i(m_K - m_K^\dagger) = -\Delta \Gamma_K \sigma_1 \] (54)

In terms of the definite lifetimes

\[ \Gamma_K^{av} = \frac{1}{2}(\Gamma_s + \Gamma_l) \quad \Delta \Gamma_K = \frac{1}{2}(\Gamma_s - \Gamma_l) \] (55)

For the mass splitting $\Delta m_B$ of the $B^o$

\[ \frac{1}{2}\Delta m_B = \sqrt{m_1^2 + m_2^2}, \] (56)

while for the $K^o$ the mass splitting is simply $\Delta m_K$.

9.2 Pauli matrices

We use the definition of the pauli matrices where

\[ \sigma_1 \sigma_2 = i\sigma_3 \quad \sigma_1 \sigma_2 = -\sigma_2 \sigma_1 \quad \sigma_1^2 = I \] (57)

and cyclic permutations.

We frequently use the identity

\[ e^{-ib \cdot \sigma} = \cos b - i(b \cdot \sigma) \frac{\sin b}{b} \] (58)

which for the $B^o$ time development implies

\[ e^{-im_B \tau} = \cos(\frac{1}{2}\Delta m_B \tau) - i\frac{(m_1 \sigma_1 + m_2 \sigma_2)}{\frac{1}{2}\Delta m_B} \sin(\frac{1}{2}\Delta m_B \tau) \]

\[ e^{-i2m_B \tau} = \cos(\Delta m_B \tau) - i\frac{(m_1 \sigma_1 + m_2 \sigma_2)}{\frac{1}{2}\Delta m_B} \sin(\Delta m_B \tau) \] (59)
We use the real version of Eq. 58 in the $K^o$ evolution as
\[ e^{-\Delta \Gamma_K \tau} \sigma_1 = \cosh(\Delta \Gamma_K \tau) - \sigma_1 \sinh(\Delta \Gamma_K \tau) \] (60)

From the absence of $\sigma_3$ (CPT assumption) in the $B^o$ and $K^o$ time evolutions one has, by anticommuting the $\sigma_i$ the useful identities
\[ \sigma_3 e^{-im_B \tau} = e^{+im_B \tau} \sigma_3 \quad \sigma_3 e^{-im_K \tau} = e^{+im_K \tau} \sigma_3 \] (61)

Analogous relations are
\[ \sigma_1 e^{-im_B \tau} = \sigma_1 e^{-i(m_1 \sigma_1 + m_2 \sigma_2) \tau} = e^{-i(m_1 \sigma_1 - m_2 \sigma_2) \tau} \sigma_1 = e^{-im_B \tau} \sigma_1 \]
\[ \sigma_2 e^{-im_B \tau} = \sigma_2 e^{-i(m_1 \sigma_1 + m_2 \sigma_2) \tau} = e^{+i(m_1 \sigma_1 - m_2 \sigma_2) \tau} \sigma_2 = e^{+im_B \tau} \sigma_2, \] (62)

where $\vec{m}_B$ is the transpose of $m_B$.

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