Abstract. In this paper, we will prove that a problem deciding whether there is an upper-triangular coordinate in which a character is not in the state of a Hilbert point is NP-hard. This problem is related to the GIT-semistability of a Hilbert point.

1. Introduction

Let $k$ be an algebraically closed field of characteristic zero and $rS = k[x_1, \ldots, x_r]$ be a polynomial ring of $r$ variables graded by degree. Let’s omit the superscript $r$ if there is no confusion. When non-negative integers $d$ and $b$ are fixed, there is a projective space $E_{d,b} = \mathbb{P}^b \left( \bigwedge^r S_d \right)$, which is a $GL_r(k)$-representation. Let $T_r$ be the maximal torus of $GL_r(k)$ which consists of diagonal matrices and $U_r$ be the set of all upper-triangular matrices with 1’s in the diagonal. There is a $G$-equivariant closed immersion

$$i_{r,P,d} : \text{Hilb}^P(\mathbb{P}^{r-1}) \to E_{d,Q(d)}$$

for $d \geq g_P$ where $g_P$ is the Gotzmann number associated to a Hilbert polynomial $P$, which is defined in [2]. Also $Q(d) = \binom{r+d-1}{d} - P(d)$.

For any point $v \in E_{d,b}$, the collection of states $\Xi_{G,v} = \{ \Xi_{g,v}(T) | g \in GL_r(k) \}$ (defined in [5]) of $v$ determines whether $v$ is semistable or not, as stated in [7]. If $v$ is unstable, $\Xi_{G,v}$ determines the Hesselink strata of $P\left( \bigwedge^b S_d \right)$ that contains $v$, which is stated in [9]. For an arbitrary character $\chi$ of $T$, $Z_{v,\chi} = \{ g \in GL_r(k) | \chi \notin \Xi_{g,v} \}$ is a Zariski-closed subset of $GL_r(k)$. In this paper, we will construct a solvability check problem (SC) which is equivalent to deciding if an arbitrary system of algebraic equations is solvable (SysAl) by specializing the defining equation of some $Z_{v,\chi}$ to the defining equation of $U_r \cap Z_{v,\chi}$ in $U_r$.

It’s a well-known fact that to decide whether an arbitrary system of algebraic equation is solvable is an NP-hard problem.([6]) We will show that this problem can be reduced to the problem asking whether there is a $g \in U_r$ such that $\chi \notin \Xi_{g,w}$, in polynomial time. This means that such a problem is NP-hard. This problem is...
related to the GIT-semistability of a Hilbert point. By solving finitely many such problems, we can decide whether a Hilbert point is semistable or not.

2. Definitions and notations

First of all, we need to define the notion of generalization of a system of algebraic equation.

**Definition** Suppose $I$ be an ideal of $S = k[x_1, \ldots, x_r]$. An ideal $J$ of a finitely generated $k$ algebra $R$ is a *generalization* of $I$ under $\pi$ if there is a surjective ring homomorphism $\pi : R \to S$ and a minimal generator $\{z_1, \ldots, z_r\}$ of $R$ satisfying follows:

- For any $1 \leq i \leq r'$, $\pi(z_i) \in k \cup \{x_1, \ldots, x_r\}$
- $\pi(J) = I$

$I$ is a *specialization* of $J$ if $J$ is a generalization of $I$.

For example, $I = (x^2 + y^2) \subset k[x, y]$ is a specialization of $J = \langle z(x^2 + y^2), zw \rangle \subset k[x, y, z, w]$ under the map $\pi : k[x, y, z, w] \to k[x, y]$ which satisfying $\pi(x) = x$, $\pi(y) = y$, $\pi(z) = 1$ and $\pi(w) = 0$.

We define some notation. Let $<_{\text{lex}}$ be a lexicographic monomial order satisfying $x_{i+1} <_{\text{lex}} x_i$ and let $A_{d,b}^{r} = \bigwedge_{1}^{r} S_d$. Let $M_d$ be the set of all monomials in $r S_d$ and

$$W_{d,b}^{r} = \left\{ \bigwedge_{i=1}^{b} m_{i} \bigm| m_{i} \in r M_{d}, m_{i} >_{\text{lex}} m_{i+1} \right\}. $$

$W_{d,b}^{r}$ is a basis of $A_{d,b}^{r}$. Suppose $v \in A_{d,b}^{r}$ and $w \in W_{d,b}^{r}$. We define $v_w$ to be the $w$-component of the vector $v$. That is,

$$v = \sum_{w \in W_{d,b}^{r}} v_w w.$$ 

Let $[v] \in E_{d,b}^{r}$ be the line in $A_{d,b}^{r}$ through $v$ and the origin of $A_{d,b}^{r}$. For any $g \in \text{GL}_r(k)$, $g_{ij} \in k$ is the component of $g$ in the $i$th row and $j$th column. That is,

$$g = \begin{bmatrix}
    g_{11} & \cdots & g_{1j} & \cdots \\
    \vdots & \ddots & \vdots & \ddots \\
    g_{i1} & \cdots & g_{ij} & \cdots \\
    \vdots & \cdots & \vdots & \ddots
\end{bmatrix}.$$ 

Also, $\text{GL}_r(k)$ action on $r S$ is given by $g.x_i = \sum_{1 \leq j \leq r} g_{ij} x_j$. Note that this action is a left action on $r S$. For any $v \in A_{d,b}^{r}$ and $w \in k[W_{d,b}^{r}]$, $(g.v)_w$ means $((id(\text{GL}_r(k), O_{\text{GL}_r(k)})) \otimes_k e_v) \circ \phi) (w)$ when $g$ is an indeterminate. Here $\phi$ is the co-action map

$$\phi : k[W_{d,b}^{r}] \to \Gamma(\text{GL}_r(k), O_{\text{GL}_r(k)}) \otimes_k k[W_{d,b}^{r}] \cong k[\{g_{ij}\}_{i,j=1}^{r} \det g] \otimes_k k[W_{d,b}^{r}]$$

and $e_v$ is the evaluation map $e_v : k[W_{d,b}^{r}] \to k$ at $v$. Let’s define $\chi_i \in X(T_r)$ for all $1 \leq i \leq r$ as follows:

$$\chi_i(D) = D_{ii}$$

where $D \in T_r$. Let $\xi_{d,b}^{r} = \frac{db}{r}(1, \ldots, 1) \in X(T_r)_{\mathbb{R}} = X(T_r) \otimes_{\mathbb{Z}} \mathbb{R}$. Here $X(T_r)$ is the group of characters of the algebraic torus $T_r$. 
Let $L_r = \{g \in \text{GL}_r(k)|g\text{ is lower-triangular}\}$. Let’s define a specialization map $	heta_r : \Gamma(\text{GL}_r(k), \mathcal{O}_{\text{GL}_r(k)}) \rightarrow \Gamma(U_r, \mathcal{O}_{U_r})$ as follows.

$$
\theta_r(z_{ij}) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i > j \\
z_{ij} & \text{if } i < j 
\end{cases}
$$

where $\Gamma(\text{GL}_r(k), \mathcal{O}_{\text{GL}_r(k)}) = k[\{z_{ij}\}_{i,j=1}^r]_{\text{det}}$ and $\Gamma(U_r, \mathcal{O}_{U_r}) = k[\{z_{ij}\}_{1 \leq i < j \leq r}]$. For any $C \subset k[\{z_{ij}\}_{i,j=1}^r]_{\text{det}}$, let span$C$ be the $k$-subspace of $k[\{z_{ij}\}_{i,j=1}^r]_{\text{det}}$ spanned by $C$. Let $\Sigma_r$ be the permutation group on the set $\{1, 2, \ldots, r\}$, which is a subgroup of $\text{GL}_r(k)$. Let $\Delta_v$ be the convex hull of $\Xi_v$ in $X(T_r)_R$ for all $v \in E_{d,b}^r$.

3. POLYNOMIAL COEFFICIENTS IN SOME SPECIAL CASES

Suppose $v \in A_{d,b}^r$. In this section, we will compute $v_w$ for some special $w \in W_{d,b}^r$. Let’s compute it when $b = 1$ first.

**Lemma 3.1.** Suppose $r \geq 2$. Let $p \in \mathfrak{s}^r d = A_{d,1}^r$. For any $g \in \text{GL}_r(k)$,

$$
\left(g.p\right)_{x_1^{d-i}x_2^i} = \sum_{i_1 + \ldots + i_r = j} \prod_{1 \leq a \leq r} g_{i_a} \frac{\partial^j p}{\partial x_1^{i_1} \ldots \partial x_r^{i_r}}.
$$

**Proof.** Without loss of generality, we can assume that $p$ is a monomial. When $p$ is a monomial, expanding $g.p$ proves the equality. \qed

We can generalize Lemma 3.1 using the following lemma.

**Lemma 3.2.** Suppose $r \geq 2$. Let $p_1, p_2 \in \mathfrak{s}^r d = A_{d,1}^r$. For any $g \in \text{GL}_r(k)$,

$$
\left(g.p_1 \wedge p_2\right)_{x_1^{d-j_1}x_2^j} = \begin{pmatrix} (g.p_1)_{x_1^{d-j_1}x_2^j} \\
(g.p_2)_{x_1^{d-j_1}x_2^j} \end{pmatrix}
$$

for all $1 \leq j_1 < j_2 \leq d$.

**Proof.** It can be derived from the definition. \qed

In Lemma 3.1 we see that taking $(g,\star)_m$ of $p$ separates each monomial with respect to the degrees of each variables of $p$ and $m$. Our construction would make use of this phenomenon. That is, we will control the degree of one variable, say $x_{r+1}$.

Fix $d$. Let $F$ be a sequence $\{F_i\}_{i=0}^{2l-1} \in (\mathfrak{s}^r d)^{2l} \subset (\mathfrak{s}^{r+1} d)^{2l}$. Let’s define $v_d^r(F) \in A_{2l+1}^{r+1}$ as follows.

$$
v_d^r(F) = \left\{ \sum_{i=0}^{2l-1} x_{r+1}^i x_1^{2l-i} F_i \right\} \wedge \left\{ \sum_{i=0}^{2l-1} x_{r+1}^{i+1} x_1^{2l-i-1} F_i \right\}
$$

Note that $[v_d^r(F)] \in \text{Hilb}^P(P_k)$ where

$$
P(t) = \frac{r+t}{r} - \left( \frac{r+t-2l-d+1}{r} \right) + \left( \frac{r+t-2l-d-1}{r-2} \right).
$$

Indeed, the graded ideal

$$
I_F = \left\{ \sum_{i=0}^{2l-1} x_{r+1}^i x_1^{2l-i} F_i, \sum_{i=0}^{2l-1} x_{r+1}^{i+1} x_1^{2l-i-1} F_i \right\}
$$

of $\mathfrak{s}^{r+1} d$ satisfies the following properties.
Lemma 3.3. \( r^{+1}S/I_F \) has the Hilbert polynomial

\[
P(t) = \binom{r+t}{r} - \binom{r+t-2l-d+1}{r} + \binom{r+t-2l-d-1}{r-2}.
\]

Also, \( gp = 2l + d \) so that \( i_{r+1,p,2l+d}(I_F) = [v_d^r(F)] \). If \( r \geq 2 \) then \( I_F \) is saturated.

Proof. \( I = I_F \) is isomorphic to \( (x_1, x_{r+1})(-2l - d + 1) \) as a graded \( r^{+1}S \) module. Thus, \( \dim_k(I_F)_{t+2l+d} \) is equal to the number of monomials in \( r^{+1}S_{t+1} \) which is divisible by \( x_1 \) or \( x_{r+1} \), for every \( t \geq 0 \). This implies that \( I_F \) has the Hilbert polynomial

\[
Q(t) = \binom{r+t-2l-d+1}{r} - \binom{r+t-2l-d-1}{r-2}.
\]

By the definition of \( n(Q) \) in [2, p. 65], \( gp = 2l + d \). The regularity of \( I_F \) is equal to the regularity of \( (x_1, x_{r+1})(-2l - d + 1) \), which is equal to \( 2l + d \). Let \( J \) be the saturation of \( I_F \). The Hilbert polynomial of \( J \) is \( Q \). This implies that the regularity of \( J \) is at most \( 2l + d \). Therefore, \( \dim_k J_t = Q(t) = \dim_k I_t \) for all \( t \geq 2l + d \) by [2] (1.2) Satz, (2.9) Lemma. Suppose \( r \geq 2 \). If there is a homogeneous \( q \in J \setminus I \) then \( q \in J_t \) for some \( t < 2l + d \). We derive an inequality

\[
2 = \dim_k I_{2l+d} = \dim_k J_{2l+d} \geq \dim_k(q)_{2l+d} \geq r + 1 \geq 3,
\]

which is false. \( \square \)

We can analyze the polynomial coefficient of \( g.v_d^r(F) \) as follows:

Lemma 3.4. \( \{ f_{a,r,t,F} \}_{a=0}^{l-1} \) is a basis for

\[
\text{span}\{\theta_{r+1}((g.v_d^r(F))_{x_1^{2l+d-a}x_{r+1}x_1^{d+a}x_{r+1}^{2l-a}}) | 0 \leq a \leq l-1\}
\]

where

\[
f_{a,r,t,F} = \sum_{0 \leq i \leq j \leq 2l-1} \tilde{F}_i \tilde{F}_j g_{1r+1} \left[ \binom{i}{a} \binom{j}{2l-a-1} + \binom{i}{2l-a-1} \binom{j}{a} \right]
\]

\[
+ \sum_{i=0}^{2l-1} \tilde{F}_i^2 g_{1r+1} \binom{i}{a} \binom{i}{2l-a-1}
\]

and

\[
\tilde{F}_i = F_i(1, g_{12}, \ldots, g_{1r}).
\]

Proof. Using Lemma 3.1 and Lemma 3.2 we can compute that

\[
f_{a,r,t,F} - f_{a-1,r,t,F} = \theta_{r+1}((g.v_d^r(F))_{x_1^{2l+d-a}x_{r+1}x_1^{d+a}x_{r+1}^{2l-a}})
\]

for all \( 1 \leq a \leq l-1 \) and

\[
f_{0,r,t,F} = \theta_{r+1}((g.v_d^r(F))_{x_1^{2l+d}x_1^{d}x_{r+1}^{2l}}).
\]

\( \square \)

Let \( \psi \) be a sequence \( \{ \psi_i \}_{i=0}^{l-1} \subset (r S_d)^l \). Let’s define a sequence \( F_{\psi} \in (r S_d)^{2l} \subset (r^{+1}S_d)^{2l} \) as follows:

- \( (F_{\psi})_i = 0 \) for all \( l \leq i \leq 2l - 2 \).
- \( (F_{\psi})_{2l-1} = x_1^{d} \).
Proof. Note that \( l \) for any choice of integers \( \pi \) for all Lemma 3.6. which is zero in this paper.

Lemma 3.5. \( \pi^\psi = \{ \pi_j \}_{j=0}^{l-1} \) is a basis for
\[
\text{span}(\Theta_{r+1+1}((g,F_{\psi}))(x_1^{2l-d-a,\ldots,x_{r+1}^{2l-d-a}})|0 \leq a \leq l-1)
\]
where
\[
\pi_j = \frac{(2l-1)!}{(2l-j)!} \left[ \sum_{a=0}^{l-1-j} \binom{2l-1-j}{a} (-1)^a \right] g_{1r+1}^{2l-1} + \tilde{\psi}_j g_{1r+1}^j
\]
and
\[
\tilde{\psi}_j = \psi_j(1,g_{12},\ldots,g_{1r})
\]

Proof. By the definition of \( f_{a,r,l,F_{\psi}} \),
\[
a!(2l-1)^{-1} f_{a,r,l,F_{\psi}} = \frac{(2l-1)!}{(2l-1-a)!} g_{1r+1}^{2l-1} + \sum_{i=a}^{l-1} \frac{1}{(i-a)!} \tilde{\psi}_i g_{1r+1}^i
\]
for all \( 0 \leq a \leq l-1 \). Now
\[
\sum_{a=j}^{l-1} \frac{(-1)^{a+j}}{(a-j)!} a!(2l-1)^{-1} f_{a,r,l,F_{\psi}} = (2l-1)! \sum_{a=j}^{l-1} \frac{(-1)^{a+j}}{(a-j)!(2l-1-a)!} g_{1r+1}^{2l-1}
\]
\[
+ \sum_{a=j}^{l-1} \sum_{i=a}^{l-1} \frac{(-1)^{a+j}}{(a-j)!(i-a)!} \tilde{\psi}_i g_{1r+1}^i
\]
\[
= \frac{(2l-1)!}{(2l-1-j)!} \sum_{a=0}^{l-1-j} \binom{2l-1-j}{a} (-1)^a g_{1r+1}^{2l-1}
\]
\[
+ \sum_{i=j}^{l-1} \sum_{a=0}^{l-1-j} \frac{1}{(i-j)!(i-a)!} (-1)^a \tilde{\psi}_i g_{1r+1}^i = \pi_j^\psi.
\]
Clearly \( \{ \pi_j \}_{0 \leq j \leq l-1} \) is a linearly independent set. This proves the lemma. \( \square \)
\( \pi^\psi \) has the following property. This property depends on the characteristic of \( k \), which is zero in this paper.

Lemma 3.6. The coefficient of \( g_{1r+1}^{2l-1} \) in \( \pi_j^\psi \) is non-zero. That is,
\[
\sum_{a=0}^{l-1-j} \binom{2l-1-j}{a} (-1)^a \neq 0
\]
for any choice of integers \( l \) and \( j \) satisfying \( l \geq 1 \) and \( 0 \leq j \leq l-1 \).

Proof. Note that
\[
\binom{2l-1-j}{a} \leq \binom{2l-1-j}{a+1}
\]
for all \( a \) satisfying \( 0 \leq a \leq l-1-j \).
If \( l-1-j \) is even,
\[
\sum_{a=0}^{l-1-j} \binom{2l-1-j}{a} (-1)^a = 1 + \sum_{a=1}^{l-1-j} \binom{2l-1-j}{2a} \left( \frac{2l-1-j}{2a-1} \right) > 0.
\]
Similarly, If \( l - 1 - j \) is odd, we can show that
\[
\sum_{a=0}^{l-1-j} \binom{2l-1-j}{a} (-1)^a < 0
\]
because the first term is always strictly smaller than the absolute value of the second term.

\[\square\]

4. NP-HARDNESS OF A PROBLEM JUDGING THE EXISTENCE OF AN UPPER-TRIANGULAR COORDINATE

Suppose \( l \geq 3, r \geq 2 \) and \( p = \{p_i\}_{i=0}^{l-3} \in k[x_2, \ldots, x_r]^{l-2} \). Assume that
\[
d \geq \max\{\deg(p_i) | 0 \leq i \leq l-3\}
\]
where \( \deg(p_i) \) means the non-homogeneous degree of \( p_i \). Let’s construct \( \psi(p) = \{\psi_i(p)\}_{i=0}^{l-1} \).

- Define the first two terms as follows:

\[
\psi_i(p) = \frac{(2l-1)!}{(2l-1-i)!} \left[ \sum_{a=0}^{l-1-i} \binom{2l-1-i}{a} (-1)^a \right] x_1^d
\]

for \( i \in \{0, 1\} \).

- For \( 2 \leq i \leq l-1 \), let

\[
\psi_{i-1}(p) = \frac{(2l-1)!}{(2l-1-i)!} \left[ \sum_{a=0}^{l-1-i} \binom{2l-1-i}{a} (-1)^a \right] x_1^d
\]

\[+ x_1^d p_{i-2} \left( \frac{x_2}{x_1}, \ldots, \frac{x_r}{x_1} \right).\]

Now we are ready to prove follows.

**Theorem 4.1.** Let \( l \geq 3 \). There is \( g \in U_{r+1} \) satisfying \( \chi = \chi_{0}^{2d+2l} \chi_{r+1}^{2l} \notin \Xi_{\{\psi(p)\}} \) if and only if the ideal \( J \) of \( k[x_2, \ldots, x_r] \) generated by \( \{p_i|0 \leq i \leq l-3\} \) has a solution over \( k \).

**Proof.** By definition, \( Z_{\{\psi(p)\}} \cap U_{r+1} \) is the zero set of the ideal
\[
I \subset \Gamma(U_{r+1}, O_{U_{r+1}}) = k\{\{g_{ij}\}_{1 \leq i < j \leq r+1}\}
\]
generated by
\[
\{x_1^d p_{i-2} \left( \frac{x_2}{x_1}, \ldots, \frac{x_r}{x_1} \right) | 0 \leq a \leq l-1\}.
\]

By Lemma 3.5, \( I \) is generated by
\[
\{\pi_i^\psi(p) | 0 \leq i \leq l-1\}.
\]

It suffices to show that the zero set of \( I \) is non empty if and only if the zero set of \( J \) is non empty. If there is an element \( \{x_{ij}\}_{1 \leq i < j \leq r+1} \) in the zero set of \( I \), then \( x_{1r+1} = 1 \) because \( \pi_i^\psi(p) = 0 \) for \( i \in \{0, 1\} \) if and only if \( g_{1r+1} = 1 \) and \( g_{1r+1}^2 - g_{1r+1} = 0 \) by Lemma 3.6. Note that \( \{x_{12}, \ldots, x_{1r}\} \) is a solution of the system of equation defined by
\[
\{\pi_i^\psi(p) | 0 \leq i \leq l-3\} = \{p_i(g_{12}, \ldots, g_{1r}) | 0 \leq i \leq l-3\}
\]
so that \( J \) has non-empty zero set. If there is an element \( \{x_i\}_{i=2}^{l-2} \) in the zero set of \( J \), \( \{z_{ij}\}_{1 \leq i < j \leq r+1} \) is in the zero set of \( I \) if \( z_{1i} = x_i \) for all \( 2 \leq i \leq r \) and \( z_{1r+1} = 1 \). \[\square\]
Theorem 4.1 implies follows.

**Corollary 4.2.** For any ideal $I$ of a polynomial ring, there is a Hilbert point $v \in \text{Hilb}^P(\mathbb{P}_k^r)$, a choice of closed immersion $\text{Hilb}^P(\mathbb{P}_k^r) \to \mathbb{P}(\bigwedge Q(d) S_d)$ and a character $\chi \in X(T_{r+1})$ such that there is an ideal $J$ of $\Gamma(\text{GL}_{r+1}(k), O_{\text{GL}_{r+1}(k)})$ such that $Z_{v, \chi}$ is the zero locus of $J$ and $J$ is a generalization of $I$.

Let’s consider some decision problems. Let $\text{SysAl}$ be a problem asking if a system of algebraic equations over $\mathbb{Q}$ has a solution over $k$ and $\text{HC}$ be a problem asking if a graph has a Hamiltonian cycle. Using the proof of Corollary 2.3.2 in [4] p. 21, we can prove that $\text{HC}$ is an NP-complete problem so that $\text{SysAl}$ is an NP-hard problem. Let’s describe a solvability check problem $\text{SC}$ as follows:

- **Given**: A rational Hilbert point $v \in \text{Hilb}^P(\mathbb{P}_k^r)$, a choice of closed immersion $\text{Hilb}^P(\mathbb{P}_k^r) \to \mathbb{P}(\bigwedge Q(d) S_d)$ and a character $\chi \in X(T_r)$.
- **Decide**: Is there a coordinate $g \in U_r$ satisfying $\chi \notin \Xi_{g,v}$?

Here, $v \in \text{Hilb}^P(\mathbb{P}_k^r)$ is rational if it represents a saturated homogeneous ideal of $rS$ generated by rational polynomials. Theorem 4.1 shows that there is a polynomial time reduction from $\text{SysAl}$ to $\text{SC}$. That is,

**Corollary 4.3.** The problem $\text{SC}$ is NP-hard.

There is an extended version of $\text{SC}$, which would be called $\text{ESC}$, described as follows:

- **Given**: A rational Hilbert point $v \in \text{Hilb}^P(\mathbb{P}_k^r)$, a choice of closed immersion $\text{Hilb}^P(\mathbb{P}_k^r) \to \mathbb{P}(\bigwedge Q(d) S_d)$ and a character $\chi \in X(T_r)$.
- **Decide**: Is there a coordinate $g \in U_r$ satisfying $C \cap \Xi_{g,v} = \emptyset$?

$\text{SC}$ can be reduced to $\text{ESC}$ in polynomial time so that we can prove follows:

**Corollary 4.4.** The problem $\text{ESC}$ is NP-hard.

On the other hand, we can use Buchberger’s algorithm in [1] to solve the problem $\text{ESC}$ because the zero set of an ideal $I \subset rS$ is non-empty if and only if $1 \notin I$ if and only if the Gröbner basis of $I$ respect to the lexicographic(or graded reverse-lexicographic) monomial order contains 1.

Let’s construct an example. Fix natural numbers $r$ and $d$. Suppose $l = 3$, $p_0 \in k[x_1, \ldots, x_r]$ and $\deg(p_0) \leq d$. In this case, $p$ is a sequence of length 1 and the ideal generated by $\{p_i \mid 0 \leq i \leq l - 3\}$ has empty zero locus if and only if $p_0$ is a non-zero constant polynomial. Let

$$F' = -6x_1^{d+5} + 15x_1^{d+4}x_{r+1} - 10x_1^{d+3}x_{r+1}^2 + \frac{x_1^{d+3}x_{r+1}^2}{2}p_0 \left(\frac{x_2}{x_1}, \ldots, \frac{x_r}{x_1}\right).$$

By the definition, $I_{F'(p)} = (x_1F', x_{r+1}F')$. This means that there is a $g \in U_{r+1}$ such that $\chi_{1}^{2d+6}x_{r+1}^{6} \notin \Xi_{[g,v_{E}(F'(p))]}$ if and only if $p_0$ is the zero polynomial or $\deg(p_0) \geq 1$.

5. A Relation Between the Problem $\text{ESC}$ and GIT-semistability

In this section, every GIT problem is related to the action of $\text{GL}_r(k)$ on $E_{r,h}$. It will be proved that we can decide whether a rational Hilbert point is GIT-semistable by solving finitely many $\text{ESC}$. As a consequence of [7, Criterion 3.3], we have the following lemma.
Lemma 5.1. A rational point \( v \in E_{d,b}^r \) is GIT-semistable if and only if \( \xi_{d,b} \in \Delta_{g,v} \) for all \( g \in \text{GL}_r(k) \).

Proof. \( v \) is semi-stable if and only if it is semi-stable under the action of every maximal torus of \( \text{GL}_r(k) \) by [3, Theorem 2.1]. Since every two maximal tori are conjugate, [7, Criterion 3.3] proves the lemma. \( \square \)

A point in \( X(T)_R \) is not in a polytope \( \Delta \) if and only if there is a separating hyperplane in \( X(T)_R \). That is,

Lemma 5.2. For any \( g \in \text{GL}_r(k) \) and \( v \in E_{d,b}^r \), \( \xi_{d,b} \notin \Delta_{g,v} \) if and only if there is an \( \omega \in X(T)_R^\gamma \) such that

\[
(3) \quad \omega(\xi_{d,b}) < \min \omega(\Xi_{g,v} \otimes \mathbb{R} 1).
\]

For some special choices of \( \omega \in X(T)_R^\gamma \) and \( v \), we can still guarantee (3) for every \( g \in L_r \).

Lemma 5.3. Suppose there are \( v \in E_{d,b}^r \) and \( \omega \in X(T)_R^\gamma \) satisfying

\[
\omega(\xi_{d,b}) < \min \omega(\Xi_v \otimes \mathbb{R} 1)
\]

and \( \omega(\chi_i) \leq \omega(\chi_{i+1}) \) for all \( 1 \leq i < r \). Then, for any \( l \in L_r \),

\[
\omega(\xi_{d,b}) < \min \omega(\Xi_{l,v} \otimes \mathbb{R} 1).
\]

Proof. Suppose \( \eta \in \Xi_{l,v} \otimes \mathbb{R} 1 \setminus \Xi_v \otimes \mathbb{R} 1 \). It suffices to show that \( \omega(\eta) \geq \min \omega(\Xi_v \otimes \mathbb{R} 1) \).

By definition, there is an \( m \in W_{d,b}^r \) satisfying \( \eta \in \Xi_{l,m} \) and \( \Xi_m \subset \Xi_v \). By expanding \( l,m \), we can prove that

\[
\omega(\eta) \geq \min \omega(\Xi_m \otimes \mathbb{R} 1)
\]

using the condition \( \omega(\chi_i) \leq \omega(\chi_{i+1}), \forall 1 \leq i \leq r-1 \). Since \( \Xi_m \subset \Xi_v \), we can deduce that \( \min \omega(\Xi_m \otimes \mathbb{R} 1) \geq \min \omega(\Xi_v \otimes \mathbb{R} 1) \). Thus the claimed statement is true. \( \square \)

Now, we can restate the condition for \( v \) to be unstable.

Theorem 5.4. Suppose \( v \in E_{d,b}^r \). \( v \) is unstable if and only if there are \( u \in U_r \) and \( q \in \Sigma_r \) satisfying

\[
\xi_{d,b} \notin \Delta_{uq,v}.
\]

Proof. If part is obvious by Lemma 5.1. Suppose there is \( g \in \text{GL}_r(k) \) satisfying

\[
\xi_{d,b} \notin \Delta_{g,v}.
\]

By Lemma 5.2 there is \( \omega \in X(T)_R^\gamma \) satisfying

\[
\omega(\xi_{d,b}) < \min \omega(\Xi_{g,v} \otimes \mathbb{R} 1).
\]

There is a \( p \in \Sigma_r \) satisfying \( \omega(\chi_{p(i)}) \leq \omega(\chi_{p(i+1)}) \) for all \( i \). Let’s define \( \omega_p(\chi_i) = \omega(\chi_{p(i)}) \). Then,

\[
\omega_p(\xi_{d,b}) = \omega(\xi_{d,b}) < \min \omega(\Xi_{g,v} \otimes \mathbb{R} 1) = \min \omega_p(\Xi_{p^{-1}g,v} \otimes \mathbb{R} 1).
\]

Now there are \( l \in L_r, u \in U_r \) and \( q \in \Sigma_r \) satisfying \( p^{-1}q = luq \) by the LU-decomposition of general non-singular matrix. \( p^{-1}g.v \) and \( \omega_p \) satisfies the condition of Lemma 5.3. Thus,

\[
\omega_p(\xi_{d,b}) < \min \omega_p(\Xi_{l^{-1}uq,v} \otimes \mathbb{R} 1) = \min \omega_p(\Xi_{uq,v} \otimes \mathbb{R} 1).
\]

By Lemma 5.2 \( \xi_{d,b} \notin \Delta_{uq,v} \), as desired. \( \square \)
Using Theorem 5.4 and Lemma 5.2 we can solve ESC for each choice of $\omega \in X(T)_0$ and $q \in \Sigma_r$ to check if

$$\{ \chi \in X(T) | \omega(\chi) \leq \omega(\xi_{d,Q(d)}) \} \cap \Xi_{uq,v} = \emptyset$$

for a rational $v \in \text{Hilb}^P(P^{r-1})$ and an integer $d \geq gp$. Note that we have to consider finitely many $\omega$'s because $A^r_{d,b}$ has only finitely many weights with respect to the action of $T_r$. In this way, we can check if $v$ is semistable or not. This fact implies that there is an algorithm deciding if a rational $v \in \text{Hilb}^P(P^{r-1})$ is GIT-semistable or not.

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