STABLE MAPS OF GENUS ZERO IN THE SPACE OF STABLE VECTOR BUNDLES ON A CURVE

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Abstract. Let \( X \) be a smooth projective curve with genus \( g \geq 3 \). Let \( \mathcal{N} \) be the moduli space of stable rank two vector bundles on \( X \) with a fixed determinant \( \mathcal{O}_X(-x) \) for \( x \in X \). In this paper, as a generalization of Kiem and Castravet’s works, we study the stable maps in \( \mathcal{N} \) with genus 0 and degree 3. Let \( P \) be a natural closed subvariety of \( \mathcal{N} \) which parametrizes stable vector bundles with a fixed subbundle \( L^{-1}(-x) \) for a line bundle \( L \) on \( X \). We describe the stable map space \( \mathcal{M}_{0}(P, 3) \). It turns out that the space \( \mathcal{M}_{0}(P, 3) \) consists of two irreducible components. One of them parameterizes smooth rational cubic curves and the other parameterizes the union of line and smooth conics.

1. Introduction

1.1. Stable maps in the space of stable vector bundles. Let \( i : Y \subset \mathbb{P}^r \) be a smooth projective variety with \( i^*\mathcal{O}_{\mathbb{P}^r}(1) = \mathcal{O}_Y(1) \). Let \( \mathcal{M}_{0}(Y, d) \) be the moduli space parameterizing stable maps \( [f : C \rightarrow Y] \) with genus \( g(C) = 0 \) and degree \( \text{deg}(f) := d \). It is well-known that the moduli space \( \mathcal{M}_{0}(Y, d) \) is compact and its geometry has been studied in various contexts: enumerative geometry ([11, 21]) and birational geometry ([5]). In this paper, as a continuation of [16, §3], we study the moduli space of stable maps of degree 3 when the target \( Y = \mathcal{N} \) is the moduli space of stable vector bundles of rank two over a smooth curve \( X \). Here \( \mathcal{N} \) parameterizes the Gieseker-Mumford stable vector bundles \( E \) of rank 2 with fixed determinant \( \text{det}(E) = \mathcal{O}_X(-x) \) on a smooth projective curve \( X \) (§2.1). It is well-known that the ample theta divisor \( \Theta \) generates the Picard group \( \text{Pic}(\mathcal{N}) \) ([2]). For the projective embedding \( i : \mathcal{N} \subset \mathbb{P}^{\mathcal{N}} \) provided by the divisor \( \Theta \) on \( \mathcal{N} \), one can consider the moduli space \( \mathcal{M}_{0}(\mathcal{N}, d) \) of stable maps with degree \( d \). As the degree of the maps become larger, the moduli space \( \mathcal{M}_{0}(\mathcal{N}, d) \) may have many different irreducible components. We study the problem from lower degree cases. Let us summarize the well-known results for \( d \leq 2 \).

- When \( d = 1 \), the moduli space is isomorphic to a projective bundle over the Picard group \( \text{Pic}^0(X) \) ([19, 15]).
- When \( d = 2 \), the moduli space \( \mathcal{M}_{0}(\mathcal{N}, 2) \) consists of two irreducible components: Hecke curves and the rational curves of extension types where they transversally intersect. Furthermore, the moduli space \( \mathcal{M}_{0}(\mathcal{N}, 2) \) is related to the Hilbert scheme of conics in \( \mathcal{N} \) by using the birational morphisms with some geometric meaningful centers ([16]).

2010 Mathematics Subject Classification. 14E05, 14H60, 14D22.
Key words and phrases. Stable bundles; Modification of vector bundles; Rational curves.
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KC was partially supported by NRF grant 2016R1D1A1B03930421.
When \( d = 3 \), it is quite natural to start with finding the possible irreducible components of the moduli space \( \mathcal{M}_0(\mathcal{N}, 3) \). In this direction, in \cite{KU, LN}, the authors found the two components parametrizing the degree 3 map \( f : \mathbb{P}^1 \to \mathcal{N} \). For \( L \in \text{Pic}^k(X) \), let \( \mathbb{P}_L := \mathbb{P}\text{Ext}^1(L, L^{-1}(-x)) \) be the space of non-split extensions
\[
0 \to L^{-1}(-x) \to E \to L \to 0.
\]

By the functoriality of the moduli space \( \mathcal{N} \), we have a natural rational map
\[
\Psi_L : \mathbb{P}_L \dashrightarrow \mathcal{N}.
\]

Two components parametrize one of the following types of degree 3 maps \( f : \mathbb{P}^1 \to \mathcal{N} \). (\cite[Lemma 4.10]{KU}, \cite[Proposition 3.9]{LN}).

i) When \( k = 0 \), \( f \) is a composition of degree 3 maps \( \mathbb{P}^1 \to \mathbb{P}\text{Ext}^1(L, L^{-1}(-x)) \cong \mathbb{P}^{g+1} \) with the map \( \Psi_L \). In this case \( \Psi_L \) is a linear embedding (i.e., \( \deg \Psi_L = 1 \)). Also it is well known that every \((g - 1)\)-dimensional linear space arise in this fashion. Hence one of the components of \( \mathcal{M}_0(\mathcal{N}, 3) \) is isomorphic to the relative stable map space \( \mathcal{M}_0(\mathbb{P}\mathcal{U}/\text{Pic}^0(X), 3) \) where \( \mathcal{U} \) is the universal rank \( g \) bundle over \( \text{Pic}^0(X) \). Thus, the method in \cite{KU, LN} comparing the various compactifications of rational curves can be applied.

ii) When \( k = 1 \), \( f \) is a composition of the degree one map \( \mathbb{P}^1 \to \mathbb{P}\text{Ext}^1(L, L^{-1}(-x)) \cong \mathbb{P}^{g+1} \) (i.e., line) with the map \( \Psi_L \). In this case, \( \deg \Psi_L = 3 \). But the base locus of the map \( \Psi_L \) may not be empty. Moreover, the rational map \( \Psi_L \) may not be injective.

In fact, the base locus \( \text{Bs}(\Psi_L) \) is isomorphic to \( X \) from \cite[1]{KU}. Also, the resolution of the undefined locus of the rational map \( \Psi_L \) is the first blow-up \( \pi : M_1 = b_1\mathbb{P}^{g+1} \to \mathbb{P}^{g+1} = \text{Ext}^1(L, L^{-1}(-x)) \) in \cite[Proposition 2.5]{KU}. Here the space \( M_1 \) is the moduli space of pairs on \( X \) (for definition, see \cite{KU}). If \( L \) is non-trisecant (Definition 2.1), the regular morphism \( M_1 \to \mathcal{N} \) is a closed embedding and thus the study of stable maps in \( \mathcal{N} \) boils down to the study of stable maps in \( M_1 \).

1.2. Main results. Let \( L \) be a non-trisecant line bundle on \( X \) (Definition 2.1). Let \( \pi^*[\text{line}] := \beta \in H_2(\text{bl}_L\mathbb{P}^{g+1}) \) be the pull-back of the line class along the blow-up map \( \pi : b_1\mathbb{P}^{g+1} \to \mathbb{P}^{g+1} \). Then we have a closed embedding
\[
\mathcal{M}_0(b_1\mathbb{P}^{g+1}, \beta) \hookrightarrow \mathcal{M}_0(\mathcal{N}, 3)
\]
between the moduli spaces of stable maps (Proposition 2.5). There may be many irreducible components in \( \mathcal{M}_0(b_1\mathbb{P}^{g+1}, \beta) \) (cf. \cite{KU, LN, LC}). One of the obvious components of the moduli space \( \mathcal{M}_0(b_1\mathbb{P}^{g+1}, \beta) \) is the closure of the locus of the lines \( l \subset \mathbb{P}^{g+1} \) such that \( l \cap \text{Bs}(\Psi_L) = \emptyset \) (i.e., an open subset of \( \text{Gr}(2, g + 2) \)). Let us denote it by \( \Gamma_1 \). There is another irreducible component in the moduli space \( \mathcal{M}_0(b_1\mathbb{P}^{g+1}, \beta) \) which arises from the blowing-up \( \pi \). The component consists of stable maps with reducible domain whose image is the union of a line and conic such that the conic lies in the exceptional locus of the blow-up. Let us denote it by \( \Gamma_2 \). One of our main goals of this paper is to prove that these are all of the components. That is,

**Theorem 1.1** (Theorem 1.2). Under the above notations, the moduli space \( \mathcal{M}_0(b_1\mathbb{P}^{g+1}, \beta) \) consists of the two irreducible components \( \Gamma_1 \) and \( \Gamma_2 \) of dimension \( 2g \). Furthermore, the intersection part \( \Gamma_1 \cap \Gamma_2 \) is \( 2g - 1 \)-dimensional irreducible space.
In fact, we can describe the moduli points in $\Gamma_1 \cap \Gamma_2$ with the deformation theory of maps (§3).

Regarding enumerative geometry, the moduli space of stable maps in the blown-up space of the projective space has been intensively studied in [20] and [12].

1.3. Notations. Throughout this article, we use the following notation.

• $X$: projective curve of genus $g \geq 3$.
• $x$: a fixed point of $X$.
• $V_L := \text{Ext}^1(L, L^{-1}(-x))$, $V^s_L$ is the stable part of $\text{Ext}^1(L, L^{-1}(-x))$.
• If there is no need to emphasize a line bundle $L \in \text{Pic}^1(X)$ (respectively, $L \in \text{Pic}^0(X)$) and extension groups, we sometimes abbreviate $P^V_L$ by $P^{g+1}$ (respectively, $P^g$). And we sometimes abbreviate stable locus of $P^V_L := P^{V^s_L}$ by $P^s_L$, when $L \in \text{Pic}^1(X)$ in the same situation.

Acknowledgement. We would like to thank W. Lee and A. Iliev for valuable discussion and comments. The second named author is grateful to his thesis advisor, Young-Hoon Kiem.

2. Review of the resolution of unstable bundles

2.1. Stable bundles and stable maps. Let $\mu(E) := \deg(E)/\text{rank}(E)$ be the slope of the vector bundle $E$ on $X$. A vector bundle $E$ is called stable if

$$\mu(F) < \mu(E)$$

for all non-zero, proper, subbundles $F \subset E$. Let $N$ be the moduli space of stable rank 2 vector bundles $E$ on $X$ with fixed determinant $\text{det}(E) \cong O_X(-x)$. Since $(\text{rank}(E), \deg(E)) = 1$, from geometric invariant theory ([27]), it is well-known that the moduli space $N$ is a smooth projective variety.

On the other hand, for a smooth projective variety $Y$, let $\beta \in H^2(Y, \mathbb{Z})$. Let $C$ be a projective connected reduced curve of genus $g(C) = 0$. A map $f : C \to Y$ is called stable if

• $C$ has at worst nodal singularities;
• $|\text{Aut}(f)| < \infty$.

Let $M_0(Y, \beta)$ be the moduli space of stable maps with $f_*[C] = \beta$. Then $M_0(Y, \beta)$ is a projective scheme ([11, Theorem 1]).

2.2. Some remarks about the rational map $\Psi : \mathbb{P}^{g+1} \dashrightarrow N$.

Definition-Proposition 2.1. ([16]) Let $E$ be a rank two vector bundle on $X$ and $p \in X$. Let $E^{\nu_p}$ be the kernel of the surjective map

$$(1) \quad 0 \to E^{\nu_p} \to E \to p \to 0.$$

Then $E^{\nu_p}$ is a vector bundle. Let $E^{\nu_p}$ be an elementary modification of the bundle $E$ at $p$.

Since $\text{Hom}(E, C_p) = \text{Hom}(E|_p, C_p) = \mathbb{C}^2$ and $\text{Ker}(v_p) = \text{Ker}(\lambda \cdot v_p)$ for all $\lambda \in \mathbb{C}^*$, we assume that $v_p \in (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^* \cong \mathbb{P}^1$. Let $E = \zeta \oplus \zeta'$ for line bundles $\zeta$ and $\zeta'$ on $X$. If $v_p \in \mathbb{C}^* = \mathbb{P}^1 \setminus \{[1 : 0], [0 : 1]\}$, one can easily see that the modified bundles $E^{\nu_p}$ are isomorphic to each other. Hence, from now on, let us denote...
Definition 2.2. Let us define by
\[(\zeta \oplus \zeta')^p := (\zeta \oplus \zeta')^p \]
for all \(v_p \in \mathbb{C}^*\).

Let \(K_X\) be the canonical line bundle on \(X\).

Lemma 2.3. (cf. [28 (3.4)] and [1 §3]) Let
\[f : X \to \mathbb{P}^{g+1} = \mathbb{P}\text{Ext}^1(L, L^{-1}(-x)), \quad p \mapsto (L \oplus L^{-1}(p-x))^p\]
be the map provided by the elementray modification. Then \(f\) is equal to the map
given by the complete linear system
\[i = |L^2(x) \otimes K_X| : X \subset \mathbb{P}^{g+1}.\]

Proof. In [28 (3.4)], it was proved that the map \(i\) is equivalent to the map \(g : X \to \mathbb{P}H^1(\Lambda^{-1}) = M_0 (\Lambda = L^2(x))\) where \(M_0\) is the moduli space of stable pairs on \(X\). In fact, the map \(g\) is defined by by \(\mathbb{P}W_1 \to \mathbb{P}H^1(L^{-2}(-x))\), where \(W_1\) is a line bundle on \(X\) and \(g(p) = \mathbb{P}H^0(L^{-2}(-x)|_p) \in \mathbb{P}H^1(L^{-2}(-x))\) (the final paragraph of [28 329p]). This is the projectivization of the first map \(\xi\) in the following exact sequence:
\[(2) \quad 0 \to \text{Ext}^1(L|_p, L^{-1}(-x)) \xrightarrow{\xi_1} \text{Ext}^1(L, L^{-1}(-x)) \xrightarrow{\xi_2} \text{Ext}^1(L(-p), L^{-1}(-x)) \to 0,\]
where (2) is obtained by taking the functor \(\text{Hom}(\_, L^{-1}(-x))\) to the short exact sequence \(0 \to L(-p) \to L \to L|_p \to 0\). Since \(\text{Ext}^1(L|_p, L^{-1}(-x)) \cong \mathbb{C}\), to prove that \(g(p) = f(p)\), it is enough to show that \(\gamma(f(p)) = L(-p) \oplus L^{-1}(-x)\). That is,
\[(3) \quad (L \oplus (L^{-1}(p-x)))^p \otimes L(-p) \cong L(-p) \oplus L^{-1}(-x),\]
where the left hand side is defined by the pull-back:
\[
\begin{array}{cccccc}
0 & \to & L^{-1}(-x) & \to & (L \oplus L^{-1}(p-x))^p \otimes L(-p) & \to & L(-p) & \to & 0 \\
& & & & \downarrow & & \downarrow & & \\
0 & \to & L^{-1}(-x) & \to & (L \oplus L^{-1}(p-x))^p & \to & L(-p) & \to & 0
\end{array}
\]
One can check the isomorphism in (3) locally as follows. For every open set \(U \subset X\),
\[
((L \oplus L^{-1}(p-x))^p \otimes L(-p))(U)
= \{(s_1, s_2, s_3)\mid (s_1, s_2) \in ((L \oplus L^{-1}(p-x))^p(U), s_3 \in L(-p)(U)), s_1 = s_3\}
\cong \{(s_1, s_2) \in ((L \oplus L^{-1}(p-x))^p(U))\mid s_1 \in L(-p)\}
\cong \{(s_1, s_2) \in L(U) \oplus L^{-1}(p-x)(U))\mid as_1(p) + bs_2(p) = 0, s_1(p) = 0\}
\cong \{(s_1, s_2) \in L(U) \oplus L^{-1}(p-x)(U))\mid s_1(p) = s_2(p) = 0\}
\cong \{(s_1, s_2) \in (L(-p)) \oplus L^{-1}(-x))(U)\},
\]
where the third isomorphism comes from the definition of the elementray modification and the fourth comes from the choice of \(v_p = [a : b] \in \mathbb{C}^*\) for \(ab \neq 0\). □

Remark 2.4. \(\text{deg}(X) = 2g + 1\).
Proposition 2.5. ([28], [1]) Let
\[ \Psi_L : \mathbb{P}^{g+1} \rightarrow N \]
be the rational map defined by the extension of \( L \) by \( L^{-1}(-x) \). Then

1. the undefined locus of the rational map \( \Psi_L \) is isomorphic to \( X \) (Lemma 2.3) and the blowing-up of \( \mathbb{P}^{g+1} \) along \( X \) extends to a regular morphism \( \tilde{\Psi}_L : \text{bl}_X \mathbb{P}^{g+1} := P \rightarrow N \).

2. The exceptional divisor \( P_{g-1} \) restricted to a fiber of the blowing-up \( \pi \) is exactly the degree 0 extension type and thus each restricted exceptional divisor is a linearly embedding into \( N \) by \( \tilde{\Psi}_L \).

3. If \( \Psi_L \) is injective and \( H^0(L^2(x)) = 0 \), then the morphism \( \tilde{\Psi}_L \) is a closed embedding.

Proof. Let us follow the notation of [28] by letting \( \Lambda = L^2(x) \) and thus \( 0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow L^2(x) \rightarrow 0 \). Also, \( P \cong M_1 \) where the latter space parameterizes the pairs \((s, E)\) such that \( E \) is stable and \( s \subset H^0(E) \) ([28]). Part (1) comes from Lemma 2.3 and [28, (2.1)]. Part (2) comes from item (2) of [1, Theorem 1]. In part (3), the injectiveness of \( \tilde{\Psi} \) comes from [28, (3.20)] because \( H^0(E) = \mathbb{C} \). Note that our extended map \( \tilde{\Psi}_L \) is just the forgetful map \((s, E) \mapsto E\) by forgetting the section \( s \subset H^0(E) \). Hence the tangential map \( \tilde{\Psi}_* : T_{[(s,E)]} P \rightarrow T_{[E]} N \) fits into the exact sequence ([28, (2.1)])

\[ 0 \rightarrow \text{Ext}^0(E, E) \rightarrow H^0(E) \rightarrow T_{[(s,E)]} P \xrightarrow{\tilde{\Psi}_*} T_{[E]} N. \]

Since \( \text{Ext}^0(E, E) = \mathbb{C} \) and \( H^0(E) = \mathbb{C} \), we conclude that the map \( \tilde{\Psi} \) is embedding. \( \square \)

The assumptions of the part (3) of above proposition can be satisfied by the following choice of the line bundles \( L \) (cf. Lemma 2.10).

Definition 2.6. The line bundle \( L \) is non-trisecant if \( H^0(L^2(x)) = 0 \).

If \( L \) satisfies the non-trisecant condition then we have the following property.

Corollary 2.7. ([22], Lemma 5.1]

(a) The smooth curve \( X \subset \mathbb{P}^{g+1} \) (embedded by \( |L^2(x) \otimes K_X| \)) has a trisecant line if and only if

\[ L^2(x) \cong \mathcal{O}_X(p+q+r)(\text{equivalently, } H^0(L^2(x)) \neq 0) \]

for some \( p, q, r \in X \). In this case, there is the trisecant line such that its intersection with \( X \) is \( p+q+r \) (if \( p = q \), then the line is tangential to \( X \) at \( p \) and if \( p = q = r \) then the line is tangential to \( X \) at \( p \) and meets \( p \) three times).

(b) If the curve \( X \) is neither trigonal nor hyperelliptic, the points \( p, q, r \) satisfying equation ([1]) are uniquely defined and thus there exists a unique trisecant line.
Here Because the first term is clearly zero and dimension of the kernel is 2, dimension
of the kernel is 2. Therefore $L^{-1}(p+q+r-x) \simeq \mathcal{O}_X(p+q+r)$.

Conversely, let us assume that the line bundle $L$ satisfies equation $\mathbf{1}$. By taking
the functor $\text{Hom}(L, -)$ in the short exact sequence $0 \to L^{-1}(-x) \to L^{-1}(p+q+r-x) \to L^{-1}(p+q+r-x)|_{p+q+r} \to 0$, we obtain

$$
\text{Ext}^0(L, L^{-1}(p+q+r-x)) \to \text{Ext}^0(L, L^{-1}(p+q+r-x)|_{p+q+r}) \to \text{Ext}^1(L, L^{-1}(-x))
$$

(6)

Since final term is clearly zero, dim($\text{Ext}^0(L, L^{-1}(-x))) = g+2$ and dim($\text{Ext}^1(L, L^{-1}(p+q+r-x))) = \dim \text{Ext}^1(L, L) = g$ by $\mathbf{1}$. Thus dim ker $\pi = 2$. Also, since $j_2$ is
equal to the composition of $j_1$ and an isomorphism $\text{Ext}^1(L^{-1}(-x), L^{-1}(-x)) \simeq \text{Ext}^1(L, L^{-1}(p+q+r-x))$, ker $j_2 = \ker j_1$. Therefore ker $j_2$ is exactly the same
as the affine cone of the linear subspace $l := \langle p, q, r \rangle$ in $\mathbb{P}^g$. Hence the linear
subspace $l$ is a line in $\mathbb{P}^g$ such that $X \cap l = p + q + r$.

For part (b), suppose that there are three points $\{p', q', r'\} \neq \{p, q, r\}$ such that
$L^2(x) \simeq \mathcal{O}_X(p' + q' + r')$. Then $\mathcal{O}_X(p + q + r - p' - q' - r') \equiv \mathcal{O}_X$. This means $X$
is hyperelliptic or trigonal. 

2.3. Geometry of lines in $\mathbb{P}^g$ meeting $X$. In this subsection, we give a
description of line the $\mathbb{P}^g := \langle f(p), f(q) \rangle$ passing through $p, q \in X$. $\mathbb{P}^g = \mathbb{P}L$
for $L \in \text{Pic}^1(X)$. If $p = q$, then $\mathbb{P}^g$ denotes the projectivized tangent line of $X$ at
$f(p)$. Recall that the image $f(t)$ (Lemma 2.3) fits into the following short exact
sequences:

$$
0 \to L^{-1}(-x) \to f(t) = (L \oplus L^{-1}(t-x))^t \to L \to 0.
$$

\textbf{Proposition 2.8.} Under the above notation, $M := L \oplus L^{-1}(p+q-x)$. Then the
line $\mathbb{P}^g \setminus \{p, q\}$ missing the points $p$ and $q$ is parameterized by the doubly modified
bundles $(M^*)^v = (M^*)^v$ which fit into the exact sequence:

$$
0 \to (M^*)^v \to M \to C_p \oplus C_q \to 0.
$$

Here $v_p \in C^* \subset \mathbb{P}(H^0(M|p)* = \mathbb{P}^1$ and $v_q \in C^* \subset \mathbb{P}(H^0(M|q)^* = \mathbb{P}^1$. Hence
$\mathbb{P}^g \setminus \{p, q\} = \mathbb{P}V^* \cap \mathbb{P}V_{L^{-1}(p+q-x)}$.

\textbf{Proof.} By some diagram chasing, one can easily show that the doubly modified
bundle is exactly the kernel of the map $v_p \oplus v_q$.

Let us describe the line $\mathbb{P}^g$. By taking the functor $\text{Hom}(L^{-1}(x))$ into the short
exact sequence $0 \to L(-p-q) \to L \to L|_{p+q} \to 0$, we obtain

$$
\text{Ext}^0(L(-p-q), L^{-1}(-x)) \to \text{Ext}^1(L|_{p+q}, L^{-1}(-x)) \to \text{Ext}^1(L, L^{-1}(-x))
$$

(7)

$$
\text{Ext}^1(L(-p-q), L^{-1}(-x)) \simeq \text{Ext}^1(L, L^{-1}(p+q-x)) \to 0.
$$

where \( \varphi \) is obtained from the twisting by \( \mathcal{O}(p + q) \) and the first equality holds for degree reasons. We claim that
\[
P \text{Ext}^1(L|_{p+q}, L^{-1}(-x)) = \overline{pq}.
\]
To show this, at first, we prove that the line \( P \text{Ext}^1(L|_{p+q}, L^{-1}(-x)) \subset \mathbb{P}^{q+1} \) is parameterized by the elements defined by the double modifications. Let \( E \in \text{Ext}^1(L|_{p+q}, L^{-1}(-x)) \). The image \( E \) by the map \( i \) fits into the short exact sequence:
\[
0 \to L^{-1}(-x) \to i(E) \to L \to 0.
\]
Since \( \varphi(j(i(E))) = L^{-1}(p + q - x) \oplus L \), we have the following push-out diagram.
\[
\begin{array}{ccc}
0 & \longrightarrow & L^{-1}(-x) \\
& & \downarrow^a \\
0 & \longrightarrow & L^{-1}(p + q - x) \quad L^{-1}(p + q - x) \oplus L \quad L \quad 0.
\end{array}
\]
By some diagram chasing, we have the short exact sequence:
\[
0 \to i(E) \overset{a}{\longrightarrow} L^{-1}(p + q - x) \oplus L \overset{\varphi_p \oplus \varphi_q}{\longrightarrow} \mathbb{C}_{p+q} \to 0
\]
such that \( p_1 \circ a \) is surjective where the map \( p_1 : L \oplus L^{-1}(p + q - x) \to L \) is the projection into the first factor. Also it is clear that \( p_1 \circ a \) is surjective if and only if \( v_p \neq [1 : 0] \) and \( v_q \neq [1 : 0] \).

Conversely, let us assume that the bundle \( E \) fits into the sequence \([8]\) such that \( p_1 \circ a \) is surjective. Then one can easily check that the \( E \) fits into the diagram \([7]\) by the snake lemma.

In summary, the extensions of the form \([8]\) such that \( v_p \neq [1 : 0] \) and \( v_q \neq [1 : 0] \) represent elements in \( \text{Ext}^1(L|_{p+q}, L^{-1}(-x)) \). Secondly, one can easily show that the modified bundle \( f(p) \) (resp. \( f(q) \)) fits into the diagram in \([8]\) if \( v_p = [0 : 1] \) and \( v_q \in \mathbb{C}^* \) (resp. \( v_p = [0 : 1] \) and \( v_q \in \mathbb{C}^* \)). Hence if \( p \neq q \), the two distinct points \( f(p) \) and \( f(q) \) lie on the line \( P \text{Ext}^1(L|_{p+q}, L^{-1}(-x)) \) which is equal to \( \overline{pq} \).

Since the extension \( \text{Ext}^1(L|_{2p}, L^{-1}(-x)) \) can be obtained as the limit of the extension \( \text{Ext}^1(L|_{p+q}, L^{-1}(-x)) \) by letting \( p \to q \), the same result holds for the case \( p = q \). \( \square \)

Now we describe the set-theoretic intersections of the vector bundles provided by the extensions:
1. \( PV^* \cap PV_q \) for \( \zeta \in \text{Pic}^1(X) \) and \( \eta \in \text{Pic}^0(X) \);
2. \( PV^* \cap PV_q \) for \( \zeta, \eta \in \text{Pic}^0(X) \).

We remark that all of these intersections are considered in the moduli space \( \mathcal{N} \). Case (2) has been already studied in \([23] \) 6.19. They meet at a single point cleanly unless the intersection is empty. Hence we focus on case (1).

**Proposition 2.9.** Let \( \zeta \in \text{Pic}^1(X) \) and \( \eta \in \text{Pic}^0(X) \). Then the intersection \( PV^* \cap PV_q \) in \( \mathcal{N} \) is one of the following:
1. \( \text{If } \zeta \otimes \eta = \mathcal{O}(p + q - x), \text{ then } PV^* \cap PV_q \text{ is equal to the image of } \overline{pq} \setminus \{p, q\} \text{ in } \mathcal{N} \).
2. \( \text{Otherwise, } PV^* \cap PV_q \cap \phi \)
Proof. Consider an element $E$ in $P V^* \cap P V_\eta$. Then,

$$
0 \to \zeta^{-1}(-x) \xrightarrow{a} E \xrightarrow{b} \zeta \to 0
$$

$$
0 \to \eta^{-1}(-x) \xrightarrow{c} E \xrightarrow{d} \eta \to 0.
$$

If $d \circ a = 0$, $d$ factors through $\zeta$. But since $\deg(\zeta) = 1 > \deg(\eta) = 0$, $d = 0$ which leads to contradiction. So, $d \circ a$ is injective. Since $\deg(\eta) = 0$ and $\deg(\zeta^{-1}(-x)) = 2$, $\text{Coker}(d \circ a) = C_{p+q}$ for $p, q \in X$. Therefore $\zeta^{-1}(-x) \cong \eta(-p - q)$. From the commutative diagram

(9)

we can see that $\text{Coker}(d \circ a) \cong C_{p+q}$. From this,

$$
0 \to \zeta^{-1}(-x) \xrightarrow{a} E \xrightarrow{b} \zeta \to 0
$$

$$
0 \to \zeta(-p - q) \xrightarrow{c} E \xrightarrow{d} \zeta^{-1}(p + q - x) \to 0.
$$

Hence we obtain a morphism

$$b \oplus d : E \to \zeta \oplus \zeta^{-1}(p + q - x).$$

Let us consider the diagram

(10)

$$
0 \to C_{p+q} \xrightarrow{\Delta} C_{p+q} \oplus C_{p+q} \xrightarrow{h} C_{p+q} \to 0
$$

$$
0 \to E \xrightarrow{b \oplus d} \zeta \oplus \zeta^{-1}(p + q - x) \to C_3 \to 0
$$

Where $h$ is defined by $h(u, v) := u - v$. Since $h \circ (s \oplus r)$ is surjective, $g$ is also surjective. Since degree of $C_3$ is 2 and supported at $\{p, q\}$, $g$ is an isomorphism.

In summary, $E$ fits into the diagram

(11)

$$
0 \to E \xrightarrow{b \oplus d} \zeta \oplus \zeta^{-1}(p + q - x) \xrightarrow{(v_p \oplus v_q)} C_{p+q} \to 0,
$$

such that $v_t \in \mathbb{C}^*$ for $t \in \{p, q\}$. We remark that $v_p$ and $v_q$ lie in $\mathbb{C}^* = \mathbb{P}^1 \setminus \{[1 : 0], [0 : 1]\}$ by the surjectivities of $b$ and $d$. Conversely, it is easy to check that
Consider a map \( E \) satisfying these conditions is contained in \( \mathbb{P}V_\xi \cap \mathbb{P}V_\eta \) when \( \eta \cong \zeta^{-1}(p + q - x) \). Therefore, by Proposition 2.8 we have the conclusion. \( \square \)

To study the stable maps in \( \mathbb{P} \), we need the following simple conclusion in Corollary 2.11.

**Lemma 2.10.** For \( \alpha \neq \beta \in \mathbb{P}V_\xi^* \), \( \Psi_L(\alpha) = \Psi_L(\beta) \) if and only if \( \alpha \) and \( \beta \) lie in a trisecant line.

**Proof.** Let \( \Psi_L(\alpha) = \Psi_L(\beta) \), then

\[
\begin{align*}
\alpha & : [ 0 \to L^{-1}(-x) \xrightarrow{a} E \xrightarrow{b} L \to 0 ] \\
\beta & : [ 0 \to L^{-1}(-x) \xrightarrow{c} E \xrightarrow{d} L \to 0 ].
\end{align*}
\]

If \( d \circ a = 0 \), then \( a \) factors through \( L^{-1}(-x) \), and \( a \) is clearly an isomorphism on \( L^{-1}(-x) \). Also \( d \) descends on \( L \) and thus \( d \) is an isomorphism on \( L \). So we conclude that \( \alpha = \beta \) which leads to a contradiction. Thus \( d \circ a \neq 0 \). Hence \( d \circ a \) is injective. This implies that \( \text{Coker}(d \circ a) = \mathbb{C}_{p+q+r} \) for some \( p, q, r \in \mathbb{X} \). Then \( L^{-1}(-x) \cong L(-p - q - r) \). That is, \( L^2(x) \cong \mathcal{O}_X(p + q + r) \). In a similar way, \( b \circ c \neq 0 \).

By Corollary 2.7 there is a trisecant line \( l \) in \( \mathbb{P}^{g+1} \) such that \( l \cap X = p + q + r \). Consider a map \( b \oplus d : E \to L \oplus L \). By the similar argument in equation 10, we have \( \text{Coker}(b \oplus d) = \mathbb{C}_{p+q+r} \). Consider a following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L^{-1}(-x) & \xrightarrow{d \circ a} & L & \xrightarrow{C_{p+q+r}} & 0 \\
\downarrow{a} & & \downarrow{\oplus} & & \downarrow{b \oplus d} & & \downarrow{\oplus} &
0 & \longrightarrow & E & \xrightarrow{b \oplus d} & L \oplus L & \xrightarrow{C_{p+q+r}} & 0
\end{array}
\]

By the snake lemma, we have a following diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & L^{-1}(-x) & \xrightarrow{a} & E & \xrightarrow{b} & L & \to 0 \\
\downarrow{d \circ a} & & \downarrow{b \oplus d} & & \downarrow{b \oplus d} & & \downarrow{b \oplus d} &
0 & \longrightarrow & L & \xrightarrow{b \oplus d} & L \oplus L & \xrightarrow{L \oplus L} & 0.
\end{array}
\]

Hence \( \alpha \) and \( \beta \) are contained in the projectivization of the kernel of the map \( j_2 \) in 10 because \( L \cong L^{-1}(p + q + r - x) \). Hence by Corollary 2.7 \( \alpha \) and \( \beta \) lies in the trisecant line \( l \).

To prove the necessary condition, it is enough to show that the trisecant line contracts to a point in \( N \). Let \( l \) be an arbitrary trisecant line such that \( l \cap X = p + q + r \). In this case, \( l = \mathbb{P}q \). Consider the following exact sequence.

\[
(12) \quad 0 \to \text{Ext}^0(L^{-1}(p + q - x), L) \to \text{Ext}^0(L^{-1}(p + q - x), L_{|p+q}) \to
\]

\[
\text{Ext}^1(L^{-1}(p + q - x), L(-p - q)) \xrightarrow{j_3} \text{Ext}^1(L^{-1}(p + q - x), L) \to \text{Ext}^1(L^{-1}(p + q - x), L_{|p+q}) = 0,
\]

be the long exact sequence from the short exact sequence \( 0 \to L(-p - q) \to L \to L_{|p+q} \to 0 \) by taking the functor \( \text{Hom}(L^{-1}(p + q - x), -) \). Let us regard elements
of \( \ker j_3 \) as \( \mathbb{P}^g = \mathbb{P}V_{L^{-1}(p+q-x)} \) represented by

\[
0 \to L(-p-q) \to E \to L^{-1}(p+q-x) \to 0.
\]

By the same method as the proof of Proposition 2.8, \( E \) fits into an exact sequence:

\[
\begin{array}{cccccc}
0 & \longrightarrow & E & \xrightarrow{b \otimes d} & L \oplus L^{-1}(p+q-x) & \xrightarrow{\gamma_p \oplus v_q} & \mathbb{C}_{p+q} & \longrightarrow & 0
\end{array}
\]

such that \( d \) is surjective. So by Proposition 2.8, it corresponds to a point \( \mathbb{P}^g \setminus \{p,q\} \).

Conversely, by some diagram chasing, one can easily check that a vector bundle \( E \) fits into the diagram (13) such that \( d \) is surjective corresponds to an element of \( \ker j_3 \). So \( \mathbb{P}^g \setminus \{p,q\} \) contracts to a point if and only if the dimension of \( \ker j_3 \) is \( \leq 1 \). By (12),

\[
\ker j_3 \cong \text{Ext}^0(L^{-1}(p+q-x), L|_{p+q})/\text{Ext}^0(L^{-1}(p+q-x), L) \cong \mathbb{C}^2/\mathbb{H}^0(L^2(x)(-p-q)).
\]

But \( L^2(x)(-p-q) \cong \mathcal{O}_X(r) \) (Corollary 2.7) and thus the claim holds. \( \square \)

**Corollary 2.11.** If the line \( l \) meets with \( X \) \( n \) times (possibly with multiplicity), the map \( i : l \setminus (l \cap X) \to \mathcal{N} \) is of degree \( 3 - n \) for \( n = 0, 1, 2, 3 \).

**Proof.**

\( n = 0 \): This case is clear since \( \deg \Psi_L = 3 \).

\( n = 1 \): The degree of \( \deg(i) \) is \( \{0, 1, 2\} \). If \( \deg(i) = 1 \), then it contracts to a point in \( \mathcal{N} \). Therefore by the Lemma 2.10, \( l \) is a trisecant line of \( X \) which leads to a contraction. If \( \deg(i) = 1 \), by [8], the map \( i \) factors through \( \mathbb{P}^g = \mathbb{P}V_M \) for some \( M \in \text{Pic}^0(X) \). Thus by Corollary 2.9 we conclude that \( l \) meets \( X \) twice, which leads to the contraction. Therefore \( \deg(i) = 2 \).

\( n = 2 \): Let \( l \cap X = p+q \) then \( l = \mathbb{P}^g \). Since \( l \) is not trisecant, \( \mathbb{H}^0(L^2(x)(-p-q)) = 0 \) by Proposition 2.7. Thus by the proof of Lemma 2.10, \( l \) corresponds to a line in \( \mathbb{P}^g = \mathbb{P}V_{L^{-1}(p+q-x)} \). Hence the map \( \deg(i) = 1 \) since \( \Psi_{L^{-1}(p+q-x)} \) is a linear embedding.

\( n = 3 \): Let \( l \cap X = p + q + r \). By Lemma 2.10, \( l \) contracts to a point in \( \mathcal{N} \). Therefore \( \deg(i) = 0 \).

\( \square \)

**Remark 2.12.** Consider the \( n = 1 \) case in Corollary 2.11. Since \( \deg(i) = 0 \), the closure \( \overline{l} := i(l \setminus (l \cap X)) \) in \( \mathcal{N} \) is a smooth conic. By [10] Proposition 3.6, \( \overline{l} \) should be a Hecke curve or a conic in \( \mathbb{P}V_M \cong \mathbb{P}^g \) for some \( M \in \text{Pic}^0(X) \). In the latter case, \( l \cap X = r, l \setminus r \subset \mathbb{P}V_M^* \cap \mathbb{P}V_M \) for some \( M \in \text{Pic}^0(X) \). This contradicts part i) of Proposition 2.6. Hence \( \overline{l} \) is a Hecke curve in \( \mathcal{N} \).

From now on, let us fix a non-trisecant line bundle \( L \) on \( X \). So by Corollary 2.7, \( n = 3 \) case does not occur.

3. Stable maps in the moduli space \( \mathcal{N} \)

In §1.1, we reviewed the classification of degree 3 stable maps \( \mathbb{P}^1 \to \mathcal{N} \) studied in [8, 10]. In this section, we study the closure of the component parametrizing the stable maps of type ii) in §1.1.
3.1. Stable maps in the blown-up space. Let $P = \text{bl}_X \mathbb{P}^{g+1}$ be the blown-up space for the non-trisecant line bundle $L$. Let $\beta = \pi^*[\text{line}]$ be the curve class in $H_2(P)$. In this subsection, we will study the moduli space

$$M_0(P, \beta) \subset M_0(N, 3)$$

of stable maps of genus 0 with embedded degree 3. Let us start with the topological classification of the stable maps in $P$.

**Lemma 3.1.** Stable maps parametrized by the closed points in $M_0(P, \beta)$ are one of the following types.

1. Lines in $\mathbb{P}^{g+1} \setminus X$.
2. Gluing of the proper transform of a line in $\mathbb{P}^{g+1}$ which meets $X$ at a point $p$ and a line in $\pi^{-1}(p) \cong \mathbb{P}^{g-1}$.
3. Gluing of the proper transform of a line in $\mathbb{P}^{g+1}$ which meets $X$ at two different points $p, q$, a line in $\pi^{-1}(p) \cong \mathbb{P}^{g-1}$, and a line in $\pi^{-1}(q) \cong \mathbb{P}^{g-1}$.
4. Gluing of the proper transform of a line in $\mathbb{P}^{g+1}$ which meets $X$ at two different points $p, q$ and a degree two stable map in $\pi^{-1}(p) \cong \mathbb{P}^{g-1}$.
5. Gluing of the proper transform of a line in $\mathbb{P}^{g+1}$ which is tangent to $X$ at $p$ and a degree two stable map in $\pi^{-1}(p) \cong \mathbb{P}^{g-1}$.

Furthermore, each dimension of the loci of these types (1)-(5) is $2g$, $2g-1$, $2g-2$, $2g$ and $2g-1$ respectively.

**Proof.** Since $H_2(P) \cong \mathbb{Z} \oplus \mathbb{Z}$ where $(1, 0)$ is a homology class of $\pi^*[\text{line}]$ and $(0, 1)$ is a homology class of a line in $\pi^{-1}(p)$, the homology class of the proper transform of a line $l \subset \mathbb{P}^{g+1}$ which meets $X$ at $n$ points is $(1, -n)$. Because of the non-trisecant condition on the curve $X$ (Corollary [2.7]), one can obtain the classifications of stable maps in $P$. The dimension computation is straightforward. For example, let us compute the dimension of the locus consisting of type (4). The locus of stable maps consisting of type (4) has a fibration over $X \times X \setminus \Delta$ with the fiber $Z$. Here $Z$ parameterizes the stable maps of degree two in $\mathbb{P}^{g-1}$ passing through a fixed point. But one can easily check that $Z$ is irreducible by [21] and [14] Chapter III, Corollary 9.6. Therefore the locus of stable maps consisting of the type (4) is of dimension $\dim Z + 2 = (2g - 2) + 2 = 2g$. □

Through the proof of [13] Corollary 4.6], one can regard the space $M_0(P, \beta)$ locally as the zero locus of the section of a vector bundle on a smooth space and thus all irreducible components of $M_0(P, \beta)$ have dimension at least $\int_{\beta=\pi^*[\text{line}]} c_1(T_P) + \dim P - 3 = 2g$.

**Theorem 3.2.** The moduli space $M_0(P, \beta)$ consists of two irreducible components $\Gamma_1$ and $\Gamma_2$, where:

1. $\Gamma_1$ parameterizes lines in $\mathbb{P}^{g+1} \setminus X$.
2. $\Gamma_2$ parameterizes the union of a smooth conic and a line $l$ meeting at a point where $\pi(l)$ is a line that meets $X$ at two points (allowing $l$ to be tangential to $X$).

Moreover, the intersection $\Gamma_1 \cap \Gamma_2$ consists of stable maps of type (5) of Lemma 3.1.

As we will see in the proof of this theorem, the loci of stable maps of types (1)-(3) and (5) (resp. (4) and (5)) are contained in $\Gamma_1$ (resp. $\Gamma_2$). For the proof
of Theorem 3.2, let us start with the computation of the obstruction space of the stable maps of type (4).

**Lemma 3.3.** Let \( l \subset P \) be the line such that \( \pi(l) \cap X = \{p, q\} \), \( p \neq q \). Then the normal bundle of the line \( l \) is given by

\[
N_{l/P} \cong O_l(-1)^\oplus(g-2) \oplus O_l(-1) \oplus O_l(1) \text{ or } O_l(-1)^\oplus(g-2) \oplus O_l^{\oplus 2}.
\]

**Proof.** Let \( l_0 \) be the line in \( \mathbb{P}^{g+1} \) cleanly intersecting \( X \) at two points \( \{p, q\} \). Let \( l \) be the strict transformation of the line \( l_0 \) along the blow-up map \( \pi : P = b_{X/P^{g+1}} \to \mathbb{P}^{g+1} \). By the proof of [20, Lemma 1], one can see that the bundle \( N_{l/P} \) fits into the following short exact sequence:

\[
0 \to \pi_* N_{l_0/P^{g+1}} \otimes O(-E)|_l \to N_{l/P} \to C_p \oplus C_q \to 0.
\]

In fact, the last map is locally given by the followings (cf. [10, Appendix B.6.10]).

Let \( T_1, ..., T_{g+1} \) be a local coordinate of \( \mathbb{P}^{g+1} \) at \( p \) such that locally \( I_{l_0/P^{g+1}} = \langle T_1, T_3, ..., T_{g+1} \rangle \), \( I_{l/P^{g+1}} = \langle T_2, T_3, ..., T_{g+1} \rangle \). Then, we have local coordinate \( t_1, t_2, x_3, ..., x_{g+1} \) of \( P \) at \( p \) which is a lift of \( p \) in \( l \) such that \( \pi \circ T_1 = t_1, \pi \circ T_2 = t_2, \pi \circ T_i = \tau_2 \circ x_i \) for \( 3 \leq i \leq g + 1 \). Therefore we have \( \pi_* I_{l_0/P^{g+1}} = \pi_* (t_1, t_2, ..., t_{g+1}) = \langle t_1, t_2 x_3, ..., t_{2x_{g+1}} \rangle \). Locally, \( \langle t_2 \rangle \) is an ideal of exceptional divisor \( E \). Therefore we have a short exact sequence.

\[
0 \to I_{l/P} \cdot E/P \to \pi_* (I_{l_0/P^{g+1}}) \to I_{l_0/P^{g+1}} / I_{l/P} \cdot E/P \to 0.
\]

Take pull-back this sequence to \( l \), we have

\[
0 \to I_{l/P} / I_{l_0}^2 \otimes O_P(-E) \to \pi^* (I_{l_0/P^{g+1}} / I_{l_0}^2) \otimes O_P \to 0.
\]

Here, the map \( \partial_p \) is a differentiation by a tangent vector \( \frac{\partial}{\partial t_i} \) in \( T_p P \). Dualizing this sequence we have a map \( N_{l/P} \to C_p \). Similarly, a map \( N_{l/P} \to C_q \) can be defined.

From \( N_{l_0/P^{g+1}} = O_{l_0}(1)^{\oplus g} \) and \( O(-E)|_l = O_l(-2) \), we obtain the result. \( \square \)

Similar to other Fano varieties, the normal bundle of the line can be described by the following geometric way.

**Corollary 3.4.** If the projectivized tangent line \( T_{p}X \) and \( T_{q}X \) are co-planar (resp. skew lines), then \( N_{l/P} \cong O_l(-1)^{\oplus(g-2)} \oplus O_l(-1)^{\oplus(g-2)} \oplus O_l(1)^{\oplus(g-3)} \oplus O_l^{\oplus 2} \).

**Proof.** One can see the results by using a local computation similar to that in the proof of Lemma 3.3 \( \square \)

Let \( Q \) be the smooth conic in the exceptional divisor \( E \). Since \( Q \subset \mathbb{P}^{g-1} \) for some fiber of the projective bundle \( E = \mathbb{P}(N_X/P^{g+1}) \to X \), we know that

\[
N_{Q/E} \cong N_{Q/P^{g+1} - P_{g-1}/E}|_Q \cong O_Q(2) \oplus O_Q(1)^{\oplus(g-3)} \oplus O_Q
\]

because \( H^1(O_Q(i)) = 0 \) for \( i = 1, 2 \). Therefore the nested bundle sequence

\[
0 \to N_{Q/E} \to N_{Q/P} \to N_{E/P}|_Q \cong O_Q(-1) \to 0
\]

says that

\[
N_{Q/P} \cong (O_Q(2) \oplus O_Q(1)^{\oplus(g-3)}) \oplus O_Q \oplus O_Q(-1).
\]

**Proposition 3.5.** Let \( \{C\} \in \Gamma_2 \) be the point such that \( C = l \cup Q \) for a line \( l \) and a smooth conic \( Q \) meeting at a point \( z \). Then \( H^1(N_C/P) = 0 \).
Proof. Since the curve \( C \) has only a nodal singularity, the conormal sheaf \( N_{C/P}^* := I_{C/P}/I_{C/P}^2 \) is locally free. From the two short exact sequences:

\[
\begin{align*}
&0 \to N_{C/P}^* \to \Omega_{P|C} \to \Omega_C \to 0 \\
&0 \to \mathcal{O}_{(-1)} \to \mathcal{O}_C \to \mathcal{O}_Q \to 0,
\end{align*}
\]

we have the following commutative diagram:

\[
\begin{array}{c}
\text{Ext}^1(\Omega_C, \mathcal{O}_C) \\
\downarrow \\
\text{Ext}^1(\Omega_P, \mathcal{O}_C) \\
\downarrow \\
\text{Ext}^1(\mathcal{O}_C) \\
\end{array} \quad \begin{array}{c}
\text{Ext}^1(\mathcal{O}_P, \mathcal{O}_C) \\
\downarrow \\
\text{Ext}^1(N_{C/P}^*, \mathcal{O}_C) \\
\end{array} \quad \begin{array}{c}
\text{Ext}^1(\mathcal{O}_P, \mathcal{O}_Q) \\
\downarrow \\
\text{Ext}^1(N_{C/P}^*, \mathcal{O}_Q) \\
\end{array} \quad \begin{array}{c}
0 \\
\cong \\
0 \\
\end{array}
\]

Since \( C = \emptyset \cup Q \) has the unique node point \( z \), \( \text{Ext}^1(\Omega_C, \mathcal{O}_C) \cong \mathbb{C} \). Also, \( \text{Ext}^2(\Omega_C, \mathcal{O}_L(-1)) = 0 \) implies that the first vertical map is surjective. By Lemma 3.6, the second vertical map \( H^1(T_P|C) \cong \text{Ext}^1(\Omega_P, \mathcal{O}_C) \to \text{Ext}^1(\Omega_P, \mathcal{O}_Q) \cong H^1(T_P|Q) = \mathbb{C} \) is an isomorphism. Hence the claim holds whenever \( H^1(N_{C/P}|Q) = 0 \). Let

\[
0 \to N_{C/P | Q}^* \to N_{Q/P}^* \xrightarrow{\partial_z} \mathbb{C}_z \to 0
\]

be the structure sequence where the map \( \partial_z \) is defined by a differentiation by the tangent vector \( T_l \). This can be shown by the following local computation. We can take local coordinate \( x_1, \ldots, x_{g+1} \) of \( P \) at \( z \) where \( I_Q|P = \langle x_2, x_3, \ldots, x_{g+1} \rangle, I_l|P = \langle x_1, x_3, \ldots, x_{g+1} \rangle \). Then, \( I_{C/P} = \langle x_1 x_2, x_3, \ldots, x_{g+1} \rangle \). Consider the exact sequence

\[
0 \to I_{C/P} \to I_{Q/P} \to I_{Q/P}/I_{C/P} \to 0.
\]

Taking pull-back to \( Q \), we have

\[
0 \to I_{C/P}/I_{C/P}|Q \to I_{Q/P}/I_{Q/P}^2 \to \mathbb{C}_z \to 0.
\]

Here, \( I_{Q/P}^2 \to \mathbb{C}_z \) is given by the differentiation by \( T_p l \) since it annihilates the local coordinates \( x_1, x_3, \ldots, x_{g+1} \). Then one can check that the composition map \( \mathcal{O}_Q(1) \cong N_{E/P|Q}^* \subset N_{Q/P}^* \xrightarrow{\partial_p} \mathbb{C}_p \) by \([13]\) is not zero because \( l \) transversely meets with the exceptional divisor \( E \). Hence one can easily see that \( N_{C/P | Q}^* \cong \mathcal{O}_Q(s) \oplus N_{Q/E}^* \) for some \( s \in Q \). Since \( H^1(\mathcal{O}_Q(-r)) = H^1(N_{Q/E}) = 0 \), we proved the claim. \( \square \)

Lemma 3.6. (cf. \([13]\) Lemma 6.4) Let

\[
H^0(T_P|l) \oplus H^0(T_P|Q) \xrightarrow{\alpha} H^0(T_P|P) \to H^1(T_P|C) \to H^1(T_P|l) \oplus H^1(T_P|Q) \to 0.
\]

be the long exact sequence coming from \( 0 \to \mathcal{O}_C \to \mathcal{O}_l \oplus \mathcal{O}_Q \to \mathcal{O}_P \to 0 \). Then the map \( \alpha \) is surjective and thus \( H^1(T_P|C) \cong H^1(T_P|Q) \cong \mathbb{C} \).

Proof. \( T_P P \cong T_p l \oplus T_p E \) since \( l \) and \( E \) transversely meets at the point \( p \). From \( H^0(T_P|l) = H^0(T_l) \oplus H^0(N_{l/P}) \), we see that

\[
(16) \quad H^0(T_P|l) \to H^0(T_p l)
\]

is surjective by the positive degree part \( H^0(T_l) = H^0(\mathcal{O}_l(2)) \).

On the other hand, \( H^0(T_P|Q) = H^0(T_P^{g-1}|Q) \oplus H^0(N_{P^{g-1}}|P|Q) \) from \( Q \subset P^{g-1} \subset E \). One can easily see that the first factor

\[
H^0(T_P^{g-1}|Q) \to H^0(T_P^{g-1})
\]
is surjective. Also, by a simple computation, one can see that $N_{p_1,\mathbb{P}}(\Gamma) \cong O_\mathbb{P} \oplus O_Q(-1)$ where $N_{p_1,\mathbb{P}}(\Gamma) = O_Q$ and thus the positive degree part $H^0(N_{p_1,\mathbb{P}}(\Gamma))$ goes to $H^0(N_{p_1,\mathbb{P}}(\Gamma)) = C$. Hence

$$H^0(T_{\mathbb{P}}|_Q) = H^0(T_{\mathbb{P}}|_Q)| \cong H^0(N_{p_1,\mathbb{P}}(\Gamma)) = H^0(T_{\mathbb{P}}|_Q)| \cong H^0(N_{p_1,\mathbb{P}}(\Gamma)) = H^0(T_{\mathbb{P}}|_Q)|$$

(17)

is surjective. By (16) and (17), the map $\alpha$ is surjective. The last isomorphisms comes from Lemma 3.3 and the equation (14).

□

Now we are ready to prove our main theorem.

Proof of Theorem 3.2. $\Gamma_1$ is isomorphic to an open subset of $Gr(2, g + 2)$ which corresponds to a line in $\mathbb{P}^{g+1}$ that does not meet $X$. So $\Gamma_1$ is irreducible. Also $\Gamma_2$ is irreducible by the proof of Lemma 3.1. By Lemma 3.1 both $\Gamma_1$ and $\Gamma_2$ have dimension $2g$, which agrees with the expected dimension. Moreover there is no other component which has dimension $\geq 2g$ (18. Proof of corollary 4.6). So $\Gamma_1$ and $\Gamma_2$ are all the irreducible components of $M_0(\mathbb{P}, \beta)$. The loci of stable maps of type (1), (2) and (3) are not contained in $\Gamma_2$, so it should be contained in $\Gamma_1$. The loci of stable maps of type (4) and (5) are contained in $\Gamma_2$ by definition.

Since all irreducible components of $M_0(\mathbb{P}, \beta)$ have dimension $2g$, it is a locally complete intersection by the proof of (18. Corollary 4.6). A stable map of type (3) degenerates to a gluing of proper transform of a line in $\mathbb{P}^{g+1}$ which is tangent to $X$ and a degenerate conic in $\pi^{-1}(p) \cong \mathbb{P}^{g-1}$ when $p \rightarrow g$, which is a stable map of type (4). Hence $\Gamma_1 \cap \Gamma_2 \neq \emptyset$. Also since there are only two irreducible components $\Gamma_1$ and $\Gamma_2$, $\Gamma_1 \cap \Gamma_2$ is pure of dimension $2g - 1$ by Hartshorne’s connectedness theorem (25. Theorem 3.4).

By the proof of Proposition 3.5, a stable map of type (4) whose conic component is smooth has no obstruction. So it is a smooth point in the moduli. So it cannot be contained in $\Gamma_1 \cap \Gamma_2$. The sublocus of type (4) whose conic component is singular has dimension $2g - 2$. Since the locus of stable maps of type (5) is dimension $2g - 1$ and obviously irreducible, we conclude that $\Gamma_1 \cap \Gamma_2$ parametrizes stable maps of type (5). □

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