The random walk on upper triangular matrices over $\mathbb{Z}/m\mathbb{Z}$

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Abstract
We study a natural random walk on the $n \times n$ uni-upper triangular matrices, with entries in $\mathbb{Z}/m\mathbb{Z}$, generated by steps which add or subtract a uniformly random row to the row above. We show that the mixing time of this random walk is $O(m^2 n \log n + n^2 m^{o(1)})$. This answers a question of Stong and of Arias-Castro, Diaconis, and Stanley.

1 Introduction
Let $n \geq 3$ and $m \geq 2$ be two integers, and let $G_n(m)$ denote the group of $n \times n$ upper triangular matrices with entries in $\mathbb{Z}/m\mathbb{Z}$ and ones along the diagonal, which is also known as the group of uni-upper triangular matrices. We number the rows of each matrix in $G_n(m)$ from top to bottom. We consider the following Markov chain $(X_t)_{t \geq 0}$ on $G_n(m)$: $X_t$ is derived from $X_{t-1}$ by picking a row $i \in \{2, \ldots, n\}$ uniformly at random and with probability $1/4$ adding it to row $i-1$, with probability $1/4$ subtracting it from row $i-1$, and otherwise staying fixed.

Let $P^t_A(B)$ be the probability that $X_t = B$ given that $X_0 = A$. The walk is irreducible with unique stationary measure, the uniform distribution $U$ on $G_n(m)$. Our main result studies the mixing time of the walk, defined as

$$t_{\text{mix}}(\varepsilon) := \inf\{t \geq 0 : d_n(t) \leq \varepsilon\},$$

where

$$d_n(t) := \max_{A \in G_n(m)} \|P^t_A - U\|_{\text{T.V.}} = \frac{1}{2} \max_{A \in G_n(m)} \left\{ \sum_{B \in G_n(m)} |P^t_A(B) - U(B)| \right\}$$

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is the total variation distance of $P_t^A(B)$ from $U$. Our main theorem determines the mixing time of the random walk $X_t$.

**Theorem 1.** For the random walk $X_t$ on $G_n(m)$, there exist positive constants $\gamma, \delta, c$, so that for all $n, m \geq 2$ with $m$ prime we have

$$d_n(t_{n,m}) \leq e^{-d},$$

where $t_{n,m} = t'_{n,m} + d\frac{\nu_{n,m}}{\log(n+m)}$, $t'_{n,m} = \gamma(m^2 n \log n + n^2 e^\delta \sqrt{\log m}) + cmn^2 \log \log n$, and $d > 0$.

For the case where $m$ is not prime, we are missing one of the main tools, namely [9, Theorem 3]. We are still able to prove a similar upper for the mixing time, which is slightly less tight than the one of Theorem 1.

**Theorem 2.** For the random walk $X_t$ on $G_n(m)$, there exist positive constants $\gamma, c$ and $\delta$, so that for all $n, m \geq 2$ we have

$$d_n(t_{n,m}) \leq e^{-d},$$

where $t_{n,m} = t'_{n,m} + d\frac{\nu_{n,m}}{\log(n+m)}$, $t'_{n,m} = \gamma(m^2 n \log n + n^2 e^\delta (\log m)^{2/3}) + cmn^2 \log \log n$, and $d > 0$.

This process has a long history. The case $n = 3$ was first introduced by Zack [22], and Diaconis and Saloff-Coste proved that order $m^2$ steps are necessary and sufficient for convergence to uniform [6, 7, 10]. This result was later proven again by Bump, Diaconis, Hicks, Míčlo and Widom using Fourier analysis [5]. For $n$ growing, a first upper bound of order $n^7$ was proved by Ellenberg [11], which was later improved by Stong to $n^3 m^2 \log m$ [19]. The case where $m = 2$ was treated by Peres and the second author [17], who proved the mixing time is $O(n^2)$.

Arias-Castro, Diaconis and Stanley [4] used super-character theory, introduced by Andre [1, 2, 3] and Yan [20, 21] to bound the spectrum of the random walk. This results in a bound for the mixing time of order $n^4 m^4 \log n$, for the case where $m$ is prime. The first author [16] improved their analysis to $n^4 m^2$, which gives the correct order of the mixing time in $m$, but not in $n$.

More recently, some other features of this walk have been studied. For the case where $m$ is a prime, Diaconis and Hough [9] studied how many steps an element on the $i$–th diagonal of the matrix needs to mix. Their
result, stated in detail in Section 3, works best for $m$ fixed. However, we are able to use their bounds for general $n$ and prime $m$ to prove our theorems. For $m = 2$, the projection onto the final column of the matrix is itself a well known Markov chain known as the East Model. Ganguly, Lubetzky and Martinelli [12] proved that the East model exhibits cutoff and later Ganguly and Martinelli [13] extending this to the last $k$ columns of the upper triangular matrix walk, where $k$ is fixed.

Recently, Hermon and Olesker-Taylor [14] considered a different question concerning $G_n(m)$. They sample $k$ generators uniformly at random and they prove cutoff for the case where $k$ is growing with $|G_n(m)|$.

Our strategy is to study how fast the first row mixes and proceed by induction on $n$. We do so by analysing it as a random sum of the second row at random times. It is important to understand the values that the second row takes to understand how well mixed this random sum becomes. This is easier to do in the case where $m$ is a prime, thanks to the work of Diaconis and Hough [9].

2 Preliminaries

For the next few sections of the paper we study instead the continuous version of the random walk. For each $i \in \{2, \ldots, n\}$, we consider a rate 1 Poisson clock, and when the $i$-th clock rings, we either add or subtract row $i$ to row $i-1$, each event happening with probability $1/4$ or we do nothing. Theorems 1 and 2 can be rewritten for the continuous time random walk as follows.

**Theorem 3.** For the random walk $X_t$ on $G_n(m)$, there exist positive constants $\alpha, \beta, \gamma$ and $\delta$, so that for all $n, m \geq 2$ with $m$ prime and $\bar{T}_{n,m} = \gamma(m^2 \log n + ne^{\delta \log m} \log m) + cm^2 \log \log n$, we have

$$d_n^{\text{cont}}(\bar{T}_{n,m}) \leq \beta e^{-ac},$$

for $c > 2$.

Similarly, we have the following theorem for the continuous time random walk.

**Theorem 4.** For the random walk $X_t$ on $G_n(m)$, there exist positive constants $\alpha, \beta, \gamma$ and $\delta$, so that for all $n, m \geq 2$ and $\bar{T}_{n,m} = \gamma(m^2 \log n + ne^{\delta (\log m)^{2/3}} \log m) + cm^2 \log \log n$, we have

$$d_n^{\text{cont}}(\bar{T}_{n,m}) \leq \beta e^{-ac},$$

for $c > 2$. 

Theorems 3 and 4 say that since each row has its own rate-1 clock, the time runs order $n$ steps faster in the continuous time version than in discrete time.

Also, note that the constants $\alpha, \beta, \gamma, \delta$ appearing in Theorems 3 and 4 are not necessarily the same as in Theorems 1 and 2, but we prefer to use the same letters for simplicity. Since from now on we only work with continuous time, we drop the superscript and denote $c_{n}^{\text{cont}}(t)$ by $d_{n}(t)$.

We finish this section by showing how to retrieve Theorems 1 and 2 from Theorems 3 and 4.

Proof of Theorems 1 and 2. Theorems 3 and 4 easily imply that there exist positive constants $\alpha, \beta, \gamma', \delta'$, so that for all $n, m \geq 2$ with $m$ prime and $t'_{n,m} = \gamma'(m^2n \log n + n^2e^{\delta' \sqrt{\log m}}) + cnm^2 \log t_{n,m} \log \log n$, we have

$$d_{n}(t'_{n,m}) \leq \beta' \frac{1}{\sqrt{t'_{n,m}}}.$$ (see [15, Theorem 20.3] and [18, equation (3.4)]), Using the formula

$$d(\ell t_{\text{mix}}(\varepsilon)) \leq (2\varepsilon)^{t},$$

we get

$$d \left( t'_{n,m} + \frac{t'_{n,m}}{\log(n+m)} \right) \leq \left( \frac{2\beta'}{\sqrt{t'_{n,m}}} \right)^{d \log(n+m)} \leq e^{-d}.$$ 

\[\square\]

2.1 The induction lemma

The goal of this section is to prove an inequality that relates $d_{n}(t)$ to $d_{n-1}(t)$ thus allowing us to prove Theorems 1 and 2 inductively.

Let $E(i, j)$ denote the $n \times n$ matrix whose $(i, j)$ entry is one and all other entries are equal to zero. We break $X_{t}$ in two parts: let $r_{t}$ be the $n \times n$ matrix that has the same first row as $X_{t}$ and every other entry zero, and let $Y_{t} = X_{t} - r_{t}$. Let $t_{1}, t_{2}, \ldots$ be the times that the second row is selected. Let $N(t) = \max\{j \geq 0 : t_{j} \leq t\}$. We have that

$$X_{t} = Y_{t} + \sum_{j=1}^{N(t)} a_{j}E(1,2)Y_{t_{j}},$$

(1)
where the $a_j \in \{-1, 1\}$, with probability $1/4$, or $a_j = 0$ with probability $1/2$. Equation (11) will allow us to separate the mixing time of the first row from the mixing time of the rest of the matrix.

The main idea is to prove a bound for the $\ell^2$ distance between $r_t$ and the uniform measure on $(\mathbb{Z}/m\mathbb{Z})^{n-1}$, by studying the spectrum of the transition matrix of $r_t$. These eigenvalues are indexed by vectors $y \in (\mathbb{Z}/m\mathbb{Z})^n$ whose first coordinate is zero. Let $X_t(i)$ be the $i$-th row of $X_t$. For a nonzero $y \in (\mathbb{Z}/m\mathbb{Z})^n$, let $Z_y^t(i) = X_t(i)y$ be the dot product $X_t(i)$ with the column vector $y$. From now on, we will write $y \in (\mathbb{Z}/m\mathbb{Z})^{n-1}$, though we actually mean that $y$ has $n$ coordinates, the first one of which is zero. The $\ell^2$ distance between $r_t$ and its stationary measure at time $t_{n,m}$ is given in terms of $\{Z_y^s(2) : y \in (\mathbb{Z}/m\mathbb{Z})^{n-1}, s \in \{t_1, \ldots, N(t_{n,m})\}\}$. We prove that most of these values are conveniently large. For $a, b \in \mathbb{Z}/m\mathbb{Z}$, we use the notation $|a| > b$ to mean that $a \in \{b + 1, \ldots, m - b - 1\}$.

Let $P_t$ be the Poisson point process counting how many times the clock assigned to row 2 rings. Let $t_{n,m}$ contains all the information from rows 2 to $n$ except with probability $d_{n-1}(t)$. This coupling moreover can be made $\mathcal{F}_t$ measurable. Conditional on $\mathcal{F}_t$ we can then couple the first row except with probability $\|q_t - u\|_{T.V.}$. The lemma then follows by averaging.

**Definition 5.** Let $x_y$ and $A_{y,x_y} = A(m)$ be constants that will be determined later, and let $E_{t,t,x_y}$ be the event that $A_{y,x_y}^t \geq A_{y,x_y} t$ for every $y \in W_t$, $A_{y,x_y}^t \geq A_{y,x_y} t$ for every $y \in Q_t$ and $A_{y,x_y}^t \geq A_{y,x_y} t$ for every $y \in P_2$.

The following lemma will help us prove Theorems 3 and 4 inductively. Let $\mathcal{F}_t$ be the $\sigma-$algebra generated by the all of updates except the random signs used when adding/subtracting the second row to the first up to time $t$. In particular, $\mathcal{F}_t$ contains all the information from rows 2 to $n$ as well as the times at which the clock assigned to row 2 rings. Let $q_t$ be the conditional distribution of $r_t$ at time $t$ given $\mathcal{F}_t$.

**Lemma 6.** We have

$$d_n(t) \leq d_{n-1}(t) + \mathbb{E}(\|q_t - u\|_{T.V.}),$$

where $u$ is the uniform measure on $(\mathbb{Z}/m\mathbb{Z})^{n-1}$.

**Proof.** Let $X$ be a uniformly random element of $G_n(m)$. We can couple $Y_t$ and the $n - 1$ last rows of $X$ except with probability $d_{n-1}(t)$. This coupling moreover can be made $\mathcal{F}_t$ measurable. Conditional on $\mathcal{F}_t$ we can then couple the first row except with probability $\|q_t - u\|_{T.V.}$. The lemma then follows by averaging. \[\square\]
We use Lemma 6 to prove Theorems 3 and 4 by induction. In particular, we prove the following proposition.

**Proposition 7.** Let \( t = T_{n,m} \). There exist positive constants \( a, b, \gamma \) such that for any \( n \) we can find \( x, w \) and \( I \) so that

\[
\|q_{T_{n,m}} - u\|_{T.V.} \cdot 1_{E_{T_{n,m},I,x,y}} \leq an^{-1}(\log n)^{-c},
\]

where \( c > 0 \) is the constant from Theorems 1 and 2. Furthermore, for the same \( x, y, I \),

\[
P(E_{T_{n,m},I,x,y}) \leq be^{-c}\left(\frac{1}{m^{\gamma n}} + \frac{1}{n^{\gamma m}}\right).
\]

We now show how to use Proposition 7 to prove Theorems 3 and 4.

**Proof of Theorems 3 and 4.** The proof for the case \( n = 3 \) can be found in [6, Theorem 1.1]. Combining Lemma 6 and Proposition 7 with induction, we get that

\[
d_n(t_{n,m}) \leq Be^{-dc} + \frac{(\log 2)^{1-c}}{c-1} + be^{-c}\sum_{i=1}^{n} \left(\frac{1}{m^{\gamma i}} + \frac{1}{i^{\gamma m}}\right),
\]

where the term \( Be^{-dc} \) comes from the case \( n = 3 \). Therefore,

\[
d_n(t_{n,m}) \leq Be^{-dc} + \frac{(\log 2)^{1-c}}{c-1} + be^{-fc},
\]

\[
\leq \beta e^{-ac}
\]

for \( c > 2 \), which completes the proof of Theorems 3 and 4.

**2.2 The \( \ell^2 \) bound**

The goal of this section is to establish an inequality which will be used to bound \( \|q_t - u\|_{T.V.} \). Let \( N(t) \) be the number of times that the second clock has rung by time \( t \). For \( k \in \mathbb{N} \) and \( w = (w_1, \ldots, w_k) \) with \( w_i \in (\mathbb{Z}/m\mathbb{Z})^{n-1} \) let \( G_{k,w} \) be the event that \( N(t) = k \) and that the second row \( X_t(2) \) is equal to \( w_j \) at the time of the \( j \)-th ring for \( j = 1, \ldots, k \).

Let \( q_{k,w} \) be the distribution of \( r_t \), conditional on \( G_{k,w} \). Then

\[
\|q_t - u\|_{T.V.} = \sum_{k,w} \|q_{k,w} - u\|_{T.V.} P(G_{k,w} \mid F_t).
\]
Each $q_{k,w}$ has the same orthonormal eigenbasis as the simple random walk on $\mathbb{Z}/m\mathbb{Z}$, despite the fact that at each step we are adding/subtracting a different quantity. Let $(y \cdot w_s)$ be the dot product of $y$ and $w_s$. The corresponding eigenvalues are $e^{-2(k - \sum_{s=1}^{k} \lambda_{y,w_s})}$, where the $\lambda_{y,w_s} = \cos \frac{2\pi(y \cdot w_s)}{m}$ are the eigenvalues of the discrete time Markov chain on $(\mathbb{Z}/m\mathbb{Z})^{n-1}$ that adds or subtracts $w_s$ to the current state with probability $1/2$ (see [8, Example 2.1] for a reference). Then we use the classical $\ell^2$ bound,

$$4\|q_{k,w} - u\|_{T.V.}^2 \leq \sum_{y \in (\mathbb{Z}/m\mathbb{Z})^{n-1}\setminus\{0\}} e^{-2(k - \sum_{s=1}^{k} \lambda_{y,w_s})}. \quad (3)$$

To continue with bounding (3), we will need the following technical lemma.

**Lemma 8.** We have

$$\sum_{j=1}^{m-1} e^{-2x(1 - \cos \frac{2\pi j}{m})} \leq me^{-2x} + \frac{\sqrt{3m}}{2\sqrt{2\pi x}},$$

where $x > 0$.

**Proof.** Using properties of the cosine, we get

$$\sum_{j=1}^{m-1} e^{-2x(1 - \cos \frac{2\pi j}{m})} \leq 2 \sum_{j=1}^{m/2} e^{-2x(1 - \cos \frac{2\pi j}{m})} \leq me^{-2x} + 2 \sum_{j=1}^{m/4} e^{-2x(1 - \cos \frac{2\pi j}{m})}, \quad (4)$$

where for the first term in (4), we bound the negative cosines. Using the inequality $\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$, we get that

$$\sum_{j=1}^{m/4} e^{-2x(1 - \cos \frac{2\pi j}{m})} \leq me^{-2x} + 2 \sum_{j=1}^{m/4} e^{-\frac{8j^2x^2}{3m^2}x} \leq me^{-2x} + 2 \int_0^{\infty} e^{-\frac{8w^2x^2}{3m^2}x} dw. \quad (5)$$

$$\leq me^{-2x} + \int_0^{\infty} e^{-\frac{8w^2x^2}{3m^2}x} dw. \quad (6)$$
Using the substitution \( v = \frac{4\pi\sqrt{x}}{\sqrt{3m}}w \), we get that

\[
0 \leq me^{-2x} + \frac{\sqrt{3m}}{2\sqrt{2\pi x}} \int_0^\infty e^{-v^2/2} dv
\]

\[
\leq me^{-2x} + \frac{\sqrt{3m}}{2\sqrt{2\pi x}}.
\]

\[\square\]

2.3 Coupling with Exponentials

Recall that \( Z^t_y(i) := X_t(i)y \), for \( i = 1, \ldots, n \). In this section, we study the time intervals during which \( Z^t_y(i) \neq 0 \). We want to understand for how long \( Z^t_y(i) \) remains equal to zero and for how long it does not. This way we can argue that a good number of \( \lambda_{y,w} \)'s is not equal to 1, thus contributing to the right hand side of (3).

Let \( y \in (\mathbb{Z}/m\mathbb{Z})^{n-1} \setminus \langle e_1 \rangle \), where \( \langle e_1 \rangle \) denotes the subspace of \( (\mathbb{Z}/m\mathbb{Z})^{n-1} \) generated by the vector \( e_1 = (1, 0, \ldots, 0) \). We start by proving that there is a good chance that \( Z^t_y(i) \) will be non-zero after order \( n \) steps.

Lemma 9. Let \( T_i \) denote the first time that \( Z^t_y(i) \) is non-zero. We have that

\[
P(T_i > 24(n + c)) \leq e^{-c},
\]

for \( c > 0 \).

Proof. To study the tails of \( T_i \), we will follow the position \( P_t \) of the first non-zero entry in the column dynamics \( Z^t_y \). We note that \( P_0 = n \). We are going to couple \( P_t \) with the following random walk \( S_t \) on \( \mathbb{Z} \), starting at zero. Consider the column dynamics, \( Z^t_y = (Z^t_y(i))_{i=2}^n \). Let \( Z_t \) be the first entry of the column \( Z^t_y \) that is not divisible by \( m \), when read from top to bottom. The worst starting point is at the second coordinate, in which case we want to study the first time \( \xi \) that \( Z \) returns at 2.

Whenever \( Z_t \) is added to or subtracted from the coordinate directly above, then \( Z_t \) moves up by one (at rate one). Since \( m \) is odd, \( Z_t \) can move down by one coordinate by at most one move between adding or subtracting (so at most at rate 1/4). Therefore, we can couple \( Z_t \) with a biased random walk \( S_t \) on \( \mathbb{Z} \).
When at $x$, $S_t$ moves to $x + 1$ according to a rate $1/2$ Poisson clock or to $x - 1$ according to a rate $1/4$ Poisson clock. A Chernoff bound gives that
\[
P(T_i > t) \leq P(S_t < n - 1) = P(e^{-\lambda S_t} > e^{-\lambda(n-1)}) \leq e^{\lambda(n-1)}E(e^{-\lambda S_t})
\]
We have that $S_t = M_t - N_t$, where $M_t$ is a Poisson($t/2$) random variable and $N_t$ is a Poisson($t/4$) random variable. Therefore
\[
E(e^{-\lambda M_t}) = e^{\frac{t}{2}(e^{-\lambda} - 1)} \text{ and } E(e^{\lambda N_t}) = e^{\frac{t}{4}(e^\lambda - 1)}.
\]
Using the fact that $M_t$ and $N_t$ are independent, we get
\[
P(T_i > t) \leq e^{\lambda t} e^{-\frac{3}{4}t + \frac{1}{2}xe^{-\lambda} + \frac{1}{4}xe^\lambda}.
\]
Setting $\lambda = \log \sqrt{2}$, we get that
\[
P(T_i > t) \leq 2^n e^{-t/12}.
\]
Setting $t = 24(n + c)$ we get the desired result. \qed

We want to study for how long $Z_y^i(2)$ can remain divisible by $m$, which means that the corresponding eigenvalues in the right hand side of (3) will be equal to one.

**Definition 10.** Let $\ell_1$ be a time such that $Z_y^{\ell_1}(2) = 0$ and $Z_y^{\ell_2}(2) \neq 0$. Let $\ell_2 = \inf\{t > \ell_1 : Z_y^t(2) \neq 0\}$. We will call $[\ell_1, \ell_2]$ a $(y, I)$-zero interval.

**Lemma 11.** Let $m > 2$ be an odd integer. Let $[\ell_1, \ell_2]$ be a $(y, i)$-zero interval. Then,
\[
P(\ell_2 - \ell_1 > 12k \mid F_{\ell_1}) \leq e^{-k},
\]
where $k > 0$.

**Proof.** Let $S_t$ be the random walk on $\mathbb{Z}$, which starts at 0, and moves by +1 according to a rate $1/2$ Poisson clock and by $-1$ according to a rate $1/4$ Poisson clock. Note that $S_t = M_t - N_t$, where $M_t$ is a Poisson($t/2$) random variable and $N_t$ is a Poisson($t/4$) random variable, and thus
\[
E(e^{-\theta M_t}) = e^{\frac{t}{2}(e^{-\theta} - 1)} \text{ and } E(e^{\theta N_t}) = e^{\frac{t}{4}(e^\theta - 1)}.
\]
Using the fact that $M_t$ and $N_t$ are independent, a Chernoff bound gives
\[
P(\xi > x) \leq P(S_x < 0) \leq P(e^{-\theta S_x} \geq 1) \leq E(e^{-\theta S_x}) \leq e^{-\frac{3}{4}t + \frac{1}{2}xe^{-\theta} + \frac{1}{4}xe^\theta}.
\]
Setting $\theta = \log \sqrt{2}$, we have that
$$P(\xi > x) \leq e^{-x/12}.$$  

Therefore, if $x = 12k$,
$$P(\ell_2 - \ell_1 > 12k | F_t) \leq e^{-k}.

More importantly, we will study the length of the intervals during which $Z_y^t(2) \neq 0$. This will help us understand the non-trivial terms on the right hand side of (3).

**Definition 12.** Let $\ell_3$ be a time such that $Z_y^{\ell_3}(i) \neq 0$ and $Z_y^{\ell_4}(i) = 0$. We will call $[\ell_3, \ell_4]$ a $(y, i)$–non-zero interval.

**Lemma 13.** Let $m > 2$ be an integer. Let $[\ell_3, \ell_4]$ be a $(y, i)$–non-zero interval. Then,
$$P(\ell_4 - \ell_3 \geq k | F_t) \geq e^{-k},$$
where $k > 0$.

**Proof.** The $i$–th entry of $Z_y$ can turn from zero to non-zero if the $i + 1$ clock rings and $Z_y(i+1)$ has the appropriate value. Therefore, we can couple $\ell_4 - \ell_3$ with the time it takes for the clock of the $i + 1$ row to ring and the statement follows.

We are now going to put all this information together to prove that during any interval, $Z_y^t(2)$ is non-zero for a constant fraction of the time. We break up the interval $[0, T_{n,m}]$ in intervals $[t_j, t_{j+1}]$ of length $L$, so that $t_j = jL$. Let $j \in \{1, \ldots, n\}$ and let $g = 99$.

**Definition 14.** Let $i \in \{1, \ldots, n - 1\}$. An interval $[t_j, t_{j+1}]$ is called $(y, i)$–good if $Z_y^t(i) \neq 0 \mod m$ for at least $1/g$ of $[t_j, t_{j+1}]$. Let $D_y^i$ be the set of all $(y, i)$–good intervals by time $T_{n,m}$. Let $M_{y,i}$ be the number of $(y, i)$–good intervals that have occurred by time $T_{n,m}$.

The following lemma follows from a simple Chernoff bound and will help us study how likely it is for a given interval to be $(y, i)$–good.

**Lemma 15.** Let $E_1, \ldots, E_k$ be independent, exponential random variables with mean one. We have that
(a) $\Pr\left( \sum_{i=1}^{k} E_i > 2k \right) \leq \left( \frac{2}{e} \right)^k$.

(b) $\Pr\left( \sum_{i=1}^{k} E_i < \frac{k}{2} \right) \leq \left( \frac{6}{7} \right)^k$.

The following lemma says that a constant fraction of intervals are $(y, i)$–good.

**Lemma 16.** Let $i \in \{3, \ldots, n\}$. At time $\bar{t}_{n,m}$ we have that

$$
\Pr\left( M_{y,i} \leq \frac{\bar{t}_{n,m}}{100L} \right) \leq e^{-d_1 \bar{t}_{n,m}},
$$

for a suitable constant $d_1$.

**Proof.** Consider the $(y, i)$–non-zero intervals $A_b \subset [0, \bar{t}_{n,m}]$ and the $(y, i)$–zero intervals $B_k \subset [0, \bar{t}_{n,m}]$. Let $|A_b|$ be the length of $A_b$. Let

$$
W_y = \sum_{b} |A_b|
$$

be the total time that $Z^y(i)$ is not equal to zero. Notice that

$$
W_y = \sum_{b} |A_b| \leq L \sum_{j} 1\{[t_j, t_{j+1}] \in D_y^b\} + \frac{L}{g} \sum_{j} 1\{[t_j, t_{j+1}] \notin D_y^b\} = LM_{y,i} + \frac{L}{g} \left( \frac{\bar{t}_{n,m}}{L} - M_{y,i} \right) = L \left( 1 - \frac{1}{g} \right) M_{y,i} + \frac{1}{g} \bar{t}_{n,m}.
$$

For $m$ odd, equation (11) gives that

$$
\left\{ M_{y,i} \leq \frac{\bar{t}_{n,m}}{100L} \right\} \subset \{ W_y \leq x \bar{t}_{n,m} \},
$$

where $x = \frac{1}{100} + \frac{99}{100g} = \frac{1}{50}$.

Lemmas 11 and 13 say that we can couple each $|A_b|$ and $|B_k|/12$ with exponential random variables with mean 1. Let $r = \frac{2}{49} \bar{t}_{n,m}$ and $c = \frac{1}{48} \bar{t}_{n,m}$.
Either $\bigcup_{j=1}^{r} A_j \subset [0, t_{n,m}]$ or $(\bigcup_{j=0}^{r} B_j)^c$ contains all $(y, i)$–non-zero intervals that $[0, t_{n,m}]$ contains. This is summarized in the following equation

$$W_y \geq \min \{ \sum_{b=1}^{r} |A_b|, t_{n,m} - B_0 - \sum_{k=1}^{r} |B_k| \}.$$ 

Therefore,

$$\mathbb{P}(W_y \leq x t_{n,m}) \leq \mathbb{P} \left( \sum_{b=1}^{r} |A_b| \leq x t_{n,m} \right) + \mathbb{P}(B_0 \geq 24(n + c))$$

$$+ \mathbb{P} \left( \sum_{k=1}^{r} |B_k| \geq (1 - x) t_{n,m} - 24(n + c) \right). \quad (13)$$

We choose $\gamma$ from Theorems 3 and 4 appropriately so that $(1 - x) t_{n,m} - 24(n + c) \geq 2r$. Then,

$$\mathbb{P}(W_y \leq x t_{n,m}) \leq \mathbb{P} \left( \sum_{b=1}^{r} |A_b| \leq r/2 \right) + e^{-c} + \mathbb{P} \left( \sum_{k=1}^{r} |B_k| \geq 2r \right). \quad (14)$$

Lemmas 11 and 13 say that $|A_b|$ and $|B_k|$ are stochastically dominated by appropriate exponentials above and below respectively. Lemma 15 gives

$$\mathbb{P} \left( \sum_{b=1}^{r} |A_b| \leq r/2 \right) \leq \left( \frac{6}{7} \right)^r + e^{-c} + \left( \frac{2}{e} \right)^r. \quad (15)$$

Putting (12) and (15) together, we have

$$\mathbb{P} \left( M_{y,i} \leq \frac{t_{n,m}}{100L} \right) \leq \left( \frac{6}{7} \right)^r + e^{-c} + \left( \frac{2}{e} \right)^r. \quad (16)$$

Using the definition of $r, c$ and (16) we get the desired result.

For the case where $m$ is even, project all values over $\mathbb{Z}/2\mathbb{Z}$. Equation 2.2 of [17] says that for every $\varepsilon > 0$, we have that

$$\mathbb{P} \left( \left| W_{\tilde{t}_{n,m}} - \tilde{t}_{n,m} \right| \geq \varepsilon \tilde{t}_{n,m} \right) \leq 2^{n+1} e^{-\frac{\tilde{t}_{n,m} \varepsilon^2 \lambda}{12}},$$

where $\lambda$ is a positive constant not depending on $n, m$. Setting $\varepsilon = \frac{12}{29}$ we get

$$\mathbb{P} \left( M_{y,i} \leq \frac{\tilde{t}_{n,m}}{100L} \right) \leq \mathbb{P}(W_y \leq x \tilde{t}_{n,m}) \leq 2^{n+1} e^{-\frac{12\tilde{t}_{n,m}}{625}}.$$

□
Finally, we need a lemma that guarantees that $Z_y(2)$ is sufficiently big for a constant fraction of $[0, \overline{T}_{n,m}]$ with high probability.

Let $I \in [n]$. Recall that $P^t$ is a Poisson point process according to which the clock of the second row rings and recall that $A^t_{y,x} = \int_0^t 1_{|Z^t_y(2)| > x} dP^s$. Let $\mathcal{F}_j$ be the $\sigma$–algebra generated by all the clock rings before time $t_j$. Let $\mathcal{F}_j^-$ be the $\sigma$–algebra generated by all the clock rings $I+1, \ldots n$ before time $t_j$ and set $\mathcal{F}_j^* = \sigma(\mathcal{F}_j, \mathcal{F}_j^-)$.

In later sections, we set $I = \min\{\lfloor \sqrt{\log m}\rfloor, n-1\}$ if $m$ is prime, otherwise we set $I = \min\{\lfloor \sqrt{\log m}\rfloor, n-1\}$. We consider different regimes for $y$ and study the corresponding values of $Z_y^t(2)$. Namely, we consider

1. $y \in W_I := (\mathbb{Z}/m\mathbb{Z})^{n-1} \setminus \langle e_1, \ldots, e_{I-1} \rangle$,
2. $y \in Q_I := \langle e_1, \ldots, e_{I-1} \rangle \setminus \langle e_1, e_2 \rangle$,
3. $y \in P_2 := \langle e_1, e_2 \rangle$.

The goal is to prove that $|Z_y^t(2)|$ is big very often. To quantify how big we need $|Z_y^t(2)|$ to be, we define the following quantity $x_y$. We also specify the length of the intervals for each $y$. More precisely, we define

$$x_y = \begin{cases} 
  m/8, & \text{if } m \text{ is prime and } y \in W_I \\
  me^{-k(\log m)^{2/3}}, & \text{if } m \text{ is not prime and } y \in W_I \\
  m/K, & \text{if } y \in Q_I \\
  \sqrt{\log m}/2, & \text{if } y \in P_2
\end{cases} \quad (17)$$

and

$$L_y = \begin{cases} 
  d_2 m^{4/1}, & \text{if } m \text{ is prime and } y \in W_I \\
  2fm^{2/1}/ \log h, & \text{if } m \text{ is not prime and } y \in W_I \\
  Fm, & \text{if } y \in Q_I \\
  \delta_1 [\log m], & \text{if } y \in P_2,
\end{cases}$$

where $k, K, d_2, f, F, \delta_1$ are suitable constants, $J = \min\{(\log m)^{1/3}, n-2\}$ and $h = 20(J+1)$.

Let $B_{j,y}$ denote the event that $|Z^t_y(2)| > x_y$ for at least a proportion $A^{-1}$ of $[t_{j+1}, t_{j+2}]$. Let $\mathcal{G}_j^y$ be the event that $[t_j, t_{j+1}] \in D^t_y$. The next sections are focusing on proving the following lemma.

**Lemma 17.** For $y \notin \langle e_1 \rangle$, there is a uniformly bounded, positive constant $\zeta$ such that $\mathbb{P}(B_{j,y} \mid \mathcal{F}_j^*) \geq \zeta 1_{\mathcal{G}_j^y}$ for all $j$. 

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Lemma 17 is crucial to proving the next lemma, which is the main ingredient for proving the second part of Proposition 7.

Lemma 18. At time $\bar{t}_{n,m}$, we have

$$\mathbb{P}(\tilde{A}_{y,x}^{t_{n,m}} < (400A)^{-1}\zeta \bar{t}_{n,m}) \leq \left(\frac{2}{e}\right)^{\frac{\zeta \bar{t}_{n,m}}{400t_{n,m}}} + e^{-\zeta \bar{t}_{n,m}/2} + e^{-d_1t_{n,m}} + 2e^{\frac{-\zeta^2 t_{n,m}}{400^2 L}},$$

where $d_1$ is as in Lemma 16, $\zeta$ is as in Lemma 17 and $L$ is the length of the intervals.

Proof. The event $\left\{\int_{0}^{\bar{t}_{n,m}} 1_{\{|Z_{y}^{(2)}| > x\}}ds \geq (200A)^{-1}\zeta \bar{t}_{n,m}\right\}$ is satisfied if at least $\zeta/200$ of the $B_{j,y}$ are satisfied. For this reason we want to estimate $\sum_j I_{B_{j,y}}$.

We write

$$\sum_j I_{B_{j,y}} = \sum_j \mathbb{P}(B_{j,y} | \mathcal{F}_j) + \sum_j (I_{B_{j,y}} - \mathbb{P}(B_{j,y} | \mathcal{F}_j)). \quad (18)$$

We set $U_\ell = \sum_{j \leq \ell} (I_{B_{j,y}} - \mathbb{P}(B_{j,y} | \mathcal{F}_j))$ and $S_1 = \{U_{\bar{t}_{n,m}} \geq -(200L)^{-1}\zeta \bar{t}_{n,m}\}$.

Let $S_2$ be the event that $\{\sum_j \mathbb{P}(B_{j,y} | \mathcal{F}_j) \geq (100L)^{-1}\zeta \bar{t}_{n,m}\}$. Equation (18) gives

$$\mathbb{P}\left(\sum_j I_{B_{j,y}} \geq (200L)^{-1}\zeta \bar{t}_{n,m}\right) \geq \mathbb{P}(S_1 \cap S_2). \quad (19)$$

We note that $X_t = \sum_{j=1}^{t-1} (I_{B_{2j,y}} - \mathbb{P}(B_{2j,y} | \mathcal{F}_{2j})$ is a martingale with respect to $\mathcal{F}_{2t}$. Using the Azuma-Hoeffding inequality we have

$$\mathbb{P}\left(X_{\bar{t}_{n,m}} \geq -(400L)^{-1}\zeta t_{n,m}\right) \geq 1 - e^{\frac{-\zeta^2 \bar{t}_{n,m}}{400^2 L}}.$$

Similarly, using a time shifting argument, we can get

$$\mathbb{P}\left(\bar{X}_{t_{n,m}} \geq -(400L)^{-1}\zeta t_{n,m}\right) \geq 1 - e^{\frac{-\zeta^2 t_{n,m}}{400^2 L}},$$

where $\bar{X}_t = U_{2t} - X_t$. Therefore,

$$\mathbb{P}(U_{\bar{t}_{n,m}} \geq -(200L)^{-1}\zeta \bar{t}_{n,m}) \geq 1 - 2e^{\frac{-\zeta^2 \bar{t}_{n,m}}{400^2 L}}. \quad (20)$$
To continue bounding the right hand side of (19), we also want to bound \( P\left( \sum_j P(B_{j,y} \mid F_j^*) \right) \). Recalling Lemma 17, we have that \( P(B_{j,y} \mid F_j^*) \geq \zeta 1_{G_j^*} \). Thus,

\[
P\left( \sum_j P(B_{j,y} \mid F_j^*) \right) \geq P\left( M_{y,I} \geq \frac{t_{n,m}}{100L} \right) \geq 1 - e^{-d_{1y,m}},
\]

by Lemma 16. A union bound and equations (18), (19), (20) and (21) give

\[
P\left( \int_{0}^{t_{n,m}} 1_{\{Z_{ry}(2) > x\}} ds < (200A)^{-1} \zeta t_{n,m} \right) \leq e^{-d_{1y,m}} + 2e^{-\frac{c_{t_{n,m}}}{400^2L}}.
\]

We finish the proof using a Poissonization argument. Conditioning on \( \Lambda_{t_{n,m}} := \int_{0}^{t_{n,m}} 1_{\{Z_{ry}(2) > x\}} ds = s \), we have that \( A_{y,x}^t \) is a Poisson random variable with mean \( s \).

\[
P(A_{y,x}^t < (400A)^{-1} \zeta t_{n,m}) \leq P(A_{y,x}^t < (400A)^{-1} \zeta t_{n,m} \mid \Lambda_{t_{n,m}} \geq (200A)^{-1} \zeta t_{n,m})
+ P(\Lambda_{t_{n,m}} < (200A)^{-1} \zeta t_{n,m})
\]

Using the tails of a Poisson random variable and (22), we get

\[
P(A_{y,x}^t < (400A)^{-1} \zeta t_{n,m}) \leq \left( \frac{2}{e} \right) \frac{c_{t_{n,m}}}{400^2L} + e^{-d_{1y,m}} + 2e^{-\frac{c_{t_{n,m}}}{400^2L}},
\]

which finishes the proof.

\[\square\]

### 2.4 Reducing the walk to a smaller dimension

Let \( I \in \{2, \ldots, n\} \). In this section, we develop the necessary tools to study the walk if we only focus on the top \( I \) coordinates of \( Z_{y}^t \).

Let \( s_1 < s_2 < \ldots \leq t_{n,m} \) denote the times when the \( I \)-th clock rings and \( Z_{y}^t(I) \neq 0 \). Let \( z_1 < z_2 < \ldots \leq \bar{t}_{n,m} \) denote the times when a clock other than the \( I \)-th one rings and let \( \{W_j\}_{j=1}^{\bar{t}_{n,m}} \) be the corresponding operation matrix applied. This means that if at time \( z_j \), row \( i \) is added to row \( i-1 \), then

\[W_j = I_n + E(i-1, i),\]
where \( E(i - 1, i) \) denote the matrix whose \((i - 1, i)\) entry is one and all other entries are equal to zero and \( I_n \) is the identity matrix.

Let \( L(t) = \max\{j \geq 0 : z_j \leq t\} \). For \( 0 \leq t \leq \tau_{n,m} \), we define the backwards process by \( Y_0 = I_n \) and

\[
Y_t = L(\tau_{n,m}) - L(t-M_{n,m}) - 1 \prod_{j=0}^{t-M_{n,m}} W_{L(\tau_{n,m})-j} \prod_{j=0}^{t-M_{n,m}} W_{L(\tau_{n,m})-1} \ldots W_{L(\tau_{n,m})+1} \tag{23}
\]

and for \( 0 \leq t' < t \leq \tau_{n,m} \) we let

\[
Y_{t',t} = Y_{L(\tau_{n,m})-t'} \prod_{j=0}^{t-M_{n,m}} W_{L(t)} \ldots W_{L(t')} + 1. \tag{24}
\]

Notice that the entries of \( Y_t \) and \( Y_{t',t} \) that fall on the \([1, I - 1] \times [I, n] \) box are equal to zero and that \( Y_t \) is a Markov chain on the columns of a matrix in \( G_n(m) \).

The next lemma explains the connection between \( Z_y^{s_\ell} \) and the \( Y_{t',t} \)'s.

**Lemma 19.** We have

\[
Z_y^{s_\ell} = Y_{0,s_\ell} Z_y^0 + \sum_{k=1}^{\ell-1} a_k Y_{s_k,s_\ell} E(I - 1, I) Y_{0,s_k} Z_y^0 + a_\ell E(I - 1, I) Y_{0,s_\ell} Z_y^0,
\]

where \( a_k \in \{\pm 1, 0\} \) are the random signs corresponding to the \( k \)-th time the \( I \)-th clock rings.

**Proof.** We prove the statement by induction. For \( \ell = 0 \) both sides are equal to \( Z_y^0 \). By the definition of \( s_{\ell+1} \) we have

\[
Z_y^{s_{\ell+1}} = (I_n + a_{\ell+1} E(I - 1, I)) Y_{s_\ell,s_{\ell+1}} Z_y^{s_\ell}. \tag{25}
\]

By the induction hypothesis we have

\[
(25) = (I_n + a_{\ell+1} E(I - 1, I)) Y_{s_\ell,s_{\ell+1}} \left( Y_{0,s_\ell} Z_y^0 + a_\ell E(I - 1, I) Y_{0,s_\ell} Z_y^0 \right.
\]

\[
+ \sum_{k=1}^{\ell} a_k Y_{s_k,s_\ell} E(I - 1, I) Y_{0,s_k} Z_y^0 \right).
\]

\[16\]
Using the facts that $E(I-1, I) E(I-1, I) = 0$ and $E(I-1, I) Y E(I-1, I) = 0$ for every $Y \in G$ whose $[I-1] \times [I, n]$ entries are zero, we get

\[ 26 = Y_{0, s_{t+1}} Z^0_y + a_{t+1} E(I-1, I) Y_{0, s_{t+1}} Z^0_y + \sum_{k=1}^{\ell-1} a_k Y_{s_k, s_{t+1}} E(I-1, I) Y_{0, s_k} Z^0_y \]

\[ + a_t Y_{s_t, s_{t+1}} E(I-1, I) Y_{0, s_{t+1}} Z^0_y, \]

which finishes the proof.

Since we are interested in $Z^t_y(2)$, we write a similar version of Lemma 19.

In the following observations we consider $\ell$ such that $s_1 < s_2 < \ldots < s_\ell \leq t < s_{\ell+1}$.

Using the fact that $Z^t_y = Y_{s_{t}, t} Z^s_y$, we get the following.

**Corollary 20.** We have that

\[ Z^t_y = Y_{0, t} Z^0_y + \sum_{k=1}^{\ell-1} a_k Y_{s_k, t} E(I-1, I) Y_{0, s_k} Z^0_y + a_t E(I-1, I) Y_{0, t} Z^0_y \]

and

\[ Z^t_y(2) = [Y_{0, t} Z^0_y](2) + \sum_{k=1}^{\ell-1} a_k [Y_{s_k, t} E(I-1, I) Y_{0, s_k} Z^0_y](2). \]

To study $[Y_{0, t} Z^0_y](2)$, consider a vector process starting at $y_I \cdot e_I$ and having the same updates as the original process. Then $[Y_{0, t} Z^0_y](2)$ is the second coordinate of this process.

Similarly, $[Y_{s_k, t} E(I-1, I) Y_{0, s_k} Z^0_y](2)$ is the same as the second coordinate of the vector process that starts at $y_{I-1} e_I$, where $y_{I-1} e_I$ is the $I$-th coordinate of $Z^s_y$, and whose updates from are the same as the updates that occur between times $s_k$ and $t$.

Recall that our goal is to study the terms $\cos \left( \frac{2\pi Z^t_y(2)}{m} \right)$ that appear in (3) in the form of $\lambda_{y, w}$. The following lemma introduces a condition under which $|Z^t_y(2)|$ is guaranteed to be big during the interval $[t_{j+1}, t_{j+2}]$. This will be crucial to proving that the eigenvalues of the walk are sufficiently small for a constant fraction of the time. Recall that $D_y^I$ be the set of all $(y, I)$–good intervals by time $t_{n, m}$.

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Let $t \in [t_{j+1}, t_{j+2}]$. We consider the decomposition of Corollary \[20\]

\[
N^t = \begin{cases}
[Y_{s_{j_1}, t} E(I - 1, I) Y_{0, s_{j_1}} Z^0_y(2), & \text{if } t_j \leq s_{j_1} \leq t_{j+1}, \\
0, & \text{otherwise},
\end{cases}
\]

where $s_{j_1}$ is the first time in $[t_j, t_{j+1}]$ that the $I$–th clock rings.

**Lemma 21.** Let $t \in [t_{j+1}, t_{j+2}]$. For every $x \in \{0, \ldots, m/4\}$, we have that

\[
\mathbb{P}(Z_t^y(2) \in [x, m/4] \mid F_j^*) \geq \frac{1}{4}\mathbb{P}(2N^t \in [2x, m/2] \mid F_j^*). \tag{27}
\]

**Proof.** Let $\mathcal{Y}_t$ be the event that $2N^t \in [2x, m/2]$.

\[
\mathbb{P}(Z_t^y(2) \in [x, m/4] \mid F_j^*) \geq \mathbb{P}(Z_t^y(2) \in [x, m/4] \mid \mathcal{Y}_t, F_j^*) \mathbb{P}(\mathcal{Y}_t \mid F_j^*). \tag{28}
\]

We turn to the decomposition of Corollary \[20\]. The condition $\{2N^t \in [2x, m/2]\}$ combined with the fact that $a_1 = 1$ or $a_1 = -1$ with probability $1/4$ results in $Z_t^y(2) \in [x, m/4]$ with probability $1/4$. Therefore,

\[
\mathbb{P}(Z_t^y(2) \in [x, m/4] \mid F_j^*) \geq \frac{1}{4}\mathbb{P}(2N^t \in [2x, m/2] \mid F_j^*). 
\]

This finishes the proof. \qed

We move onto applying the results established above.

### 3 The case where $m$ is a prime

In this section $m$ is a prime number. Since the case $m = 2$ was covered in [17], from now on in this section $m$ will denote an odd prime.

#### 3.1 The main lemma for $m$ prime

For $m$ prime, we can use the Diaconis–Hough lemma for the proof of Theorem \[3\]. We state the lemma below.

**Lemma 22.** ([9, Theorem 3])

Let $Z_t$ be the configuration of the rightmost corner of the upper triangular random walk at time $t$. For any set $A \subset \mathbb{Z}/m\mathbb{Z}$ we have that

\[\| \mathbb{P}(Z_t \in \cdot) - U \|_{T.V.} \leq \exp(-rt2^{-n}m^{-\frac{2}{n+1}}),\]

where $U$ is the uniform measure on $G_n(m)$ and $r$ is a universal constant.
Diaconis and Hough mainly treat the case where \( n \) is fixed. Therefore, the term \( 2^n \) is not announced in their main result, but can be found in the proof of [9, Proposition 22].

We select an index \( I \in \{1, \ldots, n-1\} \) so that the Lemma 22 gives useful estimates for \( y \in W_I \) when applied on \( G_{I-1}(m) \). Our goal is to prove that \( Z_y^I(2) \) is big often enough. For \( y \in Q_I \), we apply Lemma 22 for \( n = 3 \).

Finally, we bound the eigenvalues in \( P_2 \) using the \( n = 3 \) case.

### 3.2 The eigenvalues for \( y \in W_I \)

Let \( I = \min\{\lfloor \sqrt{\log m} \rfloor, n-1\} \), so that \( 2^I \leq m^2 \). Recall that \( W_I = (\mathbb{Z}/m\mathbb{Z})^{n-1} \setminus \langle e_1, \ldots, e_{I-1} \rangle \) and let \( y \in W_I \). The goal of this section is to prove that \( |Z_y^I(2)| \geq \frac{m}{8} \) for a constant fraction of \([0, \frac{t}{m}, \frac{n}{m}]\).

Let \( G_{y,j}^I \) be the event \{\( [t_j, t_{j+1}] \in D_y \)\} for the \( I \) that was chosen, but for simplicity we will write \( G_y^I \). The following lemma is the main tool for proving the Theorem 3.

**Lemma 23.** For \( y \in W_I \), we have

\[
\mathbb{P}\left(|Z_y^I(2)| \geq \frac{m}{8} \big| \mathcal{F}_j^*\right) \geq \frac{1}{128}1_{G_y^I},
\]

for every \( t \geq t_{j+1} \).

**Proof.** We apply Lemma 21. In particular, we bound \( \mathbb{P}(|2N^I| \geq \frac{m}{4} \big| \mathcal{F}_j^* \) from below, using Lemma 22 to get

\[
\mathbb{P}\left(|2N^I| \geq \frac{m}{4} \big| \mathcal{F}_j^* \right) \geq \left( \frac{1}{8} - \exp\left\{- \frac{r L}{2m^{2/m}} \right\} \right) 1_{G_y^I} \mathbb{P}(t_j \leq s_{j_1} \leq t_{j+1}).
\]

By our choice of \( L \) and \( I \), we have

\[
\mathbb{P}\left(|2N^I| \geq \frac{m}{4} \big| \mathcal{F}_j^* \right) \geq \frac{1}{16}1_{G_y^I} \mathbb{P}(t_j \leq s_{j_1} \leq t_{j+1}).
\]

We take \( d_2 \) big enough so that \( \mathbb{P}(t_j \leq s_{j_1} \leq t_{j+1}) \geq 1/2 \). Lemma 21 finishes the proof.

The next lemma says that the event \{\( |Z_y^I(2)| > m/8 \)\} considered in Lemma 23 has a good chance of holding for a constant fraction of the time. Let \( S \) be an appropriately chosen constant, which does not dependent on \( j, m, n \) and \( y \).
Lemma 24. Denote the event
\[ B_{j,y} := \left\{ \int_{t_{j+1}}^{t_{j+2}} 1_{\{|Z_{y}^s(2)| > m/8\}} ds \geq \frac{1}{16384} L \right\}. \]

For \( y \in W_I \), we have\[ \mathbb{P}(B_{j,y} \mid \mathcal{F}_j^*) \geq \frac{1}{129} \sigma_j^y, \]
for every \( j \geq 1 \).

Proof. Let \( R_j \) count the length of \( s \in [t_{j+1}, t_{j+2}] \), such that \(|Z_{y}^s(2)| \leq m/8\). So \( R_j = \int_{t_{j+1}}^{t_{j+2}} 1_{\{|Z_{y}^s(2)| \leq m/8\}} ds \). Lemma 23 and Fubini give\[ \mathbb{E}(R_j \mid \mathcal{F}_j^*) \leq \left( 1 - \frac{1}{128} \sigma_j^y \right) L. \]
Markov's inequality then gives\[ 1_{\sigma_j^y} \mathbb{P}\left( R_j > \frac{16383}{16384} L \mid \mathcal{F}_j^* \right) \leq \frac{128}{129}. \]
Therefore,
\[ \mathbb{P}(B_{j,y} \mid \mathcal{F}_j^*) \geq \mathbb{P}\left( R_j \leq \frac{129}{128} \left( 1 - \frac{1}{128} \sigma_j^y \right) L \mid \mathcal{F}_j^* \right) \geq \frac{1}{129} \sigma_j^y. \]

For the next lemma, we set \( A = 16384 \).

Lemma 25. Recall that \( P^t \) is the indicator function that the clock of the second row rings at time \( t \) and \( A_{y,x}^t = \int_0^t 1_{\{|Z_{y}^s(2)| > x\}} dP^s \). Let \( x = m/8 \). For \( y \in W_I \), letting \( B_y = \{A_{y,x}^{n,m} \geq (400A)^{-1} \tau_{n,m}\} \), we have\[ \mathbb{P}(B_y) \geq 1 - \frac{b}{m^g} e^{-c}, \]
where \( b, g \) are suitable constants with \( g > 1 \).

Proof. Recalling Lemma [17] and applying Lemma 24 we can take \( \zeta = 1/129 \). We have that there is a constant \( g > 1 \) such that \( \zeta \frac{\tau_{n,m}}{400^2 L_1} \geq gn \log m + c \). Lemmas [18] and [31] give the desired result.

The rest of the eigenvalues will be studied in Section 4.
4 The case where \( m \) is not a prime

In this section, we study the quantity \( Z_t(2) \) for the case where \( m \) is not necessarily prime. The strategy is similar to the \( m \) prime case, however we can no longer apply Lemma 22 directly. This is why we start by proving a lemma similar to Lemma 22, which works for \( m \in \mathbb{N} \) that is not necessarily prime.

Let \( J = \min\{(\log m)^{1/3}, n - 2\} \) and \( h = 20(J + 1) \). Let \( A_{J,h,m} = m^{2/J}/(6^{2/J}2\log h) \) and let \( p \) be a prime such that \( \frac{1}{20}A_{J,h,m} \leq 6r^{-1}2^jp^{2/J} \leq \frac{1}{2}A_{J,h,m} \). In other words, we choose a prime \( p \) such that

\[
\left( \frac{r}{120 \log h} \right)^{J/2} 2^{-(J+J^2)/2} \frac{m}{6} \leq p \leq \left( \frac{r}{6 \log h} \right)^{J/2} 2^{-(J+J^2)/2} \frac{m}{6}.
\]

Recall that \( r \) is the constant from Lemma 22. The goal of this section is to prove the following lemma, which ensures that \(|aZ_t(n - J - 1)|\) has a good chance of being big. This will be crucial to proving that \(|Z_t(2)|\) has a good chance of being big as well, forcing the eigenvalue \( \cos \left( \frac{2\pi Z_t(2)}{m} \right) \) to be bounded away from one.

**Lemma 26.** Let \( Z_t \) be the the last column of \( X_t \). There is an absolute constant \( K \) such that

\[
\mathbb{P}\left( |aZ_t(n - J - 1)| > me^{-K(\log m)^2/3} \right) \geq e^{-(\log m)^{1/3}},
\]

for any \( t \in [6r^{-1}2^jp^{2/J}, A_{J,h,m}] \) and \( a \in \{1, \ldots, m - 1\} \).

To prove Lemma 26, we will need the following lemma concerning \( Z_t \) over \( \mathbb{Z} \).

**Lemma 27.** Let \( W_t = X_t e_{n-1} \) be the column process over \( \mathbb{Z} \) which starts at \((0, \ldots, 1)^T\). Let \( t \in [0, t_{n,m}] \). Then, we have that

\[
\mathbb{P}\left( \max_{0 \leq i \leq k} \{|W_t(n - i)|\} \leq \frac{\ell^{k/2}(2\log h)^{k/2}}{k} \right) \geq \left( 1 - \frac{3}{h} \right)^k,
\]

for \( k \leq J + 1 \leq n - 1 \).
Proof of Lemma 27. We prove the result by induction on \( k \). Denote the event
\[
\mathcal{A}_k = \left\{ \max_{t \leq T, i \leq k} \{|W_t(n-i)|\} \leq \frac{\tau}{(2 \log h)^{k/2}} \right\}.
\]
For \( k = 0 \), clearly \( \mathbb{P}(\mathcal{A}_0) = 1 \). We take as our assumption hypothesis the assumption \( \mathbb{P}(\mathcal{A}_k) \geq (1 - \frac{3}{h})^k \).

Let \( R(t) \) be the number of times the \( n-k \) clock rings by time \( t \) and let \( s_1 \leq s_2 \leq \ldots \leq s_{R(t)} \leq t \) be the times that the \( n-k \) clock rings. The tails of a Poisson distribution with mean \( t \) give us that \( \mathbb{P}(R(t) \geq 2^{t/3}) \leq (1.2)^{-t/3} \).

Set \( \mathcal{M}_t := W_t(n-k-1) = \sum_{i=1}^{R(t)} a_i W_{s_i}(n-k) \), where the \( \{a_i \in \{0, \pm 1\}\} \) are the random signs, and note that it is a martingale. Let \( \tau = \inf\{t : |\mathcal{M}_t| \geq \frac{\tau}{(2 \log h)^{(k+1)/2}}\} \). Then \( \mathcal{M}_{\tau \wedge \frac{\tau}{2}} \) is also a martingale. The Azuma-Hoeffding inequality gives that
\[
\mathbb{P}\left(|\mathcal{M}_{\tau \wedge \frac{\tau}{2}}| \geq \frac{\tau}{(2 \log h)^{(k+1)/2}} | \mathcal{A}_k \right) \leq 2e^{-\frac{\tau}{2(2 \log h)^{(k+1)/2}}} = \frac{2}{h}.
\]
This gives
\[
\mathbb{P}(\mathcal{A}_{k+1} | \mathcal{A}_k) \leq \mathbb{P}
\left( R(T) \geq \frac{2t}{3} \right) + \mathbb{P}\left(|\mathcal{M}_{\tau \wedge \frac{\tau}{2}}| \geq \frac{\tau}{(2 \log h)^{(k+1)/2}} | \mathcal{A}_k \right) \leq \frac{3}{h},
\]
since the choice of \( T, h \) and \( J \) give that \((1.2)^{-T/3} \leq \frac{1}{h}\). Therefore,
\[
\mathbb{P}(\mathcal{A}_{k+1}) \geq \left(1 - \frac{3}{h}\right) \mathbb{P}(\mathcal{A}_k),
\]
which gives the desired result.

Let \( T \) be the first time that there is a \( j \leq n-J-1 \) which satisfies \(|W_t(j)| > m/6\). Setting \( \tilde{t} = A_{J,h,m} := m^{2/J}/(6^{2/J}2 \log h) \) and \( k = J+1 \), Lemma 27 says that
\[
\mathbb{P}(T > A_{J,h,m}) \geq \left(1 - \frac{3}{20(J+1)}\right)^{J+1} \geq e^{-\frac{3}{5}} \geq \frac{4}{5}.
\]

Let
\[
\theta_k(t) := \max_{A \subseteq \mathbb{Z}/m\mathbb{Z}} \left\{ \mathbb{P}(Z_t(n-I) \in A) \right\}.
\]
The following lemma uses Lemma 22 to give bounds for $\tilde{Z}_t$ over $\mathbb{Z}/p\mathbb{Z}$ and argues (among other things) that $Z_t(n - I)$ has a good chance of being bounded away from zero for appropriate values of $t$. Recall that $p$ is a prime such that $\frac{1}{20} A_{I,h,m} \leq 6r^{-1/2}p^{2/J} \leq \frac{1}{2} A_{I,h,m}.$

**Lemma 28.** For $t \in [6r^{-1/2}p^{2/J}, A_{I,h,m}]$, we have

$$\theta_{p/3}(t) \leq 3/5.$$ 

**Proof.** Let $\tilde{Z}_t$ be the process over $\mathbb{Z}/p\mathbb{Z}$ and let

$$\tilde{\theta}_k(t) := \max_{A \subseteq \mathbb{Z}/p\mathbb{Z} \mid |A| \leq k} \left\{ \mathbb{P}\left( \tilde{Z}_t(n - J - 1) \in A \right) \right\}.$$ 

Recall that $W_t$ is the column process over $\mathbb{Z}$, just as in Lemma 27. If $T > t$ then $W_t(n - J - 1)$ has not left $[-m/6, m/6]$. This allows us to couple $Z_t(n - J - 1)$ with $W_t(n - J - 1)$. Let $A \subseteq \mathbb{Z}/m\mathbb{Z}$ be a set of magnitude at most $p/3$ where $\theta_{p/3}(t)$ is realized at. The coupling of $Z_t(n - J - 1)$ with $W_t(n - J - 1)$ allows us to view $A$ as a subset of $\mathbb{Z}$. Projecting $A$ to $\mathbb{Z}/p\mathbb{Z}$ gives that $\mathbb{P}(Z_t \in A, T > t) \leq \mathbb{P}\left( \tilde{Z}_t \in B \right)$, where $B \subseteq \mathbb{Z}/p\mathbb{Z}$ is the projection of $A$. To sum up,

$$\theta_{p/3}(t) \leq \tilde{\theta}_{p/3}(t) + \mathbb{P}(T \leq t).$$

Equation (31) gives

$$\theta_{p/3}(t) \leq \tilde{\theta}_{p/3}(t) + \frac{1}{5}.$$ 

We will now prove that $\tilde{\theta}_{p/3}(t) < 2/5$. If we instead assume that $\tilde{\theta}_{p/3}(t) \geq 2/5$ then there is a set $A \subseteq \mathbb{Z}/p\mathbb{Z}$ with $|A| \leq p/3$ such that

$$\mathbb{P}\left( \tilde{Z}_t(n - J - 1) \in A \right) - \pi_p(A) \geq 1/15,$$

where $\pi_p$ is the uniform measure over $\mathbb{Z}/p\mathbb{Z}$. This implies that

$$d_{T.V.}(\tilde{Z}_t(n - J - 1), \pi_p) \geq 1/15. \quad (32)$$

However, $t \geq 6r^{-1/2}p^{2/J}$, and therefore Lemma 22 gives that

$$d_{T.V.}(\tilde{Z}_t(n - J - 1), \pi_p) \leq e^{-3}. \quad (33)$$

Equation (33) contradicts (32). Therefore, $\tilde{\theta}_{p/3}(t) < 2/5$, and

$$\theta_{p/3}(t) \leq \tilde{\theta}_{p/3}(t) + 1/5 \leq 3/5,$$

which finishes the proof.
The following corollary uses the definition of $p$ to quantify how big $Z_t(n - J - 1)$ can be.

**Corollary 29.** For $t \in [6r^{-1}2^Jp^{2/J}, A_{J,h,m}]$, we have that there is a universal constant $K$ such that

$$\mathbb{P}\left(|Z_t(n - J - 1)| > me^{-K(\log m)^{2/3}}\right) \geq 2/5.$$

**Proof.** Lemma 28 gives that

$$\mathbb{P}\left(|Z_t(n - J - 1)| > p/6\right) \geq 2/5.$$

The fact that $\frac{1}{20}A_{J,h,m} \leq 6r^{-1}2^Jp^{2/J}$ gives that

$$\frac{p}{6} \geq me^{-K(\log m)^{2/3}},$$

where $K$ is a universal constant. This completes the proof.

We are now ready to prove Lemma 26.

**Proof of Lemma 26.** Let $a \in \mathbb{Z}$ and let

$$\theta_k^a(t) := \max_{\mathcal{A} \subset \mathbb{Z}/m'\mathbb{Z}} \{|aZ_t(n - J - 1)| \in \mathcal{A}\}.$$

Let $g = \gcd(a, m)$. If $g = 1$ then $\theta_k^a(t) = \theta_k(t)$ and the statement follows from Corollary 29.

If $g \neq 1$ then let $m' = m/g$, $d' = a/g$, $J' = \min\{(\log m')^{1/3}, n - 1\}$ and $h' = 20(J' + 1)$. Then we can view $Z_t := aZ_t(n - J - 1)$ as a process over $\mathbb{Z}/m'\mathbb{Z}$.

Let $s$ be a suitably chosen universal constant. If $m' \geq s$, then let $p'$ be such that $\frac{1}{20}A_{J', h', m'} \leq 6r^{-1}2^J(p')^{2/J} \leq \frac{1}{2}A_{J', h', m'}$. We therefore have that there exists a constant $\tilde{r}$, that does not depend on $m'$, such that $p' \geq 6^{-1}\tilde{r}^{J'}/2^{-J'/2}(h')^{-J'/2}(\log h')^{-J'/2}m'$. Corollary 29 says that

$$\mathbb{P}\left(|aZ_t| > p'/6\right) \geq 2/5,$$

for $t \in [6\tilde{r}^{-1}2^{J'}p^{2/J'}, A_{J', h', m'}]$. Therefore

$$\mathbb{P}\left(|aZ_t(n - J' - 1)| > gp'/6\right) \geq 2/5.$$
Using the fact that \( gp' \geq 6^{-1} p^{2J'/2} 2^{-(J')^2/2} h^{-J'/2} (\log h')^{-J'/2} m \) we get
\[
\mathbb{P}\left( |aZ_t(n - J' - 1)| > 6^{-1} p^{2J'/2} 2^{-(J')^2/2} h^{-J'/2} (\log h')^{-J'/2} m \right) \geq 2/5,
\]
for \( t \in [6p^{-2J'} h', A_{J',h,m'}] \). The choice of \( J' \) and \( h' \) give
\[
e^{-K(\log m)^2/3} \leq 6^{-1} p^{2J'/2} 2^{-(J')^2/2} h^{-J'/2} (\log h')^{-J'/2},
\]
for a suitable constant \( K \). This implies that
\[
\mathbb{P}\left( |aZ_t(n - J' - 1)| > me^{-K(\log m)^2/3} \right) \geq 2/5,
\]
for \( t \in [6p^{-2J'} h', A_{J',h,m'}] \).

We are going to investigate what happens for all times \( t \in [A_{J',h,m'}, A_{J,h,m}] \). Consider the event \( C \) that in the time interval \([A_{J',h,m'}, A_{J,h,m}]\) there are times \( c_{n-J'-1} < \ldots < c_{n-J-2} \) during which clocks \( n - J' - 1, \ldots, n - J - 2 \) ring respectively. The expected time it takes to see such a sequence of updates is at most \( J' - J \). Markov’s inequality gives \( \mathbb{P}(C) \geq 1 - e^{-(\log m)^2/3} \).

Let \( C_{n-J'-1} \) be the first time after \( A_{J',h,m'} \) that the \( n - J' - 1 \) clock rings. Let \( i \in \{n - J', \ldots, n - J - 2\} \) and let \( C_i = \inf\{t > C_{i-1} : \text{clock } n - i \text{ rings}\} \). We now consider the event \( D_i = \{\text{clock } i \text{ does not ring in } [C_i, C_{i+1}]\} \). Let \( D = \cap D_i \).

Then \( \mathbb{P}(D|C) \geq 2^{J'-J} \). If \( |aZ_t(n - J' - 1)| > me^{-K(\log m)^2/3} \) then at least one among adding or subtracting \( |aZ_t(n - J' - 1)| \) to \( |aZ_t(n - J - 1)| \) will result in \( |aZ_t(n - J - 1)| > m/6 \). Therefore, for \( t \in [A_{J',h,m'}, A_{J,h,m}] \)
\[
\mathbb{P}(|aZ_t(n - J - 1)| > m/6) \geq 2^{J'-J-1} (1 - e^{-(\log m)^2/3}) \geq 2^{-(\log m)^1/3} (1 - e^{-(\log m)^2/3}).
\]

If \( m' \leq s \), then the walk \( aZ_t(n - J - 1) \) on \( \mathbb{Z}/m'\mathbb{Z} \) mixes in a bounded number of steps. Similarly to before, we have
\[
\mathbb{P}\left( |aZ_t(n - J - 1)| > 6^{-1} p^{2J'/2} 2^{-J^2/2} h^{-J/2} (\log h)^{-J/2} m \right) \geq 2/5.
\]
Using the fact that there is a universal constant \( K \) such that
\[
6^{-1} p^{2J'/2} 2^{-J^2/2} h^{-J/2} (\log h)^{-J/2} m \geq me^{-K(\log m)^2/3},
\]
we conclude the proof. \( \square \)
4.1 The eigenvalues $y \in W_I$

Recall that $W_I = (\mathbb{Z}/m\mathbb{Z})^{n-1} \setminus \{e_1, \ldots, e_{I-1}\}$. We are now going to consider the decomposition proved in Corollary 20. For the definition of the $(y, I)$–good intervals, we are going to consider $I = \min\{J + 3, n - 1\}$, where $J = [(\log m)^{1/3}]$. Recall that $A_{J,h,m} = (m/6)^2/2(2 \log h)$. Let $L_1 = A_{J,h,m}$ be the length of each $(y, I)$–good interval. Recall that $G_j^y$ is the event $\{[t_j, t_{j+1}] \in D_j^y\}$, i.e. that the interval $[t_j, t_{j+1}]$ is $(y, I)$–good.

The following lemma is one of the main tools for proving Proposition 7.

**Lemma 30.** For $y \in W_I$, we have

$$\mathbb{P}\left(|Z_y^t(2)| \geq me^{-K(\log m)^{2/3}} \middle| \mathcal{F}_j^*\right) \geq \frac{1}{4}e^{-(\log m)^{1/3}}1_{G_j^y},$$

(34)

for every $t \geq t_{j+1}$.

**Proof.** Using the decomposition

$$Z_y^t(2) = [Y_0,tZ_y^0](2) + \sum_{k=1}^{\ell-1} a_k[Y_{s_k,t}E(I-1, I)Y_{0,s_k}Z_y^0](2),$$

as presented in Corollary 20, we notice that $N_t = [Y_0,tE(I-1, I)Y_{0,s_1}Z_y^0](2)$ has the form $aZ_t(N - J - 1)$ for $N = J + 2$ and $a = [Y_0,s_1]Z_y^0(I)$.

Set

$$Z = [Y_0,tZ_y^0](2) + \sum_{k=1}^{\ell-1} a_k[Y_{s_k,t}E(I-1, I)Y_{0,s_k}Z_y^0](2).$$

If $|N_t - \frac{m}{2}| \leq me^{-K(\log m)^{2/3}}$ and $|Z| \geq me^{-K(\log m)^{2/3}}$ then the event $\{a_{s_1} = 0\}$ implies that $|Z_y^t(2)| \geq me^{-K(\log m)^{2/3}}$. This happens with probability at least $\frac{1}{2}1_{G_j^y}\mathbb{P}(t_j \leq s_{j+1} \leq t_{j+1})$.

If $|N_t - \frac{m}{2}| \leq me^{-K(\log m)^{2/3}}$ and $|Z| \leq me^{-K(\log m)^{2/3}}$ then the event $\{a_{s_1} = \pm 1\}$ implies that $|Z_y^t(2)| \geq me^{-K(\log m)^{2/3}}$. This happens with probability at least $\frac{1}{2}1_{G_j^y}\mathbb{P}(t_j \leq s_{j+1} \leq t_{j+1})$.

Otherwise, by our choice of $L$ and $J$, Lemma 20 gives that

$$\mathbb{P}\left(|2N_t| \geq me^{-K(\log m)^{2/3}} \middle| \mathcal{F}_j^*\right) \geq e^{-(\log m)^{1/3}}1_{G_j^y}\mathbb{P}(t_j \leq s_{j+1} \leq t_{j+1}).$$

In this case, the claim follows from Lemma 21. 

$\square$
The following lemma proves Lemma 17 for $y \in W_I$.

**Lemma 31.** Let $B_{j,y}$ denote the event

$$B_{j,y} := \left\{ \int_{t_{j+1}}^{t_{j+2}} 1_{\{|Z_{y}^{s}(2)| > m e^{-K(\log m)^{2/3}}\}} ds \geq \frac{1}{16} e^{-2(\log m)^{1/3}} L \right\}.$$

For $y \in W_I$, we have that

$$\mathbb{P}(B_{j,y} \mid \mathcal{F}_{j}^{*}) \geq \frac{1}{5} e^{-(\log m)^{1/3}} 1_{\mathcal{G}_{j}^{y}},$$

for every $j \geq 1$.

**Proof.** Let $R_j$ count the length of $s \in [t_{j+1}, t_{j+2}]$, such that $|Z_{y}^{s}(2)| \leq m e^{-K(\log m)^{2/3}}$. So $R_j = \int_{t_{j+1}}^{t_{j+2}} 1_{\{|Z_{y}^{s}(2)| \leq m e^{-K(\log m)^{2/3}}\}} ds$. Lemma 30 gives that

$$\mathbb{E}(R_j \mid \mathcal{F}_{j}^{*}) \leq \left(1 - \frac{1}{4} e^{-(\log m)^{1/3}} 1_{\mathcal{G}_{j}^{y}}\right) L_1.$$

Markov’s inequality then gives

$$1_{\mathcal{G}_{j}^{y}} \mathbb{P}\left(R_j > \left(1 - \frac{1}{16} e^{-2(\log m)^{1/3}}\right) L_1 \mid \mathcal{F}_{j}^{*}\right) \leq \frac{1}{1 + \frac{1}{4} e^{-(\log m)^{1/3}} 1_{\mathcal{G}_{j}^{y}}}.$$

The constant $S$ is chosen so that

$$\mathbb{P}(B_{j,y} \mid \mathcal{F}_{j}^{*}) \geq \mathbb{P}\left(R_j \leq \frac{1}{16} e^{-2(\log m)^{1/3}} L_1 \mid \mathcal{F}_{j}^{*}\right) \geq \frac{1}{3} e^{-(\log m)^{1/3}} 1_{\mathcal{G}_{j}^{y}}$$

$$\geq \frac{1}{1 + \frac{1}{4} e^{-(\log m)^{1/3}} 1_{\mathcal{G}_{j}^{y}}} 1_{\mathcal{G}_{j}^{y}}. \quad \square$$

For the next lemma, we set $A = 16 e^{2(\log m)^{1/3}}$. The following lemma makes use of the bound that Lemma 18 provides for $y \in W_I$. Recall that $P^t$ is the point process of clock rings of the second and $A_{y,x}^{t} = \int_{0}^{t} 1_{\{|Z_{y}^{s}(2)| > x\}} dP^{s}$.

**Lemma 32.** Let $x = m e^{-K(\log m)^{2/3}}$. For $y \in W_I$, letting

$$B_{y} = \left\{ A_{y,x}^{T_{n,m}} \geq (400 A)^{-1} t_{n,m} \right\},$$

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we have
\[ P(B_y) \geq 1 - \frac{b}{mRn} e^{-c}, \]
where \( b \) is a suitable constant and \( R > 1 \).

**Proof.** Recalling Lemma 17 and applying Lemma 31 we can take \( \zeta = \frac{1}{n} e^{-(\log m)^{1/3}} \).
We have that there is a constant \( R > 1 \) such that \( \zeta^t \geq Rn \log m + c \). Lemmas 18 and 31 give the desired result.

### 4.2 The eigenvalues \( y \in Q_I \)

Recall that \( Q_I = \langle e_1, \ldots, e_{I-1} \rangle \setminus \langle e_1, e_2 \rangle \). Here we will use the fact that \( J = 3 \) and \( p \geq m/\tilde{K} \), where \( \tilde{K} \) is a constant universal on \( n \) and \( m \). The proof of the following result is identical to the proof of Corollary 29.

**Corollary 33.** For \( t \in [48r^{-1}p^{2/3}, A_{3,b,m}] \), we have that there is a universal constant \( \tilde{K} \) such that
\[ P\left( |Z_t(n - 4)| > m/\tilde{K} \right) \geq 1/2. \]

We now want to bound \( |Z_y^t(2)| \). Let \( L_2 = m \) be the length of each \( (y, I) \)-good interval.

**Lemma 34.** For \( y \in Q_I \), we have that
\[ P\left( |Z_y^t(2)| \geq m/\tilde{K} \mid \mathcal{F}_j^* \right) \geq \frac{1}{8} g_j^y, \quad (35) \]
for every \( t \geq t_{j+1} \).

The proof of Lemma 34 is similar to the proof of Lemma 30. Similarly to Lemma 31, we have the following lemma.

**Lemma 35.** Let \( B_{j,y} \) denote the event that \( |Z_y^t(2)| > m/\tilde{K} \) for at least \( 1/64 \) of \( [t_{j+1}, t_{j+2}] \). For \( y \in Q_I \), we have
\[ P\left( B_{j,y} \mid \mathcal{F}_j^* \right) \geq \frac{1}{9} g_j^y, \]
for every \( j \geq 1 \).
For the next lemma, we set $A = 64$.

**Lemma 36.** Recall that $P_t$ is the indicator function that the clock of the second row rings at time $t$ and $A_{y,x}^t = \int_0^t 1\{|Z_s^y(2)|>x\}dP_s$. Let $x = m/\tilde{K}$. For $y \in Q_I$, letting $\tilde{D}_y = \{A_{y,x}^\tau \geq (400A)^{-1}T_{n,m}\}$, we have

$$\mathbb{P}(\tilde{D}_y) \geq 1 - b \frac{1}{\tau g m} e^{-c},$$

where $b, g$ are suitable constants with $g > 1$.

**Proof.** Recalling Lemma 17 and applying Lemma 35 we can take $\zeta = 1/65$. We have that there is a constant $R$ such that $\zeta^2 t_{n,m} \geq R m \log n + c$. Lemmas 18 and 35 give the desired result. \hfill \Box

### 4.3 The eigenvalues $y \in P_2$

Recall that $P_2 = \langle e_1, e_2 \rangle \setminus \langle e_1 \rangle$. For $y \in P_2$, we write $y = a e_1 + b e_2$ with $b \neq 0$. Therefore, we observe that $Z_y^t(2) = a + b S^t$, where $S^t$ is a lazy, simple random walk on the cycle $\mathbb{Z}/m\mathbb{Z}$ starting at zero. In this section, we consider the length of the intervals to be $L_3 = \lfloor \log m \rfloor$.

**Lemma 37.** Let $I \subset \mathbb{Z}/m\mathbb{Z}$ with $|I| = \lfloor \sqrt{\log m} \rfloor$. For every $y \in P_2$ and $t \geq t_{j+1}$, we have

$$\mathbb{P}(Z_y^t(2) - Z_y^{t_j}(2) \notin I) \geq 1/2.$$

**Proof.** Writing $y = a e_1 + b e_2$, we have $Z_y^t(2) - Z_y^{t_j}(2) = b(S^t - S^{t_j})$. Assume for a contradiction that

$$\mathbb{P}(b(S^t - S^{t_j}) \notin I) < 1/2. \quad (36)$$

Let $Q^t$ be the transition matrix of $b S^t$. We have

$$\|Q^{t-t_j} - \pi\|_2^2 = \sum_z \frac{1}{m} |mQ_0^{t-t_j}(z) - 1|^2$$

$$= \sum_z m(Q^{t-t_j}(z))^2 - 1$$

$$\geq \sum_{z \in I} m(Q^{t-t_j}(z))^2 - 1. \quad (37)$$
Cauchy–Schwartz leads to

\[
\frac{m}{|I|} \left( \sum_{z \in I} Q^{t-t_j}(z) \right)^2 - 1 
\]

\[
\geq \frac{m}{4|I|} - 1 \quad \text{(38)}
\]

\[
= \frac{m}{4 \sqrt{\log m}} - 1, \quad \text{(39)}
\]

where (38) occurs by applying (36). Let \( g \) be the gcd of \( b \) and \( m \). Given that \( bS^t \) can be viewed as simple random walk on \( \mathbb{Z}/g\mathbb{Z} \), we have

\[
\|Q^{t-t_j} - \pi\|_2^2 \leq \sum_{y=1}^{m/g-1} e^{-2 \left( t - \sum_{i=1}^t \cos \frac{2\pi y i}{m} \right)}
\]

\[
\leq \frac{m}{g} e^{-2t} + \frac{\sqrt{3m}}{2g \sqrt{2\pi(t - t_j)}}
\]

\[
\leq \frac{m}{g} e^{-2L_3} + \frac{\sqrt{3m}}{2g \sqrt{2\pi L_3}}, \quad \text{(40)}
\]

where (40) is a straightforward application of Lemma 8. Equation (41) contradicts (39) for a suitable choice of the constant \( \delta_1 \) and this completes the proof.

This implies the following corollary.

**Corollary 38.** For every \( y \in P_2 \) and \( t \geq t_{j+1} \), we have that

\[
\mathbb{P} \left( |Z_y^t(2)| \geq \sqrt{\log m/2} \right) \geq 1/4.
\]

We follow the reasoning of the previous section to conclude the following lemmas.

**Lemma 39.** For \( y \in P_2 \), let \( B_{j,y} \) denote the event that \( |Z_y^*| > \sqrt{\log m/2} \) for at least one third of \( [t_{j+1}, t_{j+2}] \). For \( y \in P_2 \), we have that

\[
\mathbb{P} \left( B_{j,y} \mid F_j^* \right) \geq \frac{1}{2},
\]

for every \( j \geq 1 \).
For the next lemma, we set $A = 3$.

**Lemma 40.** Recall that $P^t$ is the indicator function that the clock of the second row rings at time $t$ and $A^t_{y,x} = \int_0^t \mathbf{1}_{|Z^y_2(2)| > x} dP^s$ and let $x = \sqrt{\log m}/2$. For $y \in P_2$, letting $\tilde{D}_y = \{A^t_{y,x} \geq (400A)^{-1} \tilde{t}_{n,m}\}$, we have

$$
\mathbb{P}(\tilde{D}_y) \geq 1 - \frac{b}{n^gm}e^{-c},
$$

where $b$ and $g > 1$ are suitable constants.

**Proof.** There is a constant $g > 1$ such that $\tilde{t}_{n,m} \geq gm \log n + c$. Lemmas 18 and 31 give the desired result. 

\[\square\]

## 5 The proof of Proposition 7

**Proof of Proposition 7**. We first consider the case where $m$ is not prime. Let $x = me^{-K(\log m)^{2/3}}$, $w = m/K$ and $I = \min\{J+3, n\}$, where $J = \lfloor (\log m)^{1/3} \rfloor$. For the first statement of Proposition 7, we need to specify these indices, but it is only for the second part that we will justify these values, using Lemmas 25, 36, 40. We also consider

$$
\tilde{t}_{n,m} = \gamma(m^2 n \log n + n^2 e^{(\log m)^{2/3}} + cn^2 \log \log n),
$$

which satisfies

$$
\tilde{t}_{n,m} \geq \gamma n^2 L e^{K(\log m)^{2/3}} \log m + cn^2 \log \log n \quad (42)
$$

and

$$
\tilde{t}_{n,m} \geq \gamma (\log m)^{4/3} + cn^2 \log \log n. \quad (43)
$$

Recall the definition of the event $E_{\tilde{t}_{n,m},xy}$ from Definition 5. Using the classical $\ell^2$ bound as described in (3), given that $k, w_1, \ldots, w_k$ are such so that $E_{\tilde{t}_{n,m},xy}$ is satisfied, we have that for every $y \in W_I$ it is the case that $|Z^t_y(2)| \geq me^{-K(\log m)^{2/3}}$ for at least $\frac{1}{6400}e^{-2(\log m)^{1/3}\tilde{t}_{n,m}}$. Then, (4) says that

$$
4\|q_{\xi,-u}\|_{\ell^2} \cdot 1_{E_{\tilde{t}_{n,m},I,xy}} \leq \sum_{y \neq 0} \exp\left(-2 \left(k - \sum_{i=1}^{k} \cos \left(\frac{2\pi (y^T w_i(2))}{m}\right) \right)\right).
$$
The definition of $E_{t_{n,m}, x_y}$ gives that

$$4 \| q_{k, w} - u \|_{T.V.} \cdot 1_{E_{t_{n,m}, I, x_y}} \leq \sum_{y \notin 0 \atop y \in \langle e_1 \rangle} e^{-2(k - \sum_{i=1}^{k} \cos \frac{2\pi y}{m})} + \sum_{y \in P_2} e^{-2v \tau_{n,m} \left(1 - \cos \frac{\beta \log m}{m}\right)} + \sum_{y \in W_I} e^{-v \tau_{n,m} \left(1 - \cos \frac{2\pi}{K}\right)},$$

\[(44)\]

where $v$ is a universal constant on $n, m$. Equation (4) gives that

$$\sum_{y \notin 0 \atop y \in \langle e_1 \rangle} e^{-2(k - \sum_{i=1}^{k} \cos \frac{2\pi y}{m})} \leq me^{-2k} + 2 \sum_{j=1}^{\infty} e^{-\frac{4j^2\pi^2}{m^2} k}.$$

The definition of $P_2$ gives

$$\sum_{y \in P_2} e^{-2v \tau_{n,m} \left(1 - \cos \frac{\beta \log m}{m}\right)} \leq m^2 e^{-\tau_{n,m} \frac{\log m}{m^2}}.$$

We also have

$$\sum_{y \in W_I} e^{-2e^{-v(\log m)^{1/3}} \tau_{n,m} \left(1 - \cos \left(2\pi e^{-K(\log m)^{2/3}}\right)\right)} \leq m^n e^{-C(\log m)^{1/3} \tau_{n,m} e^{-K(\log m)^{2/3}}}$$

and

$$\sum_{y \in Q_I} e^{-v \tau_{n,m} \left(1 - \cos \frac{2\pi}{K}\right)} \leq m^I e^{-K \tau_{n,m}}$$

by the definition of $Q_I$.

Using the fact that $k \geq A^{-1} \tau_{n,m}$ and putting all the above terms together, we get that there is a constant $B$ such that

$$44 \leq B \frac{1}{n(\log n)^e}.$$

\[(45)\]
Combining (2) and (46), we see that there is a universal, positive constant $D$, such that

$$\|q_{t_n,m} - u\|_{T.V.} \leq D \frac{1}{n(\log n)^c}.$$  

For $m$ prime, we make a choice of $I$ that allows us to prove a sharper result. Set $I = \min\{\lfloor \sqrt{\log m} \rfloor, n - 1\}$, $x = m/8$ and $w = m/\tilde{K}$. To prove part (a), we assume that $E_{t_n,m,x}$ is satisfied for a universal constant $A$ that will be determined later in the proof.

$$4\|q_{k,w} - u\|^2_{T.V.} \leq \sum_{\substack{y \neq 0 \\text{ or } \nmid 2\pi y \in \langle e_1 \rangle}} e^{-2(k - \sum_{i=1}^k \cos \frac{2\pi y}{m})} + \sum_{y \in P_2} e^{-2\nu T_{n,m}(1 - \cos \frac{2\pi y}{m})} + \sum_{y \notin P_2} e^{-2\nu T_{n,m}(1 - \cos \frac{2\pi y}{m})}.$$  

(46)

Equation (4) gives

$$\sum_{\substack{y \neq 0 \\text{ or } \nmid 2\pi y \in \langle e_1 \rangle}} e^{-2(k - \sum_{i=1}^k \cos \frac{2\pi y}{m})} + \sum_{y \in P_2} e^{-2\nu T_{n,m}(1 - \cos \frac{2\pi y}{m})} + \sum_{y \notin P_2} e^{-2\nu T_{n,m}(1 - \cos \frac{2\pi y}{m})} \leq me^{-2k} + 2 \sum_{j=1}^{m/4} e^{-\frac{4j^2\pi^2}{m^2}k} + m^2 e^{-2\nu T_{n,m}\frac{\nu \log m}{m}} + \sum_{y \in P_2} e^{-2\nu T_{n,m}(1 - \cos \frac{2\pi y}{m})} + \sum_{y \notin P_2} e^{-2\nu T_{n,m}(1 - \cos \frac{2\pi y}{m})}.$$  

(47)

Since $k \geq A^{-1}T_{n,m}$ and choosing $D$ to be a suitable constant, we have that there is a constant $B$ such that

$$\sum_{\substack{y \neq 0 \\text{ or } \nmid 2\pi y \in \langle e_1 \rangle}} e^{-2(k - \sum_{i=1}^k \cos \frac{2\pi y}{m})} + \sum_{y \in P_2} e^{-2\nu T_{n,m}(1 - \cos \frac{2\pi y}{m})} + \sum_{y \notin P_2} e^{-2\nu T_{n,m}(1 - \cos \frac{2\pi y}{m})} \leq B \frac{1}{n(\log n)^c}.$$  

(48)

Combining (2) and (48), there is a universal, positive constant $D$, such that

$$\|q_{t_n,m} - u\|_{T.V.} \leq D \frac{1}{n(\log n)^c}.$$  

For the second part of Proposition 7, we will only focus on the case of general $m$, since the case $m$ being prime follows the same outline. Lemma
and a union bound give

\[
\mathbb{P}\left(E^c_{t,n,m,xy}\right) \leq \mathbb{P}\left(\bigcup_{y \in W_1} B^c_y\right) + \mathbb{P}\left(\bigcup_{y \in Q_1} B^c_y\right) + \mathbb{P}\left(\bigcup_{y \in P_2} B^c_y\right)
\]
\[
\leq m^n b e^{-c} \frac{1}{m^{gm}} + m^I b e^{-c} \frac{1}{n^{gm}}. \tag{49}
\]

If \( I = \lceil (\log m)^{1/3} \rceil \leq n \) then

\[
m^I b e^{-c} \frac{1}{n^{gm}} = e^{(\log m)^{1/3} - gm \log n} \leq \frac{1}{n^{(g-1)m}}.
\]

If \( I = n \leq \lceil (\log m)^{1/3} \rceil \), then

\[
m^I b e^{-c} \frac{1}{n^{gm}} = e^{n \log m - gm \log n} \leq \frac{1}{n^{(g-1)m}}.
\]

Equation (49) becomes

\[
\mathbb{P}\left(E^c_{t,n,m,xy}\right) \leq b e^{-c} \left( \frac{1}{m^{(g-1)n}} + \frac{1}{n^{(g-1)m}} \right).
\]

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