Quantum spin Hall Majorana anti-wires

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We propose a novel realization for a topologically superconducting phase hosting Majorana zero-modes on the basis of quantum spin Hall systems. Remarkably, our proposal is completely free of ferromagnets. Instead, we confine helical edge states around a narrow defect line of finite length in a two-dimensional topological insulator. We demonstrate the formation of a new topological regime, hosting protected Majorana modes in the presence of s-wave superconductivity and Zeeman coupling. Interestingly, when the system is weakly tunnel-coupled to helical edge state reservoirs, a truly unambiguous transport signature is associated with the presence of a non-Abelian Majorana zero-mode.

Introduction.– The theoretical prediction[1–3] and experimental realization[4] of two-dimensional topological insulators marked the beginning of immense research activities in view of their functionalization in spintronics[5–7], superconducting spintronics[8, 9], and topological quantum computation[10]. In particular, the formation of topological superconductivity on the basis of topological insulating systems attracted a lot of attention [11–14]. Especially, the emergence of topologically protected Majorana bound states came to the forefront of research [2]. The interest in those excitations is not only fundamental, but also practical since the obey non-Abelian statistics [16–18] and, hence, can potentially be used for topological quantum computation.

Regarding the realization of topologically confined Majoranas on the basis of topological insulators, most proposals rely on the presence of ferromagnetic ordering [12, 19–21], which turns out to be difficult to be achieved in the laboratory.

In parallel, another very promising platform for topological superconductivity was found by the prediction of Majorana zero-modes in spin-orbit coupled quantum wires under the influence of s-wave superconductivity and Zeeman coupling [22, 23]. Subsequently, several follow-up experimental works were able to confirm some of the proposed signatures [24–26]. However, the ultimate proof of the existence of Majoranas is probably still missing.

In this Letter, we propose a hybrid structure that combines the features of topological edge states and spin-orbit coupled quantum wires. The system we investigate – a quantum spin Hall (QSH) anti-wire – defines itself through a narrow slit of a given length in an otherwise continuous two-dimensional topological insulator as sketched in Fig. 1. This system shares some similarities with QSH quantum point contacts, recently realized in the laboratory [27], for which the formation of Kramers pairs of Majorana fermions and even more complex anyons were previously proposed [28–32]. Other than in former proposals, we demonstrate that the QSH anti-wire itself possesses a topological phase hosting Majorana end-modes in the presence of s-wave pairing and Zeeman coupling. This phase emerges when the slit is narrow enough such that the edge states at different sides overlap, which in turn leads to spectral properties similar to the ones found in spin-orbit coupled quantum wires. Superconductivity and Zeeman field open competing gaps, which eventually lead to a topological phase that hosts topologically protected Majorana modes at the ends of the slit. Remarkably, when the anti-wire is brought in proximity to helical edge state reservoirs (see Fig. 1 (a)), the presence of Majorana zero-modes at the ends of the anti-wire can be unambiguously detected by a qualitative transport signature, namely the non-local conductance $G_{1\rightarrow2} = dI_2/dV_1$ between contacts 1 and 2. In the presence of a state that closely resembles a Majorana for the part that is locally coupled to the helical edge, the signal is negative, while otherwise positive. Importantly, a local state for $\epsilon \rightarrow 0$
only produces a negative signal if $\gamma^\dagger = \gamma$ is satisfied.

**Topological phase transition.** - The main conceptual ingredient of our proposal is a long quantum constriction between two metallic edges of a two dimensional topological insulator, see Fig. 1 (b). In order to compute its genuine topological properties, we first consider the limit of an infinitely long constriction. The kinetic energy can be described by the effective Hamiltonian density ($\hbar = 1$)

$$H_p = \sum_{\nu,\sigma} \hat{\psi}^\dagger_{\nu,\sigma}(x) (-i v_F \sigma \nu \partial_x - \mu) \hat{\psi}_{\nu,\sigma}(x), \quad (1)$$

where $\hat{\psi}_{\nu,\sigma}(x)$ are annihilating fermionic fields carrying a spin $\sigma \in \{\uparrow, \downarrow\} = \{+, -\}$ and edge-index $\nu \in \{1, 2\} = \{+, -\}$, while $\mu$ acts as a chemical potential. We assume the width of the structure to be small enough, that a finite overlap of wave functions from states at different sites of the anti-wire is relevant. In presence of time-reversal symmetry, there are two terms that emerge at the single particle level [30, 33–36]

$$H_{t_0} = t_0 \sum_\sigma \left[ \hat{\psi}^\dagger_{1,\sigma}(x) \hat{\psi}_{2,\sigma}(x) + \text{h.c.} \right], \quad (2)$$

$$H_{t_c} = t_c \sum_\nu \left[ \nu \hat{\psi}^\dagger_{\nu,\uparrow}(x) \hat{\psi}_{\nu,\downarrow}(x) + \text{h.c.} \right]. \quad (3)$$

While Eq. (2) describes a hybridization of fermionic states with the same spin associated to different sides of the constriction and does not require further symmetry breaking with respect to $H_p$, Eq. (3) is only finite if axial spin symmetry is absent [37]. This can, for instance, be realized by Rashba spin orbit coupling, which naturally appears due to confinement or externally applied electric fields. The spectrum associated with $H_0 = \int_{-\infty}^{+\infty} dx [H_p + H_{t_0} + H_{t_c}]$ is depicted in Fig. 2 (a). Importantly, there are two branches, each with a well-defined spin, shifted with respect to each other in momentum space. The spin in $z$ direction ceases to be a good quantum number when we additionally apply a Zeeman field described by

$$H_B = B_2 \sum_{\nu,\sigma} \sigma \hat{\psi}^\dagger_{\nu,\sigma}(x) \hat{\psi}_{\nu,\sigma}(x). \quad (4)$$

Although Eq. (3) and Eq. (4) do not lead to a spectral gap separately, their interplay is able to open a partial gap around $k = 0$ [32]. The resulting band structure shares many ingredients with spin-orbit coupled quantum wires under the influence of magnetic fields. It can be expected that topological physics emerges in those systems when s-wave superconductivity is taken into account by

$$H_\Delta = \Delta \sum_\nu \left[ \hat{\psi}^\dagger_{\nu,\uparrow}(x) \hat{\psi}^\dagger_{\nu,\downarrow}(x) + \text{h.c.} \right]. \quad (5)$$

Indeed, we find that the infinitely long anti-wire described by $H_0 + \int_{-\infty}^{+\infty} dx [H_\Delta + H_B]$ undergoes a topological phase transition, indicated by a gap-closing and reopening depending on the control parameters $\mu$ and $B_z$ (see Fig. 2 (b)).

**Topologically protected Majoranas.** - Having established the existence of a topological phase, we now focus on a slit with a finite length $L$ in order to investigate the presence of topological bound states at its two ends. To model this case, it is convenient to consider the additional Hamiltonian density

$$H_T = T \left[ \delta(x) + \delta(x - L) \right] \sum_\sigma \left[ \hat{\psi}^\dagger_{1,\sigma}(x) \hat{\psi}_{2,\sigma}(x) + \text{h.c.} \right], \quad (6)$$

which describes the presence of gapped zones close to $x = 0$ and $x = L$. Indeed, in the limit $T \to \infty$, the Hamiltonian $H_{AW} = \lim_{T \to \infty} \int_0^L dx \left[ H_p + H_{t_0} + H_{t_c} + H_B + H_\Delta + H_T \right]$ defines an isolated antiwire in the region $x \in [0, L]$, whose fermionic fields obey the boundary conditions (BCs) [38, 39]

$$\hat{\psi}_{\nu,\sigma}(x) = -i \hat{\psi}_{-\nu,\sigma}(-x), \quad (7)$$

where $\hat{\psi}_{\nu,\sigma}(x) = \sum_q \psi_{\nu,\sigma,q}(x) \hat{c}_q$ with fermions $\hat{c}_q$ and the quantization condition $q_n = (\pi / L) (n - 1/2)$.

By exploiting Eq. (20), we can eliminate half of the fermionic fields at the cost of doubling the system size.
and the emergence of non-local terms. We henceforth obtain

\[ H_{AW} = \int_{-L}^{L} dx \Phi^\dagger(x) \left[ -ivF \partial_x + \tau_z \sigma_z B_z + \tau_z \sigma_0 \mu \\
+ \tau_z \sigma_0 \text{sign}(x) L \right] \Phi(x) \\
- \int_{-L}^{L} dx \Phi^\dagger(x) \left[ \tau_z \sigma_y \Delta + i \text{sign}(x) L_0 \right] \Phi(-x), \quad (8) \]

where \( \tau_j, \sigma_j \ (j \in \{x, y, z\}) \) are Pauli matrices acting on particle-hole, spin- space, respectively, and \( \Phi(x) = (\hat{\psi}_{1,\uparrow}(x), \hat{\psi}_{1,\downarrow}(x), \hat{\psi}_{2,\uparrow}(x), \hat{\psi}_{2,\downarrow}(x))^T \). Our goal is to determine the eigenfunctions \( U_\epsilon(x) \) of the Hamiltonian density in Eq. (8). We can overcome the non-locality of Eq. (8) with the ansatz

\[ U_\epsilon(x) = u_\epsilon(x) \theta(x) + v_\epsilon(-x) \theta(-x). \quad (9) \]

From the continuity of the solutions \( U_\epsilon(x) \) at \( x = 0 \) as well as from the anti-periodicity of the system with respect to \( 2L \), the solution needs to obey the BCs \( u_\epsilon(0) = v_\epsilon(0) \) and \( u_\epsilon(L) = -v_\epsilon(-L) \). By inserting \( U_\epsilon(x) \) and \( U_\epsilon(-x) \), respectively, with the ansatz (9) in the single particle problem associated with Eq. (8), we obtain a set of equations (local in \( x \)) for the functions \( u_\epsilon(x) \) and \( v_\epsilon(x) \)

\[ \left[ -ivF \partial_x \gamma_z \tau_0 \sigma_0 + \gamma_0 \tau_z \sigma_z B_z + \gamma_0 \tau_z \sigma_0 \mu + \gamma_z \tau_z \sigma_0 L \right] \chi_\epsilon(x) = \epsilon \chi_\epsilon(x), \quad (10) \]

where we define the basis function \( \chi_\epsilon(x) = (u_\epsilon(x), v_\epsilon(x))^T \) and the Pauli-matrices \( \gamma_j \) acting on the space spanned by \( u_\epsilon(x) \) and \( v_\epsilon(x) \). The general solution of (10) can be found by integration

\[ \chi_\epsilon(x) = M_\epsilon(x, x_0) \chi_\epsilon(x_0), \quad (11) \]

where \( M_\epsilon(x, x_0) = \exp \left[ \int_{x_0}^{x} dx' \frac{i}{v_F} \gamma_z \tau_0 \sigma_0 (\epsilon - (\gamma_0 \tau_z \sigma_z B_z + \gamma_0 \tau_z \sigma_0 \mu + \gamma_z \tau_z \sigma_0 L - \gamma_z \tau_z \sigma_0 L_0)) \right] \). Enforced by the anti-periodicity of the solution with respect to \( 2L \), not every energy \( \epsilon \) is compatible with the BCs. For the topological phase, however, we know that in the limit \( L \to \infty \) there should be a solution for \( \epsilon \to 0 \) of the form \( \Gamma(0) = (\zeta(0), \zeta(0)) \) (fulfilling the BCs at \( x = 0 \)). Moreover, in the limit \( L \to \infty \), normalizable solutions have to decay as \( x \to \infty \). Thus, in this limit, Eq. (11) turns into an eigenvalue problem for \( \zeta(0) \) of the form

\[ \lim_{L \to \infty} M_\epsilon(L, 0) \Gamma(0) = 0. \quad (12) \]

If we further demand the solution to be a Majorana, we require for \( L \to \infty \), \( \zeta(0) = (f(0), g(0), f^*(0), g^*(0))^T \).

For finite \( L \), Eq. (12) does not hold anymore, however, we find that an approximate Majorana solution exists, i.e. Eq. (12) possesses an eigenvalue \( \lambda_M \sim \exp(-\alpha L) \) where the corresponding eigenvector \( \nu_{\lambda_M} \) deviates by \( \delta \Gamma_{\lambda_M} = \frac{1}{2} |\gamma_0(\mathbb{1} - \tau_z \sigma_0)\text{Re}[\nu_{\lambda_M}] + \gamma_0(\mathbb{1} + \tau_z \sigma_0)\text{Im}[\nu_{\lambda_M}]| \sim \exp(-\beta L) \) ([\( \alpha, \beta \in \mathbb{R} \)]) from the Majorana form (Fig. 3 (a)).

**Transport.** - When the anti-wire is brought close to other boundaries of the sample, as depicted in Fig. 1 (a), a weak coupling between outer helical edge states and the anti-wire develops. The system is therefore open and transport experiments can be performed. Remarkably, we demonstrate below that a qualitative transport signature can be directly connected to the presence of Majorana zero-modes in the anti-wire.

In order to model the open system sketched in Fig. 1 (a), the only modification to be done is to keep the amplitude \( T \) of the boundary term in Eq. (6) finite. The Hamiltonian of the whole system then reads \( H_{open} \equiv \int_{-\infty}^{\infty} dx [H_p + H_T] + \int_{0}^{L} dx [H_{int} + H_{A} + H_{\Delta} + H_{B}] \), where the kinetic terms for \( x < 0 \) and \( x > L \) describe the two outer helical edges.

We discuss two distinct transport regimes. The first one is the two-terminal conductance, obtained when the contact pair 1 and 2 of Fig. 1, as well as 3 and 4, are each coupled to one reservoir with local chemical potentials \( \mu_{12} \) and \( \mu_{34} \). For small voltage differences \( \epsilon V \sim \mu_{12} - \mu_{34} \), the two-terminal conductance can be calculated in terms of elements of the corresponding scattering matrix \([40] \)

\[ G_{12\rightarrow34} = \frac{d(I_1 + I_2)}{dV} = 1 + \sum_{j=1,2} \left[ |r_{j}^{ev}|^2 - |r_{j}^{eh}|^2 \right], \quad (13) \]

where \( r_{j}^{ev} \) are normal (\( \nu = e \)) and Andreev reflection amplitudes (\( \nu = h \)) into contact \( j \). The elements of the scattering matrix can be numerically computed by integration of \( H_{open} \). Fig. 4 (a) shows the two-terminal conductance according to Eq. (13) as a function of excitation energy \( \epsilon \) and applied Zeeman field \( B_z \). Whenever an anti-wire bound state is on resonance, a peak in the two-terminal conductance emerges. Even though the Majorana solution is clearly visible as a zero-energy peak, this signature is not sufficient as a proof for the associated bound state to be a Majorana \([41–43] \). Indeed, for that proof, a signature exclusively attached to the Majorana is required. Remarkably, such a signature exists in the proposed system and it is given by the multi-terminal conductance between contacts 1 and 2 (see Fig. 1 (a))

\[ G_{1\rightarrow2} = \frac{dI_2}{dV_1} = |r_{2}^{ev}|^2 - |r_{2}^{eh}|^2, \quad (14) \]

Depending on which scattering mechanism is dominant, \( G_{1\rightarrow2} \) can take either positive or negative values. Below, we argue that a negative signal at zero energy can be unambiguously associated with the presence of a Majorana bound state at the end of the anti-wire.

Figs. 4 (b-c) nicely show that, when the anti-wire is in the topological phase and features Majoranas at its ends, the multi-terminal conductance \( G_{1\rightarrow2} \) at zero-energy is indeed negative. Moreover, Fig. 4 (c) shows that the
negative signal (highlighted in red) can not be assigned to any state of the anti-wire, but is prominently seen at zero energy. There are, however, also isolated scattering events at non-zero energy with negative multi-terminal conductance. To further clarify the meaning of a negative multi-terminal conductance, we resort to a toy model which allows us to study the generic coupling between a single helical edge and a system hosting Majoranas.

Toy model—At first, we prove that the coupling between a single helical edge and a single Majorana mode (with a non-trivial spin texture) always leads to a negative multi-terminal conductance. To this end, we study the toy model depicted in Fig. 5 (a). The single helical edge, described by the Hamiltonian density \( \mathcal{H}_p \) (with \( \nu = 1 \)), is locally coupled to a single Majorana mode, say \( \gamma_1 = (1/\sqrt{2})(d + d^\dagger) \), via the tunneling Hamiltonian (at \( x = 0 \)) \( \mathcal{H}_c = \sum_{\sigma} t_{\sigma} [\gamma_1 \psi_{1\sigma} (0) + h.c.] \). We consider spin-dependent coupling constants \( t_{\sigma} \) to take into account the spin-texture of the Majorana mode \([44, 45]\). Out of the two fermionic operators \( d \) and \( d^\dagger \), one can obtain a second Majorana mode \( \gamma_2 = i/\sqrt{2} (d - d^\dagger) \). While \( \gamma_2 \) is not directly coupled to the helical edge, it can (weakly) hybridize with the first Majorana via \( H_d = -i e_d \gamma_1 \gamma_2 \). To determine the transport properties according to Eq. (14), we need to compute the scattering matrix of the whole system, which is explicitly done in the supplementary material [1, 47]. For the case \( e_d = 0 \), we obtain

\[
\begin{align*}
t_{22}^a &= \frac{t_{12}^2}{t_{12}^2 + t_{14}^2 - iv_F \epsilon}, \\
t_{22}^b &= \frac{-t_{21}^2}{t_{21}^2 + t_{24}^2 - iv_F \epsilon},
\end{align*}
\]  

where \( \epsilon \) is the energy at which the scattering process takes place. For sufficiently small \( \epsilon \), we find that \( t_{22}^a > t_{22}^b \) implies a negative conductance \( G_{1\rightarrow 2} < 0 \). In the opposite case \( t_{22} > t_{22}^b \), one has \( G_{1\rightarrow 2} > 0 \) but \( G_{2\rightarrow 1} < 0 \) [48]. Hence, as long as the Majorana possesses a spin texture which is not polarized perpendicular to the spin quantization axis [49], we find one of the two non-local conductances \( G_{1\rightarrow 2} \) or \( G_{2\rightarrow 1} \) to be negative. This is confirmed by Fig. 5 (b), which shows \( G_{1\rightarrow 2} \) for \( t_{12} = 1.2 t_4 \). Without hybridization (blue line) the negative signal is centered around the Majorana energy \( \epsilon = \epsilon_d = 0 \), where the width of the dip is controlled by the magnitude of the coupling constants. Interestingly, even in presence of a finite hybridization energy \( \epsilon_d > 0 \) (orange line), the negative conductance values are still present and centered around \( \epsilon = \pm \epsilon_d \).

Finally, we prove that a negative signal at zero-energy in the non-local conductance represents an unambiguous signature of the tunneling into a Majorana mode. To do so, we consider a more generic coupling Hamilto-
nian, which models the local tunneling between the helical edge and a region of a generic particle-hole symmetric (PHS) system, featuring eigenstates at energies $\pm \epsilon_\alpha$. Instead of the Majorana operator $\gamma_1$, the new coupling Hamiltonian $\hat{H}_e$ will then contain a more general operator $\hat{\gamma}_1 \rightarrow \chi = \frac{1}{2} (\cos(\xi) \hat{d}_a + \sin(\xi) \hat{d}_a^\dagger)$ [48]. The parameter $N$, which renormalizes the coupling constants, characterizes which fraction of the eigenfunction is localized in the tunneling region, while $\xi$ controls whether such a fraction is more electron- or hole-like. The results of the conductance $G_{1\rightarrow 2}$ are shown in Fig. 5 (c). Remarkably, a negative conductance only appears when the helical edge couples to a mixture of electron and holes. In particular, for $\epsilon_\alpha \rightarrow 0$, the parameter regime in which we find negative conductance becomes sharply centered around the Majorana case ($\xi = \pi/4, 3\pi/4$ up to a phase). At zero energy, the negative signal is thus an unambiguous signature of tunneling into a Majorana mode, described by an operator which satisfies $\gamma^\dagger = \gamma$.

Conclusions. - We have proposed a novel platform hosting topologically protects Majorana modes, whose existence can be detected via an unambiguous qualitative transport signature. Due to recent technological developments, the system is within experimental reach and features scalability for topological quantum computing.

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[49] In the AW, with both the Zeeman field and the spin-flip scattering, spin-directions are not preserved and hence it is not possible to have a Majorana spin-texture perpendicular to the z-axis. This is also testified by the numerical results of Fig. 4 which indeed show the presence of negative signals.
SUPPLEMENTAL MATERIAL

In this supplelementary material, we give some more details of the calculations performed in the main text of the Letter “Quantum spin Hall Majorana anti-wires”. In particular, in Sec. A, we derive the boundary conditions of the anti-wire, in Sec. B we compute the scattering matrix of the toy model, introduced in the main text, while, in Sec. C, we compare the toy model with the numerical results. Finally, in Sec. D, we discuss in detail the appearance of negative non-local conductance and its relation to Majorana excitations.

A. DERIVATION OF THE BOUNDARY CONDITIONS FOR THE QSH ANTI-WIRE

The kinetic Hamiltonian including impurity scattering at \( x = 0 \) and \( x = L \) can be written as

\[
\hat{H}_p = \int dx \sum_{\nu,\sigma} \hat{\psi}_{\nu,\sigma}^\dagger(x)(-i\nu V\sigma\partial_x)\hat{\psi}_{\nu,\sigma}(x) + T \int dx [\delta(x) + \delta(x-L)] \sum_{\sigma} [\hat{\psi}_{2,\sigma}^\dagger(x)\hat{\psi}_{2,\sigma}(x) + \text{h.c.}] \tag{16}
\]

with the fermionic fields \( \hat{\psi}_{\nu,\sigma}(x) \) annihilating a \( \nu, \sigma \) fermion at position \( x \). We can formally diagonalize the Hamiltonian (16) with eigenfunctions from integration of the associated single particle problem

\[
\hat{h}_p(x)\Psi(x) = E\Psi(x), \tag{17}
\]

where \( \hat{h}_p(x) = -i\nu V\sigma\sigma_\nu\partial_x + T [\delta(x) + \delta(x-L)] s_\sigma s_0 \) with the Pauli matrices \( s_j, \sigma_j \) acting on edge-, spin-space, respectively, and \( \Psi(x) = (\psi_{1,\uparrow}(x), \psi_{1,\downarrow}(x), \psi_{2,\uparrow}(x), \psi_{2,\downarrow}(x))^T \). In vicinity \( \delta x \) close to the impurities with \( \delta x \to 0 \), Eq. (17) is solved by

\[
\Psi(-\delta x) = e^{Ts_\nu s_\sigma}\Psi(\delta x), \quad \Psi(L+\delta x) = e^{-Ts_\nu s_\sigma}\Psi(L-\delta x). \tag{18}
\]

In the limit \( T \to \infty \), this results in the boundary conditions

\[
\psi_{1,\sigma}(0) = i\psi_{2,\sigma}(0), \quad \psi_{1,\sigma}(L) = -i\psi_{2,\sigma}(L). \tag{19}
\]

Eq. (19) is satisfied by the functions \( \psi_{\nu,\sigma,q}(x) = -i\psi_{-\nu,\sigma,q}(-x) \) with \( \psi_{\nu,\sigma,q}(x) = (1/\sqrt{L}) \exp[iq_n x] \) and the quantization condition \( q_n = (\pi/L)(n - 1/2) \). By applying an expansion of the fermionic fields in terms of the functions \( \psi_{\nu,\sigma,q}(x) \), namely \( \psi_{\nu,\sigma}(x) = \sum_q \psi_{\nu,\sigma,q}(x)\hat{c}_q \), we obtain the boundary condition for the fields to be

\[
\hat{\psi}_{\nu,\sigma}(x) = -i\hat{\psi}_{-\nu,\sigma}(-x). \tag{20}
\]

Clearly, from the quantization of \( q = q_n \), the fields need to be anti-periodic with respect to \( 2L \)

\[
\hat{\psi}_{\nu,\sigma}(L) = -\hat{\psi}_{\nu,\sigma}(-L) \tag{21}
\]

Eq. (20) is stated in the main text as Eq. (7).

B. DERIVATION OF THE SCATTERING MATRIX

The system for which we aim to construct the scattering matrix is sketched in Fig. 6. It is composed of three parts: First, the helical edge passing by the anti-wire (\( h = 1 \))

\[
H_p(\nu = 1) = \int dx \sum_{\sigma} \hat{\psi}_{\sigma}^\dagger(x)(-i\nu V\sigma\partial_x + \mu)\hat{\psi}_{\sigma}(x), \tag{22}
\]

where, in analogy to the main text, \( \hat{\psi}_{\sigma}(x) \) are annihilating fermionic fields carrying an index \( \sigma \in \{R, L\} = \{+, -\} \). Further, we assume a point-like coupling of the fields \( \hat{\psi}_{\sigma}(x) \) to a Majorana mode \( \hat{\gamma}_1 \) of the anti-wire

\[
H_c = \int dx \delta(x)\sum_{\sigma} t_{\sigma}[\hat{\psi}_{\sigma}(x) - \hat{\psi}_{\sigma}^\dagger(x)] \tag{23}
\]

with coupling constant \( t_{\sigma} \) that might depend on \( \sigma \). Since time-reversal symmetry is absent in the anti-wire, the coupling does not obey symmetry constraints. Moreover, even though hybridization of the Majoranas is exponentially suppressed in the length of the anti-wire, they might aquire a small hybridization energy

\[
H_d = -i\epsilon_d\hat{\gamma}_1\hat{\gamma}_2. \tag{24}
\]

The two Majoranas \( \hat{\gamma}_1 \) and \( \hat{\gamma}_2 \) can be rewritten in terms of fermionic operators \( \hat{d} \) and \( \hat{d}^\dagger \) with

\[
\hat{\gamma}_1 = \frac{1}{\sqrt{2}}[\hat{d} + \hat{d}^\dagger], \tag{25}
\]

\[
\hat{\gamma}_2 = \frac{i}{\sqrt{2}}[\hat{d} - \hat{d}^\dagger]. \tag{26}
\]

Using Eq. (25) and (26), \( H = H_p + H_c + H_d \) might also be represented as...
\[
H = \frac{1}{2} \int dx \, \tilde{\Psi}^\dagger(x) \begin{pmatrix}
-iv_F \partial_x & 0 & 0 & t_\uparrow(x) & t_\downarrow(x) \\
0 & +iv_F \partial_x & 0 & 0 & t_\downarrow(x) & t_\uparrow(x) \\
0 & 0 & -iv_F \partial_x & 0 & -t_\uparrow(x) & -t^\dagger_\downarrow(x) \\
0 & 0 & 0 & +iv_F \partial_x & -t_\downarrow(x) & -t_\uparrow(x) \\
t_\uparrow(x) & t_\downarrow(x) & -t_\uparrow(x) & -t_\downarrow(x) & 0 & -\epsilon_d \\
t_\downarrow(x) & t_\uparrow(x) & -t_\downarrow(x) & -t_\uparrow(x) & 0 & -\epsilon_d
\end{pmatrix} \tilde{\Psi}(x) \tag{27}
\]

with \( \tilde{\Psi}(x) = (\hat{\psi}_\uparrow(x), \hat{\psi}_\downarrow(x), \hat{\psi}_1^\dagger(x), \hat{\psi}_d^\dagger(x), \hat{d}^\dagger) \) and \( t_\sigma(x) = t_\sigma \delta(x) \). To diagonalize Eq. (27), we expand \( \tilde{\Psi}(x) \) in eigenfunctions

\[
\tilde{\Psi}(x) = \sum_{k,d} U_{k,d}(x) \chi_{k,d} \tag{28}
\]

with matrices \( U_{k,d}(x) \) and fermionic annihilation operators \( \chi_{k,d} = (\hat{c}_k, \hat{\bar{c}}_d)^T \) with \( \hat{c}_k = (c_{\uparrow,k}, c_{\downarrow,k}, c_{\uparrow,k}^\dagger, c_{\downarrow,k}^\dagger) \) and \( \hat{\bar{c}}_d = (\bar{c}_d, \bar{c}_d^\dagger) \). Inserting Eq. (28) in (27), this yields

\[
H = \frac{1}{2} \sum_{k,d} \sum_{k',d'} \chi_{k',d'} \int dx \, U_{k',d'}^\dagger(x) U_{k,d}(x) \chi_{k,d} \tag{29}
\]

where we defined

\[
\Xi(x) = \begin{pmatrix}
A(x) & \eta \delta(x) \\
\eta^\dagger \delta(x) & \epsilon_d \sigma_z
\end{pmatrix} \tag{30}
\]

with

\[
A(x) = -iv_F \partial_x \mathbb{1}_{2 \times 2} \otimes \sigma_z \tag{31}
\]

and

\[
\eta = \begin{pmatrix}
t_\uparrow(x) & t_\downarrow(x) & -t_\uparrow(x) & -t_\downarrow(x) \\
t_\downarrow(x) & t_\uparrow(x) & -t_\downarrow(x) & -t_\uparrow(x)
\end{pmatrix}^T \tag{32}
\]

When the columns of \( U_{k,d}(x) \) are formed by orthogonal eigenfunctions of \( \Xi(x) \) the problem becomes diagonal.

Hence, we need to search for functions \((\Phi_k(x), \Phi_d)\), such that

\[
\begin{pmatrix}
A(x) \Phi_k(x) + \eta \delta(x) \Phi_d \\
\eta^\dagger \Phi_k(0) + \epsilon_d \sigma_z \Phi_d
\end{pmatrix} = \epsilon \begin{pmatrix}
\Phi_k(x) \\
\Phi_d
\end{pmatrix} \tag{33}
\]

where in the second row, we performed the integration of Eq. (27) right away as it contains no differential. From Eq. (33), we obtain an equation for the solutions \( \Phi_k(x) \) by solving the second row for \( \Phi_d \) and inserting the result in the first one

\[
A(x) \Phi_k(x) + \delta(x) \eta \begin{pmatrix}
1 & 0 \\
0 & 1 + \epsilon_d
\end{pmatrix} \eta^\dagger \Phi_k(0) = \epsilon e \Phi_k(x). \tag{34}
\]

This equation might be solved in the following way [1]. When \( x \neq 0 \) the equation reduces to \( A(x) \Phi_k(x) = \epsilon \Phi_k(x) \) which is solved by plane waves. Moreover, the \( \delta \)-distribution implies a discontinuous jump of the solutions at \( x = 0 \). Hence, for \( x > 0, x < 0 \) and \( x = 0 \), the solution takes different values. This can be incorporated by the ansatz

\[
\Phi_k(x) = (\Phi_k^\uparrow(x), \Phi_k^\downarrow(x)) \tag{35}
\]

with

\[
\Phi_k^{\epsilon/h}(x) = \begin{pmatrix}
\phi^{\epsilon/h}_\uparrow + \text{sign}(x) \delta \phi^{\epsilon/h}_\uparrow e^{ikx} \\
\phi^{\epsilon/h}_\downarrow + \text{sign}(x) \delta \phi^{\epsilon/h}_\downarrow e^{-ikx}
\end{pmatrix} \tag{36}
\]

where

\[
\phi^{\epsilon/h}_\uparrow = (\phi^{\epsilon/h}_{\uparrow_+, \uparrow_-} + \phi^{\epsilon/h}_{\uparrow_-, \uparrow_+})/2, \tag{37}
\]

\[
\phi^{\epsilon/h}_\downarrow = (\phi^{\epsilon/h}_{\downarrow_+, \downarrow_-} - \phi^{\epsilon/h}_{\downarrow_-, \downarrow_+})/2. \tag{38}
\]

Integration of Eq. (34) using Eqs. (35-38), this results in

\[
-i v_F \begin{pmatrix}
\sigma_z & 0 \\
0 & \sigma_z
\end{pmatrix} \begin{pmatrix}
\phi^{\epsilon}_\uparrow - \phi^{\epsilon}_\downarrow \\
\phi^{\epsilon}_\downarrow + \phi^{\epsilon}_\uparrow
\end{pmatrix} + \frac{1}{2} \eta \begin{pmatrix}
1 & 0 \\
0 & 1 + \epsilon_d
\end{pmatrix} \eta^\dagger \begin{pmatrix}
\phi^{\epsilon}_\uparrow + \phi^{\epsilon}_\downarrow \\
\phi^{\epsilon}_\downarrow + \phi^{\epsilon}_\uparrow
\end{pmatrix} = 0. \tag{39}
\]

Eq. (39) can be reorganized such that we obtain the scattering matrix

\[
\begin{pmatrix}
\phi^{\epsilon}_\uparrow \\
\phi^{\epsilon}_\downarrow \\
\phi^{\epsilon}_\uparrow \\
\phi^{\epsilon}_\downarrow
\end{pmatrix} = S \begin{pmatrix}
\phi^{\epsilon}_\uparrow \\
\phi^{\epsilon}_\downarrow \\
\phi^{\epsilon}_\uparrow \\
\phi^{\epsilon}_\downarrow
\end{pmatrix} \tag{40}
\]
FIG. 7. Non-local conductances \( G_{1\rightarrow 2} \) (a) and \( G_{2\rightarrow 1} \) (b) as a function of energy \( \epsilon \). The parameters are the same as given in Fig. 4 of the main text. All negative values are colored in red.

with

\[
S = \begin{pmatrix} R_{--} & T_{+-} \\ T_{-+} & R_{++} \end{pmatrix}
\]

(41)

and

\[
R_{--} = \begin{pmatrix} r_{ee} & r_{eh} \\ r_{he} & r_{hh} \end{pmatrix}, \quad R_{++} = \begin{pmatrix} r_{ee} & r_{eh} \\ r_{he} & r_{hh} \end{pmatrix},
\]

(42)

For the scattering amplitudes we find

\[
R_{--} = R_{++} \quad \text{(43)}
\]

with

\[
\begin{align*}
    r_{ee} &= r_{hh} = -r_{ch} = -r_{he} \\
    t_{ee} &= t_{hh} = \frac{t_1^2 \epsilon}{\epsilon t_1^2 + t_1^2 - iv_F \epsilon + iv_F c_d^2} \\
    t_{ch} &= t_{he} = -\frac{t_1^2 \epsilon}{\epsilon t_1^2 + t_1^2 - iv_F \epsilon + iv_F c_d^2} \\
    t_{ce} &= t_{he} = -\frac{t_1^2 \epsilon}{\epsilon t_1^2 + t_1^2 - iv_F \epsilon + iv_F c_d^2} - 1,
\end{align*}
\]

(44-48)

With Eqs. (44-48), it is easy to check that the scattering matrix of Eq. (41) is unitary. The elements of Eqs. (45) and (46) are used in the main text. For ease of notation, in the main text, we set \( T_{-+} \equiv T_2 \) (and accordingly for its elements).

C. NUMERICAL VALIDATION OF THE TOY MODEL

As discussed in the main text, for \( t_1 > t_1 \) in the above model, we find a non-local conductance \( G_{1\rightarrow 2} < 0 \). Likewise, the conductance \( G_{2\rightarrow 1} \) is then expected to satisfy \( G_{2\rightarrow 1} > 0 \). We can test the model against the latter statement by numerically computing the non-local conductances \( G_{1\rightarrow 2} \) and \( G_{2\rightarrow 1} \) using the Hamiltonian \( H_{\text{open}} \), defined in the main text. The results are shown in Fig. 7. While for \( G_{1\rightarrow 2} \) there is a dominant negative signal around \( \epsilon = 0 \), for \( G_{2\rightarrow 1} \) no such signal is obtained, but instead \( G_{2\rightarrow 1} > 0 \). This confirms the validity of the employed toy model in the main text and the previous section for low energies.

D. COUPLING TO A P-WAVE SUPERCONDUCTOR

The toy model can be extended also for higher energies, when we do not only couple to an isolated Majorana, but to a spin-less p-wave superconductor, which, in the 1D case can be modeled by a Kitaev chain [2]

\[
H_d = \sum_{j=1}^{N-1} [(-t)\hat{c}_j \hat{c}_{j+1} + \Delta \hat{c}_j \hat{c}_{j+1} + \text{h.c.}] \quad \text{(49)}
\]

with fermionic fields \( \hat{c}_j \) (\( \hat{c}_j^\dagger \) annihilating (creating) a fermion at site \( j \). Then the tunneling Hamiltonian can be written as

\[
H_c = \int dx \sum_{\nu=t,\downarrow} t_\nu \hat{\psi}_\nu(x) \hat{c}_1 + \text{h.c.,} \quad \text{(50)}
\]

where the fermions of the helical edge couple to the first site of the p-wave superconductor. Repeating the calculations of Sec. 2, with Eqs. (49) and (50) instead of Eq. (23) and (24), this results in an equation for the eigenstates of the helical edge

\[
-iv_F \left( \begin{array}{c} \sigma_2 \\ 0 \end{array} \right) \left( \begin{array}{ccc} \phi_{t,+} & -\phi_{t,-} \\ \phi_{t,-} & \phi_{t,+} \end{array} \right) + \frac{1}{2} \Gamma \Gamma^\dagger I^\dagger \left( \begin{array}{ccc} \phi_{t,+} & \phi_{t,+} \\ \phi_{t,+} & \phi_{t,-} \end{array} \right) = 0,
\]

(51)

where \( G = [\epsilon - H_d]^{-1} \). \( \Gamma \) is the Hamiltonian density of the coupling Hamiltonian \( H_c \), which can be written as

\[
H_c = \int dx (\hat{\psi}_\uparrow(x), \hat{\psi}_\downarrow(x), \hat{\psi}_\uparrow(x), \hat{\psi}_\downarrow(x)) \Gamma \left( \begin{array}{c} \hat{c}_1 \\ \hat{c}_\uparrow^\dagger \\ \cdot \\ \cdot \\ \cdot \\ \hat{c}_\downarrow^\dagger \end{array} \right) \quad \text{(52)}
\]

with

\[
\Gamma = \begin{pmatrix} t_1(x) & 0 & 0 & \cdots & 0 \\ t_1(x) & 0 & 0 & \ddots & 0 \\ 0 & -t_\downarrow(x) & \ddots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{(53)}
\]
From Eq. (51) we can compute the scattering matrix for the modes $\phi^{\sigma_{+}/-\pm}_{\tau,t}$ from which we obtain the conductance $G_{1\rightarrow 2}$. The results are depicted in Fig. 8 (a). In accordance with the main text and the toy model of Sec. B, we find for the topological regime $\mu < 2|t|$ a prominent negative signal around $\epsilon = 0$, signaling the presence of the Majorana. However, even higher energy states (in particular close to $\mu = 0$) can return a negative signal.

To understand this result, we investigate again Eq. (27), which, for the present case, takes the form

$$H = \frac{1}{2} \int dx \tilde{\Psi}^\dagger (x) \left( h_p \frac{\Gamma}{\Gamma^\dagger} h_d \right) \tilde{\Psi} (x)$$

(54)

with $h_p$ and $h_d$ the Hamiltonian density of the helical edge and Kitaev chain and $\Psi(x) = (\psi_\uparrow (x), \hat{\psi}_\downarrow (x), \hat{\psi}^\dagger_\uparrow (x), \hat{\psi}^\dagger_\downarrow (x), \hat{c}_1, \hat{c}^\dagger_1, \ldots, \hat{c}_N^\dagger)$ (with Hamiltonian density of the Kitaev chain we just mean the matrix elements of Eq. (49) in the given basis devied by the system size). We can now apply a unitary transformation to Eq. (56) that diagonalizes $h_d$

$$F = \begin{pmatrix} 1 & 0 \\ 0 & U_d \end{pmatrix}$$

(55)

Then, Eq. (56) becomes

$$H = \frac{1}{2} \int dx \tilde{\Psi}^\dagger (x) \left( h_p \frac{\Gamma}{\Gamma^\dagger} U_d \Gamma U_d^\dagger \right) F\dagger F \tilde{\Psi} (x)$$

(56)

Since $U_d$ diagonalizes $h_d$, it is formed from the eigenstates of $h_d$

$$U_d = (\zeta_1, \zeta_2, \ldots, \zeta_N)$$

(57)

where $\zeta_\alpha$ are column vectors with the property $h_d \zeta_\alpha = \epsilon_\alpha \zeta_\alpha$. The transformed coupling Hamiltonian thus contains the elements of the eigenfunctions at the first site. Consequently, in a low energy approximation around an eigenenergy $\epsilon_\alpha$ of $h_d$, the coupling only happens to the first site of the corresponding eigenstate $\zeta_\alpha$ and, if we want to preserve particle-hole symmetry, to its particle-hole partner at $-\epsilon_\alpha$. $P\zeta_\alpha$ with the particle-hole operator $P = 1 \otimes \sigma_x K$, where $K$ denotes complex conjugation.

The effective Hamiltonian thus reads

$$H_\alpha = \frac{1}{2} \int dx \tilde{\Psi}^\dagger_{\alpha} (x) \left( h_p \frac{\Gamma}{\Gamma^\dagger} \epsilon_\alpha \sigma_z \right) \tilde{\Psi}_{\alpha} (x)$$

(58)

with the basis $\tilde{\Psi}_{\alpha} = (\hat{\psi}_\uparrow (x), \hat{\psi}_\downarrow (x), \hat{\psi}^\dagger_\uparrow (x), \hat{\psi}^\dagger_\downarrow (x), \hat{d}_\alpha, \hat{d}^\dagger_\alpha)$ where $\hat{d}^\dagger_\alpha$ creates a fermion at energy $\epsilon_\alpha$. The coupling matrix $\Gamma_\alpha$ is given by

$$\Gamma_\alpha = \begin{pmatrix} t_1 (x) \zeta_{\alpha,1} & t_1 (x) \zeta_{\alpha,1} & -t_1 (x) \zeta_{\alpha,2} & -t_1 (x) \zeta_{\alpha,2} \\ t_2 (x) \zeta^*_{\alpha,1} & t_2 (x) \zeta^*_{\alpha,1} & -t_2 (x) \zeta^*_{\alpha,2} & -t_2 (x) \zeta^*_{\alpha,2} \end{pmatrix}$$

(59)

As discussed in the main text, this effectively corresponds to the coupling to a particle $\chi = \zeta_\alpha \xi d_x + \zeta_\alpha^* \xi d^\dagger_x$. In particular, for $\zeta_{\alpha,1} \equiv \zeta_{\alpha,2} = 1/\sqrt{2}$, it corresponds to the toy model of Section 2.

Now, we solve the scattering problem from the effective Hamiltonian of Eq. (58) and use the result to compute the conductance $G_{1\rightarrow 2}$. The results are depicted in Fig. 8 (c-d) as a function of $\zeta_{\alpha,1}$ and $\zeta_{\alpha,2}$ which we parametrize by $\zeta_{\alpha,1} = 1/N \cos (\xi)$ and $\zeta_{\alpha,2} = 1/N \sin (\xi)$ (we omit a complex phase since the result is completely independent of that). Since $\zeta_{\alpha,1/2}$ only represent the eigenfunction at the first site of the Kitaev chain, $N$ can differ from 1. In particular, for spatially spread eigenfunctions $N \gg 1$. For this case, $N$ effectively renormalizes the coupling constants $t_1$ and $t_2$. A negative conductance $G_{1\rightarrow 2}$ (on resonance i.e. $\epsilon = \epsilon_\alpha$) is reached when $\delta \zeta = ||\zeta_{\alpha,1}|| - ||\zeta_{\alpha,2}|| < \kappa$. However, the threshold $\kappa$ is in general not particularly small and
not directly related to the magnitude of the coupling constants \( t_\uparrow \) and \( t_\downarrow \) (see Fig. 8 (d)). This explains why there are scattering events away from zero-energy (where the corresponding wavefunction is not expected to be particularly close to the Majorana form) that yield a negative non-local conductance value.

An exclusion to that is the behavior close to \( \epsilon_\alpha = 0 \). For fixed \( N, t_\uparrow \) and \( t_\downarrow \), we find \( \kappa \to 0 \) for \( \epsilon_\alpha \to 0 \). For the situation \( \epsilon_\alpha \to 0 \), the parameter regime in which we find negative conductance becomes sharply centered around the Majorana case (see Fig. 8 (c-d)). This, in turn, implies that a local \(( N \to 1 \) mid-gap state \( \epsilon_\alpha \to 0 \) that produces a negative \( G_{1 \to 2} \) has to be a Majorana excitation, i.e. it has to obey \( \gamma^\dagger = \gamma \).

We can confirm our above analysis when analyzing the situation of the side-coupled Kitaev chain. Fig. 8 (b) visualizes the (numerically) obtained values of \( \delta \zeta \) for each eigenstate (on the first site). At \( \mu = 0 \) each eigenstate of the Kitaev chain satisfies the Majorana condition at the first site. Hence, we expect to find a negative non-local conductance, which coincides with the numerical results in Fig 8 (a). Away from \( \mu = 0 \) eigenstates at \( \epsilon \neq 0 \) successively loose the Majorana condition and the dominant negative signal in the non-local conductance is as well lost for those states.

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