ALGORITHMS FOR COMPUTING WITH NILPOTENT MATRIX GROUPS OVER INFINITE DOMAINS

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ABSTRACT. We develop methods for computing with matrix groups defined over a range of infinite domains, and apply those methods to the design of algorithms for nilpotent groups. In particular, we provide a practical algorithm to test nilpotency of matrix groups over an infinite field. We also provide algorithms that answer a number of structural questions for a given nilpotent matrix group. The algorithms have been implemented in GAP and Magma.

1. INTRODUCTION

In this paper we develop a technique for computing with matrix groups defined over infinite domains, based on changing the ground domain via congruence homomorphism. This technique has proved to be very efficient in linear group theory (see, e.g., [18, Chapter 3]). It is especially useful for finitely generated linear groups (see [32, Chapter 4]), and affords a general approach to many computational problems for infinite matrix groups.

Let \( F \) be a field and let \( G \leq \text{GL}(n, F) \) be given by a finite generating set. We obtain algorithms to carry out the following tasks:

(i) testing nilpotency of \( G \);

and, if \( G \) is nilpotent,

(ii) constructing a polycyclic presentation of \( G \);

(iii) testing whether \( G \) is completely reducible, and finding a completely reducible series of \( G \)-modules;

(iv) deciding finiteness of \( G \), and calculating \( |G| \) if \( G \) is finite;

(v) finding the \( p \)-primary decomposition of \( G \), and finding all Sylow \( p \)-subgroups if \( G \) is finite.

These algorithms address standard problems in computational group theory (i, ii, iv) and computing with matrix groups (iii), and facilitate structural analysis of nilpotent linear groups (v).

Our guiding objective has been to design algorithms that cover the broadest possible range of infinite domains. However, for convenience or reasons of practicality we sometimes restrict \( F \). For example, a preliminary reduction in nilpotency testing assumes that \( F \) is perfect, and constructing polycyclic presentations requires \( F \) to be finite or an algebraic number field. The implementation and practicality of our algorithms relies on the machinery (such as polynomial factorization) that is available for computing with the various fields.

A finitely generated subgroup of \( \text{GL}(n, F) \) lies in \( \text{GL}(n, R) \) for some finitely generated subring \( R \) of \( F \). In turn, a completely reducible solvable subgroup of \( \text{GL}(n, R) \) is finitely generated ([18, Theorem 6.4, p. 111], [32, 4.10, p. 57]). Thus, our algorithms supply information not only about nilpotent subgroups of \( \text{GL}(n, F) \), but also about nilpotent subgroups of \( \text{GL}(n, R) \) for any finitely generated integral domain \( R \). Furthermore, these algorithms serve as a platform for computing in abstract finitely generated nilpotent groups, because such a group is isomorphic to a subgroup of \( \text{GL}(n, \mathbb{Z}) \) for some \( n \). A method to construct a representation over \( \mathbb{Z} \) of a finitely presented polycyclic group may be found in [25].
Nilpotency is an important group-theoretic property, and testing nilpotency is consequently one of the basic functions of any computational group theory system. We provide the first uniform and effective solution to the problem of computing with infinite nilpotent matrix groups. Our algorithms for nilpotency testing (over finite fields, and \( \mathbb{Q} \)) have been implemented as part of the GAP package ‘Nilmat’ \([11]\) (this is joint work with Bettina Eick). Previous algorithms for nilpotency testing in GAP \([22]\) and MAGMA \([9]\) sometimes fail to decide nilpotency even for small finite matrix groups, and fail for almost all infinite matrix groups. In the paper’s final subsection we give sample experimental results and details of the ‘Nilmat’ package.

This paper is a slight revision of \([16]\), which was a first step in adapting the method of finite approximation as a computational tool. We have made substantial progress since \([16]\), particularly with regard to solvable-by-finite groups, including finite and solvable as special cases: see \([17]\). Also note that a MAGMA implementation of algorithms in this paper by Eamonn O’Brien handles more kinds of domains than \([11]\).

2. RELATED RESULTS

Computing in matrix groups over an infinite domain is a relatively new area of computational group theory. Most of the algorithms in this area are concerned with classes of solvable-by-finite groups (see \([3, 4, 7, 27]\)). Solvable-by-finite groups constitute the more optimistic class of the Tits alternative. The other class consists of groups that contain a non-abelian free subgroup. For those groups, some basic computational problems, such as membership testing and construction of presentations, are undecidable (see \([7, 19, 20]\)).

Changing the ground domain is a common technique in linear group theory. It has been used by several authors for computing with matrix groups; see, e.g., \([26]\). In \([7]\), a generalization of the technique as in \([26]\) leads to a Monte-Carlo solvability testing algorithm for potentially infinite subgroups \(G\) of \(\text{GL}(n, \mathbb{Q})\). The algorithm accepts as input a finite set \(S\) of generators of \(G\), and tests solvability of \(\psi_p(G) \leq \text{GL}(n, p)\), where \(\psi_p\) is reduction modulo a prime \(p\) not dividing the denominators of the entries of the elements in \(S \cup S^{-1}\). There are only finitely many primes \(p\) such that \(\psi_p(G)\) is solvable while \(G\) is not; so a non-solvable group will be identified as solvable by the algorithm of \([7]\) with small probability.

The ideas of \([7]\) might be applied to nilpotency testing. However, the upper bound on nilpotency class for nilpotent subgroups of \(\text{GL}(n, q)\) can be much larger than the bound for nilpotent subgroups of \(\text{GL}(n, Q)\) (see \([8, 33]\)). Hence the solvability testing arguments of \([7]\) may not be efficient when applied to nilpotency testing, if one simply replaces bounds depending on derived length by bounds depending on nilpotency class.

Let \(G\) be a finitely generated matrix group over \(\mathbb{Q}\). We can test solvability of \(G\) if we can test solvability both of the kernel \(G_p\) and of the image \(\psi_p(G)\) of a reduction mod \(p\) homomorphism \(\psi_p\). Theoretical background for doing this is laid out in \([19]\), where \(G_p\) is described for solvable-by-finite \(G \leq \text{GL}(n, \mathbb{Q})\). Using those results, a deterministic algorithm for solvability testing was proposed in \([27]\). There were two main obstacles to a full implementation of the algorithm in \([27]\): solvability testing of matrix groups over a finite field, and efficient construction of \(G_p\). Practical solutions of these problems were obtained in \([3, 1]\). Specifically, \([3]\) contains a method to construct a polycyclic presentation of \(\psi_p(G)\), and thereby to test solvability of \(\psi_p(G)\). The relators of this presentation may be used to calculate generators of a subgroup of \(G\) whose normal closure is \(G_p\). Although the algorithm has bottlenecks (see \([3]\ p. 1281\)), it has been successfully implemented for solvability testing over finite and algebraic number fields (see the GAP package ‘Polenta’ \([2]\)).

The aims of \([3]\) are to test whether a finitely generated subgroup \(G\) of \(\text{GL}(n, \mathbb{Q})\) is polycyclic, and, if so, to construct a polycyclic presentation for \(G\). Those problems are solved in a subsequent publication \([4]\). This provides an avenue for testing nilpotency of \(G\): if \(G\) is not polycyclic then it...
is not nilpotent; otherwise, nilpotency of $G$ can be tested using a polycyclic presentation of $G$ (for which see [24, Section 4]).

In this paper we propose an essentially different approach to nilpotency testing, valid over a broad range of infinite domains. In contrast to [4], our algorithms do not require a priori testing of polycyclicity and computation of polycyclic presentations, and are designed directly for nilpotency testing.

We use established linear group theory, chiefly structural results for nilpotent linear groups ([31, Chapter VII], [13, 15]). Accordingly, a feature of our algorithms is that they return structural information about input nilpotent groups. A full solution of the problem of testing nilpotency over finite fields appears in [14] (as we will see, much of [14] remains valid over any field). Nilpotency testing is transferred to groups over a finite field by means of a congruence homomorphism with torsion-free kernel; see Section 3. Other methods that transfer nilpotency testing to the case of finite groups are given in Subsection 4.5.

3. Changing the ground domain via congruence homomorphism

In this section we present some results from linear group theory that comprise the theoretical foundation of our algorithms.

First we set up some notation. Let $\Delta$ be an integral domain. For any ideal $\wp \subset \Delta$, the natural surjection $\psi_\wp : \Delta \to \Delta/\wp$ extends entrywise to a matrix ring homomorphism $\text{Mat}(n, \Delta) \to \text{Mat}(n, \Delta/\wp)$, and then restricts to a group homomorphism $\text{GL}(n, \Delta) \to \text{GL}(n, \Delta/\wp)$, which we also denote $\psi_\wp$. The map $\psi_\wp$ on $\text{GL}(n, \Delta)$ is called a Minkowski or congruence homomorphism ([31, p. 65]), and its kernel is called a (principal) congruence subgroup of $\text{GL}(n, \Delta)$. We denote the congruence subgroup corresponding to $\wp$ by $\mathcal{G}(n, \Delta, \wp)$, or $\mathcal{G}_\wp$ for short. If $G \leq \text{GL}(n, \Delta)$ then $G_\wp := G \cap \mathcal{G}_\wp$. For an integer $m$ we write $m \in \wp$ to mean that $m \cdot 1_\Delta \in \wp$.

We are interested in $\Delta$ and ideals $\wp$ such that $\mathcal{G}(n, \Delta, \wp)$ is torsion-free if $\text{char} \Delta = 0$, or each torsion element of $\mathcal{G}(n, \Delta, \wp)$ is unipotent if $\text{char} \Delta > 0$. Such domains in characteristic zero are discussed in [31, Chapter III, Section 11]. A slight modification of the proofs in [31] takes care of the positive characteristic case. To keep the account here reasonably self-contained, we give full proofs.

**Lemma 3.1.** Let $\Delta$ be a unique factorization domain, $q \in \Delta$ be irreducible, and $\wp$ be the principal ideal $q\Delta$ of $\Delta$. Suppose that $\mathcal{G}(n, \Delta, \wp)$ has non-trivial torsion elements. Then

(i) there is a unique prime $p \in \mathbb{Z}$ such that $p \in \wp$;
(ii) $pb = -\sum_{i=2}^{p} \binom{p}{i} q^{i-1}b^i$ for some $b \in \text{Mat}(n, \Delta)$; and
(iii) every torsion element of $\mathcal{G}(n, \Delta, \wp)$ has $p$-power order.

**Proof.** (Cf. [31, proof of Theorem 3, pp. 68-69].) Let $h \in \mathcal{G}_\wp$ be of prime order $p$. We have $h = 1_n + qb$ for some $b \in \text{Mat}(n, \Delta)$. Then

$$1_n = h^p = 1_n + pqb + \cdots + \binom{p}{i} q^i b^i + \cdots + q^p b^p$$

where the binomial coefficients are read modulo $\text{char} \Delta$. Hence

$$pb = -\sum_{i=2}^{p} \binom{p}{i} q^{i-1}b^i$$

and it follows that either $q$ divides $p$, or $q$ divides every entry of $b$.

Suppose that $q$ does not divide $p$. Then for some integer $\alpha \geq 1$, $q^{\alpha}$ divides every entry of $b$, whereas $q^{\alpha+1}$ does not. But [11] implies that $q^{2\alpha+1}$ divides $pb$, a contradiction. Thus $q$ divides $p$. 

If $G_\theta$ contains a non-trivial element of $p'$-order then it contains an element of prime order $r \neq p$. By the preceding, then, $q$ divides both $p$ and $r$ and hence divides $1 = px + ry$ for some $x, y \in \mathbb{Z}$. Since $q$ is not a unit by definition, every torsion element of $G_\theta$ must be a $p$-element.

**Proposition 3.2.** Let $\Delta$, $q$, and $\theta$ be as in Lemma 3.1.

(i) If $\text{char } \Delta = t > 0$ then every torsion element of $G(n, \Delta, \theta)$ is a $t$-element.

(ii) Suppose that $\text{char } \Delta = 0$, $q$ does not divide 2, and $q^2$ does not divide $p$ for any prime $p \in \mathbb{Z}$. Then $G(n, \Delta, \theta)$ is torsion-free.

**Proof.** (Cf. [31], pp. 68-69.)

(i) This follows from parts (i) and (iii) of Lemma 3.1.

(ii) If $G_\theta$ has non-trivial torsion then $p = qr$ for some odd prime $p$ and $r \in \Delta$ not divisible by $q$. By Lemma 3.1(ii), for some $b, c \in \text{Mat}(n, \Delta)$ we have $qr b = q^2 b^2 c$. Hence $q^a$ divides every entry of $b$ for some $a \geq 1$ such that $q^{a+1}$ does not divide every entry of $b$. As $q$ does not divide $r$, $rb = qb^2 c$ yields the contradiction that $q^{2a+1}$ divides every entry of $b$. □

The next result mimics Lemma 3.1.

**Lemma 3.3.** Let $\Delta$ be a Dedekind domain, and let $\theta$ be a proper prime ideal (i.e., maximal ideal) of $\Delta$. Suppose that $G(n, \Delta, \theta)$ has non-trivial torsion elements. Then

(i) there is a unique prime $p \in \mathbb{Z}$ such that $p \in \theta$;

(ii) for some $b \in \text{Mat}(n, \theta)$, $pb_{j,k} = -\sum_{i=2}^{p} \binom{p}{i} b_{j,k}^{(i)}$, where $b_{j,k}^{(i)}$ denotes the $(j, k)$th entry of $b$; and

(iii) every torsion element of $G(n, \Delta, \theta)$ has $p$-power order.

**Proof.** (Cf. the proof of [31], Theorem 4, p. 70.) If $h \in G_\theta$ has prime order $p$ then

$$pb + \cdots + \binom{p}{i} b^i + \cdots + b^p = 0_n$$

for some $b \in \text{Mat}(n, \theta)$, reading the binomial coefficients modulo $\text{char } \Delta$. Now (ii) is clear.

Let $l \geq 1$ be the integer such that $b_{j,k} \in \theta^l$ for all $j, k$, but $b_{r,s} \notin \theta^{l+1}$ for some $r, s$. (Such an integer $l$ exists, because the ideal $I$ of $\Delta$ generated by the entries of $b$ is contained in $\theta$, $I = \theta J$ where $J$ is the ideal $\theta^{-1}I$, and $J$ has a maximal power of $\theta$ in its primary decomposition.) Then (ii) and $b_{j,k}^{(i)} \in \theta^l$ imply that $pb_{j,k} \in \theta^{2l}$. Suppose that $p \notin \theta$. Since $\theta$ is a maximal ideal, $\Delta$ is generated by $p$ and $\theta$. Let $x \in \Delta$, $y \in \theta$ be such that $px + y = 1$. Then $b_{j,k} = pb_{j,k}x + \sum_{i=2}^{p} \binom{p}{i} b_{j,k}^{(i)} y \in \theta^{2l} + \theta^l \subseteq \theta^{l+1}$, a contradiction. Thus $p \in \theta$. Moreover, $p$ is the unique prime integer in $\theta$ (otherwise $1 \in \theta$ by Bézout’s lemma), so that every torsion element of $G_\theta$ is a $p$-element. □

**Proposition 3.4.** Let $\Delta$ and $\theta$ be as in Lemma 3.3.

(i) If $\text{char } \Delta = t > 0$ then every torsion element of $G(n, \Delta, \theta)$ is a $t$-element.

(ii) If $\text{char } \Delta = 0$, $2 \notin \theta$ and $p \notin \theta^2$ for all primes $p \in \mathbb{Z}$, then $G(n, \Delta, \theta)$ is torsion-free.

**Proof.** (Cf. [31], p. 70.)

(i) This follows at once from Lemma 3.3(i) and (iii).

(ii) If $G_\theta$ has non-trivial torsion then $G_\theta$ has elements of $p$-power order, where $p \in \theta$ for an odd prime $p$. By Lemma 3.3(ii), there exist an element $b$ of $\Delta$ and an integer $l$ such that $b \in \theta^l \setminus \theta^{l+1}$ (so that $\theta^l$ is the largest power of $\theta$ appearing in the primary decomposition of the ideal $b\Delta$) and $pb \in \theta^{2l+1}$. Certainly then $pb \in \theta^{l+2}$.

We now derive a contradiction. First, $pb \in \theta^{l+2}$ implies that $p\Delta \cdot b\Delta \subseteq pb\Delta \subseteq \theta^{l+2}$, so $\theta^{l+2}$ appears in the primary decomposition of $p\Delta \cdot b\Delta$. But since $p \in \theta \setminus \theta^2$, we know that $\theta^{l+1}$ is the largest power of $\theta$ appearing in this decomposition. □
To round out this section, we look briefly at how congruence homomorphisms may be applied in practice to finitely generated matrix groups. The congruence image should be a matrix group for which solutions to the specific problems are known (for example, the image is over a finite field), and the congruence kernel should be either torsion-free or consist of unipotent elements.

Let $\mathbb{F}$ be the field of fractions of the integral domain $\Delta$, and let $R$ be a finitely generated subring of $\mathbb{F}$. In particular, if $G = \langle g_1, \ldots, g_r \rangle \subseteq \text{GL}(n, \mathbb{F})$ then $R = R(G)$ denotes the ring generated by the entries of the elements of $\{g_i, g_i^{-1} | 1 \leq i \leq r\}$. Obviously $G \subseteq \text{GL}(n, R(G))$.

Let $\pi \subseteq \Delta$ be the set of denominators of the generators in a finite generating set of $R$. Denote by $\Delta_{\pi}$ the ring of fractions with denominators in the submonoid of $\Delta^x$ generated by $\pi$ (\cite[p. 311]{10}). Of course, $R \subseteq \Delta_{\pi}$. If $\Delta$ is a UFD or Dedekind domain then $\Delta_{\pi}$ is a UFD or Dedekind domain, respectively (\cite[Theorem 3.7, p. 315]{10} and \cite[Corollary 5.2, p. 322]{10}). Since the quotient of a finitely generated commutative ring by a maximal ideal is a finite field (\cite[4.1, p. 50]{32}), if $\Delta$ is finitely generated and $\varrho$ is a maximal ideal of $\Delta_{\pi}$ then $\Delta_{\pi}/\varrho$ is a finite field. Thus, if $G \subseteq \text{GL}(n, \mathbb{F})$ then $\psi_{\varrho} : \text{GL}(n, \Delta_{\pi}) \to \text{GL}(n, \Delta_{\pi}/\varrho)$ maps $G$ into some $\text{GL}(n, \mathbb{Q})$.

We give two examples to illustrate the above that are of computational interest.

**Example 3.5.** Let $\mathbb{F}$ be an algebraic number field, and let $\Delta$ be the ring of integers of $\mathbb{F}$. Since $\Delta$ is finitely generated, $\Delta_{\pi}$ is a finitely generated Dedekind domain. Let $\varrho$ be a maximal (proper prime) ideal of $\Delta_{\pi}$ not containing 2, such that $p \not\in \varrho^2$ for all primes $p \in \mathbb{Z}$; then $\mathbb{G}(n, \Delta_{\pi}, \varrho)$ is torsion-free by Proposition 3.4, and $\Delta_{\pi}/\varrho$ is a finite field. If $\mathbb{F} = \mathbb{Q}$, say, then $\Delta = \mathbb{Z}$, and if we choose an odd prime $p \in \mathbb{Z}$ which does not divide any element of $\pi$ then $\varrho = p\Delta_{\pi}$ as is required. In this case $\Delta_{\pi}/\varrho = \text{GF}(p)$.

For number fields $\mathbb{F}$ in general, to find $\varrho$ we can reduce to $\mathbb{Q}$ after fixing a $\mathbb{Q}$-basis of $\mathbb{F}$; this however has the disadvantage of blowing up the size of matrices. Alternatively we proceed as follows. Suppose that $\mathbb{F} = \mathbb{Q}(\alpha)$ contains all generators of $R$, where $\alpha$ is an algebraic integer. Let $m$ be the degree of the minimal polynomial of $\alpha$. Expressing each generator of $R$ uniquely as a $\mathbb{Q}$-linear combination of $\{1, \alpha, \ldots, \alpha^{m-1}\}$, and thereafter obtaining each generator in the form $\beta/z$ where $\beta$ is an algebraic integer and $z \in \mathbb{Z}$, we can find $\pi \subseteq \mathbb{Z}$. If $p \in \mathbb{Z}$ is an odd prime element of $\Delta$ not dividing any element of $\pi$ then $\varrho = p\Delta_{\pi}$ is a maximal ideal of $\Delta_{\pi}$ such that $\Delta_{\pi}/\varrho = \text{GF}(p^l)$ for some $l \leq m$, and $\mathbb{G}(n, \Delta, \varrho)$ is torsion-free. Note that $R \subseteq \mathbb{Z}[\alpha]$.

So the reduction mod $p$ congruence homomorphism on a finitely generated subgroup $G$ of $\text{GL}(n, \mathbb{F})$ with $R = R(G)$ is easily described. To evaluate $\psi_{\varrho}$, we reduce elements of $\mathbb{Z}[\alpha]$ mod $p$, and if $f(X) \in \mathbb{Z}[X]$ is the minimal polynomial of $\alpha$ then $\psi_{\varrho}(\alpha)$ is a root of the mod $p$-reduction $\tilde{f}(X)$ of $f(X)$. If $\tilde{f}(X)$ is irreducible over $\text{GF}(p)$ then $l = m$; otherwise $l < m$.

**Example 3.6.** Let $\mathbb{F}$ be a function field $P(X)$, and let $\Delta$ be the polynomial ring $P[X]$, where $P$ is a UFD. Then $\Delta$ is a UFD (\cite[p. 316]{10}), and therefore so too is $\Delta_{\pi}$. Let $q = X - \alpha$, where $\alpha$ is not a root of any element of $\pi$. (If $P$ is infinite then of course $\alpha$ always exists in $P$; else we can replace the finite field $P$ by a finite extension containing $\alpha$.) Then $\varrho = q\Delta_{\pi}$ is a prime ideal of $\Delta_{\pi}$. By Proposition 3.2 either $\mathbb{G}(n, \Delta_{\pi}, \varrho)$ is torsion-free, or every torsion element of $\mathbb{G}(n, \Delta_{\pi}, \varrho)$ is unipotent. The effect of $\psi_{\varrho}$ is just substitution of $\alpha$ for the indeterminate $X$ in elements of $\Delta_{\pi}$. Hence $\psi_{\varrho}(\Delta_{\pi})$ can be viewed as a subring of $P$. If $P$ is finite then $\psi_{\varrho}(\Delta_{\pi})$ is also a finite field. When $P$ has characteristic zero we apply a suitable congruence homomorphism over the finitely generated integral domain $\psi_{\varrho}(\Delta_{\pi}) \subseteq P$, in line with the following simple observation: if $\psi_{\varrho_1} : \Delta \to \Delta_{\varrho_1}$ and $\psi_{\varrho_2/\varrho_1} : \Delta_{\varrho_1} \to \Delta_{\varrho_2}$ are (natural) homomorphisms of integral domains such that $\mathbb{G}(n, \Delta, \varrho_1)$ and $\mathbb{G}(n, \Delta, \varrho_2/\varrho_1)$ are both torsion-free, then $\mathbb{G}(n, \Delta, \varrho_2)$ is torsion-free. Say $P = \mathbb{Q}$; then $\psi_{\varrho}(\Delta_{\pi}) \subseteq \mathbb{Z}_{\pi_1}$ for some finite subset $\pi_1$ of $\mathbb{Z} \setminus \{0\}$, and we are back to the situation of Example 3.5.
4. Computing with Nilpotent Matrix Groups

In this section we design algorithms for computing with matrix groups over a field \( F \), as set out in the introduction. We are guided by the algorithms and results in \[14\]. Although only finite \( F \) were treated in \[14\], most of that paper’s fundamental results are valid over any \( F \).

4.1. Splitting nilpotent linear groups. In linear group theory, we often reduce problems to the completely reducible case. This reduction is more straightforward for nilpotent linear groups than it is for arbitrary linear groups (see, e.g., \[13\], Subsection 2.1). In this subsection we consider a computational approach to the reduction.

Our starting point is the Jordan decomposition. Recall that \( h \in \mathrm{GL}(n, F) \) is diagonalizable if \( h \) is conjugate to a diagonal matrix over some extension of \( F \), and \( h \) is semisimple if \( \langle h \rangle \leq \mathrm{GL}(n, F) \) is completely reducible. A semisimple element of \( \mathrm{GL}(n, F) \) need not be diagonalizable, unless \( F \) is perfect: then the two concepts coincide. Denote the algebraic closure of \( F \) by \( \overline{F} \). For each \( g \in \mathrm{GL}(n, F) \), there is a unique unipotent matrix \( g_u \in \mathrm{GL}(n, F) \) and a unique diagonalizable matrix \( g_s \in \mathrm{GL}(n, \overline{F}) \) such that \( g = g_sg_u = g_u g_s \) (see \[32\], 7.2, p. 91). Note that this Jordan decomposition of \( g \) is the same over every extension of \( F \). If \( F \) is perfect then by \[29\] Proposition 1, p. 134], \( g_u \) and \( g_s \) are in \( \mathrm{GL}(n, \overline{F}) \).

An algorithm to compute the Jordan decomposition can be found in \[5\], Appendix A]. Systems such as GAP also have standard functions for computing the decomposition.

Let \( G = \langle g_1, \ldots, g_r \rangle \leq \mathrm{GL}(n, F) \). Define

\[
G_u = \langle (g_1)_u, \ldots, (g_r)_u \rangle \quad \text{and} \quad G_s = \langle (g_1)_s, \ldots, (g_r)_s \rangle.
\]

Since \( g_i = (g_1)_u (g_i)_s \in \langle G_u, G_s \rangle \), clearly \( G \leq G^* := \langle G_u, G_s \rangle \). In general, neither \( G_u \) nor \( G_s \) are necessarily subgroups of \( G \).

**Lemma 4.1.**

(i) \( G \) is nilpotent if and only if \( G_u, G_s \) are nilpotent and \( [G_u, G_s] = 1 \).

(ii) If \( G \) is nilpotent then \( G \leq G^* = G_u \times G_s \).

**Proof.** If \( G \) is nilpotent then the assignments \( g \mapsto g_u \) and \( g \mapsto g_s \) define homomorphisms \( G \to G_u \) and \( G \to G_s \); furthermore \( G^* = G_u \times G_s \) (see \[29\] Proposition 3, p. 136)). On the other hand, if \( G_u, G_s \) are nilpotent and \( [G_u, G_s] = 1 \), then \( G^* \) and thus \( G \leq G^* \) are nilpotent.

**Remark 4.2.** Let \( G \) be nilpotent. Then \( G_u = \{ g_u \mid g \in G \} \) and \( G_s = \{ g_s \mid g \in G \} \). Sometimes \( G = G_u \times G_s \). For example, this is true if \( F \) is finite. As another example, if \( G \) is an algebraic group (over algebraically closed \( F \)) then \( g_u, g_s \in G \) for all \( g \in G \), so that \( G = G^* \).

**Lemma 4.3.** \( G_u \) is nilpotent if and only if it is unipotent, i.e., conjugate to a subgroup of the group \( \mathrm{UT}(n, F) \) of all upper unitriangular matrices over \( F \).

**Proof.** A unipotent group is unitriangularizable (see \[32\], 1.21, p. 14]). If \( G_u \) is nilpotent then \( G_u = \{ g_u \mid g \in G \} \) is unipotent. As is well-known, \( \mathrm{UT}(n, F) \) is nilpotent, of class \( n - 1 \).

In \[14\], Subsection 2.1], a recursive procedure is given for deciding whether a group generated by unipotent matrices (over any field \( F \)) is unipotent. We label that procedure \( \text{IsUnipotent} \) here.

\[
\text{IsUnipotent}(H)
\]

Input: \( H = \langle h_1, \ldots, h_r \rangle, h_i \in \mathrm{GL}(n, F) \) unipotent, \( F \) any field.

Output: a \( \mathrm{UT}(n, F) \)-representation of \( H \), or a message ‘false’ meaning that \( H \) is not unipotent.

Lemmas \[4.1\] and \[4.3\] and \( \text{IsUnipotent} \), equate nilpotency testing of \( G \leq \mathrm{GL}(n, F) \) to testing nilpotency of \( G_s \) and testing whether \([G_u, G_s] = 1\).
If $G_u$ is unipotent then $\text{IsUnipotent}$ finds a $\text{UT}(n, \mathbb{F})$-representation of $G_u$ by constructing a series
\begin{equation}
V = V_0 > V_1 > \cdots > V_{i-1} > V_i = 0
\end{equation}
of $G_u$-submodules of the underlying space $V$ for $\text{GL}(n, \mathbb{F})$, such that $G_u$ acts trivially on each factor $V_{i-1}/V_i$. In fact, $V_{i-1}/V_i$ is the fixed point space $\text{Fix}_{G_u}(V/V_i)$. We get more when $G$ is nilpotent, by the next two lemmas.

**Lemma 4.4.** Each unipotent element of a completely reducible nilpotent subgroup of $\text{GL}(n, \mathbb{F})$ is trivial.

*Proof.* This follows from [31] Corollary 1, p. 239.

**Lemma 4.5.** Let $G \leq \text{GL}(n, \mathbb{F})$ be nilpotent, $\mathbb{F}$ a perfect field. Then
(i) $G_s$ is completely reducible over $\mathbb{F}$;
(ii) $G$ is completely reducible over $\mathbb{F}$ if and only if $G_u = 1$.

*Proof.* A solvable group of diagonalizable matrices (over any field) is completely reducible by [31] Theorem 5, p. 172. Since $G_s \leq \text{GL}(n, \mathbb{F})$ consists entirely of diagonalizable matrices, if $G_u = 1$ then $G = G_s$ is completely reducible. The converse is Lemma 4.4.

If $G$ is nilpotent and $\mathbb{F}$ is perfect then Lemmas 4.4 and 4.5 imply that each factor $V_{i-1}/V_i$ of (2) is a completely reducible $G^*$-module. As a subgroup of a nilpotent completely reducible subgroup of $\text{GL}(n, \mathbb{F})$ is completely reducible by [31] Theorem 5, p. 239, we see that if $G$ is nilpotent then $\text{IsUnipotent}$ constructs completely reducible modules not just for $G^*$ but also for $G$.

We now give a procedure for reducing nilpotency testing of $G \leq \text{GL}(n, \mathbb{F})$ to testing nilpotency of a matrix group generated by diagonalizable matrices.

**Reduction(G)**

**Input:** $G = \langle g_1, \ldots, g_r \rangle \leq \text{GL}(n, \mathbb{F})$, $\mathbb{F}$ any field.

**Output:** $G_s$, a $\text{UT}(n, \mathbb{F})$-representation of $G_u$, and a message that $[G_u, G_s] = 1$; or a message ‘false’ meaning that $G$ is not nilpotent.

for $i \in \{1, \ldots, r\}$ do
find $(g_i)_u, (g_i)_s$;
$G_u := \langle (g_i)_u : 1 \leq i \leq r \rangle$, $G_s := \langle (g_i)_s : 1 \leq i \leq r \rangle$;
if $\text{IsUnipotent}(G_u) = \text{false}$
then return ‘false’;
else $N := [G_u, G_s]$;
if $N \neq 1$
then return ‘false’;
else return $G_s$.

There are other reductions to the completely reducible case. For example, we could compute the radical $R$ of the enveloping algebra $(G)_{\mathbb{F}}$, and then the radical series
$$V \supset R V \supset R^2 V \supset \cdots \supset R^m V = 0$$
(see [28] for methods to compute $R$). Each term $R^j V$ in this series is a $G$-module, and each factor $R^j V/R^{j+1} V$ is a completely reducible $G$-module. The radical series may be used to write $G$ in block upper triangular form, thereby obtaining a homomorphism $\theta$ of $G$ onto a completely reducible subgroup of $\text{GL}(n, \mathbb{F})$. If $G$ is nilpotent then $\ker \theta$ is the unipotent radical of $G$ (the unique maximal unipotent normal subgroup of $G$), and $\ker \theta$ commutes with every diagonalizable element of $G$ (see [32] 7.11, p. 97)).
4.2. Further background. We now prepare the way for applying Section 3 to nilpotency testing over an arbitrary field $F$.

As usual, $Z_i(G)$ will denote the $i$th term of the upper central series of $G$; i.e., $Z_0(G) = 1$ and $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$.

Lemma 4.6. If $G$ is a completely reducible nilpotent subgroup of $\text{GL}(n, F)$ then $|G : Z(G)|$ is finite.

Proof. See [18, Corollary 6.5, p. 114], or [31, Theorem 1, p. 208]. Also cf. Zassenhaus’ result [32, 3.4, p. 44].

Remark 4.7. [31, Theorem 1, p. 208] is stated for irreducible groups only. The result for completely reducible nilpotent subgroups $G$ of $\text{GL}(n, F)$ follows from this, because $G/Z(G)$ is isomorphic to a subgroup of the direct product of central quotients of irreducible nilpotent linear groups (each of degree no more than $n$).

Lemma 4.8. Let $G$ be a completely reducible nilpotent subgroup of $\text{GL}(n, F)$. If $N$ is a torsion-free normal subgroup of $G$ then $N \leq Z(G)$.

Proof. Suppose that $N \not\leq Z(G)$. Then $NZ(G)/Z(G)$ is a non-trivial normal subgroup of the nilpotent group $G/Z(G)$, so has non-trivial intersection with $Z_2(G)/Z(G)$. Let $x \in N \cap Z_2(G)$, $x \not\in Z(G)$. By Lemma 4.6, $x^m \in Z(G)$ for some $m$. Choose $g \in G$ such that $x^g = x\epsilon$ for some $\epsilon \in Z(G)$, $\epsilon \neq 1$. Then $x^m = (x^g)^m = (x^g)^m = x^m\epsilon^m$ implies that $\epsilon$ is a non-trivial torsion element of $G$. But $\epsilon = x^{-1}x^g \in N$.

Now let $G$ be a finitely generated subgroup of $\text{GL}(n, F)$. Suppose that $\Delta$ is a finitely generated subgroup of $F$ such that $G \leq \text{GL}(n, \Delta)$, and let $\varrho$ be an ideal of $\Delta$. We continue with the notation $\mathcal{G}(n, \Delta, \varrho)$ and $G_{\varrho}$ adopted in Section 3 for congruence subgroups. Without loss of generality, we may assume that $F$ is the field of fractions of $\Delta$.

Lemma 4.9. Suppose that $\mathcal{G}(n, \Delta, \varrho)$ is torsion-free if $\text{char } \Delta = 0$, and all torsion elements of $\mathcal{G}(n, \Delta, \varrho)$ are unipotent if $\text{char } \Delta > 0$. Let $G$ be completely reducible as a subgroup of $\text{GL}(n, F)$. If $G$ is nilpotent then $G_{\varrho}$ is a torsion-free central subgroup of $G$.

Proof. By Lemma 4.4 and the hypotheses, $G_{\varrho}$ is torsion-free. Hence the result follows from Lemma 4.8.

Examples 3.5 and 3.6 show how to select $\varrho$ as stipulated in Lemma 4.9 for various $F$ and $\Delta$. Also, Subsection 4.1 shows how to split off a completely reducible subgroup of $\text{GL}(n, F)$ from an arbitrary finitely generated nilpotent subgroup of $\text{GL}(n, F)$.

Theorem 4.10. Suppose that $\mathcal{G}(n, \Delta, \varrho)$ is as in Lemma 4.9 and that $G$ is completely reducible. Then $G$ is nilpotent if and only if $\psi_{\varrho}(G)$ is nilpotent and $G_{\varrho} \leq Z(G)$.

Proof. One direction is elementary, the other is Lemma 4.9.

Theorem 4.10 transforms nilpotency testing of a finitely generated completely reducible subgroup $G$ of $\text{GL}(n, F)$ into an equivalent pair of problems: testing whether $G_{\varrho} \leq Z(G)$, and testing whether $\psi_{\varrho}(G)$ is nilpotent. If $\varrho$ is a maximal ideal then $\Delta/\varrho$ is a finite field, and we can test nilpotency of $\psi_{\varrho}(G)$ as in [14]. To test whether $G_{\varrho} \leq Z(G)$ we need a generating set for $G_{\varrho}$. This may be achieved if together with the input generating set $\{g_1, \ldots, g_r\}$ for $G_{\varrho}$, we know either (i) a transversal for the cosets of $G_{\varrho}$ in $G$, or (ii) a presentation of $G/G_{\varrho} \cong \psi_{\varrho}(G)$. In case (i), as long as the index $|G : G_{\varrho}| = |\psi_{\varrho}(G)|$ is not too large then the Schreier method [23, Section 2.5, pp. 41-45] is a realistic option for finding a generating set of $G_{\varrho}$. In case (ii), suppose that every releator $w_j$ in the known presentation of $\psi_{\varrho}(G)$ is a word in the $\psi_{\varrho}(g_i)$. Then by replacing each
occurrence of $\psi_{g_i}(g_i)$ in $w_j$ by $g_i$, $1 \leq i \leq r$, we get a generating set for a subgroup of $G$ whose normal closure is $G_{\overline{g}}$. (This is the ‘normal subgroup generators’ method; cf. [23] pp. 299-300.) As a consequence, the following lemma solves the problem of testing whether $G_{\overline{g}}$ is central in $G$.

**Lemma 4.11.** Let $G = \langle g_1, \ldots, g_r \rangle \leq \text{GL}(n, \mathbb{F})$ and

$$
\psi_{g}(G) = \langle \psi_{g}(g_1), \ldots, \psi_{g}(g_r) | w_1(\psi_{g}(g_i)), \ldots, w_s(\psi_{g}(g_i)) \rangle.
$$

Then $G_{\overline{g}}$ is the normal closure in $G$ of the subgroup

$$
\widetilde{G}_{\overline{g}} = \langle w_1(g_i), \ldots, w_s(g_i) \rangle.
$$

Hence $G_{\overline{g}} \leq Z(G)$ if and only if $w_j(g_i) \in Z(G)$ for all $j, 1 \leq j \leq s$, in which case $G_{\overline{g}} = \widetilde{G}_{\overline{g}}$.

4.3. **Deciding finiteness.** After testing nilpotency of $G \leq \text{GL}(n, \mathbb{F})$, we can move on to tackle other basic computational problems for $G$, such as testing whether $G$ is finite.

Deciding finiteness of matrix groups over algebraic number fields and functional fields has been considered by various authors, and a practical implementation was written by Beals in GAP for groups over $\mathbb{Q}$ (see [6]). The method for deciding finiteness that we introduce in this subsection is a general approach to the problem that is uniform with respect to the ground field. We apply it here only for nilpotent groups, while the general case is part of separate research.

Let $\text{char} \mathbb{F} = 0$, and let $\varrho$ be an ideal of the subring $\Delta$ of $\mathbb{F}$ such that $\Delta/\varrho$ is finite. Suppose that $\mathcal{G}(n, \Delta, \varrho)$ is torsion-free. Then, obviously, $G \leq \text{GL}(n, \Delta)$ is finite if and only if $G_{\varrho}$ is trivial. This suggests a very simple and general finiteness test for $G$. However, efficiency of this test depends on knowing an efficient method to decide whether $G_{\varrho}$ is trivial. If $G$ is nilpotent then we have such a method by Lemma 4.11.

**IsNilpotentFinite(G)**

Input: A nilpotent subgroup $G = \langle g_1, \ldots, g_r \rangle$ of $\text{GL}(n, \mathbb{F})$, $\text{char} \mathbb{F} = 0$.

Output: a message ‘true’ meaning that $G$ is finite; and ‘false’ otherwise.

if $G_u \neq 1$
  then return ‘false’;
if $G_{\varrho} \neq 1$
  then return ‘false’;
else return ‘true’.

Once $G$ is confirmed to be finite, then we know that $|G| = |\psi_{g}(G)|$. Computing the order of $G$ is thus reduced to the order problem for matrix groups over a finite field. Testing whether $G_{\varrho} \neq 1$ in **IsNilpotentFinite(G)** is feasible by Lemma 4.11 because we can compute a presentation of the nilpotent group $\psi_{g}(G)$ over a finite field without difficulty.

Now we consider $\mathbb{F}$ of positive characteristic.

**Lemma 4.12.** Let $G = \langle g_1, \ldots, g_r \rangle$ be a nilpotent subgroup of $\text{GL}(n, \mathbb{F})$, $\text{char} \mathbb{F} > 0$. Then $G_u$ is finite.

**Proof.** Schur’s First Theorem [31] p. 181 asserts that a periodic subgroup of $\text{GL}(n, \mathbb{F})$ is locally finite. As $G$ is nilpotent, $G_u$ is unipotent and so periodic. Then the result follows, because $G_u = \langle (g_1)_{u}, \ldots, (g_r)_{u} \rangle$ is finitely generated. \[Q.E.D.\]

By Lemmas 4.1 and 4.12 if $G$ is nilpotent then $G$ is finite precisely when $G_s$ is finite. Let $\mathbb{F}$ be perfect, and suppose that all torsion elements of $G(n, \Delta, \varrho)$ are unipotent. Then as $G_s/(G_s)_{\varrho}$ is finite, $G$ is finite if and only if $(G_s)_{\varrho}$ is trivial, by Lemma 4.9 If $\mathbb{F}$ is not perfect then we can still
find a normal unipotent subgroup $U$ of $G$ such that $G/U$ is isomorphic to a completely reducible subgroup of $GL(n, \mathbb{F})$ (see the end of Subsection 4.1), and the above reasoning goes through.

For another method to decide finiteness of $G$, incorporated with nilpotency testing of $G$, see Subsection 4.5 below.

4.4. Polycyclic presentations. A finitely generated nilpotent group is polycyclic, and therefore has a (consistent) polycyclic presentation. One benefit of possessing a polycyclic presentation for a nilpotent subgroup $G$ of $GL(n, \mathbb{F})$ is that we gain access to the numerous existing algorithms for abstract polycyclic groups (see [30, Chapter 9], [23, Chapter 8], and the GAP package ‘Polycyclic’ [21]), which may be used to further investigate the structure of $G$.

The papers [3, 4] deal with the problem of constructing a polycyclic presentation for a finitely generated subgroup $G$ of $GL(n, \mathbb{Q})$. Specifically, the algorithm PolycyclicPresentation($G$) in [3] attempts to compute polycyclic presentations for $\psi_p(G), G_p/U_p$, and $U_p$, where $p = p\mathbb{Z}_\pi$ for a finite set $\pi$ of primes not containing the odd prime $p$. $\psi_p : GL(n, \mathbb{Z}_\pi) \rightarrow GL(n, p)$ is the associated congruence homomorphism, and $U_p$ is a unipotent radical of $G_p$. If $G$ is polycyclic then the algorithm returns a polycyclic presentation of $G$. The algorithm fails to terminate if $G$ is solvable but not polycyclic (i.e., $U_p$ is not finitely generated). In this subsection we propose a modification of PolycyclicPresentation which either returns a polycyclic presentation of $G$, or detects that $G$ is not nilpotent.

The paper [4] contains another algorithm, IsPolycyclic($G$), for polycyclicity testing of a subgroup $G$ of $GL(n, \mathbb{Q})$. IsPolycyclic($G$) always terminates, returning either a polycyclic presentation of $G$, or a message that $G$ is not polycyclic. A nilpotency testing algorithm based on IsPolycyclic($G$) is also given in [4]. That algorithm has the following stages: (i) testing whether $G$ is polycyclic, (ii) testing whether $G/U$ is nilpotent, where $U$ is a unipotent radical of $G$, and (iii) testing whether $G$ acts nilpotently on $U$. Our approach in this subsection avoids the possibly time-consuming step (i), and replaces step (iii) with a simpler test.

The strategy of our algorithm is as follows. Let $G$ be a finitely generated subgroup of $GL(n, \mathbb{F})$, where for convenience $\mathbb{F}$ is assumed to be perfect. After applying Reduction($G$), we will know either that $G$ is not nilpotent, or that $G \leq \langle G_u, G_s \rangle$, $G_u$ is unipotent, and $[G_u, G_s] = 1$. In the latter event, we find polycyclic presentations of $G_u$ and $G_s$. Note that if we proceed further after Reduction($G$), then the finitely generated nilpotent group $G_u \leq UT(n, \mathbb{F})$ is definitely polycyclic. Next, we find presentations of $\psi_p(G_s)$ and $(G_s)_\varnothing$ for a suitable $\varnothing$. Recall that if $G$ is nilpotent then $(G_s)_\varnothing \leq Z(G_s)$ is abelian (see Lemma 4.9).

### PresentationNilpotent($G$)

**Input:** $G = \langle g_1, \ldots, g_r \rangle \leq GL(n, \mathbb{F})$, $\mathbb{F}$ perfect.

**Output:** a polycyclic presentation of $G$, or a message ‘false’ meaning that $G$ is not nilpotent.

1. If Reduction($G$) = ‘false’ then return ‘false’; else go to step (2).
2. Determine a polycyclic presentation of $G_u$ as a finitely generated subgroup of $UT(n, R)$, $R$ a finitely generated subring of $\mathbb{F}$.
3. Compute a generating set for $\psi_p(G_s)$, and use this to attempt to construct a polycyclic presentation of $\psi_p(G_s)$. Return ‘false’ if the attempt fails.
4. Determine a generating set for $(G_s)_\varnothing$. If $(G_s)_\varnothing$ is not central in $G_s$ then return ‘false’. Else construct a polycyclic presentation of the finitely generated abelian group $(G_s)_\varnothing$.
5. Combine the presentations of $\psi_p(G_s)$ and $(G_s)_\varnothing$ found in steps (3) and (4) to get a polycyclic presentation of $G_s$.
6. Combine the presentations of $G_u$ and $G_s$ found in steps (2) and (5) to get a polycyclic presentation of $G^* = G_uG_s$ and thence a polycyclic presentation of $G \leq G^*$.
Implementation of PresentationNilpotent depends on having algorithms for computing the polycyclic presentations in steps 2 and 4. Such algorithms are available for finite fields and number fields (see [3, 4]).

4.5. Testing nilpotency using an abelian series; the adjoint representation. Methods for testing nilpotency of matrix groups, relying on properties of nilpotent linear groups, were proposed in [14]. Although those methods were applied only to groups over finite fields, they are valid over other fields as well. In this subsection we justify this statement.

As in [14] Subsection 2.2 we define a recursive procedure SecondCentralElement(G, H) which accepts as input finitely generated subgroups G, H of GL(n, F), F any field, where H is a non-abelian normal subgroup of G. If G is nilpotent then the recursion terminates in a number of rounds no greater than the nilpotency class of G, returning an element of $Z_2(H) \setminus Z(H)$. We therefore seek an upper bound on nilpotency class of nilpotent subgroups of GL(n, F). (Such a bound exists only for certain fields F. For instance if F is algebraically closed then GL(n, F) contains nilpotent groups of every class; see [31 Corollary 1, p. 214].)

Lemma 4.13. Let G be a nilpotent completely reducible subgroup of GL(n, F) contained in GL(n, Δ), Δ a finitely generated subring of F. Let q be an ideal of Δ as in Lemma 4.9 Then the nilpotency class of G is at most the nilpotency class of $\psi_q(G)$ plus 1.

Proof. This is clear by Lemma 4.9. \qed

Example 4.14. Theorem 2 of [33] gives an upper bound $3n/2$ on the nilpotency class of subgroups of GL(n, Q). This further implies an upper bound $3mn/2$ for subgroups of GL(n, P), where P is a number field of degree m over Q. Suppose that G is a finitely generated nilpotent subgroup of GL(n, Q(X)). Since UT(n, F) has nilpotency class $n - 1$ (see [31 Theorem 13.5, p. 89]), it follows from Lemmas 4.4 and 4.13 that the nilpotency class of G is at most $3n/2 + 1$. Similar remarks pertain to groups over $Q(X_1, \ldots, X_m)$.

Example 4.15. Let $q$ be a power of a prime p. If G is a finitely generated nilpotent subgroup of GL(n, F) for $F = GF(q)(X)$ then G has nilpotency class at most $l_{n,q} + 1$, where $l_{n,q}$ is an upper bound on the class of nilpotent subgroups of GL(n, q). A formula for $l_{n,q}$ may be deduced from [8 Theorem C.3]:

$$l_{n,q} = n \cdot \max\{(t - 1)s + 1 \mid t \neq p \text{ prime, } t \leq n, t^s \text{ dividing } q - 1\}$$

That is, $n((t - 1)s + 1)$ is an upper bound on the class of a Sylow t-subgroup of GL(n, q), where $t^s$ is the largest power of the prime t dividing $q - 1$ (slightly better bounds are known for special cases, e.g., $t = 2$). We restrict to $t \leq n$ in [8] because a t-subgroup of GL(n, q) is abelian if $t \neq p$ and $t > n$ (cf. [14 Lemma 2.25]).

We assume henceforth that we are able to specify a number $k_F$ such that if termination does not happen in $k_F$ rounds or less then SecondCentralElement(G, H) reports that G is not nilpotent; otherwise, the procedure returns an element $a \in Z_2(H) \setminus Z(H)$ such that $[G, a] \leq Z(H)$.

Other procedures in [14] that were designed for finite fields F also carry over to any F. Given $a \in Z_2(G) \setminus Z(G)$, let $\varphi_a : G \to Z(G) \cap [G, G]$ be the homomorphism defined by $g \mapsto [g, a]$. If G is completely reducible then NonCentralAbelian(G, a) returns the abelian normal subgroup $A = \langle a \rangle^G = \langle a, \varphi_a(G) \rangle$ of G, and Centralizer(G, A) returns a generating set for the kernel $C_G(A)$ of $\varphi_a$. NonCentralAbelian(G, a) requires a ‘cutting procedure’ for $\langle A \rangle_F$, to reduce computations to the case of cyclic $\varphi_a(G)$. Also, [14] Lemma 2.17 and Corollary 2.18 hold for any field F; hence, as in the finite field case, we get a moderate upper bound on the index $\langle G : C_G(A) \rangle$.

The cutting procedure described in [28] Section 3 finds the simple components of a finite-dimensional commutative semisimple algebra over any field F, input by a set of algebra generators.
When \( F = \mathbb{Q} \) another method, based on [19 Lemma 5], can be applied (see [3 Section 5.2]). The main ingredient here is an efficient method for factorizing polynomials over \( F \).

The discussion above shows that the recursive procedure TestSeries of [14 Subsection 2.4] can be defined over any field \( F \). The basic steps in the recursion are outlined in [14 pp. 113-114]. If it does not detect that an input finitely generated subgroup \( G \) of \( GL(n, F) \) is nilpotent, then TestSeries\((G, l)\) returns a series

\[
(1_n) \lhd A_1 \lhd A_2 \lhd \cdots \lhd A_l \lhd C_l \lhd \cdots \lhd C_2 \lhd C_1 \lhd G
\]

where the \( A_i \) are abelian, \( C_i \lhd G \) is the centralizer of \( A_i \) in \( C_{i-1} \), and the factors \( C_{i-1}/C_i \) are abelian. That is, all factors of consecutive terms in (4) are abelian, except possibly the middle factor \( C_l/A_l \). The construction of further terms in (4) continues, with strict inclusions everywhere except possibly in the middle of the series, as long as \( C_l \) is non-abelian.

**Lemma 4.16.** For some \( l \leq n - 1 \), the term \( C_l \) in (4) is abelian.

**Proof.** Cf. the proof of [14 Lemma 2.20].

**N.B.** Until further notice in this subsection, \( G \) is completely reducible.

**Lemma 4.17.** \( Z(G) \) is contained in every term \( C_i \) of (4). Therefore, if \( G \) is nilpotent then \( G/C_l \) is finite.

**Proof.** Certainly \( Z(G) \leq C_1 = C_{G}(A) \). Assume that \( Z(G) \leq C_{k-1} \); then \( Z(G) \) is contained in the \( C_{k-1} \)-centralizer \( C_k \) of \( A_k \). The second statement is now clear by Lemma 4.6.

**Corollary 4.18.** Suppose that \( G \) is nilpotent. Then \( G \) is finite if and only if \( C_l \) in (4) is finite.

Corollary 4.18 gives another finiteness test for completely reducible nilpotent subgroups \( G \) of \( GL(n, F) \); cf. Subsection 4.3. This test requires that we are able to decide finiteness of the finitely generated completely reducible abelian matrix group \( C_l \). To that end, the next result may be useful.

**Lemma 4.19.** If \( G \) is non-abelian nilpotent then \( C_l \) has non-trivial torsion.

**Proof.** Suppose that \( C_l \) is torsion-free. Let \( a \in Z_2(G) \setminus Z(G) \). Since \( a^m \in Z(G) \) for some \( m \) by Lemma 4.6 there exists \( g \in G \) such that \( [g, a] \in Z(G) \) has finite non-trivial order (dividing \( m \)). This contradicts \( [g, a] \in A_1 \leq C_l \).

Suppose that \( G \) is finite. Then we can apply [14 Lemma 2.23] to \( G \). That is, we refine (4) to a polycyclic series of \( G \), then test nilpotency of \( G \) via prime factorization of the cyclic quotients in the refined series, and checking that factors for different primes commute. Hence the algorithm IsNilpotent from [14 Section 2] can be employed for nilpotency testing of \( G \). In the more general setting we label this algorithm IsFiniteNilpotent. This algorithm, which accepts only finite \( G \leq GL(n, F) \) as input, also yields the Sylow decomposition of nilpotent \( G \). Complexity estimation of IsFiniteNilpotent is undertaken in [14 Section 2].

Now we examine the case that \( G \) is infinite. First we state a few structural results.

**Lemma 4.20.** Let \( \pi \) be the set of primes less than or equal to \( n \). Suppose that \( G \) is nilpotent. Then every element of (the finite group) \( G/Z(G) \) has order divisible only by the primes in \( \pi \). Moreover, no element of \( G/Z(G) \) has order divisible by \( char \ F \).

**Proof.** It suffices to prove the lemma for irreducible \( G \) (cf. Remark 4.7). Proofs of the irreducible case are given in [31 Chapter 7]; see Corollary 1, p.206, and Theorem 2, p. 216, of [31].

**Corollary 4.21.** (Cf. Example 4.15) If \( G \) is nilpotent then for all primes \( p > n \), a Sylow \( p \)-subgroup of \( G \) is central.

Recall that a group \( H \) is said to be \( p \)-primary, for a prime \( p \), if \( H/Z(H) \) is a \( p \)-group.
Lemma 4.22. If $G$ is nilpotent then $G$ is a product of $p$-primary groups for $p \leq n$.

Now let $G$ be any subgroup of $\text{GL}(n, \mathbb{F})$. Set $m$ to be the $\mathbb{F}$-dimension of the enveloping algebra $\langle G \rangle_{\mathbb{F}}$. Define the adjoint representation $\text{adj} : G \rightarrow \text{GL}(m, \mathbb{F})$ by $\text{adj}(g) : x \mapsto gxg^{-1}$, $x \in \langle G \rangle_{\mathbb{F}}$. Clearly $\ker \text{adj} = Z(G)$. If $G$ is nilpotent and completely reducible then $\text{adj}(G)$ is a finite completely reducible subgroup of $\text{GL}(m, \mathbb{F})$, by Lemma 4.20 and Maschke’s theorem. If $G = \langle g_1, \ldots, g_r \rangle$ then by [7, Lemma 4.1] we can construct a basis of $\langle G \rangle_{\mathbb{F}}$ as a straight-line program of length $m$ over $\{g_1, \ldots, g_r\}$. Then we calculate $\text{adj}(G)$ by solving a system of linear equations. The adjoint representation is another way of transferring nilpotency testing to the case of finite groups.

The results presented so far in this subsection lead to the following algorithm to test nilpotency of $G$ using an abelian series and the adjoint representation. The input generators of $G$ are diagonalizable, but $G$ cannot be assumed in advance to be completely reducible. Also, as mentioned earlier, this algorithm requires knowledge of an upper bound on nilpotency class of nilpotent subgroups of $\text{GL}(n, \mathbb{F})$.

\begin{verbatim}
IsNilpotentAdjoint(G)
Input: $G = \langle g_1, \ldots, g_r \rangle \leq \text{GL}(n, \mathbb{F})$, $g_i \in \text{GL}(n, \mathbb{F})$ diagonalizable.
Output: a message ‘true’ meaning that $G$ is nilpotent, or a message ‘false’ meaning that $G$ is not nilpotent.
for $i \in \{1, \ldots, r\}$
  do $\bar{g}_i := \text{adj}(g_i)$;
  $\bar{G} := \langle \bar{g}_1, \ldots, \bar{g}_r \rangle$;
  if $(\bar{g}_i)_u \neq 1$ for some $i$
    then return ‘false’;
  else invoke TestSeries($\bar{G}$);
if $G$ is infinite
  then return ‘false’;
else invoke IsFiniteNilpotent($\bar{G}$).
\end{verbatim}

For testing whether $\bar{G}$ is infinite, see Corollary 4.18.

Parts of IsNilpotentAdjoint that use polynomial factorization (e.g., the cutting procedure) have running time dependent on the coefficient field. Also, computation of $\bar{G} = \text{adj}(G)$ entails squaring the dimension in worst-case; so may be time-consuming and efficient only for small $n$.

If IsNilpotentAdjoint($G$) returns ‘true’ then the algorithm furnishes additional information about $G$, such as its decomposition into $p$-primary subgroups. Also, knowing a generating set for $\bar{G}$ we can find a generating set for $Z(G) = \ker \text{adj}$ (by the Schreier method, or using a presentation of $G \cong G/Z(G)$ to pull back to ‘normal subgroup generators’, hence a generating set, of $Z(G)$; see before Lemma 4.11). If we can find a polycyclic presentation of the finitely generated completely reducible abelian matrix group $Z(G)$, then this can be combined with a polycyclic presentation of $G/Z(G)$. Thus we gain one more method to construct a polycyclic presentation of $G$.

4.6. Nilpotency testing via change of ground domain and abelian series. Finally, we outline the simplest and most effective combination of our ideas for nilpotency testing of finitely generated matrix groups, over a perfect field $\mathbb{F}$.

4.6.1. The algorithm. IsNilpotentMatGroup as below tests nilpotency over an infinite field $\mathbb{F}$, via Reduction($G$) (if $\mathbb{F}$ is perfect), and applying a congruence homomorphism $\psi_\beta$ to $G_s$, where $\beta$ satisfies the hypotheses of Lemma 4.9. Nilpotency of $\psi_\beta(G_s)$ is tested using an abelian series
If \( G \) is nilpotent then \( (G_s)_e \leq Z(G_s) \), and this containment can be tested via Lemma 4.11.

**IsNilpotentMatGroup**

**Input:** \( G = \langle g_1, \ldots, g_r \rangle \leq \text{GL}(n, \mathbb{F}) \).

**Output:** a message ‘true’ meaning that \( G \) is nilpotent, or a message ‘false’ meaning that \( G \) is not nilpotent.

1. Reduction\((G)\).
2. Construct \( \psi(G_s) \leq \text{GL}(n, q) \).
3. Test nilpotency of \( \psi(G_s) \).
4. Test whether \( (G_s)_e \leq Z(G_s) \).

There are several advantages of the approach embodied in \texttt{IsNilpotentMatGroup}. First, by reducing the amount of computation over the original field \( \mathbb{F} \), we hope to escape the unfortunate circumstances that may arise when computing over infinite fields (e.g., a blow-up in the size of matrix entries). Another issue relates to upper bounds on nilpotency class. If \( G \) is nilpotent then procedures used in \texttt{TestSeries} to construct the series (4) that depend on a class bound for the potentially nilpotent group \( \psi(G) \), such as SecondCentralElement, are guaranteed to terminate more quickly than they would for an arbitrary nilpotent subgroup of \( \text{GL}(n, q) \). For example, if \( \mathbb{F} = \mathbb{Q} \) then \( \psi(G) \) inherits from \( G \leq \text{GL}(n, \mathbb{Q}) \) an upper bound \( 3n/2 \) on nilpotency class; this can be compared with the general bound (3) for \( \text{GL}(n, q) \) stated in Example 4.15.

It is desirable to retain complete reducibility in step (2) of \texttt{IsNilpotentMatGroup}. That is, the Jordan decomposition over the top field \( \mathbb{F} \) is unavoidable; we do not want to repeat it in \( \text{GL}(n, q) \). Let \( q \) be a power of the prime \( p \). For the input \( \psi_g((g_i)_s) \) to \texttt{TestSeries} to be diagonalizable, these elements must have order coprime to \( p \). Equivalently \( q \) should be chosen so that \( f_i(X) \) and \( f'_i(X) \) are coprime for all \( i \), where \( f_i(X) \) is the minimal polynomial of \( \psi_g((g_i)_s) \) (note that \( f_i(X) \) is the image \( \psi_g(h_i(X)) \) of the minimal polynomial \( h_i(X) \) of \( g_i \)). Selection of \( q \) is a number theory problem if \( \mathbb{F} \) is a number field.

**Lemma 4.23.** If \( G_s \) is nilpotent and \( p > n \) then any preimage of \( (\psi_g(G_s))_u \) in \( G_s \) is central.

**Proof.** Let \( g \in G_s \). Then \( \psi_g(g)_u = \psi_g(g') \) for some \( l \), and \( \psi_g(g'^{(p^k)}) = 1 \) for some \( k \); i.e., \( g'^{(p^k)} \in (G_s)_e \leq Z(G_s) \). By Corollary 4.21, \( g' \in Z(G_s) \). \( \square \)

Lemma 4.23 indicates that we may reasonably expect \( \psi_g(G_s) \) to be completely reducible if \( q \) is chosen so that \( p > n \). However, if \( n \) is large then of course it is advisable to work with \( p \leq n \).

4.6.2. Implementation and experimental results. Our implementation of \texttt{IsNilpotentMatGroup} for groups defined over \( \mathbb{Q} \) also includes an algorithm \texttt{IsNilpotentMatGroupFF} for testing nilpotency over finite fields, according to Subsection 4.5. To construct congruence subgroups, \texttt{IsNilpotentMatGroup} uses some functions from ‘Polenta’ [2].

Table II below samples performance of \texttt{IsNilpotentMatGroup} for various input parameters: degree; size of the field if finite, or size of generator entries if the field is \( \mathbb{Q} \); number of generators. The last column of Table II gives CPU time in the format minutes : seconds : milliseconds. The computations were done on a Pentium 4 with 1.73 GHz under Windows, using \texttt{GAP 4}. The standard \texttt{GAP} function \texttt{IsNilpotent} failed for all groups in Table II.

As one might expect, the most challenging input groups are the nilpotent groups, because for these we pass through all stages of the algorithms. On the other hand, if the input is not nilpotent, then this is confirmed very quickly. For example, if the input does not have an abelian series—if it is not solvable—then the algorithm terminates at the \texttt{TestSeries} stage (see Subsection 4.5).
Thus, for proper testing of our algorithms, we need an extensive set of examples of nilpotent matrix groups. Constructing special classes of nilpotent matrix groups is a problem of interest in its own right. We have implemented an algorithm, MaximalAbsolutelyIrreducibleNilpotentMatGroup\((n, p, l)\), that constructs absolutely irreducible maximal nilpotent subgroups of GL\((n, p^l)\). If \(r\) divides \(p^l - 1\) for each prime divisor \(r\) of \(n\) then such a subgroup of GL\((n, p^l)\) is unique up to conjugacy; otherwise, such subgroups do not exist (see [31, Chapter 7]). If \(n = r^a\) and \(r\) divides \(p^l - 1\) then MaximalAbsolutelyIrreducibleNilpotentMatGroup\((n, p, l)\) returns the group generated by a Sylow \(r\)-subgroup of GL\((n, p^l)\), and all scalars. For other \(n\), this algorithm returns the group generated by the scalars and a Kronecker product of Sylow \(r_i\)-subgroups of GL\((r_i^{a_i}, p^l)\), \(n = \prod_{i=1}^k r_i^{a_i}\).

To check steps which rely on the Jordan decomposition, we implemented another procedure, ReducibleNilpotentMatGroup. This procedure returns reducible but not completely reducible nilpotent groups over finite fields and \(\mathbb{Q}\).

The groups \(G_i\) in Table 1 for \(i \leq 5\) are absolutely irreducible nilpotent groups constructed by MaximalAbsolutelyIrreducibleNilpotentMatGroup. The reducible groups \(G_6, G_7, G_8, G_9, G_{10}, G_{11}\), are constructed by ReducibleNilpotentMatGroup. Finally, \(G_{12}, G_{13},\) and \(G_{14}\) are non-nilpotent groups: \(G_{12} = \text{GL}(150, 13^3), G_{13} = \text{GL}(350, \mathbb{Z}),\) and \(G_{14}\) is the group POL_PolExamples2(40) from ‘Polenta’, an infinite solvable subgroup of GL\((25, \mathbb{Q})\).

Other functions in ‘Nilmat’ decide finiteness, compute orders of finite nilpotent groups, find the Sylow system of a nilpotent group over a finite field, and test whether a nilpotent group is completely reducible. Additionally ‘Nilmat’ contains a library of the nilpotent primitive groups over finite fields (based on [12]).

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| Group | Degree | Field | No. generators | Runtime        |
|-------|--------|-------|----------------|----------------|
| \(G_1\) | 9 \(5^3\) | 6 | 0 : 00 : 26.890 |
| \(G_2\) | 127 | \(2^5\) | 3 | 0 : 18 : 37.092 |
| \(G_3\) | 12 \(5^3\) | 9 | 0 : 12 : 35.592 |
| \(G_4\) | 30 | 11\(4^5\) | 9 | 0 : 17 : 39.749 |
| \(G_5\) | 63 \(2^5\) | 11 | 0 : 18 : 25.842 |
| \(G_6\) | 90 \(2^5\) | 54 | 0 : 21 : 28.280 |
| \(G_7\) | 96 \(5^3\) | 63 | 0 : 35 : 27.546 |
| \(G_8\) | 120 | \(11^4\) | 27 | 1 : 07 : 17.702 |
| \(G_9\) | 100 | \(\mathbb{Q}\) | 12 | 0 : 08 : 39.641 |
| \(G_{10}\) | 200 | \(\mathbb{Q}\) | 27 | 0 : 09 : 16.344 |
| \(G_{11}\) | 128 | \(\mathbb{Q}\) | 93 | 0 : 14 : 07.398 |
| \(G_{12}\) | 150 | \(13^4\) | 2 | 0 : 00 : 23.688 |
| \(G_{13}\) | 350 | \(\mathbb{Q}\) | 4 | 0 : 00 : 55.047 |
| \(G_{14}\) | 25 | \(\mathbb{Q}\) | 13 | 0 : 16 : 25.859 |

**Table 1. Running times for nilpotency testing algorithms**
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