LIMIT THEOREMS FOR MULTIVARIATE LONG-RANGE DEPENDENT PROCESSES

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ABSTRACT: We investigate limit theorems for multivariate long-range dependent (LRD) processes. Let \((X_k)_{k \in \mathbb{Z}}\) be a linear process with values in \(\mathbb{R}^d\) given by \(X_k = \sum_{j \in \mathbb{Z}} A_{j-k} \varepsilon_j\), where \((A_j)_{j \in \mathbb{Z}}\) is a sequence of matrices and \((\varepsilon_j)_{j \in \mathbb{Z}}\) are i.i.d. random vectors. We derive the central limit theorem and, under the additional assumption of general multivariate long-range dependence, a functional CLT with operator fractional Brownian motions as limit. Furthermore, we assume \((X_k)_{k \in \mathbb{N}}\) to be a Gaussian multivariate LRD process and investigate a limit theorem under componentwise subordination. Using the same setting, we find the asymptotic behavior of the vector of the sample autocovariances. Of particular interest are the matrix-valued normalization sequences.

1. Introduction

Over the last thirty years, long-range dependent stochastic processes become an important instrument for modeling phenomena in econometrics, engineering and hydrology to mention some examples. Statistical inference in time series analysis bases on asymptotic results. Therefore, limit theorems play an important rule in statistics. In this paper, we study different types of limit theorems under the assumption of multivariate long-range dependence. It includes the behavior of partial sums of multivariate linear processes, subordinated Gaussian processes and the sample autocovariances. We investigate a suitable matrix-valued normalization sequence under the assumption of multivariate LRD, which could be of particular interest for further results in this context.

In the one-dimensional case, the asymptotic behavior of LRD processes is well studied. See Davydov (1970) for the behavior of linear processes and Taqqu (1975), Taqqu (1979) and Dobrushin and Major (1979) for results about subordinated Gaussian LRD processes. Asymptotic results for the sample autocovariances are investigated by Horváth and Kokoszka (2008). Kechagias and Pipiras (2015) investigated the notion of LRD for multivariate stochastic processes, i.e. \(\mathbb{R}^d\)-valued processes \(X_k = (X_k^{(1)}, \ldots, X_k^{(d)})'\), where the prime denotes transpose. In this paper, we study limit theorems for such processes.

We define the multivariate linear process

\[
X_k = \sum_{j=-\infty}^{\infty} A_{j-k} \varepsilon_j,
\]

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where \((A_j)_{j \in \mathbb{Z}}\) is a sequence of matrices with \(A_j = (a_{tm}^j)_{t,m=1,\ldots,d} \in \mathbb{R}^{d \times d}\) and \((\varepsilon_j)_{j \in \mathbb{Z}}\) is a sequence of independently and identically distributed random vectors. We investigate the convergence behavior of the partial sum sequence of independently and identically distributed random vectors. We examine the LRD linear processes introduced by Kechagias and Pipiras (2015). They choose we have to be more specific. We reduce the class of processes to the announced multivariate \((2)\). To investigate the convergence behavior of the partial sum process \(S_n = \sum_{k=1}^n X_k\), suitable normalized by a sequence of matrices. Furthermore, we will extend the result to time-continuous processes. As well we will see that this setting includes the case of multivariate LRD processes defined by Kechagias and Pipiras (2015).

In the one-dimensional case, there is a result by Ibragimov and Linnik (1971). They proved that the process converges to a standard normal random variable. Furthermore, Merlevde (1996, Corollary 1.1.) investigated a central limit theorem for general linear processes in a Hilbert space under suitable assumptions. In particular, the partial sum converges to a Gaussian Hilbert space-valued random variable. Therefore, the process was normalized by a sequence of matrices. In contrast, we will normalize by a matrix-valued sequence and will not need any further assumptions.

To investigate the convergence behavior of the partial sum process

\[
S_{nt} = \sum_{k=1}^{nt} X_k, \quad t \in [0, 1],
\]

we have to be more specific. We reduce the class of processes to the announced multivariate LRD linear processes introduced by Kechagias and Pipiras (2015). They choose

\[
a_{tm}^j = L_{tm}(j)|j|^{-d_l \frac{1}{2}}, \quad j \in \mathbb{Z},
\]

in \((1)\), where \(d_j \in (0, \frac{1}{2})\) and \(L(j) = (L_{tm}(j))_{t,m=1,\ldots,d}\) is an \(\mathbb{R}^{d \times d}\)-valued function satisfying

\[
L(j) \sim A^+ \quad \text{as} \quad j \to \infty \quad \text{and} \quad L(j) \sim A^- \quad \text{as} \quad j \to -\infty
\]

for some matrices \(A^+, A^- \in \mathbb{R}^{d \times d}\), where \(\sim\) denotes componentwise asymptotic equivalence.

We will establish the convergence behavior of \((2)\) for a multivariate linear process with the previous long-memory property.

The result is a natural extension of the one-dimensional case, first proved by Davydov (1970). Furthermore, we are interested in subordinated Gaussian processes. We consider a multivariate, long-range dependent Gaussian process \((X_k)_{k \in \mathbb{Z}}, X_k = (X_k^{(1)}, \ldots, X_k^{(d)})^t\). Note, that we write for simplicity \(a^G = \text{diag}(a^{g_1}, \ldots, a^{g_d})\) for \(a > 0\) and a diagonal matrix \(G = \text{diag}(g_1, \ldots, g_d)\). The autocovariance matrix function is defined as \(\gamma(h) = (\gamma_{ij}(h))_{i,j=1,\ldots,d} = E(X_0 X_h^t) - E(X_0)E(X_h^t)\).

We assume that the components \(X_k^{(1)}, \ldots, X_k^{(d)}\) are independent for any fixed \(k \in \mathbb{Z}\). Between the vectors we use the time-domain long-range dependence condition, very general given by Kechagias and Pipiras (2015) in the following way.

**Definition 1.1.** A stationary \(\mathbb{R}^d\)-valued process \((X_k)_{k \in \mathbb{Z}}\) with finite second moments is called long-range dependent if

\[
\gamma(k) = k^{-D}R(k)k^{-D} = (R_{ij}(k))_{i,j=1,\ldots,d},
\]

where \(D = \text{diag}(d_1, \ldots, d_d)\) with \(d_j \in (0, \frac{1}{2})\), \(j = 1, \ldots, d\), and \(R(k) = (R_{ij}(k))_{i,j=1,\ldots,d}\) is an \(\mathbb{R}^{d \times d}\)-valued function satisfying

\[
R(k) \sim R = (R_{ij})_{i,j=1,\ldots,d} \quad \text{as} \quad k \to \infty
\]
for some $d \times d$ matrix $R$, where $R_{ij} \in \mathbb{R}$ and $R_{jj} \neq 0$, $j = 1, \ldots, d$.

Especially, we are interested in the convergence behavior of a componentwise subordinated process. Therefore, we define the function $G : \mathbb{R}^d \to \mathbb{R}^d$ by

$$G(x_1, \ldots, x_d) = (G_1(x_1), \ldots, G_d(x_d))$$

where $G_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, \ldots, d$. We investigate the convergence behavior of the $d$-dimensional process

$$A(n)^{-1} \sum_{k=1}^{\lfloor nt \rfloor} (G(X_k) - EG(X_k)),$$

where $A(n)^{-1}$ is a suitable matrix-valued normalization sequence.

There are well-known results in the one-dimensional case, see for example Beran et al. (2013) for a summary of known results. Furthermore, there is a multivariate extension by Bai and Taqqu (2013). They investigated results which describe the convergence behavior of the $d$-dimensional vector

$$\left( \frac{1}{B_j(n)} \sum_{k=1}^{\lfloor nt \rfloor} (G_j(Y_k) - E(G_j(Y_k))) \right)_{j=1, \ldots, d},$$

where $(Y_k)_{k \in \mathbb{N}}$ is a one-dimensional long-range dependent Gaussian process and $j \in \{1, \ldots, d\}$ the Hermite rank. They could prove the convergence to a multivariate process with dependent Hermite processes as marginals.

In contrast to their result, Ho and Sun (1990) considered a bivariate Gaussian vector sequence and applied the function $G$ for $d = 2$.

For the last result, we assume the same process as in the previous one. We derive the convergence behavior of the sample autocovariances defined by

$$\hat{\Gamma}_{n,h} - \Gamma_h = \frac{1}{n} \sum_{k=1}^{n} (X_kX_{k+h}^\prime(\cdot) - E(X_0X_h^\prime)(\cdot)).$$

The process takes values in $L(\mathbb{R}^d) = \{T : \mathbb{R}^d \to \mathbb{R}^d\}$.

In the one-dimensional case Horváth and Kokoszka (2008) and Hosking (1996) investigated the convergence behavior of the sample autocovariances under long-range dependence. Furthermore, Mas (2002) could prove more general results for Hilbert space-valued linear processes under weak dependence.

The rest of the paper is organized as follows. First, we present the main results. Afterwards, we give the proof of the result about general linear processes. Before we continue with the proofs of the other results, we insert a section with some preliminary statements.

### 2. Main Results

**Theorem 2.1.** Let $(\varepsilon_j)_{j \in \mathbb{Z}}$ be an $\mathbb{R}^d$-valued sequence of independent, identically distributed random variables with zero mean and $E(\varepsilon_j\varepsilon_j^\prime) = I$. Furthermore, let $A_j \in \mathbb{R}^{d \times d}$ be a sequence
of matrices such that \( A_j = (a_{lm}^j)_{l,m=1,\ldots,d} \). Let \((X_k)_{k \in \mathbb{Z}}\) be an \( \mathbb{R}^d \)-valued linear process defined by

\[
X_k = \sum_{j=-\infty}^{\infty} A_{j-k} \varepsilon_j
\]

with

\[
\sum_{j=-\infty}^{\infty} \|A_j\|_F^2 < \infty,
\]

where \( \| \cdot \|_F \) denotes the Frobenius norm. If each diagonal entry of the matrix

\[
\Sigma^2_n := E(S_n S_n') = \left( (\sigma_n^2)^2_{lm} \right)_{m=1,\ldots,d;l=1,\ldots,d}
\]

goes to infinity as \( n \to \infty \) and \( \text{Var}(a'X_k) \neq 0 \) for each \( a \in \mathbb{R}^d \setminus \{0\} \), then

\[
\Sigma^{-1}_n \sum_{k=1}^{n} X_k \xrightarrow{D} \mathcal{N}(0, I).
\]

The Theorem is easily extendable to time continuous processes.

**Corollary 2.2.** Let \((\varepsilon_t)_{t \in \mathbb{R}}\) be an \( \mathbb{R}^d \)-valued sequence of independent, identically distributed random variables with zero mean and \( E(\varepsilon_t \varepsilon_t') = I \). Furthermore, let \( A_t \in \mathbb{R}^{d \times d} \) be a sequence of matrices such that \( A_t = (a_{lm}^t)_{l,m=1,\ldots,d} \). Let \((X_t)_{t \in \mathbb{R}}\) be an \( \mathbb{R}^d \)-valued linear process defined by

\[
X(t) = \int_{-\infty}^{\infty} A(s-t) \varepsilon(s) ds
\]

with

\[
\int_{-\infty}^{\infty} \|A_s\|_F^2 ds < \infty.
\]

If each diagonal entry of the matrix

\[
\Sigma^2_T := E(S_T S_T') = \left( (\sigma_T^2)^2_{lm} \right)_{m=1,\ldots,d;l=1,\ldots,d}
\]

goes to infinity as \( T \to \infty \) and \( \text{Var}(a'X_t) \neq 0 \) for each \( a \in \mathbb{R}^d \setminus \{0\} \), then

\[
\Sigma^{-1}_T S_T = \Sigma^{-1}_T \int_0^T X(t) dt \xrightarrow{D} \mathcal{N}(0, I).
\]

For the next result, we refer to Didier and Pipiras (2011) to introduce Operator fractional Brownian motions (OFBMs). The multivariate extension of fractional Brownian motion is a vector \( B_H(t) = (B_{1,H}(t), \ldots, B_{d,H}(t))' \in \mathbb{R}^d \) with \( t \in \mathbb{R} \). OFBMs are

(i) Gaussian

(ii) operator self-similar and

(iii) stationary increment processes.
Additionally it shall be assumed, as it is ordinary in the multivariate context, that the process is proper. That means that the distribution of $B_H(t)$ is not contained in a proper subspace of $\mathbb{R}^d$ for each $t \in \mathbb{R}$. Operator self-similarity was introduced by Laha and Rohatgi (1981) and Hudson and Mason (1982). A stationary, vector-valued stochastic process $(X_k)_{k \in \mathbb{Z}}$ is called operator self-similar, if for every $a > 0$ there exists a $B(a) \in \mathbb{R}^{d \times d}$ such that

$$B(a)X_k \overset{f.d.d.}{=} X_{ak}.$$ 

Furthermore, by Didier and Pipiras (2011), in the time domain it admits the integral representation

$$\int_{\mathbb{R}} \left( (t-u)^{\frac{1}{2}}I_{-d} - (u)^{\frac{1}{2}}I_{-d} \right) M_+ + \left( (t-u)^{\frac{1}{2}}I_{-d} - (u)^{\frac{1}{2}}I_{-d} \right) M_- W(du),$$

where $W(du)$ is a suitable multivariate real-valued Gaussian measure and $M_+, M_- \in \mathbb{R}^{d \times d}$.

For simplicity we introduce the following conditions, which are related to the matrices in (4).

C1: $A^-$ and $A^+$ are invertible.
C2: $c_{ii}(A^-(-A)^') + A^+(A^+)^'_{ii} + (A^-(A^+)^')_{ii} \neq 0$ for all $i = 1,...,d$ with $c_{ii} = \frac{\sin(\pi d_i)}{\sin(2\pi d_i)}$.

**Theorem 2.3.** Assume that $(X_k)_{k \in \mathbb{Z}}$ is a stationary linear process given by (1) and satisfying the long-range dependence condition (3) and the conditions C1 and C2. Then

$$A(n)^{-1}S_{[nt]} \overset{D}{\rightarrow} AB_H(t) \quad t \in [0,1],$$

in $D[0,1]^d$, where $H = I - D$, $(B_H(t))_{t \in [0,1]}$ is an OFBM with $D = \text{diag}(d_1,...,d_d)$ and $A \in \mathbb{R}^{d \times d}$ is a suitable upper triangular matrix. Furthermore, the normalization matrix is such that

$$\lim_{n \to \infty} \text{Var}(A(n)^{-1} \sum_{k=1}^{n} X_k) = I.$$ 

In the case $t = 1$ the process satisfies the conditions of Theorem 2.1, since

$$E(S_n(S_n)^') = n\gamma^{(i,j)}(0) + \sum_{k=1}^{n-1} (n-k)\gamma^{(i,j)}(k) + \sum_{k=1}^{n-1} (n-k)\gamma^{(j,i)}(k)$$

$$= n\gamma^{(i,j)}(0) + \sum_{k=1}^{n-1} (n-k)R_{ij}(k)k^{-d_i-d_j} + \sum_{k=1}^{n-1} (n-k)R_{ji}(k)k^{-d_i-d_j}$$

$$\sim \left( \frac{R_{ij} + R_{ji}}{(1-d_i-d_j)(2-d_i-d_j)} \right) n^{2-d_i-d_j}.$$ 

Hence, each entry goes to infinity as $n \to \infty$. Furthermore,

$$\sum_{j=\infty}^{\infty} \|A_j\|^2_F = \sum_{j=\infty}^{\infty} \sum_{l=1}^{d} \sum_{m=1}^{d} |a_{lm}^j|^2 = \sum_{j=\infty}^{\infty} \sum_{l=1}^{d} \sum_{m=1}^{d} |L_{lm}(j)|^{-d_i-d_j} < \infty$$

This implies that the process is not contained in a proper subspace of $\mathbb{R}^d$. Therefore, the conditions of Theorem 2.1 are satisfied.
using Potter’s bound, see Bingham et al. (1989). Especially, the matrix-valued normalization sequence is asymptotically calculable. Therefore, set $\tau = 1$ in Lemma 4.1.

Before we continue with the next result, we need some preliminaries. As announced in the introduction we are interested in multivariate Gaussian processes $\{X_k\}_{k \in \mathbb{N}}$. Furthermore, we assume long-range dependence in the sense of Definition 1.1, i.e.

$$E(X^{(i)}_1) = 0,$$

$$E(X^{(i)}_1X^{(j)}_1) = \delta_{ij},$$

$$r^{(i,j)}(k) = R_{ij}(k)k^{-d_i-d_j}$$

for $i, j = 1, \ldots, d$ and $k \in \mathbb{N}$. We define the function $G : \mathbb{R}^d \to \mathbb{R}^d$ by

$$G(x_1, \ldots, x_d) = (G_1(x_1), \ldots, G_d(x_d))$$

where $(G_i)_{i=1,\ldots,d}$ are functions $G_i : \mathbb{R} \to \mathbb{R}$, which have all the same Hermite rank denoted by $\tau$. We are interested in the convergence behavior of the $d$-dimensional process

$$\sum_{k=1}^{[nt]}(G(X_k) - EG(X_k)).$$

We suppose that the functions $(G_i)_{i=1,\ldots,d}$ belong to $L^2(\mathbb{R}, \varphi)$ for each $i = 1, \ldots, d$, which denotes the space of measurable, square-integrable functions with respect to standard normal probability measure. Then, the Hermite expansion is given by

$$G_i = \sum_{l=\tau}^{\infty} h_{l,i} H_l,$$

where

$$H_l(x) = (-1)^l \exp \left( \frac{x^2}{2} \right) \frac{d^l}{dx^l} \exp \left( -\frac{x^2}{2} \right)$$

are the so called Hermite polynomials, see Pipiras and Taqqu (2017) for more information.

For simplicity we define further

$$(7) \quad H(x_1, \ldots, x_d) = (H_1(x_1), \ldots, H_d(x_d)).$$

Since $(X_k)_{k \in \mathbb{N}}$ is a stationary multivariate Gaussian process, it has the following spectral representation

$$X_k = \int e^{ik\lambda}dZ(\lambda),$$

where $Z$ is a right-continuous orthogonal increment process. The $d \times d$ spectral distribution $F$ is defined by

$$\Gamma(k) = \int e^{ik\lambda}dF(\lambda),$$

where each component $F_{mk}$ is a distribution function and

$$E \left( \int f(\lambda)dZ_m(\lambda) \overline{g(\lambda)dZ_k(\lambda)} \right) = \int f(\lambda)\overline{g(\lambda)}dF_{mk}(\lambda)$$
for any functions \( f \in L^2(F_{mm}) \) and \( g \in L^2(F_{kk}) \). See Brockwell and Davis (1986) Chapter 11 for more information about spectral theory of vector processes.

Referring to Major (2014), we introduce Multiple Wiener-Itô integrals with respect to a common complex Hermitian Gaussian random measure with Lebesgue control measures. In dependence of \( m \in \mathbb{N} \), we define

\[
I_m^{\tau}(K) = \int_{\mathbb{R}^\tau} K_m(\lambda_1, \ldots, \lambda_\tau) dB^{(m)}(\lambda_1) \ldots dB^{(m)}(\lambda_\tau),
\]

where \( \int_{\mathbb{R}^\tau} |K_m(\lambda_1, \ldots, \lambda_\tau)|^2 dB^{(m)}(\lambda_1) \ldots dB^{(m)}(\lambda_\tau) < \infty \) for each \( m = 1, \ldots, d \).

We are now ready to state the theorem.

**Theorem 2.4.** If the multivariate Gaussian process \((X_k)_{k \in \mathbb{N}}\) is long-range dependent in sense of Definition 1.1, such that the matrix \( R \) is invertible, and if the memory parameters fulfill

\[
\tau d_i < \frac{1}{2}, \ i = 1, \ldots, d,
\]

then

\[
A(n)^{-1} \sum_{k=1}^{|nt|} (G(X_k) - E G(X_k)) \xrightarrow{D} A(I_m^{\tau}(f_{\tau,d_m,t}))_{m=1,\ldots,d},
\]

in \( D[0,1]^d \), where \( A \in \mathbb{R}^{d \times d} \) is a suitable upper triangular matrix and

\[
f_{\tau,d_m,t}(x_1, \ldots, x_d) = e^{it\sum_{j=1}^\tau x_j} - \left( \frac{1}{i} \sum_{j=1}^\tau x_j \right) \prod_{r=1}^\tau |x_r|^{d_m - \frac{1}{2}}.
\]

The normalization matrix is such that

\[
\lim_{n \to \infty} \text{Var}(A(n)^{-1} \sum_{k=1}^n H(X_k)) = I.
\]

The last result gives the convergence behavior of the sample autocovariances of a purely non-deterministic, multivariate long-range dependent Gaussian process. Therefor, we need some preliminaries. The space \( \mathbb{R}^{d \times d} \) is equipped with the standard inner product \( \langle A, B \rangle = \text{trace}(A'B) = \sum_{i=1}^d \langle A e_i, B e_i \rangle \), associated norm \( \|A\| = \sqrt{\langle A, A \rangle} \) and basis

\[
\langle (e_i, \cdot) f_k \rangle_{1 \leq i \leq d, 1 \leq k \leq d},
\]

where \( (e_i)_{1 \leq i \leq d} \) and \( (f_k)_{1 \leq k \leq d} \) are basis in \( \mathbb{R}^d \) and \( \langle \cdot, \cdot \rangle \) its standard inner product.

The sample autocovariances are defined as

\[
\hat{\Gamma}_{n,h} = \frac{1}{n} \sum_{k=1}^n \langle X_{k+h}, \cdot \rangle X_k = \frac{1}{n} \sum_{k=1}^n X_k X'_{k+h}(\cdot)
\]

Furthermore, the autocovariances are given by

\[
\Gamma_h = E(\langle X_h, \cdot \rangle X_0) = E(X_0 X'_h)(\cdot).
\]

We are interested in the convergence behavior of

\[
(\hat{\Gamma}_{n,h} - \Gamma_h, h = 0, \ldots, H)
\]
with
\[ \hat{\Gamma}_{n,h} - \Gamma_h = \frac{1}{n} \sum_{k=1}^{n} (X_k X_{k+h}^\prime (\cdot) - E(X_0 X_h^\prime (\cdot))) . \]

**Theorem 2.5.** Suppose \((X_k)_{k \in \mathbb{N}}\) fulfills \(E \|X_k\|^4 < \infty\).

(i) If \(d_i \in \left( \frac{1}{4}, \frac{1}{2} \right)\), then
\[ n^{\frac{1}{d}} (\hat{\Gamma}_{n,h} - \Gamma_h, h = 0, ..., H) \overset{D}{\to} (G_h, h = 0, ..., H), \]
where \(G_h\) is a zero-mean Gaussian random element with values in \(L(\mathbb{R}^d)^{H+1}\).

(ii) If \(d_i \in \left( 0, \frac{1}{4} \right)\), then
\[ (B_n^{-1} (\hat{\Gamma}_{n,h} - \Gamma_h) B_n^{-1}, h = 0, ..., H) \overset{D}{\to} (Z, h = 0, ..., H), \]
where \(B_n^{-1}\) is such that there exist a \(N \in \mathbb{N}\) and a \(C_d \in L(\mathbb{R}^{d \times d})\) with
\[ \text{Cov}(B_n^{-1} \hat{\Gamma}_{n,h} B_n^{-1}, B_n^{-1} \hat{\Gamma}_{n,q} B_n^{-1}) \leq C_d \text{ for } n \geq N \]
componentwise and
\[ Z = B \tilde{Z} B' \]
where \(\tilde{Z} = (\tilde{z}_{j_1,j_2})_{j_1,j_2=1,...,d}\) is given by
\[ (\tilde{z}_{r_1,r_2})_{r_1,r_2=1,...,d} = K_{r_1,r_2} \int_{\mathbb{R}^2} \frac{e^{i(x_1+x_2)} - 1}{i(x_1 + x_2)} \prod_{i=1}^{2} |x_i|^{-(1/2-d_{ri})} B^{(r_1)}(dx_1) B^{(r_2)}(dx_2), \]
\(K_{r_1,r_2}\) are suitable constants and \(B \in \mathbb{R}^{d \times d}\) is an upper triangular matrix with entries equal to one.

### 3. Proof of Theorem 2.1 and Corollary 2.2

We start with the proof of Theorem 2.1. Therefor, we calculate the covariance matrix to find the normalization sequence.

\[ \Sigma_n^2 := (\sigma^2_{ij}(n))_{i,j=1,...,d} = E(S_n S_n^\prime) = E\left( \sum_{k=1}^{n} X_k \left( \sum_{k=1}^{n} X_k \right)' \right) = \left( \sum_{q=1}^{d} \sum_{j=-\infty}^{\infty} \left( \sum_{k=1}^{n} a_{iq}^{j-k} \right) \left( \sum_{k=1}^{n} a_{mq}^{j-k} \right) \right)_{l,m=1,...,d}. \]

We denote the inverse of the square root of the matrix by
\[ \Sigma_n^{-1} := (\Sigma_n^{-1})_{l,m=1,...,d}. \]
The inverse exists since \( \text{Var}(a'X_k) \neq 0 \) for each \( a \in \mathbb{R}\setminus\{0\} \) by assumption. Applying the normalization sequence to the partial sum, we get

\[
\Sigma_n^{-1} \sum_{k=1}^{n} X_k = \Sigma_n^{-1} \left( \sum_{k=1}^{n} \sum_{j=-\infty}^{\infty} \sum_{q=1}^{d} a_{pq}^{j-k} \varepsilon_{j}^{(q)} \right)_{p=1,\ldots,d} = \sum_{k=1}^{n} \sum_{j=-\infty}^{\infty} \sum_{p=1}^{d} \sum_{q=1}^{d} a_{pq}^{j-k} \varepsilon_{j}^{(q)}.
\]

To prove the convergence in distribution we use Cramér-Wold device, i.e. we show that

\[
(t')' \Sigma_n^{-1} \sum_{k=1}^{n} X_k \overset{D}{\to} t'Z \quad \text{as} \quad n \to \infty
\]

for each \( t = (t_1, \ldots, t_d) \in \mathbb{R}^d \). The left side may be written as

\[
(9) \quad t' \Sigma_n^{-1} \sum_{k=1}^{n} X_k = \sum_{j=-\infty}^{\infty} \sum_{p=1}^{d} \sum_{k=1}^{n} \sum_{i=1}^{d} a_{pq}^{j-k} \varepsilon_{j}^{(q)}.
\]

Calculating the variances, we get

\[
E(t' \Sigma_n^{-1} \sum_{k=1}^{n} X_k)^2 = t' E(\Sigma_n^{-1} \sum_{k=1}^{n} X_k (\Sigma_n^{-1} \sum_{k=1}^{n} X_k')) t = t' \Sigma_n^{-1} E(\sum_{k=1}^{n} X_k (\sum_{k=1}^{n} X_k')) t = t' \Sigma_n^{-1} \Sigma_n^2 (\Sigma_n^{-1})' t = t't.
\]

(10)

For further procedure we use a proof idea like Characiejus and Račkauskas (2013), first introduced in Račkauskas and Suquet (2011). It will make it possible to prove the theorem under the assumption of a Gaussian white noise process.

In order to do this, we refer to Račkauskas and Suquet (2011) for a theorem, which gives sufficient conditions for the weak convergence of linear processes with values in a Hilbert space. Let \( \mathbb{H} \) and \( \mathbb{E} \) be two Hilbert spaces and \((\varepsilon_j)_{j \in \mathbb{Z}}\) a sequence of independent, identically distributed random variables with values in \( \mathbb{E} \). Define \((X_n)_{n \in \mathbb{Z}}\) with

\[
(11) \quad X_n = \sum_{j \in \mathbb{Z}} D_{nj} \varepsilon_j
\]

and \( D_{nj} \in L(\mathbb{H}, \mathbb{E}) \), which denotes the space of bounded linear operators from \( \mathbb{H} \) to \( \mathbb{E} \). Now, we define a second process \((Y_n)_{n \in \mathbb{Z}}\) with

\[
Y_n = \sum_{j \in \mathbb{Z}} D_{nj} \bar{\varepsilon}_j,
\]

where \( D_{nj} \) is the same operator as above and \( \bar{\varepsilon}_j \) is a sequence of Gaussian random elements with values in \( \mathbb{E} \), zero-mean and the same covariance operator as \( \varepsilon_j \).

Before we state Račkauskas and Suquet’s lemma we need a definition of a metric on the space of probability measures on Hilbert spaces.
Definition 3.1. Let $X, Y$ be $\mathbb{H}$-valued random variables, then the metric $g_k$ is defined by

$$g_k(X, Y) = \sup_{f \in F_k} |Ef(X) - Ef(Y)|,$$

where $F_k$ is the set of all $k$ times Fréchet differentiable functions $f : \mathbb{H} \to \mathbb{R}$ such that $\sup_{x \in \mathbb{H}} |f^{(i)}(x)| \leq 1$ for $i = 0, \ldots, k$.

Now we are able to present the lemma.

Lemma 3.2. If the conditions

$$(12) \quad \lim_{n \to \infty} \sup_{j \in \mathbb{Z}} \|D_{nj}\|_{op} = 0 \quad \text{and} \quad \lim_{n \to \infty} \sup_{j \in \mathbb{Z}} \sum_{q=1}^{d} \|D_{nj}\|_{op}^2 < \infty$$

are fulfilled, then

$$\lim_{n \to \infty} g_3(X_n, Y_n) = 0.$$

Referring to Giné and León (1980), the processes have the same convergence behaviour if $\lim_{n \to \infty} g_3(X_n, Y_n) = 0$, since the metric induces the weak topology on the set of probability measures on $\mathbb{H}$.

Continuing with the expression (9) above and defining

$$\sum_{j=-\infty}^{\infty} \sum_{p=1}^{d} \sum_{k=1}^{n} t_i \Sigma_{ip} \sum_{q=1}^{d} a_{pq}^{j-k} \varepsilon_j^{(q)} =: \sum_{j=-\infty}^{\infty} \sum_{q=1}^{d} B_{nj}^q \varepsilon_j^{(q)}$$

$$= \sum_{j=-\infty}^{\infty} \left( B_{nj}^1, \ldots, B_{nj}^d \right) \begin{pmatrix} \varepsilon_j^{(1)} \\ \vdots \\ \varepsilon_j^{(d)} \end{pmatrix},$$

we get the wanted structure. Therefore, we have to prove the conditions (12) for $B_{nj} := (B_{nj}^1, \ldots, B_{nj}^d)$.

Since $B_{nj}^q \in \mathbb{R}^{1 \times d}$, the operator norm is easily calculable with $\|B_{nj}\|_{op} = \max_{1 \leq q \leq d} |B_{nj}^q|$. To prove the conditions, we introduce the following notation

$$(\omega_{pq}^n)^2 := E(\sum_{k=1}^{n} \sum_{j=-\infty}^{\infty} a_{pq}^{j-k} \varepsilon_j^{(p)})^2 = \sum_{j=-\infty}^{\infty} \left( \sum_{k=1}^{n} a_{pq}^{j-k} \right)^2$$

and continue with some auxiliary result.

Lemma 3.3. The sequence of matrix entries $(\omega_{pq}^n)^2 \Sigma_{ip}^{n \geq 1}$ converges to zero for each $p, q, i \in \{1, \ldots, d\}$. 

Proof. Define the matrix $\tilde{\Sigma}_n^2$ by

$$
\tilde{\Sigma}_n^2 = \begin{pmatrix}
1 & \sigma_{12}^2(n) & \cdots & \sigma_{1d}^2(n) \\
\sigma_{12}^2(n) & 1 & \cdots & \sigma_{2d}^2(n) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{1d}^2(n) & \sigma_{2d}^2(n) & \cdots & 1
\end{pmatrix}
$$

such that $\Sigma_n^2 = \text{diag}(\sigma_{11}^2(n), \ldots, \sigma_{dd}^2(n))\tilde{\Sigma}_n^2$. Furthermore, define the matrix $C_d$, which diagonal entries should be equal to one and the off-diagonal elements constants $c_{ij}$, then

$$
\lim_{n \to \infty} \Sigma_n^{-2} = \lim_{n \to \infty} (\Sigma_n^2)^{-1} = \lim_{n \to \infty} ((\sigma_{ij}^2(n))_{i,j=1,\ldots,d})^{-1} = \lim_{n \to \infty} (\text{diag}(\sigma_{11}^2(n), \ldots, \sigma_{dd}^2(n))\tilde{\Sigma}_n^2)^{-1} = (\lim_{n \to \infty} \text{diag}(\sigma_{11}^2(n), \ldots, \sigma_{dd}^2(n))C_d)^{-1} = C_d^{-1} \lim_{n \to \infty} \text{diag} \left( \frac{1}{\sigma_{11}^2(n)}, \ldots, \frac{1}{\sigma_{dd}^2(n)} \right),
$$

since the diagonal entries of a covariance matrix are of higher order than the off-diagonal ones and inverting a matrix is a continuous transformation. This leads to

$$
\lim_{n \to \infty} (\omega_{npq}^n)^{1/2} \Sigma_n^{-1} = \left( \lim_{n \to \infty} \omega_{pq}^n \Sigma_n^{-2} \right)^{1/2} = \left( C_d^{-1} \lim_{n \to \infty} \text{diag} \left( \frac{\omega_{pq}^n}{\sigma_{11}^2(n)}, \ldots, \frac{\omega_{pq}^n}{\sigma_{dd}^2(n)} \right) \right)^{1/2}
$$

and finally

$$
\lim_{n \to \infty} \left( \frac{\omega_{pq}^n}{\sigma_{pp}^n} \right)^{1/2} = \lim_{n \to \infty} \frac{\left( \sum_{j=-\infty}^{\infty} (\sum_{k=1}^{n} a_{pq}^{j-k})^2 \right)^{1/2}}{\sum_{l=1}^{d} \sum_{j=-\infty}^{\infty} (\sum_{k=1}^{n} a_{pl}^{j-k}) (\sum_{k=1}^{n} a_{pl}^{j-k})} = 0.
$$

Lemma 3.4. The sequence $B_{nj}$ fulfills the conditions (12).

Proof. First, we refer to the proof of the statement, which we get by setting $d$ equal to one in Theorem 2.1. As mentioned in the introduction it was investigated by Ibragimov and Linnik (1971). They used an inequality, which reads as follows, adapted to our situation

$$
(\sum_{k=1}^{n} a_{pq}^{j-k})^2 \leq \frac{1}{\omega_{pq}^n} \left( 2 \sum_{t=-\infty}^{\infty} a_{pq}^{t} \right)^2 + 4 \sum_{t=-\infty}^{\infty} \left( a_{pq}^{t} \right)^2.
$$

It bases on the fact that

$$
(\sum_{k=1}^{n} a_{pq}^{j-k})^2
$$
\[(a_{pq}^{j-1})^2 + (a_{pq}^{j-n-1})^2 + 2(a_{pq}^{j-1} - a_{pq}^{j-n-1}) \sum_{k=1}^{n} a_{pq}^{j-1-k-n} + (\sum_{k=1}^{n} a_{pq}^{j-1-k})^2.\]

Now, we are able to prove the conditions. For simplicity, we look on \(|B_{nj}^q|\) for each \(q \in 1, \ldots, d\) instead of \(\max_{1 \leq q \leq d} |B_{nj}^q|\). Then

\[
\sup_{j \in \mathbb{Z}} |B_{nj}^q| = \sup_{j \in \mathbb{Z}} \left| \sum_{k=1}^{n} \sum_{i=1}^{d} t_i \Sigma_{ip}^{n} \sum_{p=1}^{d} a_{pq}^{j-k} \right|
\leq \sup_{j \in \mathbb{Z}} \left( \sum_{i=1}^{d} \sum_{p=1}^{d} |t_i \Sigma_{ip}^{n}| \left| \sum_{k=1}^{n} a_{pq}^{j-k} \right| \right)
\leq \sup_{j \in \mathbb{Z}} \left( \sum_{i=1}^{d} \sum_{p=1}^{d} |t_i | \left| \Sigma_{ip}^{n} \right| \left( \omega_{pq}^{n} \right)^{\frac{1}{2}} \right) \left( \frac{2 \sum_{i=-\infty}^{\infty} (a_{pq}^{t-1})^2}{\omega_{pq}^{n}} + 4 \sum_{t=-\infty}^{\infty} (a_{pq}^{t-1})^2 \right)^{\frac{1}{2}}
\rightarrow 0,
\]

since \(\Sigma_{ip}^{n} (\omega_{pq}^{n})^{\frac{1}{2}}\) converges to zero for each \(i, p, q \in \{1, \ldots, d\}\) by Lemma 3.3.

The second condition follows from equality (10), since

\[
\lim\sup_{n \to \infty} \sum_{j \in \mathbb{Z}} |B_{nj}^q|^2 = \lim\sup_{n \to \infty} \sum_{j \in \mathbb{Z}} \left| \sum_{k=1}^{n} \sum_{i=1}^{d} t_i \Sigma_{ip}^{n} a_{pq}^{j-k} \right|^2
\leq \lim\sup_{n \to \infty} \sum_{j \in \mathbb{Z}} \sum_{q=1}^{d} \sum_{k=1}^{n} t_i \sum_{p=1}^{d} \Sigma_{ip}^{n} a_{pq}^{j-k}^2
\leq \sum_{i=1}^{d} t_i^2 < \infty.
\]

Now, we are able to finish the proof. Since the process behaves like a Gaussian one and the variances are given by (10), \(t' \Sigma_{n}^{-1} \sum_{k=1}^{n} X_k\) converges in distribution to \(t'Z\) where \(t'Z\) is \(\mathcal{N}(0, t'I)\)-distributed. This is the distribution of \(t'Z\) if \(Z\) possesses an \(\mathcal{N}(0, I)\) distribution.

**Proof of Corollary 2.2.** We only remind to the proof idea used for the one-dimensional case in Ibragimov and Linnik (1971). Define the process \((Y_j)_{j \in \mathbb{Z}}\) by

\[Y_j = \int_{j-1}^{j} X(t) dt.\]
It fulfills the conditions of Theorem 2.1 since
\[ \sum_{j=1}^{n} Y_j = \sum_{j=1}^{n} \int_{j-1}^{j} X(t) dt = \int_{0}^{n} X(t) dt. \]

Then, it remains to prove that the expression \( \Sigma_{T}^{-1} \sum_{T}^{T} X(t) dt \) has the same convergence behavior as \( \Sigma_{T}^{-1} \sum_{j=1}^{T} Y_j \).

\[ \square \]

4. Preliminary Results

Before getting into details with the limit processes, we investigate a suitable normalization matrix. We use the notation \( S_n(H) = \sum_{k=1}^{n} H(X_k) \).

**Lemma 4.1.** Let \( (X_k)_{k \in \mathbb{Z}} \) be as in Definition 1.1, such that the matrix \( R \) is invertible. Then a matrix-valued normalization sequence \( (A_{n}^{-1} A_{n}^{-1})_{n \geq 1} \), which fulfills
\[ \lim_{n \to \infty} \text{Var}(A_{n}^{-1} \sum_{k=1}^{n} H(X_k)) = I, \]
where the function \( H \) is defined in (7), is given by
\[ A_{n}^{-1} \sim \left( (-1)^{i+m} n^{\tau \max(l,m)-1} a_{lm} \right)_{l,m=1,\ldots,d} \]
with \( \tilde{m} = \min\{l,k\} \) and \( \sum_{m=1}^{\tilde{m}} a_{lm} a_{mk} = \frac{\det(X_{ki})}{\det(X)} \), where
\[ X = (x_{ij})_{i,j=1,\ldots,d} = \left( \frac{\tau!}{(2-\tau(d_i+d_j))(1-\tau(d_i+d_j))} (R_{ij}^{\tau} + R_{ji}^{\tau}) \right)_{i,j=1,\ldots,d}. \]

**Proof.** We assume that the process is LRD in sense of Definition 1.1 and satisfies \( d_1 > \cdots > d_d \).

The covariance matrix of the partials sum is given by
\[ A^2(n) := \text{E}(S_n(H)(S_n(H))^\prime) \]
\[ = \text{E}(S_n^i(H)S_n^j(H))_{i,j=1,\ldots,d} \]
\[ \sim \left( \frac{\tau!}{(2-\tau(d_i+d_j))(1-\tau(d_i+d_j))} n^{2-\tau(d_i+d_j)} (R_{ij}^{\tau} + R_{ji}^{\tau}) \right)_{i,j=1,\ldots,d}, \]
where \( \sim \) means a componentwise asymptotic equivalence.

To calculate the inverse matrix we use adjugate matrices. Therefore, the determinant of \( A^2(n) \) could be written as
\[ \det(A^2(n)) = n^{2d-2\tau} \sum_{i=1}^{d} d_i \det(X). \]

Analogously, the determinant of the adjugate is given by
\[ \det(A_{ji}^2(n)) = n^{2(d-1)} n^{-\tau} \sum_{l \in \{1,\ldots,d\} \setminus \{i\}} d_i - \tau \sum_{l \in \{1,\ldots,d\} \setminus \{j\}} d_j \det(X_{ji}). \]
Then, the inverse could be written as
\[
A^{-2}(n) = \frac{\text{adj}(A^2(n))}{\det(A^2(n))} = \left((-1)^{i+j} \frac{\det(A_{ji}^2(n))}{\det(A^2(n))}\right)_{i,j=1,...,d}
\]

Multiplying, the squared matrix by itself we get
\[
A^{-1}(n)A^{-1}(n) = \left((-1)^{l+k} \sum_{m=1}^{d} \tau_{\text{max}(l,m)}^{-1} a_{lm} \tau_{\text{max}(m,k)}^{-1} a_{mk}\right)_{l,k=1,...,d}
\]
with \(\tilde{m} = \min\{l,k\}\) and \((-1)^{l+k} \sum_{m=1}^{\tilde{m}} a_{lm} a_{mk} = (-1)^{l+k} \frac{\det(X_{lk})}{\det(X)}\). The existence of the entries follows by solving the equation system by iterative plugging in.

In the proofs of Theorem 2.3 and 2.4, we need to derive the convergence of the finite-dimensional distributions and tightness. We will anticipate the proof of tightness, since it coincides for both theorems.

**Lemma 4.2.** Let \((X_k)_{k \in \mathbb{Z}}\) be as in Definition 1.1, such that the matrix \(R\) is invertible. Then the partial sum of the componentwise subordinated process normalized by a matrix \(A(n)^{-1} = (a(n)_{pi})_{p,i=1,...,d}\) with
\[
E\left(\sum_{k=1}^{n} G_{i}(X_{ik})\right)^2 = O(a(n)^2_{pi}) \text{ for } p = 1, ..., d
\]
is tight.

**Proof.** We have to prove that the vector
\[
\left(\sum_{i=1}^{d} Z_{pi}(n)\right)_{p=1,...,d}
\]
with \(Z_{pi}(n) := a(n)_{pi} \sum_{k=1}^{nt} G_{i}(X_{ik})\) is tight. We define
\[
(Z_{11}(n), ..., Z_{1d}(n), Z_{21}(n), ..., Z_{2d}(n), ..., Z_{d1}(n), ..., Z_{dd}(n))
\]
Each component \(Z_{pi}(n)\) is tight in \(D[0,1]\) by (14) and Taqqu (1975, Theorem 2.1). Hence, the vector above is tight in \(D[0,1]^J\), where \(J = d^2\) by Lemma 3.10 in Bai and Taqqu (2013). The space \(D[0,1]\) is equipped with the uniform metric \(d\) and the product space \(D[0,1]^J\) with
\[
d_J(X,Y) := \max_{1 \leq j \leq J} d(X_j, Y_j)
\]
for \( X, Y \in D[0,1]^J \). Applying the continuous function \( f : D[0,1]^J \to D[0,1]^d \) with
\[
f(v_{11}, \ldots, v_{1d}, \ldots, v_{d1}, \ldots, v_{dj}) = \left( \sum_{i=1}^{d} v_{1i}, \ldots, \sum_{i=1}^{d} v_{di} \right)
\]
we get the assertion. The continuity of the function follows by
\[
d_d \left( \sum_{i=1}^{d} v_{1i}, \ldots, \sum_{i=1}^{d} v_{di}, \sum_{i=1}^{d} w_{1i}, \ldots, \sum_{i=1}^{d} w_{di} \right) \leq \sum_{i=1}^{d} \max_{1 \leq j \leq d} d(v_{ji}, w_{ji}) \leq d \max_{1 \leq i \leq d} \max_{1 \leq j \leq d} d(v_{ji}, w_{ji}),
\]

since \( d_f(v, w) = \max_{(j,i) \in \{(1,1),(1,2),\ldots,(J,J)\}} d(v_{ji}, w_{ji}) < \delta \).

\[\square\]

5. Proof of Theorem 2.3

By Proposition 3.1 in Kechagias and Pipiras (2015) the autocovariance function of linear processes as in (1) defined by (3) fulfill (5) and (6) with
\[
R_{ij} = \frac{\Gamma(d_i)\Gamma(d_j)}{\Gamma(d_i + d_j)} \left( c_{ij}^1 \frac{\sin(\pi d_i)}{\sin(\pi d_i + d_j)} + c_{ij}^2 \frac{\sin(\pi d_j)}{\sin(\pi d_i + d_j)} \right),
\]

where
\[
c_{ij}^1 = (A^-(A^+)')_{ij}, \quad c_{ij}^2 = (A^-(A^+)')_{ij}, \quad c_{ij}^3 = (A^+(A^+)')_{ij}.
\]

Especially it holds \( R_{ii} \neq 0 \) by condition C1. Furthermore, by condition C2 the matrix \( R \) is invertible and we could apply Lemma 4.1. Note, that \( H_i(x) = x \) for \( i = 1, \ldots, d \), i.e. \( \tau = 1 \) in the mentioned Lemma and we get the normalization sequence
\[
A^{-1}(n) \sim \left( (-1)^{l+m} n^{d_{\max(\{l,m\})} - 1} a_{lm} \right)_{l,m=1,\ldots,d}.
\]

First, we calculate the cross-covariance matrix of
\[
A^{-1}(n)S_{nt} = \left( \sum_{m=1}^{d} (-1)^{l+m} n^{d_{\max(\{l,m\})} - 1} a_{lm} \sum_{k=1}^{\lfloor nt \rfloor} X_k^m \right)_{l,m=1,\ldots,d},
\]

which leads to
\[
E(A^{-1}(n)S_{nt})(A^{-1}(n)S_{nu}')
= \left( \sum_{i=1}^{d} \sum_{j=1}^{d} (-1)^{l+i+m+j} \frac{a_{li}a_{mj}}{n^{2-d_{\max(l,i)}-d_{\max(m,j)}}} E(\sum_{k=1}^{\lfloor nt \rfloor} X_k^l \sum_{k=1}^{\lfloor nu \rfloor} X_k^j) \right)_{l,m=1,\ldots,d}
\sim \left( \sum_{i=1}^{d} \sum_{j=1}^{d} (-1)^{l+i+m+j} n^{d_{\max(l,i)} + d_{\max(m,j)} - 2} a_{li}a_{mj} \frac{n^{2-d_l-d_j}}{(1-d_l-d_j)(2-d_l-d_j)} \right)_{l,m=1,\ldots,d}
\]
\[
(R_{ij} n^{2-d_l-d_j} + R_{ji} n^{2-d_l-d_j} - R_{ij}(t-u)|t-u|^{2-d_l-d_j})_{l,m=1,\ldots,d}
\]
\[
J, J
\]
\[
J
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J
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J
\]
\[
\begin{align*}
&= \left( \sum_{i=1}^{d} \sum_{j=1}^{d} (-1)^{l+i+m+j} \frac{1}{a_{li}a_{mj}(1-d_i-d_j)(2-d_i-d_j)} (R_{ij}t^{2-d_i-d_j} + R_{ji}u^{2-d_i-d_j} - R_{ij}(t-u)|t-u|^{2-d_i-d_j}) \right)_{l,m=1,...,d} \\
&= A(t^{I-D} \tilde{R}t^{I-D} + u^{I-D} \tilde{R}u^{I-D} - |t-u|^{I-D} \tilde{R}(t-u)|t-u|^{I-D})A' \\
\end{align*}
\]

with \( \tilde{R}(t) = \left( \frac{1}{(1-d_i-d_j)(2-d_i-d_j)} \right)_{i,j=1,...,d'} \), where

\[
R_{ij}(t) = \begin{cases} 
R_{ij}, & \text{if } t > 0, \\
R_{ji}, & \text{if } t < 0 
\end{cases}
\]
and

\[
A = (A_{li})_{l,i=1,...,d} = \begin{cases} 
(-1)^{l+i}a_{li}, & \text{if } l \leq i, \\
0, & \text{otherwise.} 
\end{cases}
\]

We need to prove the convergence of the finite-dimensional distributions, i.e.

\[
A(n)^{-1}S_{[nt]} \overset{f.d.d.}{\to} A\mathcal{B}_H(t),
\]

with \( H = I - D \), which is equivalent to

\[
\lambda' A(n)^{-1}S_{[nt]} \overset{f.d.d.}{\to} \lambda' A\mathcal{B}_H(t),
\]

for all non-zero \( \lambda \in \mathbb{R}^d \) by Cramér-Wold device. The left side could be written as

\[
\lambda' A(n)^{-1}S_{[nt]} = \sum_{j=-\infty}^{\infty} \sum_{p=1}^{d} \sum_{q=1}^{d} a_{pq}^{j-k} \varepsilon_j^{(q)} =: \sum_{q=1}^{d} \sum_{j=-\infty}^{\infty} B_{nj}^q(t_z)\varepsilon_j^{(q)},
\]

where \( a_{pq}^{j-k} \) is given by (3) and \( a(n)_{ip} \) are the entries of the normalization matrix \( A(n)^{-1} \). Then, the convergence of the finite-dimensional distributions means that

\[
\left( \sum_{q=1}^{d} \sum_{j=1}^{d} B_{nj}^q(t_1)\varepsilon_j^{(q)}, \ldots, \sum_{q=1}^{d} \sum_{j=1}^{d} B_{nj}^q(t_z)\varepsilon_j^{(q)} \right)
\]

holds in \( \mathbb{R}^d \) for all \( z \in \mathbb{N} \) and all fixed \( \lambda_1, ..., \lambda_d \in \mathbb{R}, t_1, ..., t_z \in [0,1] \). We want to apply Lemma 3.2, thereto we write

\[
\left( \sum_{q=1}^{d} \sum_{j=1}^{d} B_{nj}^q(t_1)\varepsilon_j^{(q)}, \ldots, \sum_{q=1}^{d} \sum_{j=1}^{d} B_{nj}^q(t_z)\varepsilon_j^{(q)} \right)
\]
\[
= \sum_j \left( \begin{array}{ccc}
B_{nj}^1(t_1) & \cdots & B_{nj}^d(t_1) \\
\vdots & \ddots & \vdots \\
B_{nj}^1(t_z) & \cdots & B_{nj}^d(t_z)
\end{array} \right) \left( \begin{array}{c}
(\varepsilon_j)^{(1)} \\
\vdots \\
(\varepsilon_j)^{(d)}
\end{array} \right)
\]
\]
\[
= \sum_j B_{nj} \varepsilon_j.
\]

This representation makes it possible to proceed as in the proof of Theorem 2.1.

**Lemma 5.1.** The sequence of matrices \(B_{nj}^q\) fulfills the conditions (12).

**Proof.** The operator norm of the matrix is given by

\[
\|B_{nj}\|_{\text{op}} = \max_{1 \leq q \leq d} \sum_{i=1}^z |B_{nj}^q(t_i)|.
\]

It remains to prove the statement for \(|B_{nj}^q(t)|\) for all \(q \in 1, ..., d\) and \(t \in [0, 1]\). Referring to inequality (13) we have

\[
\frac{(\sum_{k=1}^{\lceil nt \rceil} a_{pq}^k)^2}{(\omega_{pq}^{|nt|})^2} \leq \frac{1}{\omega_{pq}^{|nt|}} \left( \frac{2 \sum_{l=-\infty}^{\infty} (a_{pq}^l)^2}{\omega_{pq}^{|nt|}} + 4 \sum_{l=-\infty}^{\infty} (a_{pq}^{l-1})^2 \right).
\]

Now, we are able to prove the conditions.

\[
\sup_{j \in \mathbb{Z}} |B_{nj}^q(t)|
\]
\[
= \sup_{j \in \mathbb{Z}} \sum_{i=1}^d \sum_{k=1}^{\lceil nt \rceil} \lambda_i a(n)_{ip} \sum_{p=1}^d a_{pq}^{j-k} |d|
\]
\[
\leq \sum_{i=1}^d \sum_{p=1}^d |\lambda_i| a(n)_{ip} (\omega_{pq}^{|nt|})^{\frac{1}{2}} \left( \frac{(\omega_{pq}^{|nt|})^2}{\omega_{pq}^{\lceil nt \rceil}} \right)^{\frac{1}{2}} \left( \frac{2 \sum_{l=-\infty}^{\infty} (a_{pq}^l)^2}{\omega_{pq}^{\lceil nt \rceil}} + 4 \sum_{l=-\infty}^{\infty} (a_{pq}^{l-1})^2 \right) \rightarrow 0,
\]

since

\[
\frac{\omega_{pq}^{|nt|}}{\omega_{pq}^{\lceil nt \rceil}} \sim \frac{(nt)^{2-2d_p} C_{pq}}{n^{2-2d_p} C_{pq}} = t^{2-2d_p}
\]

and \(a(n)_{ip} (\omega_{pq}^n)^{\frac{1}{2}} \) converges to zero for each \(i, p, q \in \{1, ..., d\} \) by Lemma 3.3.

The second condition follows from Lemma 10, since

\[
\lim_{n \to \infty} \sup_{j \in \mathbb{Z}} \sum_{i=1}^d |B_{nj}^q(t)|^2
\]
\[
= \lim_{n \to \infty} \sup_{j \in \mathbb{Z}} \sum_{k=1}^{\lceil nt \rceil} \sum_{i=1}^d \sum_{p=1}^d |\lambda_i| a(n)_{ip} a_{pq}^{j-k} |^2
\]
\[
\limsup_{n \to \infty} \sum_{j \in \mathbb{Z}} \sum_{q=1}^{d} | \sum_{k=1}^{d} \sum_{i=1}^{d} \lambda_i \sum_{p=1}^{d} a(n)_{ip} a_{pq}^j |^2
\]
\[= \sum_{l=1}^{d} \sum_{m=1}^{d} \lambda_l \lambda_m \sum_{i=l}^{d} \sum_{j=m}^{d} (-1)^{i+m+j} a_{li} a_{mj} (\tilde{R}_{ij} + \tilde{R}_{ji}) t^{2-d_i-d_j} < \infty.\]

The process could be treated as a linear one with Gaussian innovations. By Gikhman and Skorokhod (1969, Chapter I, Section 3, Theorem 4), it is sufficient to calculate the componentwise convergence behavior of the cross-covariances as we did at the beginning of the section. Then it converges to a multivariate Gaussian process \(AZ(t)\), where \(Z(t)\) has the covariance matrix
\[
E(Z(t)Z(u)') = t^{I-D} \tilde{R} t^{I-D} + u^{I-D} \tilde{R} u^{I-D} - |t - u|^{I-D} \tilde{R}(t - u)|t - u|^{I-D}.
\]
It remains to show that \(Z(t)\) is a OFBM \(B_H(t)\) with \(H = I - D\).

**Lemma 5.2.** The process \(Z(t)\) is an operator fractional Brownian motion.

**Proof.** We still know that the process is Gaussian. Furthermore, it has stationary increments since
\[
E(Z(t) - Z(u))(Z(t) - Z(u))' = |t - u|^{I-D} \tilde{R}(t - u)|t - u|^{I-D}.
\]
The previous equality proves also that the process is continuous in probability.

We prove that the process is proper. Therefore, we regard
\[
E(Z(t)Z(t)') = 2t^{I-D} \tilde{R} t^{I-D},
\]
this matrix is invertible, i.e. it has full rank. In fact, \(\tilde{R}\) is invertible by C1. Since \(E(Z(t)Z(t)')\) has full rank, the process is proper.

It remains to prove that the process is operator self-similar.

Since the process is continuous in probability, Theorem 5 in Hudson and Mason (1982) and the convergence
\[
A^{-1}A(n)^{-1} S_{\lfloor nt \rfloor} \to Z(t)
\]
imply that the process is operator self-similar. \(\square\)

Notice, that \(A\) is invertible since \(X^{-1} = AA'\). See Lemma 4.1 for a precise definition of \(X\).

6. **Proof of Theorem 2.4**

The proof depends strongly on the use of the multivariate techniques investigated by Bai and Taqqu (2013).

We have to prove the convergence of the finite-dimensional distributions. The asymptotic behavior of the matrix-valued normalization sequence
\[
A^{-1}(n) \sim \left( (-1)^{l+m} n^{\tau_d \max(l,m) - 1} a_{lm} \right)_{l,m=1,\ldots,d}
\]
Lemma 6.1. If the convergence in (8) holds with the first summand of its Hermite expansion replacing \( G \), i.e.
\[
\sum_{k=1}^{[nt]} h_{\tau,i} H_{\tau}(X_k^i)
\]
then it also holds for \( \left( \sum_{k=1}^{[nt]} G_i(X_k^i) \right) \).

Proof. First, notice \( A(n)^{-1} = (a(n)_{pi})_{p,i=1,...,d} \). Rewriting the vector to
\[
A(n)^{-1} \left( \sum_{k=1}^{[nt]} G_i(X_k^i) \right) = \left( \sum_{i=1}^{d} a(n)_{pi} \sum_{k=1}^{[nt]} G_i(X_k^i) \right)_{p=1,...,d}
\]
and using CramérvWold device, we get
\[
\sum_{p=1}^{d} t_p \sum_{i=1}^{d} a(n)_{pi} \sum_{k=1}^{[nt]} G_i(X_k^i)
\]
\[
= \sum_{p=1}^{d} t_p \sum_{i=1}^{d} a(n)_{pi} \sum_{k=1}^{[nt]} h_{\tau,i} H_{\tau}(X_k^i) + \sum_{p=1}^{d} t_p \sum_{i=1}^{d} a(n)_{pi} \sum_{k=1}^{[nt]} \sum_{j=\tau+1}^{\infty} h_{j,i} H_{j}(X_k^i).
\]

Therefor, it is enough to show that the variances of the second part converge to zero. We use the Jensen inequality.
\[
E \left( \sum_{p=1}^{d} t_p \sum_{i=1}^{d} a(n)_{pi} \sum_{k=1}^{[nt]} \sum_{j=\tau+1}^{\infty} h_{j,i} H_{j}(X_k^i) \right)^2
\]
\[
\leq d \sum_{p=1}^{d} t_p^2 E \left( \sum_{i=1}^{d} a(n)_{pi} \sum_{k=1}^{[nt]} \sum_{j=\tau+1}^{\infty} h_{j,i} H_{j}(X_k^i) \right)^2
\]
\[
= d \sum_{p=1}^{d} t_p^2 \sum_{i=1}^{d} \sum_{l=1}^{d} (-1)^{i+l} n^{\tau_{max}(p,i)+\tau_{max}(p,l)} a_{pi}a_{pl} \sum_{j=\tau+1}^{\infty} j^1 \frac{|nt|^{-j(d_i+d_l)}(R_{dl}^j + R_{il}^j)}{(2-j(d_i+d_l))(1-j(d_i+d_l))} \rightarrow 0
\]

Lemma 6.2. The expression
\[
\sum_{m=1}^{d} (-1)^{l+m} n^{\tau_{max}(i,m)-1} a_{im} \sum_{k=1}^{[nt]} h_{\tau,m} H_{\tau}(X_k^m)
\]
converges to zero in distribution if \( m < l \).
Proof.

\[
E\left( \sum_{m=1}^{d} (-1)^{l+m} n^{\tau d_{\max(l,m)} - 1} a_{lm} \sum_{k=1}^{\lfloor nt \rfloor} h_{\tau,m} H_{\tau}(X_k^m) \right)^2
\]

\[
= \sum_{m=1}^{l-1} \sum_{r=1}^{l-1} (-1)^{m+r} n^{\tau d_l - 2} a_{lm} a_{tr} \sum_{k_1=1}^{\lfloor nt \rfloor} \sum_{k_2=1}^{\lfloor nt \rfloor} h_{\tau,m} h_{\tau,r} E\left( H_{\tau}(X_k^m) H_{\tau}(X_k^r) \right)
\]

\[
\sim \sum_{m=1}^{l-1} \sum_{r=1}^{l-1} (-1)^{m+r} n^{\tau d_l - 2} a_{lm} a_{tr} h_{\tau,m} h_{\tau,r} \frac{2(R_{\tau mr} + R_{\tau rm}) \tau!}{(2 - \tau(d_m + d_r))(1 - \tau(d_m + d_r))} (tn)^{2 - \tau(d_m + d_r)} \to 0
\]

The sequence \( n^{\tau(2d_l - (d_m + d_r))} \) converges to zero if \( m, r < l \), since we assumed \( d_1 > \ldots > d_d \). \( \square \)

Using the previous Lemma we are able to write

\[
A^{-\frac{1}{2}}(n) \left( \sum_{k=1}^{\lfloor nt \rfloor} h_{\tau,i} H_{\tau}(X_k^i) \right)_{i=1,...,d}
\]

\[
f.d.d. = \left( \sum_{m=1}^{d} (-1)^{l+m} n^{\tau d_m - 1} a_{lm} \sum_{k=1}^{\lfloor nt \rfloor} h_{\tau,m} H_{\tau}(X_k^m) \right)_{l=1,...,d}
\]

\[
= \sum_{m=1}^{d} \tilde{Z}_m(n).
\]

Define the vectors

\[
\tilde{Z}^m(n) = (\tilde{Z}^m_i(n))_{i=1,...,d} = \begin{cases} (-1)^{l+m} a_{lm} h_{\tau,m} Z_m(n), & \text{if } m \geq l, \\ 0, & \text{if } m < l, \end{cases}
\]

where

\[
Z_m(n) := n^{\tau d_m - 1} \sum_{k=1}^{\lfloor nt \rfloor} h_{\tau,m} H_{\tau}(X_k^m).
\]

This component processes are long-range dependent with LRD-parameter \( d_m \), i.e. we could apply Theorem 3.6.1. in Pipiras and Taqqu (2017):

\[
n^{\tau d_m - 1} \sum_{k=1}^{\lfloor nt \rfloor} h_{\tau,m} H_{\tau}(X_k^m) \overset{f.d.d.}{\to} h_{\tau,m} R_{\tau mm}^{\tau \beta_{\tau,d_m} I_m(f_{\tau,d_m,t})},
\]

where

\[
\beta_{\tau,d_m} = \left( \frac{(1 - \tau d_m)(1 - 2\tau d_m)}{\tau!(2\Gamma(2d_m) \sin(\pi(\frac{1}{2} - d_m)))} \right) \frac{1}{2}.
\]
This, and applying Lemma 4.5 in Bai and Taqqu (2013), results that each vector $\tilde{Z}^m(n)$ converges jointly, i.e.

$$\tilde{Z}^m(n) \overset{f.d.d.}{\to} X(d_m)$$

where

$$X(d_m) = (X_l(d_m))_{l=1,...,d} = \begin{cases} (-1)^{l+m}a_{lm} h_{\tau,m} R^{\tau}_{mm} \beta_{\tau,d_m} I^m_{\tau}(f_{\tau,d_m,t}) & \text{if } m \geq l, \\ 0 & \text{if } m < l, \end{cases}$$

which is equivalent to

$$\sum_{i=1}^{d} \omega_i \tilde{Z}^m_i(n) \overset{f.d.d.}{\to} \sum_{i=1}^{d} \omega_i X_i(d_m)$$

for each $(\omega_1, ..., \omega_d) \in \mathbb{R}^d$ by Cramér-Wold device. It remains to show the convergence

$$\left( \sum_{i=1}^{d} \omega_i \tilde{Z}^1_i(n), ..., \sum_{i=1}^{d} \omega_i \tilde{Z}^d_i(n) \right) \overset{f.d.d.}{\to} \left( \sum_{i=1}^{d} \omega_i X_i(d_1), ..., \sum_{i=1}^{d} \omega_i X_i(d_d) \right),$$

which may be written as

$$\sum_{i=1}^{d} \omega_i \tilde{Z}^1_i(n) = \sum_{i=1}^{d} \omega_i \tilde{Z}^d_i(n)$$

$$\overset{f.d.d.}{\to} \sum_{i=1}^{d} \omega_i \left( (-1)^{l+1} a_{i1} h_{\tau,1} Z_1(n), ..., (-1)^{l+d} a_{id} h_{\tau,d} Z_d(n) \right)$$

where the convergence follows from Theorem 2 in Ho and Sun (1990) and the continuous mapping theorem. Therefore,

$$\sum_{m=1}^{d} \tilde{Z}_m \overset{f.d.d.}{\to} \left( \sum_{m=l}^{d} (-1)^{l+m} a_{lm} h_{\tau,m} R^{\tau}_{mm} \beta_{\tau,d_m} I^m_{\tau}(f_{\tau,d_m,t}) \right)_{l=1,...,d}$$

The process could be written as

$$\left( \sum_{m=l}^{d} (-1)^{l+m} a_{lm} h_{\tau,m} R^{\tau}_{mm} \beta_{\tau,d_m} I^m_{\tau}(f_{\tau,d_m,t}) \right)_{l=1,...,d} = A(I^m_{\tau}(f_{\tau,d_m,t}))_{m=1,...,d}$$
with
\[ A = (A_l)_{l,i=1,...,d} \begin{cases} (-1)^{l+i}a_l h_{r,i} R_{ri} \beta_{r,d}, & \text{if } l \leq i, \\ 0, & \text{otherwise}. \end{cases} \]

7. Proof of Theorem 2.5

We start with the proof of part (i). As well we refer to Theorem 5 in Mas (2002). For simplicity we repeat his result for linear processes with values in \( \mathbb{R}^d \) like in (1).

**Theorem 7.1.** Suppose \( \sum_j \|A_j\|_{op} < \infty \) and \( E\|\varepsilon_0\|^4 < \infty \). Then
\[ (n^{-\frac{1}{2}}(\hat{\Gamma}_{n,h} - \Gamma))_{h=1,...,H} \xrightarrow{D} (G)_{h=1,...,H} \text{ as } n \to \infty, \]
where \( G = (G_h)_{h=1,...,H} \) is a zero mean Gaussian random Element with values in \( L(\mathbb{R}^d)^{H+1} \).

This leads us to a Lemma, which proves our statement. Note, that we assumed \((X_k)_{k \in \mathbb{N}}\) to be a purely non-deterministic process.

**Lemma 7.2.** Theorem 7.1 remains true if we assume a Gaussian process like in Definition 1.1 and replace the assumption \( \sum_j \|A_j\|_{op} < \infty \) by \( d_p \in (\frac{1}{4}, \frac{1}{2}) \) for \( p = 1, ..., d \).

**Proof.** The Gaussian process \((X_k)_{k \in \mathbb{N}}\) has a linear representation by the multivariate Wold decomposition
\[ X_k = \sum_{j=0}^{\infty} A_{j-k}\varepsilon_j \]
with \( A_0 = I_d, \sum_{j=0}^{\infty} \|A_j\|^2_F < \infty \) and \((\varepsilon_j)_{j \in \mathbb{Z}}\) Gaussian i.i.d. with covariance matrix \( \Sigma \).

Again, we refer to Mas (2002) and get the interim result
\[ n\text{Cov}(\hat{\Gamma}_{n,p}, \hat{\Gamma}_{n,q}) = \frac{1}{n} \sum_{l=0}^{n-1} (n-l)(\Psi_{l+p-q,l} + \Psi_{l+p-q,-l} + \Psi_{l+q,l-p} + \Psi_{l+q,-l-p}) \]
for all \( T \in L(\mathbb{R}^d) \), where \( \Psi_{r,s}(T) = \Gamma_r T \Gamma_s \) and
\[ \sum_{i=0}^{\infty} \theta_{i+l,i+l+q}(\Lambda - \Phi)\theta_{i,l}^s \]
\[ = \sum_{i=0}^{\infty} \left( a_{i+l+q} E(\langle \varepsilon_0 \xi_0, a_{i+l}^* T a_{i+p} \rangle \xi_0) a_{i+l}^* \right) \]
\[ - (a_{i+l+q} \Sigma a_{i+l}^* (T + T^*) a_{i+p} \Sigma a_{i+l}^* + a_{i+l+q} \langle \Sigma, a_{i+l}^* T a_{i+p} \rangle \Sigma a_{i+l}^* ) \].

Mas (2002) continued by proving that the dominated convergence theorem is applicable. It is the only part of the proof which has to be modified to our situation. In the other parts it
remains to use the fact $\sum_{j=0}^{\infty} \|A_j\|_F < \infty$, given by the Wold representation. We remind to the nuclear norm, which is defined by $\|A\|_* = \text{trace}(\sqrt{A^*A})$. Then
\[
\sum_{l=0}^{\infty} \|\Psi_{t+p-q,l}\|_* \\
\leq \sqrt{d} \sum_{l=0}^{\infty} \|\Gamma_{t+p-q}\|_F \|\Gamma_l\|_F \\
\leq \sqrt{d} \left( \sum_{l=0}^{\infty} \|\Gamma_{t+p-q}\|_F^2 \sum_l \|\Gamma_l\|_F^2 \right)^{1/2} \\
= \sqrt{d} \left( \sum_{l=0}^{\infty} \sum_{i,j=1}^{d} ((l+p-q)^{-d_i} R_{ij}(l+p-q)^{-d_j})^2 \sum_{l=0}^{\infty} \sum_{i,j=1}^{d} (l^{-d_i} R_{ij} l^{-d_j})^2 \right)^{1/2} < \infty,
\]
since $d_i > \frac{1}{4}$ for $i = 1, \ldots, d$.

It remains to prove the costlier, second part, of Theorem 2.5. First, we will give the $L(\mathbb{R}^{d\times d})$-valued normalization sequence.

**Lemma 7.3.** The $L(\mathbb{R}^{d\times d})$-valued normalization sequence $B^{-1}(n)(\cdot)B^{-1}(n)$ fulfills
\[
\text{Cov}(B^{-1}(n)\hat{\Gamma}_{n,p}B^{-1}(n), B^{-1}(n)\hat{\Gamma}_{n,q}B^{-1}(n)) \leq C_d,
\]
where the inequality holds componentwise and $C_d$ is an element in $L(\mathbb{R}^{d\times d})$. The normalization sequence is defined by $B^{-1}(n)$, which is similar to the one, calculated in Lemma 4.1. More precisely
\[
B^{-1}(n) \sim \left(n^{d_{\text{max}(l,m)}-\frac{1}{2}}\right)_{l,m=1,\ldots,d}.
\]

**Proof.** We denote $B^{-1}(n) = (b_{ij})_{i,j=1,\ldots,d}$. Furthermore, we use the formula
\[
(17) \quad E(X^{(i)}X^{(i_j)}) = \sigma_{i1i} \sigma_{i3i} + \sigma_{i1i} \sigma_{i3i} + \sigma_{i1i} \sigma_{i3i}
\]
with $\sigma_{ij} = E(X^{(i)}X^{(j)})$ to calculate higher moments of multivariate Gaussian random vectors. Then for each $T \in L(\mathbb{R}^{d})$ given in form of a matrix $T = (T_{ij})_{i,j=1,\ldots,d}$ it follows
\[
\text{Cov}(B^{-1}(n)\hat{\Gamma}_{n,p}B^{-1}(n), B^{-1}(n)\hat{\Gamma}_{n,q}B^{-1}(n)) \\
= B^{-1}(n) E((B^{-1}(n)\hat{\Gamma}_{n,p}B^{-1}(n), \cdot)\hat{\Gamma}_{n,q}B^{-1}(n)) \\
= \frac{1}{n^2} \sum_{k=1}^{n} \sum_{l=1}^{n} B^{-1}(n) E((B^{-1}(n)X_lX_{l+p}B^{-1}(n), \cdot)X_kX_{k+q})B^{-1}(n) \\
= \frac{1}{n^2} \sum_{k=1}^{n} \sum_{l=1}^{n} E(\text{Tr}(B^{-1}(n)X_lX_{l+p}B^{-1}(n)T)) \sum_{j_1,j_2=1}^{d} b_{i_1j_1} X_{j_1}^{(j_2)} X_{k+q}^{(j_2)} b_{j_2i_2})
\]
where \( \phi \) defined for Gaussian vectors with zero mean and covariance matrix \( \Sigma \) as follows

\[
\lambda \text{ for all } \rho \sim C \leq \sum_{s=1, r_1, r_2, r_3 = 1}^{d} b_{s r_1} \sum_{j_1, j_2 = 1}^{d} b_{j_1, j_2} (E(X_0^{r_1}) X_0^{r_2}) + E(X_0^{r_1}) X_0^{r_2} X_0^{r_2} X_0^{r_2} b_{j_1, j_2} T_{s r_3} b_{j_2}
\]

\[
\sum_{j_1, j_2, s = 1, r_1, r_2, r_3 = 1}^{d} b_{s r_1} b_{j_1, j_2} \sum_{k = 1}^{n} \frac{n - k}{n^2} k^{d_1 - d_2 - d_1 - d_2} (R_{r_1, j_1} R_{r_2, j_2} + R_{r_1, j_2} R_{r_2, j_1}) T_{s r_3}
\]

\[
\leq C \sum_{s, r_3 = 1, r_1, r_2, r_3 = 1}^{d} b_{s r_1} b_{j_1, j_2} \sum_{k = 1}^{n} \frac{n - k}{n^2} k^{d_1 - d_2 - d_1 - d_2} (R_{r_1, j_1} R_{r_2, j_2} + R_{r_1, j_2} R_{r_2, j_1}) T_{s r_3}
\]

\[
= C (B R' B') \cdot |B' R_+ B'| (B R' B')',
\]

where \( R_+ = (|R_{i j}|)_{i, j = 1, ..., d} \) and

\[
B = \begin{cases} 1, & \text{if } l \leq i, \\ 0, & \text{otherwise}. \end{cases}
\]

The idea of the remaining proof, is to show, that the convergence behavior of \( \Gamma_h \) does not depend on \( h \). Especially, we want to verify that it remains to prove

\[
\langle B^{-1}(n) (\Gamma_{n,0} - \Gamma_0) B^{-1}(n), \sum_{u, l = 1}^{d} \lambda_{u l} f_u (\epsilon_l, \cdot) \rangle \xrightarrow{D} \langle Z, \sum_{u, l = 1}^{d} \lambda_{u l} f_u (\epsilon_l, \cdot) \rangle, \quad \text{as } n \to \infty
\]

for all \( \lambda_{u l} \in \mathbb{R} \) with \( u, l = 1, ..., d \). Therefore, we first rewrite the expression

\[
\langle B^{-1}(n) (\Gamma_{n,h} - \Gamma_h) B^{-1}(n), \sum_{u, l = 1}^{d} \lambda_{u l} f_u (\epsilon_l, \cdot) \rangle
\]

in dependence of multivariate Hermite polynomials. Multivariate Hermite polynomials are defined for Gaussian vectors with zero mean and covariance matrix \( \Sigma \) as follows

\[
H_q(x) = H_q(x, \Sigma) = \frac{(-1)^{|q|}}{\phi_{\Sigma}(x)} \left( \frac{d}{dx} \right)^q \phi_{\Sigma}(x), \quad q \in \mathbb{N}^d, \quad x \in \mathbb{R}^d,
\]

where \( \phi_{\Sigma}(x) \) denotes the density of a multivariate Gaussian distribution and

\[
\left( \frac{d}{dx} \right)^q \frac{d^{|q|}}{dx^{|q|}} = \frac{d^{q_1 + \ldots + q_d}}{dx^{q_1} \ldots dx^{q_d}}.
\]
Proof. Referring to (17), we get

$$H(1,1)(x) = H_{(1,1)}(x, \Sigma) = y_1 y_2 - E(Y_1 Y_2)$$

with $y = \Sigma^{-1} x$. Then we get

$$\sum_{i=1}^{d} (B^{-1}(n)(\hat{\Gamma}_{n,h} - \Gamma_h)B^{-1}(n)(u_i), \sum_{u,l=1}^{d} \lambda_{ul} f_{ul}(e_{l}, u_{i}))$$

$$= \sum_{i=1}^{d} (B^{-1}(n) \frac{1}{n} \sum_{k=1}^{n} (X_k X_{k+h} - E(X_0 X_h')) B^{-1}(n)(u_i), \sum_{k,l=1}^{d} \lambda_{ul} f_{ul}(e_{l}, u_{i}))$$

$$= \sum_{u,l=1}^{d} \lambda_{ul} \frac{1}{n} \sum_{k=1}^{n} \left( \sum_{r=1}^{d} n_{d_{\max}(r,t)}^{-\frac{1}{2}} X_{k}^{(r)} \sum_{t=1}^{d} n_{d_{\max}(u,t)}^{-\frac{1}{2}} X_{k+h}^{(t)} - \langle (B^{-1}E(X_0 X'_h)B^{-1})^* f_{u}, e_{l} \rangle \right)$$

$$= \sum_{u,l=1}^{d} \sum_{k=1}^{n} \sum_{r=1}^{d} n_{d_{u}+d_{l}-1} H_{(1,1)}(n_{d_{u}}^{-\frac{1}{2}} X_{k}^{(u)}, n_{d_{l}}^{-\frac{1}{2}} X_{k+h}^{(l)})$$

For simplicity we define

$$K_{u,l} = \sum_{r=1}^{d} \sum_{t=1}^{d} \lambda_{rt}.$$ 

Lemma 7.4. The convergence behavior does not depend on h, i.e.

$$\sum_{k=1}^{n} \sum_{u,l=1}^{d} K_{u,l} n_{d_{u}+d_{l}-1} H_{(1,1)}(X_{k}^{(u)} X_{k+h}^{(l)})$$

$$= \sum_{k=1}^{n} \sum_{u,l=1}^{d} K_{u,l} n_{d_{u}+d_{l}-1} H_{(1,1)}(X_{k}^{(u)} X_{k+h}^{(l)}).$$

Proof. Referring to (17), we get

$$E \left( \sum_{k=1}^{n} \sum_{u,l=1}^{d} K_{u,l} n_{d_{u}+d_{l}-1} (H_{(1,1)}(X_{k}^{(u)} X_{k+h}^{(l)}) - H_{(1,1)}(X_{k}^{(u)} X_{k}^{(l)})) \right)^2$$

$$= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{u_1,l_1=1}^{d} \sum_{u_2,l_2=1}^{d} K_{u_1,l_1} K_{u_2,l_2} n_{d_{u_1}+d_{l_1}+d_{u_2}+d_{l_2}-2}$$

$$E \left( (X_{k_1}^{(u_1)} X_{k_1+h}^{(l_1)} - X_{k_1}^{(u_1)} X_{k_1}^{(l_1)}) (X_{k_2}^{(u_2)} X_{k_2+h}^{(l_2)} - X_{k_2}^{(u_2)} X_{k_2}^{(l_2)}) \right)$$
\[ \sim \sum_{k=1}^{n} (n-k) \sum_{u_1,l_1=1}^{d} \sum_{u_2,l_2=1}^{d} K_{u_1,l_1} K_{u_2,l_2} n^{d_{u_1}+d_{l_1}+d_{u_2}+d_{l_2}-2} \]

\[ E \left( (X_0^{(u_1)} X_{k}^{(l_1)} - X_0^{(u_1)} X_{k}^{(l_1)}) (X_0^{(u_2)} X_{k}^{(l_2)} - X_0^{(u_2)} X_{k}^{(l_2)}) \right) + \]

\[ E \left( (X_k^{(u_1)} X_{k+h}^{(l_1)} - X_k^{(u_1)} X_{k}^{(l_1)}) (X_0^{(u_2)} X_{k}^{(l_2)} - X_0^{(u_2)} X_{k}^{(l_2)}) \right) \]

\[ = \sum_{k=1}^{n} (n-k) \sum_{u_1,l_1=1}^{d} \sum_{u_2,l_2=1}^{d} K_{u_1,l_1} K_{u_2,l_2} n^{d_{u_1}+d_{l_1}+d_{u_2}+d_{l_2}-2} \]

\[ E \left( X_0^{(u_1)} X_{k}^{(l_1)} X_{k}^{(l_2)} X_{k+h}^{(l_1)} X_0^{(u_2)} X_{k}^{(l_2)} - X_0^{(u_1)} X_{k}^{(l_1)} X_0^{(u_2)} X_{k}^{(l_2)} \right) + \]

\[ E \left( X_k^{(u_1)} X_{k+h}^{(l_1)} X_0^{(u_2)} X_{k}^{(l_2)} - X_k^{(u_1)} X_k^{(l_1)} X_0^{(u_2)} X_k^{(l_2)} \right) \]

\[ \rightarrow 0, \]

since the last summands are all asymptotically equal. We calculate one of them explicitly by

\[ E(X_0^{(u_1)} X_{k}^{(l_1)} X_{k}^{(l_2)} X_{k+h}^{(l_1)}) \]

\[ = h^{-d_{u_1}} R_{u_1,l_1} h^{-d_{l_1}} h^{-d_{u_2}} R_{u_2,l_2} h^{-d_{l_2}} + k^{-d_{u_1}} R_{u_1,u_2} K^{-d_{u_2}} K^{-d_{l_1}} R_{l_1,l_2} k^{-d_{l_2}} + \]

\[ (k+h)^{-d_{u_1}} R_{u_1,l_2} (k+h)^{-d_{l_1}} k^{-d_{u_2}} R_{l_1,u_2} (k+h)^{-d_{l_2}} \]

\[ \sim h^{-d_{u_1}} R_{u_1,l_2} R_{u_1,l_2} R_{u_1,u_2} R_{l_1,l_2}. \]

The other terms could be treated analogously.

To continue we rewrite the process in terms of univariate Hermite polynomials as well we set \( l_i = i(u,l) \), where \( i(u,l) \) denotes the number of \( l,u \) that are equal to \( i \). We get a situation similar to Arcones (1994)

\[ \sum_{k=1}^{n} \sum_{u,l=1}^{d} K_{u,l} n^{d_{u}+d_{l}-1} H_{(1,1)}(X_{k}^{(u)} , X_{k}^{(l)}) \]

\[ = n^{-1} \sum_{k=1}^{n} \sum_{l_1+l_2+\ldots+l_d=2} C_{l_1,\ldots,l_d} \prod_{p=1}^{d} n^{l_{p}d_{p}} H_{l_{p}}(X_{k}^{(p)}) \]
with
\[ C_{\tau_{1},...\tau_{d}} = E\left( \sum_{u,l=1}^{d} K_{u,l} H_{(1,1)}(X^{(u)}_{k}, X^{(l)}_{k}) \prod_{p=1}^{d} H_{p}(X^{(p)}_{k}) \right). \]

The following Lemma proves the statement for an arbitrary Hermitian rank.

**Lemma 7.5.** Let \( (X_{k})_{k \in \mathbb{N}} \) be a d-dimensional, long-range dependent Gaussian process as in Theorem 2.5. Then
\[
\sum_{k=1}^{n} \sum_{l_{1}+...+l_{d}=\tau}^{n} c_{\tau_{1},...\tau_{d}} \prod_{p=1}^{d} (l_{p}!)^{-1} n^{d+p} H_{p}(X^{(p)}_{k})
\]
with \( c_{\tau_{1},...\tau_{d}} = E(f(X_{1}) \prod_{i=1}^{d} H_{i}(X^{(i)}_{1})) \) converges weakly to
\[
\sum_{p=1}^{d} e_{j_{1},...j_{\tau}} H_{(1,...,1)}(X^{(j_{1})}_{1})...H_{(j_{\tau})}(X^{(j_{\tau})}_{1}),
\]
where \( B^{(1)},...,B^{(d)} \) are suitable Hermitian Gaussian random measures, \( K_{j_{1},...j_{\tau}}(\tau) \) are normalization constants and
\[
e_{j_{1},...j_{\tau}} = E(f(X_{1}) \prod_{i=1}^{d} H_{i(l_{1},...,l_{\tau})}(X^{(i)}_{1})),
\]
where \( i(l_{1},...,l_{\tau}) \) denotes the number of \( l_{1},...,l_{\tau} \) that are equal to \( i \).

**Proof.** First, we refer to Arcones (1994) for the following equality. For \( \tau \in \mathbb{N} \) and \( a_{1},...,a_{d} \in \mathbb{R} \) with \( \sum_{j=1}^{d} a_{j}^{2} = 1 \) we have
\[
(18) \quad H_{\tau} \left( \sum_{j=1}^{d} a_{j}x_{j} \right) = \sum_{p_{1}+...+p_{d}=\tau}^{\tau} \left( \prod_{j=1}^{d} a_{j}^{p_{j}} \right) H_{p_{1}}(x_{j})...H_{p_{d}}(x_{j}).
\]
See Buchsteiner (2017) for a detailed proof. To apply this equality, we introduce the matrix
\[
A = \left( \prod_{i=1}^{d} (a^{(i)}_{k_{1},...,k_{d}})^{p_{i}} \right)_{k_{1}+...+k_{d}=\tau \atop p_{1}+...+p_{d}=\tau}.
\]
There exist \( a^{(1)}_{k_{1},...,k_{d}},...a^{(d)}_{k_{1},...,k_{d}} \) for \( k_{1}+...+k_{d} = \tau \) such that the matrix is invertible by independence of multivariate monomials of degree \( \tau \). Normalizing leads to \( \sum_{p=1}^{d} (a^{(p)}_{k_{1},...,k_{d}})^{2} = 1 \) for each \( k_{1},...,k_{d} \). Furthermore, we could define a matrix \( \beta(k_{1},...,k_{d},l_{1},...,l_{d}) \), such that
\[
\sum_{k_{1}+...+k_{d}=\tau} b(k_{1},...,k_{d},l_{1},...,l_{d})(a^{(1)}_{k_{1},...,k_{d}})^{p_{1}}... (a^{(d)}_{k_{1},...,k_{d}})^{p_{d}}
\]
\[
= \begin{cases} 
(\tau!)^{-1} \prod_{p=1}^{d} l_{p}! & \text{if } (p_{1},...,p_{d}) = (l_{1},...,l_{d}), \\
0 & \text{otherwise.}
\end{cases}
\]
Now, using this, we are able to apply (18).

\[ n^{-1} \sum_{l_1 + \ldots + l_d = \tau} \prod_{p=1}^{d} (l_p)!^{-1} b(k_1, \ldots, k_d, l_1, \ldots, l_d) H_{\tau} \left( \sum_{p=1}^{d} a_{k_1, \ldots, k_d}^{(p)} n^{d_p} X_k^{(p)} \right) \]

\[ = n^{-1} \sum_{l_1 + \ldots + l_d = \tau} c_{l_1, \ldots, l_d} \prod_{p=1}^{d} (l_p)!^{-1} b(k_1, \ldots, k_d, l_1, \ldots, l_d)^* \]

\[ \prod_{i=1}^{d} (p_i!)^{-1} (a_{k_1, \ldots, k_d}^{(i)})^p H_{\tau_i} \left( n^{d_i} X_j^{(i)} \right) \]

\[ = n^{-1} \sum_{l_1 + \ldots + l_d = \tau} c_{l_1, \ldots, l_d} \prod_{p=1}^{d} (l_p)!^{-1} H_{\tau_p} \left( n^{d_p} X_k^{(p)} \right) \]

Define \( \gamma(k_1, \ldots, k_d) = \sum_{l_1 + \ldots + l_d = \tau} c_{l_1, \ldots, l_d} \prod_{p=1}^{d} (l_p)!^{-1} b(k_1, \ldots, k_d, l_1, \ldots, l_d) \), then

\[ n^{-1} \sum_{k=1}^{n} \sum_{l_1 + \ldots + l_d = \tau} c_{l_1, \ldots, l_d} \prod_{p=1}^{d} (l_p)!^{-1} H_{\tau_p} \left( n^{d_p} X_k^{(p)} \right) \]

\[ = n^{-1} \sum_{k=1}^{n} \sum_{l_1 + \ldots + l_d = \tau} \gamma(k_1, \ldots, k_d) H_{\tau} \left( \sum_{p=1}^{d} a_{k_1, \ldots, k_d}^{(p)} n^{d_p} X_k^{(p)} \right) . \]

Let \( Z = (Z^{(1)}, \ldots, Z^{(d)}) \) be the vector-valued spectral measure. See Brockwell and Davis (1986) for more details. Then, referring to Ho and Sun (1990)

\[ (n^{d_1} Z^{(1)}(n^{-1} A_1), \ldots, n^{d_d} Z^{(d)}(n^{-1} A_d)) \xrightarrow{w} (B^{(1)}(A_1), \ldots, B^{(d)}(A_d)) \]

for all bounded symmetric intervals \( A_1, \ldots, A_d \subseteq \mathbb{R} \). As well by Ho and Sun (1990), there exist Hermitian Gaussian random measures \( B^{(1)}, \ldots, B^{(d)} \), such that

\[ \left( n^{\tau_d - 1} \sum_{k=1}^{n} H_{\tau} \left( X_k^{(1)} \right), \ldots, n^{\tau_d - 1} \sum_{k=1}^{n} H_{\tau} \left( X_k^{(d)} \right) \right) \xrightarrow{D} \left( c_1 I_{\tau}^1(f_{\tau,d_1,1}), \ldots, c_d I_{\tau}^d(f_{\tau,d_d,1}) \right) . \]

Note, that for any integrable function \( h \)

\[ \int_{\mathbb{R}^\tau} h(x_1, \ldots, x_\tau) \left( \sum_{p=1}^{d} a_{k_1, \ldots, k_d}^{(p)} B^{(p)} \right) (dx_1) \ldots \left( \sum_{p=1}^{d} a_{k_1, \ldots, k_d}^{(p)} B^{(p)} \right) (dx_\tau) \]

\[ = \sum_{j_1, \ldots, j_\tau = 1}^{d} a_{k_1, \ldots, k_d}^{(j_1)} \ldots a_{k_1, \ldots, k_d}^{(j_\tau)} \int_{\mathbb{R}^\tau} h(x_1, \ldots, x_\tau) B^{(j_1)}(dx_1) \ldots B^{(j_\tau)}(dx_\tau) \]
and then
\[
\left\{ n^{-1} \sum_{k=1}^{n} \sum_{p=1}^{d} \lambda_{k}^{(p)} \mathbb{E} \left[ \sum_{i=1}^{n} n_{i} \sum_{m=1}^{d} X_{k}^{(p)} \right] k_{1} + \ldots + k_{d} = \tau \right\}
\]
\[
\rightarrow \left\{ \sum_{j_{1}, \ldots, j_{r}=1}^{d} a_{k_{1}, \ldots, k_{d}}^{(j_{1})} \ldots a_{k_{1}, \ldots, k_{d}}^{(j_{r})} Z_{j_{1}, \ldots, j_{r}}(1) | k_{1} + \ldots + k_{d} = \tau \right\},
\]
where
\[
Z_{j_{1}, \ldots, j_{r}}(1) = \tilde{K}_{j_{1}, \ldots, j_{r}}(\tau) \int_{\mathbb{R}^{r}} e^{i(x_{1} + \ldots + x_{r})} - 1 \prod_{i=1}^{r} [x_{i}]^{-(1/2-dj_{i})} B^{(j_{1})}(dx_{1}) \ldots B^{(j_{r})}(dx_{r}).
\]

Therefore,
\[
n^{-1} \sum_{k=1}^{n} \sum_{k_{1} + \ldots + k_{d} = \tau} \gamma(k_{1}, \ldots, k_{d}) H_{\tau}(\sum_{p=1}^{d} a_{k_{1}, \ldots, k_{d}}^{(p)} n_{i}^{(p)} X_{k}^{(p)})
\]
\[
\overset{D}{\rightarrow} \sum_{j_{1}, \ldots, j_{r}=1}^{d} \sum_{k_{1} + \ldots + k_{d} = \tau} \gamma(k_{1}, \ldots, k_{d}) a_{k_{1}, \ldots, k_{d}}^{(j_{1})} \ldots a_{k_{1}, \ldots, k_{d}}^{(j_{r})} Z_{j_{1}, \ldots, j_{r}}(1).
\]

It remains to rewrite the limit process as
\[
\sum_{j_{1}, \ldots, j_{r}=1}^{d} \sum_{k_{1} + \ldots + k_{d} = \tau} \gamma(k_{1}, \ldots, k_{d}) a_{k_{1}, \ldots, k_{d}}^{(j_{1})} \ldots a_{k_{1}, \ldots, k_{d}}^{(j_{r})}
\]
\[
= \sum_{j_{1}, \ldots, j_{r}=1}^{d} \sum_{k_{1} + \ldots + k_{d} = \tau} \sum_{l_{1} + \ldots + l_{d} = \tau} c_{l_{1}, \ldots, l_{d}} \prod_{p=1}^{d} (l_{p}!)^{-1} b(k_{1}, \ldots, k_{d}, l_{1}, \ldots, l_{d}) a_{k_{1}, \ldots, k_{d}}^{(j_{1})} \cdots a_{k_{1}, \ldots, k_{d}}^{(j_{r})}
\]
\[
\begin{cases} 
(\tau!)^{-1} c_{l_{1}, \ldots, l_{d}}, & \text{if } l_{i} = i(j_{1}, \ldots, j_{r}), \\
0, & \text{otherwise.}
\end{cases}
\]
Finally, \( e_{j_{1}, \ldots, j_{r}} = (\tau!)^{-1} c_{l_{1}, \ldots, l_{d}} \) for \( l_{i} = i(j_{1}, \ldots, j_{r}), 1 \leq i \leq d. \)

Applying the previous Lemma to our process and observing
\[
e_{j_{1}, \ldots, j_{r}} = (\tau!)^{-1} E(f(X_{1}) \prod_{i=1}^{d} H_{i(j_{1}, \ldots, j_{r})}(X_{1}^{(i)}))
\]
\[
= \frac{1}{2} E\left( \sum_{u, l=1}^{d} \sum_{r=1}^{l} \sum_{t=1}^{u} \lambda_{rt} H_{(1,1)}(X_{1}^{(u)}, X_{1}^{(l)}) \prod_{i=1}^{d} H_{t(j_{1}, j_{2})}(X_{1}^{(i)}) \right)
\]
\[
= \sum_{r=1}^{d} \sum_{s=1}^{d} \lambda_{rs}
\]
leads to

\[ \sum_{j_1, j_2=1}^{d} \sum_{r_1=1}^{j_1} \sum_{r_2=1}^{j_2} \lambda_{r_1 r_2} K_{j_1, j_2} \int_{\mathbb{R}^2} e^{i \sum_{j=1}^{n} x_j - 1} \prod_{i=1}^{2} |x_i|^{-(1/2-d_{r_1})} B^{(r_1)}(dx_1) B^{(r_2)}(dx_2) \]

\[ = \sum_{j_1, j_2=1}^{d} \lambda_{j_1 j_2} \sum_{r_1=1}^{j_1} \sum_{r_2=1}^{j_2} K_{r_1, r_2} \int_{\mathbb{R}^2} e^{i \sum_{j=1}^{n} x_j - 1} \prod_{i=1}^{2} |x_i|^{-(1/2-d_{r_1})} B^{(r_1)}(dx_1) B^{(r_2)}(dx_2) \]

and thus the limit process \( Z \) is given by

\[ \left( \sum_{r_1=1}^{d} K_{r_1, r_2} \int_{\mathbb{R}^2} e^{i \sum_{j=1}^{n} x_j - 1} \prod_{i=1}^{2} |x_i|^{-(1/2-d_{r_1})} B^{(r_1)}(dx_1) B^{(r_2)}(dx_2) \right)_{j_1, j_2=1, \ldots, d} \]

It could be written as

\[ Z = B \tilde{Z} B' \]

with

\[ \tilde{Z} = (\tilde{z}_{r_1, r_2})_{r_1, r_2=1, \ldots, d} \]

\[ = K_{r_1, r_2} \int_{\mathbb{R}^2} e^{i (x_1+x_2) - 1} \prod_{i=1}^{2} |x_i|^{-(1/2-d_{r_1})} B^{(r_1)}(dx_1) B^{(r_2)}(dx_2) \]

and

\[ B = (B_{i,l})_{i,l=1, \ldots, d} \begin{cases} 1, & \text{if } l \leq i, \\ 0, & \text{otherwise}. \end{cases} \]

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