THE CLOSED RANGE PROPERTY FOR THE ∂-OPERATOR ON PLANAR DOMAINS

GALLAGHER, A.-K., LEBL, J., AND RAMACHANDRAN, K.

Abstract. Let Ω ⊂ C be an open set. We show that ∂ has closed range in L^2(Ω) if and only if the Poincaré–Dirichlet inequality holds. Moreover, we give necessary and sufficient potential-theoretic conditions for the ∂-operator to have closed range in L^2(Ω). We also give a new necessary and sufficient potential-theoretic condition for the Bergman space of a Ω to be infinite dimensional.

1. Introduction

The ∂-operator is initially defined as

\[ ∂f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \]

for any function f which is differentiable on an open set in C^n. For a given open set D ⊂ C^n, n ≥ 1, the operator may be extended to an L^2(D)-operator, by first extending it to act on non-smooth functions in the sense of distributions and then restricting its domain to those L^2(D)-functions whose images under ∂ are forms with coefficients in L^2(D). A reason for considering the ∂-operator as an L^2-operator is that it allows one to employ Hilbert space methods to solve the inhomogeneous Cauchy–Riemann equations. This is of importance for the construction of holomorphic functions in higher dimension due to the lack of power series techniques which are available in one complex dimension. Because of the effectiveness of these power series techniques, the ∂-operator has not been sufficiently studied for open sets in C. Note that for planar open sets, the ∂-operator may be identified with an extension of the derivative operator \( \partial \). 

In this article, we give necessary and sufficient potential-theoretic conditions for the range of ∂ on an open set Ω ⊂ C to be closed in L^2(Ω). The closed range property is known to hold for ∂ on an open set Ω ⊂ C to be closed in L^2(Ω). The closed range property is known to hold for ∂ on an open set Ω ⊂ C to be closed in L^2(Ω). 

(1.1) \[ \|u\|_{L^2(\Omega)} \leq C\|\partial u\|_{L^2(\Omega)} \]

for all u ∈ L^2(Ω) with \( \partial u \in L^2(\Omega) \) and u orthogonal to the kernel of ∂, see [6, Theorem 1.1.1]. The kernel of ∂ is the closed subspace of L^2(Ω) consisting of functions holomorphic on Ω. This space is commonly called the Bergman space and denoted by A^2(Ω). Inequality (1.1) may be reformulated as

(1.2) \[ \|u\|_{L^2(\Omega)} \leq C\|u_ξ\|_{L^2(\Omega)} \quad \forall \ u \perp A^2(Ω) \text{ with } u_ξ \in L^2(Ω). \]
The relevance of (1.2) (or (1.1)) lies in the fact that, if the ∂-operator has closed range for an open set in dimension greater than 1, on two consecutive form levels, then the ∂-Neumann operator exists as a bounded $L^2$-operator. Characterizing such open sets in higher dimensions is an unresolved problem. A first step towards resolving this question is to establish necessary and sufficient conditions for the closed range property to hold on planar open sets.

Another point of interest of (1.2) is its formal similarity to the Poincaré–Wirtinger inequality. The latter is said to hold on an open set $\Omega \subset \mathbb{C}$, if there exists a constant $C > 0$ such that

$$\|v - v_\Omega\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}$$

for all $v$ in $H^1(\Omega)$, the $L^2$-Sobolev-1-space of $\Omega$. Here, $v_\Omega$ is the average value of $v$ on $\Omega$. Since the kernel, $\ker \nabla$, of $\nabla$ is either the set of constants or trivial, it follows that $v - v_\Omega$ is orthogonal to $\ker \nabla$. In fact, $u \in L^2(\Omega) \cap (\ker \nabla)^\perp$ iff $u_\Omega = 0$. Thus, the Poincaré-Wirtinger inequality is

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \quad \forall u \in H^1(\Omega) \cap (\ker \nabla)^\perp.$$ 

Hence the closed range property of $\partial$ may be considered a Poincaré–Wirtinger inequality for $\partial$.

It turns out that the closed range property for $\partial$ is more closely related to the Poincaré–Dirichlet inequality. That is, the inequality

$$\|v\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega)$$

for some constant $C > 0$; here $H^1_0(\Omega)$ is the completion of $C^\infty_c(\Omega)$ with respect to the Sobolev-1-norm. At first, this might seem surprising as membership to the domain of $\partial$ has no boundary condition folded in. However, the domain of the Hilbert space adjoint, $\partial^*$, of $\partial$ is contained in $H^1_0(\Omega)$. Due to the Closed Range Theorem of Banach, the $\partial$-operator has closed range if and only if its Hilbert space adjoint does. So it might be less surprising that the Poincaré–Dirichlet inequality is in fact equivalent to $\partial$ having closed range in $L^2(\Omega)$, see Theorem 1.3 below.

To describe the closed range property for $\partial$ on planar open sets in potential-theoretic terms, we use the notion of logarithmic capacity of a set in the complex plane. We denote the logarithmic capacity of a set $E \subset \mathbb{C}$ by $\text{cap}(E)$; see Section 2.2 for the definition. Following nomenclature used in describing sufficiency conditions for the Poincaré–Dirichlet inequality, see, e.g., [8, §2], [9, Proposition 2.1], and references therein, we introduce the following terminology. For a set $\Omega \subset \mathbb{C}$, define the capacity inradius of $\Omega$ by

$$\rho_{\text{cap}}(\Omega) = \sup \{R > 0 : \exists \delta > 0 \exists z \in \mathbb{C} \text{ such that } \text{cap}(B(z, R) \cap \Omega^c) < \delta\},$$

see Section 2.2 for more details on this concept. Finiteness of the capacity inradius completely characterizes those open sets for which $\partial$ has closed range:

**Theorem 1.3.** Let $\Omega \subset \mathbb{C}$ be an open set. Then the following are equivalent:

1. $\partial$ has closed range in $L^2(\Omega)$.
2. The Poincaré–Dirichlet inequality holds on $\Omega$.
3. $\rho_{\text{cap}}(\Omega) < \infty$.
4. There exists a bounded function $\varphi \in C^\infty(\Omega)$ and a constant $c > 0$ such that $\Delta \varphi(z) > c$ holds for all $z \in \Omega$. 

The implication “(4)⇒(1)” is proved by McNeal and the first author in [5, Corollary 6.1] (with \( n = 1, q = 0 \)). Our proof of “(3)⇒(4)” is constructive. In fact, the function \( \varphi \) in (4) is built from a sequence of potential functions associated to the equilibrium measures of certain compact sets in the complement of the open set.

The idea for the proof of “(3)⇒(4)” lead us to the completion of the characterization of planar open sets with infinite dimensional Bergman spaces in terms of the existence of bounded, strictly subharmonic functions, see (4) in the following theorem.

**Theorem 1.4.** Let \( \Omega \subset \mathbb{C} \) be an open set. Then the following are equivalent:

1. \( A^2(\Omega) \neq \{0\} \).
2. \( \dim A^2(\Omega) = \infty \).
3. \( \text{cap}(\Omega^c) > 0 \).
4. There exists a bounded function \( \varphi \in C^\infty(\Omega) \) such that \( \triangle \varphi(z) > 0 \) for all \( z \in \Omega \).

The equivalence of (1) and (2) is shown by Wiegerinck in [11], the equivalence of (1) and (3) by Carleson in [1] Theorem 1.a in §VI, the implication “(4)⇒(2)” by Harz, Herbort, and the first author in [4].

The paper is structured as follows. We define basic notions of the \( L^2 \)-theory for \( \partial \) and potential theory for open sets in the complex plane in Section 2. In this section, we also recall the connection between the best constant in the Poincaré–Dirichlet inequality and the lowest eigenvalue of the Dirichlet–Laplacian. Moreover, we derive basic characteristics of the closed range property of \( \partial \) and conclude the section with a proof of the equivalence of the closed range property for \( \partial \) and the Poincaré–Dirichlet inequality. Section 3 contains the proof of the implication “(1)⇒(3)”. We first give a proof of this implication under an additional assumption, since it is based on standard \( \partial \)-arguments that indicate how to approach the higher dimensional case. The general proof, also in Section 3, is based on the connection of the closed range property to the Poincaré–Dirichlet inequality for bounded open sets, a solution to the (lowest) eigenvalue problem for the Dirichlet–Laplacian on the unit disc, and so-called \( r \)-logarithmic potentials. The proofs of “(3)⇒(4)” of Theorem 1.3 and “(3)⇒(4)” of Theorem 1.4 are given in Sections 4 and 5, respectively. Both are constructive and based on using potential functions associated to certain compact sets in the complement of the open set in consideration.

2. Preliminaries

2.1. The \( \partial \)-operator and its closed range property on open sets in \( \mathbb{C} \). For an open set \( \Omega \subset \mathbb{C} \), we denote by \( C^\infty(\Omega) \) and \( C_c^\infty(\Omega) \) the family of smooth functions on \( \Omega \) and the family of smooth functions on \( \Omega \) whose (closed) support is compact in \( \Omega \), respectively. As usual, \( L^2(\Omega) \) is the space of square-integrable functions on \( \Omega \), the associated norm and inner product are denoted by \( \| . \|_{L^2(\Omega)} \) and \( ( . , . )_{L^2(\Omega)} \), respectively. The \( L^2 \)-Sobolev-1-space, \( H^1(\Omega) \), on \( \Omega \) is the subspace of functions \( f \in L^2(\Omega) \) for which the norm

\[
\| f \|_{H^1(\Omega)} = \left( \| f \|_{L^2(\Omega)}^2 + \| \nabla f \|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}
\]

is finite. Here \( \nabla f \) is meant in the sense of distributions. \( H^1_0(\Omega) \) is the closure of \( C_c^\infty(\Omega) \) with respect to \( \| . \|_{H^1(\Omega)} \).
The \( \overline{\partial} \)-operator on \( \Omega \) is defined as \( \overline{\partial} u = u_\partial \overline{d\zeta} \) for any \( u \in \mathcal{C}_c^\infty(\Omega) \). Since \((0,1)\)-forms on \( \Omega \) may be identified with functions on \( \Omega \), we henceforth identify \( \overline{\partial} u \) with \( u_\partial \). The maximal extension of the \( \overline{\partial} \)-operator, still denoted by \( \overline{\partial} \), is defined as follows: we first allow \( \overline{\partial} \) to act on functions in \( L^2(\Omega) \) in the sense of distributions and then restrict its domain to those functions whose image under \( \overline{\partial} \) lies in \( L^2(\Omega) \). That is,

\[
\text{Dom}(\overline{\partial}) = \left\{ u \in L^2(\Omega) : \overline{\partial} u \text{ (in the sense of distributions) in } L^2(\Omega) \right\}.
\]

As \( \mathcal{C}_c^\infty(\Omega) \) is dense in \( L^2(\Omega) \) with respect to the \( L^2(\Omega) \)-norm, it follows that \( \overline{\partial} \) is a densely defined operator on \( L^2(\Omega) \); moreover, it is a closed operator. To define the Hilbert space adjoint, \( \overline{\partial}^* \), of \( \overline{\partial} \) we first define its domain \( \text{Dom}(\overline{\partial}^*) \) to be the space of those \( v \in L^2(\Omega) \) for which there exists a positive constant \( C = C(v) \) such that

\[
| (\overline{\partial} u, v)_{L^2(\Omega)} | \leq C \| u \|_{L^2(\Omega)} \quad \forall u \in \text{Dom}(\overline{\partial}),
\]

i.e., for any \( v \in \text{Dom}(\overline{\partial}^*) \), the map \( u \mapsto (\overline{\partial} u, v)_{L^2(\Omega)} \) is a bounded linear functional on \( \text{Dom}(\overline{\partial}) \). Hence, by Hahn–Banach, the map extends to a bounded linear functional on \( L^2(\Omega) \). It then follows from the Riesz Representation theorem that for any \( v \in \text{Dom}(\overline{\partial}^*) \) there exists a \( w \in L^2(\Omega) \) such that

\[
(\overline{\partial} u, v)_{L^2(\Omega)} = (u, w)_{L^2(\Omega)} \quad \forall u \in \text{Dom}(\overline{\partial}).
\]

Set \( \overline{\partial}^* v = w \). If \( \Omega \) has smooth boundary, it follows from an integration by parts argument, that whenever \( v \in \text{Dom}(\overline{\partial}^*) \cap \mathcal{C}_c^\infty(\Omega) \), then \( v|_{\partial\Omega} = 0 \) and \( \overline{\partial}^* v = -v_\partial \). Furthermore the following density result holds.

**Lemma 2.1.** Let \( \Omega \subset \mathbb{C} \) be an open set, then \( \mathcal{C}_c^\infty(\Omega) \) is dense in \( \text{Dom}(\overline{\partial}^*) \) with respect to the graph norm

\[
v \mapsto \left( \| v \|^2_{L^2(\Omega)} + \| \overline{\partial}^* v \|^2_{L^2(\Omega)} \right)^{1/2}.
\]

This density result could be expected considering that elements of \( \text{Dom}(\overline{\partial}^*) \) in some sense vanish on the boundary while the above graph norm restricted to \( \mathcal{C}_c^\infty(\Omega) \) is equivalent to \( \| v \|_{\mathcal{H}^1(\Omega)} \), see the argument in the proof of Lemma 2.10. A concise proof of Lemma 2.1 may be found in [10, Proposition 2.3]. For the convenience of the reader, we reproduce the proof here.

**Proof.** Let \( u \in \text{Dom}(\overline{\partial}^*) \) be given. Suppose \( u \) is orthogonal to all functions on \( \mathcal{C}_c^\infty(\Omega) \) with respect to the inner product associated to the graph norm, i.e.,

\[
(u, v)_{L^2(\Omega)} + (\overline{\partial} v, \overline{\partial}^* v)_{L^2(\Omega)} = 0 \quad \forall v \in \mathcal{C}_c^\infty(\Omega).
\]

If this forces \( u \) to be zero, then the claim follows. Note first that \( (\overline{\partial} u, \overline{\partial}^* v)_{L^2(\Omega)} \), \( v \in \mathcal{C}_c^\infty(\Omega) \), defines \( \overline{\partial} \overline{\partial}^* u \) in the sense of distributions. In particular, it follows that \( u + \overline{\partial} \overline{\partial}^* u \) is zero as a distribution. As \( u \in L^2(\Omega) \), it then follows that \( \overline{\partial} \overline{\partial}^* u \in L^2(\Omega) \). Thus \( (u + \overline{\partial} \overline{\partial}^* u, u)_{L^2(\Omega)} = 0 \), hence

\[
\| u \|^2_{L^2(\Omega)} + \| \overline{\partial}^* u \|^2_{L^2(\Omega)} = 0.
\]

Therefore \( u = 0 \), which proves the claim. \( \square \)
The next proposition gives basic, equivalent descriptions for the \( \partial \)-operator to have closed range, which is the property that whenever \( \{ \partial u_n \}_{n \in \mathbb{N}} \) converges in \( L^2(\Omega) \) for \( \{ u_n \}_{n \in \mathbb{N}} \subset \text{Dom}(\partial) \), then \( \lim_{n \to \infty} \partial u_n = \partial u \) for some \( u \in \text{Dom}(\partial) \).

**Proposition 2.2.** Let \( \Omega \subset \mathbb{C} \) be an open set. Then the following are equivalent:

(i) \( \partial \) has closed range in \( L^2(\Omega) \).

(ii) There exists a constant \( C > 0 \) such that \( \|u\|_{L^2(\Omega)} \leq C \|\partial u\|_{L^2(\Omega)} \) holds for all \( u \in \text{Dom}(\partial) \) with \( u \perp \ker \partial \).

(iii) There exists a constant \( C > 0 \) such that \( \|v\|_{L^2(\Omega)} \leq C \|\partial^* v\|_{L^2(\Omega)} \) holds for all \( v \in \text{Dom}(\partial^*) \) with \( v \perp \ker \partial^* \).

(iv) There exists a constant \( C > 0 \) such that for all \( f \in L^2(\Omega) \) there exists a \( v \in (A^2(\Omega))^\perp \) such that \( \partial v = f \) holds in the distributional sense and

\[
\|v\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.
\]

These equivalences are well-known, and, in fact, higher dimensional analogs of (i)–(iv) are true. For the convenience of the reader, we give either references or short arguments for the proofs of Proposition 2.2.

**Proof.** The equivalences of (i)–(iii) are proved in \([6, \text{Theorem 1.1.1}]\).

To see that (iv) implies (ii), let \( u \in \text{Dom}(\partial) \) with \( u \perp \ker \partial \) be given. By (iv) there exists a \( v \in A^2(\Omega)^\perp \) such that \( \partial v = \partial u \) in the distributional sense and

\[
\|v\|_{L^2(\Omega)} \leq C \|\partial u\|_{L^2(\Omega)}.
\]

Hence \( \partial(u - v) = 0 \) in the distributional sense. The ellipticity of \( \partial \) on functions implies that \( \ker \partial = A^2(\Omega) \). Thus \( u - v \in A^2(\Omega) \). Since \( u - v \) is also in \( (A^2(\Omega))^\perp \), it follows that \( u = v \) so that (ii) holds.

The implication (iii) \( \Rightarrow \) (iv) follows from a standard duality argument, see Theorem 1.1.4 in \([6, \text{Theorem 1.1.4}]\) with \( A = \text{Id}, T = \partial^*, S = 0, H_1 = L^2(\Omega) = H_2, H_3 = 0 \) and \( F = (\ker \partial^*)^\perp \).

The constants in (ii)–(iv) of Proposition 2.2 may be chosen to be the same. For the best possible constant, we introduce the following notation.

**Definition 2.3.** Let \( \Omega \) be an open set in \( \mathbb{C} \). Then \( \partial \) is said to have closed range in \( L^2(\Omega) \) with constant \( C \) if

\[
\|u\|_{L^2(\Omega)} \leq C \|\partial u\|_{L^2(\Omega)}
\]

holds for all \( u \in \text{Dom}(\partial) \cap A^2(\Omega)^\perp \). In that case, set

\[
\mathcal{C}(\Omega) = \inf \{ C : \|u\|_\Omega \leq C \|\partial u\|_\Omega \forall u \in \text{Dom}(\partial) \cap A^2(\Omega)^\perp \}.
\]

If the closed range property for \( \partial \) does not hold in \( L^2(\Omega) \), we say that \( \mathcal{C}(\Omega) = \infty \).

2.2. **Terminology from potential theory in the plane.** Let \( \mu \) be a finite Borel measure with compact support in \( \mathbb{C} \). The potential, \( p_\mu \), associated to \( \mu \) is defined by

\[
p_\mu(z) = \int_\mathbb{C} \ln |z - w| \, d\mu(w).
\]

The energy, \( I_\mu \), of \( \mu \) is given by

\[
I_\mu = \int_\mathbb{C} p_\mu(z) \, d\mu(z) = \int_\mathbb{C} \int_\mathbb{C} \ln |z - w| \, d\mu(w) \, d\mu(z).
\]
A set $E \subset \mathbb{C}$ is called polar if the energy of every non-trivial, finite Borel measure with compact support in $E$ is $-\infty$. If for a compact set $K \subset \mathbb{C}$, there is a finite Borel probability measure $\nu$ with support in $K$ such that

\[ I_\nu = \sup \{ I_\mu : \mu \text{ finite Borel probability measure with support in } K \}, \]

then $\nu$ is said to be an equilibrium measure for $K$. Any compact set has an equilibrium measure, see, e.g., [7, Theorem 3.3.2]. Moreover, this equilibrium measure is unique for any non-polar compact set, see [7, Theorem 3.7.6]. The logarithmic capacity of a compact, non-polar set $K$ is defined as

\[ \text{cap}(K) = e^{-I_\nu}, \]

where $\nu$ is the equilibrium measure of $K$. If $K$ is compact and polar, then $\text{cap}(K) = 0$. For a general set $E \subset \mathbb{C}$, the logarithmic capacity, $\text{cap}(E)$, of $E$ is defined as $\sup e^{-I_\mu}$ for $\mu$ a finite Borel probability measure with compact support in $E$. Note that a set $E$ is polar if and only if $\text{cap}(E) = 0$.

That the notion of positive logarithmic capacity comes into play for the description of the dimension of the Bergman space can be seen through the following observation. Let $K \subset \mathbb{C}$ be a compact, non-polar set, and $\mu$ the associated equilibrium measure. Then $p_\mu$ is a non-constant function, which is harmonic on $K^c$, and bounded from below by $\ln(\text{cap}(K))$ by Frostman’s theorem. Thus $e^{-p_\mu}$ is a bounded, smooth, subharmonic, non-harmonic function on $K^c$. Hence it is a good candidate for the construction of subharmonic functions in part (4) of 1.4.

This construction is also used in the proof of the necessity of the existence of bounded, strictly subharmonic functions for the closed range property to hold for $\overline{\Omega}$, see part (4) of Theorem 1.3. To achieve this strict subharmonicity we need compact sets, contained in the complement, and of sufficiently large logarithmic capacity, to be somewhat regularly distributed over the complex plane. This vague description can be made precise using the terminology of capacity inradius as introduced in the first section. Recall that for a set $\Omega \subset \mathbb{C}$, the capacity inradius of $\Omega$ is defined by

\[ \rho_{\text{cap}}(\Omega) = \sup \{ R \geq 0 : \forall \delta > 0 \exists z \in \mathbb{C} \text{ such that } \text{cap}(\mathbb{D}(z,R) \cap \Omega^c) < \delta \}. \]

Note that finiteness of the capacity inradius of $\Omega$ means that for any $M > \rho_{\text{cap}}(\Omega)$ there is a $\delta > 0$ such that for any point in $z \in \Omega$ there is a set in the complement of $\Omega$, whose logarithmic capacity is larger than $\delta$ while its distance to $z$ is less than $M$. For instance, both $\rho_{\text{cap}}(\mathbb{C})$ and $\rho_{\text{cap}}(\mathbb{C} \setminus (\mathbb{Z} + \sqrt{-1}\mathbb{Z}))$ are infinite. However, if for given $\epsilon > 0$, $K_{j,\ell}$ is the disc of radius $\epsilon$ centered at $j + \sqrt{-1}\ell$ or a line segment of length $\epsilon$ containing $j + \sqrt{-1}\ell$, then $\rho(\mathbb{C} \setminus \bigcup_{j,\ell \in \mathbb{Z}} K_{j,\ell})$ is finite. We note that in the case of the removed discs, the Poincaré–Dirichlet inequality is known to hold, see part (ii) in Proposition 2.1 in [9] and references therein. It appears to be new that, as a consequence of Theorem 1.3, the Poincaré–Dirichlet inequality is true in the removed line segments case as well.

### 2.3. The Poincaré–Dirichlet inequality

Let $\Omega \subset \mathbb{C}$ be an open set. The Poincaré–Dirichlet inequality is said to hold on $\Omega$, if there exists a constant $C > 0$ such that

\[ \|v\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega). \]

Whenever (2.4) holds, it is customary to consider

\[ \lambda_1(\Omega) = \min \left\{ \frac{\|\nabla \varphi\|_{L^2(\Omega)}^2}{\|\varphi\|_{L^2(\Omega)}^2} : \varphi \in H^1_0(\Omega), \varphi \neq 0 \right\}. \]
This notation stems from the fact that \( \lambda_1(\Omega) \) is the smallest eigenvalue for the Dirichlet–Laplacian. In fact, if \( \lambda_1(\Omega) \) in (2.5) is attained at \( \psi \in H^1_0(\Omega) \), then, for fixed \( \varphi \in H^1_0(\Omega) \), the function

\[
 f_\varphi(t) = \frac{\| \nabla (\psi + t\varphi) \|_{L^2(\Omega)}^2}{\| \psi + t\varphi \|_{L^2(\Omega)}^2}
\]

is differentiable near \( t = 0 \) and has a critical point there. Unraveling the equation \( f'_\varphi(0) = 0 \) for all \( \varphi \in H^1_0(\Omega) \) then leads to observing that \( \psi \) is a distributional solution to the boundary value problem

\[
\begin{cases}
\Delta \psi + \lambda \psi = 0 & \text{on } \Omega \\
\psi|_{\partial \Omega} = 0
\end{cases}
\]  

(2.6)

for \( \lambda = \lambda_1(\Omega) \).

Furthermore, we note that, if \( \Omega \) has smooth boundary and \( \psi \in \mathcal{C}^\infty(\overline{\Omega}) \) with \( \psi = 0 \) on \( \partial D \), then integration by parts yields

\[
\lambda_1(\Omega) \leq \frac{\| \nabla \psi \|_{L^2(\Omega)}^2}{\| \psi \|_{L^2(\Omega)}^2} = \frac{\langle -\Delta \psi, \psi \rangle_{L^2(\Omega)}}{\| \psi \|_{L^2(\Omega)}^2} \leq \frac{\| \Delta \psi \|_{L^2(\Omega)}^2}{\| \psi \|_{L^2(\Omega)}^2}. 
\]

(2.7)

2.4. Basic characteristics of the closed range property for \( \overline{\partial} \).

**Proposition 2.8.** Let \( \Omega \subset \mathbb{C} \) be an open set.

(a) Invariance under rigid transformations: If \( \Omega' \) is obtained from \( \Omega \) by translation, rotation, reflection, or a combination thereof, then \( \mathcal{E}(\Omega') = \mathcal{E}(\Omega) \).

(b) Linearity under scaling: \( r\mathcal{E}(\Omega) = \mathcal{E}(r\Omega) \) for \( r\Omega = \{rz : z \in \Omega\} \) and \( r > 0 \).

(c) Monotonicity: \( \mathcal{E}(\Omega') \leq \mathcal{E}(\Omega) \) for any open set \( \Omega' \subset \Omega \).

(d) Invariance under polar sets: \( \mathcal{E}(\Omega) = \mathcal{E}(\Omega') \) for any open set \( \Omega' \subset \Omega \) such that \( \Omega \setminus \Omega' \) is polar.

**Remark.** (i) If \( \Omega \subset \mathbb{C} \) is a bounded open set, then \( \mathcal{E}(\Omega) < \infty \) by the implication “(4) ⇒ (1)” Theorem 1.3 with, say, \( \varphi(z) = |z|^2 \), see \[5\] Corollary 6.1.

(ii) Note that (b) and (c) imply that \( \overline{\partial} \) for \( \mathbb{C} \) does not have closed range in \( L^2(\mathbb{C}) \).

To wit, denote by \( \mathbb{D}(z, M) \) the disc of radius \( M \) centered at \( z \). Then

\[
M \mathcal{E}(\mathbb{D}(0, 1)) = \mathcal{E}(\mathbb{D}(0, M)) \leq \mathcal{E}(\mathbb{C}) \quad \forall \ M \in \mathbb{N}.
\]

As \( \mathcal{E}(\mathbb{D}(0, 1)) > 0 \), it follows that \( \mathcal{E}(\mathbb{C}) = \infty \), i.e., \( \overline{\partial} \) does not have closed range in \( L^2(\mathbb{C}) \).

Since \( \mathbb{Z} + \sqrt{-1} \mathbb{Z} \) is a polar set, (d) implies that \( \overline{\partial} \) for

\[
\Omega = \mathbb{C} \setminus \bigcup_{m,n \in \mathbb{Z}} (m + \sqrt{-1} n)
\]

also does not have closed range.

(iii) Using translation invariance and linearity under scaling one can in fact show that if \( \Omega \) contains arbitrarily large discs, i.e., for any \( M \in \mathbb{N} \) there exists a \( z_M \in \Omega \) such that \( \mathbb{D}(z_M, M) \subset \Omega \), then \( \overline{\partial} \) does not have closed range in \( L^2(\Omega) \) as

\[
M \mathcal{E}(\mathbb{D}(0, 1)) = \mathcal{E}(\mathbb{D}(0, M)) = \mathcal{E}(\mathbb{D}(z_M, M)) \leq \mathcal{E}(\Omega) \quad \forall \ M \in \mathbb{N}.
\]

For instance, \( \overline{\partial} \) for the upper half plane does not have closed range. Since \( \mathcal{E}(\mathbb{D}(0, 1)) \) is finite by (i), it follows that the closed range property is not invariant under biholomorphic equivalences.
Proof. Translations and rotations are biholomorphic maps for which the absolute value of its Jacobian is 1. Hence, invariance readily follows. Any reflection may be written as a composition of translations, rotations, and complex conjugation. So it remains to show the invariance under complex conjugation. Denote complex conjugation by \( T \), i.e., \( Tz = \bar{z} \) for \( z \in \mathbb{C} \). Write \( \Omega_T = \{ z \in \mathbb{C} : Tz \in \Omega \} \). Observe that the map \( u \mapsto T \circ u \circ T \) yields an isometry of \( L^2(\Omega) \) and \( L^2(\Omega_T) \) as well as \( A^2(\mathbb{C}) \) and \( A^2(\Omega_T) \). Further, one easily verifies that \( \overline{T(\partial u \circ T)} = \partial (\overline{T u}) \). Hence \( \partial \) has closed range in \( L^2(\Omega) \) iff it has closed range in \( L^2(\Omega_T) \), and \( \mathcal{E}(\Omega) = \mathcal{E}(\Omega_T) \).

Part (b) follows straightforwardly from the fact that \( \partial(u(rz)) = r(\partial u)(rz) \) for any scalar \( r \).

Monotonicity holds trivially whenever \( \mathcal{E}(\Omega) = \infty \). If \( \mathcal{E}(\Omega) < \infty \), then the monotonicity result (c) is a straightforward consequence of the equivalence of conditions (ii) and (iv) in Proposition 2.2 and the fact that, unlike in higher dimensions, the inhomogeneous \( \overline{\partial} \)-equation may be solved without the data having to satisfy a compatibility condition. In fact, it suffices to show that condition (iv) of Proposition 2.2 holds on \( \Omega \) whenever (iv) holds on \( \Omega \) with constant \( C \). To that end, let \( f \in L^2(\Omega) \) and define \( F = \chi_{\Omega'} f \), where \( \chi_{\Omega'} \) is the characteristic function of \( \Omega' \). Then \( F \in L^2(\Omega) \). Since (iv) holds for \( \Omega \), there exists an \( u \in L^2(\Omega) \) such that \( \overline{\partial} u = F \) in the distributional sense. Hence for all \( w \in C_\infty^c(\Omega) \)

\[
\left( \overline{\partial} w, u \right)_{L^2(\Omega)} = (w, F)_{L^2(\Omega)} = (w, f)_{L^2(\Omega')}.
\]

In particular,

\[
\left( \overline{\partial} w, u \right)_{L^2(\Omega')} = (w, f)_{L^2(\Omega')} \quad \forall \ w \in C_\infty^c(\Omega'),
\]

i.e., \( \overline{\partial} u = f \) holds on \( \Omega' \) in the distributional sense. Furthermore, (iv) for \( \Omega \) yields the estimate

\[
\|u\|_{L^2(\Omega')} \leq C\|F\|_{L^2(\Omega)}.
\]

Since \( \|u\|_{L^2(\Omega')} \leq \|u\|_{L^2(\Omega)} \) and \( \|F\|_{L^2(\Omega')} = \|f\|_{L^2(\Omega')} \), it follows that condition (iv) of Proposition 2.2 holds on \( \Omega' \).

Proof of (d): Since \( \Omega \setminus \Omega' \) is of Lebesgue measure zero, it follows that \( u, \overline{\partial} u \in L^2(\Omega) \) whenever \( u, \overline{\partial} u \in L^2(\Omega) \). Moreover, since \( \Omega \setminus \Omega' \) is polar, \( A^2(\Omega) = A^2(\Omega') \), see e.g., part (c) of Theorem 9.5 in [2]. Hence \( A^2(\Omega)^+ = A^2(\Omega')^+ \). Therefore, \( \mathcal{E}(\Omega) = \mathcal{E}(\Omega') \). \( \square \)

Proposition 2.9. Let \( \Omega \subset \mathbb{C} \) be an open set. If the Poincaré–Dirichlet inequality holds, then \( \overline{\partial} \) has closed range in \( L^2(\Omega) \), and

\[
\mathcal{E}(\Omega) = \frac{2}{\sqrt{\lambda_1(\Omega)}}.
\]

We shall show first that the Poincaré–Dirichlet inequality implies a closed range inequality for \( \overline{\partial} \) restricted to \( C_\infty^c(\Omega) \).

Lemma 2.10. Let \( \Omega \subset \mathbb{C} \) be an open set. If the Poincaré–Dirichlet inequality holds, then

\[
\|\varphi\|_{L^2(\Omega)} \leq \frac{2}{\sqrt{\lambda_1(\Omega)}} \left\| \overline{\partial} \varphi \right\|_{L^2(\Omega)} \quad \forall \ \varphi \in C_\infty^c(\Omega).
\]
Proof of Lemma 2.10. Let \( \Omega \subset \mathbb{C} \) be an open set, \( \lambda_1(\Omega) \) as in (2.5). Then for \( \varphi \in C_c^\infty(\Omega) \), it follows from integration by parts that
\[
\|\varphi\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda_1(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}^2 = \frac{1}{\lambda_1(\Omega)} (-\Delta \varphi, \varphi)_{L^2(\Omega)} = \frac{4}{\lambda_1(\Omega)} (\overline{\partial \varphi}, \varphi)_{L^2(\Omega)} = \frac{4}{\lambda_1(\Omega)} \|\overline{\partial \varphi}\|_{L^2(\Omega)}^2.
\]

We are now set to prove Proposition 2.9.

Proof of Proposition 2.9. The proof for \( \mathcal{S} \) having closed range is based on Lemmata 2.10 and 2.1. To wit, let \( u \in \text{Dom}(\mathcal{S}^*) \cap (\ker \mathcal{S}^*)^\perp \) and \( \epsilon > 0 \) be given. Then by the density result, there exists a \( \varphi \in C_c^\infty(\Omega) \) such that
\[
\left( \|u - \varphi - \mathcal{S}^* u\|_{L^2(\Omega)}^2 + \|\mathcal{S}^* u - \varphi\|_{L^2(\Omega)}^2 \right) < \epsilon.
\]
Hence, using Lemma 2.10 after an application of triangle inequality, yields
\[
\|u\|_{L^2(\Omega)} \leq \|u - \varphi\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)} \leq \epsilon + \frac{2}{\lambda_1(\Omega)} \|\mathcal{S}^* \varphi\|_{L^2(\Omega)}.
\]
It then follows from the density result that
\[
(2.11) \quad \|u\|_{L^2(\Omega)} \leq \epsilon \left( 1 + \frac{2}{\sqrt{\lambda_1(\Omega)}} \right) + \frac{2}{\sqrt{\lambda_1(\Omega)}} \|\overline{\partial \varphi}\|_{L^2(\Omega)}.
\]
Since \( \epsilon \) was chosen arbitrarily, it follows from Proposition 2.2 that \( \mathcal{S} \) has closed range in \( L^2(\Omega) \) and \( \mathcal{C}(\Omega) \leq \frac{2}{\sqrt{\lambda_1(\Omega)}} \).

For the proof of \( \geq \), let \( \psi \in H^1_0(\Omega) \) be a weak eigenfunction corresponding to \( \lambda_1(\Omega) \). Since \( \Delta \psi = -\lambda_1(\Omega) \psi \) in the distributional sense, it follows that \( \Delta \psi \in L^2(\Omega) \). Hence \( \psi \in \text{Dom}(\overline{\partial}) \). Moreover, \( \psi \) is orthogonal to \( A^2(\Omega) \). To wit, let \( \{\varphi_n\}_{n \in \mathbb{N}} \subset C_c^\infty(\Omega) \) be convergent to \( \psi \) in \( H^1_0(\Omega) \). Then for \( h \in A^2(\Omega) \), it follows that
\[
\int_{\Omega} \psi \overline{\psi} \, dV(z) = \int_{\Omega} (\psi - \varphi_n) \overline{\psi} \, dV(z) + \int_{\Omega} (\varphi_n) \overline{\psi} \, dV(z).
\]
The second term on the right vanishes by an integration by parts argument on a smoothly bounded open set \( \Omega_n \) with \( \text{supp} \varphi_n \subset \Omega_n \subset \Omega \). Since \( \|(\psi - \varphi_n)\|_{L^2(\Omega)} \) converges to 0 as \( n \) goes to \( \infty \), it follows that
\[
\int_{\Omega} \psi \overline{\psi} \, dV(z) = 0,
\]
i.e., \( \psi \) is orthogonal to \( A^2(\Omega) \). Thus
\[
(2.12) \quad \|\psi\|_{L^2(\Omega)} \leq C \|\psi\|_{L^2(\Omega)} \leq \frac{C}{4} \|\Delta \psi\|_{L^2(\Omega)}
\]
holds for all \( C > \mathcal{C}(\Omega) \). Note that
\[
\|\psi\|_{L^2(\Omega)}^2 = (\psi, (\psi - \varphi_n)\psi)_{L^2(\Omega)} + (\psi, (\varphi_n)\psi)_{L^2(\Omega)} = (\psi, (\psi - \varphi_n)\psi)_{L^2(\Omega)} + (\psi, (\varphi_n)\psi)_{L^2(\Omega)} = -(\psi, (\psi - \varphi_n)\psi)_{L^2(\Omega)} + O \left( \|\psi\|_{H^1_0(\Omega)} \cdot \|\psi - \varphi_n\|_{H^1_0(\Omega)} \right).
\]
Proposition 2.13. Let \( \psi \) and hence \( \psi_{zz} \), is \( C^\infty(\Omega) \) by interior regularity for the elliptic boundary value problem (2.6). Therefore,
\[
\|\psi_z\|^2_{L^2(\Omega)} = - (\psi_{zz}, \psi)_{L^2(\Omega)} = \frac{1}{4} \lambda_1(\Omega) \|\psi\|^2_{L^2(\Omega)}
\]
while
\[
\|\Delta \psi\|^2_{L^2(\Omega)} = (\lambda_1(\Omega))^2 \|\psi\|^2_{L^2(\Omega)}
\]
By (2.12) it follows that
\[
\lambda_1(\Omega) \|\psi\|^2_{L^2(\Omega)} \leq \frac{C^2}{4} (\lambda_1(\Omega))^2 \|\psi\|^2_{L^2(\Omega)}
\]
holds for any \( C > C(\Omega) \). Hence \( C(\Omega) \geq \frac{2}{\sqrt{\lambda_1(\Omega)}} \).
\( \square \)

Remark. The proof of “\( \leq \)” in the above might give the impression that \( \|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \) holds for any \( u \in \text{Dom} \nabla \), in particular for \( u \neq 0 \) satisfying \( \nabla u = 0 \). However, if \( u \) is such a function and \( \{\varphi_n\}_{n \in \mathbb{N}} \subset C_c(\Omega) \) is a sequence such that \( \|u - \varphi_n\|^2_{L^2(\Omega)} + \|\nabla (u - \varphi_n)\|^2_{L^2(\Omega)} \) converges to 0, then \( \nabla \varphi_n \) converges to 0 in \( L^2(\Omega) \). By Lemma 2.10 it then follows that \( \varphi_n \) converges to 0 in \( L^2(\Omega) \). Hence \( u \) was identically 0 to begin with.

In the following proposition, we observe that the best closed range constant \( C(\cdot) \) satisfies a continuity from below property. This plays a crucial role in our proof of the Poincaré–Dirichlet inequality being implied by the closed range property, see Corollary 2.17.

Proposition 2.13. Let \( \{\Omega_j\}_{j \in \mathbb{N}} \) be an increasing sequence of open sets, set \( \Omega = \bigcup_{j \in \mathbb{N}} \Omega_j \). If \( \nabla \) has closed range in \( L^2(\Omega_j) \) with constant \( C \) for all \( j \in \mathbb{N} \), then \( \nabla \) has closed range in \( L^2(\Omega) \) with constant \( C \). In particular, \( C(\Omega) = \lim_{j \to \infty} C(\Omega_j) \).

For the proof of Proposition 2.13 we shall use the Bergman projection. Recall that for \( \Omega \subset \mathbb{C} \), the Bergman projection is the orthogonal projection \( B^\Omega \) of \( L^2(\Omega) \) onto its closed subspace \( A^2(\Omega) \). That is, \( B^\Omega \) satisfies

(a) \( B^\Omega \circ B^\Omega = B^\Omega \),
(b) \( B^\Omega h = h \) for all \( h \in A^2(\Omega) \),
(c) \( B^\Omega f - f \) is orthogonal to \( A^2(\Omega) \).

Proof. Let \( u \in \text{Dom}(\nabla) \cap (A^2(\Omega))^\perp \) be given. Throughout, write \( B \) for \( B^\Omega \), \( B_j \) for \( B^{\Omega_j} \) and \( \|\cdot\| \) for \( \|\cdot\|_{L^2(\Omega)} \), \( \|\cdot\|_{\Omega_j} \) for \( \|\cdot\|_{L^2(\Omega_j)} \). Then for \( j \in \mathbb{N} \),
\[
\|u\|_{\Omega} \leq \|u\|_{\Omega_j} + \|u\|_{\Omega_j/\Omega_j}
\]
Let \( \chi_j \) be the characteristic function of \( \Omega_j \), and set \( f_j = (1 - \chi_j)u \). Then \( f_j \) converges to 0 almost everywhere in \( \Omega \). Moreover, \( |f_j| \leq |u|^2 \) for all \( j \in \mathbb{N} \) and \( |u|^2 \) is in \( L^1(\Omega) \). It follows from the dominated convergence theorem, that
\[
\lim_{j \to \infty} \int_{\Omega_j} f_j dV = \int_{\Omega} \lim_{j \to \infty} f_j < \infty, \text{ i.e., } \lim_{j \to \infty} \|u\|_{\Omega_j} = 0.
\]
Therefore, for a given \( \epsilon > 0 \) there exists a \( j_0 \in \mathbb{N} \), such that
\[
\|u\|_{\Omega} \leq \|u\|_{\Omega_j} + \epsilon
\]
holds for all \( j \geq j_0 \). Hence
\[
\|u\|_{\Omega} \leq \|u - B^j u\|_{\Omega_j} + \|B^j u\|_{\Omega_j} + \epsilon
\]
for all \( j \geq j_0 \). Note that \( u - B^j u \) is orthogonal to \( A^2(\Omega_j) \). Since \( u, B^j u \in \text{Dom}(\partial) \), so is \( u - B^j u \). As \( \partial \) has closed range in \( L^2(\Omega_j) \) with constant \( C \), it now follows that
\[
\|u - B^j u\|_{\Omega_j} \leq C\|\partial u\|_{\Omega_j},
\]
which yields
\[(2.14) \quad \|u\|_{\Omega} \leq C\|\partial u\|_{\Omega} + \|B^j u\|_{\Omega_j} + \epsilon \]
for all \( j \geq j_0 \).

It remains to estimate the term \( \|B^j u\|_{\Omega_j} \). To that end, notice first that the sequence \( \{\chi_j B^j u\}_{j \in \mathbb{N}} \) is uniformly bounded in \( L^2(\Omega) \) by \( \|u\|_{L^2(\Omega)} \). Thus it has a weakly convergent subsequence, say, \( \{\chi_{j_k} B^{j_k} u\}_{k \in \mathbb{N}} \). That is, there exists a \( g \in L^2(\Omega) \) such that
\[(2.15) \quad \lim_{k \to \infty} (\chi_{j_k} B^{j_k} u - g, v)_{\Omega} = 0 \quad \forall \ v \in L^2(\Omega). \]

We shall first show that \( g \) is holomorphic, and then use this fact to derive that \( \|B^{j_k} u\|_{\Omega_{j_k}} \) converges to 0 as \( k \) tends to \( \infty \). It follows from \( (2.15) \) that
\[(2.16) \quad \lim_{k \to \infty} (\chi_{j_k} B^{j_k} u - g, \varphi)_{\Omega} = 0 \quad \forall \ \varphi \in C_c^\infty(\Omega). \]

Since \( \{\Omega_{j_k}\}_{k} \) is an increasing sequence of open sets, it follows that for each \( \varphi \in C_c^\infty(\Omega) \) there exists a \( j_{k_0} \in \mathbb{N} \) such that \( \text{sup} \varphi \in \Omega_{j_k} \) for all \( j_k \geq j_{k_0} \). Let \( S_\varphi \) be a smoothly bounded open set such that \( \text{sup} \varphi \in S_\varphi \subseteq \Omega_{j_k} \) for all \( j_k \geq j_{k_0} \). Then it follows from integration by parts that
\[
(\chi_{j_k} B^{j_k} u, \varphi)_{\Omega} = (B^{j_k} u, \varphi)_{S_\varphi} = -((B^{j_k} u)_{\bar{z}}, \varphi)_{S_\varphi} = 0
\]
for all \( j_k \geq j_{k_0} \). This, together with \( (2.16) \), implies that \( (g, \varphi)_{\Omega} = 0 \) for all \( \varphi \in C_c^\infty(\Omega) \). Thus, \( g \in A^2(\Omega) \), in particular \( (g, u)_{\Omega} = 0 \). The weak convergence \( (2.15) \) yields
\[
0 = \lim_{k \to \infty} (\chi_{j_k} B^{j_k} u - g, u)_{\Omega} = \lim_{k \to \infty} (\chi_{j_k} B^{j_k} u, u)_{\Omega}
= \lim_{k \to \infty} (B^{j_k} u, u)_{\Omega_{j_k}}.
\]

Since \( u - B^{j_k} u \) is orthogonal to \( A^2(\Omega_{j_k}) \) while \( B^{j_k} u \in A^2(\Omega_{j_k}) \), it follows that
\[
0 = \lim_{k \to \infty} (B^{j_k} u, u)_{\Omega_{j_k}} = \lim_{k \to \infty} \|B^{j_k} u\|_{\Omega_{j_k}}^2.
\]
Now repeat all arguments leading up to the estimate \( (2.14) \) with \( B^{j_k} \) instead of \( B^j \). Since \( \|B^{j_k} u\|_{\Omega_{j_k}} \) tends to 0 as \( k \to \infty \), it follows that \( \|u\|_{\Omega} \leq C\|\partial u\|_{\Omega} \).

To show that \( \mathcal{C}(\Omega_j) \) converges to \( \mathcal{C}(\Omega) \) as \( j \to \infty \), set \( C := \sup\{\mathcal{C}(\Omega_j) : j \in \mathbb{N}\} \). By hypothesis, \( C < \infty \). The above argument then yields \( \mathcal{C}(\Omega) \leq C \). However, the monotonicity property in Proposition \( 2.8 \) part (c), yields \( \mathcal{C}(\Omega_j) \leq \mathcal{C}(\Omega) \). Hence \( C \leq \mathcal{C}(\Omega) \). Thus \( C = \mathcal{C}(\Omega) \) holds, which completes the proof. \( \square \)
Corollary 2.17. Let $\Omega \subset \mathbb{C}$ be an open set. Then $\overline{\mathcal{D}}$ has closed range in $L^2(\Omega)$ if and only if the Poincaré–Dirichlet inequality holds on $\Omega$. Moreover,
\[
\mathcal{E}(\Omega) = \frac{2}{\sqrt{\lambda_1(\Omega)}}.
\]

Proof. By Proposition 2.9, we only need to show that $\partial$ having closed range in $L^2(\Omega)$ implies that the Poincaré–Dirichlet inequality holds on $\Omega$. So suppose $\partial$ has closed range. Let $\{\Omega_n\}_{n \in \mathbb{N}} \subset \Omega$ be an increasing sequence of bounded open sets such that $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$. Hence by monotonicity, part (c) of Proposition 2.8:
\[
\mathcal{C}(\Omega_n) \leq \mathcal{C}(\Omega) \quad \forall \ n \in \mathbb{N}.
\]
Since the Poincaré–Dirichlet inequality holds on any bounded open set, see, e.g., [3, Theorem 3, §5.6], Proposition 2.9 implies that
\[
\lambda_1(\Omega_n) = 4 \left( \frac{\mathcal{C}(\Omega_n)}{\mathcal{C}(\Omega)} \right)^2 \geq 4 \left( \frac{\mathcal{C}(\Omega)}{\mathcal{C}(\Omega)} \right)^2 > 0.
\]
It follows from Proposition 2.13 that $\{\lambda_1(\Omega_n)\}_{n \in \mathbb{N}}$ converges to $4 \left( \frac{\mathcal{C}(\Omega)}{\mathcal{C}(\Omega)} \right)^2$. It remains to be shown that $\lambda_1(\Omega)$ equals $4 \left( \frac{\mathcal{C}(\Omega)}{\mathcal{C}(\Omega)} \right)^2$. Let $\varphi \in C^\infty_c(\Omega)$. Then there exists an $n_0 \in \mathbb{N}$ such that $\text{supp}(\varphi) \subset \Omega_n$ for all $n \geq n_0$. Since the Poincaré–Dirichlet inequality holds on $\Omega_n$ for any $n \in \mathbb{N}$, it follows that
\[
\|\varphi\|_{L^2(\Omega_n)} \leq \lambda_1(\Omega_n) \|\nabla \varphi\|_{L^2(\Omega_n)} \quad \forall \ n \geq n_0.
\]
Hence
\[
\|\varphi\|_{L^2(\Omega)} \leq \lambda_1(\Omega_n) \|\nabla \varphi\|_{L^2(\Omega)} \quad \forall \ n \geq n_0.
\]
Since $\{\lambda_1(\Omega_n)\}_{n \in \mathbb{N}}$ decreases to $4 \left( \frac{\mathcal{C}(\Omega)}{\mathcal{C}(\Omega)} \right)^2$, we get that
\[
\|\varphi\|_{L^2(\Omega)} \leq (\frac{\mathcal{C}(\Omega)}{4})^2 \|\nabla \varphi\|_{L^2(\Omega)}.
\]
By density of $C^\infty_c(\Omega)$ in $H^1_0(\Omega)$ with respect to $\|\cdot\|_{H^1(\Omega)}$, it follows that the Poincaré–Dirichlet inequality holds for $\Omega$ with constant $4 \left( \frac{\mathcal{C}(\Omega)}{\mathcal{C}(\Omega)} \right)^2$. If $\lambda_1(\Omega)$ were greater than $4 \left( \frac{\mathcal{C}(\Omega)}{\mathcal{C}(\Omega)} \right)^2$, then $\lambda_1(\Omega)$ would be greater than $\lambda_1(\Omega_n)$ for some $n$. This is a contradiction to $\lambda_1(\Omega_n)$ being the reciprocal of the best constant in the Poincaré–Dirichlet inequality as $C^\infty_c(\Omega_n) \subset C^\infty_c(\Omega)$. \qed

3. Proof of “$1 \Rightarrow 3$” of Theorem 1.3

3.1. Special case. The proof of “$1 \Rightarrow 3$” is done by contraposition, i.e., we assume that $1$ holds while $3$ does not. We shall first consider cases of open sets for which $3$ does not hold in a particular manner, see hypothesis of Lemma 3.1 and the example below. The proof of “$1 \Rightarrow 3$” for these open sets is a straightforward consequence of Proposition 2.13.

Lemma 3.1. Let $\Omega \subset \mathbb{C}$ be an open set. Suppose that for all $M > 0$, there exist a sequence $\{\delta_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^+$ with $\lim_{j \to \infty} \delta_j = 0$ and $\{z_{M,j}\}_{j \in \mathbb{N}} \subset \mathbb{C}$ such that
\begin{enumerate}
  \item $\text{cap}(\Omega^c \cap \mathbb{D}(z_{M,j}, M)) < \delta_j$, \label{item1}
  \item the sequence of open sets given by $(\Omega \cap \mathbb{D}(z_{M,j}, M)) - z_{M,j} \subset \mathbb{D}(0, M)$ is increasing in $j$. \label{item2}
\end{enumerate}
Then $\overline{\mathcal{D}}$ does not have closed range in $L^2(\Omega)$.
Example 1. For each \((j, \ell) \in \mathbb{Z} \times \mathbb{Z}\), let \(K_{j,\ell}\) be the closed horizontal line segment of length \(\frac{\arctan(j)}{\pi} + \frac{1}{2}\) centered at \(j + \sqrt{-1} \ell\). Let
\[
\Omega = \mathbb{C} \setminus \bigcup_{(j,\ell) \in \mathbb{Z} \times \mathbb{Z}} K_{j,\ell}.
\]
Then \(\overline{T}\) does not have closed range in \(L^2(\Omega)\) by Lemma 3.1. In fact, for any \(m \in \mathbb{N}\),
\[
(\Omega \cap \mathbb{D}(-m, M)) + m \subset (\Omega \cap \mathbb{D}(-m - 1, M)) + (m + 1) \subset \mathbb{D}(0, M).
\]
Moreover, for any \(j \in \mathbb{N}\) there exists an \(m_j \in \mathbb{N}\) such that
\[
\text{cap} (\Omega^c \cap \mathbb{D}(-m_j, M)) \leq \frac{1}{j}.
\]
Hence, conditions (i) and (ii) of Lemma 3.1 are satisfied with \(z_{M,j} = -m_j\) and \(\delta_j = \frac{1}{j}\).

That \(\overline{T}\) does not have closed range in \(L^2(\Omega)\) can also be seen more directly from Propositions 2.8 and 2.13. To wit, set \(\Omega_m\) to be the shift of \(\Omega\) to the right by \(m\) units. It follows from part (a) of Proposition 2.8 that \(\mathcal{C}(\Omega) = \mathcal{C}(\Omega_m)\). Further, \(\Omega_m \subset \Omega_{m+1}\) for any \(m \in \mathbb{Z}\). By Proposition 2.13 it now follows that
\[
\mathcal{C}(\Omega) = \mathcal{C}(\Omega_m) = \mathcal{C} \left( \bigcup_{m \in \mathbb{Z}} \Omega_m \right).
\]
However, the complement of \(\bigcup_{m \in \mathbb{Z}} \Omega_m\) is the lattice \(\mathbb{Z} + \sqrt{-1} \mathbb{Z}\), which is a polar set. By part (d) of Proposition 2.8 it follows that
\[
\mathcal{C}(\mathbb{C}) = \mathcal{C} \left( \bigcup_{m \in \mathbb{Z}} \Omega_m \right).
\]
Since \(\mathcal{C}(\mathbb{C}) = \infty\), we obtain from (3.2) that \(\mathcal{C}(\Omega) = \infty\).

Proof of Lemma 3.1. We suppose that \(\overline{T}\) has closed range in \(L^2(\Omega)\) with constant \(C\). Choose \(M > 0\) such that \(\frac{C}{M} < \mathcal{C}(\mathbb{D}(0,1))\). By hypothesis, there exist a sequence of positive scalars \(\{\delta_j\}_{j \in \mathbb{N}}\) with \(\lim_{j \to \infty} \delta_j = 0\) and \(\{z_j\}_{j \in \mathbb{N}} \subset \mathbb{C}\) such that
\[
\text{cap} (\Omega^c \cap \mathbb{D}(z_j, M)) < \delta_j.
\]
Define \(D_j\) to be the set obtained from translating \(\Omega \cap \mathbb{D}(z_j, M)\) by \(-z_j\) and then scaling it by a factor of \(\frac{1}{M}\), i.e.,
\[
D_j = \{ z \in \mathbb{D}(0,1) : Mz + z_j \in \Omega \}.
\]
Then by properties (a),(b) and (c) of Proposition 2.8
\[ C(D_j) \leq \frac{C}{M}. \]
(3.3)
As the logarithmic capacity satisfies analogous properties, see part (c) of Theorem 5.1.2 in [7], we also have \( \text{cap}(\mathbb{D}(0,1) \setminus D_j) = \frac{\delta_j}{M} \). Since \( D_j \subset D_{j+1} \), it follows from monotonicity, see Theorem 5.1.2 (a) in [7], that
\[
\text{cap} \left( \mathbb{D}(0,1) \setminus \bigcup_{j=1}^{\infty} D_j \right) \leq \text{cap}(\mathbb{D}(0,1) \setminus D_k) \leq \frac{\delta_k}{M} \quad \forall k \in \mathbb{N}.
\]
By part (d) of Proposition 2.8 we then get that
\[ C(\bigcup_{j=1}^{\infty} D_j) = C(\mathbb{D}(0,1)). \]
Moreover, Proposition 2.13 yields for any given \( \epsilon > 0 \) a \( j \in \mathbb{N} \) such that
\[ C(D_j) > C \left( \bigcup_{j=1}^{\infty} D_j \right) - \epsilon. \]
By (3.3), it then follows that
\[ \frac{C}{M} > C(\mathbb{D}(0,1)) - \epsilon. \]
This is a contradiction to the choice of \( M \) for \( \epsilon > 0 \) sufficiently small. \( \square \)

3.2. General case.

Proposition 3.4. Let \( \Omega \subset \mathbb{C} \) be an open set. Suppose \( \overline{\Omega} \) has closed range in \( L^2(\Omega) \). Then \( \rho_{\text{cap}}(\Omega) < \infty \), i.e., there exist positive constants \( M \) and \( \delta \) such that for each \( z \in \mathbb{C} \) there exists a compact set \( K \subset \Omega^c \) such that
\[ \text{cap}(K \cap D(z, M)) \geq \delta. \]

Lemma 3.5. Let \( K \subset \mathbb{C} \) be a compact set and \( \epsilon > 0 \).

(i) There exists a compact set \( K_\epsilon \subset \mathbb{C} \) such that \( \mathbb{C} \setminus K_\epsilon \) has smooth boundary, \( K \subset K_\epsilon \), and \( \text{cap}(K_\epsilon) \leq \text{cap}(K) + \epsilon \).

(ii) Suppose that \( K \) has smooth boundary and \( K \cap \mathbb{D}(0,1) \) is non-empty. Then there exists a compact set \( K_\epsilon \subset \mathbb{D}(0,1) \) such that \( K \cap \mathbb{D}(0,1) \subset K_\epsilon \), \( \text{cap}(K_\epsilon) \leq \text{cap}(K) + \epsilon \) and \( \mathbb{D}(0,1) \setminus K_\epsilon \) is a smoothly bounded set.

Proof. (i) Let \( K \subset \mathbb{C} \) be a compact set, \( \epsilon > 0 \) be given. Then, by a result due to Choquet, there exists an open, bounded set \( U_\epsilon \) such that \( K \subset U_\epsilon \) and \( \text{cap}(U_\epsilon) \leq \text{cap}(K) + \epsilon \). Moreover, there exists a smoothly bounded, open set \( U'_\epsilon \) such that \( K \subset U'_\epsilon \subset U_\epsilon \). Set \( K_\epsilon = \overline{U'_\epsilon} \). It then follows from monotonicity of the logarithmic capacity that \( \text{cap}(K_\epsilon) \leq \text{cap}(K) + \epsilon \).

(ii) Let \( K \subset \mathbb{C} \) be a compact set with smooth boundary such that \( K \cap \mathbb{D}(0,1) \) is non-empty. Let \( \epsilon > 0 \) be given. As in part (i) there exists an open set \( U_\epsilon \) such that \( K \subset U_\epsilon \) and \( \text{cap}(U_\epsilon) \leq \text{cap}(K) + \epsilon \). Then there exist an open set \( U'_\epsilon \) such that \( K \cap \mathbb{D}(0,1) \subset U'_\epsilon \cap \mathbb{D}(0,1) \subset U_\epsilon \cap \mathbb{D}(0,1) \)
and \( \mathbb{D}(0,1) \setminus U'_\epsilon \) has smooth boundary. With \( K_\epsilon := \overline{U'_\epsilon} \), (ii) follows. \( \square \)
Proof of Proposition 3.4. The proof is done by contraposition, i.e., we assume that \( \mathcal{D} \) has closed range in \( L^2(\Omega) \) with constant \( C \) while \( \rho_{\text{cap}}(\Omega) = \infty \). That is, we assume that for each \( M, \delta > 0 \), there is a point \( z_{M, \delta} \in \mathbb{C} \) such that for any compact set \( K \subset \Omega^c \)

\[
\text{cap}(K \cap \mathbb{D}(z_{M, \delta})) < \delta.
\]

Choose an \( M > 0 \) such that

\[
\frac{4M^2}{C^2} > \lambda_1(\mathbb{D}(0,1)).
\]

Let \( \{\delta_j\}_{j \in \mathbb{N}} \) be a sequence of positive scalars with \( \lim_{j \to \infty} \delta_j = 0 \). For \( N > M \) and positive \( \epsilon_j < \delta_j \) we may choose \( z_{N, \epsilon_j} \) such that any compact set contained in \( \Omega^c \cap \mathbb{D}(z_{N, \epsilon_j}, N) \) has logarithmic capacity less than \( \epsilon_j \). For \( j \in \mathbb{N} \), set

\[
K_j = \Omega^c \cap \mathbb{D}(z_{N, \epsilon_j}, M).
\]

Then \( K_j^1 \subset \mathbb{D}(z_{N, \epsilon_j}, M) \) is compact and, by inner regularity of the logarithmic capacity, see [7, Theorem 5.1.2(b)], \( \text{cap}(K_j^1) \leq \epsilon_j \) for all \( j \in \mathbb{N} \). By Lemma 3.5, for any \( j \in \mathbb{N} \) there exists a compact set \( K_j^2 \) such that \( \mathbb{D}(z_{N, \epsilon_j}, N) \setminus K_j^2 \) has smooth boundary, \( K_j^1 \subset K_j^2 \) and \( \text{cap}(K_j^2) < \delta_j \). Now set

\[
K_j = \frac{1}{M} \left( K_j^2 \setminus \mathbb{D}(z_{N, \epsilon_j}, M) - z_{N, \epsilon_j} \right)
\]

\[
= \left\{ z \in \mathbb{D}(0,1) : Mz + z_{N, \epsilon_j} \in K_j^2 \cap \mathbb{D}(z_{N, \epsilon_j}, M) \right\}.
\]

Then \( K_j \subset \mathbb{D}(0,1) \) is compact such that \( \mathbb{D}(0,1) \setminus K_j \) has smooth boundary and \( \text{cap}(K_j) < \frac{\delta_j}{M^2} \) for \( j \in \mathbb{N} \). Next, set \( D_j = \mathbb{D}(0,1) \setminus K_j \). Then \( D_j \) is open, has smooth boundary, and \( \mathcal{C}(D_j) \leq \frac{C}{M} \) for all \( j \in \mathbb{N} \), hence

\[
\lambda_1(D_j) \geq \frac{4M^2}{C^2} > \lambda_1(\mathbb{D}(0,1)) + \epsilon
\]

for some \( \epsilon > 0 \) by (3.6). In the following, we will show that

\[
\lim_{j \to \infty} \lambda_1(D_j) = \lambda_1(\mathbb{D}(0,1))
\]

This would conclude the proof as (3.8) is a contradiction to (3.7).

Let \( \varphi \) be an eigenfunction corresponding to the first eigenvalue for the Dirichlet problem in \( \mathbb{D}(0,1) \). Then \( \varphi \in \mathcal{C}^\infty(\mathbb{D}(0,1)) \), see e.g., the remark following Theorem 1 in [3, Section 6.5]. Moreover, we may assume that \( 0 \leq \varphi \leq 1 \) on \( \mathbb{D}(0,1) \). Next, let \( h_j \) be the harmonic function in \( D_j \) such that \( h_j = \varphi \) on \( bD_j \). Since \( bD_j \) is smooth and the boundary data \( \varphi \) is smooth up to the boundary of \( \mathbb{D}(0,1) \), it follows that \( h_j \) is smooth up to the boundary of \( D_j \) as well.

Set \( \psi_j(z) = \varphi(z) - h_j(z) \) for \( z \in \overline{D_j} \). Then \( \psi_j \in \mathcal{C}^\infty(\overline{D_j}) \), \( \psi_j = 0 \) on \( bD_j \) and \( \Delta \psi_j = \Delta \varphi \) on \( D_j \). By (2.7)

\[
0 < \lambda_1(D_j) \leq \frac{\| \Delta \psi_j \|_{L^2(D_j)}}{\| \psi_j \|_{L^2(D_j)}}.
\]
It then follows from monotonicity of the first Dirichlet eigenvalue, that

\[
\frac{1}{\lambda_1(\mathbb{D}(0,1))} \geq \frac{1}{\lambda_1(D_j)} \geq \frac{\|\psi_j\|_{L^2(D_j)}}{\|\Delta \psi_j\|_{L^2(D_j)}} = \frac{\|\varphi - h_j\|_{L^2(D_j)}}{\|\Delta \varphi\|_{L^2(D_j)}} \geq \frac{\|\varphi\|_{L^2(D_j)}}{\lambda_1(\mathbb{D}(0,1))} - \frac{\|h_j\|_{L^2(D_j)}}{\lambda_1(\mathbb{D}(0,1))}.
\]

As \(\text{cap}(\mathbb{D}(0,1) \setminus D_j) = \text{cap}(K_j)\) goes to zero as \(j \to \infty\), it follows that \(\|\varphi\|_{L^2(D_j)}\) approaches \(\|\varphi\|_{L^2(D)}\). That is,

\[
1 \geq \frac{\lambda_1(\mathbb{D}(0,1))}{\lambda_1(D_j)} \geq 1 - \frac{\|h_j\|_{L^2(D_j)}}{\|\varphi\|_{L^2(D_j)}}.
\]

To prove that (3.8) holds, it remains to show that \(\lim_{j \to \infty} \|h_j\|_{L^2(D_j)} = 0\). Note first that the maximum principle yields \(0 < h_j \leq \varphi\) on \(D_j\), hence \(\|h_j\|^2 \leq h_j\). Moreover, if \(g_j \in C(D_j)\) is a positive, harmonic function on \(D_j\) such that \(g_j = 1\) on \(bK_j \setminus b\mathbb{D}(0,1)\) and \(g_j \geq 0\) on \(b\mathbb{D}(0,1) \cap bD_j\), then \(0 < h_j \leq g_j\) on \(D_j\). In particular,

\[
\int_{D_j} |h_j|^2 \, dA \leq \int_{D_j} h_j \, dA \leq \int_{D_j} g_j \, dA.
\]

So it suffices to show that there is such a sequence \(\{g_j\}_{j \in \mathbb{N}}\) whose \(L^1(D_j)\)-integral converges to 0.

To construct such \(g_j\), let \(\nu_j\) be the equilibrium measure for \(K_j\), and set

\[
J(\nu_j) = \int_{\mathbb{C}} \int_{\mathbb{C}} \ln \left( \frac{2}{|z - w|} \right) \, d\nu_j(w) \, d\nu_j(z).
\]

Note that \(J(\nu_j) = \ln(2) - I(\nu_j)\). Furthermore, as \(\lim_{j \to \infty} \text{cap}(K_j) = 0\), it follows that \(\lim_{j \to \infty} I(\nu_j) = -\infty\), hence \(\lim_{j \to \infty} J(\nu_j) = \infty\). Hence for \(j\) sufficiently large we may define

\[
g_j(z) = \frac{1}{J(\nu_j)} \int_{\mathbb{C}} \ln \left( \frac{2}{|z - w|} \right) \, d\nu_j(w).
\]

Then \(g_j\) is a positive, harmonic function on \(D_j\), which is non-negative on \(b\mathbb{D}(0,1) \cap bD_j\). We claim that \(g_j\) equals 1 on \(bK_j \setminus b\mathbb{D}(0,1)\) and is continuous on \(\overline{D_j}\). To show the former, we first note that any boundary point of \(D_j\) is a regular boundary point since any smooth defining function of \(D_j\) serves as a subharmonic barrier function, see [7] Def. 4.1.4. This implies that the potential function \(p_j\) associated to the equilibrium measure \(\nu_j\) of \(K_j\) is equal to \(I(\nu_j)\) on \(bK_j \setminus b\mathbb{D}(0,1)\), see [7] Theorem 4.2.4. However, this implies that \(g_j = 1\) on \(bK_j \setminus b\mathbb{D}(0,1)\). It also implies that \(p_j \in C(D_j)\), see [7] Theorem 3.1.3. Therefore, \(g_j \in C(D_j)\). It remains to be shown that \(\int_{D_j} g_j \, dA\) converges to 0 as \(j \to \infty\).

We compute

\[
\int_{D_j} g_j(z) \, dA(z) = \frac{1}{J(\nu_j)} \int_{D_j} \int_{\mathbb{C}} \ln \left( \frac{2}{|z - w|} \right) \, d\nu_j(w) \, dA(z) = \frac{1}{J(\nu_j)} \int_{D_j} \int_{\mathbb{C}} \ln \left( \frac{2}{|z - w|} \right) \, dA(z) \, d\nu_j(w).
\]
Note that
\[ \int_{D_j} \ln \left( \frac{2}{|z-w|} \right) \, dA(z) \leq \int_{\mathbb{D}(0,1)} \ln \left( \frac{2}{|z-w|} \right) \, dA(z) \]
\[ \leq \int_{\mathbb{D}(w,2)} \ln \left( \frac{2}{|z-w|} \right) \, dA(z) \]
\[ = \int_{\mathbb{D}(0,2)} \ln \left( \frac{2}{|z|} \right) \, dA(z) = 2\pi. \]

Therefore \( \lim_{j \to \infty} \int_{D_j} g_j(z) \, dA(z) = 0 \), i.e., \( \lim_{j \to \infty} \| h_j \|_{L^2(D_j)} = 0 \), and hence \( (3.8) \) holds, which concludes the proof. \( \square \)

4. Proof of “(3) \( \Rightarrow \) (4)” of Theorem 1.3

Proposition 4.1. Let \( \Omega \subset \mathbb{C} \) be an open set. Suppose \( \rho_{\text{cap}}(\Omega) < \infty \), i.e., there exist positive constants \( M \) and \( \delta \) such that for each \( z \in \mathbb{C} \) there exists a compact set \( K \subset \Omega^c \) such that
\[ \cap (K \cap D(z, M)) \geq \delta. \]

Then there exists a bounded function \( \varphi \in C^\infty(\Omega) \) and a positive constant \( c \) such that \( \varphi \geq c \) for all \( z \in \Omega \).

Proof. We first observe that whenever there exists a compact, non-polar set \( K \subset \Omega^c \), then \( \Omega \) admits a non-constant, bounded, real-analytic, subharmonic function.

In fact, let \( \nu_K \) be the equilibrium measure of \( K \) such that \( \text{supp}(\nu_K) \subset K \). The associated potential \( p_K \) is given by
\[ p_K(z) = \int_{\mathbb{C}} \ln |z-w| \, dv_K(w). \]

By Frostman’s Theorem, \( p_K(z) \geq \ln(\text{cap}(K)) \) for any \( z \in \Omega \). Thus the values of \( e^{-p_K(z)} \) are in \( (0, 1/\text{cap}(K)) \). Moreover, \( p_K \) is harmonic, hence real-analytic. As \( p_K \) is also non-constant, it follows that \( e^{-p_K} \) is a non-constant, bounded, real-analytic, subharmonic function on \( \Omega \).

These kinds of functions will be the building blocks for the construction of \( \varphi \).

In fact, we will show that there exists a sequence \( \{ K_j \}_{j \in \mathbb{N}} \) in \( \Omega^c \) and constants \( c_1, c_2 > 0 \) such that
\begin{align*}
(\text{i}) & \quad \varphi(z) := \sum_{j \in \mathbb{N}} e^{-4p_{K_j}(z)} \text{ is a smooth function on } \Omega, \\
(\text{ii}) & \quad 0 \leq \varphi(z) \leq c_1 \text{ for all } z \in \Omega, \\
(\text{iii}) & \quad \Delta \varphi(z) \geq c_2 \text{ for all } z \in \Omega.
\end{align*}

Without loss of generality, we may assume that \( 2\delta < M \). We claim that for each \( (j, k) \in \mathbb{Z} \times \mathbb{Z} \), we may choose a compact set \( K_{j,k} \) such that \( \cap (K_{j,k}) \geq \delta \) and
\[ K_{j,k} \subset \mathbb{D}((2jM, 2kM), M + 2\delta) \cap \Omega^c. \]

This can be seen as follows. Suppose that for a given \( (j, k) \) there was no such compact set. If there exists a
\[ z \in \Omega \cap \mathbb{D}((2jM, 2kM), \delta), \]
then, by hypothesis, the logarithmic capacity of \( \mathbb{D}(z, M) \cap \Omega^c \) is at least \( \delta \). This is a contradiction to our assumption since
\[ \mathbb{D}(z, M) \cap \Omega^c \subset \mathbb{D}((2jM, 2kM), M + 2\delta) \cap \Omega^c. \]
See Figure 2. Thus $\Omega^{c}$ contains $D((2jM, 2kM), \delta)$ which is a compact set of logarithmic capacity $\delta$. This proves the claim.

Figure 2. The disc of radius $M + \delta$ contains the disc of radius $M$ centered at $z$.

For each $(j, k) \in \mathbb{Z} \times \mathbb{Z}$, choose a compact set $K_{j, k}$ as described above. Let $p_{K_{j, k}}$ be the associated potential; for the sake of brevity, write $p_{j, k}$ in place of $p_{K_{j, k}}$. We shall show that the series

$$
\sum_{(j, k) \in \mathbb{Z} \times \mathbb{Z}} e^{-4p_{j, k}(z)}
$$

converges for any $z \in \Omega$. To do this, we will fix a $z \in \Omega$ and show convergence of the series for a particular enumeration of $\mathbb{Z} \times \mathbb{Z}$. As the terms of the series are non-negative, it will then follow that the series converges (to the same value) for any choice of enumeration.

For given $z \in \mathbb{C}$, write $Q(z, L)$ for the closed square with center $z$ and side length $2L$. For fixed $z \in \Omega$, let $(j_0, k_0) \in \mathbb{Z} \times \mathbb{Z}$ such that $z \in Q((2j_0M, 2k_0M), M)$. For $\lambda \in \mathbb{N}$, set

$$
A_{\lambda} = \{(j, k) \in \mathbb{Z} \times \mathbb{Z} : \max\{|j - j_0|, |k - k_0|\} = \lambda\}.
$$

A straightforward computation yields that $\text{card}(A_{\lambda}) = 8\lambda$ for $\lambda \geq 1$. Next note that

$$
p_{j, k}(z) \geq \ln \delta \quad \forall (j, k) \in A_{1}.
$$

Furthermore, if $(j, k) \in A_{\lambda}$ for some integer $\lambda \geq 2$ and $w \in K_{j, k}$, then $|z - w| \geq (\lambda - 1)M$. Hence

$$
p_{j, k}(z) \geq \ln ((\lambda - 1)M) \int_{C} 1 \, dv_{j, k}(w) = \ln ((\lambda - 1)M).
$$

Therefore,

$$
\sum_{\lambda=1}^{\infty} \sum_{(j, k) \in A_{\lambda}} e^{-4p_{j, k}(z)} \leq \text{card}(A_{1})\delta^{-4} + \sum_{\lambda=2}^{\infty} \text{card}(A_{\lambda}) \frac{1}{M^4(\lambda - 1)^4}
$$

$$
= 8\delta^{-4} + \frac{8}{M^2} \sum_{\lambda=2}^{\infty} \frac{\lambda}{(\lambda - 1)^4}.
$$
Hence, by the Weierstrass $M$-test, the series is uniformly convergent on $\Omega$. In particular, $\varphi(z) := \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}} e^{-4p_{j,k}(z)}$ is a well-defined function which is continuous and bounded on $\Omega$.

To see that $\varphi$ is in fact in $C^\infty(\Omega)$, it suffices to note that all derivatives of the $p_{j,k}$'s are locally, uniformly bounded. For instance, after computing

$$\frac{\partial}{\partial z} (p_{j,k}(z)) = \frac{1}{2} \int_C \frac{1}{z-w} \ dv_{j,k}(w),$$

we note that for any $z_0 \in \Omega$, there exists a neighborhood $U_{z_0} \subset \Omega$ of $z_0$ and a constant $c > 0$ such that

$$|z-w| > c \quad \forall \ z \in U_{z_0}, \ \forall \ w \in \Omega^c.$$ 

Hence for $z \in U_{z_0}$, it follows that

$$\left| \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}} \frac{\partial}{\partial z} \left( e^{-4p_{j,k}(z)} \right) \right| = 4 \left| \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}} e^{-4p_{j,k}(z)} \frac{\partial}{\partial z} (p_{j,k}(z)) \right| 
\leq 2c_{z_0} \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}} e^{-4p_{j,k}(z)} = 2c_{z_0} \varphi(z),$$

i.e., the series $\sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}} \frac{\partial}{\partial z} \left( e^{-4p_{j,k}(z)} \right)$ converges locally uniformly. Thus it converges to $\frac{\partial \varphi}{\partial z}$. Similarly, if $D$ represents any differential operator, the series

$$\sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}} D \left( e^{-4p_{j,k}(z)} \right)$$

is convergent locally uniformly. In particular, the series converges to $D\varphi$, and $\varphi \in C^\infty(\Omega)$.

The last observation yields in particular

$$\triangle \varphi(z) = 16 \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}} e^{-4p_{j,k}(z)} |\nabla p_{j,k}(z)|^2 \quad \forall \ z \in \Omega,$$

i.e., $\varphi$ is subharmonic on $\Omega$. It remains to be shown that $\varphi$ satisfies condition (iii). For that, it suffices to show that there exists a constant $c > 0$ such that for each $z \in \Omega$ there exists a $(j,k) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$e^{-p_{j,k}(z)} |\nabla p_{j,k}(z)|^2 \geq c.$$ 

A straightforward computation yields

$$|\nabla p_{j,k}(z)| = \frac{1}{2} \left| \int_C \frac{\text{Re}(\bar{z} - \bar{w}) + i \text{Im}(\bar{z} - \bar{w})}{|z-w|^2} \ dv_{j,k}(w) \right| 
\geq \frac{1}{2} \left| \int_C \frac{\text{Re}(z-w)}{|z-w|^2} \ dv_{j,k}(w) \right|$$

for any given $z \in \Omega$ and $(j,k) \in \mathbb{Z} \times \mathbb{Z}$. Now let $z \in \Omega$ be given. Then, as before, $z \in \mathcal{Q}(2j_{0}M, 2k_{0}M, M)$ for some $(j_{0}, k_{0}) \in \mathbb{Z} \times \mathbb{Z}$. Let $(j_{1}, k_{1}) = (j_{0} - 2, k_{0} - 2)$. Note that for any $w \in K_{j_{1}, k_{1}}$

$$\text{Re}(z-w) = \text{Re}(z) - \text{Re}(w) \geq M \quad \text{and} \quad \sqrt{2}M < |z-w| < \sqrt{98}M$$

holds. Thus $|\nabla p_{j_{1}, k_{1}}(z)| \geq \frac{1}{2M^{2}}$. Moreover, $e^{-4p_{j_{1}, k_{1}}(z)} \geq \frac{1}{2}M^{-4}$. Hence, $\Delta \varphi(z) \geq \frac{2}{4M^{4}}$ for all $z \in \Omega$. See Figure 3.
5. Proof of “(3) ⇒ (4)” of Theorem 1.4

Proposition 5.1. Let \( \Omega \subset \mathbb{C} \) be an open set such that \( \text{cap}(\Omega^c) > 0 \). Then there exists a bounded \( \varphi \in C^\infty(\Omega) \) such that \( \Delta \varphi > 0 \) on \( \Omega \).

Proof. Note that there exists a compact set \( K \subset \Omega^c \) such that \( \text{cap}(K) > 0 \). Let \( \nu \) be the equilibrium measure of \( K \), and \( p \) be the associated potential function, i.e.,

\[
p(z) = \int_C \ln |z - w| \, d\nu(w).
\]

Recall that \( p \) is harmonic in \( \Omega \) and \( p \geq \ln(\text{cap}(K)) \). Hence \( \varphi := e^{-p} \) is smooth and bounded on \( \Omega \). Moreover, \( \Delta \varphi = e^{-p} |\nabla p|^2 \).

We shall show first that there is an open ball \( B \) such that \( K \subset B \) and \( |\frac{\partial p}{\partial z}(z)| > 0 \) for all \( z \in B \cap \Omega \). Let \( B, B' \) be concentric open balls, such that \( K \subset B' \), and the radius of \( B \) is twice the radius of \( B' \). Let \( z \in \Omega \cap B^c \) be given, \( \theta \in [0, 2\pi) \) fixed and chosen later. See Figure 3. Then

\[
\left| \frac{\partial p}{\partial z}(z) \right| = \frac{1}{2} \left| \int_C \frac{\bar{z} - \bar{w}}{|z - w|^2} e^{-i\theta} \, d\nu(w) \right| \\
\geq \frac{1}{2} \left| \int_C \text{Re} \left( (z - w) e^{-i\theta} \right) \frac{1}{|z - w|^2} \, d\nu(w) \right| \\
= \frac{1}{2} \left| \int_C \frac{\cos(\alpha(z - w) - \theta)}{|z - w|} \, d\nu(w) \right|,
\]

where \( \alpha(z - w) \) is the branch of the argument of \( z - w \) in \([0, 2\pi)\). Since the radius of \( B \) is twice the radius of \( B' \), it follows that there is some \( c \in (0, \pi/2) \) such that

\[
|\alpha(z - w) - \alpha(z - w')| < c \quad \forall \ w, w' \in K.
\]

This allows us to choose \( \theta \) such that \( \alpha(z - w) - \theta \in (0, c) \subset (0, \pi) \) for all \( w \in K \). Hence \( \cos(\alpha(z - w) - \theta) > 0 \) for all \( w \in K \). Thus \( |\frac{\partial p}{\partial z}(z)| > 0 \) for all \( z \in \Omega \cap B^c \).

Let \( B'' \) be an open set containing \( \overline{B} \). Then, since \( p \) is smooth in \( \Omega \) and \( |\nabla p| > 0 \) on \( \Omega \cap B^c \) there exists a constant \( C > 0 \) such that \( \Delta \varphi > C \) on \( \overline{\Omega} \cap (\overline{B'' \setminus B}) \). Let \( z_0 \) be the center of \( B' \), let \( \chi \in C^\infty(\mathbb{R}) \) such that \( \chi(|z - z_0|^2) = |z - z_0|^2 \) for all \( z \in B \) and \( \text{supp}(\chi(|z - z_0|^2)) \subset B'' \). Then \( \varphi + \epsilon \chi(|z - z_0|^2) \) is a smooth, bounded function on \( \Omega \) which is strictly subharmonic everywhere as long as \( \epsilon > 0 \) is sufficiently small. □
Figure 4. From \( z \) outside \( B \), the disc \( B' \), and hence \( K \), subtends an angle less than \( \frac{\pi}{2} \).

REFERENCES

[1] Carleson, L. Selected problems on exceptional sets. Van Nostrand Mathematical Studies, No. 13. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[2] Conway, J. B. Functions of one complex variable. II, vol. 159 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[3] Evans, L. C. Partial differential equations, vol. 19 of Graduate Studies in Mathematics. American Mathematical Society, 1998.

[4] Gallagher, A.-K., Harz, T., and Herbort, G. On the dimension of the Bergman space for some unbounded domains. J. Geom. Anal. 27, 2 (2017), 1435–1444.

[5] Herbig, A.-K., and McNeal, J. D. On closed range for \( \overline{\partial} \). Complex Var. Elliptic Equ. 61, 8 (2016), 1073–1089.

[6] Hörmander, L. \( L^2 \) estimates and existence theorems for the \( \overline{\partial} \) operator. Acta Math. 113 (1965), 89–152.

[7] Ransford, T. Potential theory in the complex plane, vol. 28 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1995.

[8] Souplet, P. Geometry of unbounded domains, Poincaré inequalities and stability in semilinear parabolic equations. Comm. Partial Differential Equations 24, 5-6 (1999), 951–973.

[9] Souplet, P. Decay of heat semigroups in \( L^\infty \) and applications to nonlinear parabolic problems in unbounded domains. J. Funct. Anal. 173, 2 (2000), 343–360.

[10] Straube, E. J. Lectures on the \( L^2 \)-Sobolev theory of the \( \overline{\partial} \)-Neumann problem. ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2010.

[11] Wiegerinck, J. J. O. O. Domains with finite-dimensional Bergman space. Math. Z. 187, 4 (1984), 559–562.