REGULAR EMBEDDINGS OF MANIFOLDS AND TOPOLOGY OF CONFIGURATION SPACES

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Abstract. For a topological space \(X\) we study continuous maps \(f : X \to \mathbb{R}^m\) such that images of every pairwise distinct \(k\) points are affinely (linearly) independent. Such maps are called affinely (linearly) \(k\)-regular embeddings.

We investigate the cohomology obstructions to existence of regular embeddings and give some new lower bounds on the dimension \(m\) as function of \(X\) and \(k\), for the cases \(X\) is \(\mathbb{R}^n\) or \(X\) is an \(n\)-dimensional manifold. In the latter case, some nonzero Stiefel–Whitney classes of \(X\) help to improve the bound.

1. Introduction

Let \(X\) be a topological space. We study continuous maps \(f : X \to \mathbb{R}^m\) such that images of every pairwise distinct \(k\) points are affinely (linearly) independent. We call such maps affinely (linearly) \(k\)-regular embeddings. This concept was introduced in [3], it is closely related to some questions of approximation by a system of functions (the Chebyshev approximation), see [6] for detailed explanations.

For a given space \(X\) there are some lower bounds on the dimension \(m\), and there are also some existence theorems for large enough \(m\). In this paper we consider lower bounds for the cases \(X = \mathbb{R}^n\) or \(X\) is some other \(n\)-dimensional manifold. Obviously, the lower bounds for the case of \(\mathbb{R}^n\) give lower bounds for any \(n\)-manifolds. In [2] the first lower bound from the dimension considerations (nevertheless, nontrivial) was made:

\[
m \geq \left\lceil \frac{k}{2} \right\rceil n + \left\lceil \frac{k - 1}{2} \right\rceil.
\]

Then some essentially topological methods were applied to improve this lower bound. In the paper [5] (the case of \(n = 2\) was considered previously in [3]) this bound was improved in the case \(n = 2^l\).

Definition 1.1. Denote by \(\alpha_p(n)\) the sum of digits in \(p\)-adic representation of an integer \(n\).

Theorem (Chisholm, 1979). If there is a linearly \(k\)-regular map from \(\mathbb{R}^n\) to \(\mathbb{R}^m\), where \(n\) is a power of two, then

\[
m \geq n(k - \alpha_2(k)) + \alpha_2(k).
\]

Remark 1.2. This theorem was also rediscovered in [14, 15].
The Chisholm theorem was established by considering some equivariant t maps, and this method does not distinguish between the affine and linear cases (see Lemma 2.3 below). From here on we state the results for linearly $k$-regular maps, noting that for affinely $k$-regular maps the lower bound is 1 less. The Chisholm theorem gives a good estimate for the growth of $m$ as a function of $k$, since every $n$-manifold can be linearly $k$-regularly mapped to $\mathbb{R}^m$ with $m = (n+1)k + 1$ (see [9, 15] for the explanation of the upper bound, in fact every “general position” map in such dimension is $k$-regular).

For the case of manifolds other than $\mathbb{R}^n$ there is a result from [9], using characteristic classes of the manifold:

**Theorem** (Handel, 1996). Suppose $M$ is an $n$-dimensional manifold, $k$ is even, and suppose that the $d$-th dual Stiefel–Whitney class of $M$ is nonzero. If there is a $k$-regular map of $M$ to $\mathbb{R}^m$ then

$$m \geq \frac{k}{2}(n+d+1).$$

Moreover, if $M$ is compact then

$$m \geq \frac{k}{2}(n+d+1) + 1.$$

The proof of this theorem in [9] was incorrect, the map between the configuration spaces (the third formula from the page bottom in [9, page 1611]) was defined incorrectly. Informally, $k$ pairwise distinct pairs of points do not necessarily constitute $2k$ pairwise distinct points. Still, in this paper this theorem is rehabilitated and a slightly stronger result (Theorem 4.4 in Section 4) is proved.

We start from an observation (see Lemma 4.2) that it is important to decompose $k$ into a sum of powers of two. It was already known (and obvious) for $M = \mathbb{R}^n$, but it also works for arbitrary manifold $M$, if we define the appropriate subspaces of the configuration space (see Section 3). The power of two sum may be the standard binary expansion, as in the theorem of Chisholm, or it may be the sum of 2’s, as in the theorem of Handel, or something between these two cases.

Using the above remark, we concentrate on the case $k$ a power of two, and denote it by $q$ in this case. In Sections 6 and 7 the external Steenrod square construction is used to describe the cohomology mod 2 of the Sylow subgroups of the symmetric group and corresponding configuration subspaces. The results of these sections were previously obtained in [10], but we give a self-contained explanation of them.

In Sections 8, 9, and 11 we give some explicit formulas that allow us to calculate the lower bounds on $m$ for given $M$. In the case $M = \mathbb{R}^n$ the formulas are almost explicit (see Theorem 11.3). For arbitrary manifolds and $q \geq 4$ there is no general explicit formula (except for the case $q = 2$ in the theorem of Handel), but the problem is reduced to some straightforward algebraic calculations with the cohomology of configuration spaces and the Stiefel–Whitney classes of $M$.

In Section 11 some particular manifolds $M$ (with a restriction on the dimension and the dual Stiefel–Whitney classes) are considered, and explicit bounds for the dimension of a regular embedding are given. In particular, some products of a projective space with a circle can be taken as $M$.

In Section 13 we apply the computations in the cohomology of the configuration spaces to another problem. We prove the existence of multiple points of continuous maps of a projective space to a Euclidean space, generalizing previous results of the author [11].

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2. Configuration spaces

In order to study the images of $k$-tuples of points under some continuous map $f : M \to \mathbb{R}^n$, it is natural to introduce the configuration space:

**Definition 2.1.** For a topological space $X$ define the configuration space by

$$F_k(X) = \{(x_1, \ldots, x_k) \in X^{\times k} : x_i \neq x_j \text{ if } i \neq j\}.$$ 

Note that the permutation group $\Sigma_k$ acts freely on $F_k(X)$.

Denote by $V_k$ the natural $k$-dimensional representation of $\Sigma_k$ by permuting the basis vectors. This representation induces a $\Sigma_k$-equivariant vector bundle 

$$V_k \times Y \to Y$$

over any space $Y$ with action of $\Sigma_k$ (or $\Sigma_k$-space).

**Definition 2.2.** For an (equivariant) vector bundle $\xi : E(\xi) \to X$, denote by $\xi^\perp$ an (equivariant) vector bundle of minimal dimension, such that the bundle $\xi \oplus \xi^\perp$ is trivial. This bundle need not be uniquely determined up to isomorphism, but its stable isomorphism class is determined uniquely.

The main tool in proving the lower bounds for the dimension of $k$-regular maps is the following lemma from [6]:

**Lemma 2.3.** Consider the equivariant bundle $\nu_k(X) : V_k \times F_k(X) \to F_k(X)$. If $\dim \nu_k(X)^\perp = l$ then there is no linearly $k$-regular map $X \to \mathbb{R}^{k+l-1}$, and no affinely $k$-regular map $X \to \mathbb{R}^{k+l-2}$.

It follows from this lemma that we have to study the Stiefel–Whitney (or Pontryagin) classes of $\nu_k(X)^\perp$ and prove that these classes are nonzero in $H^*_{\Sigma_k}(F_k(X))$ (with coefficients $\mathbb{F}_2$ or $\mathbb{Z}$ respectively) for large enough $l$. The characteristic classes of $\nu_k(X)^\perp$ are usually called the dual Stiefel–Whitney (or Pontryagin) classes of $\nu_k(X)$ and denoted by $\bar{w}(\nu_k(X))$ and $\bar{p}(\nu_k(X))$ respectively.

3. Special configuration subspaces

Let us define a subspace $Q_q(M)$ of the configuration space $F_q(M)$; this subspace is a smooth manifold provided $M$ is a smooth manifold. Such subspaces were introduced for $M = \mathbb{R}^n$ in [10] and proved to be useful in determining the cohomology of the symmetric group. They were also used in [11] to establish some theorems on multiple points of continuous maps, see also Section 13. We start with the case $M = \mathbb{R}^n$.

**Definition 3.1.** Let $q = 2^l$ and $\delta > 0$. Let $\hat{Q}_1(\mathbb{R}^n)$ be the configuration, consisting of one point at the origin.

Let by induction $\hat{Q}_q(\mathbb{R}^n, \delta)$ be the set of all $q$-point configurations, such that the first $q/2$ points form a configuration of $\hat{Q}_{q/2}(\mathbb{R}^n, \delta/3)$, shifted by a vector $u$ of length $\delta$, and the other $q/2$ points form a configuration of $\hat{Q}_{q/2}(\mathbb{R}^n, \delta/3)$, shifted by the vector $-u$.

Note that $\hat{Q}_q(\mathbb{R}^n) = \hat{M}(u, \log_2 q)$ in the notation of [10].

**Definition 3.2.** A configuration in $\hat{Q}_{q/2}(\mathbb{R}^n, \delta)$ can also be described inductively as $x_1, \ldots, x_q \in \mathbb{R}^n$ such that all the distances $\text{dist}(x_{2i-1}, x_{2i}) = \frac{2\delta}{3^{i-1}}$ and the midpoints of $[x_{2i-1}, x_{2i}]$ form a configuration of $\hat{Q}_{q/2}(\mathbb{R}^n, \delta)$.
Note that $\check{Q}_q(\mathbb{R}^n)$ is always a product of $q-1$ spheres of dimension $n-1$. We shall omit $\delta$ from the notation since it does not change the diffeomorphism type of $Q_q(\mathbb{R}^n)$. Then we can naturally define the fiberwise configuration space $Q_q(\xi)$ for any vector bundle $\xi$ as a bundle of corresponding to the union of $\check{Q}_q(\xi^{-1}(x))$ for all $x \in M$. This is a subspace of the full fiberwise configuration space $F_\eta(\xi)$, defined in a similar manner.

Note that Definition 3.2 (distance and midpoint characterization) can be applied to any Riemannian manifold $M$, if we allow the last center point (configuration $\check{Q}_1$) to be any $x \in M$.

**Definition 3.3.** Let $M$ be a Riemannian manifold. Define $Q_q(M, \delta) \subset F_q(M)$ for $q = 2^l$ inductively as follows:

1. $Q_1(M) = M$;
2. For $q \geq 2$ let $Q_q(M, \delta)$ be the set of $q$-tuples $x_1, \ldots, x_q \in M$ such that all the distances $\text{dist}(x_{2i-1}, x_{2i}) = \frac{2\delta}{3^{i-1}}$ and the midpoints of $[x_{2i-1}, x_{2i}]$ form a configuration of $Q_{q/2}(M, \delta)$.

The following lemma describes $Q_q(M, \delta)$ as a bundle over $M$.

**Lemma 3.4.** Let the injectivity radius of $M$ be $r$ and $2\delta < r$. Then $Q_q(M, \delta)$ is a fiber bundle (the bundle map is the last stage midpoint) over $M$, and is naturally homeomorphic to $Q_q(\tau M)$.

**Proof.** We prove this by induction. For any configuration $(x_1, \ldots, x_q) \in Q_q(M, \delta)$ the midpoints of pairs $[x_1, x_2], [x_2, x_3], \ldots, [x_{q-1}, x_q]$ form a configuration in $Q_{q/2}(M, \delta)$. Since $2\delta < r$, then knowing the midpoint of $[x_1, x_2]$, the possible positions of the points $x_1, x_2$ form a sphere.

So $Q_q(M, \delta)$ is a product-of-spheres bundle over $Q_{q/2}(M, \delta)$. Moreover, these spheres are spheres of the pullbacks of the tangent bundle $\pi_i^*(\tau M)$, where $\pi_i : Q_{q/2}(M, \delta/3) \to M$ is the map, assigning to a configuration its $i$-th point. Note that the maps $\pi_i$ are all homotopic to the centerpoint map $\pi : Q_{q/2}(M, \delta/3) \to M$ (the homotopy can be obtained by deforming a point $x_{2i-1}$ or $x_{2i}$ to the midpoint of $[x_{2i-1}, x_{2i}]$, and then repeating inductively), hence all the vector bundles are equivalent to $\pi^*(\tau M)$. Now the proof is completed by applying the inductive assumption.

The space $\check{Q}_q$ (or $Q_q$) is not invariant under the natural action of $\Sigma_q$, but it is invariant under the action of its $2$-Sylow subgroup.

**Definition 3.5.** Let $q = 2^k$. Denote by $\Sigma_q^{(2)}$ the Sylow subgroup of $\Sigma_q$, generated by all permutations of two consecutive blocks $[a2^l + 1, a2^l + 2^{l-1}]$ and $[a2^l + 2^{l-1} + 1, (a + 1)2^l]$, where $1 \leq l \leq k$ and $0 \leq a \leq 2^{k-l} - 1$.

Denote by $A_q$ the subspace of the natural $q$-dimensional representation $V_q$ of $\Sigma_q$, consisting of the vectors with zero coordinate sum. As in the previous section, $A_q$ induces the equivariant bundle $\alpha_q(X)$ over any $\Sigma_q$-space $X$, the group $\Sigma_q$ can be changed to $\Sigma_q^{(2)}$. The following lemma is proved in [11] by a simple geometric reasoning, it also follows from the results in [10].

**Lemma 3.6.** The manifold $\check{Q}_q(\mathbb{R}^n)$ is $\Sigma_q^{(2)}$-invariant. The cohomology $H^{(q-1)(n-1)}_{\Sigma_q^{(2)}}(\check{Q}_q(\mathbb{R}^n); \mathbb{F}_2)$ is generated by the Euler class (the topmost Stiefel–Whitney class)

$$e \left( \alpha_q(\check{Q}_q(\mathbb{R}^n)) \right)^{n-1}.$$
In is well known [1] that the $\Sigma_q$-equivariant cohomology with coefficients $\mathbb{F}_2$ is mapped injectively to the $\Sigma_q^{(2)}$-equivariant cohomology; so we do not lose anything. Actually, we could consider arbitrary $q$, not necessarily a power of two, and define the corresponding subspace $Q_q(M, \delta)$ inductively. It is again invariant under the 2-Sylow subgroup of $\Sigma_q$, but its topmost cohomology is not generated by a power of $e(A_q)$, since the latter class is zero already in $H^*(\Sigma_q; \mathbb{F}_2)$.

4. Generalization of Lemma 2.3

We are going to generalize Lemma 2.3, in order to prove the strengthening of the theorem of Handel [9] and some more results.

Let us introduce some notation, needed to state the generalizations of Lemma 2.3.

Definition 4.1. For an (equivariant) vector bundle $\xi : E(\xi) \to X$ denote by $\bar{\ell}(\xi)$ the maximum $k$ such that the dual (equivariant) Stiefel–Whitney class $\bar{w}_k(\xi)$ is nonzero.

It follows from the Künneth formula and the multiplicativity of the Stiefel–Whitney classes that for the $\times$-product of vector bundles we have

$$\bar{\ell}(\xi \times \zeta) = \bar{\ell}(\xi) + \bar{\ell}(\zeta).$$

Now we are going to state the lemma. It is stated for linearly $k$-regular maps, for affinely $k$-regular maps the lower bound is less by 1.

Lemma 4.2. Let $k = q_1 + \cdots + q_l$, where $q_i$ are powers of two. Let $M$ be a smooth manifold. If there exists a linearly $k$-regular map $f : M \to \mathbb{R}^m$, then

$$m \geq k - l + 1 + \bar{\ell} \left( \prod_{i=1}^{l} \alpha_{q_i}(Q_{q_i}(M)) \right) = k - l + 1 + \sum_{i=1}^l \bar{\ell}(\alpha_{q_i}(Q_{q_i}(M))).$$

We postpone the proof of Lemma 4.2 till the next section. Now let us discuss its consequences. If we apply this lemma to the case $M = \mathbb{R}^n$, $n$ is a power of two, $k = q_1 + \ldots + q_l$ is the binary expansion, then we obtain a slightly weaker result than the Chisholm theorem with the inequality

$$m \geq n(k - \alpha_2(k)) + 1.$$

It follows from the fact that $\mathbb{R}^n$ contains any number of copies of $\mathbb{R}^n$, and the configuration space $F_2(\mathbb{R}^n)$ contains the product $\prod_{i=1}^l F_{q_i}(\mathbb{R}^n)$. In other words, in the case $M = \mathbb{R}^n$ Lemma 4.2 can be modified as follows.

Lemma 4.3. Let $k = q_1 + \cdots + q_l$, where $q_i$ are powers of two. Let $n$ be an integer. If there exists a linearly $k$-regular map $f : \mathbb{R}^n \to \mathbb{R}^m$, then

$$m \geq k + \sum_{i=1}^l \bar{\ell}(\alpha_{q_i}(Q_{q_i}(\mathbb{R}^n))).$$

Let us give another application of Lemma 4.2, which is a stronger version of the theorem of Handel [9].

Theorem 4.4. Suppose $M$ is an $n$-dimensional manifold, $k$ is an even number. If there is a $k$-regular map of $M$ to $\mathbb{R}^m$ then

$$m \geq \frac{k}{2}(n + \bar{\ell}(\tau M) + 1) + 1.$$

Moreover, if $M$ is compact then

$$m \geq \frac{k}{2}(n + \bar{\ell}(\tau M) + 1) + 2.$$
Proof. In [7] it is shown that
\[ \bar{\ell}(\alpha_2(Q_2(M))) = n + \bar{\ell}(\tau M), \]
and the case of non-compact \( M \) follows from Lemma 4.2. For compact \( M \) (see also [9]) it is possible to replace the last \( Q_2(M) \) (see the proof of Lemma 4.2 below), which is the space of pairs in \( M \) with distance \( \delta \), by the space \( R^2(M) \), which is the space of pairs in \( M \) with distance \( \geq \delta \). In [16, 12] it is shown that
\[ \bar{\ell}(\alpha_2(R^2(M))) = \dim M + \bar{\ell}(\tau M) + 1, \]
and the estimate on \( m \) increases by 1. \( \square \)

5. PROOF OF LEMMA 4.2

If the manifold \( M \) is compact, then \( Q_q(M) \) is a compact manifold again. If \( M \) is not compact then we use the following convention. We assume that there exists a compact subset \( C \subset M \) (it can be chosen to be a compact manifold with boundary) such that the cohomology map \( H^*(M; \mathbb{F}_2) \rightarrow H^*(C; \mathbb{F}_2) \) is injective, at least on some given finite-dimensional subspace of \( H^*(M; \mathbb{F}_2) \). If we have to make \( Q_q(M) \) compact, we consider it as a bundle over \( M \), and restrict it to a bundle over \( C \).

We also suppose \( M \) to have some Riemannian metric.

**Definition 5.1.** Denote by \( d(Q_q(M, \delta)) \) and \( D(Q_q(M, \delta)) \) the minimum and the maximum distance between some pair of points in a configuration from \( Q_q(M, \delta) \). They exist and they are positive under the compactness assumptions above, and they depend continuously on \( \delta \).

We have some freedom to choose \( \delta_i \)’s in the definitions of \( Q_q(M, \delta_i) \). We are going to choose them in such a way that for each \( i = 2, \ldots, l \)

\[ d(Q_q(M, \delta_i)) > \sum_{j=1}^{i-1} q_i D(Q_q(M, \delta_j)), \tag{5.1} \]

and so that the last \( \delta_l \) is less than the injectivity radius of \( M \). From the continuous dependance of \( d(Q_q(M, \delta_i)) \) and \( D(Q_q(M, \delta_i)) \) it is possible to satisfy these inequalities if the first \( \delta_1 \) is chosen small enough.

Denote by \( G = \Sigma^2_q \times \cdots \times \Sigma^2_q \) the natural symmetry group of \( Q_q(M) \times \cdots \times Q_q(M) \). Now Lemma 4.2 is deduced from the following.

**Lemma 5.2.** Under the above assumptions there exists a fiberwise \( G \)-equivariant map of vector bundles
\[ g : \mathbb{R} \times \prod_{i=1}^{l} \alpha_{q_i}(Q_{q_i}(M_i, \delta_i)) \rightarrow \mathbb{R}^m, \]
where \( G \) acts trivially on \( \mathbb{R} \) and \( \mathbb{R}^m \), these two spaces are considered as bundles over one point.

Proof. Remind that the bundles \( \alpha_{q_i}(Q_{q_i}(M_i, \delta_i)) \) are simply the products \( A_{q_i} \times Q_{q_i}(M_i, \delta_i) \). More precisely, an element of \( A_{q_i} \times Q_{q_i}(M_i, \delta_i) \) is a \( q_i \)-tuple of points \( (x_{1}^{i}, \ldots, x_{n}^{i}) \) in \( M \) along with a set of real coefficients \( w_{1}^{i}, \ldots, w_{q_i}^{i} \) with zero sum. The action of \( \Sigma^2_q \) is given by the permutation of the points, and the corresponding permutation of the coefficients.

The required map \( g \) of bundles is defined as follows. Suppose we have \( l \) sets of \( q_i \) \( (i = 1, \ldots, l) \) points each, let the points \( x_{j}^{i} \) be as above. Suppose we also have the
respective coefficients \( w^i_j \), and another coefficient \( t \). Denote the map \( g \) on this combination by
\[
g(\ldots) = t \sum_{i,j} f(x^i_j) + \sum_{i,j} w^i_j f(x^i_j).\]

This map is obviously \( G \)-equivariant (if the points are permuted, the coefficients are permuted accordingly), so it is left to show its injectivity on fibers.

The fiber of the vector bundle
\[
\eta : \mathbb{R} \times \prod_{i=1}^l \alpha_{q_i}(Q_{q_i}(M, \delta_i)) \rightarrow \prod_{i=1}^l Q_{q_i}(M, \delta_i)
\]
over the point set \( x^i_j \) has the following base: the first vector \( e_0 \) is given by \( t = 1, w^i_j = 0 \), then for a given \( i \) and \( j = 1, \ldots, q_i - 1 \) we have a vector \( e^i_j \) with coordinates \( w^i_j = 1, \ w^i_{j+1} = -1 \), the other coordinates being zero. Let us show that the images of the system \( \{e_0\} \cup \{e^i_j\}_{i=1, \ldots, l, j=1, \ldots, q_i - 1} \) are linearly independent.

Suppose that the points \( x^i_j \) constitute the point set \( V \subset M \), note that the points \( x^i_j \) may coincide for different index pairs \( (i, j) \). Define the graph \( T \) on vertices \( V \) as follows: the images of all the pairs \((x^i_j, x^i_{j+1})\) form an edge.

We claim that \( T \) is a tree. Indeed, suppose \( T \) has a simple cycle \( C \subset V \), consider the maximum index \( i \) appearing in an edge \( (u, v) = (x^i_j, x^i_{j+1}) \) of this cycle. Let \( (w, y) \) be the next edge of \( C \) with the same index \( i \), it may happen that \( (w, y) = (u, v) \). The segment of \( C \) between \( v \) and \( w \) goes by the edges \((x^m_j, x^m_{j+1})\) with \( m < i \), hence its total length is at most
\[
\sum_{j=1}^{i-1} q_j D(Q_{q_j}(M, \delta_j)),
\]
but the points \( v \) and \( w \) are different points in some configuration of \( Q_{q_i}(M, \delta_i) \), and the distance between them is at least \( d(Q_{q_i}(M, \delta_i)) \), which contradicts \([5,1]\). Thus \( T \) is a tree.

Now we see that the images \( g(e^i_j) \), written in the basis \( f(V) \) (it is a basis since \( f \) if linearly \( k \)-regular), have nonzero coordinate pairs that form a tree \( T \). The tree can be reconstructed by adding one new edge and one new vertex at a time, hence the coordinates of these vectors form an upper triangular matrix with nonzero diagonal, after such a reordering of vertices and vectors (edges). In follows that \( e^i_j \) are linearly independent. The vector \( e_0 \) is orthogonal to all of them (if the scalar product is Euclidean in the basis \( f(V) \)), hence it is independent of the other \( e^i_j \).

\[\square\]

6. External Steenrod squares

In order to describe the \( \Sigma_q^{(2)} \)-equivariant cohomology of \( \hat{Q}_q(\mathbb{R}^n) \) and the similar spaces, we have to use the construction of external Steenrod squares. We mostly follow [4 Ch. V], where the Steenrod squares were defined in the unoriented cobordism. The cobordism was defined using mock bundles, if we allow the mock bundles to have codimension 2 singularities, we obtain ordinary cohomology modulo 2. In the sequel we consider the cohomology mod 2 and omit the coefficients from notation. The similar construction was used in [10] to calculate the cohomology of \( \hat{Q}_q(\mathbb{R}^n) \), based on the Steenrod decomposition theorem for the cohomology of \((K \times K \times S^n)/\mathbb{Z}_2\) instead of mock bundles.

The construction of the external Steenrod squares on a polyhedron \( K \) starts with the fiber bundle (for some integer \( n > 0 \))
\[
\sigma_{K,n} : (K \times K \times S^n)/\mathbb{Z}_2 \rightarrow S^n/\mathbb{Z}_2 = \mathbb{R}P^n.
\]
The group $\mathbb{Z}_2$ acts by permuting $K \times K$, and antipodally on $S^n$. Consider a cohomology class $\xi \in H^*(K)$, represented by a mock bundle $\xi : E(\xi) \to K$. Then the mock bundle

$$(\xi \times \xi \times S^n)/\mathbb{Z}_2 \to (K \times K \times S^n)/\mathbb{Z}_2$$

is the external Steenrod square $Sq_e \xi$. The operation $Sq_e$ is evidently multiplicative, in [1, Ch. V, Proposition 3.3] it is claimed that $Sq_e$ is also additive. We are going to show that it is not true, first we need a definition.

**Definition 6.1.** The difference $Sq_e(\xi + \eta) - Sq_e \xi - Sq_e \eta$ is represented by the mock bundle

$$\xi \circ \eta = (\xi \times \eta \times S^n + \eta \times \xi \times S^n)/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ exchanges the components $\xi \times \eta$ and $\eta \times \xi$.

Since the fiber of $\sigma_{K,n}$ is $K \times K$, the restriction of $\xi \circ \eta$ to the fiber is $\xi \times \eta + \eta \times \xi$, which is nonzero if $\eta \neq \xi$ as cohomology classes. Thus the operation $\circ$ is not trivial.

We need a lemma about the $\circ$-multiplication.

**Lemma 6.2.** Denote by $c$ the hyperplane class in $H^1(\mathbb{R}P^n)$. Then for any $\xi, \eta \in H^*(K)$ the product

$$\xi \circ \eta = \sigma_{K,n}^*(c) = 0$$

in $H^*((K \times K \times S^n)/\mathbb{Z}_2)$.

**Proof.** Consider the mock bundle

$$\alpha = \xi \times \eta \times S^{n-1} + \eta \times \xi \times S^{n-1},$$

which has the natural $\mathbb{Z}_2$-action, it represents $(\xi \circ \eta) = \sigma_{K,n}^*(c)$ after taking the quotient by the $\mathbb{Z}_2$-action.

Now divide $S^n$ into the upper and the lower half-spheres $H^+$ and $H^-$. Consider the mock bundle (with boundary)

$$\beta = \xi \times \eta \times H^+ + \eta \times \xi \times H^-$$

over $K \times K \times S^n$. The action of $\mathbb{Z}_2$ on $\beta$ is defined by permuting the summands and the antipodal identification of $H^+$ and $H^-$. Now it is clear that $\alpha$ is the boundary of $\beta$, and $\alpha/\mathbb{Z}_2$ is the boundary of $\beta/\mathbb{Z}_2$. Hence it is zero in the cohomology, and the similar statement is true for the unoriented bordism. \hfill \Box

We have to introduce another operation.

**Definition 6.3.** Let $\xi : E(\xi) \to K$, $\eta : E(\eta) \to K$ be two mock bundles. Let $p_+, p_-$ be the north and the south poles of $S^n$. Denote the mock bundle over $(K \times K \times S^n)/\mathbb{Z}_2$ by

$$\iota(\xi \times \eta) = (\xi \times \eta \times \{p_+\}) + (\eta \times \xi \times \{p_-\})/\mathbb{Z}_2.$$

It is obvious from the definition that we have relation

$$\iota(\xi \times \eta) = \sigma_{K,n}^*(c) = 0,$$

it is also obvious that

$$\iota(\xi \times \xi) = Sq_e \xi - \sigma_{K,n}^*(c)^n.$$
\[
\text{Sq}_n \xi \sim \iota(\eta \times \zeta) = \iota((\xi \sim \eta) \times (\xi \sim \zeta)),
\]
\[
\iota(\xi \times \eta) \sim \iota(\xi \times \chi) = 0.
\]
Now we can describe the structure of the cohomology \(H^*((K \times K \times S^n)/\mathbb{Z}_2)\).

**Definition 6.4.** Consider a graded \(\mathbb{F}_2\)-algebra \(A\) with linear basis \(v_1, \ldots, v_m\). Denote by \(A \otimes A\) the subalgebra of \(A \otimes A\), invariant w.r.t. \(\mathbb{Z}_2\)-action by permuting the factors. The linear base of \(A\) is
\[
\{v_i \otimes v_j\}_{i=1}^n, \quad \{v_i \otimes v_j + v_j \otimes v_i\}_{i<j}.
\]

**Definition 6.5.** Consider a graded \(\mathbb{F}_2\)-algebra \(A\) with linear basis \(v_1, \ldots, v_m\). Denote by \(\iota(A \otimes A)\) the quotient vector space \(A / (v_1 \otimes v_2 + v_2 \otimes v_1)\). As \(\mathbb{F}_2\)-algebra it has zero multiplication.

**Lemma 6.6.** The maps \(\text{Sq}_n\) and \(\otimes\) map the algebra \(H^*(K) \otimes H^*(K)\) to \(H^*((K \times K \times S^n)/\mathbb{Z}_2)\). The map \(\iota\) maps \(\iota(H^*(K) \otimes H^*(K))\) to \(H^*((K \times K \times S^n)/\mathbb{Z}_2)\). The images of these maps together with the generator \(c \in H^1(S^n/\mathbb{Z}_2)\) multiplicatively generate the cohomology \(H^*((K \times K \times S^n)/\mathbb{Z}_2)\).

The latter cohomology can be described as the quotient of \(H^*(K) \otimes H^*(K) \otimes \mathbb{F}_2[c] \oplus \iota(H^*(K) \otimes H^*(K))\) by the relations
\[
e^{n+1} = 0, \quad (\xi \otimes \eta) \otimes c = 0, \quad \text{Sq}_n \xi \otimes e^n = \iota(\xi \otimes \xi).
\]

Compare this lemma with [10, Theorem 2.1], see also [13]. Note the important particular case: if \(n \to \infty\), we image of \(\iota(\ldots)\) disappears, and we also can take the quotient of \(H^*(K) \otimes H^*(K)\) by the linear span of all \(\xi \otimes \eta\) for \(\xi, \eta \in H^*(K)\). Hence, the cohomology \(H^*((K \times K \times S^\infty)/\mathbb{Z}_2)\) has a quotient isomorphic to \(\text{Sq}_n(H^*(K)) \otimes \mathbb{F}_2[c]\). Here \(\text{Sq}_n(H^*(K))\) is the same algebra as \(H^*(K)\), but with twice larger degrees.

**Proof.** The Leray–Serre spectral sequence for \(\sigma_{K,n}\) starts with
\[
E_2^{p,q} = H^p(\mathbb{R}P^n; H^q(K \times K)),
\]
where \(\mathbb{Z}_2 = \pi_1(\mathbb{R}P^n)\) permutes the factors of \(H^q(K \times K) = H^q(K) \otimes H^q(K)\). Let us decompose the coefficient sheaf \(H^q(K \times K)\). If \(v_1, \ldots, v_m\) is the linear basis of \(H^q(K)\), then an element \(v_i \otimes v_i\) gives a subsheaf, isomorphic to the constant sheaf \(\mathbb{F}_2\). The two elements \(v_i \otimes v_j\) and \(v_j \otimes v_i\) generate a non-constant sheaf \(\mathcal{A} = \mathbb{F}_2 \boxplus \mathbb{F}_2\) with permutation action of \(\pi_1(\mathbb{R}P^n)\). The cohomology \(H^*(\mathbb{R}P^n; \mathcal{A}) = H^*(S^n; \mathbb{F}_2)\), since \(\mathcal{A}\) is the direct image of \(\mathbb{F}_2\) under the natural projection \(\pi : S^n \to \mathbb{R}P^n\). Thus we know the additive structure of \(E_2^*\).

The first column of \(E_2\) consists of \(\mathbb{Z}_2\)-invariant elements of \(H^*(K \times K)\), and all those elements are the restrictions of either \(\text{Sq}_n \xi\) or \(\xi \otimes \eta\) to the fiber. Hence all the differentials of the spectral sequence are zero on the first column. The columns between the first and the last (\(n\)-th) are generated by multiplication with \(c\), and the differentials are zero on them too. The last column is isomorphic to \(\iota(H^*(K) \otimes H^*(K))\) and the differentials are zero on it from the dimension considerations.

Hence this spectral sequence collapses, that is \(E_2 = E_\infty\). Let \(v_1, \ldots, v_m\) be a linear base of \(H^*(K)\). The first column of \(E_2\) has the linear base
\[
\{v_i \times v_1\}_{i=1}^m, \quad \{v_i \times v_j + v_j \times v_i\}_{1 \leq i < j \leq m},
\]
the columns \(E_2^{j,*}\) with \(j = 1, 2, \ldots, n-1\) have the linear base
\[
\{(v_i \times v_j)c_j\}_{i=1}^m,
\]
and the last column has the linear base
\[
\{\iota(v_i \times v_j)\}_{i,j=1}^m.
\]
From the definition of $\text{Sq}_e$, $\odot$, and $i(\ldots)$ the final cohomology $H^\ast((K \times K \times S^n)/\mathbb{Z}_2)$ is described the same way with $v_i \times v_i$ replaced by $\text{Sq}_e v_i$, and $v_i \times v_j + v_j \times v_i$ replaced by $v_i \odot v_j$. From the relations on $\text{Sq}_e$, $\odot$, and $i$ it follows that the isomorphism $E_2 \cong H^\ast((K \times K \times S^n)/\mathbb{Z}_2)$ is an isomorphism of graded algebras.

Now consider a vector bundle $\nu : E(\nu) \to K$ and define

$$\text{Sq}_e \nu : (E(\nu) \times E(\nu) \times S^n)/\mathbb{Z}_2 \to (K \times K \times S^n)/\mathbb{Z}_2.$$  

The Stiefel–Whitney classes of $\text{Sq}_e \nu$ are described by the following lemma.

**Lemma 6.7.** Let $\dim \nu = k$, and let the Stiefel–Whitney class of $\nu$ be

$$w(\nu) = w_0 + w_1 + \cdots + w_k.$$  

Then

$$w(\text{Sq}_e \nu) = \sum_{0 \leq i < j \leq k} w_i \odot w_j + \sum_{i=0}^k (1 + c)^{k-i} \text{Sq}_e w_i,$$

where $c$ is the image of the hyperplane class in $H^1(\mathbb{R}P^n)$.

**Proof.** Consider the case of one-dimensional $\nu$ first. Taking $n$ large enough we do not have to consider the image of $i(\ldots)$, and then we can return to lesser $n$ by the natural inclusion

$$(K \times K \times S^n)/\mathbb{Z}_2 \to (K \times K \times S^{n+m})/\mathbb{Z}_2.$$  

The restriction of $\text{Sq}_e \nu$ to the fiber $K \times K$ has the Stiefel–Whitney class

$$w(\nu \times \nu) = 1 + w_1(\nu) \times 1 + 1 \times w_1(\nu) + w_1(\nu) \times w_1(\nu).$$  

Hence $w(\text{Sq}_e \nu)$ is either $1 + w_1(\nu) \odot 1 + \text{Sq}_e w_1(\nu)$, or $1 + w_1(\nu) \odot 1 + 1 + \text{Sq}_e w_1(\nu)$. Any point $x \in K$ gives a natural section

$$s : S^n/\mathbb{Z}_2 \to (\{x\} \times \{x\} \times S^n)/\mathbb{Z}_2$$  

of the bundle $\sigma_{K,n}$, and the bundle $s^*(\text{Sq}_e \nu)$ over $\mathbb{R}P^n$ is isomorphic to $\gamma \oplus \varepsilon$, where $\gamma$ is the canonical bundle of the projective space, $\varepsilon$ is the trivial bundle. Hence we must have

$$w(\text{Sq}_e \nu) = 1 + w_1(\nu) \odot 1 + 1 + \text{Sq}_e w_1(\nu).$$  

The general formula for $k > 1$ follows from the splitting principle, suppose $\nu = \tau_1 \oplus \cdots \oplus \tau_k$, then

$$w(\text{Sq}_e \nu) = \prod_{i=1}^k (1 + w_1(\tau_i) \odot 1 + 1 + \text{Sq}_e w_1(\tau_i)),$$

and the result follows by removing parentheses. \hfill \Box

7. Cohomology mod 2 of the symmetric group

There are several approaches to the cohomology of the symmetric group, see the books [1] [15]. Here we apply the results of the previous section to describe the cohomology $H^\ast(\Sigma_q^\ast; \mathbb{F}_2)$. This description was obtained by the same method in [10] but we reproduce it here for completeness.

Consider the groups $\Sigma_q^\ast$, where $q$ is a power of two. They have an inductive definition as

$$\Sigma_{2q}^\ast = (\Sigma_q^\ast \times \Sigma_q^\ast) \rtimes \mathbb{Z}_2,$$

where the last factor $\mathbb{Z}_2$ acts by permuting the first two factors $\Sigma_q^\ast$. This construction is also known as the wreath product

$$\Sigma_{2q}^\ast = \Sigma_q^\ast \wr \mathbb{Z}_2.$
Hence, the mod 2 cohomology of $\Sigma^{(2)}_{2q}$ can be approximated by the Cartan–Leray spectral sequence (see \[1\]) with initial term

$$E_2^{p,q} = H^p(\mathbb{Z}_2; H^*(\Sigma^{(2)}_{q}) \otimes H^*(\Sigma^{(2)}_{q})),$$

where $\mathbb{Z}_2$ acts on $H^*(\Sigma^{(2)}_{q}) \otimes H^*(\Sigma^{(2)}_{q})$ by permuting the factors. It can be easily seen that this spectral sequence corresponds to the fiber bundle

$$B\Sigma^{(2)}_{q_1} \times B\Sigma^{(2)}_{q_2} \rightarrow B\Sigma^{(2)}_{q_1 q_2} \downarrow \mathbb{BZ}_2,$$

which is the limit case $n \rightarrow \infty$ of the external Steenrod square fiber bundle of Section 6.

Hence the cohomology $H^*(\Sigma^{(2)}_{q})$ is generated by $H^*(\Sigma^{(2)}_{q_1}) \otimes H^*(\Sigma^{(2)}_{q_2})$ and $H^*(\mathbb{Z}_2)$ with the relations of the form $x \otimes y \otimes c = 0$, where $c$ is the generator of $H^1(\mathbb{Z}_2)$.

We obtain a way to describe the cohomology of $\Sigma^{(2)}_{q}$ by applying repeatedly the external Steenrod square construction. Denote the cohomology algebras of the respective $\Sigma^{(2)}_{q}$ and $[10, \text{Proposition 2.8}])$:

$$H^*(\Sigma^{(2)}_{q}) = \left( H^*(\Sigma^{(2)}_{q_1}) \otimes H^*(\Sigma^{(2)}_{q_2}) \right) \otimes \mathbb{F}_2[c]/(x \otimes y \otimes c).$$

The following statement follows from Lemma 6.6 and gives an explicit description of certain quotient algebra of $H^*(\Sigma^{(2)}_{q})$ (compare with the definition of $\mathcal{M}(\ldots)$ in [10] and [10, Proposition 2.8]):

**Definition 7.1.** Define inductively the ideal $I_q \subset H^*(\Sigma^{(2)}_{q})$ as generated by the sets

$$\{x \otimes y : x, y \in H^*(\Sigma^{(2)}_{q/2})\}$$

and $\text{Sq}^* I_{q/2}$.

**Lemma 7.2.** The algebra $H^*(\Sigma^{(2)}_{q})/I_q$ (for $q = 2^l$) is the polynomial ring

$$H^*(\Sigma^{(2)}_{q})/I_q = \mathbb{F}_2[\text{Sq}^l c_1, \text{Sq}^{l-2} c_2, \ldots, c],$$

the subalgebra $\mathbb{F}_2[\text{Sq}^l c_1, \text{Sq}^{l-2} c_2, \ldots, c] \subset H^*(\Sigma^{(2)}_{q})$ is projected onto $H^*(\Sigma^{(2)}_{q})/I_q$ isomorphically.

If we consider some $\Sigma^{(2)}_{q}$-space $X$ then the natural equivariant map $\pi_X : X \rightarrow pt$ induces the natural map

$$\pi_X^* : H^*(B\Sigma^{(2)}_{q}) = H^*(\Sigma^{(2)}_{q}) \rightarrow H^*_X(X),$$

thus we speak informally that $\pi_X^*(H^*(\Sigma^{(2)}_{q}))$ is the image of $H^*(\Sigma^{(2)}_{q})$ in $H^*_X(X)$. In the sequel we usually consider the subquotient of the cohomology $H^*_X(X)$, defined as follows:

**Definition 7.3.**

$$\Xi_{\Sigma^{(2)}_{q}}(X) = \pi_X^*(H^*(\Sigma^{(2)}_{q}))/\pi_X^*(I_q).$$

Actually, the above reasoning also allows to describe the cohomology of $\Sigma_k$ with coefficients $\mathbb{F}_2$ for $k$ not a power of two. If we consider the binary decomposition $k = q_1 + \cdots + q_m$, then the Sylow subgroup $\Sigma^{(2)}_{k} = \Sigma^{(2)}_{q_1} \times \cdots \times \Sigma^{(2)}_{q_m}$, and the cohomology algebra is the tensor product of the respective algebras $H^*(\Sigma^{(2)}_{q_i})$, described above.
This approach can be applied similarly to the case of cohomology modulo $p$ for odd prime $p$ (compare [1, IV.1, Theorem 1.7]). Instead of⊙-product we have to use the cyclic product, defined on mock bundles over $K$ as (indexes are modulo $p$)

$$c(\xi_1, \ldots, \xi_p) = \left( \sum_{i=1}^{p} \xi_i \times \xi_{i+1} \times \cdots \times \xi_{i-1} \right) / \mathbb{Z}_p.$$ 

These cyclic products along with the ordinary external Steenrod $p$-th powers generate the cohomology of $K^{\times p} \times_{\mathbb{Z}_p} B\mathbb{Z}_p$. This is obvious at the level of spectral sequences; and it is true on the level of cohomology, since the leftmost column of the spectral sequence survives and multiplicatively generates (along with $H^*(\mathbb{Z}_p; \mathbb{F}_p)$) the entire spectral sequence. Then we note that for the $p$-adic decomposition $n = \sum_i p^{k_i}$ we have

$$\Sigma_n^{(p)} = \prod_i \Sigma_{p^{k_i}}^{(p)}$$

and

$$\Sigma_{p^{k}} = \mathbb{Z}_p \wr \cdots \wr \mathbb{Z}_p, k.$$ 

8. EQUIVARIANT COHOMOLOGY OF SPACES $\ddot{Q}_q(\mathbb{R}^n)$

The results of this section describe the cohomology $H^*(\ddot{Q}_q(\mathbb{R}^n); \mathbb{F}_2)$ in terms of external Steenrod squares, following mostly [10].

The space $\ddot{Q}_q(\mathbb{R}^n)$ is a product of $(n - 1)$-dimensional spheres and when $n \to \infty$ we obtain a homotopy trivial space with free $\Sigma_q^{(2)}$-action, i.e. a realization of $E\Sigma_q^{(2)}$. Denote $\ddot{Q}_q(\mathbb{R}^n)/\Sigma_q^{(2)} = \ddot{P}_q(\mathbb{R}^n)$ for brevity, for $q = 2$ it is the $(n - 1)$-dimensional projective space.

It can be easily seen that the inclusion $\ddot{Q}_q(\mathbb{R}^n) \to E\Sigma_q^{(2)}$ along with the Steenrod square fibration of the classifying spaces gives a fiber bundle

$$\ddot{P}_q(\mathbb{R}^n) \times \ddot{P}_q(\mathbb{R}^n) \longrightarrow \ddot{P}_{2q}(\mathbb{R}^n)$$

(8.1)

which is also a particular case of the external Steenrod square fiber bundle. Note that we have the natural cohomology map

$$\pi_{\ddot{Q}_q(\mathbb{R}^n)}: H^*(\Sigma_q^{(2)}) \to H^*(\ddot{P}_q(\mathbb{R}^n)),$$

whose image spans a “large part” of $H^*(\ddot{P}_q(\mathbb{R}^n))$, but there are also some cohomology classes generated by $\iota(\ldots)$ operation that are not in this image. Note that if we replace $\ddot{Q}_q(\mathbb{R}^n)$ by $F_q(\mathbb{R}^n)$, then we have the surjectivity for the map $H^*(\Sigma_q) \to H^*(F_q(\mathbb{R}^n)/\Sigma_q)$ using the certain cellular structure on $F_q(\mathbb{R}^n)$, see [3, 13], this fact was used in [11], but we do not use this fact in this paper. Another interesting fact (not used here) is that the natural restriction $H^*(F_q(\mathbb{R}^n)/\Sigma_q) \to H^*(\ddot{P}_q(\mathbb{R}^n))$ is injective, see [10, Theorem D].

Still we can describe the subquotient of the cohomology algebra.

**Lemma 8.1.** Let $q = 2^l$. The subquotient

$$\Xi_{\Sigma_q^{(2)}}(\ddot{Q}_q(\mathbb{R}^n)) = \Xi(\ddot{P}_q(\mathbb{R}^n))$$
is the polynomial ring \( \mathbb{F}_2[\text{Sq}_e^{l-1} c_1, \text{Sq}_e^{l-2} c_2, \ldots, c_l] \) with relations
\[
\forall i = 1, \ldots, l, \ (\text{Sq}_e^{l-i} c_i)^n = 0,
\]
where \( c_1, \ldots, c_l \) are the generators of the respective \( H^1(\mathbb{Z}_2) \) in the representation
\[
\Sigma_q^{(2)} = \mathbb{Z}_2 \wr \cdots \wr \mathbb{Z}_2.
\]

**Proof.** The cohomology \( H^*(\check{P}_q(\mathbb{R}^n)) \) is obtained from \( l \) copies of \( H^*(\mathbb{R} P^{n-1}) \) by successive external Steenrod square construction.

Let us use induction and Lemma 5.6. From the description of the cohomology of the group \( \Sigma_q^{(2)} \), the cohomology \( \Xi^*(\check{P}_q(\mathbb{R}^n)) \) is generated by \( \Xi^*(\check{P}_{q/2}(\mathbb{R}^n)) \) with \( \text{Sq}_e \) and \( \circ \) operations, \( (\ldots) \) operation is not used.

The relation on \( n \)-th powers is obvious, since in every \( H^*(\mathbb{R} P^{n-1}) \) we have \( c_i^n = 0 \). Let us prove that there are no other relations in \( \Xi^*(\check{P}_q(\mathbb{R}^n)) \). Denote
\[
h_{k_1 k_2 \ldots k_l} = (\text{Sq}_e^{l-1} c_1)^{k_1} (\text{Sq}_e^{l-2} c_2)^{k_2} \ldots (\text{Sq}_e c_l)^{k_l}
\]
and assume the contrary
\[
\sum_{0 \leq k_1, \ldots, k_l \leq n-1} c(k_1, \ldots, k_l)h_{k_1 \ldots k_l} = x,
\]
where \( x \) is an element from the ideal \( I_q \). Choose the lexicographically smallest index \((k_1, \ldots, k_l)\) with nonzero \( c(k_1, \ldots, k_l) \) and multiply by \( h_{n-1-k_1 \ldots n-1-k_l} \), from the \( n \)-th power relations we have
\[
h_{n-1 \ldots n-1} = y
\]
for some \( y \in I_q \). It may be proved by induction that all elements \( y \in I_q \) of dimension \((q-1)(n-1)\) are mapped to zero under the natural map \( \pi_{\check{P}_q(\mathbb{R}^n)} : H^*(\Sigma_q^{(2)}) \to H^*(\check{P}_q(\mathbb{R}^n)) \), informally it follows from the fact that the elements \( x \circ y \) do not have the largest possible dimension in the cohomology \( H^*((K \times K \times S^{n-1})/\mathbb{Z}_2), \) if \( K \) is a manifold. Thus we have obtained a contradiction. \( \square \)

Now consider the equivariant bundles over \( \check{Q}_q(\mathbb{R}^n) \). Denote the bundle
\[
\alpha_q(\check{Q}_q(\mathbb{R}^n)) = \check{Q}_q(\mathbb{R}^n) \times \check{A}_q
\]
simply by \( \alpha_q \), it is \( \Sigma_q^{(2)} \)-equivariant and can be also considered as a vector bundle over \( \check{P}_q(\mathbb{R}^n) \), after going to the quotient by \( \Sigma_q^{(2)} \) action.

**Lemma 8.2.** Let \( q = 2^l \). We have the inductive formula \( \alpha_{2^l} = \text{Sq}_e(\alpha_{2^{l-1}}) \oplus \gamma_l \), where \( \gamma_l \) is the pullback of the canonical bundle over \( \mathbb{R} P^{n-1} \) under the natural projection \( \check{P}_{2^l}(\mathbb{R}^n) \to \mathbb{R} P^{n-1} \). Applying it repeatedly we obtain
\[
\alpha_{2^l} = \bigoplus_{i=1}^l \text{Sq}_e^{l-i} \gamma_i,
\]
with \( \gamma_i \) being the appropriate pullback on the \( i \)-th stage of squaring.

**Proof.** The representation \( A_{2q} \) has a linear summand, consisting of vectors with the first \( q \) coordinates equal, and the last \( q \) coordinates equal. This summand is induced from the antipodal action of the quotient \( \mathbb{Z}_2 = \Sigma_2^{(2)} / (\Sigma_2^{(2)} \times \Sigma_2^{(2)}) \) on \( \mathbb{R} \).
The rest of $A_{2q}$ is the direct sum of $A_q$ for the first factor $\Sigma_q^{(2)}$ and $A_q$ for the second factor $\Sigma_q^{(2)}$, the quotient $\mathbb{Z}_2 = \Sigma_q^{(2)}/(\Sigma_q^{(2)} \times \Sigma_q^{(2)})$ acting on it by permuting the summands. This construction corresponds to the $\text{Sq}_e$ operation for vector bundles of the form $(X \times A_q)/\Sigma_q^{(2)} \to X/\Sigma_q^{(2)}$.

It follows from Lemmas 8.2 and 6.7 that in the above terms (at $i$-th stage $w(\gamma) = 1 + c_i$)
\[
e(\alpha_q) = \text{Sq}_e^{l-1} c_1 \text{Sq}_e^{l-2} c_2 \ldots c_l
\]
and
\[
e(\alpha_q)^{n-1} = \text{Sq}_e^{l-1} c_1^{n-1} \text{Sq}_e^{l-2} c_2^{n-2} \ldots c_l^{n-1}.
\]
Note that now Lemma 3.6 follows from these formulas and Lemma 6.6. We can also describe the full Stiefel–Whitney class of $\alpha_q$, at least modulo the ideal $I_q$. But according to Lemma 4.3, we have to describe the bundle $\alpha_q^+$ and give a formula for its Stiefel–Whitney class. We need a lemma first.

**Lemma 8.3.** Let $n$ be a positive integer, and let $N$ be the least power of two such that $N \geq n$. Then the operator $F_N : x \mapsto x^N$ is zero on the reduced cohomology $\tilde{H}^*(\mathbb{R}^n)$.

**Proof.** For $q = 2$ it is clear that the $N$-th power operator is zero on $\tilde{H}^*(\mathbb{R}P^n)$. Then we proceed by induction. Using the fibre bundle (8.1) we see that all the generators of $\tilde{H}^*(\mathbb{R}P^n)$ (external Steenrod squares, $\circ$-products, and $\tilde{H}^*(\mathbb{R}P^n)$) are annihilated by $F_N$. Since $F_N$ is an algebra homomorphism, then all the reduced cohomology is annihilated by $F_N$.

**Lemma 8.4.** Let $q = 2^l$, $X$ be a $\Sigma_q^{(2)}$-space, and let $N$ be the least power of two such that the map $F_N : x \mapsto x^N$ is zero on the image of $H^*(\Sigma_q^{(2)})$ in $H^*(X/\Sigma_q^{(2)})$. Then
\[
w(\alpha_q^+(X)) = w(\alpha_q(X))^{N-1},
\]
and the Stiefel–Whitney class $w(\alpha_q^+(X))$ in the subquotient $\Sigma_q^{(2)}(X)$ is expressed in terms of the generators of $H^*(\Sigma_q^{(2)})/I_q$ as follows
\[
w(\alpha_q^+(X)) = \sum_{k_1, \ldots, k_l \geq 0} c(k_1, \ldots, k_l) h_{k_1 \ldots k_l},
\]
where the coefficient $c(k_1, \ldots, k_l)$ is defined by
\[
c(k_1, \ldots, k_l) = \binom{N-1}{k_1} \prod_{j=2}^l \left( 2^{j-1}(N-1) - k_{j-1} - 2k_{j-2} - \cdots - 2^{j-2}k_1 \right),
\]
if the binomial coefficients are not defined, we assume they are zero.

**Proof.** It is enough to calculate $w(\alpha_q)^{N-1}$ for $X = B\Sigma_q^{(2)}$.

In this case the formula is obtained by applying Lemma 6.7 repeatedly, starting from the class
\[
w(\alpha_q^+) = (1 + c_1)^{N-1}.
\]

Note that in (8.2) we can substitute any $m$ instead of $N - 1$ and obtain the formula for $w(\alpha_q)^m$ over any $\Sigma_q^{(2)}$-space $X$. When applying this lemma to the case $X = \tilde{Q}_q(\mathbb{R}^n)$ we choose $N$ to be the least power of two $\geq n$ by Lemma 8.3 and impose the natural conditions $k_1, \ldots, k_l \leq n - 1$. 

\[\square\]
9. Regular embeddings of $\mathbb{R}^n$

Now we are prepared to consider regular embeddings of $\mathbb{R}^n$. First, consider one of the simplest cases $q = 4$.

**Definition 9.1.** Denote the function

$$N(x) = \min\{2^l : 2^l \geq x\}.$$

Let $\alpha_4 = \alpha_4(Q_4(\mathbb{R}^n))$. Lemma 8.4 shows that

$$\bar{w}(\alpha_4) = \sum_{0 \leq k_1, k_2 \leq n-1} c(k_1, k_2)(\text{Sq}_k c_1)^{k_1} c_2^{k_2},$$

modulo the ideal $I_q$ (generated by $c_1 \circ 1$ in this case). The coefficients are

$$(9.1) \quad c(k_1, k_2) = \binom{N - 1}{k_1} \binom{2(N - 1) - k_1}{k_2} = \binom{2N - 1 - k_1 - 1}{k_2},$$

where $N = N(n)$.

It is well-known that the binomial coefficients $\binom{x+y}{y}$ are nonzero iff in the binary representation of $x$ and $y$ none of the positions is taken by 1 in both $x$ and $y$. Call such two numbers binary disjoint and write $x \& y = 0$. Since $2N - 1$ is a large enough string of 1’s in the binary representation, then $c(k_1, k_2) \neq 0$ iff $(k_1 + 1) \& k_2 = 0$. Thus we have

$$(9.2) \quad \bar{\ell}(\alpha_4) = \max\{2k_1 + k_2 : 0 \leq k_1, k_2 \leq n - 1, (k_1 + 1) \& k_2 = 0\}.$$  

**Definition 9.2.** Define

$$\nu(x) = \max\{y \in \mathbb{Z}^+ : y \leq x, x \& y = 0\},$$

note that for any positive integer $x$

$$x + \nu(x) = N(x + 1) - 1.$$

**Theorem 9.3.** In the cohomology $H^*(P_4(\mathbb{R}^n))$ modulo $I_q$ we have

$$\bar{\ell}(\alpha_4(Q_4(\mathbb{R}^n))) = 2n - 2 + \nu(n) = n + N(n + 1) - 3.$$

**Proof.** Let us analyze (9.2). If $k_1 + 1 \geq k_2$, then we can assume $k_2 = \nu(k_1 + 1)$, then $\bar{\ell}(\alpha_4) \geq 2k_1 + \nu(k_1 + 1)$, and the maximum is attained for $k_1 = n - 1$.

If $k_1 + 1 < k_2$, then we can assume $k_1 = \nu(k_2) - 1$, in this case we have an estimate $\bar{\ell}(\alpha_4) \geq 2\nu(k_2) - 2 + k_2 = \nu(k_2) + N(k_2 + 1) - 3$, the maximum is $\nu(n - 1) + N(n) - 3$, which is less than the previous estimate. $\square$

Now we apply Lemma 8.3 and deduce the following.

**Corollary 9.4.** Let $k$ be divisible by 4. If there exists a linearly $k$-regular map $f : \mathbb{R}^n \to \mathbb{R}^m$, then

$$m \geq k + \frac{k}{4}(n + N(n + 1) - 3).$$

The “greedy” lower bound in Theorem 9.3 using Lemma 8.4 and (8.3) can be reproduced for any $q = 2^l$. Let us state the appropriate result. There is no explicit formula in this theorem, but it can be easily computed in any particular case.
Theorem 9.5. In the cohomology $H^*(P_q(R^n))$ we have
\[ \ell \left( \alpha_q(Q_q(R^n)) \right) \geq \sum_{i=1}^{l} 2^{l-i}k_i, \]
where $k_i$ are defined recursively as follows:
\[ k_1 = n - 1, \]
and for $i \geq 2$
\[ k_i = \max\{x \in \mathbb{Z}^+: x \leq n - 1, \ x \& (k_{i-1} + 1 + 2(k_{i-2} + 1) + \cdots + 2^{i-2}(k_1 + 1)) = 0\}, \]
where $\&$ denote the bitwise ‘and’ operation.

It is not known whether this bound is the best possible that can be obtained from (8.2). In case $n$ is a power of two this theorem gives $k_i = n - 1$, i.e. the Chisholm theorem. Applying Lemma 4.3 we obtain the following generalization of the Chisholm theorem.

Corollary 9.6. Denote the lower bound in Theorem 9.5 by $l(q,n)$. Suppose $k = q_1 + \ldots + q_s$ is a partition of $k$ into powers of two (e.g. the binary representation). If there exists a linearly $k$-regular map $f: R^n \to R^m$, then
\[ m \geq k + \sum_{i=1}^{s} l(q_i, n). \]

10. Cohomology of bundles $P_q(\xi)$

Now consider the bundle $Q_q(\xi) \to M$ associated with some vector bundle $\xi: E(\xi) \to M$. Put $P_q(\xi) = Q_q(\xi)/\Sigma_q^{(2)}$. We have the following statement about the equivariant cohomology of $P_q(\xi)$.

Lemma 10.1. Let $q = 2^l$. Suppose $\xi \to M$ is an $n$-dimensional vector bundle. The quotient $H^*(P_q(\xi))/(I_qH^*(M))$ has a free $H^*(M)$-submodule, generated by the classes $h_{k_1, \ldots, k_l}$ with $0 \leq k_1, \ldots, k_l \leq n - 1$ from $H^*(\Sigma_q^{(2)})$.

Proof. Compare the proof with the proof of Lemma 8.1. Suppose we have a nontrivial relation
\[ \sum_{0 \leq k_1, \ldots, k_l \leq n-1} m(k_1, \ldots, k_l) h_{k_1, \ldots, k_l} = xm, \]
where $m(k_1, \ldots, k_l), m \in H^*(M)$ and $x \in I_q$. Note that
\[ \pi: P_q(\xi) \to M \]
is a bundle of manifolds, and the cohomology map $\pi_\ast: H^*(P_q(\xi)) \to H^*(M)$ of degree $-(q-1)(n-1)$ is defined. Applying this map to (10.1) we obtain
\[ m(n-1, \ldots, n-1) = 0. \]
Now consider the lexicographically largest index $(k_1, \ldots, k_l)$ with nonzero $m(k_1, \ldots, k_l)$, multiply (10.1) by $h_{n-1-k_1, \ldots, n-1-k_l}$, and then apply $\pi_\ast$. Using Lemma 8.1 we again obtain $m(k_1, \ldots, k_l) = 0$.

Now consider the $(\Sigma_q^{(2)}$-equivariant) dual Stiefel–Whitney class of the bundle $\alpha_q(Q_q(\xi))$ in $H^*(P_q(\xi))$, actually we consider it in $\Xi^*(P_q(\xi))$. From naturality of this class it is
Lemma 10.2. Equation (10.2) defines the characteristic classes $t_{k_1 \ldots k_l}(\xi)$ of a vector bundle $\xi$, with the following property:

$$\tilde{e}(\alpha_q(Q_q(\xi))) \geq \max_{0 \leq k_1, \ldots, k_l \leq n-1} \{ \dim h_{k_1 \ldots k_l} + \dim t_{k_1 \ldots k_l}(\xi) \},$$

where by the dimension of a cohomology class $t_{k_1 \ldots k_l}(\xi)$ we mean the maximum dimension of a nonzero homogeneous component of $t_{k_1 \ldots k_l}(\xi)$.

The computation may be simpler for the following subset of these characteristic classes:

Definition 10.3. Define the characteristic classes

$$T_q(\xi) = t_n^{n-1 \ldots n-1}(\xi).$$

We have

$$T_q(\xi) = \pi_!(\tilde{w}(\alpha_q(Q_q(\xi)))),$$

since $h_{n-1 \ldots n-1}$ is the fundamental class of the fiber manifold $\tilde{P}_q(\mathbb{R}^n)$.

By Lemma 10.2 for the class $T_q(\xi)$ we have

$$\tilde{e}(\alpha_q(Q_q(\xi))) \geq (q - 1)(n - 1) + \dim T_q(\xi),$$

noting that if $T_q(\xi)$ is zero we put $\dim T_q(\xi) = -\infty$.

In the case $q = 2$ the class $T_2(\xi)$ is the dual Stiefel–Whitney class of $M$, as it was already used in the proof of Theorem 11.3. The calculations are harder even in the case $q = 4$, compare the calculation of coincident $q$-tuple characteristic classes in [11]. Note also that $T_q(\xi)$ (unlike the classes in [11]) is not stable under summation with a trivial bundle and depends on the dimension of $\xi$.

Now consider a way to calculate the above characteristic classes. Take a power of two $N$ such that the map $x \mapsto x^N$ is zero on $H^*(P_q(\xi))$. In this case the dual Stiefel–Whitney class of $\alpha_q(Q_q(\xi))$ modulo the ideal $I_q$ is given by $[S_2]$, because we have

$$w(\alpha_q(Q_q(\xi)))^N = 1.$$ 

In order to have the decomposition (10.2) we have to express the monomials ($q = 2^l$)

$$(S_{q_1} \delta c_{1})^{k_1} \ldots (S_{q_1} \delta c_{l})^{k_{l-1}} (\gamma_i)^{k_l}$$

with some $k_i \geq n = \dim \xi$ in terms of the similar monomials with all $k_i \leq n - 1$. This expression is done modulo $I_q H^*(M)$. The following lemma gives the needed relations. The bundles $S_{q_1} \delta \gamma_i$ were defined in Lemma 8.2 over the space $\tilde{Q}_q(\mathbb{R}^n)$, but actually they arise from the corresponding representation of $\Sigma^{(2)}_q$, and therefore they are defined over any $\Sigma^{(2)}_q$-space.

Lemma 10.4. Let $q = 2^l$, and $\pi : Q_q(\xi) \to M$ be the natural projection. Then we have the relations for $i = 1, \ldots, l$

$$e(S_{q_1} \delta \gamma_i \otimes \pi^*(\xi)) = 0$$

in the cohomology $H^*(P_q(\xi))$. 

Proof. Consider the natural map \( s : Q_q(\xi) \to \xi^q \), it can be considered as an equivariant section of the vector bundle \( \pi^*(\xi)^q \) over \( Q_q(\xi) \). We have
\[
s(p_1, \ldots, p_q) = p_1 \oplus \cdots \oplus p_q,
\]
and it is readily seen from the definition of \( Q_q(\mathbb{R}^n) \), that the sum of \( p_i \) is zero. Hence we have a section of \( \alpha_q(Q_q(\xi)) \otimes \pi^*(\xi) \). By Lemma \([8.2]\) (which is true for the representations) we have
\[
\alpha_q(Q_q(\xi)) = \bigoplus_{i=1}^l \text{Sq}_e^{l-i} \gamma_i,
\]
and therefore
\[
\alpha_q(Q_q(\xi)) \otimes \pi^*(\xi) = \bigoplus_{i=1}^l \text{Sq}_e^{l-i} \gamma_i \otimes \pi^*(\xi).
\]
It is easily seen that \( s \) gives a nonzero section for all the summands, after the corresponding projection. Thus their equivariant Euler classes are zero. \( \square \)

Note that this lemma expresses \((\text{Sq}_e^{l-i} c_i)^n \) (modulo \( I_q H^*(M) \)) through the combinations of \((\text{Sq}_e^{l-i} c_i)^{k_i} \) with \( k_i < n \) and \((\text{Sq}_e^{l-j} c_j)^{k_j} \) with \( j > i \) and \( k_j \) not necessarily < \( n \).

Combining \([8.4]\) and the above lemma, we obtain a way to calculate \( \bar{\ell}(\alpha_q(Q_q(\xi))) \) (modulo \( I_q H^*(M) \)) in every particular case.

11. SOME EXPLICIT BOUNDS FOR REGULAR EMBEDDINGS OF MANIFOLDS

Let us give more explicit examples of lower bounds for regular embeddings of manifolds in some particular cases.

Consider a vector bundle \( \xi : E(\xi) \to M \) and its spaces \( Q_q(\xi) \) and \( P_q(\xi) \). We need a claim about the nilpotence degree of the classes \( \text{Sq}_e^{l-i} c_i \) in \( \Xi^*(P_q(\xi)) \). The first lemma is a general statement, the second is its application to the cohomology of \( P_q(\xi) \).

Lemma 11.1. Let \( \xi : E(\xi) \to X \) and \( \eta : E(\eta) \to X \) be two vector bundles over a topological space \( X \). If \( e(\xi \otimes \eta) = 0 \), then
\[
e(\eta)^{\dim \xi + \bar{\ell}(\xi)} = 0.
\]

Lemma 11.2. Let \( q = 2^l \), and let \( \xi : E(\xi) \to M \) be a vector bundle over \( M \). Then we have the relations for \( i = 1, \ldots, l \)
\[
(Sq_e^{l-i} c_i)^{\dim \xi + \bar{\ell}(\xi)} = 0
\]
in the cohomology \( H^*(P_q(\xi)) \).

Lemma \([11.2]\) follows from Lemma \([10.4]\) and Lemma \([11.1]\). So we have to prove Lemma \([11.1]\)

Proof of Lemma \([11.1]\). By the splitting principle we can assume that the bundle \( \eta \) is a sum of line bundles
\[
\eta = \eta_1 \oplus \cdots \oplus \eta_m.
\]
Denote the characteristic classes
\[
e(\eta_i) = y_i, \quad w(\xi) = 1 + w_1 + \cdots + w_n, \quad \bar{w}(\xi) = 1 + \bar{w}_1 + \cdots + \bar{w}_k.
\]
We have the equation
\[
e(\xi \otimes \eta) = \prod_{i=1}^m (y_i^n + w_1 y_i^{n-1} + \cdots + w_n) = 0.
\]
By Theorem 11.4, if \( k \) Section 13) show that

\[
\text{Proof.}
\]

The result follows from Lemmas 11.3 and 4.2.

Multiplying by

\[
\prod_{i=1}^{m} (y_i^k + \bar{w}_1 y_i^{k-1} + \cdots + \bar{w}_k)
\]

we obtain

\[
e(\eta)^{n+k} = \prod_{i=1}^{m} y_i^{n+k} = 0.
\]

Now we can calculate \( \bar{\ell}(\alpha_q(Q_q(\xi))) \) in a particular case (compare the Chisholm theorem).

**Lemma 11.3.** Suppose \( \xi : E(\xi) \to M \) is a vector bundle over \( M \). Let \( q = 2^l \), \( \dim \xi = n \), \( \bar{\ell}(\eta) = d \). Suppose that either \( n + d \) is a power of two and \( \bar{w}_d(\xi)^{q-1} \neq 0 \), or \( q = 2 \). Then

\[
\bar{\ell}(\alpha_q(Q_q(\xi))) = (q - 1)(n + d - 1).
\]

**Proof.** Denote \( \alpha_q = \alpha_q(Q_q(\xi)) \). The class \( \bar{w}(\alpha_q) \) modulo the ideal \( I_q \) is given by (8.4), we can take \( N = n + d \) in this equation since the map \( x \mapsto x^{n+d} \) sends the Stiefel–Whitney classes \( w_1(\alpha_q), \ldots, w_{q-1}(\alpha_q) \) to zero (modulo \( I_q \)). Hence we have to prove that the class (the leading term of (8.4))

\[
(Sq^{l-1} c_1 \ldots Sq^{1} c_{l-1} c_l)^{n+d-1}
\]

is not zero in \( H^*(P_q(\xi)) \). From [11, Theorem 1] it follows that under the assumption \( \bar{\ell}(\eta) = d \) we have a relation

\[
(Sq^{l-1} c_1 \ldots Sq^{1} c_{l-1} c_l)^{n+d-1} = (Sq^{l-1} c_1 \ldots Sq^{1} c_{l-1} c_l)^{n-1} \bar{w}_d(\xi)^{q-1}
\]

and the result follows.

Now an estimate for the dimension of regular embeddings follows from Lemma 11.3.

**Theorem 11.4.** Let \( M \) be an \( n \)-dimensional manifold. Let \( \bar{\ell}(\tau M) = d \), and \( n + d \) be a power of two. Suppose that \( k = q_1 + \ldots + q_l \) is a sum of powers of two and \( \bar{w}_d(\xi)^{q_i-1} \neq 0 \) for any \( i \).

Under the above assumptions, if there exists a linearly \( k \)-regular embedding \( M \to \mathbb{R}^m \) then

\[
m \geq k - l + 1 + \sum_{i=1}^{l} (q_i - 1)(n + d - 1) = (k - l)(n + d) + 1.
\]

**Proof.** The result follows from Lemmas 11.3 and 11.2.

To give an explicit application of Theorem 11.4 consider \( M = \mathbb{R} P^{n-1} \times S^1 \), \( n = 2^p - d \), \( p \) and \( d \) some positive integers such that \( 2^p - 1 - d > 0 \). The direct calculations (see also Section 13) show that

\[
\bar{\ell}(\tau M) = d, \text{ if } (q - 1)d \leq n - 1, \text{ then } \bar{w}_d(\tau M)^{q-1} \neq 0.
\]

By Theorem 11.4 if \( k = q_1 + \cdots + q_l \) is a sum of powers of two, and any \( q_i \leq \frac{n - 1}{d} + 1 \), then the dimension of linearly \( k \)-regular embedding of \( M \) is at least \( m \geq (k - l)(n + d) + 1 \). If the number \( k \) is itself at most \( \frac{n - 1}{d} + 1 \), then we can take its binary representation, in this case \( m \geq (k - \alpha_2(k))(n + d) + 1 \).
12. Regular embeddings and the tangent bundle of \( P_q(M) \)

Let us describe another approach to lower bounds for the dimension of \( k \)-regular embedding, not using Lemma 11.2. The method of Boltyanskii–Ryshkov–Shashkin actually shows that any affinely (linearly) \( 2q \)-regular embedding \( M \to \mathbb{R}^m \) gives a continuous injective map

\[
F_q(M) \times D^{q-1} \to \mathbb{R}^m
\]
in the affine case, or

\[
F_q(M) \times D^{q-1} \to S^{m-1}
\]
in the linear case. Here \( D^l \) is an \( l \)-dimensional open disc, \( q \) is not necessarily a power of two. Then the dimension considerations give either \( m \geq n(q + 1) - 1 \) or \( m \geq n(q + 1) \) respectively.

This reasoning can be improved in some cases. Consider linear embeddings and let \( q \) be a power of two. The above map is restricted to an injective map

\[
F_q(M)/\Sigma_q \to S^{m-1}.
\]

The space \( F_q(M) \) contains a submanifold \( Q_q(M) \), and we obtain an injective continuous map

\[
P_q(M) = Q_q(M)/\Sigma_q^{(2)} \to S^{m-1}.
\]

According to [16, 7], the existence of such a map implies the inequality (\( \tau \) denotes the tangent bundle of a manifold)

\[
m - 1 \geq \dim P_q(M) + \bar{\ell}(\tau P_q(M)),
\]
or (for compact \( M \))

\[
m - 1 \geq \dim P_q(M) + \bar{\ell}(\tau P_q(M)) + 1.
\]

We obviously have to describe the tangent bundle of \( Q_q(M) \) and the action of \( \Sigma_q^{(2)} \) on it. Consider the case of \( \hat{Q}_q(\mathbb{R}^n) \) first. Denote \( \mathbb{R}^n = L \) for brevity. One of the descriptions of \( \hat{Q}_q(L) \) identifies it with the product of \( q - 1 \) spheres of \( L \), hence we have an embedding

\[
\hat{Q}_q(L) \subset L^{q-1}.
\]

The tangent vector of \( \hat{Q}_q(L) \) at a point \((p_1, \ldots, p_{q-1}) \in L^{q-1}\) is a vector \((v_1, \ldots, v_{q-1}) \in L^{q-1}\), such that \( p_i \) and the respective \( v_i \) are orthogonal for any \( i \). The action of the generators (block permutations) of \( \Sigma_q^{(2)} \) is given by reversing one \( p_i \) and \( v_i \), and permuting some other \( p_j \)'s and \( v_j \)'s, according to the binary tree structure. Consider also the \( q - 1 \)-dimensional bundle \( \eta \) over \( \hat{Q}_q(L) \) such that the fiber of \( \eta \) over \((p_1, \ldots, p_{q-1}) \) is the set of \( q - 1 \)-tuples \((u_1, \ldots, u_{q-1})\), such that any \( u_i \) is parallel to the respective \( p_i \). Let \( \Sigma_q^{(2)} \) act on \((u_1, \ldots, u_{q-1})\) in the same way, as on \((p_1, \ldots, p_{q-1})\). The numbers \( u_i/p_i \) give an \( \Sigma_q^{(2)} \)-invariant identification with the trivial bundle

\[
\eta = \varepsilon^{q-1},
\]

and from the obvious identification \( L^{q-1} = A_q \otimes L \) we have

\[
\tau\hat{Q}_q(L) \oplus \varepsilon^{q-1} = \alpha_q(\hat{Q}_q(L)) \otimes L.
\]

For an arbitrary manifold \( M \) we similarly obtain \((\pi : Q_q(M) \to M \) is the natural projection\)

\[
\tau Q_q(M) \oplus \varepsilon^{q-1} = \nu_q(Q_q(M)) \otimes \pi^*(\tau M),
\]
since the fiberwise tangent bundle is an \(\alpha\)-bundle is \(\pi\)-bundle is \(\pi^*(\tau M)\) — the fiberwise orthogonal bundle is \(\pi^*(\tau M)\), and \(\nu_q = \alpha \oplus \varepsilon\) by definition. Thus we have proved the following.

**Theorem 12.1.** Let \(k\) be a power of two, \(M\) be an \(n\)-dimensional manifold. If there exists a linearly \(k\)-regular map \(f : M \to \mathbb{R}^m\), then
\[
m \geq (n - 1)(k/2 - 1) + \ell (\nu_{k/2}(Q_{k/2}(M)) \otimes \pi^*(\tau M)) + 1,
\]
or (for compact \(M\))
\[
m \geq (n - 1)(k/2 - 1) + \ell (\nu_{k/2}(Q_{k/2}(M)) \otimes \pi^*(\tau M)) + 2.
\]
In the case \(M = \mathbb{R}^n\) this theorem gives a worse estimate, compared to the Chisholm theorem, but for other manifolds this bound can be useful.

### 13. Multiplicity of maps from projective spaces to Euclidean spaces

In \([11]\) it was shown that continuous maps \(f : \mathbb{R}^m \to \mathbb{R}^n\) must have coincident \(q\)-tuples under certain restrictions on \(q, m, n\). This was proved without any computation in the cohomology of the symmetric group by some geometric reasoning. Using the above description of the cohomology of the space \(Q_q(\mathbb{R}^n)\) modulo the ideal \(I_q\), it is possible to generalize the result.

**Theorem 13.1.** Let \(q = 2^l\), \(n \geq m\) be positive integers. Put \(d = n - m, p = N(m + 1) - m - 1\). Suppose for certain \(0 \leq k_1, \ldots, k_l \leq p - d - 1\) such that
\[
(2^l - 1)p - k_1 - 2k_2 - \cdots - 2^{l-1}k_l \leq m
\]
the coefficient
\[
(13.1) \quad c(k_1, \ldots, k_l) = \binom{p}{k_1} \cdot \prod_{j=2}^{l} \binom{2^j - 1 - p - k_{j-1} - 2k_{j-2} - \cdots - 2^{j-2}k_1}{k_j}
\]
is odd. Then any continuous map \(f : \mathbb{R}^m \to \mathbb{R}^n\) has a coincident \(q\)-tuple from \(Q_q(\mathbb{R}^m)\).

The result of \([11]\) follows from this theorem by putting \(k_1 = k_2 = \cdots = k_l = 0\).

**Proof.** Put \(M = \mathbb{R}^m\) for brevity, and let \(\tau M\) have dimension \(p'\). It was shown in \([11]\) that a coincident \(q\)-tuple of \(f : M \to \mathbb{R}^n\) from \(Q_q(M)\) is guaranteed by the Euler class of the vector bundle \(\alpha_q \otimes (\varepsilon^n \oplus \tau M)\) over \(\mathbb{R}^m(\mathbb{R}^{m+p'})\) \(\times M\). It is well-known that the Stiefel–Whitney class of \(\tau M\) is
\[
w(\tau M) = (1 + c)^p,
\]
where \(c\) in the generator of \(H^1(M)\) and \(p = N(m + 1) - m - 1 \leq p'\). Then by Lemma \([8.4]\) (note the remark after it) we have the equation modulo \(I_qH^*(M)\)
\[
e(\alpha_q \otimes (\tau M \otimes \varepsilon^n)) = (\text{Sq}_e^{l-1} c_1 \cdots c_l)^{n+p'} - p \cdot \sum_{k_1, \ldots, k_l \geq 0} c(k_1, \ldots, k_l)(\text{Sq}_e^{l-1} c_1)^{k_1} \cdots (c_l)^{k_l} \otimes \varepsilon^{p(2^l - 1) - k_1 - 2k_2 - \cdots - 2^{l-1}k_l},
\]
where the coefficients \(c(k_1, \ldots, k_l)\) are as in \([13.1]\).

Now we note that in the cohomology \(H^*(\mathbb{R}^m(\mathbb{R}^{m+p'}) \times M)\) we have relations
\[
\forall i = 1, \ldots, l \quad (\text{Sq}_e^{l-1} c_i)^{m+p'} = 0, \quad c^{m+1} = 0,
\]
that imply the inequalities \(k_1, \ldots, k_l \leq p - d - 1\) and
\[
(2^l - 1)p - k_1 - 2k_2 - \cdots - 2^{l-1}k_l \leq m
\]
respectively. Thus the result follows. \(\square\)
References

[1] A. Adem, R.J. Milgram. Cohomology of finite groups. 2nd ed. Springer Verlag, Berlin-Heidelberg, 2004.
[2] V.G. Boltvanski, S.S. Ryzhkov, Yu.A. Shashkin. On $k$-regular imbeddings and on applications to theory of function approximation (In Russian). // Uspehi Mat. Nauk, 15:6 (1960), 125–132.
[3] K. Borsuk. On the $k$-independent subsets of the Euclidean space and of the Hilbert space. // Bull. Acad. Polon. Sci. Cl. III, 5 (1957), 351–356.
[4] S. Buoncristiano, C.P. Rourke, B.J. Sanderson. A geometric approach to homotopy theory. Cambridge University Press, 1976.
[5] M.E. Chisholm. $k$-regular mappings of $2^n$-dimensional Euclidean space. // Proc. Amer. Math. Soc, 74:1 (1979), 187–190.
[6] F.R. Cohen, D. Handel. $k$-regular embeddings of the plane. // Proc. Amer. Math. Soc., 72:1 (1978), 201–204.
[7] P.E. Conner, E.E. Floyd. Fixed point free involutions and equivariant maps. // Bull. Amer. Math. Soc., 66:6 (1960), 416–441.
[8] D.B. Fuks. Cohomologies of the braid group mod 2 (In Russian). // Functional Analysis and Its Applications, 4:2 (1970), 62–73.
[9] D. Handel. $2k$-regular maps on smooth manifolds. // Proc. Amer. Math. Soc., 124:5 (1996), 1609–1613.
[10] Nguyễn H.V. Hung. The mod 2 equivariant cohomology algebras of configuration spaces. // Pacific Jour. Math., 143:2 (1990), 251–286.
[11] R.N. Karasev. Multiplicity of continuous maps between manifolds. // arXiv:1002.0660, 2010.
[12] C. McCrory. Geometric homology operations. // Studies in Algebraic Topology, Advances in Math. Suppl. Studies, 5 (1978), 119–141.
[13] N.E. Steenrod, D.B. Epstein. Cohomology operations. Princeton University Press, 1962.
[14] V.A. Vasil’ev. On function spaces that are interpolating at any $k$ nodes. // Functional Analysis and Its Applications, 26:3 (1992), 209–210.
[15] V.A. Vasil’ev. Complements of discriminants of smooth maps: topology and applications. Translations of Math. Monographs, 98, AMS, Providence, RI, 1994; cited by the extended Russian edition, 1997.
[16] W.-T. Wu. On the realization of complexes in Euclidean space, II. // Scientia Sinica, 7 (1958), 365–387.

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