HILBERT-KUNZ DENSITY FUNCTIONS AND $F$-THRESHOLDS

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Abstract. We had shown earlier that for a ring $R$ and an ideal $I$ in char $p > 0$, given in a graded setup, with $ℓ(R/I) < ∞$, there exists a compactly supported continuous function $f_{R,I}$ with the integral equal to the HK multiplicity $e_{HK}(R,I)$. We explore further some other invariants, namely the shape of $f_{R,m}$ and the maximum support (denoted as $α(R,I)$) of $f_{R,I}$.

In case $R$ is a domain of dimension $d ≥ 2$, we prove that $(R,m)$ is a regular ring if and only if $f_{R,m}$ has a symmetry $f_{R,m}(x) = f_{R,m}(d-x)$, for all $x$.

If $R$ is $F$-regular on the punctured spectrum then we prove that the $F$-threshold $c'(m) = α(R,I)$.

As a consequence, if $R$ is a two dimensional domain then we have a formula for the $F$-threshold $c'(m)$ in terms of the strong Harder-Narasimhan slopes of a strong $μ$-reduction bundle for $(R,I)$. We also formulate, the $F$-threshold in char 0, in terms of the Harder-Narasimhan slopes of a $μ$-reduction bundle for $(R,I)$.

This characterisation readily computes $c'^n(m)$, for the set of all irreducible plane trinomials $k[x,y,z]/(h)$, where $m = (x,y,z)$ and $I_n = (x^n, y^n, z^n)$.

Let $R$ be a standard graded ring of dimension $d ≥ 2$ over a perfect field of characteristic $p > 0$, and let $I ⊂ R$ be a graded ideal such that $ℓ(R/I) < ∞$. Let $m$ be the graded maximal ideal of $R$. Let $M$ be a finitely generated graded $R$-module. Then for given such pair $(M,I)$, we have introduced the HK density function $f_{M,I} : ℝ → [0,∞)$, a compactly supported continuous function (see [T3]). We realize this function as the limit of a uniformly convergent sequence of compactly supported functions $\{f_n(M,I) : ℝ → [0,∞)\}_{n∈ℕ}$, where

$$f_n(M,I)(x) = \frac{1}{q^{d-1}}\ell(M/I^nM)_{[xq]}, \text{ for } q = p^n.$$  

We know $\int_0^∞ f_{M,I}(x)dx = e_{HK}(M,I)$, where $e_{HK}(M,I)$ is the HK multiplicity of $M$ with respect to $I$.

This interpretation, of $e_{HK}(M,I)$ as an integral of $f_{M,I}$ and the uniform convergence property of the sequence $f_n(M,I)$ to $f_{M,I}$, gave a better tool to study the asymptotic growth of $e_{HK}(M,m^k)$, as $k → ∞$ (Theorem 3.6 in [T4]) and also gave an approach to $e_{HK}$ in char 0 (Theorem 1.1 in [T6]) (so that one needs to check the existence of a single limit rather than the existence of the double limit).

Moreover, the HK density function is additive and multiplicative. Hence (1) we can replace a module by an integral domain, and (2) the HK density functions of the rings gives the HK density function of their Segre product.

In this paper we explore other invariants associated to this function for a standard graded pair $(R,I)$ (i.e., $R$ is a standard graded ring and $I$ is graded ideal of finite colength), more specifically the maximum support of $f_{R,I}$, that is

$$α(R,I) = \text{Sup } \{x | f_{R,I}(x) ≠ 0\}.$$  

First we prove that the shape of the HK density function (and $α(R,I)$) determines the regularity of $(R,m)$:
**Theorem A.** If \((R, \mathfrak{m})\) is a standard graded domain of dimension \(d \geq 2\) then

1. \(f_{R, \mathfrak{m}}(x) = f_{R, \mathfrak{m}}(d - x), \) for all \(x\) if and only if the ring \((R, \mathfrak{m})\) is regular.
2. In fact \(\alpha(R, \mathfrak{m}) = d\) if and only if \((R, \mathfrak{m})\) is a regular ring.
3. If \(\dim \mathbb{R} = 2\) then either
   - \(f_{R, \mathfrak{m}}(1 - y) = f_{R, \mathfrak{m}}(1 + y),\) for all \(y \in (0, 1),\) or
   - \(f_{R, \mathfrak{m}}(1 - y) > f_{R, \mathfrak{m}}(1 + y),\) for all \(y \in (0, 1).\)

Next we relate the invariant \(\alpha(R, I)\) to \(c'(\mathfrak{m})\), the \(F\)-threshold of \(\mathfrak{m}\) with respect to the ideal \(I\).

The \(F\)-thresholds were introduced and studied in [MTW], in the case of regular rings. In more general setting (when \(R\) is not regular) it was further studied in [HMTW]. In [MTW] the following (see Question 1.4) was posed:

**Question.** Is it true that for all nonzero ideals \(J \) and \(I \) with \(J \subseteq \text{Rad}(I) \subseteq \mathfrak{m}\), the \(F\)-threshold \(c'(J)\) is a rational number?

One gets a positive answer from a series of papers ([KLZ], [BMS1], [BMS2]) over regular rings, and in the following way: the \(F\)-thresholds of \(I\) are also \(F\)-jumping numbers of \(I\), and all \(F\)-jumping numbers are rational.

Moreover, by Proposition 4.17 of [HMNb], if \(R\) is a direct summand of a regular \(F\)-finite domain \(S\), then \(c'(J)\) is a rational number. Here \(c'(J)\) was identified with \(c_{JS}^{J^S}(JS)\) and hence is an \(F\)-jumping number of \(JS\). We recall that in case \((R, \mathfrak{m})\) is regular, either local or standard graded, and \(J \subset R\) an ideal, then \(c_{\mathfrak{m}}(J)\) is the first jumping number \(\text{fpt}(J)\) (the \(F\)-pure threshold). However, in singular cases, \(F\)-thresholds may differ from \(F\)-jumping numbers, for example (1) when \(R\) is the coordinate ring of the Segre product \(\mathbb{P}^m \times \mathbb{P}^n\), where \(m \neq n\), we have \(\text{fpt}(\mathfrak{m}) < c_{\mathfrak{m}}(\mathfrak{m})\) (see [CM] and [HWY]), (2) when \(R = k[x, y, z]/(xy - z^2)\), we have \(\text{fpt}(\mathfrak{m}) = 1 < c_{\mathfrak{m}}(\mathfrak{m}) = 3/2\) (see [TW] and [HMTW]).

In fact for a standard graded pair \((R, \mathfrak{m})\), which is Gorenstein, \(F\)-finite and \(F\)-pure, the equality \(\text{fpt}(\mathfrak{m}) = c_{\mathfrak{m}}(\mathfrak{m})\) implies \(R\) is strongly \(F\)-regular (see Theorem 6.13 of [DsNbP]).

In general, to the best of our knowledge, it is not known whether \(c_{\mathfrak{m}}(\mathfrak{m})\) is rational, even in graded cases.

As one of the consequences of Theorem 4.12 we prove that the \(F\)-thresholds \(c'(\mathfrak{m})\) are rational numbers for standard graded pairs \((R, I)\) of dimension two.

Next we look at the reduction mod \(p\) behaviour of these \(F\)-thresholds. We recall that Theorem 3.4 and Proposition 3.8 of [HY] describe the behaviour of \(c_{\mathfrak{m}}(I)\) under reduction mod \(p\) (in their notation \(c_{\mathfrak{m}}(I) = \text{fpt}_{\mathfrak{m}}(I)\)) as follows: If \(R = A[X_1, \ldots, X_d]\) is a polynomial ring over \(A\), where \(A\) is a localization of \(\mathbb{Z}\) at some nonzero integer and \(I \subset \mathfrak{m} = (X_1, \ldots, X_d)\) is an ideal then

1. \(\lim_{p \to \infty} c_{\mathfrak{m}}^p(I_p)\) exists and \(\lim_{p \to \infty} c_{\mathfrak{m}}^p(I_p) = \text{lct}_{\mathfrak{m}}(I)\), where \(\text{lct}_{\mathfrak{m}}(I)\) is the log canonical threshold of \(I_{\mathfrak{m}}\) at \(\mathfrak{m}\).
2. Moreover for \(p_s \gg 0\), we have \(\text{lct}_{\mathfrak{m}}(I) = \lim_{p \to \infty} c_{\mathfrak{m}}^p(I_s) \geq c_{\mathfrak{m}}(I_s)\).

In this paper our approach, to study \(c'(\mathfrak{m})\), is via the HK density function \(f_{R, I}\). We relate the two as follows.

**Theorem B.** Let \((R, I)\) be a standard graded pair and \(\mathfrak{m}\) be the graded maximal ideal of \(R\). Then

1. \(\alpha(R, I) \leq c'(\mathfrak{m}),\) for dimension \(R = d \geq 2\). Moreover
2. the equality \(\alpha(R, I) = c'(\mathfrak{m})\) holds if either
(a) \( I \) is generated by a system of parameters, or
(b) \( R \) is \( F \)-regular on the punctured spectrum \( \text{Spec} R \setminus \{ \mathfrak{m} \} \).

In particular, the same equality holds for the Segre products of all such rings and also for any standard graded pair \( (R, I) \), where \( R \) is a two dimensional domain.

As a consequence of identifying \( \epsilon^I(\mathfrak{m}) \) with \( \alpha(R, I) \), in the case of two dimensional domains, we are able to study the invariant \( \epsilon^I(\mathfrak{m}) \) from a different perspective. We elaborate this as follows.

For a given pair \( (R, I) \), and \( X = \text{Proj} \, S \), where \( S \) is the normalization of \( R \) in the quotient field \( Q(R) \), we construct a \( \mu \)-reduction bundle (this notion makes sense in char 0 also) and a strong \( \mu \)-reduction bundle (this notion makes sense in char \( p > 0 \)), on \( X \). For example, if \( h_1, \ldots, h_\mu \) are homogeneous generators of \( I \), of degrees \( d_1, \ldots, d_\mu \) respectively, then we have the canonical short exact sequence (the Sequence \([4.3] \) in Section 4) of \( \mathcal{O}_X \)-modules

\[
0 \rightarrow V_0 \rightarrow M_0 = \bigoplus_i \mathcal{O}_X(1 - d_i) \rightarrow \mathcal{O}_X(1) \rightarrow 0.
\]

Then the strong \( \mu \)-reduction bundle \( V_{t_0} \) is a subbundle of \( V_0 \) with a short exact sequence of \( \mathcal{O}_X \)-modules (Definition \([4.2] \) ) \( 0 \rightarrow V_{t_0} \rightarrow M_{t_0} \rightarrow \mathcal{O}_X(1) \rightarrow 0 \), where \( M_{t_0} \subseteq M_0 \), so that

\[
f_{R,I}(x) = f_{V_0,\mathcal{O}_X(1)} - f_{M_0,\mathcal{O}_X(1)} = f_{V_{t_0},\mathcal{O}_X(1)} - f_{M_{t_0},\mathcal{O}_X(1)}
\]

and \( \alpha(R, I) = 1 - a_{\text{min}}(V_{t_0})/\bar{d} \), where for a bundle \( V \) the function \( f_{V,\mathcal{O}_X(1)} \) denotes the HK density function of \( V \) with respect to the ample line bundle \( \mathcal{O}_X(1) \), and \( a_{\text{min}}(V) \) denotes the minimum strong Harder-Narasimhan (HN) slopes for \( V \) and \( \bar{d} = \deg X \).

For two dimensional rings, the main results of this paper can be summarized as below.

**Theorem C.** For a standard graded pair \( (R, I) \), where \( R \) is a two dimensional domain, the following statements hold.

1. \( \epsilon^I(\mathfrak{m}) = \alpha(R, I) \).
2. For a given pair \( (R, I) \) in char 0, if \( (R_p, I_p) \) is a reduction mod \( p \) of the pair \( (R, I) \) then \( \lim_{p \to \infty} \epsilon^{I_p}(\mathfrak{m}_p) \) exists and

\[
\epsilon^{I_p}(\mathfrak{m}) := \lim_{p \to \infty} \epsilon^{I_p}(\mathfrak{m}_p) = 1 - \mu_{\text{min}}(V_t)/\bar{d},
\]

where \( V_t \) is a \( \mu \)-reduction bundle for \( (R, I) \) (see Notation \([4.7] \) ).
3. Moreover \( \epsilon^{I_p}(\mathfrak{m}) = \sup\{ x | f^\infty_{R,I}(x) \neq 0 \} \), where \( f^\infty_{R,I}(x) = \lim_{p \to \infty} f_{R_p,I_p}(x) \).
4. For \( p >> 0 \), \( \epsilon^{I_p}(\mathfrak{m}_p) \geq \epsilon^{I_p}(\mathfrak{m}) \). Moreover, if the bundle \( V_0 \) is semistable then \( \epsilon^{I_p}(\mathfrak{m}_p) = \epsilon^{I_p}(\mathfrak{m}) \) if and only if \( V_0 \) reduction mod \( p \) is strongly semistable.
5. In particular, for the pair \( (R, \mathfrak{m}) \) and \( X = \text{Proj} \, R \), if \( \deg \mathcal{O}_X(1) > 2 \) genus \( X \), then for \( p >> 0 \), we have

\[
\epsilon^{I_p}(\mathfrak{m}_p) = \epsilon^{I_p}(\mathfrak{m}) \iff V_0 \text{ reduction mod } p \text{ is strongly semistable}.
\]

In the statement (2) of the above theorem, the existence of \( \epsilon^I(\mathfrak{m}) \) in char 0, is proved by showing that, there exists a strong \( \mu \)-reduction bundle \( V^p_{t_0} \) for \( (R_p, I_p) \) and a \( \mu \)-reduction bundle \( V_t \) for \( (R, I) \) such that \( a_{\text{min}}(V^p_{t_0}) \) converges to \( \mu_{\text{min}}(V_t) \), as \( p \to \infty \) (note that \( V^p_{t_0} \) may not be \( V_t \) reduction mod \( p \)).

The above statements (2) and (3) (along with the fact that in char 0, a strong \( \mu \)-reduction bundle is same as a \( \mu \)-reduction bundle for \( (R, I) \) and \( a_{\text{min}}(V) = \mu_{\text{min}}(V) \) ),
give a well defined and uniform notion for $\alpha(R, I)$ and $c^I_{\infty}(m)$ in both positive and 0 characteristics. Hence in all the characteristics, we have the equality

$$\alpha(R, I) = c^I(m) = 1 - \frac{a_{\min}(V_{ts})}{d},$$

where $V_{ts}$ is a strong $\mu$-reduction bundle for $(R, I)$.

However, we have a reverse inequality, unlike the earlier mentioned results of [HY],

for $p_s > 0$ we have $c^I_{\infty}(m) = \lim_{p \to \infty} c^{I_s}(m_s) \leq c^I_{\infty}(m_s).$

For irreducible plane trinomials $k[x, y, z]/(h)$, the equality $c^I(m) = \alpha(R, I)$ gives an explicit formula for $c^{I_s}(m)$, where $I_n = (x^n, y^n, z^n)$. If we denote $c^{I_s}(m)$, by $c^{I_{np}}(m_p)$, where $p = \text{char } k$, then we find that $c^{I_{np}}(m_p)$ is just a function of the congruence class of $p \mod 2\lambda_h$ and the integer $n$, where $\lambda_h$ is the integer given in terms of the exponents of the trinomial $h$ (see (6.1)). In particular we have the following (see Section 6 for details)

**Example.** If $R = k[x, y, z]/(h)$ is an irreducible trinomial of degree $d \geq 3$ then

for all $p \geq d^2$ and $p \equiv \pm 1 \pmod{2\lambda_h}$ we have $c^{I_{np}}(m_p) = c^I_{\infty}(m)$.

If $(R, m)$ is a Segre product of irreducible trinomials then

1. there are infinitely many primes $p > 0$, for which $c^{I_{np}}(m_p) = c^I_{\infty}(m)$.
2. Moreover, if one of the trinomials, occurring in the product, is a symmetric curve (see Corollary [6.1]) of degree $\geq 5$ then there are also infinitely many primes $p > 0$, for which $c^{I_{np}}(m_p) > c^I_{\infty}(m)$.

We write down the computations made for some explicit trinomials to demonstrate the complexity of $c^m(m)$.

Recall (see Examples 4.3 and 4.6 of [MTW]) that $c^m(f)$ was computed and such phenomena were exhibited, when $R = k[x, y]$ and $f = x^2 + y^3$, or when $R = k[x, y, z]$ and $f$ is a homogeneous polynomial of degree 3 with isolated singularity at $(x, y, z)$. In Corollary 3.9, Hara and Monsky (see [H]) independently described (using syzygy gaps) the possible values of $c^{(x,y)}(f)$, whenever $f \in k[x, y]$ is homogeneous of degree 5 with an isolated singularity at the origin, when $p \neq 5$. Theorem 4.2 of [Vr] computes $c^m(m)$, for diagonal hypersurfaces.

The organisation of this paper is as follows.

In Section 2, we compute the HK density function for $(R, I)$, where $I$ is generated by a system of parameters. This turns out to be a volume function, depending only on the degrees of the generators of $I$. Here again we use the uniform convergence of the sequence $\{f_n(R, I)\}_n$ to $f_{R,I}$ and the fact that $f_{R,I}$ is a continuous function.

In Section 3, we relate $\alpha(R, I)$ with the $F$-threshold $c^I(m)$ and prove that when the ring or its normalization is $F$-regular on the punctured spectrum then $\alpha(R, I) = c^I(m)$. Here we also characterize the regularity in terms of the shape of the function $f_{R,m}$ and also in terms of the number $\alpha(R, m)$.

From Sections 4 onwards we consider dimension two standard graded pair.

In the subsection 4.1, we define an analogous notion of the HK density function for a pair $(V, O_X(1))$, where $V$ is a vector-bundle on a nonsingular curve $X$ and $O_X(1)$ is a very ample line-bundle on $X$.

In the subsection 4.2, we define the notion of $\mu$-reduction and strong $\mu$-reduction sequences (and in Notations [1.10] we define the notion of a strong $\mu$-reduction bundle and a $\mu$-reduction bundle for a given pair $(R, I)$). We prove their existence (for the
short exact sequence with the (*) property) and check the relevant properties (see Proposition 7.1 and Lemma 5.2), e.g., the relation between the \( \mu \)-reduction of the sequence and the \( \mu \)-reduction of the Frobenius pull backs of the sequence.

In Section 5, we prove the convergence of \( f(R_p, I_p) \) and \( \alpha(R_p, I_p) \) (and hence of \( c^t_p(m_p) \)) as \( p \to \infty \). Here we have to analyse the relation between the \( \mu \)-reduction bundles for all the Frobenius pull backs of the sequence and the \( \mu \)-reduction bundle of the sequence itself.

In Section 6, we give the computations of \( F \)-thresholds for plane trinomials.

In Section 7, we list the necessary results for the paper, about vector bundles over nonsingular projective curves.

In the light of Theorem B, we can ask the following natural question

**Question.** Let \( R \) be a standard graded ring in char \( p > 0 \), and let \( I \) be a graded ideal of finite colength and \( m \) be the graded maximal ideal of \( R \). Then, is \( \alpha(R, I) = c^{I}(m) \)?

### 1. Preliminaries

We first recall the following known properties of \( f_{M,I} \) from [T3], where \( M \) and \( I \) are as given above.

1. **Additive property:** Like HK multiplicity, the HK density function too have the additive property, which reduces the theory of \( f_{M,I} \) to the theory of \( f_{R,I} \), where \( R \) is a normal domain: Let \( \Lambda \) be the set of minimal prime ideals \( P \) of \( R \) such that \( \dim R/P = \dim R \). Then
   \[
   f_{M,I} = \sum_{\lambda} f_{R/P,I} \lambda(M_P).
   \]
   As a consequence, we have
   (a) \( f_{M,I} = 0 \), if \( \dim M < \dim R \).
   (b) If \( R \) is an integral domain then \( f_{R,I} = f_{S,IS} \), where \( S \) is the normalization of \( R \), regarded as a graded \( R \)-module.
   (c) \( f_{M,I} = f_{M(n),I} \), for every \( n \in \mathbb{Z} \).

2. **Multiplicative property:** The multiplicative property expresses the HK density function of the Segre product of rings in terms of the HK density function of the individual rings: If \((R, I)\) and \((S, J)\) are two pairs and \( F_{R,I}(x) = e_0(R)x^{d-1}(d-1)! \), where \( e_0(R) \) denotes the Hilbert-Samuel multiplicity of \( R \) with respect to its irrelevant maximal ideal \( m \) and \( d = \dim R \) then their Segre product satisfies
   \[
   F_{R#S,m_1#m_2}(x) = f_{R#S,I#J}(x) = [F_{R,m_1}(x) - f_{R,I}(x)][F_{S,m_2}(x) - f_{S,J}(x)].
   \]

3. **Let \( I \subseteq I' \) such that \( I' \) is homogeneous then**
   \[
   e_{HK}(R, I) = e_{HK}(R, I') \iff f_{R,I}(x) = f_{R,I'}(x), \text{ for all } x.
   \]
   In particular, if \( R \) is equidimensional then
   \[
   f_{R,I} = f_{R,I'} \iff I' \subseteq I^*,
   \]
   where \( I^* \) denotes the tight closure of \( I \) in \( R \).

4. **If \( n_0 \in \mathbb{N} \) such that \( m^{n_0} \subseteq I \) and the ideal \( I \) is generated by \( \mu \) generators then** the support of \( f_{R,I} \subseteq [0, n_0\mu] \).
2. HK Density Functions for Parameter Ideals

2.1. HK Density Functions for Parameter Ideals. Here we give an explicit formula for the HK density function $f_{R,I}$, when $I$ is generated by a system of parameters. As expected, we find that $f_{R,I}$ solely depends on the degrees of the generators of $I$.

Definition 2.1. Given nonnegative integers $n_1, \ldots, n_m$, consider a $m$-parallelopiped $P = [0, n_1] \times \cdots \times [0, n_m]$. We define a volume function

$$V_{m-1}(n_1, \ldots, n_m) : [0, \infty) \rightarrow [0, \infty)$$

given by $x \rightarrow \text{Vol}_{m-1}(P \cap H_x)$, where $H_x = \{(y_1, \ldots, y_m) \in \mathbb{R}^m | \sum_i y_i = x\}$ is a $m - 1$-dimensional hyperplane in $\mathbb{R}^m$ and $\text{Vol}_{m-1}$ is the $(m - 1)$-dimensional Euclidean volume.

Lemma 2.2. Let $R$ be a standard graded ring of dimension $d \geq 2$ over a perfect field of characteristic 0 and $I$ be generated by homogeneous system of parameters $f_1, \ldots, f_d$ of degree $n_1, n_2, \ldots, n_d$ respectively. Then

$$f_{R,I}(x) = e(R,m)\text{Vol}_{d-1}(n_1, \ldots, n_d)(x),$$

where the function $\text{Vol}_{d-1}(n_1, \ldots, n_d)$ is given as in Definition 2.1.

Proof. By the additive property of the HK density function, we may assume that $R$ is a normal domain and $f_1$ is a regular element. For $d = 2$ the lemma is easy to check as $\{f_1, f_2\}$ form a regular sequence. We prove the lemma, by induction on $d$. Henceforth we assume $d \geq 3$. Therefore, for the ring $S = R/f_1R$ and the ideal $J = I/f_1R$, we have

$$f_{S,J}(x) = e(S,mS)\text{Vol}_{d-2}(n_2, \ldots, n_d)(x),$$

for all $x \in \mathbb{R}$.

Now, for every $k \geq 1$ and $q = p^n$, we have the canonical degree 0 surjective map of graded $R$-modules.

(2.1) \[ \frac{S}{J[q]}(-kn_1) \rightarrow \frac{f_1^k R}{f_1^{k+1}R + f_1^k R \cap J[q]} = \frac{f_1^k R + J[q]}{f_1^{k+1}R + J[q]} \]

Hence, for any $x \geq 0$, we have the surjective map

\[ \otimes_{k=0}^{q-1} \left( \frac{S}{J[q]} \right)(-kn_1 + [xq]) \rightarrow \left( \frac{R}{I[q]} \right)_{[xq]}, \]

which gives

(2.2) \[ f_n(R,I)(x) \leq \frac{1}{q} \sum_{k=0}^{q-1} f_n(S,J) \left( \frac{[xq] - kn_1}{q} \right) \]

Henceforth, throughout the proof, we denote $f_n(S,J)$ by $g_n$. We know (Lemma 2.8 of [T3]) that, for every $q = p^n$, $k \geq 1$ and for every $kn_1/q < \lambda \leq (kn_1 + n_1)/q$, we have

$$g_n \left( \frac{[xq] - kn_1}{q} \right) = g_n \left( \frac{[xq]}{q} - \lambda \right) + O \left( \frac{1}{q} \right).$$

Hence

$$\frac{1}{q} g_n \left( \frac{[xq] - kn_1}{q} \right) = \frac{1}{n_1} \int_{kn_1/q}^{(k+1)n_1/q} g_n \left( \frac{[xq]}{q} - \lambda \right) d\lambda + O \left( \frac{1}{q^2} \right).$$

Case (1). If $n_1 < x$. Then for $q \gg 0$, we have $(q-1)n_1 \leq [xq]$.

Therefore

R.H.S. of (2.2) = \[ \left( \frac{1}{q} \right) \sum_{k=0}^{q-1} g_n \left( \frac{[xq] - kn_1}{q} \right) = \left( \frac{1}{n_1} \right) \int_{[xq]/q-n_1}^{[xq]/q} g_n (\lambda) d\lambda + O \left( \frac{1}{q} \right). \]
Now taking limit for (2.2) as \( q \to \infty \), and by the induction hypothesis on \( d \), we get
\[
f_{R,I}(x) \leq (1/n_1) \int_x^{x-n_1} f_{S,J}(\lambda)d\lambda = e(R) \int_x^{x-n_1} V_{d-2}(n_2, \ldots, n_d)(\lambda)d\lambda.
\]
Since \( n_1 < x \), this gives \( f_{R,I}(x) \leq e(R)V_{d-1}(n_1, n_2, \ldots, n_d)(x) \).

**Case (2)** If \( n_1 \geq x \) then \( [xq] = n_1\tilde{m} + r \), where \( 0 \leq r < n_1 \) and \( \tilde{m} < q - 1 \).

R.H.S. of (2.2) is
\[
= (1/q) \int_{0}^\tilde{m} n_{1/q} g_n(\lambda)d\lambda + \tilde{m}O(1/q^2)
\]
Now taking limit for (2.2) as \( q \to \infty \), and applying induction, we get
\[
f_{R,I}(x) \leq (1/n_1) \int_x^{x-n_1} f_{S,J}(\lambda)d\lambda = e(R)V_{d-1}(n_1, \ldots, n_d)(x).
\]
Hence \( f_{R,I}(x) \leq e(R)V_{d-1}(n_1, \ldots, n_d)(x) \) for all \( x \in \mathbb{R} \).

Now \( e(R)V_{d-1}(n_1, \ldots, n_d)(x) - f_{R,I}(x) \) is a nonnegative continuous function with integral = 0. Therefore \( f_{R,I}(x) = e(R)V_{d-1}(n_1, \ldots, n_d)(x) \), for all \( x \). This proves the lemma. \( \square \)

**Corollary 2.3.** Let \( (R, I) \) be a pair as above. If \( I \) is a parameter ideal of \( R \) generated by elements of same degree, say \( n_0 \) then \( f_{R,I} \) is a symmetric function around \( dn_0 \), i.e.,
\[
f_{R,I}(x) = f_{R,I}(dn_0 - x), \text{ for all } x \geq 0.
\]

3. The HK density function \( f_{R,I} \) versus the \( F \)-threshold \( c^f(\textbf{m}) \) and the regularity

3.1. **Support of the HKd function and \( F \)-threshold.**

**Definition 3.1.** For the pair \( (R, I) \) and its HK density function \( f_{R,I} \), let
\[
\alpha(R, I) = \text{Sup} \{ x \mid f_{R,I}(x) > 0 \}.
\]

**Remark 3.2.** It is easy to see, from the definition that, for \( 0 \leq x < 1 \), the function \( f_{R,I}(x) = e_0(R, \textbf{m})x^{d-1}/((d-1)!)) \). In particular, \( f_{R,I}[0,1] \) is a strictly monotonic increasing function and \( \alpha(R, I) \) is a positive real number with \( \alpha(R, I) > 1 \).

We recall the following notion of \( F \)-threshold, as defined in [HMTW] and proved in full generality in [DsNbP].

**Definition 3.3.** Let \( I \) and \( J \) be two ideals such that \( J \subseteq \sqrt{I} \). Then the \( F \)-threshold of \( J \) with respect to \( I \) is
\[
c^f(J) = \lim_{q \to \infty} \frac{\nu^I_f(q)}{q} = \lim_{q \to \infty} \frac{\min \{ r \mid J^{r+1} \subseteq I^q \}}{q}.
\]

**Proposition 3.4.** Let \( (R, I) \) be a standard graded pair of dimension \( \geq 2 \). Then \( \alpha(R, I) \leq c^f(\textbf{m}) \).

**Proof.** Let \( c^f(\textbf{m}) = c \). Then, by the definition of \( c^f(\textbf{m}) \), it follows that, given \( \epsilon > 0 \), there is a \( q(\epsilon) \) such that for all \( q \geq q(\epsilon) \), we have \( \textbf{m}^{(c+\epsilon)q} \subseteq I^q \). Since \( R \) is a standard graded ring this implies \( (R/I^q)_m = 0 \), for \( m \geq (c+\epsilon)q \).
Therefore, for all \( x \geq c + \epsilon \) and for \( q \geq q(\epsilon) \),
\[
\ell(R/I^{[q]})_{[xq]} = 0 \implies f_n(x) = \frac{1}{q^{d-1}}\ell(R/I^{[q]})_{[xq]} = 0,
\]
Hence, for every \( \epsilon > 0 \), we have \( f(x) = \lim_{n \to \infty} f_n(x) = 0 \), for all \( x \geq (c + \epsilon) \).

Since \( f : [0, \infty) \to (0, \infty) \) is a continuous function, we deduce that \( f(x) = 0 \) for all \( x \geq c \). Therefore \( \alpha(R, I) \leq c^f(m) \).

\[\square\]

**Remark 3.5.** For standard graded pair \((R, I)\) of dimension 1, where \( R \) is reduced, (see Theorem 2.9 [T3]) the HK density function \( f_{R,I} \) is the pointwise limit of \( f_n(R, I) \) (here the convergence may not be a uniform convergence). Moreover, for given \( x \geq 0 \), there is \( n_0 \) such that for all \( q = p^n \geq p^{n_0} \), we have \( f_n(R, I)(x) = f_{R,I}(x) \). Hence \( f_{R,I}(x) = 0 \) implies \( m^{[xq]} \subseteq I^{[q]} \), for \( q \geq q_0 \). Therefore \( c^f(m) \leq x \). This proves \( c^f(m) = \alpha(R, I) \).

**Theorem 3.6.** Let \( R \) be a standard graded pair of dimension \( d \geq 2 \) over a perfect field of characteristic \( p \) and \( I \) be generated by homogeneous system of parameters \( f_1, \ldots, f_d \) of degrees \( n_1, n_2, \ldots, n_d \) respectively. Then
\[
\alpha(R, I) = c^f(m) = n_1 + n_2 + \cdots + n_d.
\]

**Proof.** By Lemma 3.2 we have \( n_1 + n_2 + \cdots + n_d = \alpha(R, I) \). Enough to prove the following claim.

**Claim.** There exists a constant \( c \) such that \( \ell(R/(f_1^q, \ldots, f_d^q))_{[xq]} = 0 \), for all \( |xq| \geq (n_1 + \cdots + n_d)q + c \).

Proof of the claim: If \( \dim \ R = 1 \) then we choose \( c \) such that \( R_c \subseteq (f_i) \). Hence \( \ell(R/(f_1^q))_{[xq]} = 0 \), for all \( |xq| \geq n_1q + c \). Now the claim follows from the induction and the surjective map (2.1) of Section 2.

We state the following lemma without the proof as it is easy to check.

**Lemma 3.7.** For a standard graded pair \((R, I)\) and for a fixed power \( q_0 \) of \( p \) we have
\[
f_{R, I^{[q_0]}(q_0x)} = q_0^{d-1}f_{R, I}(x), \quad \text{for all } x \geq 0.
\]

In particular \( \alpha(R, I^{[q_0]}) = q_0\alpha(R, I) \).

**Theorem 3.8.** For a standard graded pair \((R, I)\) of dimension \( \geq 2 \),
\[
R \text{ is } F\text{-regular on } \text{Spec } R \setminus \{m\} \implies \alpha(R, I) = c^f(m).
\]

In particular for a graded pair \((R, I)\), where \( X = \text{Proj } R \) is \( F\)-regular, the above equality follows.

**Proof.** Since \( R \) is \( F\)-regular on the punctured spectrum, there is \( n_0 \) such that \( m^{n_0} \subseteq \tau(R) \), where \( \tau(R) \) is the test ideal of \( R \). Hence, for every ideal \( J \) of \( R \) we have \( m^{n_0} J^* \subseteq J \) (see Definition 1.1 of [HMTW]), where \( J^* \) is the tight closure of \( J \) in \( R \).

To prove \( \alpha(R, I) = c^f(m) \), it is enough to show that, if \( \beta \in \mathbb{N}[1/p] \) such that \( \beta < c^f(m) \) then \( \beta < \alpha(R, I) \). Let \( 2\epsilon = c^f(m) - \beta > 0 \).

Now we can choose a power \( q_0 \) of \( p \) such that, for \( q \geq q_0 \), we have \( \beta q \in \mathbb{N}, \epsilon q \geq n_0 \) and \( m^{\beta q + [xq]} \not\subseteq I^{[q]} \).

In particular \( m^{\beta q_0 + n_0} \not\subseteq I^{[q_0]} \) and therefore \( m^{\beta q_0} \not\subseteq I^{[q_0]} \).

We choose a homogeneous element \( z \in m^{q_0 \beta} \setminus I^{[q_0]} \) and an ideal \( J = (z, I^{[q_0]}) \) then \( e_{HK}(I^{[q_0]}) - e_{HK}(J) > 0 \) and (for the first equality see Remark 2.15 of [T3])
\[
f_{R, I^{[q_0]}(x)} = f_{R, I^{[q_0]}(x)} \geq f_{R, I}(x) \text{ for all } x.
\]
Moreover, if \( x < \beta q_0 = \lfloor \beta q_0 \rfloor \), then
\[
\deg z^q \geq q_0 \beta \implies (R/I^{[q_0]} )_{xq} = (R/(z^q R + I^{[q_0]}))_{xq}, \quad \text{for all } q.
\]
Hence \( f_{R,I^{[q_0]}}(x) = f_{R,J}(x) \), for \( x < \beta q_0 \). In particular
\[
e_{HK}(R,I^{[q_0]}) - e_{HK}(R,J) = \int_{\beta q_0}^{\infty} (f_{R,I^{[q_0]}}(x) - f_{R,J}(x))dx > 0.
\]
This implies \( \alpha(R,I^{[q_0]}) > \beta q_0 \) and hence, by Lemma 3.7, we have \( \alpha(R,I) > \beta \). This proves the theorem.

**Corollary 3.9.** Let \((R,I)\) be a standard graded pair with \( R \) a domain. Let \( S \) be the normalization of \( R \) in the quotient field \( Q(R) \). Then
\[
S \text{ is } F\text{-regular on } \text{Spec } S \setminus \{n\} \implies \alpha(R,I) = c^f(m),
\]
where \( n \) is the graded maximal ideal of \( S \). In particular the equality \( \alpha(R,I) = c^f(m) \) holds for any dimension 2 standard graded pair \((R,I)\), where \( R \) is a domain.

**Proof.** We note that \( S \) is \( \mathbb{N} \)-graded finite \( R \)-module and the canonical graded map \( \pi : R \rightarrow S \) is of degree 0. By Theorem 3.8, we have \( \alpha(S, IS) = c^f(S(n)) \). Now by the additivity property of the HK density function and by Proposition 2.2 of [HMTW]
\[
\alpha(R,I) = \alpha(S, IS) = c^fIS(n) \geq c^f_IS(mS) = c^f(m).
\]
This proves the corollary.

**Lemma 3.10.** If \((R_1,I_1), \ldots, (R_r,I_r)\) are standard graded pairs of dimension \( \geq 2 \), with their respective graded maximal ideals \( m_1, \ldots, m_r \), then, for every \( i \),
\[
c^f_i(m_i) = \alpha(R_i,I_i) \implies c^{\#I_1 \cdots \#I_r}(m_1 \# \cdots \# m_r) = \alpha(R_1 \# \cdots \# R_r, I_1 \# \cdots \# I_r)
\]
and is equal to
\[
\max\{c^f_i(m_i) \mid 1 \leq i \leq r\} = \max\{\alpha(R_i,I_i) \mid 1 \leq i \leq r\}.
\]

**Proof.** Note that if \((R,m)\) and \((S,n)\) are two standard graded rings of dimensions \( d_1 \) \( d_2 \), respectively, over a perfect field \( k \) of char \( k \) > 0 with homogeneous ideals \( I \) and \( J \) of finite colengths in \( R \) and \( S \) respectively, then, by Proposition 2.17 of [T3], we have
\[
f_{R\#S,I\#J} = (F_{R,m})(f_{S,n}) + (F_{S,n})(f_{R,I}) - (f_{R,I})(g_{S,J}),
\]
where \( F_{R,m}(x) = e_0(R,m)x^{d_1-1}/(d_1-1)! \) and \( F_{S,n}(x) = e_0(S,n)x^{d_2-1}/(d_2-1)! \). Therefore \( \alpha(R\#S, I\#J) = \max\{\alpha(R,I), \alpha(S,J)\} \}. Moreover, it follows from the definition that \( c^{\#I\#J}(m\#n) \leq \max\{c^f(m), c^f(n)\} \). Hence, for \((R_i,m_i)\) and \( I_i \) as given above, we have
\[
c^{\#I_1 \cdots \#I_r}(m_1 \# \cdots \# m_r) \leq \max\{c^f(m_i) \mid 1 \leq i \leq r\} = \alpha(R_1 \# \cdots \# R_r, I_1 \# \cdots \# I_r).
\]
On the other hand, by Proposition 3.1, we have
\[
\alpha(R_1 \# \cdots \# R_r, I_1 \# \cdots \# I_r) \leq c^{\#I_1 \cdots \#I_r}(m_1 \# \cdots \# m_r).
\]
This proves the lemma.

Now the following corollary follows from applying Lemma 3.10 to Theorem 3.6, Theorem 3.8 and Corollary 3.9.

**Corollary 3.11.** If \( R \) is a Segre product of rings \( \{R_i\}_i \) of dimension \( \geq 2 \), where \((R_i,I_i)\) denotes a standard graded pair with \( m_i \) as the irrelevant maximal ideal satisfying one of the following conditions:
Proof. (1) Let \( f \) be an ideal function with \( \alpha \) and \( c \). We have \( \text{Theorem 3.14} \).

(2) Suppose \( \alpha \) is generated by s.o.p. \( \{ h_1, \ldots, h_s \} \subset \{ h_1, \ldots, h_s \} \).

(3) Assertion (3) is a particular case of Assertion (2). This proves the theorem.

2.3. Shape of the HK density function and regularity.

Theorem 3.13. Let \( R \) be a standard graded ring of dimension \( d \) then

1. \( \alpha(R, m) \leq d \). Hence \( \text{Support} \ (f_{R,m}) \subseteq \text{the interval} \ [0, d] \). Moreover,
2. \( \alpha(R, m) = d \) if and only if \( R/P \) is regular, for some \( P \in \text{Assh}(R) = \{ P \in \text{Spec} \ R \mid \dim R/P = d \} \). In particular,
3. if \( R \) is a domain then \( \alpha(R, m) = d \) if and only if \( R \) is regular.

Proof. (1) Let \( J \) be a parameter ideal generated by elements of degree 1. Then \( J \subseteq m \) implies \( \alpha(R, m) \leq \alpha(R, J) \). But, by Lemma 2.2, \( \alpha(R, J) = d \). This proves the first assertion.

(2) Suppose \( R/P \), for some \( P \in \text{Assh}(R) \), is regular. Then the ideal \( m/P \) is generated by a system of parameters of degree 1, therefore, by Corollary 2.3, we have \( \alpha(R/P, m/P) = d \). But, by the additivity of the HK density function, we have \( \alpha(R, m) = \max \{ \alpha(R/P, m/P) \mid P \in \text{Assh}(R) \} \geq d \). Hence, by Assertion (1) we have \( \alpha(R, m) = d \).

Conversely, suppose \( \alpha(R, m) = d \). Then, since \( \text{Assh}(R) \) is a finite set, there is a \( P \in \text{Assh} (R) \) such that \( \alpha(R/P, m/P) = d \).

Suppose \( R/P \) is not regular. We choose a system of parameter ideal \( J = (x_1, \ldots, x_d) \) of linear forms such that \( \{ x_1, \ldots, x_d \} \) is a part of a minimal set of generators of \( m/P \). By Theorem 2.2 of [S], \( (J^*) \cap (R/P)_1 = J \cap (R/P)_1 \). But \( R/P \) is not regular implies that \( J \cap (R/P)_1 \neq m/P \cap (R/P)_1 \). Hence, by Corollary 3.2 of [HMTW], we have \( e^{m/P}(J) < d \). Since \( m \subseteq J \), the integral closure of \( J \) in \( R/P \), by Proposition 1.7 of [MTW], we have \( e^{m/P}(J) = e^{m/P}(m/P) < d \). Now, Proposition 3.3 implies \( \alpha(R/P, m/P) \leq e^{m/P}(m/P) < d \), which is a contradiction. This proves Assertion (2).

(3) Assertion (3) is a particular case of Assertion (2). This proves the theorem.

Theorem 3.14. Let \( R \) be a standard graded domain of dimension \( d \geq 2 \). Then the function \( f_{R,m} \) is symmetric at \( x = d/2 \), i.e.,

\[ f_{R,m}(x) = f_{R,m}(d - x), \quad \text{for all } x \text{ iff } R \text{ is a regular ring}. \]

Moreover, if \( \dim R = 2 \) then either

1. \( f_{R,m} \) is symmetric at \( x = 1 \), or
(2) \( f_{R,m}(1-y) > f_{R,m}(1+y) \), for all \( y \in (0,1) \).

Proof. If \( R \) is regular then \( m \) is generated by s.o.p. of degree 1 and therefore, by Corollary \([2.3]\) the function \( f_{R,m} \) is symmetric at \( d/2 \).

Conversely, if \( f_{R,m} \) is symmetric at \( d/2 \) then \( \alpha(R,m) = d \), hence, by Corollary \([3.13]\) the ring \( R \) is regular.

Let \( \dim R = 2 \) which is not regular. Suppose \( f_{R,m}(1-y_0) \leq f_{R,m}(1+y_0) \), for some \( y_0 \in (0,1) \).

(Now we follow the notations of the subsection 4.1 of Section 4), for the pair \((R,m)\), consider the normalization \( \pi : R \to S \), the curve \( X = \text{Proj} S \) and the associated sequence

\[
0 \to V_0 \to M_0 = \oplus^s \mathcal{O}_X \to \mathcal{O}_X(1) \to 0,
\]

for \( s = \mu(m) \). Note that \( f_{R,m}(x) = f_{V_0, \mathcal{O}_X(1)} - f_{M_0, \mathcal{O}_X(1)} \), for \( 0 \leq x \leq 2 \).

Since \( R \) is not regular we have rank \( V_0 = r = s - 1 \geq 2 \). Let \( \{a_1 > \ldots > a_{i+1}, \{r_1, \ldots, r_{i+1}\} \} \) be the strong HN data for \( V_0 \). Note that \( l + 1 \geq 1 \) and the strong HN data for \( M_0 \) is \( \{(0), \{s\}\} \). This implies that for \( y \in [0, -a_1/d) \) we have \( f_{R,m}(1+y) = d - y d y < d - y d y = f_{R,m}(1-y) \). Therefore \( -a_j/d \leq y_0 < -a_{j+1}/d \), for some \( 1 \leq j \leq l \). Hence \( \sum_{i \geq j+1} a_i r_i - (d) y_0 r_i \geq d - (d) y_0 \). This gives

\[
dy_0(1 - \sum_{i \geq j+1} r_i) \geq d + \sum_{i \geq j+1} a_i r_i = - \sum_{i=1}^{l+1} a_i r_i + \sum_{i \geq j+1} a_i r_i = - \sum_{i=1}^{j} a_i r_i.
\]

Now \( j+1 \leq l+1 \) and \( -a_i \geq 0 \), for \( i \), implies that both the sides of the above equation are \( 0 \). In particular \( j = l = 1 \) and \( r_{l+1} = r_2 \) and \( a_1 = 0 \). Hence, if for some \( m_1 \), \( F^{m_1}(V_0) \) has the strong HN filtration then it is given by \( 0 = E_0 \subset E_1 \subset E_2 = F^{m_1}V_0 \), where rank \( E_1 = \mu(m) - 2 \geq 1 \) and deg \( E_1 = 0 \). Since \( E_1 \subset \oplus^s \mathcal{O}_X \), we deduce that \( E_1 \) is trivial vector bundle of rank \( \geq 1 \). Hence \( h^0(X, F^{m_1}V_0) \geq h^0(X, E_1) \geq 1 \).

On the other hand, the linear independence of the \( \{h_1, \ldots, h_s\} \in h^0(X, \mathcal{O}_X(1)) \) over \( k \) implies that the set \( \{h^1_1, \ldots, h^1_s\} \in h^0(X, \mathcal{O}_X(q)) \) is a linearly independent set over \( k \). Therefore the map

\[
\oplus^s h^0(X, \mathcal{O}_X) \to h^0(X, \mathcal{O}_X(q))
\]
is injective. But then \( h^0(X, F^{m_1}V_0) = 0 \), which is a contradiction. Hence we conclude that \( f_{R,m}(1+y) < f_{R,m}(1-y) \), for every \( y \in (0,1) \). This proves the last assertion and hence the thm. \( \square \)

4. The HK density function in dimension 2

4.1. The HK density functions for vector bundles on curves. Let \( X \) be a non-singular projective curve over a perfect field of characteristic \( p > 0 \). Let \( \mathcal{O}_X(1) \) be a very ample line bundle on \( X \). Let \( V \) be a vector bundle \( X \). For the notion of HN data, (strong) HN slopes and \( a_{min}(V) \), for a vector bundle \( V \) on \( X \), we refer to Notations \([7.3]\).

We define the HK density function of \( V \) with respect to \( \mathcal{O}_X(1) \) (where \( q = p^n \))

\[
f_{V, \mathcal{O}_X(1)}: \mathbb{R} \to [0, \infty) \text{ given by } x \to \lim_{n \to \infty} \frac{1}{q} h^1(X, F^{n*}V(||(x-1)q||)).
\]

This function is well defined and can be given explicitly as follows. Choose \( m_1 \) such that \( F^{m_1}V \) has the strong HN filtration (see Theorem 2.7 of [L])

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = F^{m_1}V.
\]

Let \( a_i = a_i(V) = (1/p^{m_1}) \mu(E_i/E_{i-1}) \) and \( r_i = r_i(V) = \text{rank}(E_i/E_{i-1}) \) then
We choose \( q > 0 \) such that
\[
- \frac{a_1q}{d} < \frac{a_1q}{d} + (d-3) < - \frac{a_2q}{d} < - \frac{a_2q}{d} + (d-3) < \cdots < - \frac{a_t+1q}{d}.
\]

Applying Remark [7.3] to \( F^{n*}(E_i/E_{i-1}) \), where \( q = p^n \), we have
\[
h^1(X, F^{n+m1*}V([(x-1)q])) = \sum_{i=1}^{t+1} h^1(X, F^{n*}(E_i/E_{i-1})([(x-1)q])).
\]

Taking limit as \( n \to \infty \), we get
\[
x < 1 - \frac{a_1}{d} \quad \implies \quad f_{V,O_X(1)}(x) = - \left[ \sum_{i=1}^{t+1} a_ir_i + (d-1)r_i \right]
x - 1 - \frac{a_i}{d} \leq x < 1 - \frac{a_{i+1}}{d} \quad \implies \quad f_{V,O_X(1)}(x) = - \left[ \sum_{k=i+1}^{t+1} a_k r_k + (d-1)r_k \right].
\]

This implies, if we denote \( a_{\min}(V) = a_{t+1}(V) \),
\[
\text{Support } f_{V,O_X(1)} \subseteq (-\infty, 1 - a_{\min}(V)/d]
\]
and
\[
\alpha(V, O_X(1)) := \text{Sup } \{ x \mid f_{V,O_X(1)}(x) > 0 \} = 1 - a_{\min}(V)/d.
\]

Now we look at the HK density functions of the vector bundles arising in the following situation and relate them to the HK density function \( f_{R,I} \), where \( R \) is a standard graded domain of dimension 2 with a graded ideal \( I \) of finite colength. Note that the HK density function (and hence the maximum support of \( f(R, I) \)) has been given in [T3], when \( I \) is generated by degree 1 elements.

Let \( h_1, \ldots, h_\mu \) be a set of (nonzero) homogeneous generators of \( I \), of degrees \( d_1, \ldots, d_\mu \), respectively.

Let \( \pi : R \to S \) be the normalization of \( R \). Therefore it is a finite graded map of degree 0, where \( S \) is a normal domain and \( Q(R) = Q(S) \). Hence, the additivity of the HK density function implies that
\[
f_{R,I}(x) = f_{S,I}(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{q} \ell \left( \frac{S}{I^q[S]} \right)_{[xq]}.
\]

Note that \( X = \text{Proj } S \) is a nonsingular projective curve and the associated canonical exact sequence of locally free sheaves of \( O_X \)-modules (moreover the sequence is locally split exact) is given by
\[
0 \to V_0 \to M_0 = \bigoplus_{i=1}^\mu O_X(1 - d_i) \to O_X(1) \to 0,
\]
where \( O_X(1 - d_i) \to O_X(1) \) is given by the multiplication by the element \( h_i \). We can choose \( m_0 > 0 \) such that, for \( m \geq m_0 \), we have \( H^1(X, O_X(m)) = 0 \) and \( H^0(X, O_X((m)) = S_m \).

Hence, for \( m \geq m_0 \), \( n \geq 0 \) and \( q = p^n \), the long exact sequence of cohomologies
\[
0 \to H^0(X, (F^{n*}V_0)(m)) \to \bigoplus H^0(X, (F^{n*}M_0)(m)) \to H^0(X, O_X(q + m))
\]
\[
\to H^1(X, (F^{n*}V_0)(m)) \to \bigoplus H^1(X, (F^{n*}M_0)(m)) \to 0
\]
gives
\[
f_n \left( \frac{m + q}{q} \right) = \frac{1}{q} \ell \left( \frac{S}{I^q[S]} \right)_{[xq]} = \frac{1}{q} \left[ h^1(X, (F^{n*}V_0)(m)) - h^1(X, (F^{n*}M_0)(m)) \right].
\]

Hence,
\[
f_{R,I}(x) = f_{V_0,O_X(1)}(x) - f_{M_0,O_X(1)}(x), \text{ for } x \geq 0.
\]

In particular, this expresses \( f_{R,I} \) as a difference of the HK density functions of the vector bundles \( V_0 \) and \( M_0 \) (hence \( f_{R,I} \) is a difference of two piecewise linear polynomials
which are given in terms of the strong HN slopes of the vector bundles $V_0$ and $M_0$. It is easy to see that the strong HN slopes of $V_0$ and $M_0$ are all nonpositive. Hence, if $a_{\text{min}}(V_0) < a_{\text{min}}(M_0)$ then
\[
\alpha(R, I) := \text{Sup} \{ x \mid f_{R,I}(x) > 0 \} = 1 - a_{\text{min}}(V_0)/d.
\]

Note that this holds if $I$ has generating set of degree 1, as in that case $\mu(M_0) = a_{\text{min}}(M_0) = 0$ and $a_{\text{min}}(V_0) \leq \mu_{\text{min}}(V_0) = 0 = a_{\text{min}}(M_0)$.

Now, in the next section we introduce the notion of $\mu$-reduction (for a short exact sequence of type \(1.3\)). This allows us to replace the short exact sequence \(1.3\), by another one, for which we have, $a_{\text{min}}(V_0) < a_{\text{min}}(M_0)$.

In particular, we can express the maximum support of $f_{R,I}$, namely $\alpha(R, I)$, in terms of the maximum support of the HK density function of one single vector bundle. Now using the equality $\alpha(R, I) = c^l(m)$, we would be able to study various properties of the $F$-threshold $c^l(m)$ in dimension two (see Theorem \(1.2\) Theorem \(1.5\)).

4.2. $\mu$-reduction for a syzygy bundle. In this section $X$ is a nonsingular projective curve of degree $d$ over a field $k$ (not necessarily of positive characteristic) and
\[
0 \to V_0 \xrightarrow{f_0} M_0 \to \mathcal{L} \to 0
\]
is a short exact sequence of locally free $\mathcal{O}_X$-modules and $\mathcal{L}$ is a line bundle.

The appendix of the paper states the relevant generalities about vector bundles on curves.

**Notations 4.1.** For the sequence \(1.6\), we denote the HN filtiration of $M_0$ by
\[
0 = M_{l_1} \subset M_{l_1-1} \subset \cdots \subset M_0 = M
\]
and
\[
0 = V_{l_1} \subset V_{l_1-1} \subset \cdots \subset V_1 \subset V_0
\]
denote the induced filtration, where $V_i = M_i \cap V_0$. For every $0 \leq i \leq l_1 - 1$, let $f_i : V_i \to M_i$ be the canonical inclusion map.

**Definition 4.2.** Following Notations \(1.1\)

1. we say the sequence \(1.6\) has the $\mu$-reduction at $t$ if there exists $0 \leq t < l_1$ such that
   (a) for every $0 \leq i < t$, the canonical sequence
   \[
   0 \to V_i \to M_i \to \mathcal{L} \to 0
   \]
is exact and $\mu_{\text{min}}(V_i) = \mu_{\text{min}}(M_i)$, and
   (b) $\mu_{\text{min}}(V_t) < \mu_{\text{min}}(M_t)$.

We call $V_t$ the $\mu$-reduction bundle and $0 \to V_t \to M_t \to \mathcal{L} \to 0$ the $\mu$-reduction sequence of \(1.6\). (We will see that the $\mu$-reduction sequence is indeed exact from Lemma \(1.5\).

2. We say (provided $\text{char } p > 0$), the sequence \(1.6\) has the strong $\mu$-reduction at $t_0$, if for some choice of $m_1 > 0$ such that $F^{m_1*}(V_0)$ has the strong HN filtration, the sequence
\[
0 \to F^{m_1*}V_0 \xrightarrow{F^{m_1*}(f_0)} F^{m_1*}M_0 \to F^{m_1*}\mathcal{O}_X(1) \to 0
\]
has $\mu$-reduction sequence at $t_0$. We will prove (Proposition \(1.6\)) that for the sequence \(1.3\), the $\mu$-reduction and strong $\mu$-reduction exist and $t_0 \leq t$.

**Definition 4.3.** The sequence \(1.6\) has the (*) property, if

1. the induced map $M_{l_1-1} \to \mathcal{L}$ is nonzero and $\mu_{\text{max}}(M_0) < \mu(\mathcal{L})$. 
(2) If \( \text{char } k = p > 0 \) then the HN filtration of \( M_0 \) is the strong HN filtration.

**Remark 4.4.** The sequence \((4.3)\) has the (*) property.

**Lemma 4.5.** If the sequence \((4.0)\) has the (*) property then

(1) \( 0 = V_{l_1} \subset V_{l_1-1} \subset \cdots \subset V_1 \subset V_0 \)

is a sequence of proper subbundles and

(2) \( \mu_{\text{min}}(V_j) \leq \mu_{\text{min}}(M_j) \), for \( 0 \leq j < l_1 \).

(3) If \( i < l_1 - 1 \) such that \( \mu_{\text{min}}(V_j) = \mu_{\text{min}}(M_j) \), for \( 0 \leq j \leq i \), we have the short exact sequence

\[
0 \rightarrow V_{i+1} \stackrel{f_{i+1}}{\rightarrow} M_{i+1} \rightarrow \mathcal{L} \rightarrow 0
\]

and \( V_j/V_{j+1} \simeq M_j/M_{j+1} \), for all \( 0 \leq j \leq i \).

**Proof.** Note that, for any \( 0 \leq i < l_1 - 1 \), we have \( \text{coker } f_i \neq 0 \) and \( \text{coker } f_i \subseteq \mathcal{L} \), as, by part (1) of the definition of the (*) property, the induced map \( M_i \rightarrow \mathcal{L} \) is nonzero and factors through the injective map \( M_i/f_i(V_i) \rightarrow \mathcal{L} \).

(1) If \( V_i = V_{i+1} \), for some \( i < l_1 - 1 \) then we have \( M_i/M_{i+1} \simeq \text{coker } f_i/\text{coker } f_{i+1} \), where \( \text{coker } f_i/\text{coker } f_{i+1} \) is a subquotient of \( \mathcal{L} \), and hence a torsion-sheaf of \( \mathcal{O}_X \)-modules, on the other hand \( M_i/M_{i+1} \) is a nonzero locally free sheaf.

(2) This follows as \( 0 \rightarrow V_i/V_{i+1} \rightarrow M_i/M_{i+1} \) implies

\[
\mu_{\text{min}}(V_i) \leq \mu(V_i/V_{i+1}) \leq \mu(M_i/M_{i+1}) = \mu_{\text{min}}(M_i).
\]

(3) We prove this by induction on \( i \). By hypothesis \( \text{coker } f_0 = \mathcal{L} \). For \( i \geq 0 \), we assume \( \text{coker } f_i = \mathcal{L} \). Now \( \text{coker } f_{i+1} \neq 0 \) implies \( \mathcal{L}/\text{coker } f_{i+1} \) is a torsion sheaf, which implies

\[
\text{rank } \frac{V_i}{V_{i+1}} = \text{rank } \frac{M_i}{M_{i+1}} \implies \text{deg } \frac{V_i}{V_{i+1}} = \text{deg } \frac{M_i}{M_{i+1}}.
\]

Hence \( \text{deg } (\mathcal{L}/\text{coker } f_{i+1}) = \ell(\mathcal{L}/\text{coker } f_{i+1}) = 0 \) which implies \( \text{coker } f_{i+1} = \mathcal{L} \). This proves Assertion (3) and hence the lemma. \( \square \)

**Proposition 4.6.** If the sequence \((4.0)\) has the (*) property then

(1) the sequence \((4.0)\) has \( \mu \)-reduction at \( t \), for some \( 0 \leq t < l_1 \).

(2) Moreover, in the case \( \text{char } k = p > 0 \), then, for any \( s \)-th iterated Frobenius map \( F^s : X \rightarrow X \), the canonical short exact sequence of \( \mathcal{O}_X \)-modules

\[
(4.8) \quad 0 \rightarrow F^{s*}V_0 \stackrel{F^{s*}(f_0)}{\rightarrow} F^{s*}M_0 \rightarrow F^{s*}\mathcal{L} \rightarrow 0
\]

has \( \mu \)-reduction for some \( t_0 \leq t \) and the \( \mu \)-reduction sequence is

\[
(4.9) \quad 0 \rightarrow F^{s*}V_{t_0} \stackrel{F^{s*}(f_{t_0})}{\rightarrow} F^{s*}M_{t_0} \rightarrow F^{s*}\mathcal{L} \rightarrow 0.
\]

**Proof.** (1) Suppose \( \mu_{\text{min}}(V_i) = \mu_{\text{min}}(M_i) \), for every \( 0 \leq i \leq l_1-1 \). Then, by Lemma 4.5 (3), the sequence \( 0 \rightarrow V_{l_1-1} \rightarrow M_{l_1-1} \rightarrow \mathcal{L} \rightarrow 0 \) is exact. Now, as \( M_{l_1-1} \) is semistable, we have

\[
\mu_{\text{min}}(V_{l_1-1}) \leq \mu(V_{l_1-1}) \leq \mu(M_{l_1-1}) = \mu_{\text{min}}(M_{l_1-1}) = \mu_{\text{min}}(V_{l_1-1}).
\]

But then, \( \text{deg } (\mathcal{L}) = \mu(M_{l_1-1}) = \mu_{\text{max}}(M) \), which is contradicts the (*) property. Hence we have \( 0 \leq t < l_1 \) such that \( \mu_{\text{min}}(V_t) < \mu_{\text{min}}(M_t) \). Therefore any sequence with the (*) property achieves the \( \mu \)-reduction. This proves (1).
(2) Since the HN filtration of $M_0$ is a strong HN filtration, the sequence (1.8) has the (*) property. Note that $F^s$ is a flat map. Therefore $F^{s*}M_i/F^{s*}M_{i+1} \simeq F^{s*}(M_i/M_{i+1})$ and $F^{s*}M_i \cap F^{s*}V_0 = F^{s*}V_i$. If $\mu_{\min}(V_i) < \mu_{\min}(M_i)$ then

$$
\mu_{\min}(F^{s*}V_i) \leq p^s \mu_{\min}(V_i) < p^s \mu_{\min}(M_i) = \mu_{\min}(F^{s*}M_i).
$$

Therefore the sequence (1.8) has a $\mu$-reduction at $t_0$, where $0 \leq t_0 \leq t$. \hfill $\Box$

In the computation of the HK density function (Theorem 4.12), we use the following lemma to replace the sequence (1.6) by its strong $\mu$-reduction sequence. This allows us to interpret the support of the HK density function in terms of the HN slopes of a single vector bundle.

**Lemma 4.7.** If the sequence (4.6) has the $\mu$-reduction sequence at $t$ then $V_0$ has the HN filtration

$$
\cdots \subset W_{t+1} \subset W_t \subset V_{t-1} \subset V_{t-2} \cdots \subset V_1 \subset V_0,
$$

where

(1) $W_i \subset V_i$ and $\mu(V_{t-1}/W_i) = \mu(V_{t-1}/V_i)$ and

(2) $V_i/V_{i+1} \simeq M_i/M_{i+1}$, for $0 \leq i < t$,

where $V_i$ and $M_i$ are as in Notations 4.7. Moreover the HN filtration of $V_i$ is either

(1) $\cdots \subset W_{t+1} \subset V_t$, when $W_t = V_t$ (equivalently $\mu_{\min}(V_t) > \mu_{\min}(V_{t-1})$), or

(2) $\cdots \subset W_{t+1} \subset W_t \subset V_t$, when $W_t \subset V_t$ (equivalently $\mu_{\min}(V_t) = \mu_{\min}(V_{t-1})$).

**Proof.** By Lemma 4.5 (3), we have $V_i/V_{i+1} \simeq M_i/M_{i+1}$, for all $0 \leq i < t$. Let the HN filtration of $V_{t-1}$ be $\cdots \subset W_{t+1} \subset W_t \subset V_{t-2} \subset \cdots \subset V_1 \subset V_0$. Then

$$
\mu \left( \frac{V_{t-1}}{W_t} \right) = \mu_{\min}(V_{t-1}) = \mu_{\min}(M_{t-1}) = \mu \left( \frac{V_{t-1}}{V_t} \right) > \mu \left( \frac{V_{t-2}}{V_{t-1}} \right) > \cdots > \mu \left( \frac{V_0}{V_1} \right),
$$

Hence the HN filtration of $V_0$ is $\cdots \subset W_{t+1} \subset W_t \subset V_{t-1} \subset V_{t-2} \subset \cdots \subset V_1 \subset V_0$.

Now, by Remark 7.3 (6), semistability of $V_{t-1}/V_t$ implies $W_t \subseteq V_t$. If $W_t = V_t$ then $\mu_{\min}(V_t) = \mu_{\min}(W_t) > \mu_{\min}(V_{t-1})$, which implies Assertion (1). If $W_t \subset V_t$ then $\mu_{\min}(V_t) > \mu_{\min}(V_{t-1}) = \mu(V_t/W_t)$ implies Assertion (2). This completes the proof of the lemma. \hfill $\Box$

**Remark 4.8.** If the sequence (4.6) has $\mu$-reduction at $t$ (as in Definition 4.2) then $\mu(M_t/M_{t+1}) \geq \mu_{\min}(V_t)$.

For the notion of HN data, (strong) HN slopes and $a_{\min}(V)$, for a vector bundle $V$ on $X$, we refer to Notations 7.3.

**Lemma 4.9.** Let $X$ be a nonsingular projective curve in char $p > 0$, of degree $d$ with the sequence

$$
0 \longrightarrow V_0 \longrightarrow M_0 \longrightarrow \mathcal{O}_X(1) \longrightarrow 0
$$

which has the (*) property and has the strong $\mu$-reduction at $t_0$. Then

(1) $f_{V_0,\mathcal{O}_X(1)} - f_{M_0,\mathcal{O}_X(1)} = f_{V_{t_0},\mathcal{O}_X(1)} - f_{M_{t_0},\mathcal{O}_X(1)}$, and

(2) $\max \left\{ x \mid f_{V_0,\mathcal{O}_X(1)}(x) - f_{M_0,\mathcal{O}_X(1)}(x) \neq 0 \right\} = 1 - \frac{a_{\min}(V_0)}{d}$,

we recall that for a choice of $m_1$, as in Definition 4.2, $a_{\min}(V_0) = \mu_{\min}(F^{m_1*}V_0)/p^{m_1}$.

**Proof.** Throughout the proof, we denote $F^{m_1*}(V_i) = \tilde{V}_i$ and $F^{m_1*}(M_i) = \tilde{M}_i$, for all $i \leq t$ (where $\tilde{V}_i$ and $\tilde{M}_i$ are as in Notations 4.1). Let

$$
0 = \tilde{M}_{t_1} \subset \cdots \subset \tilde{M}_{t_0} \subset \cdots \subset \tilde{M}_0
$$

be the HN filtration of $\tilde{M}_0$. Therefore, by Lemma 4.7
(1) $\mu_{\min}(\tilde{V}_0) < \mu_{\min}(\tilde{M}_0)$ and
(2) the HN filtration of $V_0$ is $\cdots \subset \tilde{W}_{l+1} \subset \tilde{W}_l \subset \tilde{V}_{t_0-1} \subset \tilde{V}_{t_0-2} \subset \cdots \subset \tilde{V}_1 \subset \tilde{V}_0$
and the HN filtration of $V_{t_0}$ is
(a) either $\cdots \subset \tilde{W}_{l+1} \subset \tilde{W}_l = \tilde{V}_0$,
(b) or $\cdots \subset \tilde{W}_{l+1} \subset \tilde{W}_l \subset \tilde{V}_0$ if $\tilde{W}_l \subset \tilde{V}_0$.
(3) $\frac{\tilde{V}_i}{\tilde{V}_{i+1}} \simeq \frac{M}{M_{i+1}}$, for $i \leq t_0 - 1$ and $\mu(\tilde{V}_{i+1}/\tilde{V}_0) = \mu(\tilde{V}_{i+1}/\tilde{W}_i) = \mu(\tilde{V}_{i+1}/\tilde{M}_{t_0})$.

It is easy to check that the HN filtration of $\tilde{V}_0$ is the strong HN filtration. Therefore, if $a_1 > a_2 > \cdots > a_{k+1}$ are the strong HN slopes for $V_{t_0}$, then the HN slopes for $\tilde{V}_0$ are $a_1q_1 > a_2q_1 > \cdots > a_kq_1 > a_{k+1}q_1$. Similarly, if $b_1 > b_2 > \cdots > b_{t_1-t_0}$ are the strong HN slopes for $\tilde{M}_{t_0}$ then $b_1q_1 > b_2q_1 > \cdots > b_{t_1-t_0}q_1$ are the HN slopes for $\tilde{M}_{t_0}$. Note that $a_{k+1} = a_{\min}(V_0)$ and $b_{t_1-t_0} = a_{\min}(M_{t_0})$. Let
$$A_n(m) = h^1(X, F^{n*}\tilde{V}_0(m)) - h^1(X, F^{n*}(\tilde{M}_0)(m)),$$
$$B_n(m) = h^1(X, F^{n*}\tilde{V}_a(m)) - h^1(X, F^{n*}(\tilde{M}_{t_0})(m)).$$

Claim

(1) For $q = p^n$, $q_1 = p^{m_1}$ and $|C| \leq (\text{rank } M_0)d(d-3)$, we have
$$A_n(m) = B_n(m) + C,$$
for $\frac{m}{qq_1} \in \left[0, \frac{(d-3)}{qq_1} - \frac{a_{\min}(V_0)}{d}\right]$,.

(2) $A_n(m) = B_n(m) = h^1(X, F^{n*}\tilde{V}_a(m)) = - r_{k+1}[a_{\min}(V_0)qq_1 + md + d(d-3)]$;
for $\frac{m}{qq_1} \in \left(\frac{(d-3)}{qq_1} - \frac{\min\{a_k, a_{\min}(M_{t_0})\}}{d}, - \frac{a_{\min}(V_0)}{d}\right)$.

Proof of the claim: We prove the claim when $\tilde{W}_l \subset \tilde{V}_0$. The case $\tilde{W}_l = \tilde{V}_0$ can be argued similarly.

We choose $q >> 0$ such that
$$\frac{a_1qq_1}{d} < \frac{a_1qq_1}{d} + (d-3) < \frac{a_2qq_1}{d} + (d-3) < \frac{a_2qq_1}{d} + (d-3) \cdots < \frac{a_{k+1}qq_1}{d}.$$ Since $\mu_{\min}(\tilde{V}_0) < \mu_{\min}(\tilde{M}_{t_0})$, for sufficiently large $q$, we also have
$$\frac{a_{k+1}qq_1}{d} > d - 3 - \frac{\min\{a_kqq_1, b_{t_1-t_0}qq_1\}}{d}.$$ Note applying Remark 7.4 one can easily check that
$$A_n(m) - B_n(m) = h^1(X, F^{n*}(\tilde{V}_{t_0-1}/\tilde{W}_l)(m)) - h^1(X, F^{n*}(\tilde{V}_{t_0}/\tilde{W}_l)(m)) - h^1(X, F^{n*}(\tilde{M}_{t_0-1}/\tilde{M}_{t_0})(m)).$$

Now Part (1) of the claim follows as $\tilde{V}_{t_0-1}/\tilde{W}_l$ and $\tilde{V}_{t_0}/\tilde{W}_l$ are strongly semistable sheaves and
$$\text{rank } (\tilde{V}_{t_0-1}/\tilde{W}_l) - \text{rank } (\tilde{V}_{t_0}/\tilde{W}_l) = \text{rank } (\tilde{M}_{t_0-1}/\tilde{M}_{t_0});$$
$$
\mu (F^{n*}(\tilde{V}_{t_0-1}/\tilde{W}_l)) = \mu (F^{n*}(\tilde{V}_{t_0}/\tilde{W}_l)) = \mu (F^{n*}(\tilde{V}_{t_0-1}/\tilde{V}_{t_0})) = a_{k+1}q.$$ The part (2) of the claim also easily follows from the above details.
Now the assertion (1) of the lemma follows by part (1) of the claim, as
\[ f_{V_0, \mathcal{O}_X(1)}(x) - f_{M_0, \mathcal{O}_X(1)}(x) = \lim_{q \to \infty} \frac{1}{q \cdot a_0} A_n([xqq_1]) \]
\[ f_{V_0, \mathcal{O}_X(1)}(x) - f_{M_0, \mathcal{O}_X(1)}(x) = \lim_{q \to \infty} \frac{1}{q \cdot a_0} B_n([xqq_1]). \]

By part (2) of the claim \[ f_{V_0, \mathcal{O}_X(1)}(x) - f_{M_0, \mathcal{O}_X(1)}(x) = -r_{k+1} [a_{k+1} + d(x-1)], \]
for \( x \in (1 - \min \{a_k/d, a_{\min}(M_0)/d\}, 1 - a_{k+1}/d) \), and \( f_{V_0, \mathcal{O}_X(1)}(x) - f_{M_0, \mathcal{O}_X(1)}(x) = 0 \), for \( x \in [1 - \frac{a_{k+1}}{d}, \infty) \). This proves the second assertion and hence the lemma. \( \Box \)

4.3. Formula for \( c^I(m) \) and \( \alpha(R, I) \).

**Notations 4.10.** Let \((R, I)\) be a standard graded pair, where \( R \) is a two dimensional domain and \( I \) is generated by homogeneous elements of degrees \( d_1, \ldots, d_s \). Let \( X = \text{Proj} \, S \), where \( S \) is the normalization of \( R \) in its quotient field. Let
\[ 0 \to V_0 \to M_0 = \oplus_{i=1}^s \mathcal{O}_X(1-d_i) \to \mathcal{O}_X(1) \to 0 \]
be the canonical sequence of \( \mathcal{O}_X \)-modules. If the sequence has \( \mu \)-reduction at \( t \) and the strong \( \mu \)-reduction at \( t_0 \) (which exist by Proposition 4.16) then we say \( V_t \) is a \( \mu \)-reduction bundle and \( V_{t_0} \) is a strong \( \mu \)-reduction bundle for \((R, I)\).

**Remark 4.11.** Note that, given a choice of a set of generators of \( I \), these bundles are unique, but need not be unique for the pair \((R, I)\).

The notion of \( \mu \)-reduction makes sense in any characteristic.

**Theorem 4.12.** For a given standard graded pair \((R, I)\), where \( R \) is a two dimensional domain over a perfect field of char \( p > 0 \), (following the Notations 4.10), we have
1. \( f_{R, I}(x) = f_{V_0, \mathcal{O}_X(1)}(x) - f_{M_0, \mathcal{O}_X(1)}(x). \)
2. Moreover
\[ c^I(m) = \alpha(R, I) := \text{Sup} \{ x \mid f_{R, I}(x) > 0 \} = 1 - a_{\min}(V_0)/d, \]
where \( V_0 \) is a strong \( \mu \)-reduction bundle for \((R, I)\) and \( d = \deg \, X \).

**Proof.** Assertion (1) follows from Lemma 4.9 and the discussion of the subsection 4.1. Assertion (2) follows from Corollary 3.9 and hence the theorem. \( \Box \)

**Remark 4.13.** With Notations 4.10
1. If \( I \) has a set of generators of the same degrees then \( V_0 \) is a \( \mu \)-reduction and a strong \( \mu \)-reduction bundle for \((R, I)\); Because then \( M_0 \) is semistable and hence \( \mu_{\min}(V_0) < \mu(M_0) \). Therefore, in this case
\[ \alpha(R, I) = c^I(m) = 1 - a_{\min}(V_0)/d. \]

In particular, if \( \mu(I) = s \) and the degree of the generators is \( d_s \) and \( d = \deg \, \mathcal{O}_X(1) \) then \( a_{\min}(V_0) \leq \mu(V_0) = -d(d_s)/s(s-1) \) and therefore,
\[ (d_s)s/(s-1) \leq \alpha(R, I) = c^I(m) \leq 2(d_s). \]
2. By Theorem 4.12, the existence of strong \( \mu \)-reduction sequence gives a graded ideal \( J \subseteq I \) such that \( J^* = J^s \).

If \( I \) itself is the minimal graded tight closure reduction for \( I \), i.e., \( \{I\} = \min \{J \subseteq I \mid J \text{ graded}, J^* = I^* \} \), then by choosing a minimal generating set \( \{h_1, \ldots, h_s\} \) in the short exact sequence 4.12, we can ensure that \( V_0 \) itself is a strong \( \mu \)-reduction bundle for \((R, I)\). However we do not know the existence of a minimal tight closure reduction for an ideal.
5. $F$-threshold $\alpha'(m)$ and $\alpha(R, I)$ in characteristic 0

Notations 5.1. Let $(R, I)$ be a standard graded pair, where $R$ is a two dimensional domain over an algebraically closed field $k$ of char 0. Let $I \subset R$ be generated by homogeneous elements $h_1, \ldots, h_\mu$ of degrees $d_1, \ldots, d_\mu$, respectively. Let $X = \text{Proj} S$ be the corresponding nonsingular projective curve of degree $d$, where $\pi : R \to S$ be the normalization of $R$.

For given $(R, I)$ we have a vector bundle $V$ and an associated canonical exact sequence of locally free sheaves of $\mathcal{O}_X$-modules, as in (4.3), (moreover the sequence is locally split exact).

$$0 \to V \to \oplus_{i=1}^{\mu} \mathcal{O}_X(1 - d_i) \to \mathcal{O}_X(1) \to 0,$$

where $M = \oplus_{i=1}^{\mu} \mathcal{O}_X(1 - d_i)$. Let the HN filtration of $M$ be

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M,$$

and $V_t = V \cap M_t$.

Then, by Proposition 4.6 (1), the sequence (5.1) has the $\mu$-reduction at $t$ for some $0 \leq t < l_1$. Moreover, we recall (Lemma 4.7) that the HN filtration of $V$ is

$$\cdots \subset W_{l+1} \subset W_l \subset V_{l-1} \subset \cdots \subset V_0 = V,$$

where (1) $W_t = V_t$ if $\mu_{\min}(V_t) > \mu_{\min}(V_{t-1})$ and (2) $W_t \subset V_t$, if $\mu_{\min}(V_t) = \mu_{\min}(V_{t-1})$.

One can choose a finitely generated $\mathbb{Z}$-algebra $A \subset k$ such that there exist spreads $(R_A, I_A, A)$ and $(X_A, V_A, A)$.

Restricting to the fiber $X_s$, where $s \in \text{Spec } A$ is a closed point, we have the following exact sequence of locally free sheaves of $\mathcal{O}_{X_s}$-modules, where $X_s = X_A \otimes_A \overline{k(s)}$ and $V^s = V_A \otimes_A \overline{k(s)}$.

$$0 \to V^s \to \oplus_{i=1}^{\mu} \mathcal{O}_{X_s}(1 - d_i) \to \mathcal{O}_{X_s}(1) \to 0,$$

As a consequence of the openness of the semistability property of sheaves (see Remark 7.6), we can further choose a spread $A$ such that for $s \in \text{Spec } A$, the sequence

$$0 \to V^s_0 = V^s \to M^s_0 = M^s \to \mathcal{O}_{X_s}(1) \to 0$$

has the $\mu$-reduction at the same integer $t$. Moreover the HN filtration of $V^s$ is

$$\cdots \subset W^s_{l+1} \subset W^s_l \subset V^s_{l-1} \subset \cdots \subset V^s_0 = V^s.$$

We will use the following lemma to define $\alpha(R, I)$ in char 0.

Lemma 5.2. We have a spread $A$ such that

1. $\mu_{\min}(V^s_{l-1}) < \mu_{\min}(V^s_l) \implies \alpha(R_s, I_s) = 1 - a_{\min}(V^s_t)/d, \forall s \in \text{Spec } (A)$,
2. $\mu_{\min}(V^s_{l-1}) = \mu_{\min}(V^s_l) \implies \alpha(R_s, I_s) = 1 - a_{\min}(V^s_{l-1})/d, \forall s \in \text{Spec } (A)$.

Proof. We can choose a spread $A$ which satisfies the property as in Notations 5.1 and also char $R_s > 4(\text{genus } X)(\text{rank } V)^3$. Then, by Lemma 1.8 of [T1], for every $m \geq 1$, the HN filtration of the $m^{th}$ Frobenius pull back is a refinement of the $m^{th}$ Frobenius pull back of the HN filtration of $V^s$. Recall that $W_l \subset V_l \subset V^s_{l-1}$.

We fix $s \in \text{Spec } A$ and let $m_1$ (this may depend on $s$) such that $F^{m_1}V^s_l$ has strong HN filtration. Let the sequence (5.3) has the strong $\mu$-reduction at $t_0$, i.e., $t_0 \leq t$ is the integer where the sequence

$$0 \to F^{m_1}V^s_0 \to F^{m_1}M^s_0 \to F^{m_1}\mathcal{O}_{X_s}(1) \to 0$$

has the $\mu$-reduction at $t_0$. 
1. If \( \mu_{\min}(V_{t-1}) < \mu_{\min}(V_t) \). Then
\[
\cdots \subset F^{m_1}V^{t}_{i+1} \subset \cdots \subset F^{m_1}(V^s_{i}) \subset F^{m_1}(V^s_{t-1}) \subset \cdots \subset F^{m_1}(V^s_{0}),
\]
is the HN filtration for \( F^{m_1}(V^s_{0}) \) as \( V^s_{i}/V^s_{i+1} \approx M^s_i/M^s_{i+1} \) is strongly semistable for \( i < t \). Hence
\[
\mu_{\min}(F^{m_1}V^s_{i}) = \mu_{\min}(F^{m_1}M^s_{i}), \text{ for all } i < t.
\]
In particular, by Proposition 4.6 (2), we have \( t_0 = t \). Therefore \( a_{\min}(V^s_{0}) = a_{\min}(V^s_{t}) \).

Now Assertion (1) follows by Theorem 4.12.

2. If \( \mu_{\min}(V_{t-1}) = \mu_{\min}(V_t) \). Then the HN filtration for \( F^{m_1}(V^s_{0}) \) is
\[
F^{m_1}(W^s_t) \subset \cdots \subset F^{m_1}(V^s_{t-1}) \subset F^{m_1}(V^s_{t-2}) \subset \cdots \subset F^{m_1}(V^s_{0}).
\]
Therefore, we have
\[
\mu_{\min}(F^{m_1}V^s_{t}) = \mu_{\min}(F^{m_1}M^s_{t}), \text{ for all } i < t - 1.
\]
Hence, \( t_0 = t-1 \) or \( t_0 = t \). The second assertion of the lemma follows, by Theorem 4.12 if \( t_0 = t-1 \).

Hence, we can assume \( t_0 = t \), and therefore \( \mu_{\min}(F^{m_1}V^s_{t-1}) = \mu_{\min}(F^{m_1}M^s_{t-1}) \).

By a choice of \( \text{Spec } A \), we have \( \mu_{\min}(V^s_{t}) = \mu_{\min}(V^s_{t-1}) \).

Now, by Proposition 4.7, it is enough to prove that \( \mu_{\min}(F^{m_1}V^s_{t}) \leq \mu_{\min}(F^{m_1}V^s_{t-1}) \), which follows as
\[
\mu_{\min}(F^{m_1}V^s_{t-1}) = \mu_{\min}(F^{m_1}M^s_{t-1}) = p^m_{\min}(V^s_{t-1}) = p^m_{\min}(V^s_{t}) \geq \mu_{\min}(F^{m_1}V^s_{t}).
\]

Hence \( a_{\min}(V^s_{t}) = a_{\min}(V^s_{t-1}) \), which implies \( \alpha(R_s, I_s) = 1 - a_{\min}(V^s_{t})/d = 1 - a_{\min}(V^s_{t-1})/d \). This proves Assertion (2) of the lemma.

\[\square\]

**Remark 5.3.** For a standard graded pair \((R, I)\), where \( R \) is a two-dimensional domain, and for a vector bundle \( \tilde{E} \) on \( X = \text{Proj } S \), where \( S \) is the normalization of \( R \), let \((A, R_A, S_A, IS_A)\) and \((A, X_A, \tilde{E}_A)\) be a spread as given above. Then (see proof of Theorem 4.6 in [T6])
\[
f_{E, \mathcal{O}_X(1)}^{\infty} := \lim_{p_s \to \infty} f_{E^s, \mathcal{O}_{X^s}(1)}
\]
exists, where \( s \in \text{Spec}(A) \) is closed point and \( p_s = \text{char } R_s \). Moreover, if
\[
0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = \tilde{E}
\]
is the HN filtration of \( \tilde{E} \) such that \( \mu_i = \mu(E_i/E_{i-1}) \) and \( r_i = \text{rank}(E_i/E_{i-1}) \) then
\[
\begin{align*}
1 \leq x < 1 - \mu_i/d & \quad \Rightarrow \quad f_{E, \mathcal{O}_X(1)}^{\infty}(x) = -\left[ \sum_{i \geq 1} \mu_i r_i + d(x-1)r_i \right] \\
1 - \mu_i/d \leq x < 1 - \mu_{i+1}/d & \quad \Rightarrow \quad f_{E, \mathcal{O}_X(1)}^{\infty}(x) = -\left[ \sum_{k \geq i+1} \mu_k r_k + d(x-1)r_k \right].
\end{align*}
\]

**Remark 5.4.** Note that the uniform convergence of a sequence \( \{g_s\}_s \), of compactly supported continuous functions, to a function \( g \) does not imply \( \lim_{s \to \infty} \sup \{x \mid g_s(x) \neq 0\} = \sup \{x \mid g(x) \neq 0\} \).

For example, let \( \{g_p : \mathbb{R}_0^+ \to \mathbb{R}_0^+\}_p \) given by \( g_p(x) = x/p^2 \), for \( x \in [0,1] \) and \( g_p(x) = (2 - x)/p^2 \), for \( x \in [1,2] \). Then \( \lim_{p \to \infty} g_p \equiv 0 \) and \( \lim_{p \to \infty} \sup \{x \mid g_p(x) \neq 0\} = 2 \).

However, in Theorem 5.5, we show that the \( \lim_{p \to \infty} \alpha(R_s, I_s) \) exists and is equal to \( \sup \{x \mid \lim_{p \to \infty} f_{R_s, I_s}(x) \neq 0\} \). Note that if \( I \) has generators of degree 1, then the assertion, follows by Theorem 4.6 of [T6].
**Theorem 5.5.** Let \((R, I)\) be a standard graded pair, where \(R\) is a two dimensional domain defined over a field of char 0. For the pair \((R, I)\), let \((A, R_A, I_A)\) and \((A, X_A, V_A)\) be associated spreads, as in Notations 5.1. Let \(d\) be the degree of the curve \(X\) and \(t\) be the integer where the sequence \((5.1)\) has \(\mu\)-reduction (see Definition 4.2). Then, for closed points \(s \in \text{Spec}(A)\), we have

1. \(f_{R,I}^\infty := \lim_{p_s \to \infty} f_{R_s,I_s} \) exists and
   \[
   \alpha^\infty(R, I) := \text{Sup} \{ x \mid f_{R,I}^\infty(x) \neq 0 \} = 1 - \frac{\mu_{\min}(V_t)}{d}.
   \]
2. \(\lim_{p_s \to \infty} \alpha(R_s, I_s) = \alpha^\infty(R, I)\).
3. In particular \(c^d(R, I) = c^d(I) = 1 - \frac{\mu_{\min}(V_t)}{d}\).

**Proof.** Let \(M_0 = M = \oplus_i \mathcal{O}_X(1 - d_i) \) and \(V_0 = V\).

1. We know that \(f_{R_s,I_s}(x) = f_{V_s, \mathcal{O}_X(1)}(x) - f_{M_s, \mathcal{O}_X(1)}(x)\). Hence
   \[
   \lim_{p_s \to \infty} f_{R_s,I_s}(x) = f_{V_s, \mathcal{O}_X(1)}(x) - f_{M_s, \mathcal{O}_X(1)}(x) = f_{V_t, \mathcal{O}_X(1)}(x) - f_{M_t, \mathcal{O}_X(1)}(x),
   \]
   exists, where the last equality follows from Remark 5.3 and from the similar arguments as in Lemma 4.3. Also Remark 5.3 implies that \(\text{Sup} \{ x \mid f_{R,I}^\infty(x) \neq 0 \} = 1 - \mu_{\min}(V_t)/d\). This proves Assertion (1).

2. We recall that (Proposition 1.16 of [T1]), for any vector bundle \(W\) on \(X\) and a spread \((X_A, W_A, A)\), we have \(\lim_{p_s \to \infty} a_{\min}(W_s) = \mu_{\min}(W)\). Now, if \(\mu_{\min}(V_{t-1}) < \mu_{\min}(V_t)\) then, by Lemma 5.2
   \[
   \lim_{p_s \to \infty} \alpha(R_s, I_s) = \lim_{p_s \to \infty} a_{\min}(V_t^s)/d = 1 - \mu_{\min}(V_t)/d.
   \]
   If \(\mu_{\min}(V_{t-1}) = \mu_{\min}(V_t)\) then
   \[
   \lim_{p_s \to \infty} \alpha(R_s, I_s) = \lim_{p_s \to \infty} a_{\min}(V_{t-1}^s)/d = 1 - \mu_{\min}(V_{t-1})/d = 1 - \mu_{\min}(V_t)/d.
   \]
   This prove Assertion (2) of the theorem.

3. The last assertion follows by Theorem 4.12 and Assertion (2). This completes the proof of the theorem. 

**Proposition 5.6.** Let \((R, I)\) be a standard graded pair, where \(R\) is a domain of dimension 2 over a field of char 0 and let \(h_1, \ldots, h_{\mu}\) be homogeneous generators of \(I\). Let \(R_s, I_s\) and \(m_s\) denote a reduction mod \(p_s\) of \(R, I\) and \(m\) respectively, where char \(R_s = p_s\). Then

1. for \(p_s \gg 0\), we have \(c^d(m_s) \geq c^d(m)\).
2. If the associated syzygy bundle \(V_0\) (as in (5.1)) is semistable then
   \[
   (a) \quad c^d_m(m) = (\mu - 1)/\mu \quad \text{and}
   (b) \quad \text{for } p_s \gg 0,
   \]
   \[
   c^d_s(m_s) = c^d_s(m) \iff V_0^s \text{ is strongly semistable}.
   \]

**Proof.** (1) By Theorem 4.12 for every \(s \in \text{max spec}(A)\) we have \(\alpha(R_s, I_s) = c^d_s(m_s)\). If the sequence \((5.1)\) has \(\mu\)-reduction at \(t\). Then we know \(\mu_{\min}(V_t) \geq \mu_{\min}(V_{t-1})\). Now, if \(\mu_{\min}(V_t) > \mu_{\min}(V_{t-1})\). Then
   \[
   \alpha^\infty(R, I) = 1 - \frac{\mu_{\min}(V_t)/d}{c^d_m(m)} = 1 - \frac{\mu_{\min}(V_t^s)/d}{c^d_s(m_s)} \leq 1 - \frac{a_{\min}(V_t^s)/d}{c^d_s(m_s)} = \alpha(R_s, I_s).
   \]
If $\mu_{\text{min}}(V_t) = \mu_{\text{min}}(V_{t-1})$. Then

$$\alpha^\infty(R, I) = 1 - \frac{\mu_{\text{min}}(V_{t-1})}{d} = 1 - \frac{\mu_{\text{min}}(V^s_{t-1})}{d} \leq 1 - \frac{a_{\text{min}}(V^s_{t-1})}{d} = \alpha(R, I_s).$$

This proves Assertion (1).

(2) Suppose $V_0$ is semistable. Then we

**Claim.** The sequence (5.1) has $\mu$-reduction at $t = 0$ or $t = 1$. Moreover if the $\mu$-reduction is at $t = 1$ then $\mu_{\text{min}}(V_0) = \mu_{\text{min}}(V_1)$.

**Proof of the claim:** Since the sequence (5.1) has the (*) property, we have

$$\mu(V_0) = \mu_{\text{min}}(V_0) \leq \mu_{\text{min}}(M_0) = \mu(M_0/M_1).$$

(a) If $\mu(V_0) < \mu(M_0/M_1)$ then the sequence has the $\mu$-reduction at $t = 0$.

(b) If $\mu(V_0) = \mu(M_0/M_1)$. Then $M_1 \neq 0$, otherwise $\mu(V_0) = \mu(M_0) = \mu(L)$. Hence $V_1 \neq 0$, by Lemma 4.5. Therefore

$$\mu_{\text{min}}(V_1) \leq \mu(V_1) \leq \mu(V_0) < \mu(M_1/M_2) = \mu_{\text{min}}(M_1).$$

This implies the sequence (5.1) has reduction at $t = 1$. Moreover by Proposition 4.7 this implies $\mu_{\text{min}}(V_1) = \mu_{\text{min}}(V_0)$. This proves the claim.

Now, Theorem 5.3 implies $\alpha^\infty(R, I) = 1 - \mu(V_0)/d = (\mu - 1)/\mu$. This proves Assertion (2) (a) of the proposition. Now Lemma 5.2 implies $\alpha(R^s, I^s) = 1 - a_{\text{min}}(V^s_0)/d$. Therefore

$$\alpha^\infty(R, I) = \alpha(R^s, I^s) \iff \mu(V_0) = a_{\text{min}}(V^s_0) \iff V^s_0 \text{ is strongly semistable}.$$

This proves the proposition. \qed

**Remark 5.7.**

(1) If $I$ has a set of generators of same degree then, by Remark 4.13

$$c^I_\infty(m) = 1 - \frac{\mu_{\text{min}}(V_0)}{d} \quad \text{and, for } p_s >> 0, \quad c^I_s(m_s) = 1 - \frac{a_{\text{min}}(V_0)}{d}.$$  

(2) If $\deg \mathcal{O}_X(1) > 2(\text{genus } X)$ then, for $p_s >> 0$, we have

$$c^{m_s}(m_s) = c^{m}(m) \iff V_0 \text{ reduction mod } p_s \text{ is strongly semistable},$$

where the syzygy bundle $V_0$ is given by the short exact sequence

$$0 \to V_0 \to M_0 = H^0(X, \mathcal{O}_X(1)) \otimes_k \mathcal{O}_X \to \mathcal{O}_X(1) \to 0.$$

This follows by Proposition 5.3(2), as the bundle $V_0$ is semistable (see [PR] and Lemma 2.1 of [T2]). Note that $c^{m}(m) = h/(h - 1)$, where $h = h^0(X, \mathcal{O}_X(1))$.

(3) We will see in the next section that the set of primes $p_s$, where $c^{m_s}(m_s) = c^{m}(m)$, is a dense set whenever $X$ is an irreducible plane trinomial.

6. **F-thresholds of Plane Trinomials**

For a pair $(R, I)$, where $R$ is a two dimensional ring, the results of the previous section formulated the $F$-threshold $c^I(m)$ in terms of the strong HN slopes of the associated syzygy bundle $V_0$. In the case of plane trinomials, where $I_n = (x^n, y^n, z^n)$, we have a group theoretic interpretations of the strong HN data of $V_0$. Since, by Remark 5.7(1), the bundle $V_0$ itself is a strong $\mu$-reduction bundle for $(R, I_n)$, we can give an explicit formula for the $F$-threshold $c^I_n(m)$ as follows.

Let $R = k[x, y, z]/(h)$, where $h$ is an irreducible trinomial of degree $d \geq 3$. Note that such a plane curve is either ‘regular’ or ‘irregular’ (this terminology is taken from [M]), where a plane curve is called irregular if it has a singular point of multiplicity $r \geq d/2$, otherwise it is called regular.
Theorem 6.1. If \( R = k[x, y, z]/(h) \), where \( h \) is an irregular trinomial of degree \( d \). Let the irregular point of \( R \) be of multiplicity \( r \) (therefore \( r \geq d/2 \)). Then for \( m = (x, y, z) \) and \( I_n = (x^n, y^n, z^n) \), where \( n \geq 1 \), we have

\[
\alpha(R, I_n) = c^{I_n}(m) = \frac{n + 2}{2} + \left(\frac{(2r - d)n}{2d}\right)^2.
\]

For regular trinomial plane curves, we recall following from [M].

Notations 6.2. Given a regular trinomial \( h \) of degree \( d \) (upto linear change of variables, any such trinomial is of type I or type II, as given below), we can associate positive integers \( \alpha, \beta, \nu, \lambda > 0 \) as follows:

1. Type (I) \( h = x^{a_1}y^{a_2} + y^{b_1}z^{b_2} + z^{c_1}x^{c_2} \), we denote
   \[
   \alpha = a_1 + b_1 - d, \quad \beta = \alpha, \quad \nu = b_1 + c_1 - d, \quad \lambda = a_1b_1 + a_2c_2 - b_1c_2.
   \]

2. Type (II) \( h = x^d + x^{a_1}y^{a_2}z^{a_3} + y^b z^c \), we denote
   \[
   \alpha = a_2, \quad \beta = c, \quad \nu = a_2 + c - d \quad \text{and} \quad \lambda = a_2c - a_3b.
   \]

Moreover we denote

(6.1) \( t_h = (\alpha/\lambda, \beta/\lambda, \nu/\lambda) \), and \( a = \gcd(\alpha, \beta, \nu, \lambda) \) and \( \lambda_h = \lambda/a \).

Definition 6.3. For a given regular trinomial \( h \), we recall the following definition given in [HM] and [M]. Let \( L_{odd} = \{u = (u_1, u_2, u_3) \in Z^3 \mid \sum u_i \text{ odd}\} \). For any \( u \in L_{odd} \) and for \( l, s \in Z \) and \( n \geq 1 \), the taxicab distance

\[
Td(l^s t_h n, u) = Td((\frac{l^s \alpha n}{\lambda}, \frac{l^s \beta n}{\lambda}, \frac{l^s \nu n}{\lambda}), (u_1, u_2, u_3)) = |\frac{l^s \alpha n}{\lambda} - u_1| + |\frac{l^s \beta n}{\lambda} - u_2| + |\frac{l^s \nu n}{\lambda} - u_3|.
\]

We recall the following Theorem from [T5].

Theorem 6.4. For given regular trinomial \( h \in k[x, y, z] \) over a field of char \( p > 0 \) and given \( n \geq 1 \), there is a well defined set theoretic map (where \( t_h \) and \( \lambda_h \) are as in (6.1)),

\[
\Delta_{h,n} : \frac{(Z/2\lambda_h Z)^*}{\{1, -1\}} \to \left\{ \frac{1}{\lambda_h}, \frac{2}{\lambda_h}, \ldots, \frac{\lambda_h - 1}{\lambda_h} \right\} \times \{0, 1, \ldots, \phi(2\lambda_h) - 1\} \bigcup \{(1, \infty)\},
\]

given by \( l \to (T_l, D_l) \), where \( D_l = s \geq 0 \) is the smallest integer, for which \( Td(l^s t_h n, u) < 1 \) has a solution for some \( u \in L_{odd} \) and in that case \( T_l = Td(l^s t_h n, u) \). If there is no such \( s \) then \( \Delta_{h,n}(l) = (1, \infty) \). Moreover

1. \( \Delta_{h,n} \equiv \Delta_{h,n+2\lambda_h} \).

2. Either \( D_l = \infty \) or \( D_l < \text{ the order of the element } l \) in the group \( (Z/2\lambda_h Z)^* \).

Following result and explicit examples can be obtained easily from [T5].

Theorem 6.5. Let \( R = k[x, y, z]/(h) \) and \( m = (x, y, z) \) where \( h \) is a regular trinomial of degree \( d \) over a field of char \( p > 0 \). Let \( \Delta_{h,n} \) be the set theoretic map given as in Theorem 6.4. Then, for \( p \geq \max\{n, d^2\} \) and \( I_n = (x^n, y^n, z^n) \), where \( n \geq 1 \), we have the following:

1. \( p \equiv \pm 1 \pmod{2\lambda_h} \) then
   \[
   \alpha(R, I_n) = c^{I_n}(m) = \frac{n + 2}{2}.
   \]
Example 6.9. Let 

\[ R = \frac{k_p[x, y, z]}{(x^{d-1}y + y^{d-1}z + z^{d-1}x)}, \quad m = (x, y, z) \]

where \( k_p \) denotes a field of characteristic \( p \geq d^2 \). Then, again by Corollary 4.4 and Theorem 4.5 of \[T5\], where \( \lambda = (d^2 - 3d + 3) \), we have
(1) If \( p \equiv \pm 1 \pmod{\lambda} \) then
\[
\alpha(R, m) = c^m(m) = \frac{3}{2}.
\]

(2) If \( p \equiv \lambda \pm 2 \pmod{2\lambda} \) and
(a) if \( d \) even and \( d \geq 6 \) then
(i) for \( 3 \cdot 2^{m-2} \leq d - 1 < 2^m \), where \( m \geq 1 \), we have
\[
\alpha(R, m) = c^m(m) = \frac{3}{2} + \frac{2(d-2)(d-1-3 \cdot 2^{m-2}) + 2}{dp^m},
\]
(ii) for \( 2^m \leq d - 1 < 3 \cdot 2^{m-1} \), where \( m \geq 1 \), we have
\[
\alpha(R, m) = c^m(m) = \frac{3}{2} + \frac{(d-2)(3 \cdot 2^{m-1} - (d-1)) - 1}{dp^m}.
\]
(b) For \( d \) odd and \( d \geq 7 \),
\[
\alpha(R, m) = c^m(m) = \frac{3}{2} + \frac{\lambda - 6(d-2)}{2dp},
\]
(c) for \( d = 5 \),
\[
\alpha(R, m) = c^m(m) = \frac{3}{2} + \frac{7}{2dp^3}.
\]

7. APPENDIX

Definition 7.1. A vector bundle \( V \) on a nonsingular curve over a field \( k \) is semistable (stable) if for every proper subbundle \( W \subset V \) we have \( \mu(W) \leq \mu(V) \) (\( \mu(W) < \mu(V) \)), where \( \mu(V) = \deg(V) / \text{rank}(V) \).

Definition 7.2. Every vector bundle \( V \) on a nonsingular projective curve \( X \) has the unique filtration of subbundles
\[
0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = V,
\]
called the Harder-Narasimhan filtration (HN filtration) of \( V \), satisfying the following conditions
(1) for every \( i \), the bundles \( E_i/E_{i-1} \) is semistable and
(2) \( \mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E_{l+1}/E_l) \).

Suppose \( \text{char} k = p > 0 \). If for every \( i \), the bundles \( E_i/E_{i-1} \) is strongly semistable (i.e., for every \( m^{th} \) iterated Frobenius \( F^m : X \to X \) the bundle \( F^{m*}(E_i/E_{i-1}) \) is semistable) then we call the filtration (7.1), a strong Harder-Narasimhan filtration (or strong HN filtration).

Notations 7.3. Let \( V \) be a vector bundle on a nonsingular projective curve \( X \).
(1) If \( V \) has the HN filtration as in (7.1) then we define \( \mu_i(V) = \mu(E_i/E_{i-1}) \), the HN slopes of \( V \) and denote \( \mu_{\text{min}}(V) = \mu(V/E_l) \). We call the set
\[
\{\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(V/E_l)\}, \{\text{rank}(E_1), \ldots, \text{rank}(V/E_l)\}
\]
the HN data for \( V \).
(2) By Theorem 2.7 of [L], if \( \text{char} k > 0 \), then for a given bundle \( V \), there exists \( m > 0 \) such that the HN filtration of \( F^{m*}V \) is the strong HN filtration
\[
0 = F_0 \subset F_1 \subset \cdots \subset F_l \subset F_{l+1} = F^{m*}V.
\]
We define $a_i(V) = (1/p^m)\mu(F_i/F_{i-1})$ the strong HN slopes of $V$ and denote $a_{\text{min}}(V) = (1/p^m)\mu(F^{\text{ms}*}V/F_i)$. We call the set

$$\{\{a_1(V), \ldots, a_{i+1}(V)\}, \{\text{rank } F_1, \ldots, \text{rank}(F_{i+1}/F_i)\}\}$$

the strong HN data for $V$. The notion of the strong HN slopes and the strong HN data, for a bundle $V$, are well defined because if $V$ has a strong HN filtration, then for every $n \geq 1$, the $n^{th}$ Frobenius pull back of the HN filtration of $V$ is the strong HN filtration for $F^{n*}V$.

**Remark 7.4.** If $\tilde{E}$ is a semistable vector-bundle on $X$ with $\mu(\tilde{E}) = \mu$ and $\text{rank}(\tilde{E}) = r$ then

$$m < -\mu/d \implies h^1(X, \tilde{E}(m)) = -r(\mu + dn + (g-1))$$

$$-\mu/d \leq m \leq -\mu/d + (d-3) \implies h^1(X, \tilde{E}(m)) = C$$

$$-\mu/d + (d-3) < m \implies h^1(X, \tilde{E}(m)) = 0,$$

where $|C| \leq r(\text{genus}(X) - 1)$.

**Remark 7.5.** Let $V$ and its HN filtration be given as in Definition 7.2

1. If $V$ has a nontrivial HN filtration (i.e., $V$ is not semistable) then $\mu(V) > \mu_{\text{min}}(V)$.
2. If $V \to W$ is a nonzero surjective map of bundles then $\mu(W) \geq \mu_{\text{min}}(V)$: To show this, without loss of generality we can assume that $W$ is semistable. Now we may choose $i \geq 1$ (for $E_i$ as in (7.1)) such that there is a nonzero induced map $E_i/E_{i+1} \to W$.

In particular, for $m \geq 1$, we always have $p^m(\mu_{\text{min}}(V)) \geq \mu_{\text{min}}(F^{m*}(V))$.
3. If $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of nonzero vector bundles on $X$, then either
   (a) $\mu(V') \leq \mu(V) \leq \mu(V'')$ or
   (b) $\mu(V') \geq \mu(V) \geq \mu(V'')$.

4. Further, if the slopes of any two of the bundles are same then the slopes of all the three are same. Moreover, in that case semistability of any two bundles implies the semistability of the third one.
4. If $V' \subset V$ and $W \subset V$ are nonzero bundles such that $\mu_{\text{min}}(W) > \mu(V/V')$ and $V/V'$ is semistable, then $W \subset V'$. Otherwise, the existence of the canonical nonzero map $W \to V/V'$ would imply $\mu_{\text{min}}(W) \leq \mu(V/V')$.
5. It is easy to check the following assertion: If $0 \to W \to V \to V' \to 0$ is a short exact sequence of nonzero vector bundles on $X$ such that $V'$ is semistable and if $\cdots \subset F_{i-1} \subset F_i \subset V$ denotes the HN filtration of $V$. Then
   (a) $\mu_{\text{min}}(W) = \mu(V')$ implies $F_i \subset W \subset V$ and
   (b) $\mu_{\text{min}}(W) > \mu(V')$ implies $F_i = W \subset V$.

**Remark 7.6.** Given a a nonsingular projective curve $X$ over a field of characteristic 0 and a vector bundle $\tilde{E}$ on $X$, there exists a finitely generated $\mathbb{Z}$-algebra $A$ and a projective scheme $X_A$ over $A$ and a locally free sheaf of $\mathcal{O}_{X_A}$-modules $\tilde{E}_A$ such that if $0 = E_0 \subset E_1 \subset \cdots \subset E_l = \tilde{E}$ is the HN filtration of $\tilde{E}$ then there is a spread $(E_{lA}, A)$ of $E_i$ such that there is a filtration $0 = E_{0A} \subset E_{1A} \subset \cdots \subset E_{lA} \subset E_{(l+1)A} = \tilde{E}_A$, of locally free sheaves of $\mathcal{O}_{X_A}$-modules with the property that $0 = E_{0s} \subset E_{1s} \subset \cdots \subset E_{ls} \subset E_{(l+1)s} = \tilde{E}_s$.
is the Harder-Narasimhan filtration of the vector bundle $\tilde{E}_s$ over $X_s$ for $s \in \text{Spec } A$, (where, for a sheaf $E_A$ of $O_{X_A}$-modules, the sheaf $E_s = E_A \otimes_A k(s)$ denotes the sheaf over the nonsingular projective curve $X_s = X_A \otimes_A k(s)$).

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