Eccentricity in Trees

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Abstract

The eccentricity of a vertex, \( \text{ecc}(v) = \max_{u \in T} d_T(v, u) \), was one of the first, distance-based, tree invariants studied [Jordan, J. Reine Angew. Math., 70(1869), 185-190]. The total eccentricity of a tree, \( \text{Ecc}(T) \), is the sum of eccentricities of its vertices. We determine extremal values and characterize extremal tree structures for the ratios \( \frac{\text{Ecc}(T)}{\text{ecc}(u)} \), \( \frac{\text{Ecc}(T)}{\text{ecc}(v)} \), \( \frac{\text{ecc}(u)}{\text{ecc}(v)} \), and \( \frac{\text{ecc}(u)}{\text{ecc}(w)} \) where \( u, w \) are leaves of \( T \) and \( v \) is in the center of \( T \). Analogous problems have been resolved for other tree invariants including distance [Barefoot, Entringer, Székely, Discrete Appl. Math., 80(1997), 37-56] and number of subtrees [Székely, Wang, Electron. J. Combin., 20(1)(2013), P67]. In addition, we determine the tree structures that minimize and maximize total eccentricity among trees with a given degree sequence.

Keywords: eccentricity; extremal problems; degree sequence; greedy caterpillar; greedy tree; level-greedy tree

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1 Introduction

The eccentricity of a vertex $v$ in a connected graph $G$ is defined in terms of the distance function as

$$\text{ecc}_G(v) := \max_{u \in V(G)} d(u, v).$$

The radius of $G$, $\text{rad}(G)$, is the minimum eccentricity while the diameter, $\text{diam}(G)$, is the maximum. The center, $C(G)$, is the collection of vertices whose eccentricity is exactly $\text{rad}(G)$.

We focus our attention on trees where the center has at most two vertices [6] and the diameter is realized by a leaf. In addition, we explore the total eccentricity of a tree $T$, defined as the sum of the vertex eccentricities:

$$\text{Ecc}(T) := \sum_{v \in V(T)} \text{ecc}_T(v).$$

For a fixed tree $T$ with $v \in C(T)$ and any $z \in V(T)$,

$$\min_{u \in L(T)} \frac{\text{Ecc}(T)}{\text{ecc}_T(u)} \leq \frac{\text{Ecc}(T)}{\text{ecc}_T(z)} \leq \frac{\text{Ecc}(T)}{\text{ecc}_T(v)},$$

where $L(T)$ denotes the set of leaves. This motivates the study in Section 2 of the extremal values and structures for the following ratios where $u, w \in L(T)$ and $v \in C(T)$,

$$\frac{\text{Ecc}(T)}{\text{ecc}_T(v)}, \quad \frac{\text{Ecc}(T)}{\text{ecc}_T(u)}, \quad \frac{\text{ecc}_T(u)}{\text{ecc}_T(v)}, \quad \text{and} \quad \frac{\text{ecc}_T(u)}{\text{ecc}_T(w)}.$$

The results are analogous to similar studies made for distance in [2] and for the number of subtrees in [9, 10]. As in the cited papers, the behavior of ratios is more delicate than that of their numerators or denominators.

Returning our attention to the total eccentricity of a graph, Dankelmann, Goddard, and Swart [5] showed that the path maximizes $\text{Ecc}(T)$ among trees with a given order. Their paper also provides many other upper bounds on the average eccentricity of a graph. In Section 3, we give a short proof to show that the star minimizes $\text{Ecc}(T)$ among trees with a given order. Turning our attention to trees with a fixed degree sequence, we discover that the greedy caterpillar maximizes $\text{Ecc}(T)$ while the greedy tree minimizes $\text{Ecc}(T)$. This provides further information about the total eccentricity of greedy trees across degree sequences.

From here forward, we assume that $T$ is a tree and $|V(T)| = n$. Given two vertices $a, b \in V(T)$, $P(a, b)$ will be the unique path between $a$ and $b$.

2 Extremal ratios

In this section, we consistently use the letters $u, w$ to denote leaf vertices while $v$ is a center vertex. Before delving into ratios, the following simple observation from [6] is given without proof, and will be used many times.
Observation 1. The center, $C(T)$, contains at most 2 vertices. These vertices are located in the middle of a maximum length path, $P$. If $\{v\} = C(T)$, $v$ divides $P$ into two paths, each of length $\text{rad}(T)$. If $\{v, z\} = C(T)$, the removal of $vz \in E(T)$ will divide $P$ into two paths, each of length $\text{rad}(T) - 1$.

2.1 On the extremal values of $\frac{\text{Ecc}(T)}{\text{ecc}_T(v)}$ where $v \in C(T)$

Theorem 2. For any tree $T$ on $n \geq 2$ vertices, and $v \in C(T)$, we have

$$\frac{\text{Ecc}(T)}{\text{ecc}_T(v)} \leq 2n - 1. \quad (1)$$

Consequently, the bound holds for all $v \in V(T)$. For $n \geq 3$, equality holds in (1) if and only if $T$ is a star centered at $v$.

Proof. For each vertex $z \in V(T)$, there is a leaf $w_z \in L(T)$ with $\text{ecc}_T(z) = d(z, w_z)$. For any $y, z \in V(T)$,

$$\text{ecc}_T(y) \geq \max\{d(z, y), d(y, w_z)\} \geq \frac{1}{2}d(z, w_z) = \frac{1}{2}\text{ecc}_T(z). \quad (2)$$

Applying (2) with $v$ written to the place of $y$,

$$\text{Ecc}(T) = \text{ecc}_T(v) + \sum_{z: z \neq v} \text{ecc}_T(z) \leq \text{ecc}_T(v) + (n - 1)2\text{ecc}_T(v) \overset{(3)}{=} (2n - 1)\text{ecc}_T(v).$$

In order for (3) to hold with equality, $\text{ecc}_T(v) = \frac{1}{2}\text{ecc}_T(z)$ for all $z$ different from $v \in C(T)$. Hence $\{v\} = C(T)$, specifically for $n \neq 2$. Further, the definition dictates that the eccentricities of adjacent vertices differ by at most 1. For any neighbor $z$ of $v$, we have

$$\text{ecc}_T(v) + 1 \geq \text{ecc}_T(z) = 2\text{ecc}_T(v),$$

and hence $\text{ecc}_T(v) \leq 1$ which dictates the star.

Theorem 3. For any tree $T$ on $n \geq 2$ vertices, there are natural numbers $k, i$, $0 \leq i \leq 2k$, such that $n = k^2 + i$. For any $v \in C(T)$, we have

$$\frac{\text{Ecc}(T)}{\text{ecc}_T(v)} \geq \begin{cases} n - 3 + 2k + \frac{i}{k} & \text{if } 0 \leq i \leq k \\ n - 3 + 2k + \frac{i+1}{k+1} & \text{if } k + 1 \leq i \leq 2k. \end{cases} \quad (4)$$

For $n \geq 4$, equality holds in (4) if and only if $T$ is the tree in Fig. 1, $x = k$ in the first case and $x = k + 1$ in the second case. For $i = k$, the two bounds agree and both values for $x$ provide an extremal tree.
Proof. One can verify by hand that the bounds hold for trees on 2 or 3 vertices. Now when \( n \geq 4 \) and \( T \) is a star, \( \frac{\text{Ecc}(T)}{\text{ecc}_T(v)} = 2n - 1 \). In particular, \( k \geq 2 \) and thus \( 2k \leq k^2 + i = n \) which yields the following comparisons with the bounds in (4). (Note that \( i \leq k \) in the first and \( i \leq 2k \) in the second.)

\[
n - 3 + 2k + \frac{i}{k} \leq n - 3 + 2k + 1 = n + 2k - 2 < 2n - 1.
\]

\[
n - 3 + 2k + \frac{i + 1}{k + 1} < n - 3 + 2k + 2 = n + 2k - 1 \leq 2n - 1.
\]

In particular, the bounds in (4) are not tight when \( T \) is a path.

Let \( T \) be a tree which is not a star. Fix \( v \in C(T) \). By Observation 1, there is a longest path \( P(u, w) \) with \( v \) in the middle. Without loss of generality, \( \text{ecc}_T(v) \) is realized by the path \( P(v, u) \).

If \( C(T) = \{v\} \), then \( P(v, w) \) also has length \( \text{ecc}_T(v) \). Because \( T \) is not a star, we can create a new tree \( F \) from \( T \) by detaching leaf \( u \) and joining it to vertex \( v \). Clearly \( \text{Ecc}(T) > \text{Ecc}(F) \), while \( \text{ecc}_F(v) = \text{ecc}_T(v) \) because \( P(v, w) \) remained untouched. If \( \{v\} = C(F) \), then we can repeat this process on another longest path containing \( v \) until there are two center vertices. (Each iteration decreases the number of leaves \( u \) such that \( \text{ecc}_T(v) = d(v, u) \). Once there is only one leaf with this property, the neighbor of \( v \) on \( P(v, u) \) will also be in the center.) Hence in minimizing \( \frac{\text{Ecc}(T)}{\text{ecc}_T(v)} \), it suffices to consider the trees with two center vertices.

For any \( T \) with \( |C(T)| = 2 \), select \( v \in C(T) \) and let \( x := \text{ecc}_T(v) \). The longest path \( P(w, u) \) guaranteed by Observation 1 has length \( 2x - 1 \). The vertices on \( P(w, u) \) realize their eccentricities along this path since it has maximum length. So we can precisely calculate the sum of their eccentricities as

\[
2(x + (x + 1) + \ldots + (2x - 1)) = 3x^2 - x.
\]

For vertices not on \( P(w, u) \), the eccentricity is at least \( x + 1 \). Hence

\[
\text{Ecc}(T) \geq 3x^2 - x + (n - 2x)(x + 1),
\]

yielding

\[
\frac{\text{Ecc}(T)}{\text{ecc}_T(v)} \geq \frac{x^2 + (n - 3)x + n}{x} = x + (n - 3) + \frac{n}{x} =: f(x).
\]

Equality holds in (5) if and only if all vertices not on \( P(w, u) \) are neighbors of one of the center vertices as in Fig. 1.

It remains to determine which value of \( x \), which is a function of \( T \), minimizes \( f(x) \). For this, we use the first derivative test. Noting that

\[
f'(x) = 1 - \frac{n}{x^2}
\]
is negative for $x < \sqrt{n}$ and positive for $x > \sqrt{n}$, the minimum of $f(x)$ is obtained when

$$x \in \{ \lfloor \sqrt{n} \rfloor, \lceil \sqrt{n} \rceil \} \subseteq \{k, k+1\}.$$ 

By the following computations, we see $f(k) \leq f(k+1)$ precisely when $i \geq k$, as stated in (4).

$$f(k + 1) - f(k) = \frac{n}{k+1} + 1 - \frac{n}{k} = \frac{nk + k(k+1) - n(k+1)}{n(k+1)} = \frac{k^2 + k - n}{k(k+1)} = \frac{k^2 + k - k^2 - i}{k(k+1)} = \frac{k - i}{k(k+1)}.$$ 

Hence $f(k) = f(k + 1)$ precisely when $i = k$.

The eccentricity of the center vertices always dictates a maximum length path. However, equality in (4) requires that each vertex not on this maximum length path has minimum eccentricity, exactly $\text{ecc}_T(v) + 1$. This is realized when they are all adjacent to one of the center vertices. The result is the tree in Fig. 1.

2.2 On the extremal values of $\frac{\text{Ecc}(T)}{\text{ecc}_T(u)}$ where $u \in L(T)$

**Theorem 4.** For any tree $T$ with $n \geq 8$ vertices, let $k, i$ be natural numbers, $0 \leq i \leq 2k$, so that $2n - 1 = k^2 + i$. Then for any $u \in L(T)$, we have

$$\frac{\text{Ecc}(T)}{\text{ecc}_T(u)} \leq \begin{cases} 2n + 1 - 2k - \frac{i}{k} & \text{if } 0 \leq i \leq k \\ 2n + 1 - 2k - \frac{i+1}{k+1} & \text{if } k + 1 \leq i \leq 2k. \end{cases} \tag{6}$$

Equality holds in (6) if and only if $T$ is the tree in Fig. 2 with $x = k$ in the first case and $x = k + 1$ in the second. For $i = k$, the two bounds agree and both values of $x$ will provide an extremal tree.
Proof. First consider the value of $\frac{Ecc(T)}{ecc_T(u)}$ when $T$ is a path on $n$ vertices. There are two cases to consider. If $n$ is odd,

$$\frac{Ecc(T)}{ecc_T(u)} = \frac{2 \left( (n-1) + \ldots + \frac{n+1}{2} \right) + \frac{n-1}{2}}{n-1} = \frac{n(n-1) - \frac{1}{4}(n+1)(n-1) + \frac{1}{2}(n-1)}{n-1} = n - \frac{1}{4}(n+1) + \frac{1}{2} = \frac{3}{4}n + \frac{1}{2}.$$ 

On the other hand, if $n$ is even,

$$\frac{Ecc(T)}{ecc_T(u)} = \frac{2 \left( (n-1) + \ldots + \frac{n}{2} \right)}{n-1} = \frac{n(n-1) - \frac{1}{4}n(n-2)}{n-1} = \frac{n(n-2)}{4n} = \frac{3}{4}n + \frac{1}{2}.$$ 

Therefore, for $T$ a path on $n \geq 9$ vertices, $k \geq 4$ because $2n - 1 = k^2 + i$ and

$$\frac{Ecc(T)}{ecc_T(u)} < \frac{3}{4}n + \frac{1}{2} = \frac{3}{4}n + \frac{1}{2} + \frac{5}{4}n - \frac{5}{4}n = 2n + \frac{1}{2} - \frac{5}{8}(2n - 1) - \frac{5}{8} < 2n - \frac{5}{8}(2n - 1) \leq 2n - \frac{5}{8}k^2 \leq 2n - (2k + 1).$$

This strict upper bound can also be verified by hand for $n = 8$. We have shown that when $T$ is a path on $n \geq 8$ vertices, $\frac{Ecc(T)}{ecc_T(u)} < 2n - 2k - 1$. Now
compare this with the bounds in (6) where \( i \leq k \) in the first and \( i \leq 2k \) in the second.

\[
2n + 1 - 2k - \frac{i}{k} \geq 2n + 1 - 2k - 1 = 2n - 2k > 2n - 2k - 1.
\]

\[
2n + 1 - 2k - \frac{i + 1}{k + 1} > 2n + 1 - 2k - 2 = 2n - 2k - 1.
\]

Therefore, the bounds in (6) will not be tight if \( T \) is a path.

However, when \( T \) is not a path, let \( P(w, w') \) be a longest path in \( T \) and \( u \in L(T) \) be different from \( w \) and \( w' \). Then \( \frac{Ecc(T)}{ecc_T(u)} \leq \frac{Ecc(T)}{ecc_T(u)} \). Hence, it suffices to consider leaves not on the longest path in evaluating the upper bound of \( \frac{Ecc(T)}{ecc_T(u)} \). We proceed with a hypothetical case analysis on the parity of the length of \( P(w, w') \). Either this longest path has length \( 2(x - 1) \) or \( 2x - 3 \), i.e. even or odd. In either case, \( ecc_T(u) \geq x \) and

\[
\frac{Ecc(T)}{ecc_T(u)} \leq 1 + \frac{1}{x} \left( \sum_{v \in V(P(w, w'))} ecc_T(v) + \sum_{v \notin u \in V(P(w, w'))} ecc_T(v) \right).
\]

When longest path \( P(w, w') \) has length \( 2(x - 1) \), we explicitly calculate the eccentricities of vertices on the path and bound the eccentricities of all vertices not on this path by \( 2(x - 1) \). Hence,

\[
\frac{Ecc(T)}{ecc_T(u)} \leq 1 + \frac{1}{x} \left( [(x - 1) + 2(x + \ldots + 2(x - 1))] + (n - 2x)(2(x - 1)) \right) = \frac{-x^2 + (2n + 1)x - (2n - 1)}{x}.
\]

On the other hand, when \( P(w, w') \) has length \( 2x - 3 \),

\[
\frac{Ecc(T)}{ecc_T(u)} \leq 1 + \frac{1}{x} \left( 2 [(x - 1) + x + \ldots + (2x - 3)] + (n - 2x + 1)(2x - 3) \right) = \frac{-x^2 + (2n + 2)x - (3n - 1)}{x}.
\]

It is easy to see that the bound obtained in the even case is, pointwise in \( x \), the larger of the two because \( x < n \). Further, equality can be achieved in this case exactly when \( u \) is adjacent to the middle vertex of \( P(w, w') \) and all other vertices have eccentricity \( 2(x - 1) \). This is exactly the tree structure described in Fig. 2. Thus it suffices to consider only those trees with an even length longest path.

It remains to determine the value of \( x \) that will maximize \( \frac{Ecc(T)}{ecc_T(u)} \) for trees with the structure in Fig. 2. The first derivative test applied to

\[
f(x) := \frac{-x^2 + (2n + 1)x - (2n - 1)}{x}
\]
Figure 2: A maximizing tree with respect to $\frac{Ecc(T)}{ecc_T(u)}$. 

shows that $f(x)$ is maximized when 

$$x \in \{\lfloor \sqrt{2n-1} \rfloor, \lceil \sqrt{2n-1} \rceil \} \subseteq \{k, k+1\}.$$ 

The larger of $f(k)$ and $f(k+1)$ gives the appropriate upper bound in (6). However, we must require $2x \leq n$ in order to have a realizable tree. One can individually check that this is the case for $n \in \{8, 9, \ldots, 12\}$. When $n \geq 13$, we have $k \geq 5$ which implies 

$$0 \leq k^2 - 4k - 3 \leq k^2 + i + 1 - 4(k + 1)$$

$$2x \leq 2(k + 1) \leq \frac{1}{2}(k^2 + i + 1) = n$$

If equality holds in (6), then (7) must be equality for the appropriate value of $x$. Because all values in (7) are maximum, the bound is achieved exactly when $T$ has the structure shown in Fig. 2 with $x = k$ or $x = k + 1$ according to the value of $n$. \hfill \Box

**Theorem 5.** Let $T$ be a tree with $n \geq 5$ vertices and leaf $u \in L(T)$. There are natural numbers $k, i$, $0 \leq i \leq 2k$, so that $4n - 4 = k^2 + i$. Then 

$$\frac{Ecc(T)}{ecc_T(u)} \geq \begin{dcases} 
\frac{n-1}{2} + \frac{k}{2} + \frac{i}{4(k+1)} & \text{if } k \text{ is even} \\
\frac{n-1}{2} + \frac{k}{2} + \frac{i+1}{4(k+1)} & \text{if } k \text{ is odd.}
\end{dcases}$$

(8)

Equality holds if and only if $T$ consists of a path $P$ of length $k$ for the first case and $k + 1$ for the second with all other vertices adjacent to the middle vertex of $P$ as shown in Fig. 3. When $i = k$, both bounds in (8) give the same value and both give extremal structures.

**Proof.** Let $T$ be a tree, and for arbitrary $u \in L(T)$, choose $w \in L(T)$ so that $x := ecc_T(u)$ is realized along path $P(u, w)$. The vertices along this path have eccentricity in $T$ at least the size of their eccentricity on the path. The eccentricity of any vertex not on $P(u, w)$ is at least one more than the smallest eccentricity among the path’s vertices. Therefore, we can lower bound $Ecc(T)$ according to the parity of $x$.

If $x$ is even, the path has an odd number of vertices and a single middle vertex. A lower bound on the total eccentricity in this case is

$$Ecc(T) \geq 2 \left( x + (x-1) + \ldots + \frac{x}{2} \right) - \frac{x}{2} + (n - (x + 1)) \left( 1 + \frac{x}{2} \right)$$

8
where equality holds when all vertices not on $P(u, w)$ are adjacent to the single middle vertex of this path as in Fig. 3. (We will need $x \leq n - 1$ for this tree structure to be realizable.) Again, simplifying the lower bound on $\text{Ecc}(T)$ gives

$$\frac{\text{Ecc}(T)}{\text{ecc}_T(u)} \geq \frac{x^2 + (2n - 2)x + (4n - 4)}{4x} =: f(x)$$

Examination of $f'(x)$ show that the ratio is minimized when

$$x \in \{\lfloor \sqrt{4n - 4} \rfloor, \lceil \sqrt{4n - 4} \rceil \} \subseteq \{k, k + 1\}.$$

The lower bounds in (8) are exactly $f(k)$ and $f(k + 1)$ per our assumption that $x$ is even. In fact, assigning $k$ or $k + 1$ to $x$ as described in (8) has the property that $x \leq n - 1$ for $n \geq 5$ for realizability. It is also important to note that if $4n - 4$ is a perfect square, in which case $\lfloor \sqrt{4n - 4} \rfloor = \lceil \sqrt{4n - 4} \rceil$, then $x = k$ is appropriate since $k^2 = 4n - 4$ and hence $k$ is even.

For thoroughness, it is shown below that $f(k) \leq f(k + 2)$ and $f(k + 1) \leq f(k - 1)$, for $k > 1$. (One can quickly check that the bound also holds for $n = 3$ when $k = 1$.) Thus our choice of the even integer nearest the minimum $x$-value was correct for this concave up function.

$$f(k + 2) - f(k) = \frac{n - 1}{k + 2} + \frac{1}{2} - \frac{n - 1}{k}$$

$$= \frac{2k(n - 1) + k(k + 2) - 2(k + 2)(n - 1)}{2k(k + 2)}$$

$$= \frac{k^2 + 2k - (4n - 4)}{2k(k + 2)}$$

$$= \frac{k^2 + 2k - (k^2 + i)}{2k(k + 2)}$$

$$\geq 0$$

$$f(k - 1) - f(k + 1) = \frac{-1}{2} + \frac{n - 1}{k - 1} - \frac{n - 1}{k + 1}$$

$$= \frac{-(k - 1)(k + 1) + 2(n - 1)(k + 1) - 2(n - 1)(k - 1)}{2(k - 1)(k + 1)}$$

$$= \frac{-k^2 + 1 + 4(n - 1)}{2(k - 1)(k + 1)}$$

$$= \frac{-k^2 + 1 + k^2 + i}{2(k - 1)(k + 1)}$$

$$= \frac{i + 1}{2(k - 1)(k + 1)}$$

$$> 0$$
If \( x \) is odd, \( P(u, w) \) has an even number of vertices. Thus

\[
Ecc(T) \geq 2 \left( x + (x - 1) + \ldots + \frac{x+1}{2} \right) + (n-(x+1)) \left( 1 + \frac{x+1}{2} \right)
\]

where equality holds when all vertices not on \( P(u, w) \) are adjacent to one of the two middle vertices. Recalling that \( ecc_T(u) = x \), some simplification of the lower bound on \( Ecc(T) \) yields

\[
\frac{Ecc(T)}{ecc_T(u)} \geq \frac{x^2 + (2n-4)x + (6n-5)}{4x} =: g(x).
\]

The function \( g(x) \) obtains a minimum when \( x = \sqrt{6n-5} \). So we compare \( g(\sqrt{6n-5}) \) with \( f(k) \) and \( f(k+1) \) to determine which scenario gives the proper lower bound for \( \frac{Ecc(T)}{ecc_T(u)} \).

Since \( 4n-4 = k^2 + i \), we have \( 4n-4 > 8k + 6 \) when \( k \geq 9 \). Recall \( i \leq 2k \), so \( \frac{i}{4} \leq \frac{i+1}{4(k+1)} \leq \frac{1}{2} \). Now consider the following argument:

\[
\begin{align*}
4n - 4 &> 8k + 6 \\
2n - 2 &> 4k + 3 \\
2n - 1 &> 4k + 4 \\
(2n - 1) + (4n - 4) &> (4n - 4) + 4k + 4 \geq k^2 + 4k + 4 \\
6n - 5 &> (k + 2)^2 \\
\sqrt{6n-5} &> k + 2 \\
\sqrt{6n-5} + n - 2 &> k + n \\
\frac{\sqrt{6n-5} + n - 2}{2} &> \frac{k + n}{2}
\end{align*}
\]

\[
g(\sqrt{6n-5}) = \frac{\sqrt{6n-5}}{4} + \frac{n-2}{2} + \frac{\sqrt{6n-5}}{4} > \frac{n-1}{2} + \frac{k}{2} + \frac{1}{2} \geq f(k), f(k+1)
\]

This holds for \( k \geq 9 \) and thus \( n \geq 22 \).

For \( n = 6, 7, 10 - 13, 17 - 21 \), the corresponding value of \( k \) is even. For each \( n \) in the list, one can individually verify that \( f(k) < g(\sqrt{6n-5}) \) where \( f(k) \) is the lower bound in (8). For \( n = 8, 9, 14, 15, 16 \), the value of \( k \) is odd and one can verify \( f(k+1) < g(\sqrt{6n-5}) \). This shows that the bounds in (8) are the proper ones for \( n \geq 6 \) and the extremal tree...
structures characterized in the theorem are unique because the inequalities comparing \( f \) and \( g \) are strict.

Now when \( n = 5 \), we have \( k = 4 \). So the bound in (8) gives \( f(4) = 4 \) which is realized by the path on 5 vertices, the extremal tree described in the theorem. On the other hand, \( g(\sqrt{6n-5}) = g(5) = 4 \). So (8) is still the correct bound. However, we may now ask if the path on 5 vertices is the unique extremal example. Notice that the bound \( g(5) \) is obtained when \( x = 5 \). In other words, ecc\(_T(u) = 5 \). But this is not possible because no vertex can have eccentricity equal to the number of vertices in the tree. So the path on 5 vertices is the unique extremal tree when \( n = 5 \). This completes the proof for \( n \geq 5 \). □

2.3 On the extremal values of \( \frac{ecc_T(u)}{ecc_T(v)} \) where \( u \in L(T) \), \( v \in C(T) \)

Theorem 6. Let \( T \) be a tree on \( n \geq 3 \) vertices with \( u \in L(T) \) and \( v \in C(T) \). Then

\[
\frac{ecc_T(u)}{ecc_T(v)} \leq 2, \tag{9}
\]

where the upper bound is tight for stars, even length paths, and more. If \( n \geq 5 \),

\[
1 + \frac{1}{\lceil \frac{n-1}{2} \rceil} \leq \frac{ecc_T(u)}{ecc_T(v)}. \tag{10}
\]

Equality holds if and only if \( T \) is one of the following trees: (1) \( T \) consists of a path on \( n - 1 \) vertices and a single vertex \( u \) adjacent to \( v \), one of the path's midvertices. (2) For even \( n \), \( T \) consists of a path with two pendant vertices not on the path, one of which is \( u \) adjacent to \( v \) a middle vertex on the path. These structures are demonstrated in Fig. 4.

Proof. The upper bound in (9) is a direct consequence of (2). As stated, it is tight for stars, paths of even length, and many other trees.

Turning our attention to the lower bound in (10), we first show that it holds for paths. If \( T \) is a path with an odd number of vertices, \( n \geq 3 \), then

\[
\frac{ecc_T(u)}{ecc_T(v)} = \frac{n-1}{\frac{n-1}{2}} = 2 \geq 1 + \frac{1}{\lceil \frac{n-1}{2} \rceil}.
\]

On the other hand, if \( T \) is a path with an even number of vertices, \( n \geq 6 \),

\[
\frac{ecc_T(u)}{ecc_T(v)} = \frac{n-1}{\frac{n}{2}} = \frac{2n-2}{n} = 1 + \frac{n-2}{n} = 1 + \frac{1}{\frac{n-2}{n}} \geq 1 + \frac{1}{\frac{n-2}{2}} \geq 1 + \frac{1}{\lceil \frac{n-1}{2} \rceil}.
\]

Now let \( n \geq 5 \) and suppose \( T \) is not a path. When there are more than 2 vertices, it is evident that \(ecc_T(u) \geq ecc_T(v) + 1 \) with equality exactly when \( uv \in E(T) \). Equivalently,

\[
\frac{ecc_T(u)}{ecc_T(v)} \geq 1 + \frac{1}{ecc_T(v)}. \tag{11}
\]
Given the size of the center, Observation 1 determines the length of the longest path. If \( \{v\} = C(T) \), then the maximum length path which contains \( v \) has \( 2ecc_T(v) + 1 \) vertices. We assumed \( T \) was not a path, so there is a vertex not on \( P \). Thus \( 2ecc_T(v) + 1 \leq n - 1 \), which together with (11) gives

\[
\frac{ecc_T(u)}{ecc_T(v)} \geq 1 + \frac{1}{\frac{n-2}{2}} = 1 + \frac{1}{\left\lceil \frac{n-1}{2} \right\rceil}.
\]

Notice that the last inequality is strict for odd \( n \) and tight when \( n \) is even and exactly one vertex does not lie on the path.

If instead \( C(T) \) has two vertices and \( n \) is odd, then the longest path will have \( 2ecc_T(v) \) vertices and \( 2ecc_T(v) \leq n - 1 \) which implies

\[
\frac{ecc_T(u)}{ecc_T(v)} \geq 1 + \frac{1}{\frac{n-1}{2}} = 1 + \frac{1}{\left\lceil \frac{n-1}{2} \right\rceil}.
\]

This is also tight when there is exactly one vertex which does not lie on the path. However if \( n \) is even, we can more precisely conclude \( 2ecc_T(v) \leq n - 2 \) in which case

\[
\frac{ecc_T(u)}{ecc_T(v)} \geq 1 + \frac{1}{\frac{n-2}{2}} = 1 + \frac{1}{\left\lceil \frac{n-1}{2} \right\rceil}.
\]

This bound is different in that tightness holds whenever there are two vertices off the path.

By our earlier observation that equality holds in (11) when \( uv \in E(T) \), the pendant vertex in each case will be \( u \) as in Fig. 4. But in the last case when \( n \) is even and two vertices are not on the path, the second one can be adjacent to any internal vertex.

![Figure 4: Minimizing trees with respect to \( \frac{ecc_T(u)}{ecc_T(v)} \), the second being for even \( n \) only.](image-url)

\[\square\]
2.4 On the extremal values of $\frac{ecc_T(u)}{ecc_T(w)}$ where $u, w \in L(T)$

First note that since the maximum and minimum values of $\frac{ecc_T(u)}{ecc_T(w)}$ are reciprocals of each other, we only consider the maximum.

**Theorem 7.** For tree $T$ with $n \geq 4$ vertices and $u, w \in L(T)$,

$$\frac{ecc_T(u)}{ecc_T(w)} \leq 2 - \frac{2}{\lceil \frac{n}{2} \rceil}.$$  

Equality holds when $n$ is even if and only if $T$ is a length $n - 2$ path, $u$ at one end, with a pendant edge added at the path’s midvertex to $w$. When $n$ is odd, equality holds if and only if $T$ is a length $n - 3$ path, $u$ at one end, with a pendant edge added at the path’s midvertex to $w$, and another leaf is joined to a non-leaf vertex. This construction can be seen in Fig. 5.

**Proof.** In seeking an upper bound, it is reasonable to assume $ecc_T(u) \geq ecc_T(w)$.

Let $P := P(u, y)$ realize $ecc_T(u)$. In order to maximize the ratio, we seek to minimize $ecc_T(w)$. Notice that if $T$ is merely a path, then the ratio would be 1 which is less than our target value for $n \geq 4$. So assume $w$ is not on $P$. There is a unique path $Q$ between $w$ and $P$ that is disjoint from $P$ other than a single vertex $z$.

Observe $ecc_T(w) \geq d(w, z) + \max\{d(z, u), d(z, y)\}$ which is minimum when both $d(w, z) = 1$ and $|d(z, u) - d(z, y)| \leq 1$. In other words, $w$ is adjacent to a middle vertex of $P$. If $ecc_T(u)$ is even, $ecc_T(w) \geq 1 + \frac{1}{2}ecc_T(u)$ and if $ecc_T(u)$ is odd, $ecc_T(w) \geq 1 + \frac{1}{2}(ecc_T(u) + 1)$.

We divide our analysis into three cases. First assume that $n$ is even and so is $ecc_T(u)$. Since $w$ is not on $P$, $ecc_T(u) \leq n - 2$. Putting the pieces together,

$$\frac{ecc_T(u)}{ecc_T(w)} \leq \frac{ecc_T(u)}{1 + \frac{1}{2}ecc_T(u)} = 2 - \frac{4}{ecc_T(u) + 2} \leq 2 - \frac{4}{n} = 2 - \frac{2}{\lceil \frac{n}{2} \rceil}.$$  

This bound is tight exactly when $ecc_T(u) = n - 2$ and $w$ is adjacent to the middle vertex of $P$ as pictured in Fig. 5, without the gray pendant vertex on the end.

When $n$ is even and $ecc_T(u)$ is odd. We previously decided $ecc_T(u) < n - 1$. Hence $ecc_T(u) \leq n - 3$ in order to maintain that $ecc_T(u)$ is odd. Like in (12),

$$\frac{ecc_T(u)}{ecc_T(w)} \leq \frac{ecc_T(u)}{1 + \frac{1}{2}(ecc_T(u) + 1)} = 2 - \frac{6}{ecc_T(u) + 3} \leq 2 - \frac{6}{n} < 2 - \frac{2}{\lceil \frac{n}{2} \rceil}.$$  

Now assume that $n$ is odd. Again, $ecc_T(u) < n - 1$. When $ecc_T(u)$ is odd, it is at most $n - 2$. Repeating the argument above,

$$\frac{ecc_T(u)}{ecc_T(w)} \leq \frac{ecc_T(u)}{1 + \frac{1}{2}(ecc_T(u) + 1)} = 2 - \frac{6}{ecc_T(u) + 3} \leq 2 - \frac{6}{n + 1} < 2 - \frac{4}{n} = 2 - \frac{2}{\lceil \frac{n}{2} \rceil}.$$  

where the last inequality is strict for $n \geq 3$. So this does not provide an extremal tree.
If \( n \) is odd and \( ecc_T(u) \) is even, it is at most \( n - 3 \), and there are two vertices not on \( P \). Once again,

\[
\frac{ecc_T(u)}{ecc_T(w)} \leq \frac{ecc_T(u)}{1 + \frac{1}{2}ecc_T(u)} = 2 - \frac{4}{ecc_T(u) + 2} \leq 2 - \frac{4}{n-1} = 2 - \frac{2}{\lfloor \frac{n}{2} \rfloor}.
\]

This time equality can be achieved by joining a leaf to any internal vertex of the length \( n - 3 \) path, in addition to the leaf \( w \) joined to the midpoint. This can be seen in Fig. 5, along with the additional leaf that occurs only for odd \( n \) in gray.

![Figure 5: A maximizing tree with respect to \( \frac{ecc_T(u)}{ecc_T(w)} \).](image)

Table 1 summarizes the results in section 2. Vertex labels appear as in the theorems. In other words, each \( v \) is in \( C(T) \) while each \( u \) and \( w \) are leaves.

### 3 Extremal structures

#### 3.1 General trees

For most other indices of interests such as the sum of distances and the number of subtrees, the star and the path are extremal. Dankelmann, Goddard, and Swart [5] showed that the path maximizes \( Ecc(T) \) among trees with given order. We show that the star minimizes \( Ecc(T) \) among trees with given order.

**Proposition 8.** For any tree \( T \) with \( n > 2 \) vertices,

\[
Ecc(T) \geq 1 + 2(n - 1) = 2n - 1
\]

with equality if and only if \( T \) is a star.

**Proof.** Any tree with at least three vertices has at most one vertex which is adjacent to every other vertex (hence with eccentricity 1). Thus we have

\[
Ecc(T) \geq 1 + 2(n - 1) = 2n - 1.
\]

Equality holds if and only if the single center vertex has eccentricity 1 and all other vertices have eccentricity 2. This describes a star. \( \square \)
\[
\begin{array}{|c|c|c|}
\hline
\text{Bound} & \text{Extremal Tree} \\
\hline
\frac{\text{Ecc}(T)}{\text{ecc}(v)} & \leq 2n - 1 \\
\hline
\frac{\text{Ecc}(T)}{\text{ecc}(v)} & \geq n + 2\sqrt{n} - a(n) \\
\hline
\frac{\text{Ecc}(T)}{\text{ecc}(u)} & \leq 2n - 2\sqrt{2n} + b(n) \\
\hline
\frac{\text{Ecc}(T)}{\text{ecc}(u)} & \geq \frac{1}{2}n + \sqrt{n} - c(n) \\
\hline
\frac{\text{ecc}(u)}{\text{ecc}(v)} & \leq 2 \\
\hline
\frac{\text{ecc}(u)}{\text{ecc}(v)} & \geq 1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right) \\
\hline
\frac{\text{ecc}(u)}{\text{ecc}(w)} & \leq 2 - \frac{4}{n} + O\left(\frac{1}{n^2}\right) \\
\hline
\end{array}
\]

Table 1: A summary of results with \( v \in C(T) \) and \( u, w \in L(T) \). The quantities \( a(n), b(n), c(n) \) are bounded as follows: \( 1 \leq a(n) \leq 5, -1 \leq b(n) \leq 5, 0 \leq c(n) \leq \frac{3}{2} \).
3.2 Trees with given degree sequences

Tree structures with given degree sequences are of particular interest for some other distance-based indices. As an example, there are applications in predicting a chemical compound’s physical properties from molecular graphs where degrees correspond to the valences of given atoms. Given a degree sequence, we determine the tree structures that maximize and minimize $\text{Ecc}(T)$. It is useful to note that a sequence $(d_1, d_2, \ldots, d_n)$ is the degree sequence for a tree if and only if $\sum_{i=1}^{n} d_i = 2(n - 1)$ and each $d_i$ is a positive integer.

3.2.1 General Caterpillars

Among all trees with a given degree sequence, the sum of distances is maximized by a caterpillar [14] and the number of subtrees is minimized by a caterpillar [8, 15]. However, completely characterizing the extremal caterpillar turns out to be a very difficult question in both cases. For the sum of distances, it turned out to be a quadratic assignment problem that is NP-hard in the ordinary sense and solvable in pseudo-polynomial time [4].

Definition 9 (Greedy Caterpillar [11]). For $n \geq 3$, let $(d_1, d_2, \ldots, d_n)$ be a degree sequence for a tree with $d_k \geq d_{k+1}$ for $1 \leq k < n$ and $d_i > d_{i+1} = \ldots = d_n = 1$. (Incidentally, $1 \leq i < n - 1$ because this is a tree degree sequence on more than 2 vertices.) The greedy caterpillar, $T$, is constructed as follows:

- Create a path $P$ with $i$ vertices which will be the caterpillar’s backbone.
- Assign degrees $d_1, \ldots, d_i$ to the vertices of $P$, such that $\text{ecc}_P(u) \geq \text{ecc}_P(v)$ whenever $\text{degr}_T(u) \geq \text{degr}_T(v)$ for each pair $u, v \in V(P)$.
- Attach pendant vertices to realize the degrees.

For an example, Fig. 6 shows a greedy caterpillar with degree sequence

$$(6, 6, 5, 4, 4, 1, \ldots, 1).$$

Note that greedy caterpillars are not unique. For example, the two sequences $(9, 7, 8, 10)$ and $(9, 8, 7, 10)$ are potential arrangements for degrees on the backbone which create non-isomorphic greedy caterpillars.

![Figure 6: A greedy caterpillar.](image-url)
Proposition 10. Given a tree degree sequence, $Ecc(T)$ is maximum when $T$ is a greedy caterpillar.

Proof. We first show that $Ecc(T)$ is indeed maximized by a caterpillar. For contradiction, suppose $T$ is a tree with degree sequence $\overrightarrow{d}$ that maximizes $Ecc(T)$ and is not a caterpillar. Consider a longest path $P_T(u, v) = uu_1u_2 \ldots u_{k-1}v$ in $T$. Let $x$ be the smallest subscript such that $u_x$ has a nonleaf neighbor $w$ not on $P_T(u, v)$ (note that $x \geq 2$). Let $W$ be the component containing $w$ in $T - \{u_xw\}$.

Figure 7: Generating $T'$ from $T$.

Define another tree $T'$ which will be obtained from $T$ by replacing each edge in $W$ of the form $zw$ with $zu$. (Fig. 7). Notice that the degree sequence did not change. However, for any vertex $s \in V(T) - V(W)$, $ecc_{T'}(s) \geq ecc_T(s)$ because $P_T(u, v)$ is a longest path in $T$. Similarly, $ecc_{T'}(w) \geq ecc_T(w)$. For any vertex $r \in V(W) - w$, we have

$$ecc_{T'}(r) = d(r, u) + d(u, v) > d(u, v) \geq ecc_T(r).$$

Thus $Ecc(T') > Ecc(T)$, which contradicts the extremality of $T$.

At this point, we can safely assume $T$ is a caterpillar with internal vertices forming path $P = u_1u_2 \ldots u_k$. Observe that the eccentricity of any internal vertex is independent of the degrees on those internal vertices. As for the leaves, each neighbor $w$ of $u_i$ has

$$ecc_T(w) = \max\{k - i, i - 1\} + 2.$$

Therefore, the sum of eccentricities is maximized when $T$ is a greedy caterpillar. \qed

3.2.2 Greedy trees and level-greedy trees

In this section, each tree is rooted at a single vertex. While the root has no bearing on the total eccentricity, we use the added structure to direct our conversation. The height of a vertex is the distance to the root and the tree’s height, $h(T)$, is the maximum of all vertex heights. We start with some definitions.

Definition 11 (Level-degree sequence [7]). In a rooted tree, the list of multisets $L_i$ of degrees at height $i$, starting with $L_0$ containing the degree of the root vertex, is called the level-degree sequence of the rooted tree.

It is easy too see that a list of multisets is the level degree sequence of a rooted tree if and only if (1) all entries together make a tree degree sequence, (2) $|L_0| = 1$, and (3) $\sum_{d \in L_0} d = |L_1|$, and for all $i \geq 1$, $\sum_{d \in L_i} (d - 1) = |L_{i+1}|$. 

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For convenience, we speak about down-degrees in level-degree sequences. The down-degree is degree minus one, except for the down-degree of the root which equals the degree.

**Definition 12** (Level-greedy trees [7]). Let us be given the level-degree sequence of a rooted tree. The level-greedy tree with the same level-degree sequence is defined by the following algorithm:
In each level place $|L_i|$ vertices and, from left to right, assign to each a degree from $L_i$ in non-increasing order.
Repeat for $i = 0, 1, ...$
From left to right, join the next vertex in $L_i$ whose down-degree is $f$ to the first $f$ so far unconnected vertices on level $L_{i+1}$.

**Definition 13** (Greedy trees [12]). Given a tree degree sequence and put it in non-increasing order $d_1 \geq d_2 \geq ... \geq d_n$. The greedy tree is the level-greedy tree for a particular level-degree sequence that we define below. Let $L_0$ contain only $d_1$. Let $L_1$ contain $d_2, ..., d_{d_1+1}$. Repeat for $i \geq 1$: put the next $\sum_{d \in L_i} (d - 1)$ entries from the degree sequence into $L_{i+1}$.

For an example, Fig. 8 shows a greedy tree with degree sequence

$$\{4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 1, \ldots, 1\}.$$ 

![Figure 8: A greedy tree.]

By definition, every greedy tree is also level-greedy. However, Fig. 9 shows a level-greedy tree that is not greedy. It has level degree sequence:

$$\{3\}, \{5, 3, 2\}, \{3, 3, 3, 2, 2, 1, 1\}, \{2, 2, 1, 1, 1, 1, 1, 1\}, \{1, 1\}.$$ 

For a fixed degree sequence, greedy trees minimize the sum of distances [7, 12, 14] and maximize the number of subtrees [1, 16]. We will show that they also minimize $Ecc(T)$ among trees with a given degree sequence.

We provide some set-up for the proofs of the next two theorems. See Fig. 10 for an illustration. Given a tree $T$ rooted at $v$, let $T_1$ be the subtree, rooted at child $v_1$ of $v$, ...
containing some leaves of height $h := h(T)$. Then any vertex $u \in V(T - T_1)$ with height $j$ has

$$ecc_T(u) = j + h,$$  \hspace{1cm} (13)

being dependent only upon the height of $T$. For $h' := h(T - T_1)$, any vertex $w \in V(T_1)$ with $h_T(w) = j$ has

$$ecc_T(w) = \max\{j + h', ecc_{T_1}(w)\} \leq j + h$$  \hspace{1cm} (14)

depending only on $h'$ and the structure of $T_1$.

The following claim implies that the level-greedy tree minimizes the total eccentricity among trees with the same level-degree sequence.

**Claim 14.** Among trees with a given level-degree sequence, the level-greedy tree maximizes the number of vertices having eccentricity at most $\ell$, for each $\ell \geq 0$.

**Proof.** We proceed by induction on the number of vertices. The base case with one vertex is trivial.

The following observations for any $w \in V(T_1)$ and $u \in V(T - T_1)$ are consequences of (13) and (14). The level degree sequence determines $h$, so $ecc_T(u)$ is decided. Since
$ecc_T(w) \leq ecc_T(u)$ whenever $h_T(w) = h_T(u)$, we should maximize the number of vertices in $T_1$. Further, according to (14), we should minimize $h'$ and maximize the number of vertices in $T_1$ with $ecc_{T_1}(w) \leq \ell$ for each $\ell \geq 0$, the later of which makes use of the inductive hypothesis.

For vertices $u, w$ pictured in Figure 10, suppose $deg(u) > deg(w)$. Consider moving $deg(u) - deg(w)$ children of $u$ and their descendants to adoptive parent $w$. This effectively switches the degrees of $u$ and $w$ while all other degrees and the level degree sequence remain constant. Consequently, the number of vertices in $T_1$ increased and $h'$ did not increase. In addition, $ecc_T(w)$ does not increase for $w \in V(T_1)$. This advances our goals.

Repeating this procedure on each level, starting with vertices of height 1 and working down, we find that the larger degrees in each level’s multiset should be used in $T_1$. This will maximize the number of vertices in $T_1$ and minimize $h'$. As mentioned earlier, it is desired to apply the inductive hypothesis to $T_1$, thereby making it a level-greedy tree. In $T - T_1$, the number of vertices of each height is determined by the level-degree sequence $T - T_1$ inherited. So any arrangement within the levels will give a desired tree. In particular, the level-greedy tree suffices.

This theorem also yields a stronger result than merely minimizing total eccentricity among trees with a given degree sequence.

**Theorem 15.** Given a degree sequence, the greedy tree maximizes the number of vertices with eccentricity at most $\ell$ for each $\ell \geq 0$.

**Proof.** Suppose $T$ is a tree with the given degree sequence which maximizes the number of vertices with eccentricity at most $\ell$, for each $\ell \geq 0$. Let us root $T$ so that $h - 1 \leq h' \leq h$. Now for any $w \in V(T_1)$ with $h_T(w) = j$, $ecc_{T_1}(w) \leq (j - 1) + (h - 1) \leq j + h' - 1$. In light of (14),

$$ecc_T(w) = \max\{j + h', ecc_{T_1}(w)\} = j + h'.$$

For $w, x \in V(T_1)$, if $h_T(w) < h_T(x)$ then $ecc(w) < ecc(x)$ which implies $deg(w) \geq deg(x)$ in $T$ because $T$ maximizes the number of vertices with small eccentricities.

Vertices in $T - T_1$ with height $j$ have eccentricity $j + h$ by (13). So for $u, v \in V(T - T_1)$, when $h_T(u) < h_T(v)$, we can conclude $deg(u) \geq deg(v)$ in $T$.

These observations establish the fact that either the root of $T - T_1$ or the root of $T_1$ has the largest degree.

We now examine two cases based upon the value of $h'$. When $h = h'$, we have $ecc_T(w) = j + h = ecc_T(u)$ for any $w \in V(T_1)$, $u \in V(T - T_1)$ with $h_T(w) = h_T(u) = j$. Therefore, for $x, y \in V(T)$, if $h_T(x) < h_T(y)$, then $deg(x) \geq deg(y)$ in $T$. As an immediate consequence, the root of $T$ has the largest degree.

When $h' = h - 1$, we may assume that the root of $T$ has the largest degree, for otherwise, we could reroot $T$ at $v_1$ which would not change the vertex eccentricities or the difference between $h$ and $h'$.

Continuing in the setting with $h' = h - 1$, for $w \in V(T_1)$ and $u, y \in V(T - T_1)$, if $h_T(w) = h_T(u)$, then $ecc(w) = ecc(u) - 1$. So $deg(w) \geq deg(u)$ in $T$. However, if
\( h_T(w) \geq h_T(y) + 1 \), then \( ecc(w) \geq ecc(y) \). So it is acceptable to assume \( deg(w) \leq deg(y) \) in \( T \).

In both cases, we may assume that vertices of smaller height have larger degrees. Consequently, this is information to determine the level degree sequence of \( T \). In fact, the level degree sequence realized by the greedy tree fits all the requirements in both cases. Given this level degree sequence, the previous claim implies that \( T \) is level-greedy. Therefore, \( T \) is in fact the greedy tree.

**Remark 16.** Such extremal trees are not necessarily unique. In fact, the greedy tree gave a much stronger restriction than what we needed, as stated in the theorem, while still not being the unique structure.

### 3.2.3 Greedy trees with different degree sequences

As a final remark on greedy trees, consider ordering the greedy trees with different degree sequences according to their total eccentricity. This observation, similar to previous works on other indices, yields many extremal results as immediate corollaries. For an example of such applications see [16].

**Definition 17.** Given non-increasing sequences \( \pi' = (d'_1, \ldots, d'_n) \) and \( \pi'' = (d''_1, \ldots, d''_n) \), \( \pi'' \) is said to majorize \( \pi' \) if for \( k = 1, \ldots, n-1 \)

\[
\sum_{i=0}^{k} d'_i \leq \sum_{i=0}^{k} d''_i \quad \text{and} \quad \sum_{i=0}^{n} d'_i = \sum_{i=0}^{n} d''_i.
\]

This is denoted by

\( \pi' \prec \pi'' \).

**Lemma 18.** [13] Let \( \pi' = (d'_1, \ldots, d'_n) \) and \( \pi'' = (d''_1, \ldots, d''_n) \) be two non-increasing tree degree sequences. If \( \pi' \prec \pi'' \), then there exists a series of (non-increasing) tree degree sequences \( \pi^{(i)} = (d^{(i)}_1, \ldots, d^{(i)}_n) \) for \( 1 \leq i \leq m \) such that

\[
\pi' = \pi^{(1)} \prec \pi^{(2)} \prec \cdots \prec \pi^{(m-1)} \prec \pi^{(m)} = \pi''.
\]

In addition, each \( \pi^{(i)} \) and \( \pi^{(i+1)} \) differ at exactly two entries, say the \( j \) and \( k \) entries, \( j < k \). More specifically, \( d^{(i+1)}_j = d^{(i)}_j + 1 \) and \( d^{(i+1)}_k = d^{(i)}_k - 1 \).

**Remark 19.** Lemma 18 is in fact a breaking down of the original statement in [13]. It is indeed the case that in this process, each entry stays positive and the degree sequences remain non-increasing. Thereby, each obtained sequence is a tree degree sequence that is non-increasing without rearrangement.

**Theorem 20.** Given two tree degree sequences \( \pi' \) and \( \pi'' \) such that \( \pi' \prec \pi'' \),

\[
Ecc(T^*_\pi') \geq Ecc(T^*_\pi'')
\]

where \( T^*_\nu \) is the greedy tree for degree sequence \( \nu \).
Proof. According to Lemma 18, it suffices to compare the total eccentricity of two greedy trees whose degree sequences differ by one in two entries, i.e., assume
\[
\pi' = (d'_1, \ldots, d'_n) \triangleleft (d''_1, \ldots, d''_n) = \pi''
\]
with \(d''_j = d'_j + 1\), \(d''_k = d'_k - 1\) for some \(j < k\) and all other entries the same.

Let \(u\) and \(v\) be the vertices corresponding to \(d_j\) and \(d_k\) respectively and \(w\) be a child of \(v\) in \(T_{\pi'}\) (Fig. 11). Construct \(T_{\pi''}\) from \(T_{\pi'}\) by removing the edge \(vw\) and adding edge \(uw\). Note that \(T_{\pi''}\) has degree sequence \(\pi''\) and by Theorem 15
\[
Ecc(T_{\pi''}) \leq Ecc(T_{\pi'}).
\]

The height of any vertex in \(T_{\pi''}\) is at most that of its counterpart in \(T_{\pi'}\). An argument similar to that used in the proof of Claim 14 shows
\[
Ecc(T_{\pi''}) \leq Ecc(T_{\pi'}) \leq Ecc(T_{\pi'}).
\]

Remark 21. As in the proof of the extremality of greedy trees, equality holds more often in (15) compared with its analogue for many other graph invariants. This also serves as some indication that \(Ecc(T)\) is not as strong of a graph invariant as compared to others in terms of characterizing the structures.

By comparing greedy trees with different degree sequences, the extremality of trees with respect to minimizing \(Ecc(\cdot)\) under various restrictions easily follows. For instance, the “star-like trees” (trees which are subdivisions of stars, as in Fig. 12) among trees with given \(|V(T)|\) and \(|L(T)|\), the “extended good \(k\)-ary trees” (defined in a similar way as the greedy tree, except that all vertices take the maximum degree \(k\) until there are not enough vertices, Fig. 13) for trees with given \(|V(T)|\) and maximum degree. See for instance, [3, 16] for details.
Figure 12: A star-like tree with $|V(T)| = 11$ and $|L(T)| = 4$.

Figure 13: An extended good tree with maximum degree 4.

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