ORIENTATION-REVERSING INVOLUTIONS OF THE GENUS 3
ARNOUX–YOCOCZ SURFACE AND RELATED SURFACES

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Abstract. We present a new description of the genus 3 Arnoux–Yoccoz translation surface
in terms of its Delaunay polygons and show that, up to affine equivalence, it belongs to two
families of surfaces whose isometry groups include the dihedral group of the square.

1. Introduction

1.1. Flat surfaces. For our purposes, a flat surface is a pair \((X, q)\), where \(X\) is a Riemann
surface and \(q\) is a non-zero meromorphic quadratic differential of finite area on \(X\). We will
also speak of the flat surface \((X, \omega)\) instead of \((X, \omega^2)\) when \(q = \omega^2\) is the global square of
an abelian differential \(\omega\); in this case, \((X, \omega)\) is also called a translation surface. A quadratic
differential determines a canonical metric structure on the underlying surface (cf. [HM, EG]);
we will consider two flat surfaces to be the same when there is an isometry between them.
Recall that \(\text{SL}_2(\mathbb{R})\) and \(\text{GL}_2(\mathbb{R})\) act on the space of translation surfaces, while \(\text{PSL}_2(\mathbb{R})\)
and \(\text{PGL}_2(\mathbb{R})\) act on the space of flat surfaces. If \(A \cdot (X, \omega) = (X, \omega)\) or \([A] \cdot (X, q) = (X, q)\),
then \(\det A = \pm 1\). The set of all \([A] \in \text{PGL}_2(\mathbb{R})\) such that \([A] \cdot (X, q) = (X, q)\) form the
projective generalized Veech group of \((X, q)\), which we will simply call the Veech group (cf.
Ve1, HS).
The results in this article began with a study of Delaunay polygons on the surface described
in \([1.2]\) and so we recall their definition (cf. MS, Ri, Ve2, BS). A Delaunay triangle on
\((X, q)\) is the image of a 2-simplex on \(X\), embedded on its interior, whose vertices lie in the
set of cone points of \(q\), whose edges are geodesic with respect to the metric of \(q\), and whose
(possibly immersed) circumcircle contains no cone points of \(q\) in its interior. A Delaunay
triangulation of \((X, q)\) is a cell structure on \(X\) whose 2-cells are Delaunay triangles. A
Delaunay polygon is a Delaunay triangle or the union of two or more adjacent Delaunay
triangles that share the same circumcircle.
A generic flat surface has a unique Delaunay triangulation. When it is not unique, we
can start with any Delaunay triangulation and join two or more adjacent Delaunay triangles
to form a Delaunay polygon. After forming all of the maximal Delaunay polygons from a
Delaunay triangulation, we obtain the Delaunay decomposition of \((X, q)\), which is invariant
under isometries of \((X, q)\).

1.2. The Arnoux–Yoccoz surface. In [AY], P. Arnoux and J.-C. Yoccoz introduced a
family of pseudo-Anosov diffeomorphisms, one for each genus at least 3. Hubert–Lanneau
[HL] showed that none of these surfaces has a Veech group containing any parabolic ele-
ments. The genus 3 example, also appearing in [Ar], has received much attention recently.
Hubert–Lanneau–Möller [HLM] showed that the relevant abelian differential has a second,
Figure 1. The decomposition of \((X_{AY}, \omega_{AY})\) into its Delaunay polygons. Edges with the same label are identified by translation.

independent pseudo-Anosov element in its stabilizer, and using techniques introduced by C. McMullen \cite{Mc} in the study of genus 2 orbits they showed that the \(SL_2(\mathbb{R})\)-orbit of the genus 3 example is dense in the largest possible region of the moduli space of abelian differentials. Here we give a new description of the genus 3 surface in terms of its Delaunay polygons (of which there are only two kinds, up to isometry) and very simple gluing instructions.

Let \(\alpha \approx 0.543689\) be the real root of the polynomial \(x^3 + x^2 + x - 1\). Let \(S_0\) be the square with vertex set \(\{(0, 0), (\alpha^2, \alpha), (\alpha^2 - \alpha, \alpha^2 + \alpha), (-\alpha, \alpha^2)\}\), and let \(T_0\) be the trapezoid with vertex set \(\{(0, 0), (1 - \alpha, 1 - \alpha), (1 - \alpha - \alpha^2, 1), (-\alpha, \alpha^2)\}\). We form a flat surface \((X_{AY}, \omega_{AY})\) from two copies of \(S_0\) and four copies of \(T_0\): reflecting \(S_0\) across either a horizontal or vertical axis yields the same square \(S_1\) (up to translation); we denote by \(T_{1,0}, T_{0,1}\), and \(T_{1,1}\) the reflections of \(T_0\) across a vertical axis, across a horizontal axis, and across both, respectively. (In fact, \(T_{1,0}, T_{0,1}\), and \(T_{1,1}\) are all rotations of \(T_0\) by multiples of \(\pi/2\), but this description via reflections will be invariant under horizontal and vertical scaling, i.e., the Teichmüller geodesic flow.) Identify the long base of \(T_0\) with the long base of \(T_{1,1}\), as well as their short bases; do the same with \(T_{1,0}\) and \(T_{0,1}\). Each remaining side of a trapezoid is parallel to exactly one side of \(S_0\) or \(S_1\); identify by translations those sides which are parallel. (See Figure 1)

The resulting flat surface \((X_{AY}, \omega_{AY})\) has genus 3 and two singularities each with cone angle \(6\pi\). The images of \(S_0\) and \(T_0\) are the Delaunay polygons of \(\omega_{AY}\). \(X_{AY}\) is hyperelliptic; the hyperelliptic involution \(\tau : X_{AY} \to X_{AY}\) is evident in Figure 1 as rotation by \(\pi\) around the centers of the squares and the midpoints of the edges joining two trapezoids; these six points together with the cone points are therefore the Weierstrass points of the surface. (See Figure 3 for the quotient of \(X_{AY}\) by \(\tau\).) Moreover, \(\omega_{AY}\) is odd with respect to \(\tau\), i.e., \(\tau^* \omega_{AY} = -\omega_{AY}\).

The pseudo-Anosov diffeomorphism \(\psi_{AY}\) constructed by Arnoux–Yoccoz scales the surface by a factor of 1/\(\alpha\) in the horizontal direction and by \(\alpha\) in the vertical direction. In Figure 2 we show the result of applying this affine map to Figure 1 along with the new Delaunay
Figure 2. The result of applying the Arnoux–Yoccoz pseudo-Anosov diffeomorphism to $\omega_{AY}$. The original shapes of $S_0$ and $T_0$ and their copies can be reconstructed by matching edges.

Figure 3. Quotient surfaces of $X_{AY}$. LEFT: $\mathbb{CP}^1$ as the quotient of $X_{AY}$ by $\langle \tau \rangle$. The edges marked $b$ are identified by translation. RIGHT: $\mathbb{RP}^2$ as the quotient of $X_{AY}$ by the group $\langle \sigma_1, \sigma_2 \rangle$. Edges with the same label are identified by a glide reflection along either a horizontal or a vertical axis. Dashed lines in the left picture indicate preimages of the segments labeled $B$ on the right.

1.3. $X_{AY}$ as a cover of $\mathbb{RP}^2$. The reflections applied to $S_0$ and $T_0$ in §1.2 induce a pair of orientation-reversing involutions without fixed points on $X_{AY}$. These can be visualized (as in Figure 1) as “glide-reflections”, one along a horizontal axis and the other along a vertical axis. Both exchange $S_0$ and $S_1$. Let $\sigma_1$ be the involution that exchanges $T_0$ and $T_{1,0}$; i.e., its derivative is reflection across the horizontal axis. Let $\sigma_2$ be the involution that exchanges $T_0$ and $T_{0,1}$; i.e., its derivative is reflection across the vertical axis. The product of $\sigma_1$ and $\sigma_2$ is the hyperelliptic involution $\tau$, and neither sends any point of $X_{AY}$ to its image by $\tau$. They therefore descend to a single involution $\sigma$ on $\mathbb{CP}^1$ without fixed points. The quotient of $\mathbb{CP}^1$ by $\sigma$ is homeomorphic to $\mathbb{RP}^2$.

In fact, the presence of $\sigma_1$ and $\sigma_2$ is implicit in the work of Arnoux–Yoccoz. The original paper [AY] begins with a measured foliation of $\mathbb{RP}^2$ with one “tripod” (a singular point of valence three) and three “thorns” (singular points of valence 1), which is then lifted to the genus 3 example we have described. In Figure 3 right, we illustrate $\mathbb{RP}^2$ as the quotient of $X_{AY}$ by the group generated by $\sigma_1$ and $\sigma_2$. In Figure 3 left, we see $\mathbb{CP}^1$, on which $\sigma$ acts again by a “glide-reflection”, which is the sheet exchange for the cover $\mathbb{CP}^1 \to \mathbb{RP}^2$. In
both pictures we have drawn vertices that become tripods as open circles, and the vertices that become thorns as filled-in circles. The vertical foliation of the surface on the right of Figure 3 is the starting point of [AY].

In [2] we will show that both \((X_{AY}, \omega_{AY})\) and another affinely equivalent surface have real structures (orientation-reversing involutions whose fixed-point set is 1-dimensional) that are not evident in the original construction. These additional structures will allow us to write equations for the surfaces and fit them into families of flat surfaces with a common group of isometries. In [2] we will transfer these results to genus 2 quadratic differentials. In [4] we will conclude by showing that we have found all the surfaces that are obtained by applying the geodesic flow to \((X_{AY}, \omega_{AY})\) and have real structures.

2. Two families of surfaces

2.1. Labeling the Weierstrass points of \(X_{AY}\). As before, we denote the hyperelliptic involution of \(X_{AY}\) by \(\tau\), and we let \(\sigma_1\) and \(\sigma_2\) be the involutions described in [1.3] with \(\sigma: \mathbb{CP}^1 \to \mathbb{CP}^1\) the involution covered by both \(\sigma_1\) and \(\sigma_2\).

The purpose of this section is to show the following.

Theorem 2.1. The surface \((X_{AY}, \omega_{AY})\) belongs to a family \((X_{t,u}, \omega_{t,u})\), with \(t > 1\) and \(u > 0\), such that \(X_{t,u}\) has the equation

\[
y^2 = x(x - 1)(x - t)(x + u)(x + tu)(x^2 + tu),
\]

and \(\omega_{t,u}\) is a multiple of \(x \, dx/y\) on \(X_{t,u}\).

Each of the surfaces in Theorem 2.1 has a pair of real structures \(\rho_1\) and \(\rho_2\) whose product is again \(\tau\), and which therefore descend to a single real structure \(\rho\) on \(\mathbb{CP}^1\). Any product of the form \(\rho_i \sigma_j\) \((i, j \in \{1, 2\})\) is a square root of \(\tau\), and therefore the group generated by \(\{\sigma_1, \sigma_2, \rho_1, \rho_2\}\) is the dihedral group of the square. We will exhibit these isometries in our presentation of \((X_{AY}, \omega_{AY})\). In [2.3] we will look at surfaces in this family that have additional symmetries.

Let \(\varpi: X_{AY} \to \mathbb{CP}^1\) be the degree 2 map induced by \(\tau\), i.e., \(\varpi \circ \tau = \varpi\). We can normalize \(\varpi\) so that the zeroes of \(\omega_{AY}\) are sent to 0 and \(\infty\), and the midpoint of the short edge between \(T_0\) and \(T_{1,1}\) is sent to 1. We wish to find the images of the remaining Weierstrass points, so that we can write an affine equation for \(X_{AY}\) in the form \(y^2 = P(x)\), where \(P\) is a degree 7 polynomial with roots at 0 and 1. Hereafter we assume that \(\varpi\) is the restriction to \(X_{AY}\) of the coordinate projection \((x, y) \mapsto x\). Consequently, we may consider each Weierstrass point as either a point \((w, 0)\) or simply as a point \(w\) on the \(x\)-axis.

Each of the real structures \(\rho_1\) and \(\rho_2\) has a fixed-point set with three components: in one case, say \(\rho_1\), the real components are the line of symmetry shared by \(T_0\) and \(T_{1,1}\), and the two bases of \(T_{1,0}\) and \(T_{0,1}\). The fixed-point set of \(\rho_2\) is then the union of the corresponding lines in the orthogonal direction. Because \(\rho_1\) and \(\rho_2\) fix the points 0, 1, and \(\infty\), \(\rho\) fixes the real axis; therefore \(\rho\) is simply complex conjugation. With this normalization, the involution \(\sigma\) on \(\mathbb{CP}^1\) exchanges 0 and \(\infty\) and preserves the real axis; therefore \(\sigma\) has the form \(x \mapsto -r/\overline{x}\) for some real \(r > 0\).

Let \(s = (s, 0)\) be the center of \(S_0\). Then \(\rho_1(s) = \rho_2(s) = \sigma_1(s) = \sigma_2(s)\) is the center of \(S_1\), which implies \(\rho(s) = \sigma(s)\), i.e., \(\overline{s} = -r/\overline{s}\). The solutions to this equation are \(\pm i\sqrt{r}\). By considering the location of the fixed-point sets of \(\rho_1\) and \(\rho_2\), we see that the image of \(S_0\) by \(\varpi\) lies in the upper half-plane; therefore \(s = i\sqrt{r}\), and \(-i\sqrt{r}\) is the center of \(S_1\).
Figure 4. The Weierstrass points of $X_{AY}$, following normalization $(t > 1,
 u > 0)$. The real structures $\rho_1$ and $\rho_2$ appear as reflections in the lines of slope
$\pm 1$.

Let $t$ be the midpoint of the long edge of $T_0$. Applying $\sigma_1$ or $\sigma_2$ shows that the midpoint
of the long edge of $T_{1,0}$ is at $-r/t$.

We already know that 1 is the center of the short edge of $T_0$. Since the short edge of $T_{0,1}$
is the image of this edge by $\sigma_1$ or $\sigma_2$, the midpoint of this edge must be at $-r$.

To simplify notation, let us make the substitution $u = r/t$, so that $r = tu$ (hence $\sigma$ has
the form $\sigma(x) = -tu/x$). Thus $X_{AY}$ has the equation (1) for some $(t,u) = (t_{AY},u_{AY})$.
Furthermore, $\omega_{AY}$ is the square root of a quadratic differential on $\mathbb{C}P^1$ with simple zeroes at
0 and $\infty$ and simple poles at 1, $t$, $-u$, $-tu$, and $\pm i\sqrt{tu}$. There is therefore some complex
constant $c$ such that

$$\omega_{AY}^2 = \varpi^*(\frac{cx}{(x-1)(x-t)(x+u)(x+tu)(x^2+tu)} \, dx^2),$$

i.e., $\omega_{AY} = \pm \sqrt{c} \, x \, dx/y$. This establishes Theorem 2.1.

2.2. Integral equations. To find $t_{AY}$ and $u_{AY}$ requires solving a system of equations in-
volving hyperelliptic integrals, which we establish in this section using relative periods of $\omega_{AY}$. Choose a square root of

$$f_{t,u}(x) = \frac{x}{(x-1)(x-t)(x+u)(x+tu)(x^2+tu)}$$
in the open first quadrant such that its extension $\sqrt{f_{t,u}(x)}$ to the complement of $\{1,t,i\sqrt{tu}\}$
in the closed first quadrant is positive on the open interval $(0,1)$. Let $\eta_0$ be the Delaunay
edge between $S_0$ and $T_0$; $\varpi(\eta_0)$ is then a curve from 0 to $\infty$ in the first quadrant. Integrating
$\sqrt{cf_{t,u}(x)} \, dx$ on the portion of the first quadrant below $\varpi(\eta_0)$ will then give a conformal
map to half of $T_0$. We will be interested in integrals along the real axis.

The vector from 0 to 1 along the short side of $T_0$ is $\frac{1}{2}(1-\alpha)(1+i)$, while the the line of
symmetry of $T_0$ from 1 to $t$ gives the vector $\frac{1}{2}(\alpha + \alpha^2)(-1+i)$. Observe that

$$i \cdot (1-\alpha)(1+i) = \alpha \cdot (\alpha + \alpha^2)(-1+i),$$
and therefore
\begin{equation}
 i \int_0^1 \sqrt{c f_{t,u}(x)} \, dx = \alpha \int_1^t \sqrt{c f_{t,u}(x)} \, dx.
\end{equation}

Similarly, the vector from \( t \) to \( \infty \) along the long side of \( T_0 \) is \( \frac{1}{2}(1 - \alpha^2)(-1 - i) \), and because \( 1 - \alpha^2 = (1 + \alpha)(1 - \alpha) \), we have
\begin{equation}
-(1 + \alpha) \int_0^1 \sqrt{c f_{t,u}(x)} \, dx = \int_t^\infty \sqrt{c f_{t,u}(x)} \, dx.
\end{equation}

In both equations we can cancel out the \( c \), which was ever only a global (complex) scaling factor anyway. Now bring \( i \) under the square root on the right-hand side of (2) in order to make the radicand positive. We thus obtain from (2) and (3) the system of (real) integral equations
\begin{equation}
\begin{cases}
\int_0^1 \sqrt{f_{t,u}(x)} \, dx = \alpha \int_1^t \sqrt{-f_{t,u}(x)} \, dx \\
(1 + \alpha) \int_0^1 \sqrt{f_{t,u}(x)} \, dx = -\int_t^\infty \sqrt{f_{t,u}(x)} \, dx
\end{cases}
\end{equation}
whose solution is the desired pair \((t_{AY}, u_{AY})\). Using numeric methods, we find
\[ t_{AY} \approx 1.91709843377 \quad \text{and} \quad u_{AY} \approx 2.07067976690. \]

We conjecture that \( t_{AY} \) and \( u_{AY} \) lie in some field of small degree over \( \mathbb{Q}(\alpha) \).

2.3. Other exceptional surfaces in this family. An examination of the geometric arguments in §2.1 and an application of the principle of continuity to \( t \) and \( u \) show the following:

**Theorem 2.2.** Every \((X_{t,u}, \omega_{t,u})\) as in Theorem 2.1 can be formed by replacing \( T_0 \) in the description from §1.2 with an isosceles trapezoid \( T, S_0 \) with the square built on a leg of \( T \), and the copies of \( T_0 \) with the rotations of \( T \) by \( \pi/2 \).

The placement of \( t \) and \( u \) on \( \mathbb{R} \) determines the shape of the trapezoid \( T \), and any isosceles trapezoid may be obtained by an appropriate choice of \( t \) and \( u \). In this section, we examine certain shapes that give \((X_{t,u}, \omega_{t,u})\) extra symmetries and determine the corresponding values of \( t \) and \( u \). We continue to use \( \tau \) to denote the hyperelliptic involution of \( X_{t,u} \).

Suppose that \( T \) is a rectangle. Then there are two orthogonal closed trajectories, running parallel to the sides of \( T \) and connecting the centers \( \pm i \sqrt{tu} \) of the squares, and either of these can be made into the fixed-point set of a real structure on \( X_{t,u} \). The product of these two real structures is again \( \tau \), so they descend to a single real structure on \( \mathbb{C}P^1 \). This real structure exchanges 0 with \( \infty \) and fixes \( \pm i \sqrt{tu} \), so it must be inversion in the circle \(|x|^2 = tu \).

It also exchanges 1 with \( t \), which implies \( 1 \cdot t = tu \), i.e., \( u = 1 \). The remaining parameter \( t \) is determined by solving the single integral equation
\[ \int_0^1 \frac{x}{(x^2 - 1)(x^2 - t^2)(x^2 + t)} \, dx = \mu \int_1^t \frac{-x}{(x^2 - 1)(x^2 - t^2)(x^2 + t)} \, dx \]
where \( 2\mu \) is the ratio of the width of \( T \) to its height. Recall that an origami, also called a square-tiled surface, is a flat surface that covers the square torus with at most one branch point (cf. [Sch, EO, Zo]). By looking at rational values of \( \mu \), we have the following result:
Corollary 2.3. The family \((X_{t,1}, \omega_{t,1})\) contains a dense set of origamis.

These are not the only \((X_{t,u}, \omega_{t,u})\) that are origamis, however. If \(T\) is a trapezoid whose legs are orthogonal to each other, then \((X_{t,u}, \omega_{t,u})\) is again an origami.

2.4. Second family of surfaces. Conjugating \(\rho_1\) by the pseudo-Anosov element \(\psi_{AY}\) guarantees the existence of another orientation-reversing involution in the affine group of \(\omega_{AY}\). This element fixes a point "half-way" (in the Teichmüller metric, for instance) between \(\omega_{AY}\) and its image by \(\psi_{AY}\), lying in the Teichmüller disk of \((X_{AY}, \omega_{AY})\). This surface can be found either by scaling the vertical direction by \(\sqrt{\alpha}\) and the horizontal direction by \(1/\sqrt{\alpha}\) or, to keep our coordinates in the field \(\mathbb{Q}(\alpha)\), just by scaling the horizontal by \(1/\alpha\). This surface, which we will denote \((X'_{AY}, \omega'_{AY})\), is shown in Figure 5, along with its Delaunay polygons.

Theorem 2.4. The surface \((X'_{AY}, \omega'_{AY})\) belongs to a family \((X_s, \omega_s)\), with \(\text{Im } s > 0\) and \(s \neq i\), such that \(X_s\) has the equation

\[
y^2 = x(x^2 + 1)(x - s)(x - \overline{s})(x + 1/s)(x + 1/\overline{s}),
\]

and \(\omega_s\) is a multiple of \(x \, dx/y\) on \(X_s\).

Again, we have two real structures \(\rho'_1\) and \(\rho'_2\) whose product is the hyperelliptic involution \(\tau\). Each of these only has one real component, however: the union of the sides of the parallelograms running parallel to the axis of reflection. The only Weierstrass points that lie on these components are 0 and \(\infty\); the remaining Weierstrass points are the centers of the squares and of the parallelograms. We again let \(\rho'\) be the induced real structure on \(\mathbb{CP}^1\) and assume that it fixes the real axis (this we can do because we have only fixed the positions of two points on \(\mathbb{P}^1\)), so that the remaining Weierstrass points come in conjugate pairs.

The fixed-point free involutions \(\sigma_1\) and \(\sigma_2\) from §1.3 again preserve the union of the real loci of \(\rho'_1\) and \(\rho'_2\), and therefore they descend to a fixed-point free involution \(\sigma\) of the form \(x \mapsto -r/\overline{x}\), with \(r\) real and positive. We have one more free real parameter for normalization, so we can assume \(r = 1\). This implies that the centers of the squares are at \(\pm i\). Let \(s\) be the center of one of the parallelograms; then applying \(\rho'_1\) and \(\sigma_1\) shows that the remaining
Weierstrass points are $\bar{s}$, $1/s$, and $1/\bar{s}$. Using developing vectors again, we can find equations that define $s$, in a manner analogous to finding (4).

As an analogue to Theorem 2.2, we have:

**Theorem 2.5.** Every $(X_s, \omega_s)$ as in Theorem 2.4 can be formed from a parallelogram $P$, a square built on one side of $P$, the rotation of $P$ by $\pi/2$, and the images of $P$ and its rotation by reflection across their remaining sides.

The shape of $P$ is determined by the value of $s$. If $s = \frac{1}{2} (\sqrt{3} + i)$, then $P$ becomes a square, and we obtain one of the “escalator” surfaces in [LS]. More generally, if $s$ is any point of the unit circle, then $P$ is a rectangle, and inversion in the unit circle corresponds to another pair of real structures on $X$, which are the reflections across the axes of symmetry of $P$. By considering those rectangular $P$ whose side lengths are rationally related, we have as before:

**Corollary 2.6.** The family $(X_{e^{i\theta}} \omega_{e^{i\theta}})$ (with $0 < \theta < \pi/2$) contains a dense set of origamis.

Another origami appears when $P$ is composed of a pair of right isosceles triangles so that $s$ lies not on the hypotenuse, but on a leg of each.

### 3. Quadratic differentials and periods on genus 2 surfaces

We do not know how to compute the rest of the periods for $X_{t,u}$ or $X_s$, apart from those of $\omega_{t,u}$ or $\omega_s$, respectively. In this section, however, we consider the periods of certain related genus 2 surfaces, which demonstrate remarkable relations.

Let $X$ be any hyperelliptic genus 3 surface with an abelian differential $\omega$ that is odd with respect to the hyperelliptic involution and has two double zeroes. The pair $(X, \omega)$ has a corresponding pair $(\Xi, q)$, where $\Xi$ is a genus 2 surface and $q$ is a quadratic differential on $\Xi$ with four simple zeroes. Geometrically, the correspondence may be described as follows: two of the zeroes of $\omega$ are at Weierstrass points of $X$, hence $(X, \omega^2)$ covers a flat surface $(\mathbb{C}P^1, \tilde{q})$ where $\tilde{q}$ has six poles and two simple zeroes (Figure 3). Then $(\Xi, q)$ is the double cover of $(\mathbb{C}P^1, \tilde{q})$ branched at the poles of $\tilde{q}$. In our cases, the genus 2 surface may be obtained by cutting along opposite sides of one of the squares in Figure 1 or 5, then regluing each of these via a rotation by $\pi$ to the free edge provided by cutting along the other (cf. [La, Va]).

First we consider the family $(X_{t,u}, \omega_{t,u})$ and the related genus 2 flat surfaces $(\Xi_{t,u}, q_{t,u})$. To be explicit, the defining expressions of both types of surfaces are:

$$X_{t,u} : y^2 = x(x-1)(x-t)(x+u)(x+tu)(x^2+tu)$$

$$\omega_{t,u} = \frac{x \, dx}{y};$$

$$\Xi_{t,u} : y^2 = (x-1)(x-t)(x+u)(x+tu)(x^2+tu)$$

$$q_{t,u} = \frac{x \, dx^2}{y^2}.$$ 

The order 4 rotation $\rho_1\sigma_1$ of $X_{t,u}$ persists on $\Xi_{t,u}$. Following R. Silhol [Si], we find a new parameter $a$, depending on $t$ and $u$, so that the Riemann surface

$$\Xi_a : y^2 = x(x-1)(x-a)(x-1/a)$$

is isomorphic to $\Xi_{t,u}$. Doing so simply requires a change of coordinates in $x$, namely

$$\Phi(x) = i\sqrt{tu} \frac{(x-1)}{(x+tu)}.$$

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Then $\Phi(1) = 0$, $\Phi(-tu) = \infty$, and $\Phi(\pm i\sqrt{tu}) = \mp 1$. The images of $t$ and $u$ by $\Phi$ are

$$a = i\sqrt{\frac{u(t-1)}{t(u+1)}} \quad \text{and} \quad \frac{1}{a} = i\sqrt{\frac{t(1+u)}{u(1-t)}}.$$ 

Because $t > 1$ and $u > 0$, $a$ lies on the positive imaginary axis and $1/a$ lies on the negative imaginary axis. The involution $\rho$ becomes reflection across the imaginary axis. The images of 0 and $\infty$ by $\Phi$ are $\Phi(0) = i\sqrt{tu}$ and $\Phi(\infty) = \sqrt{tu}i$, so the image of $q_{t,u}$ on $\Xi_a$ is a scalar multiple of

$$\left(x - i\frac{1}{\sqrt{tu}}\right)\left(x - \sqrt{tu}\right)\frac{dx^2}{y^2} = \left(x^2 + i\left(\frac{tu-1}{\sqrt{tu}}\right)x + 1\right)\frac{dx^2}{y^2}.$$ 

These calculations imply that, for each pair $(t_0, u_0)$, there is a one-parameter family of surfaces $(\Xi_{t,u}, q_{t,u})$ such that $\Xi_{t,u}$ is isomorphic to $\Xi_{t_0,u_0}$ while $q_{t,u}$ and $q_{t_0,u_0}$ represent different differentials on the abstract Riemann surface.

Now we apply the same analysis to the second family. This time we are moving from $(X_s, \omega_s)$ to $(\Sigma_s, q_s)$, as defined below:

$$X_s : y^2 = x(x^2+1)(x-s)(x-\overline{s})(x+1/s)(x+1/\overline{s}), \quad \omega_s = \frac{x\,dx}{y};$$

$$\Sigma_s : y^2 = (x^2+1)(x-s)(x-\overline{s})(x+1/s)(x+1/\overline{s}), \quad q_s = \frac{x\,dx^2}{y^2}.$$ 

We change coordinates in $x$ using

$$\Psi(x) = i\left(\frac{x-s}{sx+1}\right)$$

so that $\Psi(s) = 0$, $\Psi(-1/s) = \infty$, and $\Psi(\pm i) = \mp 1$. This time we get the curve $y^2 = x(x^2-1)(x-a)(x-1/a)$, where

$$a = \Phi(\overline{s}) = \frac{2\text{Im} \, s}{1+|s|^2} \quad \text{and} \quad \frac{1}{a} = \Phi\left(-\frac{1}{\overline{s}}\right) = \frac{1+|s|^2}{2\text{Im} \, s}.$$ 

Here we have $0 < a < 1$ and $1/a > 1$; $\rho'$ becomes inversion in the unit circle. The points 0 and $\infty$ on $\Sigma_s$ become $\Phi(0) = -is$ and $\Phi(\infty) = i/s$. Again, we find just a one-parameter family of genus 2 Riemann surfaces, each carrying a one-parameter family of quadratic differentials corresponding to distinct surfaces $X_s$.

In [Si], it is shown that the full period matrix for any of the surfaces $\Xi_a$ can be expressed in terms of a single parameter, thanks to the fourfold symmetry of the surface. This parameter is the ratio of $\int_{-1}^{0} \varphi$ and $\int_{1/a}^{1} \varphi$, where $\varphi = \frac{dx}{y} - \frac{x\,dx}{y}$. This ratio is real precisely when $a$ lies on the positive imaginary axis, as in our first family, and in these cases the period matrix of $\Xi_a$ is purely imaginary.
4. Final remarks

The involutions we have exhibited also act on the Teichmüller disk generated by \((X_{AY}, \omega_{AY})\), and their effects can be seen via the iso-Delaunay tessellation shown in Figure 6. The open regions in this picture correspond to combinatorial classes of Delaunay triangulations of surfaces in the \(SL_2(\mathbb{R})\)-orbit of \((X_{AY}, \omega_{AY})\); points of the 1-skeleton correspond to surfaces with two or more Delaunay triangulations. Because Delaunay triangulations are not changed by any rotation of the surface, this picture can be drawn in the upper half-plane \(\mathbb{H}\) rather than its unit tangent bundle. Iso-Delaunay tessellations have been described previously in [Bo] and [Ve3]. We will not define them here, but simply illustrate how elements of the generalized Veech group \(\Gamma \subset GL_2(\mathbb{R})\) of \((X_{AY}, \omega_{AY})\) may be seen to act on the tessellation in Figure 6. (The hyperelliptic involution, having derivative \(-id\), acts trivially.)

Each element of \(\Gamma\) acts on \(\mathbb{H}\) by an isometry, preserving or reversing orientation according to the sign of its determinant. Figure 6 is symmetric with respect to the central axis (the imaginary axis in \(\mathbb{C}\)); both \(\sigma_1\) and \(\sigma_2\) yield elements of \(\Gamma\) that reflect \(\mathbb{H}\) across this axis. The hyperbolic element of \(\Gamma\) corresponding to \(\psi_{AY}\) fixes the points 0 and \(\infty\) in \(\partial \mathbb{H}\) and translates points along the imaginary axis by \(z \mapsto z/\alpha^2\). A sequence of concentric circles is visible in the tessellation; these are related by \(\psi_{AY}\), and one is the unit circle, so their radii are all powers of \(1/\alpha^2 \approx 3.38\).

There are two kinds of distinguished points on the central axis: ones where two geodesics meet and ones where three geodesics meet. The latter are those whose corresponding surface is isometric to \((X_{AY}, \omega_{AY})\), while the former correspond to \((X'_{AY}, \omega'_{AY})\). The real structures \(\rho_1\) and \(\rho_2\) (resp. \(\rho'_1\) and \(\rho'_2\)) yield an element of \(\Gamma\) that reflects \(\mathbb{H}\) across the unit circle (resp. across the circle \(|z| = 1/\alpha|\)). The order 4 rotations of \((X_{AY}, \omega_{AY})\) and \((X'_{AY}, \omega'_{AY})\) are thus visible as the order 2 rotations of \(\mathbb{H}\) around these distinguished points.

If any other flat surface on the central axis had real structures, then its symmetries, too, would have to induce a reflection of \(\mathbb{H}\) that preserves the tessellation. No such point exists; therefore we have described all the surfaces within the orbit of \((X_{AY}, \omega_{AY})\) under the geodesic flow that demonstrate additional symmetries.
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