A geometric action for non-geometric fluxes

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We give a geometrical interpretation of the non-geometric $Q$ and $R$ fluxes. To this end we consider double field theory in a formulation that is related to the conventional one by a field redefinition taking the form of a T-duality inversion. The $R$ flux is a tensor under diffeomorphisms and satisfies a non-trivial Bianchi identity. The $Q$ flux can be viewed as part of a connection that covariantizes the winding derivatives with respect to diffeomorphisms. We give a higher-dimensional action with a kinetic term for the $R$ flux and a ‘dual’ Einstein–Hilbert term containing the connection $Q$.

PACS numbers: 04.65.+e, 11.25.-w

String theory is a consistent theory in ten dimensions, and it is important to understand which four-dimensional theories it can give rise to. Phenomenologically, the so-called gauged supergravities are particularly interesting: the gaugings fix compactification moduli and can allow de Sitter vacua. Some gauged supergravities can be obtained through flux compactifications of string theory or de Sitter vacua. Some gauged supergravities can be obtained through flux compactifications of string theory or its low-energy supergravity limit. For instance, in the Neveu–Schwarz–Neveu–Schwarz (NS-NS) low-energy effective action for superstring theory,

\[ S = \int dx \sqrt{-g} e^{-2\phi} \left[ R(g) + 4\left(\partial\phi\right)^2 - \frac{1}{12} H^{ijk} H_{ijk} \right], \tag{1} \]

where $H_{ijk} = 3\hat{c}_{[i|b} b_{jk]}$ is the field strength of the NS-NS two-form $b_{ij}$, we may give a vacuum expectation value to this three-form. This results in massive or gauged supergravities in four dimensions. It is known, however, that there are more gauged supergravities in four dimensions that cannot be obtained through any conventional (flux) compactification. This is part of the motivation to consider non-geometric fluxes that are related to the conventional fluxes via T-duality \textsuperscript{1}. The T-duality rules suggest a chain

\[ H_{abc} \rightarrow f^{a}_{\ bc} \rightarrow Q_{c\ ab} \rightarrow R^{abc}. \tag{2} \]

Here, $H$ is the conventional three-form field strength discussed above, and $f^{a}_{\ bc} = -2\epsilon^{l}_{\ [b} e_{l}^{\ a} n_{c} e_{n}^{\ a}$, with the vielbein $e_{m}^{\ a}$, are the so-called ‘geometric fluxes’, which are related to the Levi-Civita spin connection. The geometrical interpretation of the non-geometric fluxes $Q$ and $R$ and the formulation of a higher-dimensional action will be the content of this letter. A more detailed exposition will appear elsewhere \textsuperscript{2}.

Recently, some of us have performed a field redefinition in ten-dimensional supergravity and obtained an action that captures part of the $Q$ flux \textsuperscript{3}. Introducing the field $\hat{c}_{ij} = g_{ij} + b_{ij}$ encoding the spacetime metric and the NS-NS 2-form, this field redefinition takes the form of a T-duality inversion in all ten directions:

\[ \mathcal{E} \rightarrow \tilde{\mathcal{E}} = \mathcal{E}^{-1}, \quad \tilde{\mathcal{E}}^{ij} = \tilde{g}^{ij} + \beta^{ij}, \tag{3} \]

where $\beta \rightarrow -\beta$ compared to \textsuperscript{3}. Being the inverse of $\mathcal{E}$, $\tilde{\mathcal{E}}$ naturally carries upper indices, satisfying $\tilde{\mathcal{E}}^{ik} \mathcal{E}_{kj} = \delta^{i}_{j}$, and we have decomposed $\tilde{\mathcal{E}}^{ij}$ into its symmetric and antisymmetric part. This gives rise to (the inverse of) a new metric $\tilde{g}_{ij}$ and an antisymmetric bivector $\beta^{ij}$. Moreover, a new dilaton was introduced via

\[ e^{-2\phi} \sqrt{-g} = e^{-2\tilde{\phi}} \sqrt{-\tilde{g}}, \tag{4} \]

where $\tilde{g} = \det \tilde{g}_{ij}$. In \textsuperscript{3} the simplifying assumption has been made that $\beta^{ij} \tilde{c}_{ij} = 0$ when acting on arbitrary fields. The action \textsuperscript{1} then reads in the new variables, up to total derivatives,

\[ S = \int dx \sqrt{-\tilde{g}} e^{-2\tilde{\phi}} \left[ R(\tilde{g}) + 4(\tilde{\partial}\tilde{\phi})^{2} - \frac{1}{2} \left| Q \right|^{2} \right], \tag{5} \]

where

\[ Q_{m}^{\ \ nk} = \tilde{c}_{m}^{\\ \beta nk}. \tag{6} \]

Being a partial derivative of a bivector, $Q$ is not a tensor, but it can be checked that its failure to transform covariantly becomes irrelevant upon using the simplifying assumption $\beta^{ij} \tilde{c}_{ij} = 0$. Let us now relax this assumption and consider the full field redefinition of \textsuperscript{1}. We then find that this gives rise to a new term involving part of the $R$ flux, which is a tensor, but the role of $Q$ in the full action is somewhat obscure. In particular, there does not appear to be a covariant tensor that reduces to \textsuperscript{3} upon using the assumption. We will show in this letter that the proper geometric interpretation of $Q$ becomes apparent once we consider the field variables $\tilde{g}_{ij}$ and $\beta^{ij}$ in the context of double field theory (DFT), where $Q$ will play the role of a connection rather than a tensor. This allows us to write a geometric action for $Q$ and $R$ fluxes.

We begin by reviewing DFT, which is an approach to make T-duality a manifest symmetry by doubling the coordinates at the level of the spacetime action for string theory \textsuperscript{4, 5}. (See also earlier work by Siegel and Tseytlin \textsuperscript{6, 7}.) In addition to the usual coordinates $x^{i}$ associated
to momentum modes there are new coordinates \( \tilde{x}_i \) associated to winding modes, which combine into a fundamental vector \( X^M = (\tilde{x}_i, x^i) \) under the T-duality group \( O(10, 10) \). Although the coordinates are formally doubled we have to impose the ‘strong constraint’

\[
\eta^{MN} \partial_M \tilde{\partial}_N = 0 \quad , \quad \eta^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

where \( \eta^{MN} \) denotes the \( O(10, 10) \) invariant metric and \( \partial_M = (\partial^i, \partial_i) \). This constraint holds on arbitrary fields, parameters and their products, so that in particular

\[
\partial_i \tilde{A} \partial^i B + \tilde{\partial}_i A \partial_i B = 0,
\]

for any \( A, B \). This constraint implies that for any solution the fields depend only on half of the coordinates. The double field theory formulation that is most convenient for our present purposes is based on the field \( E_i \) discussed above and a dilaton density \( d \), which is related to the scalar dilaton \( \phi \) via the field redefinition \( e^{-2d} = e^{-2\phi} \sqrt{-g} \). Its action reduces to

\[
\int dx^M d\tilde{x}^M e^{-2d} \left( \mathcal{R}(\tilde{g}, \tilde{\partial}) + \mathcal{R}(\tilde{g}^{-1}, \tilde{\partial}) + \cdots \right).
\]

DFT is invariant under a ‘generalized diffeomorphism’ symmetry parametrized by an \( O(10, 10) \) vector parameter \( \xi^M = (\xi_i, \xi^i) \). For \( \partial^i = 0 \) this reduces to conventional general coordinate transformations \( x^i \rightarrow x^i - \xi^i(x) \) and \( b \)-field gauge transformations parametrized by \( \xi_i \). Conversely, keeping \( \tilde{\partial}^i \) non-zero but setting \( \partial_i = 0 \), the gauge transformations of DFT reduce in particular to general coordinate transformations in the dual coordinates, \( \tilde{x}_i \rightarrow \tilde{x}_i - \xi_i(\tilde{x}) \). Prior to solving the strong constraint by setting half of the derivatives to zero, the full \( \xi^M \) gauge symmetry is not manifest in terms of the conventional fields \( g_{ij} \) and \( b_{ij} \). In particular, the \( \xi^i \) transformations act non-linearly, and originally the gauge invariance has only been verified through a lengthy computation \( [8] \). Afterwards, a manifestly \( O(10, 10) \) invariant formulation has been found in \( [9] \), which linearizes the gauge transformations in terms of a generalized metric. This gives rise to a more geometrical formulation which, employing earlier work by Siegel \( [7] \), has been further developed in \( [4, 11] \). The geometrical formulations developed so far involve \( O(10, 10) \) covariant tensors that combine the metric \( g \) and the \( b \)-field into a single object. Here, we will give a formulation in terms of the ‘component’ fields \( \tilde{g} \) and \( \beta \), whose tilde derivative transforms as

\[
\delta \tilde{g}_{ij} = \mathcal{L}_\xi \tilde{g}_{ij}, \quad \delta \beta^{ij} = \tilde{\partial}^i \xi - \tilde{\partial}^j \xi^i + \mathcal{L}_\xi \beta^{ij},
\]

where \( \mathcal{L}_\xi \) denotes the Lie derivative, which acts in the usual way on tensors,

\[
\mathcal{L}_\xi \beta^{ij} = \xi^k \partial_k \beta^{ij} - \beta^{ik} \partial_k \xi^j - \beta^{jk} \partial_k \xi^i,
\]

and similarly on objects with an arbitrary index structure. We call a transformation covariant if it involves only the Lie derivative, and we denote the non-covariant part of a variation by \( \Delta \xi \equiv \delta \xi - \mathcal{L}_\xi \), so that from

\[
\Delta \tilde{g}_{ij} = 0 \quad , \quad \Delta \beta^{ij} = \tilde{\partial}^i \xi - \tilde{\partial}^j \xi^i.
\]

Let us now develop a tensor calculus for the winding derivatives but with respect to the ‘momentum’ diffeomorphisms generated by \( \xi^i \). We will define covariant derivatives and invariant curvatures. We start by considering a scalar like the dilaton \( \phi \), whose tilde derivative transforms as

\[
\delta \tilde{\partial}^i (\tilde{g}^p \tilde{\partial}^q \phi) = \tilde{\partial}^i (\tilde{g}^p \partial^q \phi) + \tilde{\partial}^i \xi^p \tilde{\partial}^q \phi.
\]

In order to bring this into a form that is closer to the Lie derivative of a vector we add on the right-hand side

\[
- \tilde{\partial}^p \xi^i \tilde{\partial}^q \phi - \tilde{\partial}^q \xi^i \tilde{\partial}^p \phi = 0,
\]

which is zero due to the strong constraint \( [8] \), and obtain

\[
\delta \tilde{\partial}^i \phi = \mathcal{L}_\xi (\tilde{\partial}^i \phi) + (\tilde{\partial}^i \mathcal{E}^p - \tilde{\partial}^p \xi^i) \tilde{\partial}^q \phi.
\]

Thus, \( \tilde{\partial} \phi \) does not transform covariantly, but its non-covariant variation contains the same inhomogeneous term as the transformation \( [12] \) of \( \beta^{ij} \). Therefore, introducing the derivative operator

\[
\tilde{D}^i \equiv \tilde{\partial}^i - \beta^{ij} \partial_j,
\]

we see that \( \tilde{D}^i \phi \) is a fully covariant derivative of a scalar. In the following, the derivative \( [10] \) will play the role of a partial but anholonomic derivative. The \( \tilde{D}^i \) are non-commuting, and their commutator reads

\[
[\tilde{D}^i, \tilde{D}^j] = -R^{ijk} \tilde{\partial}_k - Q_{kij} \tilde{D}^k,
\]
where
\[ R^{ijk} = 3\tilde{D}^{[i} \beta^{jk]} = 3(\tilde{\epsilon}^{[i} \beta^{jk]} + \beta^{pl} \tilde{\epsilon}_p \beta^{jk]}), \] (18)
and \( Q \) is still given by \( \Phi \). The proof of (17) requires the
strong constraint (3). One may verify, using again (3), that \( R^{ijk} \) transforms covariantly under (10). Thus, (18)
represents the covariant field strength of \( \beta^{ij} \) that we will refer to as \( R \) flux in the following and which coincides
with the expression found in [12].

Next, we define derivatives that are covariant when acting on arbitrary tensors. For a vector \( V \) we set
\[ \tilde{\nabla}^i V_j = \tilde{D}^i V_j - \tilde{\Gamma}^i_{kj} V_k, \quad \tilde{\nabla}^i V_j = \tilde{D}^i V_j + \tilde{\Gamma}^i_{jk} V_k, \] (19)
and similarly for tensors with an arbitrary number of upper and lower indices. In order for (19) to transform
covariantly, the connection \( \tilde{\Gamma} \) needs to transform as
\[ \Delta_\xi \tilde{\Gamma}^{ij} = -\tilde{D}^i \tilde{\partial}_k \xi^j. \] (20)
We note that the antisymmetric part \( \tilde{\Gamma}^{[ij]} \) does not transform as a tensor and therefore cannot be set to zero.
In order to express the connection in terms of the physical fields we impose two covariant constraints. First, we
require the metricity condition
\[ \tilde{\nabla}^i \tilde{g}^{jk} = 0. \] (21)
This determines the symmetric part \( \tilde{\Gamma}^{(ij)} \) in terms of the antisymmetric part \( \tilde{\Gamma}^{[ij]} \) and \( \tilde{D}^i \tilde{g}^{jk} \). The antisymmetric part in turn is naturally fixed by requiring that the
commutator of covariant derivatives on a scalar is only given by the covariant term, involving the \( R \) flux,
\[ \left[ \tilde{\nabla}^i, \tilde{\nabla}^j \right] \tilde{\phi} = -R^{ijk} \tilde{\partial}_k \tilde{\phi}. \] (22)
This implies with (17)
\[ \tilde{\Gamma}^{[ij]} = -\frac{1}{2} Q^{ij}. \] (23)
Thus, \( Q \) transforms as the antisymmetric part of a connection,
\[ \Delta_\xi Q^{ij} = 2\tilde{D}^{[i} \tilde{\partial}_k \xi^{j]}, \] (24)
which can also be verified explicitly with (3) and (10). The full connection is then given by
\[ \tilde{\Gamma}^{i}_{kj} = \tilde{\Gamma}^{i}_{kj} + \tilde{g}_{kl} \tilde{g}^{p[q} Q^{i]}_{p} - \frac{1}{2} Q^{ij}, \] (25)
where
\[ \tilde{\Gamma}^{i}_{kj} = \frac{1}{2} \tilde{g}_{kl} \left( \tilde{D}^i \tilde{g}^{jl} + \tilde{D}^j \tilde{g}^{il} - \tilde{D}^l \tilde{g}^{ij} \right) \] (26)
are the conventional Christoffel symbols in the winding coordinates, but with \( \tilde{\partial}^i \) replaced by \( \tilde{D}^i \). The \( R \) flux [13]
satisfies a Bianchi identity that can be written in terms of these covariant derivatives as
\[ \tilde{\nabla}^{[i} R^{jkl]} = 0, \] (27)
or, explicitly,
\[ 4\tilde{\epsilon}^{[i} R^{jkl]} + 4\beta^{[i} \tilde{\epsilon}_p R^{jkl]} + 6Q_p^{[ij} R^{k]l]p} = 0. \] (28)
We finally note from (20) that the trace of the connection,
\[ \mathcal{T}^i \equiv \tilde{\Gamma}^i_{kj}, \] (29)
transforms as a tensor,
\[ \Delta_\xi \mathcal{T}^i = \Delta_\xi \tilde{\Gamma}^{i}_{kj} = -\tilde{\partial}_k \tilde{\partial}_j \xi^i - \beta^{pk} \tilde{\partial}_p \xi^j \xi^i = 0, \] (30)
by the strong constraint and the antisymmetry of \( \beta \). This is analogous to the antisymmetric part of the conve-
tional connection, i.e., the torsion tensor. Thus, we can think of \( \mathcal{T}^i \) as a new torsion, and we stress that it is
non-zero for (25).

Having defined covariant derivatives we next construct a Riemann tensor through the commutator of covariant
derivatives,
\[ [\tilde{\nabla}^i, \tilde{\nabla}^j] V_k = -R^{ijkp} \nabla_p V_k + \tilde{\Gamma}^{ij}_k V^p, \] (31)
where
\[ \tilde{\Gamma}^{ij}_k p = \tilde{D}^i \tilde{\Gamma}^{jk} p - \tilde{D}^j \tilde{\Gamma}^{ik} p + \tilde{\Gamma}^{kq} \tilde{\Gamma}^{jpq} - \tilde{\Gamma}^{jk} \tilde{\Gamma}^{ipq} + Q^i_{pq} \tilde{\Gamma}^{jk} p - R^{iq} \Gamma^p_{qk}. \] (32)
Here we have used the conventional covariant derivative for \( \tilde{\partial}_i \), with Christoffel symbols \( \Gamma^{ij}_k \) based on the metric \( \tilde{g}_{ij} \). As the \( R \) flux is fully covariant, the two terms on the right-hand side of (31) are separately covariant and therefore (32) defines a covariant curvature. From this we can define a Ricci tensor in the usual way,
\[ \tilde{\mathcal{R}}^{ij} \equiv \tilde{\mathcal{R}}^{ki}_j. \] (33)
or, explicitly,
\[ \tilde{\mathcal{R}}^{ij} = \tilde{D}^k \tilde{\Gamma}^{ij}_k - \tilde{D}^k \tilde{\Gamma}^{kj}_i + \tilde{\Gamma}^{ij}_k \tilde{g}^{pq} - \tilde{\Gamma}^{kij} \tilde{g}^{pq} \] (34)
which in general will not be symmetric in \( i,j \). Here, we used in the second equation that the trace of \( \tilde{\Gamma} \) yields the tensor (29) with a well-defined covariant derivative. Thus, curiously, the Ricci tensor decomposes into two tensors that are separately covariant. Finally, we can define a Ricci scalar,
\[ \tilde{\mathcal{R}} = \tilde{g}_{ij} \tilde{\mathcal{R}}^{ij}. \] (35)

We are now ready to give as our main result the full DFT action for \( \tilde{g}_{ij} \), \( \beta^{ij} \) and \( \phi \) in terms of the geometrical quantities defined above,
\[ S_{\text{DFT}} = \int d^dx \sqrt{-\tilde{g}} e^{-\tilde{\Phi}} \left[ R + \tilde{\mathcal{R}} - \frac{1}{12} R_{ijk} R^{ijk} + 4 \left( (\tilde{\partial} \phi)^2 + (\tilde{D} \phi)^2 + \tilde{\nabla}^i T_i - \mathcal{T}^i \mathcal{T}_i \right) \right]. \] (36)
This action is the precise version of the schematic form \([9]\). It involves a kinetic term for the \(R\) flux and two Einstein–Hilbert terms. The first is the conventional one for \(\tilde{g}_{ij}\) based on the usual derivatives \(\xi^i\). The second one is based on the winding derivatives but involving the novel connection \((25)\) including the \(Q\) flux. Moreover, the new torsion \(T^2\) is required in order to reproduce the full DFT. In \((26)\) every term is manifestly invariant under the diffeomorphisms generated by \(\xi^i\).

Upon setting \(D^i = 0\) and \(\kappa^i = 0\) we recover the situation analyzed in \([3]\), and the \(Q^2\) term in \((5)\) is the only remnant left of the ‘dual’ Einstein–Hilbert term. We may also set \(\tilde{c}^i = 0\) but keep \(D^i = -\beta^i{}^j \partial_j\) and \(\kappa^i\), for which \((36)\) reduces to the action obtained from the standard NS-NS action \((1)\) by performing the field redefinition \((3)\) without the simplifying assumption. This ten-dimensional supergravity action contains the \(R^2\) term, in which the \(R\) flux is reduced to the second term in \((18)\). It also contains \(Q^2\) terms and various couplings of \(\beta\) to dilaton and metric, but its geometric form and invariance is obscured in absence of the winding derivatives.

On a technical level the action \((36)\) provides an alternative formulation of DFT that makes half of the gauge symmetries, those parametrized by \(\xi^i\), manifest. The remaining gauge symmetries spanned by \(\xi^i\) are hidden in the formulation \((3)\), but we may return to the original fields \(g_{ij}\) and \(b_{ij}\) and employ the ‘T-dual’ of the geometrical structures discussed here. The \(R\) flux is then replaced by a covariantized \(H\) flux,

\[
H_{ijk} = 3(c^i b_{jk} + b^i p_{jk}) ,
\]

and similarly all other objects result from those introduced here by replacing the fields by the original fields, sending \(\tilde{c}^i \leftrightarrow \tilde{c}^i\) and, generally, consistently interchanging upper with lower indices. The resulting action will then be manifestly invariant under \(\xi^i\) gauge transformations. Thus, for each half of the gauge transformations there is a field basis where this symmetry can be made manifest, which in total yields an alternative proof for the full gauge invariance of DFT.

We believe that DFT provides a promising framework for compactifications with non-geometric fluxes. Indeed, the recent papers \([12, 13]\) showed already how \(\mathcal{N} = 4\) gauged supergravities in four dimensions result from DFT, but it turns out that the most general gaugings can only be obtained if the strong constraint is relaxed. Recently, mild relaxations of this constraint have indeed been found to be consistent \([14, 15]\).

In this letter we worked with the strongly constrained DFT, but we hope that the geometrical structures found here will provide a guide for non-geometric compactifications more generally. Finally, the action constructed here may also be closely related to the effective action of non-commutative and non-associative gravity, which describes closed strings on backgrounds with non-geometric fluxes \([16, 18]\). It would be particularly interesting to derive the action \((36)\) from the scattering amplitudes computed in \((18)\).

Acknowledgments

We would like to thank Barton Zwiebach for comments. This work is supported by the Alexander-von-Humboldt foundation, the DFG Transregional Collaborative Research Centre TRR 33 and the DFG cluster of excellence ‘Origin and Structure of the Universe’. DL thanks the Simons Center for Geometry and Physics, and ML thanks the Isaac Newton Institute in Cambridge for hospitality.

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