On Independent Transversal Dominating Sets in Graphs

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Abstract. A set $S \subseteq V(G)$ is an independent transversal dominating set of a graph $G$ if $S$ is a dominating set of $G$ and intersects every maximum independent set of $G$. An independent transversal dominating set which is a total dominating set is an independent transversal total dominating set. The minimum cardinality $\gamma_{it}(G)$ (resp. $\gamma_{itt}(G)$) of an independent transversal dominating set (resp. independent transversal total dominating set) of $G$ is the independent transversal domination number (resp. independent transversal total domination number) of $G$. In this paper, we show that for every positive integers $a$ and $b$ with $5 \leq a \leq b \leq 2a - 2$, there exists a connected graph $G$ for which $\gamma_{it}(G) = a$ and $\gamma_{itt}(G) = b$. We also study these two concepts in graphs which are the join, corona or composition of graphs.

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1. Introduction

Throughout this paper, by a graph $G = (V(G), E(G))$ is meant a finite, simple and connected graph with $V(G)$ and $E(G)$ being the vertex set and edge set, respectively. For $S \subseteq V(G)$, $|S|$ is the cardinality of $S$. In particular, $|V(G)|$ is called the order of $G$. All basic terminologies used here are adapted from [4].

Given two graphs $G$ and $H$ with disjoint vertex sets,

- the union of $G$ and $H$, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$;

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• the join of $G$ and $H$ is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{ uv : u \in V(G), v \in V(H) \}$;

• the corona of $G$ and $H$ is the graph $G \circ H$ obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$, and then joining the $i^{th}$ vertex of $G$ to every vertex in the $i^{th}$ copy of $H$; and

• the composition $G[H]$ of $G$ and $H$ is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u,v)(u',v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

For corona of graphs $G \circ H$, if $H = K_1$, we write $G \circ H = cor(G)$. It is customary to denote by $H^v$ that copy of $H$ whose vertices are joined with the vertex $v$ of $G$. Similarly, we also write $u^v$ to denote that copy of $u \in V(H)$ in $H^v$.

For vertices $u$ and $v$ of $G$, a $u$-$v$ geodesic is any shortest $u$-$v$ path. The distance between $u$ and $v$ is the length of a $u$-$v$ geodesic, and is denoted by $d_G(u,v)$. The eccentricity of $v$, denoted by $e(v)$, is given by $e(v) = \max \{d_G(u,v) : u \in V(G)\}$. The diameter of $G$, denoted by $\text{diam}(G)$, is defined by $\text{diam}(G) = \max \{d_G(u,v) : u, v \in V(G)\}$. Any geodesic of length $\text{diam}(G)$ is called a geodetic path of $G$.

Vertices $u$ and $v$ of a graph $G$ are neighbors if $uv \in E(G)$. The open neighborhood of $v$ refers to the set $N_G(v)$ consisting of all neighbors of $v$. The closed neighborhood of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v]$. A subset $S \subseteq V(G)$ is a dominating set of $G$ if $N_G[S] = V(G)$. The minimum cardinality $\gamma(G)$ of a dominating set in $G$ is the domination number of $G$. We refer to [1, 2, 5, 7, 9, 11, 14, 16, 19] for the history, fundamental concepts and the subsequent developments of domination in graphs as well as its various applications.

Provided $G$ has no isolated vertices, a dominating set $S$ is a total dominating set of $G$ if every vertex in $S$ is adjacent to another vertex in $S$. The minimum cardinality of a total dominating set of $G$ is the total domination number of $G$ denoted by $\gamma_t(G)$. References [6, 8, 12] are excellent studies on total domination in graphs.

A subset $S \subseteq V(G)$ is an independent set of $G$ if for every distinct vertices $u$ and $v$ in $S$, $uv \notin E(G)$. The maximum cardinality of an independent set is called the independence number of $G$, and is denoted by $\beta_0(G)$. Any independent set of cardinality $\beta_0(G)$ is referred to as a $\beta_0$-set of $G$. In [13], it is called a maximum independent set. It is worth noting that if $G$ is complete, then $\{v\}$ is a $\beta_0$-set for all $v \in V(G)$. The symbol $XI(G)$ denotes the family of all $\beta_0$-sets of $G$, and $\xi(G) = |XI(G)|$.

A subset $S \subseteq V(G)$ of $G$ is said to be an independent transversal set of $G$ if $S$ intersects every $\beta_0$-set of $G$. The minimum cardinality of an independent transversal set of $G$ is called the independent transversal number, denoted by $\beta_{ITD}(G)$. In particular, $\beta_{ITD}(K_n) = n$; for path $P_n$, $\beta_{ITD}(P_n) = 1$ when $n$ is odd and $\beta_{ITD}(P_n) = 2$ otherwise; for cycle $C_n$ on $n \geq 3$ vertices, $\beta_{ITD}(C_n) = 2$ when $n$ is even and $\beta_{ITD}(C_n) = 3$ otherwise; and for the complete bipartite $K_{m,n}$, $\beta_{ITD}(K_{m,n}) = 1$ if $m \neq n$, and $\beta_{ITD}(K_{m,n}) = 2$ otherwise.

An independent transversal set of $G$ is an independent transversal dominating set or (ITD-set) if $S$ is a dominating set of $G$. The minimum cardinality of an ITD-set of $G$ is called the independent transversal domination number of $G$ and is denoted by $\gamma_{ITD}(G)$. An
ITD-set $S$ of $G$ with $|S| = \gamma_{it}(G)$ is called a $\gamma_{it}$-set. The study on independent transversal domination in graphs was initiated by I. Hamid [10] in 2012. It was studied further by Yero et al. [17, 18, 20] in 2016, 2017 and 2021, and by Ozeki et al. [3] in 2018.

Provided $G$ has no isolated vertex, a subset $S \subseteq V(G)$ is an independent transversal total dominating set or (ITTD-set) if $S$ is both an ITD-set and a total dominating set of $G$. We denote by $\gamma_{itt}(G)$ the minimum cardinality of an ITTD-set of $G$, and is called the independent transversal total domination number of $G$. In [17], Cabrera Martinez et al. made a good introduction of this concept. The authors prove that the complexity of the decision problem associated to the computation of the value of $\gamma_{itt}(G)$ is NP-complete, under the assumption that the independence number is known.

Henceforth, all discussions of total domination or independent transversal total domination are always with respect to graphs without isolated vertices.

For graphs $G$ of order $n \geq 2$, $\gamma_{it}(G) \leq \gamma_{itt}(G)$.

2. Preliminaries and Known Results

**Observation 1.** (i) For the complete bipartite $K_{m,n}$, $\gamma_{it}(K_{m,n}) = \gamma_{itt}(K_{m,n}) = 2$.

(ii) For any path $P_n$ of order $n \geq 2$, we have

(a) [10] $\gamma_{it}(P_n) = \begin{cases} 2 & \text{if } n = 2, 3, \\ 3 & \text{if } n = 6, \\ \lceil \frac{n}{3} \rceil & \text{otherwise}, \end{cases}$

and

(b) $\gamma_{itt}(P_n) = \begin{cases} 3 & \text{if } n = 4, \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise} \end{cases}$

(iii) For any cycle $C_n$ of order $n \geq 3$, we have

(a) [10] $\gamma_{it}(C_n) = \begin{cases} 3 & \text{if } n = 3, 5, \\ \lceil \frac{n}{3} \rceil & \text{otherwise}, \end{cases}$

and

(b) $\gamma_{itt}(C_n) = \begin{cases} 3 & \text{if } n = 3, \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise}. \end{cases}$

**Theorem 1.** For any graph $G$, we have

(i) [10] $\gamma(G) \leq \gamma_{it}(G) \leq \gamma(G) + \delta(G)$; and

(ii) [17] $\gamma_{it}(G) \leq \gamma_{itt}(G) \leq \gamma_{it}(G) + \delta(G)$.

**Corollary 1.** Let $G$ be a graph.
(i) [10] If $G$ has an isolated vertex, then $\gamma_{it}(G) = \gamma(G)$.

(ii) If $G$ has $K_2$ as a component, then $\gamma_{it}(G) = \gamma_t(G)$.

If follows from Observation 1 that if $G = P_{3n-2}$, then $\gamma_{it}(G) = \gamma(G) = n$. Thus, the following corollary is clear.

**Corollary 2.** For all positive integers $n$, there exists a connected graph $G$ for which $\gamma(G) = \gamma_{it}(G) = n$.

**Observation 2.** Let $G$ be a connected graph of order $n$. Then

(i) $\gamma_{it}(G) = n$ if and only if $G = K_n$. Provided $n \geq 2$, $\gamma_{it}(G) = n$ if and only if $G = K_n$.

(ii) For $n \geq 3$,

(a) $\gamma_{it}(G) = n - 1$ if and only if $G = P_3$.

(b) $\gamma_{it}(G) = n - 1$ if and only if $G = P_3$ or $G = P_4$.

(iii) $\gamma_{it}(G) = 2$ if and only if either $G = K_2$ or $G$ has a dominating set $\{u,v\}$ such that every $\beta_0$-set contains $u$ or $v$.

(iv) $\gamma_{it}(G) = 2$ if and only if either $G = K_2$ or $G$ has a total dominating set $\{u,v\}$ such that every $\beta_0$-set contains either $u$ or $v$.

(v) $\gamma_{it}(G) = 1$ if and only if $n = 1$.

In Observation 2(iii), a $\beta_0$-set may contain both $u$ and $v$. Consider, for example, the graph $G = K_{n-1} + v$ for $n \geq 2$.

3. Realization Problems

**Theorem 2.** [20] For every positive integers $a$, $b$, $c$ such that $c \geq 2$ and $a \leq b \leq a + c$, there exists a graph $G$ such that $\delta(G) = c$, $\gamma(G) = a$ and $\gamma_{it}(G) = b$.

**Theorem 3.** [17] For any two positive integers $a$ and $b$ such that $2 \leq a \leq \frac{2b}{3}$, there is a graph of order $b$ such that $\gamma_{it}(G) = a$.

**Theorem 4.** For any positive integers $a$ and $b$ with $3 \leq a \leq b$ there exists a connected graph $G$ for which $\gamma_t(G) = a$ and $\gamma_{it}(G) = b$.

**Proof.** We consider the following cases:

**Case 1:** Suppose that $a = b$. If $a = b = 3$, then we choose $G = C_5$. Suppose that $a = b \geq 4$. If $a \equiv 0, 2 \pmod{4}$, then choose $G = P_{2a}$. If $a \equiv 1, 3 \pmod{4}$, then choose $G = P_{2a-1}$. In any case, $\gamma_t(G) = \gamma_{it}(G) = a$. 

Figure 1: Graph $G = C_a \circ \{K_n \cup K_n\}$ with $a \geq 3$

**Case 2:** Suppose that $a < b$. The $b = a + n$ for some positive integer $n$. Consider the corona $G = C_a \circ \{K_n \cup K_n\}$ as shown in Figure 1. For convenience, write $(K_n \cup K_n)^v = K_n^v \cup K_n^v$. Clearly, $V(C_a)$ is a $\gamma$-set of $G$ so that $\gamma(G) = a$. Let $v \in V(C_a)$ and define $S = V(C_a) \cup V(K_n^v)$. Then, $S$ is a total dominating set of $G$. Now a $\beta_0$-set $M$ of $G$ is of the form $M = \bigcup_{v \in V(C_a)} \{a^v, b^v\}$, where $a^v$ and $b^v$ belong to different components of $K_n^v \cup K_n^v$. Then $M \cap S \neq \emptyset$. Thus, $S$ is an ITTD-set of $G$. Thus, $\gamma_{it}(G) \leq |S| = a + n = b$. Let $T \subseteq V(G)$ be a total dominating set with $|T| < b$. Then $V(K_n^v) \setminus T \neq \emptyset$ for all $v \in V(C_a)$. For each $v \in V(C_a)$, pick $a^v, b^v \in V(K_n^v \cup K_n^v) \setminus T$ such that $a^v$ and $b^v$ are from different components. Then $M = \{a^v, b^v : v \in V(G)\}$ is a maximum independent set of $G$. Since $M \cap T = \emptyset$, $T$ is not an ITTD-set of $G$. Since $T$ is arbitrary, $\gamma_{itt}(G) = b$. \qed

**Theorem 5.** For every positive integers $n, a$ and $b$ with $5 \leq a \leq b \leq 2a - 2$, there exists a connected graph $G$ for which $\gamma_{it}(G) = a$ and $\gamma_{itt}(G) = b$.

**Proof.** For $k \geq 2$, denote by $G(k)$ the graph given in Figure 2.

![Figure 2: Graph $G(k)$](image)

Suppose that $a = b$. Take the graph $G$ as in Figure 3 obtained from the graph $G(a-1)$ by adding the path $[x_{a-1}, x_a, x_{a+1}, x_{a+2}, x_{a+2}]$. Observe that $\{x_2, x_3, \ldots, x_{a-1}, x_{a+2}\}$ is a $\gamma$-set of $G$ and every maximum independent set of $G$ contains either $x_{a+2}$ or $x_{a+2}$. Thus,
the set \( D = \{x_2, x_3, \ldots, x_{a-1}, x_{a+2}, x^{a+2}\} \) is a \( \gamma_G \)-set of \( G \). Note that \([x_2, x_3, \ldots, x_{a-1}]\) is a path of length at least 2. Hence \( D \) is also a \( \gamma_{ut} \)-set of \( G \). Therefore \( \gamma_{ut}(G) = \gamma_{ut}(G) = (a - 2) + 2 = a \).

Suppose now that \( a < b \). Then \( b = a + k \) for some positive integer \( 1 \leq k \leq a - 2 \).

For \( k = 1 \), consider the graph \( G \) in Figure 4 obtained from \( G(a-2) \) by adding the paths \([x_{a-2+3j}, x_{a-1+3j}, x_{a+3j}, x_{a+1+3j}, x^{a+1+3j}]\) where \( j = 0, 1 \). By a similar argument

\[
D = \{x_2, x_3, \ldots, x_{a-2}, x_{a+1}, x_{a+4}, x^{a+4}\} \text{ is a } \gamma_G \text{-set of } G. \text{ This implies that } \gamma_G(G) = (a - 3) + 3 = a. \text{ It also follows that } D \cup \{x^{a+1}\} \text{ is a } \gamma_{ut} \text{-set of } G. \text{ Thus, } \gamma_{ut}(G) = a + 1 = b.

For \( 2 \leq k \leq a - 4 \), consider the graph \( G \) in Figure 5 obtained from \( G(a-k-1) \) by adding \((k + 1)\) paths \([x_{a-k-1+3j}, x_{a-k+3j}, x_{a-k+1+3j}, x_{a-k+2+3j}, x^{a-k+2+3j}]\) \((j = 0, 1, \ldots, k - 1)\) and \([x_{a+2k}, x_{a+2k+1}, x_{a+2k+2}, x_{a+2k+3}, x_{a+2k+4}, x_{a+2k+5}]\). Then \( D = \{x_2, x_3, \ldots, x_{a-k-1}\} \cup \{x_{a-k+2+3j} : j \in \{0, 1, 2, \ldots, k - 1\}\} \cup \{x_{a+2k+3}\} \text{ is a } \gamma_{ut} \text{-set of } G \). Thus, \( \gamma_{ut}(G) = (a - k - 2) + k + 2 = a \) and \( \gamma_{ut}(G) = a + k = b \).

For \( k = a - 3 \), consider the graph \( G \) in Figure 6 obtained from \( G(2) \), by adding \((a - 2)\) paths \([x_2, x_3, x_4, x^4], [x_4+3j, x_5+3j, x_6+3j, x_7+3j, x^{7+3j}], j = 0, 1, 2, \ldots, a - 5, \) and \([x_{a+2k-2}, x_{a+2k-1}, x_{a+2k}, x_{a+2k+1}, x_{a+2k+2}, x_{a+2k+3}]\). Then \( D = \{x_2\} \cup \{x_4+3j : j = 0, 1, 2, \ldots, a - 4\} \cup \{x_{a+2k+1}, x_{a+2k+2}\} \text{ is a } \gamma_G \text{-set of } G \). On the other hand, \( D \cup \{x_3\} \cup \{x^{a+3j} : j = 0, 1, 2, \ldots, a - 4\} \text{ is a } \gamma_{ut} \text{-set of } G \). Thus, \( \gamma_G(G) = 1 + a - 3 + 2 = a \) and \( \gamma_{ut}(G) = a + (a - 3) = a + k \).

Now, for \( k = a - 2 \), consider the graph \( G \) as in Figure 7 obtained by adding \((a - 2)\) paths \([x_{2+3j}, x_{3+3j}, x_{4+3j}, x_{5+3j}, x^{5+3j}], j = 0, 1, \ldots, a - 3\) to the path \( G(2) \). Then \( D = \{x_{3a-4}, x^{3a-4}, x_{2+3j} : j \in \{0, 1, 2, \ldots, a - 3\}\} \text{ is a } \gamma_G \text{-set of } G \) and \( D \cup \{x^{2+3j} : j \in \{0, 1, 2, \ldots, a - 3\}\} \text{ forms a } \gamma_{ut} \text{-set of } G \). Therefore, \( \gamma_G(G) = 2 + (a - 2) \) and \( \gamma_{ut}(G) = a + (a - 2) = b \). \( \square \)
4. On join of graphs

For an independent subset $D \subseteq V(G + H)$, either $D \subseteq V(G)$ or $D \subseteq V(H)$. In particular, if $\beta_0(H) < \beta_0(G)$, then $D$ is a $\beta_0$-set of $G + H$ if and only if $D \subseteq V(G)$ and is a $\beta_0$-set of $G$.

**Proposition 1.** Let $G$ and $H$ be connected graphs with $\beta_0(H) < \beta_0(G)$. Then $D \subseteq V(G + H)$ is an independent transversal (total) dominating set of $G + H$ if and only if one of the following holds:

(i) $D \subseteq V(G)$ and $D$ is an independent transversal (total) dominating set of $G$;

(ii) $D \cap V(G)$ is an independent transversal set of $G$ and $D \cap V(H) \neq \emptyset$.

**Proof.** Let $D \subseteq V(G + H)$. Assume that $D$ is an independent transversal (total) dominating set of $G + H$. In view of the preceding remark, $D \cap V(H) \neq \emptyset$. Suppose that $D \subseteq V(G)$. Then $D$ is a (total) dominating set of $G$. Let $M \subseteq V(G)$ be a $\beta_0$-set of $G$. Then $M$ is a $\beta_0$-set of $G + H$. Thus, $D \cap M \neq \emptyset$. This shows that $D$ is an independent transversal (total) dominating set of $G$. Suppose that $D$ intersects both $V(G)$ and $V(H)$, and let $M \subseteq V(G)$ be a $\beta_0$-set of $G$. Since $M$ is a $\beta_0$-set of $G + H$, $M \cap (D \cap V(G)) = M \cap D \neq \emptyset$. Thus, $D \cap V(G)$ is an independent transversal set of $G$.

Conversely, since (total) dominating sets of $G$ are (total) dominating sets of $G + H$, in view of the preceding remark, if (i) holds for $D \subseteq V(G)$, then $D$ is an independent transversal (total) dominating set of $G + H$. Suppose that (ii) holds for $D$. Then $D$ is a (total) dominating set of $G + H$. Let $M \subseteq V(G + H)$ be a $\beta_0$-set of $G + H$. Then $M \subseteq V(G)$ and is a $\beta_0$-set of $G$. Thus, $D \cap M = (D \cap V(G)) \cap M \neq \emptyset$. This shows that $D$ is an independent transversal (total) dominating of $G + H$. \qed

**Proposition 2.** Let $G$ and $H$ be connected graphs with $\beta_0(G) = \beta_0(H)$. Then $D \subseteq V(G + H)$ is an independent transversal (total) dominating set of $G + H$ if and only if $D \cap V(G)$ and $D \cap V(H)$ are independent transversal sets of $G$ and $H$, respectively.
Proof. Let $D \subseteq V(G+H)$. Suppose that $D$ is an ITD-set of $G+H$. We claim that $D$ intersects both $V(G)$ and $V(H)$ so that $D$ is in fact an ITTD-set of $G+H$. Suppose not, say $D \subseteq V(G)$. Pick a $\beta_0$-set $M \subseteq V(H)$ of $H$. Since $M$ is a $\beta_0$-set of $G+H$, $D \cap M \neq \emptyset$, which is impossible.

Now, for all $\beta_0$-sets $M \subseteq V(G)$ of $G$, $M$ is a $\beta_0$-set of $G+H$ so that $(D \cap V(G)) \cap M = D \cap M \neq \emptyset$. This means that $D \cap V(G)$ is an independent transversal set of $G$. Similarly, $D \cap V(H)$ is an independent transversal set of $H$.

Conversely, suppose that $D \cap V(G)$ and $D \cap V(H)$ are independent transversal sets of $G$ and $H$, respectively. Then $D$ is a total dominating set of $G+H$. Let $M \subseteq V(G+H)$ be a $\beta_0$-set of $G+H$. Either $M \subseteq V(G)$ and is a $\beta_0$-set of $G$ or $M \subseteq V(H)$ and is a $\beta_0$-set of $H$. Either case yields $D \cap M \neq \emptyset$. Thus, $D$ is an independent transversal (total) dominating set of $G+H$.

Corollary 3. Let $G$ and $H$ be connected graphs with $\beta_0(H) \leq \beta_0(G)$. Then the following hold:

(i) If $\beta_0(H) = \beta_0(G)$, then $\gamma_{it}(G+H) = \gamma_{it}(G+H) = \beta_{ut}(G) + \beta_{ut}(H)$.

(ii) If $\beta_0(H) < \beta_0(G)$ and $\beta_{ut}(G) < \gamma_{it}(G)$, then $\gamma_{it}(G+H) = \gamma_{it}(G+H) = 1 + \beta_{ut}(G)$.

(iii) If $\beta_0(H) < \beta_0(G)$ and $\beta_{ut}(G) = \gamma_{it}(G)$, then $\gamma_{it}(G+H) = \gamma_{it}(G+H) = \gamma_{it}(G)$.

(iv) If $\beta_0(H) < \beta_0(G)$ and $\beta_{ut}(G) = \gamma_{it}(G) < \gamma_{it}(G)$, then $\gamma_{it}(G+H) = \gamma_{it}(G)$ and $\gamma_{it}(G+H) = 1 + \beta_{ut}(G)$.

Proof. The case where $\beta_0(G) = \beta_0(H)$ is immediate from Proposition 2. Assume that $\beta_0(G) > \beta_0(H)$. It follows from Proposition 1 that $\gamma_{it}(G+H) = \min\{\gamma_{it}(G), 1 + \beta_{ut}(G)\}$. Suppose that $\gamma_{it}(G) > \beta_{ut}(G)$. Then $\gamma_{it}(G+H) = 1 + \beta_{ut}(G)$ and any $\gamma_{it}$-set of $G+H$ is a total dominating set of $G+H$. Thus, $\gamma_{it}(G+H) = 1 + \beta_{ut}(G)$. Suppose that $\gamma_{it}(G) < \beta_{ut}(G)$. If $\gamma_{it}(G) = \gamma_{it}(G)$, then $\gamma_{it}(G+H) = \gamma_{it}(G+H) = \gamma_{it}(G)$. Suppose that $\gamma_{it}(G) < \gamma_{it}(G)$. Then $1 + \beta_{ut}(G) \leq \gamma_{it}(G)$. In view of Proposition 1, $\gamma_{it}(G+H) = 1 + \beta_{ut}(G)$.

Example 1. (1) For positive integers $m, n$ and $p$ with $p > \max\{m, n\}$,

$$
\gamma_{it}(K_{m,p} + K_{n,p}) = \gamma_{it}(K_{m,p} + K_{n,p}) = 2.
$$

(2) For the fan $F_n$ on $n + 1 \geq 3$ vertices,

$$
\gamma_{it}(F_n) = \begin{cases} 2, & \text{if } n = 4 \text{ or } n \text{ is odd;} \\
3, & \text{if } n \text{ is even and } n \neq 4, \end{cases} \quad \text{and} \quad \gamma_{ut}(F_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\
3, & \text{if } n \text{ is even.}
\end{cases}
$$
For the wheel $W_n$ on $n + 1$ vertices,

$$\gamma_{it}(W_n) = \begin{cases} 2, & \text{if } n = 4; \\ 3, & \text{if } n \in \{3, 5, 7, 9\} \text{ or } n \text{ is even and } n \neq 4; \\ 4, & \text{otherwise}, \end{cases}$$

and

$$\gamma_{itt}(W_n) = \begin{cases} 2, & \text{if } n = 4; \\ 3, & \text{if } n \in \{3, 5\} \text{ or } n \text{ is even and } n \neq 4; \\ 4, & \text{otherwise}. \end{cases}$$

(4) For all positive integers $n \geq 2$ and $p \geq 2$, $\gamma_{it}(K_{n,n} + K_p) = \gamma_{itt}(K_{n,n} + K_p) = 2$.

5. On corona of graphs

It is worth noting that $G \circ H$ is composed of the joins $H^v + v = H^v + \langle v \rangle$, $v \in V(G)$, joined together by the edges of $G$. Thus,

$$V(G \circ H) = V(G) \cup (\bigcup_{v \in V(G)} V(H^v)) = \bigcup_{v \in V(G)} V(H^v + v).$$

Theorem 6. [5] Let $G$ be a connected graph and $H$ any graph. Then $S \subseteq V(G \circ H)$ is a dominating set of $G \circ H$ if and only if $S \cap V(H^v + v)$ is a dominating set of $H^v + v$ for each $v \in V(G)$.

Observation 3. Let $G$ be a nontrivial connected graph of order $n$, and let $p$ be a positive integer. Then

$$\gamma_{it}(G \circ K_p) = \begin{cases} n, & \text{if } p = 1; \\ n + p, & \text{if } p \geq 2, \end{cases} \quad \text{and} \quad \gamma_{itt}(G \circ K_p) = n + p.$$

A sharp bound for $\gamma_{it}(G \circ H)$ is provided in [20]

Theorem 7. [20] Let $G$ be a graph of order $n \geq 2$. Then for any graph $H$ such that $\beta_0(H) \geq 2$,

$$n - 1 + d_\beta(H) \leq \gamma_{it}(G \circ H) \leq n + d_\beta(H),$$

where $d_\beta(H)$ is the largest number of pairwise disjoint $\beta_0$-sets of $H$. Moreover, if there is a $d_\beta(H)$-set which is a dominating set in $H$, then $\gamma_{it}(G \circ H) = n - 1 + d_\beta(H)$.

Observe that $d_\beta(H) \leq \beta_{0it}(H)$ for any graph $H$. In what follows, we determine $\gamma_{it}(G \circ H)$ in terms of $\beta_{0it}(H)$.

Lemma 1. Let $G$ be a connected graph and $H$ a noncomplete graph. Then a subset $S \subseteq V(G \circ H)$ is a $\beta_0$-set of $G \circ H$ if and only if $S = \bigcup_{v \in V(G)} S_v$, where $S_v \subseteq V(H^v)$ is a $\beta_0$-set of $H^v$ for each $v \in V(G)$. 

**Proposition 3.** Let $G$ and $H$ be nontrivial connected graphs with $H$ noncomplete. Then $S \subseteq V(G \circ H)$ is an ITD-set of $G \circ H$ if and only if $S = A \cup (\cup_{v \in V(G)} S_v)$, where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for all $v \in V(G)$ satisfying the following:

(i) For each $v \in V(G) \setminus A$, $S_v$ is a dominating set of $H^v$.

(ii) There exists $v \in V(G)$ for which $S_v$ is an independent transversal set of $H^v$.

**Proof.** Assume that $S \subseteq V(G \circ H)$ is an ITD-set of $G \circ H$. Then $S = A \cup (\cup_{v \in V(G)} S_v)$, where $A = S \cap V(G)$ and $S_v = S \cap V(H^v)$ for all $v \in V(G)$. Statement (i) follows immediately from Theorem 6. To prove (ii), suppose that for each $v \in V(G)$ there exists a $\beta_0$-set $M_v$ of $H^v$ for which $S_v \cap M_v = \emptyset$. By Lemma 1, $M = \cup_{v \in V(G)} M_v$ is a $\beta_0$-set of $G \circ H$. Since $S$ is an independent transversal set of $G \circ H$, $\cup_{v \in V(G)} (S_v \cap M_v) = S \cap M \neq \emptyset$, which is impossible. This proves (ii).

Conversely, assume that (i) and (ii) hold for $S$. By condition (i) and the fact that $S \cap V(H^v + v)$ is a dominating set in $H^v + v$ whenever $v \in A$, $S$ is a dominating set of $G \circ H$. Let $M \subseteq V(G \circ H)$ be a $\beta_0$-set of $G \circ H$. By (ii), there exists $v \in V(G)$ for which $S_v$ is an independent transversal set of $H^v$. Since $M_v = M \cap V(H^v)$ is a $\beta_0$-set of $H^v$, $S_v \cap M_v \neq \emptyset$. Thus, $S \cap M \neq \emptyset$. Therefore, $S$ is an ITD-set of $G \circ H$.

In view of Proposition 3, the following assertion is clear.

**Proposition 4.** Let $G$ and $H$ be nontrivial connected graphs with $H$ noncomplete. Then $S \subseteq V(G \circ H)$ is an independent transversal total dominating set of $G \circ H$ if and only if $S = A \cup (\cup_{v \in V(G)} S_v)$, where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for all $v \in V(G)$ satisfying the following:

(i) For each $v \in V(G) \setminus A$, $S_v$ is a total dominating set of $H^v$.

(ii) For each $v \in A$, $S_v \neq \emptyset$ or $N_G(v) \cap S \neq \emptyset$.

(iii) There exists $v \in V(G)$ for which $S_v$ is an independent transversal set of $H^v$.

**Corollary 4.** Let $G$ and $H$ be nontrivial connected graphs, where $G$ is of order $n$ and $H$ noncomplete. Then

$$\gamma_{it}(G \circ H) = n - 1 + \min\{\gamma_{it}(H), 1 + \beta_0t(H)\},$$

and

$$\gamma_{it}(G \circ H) = n - 1 + \min\{\gamma_{it}(H), 1 + \beta_0t(H)\}.$$
Let $S = A \cup (\cup_{v \in V(G)} S_v) \subseteq V(G \circ H)$ where $A \subseteq V(G)$ be a $\gamma_{it}$-set of $G \circ H$. By Proposition 3, $|S_w| \geq 1$ for all $w \in V(G) \setminus A$ and there exists $v \in V(G)$ for which $S_v$ is an independent transversal set of $H^v$. We consider two cases:

**Case 1:** Suppose that $v \in A$. Then
\[
\gamma_{it}(G \circ H) = |S| = |A| + \sum_{w \in V(G)} |S_w|
\]
\[
\geq n + |S_v|
\]
\[
\geq n - 1 + \min\{\gamma_{it}(H), 1 + \beta_{0t}(H)\}.
\]

**Case 2:** Suppose that $v \notin A$. Then $S_v$ is an ITD-set of $H^v$. Thus,
\[
\gamma_{it}(G \circ H) = |S| = |A| + \sum_{w \in V(G)} |S_w|
\]
\[
\geq n - 1 + |S_v|
\]
\[
\geq n - 1 + \min\{\gamma_{it}(H), 1 + \beta_{0t}(H)\}.
\]

This proves Equation 1.

Similar arguments will prove Equation 2. □

### 6. On composition of graphs

For a subset $C \subseteq V(G[H])$, we can always write $C = \cup_{x \in A}(\{x\} \times A_x)$ for some $A \subseteq V(G)$ and $A_x = \{y \in V(H) : (x, y) \in A\}$.

**Theorem 8.** [15] *Let $G$ and $H$ be connected graphs, and $C = \cup_{x \in S}(\{x\} \times S_x) \subseteq V(G[H])$. Then $C$ is a dominating set of $G[H]$ if and only if one of the following holds:*

(i) $S$ is a total dominating set of $G$;

(ii) $S$ is a dominating set of $G$ and $S_x$ is a dominating set of $H$ for each $x \in S \setminus N_G(S)$.

**Lemma 2.** Let $G$ and $H$ be nontrivial connected graphs, and $C = \cup_{x \in S}(\{x\} \times S_x) \subseteq V(G[H])$. Then $C$ is an (maximum) independent set of $G[H]$ if and only if $S$ is an (maximum) independent set of $G$ and $S_x$ is an (maximum) independent set of $H$ for each $x \in S$.

**Proposition 5.** Let $G$ and $H$ be nontrivial connected graphs, and $C = \cup_{x \in S}(\{x\} \times S_x) \subseteq V(G[H])$. Then $C$ is an independent transversal dominating set of $G[H]$ if and only if each of the following holds:

(i) One of the following holds:

(a) $S$ is an ITTD-set of $G$.
(b) $S$ is an ITD-set of $G$ and $S_x$ is a dominating set of $H$ for each $x \in S \setminus N_G(S)$.

(ii) For every pair of $\beta_0$-sets $A$ and $B$ of $G$ and $H$, respectively, there exists $x \in S \cap A$ for which $B \cap S_x \neq \emptyset$.

**Proof.** Suppose that conditions (i) and (ii) hold for $S$. Condition (i) implies that $C$ is a dominating set of $G[H]$ by Theorem 8. Let $D = \cup_{x \in A} \{x\} \times A_x \subseteq V(G[H])$ be a $\beta_0$-set of $G[H]$. By Lemma 2, $A$ is a $\beta_0$-set of $G$ and $A_x$ is a $\beta_0$-set of $H$ for each $x \in A$. Since $S$ is an ITD-set of $G$, $S \cap A \neq \emptyset$. By condition (ii), there exists $x \in S \cap A$ for which $S_x \cap A_x \neq \emptyset$. Let $y \in S_x \cap A_x$. Then $(x, y) \in C \cap D$. Since $D$ is arbitrary, $C$ is an ITD-set of $G[H]$.

Conversely, assume that $C$ is an ITD-set of $G[H]$. By Theorem 8, since $C$ is a dominating set of $G[H]$, $S$ is a dominating set of $G$ and $S_x$ a dominating set of $H$ for each $x \in S \setminus N_G(S)$ or $S$ is a total dominating set of $G$. Let $A \subseteq V(G)$ be a $\beta_0$-set of $G$ and $B \subseteq V(H)$ a $\beta_0$-set of $H$. Since $C^* = \cup_{x \in A} \{x\} \times B$ is a $\beta_0$-set of $G[H]$, $C \cap C^* \neq \emptyset$. Let $(x, y) \in C \cap C^*$. Then $x \in S \cap A$. So far, we have shown that $S$ is an ITD-set of $G$, thus (i) holds. Moreover, $y \in S_x \cap B$ so that $S_x \cap B \neq \emptyset$. This proves (ii). $\square$

It should be noted that if $S \cap A = \{x\}$ (singleton) in Theorem 5(ii), then $S_x$ is an independent transversal set of $H$.

Consider $G = P_5[P_4]$. Write $P_5 = [x_1, x_2, x_3, x_4, x_5]$ and $P_4 = [y_1, y_2, y_3, y_4]$. Then $C = \{(x_2, y_2), (x_3, y_1), (x_4, y_2), (x_4, y_2)\}$ is a $\gamma_{it}$-set of $G$. Put $S = \{x_2, x_3, x_4\}$ and $A = \{x_1, x_3, x_5\}$. Then $S$ is a $\gamma_{it}$-set of $P_5$ and $A$ is the unique $\beta_0$-set of $P_5$. Further, $S \cap A = \{x_3\}$ and $S_x = \{x_1, y_2\}$ is an independent transversal set of $P_4$.

The following is immediate from Proposition 5.

**Proposition 6.** Let $G$ and $H$ be nontrivial connected graphs, and $C = \cup_{x \in S} \{x\} \times S_x \subseteq V(G[H])$. Then $C$ is an independent transversal total dominating set of $G[H]$ if and only if each of the following holds:

(i) $S$ is an independent transversal total dominating set of $G$.

(ii) For every pair of maximum independent sets $A$ and $B$ of $G$ and $H$, respectively, there exists $x \in S \cap A$ for which $B \cap S_x \neq \emptyset$.

Given an ITD-set (resp. ITTD-set) $S \subseteq V(G)$ of $G$, define $M_G(S)$ (resp. $M'_G(S)$) to be any subset of $S$ of minimum cardinality such that for each $x \in M_G(S)$ (resp. $x \in M'_G(S)$), $x \in M$ for some $M \in XI(G)$, and put

$$\eta(G) = \min\{|M_G(S)| : S \text{ is a } \gamma_{it}\text{-set of } G\},$$

$$\eta_l(G) = \min\{|M'_G(S)| : S \text{ is a } \gamma_{itl}\text{-set of } G\}.$$
Corollary 5. For all nontrivial connected graphs \( G \) and positive integers \( p \geq 2 \),
\[
\gamma_{it}(G[K_p]) \leq (p - 1)\eta(G) + \gamma_{it}(G),
\]
and
\[
\gamma_{it}(G[K_p]) \leq (p - 1)\eta(G) + \gamma_{it}(G).
\]
Equality in each is attained if \( XI(G) \) consists of pairwise disjoint \( \beta_0 \)-sets of \( G \).

Proof. Let \( S \subseteq V(G) \) be a \( \gamma_{it} \)-set of \( G \) for which \( |M_G(S)| = \eta(G) \), and let \( y \in V(K_p) \).

Define
\[
C = \left[ \bigcup_{x \in M_G(S)} \{(x) \times V(K_p)\} \bigcup \bigcup_{x \in S \setminus M_G(S)} \{(x, y)\} \right].
\]
By Proposition 5, \( C \) is an ITD-set of \( G[K_p] \). Consequently,
\[
\gamma_{it}(G[K_p]) \leq |C| = p \cdot \eta(G) + (\gamma_{it}(G) - \eta(G)) = (p - 1)\eta(G) + \gamma_{it}(G).
\]
Now suppose that \( XI(G) \) consists of pairwise disjoint \( \beta_0 \)-sets of \( G \). Then \( |M_G(S)| = \eta(G) = x_i(G) \) for all ITD-sets \( S \) of \( G \).

Let \( C = \bigcup_{x \in S} \{(x) \times S_x\} \) be a \( \gamma_{it} \)-set of \( G[K_p] \).

In view of Proposition 5, \( S \) is an ITD-set of \( G \). Write
\[
C = \left( \bigcup_{x \in M_G(S)} \{(x) \times S_x\} \bigcup \bigcup_{x \in S \setminus M_G(S)} \{(x) \times S_x\} \right),
\]
and we claim that \( S_x = V(K_p) \) for all \( x \in M_G(S) \). Let \( x \in M_G(S) \), and let \( y \in V(K_p) \).

Pick a \( \beta_0 \)-set \( A \subseteq V(G) \) of \( G \) for which \( x \in S \cap A \). By Proposition 5(ii), since \( \{y\} \) is a \( \beta_0 \)-set of \( K_p \), there exists \( u \in S \cap A \), consequently \( u \in M_G(S) \), such that \( S_u = \{y\} \). By the minimality of the cardinality of \( M_G(S) \), \( x = u \) so that \( y \in S_x \). Accordingly, \( S_x = V(K_p) \).

Thus,
\[
\gamma_{it}(G \circ K_p) = |C| \geq p \cdot \eta(G) + (|S| - \eta(G)) \geq (p - 1)\eta(G) + \gamma_{it}(G).
\]

Similar arguments will prove the desired results for \( \gamma_{it}(G \circ K_p) \).

If, in particular, \( G = K_{1,n} \) on \( n + 1 \) vertices, then \( \eta(G) = 1 \), \( \gamma_{it}(G) = \gamma_{itt}(G) = 2 \), and
\[
\gamma_{it}(G[K_p]) = \gamma_{it}(G[K_p]) = p + 1.
\]

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