The containment profile of hyperrecursive trees

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Abstract We investigate vertex levels of containment in a random hypergraph grown in the spirit of a recursive tree. We consider a local profile tracking the evolution of the containment of a particular vertex over time, and a global profile concerned about counts of the number of vertices of a particular containment level.

For the local containment profile, we obtain the exact mean, variance and probability distribution in terms of standard combinatorial quantities like generalized harmonic numbers and Stirling numbers of the first kind. Asymptotically, we observe phases: the early vertices have an asymptotically normal distribution, intermediate vertices have a Poisson distribution, and late vertices have a degenerate distribution.

As for the global containment profile, we establish an asymptotically normal distribution for the number of vertices at the smallest containment level as well as their covariances with the number of vertices at the second smallest containment level and the variances of these numbers. The orders in the variance-covariance matrix establish concentration laws.

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1 Hyperrecursive trees

Hypergraphs are generalizations of graphs. In a hypergraph, we have vertices and hyperedges consisting of collections of vertices. Also, the recursive tree is a well-studied structure; see [2, 5, 7, 12], among many other sources. We propose in this paper a generalization of recursive trees to become hypergraphs.

A hyperrecursive tree with parameter (hyperedge size) \( \theta \) grows as follows. Initially, there are \( \theta \) originating vertices, all labeled with 0. These \( \theta \) vertices constitute the first hyperedge. At each subsequent step, a vertex is added to the structure. The incoming vertex chooses \( \theta - 1 \) existing vertices to co-share a hyperedge. A vertex joining at time \( n \) is labeled with \( n \). The choice of the vertices for the new hyperedge is done uniformly at random, with all subsets of vertices of size \( \theta - 1 \) being equally likely. The usual recursive tree is one with the parameter \( \theta = 2 \).

Figure 1 illustrates the growth of a hyperrecursive tree in two steps (i.e., at time \( n = 2 \)). In this example, \( \theta = 3 \), and the hyperedge appearing at step \( i \) is labeled \( e_i \).

Remark 1.1. For \( \theta \geq 3 \), the hyperrecursive tree is not a tree at all. We call such a hypergraph by this name only to preserve the historic origin of these structures and frame them as a generalization of genuine recursive trees (\( \theta = 2 \)).

![Figure 1: A hyperrecursive tree grown in two steps (\( \theta = 3 \)).](image)
2 Scope

Our interest is in a local profile of the level of containment for a vertex, which is the number of hyperedges containing it. Let $C_{n,k}^{(θ)}$ be the number of hyperedges containing vertex $k \geq 1$ at time $n \geq 0$. For instance, in the hyperrecursive tree in Figure 1 at $n = 2$, the vertex labeled 1 is at containment level 2, while the vertex labeled 2 is at containment level 1, and so $C_{2,1}^{(3)} = 2$ and $C_{2,2}^{(3)} = 1$. Each of the originators (all labeled with 0) may evolve differently and have different levels of containment at time $n$. However, the levels of these originators have the same distribution at time $n$, so we can choose a representative among them to include the case $k = 0$. For example, the representative of the originators in Figure 1 may be taken to be the rightmost vertex in the sole hyperedge at time 0. In the series of hypergraphs shown, the representative experiences the evolution $C_{0,0}^{(3)} = 1$, $C_{1,0}^{(3)} = 1$, and $C_{2,0}^{(3)} = 2$.

We later see that a vertex labeled $k = k(n)$ has an asymptotic distribution depending on the relation between $k$ and $n$. Asymptotically, earlier vertices have Gaussian distributions and later vertices have shifted Poisson distributions. The notions of “early” and “late” are made precise in the sequel, with nuances obtained from refinements into very early, early, intermediate, late, and very late vertices. We also discuss how the phases merge at the seam lines.

Moreover, we investigate a global profile of containment. Let $X_{n,i}^{(θ)}$ be the number of vertices at containment level $i$, that is, the number of vertices contained in $i$ edges after $n$ evolutionary steps. For instance, in the example shown in Figure 1 we have $X_{2,1}^{(3)} = 1$, $X_{2,2}^{(3)} = 4$, and $X_{2,i}^{(3)} = 0$, for $i \geq 3$. To get a glimpse into the interaction between different levels of containment, we compute the variance-covariance matrix of $(X_{n,1}^{(θ)}, X_{n,2}^{(θ)})$. Furthermore, we develop a Gaussian distribution for $X_{n,1}^{(θ)}$.

3 Notation

Let $τ_{n}^{(θ)}$ be the cardinality of the vertex set (size) of the hyperrecursive tree at time $n$ (right after the insertion of vertex $n$). We sometimes refer to $n$ as the age of the hyperrecursive tree. Note that

$$τ_{n}^{(θ)} = n + θ.$$  (1)
The exact results are represented in terms of Pochhammer’s symbol for the rising factorial. The \( m \)-times rising factorial of a real number \( x \) is

\[
\langle x \rangle_m = x(x + 1) \ldots (x + m - 1),
\]

with the interpretation that \( \langle x \rangle_0 = 1 \). We make use of such an expression in its form as a generating function of the signless Stirling numbers of the first kind, which is namely

\[
\langle x \rangle_m = \sum_{i=1}^{m} \left[ \begin{array}{c} m \\ i \end{array} \right] x^i,
\]

where \( \left[ \begin{array}{c} m \\ i \end{array} \right] \) is the \( i \)th (signless) Stirling number of order \( m \) of the first kind, a count of the number of permutations of \( \{1, 2, \ldots, m\} \) that have \( i \) cycles. For properties of Stirling numbers, we refer the reader to \cite{1, 3}.

Average values and variances contain generalized harmonic numbers. These are \( H_n^{(s)}(x) = \sum_{k=1}^{n} 1/(k + x)^s \), for integer \( n \geq 0 \) and real \( s, x \geq 0 \). The superscript is often dropped when it is 1; we follow this convention. It is well known that, for any fixed \( x \), as \( n \to \infty \), we have

\[
H_n(x) = H_n^{(1)}(x) = \ln(n) - \psi(x) - \frac{1}{x} + O\left(\frac{1}{n}\right); \quad (3)
\]

\[
H_n^{(s)}(x) = O(1) \text{ for } s > 1, \quad (4)
\]

where \( \psi(.) \) is the digamma function. Note that \( -\psi(x) - \frac{1}{x} \) converges to Euler’s constant \( \gamma \), as \( x \to 0 \).

In the asymptotic analysis, we utilize the Stirling approximation of the ratio of growing Gamma functions, as detailed in \cite{13}. Namely, for fixed \( a \) and \( b \) in \( \mathbb{R} \), we have

\[
\frac{\Gamma(x + a)}{\Gamma(x + b)} \sim x^{a-b} \left( 1 + \frac{(a - b)(a + b - 1)}{2x} + O\left(\frac{1}{x^2}\right) \right), \quad \text{as } x \to \infty. \quad (5)
\]

This approximation is applicable, even if \( a = a(x) \) and \( b = b(x) \) grow slowly with \( x \).

The sample taken at time \( n \) is drawn without replacement, and so the number of vertices in it at containment levels \( 1, \ldots, k \) have a (conditional) hypergeometric distribution.

\footnote{The number \( H_n^{(1)}(0) \) is often written as \( H_n \).}
We use the notation \( \text{Hypergeo}(\tau, n_1, n_2, \ldots, n_r; s) \) for the multivariate hypergeometric random vector, in which the \( i \)th component is the number of balls of color \( i \) that appear in a sample of size \( s \) drawn from an urn containing \( \tau \) balls, of which \( n_i \) balls are of color \( i \), for \( i = 1, \ldots, r \). This multivariate hypergeometric distribution is standard and can be found in classic books on distribution theory, such as [8]. In particular, we need the mean, variances and covariance for a bivariate marginal distribution. Suppose \( Y_i \) is the number of balls of color \( i \), for \( i = 1, \ldots, r \), that appear in the sample. Then, \((Y_i, Y_j, \tau - Y_i - Y_j)\) have a trivariate hypergeometric distribution like \( \text{Hypergeo}(\tau, n_i, n_j; \tau - n_i - n_j; s) \), with \( Y_i \) distributed like \( \text{Hypergeo}(\tau, n_i; \tau - n_i; s) \). Later, we utilize the formulas

\[
\begin{align*}
\mathbb{E}[Y_i] &= \frac{n_i}{\tau} s, \\
\text{Var}[Y_i] &= \frac{n_i(\tau - n_i)(\tau - s)}{\tau^2(\tau - 1)} s, \\
\text{Cov}[Y_i, Y_j] &= -\frac{n_i n_j(\tau - s)}{\tau^2(\tau - 1)} s;
\end{align*}
\]

see [8].

To develop martingale differences, we use the backward difference operator \( \nabla \). Acting on a function \( h_n \), this operator stands for \( \nabla h_n = h_n - h_{n-1} \).

4 Local containment profile

Let \( I_{n,k}^{(\theta)} \) be an indicator of the event that vertex \( n \) chooses vertex \( k \) in the hyperedge appearing at time \( n \). The indicator \( I_{n,k}^{(\theta)} \) is a Bernoulli random variable that assumes the value 1 with probability \( \frac{\binom{\tau_{\theta-1}^{(\theta)}}{\tau_{\theta-1}^{(\theta)}}}{\tau_{n-1}^{(\theta)}} = (\theta - 1)/\tau_{n-1}^{(\theta)} \), otherwise it assumes the value 0 with the complement probability. This indicator has the moment generating function

\[
\psi_{n,k}^{(\theta)}(t) := \mathbb{E}[e^{I_{n,k}^{(\theta)}}] = 1 - \frac{\theta - 1}{\tau_{n-1}^{(\theta)}} + \frac{(\theta - 1)e^t}{\tau_{n-1}^{(\theta)}}. \tag{6}
\]

We have a stochastic recurrence relation for \( C_{n,k}^{(\theta)} \). At time \( n \), the vertex labeled \( k \) either retains its level of containment at time \( n - 1 \) (if it is not
chosen for the $n^{th}$ hyperedge), or its level of containment increases by 1 (if chosen for the $n^{th}$ hyperedge). We thus have

$$C_{n,k}^{(\theta)} = C_{n-1,k}^{(\theta)} + I_{n,k}^{(\theta)}.$$  \hspace{1cm} (7)

The earliest time at which vertex $k$ is in the hyperrecursive tree is $k$, at which point it is contained in exactly one hyperedge. Therefore, the boundary condition is $C_{k,k}^{(\theta)} = 1$. Note that $C_{n-1,k}^{(\theta)}$ and $I_{n,k}^{(\theta)}$ are independent.

Unwinding the recurrence (7) back to the boundary condition, we get a representation

$$C_{n,k}^{(\theta)} = 1 + I_{k+1,k}^{(\theta)} + I_{k+2,k}^{(\theta)} + \cdots + I_{n,k}^{(\theta)}$$  \hspace{1cm} (8)

into independent (but not identically distributed) indicator random variables.

**Proposition 4.1.** Let $C_{n,k}^{(\theta)}$ be the containment level of the vertex labeled $k \geq 0$ in a hyperrecursive tree of edge size $\theta$ at age $n$.\footnote{Recall that when $k = 0$, we are tracking a chosen representative among the originators (all labeled with 0).} We have

$$\mathbb{E}[C_{n,k}^{(\theta)}] = 1 + (\theta - 1)\left(H_n(\theta - 1) - H_k(\theta - 1)\right)$$

$$= \begin{cases} 
(\theta - 1)(\ln(n) - \psi(\theta - 1) - \frac{1}{\theta - 1}) \\
\quad + 1 - (\theta - 1)H_k(\theta - 1) + O\left(\frac{1}{n}\right), & \text{if } k \geq 0 \text{ is fixed;} \\
(\theta - 1) \ln \left(\frac{n}{k}\right) + 1 + O\left(\frac{1}{k}\right), & \text{if } n \geq k \to \infty.
\end{cases}$$

and

$$\mathbb{V}[C_{n,k}^{(\theta)}] = (\theta - 1)\left(H_n(\theta - 1) - H_k(\theta - 1)\right)$$

$$- (\theta - 1)^2\left(H_n^{(2)}(\theta - 1) - H_k^{(2)}(\theta - 1)\right)$$

$$= \begin{cases} 
(\theta - 1) \ln(n) + O(1), & \text{if } k \geq 0 \text{ is fixed;} \\
(\theta - 1) \ln \left(\frac{n}{k}\right) + O\left(\frac{1}{k}\right), & \text{if } n \geq k \to \infty.
\end{cases}$$

**Proof.** Taking expectations of (8), we find

$$\mathbb{E}[C_{n,k}^{(\theta)}] = 1 + \mathbb{E}[I_{k+1,k}^{(\theta)}] + \mathbb{E}[I_{k+2,k}^{(\theta)}] + \cdots + \mathbb{E}[I_{n,k}^{(\theta)}]$$

$$= 1 + \sum_{i=k+1}^{n} \frac{\theta - 1}{\tau_i^{(\theta)}}$$

$$= 1 + (\theta - 1)\sum_{i=k+1}^{n} \frac{1}{i + \theta - 1}$$

$$= 1 + (\theta - 1)\left(H_n(\theta - 1) - H_k(\theta - 1)\right).$$

$$\mathbb{V}[C_{n,k}^{(\theta)}] = (\theta - 1)\left(H_n(\theta - 1) - H_k(\theta - 1)\right)$$

$$- (\theta - 1)^2\left(H_n^{(2)}(\theta - 1) - H_k^{(2)}(\theta - 1)\right)$$

$$= \begin{cases} 
(\theta - 1) \ln(n) + O(1), & \text{if } k \geq 0 \text{ is fixed;} \\
(\theta - 1) \ln \left(\frac{n}{k}\right) + O\left(\frac{1}{k}\right), & \text{if } n \geq k \to \infty.
\end{cases}$$
The asymptotic average follows from the approximation in (3).

By the independence of the indicators in (8), we similarly have

\[ \text{Var}[C_{n,k}^{(\theta)}] = \sum_{i=k+1}^{n} \text{Var}[I_{n,k}^{(\theta)}] = \sum_{i=k+1}^{n} \frac{\theta - 1}{i + \theta - 1} - \sum_{i=k+1}^{n} \frac{(\theta - 1)^2}{(i + \theta - 1)^2} = (\theta - 1)(H_n(\theta - 1) - H_k(\theta - 1)) - (\theta - 1)^2(H_n^{(2)}(\theta - 1) - H_k^{(2)}(\theta - 1)). \]

The asymptotic variance follows from the approximations in (3)–(4). \(\square\)

**Lemma 4.1.** The moment generating function \( \phi_{n,k}^{(\theta)}(t) = \mathbb{E}[e^{C_{n,k}^{(\theta)}t}] \) of \( C_{n,k}^{(\theta)} \) is given by

\[ \phi_{n,k}^{(\theta)}(t) = e^t \prod_{i=k}^{n-1} \frac{i + 1 + (\theta - 1)e^t}{i + \theta}. \]

**Proof.** The representation (8) as a sum of independent random variables gives rise to

\[ \phi_{n,k}^{(\theta)}(t) = \mathbb{E}[e^{C_{n,k}^{(\theta)}t}] = \mathbb{E}[e^{(1+I_{k+1,k}^{(\theta)},k+1+I_{k+2,k}^{(\theta)},\ldots,I_{n,k}^{(\theta)})t}] = e^t \mathbb{E}[e^{I_{k+1,k}^{(\theta)}t}] \mathbb{E}[e^{I_{k+2,k}^{(\theta)}t}] \ldots \mathbb{E}[e^{I_{n,k}^{(\theta)}t}] \text{ (by independence)} = e^t \psi_{k+1,k}^{(\theta)}(t) \psi_{k+2,k}^{(\theta)}(t) \ldots \psi_{n,k}^{(\theta)}(t). \]

Plug in (6) and the sizes of the hyperrecursive trees in (1), and the statement follows after simplification. \(\square\)

From Lemma 4.1 we develop an exact distribution.

**Theorem 4.1.** For \( n \geq 1 \) and \( 0 \leq k \leq n \), let \( C_{n,k}^{(\theta)} \) be the level of containment of the vertex \( k \) in a hyperrecursive tree with edge size \( \theta \) at age \( n \). For \( r \geq 1 \), we have

\[ \mathbb{P}(C_{n,k}^{(\theta)} = r) = \frac{(\theta - 1)^{r-1}}{(k + \theta)^{n-k}} \sum_{i=r-1}^{n-k} \binom{n-k}{i} \binom{i}{r-1} (k + 1)^{i-r+1}. \]
Proof. To obtain a probability generating function \( \zeta_{n,k}(u) = \sum_{i=0}^{\infty} \mathbb{P}(C_{n,k} = i) u^i \), we substitute \( \ln(u) \) for \( t \) in the generating function of Lemma (4.1). We obtain
\[
\zeta_{n,k}(u) = u \frac{(k + 1 + (\theta - 1)u) \cdots (n + (\theta - 1)u)}{(k + \theta) \cdots (n + \theta - 1)} \\
= u \frac{(k + 1 + (\theta - 1)u)_{n-k}}{(k + \theta)_{n-k}}.
\]
Using the generating function (2), we write
\[
\zeta_{n,k}(u) = \frac{u}{(k + \theta)_{n-k}} \sum_{i=0}^{n-k} \binom{n-k}{i} (k + 1 + (\theta - 1)u)^i \\
= \frac{u}{(k + \theta)_{n-k}} \sum_{i=0}^{n-k} (k + 1)^i (\theta - 1)^m u^m \binom{i}{m} \\
= \frac{u}{(k + \theta)_{n-k}} \sum_{m=0}^{\infty} (\theta - 1)^m u^m \sum_{i=m}^{n-k} \binom{n-k}{i} (k + 1)^{i-m}.
\]
The exact distribution in the statement of the theorem follows upon extracting coefficients. \( \square \)

4.1 Phases in the local containment profile of a vertex

Proposition 4.1 shows that the mean and variance of the hyperrecursive trees experience a phase change, as \( k \) increases relative to \( n \). For instance, for fixed \( k \), the mean is asymptotic to \((\theta - 1) \ln(n)\), as \( n \to \infty \), and \( k \) can only alter lower-order asymptotics. Such is the case for all fixed \( k \), as \( n \to \infty \).

However, a phase transition occurs when \( k \) grows to infinity with \( n \), but remains \( o(n) \), such as the case \( k = k(n) = \lceil 3n^2 + 2\sqrt{n} - \pi \rceil \). In this range, \( k \) provides essential leading-term asymptotics. The vertices that appear in the entire range in which \( k = o(n) \) are “early.”

For the linear “intermediate” range, \( k \sim \alpha n \), for \( 0 < \alpha < 1 \), such as the case \( \lceil \frac{2}{5} n + 3\sqrt{n} + 6 \rceil \), the asymptotic mean is \( 1 + (\theta - 1) \ln(\frac{1}{\alpha}) \). Vertices in the range \( j \sim n \) are considered “late”. The asymptotic mean of late vertices is just 1, showing that the late arrivals, such as the case \( k = \lfloor n - 5 \ln(n+1) + 14 \rfloor \), have negligible probability of participating in the recruiting events.
It is possible to conceive of a bizarre relation between \( k \) and \( n \), such as, for example, \( k = \lfloor \left( \frac{1}{2} + \frac{(-1)^n}{3} \right) n \rfloor \), for which the mean containment level oscillates, without settling on any asymptotic average. In these cases, we have no convergence in the mean, variance or distribution. Such an oscillating case is not likely to appear in practice.

Naturally, these phases in the mean are reflected in the asymptotic distributions. From Lemma 4.1, we can get asymptotic distributions. It is beneficial for the asymptotic analysis to represent the product in Lemma 4.1 in terms of Gamma functions:

\[
\phi_{n,k}^{(\theta)}(t) = e^{t/\sqrt{\ln(n)}} \frac{\Gamma(n + 1 + (\theta - 1)e^t) \Gamma(k + \theta)}{\Gamma(n + \theta) \Gamma(k + 1 + (\theta - 1)e^t)}.
\]  

**Theorem 4.2.** Let \( 0 \leq k \leq n \) and \( C_{n,k}^{(\theta)} \) be the containment level of the vertex \( k \) in a hyperrecursive tree with edge size \( \theta \) at age \( n \). We have

(i) For \( k \) fixed:

\[
\frac{C_{n,k}^{(\theta)} - (\theta - 1) \ln(n)}{\sqrt{\ln(n)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \theta - 1),
\]

(ii) For \( k \to \infty \) and \( k = o(n) \):

\[
\frac{C_{n,k}^{(\theta)} - (\theta - 1) \ln \left( \frac{n}{k} \right)}{\sqrt{\ln \left( \frac{n}{k} \right)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \theta - 1),
\]

(iii) For \( k \sim \alpha n \), and \( 0 < \alpha < 1 \):

\[
C_{n,k}^{(\theta)} \xrightarrow{\mathcal{D}} 1 + \text{Poi}\left( (\theta - 1) \ln \left( \frac{1}{\alpha} \right) \right).
\]

(iv) For \( k = n - o(n) \):

\[
C_{n,k}^{(\theta)} \xrightarrow{\mathcal{P}} 1.
\]

**Proof.** We start with the phase in which \( k \) is fixed. For this case, by the Stirling approximation in (5), we write (9) with \( t \) scaled:

\[
\phi_{n,k}^{(\theta)} \left( \frac{t}{\sqrt{\ln(n)}} \right) = e^{t/\sqrt{\ln(n)}} \frac{\Gamma(n + 1 + (\theta - 1)e^{t/\sqrt{\ln(n)}}) \Gamma(k + \theta)}{\Gamma(n + \theta) \Gamma(k + 1 + (\theta - 1)e^{t/\sqrt{\ln(n)}})}
\]

\[
\sim n^{(\theta-1)(e^{t/\sqrt{\ln(n)}} - 1)}.
\]
Going through a local expansion of the exponential, we write
\[
\phi_{n,k}(t) \sim e^{\left(\theta - 1\right) t \sqrt{\ln(n)}} + \frac{t^2}{2} \ln(n) + O\left(\frac{t^3}{\ln(n)}\right) \ln(n) - (\theta - 1) \ln(n).
\]

We can reorganize this relation as
\[
\phi_{n,k}(t) \sim e^{\theta - 1} t \sqrt{\ln(n)} \sim e^{(\theta - 1) t^2 / 2} + O\left(\frac{t^3}{\ln(n)}\right).
\]

At any fixed \(t \in \mathbb{R}\), we have convergence
\[
E\left[ e^{\frac{\phi_{n,k}(t)}{\sqrt{\ln(n)}}} \right] \to e^{(\theta - 1) t^2 / 2}.
\]

The right-hand side is the moment generating function of a centered normal distribution with variance \(\theta - 1\). By Lévy’s continuity theorem [14], we establish convergence in distribution as stated in Part (i).

The analysis of the rest of the phase of early \(k\), a phase in which \(k \to \infty\), but \(k = o(n)\), is not much different from the fixed \(k\) phase. It only requires some minor tweaks to bring in the role of \(k\), which is now pronounced. In this phase, we apply the Stirling approximation in (5) to all four gamma functions in (9). Consequently, we have
\[
\phi_{n,k}(t) \sim e^{\theta - 1} t \sqrt{\ln(n/k)} \ln(n/k) - (\theta - 1) \ln(n/k).
\]

From here, steps follow as in the case of fixed \(k\). We get convergence in distribution as stated in Part (ii) of the theorem.

In the intermediate and late phases \(j \sim \alpha n\), for \(\alpha \in (0,1]\), no scaling is required to get convergence in distribution. Instead, we have
\[
\phi_{n,k}(t) = e^{\theta - 1} t \sqrt{\ln(n/k)} \ln(n/k) - (\theta - 1) \ln(n/k) \sim e^{\theta - 1} t \ln(n/k) - (\theta - 1).
\]
The moment generating function on the right-hand side is that of $1$ added to a Poisson random variable with mean $(\theta - 1) \ln(1/\alpha)$. By Lévy’s continuity theorem (Theorem 18.1 in [14]), we establish convergence in distribution as stated in Part (iii). Then, $C_{n,k}^{(\theta)}$ degenerates to a constant in the case $\alpha = 1$, where we get

\[ C_{n,k}^{(\theta)} \xrightarrow{D} 1. \]

Convergence in distribution to a constant implies convergence in probability, as stated in Part (iv) of the theorem.

\[ \square \]

Remark 4.1. Phase changes of the type in Theorem 4.2 have been observed in [6] and [9].

Remark 4.2. At the seam lines between the phases, the change is not abrupt. In fact, the phases “flow into each other” in a natural way. For instance, in the case of Part (ii), we can write $\ln(n/k)$ as $\ln(n) - \ln(k)$. Then, we see that for $k$ fixed, $\ln(k)/\sqrt{\ln(n)} \to 0$, and by Slutsky’s theorem [11], we get the statement adjusted as in Part (i). The role of $k$ is negligible, so long as $\ln(k) = o(\sqrt{\ln(n)})$. Past this threshold, $\ln(k)$ cannot be neglected relative to $\sqrt{\ln(n)}$, and $k$ must be included in the convergence. Again, the limit in Part (iii) converges to that in Part (iv), as $\alpha \to 1$.

5 Global containment profile

In this section, we look at a profile of the hyperrecursive tree determined by a raw count of the number of vertices at a particular containment level. Such a profile is global, as it cannot be determined without looking at all the vertices in the entire hyperrecursive tree.

We defined $X_{n,i}^{(\theta)}$ to be the number of vertices contained in exactly $i$ hyperedges. To discern how the different containment levels interact, we investigate the mean of the row vector $(X_{n,1}^{(\theta)}, X_{n,2}^{(\theta)})$ and its covariance matrix.

We start with stochastic recurrences from which we proceed to a calculation of the exact mean and variances, which lead us to concentration laws. Eventually, we establish a central limit theorem for the vertices at the smallest level of containment, i.e., the vertices contained in one hyperedge. Note that when $\theta = 2$, in which case the hypergraph becomes the uniform recursive tree, vertices at containment level 1 are simply the leaves.
5.1 Stochastic recurrences

We discuss here recurrence equations that hold on the stochastic path. Let $Q_{n,i}^{\theta}$ be the number of vertices at containment level $i = 1, 2$ that appear in the sample chosen to construct the $n^{th}$ hyperedge. The row vector

$$(Q_{n,1}^{\theta}, Q_{n,2}^{\theta}, \tau_{n-1}^{\theta} - Q_{n,1}^{\theta} - Q_{n,2}^{\theta})$$

has a (conditional) trivariate hypergeometric distribution, that selects a sample of size $\theta - 1$ from among $\tau_{n-1}^{\theta}$ vertices, of which $X_{n,i}^{\theta}$ are at containment level $i$, for $i = 1, 2$. That is, the components of this row vector have the conditional joint distribution

$$P(Q_{n,1}^{\theta} = q_1, Q_{n,2}^{\theta} = q_2 \mid X_{n-1,1}^{\theta}, X_{n-1,2}^{\theta}) = \frac{\binom{X_{n-1,1}^{\theta}}{q_1} \binom{X_{n-1,2}^{\theta}}{q_2} \binom{\tau_{n-1}^{\theta} - X_{n-1,1}^{\theta} - X_{n-1,2}^{\theta}}{\theta - 1 - q_1 - q_2}}{\binom{\tau_{n-1}^{\theta}}{\theta - 1}}.$$

The conditional means, variances, and the covariance are specified in Section 3.

In the construction of the $n^{th}$ hyperedge, each vertex at containment level 1 in the sample becomes upgraded to containment level 2, and the newly added vertex at step $n$ is at containment level 1. Whence, we have the stochastic recurrence

$$X_{n,1}^{\theta} = X_{n-1,1}^{\theta} - Q_{n,1}^{\theta} + 1. \quad (10)$$

Each vertex at containment level 2 in the sample becomes upgraded to containment level 3. However, the $Q_{n,1}^{\theta}$ vertices at containment level 1 in the sample all become at containment level 2, giving rise to the stochastic recurrence

$$X_{n,2}^{\theta} = X_{n-1,2}^{\theta} - Q_{n,2}^{\theta} + Q_{n,1}^{\theta}. \quad (11)$$

5.2 The mean and covariance matrix

The pair of stochastic equations (10)–(11) is sufficient to determine the means exactly and the quadratic order moments asymptotically.

**Proposition 5.1.** Let $X_{n,i}^{\theta}$ be the number of vertices contained in exactly $i$ hyperedges, for $i = 1, 2$, of a recursive hyperrecursive tree of edge size $\theta$. We
have the mean vector

\[
\begin{pmatrix}
\mathbb{E}[X_{n,1}^{(\theta)}] \\
\mathbb{E}[X_{n,2}^{(\theta)}]
\end{pmatrix} = 
\begin{pmatrix}
\frac{1}{\theta} n + 1 + (\theta - 1) \Gamma(\theta) \frac{\Gamma(n + 1)}{\Gamma(n + \theta)} \\
(\theta - 1) \frac{n}{\theta^2} (n + \theta) + \Gamma(\theta) \frac{\Gamma(n + 1)}{\Gamma(n + \theta)} ((\theta - 1)^2 H_n - \frac{\theta - 1}{\theta})
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{n}{\theta} + 1 + O\left(\frac{1}{n^{\theta-1}}\right) \\
(\theta - 1) \frac{n}{\theta^2} (n + \theta) + O\left(\frac{\ln(n)}{n^{\theta-1}}\right)
\end{pmatrix}.
\]

**Proof.** Let \( \mathbb{F}_n^{(\theta)} \) be the sigma field generated by the first \( n \) hyperedge additions. Conditioning the stochastic relations (10) and (11) on \( \mathbb{F}_n^{(\theta)} \), we obtain

\[
\mathbb{E}[X_{n,1}^{(\theta)} | \mathbb{F}_n^{(\theta)}] = X_{n-1,1} - \mathbb{E}[Q_{n,1}^{(\theta)} | \mathbb{F}_n^{(\theta)}] + 1,
\]

\[
\mathbb{E}[X_{n,2}^{(\theta)} | \mathbb{F}_n^{(\theta)}] = X_{n-1,2} - \mathbb{E}[Q_{n,2}^{(\theta)} | \mathbb{F}_n^{(\theta)}] + \mathbb{E}[Q_{n,1}^{(\theta)} | \mathbb{F}_n^{(\theta)}].
\]

As discussed, the random variables \( Q_{n,1}^{(\theta)} \), \( Q_{n,2}^{(\theta)} \), and \( \tau_{n-1}^{(\theta)} - Q_{n,1}^{(\theta)} - Q_{n,2}^{(\theta)} \) have a (conditional) trivariate hypergeometric marginal distribution; the conditional expectations are

\[
\mathbb{E}[X_{n,1}^{(\theta)} | \mathbb{F}_n^{(\theta)}] = X_{n-1,1} - \frac{X_{n-1,1}^{(\theta)}}{\tau_n^{(\theta)}} (\theta - 1) + 1, \tag{12}
\]

\[
\mathbb{E}[X_{n,2}^{(\theta)} | \mathbb{F}_n^{(\theta)}] = X_{n-1,2} - \frac{X_{n-1,2}^{(\theta)}}{\tau_n^{(\theta)}} (\theta - 1) + \frac{X_{n-1,1}^{(\theta)}}{\tau_n^{(\theta)}} (\theta - 1);
\]

see the formulas in Section 3. Taking an iterated average, simultaneous recurrences can be written:

\[
\mathbb{E}[X_{n,1}^{(\theta)}] = \left(1 - \frac{\theta - 1}{\tau_n^{(\theta)}}\right) \mathbb{E}[X_{n-1,1}^{(\theta)}] + 1
\]

\[
= \frac{n}{n + \theta - 1} \mathbb{E}[X_{n-1,1}^{(\theta)}] + 1, \tag{13}
\]

\[
\mathbb{E}[X_{n,2}^{(\theta)}] = \left(1 - \frac{\theta - 1}{\tau_n^{(\theta)}}\right) \mathbb{E}[X_{n-1,2}^{(\theta)}] + \frac{\theta - 1}{\tau_n^{(\theta)}} \mathbb{E}[X_{n-1,1}^{(\theta)}]
\]

\[
= \frac{n}{n + \theta - 1} \mathbb{E}[X_{n-1,2}^{(\theta)}] + \frac{\theta - 1}{n + \theta - 1} \mathbb{E}[X_{n-1,1}^{(\theta)]}. \tag{14}
\]
The first of these two equations is self contained, while the second has to await for the solution of the first to be bootstrapped into it. The first equation has the standard form

\[ y_n = g_n y_{n-1} + h_n, \]  

with solution

\[ y_n = \sum_{i=1}^{n} h_i \prod_{j=i+1}^{n} g_j + y_0 \prod_{j=1}^{n} g_j. \]

We solve (13), with \( g_n = \frac{n}{(n+\theta-1)} \) and \( h_n = 1 \), and obtain

\[ \mathbb{E}[X^{(\theta)}] = \frac{n^2}{(n+\theta)} \sum_{i=1}^{n} \frac{\Gamma(i+\theta)}{\Gamma(i+1)} + \frac{n}{(n+\theta)} \frac{\Gamma(\theta)}{\Gamma(\theta)} \frac{\Gamma(n+1)}{\Gamma(n+\theta)}. \]

To simplify the sum, we use a known identity, which is namely

\[ \sum_{i=1}^{n} \frac{\Gamma(i+\alpha)}{\Gamma(i+\beta)} = \frac{\Gamma(n+\alpha+1)}{(\alpha - \beta + 1)\Gamma(n+\beta)} - \frac{\Gamma(\alpha+1)}{(\alpha - \beta + 1)\Gamma(\beta)}, \]  

for any given \( \alpha, \beta \in \mathbb{R}^+ \), such that \( \beta \neq \alpha + 1 \). Applying this identity with \( \alpha = \theta \geq 2 \) and \( \beta = 1 \), we obtain the stated result after some straightforward simplification.

The asymptotic formula given is a consequence of the Stirling approximation of the ratio of gamma functions (cf. (5)).

With \( \mathbb{E}[X^{(\theta)}] \) in hand, we can bootstrap it into the recurrence for \( \mathbb{E}[X^{(\theta)}] \) to also put that recursion in the form (15). The solution follows similar steps as those used in solving the recurrence for \( \mathbb{E}[X^{(\theta)}] \), and we only highlight the chief steps. The recurrence (15) is

\[ \mathbb{E}[X^{(\theta)}] = \frac{n}{n+\theta-1} \mathbb{E}[X^{(\theta)}] \]

\[ \quad + \left( \frac{\theta - 1}{n+\theta-1} \right) \left( \frac{n-1}{\theta} + 1 + \frac{(\theta-1)\Gamma(\theta)\Gamma(n)}{\Gamma(n+\theta-1)} \right) \]

\[ = \frac{n}{n+\theta-1} \mathbb{E}[X^{(\theta)}] + \left( \frac{\theta - 1}{\theta} + \frac{(\theta-1)^2\Gamma(\theta)\Gamma(n)}{\Gamma(n+\theta)} \right). \]
Here, we have for \((15)\)

\[
y_0 = \mathbb{E}[X_{0,2}^{(\theta)}] = 0, \quad g_n = \frac{n}{n + \theta - 1}, \quad h_n = \left(\frac{\theta - 1}{\theta} + \frac{(\theta - 1)^2 \Gamma(\theta) \Gamma(n)}{\Gamma(n + \theta)}\right).
\]

An exact solution follows after simplification. The asymptotic formula given is a consequence of the Stirling approximation of the ratio of gamma functions (cf. \((5)\)).

**Theorem 5.1.** Let \(X_{n,i}^{(\theta)}\) be the number of vertices contained in exactly \(i\) hyperedges in a hyperrecursive tree with parameter \(\theta\) at age \(n\), for \(i = 1, 2\). Let \(\Sigma_n\) be the corresponding covariance matrix. Upon scaling by \(n\), the covariance matrix converges (as \(n \to \infty\)) as given below:

\[
\frac{1}{n} \Sigma_n \to \begin{pmatrix}
\frac{(\theta - 1)^2}{\theta^2(2\theta - 1)} & -\frac{(\theta - 1)^2(\theta^2 + 2\theta - 1)}{\theta^3(2\theta - 1)^2} \\
-\frac{(\theta - 1)^2(\theta^2 + 2\theta - 1)}{\theta^3(2\theta - 1)^2} & \frac{(\theta - 1)^2(6\theta^4 - 6\theta^3 + 8\theta^2 - 5\theta + 1)}{\theta^4(2\theta - 1)^3}
\end{pmatrix}.
\]

**Proof.** It is folklore that variance computation is very lengthy, a phenomenon called the *combinatorial explosion*. We only highlight the salient points.

The starting point for variance-covariance computation is the pair of stochastic recurrences \((10)\) and \((11)\), from which we can get stochastic recurrence relations for the second-order moments. We take the square of each of these equations, as well as their product:

\[
\begin{align*}
(X_{n,1}^{(\theta)})^2 &= (X_{n-1,1}^{(\theta)})^2 + (Q_{n,1}^{(\theta)})^2 + 1 - 2X_{n-1,1}^{(\theta)}Q_{n,1}^{(\theta)} + 2X_{n-1,1}^{(\theta)} - 2Q_{n,1}^{(\theta)}; \\
X_{n,1}^{(\theta)}X_{n,2}^{(\theta)} &= X_{n-1,1}^{(\theta)}X_{n-1,2}^{(\theta)} - X_{n-1,1}^{(\theta)}Q_{n,2}^{(\theta)} + X_{n-1,1}^{(\theta)}Q_{n,1}^{(\theta)} \\
&\quad - X_{n-1,2}^{(\theta)}Q_{n,1}^{(\theta)} + Q_{n-1,1}^{(\theta)}Q_{n,2}^{(\theta)} - (Q_{n,1}^{(\theta)})^2 \\
&\quad + X_{n-1,2}^{(\theta)} - Q_{n,2}^{(\theta)} + Q_{n,1}^{(\theta)}; \\
(X_{n,2}^{(\theta)})^2 &= (X_{n-1,2}^{(\theta)})^2 + (Q_{n,2}^{(\theta)})^2 + (Q_{n,1}^{(\theta)})^2 - 2X_{n-1,1}^{(\theta)}Q_{n,2}^{(\theta)} \\
&\quad + 2X_{n-1,2}^{(\theta)}Q_{n,1}^{(\theta)} - 2Q_{n,1}^{(\theta)}Q_{n,2}^{(\theta)}.
\end{align*}
\]

We next take the expectation (conditioned on \(\mathcal{F}_{n-1}^{(\theta)}\)) and use the conditional trivariate hypergeometric distribution of \((Q_{n,1}^{(\theta)}, Q_{n,2}^{(\theta)}, Q_{n-1}^{(\theta)} - Q_{n,1}^{(\theta)} - Q_{n,2}^{(\theta)})\), which comes in terms of \((X_{n-1,1}^{(\theta)}, X_{n-1,2}^{(\theta)})\). So, an iterated expectation on each
conditional recurrence gives us three unconditional recurrence equations in 
\( \mathbb{E}(X_{n,1}^{(\theta)})^2), \mathbb{E}(X_{n,1}^{(\theta)}X_{n,2}^{(\theta)}) \) and \( \mathbb{E}(X_{n,2}^{(\theta)})^2). 

It is evident that we need a bootstrapping technique: the recurrence 

for \( \mathbb{E}(X_{n,1}^{(\theta)})^2 \) is self contained, going back only to \( \mathbb{E}(X_{n-1,1}^{(\theta)})^2 \). So, we 
can start with it. It has the form (15), with solution 

\[
\mathbb{E}(X_{n,1}^{(\theta)})^2 = \frac{n^2}{\theta^2} + \frac{5\theta^2 - 4\theta + 1}{\theta^2(2\theta - 1)} n + O(1).
\]

The recurrence equation for \( \mathbb{E}(X_{n,1}^{(\theta)}X_{n,2}^{(\theta)}) \) involves \( \mathbb{E}(X_{n,1}^{(\theta)}X_{n,1}^{(\theta)}) \), which is 
now available. So, after all, the equation is of the form (15) (with some of 
the terms in \( h_n \) specified only asymptotically). We obtain the asymptotic 
solution 

\[
\mathbb{E}(X_{n,1}^{(\theta)}X_{n,2}^{(\theta)}) = \frac{(\theta - 1)^2}{\theta^4} n^2 + \frac{22\theta^4 - 30\theta^3 + 20\theta^2 - 7\theta + 1}{\theta^4(2\theta - 1)^3} (\theta - 1)^2 n + O(\ln(n)).
\]

Similarly, the recurrence equation for \( \mathbb{E}(X_{n,2}^{(\theta)})^2 \) involves \( \mathbb{E}(X_{n-1,1}^{(\theta)})^2 \), as 
well as \( \mathbb{E}(X_{n-1,1}^{(\theta)}X_{n-1,2}^{(\theta)}) \), which are both available now. Again, the equation 
is of the form (15) (with \( h_n \) specified only asymptotically). This gives the 
solution 

\[
\mathbb{E}(X_{n,2}^{(\theta)})^2 = \frac{(\theta - 1)^2}{\theta^4} n^2 + \frac{22\theta^4 - 30\theta^3 + 20\theta^2 - 7\theta + 1}{\theta^4(2\theta - 1)^3} (\theta - 1)^2 n + O(\ln(n)).
\]

Toward the variance, from the second moments we subtract the expecta-
tions of the squares of the first moments, and toward the covariance, we 
subtract \( \mathbb{E}(X_{n,1}^{(\theta)}) \mathbb{E}(X_{n,2}^{(\theta)}) \) from the mixed moment. Huge cancellations take 
place, removing the \( n^2 \) term from the expression and leaving the covariance 
matrix convergence as stated. We relegate all the details to the appendix. \( \square \)

5.3 Concentration laws

We determine approximations of \( X_{n,1}^{(\theta)} \) and \( X_{n,2}^{(\theta)} \) by the leading asymptotic 
equivalents of their means. Errors in the \( O_L \) sense\(^6\) are of lower order.

---

\(^6\)A sequence of random variables \( Y_n \) is \( O_L(g(n)) \), when there exist a positive constant \( A \) and a positive integer \( n_0 \), such that \( \mathbb{E}[|Y_n|] \leq A|g(n)| \), for all \( n \geq n_0 \).
Lemma 5.1. As \( n \to \infty \), we have the asymptotic approximation

\[
\begin{pmatrix}
X_{n,1}^{(\theta)} \\
X_{n,2}^{(\theta)}
\end{pmatrix} = \begin{pmatrix}
\frac{n}{\theta} + O_{L_1}(\sqrt{n}) \\
(\theta - 1)n \frac{1}{\theta^2} + O_{L_1}(\sqrt{n})
\end{pmatrix}.
\]

Proof. From the asymptotics of the mean and variance, as given in Proposition 5.1 and Theorem 5.1, we have

\[
\mathbb{E}\left[\left(X_{n,1}^{(\theta)} - \frac{n}{\theta}\right)^2\right] = \mathbb{E}\left[\left((X_{n,1}^{(\theta)} - \mathbb{E}[X_{n,1}^{(\theta)}]) + \left(\mathbb{E}[X_{n,1}^{(\theta)}] - \frac{n}{\theta}\right)\right)^2\right]
\]

\[
= \text{Var}[X_{n,1}^{(\theta)}] + \left(\mathbb{E}[X_{n,1}^{(\theta)}] - \frac{n}{\theta}\right)^2
\]

\[
= O(n).
\]

So, by Jensen’s inequality

\[
\mathbb{E}\left[\left|X_{n,1}^{(\theta)} - \frac{n}{\theta}\right|\right] \leq \sqrt{\mathbb{E}\left[\left(X_{n,1}^{(\theta)} - \frac{n}{\theta}\right)^2\right]} = O(\sqrt{n}).
\]

It follows that

\[
X_{n,1}^{(\theta)} = \frac{n}{\theta} + O_{L_1}(\sqrt{n}).
\]

The proof of the asymptotic approximation for \( X_{n,2}^{(\theta)} \) is quite similar. \( \square \)

Corollary 5.1. As \( n \to \infty \), we have

\[
\frac{1}{n} \begin{pmatrix}
X_{n,1}^{(\theta)} \\
X_{n,2}^{(\theta)}
\end{pmatrix} \xrightarrow{p} \begin{pmatrix}
\frac{1}{\theta}
\\
\theta - 1 \frac{1}{\theta^2}
\end{pmatrix}.
\]

5.4 Martingalization

We perform a martingale transform on \( X_{n,1}^{(\theta)} \). Let \( M_{n}^{(\theta)} = r_{n}^{(\theta)} X_{n,1}^{(\theta)} + s_{n}^{(\theta)} \), for deterministic, but yet-to-be specified, factors \( r_{n}^{(\theta)} \) and \( s_{n}^{(\theta)} \) that render \( M_{n}^{(\theta)} \)
a martingale. Toward such martingalization, using (12), we write

\[
\mathbb{E} \left[ M_n^{(\theta)} \mid \mathcal{F}_{n-1}^{(\theta)} \right] = \mathbb{E} \left[ r_n^{(\theta)} X_{n,1}^{(\theta)} + s_n^{(\theta)} \mid \mathcal{F}_{n-1}^{(\theta)} \right] \\
= \left(1 - \frac{\theta - 1}{\tau_{n-1}^{(\theta)}}\right) r_n^{(\theta)} X_{n-1,1}^{(\theta)} + r_n^{(\theta)} + s_n^{(\theta)} \\
= M_n^{(\theta)} \\
= r_{n-1}^{(\theta)} X_{n-1,1}^{(\theta)} + s_{n-1}^{(\theta)}.
\]

This is possible, if

\[
\begin{align*}
    r_{n-1}^{(\theta)} &= \left(1 - \frac{\theta - 1}{\tau_{n-1}^{(\theta)}}\right) r_n^{(\theta)}; \\
    s_{n-1}^{(\theta)} &= r_n^{(\theta)} + s_n^{(\theta)}.
\end{align*}
\]

The factor \( r_n^{(\theta)} \) should satisfy the recurrence

\[
r_n^{(\theta)} = \left(\frac{\tau_{n-1}^{(\theta)}}{\tau_{n-1}^{(\theta)} - \theta + 1}\right) r_{n-1}^{(\theta)} = \left(\frac{n + \theta - 1}{n}\right) r_{n-1}^{(\theta)},
\]

which unwinds into

\[
r_n^{(\theta)} = \frac{(n + \theta - 1)(n + \theta - 2) \cdots \theta}{n(n-1) \times \cdots \times 1} r_0^{(\theta)} = \frac{\Gamma(n + \theta)}{\Gamma(\theta) \Gamma(n + 1)} r_0^{(\theta)},
\]

for any arbitrary \( r_0^{(\theta)} \in \mathbb{R} \); for simplicity, we take \( r_0^{(\theta)} = 1 \).

The factor \( s_n^{(\theta)} \) should satisfy the recurrence

\[
s_n^{(\theta)} = s_{n-1}^{(\theta)} - r_n^{(\theta)},
\]

which unwinds into

\[
s_n^{(\theta)} = s_0^{(\theta)} - \sum_{i=1}^{n} r_i^{(\theta)},
\]

for any arbitrary \( s_0^{(\theta)} \in \mathbb{R} \); we take \( s_0^{(\theta)} = 0 \). Using the identity (16) once again, we simplify \( s_n^{(\theta)} \) to

\[
s_n^{(\theta)} = -\frac{\Gamma(n + \theta + 1)}{\theta \Gamma(\theta) \Gamma(n + 1)} + 1.
\]
Thus, we have

$$M_n^{(\theta)} = \frac{\Gamma(n + \theta)}{\Gamma(\theta) \Gamma(n + 1)} X_{n,1}^{(\theta)} - \frac{\Gamma(n + \theta + 1)}{\theta \Gamma(\theta) \Gamma(n + 1)} + 1$$

is a martingale.

Asymptotics of $r_n^{(\theta)}$, $s_n^{(\theta)}$ and their backward differences are useful in the ensuing analysis.

Lemma 5.2. As $n \to \infty$, we have the asymptotics:

$$r_n^{(\theta)} \sim \frac{n^{\theta-1}}{\Gamma(\theta)}, \quad r_n^{(\theta)} - r_{n-1}^{(\theta)} = O(n^{\theta-2});$$

$$s_n^{(\theta)} \sim -\frac{n^{\theta}}{\theta \Gamma(\theta)}, \quad s_n^{(\theta)} - s_{n-1}^{(\theta)} = O(n^{\theta-1}).$$

Proof. Examine the forms of $r_n^{(\theta)}$ and $s_n^{(\theta)}$. Their asymptotic equivalents follow from the Stirling approximation in (5).

Further, we have

$$r_n^{(\theta)} - r_{n-1}^{(\theta)} = \frac{\Gamma(n + \theta)}{\Gamma(\theta) \Gamma(n + 1)} - \frac{\Gamma(n - 1 + \theta)}{\Gamma(\theta) \Gamma(n)}$$

$$= \frac{\Gamma(n - 1 + \theta)}{\Gamma(\theta) \Gamma(n)} \left( \frac{n - 1 + \theta}{n} - 1 \right)$$

$$\sim \frac{\theta - 1}{\Gamma(\theta)} n^{\theta-2},$$

and

$$s_n^{(\theta)} - s_{n-1}^{(\theta)} = -\frac{\Gamma(n + \theta + 1)}{\theta \Gamma(\theta) \Gamma(n + 1)} + \frac{\Gamma(n + \theta)}{\theta \Gamma(\theta) \Gamma(n)}$$

$$= -\frac{\Gamma(n + \theta)}{\theta \Gamma(\theta) \Gamma(n)} \left( \frac{n + \theta}{n} - 1 \right)$$

$$= -\frac{\Gamma(n + \theta)}{\Gamma(\theta) \Gamma(n + 1)}$$

$$\sim \frac{n^{\theta-1}}{\Gamma(\theta)}.$$
Corollary 5.2. For large enough positive constants $K_1, K_2, \text{ and } K_3$, we have
\begin{align*}
r_n^{(\theta)} &\leq K_1 n^{\theta - 1}, \\
r_n^{(\theta)} - r_{n-1}^{(\theta)} &\leq K_2 n^{\theta - 2}, \\
\tau^{(\theta)} - \tau_{n-1}^{(\theta)} &\leq K_3 n^{\theta - 1},
\end{align*}
for all $n \geq 1$.

5.5 Gaussian limit law

In this subsection, we obtain an asymptotic Gaussian law for $X_n^{(\theta)}$ by verifying the conditions of the martingale central limit theorem for $M_n^{(\theta)}$. There are several sets of such conditions. We use conditional Lindeberg’s condition and the conditional variance condition in [11], pages 57–59.

Conditional Lindeberg’s condition requires that, for some positive sequence $\xi_n^{(\theta)}$, and for any $\varepsilon > 0$, we have
\[
U_n^{(\theta)} := \sum_{j=1}^{n} \mathbb{E} \left[ \left( \frac{\nabla M_{j}^{(\theta)}}{\xi_n^{(\theta)}} \right)^2 \mathbb{I} \left\{ \left| \frac{\nabla M_{j}^{(\theta)}}{\xi_n^{(\theta)}} \right| > \varepsilon \right\} \right] \xrightarrow{p} 0,
\]
and the conditional variance condition requires that, for some random variable $G_{\theta} \neq 0$, we have
\[
V_n^{(\theta)} := \sum_{j=1}^{n} \mathbb{E} \left[ \left( \frac{\nabla M_{j}^{(\theta)}}{\xi_n^{(\theta)}} \right)^2 \left| \frac{\nabla M_{j}^{(\theta)}}{\xi_n^{(\theta)}} \right| \mathbb{I} \left\{ \frac{\nabla M_{j}^{(\theta)}}{\xi_n^{(\theta)}} \right\} \right] \xrightarrow{p} G_{\theta}.
\]

When these conditions are satisfied, we get
\[
\frac{M_n^{(\theta)}}{\xi_n^{(\theta)}} \xrightarrow{p} \mathcal{N}(0, G_{\theta}),
\]
where the right-hand side is a mixture of normally distributed random variates, with mixing variance $G_{\theta}$. In our case, we find out that $G_{\theta}$ is a constant, so the mixture has only one normal random variate in it, which has a deterministic variance. In the case of the number of vertices at containment level 1, it turns out that $\xi_n^{(\theta)}$ is $n^{\theta - \frac{1}{2}}$.

The following uniform bound paves the way to the verification of the two conditions of the martingale central limit theorem.
Lemma 5.3. The absolute differences $|\nabla M_j^{(\theta)}|/n^{\theta-1}$ are uniformly bounded in $j = 1, \ldots, n$.

Proof. By the construction of the martingale, for each $1 \leq j \leq n$, we have
\[
|\nabla M_j^{(\theta)}| = |M_j^{(\theta)} - M_{j-1}^{(\theta)}| = \left| (r_j^{(\theta)} X_{j,1}^{(\theta)} + s_j^{(\theta)}) - (r_{j-1}^{(\theta)} X_{j-1,1}^{(\theta)} + s_{j-1}^{(\theta)}) \right| \\
\leq |r_j^{(\theta)} (X_{j-1,1}^{(\theta)} - Q_{j,1}^{(\theta)} + 1) - r_{j-1} X_{j-1,1}^{(\theta)}| + |s_j^{(\theta)} - s_{j-1}^{(\theta)}| \\
\leq |r_j^{(\theta)} - r_{j-1}^{(\theta)}| r_{j-1}^{(\theta)} + r_j^{(\theta)} Q_{j,1}^{(\theta)} + r_j^{(\theta)} + |s_j^{(\theta)} - s_{j-1}^{(\theta)}|.
\]

Recall that $Q_{n,1}^{(\theta)}$ is a hypergeometric random variable representing the number of vertices at containment level 1 in a sample of size $\theta - 1$. Its maximum value is $\theta - 1$.

From the bounds in Corollary 5.2, we obtain
\[
\left| \frac{\nabla M_j^{(\theta)}}{n^{\theta-1}} \right| \leq \frac{1}{n^{\theta-1}} (K_2 j^{\theta-2} r_{n-1}^{(\theta)} + (\theta - 1) K_1 j^{\theta-1} + K_3 n^{\theta-1}) \\
\leq \frac{1}{n^{\theta-1}} (K_2 n^{\theta-2} (n + \theta - 1) + \theta K_1 n^{\theta-1}) + K_3 \\
= K_2 + \frac{(\theta - 1) K_2}{n} + \theta K_1 + K_3 \\
\leq K_2 + (\theta - 1) K_2 + \theta K_1 + K_3.
\]

\[\square\]

Lemma 5.4. For any $\varepsilon > 0$, we have
\[
U_n^{(\theta)} = \sum_{j=1}^{n} \mathbb{E} \left[ \left( \frac{\nabla M_j^{(\theta)}}{n^{\theta-\frac{1}{2}}} \right)^2 \mathbb{I}_{\left\{ \left| \nabla M_j^{(\theta)} \right| > \varepsilon n^{\theta-\frac{1}{2}} \right\}} \right] \xrightarrow{p} 0.
\]

Proof. By the uniform bound established in Lemma 5.3 for every $\varepsilon > 0$, there exists a natural number $n_0(\varepsilon)$, such that for all $n \geq n_0(\varepsilon)$, the sets $\{ |\nabla M_j^{(\theta)}| > \varepsilon n^{\theta-\frac{1}{2}} \}$ are empty, which implies that the sequence $U_n$ converges almost surely to 0. This almost-sure convergence is stronger than the required in-probability convergence. \[\square\]

Lemma 5.5.
\[
V_n^{(\theta)} = \sum_{j=1}^{n} \mathbb{E} \left[ \left( \frac{\nabla M_j^{(\theta)}}{n^{\theta-\frac{1}{2}}} \right)^2 \mathbb{I}_{\left\{ \nabla M_j^{(\theta)} \right\}} \right] \xrightarrow{p} \frac{\theta - 1}{\theta^2 (2\theta - 1) \Gamma^2(\theta)}.
\]
Proof. This is a rather lengthy calculation, but for the large part it goes in the same vein as the computations we encountered in the proof of Theorem 5.1. So, we only outline the salient features in the long chain of calculations.

Write

\[ V^{(\theta)}_n = \frac{1}{n^{2\theta - 1}} \sum_{j=1}^n \mathbb{E} \left[ (\nabla(r^{(\theta)}_j X^{(\theta)}_{j,1}) + \nabla s^{(\theta)}_j)^2 \big| \mathbb{F}^{(\theta)}_{j-1} \right] \]

\[ = \frac{1}{n^{2\theta - 1}} \sum_{j=1}^n \mathbb{E} \left[ (\nabla(r^{(\theta)}_j X^{(\theta)}_{j,1}))^2 + 2(\nabla(r^{(\theta)}_j X^{(\theta)}_{j,1})) \nabla s^{(\theta)}_j + (\nabla s^{(\theta)}_j)^2 \big| \mathbb{F}^{(\theta)}_{j-1} \right] \]

\[ := \frac{1}{n^{2\theta - 1}} \sum_{j=1}^n (A^{(\theta)}_j + B^{(\theta)}_j + D^{(\theta)}_j). \]

We take up each part separately, starting with

\[ A^{(\theta)}_j := \mathbb{E} \left[ (\nabla(r^{(\theta)}_j X^{(\theta)}_{j,1}))^2 \big| \mathbb{F}^{(\theta)}_{j-1} \right] \]

\[ = \mathbb{E} \left[ (r^{(\theta)}_j X^{(\theta)}_{j,1} - r^{(\theta)}_{j-1} X^{(\theta)}_{j-1,1})^2 \big| \mathbb{F}^{(\theta)}_{j-1} \right] \]

\[ = (r^{(\theta)}_j)^2 \mathbb{E} \left[ (X^{(\theta)}_{j,1})^2 \big| \mathbb{F}^{(\theta)}_{j-1} \right] + (r^{(\theta)}_{j-1})^2 \mathbb{E} \left[ (X^{(\theta)}_{j-1,1})^2 \big| \mathbb{F}^{(\theta)}_{j-1} \right] \]

\[ - 2r^{(\theta)}_j r^{(\theta)}_{j-1} \mathbb{E} \left[ X^{(\theta)}_{j,1} X^{(\theta)}_{j-1,1} \big| \mathbb{F}^{(\theta)}_{j-1} \right], \]

where we substituted the right-hand side of (10) for \( X^{(\theta)}_{j,1} \). Upon expanding, we get conditional expectations (given \( \mathbb{F}^{(\theta)}_{j-1} \)) of both \( (Q^{(\theta)}_{j,1})^2 \) and \( Q^{(\theta)}_{j,1} \). The variable \( Q_{j,1}^{(\theta)} \) is Hypergeo(\( r^{(\theta)}_{j-1}, X^{(\theta)}_{j-1,1}, \theta - 1 \)). The required conditional expectations are obtained from the hypergeometric distribution; see Section 3.

The second part is

\[ B^{(\theta)}_j = \mathbb{E} \left[ 2(\nabla(r^{(\theta)}_j X^{(\theta)}_{j,1})) \nabla s^{(\theta)}_j \big| \mathbb{F}^{(\theta)}_{j-1} \right] \]

\[ = 2 \mathbb{E} \left[ (r^{(\theta)}_j X^{(\theta)}_{j,1} - r^{(\theta)}_{j-1} X^{(\theta)}_{j-1,1})(s^{(\theta)}_j - s^{(\theta)}_{j-1}) \big| \mathbb{F}^{(\theta)}_{j-1} \right] \]

\[ = 2 r^{(\theta)}_j (s^{(\theta)}_j - s^{(\theta)}_{j-1}) \mathbb{E} \left[ X^{(\theta)}_{j,1} \big| \mathbb{F}^{(\theta)}_{j-1} \right] - 2r^{(\theta)}_{j-1} (s^{(\theta)}_j - s^{(\theta)}_{j-1}) X^{(\theta)}_{j-1,1}. \]

The third part is

\[ D^{(\theta)}_j = \mathbb{E} \left[ (\nabla s^{(\theta)}_j)^2 \big| \mathbb{F}^{(\theta)}_{j-1} \right] = (s^{(\theta)}_j - s^{(\theta)}_{j-1})^2. \]
We now put the three parts together and get an expression for $V_n^{(\theta)}$ as a sum, in which the summand is in terms of $r_j^{(\theta)}$ and $s_j^{(\theta)}$, and their backward differences, as well as $X_{j-1,1}^{(\theta)}$. Toward simplified asymptotics, we use the asymptotic equivalents in Corollary 5.1 for $X_{j-1,1}^{(\theta)}$, and for $r_j^{(\theta)}$, $s_j^{(\theta)}$, $\nabla r_j^{(\theta)}$, $\nabla s_j^{(\theta)}$, we use the asymptotics in Lemma 5.2.

Huge cancellations take place, leaving

$$V_n^{(\theta)} = \frac{1}{n^{2\theta-1}} \left( \sum_{j=1}^{n} \frac{(\theta - 1)^2}{\theta^2 \Gamma^2(\theta)} j^{2\theta-2} + O_{\mathcal{L}_1}(j^{2\theta-3}) \right)$$

$$= \frac{(\theta - 1)^2}{\theta^2 (2\theta - 1) \Gamma^2(\theta)} + O_{\mathcal{L}_1} \left( \frac{1}{n} \right)$$

$$\Rightarrow \frac{\mathcal{L}_1}{\theta^2 (2\theta - 1) \Gamma^2(\theta)} \left( \theta - 1 \right)^2$$

This $\mathcal{L}_1$ convergence is stronger than the required in-probability convergence.

\[ \square \]

**Theorem 5.2.** Let $X_{n,1}^{(\theta)}$ be the number of vertices in hyperrecursive tree with hyperedges of size $\theta$ at age $n$. Then, as $n \to \infty$, we have

$$\frac{X_{n,1}^{(\theta)} - \frac{n}{\theta}}{\sqrt{n}} \overset{\mathcal{D}}{\longrightarrow} \mathcal{N} \left( 0, \frac{(\theta - 1)^2}{\theta^2 (2\theta - 1)} \right).$$

**Proof.** Having checked the conditions for the martingale central limit theorem, we can ascertain that

$$\frac{M_n^{\theta}}{n^{\theta - \frac{1}{2}}} = \frac{\Gamma(n+\theta)}{\Gamma(\theta) \Gamma(n+1)} \frac{X_{n,1}^{(\theta)} - \frac{\Gamma(n+\theta+1)}{\theta \Gamma(\theta) \Gamma(n+1)}}{n^{\theta - \frac{1}{2}}} + 1$$

$$= \frac{\Gamma(n+\theta)}{\Gamma(\theta) \Gamma(n+1)} \frac{n^{\theta - \frac{1}{2}}}{\Gamma(n+\theta)} \frac{X_{n,1}^{(\theta)} - \frac{\Gamma(n+\theta)}{\Gamma(\theta) \Gamma(n+1)} \left( \frac{n}{\theta} \right) - \frac{\Gamma(n+\theta)}{\Gamma(\theta) \Gamma(n+1)}}{n^{\theta - \frac{1}{2}}} + 1$$

$$\overset{\mathcal{D}}{\longrightarrow} \mathcal{N} \left( 0, \frac{(\theta - 1)^2}{\theta^2 (2\theta - 1) \Gamma^2(\theta)} \right).$$

We have \(- \frac{\Gamma(n+\theta)}{\Gamma(n+1) \Gamma(\theta)} + 1 \left(n^{\frac{1}{2} - \theta} \right) \to 0\), and so an application of Slutsky theorem \[\square\] allows us to remove this term. By the Stirling approximation in \[\square\], we have

$$\frac{n^{\theta - 1} \Gamma(n+1)}{\Gamma(n+\theta)} \to 1.$$
An application of Slutsky theorem \([11]\) yields

\[
\frac{1}{\Gamma(\theta)} \left( X^{(\theta)}_{n,1} - \frac{n}{\theta} \right) \sqrt{n} \xrightarrow{D} N \left( 0, \frac{(\theta - 1)^2}{\theta^2(2\theta - 1) \Gamma^2(\theta)} \right),
\]

which is an equivalent statement to the one given in the theorem. \(\Box\)

**Remark 5.1.** In the very special case \(\theta = 2\), the hyperrecursive tree is the standard uniform recursive tree. In this case, \(X^{(2)}_{n,1}\) is just a count of the leaves in the tree. Theorem 5.2 recovers the result in \([10]\) and generalizes it.

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Appendix

To find the variance of $X_{n,1}^{(\theta)}$, we need to solve first for $\mathbb{E}[(X_{n,1}^{(\theta)})^2]$. We can do so through the following recurrence equation:

\[
\mathbb{E}[(X_{n,1}^{(\theta)})^2] = \mathbb{E}[(X_{n-1,1}^{(\theta)} - Q_{n,1}^{(\theta)} + 1)^2]
\]
\[
= \mathbb{E}[(X_{n-1,1}^{(\theta)})^2] + (Q_{n,1}^{(\theta)})^2 - 2X_{n-1,1}^{(\theta)}Q_{n,1}^{(\theta)} - 2Q_{n,1}^{(\theta)} + 2X_{n-1,1}^{(\theta)} + 1
\]
\[
= \mathbb{E}[(X_{n-1,1}^{(\theta)})^2] + \mathbb{E}[(Q_{n,1}^{(\theta)})^2] - 2\mathbb{E}[X_{n-1,1}^{(\theta)}Q_{n,1}^{(\theta)}]
\]
\[
= \mathbb{E}[(X_{n-1,1}^{(\theta)})^2] + 2\mathbb{E}[Q_{n,1}^{(\theta)}] + 2\mathbb{E}[X_{n-1,1}^{(\theta)}] + 1
\]
\[
= \mathbb{E}[(X_{n-1,1}^{(\theta)})^2] + \mathbb{E}[(Q_{n,1}^{(\theta)})^2 | F_{n-1}^{(\theta)}] - 2\mathbb{E}[X_{n-1,1}^{(\theta)}Q_{n,1}^{(\theta)} | F_{n-1}^{(\theta)}]
\]
\[
= \mathbb{E}[(X_{n-1,1}^{(\theta)})^2] + 2\mathbb{E}[Q_{n,1}^{(\theta)} | F_{n-1}^{(\theta)}] + 2\mathbb{E}[X_{n-1,1}^{(\theta)}] + 1
\]

We obtain

\[
\mathbb{E}[(X_{n,1}^{(\theta)})^2] = \frac{n(n-1)}{(n+\theta-1)(n+\theta-2)} \mathbb{E}[(X_{n-1,1}^{(\theta)})^2] + \left(\frac{n(2n+3\theta-5)}{(n+\theta-1)(n+\theta-2)}\right) \mathbb{E}[X_{n-1,1}^{(\theta)}] + 1.
\]

Plugging in $\mathbb{E}[X_{n-1,1}^{(\theta)}] = \frac{(n-1)}{\theta} + 1 + O(n^{-\theta+1})$, we obtain

\[
\mathbb{E}[(X_{n,1}^{(\theta)})^2] = \frac{n(n-1)}{(n+\theta-1)(n+\theta-2)} \mathbb{E}[(X_{n-1,1}^{(\theta)})^2] + \left(\frac{n(2n+3\theta-5)}{(n+\theta-1)(n+\theta-2)}\right) \left(\frac{n-1}{\theta}\right) + 1 + O(n^{1-\theta}) + 1
\]
\[
= \frac{n(n-1)}{(n+\theta-1)(n+\theta-2)} \mathbb{E}[(X_{n-1,1}^{(\theta)})^2] + \frac{2n}{\theta} + \frac{2\theta-1}{\theta} + O\left(\frac{1}{n}\right).
\]

Take $g_n = \frac{n(n-1)}{(n+\theta-1)(n+\theta-2)}$, and $h_n = \frac{2\theta-1}{\theta} + O\left(\frac{1}{n}\right)$ in (15). Under the
The initial condition $\mathbb{E}[(X_{0,1}^{(\theta)})^2] = \theta^2$, we can solve the preceding recurrence:

$$
\mathbb{E}[(X_{n,1}^{(\theta)})^2] = \theta^2 \prod_{i=1}^{n} \frac{i(i-1)}{(i+\theta-1)(i+\theta-2)} \\
+ \sum_{i=1}^{n} \left( \prod_{j=i+1}^{n} \frac{j(j-1)}{(j+\theta-1)(j+\theta-2)} \right) \left( \frac{2}{\theta} i + \frac{2\theta - 1}{\theta} + O\left(\frac{1}{i}\right) \right)
$$

$$
= \theta^2 \left( \frac{\Gamma(n+1)}{\Gamma(n+\theta)/\Gamma(\theta+1)} \right) \left( \frac{\Gamma(n)}{\Gamma(n+\theta-1)/\Gamma(\theta)} \right) \\
+ \sum_{i=1}^{n} \left( \frac{\Gamma(n+1)/\Gamma(i+1)}{\Gamma(n+\theta)/\Gamma(i+\theta)} \right) \left( \frac{\Gamma(n)/\Gamma(i)}{\Gamma(n+\theta-1)/\Gamma(i+\theta-1)} \right) \\
\times \left( \frac{2}{\theta} i + \frac{2\theta - 1}{\theta} + O\left(\frac{1}{i}\right) \right)
$$

$$
= O(n^{2-2\theta}) + \left( \frac{\Gamma(n+1)}{\Gamma(n+\theta)} \right)^2 \left( \frac{n+\theta-1}{n} \right) \sum_{i=1}^{n} \left( \frac{\Gamma(i+\theta)}{\Gamma(i+1)} \right)^2 \\
\times \left( \frac{2}{\theta} i + \frac{2\theta - 1}{\theta} + O\left(\frac{1}{i}\right) \right)
$$

$$
= O(n^{2-2\theta}) + \left( \frac{n^{1-\theta} + (1-\theta)(\theta)}{2} \right) \frac{n-\theta}{n} + O\left(n^{-\theta-1}\right) \left( \frac{n+\theta-1}{n} \right) \\
\times \sum_{i=1}^{n} \left( i^{\theta-1} + \frac{\theta(\theta-1)}{2} i^{\theta-2} + O\left(i^{\theta-3}\right) \right)^2 \left( \frac{i}{i+\theta-1} \right) \\
\times \left( \frac{2}{\theta} i + \frac{2\theta - 1}{\theta} + O\left(\frac{1}{i}\right) \right)
$$

$$
= O(n^{2-2\theta}) + \left( \frac{n^{2-2\theta} + \theta(1-\theta)n^{1-2\theta}}{n} + O\left(n^{-2\theta}\right) \right) \left( \frac{n+\theta-1}{n} \right) \\
\times \sum_{i=1}^{n} \left( \frac{2}{\theta} i^{2\theta-1} + \frac{2\theta^2 - 2\theta + 1}{\theta(2\theta - 1)} i^{2\theta-2} + O\left(i^{2\theta-3}\right) \right)
$$

$$
= O(n^{2-2\theta}) + \left( \frac{n^{2-2\theta} + \theta(1-\theta)n^{1-2\theta}}{n} + O\left(n^{-2\theta}\right) \right) \left( \frac{n+\theta-1}{n} \right) \\
\times \left( \frac{n^{2\theta}}{\theta^2} + \frac{n^{2\theta-1}}{\theta} + \frac{2\theta^2 - 2\theta + 1}{\theta(2\theta - 1)} n^{2\theta-1} + O\left(n^{2\theta-2}\right) \right)
$$

$$
= O(n^{2-2\theta}) + \left( \frac{n^{2-2\theta} - (\theta - 1)^2 n^{1-2\theta}}{n} + O\left(n^{-2\theta}\right) \right) \\
\times \left( \frac{n^{2\theta}}{\theta^2} + \frac{2\theta}{2\theta - 1} n^{2\theta-1} + O\left(n^{2\theta-2}\right) \right)
$$
\[
\frac{n^2}{\theta^2} + \frac{5\theta^2 - 4\theta + 1}{\theta^2(2\theta - 1)} n + O(1).
\]

We can now attain the variance as such:

\[
\text{Var}[X_{n,1}^{(\theta)}] = \mathbb{E} \left[ (X_{n,1}^{(\theta)})^2 \right] - (\mathbb{E}[X_{n,1}^{(\theta)}])^2
\]

\[
= \frac{n^2}{\theta^2} + \frac{5\theta^2 - 4\theta + 1}{\theta^2(2\theta - 1)} n + O(1) - \left( \frac{n}{\theta} + 1 + O(n^{1-\theta}) \right)^2
\]

\[
= \frac{(\theta - 1)^2}{\theta^2(2\theta - 1)} n + O(1)
\]

\[
\sim \frac{(\theta - 1)^2}{\theta^2(2\theta - 1)} n.
\]

To solve for the covariance of \(X_{n,1}^{(\theta)}\) and \(X_{n,2}^{(\theta)}\), we need to first solve for \(\mathbb{E}[X_{n,1}^{(\theta)}X_{n,2}^{(\theta)}]\). We can do so through the following recurrence equation:

\[
\mathbb{E}[X_{n,1}^{(\theta)}X_{n,2}^{(\theta)}] = \mathbb{E} \left[ (X_{n-1,1}^{(\theta)} - Q_{n,1}^{(\theta)} + 1)(X_{n-1,2}^{(\theta)} - Q_{n,2}^{(\theta)} + Q_{n,1}^{(\theta)}) \right]
\]

\[
= \mathbb{E}[X_{n-1,1}^{(\theta)}X_{n-1,2}^{(\theta)} - Q_{n,1}^{(\theta)}X_{n-1,2}^{(\theta)} + X_{n-1,2}^{(\theta)} - X_{n-1,1}^{(\theta)}Q_{n,2}^{(\theta)}
\]

\[
+ Q_{n,1}^{(\theta)}Q_{n,2}^{(\theta)} - Q_{n,2}^{(\theta)} + X_{n-1,1}^{(\theta)}Q_{n,1}^{(\theta)} - (Q_{n,1}^{(\theta)})^2 + Q_{n,1}^{(\theta)}]
\]

\[
= \mathbb{E}[X_{n-1,1}^{(\theta)}X_{n-1,2}^{(\theta)}] - \mathbb{E}[Q_{n,1}^{(\theta)}X_{n-1,2}^{(\theta)}] + \mathbb{E}[X_{n-1,2}^{(\theta)}]
\]

\[
- \mathbb{E}[X_{n-1,1}^{(\theta)}Q_{n,2}^{(\theta)}] + \mathbb{E}[Q_{n,1}^{(\theta)}Q_{n,2}^{(\theta)}] - \mathbb{E}[Q_{n,2}^{(\theta)}]
\]

\[
+ \mathbb{E}[X_{n-1,1}^{(\theta)}Q_{n,1}^{(\theta)}] - \mathbb{E}[(Q_{n,1}^{(\theta)})^2] + \mathbb{E}[Q_{n,1}^{(\theta)}]
\]

\[
= \mathbb{E}[X_{n-1,1}^{(\theta)}X_{n-1,2}^{(\theta)}] - \mathbb{E}[Q_{n,1}^{(\theta)}X_{n-1,2}^{(\theta)}] + \mathbb{E}[X_{n-1,2}^{(\theta)}]
\]

\[
- \mathbb{E}[X_{n-1,1}^{(\theta)}Q_{n,2}^{(\theta)}] + \mathbb{E}[Q_{n,1}^{(\theta)}Q_{n,2}^{(\theta)}] - \mathbb{E}[Q_{n,2}^{(\theta)}]
\]

\[
- \mathbb{E}[Q_{n,1}^{(\theta)}Q_{n,1}^{(\theta)}] + \mathbb{E}[Q_{n,1}^{(\theta)}] + \mathbb{E}[Q_{n,1}^{(\theta)}]
\]

\[
= \mathbb{E}[X_{n-1,1}^{(\theta)}X_{n-1,2}^{(\theta)}] - \frac{\theta - 1}{n + \theta - 1} \mathbb{E}[X_{n-1,1}^{(\theta)}X_{n-1,2}^{(\theta)}] + \mathbb{E}[X_{n-1,2}^{(\theta)}]
\]

\[
+ \frac{\theta - 1}{n + \theta - 1} \mathbb{E}[X_{n-1,1}^{(\theta)}X_{n-1,2}^{(\theta)}] + \left( \frac{\theta - 1}{n + \theta - 1} \right)^2
\]

\[
- \frac{(\theta - 1)(n + \theta - 1 - (\theta - 1)}{(n + \theta - 1)^2(n + \theta - 1 - 1)} \mathbb{E}[X_{n-1,1}^{(\theta)}X_{n-1,2}^{(\theta)}]
\]

28
Again, we will rely upon asymptotic equivalents of $E[X_{n,1}^{(\theta)}]$, $E[X_{n,2}^{(\theta)}]$, and $E[(X_{n,1}^{(\theta)})^2]$ to reduce the recursive equation to that of computable order. Plugging in the following relationships, we attain the following asymptotic relationship:

\[
E[X_{n,1}^{(\theta)}X_{n,2}^{(\theta)}] = \frac{n(n-1)}{(n+\theta-1)(n+\theta-2)} \mathbb{E}[X_{n,1}^{(\theta)}X_{n,2}^{(\theta)}] \\
+ \frac{n}{n+\theta-1} \left( \frac{\theta-1}{\theta^2} (n-1+\theta) + O\left(\frac{\ln(n)}{n^{\theta-1}}\right) \right) \\
+ \frac{n(\theta-1)}{(n+\theta-1)(n+\theta-2)} \left( \frac{1}{\theta^2} (n-1)^2 \right) \\
+ \frac{5\theta^2 - 4\theta + 1}{\theta^2(2\theta-1)} (n-1) \\
+ \frac{10\theta^4 - 25\theta^3 + 29\theta^2 - 14\theta + 2}{2\theta^2(2\theta-1)} + O\left(\frac{1}{n}\right) \\
+ \frac{(\theta-1)(\theta-2)}{(n+\theta-1)(n+\theta-2)} \left( \frac{1}{\theta} (n-1) + 1 + O\left(n^{-\theta+1}\right) \right) \\
= \frac{n(n-1)}{(n+\theta-1)(n+\theta-2)} \mathbb{E}[X_{n,1}^{(\theta)}X_{n,2}^{(\theta)}]
\]
\[ + \frac{2(\theta - 1)}{\theta^2}n + \frac{\theta - 1}{2\theta - 1} + O\left(\frac{\ln(n)}{n}\right). \]

Take \( g_n = \frac{n(n-1)}{(n+\theta-1)(n+\theta-2)} \), and \( h_n = \frac{2(\theta-1)}{\theta^2}n + \frac{\theta - 1}{2\theta - 1} + O\left(\frac{\ln(n)}{n}\right) \) in (15). Noting that \( E[\mathcal{X}_{0,1}^2(\mathcal{X}_{0,2}^\theta)] = \theta(0) = 0 \), we can solve the preceding recurrence:

\[
E[\mathcal{X}_{n,1}^\theta \mathcal{X}_{n,2}^\theta] = 0 \times \left( \prod_{i=1}^{n} \frac{i(i-1)}{(i+\theta - 1)(i+\theta - 2)} \right)
+ \sum_{i=1}^{n} \left( \prod_{j=i+1}^{n} \frac{j(j-1)}{(j+\theta - 1)(j+\theta - 2)} \right) \left( \frac{2(\theta - 1)}{\theta^2}i + \frac{\theta - 1}{2\theta - 1} + O\left(\frac{\ln(i)}{i}\right) \right)
= \left( \frac{\Gamma(n+1)}{\Gamma(n+\theta)} \right)^2 \left( \frac{n + \theta - 1}{n} \right) \sum_{i=1}^{n} \left( \frac{\Gamma(i+\theta)}{\Gamma(i+1)} \right)^2 \left( \frac{i}{i+\theta - 1} \right)
\times \left( \frac{2(\theta - 1)}{\theta^2}i + \frac{\theta - 1}{2\theta - 1} + O\left(\frac{\ln(i)}{i}\right) \right)
= (n^{2\theta} + \theta(1 - \theta)n^{1-2\theta} + O(n^{3\theta}))(\frac{n + \theta - 1}{n})
\times \sum_{i=1}^{n} (i^{2\theta-2} + (\theta^2 - \theta)i^{2\theta-3} + O(i^{2\theta-4}))
\times \left( \frac{2(\theta - 1)}{\theta^2}i + \frac{2 - 8\theta + 9\theta^2 - 3\theta^3}{\theta^2(2\theta - 1)} + O\left(\frac{\ln(i)}{i}\right) \right)
= (n^{2\theta} - (\theta - 1)^2n^{1-2\theta} + O(n^{-2\theta})) \sum_{i=1}^{n} \left( \frac{2(\theta - 1)}{\theta^2}i^{2\theta-1}
+ \frac{4\theta^2 - 13\theta^3 + 17\theta^2 - 10\theta + 2}{\theta^2(2\theta - 1)}i^{2\theta-2} + O\left(i^{2\theta-3}\ln(i)\right) \right)
= (n^{2\theta} - (\theta - 1)^2n^{1-2\theta} + O(n^{-2\theta})) \left( \frac{(\theta - 1)}{\theta^3}n^{2\theta} + \frac{\theta - 1}{\theta^3}n^{2\theta - 1}
+ \frac{4\theta^2 - 13\theta^3 + 17\theta^2 - 10\theta + 2}{\theta^2(2\theta - 1)^2}n^{2\theta - 1} + O\left(n^{2\theta-2}\ln(n)\right) \right)
= (n^{2\theta} - (\theta - 1)^2n^{1-2\theta} + O(n^{-2\theta}))
\times \left( \frac{\theta - 1}{\theta^3}n^{2\theta} + \frac{4\theta^4 - 9\theta^3 + 9\theta^2 - 5\theta + 1}{\theta^2(2\theta - 1)^2}n^{2\theta - 1} + O(n^{2\theta-2}\ln(n)) \right)
\]
We can do so through the following recurrence equation:

\[ C(n, \theta) = \left( \frac{\theta - 1}{\theta^3} \right) n^2 + \frac{7\theta^4 - 16\theta^3 + 14\theta^2 - 6\theta + 1}{\theta^3(2\theta - 1)^2} n + O(\ln(n)) \]

We can now attain the covariance:

\[
\text{Cov}[X_{n,1}^{(\theta)}, X_{n,2}^{(\theta)}] = \mathbb{E}[X_{n,1}^{(\theta)}X_{n,2}^{(\theta)}] - \mathbb{E}[X_{n,1}^{(\theta)}]\mathbb{E}[X_{n,2}^{(\theta)}]
\]

\[ = \left( \frac{\theta - 1}{\theta^3} \right) n^2 + \frac{7\theta^4 - 16\theta^3 + 14\theta^2 - 6\theta + 1}{\theta^3(2\theta - 1)^2} n + O(\ln(n)) \]

\[ - \left( \frac{n}{\theta} + 1 + O(n^{1-\theta}) \right) \left( \left( \frac{\theta - 1}{\theta^2} \right)(n + \theta) + O\left( \frac{\ln(n)}{n} \right) \right) \]

\[ = -\frac{\theta^4 - 4\theta^2 + 4\theta - 1}{\theta^3(2\theta - 1)^2} n + O(\ln(n)) \]

\[ \sim -\frac{(\theta - 1)^2(\theta^2 + 2\theta - 1)}{\theta^3(2\theta - 1)^2} n. \]

Finally, to solve for the variance of \( X_{n,2}^{(\theta)} \), we need to solve for \( \mathbb{E}[(X_{n,2}^{(\theta)})^2] \). We can do so through the following recurrence equation:

\[
\mathbb{E}[(X_{n,2}^{(\theta)})^2] = \mathbb{E}[(X_{n-1,2}^{(\theta)} - Q_{n,2}^{(\theta)} + Q_{n,1}^{(\theta)})^2]
\]

\[ = \mathbb{E}[(X_{n-1,2}^{(\theta)})^2 + (Q_{n,2}^{(\theta)})^2 + (Q_{n,1}^{(\theta)})^2 + 2X_{n-1,2}^{(\theta)}Q_{n,2}^{(\theta)} - 2X_{n-1,2}^{(\theta)}Q_{n,1}^{(\theta)} - 2Q_{n,2}^{(\theta)}Q_{n,1}^{(\theta)}] \]

\[ = \mathbb{E}[(X_{n-1,2}^{(\theta)})^2] + \mathbb{E}[(Q_{n,2}^{(\theta)})^2 | \mathcal{F}_{n-1}] + \mathbb{E}[(Q_{n,1}^{(\theta)})^2 | \mathcal{F}_{n-1}] + \mathbb{E}[2X_{n-1,2}^{(\theta)}Q_{n,1}^{(\theta)} | \mathcal{F}_{n-1}] - \mathbb{E}[2X_{n-1,2}^{(\theta)}Q_{n,2}^{(\theta)} | \mathcal{F}_{n-1}] \]

\[ = \mathbb{E}[(X_{n-1,2}^{(\theta)})^2] + \frac{2(\theta - 1)}{n + \theta - 1} \mathbb{E}[X_{n-1,1}^{(\theta)}X_{n-1,2}^{(\theta)}] - \frac{2(\theta - 1)}{n + \theta - 1} \mathbb{E}[X_{n-1,1}^{(\theta)}X_{n-1,1}^{(\theta)}] \]

\[ - 2\left( \frac{\theta - 1}{n + \theta - 1} \right)^2 \frac{(\theta - 1)n}{(n + \theta - 1)^2(n + \theta - 2)} \mathbb{E}[X_{n-1,1}^{(\theta)}X_{n-1,1}^{(\theta)}]. \]
\[
\begin{align*}
&= \frac{n(n-1)}{(n+\theta-1)(n+\theta-2)} \mathbb{E}[\left(X_{n-1,2}\right)^2] \\
&\quad + \frac{(\theta-1)(\theta-2)}{(n+\theta-1)(n+\theta-2)} \mathbb{E}[\left(X_{n-1,1}\right)^2] \\
&\quad + \frac{2n(\theta-1)}{(n+\theta-1)(n+\theta-2)} \mathbb{E}[X_{n-1,1}^{(\theta)}X_{n-1,2}^{(\theta)}] \\
&\quad + \frac{n(\theta-1)}{(n+\theta-1)(n+\theta-2)} (\mathbb{E}[X_{n-1,1}^{(\theta)}] + \mathbb{E}[X_{n-1,2}^{(\theta)}]).
\end{align*}
\]

Plugging in the following asymptotic relationships for \( \mathbb{E}[X_{n,1}^{(\theta)}] \), \( \mathbb{E}[X_{n,2}^{(\theta)}] \), \( \mathbb{E}[(X_{n,1}^{(\theta)})^2] \), and \( \mathbb{E}[X_{n,1}^{(\theta)}X_{n,2}^{(\theta)}] \), we attain the following asymptotic recurrence:

\[
\begin{align*}
\mathbb{E}[(X_{n,2}^{(\theta)})^2] &= \frac{n(n-1)}{(n+\theta-1)(n+\theta-2)} \mathbb{E}[(X_{n-1,2}^{(\theta)})^2] \\
&\quad + \frac{(\theta-1)(\theta-2)}{(n+\theta-1)(n+\theta-2)} \\
&\quad \times \left( \frac{1}{\theta^2} (n-1)^2 + \frac{5\theta^2 - 4\theta + 1}{\theta^2(2\theta - 1)} (n-1) + O(1) \right) \\
&\quad + \left( \frac{2n(\theta-1)}{(n+\theta-1)(n+\theta-2)} \right) \left( \frac{1}{\theta^3} (n-1)^2 \\
&\quad \quad + \frac{1 - 6\theta + 14\theta^2 - 16\theta^3 + 7\theta^4}{(1-2\theta)^2\theta^3} (n-1) + O\left(\ln(n)\right) \right) \\
&\quad + \frac{(\theta-1)n}{(n+\theta-1)(n+\theta-2)} \left( \frac{1}{\theta} (n-1) + 1 + O(n^{-\theta+1}) \\
&\quad \quad + \left( \frac{\theta - 1}{\theta^2} (n-1 + \theta) + O\left(\frac{\ln(n)}{n^{\theta-1}}\right) \right) \right) \\
&= \left( \frac{n(n-1)}{(n+\theta-1)(n+\theta-2)} \right) \mathbb{E}[(X_{n-1,2}^{(\theta)})^2] \\
&\quad + \frac{2(\theta-1)^2}{\theta^3} n + \frac{(10\theta^3 - 16\theta^2 + 7\theta - 1)(\theta-1)}{\theta^2(2\theta - 1)^2} + O\left(\frac{\ln(n)}{n}\right)
\end{align*}
\]

Take \( g_n = \frac{n(n-1)}{(n+\theta-1)(n+\theta-2)} \), and \( h_n = \frac{2(\theta-1)^2}{\theta^4} n + \frac{(10\theta^3 - 16\theta^2 + 7\theta - 1)(\theta-1)}{\theta^2(2\theta - 1)^2} + O\left(\frac{\ln(n)}{n}\right) \) in (1.3). Noting that \( \mathbb{E}[(X_{0,2}^{(\theta)})^2] = 0 \), we can solve the preceding
recurrence:

\[
\mathbb{E}[(X_{n,2}^{(\theta)})^2] = 0 \times \prod_{i=1}^{n} \frac{i(i-1)}{(i+\theta-1)(i+\theta-2)} \\
+ \sum_{i=1}^{n} \left( \prod_{j=i+1}^{n} \frac{j(j-1)}{(j+\theta-1)(j+\theta-2)} \right) \left( \frac{2(\theta-1)^2}{\theta^3} i \\
+ \frac{(10\theta^3 - 16\theta^2 + 7\theta - 1)(\theta - 1)}{\theta^2(2\theta - 1)^2} + O\left(\frac{\ln(i)}{i}\right) \right) \\
= \left( \frac{\Gamma(n+1)}{\Gamma(n+\theta)} \right)^2 \left( \frac{n + \theta - 1}{n} \right) \sum_{i=1}^{n} \left( \frac{\Gamma(i + \theta)}{\Gamma(i + 1)} \right)^2 \left( \frac{i}{i + \theta - 1} \right) \\
\times \left( \frac{2(\theta-1)^2}{\theta^3} i + \frac{(10\theta^3 - 16\theta^2 + 7\theta - 1)(\theta - 1)}{\theta^2(2\theta - 1)^2} + O\left(\frac{\ln(i)}{i}\right) \right) \\
= \left( n^{2-2\theta} - (\theta - 1)^2 n^{1-2\theta} + O\left(n^{-2\theta}\right) \right) \\
\times \sum_{i=1}^{n} \left( i^{2\theta-2} + (\theta^2 - \theta)i^{2\theta-3} + O\left(i^{2\theta-4}\right) \right) \\
\times \left( \frac{2(\theta-1)^2}{\theta^3} i + \frac{2\theta^5 + 6\theta^4 - 27\theta^3 + 30\theta^2 - 13\theta + 2}{\theta^3(2\theta - 1)^2} + O\left(\frac{\ln(i)}{i}\right) \right) \\
= \left( n^{2-2\theta} - (\theta - 1)^2 n^{1-2\theta} + O\left(n^{-2\theta}\right) \right) \\
\times \frac{(\theta - 1)^2}{\theta^3} \left( \frac{n^{2\theta}}{\theta} + n^{2\theta-1} \right) \left( \frac{8\theta^4 - 14\theta^3 + 20\theta^2 - 11\theta + 2}{(2\theta - 1)^2} \right) \\
\times \left( \frac{\ln(i)}{i} \right) \\
+ O\left(n^{2\theta-3}\ln(n)\right) \\
= \frac{(\theta - 1)^2}{\theta^3} \left( n^{2-2\theta} - (\theta - 1)^2 n^{1-2\theta} + O\left(n^{-2\theta}\right) \right) \\
\times \left( \frac{1}{\theta} n^{2\theta} + \frac{8\theta^4 - 6\theta^3 + 8\theta^2 - 5\theta + 1}{(2\theta - 1)^3} n^{2\theta-1} + O\left(n^{2\theta-2}\ln(n)\right) \right) \\
= \frac{(\theta - 1)^2}{\theta^4} n^2 + \frac{(22\theta^4 - 30\theta^3 + 20\theta^2 - 7\theta + 1)(\theta - 1)^2}{\theta^4(2\theta - 1)^3} n + O\left(\ln(n)\right) .
\]
We can now attain the variance as such:

\[
\text{Var}[X_{n,2}^{(\theta)}] = \mathbb{E}[(X_{n,2}^{(\theta)})^2] - (\mathbb{E}[X_{n,2}^{(\theta)}])^2 \\
= \frac{(\theta - 1)^2}{\theta^4} n^2 + \frac{(22\theta^4 - 30\theta^3 + 20\theta^2 - 7\theta + 1)(\theta - 1)^2}{\theta^4(2\theta - 1)^3} n \\
+ O(\ln(n)) - \left(\frac{\theta - 1}{\theta^2}(n + \theta) + O\left(\frac{\ln(n)}{n^{0.7}}\right)\right)^2 \\
= \left(\frac{(22\theta^4 - 30\theta^3 + 20\theta^2 - 7\theta + 1)(\theta - 1)^2}{\theta^4(2\theta - 1)^3} - \frac{2(\theta - 1)^2}{\theta^3}\right) n \\
+ O(\ln(n)) \\
= \frac{(\theta - 1)^2(6\theta^4 - 6\theta^3 + 8\theta^2 - 5\theta + 1)}{\theta^4(2\theta - 1)^3} n + O(\ln(n)) \\
\sim \frac{(\theta - 1)^2(6\theta^4 - 6\theta^3 + 8\theta^2 - 5\theta + 1)}{\theta^4(2\theta - 1)^3} n.
\]