The total rainbow connection on comb product of cycle and path graphs

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Abstract. All graphs are finite, connected, simple, non-trivial, and undirected. The graph in this study with $V(G)$ as vertex set and $E(G)$ as edge set. Total-colored graph $G$ is a total connected rainbow of any two vertices which are connected by, at least, one total rainbow path which is the edges and internal vertices which have distinct colors. The total rainbow coloring is a natural extension of edge rainbow coloring and vertex rainbow coloring. The total rainbow connection number of $G$, denoted by $trc(G)$, is the smallest number of colors required to color the edges and vertices of $G$ in order to make $G$ total rainbow connected. This paper shows the lower and upper bounds of $trc(G)$ and the result of the total rainbow connection number of comb product of cycle and path graphs.

1. Introduction

Indonesia is a country led by the President and Vice President. Previously, the election of President and Vice President is only carry out by the Peoples Consultative Assembly as the embodiment of sovereignty in Indonesia; however, since 2004 it has changed so that people can vote directly. The thing that really needs to be considered in the direct election of the President and Vice President is the implementation of the voting card distribution. The voting card will be a cheating gap if it is not well distributed. Today, the development of graph theory is not only theoretical, but also can be applicable in communication, network science, computer science, music, transportation, and other science [3]. One of the topics studied in graph theory is a rainbow connection. By using rainbow connections, we can get the easiest, shortest, and safest way for the voting card to be securely distributed and the election of the President and Vice President can go well and smooth. As the illustration, we consider each province as a vertex and each inter-provincial sending process of voting card as an edge, therefore it becomes a graph.
Figure 1. The example of voting cards distribution process

which is illustrated in figure 1 above. We give a black vertex for all provinces except DKI Jakarta. As the center of voting card distribution, DKI Jakarta is given a different vertex color that is red. Indonesia is a vast country with many separate provinces which causes the voting card distribution process is done by various lines such as air lines, land routes, and sea lanes. The air lines are marked with yellow sides, land routes are marked with green sides, and for small island areas, that cannot be reached by land and air transportation but sea lanes, are marked with blue.

The concept of rainbow connection in graphs is introduced by Chartrand et al. [1]. An edge colored path is a rainbow if its edges have distinct colors. The graph $G$ is rainbow-connected if $G$ contains a rainbow $u-v$ path for every two vertices $u$ and $v$ of $G$ [2]. The rainbow connection number of $G$, denoted by $rc(G)$, is the minimum number of colors in a rainbow connected edge-coloring of $G$.

As a natural counterpart to the rainbow connection of edge-colored graphs, [6, 7] proposes the concept of rainbow vertex-connection. Vertex-coloured path is a vertex-rainbow if its internal vertices have distinct colors. The rainbow vertex-connection number of $G$, denoted by $rvc(G)$, is the minimum number of colors in a rainbow vertex-connected vertex coloring of $G$.

Based on these two inventions [8, 5], proposes the concept of total rainbow connection. A total-coloured path is total-rainbow if its edges and internal vertices have distinct colours [4]. Total-coloring of a connected graph $G$ is a total rainbow connected if any two vertices are connected by a total-rainbow path. The total rainbow connection number of $G$, denoted by $trc(G)$, is the minimum number of colors in a total rainbow connected total-coloring of $G$.

2. Main Results

In this section, we determine the total rainbow connection number of some special graphs and its operations namely comb graph $C_n \triangleright C_m$, $C_n \triangleright P_r$, $P_q \triangleright C_m$, and $P_q \triangleright P_r$ for every $n$, $m$, $r$, and $q$ are elements of natural number with $n \geq 4$, $m \geq 4$, $r \geq 3$, $q \geq 3$.

**Lemma 1.** Let $G$ be any connected graph of order $n$ and $H$ be any connected graph of order $m$. For comb product of $G$ and $H$ the following equation is hold:

$$2diam(G \triangleright H) - 1 \leq trc(G \triangleright H) \leq n(trc(H) + trc(G)) - n.$$ 

**Proof.** Let $H$ be any graphs of order $m$ and size $|E(H)|$ and $G$ be any graphs of order $n$ and size $|E(G)|$. A comb product of $G$ and $H$, denoted by $G \triangleright H$, is a graph obtained by
Theorem 1. Let $G$ be a comb product graph of $C_n$ and $C_m$, the total rainbow connection number of $C_n \bowtie C_m$ is

$$trc(C_n \bowtie C_m) = n + \lceil \frac{n}{2} \rceil + 2m - 1.$$ 

Proof. $C_n \bowtie C_m$ is a graph with vertex set $V(C_n \bowtie C_m) = \{x_i^j; 1 \leq i \leq n; 1 \leq j \leq m\}$, edge set $E(C_n \bowtie C_m) = \{x_i^j x_{i+1}^j; 1 \leq i \leq n - 1; j = 1\} \cup \{x_i^j x_{i}^{j+1}; i = n; j = 1\} \cup \{x_i^j x_{i+1}^{j+1}; 1 \leq i \leq n - 1\} \cup \{x_i^j x_{i}^{j+1}; 1 \leq i \leq n; j = m\}$, $|V(C_n \bowtie C_m)| = nm$, $|E(C_n \bowtie C_m)| = nm + n$, and $diam(C_n \bowtie C_m) = 2\lceil \frac{n}{2} \rceil + \lceil \frac{n}{2} \rceil$.

The following figure 2 below illustrates $C_4 \bowtie C_4$.

By using Proposition 1, we have the following equations

$$trc(C_n \bowtie C_m) \geq 2diam(C_n \bowtie C_m) - 1$$

$$= 4\lceil \frac{n}{2} \rceil + 2\lceil \frac{n}{2} \rceil - 1$$

We cannot attain this lower bound. Assume that $trc(C_n \bowtie C_m) < n + \lceil \frac{n}{2} \rceil + 2m - 1$, we have $trc(C_n \bowtie C_m) = n + \lceil \frac{n}{2} \rceil + 2m - 1$.

We take a total rainbow path $x_i^r \rightarrow x_i^{r+3}$:

$$x_i^r x_i^{r-1}, x_i^{r-1} x_i^{r-2}, x_i^{r-2} \ldots, x_i^1, x_i^1 x_i^{1+1}, x_i^{1+1} x_i^{1+1+2}, x_i^{1+2} x_i^{1+3}, x_i^{1+3}$$

by giving the same color to

innerpath($C_i$)

innerpath($C_{i+3}$)

innercycle($C_i$)

innercycle($C_{i+3}$)

Figure 2. The example of graph $C_4 \bowtie C_4$.
vertices $x_i^{r-1}$ on subgraph $P_i$ and $x_i^{r-1}$ on subgraph $P_{i+3}$ it is exactly we cannot make a total rainbow path.

Furthermore, we will prove that $trc(C_n \triangleright C_m) \leq n + \lceil \frac{n}{2} \rceil + 2m - 1$ by the total coloring $c$ as follow:

$$c(v) = \begin{cases} 
  i, & \text{for } v = x_i^j; 1 \leq i \leq n, j = 1 \\
  n + j - 1, & \text{for } v = x_i^j; 1 \leq i \leq n, 1 \leq j \leq m \\
  n + m + i - 1, & \text{for } e = x_i^jx_i^{j+1}; 1 \leq i \leq \lceil \frac{n}{2} \rceil, j = 1 \\
  n + \lceil \frac{2i-n}{2} \rceil + m - 1, & \text{for } e = x_i^jx_i^{j+1}; \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, j = 1 \\
  n + \lceil \frac{n}{2} \rceil + m - 1, & \text{for } e = x_i^jx_i; i = n, j = 1 \\
  n + \lceil \frac{n}{2} \rceil + m + j - 1, & \text{for } e = x_i^jx_i^{j+1}; 1 \leq i \leq n, 1 \leq j \leq m - 1 \\
  n + \lfloor \frac{n}{2} \rfloor + 2m - 1, & \text{for } e = x_i^jx_i^j; 1 \leq i \leq n, j = m
\end{cases}$$

Easily, they give a rainbow path between two vertices on total colored of $(C_n \triangleright C_m)$ and $V(C_n \triangleright C_m) \rightarrow (1, 2, ..., n + m - 1) \rightarrow E(C_n \triangleright C_m) \rightarrow (n + m, n + m + 1, ..., n + 2m + \lceil \frac{n}{2} \rceil - 1)$, thus the maximum color is $n + 2m + \lceil \frac{n}{2} \rceil - 1$ for $i = n$ and $j = m$.

The following figure 3 below illustrates Theorem 1.

**Theorem 2.** Let $G$ be a comb product graph of $C_n$ and $P_r$, the total rainbow connection number of $C_n \triangleright P_r$ is

$$trc(C_n \triangleright P_r) = 2nr - 2n + \lceil \frac{n}{2} \rceil.$$ 

**Proof.** $C_n \triangleright P_r$ is a graph with vertex set $V(C_n \triangleright P_r) = \{x_i^j; 1 \leq i \leq n; 1 \leq j \leq r\}$, edge set $E(C_n \triangleright P_r) = \{x_i^jx_i^{j+1}; 1 \leq i \leq n - 1; j = 1\} \cup \{x_i^jx_i^{j+1}; i = n; j = 1\} \cup \{x_i^jx_i^{j+1}; 1 \leq i \leq n; 1 \leq i \leq r - 1\}$, $|V(C_n \triangleright P_r)| = nr$, $|E(C_n \triangleright P_r)| = nr$, and $diam(C_n \triangleright P_r) = 2(r - 1) + \lceil \frac{n}{2} \rceil$.

The following figure 4 below illustrates $C_4 \triangleright P_3$

By using Proposition 1, we have the following equations

![Figure 3. The total rainbow coloring of graph $C_4 \triangleright C_4$](image)
Figure 4. The example of graph $C_4 \bowtie P_3$

$$trc(C_n \bowtie P_r) \geq 2diam(C_n \bowtie P_r) - 1$$

$$= 4(r - 1) + 2\lceil \frac{n}{2} \rceil - 1$$

We cannot attain this lower bound. Assume that $trc(C_n \bowtie P_r) < 2nr - 2n + \lceil \frac{n}{2} \rceil - 1$. We take a total rainbow path $x_r^i \rightarrow x_{i+3}^r$:

- $x_i^1, x_i^1, x_i^2, x_i^2, \ldots, x_i^2$;
- $x_i^1, x_i^1+1, x_i^1+2, x_i^1+2, x_i^1$;
- $x_{i+3}, x_{i+3}, x_{i+3}, x_{i+3}, x_{i+3}$.

by giving the same color to vertices $x_i^{r-1}$ on subgraph $P_i$ and $x_{i+3}^{r-1}$ on subgraph $P_{i+3}$ it is exactly we cannot make a total rainbow path.

Furthermore, we will prove that $trc(C_n \bowtie P_r) \leq 2nr - 2n + \lceil \frac{n}{2} \rceil$ by the total coloring $c$ as follow:

$$c(v) = \begin{cases} 
  i, & \text{for } v = x_i^j; 1 \leq i \leq n, j = 1 \\
  n + (i - 1)(r - 2) + j - 1, & \text{for } v = x_i^j; 1 \leq i \leq n; 1 \leq j \leq r - 1 \\
  i, & \text{for } v = x_i^j; 1 \leq i \leq n, j = r 
\end{cases}$$
Proof. Let $j$.

Easily, they give a rainbow path between two vertices on total colored of $(n_\mathcal{R})$, $\sum_{i=1}^{n} (i-1)(r-1) + j$, thus the maximum color is $2nr-2n+\left\lceil \frac{n}{2} \right\rceil$ for $i=n$ and $j=r-1$. The following figure 5 below illustrates Theorem 2.

**Theorem 3.** Let $G$ be a comb product graph of $P_q$ and $C_m$, the total rainbow connection number of $P_q \bowtie C_m$ is

$$trc(P_q \bowtie C_m) = 2q + 2m - 2.$$  

**Proof.** $P_q \bowtie C_m$ is a graph with vertex set $V(P_q \bowtie C_m) = \{x_i^j; 1 \leq i \leq q; 1 \leq j \leq m\}$, edge set $E(P_q \bowtie C_m) = \{x_{i-1}^j, x_i^{j+1}; 1 \leq i \leq q - 1; j = 1\} \cup \{x_i^{j-1}, x_{i+1}^j; 1 \leq i \leq q - 1; 1 \leq j \leq m - 1\}$ \cup \{x_1^j, x_q^{j+1}; 1 \leq i \leq q; j = m\}, $|V(P_q \bowtie C_m)| = qm$, $|E(P_q \bowtie C_m)| = qm + q - 1$, and $diam(P_q \bowtie C_m) = \left\lceil \frac{m}{2} \right\rceil + q - 1$. The following figure 6 below illustrates $P_3 \bowtie C_4$.

By using Proposition 1, we have the following

**Figure 5.** The total rainbow coloring of graph $C_4 \bowtie P_3$.
Proof. \( P \otimes P \) is a graph with vertex set \( V(P \otimes P) = \{x_i^j; 1 \leq i \leq q; 1 \leq j \leq r\} \), edge set \( E(P \otimes P) = \{x_i^j x_{i+1}^j; 1 \leq i \leq q-1; j = 1\} \cup \{x_i^j x_{i+1}^{j+1}; 1 \leq i \leq q; 1 \leq j \leq r - 1\} \),
Figure 7. The total rainbow coloring of graph $P_3 \triangleright C_4$

Figure 8. The example of graph $P_3 \triangleright P_4$

$|V(P_q \triangleright P_r)| = qr$, $|E(P_q \triangleright P_r)| = qr - 1$, and $diam(P_q \triangleright P_r) = 2r + q - 3$.

The following figure 8 below illustrates $P_3 \triangleright P_4$

By using Proposition 1, we have the following equation

$$trc(P_q \triangleright P_r) \geq 2diam(P_q \triangleright P_r) - 1$$

$$= 4r + 2q - 7$$

We cannot attain this lower bound. Assume that $trc(P_q \triangleright P_r) < 2qr - q - 1$, we have $trc(C_n \triangleright P_r) = 2nr - 2n + \left\lceil \frac{n}{2} \right\rceil - 1$. We take a total rainbow path $x_i^r \rightarrow x_{i+3}^r$:

innerpath($P_i$)

$$(x_i^{r-1}, x_i^{r-1}, x_i^{r-2}, x_i^{r-2}, \ldots, x_1, x_1^{r+1}, x_1^{r+1}, x_1^{r+2}, x_1^{r+2}, x_1^{r+3}, x_1^{r+3})$$

innercycle($P_i$)

innerpath($P_{i+3}$)

vertices $x_i^{r-1}$ on subgraph $P_i$ and $x_{i+3}^{r-1}$ on subgraph $P_{i+3}$ it is exactly we cannot make a total rainbow path.

Furthermore, we will prove that $trc(P_q \triangleright P_r) \leq 2qr - q - 1$ by the total coloring $c$ as follow:
Figure 9. The total rainbow coloring of graph $P_3 \triangleright P_4$

\[c(v) = \begin{cases} 
  i, & \text{for } v = x_i^j; \ 1 \leq i \leq q, \ j = 1 \\
  n + (i - 1)(r - 2) + j - 1, & \text{for } v = x_i^j; \ 1 \leq j \leq r - 1 \\
  i, & \text{for } v = x_i^j; \ 1 \leq i \leq q, \ j = r
\end{cases}\]

\[c(e) = \begin{cases} 
  qr - q + i, & \text{for } e = x_i^jx_{i+1}^j; \ 1 \leq i \leq q - 1, \ j = 1 \\
  qr - 1 + (i - 1)(r - 1) + j, & \text{for } e = x_i^jx_{i+1}^j; \ 1 \leq i \leq q, \ 1 \leq j \leq r - 1
\end{cases}\]

Easily, they give a rainbow path between two vertices on total colored of $(P_q \triangleright P_r)$ and $V(P_q \triangleright P_r) \rightarrow (1, 2, \ldots, qr - q) \rightarrow E(P_q \triangleright P_r) \rightarrow (qr - q + 1, qr - q + 2, \ldots, qr - 1 + (i - 1)(r - 1) + j)$, thus the maximum color is $2qr - q - 1$ for $i = q$ and $j = r - 1$. The following figure 9 below illustrates Theorem 4.

3. Concluding Remarks
In this paper, we have determined the exact values of total rainbow connection number of some comb products of graphs, namely $(C_n \triangleright C_m)$, $(C_n \triangleright P_r)$, $(P_q \triangleright C_m)$, and $(P_q \triangleright P_r)$. As we did this proof, it was difficult to get a minimum total rainbow connection number. Thus, it still gives the following open problem.

Open Problem 1. Let $G$ be any comb product of graphs, determine sharper lower bounds of $trc(G)$?

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