Continuum quantum ferromagnets at finite temperature
and the quantum Hall effect

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Abstract

We study finite temperature (T) properties of the continuum quantum field theory of systems with a ferromagnetic ground state. A scaling theory of the T = 0 system is discussed carefully, and its consequences for crossovers between different finite T regimes in dimensions 1, 2, and 3 are described. The results are compared with recent NMR measurements of the magnetization of a quantum Hall system with filling factor ν = 1; we predict that the relaxation rate 1/T₁ of this system may have a finite T “ferromagnetic coherence peak”.

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The recent availability of nuclear magnetic resonance (NMR) measurements \cite{1} of quantum Hall systems has opened a new window onto the magnetic properties of a strongly correlated two-dimensional electronic system. Initially, the filling factor (\(\nu\)) dependence of the zero temperature (\(T\)) magnetization in the vicinity of \(\nu = 1\) attracted attention because it indicated that the low energy charged excitations of the system were spin textures (skyrmions) \cite{2,3,4}. In this paper, we examine instead the \(T\) dependence of the magnetic properties exactly at \(\nu = 1\) \cite{5}. We use a continuum quantum field theory of a ferromagnet as a model and describe its finite \(T\) properties. From a field-theoretical perspective, some features of this quantum field theory are rather unusual and lead to a noteworthy universality in the crossover functions. We will consider all values of the spatial dimension \(d > 0\), although the regime of validity of the continuum limit becomes larger as \(d\) is lowered and it is most useful for \(d \leq 2\). Our theory can also be applied to other low-dimensional ferromagnets (like the ferromagnetic layer of \(^3\)He on Grafoil \cite{6}) but we will limit our discussion here to the quantum Hall system. Some limitations of the model as applied to the quantum Hall effect will also be discussed.

The required quantum field theory is obtained from the naive continuum limit of the coherent-state path integral of an insulating, lattice ferromagnet:

\[
Z = \int \mathcal{D}\vec{n} \delta(\vec{n}^2 - 1) \exp \left( -\int d^d x \int_0^{1/T} d\tau (\mathcal{L}_0[\vec{n}] + \mathcal{L}_1[\vec{n}]) \right)
\]

\[
\mathcal{L}_0[\vec{n}] = iM_0 \vec{A}(\vec{n}) \cdot \partial_x \vec{n} + \left( \rho_s/2 \right)(\nabla_x \vec{n})^2 - M_0 \vec{H} \cdot \vec{n}.
\]

Here \(\vec{n}(x, \tau)\) is the 3-component unit vector field identifying the local orientation of the ferromagnetic order (it is periodic in the Matsubara time \(\tau\)), and \(M_0 \geq 0\) is the magnetization per unit volume in the ferromagnetic ground state. The first term in \(\mathcal{L}_0\) is the kinematical Berry phase \cite{7} which accounts for the commutation relations between the components of the order parameter; \(\vec{A}\) is the vector potential of a unit Dirac monopole at the origin of spin space with \(\epsilon_{ijk} \partial A_k / \partial n_j = n_i\), \(\rho_s\) is the ground state spin stiffness, and \(H\) is the magnetic field; \(\mathcal{L}_1\) contains local higher-gradient terms, with no time derivatives, that will be discussed.
below. (The Hopf term, which does contain a time derivative, will be discussed separately later.) We are using units in which \( k_B = \hbar = 1 \) and have absorbed a factor of \( g\mu_B \) into \( H \) (\( \mu_B \) is the Bohr magneton). For a ferromagnet on a hypercubic lattice with spacing \( a \), spin per site \( S \), and nearest-neighbor exchange \( J \), \( M_0 = Sa^{-d} \) and \( \rho_s = JS^2a^{2-d} \). In the quantum Hall effect at \( \nu = 1 \), \( M_0 = 1/(4\pi\ell_B^2) \), and (neglecting layer finite-thickness corrections which are expected to reduce \( \rho_s \)) \( \rho_s = e^2/(16\sqrt{2}\pi\varepsilon\ell_B) \), where \( \ell_B \) is the magnetic length. In the experiment of Ref. [1], \( H \approx 2K \) (note that \( g \approx 0.5 \) in GaAs), while we estimate that \( \rho_s \approx 3K \).

Since \( \mathcal{L}_1 \) contains no time derivatives, the Hilbert space is fully determined (through canonical arguments) by \( M_0 \), and the remainder of the action describes the Hamiltonian acting in this space. For \( M_0 = 0 \), there are no degrees of freedom in the system (the Hilbert space is one dimensional), so the Hamiltonian is immaterial. For \( M_0 \neq 0 \), the explicit quantization of the continuum quantum ferromagnet (CQFM) defined by Eqn (1) is difficult, but it is not hard to establish the quantization condition that \( 2M_0L^d \) must be integral; \( M_0L^d \) is the total spin of the fully polarized state \( (L^d \) is the volume of the system). Since all states must have half-integral spin, we can associate a length \( \xi_0 \) with \( M_0 \), \( 2M_0 = \xi_0^{-d} \). We expect that the degrees of freedom of the CQFM correspond roughly to independent spins \( 1/2 \) per volume \( \xi_0^d \), and the scale \( \xi_0 \) shows up naturally in the quantum theory [3]; for example, it is likely that the commutation relations for the spin density operators are smeared over the scale \( \xi_0 \) (a similar effect occurs in the quantum Hall system as a result of restriction to the lowest Landau level), and that the correlation length is never less than \( \xi_0 \).

In determining the applicability of the CQFM to a real system, we must consider a renormalization group (RG) analysis. There is a fixed point at \( M_0 = 0 \), and the terms in \( \mathcal{L}_0 \) are the most relevant perturbations. This can be seen by simple power counting: under a rescaling \( x \to e^{-\ell}x \), \( \tau \to e^{-z\ell}\tau \) (with \( z = 2 \) so that the long wavelength spin wave dispersion is invariant), we find that \( M_0 \) has dimension \( d \) (corresponding to its \( -d \) powers of \( \xi_0 \)), \( \rho_s \) has dimension \( d + z - 2 = d \), and \( T \) and \( H \) have dimension \( z = 2 \). Terms in \( \mathcal{L}_1 \) have \( k \geq 4 \) gradients, and their coefficients have dimension \( d + z - k \); for \( d < 2 \),
all such terms are irrelevant. To go beyond power-counting requires a diagrammatic RG which will be described elsewhere; the results include an RG reinterpretation of earlier spin wave calculations in $d = 3$ \[10\] and $d = 2$ \[11\]. At $T = 0$, the power-counting flows for the couplings in $\mathcal{L}_0$ are exact, but these couplings do generate a term in $\mathcal{L}_1$, $\lambda(\partial_a n_i \partial_a n_i \partial_b n_j \partial_b n_j - 2 \partial_a n_i \partial_b n_j \partial_a n_j \partial_b n_j)$, associated with spin wave scattering; this is described by the RG flow $d\lambda/d\ell = (d - 2)\lambda + c\rho_s/M_0$ (with $c$ a positive constant), which sets in at scales $> \xi_0$. For $d < 2$, $\lambda$ flows to a fixed point value $c\rho_s/(2 - d)M_0$. Similar phenomena are expected for other, even less relevant, interactions. Thus all the irrelevant couplings actually flow either to zero or to nonzero fixed point values, and approach these values with eigenvalues given by their dimensions established above, $\gamma_k = d + 2 - k$. The simple form of these results, compared with more familiar field theories, is due to the fluctuationless fully-polarized ground state, so that contributions come only from scattering of already-existing spin waves (similar to the dilute Bose gas \[12\]), and to the rotational symmetry requirements.

For $d < 2$, these considerations imply that all observables should be universal functions of the bare couplings $M_0$, $\rho_s$ and $H$, realizing a no-scale-factor universality similar to that discussed in Ref \[12\] for the dilute Bose gas in $d < 2$. Scaling forms can therefore be deduced from a naive dimensional analysis of the length and time scales in the CQFM; for the free energy density $F$ we obtain

$$F = TM_0 \Phi_F(\bar{\rho}_s/T, H/T) \tag{2}$$

where $\bar{\rho}_s \equiv \rho_s/M_0^{(d-2)/d}$ is a rescaled stiffness and $\Phi_F(r, h)$ is a universal scaling function with no arbitrary scale-factors and dependent only on $d$ and the symmetry group ($O(3)$) of the ferromagnet (in particular, for the lattice ferromagnet, $S$ enters only indirectly, through $\rho_s$ and $M_0$). Scaling forms for other thermodynamic observables can be obtained by taking derivatives of $F$. Because the scaling form is obtained by setting the irrelevant couplings to their fixed point values, it is valid only when the deviation of those couplings from their fixed points are negligible. If the bare values of the irrelevant couplings (defined at the scale $\xi_0$) are set to their fixed point values in \[1\], then the free energy is given by \[2\] at
all values of \( T, \rho_s, H \). For a real system, this tuning of parameters does not occur, and the behavior approaches the universal form only for \( T < T_{\text{max}}(H) \) or \( H < H_{\text{max}}(T) \); we expect \( T_{\text{max}} \sim \tilde{\rho}_s \) as \( H \to 0 \) for systems with small \( S \). For \( d > 2 \) additional scaling variables, associated with other relevant couplings, will be necessary in a generalization of (2).

We now consider the different \( T \) regimes of the CQFM, ignoring \( \mathcal{L}_1 \), so this will be universal for \( d < 2 \) and marginally so for \( d = 2 \). Fig 1 shows a phase diagram as a function of the three dimensionless ratios of the energy scales \( T, H \), and \( \tilde{\rho}_s \) plotted in the projective plane. All boundaries are smooth crossovers with the exception (for \( d > 2 \)) of the ferromagnetic phase transition at the single point \( H = 0, T = T_c \sim \tilde{\rho}_s \). The regimes in Fig 1 are:

(i) \textit{Quantum activated (QA)}, \( T < H \)—most spins are aligned as in the ground state along \( H \), with thermal corrections associated with a thermal activation factor \( e^{-H/T} \). There is also a crossover (indicated by the dotted line) between \( T < \tilde{\rho}_s < H \) and \( \tilde{\rho}_s < T < H \), but it only affects the prefactor of \( e^{-H/T} \).

(ii) \textit{Renormalized classical (RC)}, \( H < T < \tilde{\rho}_s \)—the behavior is dominated by fluctuations of classical Goldstone modes with energies smaller than \( T \).

(iii) \textit{Quantum critical (QC)}, \( T > H, \tilde{\rho}_s \)—this regime was proposed recently in \( d = 2 \) in Ref [13]. It is the high \( T \) limit of the CQFM. One may interpret the behavior here as the response of the \( M_0 \neq 0 \) system with zero Hamiltonian to a finite size, \( 1/T \), along the time direction. Ref [13] also suggested that, for \( S = 1/2 \), \( T_{\text{max}} \) is large enough for the square lattice ferromagnet to exhibit QC behavior.

In \( d = 1 \), Nakamura and Takahashi [14] have studied the magnetization of the spin \( S \) chain in the RC region, and their results are described by the CQFM. The expected scaling form is \( M = M_0 \phi_M \), where \( \phi_M \) is a function similar to \( \Phi_F \); they find a scaling function, \( \phi_M \), to which our function \( \Phi_M \) reduces in a limit appropriate for the RC region: \( \Phi_M(r \to \infty, h \to 0) = \phi_M(\rho h) \) and they computed \( \phi_M(y) = 2y/3 - 44y^3/135 + \ldots \) for small \( y \). For \( d < 2 \) the function \( \Phi_M(r, h) \) can also be computed in the usual spin wave expansion [10,11] which yields a universal series containing integral powers of \( r^{-d/2} \) times
functions of $h$.

In $d=2$, the flow of $\lambda$ is logarithmic, and universality at low $T$ is violated by logarithms, unlike the situation in antiferromagnets [13]. As a result, the QC regime lies at the edge of where quasi-universality holds. However the logarithmic terms contain prefactors of powers of $T$ at low $T$, and are absent in the leading low $T$ behavior. The flow of $\lambda$, in particular, has been overlooked in previous analyses of $d=2$ ferromagnets [16].

Before turning to calculations of the scaling functions, we discuss other aspects of the $d=2$ case more fully. For the CQFM in general in $d=2$, a conserved topological current, defined as $j_\mu = \epsilon_{\mu\nu\lambda} i_{ijk} n_i \partial_\nu n_j \partial_\lambda n_k / 8\pi$ ($\mu = x, y, \tau$), exists and represents the number density and current of skyrmions. The skyrmions experience an effective orbital magnetic field of strength $4\pi M_0$, produced by the Berry phase term, as can be seen from the identity $M_0 \int \vec{A}(\vec{n}) \cdot \partial_\tau \vec{n} = \int j \cdot \vec{A}$, where $\nabla \times \vec{A} = 4\pi M_0$ represents the uniform field. Thus there is an effective magnetic length for the skyrmions, that is related to $\xi_0$; moreover skyrmions come in quantized sizes that are multiples of $\xi_0^2$. In the quantum Hall system, skyrmions carry real electric charge [2,3], so the use of the CQFM does not exclude charged excitations. However, in this case, there are also other terms that are known to appear in the long-wavelength description [2,3] but are not included in the CQFM thus far. The extra terms are (i) the Hopf term $2\pi i \int j_\mu a_\mu$, where $a_\mu$ obeys $\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda = j_\mu$, which endows the skyrmions with Fermi statistics and half-integral spin [17]; (ii) the Coulomb interaction $\int \int j_\tau(x) j_\tau(x') e^2 / (2\varepsilon |x-x'|)$. The Hopf term is marginal, but since it contains a time derivative, it affects the quantization directly, and may change the dimension of the Hilbert space in a finite system. It will not affect the discussion of universality and its violation by logarithms, but it will change the precise scaling functions in general. The Coulomb interaction has dimension 1, so is relevant, though less so than $\rho_s$, and in principle requires that an additional scaling variable appears in the scaling functions. However, both terms enter only through skyrmions which, in the large $\rho_s$ region, always have an energy $> \rho_s$, and their contributions are exponentially small at low $T$.

Finally, we present our large $N$ results for the CQFM. These are valid over the entire
phase diagram of Fig 1, and exhibit all the crossovers. We discuss two different large $N$ limits; the first generalizes the symmetry group from $SU(2) \cong O(3)$ to $SU(N)$ and the second to $O(N)$. To obtain the $SU(N)$ theory we write $\vec{n} = z_\alpha^* \vec{\sigma}_{\alpha\beta} z_\beta$, where $\vec{\sigma}$ are the Pauli matrices, $z_\alpha$ is a 2-component complex field, and $\sum_\alpha |z_\alpha|^2 = 1$. The Berry phase in $\mathcal{L}$ now becomes $2M_0 \sum_\alpha z_\alpha^* \partial_\tau z_\alpha$. We can now obtain $SU(N)$ symmetry by allowing $\alpha$ to run from 1 to $N$ (for $N$ even, the field $H$ is taken to couple to the generator $\text{diag}(1_\frac{N}{2}, -1_\frac{N}{2})$); the gradient terms are as in the $CP^{N-1}$ model [2,18]. For the $O(N)$ generalization we parametrize $n_i = i \epsilon_{ijk} w_j^* w_k$ where $w_i$ is a 3-component complex field obeying $\sum_i |w_i|^2 = 1$ and $\sum_i w_i^2 = 0$. The Berry phase is now $M_0 \sum_i w_i^* \partial_\tau w_i$ and $O(N)$ symmetry is achieved by allowing $i$ to run from 1 to $N$ (for $N$ divisible by 3, $H$ couples to a generator which contains $N/3$ copies of the $O(3)$ generator). The $1/N$ expansion of both theories is standard and we omit all details: the constraints are imposed by Lagrange multipliers, and $\rho_s$ and $M_0$ should be of order $N$ as $N \to \infty$. We present below $N = \infty$ results from both theories for some observables in $d = 2$ (although results can be obtained for arbitrary $d$); the results are universal as the logarithmic violations of universality appear only at higher orders in $1/N$.

(a) Magnetization: From the $SU(\infty)$ theory we obtain:

$$
\Phi_M(r, h) = \ln \left[ \frac{(q_1 - e^{-h/2})/(q_1 - e^{h/2})}{(8\pi r)} \right]
$$

where $q_1 > 1$ is the solution of $(q_1 - e^{-h/2})(q_1 - e^{h/2}) = q_1^2 e^{-8\pi r}$. Similarly we obtain from the $O(\infty)$ theory:

$$
\Phi_M(r, h) = \ln \left[ \frac{(q_2 - e^{-h})/(q_2 - e^{h})}{(4\pi r)} \right]
$$

where $q_2 > 1$ is the solution of $(q_2 - e^{-h})(q_2 - 1)(q_2 - e^{h}) = q_2^3 e^{-4\pi r}$. We show in Fig 2 a plot of these results for $M/M_0$ as a function of $T/H$ for a few values of $\rho_s/H$, including $\rho_s/H = 0$. For $\rho_s \gg H$ it is possible, in principle, to use simpler functions characteristic of the different regions of Fig 1, punctuated by crossovers between them. At the lowest $T$ we have QA behavior with $\Phi_M \sim 1 \propto e^{-h}$. At larger $T$ we have RC behavior described by the scaling function of Ref [19]; in the $SU(\infty)$ theory we can obtain this function from (3):
\[ \Phi_M = 1 + \ln[h/2 + ((h/2)^2 + e^{-8\pi r})^{1/2}]/(4\pi r) \]. At the largest \( T \) we have QC behavior in which we expect \( \Phi_M \propto h \). Although the analytic forms are rather different in the 3 regimes, the qualitative trends in an \( M \) vs. \( T \) plot are similar; this makes picking out the regimes from experimental data rather difficult.

(b) NMR relaxation rate \( 1/T_1 \): Unlike the static magnetization, the dynamic susceptibility has significantly different behavior in the regions of Fig 1, and this leads to clear signatures of them in \( 1/T_1 \). We model the nuclear-electron contact coupling by \( A\bar{M}_0 \vec{I}(\tau)\cdot \vec{n}(0, \tau) \). Then the relaxation rate is given by \( 1/T_1 = A^2 T \lim_{\omega \to 0} \text{Im} \chi_{L+} / \omega \) where \( \chi_{L+} \) is the local transverse susceptibility. Dimensional analysis shows that \( 1/T_1 \) satisfies the scaling form \( 1/T_1 = (A^2 M_0^2 / T) \Phi_{T_1} \), where \( \Phi_{T_1} \) is a universal function like \( \Phi_F \). We evaluated \( \Phi_{T_1} \) in both large \( N \) limits and found

\[ \Phi_{T_1}(r, h) = 1/[16\pi r^2(q_1 e^{h/2} - 1)] \] (5)

in the \( SU(\infty) \) theory (with \( q_1 \) defined below (3)) and

\[ \Phi_{T_1}(r, h) = (1/8\pi r^2)[1/(q_2 e^h - 1) + 1/(q_2 - 1)] \] (6)

in the \( O(\infty) \) theory (with \( q_2 \) defined below (4)). A plot of these results for \( 1/T_1 \) is shown in Fig 3 as a function of \( T/\rho_s \) for some values of \( \rho_s/H \). The most notable feature is the “ferromagnetic coherence peak” which signals a crossover between the QA and RC regimes. This becomes clear from the asymptotic behavior for \( \rho_s \gg H \). In the low \( T \), QA regime we have activated behavior \( 1/T_1 \sim e^{-H/T} \) ( \( \Phi_{T_1} = e^{-h}/16\pi r^2 \) for \( SU(\infty) \) and \( \Phi_{T_1} = e^{-h}/8\pi r^2 \) for \( O(\infty) \)). In contrast, in the RC regime, \( 1/T_1 \) decreases exponentially fast with increasing \( T \) ( \( \Phi_{T_1} = e^{4\pi r}/16\pi r^2 \) for \( SU(\infty) \) and \( \Phi_{T_1} = e^{4\pi r/3}/4\pi r^2 \) for \( O(\infty) \)) due to the rapid decrease in the ferromagnetic correlation length; this behavior of \( 1/T_1 \) is similar to that observed in the RC region of \( d = 2 \) quantum antiferromagnets [20]. Finally in the large \( T \) QC region we find \( 1/T_1 \sim \text{const} \) ( \( \Phi_{T_1} = 1/4r \) for \( SU(\infty) \) and \( \Phi_{T_1} = 1/3r \) for \( O(\infty) \)). Notice from Fig 3 that the coherence peak survives even for moderate values of \( \rho_s/H \), though it may be absent for \( \rho_s/H \) sufficiently small. We emphasize that for \( \rho_s/H \) large, this peak is not dependent
upon a large value of $T_{\text{max}}$, as it occurs at the crossover between the low $T$ QA and RC regimes.

Nontrivial textures (skyrmions) exist, and the Hopf and Coulomb interaction terms can be included, for all $N$ in both the $SU(N)$ and $O(N)$ models, but have no effect in the $N \to \infty$ limit. While agreement with the quantum Hall experiments [1] is fair, at this point it is not clear whether the differences between theory and experiment are due to these effects, other differences between $N = \infty$ and $N = 2$ or 3, the possibility that $T_{\text{max}}$ is small, or the uncertainty in the value of $\rho_s$. More complete measurements of the $T$ dependence of $1/T_1$, particularly in samples with a larger $\rho_s/H$, could help answer these questions.

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FIGURES

FIG. 1. Phase diagram of the CQFM as a function of dimensionless ratios of the 3 energies $T$, $H$ and $\tilde{\rho}_s \equiv \rho_s/M_0^{(d-2)/d}$. Each energy becomes infinite at one of the vertices, and equals zero on the opposite side. Dashed and dotted lines are crossovers. Increasing $T$ from zero to $\infty$ at fixed $\tilde{\rho}_s/H$ corresponds to moving along a straight line from the base to the apex.

FIG. 2. Magnetization of the CQFM in $d = 2$, computed in the $N = \infty$ limit of $SU(N)$ and $O(N)$ theories for a number of values of $\rho_s/H$, and compared with the experiments of Ref [1]. The $\rho_s = 0$ limit of the large $N$ results yield the spin $S$ Brillouin function, with $S = 1/2$ in the $SU(N)$ model, and $S = 1$ for $O(N)$.

FIG. 3. As in Fig. 2 but for $1/T_1$ vs. $T/\rho_s$. The constant $A' = \rho_s/A^2M_0^2$. 
\[ T/H = \frac{T}{\rho_s} \]

\[ \text{T/Critical} \]

\[ \text{Quantum Activated} \]

\[ H = \infty \]

\[ \frac{\rho_s}{H} \]

\[ \rho_s = \infty \]

\[ \text{(only for d>2)} \]

Read and Sachdev Fig 1
Read and Sachdev Fig 2

$M/M_0$ vs $T/H$

- $\rho_s/H$
- SU(N) 1.5
- SU(N) 0.5
- SU(N) 0
- O(N) 1.5
- O(N) 0.5
- O(N) 0
- Expts

Read and Sachdev Fig 2
Read and Sachdev Fig 3

\[ \frac{A'}{T_1} \]

\[ \frac{\rho_s}{H} \]

- SU(N) 4
- SU(N) 1.5
- SU(N) 0.5
- O(N) 4
- O(N) 1.5
- O(N) 0.5

\[ T/\rho_s \]

\[ \rho_s \]

Read and Sachdev Fig 3