Exact solution of the quantum integrable model associated with the twisted $D_3^{(2)}$ algebra

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Abstract

We generalize the nested off-diagonal Bethe ansatz method to study the quantum chain associated with the twisted $D_3^{(2)}$ algebra (or the $D_3^{(2)}$ model) with either periodic or integrable open boundary conditions. We obtain the intrinsic operator product identities among the fused transfer matrices and find a way to close the recursive fusion relations, which makes it possible to determine eigenvalues of transfer matrices with an arbitrary anisotropic parameter $\eta$. Based on them, and the asymptotic behaviors and values at certain points, we construct eigenvalues of transfer matrices in terms of homogeneous $T - Q$ relations for the periodic case and inhomogeneous $T - Q$ relations for the open case with some off-diagonal boundary reflections. The associated Bethe ansatz equations are also given. The method and results in this paper can be generalized to the $D_{n+1}^{(2)}$ model and other high rank integrable models.

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1 Introduction

One of the major achievements of one-dimensional quantum integrable systems is that it can supply us some believable results of strong interacting quantum many-body systems. The coordinate \[1, 2\] and algebraic Bethe ansatz \[3–7\] as well as the \(T - Q\) relations \[8–10\] are the powerful methods to obtain the exact solutions of the systems and have made great achievements in the past several decades. Based on the exact solution, many interesting physical concepts and mathematical structures in the quantum field theory, quantum group and quantum algebra are obtained \[11\].

Recently, much attention has been paid on the investigation of quantum integrable systems with \(U(1)\) symmetry broken, because this kind of systems has many important applications in the open string, quantum magnetism and non-equilibrium statistical mechanics. The typical models are the systems with twisted boundary conditions \[12\] or non-diagonal boundary magnetic fields \[13 – 16\]. Due to the fact that the traditional Bethe ansatz does not work, many interesting methods such as q-Qnsager algebra \[17\], separation of variables \[18–21\], off-diagonal Bethe ansatz (ODBA) \[22–24\], and modified algebraic Bethe ansatz \[25–28\] were developed.

Motivated by the applications in AdS/CFT, string theory and conformal field theory, the study of quantum integrable models with high rank Lie algebras becomes a very important issue \[29–31\]. Many efforts have been made to investigate this kind of systems. For example, the exact solutions of open boundary quantum integrable models associated with \(A_n\) \[32\], \(B_2\) \[33\], \(C_n\) \[34\], \(D^{(1)}_3\) \[35\] and \(A^{(2)}_2\) \[36\] Lie algebras are obtained. Based on the \(su(2) \times su(2)\) symmetry, some new integrable strongly correlated electronic systems are very recently constructed \[37\].

The \(D^{(2)}_{n+1}\) is a typical twisted Lie affine algebra and the related quantum integrable models have been attracted many attentions \[38–44\]. Reshetikhin obtained the Bethe ansatz solutions of the \(D^{(2)}_{n+1}\) model with periodic boundary condition \[38\]. Martins and Guan studied the integrability of the \(D^{(2)}_{n+1}\) model with open boundary condition \[39\]. Further, Nepomechie, Pimenta and Retore studied the integrable quantum group invariant and \(D^{(2)}_{n+1}\) open spin chains, where an interesting result is that the \(R\)-matrix can be constructed by two six-vertex \(R\)-matrices \[40\]. The integrable \(D^{(2)}_{n+1}\) reflection matrices with quantum group symmetry are given in \[41, 42\]. Another important progress is the Bethe ansatz solution of
the $D^{(2)}_2$ model \cite{43,44}, which has application in the black hole theory.

In this paper, we study the quantum integrable spin chain associated with the twisted $D^{(2)}_3$. We generalize the nested ODBA method to the chain with either the periodic or the open boundary condition. By using the fusion technique \cite{45–50}, we systemically analyze the fusion structure of the system. We provide a way to close the recursive fusion relations, which make the fusion relations can be used to construct the energy spectrum without any additional constraints. We obtain the closed intrinsic operator product identities among the fused transfer matrices. Based on them, and the asymptotic behaviors and values at certain points, we obtain the eigenvalues of fused transfer matrices, which are expressed as the homogeneous $T - Q$ relations for the periodic case and inhomogeneous ones for the open case with off-diagonal reflection matrices. The associated Bethe ansatz equations are also given.

The plan of the paper is as follows. In section 2, we study the $D^{(2)}_3$ spin chain with the periodic boundary condition. The closed operator product identities among the fused transfer matrices are given. By constructing the homogeneous $T - Q$ relations, we obtain the eigenvalues and Bethe ansatz equations of the system. Section 3 is devoted to diagonalize the model with some non-diagonal boundary reflections. We obtain the recursive fusion relations, the eigenvalues of transfer matrices in terms of inhomogeneous $T - Q$ relations, and the Bethe ansatz equations. The summary of main results and some concluding remarks are presented in section 4. Some details deriving the fusions of the $R$-matrices and related $K$-matrices are given in Appendices A and B.

## 2 $D^{(2)}_3$ model with the periodic boundary condition

Throughout this paper, we adopt the standard notations. $V$ denotes a $n$-dimensional linear space with the orthogonal basis \{
\begin{align*}
|i\rangle, \quad i = 1, 2, \ldots, n
\end{align*}
\}\}. For any matrix $A \in \text{End}(V)$, $A_j$ is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as $A$ on the $j$-th space and as identity on the other factor spaces. For any matrix $B \in \text{End}(V \otimes V)$, $B_{ij}$ is an embedding operator of $B$ in the tensor space, which acts as identity on the factor spaces except for the $i$-th and $j$-th ones.

Let us consider the fundamental $R$-matrix associated with the vectorial representation
of the twisted algebra $D^{(2)}_3$ given by

$$R^{uv}_{12}(u) = U_1 U_2 R_{12}(u) U_2^{-1} U_1^{-1}, \quad (2.1)$$

where $U$ is the gauge transformation with the form of

$$U = \begin{pmatrix}
\cosh \eta & 0 & 0 & 0 & 0 & 0 \\
0 & \cosh \eta & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\cosh 3\eta} & 0 & 0 & 0 \\
0 & 0 & 2 \sinh^2 \eta \sqrt{\cosh \eta} - \cosh 2\eta \sqrt{\cosh \eta} & 0 & 0 & 0 \\
0 & 0 & 0 & \cosh 2\eta & 0 & 0 \\
0 & 0 & 0 & 0 & \cosh 2\eta & 0
\end{pmatrix}, \quad (2.2)$$

and the matrix $R_{12}(u)$ is given by [40]

$$R_{12}(u) = e^{-2(u+3\eta)} \left\{ (e^{2u} - e^{4\eta}) \left( e^{2u} - e^{8\eta} \right) \sum_{\alpha \neq 3,4 \atop \alpha \text{ or } \beta \neq 3,4} [e_1]_{\alpha}^{\alpha} \otimes [e_2]_{\beta}^{\beta} - (e^{4\eta} - 1) (e^{2u} - e^{8\eta}) \left( \sum_{\alpha < \beta, \alpha \neq \beta' \atop \alpha, \beta \neq 3,4} [e_1]_{\alpha}^{\alpha} \otimes [e_2]_{\beta}^{\beta} \right) + \sum_{\alpha \neq 3, \beta = 3,4} (e^{u} + 1) \left( \sum_{\alpha < 3, \beta = 3,4} e^{u} \sum_{\alpha > 4, \beta = 3,4} \right) \right\}$$

$$\times \left[ \sum_{\alpha \neq \beta', \alpha \text{ or } \beta \neq 3,4} (e^{u} - 1) \left( \sum_{\alpha < 3, \beta = 3,4} e^{u} \sum_{\alpha > 4, \beta = 3,4} \right) \right]$$

$$\times \left[ (e_1)_{\beta}^{\beta} \otimes (e_2)_{\alpha}^{\alpha} + (e_1)_{\beta}^{\beta'} \otimes (e_2)_{\alpha'}^{\alpha'} \right] + \sum_{\alpha, \beta \neq 3,4} a_{\beta}^{\alpha}(u) [e_1]_{\beta}^{\beta} \otimes [e_2]_{\alpha'}^{\alpha'} + \frac{1}{2} \sum_{\alpha, \beta \neq 3,4, \beta = 3,4}$$

$$\times \left[ b_{\alpha}(u) \left( (e_1)_{\beta}^{\beta} \otimes (e_2)_{\alpha'}^{\alpha'} + (e_1)_{\beta}^{\beta'} \otimes (e_2)_{\alpha}^{\alpha} \right) + c_{\alpha}(u) \left( (e_1)_{\beta}^{\beta} \otimes (e_2)_{\alpha'}^{\alpha'} + (e_1)_{\beta}^{\beta'} \otimes (e_2)_{\alpha}^{\alpha} \right) \right]$$

$$+ \sum_{\alpha = 3,4} \left[ d_{\alpha}(u) [e_1]_{\alpha}^{\alpha} \otimes [e_2]_{\alpha'}^{\alpha'} + d_{\alpha}(u) [e_1]_{\alpha}^{\alpha} \otimes [e_2]_{\alpha'}^{\alpha'} \right]. \quad (2.3)$$

Here $u$ is the spectral parameter and $\eta$ is the crossing or anisotropic parameter. The subscripts $\alpha, \beta = 1, \cdots, 6, \alpha' = 7 - \alpha$ and $\beta' = 7 - \beta$. Each of matrices $[e_{k}]^{\alpha}_{\beta}(k = 1, 2)$ is the Weyl basis of the $6 \times 6$ representation matrix of the $k$-th space. The coefficients $a_{\beta}^{\alpha}(u)$ are

$$a_{\beta}^{\alpha}(u) = \begin{cases}
(e^{4\eta}e^{2u} - e^{8\eta})(e^{2u} - 1), & \alpha = \beta, \\
(e^{4\eta} - 1)(e^{8\eta}e^{2(n-\alpha - \beta)} - e^{2u} - 1) + \delta_{\alpha,\beta'}(e^{2u} - e^{8\eta}), & \alpha < \beta, \\
(e^{4\eta} - 1)e^{2u}(e^{2(n-\alpha - \beta)} - e^{8u} - 1) - \delta_{\alpha,\beta'}(e^{2u} - e^{8\eta}), & \alpha > \beta, 
\end{cases} \quad (2.4)$$
where $\alpha, \beta \neq 3, 4$ is supposed and we have used the notation

\[
\bar{\alpha} = \begin{cases} 
\alpha + 1, & 1 \leq \alpha < 3, \\
\frac{7}{2}, & \alpha = 3, 4, \\
\alpha - 1, & 4 < \alpha \leq 6.
\end{cases} 
\tag{2.5}
\]

The functions $b^\pm_\alpha(u)$, $c^\pm(u)$ and $d^\pm(u)$ are

\[
b^\pm_\alpha(u) = \begin{cases} 
\pm e^{2\eta (\alpha - 1/2)} (e^{4\eta} - 1)(e^{2u} - 1)(e^\alpha \pm e^{4\eta}), & \alpha < 3, \\
ed^{2\eta (\alpha - 9/2)} (e^{4\eta} - 1)(e^{2u} - 1)e^\alpha (e^\alpha \pm e^{4\eta}), & \alpha = 4, \\
ed^{2\eta (\alpha - 1/2)} (e^{4\eta} - 1)(e^{2u} - 1)(e^{2u} - e^{8\eta}), & \alpha > 4.
\end{cases}
\]

The $R$-matrix (2.1) satisfies the properties

regularity : $R^\nu_{\nu 12}(0) = \rho_1(0)^{1/2} \mathcal{P}_{12}$, 

unitarity : $R^\nu_{\nu 12}(u) R^\nu_{\nu 21}(-u) = \rho_1(u) = a_1(u) a_1(-u)$, 

crossing symmetry : $R^\nu_{\nu 12}(u) = V_1 \{R^\nu_{\nu 12}(-u + 4\eta)\}^{t_2} V_1 = V_2 \{R^\nu_{\nu 21}(-u + 4\eta)\}^{t_1} V_2^{t_2}$, 

where $\mathcal{P}_{12}$ is the permutation operator with the matrix elements $[\mathcal{P}_{12}]_{\alpha\beta} = \delta_{\alpha\beta} \delta^{\alpha\beta}$, $t_k$ denotes the transposition in the $k$-th space, $a_1(u) = 4 \sinh(u - 4\eta) \sinh(u - 2\eta)$, $R_{21}(u) = \mathcal{P}_{12} R_{12}(u) \mathcal{P}_{12}$, and the crossing matrix $V$ is defined as

\[
V = \begin{pmatrix} 
1 & e^{-\eta} & e^{-3\eta} \\
e^\eta & 1 \\
e^{3\eta} & e^\eta & 1
\end{pmatrix}, 
\quad V^2 = \text{id}. 
\tag{2.10}
\]

Combining the unitarity (2.8) and the crossing symmetry (2.9), one can derive that the $R$-matrix (2.1) has the crossing unitarity relation

\[
crossing unitarity : \quad R^\nu_{\nu 12}(u)^{t_1} M_1 R^\nu_{\nu 21}(-u + 8\eta)^{t_1} M_1^{-1} = \rho_1(u - 4\eta), 
\tag{2.11}
\]

where the matrix $M$ reads

\[
M = V^t V = \text{diag}[e^{6\eta}, e^{2\eta}, 1, 1, e^{-2\eta}, e^{-6\eta}]. 
\tag{2.12}
\]
The $R$-matrix (2.1) also has the periodicity

$$R_{12}^{vw}(u + i\pi) = \bar{V}_2 R_{12}^{vw}(u) \bar{V}_2^{-1},$$

where

$$\bar{V} = \begin{pmatrix} \cosh 2\eta & \cosh 2\eta \\ \cosh 2\eta & -2 \sinh^2 \eta - \sqrt{\cosh 3\eta \cosh \eta} \\ 2 \sinh^2 \eta - \sqrt{\cosh 3\eta \cosh \eta} & \cosh 2\eta \\ -\sqrt{\cosh 3\eta \cosh \eta} & -2 \sinh^2 \eta & \cosh 2\eta \end{pmatrix}. \tag{2.14}$$

Besides the above properties, the $R$-matrix (2.1) satisfies the Yang-Baxter equation

$$R_{12}^{vw}(u - v) R_{13}^{vw}(u) R_{23}^{vw}(v) = R_{23}^{vw}(v) R_{13}^{vw}(u) R_{12}^{vw}(u - v). \tag{2.15}$$

The monodromy matrix of the system is constructed by the vectorial $R$-matrix (2.1) as

$$T_0^v(u) = R_{01}^{vw}(u - \theta_1) R_{02}^{vw}(u - \theta_2) \cdots R_{0N}^{vw}(u - \theta_N), \tag{2.16}$$

where the index 0 indicates the auxiliary space and the indices $\{1, \cdots, N\}$ denote the physical or quantum spaces, $N$ is the number of sites and $\{\theta_j| j = 1, \cdots, N\}$ are the inhomogeneous parameters. The monodromy matrix satisfies the Yang-Baxter relation

$$R_{12}^{vw}(u - v) T_1^v(u) T_2^v(v) = T_2^v(v) T_1^v(u) R_{12}^{vw}(u - v). \tag{2.17}$$

The transfer matrix is defined as the partial trace of monodromy matrix in the auxiliary space

$$t^{(p)}(u) = \text{tr}_0 T_0^v(u). \tag{2.18}$$

From the Yang-Baxter relation (2.17), one can prove that the transfer matrices with different spectral parameters commute with each other, i.e., $[t^{(p)}(u), t^{(p)}(v)] = 0$. Therefore, $t^{(p)}(u)$ serves as the generating functional of the conserved quantities of the system. The Hamiltonian of the $D_3^{(2)}$ spin chain with the periodic boundary condition can be given in terms of the transfer matrix (2.18) as

$$H_p = \frac{\partial \ln t^{(p)}(u)}{\partial u}|_{u=0,\{\theta_j\}=0} = \sum_{k=1}^{N} H_{k \, k+1}. \tag{2.19}$$
with
\[ H_{kk+1} = \mathcal{P}_{kk+1} \frac{\partial}{\partial u} R_{kk+1}(u) |_{u=0}. \] (2.20)

The periodic boundary condition reads
\[ H_{NN+1} = H_{N1}. \] (2.21)

### 2.1 Operator product identities among the fused transfer matrices

In order to obtain eigenvalues of the fundamental transfer matrix \( t^{(p)}(u) \), we need further to introduce some fused transfer matrices [24]. For the \( D_3^{(2)} \) spin chain with the periodic boundary condition, we introduce the fused monodromy matrices via the fused \( R \)-matrices \( R_{\theta j}^{s_\pm}(u) \) given by (A.1) and (A.2) in the Appendix A,
\[ T_0^\pm(u) = R_{\tilde{0}1}^{s_\pm} (u - \theta_1) R_{\tilde{0}2}^{s_\pm} (u - \theta_2) \cdots R_{\tilde{0}N}^{s_\pm} (u - \theta_N). \] (2.22)

Here and after, \( \tilde{0}' = 0' \) denotes the auxiliary space for the spinorial representation \( s_\pm \) and \( \tilde{0}' = \bar{0}' \) denotes the auxiliary space for the spinorial representation \( s_{\pm} \). We note that the quantum spaces of \( T_0^+(u) \) and \( T_0^-(u) \) are the same, which are also the quantum spaces of \( T_0(u) \). The fused \( R \)-matrices \( R_{\tilde{0}j}^{s_\pm} \) satisfy the Yang-Baxter relations
\[ R_{\tilde{0}j}^{s_\pm} (u - v) T_{\tilde{0}}^\pm (u) T_{\tilde{0}}^\pm (v) = T_{\tilde{0}}^\pm (v) T_{\tilde{0}}^\pm (u) R_{\tilde{0}j}^{s_\pm} (u - v), \] (2.23)
\[ R_{\tilde{0}j}^{s_\pm,s_-} (u - v) T_{\tilde{0}}^+ (u) T_{\tilde{0}}^- (v) = T_{\tilde{0}}^- (v) T_{\tilde{0}}^+ (u) R_{\tilde{0}j}^{s_\pm,s_-} (u - v), \] (2.24)

where \( R_{\tilde{0}j}^{s_\pm,s_-} (u) \) is defined by Eq.(B.5). Taking the partial trace in the auxiliary space, we obtain the fused transfer matrices
\[ t^{(p)}_\pm(u) = tr_{\tilde{0}} T_{\tilde{0}}^\pm(u). \] (2.25)

From the Yang-Baxter relation (2.24), we can prove that the transfer matrices \( t^{(p)}(u) \) and \( t^{(p)}_\pm(u) \) commute with each other, i.e.,
\[ [t^{(p)}_\pm(u), t^{(p)}_\pm(u)] = [t^{(p)}_\pm(u), t^{(p)}(u)] = 0. \] (2.26)

Thus they have the common eigenstates.

From the Yang-Baxter relations (2.17), (2.23) and (2.24) at certain points and using the properties of projectors, we obtain
\[ T_1(\theta_j) T_2(\theta_j + 4\eta) = P_{21}^{v_1v_2} T_1(\theta_j) T_2(\theta_j + 4\eta), \]
\[ T_2(\theta_j)T_1(\theta_j + 2\eta + i\pi) = P_{12}^{v(16)} T_2(\theta_j) T_1(\theta_j + 2\eta + i\pi), \]
\[ T_2(\theta_j)T_1^{\pm}(\theta_j + 3\eta + i\pi) = P_{12}^{(\pm)} T_2(\theta_j) T_1^{\pm}(\theta_j + 3\eta + i\pi), \quad j = 1, \ldots, N, \quad (2.27) \]

where the projectors: \( P_{21}^{v(1)}, P_{12}^{v(16)}, P_{12}^{(\pm)} \) are given by \((A.19), (A.25), (A.7)\) and \((A.13)\). By using the fusion identities \((A.10), (A.13), (A.22)\) and \((A.28)\), we obtain the fusion identities:

\[ P_{21}^{v(1)} T_1(u) T_2(u + 4\eta) P_{21}^{v(1)} = P_{21}^{v(1)} \prod_{j=1}^{N} a_1(u - \theta_j) e_1(u - \theta_j + 4\eta) \times \text{id}, \]
\[ P_{12}^{v(16)} T_2(u) T_1(u + 2\eta + i\pi) P_{12}^{v(16)} = \prod_{j=1}^{N} \tilde{\rho}_0(u - \theta_j) T_{12}^+(u + \eta + i\pi) T_{21}^-(u + \eta + i\pi) S_{12}^{-1}, \]
\[ P_{12}^{(\pm)} T_2(u) T_1^{\pm}(u + 3\eta + i\pi) P_{12}^{(\pm)} = \prod_{j=1}^{N} \tilde{\rho}_0(u - \theta_j) T_{12}^-\tilde{S}_{12}^+(u + \eta + i\pi) S_{12}^{-1}, \quad (2.28) \]

Taking the partial trace of Eq. \((2.28)\) in the auxiliary spaces and using the relation \((2.27)\), we obtain the operator product identities

\[ t^{(p)}(\theta_j) t^{(p)}(\theta_j + 4\eta) = \prod_{l=1}^{N} a_1(\theta_j - \theta_l) e_1(\theta_j - \theta_l + 4\eta) \times \text{id}, \]
\[ t^{(p)}(\theta_j) t^{(p)}(\theta_j + 2\eta + i\pi) = \prod_{l=1}^{N} \tilde{\rho}_0(\theta_j - \theta_l) t_+^{(p)}(\theta_j + \eta + i\pi) t_-^{(p)}(\theta_j + \eta + i\pi), \]
\[ t^{(p)}(\theta_j) t^{(p)}_\pm(\theta_j + 3\eta + i\pi) = \prod_{l=1}^{N} \tilde{\rho}_0(\theta_j - \theta_l) t^{(p)}_\pm(\theta_j + \eta + i\pi), \quad j = 1, \ldots, N. \quad (2.29) \]

Next, we consider the asymptotic behaviors of fused transfer matrices. According to the definitions, the direct calculation gives

\[ \left. t^{(p)}(u) \right|_{u \to \pm \infty} = e^{\pm(2N u - \sum_{j=1}^{N} \theta_j)} \sum_{\alpha=1}^{6} [T_{\pm}^{\alpha}]_{\alpha} + \cdots, \]
\[ \left. t^{(p)}_\pm(u) \right|_{u \to \pm \infty} = e^{\pm(2N u - \sum_{j=1}^{N} \theta_j)} \sum_{\alpha=1}^{4} [T_{\pm}^{\alpha}]_{\alpha} + \cdots, \quad (2.30) \]

where \( \sum_{\alpha=1}^{6} [T_{\pm}^{\alpha}]_{\alpha} \) and \( \sum_{\alpha=1}^{4} [T_{\pm}^{\alpha}]_{\alpha} \) are the conserved quantities acting on the quantum space \( V \otimes V \otimes \cdots \otimes V \). The related operators are defined as

\[ [T_{\pm}^{\alpha}]_{\beta} = \sum_{\{\delta_i\}=1,\{\gamma_i\}=1}^{6} [R_{01}^{v(\pm)}]_{\alpha \gamma_1}^\beta [R_{02}^{v(\pm)}]_{\delta_1 \gamma_1}^\alpha [R_{03}^{v(\pm)}]_{\delta_2 \gamma_2}^\beta \cdots [R_{0N}^{v(\pm)}]_{\delta_N \gamma_N}^\alpha N - 1 \gamma N, \delta N. \]
Here the repeated indicators should be summarized. $R_{0j}^{\nu v(\pm)}$ and $R_{\theta j}^{\nu v(\pm)}$ are the leading terms of $e^{\pm 2u} R_{0j}^{\nu v(\pm)}(u)$ and $e^{\mp u} R_{\theta j}^{\nu v(\pm)}(u)$ with $u \to \pm \infty$, respectively. From the direct calculation, we find that the eigenvalues of conserved quantities $\sum_{\alpha=1}^{6} [T^{v}]_{\alpha}^{\beta}$ and $\sum_{\alpha=1}^{4} [T^{s\pm}]_{\alpha}^{\beta}$ can be expressed by two quantum numbers $m_1$ and $m_2$ as $2[1 + \cosh(2m_1 \eta) + \cosh(2m_2 \eta)] e^{\pm 2N \eta}$ and $2\{\cosh [(m_1 + m_2) \eta] + \cosh [(m_1 - m_2) \eta]\} e^{\mp 2N \eta}$, respectively, where $m_1 \in [0, N]$ and $0 \leq m_2 \leq N - m_1$. Then the asymptotic behaviors of fused transfer matrices read

\begin{equation}
\begin{split}
t^{(p)}(u)\big|_{u \to \pm \infty} &= 2[1 + \cosh(2m_1 \eta) + \cosh(2m_2 \eta)] e^{\pm (2N \eta - \sum_{j=1}^{N} \theta_j - 4N \eta)} + \ldots, \\
t_{\pm}^{(p)}(u)\big|_{u \to \pm \infty} &= 2\{\cosh [(m_1 + m_2) \eta] + \cosh [(m_1 - m_2) \eta]\} e^{\pm (N \eta - \sum_{j=1}^{N} \theta_j - 2N \eta)} + \ldots. 
\end{split}
\end{equation}

Acting the transfer matrices on the common eigenstate, we obtain the corresponding eigenvalues. Denote the eigenvalues of $t^{(p)}(u)$ and $t_{\pm}^{(p)}(u)$ as $\Lambda^{(p)}(u)$ and $\Lambda_{\pm}^{(p)}(u)$, respectively.

As mentioned previously, the eigenvalues $\Lambda^{(p)}(u)$ and $\Lambda_{\pm}^{(p)}(u)$ are the trigonometric polynomials of $u$ with degrees $2N$ and $N$, respectively. Therefore, we need $4N + 3$ conditions to determine the values of $\Lambda^{(p)}(u)$ and $\Lambda_{\pm}^{(p)}(u)$.

From the operator product identities \([2.29]\), we have the functional relations among the eigenvalues

\begin{equation}
\begin{split}
\Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j + 4 \eta) &= \prod_{l=1}^{N} a_1(\theta_j - \theta_l) e_1(\theta_j - \theta_l + 4 \eta), \\
\Lambda^{(p)}(\theta_j) \Lambda^{(p)}(\theta_j + 2 \eta + i \pi) &= \prod_{l=1}^{N} \tilde{\rho}_0(\theta_j - \theta_l) \Lambda^{(p)}(\theta_j + \eta + i \pi) \Lambda^{(p)}(\theta_j + \eta + i \pi), \\
\Lambda^{(p)}(\theta_j) \Lambda_{\pm}^{(p)}(\theta_j + 3 \eta + i \pi) &= \prod_{l=1}^{N} \tilde{\rho}_0(\theta_j - \theta_l) \Lambda_{\pm}^{(p)}(\theta_j + \eta + i \pi), \quad j = 1, \ldots, N. 
\end{split}
\end{equation}

The corresponding asymptotic behaviors are

\begin{equation}
\begin{split}
\Lambda^{(p)}(u)\big|_{u \to \pm \infty} &= 2[1 + \cosh(2m_1 \eta) + \cosh(2m_2 \eta)] e^{\pm (2N \eta - \sum_{j=1}^{N} \theta_j - 4N \eta)} + \ldots, \\
\Lambda_{\pm}^{(p)}(u)\big|_{u \to \pm \infty} &= 2\{\cosh [(m_1 + m_2) \eta] + \cosh [(m_1 - m_2) \eta]\} \\
&\quad \times e^{\pm (N \eta - \sum_{j=1}^{N} \theta_j - 2N \eta)} + \ldots. 
\end{split}
\end{equation}

Then we arrive at that $4N$ functional relations \([2.33]\) together with 6 asymptotic behaviors \([2.34]\) give us sufficient conditions to determine the eigenvalues of transfer matrices.
2.2 \( T-Q \) relations for eigenvalues

The function relations (2.33) and asymptotic behaviors (2.34) allow us to parameterize the eigenvalues \( \Lambda^{(p)}(u) \) and \( \Lambda^{(p)}_{\pm}(u) \) in terms of the \( T-Q \) relations as

\[
\Lambda^{(p)}(u) = \prod_{j=1}^{N} a_1(u - \theta_j) \frac{Q_p^{(1)}(u + 2\eta)}{Q_p^{(1)}(u)} + \prod_{j=1}^{N} b_1(u - \theta_j) \left\{ \frac{Q_p^{(2)}(u + 2\eta)Q_p^{(3)}(u - 2\eta)}{Q_p^{(2)}(u)Q_p^{(3)}(u)} \right\} \tag{2.35}
\]

\[
\Lambda^{(p)}_{\pm}(u) = \prod_{j=1}^{N} a_2(u - \theta_j) \left\{ \frac{Q_p^{(2)}(u + \eta)}{Q_p^{(2)}(u)} \pm \frac{Q_p^{(3)}(u + \eta)}{Q_p^{(3)}(u)} \right\} + \prod_{j=1}^{N} b_2(u - \theta_j) \left\{ \frac{Q_p^{(1)}(u + 3\eta)}{Q_p^{(1)}(u)} \pm \frac{Q_p^{(1)}(u + \eta)}{Q_p^{(1)}(u)} \right\}, \tag{2.36}
\]

\[
\Lambda^{(p)}_{\mp}(u) = \prod_{j=1}^{N} a_2(u - \theta_j) \left\{ \frac{Q_p^{(3)}(u + 3\eta)}{Q_p^{(3)}(u)} \pm \frac{Q_p^{(3)}(u + \eta)}{Q_p^{(3)}(u)} \right\} + \prod_{j=1}^{N} b_2(u - \theta_j) \left\{ \frac{Q_p^{(1)}(u + 3\eta)}{Q_p^{(1)}(u)} \pm \frac{Q_p^{(1)}(u + \eta)}{Q_p^{(1)}(u)} \right\}, \tag{2.37}
\]

where

\[
Q_p^{(1)}(u) = \prod_{k=1}^{L_1} \sinh(u - \mu_k^{(1)} - \eta), \quad Q_p^{(2)}(u) = Q_p^{(3)}(u - i\pi) = \prod_{l=1}^{L_2} \sinh \frac{1}{2}(u - \mu_l^{(2)} - 2\eta),
\]

\[
b_1(u) = 4 \sinh u \sinh(u - 4\eta), \quad b_2(u) = 2 \sinh(u - \eta), \quad a_2(u) = 2 \sinh(u - 3\eta), \quad \eta \in (0, \infty) \tag{2.38}
\]

\( L_1 \) is the number of Bethe roots \( \{\mu_k^{(1)}\} \) and \( L_2 \) is the number of Bethe roots \( \{\mu_l^{(2)}\} \).

Because the eigenvalues \( \Lambda^{(p)}(u) \) and \( \Lambda^{(p)}_{\pm}(u) \) are the polynomials. The regularity analyses of the \( T-Q \) relations (2.33)-(2.37) lead to that the Bethe roots \( \{\mu_k^{(1)}\} \) and \( \{\mu_l^{(2)}\} \) should satisfy the Bethe ansatz equations (BAEs)

\[
\frac{Q_p^{(1)}(\mu_k^{(1)} + 3\eta)Q_p^{(2)}(\mu_k^{(1)} + \eta)Q_p^{(3)}(\mu_k^{(1)} + \eta)}{Q_p^{(1)}(\mu_k^{(1)} - \eta)Q_p^{(2)}(\mu_k^{(1)} + 3\eta)Q_p^{(3)}(\mu_k^{(1)} + 3\eta)} = -\prod_{j=1}^{N} \sinh(\mu_k^{(1)} + \eta - \theta_j),
\]

\( k = 1, \cdots, L_1, \tag{2.39} \)

\[
\frac{Q_p^{(1)}(\mu_l^{(2)} + 4\eta)Q_p^{(2)}(\mu_l^{(2)} + 2\eta)}{Q_p^{(1)}(\mu_l^{(2)} + 2\eta)Q_p^{(2)}(\mu_l^{(2)})} = -1, \quad l = 1, \cdots, L_2, \tag{2.40}
\]
where \( L_1 \leq N \) and \( L_2 \leq L_1 \).

Some remarks are in order. We note that the BAEs (2.39) and (2.40) are homogeneous. This is because that the periodic boundary condition does not break the \( U(1) \) symmetry of the system. The BAEs obtained from the regularity of \( \Lambda^{(p)}(u) \) are the same as those obtained from the regularities of \( \Lambda^{(p)}(u) \). From the asymptotic behaviors of \( \Lambda^{(p)}(u) \) and \( \Lambda^{(p)}(u) \), we know that the quantum numbers \( m_1 \) and \( m_2 \) characterizing the conserved quantities

\[
\sum_{\alpha=1}^{6}[T^{(v)}]_{\alpha}^{\alpha} \quad \text{and} \quad \sum_{\alpha=1}^{4}[T^{(s)}]_{\alpha}^{\alpha}
\]

are related with the numbers of Bethe roots as

\[
m_1 = N - L_1, \quad m_2 = L_1 - L_2.
\]

The existence of two good quantum numbers consists with the fact that there are two sets of homogeneous BAEs. It is easy to check that \( \Lambda^{(p)}(u) \) and \( \Lambda^{(p)}(u) \) satisfy the functional relations (2.33) and the asymptotic behaviors (2.34). Therefore, we conclude that \( \Lambda^{(p)}(u) \) and \( \Lambda^{(p)}(u) \) are the eigenvalues of the transfer matrices \( t^{(p)}(u) \) and \( t^{(p)}(u) \), respectively. We should note that the \( T - Q \) relations (2.35)-(2.37) and associated BAEs (2.39)-(2.40) have the well-defined homogeneous limit. These results with the constraint \( \{\theta_j\} = 0 \) are coincide with the previous results [29, 38].

The eigenvalues of the Hamiltonian (2.19) can be obtained by the \( \Lambda^{(p)}(u) \) as

\[
E_p = \frac{\partial \ln \Lambda^{(p)}(u)}{\partial u} \Big|_{u=0,\{\theta_j\}=0}.
\]

### 3 \( D_3^{(2)} \) model with non-diagonal boundary condition

In this section, we study the system with general integrable open boundary condition. The boundary reflection at one side is quantified by the reflection matrix \( K^v(u) \) which satisfies the reflection equation

\[
R_{12}^{uv}(u - v)K_1^v(u)R_{21}^{uv}(u + v)K_2^v(v) = K_2^v(v)R_{12}^{uv}(u + v)K_1^v(u)R_{21}^{uv}(u - v).
\]

The boundary reflection at the other side is described by the dual reflection matrix \( \tilde{K}^v(u) \), which satisfies the dual reflection equation

\[
R_{12}^{uv}(-u + v)\tilde{K}_1^v(u)M_1^{-1}R_{21}^{uv}(-u - v + 8\eta)M_1\tilde{K}_2^v(v) = \tilde{K}_2^v(v)M_1R_{12}^{uv}(-u - v + 8\eta)M_1^{-1}\tilde{K}_1^v(u)R_{21}^{uv}(-u + v).
\]
In the open boundary condition, besides the monodromy matrix $T^v_0(u)$ given by (2.16), we should also consider the reflecting monodromy matrix

$$
\hat{T}^v_0(u) = R^v_{N0}(u + \theta_N) \cdots R^v_{20}(u + \theta_2)R^v_{10}(u + \theta_1),
$$

which satisfies the Yang-Baxter relation

$$
R^v_{21}(u - v)\hat{T}^v_1(u)\hat{T}^v_2(v) = \hat{T}^v_2(v)\hat{T}^v_1(u)R^v_{21}(u - v).
$$

The transfer matrix $t(u)$ of the model with boundary reflections is defined as

$$
t(u) = tr_0\{\bar{K}^v_0(u)T^v_0(u)K^v_0(u)\hat{T}^v_0(u)\}.\tag{3.5}
$$

From the Yang-Baxter relations (2.17), (3.4), reflection equation (3.1) and dual one (3.2), one can prove that the transfer matrices with different spectral parameters commute with each other, $[t(u), t(v)] = 0$. Therefore, $t(u)$ serves as the generating functional of all the conserved quantities of the system. The Hamiltonian is constructed by taking the derivative of the logarithm of the transfer matrix

$$
H = \frac{\partial \ln(t(u))}{\partial u}_{u=0,\{\theta_j\}=0}
= \sum_{k=1}^{N-1} H_{kk+1} + \frac{K^v_1(0)'}{K^v_1(0)} + \frac{tr_0\{\bar{K}^v_0(0)H_{N0}\}}{tr_0K^v_0(0)} + \text{constant},\tag{3.6}
$$

where $H_{kk+1}$ is given by (2.20).

### 3.1 Reflection matrix

In this paper, we consider the integrable open boundary condition where the reflection matrices have the non-diagonal elements, which break the $U(1)$-symmetry of the system. The non-diagonal reflection matrix for $D^{(2)}_{n+1}$ vertex model has been constructed by Malara et al [41] and Nepomechie et al [42]. Without losing generality, we here chose a reflection matrix $K^v(u)$ with off-diagonal elements to demonstrate how our method works, namely,

$$
K^v(u) = \begin{pmatrix}
k_1(u) & 0 & 0 & 0 & k_4(u) & 0 \\
0 & k_1(u) & 0 & 0 & 0 & -k_4(u) \\
0 & 0 & k_2(u) & 0 & 0 & 0 \\
0 & 0 & 0 & k_2(u) & 0 & 0 \\
k_5(u) & 0 & 0 & 0 & k_3(u) & 0 \\
0 & -k_5(u) & 0 & 0 & 0 & k_3(u)
\end{pmatrix},\tag{3.7}
$$
where the non-zero matrix elements are

\[ k_1(u) = e^{-u} \cosh \eta, \quad k_2(u) = \cosh(u + \eta) \cosh \eta, \quad k_3(u) = e^u \cosh \eta, \]
\[ k_4(u) = -c \sinh ucosh 2\eta, \quad k_5(u) = \frac{\sinh u}{c \cosh 2\eta}, \]

and \( c \) is a free boundary parameter. The dual reflection matrix \( \tilde{K}^v(u) \) is obtained by the mapping

\[ \tilde{K}^v(u) = MK^v(-u + 4\eta)|_{c \to c'}, \]

where \( c' \) is the boundary parameters at the other side. For a generic choice of the boundary parameters \( \{ c, c' \} \), it is easily to check that \( [K^v(u), \tilde{K}^v(v)] \neq 0 \), which implies that the matrices \( K^v(u) \) and \( \tilde{K}^v(u) \) cannot be diagonalized simultaneously. Thus the \( U(1) \) symmetry of the system is broken while the integrability is still held.

Substituting the expressions of \( R \)-matrix (2.1) and reflection matrices (3.7) and (3.9) into (3.6), we obtain the integrable Hamiltonian of \( D_3^{(2)} \) model with the non-diagonal boundary reflections given by (3.7) and (3.9).

### 3.2 Operators product relations

Similarly as the periodic case in previous section, we need further to introduce some fused transfer matrices (see below (3.13)) besides the fundamental one \( t(u) \). Due to the boundary reflection, besides the fused monodromy matrices \( T_{\tilde{\theta}^-}^0(u) \) given by (2.22), we should define the reflecting fused monodromy matrices \( \hat{T}^{\pm}_{\tilde{\theta}^-}(u) \) in terms of fused \( R \)-matrix \( R^{u\pm}_{\tilde{\theta}^-}(u) \) as

\[ \hat{T}^{\pm}_{\tilde{\theta}^-}(u) = R^{u\pm}_{N\tilde{\theta}^-}(u + \theta_N) \cdots R^{u\pm}_{2\tilde{\theta}^-}(u + \theta_2) R^{u\pm}_{1\tilde{\theta}^-}(u + \theta_1), \]

which satisfy the Yang-Baxter relations

\[ R^{u\pm}_{0\tilde{\theta}^-}(u - v) \hat{T}^{\pm}_{\tilde{\theta}^-}(v) \hat{T}_0(u) = \hat{T}_0(u) \hat{T}^{\pm}_{\tilde{\theta}^-}(v) R^{u\pm}_{0\tilde{\theta}^-}(u - v), \]
\[ R^{s\pm,s^-}_{0\tilde{\theta}^-}(u - v) \hat{T}^{s^-}_{\tilde{\theta}^-}(v) \hat{T}^{s^+}_{\tilde{\theta}^-}(u) = \hat{T}^{s^+}_{\tilde{\theta}^-}(u) \hat{T}^{s^-}_{\tilde{\theta}^-}(v) R^{s\pm,s^-}_{0\tilde{\theta}^-}(u - v). \]

The fused transfer matrices are constructed as

\[ t_{\pm}(u) = tr_{\tilde{\theta}^-}\{ \hat{K}^{\pm}_{\tilde{\theta}^-}(u) T_{\tilde{\theta}^-}^{\pm}(u) \hat{K}^{\pm}_{\tilde{\theta}^-}(u) \hat{T}^{\pm}_{\tilde{\theta}^-}(u) \}, \]

where the fused K-matrices \( K^{\pm}_{\tilde{\theta}^-}(u) \) and \( \hat{K}^{\pm}_{\tilde{\theta}^-}(u) \) are given by (B.7) and (B.8). From the Yang-Baxter relations (2.24), (3.11)-(3.12) and reflection equations (B.11)-(B.12), one can prove that the transfer matrices \( t(u) \) and \( t_{\pm}(u) \) commute with each other,

\[ [t(u), t_{\pm}(v)] = [t_{\pm}(u), t_{\pm}(v)] = [t_{\pm}(u), t(v)] = 0. \]
Thus $t(u)$ and $t_{\pm}(u)$ have the common eigenstates.

Considering the Yang-Baxter relations (3.4) at the points of $u = -\theta_j$, $v = \{-\theta_j + 4\eta, -\theta_j + 2\eta + i\pi\}$, (3.11) at the points of $u = -\theta_j$, $v = -\theta_j + 3\eta + i\pi$ and using the properties of projector, we obtain

\[
\begin{align*}
\hat{T}_1(-\theta_j) \hat{T}_2(-\theta_j + 4\eta) &= P_{12}^{\text{ev}(1)} \hat{T}_1(-\theta_j) \hat{T}_2(-\theta_j + 4\eta), \\
\hat{T}_2(-\theta_j) \hat{T}_1(-\theta_j + 2\eta + i\pi) &= P_{21}^{\text{ev}(16)} \hat{T}_2(-\theta_j) \hat{T}_1(-\theta_j + 2\eta + i\pi), \\
\hat{T}_2(-\theta_j) \hat{T}_1^\pm(-\theta_j + 3\eta + i\pi) &= P_{21}^{(\pm)} \hat{T}_2(-\theta_j) \hat{T}_1^\pm(-\theta_j + 3\eta + i\pi),
\end{align*}
\]  
(3.15)

where $j = 1, \ldots, N$, and the projectors: $P_{12}^{\text{ev}(1)}$, $P_{12}^{\text{ev}(16)}$, $P_{21}^{(\pm)}$ are given by (A.19), (A.25), (A.7) and (A.13). Taking the fusion of reflecting monodromy matrices $\hat{T}(u)$, $\hat{T}^\pm(u)$ with the projectors $P_{12}^{\text{ev}(1)}$, $P_{12}^{\text{ev}(16)}$, $P_{21}^{(\pm)}$ and using Eqs. (A.23), (A.29), (A.11), (A.16), we obtain the fusion identities

\[
P_{12}^{\text{ev}(1)} \hat{T}_1(u) \hat{T}_2(u + 4\eta) P_{12}^{\text{ev}(1)} = P_{12}^{\text{ev}(1)} \prod_{j=1}^{N} a(u + \theta_j) e(u + \theta_j + 4\eta) \times \text{id},
\]

\[
P_{21}^{\text{ev}(16)} \hat{T}_2(u) \hat{T}_1(u + 2\eta + i\pi) P_{21}^{\text{ev}(16)} = \prod_{j=1}^{N} \rho_0(u + \theta_j) S_{12}^\tau \hat{T}_v^+(u + \eta + i\pi) \hat{T}_v^-(u + \eta + i\pi) S_{12}^{-1},
\]

\[
P_{21}^{(\pm)} \hat{T}_2(u) \hat{T}_1^\pm(u + 3\eta + i\pi) P_{21}^{(\pm)} = \prod_{j=1}^{N} \rho_0(u + \theta_j) \hat{T}_{12}^\pm(u + \eta + i\pi),
\]

\[
P_{21}^{(\pm)} \hat{T}_2(u) \hat{T}_1^\pm(u + 3\eta + i\pi) P_{21}^{(\pm)} = \prod_{j=1}^{N} \rho_0(u + \theta_j) \hat{T}_{12}^\pm(u + \eta + i\pi) S_{12}^{-1}. \tag{3.16}
\]

Next, we should combine the fusion relations (2.27)-(2.28) of monodromy matrices and the relations (3.13)-(3.16) of reflecting monodromy matrices, which can be achieved by the fusion of reflection matrices and that of dual ones. All the necessary fusion identities of reflection matrices have be deduced in appendices A and B as Eqs. (B.4)-(B.14), (B.15)-(B.16).

According to the definitions of fused transfer matrices and through the direct calculation, we have

\[
t(u)t(u + \Delta) = [\rho_1(2u + \Delta - 4\eta)]^{-1} tr_{12} \{ \hat{K}_2^v(u + \Delta) M_2^{-1} R_{12}^{\text{ev}}(-2u + 8\eta - \Delta) \times M_2 \hat{K}_2^v(u) T_1(u) T_2(u + \Delta) K_2^v(u) R_{21}^{\text{ev}}(2u + \Delta) K_2^v(u + \Delta) \hat{T}_1(u) \hat{T}_2(u + \Delta) \}, \tag{3.17}
\]

\[
t(u)t_{\pm}(u + \Delta) = [\rho_s(2u + \Delta - 4\eta - i\pi)]^{-1} tr_{12} \{ \hat{K}_2^{s\pm}(u + \Delta) M_2^{-1}
\]

14
\[ \times R_{12}^{s, \pm}(2u + \Delta)\tilde{M}_2 \tilde{K}_1^{i}(u)T_1(u)T_2^{\pm}(u + \Delta)K_1^{i}(u) \]
\[ \times R_{21}^{s, \pm}(-2u + 8\eta - 2i\pi - \Delta)\tilde{M}_2 \tilde{K}_1^{i}(u)T_1(u)T_2^{\pm}(u + \Delta) \],
\[ t_{\pm}(u_\eta, u + \Delta) = \left[ \rho_{ss}(2u - 4\eta - i\pi + \Delta) \right]^{-1} tr_{12} \left\{ K_2^{s, -}(u + \Delta)M_2^{-1} \right\} \]
\[ \times R_{12}^{s, \pm}(-2u + 8\eta + 2i\pi - \Delta)\tilde{M}_2 \tilde{K}_1^{i}(u)T_1^{\pm}(u)T_2^{\pm}(u + \Delta)K_1^{i}(u) \]
\[ \times R_{21}^{s, \pm}(2u + \Delta)\tilde{M}_2 \tilde{K}_1^{i}(u)T_1^{\pm}(u)T_2^{\pm}(u + \Delta) \}, \] (3.18)

where \( \Delta \) is the shift of spectral parameter. We find that if \( \Delta \) is chosen as some suitable values such as \( 4\eta, 2\eta + i\pi, 3\eta + i\pi \) and \( 0 \) in Eqs. (3.17)-(3.19), we can connect the fusions of monodromy matrices and that of the reflecting ones, and obtain the correct fusion relations among the fused transfer matrices \( t(u) \) and \( t_{\pm}(u) \).

Substituting Eqs. (2.27)-(2.28), (B.1)-(B.4), (B.15)-(B.16), (3.15)-(3.16) into Eq. (3.17) and considering \( \{ u = \pm \theta_j, \Delta = 4\eta \} \), into Eq. (3.17) with \( \{ u = \pm \theta_j, \Delta = 2\eta + i\pi \} \) and using the result of Eq. (3.19) with the shift \( \Delta = 0 \), into Eq. (3.18) with \( \{ u = \pm \theta_j, \Delta = 3\eta + i\pi \} \), we arrive at

\[ t(\pm \theta_j)t(\pm \theta_j + 4\eta) = \cosh^2(\pm \theta_j) \frac{\sinh(\pm \theta_j - 4\eta) \sinh(\pm \theta_j + 4\eta)}{\sinh(\pm \theta_j - \eta) \sinh(\pm \theta_j + \eta)} \]
\[ \times \frac{\sinh(2\theta_j - 6\eta) \sinh(2\theta_j + 6\eta)}{\sinh(2\theta_j - 4\eta) \sinh(2\theta_j + 4\eta)} \cosh(\pm \theta_j - \eta) \cosh(\pm \theta_j + \eta) \]
\[ \times \prod_{l=1}^{N} a_1(\pm \theta_j - \theta_l)e_1(\pm \theta_j - \theta_l + 4\eta)a_1(\pm \theta_j + \theta_l)e_1(\pm \theta_j + \theta_l + 4\eta) \times \text{id}, \]
\[ t(\pm \theta_j)t(\pm \theta_j + 2\eta + i\pi) = \frac{\sinh(\pm \theta_j + 2\eta) \sinh(\pm \theta_j - 4\eta) \cosh(\pm \theta_j + \eta) \cosh(\pm \theta_j - 3\eta)}{\sinh(\pm \theta_j + \eta) \sinh(\pm \theta_j - 3\eta) \cosh(\pm \theta_j) \cosh(\pm \theta_j - 2\eta)} \]
\[ \times \cosh^4 2\eta \prod_{l=1}^{N} \tilde{\rho}_0(\pm \theta_j - \theta_l)\tilde{\rho}_0(\pm \theta_j + \theta_l)t_{\mp}(\pm \theta_j + \eta + i\pi) t_{\pm}(\pm \theta_j + \eta + i\pi), \]
\[ t(\pm \theta_j)t_{\pm}(\pm \theta_j + 3\eta + i\pi) = 4 \frac{\cosh(\pm \theta_j) \cosh(\pm \theta_j + \eta) \sinh(\pm \theta_j + 3\eta) \sinh(\pm \theta_j - 4\eta)}{\sinh(\pm 2\theta_j + 2\eta) \sinh(\pm 2\theta_j - 4\eta)} \]
\[ \times \cosh(\pm \theta_j + \eta) \cosh(\pm \theta_j - 3\eta) \prod_{l=1}^{N} \tilde{\rho}_0(\pm \theta_j - \theta_l)\tilde{\rho}_0(\pm \theta_j + \theta_l)t_{\mp}(\pm \theta_j + \eta + i\pi), \] (3.20)

where \( j = 1, \cdots, N \).

The values of transfer matrices \( t(u) \) and \( t_{\pm}(u) \) at the point of \( u = 0 \) can be calculated directly

\[ t(0) = \frac{\sinh 6\eta}{\sinh \eta} \prod_{j=1}^{N} \rho_1(-\theta_j) \times \text{id}, \]
\[ t_{\pm}(0) = -4 \cosh^2 \eta \prod_{j=1}^{N} \rho_\pm(-\theta_j) \times \text{id}. \] (3.21)
In the derivation, we have used the relations

$$tr[\tilde{K}^v(0)]K^v(0) = \frac{\sinh \delta \eta}{\sinh \eta} \times \text{id}, \quad tr[\tilde{K}^{s\pm}(0)]K^{s\pm}(0) = -4 \cosh^2 \eta \times \text{id}.$$  

The asymptotic behaviors of $t(u)$ and $t_{\pm}(u)$ read

$$t(u)|_{u \to \pm \infty} = Q^v_{\pm} e^{\pm(4N+2)u} + \cdots, \quad t_{\pm}(u)|_{u \to \pm \infty} = Q^{s\pm}_{\pm} e^{\pm(2N+2)u} + \cdots,$$  

where $Q^v_{\pm}$ and $Q^{s\pm}_{\pm}$ are the conserved quantities with the definitions

$$Q^v_{\pm} = \frac{1}{4} \left\{ \frac{c'}{c} \left( e^{2\eta} [T^v_{\pm}]^5_1 [\hat{T}^v_{\pm}]^1_1 - e^{-2\eta} [T^v_{\pm}]^0_1 [\hat{T}^v_{\pm}]^2_1 + e^{-4\eta} [T^v_{\pm}]^0_0 [\hat{T}^v_{\pm}]^2_2 \right) + e^{-4\eta} \left( [T^v_{\pm}]^3_3 [\hat{T}^v_{\pm}]^3_3 + [T^v_{\pm}]^4_4 [\hat{T}^v_{\pm}]^4_4 + [T^v_{\pm}]^4_4 [\hat{T}^v_{\pm}]^4_4 \right) + \frac{c}{c'} \left( e^{-6\eta} [T^v_{\pm}]^1_1 [\hat{T}^v_{\pm}]^5_5 - e^{-10\eta} [T^v_{\pm}]^2_2 [\hat{T}^v_{\pm}]^2_2 + e^{-10\eta} [T^v_{\pm}]^2_2 [\hat{T}^v_{\pm}]^2_2 \right) \right\};$$

$$Q^{s\pm}_{\pm} = -\frac{1}{4c' e^{4\eta} \cosh^2 2\eta} \left\{ \left( \frac{c}{c'} e^{4\eta} [T_{\pm}]^4_1 [\hat{T}^{s\pm}_{\pm}]^1_1 + \frac{c}{c'} e^{-4\eta} [T_{\pm}]^4_1 [\hat{T}^{s\pm}_{\pm}]^4_4 \right) + e^{2\eta} [T_{\pm}]^2_2 [\hat{T}^{s\pm}_{\pm}]^2_2 + e^{2\eta} [T_{\pm}]^2_2 [\hat{T}^{s\pm}_{\pm}]^2_2 + e^{-2\eta} [T_{\pm}]^2_2 [\hat{T}^{s\pm}_{\pm}]^2_2 \right\}. \quad \quad (3.23)$$

Here $[T^v_{\pm}]^\alpha_\beta$ and $[T^{s\pm}_{\pm}]^\alpha_\beta$ are given by (2.31), $[\hat{T}^v_{\pm}]^\alpha_\beta$ and $[\hat{T}^{s\pm}_{\pm}]^\alpha_\beta$ are the operators acting on the quantum space $V \otimes V \otimes \cdots \otimes V$ with the explicit expressions

$$[\hat{T}^v_{\pm}]^\alpha_\beta = \sum_{\{\delta\}_{1} = 1, \{\gamma\}_{1}}^6 \left[ R_{10}^{v\alpha} \right]_{\delta_1 \alpha_1} \left[ R_{20}^{v\beta} \right]_{\delta_2 \alpha_2} \cdots \left[ R_{N0}^{v\gamma} \right]_{\delta_N \beta},$$

$$[\hat{T}^{s\pm}_{\pm}]^\alpha_\beta = \sum_{\{\delta\}_{1} = 1, \{\gamma\}_{1}}^6 \left[ R_{10}^{s\pm \alpha} \right]_{\delta_1 \alpha_1} \left[ R_{20}^{s\pm \beta} \right]_{\delta_2 \alpha_2} \cdots \left[ R_{N0}^{s\pm \gamma} \right]_{\delta_N \beta}, \quad \quad (3.24)$$

$R_{j0}^{v\alpha}$ and $R_{j0}^{s\pm \alpha}$ are the leading terms of $e^{\mp 2u} R_{j0}^{v\alpha}(u)$ and $e^{\pm u} R_{j0}^{s\pm \alpha}(u)$ with $u \to \pm \infty$, respectively, and the repeated indicators should be summarized. The detailed calculation shows that the eigenvalues of conserved quantities $Q^v_{\pm}$ and $Q^{s\pm}_{\pm}$ can be characterized by the quantum number $m$ as

$$\Lambda_{Q^v_{\pm}} = \frac{1}{2} \left[ \left( \frac{c}{c'} e^{-4\eta} + \frac{c'}{c} e^{4\eta} \right) \cosh(2m\eta) + 1 \right] e^{\pm(-8N\eta - 4\eta)},$$

$$\Lambda_{Q^{s\pm}_{\pm}} = -\frac{1}{4 \cosh^2 2\eta} \left[ \left( \frac{c}{c'} e^{-4\eta} + \frac{c'}{c} e^{4\eta} + 2 \cosh(2m\eta) \right) e^{-4\eta} \right]. \quad \quad (3.25)$$
where $m \in [1, N+1]$. Then we obtain the asymptotic behaviors of $t(u)$ and $t\pm(u)$ as

$$
t(u)\big|_{u \to \pm \infty} = \frac{1}{2} \left[ \left( \frac{c}{c'} e^{-4\eta} + \frac{c'}{c} e^{4\eta} \right) \cosh(2m\eta) + 1 \right] e^{\pm(4Nu+2u-8N\eta-4\eta)} + \ldots,$$

$$
t\pm(u)\big|_{u \to \pm \infty} = -\frac{c}{c'} e^{-4\eta} + \frac{c'}{c} e^{4\eta} + 2 \cosh(2m\eta) e^{\pm(2Nu+2u-4N\eta-4\eta)} + \ldots. \quad (3.26)
$$

Acting the fused transfer matrices $t(u)$ and $t\pm(u)$ on a common eigenstate, we obtain the eigenvalues. Denote the eigenvalues of $t(u)$ and $t\pm(u)$ as $\Lambda(u)$ and $\Lambda_\pm(u)$, respectively. From the operators product identities (3.20), we obtain the functional relations among the eigenvalues $\Lambda(u)$ and $\Lambda_\pm(u)$ as

$$
\Lambda(\pm \theta_j)\Lambda(\pm \theta_j + 4\eta) = \cosh^2(\pm \theta_j) \frac{\sinh(\pm \theta_j - 4\eta) \sinh(\pm \theta_j + 4\eta)}{\sinh(\pm \theta_j - \eta) \sinh(\pm \theta_j + \eta)} \times \frac{\sinh(\pm 2\theta_j - 6\eta) \sinh(\pm 2\theta_j + 6\eta)}{\sinh(\pm 2\theta_j - 4\eta) \sinh(\pm 2\theta_j + 4\eta)}
\times \prod_{l=1}^N a_1(\pm \theta_j - \theta_l) e_1(\pm \theta_j - \theta_l + 4\eta) a_1(\pm \theta_j + \theta_l) e_1(\pm \theta_j + \theta_l + 4\eta),
$$

$$
\Lambda(\pm \theta_j)\Lambda(\pm \theta_j + 2\eta + i\pi) = \frac{\sinh(\pm \theta_j + 2\eta) \sinh(\pm \theta_j - 4\eta) \cosh(\pm \theta_j + \eta) \cosh(\pm \theta_j - 3\eta)}{\sinh(\pm \theta_j + \eta) \sinh(\pm \theta_j - 3\eta) \cosh(\pm \theta_j) \cosh(\pm \theta_j - 2\eta)}
\times \cosh^4 2\eta \prod_{l=1}^N \tilde{\rho}_0(\pm \theta_j - \theta_l) \tilde{\rho}_0(\pm \theta_j + \theta_l) \Lambda_+(\pm \theta_j + \eta + i\pi) \Lambda_-(\pm \theta_j + \eta + i\pi),
$$

$$
\Lambda(\pm \theta_j)\Lambda_\pm(\pm \theta_j + 3\eta + i\pi) = \frac{4 \cosh(\pm \theta_j) \cosh(\pm \theta_j + \eta) \sinh(\pm \theta_j + 3\eta) \sinh(\pm \theta_j - 4\eta)}{\sinh(\pm 2\theta_j + 2\eta) \sinh(\pm 2\theta_j - 4\eta)}
\times \cosh(\pm \theta_j + \eta) \cosh(\pm \theta_j - 3\eta) \prod_{l=1}^N \tilde{\rho}_0(\pm \theta_j - \theta_l) \tilde{\rho}_0(\pm \theta_j + \theta_l) \Lambda_\pm(\pm \theta_j + \eta + i\pi), \quad (3.27)
$$

where $j = 1, \ldots, N$. According to Eq. (3.21), the values of $\Lambda(u)$ and $\Lambda_\pm(u)$ at the point of $u = 0$ are

$$
\Lambda(0) = \frac{\sinh 6\eta}{\sinh \eta} \prod_{j=1}^N \rho_1(-\theta_j), \quad \Lambda_+(0) = \Lambda_-(0) = -4 \cosh^2 \eta \prod_{j=1}^N \rho_1(-\theta_j). \quad (3.28)
$$

Eq. (3.22) gives the asymptotic behaviors of $\Lambda(u)$ and $\Lambda_\pm(u)$

$$
\Lambda(u)\big|_{u \to \pm \infty} = \frac{1}{2} \left[ \left( \frac{c}{c'} e^{-4\eta} + \frac{c'}{c} e^{4\eta} \right) \cosh(2m\eta) + 1 \right] e^{\pm(4Nu+2u-8N\eta-4\eta)} + \ldots,
$$

$$
\Lambda_\pm(u)\big|_{u \to \pm \infty} = -\frac{c}{c'} e^{-4\eta} + \frac{c'}{c} e^{4\eta} + 2 \cosh(2m\eta) e^{\pm(2Nu+2u-4N\eta-4\eta)} + \ldots. \quad (3.29)
$$

From the definition, we know that the eigenvalues $\Lambda(u)$ and $\Lambda_\pm(u)$ are the polynomials of $e^u$ with degrees $4N+2$ and $2N+2$, respectively. Therefore, the $8N+9$ constraints (3.27)-(3.29) can completely determine the values of $\Lambda(u)$ and $\Lambda_\pm(u)$. 

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3.3 Inhomogeneous T-Q relations

The function relations (3.27), the values (3.28) of eigenvalues at \( u = 0 \) and the asymptotic behaviors (3.29) allow us to determine the eigenvalues of the corresponding transfer matrices, which are given in terms of some inhomogeneous \( T - Q \) relations. The explicit expressions of the eigenvalues \( \Lambda(u) \) and \( \Lambda_+(u) \) read

\[
\Lambda(u) = 2 \frac{\cosh u \sinh(u - 4\eta) \sinh(u - 3\eta)}{\sinh(u - \eta) \sinh(2u - 4\eta)} \prod_{j=1}^{N} a_1(u - \theta_j) a_1(u + \theta_j)
\]
\[
\times \cosh(u + \eta) \cosh(u - 3\eta) \frac{Q^{(1)}(u + 2\eta)}{Q^{(1)}(u)}
\]
\[
+ 2 \frac{\sinh u \cosh(u - 4\eta) \sinh(u - \eta)}{\sinh(u - 3\eta) \sinh(2u - 4\eta)} \prod_{j=1}^{N} e_1(u - \theta_j) e_1(u + \theta_j)
\]
\[
\times \cosh(u - \eta) \cosh(u - 5\eta) \frac{Q^{(1)}(u - 4\eta)}{Q^{(1)}(u - 2\eta)}
\]
\[
+ \frac{\sinh u \sinh(u - 4\eta)}{\sinh(u - \eta) \sinh(u - 3\eta)} \prod_{j=1}^{N} b_1(u - \theta_j) b_1(u + \theta_j) \frac{1}{Q^{(1)}(u) Q^{(1)}(u - 2\eta)}
\]
\[
\times \left[ \frac{\sinh(2u - 6\eta)}{\sinh(2u - 4\eta)} \frac{Q^{(1)}(u - 2\eta) Q^{(2)}(u + 2\eta)}{Q^{(2)}(u)} \cosh u \right.
\]
\[
+ \frac{\sinh(2u - 2\eta)}{\sinh(2u - 4\eta)} \frac{Q^{(1)}(u) Q^{(2)}(u - 2\eta)}{Q^{(2)}(u)} \cosh(u - 4\eta) \right]
\]
\[
\times \left[ \frac{\sinh(2u - 6\eta)}{\sinh(2u - 4\eta)} \frac{Q^{(1)}(u - 2\eta) Q^{(3)}(u + 2\eta)}{Q^{(3)}(u)} \cosh u \right.
\]
\[
+ \frac{\sinh(2u - 2\eta)}{\sinh(2u - 4\eta)} \frac{Q^{(1)}(u) Q^{(3)}(u - 2\eta)}{Q^{(3)}(u)} \cosh(u - 4\eta) \right]
\]
\[
+ 4^N h \prod_{j=1}^{N} a_1(u - \theta_j) a_1(u + \theta_j) \sinh(u - \theta_j) \sinh(u + \theta_j)
\]
\[
\times \frac{\sinh u \sinh(u - 4\eta)}{\sinh(2u - 4\eta)} \left[ \cosh^2 u \sinh(2u - 6\eta) \frac{Q^{(2)}(u + 2\eta) Q^{(3)}(u + 2\eta)}{Q^{(1)}(u)} \right.
\]
\[
+ \cosh^2(u - 4\eta) \sinh(2u - 2\eta) \frac{Q^{(2)}(u - 2\eta) Q^{(3)}(u - 2\eta)}{Q^{(1)}(u - 2\eta)} \right],
\]

\[
\Lambda_+(u) = - \frac{1}{\cosh^2 2\eta} \left\{ \prod_{j=1}^{N} a_2(u - \theta_j) a_2(u + \theta_j) \sinh(u - 4\eta) \sinh(u - \eta) \right.
\]
\[
\times \left[ \cosh(u + \eta) \cosh(u - 2\eta) \frac{Q^{(2)}(u + 3\eta)}{Q^{(2)}(u + \eta)} \right]
\]

(3.30)
where $h$ is an undetermined parameter, the $Q$-functions are defined by

$$Q^{(1)}(u) = \prod_{k=1}^{L_1} \sinh(u - \mu_k^{(1)} - \eta) \sinh(u + \mu_k^{(1)} - \eta),$$

$$Q^{(2)}(u) = Q^{(3)}(u - i\pi) = \prod_{l=1}^{L_2} \sinh \left( \frac{1}{2} (u - \mu_l^{(2)} - 2\eta) \right) \sinh \left( \frac{1}{2} (u + \mu_l^{(2)} - 2\eta) \right), \quad (3.33)$$

$L_1$ is the number of Bethe roots $\{\mu_k^{(1)}\}$ and $L_2$ is the number of Bethe roots $\{\mu_k^{(2)}\}$.

The regularities of eigenvalues $\Lambda(u)$ and $\Lambda_\pm(u)$ require that the Bethe roots $\{\mu_k^{(1)}\}$ and $\{\mu_k^{(2)}\}$ satisfy the following equations:

\[
\begin{align*}
\sinh u & \cosh u \cosh(u - 3\eta) \frac{Q^{(1)}(u + \eta)Q^{(2)}(u - \eta)}{Q^{(1)}(u - \eta)Q^{(2)}(u + \eta)} \\
+ \prod_{j=1}^{N} b_2(u - \theta_j) b_2(u + \theta_j) \frac{\sinh u}{\sinh(u - 3\eta)} \times \\
& \left[ \cosh(u - 5\eta) \cosh(u - 2\eta) \frac{Q^{(3)}(u - 3\eta)}{Q^{(3)}(u - \eta)} \\
+ \frac{\sinh(u - 4\eta)}{\sinh(u - 2\eta)} \cosh(u - \eta) \cosh(u - 4\eta) \frac{Q^{(1)}(u - 3\eta)Q^{(3)}(u + \eta)}{Q^{(1)}(u - \eta)Q^{(3)}(u + \eta)} \right] \\
& + h \sinh u \sinh(u - 4\eta) \cosh(u - \eta) \cosh(u - 3\eta) \times \\
& \prod_{j=1}^{N} a_2(u - \theta_j) a_2(u + \theta_j) b_2(u - \theta_j) b_2(u + \theta_j) \frac{Q^{(2)}(u - \eta)Q^{(3)}(u + \eta)}{Q^{(1)}(u - \eta)}, \quad (3.31)
\end{align*}
\]

\[
\begin{align*}
\Lambda_-(u) &= -\frac{1}{\cosh^2 2\eta} \left\{ \prod_{j=1}^{N} a_2(u - \theta_j) a_2(u + \theta_j) \frac{\sinh(u - 4\eta)}{\sinh(u - \eta)} \times \\
& \left[ \cosh(u + \eta) \cosh(u - 2\eta) \frac{Q^{(3)}(u + 3\eta)}{Q^{(3)}(u + \eta)} \\
+ \frac{\sinh u}{\sinh(u - 2\eta)} \cosh(u - 3\eta) \frac{Q^{(1)}(u + \eta)Q^{(3)}(u - \eta)}{Q^{(1)}(u - \eta)Q^{(3)}(u + \eta)} \right] \\
& + \prod_{j=1}^{N} b_2(u - \theta_j) b_2(u + \theta_j) \frac{\sinh u}{\sinh(u - 3\eta)} \times \\
& \left[ \cosh(u - 5\eta) \cosh(u - 2\eta) \frac{Q^{(2)}(u - 3\eta)}{Q^{(2)}(u - \eta)} \\
+ \frac{\sinh(u - 4\eta)}{\sinh(u - 2\eta)} \cosh(u - \eta) \cosh(u - 4\eta) \frac{Q^{(1)}(u - 3\eta)Q^{(2)}(u + \eta)}{Q^{(1)}(u - \eta)Q^{(2)}(u + \eta)} \right] \\
& + h \sinh u \sinh(u - 4\eta) \cosh(u - \eta) \cosh(u - 3\eta) \times \\
& \prod_{j=1}^{N} a_2(u - \theta_j) a_2(u + \theta_j) b_2(u - \theta_j) b_2(u + \theta_j) \frac{Q^{(3)}(u - \eta)Q^{(2)}(u + \eta)}{Q^{(1)}(u - \eta)} \right\}, \quad (3.32)
\end{align*}
\]
\{\mu_l^{(2)}\} satisfy the BAEs

\[
\frac{2 \sinh(2\mu_k^{(1)} - 2\eta) \cosh(\mu_k^{(1)} + 2\eta)}{4^N \prod_{j=1}^{N} \sinh(\mu_k^{(1)} + \eta - \theta_j) \sinh(\mu_k^{(1)} + \eta + \theta_j)} \cdot \frac{Q_1^{(1)}(\mu_k^{(1)} + 3\eta)}{Q_2^{(2)}(\mu_k^{(1)} + 3\eta)Q_3^{(3)}(\mu_k^{(1)} + 3\eta)}
\]
\[
+ \frac{2 \sinh(2\mu_k^{(1)} + 2\eta) \cosh(\mu_k^{(1)} - 2\eta)}{4^N \prod_{j=1}^{N} \sinh(\mu_k^{(1)} - \eta - \theta_j) \sinh(\mu_k^{(1)} - \eta + \theta_j)} \cdot \frac{Q_1^{(1)}(\mu_k^{(1)} - \eta)}{Q_2^{(2)}(\mu_k^{(1)} + \eta)Q_3^{(3)}(\mu_k^{(1)} + \eta)}
\]
\[
= -h \sinh(\mu_k^{(1)} + 2\eta) \sinh(\mu_k^{(1)} + 2\eta), \quad k = 1, \cdots, L_1,
\]
\[
\frac{Q_1^{(1)}(\mu_l^{(2)})Q_2^{(2)}(\mu_l^{(2)} + 4\eta)}{Q_1^{(1)}(\mu_l^{(2)} + 2\eta)Q_2^{(2)}(\mu_l^{(2)})} = -\frac{\sinh(2\mu_l^{(2)} - 2\eta) \cosh(\mu_l^{(2)} - 2\eta)}{\sinh(2\mu_l^{(2)} - 2\eta) \cosh(\mu_l^{(2)} + 2\eta)}, \quad l = 1, \cdots, L_2,
\]

where the numbers of Bethe roots should satisfy the constraint

\[
L_1 = L_2 + N + 1,
\]

and the parameter \(h\) is

\[
h = 2^{2L_2 - 2N} \left\{ \frac{c}{c'} e^{-4\eta} + \frac{c'}{c} e^{4\eta} - 2 \cosh[2(L_1 + 1)\eta] \right\}.
\]

Some remarks are in order. The BAEs (3.34) are inhomogeneous while the BAEs (3.35) are homogeneous. This is because that the reflection matrices (3.7) and (3.9) can be divided into the direct summation of a 4 \times 4 non-diagonal and a 2 \times 2 diagonal submatrices. The boundary reflection in the non-diagonal subspace breaks the \(U(1)\) symmetry of the system. While in the diagonal subspace, there exists a conserved charge, which leads to the homogeneous BAEs (3.35). From the asymptotic behavior of eigenvalues, we obtain that the quantum number \(m\) of conserved quantities \(Q_\pm\) is related with the number of Bethe roots \(\{\mu_l^{(2)}\}\) as \(m = L_2 - N\), which is consistent with the conclusion that BAEs (3.35) are homogeneous. We shall also note that the BAEs obtained from the regularities of \(\Lambda(u)\) are the same as those obtained from the regularities of \(\Lambda_\pm(u)\). The function \(Q_l^{(i)}(u)\) has two sets of zero roots, i.e., \(\{\mu_k^{(l)} + l\eta|k = 1, \cdots, L_l\}\) and \(\{\mu_k^{(l)} + l\eta|k = 1, \cdots, L_l\}\), where \(l = 1, 2\). The BAEs obtained from these two sets of zero roots are also the same. It is easy to check that \(\Lambda(u)\) (3.30) and \(\Lambda_\pm(u)\) (3.31)-(3.32) satisfy the functional relations (3.27), the values at the special points (3.28) and the asymptotic behaviors (3.29). Therefore, we conclude that the inhomogeneous \(T - Q\) relations (3.30)-(3.32) give the eigenvalues of the transfer matrices \(t(u)\) and \(t_\pm(u)\). All the eigenvalues (3.31)-(3.32) and BAEs (3.34)-(3.35) have the well-defined homogeneous limit.
The energy spectrum of the Hamiltonian (3.6) can be obtained by $\Lambda(u)$ as

$$E = \frac{\partial \ln \Lambda(u)}{\partial u} \bigg|_{u=0,\{\theta_j\}=0}. \quad (3.38)$$

## 4 Discussion

In this paper, we have studied the quantum integrable model associated with the twisted $D_3^{(2)}$ Lie algebra by generalizing the nested off-diagonal Bethe ansatz. We obtain the exact solutions of the system with either periodic or non-diagonal open boundary conditions. With the help of fusion, we obtain the closed recursive operator product identities among the fused transfer matrices. Based on them and the asymptotic behaviors as well as the special values at certain points, we obtain the eigen-spectrum and Bethe ansatz equations. For the periodic case, the eigenvalues of transfer matrices are described by the homogeneous $T - Q$ relations. While for the open boundary case, the eigenvalues are characterized by the inhomogeneous $T - Q$ relations due to the off-diagonal $K$-matrices (3.7)-(3.9). The method can be generalized to the model with other high rank twisted algebras.

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## Appendix A: Fusion of the $R$-matrices

### A.1 Spinorial $R$-matrix

In this appendix, we shall give the $R$-matrices $R^{\pm v}(u)$ which has the same quantum space as that of (2.1) but the spinorial representations of $D_3^{(2)}$ as their auxiliary spaces. Namely,
we provide $R^{s+v}(u)$ as

\[
\begin{pmatrix}
    r_1^+ & r_1^+ & r_3^+ & r_5^+ & r_9^+ & r_{11}^+ & r_1^+ \\
    r_3^+ & r_3^+ & r_3^+ & r_7^+ & r_{11}^+ & r_{13}^+ & r_1^+ \\
    r_6^+ & r_4^+ & r_{14}^+ & r_8^+ & r_{14}^+ & r_{14}^+ & r_1^+ \\
    r_2^+ & r_2^+ & r_2^+ & r_2^+ & r_2^+ & r_2^+ & r_1^+ \\
    r_{22}^+ & r_{22}^+ & r_{22}^+ & r_{22}^+ & r_{22}^+ & r_{22}^+ & r_1^+ \\
    r_{24}^+ & r_{24}^+ & r_{24}^+ & r_{24}^+ & r_{24}^+ & r_{24}^+ & r_1^+ \\
    r_{15}^+ & r_{15}^+ & r_{15}^+ & r_{15}^+ & r_{15}^+ & r_{15}^+ & r_1^+ \\
\end{pmatrix}
\]

(A.1)

where

\[
\begin{align*}
    r_1^+ &= 2 \sinh(u - 3\eta), \\
    r_2^+ &= 2 \sinh(u - \eta), \\
    r_3^+ &= 2 \sinh(u - 2\eta) + \frac{e^\eta \sinh 4\eta - 2e^{2\eta} \sinh \eta}{\cosh 2\eta}, \\
    r_4^+ &= 2 \sinh(u - 2\eta) - \frac{e^\eta \sinh 4\eta - 2e^{2\eta} \sinh \eta}{\cosh 2\eta}, \\
    r_5^+ &= -2 \sinh \eta \tanh 2\eta \sqrt{\frac{\cosh 3\eta}{\cosh \eta}}, \\
    r_6^+ &= \frac{2 \sinh^2 \eta(1 + e^{4\eta} \cosh 2\eta)}{\cosh 2\eta \sqrt{\cosh \eta \cosh 3\eta}}, \\
    r_7^+ &= 4 \sinh \frac{1}{2}(u - 3\eta) \cosh \frac{1}{2}(u - \eta), \\
    r_8^+ &= 4 \sinh \frac{1}{2}(u - \eta) \cosh \frac{1}{2}(u - 3\eta), \\
    r_9^+ &= 2 \sinh(u - 2\eta) + \frac{e^{-\eta} \sinh 4\eta - 2e^{-2\eta} \sinh \eta}{\cosh 2\eta}, \\
    r_{10}^+ &= 2 \sinh(u - 2\eta) - \frac{e^{-\eta} \sinh 4\eta - 2e^{-2\eta} \sinh \eta}{\cosh 2\eta}, \\
    r_{11}^+ &= -4e^{\frac{\eta}{2} - \eta} \sinh \eta \cosh \frac{1}{2}(u - \eta) \sqrt{\frac{\cosh 3\eta}{\cosh 2\eta}}, \\
    r_{12}^+ &= -2e^{-\frac{\eta}{2}} \sinh(2\cosh 2\eta - 1 - e^{u-\eta}) \sqrt{\frac{\cosh \eta}{\cosh 2\eta}}, \\
    r_{13}^+ &= -\frac{2 \sinh^2 \eta(1 + e^{-4\eta} \cosh 2\eta)}{\cosh 2\eta \sqrt{\cosh \eta \cosh 3\eta}}, \\
    r_{14}^+ &= 4 \sinh 2\eta \sinh^2 \eta \sqrt{\cosh \eta \cosh 3\eta}, \\
\end{align*}
\]
\[ r_{18}^{+} = 2e^{u} \sin 2\eta, \quad r_{19}^{+} = e^{-u+2\eta}r_{11}^{+}, \quad r_{20}^{+} = -2e^{\frac{u}{2}} \sinh \eta (2 \cosh 2\eta - 1 - e^{\eta-u}) \sqrt{\frac{\cosh \eta}{\cosh 2\eta}}, \]

\[ r_{21}^{+} = \frac{e^{-u+3\eta}}{\sqrt{\cosh 2\eta \cosh 3\eta}} \sinh 2\eta (2 \cosh 2\eta - 1 - e^{u-3\eta}), \]

\[ r_{22}^{+} = -4e^{\frac{u}{2}} \sinh \eta \cosh \left(\frac{1}{2}(u - 3\eta)\right) \sqrt{\frac{\cosh \eta}{\cosh 2\eta}}, \quad r_{23}^{+} = -e^{-u}r_{11}^{+}, \quad r_{24}^{+} = -e^{-2\eta}r_{20}^{+}, \]

\[ r_{25}^{+} = -\frac{e^{u-2\eta}}{\sqrt{\cosh 2\eta \cosh 3\eta}} \sinh 2\eta (2 \cosh 2\eta - 1 + e^{\eta-u}) \sqrt{\frac{\cosh \eta}{\cosh 2\eta}}, \quad r_{26}^{+} = 4e^{\frac{u}{2}} \sinh \eta \sinh \left(\frac{1}{2}(u - \eta)\right) \sqrt{\frac{\cosh \eta}{\cosh 2\eta}}, \]

\[ r_{27}^{+} = 4e^{\frac{u}{2}} \sinh \eta \sinh \left(\frac{1}{2}(u - 3\eta)\right) \sqrt{\frac{\cosh 3\eta}{\cosh 2\eta}}, \quad r_{28}^{+} = -\frac{e^{3\eta}}{\sqrt{\cosh \eta \cosh 2\eta}} (2 \cosh 2\eta - 1 + e^{u-3\eta}), \]

\[ r_{29}^{+} = -e^{2\eta}r_{25}^{+}, \quad r_{30}^{+} = -e^{2\eta}r_{26}^{+}, \quad r_{31}^{+} = -e^{-u}r_{26}^{+}, \]

\[ r_{32}^{+} = -\frac{e^{u-2\eta}}{\sqrt{\cosh \eta \cosh 2\eta}} (2 \cosh 2\eta - 1 + e^{3\eta-u}), \]

\[ r_{33}^{+} = -\frac{e^{-3\eta}}{\sqrt{\cosh 2\eta \cosh 3\eta}} (2 \cosh 2\eta - 1 + e^{\eta-u}), \quad r_{34}^{+} = -e^{-u}r_{26}^{+}, \quad r_{35}^{+} = e^{2\eta}r_{33}^{+}, \]

\[ r_{36}^{+} = -e^{-u+2\eta}r_{26}^{+}, \quad r_{37}^{+} = \frac{e^{u-3\eta}}{\sqrt{\cosh 2\eta \cosh 3\eta}} (2 \cosh 2\eta - 1 - e^{3\eta-u}), \]

\[ r_{38}^{+} = e^{u}r_{22}^{+}, \quad r_{39}^{+} = e^{2\eta}r_{11}^{+}, \quad r_{40}^{+} = e^{2\eta}r_{12}^{+}, \]
and $R_{s-v}(u)$ as

$$
\begin{aligned}
\left( r_1^- 
\begin{array}{c}
\bar{r}_3^- \\
\bar{r}_5^- \\
\bar{r}_6^- \\
\bar{r}_4^- \\
\bar{r}_2^- \\
\bar{r}_1^- \\
\bar{r}_{20}^- \\
\bar{r}_{21}^- \\
\bar{r}_{22}^- \\
\bar{r}_{23}^- \\
\bar{r}_{24}^- \\
\bar{r}_{15}^- \\
\bar{r}_{25}^- \\
\bar{r}_{26}^- \\
\bar{r}_{27}^- \\
\bar{r}_{28}^- \\
\bar{r}_{17}^- \\
\bar{r}_{29}^- \\
\bar{r}_{30}^- \\
\bar{r}_{19}^- \\
\bar{r}_{11}^- \\
\bar{r}_{12}^- \\
\bar{r}_{13}^- \\
\bar{r}_{10}^- \\
\bar{r}_{1}^- \\
\bar{r}_{27}^- \\
\bar{r}_{28}^- \\
\bar{r}_{29}^- \\
\bar{r}_{30}^- \\
\bar{r}_{31}^- \\
\bar{r}_{32}^- \\
\bar{r}_{33}^- \\
\bar{r}_{34}^- \\
\bar{r}_{35}^- \\
\bar{r}_{36}^- \\
\bar{r}_{37}^- \\
\bar{r}_{38}^- \\
\bar{r}_{39}^- \\
\bar{r}_{40}^- \\
\bar{r}_{18}^-
\end{array}
\right)
\end{aligned}
$$

(A.2)

with

$$
\begin{aligned}
r_1^- = r_1^+ , \quad r_2^- = r_2^+ , \quad r_3^- = r_4^+ , \quad r_4^- = r_3^+ , \quad r_5^- = -r_5^+ , \quad r_6^- = -r_6^+ , \quad r_7^- = r_8^+ , \\
r_8^- = r_7^+ , \quad r_9^- = r_{10}^+ , \quad r_{10}^- = r_9^+ , \quad r_{11}^- = \sqrt{\frac{\cosh \eta}{\cosh \eta}} r_{26}^+ , \quad r_{12}^- = e^u \sqrt{\frac{\cosh \eta}{\cosh \eta}} r_{35}^+ , \\
r_{13}^- = -r_{13}^+ , \quad r_{14}^- = -r_{14}^+ , \quad r_{15}^- = r_{15}^+ , \quad r_{16}^- = -r_{16}^+ , \quad r_{17}^- = -r_{17}^+ , \quad r_{18}^- = r_{18}^+ , \\
r_{19}^- = e^{-u+2\eta} r_{11}^- , \quad r_{20}^- = -e^{2\eta-u} \sqrt{\frac{\cosh \eta}{\cosh \eta}} r_{25}^+ , \quad r_{21}^- = -e^{-u} \sqrt{\frac{\cosh \eta}{\cosh \eta}} r_{28}^+ , \\
r_{22}^- = e^{-u} \sqrt{\frac{\cosh \eta}{\cosh \eta}} r_{27}^+ , \quad r_{23}^- = e^{-u} r_{11}^- , \quad r_{24}^- = e^{2\eta} r_{20}^- , \quad r_{25}^- = e^{u-2\eta} \sqrt{\frac{\cosh \eta}{\cosh \eta}} r_{20}^+ , \\
r_{26}^- = -\sqrt{\frac{\cosh \eta}{\cosh \eta}} r_{11}^+ , \quad r_{27}^- = -e^u \sqrt{\frac{\cosh \eta}{\cosh \eta}} r_{22}^+ , \quad r_{28}^- = e^u \sqrt{\frac{\cosh \eta}{\cosh \eta}} r_{21}^+ , \\
r_{29}^- = e^{2\eta} r_{25}^- , \quad r_{30}^- = e^{2\eta} r_{26}^- , \quad r_{31}^- = e^{-u} r_{27}^- , \quad r_{32}^- = e^{-u} \sqrt{\frac{\cosh \eta}{\cosh \eta}} r_{37}^+ , \\
r_{33}^- = e^{-u} \sqrt{\frac{\cosh \eta}{\cosh \eta}} r_{12}^+ , \quad r_{34}^- = e^{-u} r_{26}^- , \quad r_{35}^- = -e^{2\eta} r_{33}^- , \quad r_{36}^- = -e^{-u+2\eta} r_{26}^- .
\end{aligned}
$$
We remark that the fused $R$-matrices $R_{1/2}^{s,v}(u)$ are necessary to derive the exact solution of the $D_3^{(2)}$ model. For simplicity, throughout this paper, we denote $I' = 1'$ for the representation $s_+$ and $\bar{I}' = \bar{1}'$ for the representation $s_-$. Here we list some useful relations among the $R$-matrices:

unitarity : \[ R_{1/2}^{s,v}(u) R_{21/2}^{s,v}(-u) = \rho_s(u) = -4 \sinh(u - 3\eta) \sinh(u + 3\eta), \]
crossing unitarity : \[ R_{1/2}^{s,v}(u) t_{1/2} \bar{M}_1 R_{21/2}^{s,v}(-u + 8\eta) t_{1/2} \bar{M}_1^{-1} = \rho_s(u - 4\eta - i\pi), \]
periodicity : \[ R_{1/2}^{s,v}(u + i\pi) = -\bar{V}_2 R_{1/2}^{s,v}(u) \bar{V}_2^{-1}, \]
where $\bar{M}$ is the diagonal matrix given by

\[ \bar{M} = \text{diag}[e^{4\eta}, e^{2\eta}, e^{-2\eta}, e^{-4\eta}], \]

and the Yang-Baxter relations

\[ R_{1/2}^{s,v}(u_1 - u_2) R_{1/2}^{s,v}(u_1 - u_3) R_{23}^{s,v}(u_2 - u_3) = R_{23}^{s,v}(u_2 - u_3) R_{1/2}^{s,v}(u_1 - u_3) R_{1/2}^{s,v}(u_1 - u_2). \]

**A.2 Fusion and the fused $R$-matrices**

By using the fusion technique [45–50], we systemically analyze the fusion structure of the $R$-matrices. For this purpose, we consider the fusion of $R_{1/2}^{s,v}(u)$ and $R_{23}^{s,v}(u)$ in the spaces $V_1'$ and $V_2$. This can be realized is because that the fused $R$-matrix $R_{1/2}^{s,v}(u)$ (A.1) has the degenerate point of $u = 3\eta + i\pi$. At which, $R_{1/2}^{s,v}(u)$ becomes a $4 \times 4$ matrix

\[ R_{1/2}^{s,v}(3\eta + i\pi) = P_{1/2}^{(+)} S_{1/2}^{(+)}, \]

where $P_{1/2}^{(+)}$ is a $4$-dimensional projector

\[ P_{1/2}^{(+)} = \sum_{i=1}^{4} |\phi_i^{(+)}\rangle \langle \phi_i^{(+)}|, \]

with the basis vectors

\[ |\phi_1^{(+)}\rangle = e^{\eta} \frac{\sinh \eta \sqrt{\cosh 3\eta}}{x_1 \sqrt{\cosh \eta \cosh 2\eta}} |13\rangle + e^{\eta} \frac{(e^{\eta} \cosh 2\eta - \sinh \eta)}{x_1 \sqrt{\cosh 2\eta}} |14\rangle + e^{\eta} \frac{\sqrt{\cosh \eta}}{x_1} |22\rangle - e^{3\eta} \frac{\sqrt{\cosh \eta}}{x_1} |31\rangle, \quad x_1 = \sqrt{e^{3\eta}(2 \cosh 2\eta + \cosh 4\eta)}, \]
and $S^{(+)}_{12}$ is a constant matrix omitted here. Exchanging the two spaces $V_{1'}$ and $V_2$, we obtain the fused $R$-matrix $R^{s,v+}_{21'}(u)$. From it, we deduce another 4-dimensional projector

$$P^{(+)}_{21'} = \sum_{i=1}^{4} |\phi^{(+)}_i⟩⟨\phi^{(+)}_i|, \quad |\phi^{(+)}_i⟩ = |\phi^{(+)}_i⟩|_{\eta \rightarrow -\eta, |k\rangle \rightarrow |k\rangle}.$$ (A.8)

Taking the fusion of $R^{s,v+}_{13'}(u)$ and $R^{s,v}_{23}(u)$ by using the projectors $P^{(+)}_{12}$ and $P^{(+)}_{21'}$, we obtain

$$P^{(+)}_{12} R^{s,v}_{23}(u) R^{s,v+}_{13'}(u + 3\eta + i\pi) P^{(+)}_{12} = \tilde{\rho}_0(u) R_{13'}^{s,v}(u + \eta + i\pi),$$ (A.10)

$$P^{(+)}_{21'} R^{s,v}_{32}(u) R^{s,v+}_{13'}(u + 3\eta + i\pi) P^{(+)}_{21'} = \tilde{\rho}_0(u) R_{13'}^{s,v-}(u + \eta + i\pi),$$ (A.11)

where $\tilde{\rho}_0(u) = 4 \sinh(u + 2\eta) \sinh(u - 4\eta)$, $1'2$ denotes the fused space $V_{1'2}$, and $R^{s,v}_{13'}(u)$ and $R^{s,v}_{31'}(u)$ are the new fused $R$-matrices. For simplicity, we define $1' \equiv 1'2$. We shall note that although the dimension of fused space $V_{1'}$ is 4, the $V_{1'}$ is not the original spinorial representation space $V_{1'}$, i.e., $V_{1'} \neq V_{12}$. In fact, $V_{1'}$ is the space of another spinorial representation $s_{-}$ of $D_3^{(2)}$ Lie algebra. Thus the fused $R$-matrix $R^{s,v}_{13'}(u)$ is defined in the tensor spaces of $V_{1'} \otimes V_2$ and can be expressed as a $24 \times 24$ matrix.

The fused $R$-matrices [A.10] and [A.11] can also be used to make the further fusion. At the point of $u = 3\eta + i\pi$, $R^{s,v}_{13'}(u)$ reduces into a $4 \times 4$ matrix

$$R^{s,v}_{13'}(3\eta + i\pi) = P^{(-)}_{12} S^{(-)}_{12},$$ (A.12)

where $P^{(-)}_{12}$ is a 4-dimensional projector

$$P^{(-)}_{12} = \sum_{i=1}^{4} |\phi^{(-)}_i⟩⟨\phi^{(-)}_i|,$$ (A.13)

$$\tilde{\rho}_0(u) = 4 \sinh(u + 2\eta) \sinh(u - 4\eta)$$

and $S^{(+)}_{12}$ is a constant matrix omitted here. Exchanging the two spaces $V_{1'}$ and $V_2$, we obtain the fused $R$-matrix $R^{s,v+}_{21'}(u)$. From it, we deduce another 4-dimensional projector

$$P^{(+)}_{21'} = \sum_{i=1}^{4} |\phi^{(+)}_i⟩⟨\phi^{(+)}_i|, \quad |\phi^{(+)}_i⟩ = |\phi^{(+)}_i⟩|_{\eta \rightarrow -\eta, |k\rangle \rightarrow |k\rangle}.$$ (A.8)

Taking the fusion of $R^{s,v+}_{13'}(u)$ and $R^{s,v}_{23}(u)$ by using the projectors $P^{(+)}_{12}$ and $P^{(+)}_{21'}$, we obtain

$$P^{(+)}_{12} R^{s,v}_{23}(u) R^{s,v+}_{13'}(u + 3\eta + i\pi) P^{(+)}_{12} = \tilde{\rho}_0(u) R_{13'}^{s,v}(u + \eta + i\pi),$$ (A.10)

$$P^{(+)}_{21'} R^{s,v}_{32}(u) R^{s,v+}_{13'}(u + 3\eta + i\pi) P^{(+)}_{21'} = \tilde{\rho}_0(u) R_{13'}^{s,v-}(u + \eta + i\pi),$$ (A.11)

where $\tilde{\rho}_0(u) = 4 \sinh(u + 2\eta) \sinh(u - 4\eta)$, $1'2$ denotes the fused space $V_{1'2}$, and $R^{s,v}_{13'}(u)$ and $R^{s,v}_{31'}(u)$ are the new fused $R$-matrices. For simplicity, we define $1' \equiv 1'2$. We shall note that although the dimension of fused space $V_{1'}$ is 4, the $V_{1'}$ is not the original spinorial representation space $V_{1'}$, i.e., $V_{1'} \neq V_{12}$. In fact, $V_{1'}$ is the space of another spinorial representation $s_{-}$ of $D_3^{(2)}$ Lie algebra. Thus the fused $R$-matrix $R^{s,v}_{13'}(u)$ is defined in the tensor spaces of $V_{1'} \otimes V_2$ and can be expressed as a $24 \times 24$ matrix.

The fused $R$-matrices [A.10] and [A.11] can also be used to make the further fusion. At the point of $u = 3\eta + i\pi$, $R^{s,v}_{13'}(u)$ reduces into a $4 \times 4$ matrix

$$R^{s,v}_{13'}(3\eta + i\pi) = P^{(-)}_{12} S^{(-)}_{12},$$ (A.12)

where $P^{(-)}_{12}$ is a 4-dimensional projector

$$P^{(-)}_{12} = \sum_{i=1}^{4} |\phi^{(-)}_i⟩⟨\phi^{(-)}_i|,$$ (A.13)
with the basis vectors

\[ |\phi_1^{(-)}\rangle = e^{\frac{\eta}{3}} \sqrt{\cosh \eta \cosh 3\eta} |13\rangle + e^{\frac{2\eta}{3}} \frac{\sinh \eta}{\sqrt{x_1}} \cosh 2\eta |43\rangle \]

\[ -e^{\frac{2\eta}{3}} \sqrt{\cosh \eta \cosh 2\eta} |22\rangle - e^{3\eta} \frac{\sqrt{\cosh \eta}}{x_1} |31\rangle , \]

\[ |\phi_2^{(-)}\rangle = \frac{\sqrt{\cosh \eta}}{x_1} |15\rangle - e^{\frac{2\eta}{3}} \frac{\sqrt{\cosh 2\eta}}{x_1} |24\rangle + e^{3\eta} \frac{\sqrt{\cosh \eta}}{x_1} |41\rangle , \]

\[ |\phi_3^{(-)}\rangle = \frac{\sqrt{\cosh \eta}}{x_1} |16\rangle - e^{\frac{2\eta}{3}} \frac{\sqrt{\cosh 2\eta}}{x_1} |34\rangle - e^{3\eta} \frac{\sqrt{\cosh \eta}}{x_1} |42\rangle , \]

\[ |\phi_4^{(-)}\rangle = \frac{\sqrt{\cosh \eta}}{x_1} |26\rangle - e^{2\eta} \frac{\sqrt{\cosh \eta}}{x_1} |35\rangle - e^{3\eta} \frac{\sqrt{\cosh \eta \cosh 3\eta}}{x_1 \sqrt{\cosh 2\eta}} |43\rangle + \frac{\eta}{x_1 \sqrt{\cosh 2\eta}} |44\rangle , \]

and \( S_{12}^{(-)} \) is a constant matrix omitted here. From Eq. (A.11), we know that the fused \( R \)-matrix \( R_{21'}^{s_0} (3\eta + i\pi) \) degenerates into the 4-dimensional projector

\[ P_{21'}^{(-)} = \sum_{i=1}^{4} |\varphi_i^{(-)}\rangle \langle \varphi_i^{(+)}|, \quad |\varphi_i^{(-)}\rangle = |\phi_i^{(-)}\rangle |\eta \rightarrow -\eta, |k\rangle \rightarrow |k\rangle . \]  

Taking the fusion of \( R_{13}^{s_0} (u) \) and \( R_{23}^{s_0} (u) \) by using the projectors \( P_{12}^{(-)} \) and \( P_{21'}^{(-)} \), we obtain

\[ P_{12}^{(-)} R_{23}^{s_0} (u) R_{13}^{s_0} (u + 3\eta + i\pi) P_{12}^{(-)} = \tilde{\rho}_0 (u) \tilde{S}^{(1)}_{|12\rangle \langle 12|} , \]  

\[ P_{21'}^{(-)} R_{34}^{s_0} (u) R_{34}^{s_0} (u + 3\eta + i\pi) P_{21'}^{(-)} = \tilde{\rho}_0 (u) \tilde{S}^{(1)}_{|12\rangle \langle 12|} , \]

where \( \langle 12\rangle \) denotes the 4-dimensional fused space \( V_{12} \) and \( \tilde{S} \) is a 4 \times 4 diagonal matrix

\[ \tilde{S} = \text{diag}[1, -1, 1, -1] . \]

We shall note that the space \( V_{12} \) is indeed the original spin representation space \( V_{1'} \). Then we have \( 1' = \langle 12\rangle \). From Eqs. (A.15) and (A.16), we know that the fusion of \( R_{13}^{s_0} (u) \) and \( R_{23}^{s_0} (u) \) gives the already obtained fused \( R \)-matrix \( R_{12}^{s_0} (u) \) (A.1) and (A.2). Therefore, the fusion processes are closed.

At the point of \( u = 4\eta \), the 36 \times 36 dimensional \( R \)-matrix (2.1) degenerates into

\[ R_{12}^{s_0} (4\eta) = P_{12}^{s_0} S_{12}^{(1)} , \]

where \( P_{12}^{s_0} \) is the one-dimensional projector

\[ P_{12}^{s_0} = |\psi_0\rangle \langle \psi_0| , \]

(A.19)
with the basis vector
\[ |\psi_0\rangle = \frac{1}{x_0} \left( e^{-3\eta}|16\rangle + e^{-\eta}|25\rangle - \sqrt{\frac{\cosh 3\eta}{\cosh \eta}}|34\rangle - \sqrt{\frac{\cosh 3\eta}{\cosh \eta}}|43\rangle + e^{\eta}|52\rangle + e^{3\eta}|61\rangle \right), \]
\[ x_0 = \frac{\sqrt{\cosh \eta}}{\sqrt{2 \cosh 3\eta(2 \cosh 2\eta + \cosh 4\eta)}}. \]  
(A.20)

and \( S_{12}^{(1)} \) is a constant omitted here. Exchanging two spaces \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \), we obtain
\[ P_{21}^{\text{ev}} = |\tilde{\psi}_0\rangle\langle \tilde{\psi}_0|, \quad |\tilde{\psi}_0\rangle = |\psi_0\rangle|_{kl} \to |lk\rangle. \]  
(A.21)

The fusion of two vectorial \( R \)-matrices by using the projectors \( P_{12}^{\text{ev}} \) and \( P_{21}^{\text{ev}} \) gives
\[ P_{21}^{\text{ev}} R_{13}^{\text{ev}}(u) R_{23}^{\text{ev}}(u - 2) P_{21}^{\text{ev}} = a_1(u) e_1(u + 4\eta) P_{21}^{\text{ev}}, \]  
(A.22)
\[ P_{12}^{\text{ev}} R_{31}^{\text{ev}}(u) R_{32}^{\text{ev}}(u - 2) P_{12}^{\text{ev}} = a_1(u) e_1(u + 4\eta) P_{12}^{\text{ev}}, \]  
(A.23)

where \( e_1(u) = 4 \sinh(u - 2\eta) \sinh(u) \). We see that after taking fusion, we obtain a one-dimensional vector.

At the point of \( u = 2\eta + i\pi \), the \( R \)-matrix (2.1) reduces to a \( 16 \times 16 \) matrix
\[ R_{12}^{\text{ev}}(2\eta + i\pi) = P_{12}^{\text{ev}(16)} S_{12}^{(16)}, \]  
(A.24)

where \( P_{12}^{\text{ev}(16)} \) is a 16-dimensional projector
\[ P_{12}^{\text{ev}(16)} = \sum_{i=1}^{16} |\phi_i^{(16)}\rangle\langle \phi_i^{(16)}|, \]  
(A.25)

with the bases
\[ |\phi_1^{(16)}\rangle = \frac{e^{-\eta} \sqrt{2 \cosh 2\eta}}{\sqrt{2 \cosh 2\eta}}|12\rangle - \frac{e^{\eta} \sqrt{2 \cosh 2\eta}}{\sqrt{2 \cosh 2\eta}}|21\rangle, \]
\[ |\phi_2^{(16)}\rangle = \frac{e^{-2\eta} \sqrt{\cosh 2\eta}}{x_1} |13\rangle - \frac{e^{2\eta} \sinh \eta}{x_1 \sqrt{\cosh 2\eta}} |31\rangle + \frac{\sqrt{\cosh \eta \cosh 3\eta}}{x_1 \sqrt{\cosh 2\eta}} |41\rangle, \]
\[ |\phi_3^{(16)}\rangle = \frac{\sinh \eta \sqrt{\cosh \eta \cosh 3\eta}}{x_1 \cosh 2\eta} |13\rangle + \frac{\cosh \eta \sqrt{\cosh \eta \cosh 3\eta}}{x_1 \cosh 2\eta} |31\rangle \]
\[ + \frac{x_1}{2 \cosh 2\eta} |14\rangle + \frac{\sinh \eta \sinh 3\eta}{x_1 \cosh 2\eta} |41\rangle, \]
\[ |\phi_4^{(16)}\rangle = \frac{e^{-\eta}}{\sqrt{2 \cosh 2\eta}} |15\rangle - \frac{e^{\eta}}{\sqrt{2 \cosh 2\eta}} |51\rangle, \]
\[ |\phi_5^{(16)}\rangle = \frac{e^{-\frac{\eta}{2}} \cosh \eta}{x_2 \sqrt{\cosh 2\eta}} |16\rangle + \frac{e^{-\frac{\eta}{2}} \cosh 2\eta}{x_2 \sqrt{\cosh 2\eta}} |44\rangle + \frac{e^{\frac{3\eta}{2}} \cosh \eta}{x_2 \sqrt{\cosh 2\eta}} |52\rangle, \]
\(|\phi_6^{(16)}\rangle = \frac{e^{-\eta \sqrt{\cosh 2\eta}}}{x_1}|23\rangle - \frac{e^{2\eta \cosh \eta}}{x_1 \sqrt{\cosh 2\eta}}|32\rangle + \frac{\sqrt{\cosh \eta \cosh 3\eta}}{x_1 \sqrt{\cosh 2\eta}}|42\rangle,
\)|

\(|\phi_7^{(16)}\rangle = \frac{\sinh \eta \sqrt{\cosh \eta \cosh 3\eta}}{x_1 \cosh 2\eta}|23\rangle + \frac{\cosh \eta \sqrt{\cosh \eta \cosh 3\eta}}{x_1 \cosh 2\eta}|32\rangle
+ \frac{x_1}{2 \cosh 2\eta}|24\rangle + \frac{\sinh \eta \sinh 3\eta}{x_1 \cosh 2\eta}|42\rangle,
\)

\(|\phi_8^{(16)}\rangle = -\frac{e^{-\frac{2\eta}{3} \cosh 2\eta}}{2x_2x_3 \sqrt{\cosh 2\eta}}|16\rangle + \frac{e^{-\frac{2\eta}{3} x_2}}{2x_3 \sqrt{\cosh 2\eta}}|25\rangle + \frac{e^{\frac{2\eta}{3} \sinh 4\eta}}{4x_2x_3 \sinh \eta \sqrt{\cosh 2\eta}}|44\rangle,
\)

\(|\phi_9^{(16)}\rangle = \frac{e^{-\eta \sqrt{2 \cosh 2\eta}}}{2\sqrt{2 \cosh 2\eta}}|26\rangle - \frac{e^{\eta \sqrt{2 \cosh 2\eta}}}{2 \sqrt{2 \cosh 2\eta}}|62\rangle,
\)

\(|\phi_{10}^{(16)}\rangle = \frac{(e^n \cosh^2 2\eta + \sinh \eta) \sqrt{\cosh \eta}}{2x_3x_4 \cosh^{\frac{5}{2}} 2\eta}|16\rangle + \frac{e^{2\eta(n \cosh 2\eta + \sinh \eta) \sqrt{\cosh \eta}}}{2x_3x_4 \cosh^{\frac{5}{2}} 2\eta}|25\rangle
+ \frac{(e^{-3\eta}(\cosh^2 2\eta - e^n \sinh \eta) \sqrt{\cosh \eta}}{2x_3x_4 \cosh^{\frac{5}{2}} 2\eta}|52\rangle
+ \frac{e^{-\eta}(\cosh^2 2\eta - e^n \sinh \eta) \sqrt{\cosh \eta}}{2x_3x_4 \cosh^{\frac{5}{2}} 2\eta}|61\rangle
+ \frac{x_3 \cosh 3\eta}{x_3 \sqrt{\cosh \eta \cosh 2\eta}}|33\rangle - \frac{\cosh^{\frac{5}{2}} \eta}{x_3x_4 \sqrt{\cosh 2\eta}}|44\rangle,
\)

\(|\phi_{11}^{(16)}\rangle = -\frac{e^{-\eta}(2e^n + \cosh 5\eta - \sinh 3\eta) \sqrt{\cosh \eta \cosh 3\eta}}{2x_4x_5 \cosh^{\frac{5}{2}} 2\eta}|16\rangle
- \frac{e^{\eta}(2e^n + \cosh 5\eta - \sinh 3\eta) \sqrt{\cosh \eta \cosh 3\eta}}{2x_4x_5 \cosh^{\frac{5}{2}} 2\eta}|25\rangle
+ \frac{e^{-\eta}(2e^{-n} + \cosh 5\eta + \sinh 3\eta) \sqrt{\cosh \eta \cosh 3\eta}}{2x_4x_5 \cosh^{\frac{5}{2}} 2\eta}|52\rangle
+ \frac{e^{\eta}(2e^{-n} + \cosh 5\eta + \sinh 3\eta) \sqrt{\cosh \eta \cosh 3\eta}}{2x_4x_5 \cosh^{\frac{5}{2}} 2\eta}|61\rangle
+ \frac{2 \sinh \eta \sqrt{\cosh \eta \cosh 2\eta \cosh 3\eta}}{x_4x_5}|33\rangle + \frac{x_4 \sinh \eta}{x_5 \sqrt{\cosh 2\eta}}|34\rangle
+ \frac{x_4 \sinh \eta}{x_5 \sqrt{\cosh 2\eta}}|43\rangle - \frac{2 \cosh \eta \sinh 2\eta \sqrt{\cosh \eta \cosh 3\eta}}{x_4x_5 \sqrt{\cosh 2\eta}}|44\rangle,
\)

\(|\phi_{12}^{(16)}\rangle = \frac{e^{-\eta \sqrt{2 \cosh 2\eta}}}{x_1}|35\rangle - \frac{e^{2\eta \sinh \eta}}{x_1 \sqrt{\cosh 2\eta}}|53\rangle + \frac{\sqrt{\cosh \eta \cosh 3\eta}}{x_1 \sqrt{\cosh 2\eta}}|54\rangle,
\)

\(|\phi_{13}^{(16)}\rangle = \frac{e^{-\eta \sqrt{2 \cosh 2\eta}}}{x_1}|36\rangle - \frac{e^{2\eta \sinh \eta}}{x_1 \sqrt{\cosh 2\eta}}|63\rangle + \frac{\cosh \eta \sqrt{\cosh \eta \cosh 3\eta}}{x_1 \sqrt{\cosh 2\eta}}|64\rangle,
\)

\(|\phi_{14}^{(16)}\rangle = \frac{\sinh \eta \sqrt{\cosh \eta \cosh 3\eta}}{x_1 \cosh 2\eta}|35\rangle + \frac{\cosh \eta \sqrt{\cosh \eta \cosh 3\eta}}{x_1 \cosh 2\eta}|53\rangle
\)
\[ |\phi_{15}^{(16)}\rangle = \frac{x_1}{2\cosh 2\eta} |45\rangle + \frac{\sinh \eta \sinh 3\eta}{x_1 \cosh 2\eta} |54\rangle, \]
\[ |\phi_{16}^{(16)}\rangle = \frac{e^{-\eta}}{\sqrt{2 \cosh 2\eta}} |56\rangle - \frac{e^\eta}{\sqrt{2 \cosh 2\eta}} |65\rangle, \]
\[ x_1 = \sqrt{1 + \cosh 4\eta - \sinh 2\eta}, \quad x_2 = \sqrt{e^{\eta} + \cosh \eta \cosh 2\eta}, \]
\[ x_3 = \sqrt{1 + 2 \cosh \eta \cosh 3\eta}, \quad x_4 = \sqrt{\cosh 3\eta + 2 \cosh \eta \cosh 2\eta \cosh 4\eta}, \]
\[ x_5 = \sqrt{3 \cosh 3\eta + 2 \sinh \eta \sinh 2\eta \cosh 4\eta}, \] (A.26)

and \( S_{12}^{(16)} \) is a constant matrix omitted here. Exchanging the spaces \( V_1 \) and \( V_2 \), we obtain
\[ P_{21}^{\nu(16)} = \sum_{i=1}^{16} |\varphi_i^{(16)}\rangle \langle \varphi_i^{(16)}|, \quad |\varphi_i^{(16)}\rangle = |\phi_i^{(16)}\rangle \langle \eta \rightarrow -\eta, |k\rangle \rightarrow |k\rangle. \] (A.27)

Taking the fusion of two vectorial \( R \)-matrices by using the 16-dimensional projectors \( P_{12}^{\nu(16)} \) and \( P_{21}^{\nu(16)} \), we obtain
\[ P_{12}^{\nu(16)} R_{23}^{\nu(16)}(u) P_{13}^{\nu(16)}(u + 2\eta + i\pi) P_{12}^{\nu(16)} = \tilde{\rho}_0(u) S_{12}^{\nu(16)} R_{13}^{s,v}(u + \eta + i\pi) R_{23}^{s,v}(u + \eta + i\pi) S_{12}^{-1}, \] (A.28)
\[ P_{21}^{\nu(16)} R_{32}^{\nu(16)}(u) P_{32}^{\nu(16)}(u + 2\eta + i\pi) P_{12}^{\nu(16)} = \tilde{\rho}_0(u) S_{12}^{\nu(16)} R_{32}^{s,v}(u + \eta + i\pi) R_{32}^{s,v}(u + \eta + i\pi) S_{12}^{-1}. \] (A.29)

Here, two 16-dimensional spaces \( V_1 \) and \( V_2 \) is fused into a 16-dimensional fused space \( V_{(12)} \). We find that the fused space \( V_{(12)} \) can be divided into two 4-dimensional spaces \( V_1' \) and \( V_2' \), where \( V_1' \) is the space of spinorial representation \( s_+ \) and \( V_2' \) is the space of spinorial
The representation $s_-$ is a $16 \times 16$ matrix defined in the tensor space $V_1' \otimes V_2'$. The matrix $S_{1/2}'$ is a $16 \times 16$ matrix defined in the tensor space $V_1' \otimes V_2'$. The matrix $\bar{S}_{1/2}'$ can be obtained from the $S_{1/2}'$ by using the mapping $\bar{S}_{1/2}' = S_{1/2}'|_{\eta \rightarrow -\eta}$. The nonzero matrix elements are

$$
S_{1/2}' = \begin{pmatrix}
1 & s_1 & s_2 & -1 \\
\bar{s}_1 & s_3 & s_4 & -s_1 \\
-\bar{s}_1 & -s_3 & -s_4 & s_1 \\
\bar{s}_9 & s_{10} & -1 & s_1 \\
s_{12} & \bar{s}_{14} & s_3 & -s_3 \\
\bar{s}_{13} & \bar{s}_{15} & s_4 & s_4
\end{pmatrix}, \quad (A.30)
$$

where the nonzero matrix elements are

$$
\begin{align*}
    s_1 &= -\frac{cosh\eta}{x_1}\sqrt{\frac{2\cosh 3\eta}{e^{\eta} \cosh 2\eta}}, \\
    s_2 &= \frac{sinh\eta}{x_1}\sqrt{\frac{2e^{\eta}\cosh 3\eta}{\cosh 2\eta}}, \\
    s_3 &= \frac{sinh\eta}{x_1}\sqrt{\frac{2e^{3\eta}}{\cosh \eta}}, \\
    s_4 &= -\frac{\cosh 2\eta}{x_1}\sqrt{\frac{2}{e^{\eta} \cosh \eta}}, \\
    s_5 &= -\frac{\sinh(\eta - \cosh 3\eta - \sinh \eta)}{2x_2 \cosh 4\eta}, \\
    s_6 &= \frac{\cosh 2\eta}{x_2 \cosh \eta}\sqrt{\frac{e^{\eta}}{2}}, \\
    s_7 &= -\frac{x_2}{\cosh \eta}\sqrt{\frac{1}{2e^{\eta}}}, \\
    s_8 &= \frac{\sinh(\eta + \cosh 3\eta + \sinh \eta)}{x_2 \cosh 4\eta}\sqrt{\frac{2}{e^{3\eta}}}, \\
    s_9 &= \frac{e^{2\eta} \cosh \eta}{x_3}, \\
    s_{10} &= \frac{x_2}{x_3}\sqrt{2e^{\eta}}, \\
    s_{11} &= \frac{e^{2\eta} \cosh \eta}{x_3}, \\
    s_{12} &= -\frac{\cosh 3\eta + \cosh 2\eta(e^{\eta} + \cosh 5\eta - \sinh 3\eta)}{x_3 x_4 \cosh 2\eta}\sqrt{\cosh \eta}, \\
    s_{13} &= \frac{\cosh 3\eta + \cosh 2\eta(e^{-\eta} + \cosh 5\eta + \sinh 3\eta)}{x_3 x_4 \cosh 2\eta}\sqrt{\cosh \eta}, \\
    s_{14} &= -\frac{e^{2\eta} \cosh 3\eta \sinh 6\eta + 2 \sinh 2\eta \sinh^2 \eta(2 \cosh 2\eta - 1)x_4^2}{x_4 x_5 \cosh 2\eta \sinh 4\eta}\sqrt{\frac{2 \cosh \eta}{\cosh 3\eta}}, \\
    s_{15} &= -\frac{x_5}{2e^{2\eta} x_4 \cosh 2\eta}\sqrt{\frac{2 \cosh 3\eta}{\cosh \eta}}.
\end{align*}
$$

The matrix $\bar{S}_{1/2}'$ can be obtained from the $S_{1/2}'$ by using the mapping

$$
\bar{S}_{1/2}' = S_{1/2}'|_{\eta \rightarrow -\eta}, \quad (A.31)
$$
Appendix B: Fusion of the reflection matrices

In this appendix, we shall give the related fusion of the reflection matrices \[51,52\] associated with those of the \(R\)-matrices in Appendix A. By using the one-dimensional projector \(P^{12^{(1)}}\), we obtain the fusion relation of reflection matrix \(K^u(u)\) and that of \(\bar{K}^u(u)\) as

\[
P_{21}^{12^{(1)}} K_1^u(u) R_{21^{(2)}}^{12^{(1)}}(2u + 4\eta) K_1^u(u + 4\eta) R_{12^{(1)}}^{12^{(1)}}
\]

\[
= 8 \sinh(u + 4\eta) \sinh(2u + 6\eta) \cosh(u) \cosh(u + \eta) \cosh(u - \eta) P_{12}^{12^{(1)}}, \tag{B.1}
\]

\[
P_{12}^{12^{(1)}} \bar{K}_2^u(u + 4\eta) M_1 R_{12^{(2)}}^{12^{(1)}}(-2u + 4\eta) M_1^{-1} \bar{K}_2^u(u) P_{21}^{12^{(1)}}
\]

\[
= 8 \sinh(u - 4\eta) \sinh(2u - 6\eta) \cosh(u) \cosh(u + \eta) \cosh(u - \eta) P_{21}^{12^{(1)}}. \tag{B.2}
\]

We see that the fused results are the one-dimensional vectors. Taking the fusion of reflection matrices by using the 16-dimensional projector \(P_{12}^{12^{(16)}}\), we have

\[
P_{12}^{12^{(16)}} K_2^u(u) R_{12^{(2)}}^{12^{(16)}}(2u + 2\eta + i\pi) K_1^u(u + 2\eta + i\pi) R_{21^{(2)}}^{12^{(16)}}
\]

\[
= -16 \cosh^2 2\eta \cosh(u + \eta) \sinh(u + 2\eta) \times S_{1/2^{1}} K_{1/2^{1}}^{s_+} (u + \eta + i\pi) R_{12^{(2)}}^{12^{(16)}} (2u + 2\eta + 2i\pi) K_{2/1}^{s_-} (u + \eta + i\pi) \bar{S}_{1/2^{1}}^{-1}, \tag{B.3}
\]

\[
P_{21}^{12^{(16)}} \bar{K}_1^u(u + 2\eta + i\pi) M_1^{-1} R_{21^{(2)}}^{12^{(16)}}(-2u + 6\eta - i\pi) M_1 \bar{K}_2^u(u) P_{12}^{12^{(16)}}
\]

\[
= 16 \cosh^2 2\eta \cosh(u - 3\eta) \sinh(u - 4\eta) \times \bar{S}_{1/2^{1}} \bar{K}_{2/1}^{s_-} (u + \eta + i\pi) M_2^{-1} \bar{R}_{12^{(2)}}^{12^{(16)}} (2u + 6\eta) M_2 \bar{K}_{1/2^{1}}^{s_+} (u + \eta + i\pi) S_{1/2^{1}}^{-1}. \tag{B.4}
\]

From Eqs. (B.3) and (B.4), we know that the 36-dimensional auxiliary spaces \(V_1\) and \(V_2\) is projected into a 16-dimensional fused space \(V_{(12)}\), which can be divided into two 4-dimensional spaces \(V_{1'}\) and \(V_{2'}\), i.e., \(V_{(12)} = V_{1'} \otimes V_{2'}\). As mentioned before, \(V_{1'}\) is the space of spinorial representation \(s_+\) and \(V_{2'}\) is the space of spinorial representation \(s_-\). All the matrices in the right hand side of Eqs. (B.3) and (B.4) are defined in the tensor space \(V_{1'} \otimes V_{2'}\). The matrices \(S_{1/2'}\) and \(\bar{S}_{1/2'}\) are given by (A.30) and (A.32), respectively. The \(R_{1/2'}^{s_+} (u)\) is another 16 \(\times\) 16 spinorial \(R\)-matrix of \(D_3^{(12)}\) model
where the non-zero matrix elements are

\[ r_1 = \sinh\left(\frac{u}{2} - 2\eta\right) \cosh\left(\frac{u}{2} - \eta\right), \quad r_2 = \sinh\left(\frac{u}{2} - 2\eta\right) \cosh\frac{u}{2}, \]

\[ r_3 = e^{-\frac{u}{2}} \sinh\eta \sinh\left(\frac{u}{2} - 2\eta\right), \quad r_4 = -e^{\frac{u}{2}} \sinh\eta \sinh\left(\frac{u}{2} - 2\eta\right), \]

\[ r_5 = \sinh\left(\frac{u}{2} - \eta\right) \cosh\frac{u}{2}, \quad r_6 = -e^{\frac{u}{2} - \eta} \sinh\eta \cosh\frac{u}{2}, \]

\[ r_7 = e^{\frac{u}{2} + \eta} \sinh\eta \cosh\frac{u}{2}, \quad r_8 = e^u r_7, \quad r_9 = e^{u-2\eta} r_7, \]

\[ r_{10} = -e^{-u} \sinh\eta \cosh(2\eta), \quad r_{11} = e^{2u} r_{10}, \quad r_{12} = e^{-\frac{u}{2}} \sinh\eta[cosh\frac{u}{2} - \sinh\left(\frac{u}{2} - 2\eta\right)], \]

\[ r_{13} = e^{\frac{u}{2}} \sinh\eta[cosh\frac{u}{2} + \sinh\left(\frac{u}{2} - 2\eta\right)]. \]  

The \(K^{s\pm}(u)\) are the spinorial reflection matrices defined in the space \(\mathbf{V}_k\)  

\[ K^{s\pm}(u) = \begin{pmatrix} e^{-u} & 0 & 0 & c \sinh u \\ 0 & \frac{\cosh(u-2\eta)}{\cosh 2\eta} & 0 & 0 \\ 0 & 0 & \frac{\cosh(u-2\eta)}{\cosh 2\eta} & 0 \\ -\frac{\sinh u}{c \cosh^2 2\eta} & 0 & 0 & e^u \end{pmatrix}. \] (B.7)

From the above equation, we see that two spinorial reflection matrices \(K^{s\pm}(u)\) are the same, \(K^{s+}(u) = K^{s-}(u)\), although they are defined in different spaces. The \(\tilde{K}^{s\pm}(u)\) are the dual reflection matrices with the definition

\[ \tilde{K}^{s\pm}(u) = \tilde{M} K^{s\pm}(-u + 4\eta + i\pi) |_{e \rightarrow e'} . \] (B.8)

The spinorial R-matrix \(R^{s\pm,s\pm}_{1'2'}(u)\) has following properties

unitarity : \(R^{s\pm,s\pm}_{1'2'}(u) R^{s\pm,s\pm}_{2'1'}(-u) = \rho_{ss}(u)\),  

\[ (B.9) \]

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\[ \rho_{ss}(u) = -\sinh\left(\frac{u}{2} + 2\eta\right) \sinh\left(\frac{u}{2} - 2\eta\right) \cosh\left(\frac{u}{2} + \eta\right) \cosh\left(\frac{u}{2} - \eta\right), \]
crossing unitarity:

\[ R_{1/2}^{s+s}(u_1 - u_2) R_{1/3}^{s-v}(u_1 - u_3) R_{2/3}^{s-v}(u_2 - u_3) = R_{2/3}^{s-v}(u_2 - u_3) R_{1/3}^{s-v}(u_1 - u_3) R_{1/2}^{s+s}(u_1 - u_2). \]

The matrices \( K^{s\pm}(u) \) satisfy the reflection equation

\[ R_{1/2}^{s\pm}(u - v) K_{1/1}^{s\pm}(u) R_{2/1}^{s\pm}(u + v) K_{2/1}^{v}(v) = K_{2/1}^{v}(v) R_{1/2}^{s\pm}(u + v) K_{1/1}^{s\pm}(u) R_{2/1}^{s\pm}(u - v). \]

The matrices \( \bar{K}^{s\pm}(u) \) satisfy the dual reflection equation

\[ R_{1/2}^{s\pm}(u + v) K_{1/1}^{s\pm}(u) \bar{M}_{1/1}^{-1} R_{2/1}^{s\pm}(-u - v + 8\eta + 2i\pi) \bar{M}_{2/1} K_{2/1}^{v}(v) = \bar{K}_{2/1}^{v}(v) \bar{M}_{1/1} R_{1/2}^{s\pm}(-u + v + 8\eta + 2i\pi) \bar{M}_{2/1}^{-1} \bar{K}_{2/1}^{s\pm}(u) R_{2/1}^{s\pm}(-u + v). \]

Taking the fusion of reflection matrices \( K^{v}(u) \) and \( K^{s\pm}(u) \) by using the 4-dimensional projector \( P_{1/2}^{(s)} \), we obtain

\[ P_{1/2}^{(+)} K_{1/1}^{v}(u) R_{1/2}^{s\pm}(2u + 3\eta + i\pi) K_{1/1}^{s\pm}(u + 3\eta + i\pi) P_{1/2}^{(+)} = -4 \cosh(u) \cosh(u + \eta) \sinh(u + 3\eta) K_{1/2}^{s-}(u + \eta + i\pi), \]

\[ P_{1/2}^{(-)} K_{1/1}^{v}(u) R_{1/2}^{s\pm}(2u + 3\eta + i\pi) K_{1/1}^{s-}(u + 3\eta + i\pi) P_{1/2}^{(-)} = -4 \cosh(u) \cosh(u + \eta) \sinh(u + 3\eta) \tilde{S}_{1/2} K_{1/2}^{s+}(u + \eta + i\pi) \tilde{S}_{1/2}^{-1}. \]

From Eqs. (B.13) and (B.14), we see that the reflection matrix \( K^{s\pm}(u) \) can be obtained from \( K^{v}(u) \) and \( K^{s\pm}(u) \). Similarly, the fusion of dual reflection matrices \( \bar{K}^{v}(u) \) and \( \bar{K}^{s\pm}(u) \) by using the 4-dimensional projector \( P_{2/1}^{(s)} \), gives

\[ P_{2/1}^{(+)} \bar{K}_{1/1}^{s\pm}(u + 3\eta + i\pi) \bar{M}_{1/1}^{-1} R_{2/1}^{s\pm}(-2u + 5\eta + i\pi) \bar{M}_{2/1} \bar{K}_{2/1}^{v}(u) P_{2/1}^{(+)} = 4 \cosh(u + \eta) \cosh(u - 3\eta) \sinh(u - 4\eta) \bar{K}_{1/2}^{s-}(u + \eta + i\pi), \]

\[ P_{2/1}^{(-)} \bar{K}_{1/1}^{s\pm}(u + 3\eta + i\pi) \bar{M}_{1/1}^{-1} R_{2/1}^{s\pm}(-2u + 5\eta + i\pi) \bar{M}_{2/1} \bar{K}_{2/1}^{v}(u) P_{2/1}^{(-)} = 4 \cosh(u + \eta) \cosh(u - 3\eta) \sinh(u - 4\eta) \tilde{S}_{1/2} K_{1/2}^{s+}(u + \eta + i\pi) \tilde{S}_{1/2}^{-1}. \]

From Eqs. (B.15) and (B.16), we see that the dual reflection matrix \( \bar{K}^{s\pm}(u) \) can be obtained from \( \bar{K}^{v}(u) \) and \( \bar{K}^{s\pm}(u) \), which means that the fusion process now is closed.
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