ON THE SUM OF SUPEROPTIMAL SINGULAR VALUES

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Abstract. In this paper, we study the following extremal problem and its relevance to the sum of the so-called superoptimal singular values of a matrix function: Given an $m \times n$ matrix function $\Phi$, when is there a matrix function $\Psi_*$ in the set $A_{k,m}^n$ such that

$$\int_T \text{trace}(\Phi(\zeta)\Psi_*(\zeta))d\mu(\zeta) = \sup_{\Psi \in A_{k,m}^n} \int_T \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mu(\zeta)?$$

The set $A_{k,m}^n$ is defined by

$$A_{k,m}^n \overset{\text{def}}{=} \{ \Psi \in H^1_0(M_{n,m}) : \|\Psi\|_{L^1(M_{n,m})} \leq 1, \text{rank } \Psi(\zeta) \leq k \text{ a.e. } \zeta \in T \}.$$ 

To address this extremal problem, we introduce Hankel-type operators on spaces of matrix functions and prove that this problem has a solution if and only if the corresponding Hankel-type operator has a maximizing vector. The main result of this paper is a characterization of the smallest number $k$ for which

$$\int_T \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mu(\zeta)$$

equals the sum of all the superoptimal singular values of an admissible matrix function $\Phi$ (e.g. a continuous matrix function) for some function $\Psi \in A_{k,m}^n$. Moreover, we provide a representation of any such function $\Psi$ when $\Phi$ is an admissible very badly approximable unitary-valued $n \times n$ matrix function.

1. Introduction

The problem of best analytic approximation for a given $m \times n$ matrix-valued bounded function $\Phi$ on the unit circle $T$ is to find a bounded analytic function $Q$ such that

$$\|\Phi - Q\|_{L^\infty(M_{m,n})} = \inf\{\|\Phi - F\|_{L^\infty(M_{m,n})} : F \in H^\infty(M_{m,n})\}.$$

Throughout,

$$\|\Psi\|_{L^\infty(M_{m,n})} \overset{\text{def}}{=} \text{ess sup}_{\zeta \in T} \|\Psi(\zeta)\|_{M_{m,n}}.$$

$M_{m,n}$ denotes the space of $m \times n$ matrices equipped with the operator norm $\|\cdot\|_{M_{m,n}}$ (of the space of linear operators from $\mathbb{C}^n$ to $\mathbb{C}^m$), and $H^\infty(M_{m,n})$ denotes the space of bounded analytic $m \times n$ matrix-valued functions on $T$.

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It is well-known that, unlike scalar-valued functions, a polynomial matrix function $\Phi$ may have many best analytic approximants. Therefore it is natural to impose additional conditions in order to distinguish a “very best” analytic approximant among all best analytic approximants. To do so here, we use the notion of superoptimal approximation by bounded analytic matrix functions.

1.1. Superoptimal approximation and very badly approximable matrix functions. Recall that for an $m \times n$ matrix $A$, the $j$th-singular value $s_j(A)$, $j \geq 0$, is defined to be the distance from $A$ to the set of matrices of rank at most $j$ under the operator norm. More precisely,

$$s_j(A) = \inf \{ \| A - B \|_{M_{m,n}} : B \in M_{m,n} \text{ such that } \text{rank } B \leq j \}.$$ 

Clearly, $s_0(A) = \| A \|_{M_{m,n}}$.

**Definition 1.1.** Let $\Phi \in L^\infty(M_{m,n})$. For $k \geq 0$, we define the sets $\Omega_k = \Omega_k(\Phi)$ by

$$\Omega_0(\Phi) = \left\{ F \in H^\infty(M_{m,n}) : F \text{ minimizes } \text{ess sup}_{\zeta \in T} \| \Phi(\zeta) - F(\zeta) \|_{M_{m,n}} \right\},$$

and

$$\Omega_j(\Phi) = \left\{ F \in \Omega_{j-1} : F \text{ minimizes } \text{ess sup}_{\zeta \in T} s_j(\Phi(\zeta) - F(\zeta)) \right\} \text{ for } j > 0.$$ 

Any function $F \in \bigcap_{k \geq 0} \Omega_k = \Omega_{\min\{m,n\}-1}$ is called a superoptimal approximation to $\Phi$ by bounded analytic matrix functions. In this case, the superoptimal singular values of $\Phi$ are defined by

$$t_j = t_j(\Phi) = \text{ess sup}_{\zeta \in T} s_j((\Phi - F)(\zeta)) \text{ for } j \geq 0.$$

Moreover, if the zero matrix function $O$ belongs to $\Omega_{\min\{m,n\}-1}$, we say that $\Phi$ is very badly approximable.

Notice that any function $F \in \Omega_0$ is a best analytic approximation to $\Phi$. Also, any very badly approximable matrix function is the difference between a bounded matrix function and its superoptimal approximant.

It turns out that Hankel operators on Hardy spaces play an important role in the study of superoptimal approximation. For a matrix function $\Phi \in L^\infty(M_{m,n})$, we define the **Hankel operator** $H_\Phi$ by

$$H_\Phi f = \mathbb{P}_- \Phi f, \text{ for } f \in H^2(\mathbb{C}^n),$$

where $\mathbb{P}_-$ denotes the orthogonal projection from $L^2(\mathbb{C}^m)$ onto $H^2(\mathbb{C}^m)$ defined as $L^2(\mathbb{C}^m) \subset H^2(\mathbb{C}^m)$.

When studying superoptimal approximation, we only consider bounded matrix functions that are admissible. A matrix function $\Phi \in L^\infty(M_{m,n})$ is said to be **admissible** if the essential norm $\| H_\Phi \|_e$ of the Hankel operator $H_\Phi$ is strictly less than the smallest non-zero superoptimal singular value.
of $\Phi$. As usual, the essential norm of a bounded linear operator $T$ between Hilbert spaces is defined by
\[
\|T\|_e \overset{\text{def}}{=} \{\|T - K\| : K \text{ is compact}\}.
\]
Note that any continuous matrix function $\Phi$ is admissible, as the essential norm of $H_\Phi$ equals zero in this case. Moreover, in the case of scalar-valued functions, to say that a function $\varphi$ is admissible simply means that $\|H_\varphi\|_e < \|H_\varphi\|$.

It is known that if $\Phi$ is an admissible matrix function, then $\Phi$ has a unique superoptimal approximation $Q$ by bounded analytic matrix functions. Moreover, the functions $\zeta \mapsto s_j((\Phi - Q)(\zeta))$ equal $t_j(\Phi)$ a.e. on $T$ for each $j \geq 0$. These results were first proved in [PY] for the special case $\Phi \in (H^\infty + C)(M_{m,n})$ (i.e. matrix functions which are a sum of a bounded analytic matrix function and a continuous matrix function), and shortly after proved for the class of admissible matrix functions in [PT].

While it is possible to compute the superoptimal singular values of a given matrix function in concrete examples, it is not known how to verify if a matrix function that is not continuous is admissible or not. Thus a complete characterization of the smallest non-zero superoptimal singular value of a given matrix function is an important problem for superoptimal approximation. This remains an open problem.

We refer the reader to Chapter 14 of [Pe1] which contains proofs to all of the previously mentioned results and many other interesting results concerning superoptimal approximation.

1.2. An extremal problem. Throughout this note, $m$ denotes normalized Lebesgue measure on $T$ so that $m(T) = 1$.

**Definition 1.2.** Let $m, n > 1$ and $1 \leq k \leq \min\{m, n\}$. For $\Phi \in L^\infty(M_{m,n})$, we define $\sigma_k(\Phi)$ by
\[
\sigma_k(\Phi) \overset{\text{def}}{=} \sup_{\Psi \in A_k^{n,m}} \left| \int_T \text{trace}(\Phi(\zeta)\Psi(\zeta))dm(\zeta) \right|,
\]
where
\[
A_k^{n,m} = \{\Psi \in H_0^1(M_{n,m}) : \|\Psi\|_{H_0^1(M_{n,m})} \leq 1 \text{ and } \text{rank } \Psi(\zeta) \leq k \text{ a.e. } \zeta \in T\}.
\]
Whenever $n = m$, we use the notation $A_k^n = A_k^{n,m}$.

We are interested in the following extremal problem:

**Extremal Problem 1.1.** For a matrix function $\Phi \in L^\infty(M_{m,n})$, when is there a matrix function $\Psi \in A_k^{n,m}$ such that
\[
\int_T \text{trace}(\Phi(\zeta)\Psi(\zeta))dm(\zeta) = \sigma_k(\Phi)?
\]

The importance of this problem arose from the following observation due to Peller [Pe3].
Theorem 1.3. Let $1 \leq k \leq \min\{m,n\}$. If $\Phi \in L^\infty(\mathbb{M}_{m,n})$ is admissible, then

$$\sigma_k(\Phi) \leq t_0(\Phi) + \ldots + t_{k-1}(\Phi). \quad (1.2)$$

Proof. Let $\Psi \in A_{k,m}^n$. We may assume, without loss of generality, that $\Phi$ is very badly approximable. Indeed,

$$\int_T \text{trace}(\Phi(\zeta)\Psi(\zeta))dm(\zeta) = \int_T \text{trace}((\Phi - Q)(\zeta)\Psi(\zeta))dm(\zeta)$$

holds for any $Q \in H^\infty(\mathbb{M}_{m,n})$, and so we may replace $\Phi$ with $\Phi - Q$ if necessary, where $Q$ is the superoptimal approximation to $\Phi$ in $H^\infty(\mathbb{M}_{m,n})$.

Let $S_m^n$ denote the collection of $m \times m$ matrices equipped with the trace norm $\|A\|_{S_m^n} = \text{trace}(A^*A)^{1/2} = \sum_{j=0}^\infty s_j(A)$.

It follows from the well-known identity $|\text{trace}(A)| \leq \|A\|_{S_m^n}$ that the inequalities

$$|\text{trace}(\Phi(\zeta)\Psi(\zeta))| \leq \|\Phi(\zeta)\Psi(\zeta)\|_{S_m^n} \leq \left(\sum_{j=0}^{k-1} s_j(\Phi(\zeta))\right) \|\Psi(\zeta)\|_{\mathbb{M}_{m,n}}$$

hold for a.e. $\zeta \in T$. Thus,

$$\left|\int_T \text{trace}(\Phi(\zeta)\Psi(\zeta))dm(\zeta)\right| \leq \int_T \left(\sum_{j=0}^{k-1} s_j(\Phi(\zeta))\right) \|\Psi(\zeta)\|_{\mathbb{M}_{m,n}} dm(\zeta)$$

$$\leq \int_T \left(\sum_{j=0}^{k-1} t_j(\Phi)\right) \|\Psi(\zeta)\|_{\mathbb{M}_{m,n}} dm(\zeta)$$

$$\leq \left(\sum_{j=0}^{k-1} t_j(\Phi)\right) \|\Psi\|_{L^1(\mathbb{M}_{m,n})}$$

$$\leq \sum_{j=0}^{k-1} t_j(\Phi), \quad (1.3)$$

because the singular values of $\Phi$ satisfy $s_j(\Phi(\zeta)) = t_j(\Phi)$ for a.e. $\zeta \in T$ since $\Phi$ is very badly approximable. \qed

Before proceeding, let us observe that equality holds in (1.2) for some simple cases. Let $r$ be a positive integer and $t_0, t_1, \ldots, t_{r-1}$ be positive numbers satisfying

$$t_0 \geq t_1 \geq \ldots \geq t_{r-1}.$$
Suppose $\Phi$ is an $n \times n$ matrix function of the form

$$
\Phi \overset{\text{def}}{=} \begin{pmatrix}
  t_0 u_0 & 0 & \cdots & 0 & 0 \\
  0 & t_1 u_1 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & t_{r-1} u_{r-1} & 0 \\
  0 & 0 & \cdots & 0 & \Phi_0
\end{pmatrix}, \quad (1.4)
$$

where $\|\Phi_0\|_{L^\infty} \leq t_{r-1}$ and $u_j$ is a unimodular function of the form $u_j = \bar{z} \tilde{h}_j / h_j$ with $\theta_j$ an inner function and $h_j$ an outer function in $H^2$ for $0 \leq j \leq r - 1$. Without loss of generality, we may assume that $\|h_j\|_{L^2} = 1$ for each $j$. By setting

$$
\Psi \overset{\text{def}}{=} \begin{pmatrix}
  z\theta_0 h_0^2 & 0 & \cdots & 0 & 0 \\
  0 & z\theta_1 h_1^2 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & z\theta_{r-1} h_{r-1}^2 & 0 \\
  0 & 0 & \cdots & 0 & \Psi_0
\end{pmatrix}, \quad (1.5)
$$

it can be seen that $\Psi \in H^1_0(M_n)$, rank $\Psi(\zeta) = r$ a.e. on $T$, $\|\Psi\|_{L^1(M_n)} = 1$, and

$$
\int_T \text{trace}(\Phi(\zeta) \Psi(\zeta)) dm(\zeta) = t_0 + \ldots + t_{r-1}.
$$

Thus we obtain that

$$
\sigma_r(\Phi) = t_0(\Phi) + \ldots + t_{r-1}(\Phi).
$$

On the other hand, one cannot expect the inequality (1.2) to become an equality in general. After all, by the Hahn-Banach Theorem,

$$
dist_{L^\infty(S^1_2)}(\Phi, H^\infty(M_n)) = \sigma_n(\Phi), \quad (1.6)
$$

and there are admissible very badly approximable $2 \times 2$ matrix functions $\Phi$ for which the strict inequality

$$
dist_{L^\infty(S^1_2)}(\Phi, H^\infty(M_2)) < t_0(\Phi) + t_1(\Phi)
$$

holds. For instance, consider the matrix function

$$
\Phi = \begin{pmatrix}
  \bar{z} & 0 \\
  0 & \bar{z}
\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix}
  1 & \bar{z} \\
  1 & -z
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
  \bar{z} & \bar{z}^2 \\
  -1 & \bar{z}
\end{pmatrix}.
$$

Clearly, $\Phi$ has superoptimal singular values $t_0(U) = t_1(U) = 1$. Let

$$
F = \frac{1}{\sqrt{2}} \begin{pmatrix}
  0 & 0 \\
  -1 & 0
\end{pmatrix}.
$$

It is not difficult to verify that

$$
s_0((\Phi - F)(\zeta)) = \frac{1}{2} \sqrt{3 + \sqrt{5}} \quad \text{and} \quad s_1((\Phi - F)(\zeta)) = \frac{1}{2} \sqrt{3 - \sqrt{5}}
$$

for all $\zeta \in T$. Therefore

$$
dist_{L^\infty(S^1_2)}(\Phi, H^\infty(M_2)) \leq \|\Phi - F\|_{L^\infty(S^1_2)} \leq 2 = t_0(\Phi) + t_1(\Phi). \quad (1.7)
$$
1.3. **What is done in this paper?** In virtue of Theorem 1.3 and the remarks proceeding it, one may ask whether it is possible to characterize the matrix functions $\Phi$ for which (1.2) becomes an equality. So let $\Phi$ be an admissible $n \times n$ matrix function with a superoptimal approximant $Q$ in $H^\infty(M_n)$ for which equality in Theorem 1.3 holds with $k = n$. In this case, it must be that

$$\text{dist}_{L^\infty(S_1^n)}(\Phi, H^\infty(M_n)) = \sum_{j=0}^{n-1} t_j(\Phi) = \sum_{j=0}^{n-1} s_j((\Phi - Q)(\zeta)) = \|\Phi - Q\|_{L^\infty(S_1^n)}$$

by (1.6) and thus the superoptimal approximant $Q$ must be a best approximant to $\Phi$ under the $L^\infty(S_1^n)$ norm as well. Hence, we are led to investigate the following problems:

1. For which matrix functions $\Phi$ does Extremal problem 1.1 have a solution?
2. If $Q_\Phi$ is a best approximant to $\Phi$ under the $L^\infty(S_1^n)$-norm, when does it follow that $Q_\Phi$ is the superoptimal approximant to $\Phi$ in $L^\infty(M_n)$?
3. Can we find necessary and sufficient conditions on $\Phi$ to obtain equality in (1.2) of Theorem 1.3?

Before addressing these problems, we recall certain standard principles of functional analysis in Section 2 that are used throughout the paper. In particular, we give their explicit formulation for the spaces $L^p(S_{mn}^q)$. In Section 3 we introduce the Hankel-type operators $H_{\Phi}^{(k)}$ on spaces of matrix functions and $k$-extremal functions, and prove that the number $\sigma_k(\Phi)$ equals the operator norm of $H_{\Phi}^{(k)}$. We also show that Extremal problem 1.1 has a solution if and only if the Hankel-type operator $H_{\Phi}^{(k)}$ has a maximizing vector, and thus answer question 1 in terms Hankel-type operators.

In Section 4 we establish the main results of this paper concerning best approximation under the $L^\infty(S_1^n)$ norm (Theorem 4.7) and the sum of superoptimal singular values (Theorem 4.13). The latter result characterizes the smallest number $k$ for which

$$\int_T \text{trace}(\Phi(\zeta)\Psi(\zeta))dm(\zeta)$$

equals the sum of all non-zero superoptimal singular values for some function $\Psi \in A_{k,m}^{n,m}$. These results serve as partial solutions to problems 2 and 3.

Lastly, in Section 5 we restrict our attention to unitary-valued very badly approximable matrix functions. For any such matrix function $U$, we provide a representation of any function $\Psi$ for which the formula

$$\int_T \text{trace}(U(\zeta)\Psi(\zeta))dm(\zeta) = n$$

holds.
2. BEST APPROXIMATION AND DUAL EXTREMAL PROBLEMS

We now provide explicit formulation of some basic results concerning best approximation in $H^q(S_p^{m,n})$ for functions in $L^q(S_p^{m,n})$ and the corresponding dual extremal problem. We first consider the general setting.

2.1. Best approximation.

Definition 2.1. Let $X$ be a normed space, $M$ be a closed subspace of $X$, and $x_0 \in X$. We say that $m_0$ is a best approximant to $x_0$ in $M$ if $m_0 \in M$ and

$$\|x_0 - m_0\|_X = \text{dist}(x_0, M) \overset{\text{def}}{=} \inf\{\|x_0 - m\|_X : m \in M\}.$$ 

It is known that if $X$ is a reflexive Banach space and $M$ is a closed subspace of $X$, then each $x_0 \in X \setminus M$ has a best approximant $m_0$ in $M$.

Two standard principles from functional analysis are used throughout this note. Namely, if $X$ is a normed space with a linear subspace $M$, then

$$\sup_{m \in M, \|m\| \leq 1} |\Lambda_0(m)| = \min \left\{\|\Lambda_0 - \Lambda\| : \Lambda \in M^\perp\right\} \quad \text{and}$$

$$\max_{\Lambda \in M^\perp, \|\Lambda\| \leq 1} |\Lambda(x_0)| = \text{dist}(x_0, M) \quad \text{whenever } M \text{ is closed}.$$ 

We now discuss these results in the case of the spaces $L^q(S_p^{m,n})$.

2.2. The spaces $L^q(S_p^{m,n})$. Let $1 \leq q < \infty$ and $1 \leq p \leq \infty$. Let $p'$ denote the conjugate exponent to $p$, i.e. $p' = p/(p - 1)$.

Let $S_p^{m,n}$ denote the space of $m \times n$ matrices equipped with the Schatten-von Neumann norm $\| \cdot \|_{S_p^{m,n}}$, i.e. for $A \in M_{m,n}$

$$\|A\|_{S_p^{m,n}} \overset{\text{def}}{=} \|A\|_{S_m^{m,n}} \quad \text{and} \quad \|A\|_{S_p^{m,n}} \overset{\text{def}}{=} \left(\sum_{j \geq 0} s^p_j(A)\right)^{1/p} \quad \text{for } 1 \leq p < \infty.$$ 

We also use the notation $S_p^n \overset{\text{def}}{=} S_p^{n,n}$.

If $X$ is a normed space of functions on $T$ with norm $\| \cdot \|_X$, then $X(S_p^{m,n})$ denotes the space of $m \times n$ matrix functions whose entries belong to $X$. For $\Phi \in X(S_p^{m,n})$, we define

$$\|\Phi\|_{X(S_p^{m,n})} \overset{\text{def}}{=} \|\rho\|_X, \quad \text{where } \rho(\zeta) \overset{\text{def}}{=} \|\Phi(\zeta)\|_{S_p^{m,n}} \quad \text{for } \zeta \in T.$$ 

It is known that the dual space of $L^q(S_p^{m,n})$ is isometrically isomorphic to $L^{q'}(S_{p'}^{m,n})$ via the mapping $\Phi \mapsto \Lambda_\Phi$, where $\Phi \in L^{q'}(S_{p'}^{m,n})$ and

$$\Lambda_\Phi(\Psi) = \int_T \text{trace}(\Phi(\zeta)\Psi(\zeta))\,d\mu(\zeta) \quad \text{for } \Psi \in L^q(S_p^{m,n}).$$
In particular, it follows that the annihilator of \( H^q(S^m_{p,n}) \) in \( L^q(S^m_{p,n}) \) is given by \( H^q_0(S^m_{p,n}) \), and so

\[
\text{dist}_{L^q(S^{m,n}_p)}(\Phi, H^q(S^{m,n}_p)) = \max_{\|\Psi\|_{H^q(S^{m,n}_p)} \leq 1} \left| \int_T \text{trace}(\Phi(\zeta)\Psi(\zeta)) \, dm(\zeta) \right|
\]

by our remarks in Section 2.1. Moreover, if \( 1 < q < \infty \), then \( \Phi \in L^q(S^m_{p,n}) \) has a best approximant \( Q \in H^q(S^m_{p,n}) \) (as \( L^q(S^m_{p,n}) \) is reflexive); that is,

\[
\|\Phi - Q\|_{L^q(S^m_{p,n})} = \text{dist}_{L^q(S^m_{p,n})}(\Phi, H^q(S^m_{p,n})).
\]

The situation is similar in the case of \( L^\infty(S^m_{p,n}) \). Indeed, \( L^\infty(S^m_{p,n}) \) is a dual space, and so there is a \( Q \in H^\infty(S^m_{p,n}) \) such that

\[
\|\Phi - Q\|_{L^\infty(S^m_{p,n})} = \text{dist}_{L^\infty(S^m_{p,n})}(\Phi, H^\infty(S^m_{p,n})).
\]

Again, it also follows from our remarks in Section 2.1 that

\[
\text{dist}_{L^\infty(S^m_{p,n})}(\Phi, H^\infty(S^m_{p,n})) = \sup_{\|\Psi\|_{H^1(S^{m,n}_p)} \leq 1} \left| \int_T \text{trace}(\Phi(\zeta)\Psi(\zeta)) \, dm(\zeta) \right|
\]

However, an extremal function may fail to exist in this case even if \( \Phi \) is a scalar-valued function. An example can be deduced from Section 1 of Chapter 1 in [Pe1].

3. \( \sigma_k(\Phi) \) as the norm of a Hankel-type operator and \( k \)-extremal functions

We now introduce the Hankel-type operators \( H^{(k)}_\Phi \) which act on spaces of matrix functions. We prove that the number \( \sigma_k(\Phi) \) equals the operator norm of \( H^{(k)}_\Phi \) and characterize when \( H^{(k)}_\Phi \) has a maximizing vector. Recall that for an operator \( T : X \to Y \) between normed spaces \( X \) and \( Y \), a vector \( x \in X \) is called a a maximizing vector of \( T \) if \( x \) is nonzero and

\[
\|Tx\|_Y = \|T\| \cdot \|x\|_X.
\]

We begin by establishing the following lemma.

**Lemma 3.1.** Let \( 1 \leq k \leq \min\{m,n\} \). If \( \Psi \in H^1(M_{n,m}) \) is such that \( \text{rank} \, \Psi(\zeta) = k \) for a.e. \( \zeta \in \mathbb{T} \), then there are functions \( R \in H^2(M_{n,k}) \) and \( Q \in H^2(M_{k,m}) \) such that \( R(\zeta) \) has rank equal to \( k \) for almost every \( \zeta \in \mathbb{T} \),

\[
\Psi = RQ \quad \text{and} \quad \|R(\zeta)\|_{M_{n,k}}^2 = \|Q(\zeta)\|_{M_{k,m}}^2 = \|\Psi(\zeta)\|_{M_{n,m}}^2 \quad \text{for a.e.} \ \zeta \in \mathbb{T}.
\]

**Proof.** Consider the set

\[
\mathcal{A} = \text{clos}_{L^1(\mathbb{C}^n)} \{ f \in H^1(\mathbb{C}^n) : f(\zeta) \in \text{Range} \, \Psi(\zeta) \text{ a.e. on } \mathbb{T} \}.
\]

Since \( \mathcal{A} \) is a non-trivial completely non-reducing (closed) invariant subspace of \( L^1(\mathbb{C}^n) \), there is an \( n \times r \) inner function \( \Theta \) such that \( \mathcal{A} = \Theta H^1(\mathbb{C}^r) \). We first show that \( r = k \). Let \( \{e_j\}_{j=1}^r \) be an orthonormal basis for \( \mathbb{C}^r \). Then for almost every \( \zeta \in \mathbb{T} \), we have that \( \{\Theta(\zeta)e_j\}_{j=1}^r \) is a linearly independent set, since \( \Theta \) is inner. Moreover, \( \{\Theta(\zeta)e_j\}_{j=1}^r \) is a basis for \( \text{Range} \, \Theta(\zeta) = \)
Range $\Psi(\zeta)$ for a.e. $\zeta \in \mathbb{T}$. Since $\dim \text{Range } \Psi(\zeta) = k$ a.e. on $\mathbb{T}$, it follows that $r = \dim \text{Range } \Theta(\zeta) = \dim \text{Range } \Psi(\zeta) = k$. In particular, we obtain that

$$\mathcal{A} = \Theta H^1(\mathbb{C}^k).$$

By considering the columns of $\Psi$, it is easy to see that $\Psi = \Theta F$ for some $k \times m$ matrix function $F \in H^1(\mathbb{M}_{k,m})$ as these columns belong to $\mathcal{A}$. Let $h$ be an outer function in $H^2$ such that $|h(\zeta)| = \|\Psi(\zeta)\|_{\mathbb{M}_{n,m}}^{1/2}$ for a.e. $\zeta \in \mathbb{T}$. The conclusion of the lemma now follows by considering the functions $R = h\Theta$ and $Q = h^{-1}F$. □

**Definition 3.2.** Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$, $1 \leq k \leq \min\{m, n\}$, and $\rho : L^2(S^{m,k}_1) \rightarrow L^2(S^{m,k}_1)/H^2(S^{m,k}_1)$ denote the natural quotient map. We define the Hankel-type operator $H^{(k)}_\Phi : H^2(\mathbb{M}_{n,k}) \rightarrow L^2(S^{m,k}_1)/H^2(S^{m,k}_1)$ by setting

$$H^{(k)}_\Phi F \overset{\text{def}}{=} \rho(\Phi F) \text{ for } F \in H^2(\mathbb{M}_{n,k}).$$

The norm in the quotient space $L^2(S^{m,k}_1)/H^2(S^{m,k}_1)$ is the natural one; that is, the norm of a coset equals the infimum of the $L^2(S^{m,k}_1)$-norms of its elements.

**Theorem 3.3.** Let $1 \leq k \leq \min\{m, n\}$. If $\Phi \in L^\infty(\mathbb{M}_{m,n})$, then

$$\sigma_k(\Phi) = \|H^{(k)}_\Phi\|_{H^2(\mathbb{M}_{n,k}) \rightarrow L^2(S^{m,k}_1)/H^2(S^{m,k}_1)}.$$

**Proof.** Consider the collection

$$\mathcal{B}^{n,m}_k = \{RQ : \|R\|_{H^2(\mathbb{M}_{n,k})} \leq 1, \|Q\|_{H^2(\mathbb{M}_{k,m})} \leq 1\}.$$

We claim that $\mathcal{B}^{n,m}_k = \mathcal{A}^{n,m}_k$. Indeed if $\Psi \in \mathcal{A}_k$ satisfies rank $\Psi(\zeta) = j$ for $\zeta \in \mathbb{T}$, where $1 \leq j \leq k$, then by Lemma 3.1 there are functions $R \in H^2(\mathbb{M}_{n,j})$ and $Q \in H^2(\mathbb{M}_{j,m})$ such that $R(\zeta)$ has rank equal to $j$ for almost every $\zeta \in \mathbb{T}$,

$$\Psi = RQ \text{ and } \|R(\zeta)\|_{\mathbb{M}_{n,j}}^2 = \|Q(\zeta)\|_{\mathbb{M}_{j,m}}^2 = \|\Psi(\zeta)\|_{\mathbb{M}_{n,m}} \text{ for a.e. } \zeta \in \mathbb{T}.$$

We may now add zeros, if necessary, to obtain $n \times k$ and $k \times m$ matrix functions

$$R_\# = (R \ 0) \text{ and } Q_\# = \begin{pmatrix} Q \\ 0 \end{pmatrix},$$

respectively, from which it follows that $\Psi = R_\#Q_\# \in \mathcal{B}^{n,m}_k$. Therefore $\mathcal{A}^{n,m}_k \subseteq \mathcal{B}^{n,m}_k$. The reverse inclusion is trivial and so these sets are equal.
Let \( \Phi \) be a matrix function in \( A_k^n \). We can now describe matrix functions that have a maximizing vector.

**Theorem 3.5.** Let \( \Phi \in L^\infty(M_{m,n}) \). The matrix function \( \Phi \) has a k-extremal function if and only if the Hankel-type operator \( H^k_\Phi : L^2(M_{n,k}) \rightarrow L^2(S^{m,k}_1) / H^2(S^{m,k}_1) \) has a maximizing vector.

**Proof.** To simplify notation, let

\[
\| H^k_\Phi \| = \sup_{\| R \|_{H^2(M_{n,k})} \leq 1, \| Q \|_{H^2_0(M_{k,m})} \leq 1} \left| \int \text{trace}(\Phi(\zeta)R(\zeta)Q(\zeta))d\mathbf{m}(\zeta) \right|
\]

Since \( H^k_\Phi \) is a Hankel-type operator, we have that for any \( F \in H^2(S^{m,k}_1) \)

\[
\sigma_k(\Phi) = \int \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mathbf{m}(\zeta) = \int \text{trace}(\Phi(\zeta)R(\zeta)Q(\zeta))d\mathbf{m}(\zeta)
\]

Thus a matrix function \( \Phi \) has a k-extremal function if and only if Extremal Problem 1 has a solution.

We can now describe matrix functions that have a k-extremal function in terms of Hankel-type operators.

**Definition 3.4.** Let \( \Phi \in L^\infty(M_{m,n}) \) and \( 1 \leq k \leq \min\{m, n\} \). We say that \( \Psi \) is a k-extremal function for \( \Phi \) if \( \Psi \in A_k^n \) and

\[
\sigma_k(\Phi) = \int \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mathbf{m}(\zeta).
\]

Hence

\[
\sigma_k(\Phi) = \sup_{\| R \|_{H^2(M_{n,k})} \leq 1} \sup_{\| Q \|_{H^2_0(M_{k,m})} \leq 1} \left| \int \text{trace}(\Phi(\zeta)R(\zeta)Q(\zeta))d\mathbf{m}(\zeta) \right|
\]

\[
= \sup_{\| R \|_{H^2(M_{n,k})} \leq 1} \text{dist}_{L^2(S^{m,k}_1)}(\Phi R, H^2(M_{n,k}))
\]

\[
= \| H^k_\Phi \|_{H^2(M_{n,k}) \rightarrow L^2(S^{m,k}_1) / H^2(S^{m,k}_1)}.
\]

**Theorem 3.5.** Let \( \Phi \in L^\infty(M_{m,n}) \). The matrix function \( \Phi \) has a k-extremal function if and only if the Hankel-type operator \( H^k_\Phi : H^2(M_{n,k}) \rightarrow L^2(S^{m,k}_1) / H^2(S^{m,k}_1) \) has a maximizing vector.

**Proof.** To simplify notation, let

\[
\| H^k_\Phi \| = \sup_{\| R \|_{H^2(M_{n,k})} \leq 1, \| Q \|_{H^2_0(M_{k,m})} \leq 1} \left| \int \text{trace}(\Phi(\zeta)R(\zeta)Q(\zeta))d\mathbf{m}(\zeta) \right|
\]

Suppose \( \Psi \) is a k-extremal function for \( \Phi \). Let \( j \in \mathbb{N} \) be such that \( j \leq k \) and

\[
\text{rank } \Psi(\zeta) = j \text{ for a.e. } \zeta \in \mathbb{T}.
\]

By Lemma 3.1 there is an \( R \in H^2(M_{n,j}) \) and a \( Q \in H^2_0(M_{j,m}) \) such that

\[
\Psi = RQ \quad \text{and} \quad \| R(\zeta) \|_{M_{n,j}}^2 = \| Q(\zeta) \|_{M_{j,m}}^2 = \| \Psi(\zeta) \|_{M_{n,m}} \text{ for a.e. } \zeta \in \mathbb{T}.
\]

As before, adding zeros if necessary, we obtain \( n \times k \) and \( k \times m \) matrix functions

\[
R_\# = (R \ 0) \quad \text{and} \quad Q_\# = \left( \begin{array}{c} Q \ 0 \end{array} \right),
\]

respectively, so that \( \Psi = R_\#Q_\# \) and

\[
\| Q_\#(\zeta) \|_{M_{k,m}}^2 = \| Q(\zeta) \|_{M_{j,m}}^2 = \| \Psi(\zeta) \|_{M_{n,m}} \text{ for a.e. } \zeta \in \mathbb{T}.
\]

Let us show that \( R_\# \) is a maximizing vector for \( H^k_\Phi \). Since \( Q_\# \) belongs to \( H^2_0(M_{k,m}) \), we have that for any \( F \in H^2(S^{m,k}_1) \)

\[
\sigma_k(\Phi) = \int \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mathbf{m}(\zeta) = \int \text{trace}(\Phi(\zeta)R_\#(\zeta)Q_\#(\zeta))d\mathbf{m}(\zeta)
\]

\[
= \int \text{trace}(\Phi R_\#(\zeta) - F(\zeta)Q_\#(\zeta))d\mathbf{m}(\zeta),
\]
and so
\[\sigma_k(\Phi) = \left| \int_T \text{trace}((\Phi R_\# - F)(\zeta)Q_\#(\zeta))dm(\zeta) \right|\]
\[\leq \int_T |\text{trace}((\Phi R_\# - F)(\zeta)Q_\#(\zeta))| dm(\zeta)\]
\[\leq \int_T \|(\Phi R_\# - F)(\zeta)Q_\#(\zeta)\|_{S_1^n} dm(\zeta)\]
\[\leq \int_T \|(\Phi R_\# - F)(\zeta)\|_{S_1^n,k} \|Q_\#(\zeta)\|_{M_{n,m}} dm(\zeta)\]
\[\leq \|\Phi R_\# - F\|_{L^2(S_1^{m,k})} \|Q_\#\|_{L^2(M_{n,m})}\]
\[= \|\Phi R_\# - F\|_{L^2(S_1^{m,k})} \|\Psi\|_{L^2(M_{n,m})}\]
\[\leq \|\Phi R_\# - F\|_{L^2(S_1^{m,k})}.\]

By Theorem 3.3, we obtain that
\[\sigma_k(\Phi) \leq \left\| H_\Phi^{(k)} R_\# \right\|_{L^2(S_1^{m,k})} \leq \left\| H_\Phi^{(k)} \right\| = \sigma_k(\Phi),\]

and therefore
\[\left\| H_\Phi^{(k)} \right\| = \left\| H_\Phi^{(k)} R_\# \right\|_{L^2(S_1^{m,k})} \leq \left\| H_\Phi^{(k)} \right\|.\]

Thus, \(R_\#\) is a maximizing vector of \(H_\Phi\).

Conversely, suppose the Hankel-type operator \(H_\Phi^{(k)}\) has a maximizing vector \(R \in H^2(M_{n,k})\). Without loss of generality, we may assume that \(\|R\|_{L^2(M_{n,k})} = 1\). Then
\[\text{dist}_{L^2(M_{n,k})}(\Phi R, H^2(S_1^{m,k})) = \left\| H_\Phi^{(k)} \right\|.\]

By the remarks in Section 2.2, there is a function \(G \in H_0^2(M_{k,m})\) such that \(\|G\|_{L^2(M_{k,m})} \leq 1\) and
\[\int_T \text{trace}((\Phi R)(\zeta)G(\zeta))dm(\zeta) = \text{dist}_{L^2(M_{n,k})}(\Phi R, H^2(S_1^{m,k})).\]

On the other hand, since \(R\) is a maximizing vector of \(H_\Phi^{(k)}\), it follows from Theorem 3.3 that
\[\int_T \text{trace}(\Phi(\zeta)(RG)(\zeta))dm(\zeta) = \left\| H_\Phi^{(k)} \right\| = \sigma_k(\Phi).\]

Hence \(\Psi \overset{\text{def}}{=} RG\) is a \(k\)-extremal function for \(\Phi\).

Before stating the next result, let us recall that the Hankel operator \(H_\Phi : H^2(C^n) \to H^2(C^n)\) is defined by \(H_\Phi f = \mathbb{P}_+ \Phi f\) for \(f \in H^2(C^n)\). The following is an immediate consequence of the previous theorem when \(k = 1\).

**Corollary 3.6.** Let \(\Phi \in L^\infty(M_{m,n})\). The Hankel operator \(H_\Phi\) has a maximizing vector if and only if \(\Phi\) has a 1-extremal function.
Proof. By Theorem 3.5, $\Phi$ has a 1-extremal function if and only if the Hankel-type operator $H_{\Phi}^{(1)} : H^2(\mathbb{C}^m) \to L^2(\mathbb{C}^m)/H^2(\mathbb{C}^m)$ has a maximizing vector. The conclusion now follows by considering the "natural" isometric isomorphism between the spaces $H^2(\mathbb{C}^m) = L^2(\mathbb{C}^m) \ominus H^2(\mathbb{C}^m)$ and $L^2(\mathbb{C}^m)/H^2(\mathbb{C}^m)$. □

Remark 3.7. It is worth mentioning that if a matrix function $\Phi$ is such that the Hankel operator $H_{\Phi}$ has a maximizing vector (e.g. $\Phi \in (H^\infty + C)(M_n)$), then any 1-extremal function $\Psi$ of $\Phi$ satisfies

$$\int_T \text{trace}(\Phi(\zeta)\Psi(\zeta)) \, dm(\zeta) = \|H_{\Phi}\| = t_0(\Phi).$$

This is a consequence of Corollary 3.6 and Theorem 3.3.

Remark 3.8. There are other characterizations of the class of bounded matrix functions $\Phi$ such that the Hankel operator $H_{\Phi}$ has a maximizing vector. These involve "dual" extremal functions and "thematic" factorizations. We refer the interested reader to [Pe2] for details.

Corollary 3.9. Let $1 \leq k \leq \ell \leq n$ and $\Phi \in L^\infty(M_n)$. Suppose that $\sigma_k(\Phi) = \sigma_\ell(\Phi)$. If $H_{\Phi}^{(k)}$ has a maximizing vector, then $H_{\Phi}^{(\ell)}$ also has a maximizing vector.

Proof. This is an immediate consequence of Theorem 3.5 □

4. HOW ABOUT THE SUM OF SUPEROPTIMAL SINGULAR VALUES?

In this section, we prove in Theorem 4.7 that equality is obtained in (1.2) under some natural conditions.

For the rest of this note, we assume that $m = n$.

Consider the non-decreasing sequence $\sigma_1(\Phi), \ldots, \sigma_n(\Phi)$. Recall that

$$\sigma_n(\Phi) = \text{dist}_{L^\infty(S_1^n)}(\Phi, H^\infty(M_n))$$

and the distance on the right-hand side is in fact always attained, i.e. a best approximant $Q$ to $\Phi$ under the $L^\infty(S_1^n)$-norm always exists as explained in Section 2.2.

Theorem 4.1. Let $\Phi \in L^\infty(M_n)$ and $1 \leq k \leq n$. Suppose $Q$ is a best approximant to $\Phi$ in $H^\infty(M_n)$ under the $L^\infty(S_1^n)$-norm. If the Hankel-type operator $H_{\Phi}^{(k)}$ has a maximizing vector $F$ in $H^2(M_{n,k})$ and $\sigma_k(\Phi) = \sigma_n(\Phi)$, then

1. $QF$ is a best approximant to $\Phi F$ in $H^2$ under the $L^2(S_1^{n,k})$-norm,
2. for each $j \geq 0$,

$$s_j((\Phi - Q)(\zeta)F(\zeta)) = s_j((\Phi - Q)(\zeta)) \|F(\zeta)\|_{M_{n,k}}$$

for a.e. $\zeta \in T$,

3. $\sum_{j=0}^{k-1} s_j((\Phi - Q)(\zeta)) = \sigma_k(\Phi)$ holds for a.e. $\zeta \in T$, and

4. $s_j((\Phi - Q)(\zeta)) = 0$ holds for a.e. $\zeta \in T$ whenever $j \geq k$. 

Lemma 4.2. Let \( H^2(\Omega) \) be the Sobolev space of functions with square-integrable first derivatives on \( \Omega \). Suppose that \( A, B \in \mathbb{M}_n \) and \( \text{rank}(A) \leq k \). Then \( \text{rank}(AB) \leq k \) for a.e. \( \zeta \in \mathbb{T} \).

Proof. By our assumptions,
\[
\|H^2(\Omega)\|^2 \|F\|^2_{L^2(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)} = \|H^2(\Omega)\|^2_{L^2(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)} = \rho(\Phi F) = \rho(\Phi Q) F
\]
\[
\leq \|\Phi - Q\|_{L^2(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)}^2 = \int_\mathbb{T} \|\Phi - Q\|_{L^2(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)}^2 \,dm(\zeta)
\]
\[
\leq \|\Phi - Q\|_{L^\infty(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)}^2 = \sigma_k(\Phi^2) = \sigma_k(\mathbb{M}_n).
\]
It follows from Theorem 3.3 that all inequalities are equalities. In particular, we obtain that \( Q F \) is a best approximant to \( \Phi F \) under the \( L^2(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n) \)-norm since the first inequality is actually an equality. For almost every \( \zeta \in \mathbb{T} \),
\[
\|\Phi - Q\|_{L^\infty(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)} = \sigma_k(\mathbb{M}_n),
\]
because the second and third inequalities are equalities as well. It follows from (4.1) that for each \( j \geq 0 \),
\[
s_j(\Phi - Q)(\zeta) F(\zeta)) = s_j(\Phi - Q)(\zeta) \|F(\zeta)\|_{\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n} \text{ for a.e. } \zeta \in \mathbb{T}.
\]

We claim that if \( j \geq k \), then \( s_j(\Phi - Q)(\zeta) = 0 \) for a.e. \( \zeta \in \mathbb{T} \). By Theorem 3.3 we can choose a \( k \)-extremal function, say \( \Psi \), for \( \Phi \). Since \( \Psi \) belongs to \( H^1(\mathbb{M}_n) \),
\[
\sigma_k(\Phi) = \int_\mathbb{T} \text{trace}(\Phi(\zeta) \Psi(\zeta)) dm(\zeta) = \int_\mathbb{T} \text{trace}(\Phi(\zeta) \Psi(\zeta)) dm(\zeta)
\]
\[
\leq \int_\mathbb{T} \|\Phi - Q\|_{L^\infty(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)} \|\Psi\|_{L^1(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)} dm(\zeta)
\]
\[
\leq \|\Phi - Q\|_{L^\infty(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)} \|\Psi\|_{L^\infty(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)} = \sigma_k(\mathbb{M}_n),
\]
and so all inequalities are equalities. It follows that
\[
|\text{trace}(\Phi - Q)(\zeta) \Psi(\zeta))| = \|\Phi - Q\|_{L^\infty(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)} \|\Psi\|_{L^\infty(\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n)} \text{ for a.e. } \zeta \in \mathbb{T}.
\]
In order to complete the proof, we need the following lemma.

Lemma 4.2. Let \( A \in \mathbb{M}_n \) and \( B \in \mathbb{M}_n \). Suppose that \( A \) and \( B \) satisfy
\[
|\text{trace}(AB)| = \|A\|_{\mathbb{M}_n} \|B\|_{\mathbb{M}_n,\mathbb{M}_n,\mathbb{M}_n}.
\]
If \( \text{rank}(A) \leq k \), then \( \text{rank}(B) \leq k \) as well.

We first finish the proof of Theorem 4.1 before proving Lemma 4.2. It follows from (4.2) and Lemma 4.2 that
\[
\text{rank}((\Phi - Q)(\zeta)) \leq k \text{ for a.e. } \zeta \in \mathbb{T}.
\]
In particular, if \( j \geq k \), then
\[
s_j((\Phi - Q)(\zeta)) = 0 \text{ for a.e. } \zeta \in \mathbb{T},
\]
and so
\[\sum_{j=0}^{k-1} s_j((\Phi - Q)(\zeta)) = \|((\Phi - Q)(\zeta))\|_{\mathcal{S}_1^k} = \sigma_k(\Phi)\text{ for a.e. } \zeta \in \mathbb{T}.\]

This completes the proof. \[\square\]

**Remark 4.3.** Lemma 4.2 is a slight modification of Lemma 4.6 in [BNP]. Although the proof of Lemma 4.2 given below is almost the same as that given in [BNP] for Lemma 4.6, we include it for the convenience of the reader.

**Proof of Lemma 4.2.** Let \( B \) have polar decomposition \( B = UP \) and set \( C = AU \), where \( P = (B^*B)^{1/2}. \) Let \( e_1, \ldots, e_n \) be an orthonormal basis of eigenvectors for \( P \) and \( Pe_j = \lambda_j e_j \). It is easy to see that the following inequalities hold:

\[|\text{trace}(AB)| = |\text{trace}(CP)| = \sum_{j=1}^{n} \lambda_j \langle C e_j, e_j \rangle \leq \sum_{j=1}^{n} \lambda_j |\langle Ce_j, e_j \rangle| \leq \sum_{j=1}^{n} \lambda_j \|Ce_j\| \]

\[\leq \|C\|_{\mathcal{M}_n} \sum_{j=1}^{n} \lambda_j.\]

On the other hand,

\[\|A\|_{\mathcal{M}_n} \|B\|_{\mathcal{S}_1^k} = \|C\|_{\mathcal{M}_n} \|P\|_{\mathcal{S}_1^k} = \|C\|_{\mathcal{M}_n} \sum_{j=1}^{n} \lambda_j\]

and so, by the assumption \(|\text{trace}(AB)| = \|A\|_{\mathcal{M}_n} \|B\|_{\mathcal{S}_1^k}\), it follows that

\[\sum_{j=1}^{n} \lambda_j \|Ce_j\| = \|C\|_{\mathcal{M}_n} \sum_{j=1}^{n} \lambda_j.\]

Therefore \( \lambda_j \|Ce_j\| = \|C\|_{\mathcal{M}_n} \lambda_j \) for each \( j \). However, if rank \( A \leq k \), then rank \( C \leq k \). Thus there are at most \( k \) vectors \( e_j \) such that \( \|Ce_j\| = \|C\|_{\mathcal{M}_n}. \) In particular, there are at least \( n - k \) vectors \( e_j \) such that \( \|Ce_j\| < \|C\|_{\mathcal{M}_n}. \) Thus, \( \lambda_j = 0 \) for those \( n - k \) vectors \( e_j \), rank \( P \leq k \), and so rank \( B \leq k \). \[\square\]

**Remark 4.4.** Note that the distance function \( d_\Phi \) defined on \( \mathbb{T} \) by

\[d_\Phi(\zeta) \overset{\text{def}}{=} \|((\Phi - Q)(\zeta))\|_{\mathcal{S}_1^k}\]

equals \( \sigma_k(\Phi) \) for almost every \( \zeta \in \mathbb{T} \) and is therefore independent of the choice of the best approximant \( Q \). This is an immediate consequence of Theorem 4.1. A similar phenomenon occurs in the case of matrix functions \( \Phi \in L^p(\mathcal{M}_n) \) for \( 2 < p < \infty \). We refer the reader to [BNP] for details.
Corollary 4.5. Let $\Phi \in L^\infty(\mathbb{M}_n)$ be an admissible matrix function and $1 \leq k \leq n$. If the Hankel-type operator $H^{\{k\}}_{\Phi}$ has a maximizing vector and $\sigma_k(\Phi) = \sigma_n(\Phi)$, then
\[
\sum_{j=0}^{k-1} s_j((\Phi - Q)(\zeta)) \leq \sum_{j=0}^{k-1} t_j(\Phi)
\]
for any best approximation $Q$ of $\Phi$ in $H^\infty(\mathbb{M}_n)$ under the $L^\infty(S^1)$-norm.

Proof. This is an immediate consequence of Theorems 1.3 and 4.1. \qed

Definition 4.6. A matrix function $\Phi \in L^\infty(\mathbb{M}_n)$ is said to have order $\ell$ if $\ell$ is the smallest number such that $H^{\{\ell\}}_{\Phi}$ has a maximizing vector and $\sigma_\ell(\Phi) = \text{dist}_{L^\infty(S^1)}(\Phi, H^\infty(\mathbb{M}_n))$.

If no such number $\ell$ exists, we say that $\Phi$ is inaccessible.

The interested reader should compare this definition of “order” with the one made in [BNP] for matrix functions in $L^p(\mathbb{M}_n)$ for $2 < p < \infty$. Also, due to Corollary 3.9, it is clear that if $\Phi \in L^\infty(\mathbb{M}_n)$ has order $\ell$, then the Hankel-type operator $H^{\{k\}}_{\Phi}$ has a maximizing vector and $\sigma_k(\Phi) = \text{dist}_{L^\infty(S^1)}(\Phi, H^\infty(\mathbb{M}_n))$ holds for each $k \geq \ell$.

Theorem 4.7. Let $\Phi \in L^\infty(\mathbb{M}_n)$ be an admissible matrix function of order $k$. The following statements are equivalent.

1. $Q \in H^\infty$ is a best approximant to $\Phi$ under the $L^\infty(S^1)$-norm and the functions
\[
\zeta \mapsto s_j((\Phi - Q)(\zeta)), \quad 0 \leq j \leq k - 1,
\]
are constant almost everywhere on $\mathbb{T}$.

2. $Q$ is the superoptimal approximant to $\Phi$, $t_j(\Phi) = 0$ for $j \geq k$, and $\sigma_k(\Phi) = t_0(\Phi) + \ldots + t_{k-1}(\Phi)$.

Proof. We first prove that 1 implies 2. By Corollary 4.5, we have that, for almost every $\zeta \in \mathbb{T}$,
\[
\sum_{j=0}^{k-1} s_j((\Phi - Q)(\zeta)) \leq \sum_{j=0}^{k-1} t_j(\Phi) \leq \sum_{j=0}^{k-1} \text{ess sup}_{\zeta \in \mathbb{T}} s_j((\Phi - Q)(\zeta)) = \sum_{j=0}^{k-1} s_j((\Phi - Q)(\zeta)).
\]
This implies that
\[
t_j(\Phi) = \text{ess sup}_{\zeta \in \mathbb{T}} s_j((\Phi - Q)(\zeta)) = s_j((\Phi - Q)(\zeta)) \quad \text{for } 0 \leq j \leq k - 1,
\]
$Q \in \Omega_{k-1}(\Phi)$, and
\[
\sum_{j=0}^{k-1} t_j(\Phi) = \sum_{j=0}^{k-1} s_j((\Phi - Q)(\zeta)) = \sigma_k(\Phi).
\]
Moreover, Theorem 4.1 gives that

\[ s_j((\Phi - Q)(\zeta)) = 0 \text{ a.e. on } \mathbb{T} \text{ for } j \geq k, \]

and so \( t_j(\Phi) = 0 \) for \( j \geq k \), as \( Q \in \Omega_{k-1}(\Phi) \). Hence, \( Q \) is the superoptimal approximant to \( \Phi \).

Let us show that 2 implies 1. Clearly, it suffices to show that if 2 holds, then \( Q \) is a best approximant to \( \Phi \) under the \( L^\infty(S^n_1) \)-norm. Suppose 2 holds. In this case, we must have that

\[
\sigma_k(\Phi) = \sum_{j=0}^{k-1} t_j(\Phi) = \sum_{j=0}^{k-1} s_j((\Phi - Q)(\zeta)) = \|\Phi - Q\|_{L^\infty(S^n_1)}.
\]

Since \( \Phi \) has order \( k \), it follows that

\[
\sigma_n(\Phi) = \|\Phi - Q\|_{L^\infty(S^n_1)}
\]

and so the proof is complete. \( \Box \)

For the rest of this section, we restrict ourselves to admissible matrix functions \( \Phi \) which are also very badly approximable. Recall that, in this case, the function \( \zeta \mapsto s_j(\Phi(\zeta)) \) equals \( t_j(\Phi) \) a.e. on \( \mathbb{T} \) for \( 0 \leq j \leq n - 1 \), as mentioned in Section 1.1. The next result follows at once from Theorem 4.7.

**Corollary 4.8.** Let \( \Phi \) be an admissible very badly approximable \( n \times n \) matrix function of order \( k \). The zero matrix function is a best approximant to \( \Phi \) under the \( L^\infty(S^n_1) \)-norm if and only if \( t_j(\Phi) = 0 \) for \( j \geq k \) and

\[
\sigma_k(\Phi) = t_0(\Phi) + \ldots + t_{k-1}(\Phi).
\]

It is natural to question at this point whether or not the collection of admissible very badly approximable matrix functions of order \( k \) is non-empty. It turns out that one can easily construct examples of admissible very badly approximable matrix functions of order \( k \) (see Examples 4.14 and 4.15). Theorem 4.10 below gives a simple sufficient condition for determining when a very badly approximable matrix function has order \( k \). We first need the following lemma.

**Lemma 4.9.** Let \( \Phi \in L^\infty(M_n) \). Suppose there is \( \Psi \in A^n_k \) such that

\[
\int_{\mathbb{T}} \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mu(\zeta) = \|\Phi\|_{L^\infty(S^n_1)}.
\]

Then \( \Psi \) is a \( k \)-extremal function for \( \Phi \), \( \sigma_k(\Phi) = \sigma_n(\Phi) \), and the zero matrix function is a best approximant to \( \Phi \) under the \( L^\infty(S^n_1) \)-norm.

**Proof.** By the assumptions on \( \Psi \), we have

\[
\|\Phi\|_{L^\infty(S^n_1)} = \int_{\mathbb{T}} \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mu(\zeta) \leq \sigma_k(\Phi).
\]

On the other hand,

\[
\sigma_k(\Phi) \leq \text{dist}_{L^\infty(S^n_1)}(\Phi, H^\infty) \leq \|\Phi\|_{L^\infty(S^n_1)}
\]

always holds. Since all the previously mentioned inequalities are equalities, the conclusion follows. \( \Box \)
Theorem 4.10. Let $\Phi \in L^\infty(M_n)$ be an admissible very badly approximable matrix function. Suppose there is $\Psi \in A_n^k$ such that

$$\int_T \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mu(\zeta) = t_0(\Phi) + \ldots + t_n(\Phi).$$

If $t_{k-1}(\Phi) > 0$, then $\Phi$ has order $k$ and the zero matrix function is a best approximant to $\Phi$ under the $L^\infty(S_n^1)$-norm.

Proof. By the remarks preceding Corollary 4.8, it is easy to see that $\|\Phi\|_{L^\infty(S_n^1)} = t_0(\Phi) + \ldots + t_n(\Phi)$. It follows from Lemma 4.9 that $\Psi$ is a $k$-extremal function for $\Phi$, $\sigma_k(\Phi) = \sigma_n(\Phi)$, and the zero matrix function is a best approximant to $\Phi$ under the $L^\infty(S_n^1)$-norm. Thus $\|\Phi\|_{L^\infty(S_n^1)} = \sigma_k(\Phi)$. Moreover, by Theorem 1.3

$$\sigma_{k-1}(\Phi) \leq t_0(\Phi) + \ldots + t_{k-2}(\Phi) < t_0(\Phi) + \ldots + t_{k-1}(\Phi) \leq \|\Phi\|_{L^\infty(S_n^1)}.$$

Therefore $\sigma_{k-1}(\Phi) < \sigma_k(\Phi)$. \[\square\]

Remark 4.11. Notice that under the hypotheses of Theorem 4.10, one also obtains that $t_{k-1}(\Phi)$ is the smallest non-zero superoptimal singular value of $\Phi$. This is an immediate consequence of Corollary 4.8.

We now formulate the corresponding result for admissible very badly approximable unitary-valued matrix functions. These functions are considered in greater detail in Section 5.

Corollary 4.12. Let $U \in L^\infty(M_n)$ be an admissible very badly approximable unitary-valued matrix function. If there is $\Psi \in A_n^k$ such that

$$\int_T \text{trace}(U(\zeta)\Psi(\zeta))d\mu(\zeta) = n,$$

then $U$ has order $n$ and the zero matrix function is a best approximant to $U$ under the $L^\infty(S_n^1)$-norm.

Proof. This is a trivial consequence of Theorem 4.10 and the fact that $t_j(U) = 1$ for $0 \leq j \leq n - 1$. \[\square\]

We are now ready to state the main result of this section.

Theorem 4.13. Let $\Phi$ be an admissible very badly approximable $n \times n$ matrix function. The following statements are equivalent:

1. $k$ is the smallest number for which there exists $\Psi \in A_n^k$ such that

$$\int_T \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mu(\zeta) = t_0(\Phi) + \ldots + t_{n-1}(\Phi);$$

2. $\Phi$ has order $k$, $t_j(\Phi) = 0$ for $j \geq k$ and

$$\sigma_k(\Phi) = t_0(\Phi) + \ldots + t_{k-1}(\Phi).$$
Proof. Let
\[
\kappa(\Phi) \overset{\text{def}}{=} \inf \{ j \geq 0 : \text{there exists a } \Psi \in \mathcal{A}_j^{\phi} \text{ such that } \\
\int_{\mathbb{T}} \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mu(\zeta) = t_0(\Phi) + \ldots + t_{n-1}(\Phi) \}
\]
Clearly, \(\kappa(\Phi)\) may be infinite for arbitrary \(\Phi\).

Suppose \(\kappa = \kappa(\Phi)\) is finite. Then Lemma 4.9 implies that \(\Phi\) has a \(\kappa\)-extremal function, \(\sigma_{\kappa}(\Phi) = \sigma_{\kappa}(\Phi)\), and the zero matrix function is a best approximant to \(\Phi\) under the \(L^\infty(S_1^n)\)-norm. In particular, \(\Phi\) has order \(k \leq \kappa(\Phi)\), \(t_j(\Phi) = 0\) for \(j \geq k\), and
\[
\sigma_k(\Phi) = t_0(\Phi) + \ldots + t_{k-1}(\Phi),
\]
by Corollary 4.18.

On the other hand, if \(\Phi\) has order \(k\), \(t_j(\Phi) = 0\) for \(j \geq k\), and
\[
\sigma_k(\Phi) = t_0(\Phi) + \ldots + t_{k-1}(\Phi),
\]
then \(\Phi\) has a \(k\)-extremal function \(\Psi \in \mathcal{A}_k^{\phi}\) such that
\[
\int_{\mathbb{T}} \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mu(\zeta) = \sigma_k(\Phi) = t_0(\Phi) + \ldots + t_{k-1}(\Phi).
\]
Since \(t_j(\Phi) = 0\) for \(j \geq k\), it follows that
\[
\int_{\mathbb{T}} \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mu(\zeta) = t_0(\Phi) + \ldots + t_{n-1}(\Phi).
\]
Thus \(\kappa(\Phi) \leq k\).

Hence, if either \(\kappa(\Phi)\) is finite or \(\Phi\) satisfies 2, then \(k = \kappa(\Phi)\). \(\square\)

We end this section by illustrating existence of very badly approximable matrix functions of order \(k\) by giving two simple examples; a \(2 \times 2\) matrix function of order 2 and a \(3 \times 3\) matrix function of order 2.

**Example 4.14.** Let
\[
\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
1 & 1 \\
\end{pmatrix} \begin{pmatrix}
z^2 & \circ \\
\circ & \bar{z} \\
\end{pmatrix}.
\]
It is easy to see that \(\Phi\) is a continuous (and hence admissible) unitary-valued very badly approximable matrix function with superoptimal singular values \(t_0(\Phi) = t_1(\Phi) = 1\). We claim that \(\Phi\) has order 2. Indeed, the matrix function
\[
\Psi = \begin{pmatrix}
z^2 & \circ \\
\circ & z \\
\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
-1 & 1 \\
\end{pmatrix}
\]
satisfies
\[
\int_{\mathbb{T}} \text{trace}(\Phi(\zeta)\Psi(\zeta))d\mu(\zeta) = 2,
\]
and so \(\Phi\) has order 2 by Corollary 4.12.
Example 4.15. Let $t_0$ and $t_1$ be two positive numbers satisfying $t_0 \geq t_1$. Let

$$\Phi = \begin{pmatrix} t_0 z^a & \circ & \circ \\ \circ & t_1 z^b & \circ \\ \circ & \circ & \circ \end{pmatrix}$$

where $a$ and $b$ are positive integers. It is easy to see that $\Phi$ is a continuous (and hence admissible) very badly approximable matrix function with superoptimal singular values $t_0(\Phi) = t_0$, $t_1(\Phi) = t_1$, and $t_2(\Phi) = 0$. Again, we have that $\Phi$ has order 2. After all, the matrix function

$$\Psi = \begin{pmatrix} z^a & \circ & \circ \\ \circ & z^b & \circ \\ \circ & \circ & \circ \end{pmatrix}$$

satisfies

$$\int_T \text{trace}(\Phi(\zeta)\Psi(\zeta))d\nu(\zeta) = t_0 + t_1 = t_0(\Phi) + t_1(\Phi) + t_2(\Phi),$$

and so $\Phi$ has order 2 by Theorem 4.10, since $t_1(\Phi) = t_1 > 0$.

5. **Unitary-valued very badly approximable matrix functions**

We lastly consider the class $U_n$ of admissible very badly approximable unitary-valued matrix functions of size $n \times n$ and provide a representation of any $n$-extremal function $\Psi$ for a function $U \in U_n$ such that

$$\int_T \text{trace}(U(\zeta)\Psi(\zeta))d\nu(\zeta) = t_0(U) + \ldots + t_{n-1}(U) \quad (5.1)$$

holds. Note that for any such $U$ we have that $t_j(U) = 1$ for $0 \leq j \leq n - 1$.

When studying functions in $U_n$, it turns out that Toeplitz operators on Hardy spaces are quite useful. For a matrix function $\Phi \in L^\infty(M_{m,n})$, we define the Toeplitz operator $T_\Phi$ by

$$T_\Phi f = P_+ \Phi f, \quad f \in H^2(\mathbb{C}^n),$$

where $P_+$ denotes the orthogonal projection from $L^2(\mathbb{C}^n)$ onto $H^2(\mathbb{C}^n)$.

It is well-known that, for any function $U \in U_n$, the Toeplitz operator $T_U$ is Fredholm and $\text{ind} T_U > 0$. (As usual, for a Fredholm operator $T$, its index, $\text{ind} T$, is defined by $\text{dim} \ker T - \text{dim} \ker T^*.$) In particular, the Toeplitz operator $T_{\det U}$ is Fredholm and

$$\text{ind} T_{\det U} = \text{ind} T_U.$$

This latter fact can be easily deduced by considering any thematic factorization of $U$. We refer the reader to Chapter 14 in [Pe1] for more information concerning functions in $U_n$ and thematic factorizations.

In order to state the main result of this section, we first discuss the notion of Blaschke-Potapov products. A matrix function $B \in H^\infty(M_n)$ is called a finite Blaschke-Potapov product if it admits a factorization of the form

$$B = UB_1 B_2 \ldots B_m,$$
where $U$ is a unitary matrix and, for each $1 \leq j \leq m$,

$$B_j = \frac{z - \lambda_j}{1 - \lambda_jz} P_j + (I - P_j)$$

for some $\lambda_j \in \mathbb{D}$ and orthogonal projection $P_j$ on $\mathbb{C}^n$. The degree of the Blaschke-Potapov product $B$ is defined to be

$$\deg B \equiv \sum_{j=1}^m \text{rank } P_j.$$

It turns out that every invariant subspace $\mathcal{L}$ of multiplication by $z$ on $H^2(\mathbb{C}^n)$ of finite codimension is of the form $\mathcal{B}H^2(\mathbb{C}^n)$ for some Blaschke-Potapov product of finite degree $\text{codim } \mathcal{L}$. A proof of this fact may be found in Lemma 2.5.1 of [Pe1].

We now state the main result.

**Theorem 5.1.** Suppose $U \in \mathcal{U}_n$ has an $n$-extremal function $\Psi$ such that (5.1) holds. Then $\Psi$ admits a representation of the form

$$\Psi = zh^2\Theta,$$

where $h \in H^2$ is an outer function such that $\|h\|_{L^2} = 1$ and $\Theta$ is a finite Blaschke-Potapov product. Moreover, the scalar functions $\det(U\Theta)$ and $\text{trace}(U\Theta)$ are admissible badly approximable functions that admit the factorizations

$$\det(U\Theta) = \bar{z}^n\bar{h}^n \quad \text{and} \quad \text{trace}(U\Theta) = n\bar{z}\bar{h}.$$

**Proof.** It follows from (5.1) that all inequalities in (1.3) are equalities and so

$$\text{trace}(U(\zeta)\Psi(\zeta)) = \|U(\zeta)\Psi(\zeta)\|_{S_n^1} = n\|\Psi(\zeta)\|_{M_n}$$

holds for a.e. $\zeta \in \mathbb{T}$. Since $U$ is unitary-valued, then

$$\|U(\zeta)\Psi(\zeta)\|_{S_n^1} = \|\Psi(\zeta)\|_{S_n^1},$$

and so

$$\|\Psi(\zeta)\|_{S_n^1} = n\|\Psi(\zeta)\|_{M_n}$$

must hold for a.e. $\zeta \in \mathbb{T}$. Therefore

$$s_j(\Psi(\zeta)) = \|\Psi(\zeta)\|_{M_n} \quad \text{for a.e. } \zeta \in \mathbb{T}, \; 0 \leq j \leq n - 1.$$

By the Singular Value Decomposition Theorem for matrices (or, more generally, the Schmidt Decomposition Theorem), it follows that

$$\Psi(\zeta) = \|\Psi(\zeta)\|_{M_n} V(\zeta)$$

for some unitary-valued matrix function $V$. Let $h \in H^2$ be an outer function such that

$$|h(\zeta)| = \|\Psi(\zeta)\|_{M_n}^{1/2} \text{ on } \mathbb{T}.$$

Consider also the matrix function $\Xi \equiv h^{-2}\Psi$. It follows from (5.3) that

$$(\Xi^*\Xi)(\zeta) = \frac{1}{|h(\zeta)|^2} (\Psi^*\Psi)(\zeta) = I_n \text{ for a.e. } \zeta \in \mathbb{T},$$

and so

$$\Xi(\zeta) = \|\Xi(\zeta)\|_{M_n} V(\zeta)$$

for some unitary-valued matrix function $V$. Let $h \in H^2$ be an outer function such that

$$|h(\zeta)| = \|\Xi(\zeta)\|_{M_n}^{1/2} \text{ on } \mathbb{T}.$$
and so $\Xi$ is an inner function. Thus $\Psi$ admits the factorization
$$\Psi = zh^2\Theta$$
for some $n \times n$ unitary-valued inner function $\Theta$ and an outer function $h \in H^2$ such that $\|h\|_{L^2} = 1$.

Note that the first equality in (5.2) indicates that the scalar function $\varphi \overset{\text{def}}{=} \text{trace}(U\Theta)$ satisfies
$$zh^2\varphi = n|h|^2$$
on $T$, or equivalently
$$\varphi = n\bar{z}\frac{h}{h}.$$  
Moreover, $\|H_U\Theta\|_e \leq \|H_U\|_e < 1$, hence $\|H_\varphi\|_e < n = \|H_\varphi\|$ implying that $\varphi$ is an admissible badly approximable scalar function on $T$. We conclude that the Toeplitz operator $T_\varphi$ is Fredholm and $\text{ind} T_\varphi > 0$ by the following well-known fact (c.f. Theorem 7.5.5 in [Pe1].)

**Fact.** Let $\varphi \in L^\infty$ be admissible. Then $\varphi$ is badly approximable (i.e. the zero scalar function is a best approximant) if and only if $\varphi$ has constant modulus, the Toeplitz operator $T_\varphi$ is Fredholm, and $\text{ind} T_\varphi > 0$.

Returning to (5.2), it also follows that each eigenvalue of $U(\zeta)\Psi(\zeta)$ equals $\|\Psi(\zeta)\|_{\mathcal{M}_n} = |h(\zeta)|^2$ for a.e. $\zeta \in T$. In particular,
$$|h(\zeta)|^{2n} = \det U(\zeta)\Psi(\zeta) = (z^nh^{2n})(\zeta) \cdot \det U(\zeta) \cdot \det \Theta(\zeta)$$
holds a.e. $\zeta \in T$. By setting
$$\theta \overset{\text{def}}{=} \det \Theta \quad \text{and} \quad u \overset{\text{def}}{=} \det U,$$
we have that $u$ admits the factorization
$$u = \bar{\theta}z^n\frac{h^n}{h} = \bar{\theta}\omega^n,$$
where $\omega \overset{\text{def}}{=} \bar{z}h/h = \varphi/n$. Since the Toeplitz operator $T_\omega$ is Fredholm with positive index, $T_{\omega^n}$ is Fredholm as well. Since $\ker T_\theta = \{0\}$ and $u\omega^n = \theta$, then
$$\dim(H^2 \ominus \theta H^2) = \dim \ker T_\theta^* = \dim \ker T_\bar{\theta} = \text{ind} T_\bar{\theta} < \infty$$
and so $\theta$ is a finite Blaschke product. The conclusion follows from the well-known lemma stated below. \hfill $\Box$

**Lemma 5.2.** If $\Theta$ is a unitary-valued inner function such that $\det \Theta$ is a finite Blaschke product, then $\Theta$ is a Blaschke-Potapov product.

**Proof.** Let $\theta = \det \Theta$. It is easy to see that $\Theta^*\theta$ is an inner function. Since $B \overset{\text{def}}{=} \theta I_n$ is a finite Blaschke-Potapov product and $BH^2(C^n) \subset \Theta H^2(C^n)$, then $\Theta H^2(C^n)$ has finite codimension, and so $\Theta$ must be a finite Blaschke-Potapov product. \hfill $\Box$
Corollary 5.3. Suppose $U \in U_2$ has a 2-extremal function $\Psi$ such that (5.1) holds. If $U$ is a rational matrix function such that $\text{ind} T_U = 2$, then $\Theta$ is a unitary constant on $\mathbb{T}$.

Proof. Due to the results of [PY], $U$ admits a (thematic) factorization of the form

$$U = \begin{pmatrix} \bar{w}_1 & -w_2 \\ \bar{w}_2 & w_1 \end{pmatrix} \begin{pmatrix} u_0 & \Theta \\ \Theta^* & u_1 \end{pmatrix} \begin{pmatrix} \bar{v}_1 & \bar{v}_2 \\ -v_2 & v_1 \end{pmatrix},$$

where $v_1, v_2, w_1$ and $w_2$ are scalar rational functions such that

$$|v_1|^2 + |v_2|^2 = |w_1|^2 + |w_2|^2 = 1 \text{ a.e. on } \mathbb{T},$$

$v_1$ and $v_2$ have no common zeros in the unit disk $\mathbb{D}$, $w_1$ and $w_2$ have no common zeros in $\mathbb{D}$, and $u_0$ and $u_1$ are scalar badly approximable rational unimodular functions on $\mathbb{T}$. These results may also be found in Sections 5 and 12 from Chapter 14 of [Pe1].

Suppose $\Psi = zh^2\Theta$ is an $n$-extremal function for $U$ such that (5.1) holds as in the conclusion of Theorem 5.1. Assume, for the sake of contradiction, that $\Theta$ is not a unitary constant.

Since $u_j$ is a scalar badly approximable rational unimodular function on $\mathbb{T}$, it admits a factorization of the form

$$u_j = c_j z^{k_j} \frac{\bar{h}_j}{h_j},$$

where $c_j$ is a unimodular constant, the function $h_j$ is $H^\infty$-invertible, and $k_j = \text{ind} T_{u_j}$, for $j = 0, 1$. In particular, we have

$$u \theta = c_0 c_1 z^2 \frac{\bar{h}_0 h_1}{h_0 h_1},$$

as $k_0 + k_1 = \text{ind} T_U = 2$, where $\theta \overset{\text{def}}{=} \det \Theta$ and $u \overset{\text{def}}{=} \det U$.

On the other hand, by Theorem 5.1

$$u \theta = z^2 \frac{\bar{h}^2}{h^2},$$

and so the function $h^2 h_0^{-1} h_1^{-1}$ and its conjugate

$$\frac{\bar{h}^2}{h_0 h_1} = c_0 c_1 \theta \frac{h^2}{h_0 h_1}$$

belong to $H^1$. Therefore $h^2 h_0^{-1} h_1^{-1}$ equals a constant and so $\theta$ equals a constant as well. Thus, the conclusion follows from the fact that $\theta \Theta^*$ is an inner function. □

We end this section with an example to illustrate some of our main results.

Example 5.4. Consider the matrix function

$$U = \begin{pmatrix} \bar{z} & \Theta \\ \Theta^* & z \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \bar{z} \\ -z & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z} & z^2 \\ -1 & \bar{z} \end{pmatrix}.$$
Clearly, $U$ belongs to $\mathcal{U}_2$ and it has superoptimal singular values $t_0(U) = t_1(U) = 1$.

We ask the question, *is there a 2-extremal function $\Psi$ for $U$ such that (5.1) holds with $n = 2$?* Let us assume for the moment that such a function $\Psi$ exists. In this case, Corollary 5.3 implies that $\Psi$ must be of the form

$$\Psi = z h^2 \Theta,$$

where

$$\Theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a unitary constant and $h$ is an outer function in $H^2$ such that $\|h\|_{L^2} = 1$. Since

$$\frac{z^2 \bar{h}^2}{h^2} = \text{det}(U\Theta) = z^2(ad - bc),$$

it is easy to see that $h^2$ and its conjugate belong to $H^1$, and so $h^2$ is a constant of modulus 1. Relabeling the scalars $a, b, c,$ and $d$, we may assume that $h^2$ equals 1 a.e. on $\mathbb{T}$. Thus,

$$2\zeta = \text{trace}(U(\zeta)\Theta(\zeta)) = \frac{1}{\sqrt{2}} (a\zeta + c\bar{\zeta}^2 - b + d\zeta)$$

holds for a.e. $\zeta \in \mathbb{T}$, and so $b = c = 0$ and $a + d = 2\sqrt{2}$. However, $\Theta$ is unitary valued so it must be the case that $|a| = |d| = 1$, and so

$$2\sqrt{2} = a + d = |a + d| \leq |a| + |d| = 2,$$

which is a contradiction. Thus no such $\Psi$ exists. In particular, we must have that $\Phi$ does not have order 2 or $\sigma_2(\Phi) < t_0(\Phi) + t_1(\Phi) = 2$ by Theorem 4.13.

Actually, we have already shown that *the zero matrix function is not a best approximant to $U$ under the $L^\infty(\mathcal{S}_1^2)$ norm*, i.e. $\sigma_2(\Phi) < 2$. Indeed, we have

$$\text{dist}_{L^\infty(\mathcal{S}_1^2)}(U, H^\infty(\mathcal{M}_2)) < t_0(U) + t_1(U) = \|U\|_{L^\infty(\mathcal{S}_1^2)},$$

by (1.7).

We now ask, *does $U$ have order 1, order 2, or is $U$ inaccessible?* It is clear that $U$ has a 1-extremal function by Remark 3.7. In fact, it is easy to check that the matrix function

$$\Psi_1 = \frac{z}{\sqrt{2}} \begin{pmatrix} 1 & \Theta \\ \Theta & \Theta \end{pmatrix}$$

defines a 1-extremal function for $U$ and

$$\sigma_1(U) = \int_T \text{trace}(U(\zeta)\Psi_1(\zeta)) dm(\zeta) = \|H_U\| = t_0(U) = 1.$$

However, $U$ does *not have order 1*. Indeed, one can see that the matrix function

$$\Psi_* = \frac{z}{\sqrt{3}} \begin{pmatrix} 1 & \Theta \\ \Theta & 1 \end{pmatrix}$$

defines a 1-extremal function for $U$ and

$$\sigma_1(U) = \int_T \text{trace}(U(\zeta)\Psi_1(\zeta)) dm(\zeta) = \|H_U\| = t_0(U) = 1.$$
belongs to $H^1_0(M_2)$, $\|\Psi_s\|_{L^1(M_2)} \leq 1$, and

$$1 < \sqrt{\frac{3}{2}} = \int_T \text{trace}(U(\zeta)\Psi(\zeta))\,dm(\zeta) \leq \sigma_2(U).$$

Therefore, either $U$ has order 2 or $U$ is inaccessible. This matter requires further investigation.

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