Dynamical Chiral Symmetry Breaking from Variationally Improved Perturbative Expansion

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We review how a specific resummation of the so-called “delta-expansion”, applied to the QCD Lagrangian, transforms the ordinary perturbative expansion in $\alpha_s$ into an expansion in an arbitrary mass parameter, around the basic scale $\bar{\Lambda}$. When applied to the pole mass, the resulting expression may be interpreted as a dynamical mass ansatz, to be optimized with respect to the new expansion (mass) parameter. The construction is generalized to obtain estimates of the order parameters of the $SU(n_f)_L \times SU(n_f)_R$ ($n_f = 2, 3$) symmetry breakdown.

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1. Introduction

There is a common lore that a “first principle” determination of the order parameters characterizing the (chiral) dynamical symmetry breaking (DSB), such as the $\langle \bar{q}q \rangle$ condensate typically, is definitively out of the reach of the basic QCD perturbation theory. This is largely justified, traditionally, by the fact that DSB is an essentially non-perturbative mechanism. However, it may depend on what exactly one means by perturbation theory. For instance, since the pioneering work of Nambu and Jona-Lasinio (NJL) [1], it has been understood how it is possible to resum a relevant class of graphs to obtain the qualitative (and some quantitative as well) properties of DSB explicitly, at least in specific approximations and/or models. Also, independently of the NJL idea, the modern Chiral Perturbation Theory (ChPT) [2] gives a consistent effective description of data at low energies where the QCD perturbative series is not applicable. Indeed, definite progress have been made to relate ChPT with generalized NJL models [3], although a precise connection between the (numerous) ChPT parameters, and the basic QCD coupling and quark mass parameters is far from being resolved at present. With a more formal (but related) motivation, it has been also explored since long ago how definite non-perturbative informations may be inferred from appropriately modified perturbation series [4], at least in simplified or exactly solvable models. In particular, the convergence of ordinary perturbation can be systematically improved by a variational-like procedure, in which the separation of the action into “free” and “interaction” parts is made to depend on a set of auxiliary parameters, to be fixed by some optimization procedure [1].

As a partial attempt to merge some of these ideas, we have re-examined [5,6] with a new approach the above mentioned old problem of generating from the basic QCD Lagrangian non-trivial values for the quark condensate, pion decay constant, or dynamical quark masses. The basic point is to transform the ordinary perturbative expansion, in $\alpha_s$, into an expansion in an arbitrary mass parameter, around a non-trivial (fixed-point) solution of the renormalization group evolution, proportional to the basic scale $\bar{\Lambda}$. In some sense it may be viewed as a systematic, order by order, improvement of the original NJL construction, but with a consistent treatment of the renormalization (and directly applied to the QCD quark-gluon interactions).

2. A crude dynamical mass ansatz

As a crude first illustration of the mechanism, consider the renormalization group (RG) evolu-

\footnote{In $D = 1$ field theories, this optimized perturbation theory (“delta-expansion”) gives a rigorously convergent series of approximations, even in strong coupling cases.}
tion of the running mass,

\[ m(\mu') = m(\mu) \exp\left\{ -\int_{\mu}^{\mu'} \frac{dg(\mu)}{\beta(g)} \gamma_m(g) \right\}, \quad (1) \]

where \( \beta(g) \), \( \gamma_m(g) \) drive the running of the coupling \( g(\mu) \) and mass \( m(\mu) \), respectively. Solving (1) for the “fixed point” boundary condition: \( M \equiv m(M) \), gives (to first RG order)

\[ M_1 = \frac{m(\mu)}{\left[1 + 2b_0g^2(\mu)\ln(\frac{M_1}{\mu})\right]^{\gamma_0}}, \quad (2) \]

where \( b_0 \), \( \gamma_0 \) are the one-loop RG-coefficients (normalization is such that \( b_0 = -b_0g^3 - b_2g^5 - \cdots \), \( \gamma_m(g) = \gamma_0g^2 + \gamma_1g^4 + \cdots \)).

Although expression (2) is initially related to the “current” mass \( m(\mu) \) via (1), it has the trademarks of a pole mass, thanks to the boundary condition \( M_1 \equiv m(M_1) \). This coincidence between the pole mass \( M \) and the current mass \( m(\mu = M) \), is, however, only an artifact of our crude approximation, neglecting at the moment the non-logarithmic perturbative corrections.

Now, the most important property of expression (2) is that it is non-zero in the chiral limit, \( m(\mu) \rightarrow 0 \). Indeed, (2) identically reads

\[ M_1\left(\ln\left(M_1/\bar{\Lambda}\right)\right)^{\gamma_0} = \hat{m}, \quad (3) \]

where for convenience we introduced the RG invariant scale \( \bar{\Lambda} = \hat{\mu} e^{-\frac{1}{\gamma_0}} \) (at first RG order), and the scale-invariant mass \( \hat{m} \equiv m(\mu)(2b_0g^2(\mu))^{-\gamma_0} \). May then be seen as a function \( \hat{m}(M_1) \), and requiring its inverse, \( M_1(\hat{m}) \), to be defined on the whole physical domain \( 0 < \hat{m} < \infty \), and to match the ordinary perturbative asymptotic behavior for \( \hat{m} \rightarrow \infty \), implies \( M_1(\hat{m} \rightarrow 0) \rightarrow \bar{\Lambda} \). It is of course desirable to go beyond the one-loop RG approximation, and to take into account as well the non-logarithmic corrections, necessary to make contact with the usual perturbative pole mass.

\[ \text{Our aim is to obtain a variational “mass gap” where the non-trivial chiral limit property of } 3 \text{ is preserved, while at the same time providing us with a systematically (order by order) improvable ansatz, thanks to a particular reorganization of the basic perturbative expansion, as will be explained in the next section.} \]

3. Resumming the delta-expansion

In the present context, a simplest form of the so-called delta-expansion \( 4 \) may be defined by formally substituting everywhere in the bare QCD Lagrangian:

\[ m_0 \rightarrow m_0(1 - x); \quad g_0 \rightarrow g_0(x)^{1/2}. \quad (4) \]

The parameter \( x \) in \( 4 \) just interpolates between the free Lagrangian, for \( x = 0 \), and the interacting but massless Lagrangian, for \( x = 1 \). In the simplest field-theoretical applications, one would then use \( 4 \) to expand any perturbative expression of \( (m_0, g_0^2) \) to a given order \( x^q \), and try to apply some optimization prescription with respect to the (arbitrary) mass, \( m_0 \). Accordingly, the somewhat empirical but most often successful idea \( 4 \) is that the least sensitive region with respect to \( m_0 \) (entering at any fixed order \( q \)) should give the best approximation to the exact result, which is independent of \( m_0 \). But, in many non-trivial field theories, and in particular in the present QCD case, before anything the whole procedure should be made consistent with renormalization. As it turns out, the only way to get a finite and non-zero result (e.g., \( M(\hat{m} \rightarrow 0) \neq 0 \)) is to resum the \( x \)-series, using an appropriately constructed contour integral transform. At first RG order, this essentially gives a mass as an integral over expression (2) (with substitution (3) understood). Beyond the one-loop approximation, our final mass ansatz reads:

\[ \frac{M_2'(\hat{m}'')}{\bar{\Lambda}} = \frac{2 - C}{2\pi i} \int dv e^v \frac{e^v}{F^A(v)[C + F(v)]^2} \left(1 + \frac{M_1}{F(v)} + \frac{M_2}{F^2(v)} + \cdots \right), \quad (5) \]

\[ 4 \text{ in } (5) \text{ is related to the original expansion parameter } x \text{ as } x = 1 - v/q, q \text{ being the order of the } x \text{-expansion.} \]
where the contour is around the $]-\infty, 0]$ axis; 
$$F(v) \equiv \ln[m_0''v] - A \ln F - (B - C) \ln(C + F), \quad (6)$$
with $A = \gamma_1/(2b_1)$, $B = \gamma_0/(2b_0) - \gamma_1/(2b_1)$, $C = b_1/(2b_0^2)$; $\Lambda$ is the (RG-invariant) scale at two-loop order; and finally
$$m'' \equiv \frac{(m(\bar{\mu})}{\Lambda}) 2^C [2b_0\bar{g}^2]^{-\frac{b_1}{b_0}} \left[1 + \frac{b_1}{b_0} \bar{g}^2\right]^B \quad (7)$$
is the scale-invariant, arbitrary (dimensionless) “mass” parameter. By construction, $F(1)$ in the integrand of (6) resums the leading and next-to-leading logarithmic dependence in $m(\bar{\mu})$ to all orders (4). The non-logarithmic perturbative coefficients, $M_1 \equiv (2/3)(\gamma_0/2b_0)$ and $M_2$, connect the pole mass with the running mass $m(M)$.

Note that it is implicitly always possible to choose a renormalization scheme (RS) such that $b_i = \gamma_i = 0$ for $i \geq 2$, since, by then, $\gamma_i$ are then RS-dependent. In that sense, eq. (6) resums the full RG dependence in $\ln(m''v)$. In contrast, the purely perturbative (non-logarithmic) information, contained in $M_1$, $M_2$, is limited by present knowledge to two-loop order. This is where the variational principle and optimization play their role, whereby we hope to obtain a sensible approximation to the true dynamical mass. Observe first that, were we in a simplified theory where $M_1 = M_2 = \cdots = 0$, (4) would have a very simple behaviour near its optimum (located at $m'' \to 0$), giving a simple pole with residue $M_2 = (2C)^{-C} \Lambda$. Now, in the more realistic cases, $M_1$, $M_2$,... cannot be neglected, but we can obtain a series of approximants to the dynamical mass, by expanding (4) in successive powers of $m''v$, using the standard relation
$$\frac{1}{2\pi} \int dv \, e^v \, v^\alpha = \frac{1}{\Gamma[-\alpha]}, \quad (8)$$
and then looking for optima $m''_\text{opt}(m''_\text{opt})$, $m''_\text{opt} \neq 0$. The previous construction is quite general and therefore directly applicable to any (asymptotically free) model, taking obviously its appropriate values of the RG coefficients, $b_i$, $\gamma_i$. The ansatz (4) was confronted (3) to the exactly known mass gap (1) for the $O(N)$ Gross-Neveu (GN) model, for arbitrary $N$. The results of different optimization prescriptions gave estimates with errors of $\mathcal{O}(5\%)$ or less, depending on $N$ values (9).

It is important to note that expression (3), for arbitrary $N$ in the GN model, uses exactly the same amount of (perturbative plus RG) information than the one at disposal at present for a QCD quark mass: namely, the exact two-loop RG-resumed plus perturbative $M_1$, $M_2$ dependence. Since our construction essentially relies on RG-properties (and analytic continuation), going from 2 to 4 dimensions is not expected to cause major changes, at least naively.

4. Hidden singularities of the mass ansatz

One complication, actually, does occur: as a more careful examination of relation (3) indicates, there are branch cuts in the $v$ plane, with Re$[v_{\text{cut}}] > 0$ for the relevant case of $N_f = 2$ or 3 in QCD. These make the expansion undefined when approaching the origin, $v = 0$, and simply indicate the non single-valuedness of (3) below those branch points. The origin of those singularities has some similarity with the renormalon ones (1), as they also appear when extrapolating a RG-resummed expression down to an infrared scale $m'' \simeq 0$. However, a main difference with renormalons is that in our construction it is possible (4) to move those extra cuts to a safe location, Re$[v_{\text{cut}}] \leq 0$, observing that the actual position of those cuts depends, at second order, on the RS, via $\gamma_1$. Performing thus a second-order perturbative RS change in $m(\bar{\mu})$, $g(\mu)$, which changes $\gamma_1(M_S)$ to a (singularity-safe) $\gamma_1'$, it is then sensible, in the present context, to invoke a variant of the “principle of minimal sensitivity” (PMS) (2), requiring a flat optimum (plateau) of (3) with respect to the remnant RS arbitrariness (4).

One may perhaps legitimately wonder why the ordinary renormalon singularities of the pole mass (2) do not seem to appear in our construction. In fact, the usual renormalon singularities always appear as a result of crossing the Landau pole (4) which simply reflects an ambiguity from

5Actually, this is an oversimplified picture, valid at one-loop RG level only (4). However, higher order properties of renormalons do not affect, qualitatively, our argument.
perturbation theory, calling for non-perturbative corrections which are typically in the form of power corrections \( \frac{1}{n!} \). In contrast, (3) is such that the Landau pole (corresponding to \( F = 0 \) in our language) is not crossed, but only smoothly reached from above, \( \text{Re}F > 0 \). (Moreover, due to the recurrent dependence in \( F \), (3), implying that \( F(v) \approx m' v \) for \( m' v \rightarrow 0 \), the poles of (3) at \( F = 0 (v = 0) \) entirely come from the purely perturbative part, i.e. due to \( M_1, M_2 \neq 0 \).)

Note that, on more phenomenological grounds, there is no strong contradiction with the usual consequences of the presence of renormalons: while the latter indicate, in the pole mass case, an arbitrary \( F \) to the recurrent dependence in (5) for \( \gamma \) dependence, via the above mentioned \( \gamma_1 \) coefficient, calling for optimization. Practically we have obtained:

\[
M_{\text{opt}}^2(m''_{\text{opt}} \rightarrow 0) \approx 2.97 \bar{\Lambda}(2) \tag{9}
\]

for \( n_f = 2 \), and a similar result for \( n_f = 3 \).

5. Order parameters: \( F \) and \( \langle \bar{q}q \rangle \)

The previous dynamical quark mass, although it has some meaning as regards DSB in QCD, hardly has a direct physical interpretation, e.g. as a pole of the S-matrix, due to the confinement. In other words, it is not a properly defined order parameter. It is however possible to apply the same construction as the one leading to (3), to obtain a determination of the ratios \( F_\pi/\bar{\Lambda} \) and \( \langle \bar{q}q \rangle/\mu/\bar{\Lambda}^2 \). The latter gauge-invariant quantities are unambiguous order parameters, i.e. \( F_\pi \neq 0 \) or \( \langle \bar{q}q \rangle \neq 0 \) imply DSB. The appropriate generalization of (7) for \( F_\pi \) is

\[
\frac{F_\pi^2}{\bar{\Lambda}^2} = (2\hbar_0) \frac{2C(m'')^2}{2\pi} \left( \frac{dv}{u} \right) v^2 e^v \frac{1}{F^{2\bar{\Lambda} - 1}[C + F]^{2\bar{\Sigma}}} \bar{\Sigma} \left( 1 + \frac{\alpha_\pi}{F} + \frac{\beta_\pi}{F^2} \right) \tag{10}
\]

in terms of the same \( F(v) \) defined in eq. (3) (therefore leading to the same extra cut locations as in the mass case), and where \( \delta_\pi, \alpha_\pi \) and \( \beta_\pi \) are fixed by matching the perturbative \( MS \) expansion, known to 3-loop order \([13]\). A numerical optimization with respect to the RS-dependence, in a way similar to the mass case, gives e.g for \( n_f = 2 \):

\[
F_{\pi,\text{opt}}(m''_{\text{opt}} \rightarrow 0) \approx 0.55 \bar{\Lambda}(2) \tag{11}
\]

Concerning \( \langle \bar{q}q \rangle \), an ansatz similar to (10) can be derived (with coefficients \( \delta, \alpha, \beta \) specific to \( \langle \bar{q}q \rangle \) and appropriate changes in the \( m'' \), \( F \) and \( v \) powers), but for the RG-invariant combination \( m(\bar{q}q) \), due to the fact that our construction only apply to RG-invariant quantities. To extract an estimate of the (scale-dependent) condensate \( \langle \bar{q}q \rangle(\mu) \) is only possible by introducing an explicit symmetry-breaking quark mass \( m_{\text{exp}} \) (i.e. \( m_{\text{exp}} \neq m \)), and expanding the \( m(\bar{q}q) \) ansatz to first order in \( m_{\text{exp}} \). This gives for \( n_f = 2 \) \([7]\):

\[
\langle \bar{q}q \rangle^{1/3}(\mu = 1 \text{ GeV}) \approx 0.52 \bar{\Lambda}(2) \tag{12}
\]

Confronting (3), (11) and (12) gives a fairly small value of the quark condensate \( \langle \bar{q}q \rangle \) (and a fairly high value of the dynamical mass), as compared to other non-perturbative determinations \([15]\). Although small values of \( \langle \bar{q}q \rangle \) are not experimentally excluded at present \([14]\), it is also clear that our relatively crude approximation deserves more refinements for more realistic QCD predictions.

6. Conclusion and discussion

The variationally improved expansion in arbitrary \( m'' \), first developed in the GN model \([3]\), has been formally extended to the QCD case. It gives non-trivial relationships between \( \bar{\Lambda} \) and the dynamical masses and order parameters, \( F_\pi \) and \( \langle \bar{q}q \rangle \).

To make progress, what is certainly restrictive is the relatively poor knowledge of the purely perturbative part of the expansion (only known to two-loop order in most realistic field theories). Accordingly, our final numerical results crucially depend on the optimization \([7]\). Apart from a few models where the series is known to large orders (as in the anharmonic oscillator \([4,17]\), or in the

\footnote{The smallness of \( \langle \bar{q}q \rangle \) is however essentially correlated with the smallness of the \( F_\pi/\bar{\Lambda} \) ratio estimate in our framework, eq. (11).}

\footnote{For instance, results for (3), (10) are substantially different \([16]\) in the unoptimized \( MS \) scheme.
GN model for $N \to \infty$), we can hardly compare successive orders of this expansion to estimate, even qualitatively, the \textit{intrinsic} error of such a method. Invoking the PMS principle [4], although physically motivated, may artificially force the series to converge, with no guarantees that it is toward the right result.

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Questions

E. de Rafael: Can one simply see, just at first RG order approximation already, what type of graphs are resummed by this mass ansatz?

J.L.K: It contains the “bubble” chain (the one-loop insertions in the gluon line) but, in addition, there is an iteration of those dressed gluon lines (the so-called Ladder graphs).

S. Narison: Your calculation is essentially perturbative. Can you include in it the truly non perturbative contributions, like condensates as they appear in the operator product expansion typically?

J.L.K: The aim here is to try to estimate these NP quantities from the basic QCD interactions only, with of course this peculiar resummation. To include OPE-like condensates from the start would be a kind of double-counting.