Distribution of partitions of \( n \) in which no part appears exactly once

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Abstract

In this paper we study the function \( g(n) \), which denotes the number of partitions of \( n \) in which no part appears exactly once. We prove that for each prime \( m \geq 5 \), there exist Ramanujan-type congruences of \( g(n) \) and give an example of such congruences.

Moreover, we also consider the case \( g(n) \equiv r \pmod{m} \) where \( r \not\equiv 0 \pmod{m} \) and give a useful criterion to estimate the distribution of \( g(n) \) modulo \( r \).

1 Introduction and statement of results

1.1 Introduction

In partition theory, we usually let \( p(n) \) denote the number of partitions of \( n \). We are not only interested in \( p(n) \), but also the number of partitions of \( n \) with some restriction. We denote such a number by \( p(n \mid [\text{condition}]) \). For example, Euler obtain that

\[
p(n \mid \text{odd parts}) = p(n \mid \text{distinct parts}).
\]

Let \( Q(n) \) denote \( p(n \mid \text{distinct parts}) \) throughout this paper.

In this paper, we are mainly interested in the number of partitions of \( n \) in which no part appears exactly once, which denoted by \( g(n) \). For example \( g(6) = 4 \), since

\[
6 = 3 + 3 = 2 + 2 + 2 = 2 + 2 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1.
\]

We define \( p(0) = Q(0) = g(0) = 1 \) by convention. If \( n \) is not a nonnegative integer, define \( p(n \mid [\text{condition}]) = 0 \).

In 1919, Ramanujan found three remarkable congruences of \( p(n) \) as follows

\[
p(5n + 4) \equiv 0 \pmod{5},
p(7n + 5) \equiv 0 \pmod{7},
p(11n + 6) \equiv 0 \pmod{11}.
\]

In 2000, Ono[5] proved that for each prime number \( m \), there exists some arithmetic sequences \( An + B \) such that

\[
p(An + B) \equiv 0 \pmod{m}.
\]

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Lovejoy\[3\] gave similar results for the function $Q(n)$. For example,

$$Q(26645n + 76) \equiv 0 \pmod{5},$$
$$Q(3170503n + 1374) \equiv 0 \pmod{7}.$$

Such congruences are called Ramanujan-type congruences. We will give one such congruence for function $g$ in this paper, which is

$$g(102487n + 1941) \equiv 0 \pmod{7}. \quad (1.1)$$

1.2 Statement of results

Using the same technique of Ono\[5\] and Lovejoy\[3\], we prove the following theorem:

**Theorem 1.** Let $m \geq 5$ be prime. A positive density of the primes $l$ have the property that

$$g \left( \frac{ml^2n - 1}{24} \right) \equiv 0 \pmod{m} \quad (1.2)$$

for each nonnegative integer $n$ coprime to $l$.

Since the number of selections of $l$ is infinite, choose $l > 3$. Replacing $n$ by $24nl + ml + 24$, then we prove that there exists Ramanujan-type congruences for $g(n)$ modulo prime $m \geq 5$.

**Corollary 1.** If $m$ is a prime, then there exists an integer $n > 1$ for which

$$g(n) \equiv 0 \pmod{m}.$$ 

**Remark.** If $m \geq 5$ is a prime, then there exist infinitely many integers $n \geq 1$ for which

$$g(n) \equiv 0 \pmod{m}.$$ 

More precisely, we have

$$\#\{0 \leq n \leq X : g(n) \equiv 0 \pmod{m}\} \gg X.$$ 

For other residue class $r \not\equiv 0 \pmod{m}$, we provide a useful criterion to verify whether there is infinitely many $n$ such that $g(n) \equiv r \pmod{m}$. We shall call a prime $m \geq 5$ good if for each $r = 1, 2, \cdots, m - 1$, there exists an integer $n_r$ for which

$$g \left( mn_r + \frac{m^2 - 1}{24} \right) \equiv r \pmod{m}.$$ 

**Theorem 2.** If $m \geq 5$ is good, then for each $r = 1, 2, \cdots, m - 1$, we have

$$\#\{0 \leq n \leq X : g(n) \equiv r \pmod{m}\} \gg \frac{\sqrt{X}}{\log X}.$$ 

**Remark.** The computation yields that primes $m$ are good primes if $5 \leq m \leq 100$.

It’s very likely true if we cancel the restriction of good in Theorem 2, so we conjecture that

**Conjecture 1.** If $m$ is a prime, then for each $r = 0, 1, \cdots, m - 1$, there exists infinitely many integers $n$ for which

$$g(n) \equiv r \pmod{m}.$$
Theorem 3. For all positive integer \( n \),
\[ g(n) = p(n \mid \text{no part appears exactly once}) = p(n \mid \text{parts } \equiv 0, 2, 3, 4 \pmod{6}) \].

Theorem 4. For all positive integer \( n \),
\[ g(n) = \begin{cases} 
\sum_{i=0}^{\lfloor n/6 \rfloor} Q(2i)p \left( \frac{n - 6i}{2} \right), & \text{if } n \text{ is even,} \\
\sum_{i=0}^{\lfloor (n-3)/6 \rfloor} Q(2i+1)p \left( \frac{n - 6i - 3}{2} \right), & \text{if } n \text{ is odd.}
\end{cases} \tag{1.3} \]

We’ll use Theorem 4 to compute some examples for Ramanujan-type congruences.

2 Preliminaries

First we introduce the \( U, V \) operators which act on power series. If \( j \) is a positive integer, then
\[ \left( \sum_{n=0}^{\infty} a(n)q^n \right) | U(j) := \sum_{n=0}^{\infty} a(jn)q^n. \tag{2.1} \]
\[ \left( \sum_{n=0}^{\infty} a(n)q^n \right) | V(j) := \sum_{n=0}^{\infty} a(n)q^{jn}. \tag{2.2} \]

Recalling that Dedekind’s eta function is defined by
\[ \eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \tag{2.3} \]
where \( q = e^{2\pi iz} \) and Ramanujan’s Delta function is defined by
\[ \Delta(z) = \eta^{24}(z) \in S_{12}(SL_2(\mathbb{Z})). \tag{2.4} \]

If \( l \) is a prime, then let \( M_k(\Gamma_0(N), \chi)_m \) (resp. \( S_k(\Gamma_0(N), \chi)_m \)) denote the \( \mathbb{F}_m \)-vector space of the reductions mod \( m \) of the \( q \)-expansions of modular forms (resp. cusp forms) in \( M_k(\Gamma_0(N), \chi) \) (resp. \( S_k(\Gamma_0(N), \chi) \)) with integer coefficients.

We define \( F(m; z) \) by
\[ F(m; z) := \sum_{n=0}^{\infty} g \left( \frac{mn - 1}{24} \right) q^n. \tag{2.5} \]

3 Proof of Theorem 3 and 4

We will use the idea of generating function in the following proof:
Proof of Theorem 3.

\[
\sum_{n=0}^{\infty} g(n)q^n = \prod_{n=1}^{\infty} \left( \frac{1}{1 - q^n - q^n} \right) \\
= \prod_{n=1}^{\infty} \frac{1 - q^n + q^{2n}}{1 - q^n} \\
= \prod_{n=1}^{\infty} \frac{1 - q^{6n}}{(1 - q^{2n})(1 - q^{3n})} \\
= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{6n+2})(1 - q^{6n+3})(1 - q^{6n+4})(1 - q^{6n+6})} \\
= \sum_{n=0}^{\infty} p(n \mid \text{parts } \equiv 0, 2, 3, 4 \pmod{6})q^n.
\]

Comparing the coefficients on both sides yields the result. \(\Box\)

Proof of Theorem 4.

\[
\sum_{n=0}^{\infty} g(n)q^n = \prod_{n=1}^{\infty} \frac{1 - q^{6n}}{(1 - q^{2n})(1 - q^{3n})} \\
= \prod_{n=1}^{\infty} (1 + q^{3n}) \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n}} \\
= \left( \sum_{i=0}^{\infty} Q(i)q^{3i} \right) \left( \sum_{j=0}^{\infty} p(j)q^{2j} \right) \\
= \left( \sum_{i=0}^{\infty} Q \left( \frac{i}{3} \right) q^{i} \right) \left( \sum_{j=0}^{\infty} \left( \frac{j}{2} \right) p \left( \frac{n-j}{2} \right) q^{j} \right) \\
= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} Q \left( \frac{i}{3} \right) p \left( \frac{n-i}{2} \right) \right) q^n.
\]

Comparing the coefficients on both sides, we obtain

\[
g(n) = \sum_{i=0}^{n} Q \left( \frac{i}{3} \right) p \left( \frac{n-i}{2} \right)
\]

\[
= \sum_{i=0}^{\lfloor n/3 \rfloor} Q(i) p \left( \frac{n-3i}{2} \right).
\]

If \(n\) is even, then

\[
g(n) = \sum_{i=0}^{\lfloor n/6 \rfloor} Q(2i) p \left( \frac{n-6i}{2} \right).
\]

We use \(\lfloor [n/3] /2 \rfloor = [n/6]\) in the proof above.

Similarly, if \(n\) is odd, then

\[
g(n) = \sum_{i=0}^{\lfloor (n-3)/6 \rfloor} Q(2i+1) p \left( \frac{n-6i-3}{2} \right).
\]

We use \(\lfloor (n/3 - 1)/2 \rfloor = ([n-3]/6\) in the proof above. \(\Box\)
4 Proof of Theorem 1 and Theorem 2

4.1 Some useful results

We need the following theorem to construct modular forms:

**Theorem 5** (B. Gordon, K. Hughes). Let

\[ f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z) \]

be a \( \eta \)-product provided

(i) \[ \sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24}; \]

(ii) \[ \sum_{\delta | N} \frac{Nr_\delta}{\delta} \equiv 0 \pmod{24}; \]

(iii) \[ k := \frac{1}{2} \sum_{\delta | N} r_\delta \in \mathbb{Z}_{\geq 1}; \]

(iv) For each \( d | N \),

\[ \sum_{\delta | N} \frac{(d, \delta)^2 r_\delta}{\delta} \geq 0, \]

then \( f(z) \in M_k(\Gamma_0(N), \chi) \), where

\[ \chi(n) := \left( \frac{-1}{n} \prod_{\delta | N} \frac{\delta r_\delta}{n} \right). \]

**Theorem 6.** \( \eta(48z) \in S_{1/2}(\Gamma_0(1152), \chi_{24}) \), where \( \chi_{24}(n) = \left( \frac{6}{n} \right) \) is the usual Jacobi symbol.

**Proof.** It’s well-known that \( \eta(24z) = \sum_{n=1}^{\infty} \chi_{12}(n) q^{n^2} \), where

\[ \chi_{12}(n) = \begin{cases} 1, & n \equiv \pm 1 \pmod{12}, \\ -1, & n \equiv \pm 5 \pmod{12}, \\ 0, & \text{else}. \end{cases} \]

Moreover, if \( n \) is odd, then \( \chi_{12}(n) = \left( \frac{4}{n} \right) \).

By Proposition 2.2 of [9], we obtain \( \sum_{n=1}^{\infty} \chi_{12}(n) q^{n^2} \in S_{1/2}(\Gamma_0(576), \chi_{12}) \). Hence \( \eta(24z) \in S_{1/2}(\Gamma_0(576), \chi_{12}) \). Let

\[ \gamma = \begin{bmatrix} a & b \\ 2c & d \end{bmatrix} \in \Gamma_0(1152). \]

Then
\[ \eta(48\gamma z) = \eta \left( 24 \left[ \begin{array}{cc} a & 2b \\ c & d \end{array} \right] (2z) \right) = \chi_{12}(d) \left( \frac{c}{d} \right) \varepsilon_d^{-1}(2cz + d)^{\frac{1}{2}} \eta(48z) = \left( \frac{6}{d} \right) \left( \frac{2n}{d} \right) \varepsilon_d^{-1}(2cz + d)^{\frac{1}{2}} \eta(48z). \]

where we let \( z^{1/2} \) be the branch of the square root having argument in \( (-\pi/2, \pi/2] \). For the definitions of \( (\frac{c}{d}) \) and \( \varepsilon_d \), one can refer to Definition 1.36 of [6].

We need an important theorem due to Serre([6.4] of [7]), which is the critical factor of the existence of Ramanujan-type congruences.

**Theorem 7** (J-P. Serre). The set of primes \( l \equiv -1 \pmod{N} \) for which \( f \mid T(l) \equiv 0 \pmod{m} \)

for each \( f(z) \in S_k(\Gamma_0(N), \psi)_m \) has positive density, where \( T(l) \) denotes the usual Hecke operator acting on \( S_k(\Gamma_0(N), \psi) \).

### 4.2 Shimura lifting

Here we will give a brief introduction to Shimura lifting. For the theory of Shimura lifting and its generalization, one can refer to [1, 4, 9]. The following introduction is due to Ono[5].

Suppose that \( f(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{\lambda + \frac{1}{2}}(\Gamma_0(4N), \psi) \) is a cusp form of half-integral weight where \( \lambda \in \mathbb{Z}_{>1} \). Let \( t \) be any squarefree integer, then define \( A_t(n) \) by

\[ \sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} := L(s - \lambda + 1, \psi \chi_{-1}^\lambda \chi_t) \cdot \sum_{n=1}^{\infty} \frac{b(tn^2)}{n^s}, \]

where \( \chi_{-1} \) (resp. \( \chi_t \)) denotes the Kronecker character for \( \mathbb{Q}(i) \) (resp. \( \mathbb{Q}(\sqrt{t}) \)).(See [8] for the definition of Kronecker character).

Next we define the Fourier expansion of \( S_t(f(z)) \) by \( A_t(n) \) as follows:

\[ S_t(f(z)) := \sum_{n=1}^{\infty} A_t(n)q^n. \]

By the theory of Shimura lifting, \( S_t(f(z)) \in S_{2\lambda}(\Gamma_0(4N), \psi^2) \). Moreover, the Shimura lifting \( S_t \) commutes with Hecke operator \( T(m) \). More specifically, if \( m \not| 4N \) is a prime, then

\[ S_t(f \mid T(m^2)) = S_t(f) \mid T(m), \]

where \( T(m)(\text{resp. } T(m^2)) \) denotes the usual Hecke operator acting on \( S_{2\lambda}(\Gamma_0(4N), \psi^2) \) (resp. \( S_{\lambda + \frac{1}{2}}(\Gamma_0(4N), \psi) \)).

### 4.3 Proof of Theorem 1

The main idea of the proof is to construct a cusp form correspond to the generating function of \( g(n) \), then we can apply the theory of modular form to study its properties.

Sometimes, we will use the notation \( a \equiv_m b \) in the place of \( a \equiv b \pmod{m} \) for convenience.
Proof of Theorem 1. We begin with an \( \eta \)-product \( \eta(6z) \eta^a(6mz) \eta^b(3mz) / \eta(3z) \), where \( m \geq 5 \) is a prime and \( a := 16 - (m \bmod 24) \), \( b := (m \bmod 24) - 8 \). It is easy to verify that

\[
\frac{\eta(6z)}{\eta(3z)} \eta^a(6mz) \eta^b(3mz) = \eta^a(6z) \eta^{b-1}(3z),
\]

therefore

\[
\frac{\eta(6z)}{\eta(3z) \eta(2z)} \eta^a(6mz) \eta^b(3mz) \eta^m(2mz) = \eta^a(6z) \eta^{b-1}(3z) q^{m^2 - \frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{2mn})^m
\]

\[
= \eta^a(6z) \eta^{b-1}(3z) \Delta_{24}^m(2z).
\]

Moreover, by Theorem 5 and computation of cusps we obtain \( \eta^{a+1}(2z) \eta^{b-1}(3z) \in S_{4m}(\Gamma_0(2)) \). Then we act on both sides of (4.2) by operator \( U(m) \) to obtain

\[
\frac{\eta(6z)}{\eta(3z) \eta(2z)} \eta^a(6mz) \eta^b(3mz) \eta^m(2mz) \mid U(m) = \eta^a(6z) \eta^{b-1}(3z) \Delta_{24}^m(2z) \mid U(m)
\]

(4.3)

where \( T(m) \) denotes the usual Hecke operator acting on \( S_{4m}(\Gamma_0(2)) \). By Lemma 7 of [3], we can write

\[
\eta^{a+1}(2z) \eta^{b-1}(3z) \mid T(m) = \eta^8(z) \eta^8(2z) h(z),
\]

where \( h(z) \in M_{4m-8}(\Gamma_0(2)) \).

On the other hand,

\[
\frac{\eta(6z)}{\eta(3z) \eta(2z)} \eta^a(6mz) \eta^b(3mz) \eta^m(2mz) \mid U(m) = \left( \sum_{n=0}^{\infty} g(n) q^{m(6a + 3b + 2m) + 1} \right) \prod_{n=1}^{\infty} (1 - q^{2n})^m (1 - q^{2m})^b (1 - q^{6n})^a.
\]

(4.4)

To sum up,

\[
\sum_{n \geq 0}^* g(n) q^{24n + m(6a + 3b + 2m) + 1} = \frac{\eta^8(z) \eta^8(2z) h(z)}{\prod_{n=1}^{\infty} (1 - q^{3n})^b (1 - q^{6n})^a},
\]

where \( \sum^* \) means take integral power coefficients of \( q \), i.e.

\[
24n + m(6a + 3b) + 2m^2 + 1 \equiv 0 \pmod{24m}.
\]

It is easy to check that \( 24n + m(6a + 3b) + 2m^2 + 1 \equiv 0 \pmod{24} \). Then the condition becomes \( m \mid 24n + 1 \). Consequently,

\[
\sum_{n \geq 0 \text{ even}} \frac{g(n) q^{24n + m(6a + 3b + 2m) + 1}}{24^m} = \frac{\eta^8(3z) \eta^8(6z) h(3z)}{\prod_{n=1}^{\infty} (1 - q^{3n})^b (1 - q^{6n})^a}.
\]

\[
\sum_{n \geq 0 \text{ odd}} \frac{g(n) q^{24n + m(6a + 3b + 2m) + 1}}{24^m} = \frac{\eta^8(3z) \eta^8(6z) h(3z) \eta^{8-a}(6z)}{\prod_{n=1}^{\infty} (1 - q^{3n})^b (1 - q^{6n})^a},
\]

(4.6)
Replacing \( q \) by \( q^{24} \) on both sides of (4.7) and then multiplying by \( q^{-(6\alpha+3\beta)} \), we obtain

\[
\sum_{\substack{m \geq 0 \\ m \mid 24n+1}} g(n)q^{\frac{24m+24n+1}{m}} = m \frac{\Delta \frac{m^2-1}{m}(48z) \mid U(m) \eta^a(144z)h(72z)}{\prod_{n=1}^{\infty}(1 - q^{48n})m} \eta^{b}(72z) \eta^{a}(144z)h(72z).
\]

(4.8)

Equivalently,

\[
\sum_{\substack{n \geq 0 \\ m \mid 24n+1}} g(n)q^{\frac{24n+1}{m}} = m \frac{\Delta \frac{m^2-1}{m}(48z) \mid T(m) \eta^{b}(72z) \eta^{a}(144z)h(72z)}{\eta^m(48z)}
\]

(4.9)

where \( T(m) \) denotes the usual Hecke operator acting on \( S_{\frac{m^2-1}{2}}(\Gamma_0(48)) \).

As for LHS of (4.9), we have

\[
\sum_{\substack{n \geq 0 \\ m \mid 24n+1}} g(n)q^{\frac{24n+1}{m}} = \sum_{n=0}^{\infty} g\left(\frac{mn-1}{24}\right) q^n = F(m; z).
\]

(4.10)

As for RHS of (4.9), we have \( h(72z) \in M_{4m-8}(\Gamma_0(144)) \) and \( \eta^{b}(72z) \eta^{a}(144z) \in S_{1}(\Gamma_0(3456), \chi_8) \) by Theorem 5, where \( \chi_8(n) = (\frac{2}{n}) = (-1)^{(n^2-1)/8} \). Now we deal with the last one of RHS of (4.9). Let

\[
f(z) := \Delta \frac{m^2-1}{m}(48z) \mid T(m) \in S_{\frac{m^2-1}{2}}(\Gamma_0(48)),
\]

then for \( \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(1152), \) we have

\[
f(\gamma z) = \frac{(cz+d)^{(m^2-1)/2}f(z)}{\eta^m(48z)} = \frac{(\frac{c}{d})^m (\frac{a}{d})^m \varepsilon_d^{-m}(cz+d)^{m^2/2} \eta^m(48z)}{\eta^m(48z)}.
\]

(4.11)

Thus

\[
\Delta \frac{m^2-1}{m}(48z) \mid T(m) \in S_{\frac{m^2-1}{2}}(\Gamma_0(1152), \chi_{24}).
\]

In summary,

\[
F(m; z) \in S_{\frac{m^2+7m-9}{2}}(\Gamma_0(3456), \chi_{12}),
\]

(4.12)

where \( \chi_{12}(n) = (\frac{2}{n}) \). Hence we can choose \( F'(m; z) \in S_{\frac{m^2+7m-9}{2}}(\Gamma_0(3456), \chi_{12}) \) such that \( F'(m; z) \equiv F(m; z) \) (mod \( m \)).

For a fix prime \( m \geq 5 \), let \( S(m) \) denote the set of primes \( l \) such that

\[
f \mid T(l) \equiv 0 \pmod{m}
\]

for each \( f \in S_{\frac{m^2+7m-9}{2}}(\Gamma_0(3456)) \). By Theorem 7, \( S(m) \) contains a positive density of primes. Thus if \( l \in S(m) \), by the commutativity of Shimura lifting we have

\[
S_t(F'(m; z) \mid T(l^2)) = S_t(F'(m; z)) \mid T(l) \equiv 0 \pmod{m}
\]

satisfied for each squarefree integers \( t \), where \( T(l^2) \) (resp. \( T(l) \)) denotes the usual Hecke operator acting on \( S_{\frac{m^2+7m-9}{2}}(\Gamma_0(3456), \chi_{12}) \) (resp. \( S_{\frac{m^2+7m-10}{2}}(\Gamma_0(3456)) \)).
Suppose $F'(m; z) = T(l^2) = \sum_{n=1}^{\infty} a(n)q^n$, then by the definition of Shimura lifting we obtain

$$b(n^2) = A_t(n) * \alpha(n) \equiv 0 \pmod{m}$$

satisfied for all $n \in \mathbb{Z}$, where $*$ denotes Dirichlet convolution and $\alpha(n)$ is the Möbius inversion of $\chi_{12}\chi_{-1}^{(m^2+7m-10)/2} \chi_t(n)^{(m^2+7m-12)/2}$. Since the choice of the squarefree integer $t$ is arbitrary, we obtain

$$b(n) \equiv 0 \pmod{m}, \ n \in \mathbb{Z}.$$

Therefore $F'(m; z) = T(l^2) \equiv 0 \pmod{m}$. In particular, if we suppose that $F'(m; z) = \sum_{n=1}^{\infty} a(n)q^n$, then by the theory of Hecke operator we have

$$F' | T(l^2) = \sum_{n=1}^{\infty} \left( a(l^2n) + \chi_{12}(l) \left( \frac{-1}{l} \right) \frac{n^2+7m-10}{l} a(n) + \chi_{12}(l^2)m^2+7m-11 \left( \frac{N}{l^2} \right) \right) q^n$$

$$= \sum_{n=1}^{\infty} \left( a(l^2n) + \left( \frac{-1}{l} \right) \frac{n^2+7m-10}{l} 3n a(n) + m^2+7m-11 m^2+7m-11 \left( \frac{N}{l^2} \right) \right) q^n. \quad (4.13)$$

Hence for $l \in S(m)$, we replace $n$ by $nl$ with $(n, l) = 1$, obtaining

$$a(l^3n) \equiv 0 \pmod{m} \quad (4.14)$$

Since $F'(m; z) \equiv F(m; z) \pmod{m}$, finally we obtain

$$g \left( \frac{ml^3n - 1}{24} \right) \equiv 0 \pmod{m}$$

satisfied for all integers $n$ coprime to $l$. Moreover, for a fixed prime $m \geq 5$, the set of primes $l$ is of positive density.

4.4 Proof of Theorem 2

Proof of Theorem 2. First we recall that

$$F'(m; z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\frac{m^2+7m-9}{2}}(\Gamma_0(3456), \chi_{12}).$$

Since $m$ is good, for each $1 \leq r \leq m-1$ we fix an $n_r$ for which

$$g \left( mn_r + \frac{m^2 - 1}{24} \right) \equiv r \pmod{m}.$$

Let $k_r = 24n_r + m$, first we show that $k_r > 0$ for each $r = 1, 2, \ldots, m-1$. Since $g(n)$ vanishes for negative $n$, we notice that $mn_r + (m^2 - 1)/24 \geq 0$, so $k_r = 24n_r + m \geq m - (m^2 - 1)/m > 0$.

Let $K = \prod_{i=1}^{m-1} k_r \in \mathbb{Z}_{>0}$, then $F'(m; z) \in S_{\frac{m^2+7m-9}{2}}(\Gamma_0(3456mK), \chi_{12})$. Again by Theorem 7 and Shimura lifting, there are a positive density of primes $l \equiv -1 \pmod{3456mK}$ such that

$$F'(m; z) | T(l^2) \equiv 0 \pmod{m}.$$

By (4.13) we obtain

$$a(l^2k_r) + \left( \frac{-1}{l} \frac{m^2+7m-10}{l} 3k_r \right) l^m+7m-11 a(k_r) + l^m+7m-11 a \left( \frac{k_r}{l^2} \right) \equiv 0 \pmod{m} \quad \text{for which } k_r = 24n_r + m \geq 0. \quad (4.15)$$

9
for \( r = 1, 2, \cdots, m - 1 \). Since \( l \equiv -1 \pmod{3456mK} \), we have \((l, k_r) = 1\), so
\[
a(l^2 k_r) \equiv \left( \frac{-1}{l} \right) \left( \frac{m^2 + 7m - 10}{l} \right) 3k_r \left( -1 \right)^{m^2 + 7m - 10} a(k_r) \pmod{m}.
\]
(4.16)

We have \( \left( \frac{-1}{l} \right) = -1 \) since \( l \equiv -1 \pmod{4} \) and \( \left( \frac{4}{l} \right) = 1 \) since \( l \equiv -1 \pmod{12} \), therefore
\[
a(l^2 k_r) \equiv \left( \frac{k_r}{l} \right) a(k_r) \pmod{m}.
\]
(4.17)

Since \( k_r \) is odd, \( l \equiv -1 \pmod{4} \) and \( -l \equiv 1 \pmod{k_r} \), we find by quadratic reciprocity that
\[
\left( \frac{k_r}{l} \right) = \left( \frac{l}{k_r} \right) \left( -1 \right)^{k_r - 1} = \left( \frac{-l}{k_r} \right) = 1.
\]

Finally we have
\[
a(l^2 k_r) \equiv a(k_r) \pmod{m}.
\]
(4.18)

Recalling that \( a(n) \equiv g \left( \frac{mn - 1}{24} \right) \pmod{m} \), we obtain
\[
g \left( \frac{m^2 k_r - 1}{24} \right) \equiv g \left( \frac{mk_r - 1}{24} \right) \pmod{m},
\]
(4.19)
or
\[
g \left( \frac{mt^2 n_r + m^2 l^2 - 1}{24} \right) \equiv g \left( \frac{mn_r + m^2 - 1}{24} \right) \equiv r \pmod{m}.
\]
(4.20)

The \( \sqrt{X}/\log X \) estimate is easily derived from Theorem 7 and prime number theorem.
\[\Box\]

5 Example of Ramanujan-type congruences

Here we introduce a theorem of Sturm(Theorem 1 of [10]), which provide a useful criterion for deciding when modular forms with integer coefficients are congruent to zero modulo a prime via finite computation.

Theorem 8 (J. Sturm). Suppose \( f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)_m \) such that
\[a(n) \equiv 0 \pmod{m}\]
for all \( n \leq \frac{kN}{12} \prod_{p|N} \left( 1 + \frac{1}{p} \right) \). Then \( a(n) \equiv 0 \pmod{m} \) for all \( n \in \mathbb{Z} \).

A finite computation yields that \( F'(7; z) | T(11^2) \equiv 0 \pmod{7} \), i.e.
\[
g \left( \frac{7 \cdot 11^3 \cdot n - 1}{24} \right) \equiv 0 \pmod{7}
\]
(5.1)
for each nonnegative integer \( n \) coprime to 11. Replacing \( n \) by \( 24n + 5 \) yields
\[
g(9317n + 1941) \equiv 0 \pmod{7}
\]
(5.2)
for each nonnegative integers \( n \) with \( n \neq 3 \pmod{11} \). Replacing \( n \) again by \( 11n \) yields

Example.
\[
g(102487n + 1941) = g \left( 7 \cdot 11^4 \cdot n + 3 \cdot 647 \right) \equiv 0 \pmod{7}, \ n \in \mathbb{Z}.
\]
(5.3)
We have also tried the case \( m = 5 \). However, for all primes \( 5 \leq l \leq 3500 \), \( F'(5;z) \mid T(l^2) \not\equiv 0 \pmod{5} \). We are also expecting that someone who is interested in our results can provide more examples of Ramanujan-type congruences modulo 5 and other primes.

We have not discussed Ramanujan-type congruence when \( m = 2, 3 \). In fact, we have the following conjecture:

**Conjecture 2.** If \( m = 2 \) or 3, then for all arithmetic sequences \( An + B \) and each \( r = 0, 1, \cdots, m - 1 \), there exist infinitely many integers \( n \) such that \( g(An + B) \equiv r \pmod{m} \).

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