Inverse Born approximation for the generalized nonlinear Schrödinger operator in two dimensions

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Abstract. This work deals with the inverse scattering problems for two nonlinear Schrödinger operators in two dimensions. One operator has a finite combinations of any power-like nonlinearity and another has a saturation-like nonlinearity. Both these operators can be met in nonlinear optics. The coefficients of these operators are assumed to be the real-valued functions from $L^p_{\text{loc}}(\mathbb{R}^2)$ with certain behaviour at infinity. We prove Saito’s formula for both operators which implies the uniqueness result and a representation formula for a sum of the unknown coefficients in the sense of tempered distributions. What is more, we prove that the leading order singularities of the sum can be obtained exactly by the inverse Born approximation method from general scattering data at arbitrarily large energies. Especially, we show for the functions in $L^p_{\text{loc}}(\mathbb{R}^2)$, for certain values of $p$, that the approximation agrees with the true sum up to the functions from the Sobolev spaces $H^s(\mathbb{R}^2)$. In particular, for the sum being the characteristic function of a smooth bounded domain this domain is uniquely determined by this scattering data.

1. Introduction
This work is concerned with the two generalized nonlinear Schrödinger equations in two dimensions

\[-\Delta u + \sum_{l=0}^{m} \alpha_l(x)|u|^l u = k^2 u\tag{1}\]

and

\[-\Delta u + q(x)u + \frac{\alpha(x)|u|^2 u}{1 + r(x)|u|^2} = k^2 u,\tag{2}\]

where $k \in \mathbb{R}$ and $r(x) \geq 0$. For integrable coefficients and $u \in L^p_{\text{loc}}(\mathbb{R}^2)$, we understand these equations in the sense of distributions, i.e., continuous linear functionals on the test function space $C_0^\infty(\mathbb{R}^2)$.

Form (1) appears quite naturally in applications. It includes the linear case ($m = 0$) and the basic nonlinearities of cubic and cubic-quintic type. The latter two equations can be met, e.g., in optics. In addition to these cases, we allow here any finite combination of such powers of nonlinearity. In the saturation model (see Eq. (2)), the quantity $r$ is often a positive constant but we allow it to vary as a function of the space coordinates. More precisely, we assume that $r(x) \geq 0$ is a bounded continuous function. We also assume that all others coefficients in the
equations (1) and (2) are real-valued functions and belong to the weighted space \(L^p_\sigma(R^2)\) defined by the norm

\[
\|f\|_{L^p_\sigma(R^2)} = \left( \int_{R^2} (1 + |x|)^{p\sigma} |f(x)|^p \, dx \right)^{1/p},
\]

where \(1 < p \leq \infty\) and \(\sigma\) is a nonnegative number that will later be seen to depend on \(p\).

We consider direct and inverse scattering problems concerning equations (1) and (2). For given coefficients and wave number \(k \in R\), the direct scattering problem asks to find the solution \(u\) of equations (1) and (2) such that

\[
u(x, k, \theta) = u_0(x, k, \theta) + u_{sc}(x, k, \theta),
\]

where \(u_0(x, k, \theta) = e^{i k(x, \theta)}\) is the incident plane wave with direction \(\theta \in S^1 := \{x \in R^2 : |x| = 1\}\) and \(u_{sc}(x, k, \theta)\) is the scattered wave. The inverse scattering problem is to extract information about the unknown coefficients of equations (1) and (2) from scattering data, which is obtained from the scattered wave \(u_{sc}\) as a response to a given incident wave \(u_0\) (given wave number \(k\)).

Inverse scattering problem for nonlinear Schrödinger equations have recently been studied in dimensions one and three or higher, see [1], [13], [14], [15]. These works were concerned with the unique recovery of the potentials from appropriate scattering data.

In this article we follow partly the outlines of works [10] and [11].

2. Formulation of the problems and results

Solutions of the equations (1) and (2) of the particular form (3) are the unique solutions of the Lippmann-Schwinger equations

\[
u(x, k, \theta) = u_0(x, k, \theta) - \int_{R^2} G^+_k(|x - y|) \sum_{l=0}^{m} \alpha_l(y) \|u_l\|^2 u(y, k, \theta) \, dy
\]

and

\[
u(x, k, \theta) = u_0(x, k, \theta) - \int_{R^2} G^+_k(|x - y|) \left( q(y) u(y, k, \theta) + \frac{\alpha(y) \|u\|^2 u(y, k, \theta)}{1 + r(y) \|u(y, k, \theta)\|^2} \right) \, dy,
\]

respectively. Here the outgoing fundamental solution \(G^+_k\) of the Helmholtz operator \(-\Delta - k^2\) is defined as

\[
G^+_k(|x|) = \frac{i}{4} H_0^{(1)}(|k||x|),
\]

where \(H_0^{(1)}\) is the Hankel function of the first kind and zero order. Recall that \(G^+_k\) is the kernel of the integral operator \((-\Delta - k^2 - i0)^{-1}\). Our additional assumptions on the functions \(q, \alpha\) and \(\alpha_l, l = 0, 1, 2, ..., m,\) are that

\[
c_k := \sup_{x \in R^2} \int_{R^2} G^+_k(|x - y|) h(y) \, dy \to 0, \quad k \to +\infty,
\]
where \( h(x) = \sum_{l=0}^{m} |\alpha_l(x)| \) for equation (1) and \( h(x) = |q(x)| + |\alpha(x)| \) for equation (2), respectively. This condition (6) is satisfied if e.g. the functions \( q(x) \), \( \alpha(x) \), or \( \alpha_l(x) \), \( l = 0, 1, 2, ..., m \), belong to \( L^p(R^2) \) for some \( 1 < p \leq \infty \) and have the special behaviour at the infinity

\[
|f(x)| \leq \frac{c}{|x|^\mu}, \quad |x| \to +\infty
\]

with some \( \mu > 2 \). In particular, compactly supported functions from \( L^p(R^2) \) with \( 1 < p \leq \infty \) satisfy the condition (6).

The existence of a unique bounded scattering solution of (1) and (2) can be established by introducing the sequences

\[
u_{j+1}(x, k, \theta) := u_0(x, k, \theta) - \int_{R^2} G_k^+(|x-y|) \sum_{l=0}^{m} \alpha_l(y) u_j(y, k, \theta) \, dy
\]

and

\[
u_{j+1}(x, k, \theta) = u_0(x, k, \theta) - \int_{R^2} G_k^+(|x-y|) \left( q(y) u_j(y, k, \theta) + \frac{\alpha(y) |u_j|^2 u_j(y, k, \theta)}{1 + r(y) |u_j(y, k, \theta)|^2} \right) \, dy
\]

for \( j = 0, 1, 2, ..., \) respectively to the equations (1) and (2).

**Lemma 1.** If \( k \) is chosen large enough so that

\[
c_k \leq \frac{\beta}{(1 + \beta)^{m+1}}
\]

with some \( \beta \leq \frac{1}{m} \) for \( m = 1, 2, ... \), and \( \beta \leq 1 \) for \( m = 0 \) for equation (1) and with some \( \beta \leq \frac{1}{2} \) and \( m = 2 \) for equation (2) then for each \( j = 0, 1, 2, ... \),

\[
\|u_j\|_{L^\infty(R^2)} \leq 1 + \beta
\]

uniformly with respect to \( \theta \in S^1 \), where \( u_j \) is defined by (7) and (8), respectively to the equations (1) and (2).

**Lemma 2.** There exists \( k_0 \) such that the sequences (7) and (8) converge in \( L^\infty(R^2) \) uniformly with respect to \( |k| \geq k_0 \) and \( \theta \in S^1 \) to some \( L^\infty \)–functions \( u(x, k, \theta) \) which are the unique solutions of the corresponding Lippmann-Schwinger equations (4) and (5). These functions \( u \) have the form (3) and

\[
\|u_{sc}\|_{L^\infty(R^2)} \to 0, \quad |k| \to +\infty.
\]

**Lemma 3.** Assume that the functions \( q(x) \), \( \alpha(x) \), and \( \alpha_l(x) \), \( l = 0, 1, 2, ..., m \), belong to \( L^p_{\delta}(R^2) \cap L^1(R^2) \) with \( \delta = 0 \) for \( 1 < p \leq 3/2 \) and with \( \delta > 1/2 - 3/4p \) for \( 3/2 < p \leq \infty \) and condition (9) is satisfied. Then there exists \( k_0 \) such that the sequences (7) and (8) converge in the weighted space \( L^{\frac{2p}{2p-\delta}}_{\delta}(R^2) \) to the functions \( u \) and for each \( j = 0, 1, 2, ... \),

\[
u_{sc}^{(j)}(x, k, \theta) := u(x, k, \theta) - u_j(x, k, \theta)
\]

satisfy the estimate

\[
\|\nu_{sc}^{(j)}\|_{L^{\frac{2p}{2p-\delta}}_{\delta}(R^2)} \leq \frac{c}{|k|^\gamma(j+1)}
\]

where \( \gamma = 2 - 2/p \) for \( 1 < p \leq 3/2 \) and \( \gamma = 1 - 1/2p \) for \( 3/2 < p \leq \infty \).
Saito’s formula. The next result is the uniqueness theorem of the reconstruction of unknown function $g$. Under the same assumptions for $q(x)$, $\alpha(x)$ and $\alpha_l(x)$, $l = 0, 1, 2, ..., m$, as in Lemma 3

$$g(x) = \sum_{l=0}^{m} \alpha_l(x),$$

or

$$g(x) = q(x) + \frac{\alpha(x)}{1 + r(x)},$$

respectively to the equations (1) and (2), given the continuous function $r(x) \geq 0$ and the scattering amplitude $A(k, \theta', \theta)$ for all $k \geq k_0$ with an arbitrary large $k_0$ and for all angles $\theta$ and $\theta'$ from $S^1$.

We achieve one of our goals as consequence of the following important result.

**Theorem 1 (Saito’s formula).** Under the same assumptions for $q(x)$, $\alpha(x)$ and $\alpha_l(x)$, $l = 0, 1, 2, ..., m$, as in Lemma 3

$$\lim_{k \to +\infty} \int_{S^1 \times S^1} e^{-ik\theta - ik'\phi} A(k, \theta', \theta) \, d\theta \, d\theta' = 4\pi \int_{\mathbb{R}^2} \frac{q(y)}{|x - y|} \, dy,$$

where the function $g$ is defined by (15) and (16), respectively to the equations (1) and (2), and where the limit is valid in the sense of distributions for $4/3 < p \leq 2$ and pointwise (even uniformly) for $2 < p \leq \infty$.

The next result is the uniqueness theorem of the reconstruction of unknown function $g(x)$ defined by (15) and (16), respectively by the scattering amplitude and it is a corollary from Saito’s formula.

**Corollary 1 (Uniqueness).** Assume that the functions $q'(x), \alpha'(x)$ and $\alpha_l'(x), l = 0, 1, 2, ..., m$, and $q''(x), \alpha''(x)$ and $\alpha_l''(x), l = 0, 1, 2, ..., m$, satisfy the conditions of Lemma 3
and the corresponding scattering amplitudes $A'(k, \theta', \theta)$ and $A''(k, \theta', \theta)$ (13), (14) coincide for some sequence $k_j \to +\infty$ and for all $\theta$ and $\theta'$ from $S^1$. Then

$$g'(x) = g''(x)$$

holds in the sense of tempered distributions for $4/3 < p \leq \infty$, where the functions $g'(x)$ and $g''(x)$ are defined by (15) and (16), respectively.

The inversion of Saito’s formula gives us also the representation formula for the unknown function $g(x)$ (15), (16).

**Corollary 2 (Representation formula).** Under the same assumptions for $q(x)$, $\alpha(x)$ and $\alpha_l(x), l = 0, 1, 2, \ldots, m$, as in Lemma 3

$$g(x) = \lim_{k \to +\infty} \frac{k^2}{8\pi^2} \int_{S^1 \times S^1} e^{-ik(\theta - \theta', x)} A(k, \theta', \theta) |\theta' - \theta| \, dk \, d\theta \, d\theta'$$

(18)

holds in the sense of tempered distributions for $4/3 < p \leq \infty$, where the function $g(x)$ is defined by (15) and (16), respectively.

The definition of the scattering amplitude $A(k, \theta', \theta)$ and Lemma 2 allow us to conclude that for large $k$ and uniformly with respect to $\theta$ and $\theta'$ from $S^1$

$$A(k, \theta', \theta) \approx \mathcal{F}(g(x))(k(\theta - \theta'))$$

(19)

where $\mathcal{F}$ is the ordinary Fourier transform in $R^2$ and $g(x)$ is the function which is defined by (15) and (16), respectively. If we keep $\theta$ fixed and write $\xi = k(\theta - \theta')$ then $k$ and $\theta'$ can be obtained back as

$$k = \frac{|\xi|^2}{2(\theta, \xi)}, \quad \theta' = \theta - \frac{2(\theta, \xi)\xi}{|\xi|^2}.$$

Furthermore, the coordinate transformation $\xi \to (k, \theta')$ has the Jacobian

$$\left| \frac{\partial \xi}{\partial (k, \theta')} \right| = \frac{1}{2} |k||\theta - \theta'|^2.$$

Using this fact and inverting (19) with integrating once over $\theta'$s motivates the following definition.

**Definition.** The inverse Born approximation $q_B(x)$ of the function $g(x)$ (15), (16) is defined as

$$q_B(x) := \frac{1}{32\pi^3} \int_{R \times S^1 \times S^1} e^{-ik(\theta - \theta', x)} A(k, \theta', \theta) |k||\theta' - \theta|^2 \, dk \, d\theta \, d\theta'$$

(20)

and this equality must be considered in the sense of tempered distributions.

Note also that the latter integral is just the inverse Fourier transform of $A(k, \theta', \theta)$ on the manifold $R \times S^1 \times S^1$. It is very easy to see that within the Born approximation, the scattering amplitude is simply the Fourier transform of the unknown function which is connected to the coefficients of the equations (1) and (2). The weaker the coefficients, the better is this approximation. But even when the coefficients are not weak the Fourier transform of a scattering amplitude contains the essential information of the coefficients.

In what follows we attempt to conclude that the difference $q_B(x) - g(x)$ is ”smoother” than the unknown function $g(x)$ (15), (16). As we cannot estimate the smoothness of $q_B(x) - g(x)$ directly we use the following Born sequence

$$q_{B,j}(x) := \frac{1}{32\pi^3} \int_{R \times S^1 \times S^1} e^{-ik(\theta - \theta', x)} A_j(k, \theta', \theta) |k||\theta' - \theta|^2 \, dk \, d\theta \, d\theta',$$

(21)
where $A_j, j = 0, 1, 2, \ldots$, is defined with respect to the sequences $u_j$ (7), (8) as

$$A_j(k, \theta', \theta) := \int_{R^2} e^{-ik(\theta', y)} \sum_{l=0}^{m} a_l(y) |u_j|^l u_j(y, k, \theta) \, dy$$

and

$$A_j(k, \theta', \theta) := \int_{R^2} e^{-ik(\theta', y)} \left( q(y) u_j(y, k, \theta) + \frac{\alpha(y)}{1 + r(y)} |u_j(y, k, \theta)|^2 \right) \, dy,$$

respectively to the equations (1) and (2). For the sequence (21) the following lemmata hold.

**Lemma 4.** Assume that the functions $q(x)$, $\alpha(x)$ and $\alpha_l(x), l = 0, 1, 2, \ldots, m$, satisfy all conditions of Lemma 3. Then the difference

$$q_B(x) - q_{B,j}(x)$$

is a bounded continuous function for $j \geq 3$ and for $4/3 < p \leq 3/2$, and for $j \geq 2$ and for $3/2 < p \leq \infty$.

**Lemma 5.** Under the same assumptions for $q(x)$, $\alpha(x)$ and $\alpha_l(x), l = 0, 1, 2, \ldots, m$, as in Lemma 3

$$q_B(x) - q_{B,1}(x) \in H^t(R^2),$$

where $g(x)$ is defined by (15) or (16), respectively, $t < 3 - 4/p$ if $1 < p \leq 3/2$ and $t < 1 - 1/p$ if $3/2 < p \leq \infty$ and where $H^t(R^2)$ denotes the $L^2$-based Sobolev space with smoothness index $t \in R$.

Since $q_{B,0}(x) = g(x) (modC^\infty(R^2))$ then these lemmata show that in order to estimate the smoothness of the difference $q_B(x) - g(x)$ it is necessary to investigate more accurately $q_{B,1}(x)$. The following important result holds.

**Lemma 6.** Assume that the functions $q(x)$, $\alpha(x)$ and $\alpha_l(x), l = 0, 1, 2, \ldots, m$, satisfy all conditions of Lemma 3. Then in the sense of tempered distributions the following representation holds:

$$q_{B,1}(x) = g(x) + g_1(x) + \tilde{g}(x) + g_{rest}(x),$$

where $g(x)$ is defined by (15) or (16), respectively, $\tilde{g}(x)$ belongs to $C^\infty(R^2)$, $g_{rest}(x)$ belongs to the Sobolev space $H^t(R^2)$ with $t < 3 - 4/p$ for $1 < p \leq 3/2$ and with $t < 1 - 1/p$ for $3/2 < p \leq \infty$, and the term $g_1(x)$ admits the following integral representation:

$$g_1(x) = \frac{1}{32\pi^2} \int_{R^2 \times R^2} \frac{(x-y, x-z)}{|x-y|^2 |x-z|^2} g^{(1)}(y) g(z) \, dy \, dz$$

$$- \frac{1}{32\pi^2} \int_{R^2 \times R^2} \frac{(x-y, x+z+2y)}{|x-y|^2 |x+z+2y|^2} g^{(2)}(y) g(z) \, dy \, dz,$$

where $g(z)$ is as above and

$$g^{(1)}(y) = \sum_{l=0}^{m} \frac{l}{2} + 1 \alpha_l(y), \quad g^{(2)}(y) = \sum_{l=0}^{m} \frac{l}{2} \alpha_l(y)$$

for the equation (1), and

$$g^{(1)}(y) = q(y) + \frac{\alpha(y)(2 + r(y))}{(1 + r(y))^2}, \quad g^{(2)}(y) = \frac{\alpha(y)}{(1 + r(y))^2}.$$
for the equation (2).

Due to this integral representation the smoothness of \( g_1(x) \) can now be studied in terms of weighted Sobolev spaces \( W^1_{p,\sigma}(R^2) \) consisting of functions \( f(x) \) from \( L^p(R^2) \) such that \( \nabla f(x) \) is in \( L^p(R^2) \). Since this result is so important for our considerations we formulate it as the following theorem.

**Theorem 2.** Assume that the functions \( q(x), \alpha(x) \) and \( \alpha_l(x), l = 0, 1, 2, ..., m, \) satisfy all conditions of Lemma 3. Then

1. for \( 1 < p \leq 3/2 \), the function \( g_1(x) \) (26) belongs to \( (W^1_{p,1}(R^2)) \); \( 2 \)
2. for \( 3/2 < p < \infty \), the function \( g_1(x) \) (26) belongs to \( (W^1_{p,2\delta-1}(R^2)) \) with \( 1/2 - 3/4p < \delta < 1 - 1/p \);
3. for \( p = \infty \), the function \( g_1(x) \) (26) belongs to \( (\Lambda^1(R^2)) \); the classical Zygmund space with smoothness index 1.

We are now in the position to formulate the sharpest result concerning the smoothness of the difference \( q_B(x) - g(x) \). The next theorem is the immediate consequence of Lemmas 5 and 6 and Theorem 2.

**Theorem 3.** Assume that the functions \( q(x), \alpha(x) \) and \( \alpha_l(x), l = 0, 1, 2, ..., m, \) satisfy all conditions of Lemma 3. Then

\[
q_B(x) - g(x) \in H^1_{\text{loc}}(R^2),
\]

where \( t < 3 - 4/p \) if \( 1 < p \leq 3/2 \) and \( t < 1 - 1/p \) if \( 3/3 < p \leq \infty \).

**Remark.** Theorem 3 means that the Born approximation recovers the main singularities of the unknown function \( g(x) \in L^p(R^2) \) for \( 4/3 < p < \infty \) which is defined by (15) and (16), respectively to the equations (1) and (2). More importantly for applications, for the function \( g(x) \) being the characteristic function of a smooth bounded domain it belongs to \( H^s_{\text{comp}}(R^2) \) for \( s < 1/2 \). But by Theorem 3, \( q_B(x) - g(x) \in H^1_{\text{loc}}(R^2) \) for any \( t < 1/2 \). Since no function in \( H^1_{\text{loc}}(R^2) \), \( t > 1/2 \), can have conormal jumps we can conclude that this domain is uniquely determined by this scattering data. However, we cannot distinguish the singularities or jumps of the coefficients \( q(x), \alpha(x) \) and \( \alpha_l(x), l = 0, 1, 2, ..., m, \) in the equations (1) and (2) if all are discontinuous.

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**References**

[1] Aktosun T, Papanicolau V G and Zisis V 2004 Inverse scattering on the line for a generalized nonlinear Schrödinger equation *Inverse Problems* **20** 1267

[2] Ola P, Päivärinta L and Serov V 2001 Recovering singularities from backscattering in two dimensions *Comm. Partial Diff. Eqns.* **26** 697

[3] Päivärinta L, Serov V and Somersalo E 1994 Reconstruction of singularities of a scattering potential in two dimensions *Adv. in Appl. Math.* **15** 97

[4] Päivärinta L and Serov V 2007 Recovery of jumps and singularities in the multidimensional Schrödinger operator from limited data *Inverse Problems and Imaging* **1** 525

[5] Reyes J M 2007 Inverse backscattering for the Schrödinger equation in 2D *Inverse Problems* **23** 625

[6] Ruiz A 2001 Recovery of the singularities of a potential from fixed angle scattering data *Comm. Partial Diff. Eqns.* **26** 1721

[7] Ruiz A and Vargas A 2005 Partial recovery of a potential from backscattering data *Comm. Partial Diff. Eqns.* **30** 67

[8] Serov V and Päivärinta L 2005 New estimates of the Green-Faddeev function and recovering of singularities in the two-dimensional Schrödinger operator with fixed energy *Inverse Problems* **21** 1291
[9] Serov V and Harju M 2007 Reconstruction of discontinuities in the nonlinear one-dimensional Schrödinger equation from limited data Inverse Problems 23 493
[10] Serov V 2007 Inverse Born approximation for the nonlinear two-dimensional Schrödinger operator Inverse Problems 23 1259
[11] Serov V and Harju M 2007 Partial recovery of potentials in generalized nonlinear Schrödinger equations on the line J. Math. Phys. 48 18 pp
[12] Sun Z and Uhlmann G 1993 Recovery of singularities for formally determined inverse problems Comm. Math. Phys. 153 431
[13] Weder R 1997 Inverse scattering for the nonlinear Schrödinger equation Comm. Partial Diff. Eqns. 22 2089
[14] Weder R 2001 Inverse scattering for the nonlinear Schrödinger equation: reconstruction of the potential and the nonlinearity Math. Meth. Appl. Sci. 24 245
[15] Weder R 2001 Inverse scattering for the nonlinear Schrödinger equation II. Reconstruction of the potential and the nonlinearity in the multidimensional case Proc. Amer. Math. Soc. 129 3637