On the Scaling and Spacing of Extra-Solar Multi-Planet Systems

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ABSTRACT

We investigate whether certain extra-solar multi-planet systems simultaneously follow the scaling and spacing rules of the angular-momentum-deficit model. The masses and semi-major axes of exoplanets in ten multi-planet systems are considered. It is found that GJ 667C, HD 215152, HD 40307, and Kepler-79 systems are currently close to configurations of the angular-momentum-deficit model. In a gas-poor scenario, GJ 3293, HD 141399, and HD 34445 systems are those which had a configuration of the angular-momentum-deficit model in the past and get scattered away due to post gaseous effects. In addition, no matter in gas-free or gas-poor scenario, 55 Cnc, GJ 876, and WASP-47 systems do not follow the angular-momentum-deficit model. Therefore, our results reveal important formation histories of these multi-planet systems.

Subject headings: planetary systems – stars: individual

1. Introduction

It is well known that the relative spacing of most planets in the Solar System approximately follow an empirical rule, i.e. the Titus-Bode Rule (TBR). Although the physical mechanism of this rule was unclear, it successfully predicted the existence of Uranus and Ceres. The discoveries of these celestial bodies were remarkable and regarded as a great success of TBR. The significance of the TBR for the distribution of the planets in the Solar system was later statistically investigated in Lynch (2003).

The above success motivated some theoretical investigations on possible mechanisms which could explain the planet spacing. This work was based on the physics of planetary formation.
For example, Rawal (1986) studied the spacing relations through the formation and evolution of gaseous rings. Patton (1988) tried to obtain the planet spacing through an algorithm derived from the principle of least action interaction. Though Patton (1988) did not provide an analytic formula, the numerical results of planet spacing were very close to the TBR. This implied that the least action interaction could be a physical reason that the TBR gave a good approximation for planet and satellite spacing in the Solar System. In addition, Hayes & Tremaine (1998) employed a stability condition which was based on Hill radii to address the spacing of the Solar system, and concluded that the planets are not randomly but regularly spaced in a stable system. Later, Griv & Gedalin (2005) found that a theory of disk instability could lead to another rule of planet spacing, which is different though similar to the TBR.

More recently, Christodoulou & Kazanas (2017) considered the geometric TBR for the semi-major axes of planetary orbits. They interpreted it in terms of the work done in the gravitational field of the central star by particles whose orbits are perturbed around each planetary orbit. Further, Laskar (2000) and Laskar & Petit (2017) developed a model of planet formation based on an angular-momentum-deficit theory. Their model starts from a phase that the disk is composed of planet embryos and planetesimals after the gas is depleted. Two main analytic laws are obtained and presented in their papers. The first one gives a scaling relation between the period-ratio and mass-ratio of adjacent planets (the scaling rule hereafter). The second one leads to the spacing between adjacent planets (the spacing rule hereafter). This spacing rule is equivalent to the TBR for a particular mass distribution of planetesimal disk, and thus can be regarded as a generalized TBR.

The discoveries of extra-solar planets (exoplanets) gave excellent opportunities to further examine the TBR. For example, the 55 Cancri planetary system was studied by Poveda & Lara (2008) and Cuntz (2012). The TBR was employed to compare its predictions with the planet spacing. Interestingly, Cuntz (2012) further predicted possible new planet candidates in the 55 Cancri planetary system through the TBR, and claimed one of them to be located in the outskirts of the habitable zone. Bovaird & Lineweaver (2013) studied those extra-solar planetary systems with higher multiplicity. They found that these planetary systems do not exactly follow the TBR. Nevertheless, they obtained a new two-parameter relation empirically. As a test of this new relation, Huang & Bakos (2014) then used Kepler data to search for possible new planets predicted by that relation. They successfully identified five planetary candidates around predicted positions. However, among most systems in their study, there were no signals of new planets. Moreover, Aschwanden (2018) developed a rule of quantized harmonic ratios and applied to the planet spacing of extra-solar planetary system and also moon spacings in the Solar system. Finally, Pletser (2019) found that the spacing of the above systems could be related to Fibonacci numbers.

On the other hand, because the mass and orbital period are the most important parame-
ters, there have been many studies about the distributions of exoplanets in the period-mass plane (Zucker & Mazeh 2002, Tabachnik & Tremaine 2002, Jiang et al. 2006). Additionally, the coupled period-mass functions were first explored in Jiang et al. (2007, 2009), and then further investigated with proper treatments of the selection effect in Jiang et al. (2010). Moreover, Jiang et al. (2015) studied the period-ratio-mass-ratio correlation of adjacent planet pairs in 166 multiple planetary systems. A moderate correlation between the period-ratio and mass-ratio was found with a correlation coefficient 0.5303.

This correlation between adjacent planets strongly indicates that planets might follow the scaling or spacing rule. It motivates us to investigate whether there are any known multi-planet systems that satisfy both the scaling and spacing rules in the angular-momentum-deficit (AMD) model, which has a clear physical foundation and provides analytic expressions.

The main expressions of the AMD model are summarized in Section 2. The employed data of planetary systems is described in Section 3. The data-model fitting results are in Section 4. We propose a scenario with gaseous effects to explain the data-model deviations and do another fitting in Section 5. We convey conclusions in Section 6.

2. The AMD Model

Based on Eq. (10) in Laskar & Petit (2017), the total angular momentum of a planetary system would have its maximum value when all bodies are moving in circular orbits in a coplanar system. For a realistic system, the bodies’ orbits have different eccentricities and inclinations, and thus the total angular momentum is smaller. The AMD is defined to be the difference between the maximum and the real value of total angular momentum. The AMD could evolve with time when there are collisions between bodies, i.e. planetesimals or embryos. In fact, the AMD was used to study the evolution of inner Solar system (Chambers 2001).

During the stage of planetary formation, collisions between planetesimals and embryos could happen continuously in the disk until the system is settled and becomes more stable. A planetary system is called AMD-stable if its AMD is not sufficient to allow for further collisions. Laskar & Petit (2017) derived the predicted planetary distribution in an AMD-stable system. Their results lead to a scaling rule for adjacent planets, and a spacing rule for the planets in the system.

Using semi-major axis $a$ as the variable, the mass distribution of a planetesimal disk is set to be

$$\frac{dm}{da} \equiv \rho(a) = \rho_0 a^p,$$

where $m$ is the planetesimal mass as a function of $a$, $\rho(a)$ is the linear mass density as a function
of $a$, $\rho_0$ is a constant, and $p$ is the power index. Depending on the disk structure, the value of $p$ could be negative, zero, or positive. According to Laskar & Petit (2017), for a given $p$, the disk surface density is proportional to $r^{p-1}$. For example, when $p = 0$, it corresponds to a surface density proportional to $r^{-1}$. Laskar (2000) discussed the cases when $p=0$, -1/2, -1, and -3/2. In this paper, $p$ is allowed to be any real number, which would be determined through the data-model fitting. However, for those models with $p > 1$, their disk surface density would increase with the radius $r$, and thus are considered as unphysical models.

The scaling rule for a pair of consecutive planets is stated as:

$$\frac{m_i}{m_o} = \left(\frac{a_i}{a_o}\right)^{\frac{1}{2} + \frac{2}{3}p},$$

(2)

or equivalently

$$\ln\left(\frac{m_o}{m_i}\right) = \left[\frac{1}{2} + \frac{2}{3}p\right] \ln\left(\frac{a_o}{a_i}\right),$$

(3)

where $m_o$ and $m_i$ are the masses of the outer and inner planets and $a_o$ and $a_i$ are the semi-major axes of the outer and inner planets, respectively.

On the other hand, the spacing rule is separated into two cases in terms of the power index $p$. When $p \neq -3/2$, the semi-major axis $a$ of the $n$-th planet satisfies

$$a^b = a_0^b + bD_0n,$$

(4)

with

$$b \equiv \frac{2p + 3}{6},$$

(5)

$$D_0 \equiv \left(\frac{4C}{\rho_0 \sqrt{GM_s}}\right)^{1/3},$$

(6)

where $a_0$ is a constant, $C$ is the AMD, $G$ is the gravitational constant, and $M_s$ is the mass of the central star of the multi-planet system. When $p = -3/2$, the relation is

$$\ln(a) = \ln(a_0) + D_0n.$$  

(7)

In this paper, we will examine whether those planets in multi-planet systems satisfy both scaling and spacing rules of the AMD model. These results would lead to important information on planet formation in multi-planet systems.
3. The Exoplanet Data

To obtain suitable samples for our study, the exoplanet catalog in http://exoplanet.eu/catalog/ was utilized. The conditions of selecting multi-planet systems were set as:
(a) the number of planets is three or larger;
(b) the values of minimum masses and the corresponding error-bars are available;
(c) the values of semi-major axes and the corresponding error-bars are available;
(d) the error of minimum mass cannot be larger than the minimum mass itself;
(e) the error of semi-major axis cannot be larger than the semi-major axis itself.

In the end, ten planetary systems are chosen as the samples. The names, the numbers of planets, and the references of these systems are shown in Table 1.

| System ID | Name       | Planet Number | Reference                  |
|-----------|------------|---------------|----------------------------|
| 1         | 55 Cnc     | 5             | Nelson et al. (2014)       |
| 2         | GJ 3293    | 4             | Astudillo-Defru et al. (2017) |
| 3         | GJ 667C    | 6             | Anglada-Escude et al. (2013) |
| 4         | GJ 876     | 4             | Jenkins et al. (2014)      |
| 5         | HD 141399  | 4             | Hebrard et al. (2016)      |
| 6         | HD 215152  | 4             | Delisle et al. (2018)      |
| 7         | HD 34445   | 6             | Vogt et al. (2017)         |
| 8         | HD 40307   | 6             | Diaz et al. (2016)         |
| 9         | Kepler-79  | 4             | Jontof-Hutter et al. (2014) |
| 10        | WASP-47    | 4             | Weiss et al. (2017)        |

The minimum masses, semi-major axes, and their corresponding errors regarding the planets in the above ten multi-planet systems, originally from the references in Table 1, are listed in Appendix A. Because the orbital inclinations of many planets in the above ten systems are not known, the confirmed planetary masses are not available in general. Nevertheless, as presented in the previous section, only mass ratios of two adjacent planets are involved in the AMD model, we thus use the values of minimum mass to represent the real masses in this paper. This leads to correct mass ratios under the assumption that these multi-planet systems are coplanar systems. This shall be a good approximation here. In addition, the values of planetary mass hereafter means the values of minimum masses shown in Appendix A.
4. A Gas-Free Scenario

The scaling rule and spacing rule of the AMD model were obtained based on the assumption that the gas was depleted during the early stage of planet formation. In a gas-free environment, the planetesimals and embryos dynamically involved to a settled configuration. If that process takes place and such a configuration exists, the planets shall follow what the AMD model predicts. Thus, what we aim to investigate here is whether the masses and semi-major axes of exoplanets are governed by the scaling and spacing rules.

Ideally, the data can be taken to fit both rules simultaneously. However, the spacing rule is divided into two different equations based on the value of the power index $p$. Thus, we first examine whether the planetary masses and semi-major axes follow the scaling rule, and through the fitting, we obtain the best-fit $p$. Using this value of $p$, the spacing rule is then fitted with data accordingly.

In order to simplify the notation of the scaling rule in Eq.(3), we set $y = \ln(m_o/m_i)$ and $x = \ln(a_o/a_i)$. Thus, the scaling rule, $y = (1/2 + 2p/3)x$, is a straight line passing through the origin. For the corresponding observational data, we use $y_j$ and $x_j$, $j = 1, 2, ..., N$, where $N$ is the number of pairs. For example, $y_1$ is the logarithm of planetary mass ratio of innermost adjacent pair, and $x_1$ is the logarithm of the ratio of semi-major axes of the innermost adjacent pair.

The values of $y_j$ and $x_j$ can be calculated from observational values of planetary masses and semi-major axes listed in Appendix A. Through the method of error propagation (see Appendix B), the error bars of both $y_j$ and $x_j$ can be determined.

For the data-model fitting, the $\chi^2$ minimization is employed (Press et al. 1992). Through a Markov Chain Monte Carlo (MCMC) sampling to minimize the $\chi^2$ function, the reduced $\chi^2$ which measures the agreement between the data and the rule is determined. In this paper, the emcee code (Foreman-Mackey et al. 2013) is chosen as the tool for the MCMC sampling.

For convenience, we define $\beta \equiv 1/2 + 2p/3$, and thus the scaling rule is simply $y = \beta x$. During the MCMC sampling of $\chi^2$ minimization, $y$ is the fitting function, $x$ is the variable, and $\beta$ is the fitting parameter. Through the $\chi^2$ minimization, the $\beta$ and the corresponding reduced $\chi^2$, i.e. $\chi^2_{r1}$, would be determined. Then, the best-fit parameter $\beta$ leads to the best-fit value of $p$. Once the best-fit power-index $p$ is obtained, the value $p$ is substituted into the spacing rule. Then, the spacing rule is also fitted in a similar way. The best-fit parameters $D_0$, $a_0^b$, and the corresponding reduced $\chi^2$, i.e. $\chi^2_{r2}$, can be obtained.

All the above results are presented in Table 2. For System 1 and 10, it is found that $p$ is larger than one and $a_0^b$ is negative, and thus the corresponding models are unphysical. Fig. 1-2 show the observational data with error bars, and the best-fit straight line of the scaling rule and the spacing rule, respectively. In addition, the values of reduced $\chi^2$ indicate whether the fittings
are satisfactory. For a good fitting, the reduced $\chi^2$ shall be less than one or two (Wall & Jenkins 2003). From Table 2, it is clear that the $\chi^2_{r1}$ of System 3, 6, 9 are all smaller than one, the $\chi^2_{r1}$ of System 8 is around two, and the $\chi^2_{r1}$ of the rest indicate bad fittings. As shown in Fig. 1, System 3, 6, 8, 9 are consistent with the scaling rule approximately.

For the spacing rule, we only need to discuss the results of System 3, 6, 8, 9 because the other systems already fail to fit the scaling rule. It is found that the $\chi^2_{r2}$ of System 3, 6, 9 are smaller than one, and the $\chi^2_{r2}$ of System 8 is around two. They follow the spacing rule as presented in Fig. 2. Considering both scaling and spacing rules, we conclude that System 3, 6, 8, 9, i.e. GJ 667C, HD 215152, HD 40307, Kepler-79 are consistent with the AMD model. It is noted that their corresponding AMD models are all physically acceptable.

### Table 2

| ID | $p$  | $\chi^2_{r1}$ | $D_0$ (AU) | $a_0^b$ (AU) | $a_0$ (AU) | $\chi^2_{r2}$ |
|----|------|--------------|------------|-------------|------------|--------------|
| 1  | 1.16$^{+0.10}_{-0.10}$ | 63.88 | 0.27$^{+0.003}_{-0.003}$ | $-0.23^{+0.0053}_{-0.0053}$ | $-$ | 193.33 |
| 2  | 1.21$^{+0.33}_{-0.33}$ | 56.46 | 0.098$^{+0.018}_{-0.018}$ | 0.0052$^{+0.039}_{-0.039}$ | 0.0029 | 1.03 |
| 3  | $-1.23^{+0.09}_{-0.91}$ | 0.20 | 0.37$^{+1.02}_{-1.01}$ | 0.75$^{+0.46}_{-0.47}$ | 0.038 | 0.0013 |
| 4  | $-1.26^{+0.09}_{-0.06}$ | 6714.12 | 0.58$^{+0.18}_{-0.18}$ | 0.74$^{+0.048}_{-0.048}$ | 0.021 | 0.66 |
| 5  | $-0.39^{+0.20}_{-0.19}$ | 38.65 | 0.70$^{+0.11}_{-0.11}$ | 0.40$^{+0.080}_{-0.080}$ | 0.082 | 4.06 |
| 6  | 0.15$^{+0.75}_{-1.74}$ | 0.27 | 0.085$^{+0.30}_{-0.30}$ | 0.15$^{+0.43}_{-0.43}$ | 0.030 | 0.0056 |
| 7  | $-0.46^{+0.38}_{-0.36}$ | 15.01 | 0.52$^{+0.087}_{-0.086}$ | 0.39$^{+0.093}_{-0.094}$ | 0.067 | 0.70 |
| 8  | $0.34^{+0.28}_{-0.28}$ | 1.80 | 0.16$^{+0.021}_{-0.021}$ | 0.020$^{+0.046}_{-0.046}$ | 0.0017 | 1.87 |
| 9  | $-1.78^{+1.57}_{-1.58}$ | 0.20 | 0.45$^{+3.33}_{-4.34}$ | 1.26$^{+1.35}_{-1.35}$ | 0.085 | $9.96 \times 10^{-5}$ |
| 10 | $4.10^{+1.46}_{-0.92}$ | 13.42 | 0.0025$^{+0.0026}_{-0.0026}$ | $-0.0038^{+0.0001}_{-0.0052}$ | $-$ | 1.74 |

5. A Gas-Poor Scenario

Here we consider a situation that the gaseous disk is not completely depleted while the planetesimals and planetary embryos start to grow through collisions. Because the collisions and interactions between objects are completely random, the net result could be very close to what the AMD model predicts. Statistically, the presence of a small amount of gas might only make the process described in the AMD model slower. Therefore, there could be a phase that the planetary system is consistent with a configuration described by the AMD model. After planets settle to that configuration with particular masses and semi-major axes, the existence of gas would cause two further slow processes described below.
Firstly, because planets have grown to certain sizes, their gravitational force can attract the nearby gas efficiently. The gas would accrete onto the planets and increase the planetary masses. Once this process finishes, the planets are surrounded by gaps. They would stop growing and reach the final masses as observed today. Secondly, after the planetary masses are settled and the gaps are opened, planets might migrate slightly due to the torque caused by the gaseous disk. Their current semi-major axes are thus different from what the AMD model predicts. Thus, the system would change from the AMD-model configuration due to post gaseous effects. This could produce the observed deviation from the AMD model.

Therefore, we investigate whether there was such an AMD-model phase. In order to move backwards in time for the above mentioned two processes, we first allow the semi-major axes of the planets to be different from the current observed values, and inspect which combination could improve the fitting of the scaling and spacing rules. We then assign several masses, which are the same or less than the observed values, to each planet of a multi-planet system and examine which combination leads to the best-fit configuration.

There is no general rule how far each planet could be deviated from the current orbit in the past. For our purpose, it could be good enough if we approximately allow one-third of the current semi-major-axis difference. For a given multi-planet system, the innermost planet, i.e. the 1st planet, has a current semi-major axis $a_1$, the 2nd planet has a current semi-major axis $a_2$, and the $i$th planet has a current semi-major axis $a_i$, $i = 3, \cdots, n$, where $n$ is the total number of planets in this system. The five values of semi-major axis assigned to the innermost planet are $a_1 - a_1 / 3$, $a_1 - a_1 / 3 + a_1 / 6$, $a_1, a_1 + a_{2,1} / 6, a_1 + a_{2,1} / 3$, where $a_{2,1} \equiv a_2 - a_1$. The five values of semi-major axis assigned to the 2nd planet are $a_2 - a_{2,1} / 3, a_2 - a_{2,1} / 3 + a_{2,1} / 6, a_2, a_2 + a_{3,2} / 6, a_2 + a_{3,2} / 3$, where $a_{3,2} \equiv a_3 - a_2$. The five values of semi-major axis assigned to the the $i$th planet, $i = 3, \cdots, n - 1$ can be worked out similarly. Finally, the five values of semi-major axis assigned to the $n$th planet, i.e. the outermost planet, are $a_n - a_{n,n-1} / 3, a_n - a_{n,n-1} / 3 + a_{n,n-1} / 6, a_n, a_n + a_{n,n-1} / 6, a_n + a_{n,n-1} / 3$, where $a_{n,n-1} \equiv a_n - a_{n-1}$.

Each planet is assigned one of the above semi-major axes and the combination is considered as one of the possible past configuration of planets. The error bars of semi-major axes, planetary masses, and the error bars of masses are set to be the observed values here. For each of the possible configuration, the reduced $\chi^2$ and the best-fit power-index $p$ of the scaling rule are determined. After all reduced $\chi^2$ of different combinations are obtained, the configuration with the smallest reduced $\chi^2$ is the most likely one at this stage.

Secondly, in order to consider the mass accretion process back in time, we now allow each planet to have five possible values of mass. The maximum is set to be $m + m_+$, where $m$ is the current observed mass and $m_+$ is the upper bound of the error bar (Appendix A). The minimum is set to be $(m + m_+)/3$. Then, three more values are set between them with uniform gaps.
Similarly, each planet is assigned one of the above masses and the combination is considered as one of the possible past configuration of planets. Their semi-major axes are those from the most likely choices we just determined. Through exactly the same process, the configuration with the smallest reduced $\chi^2$, which is set as $\chi^2_{r1}$, is regarded as the past configuration of the planetary system in this gas-poor scenario. With the above best-fit power-index $p$, the spacing rule is fitted in a similar way. The best-fit parameters $D_0$, $a_0^b$, and the corresponding $\chi^2_{r2}$ are therefore determined.

For all considered planetary systems, the corresponding parameters of past configurations are presented in Table 3. For System 1, it is found that $a_0^b$ is negative; for System 10, $p$ is larger than one and $a_0^b$ is negative. Their corresponding models are thus unphysical. These past configurations and the best-fit straight lines of scaling and spacing rules are shown in Fig. 3 and Fig. 4, respectively.

According to Table 3, the values of both $\chi^2_{r1}$ and $\chi^2_{r2}$ are smaller or around two for System 2, 3, 5, 6, 7, 8, 9. Indeed, as shown in Fig. 3-4, these systems follow both scaling and spacing rules and can be regarded as the multi-planet systems which have AMD-model phases.

We also plot the comparison of the semi-major axes and the masses between the observational data and the past configurations in Fig. 5 and Fig. 6, respectively. The deviations caused by the orbital migrations and mass accretions are clearly presented.

### Table 3

| ID | $p$ | $\chi^2_{r1}$ | $D_0$ (AU) | $a_0^b$ (AU) | $a_0$ (AU) | $\chi^2_{r2}$ |
|----|----|-------------|------------|--------------|------------|-------------|
| 1  | 0.51$^{+0.16}_{-0.16}$ | 3.17 | $0.32^{+0.01}_{-0.01}$ | $-0.17^{+0.016}_{-0.016}$ | – | 21.53 |
| 2  | 0.53$^{+0.31}_{-0.31}$ | $9.16 \times 10^{-3}$ | $0.15^{+0.03}_{-0.03}$ | $0.041^{+0.056}_{-0.056}$ | 0.009 | 0.623 |
| 3  | $-0.72^{+0.95}_{-0.95}$ | $2.18 \times 10^{-3}$ | $0.27^{+0.031}_{-0.031}$ | $0.39^{+0.036}_{-0.037}$ | 0.027 | $9.76 \times 10^{-3}$ |
| 4  | $-4.90^{+0.12}_{-0.12}$ | 447.84 | $4.04^{+0.30}_{-0.30}$ | $21.32^{+1.33}_{-1.34}$ | 0.067 | 5.80 |
| 5  | $-0.79^{+0.23}_{-0.24}$ | 0.057 | $0.66^{+0.18}_{-0.18}$ | $0.72^{+0.087}_{-0.087}$ | 0.241 | 2.17 |
| 6  | $-0.56^{+3.38}_{-3.42}$ | $1.49 \times 10^{-3}$ | $0.14^{+1.78}_{-1.78}$ | $0.35^{+1.57}_{-1.57}$ | 0.037 | $2.68 \times 10^{-4}$ |
| 7  | $-0.07^{+0.39}_{-0.39}$ | $8.68 \times 10^{-3}$ | $0.39^{+0.03}_{-0.05}$ | $0.44^{+0.08}_{-0.08}$ | 0.178 | 1.54 |
| 8  | $-0.008^{+0.46}_{-0.48}$ | 0.055 | $0.263^{+0.04}_{-0.04}$ | $0.005^{+0.09}_{-0.09}$ | $3.18 \times 10^{-5}$ | 1.05 |
| 9  | $-2.00^{+0.09}_{-0.09}$ | $1.47 \times 10^{-3}$ | $0.51^{+1.05}_{-0.06}$ | $1.50^{+0.97}_{-0.96}$ | 0.090 | $3.83 \times 10^{-3}$ |
| 10 | $3.26^{+1.40}_{-0.84}$ | 4.46 | $0.0057^{+0.0061}_{-0.0062}$ | $-0.0066^{+0.012}_{-0.012}$ | – | 1.87 |

6. Conclusions

The origin of our Solar System is one of the most important topics in astronomy. The existence of extra-solar multi-planet systems has given a new opportunity to understand the formation
of planetary systems in general. Since the AMD model is the only theory which gives analytic expressions for both the mass and spacing distributions of planets, it is essential to know whether the detected extra-solar systems follow the AMD model.

In this paper, the masses and semi-major axes of planets in ten multi-planet systems are taken to be examined. It is found that four systems, i.e. GJ 667C, HD 215152, HD 40307, and Kepler-79, follow the scaling and spacing rules of the AMD model.

Moreover, for the gas-poor scenario in which the mass accretion onto planets from the gaseous disk and the orbital migrations caused by the gas were taken into account, we find that GJ 3293, HD 141399, and HD 34445 systems were also close to the AMD-model configurations and got scattered away later.

The embryos and planetesimals of the above seven systems were probably mainly grow is a gas-free or gas-poor environment. The limited gaseous effects explain why these seven systems are consistent with the AMD model.

However, if the embryos and planetesimals grow in gas-rich environments, they would not be close to AMD model. This could be why the configurations of other three systems, i.e. 55 Cnc, GJ 876, and WASP-47 systems do not follow the AMD model. Note that some of these three systems, for example, 55Cnc, could still follow the TBR (Cuntz 2012), which could be driven by other unrelated mechanisms. To conclude, our results provide important information on the formation histories of these multi-planet systems.

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Fig. 1.— The scaling law with the current data. The circles are the observational data in the $\ln(a_o/a_i)$-$\ln(m_o/m_i)$ plane. The error bars are determined by equations in Appendix B. The straight dashed lines are the best-fit models. Panel (i) is for System i, where $i=1,2,...,10$. 
Fig. 2.— The spacing law with the current data. The circles are the observational data in the $n-a^{(2p+3)/6}$ plane, where $n$ is the planet order. The error bars are determined by equations in Appendix B. The straight dashed lines are the best-fit models. Panel (i) is for System i, where $i=1,2,\ldots,10$. 
Fig. 3.— The scaling law with hypothetical data. The crosses are the data in the \( \ln(a_o/a_i) - \ln(m_o/m_i) \) plane. The error bars are determined by equations in Appendix B. The straight dashed lines are the best-fit models. Panel (i) is for System i, where i=1,2,...,10.
Fig. 4.— The spacing law with hypothetical data. The crosses are the data in the \( n - a^{(2p+3)/6} \) plane, where \( n \) is the planet order. The error bars are determined by equations in Appendix B. The straight dashed lines are the best-fit models. Panel (i) is for System i, where \( i = 1,2,\ldots,10 \).
Fig. 5.— The observed and hypothetical values of semi-major axes of all planets. The circles are the observational and crosses are the hypothetical. Panel (i) is for System i, where i=1,2,...,10.
Fig. 6.— The observed and hypothetical values of masses of all planets. The circles are the observational and crosses are the hypothetical. Panel (i) is for System i, where $i=1,2,...,10$. 
Appendix A

The data of ten systems are listed here, where $m$ is the planetary minimum mass with lower error bound $m_-$ and upper error bound $m_+$; $a$ is the planetary semi-major axis with lower error bound $a_-$ and upper error bound $a_+$.

| ID | Name       | $m$  | $m_-$ | $m_+$  | $a_-$ | $a_+$ |
|----|------------|------|-------|--------|-------|-------|
|    |            | ($M_J$) | ($M_J$) | (AU) | ($M_J$) | (AU) |
| 1  | 55 Cnc b   | 0.84  | 0.031 | 0.23   | 0.1139 | 0.00011 |
|    | 55 Cnc c   | 0.1784 | 0.0078 | 0.0275 | 0.23735 | 0.00024 |
|    | 55 Cnc d   | 3.86  | 0.15  | 0.6    | 5.446 | 0.02 |
|    | 55 Cnc e   | 0.02547 | 0.00081 | 0.00089 | 0.015439 | 1.50 E−5 |
|    | 55 Cnc f   | 0.1479 | 0.0093 | 0.0219 | 0.7733 | 0.001 |
| 2  | GJ 3293 b  | 0.07406 | 0.0028 | 0.0028 | 0.14339 | 0.0003 |
|    | GJ 3293 c  | 0.06636 | 0.00396 | 0.00396 | 0.36175 | 0.00048 |
|    | GJ 3293 d  | 0.024  | 0.0031 | 0.0031 | 0.194 | 0.00018 |
|    | GJ 3293 e  | 0.0103 | 0.002 | 0.002 | 0.0821 | 4.00 E−5 |
| 3  | GJ 667C b  | 0.0176 | 0.0041 | 0.0044 | 0.0505 | 0.0053 |
|    | GJ 667C c  | 0.012  | 0.0038 | 0.0047 | 0.125 | 0.013 |
|    | GJ 667C d  | 0.01604 | 0.00535 | 0.00566 | 0.276 | 0.03 |
|    | GJ 667C e  | 0.0085 | 0.0044 | 0.005 | 0.213 | 0.02 |
|    | GJ 667C f  | 0.0085 | 0.0038 | 0.0044 | 0.156 | 0.017 |
|    | GJ 667C g  | 0.0145 | 0.0072 | 0.0082 | 0.549 | 0.058 |
| 4  | GJ 876 b   | 1.938 | 0.014 | 0.036 | 0.208317 | 2.00 E−5 |
|    | GJ 876 c   | 0.856 | 0.029 | 0.032 | 0.12959 | 2.40 E−5 |
|    | GJ 876 d   | 0.022 | 0.001 | 0.001 | 0.02080665 | 1.50 E−7 |
|    | GJ 876 e   | 0.045 | 0.001 | 0.001 | 0.3343 | 0.0013 |
| 5  | HD 141399 b| 0.451 | 0.03 | 0.03 | 0.415 | 0.011 |
|    | HD 141399 c| 1.33 | 0.08 | 0.08 | 0.689 | 0.02 |
|    | HD 141399 d| 1.18 | 0.08 | 0.08 | 2.09 | 0.06 |
|    | HD 141399 e| 0.66 | 0.1 | 0.1 | 5 | 1.5 |
| 6  | HD 215152 b| 0.00628 | 0.00235 | 0.0013 | 0.057635 | 0.000755 |
|    | HD 215152 c| 0.004801 | 0.00177 | 0.00249 | 0.067399 | 0.000898 |
|    | HD 215152 d| 0.008511 | 0.00259 | 0.00287 | 0.08799 | 0.00115 |
|    | HD 215152 e| 0.01091 | 0.006711 | 0.00145 | 0.15417 | 0.00204 |
| ID | Name     | $m$  | $m_-$ | $m_+$ | $a$  | $a_-$ | $a_+$ |
|----|----------|------|-------|-------|------|-------|-------|
| 7  | HD 34445 b | 0.629 | 0.028 | 0.028 | 2.075 | 0.016 | 0.016 |
|    | HD 34445 c | 0.168 | 0.016 | 0.016 | 0.7181 | 0.0049 | 0.0049 |
|    | HD 34445 d | 0.097 | 0.037 | 0.13  | 0.4817 | 0.0033 | 0.0033 |
|    | HD 34445 e | 0.0529 | 0.0089 | 0.0089 | 0.2687 | 0.0019 | 0.0019 |
|    | HD 34445 f | 0.119 | 0.021 | 0.021 | 1.543 | 0.016 | 0.016 |
|    | HD 34445 g | 0.38 | 0.13 | 0.13 | 6.36 | 1.02 | 1.02 |
| 8  | HD 40307 b | 0.012 | 0.00094 | 0.00094 | 0.0475 | 0.0011 | 0.0011 |
|    | HD 40307 c | 0.0202 | 0.0014 | 0.0014 | 0.0812 | 0.0018 | 0.0018 |
|    | HD 40307 d | 0.0275 | 0.0018 | 0.0018 | 0.134 | 0.0029 | 0.0029 |
|    | HD 40307 e | 0.011 | 0.0044 | 0.0044 | 0.1886 | 0.0104 | 0.0083 |
|    | HD 40307 f | 0.0114 | 0.0019 | 0.0019 | 0.2485 | 0.0054 | 0.0054 |
|    | HD 40307 g | 0.0223 | 0.0082 | 0.0082 | 0.6 | 0.034 | 0.034 |
| 9  | Kepler-79 b | 0.034285 | 0.018872 | 0.023276 | 0.117 | 0.002 | 0.002 |
|    | Kepler-79 c | 0.018558 | 0.007234 | 0.005976 | 0.187 | 0.003 | 0.003 |
|    | Kepler-79 d | 0.018872 | 0.005033 | 0.006605 | 0.287 | 0.004 | 0.004 |
|    | Kepler-79 e | 0.0129 | 0.0034 | 0.0037 | 0.386 | 0.005 | 0.005 |
| 10 | WASP-47 b | 1.13 | 0.038 | 0.038 | 0.052 | 0.011 | 0.011 |
|    | WASP-47 c | 1.31 | 0.05 | 0.72 | 1.41 | 0.3 | 0.3 |
|    | WASP-47 d | 0.0428 | 0.0063 | 0.0063 | 0.088 | 0.019 | 0.019 |
|    | WASP-47 e | 0.029 | 0.0031 | 0.0031 | 0.0173 | 0.0038 | 0.0038 |
Appendix B

(1)

Let the parameter \( p = X^+_{-x} \), i.e. a value \( X \) with upper error bound \( x_+ \) and lower error bound \( x_- \); the parameter \( q = Y^+_{-y} \), i.e. a value \( Y \) with upper error bound \( y_+ \) and lower error bound \( y_- \).

In order to determine the upper and lower error bounds of the parameter ratio \( p/q \), we need the expression that when \( |s| << 1 \),
\[
\frac{p}{q} \approx \frac{X^+_{-x}}{Y^+_{-y}},
\]
we know the maximum is
\[
\frac{p}{q} = \frac{X + x_+}{Y - y_-} = \frac{X \left(1 + \frac{x_+}{X}\right)}{Y \left(1 - \frac{y_-}{Y}\right)} \approx \frac{X}{Y} \left(1 + \frac{x_+}{X}\right) \left(1 + \frac{y_-}{Y}\right)
\]
\[
\approx \frac{X}{Y} \left(1 + \frac{x_+}{X} + \frac{y_-}{Y}\right),
\]
and the upper error bound is
\[
\frac{X}{Y} \left(\frac{x_+}{X} + \frac{y_-}{Y}\right).
\]
Similarly, the minimum is
\[
\frac{p}{q} = \frac{X - x_-}{Y + y_+} = \frac{X \left(1 - \frac{x_-}{X}\right)}{Y \left(1 + \frac{y_+}{Y}\right)} \approx \frac{X}{Y} \left(1 - \frac{x_-}{X}\right) \left(1 + \frac{y_+}{Y}\right)
\]
\[
\approx \frac{X}{Y} \left(1 - \frac{x_-}{X} - \frac{y_+}{Y}\right),
\]
and the lower error bound is
\[
\frac{X}{Y} \left(\frac{x_-}{X} + \frac{y_+}{Y}\right).
\]

(2)

Given that the value of \( x \) has an upper error bound \( x_+ \) and a lower error bound \( x_- \), the upper and lower error bounds of a general function \( f(x) \) can be determined through Taylor series. When \( x_+ \) and \( x_- \) are small, using \( \Delta x \) to denote \( x_+ \) or \( -x_- \), we have
\[
f(x + \Delta x) \approx f(x) + f'(x)\Delta x + O((\Delta x)^2).
\]
The term \( f'(x)\Delta x \) is thus used as the error estimation. It equals to \( f'(x)x_+ \) or \( -f'(x)x_- \). When \( f'(x) \) is positive, the upper error bound of \( f(x) \) is \( f'(x)x_+ \), and the lower error bound of \( f(x) \) is
When \( f'(x) \) is negative, the upper error bound of \( f(x) \) is \(-f'(x)x_-\), and the lower error bound of \( f(x) \) is \(-f'(x)x_+\). For example, when the function \( f(x) = \ln(x) \), \( f'(x) > 0 \), the upper and lower error bounds are \( x_+/x \) and \( x_-/x \).

(3)

Given that the value of \( x \) has an upper error bound \( x_+ \) and a lower error bound \( x_- \), the value of \( y \) has an upper error bound \( y_+ \) and a lower error bound \( y_- \), the upper and lower error bounds of a general function \( f(x, y) \), where \( x \) and \( y \) are two independent variables, can be determined through Taylor series. When \( x_+, x_-, y_+, \) and \( y_- \) are small, using \( \Delta x \) to denote \( x_+ \) or \( -x_- \) and \( \Delta y \) to denote \( y_+ \) or \( -y_- \), we have

\[
f(x + \Delta x, y + \Delta y) \approx f(x, y) + \frac{\partial f(x, y)}{\partial x} \Delta x + \frac{\partial f(x, y)}{\partial y} \Delta y + O((\Delta x)^2, (\Delta y)^2, (\Delta x)(\Delta y)).
\]

(13)

The summation of below two terms, i.e.

\[
\frac{\partial f(x, y)}{\partial x} \Delta x + \frac{\partial f(x, y)}{\partial y} \Delta y,
\]

(14)

is thus used as the error estimation. Its four possible values can be determined after \( \Delta x \) is substituted by \( x_+ \) or \( -x_- \), and \( \Delta y \) is substituted by \( y_+ \) or \( -y_- \). The maximum value would be positive, and the minimum value would be negative. The upper error bound of \( f(x, y) \) is then equal to this maximum, and the lower error bound of \( f(x, y) \) is the absolute value of the minimum.