Active particles are driven out of equilibrium through a dissipative exchange of energy and momentum with their environment, which endows them with anomalous thermomechanical properties [11, 12]. Among these, the pressure exerted by active systems on their confining vessels has recently attracted a lot of interest [3, 5, 8, 9, 11, 13, 18]. In particular, the existence of an equation of state for the pressure was established for a large subclass of these non-equilibrium and momentum non-conserving systems. The defining feature which allows for an equation of state is a self-propulsion force whose dynamics is independent of the positional and angular degrees of freedom [11]. This encompasses standard models of active particles, such as run-and-tumble particles, active Brownianian particles (ABPs), and active Ornstein-Uhlenbeck particles (AOUPs), both non-interacting and in the presence of pairwise conserving forces. For such systems, which are the focus of this article, macroscopic mechanical properties echo the equilibrium case with, for instance, equality of pressures in coexisting phases [13]. This raises the hope of a generalized thermodynamics [19, 22].

While pressure controls the bulk thermodynamic state of a system in equilibrium, its interfacial properties rely on surface tension, perhaps the most elusive thermomechanical quantity whose microscopic origin has been the topic of long-standing debates [23, 25]. Surface tension indeed controls a wealth of phenomena, from the demixing of binary mixtures to the wetting of surfaces, and the instability of thin films and jets [26–29]. Given the atypical properties of active matter at interfaces, from the accumulation at boundaries [30, 32] to the wetting of soft gels by swarming bacteria [33–35], the study of surface tension in active systems has naturally attracted attention [7, 36]. In particular, the surface tension of active particles at a liquid-gas interface in a system undergoing motility-induced phase separation has been measured using an expression derived by Kirkwood and Buff for Hamiltonian systems [37, 38]. Despite stable interfaces, this liquid-gas surface tension was, somewhat surprisingly, found to be negative [7, 12, 22]. Its thermodynamic role has nevertheless been confirmed: it controls the pressure drop through the boundary of a circular droplet of radius \( R \) through a Laplace pressure given by \( \Delta P = \frac{\gamma R}{R} \) [22]. This combination of surprising and familiar aspects of interfacial physics enjoins us to clarify the microscopic origin of surface tension in these active systems and in particular its mechanical implications, which have been little discussed so far. Accounting for capillary and wetting phenomena in active matter is a formidable program which we attack by elucidating the interfacial properties of active fluids in contact with a solid boundary. For simplicity, we work in two dimensions throughout the article, the generalization to higher dimensions being straightforward.

In the absence of an established thermodynamic route, it is natural to revert to mechanics. The Langmuir balance [39] is a standard tool to measure the tangential force exerted on an interface between a fluid and a solid in equilibrium, and hence to access surface tension: Take a mobile wall made of two different materials confining a fluid in a cavity (see Fig 1-a);
Moving the wall changes the contact area between the fluid and each material; in equilibrium, the corresponding tangential force exerted by the fluid on the wall is thus given by the difference of surface tensions between the fluid and each material \( \Delta \gamma = \gamma_2 - \gamma_1 \). This highlights an important property of equilibrium systems: the tangential force is independent of the details of the junction between the two walls, as illustrated in Fig. 1-b. Strikingly, similar computations carried out for non-interacting ABPs depict a very different physics: the tangential force now strongly depends on the details of the junction between the two materials. The existence of an equation of state for the pressure of an active fluid thus does not entail a similar property for the fluid-solid surface tension.

In this article we indeed show that, in addition to an intrinsic, bare, equation-of-state-abiding contribution, the emergence of junction-dependent currents leads to an extra contribution that prevents the existence of a global equation of state for the tangential force. The Langmuir setup of Fig. 1 hence only accesses a surface tension dressed by these junction-dependent currents: the reading of the surface tension depends on the probe used for its measurement. In addition to its intrinsic, junction-independent nature, the bare surface tension plays an important, versatile role in other surface phenomena, much like in equilibrium. For example, we show how it controls the Young-Laplace corrections to the normal pressure felt by a circular confining cavity. We thus suggest a second setting designed to make the contribution of the currents vanish, hence providing a direct mechanical measurement of the bare surface tension. Equations (6) and (7) work in the color encodes its directions. In both figures, \( \lambda_1 = 5, \lambda_2 = 0.05, \ell = \ell_c/2, v = 5, D_x = 1, L_x = 2L_y = 1280 \). For the current field, \( w = 0.01 \).

**Current-induced tangential forces.** To model the experiment depicted in Fig. 1-a, we follow Navascués and Berry [41] and disregard the internal dynamics of the wall, which we model as a confining potential smoothly interpolating between the two materials. This allows us to focus on the new features due to activity and leads to a straightforward expression for the tangential force exerted by the particles on the wall:

\[
\mathbf{F}_x = \int_{x_1}^{x_2} dx \int_{y_b}^{\infty} dy \rho(x, y) \partial_x V(x, y)
\]

where \( \rho \) is the mean density of particles, \( x_1 \) and \( x_2 \) are abscissas far away from the junction between the two materials, and \( y_b \) is an ordinate in the ‘bulk’ of the system, at a distance \( \ell_w \equiv y_w - y_b \) from the upper confining wall (See Fig. 1). For an equilibrium ideal gas, a direct computation leads to

\[
\mathbf{F}_x = \gamma_2 - \gamma_1 \quad \text{with} \quad \gamma_k \equiv -\int_{y_w}^{\infty} dy k T \rho(x, y).
\]

\( \mathbf{F}_x \) is indeed independent of the details of the junction between the two walls. This is expected from thermodynamical considerations, from which \( \gamma_k \) could have been derived [42].

To elucidate analytically the dependence of \( \mathbf{F}_x \) on the width of the junction \( w \) reported for ABPs in Fig. 1-b, we start from the dynamics of \( N \) non-interacting active particles:

\[
\dot{\mathbf{r}}_i = v_0 \mathbf{u}_i - \mu \nabla_x V(\mathbf{r}_i),
\]

where the active force on particle \( i \) enters through \( v_0 \mathbf{u}_i / \mu \). We consider tumbles and rotational diffusion of the particle orientations \( \mathbf{u}_i = (\cos \theta_i, \sin \theta_i) \), and denote \( \tau \) the persistence time. The generalization to other models of active particles like AOUPS is straightforward. To compute the tangential force on the upper wall, we introduce the average density field \( \rho(\mathbf{r}) = \langle \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \rangle \), where the brackets represent averages over the active-force statistics. It evolves according to the continuity equation

\[
\partial_t \rho = -\nabla \cdot \mathbf{J}, \quad \mathbf{J} = v_0 \mathbf{m} - \rho \mu \nabla V \quad (4)
\]

where the local orientation field \( \mathbf{m}(\mathbf{r}) = \langle \sum_i \mathbf{u}_i \delta(\mathbf{r} - \mathbf{r}_i) \rangle \) controls the contribution of self-propulsion to the particle flow. In steady-state (see, e.g., [6]),

\[
m_{\alpha} = -\partial_\beta \left( v_0 \tau \left( Q_{\alpha\beta} + \frac{\rho \delta_{\alpha\beta}}{2} \right) - \mu \tau m_{\alpha} \partial_\beta V \right)
\]

where \( Q \) measures the local nematic order through \( Q_{\alpha\beta}(\mathbf{r}) = \langle \sum_i (\mathbf{u}_{i,\alpha} \mathbf{u}_{i,\beta} - \delta_{\alpha\beta}/2) \delta(\mathbf{r} - \mathbf{r}_i) \rangle \), with Greek letters referring to space directions. From Eqs. (1) and (4), the tangential force \( \mathbf{F}_x \) exerted on the upper wall can be written as

\[
\mathbf{F}_x = \mathbf{F}_D + \int_{x_1}^{x_2} \int_{y_b}^{\infty} \frac{v_0}{\mu} m_{\alpha}(x, y) dx dy
\]

where \( \mathbf{F}_D \) is the total drag force experienced by particles around the junction:

\[
\mathbf{F}_D = \frac{1}{\mu} \int_{x_1}^{x_2} dx \int_{y_b}^{\infty} dy J_x = -\mu \sum_i \langle \mathbf{r}_i \cdot \mathbf{e}_x \rangle
\]

Equation (5) has an appealing physical interpretation in terms of force balance: the total active force exerted by the particles,
FIG. 3. Left: Setup used to directly measure the bare surface tension. Right: The measurement of γ for two different confining materials are shown to be independent of the piston’s stiffnesses λw. Measuring γ from the force on the piston (blue squares) or from the ‘bulk correlator’ [5] evaluated at x = x0 (red triangles) lead to consistent results. Parameters: Lx = Ly = 50, v0 = 5, τ = 1. Upper and lower results correspond to confining materials of stiffness λ = 5 and λ = 0.05, respectively. The distance between the two measurements corresponds to ∆γ in Fig. 2.

which is the second term on the r.h.s. of (6), is split between the force exerted on the upper wall $F_x$ and the drag force $F_D$ exerted on the environment. Using Eq. (5), the isotropy of the bulk, translational invariance far from the junction, and that $\int_{y_b}^{y_0} dy \rho(x,y) = 0$ for $x = x_1$ or $x = x_2$ [43], the force balance equation (6) can be rewritten as

$$F_x = F_D + \gamma_2 - \gamma_1$$

where we define $\gamma_k$ as

$$\gamma_k = -\frac{v_0^2 \tau}{2\mu} \left( \int_{y_b}^{\infty} dy \rho(x,k,y) - \int_{y_b}^{y_0} dy \rho_0 \right).$$

Note that the final integral in Eq. (9) cancels out in Eq. (8); it ensures that $\gamma_k$ does not depend on $y_b$. Equations (8) and (9) form a central result of this letter; they are exact in the macroscopic limit $L_x, L_y, \ell_w \to \infty$ with $\rho_0 = N/L^2$ finite and $\ell_p \ll \ell_w \ll L_x, L_y$, where $\ell_p = v_0 \tau$ is the persistence length. The result has several implications. As in equilibrium, $\Delta \gamma = \gamma_2 - \gamma_1$ is a contribution to $F_x$ that is independent of the details of the junction; Eq. (9) is the equation of state for $\gamma_k$. It is determined by the excess density accumulated at the wall compared to the bulk values. Note that $\gamma_k$ can be measured independently in a translationally invariant system and thus solely depends on the interaction between the active fluid and wall $k$. Conversely, $F_D$ strongly depends on the details of the junction: the presence of an asymmetric potential combined with the nonequilibrium dynamics of the active particles generates, as in a ratchet, steady currents which are bound to be junction-dependent; the latter give rise to a non-zero drag force which dresses $F_x$. Fig. 2a indeed shows that the dependence of $F_x$ on the width $w$ of the junction is entirely due to the variations of $F_D$, whereas $\Delta \gamma$ is independent of $w$, and can be obtained from independent measurements of $\gamma_{1,2}$. Figure 2b suggests that the junction generates a current field which decays slowly with the distance to the junction. This is consistent with recent results on the power-law decay of currents generated by asymmetric obstacles located in the bulk of active fluids [44]. Note that the macroscopic limit defined above, in which $\ell_w \ll L_x, L_y$ ensures that $\Delta \gamma$ is indeed independent of the junction. Otherwise, if $\ell_w \sim L_x, L_y$, the measurement of $\Delta \gamma$ itself could be modified by the currents. In our numerics, we found that $\ell_w = 2\ell_p$ was sufficient for the system to be isotropic at $y = y_b$ and gives a reliable measurement of the bare $\Delta \gamma$.

Finally, in the $\tau \to 0$ limit, with $v_0^2 \tau$ fixed, active particles behave as equilibrium colloids at an effective temperature $kT_{eff} = v_0^2 \tau/(2\mu)$ and Eq. (9) reduces to its equilibrium counterpart [2]. The partial cancellation of the two integrals in (9) then stems from the fact that, unlike active particles, passive ones have a steady-state distribution which is a local function of the external potential $V(x,y)$. In this limit, as for true equilibrium systems, no currents survive in the steady state; $F_D$ vanishes and $F_x$ is then a direct measurement of the bare, equation-of-state-abiding contribution $\Delta \gamma$.

Finally, the existence of an equation of state for surface tension extends to equilibrium fluids with interactions [41]. Similarly, all our derivations extend to active fluids with pairwise forces at the cost of a lengthier expression for $\gamma$.

**Bare surface tension and Laplace pressure corrections.**

The use of surface tension is not limited, in equilibrium, to the measurement of tangential forces. A famous example is that of the Young-Laplace law which shows how surface tension impacts the pressure on a circular interface. In our solid-liquid framework, this amounts to computing the normal pressure exerted by an active fluid on a circular confinement. This follows from the Young-Laplace law $P = \rho_0 \gamma - \mu \partial_r V$. In the flux-free steady state observed in such a system, one has $\rho(r)\partial_r V = v_0 \tau\partial_r V$. Since Eq. (5) shows the local orientation to the divergence of a local tensor $\nabla \cdot \mathbf{T}$, the pressure is given, in circular coordinates, by

$$P(R) = -\frac{1}{R} \int_0^R \partial_r \left[ v_0 \tau (Q_{rr} + \rho/2) - \mu \tau \partial_r V \right] r dr = -\frac{1}{R} \int_0^R \left[ v_0 \tau (Q_{rr} - Q_{\theta\theta}) - \mu \tau \partial_r V \right] r dr.$$

Integrating by part the first term on the r.h.s. leads to $P(R) = R^{-1} \int_0^R v_0 \tau (Q_{\theta\theta} + \rho/2) dr$. Adding and subtracting the bulk pressure $P_b = \rho v_0^2 \tau/(2\mu)$ then leads, in the large $R$ limit, to

$$P(R) \simeq P_b - R^{-1} \int_0^R \frac{2\mu v_0^2 \tau \rho_0}{2\mu} dr - \int_0^R \frac{2\mu v_0^2 \tau \rho(r)}{2\mu} dr$$

where $\rho_0$ is the density at $r = 0$. Importantly, Eq. (10) shows that the pressure can be written as a generalized Young-Laplace law $P(R) = P_b - \frac{\gamma}{R} + \sigma(R)$, with $\gamma$ identified as the bare surface tension defined in [9], a connection not made previously. Despite its dressing by currents in the microscopic setting of Fig. 1, the bare surface tension thus plays an important role in active fluids. We now tackle the question of its direct mechanical measurement.
Direct measurement of the bare surface tension. The currents arising in the Langmuir setup, which prevents the direct measurement of the bare surface tension, are due to the junction. Furthermore, the Langmuir setup only gives access to the difference between the two fluid-solid surface tensions $\Delta \gamma$. We now consider instead the setup depicted in Fig. 3 in which, as we show below, the force exerted by the active fluid on a piston allows one to directly access the (bare) surface tension between the fluid and the walls of the container. Using Eq (3), the total force $F_y$ is

$$ F_y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \rho(x,y) \partial_y V(x,y) $$

exerted on the piston can be rewritten as

$$ F_y = -\frac{1}{\mu} \left( \sum_{i|x_i > x_{b}} \dot{x}_i \right) + \int_{-\infty}^{\infty} \int_{x_{b}}^{\infty} dx dy \frac{v_0}{\mu} m_x. $$

(11)

Equation (5) then allows us to simplify (11) into

$$ F_y = F_D + \int_{-\infty}^{\infty} dy \rho(x,y) \frac{v_0}{2\mu}x $$

(12)

where $F_D$ is defined as the first term in the r.h.s. of (11) and we have used that all fields vanish at infinity and that $\int_{-\infty}^{\infty} Q_{xx}(x,y) dy = 0$. Introducing the bulk pressure $P_b = \rho v_0^2 \tau / 2\mu$, $F_y$ is finally given by

$$ F_y = F_D + P_b y_b - 2\gamma $$

(13)

with $\gamma$ defined as in Eq (9), assuming for simplicity that the upper and lower walls are identical. At first glance, Eq. (13) seems to imply that, once again, the bare surface tension is dressed by currents through the drag force $F_D$. Note, however, that the average density of particles in the half-plane defined by $x > x^*$ is constant, for all $x^*$, in steady-state. Since $\dot{\rho} = -\nabla \cdot \mathbf{J}$, this implies that $\int_{-\infty}^{\infty} J_x(x^*,y) dy = 0$. Integrating this from $x^* = x_b$ up to infinity shows that $F_D = 0$ in this setup: $F_x$ thus provides a direct measurement of the bare surface tension between the fluid and the upper and lower walls, as shown in Fig. 3 right.

A Virial route to surface tension. In equilibrium, the mechanical route to surface tension can be circumvented by free energy considerations, a path which is not accessible to active systems. Another historical route to equations of states was proposed by Clausius through the Virial. This has recently been applied successfully to determine the pressure of active fluids \[5\] 15 18 40 and we now extend this approach to surface tension.

We consider $N$ active particles in a box of dimensions $L_x \times L_y$, evolving according to:

$$ \dot{\mathbf{r}}_i = v_0 \mathbf{u}_i + \mu \mathbf{F}_{i}^{\text{ext}} + \sum_j \mu \mathbf{F}_{j \rightarrow i} $$

(14)

where $\mathbf{F}_{i}^{\text{ext}}$ is the force exerted by the walls to confine particle $i$. Up to now, we had for clarity considered non-interacting particles. Our results however extend to particles interacting via pairwise forces $\mathbf{F}_{j \rightarrow i} = -\nabla \Phi(\mathbf{r}_i - \mathbf{r}_j)$, a case that we now consider more explicitly. To develop a Virial approach to surface tension, we introduce a modified Virial $V(t) = \frac{1}{2} \sum_i \langle \mathbf{r}_i^* \cdot \mathbf{r}_i^* \rangle$, where $\mathbf{r}_i^* = (x_i - y_i)$ are divergence-free vectors. We now use that $\dot{V}(t) = 0$ in the steady-state to get

$$ 0 = v_0 \sum_i \langle \mathbf{u}_i \cdot \mathbf{r}_i^* \rangle + \mu \sum_i \langle \mathbf{F}_{i}^{\text{ext}} \cdot \mathbf{r}_i^* \rangle + \mu \sum_{i,j} \langle \mathbf{F}_{j \rightarrow i} \cdot \mathbf{r}_i^* \rangle. $$

(15)

In the limit of hard walls, the external force density along a side of the box is naturally decomposed into pressure and surface tension contributions, as depicted in Fig. 3. Using this decomposition, a direct evaluation shows that $\langle \sum_i \mathbf{F}_{i}^{\text{ext}} \cdot \mathbf{r}_i^* \rangle = \gamma L_x + \gamma (L_x - L_y) - \gamma L_y - P_b L_x L_y - P_b L_y L_x$. The pressure contribution cancels out, since $\mathbf{r}_i^*$ is divergenceless, and Eq. (15) leads to our modified Virial for the surface tension:

$$ \gamma = -\frac{1}{2(L_x - L_y)} \sum_i \langle \mathbf{r}_i^* \cdot \left( \frac{v_0}{\mu} \mathbf{u}_i + \sum_j \mathbf{F}_{j \rightarrow i} \right) \rangle. $$

(16)

Much like the usual Virial formula, the presence of $\mathbf{r}_i^*$ in Eq. (16) makes its numerical measurement difficult 15. Using that $\partial_t \langle \mathbf{u}_i \cdot \mathbf{r}_i^* \rangle = 0$ in the steady state, we arrive at

$$ \tau \langle \mathbf{u}_i \cdot \mathbf{r}_i^* \rangle = \mu \langle \mathbf{F}_{i}^{\text{ext}} \cdot \mathbf{u}_i \rangle + \sum_j \mu \langle \mathbf{F}_{j \rightarrow i} \cdot \mathbf{u}_i \rangle, $$

(17)

where we have used $\langle \mathbf{u}_i \cdot \mathbf{u}_i^* \rangle = 0$ in steady-state, with $\mathbf{u}_i^* = (\cos \theta_i, -\sin \theta_i)$. All in all, the surface tension can thus be written in a more numerically-friendly way as

$$ \gamma = -\frac{1}{2(L_x - L_y)} \left[ v_0 \tau \left( \sum_i \langle \mathbf{F}_{i}^{\text{ext}} \cdot \mathbf{u}_i \rangle + \sum_{ij} \langle \mathbf{F}_{j \rightarrow i} \cdot \mathbf{u}_i \rangle \right) \right] + \frac{1}{2} \sum_{ij} \langle \mathbf{F}_{j \rightarrow i} \cdot (\mathbf{r}_i^* - \mathbf{r}_j^*) \rangle. $$

(18)

Equation (16) of course also holds for a passive dynamics in which it offers an alternative to the Kirkwood-Buff formula 37, of which Eq. (15) is a generalization to active systems for a solid-liquid interface. In the noninteracting limit, dimensional analysis shows the surface tension between an active gas and a hard wall to be given by

$$ \gamma = -\frac{v_0^3}{\mu^2} \rho_0, $$

(19)
where \( c \) is a dimensionless constant. Figure 4 shows measurements of the bare surface tension using Eq. 9 and the Virial (18) to coincide, and confirm the scaling of \( \gamma \) with \( \nu_0 \).

**Conclusion.** We have shown that, as in equilibrium, surface tension can be defined in active fluids by considering the tangential forces exerted on an interface. At a solid-fluid interface, any anisotropy of the boundary is expected to lead to steady currents that contribute to the force balance. We nevertheless identified a bare, equation-of-state abiding constant with its equilibrium counterpart. We hope this work will trigger new theoretical and experimental endeavours into interfacial problems in active matter, from the role of currents to that of wetting and capillarity in multiphase systems.

FvW, JT & RZ thank K. Mandadapu at UC Berkeley for hospitality and discussions, JT acknowledges support from ANR Bactterns, YK acknowledges support from I-CORE Program of the Planning and Budgeting Committee of the Israel Science Foundation, from an Israel Science Foundation grant and from the NSF-BSF. JT & YK acknowledge support from a joint CNRS-MOST grant. MK acknowledges support from the Alexander von Humboldt Foundation.

---

* These authors contributed equally.

[1] R. Di Leonardo, L. Angelani, D. Dell’Arciprete, G. Ruocco, V. Iebba, S. Schippa, M. Conte, F. Mecarini, F. De Angelis, and E. Di Fabrizio, Proceedings of the National Academy of Sciences 107, 9541 (2010).

[2] A. Sokolov, M. M. Apodaca, B. A. Grzybowski, and I. S. Aran, Proceedings of the National Academy of Sciences 107, 969 (2010).

[3] S. Mallory, A. Šarić, C. Valeriani, and A. Cacciuto, Physical Review E 89, 052303 (2014).

[4] S. Takatori, W. Yan, and J. F. Brady, Physical review letters 113, 028002 (2014).

[5] X. Yang, M. L. Manning, and M. C. Marchetti, Soft matter 10, 6477 (2014).

[6] A. P. Solon, Y. Fily, A. Baskaran, M. E. Cates, Y. Kafri, M. Kardar, and J. Tailleur, Nature Physics 11, 673 (2015).

[7] J. Bialké, J. T. Siebert, H. Löwen, and T. Speck, Phys. Rev. Lett. 115, 098301 (2015).

[8] C. Sandford, A. Y. Grosberg, and J.-F. Joanny, Physical Review E 96, 052605 (2017).

[9] J. Junot, G. Briand, R. Ledesma-Alonso, and O. Dauchot, Physical review letters 119, 028002 (2017).

[10] G. Vizsnyiczai, G. Frangipane, C. Maggi, F. Saglimbeni, S. Bianchi, and R. Di Leonardo, Nature communications 8, 15974 (2017).

[11] Y. Fily, Y. Kafri, A. P. Solon, J. Tailleur, and A. Turner, Journal of Physics A: Mathematical and Theoretical 51, 044003 (2017).

[12] A. Patch, D. M. Sussman, D. Villeda, and M. C. Marchetti, Soft matter 14, 7435 (2018).

[13] A. P. Solon, J. Stenhammar, R. Wittkowski, M. Kardar, Y. Kafri, M. E. Cates, and J. Tailleur, Physical review letters 114, 198301 (2015).

[14] N. Nikola, A. P. Solon, Y. Kafri, M. Kardar, J. Tailleur, and R. Voituriez, Phys. Rev. Lett. 117, 098001 (2016).

[15] T. Speck and R. L. Jack, Physical Review E 93, 062605 (2016).

[16] U. Marini Bettolo Marconi, C. Maggi, and S. Melchionna, Soft Matter 12, 5727 (2016).

[17] F. Ginot, A. Solon, Y. Kafri, C. Ybert, J. Tailleur, and C. Cottin-Bizonne, New Journal of Physics 20, 115001 (2018).

[18] G. Falasco, F. Baldovin, K. Kroy, and M. Baiesi, New Journal of Physics 18, 093043 (2016).

[19] F. Ginot, I. Theurkauff, D. Levis, C. Ybert, L. Bocquet, L. Berthier, and C. Cottin-Bizonne, Physical Review X 5, 011004 (2015).

[20] A. P. Solon, J. Stenhammar, M. E. Cates, Y. Kafri, and J. Tailleur, Physical Review E 97, 020602 (2018).

[21] J. Rodenburg, M. Dijkstra, and R. van Roij, Soft matter 13, 8957 (2017).

[22] A. P. Solon, J. Stenhammar, M. E. Cates, Y. Kafri, and J. Tailleur, New Journal of Physics 20, 075001 (2018).

[23] M. V. Berry, Physics Education 6, 79 (1971).

[24] A. Marchand, J. H. Weijs, J. H. Snoeijer, and B. Andreotti, American Journal of Physics 79, 999 (2011).

[25] J. Rowlinson and B. Widom, Clarendon: Oxford , 56 (1982).

[26] P.-G. De Gennes, F. Brochard-Wyart, and D. Quéré, Capillarity and wetting phenomena: drops, bubbles, pearls, waves (Springer Science & Business Media, 2013).

[27] D. Bonn, J. Eggers, J. Indekeu, J. Meunier, and E. Rolley, Rev. Mod. Phys. 81, 759 (2009).

[28] J. Eggers and E. Villermaux, Rep. Prog. Phys. 71, 036601 (2008).

[29] R. V. Craster and O. K. Matar, Rev. Mod. Phys. 81, 1131 (2009).

[30] J. Elgeti and G. Gompper, EPL (Europhysics Letters) 85, 38002 (2009).

[31] J. Tailleur and M. Cates, EPL (Europhysics Letters) 86, 60002 (2009).

[32] J. Elgeti and G. Gompper, EPL (Europhysics Letters) 101, 48003 (2013).

[33] B. B. Kearns, Nature Reviews Microbiology 8, 634 (2010).

[34] N. C. Darnton, L. Turner, S. Rojevsky, and H. C. Berg, BioPhysical Journal 98, 2082 (2010).

[35] M. Hennes, J. Tailleur, G. Charron, and A. Daerr, Proceedings of the National Academy of Sciences 114, 5958 (2017).

[36] S. Paliwal, V. Prymidis, L. Filion, and M. Dijkstra, The Journal of chemical physics 147, 084902 (2017).

[37] J. G. Kirkwood and F. P. Buff, The Journal of Chemical Physics 17, 338 (1949).

[38] F. P. Buff, The Journal of Chemical Physics 23, 419 (1955).

[39] I. Langmuir, Journal of the American chemical society 39, 1848 (1917).

[40] R. G. Winkler, A. Wysocki, and G. Gompper, Soft matter 11, 6680 (2015).

[41] G. Navascués and M. Berry, Molecular Physics 34, 649 (1977).

[42] Consider the fluid in a rectangular cavity of linear sizes \( L_x \) and \( L_y \), bounded by a potential of stiffness \( \lambda_i \). A direction computation leads to \( \gamma = \frac{\partial P}{\partial x} \bigg|_{A=L_x, L_y} = - \frac{1}{\nu_y} kT \rho(x, y) \) where \( x \) is a position in the bulk of the system and the free energy [46].

[43] This stems from \( \langle Q_{x,x} \rangle \propto \nabla \cdot \left( \sum_{i=0}^{n} \nu_i \cos 2\theta \right) \) in steady state. For a system invariant by translation along \( x \), this reduces to \( \langle Q_{x,x} \rangle \propto \sin \delta \left( \sum_{i=0}^{n} \nu_i \cos 2\theta \right) \), and \( f_{\nu_y}(\langle Q_{x,x} \rangle dy \propto \sin \delta (0 - r_i) \) = 0 \) in an isotropic bulk.

[44] Y. Baek, A. P. Solon, X. Xu, N. Nikola, and Y. Kafri, Phys. Rev. Lett. 120, 058002 (2018).

[45] W. Yan and J. F. Brady, Journal of Fluid Mechanics 785 (2015).

[46] M. J. P. Nijmeijer, van, and J. M. J. Leeuwen, Journal of Physics A: Mathematical and Theoretical 41, 385002 (2008).
Physics A: Mathematical and General 23, 4211 (1990).