Event Conditional Correlation
Or How Non-Linear Linear Dependence Can Be

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Abstract: Given two random variables we study their correlation conditional on a given event, and call this parameter “Event Conditional Correlation”. This parameter can be used to describe conditional dependence and to produce local linear approximations of the overall dependence. To this end we introduce a new estimator of event conditional correlation, and a new estimator of the unconditional correlation based on a partial sample. In both cases we provide proof of consistency, asymptotic normality, and present simulations where the proposed estimators have mean square errors close to that induced by the Cramér-Rao bound.

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1. Introduction

Correlation has been invented more than a century ago (Galton, 1888; Pearson, 1897; Yule, 1907), and is today maybe the most used measure of dependence in all fields of statistics. We here improve on its estimation in the case of conditional dependence and partial samples.

Event conditional correlation is defined as the correlation of two variables $X$ and $Y$ conditionally to an event $A$; we write it $\rho_{X|A}$. It depends strongly on the event according to which it is defined, and hence differs greatly from the classical, or unconditional correlation parameter, which is denoted $\rho_{XY}$. When considering partial samples, event conditional correlation is the natural correlation parameter: if $A$ is verified for all realisations in the sample, directly using the ordinary least squares estimator of correlation produces an estimate of $\rho_{X|A}$, and not of $\rho_{XY}$.

It is then crucial to relate the two measures, both to estimate $\rho_{XY}$ given a partial sample, and to estimate $\rho_{X|A}$ given a full sample without discarding all the realisations where $A$ is not verified. Fields where these problems arise are numerous: one is concerned with the use of copula methods involving co-variates (Gijbels, Omelka and Veraverbeke (2012) and Acar, Craiu and Yao (2013)), an other aims to compare and predict the correlation levels occurring in different regimes, and especially before and during a financial crisis (Forbes and Rigobon (2002); Campbell et al. (2008); Preis et al. (2012) and Kalkbrener and Packham (2013)).
We present two estimators that address under minimal assumptions these two problems. First, given a full sample, we propose an admissible estimator of $\rho_{XY|A}$ for any $A$ such that $\Pr(A) > 0$. Second, we present an estimator of $\rho_{XY}$ relying on a sample where for all realisations an event $A$ is verified (henceforth referred to as an $A$-sample). Using both estimators together allows one to estimate $\rho_{XY|A}$ given an $A'$-sample, this for any two events $A$ and $A'$ with non zero probabilities of occurring.

This is non trivial since $\rho_{XY|A}$ has a strikingly far from linear behaviour even in the Gaussian case, as shown in Figure 1. This underlines how non-linear linear dependence can be and contrasts with the more common measures of conditional linear dependence: the partial and conditional correlations, which describe an homogeneous effect across the support of the variables (Baba, Shibata and Sibuya, 2004). As the analysis will show, it is the homogeneous nature of correlation that allows to transport estimates under one condition to another, however the scale at which it is observed under a given condition is driven by the conditional variances of the variables under $A$.

The proposed estimators can directly be used to complement many statistical approaches—such as segmented regression models, Markov switching models, graphical models and asymmetric dependence models, among others—, either to extend them to non-linear cases or to address sampling and censoring problems. We present such an example for segmented linear regression models (Liu, Wu and Zidek, 1997).

2. New Event Conditional Correlation Estimator

In this section we present an admissible estimator of event conditional correlation that uses all the available sample rather than the subsample where the condition is verified, as is common practice. We will proceed in two steps, first introducing the hypotheses and notations, before presenting the general result supported by a small-sample study (see Figure 2).
Definition 1. Let $X$ and $Y$ be two centered and normed real valued univariate random variables with finite second moments. Let $Z_1$ and $Z_2$ be two real random vectors of the same dimension, possibly containing (or equal) to $X$ or $Y$. Finally let $A$ be an event related to $Z_1$ and $Z_2$ ($Z_1$ and $Z_2$-measurable) of non-zero probability of occurring.

We use classical notations: $\rho$-s are the correlations and $\Sigma$-s are the covariance matrices of the variables in index (we use $\sigma$-s for the standard deviation in the univariate case), finally $\beta$-s are regression parameters and $\epsilon$-s are the regression residuals. For instance for $Z$ a scalar random variable we have:

$$X = Z \beta_X + \epsilon_X,$$

with $\beta_X$ the classical ordinary least squares estimator, equal to $\rho_{XZ} \sigma_X / \sigma_Z$.

Assumption 1. We define here the two hypotheses $H_1$ and $H_2$:

$H_1(X, Y, Z_1, Z_2, A) : \Sigma_{\epsilon X Z_1, \epsilon Y Z_2} = \Sigma_{\epsilon X Z_1, \epsilon Y Z_2} | A$.

$H_2(X, Y, Z_1, Z_2, A) : \text{cov}(Z_1 \beta_{X Z_1}, \epsilon_{Y Z_2} | A) = - \text{cov}(Z_2 \beta_{Y Z_2}, \epsilon_{X Z_1} | A)$.

These assumptions should be seen as minimal since $H_1$ is necessary according to Baba, Shibata and Sibuya (2004) and if $Z_1 = Z_2$, $H_2$ is automatically verified. Meeting $H_1$ can be attained by adding the sufficient number of covariates in $Z_1$ and $Z_2$, something that was not possible before our contribution. Finally, if $H_1$ remains falsified, the bias induced in the following estimators is controlled by $\| \Sigma_{Z_1, Z_2} | A - \Sigma_{Z_1, Z_2} \|^2$, making them still of interest in cases where this value is small.

Theorem 1. Under $H_1$ and $H_2$, we have that:

$$\rho_{XY | A} = \frac{\rho_{XY} + \beta_X^T \delta_A(Z_1, Z_2) \beta_Y Z_2}{\left[1 + \beta_X^T \delta_A(Z_1, Z_1) \beta_X Z_1\right]^{1/2} \left[1 + \beta_Y^T \delta_A(Z_2, Z_2) \beta_Y Z_2\right]^{1/2}},$$

with for all $i, j \leq 2$ $\delta_A(Z_i, Z_j) = \text{cov}(Z_i, Z_j | A) - \text{cov}(Z_i, Z_j)$.

Proof. To be found in Appendix 4. $\square$

To obtain a better intuition of the result, we simplify the problem and assume that the variables are scaled and such that $Z_1 = Z_2 = Z$, with $Z$ univariate. Then Formula 1 becomes

$$\rho_{XY | A} = \frac{\rho_{XY} + \rho_{XZ} \rho_{Y Z} \delta}{\left[1 + \rho_{XZ}^2 \delta^2\right]^{1/2} \left[1 + \rho_{Y Z}^2 \delta^2\right]^{1/2}},$$

with $\delta = \sigma_{Z_1 A}^2 / \sigma_Z^2 - 1$. This form shows that $\rho_{XY | A}$ is driven by the conditional variance, and more precisely by $\delta$, the normalised shift in conditional variance between inside and outside of $A$. In the limit case where $\text{pr}(A) = 0$, we recover the recursive formula to compute partial correlation, allowing us to relate the two dependence measures.
Finally, this formula also recover that presented in Boyer, Gibson and M. (1997); Avonyi-Dovi, Guégan and Ladoucette (2002); Forbes and Rigobon (2002) and Kalkbrener and Packham (2013), connecting our result with theirs. However, because all these works focus on risk measures in finance, they make field specific assumptions on the nature of the condition $A$ and on the joint distribution of $X$ and $Y$, while we work under minimal assumptions.

We now draw from Theorem 1 a new estimator of $\rho_{XY|A}$. In the following we denote using hat estimators: for instance $\hat{\beta}_{XY}$ is an estimator of $\beta_{XY}$.

**Corollary 1.** Under $H1$, $H2$ and the additional hypothesis that $\hat{\rho}_{XY}$, $\hat{\beta}_{XZ_1}$, $\hat{\beta}_{YZ_2}$, $\hat{\delta}_A(Z_1, Z_1)$, $\hat{\delta}_A(Z_2, Z_2)$ and $\hat{\delta}_A(Z_1, Z_2)$ are consistent, asymptotically normal estimators that converge at rate $n^{1/2}$, we have that

$$
\hat{\rho}_{XY} + \hat{\beta}_{XZ_1} \hat{\delta}_A(Z_1, Z_2) \hat{\delta}_{YZ_2}
\left[1 + \hat{\beta}_{XZ_1}^\top \hat{\delta}_A(Z_1, Z_1) \hat{\beta}_{XZ_1}\right]^{1/2}
\left[1 + \hat{\beta}_{YZ_2}^\top \hat{\delta}_A(Z_2, Z_2) \hat{\beta}_{YZ_2}\right]^{1/2}
$$

is a consistent, asymptotically normal, estimator of $\rho_{XY|A}$ converging at rate $n^{1/2}$. 

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**Fig 2.** Root mean squared error (RMSE) of an event conditional correlation curve estimate, as shown in Figure 1, to its true value for different sample sizes $n$ for: the proposed method (dashed line), using sub-sampling (dotted line) and using well specified maximum likelihood initiated at the true value (solid line); each point is produced using 1000 simulations. We consider three distributions with two sets of parameters $\theta = (\rho_{XY}, \rho_{XZ}, \rho_{YZ}, \eta)$. 

Finally, this formula also recover that presented in Boyer, Gibson and M. (1997); Avonyi-Dovi, Guégan and Ladoucette (2002); Forbes and Rigobon (2002) and Kalkbrener and Packham (2013), connecting our result with theirs. However, because all these works focus on risk measures in finance, they make field specific assumptions on the nature of the condition $A$ and on the joint distribution of $X$ and $Y$, while we work under minimal assumptions.
Proof. The result is a direct application of Theorem 1 and the delta method. We detail the proof in Appendix 4. Figure 2 shows a small-sample study corroborating this result. To obtain an estimator of the variances, the $\hat{\delta}_A$-s, that converges at rate $n^{1/2}$ in Figure 2, we estimate the joint distribution of $(Z_1, Z_2)$ using the whole sample and infer from the estimated distribution the conditional covariance matrices. In all six considered cases our estimator almost realises the Cramér-Rao bound: on average it only adds a 0.04 error in correlation estimates compared to the Cramér-Rao case.

It is of interest to tell whether two event conditional correlation estimates are significantly different or not. To this end we must produce confidence intervals for our estimator. Since Corollary 1 is derived through the delta method, the variance stabilising transformation, or inverse delta method, should be used (Fisher, 1915; Hotelling, 1953). However, this method is not applicable here since it cannot be computed in closed form (van der Vaart, 1998). Nevertheless, resampling methods can be used, which we recommend (Efron, 1981; Young, 1988).

3. Implied Unconditional Correlation

In this section we present an estimator of unconditional correlation based on an $A$-sample. The formulation of this estimator is not intuitive and requires technical assumptions, but as detailed below it is in fact driven by the same adjustment for conditional variance shift as Formula 1. We are not aware of any other estimator to compare our result with, but in Figure 3 we present a small-sample study where our estimator almost realises the Cramér-Rao bound: on average it only adds a 0.02 error in correlation estimates compared to the Cramér-Rao case.

Definition 2. We write $R_{XZ}$ the vector $(\rho_{XZ_i})_{i \leq \text{dim}(Z)}$, $\bar{\delta}_A(Z)$ the matrix

$$\text{diag}(\delta_A(Z, Z) + Id)^{-\frac{1}{2}} (\delta_A(Z, Z) + Id),$$

and $\bar{\delta}_A^{-1}(Z)$ its inverse if it exists. In this later case we will write for a vector $R$:

$$\|R\|_{\bar{\delta}_A(Z)}^2 = R^\top \left[ \bar{\delta}_A^{-1}(Z) \delta_A(Z, Z) \bar{\delta}_A^{-1}(Z)^\top \right] R.$$  

Corollary 2. Assuming $H_1$, $H_2$ and that $\|R_{XZ}\|_{\bar{\delta}_A(Z_1)}$ and $\|R_{YZ}\|_{\bar{\delta}_A(Z_2)}$ exist and are different from 1, we have that

$$\rho_{XY} = \rho_{XY|A} \left( 1 + \beta_{XZ_1}^\top \bar{\delta}_A^{-1}(Z_1) \beta_{XZ_1} \right)^\frac{1}{2} \left( 1 + \beta_{YZ_2}^\top \bar{\delta}_A^{-1}(Z_2) \beta_{YZ_2} \right)^\frac{1}{2} - \beta_{XZ_1}^\top \bar{\delta}_A^{-1}(Z_1) \beta_{YZ_2}, \quad (4)$$

where:

$$\begin{cases}
\beta_{XZ_1} = \left(R_{XZ_i|A} \bar{\delta}_A^{-1}(Z_1) \left( 1 - \|R_{XZ_i|A}\|_{\bar{\delta}_A(Z_1)}^2 \right)^{-\frac{1}{2}} 
\beta_{YZ_2} = \left(R_{YZ_i|A} \bar{\delta}_A^{-1}(Z_2) \left( 1 - \|R_{YZ_i|A}\|_{\bar{\delta}_A(Z_2)}^2 \right)^{-\frac{1}{2}}.
\end{cases}$$
Normal(0, η): \( \theta = (0.2, 0.4, 0.6, 1) \)

Student-\( t \): \( \theta = (0.2, 0.6, 0.8, 1) \)

Normal(0, \( \chi^2 \eta \)): \( \theta = (0.2, 0.6, 0.6, 1) \)

Student-\( t \): \( \theta = (0.2, 0.3, 0.4, 3) \)

Fig 3. Root mean squared error (RMSE) of ten estimates based on sub-samples constructed as in Figure 1 to the true value for different sample sizes \( n \) for: the proposed method (dashed line) and using well specified maximum likelihood initiated at the true value (solid line). Otherwise this small-sample study is structured as that of Figure 2.

**Proof.** To be found in Appendix 4. The proof consists in two steps: i) inverting Formula 1 gives Formula 4, ii) inverting it again where \( Y \) is equal to one of the components of \( Z_1 \) to compute \( \tilde{\beta}_{XZ} \) and \( X \) is equal to one of the components of \( Z_2 \) to compute \( \tilde{\beta}_{YZ} \) gives the result.

Formula 4 does not make the link between conditional variance and event conditional correlation explicit. Rewriting it in the simplified case presented after Theorem 1 shows that it is in fact based on exactly the same transformation as that in Formula 1: with \( \delta = \sigma^2_{Z}/\bar{\sigma}^2_{Z|A}(\sigma^2_{X}/\bar{\sigma}^2_{Z|A} - 1) \) we have

\[
\rho_{XY} = \frac{\rho_{XY|A} + \rho_{XZ|A}\rho_{YZ|A}\delta}{\left[1 + \rho^2_{XZ|A}\delta\right]^{1/2} \left[1 + \rho^2_{YZ|A}\delta\right]^{1/2}}.
\]

**Corollary 3.** Using an \( A \)-sample, under \( H1, H2 \) and the additional hypothesis that \( \hat{\sigma}_X, \hat{\sigma}_Y, \hat{\rho}_{XY|A}, \hat{\beta}_{XZ|A}, \hat{\beta}_{YZ|A}, \hat{\delta}(Z_1, Z_1), \hat{\delta}(Z_2, Z_2) \) and \( \hat{\delta}(Z_1, Z_2) \) are asymptotically normal estimators that converge at rate \( n^{1/2} \), using Formula 4 yields a centered asymptotically normal estimator converging at rate \( n^{1/2} \) of \( \rho_{XY} \).

**Proof.** The result is a direct consequence of Corollary 2 and the delta method.
We compare three regression methodologies in a simulated dataset where the response variable \( y \) is equal to \( \tanh(x) + \epsilon \), with \( x \) a covariate and \( \epsilon \) a centered Gaussian noise. We compare the generalized additive regression (dotted line, Hastie and Tibshirani (1986)), the tree regression (dashed lines, Breiman (1984)), and one produced using event conditional correlation (solid line, different regression slopes based on event conditional correlation are estimated for different parts of the support of \( x \)). On the left we present the raw response function estimates and on the right we present smoothed versions of the same estimates for fair comparison, in both cases with the true response function in continuous grey.

and is to be found in Appendix 4. To estimate the conditional covariances, the \( \hat{\delta}_A \)-s, at rate \( n^{1/2} \) in Figure 3 we use maximum likelihood with a truncated distribution and infer from the estimate the corresponding value.

4. Discussion

Taken together our result show that event conditional correlation is a function of the conditional variance shift of \( (Z_1, Z_2) \) under \( A \). It follows that by estimating the conditional variance shift, or by assuming a given value for it, we can compute and estimate correlation conditionally to any event using any partial sample. This is done while making no assumption on the dependence occurring between \( X \) and \( Y \), but may require some parametric assumption on the distribution of \( (Z_1, Z_2) \) as was done in Figure 2 and Figure 3.

By stringing \( \rho_{XY|A} \) estimates together we can produce a description of non-linear dependence structures, in the same fashion that locally affine functions can approximate smooth functions. A practical example of this concept is presented in Figure 4 in the case of linear regression: as the slope of the regression line is driven by the correlation parameter, we can compute different such slopes for different parts of the support of the regressor using event conditional correlations. By connecting all these lines we obtain a description of non-linear dependence. In Figure 4 the event conditional correlation based approach proves more efficient than comparable methods, especially in the tails of the distribution.
Appendix

Proof of Theorem 1. We will proceed in two steps: first we will compute the covariance of \(X\) and \(Y\) knowing \(A\) and then proceed to compute the variance of \(X\) and \(Y\) knowing \(A\). In the following we will put \(A\) as index to operators used conditionally to \(A\): for instance \(E_A[X] = E[X | A]\).

Covariance

\[
\text{cov}_A(X, Y) = E_A \left[ (X - E_A[X])^\top (Y - E_A[Y]) \right]
\]

\[
= E_A \left[ (Z_1 \beta_{XZ_1} + \epsilon_{XZ_1} - E_A[Z_1 \beta_{XZ_1} + \epsilon_{XZ_1}]) \right] \\
(Z_2 \beta_{YZ_2} + \epsilon_{YZ_2} - E_A[Z_2 \beta_{YZ_2} + \epsilon_{YZ_2}])^\top
\]

\[
= \beta_{XZ_1}^\top E_A \left[ (Z_1 - E_A[Z_1])^\top (Z_2 - E_A[Z_2]) \right] \beta_{YZ_2}
+ E_A \left[ (Z_1 - E_A[Z_1]) (\epsilon_{YZ_2} - E_A[\epsilon_{YZ_2}]) \right] \beta_{XZ_1}
+ E_A \left[ (Z_2 - E_A[Z_2]) (\epsilon_{XZ_1} - E_A[\epsilon_{XZ_1}]) \right] \beta_{YZ_2}
+ E_A \left[ (\epsilon_{XZ_1} - E_A[\epsilon_{XZ_1}]) (\epsilon_{YZ_2} - E_A[\epsilon_{YZ_2}]) \right].
\]

Under \(H_1\) and \(H_2\), we can simplify the above formula to:

\[
\text{cov}_A(X, Y) = \beta_{XZ_1}^\top \text{cov}_A(Z_1, Z_2) \beta_{YZ_2} + \text{cov}(\epsilon_{XZ_1}, \epsilon_{YZ_2}). \tag{5}
\]

Let use now compute \(\text{cov}(\epsilon_{XZ_1}, \epsilon_{YZ_2})\). To do so we use the fact that (5) is verified for any event \(A\) such that \(H_1(X, Y, Z_1, Z_2, A)\) and \(H_2(X, Y, Z_1, Z_2, A)\) are verified. This is the case if \(\text{pr}(A) = 1\), so that:

\[
\text{cov}(X, Y) = \beta_{XZ_1}^\top \text{cov}(Z_1, Z_2) \beta_{YZ_2} + \text{cov}(\epsilon_{XZ_1}, \epsilon_{YZ_2}),
\]

and we obtain:

\[
\text{cov}(\epsilon_{XZ_1}, \epsilon_{YZ_2}) = \text{cov}(X, Y) - \beta_{XZ_1}^\top \text{cov}(Z_1, Z_2) \beta_{YZ_2}. \tag{6}
\]

Hence, merging the formulas (5) and (6) we have:

\[
\text{cov}_A(X, Y) = \text{cov}(X, Y) + \beta_{XZ_1}^\top \delta_A(Z_1, Z_2) \beta_{YZ_2}. \tag{7}
\]

Variance

\[
\text{var}_A(X) = \text{var}_A[Z_1 \beta_{XZ_1} + \epsilon_{XZ_1}]
\]

\[
= \beta_{XZ_1}^\top \text{var}_A(Z_1) \beta_{XZ_1} + \text{var}(\epsilon_{XZ_1}). \tag{8}
\]

We make the simplification using \(H_1\). Let us now compute the variance of \(\epsilon_{XZ_1}\). To do so we use the fact that (8) is verified for any event \(A\) such that
H1(X, Y, Z1, Z2, A) and H2(X, Y, Z1, Z2, A) are verified. This is the case if pr(A) = 1, then the above formulas writes:

\[ \text{var}(X) = \beta_{XZ_1}^\top \text{var}[Z_1] \beta_{XZ_1} + \text{var}(\epsilon_{XZ_1}), \]

so that:

\[ \text{var}(\epsilon_{XZ_1}) = \text{var}(X) - \beta_{XZ_1}^\top \text{var}[Z_1] \beta_{XZ_1}. \tag{9} \]

By merging formulas (8) and (9):

\[ \text{var}_A(X) = \text{var}(X) + \beta_{XZ_1}^\top \delta_A(Z_1, Z_1) \beta_{XZ_1}, \]
\[ \text{var}_A(Y) = \text{var}(Y) + \beta_{YZ_2}^\top \delta_A(Z_2, Z_2) \beta_{YZ_2}. \tag{10} \]

The result for Y is obtained similarly.

Merging the formula for the covariance (7) as well as those for the variance (10) and using that both X and Y are normed we obtain Formula 1.

Proof of Corollary 1. For simplicity we will work here in the simplified case presented after Theorem 1, where Z1 = Z2 = Z and with Z univariate. The result and the proof extends to the multivariate setting.

We call \( \theta \) the vector composed of the parameters required to compute the \( \rho_{XY|A}: \theta = (\rho_{XY}, \rho_{XZ}, \rho_{YZ}, \delta) \). We denote \( \hat{\theta} \) an estimator of \( \theta \) that converges at rate \( n^{1/2} \) as in the statement of the result and \( \Sigma_{\theta} \) the asymptotic covariance matrix of \( \theta \).

Let \( \phi: \mathbb{R}^4 \rightarrow \mathbb{R} \) be the map defined by:

\[ \phi(a, b, c, d) = \frac{a + bcd}{(1 + b^2d)^{1/2}(1 + c^2d)^{1/2}}. \]

The map \( \phi \) is such that \( \phi(\theta) = \rho_{XY|A} \). We denote \( \nabla \phi \) the gradient of \( \phi \):

\[ \nabla \phi = \left( \frac{\partial \phi}{\partial a}, \frac{\partial \phi}{\partial b}, \frac{\partial \phi}{\partial c}, \frac{\partial \phi}{\partial d} \right). \]

Then under our assumptions we have that \( \phi(\hat{\theta}) \) is an asymptotically normal estimator of \( \phi(\theta) \) converging at the same rate \( \nu \), and of asymptotic variance \( \nabla \phi \Sigma_{\theta} \nabla \phi^\top \) as a direct application of the Delta Method, see van der Vaart (1998) pp 30 (with the same notations).

Proof of Corollary 2. To simplify notation, without loss of generality we will assume all the component of both \( Z_1 \) and \( Z_2 \) to be normed (e.g. \( Z_1 \) is replaced by \( \Sigma_{Z_1}^{-1/2}Z_1 \)). We first produces Formula 4 before computing \( \hat{\beta}_{XZ_1} \) and \( \hat{\beta}_{YZ_2} \).

According to Formula 1, we have that:

\[ \rho_{XY} = \rho_{XY|A} \left( 1 + \beta_{XZ_1}^\top \delta_A(Z_1, Z_1) \beta_{XZ_1} \right)^{1/2} \left( 1 + \beta_{YZ_2}^\top \delta_A(Z_2, Z_2) \beta_{YZ_2} \right)^{1/2} - \beta_{XZ_1}^\top \delta_A(Z_1, Z_2) \beta_{YZ_2}. \]
We observe that here \( \tilde{\beta}_{XZ_1} = \beta_{XZ_1} \) and \( \tilde{\beta}_{YZ_2} = \beta_{YZ_2} \). As they cannot be directly estimated using a \( A \)-sample we will rewrite them using Formula 1.

We denote \( Z_{1,i} \) the \( i \)-th component of the vector \( Z_1 \). Using Formula 1 and regressing \( Z_{1,i} \) against \( Z_1 \) (the hypotheses are verified in this particular case), we have that for all \( i \leq \text{dim}(Z_1) \),

\[
\rho_{XZ_1,i|A} = \frac{\rho_{XZ_1,i} + \beta_{XZ_1}^T \delta_A(Z_1,Z_1) 1_i}{\left[ 1 + \beta_{XZ_1}^T \delta_A(Z_1,Z_1) \right]^{1/2} \left[ 1 + 1_i^T \delta_A(Z_1,Z_1) 1_i \right]^{1/2}}.
\]

where \( 1_i \) is the vector composed of zeros except for its \( i \)-th component. To simplify notations we will write in the following \( \delta_A \) instead of \( \delta_A(Z_1,Z_1) \).

Using Formula 1 and\( \rho_{XZ_1,i} = \beta_{XZ_1,i} \), so that we can write:

\[
\rho_{XZ_1,i|A} = \frac{\beta_{XZ_1}^T (\delta_A + \text{Id}) 1_i}{\left[ 1 + \beta_{XZ_1}^T \delta_A \beta_{XZ_1} \right]^{1/2} \left[ 1 + 1_i^T (\delta_A + \text{Id}) 1_i \right]^{1/2}}.
\]

So that finally we obtain:

\[
R_{XZ_1|A}^T = \frac{\tilde{\delta}_A(Z_1) \beta_{XZ_1}}{\left[ 1 + \beta_{XZ_1}^T \delta_A \beta_{XZ_1} \right]^{1/2}}.
\]

Our aim is now to compute \( \beta_{XZ_1}^T \delta_A \beta_{XZ_1} \), as it will allow us to write \( \beta_{XZ_1} \) using only known terms. To do so we make it appear at the numerator by successively multiplying by \( \tilde{\delta}_A^{-1}(Z_1) \) and by \( \delta_A^{1/2} \) and then squaring both sides of the equation. The produced results is that:

\[
\beta_{XZ_1}^T \delta_A \beta_{XZ_1} = \frac{\|R_{XZ_1|A}\|_2^2}{\tilde{\delta}_A(Z_1)} \delta_A^{-1}(Z_1).
\]

And finally that:

\[
\beta_{XZ_1} = \frac{R_{XZ_1|A} \delta_A^{-1}(Z_1)}{\left[ 1 - \|R_{XZ_1|A}\|_2^2 \delta_A^{-1}(Z_1) \right]^{1/2}}.
\]

Using a similar approach to compute \( \beta_{YZ_2} \) permits to obtain the result presented in Corollary 2.

**Proof of Corollary 3.** In the same fashion as for Corollary 2 we will consider here only the simplified framework where \( Z_1 = Z_2 = Z \) and with \( Z \) univariate. The result and the proof extends to the multivariate setting. Then the proof is the same as that of Corollary 2 using \( \theta = (\rho_{XY}, \rho_{XZ}, \rho_{YZ}, \delta) \) with the exact same function \( \phi \).

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