Generalised triangle groups of type \((3, q, 2)\)

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Abstract. If \(G\) is a group with a presentation of the form 
\[ \langle x, y \mid x^3 = y^q = W(x, y)^2 = 1 \rangle \]
then either \(G\) is virtually soluble or \(G\) contains a free subgroup of rank 2. This provides additional evidence in favour of a conjecture of Rosenberger.

1. Introduction

A generalised triangle group is a group \(G\) with a presentation of the form
\[ \langle x, y \mid x^p = y^q = W(x, y)^r = 1 \rangle \]
where \(p, q, r \geq 2\) are integers and \(W(x, y)\) is a word of the form
\[ x^{\alpha(1)} y^{\beta(1)} \cdots x^{\alpha(k)} y^{\beta(k)} \]
\((0 < \alpha(i) < p, 0 < \beta(i) < q)\). We say that \(G\) is of type \((p, q, r)\). The parameter \(k\) is called the length-parameter. (The syllable-length, or free-product length, of \(W\) regarded as a word in \(\mathbb{Z}_p \ast \mathbb{Z}_q\) is \(2k\).) Without loss of generality, we assume that \(p \leq q\).

A conjecture of Rosenberger [20] asserts that a Tits alternative holds for generalised triangle groups:

Conjecture A (Rosenberger). Let \(G\) be a generalised triangle group. Then either \(G\) is soluble-by-finite or \(G\) contains a non-abelian free subgroup.

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This conjecture has been verified in a large number of special cases. (See for example the survey in [10].) In particular it is now known:

- when $r \geq 3$ [9];
- when $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$ [3, 14];
- when $q \geq 6$ [19, 22, 4, 5, 1, 7, 17];
- when $k \leq 6$ [20, 19, 21];
- for $(p, q, r) = (3, 4, 2)$ [2, 18];
- for $(p, q, r) = (2, 4, 2)$ and $k$ odd [6].

In the present article we describe a proof of the Rosenberger Conjecture for the cases $(p, q, r) = (3, 3, 2)$ and $(p, q, r) = (3, 5, 2)$, hence completing the proof of the following

**Theorem B.** Let $G$ be a generalised triangle group of type $(p, q, r)$ with $p, q \geq 3$ and $r \geq 2$. Then either $G$ is soluble-by-finite or $G$ contains a non-abelian free subgroup.

Thus the Rosenberger Conjecture is now reduced to three cases, where $p = r = 2$ and $q \in \{3, 4, 5\}$.

The results presented here have been posted online at [15, 16], where the arguments are given in more detail. In particular, the proof in the case $q = 3$ requires a certain amount of computer calculation using GAP [11]: [15] provides full details of the computations involved, including code and output. Also included in [15] are some partial results on the case $(p, q, r) = (2, 3, 2)$ of the Rosenberger Conjecture.

Our strategy of proof is essentially the same for the two cases $q = 3$ and $q = 5$, but the details differ substantially. A theoretical analysis of the trace polynomial (see § 2.2 for details) reduces the problem to a finite set of candidate words $W$ by finding an upper bound for the length-parameter $k$. In the case $q = 5$ the analysis is more detailed and yields the bound $k \leq 4$; the conjecture has already been proved when $k \leq 4$ by Levin and Rosenberger [19].

In the case $q = 3$ the analysis yields only the bound $k \leq 20$. This however is sufficient for a computer-based attack on the problem: a computer search using GAP [11] refines the set of candidates to a list of 19 words. The conjecture is known for the 8 shortest words in the list, by work of Levin and Rosenberger [19] and Williams [21]. The remainder of the words satisfy a small cancellation condition, which ensures the existence of nonabelian free subgroups.
Section 2 below contains some preliminary results on trace polynomials and equivalence of words. The proof of the main result for the case \( q = 3 \) is contained in Section 3, and for the case \( q = 5 \) in Section 4.

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\section{Preliminaries} \label{sec:prelim}

\subsection{Equivalence of words} \label{eqn:equiv}

Our object of study is a group
\[ G = \langle x, y \mid x^p = y^q = W(x, y)^r = 1 \rangle \]
where
\[ W(x, y) = x^{\alpha(1)}y^{\beta(1)} \cdots x^{\alpha(k)}y^{\beta(k)} , \]
and \( 0 < \alpha(i) < p, 0 < \beta(i) < q \) for each \( i \).

We think of the word \( W \) as a cyclically reduced word in the free product
\[ \mathbb{Z}_p \ast \mathbb{Z}_q = \langle x, y \mid x^p = y^q = 1 \rangle . \]

We regard two such words \( W, W' \) as \textit{equivalent} if one can be transformed to the other by moves of the following types:

- cyclic permutation of \( W \),
- inversion of \( W \),
- automorphism of \( \mathbb{Z}_p \) or of \( \mathbb{Z}_q \), and
- (if \( p = q \)) interchange of \( x, y \).

It is clear that, if \( W, W' \) are equivalent words, then the resulting groups
\[ G = \langle x, y \mid x^p = y^q = W(x, y)^r = 1 \rangle \]
and
\[ G' = \langle x, y \mid x^p = y^q = W'(x, y)^r = 1 \rangle \]
are isomorphic. Hence for the purposes of studying the Rosenberger Conjecture (Conjecture A) it is enough to consider words up to equivalence.
2.2. Trace Polynomials

Suppose that $X, Y \in SL(2, \mathbb{C})$ are matrices, and $W = W(X, Y)$ is a word in $X, Y$. Then the trace of $W$ can be calculated as the value of a 3-variable polynomial, where the variables are the traces of $X, Y$ and $XY$ [12]. We can use this to find and analyse essential representations from $G$ to $PSL(2, \mathbb{C})$. (A representation of $G$ is essential if the images of $x, y, W(x, y)$ have orders $p, q, r$ respectively.)

We can force the images $x, y$ to have orders $p, q$ in $PSL(2, \mathbb{C})$ by mapping them to matrices $X, Y \in SL(2, \mathbb{C})$ of trace $2 \cos(\pi/p)$ and $2 \cos(\pi/q)$ respectively. Then the trace of $W(X, Y) \in SL(2, \mathbb{C})$ is given by a one-variable polynomial $\tau_W(\lambda)$, where $\lambda$ denotes the trace of $XY$. We will refer to $\tau_W$ as the trace polynomial of $W$. Since we are in practice interested in the case where $r = 2$, we obtain an essential representation by choosing $\lambda$ to be a root of $\tau_W$.

We recall here some properties of $\tau_W$. Details can be found, for example, in [10]. (Complete formulae for the coefficients of $\tau_W$ are given in [17, Appendix].)

Lemma 2.1. • $\tau_W$ has degree $k$;
• when $p, q \leq 3$, $\tau_W(\lambda)$ is monic and has integer coefficients;
• in general, the coefficients of $\tau_W$ are real algebraic integers.

We also note a few more elementary properties.

Lemma 2.2. 1) Let $J$ denote the interval

$$J = [2 \cos(\pi/p + \pi/q), 2 \cos(\pi/p - \pi/q)] \subset \mathbb{R}.$$ 

Then $\tau_W(J) \subset [-2, 2]$.

2) If $p = q = 3$ and $G$ does not contain a non-abelian free subgroup, then the roots of $\tau_W$ belong to $\{0, 1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$.

3) If $p = 3$, $q = 5$ and $G$ does not contain a non-abelian free subgroup, then the roots of $\tau_W$ belong to $\{0, 1, \frac{1+\sqrt{5}}{2}, -\frac{1+\sqrt{5}}{2}\}$.

Proof. The space $M_q$ of matrices in $SU(2) \subset SL_2(\mathbb{C})$ with trace $2 \cos(\pi/q)$ is path-connected. (Indeed, it is homeomorphic to the 2-sphere $S^2$.) Writing $D_p$ for the diagonal matrix with diagonal entries $e^{i\pi/p}, e^{-i\pi/p}$, fix $X = D_p \in M_p$ and let $Y$ vary continuously in $M_q$ from $D_q$ to $D_q^{-1}$. Then $\lambda = tr(XY)$ will vary continuously from $tr(D_pD_q) = 2 \cos(\pi/p + \pi/q)$ to $tr(D_pD_q^{-1}) = 2 \cos(\pi/p - \pi/q)$. By the Intermediate Value Theorem
every $j \in J$ arises as $\text{tr}(XY)$ for some $Y \in M_q$ and $X = D_p \in M_p$, so $\tau_W(j) = \text{tr}(W(X,Y)) \in [-2, 2]$ since $W(X,Y) \in SU(2)$.

Any root $\lambda$ of $\tau_W$ corresponds to an essential representation $\rho : G \to PSL(2, \mathbb{C})$. When $p = q = 3$, the image of $\rho$ is a subgroup of $PSL(2, \mathbb{C})$ generated by two elements of order 3 and containing an element of order 2. Any such subgroup contains a free non-abelian subgroup unless it is isomorphic to $A_4$ or $A_5$, in which case each of $\rho(xy), \rho(xy^{-1})$ has order 2, 3 or 5. In this case $\text{tr}(\rho(x))\text{tr}(\rho(y)) = 2\cos(\pi/5) = 1 + \sqrt{5}/2$, so $\lambda, 1 + \sqrt{5}/2 - \lambda \in \{0, \pm 1, \pm 1 + \sqrt{5}/2\}$, which is possible only if $\lambda \in \{0, 1, \pm 1 + \sqrt{5}/2\}$.

A similar argument applies when $p = 3$ and $q = 5$. Here the image of $\rho$ is generated by an element of order 3 and an element of order 5, and it contains an element of order 2. Such a subgroup of $PSL(2, \mathbb{C})$ contains a non-abelian free subgroup unless it is isomorphic to $A_5$, in which case each of $\rho(xy), \rho(xy^\pm 1)$ has order 2, 3 or 5. In this case $\text{tr}(\rho(x))\text{tr}(\rho(y)) = 2\cos(\pi/5) = 1 + \sqrt{5}/2$, so $\lambda, 1 + \sqrt{5}/2 - \lambda \in \{0, \pm 1, \pm 1 + \sqrt{5}/2\}$, which is possible only if $\lambda \in \{0, 1, \pm 1 + \sqrt{5}/2\}$, as claimed.

**Lemma 2.3.** If $p = q = 3$ and $W, W'$ are equivalent with length-parameter $k$, then either $\tau_W(\lambda) = \tau_{W'}(\lambda)$ or $\tau_W(\lambda) = (-1)^k\tau_{W'}(1 - \lambda)$.

**Proof.** Since the trace of a matrix is a conjugacy invariant, it follows that the trace polynomial is unchanged by cyclically permuting $W$. Moreover, if $X \in SL(2, \mathbb{C})$ then the traces of $X, X^{-1}$ are equal, so the trace polynomial is unchanged by inverting $W$.

If $\text{tr}(X) = 1 = \text{tr}(Y)$, then $\text{tr}(Y^{-1}) = 1$ also. Interchanging $x, y$ in $W$ has the effect on $\tau_W(\lambda) = \text{tr}(W(X,Y))$ of replacing $\lambda = \text{tr}(XY)$ by $\text{tr}(YX) = \lambda$ in other words, no change.

Finally, $\text{tr}(XY^{-1}) + \text{tr}(XY) = \text{tr}(X)\text{tr}(Y) = 1$.

Hence replacing $y$ by $y^2$ has the effect of replacing $\tau_W(\lambda) = \text{tr}(W(X,Y))$ by

$$\text{tr}(W(X,Y^2)) = \text{tr}(W(X, -Y^{-1})) = (-1)^k\text{tr}(W(X,Y^{-1})) = (-1)^k\tau_W(1 - \lambda),$$

as claimed. \qed
Theorem 2.4. Let $G = \langle x, y | x^3 = y^3 = W(x,y)^2 = 1 \rangle$ where $W = x^{\alpha(1)}y^{\beta(1)} \cdots x^{\alpha(k)}y^{\beta(k)}$ with $\alpha(i), \beta(i) \in \{1, 2\}$ for each $i$. If $G$ does not contain a free subgroup of rank 2, then $\tau_W(\lambda)$ has the form

$$\tau_W(\lambda) = \lambda^a(\lambda - 1)^b(\lambda^2 - \lambda - 1)^c$$

with $a, b \leq 1$ and $c \leq 3(a + b + 1)$. In particular $k = a + b + 2c \leq 20$.

Proof. By Lemma 2.2 we may assume that the roots of $\tau_W$ all lie in $\{0, 1, 1 + \sqrt{5}/2, 1 - \sqrt{5}/2\}$. Moreover, $\tau_W$ is monic with integer coefficients by Lemma 2.1, so the the two potential roots $\frac{1 \pm \sqrt{5}}{2}$ occur with equal multiplicity $c$ say, and $\tau_W$ has the form

$$\tau_W(\lambda) = \lambda^a(\lambda - 1)^b(\lambda^2 - \lambda - 1)^c$$

for some non-negative integers $a, b, c$.

We deduce the desired bounds on $a, b, c$ from Lemma 2.2 as follows. In this case the interval $J$ in Lemma 2.2 is $J = [-1, 2]$. Hence $2^a = |\tau_W(2)| \leq 2$ and $2^b = |\tau_W(-1)| \leq 2$, so $a \leq 1$ and $b \leq 1$. In addition,

$$\left(\frac{5}{4}\right)^c \left(\frac{1}{2}\right)^{a+b} = |\tau_W\left(\frac{1}{2}\right)| \leq 2.$$ 

It follows that

$$c \ln(5) \leq (a + b + 2c + 1) \ln(2),$$

which implies the desired conclusion

$$c \leq 3(a + b + 1)$$

given that $a + b \in \{0, 1, 2\}$. \qed

3. The case $q = 3$

3.1. Small Cancellation

In this section we prove a result on one-relator products of groups where the relator satisfies a certain small cancellation condition. We will apply this specifically to generalised triangle groups of types $(3, 3, 2)$, but as the result seems of independent interest, we prove it in the widest generality available.

Suppose that $\Gamma_1, \Gamma_2$ are groups, and $U \in \Gamma_1 \ast \Gamma_2$ is a cyclically reduced word of length at least 2. (Here and throughout this section, length means
length in the free product sense.) A word \( V \in \Gamma_1 \ast \Gamma_2 \) is called a piece if there are words \( V', V'' \) with \( V' \neq V'' \), such that each of \( V \cdot V' \), \( V \cdot V'' \) is cyclically reduced as written, and each is equal to a cyclic conjugate of \( U \) or of \( U^{-1} \). A cyclic subword of \( U \) is a non-piece if it is not a piece.

By a one-relator product \((\Gamma_1 \ast \Gamma_2)/U\) of groups \( \Gamma_1, \Gamma_2 \) we mean the quotient of their free product \( \Gamma_1 \ast \Gamma_2 \) by the normal closure of a cyclically reduced word \( U \) of positive length. Recall [13] that a picture over the one-relator product \( G = (\Gamma_1 \ast \Gamma_2)/U \) is a graph \( P \) on a surface \( \Sigma \) (which for our purposes will always be a disc) whose corners are labelled by elements of \( \Gamma_1 \cup \Gamma_2 \), such that

1) the labels around any vertex, read in clockwise order, spell out a cyclic permutation of \( U \) or \( U^{-1} \);

2) the labels in any region of \( \Sigma \setminus P \) either all belong to \( \Gamma_1 \) or all belong to \( \Gamma_2 \);

3) if a region has \( k \) boundary components labelled by words \( W_1, \ldots, W_k \in \Gamma_i \) (read in anti-clockwise order; with \( i = 1, 2 \)), then the quadratic equation

\[
\prod_{j=1}^k X_j W_j X_j^{-1} = 1
\]

is solvable for \( X_1, \ldots, X_k \) in \( \Gamma_i \). (In particular, if \( k = 1 \) then \( W_1 = 1 \) in \( \Gamma_i \)).

Note that edges of \( P \) may join vertices to vertices, or vertices to the boundary \( \partial \Sigma \), or \( \partial \Sigma \) to itself, or may be simple closed curves disjoint from the rest of \( P \) and from \( \partial \Sigma \).

The boundary label of \( P \) is the product of the labels around \( \partial \Sigma \). By a version of van Kampen’s Lemma, there is a picture with boundary label \( W \in \Gamma_1 \ast \Gamma_2 \) if and only if \( W \) belongs to the normal closure of \( U \).

A picture is minimal if it has the fewest possible vertices among all pictures with the same (or conjugate) boundary labels. In particular every minimal picture is reduced: no edge \( e \) joins two distinct vertices in such a way that the labels of these two vertices that start and finish at the endpoints of \( e \) are mutually inverse.

In a reduced picture, any collection of parallel edges between two vertices (or from one vertex to itself) corresponds to a collection of consecutive 2-gonal regions, and the labels within these 2-gonal regions spell out a piece.

Since \( U \) is cyclically reduced, no corner of an interior vertex is contained in a 1-gonal region.
**Theorem 3.1.** Let \( \ell \) be an even positive integer. Suppose that \( U \equiv U_1 \cdot U_2 \cdot U_3 \cdot U_4 \cdot U_5 \cdot U_6 \in \Gamma_1 \ast \Gamma_2 \) with each \( U_i \) a non-piece of length at least \( \ell \). Suppose also that \( A, B \in \Gamma_1 \ast \Gamma_2 \) are reduced words of length \( \ell \) such that \( A \) is not equal to any cyclic conjugate of \( B^{\pm 1} \) and such that no \( U_i \) is equal to a subword of a power of \( A \). Then \( G := (\Gamma_1 \ast \Gamma_2)/\langle\langle U \rangle\rangle \) contains a non-abelian free subgroup.

**Proof.** Since \( \ell \) is even and positive, any reduced word of length \( \ell \) in \( \Gamma_1 \ast \Gamma_2 \) is cyclically reduced. Replacing \( A \) by \( A^{-1} \) and/or \( B \) by \( B^{-1} \) if necessary, we may assume that each of \( A, B \) begins with a letter from \( \Gamma_1 \) and ends with a letter from \( \Gamma_2 \). Choose a large positive integer \( N > 20K\ell \), where \( K \) is the length of \( U \), and define \( X := A^NB^N \), \( Y := B^NA^N \). We claim that \( X, Y \) freely generate a free subgroup of \( G \).

We prove this claim by contradiction. Suppose that \( Z(X,Y) \) is a non-trivial reduced word in \( X, Y \) such that \( Z(X,Y) = 1 \) in \( G \). Then there exists a picture \( \mathcal{P} \) on the disc \( D^2 \) over the one-relator product \( G \) with boundary label \( Z(X,Y) \). Without loss of generality, we may assume that \( \mathcal{P} \) is minimal, hence reduced.

Suppose that \( v \) is an interior vertex of \( \mathcal{P} \). The vertex label of \( v \) is \( U \) or \( U^{-1} \) – by symmetry we can assume it is \( U \). The subword \( U_1 \) of \( U \) corresponds to a sequence of consecutive corners of \( v \); at least one of these corners does not belong to a 2-gonal region of \( \mathcal{P} \), since \( U_1 \) is a non-piece. It follows that at least one of the corners of \( v \) within the subword \( U_1 \) of the vertex label does not belong to a 2-gonal region. The same follows for the subwords \( U_2, \ldots, U_6 \), so \( v \) has at least 6 non-2-gonal corners.

Now consider the (cyclic) sequence of boundary (that is, non-interior) vertices of \( \mathcal{P} \), \( v_1, \ldots, v_n \) say. This is intended to mean that the closed path \( \partial D^2 \), with an appropriate choice of starting point, meets a sequence of arcs that go to \( v_1 \), separated by 2-gons, then a sequence of arcs that go to \( v_2 \), separated by 2-gons, and so on, finishing with a sequence of arcs that go to \( v_n \), separated by 2-gons, before returning to its starting point. Note that it is possible that an arc of \( \mathcal{P} \) joins two points on \( \partial D^2 \); any such arc is ignored here. Note also that we do not insist that \( v_i \neq v_j \) for \( i \neq j \) in general. It is possible for the sequence of boundary vertices to visit a vertex \( v \) several times. Nevertheless it is important to regard such visits as pairwise distinct, so the notation \( v_1, v_2, \ldots \) is convenient. We say that a boundary vertex is *simple* if it appears only once in this sequence.

If \( v_j \) is connected to \( \partial D^2 \) by \( k \) arcs separated by \( k - 1 \) 2-gons, then this corresponds to a common (cyclic) subword \( W_j \) of \( Z(X,Y) \) and \( U \), of length \( k - 1 \). Let \( \kappa(j) \leq 6 \) be the maximum integer \( t \) such that, for
some \( s \in \{1, \ldots, 6\} \), \( W_j \) contains a subword equal to \( (U_s, U_{s+1} \cdots U_{s+t})^{\pm 1} \) (indices modulo 6). If no such \( t \) exists, we define \( \kappa(j) = -1 \).

If \( v_j \) is a simple boundary vertex with only \( r \leq 4 \) corners not belonging to 2-gons, then it is easy to see that \( \kappa(j) \geq 5 - r \):

There are more complex rules for non-simple boundary vertices. Nevertheless, it is an easy consequence of Euler’s formula, together with the fact that interior vertices have 6 or more non-2-gonal corners, that

\[
\sum_{j=1}^{n} \kappa(j) \geq 6.
\]

Now consider the word \( Z(X, Y) \) as a cyclic word in \( \Gamma_1 \ast \Gamma_2 \). Where a letter \( X = A^N B^N \) or \( Y = B^N A^N \) is followed by another letter \( X \) or \( Y \), then there is no cancellation in \( \Gamma_1 \ast \Gamma_2 \). Similarly there is no cancellation where \( X^{-1} \) or \( Y^{-1} \) is followed by \( X^{-1} \) or \( Y^{-1} \). Where \( X \) is followed by \( Y^{-1} \) or vice versa, or where \( Y \) is followed by \( X^{-1} \) or vice versa, then there is possible cancellation, but since \( A \neq B \) the amount of cancellation is limited to at most \( \ell \) letters from either side.

If \( Z \) has length \( L \) as a word in \( \{X^{\pm 1}, Y^{\pm 1}\} \), then after cyclic reduction in \( \Gamma_1 \ast \Gamma_2 \) it consists of \( L \) subwords of the form \( A^{\pm(N-1)} \), \( L \) subwords of the form \( B^{\pm(N-1)} \), and \( L \) subwords \( V_1, \ldots, V_L \), each of length at most \( 2\ell \).

Now suppose that \( v_j \) is a boundary vertex of \( P \) with \( \kappa(j) \geq 0 \). Then \( U_i^{\pm 1} \) is equal to a subword of \( W_j \) for some \( i \). Since \( U_i \) cannot be a subword of a power of \( A \), \( W_j \) is not entirely contained within one of the segments labelled \( A^{\pm(N-1)} \).

If, in addition, \( \kappa(j) > 0 \), then \( W_j \) has a subword of the form \( (U_i U_{i+1})^{\pm 1} \) (subscripts modulo 6) As above, \( W_j \) cannot be contained in one of the subwords \( A^{\pm(N-1)} \). If it is contained in a subword of \( B^{\pm(N-1)} \), then it is a periodic word of period \( \ell \) (that is, its \( i \)-th letter is equal to its \( (i + \ell) \)-th letter for all \( i \) for which this makes sense). Since \( U_{i+1} \) has length at least \( \ell \), there are at least two distinct subwords of \( U_i U_{i+1} \) equal to \( U_i \), contradicting the fact that \( U_i \) is a non-piece in \( U \).

Thus we see that the subwords \( W_j \) of \( Z(X, Y) \) corresponding to boundary vertices \( v_j \) with \( \kappa(j) > 0 \) can occur only at certain points of \( Z(X, Y) \): where an \( A^{\pm(N-1)} \)-segment meets a \( B^{\pm(N-1)} \)-segment; or at part of one of the words \( V_i \).

In particular, the number of boundary vertices \( v_j \) with \( \kappa(j) > 0 \) is bounded above by \( L(2\ell + 1) \). It follows that

\[
\kappa := \sum_j \kappa(j) \leq 5L(2\ell + 1),
\]
where the sum is taken over those boundary vertices $v_j$ with $\kappa(j) \geq 0$.

The goal is to show that the total positive contribution to the sum $\kappa$ from those $v_j$ with $\kappa(j) > 0$ is cancelled out by negative contributions to $\kappa$ from other boundary vertices. This will show that $\kappa \leq 0$, contradicting the assertion above that $\kappa \geq 6$.

Recall that $K$ is the length of $U$. Thus each $A^{\pm(N-1)}$-segment of $\partial \mathcal{P}$ is joined to at least $(N - 1)\ell/K$ boundary vertices, at most 2 of which (those at the ends of the segment) can make non-negative contributions to $\kappa$. The remaining vertices each contribute at most $-1$ to $\kappa$. Since $N > 20K\ell$, it follows that the negative contributions outweigh the positive contributions, as required.

This gives the desired contradiction, which proves the theorem.

\section*{Corollary 3.2.} Let $\Gamma_1$ and $\Gamma_2$ be groups, and suppose $x \in \Gamma_1$ and $y \in \Gamma_2$ are elements of order greater than 2. Suppose that $W \equiv U_1 \cdot U_2 \cdot U_3 \in \Gamma_1 \ast \Gamma_2$ with each $U_i$ a non-piece of length at least 4. Then $G = (\Gamma_1 \ast \Gamma_2)/\langle \langle W^2 \rangle \rangle$ contains a non-abelian free subgroup.

**Proof.** Let $A_1 = xyxy$, $A_2 = xy^{-1}xy^{-1}$, $A_3 = xyxy^{-1}$ and $A_4 = xyyx^{-1}y^{-1}$. Then for $i \neq j$, $A_i$ is not equal to a cyclic conjugate of $A_j^{\pm 1}$. Hence if (say) $U_1$ is equal to a subword of a power of $A_i$, it cannot be equal to a subword of a power of $A_j$. Hence there is at least one $A \in \{A_i, 1 \leq i \leq 4\}$ with the property that no $U_i$ is equal to a subword of a power of $A$. Now choose $B \in \{A_i, 1 \leq i \leq 4\} \setminus \{A\}$ and apply the theorem, with $U_4 = U_1$, $U_5 = U_2$ and $U_6 = U_3$. \hfill \qed

\section{Conclusion}

**Theorem 3.3.** Let $G = \langle x, y | x^3 = y^3 = W(x, y)^2 = 1 \rangle$ be a generalised triangle group of type $(3, 3, 2)$. Then the Rosenberger Conjecture holds for $G$: either $G$ is soluble-by-finite, or $G$ contains a non-abelian free subgroup.

**Proof.** Write $W = x^{\alpha(1)}y^{\beta(1)} \cdots x^{\alpha(k)}y^{\beta(k)}$.

A computer search using GAP [11] (see [15] for details) produces a list of all words $W$, up to equivalence, for which the trace polynomial $\tau_W$ has the form indicated in Theorem 2.4: see Table 1. If $W$ is not equivalent to a word in the list, then $G$ has a nonabelian free subgroup by Theorem 2.4, so we may restrict our attention to the words $W$ in Table 1.
\begin{table}
\begin{tabular}{|c|c|c|}
\hline
W(x, y) & SCC \\
\hline
1 & xy & NO \\
2 & xyxy^2 & NO \\
3 & xyx^2y^2 & NO \\
4 & xyxyx^2 y^2 & NO \\
5 & xyxy^2yx^2 y^2 & NO \\
6 & xyxy^2x^2yx^2 y^2 & NO \\
7 & xyxy^2x^2y^2xy^2 & NO \\
8 & xyxyx^2x^2yx^2 yxy^2 & NO \\
9 & (xyxy)(y^2x^2y^2x)(yx^2yx^2y^2) & YES \\
10 & (xyxy)(x^2y^2x^2xy)(y^2x^2yx^2y^2x) & YES \\
11 & (xyxy)(x^2y^2x^2yx^2x)(y^2x^2yx^2y^2x) & YES \\
12 & (xyxy)(x^2y^2x^2y^2x)(xyx^2x^2y^2) & YES \\
13 & (xyxy)(x^2y^2x^2y^2x)(xyx^2x^2yx^2y^2) & YES \\
14 & (xyxy)(x^2y^2x^2y^2x^2xy)(x^2y^2x^2yx^2y^2x^2) & YES \\
15 & (xyxy)(x^2y^2x^2y^2x)(xyx^2x^2yx^2y^2x^2) & YES \\
16 & (xyxy)(x^2y^2x^2y^2x)(xyx^2x^2yx^2y^2x^2) & YES \\
17 & (xyxy)(x^2y^2x^2y^2x)(xyx^2x^2yx^2y^2x^2) & YES \\
18 & (xyxy)(x^2y^2x^2y^2x)(xyx^2x^2yx^2y^2x^2) & YES \\
\hline
\end{tabular}
\end{table}

Table 1. Words in \( \mathbb{Z}_3 \ast \mathbb{Z}_3 \) with trace polynomial as in Theorem 2.4. The final column indicates whether or not W satisfies the small-cancellation hypotheses of Corollary 3.2. In those cases where it does, the bracketing indicates a subdivision of W into three non-pieces of length \( \geq 4 \): \( W \equiv U_1 \cdot U_2 \cdot U_3 \).

For those W in Table 1 for which \( k \geq 7 \) (namely, numbers 9-19) the small cancellation hypotheses of Corollary 3.2 are satisfied, and so G contains a nonabelian free subgroup.

For \( k \leq 6 \) (words 1-8) in the table, the result is known. Specifically, groups 1-3 are well-known to be finite of orders 12, 180 and 288 respectively; groups 4-6 were proved to have nonabelian free subgroups in [19]; and finally groups 7 and 8 were shown in [21] (see also [15]) to be large. (That is, each contains a subgroup of finite index which admits an epimorphism onto a non-abelian free group.)

This completes the proof.
4. The case $q = 5$

To prove the result in the case $q = 5$, we first prove a number of preliminary results.

**Lemma 4.1.** Let $p : \overline{K} \rightarrow K$ be a regular covering of connected 2-complexes with $K$ finite, with covering transformation group abelian of torsion-free rank at least 2. Let $F$ be a field. If

$$H_2(\overline{K}, F) = 0 \neq H_1(\overline{K}, F),$$

then

$$\dim_F H_1(\overline{K}, F) = \infty.$$ 

**Proof.** Let $\{a, b\}$ be a basis for a free abelian subgroup $A$ of the group of covering transformations of $p : \overline{K} \rightarrow K$, and let $\alpha$ be a cellular 1-cycle of $\overline{K}$ over $F$ that represents a non-zero element of $H_1(\overline{K}, F)$.

If the $F[a]$-submodule of $H_1(\overline{K}, F)$ generated by $\alpha$ is free, then $H_1(\overline{K}, F)$ is infinite-dimensional over $F$, as claimed. So we may assume that there is a cellular 2-chain $\beta$ of $\overline{K}$ with $d(\beta) = f(a)\alpha$ for some non-zero polynomial $f(a) \in F[a]$.

For similar reasons, we may also assume that $d(\gamma) = g(b)\alpha$ for some cellular 2-chain $\gamma$ of $\overline{K}$ and some non-zero polynomial $g(b) \in F[b]$.

Now $f(a)\gamma - g(b)\beta \in H_2(\overline{K}, F) = 0$. In other words $f(a)\gamma = g(b)\beta$ in the group $C_2(\overline{K}, F)$ of cellular 2-chains of $\overline{K}$, which is a free module over the unique factorisation domain $FA \cong F[a^\pm 1, b^\pm 1]$. Since $f(a), g(b)$ are coprime in $F[a^\pm 1, b^\pm 1]$, it follows that there is a 2-chain $\delta$ with $f(a)\delta = \beta$ and $g(b)\delta = \gamma$. Hence $f(a)(d(\delta) - \alpha) = d(\beta) - f(a)\alpha = 0$, in the group $C_1(\overline{K}, F)$ of cellular 1-chains of $\overline{K}$. But $C_1(\overline{K}, F)$ is also a free module over the domain $F[a^\pm 1, b^\pm 1]$, and $f(a) \neq 0$, so $d(\delta) = \alpha$, contradicting the hypothesis that $\alpha$ represents a non-zero element of $H_1(\overline{K}, F)$.

This contradiction completes the proof. 

**Lemma 4.2.** Let $E$ be the set of midpoints of edges of a regular icosahedron $I \subset \mathbb{R}^3$ centred at the origin, and let $M = \mathbb{Z}E$ its $\mathbb{Z}$-span in $\mathbb{R}^3$. Let $V = \{1, a, b, c\} \subset \text{Isom}^+(I) \subset SO(3)$ be the Klein 4-group, and let $C = \{1, c\} \subset V$. Then, regarding $M$ as a $\mathbb{Z}V$-module via the action of $V$ by isometries of $I$, we have the following.

1) $M \cong \mathbb{Z}^6$ as an abelian group.

2) $H_0(C, M) = \mathbb{Z} \otimes_{\mathbb{Z}C} M \cong \mathbb{Z}_2^4 \oplus \mathbb{Z}^2$. 
3) *The induced action of $V/C$ on $H_0(C, M)/(\text{torsion})$ is multiplication by $-1$.*

Proof. If $e$ is the midpoint of the edge joining two vertices $u, v$ of $I$, then $e = (u + v)/2$. Thus $E$ is contained in the $\mathbb{Q}$-span $W$ of the set of vertices of $I$. Since the vertices occur in 6 antipodal pairs, the $\mathbb{Q}$-span $\mathbb{Q}M$ of $E$ has dimension at most 6 over $\mathbb{Q}$.

On the other hand, for any vertex $v$, $\sqrt{5} \cdot v$ is the sum of the 5 vertices adjacent to $v$ in $I$. Thus $\sqrt{5} \cdot v \in W$. It also follows that $\sqrt{5} \cdot e \in M$ for any $e \in E$: specifically, $(\sqrt{5} + 3) \cdot e$ is the sum of the midpoints of the eight edges of $I$ that share a vertex with the edge containing $e$. If $e_1, e_2, e_3 \in E$ are chosen to be linearly independent over $\mathbb{R}$ – and hence over $\mathbb{Q}[\sqrt{5}]$ – then $e_1, e_2, e_3, \sqrt{5} \cdot e_1, \sqrt{5} \cdot e_2, \sqrt{5} \cdot e_3 \in M$ are linearly independent over $\mathbb{Q}$. Thus $\mathbb{Q}M = \mathbb{Q} \otimes \mathbb{Z} M$ has dimension exactly 6 over $\mathbb{Q}$. Since $M \subset \mathbb{Q}M$ is torsion-free and finitely generated, it follows that $M \cong \mathbb{Z}^6$, as claimed.

If, in the above, we choose $e_1, e_2, e_3$ to lie on the axes of the rotations $a, b, c \in V$ respectively, then we obtain a decomposition

$$\mathbb{Q}M = \mathbb{Q}[\sqrt{5}]e_1 \oplus \mathbb{Q}[\sqrt{5}]e_2 \oplus \mathbb{Q}[\sqrt{5}]e_3$$

of $\mathbb{Q}M$ as a $\mathbb{Q}[\sqrt{5}]$-vector space, with respect to which $a, b, c$ act as the diagonal matrices $\text{diag}(1, -1, -1)$, $\text{diag}(-1, 1, -1)$ and $\text{diag}(-1, -1, 1)$ respectively. Let

$$M_+ := M \cap \mathbb{Q}[\sqrt{5}]e_3 \quad \text{and} \quad M_- := M \cap (\mathbb{Q}[\sqrt{5}]e_1 \oplus \mathbb{Q}[\sqrt{5}]e_2).$$

Then $M_- \cap M_+ = \{0\}$, while $e_1, e_2, \sqrt{5}e_1, \sqrt{5}e_2 \in M_-$ and $e_3, \sqrt{5}e_3 \in M_+$, so $M_-, M_+$ are free abelian of ranks 4 and 2 respectively.

Moreover, $M/M_-$ is naturally embedded in the vector space $\mathbb{Q}M/\mathbb{Q}M_-$, so is also free abelian – necessarily of rank 2. Note that $M_-$ is closed under the action of $V$ on $M$. Under the induced action on $M/M_-$, each of $a, b$ acts as the antipodal map, multiplication by $-1$, and $c$ acts as the identity. Clearly also $c$ acts on $M_-$ as the antipodal map.

Hence $(1 - c)M = 2M_-$, so

$$H_0(C, M) = M/(1 - c)M = M/2M_- \cong \mathbb{Z}_2^4 \oplus \mathbb{Z}^2,$$

as claimed.

Finally, the quotient of $H_0(C, M)$ by its torsion subgroup is naturally isomorphic to $M/M_-$, and the induced action of $V/C$ on this quotient is via the antipodal map. \qed
Lemma 4.3. Let \( G = \langle x, y | x^3 = y^5 = W(x, y)^2 = 1 \rangle \) and suppose that \((\lambda - \alpha)^2\) divides the trace polynomial \( \tau_W(\lambda) \) of \( W \), for some \( \alpha \in \{0, 1, (1+\sqrt{5})/2, (1-\sqrt{5})/2 \} \). Let \( \rho : G \to A_5 \) be the natural epimorphism corresponding to the root \( \alpha \) of \( \tau_W(\lambda) \). Let \( C \subset A_5 \) be a subgroup of order 2 and \( V \subset A_5 \) its centraliser of order 4. Then \( G \) has subgroups \( N_1 \triangleleft N_2 \triangleleft \rho^{-1}(V) \) such that

1) \( \rho(N_2) = \{1\} \);
2) \( \rho^{-1}(C)/N_2 \cong \mathbb{Z}^2 \);
3) \( \rho^{-1}(V)/N_2 \cong \mathbb{Z}^2 \rtimes \langle \alpha \rangle \);
4) \( N_2/N_1 \) is a non-zero vector space over \( \mathbb{Z}_2 \).

Proof. Let \( \Lambda = \mathbb{C}[\lambda]/((\lambda - \alpha)^2) \), and choose matrices

\[
X = \begin{pmatrix}
e^{i\pi/3} & 0 \\
1 & e^{-i\pi/3}
\end{pmatrix}, \quad Y = \begin{pmatrix}e^{i\pi/5} & \lambda - \alpha - 2\cos(8\pi/15) \\
0 & e^{-i\pi/5}\end{pmatrix} \in SL_2(\Lambda)
\]

so that

\[
tr(X) = 1, \quad tr(Y) = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad tr(XY) = \lambda - \alpha.
\]

Then \( X, Y \) determine a representation \( \hat{\rho} : G \to PSL_2(\Lambda) \), since \( tr(W(X, Y)) = \tau_W(\lambda) = 0 \) in \( \Lambda \). If \( \phi : PSL_2(\Lambda) \to PSL_2(\mathbb{C}) \) is the natural epimorphism obtained by setting \( \lambda = \alpha \), then the image of \( \rho = \phi \circ \hat{\rho} \) is isomorphic to \( A_5 \). Let \( K \) denote the kernel of \( \rho \) and let \( L \) denote the kernel of \( \hat{\rho} \).

Clearly \( G/K \cong A_5 \). Now \( K/L \cong \hat{\rho}(K) \) is the normal closure of \((xy)^2.L\), so it is isomorphic to the subgroup of \( PSL(2, \Lambda) \) generated by

\[
(XY)^2 = -I + (\lambda - \alpha)(XY)
\]

together with its conjugates by elements of \( \hat{\rho}(G) \). Let \( Z = \phi(XY) \in A_5 \subset SU(2) \) denote the matrix obtained from \( XY \) by substituting \( \lambda = \alpha \). Note that \( tr(Z) = 0 \), in other words, \( Z \in sl_2(\mathbb{C}) \). Since \((\lambda - \alpha)^2 = 0 \) in \( \Lambda \), we also have

\[
(XY)^2 = -I + (\lambda - \alpha)Z.
\]

For similar reasons, for any \( M \in \hat{\rho}(G) \) we have

\[
M(XY)^2M^{-1} = -I + \phi(M)Z\phi(M)^{-1}.
\]
Moreover, since \((\lambda - \alpha)^2 = 0\) in \(\Lambda\) we have, for any \(A, B \in \mathfrak{sl}_2(\mathbb{C})\),
\[
(I - (\lambda - \alpha)A)(I - (\lambda - \alpha)B) = I - (\lambda - \alpha)(A + B).
\]
Thus \(K/L \cong \rho(K)\) is isomorphic to the additive subgroup of \(\mathfrak{sl}_2(\mathbb{C})\) generated by \(MZM^{-1}\) for all \(M \in \hat{A}_5 \subset SU(2)\). There are precisely 30 such conjugates of \(Z\); geometrically they correspond to the midpoints of the edges of a regular icosahedron centred at the origin in \(\mathbb{R}^3\), where we identify \(SU(2)\) with the 3-sphere of unit-norm quaternions, and \(\mathbb{R}^3\) with the space of purely imaginary quaternions. As an abelian group, therefore, 
\[K/L \cong \rho(K) \cong \mathbb{Z}^6\] by Lemma 4.2.

Now \(K/L\) is also an \(A_5\)-module. Its structure as an \(A_5\)-module does not need to concern us, but Lemma 4.2 gives us some information about its structure as a \(C\)-module and as a \(V\)-module. This in turn gives information on the structure of \(\Delta := (\rho)^{-1}(C)\).

Specifically, \(H_0(C, K/L) = H_0(\Delta/K, K/L) \cong \mathbb{Z}_2^4 \oplus \mathbb{Z}_2^2\). It follows from the 5-term exact sequence
\[
H_2(\Delta/L) \to H_2(\Delta/K) \to H_0(\Delta/K, K/L) \to H_1(\Delta/L) \to H_1(\Delta/K) \to 0
\]
and the fact that \(\Delta/K \cong \mathbb{Z}_2\) that \(H_1(\Delta/L)\) has torsion-free rank 2, and that the torsion subgroup of \(H_1(\Delta/L)\) is a non-zero finite abelian 2-group.

Now let \(N_0 = [\Delta, \Delta].L\) and define \(N_2 \supset N_1 \supset N_0\) such that \(N_2/N_0\) is the torsion-subgroup of \(\Delta/N_0 = H_1(\Delta/L)\) and \(N_1/N_0 = 2(N_2/N_0)\). Then \(N_0 \lhd \rho^{-1}(V)\) since \([\Delta, \Delta]\) and \(L\) are both normal in \(\rho^{-1}(V)\). Hence also \(N_0 \lhd \rho^{-1}(V)\) and \(\rho^{-1}(V)\) is characteristic in \(\Delta\), hence is normal in \(\rho^{-1}(V)\).

By construction, \(\Delta/N_2 \cong \mathbb{Z}_2^2\), while \(N_2/N_1\) is a non-zero \(\mathbb{Z}_2\)-vector space.

Finally, since \(V/C\) acts on \(\mathbb{Z}_2^2 \cong \Delta/N_2\) by the antipodal map, it follows that \(\rho^{-1}(V)/N_2 \cong \mathbb{Z}_2^2 \rtimes (-1) \mathbb{Z}_2\), as required.

\(\square\)

5. Conclusion

**Theorem 5.1.** Let \(G = \langle x, y|x^3 = y^5 = W(x, y)^2 = 1\rangle\). If the trace polynomial \(\tau_W(\lambda)\) of \(W\) has a multiple root, then \(G\) contains a nonabelian free subgroup.

**Proof.** We may assume that the root \(\alpha\) is one of 0, 1, \((\pm 1 + \sqrt{5})/2\), for otherwise the result is immediate from Lemma 2.2. Let \(\rho : G \to A_5\) be the essential representation corresponding to \(\alpha\), let \(c = \rho(W) \in A_5\),
\[ C = \{1,c\} \subset A_5 \] is the subgroup generated by \( c \), and \( V = \{1,a,b,c\} \subset A_5 \) its centraliser in \( A_5 \).

Let \( N_1 \triangleleft N_2 \triangleleft \rho^{-1}(V) < G \) be the subgroups promised by Lemma 4.3.
Let \( \Gamma = \rho^{-1}(C) < \rho^{-1}(V) \) be the unique index 2 subgroup such that \( N_2 \subset \Gamma \) and \( \Gamma/N_2 \cong \mathbb{Z}^2 \). Then \( \Gamma \) has index 30 in \( G \) and contains no conjugate of \( x \) or of \( y \).

Applying the Reidemeister-Scheier process to the presentation of \( G \) in the statement of the Theorem, we obtain a presentation of \( \Gamma \) of the form
\[
\Gamma = \langle k_1, \ldots, k_{31} | r_1, \ldots, r_{30}, s_1^2, s_2^2 \rangle,
\]
where \( r_1, \ldots, r_{10} \) are rewrites of conjugates of \( x^3 \); \( r_{11}, \ldots, r_{16} \) are rewrites of conjugates of \( y^5 \); and \( r_{17}, \ldots, r_{30} \) and \( s_1^2 = W^2, s_2^2 = \hat{a}W^2\hat{a}^{-1} \) are rewrites of conjugates of \( W^2 \), with \( \rho(\hat{a}) = a \) and so \( s_1 = W, s_2 = \hat{a}W\hat{a}^{-1} \in \Gamma \).

Let \( K \) be the 2-complex model of this presentation, \( F = \mathbb{Z}_2 \), and \( p : \overline{K} \to K \) the regular cover corresponding to the normal subgroup \( N_2 \triangleleft \Gamma \). Let \( L \subset \overline{K} \) be the subcomplex obtained by omitting the 2-cells corresponding to the relators \( s_1^2, s_2^2 \), and let \( \overline{L} := p^{-1}(L) \subset \overline{K} \).

Now, since \( \Gamma/N_2 \) is torsion-free, and since \( s_1^2 = 1 = s_2^2 \) in \( \Gamma \), \( s_1, s_2 \in N_2 \). Hence each lift of each 2-cell \( s_i^2 \) (\( i = 1,2 \)) to \( \overline{K} \) is bounded by the square of some path in \( \overline{K}^{(1)} \). As a consequence, the 2-cells in \( \overline{K} \setminus \overline{L} \) represent elements of \( H_2(\overline{K}, F) \), and it follows that the inclusion-induced map \( H_1(\overline{L}, F) \to H_1(\overline{K}, F) \) is an isomorphism.

Since \( N_2/N_1 \) is a nonzero \( F \)-vector space, we have
\[
H_1(\overline{L}, F) \cong H_1(\overline{K}, F) = H_1(N_2, F) \neq 0.
\]
If \( H_2(\overline{L}, F) = 0 \), then by Lemma 4.1 it follows that \( \text{dim}_F H_1(N_2, F) = \infty \).
On the other hand, if \( H_2(\overline{L}, F) \neq 0 \) then \( H_2(\overline{L}, F) \) contains a free \( F(\Gamma/N_2) \)-module of rank \( > 0 = \chi(L) \), since \( F(\Gamma/N_2) \) is an integral domain. In this case \( H_1(\overline{L}, F) \) contains a non-zero free \( F(\Gamma/N_2) \)-submodule, by [14, Proposition 2.1 and Theorem 2.2]. Again we deduce that \( \text{dim}_F H_1(N_2, F) = \infty \).

Thus the Bieri-Strebel invariant \( \Sigma \) of the \( F(\Gamma/N_2) \)-module \( N_2/N_1 \) is a proper subset of \( S^1 \) [8, Theorem 2.4]. But by Lemma 4.2 (3) it follows that \( \Sigma \) is invariant under the antipodal map: \( \Sigma = -\Sigma \). Hence \( \Sigma \cup -\Sigma \neq S^1 \), and it follows [8, Theorem 4.1] that \( \Gamma \) contains a nonabelian free subgroup, as claimed.

\[ \square \]

**Corollary 5.2 (Main Theorem).** Let \( G \) be a generalised triangle group of type \((3, 5, 2)\). Then either \( G \) is virtually soluble or \( G \) contains a nonabelian free subgroup.
Proof. By Theorem 5.1 and Lemma 2.2 the result follows unless $\tau_W(\lambda)$ has only simple roots in the set \{0, 1, $(1 + \sqrt{5})/2, (-1 + \sqrt{5})/2$\}, in which case the degree $k$ of $\tau_W(\lambda)$ is at most equal to 4.

But the Rosenberger Conjecture is known for $k \leq 4$ [19].

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