SPECTRAL ASYMPTOTICS FOR MAGNETIC SCHRÖDINGER OPERATORS IN DOMAINS WITH CORNERS

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Abstract. This paper is on magnetic Schrödinger operators in two dimensional domains with corners. Semiclassical formulas are obtained for the sum and number of eigenvalues. The obtained results extend former formulas for smooth domains in \cite{11, 10} to piecewise smooth domains.

1. Introduction

The spectral analysis of magnetic Schrödinger operators in domains with boundary has been the subject of many research papers in the last two decades. Apart from the mathematical interest behind the study of their spectra, magnetic Schrödinger operators in interior/exterior domains with various boundary conditions arise in several models of condensed matter physics, as superconductivity \cite{14, 9, 15, 16}, liquid crystals \cite{12, 18} and Fermi gases \cite{6}.

The present paper is devoted to the study of magnetic Schrödinger operators in domains with corners (piecewise smooth domains). The presence of corners in the domain has a strong effect on the spectrum of the operator. In particular, it is shown in \cite{2, 13} that the presence of corners decreases the value of the ground state energy of the operator compared with the case of smooth domains. Discussion of this effect in the framework of superconductivity is given in \cite{3}.

We give in a simple particular case, a brief presentation of the semiclassical results proved in this paper. Suppose for simplicity that $\Omega$ is a simply connected bounded domain in $\mathbb{R}^2$ and that $A$ is a vector field such that $b = \text{curl} A$ is constant. Let $P_{h,\Omega} = -(h\nabla - iA)^2$ be the magnetic Laplacian in $L^2(\Omega)$ with magnetic Neumann boundary condition, $N(\lambda h)$ and $E(\lambda h)$ be the number and sum of negative eigenvalues of $P_{h,\Omega} - \lambda I_d$. If the boundary of $\Omega$ is smooth, it is proved in \cite{11, 10} that, as $h \to 0$,

$$N(\lambda h) \sim c_1(b,\lambda)|\partial \Omega|h^{-1/2}, \quad E(\lambda h) \sim c_2(b,\lambda)|\partial \Omega|h^{1/2},$$

(1.1)

where $c_1$ and $c_2$ are two explicit constants. The formula for the number $N(\lambda h)$ is valid for all $\lambda < b$, while that for the energy $E(\lambda h)$ is valid for all $\lambda \leq b$.

In Theorem 1.1 of this paper, we prove that (1.1) is still holding true when the domain $\Omega$ is only piecewise smooth. The key to prove this extension is a rough estimate on the number of negative eigenvalues of a Schrödinger operator with constant magnetic field in a sectorial domain (see Lemma 2.5).

The result of Theorem 1.1 shows that corners only affect the low-lying eigenvalues of the operator. As we will see in Theorem 3.1 corners create a few additional eigenvalues compared with smooth domains.

As a consequence, we may say that in the semi-classical limit and the regime considered in this paper, the number of eigenvalues of magnetic Schrödinger operators for smooth and non-smooth domains are asymptotically the same.

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The paper is organized in the following way. In Section 2, we collect some key results that will be used throughout the paper. Section 3 is about the number of low-lying eigenvalues of the operator in a domain with corners (here the regularity assumption on the domain is precisely stated). Finally, Section 4 contains the semi-classical analysis that extends (1.1) to domains with corners.

Throughout the paper, the notation $N(\lambda, P)$ will be often used to denote the number of eigenvalues (counting multiplicities) of the operator $P$ that are below $\lambda$.

2. Preliminaries

2.1. Variational principles. Let $H$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$ (of domain $D(H)$) such that

\[
\inf_{\sigma_{\text{ess}}(H)} \geq 0 \\
H_{1(-\infty,0)}(H) \text{ is trace class}.
\]

We shall need the following two simple variational principles concerning the operator $H$, which are frequently used in \[8\].

**Lemma 2.1.** Let $\gamma$ be a bounded operator such that $0 \leq \gamma \leq 1$ (in the sense of quadratic forms) and the operator $H\gamma$ is trace class. Then it holds that,

\[ \text{tr} \left( H_{1(-\infty,0)}(H) \right) \leq \text{tr}(H\gamma). \]

**Lemma 2.2.** Assume that the operator $H$ satisfies the hypothesis (H). Then it holds that,

\[ \text{tr} \left( H_{1(-\infty,0)}(H) \right) = \inf \sum_{j=1}^{N} \langle f_j, H f_j \rangle, \]

where the infimum is taken over all orthonormal families $\{f_1, f_2, \ldots, f_N\} \subset D(H)$ and $N \geq 1$.

2.2. A family of one-dimensional differential operators. Let us recall the main results obtained in \[7, 14\] concerning the family of harmonic oscillators with Neumann boundary condition. Given $\xi \in \mathbb{R}$, we define the quadratic form,

\[
B^1(\mathbb{R}_+) \ni u \mapsto q(\xi)(u) = \int_{\mathbb{R}_+} |u'(t)|^2 + |(t - \xi)u(t)|^2 dt,
\]

where, for a positive integer $k \in \mathbb{N}$ and a given interval $I \subset \mathbb{R}$, the space $B^k(I)$ is defined by:

\[
B^k(I) = \{ u \in H^k(I); \quad t^j u(t) \in L^2(I), \quad \forall j = 1, \ldots, k \}.
\]

(2.2)

Since the quadratic form (2.1) is closed and symmetric it defines a unique self-adjoint operator $L[\xi]$. This operator has domain,

\[
D(L[\xi]) = \{ u \in B^2(\mathbb{R}_+); \quad u'(0) = 0 \},
\]

and is the realization of the differential operator,

\[
L[\xi] = -\partial_t^2 + (t - \xi)^2,
\]

on the given domain. We denote by $\{\mu_j(\xi)\}_{j=1}^{+\infty}$ the increasing sequence of eigenvalues of $L[\xi]$, which are all simple. By the min-max principle, we have,

\[
\mu_1(\xi) = \inf_{u \in B^1(\mathbb{R}_+), u \neq 0} \frac{q(\xi)(u)}{\|u\|^2_{L^2(\mathbb{R}_+)}}.
\]

(2.4)

It follows from analytic perturbation theory (see \[7\]) that the functions

\[
\mathbb{R} \ni \xi \mapsto \mu_j(\xi)
\]
are analytic. Furthermore, \( \mu_1(0) = 1 \) and \(|\mu_1(\xi) - 1|\) decays like \( \exp(-\xi^2) \) as \( \xi \to +\infty \) (see [4]) thus yielding that
\[
\int_0^\infty (\mu_1(\xi) - 1) \, d\xi = -\int_\mathbb{R} \left[\mu_1(\xi) - 1\right] \, d\xi \quad \text{is finite.}
\]
We define the constant:
\[
\Theta_0 = \inf_{\xi \in \mathbb{R}} \mu_1(\xi). \tag{2.5}
\]
In [7], it is proved that \( \Theta_0 = \mu_1(\xi_0) \), that \( \xi_0 \) is the unique value at which the minimum \( \Theta_0 \) is attained and that \( \mu''(\xi_0) > 0 \).

An important consequence of standard Sturm-Liouville theory is recalled below (c.f. [11, Lemma 2.1]).

**Lemma 2.3.** The second eigenvalue of \( L[\xi] \) satisfies,
\[
\inf_{\xi \in \mathbb{R}} \mu_2(\xi) > 1.
\]

Notice that part of this conclusion is a consequence of the analysis of Dauge-Helffer [7], who show that the infimum of \( \mu_2(\xi) \) is attained for a unique \( \xi_2 \in \mathbb{R} \).

### 2.3. Rough energy bound for the cylinder.

Let us consider the operator
\[
P_{\Omega} = -(h \nabla - ibA_0)^2 \quad \text{in} \quad L^2(\Omega),
\]
with
\[
\Omega = [0,S] \times (0,h^{1/2}T).
\]
Functions in the domain of \( P_{\lambda,\Omega} \) satisfy Neumann condition at \( t = 0 \), periodic conditions at \( s \in \{0,S\} \) and Dirichlet condition at \( t = h^{1/2}T \). We assume that the vector field \( A_0 \) is given by
\[
A_0(x_1, x_2) = (-x_2, 0).
\]
In this particular case, the operator has compact resolvent, hence the spectrum consists of an increasing sequence of eigenvalues \((e_j)_{j\geq 1}\) converging to \(+\infty\). Note that the terms of the sequence \((e_j)\) are listed with multiplicities counted. Given \( \lambda \in \mathbb{R} \), the energy
\[
E(\lambda, b, S, T) = \sum_j [hb(1 + \lambda) - e_j]_+
\]
is finite. The number of eigenvalues below \((1 + \lambda)hb\) is also of particular interest:
\[
N(\lambda, b, S, T) = \text{Card}\{j : e_j \leq hb(1 + \lambda)\}. \tag{2.7}
\]

In the lemma below, we state estimates of the energy \( E(\lambda, b, S, T) \) and the number \( N(\lambda, b, S, T) \).

**Lemma 2.4.** There exist positive constants \( T_0 \) and \( \lambda_0 \) such that, for all \( S > 0, b > 0, T \geq \sqrt{b}T_0 \) and \( \lambda \in (0, \lambda_0] \), we have,
\[
E(\lambda, b, S, T) \leq (1 + \lambda)hb \left( \frac{ST}{2\pi \sqrt{h}} + 1 \right), \tag{2.8}
\]
\[
N(\lambda, b, S, T) \leq \frac{ST}{2\pi \sqrt{h}} + 1. \tag{2.9}
\]

The lemma is proved in [10], although the estimate (2.9) is not stated explicitly. Actually, (2.8) is proved as follows. First (2.9) is established by separation of variables and the variational min-max principle then the energy \( E(\lambda, b, S, T) \) is easily estimated as \( N(\lambda, b, S, T) \times (1 + \lambda)bh \), from which (2.8) follows.
2.4. **Rough bounds for the sector.** Let $\alpha \in (0, 2\pi)$, $R > 0$, $h > 0$, $b > 0$ and
\[
\Omega_{R,h,\alpha} = \{ (r \cos \theta, r \sin \theta) \in \mathbb{R}^2 : 0 < \theta < \alpha, \quad 0 \leq r < h^{1/2}R \}. \tag{2.10}
\]

Consider the self-adjoint operator $P_{h,\alpha} = -(h\nabla - ibA_0)^2$ in $L^2(\Omega_{R,h,\alpha})$. Functions in the domain of $P_{h,\alpha}$ satisfy Dirichlet condition on the boundary $r = R$ and Neumann condition
\[
\nu \cdot (h\nabla - ibA_0)u = 0
\]
on the boundary defined by $\theta = 0$ or $\theta = \alpha$. Here $\nu$ is the unit outward normal vector on the boundary $\partial \Omega_{R,h,\alpha}$ and $A_0$ is the magnetic potential introduced in (3.2).

The operator $P_{h,\alpha}$ has compact resolvent and its spectrum is discrete and consists of isolated eigenvalues $(e_j)$ counted with multiplicities. Let $\lambda \in \mathbb{R}$ and define,
\[
\mathcal{E}_{\text{corn}}(\alpha, b, R, \lambda) = \sum_j [hb(1 + \lambda) - e_j]_+ , \tag{2.11}
\]
and
\[
\mathcal{N}_{\text{corn}}(\alpha, b, R, \lambda) = \text{Card} \{ j : e_j \leq hb(1 + \lambda) \} . \tag{2.12}
\]

We give rough estimates of $\mathcal{E}_{\text{corn}}(\alpha, b, R, \lambda)$ and $\mathcal{N}_{\text{corn}}(\alpha, b, R, \lambda)$ in the next lemma.

**Lemma 2.5.** Given $b > 0$ and $\alpha \in [0, 2\pi)$, there exist positive constants $C, h_0, R_0$ and $\lambda_1$ such that, for all $R \geq R_0$, $h \in (0, h_0]$, and $\lambda \in (-\infty, \lambda_1]$, we have,
\[
\mathcal{N}_{\text{corn}}(\alpha, b, R, \lambda) \leq C \left(R^2 + 1\right), \tag{2.13}
\]
and
\[
\mathcal{E}_{\text{corn}}(\alpha, b, R, \lambda) \leq Chb \left(R^2 + 1\right). \tag{2.14}
\]

**Proof.** Observe that the upper bound (2.13) on the energy $\mathcal{E}_{\text{corn}}(\alpha, b, R, \lambda)$ follows immediately from the definition of $\mathcal{E}_{\text{corn}}(\alpha, b, R, \lambda)$ and the upper bound in (2.13). Therefore, we establish the bound in (2.13). The method that will be used is introduced in [5] and based on a decomposition of the operator via a partition of unity and on the min-max variational principle.

To simplify the notation, we denote by $\Omega$ and $P$ the set $\Omega_{R,h,\alpha}$ introduced in (2.10) and the operator $P_{h,\alpha}$ respectively.

Let $A_0$ be a positive real number whose choice will be specified later. Suppose $R > A_0$. We cover $\Omega$ by two sets,
\[
U_1(h) = \{ x \in \Omega : |x| < A_0h^{1/2} \}, \quad U_2(h) = \{ x \in \Omega : \frac{A_0h^{1/2}}{2} < |x| < Rh^{1/2} \}.
\]

Consider a partition of unity of $\Omega$,
\[
\sum_{j=1}^2 \chi_j^2(x) = 1 \quad \text{and} \quad \sum_{j=1}^2 \nabla \chi_j(x)^2 \leq \frac{C_4}{A_0^2h} \quad \text{in} \ \Omega,
\]
where $C_4 > 0$ is a universal constant and $\text{supp} \chi_j \subset U_j(h^{1/2})$.

Let $u \in H^1(\Omega)$. The following decomposition formula holds true,
\[
\int_\Omega |(\nabla - ibA_0)u|^2 \, dx = \sum_{j=1}^2 \int_\Omega \left( |(\nabla - ibA_0)\chi_j u|^2 - h^2|\nabla \chi_j|^2 |u|^2 \right) \, dx.
\]

This decomposition yields the inequality stated below by using the upper bound on $|\nabla \chi_j|$:
\[
\int_\Omega |(\nabla - ibA_0)u|^2 \, dx \geq \sum_{j=1}^2 \int_\Omega \left( |(\nabla - ibA_0)\chi_j u|^2 - C_4A_0^{-2}h |u|^2 \right) \, dx.
\]
By using the method in [3] and the variational min-max principle, we get that,

$$N_{\text{corn}}(\alpha, b, R, \lambda) \leq N(\Lambda, P_{U_1(h)}) + N(\Lambda, P_{U_2(h)})$$

(2.15)

where $\Lambda = (1 + \lambda)b + C_1 A_0^{-2}$, $P_{U_1(h)} = -(h \nabla - ibA_0)^2$ is the operator in $L^2(U_1(h))$ with Dirichlet condition on $r = h^{1/2}$ and Neumann condition on the other parts of the boundary of $U_1(h)$, and $P_{U_2(h)} = -(h \nabla - ibA_0)^2$ is the operator in $L^2(U_2(h))$ with Neumann boundary condition on $\theta = 0$ and $\theta = \alpha$ and Dirichlet condition on the other parts of the boundary. Notice that we use polar coordinates $(r, \theta)$ in the definition of the domains.

Let $\lambda_0$ be as in the statement of Lemma 2.4 and select $\lambda_1 \in (0, \lambda_0)$. In light of (2.15), Theorem 2.3 will be proved once the statement that follows below is shown to be true:

$$\exists C > 0, N(\Lambda, P_{U_1(h)}) \leq C$$

(2.16)

and

$$N(\Lambda, P_{U_2(h)}) \leq C \left(R^2 + 1\right).$$

(2.17)

Proof of (2.16). Applying the re-scaling $y = h^{-1/2}x$ changes the domain $U_1(h)$ to $U_1(A_0)$ and the operator $P_{U_1(h)}$ to the operator in $L^2(U_1(A_0))$,

$$P_{U_1(A_0)} = -(\nabla - ibA_0)^2.$$

More precisely, a number $c$ is in the spectrum of $P_{U_1(h)}$ if and only if $c/h$ is in the spectrum of $P_{U_1(A_0)}$. Thus, the term we wish to estimate is given as follows,

$$N(\Lambda, P_{U_1(h)}) = N(\Lambda, P_{U_1(A_0)}) .$$

Since the operator $P_{U_1(1)}$ has compact resolvent and the number $\Lambda$ is bounded independently of $h \in (0, 1]$, it follows that $N(\Lambda, P_{U_1(1)})$ is bounded as $h$ varies in $(0, 1]$ too. This establishes the upper bound in (2.16).

Proof of (2.17). As we shall see, the upper bound in (2.17) follows mainly from Lemma 2.4. Let $\epsilon$ be a positive real number such that,

$$0 < \epsilon < \min \left(\frac{\alpha}{8}, \frac{\pi}{4}, \frac{\alpha}{8}\right).$$

Recall the definition of the set

$$U_2(h) = \{x \in \Omega : \frac{A_0 h^{1/2}}{2} < |x| < Rh^{1/2}\} .$$

We cover $U_2(h)$ by four sets,

$$V_1(h) = \{(r \cos \theta, r \sin \theta) \in \overline{U_2(h)} : 0 \leq \theta < \frac{\alpha}{4}\} ,$$

$$V_2(h) = \{(r \cos \theta, r \sin \theta) \in \overline{U_2(h)} : \frac{\alpha}{4} - \epsilon < \theta < \frac{\alpha}{2} + \epsilon\} ,$$

$$V_3(h) = \{(r \cos \theta, r \sin \theta) \in \overline{U_2(h)} : \frac{\alpha}{2} - \epsilon < \theta < \frac{3\alpha}{4} + \epsilon\} ,$$

$$V_4(h) = \{(r \cos \theta, r \sin \theta) \in \overline{U_2(h)} : \frac{3\alpha}{4} < \theta \leq \alpha\} .$$

Define four operators $P_{V_1(h)}$, $P_{V_2(h)}$, $P_{V_3(h)}$ and $P_{V_4(h)}$ in $L^2(V_1(h))$, $L^2(V_2(h))$, $L^2(V_3(h))$ and $L^2(V_4(h))$ respectively. The four operators are self-adjoint realizations of the differential operator $-(h \nabla - ibA_0)^2$. Functions in the domain of $P_{V_1(h)}$ satisfy Neumann condition on $\theta = 0$ and Dirichlet condition on the other parts of the boundary. Similarly, functions in the domain of $P_{V_4(h)}$ satisfy Neumann condition on $\theta = \alpha$ and Dirichlet condition on the other parts of the boundary. Functions in the domains of $P_{V_2(h)}$ and $P_{V_3(h)}$ satisfy Dirichlet boundary condition. Thus, the bottoms of the spectra of the operators $P_{V_2(h)}$ and $P_{V_3(h)}$ are larger or equal to $bh$. 

Notice that the operators $P_{V_1(h)}$ and $P_{V_4(h)}$ are unitary equivalent and hence have same spectra. Also, the operators $P_{V_2(h)}$ and $P_{V_3(h)}$ are unitary equivalent and have same spectra.

We apply an argument similar to the one we did to obtain (2.15). By introducing a partition of unity supported in $V_1(h)$, $V_2(h)$, $V_3(h)$ and $V_4(h)$, using the IMS decomposition formula and the variational min-max principle, we get a constant $C_2 > 0$ that is allowed to depend on $\epsilon$ but not on $h$ and such that,

$$N(\Lambda h, P_{U_2(h)}) \leq \sum_{j=1}^{4} N(\widetilde{\Lambda} h, P_{V_j(h)})$$

$$= 2 \sum_{j=1}^{2} N(\widetilde{\Lambda} h, P_{V_j(h)}) ,$$

(2.18)

with $\widetilde{\Lambda} = \Lambda + C_1 A_0^{-2} + C_2 h$.

Recall the number $N(\lambda, b, S,T)$ introduced in (2.17). This number counts the eigenvalues of the operator $P_U$ in the cylinder, where $U = [0, S] \times (0, h^{1/2}T)$.

Since the angle $\alpha/4$ is acute and Dirichlet condition is imposed in the form domain of $P_{V_1(h)}$, functions in the form domain of $P_{V_1(h)}$ can be extended by zero to functions defined in the cylinder,

$$U = [0, Rh^{1/2}] \times (0, Rh^{1/2}) .$$

The extended functions are in the form domain of the operator $P_U$. Thus, by the variational min-max principle, it is easy to see that,

$$N(\Lambda h, P_{V_1(h)}) \leq N(\widetilde{\Lambda}, b, Rh^{1/2}, R) ,$$

with $\widetilde{\lambda} = \lambda + 2b^{-1}C_1 A_0^{-2} + C_2 b^{-1}h$. Similarly, we have,

$$N(\Lambda h, P_{V_2(h)}) \leq N(\widetilde{\lambda}, b, Rh^{1/2}, R) .$$

Thus we get from (2.18),

$$N(\Lambda h, P_{U_2(h)}) \leq 2N(\widetilde{\lambda}, b, Rh^{1/2}, R) .$$

(2.19)

Select $h_0$ sufficiently small and $A_0$ sufficiently large such that

$$2b^{-1}C_1 A_0^{-2} + C_2 b^{-1}h_0 < \lambda_0 - \lambda_1 .$$

In this way, we get for all $\lambda \in (0, \lambda_1)$ and $h \in (0, h_0)$, $\widetilde{\lambda} \in (0, \lambda_0]$. Consequently, it follows from Lemma 2.4 that $N(\widetilde{\lambda}, b, Rh^{1/2}, R)$ is bounded by a constant times $(R^2 + 1)$, and thereby get the upper bound in (2.17).

3. Asymptotic number of low-lying eigenvalues near corners

3.1. The operator in an infinite sector. Let $\alpha \in (0, 2\pi)$ and

$$\Omega_\alpha = \{ (r \cos \theta, r \sin \theta) \in \mathbb{R}^2 : r \geq 0, \ 0 < \theta < \alpha \} .$$

(3.1)

Define the magnetic potential,

$$\forall \ (x_1, x_2) \in \mathbb{R}^2, \ \ A_0(x_1, x_2) = (-x_2, 0) ,$$

(3.2)

whose curl is constant and equal to 1.

Consider the self-adjoint operator $P_{1_{\alpha}} = - (\nabla - i A_0)^2$ in $L^2(\Omega_\alpha)$ whose domain is:

$$D(P_{1_{\alpha}}) = \{ u \in H^1(\Omega_\alpha) : P_{1_{\alpha}} u \in L^2(\Omega_\alpha), \ \nu \cdot (\nabla - i A_0)u = 0 \text{ on } \partial \Omega_\alpha \setminus \{0\} \} ,$$

where $\nu$ is the unit outward normal vector of $\partial \Omega_\alpha$. 
It is proved in [2] that the bottom of the essential spectrum of $P_{\Omega_\alpha}$ is a universal constant $\Theta_0 \in (\frac{\pi}{2}, 1)$. Furthermore, when $\alpha \in (0, \frac{\pi}{2}]$, the operator $P_{\Omega_\alpha}$ has discrete spectrum below $\Theta_0$.

Many questions connected with the spectrum of the operator $P_{\Omega_\alpha}$ are left open. Among these questions are the following:

- Is the spectrum of $P_{\Omega_\alpha}$ below $\Theta_0$ finite?
- When $\alpha = \pi$, the operator $P_{\Omega_\alpha}$ is the half-plane and its spectrum is purely essential and consists of the interval $[\Theta_0, \infty)$. Is $[\Theta_0, \infty)$ the essential spectrum of $P_{\Omega_\alpha}$ for any $\alpha$?
- It is conjectured in [2] that the ground state energy (bottom of the spectrum) of $P_{\Omega_\alpha}$ is constant and equal to $\Theta_0$ when $\alpha \in [\pi, 2\pi)$. This conjecture has not been proved or disproved yet (see [2]).

### 3.2. Assumptions on the domain.

We describe precisely the regularity properties of the domain $\Omega$. The assumptions will be the same as those made in [2, 3].

In this and the subsequent sections, $\Omega$ is an open and connected set in $\mathbb{R}^2$ whose boundary is compact and consists of a curvilinear polygon of class $C^3$. By saying that the boundary $\Gamma$ of $\Omega$ is a curvilinear polygon (of class $C^3$) we mean the following (see [Gr] p.34-42). For every $x \in \Gamma$, there exists a neighborhood $V$ of $x$ in $\mathbb{R}^2$ and a mapping $\psi$ from $V$ to $\mathbb{R}^2$ such that

1. $\psi$ is injective,
2. $\psi$ together with $\psi^{-1}$ (defined on $\psi(V)$) belongs to the class $C^3$,
3. $\Omega \cap V$ is either $\{y \in \Omega : \psi_2(y) < 0\}$, $\{y \in \Omega : \psi_1(y) < 0 \text{ and } \psi_2(y) < 0\}$, or $\{y \in \Omega : \psi_1(y) < 0 \text{ or } \psi_2(y) < 0\}$, where $\psi_j$ denotes the components of $\psi$.

The boundary $\Gamma$ of $\Omega$ is a piecewise smooth curve. We work under the assumption that the boundary $\Gamma$ consists of a finite number of smooth curves $\Gamma_k$ for $k = 1, \ldots, m$. The family $(\Gamma_k)$ is the minimal family of curves making up the boundary $\Gamma$. The curve $\Gamma_{k+1}$ follows $\Gamma_k$ according to a positive orientation, on each connected component of $\Gamma$. Let $s_k$ denotes the vertex which is the end point of $\Gamma_k$. In a neighborhood of $\partial \Omega$, define a vector field $\nu_k$. For each $k$, the vector $\nu_k$ is the unit normal a.e. on $\Gamma_k$.

Let $\Sigma$ be the set of vertices the domain $\Omega$ has. Suppose that $\Sigma \neq \emptyset$ and consists exactly of $N$ vertices. This assumption certifies that the domain $\Omega$ does not have a smooth boundary but only a piecewise smooth boundary. When $\Sigma = \emptyset$, the boundary of $\Omega$ is smooth and the operator in $L^2(\Omega)$ is the one studied in [11, 10].

At each vertex $s_k \in \Sigma$, let $\alpha_{s_k}$ denotes the angle between $\Gamma_k$ and $\Gamma_{k+1}$ measured towards the interior.

### 3.3. Main result.

For each angle $\alpha$, recall the sectorial domain $\Omega_\alpha$ introduced in (3.1). Let $P_{b, \Omega_\alpha} = -(\nabla - ibA_0)^2$ be the operator in $L^2(\Omega_\alpha)$ introduced in Section 3.1 (with Neumann boundary condition on the smooth part of the boundary of the sector $\Omega_\alpha$). The number

$$n(\alpha, \lambda; b) = \text{tr} \left( 1_{(-\infty, \lambda)}(P_{\Omega_\alpha}) \right) = \sum_{\Lambda < \lambda} \dim \left( \text{Ker} (P_{\Omega_\alpha} - \Lambda \text{Id}) \right)$$

(3.3)

is finite for all $\alpha$ and $\lambda < \Theta_0$.

If $b = 1$, we write,

$$n(\alpha, \lambda) = n(\alpha, \lambda; b = 1).$$

(3.4)

By a scaling argument, it is easy to see that,

$$n(\alpha, \lambda; b) = n(\alpha, \lambda/b).$$

Let the domain $\Omega$ be as described in Section 3.2. Define the magnetic Schrödinger operator,

$$P_{h, \Omega} = -(h \nabla - iA)^2, \quad \text{in} \quad L^2(\Omega),$$

(3.5)
where $A \in C^2(\Omega; \mathbb{R}^2)$ is the magnetic potential, $h > 0$ is the semi-classical parameter and $B = \text{curl } A$ is the magnetic field. The domain of the operator $P_{h,\Omega}$ is,

$$D(P_{h,\Omega}) = \{ u \in L^2(\Omega) : (h\nabla - iA)^k \in L^2(\Omega), \ k = 1, 2, \ \nu_j \cdot (h\nabla - iA)u = 0 \text{ on } \Gamma_k, \ k = 1, \cdots, m \}.$$ 

Define two constants,

$$b = \inf_{x \in \Omega} B(x), \ b' = \inf_{x \in \partial\Omega} B(x). \quad (3.6)$$

The main result of this section is:

**Theorem 3.1.** Let the constant $\Theta_0$ be as defined in [25], $\lambda \in (-\infty, \min(\Theta_0b', b))$ and $N(\lambda h)$ the number of eigenvalues of the operator $P_{h,\Omega}$ below $\lambda h$ counting multiplicities. Suppose that the magnetic field $B(x)$ is selected such that $b > 0$. There exists a positive number $h_0$ such that, for all $h \in (0, h_0)$, the following equality holds true,

$$N(\lambda h) = \sum_{k=1}^N n\left(\alpha_{s_k}, \frac{\lambda}{B(s_k)}\right). \quad (3.7)$$

The number $h_0$ depends only on the angles $\alpha_{s_k}$ and the domain $\Omega$.

Theorem 3.1 gives the exact number of low-lying eigenvalues of $P_{h,\Omega}$ corresponding to corners in the domain $\Omega$. When $\lambda > \Theta_0$, we will see in Section 4 that corners no more affect the leading order term of $N(\lambda h)$.

### 3.4. Proof of Theorem 3.1

**Upper bound.** Let $\rho$ be a positive constant satisfying $0 < \rho < 1$. We start by choosing a partition of unity $\chi_{k,h}$ introduced in [2] Prop. 11.2] satisfying:

$$\sum_k |\chi_{k,h}|^2 = 1, \ \sum_k |\nabla \chi_{k,h}|^2 \leq C h^{-2\rho} \text{ in } \mathbb{R}^2, \ \text{supp } \chi_{k,h} \subset B(z_j, c_k h^\rho),$$

with the choice of indices such that

$$\begin{align*}
z_k = s_k \text{ and } c_k &= 1 \text{ for all } k = 1, 2, \cdots, N, \\
\text{If } k \notin \{1, \cdots, m\} \text{ and } z_k \notin \partial \Omega, \text{ then } B(z_k, c_k h^\rho) \cap \partial \Omega = \emptyset \text{ and } c_k = 1/2 \min(\tan \alpha_{s_k}, 1), \\
\text{If } k \notin \{1, \cdots, m\} \text{ and } z_k \in \partial \Omega, \text{ then } B(z_k, c_k h^\rho) \cap \Sigma = \emptyset, \quad B(z_k, c_k h^\rho) \cap \partial \Omega \text{ is connected and } c_k = \frac{1}{2} \min(|\tan \alpha_{s_k}|, 1). 
\end{align*}$$

Recall that $\Sigma$ is the set of vertices of the domain $\Omega$. If $u$ is a function in the form domain of $P_{h,\Omega}$, define,

$$q(u) = \int_\Omega |(h\nabla - iA)u|^2 \, dx.$$ 

The following decomposition formula holds true for every function $u$ in the form domain of the operator $P_{h,\Omega}$,

$$q(u) = \sum_{k=1}^m q(\chi_{k,h}u) + \sum_{k \geq m} q(\chi_{k,h}u) + \sum_{j \geq m} q(\chi_{k,h}u) - h^2 \sum_k \int_\Omega |\nabla \chi_{k,h}|^2 |u|^2 \, dx.$$ 

Using the upper bound on $|\nabla \chi_{k,h}|^2$, we get the lower bound,

$$q(u) \geq \sum_{k=1}^m q(\chi_{k,h}u) + \sum_{k \geq m} q(\chi_{k,h}u) + \sum_{j \geq m} q(\chi_{k,h}u) - C h^{2-2\rho} \int_\Omega |u|^2 \, dx. \quad (3.8)$$
Let \( k \in \{1, 2, \cdots, m\} \). It is proved in \cite{2} p. 252] that by performing a change of variable \( y = \psi_k(x) \) and a gauge transformation (defined by a function \( \phi_k \)), the following lower bound holds true,

\[
q(\chi_{k,h} u) \geq \int_{\Omega_{\alpha_k}} \left( (1 - Ch^\rho - Ch^{2\theta})(1 + 2\theta)(h\nabla - iB_k A_0) v_k \right)^2 \, dy. \tag{3.9}
\]

Here \( \theta \in (0,1) \) is any constant, \( B_k = B(z_k) \), \( A_0 \) is the magnetic potential in \cite{3} and

\[
v_k(y) = e^{-i\phi_k(y)}(\chi_{k,h} u) \circ \psi_k^{-1}(y).
\]

The optimal choice of \( \rho \) and \( \theta \) is \( \rho = 3/8 \) and \( \theta = 1/8 \). This produces the lower bound:

\[
q(\chi_{k,h} u) \geq \int_{\Omega_{\alpha_k}} \left( (1 - Ch^{1/4})(1 + 2\theta)(h\nabla - iB_k A_0) v_k \right)^2 \, dy. \tag{3.10}
\]

Similarly, when \( k \not\in \{1, 2, \cdots, m\} \) and \( z_k \in \partial \Omega \), by applying a change of variable and a gauge transformation, we obtain the lower bound (see \cite{11}),

\[
q(\chi_{k,h} u) \geq \int_{U_h} \left( (1 - Ch^{1/4})(1 + 2\theta)(h\nabla - iB_k A_0) v_k \right)^2 \, dy, \tag{3.11}
\]

where \( U_h = \{ y = (y_1, y_2) \in \mathbb{R}^2 : y_1 \in (0, h^{3/8}) \) and \( y_2 \in (0, \infty) \} \). By inserting the lower bounds in (3.10) and (3.11) into (3.8) we obtain,

\[
q(u) \geq \sum_{k=1}^{m} \int_{\Omega_{\alpha_k}} \left( (1 - Ch^{1/4})(1 + 2\theta)(h\nabla - iB_k A_0) v_k \right)^2 \, dy \tag{3.12}
\]

\[+ \sum_{k \geq m, \, \alpha_k \in \partial \Omega} \int_{U_h} \left( (1 - Ch^{1/4})(1 + 2\theta)(h\nabla - iB_k A_0) v_k \right)^2 \, dy
\]

\[+ \sum_{k \geq m, \, \alpha_k \not\in \partial \Omega} \int_{\Omega} |u|^2 \, dx.
\]

By the variational min-max principle, we deduce the following upper bound on the number \( N(\lambda h) \),

\[
N(\lambda h) \leq \sum_{k=1}^{m} \left( \alpha_{s_k} \cdot \frac{\lambda + Ch^{1/4}}{1 - Ch^{1/4}B(s_k)} \right) + \sum_{k \geq m, \, \alpha_k \in \partial \Omega} N\left( P_{B_k, U_h}, \frac{\lambda h + Ch^{5/4}}{1 - Ch^{1/4}} \right) + \sum_{k \geq m, \, \alpha_k \not\in \partial \Omega} N\left( P_{h, \Omega}, \lambda h + Ch^{5/4} \right).
\]

Here \( P_{B_k, U_h} = -(h \nabla - iB_j A_0)^2 \) is the operator in \( L^2(U_h) \) with Neumann boundary condition at \( y_2 = 0 \) and Dirichlet condition elsewhere, and \( P_{h, \Omega} = -(h \nabla - iA)^2 \) is the operator in \( L^2(\Omega) \) with Dirichlet boundary condition. The spectrum of \( P_{B_k, U_h} \) starts at \( \Theta_0 B_k h \) and that of \( P_{h, \Omega} \) starts above \( bh \) (cf. \cite{11}). Thus, if \( \lambda < \min(\Theta_0 b', b) \) and \( h \) is selected sufficiently small, we get,

\[
N\left( P_{B_k, U_h}, \frac{\lambda h + Ch^{5/4}}{1 - Ch^{1/4}} \right) = N\left( P_{h, \Omega}, \lambda h + Ch^{5/4} \right) = 0.
\]

Since the spectrum of the operator \( P_{\Omega_{\alpha_k}} \) below \( \Theta_0 \) consists of isolated eigenvalues, we get for \( h \) sufficiently small that,

\[
n\left( \alpha_{s_k}, \frac{\lambda + Ch^{1/4}}{1 - Ch^{1/4}} \right) = n\left( \alpha_{s_k}, \frac{\lambda}{B(s_k)} \right).
\]

This finishes the proof of the upper bound. \qed
Lower bound. The proof of the lower bound is similar to the one in [11] and uses a bracketing technique. Let \( \widetilde{P}_{h, \Omega} \) be the self-adjoint realization of the restriction of the operator \( P_{h, \Omega} \) on functions vanishing outside the set,
\[
\bigcup_{k=1}^{N} B(s_k, h^{3/8}).
\]
By the variational min-max principle, the eigenvalues of \( \widetilde{P}_{h, \Omega} \) are larger than those of \( P_{h, \Omega} \). Thus,
\[
N(\lambda h) \geq N(\lambda h, \widetilde{P}_{h, \Omega}).
\]
We will show next that there exist constants \( C > 0 \) and \( h_0 > 0 \) such that, for all \( h \in (0, h_0] \), we have,
\[
N(\lambda h, \widetilde{P}_{h, \Omega}) \geq \sum_{k=1}^{N} n\left( \alpha_{s_k}, \frac{\lambda - Ch^{1/4}}{(1 + Ch^{1/4})B(s_k)} \right).
\]
If \( h \) is made sufficiently small, then
\[
n\left( \alpha_{s_k}, \frac{\lambda - Ch^{1/4}}{(1 + Ch^{1/4})B(s_k)} \right) = n\left( \alpha_{s_k}, \frac{\lambda}{B(s_k)} \right),
\]
and we get the lower bound we wish to prove.

Derivation of (3.13) is easy. Let \( \bar{q} \) be the quadratic form associated to \( \widetilde{P}_{h, \Omega} \).

Select an arbitrary function \( u \) in the form domain of \( \widetilde{P}_{h, \Omega} \). A matching asymptotic upper bound to (3.10) is proved in [2, p. 252]:
\[
\int_{B(s_k, h^{3/8})} |(\nabla - iA)u|^2 \, dx \leq \int_{\Omega_{\alpha s_k}} \left( (1 - Ch^{1/4})|h\nabla - iB_kA_0)v_j|^2 - Ch^{5/4}|v_k|^2 \right) \, dy,
\]
where \( B_k = B(s_k) \) and \( v_k \) is obtained from \( u \) by the change of variables \( y = \psi_k(x) \) times a gauge transformation. Summing over \( k \), we get,
\[
\bar{q}(u) \geq \int_{\Omega_{\alpha s_k}} \left( (1 - Ch^{1/4})|h\nabla - iB_kA_0)v_k|^2 - Ch^{5/4}|v_k|^2 \right) \, dy.
\]
By the variational min-max principle, we deduce that the eigenvalues of the operator \( \widetilde{P}_{h, \Omega} \) are larger than those of the direct sum of the operators \( P_{h, \alpha s_k, B_k} \), thereby proving (3.13). \( \square \)

4. Energy and number of eigenvalues in piecewise smooth domains

4.1. Main results. We will state other results concerning the number and sum of eigenvalues of the operator \( P_{h, \Omega} \) introduced in (3.5). The assumptions on the domain \( \Omega \) is as described in Section 3.2. The notation and assumption on the magnetic field \( B = \text{curl} A \) is as in (3.6) and Section 3.4.

Let \( (e_j) \) be the increasing sequence of eigenvalues of the operator \( P_{h, \Omega} \) in the interval \( (-\infty, bh] \), counting multiplicities. If \( \lambda \in (-\infty, b] \), define,
\[
N(\lambda h) = \text{tr} \left( 1_{(-\infty, \lambda h]}(P_{h, \Omega}) \right) = \text{Card} \{ e_j : e_j \leq \lambda h \}, \quad (4.1)
\]
\[
E(\lambda h) = \text{tr} \left( 1_{(-\infty, \lambda h]}(P_{h, \Omega} - \lambda h\text{Id}) \right) = \sum_{e_j \leq \lambda h} (e_j - \lambda h). \quad (4.2)
\]

The main result we prove in this section is Theorem 4.1. Its statement requires the notation \( [x]_+ = \max(x, 0) \) for any real number \( x \), the eigenvalue \( \mu_1(\xi) \) introduced in (2.4), and the arc-length measure \( ds(x) \) along the boundary of \( \Omega \).
Theorem 4.1. For any real number \( \lambda \leq b \) and as \( h \to 0 \), it holds true that

\[
E(\lambda h) = -\frac{h^{1/2}}{2\pi} \int_{\partial \Omega} \int_{-\infty}^{\infty} B(x)^{3/2} \left[ \frac{\lambda}{B(x)} - \mu_1(\xi) \right] d\xi ds(x) + o(h^{1/2}).
\] (4.3)

Furthermore, if \( \lambda < b \), then,

\[
N(\lambda h) = \frac{1}{2\pi h^{1/2}} \int_{\{(x,\xi) \in \partial \Omega \times \mathbb{R} : B(x) \mu_1(\xi) \leq \lambda\}} B(x)^{1/2} d\xi ds(x) + o \left( h^{-1/2} \right). \] (4.4)

While proving Theorem 4.1 we only give the new ingredients and constructions required to adapt the proof given in [10]. We will refer to [10] for the detailed calculations.

4.2. Proof of Theorem 4.1. Notice that (4.4) is a consequence of (4.3). Actually, the term on the right side of (4.4) is the derivative with respect to \( \lambda h \) of that on the right side of (4.3). On the other hand, using the variational principle of Lemma 2.2, the number \( N(\lambda h) \) can be seen as the derivative of the energy \( E(\lambda h) \) with respect to \( \lambda h \).

Therefore, to prove Theorem 4.1 it is sufficient to establish the asymptotic formula in (4.3). Using a magnetic Lieb-Thirring inequality and a decomposition of the operator by a partition of unity, it is possible to prove that the energy \( E(\lambda h) \) is finite for all \( \lambda \leq b \). Details regarding this proof are given in [10] Section 5.1, pp. 242-243.

We start by proving the asymptotic lower bound:

\[
E(\lambda h) \geq -\frac{h^{1/2}}{2\pi} \int_{\partial \Omega} \int_{-\infty}^{\infty} B(x)^{3/2} \left[ \frac{\lambda}{B(x)} - \mu_1(\xi) \right] d\xi ds(x) + o(h^{1/2}).
\] (4.5)

Let \( H = P_{h,\Omega} - \lambda h Id \) and notice that the energy \( E(\lambda h) \) can be expressed in the more useful form,

\[
E(\lambda h) = \text{tr} \left( H 1_{(-\infty,0)}(H) \right).
\]

Let \( \{f_1, f_2, \ldots, f_N\} \) be any \( L^2 \) orthonormal set in \( D(H) \). We will give a uniform lower bound to

\[
\sum_{j=1}^{N} \langle f_j, H f_j \rangle.
\]

Using Lemma 2.2 this will imply a lower bound to \( \text{tr} \left( H 1_{(-\infty,0)}(H) \right) \).

Consider a partition of unity of \( \mathbb{R} \),

\[
\psi_1^2 + \psi_2^2 = 1, \quad \text{supp} \, \psi_1 \subset ]-\infty, 1[, \quad \text{supp} \, \psi_2 \subset ]\frac{1}{2}, \infty[,
\] (4.6)

such that,

\[
|\nabla \psi_1|^2 + |\nabla \psi_2|^2 \leq C,
\] (4.7)

and \( C > 0 \) is a universal constant.

Let

\[
\tau(h) = \frac{1}{2} \min(\tan \alpha, 1) h^{3/8},
\] (4.8)

where \( \alpha = \min\{\alpha_{k} : k = 1, 2, \ldots, m\} \).

Using the partition of unity in (4.6), we put

\[
t(x) = \text{dist}(x, \partial \Omega), \quad \psi_{1,h}(x) = \psi_1 \left( \frac{t(x)}{\tau(h)} \right), \quad \psi_{2,h}(x) = \psi_2 \left( \frac{t(x)}{\tau(h)} \right), \quad \forall \, x \in \overline{\Omega}.
\] (4.9)

We introduce the potential

\[
V_h = h^2 \left( |\nabla \psi_{1,h}|^2 + |\nabla \psi_{2,h}|^2 \right).
\] (4.10)
It is possible to prove that (see [10] Proof of (5.10), pp. 243):

$$\sum_{j=1}^{N} \langle f_j, H f_j \rangle \geq \sum_{j=1}^{N} \langle f_j, \psi_{1,h}(H - V_h)\psi_{1,h} f_j \rangle - C \frac{h}{\tau(h)} \left(1 + \frac{h}{\tau(h)^2}\right). \quad (4.11)$$

With our choice of \( \tau(h) \), the remainder terms in (4.11) are of the order of \( h^{5/8} = o(h^{1/2}) \). Thus, we have,

$$\sum_{j=1}^{N} \langle f_j, H f_j \rangle \geq \sum_{j=1}^{N} \langle f_j, \psi_{1,h}(H - V_h)\psi_{1,h} f_j \rangle + o(h^{1/2}). \quad (4.12)$$

Next, we estimate from below the term on the right side of (4.12). Recall the vertices \( s_k, k = 1, 2, \cdots, m \), of the domain \( \Omega \). Recall also that the boundary of the domain \( \Omega \) consists of smooth curves (\( \Gamma_k \)). For each \( k \), define,

$$\Gamma_k(h) = \{ x \in \overline{\Omega} : \text{dist}(x, \Gamma_k) \leq \tau(h) \ \text{and} \ \text{dist}(x, s_k) \geq \frac{1}{2} h^{3/8}\}, \quad (4.13)$$

where \( \tau(h) \) is defined in (1.8). Consider a partition of unity of \( \mathbb{R}^2 \):

$$\sum_{k=1}^{m} g_k^2 + \sum_{k=1}^{m} h_k^2 = 1, \quad \sum_{k=1}^{m} (|\nabla g_k|^2 + |\nabla h_k|^2) \leq \frac{C}{h^{3/4}},$$

where supp \( g_k \subset B(s_k, h^{3/8}) \), supp \( h_k \subset \Gamma_k(h) \) for all \( k \), and \( C \) is a universal constant.

Then, we have the decomposition formula,

$$\sum_{j=1}^{N} \langle f_j, \psi_{1,h}(H - V_h)\psi_{1,h} f_j \rangle = \sum_{j=1}^{N} \sum_{k=1}^{m} \langle f_j, g_k \psi_{1,h}(H - V_h)g_k \psi_{1,h} f_j \rangle$$

$$+ \sum_{j=1}^{N} \sum_{k=1}^{m} \langle f_j, h_k \psi_{1,h}(H - V_{1,h})h_k \psi_{1,h} f_j \rangle, \quad (4.14)$$

where

$$V_{1,h} = V_h + h^2 \sum_{k=1}^{m} |\nabla g_k|^2 + h^2 \sum_{k=1}^{m} |\nabla h_k|^2,$$

and

$$|V_{1,h}| \leq C h^{5/4}.$$  

We will show that the first term on the right side of (4.14) is of order \( o(h^{1/2}) \). Recall the definition of the energy \( \mathcal{E}_{\text{corn}}(\alpha, b, R, \lambda) \) in (2.11). We use (3.11) with \( u = \psi_{1,h} f_j \) and \( \chi_{k,h} = g_k \) to bound the term \( \langle f_j, g_k \psi_{1,h}(H - V_{1,h})g_k \psi_{1,h} f_j \rangle \) from below by a quadratic form defined over the sector \( \Omega_{\alpha_j} \). Then we use the variational principle of Lemma 2.2 and the lower bound of \( V_{1,h} \) to get,

$$\sum_{j=1}^{N} \langle f_j, g_k \psi_{1,h}(H - V_{1,h})g_k \psi_{1,h} f_j \rangle \geq -(1 - Ch^{1/4}) \mathcal{E}_{\text{corn}}(\alpha_s, B_k, h^{-1/8}, \tilde{\lambda}).$$

Here \( \tilde{\lambda} = \lambda b^{-1} - 1 + Ch^{1/4} \leq Ch^{1/4} \). Clearly, \( \tilde{\lambda} \) can be made smaller than an arbitrary positive number. Thus, we can apply Lemma 2.5 and use (2.14). In this way, we get the following lower bound that holds uniformly with respect to \( k \),

$$\sum_{j=1}^{N} \langle f_j, g_k \psi_{1,h}(H - V_{1,h})g_k \psi_{1,h} f_j \rangle \geq -Ch^{3/4} = o(h^{1/2}). \quad (4.15)$$
The second term on the right side of (4.14) is bounded as in [10] Proof of (5.26); pp. 244-248. The following lower bound holds uniformly in $k$, $N$ and the orthonormal family $\{f_j\}$:
\[
\sum_{j=1}^{N} \langle f_j, h_k \psi_{1,h}(H - V_{1,h}) h_k \psi_{1,h} f_j \rangle \geq - \frac{h^{1/2}}{2\pi} \int_{\Omega} \int_{\mathbb{R}} B(x)^{3/2} \left[ \mu_1(\xi) - \frac{\lambda}{B(x)} \right] \, d\xi \, dx - C h^{5/8}.
\]
(4.16)

Substitution of (4.15) and (4.16) into (4.14) establishes the lower bound (4.5).

**Upper bound.** The upper bound will be obtained by constructing a specific density matrix $\gamma$ and computing the energy of $\text{tr} \left( (P_{h,\Omega} - \lambda \text{Id}) \gamma \right)$. The calculations follow closely those in [10] Section 5.3, pp. 248.

Recall the sets $\Gamma_k(h)$ introduced in (4.13). Let $\tau(h)$ be as defined in (4.3). For each $k$, we cover $\Gamma_k(h)$ by disjoint squares,
\[
\Gamma_{j,k} = \{ x \in \mathbb{R}^2 : |x - z_{j,k}| < \tau \}, \quad 1 \leq j \leq N_k,
\]
where the points $(z_{j,k})$ are on $\Gamma_k$ and equally spaced. In each $\Gamma_{j,k}$, it is possible to apply a transformation $\Phi_{j,k}(x) = (s, t)$ that straightens $\partial \Omega$ ($t = 0$ defines the boundary of $\Omega$ and $s$ measures the arc-length along $\partial \Omega$).

Recall the eigenvalues $(\mu_p(\xi))$ of the harmonic oscillator in (2.3). We introduce an orthonormal basis $(u_p)$ of $L^2(\mathbb{R}_+)$:
\[
\left\{ \begin{array}{ll}
- \partial_t^2 + (t - \xi)^2 u_p(t; \xi) = \mu_p(\xi) u_p(t; \xi), & t > 0, \\
\int_0^\infty |u_p(t; \xi)|^2 \, dt = 1.
\end{array} \right.
\]
(4.17)

Let $\chi \in C^\infty(\mathbb{R}^2)$ be positive, smooth, have values in $[0, 1]$, supported in $B(0, 2)$ and equal to 1 on $B(0, 1)$. Define $\chi_{j,k}(x)$ to be
\[
\chi_{j,k}(x) := \chi\left( \frac{x - z_{j,k}}{\tau(h)} \right).
\]
We denote by $B_{j,k} = B(z_{j,k})$. For each $(j, k)$, there exists a function $\phi_{j,k}$ such that
\[
A - \nabla \phi_{j,k} = \left( - B_{j,k} x_2, 0 \right) + \mathcal{O}(|x - z_{j,k}|) \quad \text{in } K_{j,k}.
\]
Define
\[
\tilde{f}_{j,p,k}(s,t;h,\xi) := \left( \frac{B_{j,k}}{h} \right)^{1/4} e^{-i\xi s} \sqrt{B_{j,k}/h} u_p \left( \sqrt{\frac{B_{j,k}}{h}} t; \xi \right).
\]
(4.18)

We get—by the coordinate transformation $\Phi_{j,k}$—the following function in $\Omega$,
\[
\tilde{f}_{j,p,k}(x;h,\xi) = \tilde{f}_{j,p,k}(\Phi_{j,k}(x);h,\xi) \chi_{j,k}(x) \psi_{1,h}(x).
\]
(4.19)

Next, let $K > 0$ and define the function,
\[
f_p(x; \xi) = \sum_{k=1}^{m} \sum_{j=1}^{N_k} M_p(h, \xi, j, k; K) e^{i\phi_{j,k}(x)/h} \chi_{j,k}(x) \psi_{1,h}(x) \tilde{f}_{j,p,k}(\Phi_{j,k}(x); h, \xi),
\]
where $M_p(h, \xi, j, k; K) = M^K(h, \xi, j, k) 1_{\{p=1\}}$ and $M^K(h, \xi, j, k)$ is the characteristic function of the set:
\[
\{(\xi, j, k) : \frac{\lambda}{B_{j,k}} - \mu_1(\xi) \geq 0, \ |\xi| \leq K \}.
\]
Let $\Pi_p^{\text{bnd}}(h, \xi)$ be the operator with integral kernel in $L^2(\Omega)$ whose integral kernel is defined by,
\[
\Pi_p^{\text{bnd}}(h, \xi)(x, x') = f_p(x; \xi) \overline{f_p(x'; \xi)}.
\]
(4.20)
Define the density matrix,
\[
\gamma = \frac{1}{2\pi \sqrt{h/B}} \sum_{p=1}^{\infty} \Pi_p^{\text{bnd}}(h, \xi).
\]
(4.21)
Clearly $0 \leq \gamma$ as an operator on $L^2(\Omega)$. Furthermore, there exists a constant $C > 0$ such that,
\[
\gamma \leq (1 + C\tau(h)),
\]
(4.22)
in the quadratic form sense. Details concerning the derivation of (4.22) are given in [10, Proof of (5.34)]. Moreover, (see [10, (5.37)]),
\[
\text{tr}[(P_{h,\Omega} - \lambda h)\gamma] \leq -\frac{h^{1/2}}{2\pi} \int_{-K}^{K} \int_{\partial\Omega} B(x)^{3/2} \left[ \frac{\lambda}{B(x)} - \mu_1(\xi) \right] ds(d\xi) + CKh^{3/4}.
\]
(4.23)
Let us mention that, while estimating the term on the left side of (4.23), the discrete sum over $j$ and $k$ becomes a Riemann sum as $h \to 0$, thereby resulting in an integral on the right side of (4.23). Since $K$ can be chosen arbitrarily large, (4.23) implies the asymptotic upper bound,
\[
\text{tr}[(P_{h,\Omega} - \lambda h)\gamma] \leq -\frac{h^{1/2}}{2\pi} \int_{-\infty}^{\infty} \int_{\partial\Omega} B(x)^{3/2} \left[ \frac{\lambda}{B(x)} - \mu_1(\xi) \right] ds(d\xi) + o(h^{1/2}).
\]

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