Some definable types that cannot be amalgamated

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We exhibit a theory where definable types lack the amalgamation property.

If $q_0(x, y)$ and $q_1(x, z)$ are types over a given set $A$, both extending the same type $p(x) \in S(A)$, it is an easy exercise to show that there is $r(x, y, z) \in S(A)$ extending $q_0(x, y)$ and $q_1(x, z)$ simultaneously. In other words, in every theory, types have the amalgamation property. Suppose now that $p, q_0, q_1$ all belong to some special class $\mathcal{K}$ of types, and consider the following question: among the amalgams $r$ as above, is there always one which belongs to $\mathcal{K}$? In this short note, we prove that, for $\mathcal{K}$ the class of definable types, the answer is in general negative.

By a fundamental result of Shelah, a complete theory is stable if and only if all types over models are definable. Definable types, and the tightly related notion of stable embeddedness, recently attracted considerable attention in unstable contexts as well. For instance, Hrushovski isolated a criterion for elimination of imaginaries in terms of density of definable types, which yielded a simplified proof of the classification of imaginaries in algebraically closed valued fields [7], and similar classification results in other (enriched) henselian valued fields [4, 5, 11]. In o-minimal theories, stable embeddedness of elementary substructures corresponds to relative Dedekind completeness [9], and in benign theories of henselian valued fields stable embeddedness obeys an Ax–Kochen–Ershov principle [2, 12]. Definable types are also central in Hrushovski and Loeser’s celebrated work on Berkovich analytifications [6]: their stable completions of algebraic varieties are certain spaces of definable types which, crucially, form strict pro-definable sets.

This brings us to the main motivation for the present paper. If $T$ is stable, then definable types over $M$ may be seen as a pro-definable set in $M^{eq}$ (this is a special case of [6, Lemma 2.5.1]), albeit this pro-definability need not be strict: it follows from work of Poizat on belles paires [10] that, if $T$ is stable, then definable types over models form strict pro-definable sets if and only if $T$ is nf cp, if and only if all belles paires of models of $T$ are $\omega$-saturated (cf. [3, § 3.1]). In order to establish strict pro-definability for other spaces of definable types, Cubides, Ye and the first author [3] recently introduced beautiful pairs in an arbitrary $L$-theory $T$. Poizat's belles paires are beautiful, and his theory generalises smoothly to unstable $T$,
provided the latter satisfies certain assumptions: an extension property called (EP), and the amalgamation property (AP) for definable types.

In stable theories, (AP) and (EP) always hold. Similarly in o-minimal theories, where (AP) follows from a result of Baisalov–Poizat [1] (cf. [3, §3.2]). In benign valued fields, there is an Ax–Kochen–Ershov type reduction for (AP) [3, §8]. In [3, Corollary 2.4.16] an example of a (dp-minimal) theory satisfying (AP) but not (EP) was found, namely the levelled binary tree with level set \( \omega \). Whether there is a theory where (AP) fails is left open in [3].

Building on the aforementioned tree, we construct such an example.

1. THE THEORY

Models \( M \) of the theory in which our counterexample lives are four-sorted, and are roughly obtained as follows. We start with a binary tree \( T(M) \), with discrete level set \( L(M) \). We then introduce two levelled sets \( A(M) \) and \( B(M) \), both with the same levelset as the tree, namely \( L(M) \), and cover each level \( x \) of \( T(M) \) with a generic surjection from the cartesian product of the \( x \)th levels of \( A(M) \) and \( B(M) \).

### Definition 1.1

Let \( L \) be the following language.

1. \( L \) has four sorts \( A, B, T, \) and \( L \).
2. \( T \) has a binary relation \( \leq_T \), a binary function \( \sqcap \), a unary function \( \text{pred}_T \), and constants \( g_T, r \).
3. \( L \) has a binary relation \( \leq_L \), a unary function \( \text{pred}_L \), and constants \( g_L, 0 \).
4. There are functions \( \ell_T : T \to L \), \( \ell_A : A \to L \), and \( \ell_B : B \to L \).
5. There is a function \( f : A \times B \to T \).

### Definition 1.2

Let \( T \) be the \( L \)-theory expressing the following properties.

(i) \( 0 \neq g_L \), and \( (L \setminus \{g_L\}, \leq_L) \) is a discrete linear order with smallest element \( 0 \) and no largest element, with predecessor function \( \text{pred}_L \), with the convention that \( \text{pred}_L(0) = 0 \). The “garbage” point \( g_L \) is not \( \leq_L \)-related to anything, and \( \text{pred}_L(g_L) = g_L \).

(ii) \( (T \setminus \{g_T\}, \leq_T, \sqcap) \) is a meet-tree, viewed as a semilinear order \( \leq_T \) with associated meet function \( \sqcap \), root \( r \), binary ramification,\(^1\) and, for every fixed element, its set of predecessors is a discrete linear order, with predecessor function \( \text{pred}_L \), with the convention that \( \text{pred}_L(r) = r \). The “garbage” point \( g_T \) behaves similarly to the garbage point \( g_L \).

(iii) \( \ell_T \) is a surjective level function \( T \setminus \{g_T\} \to L \setminus \{g_L\} \), extended by \( \ell_T(g_T) = g_L \). For every fixed element \( t \in T \setminus \{g_T\} \), the restriction of \( \ell_T \) to the set of predecessors of \( t \) defines an order isomorphism onto an initial segment of \( L \setminus \{g_L\} \) (in particular, \( \ell_T \circ \text{pred}_L = \text{pred}_L \circ \ell_T \)). Moreover, for any \( t \) from \( T \setminus \{g_T\} \) and any \( y \) in \( L \) with \( \ell_T(t) \leq_L y \) there is \( t' \) in \( T \) with \( t \leq_T t' \) such that \( \ell_T(t') = y \).

(iv) \( g_L \) is not in the image of \( \ell_A : A \to L \), nor in that of \( \ell_B : B \to L \).

(v) For every \( c \in L \setminus \{g_L\} \) the fibers \( \ell^{-1}_A(c) \) and \( \ell^{-1}_B(c) \) are infinite.

(vi) \( f(a, b) = g_T \) if and only if \( \ell_A(a) \neq \ell_B(b) \).

(vii) If \( \ell_A(a) = \ell_B(b) \), then \( f(f(a, b)) = \ell_A(a) \).

(viii) At any level, \( f \) defines a generic surjection: for any \( c \in L \setminus \{g_L\} \), any \( t_1, \ldots, t_n \) from \( T \setminus \{g_T\} \) and any pairwise distinct \( a_1, \ldots, a_n \) from \( A \) such that \( \ell_T(t_i) = c = \ell_A(a_i) \) for all \( i \), there are infinitely many \( b \) from \( B \) such that, for all \( i \), we have \( f(a_i, b) = t_i \); similarly with the roles of \( A \) and \( B \) interchanged.

Recall that a meet-tree together with a linear order and a map satisfying (iii) above is called a levelled tree.

### Proposition 1.3

The following properties hold.

1. \( T \) is complete and admits quantifier elimination.
2. The union of the definable sets \( T \setminus \{g_T\} \) and \( L \setminus \{g_L\} \) is stably embedded, with induced structure a pure levelled (binary) meet-tree. In particular, \( L \setminus \{g_L\} \) is stably embedded with induced structure a pure ordered set.

\(^1\)We fix binary ramification for simplicity, but this is not important: any fixed finite ramification will work.
Proof. It is easy to see that $T$ is consistent. It is enough to prove quantifier elimination: (2) is a direct consequence of the latter; as for completeness, it follows (cf., e.g., [8, Proposition 18.4]) from quantifier elimination and the fact that, given an arbitrary model of $T$, the $L$-substructure with underlying set the interpretations of the closed $L$-terms over $\emptyset$ embeds in every other model.

Let $N_0$ and $N_1$ be models of $T$ with $N_0$ countable and $N_1$ $\aleph_1$-saturated, and let $M$ be a common $L$-substructure of $N_0$ and $N_1$. It is an easy exercise to $M$-embed $N_0$ into $N_1$, yielding quantifier elimination. \qed

The theory induced on $T \setminus \{g_T\}$ and $L \setminus \{g_L\}$ is thus precisely the one used in [3, Fact 2.4.15].

**Lemma 1.4.** For all $M \models T$, all linearly ordered definable subsets of $T(M)$ have a maximum.

Proof. This follows from quantifier elimination. Alternatively, one may use that no infinite branch is definable in the standard binary meet-tree $(2^{<\omega}, \omega)$, e.g., since for any $n \in \omega$ and branches $s, s' \in 2^{\omega}$ with $s|_n = s'|_n$ there is an automorphism $\sigma$ over $2^{<n}$ with $\sigma(s) = s'$. \qed

2 | THE TYPES

Let $T$ be the theory defined in the previous section, and $\mathcal{U} \models T$ a monster model. Failure of amalgamation of definable types boils down to the following phenomenon. All elements $y$ of $A$ with level larger than $L(\mathcal{U})$ have the same, definable, type, and similarly for $z$ in $B$; nevertheless, if such $y$ and $z$ have the same infinite level, then $f(y, z)$ can be used to produce an externally definable subset of $T(\mathcal{U})$ which is not definable. More formally, we proceed as follows.

**Definition 2.1.** Define the following sets of $L(\mathcal{U})$-formulas.

1. $p(x)$ is the global type of an element $x$ of sort $L$ such that $x > L(\mathcal{U})$.
2. $q_A(x, y)$ restricts to $p$ on $x$, and says that $y$ is an element of sort $A$ with $\ell_A(y) = x$.
3. $q_B(x, z)$ restricts to $p$ on $x$, and says that $z$ is an element of sort $B$ with $\ell_B(y) = x$.

**Lemma 2.2.** All of $p, q_A, q_B$ are complete types over $\mathcal{U}$ which are $\emptyset$-definable.

Proof. Consistency and $\emptyset$-definability are clear. As for completeness, we argue as follows.

1. Completeness of $p(x)$ follows from Proposition 1.3(2).
2. As for $q_A(x, y)$, note that, since $y$ has a new level, it cannot be in $A(\mathcal{U})$. Again because $\ell_A(y)$ is new, for all $b \in B(\mathcal{U})$ we must have $f(y, b) = g_T$. By quantifier elimination, this is enough to determine a complete type.
3. The argument for $q_B(x, z)$ is symmetrical. \qed

**Proposition 2.3.** The types $q_A(x, y)$ and $q_B(x, z)$ cannot be amalgamated over $p(x)$ into a definable type. In other words, no completion of $q_A(x, y) \cup q_B(x, z)$ is definable.

Proof. Suppose $r(x, y, z)$ is a completion of $q_A(x, y) \cup q_B(x, z)$. Then $r(x, y, z) \vdash \ell_A(y) = x = \ell_B(z)$, thus $r(x, y, z) \vdash \ell_T(f(y, z)) = x$. Since $p(x)$ is not realised, $f(y, z)$, having a new level, cannot be in $T(\mathcal{U})$. Consider the set $\{d \in T(\mathcal{U}) \mid r(x, y, z) \vdash f(y, z) > d\}$. If $r(x, y, z)$ is definable, then this set is definable. As a set of predecessors, it must be linearly ordered, hence have a maximum by Lemma 1.4. But then $f(y, z) \notin \mathcal{U}$ contradicts binary ramification. \qed

**Corollary 2.4.** In $T$, global definable types do not have the amalgamation property.

This partially answers [3, Question 9.3.1], asking whether there is such a theory which, additionally, has uniform definability of types; note that $T$ does not. Moreover, $T$ is easily shown to have IP.

**Question 2.5.** Is there a NIP theory where global definable types do not have the amalgamation property?
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CONFLICT OF INTEREST STATEMENT
The authors declare no conflicts of interest.

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