Physical Applications of a Generalized Clifford Calculus*
(Papapetrou equations and metamorphic curvature)

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Abstract

A generalized Clifford manifold is proposed in which there are coordinates not only for the basis vector generators, but for each element of the Clifford group, including the identity scalar. These new quantities are physically interpreted to represent internal structure of matter (e.g. classical or quantum spin). The generalized Dirac operator must now include differentiation with respect to these higher order geometric coordinates. In a Riemann space, where the magnitude and rank of geometric objects are preserved under displacement, these new terms modify the geodesics. One possible physical interpretation is natural coupling of the classical spin to linear motion, providing a new derivation of the Papapetrou equations. A generalized curvature is proposed for the Clifford manifold in which the connection does not preserve the rank of a multivector under parallel transport, e.g. a vector may be “rotated” into a scalar.

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I. Introduction

To break out of the limitations imposed by habitual thinking, we propose a different mathematical model for the structure of spacetime. We adopt the concept proposed by Chisholm\[1\] of a curved spacetime with a Clifford structure imposed upon it (in order to describe quantum particles with spin in a curved space). However we take the concept much further in proposing that it applies to classical physics (non-quantum) and that there are coordinates for each multivector element. In section III we consider derivatives with respect to bivector geometry which leads to a new form of the geodesic which is useful in describing the spinning particle problem. A new type of curvature is proposed in section IV in which the connection can modify a vector into a bivector, yielding a new law of motion for the spinning particle in a gravity field.

II. The Clifford Manifold

We propose that space is really a pandimensional continuum. In other words, it is made up of points, lines, planes, volumes etc., all together. Instead of a vector manifold, we have a “Clifford Manifold”, or if you like a polyvector manifold. True, there may only be 4 independent tangent basis vectors, but there will also be 6 tangent basis planes, a tangent scalar, pseudoscalar, and 4 tangent trivectors (pseudovectors), a total of 16 elements. For our discussion we will not use the full 16 degrees of freedom, but use simpler examples that only utilize the vector and bivector features.

A. Review of Clifford Algebra

We recommend the standard references on the subject for complete coverage. Here we present only brief definitions for notational clarity and some specialized equations. In a non-orthonormal curved space, the basis vectors (generators of the Clifford algebra) obey,

\[
\gamma_\mu \cdot \gamma_\nu = \frac{1}{2} (\gamma_\mu, \gamma_\nu) = \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu},
\]

where \(g_{\mu\nu}\) is metric tensor which may be a function of position. The direct or Clifford product of two different basis vectors yields a new object, called a (basis) bivector. This is the basis element (2-vector) for the plane spanned by the two basis vectors which we can notate,

\[
\gamma_{\mu\nu} = -\gamma_{\nu\mu} = \gamma_\mu \wedge \gamma_\nu = \frac{1}{2} (\gamma_\mu, \gamma_\nu) = \frac{1}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu).
\]

The product of three basis vectors is called a trivector. Assuming we are in a four dimensional space, then there is only one quadvector, produced by the product of all four elements.

The unique property of Clifford algebra is the ability to add together different rank geometries (e.g. scalar plus bivector). This creates a generalized type of
entity which is often called a ‘multivector’ in the literature. However this name which literally means ‘multiple vector’ is ambiguous as it could simply mean an n-vector where \( n > 1 \). We propose instead the term \textit{polyvector} to mean that it is a combination of a p-vector with a q-vector (where \( p \neq q \)). The formulation of standard physics of course does not allow for the addition of a p-form to a q-form. We propose to label the formulation of physical theories which exploit the use of polyvectors as \textit{polydimensionalism}, literally the use of many different dimensional geometric elements in one equation. As an example, consider the motion of a charged particle in an electromagnetic field. We define the \textit{momentum polyvector} as the vector momentum plus bivector spin angular momentum,

\[
\mathcal{M} = \frac{1}{\lambda} p^\mu \gamma^\mu + \frac{1}{2\lambda^2} S^{\mu\nu} \gamma_{\mu\nu}, \tag{2a}
\]

\[
p^\mu = m \frac{dx^\mu}{d\tau}, \tag{2b}
\]

\[
p_\mu S^{\mu\nu} = 0. \tag{2c}
\]

The Weyssenhoff condition eq. (2c) insures the spin to be a simple bivector, which is purely spacelike in the rest frame of the particle. The scale factor \( \lambda \) is a universal length constant that will be defined in the next section, and the proper time \( d\tau \) is given below in eq. (7). The Einstein summation convention on repeated indices will be used throughout the paper. With charge-to-mass ratio \( e/m \), the polydimensional equation of motion is simply,

\[
\dot{\mathcal{M}} \equiv \frac{d\mathcal{M}}{d\tau} = \frac{e}{2m} [\mathcal{M}, \mathbf{F}], \tag{3a}
\]

where \( \mathbf{F} = \frac{1}{2} F_{\mu\nu} \gamma_{\mu\nu} \) is the electromagnetic field tensor. The vector and bivector parts of eq. (3a) describe the linear motion and spin motion respectively,

\[
\dot{p}^\beta \equiv \frac{dp^\beta}{d\tau} = \left( \frac{e}{m} \right) p\alpha F^{\alpha\beta}, \tag{3b}
\]

\[
\dot{S}^{\mu\beta} \equiv \frac{dS^{\mu\beta}}{d\tau} = \left( \frac{e}{2m} \right) \left( F^\nu_{\mu} S^{\nu\beta} - F^\beta_{\nu} S^{\nu\mu} \right). \tag{3c}
\]

This is a somewhat trivial example because there is no coupling between the spin and momentum. In tensor form eq. (3b) and eq. (3c) do not look really look the same, while eq. (3a) shows a possible underlying symmetry between the two equations. Is this just notational economy or is there a broader relationship being revealed which is obscured in tensor form?

B. Dimensional Democracy

In quaternionic analysis, one gives coordinates \( x^\mu \) to all 4 elements, \( \mathbf{H} = x^1 \mathbf{i} + x^2 \mathbf{j} + x^3 \mathbf{k} + x^4 \mathbf{1} \). However, when viewed as a Clifford algebra, the three imaginaries are NOT all the same ‘rank’ of geometry. Two of them are basis vectors (1-vectors), and the third is really a basis plane (2-vector), while the
fourth element is a scalar (0-vector). Hence we have given coordinates to ALL
the geometric elements.

The standard formulation of physical theories might be called a vector oli-
garchy as coordinates are only associated with the basis vectors (1-vect ors) of
the spacetime manifold. Einstein’s relativity for example represents the struc-
ture of space-time with a four-dimensional vector manifold. The coordinates
of an event \( x^\mu(\mathcal{P}) \) for \( \mu = 1, 2, 3, 4 \). The fourth dimension: \( x^4 = ct \) is propor-
tional to ‘time’ \( t \), where the speed of light \( c \) is the universal constant which
converts time into equivalent spatial units. From the differential of the event
point: \( d\mathcal{P} \equiv e_\mu dx^\mu \) the tangent basis vectors can defined by applying the chain
rule of differentiation,

\[
e_\mu = \frac{\partial \mathcal{P}}{\partial x^\mu}.
\]  

In our practical world, the quantity \( x^\mu \) (more properly a finite interval \( \Delta x^\mu \))
is quantified in terms of units, such as meters or feet. However the quantity
\( d\mathcal{P} \) is purely a geometric interval which has no units. Thus by eq. (4) the basis
vectors must have units of inverse length! If we change from feet to inches these
basis vectors will need to be decreased by a corresponding factor of twelve. The
relationship of these tangent vectors to the purely algebraic basis set \( \gamma_\mu \) (which
has no units), is hence: \( \gamma_\mu \equiv e_\mu \lambda \), where \( \lambda \) is the scale constant associated with
the particular unit system in which we measure \( x^\mu \).

We will define the Clifford manifold in terms of these four tangent basis vec-
tors. Let us define the set of 16 elements: \( \{ E_A \} = \{ 1, e_\mu, e_{\mu\nu}, e_{\alpha\mu\nu} \} \) [the
notation implies antisymmetry on the indices]. We now propose that each basis
element \( E_A \) has a coordinate \( q^A \) associated with it. The bivector coordinate:
\( a^{\mu\nu} = -a^{\nu\mu} \) would for example have the units of area. An event \( \Sigma \) in pan-
dimensional space has a unique set of coordinates \( q^A(\Sigma) \). The pandimensional
differential is hence,

\[
d\Sigma \equiv E_A dq^A = d\kappa + e_\mu dx^\mu + \frac{1}{2} e_{\mu\nu} da^{\mu\nu} + \frac{1}{3!} e_{\mu\nu\rho} e^{\mu\nu\sigma} dy_\sigma + \frac{1}{4!} e_{\mu\nu\rho\sigma} e^{\mu\nu\rho\sigma} d\sigma,
\]  

where \( \epsilon^{\mu\nu\rho\sigma} \) is the totally antisymmetric tensor. By construction, eq. (5) has
no units, it is a pure geometric infinitessimal interval.

For the remainder of the paper we will restrict our treatment to only the vec-
tor and bivector coordinates. A finite interval of the vector coordinates implies
a displacement from event \( \Sigma \) to \( \Sigma' \). The simultaneous bivector displacement
might represent the amount by which the path of the particle deviated from
the straight line path, or perhaps the area traced out by its spinning motion.
The momentum of a particle defined in eq. (2b) is the rest mass \( m \) times the
rate of displacement with respect to the affine parameter, called the proper time
in relativity, defined below in eq. (7). A possible interpretation of the bivector
coordinate would be the quantity for which the spin angular momentum of the
particle is given in analogy to eq. (2b),

\[
S^{\mu\nu} = m \frac{da^{\mu\nu}}{d\tau}.
\]
The momentum polyvector eq. (2a) can now be expressed compactly as: \( M \equiv \frac{dM}{d\tau} \).

**C. Quadratic Forms**

The inner product on a vector space induces the quadratic form:

\[
\| dP \|^2 \equiv dx^\mu dx^\nu \ g_{\mu\nu}/\lambda^2 \quad \text{where we recall the metric tensor is: } \quad g_{\mu\nu} \equiv \gamma_\mu \cdot \gamma_\nu = \lambda^2 \ e_\mu \cdot e_\nu.
\]

There is an affine parameter associated with the quadratic form, known as the *proper time* in mechanics (as it is the time that the particle experiences in its own frame of reference),

\[
d\tau \equiv \frac{1}{c} \sqrt{sg_{\mu\nu}(x^\alpha) \ dx^\mu dx^\nu}, \quad (7)
\]

which has units of time. For the \((- - - +)\) metric the *metric signature* \( s \) is defined to be \( s = +1 \), while \( s = -1 \) for the \((+++ -)\) metric. Although relativity cannot distinguish between these two cases, the Clifford group associated with each case is inequivalent which may have consequences in physical theories\(^2\). Most of the results in this paper require \( s = +1 \). The basic precept of classical mechanics is that out of all the possible trajectories connecting two events, nature will “choose” the special path which extremizes the quadratic form of eq. (7) integrated over the path. The solution is a *geodesic* in a curved space (or an “autoparallel” if there is torsion).

In classical mechanics, the modulus of the momentum vector,

\[
\p^2 = (p^\mu \gamma_\mu)^2 = p^\mu p^\nu g_{\mu\nu} = p^\mu p_\mu = s(mc)^2, \quad (8)
\]

must be invariant under a change of coordinate system. The invariant \( m \) is the rest mass of the particle. Corben\(^3\) and others have argued that for a spinning particle eq. (8) gives the *effective mass*, which is not the same as the *intrinsic rest mass* \( m_0 \). Note if we naively square the momentum polyvector defined in eq. (2a) (in the \( s = +1 \) signature), we get a scalar plus trivector,

\[
M^2 = \frac{c^2}{\lambda^2} I_1 + \lambda^{-3} p \wedge S, \quad (9a)
\]

\[
I_1 \equiv \frac{p^\mu p_\mu}{c^2} - \frac{S^{\mu\nu} S_{\mu\nu}}{2\lambda^2 c^2} = sm^2 - \frac{S^2}{\lambda^2 c^2} \equiv m_0^2, \quad (9b)
\]

\[
S^2 \equiv \frac{1}{2} S^{\mu\nu} S_{\alpha\beta} g_{\mu\alpha} g_{\nu\beta} = \frac{1}{2} S^{\mu\nu} S_{\mu\nu}, \quad (9c)
\]

where eq. (2c) gives us \( S^2 \geq 0 \). The scalar part of the modulus of the momentum polyvector is proportional to the invariant \( I_1 \) which Dixon\(^4\) associates with the (square of the) intrinsic or “bare” rest mass \( m_0 \). In other words, the spin of the particle increases the effective mass \( m \) which appears in equations (2b), (6) and (8). We might also note that the modulus of the trivector part of eq. (9a) yields the second invariant \( I_2 \) of Dixon\(^4\). He also interprets the scale constant \( \lambda \) as being possibly the radius of the universe, although we think it should be
the radius of gyration of a macroscopic body, or within a geometric factor of the Compton wavelength for an elementary particle (for which the spin would be a fundamental constant).

This suggests a way to generalize eq. (7) for polydimensional coordinates. Keeping only the vector and bivector coordinates of eq. (5) and squaring it (in the \( s = +1 \) signature),

\[
(d\Sigma)^2 \equiv \left( \frac{c}{\lambda} \right)^2 d\kappa^2 + (\text{trivector}),
\]  

\[
d\kappa^2 \equiv d\tau^2 - \frac{da_{\mu\nu}da_{\mu\nu}}{2c^2\lambda^2}.
\]

The new affine parameter \( d\kappa \) is not the proper time \( d\tau \) of special relativity, as there is an additional contribution from the bivector displacement. We can be consistent with eq. (9b), showing the spin gives an increase in the effective mass, if we rewrite the definitions for the momentum and spin in the following way,

\[
p^\mu \equiv m_o \overset{\circ}{x}^\mu \equiv m_o \frac{dx^\mu}{d\kappa} = m\dot{x}^\mu,
\]

\[
S^{\mu\nu} \equiv m_o \overset{\circ}{a}^{\mu\nu} \equiv m_o \frac{da^{\mu\nu}}{d\kappa} = m\dot{a}^{\mu\nu},
\]

\[
m \equiv m_o \frac{d\tau}{d\kappa} \equiv m_o \sqrt{1 + \left( \frac{S^{\mu\nu}}{\lambda cm_o} \right)^2}.
\]

### III. Generalized Clifford Analysis

With the introduction of the bivector coordinate, we are ontologically committed to consider the meaning of a derivative with respect to this coordinate.

#### A. Differential Multiforms

The standard differential operator \( \text{d} \) applied to the vector coordinate gives \( dx^\mu = dx^\mu \) as desired, where \( dx^\mu \) denotes an infinitesimal quantity. Note however that the vector differential \( \text{d}x^\mu \equiv e_\mu dx^\mu \) must be defined as it can NOT be constructed by an application of \( \text{d} \) on \( x^\mu \). Similarly, the bivector differential \( \text{d}a^{\mu\nu} \equiv \frac{1}{2}e_\mu \wedge e_\nu da^{\mu\nu} \) must be defined rather than constructed. When integrated, this entity should yield the area displacement. Hence \( da \) is really a Leibniz form of second order, the form remaining after the magnitude of area has gone to zero. Hence we wish to associate \( da^{12} = dx dy \), which can not be constructed by applying the standard \( \text{d} \) on \( a^{\mu\nu} \). We have instead,

\[
D da^{\mu\nu} = da^{\mu\nu} = -da^{\nu\mu} \equiv dx^\mu dx^\nu, \quad (\mu < \nu),
\]

where the operator \( D \) is defined below in eq. (12b).

It remains to provide a geometric interpretation to the pandimensional differential of eq. (5). Let us consider the restricted case of the polydimensional
differential which only has the vector and bivector portion. We might interpret the finite polydimensional displacement as follows. The vector part tells the straight line vector displacement connecting the endpoints. The bivector part describes the amount by which the actual path deviated from the straight line (expressed in terms of the area enclosed between the actual path and the straight path).

The physical application for which the polydimensional differential may be useful is in describing the spinning particle. Corben for example states that the elementary particle travels in a helical path about the motion of its center of mass, the helical motion is (in part) seen as the ‘spin’ of the particle. We could hence let the vector part of the differential describe the center of mass motion, and the bivector part the area swept out by the helical motion.

B. The Clifford Polydifferential Operator

Consider the function \( f(x^\mu, a^{\mu\nu}) \) which has a value at point \( x^\mu \) that will vary according to the path \( \mathcal{P} \) taken from the origin, which in turn is described by the bivector coordinate. We introduce the Clifford Polydifferential Operator which includes the higher order coordinates (for our treatment we will consider only the vector and bivector coordinates),

\[
\mathcal{D} f \equiv dq^A \frac{\partial f}{\partial q^A} = dx^\mu \frac{\partial f}{\partial x^\mu} + \frac{1}{2} da^{\mu\nu} \frac{\partial f}{\partial a^{\mu\nu}}.
\]

The problem is to provide interpretation to the derivative with respect to the bivector coordinate. The total change in the function from the origin to the endpoint along the path \( \mathcal{P} \) is the integral of above. This can be partitioned into the integral along the straight (geodesic) path \( \mathcal{P}_o \) plus the closed integral which is the true path \( \mathcal{P} \) minus the return by path \( \mathcal{P}_o \),

\[
\Delta f \equiv \int \mathcal{D} f = \Delta x \bullet \nabla f + \oint \text{d}f,
\]

Inserting eq. (12b) into the left integral, matching terms and applying Stokes theorem for differential forms we get,

\[
\frac{1}{2} \int da^{\mu\nu} \frac{\partial f}{\partial a^{\mu\nu}} = \oint \text{d}f = \frac{1}{2} \int da^{\mu\nu} [\partial_\nu, \partial_\mu] f.
\]

We therefore propose the association of the bivector derivative with,

\[
\frac{\partial f}{\partial a^{\mu\nu}} = [\partial_\nu, \partial_\mu] f = \left[ \frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\mu} \right] f.
\]

This gives a new non-standard result for the exact differential of a basis vector in curved space,

\[
\mathcal{D} e_\mu = \left( dx^\alpha \Gamma_{\alpha\mu}^\nu - \frac{1}{2} da^{\alpha\beta} R_{\alpha\beta\mu}^\nu \right) e_\nu,
\]
\[ \Gamma_{\alpha \mu}^\nu = e^\alpha \cdot \partial_{\mu} e_\nu, \quad (13b) \]
\[ R_{\alpha \beta \mu}^\nu = e^\nu \cdot [\partial_{\alpha}, \partial_{\beta}] e_\mu = -e^\nu \cdot \partial_{\alpha} e_\beta. \quad (13c) \]

Not only does eq. (13a) contain the usual affine connection \( \Gamma_{\alpha \mu}^\nu \), it also contains the Riemann curvature tensor \( R_{\alpha \beta \mu}^\nu \). The connection of a bivector can be derived from eq. (13a),
\[ De_{\mu \nu} = \left( dx^\alpha \Gamma_{\alpha \omega}^\sigma - \frac{1}{2} d\alpha^\alpha e_{\alpha}^\beta R_{\alpha \beta \omega}^\sigma \right) \delta_{\mu \nu}^\sigma e_\sigma. \quad (13d) \]

C. Spinning Particles in Curved Space

Galileo’s famous experiment asserted that big balls fall at the same rate as small balls. The more general statement is known as the EEP (Einstein equivalence principle), that all particles follow the same geodesic path, independent of the mass or internal structure. Specifically, a spinning mass should fall at the same rate as a non-spinning one. However, in a landmark paper, Papapetrou showed that spinning masses will deviate from geodesics. His derivation was based upon taking a macroscopic mass, expanding in moments about the center, and looking at the geodesic deviations. The basic result is that the deviation is proportional to the spin of the particle, the curvature of the space and inversely proportional to the mass. It is presumed that this result is valid for fundamental particles, but there is no classical theory that yields the result. The difficulty is that there has not been a satisfactory theory which derives these equations from a simple “least action” principle based upon purely geometric concepts such that Einstein used. There have been far too numerous attempts, a good recent review is given by Frydryszak.

By introducing the bivector coordinates, we believe that we can now present a new derivation of the Papapetrou equations. They are the polygeodesic paths, which extremize the polydimensional quadratic form of eq. (10b). The proof of this statement is too long to show here, hence will be presented in a future paper. The main difficulty is that in applying Lagrange’s equations, one finds that the variation of the path does not commute with the derivative with respect to the new affine parameter,
\[ \frac{d\delta x^\mu}{dk} \neq \delta \left( \frac{dx^\mu}{dk} \right). \quad (14a) \]

We can however show here that the resulting equations of motion can be expressed in the very compact form of a polygeodesic. This is defined as the the polycurve \( \Sigma(\kappa) \) that “parallel transports” its tangent polyvector along itself,
\[ 0 = \overset{\circ}{M} \equiv \frac{dM}{dk} = \frac{\partial M}{\partial \kappa} + \overset{\circ}{x}^\mu \frac{\partial M}{\partial x^\mu} + \frac{1}{2} \overset{\circ}{a}^\alpha \overset{\circ}{a}^\beta R_{\alpha \beta \mu \nu}. \quad (14b) \]

Explicitly substituting eq. (13ad) as needed, one can split out the following equations for the momentum and spin respectively,
\[ 0 = \overset{\circ}{P} \equiv \frac{d(\overset{\circ}{p}^\mu e_\mu)}{dk} = \left( \overset{\circ}{p}^\mu + p^\nu \left[ \overset{\circ}{\alpha}^\beta \Gamma_{\beta \nu}^\mu + \frac{1}{2} \overset{\circ}{a}^\alpha \overset{\circ}{a}^\beta R_{\alpha \beta \nu} \right] \right) e_\mu, \quad (14c) \]
\[ 0 = S = \frac{1}{2} \frac{d}{dk} \left( S^{\alpha\beta} e_{\alpha\beta} \right) \]

\[ = \left\{ S^{\alpha\beta} + \frac{\alpha}{\beta} \left( S^{\sigma\beta} \Gamma_{\nu\sigma}^{\alpha} + S^{\alpha\sigma} \Gamma_{\nu\sigma}^{\beta} \right) + \frac{1}{2} \delta_{\mu\nu} \left( S^{\sigma\beta} R_{\mu\nu\sigma}^{\alpha} + S^{\alpha\sigma} R_{\mu\nu\sigma}^{\beta} \right) \right\} \frac{e_{\alpha\beta}}{2}, \]

These are equivalent to the Papapetrou equations except that the ‘open dot’ refers to differentiation by the new affine parameter \( dk \) of eq. (10b) instead of the old proper time \( d\tau \) of eq. (7).

IV. Geo-Metamorphic Curvature

We propose a radical concept of curved space in which a vector may be ‘bent’ into a bivector. These ideas were first introduced in an earlier paper. We might alternatively call this polymetamorphic curvature, or more generally pan-dimensional curvature.

A. Automorphism Invariance

A principle of special relativity is that the laws of physics must be of the same form in all frames of reference that differ only by a (global) transformation generated from the Lorentz group. Note that scalars such as eq. (7), eq. (8), eq. (9b) and even eq. (10b) are invariant under these transformations. Let us consider however the broader class of algebra automorphisms which just leave eq. (9b) invariant. They would be of the type that interchange some of the basis vectors with some of the basis bivectors, but in such a way that the overall algebra is preserved. The polymomentum vector would hence transform: \( M' = QMQ^{-1} \) where,

\[ Q = \exp \left( \frac{1}{2} \gamma_{\mu} \psi_{\mu} + \frac{1}{4} \gamma_{\mu\nu} \phi_{\mu\nu} \right). \] (15)

The parameters \( \phi_{\mu\nu} \) represent the Lorentz transformations, which preserve the rank of the geometry, while \( \psi_{\mu} \) generates the transformations that will exchange vectors with bivectors, and the pseudoscalar with one of the trivectors.

This leaves eq. (9b) invariant, and most important leaves eq. (14b) invariant in form. We have hence proposed a generalization of the covariance principle, that the laws of physics must be the same form in frames of reference that differ only by a global (restricted) automorphism transformation. This also means that there is no absolute ‘direction’ to which one can assign the geometry of ‘vector’. What is a vector in one frame of reference may be a bivector in another. Taken a step further, observers in the different frames will disagree on the interpretation of the phenomena. They will both agree on the bare mass given by the invariant of eq. (9b), but will disagree on the amount of the particle’s spin and momentum. This is because the coordinate transformation between the two frames is such that the vector coordinate \( z^{\alpha} \) in one frame may be a function of both the vector AND bivector coordinate of the other frame: \( z^{\alpha}(x^{\beta}, a^{\mu\nu}) \). For example, the vector displacement in a rotating frame is the
vector displacement of the non-rotating frame minus the contribution of the rotation (described by a bivector). The careful reader might note that eq. (3a) is NOT invariant under our automorphism transformation, but will acquire an additional term which possibly could be interpreted as giving the photon mass.

B. Polymorphically Connected Space

In relativistic quantum mechanics, Crawford\[9\] has proposed a local automorphism invariant principle as a method to generate a gauge theory of gravity and possible unified field theory. The basic feature he introduced was a Clifford connection on the abstract Dirac matrix (Clifford) algebra. The analogous form on our very concrete geometric Clifford algebra would be something like,

$$\mathcal{D}e_\mu = dx^\alpha \left( \Gamma_{\alpha \mu}^\nu e_\nu + \frac{1}{2} \Xi_{\alpha \mu}^{\nu \sigma} e_{\nu \sigma} \right).$$ \hspace{1cm} (16)

We might call $\Xi_{\alpha \mu}^{\nu \sigma}$ a metamorphic connection because under parallel transport it can change a vector into a bivector! This is a logically incomplete theory however, because only the basis vectors have been given a coordinate. Hence the metamorphic connection cannot be expressed in terms of a coordinate transformation.

We introduce a polymorphic coordinate transformation from a flat cartesian frame with coordinates $(z^j, \phi^{jk})$ to polymorphically curved coordinates $(x^\alpha, a^{\mu \nu})$ via an anholonomic local transformation,

$$dz^j \equiv h^j_\alpha dx^\alpha + \frac{1}{2} K^j_\mu \nu da^{\mu \nu}, \hspace{1cm} (17a)$$
$$d\phi^{jk} \equiv C^{jk}_\alpha dx^\alpha + \frac{1}{2} H^{jk}_\mu \nu da^{\mu \nu}. \hspace{1cm} (17b)$$

The coefficient $h^j_\alpha$ is the familiar vierbein (tetrad). The entire set of coefficients $\{h^j_\alpha, K^j_\mu \nu, C^{jk}_\alpha, H^{jk}_\mu \nu\}$, has been collectively called the geobeins or “geometry-legs”\[10\].

The generalized polymorphic Clifford connection is hence,

$$\mathcal{D}e_\mu = dx^\alpha \left( \Gamma_{\alpha \mu}^\nu e_\nu + \frac{1}{2} \Xi_{\alpha \mu}^{\nu \sigma} e_{\nu \sigma} \right) - \frac{1}{2} da^{\alpha \beta} \left( R_{\alpha \beta \mu}^\nu e_\nu + \frac{1}{2} Q_{\alpha \beta \mu}^{\nu \sigma} e_{\nu \sigma} \right), \hspace{1cm} (18a)$$

where the coefficients can now be written in terms of the geobeins. For example,

$$\Gamma_{\alpha \mu}^\nu = h^j_\alpha h_{\mu, \nu} + \frac{1}{2} K^j_\mu C^i_{\nu, \alpha} = \frac{\partial x^\nu}{\partial z^j} \frac{\partial^2 z^i}{\partial x^\mu \partial x^\nu} + \frac{1}{2} \frac{\partial x^\nu}{\partial \phi^{ij}} \frac{\partial^2 \phi^{ij}}{\partial x^\mu \partial x^\nu}, \hspace{1cm} (18b)$$
$$\Xi_{\alpha \mu}^{\nu \sigma} = C^{i \sigma}_{j} h_{\mu, \alpha} + \frac{1}{2} H^{i \sigma}_{j} C^{i \sigma}_{\mu, \alpha} = \frac{\partial a^{\nu \sigma}}{\partial z^j} \frac{\partial^2 z^i}{\partial x^\mu \partial x^\nu} + \frac{1}{2} \frac{\partial a^{\mu \nu}}{\partial \phi^{ij}} \frac{\partial^2 \phi^{ij}}{\partial x^\alpha \partial x^\nu}. \hspace{1cm} (18c)$$

where the first term in eq. (18b) is the standard part. In deriving the connection for the bivector, one discovers that the Leibniz rule does not hold for the Clifford
polydifferential operator $\mathcal{D}$ over the wedge or dot product, e.g. $\mathcal{D}(e_\mu \wedge e_\nu) \neq (\mathcal{D}e_\mu) \wedge e_\nu + e_\mu \wedge (\mathcal{D}e_\nu)$. However it works fine for the Clifford (direct) product. Hence,

$$\mathcal{D}(e_\mu \wedge e_\nu) = \frac{1}{2} [(\mathcal{D}e_\mu), e_\nu] + \frac{1}{2} [e_\mu, (\mathcal{D}e_\nu)]$$

(19)

where again $\Xi_{\alpha\omega}^{\xi\sigma}$ and $Q_{\alpha\beta\omega}^{\mu\sigma}$ are the metamorphic connections.

C. Metamorphic Polygeodesics

Parallel transporting the momentum vector around a closed loop might have it return as a pure bivector due to the metamorphic connection! However, if we transport the momentum polyvector, then it will return as the same form but with a reshuffling of the momentum and spin portions, subject to the constraint that the modulus of eq. (9b) be unchanged. Physically this would appear as additional forces due to the coupling between the spin and momentum via the metamorphic connection. Although eq. (14b) is still valid, eq. (14cd) will be modified with terms that couple the equations. In particular eq. (14c) becomes,

$$\ddot{p}^\mu + p^\nu \left( \dddot{x}^{\alpha \beta} \Gamma^\mu_{\beta \nu} + \frac{1}{2} \dddot{a}^{\alpha \beta} R_{\alpha \beta \nu}^{\mu} \right) + \frac{1}{2} S^\nu_{\sigma} \left( \ddot{x}^{\alpha \sigma} \Xi_{\alpha \omega}^{\mu \sigma} + \frac{1}{2} \dddot{a}^{\alpha \beta} Q_{\alpha \beta \omega}^{\mu \sigma} \right) = 0,$$

(20a)

Note that the metamorphic connection $\Xi_{\alpha \omega}^{\mu \sigma}$ enhances the Papapetrou term, while the other metamorphic connection $Q_{\alpha \beta \omega}^{\mu \sigma}$ adds a pure second order spin interaction. The generalization of eq. (14d) can be generated in the same way, but will be left to a more detailed future paper.

V. Summary

In introducing Dimensional Democracy we have given the bivector a coordinate and shown its utility in treating the spinning particle problem. In adopting automorphism invariance we find that the concept of what is a vector depends upon the observer’s frame of reference, leading to demanding that physical laws are covariant under this transformation. This allows us to derive Crawford’s Clifford Connection in a classical context as a local polymorphic coordinate transformation. Most important the themes we have introduced provide an entirely new general way in which to apply Clifford calculus to physical theories.

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