The Number of Maximal Independent Sets in Quasi-Tree Graphs and Quasi-Forest Graphs

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Abstract
A maximal independent set is an independent set that is not a proper subset of any other independent set. A connected graph (respectively, graph) $G$ with vertex set $V(G)$ is called a quasi-tree graph (respectively, quasi-forest graph), if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree (respectively, forest). In this paper, we survey on the large numbers of maximal independent sets among all trees, forests, quasi-trees and quasi-forests. In addition, we further look into the problem of determining the third largest number of maximal independent sets among all quasi-trees and quasi-forests. Extremal graphs achieving these values are also given.

Keywords
Maximal Independent Set, Quasi-Tree Graph, Quasi-Forest Graph, Extremal Graph

1. Introduction and Preliminary
Let $G = (V, E)$ be a simple undirected graph. An independent set is a subset $S$ of $V$ such that no two vertices in $S$ are adjacent. A maximal independent set is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of a graph $G$ is denoted by $\text{MI}(G)$ and its cardinality by $\text{mi}(G)$.

The problem of determining the largest value of $\text{mi}(G)$ in a general graph of order $n$ and those graphs achieving the largest number was proposed by Erdős and Moser, and solved by Moon and Moser [1]. It was then studied for various families of graphs, including trees, forests, (connected) graphs with at most one cycle, (connected) triangle-free graphs, ($k$-)connected graphs, bipartite graphs; for a survey see [2]. Jin and Li [3] investigated the second largest number of $\text{mi}(G)$ among all graphs of order $n$; Jou and Lin [4] further explored the same problem for trees and forests; Jin and Yan [5] solved the third largest number of
among all trees of order \( n \). A connected graph (respectively, graph) \( G \) with vertex set \( V(G) \) is called a \textit{quasi-tree graph} (respectively, \textit{quasi-forest graph}), if there exists a vertex \( x \in V(G) \) such that \( G - x \) is a tree (respectively, forest). The concept of quasi-tree graphs was mentioned by Liu and Lu in [6]. Recently, the problem of determining the largest and the second largest numbers of \( mi(G) \) among all quasi-tree graphs and quasi-forest graphs of order \( n \) was solved by Lin [7] [8].

In this paper, we survey on the large numbers of maximal independent sets among all trees, forests, quasi-trees and quasi-forests. In addition, we further look into the problem of determining the third largest number of maximal independent sets among all quasi-trees and quasi-forests. Extremal graphs achieving these values are also given.

For a graph \( G = (V, E) \), the \textit{neighborhood} \( N_G(x) \) of a vertex \( x \) is the set of vertices adjacent to \( x \) in \( G \) and the \textit{closed neighborhood} \( N_G[x] \) is \( \{x\} \cup N_G(x) \). The \textit{degree of} \( x \) is the cardinality of \( N_G(x) \), denoted by \( \deg_G(x) \). For a set \( A \subseteq V(G) \), the \textit{deletion of} \( A \) from \( G \) is the graph \( G - A \) obtained from \( G \) by removing all vertices in \( A \) and their incident edges. Two graphs \( G_1 \) and \( G_2 \) are \textit{disjoint} if \( V(G_1) \cap V(G_2) = \emptyset \). The \textit{union} of two disjoint graphs \( G_1 \) and \( G_2 \) is the graph \( G_1 \cup G_2 \) with vertex set \( V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \) and edge set \( E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \).

\( nG \) is the short notation for the union of \( n \) copies of disjoint graphs isomorphic to \( G \). Denote by \( C_n \) a \textit{cycle} with \( n \) vertices and \( P_n \) a \textit{path} with \( n \) vertices.

Throughout this paper, for simplicity, let \( r = \sqrt{2} \).

Lemma 1.1 ([9]) For any vertex \( x \) in a graph \( G \),

\[
mi(G) \leq mi(G - x) + mi(G - N_G[x]).
\]

Lemma 1.2 ([10]) If \( G \) is the union of two disjoint graphs \( G_1 \) and \( G_2 \), then \( mi(G) = mi(G_1)mi(G_2) \).

2. Survey on the Large Numbers of Maximal Independent Sets

In this section, we survey on the large numbers of maximal independent sets among all trees, forests, quasi-trees and quasi-forests. The results of the largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.1 and 2.2, respectively.

Theorem 2.1 ([10] [11]) If \( T \) is a tree with \( n \geq 1 \) vertices, then \( mi(T) \leq t_1(n) \), where

\[
t_1(n)=\begin{cases} r^{n-2} + 1, & \text{if } n \text{ is even,} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}
\]

Furthermore, \( mi(T) = t_1(n) \) if and only if \( T \in T_1(n) \), where

\[
T_1(n)=\begin{cases} B\left(2, \frac{n-2}{2}\right) \text{ or } B\left(4, \frac{n-4}{2}\right), & \text{if } n \text{ is even,} \\ B\left(1, \frac{n-1}{2}\right), & \text{if } n \text{ is odd,} \end{cases}
\]
where $B(i, j)$ is the set of batons, which are the graphs obtained from the basic path $P$ of $i \geq 1$ vertices by attaching $j \geq 0$ paths of length two to the endpoints of $P$ in all possible ways (see Figure 1).

**Theorem 2.2** ([10] [11]) If $F$ is a forest with $n \geq 1$ vertices, then $\text{mi}(F) \leq f_1(n)$, where

$$f_1(n) = \begin{cases} \frac{r^n}{2}, & \text{if } n \text{ is even,} \\ \frac{r^{n-1}}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $\text{mi}(F) = f_1(n)$ if and only if $F \in F_1(n)$, where

$$F_1(n) = \begin{cases} \frac{n}{2}P_2, & \text{if } n \text{ is even,} \\ B\left(1, \frac{n-1-2s}{2}\right) \cup sP_2 \text{ for some } s \text{ with } 0 \leq s \leq \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

The results of the second largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.3 and 2.4, respectively.

**Theorem 2.3** ([4]) If $T$ is a tree with $n \geq 4$ vertices having $T \not\in T_1(n)$, then $\text{mi}(T) \leq t_2(n)$, where

$$t_2(n) = \begin{cases} r^{n-2}, & \text{if } n \geq 4 \text{ is even,} \\ 3, & \text{if } n = 5, \\ 3r^{n-3} + 1, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

Furthermore, $\text{mi}(T) = t_2(n)$ if and only if $T = T'_2(8), T''_2(8), P_{10}$ or $T \in T_2(n)$, where $T_2(n)$ and $T'_2(8), T''_2(8)$ are shown in Figure 2 and Figure 3, respectively.

**Theorem 2.4** ([4]) If $F$ is a forest with $n \geq 4$ vertices having $F \not\in F_1(n)$, then $\text{mi}(F) \leq f_2(n)$, where

![Figure 1. The baton $B(i, j)$ with $j = j_1 + j_2$.](image1)

![Figure 2. The trees $T_i(n)$.](image2)

![Figure 3. The trees $T'_2(8)$ and $T''_2(8)$.](image3)
\[ f_2(n) = \begin{cases} \left\lfloor \frac{3r^{n-4}}{2} \right\rfloor & \text{if } n \geq 4 \text{ is even,} \\ 3 & \text{if } n = 5, \\ \left\lfloor \frac{7r^{n-7}}{2} \right\rfloor & \text{if } n \geq 7 \text{ is odd.} \end{cases} \]

Furthermore, \( mi(F) = f_2(n) \) if and only if \( F \in F_2(n) \), where
\[
F_2(n) = \begin{cases} P_4 \cup \frac{n-4}{2} P_2, & \text{if } n \geq 4 \text{ is even,} \\ T_2(5) \text{ or } P_2 \cup P_1, & \text{if } n = 5, \\ P_2 \cup \frac{n-7}{2} P_2, & \text{if } n \geq 7 \text{ is odd.} \end{cases}
\]

The results of the third largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.5 and 2.6, respectively.

**Theorem 2.5** ([5]) If \( T \) is a tree with \( n \geq 7 \) vertices having \( T \not\in T_i(n), i = 1, 2 \), then \( mi(T) \leq t_i(n) \), where
\[
t_i(n) = \begin{cases} \left\lfloor \frac{3r^{n-5}}{2} \right\rfloor & \text{if } n \geq 7 \text{ is odd,} \\ 7, & \text{if } n = 8, \\ 15, & \text{if } n = 10, \\ 7r^{n-8} + 2 & \text{if } n \geq 12 \text{ is even.} \end{cases}
\]

Furthermore, \( mi(T) = t_i(n) \) if and only if \( T = T_i(8), T_i'(10), T_i''(10) \) or \( T \in T_i(n) \), where \( T_i(8), T_i'(10), T_i''(10), T_i(n) \) are shown in Figure 4 and Figure 5, respectively.

**Theorem 2.6** ([12]) If \( F \) is a forest with \( n \geq 8 \) vertices having \( F \not\in F_i(n), i = 1, 2 \), then \( mi(F) \leq f_i(n) \), where
\[
f_i(n) = \begin{cases} \left\lfloor \frac{5r^{n-6}}{2} \right\rfloor & \text{if } n \text{ is even,} \\ 13r^{n-9} & \text{if } n \text{ is odd.} \end{cases}
\]

Furthermore, \( mi(F) = f_i(n) \) if and only if \( F \in F_i(n) \), where
\[
F_i(n) = \begin{cases} \frac{n-6}{2} P_2, & \text{if } n \text{ is even,} \\ T_2(9) \cup \frac{n-9}{2} P_2, & \text{if } n \text{ is odd.} \end{cases}
\]

![Figure 4](image4.png) The trees \( T_i(8), T_i'(10) \) and \( T_i''(10) \).

![Figure 5](image5.png) The trees \( T_i(n) \).

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The results of the largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs are described in Theorems 2.7 and 2.8, respectively.

**Theorem 2.7** ([7]) If $Q$ is a quasi-tree graph with $n \geq 5$ vertices, then $mi(Q) \leq q_1(n)$, where

$$q_1(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even}, \\ r^{n-1} + 1, & \text{if } n \text{ is odd}. \end{cases}$$

Furthermore, $mi(Q) = q_1(n)$ if and only if $Q = C_3$ or $Q \in Q_1(n)$, where $Q_1(n)$ is shown in [Figure 6].

**Theorem 2.8** ([7]) If $Q$ is a quasi-forest graph with $n \geq 2$ vertices, then $mi(Q) \leq \overline{q}_1(n)$, where

$$\overline{q}_1(n) = \begin{cases} r^3, & \text{if } n \text{ is even}, \\ 3r^{n-3}, & \text{if } n \text{ is odd}. \end{cases}$$

Furthermore, $mi(Q) = \overline{q}_1(n)$ if and only if $Q \in \overline{Q}_1(n)$, where

$$\overline{Q}_1(n) = \begin{cases} n/2 P_2, & \text{if } n \text{ is even}, \\ C_3 \cup \frac{n-3}{2} P_2, & \text{if } n \text{ is odd}. \end{cases}$$

The results of the second largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs are described in Theorems 2.9 and 2.10, respectively.

**Theorem 2.9** ([8]) If $Q$ is a quasi-tree graph with $n \geq 6$ vertices having $Q \notin Q_1(n)$, then $mi(Q) \leq q_2(n)$, where

$$q_2(n) = \begin{cases} 5r^{n-6} + 1, & \text{if } n \text{ is even}, \\ r^{n-1}, & \text{if } n \text{ is odd}. \end{cases}$$

Furthermore, $mi(Q) = q_2(n)$ if and only if $Q \in Q_2(n)$, where

$$Q_2(n) = \begin{cases} Q_1^{(1)}(n), Q_1^{(2)}(n), Q_1^{(3)}(n), Q_1^{(4)}(n), \\ B\left(1, \frac{n-1}{2}\right), Q_2^{(1)}(7), Q_2^{(2)}(7), Q_2^{(3)}(7), Q_2^{(4)}(7), \end{cases}$$

if $n$ is even,

$$Q_2(n) = \begin{cases} Q_3^{(1)}(n), Q_3^{(2)}(n), Q_3^{(3)}(n), Q_3^{(4)}(n), \\ C_3 \cup \frac{n-3}{2} P_2, \end{cases}$$

if $n$ is odd,

where $Q_1(n)$ is shown in [Figure 7] and [Figure 8].

**Theorem 2.10** ([8]) If $Q$ is a quasi-forest graph with $n \geq 4$ vertices having $Q \notin \overline{Q}_1(n)$, then $mi(Q) \leq \overline{q}_2(n)$, where

$$\overline{q}_2(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even}, \\ 5r^{n-5}, & \text{if } n \text{ is odd}. \end{cases}$$

[Figure 6. The graph $Q_1(n)$.]

[Figure 7. The graph $Q_2(n)$.]

[Figure 8. The graph $Q_3(n)$.]

[Figure 9. The graph $Q_4(n)$.]
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Figure 7. The graphs \( Q^{(i)}_{2e}(n) \), \( 1 \leq i \leq 4 \).

Figure 8. The graphs \( Q^{(i)}_{2o}(7) \), \( 1 \leq i \leq 4 \).

Furthermore, \( mi(Q) = \overline{u}_2(n) \) if and only if \( Q \in \overline{Q}_2(n) \), where
\[
\overline{Q}_2(n) = \begin{cases} 
P_2 \cup \frac{n-4}{2} \quad P_2 Q_1(n-2s) \cup sP_2, & \text{if } n \text{ is even}, \
Q_3(6) \cup \frac{n-6}{2} P_2 C_3 \cup B\left(\frac{n-4-2s}{2}\right) \cup sP_2, & \text{if } n \text{ is odd}, \
Q_5(5) \cup \frac{n-5}{2} P_2 W \cup \frac{n-5}{2} P_2 C_3 \cup \frac{n-5}{2} P_2, & \text{if } n \text{ is even}, \
\end{cases}
\]

where \( W \) is a bow, that is, two triangles \( C_3 \) having one common vertex.

A graph is said to be \textit{unicyclic} if it contains exactly one cycle. The result of the second largest number of maximal independent sets among all connected unicyclic graphs are described in Theorems 2.11.

\textbf{Theorem 2.11} ([13]) If \( U \) is a connected unicyclic graph of order \( n \geq 6 \) with \( U \neq C_3 \) and \( Q \not\in Q_2(n) \), then \( mi(G) \leq u_2(n) \), where
\[
u_2(n) = \begin{cases} 
5r^{n-6} + 1, & \text{if } n \text{ is even}, \\
3r^{n-5} + 2, & \text{if } n \text{ is odd}.
\end{cases}
\]

Furthermore, \( mi(G) = u_2(n) \) if and only if \( U \in U_2(n) \), where
\[
U_2(n) = \begin{cases} 
Q_{2e}^{(i)}(n), & \text{if } n \text{ is even,} \\
U^{(i)}_{2o}(n), U^{(i)}_{2o}(n), U^{(i)}_{2o}(n), U^{(i)}_{2o}(n), U^{(i)}_{2o}(n), U^{(i)}_{2o}(n), & \text{if } n \text{ is odd},
\end{cases}
\]

where \( U_{2o}^{(i)}(n) \) is shown in Figure 9.

\section*{3. Main Results}

In this section, we determine the third largest values of \( mi(G) \) among all quasi-tree graphs and quasi-forest graphs of order \( n \geq 7 \), respectively. Moreover, the extremal graphs achieving these values are also determined.

\textbf{Theorem 3.1} If \( Q \) is a quasi-tree graph of odd order \( n \geq 7 \) having \( Q \not\in Q_1(n), Q_2(n) \), then \( mi(Q) \leq 3r^{n-5} + 2 \). Furthermore, the equality holds if and only if \( Q = U^{(i)}_{2o}, 1 \leq i \leq 6 \), where \( U^{(i)}_{2o}(n) \) is shown in Figure 9.

\textit{Proof}: It is straightforward to check that \( mi(U^{(i)}_{2o}(n)) = 3r^{n-5} + 2, 1 \leq i \leq 6 \). Let \( Q \) be a quasi-tree graph of odd order \( n \geq 7 \) having \( Q \not\in Q_1(n), Q_2(n) \) such that \( mi(Q) \) is as large as possible. Then \( mi(Q) \geq 3r^{n-5} + 2 \). If \( Q \) is a tree, by Theorems 2.1, 2.3 and \( Q \not\in Q_2(n) \), we have that
\[3r^{a-5} + 2 \leq \text{mi}(Q) \leq t_s(n) = 3r^{a-5} + 1.\] This is a contradiction.

Suppose that \(Q\) contains at least two cycles and \(x\) is the vertex such that \(Q - x\) is a tree. Then \(\deg_G(x) \geq 3\). By Lemma 1.1, Theorems 2.1 and 2.2,
\[3r^{a-5} + 2 \leq \text{mi}(Q) \leq \text{mi}(Q - x) + \text{mi}(Q - N_Q[x]) \leq r^{(a-1)-2} + r^{(a-4)-1} = 3r^{a-5} + 1,\]
which is a contradiction. We obtain that \(Q\) is a connected unicyclic graph, thus the result follows from Theorem 2.11.

**Theorem 3.2** If \(Q\) is a quasi-tree graph of even order \(n \geq 8\) having \(Q \notin Q_e(n), Q_s(n)\), then \(\text{mi}(Q) \leq 5r^{a-6}\). Furthermore, the equality holds if and only if \(Q = Q'(8), Q^*(8), Q^*(10), Q^*_6(n), 1 \leq i \leq 12,\) where \(Q'(8), Q^*(8), Q^*(10)\) and \(Q^*_6(n)\) are shown in Figure 10.

**Proof.** It is straightforward to check that \(\text{mi}(Q'(8)) = \text{mi}(Q^*(8)) = 10,\) \(\text{mi}(Q^*(10)) = 20\) and \(\text{mi}(Q^*_6(n)) = 5r^{a-6}, 1 \leq i \leq 12.\) Let \(Q\) be a quasi-tree graph of even order \(n \geq 8\) having \(Q \notin Q_e(n), Q_s(n)\) such that \(\text{mi}(Q)\) is as large as possible. Then \(\text{mi}(Q) \geq 5r^{a-6}\). If \(Q\) is a tree, by Theorem 2.1, we have that \(5r^{a-6} \leq \text{mi}(Q) \leq t_s(n) = r^{a-2} + 1.\) This is a contradiction, so \(Q\) contains at least one cycle. Let \(x\) be the vertex such that \(Q - x\) is a tree. Then \(x\) is on some cycle of \(Q\), it follows that \(\deg_Q(x) \geq 2\). In addition, by Lemma 1.1, Theorems 2.2 and 2.5, \(\text{mi}(Q - x) \geq 5r^{a-6} - r^{(a-3)-1} = 3r^{a-6} = t_s(n-1).\) We consider the following three cases.

- **Case 1.** \(Q - x \in T_e(n-1).\) If \(\deg_Q(x) \geq 6\) then \(Q - N_Q[x]\) is a forest with at most \(n - 7\) vertices, by Lemma 1.1, Theorems 2.1 and 2.2,
\[5r^{a-6} \leq \text{mi}(Q) \leq \text{mi}(Q - x) + \text{mi}(Q - N_Q[x]) \leq r^{(a-1)-1} + r^{(a-4)-1} = 9r^{a-8}.\] This is a contradiction. So we assume that \(2 \leq \deg_Q(x) \leq 5\).

  - **deg x = 2.** There are 6 possibilities for graph \(Q\). See Figure 11. Note that \(Q'_1 = Q_e(n)\). By simple calculation, we have that \(\text{mi}(Q'_i) \leq r^{a-2} + 1\) for \(2 \leq i \leq 6,\) a contradiction to \(\text{mi}(Q) \geq 5r^{a-6}\).

  - **deg x = 3.** Suppose that there exists an isolated vertex \(y\) in \(Q - N_Q[x]\) and \(Q - N_Q[x] - y \notin F_1(n-5)\), then \(\text{mi}(Q) \leq \text{mi}(Q - x) + \text{mi}(Q - N_Q[x]) < r^{(a-1)-1} + r^{(a-4)-1} = 5r^{a-6}.\) Hence there are 4 possibilities for graph \(Q\). See Figure 12.

Note that \(Q'_3 = Q^*_3(n)\), \(Q'_4 = Q^*_4(n)\) and \(Q'_5 = Q^*_5(n)\). By simple calculation, we have \(\text{mi}(Q'_i) = r^{a-2} + 1,\) a contradiction to \(\text{mi}(Q) \geq 5r^{a-6}\).

- **deg x \leq 5.** Since \(Q - N_Q[x]\) is a forest of odd order \(n-5\) or even or-
Figure 10. The graphs $Q'(8)$, $Q''(8)$, $Q'''(10)$ and $Q_{6i}(n)$, $1 \leq i \leq 12$.

Figure 11. The graphs $Q'_i$, $1 \leq i \leq 6$.

der $n-6$, by Lemma 1.1, Theorems 2.1 and 2.2, we have $5r^{n-6} \leq mi(Q) \leq mi(Q-x) + mi(Q-N_Q[x]) \leq r^{n-2} + r^{n-6} = 5r^{n-6}$. The equalities holding imply that $Q-x = T_i(n-1)$ and $Q-N_Q[x] = F_i(n-5)$ or $F_i(n-6)$. Hence we obtain that $Q = Q_{6i}(n)$, $1 \leq i \leq 4$.

Case 2. $Q-x \in T_i(n-1)$. If $deg(x) \geq 4$ then $Q-N_Q[x]$ is a forest with at most $n-5$ vertices, by Lemma 1.1, Theorems 2.2 and 2.3, we have that $5r^{n-6} \leq mi(Q) \leq mi(Q-x) + mi(Q-N_Q[x]) \leq 3r^{(n-4)-5} + 1 + r^{(n-5)-1} = 4r^{n-6} + 1$. This is a contradiction. So we assume that $2 \leq deg_Q(x) \leq 3$.

- $deg(x) = 2$. Suppose that $Q-N_Q[x] \notin F_i(n-3)$, by Lemma 1.1, Theorems 2.3 and 2.4, we have that
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Figure 12. The graphs \( Q_i, \ 7 \leq i \leq 10 \).

\[
5r^{n-6} \leq mi(Q) \leq mi(Q-x) + mi\left(Q - N_Q[x]\right) \leq 3r^{(n-1)-5} + 1 + 7r^{(n-3)-7} = 19r^{n-10} + 1.
\]

The equalities holding imply that \( n = 10 \), that is, \( Q - x = T_i(9) \) and
\( Q - N_Q[x] = F_i(7) \). Hence we obtain that \( Q = Q'(10) \). Now we assume that
\( Q = Q_{10}(n) \) \( \in F_i(n-3) \). There are 7 possibilities for graph \( Q \). See Figure 13.

Note that \( Q_{10} = Q_{10}^{(1)}(n), \ Q_{12} = Q_{12}^{(5)}(n) \) and \( Q_{16} = Q_{16}^{(6)}(n) \). By simple calculation, we have
\[
5r^{n-6} \leq mi(Q) \leq r^{n-2} + 2 \quad \text{for} \quad 14 \leq i \leq 17,
\]
and
\[
r^{n-6} + 2 = mi(Q_{17}) = 5r^{n-6} \quad \text{when} \quad n = 8.
\]

In addition, \( r^{n-2} = mi(Q) \) when \( n \geq 10 \). In addition, \( r^{n-6} = mi(Q_{17}) \) when
\( n = 8 \), it follows that \( Q = Q_{16}^{(6)}(8) \).

Case 3. \( Q = T_i(9) \). Suppose that \( Q = T_i(9) \), by Lemma 1.1, Theorems 2.3 and 2.4, we have that
\[
5r^{n-6} \leq mi(Q) \leq mi(Q-x) + mi\left(Q - N_Q[x]\right) \leq 3r^{(n-1)-5} + 1 + 7r^{(n-3)-7} = 19r^{n-10} + 1.
\]

The equalities holding imply that \( n = 8 \), that is, \( Q - x = T_i(7) \) and
\( Q - N_Q[x] = F_i(4) \). Hence we obtain that \( Q = Q'(8), \ Q''(8) \). Now we assume that
\( Q = Q_{10}^{(1)}(n) \). Since \( Q-x = T_i(n-1) \) and
\( Q - N_Q[x] = F_i(n-4) \), it follows that \( Q = Q_{10}^{(1)}(n) \), \( 2 \leq i \leq 4 \), a contradiction to
\( Q \neq Q_{10}^{(8)}(n) \).

In the following, we will investigate the same problem for quasi-forest graphs.

**Theorem 3.3** If \( Q \) is a quasi-forest graph of odd order \( n \geq 7 \) having

$Q \notin Q(n), \overline{Q}(n)$, then $m_i(Q) \leq 9r^{n-7}$. Furthermore, the equality holds if and only if $Q = \overline{Q_i}(n)$, $1 \leq i \leq 4$, where $\overline{Q_i}(n)$ is shown in Figure 15.

Figure 13. The graphs $Q_i^r$, $11 \leq i \leq 17$.

Figure 14. The graphs $Q_i^r$, $18 \leq i \leq 24$.

Figure 15. The graphs $\overline{Q_i}(n)$, $1 \leq i \leq 4$.
Proof. It is straightforward to check that \( \text{mi}(\overline{Q}_i(n)) = 9r^{n-7}, \ 1 \leq i \leq 4 \). Let \( Q \) be a quasi-forest graph of odd order \( n \geq 7 \) having \( Q \notin \overline{Q}_i(n), \overline{Q}_j(n) \) such that \( \text{mi}(Q) \) is as large as possible. Then \( \text{mi}(Q) \geq 9r^{n-7} \). If \( Q \) is a forest, by Theorem 2.2, we have that \( 9r^{n-7} \leq \text{mi}(Q) \leq f_1(n) = r^{n-4} \). This is a contradiction, so \( Q \) contains at least one cycle. Let \( x \) be a vertex such that \( Q - x \) is a forest. Then \( x \) is on some cycle of \( Q \), it follows that \( \deg_Q(x) \geq 2 \) and \( Q - N_Q[x] \) is a forest with at most \( n - 3 \) vertices. By Lemma 1.1, Theorem 2.2 and 2.6, we obtain that \( \text{mi}(Q - x) \geq \text{mi}(Q) - \text{mi}(Q - N_Q[x]) \geq 9r^{n-7} - r^{n-3} = 5r^{n-7} = f_3(n-1) \). We consider the following three cases.

Case 1. \( Q - x \in F_i(n-1) \). If \( \deg_Q(x) \geq 7 \) then \( Q - N_Q[x] \) is a forest with at most \( n - 8 \) vertices, by Lemma 1.1 and Theorem 2.2, we have that \( 9r^{n-7} \leq \text{mi}(Q) \leq \text{mi}(Q - x) + \text{mi}(Q - N_Q[x]) \leq r^{n-7} + r^{(n-8)-1} = 17r^{n-9} \). This is a contradiction. So we assume that \( 2 \leq \deg_Q(x) \leq 6 \). There are 9 possibilities for graph \( Q \). See Figure 16.

Note that \( \overline{Q}_i \notin \overline{Q}_i(n), \overline{Q}_2 \notin \overline{Q}_2(n), \overline{Q}_3 \notin \overline{Q}_3(n), \overline{Q}_4 = \overline{Q}_4(n), \overline{Q}_5 = \overline{Q}_5(n), \overline{Q}_6 = \overline{Q}_6(n), \overline{Q}_7 = \overline{Q}_7(n), \overline{Q}_8 = \overline{Q}_8(n) \). By simple calculation, we have \( \text{mi}(\overline{Q}_i) \leq 17r^{n-9} \), \( i = 6, 8, 9 \), a contradiction to \( \text{mi}(Q) \geq 9r^{n-7} \).

Case 2. \( Q - x = F_z(n-1) \). If \( \deg_Q(x) \geq 3 \) then \( Q - N_Q[x] \) is a forest with at most \( n - 4 \) vertices, by Lemma 1.1, Theorems 2.2 and 2.4, we have that \( 9r^{n-7} \leq \text{mi}(Q) \leq \text{mi}(Q - x) + \text{mi}(Q - N_Q[x]) \leq 3r^{(n-4)-1} + r^{(n-4)-1} = 4r^{n-5} \). This is a contradiction. So we assume that \( \deg_Q(x) = 2 \). There are 5 possibilities for graph \( Q \). See Figure 17.

Note that \( \overline{Q}_6 = \overline{Q}_6(n), \overline{Q}_7 = \overline{Q}_7(n), \overline{Q}_8 = \overline{Q}_8(n) \). By simple calculation, we have \( \text{mi}(\overline{Q}_i) \leq 3r^{n-5} + 1, \ i = 11, 13 \), a contradiction to \( \text{mi}(Q) \geq 9r^{n-7} \).

Case 3. \( Q - x \in F_i(n-1) \). Since \( Q - N_Q[x] \) is a forest with at most \( n - 3 \) vertices, by Lemma 1.1, Theorems 2.2 and 2.6, we have that \( 9r^{n-7} \leq \text{mi}(Q) \leq \text{mi}(Q - x) + \text{mi}(Q - N_Q[x]) \leq 5r^{(n-3)-6} + r^{n-3} = 9r^{n-7} \). The equali-

![Figure 16](image16.png)

Figure 16. The graphs \( \overline{Q}_i, 1 \leq i \leq 9 \).
ties holding imply that \( Q-x \in F_i(n-1) \) and \( Q-N_Q[x] \in F_i(n-3) \). There are 3 possibilities for graph \( Q \). See Figure 18.

Note that \( \bar{Q}_{17} = \bar{Q}_{10}(n) \). By simple calculation, we have \( mi(\bar{Q}) = 8r^{n-7}, \quad 15 \leq i \leq 16 \), a contradiction to \( mi(Q) \geq 9r^{n-7} \).

**Theorem 3.4** If \( Q \) is a quasi-forest graph of even order \( n \geq 8 \) having \( Q \notin \bar{Q}_1(n), \bar{Q}_2(n) \), then \( mi(Q) \leq 11r^{n-8} \). Furthermore, the equality holds if and only if \( Q = Q_5(8) \cup \frac{n-8}{2}P_2 \).

**Proof.** It is straightforward to check that \( mi(\bar{Q}_5(8) \cup \frac{n-8}{2}P_2) = 11r^{n-8} \). Let \( Q \) be a quasi-forest graph of even order \( n \geq 8 \) having \( Q \notin \bar{Q}_1(n), \bar{Q}_2(n) \) such that \( mi(Q) \) is as large as possible. Then \( mi(Q) \geq 11r^{n-8} \). If \( Q \) is a forest, by Theorems 2.2, 2.4, 2.6, 2.8 and 2.10, we have that \( 11r^{n-8} \leq mi(Q) \leq f_5(n) = 5r^{n-6} \). This is a contradiction, so \( Q \) contains a component \( \hat{Q} \) with at least one cycle.

Let \( |\hat{Q}| = s \). Suppose that \( Q - \hat{Q} \neq \frac{n-s}{2}P_2 \). Since \( \hat{Q} \) is not a tree and \( Q \notin \bar{Q}_1(n), \bar{Q}_2(n) \), by Lemma 1.2, Theorems 2.2, 2.4 and 2.7, we have that

\[
mi(Q) = mi(\hat{Q}) \cdot mi(Q-\hat{Q}) \leq \begin{cases} 
3r^{s-4} \cdot 3r^{(s-1)-4}, & \text{if } s \geq 4 \text{ is even,} \\
3 \cdot 7r^{(s-1)-7}, & \text{if } s = 3, \\
(r^{s-1}+1) \cdot r^{(s-3)-1}, & \text{if } s \geq 5 \text{ is odd,} \\
9r^{n-8}, & \text{if } s \geq 4 \text{ is even,} \\
21r^{n-10}, & \text{if } s = 3, \\
5r^{n-6}, & \text{if } s \geq 5 \text{ is odd,} \\
<11r^{n-8}, & \text{if } s \geq 4 \text{ is odd.}
\end{cases}
\]

![Figure 17. The graphs \( \bar{Q}_i \), \( 10 \leq i \leq 14 \).](image1)

![Figure 18. The graphs \( \bar{Q}_i \), \( 15 \leq i \leq 17 \).](image2)
which is a contradiction. Hence we obtain that \( s \) is even and \( Q - \hat{Q} = \frac{n - s}{2} P_2 \).

Let \( x \) be the vertex in \( \hat{Q} \) such that \( \hat{Q} - x \) is a forest and \( w(\hat{Q} - x) \) be the number of components of \( \hat{Q} - x \). We consider the following two cases.

Case 1. \( w(\hat{Q} - x) = 1 \). Then \( \hat{Q} \) is a quasi-tree graph. Since \( Q \notin \overline{Q}_i(n), \overline{Q}_i(n) \) it follows that \( s \geq 8 \). By Lemma 1.2 and Theorem 2.9, it follows that \( mi(Q) = (5r^{s-6} + 1) \cdot r^{s-6} = 5r^{s-6} + r^{s-6} \leq 11r^{s-8} \). The equality holding imply that \( s = 8 \). In conclusion, \( Q = Q_2(8) \cup \frac{n - 8}{2} P_2 \).

Case 2. \( w(\hat{Q} - x) \geq 2 \). Then \( \deg x \geq 3 \). In addition, suppose that \( Q - N_{\hat{Q}}[x] \) has a isolated vertex or \( \deg_{\hat{Q}}(x) \geq 4 \), by Lemma 1.1 and Theorem 2.2, we have that \( 11r^{s-8} \leq mi(Q) \leq mi(Q - x) + mi(Q - N_{\hat{Q}}[x]) \leq r^{x-1} + r^{s-5-1} = 5r^{s-6} \). This is a contradiction, hence, we have that \( \deg_{\hat{Q}}(x) = 3 \) and \( Q - N_{\hat{Q}}[x] \) has no isolated vertex. For the case that \( Q - x \notin F_i(n-1) \), by Lemma 1.1, Theorems 2.2 and 2.4, we have that \( 11r^{s-8} \leq mi(Q) \leq mi(Q - x) + mi(Q - N_{\hat{Q}}[x]) \leq 7r^{s-7} + r^{s-4} = 11r^{s-8} \). The equalities holding imply that \( Q - x \in F_2(n-1) \) and \( Q - N_{\hat{Q}}[x] \in F_1(n-4) \). Since \( w(\hat{Q} - x) \geq 2 \), there no such graph \( Q \). For the other case that \( Q - x \in F_i(n-1) \), there are 2 possibilities for graph \( Q \). See Figure 19.

Note that \( \overline{Q}_{18} = Q_2(n) \) and \( \overline{Q}_{19} = Q_2(8) \cup \frac{n - 8}{2} P_2 \) when \( s = 8 \). On the other hand, \( mi(\overline{Q}_9) \leq 2^{-10} r^{s+10} \) when \( s \geq 10 \), a contradiction to \( mi(Q) \geq 11r^{s-8} \).

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