Abstract. We prove inequalities of Hardy type for functions in Triebel–Lizorkin spaces $F_{pq}^s(G)$ on a domain $G \subset \mathbb{R}^n$, whose boundary has the Aikawa dimension strictly less than $n - sp$.

1. Introduction

In this paper, we study Hardy-type inequalities for functions in Triebel–Lizorkin spaces $F_{pq}^s(G)$; see [26] for the case of bounded smooth domains $G$. We assume that the boundary $\partial G$ of a domain $G$ is ‘thin’, in the sense that its Aikawa dimension is strictly smaller than $n - sp$. The notion of the Aikawa dimension appears in connection with the quasiadditivity of Riesz capacity, [2]; subsequently, it has turned out to be useful in other questions in the theory of function spaces, see e.g. [10, 23]. In particular, it is known that, for every $f \in C_0^\infty(G)$, a ‘classical’ Hardy inequality

$$\int_{G} \frac{|f(x)|^p}{\text{dist}(x, \partial G)^p} \, dx \leq C \int_{G} |\nabla f(x)|^p \, dx,$$

holds if $1 < p < n$ and $\partial G$ is ‘thin’, i.e., if $\dim_A(\partial G) < n - p$. Indeed, as it is observed in [17], this result is implicitly contained in [16]. On the other hand, it is well known that inequality (1.1) holds if $\mathbb{R}^n \setminus G$ is $(1, p)$ uniformly fat with $1 < p \leq n$, we refer to [20]. These two last results exhibit a dichotomy between ‘thin’ and ‘fat’ sets which manifests in Hardy-type inequalities.

Though our main result is Theorem 1.5, we also formulate and prove the following illustrative theorem under an additional assumption that $G$ is a John domain.

1.2. Theorem. Let $n \geq 2$, $1 < p < \infty$, and $0 < s < \min\{1, n/p\}$. Suppose that $G$ is a John domain in $\mathbb{R}^n$ such that $\dim_A(\partial G) < n - sp$. Then, for every $f \in L^p(G)$,

$$\left( \int_{G} \frac{|f(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \right)^{1/p} \leq C \left\{ \|f\|_{L^p(G)} + \left( \int_{G} \int_{G} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dy \, dx \right)^{1/p} \right\},$$

where a constant $C$ depends on parameters $n$, $s$, $p$, and $G$.

Recall that bounded Lipschitz domains, and bounded domains with the interior cone condition, are John domains. Also, the Koch snowflake $G$ is a John domain with $\dim_A(\partial G) = \log 4/ \log 3$. For bounded Lipschitz domains $G$ in the ‘fat’ case of $sp > 1$, inequality (1.3) holds for every $f \in C_0^\infty(G)$ without the $L^p$-term $\|f\|_{L^p(G)}$ on the right hand side; furthermore, the $L^p$-term cannot

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be omitted if \( sp \leq 1 \), [8]. But in contrast with the ‘fat’ case, if the boundary is ‘thin’, the vanishing boundary values play no role.

In §5, we apply Theorem 1.2 to prove the boundedness of the ‘zero extension operator’ on fractional Sobolev spaces,

\[
E_0 : W^{s,p}(G) \rightarrow W^{s,p}(\mathbb{R}^n), \quad E_0 f(x) = \begin{cases} f(x), & x \in G; \\ 0, & x \in \mathbb{R}^n \setminus G. \end{cases}
\]

Further extension results for fractional Sobolev spaces can be found in [31].

The following theorem is our main result; it is a generalisation of Theorem 1.2. For the definition of Triebel–Lizorkin spaces on domains, we refer to §2.

1.5. **Theorem.** Suppose \( G \) is a bounded or unbounded domain in \( \mathbb{R}^n \), \( n \geq 2 \), with a compact boundary. Let \( 1 < p < \infty \) and \( 0 < s < n/p \). Then the following statements hold:

(A) Assume that \( \dim A(\partial G) < n - sp \). Then, for every \( 1 \leq q < \infty \) and \( f \in F_{pq}^s(G) \),

\[
\left( \int_G \frac{|f(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \right)^{1/p} \lesssim \|f\|_{F_{pq}^s(G)}.
\]

(B) Conversely, if \( \mathbb{R}^n \setminus G \) has zero Lebesgue measure and inequality (1.6) holds for some \( 1 \leq q \leq \infty \) and all \( f \in F_{pq}^s(G) \), then \( \dim A(\partial G) < n - sp \).

For the class of domains whose boundary is compact and whose complement has zero Lebesgue measure, Theorem 1.5 gives further information. First, in this class we obtain a characterization for the validity of Hardy inequality (1.6) in terms of the Aikawa dimension of \( \partial G \). In particular, the validity is seen to be independent of the microscopic parameter \( q \). Moreover, we recover a self-improving property of Hardy inequality (1.6) with respect to parameters \( s \) and \( p \); this extends the results in [16] for the classical Hardy inequality. Finally, since the notion of porosity is related to the Aikawa dimension, see Remark 2.6, we immediately obtain a characterization for porosity, Corollary 1.7; analogous statements, showing the sufficiency of porosity for the boundedness of pointwise multipliers, can be found in [27] and [30, Proposition 3.19].

1.7. **Corollary.** Suppose \( G \) is a domain in \( \mathbb{R}^n \) such that \( \partial G \) is compact and the Lebesgue measure of \( \mathbb{R}^n \setminus G \) is zero. Then the following statements are equivalent:

1. There is \( 0 < \varepsilon < n \) such that, if \( 1 < p < \infty \) and \( 0 < s < \varepsilon/p \), then inequality (1.6) holds for every \( 1 \leq q < \infty \) and every \( f \in F_{pq}^s(G) \).

2. The boundary \( \partial G \) is porous in \( \mathbb{R}^n \).

Explicit boundary conditions for \( f \) are not imposed in statement (A) of Theorem 1.5. This is related to the fact that for a bounded domain \( G \) with \( \dim A(\partial G) < n - sp \),

\[
F_{pq}^s(G) = F_{pq}^s(G) = \tilde{F}_{pq}^s(G).
\]

In particular, \( C_0^\infty(G) \) is dense in \( F_{pq}^s(G) \). These embeddings are available in the literature, some of them implicitly, [6, 10]. The validity of Hardy inequality is related to the second identification
in (1.8) or, more precisely, to a question if the pointwise multiplication \( f \mapsto \chi_G f \) is bounded on \( F^s_{pq}(\mathbb{R}^n) \). As an application of our main result, Theorem 1.5, we obtain a simple proof for certain known multiplier results in [10, §13] for spaces \( F^s_{pq}(\mathbb{R}^n) \), see also §5.

Theorem 1.5 applies in the case of ‘thin’ sets. We would also like to mention some of the known results for ‘fat’ sets. In this case the first identification in (1.8) often fails, and it seems natural to impose zero boundary conditions. For an illustration, let us first focus on the case of \( G = \mathbb{R}^n \setminus S \), where \( S \) is a closed Ahlfors \( d \)-regular set in \( \mathbb{R}^n \) with \( n - 1 < d < n \) and \( sp > n - d \). The \( d \)-regularity condition is given in terms of the Hausdorff measure, that is, for every \( x \in S \) and \( 0 < r < 1 \),

\[
\mathcal{H}^d(B(x, r) \cap S) \simeq r^d.
\]

For such sets, we have \( \dim_A(S) = d \) [17, Lemma 2.1]. In our previous work [14] we have shown that inequality (1.6) holds for functions \( f \) in the subspace

\[
\{ f \in F^s_{pq}(\mathbb{R}^n) : \text{Tr}_{\partial G} f = 0 \},
\]

where \( \text{Tr}_{\partial G} f \) is a trace of \( f \) on \( \partial G = S \).

Although the last result applies in a somewhat general setting, it is probably not yet optimal. Indeed, it seems to be natural to replace the \( d \)-regularity condition with uniform fatness, which is typically used in connection with classical Hardy inequalities. Moreover, it is known that a certain fractional Hardy-type inequality holds, if \( G \) is a bounded domain in \( \mathbb{R}^n \) for which \( \mathbb{R}^n \setminus G \) is \( (s, p) \) locally uniformly fat, [9, Theorem 1.3]. Recall that a set \( E \subset \mathbb{R}^n \) is \( (s, p) \) locally uniformly fat, if there are positive constants \( r_0 \) and \( \lambda \) such that, for every \( x \in E \) and \( 0 < r < r_0 \),

\[
R_{s,p}(B(x, r) \cap E) \geq \lambda r^{n-sp}.
\]

For example, \( R_{s,p}(B(x, r)) \simeq r^{n-sp} \) if \( x \in \mathbb{R}^n \) and \( r > 0 \). Here \( R_{s,p}(\cdot) \) is the \( (s, p) \) outer Riesz capacity of a set in \( \mathbb{R}^n \), we refer to [20].

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2. Notation and preliminaries

Basic notation. Throughout the paper, a cube means a closed cube \( Q = Q(x, r) \) in \( \mathbb{R}^n \) centred at \( x \in \mathbb{R}^n \) with side length \( \ell(Q) = 2r > 0 \), and with sides parallel to the coordinate axes. For a cube \( Q \) and for \( \rho > 0 \), we write \( \rho Q \) for the dilated cube with side length \( \rho \ell(Q) \). By \( \chi_E \) we denote the characteristic function of a set \( E \), the boundary of \( E \) is written as \( \partial E \), and \( |E| \) is the Lebesgue \( n \)-measure of a measurable set \( E \) in \( \mathbb{R}^n \). The integral average of \( f \in L^1_{loc}(\mathbb{R}^n) \) over a bounded set \( E \) with positive measure is written as \( f_E \), that is,

\[
f_E = \frac{1}{|E|} \int_E f \, dx.
\]

Various constants whose value may change even within a given line are denoted by \( C \).
The family of closed dyadic cubes is denoted by $D$. Let also $D_j$ be the family of those dyadic cubes whose side length is $2^{-j}$, $j \in \mathbb{Z}$. For a proper open set $G$ we fix its Whitney decomposition $W(G) \subset D$, and write $W_j(G) = D_j \cap W(G)$. For a Whitney cube $Q \in W(G)$ we write $Q^* = \frac{2}{3}Q$. Such dilated cubes have a bounded overlap, and they satisfy

$$\frac{3}{4} \text{diam}(Q) \leq \text{dist}(x, \partial G) \leq 6 \text{diam}(Q),$$

whenever $x \in Q^*$. For other properties of Whitney cubes we refer to [24, VI.1].

**Aikawa dimension and porous sets.** We begin with the definition of the Aikawa dimension, [17].

2.2. **Definition.** If $E \subset \mathbb{R}^n$ is a closed set with an empty interior, then define $A(E)$ to be the set of all $0 < s \leq n$ with the following property. There is a constant $C > 0$, such that

$$\int_{B(x,r)} \text{dist}(y, E)^{s-n} \, dy \leq Cr^s$$

for every $x \in E$ and all $0 < r < \infty$. The Aikawa dimension of $E$ is $\dim A(E) = \inf A(E)$.

2.4. **Remark.** Let $E$ be a compact set in $\mathbb{R}^n$ with an empty interior. Suppose that for $0 < s \leq n$ there are constants $\epsilon, C > 0$ such that (2.3) holds for all $x \in E$ and $0 < r < \epsilon$. By using compactness arguments it is straightforward to verify that $s \in A(E)$.

2.5. **Definition.** A set $S \subset \mathbb{R}^n$ is **porous** (or $\kappa$-**porous**) if for some $\kappa \geq 1$ the following statement is true: For every cube $Q(x, r)$ with $x \in \mathbb{R}^n$ and $0 < r \leq 1$ there is $y \in Q(x, r)$ such that $Q(y, r/\kappa) \cap S = \emptyset$.

2.6. **Remark.** A set $S$ is porous in $\mathbb{R}^n$ if, and only if, its so called Assouad dimension is strictly less than $n$, [21]. It has been recently shown in [18] that the dimensions of Assouad and Aikawa of a given set $S \subset \mathbb{R}^n$ coincide.

Let us then recall a reverse Hölder type inequality involving porous sets, Theorem 2.8. Similar techniques are applied in [10, §13]; we also refer to [4] and [15]. For a set $S$ in $\mathbb{R}^n$ and a positive constant $\gamma > 0$ we denote

$$C_{S, \gamma} = \{Q \in D : \gamma^{-1} \text{dist}(x_Q, S) \leq \ell(Q) \leq 1\}.$$

This is the family of dyadic cubes that are relatively close to the set.

2.8. **Theorem.** Suppose that $S \subset \mathbb{R}^n$ is porous. Let $p, q \in (1, \infty)$ and $(a_Q)_{Q \in C_{S, \gamma}}$ be a sequence of non-negative scalars. Then

$$\left\| \sum_{Q \in C_{S, \gamma}} \chi_Q a_Q \right\|_p \leq C \left\| \left( \sum_{Q \in C_{S, \gamma}} (\chi_Q a_Q)^q \right)^{1/q} \right\|_p.$$

Here a positive constant $C$ depends on $n, p, \gamma$ and the set $S$.

The proof of this theorem is based on maximal-function techniques, we refer to [13].
Function spaces. There are several equivalent characterizations for the fractional Sobolev spaces and their natural extensions, Triebel–Lizorkin spaces, see e.g. [1], [3], [25], and [28]. In this paper, we mostly use the definition based on the local polynomial approximation approach.

Let \( f \in L^u_{\text{loc}}(\mathbb{R}^n) \), \( 1 \leq u \leq \infty \), and \( k \in \mathbb{N}_0 \). Following [5], we define the normalized local best approximation of \( f \) on a cube \( Q \) in \( \mathbb{R}^n \) by

\[
E_k(f, Q)_{L^u(\mathbb{R}^n)} := \inf_{P \in \mathcal{P}_{k-1}} \left( \frac{\int_Q |f(x) - P(x)|^u \, dx}{\int_Q |f(x)|^u \, dx} \right)^{1/u}.
\]

Here and below \( \mathcal{P}_m \), \( m \geq 0 \), denotes the space of polynomials in \( \mathbb{R}^n \) of degree at most \( m \). We also denote \( \mathcal{P}_{-1} = \{0\} \). Let \( Q_1 \subset Q_2 \) be two cubes in \( \mathbb{R}^n \). Then

\[
E_k(f, Q_1)_{L^u(\mathbb{R}^n)} \leq E_k(f, Q_2)_{L^u(\mathbb{R}^n)}.
\]

This property is referred as the monotonicity of local approximation.

The following definition of Triebel–Lizorkin spaces with positive smoothness can be found in [25]. Let \( s > 0 \), \( 1 \leq p < \infty \), \( 1 \leq q \leq \infty \), and \( k \) be an integer such that \( s < k \). For \( f \in L^u_{\text{loc}}(\mathbb{R}^n) \), \( 1 \leq u \leq \min\{p, q\} \), set for all \( x \in \mathbb{R}^n \),

\[
F(x) := \left( \int_0^1 \left( \frac{E_k(f, Q(x, t))_{L^u(\mathbb{R}^n)}}{t^s} \right)^{n/u} dt \right)^{1/q}, \quad \text{if } q < \infty,
\]

and

\[
F(x) := \sup\{t^{-s}E_k(f, Q(x, t))_{L^u(\mathbb{R}^n)} : 0 < t \leq 1\}, \quad \text{if } q = \infty.
\]

A function \( f \) belongs to a Triebel–Lizorkin space \( F^{s}_{pq}(\mathbb{R}^n) \) if \( f \) and \( F \) are both in \( L^p(\mathbb{R}^n) \), and the Triebel–Lizorkin norms

\[
\|f\|_{F^{s}_{pq}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|F\|_{L^p(\mathbb{R}^n)}
\]

are equivalent if \( s < k \) and \( 1 \leq u \leq \min\{p, q\} \). In particular, if \( q \geq p \), then we can set \( u = p \).

Function spaces on domains. Let us recall the definition of the fractional order Sobolev spaces on a domain \( G \subset \mathbb{R}^n \). Let \( W^{s,p}(G) \), for \( 0 < s < 1 \) and \( 1 < p < \infty \), be the space of functions \( f \) in \( L^p(G) \) with \( \|f\|_{W^{s,p}(G)} := \|f\|_{L^p(G)} + \|f\|_{W^{s,p}(G)} < \infty \), where

\[
|f|_{W^{s,p}(G)} := \left( \int_G \int_G \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dy \, dx \right)^{1/p}.
\]

The Triebel–Lizorkin space \( F^{s}_{pq}(\mathbb{R}^n) \) coincides with the Sobolev space \( W^{s,p}(\mathbb{R}^n) \), [28, pp. 6–7]. Let us also recall some notation which is common in the literature on function spaces on domains, [28, 29]. Let \( G \) be an open set in \( \mathbb{R}^n \), \( 1 \leq p < \infty \), \( 1 \leq q \leq \infty \), and \( s > 0 \). Then

\[
F^{s}_{pq}(G) = \{f \in L^p(G) : \text{there is a } g \in F^{s}_{pq}(\mathbb{R}^n) \text{ with } g|_G = f\}
\]

\[
\|f\|_{F^{s}_{pq}(G)} = \inf \|g\|_{F^{s}_{pq}(\mathbb{R}^n)},
\]
where the infimum is taken over all \( g \in F_{pq}^s(\mathbb{R}^n) \) such that \( g|_G = f \) pointwise a.e. As usually, we also denote
\[
(2.11) \quad \widehat{F}_{pq}^s(G) = \{ f \in L^p(G) : \text{there is a } g \in F_{pq}^s(\mathbb{R}^n) \text{ with } g|_G = f \text{ and } \text{supp } g \subseteq \overline{G} \}
\]
\[\|f\|_{\widehat{F}_{pq}^s(G)} = \inf \|g\|_{F_{pq}^s(\mathbb{R}^n)},\]
where the infimum is taken over all \( g \) admitted in (2.11).

Finally, \( F_{pq}^s(G) \) is a completion of \( C_0^\infty(G) \) in \( F_{pq}^s(G) \).

3. Proof of Theorem 1.2

We apply the ideas in [12] to prove a Hardy inequality for John domains.

3.1. Definition. A bounded domain \( G \) in \( \mathbb{R}^n \), \( n \geq 2 \), is a John domain, if there exist a point \( x_0 \in G \) and a constant \( \beta_G \geq 1 \) such that every point \( x \) in \( G \) can be joined to \( x_0 \) by a rectifiable curve \( \gamma : [0, \ell] \to G \) parametrized by its arc length for which \( \gamma(0) = x \), \( \gamma(\ell) = x_0 \), \( \ell \leq \beta_G \text{ diam}(G) \), and for all \( t \in [0, \ell] \),
\[
\text{dist}(\gamma(t), \partial G) \geq t/\beta_G.
\]
The point \( x_0 \) a John center of \( G \), and the smallest constant \( \beta_G \geq 1 \) is the John constant of \( G \).

Chain decomposition. Suppose that \( G \) is a John domain and \( Q \in \mathcal{W}(G) \). Below, we obtain a chain of cubes
\[
\mathcal{C}(Q) = \{ Q_0, \ldots, Q_m \} \subset \mathcal{W}(G),
\]
joining a fixed cube \( Q_0 \) to the given cube \( Q = Q_m \), such that \( Q_i \neq Q_j \) whenever \( i \neq j \), and there exists a positive constant \( C = C(n) \) for which
\[
(3.2) \quad |Q_j^* \cap Q_j^{*-1}| \geq C \max(\{|Q_j^*|, |Q_j^{*-1}|\}), \quad j \in \{1, \ldots, m\}.
\]
A family \( \{\mathcal{C}(Q) : Q \in \mathcal{W}(G)\} \) is called a chain decomposition of \( G \), and the shadow of a Whitney cube \( R \in \mathcal{W}(G) \) is
\[
\mathcal{S}(R) = \{ Q \in \mathcal{W}(G) : R \in \mathcal{C}(Q) \}.
\]
The following proposition provides a chain decomposition for the appropriate John domains.

3.3. Proposition. (Chain decomposition) Suppose that \( n \geq 2 \), \( 1 < p < \infty \), and \( 0 < s < \frac{n}{p} \).

Let \( G \) be a John domain in \( \mathbb{R}^n \), with \( \dim_A(\partial G) < n - \text{sp} \). Then there exist constants \( \sigma, \tau \in \mathbb{N} \) and a chain decomposition \( \{\mathcal{C}(Q) : Q \in \mathcal{W}(G)\} \) of \( G \) satisfying the following conditions:
(1) \( \ell(Q) \leq 2^s \ell(R) \) for each \( Q \in \mathcal{W}(G) \) and \( R \in \mathcal{C}(Q) \);
(2) \( \ell[R \in \mathcal{W}_j(G) : R \in \mathcal{C}(Q)] \leq 2^s \) for each \( Q \in \mathcal{W}(G) \) and \( j \in \mathbb{Z} \);
(3) The following inequality holds,
\[
\sup_{j \in \mathbb{Z}} \sup_{R \in \mathcal{W}_j(G)} \frac{1}{|R|^{1-sp/n}} \sum_{k=j-\tau}^{\infty} \sum_{Q \in \mathcal{W}_k(G)} |Q|^{1-sp/n}(\tau + 1 + k-j)^p < \sigma.
\]

The constants \( \sigma \) and \( \tau \) depend only on \( n \), \( p \), \( s \), \( \partial G \), and the John constant \( \beta_G \).
Proof. For the construction of chain decomposition and the verification of conditions (1) and (2), we refer to [12]. Therein one may also find a proof of the following useful fact. There is a constant $C = C(n, \beta_G) > 0$ such that, for each $R \in \mathcal{W}(G)$,

$$\bigcup_{Q \in \mathcal{S}(R)} Q \subset B(y_R, C\ell(R)),$$

where $y_R \in \partial G$ is any point satisfying $|x_R - y_R| = \text{dist}(x_R, \partial G)$.

It remains to check condition (3). Let us fix $\epsilon > 0$, depending on the allowed parameters, such that $n - sp - \epsilon \in \mathcal{A}(\partial G)$. This can be done since, by the assumption, $\text{dim}_{\mathcal{A}}(\partial G) < n - sp$. Fix $j \in \mathbb{Z}$ and $R \in \mathcal{W}_j(G)$. Then, if $k \geq j - \tau$ and $Q \in \mathcal{W}_k(G)$,

$$\sum_{Q \in \mathcal{S}(R)} \ell(Q)^{n-sp} \leq C_2 \ell(R)^{n-sp-\epsilon},$$

where $C = C(\epsilon, p) > 0$. By inequality (3.5),

$$\sum_{k=j-\tau}^{\infty} \sum_{Q \in \mathcal{W}_k(G), Q \in \mathcal{S}(R)} \left(\frac{\ell(Q)}{\ell(R)}\right)^{n-sp} (\tau + 1 + k - j)^p \leq C_2 \ell(R)^{-\epsilon} \sum_{Q \in \mathcal{S}(R)} \ell(Q)^{n-sp-\epsilon}.$$

On the other hand, by (2.1), (3.4), and (2.3), we may conclude that

$$\sum_{Q \in \mathcal{S}(R)} \ell(Q)^{n-sp-\epsilon} \leq C \int_{B(y_R, C\ell(R))} \text{dist}(x, \partial G)^{n-sp-\epsilon} \ dx \leq C \ell(R)^{n-sp-\epsilon},$$

and condition (3) follows. \qed

Hardy inequality for John domains. We are ready to verify one of our main results.

Proof of Theorem 1.2. Let $\{C(Q)\}$ be a chain decomposition given by Proposition 3.3, with a fixed cube $Q_0 \in \mathcal{W}(G)$. Without loss of generality, we may assume the normalisation $f_{Q_0} = 0$. Indeed, if necessary, we replace $f$ with $f - f_{Q_0}1_G$ in the proof below, and use the bound on the Aikawa dimension to control the error term.

Let us estimate

$$\int_G \frac{|f(x)|^p}{\text{dist}(x, \partial G)^{sp}} \ dx \leq \sum_{Q \in \mathcal{W}(G)} \ell(Q)^{n-sp} \int_Q |f(x) - f_{Q^*}|^p \ dx + \sum_{Q \in \mathcal{W}(G)} \ell(Q)^{n-sp} |f_{Q^*}|^p.$$
By Hölder’s inequality, and the facts \(|x−y| \leq \ell(Q)\) for \((x, y) \in Q^* \times Q^*\) and \(\sum_{Q \in \mathcal{W}(G)} \chi_{Q} \preceq \chi_G\), the first term on the right hand side is bounded by

\[
\sum_{Q \in \mathcal{W}(G)} \ell(Q)^{n-sp} \int_{Q^*} \int_{Q^*} |f(x) - f(y)|^p \, dy \, dx
\]

\[
\leq \sum_{Q \in \mathcal{W}(G)} \int_{Q^*} \int_{Q^*} \frac{|f(x) - f(y)|^p}{|x−y|^{n+sp}} \, dy \, dx \leq \int_{G} \int_{G} \frac{|f(x) - f(y)|^p}{|x−y|^{n+sp}} \, dy \, dx.
\]

(3.7)

In order to control the remaining term, let us first prove some auxiliary estimates. For a Whitney cube \(Q \in \mathcal{W}(G)\), consider its chain \(C(Q)\). By inequality (3.2), if \(j \in \{1, \ldots, m\}\),

\[
|f_{Q_j^*} - f_{Q_j^*_{j-1}}| = \int_{Q_j \cap Q_{j-1}^*} |f_{Q_j^*} - f_{Q_{j-1}^*}| \, dx \leq \sum_{i=j-1}^j \int_{Q_{i}^*} |f(x) - f_{Q_i^*}| \, dx
\]

By normalisation \(f_{Q_0^*} = 0\) and the property that cubes in the chain \(C(Q)\) are distinct,

\[
|f_{Q_j^*}| = |f_{Q_m^*} - f_{Q_0^*}|
\]

\[
\leq \sum_{j=1}^{m} \sum_{i=j-1}^j \int_{Q_i^*} |f(x) - f_{Q_i^*}| \, dx \leq \sum_{R \in \mathcal{C}(Q)} \int_{R^*} |f(x) - f_{R^*}| \, dx.
\]

(3.8)

We are ready to estimate the second term in the right hand side of (3.6). First we will use inequality (3.8) and property (1) of the chain \(C(Q)\). Then we will write 1 = \((\tau + 1 + k - j)^{-1}(\tau + 1 + k - j)\) and apply Hölder’s inequality,

\[
\sum_{Q \in \mathcal{W}(G)} \ell(Q)^{n-sp} |f_{Q_j^*}|^p
\]

\[
\leq \sum_{k=\infty}^{\infty} \sum_{Q \in \mathcal{W}_k(G)} \ell(Q)^{n-sp} \left\{ \sum_{j=\infty}^{k+\tau} \sum_{R \in \mathcal{W}_j(G)} \int_{R^*} |f - f_{R^*}| \right\}^p
\]

\[
\leq \sum_{k=\infty}^{\infty} \sum_{Q \in \mathcal{W}_k(G)} \ell(Q)^{n-sp} \sum_{j=\infty}^{k+\tau} (\tau + 1 + k - j)^p \left\{ \sum_{R \in \mathcal{W}_j(G)} \int_{R^*} |f - f_{R^*}| \right\}^p.
\]

(3.9)

By property (2) of chain \(C(Q)\) and Hölder’s inequality, for any \(j \in \mathbb{Z}\),

\[
\sum_{R \in \mathcal{W}_j(G)} \int_{R^*} |f - f_{R^*}| \leq \left\{ \sum_{R \in \mathcal{W}_j(G)} \int_{R^*} |f - f_{R^*}|^p \right\}^{1/p}.
\]
We substitute the last inequality to (3.9). Next, we change the order of summation, and apply an equivalence for Whitney cubes: \( R \in \mathcal{C}(Q) \) if and only if \( Q \in \mathcal{S}(R) \),

\[
\sum_{Q \in \mathcal{W}(G)} \ell(Q)^{n-sp} |f_{Q^*}|^p \lesssim \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{W}_k(G)} \ell(Q)^{n-sp} \left( \sum_{j=-\infty}^{k+\tau} (\tau + 1 + k - j)^p \right) \sum_{R \in \mathcal{W}_j(G)} \int_{R^*} |f - f_{R^*}|^p
\]

\[
= \sum_{j=-\infty}^{\infty} \sum_{R \in \mathcal{W}_j(G)} \ell(R)^{n-sp} \int_{R^*} |f - f_{R^*}|^p \cdot A_{j,R}.
\]

The constants

\[
A_{j,R} = \sum_{k=j-\tau}^{\infty} \sum_{Q \in \mathcal{W}_k(G)} \left( \frac{\ell(Q)}{\ell(R)} \right)^{n-sp} (\tau + 1 + k - j)^p,
\]

are uniformly bounded in \( j \) and \( R \) by condition (3) in Proposition 3.3. Applying inequalities (3.7) finishes the proof. \( \square \)

### 3.10. Remark.

The main line of the proof above is similar to the proof of the following well known Hardy inequality for series. If \( 1 \leq p \leq \infty \) and \( a_j \geq 0, j = 0, 1, \ldots \), then

\[
\sum_{j=0}^{\infty} \left( \sum_{i=0}^{j} a_i \right)^p \leq c \sum_{j=0}^{\infty} 2^{\sigma j} a_j^p \quad \text{for } \sigma < 0.
\]

See e.g. [19]

### 4. Proof of Theorem 1.5

In this section, we prove our main result.

**Chain decomposition.** Suppose \( G \) is a bounded or unbounded domain in \( \mathbb{R}^n \), \( n \geq 2 \), with a compact boundary satisfying \( \dim_A(\partial G) < n - sp \), where \( 1 < p < \infty \) and \( 0 < s < n/p \). Let \( \gamma = 7 \sqrt{n} \), and recall definition (2.7) of \( \mathcal{C} := \mathcal{C}_{\partial G, \gamma} \).

By scaling and translating \( G \), if necessary, we may assume that there is a dyadic cube \( Q_0 \in \mathcal{D}_0 \) such that \( Q \subset Q_0 \) if

\[
Q \in \mathcal{W}^{G-\text{small}} := \{ Q \in \mathcal{W}(G) : \ell(Q) \leq \text{diam}(\partial G) \}.
\]

In particular, this implies a relation \( \partial G \subset Q_0 \). For a small Whitney cube \( Q \in \mathcal{W}^{G-\text{small}} \), we let

\[
\mathcal{C}(Q) = (Q_0, \ldots, Q_m) \subset \mathcal{C}
\]

be the unique chain of dyadic cubes such that \( Q_m = Q \) and \( Q_{j-1} \) is the dyadic parent of \( Q_j \), \( j = 1, \ldots, m \). In particular,

\[
Q \in \mathcal{D}_m.
\]

The shadow of a cube \( R \in \mathcal{C} \) is \( \mathcal{S}(R) = \{ Q \in \mathcal{W}^{G-\text{small}} : R \in \mathcal{C}(Q) \} \). Observe that \( \cup_{Q \in \mathcal{S}(R)} Q \subset R \) for all \( R \in \mathcal{C} \).
Projection operators. We also need certain projection operators. For a cube $Q$ in $\mathbb{R}^n$ and $k \in \mathbb{N}_0$, we let $P_{k, Q}$ be a projection from $L^1(Q)$ to $\mathcal{P}_{k-1}$ such that for every $1 \leq u \leq \infty$ and every $f \in L^u(Q)$,

\begin{equation}
\left( \int_Q |f(x) - P_{k, Q} f(x)|^u \, dx \right)^{1/u} \leq C \mathcal{E}_k(f, Q)_{L^u(\mathbb{R}^n)},
\end{equation}

where a constant $C$ depends on $n$ and $k$. For the construction of these projection operators, we refer to [22, Proposition 3.4] and [7].

4.2. Proposition. Suppose that $k \in \mathbb{N}_0$ and $Q \in \mathcal{W}^{G, \text{small}}$. Then, for every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\|P_{k, Q} f - P_{k, Q_0} f\|_{L^\infty(Q)} \leq C \sum_{R \in \mathcal{C}(Q)} \mathcal{E}_k(f, R)_{L^1(\mathbb{R}^n)},$$

where a constant $C$ depends on $n$ and $k$.

Proof. Recall that $\mathcal{C}(Q) = (Q_0, \ldots, Q_m)$, with $Q_m = Q \in \mathcal{W}^{G, \text{small}}$. We claim that, for every $j \in \{1, \ldots, m\}$,

\begin{equation}
\|P_{k, Q_j} f - P_{k, Q_{j-1}} f\|_{L^\infty(Q_j)} \leq C \mathcal{E}_k(f, Q_{j-1})_{L^1(\mathbb{R}^n)},
\end{equation}

where $C$ depends on $n$ and $k$. Indeed, let us first recall that $Q_j \subset Q_{j-1}$, and $2\ell(Q_j) = \ell(Q_{j-1})$. By a reverse Hölder inequality for polynomials [7, §3], and inequalities (4.1) and (2.9),

$$\|Q_j\| P_{k, Q_j} f - P_{k, Q_{j-1}} f\|_{L^\infty(Q_j)} \leq \|P_{k, Q_j} f - P_{k, Q_{j-1}} f\|_{L^1(Q_j)} \leq \|f - P_{k, Q_j} f\|_{L^1(Q_j)} + \|f - P_{k, Q_{j-1}} f\|_{L^1(Q_{j-1})} \leq \|Q_{j-1}\| \mathcal{E}_k(f, Q_{j-1})_{L^1(\mathbb{R}^n)}.$$

Thus, inequality (4.3) follows.

Since $Q_j \subset Q$, for every $j = 0, \ldots, m$, by inequality (4.3),

$$\|P_{k, Q} f - P_{k, Q_0} f\|_{L^\infty(Q)} \leq \sum_{j=0}^m \|P_{k, Q_j} f - P_{k, Q_{j-1}} f\|_{L^\infty(Q_j)} \leq C \sum_{j=0}^m \mathcal{E}_k(f, Q_j)_{L^1(\mathbb{R}^n)}.$$

This concludes the proof of the proposition.

Proof of Statement (A). We rewrite statement (A) as the following proposition.

4.4. Proposition. Let $n \geq 2$, $1 < p < \infty$, and $0 < s < n/p$. Suppose $G$ is a bounded or unbounded domain in $\mathbb{R}^n$, with a compact boundary satisfying $\dim_A(\partial G) < n - sp$. Then

\begin{equation}
\left( \int_G \frac{|f(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \right)^{1/p} \leq \|f\|_{F_{pq}^s(G)}
\end{equation}

for all $f \in F_{pq}^s(G)$ and $1 \leq q < \infty$. The implied constant in (4.5) depends on $n$, $s$, $p$, and $\partial G$.

Proof. The embeddings $F_{pq}^s(G) \subset F_{pq}^r(G)$ are trivially bounded if $q' \leq q$. Hence, it suffices to consider the case of $p \leq q < \infty$. This allows us to set $u = p$ and $k = [s] + 1 > s$ in the definition of Triebel–Lizorkin space $F_{pq}^s(\mathbb{R}^n)$. 
Let \( f \in F_{pq}^s(G) \). Then, by definition, it suffices to show that
\[
\int_G \frac{|g(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \lesssim \|g\|_{F_{pq}^s(\mathbb{R}^n)}^p,
\]
where \( g \in F_{pq}^s(\mathbb{R}^n) \) is any extension of \( f \), that is, \( g|_G = f \) almost everywhere. By first inequality in (2.1), we can bound the left hand side of (4.6) by a constant multiple of
\[
\sum_{Q \in W(G)} \ell(Q)^{n-sp} \int_Q |g(x) - P_{k,Q}g(x)|^p + |P_{k,Q}g(x)|^p \, dx.
\]
By inequalities (4.1) and (2.9),
\[
\sum_{Q \in W(G)} \ell(Q)^{n-sp} \int_Q |g(x) - P_{k,Q}g(x)|^p \, dx \lesssim \|g\|_{L^p(\mathbb{R}^n)}^p + \sum_{Q \in W(G)} \ell(Q)^{n-sp} \mathcal{E}_k(g, Q)^p_{L^p(\mathbb{R}^n)}
\]
\[
\lesssim \|g\|_{L^p(\mathbb{R}^n)}^p + \left\{ \sum_{Q \in W(G)} \ell(Q)^{-sp} \mathcal{E}_k(g, Q)^q_{L^p(\mathbb{R}^n)} \right\}^{1/q} \lesssim \|g\|_{F_{pq}^s(\mathbb{R}^n)}^p.
\]
In the penultimate step, we used the fact that every point in \( \mathbb{R}^n \) belongs to at most \( C = C(n) \) Whitney cubes. And, the last step follows from monotonicity (2.9) of the local approximation.

Then we estimate the remaining term in (4.7), i.e.,
\[
\sum_{Q \in W(G)} \ell(Q)^{n-sp} \int_Q |P_{k,Q}g(x)|^p \, dx.
\]
This series, when restricted to big cubes \( Q \in W(G) \) satisfying \( \ell(Q) > \text{diam}(\partial G) \), is bounded by \( C \|g\|_{L^p(\mathbb{R}^n)}^p \). Indeed, this is an easy consequence of inequality (4.1).

Let us estimate the remaining part of series (4.8), where the summation is restricted to small cubes \( W^{G\text{-small}} \). In order to do this, we write \( P_{k,Q}g = P_{k,Q_0}g + (P_{k,Q}g - P_{k,Q_0}g) \), and estimate the resulting two series, denoted by \( S_1 \) and \( S_2 \). First, by a reverse Hölder inequality for polynomials and the assumption \( \dim_A(\partial G) < n - sp \),
\[
S_1 := \sum_{Q \in W^{G\text{-small}}} \ell(Q)^{n-sp} \int_Q |P_{k,Q_0}g(x)|^p \, dx \leq \|P_{k,Q_0}g\|_{L^\infty(Q_0)}^p \sum_{Q \in W^{G\text{-small}}} \ell(Q)^{n-sp}
\]
\[
\leq \|P_{k,Q_0}g\|_{L^p(Q_0)}^p \int_{Q_0} \text{dist}(x, \partial G)^{-sp} \, dx \leq \|g\|_{L^p(\mathbb{R}^n)}^p.
\]
By Proposition 4.2,

\[ S_2 := \sum_{Q \in W^\text{G-small}} \ell(Q)^{n-\text{sp}} \int_Q |P_{k,Q}g(x) - P_{k,Q_0}g(x)|^p \, dx \]

\[ \leq \sum_{m=0}^{\infty} \sum_{Q \in W^\text{G-small}} \ell(Q)^{n-\text{sp}} \left\{ \sum_{j=0}^{m} (1 + m - j)^{-1} (1 + m - j) \int_{R_j^Q} |g(x) - P_{k,R_j^Q}g(x)| \, dx \right\}^p \]

where we use notation \( R_j^Q \) for the unique cube \( R \in C(Q) \cap D_j \). Next, proceeding as in the proof of Theorem 1.2, we obtain

\[ S_2 \leq \sum_{j=0}^{\infty} \sum_{R \in C \cap D_j} \ell(R)^{n-\text{sp}} \int_R |g(x) - P_{k,R}g(x)|^p \, dx \cdot A_{j,R} \]

\[ \leq \sum_{R \in C} \ell(R)^{n-\text{sp}} \int_R |g(x) - P_{k,R}g(x)|^p \, dx . \]

Here the uniform boundedness in \( j \) and \( R \) of the constants \( A_{j,R} \) can be easily shown as in the proof of Proposition 3.3.

By Inequality (4.1), Remark 2.6, and a reverse Hölder inequality in Theorem 2.8, we can bound term \( S_2 \) by a constant multiple of

\[ \left\| \left\{ \sum_{R \in C} \chi_R \ell(R)^{-\text{sp}} \mathcal{E}_k(g, R)^p_{\text{L}^p(R^n)} \right\}^{1/p} \right\|_p \leq \left\| \left\{ \sum_{R \in C} \chi_R \ell(R)^{-sq} \mathcal{E}_k(g, R)^q_{\text{L}^q(R^n)} \right\}^{1/q} \right\|_p , \]

where \( C = C_{\partial G, \gamma} \). The observation that the last term is dominated by the required upper bound \( C \|g\|_{F^p_{pq}(R^n)} \) finishes the proof. \( \square \)

**Proof of Statement (B).** This statement is covered by the following proposition.

4.9. **Proposition.** Let \( n \geq 2, 1 \leq p < \infty \), and \( 0 < s < n/p \). Suppose that \( G \) is domain in \( R^n \), with a compact boundary, such that \( R^n \setminus G \) has zero Lebesgue measure, and inequality

\[ \left( \int_G \frac{|f(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \right)^{1/p} \leq \|f\|_{F^p_{pq}(G)} \]

holds for some \( 1 \leq q \leq \infty \) and for all \( f \in F^s_{pq}(G) \). Then \( \dim_A(\partial G) < n - sp \).

**Proof.** We rely on the following homogeneity property, [29, Corollary 5.16]. Namely,

\[ \|f(r \cdot)\|_{F^s_{pq}(R^n)} \simeq r^{s-n/p} \|f\|_{F^s_{pq}(R^n)} \]

for every \( 0 < r \leq 1 \) and every \( f \in F^s_{pq}(R^n) \) supported in \( B(0, r) = \{ x \in R^n : |x| < r \} \).

Let us consider a point \( x \in \partial G \) and a radius \( 0 < r \leq 1 \). Without loss of generality, we may assume that \( x = 0 \). Fix a function \( \varphi \in C^\infty(\overline{R^n}) \), supported in \( B(0, 1) \), and satisfying \( \varphi(x) = 1 \) if \( x \in B(0, 1/2) \). Denote \( f(y) = \varphi(y/r) \) for \( y \in R^n \). Since the measure of \( R^n \setminus G \) is zero,

\[ \left( \int_{B(x,r/2)} \text{dist}(y, \partial G)^{-sp} \, dy \right) \leq \int_G \frac{|f(y)|^p}{\text{dist}(y, \partial G)^{sp}} \, dy \leq \|f\|_{F^p_{pq}(G)} \]

(4.10)
Hence, by the self-improving properties of reverse Hölder inequalities, \( \| f \|_{p,q} \simeq r^{n-sp} \| f(r) \|_{p,q}^p \simeq r^{n-sp} . \)

By inequality (4.10) and Remark 2.4, we have \( n - sp \in \mathcal{A}(\partial G) \). To show that the Aikawa dimension is, indeed, strictly less than \( n - sp \), we proceed as in the proof of [16, Lemma 2.4]. Since \( n - sp \in \mathcal{A}(\partial G) \) it is straightforward to verify that, for every \( x \in \mathbb{R}^n \) and \( r > 0 \),

\[
\left( \int_{B(x,r)} \text{dist}(y, \partial G)^{-sp} \, dy \right)^{1/p} \leq \int_{B(x,r)} \text{dist}(y, \partial G)^{-s} \, dy .
\]

Hence, by the self-improving properties of reverse Hölder inequalities, [11, Lemma 3], we find that \( n - sp - \delta \in \mathcal{A}(\partial G) \) for some \( \delta > 0 \), and the claim follows.

\[\square\]

5. Applications

In this section, we study two problems closely related to Hardy-inequalities: the boundedness of the zero extension operator and the boundedness of pointwise multiplier operators. First, let us consider the zero extension operator \( E_0 : W^{s,p}(G) \to W^{s,p}(\mathbb{R}^n) \), recall definition (1.4).

5.1. Lemma. Suppose that \( G \) is a proper domain in \( \mathbb{R}^n \). Let \( 0 < s < 1 \) and \( 1 < p < \infty \). Then the zero extension \( E_0 f \) of any \( f \in W^{s,p}(G) \) satisfies the following inequality

\[
\| E_0 f \|_{W^{s,p}(\mathbb{R}^n)} \leq \| f \|_{W^{s,p}(G)} + \left( \int_G \frac{|f(x)|^p}{\text{dist}(x, \partial G)^{sp}} \, dx \right)^{1/p} ,
\]

where the implied constant depends on \( n, s, \) and \( p \).

Proof. By definition (2.10), \( \| E_0 f \|_{L^p(\mathbb{R}^n)} = \| f \|_{L^p(\mathbb{R}^n)} \leq \| f \|_{W^{s,p}(G)} \). Next, let us write

\[
\| E_0 f \|_{W^{s,p}(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|E_0 f(x) - E_0 f(y)|^p}{|x - y|^{n+sp}} \, dy \, dx
\]

\[
\leq |f|^p_{W^{s,p}(G)} + 2 \int_G |f(x)|^p \int_{\mathbb{R}^n \setminus G} \frac{1}{|x - y|^{n+sp}} \, dy \, dx .
\]

It remains to apply the following estimates, which are valid for \( x \in G \),

\[
\int_{\mathbb{R}^n \setminus G} \frac{1}{|x - y|^{n+sp}} \, dy
\]

\[
\leq \int_{\mathbb{R}^n \setminus B(x, \text{dist}(x, \partial G))} \frac{1}{|x - y|^{n+sp}} \, dy \leq \int_{\text{dist}(x, \partial G)} r^{-1-sp} \, dr \leq \text{dist}(x, \partial G)^{-sp} .
\]

This completes the proof of the lemma. \[\square\]

The following theorem is a consequence of Lemma 5.1 and Theorem 1.2.

5.2. Theorem. Suppose that \( n \geq 2, 1 < p < \infty, \) and \( 0 < s < \min\{1, n/p\} \). Let \( G \) be a John domain in \( \mathbb{R}^n \) such that \( \dim_{\mathcal{A}}(\partial G) < n - sp \). Then, for every \( f \in W^{s,p}(G) \),

\[
\| E_0 f \|_{W^{s,p}(\mathbb{R}^n)} \leq C \| f \|_{W^{s,p}(G)} ,
\]

where a constant \( C \) depends on \( n, s, p, \) and \( G \).
The assumption that $G$ is a John domain can be relaxed, e.g., by Theorem 1.5 and extension results in [31]. However, the proof of Theorem 5.2 under the John assumption has the advantage of being rather simple.

Let us now study the boundedness of pointwise multiplier operators. The following proposition is proved in [14, Proposition 4.1].

5.3. **Proposition.** Let $G$ be a domain in $\mathbb{R}^n$ whose boundary is porous in $\mathbb{R}^n$. Let $f \in F^s_{pq}(\mathbb{R}^n)$, $1 < p < \infty$, $1 \leq q < \infty$, and $s > 0$. Then

$$
\|f \chi_G\|_{F^s_{pq}(\mathbb{R}^n)} \leq \|f\|_{F^s_{pq}(\mathbb{R}^n)} + \left( \int_{G} \frac{|f(x)|^p}{\text{dist}(x, \partial G)^sp} \, dx \right)^{1/p}.
$$

The implied constant depends on $p$, $q$, $s$, $n$, and $\partial G$.

5.4. **Theorem.** Let $n \geq 2$, $1 < p < \infty$, and $0 < s < n/p$. Suppose that $G$ is a domain in $\mathbb{R}^n$, with a compact boundary, such that $\dim_A(\partial G) < n - sp$. Then, for every $1 \leq q < \infty$, the pointwise multiplier operator $f \mapsto \chi_G f$ is bounded on $F^s_{pq}(\mathbb{R}^n)$.

**Proof.** Observe that $\partial G$ is porous in $\mathbb{R}^n$, see Remark 2.6. By Proposition 5.3 and Theorem 1.5, for $f \in F^s_{pq}(\mathbb{R}^n)$,

$$
\|f \chi_G\|_{F^s_{pq}(\mathbb{R}^n)} \leq \|f\|_{F^s_{pq}(\mathbb{R}^n)} + \|f|_G\|_{F^s_{pq}(G)} \leq 2\|f\|_{F^s_{pq}(\mathbb{R}^n)}.
$$

This concludes the proof. $\square$

The statement of Theorem 5.4 is not new and it is covered by [10, Theorem 13.3], whose proof is more technical, based upon atomic decompositions. We also refer to [23, §4.1].

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