MAPPING CLASS GROUPS DO NOT HAVE KAZHDAN’S PROPERTY (T)

JØRGEN ELLEGAARD ANDERSEN

ABSTRACT. We prove that the mapping class group of a closed oriented surface of genus at least two does not have Kazhdan’s property (T).

1. INTRODUCTION

In the paper [Kazh] Kazhdan introduced his property (T) for topological groups. A topological group has Kazhdan’s property (T), if the trivial representation is isolated in the Fell topology on the space of unitary representations of the group. Alternatively, we can formula Kazhdan’s property (T) as follows.

Definition 1 (Kazhdan). For a unitary Hilbert space representation $\rho$ of a topological group $G$, a unit vector $v$ is call $(\varepsilon,K)$-invariant, where $\varepsilon$ is positive real and $K$ is a compact subset of $G$, if

$$|\rho(g)v - v| < \varepsilon, \forall g \in K.$$ 

We say that $\rho$ has almost invariant vectors if there exist $(\varepsilon,K)$-invariant unit vector for all such pairs $(\varepsilon,K)$. A topological group $G$ has Kazhdan’s property (T) if every unitary Hilbert space representation which has almost invariant vectors has an actual nontrivial invariant vector.

Since Kazhdan introduced this property it has been rather extensively studied, also for discrete countable groups, as we shall be interested here. For such groups we can give an alternative formulation of Kazhdan’s property (T).

Definition 2. Let $\rho$ be a unitary Hilbert space representation of a discrete countable group $G$. By an almost fixed vector for $\rho$ we mean a sequence of unit vectors $(v_k) \subset H$ with the property that

$$\lim_{k \to \infty} |\rho(g)v_k - v_k| = 0$$

for all $g \in G$.

We see that a discrete countable group $G$ has Kazhdan’s property (T) if and only if the existence of an almost fixed vector implies the existence of a non trivial fixed vector for all unitary Hilbert space representations of $G$.

Theorem 1. The mapping class group of a closed oriented surface of genus at least two does not have Kazhdan’s property (T).

We construct a counter example to Kazhdan’s property (T) for these mapping class groups using the Reshetikhin-Turaev Topological Quantum Field Theory constructed in [RT1], [RT2] and [Tu]. These TQFT-constructions by Reshetikhin and
Tuevaev was given on the basis of the suggestions by Witten in \cite{W}, which gave a quantum field theory description of the Jones polynomial \cite{J}. Indeed we shall need the geometric constructions of these TQFT’s as proposed by Witten in that paper and by Atiyah in \cite{At}. That the geometric construction gives the same representations as the Reshetikhin-Turaev TQFT representations follow from combining the results of \cite{L} and \cite{AU1}, \cite{AU2}, \cite{AU3} and \cite{AU4}. In fact this identifies the geometrically constructed representations with the TQFT representations constructed by Blanchet, Habegger, Masbaum and Vogel in \cite{BHMV1} and \cite{BHMV2}, which is the skein theory construction of the RT-TQFT’s. Please see Theorem\textit{2} below regarding this.

Let us briefly recall the geometric construction of these representations of the mapping class group.

Let \( \Sigma \) be a closed oriented surface of genus at least two. Let \( p \) be a point on \( \Sigma \). Let \( M \) be the moduli space of flat \( SU(2) \) connections on \( \Sigma - p \) with holonomy around \( p \) equal to \( -\text{Id} \in SU(2) \). This moduli space is smooth and has a natural symplectic structure \( \omega \). There is a smooth symplectic action of the mapping class group \( \Gamma \) of \( \Sigma \) on \( M \). Moreover there is a unique prequantum line bundle \((L, \nabla, (\cdot, \cdot)) \) over \((M, \omega) \). The Teichmüller space \( T \) of complex structures on \( \Sigma \) naturally \( \Gamma \)-equivariantly parametrizes Kähler structures on \((M, \omega) \). For \( \sigma \in T \), we denote \((M,\omega) \) with its corresponding Kähler structure \( M_\sigma \).

By applying geometric quantization to the moduli space \( M \), one gets a certain finite rank bundle over Teichmüller space \( T \), which we will call the \textit{Verlinde} bundle \( V_k \) at level \( k \), where \( k \) is any positive integer. The fiber of this bundle over a point \( \sigma \in T \) is \( V_{k,\sigma} = H^0(M_\sigma, L^k) \). We observe that there is a natural Hermitian structure \( \langle \cdot, \cdot \rangle \) on \( H^0(M_\sigma, L^k) \) by restricting the \( L^2 \)-inner product on global \( L^2 \) sections of \( L^k \) to \( H^0(M_\sigma, L^k) \).

The main result pertaining to this bundle \( V_k \) is that its projectivization \( V_k \) supports a natural flat \( \Gamma \)-invariant connection \( \hat{\nabla} \). This is a result proved independently by Axelrod, Della Pietra and Witten \cite{ADW} and by Hitchin \cite{H} (see also \cite{A3}). This flat connection \( \hat{\nabla} \) induces a flat connection \( \hat{\nabla}^e \) in \( \text{End}(V_k) \). Let \( \text{End}_0(V_k) \) be the subbundle consisting of traceless endomorphisms. The connection \( \hat{\nabla}^e \) also induces a connection in \( \text{End}_0(V_k) \), which is invariant under the action of \( \Gamma \).

We get this way for each \( k \) a finite dimensional representation of \( \Gamma \), namely on the covariant constant sections, say \( \mathcal{H}_k \), of \( \text{End}_0(V_k) \) over \( T \). Let

\[
\mathcal{H} = \bigoplus_{k \text{ prime}}^\infty \mathcal{H}_k
\]

on which \( \Gamma \) acts. From the proof of the asymptotic faithfulness in \cite{A2}, one see that this representation of \( \Gamma \) is faithful.

Each of the vector spaces \( \mathcal{H}_k \) has a natural positive definite Hermitian structure \([\cdot, \cdot]\), which is preserved by the action of \( \Gamma \). This Hermitian structure is clear from the skein theory construction of \( \mathcal{H}_k \) following \cite{BHMV2}:

Consider the BHMV-TQFT (as defined in \cite{BHMV2}) at \( A = \exp(2\pi i/4(k+2)) \). The label set for this theory is then \( L_k = \{0, 1, \ldots, k\} \). We denote by \( Z_k \) the vector space this theory associates to \( \Sigma \sqcup \Sigma \) with \( p \in \Sigma \) label by the last color \( k \) in both copies of \( \Sigma \). The BHMV construction also requires us to choose a \( p_1 \)-structure on \( \Sigma \sqcup \Sigma \). However, we note that this vector space does not depend on the choice of a
$p_1$-structure, as long as the choice of the $p_1$-structure $\Sigma \sqcup \Sigma = \partial(\Sigma \times [0,1])$ extends over $\Sigma \times [0,1]$.

Since the vector space $Z_k$ is part of a TQFT, there is in particular an action of the mapping class group of $\Sigma \sqcup \Sigma$ on $Z_k$. There is a natural homomorphism of $\Gamma$ into the mapping class group of $\Sigma \sqcup \Sigma$ given by mapping $\phi \in \Gamma$ to $\phi \circ \phi$.

In [BHMY2], a Hermitian structure $\{\cdot, \cdot\}$ is constructed on $Z_k$, which is invariant under the action of the mapping class group of $\Sigma \sqcup \Sigma$ and therefore also invariant under the action of $\Gamma$. For the choice of $A$ made above, it is proved in [BHMY2], that the Hermitian structure $\{\cdot, \cdot\}$ is positive definite.

By the work of Andersen and Ueno [AU1], [AU2], [AU3] and [AU4] combined with the work of Laszlo [L], we have an identification of the two constructions.

**Theorem 2** (Andersen & Ueno). *There is a natural $\Gamma$-equivariant isomorphism*

$$I_k : Z_k \rightarrow \mathcal{H}_k$$

Using the isomorphism $I_k$, we define the the positive definite Hermitian structure $\{\cdot, \cdot\}$ on $\mathcal{H}_k$ by the formula

$$\{\cdot, \cdot\} = \{I_k^{-1}(\cdot), I_k^{-1}(\cdot)\}.$$ The norm associated to $\{\cdot, \cdot\}$ is denoted $|\cdot|$. The Hermitian structure $\{\cdot, \cdot\}$ on $\mathcal{H}_k$ induces a Hermitian structure on $\text{End}(\mathcal{V}_k)$, which is parallel with respect to $\nabla^e$ and which is $\Gamma$-invariant.

**Definition 3.** *We define $\mathcal{H}$ to be the Hilbert space completion of $\mathcal{H}$ with respect to the norm $|\cdot|$.*

This is an infinite dimensional Hilbert space, on which $\Gamma$ acts isometrically. This representation provides us with the needed counter example to Kashdan’s property (T) for $\Gamma$. Let us discuss the proof of this.

**Theorem 3** (Roberts). *The only $\Gamma$ invariant vector in $\mathcal{H}$ is 0.*

This theorem follows from the fact that the representations $\mathcal{H}_k$ are irreducible as $\Gamma$-representation for $k$, such that $k + 2$ is prime. This result was established in the un-twisted case by Roberts in [Ro] and his proof can be applied word for word also to this case.

The basic idea behind building the required almost fixed vector for $\mathcal{H}$ is to consider coherent states on $M_\sigma, \sigma \in \mathcal{T}$.

Fix a point $x \in M$. Evaluation at $x$ gives a section of $\mathcal{V}_k^*$ up to scale. Using $\langle \cdot, \cdot \rangle$ we get induced a section $e^{(k)}_x$ of $\mathcal{V}_k$ up to scale. For each $\sigma \in \mathcal{T}$, $e^{(k)}_x(\sigma)$ is the coherent state associated to $x$ on $M_\sigma$. Let $E^{(k)}_x$ be the section of $\text{End}(\mathcal{V}_k)$ obtained as the orthogonal projection (with respect to $\langle \cdot, \cdot \rangle$) onto the one dimensional subspace spanned by $e^{(k)}_x$. We observe that $E^{(k)}_x$ only depends on $x$.

**Theorem 4.** *The sections $E^{(k)}_x$ of $\text{End}(\mathcal{V}_k)$ over $\mathcal{T}$ are asymptotically covariant constant. I.e. for any pair of points $\sigma_0, \sigma_1 \in \mathcal{T}$ there exists a constant $C$ such that*

$$|P^{e}_{\sigma_0, \sigma_1}(E^{(k)}_x(\sigma_0)) - E^{(k)}_x(\sigma_1)| \leq \frac{C}{k},$$

*where $P^{e}_{\sigma_0, \sigma_1}$ is the parallel transport from $\sigma_0$ to $\sigma_1$ in $\text{End}(\mathcal{V}_k)$ and the norm $|\cdot|$ is the one associated to the Hermitian structure on $\mathcal{V}_k \otimes \mathcal{V}_k$ induced from $\langle \cdot, \cdot \rangle$ on $\mathcal{V}_k$.***
The proof of this theorem is given in section 6.

In order to relate these norm estimates to estimates for the norm associated to the Hermitian structure $[\cdot, \cdot]$, we need the following result.

Pick a point $\sigma_0 \in T$.

**Theorem 5 (Andersen).** There exist a constant $C$ such that at $\sigma_0$

$$C^{-1}|\Psi| \leq |\Psi| \leq C|\Psi|$$

for all $\Psi$ in $\text{End}(V_k)_{\sigma_0}$ and all $k \in \mathbb{N}$.

This theorem is proved in [A6].

To produce the almost fixed vectors, we now pick a finite subgroup $\Lambda$ of $SU(2)$ which contains $-1 \in SU(2)$ and we consider the finite subset $X$ of $M$, consisting of connections which reduces to $\Lambda$. As we will see in section 7, for appropriate choice of $\Lambda$, we get a non-empty finite subset of $M$ this way which is invariant under the action of the mapping class group and such that $|X| > 1$. Let now $E^{(k)}_X$ be the section of $\text{End}(V_k)$ given by

$$E^{(k)}_X = \sum_{x \in X} E^{(k)}_x.$$

Let $E^{(k)}_{X,0}$ be the traceless part of $E^{(k)}_X$. As we will see in section 7, for large enough $k$, $E^{(k)}_X \neq 0$. Hence for large enough $k$ we have a unique vector in $E^{(k)}_{X,0} \in \mathcal{H}_k$, which at $\sigma_0$ agrees with $E^{(k)}_{X,0}(\sigma_0)/|E^{(k)}_{X,0}(\sigma_0)|$.

**Theorem 6.** The sequence $\{E^{(k)}_{X,0}\}$ is an almost fixed vector for the action of $\Gamma$ on $\mathcal{H}$.

This theorem will be proved in section 7. Our main Theorem 1 is of course a consequence of Theorem 3 and Theorem 6.

Since the mapping class group of a genus one closed surface is $SL(2, \mathbb{Z})$, it is well known that this mapping class group does not have Kazhdan’s property (T). The constructions as presented here can be applied to the $U(1)$-moduli space in the genus one case to provide a counter example for $SL(2, \mathbb{Z})$ as well.

It is a result of F. Taherkhani that the mapping class group of a closed oriented surface of genus two does not have Kazhdan’s property (T) [Ta]. Taherkhani’s proof of the genus two case relies on computer aided calculations of the first cohomology of certain cofinite subgroups of the genus two mapping group and so is completely different from the arguments presented here.

The construction of the almost fixed vector applies to the rather more general setting of [A3], for which we have established the existence of Hitchin’s connection. It would be interesting to find a geometric argument for the irreducibility, which potentially could be applied in the more general setting of [A3], and thereby possibly provide counter examples to Kazhdan’s property (T) for other groups.

The methods of asymptotic analysis and the theory of Toeplitz operators has also allowed us to link these TQFT to the Nielsen-Thurston classification of mapping classes [A4] and to asymptotics of Hermitian pairings of loop operators [A5] (Toeplitz operator interpretation of [MN]).

The paper is organized as follows. In section 2 and 3 we recall the construction of the Verlinde bundle and Hitchin’s connection. Basics about coherent states and Toeplitz operators are recalled in section 4. Another geometrically defined
Hermitian structure which is asymptotically preserved by the Hitchin connection is recalled in section 5. We need this structure in section 6 to prove the asymptotic flatness of the coherent states as stated in Theorem 4, followed in section 7 by the construction of the almost fixed vector.

We would like to thank Vaughan Jones for suggesting the use of these representations to settle this property for these mapping class groups. Jones has also posted this problem on the CTQM problem list at www.ctqm.au.dk (Problem 14). At the time of this writing it remains an open problem to what extent the mapping class groups has the Haagerup property. Further we would like to thank Gregor Masbaum for valuable discussions.

2. THE GAUGE THEORY CONSTRUCTION OF THE VERLINDE BUNDLE

Let us now very briefly recall the construction of the Verlinde bundle. Only the details needed in this paper will be given. We refer e.g. to [H] for further details. As in the introduction we let Σ be a closed oriented surface of genus \( g \geq 2 \) and \( p \in \Sigma \). Let \( P \) be a principal \( SU(2) \)-bundle over \( \Sigma \). Clearly, all such \( P \) are trivializable. Let \( M \) be the moduli space of flat \( SU(2) \)-connections in \( P|_{\Sigma-p} \) with holonomy \(-1 \in SU(2)\) around \( p \). We can identify

\[
M = \text{Hom}'(\tilde{\pi}_1(\Sigma), SU(2))/SU(2).
\]

Here \( \tilde{\pi}_1(\Sigma) \) is the universal central extension

\[
0 \to \mathbb{Z} \to \tilde{\pi}_1(\Sigma) \to \pi_1(\Sigma) \to 1
\]

as discussed in [H] and in [AB] and \( \text{Hom}' \) means the space of homomorphisms from \( \tilde{\pi}_1(\Sigma) \) to \( SU(2) \) which send the image of \( 1 \in \mathbb{Z} \) in \( \tilde{\pi}_1(\Sigma) \) to \(-1 \in SU(2)\) (see [H]).

**Theorem 7** (Goldman; Atiyah & Bott). The moduli space \( M \) is a smooth compact manifold of dimension \( 6g - 6 \).

See [G] for a representation variety proof of this theorem and [AB] for a gauge theory proof.

Since there is a natural homomorphism from the mapping class group to the outer automorphisms of \( \tilde{\pi}_1(\Sigma) \), we get a smooth action of \( \Gamma \) on \( M \).

On \( \mathfrak{g} = \text{Lie}(SU(2)) \) we have the invariant symmetric bilinear form \( (X, Y) \mapsto \text{Tr}(XY) \), normalized such that \(-\frac{1}{6} \text{Tr}(\vartheta \wedge [\vartheta \wedge \vartheta])\) is a generator of the image of the integer cohomology in the real cohomology in degree 3 of \( SU(2) \), where \( \vartheta \) is the \( \mathfrak{g} \)-valued Maurer-Cartan 1-form on \( SU(2) \). This bilinear form induces a symplectic form on \( M \). At a flat connection \( A \) representing a point \([A] \in M:\)

\[
\omega(\varphi_1, \varphi_2) = \int_{\Sigma} \text{Tr}(\varphi_1 \wedge \varphi_2),
\]

where \( \varphi_i \) are \( d_A \)-closed 1-forms on \( \Sigma \) with values in \( \text{ad} P \) representing tangent vectors to \( M \) at \([A]\). See e.g. [G], [AB] or [H] for further details on this. We summarize this in the following theorem.

**Theorem 8** (Goldman; Atiyah & Bott; Narashimhan & Seshadri). On the moduli space \( M \), the form \( \omega \) is a symplectic structure and the natural action of \( \Gamma \) on \( M \) is symplectic.

Let \( \mathcal{L} \) be the Hermitian line bundle over \( M \) and \( \nabla \) the compatible connection in \( \mathcal{L} \) constructed by Freed in [Fr]. This is the content of Corollary 5.22, Proposition 5.24.
and equation (5.26) in [Fr] (see also the work of Ramadas, Singer and Weitsman [RSW]). By Proposition 5.27 in [Fr] we have that the curvature of \( \nabla \) is \( \sqrt{-\frac{1}{2\pi}} \omega \). We will also use the notation \( \nabla \) for the induced connection in \( \mathcal{L}^k \), where \( k \) is any integer.

**Theorem 9** (Ramadas, Singer & Weitsman; Freed). The Hermitian line bundle with connection \((\mathcal{L}, \nabla)\) is a prequantum line bundle over the moduli space, i.e. the curvature of \( \nabla \) is \( \sqrt{-\frac{1}{2\pi}} \omega \).

By an almost identical construction, we can lift the action of \( \Gamma \) on \( M \) to act on \( \mathcal{L}^k \) such that the Hermitian connection is preserved (See e.g. [A1]). In fact, since \( H^2(M, \mathbb{Z}) \cong \mathbb{Z} \) and \( H^1(M, \mathbb{Z}) = 0 \), it is clear that the action of \( \Gamma \) leaves the isomorphism class of \( (\mathcal{L}, \nabla) \) invariant, thus alone from this one can conclude that a \( U(1) \)-central extension of \( \Gamma \) acts on \( (\mathcal{L}, \nabla) \) covering the \( \Gamma \) action on \( M \). This is actually all we need in this paper. We will return to this point at the end of this section.

Let now \( \sigma \in \mathcal{T} \) be a complex structure on \( \Sigma \). Let us review how \( \sigma \) induces a complex structure on \( M \) which is compatible with the symplectic structure on this moduli space. The complex structure \( \sigma \) induces a \(*\)-operator on 1-forms on \( \Sigma \) with values in \( \text{ad} P \), which acts on the harmonic forms with square \(-1\). Hence we get an almost complex structure on \( M \) by letting \( I = I_\sigma = -* \) acting on harmonic forms \( \Sigma \) with values in \( \text{ad} P \).

We have the following classical result by Narasimhan and Seshadri (see [NS]),

**Theorem 10** (Narasimhan & Seshadri). The triple \((M, \omega, I_\sigma)\) is a smooth Kähler manifold for any \( \sigma \in \mathcal{T} \).

We use the notation \( M_\sigma = (M, \omega, I_\sigma) \). By using the \((0, 1)\) part of \( \nabla \) in \( \mathcal{L} \) over \( M_\sigma \), we get an induced holomorphic structure in the bundle \( \mathcal{L} \). See also [H] and [AH] for further details on this.

From a more algebraic geometric point of view, we consider the moduli space of S-equivalence classes of semi-stable bundles of rank 2 and determinant isomorphic to the line bundle \( \mathcal{O}(p) \). By using Mumford’s Geometric Invariant Theory, Narasimhan and Seshadri (see [NS]) showed that

**Theorem 11** (Narasimhan & Seshadri). The moduli space of semi-stable bundles of rank 2 and determinant isomorphic to \( \mathcal{O}(p) \) is a smooth complex algebraic projective variety, which is isomorphic as a Kähler manifold to \( M_\sigma \).

Referring to [DN] we further recall that

**Theorem 12** (Drezet & Narasimhan). The Picard group of \( M_\sigma \) is generated by the holomorphic line bundle \( \mathcal{L} \) over \( M_\sigma \) constructed above:

\[ \text{Pic}(M_\sigma) = \langle \mathcal{L} \rangle. \]

**Definition 4.** The Verlinde bundle \( \mathcal{V}_k \) over Teichmüller space is by definition the bundle whose fiber over \( \sigma \in \mathcal{T} \) is \( H^0(M_\sigma, \mathcal{L}^k) \), where \( k \) is a positive integer.

We will consider the endomorphism bundle \( \text{End}(\mathcal{V}_k) \) of \( \mathcal{V}_k \). We observe that the general argument above gives an action of a central extension of \( \Gamma \) acting on \( \mathcal{V}_k \), which then induces an action of \( \Gamma \) on \( \text{End}(\mathcal{V}_k) \). This intern induces an action of \( \Gamma \) on the sub-bundle \( \text{End}_0(\mathcal{V}_k) \) consisting of traceless endomorphisms.
3. The projectively flat connection

In this section we will review Axelrod, Della Pietra and Witten’s and Hitchin’s construction of the projective flat connection over Teichmüller space in the Verlinde bundle. We refer to [H], [ADW] and [A3] for further details.

Let \( \mathcal{V}_k \) be the trivial \( C^\infty(M, L^k) \)-bundle over \( \mathcal{T} \) which contains \( \mathcal{V}_k \), the Verlinde sub-bundle.

The Hitchin connection is a connection in \( \mathcal{V}_k \) which is of the form

\[
\hat{\nabla}_v = \hat{\nabla}_v^t - u(v),
\]

for all \( v \in T(\mathcal{T}) \). Here \( \hat{\nabla}_v^t \) is the trivial connection in the trivial bundle \( \mathcal{V}_k \) and \( u \) is a smooth map from \( T(\mathcal{T}) \) to the vector space \( D(M, L^k) \) consisting of differential operators acting on sections of \( L^k \). Hitchin constructs a specific \( u \) such that the corresponding connection preserved the Verlinde subbundle. Let us recall his construction here.

The holomorphic tangent space to Teichmüller space at \( \sigma \in \mathcal{T} \) is given by

\[
T^1_{\sigma,0}(\mathcal{T}) \cong H^1(\Sigma, K_\sigma^{-1}).
\]

If \( v \in T_\sigma(\mathcal{T}) \), then we denote its \((1,0)\)-part in \( T^1_{\sigma,0}(\mathcal{T}) \) by \( v' \). The holomorphic cotangent space to the moduli space of semi-stable bundles at the equivalence class of a stable bundle \( E \) is given by

\[
T^*_{E,M,\sigma} \cong H^0(\Sigma, \text{End}_0(E) \otimes K_\sigma).
\]

We denote the holomorphic tangent bundle of \( M_\sigma \) by \( T_\sigma \). For each \( v \in T_\sigma(\mathcal{T}) \) we now specify a \( G(v) \in \Omega^0(M_\sigma, S^2(T_\sigma)) \) as a quadratic function on \( T^*_\sigma \) by the formula

\[
G(v)(\alpha, \alpha) = \int_{\Sigma} \text{Tr}(\alpha^2)v',
\]

for all \( \alpha \in H^0(\Sigma, \text{End}_0(E) \otimes K_\sigma) \). From this formula it is clear that \( G(v) \in H^0(M_\sigma, S^2(T_\sigma)) \).

Axelrod, Della Pietra and Witten’s \( u(v) \), which by the results of [A3] agrees projectively with Hitchin’s \( u(v) \), given in [H], is

\[
u(v)(s) = \frac{1}{2(k+2)}(\Delta_{G(v)} - 2\nabla_{G(v)} \partial F(\sigma) + kv'[F](\sigma))s.
\]

The leading order term \( \Delta_{G(v)} \) is the 2'nd order operator given by

\[
\Delta_{G(v)} : C^\infty(M, L^k) \xrightarrow{\nabla^{1,0}} C^\infty(M, T^*_\sigma \otimes L^k) \xrightarrow{G(v)} C^\infty(M, T_\sigma \otimes L^k) \xrightarrow{\text{Tr}} C^\infty(M, L^k),
\]

where we have used the Chern connection in \( T_\sigma \) on the Kähler manifold \((M_\sigma, \omega)\).

Further \( F \) is the unique smooth map from \( \mathcal{T} \) to the vector space \( C^\infty(M) \), such that \( F(\sigma) \) is the Ricci potential uniquely determined as the real function with zero average over \( M \), which satisfies the following equation

\[
\text{Ric}_\sigma = 2n\omega + 2\sqrt{-1}\partial\bar{\partial}F(\sigma).
\]

The notation \( v'[F](\sigma) \in C^\infty(M) \) mean the derivative of \( F \) along the direction \( v' \) in \( \mathcal{T} \) at \( \sigma \in \mathcal{T} \).

The complex vector field \( G(v) \partial F(\sigma) \in C^\infty(M, T_\sigma) \) is simply just the contraction of \( G(v) \) with \( \partial F(\sigma) \in C^\infty(M, T^*_\sigma) \).
We observe that $\hat{\nabla}$ agrees with $\hat{\nabla}^T$ along the anti-holomorphic directions $T^{0,1}(\mathcal{T})$.

We remark that there is some finite set of vector fields $X_r(v), Y_r(v), Z(v) \in \mathcal{C}^\infty(M_{\sigma}, T)$, $r = 1, \ldots, R$ (where $v \in T_{\sigma}(\mathcal{T})$) all varying smoothly with $v \in T(\mathcal{T})$, such that

$$\Delta_{G(v)} - 2\nabla_{G(v)} \partial F = \sum_{r=1}^R \nabla X_r(v) \nabla Y_r(v) + \nabla Z(v).$$

This follows immediately from the definition of $\Delta_{G(v)}$. See also [A3].

**Theorem 13** (Axelrod, Della Pietra & Witten; Hitchin). The expression (1) above defines a connection $\hat{\nabla}$ in the bundle $V_k$, which induces a flat connection in $P(V_k)$.

We remark about genus 2, that [ADW] covers this case, but [H] excludes this case, however, the work of Van Geemen and De Jong [vGdJ] extends Hitchin’s approach to the genus 2 case. See also [A3] where the connection is constructed in a more general situation covering this moduli space application for all $g > 1$. It is clear from formula (1) that $\hat{\nabla}$ is invariant under the action of $\Gamma$ on $P(V_k)$.

We are here interested in the induced flat connection $\hat{\nabla}_e$ in the endomorphism bundle $\text{End}(V_k)$. Suppose $\Phi$ is a section of $\text{End}(V_k)$. Then for all sections $s$ of $V_k$ and all $v \in T(\mathcal{T})$ we have that

$$(\hat{\nabla}_e v)(s) = \hat{\nabla}_v \Phi(s) - \Phi(\hat{\nabla}_v(s)).$$

It is clear from the construction of $\hat{\nabla}_e$, that the subbundle $\text{End}_0(V_k)$ is preserved by $\hat{\nabla}_e$. Thus $(\text{End}_0(V_k), \hat{\nabla}_e)$ is a vector bundle over $\mathcal{T}$ with a flat mapping class group invariant connection.

**Definition 5.** Let $\mathcal{H}_k$ be the finite dimensional representation of $\Gamma$ consisting of the covariant constant sections of $(\text{End}_0(V_k), \hat{\nabla}_e)$ over $\mathcal{T}$.

The dimension of $\mathcal{H}_k$ is of course $d_g(k)^2 - 1$, where $d_g(k)$ is the rank of $V_k$, which is given by the twisted Verlinde formula [Th] (see also [AM])

$$d_g(k) = (k + 1)^{g-1} \sum_{j=1}^{2k+1} (-1)^j+1 \left(\sin\left(\frac{\pi j}{2(k+1)}\right)\right)^{2-2g}$$

4. Asymptotics of Toeplitz operators and coherent states

We shall in this section discuss the asymptotics of Toeplitz operators and coherent states as the level $k$ goes to infinity. The properties we need can all be derived from the fundamental work of Boutet de Monvel and Sjöstrand. In [BdMS] they did a microlocal analysis of the Szegö projection, which can be applied to the asymptotic analysis in the situation at hand, as it was done by Boutet de Monvel and Guillemin in [BdMG] (in fact in a much more general situation than the one we consider here) and others following them. In particular the applications developed by Schlichenmaier [Sch], [Sch1], [Sch2] and further by Karabegov and Schlichenmaier [KS] to the study of Toeplitz operators in the geometric quantization setting, is what will interest us here. Let us first describe the basis setting.

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1. This makes sense when we consider the holomorphic tangent bundle $T$ of $M_{\sigma}$ inside the complexified real tangent bundle $TM \otimes \mathbb{C}$ of $M$. 
On $C^\infty(M, \mathcal{L}^k)$ we have the $L_2$-inner product:

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M (s_1, s_2) \omega^m$$

where $s_1, s_2 \in C^\infty(M, \mathcal{L}^k)$ and $(\cdot, \cdot)$ is the fiberwise Hermitian structure in $L^k$.

Let $\sigma$ be a point in $\mathcal{T}$ and consider the Kähler manifold $M_\sigma$. Inside the space of all smooth sections $C^\infty(M, \mathcal{L}^k)$, we have the finite dimensional subspace $H^0(M_\sigma, \mathcal{L}^k)$ consisting of holomorphic sections with respect to the complex structure on $M_\sigma$.

The $L_2$-inner product on $C^\infty(M, \mathcal{L}^k)$ determines the orthogonal projection $\pi_{\sigma}^{(k)} : C^\infty(M, \mathcal{L}^k) \rightarrow H^0(M_\sigma, \mathcal{L}^k)$. For each $f \in C^\infty(M)$ consider the associated Toeplitz operator $T_{f,\sigma}^{(k)}$ given as the composition of the multiplication operator (which we also denote $f$) $f : C^\infty(M, \mathcal{L}^k) \rightarrow C^\infty(M, \mathcal{L}^k)$ with the orthogonal projection $\pi_{\sigma}^{(k)} : C^\infty(M, \mathcal{L}^k) \rightarrow H^0(M_\sigma, \mathcal{L}^k)$, so that

$$T_{f,\sigma}^{(k)}(s) = \pi_{\sigma}^{(k)}(fs).$$

Since the multiplication operator is a zero order differential operator, $T_{f,\sigma}^{(k)}$ is a zero-order Toeplitz operator.

The first result we will need is due to Bordemann, Meinrenken and Schlichenmaier (see [BMS]). The $L_2$-inner product on $C^\infty(M, \mathcal{L}^k)$ induces an inner product $(\cdot, \cdot)$ on $H^0(M_\sigma, \mathcal{L}^k)$, which in turn induces the operator norm $\|\cdot\|$ on $\text{End}(H^0(M_\sigma, \mathcal{L}^k))$.

**Theorem 14** (Bordemann, Meinrenken and Schlichenmaier). For any $f \in C^\infty(M)$ we have that

$$\lim_{k \rightarrow \infty} \|T_{f,\sigma}^{(k)}\| = \sup_{x \in M} |f(x)|.$$

This result follows also directly from the results of [BdMS] and [BdMG] as shown in [KS].

**Theorem 15** (Schlichenmaier). For each $\sigma \in \mathcal{T}$ and any pair of smooth functions $f_1, f_2 \in C^\infty(M)$, we have an asymptotic expansion

$$T_{f_1,\sigma}^{(k)} T_{f_2,\sigma}^{(k)} \sim \sum_{l=0}^{\infty} c_{l}^{(k)}(f_1, f_2, \sigma) k^{-l},$$

where $c_{l}^{(k)}(f_1, f_2) \in C^\infty(N)$ are uniquely determined since $\sim$ means the following: For all $L \in \mathbb{Z}_+$ we have that

$$\|T_{f_1,\sigma}^{(k)} T_{f_2,\sigma}^{(k)} - \sum_{l=0}^{L} c_{l}^{(k)}(f_1, f_2, \sigma) k^{-l}\| = O(k^{-(L+1)}).$$

Moreover, $c_{0}^{(k)}(f_1, f_2) = f_1 f_2$.

This Theorem was proved in [Sch] (again building on the works [BdMS] and [BdMG]) and it is published in [Sch1] and [Sch2], where it is also proved that the formal generating series for the $c_l(f_1, f_2)$’s gives a formal deformation quantization of the Poisson structure on $(M, \omega)$. By examining the proof in [Sch] (or in [Sch1] and [Sch2]) of this Theorem, one observes that for continuous families of functions, the estimates in Theorem 15 are uniform over compact subsets of $\mathcal{T}$.

Let $\sigma_0$ and $\sigma_1$ be two points in $\mathcal{T}$. For any $f \in C^\infty(M)$ we consider

$$T_{f,\sigma_0,\sigma_1}^{(k)} : H^0(M_{\sigma_0}, \mathcal{L}^k) \rightarrow H^0(M_{\sigma_1}, \mathcal{L}^k)$$
given by
\[ T_{f,(\sigma_0,\sigma_1)}^{(k)} = T_{f,\sigma_1}^{(k)} |_{H^0(M_{\sigma_0},\mathcal{L}^k)}. \]

We see that
\[ T_{f,\sigma_1}^{(k)} = \pi_{\sigma_0}^{(k)} f \pi_{\sigma_0}. \]

We will also use the notation
\[ \pi^{(k)}_{(\sigma_0,\sigma_1)} = \pi_{\sigma_1}^{(k)} |_{H^0(M_{\sigma_0},\mathcal{L}^k)}. \]

**Theorem 16.** For all \( f \in C^\infty(M) \) we have that
\[ \| T_{f,(\sigma_0,\sigma_1)}^{(k)} - \pi_{(\sigma_0,\sigma_1)}^{(k)} T_{f,\sigma_1}^{(k)} \| = O(k^{-1}) \]
and
\[ \| T_{f,\sigma_1}^{(k)} \pi_{(\sigma_0,\sigma_1)}^{(k)} - T_{f,\sigma_1} \pi_{(\sigma_0,\sigma_1)}^{(k)} \| = O(k^{-1}). \]

We will prove this theorem by using the theory of Fourier integral operators and their symbol calculus as discussed in [BdMG].

Let \( Z \) be the unit tangent bundle in \( \mathcal{L}^* \). Let \( \tilde{\Omega} \) be the volume form on \( Z \), which is invariant under the \( U(1) \) action on \( Z \) and with the property that
\[ \int_Z \tau^*(f) \tilde{\Omega} = \int_M f \Omega \]
for all \( f \in C^\infty(M) \). The vector space \( C^\infty(M,\mathcal{L}^k) \) is isomorphic to the vector space of smooth functions on \( Z \) which transforms in the \( k \)th representation of \( U(1) \) via the formula
\[ \psi_\alpha(s(x)) = \alpha^0(x) \]
for all \( \alpha \in \tau^{-1}(x), x \in M \) and all \( s \in C^\infty(M,\mathcal{L}^k) \). In fact this is an isometry between \( L_2(M,\mathcal{L}^k) \) and the \( k \)th weight space of the \( U(1) \) action on \( L_2(Z) \).

For each \( \sigma \in \mathcal{T} \), we define the Hardy space \( \mathbb{H}_\sigma \subset L_2(Z) \) consisting of the \( L_2 \) functions, which extend over the unit disc bundle in \( \mathcal{L}^* \) holomorphically with respect to the complex structure induced from the one on \( M_\sigma \). The orthogonal project from \( L_2(Z) \) to the closed subspace \( \mathbb{H}_\sigma \) is the Szegö projections and we denote it \( \Pi_\sigma \). If we denote by \( \mathbb{H}_\sigma^{(k)} \) the \( k \)th weight space of the \( U(1) \) action on \( \mathbb{H}_\sigma \). Then the above isomorphism restrict to an isometry \( H^0(M_\sigma,\mathcal{L}^k) \cong \mathbb{H}_\sigma^{(k)} \) and we (also) denote the Bargman orthogonal projection onto \( \mathbb{H}_\sigma^{(k)} \) by \( \pi_\sigma^{(k)} \).

This way, we see that every \( \sigma \in \mathcal{T} \) makes \( Z \) a pseudo-convex domain with a Toeplitz structure \( \Pi_\sigma \) in the sense of [BdMG].

**Proof of Theorem 16** The Szegö projectors \( \Pi_{\sigma_0} \) and \( \Pi_{\sigma_1} \) with respect to the complex structure \( \sigma_0 \) and \( \sigma_1 \) are Fourier integral operators by Theorem 11.1 in [BdMG] and by the composition rule of symbols of these types of operators (Theorem 9.8 in [BdMG]), we see that the leading order symbol of the operator
\[ \Pi_{\sigma_0} f \Pi_{\sigma_0} - \Pi_{\sigma_1} \Pi_{\sigma_0} f \Pi_{\sigma_0} \]
vanishes. This means this zero order operator in fact is of order \(-1\). Consider the first order differential operator \( D \) corresponding to the infinitesimal generator of the circle action on \( X \). This operator commutes with \( \Pi_{\sigma_1} \) and we have that
\[ \Pi_{\sigma_1} D f \Pi_{\sigma_0} - \Pi_{\sigma_1} D \Pi_{\sigma_0} f \Pi_{\sigma_0} \]
is a zero order operator, hence bounded. However $D$ acts on $H^0(M_{\sigma_1}, L^k)$ by multiplication by $k$. Thus there exist a constant $C$ (equal to the operator norm of the above operator) such that
\[ \|k(\pi^{(k)}_{\sigma_1} f \pi^{(k)}_{\sigma_0} - \pi^{(k)}_{\sigma_1} \pi^{(k)}_{\sigma_0} f)\| \leq C. \]

The other inequality is proved the same way.

\[ \square \]

We again observe from the proof, that the estimates in Theorem 16 are uniform for $(\sigma_0, \sigma_1)$ contained in compact subsets of $T$.

Let us now consider coherent states. Pick a point $x$ in $M$ and let $\alpha \in \mathcal{L}_r^\ast$. We think of $\alpha$ as a linear map
\[ \alpha : C^\infty(M, L^k) \rightarrow \mathbb{C} \]
given by
\[ \alpha(s) = \alpha \otimes k(s(x)). \]

For a $\sigma$ in $T$, we let $e^{(k)}_{\alpha, \sigma} \in H^0(M_\sigma, L^k)$ be the corresponding coherent state, i.e.
\[ (s, e^{(k)}_{\alpha, \sigma}) = \alpha(s) \]
for all $s \in H^0(M_\sigma, L^k)$.

**Theorem 17.** For all $f \in C^\infty(M)$, any $x \in M$ and any pair of points $(\sigma_0, \sigma_1)$ in $T$ we have that
\[ \left| T^{(k)} f(x) e^{(k)}_{\alpha, \sigma_0} - f(x) e^{(k)}_{\alpha, \sigma_1} \right| = O(k^{-1}). \]

We will present a proof of this theorem here, which builds further on results of Karabegov and Schlichenmaier [KS], which intern again uses the Boutet de Monvel and Sjöstrand expression for the Szegő kernel in [BdMS] and then stationary phase approximation.

We recall the part of the setting from [KS] we need here.

Let $B^{(k)}_{\sigma} \in C^\infty(Z \times Z)$ be the Bargman kernel. I.e. for $s \in C^\infty(M, L^k)$, we have that
\[ \pi_{\sigma}^{(k)}(\psi_s)(\alpha) = \psi_{\pi_{\sigma}^{(k)}(s)}(\alpha) = \int_Z B^{(k)}_{\sigma}(\alpha, \beta) \psi_s(\beta) \tilde{\Omega}(\beta). \]

In fact we get the relation that
\[ B^{(k)}_{\sigma}(\alpha, \beta) = (e^{(k)}_{\beta, \sigma}, e^{(k)}_{\alpha, \sigma}) = \psi_{\pi_{\sigma}^{(k)}(\alpha)}. \]

Since the Bargman Kernel decays faster than any power of $k$ of the diagonal (see e.g. [KS]), it follows immediately that
\[ |(e^{(k)}_{\alpha_1, \sigma_0}, e^{(k)}_{\alpha_2, \sigma_1})| = O(k^{-N}) \]
for all $N$ and all $\alpha_i \in \mathcal{L}_r^\ast$, $x_1 \neq x_2$, $\sigma_i \in T$, $i = 1, 2$. Furthermore, for all $\alpha \in Z$ and $\sigma \in T$ we have that
\[ |k^{-\eta}|e^{(k)}_{\alpha, \sigma}| - 1| = O(k^{-1}). \]

In fact Zelditch [Z] provides a full asymptotic expansion of $|e^{(k)}_{\alpha, \sigma}|$ to all orders in $k$, but we will not need it here.

Let us now recall the expression from [KS] for the asymptotic expansion of $B^{(k)}_{\sigma}$ near the diagonal.
Choose a \( \sigma \in \mathcal{T} \) and \( x_0 \in M \). Let \( U \) be a sufficiently small neighbourhood of \( x_0 \). Let \( s \) be a holomorphic frame of \( \mathcal{L}^* \) over \( U \). Then \( \alpha(x) = \frac{s(x)}{|s(x)|} \), \( x \in U \) is a smooth section of \( Z \) over \( U \). Furthermore \( \Phi(x) = \log|s(x)| \) is a potential for \( \omega \), i.e. \( \omega = -i\partial\bar{\partial}\Phi \). By adjusting \( s \), we can arrange that \( x_0 \) is a stationary point for \( \Phi \). Let \( \hat{\Phi} \in C^\infty(U \times \hat{U}) \) be an almost analytic extension of \( \Phi \) from the diagonal, in the sense of Hörmander (see [KS] for the definition of \( \hat{\Phi} \)). We will and may assume that \( \hat{\Phi}(y) = \Phi(y, x) \). Let \( \chi \in C^\infty(U \times \hat{U}) \) be given by

\[
\chi(x, y) = \hat{\Phi}(x, y) - \frac{1}{2}(\Phi(x) + \Phi(y))
\]

Note that \( \chi(x, x) = 0 \) and by Lemma 5.5 in [KS] we can assume that \( \text{Re}(\chi(x, y)) < 0 \) for all \( x \neq y, x, y \in U \). Furthermore the function \( y \mapsto \chi(x_0, y) \) has a non-degenerate critical point at \( y = x_0 \).

**Theorem 18** (Karabegov and Schlichenmaier). There exist a unique function \( b \in C^\infty(U \times \hat{U}) \) such that for any compact subset \( K \subset U \times U \), there exist a constant \( C \) such that

\[
\sup_{(x, y) \in K} \left| k^{-n} B^{(k)}_\sigma(\alpha(x), \alpha(y)) - e^{k\chi(x, y)} b(x, y) \right| \leq \frac{C}{k}.
\]

We remark that a full asymptotic expansion of \( B^{(k)}_\sigma \) is given in Theorem 5.6. of [KS]. We also remark that \( \alpha, \chi \) and \( b \) of course depends on \( \sigma \).

**Proof of Theorem 17** First we compute

\[
\langle T^{(k)}_{f, (\sigma_0, \sigma_1)}, f(x) \rangle = \langle e^{(k)}_{\alpha, \sigma_0}, f(x) e^{(k)}_{\alpha, \sigma_1} \rangle
\]

\[
= \langle f e^{(k)}_{\alpha, \sigma_0}, e^{(k)}_{\alpha, \sigma_1} \rangle
\]

\[
= \frac{1}{|e^{(k)}_{\alpha, \sigma_0}| |e^{(k)}_{\alpha, \sigma_1}|} \int Z \tau^* f(\beta)\psi_{e^{(k)}_{\alpha, \sigma_0}}(\beta)\overline{\psi_{e^{(k)}_{\alpha, \sigma_1}}(\beta)}\hat{\Omega}(\beta)
\]

Choose a neighbourhood \( U \) as discussed above around \( x \). We can disregard the part of the integral, which is outside a compact neighbourhood of \( x \) inside \( U \), because of the estimate (7). The remaining integral we evaluate using the stationary phase method (see e.g. [MS]). In fact the situation is very close to the one consider in [KS] in formula (5.22) and the following paragraphs. We use Theorem 1 to change the integral to an oscillatory integral, where the phase is expressed in terms of sums of \( \chi \)'s (one for each of the \( B^{(k)}_\sigma \)'s). The above discussed properties of the \( \chi \) implies the need properties of the phase in order to to apply the stationary phase method, which then yields

\[
\left| \langle T^{(k)}_{f, (\sigma_0, \sigma_1)}, f(x) e^{(k)}_{\alpha, \sigma_1} \rangle - |f(x)|^2 \right| = O\left(\frac{1}{k}\right).
\]
Then we compute
\[
\langle T^{(k)}_{f, (\sigma_0, \sigma_1)}, \frac{e_{\alpha, \sigma_0}^{(k)}}{|e_{\alpha, \sigma_0}|}, T^{(k)}_{f, (\sigma_0, \sigma_1)}, \frac{e_{\alpha, \sigma_0}^{(k)}}{|e_{\alpha, \sigma_0}|} \rangle = \frac{1}{|e_{\alpha, \sigma_0}|} \int_Z \tau^* f(\beta) \psi_{\alpha, \sigma_0}^{(k)}(\beta) \tilde{\Omega}(\beta) = \frac{1}{|e_{\alpha, \sigma_0}|} \int_{Z \times Z} \tau^* f(\beta) \psi_{\alpha, \sigma_0}^{(k)}(\beta) B_{\sigma_1}^{(k)}(\beta, \gamma) \tau^* f(\gamma) \psi_{\alpha, \sigma_0}^{(k)}(\gamma) \tilde{\Omega}(\beta) \tilde{\Omega}(\gamma)
\]
Which we treat by the same method as above (again in parallel to formula (5.22) and the following paragraphs of [KS]) to get that
\[
\left| \langle T^{(k)}_{f, (\sigma_0, \sigma_1)}, \frac{e_{\alpha, \sigma_0}^{(k)}}{|e_{\alpha, \sigma_0}|}, T^{(k)}_{f, (\sigma_0, \sigma_1)}, \frac{e_{\alpha, \sigma_0}^{(k)}}{|e_{\alpha, \sigma_0}|} \rangle - |f(x)|^2 \right| = O\left(\frac{1}{k}\right).
\]
But then we can simply compute
\[
|T^{(k)}_{f, (\sigma_0, \sigma_1)}, \frac{e_{\alpha, \sigma_0}^{(k)}}{|e_{\alpha, \sigma_0}|} - f(x) \frac{e_{\alpha, \sigma_0}^{(k)}}{|e_{\alpha, \sigma_0}|}|^2 + |f(x)|^2
- 2 \text{Re}(T^{(k)}_{f, (\sigma_0, \sigma_1)}, \frac{e_{\alpha, \sigma_0}^{(k)}}{|e_{\alpha, \sigma_0}|}, f(x) \frac{e_{\alpha, \sigma_0}^{(k)}}{|e_{\alpha, \sigma_0}|})
= O\left(\frac{1}{k}\right).
\]

\[
\text{Theorem 19. For all } f \in C^\infty(M) \text{ and any pair of points } (\sigma_0, \sigma_1) \text{ in } T \text{ we have that}
\]
\[
\lim_{k \to \infty} \|T^{(k)}_{f, (\sigma_0, \sigma_1)}\| = \sup_{x \in M} |f(x)|.
\]

We remark that this theorem is a generalization of Theorem 14 by Bordemann, Meinrenken and Schlichenmaier.

\[
\text{Proof. It is clear that } \|T^{(k)}_{f, (\sigma_0, \sigma_1)}\| \leq \sup_{x \in M} |f(x)|. \text{ Choose an } x_0 \in M \text{ such that } |f(x_0)| = \sup_{x \in M} |f(x)|. \text{ Pick an } \alpha_0 \in \mathcal{L}^*_{\sigma_0} \text{ and use Theorem 17 to conclude that}
\]
\[
\|T^{(k)}_{f, (\sigma_0, \sigma_1)}\| \geq \sup_{x \in M} |f(x)|.
\]

\[
\text{Suppose we have a smooth section } X \in C^\infty(M, T_{\sigma}) \text{ of the holomorphic tangent bundle of } M_{\sigma}. \text{ We then claim that the operator } \pi_{\sigma} \nabla_X \text{ is a zero-order Toeplitz operator. Suppose } s_1 \in C^\infty(M, \mathcal{L}^k) \text{ and } s_2 \in H^0(M_{\sigma}, \mathcal{L}^k), \text{ then we have that}
\]
\[
X(s_1, s_2) = (\nabla_X s_1, s_2).
\]
Now, calculating the Lie derivative along $X$ of $(s_1, s_2)\omega^m$ and using the above, one obtains after integration that

$$\langle \nabla_X s_1, s_2 \rangle = -\langle \Lambda d(i_X \omega)s_1, s_2 \rangle,$$

where $\Lambda$ denotes contraction with $\omega$. Thus

$$\pi_\sigma \nabla_X = T^{(k)}_{fx_1 \sigma},$$

as operators from $C^\infty(M, \mathcal{L}^k)$ to $H^0(M_\sigma, \mathcal{L}^k)$, where $f_X = -\Lambda d(i_X \omega)$. Iterating this, we find for all $X_1, X_2 \in C^\infty(T_\sigma)$ that

$$\pi_\sigma \nabla_{X_1} \nabla_{X_2} = T^{(k)}_{fx_2 f_{x_1} - X_2 (f_{x_1}), \sigma}$$

again as operators from $C^\infty(M, \mathcal{L}^k)$ to $H^0(M_\sigma, \mathcal{L}^k)$.

For $X \in C^\infty(M, TM \otimes \mathbb{C})$ and for $s_1, s_2 \in C^\infty(M, \mathcal{L}^k)$, we have that

$$\bar{X}(s_1, s_2) = (\nabla_X s_1, s_2) + (s_1, \nabla_X s_2).$$

Computing the Lie derivative along $\bar{X}$ of $(s_1, s_2)\omega^m$ and integrating, we get that

$$\langle \nabla_{\bar{X}} s_1, s_2 \rangle + \langle (\nabla_X)^* s_1, s_2 \rangle = -\langle \Lambda d(\bar{X} \omega)s_1, s_2 \rangle.$$

Hence we see that

$$\langle \bar{X} \rangle^* = -\langle \nabla_{\bar{X}} - f_{\bar{X}} \rangle$$

as operators on $C^\infty(M, \mathcal{L}^k)$. In particular if $X \in C^\infty(M, T_\sigma)$, we see that

$$\pi_\sigma (\bar{X})^* \pi_\sigma = -T^{(k)}_{f_x, \sigma} |_{H^0(M_\sigma, \mathcal{L}^k)} : H^0(M_\sigma, \mathcal{L}^k) \to H^0(M_\sigma, \mathcal{L}^k).$$

For two smooth sections $X_1, X_2$ of the holomorphic tangent bundle $T_\sigma$ and a smooth function $h \in C^\infty(M)$, we deduce from the formula for $(\nabla_X)^*$ that

$$\pi_\sigma (\nabla_{X_1})^* (\nabla_{X_2})^* h \pi_\sigma = \pi_\sigma \bar{X}_1 \bar{X}_2 (h) \pi_\sigma + \pi_\sigma f_{\bar{X}_1} \bar{X}_2 (h) \pi_\sigma + \pi_\sigma f_{\bar{X}_1} \bar{X}_2 (h) \pi_\sigma + \pi_\sigma f_{\bar{X}_1} \bar{X}_2 (h) \pi_\sigma$$

as operators on $H^0(M_\sigma, \mathcal{L}^k)$.

Suppose $X \in C^\infty(M, TM \otimes \mathbb{C})$. Since we have that $T_{\sigma_0} \cap \bar{T}_{\sigma_1} = \{0\}$, we get a decomposition

$$X = X' + X''$$

where $X' \in C^\infty(M, T_{\sigma_0})$ and $X'' \in C^\infty(M, \bar{T}_{\sigma_1})$. We then have by formula [(10)]

$$\langle \nabla_X \rangle^* = -(\nabla_{X'} + \bar{X} \nu - f_{\bar{X}}).$$

From which we conclude for any $h \in C^\infty(M)$ that

$$\pi_\sigma_1 (\nabla_X)^* h \pi_{\sigma_0} = -\pi_\sigma_1 (h(\nabla_{X''} - f_{\bar{X}}) + \bar{X}(h)) \pi_{\sigma_0}$$

$$= \pi_\sigma_1 (f_{\bar{X}} h - f_{\bar{X}} h + \bar{X}(h)) \pi_{\sigma_0}.$$
Suppose we now have \( X_1, X_2 \in C^\infty(M, TM \otimes \mathbb{C}) \). Then we compute
\[
\pi_\sigma(\nabla X_1) (\nabla X_2)^* h \pi_{\sigma_0} = \pi_\sigma(\nabla \tilde{X}_1 - f \tilde{X}_1)(\nabla \tilde{X}_2 - f \tilde{X}_2) h \pi_{\sigma_0}
\]
\[
= \pi_\sigma \nabla \tilde{X}_1 \nabla \tilde{X}_2 h \pi_{\sigma_0} - \pi_\sigma f \tilde{X}_1 \nabla \tilde{X}_2 h \pi_{\sigma_0} - \pi_\sigma \nabla \tilde{X}_1 f \tilde{X}_2 h \pi_{\sigma_0} + \pi_\sigma f \tilde{X}_1 f \tilde{X}_2 h \pi_{\sigma_0}
\]
\[
= -\pi_\sigma \nabla \tilde{X}_1 \nabla \tilde{X}_2 h \pi_{\sigma_0} - \pi_\sigma f \tilde{X}_1 \nabla \tilde{X}_2 h \pi_{\sigma_0} + \pi_\sigma \nabla \tilde{X}_1 \tilde{X}_2 h \pi_{\sigma_0}
\]
\[
= -\pi_\sigma \nabla \tilde{X}_1 \tilde{X}_2 h \pi_{\sigma_0} + \pi_\sigma f \tilde{X}_1 \tilde{X}_2 h \pi_{\sigma_0}
\]

Now
\[
\pi_\sigma \nabla \tilde{X}_1 \nabla \tilde{X}_2 h \pi_{\sigma_0} = \pi_\sigma h(\nabla \tilde{X}_1 \nabla \tilde{X}_2 + \nabla \tilde{X}_1 \nabla \tilde{X}_2) h \pi_{\sigma_0}
\]
\[
= k \pi_\sigma h \omega(\tilde{X}_1, \tilde{X}_2) h \pi_{\sigma_0} + \pi_\sigma h \nabla \tilde{X}_1 \nabla \tilde{X}_2 h \pi_{\sigma_0}
\]

Hence, by splitting \([\tilde{X}_1, \tilde{X}_2] \in C^\infty(M, TM \otimes \mathbb{C})\) with respect to the direct sum \( TM \otimes \mathbb{C} = T_{\sigma_0} \oplus T_{\sigma_1} \), and by using formula (8) and (9), we see there exists functions \( H_0(X_1, X_2)_{(\sigma_0, \sigma_1)}(h), H_1(X_1, X_2)_{(\sigma_0, \sigma_1)}(h) \in C^\infty(M) \) such that

\[
\pi_\sigma(\nabla X_1)^* (\nabla X_2)^* h \pi_{\sigma_0} = k \pi_\sigma H_0(X_1, X_2)_{(\sigma_0, \sigma_1)}(h) h \pi_{\sigma_0} + \pi_\sigma H_1(X_1, X_2)_{(\sigma_0, \sigma_1)}(h) h \pi_{\sigma_0}.
\]

In fact
\[
H_0(X_1, X_2)_{(\sigma_0, \sigma_1)}(h) = h \omega(\tilde{X}_1, \tilde{X}_2).
\]

\( H_1(X_1, X_2)_{(\sigma_0, \sigma_1)} \) is a second order differential operator as a function of \( h \).

These calculations will be applied in section 6 in the proof of Proposition 2.

5. Hermitian structures on \( \mathcal{V}_k \) and \( \text{End}(\mathcal{V}_k) \).

In this section we consider a further geometric Hermitian structures on \( \mathcal{V}_k \) and recall its asymptotic flatness as proved in [A2].

We will use the following Hermitian structure on \( \mathcal{H}_k \)

\[
\langle s_1, s_2 \rangle_F = \frac{1}{m!} \int_M (s_1, s_2) e^{-F} \omega^m,
\]

where \( s_1, s_2 \) are sections of \( \mathcal{H}_k \) over \( \mathcal{T} \). The associated operator norm on sections of \( \text{End}(\mathcal{V}_k) \) is denoted \( \| \cdot \|_F \).

We recall that \( F(\sigma) \) is the Ricci potential on \( M_\sigma \) for each \( \sigma \in \mathcal{T} \) determined by equation (3). From [A2] we recall that

**Lemma 1.** The Hermitian structure on \( \mathcal{H}_k \)

\[
\langle s_1, s_2 \rangle_F = \frac{1}{m!} \int_M (s_1, s_2) e^{-F} \omega^m
\]

and the constant \( L_2 \)-Hermitian structure on \( \mathcal{H}_k \)

\[
\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M (s_1, s_2) \omega^m.
\]
are equivalent uniformly in \( k \) when restricted to \( \mathcal{V}_k \) over any compact subset \( K \) of \( \mathcal{T} \).

The constant \( L_2 \)-Hermitian structure on \( \mathcal{H}_k \) is not asymptotically flat with respect to \( \hat{\nabla} \).

For any tangent vector field \( V \) on \( \mathcal{T} \) we have
\[
V \langle s_1, s_2 \rangle_F = \langle \hat{\nabla}_V s_1, s_2 \rangle_F + \langle s_1, \hat{\nabla}_V s_2 \rangle_F - \langle V[F]s_1, s_2 \rangle_F.
\]

Let now
\[
E(V)(s) = \frac{1}{k} \left( \langle \hat{\nabla}_V s, s \rangle_F - \langle s, \hat{\nabla}_V s \rangle_F \right).
\]

Then we have by Proposition 2 in [A2] that

**Proposition 1.** The Hermitian structure \([13]\) is asymptotically flat with respect to the connections \( \hat{\nabla} \), i.e. for any compact subset \( K \) of \( \mathcal{T} \) and any vector field \( V \) defined over \( K \), there exists a constant \( C \) such that for all sections \( s \) of \( \mathcal{V}_k \) over \( K \), we have that
\[
|E(V)(s)| \leq C \frac{|s|^2}{k}
\]

over \( K \).

This Proposition is used in section [9].

6. **Asymptotic Flatness of the Coherent States.**

Let us first consider the asymptotics of the parallel transport of the Hitchin connection. Let \( J = [0, 1] \).

**Theorem 20.** Let \( \sigma : J \to \mathcal{T} \) be a curve in Teichmüller space from \( \sigma_0 \) to \( \sigma_1 \) and \( P_\sigma \) the parallel transport in the bundle \( \mathcal{V}_k \) with respect to \( \hat{\nabla} \) along \( \sigma \). Then there exist a function \( g_\sigma \in C^\infty(M) \) such that
\[
\|P_\sigma - T^{(k)}_{\sigma_0, \sigma_1}\| = O(k^{-1}),
\]
where \( \| \cdot \| \) is the operator norm with respect to the \( L_2 \)-norm on \( H^0(M_{\sigma_i}, \mathcal{L}_{\sigma_i}) \), \( i = 0, 1 \).

In order to prove this theorem we first prove

**Proposition 2.** Let \( \sigma : J \to \mathcal{T} \) be a curve in Teichmüller space from \( \sigma_0 \) to \( \sigma_1 \).

Then there exist a unique curve \( g : J \to C^\infty(M) \) such that \( g(0) = 1 \) and
\[
\sup_{t \in J} |\hat{\nabla}_{\sigma'(t)}(T_{g(t), \sigma(t)}s)| = \frac{C}{k} |s|
\]

for some constant \( C \) and all \( s \in H^0(M_{\sigma_0}, \mathcal{L}^k) \).

**Proof.** We recall from [A2] the formula
\[
\pi_{\sigma(t)}(\pi_{\sigma(t)})' = \pi_{\sigma(t)}u(\sigma'(t))^* - \pi_{\sigma(t)}u(\sigma'(t))^* \pi_{\sigma(t)}.
\]

For any choice of \( g : J \to C^\infty(M) \) and \( s \in H^0(M_{\sigma_0}, \mathcal{L}^k) \), we compute using this formula that
\[
\pi_{\sigma(t)} \hat{\nabla}_{\sigma'(t)}(T_{g(t), \sigma(t)}s) = \pi_{\sigma(t)}g'(t)\pi_{\sigma_0} s
\]
\[
\pi_{\sigma(t)}u(\sigma'(t))^* g(t) \pi_{\sigma_0} s - \pi_{\sigma(t)}u(\sigma'(t))^* \pi_{\sigma(t)} g(t) \pi_{\sigma_0} s
\]
\[
- \pi_{\sigma(t)}u(\sigma'(t)) \pi_{\sigma(t)} g(t) \pi_{\sigma_0} s.
\]
By applying Theorem 13 and formulae (8), (9), (11), (12) and (13), we see that there exist a unique smooth map \( A : J \to D(M) \) and a constant \( C \) such that

\[
\pi_{\sigma(t)}\hat{\nabla}_{\sigma'(t)}(T_{g(t),\sigma(t)}s) - \pi_{\sigma(t)}(g'(t) - A(t)(g(t))(s)) \leq \frac{C}{k}|s|.
\]

Let now \( g : J \to C^\infty(M) \) be the unique smooth map, such that \( g(0) = 1 \) and which solves

\[
g'(t) = A(t)(g(t)).
\]

We then have the required estimate. Conversely any smooth \( g : J \to C^\infty(M) \) with the property that the norm estimate (15) is satisfied most solve (18) by formula (17) and Theorem 19. Hence the uniqueness of \( g \) follows from the uniqueness of solutions to ordinary differential equations.

\[
\square
\]

**Proof of Theorem 20.** Let \( \sigma : J \to T \) be a smooth curve in \( T \) from \( \sigma_0 \) to \( \sigma_1 \). Let \( g : J \to C^\infty(M) \) be given by Proposition 2. Define

\[
\Theta_k(t) : \mathcal{V}_{k,\sigma_0} \to \mathcal{V}_{k,\sigma(t)}
\]

by

\[
\Theta_k(t) = P_{\sigma,t} - T_{g(t),\sigma(t)}^{(k)},
\]

where \( P_{\sigma,t} \) is the parallel transport in \( \mathcal{V}_k \) along \( \sigma \) from \( \sigma_0 \) to \( \sigma(t) \). Let \( s \in H^0(M_{\sigma_0}, \mathcal{L}^k) \) such that \( |s|_F = 1 \) and define \( n_k : J \to [0, \infty) \) by

\[
n_k(t) = |\Theta_k(t)s|^2_F.
\]

Then the functions \( n_k \) are differentiable in \( t \) and we compute that

\[
\frac{d n_k}{dt} = (\hat{\nabla}_{\sigma'(t)}(\Theta_k(t)s), \Theta_k(t)s)_F + (\Theta_k(t), \hat{\nabla}_{\sigma'(t)}(\Theta_k(t)s))_F + E(\Theta_k(t)s) \]

\[
= -(\hat{\nabla}_{\sigma'(t)}T_{g(t),\sigma(t)}^{(k)}(\Theta_k(t)s), \Theta_k(t)s)_F - (\Theta_k(t)s, \hat{\nabla}_{\sigma'(t)}T_{g(t),\sigma(t)}^{(k)}(\Theta_k(t)s))_F + E(\Theta_k(t)s).
\]

Using the above, we get the following estimate

\[
\left| \frac{d n_k}{dt} \right| \leq 2|\hat{\nabla}_{\sigma'(t)}T_{g(t),\sigma(t)}^{(k)}(\Theta_k(t)s)|_F|\Theta_k(t)s|_F + |E(\Theta_k(t)s)| \]

\[
\leq 2|\hat{\nabla}_{\sigma'(t)}T_{g(t),\sigma(t)}^{(k)}(\Theta_k(t)s)|_F n_k^{1/2} + |E(\Theta_k(t)s)|.
\]

Now we use Lemma 1 and Propositions 1 and 2 to obtain that there exists a constant \( C \) independent of \( s \) such that

\[
\left| \frac{d n_k}{dt} \right| \leq \frac{C}{k}(n_k^{1/2} + n_k).
\]

This estimate implies that

\[
n_k(t) \leq (\exp(\frac{Ct}{2k}) - 1)^2.
\]

Thus

\[
|P_{\sigma,t} - T_{g(t),\sigma_1}^{(k)} s|_F = |\Theta_k(1)s|_F \leq C_1 n_k(1)^{1/2}.
\]

The Theorem then follows from these estimates combined with Lemma 1.

\[
\square
\]
Let $x \in M$ be a point in the moduli space. Let $\alpha$ be a point in $\mathcal{L}_x - \{0\}$. As in section 4 we consider the associated section $e^{(k)}_\alpha$ of $\mathcal{V}_k$ determined by $\alpha$ by formula (6).

**Theorem 21.** Let $\sigma : J \to \mathcal{T}$ be a curve from $\sigma_0$ to $\sigma_1$. Let $g_\sigma \in C^\infty(M)$ be the function determined by $\sigma$ in Theorem 20. Then

$$|P_\sigma(e^{(k)}_{\alpha,\sigma_0}) - g_\sigma(x)e^{(k)}_{\alpha,\sigma_0}| = O(k^{-1}).$$

This theorem follows directly from Theorem 20 and Theorem 17.

Define a section $E^{(k)}_x$ of $\text{End}(\mathcal{V}_k)$ from the section $e^{(k)}_\alpha$ of $\mathcal{V}_k$ as follows

$$E^{(k)}_x(s) = \langle s, e^{(k)}_\alpha \rangle e^{(k)}_\alpha.$$

**Proof of Theorem 4.** Let $\sigma : J \to \mathcal{T}$ be a curve from $\sigma_0$ to $\sigma_1$. For $s \in H^0(M_{\sigma_1}, \mathcal{L}^k)$, we have that

$$P_\sigma(E^{(k)}_x(\sigma_0))(s) = P_\sigma \circ E^{(k)}_x(\sigma_0)P_\sigma^{-1}(s)$$

$$= \langle P_\sigma^{-1}(s), e^{(k)}_{\alpha,\sigma_0} \rangle P_\sigma(e^{(k)}_{\alpha,\sigma_0})$$

$$= \langle s, (P_\sigma^{-1})^*(e^{(k)}_{\alpha,\sigma_0}) \rangle P_\sigma(e^{(k)}_{\alpha,\sigma_0}).$$

But then by theorem 21 we get that

$$|P_\sigma(E^{(k)}_x(\sigma_0)) - \langle s, g_\sigma(x)^{-1}e^{(k)}_{\alpha,\sigma_1} \rangle \otimes g_\sigma(x)e^{(k)}_{\alpha,\sigma_1}| = O(k^{-1}).$$

Which implies the theorem since

$$E^{(k)}_x(\sigma_1) = \langle s, g_\sigma(x)^{-1}e^{(k)}_{\alpha,\sigma_1} \rangle \otimes g_\sigma(x)e^{(k)}_{\alpha,\sigma_1}.$$

□

7. **The almost fixed vector**

Let $\Lambda \subset SU(2)$ be the finite subgroup

$$\Lambda = \left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Let $X$ be the image

$$X = \text{Im}(\text{Hom}'(\tilde{\pi}_1(\Sigma), \Lambda) \to M).$$

Choosing a standard set of generators for $\pi_1(\Sigma - \{p\})$ and mapping the $i$'th pair to the two generators of $\Lambda$ given above and the rest to the identity, we see that $X$ contains at least $g$ points. Moreover, $X$ is clearly a $\Gamma$-invariant subset of $M$.

As stated in the introduction, we then define the section $E_{\alpha}^{(k)}$ of $\text{End}(\mathcal{V}_k)$ by

$$E_{\alpha}^{(k)} = \sum_{x \in X} E^{(k)}_x.$$
and let $E_{X,0}^{(k)}$ be the traceless part of $E_X^{(k)}$. Thus

$$E_{X,0}^{(k)} = E_X^{(k)} - \frac{\text{Tr}(E_X^{(k)})}{d_g(k)} \text{Id}.$$  

By formula (7) we have that $\text{Tr}(E_X^{(k)}(\sigma_0))$ converges to $|X|$ and therefore so does

$$|E_{X,0}^{(k)}|^2 = \langle E_{X,0}^{(k)}(\sigma_0), E_{X,0}^{(k)}(\sigma_0) \rangle = \text{Tr}((E_{X,0}^{(k)}(\sigma_0))^2),$$

since $d_g(k)$ is a polynomial of degree $3g - 3$. Hence we see that for sufficiently large $k$, $E_{X,0}^{(k)} \neq 0$.

Hence for large enough $k$ we have a unique unit vector $E_{X,0}^{(k)} \in H_k$, which at $\sigma_0$ agrees with $E_{X,0}^{(k)}(\sigma_0)/[E_{X,0}^{(k)}(\sigma_0)]$.

**Proof of Theorem 6**. Suppose we have a $\phi \in \Gamma$. We then have that

$$\phi^*(E_{X,0}^{(k)}) = E_{X,0}^{(k)}$$

which means that

$$\phi^* \circ E_{X,0}^{(k)}(\sigma_0) \circ (\phi^*)^{-1} = E_{X,0}^{(k)}(\phi(\sigma_0)).$$

Hence

$$\phi(E_{X,0}^{(k)}) = \frac{1}{[E_{X,0}^{(k)}(\sigma_0)]} P_{\phi(\sigma_0),\sigma_0}^{(k)}(E_{X,0}^{(k)}(\phi(\sigma_0)))$$

$$= \frac{1}{[E_{X,0}^{(k)}(\sigma_0)]} (P_{\phi(\sigma_0),\sigma_0}^{(k)}(E_{X}^{(k)}(\phi(\sigma_0)))) - \frac{\text{Tr}(E_X^{(k)})}{d_g(k)} \text{Id}.$$  

But from Theorem 5 and Theorem 4 we then get that there exist a constant $\tilde{C}$ such that

$$[E_{X,0}^{(k)} - \phi(E_{X,0}^{(k)})] \leq \frac{C}{[E_{X,0}^{(k)}(\sigma_0)]} |E_{X,0}^{(k)}(\sigma_0)| - P_{\phi(\sigma_0),\sigma_0}^{(k)}(E_{X}^{(k)}(\phi(\sigma_0))) \leq \frac{\tilde{C}}{k}$$

for all sufficiently large $k$.  

\[\square\]

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Department of Mathematics, University of Aarhus, DK-8000, Denmark
E-mail address: andersen@imf.au.dk