Two- and Three-dimensional Generalisation of Lower Order Local Wave Velocities

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Abstract

A general local approach for the definition of velocities and especially phase velocities for waves recently proposed for one-dimensional waves is generalized for 2 and 3 dimensional scalar wave. A geometrically consistent generalization has been found for the local wave velocities of order zero and one.

1 Introduction

This paper is an extension of our recent work [1] being devoted to the definition and basic properties of local velocities spectrum of arbitrary waves in 1+1 space-time dimensions. There it has been mentioned, that the local concept proposed allows a straightforward and unique generalization on an arbitrary dimension \( N \), at least for the physically most important cases of the ordinary zero order phase velocity (0-PV or attribute velocity) and the first order velocity (1-PV or peak velocity).

The formalism performed here is still developed for an object defined by a distribution of some one-valued ”attribute” in the space-time as a single valued function of coordinates \( \{ x, t \} \) that can be considered therefore as a scalar field.

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Hence, the velocities determined below describe propagation of perturbations of this field.

2 Velocity of zero-order (0-PV)

Let the state of a medium be now described in 2-dimensional space \((x, y)\) as a scalar field \(\psi(x, y, t)\). As an illustration one can imagine a propagating perturbation on a two-dimensional surface, e.g. a brane.

We follow the approach developed in Sec.2-4 of [1] and assume, the traced attribute is the certain fixed value of the field: \(\psi(x, y, t) = \psi_0 = \text{const}\). This condition leads to the requirement:

\[
d\psi \equiv \psi_x dx + \psi_y dy + \psi_t dt = 0,
\]

that means for velocity components \(v^{(00)}\):

\[
\psi_x \frac{dx}{dt} + \psi_y \frac{dy}{dt} = \psi_x v_x^{(00)} + \psi_y v_y^{(00)} = -\psi_t
\]

(2.2)

The lower indices denote conventionally the partial derivative with respect to the corresponding variable.

Since the forms of kind

\[
v_x^{(00)} = -C_x \frac{\psi_t}{\psi_x}, \quad v_y^{(00)} = -C_y \frac{\psi_t}{\psi_y}
\]

(2.3)

satisfy Eq. (2.2) with arbitrary coefficients \(C_x, C_y\) obeying the relation

\[
C_x + C_y = 1
\]

(2.4)

the definition (2.3) seems to be ambiguous. However, the coefficients \(C_x, C_y\) can be fixed through the meaningful geometrical requirement of transformation properties. Indeed, if we set \(C_x = C_y = 1/2\), we make sure that the doublet of reciprocal velocities: \(\tilde{v}^{(00)} \equiv \{1/v_x^{(00)}; 1/v_y^{(00)}\}\) transforms under coordinate change \((x, y) \to (X, Y)\) as a covariant vector (co-vector) in the basis of the coordinate system.

\[
v_x^{(00)} = -\frac{1}{2} \frac{\psi_t}{\psi_x} X_x + \frac{\psi_t}{\psi_y} Y_x, \quad v_y^{(00)} = -\frac{1}{2} \frac{\psi_t}{\psi_x} X_y + \frac{\psi_t}{\psi_y} Y_y
\]

(2.5)

that means:

\[
\tilde{v}^{(00)} \equiv \left\{ \begin{array}{c} 1/v_x^{(00)} \\ 1/v_y^{(00)} \end{array} \right\} = \left[ \begin{array}{cc} \frac{\partial X}{\partial x} & \frac{\partial Y}{\partial x} \\ \frac{\partial X}{\partial y} & \frac{\partial Y}{\partial y} \end{array} \right] \left\{ \begin{array}{c} 1/v_X^{(00)} \\ 1/v_Y^{(00)} \end{array} \right\}
\]

(2.6)
Similarly, for the 0-PV components of three-dimensional scalar wave \( \psi(x, y, z, t) \) (e.g. pressure in gas or liquid) we will obtain

\[
v_x^{(000)} = -\frac{1}{3} \frac{\psi_t}{\psi_x}; \quad v_y^{(000)} = -\frac{1}{3} \frac{\psi_t}{\psi_y}; \quad v_z^{(000)} = -\frac{1}{3} \frac{\psi_t}{\psi_z}
\] (2.7)

and the reciprocal velocity triplet \( \tilde{v}^{(000)} \equiv \{1/v_x^{(000)}; 1/v_y^{(000)}; 1/v_z^{(000)}\} \) transforms as a 3-dimensional co-vector.

Generally, the components of N-dimensional 0-PV for scalar wave \( \psi(x_1, x_2, ..., x_N, t) \) are defined as

\[
v_i^{(N\times0)} = -\frac{1}{N} \frac{\partial \psi}{\partial t} \frac{\partial \psi}{\partial x_i}
\] (2.8)

with the corresponding transformation properties for \( N \)-multiplet of reciprocal components \( 1/v_i^{(N\times0)} \) as an \( N \)-dimensional co-vector.

3 First-order velocity (1-PV)

3.1 Two-dimensional case

Suppose, the traced point is labelled by an attribute of first order, i.e. a fixed value of derivative. To this end we fix two values for spatial partial derivatives of \( \psi \)

\[
\{ \psi_x = C_1; \quad \psi_y = C_2, \}
\] (3.1)

where \( C_1, C_2 \)— some constants. Especially it can be zero, if we trace propagation of a positive/negative peak or a saddle point of the surface.

\[
d \left\{ \begin{array}{l} \psi_x \\ \psi_y \end{array} \right\} = 0 = \left\{ \begin{array}{l} \psi_{xx}dx + \psi_{xy}dy + \psi_{xt}dt \\ \psi_{xy}dx + \psi_{yy}dy + \psi_{yt}dt \end{array} \right\}
\] (3.2)

This condition leads to the system of two equations

\[
\psi_{xx}v_x^{(11)} + \psi_{xy}v_y^{(11)} = -\psi_{xt} \\
\psi_{xy}v_x^{(11)} + \psi_{yy}v_y^{(11)} = -\psi_{yt},
\] (3.3)

and the concerning two components of 1-PV are the solutions:

\[
v_y^{(11)} = \begin{vmatrix} \psi_{xy} & \psi_{xt} \\ \psi_{yy} & \psi_{yt} \end{vmatrix} \equiv \frac{\psi_{xy}\psi_{xt} - \psi_{xx}\psi_{yt}}{\psi_{xx}\psi_{yy} - \psi_{xy}^2} \quad v_x^{(11)} = \begin{vmatrix} \psi_{xy} & \psi_{xt} \\ \psi_{yy} & \psi_{yt} \end{vmatrix} \equiv \frac{\psi_{xy}\psi_{yt} - \psi_{yy}\psi_{xt}}{\psi_{xx}\psi_{yy} - \psi_{xy}^2}.
\] (3.4)
To recover a geometric nature of this doublet, we perform a transformation of coordinates as in the former case \((x, y) \rightarrow (X, Y)\).

Obviously, the transformation of \(v^{(11)} = \{v_x^{(11)}, v_y^{(11)}\}\) is non-linear and involves second derivatives of coordinates as well as first derivatives of the function \(\psi\). In other words, the finite transformation law shows, that, for an arbitrary coordinate change, the doublet \(v^{(11)}\) transforms not solely through its components itself, but also through the components of lower order velocity \(v^{(00)}\).

On account of this we consider the infinitesimal coordinate transformation and respect only the linear part of it, it means, we omit all the terms containing second derivatives of coordinates. It arranges to establish the transformation law in the form:

\[
\begin{pmatrix}
 v_x^{(11)} \\
v_y^{(11)}
\end{pmatrix} = \left[\frac{\partial(X, Y)}{\partial(x, y)}\right]^{-1} \begin{pmatrix}
v_X^{(11)} \\
v_Y^{(11)}
\end{pmatrix}
\]

where the Jacobi-matrix:

\[
J = \left[\frac{\partial(X, Y)}{\partial(x, y)}\right] = \begin{bmatrix}
\frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\
\frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y}
\end{bmatrix},
\]

and we assured, that the doublet \(v^{(11)}\) transforms as a true vector (contra-variant vector) in the basis of spatial coordinates \(\{x, y\}\). It follows from the Sec.2, that the contraction product

\[
Y^{(11)}_{(00)} = \tilde{v}^{(00)} \cdot v^{(11)} \equiv \frac{1}{v_x^{(00)}} v_x^{(11)} + \frac{1}{v_y^{(00)}} v_y^{(11)}
\]

is a dimensionless value and behaves as a true scalar under transformations of spatial coordinates.

The both vectors respect the mirror transformations of the coordinates \((x, y) \rightarrow (-x, -y)\) as true vectors, as it is easy to see from their definitions.

### 3.2 Three- and higher-dimensional

A quite analogous procedure holds for the definition of 1-PV \(v^{(111)}\) in a three-dimensional case. As an example a scalar field of potential \(\psi(x, y, z)\) can be considered. The velocity of propagation of a gradient has to be defined. It follows from the condition:

\[
d\begin{pmatrix}
 \psi_x \\
 \psi_y \\
 \psi_z
\end{pmatrix} = \begin{pmatrix}
 \psi_{xx} dx + \psi_{xy} dy + \psi_{xz} dz + \psi_{xt} dt \\
 \psi_{xy} dx + \psi_{yy} dy + \psi_{yz} dz + \psi_{yt} dt \\
 \psi_{xz} dx + \psi_{yz} dy + \psi_{zz} dz + \psi_{zt} dt
\end{pmatrix} = 0
\]
With the determinants of this system:

\[
D = \begin{vmatrix}
\psi_{xx} & \psi_{xy} & \psi_{xz} \\
\psi_{xy} & \psi_{yy} & \psi_{yz} \\
\psi_{xz} & \psi_{yz} & \psi_{zz}
\end{vmatrix},
\]

\[
D_x = \begin{vmatrix}
-\psi_{xt} & \psi_{xy} & \psi_{xz} \\
-\psi_{yt} & \psi_{yy} & \psi_{yz} \\
-\psi_{zt} & \psi_{yz} & \psi_{zz}
\end{vmatrix},
\]

\[
D_y = \begin{vmatrix}
\psi_{xx} & -\psi_{xt} & \psi_{xz} \\
\psi_{xy} & -\psi_{yt} & \psi_{yz} \\
\psi_{xz} & -\psi_{zt} & \psi_{zz}
\end{vmatrix},
\]

\[
D_z = \begin{vmatrix}
\psi_{xx} & \psi_{xy} & -\psi_{xt} \\
\psi_{xy} & \psi_{yy} & -\psi_{yt} \\
\psi_{xz} & \psi_{yz} & -\psi_{zt}
\end{vmatrix},
\]

(3.9)

the components of the 3-dimensional 1-PV vector are defined as:

\[
v_x^{(111)} = \frac{D_x}{D}, \quad v_y^{(111)} = \frac{D_y}{D}, \quad v_z^{(111)} = \frac{D_z}{D}
\]

(3.10)

and it follows for the dimensionless scalar velocity

\[
V_{(000)}^{(111)} = \tilde{v}_{(000)} \cdot v^{(111)} \equiv \frac{1}{v_{x}^{(000)}} v_{x}^{(111)} + \frac{1}{v_{y}^{(000)}} v_{y}^{(111)} + \frac{1}{v_{z}^{(000)}} v_{z}^{(111)}
\]

(3.11)

respectively.

The similar analysis can be performed and remains to be valid for an arbitrary space dimension \(N\).

References

[1] I. Drozdov, A. Stahlfen, arXiv/ math-ph /0612023.