SOME DIOPHANTINE EQUATIONS AND INEQUALITIES WITH PRIMES

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Abstract. We consider the solutions to the inequality

$$|p_{c_1}^c + \cdots + p_{c_s}^c - R| < R^{-\eta}$$

(where $c > 1$, $c \notin \mathbb{N}$ and $\eta$ is a small positive number; $R$ is large).

We obtain new ranges of $c$ for which this has many solutions in primes $p_1, \ldots, p_s$, for $s = 2$ (and ‘almost all’ $R$), $s = 3, 4$ and $5$.

We also consider the solutions to the equation in integer parts

$$[p_{c_1}^c] + \cdots + [p_{c_s}^c] = r$$

where $r$ is large. Again $c > 1$, $c \notin \mathbb{N}$. We obtain new ranges of $c$ for which this has many solutions in primes, for $s = 3$ and $5$.

1. Introduction

Let $c > 1$, $c \notin \mathbb{N}$. Let $\eta$ be a small positive number depending on $c$. Let $R$ be a large positive number. We consider solutions in primes of the inequality

$$(1)_s \quad |p_{c_1}^c + \cdots + p_{c_s}^c - R| < R^{-\eta}$$

first studied by Šapiro-Pyateckiĭ [38]. We also consider the equation in integer parts

$$(2)_s \quad [p_{c_1}^c] + \cdots + [p_{c_s}^c] = r.$$ 

We give results providing large numbers of solutions of $$(1)_s$$ ($s = 2, 3, 4, 5$) and $$(2)_s$$ ($s = 3, 5$) for new ranges of $c$. For $s = 2$, one needs to restrict $R$ to ‘almost all’ real numbers in an interval $[V, 2V]$.

Following a nice innovation in a paper of Cai [10], there has been recent progress in all these cases; see below for details. In the present paper progress is made by combining this innovation with the powerful

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exponential sum bounds of Huxley [23], Bourgain [6], and Heath-Brown [22]. When discussing (1), (1)\textsubscript{3}, (1)\textsubscript{5} and (2)\textsubscript{5}, we use a vector sieve in conjunction with the Harman sieve. The other cases are simpler and Heath-Brown’s generalized Vaughan identity replaces the sieve method.

We write ‘\( n \sim N \)’ to signify \( N < n \leq 2N \). Let \( X = R^{1/c} \).

Let \( A_s(R) \) denote the number of solutions of (1)\textsubscript{s} with \( \frac{X}{s} < p_j \leq X \) \( (j = 1, \ldots, s) \). Let \( B_s(r) \) denote the number of solutions of (2)\textsubscript{s} with \( \frac{X}{s} < p_j \leq X \) \( (j = 1, \ldots, s) \). One expects heuristically to obtain (at least for \( c \) not too large) the bounds

\[
(3)_s \quad A_s(R) \gg \frac{R^\frac{s}{c} - 1 - \eta}{(\log R)^s}
\]

and

\[
(4)_s \quad B_s(r) \gg \frac{r^\frac{s}{c} - 1}{(\log r)^s}.
\]

**Theorem 1.** Let \( V \) be large. Suppose that \( c < \frac{39}{29} = 1.3448 \ldots, c \neq 4/3 \). We have (3)\textsubscript{2} for all \( R \) in \([V, 2V]\) except for a set of \( R \) having measure \( O(V \exp(-C(\log V)^{1/4})) \).

(We denote by \( C \) a positive absolute constant, not the same at each occurrence.)

Previous upper bounds for permissible \( c \):

\[ 17/16 \ [26], \ 15/14 \ [27], \ 43/36 \ [16], \ 59/44 = 1.3409 \ldots \ [12]. \]

**Theorem 2.** Let \( R \) be large. Suppose that \( c < 6/5 \). Then (3)\textsubscript{3} holds.

Previous upper bounds:

\[ 15/14 \ [26], \ 13/12 \ [8], \ 11/10 \ [9, 25], \ 237/214 \ [13], \]
\[ \frac{61}{55} \ [24], \ \frac{10}{9} \ [5], \ \frac{43}{36} = 1.1944 \ldots \ [10]. \]

**Theorem 3.** Let \( R \) be large. Suppose that \( c < 39/29 \). Then (3)\textsubscript{4} holds.

Previous upper bounds:

\[ \frac{97}{81} \ [35], \ \frac{6}{5} \ [31], \ \frac{59}{44} \ [12], \ \frac{1198}{889} = 1.3419 \ldots \ [33]. \]

**Theorem 4.** Let \( R \) be large. Suppose that \( c < \frac{478}{181} = 2.6549 \ldots \). Then (3)\textsubscript{5} holds.
Previous upper bounds:

\[
1.584 \ldots [14], \quad \frac{1 + \sqrt{5}}{2} \quad [18], \quad \frac{81}{40} \quad [14], \quad \frac{108}{53} \quad [35], \quad 2.041 \quad [4],
\]
\[
2.08 \quad [12], \quad \frac{66576}{319965} = 2.0801 \ldots [30].
\]

**Theorem 5.** Let \( n \) be large. Suppose that \( c < \frac{3581}{3106} = 1.1529 \ldots \). Then (4)_3 holds.

Previous upper bounds:

\[
\frac{17}{16} \quad [28], \quad \frac{12}{11} \quad [25], \quad \frac{258}{235} \quad [13], \quad \frac{137}{119} \quad [11], \quad \frac{3113}{2703} = 1.1516 \ldots [32].
\]

**Theorem 6.** Let \( n \) be large. Suppose that \( c < \frac{609}{293} = 2.0784 \ldots \). Then (4)_5 holds.

Previous upper bounds:

\[
\frac{4109054}{1999527} \quad [29], \quad \frac{408}{197} = 2.071 \ldots [34].
\]

Along usual lines, we employ a continuous function \( \phi : \mathbb{R} \to [0,1] \) such that

\[
(1.1) \quad \phi(y) = 0 \quad (|y| \leq R^{-\eta}), \quad \phi(y) = 1 \quad \left( |y| \leq \frac{4R^{-\eta}}{5} \right),
\]

with Fourier transform

\[
\Phi(x) := \int_{-\infty}^{\infty} e(-xy)\phi(y)dy , \text{ where } e(\theta) := e^{2\pi i \theta},
\]

satisfying

\[
(1.2) \quad \int_{|x|>X^{2\eta}} |\Phi(x)| \, dx \ll X^{-3}.
\]

We define

\[
\tau = X^{8n-c}, \quad K = X^{2\eta}, \quad \mathcal{L} = \log X, \quad P(z) = \prod_{p<z} p.
\]
Let \( \rho(n) \) denote the indicator function of the prime numbers. For \( u \in \mathbb{N}, z > 1, \) let

\[
\rho(u, z) = 1 \text{ if } (u, P(z)) = 1.
\]

Let \( \rho(u, z) = 0 \) otherwise. For a vector sieve one usually uses functions \( \rho^-(\ldots), \rho^+(\ldots) \) with

\[
\rho^-(n) \leq \rho(n) \leq \rho^+(n);
\]

but (without loss) we shall take \( \rho^- = \rho, \) so that the inequality basic to [7] becomes

\[
(1.3) \quad \rho(m)\rho(\ell) \geq \rho^+(m)\rho(\ell) + \rho(m)\rho^+(\ell) - \rho^+(m)\rho^+(\ell).
\]

We shall need exponential sums

\[
S(x) = \sum_{\frac{X}{8} < n \leq X} \rho(n)e(n^c x), S_1(x) = \sum_{\frac{X}{8} < n \leq X} \rho(n) \log n \ e(n^c x),
\]

\[
S^+(x) = \sum_{\frac{X}{8} < n \leq X} \rho^+(n)e(n^c x),
\]

\[
T(x) = \sum_{\frac{X}{8} < n \leq X} \rho(n)e([n^c] x), T_1(x) = \sum_{\frac{X}{8} < n \leq X} \rho(n) \log n \ e([n^c] x),
\]

\[
T^+(x) = \sum_{\frac{X}{8} < n \leq X} \rho^+(n)e([n^c] x),
\]

and approximating functions

\[
I(x) = \int_{X/8}^X e(t^c x) dt,
\]

\[
J(x) = \sum_{(\frac{X}{8})^{c} < m \leq X^c} \frac{1}{c} m^{\frac{1}{c}-1} e(xm).
\]

In using Cai’s idea we also need the sums

\[
A(x) = \sum_{\frac{X}{8} < n \leq X} e(n^c x), \quad B(x) = \sum_{\frac{X}{8} < n \leq X} e([n^c] x).
\]
We now describe briefly the underlying principle of the proofs. For (3) we use
\[ L^s A_s(R) \gg \sum_{\frac{R}{s} < p_j \leq X \quad (j=1,\ldots,s)} \log p_1 \cdots \log p_s \phi(p_1^c + \cdots + p_s^c - R) \]
\[ = \sum_{\frac{R}{s} < p_j \leq X \quad (j=1,\ldots,s)} \log p_1 \cdots \log p_s \int_{\mathbb{R}} \Phi(x)e((p_1^c + \cdots + p_s^c)x)e(-Rx)dx \]
\[ = \int_{\mathbb{R}} S_1(x)^s \Phi(x)e(-Rx)dx. \]
For (4) we use
\[ L^s B_s(R) \gg \sum_{\frac{R}{s} < p_j \leq X \quad (j=1,2,3)} \log p_1 \cdots \log p_s \int_{-1/2}^{1/2} e(([p_1^c] + [p_2^c] + [p_3^c] - r)x)dx \]
\[ = \int_{-1/2}^{1/2} T_1(x)^3 e(-rx)dx. \]
Modifying this for (3), (3), we use
\[ A_s(R) \geq \sum_{\frac{R}{s} < m_1,\ldots,m_{s-2},m,\ell \leq X} \rho(m_1) \cdots \rho(m_{s-2}). \]
\[ (\rho^+(m)\rho(\ell) + \rho(m)\rho^+(\ell) - \rho^+(m)\rho^+(\ell)). \]
\[ \phi(m_1^c + \cdots + m_{s-2}^c + m^c + \ell^c - R) \]
\[ = \int_{-\infty}^{\infty} S(x)^{m-2}(2S(x)S^+(x) - S^+(x)^2)\Phi(x)e(-Rx)dx \]
and for (4),
\[(1.7)\]
\[\mathcal{B}_s(n) \geq \sum_{\frac{X}{\Delta} < m_1, m_2, m_3, n \leq X} \rho(m_1) \rho(m_2) \rho(m_3) \]
\[= \int_{-1/2}^{1/2} e((\lfloor m_1 \rfloor + \lfloor m_2 \rfloor + \lfloor m_3 \rfloor + \lfloor \ell \rfloor - r)x)dx\]
\[= \int_{-1/2}^{1/2} (2T(x)^4 T^+(x) - T(x)^3 T^+(x)^2)e(-rx)dx.\]

Note that the function \(\rho^+\) in Theorem 3 is different (although we still write \(\rho^+\)) from the function in Theorems 4 and 6.

We give a few more indications of method in the simplest case (3)_4. The 'major arc' \(\mathcal{M}\) is \((-\tau, \tau)\), and \(\mathbb{R}\setminus\mathcal{M}\) is the 'minor arc'.

(i) Show that the last integral in (1.4) reduces to the corresponding integral over \((-\tau, \tau)\) with acceptable error (the 'minor arc' stage).

(ii) Show that the integrand over \((-\tau, \tau)\) can be replaced by
\[\Phi(x)e(-Rx)\] with acceptable error (first part of 'major arc' stage).

(iii) Extend the integral from \((-\tau, \tau)\) to \(\mathbb{R}\) with acceptable error and obtain the lower bound (3)_3 for this last integral (second part of 'major arc' stage).

The other cases (3)_4, (4)_4 are similar in principle, but more complicated. The innovations are in part (i) in each case.

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2. Lemmas for the minor arc.

For \(N \geq 1\), we write \(I(N)\) for a subinterval of \((N, 2N]\), not the same at each occurrence. For a real function \(f\) on \([N, 2N]\) we write
\[S(f, N) = \sum_{n \in I(N)} e(f(n)).\]

The fractional part of \(x\) is written as \(\{x\}\). We write \(A \asymp B\) for \(A \ll B\). Implied constants in the conclusions of the lemmas depend
Lemma 1. Let $x > 0$. For real numbers $a_n, |a_n| \leq 1$, let

$$W(X, x) = \sum_{\frac{X}{x} < n \leq X} a_n e(x[n^c]).$$

For $2 \leq H \leq X$, we have

$$W(X, x) \ll \frac{X L}{H} + \sum_{0 \leq h < H} \min \left(1, \frac{1}{h}\right) \left| \sum_{\frac{X}{x} < n \leq X} a_n e((h + \gamma)m^c) \right|$$

$$+ \sum_{1 \leq h \leq H} \frac{1}{h} \left| \sum_{\frac{X}{x} < n \leq X} e(hn^c) \right| + \sum_{h > H} \frac{H}{h^2} \left| \sum_{\frac{X}{x} < n \leq X} e(hn^c) \right|.$$

Here $\gamma = \{x\}$ or $-\{x\}$.

Proof. Let $\alpha = \{x\}$. By [2], Lemma 2.3, we have

$$e(-\alpha\{t\}) = \sum_{|h| \leq H} c_h(\alpha) e(ht) + O \left( \min \left(1, \frac{1}{H\|t\|}\right) \right)$$

($t \in \mathbb{R}$), where

$$c_h(\alpha) = \frac{1 - e(-\alpha)}{2\pi i(h + \alpha)}.$$

Moreover,

$$\min \left(1, \frac{1}{H\|t\|}\right) = \sum_{h = -\infty}^{\infty} c_h e(ht)$$

where

$$c_h \ll \min \left(\frac{\log H}{H}, \frac{1}{|h|}, \frac{H}{h^2}\right);$$

see e.g. [21]. Note that for real $u$, this implies

$$e(\alpha[u]) = e(\alpha u) e(-\alpha\{u\})$$

$$= e(\alpha u) \sum_{|h| \leq H} c_h(\alpha) e(hu) + O \left( \sum_{h = -\infty}^{\infty} c_h e(hu) \right).$$

Set $u = n^c$, so that

$$e(x[n^c]) = e(\alpha[n^c]).$$
Summing over $n$,

\[
W(X, x) = \sum_{x^c n \leq X} a_n e(\alpha [n^c]) = \sum_{|h| \leq H} c_h(\alpha) \sum_{\frac{x^c}{h} n \leq X} a_n e((h + \alpha) n^c)
\]

\[
+ O \left( \sum_{h = -\infty}^{\infty} c_h \sum_{\frac{x^c}{h} n \leq X} e(h n^c) \right).
\]

The ‘$O$’ term yields

\[
\ll \frac{X \log X}{H} + \sum_{1 \leq h \leq H} \frac{1}{h} \left| \sum_{\frac{x^c}{h} n \leq X} e(h n^c) \right| + \sum_{h > H} \frac{H}{h^2} \left| \sum_{\frac{x^c}{h} n \leq X} e(h n^c) \right|.
\]

For the remaining terms we use

\[
c_h(\alpha) \ll \begin{cases} 
1 & \text{if } h = 0 \\
\frac{1}{h} & \text{(} h \neq 0 \text{)}
\end{cases}
\]

and note that for $h = -1, -2, \ldots, -[H]$ we have

\[
h + \alpha = -(|h| - \alpha). \quad \square
\]

**Lemma 2** (Kusmin-Landau). If $f$ is continuously differentiable, $f'$ is monotonic, and $|f'| \geq \lambda$ on $[N, 2N]$, then

\[
S(f, N) \ll \lambda^{-1}.
\]

**Proof.** [19, Theorem 2.1]. \quad \square

**Lemma 3** (A process). For $1 \leq Q \leq N$,

\[
S(f, N)^2 \ll \frac{N}{Q} \sum_{|q| < Q} |S(f_q, N)|
\]

where $f_q(x) = f(x + q) - f(x)$.

**Proof.** [19, p. 10]. \quad \square

**Lemma 4** (B process). Suppose that $f'' \asymp FN^{-2}$ on $[N, 2N]$, where $F > 0$, and

\[
f^{(j)}(x) \ll FN^{-j} \quad (j = 3, 4) \text{ for } x \in [N, 2N].
\]
Define $x_\nu$ by $f'(x_\nu) = \nu$ and let

$$\phi(\nu) = -f(x_\nu) + \nu x_\nu.$$  

Then for $I(N) = [a, b]$,  

$$S(f, N) = \sum_{f'(b) \leq \nu \leq f'(a)} \frac{e(-\phi(\nu) - 1/2)}{|f''(x_\nu)|^{1/2}} + O(\log(FN^{-1} + 2) + F^{-1/2}N).$$

Proof. [19, Lemma 3.6]. □

Lemma 5. (i) Let $\ell \geq 0$ be a given integer, $L = 2^\ell$. Suppose that $f$ has $\ell + 2$ continuous derivatives on $[N, 2N]$ and  

$$|f^{(r)}(x)| \asymp FN^{-r}(r = 1, \ldots, \ell + 2, x \in [N, 2N]).$$

Then  

$$S(f, N) \ll F^{1/(4L-2)}N^{1-(\ell+2)/(4L-2)} + F^{-1}N.$$  

(ii) Let $f(x) = yx^b$ where $y \neq 0$, $\frac{b}{\ell+1} \not\in \{2, 3, \ldots, \ell + 1\}$. With $F = |y|N^c$ and $\ell, L$ as in (i),  

$$S(f, N) \ll F^{1/2 - \frac{\ell+1}{4L-2}}N^{\frac{\ell+2}{4L-2}} + NF^{-1}.$$  

Proof. Part (i) is Theorem 29 of [19]. If $FN^{-1} < \eta$, part (ii) follows from the Kusmin-Landau theorem. Suppose now that $FN^{-1} \geq \eta$. We apply the $B$ process to $S(f, N)$ and then part (i) of the lemma to the resulting sum (after a partial summation). This yields  

$$S(f, N) \ll NF^{-1/2} \left( F^{1/(4L-2)}(FN^{-1})^{1-\frac{\ell+2}{4L-2}} + (FN^{-1})F^{-1} \right) + NF^{-1/2} + 1.$$  

The last three terms can be absorbed into the first term. □

Lemma 6. Suppose that $g^{(6)}$ is continuous on $[1, 2]$ and  

$$g^{(j)}(x) \ll 1 \quad (5 \leq j \leq 6)$$  

$$g^{(j)}(x) \asymp 1 \quad (2 \leq j \leq 4).$$

Let $T, N$ be positive with  

$$T^{1/3} \ll N \ll T^{1/2}.$$
Let

\[ S_h = \sum_{m \in I_h} e \left( Ty_h g \left( \frac{m}{N} \right) \right) \]

where \( I_h \) is a subinterval of \([N, 2N]\) and \( y_1, \ldots, y_H \in [1, 2] \) with

\[ y_{j+1} - y_j \gg \frac{1}{H} \quad (j = 1, \ldots, H - 1). \]

Then

\[
\sum_{h=1}^H |S_h| \ll H^{319/345} N^{449/690} T^{449/690} + \eta
\]

\[ + H N^{1/2} T^{141/950} + \eta \]

Proof. We combine a special case of [23, Theorem 2] with Hölder’s inequality. \qed

**Lemma 7.** Let \( g \) be a function with derivatives of all orders on \([\frac{1}{2}, 1]\) and

\[ |g^{(j)}(x)| \gg 1 \quad \left( x \in \left[ \frac{1}{2}, 1 \right], 2 \leq j \leq 4 \right). \]

Let \( T, N \) be positive with

\[ T^{17/42} \leq N \leq T^{25/42}. \]

Then

\[
\sum_{n \in I(N)} e \left( Tg \left( \frac{n}{N} \right) \right) \ll N^{1/2} T^{141/950} + \eta.
\]

Proof. Theorem 4 of [6] is the case

\[ T^{17/42} \leq N \leq T^{15}, \quad I(N) = [N, 2N]. \]

On pages 222–223 of [6] it is indicated how to extend this to \( T^{15}, T^{25/42} \) using the \( B \) process. An application of [19, Lemma 7.3] enables one to replace \([N, 2N]\) by \( I(N) \) with the loss of a log factor. \qed

**Lemma 8.** Let \( k \in \mathbb{N}, \ k \geq 3. \) Let \( f \) have continuous derivatives \( f^{(j)} \) \((1 \leq j \leq k)\) on \([0, N],\)

\[ |f^{(k)}(x)| \asymp \lambda_k \quad \text{on} \ (0, N). \]
Then
\[(2.1) \quad S(f, N) \ll N^{1+\eta}\left(\lambda_k^{1 \over (k-1)} + N^{-1 \over 4(k-1)} + N^{-2 \over 4(k-1)} \lambda_k^{2 \over (k-1)} \right).
\]

Proof. \[22\] Theorem 1]. \[\square\]

**Lemma 9.** Let \(\theta, \phi\) be real constants,
\[\theta(\theta - 1)(\theta - 2)\phi(\phi - 1)(\theta + \phi - 2)(\theta + \phi - 3)(\theta + 2\phi - 3)(2\theta + \phi - 4) \neq 0.
\]
Let \(F \geq 1\) and let \(|a_m| \leq 1\). Let
\[T(\theta, \gamma) = \sum_{m \sim M} a_m \sum_{n \in I_m} e\left(\frac{Fm^\theta n^\phi}{MN^\gamma}\right)
\]
where \(I_m\) is a subinterval of \([N, 2N]\). Then
\[T(\theta, \gamma) \ll (MN)^\eta(F^{3/14}M^{41/56}N^{29/56} + F^{1/5}M^{3/4}N^{11/20}
\]
\[+ F^{1/8}M^{13/16}N^{11/16} + M^{3/4}N + MN^{3/4} + MNF^{-1}).
\]

Proof. \[3\] Theorem 2]. \[\square\]

We write \((\alpha)_0 = 1\), \((\alpha)_s = (\alpha)_{s-1}(\alpha + s - 1)\) \((s = 1, 2, \ldots)\).

**Lemma 10.** Let \(\theta, \phi\) be real
\[(\theta)_{4}(\phi)_{4}(\theta + \phi + 2) \neq 0.
\]
Let \(MN \asymp X\), \(F \geq 1\), \(|a_m| \leq 1\). Let \(N_0 = \min(M, N)\). Then in the notation of Lemma \[7\]
\[T(-\theta, -\phi) \ll X^\eta(X^{11/12} + XN^{-1/2} + F^{1/8}X^{13/16}N^{-1/2}
\]
\[+ (FX^5N^{-1}N_0^{-1})^{1/6} + XF^{-1}).
\]

Proof. \[4\] Lemma 9]. \[\square\]

**Lemma 11.** Let \(|a_m| \leq 1\), \(|b_n| \leq 1\). Let
\[S = \sum_{m \sim M} a_m \sum_{n \sim N} b_n e(Bm^\beta n^\alpha)
\]
where \(M \geq 1\), \(N \geq 1\), \(\alpha(\alpha - 1)(\alpha - 2)\beta(\beta - 1)(\beta - 2) \neq 0\). Suppose that
\[F := BM^\beta N^\alpha \gg X.
\]
Then
\[ SX^{-\eta} \leq F^{1/20}N^{19/20}M^{29/40} + F^{3/46}N^{43/46}M^{16/23} + F^{1/10}N^{9/10}M^{3/5} + F^{3/28}N^{23/28}M^{41/56} + F^{1/11}N^{53/66}M^{17/22} + F^{2/21}N^{31/42}M^{17/21} + F^{1/5}N^{7/10}M^{3/5} + N^{1/2}M + F^{1/8}(NM)^{3/4}. \] 

\[(2.2)\]

Proof. This is due to Sargos and Wu \[39\]. Full details are given in \[5, proof of Theorem 3\]. \[\square\]

Lemma 12. Let \(0 < B < K\) and \(|c_n| \leq 1\). Let 
\[ V(x) = \sum_{\frac{X}{2} < n \leq X} c_n e(n^{\varepsilon}x) \text{ or } \sum_{\frac{X}{2} < n \leq X} c_n e([n^{\varepsilon}]x). \]

Then
\[(i)\] 
\[ \int_{B}^{2B} |V(y)|^2 dy \ll XB + X^{2-\varepsilon}\mathcal{L}, \]

\[(ii)\] 
\[ \int_{B}^{2B} |V(y)|^4 dy \ll (X^2B + X^{4-\varepsilon})X^\eta (c > 2). \]

Proof. \(i\) It suffices to give the details for \[(2.3)\] 
\[ V(x) = \sum_{\frac{X}{2} < n \leq X} c_n e([n^{\varepsilon}]x). \]

The left-hand side in \(i\) is
\[ \int_{B}^{2B} \left\{ \sum_{\frac{X}{2} < n \leq X} |c_n|^2 + 2 \sum_{\frac{X}{2} < n, n+j \leq X, j \neq 0} c_n \bar{c}_{n+j}e(([n^{\varepsilon}] - [(n+j)^{\varepsilon}])x)dx \right\} \]
\[ \ll XB + \sum_{\frac{X}{2} < n \leq X} \sum_{j \leq X} \frac{1}{jX^{\varepsilon-1}} \ll XB + X^{2-\varepsilon}\mathcal{L}, \]

since (assuming \(X\) is large and \(j > 0\)) \([n+j]^{\varepsilon} - [n^{\varepsilon}] = (n+j)^{\varepsilon} - n^{\varepsilon} + O(1) \sim jn^{\varepsilon-1} \).

\(ii\) Again, we give details only for \[(2.3)\]. The left-hand side in \(ii\) is
\[\sum_{n_j \leq X, 1 \leq j \leq 4} \left( \sum_{\frac{n}{8} < n_j \leq X} \int_{B}^{2B} e\left( \left( [n_1] + [n_2] - [n_3] - [n_4] \right) x \right) dx \right) \]

\[\leq \sum_{\frac{n}{8} < n_j \leq X, j=1,\ldots,4} \min \left( B, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c + \theta|} \right) \]

\[= \sum_{1}, \text{ say.} \]

Here \(\theta\) depends on the \(n_i\), \(\theta \in (-2, 2)\). The number of terms in the sum with

\[|n_1^c + n_2^c - n_3^c - n_4^c| \leq 4^j\]

is

\[\ll X^{n/4} (X^{4-j} + X^2)\]

for \(4^j \ll X^c\), by [37] Theorem 2. Thus

\[\sum_{1} = \sum_{4^j \ll X^c} W_j\]

with \(W_1\) corresponding to

\[|n_1^c + n_2^c - n_3^c - n_4^c| \leq 4\]

and \(W_j\) corresponding to

\[4^{j-1} < |n_1^c + n_2^c - n_3^c - n_4^c| \leq 4^j.\]

We see that

\[W_1 \ll X^{n/2} (X^2 + X^{4-c}) B \ll X^{2+n/2} B\]

while for \(j \geq 2\),

\[W_j \ll X^{n/2} \min(4^{-j}, B) (X^2 + X^{4-c} 4^j).\]

The desired bound follows at once. \(\square\)

The following result abstracts the idea of Cai mentioned in Section 1.

**Lemma 13.** Let \(\mu\) be a complex Borel measure on \([X^{1-c}, K]\). Let \(\lambda_1, \ldots, \lambda_N \in \mathbb{R}\). Let \(a_n \left( \frac{X}{8} < n \leq X \right)\) be real numbers,
\[ S(x) = \sum_{\frac{X}{8} < n \leq X} a_n e(\lambda_n x), \quad J(x) = \sum_{\frac{X}{8} < n \leq X} e(\lambda_n x). \]

Then
\[ (i) \left| \int_{X^{1-c}}^K S(x) d\mu(x) \right|^2 \ll \left( \sum_{\frac{X}{8} < n \leq X} |a_n|^2 \right) \int_{X^{1-c}}^K d\bar{\mu}(y) \int_{X^{1-c}}^K J(x-y) d\mu(x). \]

(ii) Suppose further that \( a_n \ll \mathcal{L} \) and for some \( U > 0 \),
\[ (2.4) \quad J(x) \ll U + \mathcal{L} X^{1-c} |x|^{-1} \quad (0 < |x| \leq 2K). \]

Then for any Borel measurable bounded function \( G \) on \([X^{1-c}, K]\) we have
\[ \left| \int_{X^{1-c}}^K S(x) G(x) dx \right|^2 \ll \mathcal{L}^4 X^{2-c} \int_{X^{1-c}}^K |G(x)|^2 dx + UX \mathcal{L}^2 \left( \int_{X^{1-c}}^K |G(x)| dx \right)^2. \]

Proof. (i) We have (summations over \( n \) corresponding to \( \frac{X}{8} < n \leq X \))
\[ \left| \int_{X^{1-c}}^K S(x) d\mu(x) \right| = \sum_n a_n \int_{X^{1-c}}^K e(\lambda_n x) d\mu(x) \]
\[ \leq \sum_n |a_n| \left| \int_{X^{1-c}}^K e(\lambda_n x) d\mu(x) \right|. \]

By Cauchy’s inequality,
\[ \left| \int_{X^{1-c}}^K S(x) d\mu(x) \right|^2 \leq \sum_n |a_n|^2 \sum_n \left| \int_{X^{1-c}}^K e(\lambda_n x) d\mu(x) \right|^2 \]
\[ = \sum_n |a_n|^2 \sum_n \int_{X^{1-c}}^K e(\lambda_n x) d\mu(x) \int_{X^{1-c}}^K e(-\lambda_n y) d\bar{\mu}(y) \]
\[ = \sum_n |a_n|^2 \sum_n \int_{X^{1-c}}^K d\bar{\mu}(y) \int_{X^{1-c}}^K J(x-y) d\mu(x). \]

(ii) We apply (i) with \( d\mu(x) = G(x) dx \). The right-hand side is
\[
\ll X L^2 \int_{X_1-c}^K |G(y)| dy \int_{X_1-c}^K |G(x)| U dx \\
+ X L^3 \int_{X_1-c}^K |G(y)| \int_{X_1-c}^K |G(x)| \min \left(X, \frac{X_1-c}{|x-y|}\right) dx dy.
\]

It now suffices to show that
\[
(2.5) \quad \int_{X_1-c}^K |G(y)| \int_{X_1-c}^K |G(x)| \min \left(X, \frac{X_1-c}{|x-y|}\right) dx dy 
\ll X^{1-c} L \int_{X_1-c}^K |G(x)|^2 dx.
\]

The left-hand side of (2.5) is
\[
\leq \frac{1}{2} \int_{X_1-c}^K \int_{X_1-c}^K (|G(x)|^2 + |G(y)|^2) \min \left(X, \frac{X_1-c}{|x-y|}\right) dx dy \\
= \int_{X_1-c}^K |G(x)|^2 \int_{X_1-c}^K \min \left(X, \frac{X_1-c}{|x-y|}\right) dy dx.
\]

The contribution to the inner integral from \(|y-x| \leq X^{-c}\) is \(\leq 2 X^{1-c}\) and the contribution from \(|y-x| > X^{-c}\) is \(\leq 2 X^{1-c} \log(KX^{c-1})\). Now (2.5) follows. \(\square\)

We write \(d(n)\) for the divisor function.

**Lemma 14.** Let \(G\) be a complex function on \([X, 2X]\). Let \(u \geq 1, v, z\) be numbers satisfying \(u^2 \leq z\), \(128 uz^2 \leq X\) and \(2^{20} X \leq v^3\). Then
\[
\sum_{\frac{X}{2} < n \leq X} \Lambda(n) G(n)
\]
is a linear combination (with bounded coefficients) of \(O(L)\) sums of the form
\[
\sum_m a_m \sum_{\frac{X}{2} \leq mn \leq X} (\log n)^h G(mn)
\]
with \(h = 0\) or \(1\), \(|a_m| \leq d^5(m)\) together with \(O(L^3)\) sums of the form
\[
\sum_{m} a_m \sum_{u \leq n \leq v} b_n G(mn)
\]
in which \(|a_m| \leq d(m)^5, |b_n| \leq d(n)^5\).

**Proof.** [20, pp. 1367–1368]. □

The following lemma encapsulates the ‘Harman sieve’ in the version we need.

**Lemma 15.** Let \(w(\ldots)\) be a complex function with support on \([X/\delta, X] \cap \mathbb{Z}\), \(|w(n)| \leq X^{1/\eta}\) for all \(n\). For \(m \in \mathbb{N}, z \geq 2\) let

\[S(m, z) = \sum_{(n, P(z))=1} w(mn).\]

Let \(\alpha > 0, 0 < \beta \leq 1/2, M \geq 1, Y > 0\). Suppose that whenever \(|a_m| \leq 1, |b_n| \leq d(n),\) we have

\[(2.6) \quad \sum_{m \leq M} a_m \sum_n w(mn) \ll Y,\]

\[(2.7) \quad \sum_{X^{\alpha} \leq m \leq X^{\alpha+\beta}} a_m \sum_n b_n w(mn) \ll Y.\]

Let \(|u_\ell| \leq 1, |v_s| \leq 1, \) for \(\ell \leq R, s \leq S,\) also \(u_\ell = 0 \) for \((\ell, P(X^\eta)) > 1, v_s = 0 \) for \((s, P(X^\eta)) > 1.\) Suppose that

\(R < X^\alpha, S < M X^{-\alpha}.\)

Then

\[\sum_{\ell \leq R} \sum_{s \leq S} u_\ell v_s S(\ell s, X^\beta) \ll Y L^3.\]

**Proof.** [5, Lemma 14]. □

**Lemma 16.** Let \(\alpha > 0, 0 < \beta \leq 1/2, Y > 1, y > 0.\) Suppose that whenever \(|a_m| \leq 1, |b_n| \leq 1\) we have

\[S := \sum_{X^\alpha \leq m \leq X^{\alpha+\beta}} \sum_{X^\delta \leq mn \leq X} b_n e(y(mn)^\delta) \ll Y.\]
Let $|a_m| \leq 1$, $|b_n| \leq 1$. Let
\[
S_1 = \sum_{p_1, \ldots, p_s} \sum_{X^\alpha \leq mp_1 \ldots p_r \leq X^{\alpha + \beta}} a_m b_n e(y(mp_1 \ldots p_s)^c)
\]
where the asterisk indicates that $X^\eta \leq p_1 < p_2 < \cdots < p_r$ together with no more than $\eta^{-1}$ conditions of the form
\[
A(\mathcal{F}) \leq \prod_{j \in \mathcal{F}} p_j \leq B(\mathcal{F})
\]
(\mathcal{F} \subseteq \{1, \ldots, s\}). Then
\[
S_1 \ll YX^\eta.
\]
Corresponding bounds hold when $S$, $S_1$ are replaced by sums containing (e.g.) \((mn)^c\) in place of \((mn)^c\).

Proof. This is a variant of Lemma 10 of [3]. Each condition implied by * can be removed using repeatedly the truncated Perron formula
\[
\frac{1}{\pi} \int_{-T}^{T} e^{\tau t} \sin t \beta \frac{1}{t} \, dt = \begin{cases} 
1 + O(T^{-1}(|\alpha|^{-1}) & \text{if } |\alpha| \leq \beta \\
O(T^{-1}(|\alpha| - \beta)^{-1}) & \text{if } \alpha > \beta.
\end{cases}
\]
We can keep the error term negligible by suitable choice of $T$, the main term being a multiple integral of a multiple sum with coefficients of absolute value at most $X^{\eta/2}$, and with no interaction between the summation variables. For more details see [3, pp. 270–272].

3. The minor arcs: small $x$ and large $x$.

We can disregard the contribution to the minor arc from $x > K$ by (1.2). In the present section we show that
\[
\int_{\tau}^{X^{1-c}} |U(x)|^s |\Phi(x)| \, dx \ll X^{s-c-3}\eta \quad (3 \leq s \leq 4)
\]
and
\[
\int_{\tau}^{X^{2-c}} |U(x)|^5 |\Phi(x)| \, ds \ll X^{5-c-3}\eta \quad (s = 5),
\]
where $U(x)$ is any of $A(x)$, $B(x)$, $S(x)$, $S_1(x)$, $S^+(x)$, $T(x)$, $T^+(x)$. For Theorems 1–6 this takes care of the (positive) left-hand part of
the minor arc. (We need no separate discussion for the part of the minor arc in \((-\infty, -\tau)\), here or later.) This is not quite obvious for \(s = 2\), but see the discussion at the beginning of Section 4.

**Lemma 17.** Let \(c < 2.1\) and \(x \in (\tau, X^{2-c}]\). Let

\[
V(x) = \sum_{m \sim M} \sum_{n \sim N} b_m c_n e(x(mn)^c) \tag{3.3}
\]

or

\[
V(x) = \sum_{m \sim M} \sum_{n \sim N} b_m c_n e(x([mn]^c)), \tag{3.4}
\]

where \(|b_m| \leq 1, |c_n| \leq 1\). Then

\[
V(x) \ll X^{1-3\eta} \quad \text{whenever } X^{10\eta} \ll N \ll X^{1/2}. \tag{3.5}
\]

The bound \(3.5\) also holds when \(b_n = 1\) for all \(n\) and \(n \gg X^{1-10\eta}\).

**Proof.** We prove this for \(3.4\); the details for \(3.3\) are similar but simpler. We apply Lemma 1 with

\[
a_n = \frac{1}{X^{\eta/2}} \sum_{\ell s = n} b_{\ell} c_s.
\]

We take \(H = X^{\eta/10}\). Now it suffices to obtain

\[
S(\gamma) := \sum_{m \sim M} \sum_{n \sim N} b_m c_n e((h + \gamma)(mn)^c) \ll X^{1-7\eta/2} \tag{3.6}
\]

with \(\gamma = \{x\} \text{ or } -\{x\}\) and \(|h| < H\), together with

\[
\sum_{1 \leq h \leq H} \left| \frac{1}{h} \right| \left| \sum_{\frac{X}{s} < n \leq X} e(hn^c) \right| \ll X^{1-3\eta} \tag{3.7}
\]

and

\[
\sum_{h > H} \left| \frac{H}{h^2} \right| \left| \sum_{\frac{X}{s} < n \leq X} e(hn^c) \right| \ll X^{1-3\eta}. \tag{3.8}
\]
In each case we use Lemma 4 with \( \ell = 2 \). Treating the simpler bounds (3.7), (3.8) first, we bound the sum over \( n \) in (3.7), (3.8) by

\[
\ll F^{1/14}X^{10/14} + F^{-1}X
\]

\[
\ll h^{1/14}X^{13/14},
\]

which yields (3.7), (3.8).

For (3.6), take \( Q = \eta N \). Arguing as in [1, proof of Theorem 5], we have

\[
|S(\gamma)|^2 \ll \frac{X^2}{Q} + \frac{X}{Q} \sum_{q \leq Q} \sum_{n \leq N} \left| \sum_{m \in I(M)} e((h+\gamma)m^c((n+q)^c-n^c)) \right|.
\]

For the inner sum on the right-hand side we use Lemma 4 with \( \ell = 2 \), obtaining the bound

\[
\ll (qN^{-1}HX^c)^{1/14}M^{10/14} + \frac{M}{|\gamma|qN^{-1}X^c}.
\]

Thus

\[
|S(\gamma)|^2 \ll \frac{X^2}{N} + XN(HX^c)^{1/14} + X^{2-7\eta}
\]

since \( |\gamma|X^c > X^{8\eta} \). This gives the desired bound (3.5).

For the case \( b_n = 1 \) identically, it suffices to add the bound

\[
\sum_{n \in I(N)} e((h+\gamma)m^c n^c) \ll NX^{-4\eta}
\]

whenever \( N \geq X^{1-10\eta} \). We bound the left-hand side by

\[
\ll (HX^c)^{1/14}N^{10/14} + (X^c)^{-1}N
\]

and obtain (3.10) at once.

We now deduce (3.11) for \( 2 \leq s \leq 4 \). We find that Lemma 17 yields

\[
U(x) \ll X^{1-2\eta} \quad (\tau < x < X^{1-c})
\]

(here we require the reader to look ahead to the form of \( S^+(x) \) and \( T^+(x) \) in later sections, or else use Heath-Brown’s identity if appropriate). Now (recalling Lemma 12(i)) a simple splitting-up argument yields (for some \( B < X^{9} \))
\[
\int_{\tau}^{X^{1-c}} |U(x)|^s \Phi(x) dx \ll \mathcal{L} X^{-c\eta} \sup_{y \in [\tau,K]} |U(y)|^{s-2} \int_{\tau}^{K} |U(x)|^2 dx \\
\ll \mathcal{L} X^{-c\eta} X^{s-2-3\eta} X^{2-c+\eta} \ll X^{s-c-3\eta}.
\]

The argument for (3.2) is very similar using Lemma 12 (ii). □

4. MINOR ARC IN THEOREMS 1 AND 3

We shall show that

\begin{equation}
\int_{X^{1-c}}^{K} |S_1(x)|^4 dx \ll X^{4-c-3\eta}.
\end{equation}

Since \( \Phi(x) \ll X^{-c\eta} \), this reduces the integral in (1.4), in effect, to the major arc in Theorem 3. For Theorem 1 let

\[ E_0(x) = \begin{cases} 
S^2(x)\Phi(x) & x \in [\tau, K] \\
0 & \text{otherwise.}
\end{cases} \]

By Parseval’s formula,

\[
\int_{V}^{2V} |\hat{E}_0(R)|^2 dR < \int_{\tau}^{K} |E_0(x)|^2 dx \ll (V^{1/c})^{4-c-5c\eta}.
\]

Hence

\[ \hat{E}_0(R) = \int_{R} S^2(x)\Phi(x)e(-Rx) dx \]

satisfies

\[ |\hat{E}_0(R)| < V^{\frac{2}{c} - 1 - 2\eta} \]

except for a set of measure \( O(V^{1-\eta}) \) in \([V, 2V]\). Again, this gives the desired ‘reduction to the major arc.’

To prove (4.1) we first apply Lemma 13 (ii) with

\[ S(x) = S_1(x), \mathcal{J}(x) = A(x), G(x) = \tilde{S}_1(x)S_1(x)^2. \]

From Lemma 12 and the Cauchy–Schwarz inequality,

\[
\int_{\tau}^{K} |G(y)| dy \ll \mathcal{L} \left( \int_{B}^{2B} |S_1(x)|^2 dx \right)^{1/2} \left( \int_{B}^{2B} |S_1(x)|^4 dx \right)^{1/2}
\]
SOME DIOPHANTINE EQUATIONS AND INEQUALITIES WITH PRIMES

(for some \( B \in [\tau, K] \)) 

\[ \ll X^{\frac{2c+2}{3} + 2\eta}. \]

We shall show below that \( (4.2) \) holds with 

\[ U = X^{2-c-12\eta} \]

and that 

\[ S_1(x) \ll X^{(7-c)/6-5\eta}. \]

Hence 

\[ \int_{X^{1-c}}^{K} |G(y)|^2 \, dy \ll X^{(7-c)/2-15\eta} \int_{X^{1-c}}^{K} |G(y)| \, dy \ll X^{6-c-13\eta}. \]

Now Lemma \( \text{[13]}(\text{ii}) \) yields 

\[ \left( \int_{\tau}^{K} |S_1(x)|^4 \, dx \right)^2 \ll X^{2-c+6-c-10\eta} \]

\[ + X^{2-c-12\eta+6-c+2\eta} \ll X^{8-2c-10\eta} \]

as required for \( (4.1) \).

Turning to the proof of \( (4.2) \), we apply the \( B \) process first, followed by a partial summation and then Lemma \( \text{[8]} \) with \( k = 5 \). This is legitimate since the \( B \) process produces a sum of the form 

\[ \sum_{n \in I} e(yn^{c/(c-1)}) \]

where 

\[ yn^{c-1} \asymp F := xX^c; \]

and the five differentiations required in Lemma \( \text{[8]} \) are permissible unless 

\[ \frac{c}{c-1} = m \in \mathbb{N}, \ m \leq 4. \]

We have excluded \( c = 4/3 \), so \( (4.4) \) cannot hold. The error term in Lemma \( \text{[4]} \) is 

\[ \ll \mathcal{L} + F^{-1/2}X \ll X^{1/2} \]
(since \( x \geq X^{1-c} \)), which is acceptable. We may take
\[
U \ll X F^{-\frac{1}{2}} N_1^{1+\eta} \left\{ (FN_1^{-5})^{\frac{1}{20}} + N_1^{-\frac{1}{20}} + (F_1 N^{-5})^{-\frac{1}{20}} N_1^{-\frac{1}{20}} \right\}
\]
where \( N_1 = FX^{-1} \). Here
\[
XF^{-\frac{1}{2}} N_1^{1+\eta} (FN_1^{-5})^{\frac{1}{20}} \ll X^{\frac{1}{4}+\eta} F^{3/10} \ll X^{2-c-12\eta}
\]
since \( c < \frac{39}{29} < \frac{35}{26} \). Next,
\[
XF^{-1/2} N_1^{\frac{19}{20}+\eta} \ll X^{\frac{26}{20}+2\eta} \ll X^{2-c-12\eta}
\]
since \( c < \frac{39}{29} \). Finally
\[
XF^{-\frac{1}{4}} (FN_1^{-5})^{-\frac{1}{50}} N_1^{\frac{24}{50}+\eta} \ll X^{\eta} F^{24/50} \ll X^{2-c-12\eta}
\]
since \( c < \frac{39}{29} < \frac{50}{37} \).

We now use Lemma 14 to prove (4.3). Here and below, we take \( G(n) = e(xn^c) \) in Lemma 14. A Type I sum will be of the form
\[
S_I(x) = \sum_{m \sim M} a_m \sum_{n \sim N} e((mn)^c x) \quad \text{for} \quad \frac{X}{4} < mn \leq X
\]
and a Type II sum will be of the form
\[
S_{II}(x) = \sum_{m \sim M} a_m \sum_{n \sim N} b_n e((mn)^c x) \quad \text{for} \quad \frac{X}{4} < mn \leq X
\]
Here \( |a_m| \leq 1, |b_n| \leq 1 \). Taking
\[
v = X^{0.36}, \; u = X^{0.12}, \; z = X^{0.35}
\]
in Lemma 14 it suffices to show that
(4.5) \( S_I(x) \ll X^{(7-c)/6-6\eta} \) for \( N \geq z \)
and
(4.6) \( S_{II}(x) \ll X^{(7-c)/6-6\eta} \) for \( u \leq N \leq v \).
For (4.5), we appeal to Lemma 10. We have \((7-c)/6 > 0.942\). The first two terms in the bound for \(S_I(x)\) are acceptable, while

\[XF^{-1} \ll 1.\]

Next, for \(N \geq z\),

\[F^{1/8}X^{13/16}N^{-1/8} \ll X^{1.345/8+13/16-0.35/8} \ll X^{0.94}.\]

Finally, we have a term that is

\[\ll (FX^4)^{1/6} + (FX^5N^{-2})^{1/6} \ll X^{(1.345+4)/6} + X^{(1.345+5-0.7)/6} \ll X^{0.941}.\]

For (4.6), we apply the obvious variant of (3.9), taking \(Q = X^{0.116}\) to give an acceptable term \(X^2/Q\). It remains to show that for \(q \leq Q, n \leq N,\)

\[(4.7) \quad S_{n,q} := \sum_{m} e(xm^c((n + q)^c - n^c)) \ll MQ^{-1}.\]

Here \(F\) is replaced by \(F_1 := xqX^cN^{-1}\). We apply Lemma 5 (ii) with \(\ell = 4\) to obtain

\[S_{n,q} \ll F_1^{13/31}M^{3/31} \ll MX^{-0.116}\]

since \(X^{28/31}N^{-15/31} > X^{0.729} > X^{(39/29+.116)13/31+.116}\). However, the six differentiations are only permissible when

\[c \neq 1 + \frac{1}{m}\]

where \(m \leq 4\). We excluded \(m = 3\), so we now need to treat \(c = \frac{5}{2}\) separately. Here we use Lemma 5 (ii) with \(\ell = 3\); the five differentiations are permissible and

\[S_{n,q} \ll F_1^{11/30}M^{1/6} \ll MX^{0.116}\]

by a similar calculation. This completes the discussion of the minor arc.

5. Minor arc in Theorem 2

We shall set up a suitable function \(\rho^+\) based on Type I and Type II information. To obtain a negligible contribution of the minor arc we
require

\[ (5.1) \int_{X^{1-c}}^{K} S(x)G(x)dx \ll X^{3-c-3\eta} \]

for the two functions

\[ (5.2) \quad G(x) = S^+(x)S(x)\Phi(x)e(-Rx), \quad S^+(x)^2\Phi(x)e(-Rx). \]

It will suffice to show that

\[ (5.3) \quad A(x) \ll X^{\frac{2}{3}+\eta} + X^{1-c}|x|^{-1} \quad (|x| < 2X^{2\eta}) \]

and

\[ (5.4) \quad S^+(x) \ll X^{\frac{1-c}{2}-4\eta} \quad (X^{1-c} \leq x \leq K). \]

We then apply Lemma 13 (ii) with \( S(x) = S(x), \ T(x) = A(x), \ G(x) \) as in (5.2) so that

\[ \int_{X^{1-c}}^{K} |G(x)|dx \ll X^{1+\eta} \]

and, using (5.4),

\[ \int_{X^{1-c}}^{K} |G(x)|^2dx \ll X^{3-c-8\eta} \int_{X^{1-c}}^{K} |G(x)|dx \ll X^{4-c-7\eta}. \]

Thus

\[ \left( \int_{X^{1-c}}^{K} S(x)G(x)dx \right)^2 \ll \mathcal{L}^4 X^{2-c} X^{4-c-7\eta} + \mathcal{L}^2 X^{\frac{3}{2}+1+\eta+2(1+2\eta)} \ll X^{6-2c-6\eta} \]

using \( c < 6/5 \), which proves (5.1).

To obtain (5.3) we use the Kusmin-Landau theorem if \( X^{c-1}|x| < \eta \). Otherwise, we use the \( B \) process, giving a main term

\[ \ll F^{\frac{1}{2}} \ll X^{\frac{2}{3}+\eta} \]
where $F = xX^c$, and error terms
\[ \ll F^{-1/2}X + \mathcal{L} \ll X^{1/2}. \]

Aiming towards the definition of $S^+(x)$, we claim that Type II sums are $\ll X^{0.9}$ for either of the alternatives
\[ 5.5 \quad X^{1/5} \ll N \ll X^{29/105} \]
and
\[ 5.6 \quad X^{1/3} \ll N \ll X^{11/25}. \]

For (5.5) we begin with (3.9), replacing $h + \beta$ by $x$, and taking $Q = X^{0.2} \ll N$. It remains to show that for given $Q_1 \in \left[ \frac{1}{2}, Q \right]$ and $n \sim N$, we have
\[ S^* := \sum_{q \sim Q_1} \sum_{n \in \mathcal{I}(M)} e(x((n + q)^c - n^c)m^c) \ll Q_1MX^{-1/5}. \]

Following the analysis on pp. 171–172 of [5], we find that for some $q \sim Q_1$, and $R$ at our disposal with $R \ll N_1$, and some $r \sim R$, we have
\[ S^* \ll \frac{N^5 M^4}{FQ} + N^4 M^2 + \frac{X^4 NQ_1}{FQ^2} \left( \frac{N_1^2}{R} + N_1|S(n, q, r)| \right). \]

Here $N_1 \approx FQ_1/X \ll X^{0.4},$
\[ t(n, q) = (n + q)^c - (n - q)^c, \]
\[ t_1(n_1, r) = (n_1 + r)^{c/(c-1)} - (n_1 - r)^{c/(c-1)}, \]
and we define
\[ S(n, q, r) = \sum_{n_1 \in \mathcal{I}(r)} e \left( C(xX^c t(n, q))^{1/c} t_1(n_1, r) \right) \]
with $I(r)$ a subinterval of $[N_1, 2N_1]$. We choose $R$ so that
\[ \frac{M^4 N^5 Q_1 N_1^2}{FQ^2 R} = X^4 \]
that is,
\[ R = \frac{NQ_1 N_1^2}{F} \ll \frac{FNQ_1^3}{X^2}. \]

We have $R \ll N$ since $NQ_1^2 \ll NXQ \ll X$. 
The terms $N^5 M^4 / F Q$ and $N^4 M^2$ in (5.8) are $\ll X^4 / Q^2$ since

$$\frac{N^5 M^4}{F Q} \frac{Q^2}{X^4} = \frac{N Q}{F} \ll \frac{N Q}{X} \ll 1,$$

$$\frac{N^4 M^2 Q^2}{X^4} \ll \frac{Q^2}{M^2} \ll 1.$$

For $S(n, q, r)$, it suffices to show that

$$S(n, q, r) \ll X^{0.6} / N.$$

For then

$$\frac{X^4}{F Q^2} N Q_1 N_1 S(n, q, r) \ll \frac{X^4}{F Q^2} X^{0.6} Q_1 \frac{F Q_1}{X} \ll \frac{X^4}{Q^2}.$$

We apply Lemma 7 to $S(n, q, r)$ with (taking $2\eta < 1.2 - c$). We have

$$T \simeq \frac{F Q_1}{N} \frac{r}{N_1} \simeq \frac{X r}{N}; T \ll \frac{X R}{N} \simeq \frac{F Q_1}{X} \ll X^{0.8 - 2\eta}.$$

Provided that

$$T^{17/42} \leq N_1 \leq T^{25/42},$$

we obtain

$$S(n, q, r) \ll X^{\eta} T^{13/84} N_1^{1/2}$$

$$\ll X^{\eta + \frac{13}{84} \frac{4}{5} + \frac{1}{2} - \eta} \ll \frac{X^{0.6}}{N} \quad (N \ll X^{29/105}).$$

We certainly have

$$N_1 \leq T^{25/42}$$

since

$$X^{0.4} < \left( \frac{X}{N} \right)^{25/42} X^{-\eta} \quad \text{as} \quad N < X^{0.3} < X^{1 - 0.4 \times \frac{44}{25} - 2\eta}.$$

We may have $N_1 < T^{17/42}$. In this case we apply Lemma 4 with $\ell = 2$ to $S(n, q, r)$. The term $T^{-1} N_1$ is $\ll 1$, so that

$$S(n, q, r) \ll T^{1/14} N_1^{10/14} \ll T^{\frac{170}{988}} \ll \frac{X^{0.6}}{N}.$$
since \( T < X^{4/5} \), \( N < X^{3/10} \). This completes the proof that \( S_{II} \ll X^{0.9} \) when (5.5) holds.

Now suppose that (5.6) holds. We apply Lemma 11. Five of the terms \( U_1, U_2, \ldots, U_9 \) on the right-hand side of (2.2) are acceptable for \( N \ll X^{0.5} \):

\[
\begin{align*}
U_1 &\ll F^{1/20} N^{9/40} X^{29/40}, \quad \frac{1.2}{20} + \frac{9/2}{40} + \frac{29}{40} = 0.8\ldots,
U_2 &\ll F^{3/46} N^{11/46} X^{32/46}, \quad \frac{3.6}{46} + \frac{11/2}{46} + \frac{32}{46} = 0.8\ldots,
U_3 &\ll F^{1/10} N^{3/10} X^{3/5}, \quad \frac{1.2}{10} + \frac{3/2}{10} + \frac{3}{5} = 0.87,
U_5 &\ll F^{1/11} N^{1/33} X^{17/22}, \quad \frac{1.2}{11} + \frac{1/2}{33} + \frac{17}{22} = 0.8\ldots,
U_7 &\ll F^{1/5} N^{1/10} X^{3/5}, \quad \frac{1.2}{5} + \frac{1/2}{10} + \frac{3}{5} = 0.89.
\end{align*}
\]

Also \( U_8 \ll X N^{-1/2} \ll X^{0.9} \) for \( N > X^{0.2} \). For the remaining terms,

\[
\begin{align*}
U_4 &\ll F^{3/28} N^{5/56} X^{41/56} \ll X^{0.9} \quad \text{for } N \ll X^{0.44},
U_6 &\ll F^{2/21} X^{17/21} N^{-1/4} \ll X^{0.9} \quad \text{for } N \gg X^{1/3},
\end{align*}
\]

while the bound

\[
U_9 = F^{1/8} X^{3/4} \ll X^{0.9} \quad (F \ll X^{6/5})
\]

actually determines our range of \( c \).

Finally we consider Type I sums, using

\[
S_I \ll M F^{1/14} N^{10/14},
\]

which follows from Lemma 4 with \( \ell = 2 \). Here

\[
MF^{1/14} N^{10/14} \ll X^{14/14} + M^{14/14} \ll X^{0.9} \quad \text{for } M \ll X^{0.35}.
\]

If \( X^{0.35} \ll M \ll X^{0.44} \) we treat \( S_I \) as a Type II sum. Hence

\[
S_I \ll X^{0.9} \quad \text{for } M \ll X^{0.44}.
\]
We now apply Lemma 15, taking \(w(n) = e(xn^c), \alpha = \frac{1}{3}, \) and \(\beta = \frac{11}{25} - \frac{1}{3} = \frac{8}{75}, M = X^{0.44}, S = 1.\) Thus
\[
\sum_{\ell \leq X^{11/25}} u_\ell S(\ell, X^\beta) \ll X^{0.9} L^3
\]
for any coefficients \(u_\ell\) with \(|u_\ell| \leq 1, u_\ell = 0\) for \((\ell, P(X^\gamma)) > 1.\) (For \(X^{1/3} < \ell \leq X\) this uses Lemma 16.) We use Buchstab’s identity
\[
\rho(u, z) = \rho(u, w) - \sum_{w \leq p < z} \rho\left(\frac{u}{p}, p\right) \quad (2 \leq w < z).
\]
Multiplying by \(e(xn^c)\) and summing over \(n,\) we obtain
\[
(5.8) \quad S(x) = \sum_{X^8 < n \leq X} \rho(n, (3X)^{1/2}) e(xn^c)
\]
\[
= \sum_{X^8 < n \leq X} \rho(n, X^\beta) e(xn^c) - \sum_{X^{\beta} \leq p_1 < (3X)^{1/2}} \sum_{X^8 < p_1 n \leq X \atop (n, P(p_1)) = 1} e(x(p_1 n)^c).
\]
In writing sums over primes, we define \(\alpha_j\) by \(p_j = X^{\alpha_j}\). We introduce intervals
\[
I_1 = \left[\beta, \frac{1}{5}\right), \quad I_2 = \left[\frac{1}{5}, \frac{29}{105}\right), \quad I_3 = \left[\frac{29}{105}, \frac{1}{3}\right), \quad I_4 = \left[\frac{1}{3}, \frac{11}{25}\right), \quad I_5 = \left[\frac{11}{25}, \frac{1}{2} + \frac{\log 3}{2 \log X}\right).
\]
Let
\[
S_j(x) = \sum_{\alpha_1 \in I_j} \sum_{\frac{X^8 < p_1 n \leq X\atop (n, P(p_1)) = 1}} e(x(p_1 n)^c) \quad (1 \leq j \leq 4)
\]
and \(S_0(x) = \sum_{X^8 < n \leq X \atop (n, P(X^\gamma)) = 1} e(xn^c),\)
\[
D_1(x) = \sum_{\alpha_1 \in I_5} \sum_{\frac{X^8 < p_1 n \leq X\atop (n, P(p_1)) = 1}} e(x(p_1 n)^c).
\]
From (5.8), we have
(5.9) \[ S(x) = S_0(x) - \sum_{j=1}^{4} S_j(x) - D_1(x). \]

We use Buchstab’s identity again for \( S_1(x) \):

(5.10) \[ S_1(x) = \sum_{\alpha_1 \in I_1} \sum_{\substack{\beta \leq \alpha_2 < \alpha_1 \atop \frac{x}{p_1} < p_1p_2n < X \atop (n, P(X^\beta)) = 1}} e(x(p_1n)^c) \]

\[ - \sum_{\alpha_1 \in I_1} \sum_{\beta \leq \alpha_2 < \alpha_1} \sum_{\substack{\delta \leq \alpha_3 < \alpha_2 \atop \frac{x}{p_1p_2} < p_1p_2p_3n < X \atop (n, P(p_2)) = 1}} e(x(p_1p_2n)^c) \]

\[ = S_5(x) - S_6(x), \text{ say.} \]

Next,

(5.11) \[ S_6(x) = S_7(x) - S_8(x) \]

where

\[ S_7(x) = \sum_{\alpha_1 \in I_1} \sum_{\beta \leq \alpha_2 < \alpha_1} \sum_{\substack{\delta \leq \alpha_3 < \alpha_2 \atop \frac{x}{p_1p_2} < p_1p_2p_3n < X \atop (n, P(X^\beta)) = 1}} e(x(p_1p_2n)^c), \]

\[ S_8(x) = \sum_{\alpha_1 \in I_1} \sum_{\beta \leq \alpha_3 < \alpha_2 < \alpha_1} \sum_{\frac{x}{p_1p_2} < p_1p_2p_3n < X \atop (n, P(p_3)) = 1}} e(x(p_1p_2p_3n)^c). \]

We write \( D_2(x) \) for the part of the right-hand side of (5.11) with \( \alpha_1 + \alpha_2 + \alpha_3 \in I_3 \), \( \alpha_2 + \alpha_3 \in I_3 \), \( \alpha_1 + \alpha_2 + \alpha_3 \in I_5 \), and define \( K_2(x) \) by

(5.12) \[ S_8(x) = D_2(x) + K_2(x). \]

Next, considering the possible decompositions \( n = p_2 \) and \( n = p_2p_3 \) in the definition of \( S_3(x) \), we have

(5.13) \[ S_3(x) = D_3(x) + S_9(x) \]

where

\[ D_3(x) = \sum_{\alpha_1 \in I_3} \sum_{\alpha_2 > \alpha_1} \sum_{1 \leq \alpha_1 + \alpha_2 < 1 + \frac{\log 2X}{\log X}} e(x(p_1p_2)^c), \]
$$S_9(x) = \sum_{\alpha_1 \in I_3} \sum_{\alpha_3 \geq \alpha_2 > \alpha_1 \atop 1 \leq \alpha_1 + \alpha_2 + \alpha_3 < 1 + \frac{\log 2X}{\log X}} e(x(p_1p_2p_3)^c).$$

Finally,

$$(5.14) \quad S_9(x) = D_4(x) + K_4(x),$$

where $D_4(x)$ is the part of the sum defining $S_9(x)$ for which $\alpha_1 + \alpha_2 \leq \frac{14}{25} + \frac{\log 2}{\log X}$, and $K_4 := S_9 - D_4$. Combining (5.9)–(5.14), our decomposition of $S(x)$ is

$$S = S_0 - S_5 + S_7 - D_2 - K_2 - S_2 - D_3 - D_4 - K_4 - S_4 - D_1.$$  

We define

$$(5.15) \quad S^+ = S_0 - S_5 + S_7 - K_2 - S_2 - K_4 - S_4$$

$$= S + \sum_{j=1}^4 D_j.$$  

We observe firstly that

$$S^+(x) = \sum_{\frac{4}{3} < n \leq X} \rho^+(n) e(xn^c)$$

with $\rho^+ \geq \rho$, since the $D_j$ have non-negative coefficients. Secondly, all of $S_0, S_5, S_7, K_2, S_2, K_4, S_4$ have values $\ll X^{3-c-\eta}$, hence so does $S^+$. For $S_0, S_5, S_7$, this follows from Lemma 15. For $S_2, S_4, K_2$ and $K_4$ we appeal to Lemma 16 and the following observations.

(i) If $\beta \leq \alpha_3 < \alpha_2 < \alpha_1 < 1/5$ and $\alpha_2 + \alpha_3 \not\in I_3$, then $\alpha_2 + \alpha_3 > \frac{16}{75} > \frac{1}{5}$, $\alpha_2 + \alpha_3 < 2/5$. Hence $\alpha_2 + \alpha_3 \in I_2 \cup I_4$. Similarly for $\alpha_1 + \alpha_2$.

(ii) If $\beta \leq \alpha_3 < \alpha_2 < \alpha_1 < 1/5$ and $\alpha_2 + \alpha_3 \in I_3$, $\alpha_1 + \alpha_3 \in I_3$, $\alpha_1 + \alpha_2 + \alpha_3 \not\in I_5$, then $\alpha_1 + \alpha_2 + \alpha_3 \leq \frac{3}{2} \cdot \frac{1}{3} = \frac{1}{2}$, hence $\alpha_1 + \alpha_2 + \alpha_3 < \frac{11}{25}$, while $\alpha_1 + \alpha_2 + \alpha_3 \geq \frac{3}{2} \cdot \frac{29}{105} > \frac{1}{3}$; $\alpha_1 + \alpha_2 + \alpha_3 \in I_4$.

(iii) If $\alpha_1 < \alpha_2 \leq \alpha_3$, $\frac{29}{105} \leq \alpha_1 < \frac{1}{3}$, $1 \leq \alpha_1 + \alpha_2 + \alpha_3 < \frac{\log 2X}{\log X}$, and $\alpha_1 + \alpha_2 \geq \frac{14}{25} + \frac{\log 2}{\log X}$, then $\alpha_3 < \frac{11}{25}$. Moreover $\alpha_3 \geq \frac{1}{3} (\alpha_1 + \alpha_2 + \alpha_3) \geq \frac{1}{3}$, hence $\alpha_3 \in I_4$.

In Section 9 we shall quantify the contribution of the functions $D_j(x)$ to the integral in (1.6); similarly for Theorem ??.
6. Minor arc in Theorem \[5\]

In the present section we show that for \(c < \frac{3581}{3106}\), we have

\[
\int_{X^{1-c}}^{\frac{1}{2}} T(x)G(x)dx \ll X^{3-c-3\eta},
\]

where

\[
G(x) = T^2(x)\Phi(x)e(-Rx).
\]

We apply Lemma \[13\] (ii) with \(S(x) = T(x), J(x) = B(x)\). To prove (6.1) it suffices to show that

\[
B(x) \ll X^{3-2c-15\eta} + \mathcal{L}X^{1-c}|x|^{-1} \quad (0 < |x| < 2X^{2\eta})
\]

and

\[
T(x) \ll X^{\frac{3-c}{2}-5\eta} \quad \left(X^{1-c} \leq x < \frac{1}{2}\right).
\]

For then Lemma \[13\] (ii) (with \(G(x) = 0\) for \(x > \frac{1}{2}\)) yields

\[
\left| \int_{X^{1-c}}^{\frac{1}{2}} T(x)G(x)dx \right|^2 \ll X^{2-c}\mathcal{L}^4 \max_{x \in [X^{1-c}, \frac{1}{2}]} |T(x)|^2 \int_{X^{1-c}}^{\frac{1}{2}} |T(x)|^2 dx
\]

\[
+ X^{4-2c-15\eta}\mathcal{L}^2 \left( \int_{X^{1-c}}^{\frac{1}{2}} |T(x)|^2 dx \right)^2.
\]

The first summand on the right-hand side is

\[
\ll X^{2-c+3-c-10\eta+1+3\eta} \ll X^{6-2c-7\eta}
\]

by (6.3) and Lemma \[12\] (i). The second summand is

\[
\ll X^{4-2c-14\eta}X^{2(1+2\eta)} \ll X^{6-2c-6\eta}
\]

by (6.2) and Lemma \[12\] (i).

For (6.2), we use Lemma \[11\] with \(a_n = 1\). We take

\[
H = X^{2c-2+16\eta}.
\]

Since \(\{x\} = x\) in the sum, our objective is to show that we have
\[ \sum_{0 \leq h \leq H} \min \left( 1, \frac{1}{h} \right) \left| \sum_{\frac{X}{h} < n \leq X} e((h \pm x)n^c) \right| \ll X^{3-2c-16\eta} + X^{1-c}|x|^{-1}, \]

(6.4)

\[ \sum_{1 \leq h \leq H} \frac{1}{h} \left| \sum_{\frac{X}{h} < n \leq X} e(hn^c) \right| \ll X^{3-2c-16\eta} \]

(6.5)

\[ \sum_{h > H} \frac{H}{h^2} \left| \sum_{\frac{X}{h} < n \leq X} e(hn^c) \right| \ll X^{3-2c-16\eta} . \]

(6.6)

We begin with the contribution from \( h = 0 \) in (6.4). If \( X^{c-1}|x| < \eta \), we obtain the desired bound from the Kusmin-Landau theorem. Otherwise Lemma 5 (ii) with \( \ell = 2 \) yields the bound
\[ \ll (x X^c)^{2/7} X^{2/7} , \]

which is acceptable since \( 16c < 19 \).

For the terms in (6.6) with \( h \geq X^{3/2-c} \) we use Lemma 5 (ii) with \( \ell = 2 \). It is clear that these sums produce a contribution
\[ \ll X^a \ll X^{3-2c} \]

where
\[ a = 2c - 2 + \eta - \frac{5}{7} \left( \frac{3}{2} - c \right) + \frac{2c + 2}{7} , \]

(6.7)

and \( a < 3 - 2c \) follows from \( c < \frac{81}{70} \).

We can treat together the terms in (6.4) with \( 1 \leq h \leq H \), the sum (6.5), and the remaining part of (6.6) by estimating (for \( \frac{1}{2} \leq H_1 \ll H^{\frac{3}{2}-c}, H_1 = 2^j \))

\[ S(H_1) := H_1^{-1} \sum_{h \sim H_1} \left| \sum_{\frac{X}{h} < n \leq X} e((h + \gamma)n^c) \right| \]
where $\gamma \in \{x, -x, 0\}$. Let $F = H_1 X^c$. We apply the $B$ process, followed by the $A$ process, to the sum over $n$, choosing

$$Q = H_1 X^{5c - 6 + 34\eta}$$

so that

$$XF^{-1/2} \left(\frac{FX^{-1}}{Q^{1/2}}\right) \ll X^{3 - 2c - 17\eta}.$$  

The errors from the $B$ process contribute (for some $H_1$)

$$\ll \mathcal{L} H_1^{-1} \sum_{h \sim H_1} (XF^{-1/2} + 1) \ll X^{1 - c/2} H_1^{-\frac{1}{2}} + \mathcal{L},$$

which is acceptable.

After the $A$ process we arrive at sums

$$S^*(h) = \sum_{n \in I} e \left((h + \gamma) \frac{1}{1-c} \left((n + q) \frac{c}{c-1} - n \frac{c}{c-1}\right)\right),$$

where the interval $I$ has endpoints $\approx FX^{-1} \approx H_1 X^{c-1} = N_1$, say. It suffices to show that, for fixed $q \in [1, Q]$,

$$\sum_{h \sim H_1} |S^*(h)| \ll H_1 N_1 Q^{-1}.$$

We apply Lemma 6 with $y_h = \left(\frac{h + \gamma}{H_1 + \gamma}\right)^{1-c}$. This gives, with $F_1 = H_1 X^{c} qN_1^{-1} \approx qX$, and assuming initially that $N_1 \in [F_1^{1/3}, F_1^{1/2}]$,

$$\sum_{h \sim H_1} |S^*(h)| \ll H_1^{\frac{319}{319}} N_1^{\frac{449}{690}} F_1^{\frac{63}{690} + \eta} + H_1 N_1^{\frac{1}{141} F_1^{\frac{141}{690} + \eta}}. \tag{6.8}$$

It suffices to show that

$$N_1^{\frac{449}{690}} F_1^{\frac{63}{690}} \ll H_1^{\frac{36}{345}} X^{-2\eta} N_1 Q^{-1} \quad \tag{6.9}$$

and

$$N_1^{\frac{1}{2}} F_1^{\frac{141}{690}} N_1^{-1} Q \ll X^{-2\eta}. \quad \tag{6.10}$$

The worst case in each of (6.8), (6.9) is $q = Q, H_1 = H$. (The factor lost for $H_1 > H$ is outweighed by the factor $HH_1^{-1}$ arising from $HH_1^{-2}$).
For \((6.9)\) we must show
\[
(HX^{c-1}) \frac{1}{2}(QX)^{\frac{141}{350}} HX^{5c-6} \ll X^{-2\eta}.
\]
This follows after a short computation from \(c < \frac{3581}{3106}\), determining our upper bound for \(c\).

We certainly have \(N_1 \leq \frac{1}{2}\) (using \(c < 7/6\)). If \(N_1 < \frac{1}{3}\), we apply Lemma 1 with \(\ell = 2\):
\[
S^* (h) \ll \frac{11}{14} N_1^{10/14} \ll \frac{13}{42}.
\]
It suffices to show that \(S^* (h) \ll N_1 Q^{-1}\), or
\[
(6.12) \quad \frac{13}{42} N_1^{-1} Q \ll X^{-2\eta}.
\]
The worst case is \(q = Q, H_1 = H\) and in this case the left-hand side of \((6.11)\) is
\[
\ll (X^{7c-7}) \frac{141}{350} X^{4c-5+C\eta} \ll X^{-2\eta}.
\]
This completes the discussion of \((6.2)\).

In view of Lemma 15, in order to prove \((6.3)\) it suffices to show that
\[
(6.13) \quad S_{II} := \sum_{m \sim M} \sum_{\ell \sim N} b_m c_\ell e(x\lceil m\ell \rceil^c) \ll X^{0.92353}
\]
for \(X^{0.16} \ll N \ll X^{0.38}\), and that
\[
(6.14) \quad S_I := \sum_{m \sim M} \sum_{\ell \sim N} b_m e(x\lceil m\ell \rceil^c) \ll X^{0.92353}
\]
for \(N \gg X^{0.38}\). In both cases we apply Lemma 1 (adapted to allow \(a_n \ll X^\eta\)) with (e.g.)
\[
a_n = \sum_{m\ell = n} b_m c_\ell \quad \text{in case (6.12)}.
\]
We choose \(H = X^{0.07648}\) in both cases.
For (6.13), it suffices to show that

\[ \sum_{m \sim M} \sum_{\ell \sim N, b \leq m \ell \leq X} b_m c_\ell e(x m^c n^c) \ll X^{0.92353} \]

(corresponding to \( h = 0 \) in Lemma 1) and that

\[ \sum_{m \sim M} \sum_{\ell \sim N, b \leq m \ell \leq X} b_m c_\ell e((h + \gamma) m^c n^c) \ll X^{0.92352}. \]

(The terms with \( h > H \) in Lemma 1 are covered by (6.7).) Proceeding as in (3.9), and taking \( Q = X^{0.15296} \), we require the bound

\[ S(q, M) := \sum_{m \sim M} e(\lambda m^c((n + q)^c - m^c)) \ll MQ^{-1} \]

\((1 \leq q \leq Q, Q \ll N \ll X^{0.38})\). Here \( \lambda \in \{x, h + \gamma\} \). We apply Lemma 14 (ii) with \( \ell = 2 \) to obtain

\[ S(q, M) \ll (qN^{-1}\lambda X^c)^{2/7}M^{2/7} + (qN^{-1}\lambda X^c)^{-1}. \]

The first term is bounded by \( MQ^{-1} \), as we easily verify. Since \( qN^{-1}x X^c \geq XN^{-1} \), the second term is acceptable.

For (6.14) it suffices with the same \( \lambda \) to show that

\[ \sum_{m \sim M} \sum_{n \sim N, b \leq mn \leq X} a_m e(\lambda m^c n^c) \ll X^{0.92352} \]

whenever \( N \geq X^{0.38} \). We appeal to Lemma 10 with \( F = \lambda X^c \). As above, the term \( X^{1+\eta}F^{-1} \) causes no difficulty, and the terms \( X^{11/12+\eta}, X^{1+\eta}N^{-1/2} \) are also acceptable. We have

\[ F^{1/8}X^{13/16}N^{-1/8} \ll \left( X^{35/80} + 0.07648 \right)^{1/8} X^{13/16 - 0.38/8} \ll X^{0.92}, \]

and

\[ (FX^5N^{-1}N_0^{-1})^{1/6} \ll (FX^{5-0.76})^{1/6} \ll X^{(35/80 + 0.07648 + 4.24)/6} \ll X^{0.92}. \]

This completes the proof of (6.3) and the discussion of the minor arc.
Here we use (1.6), so in the present section we show that (with $S^+$ to be specified below)

\begin{equation}
\int_{X^{2-c}}^{K} S(x)G(x)dx \ll X^{5-c-3\eta}
\end{equation}

where $G(x)$ is either $S^3(x)S^+(x)\Phi(x)e(-Rx)$ or $S^2(x)S^+(x)^2\Phi(x)e(-Rx)$.

Let us write $\|\ldots\|$ for sup norm on $[X^{2-c},K]$. It suffices to show that

\begin{equation}
A(x) \ll X^{5-2c-14\eta} + X^{1-c}x^{-1} \quad (0 < x \leq 2X^{2\eta})
\end{equation}

and that

\begin{equation}
\|S\|_\infty \ll X^{79/80+2\eta},
\end{equation}

\begin{equation}
\|S^+\|_\infty \ll X^{0.968+2\eta}.
\end{equation}

Using the bounds in Lemma 13 (ii), together with Lemma 12 (ii),

\begin{align*}
\left| \int_{X^{2-c}}^{K} S(x)G(x)dx \right|^2 & \ll L^4 X^{2-c}(\|S\|_\infty^2 \|S^+\|_\infty^2 + \|S^+\|_\infty^4)X^{2+3\eta} \\
& \quad + X^{5-c-14\eta} X L^2 X^{4+6\eta} \\
& \ll X^{2-c+2(\frac{79}{80}+0.968)+2+4\eta} + X^{10-2c-7\eta} \\
& \ll X^{10-2c-6\eta}
\end{align*}

as required for (7.1).

We turn to (7.2). This is obtained from Lemma 4, with $xX^c$, $X$ in place of $T$, $N$. The Kusmin-Landau theorem gives

\[ A(x) \ll X^{1-c}|x|^{-1} \]

unless $X^{c-1}x \gg 1$, which we now assume. If

\[ X \leq (xX^c)^{25/42} \]

we can use Lemma 7, since

\[ (xX^c)^{17/42} \ll X^{2.09\times17/42} \ll X^{1-\eta}; \]
we obtain
\[ A(x) \ll (X^{c+2\eta})^{\frac{4}{7}+\eta}X^{\frac{4}{7}} \ll X^{5-2c-14\eta} \]
since \(13c + 42 < 420 - 168c\) (this inequality determines the range of \(c\) in the theorem).

In the remaining case \(X > (xXc)^{25/42}\), we apply Lemma 5 (i) with \(\ell = 1:\)
\[ A(x) \ll (xXc)^{1/6}X^{\frac{4}{9}} \ll X^{7/25 + \frac{4}{9}+\eta} = X^{0.78+\eta} \]
which suffices for (7.2).

Turning to (7.3), we first show that Type II sums are \(O(X^{79/80+\eta})\) whenever
\[ X^{1/40} \ll N \ll X^{\frac{4}{5}} \]
(and hence whenever \(X^{1/40} \ll N \ll X^{39/40}\)). Proceeding as in (3.9), we need to show
\[ \sum_{x < \min \leq 2x} e(xm^c((n + q)^c - n^c)) \ll MX^{1-\frac{4}{5}+\eta} \]
whenever \(X^{39/40} \gg M \gg X^{1/2}\); here \(Q = X^{1/40}\), \(1 \leq q \leq Q\). We apply Lemma 8 with \(k = 5\); here
\[ f(5)(x) \asymp X^c q N^{-1} M^{-5}. \]
For the second term on the right-hand side in (2.1) we have the bound
\[ \ll M^{19/40 + \eta} \ll MX^{1-\frac{4}{5}+\eta} \]
since \(M \gg X^{1/2}\). We can absorb the first term into the second:
\[ x X^c q N^{-1} M^{-5} \ll M^{-1}, \]
because \(M^4 N \gg X^{5/2}\). For. the third term, we have
\[ M^{1+\eta}(x X^c q N^{-1} M^{-5})^{-\frac{4}{7} M^{-1/10}} \ll MX^{1-\frac{4}{7}} \]
since \(x X^c q N^{-1} \gg X^{3/2}\).

We claim that Type I sums are \(\ll X^{\frac{5}{7}}\) whenever \(N \geq X^{39/40}\). Using a familiar estimate,
\[ S_I \ll M(xXc)^{1/14}N^{10/14} \ll X^{1+\frac{4}{7}+\eta}N^{-\frac{2}{7}} \ll X^{79/80}. \]
It is now clear from Lemma 14 that (7.3) holds.
As for (7.4), we take (7.5)
\[ S^+(x) = \sum_{\frac{x}{2} < n \leq x} \rho(n, X^{0.064}) - \sum_{X^{0.064} < p_1 \leq X^{0.317}} \sum_{\frac{x}{2} < p_1 n \leq x} e(x(p_1 n)^e). \]

Using Buchstab’s identity, we have \( \rho^+ \geq \rho \) since (7.6)
\[ S^+(x) = \sum_{\frac{x}{2} < n \leq x} \rho(n)e(xn^e) + \sum_{X^{0.317} < p_1 < (3X)^{\frac{1}{2}}} \sum_{\frac{x}{2} < p_1 n \leq x} e(x(p_1 n)^e). \]

We show that (7.4) holds using Lemmas 15, 16. We claim first that
\[ S_I(x) \ll X^{0.968} \]
for \( N \gg X^{7/10} \). To see this,
\[ S_I(x) \ll M(xX^c)^{1/14}N^{10/14} \]
\[ \ll X^{1+2.09/14}(X^{7/10})^{-2/7} \ll X^{0.95}. \]

Next, we claim that
\[ S_{II}(x) \ll X^{0.968} \]
for
\[ X^{0.064} \ll N \ll X^{0.317}. \]

By a familiar argument, we need to show that for \( 1 \leq q \leq Q := X^{0.064} \), \( n \sim N \) we have
\[ S_* := \sum_{m \in I} e(x((n + q)^c - n^c)m^c) \ll MX^{-0.064} \]
\( (I \) is a subinterval of \((M, 2M)\). We have
\[ S_*M^{-1}X^{0.064} \ll (xqN^{-1}X^c)^{1/14}M^{10/14}M^{-1}X^{0.064} \]
\[ \ll X^{(0.064\times15+2.0884)/14}X^{-(1+3\times0.683)/14} \ll 1, \]
proving (7.7).
We may now apply Lemma 15 with $\alpha = 0.064$, $\beta = 0.064$, $M = X^{0.3}$, $R = S = 1$, $w(n) = e(xn^c)$. We obtain the desired bound
\[ \sum_{n \sim X} \rho(n, X^{0.064}) e(xn^c) \ll X^{0.968+\eta}, \]
while the sum
\[ \sum_{X^{0.064} \leq p_1 \leq X^{0.317}} \sum_{\frac{X}{p_1 n} \leq X \atop (n, P(p_1))=1} e(x(p_1 n)^c) \]
satisfies the same bound from Lemma 16. This completes the discussion of the minor arc.

8. Minor arc in Theorem 6

We shall show, for suitably chosen $T^+(x)$, that
\[ (8.1) \int_{x^{2-c}}^{x^{2-c}} T(x) G(x) dx \ll X^{5-c-3\eta} \]
where $G(x)$ is either of $T(x)^3 T^+(x) \Phi(x) e(-Rx)$ or $T(x) T^+(x)^2 \Phi(x) e(-Rx)$.

In Lemma 13 (ii) we take $S(x) = T(x)$, $J(x) = B(x)$ and $G$ as above. Suppose for the moment that
\[ (8.2) B(x) \ll X^{5-2c-20\eta} + L X^{1-c} x^{-1} \quad (0 < x \leq 2K) \]
and that, with
\[ T^+(x) = \sum_{\frac{X}{n} \leq X} \rho(n, X^{0.064}) e(x[n^c]) - \sum_{X^{0.064} \leq p_1 \leq X^{0.317}} \sum_{\frac{X}{p_1 n} \leq X \atop (n, P(p_1))=1} e(x[(p_1 n)^c]), \]
(as in Section 7, mutatis mutandis), we have
\[ (8.3) \| T \| \ll X^{2-\eta+3\eta}, \]
\[ (8.4) \| T^+ \| \ll X^{0.97095+3\eta}. \]
This implies (using Lemma 13 and Lemma 12 (ii)) that
\[
\left( \int_{X^{2-c}} T(x)G(x)dx \right)^2 \ll X^{2-c+7\eta+\frac{79}{60}+1.9419+2} + X^{5-2c-20\eta+5+8\eta} \ll X^{10-2c-6\eta}
\]
as required for (8.1).
Let \( H = X^{2c-4+60\eta} \). In order to prove (8.2) it suffices to obtain
\[
\sum_{0 \leq h \leq H} \min \left( 1, \frac{1}{h} \right) \left| \sum_{\frac{X}{h} < n \leq X} e((h + \gamma)n^c) \right| \ll X^{5-2c-20\eta}
\]
for \( \gamma \in \{x, -x, 0\} \), and
\[
\sum_{h > H} \frac{H}{h^2} \left| \sum_{\frac{X}{h} < n \leq X} e(hn^c) \right| \ll X^{5-2c-20\eta}.
\]
For the contribution from \( h = 0 \) in (8.5), we use the analysis leading to (7.2).
For the contribution from \( h \sim H \) in (8.5), we apply Lemma 6 with \( T \gg H_1X^c \) and \( N = X \), with \( y_h = \frac{h+\gamma}{H_1+\gamma} \). The condition \( T^{1/3} \leq X \leq T^{1/2} \) is obviously satisfied. We must show that
\[
H_1^{\frac{319}{309}} X^{\frac{449}{630}} (H_1X^c)^{\frac{63}{690}+\eta} \ll H_1 X^{5-c-20\eta},
\]
and that
\[
X^{\frac{1}{2}} (H_1X^c)^{\frac{141}{950}+\eta} \ll X^{5-c-20\eta}.
\]
The worst case in (8.7) is clearly \( H_1 = H \). We verify that
\[
\frac{11}{690} (2c - 4) + \frac{449}{690} + \frac{63c}{690} < 5 - 2c,
\]
which reduces to \( c < \frac{609}{293} \). (This determines the range of \( c \) in Theorem 5). In (8.8) we require
\[
\frac{1}{2} + (3c - 4) \frac{141}{950} < 5 - 2c,
\]
which holds for \( c < \frac{609}{293} \) with a little to spare.
The contribution to the left-hand side of (8.6) from $h \sim H_1$, $H \leq H_1 < \frac{1}{2} X^{3-c}$, can be handled using (8.7), (8.8) since we have

$$(2H_1X^c)^{\frac{3}{2}} \leq X \leq (H_1X^c)^{\frac{3}{2}}.$$ 

The additional factor $H/H_1$ arising from $H/h^2$ leads to a negative exponent of $H_1$ in using (8.7), (8.8).

For $H_1 \geq \frac{1}{2} X^{3-c}$, we use Lemma 5 (i) with $\ell = 2$: we need to verify that

$$\frac{H}{H_1}(H_1X^c)^{\frac{1}{2} \gamma} X^{\frac{12}{14}} < X^{5-2c-20\eta}.$$ 

The worst case is $H_1 = \frac{1}{2} X^{3-c}$. Here

$$H^{14}H_1X^c X^{10} < H_1^{14} X^{70-28c-300\eta}$$

since $70c < 155$. This completes the discussion of (8.2).

We also treat $T(x)$ and $T^+(x)$ using Lemma 1. For $T(x)$, we choose $H = X^{1/80}$. Now for (8.3) it suffices to show that for $X^{1/40} \ll N \ll X^{1/2}$, $X < X' \leq 2X$, $1 \leq h \leq H$, we have

$$(8.9) \quad S_{II} := \sum_{m \sim M} a_m \sum_{n \sim N} b_n e((h + \gamma)(mn)^c) \ll X^{79/80+2\eta}$$

for $\gamma \in \{x, -x, 0\}$; and, with the same ranges of $h$, $x$, $\gamma$ and $N \gg X^{9/10}$,

$$(8.10) \quad S_I := \sum_{m \sim M} \sum_{n \sim N} e((h + \gamma)m^c n^c) \ll X^{79/80+2\eta}.$$ 

(We already have a satisfactory bound for the sum

$$\sum_{h>H} \frac{H}{h^2} \left| \sum_{\frac{1}{X} < n \leq X} e(hn^c) \right|.$$ 

We begin with (8.9). By a familiar argument, we must show that, for $n \sim N$, $q \leq Q := x^{1/40}$, we have

$$\sum_{m \sim M} e((h + \gamma)m^c((n + q)^c - n^c)) \ll MQ^{-1}X^\eta.$$
We apply Lemma 8 with $k = 5$,

$$\lambda_5 = (h + \gamma)q X^c N^{-1} M^{-5}.$$  

Note that

$$\lambda_5 \ll M^{-1}$$  

since $NM^4 \gg X^{5/2}$. As for the second term in the bound in (2.1), it is

$$\ll M^{19/20+\eta} \ll MX^{-\frac{1}{40}+\eta}.$$  

For the third term,

$$M^{1-1/10} ((h + \gamma)q X^c N^{-1} M^{-5})^{-1/50} \ll MX^{-1/40}$$

since $(h + \gamma)X^c \gg X^2$ and $X^2 N^{-1} \gg X^{3/2}$. This proves (8.9).

Now we readily verify (8.10) on bounding $S_I$ by

$$\ll M (HX^c)^{1/14} N^{10/14} \ll X^{(1/80+c+3\eta)/14} (X^{9/10})^{-2/7} X \ll X^{0.9}.$$  

This establishes (8.10).

For $T^+(x)$ we proceed similarly, except that we now take $H = X^{0.02905}$, and instead of the range $[X^{\frac{1}{10}}, X^{\frac{1}{2}}]$, we have

$$Q := X^{0.0581} \ll N \ll X^{0.317}.$$  

The discussion of (8.9) goes as before, and it only remains to obtain the bound corresponding to (8.10). It suffices to show that, for $N \gg X^{9/10}$,

$$(HX^c)^{1/14} N^{10/14} \ll NX^{-0.06}$$

which is true with something to spare. This completes the proof of (8.4) and the treatment of the minor arc.

9. Major arc in Theorems 1–6.

The arguments in the present section are adapted from [4, 24, 28].

We begin with a number of lemmas. Let
\begin{align*}
v_1(X, x) &= \int_0^X e(x\gamma^c) d\gamma, \\
v(X, x) &= \sum_{1 \leq m \leq X} \frac{1}{c} m^{1/c-1} e(xm).
\end{align*}

Proofs of Lemmas 18 and 19 (ii) can be found in Vaughan [41, Sections 2.4, 2.5] with the unimportant difference that \( c \in \mathbb{N} \) in [41], while Lemma 19 (i) follows from [19] Lemma 3.1.

**Lemma 18.** We have
\[ v(X, x) = v_1(X, x) + O(1 + X^c|x|). \]

**Lemma 19.**

(i) We have
\[ v_1(X, x) - v_1(X/8, x) \ll (|x| X^{c-1})^{-1}. \]

(ii) For \( |x| \leq 1/2 \), we have
\[ v(X, x) \ll |x|^{-1/c}. \]

**Lemma 20.** For \( 2 \leq s \leq 5 \) and \( r \) large, let \( X = r^{1/c} \). We have
\[ L_s := \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( v(X^c, x) - v \left( \left( \frac{X}{8} \right)^c, x \right) \right)^s e(-rx) dx \gg r^{s/c-1}. \]

**Proof.** The integral is
\[ \frac{1}{c^s} \sum_{\left( \frac{X}{8} \right)^c < m_j \leq X^c \ (1 \leq j \leq s)} (m_1 \ldots m_s)^{\frac{1}{c}-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} e(x(m_1 + \cdots + m_s - r)) dx \]
\[ = \frac{1}{c^s} \sum_{\left( \frac{X}{8} \right)^c < m_j \leq X^c \ m_1 + \cdots + m_s = r} (m_1 \ldots m_s)^{\frac{1}{c}-1} \]
\[ \gg r^{\frac{c}{c-1} - s} \sum_{\frac{X}{8} \leq m_j \leq X \ (1 \leq j \leq s-1)} 1. \]
The last step is valid because for each choice of $m_1, \ldots, m_{s-1}$ in the last sum we have
\[ m_1 + \cdots + m_{s-1} \leq \frac{r}{s}(s - 1), \quad r \geq r - (m_1 + \cdots + m_{s-1}) > \frac{r}{8}, \]

hence $r - (m_1 + \cdots + m_{s-1}) = m_s$ with $\frac{r}{8} < m_s \leq r$. Now the desired lower bound follows at once. □

Lemma 21. For $2 \leq s \leq 5$, we have
\[ H_s = \int_{X/8}^{X} \cdots \int_{X/8}^{X} \phi(t_1^c + \cdots + t_s^c - R)dt_1 \cdots dt_s \gg X^{s-c-c\eta}. \]

Proof. One verifies easily that for each choice of $t_1, \ldots, t_{s-1}$ from $[\frac{X}{8}, \frac{X}{7}]$, there is an interval of $t_s$ in $[X, 2X]$ of length $\gg X^{1-c-c\eta}$ on which
\[ \phi(t_1^c + \cdots + t_s^c - R) = 1. \] □

A polytope means a bounded intersection of half-spaces in $\mathbb{R}^j$. The polytope $P_j$ is defined by
\[ P_j = \left\{ (y_1, \ldots, y_j) : \beta \leq y_j < y_{j-1} < \cdots < y_1, \right. \]
\[ \left. y_1 + \cdots + y_{j-1} + 2y_j \leq 1 + \frac{\log 3}{L} \right\}, \]

where $\beta = 8/75$. In writing sums containing $p_1, \ldots, p_j$, it is convenient to set
\[ \alpha_j = (\alpha_1, \ldots, \alpha_j) := \frac{1}{L} (\log p_1, \ldots, \log p_j), \]
\[ f_1(\alpha_1) = \alpha_1^{-2}, \quad f_j(\alpha_j) = (\alpha_1 \cdots \alpha_{j-1})^{-1} \alpha_j^{-2} (j \geq 2), \]
\[ \pi_j = p_1 \cdots p_j, \quad \pi_j' = p_1' \cdots p_j'. \] Let $\omega(\ldots)$ denote Buchstab’s function.

Lemma 22. Let $E$ be a polytope, $E \subseteq P_j$. Let $j + 1 \leq k \leq 9$.

(i) Let
\[ S_k(E) = \sum_{\alpha_j \in E} \sum_{p_j \leq p_{j+1} \leq \cdots \leq p_{k-1}} \frac{1}{\pi_{k-1}}, \]

Then
\[ S_k(E) \ll 1. \]
(ii) Let
\[ S_k^*(E) = \sum_{\alpha_j \in E} \sum_{p_1 \leq \cdots \leq p_{k-1}} \frac{1}{\pi_{k-1}}. \]
Then
\[ S_k^*(E) \ll L^{-1}. \]

Proof. Mertens' formula [7, Chapter 7] implies
\[ \sum_{A < p \leq B} \frac{1}{p} = \log \log B \log \frac{1}{A} + O(L^{-1}) \quad (X^\beta \leq A < B \leq 2X). \]

Now
\[ S_k(E) \leq \sum_{X^\beta \leq p_1 \leq X^{1-\beta}} \frac{1}{p_1} \cdots \sum_{X^\beta \leq p_{k-1} \leq X^{1-\beta}} \frac{1}{p_{k-1}} \]
\[ \leq \left( \log \frac{\log X^{1-\beta}}{\log X^\beta} + O(L^{-1}) \right)^{k-1} \ll 1. \]

In \( S_k^*(E) \) we replace the factor \( \sum_{X^\beta \leq p_{k-1} \leq X^{1-\beta}} \frac{1}{p_{k-1}} \) by
\[ \sum_{X^\beta \leq p_{k-1} \leq X^{1-\beta}} \frac{1}{p_{k-1}} \]
and use
\[ \log \left( \frac{\log \frac{X}{\pi_{k-1}}}{\log \frac{X}{8\pi_{k-1}}} \right) = \log \left( 1 + \frac{\log 8}{\log \frac{X}{8\pi_{k-1}}} \right) \ll L^{-1}. \]

Lemma 23. (i) Let \( 2 \leq Z < Z' \leq 2Z \). We have, for \( 0 < y < X^{1-c-2\eta} \),
\[ \sum_{Z \leq p < Z'} e(p^c y) = \int_Z^{Z'} \frac{e(u^c y)}{\log u} \, du + O(Z \exp(-C(\log Z)^{1/4})). \]

(ii) Let \( E \) be a polytope, \( E \subseteq P_j \). Let \( j + 1 \leq k \leq 9 \). Then for \( 0 < x \leq \tau \), \( X < X' \leq 2X \), we have
\[
\sum_{\alpha_j \in E} \sum_{\frac{\alpha}{\pi} < p_1 \ldots p_k \leq X} e(\pi_k^c x) = \sum_{\alpha_j \in E} \sum_{\frac{\alpha}{\pi} < p_1 \ldots p_k \leq X} \frac{1}{\pi_{k-1}}
\]

\[
\int_X^{\max(\pi_{k-1}p_{k-1}, \frac{a}{\pi})} \frac{e(t^c x)}{\log(t/\pi_{k-1})} \, dt + O(X \exp(-C\mathcal{L}^{1/4})).
\]

(iii) The assertions of (i), (ii) remain valid if \(e(p^c x), e(\pi_k^c x)\) are replaced respectively by \(e([p^c] x), e([\pi_k^c] x)\).

**Proof.** (i), (ii) are slight variants of [4, Lemma 24] and [5, Lemma 21] respectively. For (iii) we note that, when \(a_n \ll 1, x \ll \tau\),

\[
\sum_{n \leq 2X} a_n e(n^c x) - \sum_{n \leq 2X} a_n e([n^c] x) \ll X \tau \ll X^{1-c/2}.
\]

**Lemma 24.** Let \(E\) be a polytope, \(E \subseteq P_j\). Let

\[
f(E; X) = \sum_{\alpha_j \in E} \sum_{j+1 \leq k \leq 9} \sum_{\frac{\alpha_j}{\pi_{k-1}} \leq p_1 \ldots p_k \leq X} \frac{1}{\pi_{k-1} \log(X/\pi_{k-1})}.
\]

As \(X \to \infty\), we have

\[
f(E; X) = (1 + o(1)) \frac{1}{E} \int_{E} f_j(z_j) \omega \left( \frac{1 - z_1 - \cdots - z_j}{z_j} \right) \, dz_1 \ldots dz_j.
\]

**Proof.** This is a slight variant of [5, Lemma 20].

We now discuss the major arc for Theorems 1–6, and complete the proofs of the theorems.

(i) **Theorem** Let \(1 \leq j \leq s, 3 \leq s \leq 5\) and \(g\) having

\[
\sup_{x \in [-\tau, \tau]} |f_j| \ll X, \quad \int_{-\tau}^{\tau} |f_j|^2 \, dx \ll X^{2-c} \mathcal{L},
\]

\[
\sup_{x \in [-\tau, \tau]} |f(x) - g(x)| \ll X \exp(-C\mathcal{L}^{1/4}),
\]
we have

\[ \int_{-\tau}^{\tau} g(x)f_2(x) \ldots f_n(x)\Phi(x)e(-Rx)dx \]

\[ - \int_{-\tau}^{\tau} f_1(x)f_2(x) \ldots f_n(x)\Phi(x)e(-Rx)dx \]

\[ \ll X^{s-c-\eta}\exp(-C\mathcal{L}^{1/4}). \]

Thus in view of Lemma 23 (i), we can replace \[ \int_{-\tau}^{\tau} S_1(x)^4\Phi(x)dx \] by \[ \int_{-\tau}^{\tau} I(x)^4\Phi(x)e(-Rx)dx, \] replacing factors one at a time, with error \[ \ll X^{4-c-\eta}\exp(-C\mathcal{L}^{1/4}). \] Now we extend the integral to \( \mathbb{R} \) with total error \[ \ll X^{4-c-\eta}\exp(-C\mathcal{L}^{1/4}) \] using Lemma 19 (i) and the case \( s = 4 \) of

\[ \int_{-\tau}^{\tau} |x|^{-s}X^{s-c}a^4\Phi(x)dx \ll (X^{-c+8\eta})^{-s+1}X^{s-c-\eta} \]

\[ = X^{s-c-\eta-8(s-1)\eta}. \]

We find using (1.4) and the bound (4.1) that

\[ (9.2) \]

\[ \mathcal{L}^4 A_4(R) \gg \int_{-\infty}^{\infty} I(x)^4\Phi(x)e(-Rx)dx + O(X^{4-c-\eta}\exp(-C\mathcal{L}^{1/4})). \]

The integral here is

\[ (9.3) \]

\[ \int_{-\tau}^{\tau} (f_1(x) - g(x))f_2(x)\Phi(x)e(-Rx)dx \ll X^{s-c-\eta} \]

by Lemma 21. This yields Theorem 3 at once.

(ii) **Theorem 1** If \( f_1, f_2, g \) satisfy \( f_j \ll X, \int_{-\tau}^{\tau} f_j^2 \ll X^{2-c}\mathcal{L}, \) then the integral

\[ \int_{-\tau}^{\tau} (f_1(x) - g(x))f_2(x)\Phi(x)e(-Rx)dx \ll X \exp(-C\mathcal{L}^{1/4}), \]
is of the form \( \widehat{E}(R) \) where
\[
E(y) = \begin{cases} 
(f_1(y) - g(y))f_2(y)\Phi(y) & (y \in [-\tau, \tau]) \\
0 & \text{otherwise.}
\end{cases}
\]

By Parseval’s formula
\[
\int_\mathbb{V}^2 |\widehat{E}(R)|^2 dR < \int_\mathbb{R} |E(y)|^2 dy
\]
\[
= \int_{-\tau}^{\tau} (f_1(y) - g(y))^2 f_2^2(y)\Phi^2(y)dy
\]
\[
\ll X^{2-c-2\eta}X^2 \exp(-CL^{1/4})
\]
\[
\ll X^{4-c-\eta} \exp(-CL^{1/4}).
\]

Thus in two steps we can replace \( \int_{-\tau}^{\tau} S_1(x)\Phi(x)e(-Rx)dx \) by \( \int_{-\tau}^{\tau} I(x)^2 \Phi(x)e(-Rx)dx \) with an error that is acceptable for Theorem 1. (Compare the discussion of \( E_0(x) \) in Section 4.) Similarly in replacing
\[
\int_{-\tau}^{\tau} I(x)^2 \Phi(x)e(-Rx)dx \quad \text{by} \quad \int_{\mathbb{R}} I(x)^2 \Phi(x)e(-Rx)dx
\]
we incur an error \( E_1(x) \) with
\[
\int_\mathbb{V}^2 |\widehat{E}_1(R)|^2 dR < \int_{\tau}^{\infty} |I(y)|^4 |\Phi(y)|^2 dy
\]
which from (9.2) is \( \ll X^{4-c-\eta-24\eta} \). Now we easily adapt the argument leading to (9.3) to obtain
\[
\mathcal{L}^2 \mathcal{A}_2(R) \gg X^{2-c-\eta}
\]
extcept for a set of \( R \) in \([V, 2V] \) whose measure is \( \ll V \exp(-C(\log V)^{1/4}) \), proving Theorem 1.

(iii) Theorem 5. In the minor arc for Theorem 5 \( T_1(x) - S_1(x) = O(X\tau) \). We may replace \( \int_{-\tau}^{\tau} T_1(x)^3 \Phi(x)e(-Rx)dx \) by \( \int_{-\tau}^{\tau} S_1(x)^3 \Phi(x)e(-Rx)dx \) with error \( O(X^{3-c-3\eta}) \) since
\[
9.5 \quad T_1^3 - S_1^3 \ll X^2X\tau, \quad \int_{-\tau}^{\tau} |T_1^3 - S_1^3| dx \ll X^{3-2c+16\eta}.
\]
Now we replace \( \int_{\tau}^{1} \frac{1}{x^{3/4}} \log(x) \Phi(x) \) by \( \int_{\tau}^{1} J(x) \Phi(x) e(-Rx)dx \) with error \( O(X^{3-1/4} \exp(-C/L^{1/4})) \) using Lemmas 18 and 23 (i). We then extend the integral to \([-\frac{1}{2}, \frac{1}{2}]\) using Lemma 19 (ii); here we note that
\[
\int_{\tau}^{1/2} x^{-3/4} dx < (X-c+8\eta)^{1/4} < X^{3-8\eta}.
\]
Now we can complete the proof of Theorem 3 by drawing on Lemma 20 together with (1.5) and the result of Section 6.

(iv) **Theorem 2** We consider the sum \( S^+ \) on the major arc. We decompose \( S^+ \) into \( S \) plus \( O(1) \) sums of the type
\[
U_k(E, x) := \sum_{\alpha_j \in E} \sum_{p_j \leq p_{j+1} \leq \cdots \leq p_k} e(x \pi_k^c) \quad \text{where} \quad 1 \leq j \leq 3 \quad \text{and} \quad E \subseteq P_j.
\]
Recalling Lemma 23 (ii), we replace \( U_k(E, x) \) by
\[
V_k(E, x) := \sum_{\alpha_j \in E} \sum_{p_j \leq p_{j+1} \leq \cdots \leq p_k} \frac{1}{\pi_k \log(X/\pi_k)} \quad \text{where} \quad \pi_k \leq p_{k+1} \leq X
\]
\[
\left\{ v_1(X, x) - v_1 \left( \max \left( \pi_k P_{k-1}, \frac{X}{8} \right) \right) \right\}
\]
with error \( O(XL^{-1}) \). (We include \( L^{-1} I(x) \) as a term \( V_k(E, x) \) for convenience.)

By an obvious variant of the argument leading to (9.1), we can replace
\[
\int_{\tau}^{1} S^2(x) S^+(x) \Phi(x) e(-Rx) dx
\]
by
\[
\sum_{(k, E, x)} \frac{1}{L^2} \int_{-\tau}^{\tau} I(x)^2 V_k(E, x) \Phi(x) e(-Rx) dx,
\]
and replace
\[
\int_{\tau}^{1} S(x) S^+(x)^2 \Phi(x) e(-Rx) dx
\]
by

$$\sum_{(k,E)} \sum_{(k',E')} \frac{1}{\mathcal{L}} \int_{-\tau}^{\tau} I(x)V_k(E,x)V_{k'}(E',x)\Phi(x)e(-Rx)dx$$

with error $O(X^{3-c-c_3}\mathcal{L}^{-4})$. We can extend the integrals in (9.9) and (9.10) to $\mathbb{R}$ with error $O(X^{3-c-c_3}\mathcal{L}^{-4})$ using Lemma 19 (i) and (9.2).

We now observe that (omitting regions of summation)

$$\frac{1}{\mathcal{L}} \int_{-\infty}^{\infty} I(x)V_k(E,x)V_{k'}(E',x)\Phi(x)e(-Rx)dx$$

$$= \sum_{p_1,\ldots,p_{k-1}} \sum_{p'_1,\ldots,p'_{\ell-1}} \frac{1}{\pi_{k-1}\pi'_{\ell-1}} \frac{1}{(\log X)(\log X/\pi_{k-1})(\log X/\pi'_{\ell-1})}$$

$$\int_{X_3}^{X} \int_{X_2}^{X} \int_{X/8}^{X/2} e(x(t_1^c + t_2^c + t_3^c - R))\Phi(x)dx\,dt_1\,dt_2\,dt_3,$$

where $X_3 = \max(\pi_{k-1}p_{k-1}, \frac{X}{8})$, $X_2 = \max(\pi'_{\ell-1}p'_{\ell-1}, \frac{X}{8})$.

We rewrite the last expression as

$$\sum_{p_1,\ldots,p_{k-1}} \sum_{p'_1,\ldots,p'_{\ell-1}} \frac{1}{\pi_{k-1}\pi'_{\ell-1}} \frac{1}{(\log X)(\log X/\pi_{k-1})(\log X/\pi'_{\ell-1})}$$

$$\int_{X_3}^{2X} \int_{X_2}^{2X} \int_{X}^{2X} \phi(t_1^c + t_2^c + t_3^c - R)\,dt_1\,dt_2\,dt_3.$$

We replace $X_2, X_3$ by $X/8$, inducing an error that is $O(\mathcal{L}^{-4}H_3)$ by Lemma 22 (ii). This produces the quantity

$$\frac{H_3}{\mathcal{L}} \left( \sum_{p_1,\ldots,p_{k-1}} \frac{1}{\pi_{k-1}\log X/\pi_{k-1}} \right) \cdot \left( \sum_{p'_1,\ldots,p'_{\ell-1}} \frac{1}{\pi'_{\ell-1}\log X/\pi'_{\ell-1}} \right),$$

which can be calculated to within a factor $1 + o(1)$ using Lemma 24.

Following a similar argument with the integrals in (9.9), we arrive at representations, to within a factor $1 + o(1)$, of

$$\int_{-\tau}^{\tau} S^2(x)S^+(x)\Phi(x)e(-Rx)dx, \quad \int_{-\tau}^{\tau} S(x)S^+(x)^2\Phi(x)e(-Rx)dx$$
of the respective forms

\[ \frac{u^+ H_3}{L^3}, \quad \frac{(u^+)^2 H_3}{L^3}, \]

where (recalling (5.15)), \( u^+ = 1 + d_1 + d_2 + d_3 + d_4 \),

\[
\begin{align*}
    d_1 &= \int_{11/25}^{1/2} \frac{dx}{x(1-x)}, \\
    d_3 &= \int_{29/105}^{1/3} \frac{dx}{x(1-x)}, \\
    d_2 &= \int_{11/75}^{1/5} \int_{\max(\frac{29}{105}, \frac{11}{25} - x - y)}^{\frac{1}{z}} \int_{\max(\frac{11}{25} - x - y, \frac{29}{105} - y)}^{y} \frac{1}{x y^z} \\
    d_4 &= \int_{29/105}^{7/25} \int_{x}^{14/25 - x} \frac{dydx}{xy(1-x-y)}.
\end{align*}
\]

Taking into account (1.6) and the result of Section 5, we find that

\[ (9.11) \quad A_3(R) \gg (1 + o(1))(2u^+ - (u^+)^2) \frac{H_3}{L^3}. \]

Using a computer calculation for \( d_3 \) and \( d_4 \), we find that

\[
    d_1 < 0.242, \quad d_2 < 0.016, \quad d_3 < 0.272, \quad d_4 < 0.001.
\]

Thus \( u^+ \in (1, 2) \), and Theorem 2 follows from (9.11).

(v) **Theorem 4**  The discussion of the major arc is similar to that for Theorem 2. We decompose \( S^+(x) \) as \( S(x) \) plus \( O(1) \) sums of the form \( U_k(E,x) \). We replace \( U_k(E,x) \) by \( V_k(E,x) \) with error \( O(XL^{-1}). \) By a variant of the argument leading to (9.1), we can replace

\[
\begin{align*}
    \int_{-\tau}^{\tau} S^4(x)S^+(x)\Phi(x)e(-Rx)dx, \\
    \int_{-\tau}^{\tau} S^3(x)S^+(x)^2\Phi(x)e(-Rx)dx
\end{align*}
\]

respectively by

\[ (9.12) \quad \sum_{(k,E)} \int_{-\tau}^{\tau} I^4(x)U_k(E,x)\Phi(x)e(-Rx)dx \]

and
\[ (9.13) \quad \sum_{(k,E)} \sum_{(l,E')} \int_{-\tau}^{\tau} I^3(x) U_k(E, x) U_l(E', x) \Phi(x) e(-Rx) \, dx \]

with error \( O(X^{5-c-cn}L^{-6}) \). We extend the integrals in (9.12), (9.13) to \( \mathbb{R} \) with error \( O(X^{5-c-cn}L^{-6}) \) using Lemma 19 (ii).

Omitting regions of summation, we have

\[
\int_\mathbb{R} I(x)^3 U_k(E, x) U_l(E', x) \Phi(x) e(-Rx) \, dx
= \sum_{p_1, \ldots, p_{k-1}} \sum_{p'_1, \ldots, p'_{l-1}} \frac{1}{\pi_{k-1} \pi_{l-1}} \frac{1}{L^3} \frac{1}{(\log X / \pi_{k-1}) (\log X / \pi_{l-1})}
\int_{X_5}^{X} \int_{X_4}^{X} \int_{X}^{X} \int_{X}^{X} \int_{-\infty}^{\infty} e(x(t_1^c + t_2^c + t_3^c + t_4^c + t_5^c - R)) \Phi(x) \, dt_1 \ldots \, dt_5
\]

where \( X_5 = \max (\pi_{k-1}p_{k-1}, \frac{X}{8}) \), \( X_4 = \max (\pi'_{k-1}p'_{k-1}, \frac{X}{8}) \). We write the inner integral as \( \phi(t_1^c + \cdots + t_5^c - R) \) and replace \( X_4, X_5 \) by \( X/8 \), incurring an error that is \( O(L^{-6}H_5) \), by Lemma 22 (ii). This produces the quantity

\[
\frac{H_5}{L^3} \left( \sum_{p_1, \ldots, p_{k-1}} \frac{1}{\pi_{k-1} \log \frac{X}{\pi_{k-1}}} \right) \left( \sum_{p'_1, \ldots, p'_{l-1}} \frac{1}{\pi'_{l-1} \log \frac{X}{\pi'_{l-1}}} \right).
\]

Arguing as in the preceding proof, we arrive at representations of

\[
\int_{-\tau}^{\tau} S(x)^4 S^+(x) \Phi(x) e(-Rx) \, dx , \int_{-\tau}^{\tau} S^3(x) S^+(x)^2 \Phi(x) e(-Rx) \, dx
\]

to within a factor \( 1 + o(1) \), of the respective forms

\[
\frac{u^+ H_5}{L^5} , \quad \frac{(u^+)^2 H_5}{L^5}.
\]

Here, recalling (7.6),

\[
u^+ = 1 + d_1 + d_2;
\]

the integrals

\[
d_1 = \int_{0.317}^{0.5} \frac{dx}{x(1-x)} \quad \text{and} \quad d_2 = \int_{0.317}^{1/3} \int_{x}^{1/2(1-x)} \frac{dy \, dx}{xy(1-x-y)}
\]
take account of the products $p_1p_2 \sim X$ ($p_1 \leq p_2$) and $p_1p_2p_3 \sim X$ ($p_1 \leq p_2 \leq p_3$) respectively. Simple estimations yield

$$1 < u^+ < 1.8,$$

and Theorem 5 follows from (1.6) combined with the minor arc bound of Section 7.

(vi) **Theorem 6.** As in the discussion of the major arc for Theorem 5 we replace $T(x), T^+(x)$ respectively by $S(x), S^+(x)$ with acceptable error. We decompose $S^+$ as $S$ plus two sums of the form $U_k(E, x)$ in (9.7), $k = 2, 3$. Using Lemma 22 (ii), we replace $V_{k}(E, x)$ by

$$W_{k}(E, x) = \sum_{\alpha_{1} \in E} \sum_{p_{1} \leq p_{2} \leq \cdots \leq p_{k-1}} \frac{1}{\pi_{k-1} \log \frac{x}{\pi_{k-1}}} (v(X^c, x))$$

$$- v\left(\max\left(\frac{\pi_{k-1} p_{k-1}^c}{X^c}, \left(\frac{X}{8}\right)^{c}\right)\right)$$

with error $O(XL^{-1})$; similarly for $S(x)$. By a variant of the argument leading to (9.1), we replace

$$\int_{-\tau}^{\tau} S^4(x)S^+(x)e(-rx)dx, \int_{-\tau}^{\tau} S^3(x)S^+(x)^3e(-rx)dx$$

respectively by

(9.14) \[ \sum_{(k, E)} \frac{1}{L^4} \int_{-\tau}^{\tau} J^4(x)W_k(E, x)e(-rx)dx, \]

(9.15) \[ \sum_{(k, E)} \sum_{(\ell, E')} \frac{1}{L^3} \int_{-\tau}^{\tau} J^3(x)W_k(E, x)W_\ell(E', x)e(-rx)dx \]

with error $O(X^{5-c-\epsilon}L^{-6})$. Using Lemma 19 (ii), with the same error we can extend the integrals in (9.14), (9.15) to $[-\frac{1}{2}, \frac{1}{2}]$. Thus the expression in (9.15) has been replaced by

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} J^3(x)v(X^c, x) - v(X^c_4, x) v(X^c, x) - v(X^c_5, x))e(-rx)dx,$$
where \(X_4 = \max\left(\pi_{k-1}\pi_{k-1}, \frac{X}{8}\right)\) \(X_5 = \max\left(\pi'_{\ell-1}\pi'_{\ell-1}, \frac{X}{8}\right)\). We replace \(X_4, X_5\) by \(\frac{X}{8}\), incurring an error that is \(O(\mathcal{L}^{-6}L_5)\). This produces the quantity

\[
\frac{L_5}{\mathcal{L}^3} \sum_{p_1, \ldots, p_{k-1}} \frac{1}{\pi_{k-1}} \left(\frac{\log \frac{X}{\pi_{k-1}}}{\log \frac{X}{\pi_{k-1}}}\right) \sum_{p'_{1}, \ldots, p'_{\ell-1}} \frac{1}{\pi'_{\ell-1}} \left(\frac{\log \frac{X}{\pi'_{\ell-1}}}{\log \frac{X}{\pi'_{\ell-1}}}\right).
\]

We can now complete the proof of Theorem 6 in a similar manner to that of Theorem 4, with \(L_5\) in place of \(H_5\), since \(S^+\) is the same as in Theorem 4 and we have

\[
L_5 \gg r^{2/3} - 1
\]

by Lemma 20.

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