Solving Single and Many-body Quantum Problems: A Novel Approach

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Abstract

A unified approach, for solving a wide class of single and many-body quantum problems, commonly encountered in literature is developed based on a recently proposed method for finding solutions of linear differential equations. Apart from dealing with exactly and quasi-exactly solvable problems, the present approach makes transparent various properties of the familiar orthogonal polynomials and also the construction of their respective ladder operators. We illustrate the procedure for finding the approximate eigenvalues and eigenfunctions of non-exactly solvable problems.

1 Introduction

The familiar series solution approach [1] is routinely employed, when one encounters differential equations in quantum mechanical eigenvalue problems. This approach, not only is tedious to implement, but also throws little light on the underlying symmetries of the problem at hand. Hence, one looks for alternate simpler methods, a classic example being the elegant ladder operator approach to the harmonic oscillator problem. Another often used approach is the factorization technique pioneered by Schrödinger [2], Infeld and Hull [3]. In a number of cases, the underlying symmetries of the equation under consideration, have led to algebraic approaches based on group theory [4] and supersymmetry [5]. Most of these methods fail to generalize to many-body quantal systems and are also not easily applicable to non-exactly solvable problems.

In this paper, we give a pedagogical description of a recently developed method for solving linear differential equations [6, 7] and apply it to a number of single and many-body quantum problems. The proposed method is simple and assumes no special symmetry of the differential equation (DE) under consideration. The symmetries and the algebraic structure of the solution space emerge in a natural manner [8].

The paper is organized as follows. In the following section, we give a brief description of the proposed method of solving linear DEs, both, for the single and the many variable cases. For the purpose of illustration, we then consider the familiar harmonic oscillator problem, as also the related Hermite DE. Novel expressions for the solutions of the confluent hypergeometric and hypergeometric equations are then provided for later use.

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Section 3 is devoted to the construction of the ladder operators. It is shown that the novel form of the solutions, in conjunction with the Baker-Campbell-Hausdorff (BCH) formula, lead to a straightforward construction of the ladder operators, much akin to the harmonic oscillator case. We then proceed to the quasi-exactly solvable problem in section 4, wherein, the utility of the present approach for finding approximate eigenvalues and eigenfunctions for non-exactly solvable problems is also illustrated. Section 5 is devoted to the many-body correlated systems, which are currently under intense study. The procedure and the subtleties involved in dealing with such multi-variate Hamiltonians are explicitly pointed out. We then conclude in section 6 after pointing out other areas, where this approach may find fruitful application.

For the purpose of facilitating comparison with the standard literature, units have been appropriately chosen in different sections.

2 A simple approach to familiar differential equations

For simplicity, we will first consider the case of single variable linear DEs and point out its multi-variate generalization later. A single variable linear DE, as will become clear from the examples of later sections, can be cast in the form

\[ [F(D) + P(x, d/dx)] y(x) = 0 \]  

where, \( D \equiv xd/dx \) is the Euler operator, \( F(D) \equiv \sum_{n=-\infty}^{\infty} a_n D^n \) and \( a_n \)'s are some parameters; \( P(x, d/dx) \) can be an arbitrary polynomial function of \( x, d/dx \) and other operators. The solution to Eq. (1) can be written as \[6, 7\],

\[ y(x) = C_\lambda \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} P(x, d/dx) \right]^m \right\} x^\lambda \]  

provided, \( F(D)x^\lambda = 0 \); here \( C_\lambda \) is constant. Before proceeding further, we list two important properties of the Euler operator to be extensively used in the text. Euler operator is diagonal in the space of monomials i.e., \( Dx^\lambda = \lambda x^\lambda \) and other operators carry definite degrees, with respect to the Euler operator i.e., \[D, O^d] = dO^d\), where \( d \) is the degree of the operator \( O^d \). For example, \[D, x^2] = 2x^2 \) and \[D, d^2/dx^2] = -2 d^2/dx^2\). Using the above results, it is easy to see that, the operator \( 1/F(D) \) is well defined in the above expression and will not lead to any singularity, if \( P(x, d/dx) \) does not contain any degree zero operator.

The proof of Eq. (2) is straightforward and follows by direct substitution \[6\]. Alternatively, since \( F(D)x^\lambda = 0 \), equating \( F(D)x^\lambda \) modulo \( C_\lambda \) and Eq. (1), one finds,

\[ [F(D) + P(x, d/dx)] y(x) = C_\lambda F(D)x^\lambda \]  

Rearranging the above equation in the form

\[ F(D) \left[ 1 + \frac{1}{F(D)} P(x, d/dx) \right] y(x) = C_\lambda F(D)x^\lambda \]

and cancelling \( F(D) \), we obtain

\[ \left[ 1 + \frac{1}{F(D)} P(x, d/dx) \right] y(x) = C_\lambda x^\lambda \].
This yields
\[ y(x) = C_\lambda \frac{1}{1 + \frac{1}{F(D)} P(x, d/dx)} x^\lambda, \tag{6} \]
which can be cast in the desired series form:
\[ y(x) = C_\lambda \sum_{m=0}^{\infty} (-1)^m \left( \frac{1}{F(D)} P(x, d/dx) \right)^m x^\lambda. \]

It is explicit that, the above procedure connects the solution \( y(x) \) to the space of the monomials \( x^\lambda \). This fact will be exploited in the later sections for obtaining the ladder operators and explicate various properties of the solution space.

The generalization of this method to a wide class of many-variable problems is immediate. Using the fact that, \( F(\bar{D}) X_\lambda = 0 \) has solutions, in the space of monomial symmetric functions [9], where \( \bar{D} = \sum_i D_i \equiv \sum_i x_i \frac{d}{dx_i} \), the solutions of those multi-variate DEs, which can be separated into the form given in Eq. (1), can be solved like the single variable case. As will be seen later, this procedure enables one to solve a number of correlated many-body problems.

For illustration, we consider the harmonic oscillator problem. The Schrödinger eigenvalue equation (in the units, \( \hbar = \omega = m = 1 \))
\[ \left[ \frac{d^2}{dx^2} + (2E_n - x^2) \right] \psi_n = 0, \tag{7} \]
can be written in the form given in Eq. (1), after multiplying it by \( x^2 \) :
\[ \left[ (D - 1)D + x^2(2E_n - x^2) \right] \psi_n = 0. \tag{8} \]

Here, \( F(D) = (D - 1)D \) and the condition \( F(D)x^\lambda = 0 \) yields, \( \lambda = 0 \) or 1. Using Eq. (8), the solution for \( \lambda = 0 \) is,
\[
\psi_0 = C_0 \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{(D - 1)D} (x^2(2E_0 - x^2)) \right]^m \right\} x^0
= C_0 \left[ 1 - \frac{[2E_0]}{2!} x^2 + \frac{(2! + [2E_0]^2)}{4!} x^4 - \frac{(4! + (2!)^2[2E_0] + (2!)^2[2E_0]^2)}{2!6!} x^6 + \cdots \right]. \tag{9}
\]
Here \( \psi_0 \) is an expansion in powers of \( x \), whose coefficients are polynomials in \( E_0 \). The above series can be written in a closed, square integrable form, \( C_0 \exp(-x^2/2) \), only when \( E_0 = 1/2 \). Analogously, \( \lambda = 1 \), yields the first excited state. To find the \( n \)th excited state, one has to differentiate the Schrödinger equation \( (n - 2) \) number of times and subsequently multiply it by \( x^n \) to produce a \( F(D) = x^n \frac{d^n}{dx^n} = \prod_{l=0}^{n-1} (D - l) \), and proceed in a manner similar to the ground state case.

It is clear that, our procedure yields a series solution, where additional conditions like square integrability has to be imposed to obtain physical eigenfunctions and their corresponding eigenvalues. Once, the ground state has been identified and for those cases, where \( \psi(x) = \psi_0 P(x) \), where \( P(x) \) is a polynomial, one can effortlessly obtain the polynomial part, as will be shown below. Proceeding with the harmonic oscillator case and writing
\[ \psi_\alpha(x) = \exp\left(-\frac{x^2}{2}\right) H_\alpha(x), \]
one can easily show that \( H_\alpha \) satisfies
\[
\left[D - \alpha - \frac{1}{2} \frac{d^2}{dx^2}\right] H_\alpha(x) = 0 ,
\] (10)
where \( \alpha = E_n - 1/2 \). The solution of the DE
\[
H_\alpha(x) = C_\alpha \sum_{m=0}^{\infty} (-1)^m \left[ -\frac{1}{(D - \alpha)} \right] \frac{1}{2} \frac{d^2}{dx^2} \right] x^\alpha .
\] (11)
yields a polynomial only when \( \alpha \) is an integer, since the operator \( d^2/dx^2 \) reduces the degree of \( x^\alpha \) by two, in each step. Setting \( \alpha = n \) in Eq. (10), we obtain the Hermite DE and \( E_n = (n+1/2) \) as the energy eigenvalue. Below, we give the algebraic manipulations required to cast the series solution of Eq. (11) into a form, not very familiar in the literature. For the Hermite DE, \( F(D) = D - n \) and \( P(x, d/dx) = -\frac{1}{2} \frac{d^2}{dx^2} \), the condition \( F(D)x^\lambda = 0 \) yields \( \lambda = n \), hence,
\[
H_n(x) = C_n \sum_{m=0}^{\infty} (-1)^m \left[ -\frac{1}{(D - n)} \right] \frac{1}{2} \frac{d^2}{dx^2} \right] x^n .
\] (12)
Using, \( [D, (d^2/dx^2)] = -2(d^2/dx^2) \) and making use of the fact
\[
\frac{1}{(D - n)} = \int_0^\infty ds e^{-(D-n)}
\] (13)
we can write,
\[
\left[ -\frac{1}{2} \frac{1}{(D - n)} \frac{d^2}{dx^2} \right] = -\frac{1}{2} \frac{d^2}{dx^2} \frac{1}{(D - n - 2)} ,
\]
\[
\left[ -\frac{1}{2} \frac{1}{(D - n)} \frac{d^2}{dx^2} \right] \left[ \frac{1}{2} \frac{1}{(D - n)} \frac{d^2}{dx^2} \right] = \left[ -\frac{1}{2} \frac{d^2}{dx^2} \right] \frac{1}{(D - n - 4)} \frac{1}{(D - n - 2)}
\] (14)
Hence in general,
\[
\left[ -\frac{1}{2} \frac{1}{(D - n)} \frac{d^2}{dx^2} \right] x^n = \left( -\frac{1}{2} \frac{d^2}{dx^2} \right) \prod_{l=1}^{m} \frac{1}{(-2l)} x^n ,
\]
\[
= \frac{1}{m!} \left( \frac{1}{4} \frac{d^2}{dx^2} \right) x^n .
\] (15)
Substituting Eq. (13) in Eq. (12), we obtain,
\[
H_n(x) = C_n \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!} \left( \frac{1}{4} \frac{d^2}{dx^2} \right)^m x^n ,
\]
\[
= C_n \exp \left( -\frac{1}{4} \frac{d^2}{dx^2} \right) x^n ,
\] (16)
a result, not commonly found in the literature [10]. The arbitrary constant \( C_n \) is chosen to be \( 2^n \), so that the polynomials obtained can match with the standard definition [11].
The algebraic manipulations shown above can be applied to the confluent hypergeometric DE
\[
\left[ x \frac{d^2}{dx^2} + (\gamma - x) \frac{d}{dx} - \alpha \right] \Phi(\alpha; \gamma; x) = 0 ,
\]
(17)
to give
\[
\Phi(\alpha, \gamma, x) = (-1)^{-\alpha} \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)} \exp \left( -x \frac{d^2}{dx^2} - \gamma \frac{d}{dx} \right) . x^{-\alpha} ,
\]
(18)
where the normalization has been chosen appropriately. Likewise, the solution to the hypergeometric DE
\[
\left[ x^2 \frac{d^2}{dx^2} + (\alpha + \beta + 1) x \frac{d}{dx} + \alpha \beta - x \frac{d^2}{dx^2} - \gamma \frac{d}{dx} \right] F(\alpha, \beta; \gamma; x) = 0
\]
(19)
can be written as,
\[
F(\alpha, \beta; \gamma; x) = (-1)^{-\beta} \frac{\Gamma(\alpha - \beta) \Gamma(\gamma)}{\Gamma(\gamma - \beta) \Gamma(\alpha)} \exp \left[ \frac{-1}{(D + \alpha)} \left( x \frac{d^2}{dx^2} + \gamma \frac{d}{dx} \right) \right] . x^{-\beta} .
\]
(20)

For convenience a table has been provided at the end, which lists a number of commonly encountered DEs and the novel exponential forms of their solutions. It is worth pointing out that, the solutions of confluent hypergeometric and hypergeometric DEs given above, are polynomial solutions, provided \(\alpha\) and \(\beta\) are negative integers. It should be noticed that, unlike the conventional expressions, the monomials in the above solutions are arranged in decreasing powers of \(x\).

The series solutions for the same can be obtained by a simple modification of the DE. Multiplying Eq. (17) by \(x\), we get
\[
\left[ x^2 \frac{d^2}{dx^2} + \gamma x \frac{d}{dx} - x^2 \frac{d}{dx} - \alpha x \right] \Phi(\alpha; \gamma; x) = 0
\]
(21)
for which, \(F(D) = \left( x^2 \frac{d^2}{dx^2} + \gamma x \frac{d}{dx} \right)\) and \(P(x, d/dx) = -x^2 \frac{d}{dx} - \alpha x\). The requirement \(F(D)x^\lambda = 0\), generates two roots, \(\lambda = 0, 1 - \gamma\), yielding the two linearly independent solutions. For \(\lambda = 0\), we obtain the familiar confluent hypergeometric series
\[
\Phi(\alpha; \gamma; x) = 1 + \frac{\alpha}{\gamma} x + \frac{\alpha(\alpha + 1)}{\gamma(\gamma + 1)} \frac{x^2}{2!} + \cdots .
\]
(22)
Similarly multiplying the hypergeometric DE with \(x\) yields two roots, \(\lambda = 0, 1 - \gamma\) and \(\lambda = 0\) solution gives rise to the well-known, Gauss hypergeometric series. Since a number of quantum mechanical problems can be related to confluent and hypergeometric DEs [12], we hope that the novel expressions given above and in the table will find physical applications.

3 Construction of Ladder Operators

In this section, we first derive the ladder operators for the harmonic oscillator, making use of the exponential form of the solution for the Hermite polynomials derived earlier and then proceed to the Coulomb problem and the Laguerre polynomials associated
with it. Subsequently, we outline the steps required for generalizing these results for other cases.

The advantage of the present approach, which connects the solution space to the space of the monomials, lies in the fact that, at the level of the monomials, the ladder operators can be constructed easily, which facilitates the construction of the same at the level of the polynomials and the wave functions. The raising and lowering operators for the Hermite polynomials are derived, starting respectively from \( x \) and \( d/dx \) at the level of the monomials. For example, \( xx^n = x^{n+1} \), when operated by \( \exp[-(1/4)(d^2/dx^2)] \equiv \exp(-A) \), from the left on both the sides and after introducing an identity operator suitably, yields

\[
e^{-A}2xe^A2^ne^{-A}x^n = 2^{n+1}e^{-A}x^{n+1} ,
\]

which leads to

\[
\left[2x - \frac{d}{dx}\right] H_n(x) = H_{n+1}(x) .
\]

In deriving the above relation, we have made use of the BCH formula,

\[
e^{-A}Be^A = B + [B,A] + \frac{1}{2!}[[B,A],A] + \cdots .
\]

The construction of the lowering operator is analogous to the above procedure. From \( \frac{d}{dx}x^n = nx^{n-1} \), one obtains, by performing a similarity transformation,

\[
e^{-A}\frac{d}{dx}e^A2^ne^{-A}x^n = 2n2^{n-1}e^{-A}x^{n-1} ,
\]

or

\[
\frac{d}{dx}H_n(x) = 2nH_{n-1}(x) .
\]

The creation and annihilation operators for the harmonic oscillator problem follow after one more similarity transformation:

\[
e^{-\frac{x^2}{2}} \left[2x - \frac{d}{dx}\right] e^{\frac{x^2}{2}} \sqrt{\frac{1}{2n!}} e^{-\frac{x^2}{2}} H_n(x) = \sqrt{\frac{2(n+1)}{2n+1(n+1)!}} e^{-\frac{x^2}{2}} H_{n+1}(x) ,
\]

or

\[
\left[x - \frac{d}{dx}\right] \psi_n(x) = \sqrt{2(n+1)} \psi_{n+1}(x) ;
\]

and

\[
e^{-\frac{x^2}{2}} \frac{d}{dx} e^{\frac{x^2}{2}} \sqrt{\frac{1}{2n!}} e^{-\frac{x^2}{2}} H_n(x) = \sqrt{\frac{2n}{2n-1(n-1)!}} e^{-\frac{x^2}{2}} H_{n-1}(x) ,
\]

leads to

\[
\left[\frac{d}{dx} + x\right] \psi_n(x) = \sqrt{2n} \psi_{n-1}(x) .
\]

Here

\[
\psi_n = \sqrt{\frac{1}{2n}} \exp(-\frac{x^2}{2}) H_n(x) .
\]

Proceeding in an analogous manner, we derive the ladder operators for the Coulomb problem. The polynomial solution of the Laguerre DE

\[
\left[x \frac{d^2}{dx^2} + (\alpha - x + 1) \frac{d}{dx} + n\right] L_n^\alpha(x) = 0 ,
\]

\( \alpha, n = 0, 1, 2, \ldots \)
can be written as,
\[ L_\alpha^n(x) = \frac{(-1)^n}{n!} \exp \left[ -x \frac{d^2}{dx^2} - (\alpha + 1) \frac{d}{dx} \right] x^n. \]  
(34)

With \( B = [xd^2/dx^2 + (\alpha + 1)d/dx] \) and proceeding as in the Hermite polynomial case we obtain
\[ \left[ x - 2x \frac{d}{dx} - (\alpha + 1) + x \frac{d^2}{dx^2} + (\alpha + 1) \frac{d}{dx} \right] L_\alpha^n(x) = -(n + 1)L_\alpha^{n+1}(x) . \]  
(35)

To construct the lowering operator, one should start with a lowering operator different from \( d/dx \) at the level of the monomials, since a similarity transformation on this operator does not lead to a closed form expression. The simplest one is the operator \( B \) itself, since it commutes with \( e^{-B} \). Acting \( B \) on the monomial \( x^n \), we obtain
\[ \left[ x \frac{d^2}{dx^2} + (\alpha + 1) \frac{d}{dx} \right] x^n = n(n + \alpha)x^{n-1} , \]  
and performing a similarity transformation through \( e^{-B} \), one gets
\[ \left[ x \frac{d^2}{dx^2} + (\alpha + 1) \frac{d}{dx} \right] L_\alpha^n(x) = -(n + \alpha)L_\alpha^{n-1}(x) . \]  
(36)

To obtain the ladder operators for the radial wave functions, \( R_\alpha^n(x) = x^l e^{-\frac{\alpha}{x}}L_\alpha^n(x) \) of the Coulomb Hamiltonian, one proceeds in a manner similar to the oscillator problem. Explicitly, the action of the raising and lowering operator, respectively are,
\[ \left[ \frac{x}{2} - x \frac{d}{dx} + x \frac{d^2}{dx^2} + 2 \frac{d}{dx} - \frac{l(l + 1)}{x} - 1 \right] R_\alpha^n(x) = -(n + 1)R_\alpha^{n+1}(x) , \]  
(38)

and
\[ \left[ \frac{x}{2} + x \frac{d}{dx} + x \frac{d^2}{dx^2} + 2 \frac{d}{dx} - \frac{l(l + 1)}{x} + 1 \right] R_\alpha^n(x) = -(n + \alpha)R_\alpha^{n-1}(x) . \]  
(39)

It is worth pointing out that the value of \( \alpha = (2l + 1) \) can also be changed, since
\[ \frac{d}{dx} L_\alpha^n(x) = (-1^n/n!) \exp \left[ -x \frac{d^2}{dx^2} - (\alpha + 2) \frac{d}{dx} \right] \frac{d}{dx} x^n , \]  
and from where, one obtains the standard result,
\[ \frac{d}{dx} L_\alpha^n(x) = -L_\alpha^{n+1}(x) . \]  
(41)

It is clear that a composite operator can be obtained keeping \( n \) unchanged, while changing the value of \( \alpha \). The forms of the ladder operators given above are not unique, one can construct more complicated operators starting from different ones at the monomial level.

The exponential form of the solutions enables one to construct ladder operator for hypergeometric, confluent hypergeometric and other polynomials and functions, which can be used to construct similar operators for the quantum mechanical problems associated with these polynomials, e.g., Morse, Pöschl-Teller, Eckart and other potentials.
4 Quasi-Exactly Solvable Problems

This section is devoted to the study of QES problems \cite{14, 15}. These problems are intermediate to exactly and non-exactly solvable quantum potentials, in the sense that, only a part of the spectrum can be determined analytically. These potentials have attracted considerable attention in recent times, because of their connection to various physical problems.

We illustrate our procedure, through the sextic oscillator (in the units $\hbar = 2m = \omega = 1$), whose eigenvalue equation is given by:

$$ -\frac{d^2}{dx^2} + \alpha x^2 + \gamma x^6 \right) \psi(x) = E \psi(x) \ . \quad (42) $$

Asymptotic analysis suggests a trial wave function of the form,

$$ \psi(x) = \exp(-bx^4) \tilde{\psi}(x) \ , \quad (43) $$

which leads to,

$$ \left[ -\frac{d^2}{dx^2} + 2\sqrt{\gamma}x \frac{d}{dx} + (\alpha + 3\sqrt{\gamma})x^2 \right] \tilde{\psi}(x) = E \tilde{\psi}(x) \ , \quad (44) $$

where $x^6$ term has been removed by the condition $16b^2 = \gamma$. One notices that, the operator $\tilde{O} = (\alpha + 3\sqrt{\gamma})x^2 + 2\sqrt{\gamma}x^3d/dx$ increases the degree of $\tilde{\psi}(x)$ by two, if $\tilde{\psi}(x)$ is a polynomial. Confining ourselves to polynomial solutions and assuming that the highest power of the monomial in $\tilde{\psi}(x)$ is $n$, one obtains,

$$ -\frac{\alpha}{\sqrt{\gamma}} = 2n + 3 \ , \quad (45) $$

after imposing the condition that $\tilde{O}$ does not increase the degree of the polynomial. This is the well-known relationship between the coupling parameters of the quasi-exactly solvable sextic oscillator. Taking $n = 4$ and $\gamma = 1$ for simplicity, and after multiplying the above equation with $x^2$ :

$$ \left[ D(D - 1) + E x^2 + 8x^4 - 2x^5 \frac{d}{dx} \right] \tilde{\psi}(x) = 0 \ , \quad (46) $$

we get,

$$ \tilde{\psi}_0(x) = C_0 \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{D(D - 1)} \left( x^2E_0 + 8x^4 - 2x^5 \frac{d}{dx} \right) \right]^m \right\} \psi_0(x) \ , \quad (47) $$

Modulo $C_0$, the above series can be expanded as,

$$ \tilde{\psi}_0(x) = 1 - E_0 \frac{x^2}{2!} + (E_0^2 - 16) \frac{x^4}{4!} + (64E_0 - E_0^3) \frac{x^6}{6!} + \cdots \ . \quad (48) $$

The monomials having degree greater than four vanish provided, $E_0 = 0, \pm 8$. It can be explicitly checked that, for these values of $E_0$, Eq. (48) is satisfied. The eigenfunctions
corresponding to these three values are given by,
\[
\psi_{-8}(x) = \exp\left(-\frac{x^4}{4}\right)[1 + 4x^2 + 2x^4]
\]
(49)

\[
\psi_0(x) = \exp\left(-\frac{x^4}{4}\right)[1 - \frac{2}{3}x^4]
\]
, (50)

and \[
\psi_{+8}(x) = \exp\left(-\frac{x^4}{4}\right)[1 - 4x^2 + 2x^4]
\]
. (51)

This procedure generalizes to a wide class of QES problems [16].

Below, we demonstrate the method of finding approximate eigenvalues and eigenfunctions for non-exactly solvable problems, using the well studied anharmonic oscillator as the example:

\[
\left[-\frac{d^2}{dx^2} + \alpha x^2 + \beta x^4 - E_n\right] \psi_n(x) = 0
\]
(52)

Proceeding as before, \(\psi_0(x)\) can be written as,
\[
\psi_0(x) = C_0 \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{(D-1)D} \left( E_0 x^2 - \alpha x^4 - \beta x^6 \right) \right]^m \right\} .1
\]
, (53)

which can be expanded as,
\[
\psi_0(x) = 1 - \frac{E_0 x^2}{2!} + \frac{1}{4!} \left( 2\alpha + E_0^2 \right) x^4 - \frac{1}{6!} \left( 24\beta - (14\alpha E_0 + E_0^3) \right) x^6 + \cdots
\]
(54)

Although a number of schemes can be devised for the purpose of approximation, we consider the simplest one of starting with a trial function \(\tilde{\psi}_0(x) = \exp(-\mu x^2 - \nu x^4)\) and matching it with \(\psi_0(x)\). Comparison of the first three terms yields,

\[
\mu = \frac{E_0}{2!}
\]
,

\[
\frac{\mu^2}{2!} - \nu = \frac{2\alpha + E_0^2}{4!} + \frac{E_0^2}{4!}
\]
,

and \[
\mu \nu - \frac{\mu^3}{3!} = \frac{\beta}{30} - \frac{14E_0 + E_0^3}{6!}
\]. (55)

The resulting cubic equation in energy, \(E_0^3 - E_0\alpha = 3\beta/2\), leads to one real root and two complex roots. Choosing the real root on physical grounds, one obtains,
\[
E_0 = \frac{2^{1/3}\alpha}{A} + \frac{A}{3 \cdot 2^{1/3}}
\]
, (56)

where \(A = \left[40.5\beta + (1640.25\beta^2 - 108\alpha^3)^{1/2}\right]^{1/3}\). The value of \(E_0\), obtained in the weak coupling regime, matches reasonably well with the earlier obtained results [17]. An approximate \(\psi_0\) can be obtained from Eq. (54). One can easily improve upon the above scheme by taking better trial wave functions. Similar analysis can be carried out for the excited states. The above expansion of the wave function may be better amenable for a numerical treatment. For example, an accurate numerically determined energy value can lead to a good approximate wave function.
5 Many-body interacting systems

In this section, we will be dealing with correlated many-body systems, particularly of the Calogero-Sutherland \[9\] and Sutherland type \[9\]. These models have found application in diverse branches of physics like fluid flow, random matrix theory, novel statistics, quantum Hall effect and others \[20, 21\]. A number of methods e.g., Lax pair \[22\], Bethe-ansatz techniques \[23\] and $S_N$-extended Heisenberg algebra \[7, 24, 25\] have been employed for studying these systems.

We start with the relatively difficult Sutherland model, where the particles are confined to a circle of circumference $L$. The two-body problem treated explicitly below, straightforwardly generalizes to $N$ particles. The Schrödinger equation is given by (in the units $\hbar = m = 1$)

$$
\left[-\frac{1}{2} \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} + \beta (\beta - 1) \frac{\pi^2}{L^2} \sin^2 \left(\frac{\pi (x_1 - x_2)}{L}\right) - E_\lambda \right] \psi_\lambda(\{x_i\}) = 0 \quad .
$$

Taking, $z_j = e^{2ix_j/L}$ and writing $\psi_\lambda(\{z_i\}) = \prod_{i,i \neq j} z_i^{-\beta/2} (z_i - z_j)^\beta J_\lambda(\{z_i\})$, the above equation becomes,

$$
\left[ \sum_{i=1}^{2} D_i^2 + \beta \frac{z_1 + z_2}{z_1 - z_2} (D_1 - D_2) + \tilde{E}_0 - \tilde{E}_\lambda \right] J_\lambda(\{z_i\}) = 0 \quad ,
$$

where, $D_i \equiv z_i \frac{\partial}{\partial z_i}$, $\tilde{E}_\lambda \equiv 2\left(\frac{\pi}{L}\right)^2 E_\lambda$, $\tilde{E}_0 \equiv 2\left(\frac{\pi}{L}\right)^2 E_0$ and $E_0 = (\frac{\pi}{L})^2 \beta^2$, is the ground-state energy. Here, $J_\lambda(\{z_i\})$ is the polynomial part, which in the multivariate case is the well known Jack polynomial \[9\]. Here, $\lambda$ is the degree of the symmetric function and $\{\lambda\}$ refers to different partitions of $\lambda$. $\sum_i D_i^2$ is a diagonal operator in the space spanned by the monomial symmetric functions, $m_{\{\lambda\}}$, with eigenvalues $\sum_{i=1}^{2} \lambda_i^2$. A monomial symmetric function is a symmetrized combination of monomials of definite degree. For example for two particle case, there are two monomial symmetric functions having degree two. These are $m_{2,0} = x_1^2 + x_2^2$ and $m_{1,1} = x_1 x_2$. Readers are referred to Ref. \[9\] for more details about various symmetric functions and their properties. Rewriting Eq. (58) in the form,

$$
\left[ \sum_{i} (D_i^2 - \lambda_i^2) + \beta \frac{z_1 + z_2}{z_1 - z_2} (D_1 - D_2) + \tilde{E}_0 + \sum_i \lambda_i^2 - \tilde{E}_\lambda \right] J_\lambda(\{z_i\}) = 0 \quad ,
$$

one can immediately show that,

$$
J_\lambda(\{z_i\}) = C_\lambda \left\{ \sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{\sum_i (D_i^2 - \lambda_i^2)} \left( \beta \frac{z_1 + z_2}{z_1 - z_2} (D_1 - D_2) + \tilde{E}_0 + \sum_i \lambda_i^2 - \tilde{E}_\lambda \right) \right]^n \right\} \times m_\lambda(\{z_i\}) \quad .
$$

For the sake of convenience, we define

$$
\hat{S} \equiv \left[ \frac{1}{\sum_i (D_i^2 - \lambda_i^2)} \hat{Z} \right] \quad ,
$$

and

$$
\hat{Z} \equiv \beta \frac{z_1 + z_2}{z_1 - z_2} (D_1 - D_2) + \tilde{E}_0 + \sum_i \lambda_i^2 - \tilde{E}_\lambda \quad .
$$

10
The action of $\hat{S}$ on $m_\lambda(\{z_i\})$ yields singularities, unless one chooses the coefficient of $m_\lambda$ in $\hat{Z} m_\lambda(\{z_i\})$ to be zero; this condition yields the eigenvalue equation

$$\hat{E}_\lambda = E_0 + \sum_i (\lambda_i^2 + \beta [3 - 2i] \lambda_i) .$$

Using the above, one can write down the two particle Jack polynomial as,

$$J_\lambda(\{z_i\}) = \sum_{n=0}^{\infty} (-\beta)^n \left[ \frac{1}{\prod_i (D_i^2 - \lambda_i^2)} \left( \frac{z_1 + z_2}{z_1 - z_2} (D_1 - D_2) - \sum_i (3 - 2i) \lambda_i \right) \right]_n \times m_\lambda(\{x_i\}) . \quad (62)$$

Starting from $m_{2,0} = z_1^2 + z_2^2$, it is straightforward to check that

$$\begin{align*}
\hat{Z} m_{2,0} &= 4\beta (z_1 + z_2)^2 = 4\beta m_{1,0}^2 \\
\hat{S} m_{2,0} &= \frac{1}{\sum_i (D_i^2 - 4)} (4\beta m_{1,0}^2) = -2\beta m_{1,0}^2 , \\
\hat{S}^n m_{2,0} &= -2(\beta)^n m_{1,0}^2 \text{ for } n \geq 1 .
\end{align*}$$

Substituting the above result in Eq. (60), apart from $C_2$, one obtains

$$J_2 = m_{2,0} + \left( \sum_{n=1}^{\infty} (-1)^n (-2)^n (\beta)^n \right) m_{1,0}^2$$

$$= m_{2,0} + 2\beta \left( \sum_{n=0}^{\infty} (-\beta)^n \right) m_{1,0}^2$$

$$= m_{2,0} + \frac{2\beta}{1 + \beta} m_{1,0}^2 ; \quad (63)$$

which is the desired result. The above approach can be easily generalized to the $N$-particle case.

Another class of many-body problems, which can be solved by the present approach is the Calogero-Sutherland model (CSM) and its generalizations [22]. Proceeding along the line, as for the Sutherland model, one finds the eigenvalues and eigenfunctions for the CSM.

The Schrödinger equation for the CSM in the previous units, is given by,

$$\left[ -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial}{\partial x_i^2} + \frac{1}{2} \sum_{i=1}^{N} x_i^2 + \frac{g^2}{2} \sum_{i,j}^{N} \frac{1}{(x_i - x_j)^2} - E_n \right] \psi_n(\{x_i\}) = 0 , \quad (64)$$

where the wave function is of the form, [18]

$$\psi_n(x) = \psi_0 P_n(\{x_i\}) = ZGP_n(\{x_i\}) . \quad (65)$$

Here $Z \equiv \prod_{i<j} (x_i - x_j)^{\beta}$, $G \equiv \exp \left\{-\frac{1}{2} \sum_i x_i^2 \right\}$, $g^2 = \beta(\beta - 1)$ and $P_n(\{x_i\})$ is a polynomial. After removing the ground state, the polynomial $P_n(\{x_i\})$ satisfies,

$$\left[ \sum_i x_i \frac{\partial}{\partial x_i} + E_0 - E_n - \frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} - \beta \sum_{i \neq j} \frac{1}{(x_i - x_j)^2} \frac{\partial}{\partial x_i} \right] P_n(\{x_i\}) = 0 , \quad (66)$$
where \( E_0 = \frac{1}{2} N + \frac{1}{2} \beta N (N - 1) \). Defining

\[
\hat{A}(\beta) \equiv \frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \beta \sum_{i \neq j} \frac{1}{(x_i - x_j)} \frac{\partial}{\partial x_i},
\]

one can easily see, following the procedure adopted for the Hermite DE that

\[
P_n(\{x_i\}) = C_n e^{-\hat{A}(\beta)} m_{\{n\}}(\{x_i\}), \tag{67}
\]

where \( m_{\{n\}}(\{x_i\}) \) is a monomial symmetric function of degree \( n \). The corresponding energy is given by

\[
E_n = E_0 + n, \tag{68}
\]

One can use the above procedure to solve many other interacting systems.

6 Conclusions

In conclusion, we have presented a novel scheme to treat exactly, quasi-exactly and non-exactly solvable problems, which also extends to a wide class of many-body interacting systems. The procedure was used for the construction of the ladder operators for various orthogonal polynomials and the quantum systems associated with them. The approximation scheme presented needs further refinement. It should be analyzed in conjunction with computational tools for finding its efficacy as compared to other methods. The many-body problems presented here have deep connection with diverse branches of physics and mathematics. The fact that the procedure employed for solving them, connects the solution space of the problem under study to the space of monomials, will make it useful for constructing ladder operators for the many-variable case. This will throw light on the structure of the Hilbert space of these correlated systems. Some of these questions are currently under study and will be reported elsewhere.

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TABLE I. Some frequently encountered DE and their novel solutions.

Below, the differential equations from top to bottom, respectively, are Hermite, Laguerre, Legendre, Gegenbauer, Chebyshev Type I, Chebyshev Type II, Bessel, Confluent Hypergeometric and Hypergeometric.

| Differential Equation | $F(D)$, $D \equiv x(d/dx)$ | Solution |
|-----------------------|-----------------------------|----------|
| \[
\left[ x \frac{d}{dx} - n - \frac{1}{2} \frac{d^2}{dx^2} \right] H_n(x) = 0
\]
| $(D - n)$ | $H_n(x) = C_n \exp \left[ -\frac{1}{4} \frac{d^2}{dx^2} \right] x^n$ |
| \[
\left[ x \frac{d}{dx} - n - (\alpha + 1) \frac{d}{dx} - x \frac{d^2}{dx^2} \right] L_n^\alpha = 0
\]
| $(D - n)$ | $L_n^\alpha(x) = C_n \exp \left[ -x \frac{d}{dx} - (\alpha + 1) \frac{d}{dx} \right] x^n$ |
| \[
\left[ x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} - n(n+1) - \frac{d^2}{dx^2} \right] P_n(x) = 0
\]
| $(D + n + 1)(D - n)$ | $P_n(x) = C_n \exp \left[ -\frac{1}{2(D+n+1)} \frac{d^2}{dx^2} \right] x^n$ |
| \[
\left[ x^2 \frac{d^2}{dx^2} + (2\lambda + 1)x \frac{d}{dx} - n(2\lambda + n) - \frac{d^2}{dx^2} \right] \times
\]
| $(D + n + 2\lambda)(D - n)$ | $C_n^\lambda(x) = C_n \exp \left[ -\frac{1}{2(D+n+2\lambda)} \frac{d^2}{dx^2} \right] x^n$ |
| \[
\left[ x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} - n^2 - \frac{d^2}{dx^2} \right] T_n(x) = 0
\]
| $(D + n)(D - n)$ | $T_n(x) = C_n \exp \left[ -\frac{1}{2(D+n)} \frac{d^2}{dx^2} \right] x^n$ |
| \[
\left[ x^2 \frac{d^2}{dx^2} + 3x \frac{d}{dx} - n(n+2) - \frac{d^2}{dx^2} \right] U_n(x) = 0
\]
| $(D + n + 2)(D - n)$ | $U_n(x) = C_n \exp \left[ -\frac{1}{2(D+n+2)} \frac{d^2}{dx^2} \right] x^n$ |
| \[
\left[ x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} - \nu^2 + x^2 \right] J_{\pm \nu}(x) = 0
\]
| $(D + \nu)(D - \nu)$ | $J_{\pm \nu}(x) = C_{\pm \nu} \exp \left[ -\frac{1}{2(D \pm \nu)} x^2 \right] x^{\mp \nu}$ |
| \[
\left[ x \frac{d}{dx} + \alpha - x \frac{d}{dx} - \gamma \frac{d}{dx} \right] \Phi(\alpha, \gamma, x) = 0
\]
| $(D + \alpha)$ | $\Phi(\alpha, \gamma, x) = C_{-\alpha} \exp \left[ -x \frac{d}{dx} - \gamma \frac{d}{dx} \right] x^{-\alpha}$ |
| \[
\left[ x(1-x) \frac{d^2}{dx^2} + (\gamma - [\alpha + \beta + 1]x) \frac{d}{dx} - \alpha \beta \right] \times
\]
| $(D + \alpha)(D + \beta)$ | $F(\alpha, \beta, \gamma, x) = C_{-(\alpha, \beta)} \times$
| \[
\times \exp \left[ -\frac{1}{(D + \alpha \beta)} (x \frac{d}{dx} + \gamma \frac{d}{dx}) \right] x^{-(\beta, \alpha)}
\]

The solution to the DE \[F(D) + P(x, d/dx)] y(x) = 0,\] is,
\[y(x) = C_{\lambda} \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} P(x, d/dx) \right]^m \right\} x^\lambda,\] provided \[F(D)x^\lambda = 0.\]