Truncated moment problems on positive-dimensional algebraic varieties

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This manuscript transfers the main aspects of Prony’s method from finitely-supported measures to the classes of signed or non-negative measures supported on algebraic varieties of any dimension. In particular, we show that the Zariski closure of the support of these measures is determined by finitely many moments and can be computed from the kernel of certain moment matrices.

Introduction

The truncated moment problem for finitely-supported measures asks for parameter recovery from a given finite set of moments. This problem can be addressed by a multivariate form of Prony’s method [KPRvdO16; vdOhe17; Sau17; Mou18], a widely-used tool in signal processing that is algebraic at heart. It recovers the finitely many support points of such a measure as the zero set of a family of polynomials, so it is natural to view the support as a zero-dimensional algebraic variety.

In this manuscript, we switch from finitely-supported measures to the much more general class of measures that are supported on algebraic varieties of any dimension and we analyze which features of Prony’s method can be transferred to this setting. By considering the kernels of certain moment matrices, we show that it is possible to recover the vanishing ideal of the support of a measure. In other words, given sufficiently many moments, one obtains the Zariski closure of the support, by algebraic means.

Contributions

The Vandermonde decomposition of the moment matrix of a finitely-supported signed measure is an essential ingredient of Prony’s method. Theorem 3.2 forms an analog of this decomposition that is suitable also for measures supported on positive-dimensional varieties.

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For any compactly-supported signed measure, Theorem 4.3 establishes a relationship between moment matrices and the vanishing ideal of the support. It shows that the Zariski closure of the support can be computed from finitely many moments. The theorem can be viewed as an extension of Proposition 4.10, which makes a similar statement for non-negative measures and has been considered by [LR12; PPL21], in the real affine (non-trigonometric) setting, and is also related to [OJ15; OJ16; FAV16] which have investigated the case of plane curves.

Additionally, Corollary 4.12 gives an extension for signed measures that are a product of a polynomial and a non-negative measure. This allows us to formulate a generalization of Prony’s method in Remark 4.13 for particular measures that are not necessarily finitely-supported, but are supported on an algebraic variety of any dimension. Moreover, a variant for complex linear combinations of non-negative measures is proved in Theorem 4.16.

Outline
After briefly summarizing the main ideas of the multivariate Prony method in Section 1, we start in Section 2 with a short treatment of sesquilinear forms that can be associated to a functional $\sigma$. The concept of sesquilinearity is useful in this context as it allows us to treat both cases, that of measures in affine space and on the complex torus, simultaneously. We then continue in Section 3 by transferring to the more general setting the Vandermonde factorization of Proposition 1.1 (1) that is such an essential ingredient for Prony’s method. Section 4 addresses our leading question, that of recovering the algebraic variety the measure $\mu$ is supported on. This can be achieved by using finitely many moments, both for non-negative as well as compactly-supported signed measures, in affine space and on the complex torus. In case of non-negative measures, the moment matrices are positive-semidefinite which allows for stronger statements; we illustrate this difference in some examples.

Terminology
The symbol $k$ always denotes a field; in some sections we explicitly assume that it is of characteristic 0. The (algebraic) dual space of a $k$-vector space $V$ is written as $\text{Hom}_k(V, k)$. Similarly, $\text{Hom}_{k}^{\text{semi}}(V, k)$ denotes the set of semilinear maps from $V$ to $k$ (cf. Section 2).

For an introduction to algebraic geometry, see [CLO15]. By algebraic variety, we refer to the vanishing set of a set of polynomials, also known as algebraic set, that is, we do not require irreducibility. A variety generated by an ideal $\mathfrak{a}$ is denoted by $V(\mathfrak{a})$. The vanishing ideal of a set $X \subseteq k^n$ is denoted by $I(X)$. We use multi-index notation for monomials. Thus, when working in the polynomial ring $k[x_1, \ldots, x_n]$, the monomials are denoted by $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \alpha \in \mathbb{N}^n$. The (total) degree of a polynomial $p = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha$ with coefficients $p_\alpha \in k$ is given by $\deg(p) = \max\{|\alpha| \mid \alpha \in \mathbb{N}^n, p_\alpha \neq 0\}$, where $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Similarly, we define the max-degree of a Laurent polynomial $q = \sum_{\alpha \in \mathbb{Z}^n} q_\alpha x^\alpha, q_\alpha \in k$.
as $\max\{|\alpha|_\infty \mid \alpha \in \mathbb{Z}^n, q_\alpha \neq 0\}$, where $|\alpha|_\infty := \max\{|\alpha_1|, \ldots, |\alpha_n|\}$. The same definition applies when $q$ is a polynomial. Though, note that the max-degree does not define a grading of the polynomial ring, but gives rise to a filtration (cf. Example 2.3).

Given an ideal $a \subseteq \mathbb{k}[x_1, \ldots, x_n]$, the Krull-dimension of the quotient ring $\mathbb{k}[x_1, \ldots, x_n]/a$, i.e., the supremum of the heights of all prime ideals, is the same as the dimension of the variety $V(a) \subseteq \mathbb{k}^n$ (cf. [CLO15, Theorem 9.3.8]). By abuse of language, we also refer to this as the dimension of the ideal $a$. The residue class of a polynomial $p \in \mathbb{k}[x_1, \ldots, x_n]$ modulo an ideal $a$ is denoted by $\overline{p} = p + a$. By $\langle - \rangle$, we denote the ideal spanned by a family of ring elements. We write $m_\xi := \langle x - \xi \rangle = \langle x_1 - \xi_1, \ldots, x_n - \xi_n \rangle$ for the maximal ideal associated to a point $\xi \in \mathbb{k}^n$. Furthermore, the map $ev_\xi: \mathbb{k}[x_1, \ldots, x_n] \to \mathbb{k}, p \mapsto p(\xi)$, denotes the evaluation homomorphism associated to a point $\xi \in \mathbb{k}^n$. It can naturally be viewed as a ring homomorphism to the quotient ring corresponding to the ideal $m_\xi$.

Unless otherwise noted, the term measure refers to non-negative Borel measures. Occasionally, we also work with signed measures. Over the complex numbers, the term signed measure stands for complex(-signed) measure. Every (finite) non-negative measure is, in particular, a signed measure. For details, we refer to [Sch73; Rud87].

1 Prony’s method

The following is a multivariate generalization of Prony’s method that, in its univariate form, goes back to [Pro95]. We wish to transfer its essence to the more general setting of algebraic varieties of any dimension. The variant we cite here is useful for this, but there are many alternative formulations that accentuate different points of view. For instance, it has been considered in terms of exponential sums with a focus on sign-processing in [KPRvdO16; vdOhe17; Sau17; Mou18]. Another variation of Prony’s method is Sylvester’s algorithm [Syl86]. It is also related to Macaulay inverse systems (see e.g. [Eis99, Chapter 21.2]) and apolarity theory (cf. [IK99, Lemma 1.15, algorithm in Chapter 5.4], [Sch17, Chapter 19]), which put more emphasis on algebraic and geometric aspects.

**Proposition 1.1** ([Pro95], [KPRvdO16], [vdOhe17, Remark 2.8, Corollary 2.19]). Let $k$ be a field and let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables. Let $\sigma = \sum_{j=1}^{\ell} \lambda_j \mathbb{ev}_\xi_j$ for $\lambda_j \in k$ and $\xi_j \in \mathbb{k}^n$, $1 \leq j \leq r$. Let $d, d' \in \mathbb{N}$ and define $H_{d,d'} := (\sigma(x^\alpha))_{|\alpha| \leq d', |\beta| \leq d'}$. Then the following properties hold:

1. $H_{d,d'} = V_{\leq d'}^\top A V_{\leq d}$, where $A := \text{diag}(\lambda_1, \ldots, \lambda_r)$ and $V_{\leq d} := (\xi_j^\alpha)_{1 \leq j \leq r, |\alpha| \leq d'}$.

2. If $\lambda_1, \ldots, \lambda_r \neq 0$ and $ev_{\leq d'}: R_{\leq d'} \to k^r$, $x^\alpha \mapsto (\xi_j^\alpha)_{1 \leq j \leq r}$, is surjective, then:
   (a) $\ker H_{d,d'} = \ker V_{\leq d} = I(\{\xi_1, \ldots, \xi_r\}) \cap R_{\leq d}$.
   (b) $V(\ker H_{d,d'}) = \{\xi_1, \ldots, \xi_r\}$ if $d - 1 \geq d'$. 

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Proof. The factorization $H_{d',d} = V_{d,d}^\top A V_{d,d}$ follows by direct computation. Furthermore, if $\lambda_1, \ldots, \lambda_r \neq 0$ and $\text{ev}_{d'}$ is surjective, then $V_{d,d}^\top A$ represents an injective map, so the kernels of $V_{d,d}$ and $H_{d,d}$ are the same and agree with the truncated vanishing ideal $I(\{\xi_1, \ldots, \xi_r\}) \cap R_{d,d}$, which shows (2a). Then part (2b) follows from the observation that the surjectivity of $\text{ev}_{d-1}$ implies $V(\ker V_{d,d}) = \{\xi_1, \ldots, \xi_r\}$; see [vdOhe17, Theorem 2.15].

Note that, if the points $\xi_1, \ldots, \xi_r$ are not distinct, then the map $\text{ev}_{d'} : R_{d,d} \to \mathbb{k}^r$ in (2a) can never be surjective, so the surjectivity assumption implies in particular that the points are distinct. Further, note that the matrix $H_{d,d}$ in Proposition 1.1 represents the $\mathbb{k}$-linear map into the dual space of the vector space $R_{d,d}$ given by

$$R_{d,d} \to \text{Hom}_\mathbb{k}(R_{d,d}, \mathbb{k}), \quad p \mapsto (q \mapsto \sigma(pq)),$$

as well as the $\mathbb{k}$-bilinear mapping

$$R_{d,d} \times R_{d,d} \to \mathbb{k}, \quad (q, p) \mapsto \sigma(pq).$$

A map of the form $\sigma = \sum_{j=1}^r \lambda_j \text{ev}_{\xi_j}$, where $\text{ev}_{\xi_j}$ denotes the evaluation homomorphism associated to the point $\xi_j$, can also be viewed as an exponential sum. It satisfies $\sigma(x^\alpha) = \sum_{j=1}^r \lambda_j x_j^\alpha$ for all $\alpha \in \mathbb{N}^n$, so can be interpreted as a map $\mathbb{N}^n \to \mathbb{k}$, by composing it with the $\alpha$-th moment $x^\alpha$. Also note that $\sigma$ is the moment functional of the finitely-supported measure $\mu := \sum_{j=1}^r \lambda_j \delta_{\xi_j}$, where $\delta_{\xi_j}$ denotes the Dirac measure supported at the point $\xi_j \in \mathbb{k}^n$ for $1 \leq j \leq r$. For this interpretation, we usually assume that $\mathbb{k}$ is $\mathbb{R}$ or $\mathbb{C}$. If $\mathbb{k} = \mathbb{C}$ and the weights $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ are complex, then $\mu$ is a signed (complex) measure, which is explicitly allowed in this setting. The signed measure $\mu$ satisfies $\int_{\mathbb{k}^n} x^\alpha d\mu(x) = \sum_{j=1}^r \lambda_j x_j^\alpha = \sigma(x^\alpha)$, so $\sigma(x^\alpha)$ agrees with the $\alpha$-th moment of $\mu$. On top of that, the moments $\sigma(x^\alpha)$ uniquely determine the map $\sigma$.

From this point of view, the statement of Proposition 1.1 (2b) is that the support of the finitely-supported signed measure $\mu$ is already determined by finitely many of its moments, namely the ones that are required to construct the matrix $H_{d-1,d}$. In fact, in this case, the weights $\lambda_1, \ldots, \lambda_r$ can be recovered as well, by subsequently solving a linear system of equations (cf. [vdOhe17, Algorithm 2.1]), so the measure $\mu$ is fully determined by these moments. The condition that $\text{ev}_{d-1}$ is surjective holds if $d$ is sufficiently large, a trivial bound being $d \geq r$, as can be seen by constructing Lagrange polynomials of degree $r - 1$ for the points $\xi_1, \ldots, \xi_r$; cf. [vdOhe17, Corollary 2.20]. The ideal

$$\bigcap_{j=1}^r (x - \xi_j) = \prod_{j=1}^r (x - \xi_j) = \prod_{j=1}^r (x_1 - \xi_{j1}, \ldots, x_n - \xi_{jn})$$

is clearly generated by polynomials of degree at most $r$, but in the multivariate setting with $n \geq 2$, unless the points $\xi_1, \ldots, \xi_r$ are contained in a one-dimensional subspace of $\mathbb{k}^n$, this bound can be much larger than necessary. A more practical sufficient criterion
for the evaluation map $\text{ev}_{\leq d-1}$ being surjective is obtained by checking the rank of the matrix $H_{d-1,d}$. As this rank is at most $r$, it follows from the Vandermonde factorization in Proposition 1.1 (1) that $\text{ev}_{\leq d-1}$ is surjective if and only if $\text{rk} H_{d-1,d} = r$.

**Remark 1.2.** A variation of Prony’s method works with Toeplitz matrices of the form

$$
\left( \sum_{j=1}^{r} \lambda_j \xi_j^{-\alpha+\beta} \right)_{\alpha,\beta \in \mathbb{N}^n, |\alpha|_\infty \leq d', |\beta|_\infty \leq d}
$$

instead of Hankel matrices, where the moments are usually bounded in max-degree. For this to be defined, the points $\xi_1, \ldots, \xi_r$ must have non-zero coordinates, so they are contained in the algebraic torus $(\mathbb{C}^*)^n$. This is especially common when working in a trigonometric setting, with points on the complex torus

$$
\mathbb{T}^n := \{ z \in \mathbb{C}^n \mid |z_1| = \cdots = |z_n| = 1 \}.
$$

Moreover, one can work with much more general filtrations of the polynomial ring; see the statements in [vdOhe17, Chapter 2]. See also [KRvdO20] for an approach relating Toeplitz and Hankel matrices in this context.

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**2 Sesquilinearity and filtrations**

In this section, we set up a framework that allows us to treat in a unified way the two different settings of moment problems we are primarily interested in, namely moment problems on affine space and on the torus. See [Sch17, Chapter 2] for a similar approach to these concepts.

**Definition 2.1.** Let $R$ be a ring with a map $-^\circ : R \to R$ satisfying

$$
(x + y)^\circ = x^\circ + y^\circ, \quad (xy)^\circ = y^\circ x^\circ, \quad 1^\circ = 1, \quad (x^\circ)^\circ = x
$$

for all $x, y \in R$. Then the map $-^\circ$ is called **involution** and $R$ is an **involutive ring** (also called $^*\text{-ring}$). An involutive ring $A$ with involution $-^\circ_A$ that is also an (associative) algebra over a commutative involutive ring $R$ is an **involutive algebra** (also called $^*\text{-algebra}$), if the involution satisfies $(ra)^\circ_A = r^\circ a^\circ_A$ for all $r \in R$ and $a \in A$. As this property means that there is no ambiguity, we denote the involution on $A$ by $-^\circ$ as well. A map $f : A \to A$ is $^\circ\text{-semilinear}$ if $f(a + b) = f(a) + f(b)$ and $f(ra) = r^\circ f(a)$ holds for all $r \in R$ and $a, b \in A$.

Common examples of involutive rings include the field of complex numbers $\mathbb{C}$ with complex conjugation as well as square complex matrices with conjugate transposition as involution. Another important example for our discussion is given in Example 2.5 below. Also note that any commutative ring (algebra) is an involutive ring (algebra) with respect to the trivial involution which leaves every element unchanged.
Definition 2.2. Let $\mathbb{k}$ be a field and $A$ an (associative) algebra over $\mathbb{k}$. If $F_d \subseteq A$, $d \in \mathbb{N}$, is a family of $\mathbb{k}$-vector subspaces satisfying

- $F_d \subseteq F_e$ for $d, e \in \mathbb{N}$ with $d \leq e$,
- $A = \bigcup_{d \in \mathbb{N}} F_d$, 

then $A$ is a filtered algebra over $\mathbb{k}$ and the family $\{F_d\}_{d \in \mathbb{N}}$ is called filtration of $A$. In particular, the filtrations we consider are exhaustive. For simplicity of notation, we often denote the filtered components of the filtration by $A_{\leq d} := F_d$.

Example 2.3. Let $\mathbb{k}$ be a field and $R = \mathbb{k}[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $\mathbb{k}$, for some $n \in \mathbb{N}$. Then the total degree of polynomials gives rise to a filtration of $R$ where

$$R_{\leq d} = \{p \in R \mid \deg(p) \leq d\}$$

for $d \in \mathbb{N}$. Similarly, we can define a filtration $\{F_d\}_{d \in \mathbb{N}}$, on $R$ in terms of max-degree by

$$F_d = \bigoplus_{\alpha \in \mathbb{N}^n, |\alpha|_{\infty} \leq d} \mathbb{k}x^\alpha.$$  

Note that all the filtered components of these two filtrations happen to be $\mathbb{k}$-vector spaces of finite dimension, which is a useful property when it comes to computations.

Now let $a \subseteq R$ be an ideal with $1 \notin a$ and define $S = R/a$. If $\{F_d\}_{d \in \mathbb{N}}$, is any filtration of $R$, then $G_d := F_d/(a \cap F_d)$ defines a filtration of the quotient ring $S$. For this, observe that $G_d$ can be embedded in $G_{d+1}$ via the injective map $p + a \cap F_d \mapsto p + a \cap F_{d+1}$, for all $p \in F_d$, $d \in \mathbb{N}$.

For the remainder of this section, we assume, for simplicity, that $\mathbb{k}$ is a field of characteristic 0 together with an involution $-^\circ$ that endows $\mathbb{k}$ with the structure of an involutive ring. Moreover, we denote by $R = \mathbb{k}[x_1, \ldots, x_n]$ the polynomial ring in finitely many variables and fix a filtration $\{R_{\leq d}\}_{d \in \mathbb{N}}$ that turns $R$ into a filtered algebra over $\mathbb{k}$ and has the property that $R_{\leq d}$ is a finite-dimensional $\mathbb{k}$-vector space for every $d \in \mathbb{N}$. Additionally, we assume that $R \subseteq L$ is a $\mathbb{k}$-subalgebra of an involutive commutative algebra $L$ over $\mathbb{k}$. The involution on $L$ is denoted by $-^\circ$ as well. Typical examples are the following:

Example 2.4. If $\mathbb{k}$ is any field, let $L = R$ and define the involutions on $\mathbb{k}$ and $L$ to act trivially. The filtration $\{R_{\leq d}\}_{d \in \mathbb{N}}$ on $R$ is defined by total degree as in (1). Of particular interest is the case when $\mathbb{k}$ is the field of real numbers $\mathbb{R}$ (or a subfield thereof).

Example 2.5. If $\mathbb{k}$ is any field with an involution $-^\circ$, let $L = \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ring of Laurent polynomials and define the involution on $L$ by

$$\left(\sum_{\alpha} p_\alpha x^\alpha\right)^\circ := \sum_{\alpha} p_\alpha^\circ x^{-\alpha},$$

where $p_\alpha \in \mathbb{k}$, $\alpha \in \mathbb{Z}^n$, which turns $L$ into an involutive algebra. For the filtration on $R$, in this situation we usually pick the one that is induced by max-degree as in (2), since $L$ is the coordinate ring of the algebraic torus, and denote it by $\{R_{\leq d}\}_{d \in \mathbb{N}}$ again.
Of particular interest is the case $\mathbb{k} = \mathbb{C}$ of complex numbers with complex conjugation as involution. In this case, an observation that can be significant in some applications is the following: If we restrict a Laurent polynomial $p \in L$ to the complex torus $T^n$, then the involution $p^\circ$ is the complex conjugate of $p$ as a function on $T^n$, so we have

$$p^\circ(\xi) = \overline{p(\xi)}$$

for all $\xi \in T^n$, since $\xi^{-\alpha} = \overline{\xi}^\alpha$ for all $\alpha \in \mathbb{Z}^n$. In particular, the Laurent polynomial $p$ is a real function on $T^n$ if and only if $p^\circ = p$, i.e. $p_\alpha = \overline{p_{-\alpha}}$ for all $\alpha$. Furthermore, note that, if $\mathfrak{a} \subseteq L$ is a vanishing ideal of a set contained in $T^n$, then it follows that $\mathfrak{a}^\circ = \mathfrak{a}$.

**Definition 2.6.** Let $\sigma : L \to \mathfrak{k}$ be a $\mathfrak{k}$-linear map. Then we define the $\mathfrak{k}$-sesquilinear form

$$\langle -, - \rangle_\sigma : L \times L \to \mathfrak{k}, \quad (q, p) \mapsto \sigma(q^\circ p),$$

which is $^\circ$-semilinear in the first and linear in the second argument. Defining sesquilinear forms to be semilinear in the first rather than in the second argument is an arbitrary choice. We choose this convention as it simplifies our notation later on. By restriction, we can also view this as a sesquilinear form on $R$ as well as on the finite-dimensional vector spaces $R_{\leq d}, d \in \mathbb{N}$. Note that this is a symmetric bilinear form if the involution is trivial.

A form $\langle -, - \rangle$ on a $\mathfrak{k}$-vector space $U$ is Hermitian if $\langle q, p \rangle = \langle p, q \rangle^\circ$ for all $p, q \in U$. If the involution is trivial, as in Example 2.4, then this always holds for $\langle -, - \rangle_\sigma$, as the form is symmetric in that case. When $\mathfrak{k}$ is (a subfield of) the complex numbers $\mathbb{C}$, then a Hermitian form $\langle -, - \rangle_\sigma$ on $U$ is positive-semidefinite if, additionally, $\langle p, p \rangle_\sigma \geq 0$ for all $p \in U$. Note that this never holds if $\mathfrak{k} \not\subseteq \mathbb{R}$ and the involution is linear, rather than $^\circ$-semilinear, unless the form is trivial.

**Remark 2.7.** Assume that a family of monomials $\{x^\alpha\}_{\alpha \in J} \subseteq R_{\leq d}$ for a suitable index set $J \subseteq \mathbb{N}^n$ forms a basis of the finite-dimensional vector space $R_{\leq d}$ and that the involution $^\circ$ is trivial. Then the Gramian matrix of $\langle -, - \rangle_\sigma$ with respect to this basis is of the form

$$\begin{pmatrix} \sigma(x^{\alpha + \beta})_{\alpha, \beta \in J} \end{pmatrix},$$

which is a (generalized) Hankel matrix.

Likewise, if $\{x^\alpha\}_{\alpha \in J} \subseteq R_{\leq d}$ is a basis of $R_{\leq d}$, but $L$ is the ring of Laurent polynomials with involution $^\circ : L \to L$ defined as in Example 2.5, then the Gramian matrix with respect to this basis is of the form

$$\begin{pmatrix} \sigma(x^{-\alpha + \beta})_{\alpha, \beta \in J} \end{pmatrix},$$

which is a (generalized) Toeplitz matrix.

**Lemma 2.8.** Assume that $\sigma : L \to \mathfrak{k}$ is a $\mathfrak{k}$-linear map, $\mathfrak{a} \subseteq L$ is an ideal such that $\mathfrak{a}, \mathfrak{a}^\circ \subseteq \ker \sigma$. Let $W \subseteq L$ be a $\mathfrak{k}$-vector subspace. Then the sesquilinear form $\langle -, - \rangle_\sigma$ on $L$ induces a sesquilinear form

$$W/(\mathfrak{a} \cap W) \times W/(\mathfrak{a} \cap W) \to \mathfrak{k}, \quad (\overline{q}, \overline{p}) \mapsto \langle q, p \rangle_\sigma = \sigma(q^\circ p).$$
Here, \( q, p \) denotes the residue class of polynomials \( q, p \in W \) modulo \( a \cap W \). We denote the induced sesquilinear form on \( W/(a \cap W) \) by \( \langle -,-\rangle_{\sigma} \) again. Also note that that the requirements \( a \subseteq \ker \sigma \) and \( a^\circ \subseteq \ker \sigma \) are equivalent when the sesquilinear form \( \langle -,-\rangle_{\sigma} \) on \( L \) is Hermitian.

**Proof.** Let \( p, q \in W \). If \( p \in a \cap W \), then \( q^a p \) is contained in \( a \subseteq \ker \sigma \), so \( \sigma(q^a p) = 0 \). Likewise, if \( q \in a \cap W \), then \( q^a p \in a^\circ \subseteq \ker \sigma \), so the sesquilinear form on \( W/(a \cap W) \) is well-defined. \( \square \)

**Remark 2.9.** If \( \sigma: L \rightarrow \mathbb{k} \) is \( \mathbb{k} \)-linear and \( a \subseteq L \) is an ideal such that \( a \subseteq \ker \sigma \), then the sesquilinear form \( \langle -,-\rangle_{\sigma} \) on \( L \) does not induce a sesquilinear form on the quotient spaces \( W/(a \cap W) \). (Observe that this would need \( a^\circ \subseteq \ker \sigma \) or require the form to be Hermitian, as in Lemma 2.8.) Many of our arguments here can be transferred to this setting by working with a sesquilinear map instead of a sesquilinear form; for details we refer to [Wag21, Definition 3.1.12]. \( \diamondsuit \)

### 3 Factorization properties

The Vandermonde factorization of Proposition 1.1 (1) is an essential aspect of Prony’s method. Here, we analyze how to transfer it from measures on zero-dimensional to measures on positive-dimensional algebraic varieties. The statements here are also motivated by the study of finite-rank Hankel operators as in e.g. [Mou18]. In the positive-dimensional setting, such operators are not of finite rank anymore, but some properties are still valid.

Let \( \mathbb{k}, R, L \) be as in Section 2, so \( \mathbb{k} \) is a field of characteristic 0, \( R = \mathbb{k}[x_1, \ldots, x_n] \) is the polynomial ring in \( n \) variables endowed with a filtration \( \{R_{\leq d}\}_{d \in \mathbb{N}} \) and \( L \) is an involutive commutative \( \mathbb{k} \)-algebra such that \( R \subseteq L \).

We wish to examine more closely the following situation. Let \( a \subseteq L \) be an ideal and let \( \sigma: L \rightarrow \mathbb{k} \) be a \( \mathbb{k} \)-linear map with the property that \( a \subseteq \ker \sigma \). This means that the map \( \sigma \) factors via the quotient homomorphism

\[
\pi_a: L \rightarrow L/a, \quad p \mapsto \overline{p} := p + a,
\]

which we denote by \( \pi_a \), and a \( \mathbb{k} \)-linear map \( \overline{\sigma}: L/a \rightarrow \mathbb{k} \), denoted by \( \overline{\sigma} \).

**Example 3.1.** Assume that \( L \) is the polynomial ring \( R \) and \( \xi \in \mathbb{k}^n \) (or that \( L \) is the Laurent polynomial ring in \( n \) variables and \( \xi \in (\mathbb{k}^*)^n \)). Then, for the maximal ideal \( m_\xi = \langle x - \xi \rangle \subseteq L \), this gives the evaluation homomorphism at the point \( \xi \),

\[
\pi_{m_\xi}: L \rightarrow L/m_\xi \cong \mathbb{k}, \quad x^\alpha \mapsto \overline{x}^\alpha = \xi^\alpha,
\]

for \( \alpha \in \mathbb{N}^n \) (or \( \alpha \in \mathbb{Z}^n \)), so \( \pi_{m_\xi}(p) = p(\xi) \) for \( p \in L \). Note further that, for any \( \mathbb{k} \)-linear map \( \sigma: L \rightarrow \mathbb{k} \) with \( m_\xi \subseteq \ker \sigma \), the linear map \( \overline{\sigma}: L/m_\xi \cong \mathbb{k} \rightarrow \mathbb{k} \) is determined by a single scalar \( \lambda \in \mathbb{k} \), with respect to a suitable basis. Thus, \( \sigma = \lambda \pi_{m_\xi} = \lambda \ev_\xi \in \text{Hom}_\mathbb{k}(L, \mathbb{k}) \), which we can interpret as an exponential sum of rank 1 if \( \lambda \neq 0 \) (cf. Section 1).
More generally, consider the zero-dimensional ideal \( a = \bigcap_{j=1}^r \mathfrak{m}_{\xi_j} \), for distinct points \( \xi_1, \ldots, \xi_r \). Then it follows from the Chinese Remainder Theorem (cf. [Bou06, Chapter 2.1.2, Proposition 5]) that

\[
L/a \cong \bigoplus_{j=1}^r L/\mathfrak{m}_{\xi_j} \cong k^r,
\]

where \( \pi_a(p) \) is identified with \( (p(\xi_1), \ldots, p(\xi_r)) \) for \( p \in L \). As a \( k \)-linear map with respect to the monomial basis of \( L \), we can view \( \pi_a \) as being described by an infinite Vandermonde matrix associated to the points \( \xi_1, \ldots, \xi_r \). If \( \sigma: L \to k \) is a \( k \)-linear map with \( a \subseteq \ker \sigma \), then it is of the form \( \sigma = \sum_{j=1}^r \lambda_j \text{ev}_{\xi_j} \) with suitable parameters \( \lambda_1, \ldots, \lambda_r \in k \), which corresponds to an exponential sum of rank \( r \) if \( \lambda_1, \ldots, \lambda_r \neq 0 \).

The ideal \( a \) does not need to be radical in this setup. An explicit example is given in [Wag21, Example 3.2.2]; more generally polynomial exponential series as studied in [Mou18] correspond to non-radical ideals. Later on, we will focus on the case in which \( a \) is a vanishing ideal, though.

As \( R \) is endowed with a filtration \( \{ R_{\leq d} \}_{d \in \mathbb{N}} \) for which each component \( R_{\leq d} \) is finite-dimensional and since \( R \subseteq L \), we can restrict the map \( \pi_a: L \to L/a \) to a map on finite-dimensional vector subspaces \( R_{\leq d} \to R_{\leq d}/(a \cap R_{\leq d}) \), which we denote by \( \pi_{a, \leq d} \), as explained in Example 2.3.

An important ingredient of Prony’s method is that we can extract information about the vanishing ideal from the kernel of the moment matrix, if the moment matrix is sufficiently large; see Proposition 1.1 (2a). In the following, we examine what is required to transfer this property to the setting of ideals which are possibly not of dimension zero, but are of higher dimension. This is answered by the following theorem as well as Corollary 3.3 below.

**Theorem 3.2.** Let \( a \subseteq L \) be an ideal and let \( \sigma: L \to k \) be a \( k \)-linear map with \( a \subseteq \ker \sigma \). Then the \( k \)-linear map

\[
H: R \to \text{Hom}^\text{semi}_k(R, k), \quad p \mapsto (q \mapsto \langle q, p \rangle_\sigma),
\]

factors as

\[
R \xrightarrow{\pi_a} R/(a \cap R) \xrightarrow{\text{Hom}^\text{semi}_k(R/(a \cap R), k)} H^\dagger \xrightarrow{\text{Hom}^\text{semi}_k(R, k)},
\]

\[
p + a \cap R \xrightarrow{\pi_a^\dagger} (q + a^\circ \cap R) \mapsto \langle q, p \rangle_\sigma,
\]

where \( \pi_a^\dagger(\varphi) = \varphi \circ \pi_a^\dagger \) for \( \varphi \in \text{Hom}^\text{semi}_k(R/(a^\circ \cap R), k) \).

Moreover, the truncated map between finite-dimensional vector subspaces given by

\[
H_{d+\delta,d}: R_{\leq d} \to \text{Hom}^\text{semi}_k(R_{\leq d+\delta}, k), \quad p \mapsto (q \mapsto \langle q, p \rangle_\sigma),
\]

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for $d, \delta \in \mathbb{N}$, factors as

$$
\begin{array}{ccc}
R_{\leq d} & \xrightarrow{H_{d+\delta,d}} & \text{Hom}_{\mathbb{k}}^{\text{semi}}(R_{\leq d+\delta}, \mathbb{k}) \\
\pi_{a,\leq d} & & \pi_{a^0,\leq d+\delta}^T \\
R_{\leq d}/(a \cap R_{\leq d}) & \xrightarrow{H_{d+\delta,d}} & \text{Hom}_{\mathbb{k}}^{\text{semi}}(R_{\leq d+\delta}/(a^0 \cap R_{\leq d+\delta}), \mathbb{k}), \\
p + a \cap R_{\leq d} & \mapsto & (q + a^0 \cap R_{\leq d+\delta} \mapsto \langle q, p \rangle_\sigma).
\end{array}
$$

(3)

**Proof.** Due to the inclusion $a \subseteq \ker \sigma$, we have that

$$
\sigma((q + a^0 \cap R)\sigma(p + a \cap R)) = \sigma((q^0 + a \cap R^0)(p + a \cap R)) = \sigma(q^0 p) = \langle q, p \rangle_\sigma,
$$

for all $q, p \in R$, which shows the first factorization property. The other one follows analogously.

The truncated map $H_{d+\delta,d}$ is of importance for us, since we are interested in recovery from finitely many moments. By Theorem 3.2, it holds that

$$
a \cap R_{\leq d} \subseteq \ker H_{d+\delta,d}
$$

and we ask when this is an equality. This leads to the following corollary.

**Corollary 3.3.** If the map $H_{d+\delta,d}^{-1} : R_{\leq d}/(a \cap R_{\leq d}) \to \text{Hom}_{\mathbb{k}}^{\text{semi}}(R_{\leq d+\delta}/(a^0 \cap R_{\leq d+\delta}), \mathbb{k})$ is injective, then

$$
\ker(H_{d+\delta,d}) = \ker(\pi_{a,\leq d}) = a \cap R_{\leq d}.
$$

**Proof.** Due to the factorization (3) and since the map $\pi_{a^0,\leq d+\delta}^T$ is injective, the equality holds if and only if the map $H_{d+\delta,d}$ is injective.

As the vector space dimension of the codomain of $H_{d+\delta,d}$ is finite and at least as large as the dimension of the domain, saying that $H_{d+\delta,d}$ is injective is the same as saying that the map $H_{d+\delta,d}$ has full rank. As such, this can be regarded as a variant of the statement about the Vandermonde factorization in Proposition 1.1.

**Remark 3.4.** In this formalism, $\pi_{a,\leq d}$ is always surjective, which is an important difference from the Vandermonde factorization considered in Proposition 1.1(1), as the Vandermonde matrix considered there can be non-surjective for small $d$. This is explained further in Example 3.5 below. There, for an ideal of the form $a = \bigcap_{j=1}^r m_{\xi_j}$, the dimension of $\text{im}(\pi_{a,\leq d}) = R_{\leq d}/(a \cap R_{\leq d})$ as vector space is at most $r$, but can be smaller. Equality holds if and only if the corresponding Vandermonde matrix has rank $r$, which only holds if $d$ is sufficiently large.

Moreover, we remark that the map $H_{d+\delta,d}$ is injective in particular when $\sigma$ is a moment functional of a measure and $a$ is the vanishing ideal of its support, as will be shown in Proposition 4.10.
Example 3.5. Let us revisit Example 3.1, so let \( \mathbf{a} := \bigcap_{j=1}^{r} m_{\xi_j} \subseteq L \) for distinct points \( \xi_1, \ldots, \xi_r \in \mathbb{R}^n \), where now we assume that \( L = R = k[x_1, \ldots, x_n] \) is endowed with the trivial involution and the filtration induced by total degree.

If \( d \) is sufficiently large, \( \pi_{\mathbf{a}, \leq d} \) has rank \( r \) and we have \( R_{\leq d}/(a \cap R_{\leq d}) \cong \bigoplus_{j=1}^{r} R/m_{\xi_j} \cong k^r \). Hence, we also have \( R_{\leq d+\delta}/(a \cap R_{\leq d+\delta}) \cong k^r \) for all \( \delta \in \mathbb{N} \). If \( \sigma: R \to k \) is a \( k \)-linear map with \( a \subseteq \ker \sigma \), then, by Example 3.1, it is of the form \( \sigma = \sum_{j=1}^{r} \lambda_j ev_{\xi_j} \) for some \( \lambda_1, \ldots, \lambda_r \in k \). Thus, the map \( H_{d+\delta, d} \) corresponds to the diagonal matrix \( \text{diag}(\lambda_1, \ldots, \lambda_r) \) with respect to the natural bases. Clearly, it is injective if and only if \( \lambda_1, \ldots, \lambda_r \neq 0 \), which illustrates the connection of Corollary 3.3 to Proposition 1.1 (2a).

Although for zero-dimensional ideals as in the preceding example it is enough to consider the case \( \delta = 0 \) to infer that \( \ker H_{d+\delta, d} = a \cap R_{\leq d} \) if \( d \) is sufficiently large, this does not hold in general (cf. Example 4.15). We will see a non-trivial example in Example 4.9, which involves an ideal of positive dimension. In connection to that, Theorem 4.3 will show that it can be useful to consider \( \delta \) larger than 0.

4 Recovery of the support from moments

In this section, we explore how to recover the underlying algebraic variety that a measure is supported on, by using finitely many of its moments. We consider a non-negative or signed measure \( \mu \) whose support lives in the affine space \( \mathbb{R}^n \) or the complex torus \( T^n \) and wish to find the smallest variety that contains the support. Following the notation of Section 2, we consider the following two cases, to which we also refer as affine and trigonometric cases, respectively:

1. \( \Omega = \mathbb{R}^n, \ k = \mathbb{R}, L = R = \mathbb{R}[x_1, \ldots, x_n] \) with trivial involutions (cf. Example 2.4);
2. \( \Omega = T^n, \ k = \mathbb{C}, R = \mathbb{C}[x_1, \ldots, x_n], L = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) with complex conjugation and involution \(-^0\) on \( L \) defined as in Example 2.5.

Additionally, we fix a filtration \( \{R_{\leq d}\}_{d \in \mathbb{N}} \) of \( R \) consisting of finite-dimensional vector spaces. Recall that the support of a non-negative or signed measure is defined as follows.

Definition 4.1 (cf. [Sch73, Chapter 1.3]). Let \( \mu \) be a signed measure on \( \Omega \). Then

\[ \text{supp} \mu := \{ \xi \in \Omega \mid \mu|_{U} \neq 0 \text{ for all open neighborhoods } U \subseteq \Omega, \xi \in U \} \]

is called support of \( \mu \), where \( \mu|_{U} \) denotes the restriction of \( \mu \) to \( U \).

By convention, we consider the support in terms of the standard topology on \( \Omega \). The complement of \( \text{supp} \mu \) in \( \Omega \) is the union of all open sets on which \( \mu \) is constantly zero and is open, so \( \text{supp} \mu \) is a closed set. When we consider the support in terms of the Zariski topology, we denote it by \( \text{supp}^\ast \mu \) (as a subset of \( \Omega \) or \((\mathbb{C}^*)^n \)). It is the smallest Zariski-closed set containing \( \text{supp} \mu \).
This topic has been studied in various forms, usually in the real affine case with non-negative measures (e.g. [LP15; PPL21]) and an emphasis on finitely-supported measures; see for instance [LR12]. The case of plane algebraic curves has also been investigated in [FAV16], with a focus on the presence of noise. The case of plane trigonometric curves on the torus has been considered in [OJ15; OJ16].

We unify the different noise-free settings in Proposition 4.10 and expand the existing results by Theorem 4.3, a statement for compactly-supported signed measures, as well as Corollary 4.12 and Theorem 4.16. Moreover, we give examples that highlight the differences between signed and non-negative measures.

4.1 Signed measures

Here, we consider a signed measure $\mu$ on $\Omega$. If $k = \mathbb{C}$, as in the trigonometric case, then $\mu$ is a complex measure. As a consequence of the Riesz representation theorem (see e.g. [Rud87, Theorem 6.19]), these measures can be defined as elements in the continuous dual space of the space $C^0_c(\Omega)$ of compactly-supported continuous functions from $\Omega$ to $k$. We refer to [Sch73, Chapter 1.2] for an extensive treatment of this topic.

In the trigonometric case, all the moments of $\mu$ are defined, as the torus $T^n$ is compact. In order to speak of moments $\int_{\Omega} x^\alpha d\mu$, $\alpha \in \mathbb{N}^n$, in the affine case, we need to make additional assumptions on the measure $\mu$, since the monomials $x^\alpha$ are not compactly-supported functions on $\mathbb{R}^n$. Certainly, the moments are defined when the measure $\mu$ itself is compactly supported. More generally, all the moments are defined for signed measures with a sufficiently rapid decay toward infinity, such as those that can be written as a product $\mu = g\mu_0$ of a Schwartz function $g$ and a tempered distribution $\mu_0$ (see e.g. [Gra14, Chapter 2] or [Sch73, Chapter 7]), which in particular includes Gaussians and mixtures thereof. In this section, we focus on signed measures with compact support only, as these are determined by their moments.

First, let us take note of the following elementary properties of the support of the product between a measure and a continuous function.

**Lemma 4.2.** Let $\mu$ be a signed measure on $\Omega$ and let $f, g \in C^0(\Omega)$ be continuous functions. Then:

1. $D(f) \cap \text{supp}\, \mu \subseteq \text{supp}(f\mu)$, where $D(f) \subseteq \Omega$ denotes the set of points in which $f$ does not vanish.

2. The measure $f\mu$ is zero if and only if $f$ vanishes on $\text{supp}\, \mu$.

3. If $D(f) \cap \text{supp}\, \mu = D(g) \cap \text{supp}\, \mu$, then $\text{supp}(f\mu) = \text{supp}(g\mu)$.

**Proof.** For (1), let $\xi \in \text{supp}\, \mu$ be any point such that $f(\xi) \neq 0$ and let $U \subseteq \Omega$ be an arbitrary open neighborhood of $\xi$. We need to show that $f\mu|_U \neq 0$. For this, let $U_0 \subseteq U$ be an open neighborhood of $\xi$ in which $f$ does not have any roots. Since $\xi$ is a support point of $\mu$, there exists a compactly-supported continuous function $\varphi \in C^0_0(U_0)$ such
that \( \int_{U_0} \varphi \, d\mu \neq 0 \). Then \( \psi := \frac{\varphi}{f} \in C^0(U_0) \) can be extended trivially to a compactly-supported function \( \psi \in C^0(U) \) and we have \( \int_{U} \psi \, d(f\mu) = \int_{U_0} \frac{\varphi}{f} \, d(f\mu) = \int_{U_0} \varphi \, d\mu \neq 0 \) and thus \( f\mu|_U \neq 0 \), which proves the statement. For part (2), assume that \( f\mu \) is zero. Then \( \text{supp}(f\mu) = \emptyset \), so \( f \) vanishes on \( \text{supp}(\mu) \) by (1). The converse holds by [Sch73, Chapter 3, Theorem 33, addendum]. Finally, for part (3), observe that the complement of \( \text{supp}(f\mu) \) consists of the union of all open sets \( U \subseteq \Omega \) satisfying \( f\mu|_U = 0 \). By (2), this is equivalent to \( f \) vanishing on \( \text{supp}(\mu|_U) \). By hypothesis, this is the case if and only if \( g \) vanishes on \( \text{supp}(\mu|_U) \), which in turn is equivalent to \( g\mu|_U = 0 \) and thus completes the proof. 

For the remainder of this section, we fix a filtration \( \{L_{\leq d}\}_{d \in \mathbb{N}} \) of \( L \) for which all the components are finite-dimensional vector spaces. In the affine case, we may choose \( L_{\leq d} = R_{\leq d} \). Additionally, we denote by \( B^L_d \) and \( B^R_d \) any bases of the filtered components \( L_{\leq d} \) and \( R_{\leq d} \), respectively. With this notation, we arrive at the following theorem.

**Theorem 4.3.** Let \( \mu \) be a compactly-supported signed measure on \( \Omega \), denote by \( a := I(\text{supp}\mu) \subseteq L \) the vanishing ideal of (the Zariski closure of) its support and let \( \sigma : L \to K \) be its moment functional. Let \( d \in \mathbb{N} \). Then

\[
a \cap R_{\leq d} = \ker H_{d',d}
\]

holds for all sufficiently large \( d' \in \mathbb{N} \), where \( H_{d',d} := (\langle w,v \rangle)_{w \in B^L_d, v \in B^R_d} \).

It then follows from Hilbert’s basis theorem that \( a \) is generated by \( \ker H_{d',d} \) if \( d \in \mathbb{N} \) is sufficiently large.

**Proof.** As the measure \( \mu \) is compactly supported, all its moments exist. Let \( d \in \mathbb{N} \) be arbitrary and observe that

\[
a \cap R_{\leq d} \subseteq \ker H_{d',d} = \{ p \in R_{\leq d} \mid \langle q,p \rangle_\sigma = 0 \text{ for all } q \in L_{\leq d'} \}, \tag{4}
\]

for all \( d' \in \mathbb{N} \). Indeed, if \( p \in a \), then \( p \) vanishes on the support of \( \mu \), so \( \langle q,p \rangle_\sigma = \int_{\Omega} q^*p \, d\mu = 0 \) for all \( q \in L \), by Lemma 4.2(2). More specifically, we have a descending chain

\[
R_{\leq d} \supseteq \ker H_{0,d} \supseteq \ker H_{1,d} \supseteq \cdots \supseteq a \cap R_{\leq d}
\]

which must stabilize, so we can fix \( a' \in \mathbb{N} \) such that

\[
\ker H_{d',d} = \ker H_{d'+\delta,d} \tag{5}
\]

holds for all \( \delta \in \mathbb{N} \).

Assume that \( \ker H_{d',d} \not\subseteq a \cap R_{\leq d} \). Then we can choose a polynomial \( p \in \ker H_{d',d} \) with \( p \not\in a \), so \( p \) does not vanish everywhere on \( \text{supp}\mu \). Hence, by Lemma 4.2(2), the signed measure \( \nu := p\mu \) is non-zero, so there exists a compactly-supported continuous function \( \varphi \in C^0(\Omega) \) such that \( \int_{\Omega} \varphi \, d\nu \neq 0 \). By the Weierstrass approximation theorem (see [Con90, Chapter 5, Theorem 8.1] for the affine real\(^1\) and [Gra14, Corollary 3.2.2] for the

---

\(^1\)This argument would not hold if, in the affine case, we were to work over the field of complex numbers, as the algebra of polynomials on \( \mathbb{C}^n \) is not closed under conjugation.
trigonometric version), the function \( \varphi \) can be uniformly approximated by polynomials in \( L \) on a compact set containing the support of the measure \( \nu \), which implies that not all moments of \( \nu \) can be zero. Hence, there exists a polynomial \( q \in L \) such that \( \int_{\Omega} q \, d\nu = \int_{\Omega} qp \, d\mu = \langle q^\circ, p \rangle_\sigma \neq 0 \). As \( q^\circ \in L_{<d+\delta} \) for some \( \delta \in \mathbb{N} \), this implies that \( p \notin \ker H_{d'+d,d} \), which is a contradiction to (5), by the choice of the polynomial \( p \).

**Remark 4.4.** In the proof of Theorem 4.3, the hypothesis that the support of the signed measure \( \mu \) is compact does not only guarantee that all its moments exist, but, more importantly, it asserts that the signed measure \( \nu = p\mu \) is determined by its moments, so that \( \nu \) is already zero if all its moments vanish. This does not in general hold for measures that are not compactly supported – not even for rapidly decreasing functions. For instance, let \( g \) be a non-zero Schwartz function on \( \mathbb{R}^n \) such that all its derivatives vanish at the origin, i.e. \( (\partial^\alpha g)(0) = 0 \) for all \( \alpha \in \mathbb{N}^n \). Then its Fourier transform \( \hat{g} \) is a non-zero Schwartz function satisfying

\[
(-1)^n (2\pi i)^{|\alpha|} \int_{\mathbb{R}^n} x^\alpha \hat{g}(x) dx = (\partial^\alpha g)(0) = 0
\]

for all \( \alpha \in \mathbb{N}^n \) (cf. [Gra14, Proposition 2.2.11 (10)]), so all the moments of \( \hat{g} \) are zero.

**Remark 4.5.** In the affine case of Theorem 4.3, we can choose the filtration of \( L \) as \( L_{\leq d} = R_{\leq d} \) for all \( d \in \mathbb{N} \). Let \( H_{d',d} \) be the rectangular moment matrix satisfying the statement of the theorem, so \( a \cap R_{\leq d} = \ker H_{d',d} \). By Corollary 3.3, this equality can only hold when the induced map \( H_{d',d} \) on the quotient spaces, as in (3) of Theorem 3.2, is injective. This implies \( d' \geq d \), for this choice of filtration in the affine case. This is in contrast to the statement of Proposition 1.1 (2b) in which the moment matrix \( H_{d',d} \) had a different shape.

By Theorem 4.3, we can recover the vanishing ideal of the support from finitely many moments. In particular, this means that the kernel of the non-truncated moment map also yields the vanishing ideal, as the following statement shows.

**Corollary 4.6.** Under the assumptions of Theorem 4.3, we have

\[
a \cap R = \ker H,
\]

where \( H \) denotes the map \( H: R \to \text{Hom}_{\mathbb{K}}^\text{semi}(L, \mathbb{K}), p \mapsto \langle q \mapsto \langle q, p \rangle_\sigma \rangle \).

**Proof.** To see this, first observe that we always have the inclusion \( a \cap R \subseteq \ker H \), by Lemma 4.2 (2). On the other hand, if \( p \in \ker H \), then \( p \in R_{\leq d} \) for some \( d \in \mathbb{N} \). In particular, this implies \( \langle q, p \rangle_\sigma = 0 \) for all \( q \in L_{\leq d'} \subseteq L \) and arbitrary \( d' \in \mathbb{N} \). Choosing \( d' \) as in Theorem 4.3, we therefore obtain \( p \in \ker H_{d',d} = a \cap R_{\leq d} \), so the statement follows.

**Remark 4.7.** Theorem 4.3 does not quantify what it means for \( d' \in \mathbb{N} \) to be large enough for the statement to hold. In general, the choice of \( d' \) cannot be made purely based on knowledge of the support or its vanishing ideal, but it must inherently depend on the signed measure itself. Indeed, for arbitrarily large \( d, d' \in \mathbb{N} \), one can construct a
signed measure with the following properties: its support is compact and Zariski-dense, so its vanishing ideal is zero, and all the low order moments vanish so that the matrix $H_{d,d'} = (\langle w, v \rangle_{\sigma})_{w \in B_{d'}^R, v \in B_d^R}$ is zero. Hence, the kernel of $H_{d,d'}$ is non-zero and thus is not a generating set of the zero ideal, the vanishing ideal of the support. In other words, $d'$ is not large enough for the statement of the theorem to hold. However, for particular signed measures, a bound on $d'$ is given in Corollary 4.12.

**Remark 4.8.** In the trigonometric case, we could also state Theorem 4.3 in a more symmetric fashion in terms of a matrix for which both rows and columns are indexed by $B_d^L$, a basis of the filtered component $L_{\leq d}$. We prefer to index the columns by $B_d^R$ instead because it allows for a finer filtration, i.e. the filtered components $R_{\leq d}$ can be chosen to be of smaller dimension than the components $L_{\leq d}$, and every ideal in $L$ can be generated by elements in $R$. Indexing the rows of the matrix by $B_d^L$ is needed in the proof of Theorem 4.3 due to the use of the Weierstrass approximation theorem. This leads to the question whether a statement similar to Theorem 4.3 is possible in which rows and columns are indexed by $B_d^R$, $B_d^L$, i.e. bases of components of the filtration on $R$ instead of $L$. In general, this is answered negatively by the following example, but a positive answer is possible for non-negative measures, as will be shown in Section 4.2.

**Example 4.9.** We consider the two-dimensional trigonometric case, so let $n = 2$. Let $v_1 := (2,1), v_2 := (1,2) \in \mathbb{Z}^2$ and define the functionals

$$\sigma_j : L \rightarrow \mathbb{C}, \quad x^\alpha \mapsto \begin{cases} 1 & \text{if } \langle \alpha, v_j \rangle = 0, \\ 0 & \text{otherwise}, \end{cases}$$

for $\alpha \in \mathbb{Z}^2$ and $j = 1, 2$. These are moment functionals of uniform measures supported on the one-dimensional varieties in $\mathbb{T}^2$ that are defined by the polynomials $x_1 - x_2^2$ and $x_1^2 - x_2$, respectively, and are depicted in Figure 1. Thus, the functional $\sigma := \sigma_1 - \sigma_2$ is a moment functional of a signed measure.

![Figure 1: The varieties $V(x_1 - x_2^2)$ (solid) and $V(x_1^2 - x_2)$ (dashed) on the torus $\mathbb{T}^2$ parametrized by $[0,1)^2$. The shaded region designates where the polynomial $g$ from Remark 4.17 is negative.](image)

Observe that $\langle x^\alpha, 1 \rangle_{\sigma} = \sigma(x^{-\alpha}) = 0$ holds for all $\alpha \in \mathbb{N}^2$. This implies that $\langle q, 1 \rangle_{\sigma} = 0$ for all $q \in R$. Hence, for every choice of $d,d'$, the polynomial $p := 1$ is contained in the kernel of the moment matrix $H_{d,d'} := (\langle w, v \rangle_{\sigma})_{w \in B_{d'}^R, v \in B_d^R}$, where $B_d^R$ denotes a basis of...
$R_{\leq d}$, with respect to any filtration of $R$. As $p = 1$ does not vanish on any non-empty variety, this shows that the statement of Theorem 4.3 does not hold for this matrix $H_{d',d}$ with rows indexed by $B^F_{d'}$ rather than $B^F_d$.

Additionally, this shows that the kernel of the non-truncated map $R \to \text{Hom}^\text{semi}_k(R, k)$, $p \mapsto (q \mapsto \langle q, p \rangle_\sigma)$, is not in general an ideal in $R$, in the trigonometric case. For instance, we have $\langle x_2^2, x_1 \rangle_\sigma = \sigma(x^{(1,-2)}) \neq 0$, so $x_1 \notin \ker H$, even though $1 \in \ker H$. \hfill \Box

4.2 Non-negative measures

In this section, we consider non-negative measures as well as statements about signed measures that involve non-negative measures. The non-negativity is an essential property that allows us to state the following proposition which is a stronger version of Theorem 4.3. If $W = R_{\leq d}$ is a component of the total degree filtration, then in the affine case this statement can also be obtained with a different proof by combining [LR12, Theorem 2.10] and [PPL21, Lemma 5].

**Proposition 4.10.** Let $\mu$ be a non-negative measure on $\Omega$ with finite moments, let $a := I(\supp \mu) \subseteq L$ be the vanishing ideal of (the Zariski closure of) its support and let $\sigma : L \to k$ be its moment functional. Let $W \subseteq L$ be a $k$-vector subspace. Then $\langle -,- \rangle_\sigma$ induces a positive-definite form on $W/(a \cap W)$.

In particular, if $W$ is finite-dimensional and $B$ is a basis of $W$, let $H := (\langle w,v \rangle_\sigma)_{w,v \in B}$. Then

$$a \cap W = \ker H.$$  

Furthermore, $H$ is non-singular if and only if the elements of $B$ are linearly independent modulo $a \cap W$.

For the statement, only finiteness of the moments that occur in $H$ is needed, so $\sigma$ must be defined on the subspace $W^\sigma \cdot W \subseteq L$.

**Proof.** First observe that $\langle -,- \rangle_\sigma$ is positive-semidefinite, as $\langle p,p \rangle_\sigma = \int_\Omega |p(x)|^2 \, d\mu(x) \geq 0$ for all $p \in L$. By Lemma 2.8, $\langle -,- \rangle_\sigma$ induces a form on $W/(a \cap W)$ and we need to show that it is non-degenerate. Assume that $p \in W$ is a polynomial such that $\langle p,p \rangle_\sigma = 0$. Since $|p|^2 \geq 0$ on $\Omega$, it follows from [Sch17, Proposition 1.23] that $|p|^2$ vanishes on $\supp \mu$ and thus $p \in a$. Hence, the induced form is non-degenerate and we have $\ker H \subseteq a \cap W$.

Conversely, if $p \in a \cap W$, then $(q^0 p)(\xi) = q^0(\xi)p(\xi) = 0$ for all $\xi \in \supp \mu$ and all $q \in L$. Thus, in particular, we have $\sigma(q^0 p) = \int_\Omega q^0(x)p(x) \, d\mu(x) = 0$ for all $q \in W$, so $p \in \ker H$.

From this, the addendum readily follows. If $H$ is non-singular, we have $a \cap W = \ker H = 0$, so the elements of $B$ are linearly independent modulo $a \cap W$. If $H$ is singular, we find a non-trivial linear combination $q = \sum_{w \in B} q_w w \neq 0$, $q_w \in k$, with $q \in \ker H = a \cap W$, so $q \equiv 0 \pmod{a \cap W}$. \hfill $\Box$
In particular, Proposition 4.10 holds with $W = R_{≤d}$ for any $d ∈ \mathbb{N}$, so that

$$a ∩ R_{≤d} = \ker H.$$ 

Again, by Hilbert’s basis theorem, the ideal $a$ is generated by $a ∩ R_{≤d}$ if $d$ is sufficiently large. Hence, for such a number $d$, the kernel of $H$ generates the ideal $a$, which is the statement of [LR12, Theorem 2.10], so we can fully recover the ideal $a$ from finitely many moments.

**Lemma 4.11.** Let $\{F_d\}_{d ∈ \mathbb{N}}$ be a filtration of $R$ or $L$. Let $μ$ be a signed measure on $Ω$ and $g ∈ F_δ$ for some $δ ∈ \mathbb{N}$ such that $μ_+ = g^c μ$ is a non-negative measure with finite moments satisfying $\text{supp } μ = \text{supp } μ_+$. Then

$$I(\text{supp } μ) ∩ F_d = \ker H_{d+δ,d},$$

for every $d ∈ \mathbb{N}$ with $H_{d+δ,d} := (⟨w, v⟩_σ)_{w ∈ B_{d+δ}, v ∈ B_d}$, where $σ : L → k$ denotes the moment functional of $μ$ and $B_d, B_{d+δ}$ denote finite bases of $F_d, F_{d+δ}$, respectively.

**Proof.** Observe that

$$I(\text{supp } μ) ∩ F_d ⊆ \ker H_{d+δ,d} = \left\{ p ∈ F_d \left| \int_Ω q^c p dμ = 0 \text{ for all } q ∈ F_{d+δ} \right\} \right. \subseteq \left\{ p ∈ F_d \left| \int_Ω (gq)^c p dμ = 0 \text{ for all } q ∈ F_d \right\},$$

where the last inclusion holds due to $gF_d ⊆ F_{d+δ}$. As $g^c μ = μ_+$ is a non-negative measure on $Ω$, it follows from Proposition 4.10 that the set (6) is equal to $I(\text{supp } μ_+) ∩ F_d$. Then the statement follows from $\text{supp } μ_+ = \text{supp } μ$. □

For signed measures that are a product of a polynomial and a non-negative measure, we then obtain the following result, which in contrast to Theorem 4.3 comes with an explicit bound on the size of the moment matrix and does not require compactness of the support.

**Corollary 4.12.** Let $\{F_d\}_{d ∈ \mathbb{N}}$ be a filtration of $R$ or $L$. Let $μ = gμ_+$ be a signed measure, where $μ_+$ denotes a non-negative measure on $Ω$ with finite moments and $g ∈ F_δ$ a polynomial for some $δ ∈ \mathbb{N}$. Then

$$I(\text{supp } μ) ∩ F_d = \ker H_{d+δ,d},$$

for every $d ∈ \mathbb{N}$ with $H_{d+δ,d} := (⟨w, v⟩_σ)_{w ∈ B_{d+δ}, v ∈ B_d}$, where $σ : L → k$ denotes the moment functional of $μ$ and $B_d, B_{d+δ}$ denote finite bases of $F_d, F_{d+δ}$, respectively.

**Proof.** As $g^c g$ and $g$ have the same vanishing set on $Ω$, it follows from Lemma 4.2 (3) that $\text{supp } (g^c gμ_+) = \text{supp } (gμ_+) = \text{supp } μ$. As $g^c gμ_+ = g^c μ$ is a non-negative measure on $Ω$, the result follows from Lemma 4.11. □
Remark 4.13. Under the assumptions that \( \mu_+ \) is the uniform measure on some unknown variety \( V \subseteq \Omega \), that \( g \) is non-zero on a Zariski-dense subset of \( V \) and that sufficiently large integers \( d, \delta \in \mathbb{N} \) are known such that \( g \in F_d \) and \( V \) is generated by polynomials in \( F_d \), then Corollary 4.12 gives rise to a scheme for recovering all the defining data of \( \mu \) from finitely many of its moments. In particular, this includes finitely-supported measures as a special case, for which the variety \( V \) is zero-dimensional. Hence, this may be regarded as an extension of Prony’s method to more general measures.

In this setting, we have \( V = \text{supp} \mu = \text{supp} \mu_+ \). Thus, we obtain the variety from \( V = V(a \cap F_d) = V(\ker H_{d+\delta,d}) \), where \( a := I(\text{supp} \mu) \subseteq L \) denotes the vanishing ideal. Knowing \( V \), one can compute the moments of the uniform measure \( \mu_+ \) on \( V \). Finally, finding \( g \) is a linear problem involving only the moments of \( \mu \) and \( \mu_+ \). Indeed, if \( B_\delta \subseteq F_\delta \) represents a basis of \( F_\delta/(a \cap F_\delta) \) and \( H := (\int_{\Omega} x^a v \, d\mu_+)_{w,v \in B_\delta} \) is the corresponding moment matrix, we have

\[
H \mathbf{g} = \left( \int_{\Omega} x^a \mathbf{g} \, d\mu_+ \right)_{w \in B_\delta} = \left( \int_{\Omega} x^a \, d\mu \right)_{w \in B_\delta},
\]

where \( \mathbf{g} = \sum_{v \in B_\delta} g_v v \) is the reduction of \( g \) modulo \( a \cap F_\delta \). As \( H \) is a positive-definite matrix by Proposition 4.10, this linear system has a unique solution, so the polynomial \( g \) is unique modulo \( a \cap F_\delta \).

Though, we remark that computing the moments of the uniform measure \( \mu_+ \) can be a difficult problem itself if the variety \( V \) is not zero-dimensional. An approach that proved successful for us is to find a parametrization of the variety \( V \) and then compute the moments numerically with respect to this parametrization.

We give a few examples of signed measures that illustrate that the assumption of non-negativity is crucial for Proposition 4.10.

Example 4.14. Let \( \mu \) be a signed measure supported on the real interval \([-1,1] \subseteq \mathbb{R} \) with density \( g(x) := x \) and denote its moment functional by \( \sigma : R := \mathbb{R}[x] \to \mathbb{R} \), so that \( \sigma(p) = \int_{-1}^1 p(x)g(x)\,dx \) for \( p \in R \). In particular, this means that \( \langle -,- \rangle_\mu \) is not positive-semidefinite. One checks that, due to symmetry, the even moments \( \sigma(x^{2\alpha}) = 0 \) vanish for \( \alpha \in \mathbb{N} \) and thus \( \det(\sigma(x^{2\alpha+2\beta}))_{0 \leq \alpha,\beta \leq d} = 0 \) for all \( d \in \mathbb{N} \). Then it follows that \( \det(\sigma(x^{\alpha+\beta}))_{0 \leq \alpha,\beta \leq d} = 0 \) if \( d \) is even, for example using the Leibniz formula or by a suitable permutation of rows and columns.

This means that, for every even \( d \), we find some non-zero polynomial in \( R_d \) that lies in the kernel of the square moment matrix \( (\sigma(x^{\alpha+\beta}))_{0 \leq \alpha,\beta \leq d} \) even though the variety corresponding to the Zariski closure of the support of the signed measure \( \mu \) is the entire line \( \mathbb{R} \), which is defined by the zero-ideal in \( R \), and despite the fact that the monomials are linearly independent modulo the zero-ideal. Hence, the statement of Proposition 4.10 cannot hold. However, note that, in this example, the non-truncated Hankel operator is injective nevertheless, as stated in Corollary 4.6. Moreover, as \( g \) is a polynomial of degree 1, it follows from Corollary 4.12 that the kernel of the rectangular matrix \( (\sigma(x^{\alpha+\beta}))_{0 \leq \alpha \leq d+1,0 \leq \beta \leq d} \) is zero, for every \( d \in \mathbb{N} \).
In the affine setting with \( L \leq d = R \leq d \), \( d \in \mathbb{N} \), and for a finitely-supported signed measure, it follows from Proposition 1.1 that the statement of Theorem 4.3 holds with \( d' : = d \), as long as \( d \in \mathbb{N} \) is sufficiently large. The following example shows that this can fail for small \( d \).

**Example 4.15.** Let \( R = \mathbb{k}[x] \) be the univariate polynomial ring and let \( a = m_{\xi_1} \cap m_{\xi_2} \) with two distinct points \( \xi_1, \xi_2 \in \mathbb{k} \). We consider the map \( \sigma = ev_{\xi_1} - ev_{\xi_2} \). Denote by \( H_{d',d} \) the corresponding Hankel matrix, for \( d, d' \in \mathbb{N} \). By (4), we have \( a \cap R_{\leq d} \subseteq \ker H_{d',d} \), but equality does not hold for small \( d \).

For instance, if \( d' = d = 0 \), we have 

\[
a \cap R_{\leq d} = 0 \subsetneq \ker H_{0,0} = \ker(0).
\]

However, if \( d \) is sufficiently large, namely \( d \geq 2 \), and if \( d' \geq d \), we have \( a \cap R_{\leq d} = \ker H_{d',d} \) by Proposition 1.1 (2a), regardless of the choice of \( d' \).

In contrast, we have seen in Example 4.9 that a similar statement is not possible for infinitely-supported signed measures. More precisely, it is an example in which one has \( a \cap R_{\leq d} \neq \ker H_{d,d} \) for all \( d \in \mathbb{N} \), since \( 1 \in \ker H_{d,d} \), but \( 1 \notin a \). For a non-negative measure, this would not be possible due to Proposition 4.10.

For signed measures that are a complex linear combination of non-negative measures, we obtain the following statement, which in contrast to Theorem 4.3 bounds the size of the moment matrix and does not require compactness of the support.

**Theorem 4.16.** Let \( \mu = \sum_{j=1}^{r} \lambda_j \mu_j \), where \( \lambda_j \in \mathbb{C}^* \) and \( \mu_j \) are non-negative measures on \( \Omega \) with finite moments. Assume that \( \delta \in \mathbb{N} \) such that there exist elements \( h_j \in L_{\leq \delta} \), \( 1 \leq j \leq r \), such that \( h_j \geq 0 \) on \( \Omega \) and

\[
\text{supp}(h_j \mu_k) = \begin{cases} 
\text{supp} \mu_k & \text{if } k = j, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

Then

\[
I(\text{supp} \mu) \cap L_{\leq d} = \ker H_{d+\delta,d},
\]

holds for all \( d \in \mathbb{N} \) with \( H_{d+\delta,d} := (\langle w, v \rangle)_{w \in B_{d+\delta}, v \in B_d} \) where \( \sigma : L \to \mathbb{k} \) denotes the moment functional of \( \mu \) and \( B_d, B_{d+\delta} \) denote bases of \( L_{\leq d}, L_{\leq d+\delta} \), respectively.

**Proof.** Since \( h_j \mu_k = 0 \) for all \( k \neq j \), we have

\[
h_j \mu = h_j \lambda_j \mu_j = h_j \lambda_j \mu_+,
\]

where we define \( \mu_+ := \sum_{k=1}^{r} \mu_k \). Letting \( g := \sum_{j=1}^{r} \lambda_j h_j \in L_{\leq \delta} \), we thus have \( g^0 \mu = \sum_{j=1}^{r} |\lambda_j|^2 h_j \mu_+ \), which is a non-negative measure. Its support satisfies

\[
\text{supp} \left( \sum_{j=1}^{r} |\lambda_j|^2 h_j \mu_+ \right) = \bigcup_{j=1}^{r} \text{supp}(h_j \mu) = \text{supp} \mu,
\]

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where the first equality holds due to (8) and the second due to $\text{supp} \mu_j = \text{supp}(h_j \mu_j)$. Hence, the statement follows from Lemma 4.11.

\begin{remark}
Note that elements $h_j \in L_{\leq \delta}$ satisfying (7) exist, as long as $\delta \in \mathbb{N}$ is large enough and the Zariski closures of $\text{supp} \mu_j$, $1 \leq j \leq r$, are varieties such that each pair of them does not share a common irreducible component. This allows for elements $f_j \in L$ such that $f_j$ vanishes on $\text{supp} \mu_k$ for all $1 \leq k \leq r$ with $k \neq j$ and $f_j$ is non-zero on a dense subset of $\text{supp} \mu_j$, so we can choose $h_j := f_j^0 f_j$, for $1 \leq j \leq r$.

In particular, we can apply Theorem 4.16 to Example 4.9 with $f_1 := x_1^2 - x_2$, $f_2 := x_1 - x_2^2$. With $\delta := 2$, we then have $h_1, h_2 \in L_{\leq \delta}$ in terms of the max-degree filtration and the hypotheses of the theorem are satisfied. The Laurent polynomial $g = h_1 - h_2$ constructed in the proof of the theorem is non-negative on one of the components and non-positive on the other, as depicted in Figure 1, so that $g^0 \mu$ is a non-negative measure.
\end{remark}

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\section*{References}

[\textit{Bou06}] N. Bourbaki. \textit{Algèbre commutative. Chapitres 1 à 4}. 2nd ed. Éléments de mathématique. Berlin, Heidelberg: Springer, 2006, pp. vi+356. doi: 10.1007/978-3-540-33976-2.

[\textit{CLO15}] D. A. Cox, J. Little, and D. O’Shea. \textit{Ideals, varieties, and algorithms}. 4th ed. Undergraduate Texts in Mathematics. An introduction to computational algebraic geometry and commutative algebra. Cham: Springer, 2015, pp. xvi+646. doi: 10.1007/978-3-319-16721-3.

[\textit{Con90}] J. B. Conway. \textit{A course in functional analysis}. 2nd ed. Vol. 96. Graduate Texts in Mathematics. Corr. fourth print. New York: Springer, 1990, pp. xvi+399.

[\textit{Eis99}] D. Eisenbud. \textit{Commutative algebra with a view toward algebraic geometry}. Corr. third print. Vol. 150. Graduate Texts in Mathematics. New York: Springer, 1999, pp. xvi+797. doi: 10.1007/978-1-4612-5350-1.

[\textit{FAV16}] M. Fatemi, A. Amini, and M. Vetterli. “Sampling and reconstruction of shapes with algebraic boundaries”. In: \textit{IEEE Trans. Signal Process.} 64.22 (2016), pp. 5807–5818. doi: 10.1109/TSP.2016.2591505.

[\textit{Gra14}] L. Grafakos. \textit{Classical Fourier analysis}. 3rd ed. Vol. 249. Graduate Texts in Mathematics. New York: Springer, 2014, pp. xviii+638. doi: 10.1007/978-1-4939-1194-3.

[\textit{IK99}] A. Iarrobino and V. Kanev. \textit{Power Sums, Gorenstein Algebras, and Determinantal Loci}. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1999.
