SAVAGE SURFACES

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Abstract. Let $G$ be the topological fundamental group of a given nonsingular complex projective surface. We prove that the Chern slopes $c_1^2(S)/c_2(S)$ of minimal nonsingular projective surfaces of general type $S$ with $\pi_1(S) \simeq G$ are dense in the interval $[1,3]$. 

1. Introduction

By the Lefschetz hyperplane theorem, we know that the fundamental group of any nonsingular projective variety is the fundamental group of some nonsingular projective surface. Groups that are fundamental groups of varieties are abundant. Serre proved, for example, that any finite group is realizable [S58]. For singular surfaces we know that every finitely presented group is possible as fundamental group by [KK14] (reducible surfaces) and [K13] (irreducible surfaces), but there are of course various restrictions in the case of nonsingular projective surfaces. (See the survey [A95] for more on that topic, and see the book [ABCKT96] for Kähler manifolds.) A natural geographical question is: Are there any constraints for the Chern slope of surfaces of general type after we fix the fundamental group? In more generality, this question has been studied for 4-manifolds (cf. [KL09]) with a particular focus on symplectic 4-manifolds (see e.g. [G95], [BK06], [BK07], [Park07]). For example, Park showed in [Park07] that the set of Chern slopes $c_1^2/c_2$ of minimal symplectic 4-manifolds $S$ with $\pi_1(S) \simeq G$ is dense in the interval $[0,3]$, for any fixed finitely presented group $G$.

For complex surfaces, we know that simply connected surfaces of general type have Chern slopes dense in $[1/5, 3]$ (see [P81, Ch87, PPX96, U10, RU15]), which is the largest possible interval by the Noether inequality $1/(c_2 - 36) \leq c_1^2$ and the Bogomolov-Miyaoka-Yau inequality $c_1^2 \leq 3c_2$ (cf. [BHPV04]). (See [U17] for an analogue geographical result for surfaces in positive characteristic.) In general, however, it is known that for low slopes we do have some constraints for the possible fundamental groups. For instance, from [MP07] we deduce that if $S$ is
a surface of general type with $c_1^2(S) < \frac{1}{3}c_2(S)$ and $\pi_1(S)$ finite, then the order of $\pi_1(S)$ is at most 9. We would also like to mention Reid’s conjecture: The fundamental group of a surface with $c_1^2 < \frac{1}{2}c_2$ is either finite or commensurable with the fundamental group of a compact Riemann surface (see [BHPV04, p.294] for details). Pardini’s proof [Par05] of the Severi inequality together with Xiao’s result [X87, Theorem 1] give evidence on this conjecture at the level of étale fundamental groups.

On the other hand, we remark that a similar question for pairs $(c_1^2, c_2)$ has much stronger constraints. By Gieseker [G77], there are only finitely many possibilities of $\pi_1$ for a given pair. A concrete example: It is expected that for numerical Godeaux surfaces (i.e., $c_1^2 = 1, p_g = q = 0$) the fundamental group belongs to the set $\{1, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/5\}$. Also, on the Bogomolov-Miyaoka-Yau line we have that $\pi_1(S)$ is an infinite group (since those surfaces are ball quotients by results of Miyaoka [Mi84, Prop.2.1.1] and Yau [Yau77]), and, on the opposite side, on the Noether line we have only simply connected surfaces by the classification of Horikawa [Ho75, Ho76]. In this article we prove the following.

**Theorem 1.1.** Let $G$ be the (topological) fundamental group of a non-singular complex projective surface. Then the Chern slopes $c_1^2(S)/c_2(S)$ of minimal nonsingular projective surfaces of general type $S$ with $\pi_1(S)$ isomorphic to $G$ are dense in the interval $[1, 3]$.

In this way, for instance, any finite group $G$ densely populate the wide sector $[1, 3]$. The method to prove the theorem is very different to the one used in [RU15, Theorem 6.3] for trivial $\pi_1$, but we do consider as key input the extremal simply connected surfaces constructed in that paper. An observation here is that the $\pi_1$ trivial surfaces constructed by Persson in [P81] do not work for our method, and they cannot work since, if they do, then some of them would violate Mendes-Lopes–Pardini’s theorem mentioned above for low Chern slopes. Chen surfaces in [Ch87, Theorem 1] do not work for our method either.

We now explain roughly the idea of the proof together with the central ingredients. Let $Y$ be a minimal nonsingular projective surface with $\pi_1(Y) \simeq G$, let $r \in [1, 3]$, and let $\{X_p\}$ be a sequence of simply connected surfaces as in [RU15, Theorem 6.3], so that $c_1^2(X_p)/c_2(X_p)$ approaches $r$ as $p \to \infty$. Let $A_p, B$ be very ample divisors in $X_p$ and $Y$ respectively, and consider the very ample divisor $A_p + B$ in $X_p \times Y$. As in [Cat00, Section 1], one obtains a surface $S_p$ from the intersection of two general sections in $|A_p + B|$ so that $\pi_1(S_p) \simeq G$ (Lefschetz hyperplane theorem), but it is not possible to have the result for $c_1^2(S_p)/c_2(S_p)$ since we have no control on $A_p$. On the other hand, an appropriate $A_p$
to control $c_2^2(S_p)/c_2(S_p)$ may not be even ample, so we may not have $\pi_1(S_p) \simeq G$, or even an $S_p$ to start with. To overcome both difficulties, we consider a very special $A_p$ which works for $c_2^2(S_p)/c_2(S_p)$ and it is also a lef (Lefschetz effettivamente funziona) line bundle, as introduced by de Cataldo and Migliorini [CM02]. It turns out that such an $A_p$ allows us to prove existence of $S_p$ as above which, by a generalization of the Lefschetz hyperplane theorem due to Goresky and MacPherson [GM88, Part II, Theorem 1.1], satisfy $\pi_1(S_p) \simeq G$. These surfaces are used to prove the claim on density of Chern slopes in [1, 3]. We also show that it is not possible to improve this lower bound 1 by using modifications of the surfaces $X_p$.

We finish the paper with two conjectures in relation to geography of Chern slopes for surfaces with ample canonical class, and for Brody hyperbolic surfaces, which might be proved by using the same techniques as in this paper.

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2. Semi-small morphisms, lef line bundles, Bertini and Lefschetz type theorems

Throughout this paper the ground field is $\mathbb{C}$. The following definition can be found in several places, e.g. [GM88 p.151], [Mig95 Def. 4.1] or [CM02 Def. 2.1.1].

Definition 2.1. Let $X,Y$ be irreducible varieties. For a proper surjective morphism $f : X \to Y$, we define

$$Y_f^k = \{ y \in Y \mid \dim f^{-1}(y) = k \}.$$ 

We say that $f$ is semi-small if $\dim(Y_f^k) + 2k \leq \dim X$ for every $k \geq 0$. (Note that $\dim(\emptyset) = -\infty$.) If no confusion can arise, the subscript $f$ will be suppressed.

We note that for a semi-small morphism we have $\dim(X) = \dim(Y)$.

Lemma 2.2. Let $X,Y$ be surfaces. If $f : X \to Y$ is a proper surjective morphism, then $f$ is semi-small.
Proof. It is clear that $\dim(Y^1) = 0$ and $\dim(Y^0) = 2$, since $f$ is surjective. Then the inequality $\dim(Y^k) + 2k \leq \dim(X)$ holds for any $k \geq 0$. □

**Proposition 2.3.** Let $f: X \to Y$ and $g: Z \to W$ be two semi-small morphisms. Then the product morphism $f \times g: X \times Z \to Y \times W$ is a semi-small morphism.

*Proof.* Let $n = \dim(X)$ and $m = \dim(Z)$. Since $f$ and $g$ are semi-small, then we have that $\dim(Y^k) \leq n - 2k$ for any $k \geq 0$, $\dim(Z^l) \leq m - 2l$ for any $l \geq 0$, and $\dim(Y^0) = n, \dim(W^0) = m$. We also have $(Y \times W)^q = \bigcup_{i+j=q} Y^i \times W^j$, and so

$$\dim(Y \times W)^q \leq \max_{i+j=q} \dim(Y^i \times W^j) \leq n + m - 2i - 2j = n + m - 2q.$$ 

Hence $f \times g$ is semi-small. □

**Proposition 2.4.** Let $X, Y, Z$ be nonsingular projective varieties. Assume that $f: X \to Y$ is semi-small, and that $g: Y \to Z$ is finite morphism. Then, $h = g \circ f: X \to Z$ is semi-small.

*Proof.* Since $g$ is a finite, we have $Z^k_h = g(Y^k_f)$ for each $k \geq 0$, and so $\dim(Z^k_h) = \dim(Y^k_f)$. Thus $\dim(Z^k_h) + 2k \leq \dim(X)$, and so $h$ is semi-small. □

**Definition 2.5.** ([CM02, Def. 2.3]) Let $X$ be a nonsingular projective variety, and let $M$ be a line bundle on $X$. We say that $M$ is lef if there exists $n > 0$ such that $|nM|$ is generated by global sections, and the morphism $\psi_{|nM|}$ associated to $|nM|$ is semi-small onto its image. The exponent of $M$ is the smallest $n$ so that $M$ is lef. We denote it by $\exp(M)$.

If $L$ is an ample line bundle, then $L$ is lef. If moreover $L$ is very ample, then $\exp(L) = 1$. Next we write a corollary of Proposition 2.4 which will be used later.

**Proposition 2.6.** Let $f: X \to Y$ be semi-small between nonsingular projective varieties, and let $L$ be very ample on $Y$. Then $f^*(L)$ is lef with $\exp(f^*(L)) = 1$.

A useful Bertini type theorem for lef line bundles is the following. (See [CM02] Prop. 2.1.7 or [Mig95, Lemma 4.3].)

**Theorem 2.7.** Let $X$ be a nonsingular projective variety of dimension at least 2. Let $M$ be a lef line bundle on $X$. Assume that $M$ is globally generated and with $\exp(M) = e$. Then any generic member $Y \in |M|$
is a nonsingular projective variety, and the restriction $M|_Y$ is left on $Y$ with $\exp(M|_Y) \leq e$.

We now state some Lefschetz type theorems relevant for the computation of the fundamental group, which are due to Goresky and MacPherson [GM88]. For comparison, we mention the usual Lefschetz theorem for ample line bundles. (See e.g. See [Laz17, Theo. 3.1.21].)

**Theorem 2.8** (Lefschetz theorem for homotopy groups). Let $X$ be a nonsingular projective variety of dimension $n$. Let $\iota: A \to X$ be the inclusion of an effective ample divisor $A$. Then the induced homomorphism

$$\iota^*: \pi_i(A) \to \pi_i(X)$$

is bijective if $i \leq n - 2$, and it is surjective if $i = n - 1$.

**Theorem 2.9.** Let $X$ be a nonsingular projective variety of dimension $n$. Suppose that $f: X \to \mathbb{P}^N$ is a proper morphism, and let $H$ be a linear subspace of codimension $c$. Define $\phi(k) := \dim((\mathbb{P}^N \setminus H)^k)$. Then the induced homomorphism

$$\pi_i(f^{-1}(H)) \to \pi_i(X)$$

is an isomorphism if $i < \hat{n}$, and it is surjective if $i = \hat{n}$, where

$$\hat{n} = n - 1 - \sup_k (2k - n + \phi(k) + \inf_k (\phi(k), c - 1)).$$

**Proof.** See [GM88, Part II, Theorem 1.1].

**Remark 2.10.** In the last theorem, if $H$ is a hyperplane and $f: X \to \mathbb{P}^N$ is semi-small into its image, then

$$\pi_i(f^{-1}(H)) \simeq \pi_i(X)$$

if $i < n - 1$. That is because $\hat{n} = n - 1$, since codimension of $H$ is $c = 1$ and the inequality $\phi(k) \leq \dim((f(X))^k)$ holds.

**Corollary 2.11.** Let $X$ be a nonsingular projective variety with $\dim(X) \geq 3$. Let $M$ be a left line bundle on $X$ with $\exp(M) = 1$. If $E \in |M|$, then $\pi_1(E) \simeq \pi_1(X)$.

**Corollary 2.12.** Let $X$ be a nonsingular projective variety with $\dim(X) \geq 4$. Let $M$ be a left line bundle with $\exp(M) = 1$. Then a generic member $E \in |M|$ is nonsingular projective variety, and $M_E := |M|_E$ is left. Moreover, if $F \in M_E$, then $\pi_1(F) \simeq \pi_1(X)$.

**Proof.** The first part is just Theorem 2.7. If $F \in M_E$, then by Corollary 2.11 we obtain that $\pi_1(F) \simeq \pi_1(E) \simeq \pi_1(X)$.
3. RU surfaces

In this section we recall some surfaces of general type $X_p$ from [RU15, Section 6] which are key in the main result of this paper.

Let $p \geq 5$ be a prime number, and let $\alpha > 0, \beta > 0$ be integers. Let $n = 3\alpha p$. Let $H$ be the blow-up at the 12 3-points of the dual Hesse arrangement of 9 lines $(x^3 - y^3)(y^3 - z^3)(x^3 - z^3) = 0$ in $\mathbb{P}^2$. As defined in [RU15, Section 3], we consider the very special arrangement of $\frac{4n^2-12}{3}$ elliptic curves $\mathcal{H}'_n = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_\infty + \mathcal{E}_\zeta$ in $H$. Let $\mathcal{E}'_i$ be $\beta^2p^2$ general fibers of $\pi'_i$ (defined also in [RU15, Section 3]), and let $\mathcal{A}_{2d} = L_1 + \ldots + L_{2d}$ be the strict transform of an arrangement of $2d$ general lines in $\mathbb{P}^2$, where $3 \leq 2d \leq p$. We define $a_0 = a_1 = b_i = 1$ for $1 \leq i \leq d$, and $a_\infty = a_\zeta = b_i = p - 1$ for $d + 1 \leq i \leq 2d$. Then

$\mathcal{O}_H\left( \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}'_i + \sum_{i=0,1,\zeta,\infty} a_i(F_{i,1} + F_{i,2} + F_{i,3}) + \sum_{i=1}^{2d} 3b_iL_i \right)$

is isomorphic to $\mathcal{L}_0^p$ where

$\mathcal{L}_0 := \mathcal{O}_H\left( 3p(3\alpha^2 + \beta^2)\left( \sum_{i=0,1,\zeta,\infty} a_iF_i \right) + 3dL \right)$,

and all symbols have been defined in [RU15, Section 5]. For each $i$, we denote the strict transform of $\mathcal{E}_i, \mathcal{E}'_i, L_j, F_{i,j}$ in $Z_n$ by the same symbol, where $\varphi_n: Z_n \to H$ is the blow-up of $H$ at all the $\frac{(n^2-3)(n^2-9)}{3}$ 4-points in $\mathcal{H}'_n$. Then

$\mathcal{O}_{Z_n}\left( \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}'_i + \sum_{i=0,1,\zeta,\infty} a_i(F_{i,1} + F_{i,2} + F_{i,3}) + \sum_{i=1}^{2d} 3b_iL_i \right)$

is $\mathcal{L}_1^p$ where $\mathcal{L}_1 := \varphi_n^*(\mathcal{L}_0) \otimes \mathcal{O}_{Z_n}(-6E)$, and $E$ is the exceptional divisor of $\varphi_n$. Again, we denote the strict transform of $\mathcal{E}_i, \mathcal{E}'_i, L_j, F_{i,j}, M, N_i, N$ in $Y_n$ by the same symbol, where $\sigma_n: Y_n \to Z_n$ is the blow-up at all the $4(n^2-3)$ 3-points in $\mathcal{H}'_n$. Then we have

$\mathcal{O}_{Y_n}\left( \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}'_i + \sum_{i=0,1,\zeta,\infty} 3a_iN_i + \sum_{i=1}^{2d} 3b_iL_i \right) \simeq \mathcal{L}^p$

where $\mathcal{L} := \sigma_n^*(\mathcal{L}_1) \otimes \mathcal{O}_{Y_n}(-2M - 6G)$.

With this data, we construct a $p$-th root cover of $Y_n$ branch along

$A := \sum_{i=0,1,\zeta,\infty} \mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} \mathcal{E}'_i + \sum_{i=0,1,\zeta,\infty} N_i + \sum_{i=1}^{2d} L_i$. 
Let $f : X_p \to Y_n$ be the corresponding morphism for the $p$-th root cover, as in [RU15, Section 5]. The nonsingular projective surface $X_p$ is simply connected [RU15, Prop.6.1], and minimal [RU15, Prop.6.2].

Let us write

$$A = \sum_j \nu_j A_j = \sum_{i=0,1,\infty} 3a_i \mathcal{E}_i + \sum_{i=0,1,\infty} 3a'_i \mathcal{E}'_i + \sum_{i=0,1,\infty} 3a_i N_i + \sum_{i=1}^{2d} 3b_i L_i$$

where $A_j$ are the irreducible curves in $A$. Hence $\nu_j$ is equal to either $3a_i$ or $3b_k$ for some $i, k$. The arrangement $A$ has only 2-points, and its number is

$$t_2 = 108\alpha^2 \beta^2 p^4 + 18\beta^4 p^4 + 72d\alpha^2 p^2 - 25d + 24d\beta^2 p^2 + 2d^2.$$  

By [RU15, Prop.4.1], the log Chern numbers of $A$ are

$$\bar{c}_1^2 = n^4 + 2t_2 - 10d - 48 \text{ and } \bar{c}_2 = \frac{n^4}{3} + t_2 - 4d - 12.$$  

As in [RU15, Section 5], the Chern numbers of $X_p$ are

$$c_1^2(X_p) = p\bar{c}_1^2 - 2\left(t_2 + 2 \sum_j (g(A_j) - 1)\right) + \frac{1}{p} \sum_j A_j^2 - \sum_{i<j} c(q_{i,j}, p) A_i \cdot A_j$$

and

$$c_2(X_p) = p\bar{c}_2 - \left(t_2 + 2 \sum_j (g(A_j) - 1)\right) + \sum_{i<j} l(q_{i,j}, p) A_i \cdot A_j$$

where $0 < q_{i,j} < p$ with $\nu_i + q_{i,j}\nu_j \equiv 0 \pmod{p}$,

$$c(q_{i,j}, p) := 12s(q_{i,j}, p) + l(q_{i,j}, p),$$

and $s(q_{i,j}, p)$ and $l(q_{i,j}, p)$ are the numbers we recall below.

**Definition 3.1.** Let $q, p$ be coprime integers such that $0 < q < p$.

1. The associated *Hirzebruch-Jung continued fraction* is

$$\frac{p}{q} = e_1 - \frac{1}{e_2 - \frac{1}{e_3 - \ddots - \frac{1}{e_l}}},$$

where $[e_1, \ldots, e_l]$ is the continued fraction.

We denote its *length* as $l(q, p) := l$.

2. The *Dedekind sum* associated to the pair $(q, p)$ is defined as

$$s(q, p) := \sum_{i=1}^{p-1} \left(\left(\frac{i}{p}\right)\left(\frac{iq}{p}\right)\right)$$

where $\left(\frac{x}{y}\right) = x - \lfloor x \rfloor - \frac{1}{2}$ for any rational number $x$. 

For the particular multiplicities $a_0 = a_1 = b_i = 1$ for $1 \leq i \leq d$ and $a_\infty = a_\zeta = b_i = p - 1$ for $d + 1 \leq i \leq 2d$ we chose, we have to consider only the numbers $c(p-1, p) = \frac{2p-2}{p}$ and $c(1, p) = \frac{2p-2p+2}{p}$, and $l(p-1, p) = p - 1$ and $l(1, p) = 1$. Therefore,

$$
\sum_{i<j} c(q_{i,j}, 4p) A_i \cdot A_j = \frac{(2p-2)}{p} t_{2,1} + \frac{(p^2 - 2p + 2)}{p} t_{2,2}
$$

and

$$
\sum_{i<j} l(q_{i,j}, 4p) A_i \cdot A_j = (p-1)t_{2,1} + t_{2,2}
$$

where $t_{2,1}$ and $t_{2,2}$ are the number of 2-points corresponding to the singularities $\frac{1}{p}(1, p - 1)$ and $\frac{1}{p}(1, 1)$ respectively. Hence

$$
t_{2,1} = 6\beta^4 p^4 + 36\alpha^2 \beta^2 p^4 + 36\alpha^2 p^2 - 13d + 12d\beta^2 p^2 + d^2
$$

and

$$
t_{2,2} = 12\beta^4 p^4 + 72\alpha^2 \beta^2 p^4 + 36\alpha^2 p^2 - 12d + 12d\beta^2 p^2 + d^2.
$$

By plugging in the formulas for Chern numbers, we obtain that

$$
c_1^2(X_p) = (81\alpha^4 + 144\alpha^2 \beta^2 + 24\beta^4)p^5 + l.o.t.
$$

and

$$
c_2(X_p) = (27\alpha^4 + 144\alpha^2 \beta^2 + 24\beta^4)p^5 + l.o.t.,
$$

where l.o.t. (lower order terms) is a Laurent polynomial in $p$ of degree less than 5. In this way, we obtain that

$$
\lim_{p \to \infty} \frac{c_1^2(X_p)}{c_2(X_p)} = \frac{27x^4 + 48x^2 + 8}{9x^4 + 48x^2 + 8} =: \lambda(x)
$$

where $x := \alpha/\beta$. We note that $\lambda([0, \infty^+]) = [1, 3]$. This allows to prove the following theorem (see [RU15, Theorem 6.3]).

**Theorem 3.2.** For any number $r \in [1, 3]$, there are simply connected minimal surfaces of general type $X$ with $c_1^2(X)/c_2(X)$ arbitrarily close to $r$.

**Proposition 3.3.** Let $A_p := f^*(L)$, where as before $L$ is the pull-back in $Y$ of a general line in $\mathbb{P}^2$. Then we have $A_p^2 = p$ and $A_p \cdot K_{X_p} = -3p + (p-1)(2d + 36\alpha^2 p^2 - 12 + 12\beta^2 p^2)$.

**Proof.** As $f$ is a generically finite morphism of degree $p$, we have $A_p^2 = p$. Let us consider $L$ generic, so that $f^*(L)$ is a nonsingular curve of genus $A_p$. We note that $L \cdot N_i = 0$ for all $i$, $L \cdot \sum_{i=1}^{2d} L_i = 2d$, $L \cdot \sum_{i=0,1,\infty} E_i = 36\alpha^2 p^2 - 12$, and $L \cdot \sum_{i=0,1,\infty} E'_i = 12\beta^2 p^2$. Therefore, the morphism $f_{A_p} : A_p \to L = \mathbb{P}^1$ is totally ramified at $2d + 36\alpha^2 p^2 - 12 + 12\beta^2 p^2$.
points, and so, by the Riemann-Hurwitz formula and adjunction, we obtain the desired equality for $A_p \cdot K_{X_p}$. \hfill \square

We finish this section with a proof that the best lower bound for Chern slopes in this construction is indeed 1. As it was shown above, the values of the $b_i$’s do not contribute in the asymptotic final result. We also point out that it is enough to have either $\sum_{i=0,1,\infty} a_i = p$ or $\sum_{i=0,1,\infty} a_i = 2p$ by considering $0 < a_i < p$ and multiplying by units modulo $p$. In fact, we can and do take $a_0 = 1$, $a_1 = a$, $a_\infty = b$, and $a_{\infty} = c$ with $1 + a + b + c = mp$ for $m$ either equal to 1 or 2.

Through the formulas obtained above, we have

$$
\lim_{x \to 0} \frac{c^2(X_p)}{c_2(X_p)} = \frac{12 - \frac{1}{p}C}{6 + \frac{1}{p}L}
$$

where $C := c(-a, p) + c(-b, p) + c(-c, p) + c(-ba^{-1}, p) + c(-ca^{-1}, p) + c(-cb^{-1}, p)$, $L := l(-a, p) + l(-b, p) + l(-c, p) + l(-ba^{-1}, p) + l(-ca^{-1}, p) + l(-cb^{-1}, p)$, and all the $q$’s in these expressions are taken modulo $p$ with $0 < q < p$. For example, for generic $a, b, c$ one can prove that $C/p$ and $L/p$ tend to 0 as $p$ approaches infinity, and so the limit of the Chern slopes is 2 (see [U10] for these generic behaviour).

Since $c(q, p) = 12s(q, p) + l(q, p)$, it is enough to show that

$$
6S + L \leq 3p + 3 - \frac{6}{p}
$$

any $p$, where $S := s(-a, p) + s(-b, p) + s(-c, p) + s(-ba^{-1}, p) + s(-ca^{-1}, p) + s(-cb^{-1}, p)$. The proof will use the following numerical lemma.

**Lemma 3.4.** Let $0 < q < p$ be coprime integers. Let $\frac{p}{q} = [e_1, \ldots, e_l]$. Then $\sum_{i=1}^l (e_i - 1) \leq p - 1$.

**Proof.** We do induction on $p$. Say for all coprime pairs $(q', p')$ with $p' < p$ we have that the statement is true. We write $\frac{p}{q} = [e_1, \ldots, e_l]$. Then $e_1 = [p/q] + 1$, and $\frac{p}{q} = [e_2, \ldots, e_l]$ with $(r, q)$ coprime and $q < p$. Hence

$$
\sum_{i=1}^l (e_i - 1) = [p/q] + \sum_{i=2}^l (e_i - 1) \leq [p/q] + q - 1
$$

by the induction hypothesis. Therefore, we should prove that $[p/q] + q \leq p$. Let $q \neq 1$ (otherwise we are done). Let $1 \leq r < q$ be the unique integer such that $[p/q]q + r = p$. Then $[p/q] + q \leq p$ is equivalent to

$$
\frac{r}{q-1} + q \leq p.
$$

But $\frac{r}{q-1} \leq 1$ if $r \geq 1$, and $q + 1 \leq p$. \hfill \square

**Proposition 3.5.** We have $6S + L \leq 3p + 3 - \frac{6}{p}$.
Proof. Let $0 < q < p$ integers where $p$ is a prime number. Then (see e.g. [UI0 Example 3.5])

$$12s(q,p) = \frac{q + q^{-1}}{p} + \sum_{i=1}^{l}(e_i - 3)$$

where $\frac{p}{q} = [e_1, \ldots, e_l]$ and $q^{-1}$ is the integer between 0 and $p$ such that $qq^{-1} \equiv 1$ modulo $p$. Hence $6s(q,p) + l = \frac{q + q^{-1}}{2p} + \frac{1}{2}\sum_{i=1}^{l}(e_i - 1)$. We note that always $\frac{2 + q^{-1}}{2p} \leq \frac{p - 1}{p}$. We now run this equality for each of the terms in $S$ and in $L$, and use Lemma 3.4 to conclude that

$$6S + L \leq 3p - 3 + \frac{6(p - 1)}{p} = 3p + 3 - \frac{6}{p}.$$

□

4. Key construction and density theorem

In this section, we generalize the construction used in [Cat00, Section 1] in the context of nef line bundles, which will be used for the main theorem.

Proposition 4.1. Let $X$ and $Y$ be nonsingular projective surfaces. Let $p: X \times Y \to X$ and $q: X \times Y \to Y$ be the usual projections. Let $A$ and $B$ be nef line bundles on $X$ and $Y$ respectively. Assume that $\exp(A) = \exp(B) = 1$. Then $p^*(A) \otimes q^*(B)$ is a nef line bundle on $X \times Y$ of exponent 1.

Proof. This is elementary, we briefly give an argument. Let $M := p^*(A) \otimes q^*(B)$. Let $s_0, \ldots, s_l$ be a basis of $H^0(X, A)$, and let $t_0, \ldots, t_b$ be basis of $H^0(Y, B)$. Since $H^0(X, A) \otimes H^0(Y, B) \simeq H^0(X \times Y, M)$ (see e.g. [Bea96 Fact III.22, i]), then $M$ is generated by the global sections $s_it_j$ with $0 \leq i \leq l$ and $0 \leq j \leq b$. The morphism $\psi_M: X \times Y \to \mathbb{P}(|M|)$ is $\Sigma_{l,b} \circ (\psi_{[A]} \times \psi_{[B]})$, where $\Sigma_{l,b}$ is the Segre embedding. Therefore $\psi_M$ is semi-small into its image as $\psi_{[A]} \times \psi_{[B]}$ is semi-small by Proposition 2.3. It follows that $M$ is nef and $\exp(M) = 1$. □

Theorem 4.2. Let $X$ be a nonsingular projective surface of general type with $K_X$ nef, and let $Y$ be a nonsingular projective surface with $K_Y$ nef. Let $B$ be a very ample line bundle on $Y$. Assume that there is a nef line bundle $A$ on $X$ with $\exp(A) = 1$.

Then there exist a nonsingular projective surface $S \subset X \times Y$ with the following properties:

1. $\pi_1(S) \simeq \pi_1(X) \times \pi_1(Y)$.
(2) The morphisms \( p|_S : S \to X \) and \( q|_S : S \to Y \) have degrees 
\( \deg(p|_S) = B^2 \) and \( \deg(q|_S) = A^2 \).

(3) We have
\[
c_1^2(S) = c_1^2(X)B^2 + c_1^2(Y)A^2 + 8c(A, B) - 4A^2B^2
\]
and
\[
c_2(S) = c_2(X)B^2 + c_2(Y)A^2 + 4c(A, B) + 4A^2B^2
\]
where
\[
c(A, B) = \frac{7}{2}A^2B^2 + \frac{3}{2}(A \cdot K_X)B^2 + \frac{3}{2}(B \cdot K_Y)A^2 + \frac{1}{2}(A \cdot K_X)(B \cdot K_Y).
\]

(4) \( K_S \) is big and nef.

Proof. We first construct a surface \( S \subset X \times Y \) which satisfies (1) and (2). Let \( M := p^*(A) \otimes q^*(B) \). Then, by Proposition 4.1, we have that \( M \) is lef with \( \exp(M) = 1 \). Take a generic member \( E \in |M| \), and a generic \( S \in |M|_E \). From Proposition 2.12 and Theorem 2.7, we have that \( S \) is a nonsingular projective surface with \( \pi_1(S) \cong \pi_1(X) \times \pi_1(Y) \). We also have that the degree of \( p|_S \) is \( ((p^*(A) \otimes q^*(B))|_Y)^2 = B^2 \). Similarly the morphism \( q|_S \) has degree \( A^2 \).

Now we prove (3). By the adjunction formula applied twice, and since \( K_{X \times Y} \sim p^*(K_X) + q^*(K_Y) \), we get that
\[
K_S \sim p|_S^*(K_X + 2A) + q|_S^*(K_Y + 2B),
\]
and so
\[
K_S^2 = K_X^2B^2 + K_Y^2A^2 + 24A^2B^2 + 12((A \cdot K_X)B^2 + (B \cdot K_Y)A^2) + 4(A \cdot K_X)(B \cdot K_Y).
\]

To calculate \( \chi(S) \), we use the following exact Koszul complex. Since \( S \) is a complete intersection of two sections of \( M \) and \( X \times Y \) is nonsingular, then we have (see e.g. [FL85 Pags. 76-77])
\[
0 \to O_{X \times Y}(-2M) \to O_{X \times Y}^{\oplus 2}(-M) \to O_{X \times Y} \to O_S \to 0.
\]

Then by the additivity of the Euler characteristic and the Künneth formula (see e.g. [Cut18 Theo. 17.23])
\[
H^n(X \times Y, M) = \bigoplus_{i+j=n} H^i(X, A) \otimes H^j(Y, B),
\]
we obtain
\[ \chi(\mathcal{O}_S) = \chi(\mathcal{O}_{X,Y}) + \chi(\mathcal{O}_{X,Y}(-2A - 2B)) - 2\chi(\mathcal{O}_{X,Y}(-A - B)) \\
= \chi(\mathcal{O}_X)\chi(\mathcal{O}_Y) + \chi(\mathcal{O}_X(-2A))\chi(\mathcal{O}_Y(-2B)) - 2\chi(\mathcal{O}_X(-A))\chi(\mathcal{O}_Y(B)) \\
= \chi(\mathcal{O}_X)\chi(\mathcal{O}_Y) + (\chi(\mathcal{O}_X) + \frac{1}{2}(4A^2 + 2(A \cdot K_X)))\chi(\mathcal{O}_Y) + \frac{1}{2}(4B^2 + 2(B \cdot K_Y))) - 2(\chi(\mathcal{O}_X) + \frac{1}{2}(A^2 + A \cdot K_X))\chi(\mathcal{O}_Y) + \frac{1}{2}(B^2 + B \cdot K_Y)) \\
= \chi(\mathcal{O}_X)B^2 + \chi(\mathcal{O}_Y)A^2 + c(A, B), \]
where
\[ c(A, B) = \frac{7}{2}A^2B^2 + \frac{3}{2}(A \cdot K_X)B^2 + \frac{3}{2}(B \cdot K_Y)A^2 + \frac{1}{2}(A \cdot K_X)(B \cdot K_Y). \]

Finally we show (4). Let \( \Gamma \) be an irreducible curve on \( S \). Let \( a = \deg p|_{\Gamma} = a \) and \( b = \deg q|_{\Gamma} \). Then, by the projection formula for generically finite morphisms, we have
\[ \Gamma \cdot K_S = \Gamma \cdot p|_{\mathcal{S}}(K_X + 2A) + \Gamma \cdot q|_{\mathcal{S}}(K_Y + 2B) = ap(\Gamma) \cdot (K_X + 2A) + bq(\Gamma) \cdot (K_Y + 2B). \]
We note that \( K_X, K_Y, \) and \( A \) are nef, and \( B \) is very ample, and so \( \Gamma \cdot K_S \geq 0 \). Using the formula for \( K_S^2 \) above and by the same previous reasons, we obtain \( K_S^2 > 0 \). \hfill \Box

We now present our main result, which puts together all the ingredients elaborated until now.

**Theorem 4.3.** Let \( Y \) be a nonsingular projective surface with \( K_Y \) nef, and let \( r \in [1, 3] \) be a real number. Then there are minimal nonsingular projective surfaces \( S \) with \( c_1^2(S)/c_2(S) \) arbitrarily close to \( r \), and \( \pi_1(S) \simeq \pi_1(Y) \).

**Proof.** Let \( X_p \) be the collection of simply connected surfaces described in Section 4. Let \( A_p \) be the line bundle defined in Proposition 3.3. For any \( p \) we have that \( A_p \) is lef by Proposition 2.6. (We note that \( A_p \) is not ample because of the resolution of singularities involved in the construction of the surfaces \( X_p \).) Let \( B \) be a very ample divisor on \( Y \). Note that we satisfy all the hypothesis in Theorem 4.2 with \( X = X_p \) and \( A = A_p \). Therefore, there are surfaces \( S_p := S \) such that all the conclusions in Theorem 4.2 hold. In particular, we have \( \pi_1(S_p) \simeq \pi_1(Y) \).
The formulas in Theorem 4.2 part (3) are
\[ c_1^2(S_p) = c_1^2(X_p)B^2 + c_1^2(Y)A_p^2 + 8c(A_p, B) - 4A_p^2B^2 \]
and
\[ c_2(S_p) = c_2(X_p)B^2 + c_2(Y)A_p^2 + 4c(A_p, B) + 4A_p^2B^2, \]
where
\[ c(A_p, B) = \frac{7}{2}A_p^2B^2 + \frac{3}{2}(A_p \cdot K_{X_p})B^2 + \frac{3}{2}(B \cdot K_Y)A_p^2 + \frac{1}{2}(A_p \cdot K_{X_p})(B \cdot K_Y). \]

By Proposition 3.3 we have that \( A_p^2 = p \) and \( A_p \cdot K_{X_p} \) is a polynomial in \( p \) of degree 3. Thus \( c(A_p, B) \) is a polynomial in \( p \) of degree 3. By Section 4, the invariants \( c_1^2(X_p) \) and \( c_2(X_p) \) are Laurent polynomials in \( p \) of degree 5. Therefore, by the formulas above, we have
\[ \lim_{p \to \infty} \frac{c_1^2(S_p)}{c_2(S_p)} = \lim_{p \to \infty} \frac{c_1^2(X_p)}{c_2(X_p)} = \frac{27x^4 + 48x^2 + 8}{9x^4 + 48x^2 + 8} =: \lambda(x) \]
where \( x := \alpha/\beta \), as in Section 4. In this way, just as in [RU15, Thereom 6.3], we obtain the desired surfaces \( S = S_p \) with \( c_1^2(S)/c_2(S) \) arbitrarily close to \( r \).

Corollary 4.4. Let \( G \) be the fundamental group of a nonsingular projective surface. Then the Chern slopes \( c_1^2(S)/c_2(S) \) of nonsingular projective surfaces \( S \) with \( \pi_1(S) \cong G \) are dense in \([1, 3]\).

Proof. Since \( \pi_1 \) is invariant under birational transformations between nonsingular projective surfaces, then it is enough to consider surfaces with no \((-1)\)-curves. If \( G \) is the fundamental group of \( \mathbb{P}^1 \times C \), where \( C \) is a nonsingular projective curve, then, for example, we can take as \( Y \) a surface in [RU15, Corollary 6.4] to apply Theorem 4.3. Otherwise, we have a non-ruled surface with nef canonical class, and we can directly use Theorem 4.3.

As we remarked in the introduction, the previous corollary involves the fundamental group \( G \) of any nonsingular projective variety by means of the usual Lefschetz hyperplane theorem.

One may be tempted to use the result of Persson [P81] on density of Chern slopes of simply connected minimal surfaces of general type in \([1/5, 2]\) as an input in Theorem 4.3 but the strategy does not work. It is not clear in that case how to find a suitable \( A_m \) which makes things work. On the top of that, and as it was said in the introduction, this cannot work in full generality since, for example, from [MP07] one can deduce that: If \( S \) is a surface of general type with \( c_1^2(S) < \frac{1}{3}c_2(S) \) and \( \pi_1(S) \) finite, then the order of \( \pi_1(S) \) is at most 9. In this way,
the question of “freedom” of fundamental groups remains open for the interval $[1/3, 1]$. We finish with two conjectures in relation to geography of Chern slopes for surfaces with ample canonical class, and for Brody hyperbolic surfaces. They could be proved through the theorems in this section if we can show that the projection $q|_{S_p}: S_p \to Y$ is a finite morphism (see Theorem 4.2). This depends on the line bundles $A_p$. Catanese proves in [Cat00, Lemma 1.1] that $q|_{S_p}$ is a finite morphism if $A_p$ is very ample. We note that in [RU15] it is proved that Chern slopes $c_1^2/c_2$ of simply connected minimal surfaces of general type are dense in $[1, 3]$, but canonical class for all the constructed surfaces was not ample, because of the presence of arbitrarily many $(-2)$-curves.

**Conjecture 4.5.** Let $G$ be the (topological) fundamental group of a nonsingular complex projective surface. Then Chern slopes $c_1^2(S)/c_2(S)$ of minimal nonsingular projective surfaces of general type $S$ with $\pi_1(S)$ isomorphic to $G$ and ample canonical class are dense in $[1, 3]$.

**Conjecture 4.6.** Let $Y$ be a Brody hyperbolic nonsingular projective surface. Then Chern slopes of hyperbolic nonsingular projective surfaces $S$ with $\pi_1(S)$ isomorphic to $\pi_1(Y)$ are dense in $[1, 3]$.

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