BEHAVIOR OF SOLUTIONS NONLINEAR
REACTION-DIFFUSION PDE’S RELATION TO DYNAMICS OF
PROPAGATION OF CANCER

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Abstract. In this paper, we propose a new mathematical model nonlinear reaction-diffusion PDE’s describing the dynamics of propagation of cancer. Here the mixed problem for the proposed PDE’s is investigated and by applying obtained results conclusions on the dynamics of propagation of cancer are drawn. These problems have nonlocal nonlinearity with variable exponents and possess special properties: these can be to remain either dissipative all time or become non-dissipative after a finite time. Here the solvability and behavior of solutions both when problems are yet dissipative and when become nondissipative are proved. It is shown that if the studied process gets become nondissipative can have various states, e.g. an infinite number of different unstable solutions with varying speeds, in addition, their propagation can become chaotic. The behavior of these solutions is analyzed in detail and it is explained how space-time chaos can arise. Investigation of this mathematics model allows explaining the dynamics of propagation of cancer, which are provided here as conclusions for each case.

1. Introduction

In this article, we introduce and study a mathematical model nonlinear reaction-diffusion equation describing the dynamics of propagation of cancer. The studied here partial differential equations (PDE’s) are nonlinear equations of parabolic type with variable exponent nonlocal nonlinearity that in addition, possess special properties. Here the mixed problem for this class of nonlinear reaction-diffusion equations is studied.

The theory of nonlinear PDE’s of parabolic type is of great interest both in itself and also as a useful mathematical model for a wide variety of important problems. This theory is widely used in understudies of various problems of hydrodynamics, the theory of nonlinear diffusion, and also understudies many problems in physics, chemistry, biology, etc. For further information on applications, the reader is referred to such sources as [1, 3, 4, 11, 13, 17, 23, 24, 30, 33, 35, 36, 37, 39, 40, 43, 45], etc..

Many biological processes at mathematical modeling are described by the nonlinear reaction-diffusion equations (reaction-diffusion-convection or advection). In addition, the mathematical modeling of the dynamics of the process of diseases due to infection also be described by the reaction-diffusion equation. It should be noted

2010 Mathematics Subject Classification. Primary 35K55, 35K57, 35Q92; Secondary 37G30, 37B45, 34C23, 92C15;

Key words and phrases. Reaction-diffusion equation, nonlocal nonlinearity, blow-up, chaos, dynamics of cancer.
that for the study of the dynamics of propagation of cancer usually were used the various problems for nonlinear reaction-diffusion equations ([1], [5], [6], [7], [8], [9], [10], [13], [14], [16], [18], [19], [20], [21], [22], [26], [28], [31], [34], [42], [44] and the references therein).

It needs to note the nonlinear parabolic PDE’s (also reaction-diffusion equations) comprise infinite-dimensional dynamical systems that exhibit an amazing spectrum of solution phenomena including traveling waves, dissipative solitons, spiral waves, target patterns, bifurcation cascades, chaos, and long-time dynamical configurations of great complexity ([3], [4], [11], [21], [29], [33], [35], [40], [42], [43], [44], [45] and the references therein). Especially interest represents an investigation of the long-time behavior of the solutions that concerns the dynamic regimes, which the system may settle into as growing the time. If the system is dissipative, there is typically a global strange attractor, but mathematically things are especially challenging when the system is non-dissipative inasmuch as the long-time dynamics can be much more diverse and complicated than in the dissipative case. It needs to note that long-time dynamics of solutions of nonlinear reaction-diffusion equations in the non-dissipative cases have many open problems. In this article, we obtain new results on the finite and long-time dynamics of classes of nonlinear reaction-diffusion equations, which provide additional mathematical details regarding blow-up and chaotic transitions. Such type questions were analyzed in the literature by many authors (see, [4], [8], [11], [21], [24], [29], [30], [33], [35], [40], [42], [43], [45] and the references therein).

Many authors for the study of the dynamics of propagation of cancer account that the mathematical model describing this process is a reaction-diffusion PDE’s. Moreover, is assumed that this model will feel the changes of the state of the cells according to time if the study will be to use the mathematics model in the form of the mixed problem with the free boundary that depends on time. According to medical investigations known that genes can either be activated or suppressed when signals stimulate receptors on the cell surface and are then transmitted to the nucleus of the cell. The reception of particular signals can induce a cell to reproduce itself in the form of identical descendants, that is the so-called clonal expansion or mitosis, or to die, that is the so-called apoptosis or programmed death (e.g. [1], [2], [5], [6], [8], [13], [16], [18], [20], [21], [26], [29], [42], [43], [45], [46], [52], [22] and the references therein).

We use reaction-diffusion PDE’s for the study of the same problem also, but our approach is different from the models were used in the above-mentioned works. Here we take account of the mentioned properties by selecting the coefficients and exponents of nonlinearity as the functions which are dependents of variables \((t, x)\). Since the changes of the state of the cells pass over time and position, one needs to use such a model that can feel these changes. According to these properties, it requires to take into account that the changes of cells are the spatiotemporal process that in particular, are arose in the appearance of cancer. Consequently, studying the properties of the solutions of these PDEs helps explain the dynamics of propagation of the natural processes under discussion in applications. The here obtained results we used to analyze the dynamics of cancer spread. In this paper, we provide some explanation of these dynamics and how the changes can spread over a long-time in various cases.

In what follows we study a mixed problem for an equation that changes from dissipative to non-dissipative reaction-diffusion PDEs with increasing time. Here
we study also the long-time dynamics of classes of nonlinear reaction-diffusion equations, which during time pass from dissipative to non-dissipative equations. Moreover, the corresponding steady-state problem passes from a uniquely solvable problem to a problem with infinitely many solutions. It is shown that the trajectories of solutions in the phase space depend on the choice of starting point on a sphere of the initial values. Therefore the behavior of solutions to this problem can be more complicated. Here can arise many variants, e.g. can be the blow-up at finite time, can exist such absorbing manifold, that the associated dynamics tend to be chaotic. This class of nonlinear reaction-diffusion equations we study for analyzing the dynamics of propagation of cancer.

In this article, we investigate a reaction-diffusion problem that over time is converted to the advection reaction-diffusion process that can have spatiotemporal chaos. So, we will study the mixed problem for the following equation as the first step of the approach to the study of the posed question

\[
\frac{\partial u}{\partial t} = \nabla \cdot \left( a(x) \nabla u \right) - b(x) u - f(t, x) \|u\|^{p_0(t)} u + g(t, x) \|u\|^{p_1(t)} u, \quad (t, x) \in \mathbb{R}_+ \times \Omega
\]

where \( \Omega \subset \mathbb{R}^n, \ n \geq 1 \) is the bounded domain with the sufficiently smooth boundary \( \partial \Omega \), \( u(t, x) \) is unknown function and

\[
(i) \quad a(x), b(x), f(t, x), g(t, x) > 0, p_0(t, \|u_0\|) \geq 0, p_1(t, \|u_0\|) \geq 0
\]

are known functions.

We presuppose that equation (1.1) is the mathematical model that describes the dynamics of propagation of cancer, on the other words, describes the corresponding process, which is necessary to investigate. Here the function \( u(t, x) \) denote the density (mass) of cells (as the normal cells), which are proliferating cells, but these states during time can be changed according to changes of cells from proliferating to infected (such as proliferating, quiescent or dead cells). It is necessary to note that \( \|u\| \) denote the density of all cells from the domain \( \Omega \) and \( p_0(t) \) denote the change of the range of normal cells, and \( p_1(t) \) denotes the change of all cells that do not receive signals from the immune system during at time \( t \), i.e. infected cells.

We should be noted the suggested here mathematical model also is the free boundary problem but it differs from the usual problems with free boundary since this model is such as the "two-phase" process where exists the separating boundary between the normal cells and the destructed cells that is the free boundary, which generates by the variable exponents \( p_0(t) \) and \( p_1(t) \). The equation (1.1) assumed if the function \( g(t, x) > 0 \) then the change begins to take place. The transformation of the cells from the normal state to the solid-state is the known process of vary, therefore the coefficients of the equation (1.1) can describe by the known functions. So, here for the immune and "infected" cells, we assume one and the same denotations since these both are the cells. It needs to note usually authors assume that the process of cancer already takes place and therefore separately designates each of the states of cells. According to this approach, the dynamics of the states of the cells are investigated by the system of equations.

In this article, we use another approach for the study of the same problem. We will study this process by using such a mathematical model that will feel the changes of the cells with time and state, which cells receive. Due to medical investigations is known that genes can either be activated or suppressed when signals stimulate
receptors on the cell surface (and then transmitted to the nucleus of the cell). These properties we take into account by selecting the coefficients and exponents as the functions which are dependent on variables \((t, x)\). The cellular scale refers to the main (interactive) activities of the cells: activation and proliferation of tumor cells and competition with immune cells. Since the cells from one state to another pass gradually, i.e. from the immune state to the activation and proliferation (interactive) state. The activities (interactiveness) of the cells after infections mainly are changed in the following sequence: activation and proliferation of tumor cells and competition with immune cells. As one can see in the works dedicated to cancer the different states of cells are usually denoted by different notations (see, e.g. mentioned above), but the offered here mathematical model feels of these different states by coefficients and exponents.

In the equation (1.1) \(a(x)\) is the coefficient of diffusion measuring the mobility of any cells (namely, the immune cells and proliferating cells) and the functions \(f(t, x)\) and \(g(t, x)\) are the recruitment rate of susceptible and degenerated cells, respectively.

So, we will study the following mixed problem with homogeneous boundary condition: the equation (1.1) with the initial and boundary conditions

\[
(1.2) \quad u(0, x) = u_0(x), \ u(t, x) \geq 0,
\]

\[
(1.3) \quad u(t, x) \mid_{\partial \Omega \times R^+} = 0, \ p_0(t) \geq 0, \ p_0(0) = p - 1.
\]

In this problem assumed that the external interferences are excluded.

Let the following conditions (ii)

\[ p_1(t) \geq 0; \ \exists t_0 > 0, \ t > t_0 \implies p_1(t) > p_0(t); \ \exists t_1 > t_0, \ t > t_1 \implies p_0(t) = 0; \]

hold, where \(t_0\) defined by equality \(t_0 = \inf \{ t \in R^+ \mid \frac{p_1(t)}{p_0(t)} > 1 \}\). Where, in general, \(p_0(t, \tau) \ \searrow\) and \(p_1(t, \tau) \ \nearrow\), \(g(t, x) \ \searrow\) if \(t, \tau \ \searrow\). \(b(x) > g(0, x) \geq 0.\) Here \(p > 0\) denotes the quantity of cells in the examined domain \(\Omega\).

In this article, we study a new class of the nonlinear reaction-diffusion PDE’s with the nonlocal nonlinearity with the variable exponents. Here we investigated the solvability and the behavior of the solutions for the considered problems both when these are yet dissipative and when these get become non-dissipative. These problems are possessed special properties: these can be to remain dissipative all time or can be to exists finite time after that these get become non-dissipative. The long-time behavior of the solutions things is especially challenging mathematically when the PDE is non-dissipative inasmuch as these dynamics can be much more diverse and complicated than in the dissipative case. Here we study the long-time behavior of the solutions that are necessary to study, as it gives provide additional details on solutions and especially regarding the chaotic cases. The dynamics becomes much more complicated in the case PDEs largely due to the formation of space-time chaotic modes (see e.g. [3, 4, 24, 29, 33, 38, 40, 42, 43, 44, 45], and the references therein). It should be noted that there have been many investigations on this and related topics (see, for example, [11, 13, 21, 33, 40, 42, 43, 44, 45] and the references therein). It should be noted that the corresponding steady-state problem of a non-dissipative nonlinear diffusion-reaction PDE has infinitely many solutions and because of which there is a bifurcation of solutions, consequently, appear the chaos (see, e.g. [?]). It is shown that if the studied problem becomes non-dissipative can arise an infinite number of different unstable solutions with varying speeds. In
this case, due to bifurcations will generate an infinite number of different states of spatio-temporal chaos.

We should be noted that in this article our main aim is to study the behavior of solutions to the posed problem, which is necessary for the study of the process of the dynamics of propagation of cancer, consequently to answer when could be the blow-up (or collapse) and what kind can be the long-time dynamics of propagation of cancer.

This article is constructed in the following way: In Section 2 the solvability of the problem \([1.1]-[1.3]\) is studied that has 3 subsections where the variants depending on the relation between \(p_0\) and \(p_1\) are studied. In Section 3 the behavior of solutions of the considered problem is studied and the blow-up of solutions under finite time is proved. In Section 4 the long-time dynamics of the solutions of the examined problem is investigated, where the chaotics of the behavior of solutions of this problem also is shown. Each section contains the conclusion about the dynamics of propagation of cancer, where is provided an explanation: what means to the studied process the obtained in this section results.

2. Solvability of problem \([1.1]-[1.3]\)

Let \(\Omega \subset \mathbb{R}^n\), \(n \geq 1\), is bounded domain with sufficiently smooth boundary \(\partial \Omega\) (at least from the Lipschitz class). By \(W^{1,2}_{0,1}(\Omega) \equiv H^1_0(\Omega)\) we denote the Sobolev space and by \(L^p(\Omega), r \geq 1\) the Lebesgue spaces.

Assume the known functions of the problem satisfy the condition \((i)\) and are bounded continuous functions, where \(Q_T = (0, T) \times \Omega, T > 0\) is a number. It is need noted \(\nabla \cdot (a \nabla u) : H^1_0(\Omega) \longrightarrow H^{-1}(\Omega)\).

Now we consider the solvability of this problem, which will be analyzed making use of the general results from \([37, 40, 38]\). We take \(u_0 \in B^H_{r_0}(0)\) \(\perp\) where \(r_0 > 0\), and study the operator \(A\) generated by the problem \([1.1]-[1.3]\): it acts, by definition, from \(X := W^{1,2}(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H^1_0(\Omega)) \cap \{u(t, x) \mid u(0, x) = u_0\}\) to \(Y = L^2(0, T; H^{-1}(\Omega))\). Consequently, as the solution of the problem \([1.1]-[1.3]\) we understand follows: a function \(u \in X\) called the solution of this problem if it satisfies this problem in the sense the space \(Y = L^2(0, T; H^{-1}(\Omega))\).

According to conditions \((i)\) and \((ii)\) the equation \([1.1]\) can changes its character from the dissipative equation to the non-dissipative, whence ensure that here is possible the following different cases, which necessary investigate separately: namely, there exist such times \(0 \leq t_0 \leq t_1 < \infty\) that \((1)\) \(p_0(t) > p_1(t), t \in [0, t_0]\); \(2)\) \(p_1(t) \geq p_0(t)\) for \(t \in [t_0, t_1]\); \(3)\) \(p_0(t) = 0\) for \(t > t_1\). Since the character of the problem \([1.1]-[1.3]\) also will changes depending on the analysed case.

2.1. Existence in the case \(p_0(t) > p_1(t)\). So, let \(p_0(t) > p_1(t)\). In the beginning it is needed to obtain a priori estimates for the possible solutions, therefore we consider the following expression

\[
\left\langle \frac{\partial u}{\partial t}, u \right\rangle = \left\langle \nabla \cdot (a \nabla u) - bu - f \left\|u\right\|^{p_0(t)} u + g \left\|u\right\|^{p_1(t)} u, u \right\rangle \implies
\]

\[1\text{Where } B^H_{r_0}(0) = \{u \in H^1_0 \mid \|\nabla u\| \leq r_0 \},\]

need to note \(\|\nabla u\| \equiv \|u\|_{H^1_0}\).
and in sequel having carried out some calculations we get
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 = -\langle a \nabla u, \nabla u \rangle + \left\langle a \frac{\partial u}{\partial v}, u \right\rangle |_{\partial \Omega} - \langle bu, u \rangle - \|u\|^{p_0(t)} \langle fu, u \rangle + \\
+ \|u\|^{p_1(t)} \langle gu, u \rangle = -\left\| a \frac{\partial u}{\partial v} \right\|_2^2 + \left\langle a \frac{\partial u}{\partial v}, u \right\rangle |_{\partial \Omega} - \\
- \left\| b \frac{\partial u}{\partial n} \right\|^2 - \|u\|^{p_0(t)} \left\| \int f \frac{\partial}{\partial t} (t) u \right\|^2 + \|u\|^{p_1(t)} \left\| \int (g) \frac{\partial}{\partial t} (t) u \right\|^2
\]
according of the boundary condition
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 = -\left\| a \frac{\partial u}{\partial v} \right\|^2 - \left\| b \frac{\partial u}{\partial n} \right\|^2 - \|u\|^{p_0(t)} \left\| \int f \frac{\partial}{\partial t} (t) u \right\|^2 + \\
+ \|u\|^{p_1(t)} \left\| \int g \frac{\partial}{\partial t} (t) u \right\|^2,
\]
where \(\langle \cdot, \cdot \rangle = \int_\Omega \partial \cdot \partial dx\), \(\|\cdot\|\) is the norm of \(L^2(\Omega) = H(\Omega)\).

For estimate of the last term in the equation (2.1) we consider the expression
\[
g(t, x) \cdot \|u\|^{p_1(t)} \text{ due of }
\]
\[
\|u\|^{p_1(t)} \|g \frac{\partial}{\partial t} (t) u \|^2 = \int_\Omega gu^2 \|u\|^{p_1(t)} dx = gu^2 \|u\|^{p_1(t)}.
\]

Hence use \(\varepsilon\)-Young inequality we have\(^2\)
\[
g \|u\|^{p_1(t)} \leq \frac{p_0 - p_1}{p_0} \left( \varepsilon \frac{p_0}{p_0 - p_1} g \frac{p_0}{p_0 - p_1} f^{- \frac{p_0}{p_0 - p_1}} \right) + \frac{p_1}{p_0} \left( \frac{p_0}{p_1 - p_0} \right) \left( \frac{g}{f} \right)^{\frac{p_0}{p_0 - p_1}}.
\]

and taking \(\varepsilon = \left( \frac{p_0}{p_1} \right)^{\frac{p_0}{p_0 - p_1}}\)
\[
(2.2) \quad g \|u\|^{p_1(t)} \leq f \|u\|^{p_0(t)} + \left[ \left( \frac{p_1}{p_0} \right)^{\frac{p_0}{p_0 - p_1}} - \left( \frac{p_1}{p_0} \right)^{\frac{p_0}{p_0 - p_1}} \right] \left( \frac{g}{f} \right)^{\frac{p_0}{p_0 - p_1}}.
\]

If denote by \(b_1(t, x)\) the following expression
\[
b_1(t, x) = b(x) - \left[ \left( \frac{p_1}{p_0} \right)^{\frac{p_0}{p_0 - p_1}} - \left( \frac{p_1}{p_0} \right)^{\frac{p_0}{p_0 - p_1}} \right] \left( \frac{g^{p_0(t)}(t, x)}{f^{p_1(t)}(t, x)} \right)^{\frac{1}{p_0(t) - p_1(t)}}
\]
then from (2.1) we derive the inequality
\[
(2.3) \quad \frac{1}{2} \frac{d}{dt} \|u\|^2 \leq -\int_\Omega a(x) \|\nabla u\|^2 dx - \int_\Omega b_1(t, x) u^2 dx
\]

Hence one need to examine 2 variants: (a) \(b_1(t, x) \geq 0\) and (b) \(b_1(t, x) < 0\). Let takes place (a) then inequation
\[
(2.4) \quad \int_\Omega a(x) \|\nabla u\|^2 dx + \int_\Omega b_1(t, x) u^2 dx \geq 0.
\]

\(^2\)We wish to note when degradation of cells begin \(p_1(t)\) remains less than \(p_0(t)\) until some time \(t_0\). By conditions take place: \(p_1(t) \nearrow\) and \(p_0(t) \searrow\) when \(t \nearrow\).
holds and we derive the following problem

\[ \frac{1}{2} \frac{d}{dt} \|u\|^2 \leq - \int_{\Omega} a(x) |\nabla u|^2 \, dx - \int_{\Omega} b_1(t, x) u^2 \, dx, \quad \|u\|^2(0) = \|u_0\|^2. \]

Assume there exist such numbers \(a_0, A_0, b_0, B_0 > 0\) that \(a_0 \leq a(x) \leq A_0\) and \(b_0 \leq b(x) \leq B_0\). Whence if we denote by \(b_1(t) = \inf \{ b_1(t, x) \mid x \in \Omega \} \) then we get the problem

\[ \frac{1}{2} \frac{d}{dt} \|u\|^2 \leq -a_0 \|\nabla u\|^2 - b_1(t) \|u\|^2, \quad \|u\|^2(0) = \|u_0\|^2. \]

Then one can affirm that solutions will be bounded. Moreover, maybe a solution will remained stable when \(t \to \infty\) if \(p_1(t)\) will remains less than \(p_0(t)\) for any \(t > 0\).

**Remark 1.** Consequently, the investigated process either doesn’t make worsen or will improve and remains in the bounded vicinity of zero when \(t \to \infty\), i.e. there exists a bounded absorbing subset in the phase space where tend all trajectories at \(t \to \infty\).

Now let takes place (b) then if we denote by \(\tilde{b}_1(t) = \sup \{ b_1(t, x) \mid x \in \Omega \} \) then we have the problem

\[ \frac{1}{2} \frac{d}{dt} \|u\|^2 \leq -a_0 \|\nabla u\|^2 + \tilde{b}_1(t) \|u\|^2, \quad \|u\|^2(0) = \|u_0\|^2. \]

In this case if

\[ a_0 \|\nabla u\|^2 \geq |\tilde{b}_1(t)| \|u\|^2 \]

then the previous assertion occurs.

Since the case of nonfulfillment of \((2.4)\) is particular case of the studied in the subsection 3, here we not will discuss on this.

Thus the following result is proved.

**Theorem 1.** Let \(p_1(t) < p_0(t)\) for \(t \in [0, t_0)\). Let all above conditions and \((2.4)\) are fulfilled.

Then the problem \((1.1)-(1.3)\) solvable and solutions remain in the bounded vicinity of the zero for \(t \to \infty\).

From this theorem and inequalities of such type as \((2.2)\) and \((2.4)\) follows

**Corollary 1.** If is fulfilled case (1), i.e. \(p_1(t) < p_0(t)\) and \(g(t, x) < f(t, x)\) for \(t \in [0, t_0)\) then the problem \((1.1)-(1.3)\) solvable and solutions remain in the bounded vicinity of the zero for \(t \to \infty\).

The above discussions shows that the solvability of the problem \((1.1)-(1.3)\) in this case follows from general results from \([36, 37, 39]\).

2.2. **Conclusion on the dynamics of cancer.** It is clear that the immune system of an organism act on infected cells to stop of propagation of the degeneration or reanimate of such cells. If the immune system is sufficiently strong then it can reanimate certain parts of degenerated cells or, at least, stop the continuation of the degeneration, i.e. can change the dynamics of propagation. Consequently, if this is possible then the exponent \(p_1(t)\) can vary and maybe, not will increase.

If the immune system isn’t sufficiently powerful then it can’t reanimate degenerate cells or stop the continuation of the degeneration, i.e. can’t change the dynamics of propagation. If the function \(p_1(t)\) continuously increases this shows that
the immune system of this human.

2.3. **Analysis of the case** \( p_1 (t) = p_0 (t) \). According to conditions \( p_0 (t) \searrow \) and \( p_1 (t) \nearrow \) when \( t \nearrow \) there exists such time \( t_0 > 0 \) that \( p_1 (t_0) = p_0 (t_0) \). In this case the relation between \( f (t, x) \) and \( g (t, x) \) can changed depending of points of \( \Omega \).

It is clear that if \( p_1 (t) = p_0 (t) \) and \( g (t, x) \leq f (t, x) \) for \( t > t_0 \) then the propagation of the possible solutions will rightly determinable as in the previous subsection, but the case \( g (t, x) > f (t, x) \) for \( t > t_0 \) is similar to the case \( p_1 (t) \geq p_0 (t) \) for \( t \geq t_0 \) therefore, we will be to study it later. But if the relation between \( g (t, x) \) and \( f (t, x) \) for \( t > t_0 \) is undetermined then maybe arise a chaos. Therefore, this case will be best to investigate together of the case \( p_1 (t) > p_0 (t) \).

2.4. **Solvability in the case** \( p_1 (t) > p_0 (t) \). Let the case (3) is fulfilled, i.e. \( p_1 (t) \geq p_0 (t) \) for \( t \geq t_0 \). If \( p_1 (t) > p_0 (t) \) then independent of the relation between \( g (t, x) \) and \( f (t, x) \) the behavior of the possible solutions generally speaking will indeterminable, as their behavior will has vary according of \( p_1 (t) \), of the initial data and the spectrum of the Laplace operator. It should be noted that according to conditions \( f (t, x), p_0 (t) \searrow \) and \( g (t, x), p_1 (t) \nearrow \) as \( t \nearrow \), therefore across some time \( t > t_1 \) will be: \( g (t, x) > f (t, x) \) and \( p_1 (t) \gg p_0 (t) \).

So, let us \( p_1 (t) > p_0 (t) \) for \( t \geq t_0 \) and again will examine (2.1), more exactly

\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 + \int_\Omega \left[ a (x) |\nabla u|^2 + b (x) u^2 + \| u \|^{p_0 (t)} f (t, x) u^2 \right] \, dx =
\]

\[
= \| u \|^{p_1 (t)} \int_\Omega g (t, x) u^2 \, dx, \quad \| u \| (t_0) = \| u_0 \|^2.
\]

The expression \( \| u \|^{p_0 (t)} f (t, x) \) one can estimate by use of the expression \( \| u \|^{p_1 (t)} g (t, x) \) like in subsection 2.1

\[
f \| u \|^{p_0 (t)} \leq \frac{p_1 - p_0}{p_1} \left( \frac{f^{p_1}}{g^{p_0}} \right)^{\frac{p_1 - p_0}{p_1}} + \frac{p_0}{p_1} g \| u \|^{p_1 (t)}.
\]

Consequently, there exist such functions \( b_1 (t, x) \) and \( g_1 (t, x) \) that the following inequality

\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 + \int_\Omega \left[ a (x) |\nabla u|^2 + b_1 (t, x) u^2 \right] \, dx - \| u \|^{p_1 (t)} \int_\Omega g_1 (t, x) u^2 \, dx \leq 0
\]

holds, where

\[b_1 (t, x) = b_1 (b, g, f, p_0, p_1); \quad g_1 (t, x) = g_1 (g, p_0, p_1).
\]

Thus, from (2.1) we derive

\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 + \int_\Omega \left[ a (x) |\nabla u|^2 + b_1 (t, x) u^2 \right] \, dx - \| u \|^{p_1 (t)} \int_\Omega g_1 (t, x) u^2 \, dx \leq 0,
\]

\[\]\[\]

\[\]\[\]The control of the dynamics of propagation of cancer will be discussed in the next paper, where will be provided and some numerical examples.
where exist due to conditions of this problem. We assume that the Laplace operator \( \Delta \) which permits to investigate of the behavior of solutions of the problem (2.7). 

So, from the Cauchy problem (2.5) we derive the Cauchy problem (2.7) for the initial data satisfying the condition (2.9). Under above conditions one can formulate the following result that follows from the general results of [36, 37, 39].

Thus we have

\[
\|u\|^2(t) \leq \exp \left\{ -2 (a_0 \lambda_1 + b_1) t - 2 g_{10} \int_0^t r(\tau) r_0(\tau) \, d\tau \right\} r_0^2,
\]

where \( r(t) = \|u\|(t) \) and \( r_0 = \|u_0\| \). Now if we assume

\[
g_{10} \left( e^{-(a_0 \lambda_1 + b_1) r_0} r_0^{p_1(t_0)} \right) < a_0 \lambda_1 + b_1 (t_0),
\]

where \( b_1 = \sup \{ b_1(t) \mid t \in (0, t_0) \} \), then from (2.9) we get

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 \leq - a_0 \|\nabla u\|^2 - b_1 \|u\|^2 + g_{10} \|u\|^{p_1(t)+2} \leq
\]

\[
- \left( a_0 \lambda_1 + b_1 - g_{10} \|u\|^{p_1(t)} \right) \|u\|^2, \quad \|u(t)\|^2_{t=t_0} = \|u(t_0)\|^2.
\]

Consequently, in this case on the solvability of the problem (1.1) one can formulate the following result that follows from the general results of [36, 37, 39].

**Theorem 2.** Let all above conditions on the problem (1.1)-(1.3) are fulfilled. Then this problem solvable for any \( u_0 \in H_{r_0}^{1,1} (0) \subset H_0^1 (\Omega) \) if \( r_0 \) satisfies the inequality \( g_{10} \left( e^{-(a_0 \lambda_1 + b_1) r_0} r_0^{p_1(t_0)} \right) < a_0 \lambda_1 + b_1 (t_0) \). Moreover the mapping (semi-flow) \( S(t) : u_{t_0} \rightarrow u(t) \) is such that \( H \) strongly \( S(t) : (B_{r_0}^{1,1} (0)) \rightarrow 0 \) as \( t \rightarrow \infty \).

2.5. **Conclusion on the dynamics of cancer.** This result shows the rate of the diffusion process in the body can help immune systems to improve the process of propagation of the beginning to degenerate cells at a certain moment after the start of the main process.

\[S_{r_0}^{1,1} (0) = \{ u \in H_0^1 \mid \|\nabla u\| = r_0 \} .\]
3. Behavior of solutions

Now we will study the case when \( p_0(t) \approx 0 \) for \( t \geq t_1 \). Moreover, since the inequality \((2.6)\) shows that if \( p_1(t) > p_0(t) \) then always instead of the equation from \((2.5)\) one can use the inequation \((2.8)\), which is sufficient for the investigation of the behavior of solutions. One can make use of the formal solution to the problem \((1.1)-(1.3)\) by taking account of the above assumption onto the known functions. But if the known functions depend at both of variable then the study of the posed problem becomes very complicated. Therefore, we will investigate the behavior of solutions of the problem in a somewhat weak form. As will be seen that such an approach is enough to receive the necessary information about the behavior of solutions.

Now we will begin to study the main questions.

3.1. Blow-up of Solutions. So, we will begin to investigate the behavior of solutions of posed problem when \( p_0(t) \approx 0 \).

Let us \( p_0(t) = 0 \) for \( t \geq t_1 \). We will examine again \((2.1)\) written in the following form

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \int _\Omega \left[ a(x)|\nabla u|^2 + (b(x) + f(t, x)) u^2 \right] dx - \|u\|^{p_1(t)} \int _\Omega g(t, x) u^2 dx = 0, \quad \|u\|^2(t_1) = \|u_{t_1}\|^2.
\]

According to conditions there exist such constants \( a_0, A_0, b_0, B_0, f_0, F_0, g_0, G_0 > 0 \) that the following inequalities

\[
a_0 \leq a(x) \leq A_0; \quad b_0 \leq b(x) \leq B_0; \quad f_0 \leq f(t, x) \leq F_0; \quad g_0 \leq g(t, x) \leq G_0
\]

hold.

Consequently, we will study the following problems

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + A_0 \|\nabla u\|^2 + (B_0 + F_0) \|u\|^2 - g_0 \|u\|^{p_1(t)+2} \geq 0
\]

and

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + a_0 \|\nabla u\|^2 + (b_0 + f_0) \|u\|^2 - G_0 \|u\|^{p_1(t)+2} \leq 0
\]

with the initial condition \( \|u\|^2(t_1) = \|u_{t_1}\|^2 \), which can present the behavior of the possible solutions of the problem \((3.1)\). More exactly, the trajectory of the possible solutions of the problem \((3.1)\) in phase space go-between of the boundary layers described by solutions of the above problems.

So, from problem \((3.2)\) and \((3.3)\) we derive the following problems

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + A_0 \|\nabla u\|^2 + B_1 \|u\|^2 - g_0 \|u\|^{p_1(t)+2} \geq 0
\]

and

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + a_0 \|\nabla u\|^2 + b_1 \|u\|^2 - G_0 \|u\|^{p_1(t)+2} \leq 0,
\]

with the initial condition \( \|u\|^2(t_1) = \|u_{t_1}\|^2 \), where \( B_1 = B_0 + F_0 \) and \( b_1 = b_0 + f_0 \).
Remark 2. The above inequalities show that it is necessary used the derivative of the function with the variable exponent, therefore here we deduce it. Consider the function \( y(t)^{-q(t)} \) then

\[
\frac{d}{dt} \left( y(t)^{-q(t)} \right) = \frac{d}{dt} \exp \left\{ -q(t) \ln y(t) \right\} = - \exp \left\{ -q(t) \ln y(t) \right\} \frac{d}{dt} \left( q(t) \ln y(t) \right) =
\]

\[
= - \exp \left\{ -q(t) \ln y(t) \right\} \left[ q'(t) \ln y(t) + \frac{q(t)}{y(t)} y'(t) \right] =
\]

\[
(3.6) = y(t)^{-q(t)} \left[ q'(t) \ln y(t) + \frac{q(t)}{y(t)} y'(t) \right].
\]

Consequently, for investigate of the inequality (3.5) we will rewrite it in the following form

\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 + \| u \|^2 \left( \frac{p_1'(t)}{2p_1(t)} \ln \| u \|^2 \right) \leq -a_0 \| \nabla u \|^2 - b_1 \| u \|^2 + G_0 \| u \|^{p_1(t)+2} + \| u \|^2 \left( \frac{p_1'(t)}{2p_1(t)} \ln \| u \|^2 \right).
\]

(3.7)

Remark 3. Consider the function \( h(z) = z^\frac{1}{2} - \ln z \) for \( z > 0 \), then we have \( \frac{d}{dz} h(z) = \frac{1}{2} z^{-\frac{1}{2}} - \frac{1}{z} = \frac{1}{z} \left( \frac{1}{2} z^{-\frac{1}{2}} - 1 \right) \). Whence for \( \frac{d}{dz} h(z) = 0 \) we get \( z = 4 \) or \( z = 0 \) is the minimum of function \( h(z) \), i.e. \( \min \{ h(z) \mid z > 0 \} = 2 - \ln 4 > 0. \) Consequently, \( h(z) > 0 \) for \( z > 0 \) (see, e.g. the proof of Lemma 9 of [41]).

Thus we get

\[
\| u \|^2 \left( \frac{p_1'(t)}{2p_1(t)} \ln \| u \|^2 \right) \leq \| u \|^2 \leq (b_0 + f_0) \| u \|^2 + c \| u \|^{p_1(t)+2}
\]

as \( p_1(t) \geq 1 \), where \( c = c(b_0, f_0, p_1) \). More exactly, use the \( \varepsilon \)-Young inequality and selecting

\[
\varepsilon = \frac{p_1'}{2p_1} \left( \frac{p_1}{p_1 - 1} \right) (b_0 + f_0) = \frac{p_1'}{2p_1} \left( \frac{p_1}{p_1 - 1} \right) b_1
\]

we obtain \( c = \frac{p_1}{p_1'} \).

Using the inequality (3.8) in (3.7) we derive the following problem

\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 + \| u \|^2 \left( \frac{p_1'(t)}{2p_1(t)} \ln \| u \|^2 \right) \leq -a_0 \| \nabla u \|^2 + (G_0 + c) \| u \|^{p_1(t)+2}
\]

with the initial condition

\[
\| u \|^2 (t_1) = S(t_1) \| u_0 \|^2 =
\]

\[
(3.10) = \exp \left\{ - \left[ a_0 \lambda_1 t_1 + \int_{t_0}^{t_1} b_{10}(s) ds + \int_{t_0}^{t_1} \left( b_1(s) - g_{10}(s) r_0^{p_1(s)} \right) ds \right] \right\} \| u_0 \|^2
\]

where \( u_0 \in H^1_0 \) is a given initial function, e.g. one can to choose it from the ball \( B_r^{H^1_0} (0) \subset H^1_0 \) with the radius \( r > 0 \). So, we study this problem for the initial function \( u_0 \) belonging to the \( B_r^{H^1_0} (0) \). In the previous section was proved
that problem has solution in appropriate space for \( t \in (0, t_1] \), consequently we can affirm the value of \( \|u\|^2(t) \) is defined on \( t_1 \) as in (3.10) with the semi-flow \( S(t) \).

Thus if one denote \( \|u\|^2(t) = y(t) \), \( p(t) = \frac{p_1(t)}{1} \) and account \( \|\nabla u\|^2 \geq \lambda_1 \|u\|^2 \) then (3.9)-(3.10) we derive the following Cauchy problem

\[
\frac{1}{2} \frac{dy}{dt} + \left( \frac{\dot{p}(t)}{p_1(t)} \right) \ln y \leq -a_0 \lambda_1 y + G_0 y^{p(t)+1}, \quad y(t_1) = y_{t_1}.
\]

here is denoted \( p(t) = \frac{p_1(t)}{2} \). Whence follows

\[
p(t) y^{-p(t)} \frac{dy}{dt} + y^{-p(t)} (\dot{p}(t) \ln y) \leq -a_0 \lambda_1 p_1(t) y^{-p(t)} + G_0 p_1(t)
\]

and consequently, we have the inequality

\[
- \frac{d}{dt} (y^{-p(t)}) \leq -a_0 \lambda_1 p_1(t) y^{-p(t)} + G_0 p_1(t)
\]

denoted by \( z(t) = y^{-p(t)} \) we get

\[
\frac{dz}{dt} \geq a_0 \lambda_1 p_1(t) z - G_0 p_1(t), \quad z(t_1) = y(t_1)^{-p(t_1)}.
\]

Hence follows

\[
z(t) \geq e^{a_0 \lambda_1 \int_{t_1}^{t} p_1(s) ds} z(t_1) - \int_{t_1}^{t} e^{a_0 \lambda_1 \int_{\tau}^{t} p_1(s) ds} G_0 p_1(\tau) d\tau.
\]

Assume \( \frac{d}{dt} P_1(t) = p_1(t) \) then we obtain

\[
z(t) \geq e^{a_0 \lambda_1 (P_1(t) - P_1(t_1))} z(t_1) + G_0 \int_{t_1}^{t} e^{a_0 \lambda_1 (P_1(t) - P_1(\tau))} p_1(\tau) d\tau G_0
\]

Consequently, we arrive

\[
z(t) \geq e^{a_0 \lambda_1 (P_1(t) - P_1(t_1))} z(t_1) + \left[ 1 - e^{a_0 \lambda_1 (P_1(t) - P_1(t_1))} \right] \frac{G_0}{a_0 \lambda_1} =
\]

\[
= \frac{G_0}{a_0 \lambda_1} - e^{a_0 \lambda_1 (P_1(t) - P_1(t_1))} \left[ \frac{G_0}{a_0 \lambda_1} - z(t_1) \right].
\]

Takes account that \( z(t) = y^{-p(t)} \) we get

\[
y^{-p(t)} \geq \frac{G_0}{a_0 \lambda_1} - e^{a_0 \lambda_1 (P_1(t) - P_1(t_1))} \left[ \frac{G_0}{a_0 \lambda_1} - y(t_1)^{-p(t_1)} \right]
\]

or

\[
y^{p(t)} \leq \frac{e^{-a_0 \lambda_1 (P_1(t) - P_1(t_1))} y(t_1)^{p(t_1)}}{\left\{ 1 - \left[ 1 - e^{-a_0 \lambda_1 (P_1(t) - P_1(t_1))} \right] \frac{G_0 y(t_1)^{p(t_1)}}{a_0 \lambda_1} \right\}}
\]

\[
= \left\{ a_0 \lambda_1 - \left[ 1 - e^{-a_0 \lambda_1 (P_1(t) - P_1(t_1))} \right] G_0 y(t_1)^{p(t_1)} \right\}.
\]

Therefore one need to investigate the following equation

\[
a_0 \lambda_1 - G_0 y(t_1)^{p(t_1)} + e^{-a_0 \lambda_1 (P_1(t) - P_1(t_1))} G_0 y(t_1)^{p(t_1)} = 0
\]
or
\[ P_1(t) = P_1(t_1) - \frac{1}{a_0\lambda_1} \ln \left( 1 - \frac{a_0\lambda_1}{G_0 y(t_1)^{p(t_1)}} \right) \]
since \( P_1(t) \) is the increasing function its inverse function \( (P_1(t))^{-1} \) exists therefore we can derive the upper bound time of the blow-up.

If one lead the above-mentioned calculations for the Cauchy problem posed for inequality (3.3) then we derive the following inequation
\[
y^{p(t)} \geq e^{-\left(a_0\lambda_1 + \tilde{B}_0\right)(P_1(t) - P_1(t_1))} \left(A_0\lambda_1 + \tilde{B}_0\right) y(t_1)^{p(t_1)} \left\{\left(A_0\lambda_1 + \tilde{B}_0\right) - \left[1 - e^{-\left(a_0\lambda_1 + \tilde{B}_0\right)(P_1(t) - P_1(t_1))}\right] G_0 y(t_1)^{p(t_1)}\right\}.\]

Consequently, we arrive to a result similar to the obtained above result.\(^5\) Whence by similar way as of the previous case we can determine the lower bound time of the blow-up. Therefore the time of blow-up \( t_{\text{col}} \) of the main problem can be found between these times that is a finite since \( p \) is finite number, and \( P_1(t) \) is a continuous function and satisfies condition \( P_1(t) \leq p - 1 \).

Accordingly, we have proved the following result.

**Theorem 3.** Let us the initial function \( u_0 \in H^1_0 \) is such that the condition
\[
\left|\alpha^+ \nabla u(t_1)\right|^2 - \left|g^+(t_1) u(t_1)\right|^2 \|u(t_1)\|^{p(t_1)} < 0
\]
is fulfilled. Then each solution \( u \in X \) of the problem (1.1) - (1.3) has blow-up in \( H \) a finite time.

**Notation 1.** Whence one can derive condition on \( u(0) \) using of the adduced above definition of \( u(t_1) \).

### 3.2. **Conclusion on the dynamics of cancer.** The result of Theorem 3 shows the following case is possible: There exist such initial dates that if the process of the dynamics of propagation of cancer starts with such initial values then beginning on some time almost everywhere in the part of the body where take place the process will non remain of the normal cells. Consequently, one can count that the treatment already is impossible. Whence follows that is necessary exterior interference, moreover during and before of the critic moment. In the case when all conditions of this theorem are fulfilled then according to its result one can define the approximate time when will be already late. We should note in this case also is necessary to take account of Conclusion 2.2.

### 4. **LONG-TIME BEHAVIOR OF SOLUTIONS**

Consider the equation (1.1) in the case when \( p_0(t) = 0 \)
\[
\frac{\partial u}{\partial t} = \nabla \cdot \left( a \left( x \right) \nabla u \right) - \left[ b \left( x \right) + f \left( t, x \right) \right] u + g \left( t, x \right) \|u\|^{p_1(t)} u.
\]

For study of the behavior of solutions we will investigate the following problem
\[
\frac{\partial u}{\partial t} = \tilde{a} \Delta u - \left[ \tilde{b} + \tilde{f} \right] u + \tilde{g} \|u\|^{p_1(t)} u, \quad u(t_1) = u_{t_1},
\]

\(^5\)Consequently, if we wish to obtain such a result as in the above section then necessary to select the initial function in the appropriate way as in the above section.
where instead of the functions \(a, b, f\) and \(g\) set their mean values that denoted by \(\bar{a}, \bar{b}, \bar{f}\) and \(\bar{g}\), e.g. \(\bar{a} = \frac{1}{\text{mes} \Omega} \int_{\Omega} a(x) \, dx\), in addition we denote \(\bar{b} + \bar{f}(t)\) by \(\hat{b}(t)\). We would be noted the function \(p_1(t)\) increases at \(t \not\to\) and converges to the value \(p - 1\) that was defined in the introduction.

It should be noted according to the results of [12, 36] on the differential operators of elliptic type, a study of the behavior of solutions of the problem for the equation (4.1) will give the sufficient informations about the behavior of solutions and also for the main problem.

So, for the study of the behavior of solutions to the problem (4.1) consider the following problem

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \bar{a} \|\nabla u\|^2 + \bar{b}(t) \|u\|^2 - \bar{g}(t) \|u\|^{p_1(t)+2} = 0, \tag{4.2}
\]

\[
\|u(t_1)\|^2 = \|u_{t_1}\|^2, \quad \text{where} \quad \|u(t_1)\|^2 = S(t_1) \|u_0\|^2
\]

the semi-flow \(S(t)\) defined in (3.10).

The examined here problem one can compare the studied in the article [40]. It isn’t difficult to see that these problems are different essentially since in the examined problem, unlike the above-mentioned, coefficients and exponents depend on the independent variables, which create additional difficulties.

We will use the norm \(\|\nabla u\|\) for the norm of the space \(H_0^1(\Omega)\) due to the equivalence \(\|u\|_{H_0^1(\Omega)} = \|\nabla u\|\) for \(\forall u \in H_0^1(\Omega)\). Here for simplicity we assume the domain \(\Omega\) in a geometric sense is such that Laplace operator \(-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)\) has only a point spectrum, i.e. \(\sigma(-\Delta) = \sigma_{p_1}(-\Delta) \subset (0, \infty)\) and we will denote of these eigenvalues of the Laplace operator \(-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)\) by

\[
\lambda_k, \quad k = 1, 2, \ldots; \quad \sigma_{p_1}(-\Delta) = \{\lambda_j \mid j \in \mathbb{N}\},
\]

and the corresponding eigenfunctions by

\[
w_k, \quad k = 1, 2, \ldots; \quad \{w_k\}_{k=1}^{\infty} \subset H_0^1(\Omega).
\]

Now we will study the inverse mapping of the operator \(\hat{A}\) generated by this problem that generated similarly as the inverse mapping of the operator \(A\) of the problem (4.2) - (4.3) (i.e. the mapping \(\hat{S}(t)\) is defined as the mapping \(S(t)\) in Section 2).

Moreover, we assume that the eigenfunctions and adjoint eigenfunctions are total (complete) in the space \(H_0^1(\Omega)\) and also in the dual space, respectively; in addition, we assume without loss of generality that they generate an orthogonal basis in these spaces, respectively.

We introduce the following denotation

\[
\inf \left\{ \lambda_k \in \sigma_{p_1}(-\Delta) \mid \bar{a}\lambda_k + \bar{b}(t_1) > \bar{g}(t) \lambda_k^{p_1(t_1)}, \quad k = 1, 2, \ldots \right\} = \lambda_{k_0},
\]

and use the usual representation of the space \(H_0^1(\Omega)\) ([3, 34, 40, 33, 45]) in the form

\[
H_0^1(\Omega) = H_{k_0} + H_{-k_0},
\]

where the subspace \(H_{k_0} \subset H_0^1(\Omega)\) is the span over \(\{w_k\}_{k=1}^{k_0-1}\) and has dimension \(\dim H_{k_0} = k_0 - 1\) and \(H_{-k_0}\) is a subspace of \(\text{codim} H_{-k_0} = k_0 - 1\).

Let \(Q_{k_0}\) and \(P_{k_0}\) are the projections: \(P_{k_0} : H_0^1(\Omega) \to H_{k_0} \subset H_0^1(\Omega)\) and \(Q_{k_0} : H_0^1(\Omega) \to H_{-k_0} \subset H_0^1(\Omega)\), giving rise to the splitting \(u \equiv Q_{k_0}u + P_{k_0}u\). (This is well known decomposition of the Hilbert space, see e.g., [34, 40, 33, 29], etc.).
for some \( \delta > t \). Indeed, if \( \| \cdot \|_{H^1_0} \) increasing function, by virtue of the inequality:

\[
\hat{g}(t) \| u \|^p(t) \| P_{k_0} u \|^p(t) = 0,
\]

(4.4)

\[
\frac{\partial}{\partial t} P_{k_0} u - \hat{\Delta} P_{k_0} u + \hat{b}(t) P_{k_0} u - \hat{g}(t) \| u \|^p(t) P_{k_0} u = 0,
\]

(4.5)

\[
\frac{\partial}{\partial t} Q_{k_0} u - \hat{\Delta} Q_{k_0} u + \hat{b}(t) Q_{k_0} u - \hat{g}(t) \| u \|^p(t) Q_{k_0} u = 0,
\]

(4.6)

\[
Q_{k_0} u (t_1, x) = Q_{k_0} u_{t_1}(x) \in H_{k_0} \subset H^1_0(\Omega),
\]

(4.7)

As our aim is the investigation of the behavior of solutions of the problem under the condition \( u_{t_1} \in B_{r(t_1)}^{H^1_0(\Omega)}(0) \) then it is enough to assume that \( \hat{\Delta}_{k_0-1} + \hat{b}(t_1) < \hat{g}(t_1) r(t_1)^{p_1(t_1)} < \hat{\Delta}_{k_0} + \hat{b}(t_1) \). Then for the problem (4.2) - (4.3) we obtain

\[
0 = \frac{1}{2} \frac{d}{dt} \| P_{k_0} u \|^2(t) + \hat{a} \| \nabla P_{k_0} u \|^2(t) + \hat{b}(t) \| P_{k_0} u \|^2(t) - \hat{g}(t) \| u \|^p(t) \| P_{k_0} u \|^p(t)
\]

(4.8)

\[
\langle P_{k_0} u, P_{k_0} u \rangle_{t=t_1} = \| P_{k_0} u \|^2(t_1) = \| P_{k_0} u_{t_1} \|^2.
\]

Whence, it follows that the solution \( \| u \|(t) \) and exponent \( p_1(t) \) grow and for some \( t_2 > t_1 \) for \( t \in [t_1, t_2] \) we have

\[
\hat{g}(t) \| u \|^p(t)(t) \leq \hat{g}(t_1) r(t_1)^{p_1(t_1)} + \delta < \hat{\Delta}_{k_0} + \hat{b}(t_1),
\]

for some \( \delta > 0 \), by virtue of the inequality: \( \hat{g}(t_1) r(t_1)^{p_1(t_1)} - (\hat{\Delta}_{k_0-1} + \hat{b}(t_1)) > 0 \). Indeed if \( \| u_{t_1} \| = r(t_1) \), then we have from (4.8) - (4.9) that

\[
\frac{d}{dt} \| P_{k_0} u \|^2(t) + 2 \hat{a} \| \nabla P_{k_0} u \|^2(t) + 2 \hat{b}(t) \| P_{k_0} u \|^2(t) \geq 0
\]

and we get the inequality

\[
\| P_{k_0} u \|^2(t) \geq \exp \left\{ -\int_{t_1}^t \left( \hat{\Delta}_{k_0-1} + \hat{b}(s) - \hat{g}(s) r^{p_1(s)}(t_1) \right) ds \right\} \| P_{k_0} u_{t_1} \|^2.
\]

(4.10)

Consequently if \( \| Q_{k_0} u \|^2(t) \leq \epsilon < \delta < r_0 \) for some enough small \( \epsilon > 0 \) and \( t \in [t_1, t_2] \), then the solution of problem (4.8) - (4.9) exists and is an exponentially increasing function, by virtue of the inequality: \( \hat{g}(t_1) r(t_1)^{p_1(t_1)} - (\hat{\Delta}_{k_0} + \hat{b}(t_1)) < 0 \).

Unlike of the above problem, for the problem (4.6) - (4.7) we obtain

\[
\frac{d}{dt} \| Q_{k_0} u \|^2(t) + 2 \hat{a} \| \nabla Q_{k_0} u \|^2(t) + 2 \hat{b}(t) \| Q_{k_0} u \|^2(t) \leq 0
\]

\[
\langle Q_{k_0} u, Q_{k_0} u \rangle_{t=t_1} = \| Q_{k_0} u \|^2(t_1) = \| Q_{k_0} u_{t_1} \|^2.
\]
whence we get the inequality
\[
\|Q_{k0}u\|_2^2(t) \leq \exp \left\{-2 \int_{t_1}^{t} \left( \tilde{\alpha} \lambda_{k0-1} + \tilde{b}(s) - \tilde{g}(s) r_{11}^{p_1(s)} \right) ds \right\} \|Q_{k0}u_{t_1}\|_2^2.
\]

The inequality (4.11) shows that the solution of the problem (4.6) - (4.7) exists and is an exponentially decreasing function, since \( \tilde{g}(t_1) r(t_1)^{p_1(t_1)} - \tilde{\alpha} \lambda_{k0} + \tilde{b}(t_1) < 0 \). Consequently for \( \|P_{k0}u_{t_1}\| + \|Q_{k0}u_{t_1}\| = r(t_1) \) if \( \|P_{k0}u_{t_1}\| < \|Q_{k0}u_{t_1}\| \) is enough small, then the solution \( \|u\|_{(t)} \) exists and is an increasing function up to some time.

Now we will assume the system of eigenfunctions \( \{w_k\}_{k=1}^{\infty} \subset H_0^1(\Omega) \) is an orthonormal basis of this space that can give possible for detail to investigate of the behavior of solutions. In this case each function \( u(t, x) = \sum_{k=1}^{\infty} u_k(t) w_k(x) \) and consequently, study of the problem (4.12) - (4.13) is equivalent to studying the system of equations
\[
\frac{1}{2} \frac{d}{dt} |u_k(t)|^2 + \left( \tilde{\alpha} \lambda_k + \tilde{b} \right) |u_k(t)|^2 - \tilde{g}(t) \left( \sum_{i=1}^{\infty} |u_k(t)|^2 \right)^{p_1(t)} |u_k(t)|^2 = 0, \quad k = 1, 2, ...
\]
with the initial conditions
\[
|u_k(t_1)|^2 = |u_{t_1, k}|^2, \quad k = 1, 2, ...
\]

From mentioned above reasoning follows that for \( u(t_1, x) \in B_{r(t_1)}^{H_0^1(\Omega)}(0) \) and \( \|u_{t_1}\| \leq r(t_1) \) will be fulfill
\[
\|u\|_{(t)}^2(t) = \left( \sum_{i=1}^{\infty} |u_i(t)|^2 \right) \leq r(t_1)^2 + \varepsilon,
\]
where \( t = t(\varepsilon, r(t_1), p_1(t_1)) > t_1 \) and \( \delta > 0 \) are sufficiently small. It is known that in this case \( |u_k(t)|^2 \) increases for \( k = 1, 2, ..., \tilde{k}_0 < k_0 \) and decreases for \( k = k_0, k_0+1, ... \) depending on the relationship between \( \|u_{t_1}\|_{p_1(t_1)} \) and \( \lambda_k \).

Consequently, it is need to investigate the behavior of \( |u_k(t)| \) for each \( k = 1, 2, ..., \). Assume \( u_{t_1} \in H_0^1(\Omega) \) and \( \|u_{t_1}\| \equiv r(t_1) > 0 \). Let us list all of possible cases: 1) \( \tilde{g}(t_1) r(t_1)^{p_1(t_1)} < \tilde{\alpha} \lambda_1 + \tilde{b}(t_1) \); 2) \( \exists \lambda_{k0} : \tilde{\alpha} \lambda_{k0-1} + \tilde{b}(t_1) < \tilde{g}(t_1) r(t_1)^{p_1(t_1)} < \tilde{\alpha} \lambda_{k0} + \tilde{b}(t_1) \) and 3) \( \exists \lambda_{k0} : \tilde{g}(t_1) r(t_1)^{p_1(t_1)} = \tilde{\alpha} \lambda_{k0} + \tilde{b}(t_1) \). Case 1) was already investigated, therefore we will consider here only cases 2) and 3).

Consider either the case 2) or 3), i.e.
\[
\exists \lambda_{k0} : \tilde{\alpha} \lambda_{k0-1} + \tilde{b}(t_1) < \tilde{g}(t_1) r(t_1)^{p_1(t_1)} < \tilde{\alpha} \lambda_{k0} + \tilde{b}(t_1)
\]
and \( \exists \lambda_{k0} : \tilde{g}(t_1) r(t_1)^{p_1(t_1)} = \tilde{\alpha} \lambda_{k0} + \tilde{b}(t_1) \).

So, we will investigate the Cauchy problem for the following system of equations
\[
\frac{1}{2} \frac{d}{dt} |u_k(t)|^2 + \left( \tilde{\alpha} \lambda_k + \tilde{b}(t) \right) |u_k(t)|^2 - \tilde{g}(t) r(t)^{p_1(t)} |u_k(t)|^2 =
\]
such that the cases $k_u$ with the initial conditions $u (t, x) \equiv \sum_{k=1}^{\infty} u_k (t) w_k (x)$. It is easy to see that this system of equations are such that the cases $k \geq k_0, k \leq k_0 - 1$ and $k = k_0$ is necessary to study separately after which to investigate the fact that they will be to give when these are together.

Formally, we can determine the solution of each equation from (4.14) to be

$$|u_k (t)|^2 = |u_{t_k}|^2, k = 1, 2, \ldots,$$

with the initial conditions

$$|u_k (t_1)|^2 = |u_{t_{1k}}|^2, k = 1, 2, \ldots,$$

where $u (t, x) \equiv \sum_{k=1}^{\infty} u_k (t) w_k (x)$. It is easy to see that this system of equations are such that the cases $k \geq k_0, k \leq k_0 - 1$ and $k = k_0$ is necessary to study separately after which to investigate the fact that they will be to give when these are together.

Formally, we can determine the solution of each equation from (4.14) to be

$$|u_k (t)|^2 = \exp \left\{ -2 \int_{t_1}^{t} \left( \hat{a} \lambda_k + b \left( s \right) - \hat{g} (s) r (s) p_1 (s) \right) ds \right\} |u_{t_{1k}}|^2.$$

So, let us $k \leq k_0 - 1$ then in the case 2), it follows from (4.15) that

$$|u_k (t)|^2 = \exp \left\{ -2 \int_{t_1}^{t} \left( \hat{a} \lambda_k + b \left( s \right) - \hat{g} (s) r (s) p_1 (s) \right) ds \right\} |u_{t_{1k}}|^2 \geq \exp \left\{ 2 \left( \hat{g} (t_1) r (t_1) p_1 (t_1) - \hat{a} \lambda_k - b (t_1) \right) \right\} |u_{t_{1k}}|^2,$$

for $1 \leq k \leq k_0 - 1$ and some $t > t_1$ due to $\hat{g} (t_1) r (t_1) p_1 (t_1) > \hat{a} \lambda_k + b (t_1)$ and $p_1 (t) > \hat{g} (s) r (s) p_1 (s)$. Consequently, $|u_k (t)|$ increases for each $k : 1 \leq k \leq k_0 - 1$ that lead to the increase of $r (t)$ as long as $\| P_{k_{t_1}} u_{t_1} \|$ is enough greater than $\| Q_{k_{t_1}} u_{t_1} \|$.

For the case 3), i.e. when there exists $k = k_0$ such that $\hat{g} (t_1) r (t_1) p_1 (t_1) = \hat{a} \lambda_k + b (t_1)$ then one has

$$|u_{k_0} (t)|^2 = \exp \left\{ -2 \int_{t_1}^{t} \left( \hat{a} \lambda_{k_0} + b \left( s \right) - \hat{g} (s) r (s) p_1 (s) \right) ds \right\} |u_{t_{1k_0}}|^2$$

according to (4.15). If we denote here the function $\rho (t) = \hat{a} \lambda_{k_0} + b (t) - \hat{g} (t) r (t) p_1 (t)$ then $\rho (t_1) = 0$, but in general the vary of function $\rho (t)$ is not known. Consequently, it is impossible to obtain a result about variation of a solution to the equation (4.14), as the behavior of the function $\rho (t)$ is not known. As one can see in the sequel, the behavior of $\rho (t)$ depends on the geometrical location of the initial data $u_t$ on spheres $S^{H_{k_0}}_r (0), \ 0 < r \leq r (t_1)$.

As the functional $\rho (t)$ depends on $r (t) p_1 (t), r^2 (t) = \| u \|^2 (t)$ and $\lambda_k$ it is clear that in order to study the behavior of the functional $\rho (t)$ one should investigate both of functionals $\| P_{k_{t_1}} u \|$ and $\| Q_{k_{t_1}} u \|$. Clearly, in the case 2) the functional $\| P_{k_0} u \|$ increases, and $\| Q_{k_0} u \|$ decreases at $t > t_1$ at least near zero according to (4.3) and (4.11). Using the orthogonal splitting $u = P_{k_0} u + Q_{k_0} u$, we also find that

$$\| u \|^2 = \| P_{k_0} u \|^2 + \| Q_{k_0} u \|^2.$$

Whence follows, the behavior of the function $\| u \|^2 (t)$ depends on the relationship between the values $\| P_{k_0} u_{t_1} \|$ and $\| Q_{k_0} u_{t_1} \|$. Let

$$\| u_{t_1} \| \equiv r (t_1); \ \hat{a} \lambda_{k_0-1} + b (t_1) < \hat{g} (t_1) r (t_1) p_1 (t_1) < \hat{a} \lambda_{k_0} + b (t_1)$$
and consider above equality, i.e.

\[(4.16) \quad \|u\|^2(t) = \|P_{k_0}u\|^2(t) + \|Q_{k_0}u\|^2(t) = \sum_{k \leq k_0} |u_k(t)|^2 + \sum_{k > k_0} |u_k(t)|^2.\]

It is necessary to study the following cases:

a) \[u_{t_1} = \sum_{k \leq k_0} u_{t_1,k}w_k \in P_{k_0}(H^1_0(\Omega)) \equiv H_{k_0};\]

b) \[u_{t_1} = \sum_{k \geq k_0} u_{t_1,k}w_k \in Q_{k_0}(H^1_0(\Omega)) \equiv H_{-k_0}\]

and c) \[u_{t_1} = \sum_{k \geq k_0} u_{t_1,k}w_k, \text{ when } c_1) \|Q_{k_0}u_{t_1}\| < \|P_{k_0}u_{t_1}\| \text{ and } c_2) \|Q_{k_0}u_{t_1}\| \geq \|P_{k_0}u_{t_1}\|, \text{ separately.}\]

In the case a) we have

\[|u_k(t)|^2 = \exp\left\{-2 \int_{t_1}^t \left(\hat{\alpha}\lambda_k + \hat{b}(s) - \hat{g}(s) r(s)^{p_1(s)}\right) ds\right\} |u_{t_1,k}|^2\]

for any \(k = 1, \ldots, k_0 - 1\), thus, \(u_k(t) = 0\) for \(k \geq k_0\) since \(\|u_{t_1}\|^2 = r(t_1)^2 = \sum_{k \leq k_0 - 1} (u_{t_1,k})^2\) and \(r(t)^2 = \sum_{k \geq k_0 - 1} |u_k(t)|^2\). In this case \(\hat{\alpha}\lambda_k + \hat{b}(t) - \hat{g}(t) r(t)^{p_1(t)} < 0\) as \(\hat{g}(t) r(t)^{p_1(t)} \geq \hat{\alpha}\lambda_k + \hat{b}(t)\) for each \(k = 1, \ldots, k_0 - 1\) and \(p_1(t)\), and also \(r(t)\) increase as \(t \uparrow \infty\).

In the case b) we have \(\|u_{t_1}\|^2 = r(t_1)^2 = \sum_{k \geq k_0} (u_{t_1,k})^2\), i.e. \(u_{t_1} \in Q_{k_0}(H) \equiv H_{-k_0}\). Consequently,

\[|u_k(t)|^2 = \exp\left\{-2 \int_{t_1}^t \left(\hat{\alpha}\lambda_k + \hat{b}(s) - \hat{g}(s) r(s)^{p_1(s)}\right) ds\right\} |u_{t_1,k}|^2\]

for any \(k \geq k_0\), giving rise to \(u_k(t) \searrow 0\) for \(k = 1, 2, \ldots, k_0 - 1\) and \(r(t)^{p_1(t)}\) decreases although \(p_1(t) \nearrow t \uparrow \infty\) for each \(k \geq k_0\) due to assumption that \(\hat{\alpha}\lambda_k + \hat{b}(s) - \hat{g}(s) r(s)^{p_1(s)} > 0\) as \(\hat{g}(t) r(t)^{p_1(t)} < \hat{\alpha}\lambda_k + \hat{b}(t)\). Thus, from continuity we obtain that the inequality \(\|u\|^2(t) = r(t) < r(t_1)\) and \(k \geq k_0\) is fulfilled for all \(t > t_1\) and for any solution to the problem

\[\frac{1}{2} \frac{d}{dt} |u_k(t)|^2 + \left(\hat{\alpha}\lambda_k + \hat{b}(t) - \hat{g}(t) r(t)^{p_1(t)}\right) |u_k(t)|^2 = 0,\]

\[|u_k(t_1)|^2 = |u_{t_1,k}|^2.\]

It should be noted above in this section essentially was discussed the case c) that is general case, since here taken into account both side of the following inequality

\[\hat{\alpha}\lambda_k + \hat{b}(t) > \hat{g}(t) r(t)^{p_1(t)} > \hat{\alpha}\lambda_{k_0 - 1} + \hat{b}(t_1).\]

Unlike of above cases here it is needed to consider the space decomposition \(H \equiv H_{k_0} \oplus H_{-k_0} \equiv H^1_0(\Omega) \quad (P_{k_0}(H) \equiv H_{k_0}, \ Q_{k_0}(H) \equiv H_{-k_0}),\) which is necessary for analyzes the intrinsic behavior of the corresponding solutions. Thus, using the
decomposition of the space \( H = H_0^1(\Omega) \) for the \( r^2(t_1) = \| u^-_t \|_H^2 \) we have the representation \( r^2(t_1) = \| u^+_{t_1} \|_{H_0^1}^2 + \| u^-_{t_1} \|_{H_{-k_0}}^2 \), and in the vector form
\[
H \equiv \{ u = (u^+_{k_0}, u^-_{k_0}) : u^+_{k_0} \in H_{k_0}, u^-_{k_0} \in H_{-k_0} \}.
\]

Then the following proposition follows directly from the above discussion.

**Theorem 4.** Under the above conditions if \( \| u^+_{t_1} \|_{H_{-k_0}}^2 > \| u^-_{t_1} \|_{H_{-k_0}}^2 \) and if the rate of decrease of the norm \( \| u^-_{k_0} (t) \|_{H_{-k_0}}^2 \) is greater than the rate of decrease of the norm \( \| u^+_{k_0} (t) \|_{H_0^1}^2 \), then there exists \( \tilde{t} > t_1 \) such that \( |u_k(t)| \) decreases for \( t \geq \tilde{t} \). On the other hand, if \( \| u^+_{t_1} \|_{H_{-k_0}}^2 < \| u^-_{t_1} \|_{H_{-k_0}}^2 \) and the rate of increase of the norm \( \| u^+_{k_0} (t) \|_{H_0^1}^2 \) is greater than the rate of decrease of \( \| u^-_{k_0} (t) \|_{H_{-k_0}}^2 \), then there exists \( \tilde{t} > t_1 \) such that \( |u_k(t)| \) increases for \( t \geq \tilde{t} \). Here the increases of the function \( p(t) \) loosen only somewhat of the corresponding behaviors of the solutions.

**Remark 4.** This shows how is to control in order to change of the behavior of the dynamics of cancer, i.e. to which coefficients or exponents to control is necessary.

Now we will proceed to investigate in detail the behavior of solutions beginning with the time \( t_1 \) when \( p_0(t) = 0 \) (i.e. when the degregated cells remain only). If one selects an initial function \( u_0 \in H \) then into time \( t_1 \) we have the function
\[
\| u_{t_1} \|^2 = r^2(t_1) = \exp \left\{ - \left[ a_0 \lambda t_1 + \int_0^{t_0} b_{10}(s) \, ds + \int_{t_0}^{t_1} \left( b_1(s) - g_{10}(s) (r) p_1(s) \right) \, ds \right] \right\} \| u_0 \|^2
\]
according of the expression (3.11). So, it isn’t difficult to see that arbitrarily chosen initial function \( u_0 \) in the moment \( t_1 \) will be \( u_{t_1} \) with the norm determined by the number \( r(t_1) \) that will satisfies the inequalities
\[
\tilde{a} \lambda_{k_0} + \tilde{b}(t_1) \geq \tilde{g} (t_1) r(t_1) p_1(t_1) > \tilde{a} \lambda_{k_0-1} + \tilde{b}(t_1)
\]
for some \( \lambda_{k_0} \). As \( H = H_{k_0} \oplus H_{-k_0} \) therefore each \( u \in H \) has the representation by decomposition \( u(t) = u^+(t) + u^-(t) \), where \( u^+(t) \in H_{k_0} \) and \( u^-(t) \in H_{-k_0} \) for any \( t > 0 \). As show the above discussions it suffices to study of the behavior of solutions beginning with the cases that depend of the relations between \( \| u_{t_1} \|_{H_{-k_0}} \) and \( \| u^+_{t_1} \|_{H_{k_0}} \). Indeed if we set \( \| u_{t_1} \| = r(t_1) \) and \( r(t_1) \) satisfies the inequality
\[
\tilde{a} \lambda_{k_0} + \tilde{b}(t_1) > \tilde{g} (t_1) r(t_1) p_1(t_1) > \tilde{a} \lambda_{k_0-1} + \tilde{b}(t_1),
\]
then each of solutions to the problem (4.2) - (4.3) satisfies one of the following statements:

1. If \( u_{t_1} \) lies in \( H_{k_0} \) or in a small neighborhood of the subspace \( H_{k_0} \), then \( |u_k(t)| \to 0 \) at \( t \to \infty \) for \( k = 1, k_0 - 1 \), moreover since in this case \( r(t) \) and \( p_1(t) \) increase gradually and so in time \( \tilde{g} (t) r(t) p_1(t) \) is greater than each \( \tilde{a} \lambda_k + \tilde{b}(t) \) for \( k \geq k_0 \); i.e. in this case \( |u_k(t)| \) gradually increases for all \( k \);
2. If \( u_{t_1} \) lies in \( H_{-k_0} \) or in a small neighborhood of the subspace \( H_{-k_0} \), then \( |u_k(t)| \gtrsim 0 \) at \( \tau \to \infty \) for \( k \geq k_0 \), moreover since in this case \( r(t) \) decreases although \( p_1(t) \) increases gradually so that in time \( \hat{g}(t) r(t)^{p_1(t)} \) is less than each \( \tilde{a} \lambda_k + \tilde{b}(t) \); i.e. in this case \( |u_k(t)| \) gradually decreases for all \( k \);

3. If \( u_{t_1} \in H \) such that \( \|u_{t_1}^\pm\|_{H_{-k_0}} \approx \|u_{t_1}^\pm\|_{H_{k_0}} \), then there is a relation between of \( u_{t_1}^\pm(t) \) and \( u_{t_1}^\pm \) such that the behavior of the \( u_k(t) \) is chaotic for all \( k \) for which \( u_k(t_1) \neq 0 \).

4. If \( u_{t_1} \in H \) such that \( \|u_{t_1}^-\|_{H_{-k_0}} \) and \( \|u_{t_1}^+\|_{H_{k_0}} \) are different, then the behavior of solutions impossible to define, since its can be arbitrary due to the coefficients and the exponent.

Since the claims 1 and 2 were studied above, remains to study the claim 3 and the claim 4. We will use again of the representation of the formal solutions (4.15) to the problem (4.4)

\[
|u_k(t)|^2 = \exp \left\{ -2 \left( \int_{t_1}^{t} \left( \tilde{a} \lambda_k + \tilde{b}(s) - \tilde{g}(s) r(s)^{p_1(s)} \right) ds \right) \right\} |u_{t_1}^\pm|^2,
\]

\( k = 1, 2, \ldots \). It is known that \( |u_k(t)| \) increases for \( k : 1 \leq k \leq k_0 - 1 \) and decreases for \( k \geq k_0 \) by virtue of Theorem 4 depending on the difference \( \tilde{a} \lambda_k + \tilde{b}(t_1) - \tilde{g}(t_1) r(t_1)^{p_1(t_1)} \). Since the \( \hat{g}(t) r(t)^{p_1(t)} \) grows therefore will be to take place \( \tilde{a} \lambda_k + \tilde{b}(t) = \hat{g}(t) r(t)^{p_1(t)} \) for some \( k \) and \( t > t_2 \) in this case will appear the bifurcation that bring to the appear of the chaos. Moreover as \( r_{k_0}^2(t) \equiv \|u_{k_0}^-\|_{H_{-k_0}}^2(t) + \|u_{k_0}^+\|_{H_{-k_0}}^2(t) \) holds near \( t = t_1 \), \( \hat{g}(t) r(t)^{p_1(t)} \) can change depending on the behavior of \( |u_k(t)|, \hat{g}(t) \) and \( p_1(t) \) as \( k \geq k_0 \) and \( k \leq k_0 - 1 \) vary consequently, the corresponding decomposition of \( H \) will change, i.e. \( \hat{g}(t) r(t)^{p_1(t)} \) can become greater than \( \tilde{a} \lambda_k + \tilde{b}(t) \) or less than \( \tilde{a} \lambda_{k-1} + \tilde{b}(t) \). Thus, the variation of \( \rho(t) \) and \( r(t) \), and also their coupling are very complicated, as it depends on relations among the behaviors of \( u_0(t) \) in the case when \( k \geq k_0 \) or \( k \leq k_0 - 1 \), and also when change \( k_0 \), which may give rise to chaos.

Now we will investigate the claim 4, i.e. the behavior of solutions of the problem (4.12) - (4.13) when \( u_{t_1} \in H \) such that \( \|u_{t_1}^\pm\|_{H_{-k_0}} \) and \( \|u_{t_1}^\mp\|_{H_{k_0}} \) are different moreover their relations in general not are such as in the claim 1 and claim 2. In the other words we will study properties of the resolving operator corresponding to the problem (4.12) - (4.13). Toward this end, we consider again to the following system of differential equations

\[
\frac{1}{2} \frac{d}{dt} \langle u(t), w_k \rangle + \langle \tilde{a} \nabla u(t), \nabla w_k \rangle + \tilde{b}(t) \langle u(t), w_k \rangle - \hat{g}(t) \|u(t)\|^{p_1(t)} \langle u(t), w_k \rangle = 0,
\]

where \( \{w_k(x)\}_{k=1}^{\infty} \) are eigenfunctions of the Laplacian \(-\Delta\) in \( H_0^1(\Omega) \) corresponding to the eigenvalues \( \{\lambda_k\}_{k=1}^{\infty} \), respectively, by virtue of the imposed conditions.

Whence, it follows that

\[
(4.17) \quad \frac{1}{2} \frac{d}{dt} u_k(t) + \left( \tilde{a} \lambda_k + \tilde{b}(t) \right) u_k(t) - \hat{g}(t) \|u(t)\|^{p_1(t)} u_k(t) = 0, \quad k = 1, 2, \ldots
\]

So, for the study of the behavior of solutions of the problem (4.12) - (4.13) under the condition \( u_{t_1} \in S_{r_0^2}^{H_0^1(\Omega)}(0), \tilde{a} \lambda_{k_0-1} + \tilde{b}(t) \leq \hat{g}(t) r_{k_0-1}^{p_1(t)} < \tilde{a} \lambda_{k_0-1} + \tilde{b}(t) \) it is necessary to study the Cauchy problems in the case when \( k = 1, 2, \ldots, k_0 - 1, \) and
next the case \( k \geq 1 \). In the beginning we consider the following Cauchy problem:

(4.18) \[ \frac{1}{2} \frac{d}{dt} u_k(t) + \left( \tilde{\alpha} \lambda_k + \tilde{b}(t) \right) u_k(t) - \tilde{g}(t) \| u(t) \|^p(t) u_k(t) = 0, \]

(4.19) \[ \langle u(t), w_k \rangle |_{t=t_1} = u_k(t) |_{t=t_2} = u_{t_1,k}, \quad k = 1, 2, \ldots, k_0 - 1 \]

Whence, for some \( t_2 > t_1 \) for \( t \in [t_1, t_2] \) we have

\[ \tilde{g}(t) \| u(t) \|^p(t) \leq \tilde{g}(t) r_0^{p(t)} + \varepsilon < \tilde{\alpha} \lambda_k + \tilde{b}(t), \]

for some \( \varepsilon > 0 \). Indeed, we have from (4.14) - (4.15) that

\[ \frac{d}{dt} u_k(t) + 2 \left( \tilde{\alpha} \lambda_k + \tilde{b}(t) - \tilde{g}(t) \| u(t) \|^p(t) \right) u_k(t) = 0, \]

which leads to the formal solution of the Cauchy problem

(4.20) \[ u_k(t) = \exp \left\{ -2 \int_{t_1}^{t} \left( \tilde{\alpha} \lambda_k + \tilde{b}(s) - \tilde{g}(s) r_0^{p(s)} \right) ds \right\} u_{t_1,k}. \]

Consequently, if \( \tilde{\alpha} \lambda_{k_0-1} + \tilde{b}(t_1) < \tilde{g}(t_1) r_0^{p(t_1)} \), then \( u_k(t, x) \) increases in the vicinity of \( u_{t_1,k}(x) \) if \( u_{t_1,k}(x) > 0 \) for \( k = 1, 2, \ldots, k_0 - 1 \) and the part \( \| P_{k_0} u_{t_1} \| \) is enough greater than \( \| Q_{k_0} u_{t_1} \| \).

Let \( u_{t_1} \in B_{R_0}^{H_0} (0) \) and \( \| u_t \| = r_0 \), then the above expression implies that the behavior of the solution \( u_k(t) \) depends on the relation between \( \| P_{k_0} u_{t_1} \| \) and \( \| Q_{k_0} u_{t_1} \| \), and also depends on the relationship between \( \tilde{g}(t_1) r_0^{p(t_1)} \) and \( \tilde{\alpha} \lambda_k + \tilde{b}(t_1) \).

Now for study of the behavior of the \( u(t, x) = \sum_{k=1}^{\infty} u_k(t) w_k(x) \) consider the system of equations that one can write as

(4.21) \[ u_k(t_1) = u_{t_1,k}, \quad k = 1, 2, \ldots \]

according to assumption.

Let there exists \( \lambda_{k_0}, k \geq 2 \) such that the inequalities \( \tilde{\alpha} \lambda_{k_0-1} + \tilde{b}(t_1) < \tilde{g}(t_1) r_0^{p(t_1)} < \tilde{\alpha} \lambda_{k_0} + \tilde{b}(t_1) \) hold, whence isn’t difficult to see that this system of equations is of interest only for the cases \( k \geq k_0, k \leq k_0 - 1 \) and \( k = k_0 \) separately. If \( k \geq k_0 \), this part of the system has a solution as \( t \leq t_2 (k, r_0, p_1) \) for some \( t_2 (k, r_0, p_1) > t_1 \) if \( \| Q_{k_0} u_{t_1} \| \) is sufficiently greater than \( \| P_{k_0} u_{t_1} \| \). Formally we can determine the solution of each equation from (4.12) - (4.13) in the following form:

(4.22) \[ u_k(t) = \exp \left\{ -2 \left( \tilde{\alpha} \lambda_k + \tilde{b}(s) - \tilde{g}(s) r_0^{p(s)} \right) ds \right\} u_{t_1,k}. \]

Thus, considering the problem (4.12) - (4.13) for \( k \leq k_0 - 1 \), (4.17) implies that

\[ |u_k(t)| = \exp \left\{ -2 \left( \tilde{\alpha} \lambda_k + \tilde{b}(s) - \tilde{g}(s) r_0^{p(s)} \right) ds \right\} |u_{t_1,k}| \geq \exp \left\{ 2 \left( \tilde{g}(t_1) r_0^{p(t_1)} - \tilde{\alpha} \lambda_k + \tilde{b}(t_1) \right) t \right\} |u_{t_1,k}|, \]
as $\tilde{\alpha}_k + \tilde{b}(t) < \tilde{g}(t) r_0^{p_1(t)}$ for $1 \leq k \leq k_0 - 1$ and some $t > t_1$. Consequently, the sequence $|u_k(t)|$ increases for each $k : 1 \leq k \leq k_0 - 1$, if the part $\|P_0 u_1\|$ is sufficiently greater than $\|Q_k u_0\|$, and that renders the increase of $r(t)$.

Consider the representation of the formal solutions (4.21) to the problem (4.12)–(4.13):

$$u_k(t) = \exp \left\{ -2 \left( \int_{t_1}^{t} \tilde{\alpha}_k + \tilde{b}(s) - \tilde{g}(s) r_0^{p_1(s)} ds \right) \right\} u_{1,k}, \quad k = 1, 2, ...$$

It is known that $|u_k(t)|$ increases for $1 \leq k \leq k_0 - 1$ and decreases for $k \geq k_0$ by virtue of Theorem 4 depending on the difference $\tilde{\alpha}_k + \tilde{b}(t_1) - \tilde{g}(t_1) r_0^{p_1(t_1)}$ and the relation between $\|P_0 u_1\|$ and $\|Q_k u_0\|$. Thus, in order to investigate of the system (4.21) for each $k$ we need to study the expression $\tilde{\alpha}_k + \tilde{b}(t_1) - \tilde{g}(t_1) r_0^{p_1(t_1)}$ which can be negative or positive due to the conditions imposed. But the behavior of functions $|u_k(t)|$ cannot exactly explain the behavior of functions $u_k(t)$, and also the behavior of the solution $u(t, x)$. Consequently, we need to study the behavior of functions $u_k(t)$ in greater detail.

So, from the representation (4.22) under the corresponding relation between $\|P_0 u_1\|$ and $\|Q_k u_0\|$ follows that if $u_{1,k} \geq 0$ then $u_k(t) \geq 0$ and in addition if $\tilde{\alpha}_k + \tilde{b}(t_1) - \tilde{g}(t_1) r_0^{p_1(t_1)} > 0$ then one can see that $u_k(t)$ will decreases at least in some vicinity of $u_{1,k}$, and if $\tilde{\alpha}_k + \tilde{b}(t_1) - \tilde{g}(t_1) r_0^{p_1(t_1)} < 0$, then when $u_{1,k} \geq 0$ one can see that $u_k(t)$ increases at least in a vicinity of $u_{1,k}$. But if $\tilde{\alpha}_k + \tilde{b}(t_1) - \tilde{g}(t_1) r_0^{p_1(t_1)} = 0$ then not is possible to understand how will vary the $u_k(t)$ in vicinity of $u_{1,k}$.

Thus, we need to investigate the behavior of $r(t)$ for various function $u_{1,k} (x)$ in the case when $\|u_1\| = r_0$, and more exactly for various function $u_0 (x)$ since $u_{1,k} (x)$ depends on $u_0 (x)$. Consequently, the behavior of $r(t)$ essentially depends on the geometrical selections of $u_k (t_1)$ (more exactly on the geometrical selections of initial function $u_0$).

In the above assumptions the solution of examined problem (4.12)–(4.13) and of the initial function at the moment $t_1$ that is obtained from initial data $u_0$ we can represent with the expressions $u (t, x) = \sum_{k \geq 1} u_k (t) w_k (x)$ and $u_{1,k} (x) = \sum_{k \geq 1} u_{1,k} w_k (x)$, respectively. The provided analysis shows that for the study of the behavior of solutions of the problem in detail it is necessary to use the above expressions.

So, let $r_0 > 0$, then there exists $k_0 \geq 2$ such that

$$\tilde{\alpha}_k + \tilde{b}(t_1) < \tilde{g}(t_1) r_0^{p_1(t_1)} \leq \tilde{\alpha}_k + \tilde{b}(t_1)$$

and $\|u_{1,1}\| = r_0$.

Then by use the orthogonal splitting $u (t) = P_0 u (t) + Q_k u (t)$, we obtain the following represatations:

$$u_{1,k} (x) = \sum_{k \geq 1} u_{1,k} w_k (x) = \sum_{k_0 - 1 \geq k \geq 1} u_{1,k} w_k (x) + \sum_{k \geq k_0} u_{1,k} w_k (x)$$

and

$$u (t, x) = \sum_{k \geq 1} u_k (t) w_k (x) = \sum_{k_0 - 1 \geq k \geq 1} u_k (t) w_k (x) + \sum_{k \geq k_0} u_k (t) w_k (x).$$
From the relations between the expressions $\hat{g}(t) r(t)^{p_1(t)}$ and $\hat{a}_k \lambda_k - 1 + \hat{b}(t)$ follows there exist $t_2 > t_1$ and $t_3 > t_1$ such that $P_{k_0} u(t)$ can increases in $(t_1, t_2)$ and $Q_{k_0} u(t)$ decreases in $(t_1, t_3)$ in the formula (4.23) by virtue of the (4.22). Then if $\min\{t_2, t_3\} = t_2$, then for $t > t_2$ these summands can behave quite differently. Here there exist the following possibilities: (1) the functional $\|u(t)\|$ becomes greater than $r_0$ for $t \geq t_2$, moreover $\hat{g}(t) r(t)^{p_1(t)} \geq \hat{a}_k \lambda_k - 1 + \hat{b}(t)$ for $t > t_3$, so the orthogonal splitting $u = P_{k_0} u + Q_{k_0} u$ changes and becomes, at least, $u = P_{k_0 - 1} u + Q_{k_0 - 1} u$; (2) $Q_{k_0} u$ decreases up to a point where $\|u(t)\|$ is smaller than $r_0$ for $t \geq t_2$, moreover $\hat{g}(t) r(t)^{p_1(t)} \leq \hat{a}_k \lambda_k - 1 + \hat{b}(t)$ for $t > t_2$, so the orthogonal splitting $u = P_{k_0} u + Q_{k_0} u$ changes and becomes, at least, $u = P_{k_0} u + Q_{k_0 - 1} u$; (3) there exist a $t_4 \geq t_2$ and an $R_0 \geq \|P_{k_0} u\| \geq R_1 > 0$ such that beginning at $t_4$ the changes of $P_{k_0} u$ and $Q_{k_0} u$ become such that

\[ r^2(t) = \|u(t)\|^2 = \sum_{k_0 - 1 \geq k \geq 1} |u_k(t)|^2 + \sum_{k \geq k_0} |u_k(t)|^2 \]

satisfies $R_1 \leq r(t) \leq R_0$ for $t \geq t_4$.

Consider case (i) where we have the following possibilities: (i) $P_{k_0} u$ increases with such velocity that $\|u(t)\| \not\rightarrow \infty$, which can occur when $u_{t_1}(x)$ is chosen near the subspace $H_{k_0}$ (this scenario is studied in Theorem 3); (ii) $Q_{k_0} u$ decreases beginning at time $t$ and the functional $u(t, x)$ behaves as in case 3, which we will explain in what follows. Case 2 has 2 variants: (iii) $Q_{k_0} u$ decreases with such velocity that $\|u(t)\| \not\rightarrow 0$ leading to the inequality $\hat{g}(t) r(t)^{p_1(t)} < \hat{a}_k \lambda_k - 1 + \hat{b}(t)$, which can take place when $u_{t_1}(x)$ is chosen near the subspace $H_{-k_0}$ (this variant is also studied in Theorem 3); (iv) the rate of decrease of $Q_{k_0} u$ diminishes beginning at some time $t$ and leads to case 1(ii).

We should be noted all of the above cases depends of the geometrical location of the initial function on the sphere $S_{r_0}^{H_0}(0) \subset H_1^1(\Omega)$.

Consequently, it remains only to investigate case 3. Therefore, we can choose special initial data $u(0) = u_0$ such that into the time $t_1$ the functions $u(t_1)$ be on the sphere $S_{r_0}^{H_0}(0)$ and try to explain case 3 for such functions. So, let $P_{k_0} u_{t_1} = u_{t_1, k_0 - 1} u_{k_0 - 1}$, i.e. $u_{t_1}(x) = u_{t_1, k_0 - 1} u_{k_0 - 1}(x) + Q_{k_0} u_{t_1}(x)$ and $\|u_{t_1}\| = r_0$. Then we obtain the change of the function $u(t, x)$ happen by the following way: $u_{k_0 - 1}(t)$ changes so that $|u_{k_0 - 1}(t)|$ increases with $t$ and $\hat{g}(t) |u_{k_0 - 1}(t)|^{p_1(t)} \rightarrow \hat{a}_k \lambda_k - 1 + \hat{b}(t)$ when $t \not\rightarrow \infty$, and $Q_{k_0} u(t)$ changes so that $\|Q_{k_0} u(t)\|$ decreases although $p_1(t)$ increases with increasing $t$ and therefore $\|Q_{k_0} u(t)\| \rightarrow 0$ when $t \not\rightarrow \infty$. Hence, $\hat{g}(t) |u(t)|^{p_1(t)} \not\rightarrow \hat{a}_k \lambda_k - 1 + \hat{b}(t)$ as $t \not\rightarrow \infty$. In other words, the increase of $\|P_{k_0} u(t)\|$ and decrease of $\|Q_{k_0} u(t)\|$ compensate for each other in such a way that this process leads to the case described above.

We should be noted in order to the functions $u_{t_1}$ satisfy of these conditions is necessary the initial data $u_0$ to choose by corresponding way, namely using of the representation (4.10), that isn’t difficult see. Obviously that the function $u_0$ must be has also of same representation.

Thus, it isn’t difficult to see in order to obtain the above result, we need to select $u_{t_1, k_0 - 1}$ (and also $u_{k_0 - 1}$) in near of $H_{-k_0}$, which depends on the given

\[ r_0 : \hat{a}_k \lambda_k - 1 + \hat{b}(t_1) < \hat{g}(t_1) r_0^{p_1(t_1)} \leq \hat{a}_k \lambda_k - 1 + \hat{b}(t_1). \]

Accordingly it follows in the case when $P_{k_0} u(t)$ increases, the corresponding $u_{t_1, k}$, $1 \leq k \leq k_0 - 1$, must be chosen as done previously. Moreover, there is a $\lambda_{j_0}$ such
that \( \bar{g}(t) \| P_{k_0} u(t) \|_{p_1(t)} < \hat{\alpha} \lambda_{j_0} + \hat{b}(t) \) when \( t \to \infty \), where
\[
\lambda_{j_0} = \inf \{ \lambda_k \mid 1 \leq k \leq k_0 - 1, \ u_{t_k} \neq 0 \}.
\]

Consequently, we arrive that there exists a "double cone" with the "vertex at zero" that contains the subspace \( H_{-k_0} \) and all elements are contained in some neighborhood of \( H_{-k_0} \), where the maximal distance between of the elements of this subset and the subspace \( H_{-k_0} \) depends on the given \( r_0 \), and also on the coefficients and the exponent from the inequality (4.23). If we denote this subset by \( H \subset H_{0}^1(\Omega) \), then we have that any subset of \( H \cap \{ B_{r_{10}}^H(0) - B_{r_{20}}^H(0), \ r_1 > r_2 > 0 \} \) converges to a set, which we can define as \( H_{k_0} \cap B_{r_{j0}}^H(0) \), where \( r_{j0} = \hat{\alpha} \lambda_{j0} + \hat{b}(t_1) \) and \( r_1, r_2 \) are some numbers with
\[
\hat{\alpha} \lambda_{k_1 - 1} + \hat{b}(t_1) < \bar{g}(t_1) \| P_{j_1}^{(t_1)} \|_{H_{r_{j1}}^0} \leq \hat{\alpha} \lambda_{k_1} + \hat{b}(t_1)
\]
and there is a \( \lambda_{j_1} = \inf \{ \lambda_k \mid 1 \leq k \leq k_1 - 1, \ u_{t_k} \neq 0 \} \) and \( k_1 = k_1(r_1) \). This shows that \( H_{k_0} \cap B_{r_{j1}}^H(0) \) is a subset of a finite-dimensional space and it is local attractor in some sense, where \( r_{j1} = \hat{\alpha} \lambda_{j1} + \hat{b}(t_1) \).

Thus, we have proved the following result.

**Theorem 5.** Let all the above conditions hold and \( u_{t_1} \in H_{k_0}^1(\Omega) \) satisfies the inequality \( \hat{\alpha} \lambda_{k_0 - 1} + \hat{b}(t_1) < \bar{g}(t_1) \| u(t) \|_{H_{r_{j1}}^0} \leq \hat{\alpha} \lambda_{k_1} + \hat{b}(t_1) \)**. Then each solution to the problem (4.22) - (4.23) satisfies one of the following properties:

1. If \( u_{t_1} \) lies in \( H_{k_0} \) or in a sufficiently small neighborhood of the subspace \( H_{k_0} \), then \( |u_k(t)| \to \infty \) as \( t \to \infty \) for \( k = 1, k_0 - 1 \); moreover, in this case \( r(t) = \| u(t) \| \) increases and \( \bar{g}(t) r(t) P_{k}(t) \) gradually becomes greater than \( \hat{\alpha} \lambda_k + \hat{b}(t) \) for each \( \lambda_k : k \geq k_0, |u_k(t)| \) gradually increases for all \( k \);

2. If \( u_{t_1} \) lies in \( H_{-k_0} \) or in a small neighborhood of the subspace \( H_{-k_0} \), then \( |u_k(t)| \to 0 \) as \( t \to \infty \) for \( k \geq k_0 \); moreover, since in this case \( r(t) \) decreases and although \( P_{k}(t) \) increases but \( \bar{g}(t) r(t) P_{k}(t) \) gradually becomes less than \( \hat{\alpha} \lambda_k + \hat{b}(t) \) for each \( \lambda_k, |u_k(t)| \) gradually decreases for all \( k \);

3. If \( u_{t_1} \in H, \| P_{k_0} u_{t_1} \| \ll \| Q_{k_0} u_{t_1} \| \) and if there are small numbers
\[
\delta \left( \hat{\alpha} \lambda_{k_0} + \hat{b}(t_1) \right) > \varepsilon \left( \hat{\alpha} \lambda_{k_0} + \hat{b}(t_1) \right) > 0
\]
such that the Hausdorff distance satisfies
\[
(4.26) \quad \varepsilon \leq d \left( H_{-k_0}; \{ u_{t_k}^+ \mid k = 1, k_0 - 1 \} \right) \leq \delta,
\]
then \( u(t, x) \) is chaotic for sufficient large \( t \). In addition, if \( \| P_{k_0} u_{t_1} \|_{H_{-k_0}} \approx \| Q_{k_0} u_{t_1} \|_{H_{k_0}}, \) then there is a relation between \( u_{t_{k_0}}(t) \) and \( u_{t_{k_0}}^+ \) for which the behavior of the \( u_{k}(t) \) is chaotic for all \( k \) satisfying \( u_{k}(0) \neq 0 \).

**Remark 5.** If property 3 of the above theorem obtains, then the following claim is reasonable: for any \( \lambda_{k_0} \) there is a subset \( D_{k_0} \subset H_{0}^1(\Omega) \) for which (4.26) holds and for any \( u_{t_1} \in D_{r_{k_0}} \) the corresponding solution \( u(t) \) satisfies the condition
\[
\hat{\alpha} \lambda_{j_0} + \hat{b}(t_1) \leq \bar{g}(t) r(t) P_{j_1}(t) < \hat{\alpha} \lambda_{j_0} + \hat{b}(t_1) \lambda_{k_0}
\]
\[\text{The relation between } u_{t_1} \text{ and } u_0 \text{ is given by formula (4.10).}\]
for any $t > 0$, where $r_{k_0} = \hat{\lambda}_{k_0} + \hat{b}(t_1)$ and

$$\lim_{t \to \infty} \hat{g}(t) r(t)^{p_1(t)} = \hat{\lambda}_{j_0} + \hat{b}(t_1), \quad u_{0k} \neq 0$$

then there is an absorbing chaotic set in $L^2(\Omega)$, where

$$\lambda_{j_0} = \inf \{\lambda_k \mid 1 \leq k \leq k_0 - 1\}.$$

4.1. Conclusion on the dynamics of cancer. The obtained results show that the propagation of cancer essentially depends on the initial value, i.e. what moves the start of the destruction (degeneration) and of which initial mass covers. It is naturally that the dynamics of propagation of cancer essential depends on the immune systems of the body. Naturally, the dynamics of propagation of cancer essential depends on the immune systems of the body, consequently on its powers to act and to stop the degeneration process, and also the sufficient stability of its act. So, we can classify the initial date of the space of initial dates beginning with their role in participation in the process of the propagation of cancer. Thus one can determine 3 classes of the initial dates that have a principal role in the studied process. The first class contains such initial dates, under which the process has the successful result. (The cases investigated in sections 2.1 and 2.3 contain this class.) The second class contains such initial dates, under which the process has the worse result. (The case investigated in section 3.1 contain this class.) The third class is found between classes 1 and 2 and contains such initial dates, under which the process of the dynamics of propagation of cancer is sufficiently complicated. The appearance of chaos in the dynamics at the process of the propagation of cancer is connected with the initial date and stationary part of the problem. Since at the initial value from this class due to the stationary part of the examined problem arise the bifurcations, which turn into for the examined process to the cascade of bifurcations, therefore exist such absorbing manifold, that the associated dynamics become chaotic. Consequently, in this case, the long-time behavior of the trajectory of the dynamics of propagation of cancer can be mainly chaotic, therefore it become undefinable.

5. Open Problems

The above discuss shows: in order to the mathematical model represented in this article could describe of the dynamics of the cancer more exactly is necessary that all functions in the equation (1.1) depends from the both variables $t$ and $x$.

Problem 1. Do possible to study the long-time behavior of solutions of the problem (1.1)-(1.3) in the above-mentioned conditions?

Problem 2. Do possible to study the control problem for the dynamics of the cancer using the represented in this article mathematical model: the problem (1.1)-(1.3)?

We should be noted the control function must depends from the both variables and have the variable exponent, and also it is need to determine of the corresponding class of the control functions.

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