The Self-Gravity of Pressure in Neutron Stars

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Following an earlier analysis that examined the effect of the self-gravity of pressure on big-bang nucleosynthesis (BBN), we explore the effect of pressure’s self-gravity on the structure of neutron stars. We construct an ad hoc modification of the Tolman-Oppenheimer-Volkoff equation wherein pressure’s self-gravity is parameterized by a constant, \( \chi \), with \( 0 \leq \chi \leq 1 \). The full general relativistic contribution to the gravity of pressure is recovered with \( \chi = 1 \), and is eliminated when \( \chi = 0 \). This formulation is not proposed as an alternative theory of gravity, but is merely used to quantify the extent to which the self-gravity of pressure contributes to the structure of dense objects. As can be surmised qualitatively, neutron star masses can be quite sensitive to \( \chi \), with higher values of neutron-star mass (by \( \sim 20-25 \% \)) allowed for smaller values of \( \chi \). However, for a given equation of state, neither the range of neutron star radii nor the radii at fixed central density depend sensitively on \( \chi \). Over the neutron star mass range measured so far, the presence or absence of pressure’s self-gravity yields a nearly immeasurable change in radius — much smaller than the variations in radius due to the uncertainty in the equation of state. In contrast to the result for BBN, we thus find that neutron stars are not likely to be useful testbeds for examining the self-gravity of pressure.

I. INTRODUCTION

One of the more fascinating predictions of general relativity is that pressure is self gravitating. This is a strictly non-Newtonian result, whereby the pressure of a field contributes to gravity, increasing the effective density by an amount \( 3P/c^2 \) for a homogeneous perfect fluid.

There are four prominent physical situations in which the self-gravity of pressure could potentially lead to a measurable effect (three of which involve the expansion history of the Universe): (i) the acceleration of the Universe during inflation; (ii) the expansion of the Universe during the radiation-dominated epoch; (iii) the current acceleration of the Universe due to dark energy; and (iv) the mass-radius relation for neutron stars. In an earlier paper, Rappaport et al. \[1\] addressed (ii) by setting a constraint on the self-gravity of pressure during the radiation-dominated epoch of big-bang nucleosynthesis (BBN). They introduced an ad hoc multiplicative parameter \( \chi \) to the \( 3P/c^2 \) contribution in the Einstein field equations. For \( \chi = 1 \), the full general relativistic contribution to the self-gravity of pressure is retained; for \( \chi = 0 \), there is no such contribution. This yielded a modified set of Friedmann-like equations. The contribution to \((\dot{a}/a)^2\) for a species of matter with equation of state (EOS) \( w = P/\rho c^2 \) appears as

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{\Omega}{a^{1+3w}} \left( \frac{1 + 3w\chi}{1 + 3w} \right) \frac{\Omega}{a^{1+3w}}. \tag{1}
\]

The gravitational effect of radiation \((w = 1/3)\) is therefore scaled by \((1 + \chi)/2\). The absence of pressure’s self-gravity means that the Hubble constant at the time of BBN would be smaller by a factor of \( \sqrt{2} \), potentially changing the abundance of light elements. Rappaport et al. used a standard BBN code and performed light element abundance calculations for a range of \( \chi \) to quantify this effect. When combined with current light element observations, the data were shown to be consistent with \( \chi = 1 \), and strongly exclude \( \chi = 0 \).

Our goal here is to see whether similar limits can be set by observations of neutron stars. At neutron star densities, \( P \sim (0.1 - 0.6)\rho c^2 \), suggesting that pressure is high enough that a measureable self-gravity effect might exist. We introduce a parameterization of self-gravity very similar to that used to modify the Friedmann equations. The result is a parameterized equation of stellar structure, with \( \chi = 1 \) reproducing the Tolman-Oppenheimer-Volkoff equation, and \( \chi = 0 \) “turning off” the pressure self-gravity of that equation.

Although at fixed central density \( \chi \) has a significant impact on a neutron star’s mass, it has very little impact on its radius. If measurements were to determine both mass and radius, it would still be extremely difficult to tell the difference between models with \( \chi = 0 \) and \( \chi = 1 \) due to the uncertainty in the neutron star EOS. Neutron stars thus appear to not be very useful for testing the self-gravity of pressure. This is not to say that one cannot make interesting statements about gravity with neutron stars \[2\]; but, in contrast to the situation with BBN, the pressure self-gravity aspect cannot be usefully tested. The difference between the two cases is simple: the EOS of the universe during BBN is well understood, but the EOS of neutron stars is not. Testing pressure’s self-gravity is thus degenerate with testing the EOS for neutron stars, but is not degenerate during BBN.

II. STRUCTURE OF NEUTRON STARS

In general relativity, the structure of spherically symmetric fluid equilibria such as neutron stars is governed by the Tolman-Oppenheimer-Volkoff (TOV) equation \[3\,4\]. To motivate our heuristic, it is useful to review
the contribution of pressure to $dP/dr$ by modifying the TOV equation as follows:

$$
\frac{dP}{dr} = -(\rho + P) \frac{G[m(r) + 3P(4\pi r^3/3)]}{r^2[1 - 2Gm(r)/r]}. \quad (9)
$$

In contrast to the modified Friedmann equations discussed in the Introduction, $\chi = 0$ does not correspond to a Newtonian limit. The $[1 - 2Gm(r)/r]$ term in the denominator is a statement of geometry, and the $(\rho + P)$ term arises from conservation of energy that must be valid even in the special relativity limit.

It is worth emphasizing that Eq. (9) is not derived by proposing a plausible alternative theory of gravity. Our goal is much less ambitious than reformulating gravity; we merely want to quantify the extent to which the self-gravity of pressure contributes to the structure of dense objects. The point of this exercise was to see which of the pressure terms in Eq. (8) acts as a source of gravity, and then to appropriately modify that equation in a way that tests that term’s importance.

### III. MODEL BUILDING, RESULTS, AND DISCUSSION

The first step in building neutron star models is to select an EOS, $\rho = \rho(P)$. For high densities, we use the compilation of Lattimer & Prakash [6], plus two of the “mixed” EOSs from Alford et al. [8]. Our set includes a wide range of models, representing several different theoretical approaches, which incorporate both normal nuclear matter compositions and more exotic matter (quarks, hyperons, etc.). Table 4 gives an EOS summary, listing general properties and giving relevant references. The trailing numerical labels are as given in [6]. For the “ALF” EOSs, the numerical label indicates the transition density in multiples of the nuclear saturation density. In all cases, we use the Negele-Vautherin (NV) EOS [8] for the lower baryon density range $0.08 > n_b > 6 \times 10^{-4}$ fm$^{-3}$, and we use the Baym-Pethick-Sutherland (BPS) EOS [6] for $n_b < 6 \times 10^{-4}$ fm$^{-3}$. This is similar to the low density choices made in [8].

We construct the neutron star models by numerically solving the coupled differential equations

$$
\frac{d\rho}{dr} = \frac{d\rho}{dP} \frac{dP}{dr} \quad \text{and} \quad \frac{dM}{dr} = 4\pi r^2 \rho. \quad (10)
$$

The tabulated EOSs were logarithmically interpolated and the derivatives evaluated by a simple 3-point Lagrange interpolation. Integration proceeds until the density falls below that of iron, $7.9$ g cm$^{-3}$. We constructed models with both $\chi = 0$ and $\chi = 1$ for each EOS.

Figure 1 illustrates the effect that varying $\chi$ has for a particular EOS, AP4. We show the mass-radius relation for $\chi = 1$ (black curve) and $\chi = 0$ (red curve). Notice that the radial range spanned by these two models...
is largely the same; the mass, however, can change substantially. The arrows in this figure connect models that have equal central densities. This shows that pressure’s self-gravity can have a \( \sim 10 - 30\% \) effect on the mass, but barely change the star’s radius—at a fixed central density. One might have guessed on intuitive grounds that an effect of order tens of percent must occur, since for these models \( P/\rho \sim 0.1 - 0.6 \). The fact that this change is mostly in mass, leaving the radius relatively unaffected, was not obvious in advance. Finally, in this regard, we note that for a given EOS and at a fixed neutron-star mass near the upper mass limit, the radius can vary by up to \( \sim 15\% \) when \( \chi \) is reduced to near zero. However, the differences in radii at a fixed mass, among the various plausible EOSs, are much larger than this, thereby making it difficult, at best, to constrain \( \chi \).

Figure 4 shows the effect of removing the self-gravity of pressure for several illustrative EOSs. The general trend seen in Fig. 1—an increase in mass for a given radius—holds for both “normal” (left panel) and “exotic” (right panel) compositions. In the most compact stars, \( \chi = 0 \) corresponds to a change of \( \sim 0.5M_\odot \) over the nominal \( \chi = 1 \) model, for a constant radius.

In the case of BBN, the cosmic abundance of light elements puts interesting constraints on \( \chi \). Might it be possible to place similar limits on \( \chi \) from neutron star observations? The answer, unfortunately, appears to be “no”. Neutron star masses have in some cases been measured to accuracies of greater than 1 part in 1000, most notably PSR 1913+16 [17], where the pulsar and its companion have measured masses of 1.441\( M_\odot \) and 1.387\( M_\odot \) respectively. A measurement of the radius of one of these objects would place strong constraints on the EOS. Referring to Fig. 2 in this mass range and for “normal” EOSs, it would take a radius measurement with an accuracy \( \sim 300 \) m in order to, for a given EOS, differentiate between the \( \chi = 0 \) and \( \chi = 1 \) cases. A number of proposed techniques for measuring neutron star radii have been recently summarized in [18]. While inventive and intriguing, none of these techniques has yet led to robust determinations of neutron-star radii.

| Symbol  | Reference                        | Approach      | Composition |
|---------|----------------------------------|---------------|-------------|
| AP(3-4) | Akmal & Pandharipande [10]       | Variational   | np          |
| MPA     | M"uther, Prakash, & Ainsworth [11]| Dirac-Brueckner HF | np          |
| MS(1-3) | M"uller & Serot [12]            | Field Theoretical | np          |
| WFF(1-3)| Wiringa, Fiks, & Fabrocine [13]  | Variational   | np          |
| ALF(2-3)| Alford et al. [7]               | Quark Matter  | npQ         |
| GM(1-3) | Glendenning & Moszkowski [14]    | Field Theoretical | npH         |
| GS(1-2) | Glendenning & Schaffner-Biedich [15] | Field Theoretical | npK         |
| PCL     | Prakash, Cooke, & Lattimer [16]  | Field Theoretical | npHQ        |

TABLE I: Summary of the EOSs shown in Figure 2 and 3. “Approach” characterizes the theoretical basis. “Composition” is characterized by \( (n – \text{neutrons}, p – \text{protons}, H – \text{hyperons}, K – kaons, Q – quarks}) \). The horizontal line separates “normal” and “exotic” equations of state.

Figure 3 makes this point even more clearly, including a more comprehensive sample (all EOSs listed in Table 1) and showing the ranges of possible masses and radii as shaded regions, bounded by the most extreme EOSs. The change in the region between the \( \chi = 0 \) and \( \chi = 1 \) cases clearly shows that the absence of gravitating pressure acts to stretch possible masses upward, coupled with a minimal increase in radius.

On this plot, the only plausible evidence for a model with \( \chi \neq 1 \) would be a precise measurement of both mass and radius that placed a star in a red-shaded region that is not also a black-shaded region. For neutron stars with \( M \lesssim 2M_\odot \), it would be extremely difficult to distinguish between viable EOSs on the one hand, and the effect of pressure’s self-gravity on the other. For higher masses,
Given the difficulty of interpreting such a measurement as a test of gravity, we conclude that neutron stars (in contrast to BBN) are not a good laboratory for examining the self-gravity of pressure.

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