Crossing Symmetric Dispersion Relations in QFTs

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Abstract

For 2-2 scattering in quantum field theories, the usual fixed \( t \) dispersion relation exhibits only two-channel symmetry. This paper considers a crossing symmetric dispersion relation, reviving certain old ideas in the 1970s. Rather than the fixed \( t \) dispersion relation, this needs a dispersion relation in a different variable \( z \), which is related to the Mandelstam invariants \( s, t, u \) via a parametric cubic relation making the crossing symmetry in the complex \( z \) plane a geometric rotation. The resulting dispersion is manifestly three-channel crossing symmetric. We give simple derivations of certain known positivity conditions for effective field theories, including the null constraints, which lead to two sided bounds and derive a general set of new non-perturbative inequalities. We show how these inequalities enable us to locate the first massive string state from a low energy expansion of the four dilaton amplitude in type II string theory. We also show how a generalized (numerical) Froissart bound, valid for all energies, is obtained from this approach.

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1 Introduction

Dispersion relations provide non-perturbative representations for scattering amplitudes in quantum field theories [1, 2]. The usual way to write dispersion relations in the context of 2-2 scattering of identical particles is to keep one of the Mandelstam invariants, usually \( t \), fixed and write a complex integral in the variable \( s \). This approach naturally leads to an \( s - u \) symmetric representation of the amplitude. Then, one imposes crossing symmetry as an additional condition. A similar approach can also be developed for Mellin amplitudes for conformal field theories. Recent developments in this direction include [3], [4], [5].

The amplitude’s resulting representation not having manifest three-channel crossing symmetry may appear to be a drawback. For instance, in perturbative quantum field theories, when we compute Feynman diagrams, the amplitude’s resulting expansion has manifest crossing symmetry. In the worldsheet formulation of string theory, the tree level sphere diagram, for instance, is manifestly crossing symmetric without having the explicit three-channel split. Hence, it would be somewhat disappointing if a non-perturbative approach failed to have this elegance inbuilt into its starting point.

The question naturally arises: Is there a manifestly crossing symmetric version of these dispersion relations? In the 1970s, this question was briefly considered in a few papers, for example, in 1972 by Auberson and Khuri in [6] and in 1974 by Mahoux, Roy, and Wanders in [7]. Unfortunately, due to the technical complications involved, barring for a smattering of a few papers (e.g., [8]), this approach has not been well explored in the literature. We will follow [6] and revive this line of questioning again. In the CFT context, Polyakov’s work in [9] proposed a fully crossing symmetric bootstrap, which was developed in [10, 11]. However, this approach currently lacks a non-perturbative derivation for \( d \geq 2 \).

Our methods in this paper will enable us to address this important question in the near future [12].

Dispersion relations give a window to understanding how analyticity and unitarity assumptions for the high energy behaviour of amplitudes constrain low energy physics contained in effective field theories [13]. Our manifestly crossing symmetric approach not only leads to a simpler and unifying derivation of recently considered positivity constraints in effective field theories [14, 15, 16, 17], but also enables us to write down a completely general set of positivity constraints on the Wilson coefficients. In particular, we will provide straightforward derivations of many of the upper bounds on the ratios of Wilson coefficients, as well as the null constraints listed in [16, 17], leading to the lower bounds. Our formalism will enable us to write down general formulae for the upper bounds and the independent null constraints.

We will consider two novel applications of our constraints. First, we will use them to find the location of the first massive string pole from the low energy expansion for the tree-level four-dilaton scattering in type II string theory (e.g., [18]). Second, our approach enables us to derive a numerical upper bound on the total scattering cross-section of identical particles valid at all energies. This is a generalization of the famous Froissart bound [19]. We will conclude with some future directions.

2 Crossing symmetric dispersion relation

We begin by considering cubic hypersurfaces [6] in the variables \( s_1 = s - \frac{t}{3}, \ s_2 = t - \frac{u}{3}, \ s_3 = u - \frac{t}{3} = -s_1 - s_2, \ \mu = 4m^2 \), where \( s, t, u \) are usual Mandelstam variables. Explicitly, these hypersurfaces will
be given by \((s_1(z) - a) (s_2(z) - a) (s_3(z) - a) = -a^3\), with \(a\) being a real parameter. The \(s_i\)'s can be parametrized via

\[ s_k = a - \frac{a(z-z_k)^3}{z^3 - 1}, \quad k = 1, 2, 3, \quad (2.1) \]

where \(z_k\) are cube roots of unity and we will restrict \(-\frac{\mu}{3} \leq a < \frac{2\mu}{3}\). Importantly, \(a = \frac{s_1 s_2 s_3}{s_1 s_2 + s_2 s_3 + s_3 s_1}\).

The amplitude \(\mathcal{M}(s_1, s_2)\) can be written as an analytic function of \((z, a)\) i.e., \(\overline{\mathcal{M}}(z, a) = \mathcal{M}(s_1(z), s_2(z))\). \(\mathcal{M}(s_1(z), s_2(z))\) has physical cuts for \(s_k \geq \frac{2\mu}{3}, \quad k = 1, 2, 3\). The image of the three physical cuts in the \(z\)-plane is defined via \(V(a) = V_1(a) \cup V_2(a) \cup V_3(a)\). For our region of interest \(-\frac{2\mu}{9} < a < 0\), it is

\[ V_1(a) = \left\{ z, |z| = 1, \frac{2}{3}\pi \leq |\arg z| \leq \phi_0(a) \right\}, \quad V_2(a) = e^{2i\pi/3} V_1(a), \quad V_3(a) = e^{4i\pi/3} V_1(a). \]

which gives us the fig. 1. Here \(\phi_0(a) = \tan^{-1}\left\{ \left[ \frac{2\mu - a}{a + 2\mu} \right]^{1/2} \right\}, \quad 0 < \phi_0 \leq \pi\). For a complete discussion see Appendix (A).

We now write down a twice subtracted dispersion relation across \(V(a)\):

\[
M(z,a) = f_0 + f_1(a) \frac{z}{1 - z^3} + f_2(a) \frac{z^2}{1 - z^3} + \frac{z^3}{(1 - z^3)} \int_{V(a)} dz' \frac{z'^3 - 1}{z'^3} \mathcal{A}(z', a) \quad (2.2)
\]

where \(\mathcal{A}(z', a)\) is the discontinuity of the amplitude across \(V(a)\), defined in eq (A.4).

For the completely crossing symmetric case (eg. \(\pi^0 \pi^0 \rightarrow \pi^0 \pi^0\)) the above dispersion relation simplifies

\[1\text{As argued in [6], in all three channels, enlarged Martin domains [20] can be considered. The enlarged domain of }a\text{ is worked out in [6].}\]
dramatically\footnote{This form was mentioned in the “Note added” in \cite{6} and the discussion in \cite{6} did not exploit the full power of this simple form.} in terms of the \(s_1, s_2, s_3\) variables:

\[
\mathcal{M}_0(s_1, s_2) = \alpha_0 + \frac{1}{\pi} \int_{2\mu/3}^{\infty} ds_1' \frac{\mathcal{A}(s_1'; s_2^+(s_1', a)) H(s_1'; s_1, s_2, s_3)}{s_1'^{2n+1}} \left(1 - \frac{a}{s_1'}\right)^{n-1} \left(2 - \frac{3a}{s_1'}\right) x^n. \tag{2.6}
\]

We can now extract the coefficient of \(a^m\). Notice that \(a = y/x\), which means that the coefficient of \(a^m\) will be \(\mathcal{W}_{n-m,m}\) in eq. (2.5). The s-channel discontinuity has a partial wave expansion over even spins

\[
\mathcal{A}_1(s_1, s_2) = \Phi(s_1; \alpha) \sum_{\ell=0}^{\infty} (2\ell + 2\alpha) a_\ell(s_1) C_\ell^{(a)} \left(\sqrt{\xi(s_1, a)}\right)
\]

\[
|\Phi(s_1; \alpha) = \Psi(\alpha) \frac{\sqrt{s_1 + \frac{\delta}{s_1 - 2\mu/3}}}{s_1 - 2\mu/3} |\text{ with } \Psi(\alpha) > 0 \text{ is real number and } \xi_0 = \frac{\sqrt{s_1}}{(s_1 - 2\mu/3)^{\delta/2}}. \tag{2.7}
\]

From the expansion \(C_\ell^{(a)} (\xi^{1/2}) = \sum \ell/2 p_\ell^{(j)}(\xi_0) (\xi - \xi_0)^j\) with \(p_\ell^{(j)}(\xi_0) = \left. \frac{\partial^j C_\ell^{(a)}(\sqrt{\xi})}{\partial \xi^j} \right|_{\xi=\xi_0}\), we find the coefficient of
\( a^m \) leading to the inversion formula

\[
W_{n-m,m} = \int_{2\mu/3}^{\infty} \frac{ds_1}{s_1} \Phi(s_1; \alpha) \sum_{\ell=0}^{\infty} (2\ell + 2\alpha) a_\ell(s_1) B^{(\ell)}_{n,m}(s_1), \quad n \geq 1
\]

\[
B^{(\ell)}_{n,m}(s_1) = \frac{1}{\pi} \sum_{j=0}^{m} \frac{1}{s_1^{2n+m}} \frac{p^{(j)}_\ell(\xi_0)}{j!} (4\xi_0)^2 \times \frac{(3j-n-2n)(-n)_m}{(m-j)!(-n)_{j+1}}.
\]  

Note that \( W_{0,0} = \alpha_0 \), which is the subtraction constant.

Eq. (2.8) allows two lines of investigation. (1). For \( n \geq m \) with \( n \geq 1 \), we get the coefficients in terms of \( a_\ell(s_1) \). Since partial wave unitarity implies \( 0 \leq a_\ell(s_1) \leq 1 \), we can find positivity constraints on \( W_{n-m,m} \). (2). For \( n < m \) with \( n \geq 1 \), the coefficients should vanish (as needed by eq. (2.5)), which give rise to non-trivial constraints on \( a_\ell(s_1) \). These sum rules or “null constraints” are instrumental in getting the lower bounds on \( W_{n-m,m} \) in EFTs. We will use these sum rules to put a bound on the total cross section at any \( s_1 \), generalizing the famous Froissart bound.

3 Constraining QFTs

3.1 Positivity constraints

Now from eq. (2.8), we can derive inequalities involving \( W_{p,q} \). Note that \( p^{(j)}_\ell \)'s involve derivatives of Gegenbauer. Since \( \partial_x C^{(\ell)}_\ell(x) = 2\alpha C^{(\ell+1)}_\ell(x) \), we will get linear combinations of \( C^{(\alpha+k)}_{\ell-k}(x) \). Specifically we find the useful expression for \( r \geq 1 \)

\[
p^{(r)}_\ell(\xi_0) = \sum_{k=1, k \leq \ell} (-1)^{k+1} 4^{-k-r} \Gamma(k + \alpha)(k - 2\ell + 1)_{r-1} \xi_0^{k-2r} \Gamma(\alpha) \Gamma(\ell) C^{(\alpha+k)}_{\ell-k} \left( \sqrt{\xi_0} \right). \tag{3.1}
\]

We use the unitarity constraints \( 0 \leq a_\ell(s_1) \leq 1 \) as well as properties of \( C^{(\alpha+k)}_{\ell-k} \left( \frac{2\mu}{3\delta} + 1 \right) \), namely

\[
C^{(\alpha+k)}_{\ell-k} \left( \frac{2\mu}{3\delta} + 1 \right) \geq 0, \quad \forall \ell \in 2\mathbb{Z}\geq 0, \quad \delta = s_1 - \frac{2\mu}{3} \geq 0, \quad \alpha \geq 0, \quad \mu \geq 0. \tag{3.2}
\]

Since the range of \( s_1 \) in eq. (2.8) starts at \( 2\mu/3 \), we have introduced \( \delta \) as a convenient variable.

Non-perturbative constraints on \( W_{p,q} \)

By using eq. (3.1), we find that

\[
B^{(\ell)}_{n,m}(s_1) = \mathcal{I}^{(\ell)}_{n,m,m}(s_1) C^{(\alpha+m)}_{\ell-m} \left( \frac{2\mu}{3\delta} + 1 \right) + \sum_{k=0}^{m-1} \mathcal{I}^{(\ell)}_{n,m,k}(s_1) (-1)^{k+m} C^{(\alpha+k)}_{\ell-k} \left( \frac{2\mu}{3\delta} + 1 \right), \tag{3.3}
\]
with \( s_1 = \frac{2u}{3} + \delta, \ \sqrt{\xi_0} = \frac{2u}{3\delta} + 1 \), where

\[
\mathcal{U}_{n,m,k}^{(\alpha)}(s_1) = \sum_{j=k}^{m} \frac{4^k \sqrt{\xi_0}^k (\alpha)_k (m + 2n - 3j)}{s_1^{m+2n} j! \Gamma(k) (m-k)! (j-k)! (n-m)!} (2j-k) \Gamma(2j-k)
\]

is positive for \( n \geq m \). Note that in the sum, \((-1)^{k+m}\) spoils the definite sign of \( B^{(\ell)}_{n,m}(s_1) \). We can search for \( \chi^{(r,m)}(\mu, \delta) \), such that

\[
\sum_{r=0}^{m} \chi^{(r,m)}(\mu, \delta) B^{(\ell)}_{n,m}(s_1) = \mathcal{U}_{n,m,m}^{(\alpha)}(s_1) C_{\ell-m}^{(\alpha+m)} \left( \frac{2\mu}{3\delta} + 1 \right) \geq 0,
\]

with \( \chi^{(m,m)}(\mu, \delta) = 1, \ \forall \ \ell \in 2\mathbb{Z}^> \), \( \delta \geq 0, \ \alpha \geq 0, \ \mu \geq 0, \ m \leq n \). A solution is easily found using the following recursion relation:

\[
\chi^{(r,m)}(\mu, \delta) = \sum_{j=r+1}^{m} (-1)^{j+r+1} \chi^{(j,m)}(\mu, \delta) \frac{\mathcal{U}_{n,j,r}^{(\alpha)}(\mu, \delta)}{\mathcal{U}_{n,j,j}^{(\alpha)}(\mu, \delta)}.
\]

Explicitly, for the first few cases, we have

\[
\chi^{(0,1)}(\mu, \delta) = \frac{6n+3}{6\delta + 4\mu}, \ \chi^{(0,2)}(\mu, \delta) = \frac{6n+3}{3(6\delta + 4\mu)^2}, \ \chi^{(1,2)}(\mu, \delta) = \frac{6n+3}{6\delta + 4\mu}.
\]

Using the recursion relation eq.(3.7) one can check that \( \chi^{(r,m)} > 0, \ for \ n \geq m \). Then eq.(2.8) leads to

\[
W_{n-m,m} + \sum_{r=0}^{m-1} \chi^{(r,m)}(\mu, \delta = 0) W_{n-r,r} \geq 0,
\]

which we will refer to as non-perturbative constraints to differentiate from the effective field theory constraints to be derived next.

**Constraints on \( W_{p,q} \) in EFTs**

In order to derive EFT bounds, we start with eq.(2.8) and write

\[
W_{n-m,m}^{(\delta_0)} \equiv \int_{\delta_0 + 2\mu/3}^{\infty} \frac{ds_1}{s_1} \Phi(s_1; \alpha) \sum_{\ell=0}^{\infty} (2\ell + 2\alpha) a_{\ell}(s_1) B_{n,m}^{(\ell)}(s_1), \ n \geq 1
\]

\[
\mathcal{U}_{n,m,k}^{(\alpha)}(s_1) = \frac{4^k \sqrt{\xi_0}^k (\alpha)_k (m + 2n - 3k)}{s_1^{m+2n} k! (m-k)! (n-m)!} \times 4F_3 \left( \frac{k}{2}, \frac{1}{2}, \frac{k}{2}, k - m, k - \frac{m+2n}{3}; 1; k+1, k-n+1, k - \frac{m+2n}{3}; 4 \right).
\]
which defines for us the Wilson coefficients and are the $W$'s in eq. (2.8) when $\delta_0 = 0$. In such cases, we can show that

$$B_{n,m}^{(\ell)} \left( \frac{2\mu}{3} + \delta \right) + \sum_{r=0}^{m-1} \chi_{n}^{(r,m)}(\mu, \delta_0) B_{n,r}^{(\ell)} \left( \frac{2\mu}{3} + \delta \right) \geq 0, \quad (3.11)$$

for $\delta \geq \delta_0$. See Appendix (B) for a verification. This leads to positivity constraints

$$W_{n-m,m}^{(\delta_0)} + \sum_{r=0}^{m-1} \chi_{n}^{(r,m)}(\mu, \delta = \delta_0) W_{n-r,r}^{(\delta_0)} \geq 0, \quad (3.12)$$

for $\mu \geq 0$, $\delta_0 \geq 0$, $m \leq n$. Since $B_{n,0}^{(\ell)}(\delta + 2\mu^3) = \frac{2C^{(\alpha)}_\ell(\delta + 2\mu^3) + 1}{\pi(\delta + 2\mu^3)^{\alpha}}$, we have

$$W_{n,0}^{(\delta_0)} = \frac{1}{(\delta_0 + 2\mu^3)^2} W_{n-1,0}^{(\delta_0)} \cdot \quad (3.13)$$

The $n = 2$ case of eq. (3.13) for $\delta_0 \gg \mu$, which gives [17, eq (4.2)]. The $m = 1$, $n = 1$ case of the eq. (3.12) for $\delta_0 \gg \mu$ was first derived in [14, eq (29)] and the $m = 1$, $n = 2$ cases can be found in [16, eq (6.1)] and [17, eq (4.2)]. The derivations of [14, 16, 17] are based on fixed-$t$ dispersion relation, while our derivation is manifestly crossing symmetric and directly involve $W_{n-m,m}$.

A straightforward application of our general formulae is the examination of the $n \gg m$ limit. We find simply

$$\sum_{r=0}^{m} \frac{n^{m-r}}{(m-r)!} \left( \frac{\delta_0 + 2\mu^3}{3} \right)^{m-r} \frac{W_{n-r,r}}{W_{n,0}} \geq 0, \quad (3.14)$$

for $n \gg m$. We have checked that tree level type II string theory, to be discussed below, respects this.

### 3.2 Null constraints

To derive lower bounds, we make use of the $n < m$ vanishing conditions arising from eq. (2.8). In the large $\delta$ limit, we have

$$B_{n,m}^{(\ell)}(\delta) = C^{(\alpha)}_\ell \frac{(1)}{\pi} \frac{D_{\ell,\alpha}^{(n,m)}}{\delta^{2n+m+1}} + O \left( \frac{\mu}{\delta^{2n+m+1}} \right),$$

$$D_{\ell,\alpha}^{(n,m)} = \sum_{j=0}^{m} (-1)^j (\frac{\ell}{2})_j \frac{(\ell + \alpha)_j}{(\alpha + 1/2)_j} \times \frac{4^j(3j - m - 2n)(-n)_m}{j!(m-j)!(-n)_{j+1}}, \quad (3.16)$$

\[A \text{ closed form exists}

$$D_{\ell,\alpha}^{(n,m)} = \frac{(m + 2n)(-n)_m}{n!} 4^j \left( -m, -\frac{n}{3} - \frac{2n}{3} + 1, -\frac{\ell}{2} + \frac{\ell}{2} + \alpha; 1 - n, -\frac{n}{3} - \frac{2n}{3}, \alpha + 1/2; 4 \right). \quad (3.15)$$
Then in the limit when $\delta_0 \gg \mu$ we have\footnote{In the large $\ell$ limit, $C_{\ell}^{(\alpha)}(1)D_{\ell,\alpha}^{(n,m)} \sim \ell^{2m+2\alpha-1}$. Convergence of the sum in (3.17) implies $a_\ell \sim \frac{1}{\ell^{m+2\alpha+2}}$, $\forall m$. Therefore, $a_\ell$ is “superpolynomially suppressed” in $\ell$.} \[
\int_{\delta_0}^{\infty} \frac{ds_1}{s_1^{\alpha+1/2}} \sum_{\ell=0}^{\infty} (2\ell + 2\alpha) a_\ell(s_1) C_{\ell}^{(\alpha)}(1) \frac{D_{\ell,\alpha}^{(n,m)}}{s_1^{2n+m}} = 0. \quad (3.17)\]

for $m > n$, $n \geq 1$. For example $m = 2$, $n = 1$ gives $D_{\ell,\alpha}^{(1,2)} = \frac{2\ell(\ell+2\alpha)(-11-10\alpha+2\ell(\ell+2\alpha))}{(2\alpha+1)(2\alpha+3)}$, which was first derived in \cite{16} from fixed-$t$ dispersion relations\footnote{Some of the “null constraints” listed in \cite{17} eq (3.29) are related to $D_{\ell,\alpha}^{(n,m)}$ as \[D_{\ell,\alpha}^{(1,2)} = \frac{n_4}{2(\alpha + \frac{1}{2})_2}, \quad D_{\ell,\alpha}^{(1,3)} = \frac{n_5}{12(\alpha + \frac{1}{2})_3}, \quad D_{\ell,\alpha}^{(1,4)} = \frac{n_6}{24(\alpha + \frac{1}{2})_4}, \quad D_{\ell,\alpha}^{(1,5)} - D_{\ell,\alpha}^{(2,3)} = \frac{n_7}{240(\alpha + \frac{1}{2})_5}. \quad (3.18)\]}. Once these null constraints are in place, a judicious use of Cauchy-Schwarz inequality as used in \cite{16}, or a more constraining numerical argument used in \cite{17} can be pursued to derive lower bounds. The existence of such bounds was originally emphasized in \cite{21}. Our approach gives completely general expressions for the independent null constraints.

4 Applications

We will now consider two applications. The first application will make use of the $n \geq m$ constraints while the second will use the $n < m$ constraints arising from eq.(2.8).

4.1 Tree level type II superstring theory

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{delta0_max.png}
\caption{$\delta_0^{(\text{max})}$ vs $2(2n + m)$, the derivative order, for $m = 1, 2, 3, 4, 5$.}
\end{figure}

The four dilaton closed superstring tree amplitudes, after subtracting out the massless pole contri-
In this section, we will exploit the null constraints arising from higher constraints. We can easily compute the \( W_{p,q} \), for example \([15]\), \( W_{0,0} = 2\zeta(3) \), \( W_{0,1} = -2\zeta(3)^2 \), \( W_{0,2} = \frac{2}{3} (2\zeta(3)^3 + \zeta(9)) \), \( W_{1,0} = 2\zeta(5) \), \( W_{1,1} = -4\zeta(3)\zeta(5) \). Suppose we are given the first few terms in the derivative expansion. From unitarity, we know that \( 0 < \frac{3\zeta(5)}{\delta_0} - 2\zeta(3)^2 > 0 \), which implies \( \delta_0 < 1.07644 \). Similarly from other conditions, we can show that \( \delta_0 < \delta_0^{(\text{max})} \). In figure (2), we have shown that for higher constraints\(^7\), the \( \delta_0^{(\text{max})} \) converges towards 1, which is exactly the location of the first massive pole.

### 4.2 Bound on total scattering cross-section

In this section, we will exploit the null constraints arising from \( m > n \) in (2.8) to bound total scattering cross-sections. We will be in \( d = 4 \) and we will use the standard notation \( s = s_1 + 4/3 \) with \( \mu = 4 \). The null constraints read

\[
\int_{4}^{\infty} \frac{ds}{(s - 4)^{2n + m + 1}} \sqrt{\frac{s}{s - 4}} \sum_{\ell=2n}^{L_{\text{max}}} (2\ell + 1) a_\ell(s) g^{(\ell)}_{m,n}(s) = 0, \tag{4.2}
\]

where

\[
g^{(\ell)}_{m,n}(s) \equiv \sum_{j=n}^{m} p^{(j)}_{\ell} (\xi_0)^j (4\xi_0)^{j} \frac{(3j - m - 2n)}{j!(m-j)!(j-n)!},
\]

for \( m > n, \ n \geq 1 \). For \( m + n \leq \ell, \ m > n, \ n \geq 1 \) we can verify that \( g^{(\ell)}_{m,n}(s) \geq 0 \). Then we can write (4.2) as

\[
\int_{4}^{\infty} \frac{ds}{(s - 4)^{2n + m + 1}} \sqrt{\frac{s}{s - 4}} \sum_{\ell=2n}^{L_{\text{max}}} (2\ell + 1) a_\ell(s) g^{(\ell)}_{m,n}(s) \leq 0, \tag{4.4}
\]

for \( m + n \leq L_{\text{max}}, \ m > n, \ n \geq 1 \). We have placed the contributions arising from \( \ell \geq L_{\text{max}} + 2 \) on the right which gives the inequality. From unitarity, we know that \( 0 \leq a_\ell(s) \leq 1 \). The inequalities (4.4) impose further conditions on the \( a_\ell(s) \). We convert the integral over \( s \) in (4.4) as a sum by defining \( s(k) = 4 + k \frac{(s_{\text{max}} - 4)}{N_{\text{max}}} \). Using the constraints (4.4), we want to bound the total scattering cross-section \( \sigma(s) \)\(^8\)

\[
\sigma(s) = \frac{16\pi}{s - 4} \sum_{\ell=0}^{\infty} (2\ell + 1) a_\ell(s) \approx \frac{16\pi}{s - 4} \sum_{\ell=0}^{L_{\text{max}}} (2\ell + 1) a_\ell(s). \tag{4.5}
\]

We maximize the value of \( \frac{s - 4}{16\pi} \times \sigma(s) = \bar{\sigma} = \sum_{\ell=0}^{L_{\text{max}}} (2\ell + 1) a_\ell(s) \). The bound is shown in figure (3) for various \( L_{\text{max}} = 36, 46, 60 \). A fit with \( \bar{\sigma}_0 \times s \log^2 \left( \frac{s}{\bar{\sigma}_0} \right) \) can be found. The \( L_{\text{max}} = 36 \) gives fit values

\^7 For example \( m = 3, n = 4 \) (22 derivatives), \( \delta_0^{(\text{max})} = 1.00437 \), for \( m = 1, n = 8 \) (34 derivatives), \( \delta_0^{(\text{max})} = 1.000000833 \).

\^8 See [20, 22] for conventions. We are truncating the spin sum to \( L_{\text{max}} \), assuming higher spins are suppressed.

\^9 For \( L_{\text{max}} = 36, s_{\text{max}} = 90, N_{\text{max}} = 480 \), for \( L_{\text{max}} = 46, s_{\text{max}} = 100, N_{\text{max}} = 500 \), for \( L_{\text{max}} = 60, s_{\text{max}} = 130, N_{\text{max}} = 550 \).
$\bar{\sigma} = 0.29, s_0 = 0.128$, similarly $L_{\text{max}} = 60$ gives fit values $\bar{\sigma}_0 = 0.22, s_0 = 0.066$. The convergence with the spin sum is suggested by the figure but appears to be slow for higher values of $s$. Also it is clear that the $\bar{\sigma}$ found using a typical S-matrix living on the boundary of the so-called river arising from the S-matrix bootstrap [23] or even the lake boundary in [24], is far below the numerical bound presented.

Note that Froissart bound, i.e., $\bar{\sigma} \lesssim \frac{s}{16} \log^2(s/s_0)$ (which is not valid for lower values of $s$ we are considering) is below the numerical bound. The main utility of our numerical bound is that it is valid for $s \geq 4$ unlike the Froissart bound which is valid for $s \gg 4$. It will be fascinating to derive analytic bounds using the present method and see if a stronger than Froissart bound is possible at higher energies. In existing derivations of the Froissart bound, the role of crossing symmetry has not been explored [11].

5 Discussion

The crossing symmetric dispersion relation approach, presented in this paper, promises to open up a new and efficient way to constrain field theories. It will be worth exploring these ideas further, as clearly, at least for the cases considered here, the crossing symmetric approach provides a more direct derivation of certain constraints. It will be very interesting to connect the ideas and techniques in this paper with the “EFT-hedron” picture in [26]. Another place where we expect these crossing symmetric dispersion relations to play an important role [27] is the formulation of the dual S-matrix bootstrap in higher dimensions. So far, an explicit attempt has only been made in 2 dimensions [28].

On the CFT side, we will show in [12] how the manifestly crossing symmetric method extended to CFT Mellin amplitudes leads to the sum rule constraints arising from the two-channel dispersion relation presented in [3], [4]. Furthermore, the CFT generalized null constraints admit a straightforward

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10 We have used $s_0 = 0.015$ in the Froissart bound plot in figure [3]. This is the minimum possible value of $s_0$, see [25].

11 Also note that we did not impose polynomial boundedness directly in what we did; however, we are implicitly assuming convergence of eq.(4.2).
derivation and are needed to show the equivalence. This suggests that the manifestly crossing symmetric dispersion relation will not only be more systematic but will have more constraints than what is easily derivable in the two-channel symmetric approach.

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A Crossing symmetric dispersion relation

This section reviews the derivation of a crossing symmetric dispersion relation based on [6].

Image of the physical cuts

The image of the three physical cuts in the \( z \)-plane, \( V(a) = V_1(a) \cup V_2(a) \cup V_3(a) \) can be derived from (2.1). For example, we want \( s_1 \geq \frac{2\mu}{3} \) (or \( s \geq \mu \)), which is only possible if \( \text{Im } (s_1) = 0 \). Now \( \text{Im } (s_1) = 0 \) if \( a = 0 \) or \( |z| = 1 \) or \( \arg(z) = \pi \). Let us take the case of \( |z| = 1 \), then conditions \( s_1 \geq \frac{2\mu}{3} \) and \( -\frac{\mu}{3} \leq a < \frac{2\mu}{3} \) constrain the value of \( \arg(z) \). For \( -\frac{2\mu}{3} < a < 0 \)

\[
\frac{2\pi}{3} \leq \arg(z) \leq \phi_0(a),
\]
also for \( 0 < a < \frac{2\mu}{3} \)

\[
\phi_0(a) \leq \arg(z) \leq \frac{2\pi}{3}.
\]

Another case of \( \arg(z) = \pi \) can be easily worked out similarly, which leads to [6 eq (3.11)]

\[
V_1(a) = \begin{cases} 
\{ z; |z| = 1, \frac{2}{3} \pi \leq |\arg z| \leq \phi_0(a) \} & \text{if } -\frac{2\mu}{3} < a < 0 \quad \text{(I)}, \\
\{ z; |z| = 1, \phi_0(a) \leq |\arg z| \leq \frac{2\pi}{3} \} & \text{if } 0 < a < \frac{2\mu}{3} \quad \text{(II)}, \\
\{ z; |z| = 1, \frac{2}{3} \pi \leq |\arg z| \leq \pi \} \cup \{ z; \rho_-(a) \leq |z| \leq \rho_+(a), \arg(z) = \pi \} & \text{if } a \leq -\frac{2\mu}{3} \quad \text{(III)},
\end{cases}
\]

\[
V_2(a) = e^{2\pi i/3}V_1(a), \quad V_3(a) = e^{4\pi i/3}V_1(a).
\]

(A.1)

with

\[
\phi_0(a) = \tan^{-1} \left\{ \frac{\left( \frac{2\mu}{3} - a \right) \left( a + \frac{2\mu}{9} \right)}{a - \frac{2\mu}{9}} \right\}, \quad 0 < \phi_0 \leq \pi
\]

(A.2)

\[
\rho_\pm(a) = \frac{\mu}{4\mu} \left\{ \left( \frac{2\mu}{9} - a \right) \pm \left( \frac{2\mu}{3} - a \right) \left( -a - \frac{2\mu}{9} \right) \right\}^{1/2}
\].

11
Figure 4: All three cases of $V(a)$. In the above plot we have assumed $\mu = 4$.

**Parametric Dispersion Relation**

We denote the $k$-channel discontinuity of $M(s_1, s_2)$ as $A_k(s_1, s_2)$. The usual definition, for example, in the $s$-channel is

$$A_1(s_1, s_2) \equiv \lim_{\epsilon \to 0} \frac{1}{2i} [M(s_1 + i\epsilon, s_2) - M(s_1 - i\epsilon, s_2)], \quad s \geq 2\mu/3. \quad (A.3)$$

In a similar fashion, the discontinuity across $V(a)$ is defined by

$$\mathcal{A}(z, a) \equiv \begin{cases} 
\lim_{\epsilon \to 0} \frac{1}{2i} \left[ M((1 + \epsilon)z, a) - M((1 - \epsilon)z, a) \right] & \text{for } |z| = 1 \\
\lim_{\epsilon \to 0} \frac{1}{2i} \left[ M(ze^{i\epsilon}, a) - M(ze^{-i\epsilon}, a) \right] & \text{for } \arg z = \pi \left( \text{mod } \frac{2}{3}\pi \right)
\end{cases} \quad (A.4)$$

The condition $M(s, t) = M^*(s^*, t^*)$ translates in to

$$\overline{M}(z, a) = \overline{M}^*(1/z^*, a), \quad (A.5)$$

which directly follows from \(2.1\). This relation implies

$$\mathcal{A}(z, a) = \mathcal{A}^*(z, a) \quad \text{for } |z| = 1$$

$$\mathcal{A}(z, a) = -\mathcal{A}^* \left( \frac{1}{z^*}, a \right) \quad \text{for } \arg z = \pi \left( \text{mod } \frac{2}{3}\pi \right) \quad (A.6)$$

In order to relate $\mathcal{A}(z, a)$ to $A_k(s, t)$, we first note that $V(a)$ can be split into two subsets depending on the sign of $\text{Im}(s_k)$, namely

$$V_k(a) = V^+_k(a) \cup V^-_k(a), \quad \text{with } \text{Im} \left( \frac{z}{z_k} \right) \quad \text{or } \quad \left( |z| - 1 \right) \begin{cases} 
\geq 0 & \text{in } V^+_k(a) \\
\leq 0 & \text{in } V^-_k(a)
\end{cases} \quad (A.7)$$
If \( z \) is in \( V_k^±(a) \) then the sign of \( \text{Im}(s_k) \) is positive and vice-versa, which implies

\[
\mathcal{A}(z, a) = \begin{cases} 
\mathcal{A}_k(s_1, s_2), & z \in V_k^+(a) \\
-\mathcal{A}_k(s_1, s_2), & z \in V_k^-(a) 
\end{cases} \quad a < 0 \quad \text{(cases I and III)}
\]

\[
\mathcal{A}(z, a) = \begin{cases} 
-\mathcal{A}_k(s_1, s_2), & z \in V_k^+(a) \\
\mathcal{A}_k(s_1, s_2), & z \in V_k^-(a) 
\end{cases} \quad a > 0 \quad \text{(case II)}
\]

(A.8)

Following the standard practice of allowing for at most two subtractions \[20\],

\[
\mathcal{M}(s_1, s_2) = o\left( s_1^2 \right) \quad \text{for } |s_1| \to \infty, \quad s_2 = \text{fixed}, \quad (s_1, s_2) \in D
\]

(A.9)

Here \( D \) denotes the Martin domains \[6, 20\]. Similarly for fixed \( s_1, s_3 \). We want to study a similar behaviour of \( \mathcal{M}(z, a) \) around \( z \to z_k \). We notice that

\[
s_k \to a, \quad s_j \simeq \text{constant} \times \frac{a}{z - z_k} \quad (j \neq k).
\]

(A.10)

Therefore, we have

\[
\mathcal{M}(z, a) = o\left( \frac{1}{(z - z_k)^2} \right), \quad \text{as } z \to z_k, \quad a \text{ fixed}
\]

(A.11)

We Taylor expand the amplitude (convergent for \(|z| < \rho_-(a)\) or \(|z| < 1\))

\[
\mathcal{M}(z, a) = \sum_{n=0}^{\infty} f_n(a) z^n
\]

(A.12)

Note that

\[
\mathcal{M}(0, a) = \mathcal{M}(s_1 = 0, s_2 = 0) = f_0 = \mathcal{M}(\infty, a) = \mathcal{M}^*(0, a),
\]

(A.13)

therefore \( f_0 \) is real and independent of \( a \). We can write

\[
\frac{1}{2\pi i} \oint_{\mathcal{I}} dz' \frac{z'^3 - 1}{z'^3 (z' - z)} \mathcal{M}(z', a) = \frac{z^3 - 1}{z^3} \mathcal{M}(z, a) + f_0 \frac{a}{z^3} + \frac{f_1(a)}{z^2} + \frac{f_2(a)}{z}
\]

(A.14)

where \( \mathcal{I}, \mathcal{E} \) are the interior and exterior of \( V(a) \) with \( z \notin V(a), \) \(|z| < 1\). We can subtract these two equations on \( \mathcal{I}, \mathcal{E} \). Now as \( \mathcal{I}, \mathcal{E} \) approaches to \( V(a) \), we get the dispersion relation across \( V(a) \), which is given in eq (2.2).

**Dispersion relation across \( V(a) \) in \( s_1, s_2 \) variables:**

12In case of \( \mathcal{I} \), the terms involving \( f_0, f_1, f_2 \) follows from the residue at \( z = 0 \), and the term \( \frac{z^3 - 1}{z^3} \mathcal{M}(z, a) \) comes from residue at \( z = z' \). In case of \( \mathcal{E} \), we can take the contour upto infinity, and pick up \( \infty \).
While writing dispersion relation across $V(a)$ in $s_1, s_2$ variables, simplifications occur by noticing that

\[
9 a z (1 - z^3)^{-1} = \sum_{k=1}^{3} z_k s_k
\]

(A.15)

\[
9 a z^2 (1 - z^3)^{-1} = -\sum_{k=1}^{3} z^{-1}_k s_k.
\]

Then we translate (2.2) is $s_i$ variables (for Case-I)

\[
\mathcal{M}(s_1, s_2) = f_0 + \frac{f_1(a)}{9 a} (z_1 s_1 + z_2 s_2 + z_3 s_3) - \frac{f_2(a)}{9 a} \left( \frac{s_1}{z_1} + \frac{s_2}{z_2} + \frac{s_3}{z_3} \right)
\]

\[
+ \frac{(s_1/z_1 + s_2/z_2 + s_3/z_3)^2}{9 a (z_1 s_1 + z_2 s_2 + z_3 s_3) \pi} \int_{\frac{2\mu}{3}}^{\infty} ds_1' \left[ K_+ \left( s_1'; s_1, s_2, s_3 \right) A_1 \left( s_1', s_2^+ \left( s_1', a \right) \right) \right. \\
+ K_- \left( s_1'; s_1, s_2, s_3 \right) A_1 \left( s_1', s_2^- \left( s_1', a \right) \right) \\
\left. + \int_{\frac{2\mu}{3}}^{\infty} ds_2' \left[ 1' \rightarrow 2'; (123) \rightarrow (231) \right] + \int_{\frac{2\mu}{3}}^{\infty} ds_3' \left[ 1' \rightarrow 3'; (123) \rightarrow (312) \right] \right] 
\]  

(A.16)

with

\[
s_2^{(\pm)} (s_1', a) = -\frac{s_1'}{2} \left[ 1 \pm \left( \frac{s_1' + 3a}{s_1' - a} \right)^{1/2} \right]
\]

\[
K_\pm (s_1'; s_1, s_2, s_3) = \frac{2s_1' - 3a}{2 (s_1')^2} \left[ 1 \pm \left( \frac{s_1' - a}{s_1' + 3a} \right)^{1/2} \right] \\
\times \left( z_1 s_1 + z_2 s_2 + z_3 s_3 \right) \left[ z_1 s_1 + z_2 s_2 + z_3 s_3 + \left( \frac{s_1}{z_1} + \frac{s_2}{z_2} + \frac{s_3}{z_3} \right) \frac{1}{z_\pm (s_1', a)} \right]^{-1}
\]

\[
z_\pm' (s_1', a) = -\frac{1}{2} s_1' \left\{ (s_1' - 3a) \mp i \left[ 3 \left( s_1' - a \right) \left( s_1' + 3a \right) \right]^{1/2} \right\}
\]

(A.17)

The derivation uses the fact that $\frac{z^3}{(1-z)}$ can be written in terms of $s_i$ using eq. (A.15). For simplicity take the first two terms. We write $z'$ in terms of $s_1'$ and then $s_2'$ in terms of $s_1'$. Let's call them $s_2^{(\pm)} (s_1', a)$, which are given in (A.17). Now we want to write $dz' \frac{z^3 (1-z)}{z^3 (1-z)}$ in terms of $s_1'$, $s_i$. Here, again we write $z$ in terms of $s_i$ by the formula (A.15) and replace $z'$ in terms of $s_1'$ which leads to $ds_1' K_+(s_1'; s_1, s_2, s_3)$ and $ds_1' K_-(s_1'; s_1, s_2, s_3)$. Similarly other terms follow.

### Completely crossing symmetric amplitudes

This section considers completely crossing symmetric case (for example $\pi^0\pi^0$ scattering). In such a case $\mathcal{M}(z, a)$ should be a function of $z^3$.

\[
\mathcal{M}_0(z, a) = \sum_{n=0}^{\infty} \alpha_n (a) z^{3n}.
\]

(A.18)
Therefore the dispersion relation on $V(a)$, i.e. eq (2.2) becomes

$$\mathcal{M}_0(z,a) = \alpha_0 + \frac{z^3}{(1-z^3)\pi} \int_{V(a)} dz' \frac{z'^\beta - 1}{z'^\beta (z' - z)} \mathcal{A}(z', a).$$  \hfill (A.19)

In the above equation, we are only interested in powers like $z^{3n}$, $n \in \mathbb{N}$. While other powers like $z^m, m \notin 3\mathbb{N}$ should disappear from the RHS of (A.19). We achieve this as follows. Comparing (A.19) with (A.18), we find

$$\alpha_n(a) = \frac{1}{\pi} \int_{V(a)} dz' \mathcal{A}(z', a) \frac{1 - z'^{-3n}}{z'}, \quad n \geq 1. \hfill (A.20)$$

We put the above (A.20) formula in (A.18), and doing the sum over $n$, we get

$$\mathcal{M}_0(z,a) = \alpha_0 + \frac{z^3}{(1-z^3)\pi} \int_{V(a)} dz' \frac{z'^\beta - 1}{z'^\beta (z'^3 - z^3)} \mathcal{A}(z', a).$$  \hfill (A.21)

While writing the above equation in $s_1$, $s_2$ variable, we follow the similar method as we got the second line of (A.16) (for $-2\mu/9 < a < 0$). We solve for $z'$ in terms of $s'_1$. Notice that $z'$ has two solutions in terms of $s'_1$. If one solution is $z'(s'_1)$ then other solution is $z''(s'_1)$, also $|z'(s'_1)| = 1$. First two terms are given by

$$\frac{z^3}{(1-z^3)} \frac{dz'(s'_1)}{ds'_1} \frac{z'^\beta(s'_1) - 1}{z'^\beta(s'_1) - z^3} \mathcal{A}(z'(s'_1), a) + \frac{z^3}{(1-z^3)} \frac{dz''(s'_1)}{ds'_1} \frac{z''\beta(s'_1) - 1}{z''\beta(s'_1) - z^3} \mathcal{A}(z''(s'_1), a).$$ \hfill (A.22)

Now we note that \( \mathcal{A}(z'(s'_1), a) = \mathcal{A}(z''(s'_1), a) = A_1(s'_1, s_2^+(s'_1, a)) \). One can easily see that $\mathcal{M}_0(z,a)$ is function of combinations of $z^3/(z^3-1)^3$. This property follows from complete crossing symmetry, then $\mathcal{A}(z,a)$ should also have this property. We also note $\frac{z^3}{(z^3-1)^3}$ is invariant under $z \rightarrow \frac{1}{z}$ and $z* = \frac{1}{z}$ for $|z| = 1$. We finally have (2.3).

**B Some algebraic details**

**Derivation of eq (3.11)**

We note that

$$\mathcal{B}_{n,1}^{(f)} \left( \delta + \frac{2\mu}{3} \right) + \chi^{(0,1)}(\mu, \delta_0) \mathcal{B}_{n,0}^{(f)} \left( \delta + \frac{2\mu}{3} \right)$$

$$= \frac{(\delta - \delta_0)^{3n+1}(2n + 1)(3\delta + 2\mu)^{-2n-1}}{\pi (3\delta_0 + 2\mu)} \mathcal{C}_{\ell}^{(a)} \left( \frac{2\mu}{3\delta} + 1 \right) + \frac{8\alpha}{\pi \delta} \frac{2\mu}{3\delta} \mathcal{C}_{\ell-1}^{(a+1)} \left( \frac{2\mu}{3\delta} + 1 \right) \geq 0 \quad (B.1)$$
for $\delta \geq \delta_0$. Similarly

$$\mathcal{B}_{n,2}^{(\ell)} \left( \delta + \frac{2\mu}{3} \right) + \chi^{(1,2)}(\mu, \delta_0)\mathcal{B}_{n,1}^{(\ell)} \left( \delta + \frac{2\mu}{3} \right) + \chi^{(0,2)}(\mu, \delta_0)\mathcal{B}_{n,0}^{(\ell)} \left( \delta + \frac{2\mu}{3} \right)$$

$$= \frac{(\delta - \delta_0)3^{2n+3}(3\delta + 2\mu)^{-2(n+1)}}{2\pi (3\delta_0 + 2\mu)^2} \left( 6\delta_0 + 3\delta(4n + 3) + 2\mu(4n + 5) + 6n^2(\delta - \delta_0) - 2\right) C_{\ell}^{(\alpha)} \left( \frac{2\mu}{3\delta} + 1 \right) + 16\alpha(\alpha + 1)9^n(3\delta + 2\mu)^{-2n} C_{\ell-1}^{(\alpha+1)} \left( \frac{2\mu}{3\delta} + 1 \right) + 16\alpha(\alpha + 1)9^n(3\delta + 2\mu)^{-2n} C_{\ell-2}^{(\alpha+2)} \left( \frac{2\mu}{3\delta} + 1 \right) \geq 0$$

for $\delta \geq \delta_0$. Similar strategy works up to $m = 4$. For $m = 5$ onwards we have verified these inequalities numerically.

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