The phase diagram of Lévy spin glasses

I Neri, F L Metz and D Bollé

Institute of Theoretical Physics, Katholieke Universiteit Leuven, B-3001 Heverlee, Belgium
E-mail: izaak.neri@fys.kuleuven.be, fernando@itf.fys.kuleuven.be and desire.bolle@fys.kuleuven.be

Received 8 October 2009
Accepted 11 December 2009
Published 21 January 2010

Online at stacks.iop.org/JSTAT/2010/P01010
doi:10.1088/1742-5468/2010/01/P01010

Abstract. We study a fully connected spin-glass model in which the coupling strengths are drawn from a Lévy distribution. In this model each spin can be seen as interacting through a finite number of strong bonds and an infinite number of weak bonds, due to the large tail behavior of the coupling distribution. We solve the model through the replica and the cavity method. The hybrid behavior of Lévy spin glasses becomes transparent in our solution: the local field contains a part propagating along a backbone of strong bonds and a Gaussian noise term due to weak bonds. Our method allows us to determine the complete replica symmetric phase diagram, the replica symmetry breaking line and the entropy. The results are compared with ones from simulations and previous calculations using a Gaussian ansatz for the distribution of fields.

Keywords: cavity and replica method, disordered systems (theory), spin glasses (theory)
1. Introduction

The prototype mean-field model of spin glasses is the Sherrington–Kirkpatrick (SK) model [1]–[3]. In this fully connected (FC) system any pair of spins is coupled through weak interactions, of order $O(N^{-1/2})$ in the total number of spins $N$, whose values are drawn independently from a Gaussian distribution. Within the assumption that the free-energy landscape contains one single valley, the effective field on a given spin is a sum of a large number of uncorrelated random variables with a finite variance and the usual central limit theorem (CLT) holds. As a consequence, the effective field follows a Gaussian distribution fully characterized by its first two moments, leading to a description in terms of two observables: the magnetization and the Edwards–Anderson order parameter. The CLT reflects the independence of the macroscopic behavior of the system with respect to the details of the coupling distribution. The existence of a CLT in the SK model is technically very convenient and simplifies the replica and the cavity method at high temperatures $T$. At low temperatures extreme values become important and a more complicated description is necessary. The success of the cavity and replica method lies in the detailed and exact description that they give of the intricate behavior of the SK model at low temperatures, characterized by the presence of several degenerate states separated by infinite barriers [3].

However, it was shown that materials composed of magnetic impurities, randomly distributed in a non-magnetic host and interacting through the RKKY dipolar potential,
exhibit a Cauchy distribution of effective fields. This is in particular true for a small concentration of magnetic impurities [4,5]. An analogous result was obtained for a spatially disordered system of particles with dipolar interactions [6]. These results suggest that the choice of a coupling distribution that allows a wider variation of coupling strengths would be more realistic than the traditional Gaussian assumption used in mean-field models for spin glasses.

Disordered systems in which the randomness of the disorder variable \( J \) is modeled by a distribution \( P(J) \) that has a power-law decay \( P(J) \sim |J|^{-1-\alpha} \) (\( \alpha < 2 \)), for large \( |J| \), have attracted less interest. A possible reason is the technical challenge of dealing with distributions that do not fulfill the classical CLT. The heavy tails of \( P(J) \) give rise to the divergence of the second moment of the distribution, which invalidates the application of classical CLT. In this case, the generalized CLT of Lévy and Gnedenko holds [7,8], and the sum of a large number of independent random variables drawn from \( P(J) \) follows the same distribution as the individual summands, exhibiting only different scale factors. The role of the large tails of \( P(J) \) has proven to be crucial to the long time or large size properties of different disordered systems [9]. As examples in this context, we mention the theory of random matrices [10]–[12], diffusion processes [13] and the portfolio optimization problem in theoretical finance [14].

A FC model of spin glasses with interactions drawn from a distribution with power-law tails (a Lévy spin glass) was introduced by Cizeau and Bouchaud [15]. In Lévy spin glasses every spin interacts with infinitely many weak bonds of order \( O(N^{-1/\alpha}) \) and a finite number of strong bonds of order \( O(1) \). In this sense the model is a hybrid between a FC spin glass, like the SK model, and a finitely connected (FiC) spin glass, like the Viana–Bray model [16]. The authors of [15] studied the model with the cavity method under the assumption that the distribution of effective fields is Gaussian. They found a spin-glass phase stable under replica symmetry breaking and it was conjectured that at zero temperature the stability of replica symmetry is restored for \( \alpha < 1 \). Recently, this model has been studied with replica theory [17]. The effective field distribution is not Gaussian. In [17] a complete phase diagram and a discussion of replica symmetry breaking were not given.

The purpose of this paper is to improve upon the foregoing studies by deriving the complete phase diagram without the Gaussian assumption, the entropy and the stability against replica symmetry breaking effects. We propose a method that consists in the insertion of a small cutoff in the distribution of the couplings \( P(J) \), which gives rise to a natural distinction between ‘weak bonds’ and ‘strong bonds’. This allows us to solve the problem through both the replica method and the cavity method. We obtain a solvable self-consistent equation for the distribution of effective fields. Formally this equation is similar to the self-consistent equation for the effective field distribution of a FiC spin-glass system on a random graph [18] and a straightforward implementation of the population dynamics algorithm [19] is possible. Therefore, the procedure allows us to obtain the complete phase diagram of the model for all Lévy distributions, in contrast to previous works [15,17]. We include a skewness parameter in the definition of the model, responsible for controlling the relative weight of the positive and the negative tails of the coupling distribution. The dependence of the different phases on this parameter is shown in the phase diagrams. The results are compared with ones from simulations. We calculate the entropy of the system and the stability against replica symmetry breaking. Our results
are compared with those obtained by Cizeau and Bouchaud [15]. After we submitted the paper a preprint appeared [20] where similar issues were addressed.

The paper is organized as follows. In section 2, we define the model. We explain how to calculate the self-consistent equation for the distribution of effective fields through the replica method in section 3. In section 4, we show how this equation can be derived through the cavity method, and the differences with respect to the cavity approach as presented in [15] are clarified. In these sections, we compare the numerical results for the distribution of effective fields with ones obtained making the Gaussian assumption. The theoretical results for the magnetization are compared with ones from simulations. In section 5 we derive the stability condition against replica symmetry breaking. The order parameter equations, derived in sections 3–5, are solved numerically to obtain the phase diagrams and the entropy in sections 6 and 7. In section 8 we present a conclusion. The effects of the different parameters of the Lévy distributions are shown in appendix A. Some details of the replica calculations are shown in appendix B.

2. The Lévy spin glass

We study a FC system of $N$ Ising spins $\sigma_i = \pm 1$ ($i = 1, \ldots, N$) with the Hamiltonian

$$H(\{\sigma_i\}_{i=1,\ldots,N}) = -\sum_{i<j} J_{ij} \sigma_i \sigma_j,$$

(1)

where the symmetric couplings $\{J_{ij}\}$ are i.i.d. r.v. drawn from a stable distribution $P_{\alpha,J_0}(J)$. We define the stable distributions $P_{\alpha,J_0}(J)$ through their characteristic function $L_{\alpha,J_0}(q)$:

$$P_{\alpha,J_0}(J) \equiv \int \frac{dq}{2\pi} \exp(-iqJ)L_{\alpha,J_0}(q).$$

(2)

The characteristic function is of the form

$$L_{\alpha,J_0}(q) = \exp\left[\frac{i\alpha}{N} \cdot \frac{J_0 q}{\sqrt{2}N^{1/\alpha}} - |q|^\alpha (1 - i\gamma \Phi \text{sign}(q))\right].$$

(3)

The distribution $P_{\alpha,J_0}(J)$ is characterized by four parameters: the exponent $\alpha \in (0,1) \cup (1,2]$, the skewness $\gamma \in [-1,1]$, the scale parameter $J_0 > 0$ and the shift $J_0 \in \mathbb{R}$. The quantity $\Phi$ is given by $\Phi = \tan(\alpha\pi/2)$. The scaling with $N$ in equation (3) ensures that the Hamiltonian (1) is of order $O(N)$. Lévy distributions contain two different parameters that control the bias in the couplings: $J_0$ and $\gamma$. We refer the reader to appendix A for a discussion of the role of $\alpha$ and $\gamma$. For $\alpha = 1$ and $\gamma \neq 0$ the quantity $\Phi$ has a different expression and we will not consider this case in the sequel.

The SK model is obtained for $\alpha = 2$ independent of $\gamma$: in this case the distribution $P_{\alpha,J_0}(J)$ is Gaussian with mean $J_0/N$ and variance $J_1^2/N$ [1]. For $\alpha < 2$ and $-1 < \gamma < 1$, the asymptotic behavior $\rho(J)$ of $P_{\alpha,J_0}(J)$ for $|J| \to \infty$ can be derived from the explicit form of $L_{\alpha,J_0}(q)$:

$$\rho(J) \equiv N \lim_{|J| \to \infty} P_{\alpha,J_0}(J) = (1 + \gamma \text{sign } J) \frac{C_\alpha}{|J|^{\alpha+1}},$$

(4)

doi:10.1088/1742-5468/2010/01/P01010
where

$$C_\alpha = \left( \frac{J_1}{\sqrt{2}} \right)^\alpha \frac{1}{\pi} \sin \left( \frac{\alpha \pi}{2} \right) \Gamma(\alpha + 1).$$

(5)

Accordingly, the integrals for the second and higher moments of the distribution diverge for $\alpha < 2$ due to the power-law decay illustrated by equation (4).

We can define stable distributions through (3) without losing any generality; see for example [21, 22]. We remark that there are many equivalent definitions possible for the characteristic function $L$; see [22]. Loosely speaking, a random variable $x$ is stable if the sum of a given number of independent and identical copies of $x$ is characterized by the same distribution as the original variable, exhibiting only a different scale and shift.

3. The replica method

3.1. The distribution of effective fields

In order to study the thermodynamic behavior of the Lévy spin glass we employ the replica method [3]. The partition function of the system at inverse temperature $\beta = T^{-1}$ is defined by

$$Z = \sum_{\{\sigma\}_{i=1,\ldots,N}} \exp[-\beta H(\{\sigma\}_{i=1,\ldots,N})],$$

(6)

with $H(\{\sigma\}_{i=1,\ldots,N})$ given by equation (1). In the sequel we assume self-averaging of the free energy. The averaged free energy per spin $f$ can be written as follows:

$$f = -\lim_{N \to \infty} \lim_{n \to 0} \frac{1}{\beta N n} \ln \overline{Z^n}.$$

(7)

The symbol $\langle \cdots \rangle$ denotes the average over the quenched random couplings $\{J_{ij}\}$ with the distribution $P^{J_1,\gamma,J_0}(J)$. The quantity $\overline{Z^n}$ is computed for positive integers $n$ and the limit $n \to 0$ is taken through an analytic continuation to real values.

However, the integer moments $\overline{Z^n}$ of the partition function diverge for real $\beta$ due to the power-law behavior of $P^{J_1,\gamma,J_0}(J)$ for $|J| \to \infty$. As noted in [17], the introduction of an imaginary temperature $\beta = -ik$, with a real parameter $k > 0$, allows a straightforward calculation of the average $\overline{Z^n}$ by means of the definition of the characteristic function, equation (3). However, it is not possible to write the averaged $\overline{Z^n}$ in terms of the two standard order parameters usually employed in the description of FC systems, i.e., the magnetization and the spin-glass order parameter. Therefore, it is necessary to use the replica method, as developed to deal with FiC spin glasses [23]. The macroscopic behavior is characterized in terms of a non-Gaussian effective field distribution. This procedure was followed in [17]. Following their calculations we find the equation for the free energy $f$ in the limit $N \to \infty$:

$$f = f_1 + f_2.$$  

(8)
We use the replica symmetric (RS) ansatz,
\[ \langle \alpha \rangle \]
\[ W \]
The function equation (13) over the \( W \) which defines the field distribution T
\[ T = \exp \left( -\sum_{\tau} \left( -\left( \frac{J_i k}{\sqrt{2}} \right)^{\alpha} |\sigma \cdot \tau|^{\alpha} \right) \right) \times (1 + i\gamma \text{sign}(\sigma \cdot \tau)\Phi) - ikJ_0(\sigma \cdot \tau) \right] P(\tau), \]
\[ \text{(9)} \]
\[ ikf_2 = \lim_{n \to 0} \frac{1}{n} \log \left\{ \sum_{\sigma} \exp \left[ -\sum_{\tau} \left( -\left( \frac{J_i k}{\sqrt{2}} \right)^{\alpha} |\sigma \cdot \tau|^{\alpha} \right) \times (1 + i\gamma \text{sign}(\sigma \cdot \tau)\Phi) - ikJ_0(\sigma \cdot \tau) \right] P(\tau) \right\}. \]
\[ \text{(10)} \]
The order parameter \( P(\sigma) \), with \( \sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n) \), fulfills the self-consistent equation
\[ P(\sigma) = \frac{\exp(\sum_{\tau} P(\tau)[|\text{sign}(\sigma \cdot \tau)\Phi|] \right) \times (1 + i\gamma \text{sign}(\sigma \cdot \tau)\Phi) - ikJ_0(\sigma \cdot \tau) \right] P(\tau) \left\} \).
\[ \text{(11)} \]
We use the replica symmetric (RS) ansatz,
\[ P(\sigma) = \int dh W(h) \prod_{\alpha=1}^{n} \frac{\exp(-ik\alpha\sigma)}{2\cosh(-ik\alpha\sigma)}, \]
\[ \text{(12)} \]
which defines the field distribution \( W(h) \). Substitution of (12) in (11) gives
\[ W(h) = \int \frac{ds}{2\pi} \exp(ish) \exp \left\{ -\int dh W(h) \int \frac{\tilde{J} dJ}{2\pi} \times \left[ \left( \frac{J_i k}{\sqrt{2}} \right)^{\alpha} |\tilde{J}|^{\alpha}(1 + i\gamma\Phi \text{sign}(\tilde{J}) + iJ_0\tilde{J}] \exp[is\tilde{J}f(J,h,s)] \right\}. \]
\[ \text{(13)} \]
The function \( f(J,h,s) \) is defined as
\[ f(J,h,s) \equiv \exp \left( -\frac{is}{\beta} \tanh[\tanh(\beta J) \tanh(\beta h)] \right). \]
\[ \text{(14)} \]
The analytic continuation of \( T \) to real values has been achieved by taking \( k = i\beta \) at the end of the calculation. Using equations (B.1)–(B.5) from appendix B, we integrate in equation (13) over the \( \tilde{J} \) variable to obtain the following simplified expression for \( W(h) \):
\[ \text{0 < } \alpha < 1 : \]
\[ W(h) = \int \frac{ds}{2\pi} \exp \left\{ ish - isJ_0m + \int dh W(h) \int_{-\infty}^{\infty} dJ \rho(J)[f(J,h,s) - 1] \right\}, \]
\[ \text{(15)} \]
\[ \text{1 < } \alpha < 2 : \]
\[ W(h) = \int \frac{ds}{2\pi} \exp(ish - isJ_0m) \times \exp \left\{ \int dh W(h) \int_{-\infty}^{\infty} dJ \rho(J)[f(J,h,s) - f'(0,h,s)J - 1] \right\}, \]
\[ \text{(16)} \]
where \( f'(0,h,s) = (\partial f(J,h,s)/\partial J)|_{J=0} \). The distribution of couplings \( \rho(J) \) is defined by equation (4). The RS magnetization \( m \) and the RS spin-glass order parameter \( q \) are
The phase diagram of Lévy spin glasses

determined through the averages

\[ m = \int dh W(h) \tanh(\beta h), \quad q = \int dh W(h) \tanh^2(\beta h). \]  

(17)

Only the large tail behavior of the distribution \( P^{h_1,\gamma=0}_\alpha \) appears in the equations (15) and (16). This could mean that the system exhibits a certain degree of universality: the thermodynamic behavior only depends on the large tail behavior of the coupling distribution \( P(J) \). The distribution \( \rho(J) \) is symmetric when \( \gamma = 0 \), with equations (15) and (16) reducing to a single equation, obtained previously in [17].

3.2. The normalization of the coupling distribution through a cutoff

The main difficulty in equations (15) and (16) concerns the normalization of \( \rho(J) \) since the integral \( \int dJ \rho(J) \) diverges for \( \alpha < 2 \). Therefore, it is not possible to normalize the distribution. This invalidates the numerical calculation of \( W(h) \) through the population dynamics algorithm [24] because it is not possible to sample random numbers from a non-normalizable distribution.

In this subsection we propose a simple procedure that allows us to normalize \( \rho(J) \) and to derive a self-consistent equation for \( W(h) \) which is similar to the order parameter equation of FiC spin glasses on random graphs. The numerical solution of this equation can be obtained through population dynamics.

The method consists of the insertion of a temperature dependent cutoff \( T\epsilon > 0 \) in the integrals over \( J \) occurring in equations (15) and (16), splitting each of them into an integral around zero (from \(-T\epsilon \) to \( T\epsilon \)) plus an integral over the couplings that satisfy \( |J| > T\epsilon \). Assuming \( T\epsilon \ll 1 \), the integrations around zero can be analytically performed by expanding \( f(J, h, s) \) around \( J = 0 \) up to order \( O(J^2) \), resulting in the following equations:

\[
0 < \alpha < 1 : \\
\int_{-\infty}^{\infty} dJ \rho(J)[f(J, h, s) - 1] = -2is\gamma C_\alpha \tanh(\beta h) \frac{(T\epsilon)^{1-\alpha}}{1-\alpha} - s^2 C_\alpha \tanh^2(\beta h) \frac{(T\epsilon)^{2-\alpha}}{2-\alpha} + \int_{-\infty}^{\infty} dJ \rho(J)[\Theta(J - T\epsilon) + \Theta(-J - T\epsilon)][f(J, h, s) - 1],
\]

(18)

\[
1 < \alpha < 2 : \\
\int_{-\infty}^{\infty} dJ \rho(J)[f(J, h, s) - f'(0, h, s)J - 1] = -s^2 C_\alpha \tanh^2(\beta h) \frac{(T\epsilon)^{2-\alpha}}{2-\alpha} + \int_{-\infty}^{\infty} dJ \rho(J)[\Theta(J - T\epsilon) + \Theta(-J - T\epsilon)][f(J, h, s) - f'(0, h, s)J - 1].
\]

(19)

The symbol \( \Theta(J) \) denotes the Heaviside step function: \( \Theta(J) = 1 \) if \( J > 0 \) and \( \Theta(J) = 0 \) otherwise. We define the normalized distribution \( P_\epsilon(J) \) in terms of \( \rho(J) \):

\[
P_\epsilon(J) \equiv \frac{\alpha(T\epsilon)^\alpha}{2C_\alpha} \rho(J)[\Theta(J - T\epsilon) + \Theta(-J - T\epsilon)].
\]

(20)
Substituting equations (18) and (19) in, respectively, equations (15) and (16) the integrals over $s$ can be analytically calculated:

$$W_{\epsilon}(h) = \exp(-c) \sum_{k=0}^{\infty} \frac{c^k}{k!} \int \left( \prod_{r=1}^{k} \text{d} h_r \right) \int \left( \prod_{r=1}^{k} \text{d} J_r \right) \int Dz \times \delta \left( h - \tilde{J}_0 m - \beta^{-1} \sum_{r=1}^{k} \text{atanh} \left( \tanh(\beta J_r) \tanh(\beta h_r) \right) - \sqrt{2q\Delta} z \right),$$

where $Dz = (2\pi)^{-\frac{d}{2}} \exp \left( -\frac{z^2}{2} \right) \text{d} z$ and

$$c = \frac{2C_\alpha}{\alpha(T\epsilon)^{\alpha-1}}, \quad \Delta = \frac{(T\epsilon)^{2-\alpha}C_\alpha}{2 - \alpha},$$

$$\tilde{J}_0 = \left( J_0 + 2\gamma C_\alpha \left[ \frac{(T\epsilon)^{1-\alpha}}{1 - \alpha} \right] \right).$$

To describe the thermodynamic behavior of Lévy spin glasses one has to solve the set of equations (17) and (21) for $\epsilon \to 0$.

When we compare equation (21) with the order parameter equations of FiC systems [18], it describes the effective field distribution of a FiC system of Ising spins, in which the number of connections per site $k$ follows a Poissonian distribution with connectivity $c$. The values of the $k$ couplings attached to a site are drawn from the distribution $P_\epsilon(J)$; see equation (20). In addition, the analytical calculation of the integrals over the couplings that satisfy $|J| < T\epsilon$ yields an interaction with the global magnetization with effective strength $\tilde{J}_0$ and an extra source of noise in equation (21), represented by the Gaussian random variable $z$ with zero mean and variance $\Delta$. The effective strength contains the shift parameter $J_0$ and a term linear in $\gamma$ corresponding to the center of the distribution of the couplings. The interpretation of equation (21) is clear: the effective field contains a Poissonian term coming from a finite number of strong bonds and a Gaussian term coming from an infinite number of weak bonds that it interacts with. One can take the limit $\alpha \to 2$ to find the effective field $J_0 m + J_1 \sqrt{\Delta} z$, i.e. the RS solution of the SK model. The equations (17) and (21) show explicitly how Lévy spin glasses are a hybrid between FC and FiC models. Equation (21) is formally similar to the equation describing the behavior of composite models [25, 26], where each spin interacts through a finite number of strong couplings and an infinite number of weak couplings. The specific choice of the distribution of coupling strengths in [25, 26] allows an interpolation between the SK model and the Viana–Bray model. When one takes a Gaussian ansatz for the distribution $W_\epsilon$, equation (21) becomes in the limit $\epsilon \to 0$ equal to the result derived by Cizeau and Bouchaud [15].

The population dynamics algorithm [24] can be easily adapted to solve numerically equation (21) and obtain $W_\epsilon(h)$. The idea is to obtain numerical results for sufficiently small values of $T\epsilon$ in such a way that they can be extrapolated for $T\epsilon \to 0$: the first two moments of the distribution already obtain their limiting values around $T\epsilon \lesssim 0.5$. The equations become very hard to solve around $\alpha \approx 1.5$ because the mean connectivity $c$ has a maximum there. For low values of $\alpha \lesssim 0.1$ population dynamics becomes inaccurate because of numerical imprecisions due to the larger tails of the coupling distribution.

doi:10.1088/1742-5468/2010/01/P01010
Figure 1. The magnetization $m$ as a function of the temperature $T$ for several values of $\alpha$ and $J_0$. Simulation results (markers) are compared with results from the theory (solid lines) for $J_1 = 1$, $\gamma = 0$. At low temperatures theory and simulations are in good agreement. Because of the increase in the equilibration time around the critical temperature the results from simulation overestimate the magnetization. The inset confirms this: it shows the value of the magnetization as a function of the number of Monte Carlo sweeps for $\alpha = 0.5$, $J_0 = 0.75$ and $T = 1$.

In figure 1 we compare the solution of equations (17) and (21) with results from Monte Carlo simulations. We simulated a Lévy spin glass using the algorithm described in [17] without the parallel tempering. The algorithm contains two update rules: single-spin-flip updates as usually done in Metropolis algorithms and updates of clusters of spins connected through strong bonds. For low temperatures we find a good agreement between the simulations and the theory. Around the critical temperature the magnetization obtained by the simulations is larger than the one predicted by the theory. The reason for this difference is that the simulations equilibrate very slowly. Indeed, as shown in the inset of figure 1, the magnetization decays as a power law as a function of the number of Monte Carlo sweeps. The presence of strong bonds slows down the dynamics since the effect becomes larger for smaller values of $\alpha$. For very low temperatures the simulation results for the magnetization deviate from those of the RS study. The RS ansatz (12) is invalid for very low temperatures; see section 6.

In figure 2 we plotted the solution to the self-consistent equation (21) for different values of $\alpha$. The result is compared with the Gaussian ansatz one (solid lines), used in [15]. The difference between the two approaches is clear. For $\alpha \to 2$ the distribution of fields becomes more and more Gaussian. For $\alpha < 2$ the distributions of fields are not Lévy but leptokurtic distributions where the moments converge to a finite value as a function of the size of the population. Leptokurtic distributions have a smaller kurtosis than a Gaussian distribution with the same variance.

4. The cavity method

We derive the self-consistent equation (21) for $W(h)$ through the cavity method. Although one gets the same result as was obtained through the replica method, it is interesting to see how the derivation differs from the one in [15]. In contrast with [15] our approach
Figure 2. The distribution of effective fields with $J_0 = \gamma = 0$, $J_1 = 1$, $T = 0.4$ and several values of $\alpha$. The markers give the distributions according to equation (21) while the solid lines are obtained through the theory of [15]. All the moments of the distributions are finite. Therefore we have leptokurtic distributions which are neither Gaussian nor Lévy distributions.

only applies the CLT to the field coming from the weak bonds, i.e. bonds smaller than the cutoff $T \epsilon$. The bonds larger than $T \epsilon$ form a backbone graph of strong bonds which is treated as a FiC system. For $\epsilon \to \infty$ we get back the results of [15]. For $\epsilon \to 0$ we expect to find the RS behavior of the spin glass.

The marginal $P_i(\sigma_i) \equiv \sum_{\{\sigma_j\}_{j=1,...,N} \sigma_i} P(\{\sigma_j\}_{j=1,...,N})$ of the Gibbs distribution $P(\{\sigma_j\}_{j=1,...,N}) \sim \exp[-\beta H(\{\sigma_j\}_{j=1,...,N})]$ can be written as

$$P_i(\sigma_i) \sim \sum_{\{\sigma_j\}_{j=1,...,N} \sigma_i} P^{(i)}(\{\sigma_j\}_{j=1,...,N} \setminus \sigma_i) \exp \left( \sum_k J_{ki} \sigma_j \sigma_i \right),$$

(24)

with $P^{(i)}(\{\sigma_j\}_{j=1,...,N} \setminus \sigma_i)$ the Gibbs distribution on the cavity graph $G^{(i)}$. The cavity graph is the subgraph of the original graph $G$ where one has removed the $i$th spin and all of its interactions with the other spins. We assume that the probability distribution on the cavity graph factorizes [3]:

$$P^{(i)}(\{\sigma_j\}_{j=1,...,N} \setminus \sigma_i) = \prod_{j \neq i} P^{(i)}(\sigma_j).$$

(25)

This factorization is valid when there is one pure phase in the system. The set $\overline{\omega}^{(i)}$ of all weak bonds and the set $\omega^{(i)}$ of all strong bonds are defined through

$$\overline{\omega}^{(i)} \equiv \{ j \in N \cap [1,N] | J_{ij} < T \epsilon \},$$

$$\omega^{(i)} \equiv (N \cap [1,N]) \setminus \overline{\omega}^{(i)}.$$

(26)

(27)

The cavity fields $h_j^{(i)}$ and $g_j^{(i)}$ are defined through

$$h_j^{(i)} \equiv \sum_{\sigma} \frac{\sigma}{2} \log(P_j^{(i)}(\sigma)) \quad \text{if} \quad j \in \omega^{(i)},$$

$$g_j^{(i)} \equiv \sum_{\sigma} \frac{\sigma}{2} \log(P_j^{(i)}(\sigma)) \quad \text{if} \quad j \in \overline{\omega}^{(i)}.$$

(28)

(29)
The marginal \( P_i^{(j)} \) of the \( i \)th spin on the cavity graph \( G^{(j)} \) is equal to
\[
P_i^{(j)}(\sigma_i) \sim \prod_{k \in \omega_i \setminus j} \sum_{\tau} \exp(\beta J_{ki} \sigma_i \tau + \beta g_k^{(i)} \tau) \prod_{k \in \omega_i \setminus j} \sum_{\tau} \exp(\beta J_{ki} \sigma_i \tau + \beta h_k^{(i)} \tau). \tag{30}
\]

We used the notation \( h_{ij}^{(j)} \) for cavity fields where one has removed a site \( j \) connected with \( i \) through a strong bond and the notation \( g_{ij}^{(j)} \) for fields where the site \( j \) was connected with \( i \) through a weak bond. We thus find the following set of closed equations in the cavity fields \( h_{ij}^{(j)} \) and \( g_{ij}^{(j)} \):
\[
g_{ij}^{(j)} = z_{ij}^{(j)} + \beta^{-1} \sum_{k \in \omega_i \setminus j} \text{atanh}(\text{tanh}(\beta h_k^{(i)}) \text{tanh}(\beta J_{ki})) , \tag{31}
\]
\[
h_{ij}^{(j)} = z_{ij} + \beta^{-1} \sum_{k \in \omega_i \setminus j} \text{atanh}(\text{tanh}(\beta h_k^{(i)}) \text{tanh}(\beta J_{ki})) , \tag{32}
\]
where we defined a third field containing the contributions from the weak bonds:
\[
z_{ij}^{(j)} = \beta^{-1} \sum_{k \in \omega_i \setminus j} \text{atanh}(\text{tanh}(\beta g_{kj}^{(i)}) \text{tanh}(\beta J_{ki})). \tag{33}
\]
In the limit \( N \to \infty \) we can remove the \( j \) dependence in the fields \( z_{ij}^{(j)} \) and \( g_{ij}^{(j)} \) because the sum over the weak bonds \( (k \in \omega_i) \) contains an infinite number of terms.

To take the disorder average over the couplings we define the following distributions:
\[
W_w(g) \equiv \frac{\sum_{i=1}^{N} \delta(g - g_i)}{N}, \tag{34}
\]
\[
W_h(h) \equiv \frac{\sum_{i=1}^{N} \sum_{j \in \omega_i} \delta(h - h_{ij}^{(j)})}{\sum_{i=1}^{N} \sum_{j \in \omega_i}}, \tag{35}
\]
\[
W_z(z) \equiv \frac{\sum_{i=1}^{N} \delta(z - z_i)}{N}. \tag{36}
\]
We treat the \( z \)-fields as a sum of infinitely many random variables on which we can apply the CLT:
\[
W_z(z) = \frac{1}{\sqrt{4\pi \Delta q}} \exp \left( -\frac{(z - \tilde{J}_0 m)^2}{4\Delta q} \right) , \tag{37}
\]
with \( \Delta \) and \( \tilde{J}_0 \) as defined in equations (22) and (23). The parameters \( m \) and \( q \) determine, respectively, the mean and the variance of the Gaussian distribution \( W_z(z) \). Here is the important difference with [15] where the CLT is applied on all bonds, including on the strong ones.

From equation (33) one finds, for \( N \to \infty \) and \( \epsilon \ll 1 \), the following expressions for the mean \( m \) and the variance \( q \):
\[
m = (\tilde{J}_0)^{-1} \left( N \int_{-T_{\epsilon}}^{T_{\epsilon}} dJ P_{J_{\alpha,\gamma,\delta}}(J) J \right) \int d\tau \text{tanh}(\beta g) W_w(g),
\]
\[
q = (2\Delta)^{-1} \left( N \int_{-T_{\epsilon}}^{T_{\epsilon}} dJ P_{J_{\alpha,\gamma,\delta}}(J) J^2 \right) \int d\tau \text{tanh}^2(\beta g) W_w(g). \tag{39}
\]
The integrals over the couplings in equations (38) and (39) can be calculated using methods analogous to those used to derive the integrals in appendix B:

\[ N \int_{-T \epsilon}^{T \epsilon} dJ P_{\alpha}^{J_1, J_0}(J) J = \tilde{J}_0, \tag{40} \]

\[ N \int_{-T \epsilon}^{T \epsilon} dJ P_{\alpha}^{J_1, J_0}(J) J^2 = 2 \Delta. \tag{41} \]

Using the definitions of the distributions \( W_w(g) \) and \( W_s(h) \) in equations (34) and (35) we get

\[
W_s(h) = \sum_{k=0}^{\infty} \frac{p_{\text{Poisson}}(k; c)}{c} k \prod_{r=1}^{k-1} dh_r W_s(h_r) \int \prod_{r=1}^{k-1} dJ_r P_\epsilon(J_r) \int dz W_g(z) \\
\times \delta \left( h - z - \beta^{-1} \sum_{r=1}^{k-1} \text{atanh} (\tanh(\beta h_r) \tanh(\beta J_r)) \right), \tag{42} \]

\[
W_w(g) = \sum_{k=0}^{\infty} p_{\text{Poisson}}(k; c) \int \prod_{r=1}^{k} dh_r W_s(h_r) \int \prod_{r=1}^{k} dJ_r P_\epsilon(J_r) \int dz W_g(z) \\
\times \delta \left( g - z - \beta^{-1} \sum_{r=1}^{k} \text{atanh} (\tanh(\beta h_r) \tanh(\beta J_r)) \right). \tag{43} \]

The mean connectivity \( c \) is given by

\[
c = \lim_{N \to \infty} \frac{\sum_{i=1}^{N} |\omega^{(i)}(\epsilon)|}{N} = \int_{-\infty}^{\infty} dJ \rho(J) + \int_{-T \epsilon}^{-\infty} dJ \rho(J) = \frac{2C_\alpha}{\alpha(T \epsilon)^\alpha}, \tag{44} \]

with \( \rho(J) \) the large tail behavior as defined in (4). We use the following property of Poissonian distributions: \( \frac{1}{2} p_{\text{Poisson}}(k; c) = p_{\text{Poisson}}(k - 1; c) \) to find \( W_w(g) = W_s(g) = W_s(g) \), i.e. the solutions to (42) and (43) are the same as the solution \( W_\epsilon \) of (21). Indeed, equations (43) combined with (38) and (39) are identical to equations (17) and (21) derived with the replica method. From the cavity approach the importance of the CLT in Lévy spin glasses becomes clear: the couplings have a divergent variance; therefore one cannot apply the CLT as was done in [15]. We remark that the effective coupling \( \tilde{J}_0 \) and the parameter \( 2\Delta \) appearing in the replica method are the mean and the variance of the weak couplings. The distribution \( W_\epsilon(h) \) in equation (21) is the distribution of the cavity fields propagating along the backbone graph of strong bonds.

5. Stability of the replica symmetric ansatz

As is known from a local stability analysis [27], the RS ansatz introduced in (12) is unstable at low temperatures. It is possible to calculate the regions of stability by using the two-replica method, first introduced for FiC models in [28]. For models on graphs a relevant instability condition is proved rigorously in [29]. It determines the region where the message passing algorithms stop converging; see for example the discussion in [30].

We start by considering two uncoupled replicas. Both replicas fulfill the equations (30)–(33). The replicas only become coupled when we take the average over...
the graph instance. Indeed, the effective field distribution of two sets of uncoupled spins
on the same graph with the same couplings is given by

\[
W_\epsilon(h^1, h^2) = \sum_{k=0}^{\infty} \frac{P_{\text{Poisson}}(k; \rho)}{c} \prod_{r=1}^{k-1} \int dh_r^1 dh_r^2 W_\epsilon(h_r^1, h_r^2) \times \int \prod_{r=1}^{k-1} dJ_r \, P_r(J_r) \int dz^1 dz^2 W_g(z^1, z^2) \times \delta \left( h^1 - z^1 - \beta^{-1} \sum_{r=1}^{k-1} \tanh(\beta h_r^1) \tanh(\beta J_r) \right) \times \delta \left( h^2 - z^2 - \beta^{-1} \sum_{r=1}^{k-1} \tanh(\beta h_r^2) \tanh(\beta J_r) \right). \tag{45}
\]

We assume again that we can apply the CLT on the \( z \)-fields:

\[
W_g(z^1, z^2) = \frac{1}{4\Delta \pi \sqrt{q^1 q^2 (1 - \rho^2)}} \exp \left( - \frac{1}{2(1 - \rho^2)} \left( \frac{(z^1 - \tilde{J}_0 m^1)^2}{2\Delta q^1} + \frac{(z^2 - \tilde{J}_0 m^2)^2}{2\Delta q^2} \right) \right) \times \exp \left( \frac{\rho(z^1 - \tilde{J}_0 m^1)(z^2 - \tilde{J}_0 m^2)}{2(1 - \rho^2)\Delta \sqrt{q^1 q^2}} \right). \tag{46}
\]

The order parameters become

\[
m^1 = \int dg^1 dg^2 \tanh(\beta g^1) W_\epsilon(g^1, g^2), \tag{47}
\]

\[
q^1 = \int dg^1 dg^2 \tanh^2(\beta g^1) W_\epsilon(g^1, g^2), \tag{48}
\]

\[
m^2 = \int dg^1 dg^2 \tanh(\beta g^2) W_\epsilon(g^1, g^2), \tag{49}
\]

\[
q^2 = \int dg^1 dg^2 \tanh^2(\beta g^2) W_\epsilon(g^1, g^2), \tag{50}
\]

\[
\rho \sqrt{q^1 q^2} = \int dg^1 dg^2 \tanh(\beta g^2) \tanh(\beta g^1) W_\epsilon(g^1, g^2). \tag{51}
\]

In [29] it has been proven that convergence of the message passing algorithm on the related
finitely connected graph is equivalent to the following condition: if \( W_\epsilon^*(h) \) is the solution
to equation (21) then the initial distribution \( W_\epsilon^\text{init}(h_1, h_2) = W_\epsilon^*(h_1)W_\epsilon^*(h_2) \) must converge
to the diagonal measure \( W_\epsilon(h_1, h_2) = W_\epsilon^*(h_1)\delta(h_1 - h_2) \) under iteration of \( W_\epsilon^\text{init}(h_1, h_2) \)
through equation (45). This is the criterion that we use to determine the replica symmetry
breaking lines in Lévy spin glasses. Non-convergence of the message passing algorithm is a
sufficient condition for replica symmetry breaking but not a necessary condition. For fully
connected models, like the SK model or the \( p \)-spin model [31], a local stability analysis
of the two-replica method around the diagonal measure gives the AT line and is thus
equivalent to a local stability analysis in replica space [27]. The two-replica method leads
to the same results as a Hessian calculation in replica space since the transition of the bivariate distribution \( W(h_1, h_2) \) is continuous. In [20] the replica symmetry breaking line of Lévy spin glasses is calculated through the divergence of the spin-glass susceptibility, a method which is equivalent for FC models to a local stability analysis [3].

In the limit \( \alpha \to 2 \) we find \( W_\epsilon(h_1, h_2) = W_\delta(h_1, h_2) \). An expansion around the RS solution \( m^1 = m^2 = m, q^1 = q^2 = q \) and \( 1 - |\rho| \sim O(\delta) \), with \( \delta \ll 1 \), leads to the following stability condition:

\[
\beta^{-2} = \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} \exp \left( -\frac{u^2}{2} \right) \text{sech} \left( \beta \sqrt{\rho u + \beta J_0 m} \right).
\]

The parameters \((q, m)\) in (52) are, respectively, the overlap parameter and the magnetization of the SK model. Equation (52) is precisely the AT line of the SK model; see [27].

6. The phase diagram

The system shows four phases depending on the values of the order parameters \( m \) and \( q \) defined in (17) and on the stability of the RS ansatz: a paramagnetic phase (P) with \( m = q = 0 \), a stable ferromagnetic phase (F) with \( m > 0, q > 0 \), an unstable ferromagnetic phase with \( m > 0, q > 0 \) which is called the mixed phase (M) and a spin-glass phase (SG) with \( m = 0, q > 0 \).

The P–F and P–SG transitions are determined using an expansion of the self-consistent equation (21) around the paramagnetic solution \( W_\epsilon(h) = \delta(h) \). For \( \gamma = 0 \) we find the same bifurcation lines as were derived in [17]. To determine the SG–F transition and the M to F transition one has to solve numerically, respectively, equations (21) and (45) with for instance a population dynamics algorithm.

In figure 3 the different phases in the \((J_0/J_1, T/J_1)\) phase diagram are presented for a skewness \( \gamma = 0 \) and several values of \( \alpha \). The open circles present the SG–M transitions and the stars mark the points where the F phase becomes stable with respect to RSB. These results generalize the phase diagram obtained in the seminal paper of Sherrington and Kirkpatrick [1] to coupling distributions with a large tail. For \( \gamma = 0 \) the P–F transition is independent of \( \alpha \). When \( \alpha \) increases the SG phase increases in favor of a smaller F phase. The RSB effects decrease when \( \alpha \) decreases: indeed, the M-phase becomes smaller and the re-entrance effect in the SG–F phase transition line diminishes and finally disappears. This is related to the decrease of frustration due to the presence of stronger bonds that dominate the systems behavior. We did not find a replica symmetric SG phase (i.e., a SG phase stable with respect to RSB), contrary to the conjecture made in [15]. Replica symmetry breaks continuously at the SG transition, which is similar to the behavior of the SK model. We did not find any further evidence for the conjecture in [15] that replica symmetry is restored at \( T = 0 \).

In figure 4 we present the \((T/J_1, \alpha)\) phase diagram for different values of \( \gamma \) and \( J_0 = 0 \). We consider the following regions:

- \( \gamma > 0 \) and \( \alpha < 1 \) (left figure): the F phase increases and the SG phase decreases as a function of increasing \( \gamma \). The SG phase disappears at \( \gamma = 1 \). For values of \( \gamma \) very close to 1 the SG phase is only present for very small values of \( \alpha \). The temperature of the transition between the P and F phase becomes infinite for \( \alpha \to 1^- \).

\[ \text{doi: 10.1088/1742-5468/2010/01/P01010} \]
Figure 3. The \((T/J_1, J_0/J_1)\) phase diagram for several values of \(\alpha\) and with a skewness \(\gamma = 0\). Four phases appear: P (paramagnetic), F (ferromagnetic), M (mixed) and SG (spin glass). The circles present the SG–M transitions and the stars indicate where the F phase becomes stable against replica symmetry breaking. For \(\alpha = 2\) the phase diagram coincides with that of the SK model.

Figure 4. The \((T/J_1, \alpha)\) phase diagrams for different values of the skewness \(\gamma\) and a shift \(J_0 = 0\). The figure on the left gives the phase diagram for \(\gamma > 0\) while the figure on the right gives the phase diagram for \(\gamma < 0\). For \(\gamma = 1\) and \(\alpha < 1\) there is no SG phase. The P–SG transition is independent of \(\gamma\). For \(\gamma = 0.99\) the dashed part of the transition line represents not the SG–F transition but the line of instability of the P phase with respect to the F phase.

- \(\gamma < 0\) and \(\alpha > 1\) (right figure): the F phase decreases and the SG phase increases as a function of increasing \(\gamma\). The temperature of the transition between the P and F phase becomes infinite for \(\alpha \to 1^+\).
- \(\gamma > 0\) and \(\alpha > 1\) (not shown): there is no F phase but there are P and SG phases.
- \(\gamma < 0\) and \(\alpha < 1\) (not shown): there is no F phase but there are P and SG phases.

We have some additional remarks. The transition temperature becomes very large for \(\alpha \to 1^\pm\) (for, respectively, \(\gamma < 0\) and \(\gamma > 0\)) because the effective coupling \(J_0 \to \infty\). There is no SG phase for \(\gamma = 1\) and \(\alpha < 1\) because there are no negative couplings; only
the P–F transition occurs. The P–SG transitions coincide for different values of $\gamma$. For low values of $\alpha$ the results for the population dynamics become inaccurate because of numerical imprecisions when dealing with a broad range of coupling values. In this case we used the instability line of the P phase with respect to the F phase as the location of the SG–F transition.

7. Entropy

It is possible to calculate the free energy from the saddle point equations. We use the RS ansatz and we introduce again a cutoff $\epsilon$. The entropy is given by $s = \beta^2 (\partial f / \partial \beta) = \lim_{\epsilon \to 0} s(\epsilon)$ with $s(\epsilon)$:

$$s(\epsilon) = \beta^2 \Delta (1 - q^2) - \beta^2 \Delta (1 - q) + s_{\text{site}}(\epsilon) - \frac{c}{2} s_{\text{link}}(\epsilon). \quad (53)$$

The quantity $s_{\text{link}}$ is equal to

$$s_{\text{link}}(\epsilon) = - \int dh \, dh' W^c(h) W^c(h') \int dJ P^c(J) \sum_{\sigma,\tau} \exp(\beta J \sigma \tau + \beta h \sigma + \beta h' \tau) \times \log \left( \sum_{\sigma,\tau} \exp(\beta J \sigma \tau + \beta h \sigma + \beta h' \tau) \right), \quad (54)$$

and $s_{\text{site}}$ reads

$$s_{\text{site}}(\epsilon) = - \sum_{k=0}^{\infty} \frac{c}{k!} \prod_{l=1}^{k} \left[ \int dh_l \, W(h_l) \int dJ_l \, P_l(J_l) \right] \int Dz \times \sum_{\sigma: (\tau_1, \tau_2, \ldots, \tau_k)} \left\{ \exp((\beta J_0 m + \sqrt{2 q \Delta z}) \sigma) \prod_{\ell=1}^{k} (\exp(\beta J_\ell \tau_\ell \sigma) \exp(\beta h_\ell \tau_\ell)) \right\}^{-1} \times \left\{ \sum_{\sigma: (\tau_1, \tau_2, \ldots, \tau_k)} \exp((\beta J_0 m + \sqrt{2 q \Delta z}) \sigma) \prod_{\ell=1}^{k} (\exp(\beta J_\ell \tau_\ell \sigma) \exp(\beta h_\ell \tau_\ell)) \right\}^{-1} \times \log \left[ \exp((\beta J_0 m + \sqrt{2 q \Delta z}) \sigma) \prod_{\ell=1}^{k} (\exp(\beta J_\ell \tau_\ell \sigma) \exp(\beta h_\ell \tau_\ell)) \right] \times \left\{ \sum_{\sigma: (\tau_1, \tau_2, \ldots, \tau_k)} \exp((\beta J_0 m + \sqrt{2 q \Delta z}) \sigma) \prod_{\ell=1}^{k} (\exp(\beta J_\ell \tau_\ell \sigma) \exp(\beta h_\ell \tau_\ell)) \right\}^{-1}. \quad (55)$$

For $\alpha \to 2$ one gets precisely the entropy of the SK model [2]. The entropies $s_{\text{site}}$ and $s_{\text{link}}$ correspond to the entropy differences when performing, respectively, a site addition and a link addition on the backbone graph of strong bonds; see [19, 24]. Like for the form of the self-consistent equation (21) for $W^c(h)$, we find that the entropy as given by equation (53) corresponds to the entropy of an Ising model on a Poissonian graph with mean connectivity $c$, a distribution of the bonds $P_c$ and an extra Gaussian noise $z$.

doi:10.1088/1742-5468/2010/01/P01010
The entropy $s$ as a function of the exponent $\alpha$ for different values of the temperature $T$, $J_0 = 0$, $\gamma = 0$ and $J_1 = 1$. The filled markers at $\alpha = 2$ show the SK values. The entropy converges to the SK value for $\alpha \to 2$.

We plotted the entropy $s$ as a function of $\alpha$; see figure 5. From this figure we see that the entropy gets less negative when the exponent $\alpha$ decreases for $T \to 0$. We find that for smaller values of $\alpha \lesssim 1$ the entropy becomes eventually zero for $T \to 0$. This is consistent with a decrease of RSB effects when $\alpha$ decreases.

8. Conclusion

In this paper we have shown how to derive the phase diagrams of Lévy spin glasses where the couplings between the spins are drawn from a distribution with power-law tails characterized by an exponent $\alpha$. These models are known to have a finite number of strong bonds of order $\mathcal{O}(1)$ and an infinite amount of weak bonds of order $\mathcal{O}(N^{-1/\alpha})$. The crucial difference from previous works \cite{15} and \cite{17} is that we derive the phase diagrams, the entropy and the stability against replica symmetry of Lévy spin glasses without using the Gaussian assumption for the distribution of fields. We have not found evidence for a replica symmetric spin-glass phase, or for a restoration of the replica symmetry at zero temperature, contrary to the conjecture in \cite{15}. We have solved the problem using the replica and the cavity method within, respectively, the replica symmetric assumption and the assumption of one pure phase. The resultant effective equations for the distribution of cavity fields show clearly the hybrid character of the model: it is a mixture between a finite connectivity model and a fully connected model.

The phase transitions are qualitatively similar to the ones found in the SK model. Large tails do influence the phase diagram quantitatively: the Lévy spin-glass model becomes more stable with respect to replica symmetry breaking and the SG phase decreases when the tails get larger. Moreover, the re-entrance effects in the replica symmetric phase diagram disappear for $\alpha \lesssim 1$. The replica symmetry breaking transitions are all continuous. The skewness $\gamma$ in the Lévy distribution can have a big influence on the size of the F phase. For $\alpha \to 2$ the effective distribution of fields becomes Gaussian and we have got back the results of the SK model. For $\alpha < 2$ this distribution is neither Lévy nor Gaussian, but a distribution with finite moments and a kurtosis smaller than that of a Gaussian with the same variance.

doi:10.1088/1742-5468/2010/01/P01010
Figure A.1. The Lévy distributions $P^{J_1,\gamma,J_0}_\alpha(J)$ for $J_1 = 1, 0$ and different values of $\alpha$ and $\gamma$. The distributions with $\alpha = 1.5$ approach a Gaussian while the ones for $\alpha = 0.5$ have larger tails. For $\gamma > 0$, the center of the distribution goes to $+\infty$ and $-\infty$ for $\alpha \uparrow 1$ and $\alpha \downarrow 1$, respectively. The coupling distribution fulfills $P^{J_1,\gamma,J_0}_\alpha(J) = P^{J_1,-\gamma,J_0}_\alpha(-J)$.

Acknowledgments

One of the authors (F L Metz) acknowledges a fellowship from CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), Brazil.

Appendix A. Stable distributions

The purpose of this appendix is to give some intuition on the role of the parameters $\alpha$ and $\gamma$ present in stable distributions, defined through equations (2) and (3). Both $\alpha$ and $\gamma$ are responsible for the shape of the distribution. The main role of the exponent $\alpha$ is to control the decay of the tails. Figure A.1 shows that, for a fixed $\gamma$, a decrease in $\alpha$ gives rise to a distribution $P^{J_1,\gamma,J_0}_\alpha(J)$ with larger tails and more sharply peaked around its most probable value $J$. The center of $P^{J_1,\gamma,J_0}_\alpha(J)$ is also shifted from the negative to the positive $J$-axis as $\alpha$ decreases from $\alpha > 1$ to $\alpha < 1$. A change of $\alpha$ has no effect on the position of the center when $\gamma = 0$.

The skewness parameter $\gamma$ controls the relative weight of the positive and negative tails. For $\gamma > 0$, the positive tail of $P^{J_1,\gamma,J_0}_\alpha(J)$ is larger than the negative one; for $\gamma < 0$, vice versa. The distribution is symmetric around $J_0$ when $\gamma = 0$. For increasing positive values of $\gamma$ (see figure A.1), the center of $P^{J_1,\gamma,J_0}_\alpha(J)$ shifts to the right or left provided $\alpha < 1$ or $\alpha > 1$, respectively.

Appendix B. Solution of integrals

In this appendix we show how to integrate over $\hat{J}$ in the following equations:

$$I_1 = \int_{-\infty}^{\infty} \frac{d\hat{J}}{2\pi} \hat{J}^\alpha e^{i\hat{J}J} f(J),$$

$$I_2 = \int_{-\infty}^{\infty} \frac{d\hat{J}}{2\pi} \hat{J}^\alpha e^{i\hat{J}J} f(J).$$

doi:10.1088/1742-5468/2010/01/P01010
The phase diagram of Lévy spin glasses

\[ I_2 = \int_{-\infty}^{\infty} \frac{dJ}{2\pi} |J|^{\alpha} \frac{\sin(\hat{J})}{J^{\alpha} + 1} f(J), \]  
(B.2)

where \( f(J) \) is given by equation (14) and \( \alpha \in (0, 1) \cup (1, 2) \). We obtained the following results for \( I_1 \) and \( I_2 \) after integration over \( \hat{J} \):

\[ I_1 = -\left( \frac{\sqrt{2}}{J_1} \right)^\alpha C_\alpha \int_{-\infty}^{\infty} \frac{dJ}{|J|^{\alpha + 1}} [f(J) - f(0)], \quad \text{if } 0 < \alpha < 2, \]  
(B.3)

\[ I_2 = i \left( \frac{\sqrt{2}}{J_1} \right)^\alpha \frac{C_\alpha}{\Phi} \int_{-\infty}^{\infty} \frac{dJ}{|J|^{\alpha + 1}} \sin(J)f(J), \quad \text{if } 0 < \alpha < 1, \]  
(B.4)

\[ = i \left( \frac{\sqrt{2}}{J_1} \right)^\alpha \frac{C_\alpha}{\Phi} \int_{-\infty}^{\infty} \frac{dJ}{|J|^{\alpha + 1}} \sin(J)[f(J) - f'(0)J], \quad \text{if } 1 < \alpha \leq 2, \]  
(B.5)

where the parameters \( C_\alpha \) and \( \Phi \) are defined in section 2. We have left out the dependence of \( f(J) \) with respect to \( h \) and \( s \) since it is not important here. The aim of this appendix is to show how one can derive equations (B.3)–(B.5) from equations (B.1) and (B.2).

By introducing an exponential convergence factor in equations (B.1) and (B.2), we can rewrite them as follows:

\[ I_1 = \lim_{a \to 0^+} \int_0^{\infty} dJ [\sin(\hat{J})]^a \pi \int_0^{\infty} \frac{d\hat{J}}{\pi} \hat{J}^\alpha \cos(\hat{J})e^{-a\hat{J}}, \]  
(B.6)

\[ I_2 = i \lim_{a \to 0^+} \int_0^{\infty} dJ [\sin(\hat{J})]^a \pi \int_0^{\infty} \frac{d\hat{J}}{\pi} \hat{J}^\alpha \sin(\hat{J})e^{-a\hat{J}}. \]  
(B.7)

The integrals over \( \hat{J} \) are calculated for \( a > 0 \) and, afterwards, the limit \( a \to 0^+ \) is taken. Reference [32] can be used in order to integrate over \( \hat{J} \) in equations (B.6) and (B.7), giving rise to

\[ I_1 = \frac{\Gamma(\alpha + 1)}{\pi} \lim_{a \to 0^+} \int_0^{\infty} dJ [\sin(\hat{J})]^a \pi \int_0^{\infty} \frac{d\hat{J}}{\pi} \hat{J}^\alpha \frac{\cos[(\alpha + 1)\arctan(J/a)]}{(J^2 + a^2)^{\alpha + 1/2}}, \]  
(B.8)

\[ I_2 = i \frac{\Gamma(\alpha + 1)}{\pi} \lim_{a \to 0^+} \int_0^{\infty} dJ [\sin(\hat{J})]^a \pi \int_0^{\infty} \frac{d\hat{J}}{\pi} \hat{J}^\alpha \frac{\sin[(\alpha + 1)\arctan(J/a)]}{(J^2 + a^2)^{\alpha + 1/2}}. \]  
(B.9)

In order to analyze the behavior of integrals (B.8) and (B.9) around \( J = 0 \) when \( a \to 0^+ \), we insert a cutoff \( \lambda > 0 \) and split them as follows:

\[ I_1 = \frac{\Gamma(\alpha + 1)}{\pi} U_1(\lambda) + \frac{\Gamma(\alpha + 1)}{\pi} \cos \left[ (\alpha + 1)\frac{\pi}{2} \right] \int_0^{\infty} \frac{dJ}{\lambda^{\alpha + 1}} [f(J) + f(-J)], \]  
(B.10)

\[ I_2 = i \frac{\Gamma(\alpha + 1)}{\pi} U_2(\lambda) + i \frac{\Gamma(\alpha + 1)}{\pi} \sin \left[ (\alpha + 1)\frac{\pi}{2} \right] \int_0^{\infty} \frac{dJ}{\lambda^{\alpha + 1}} [f(J) - f(-J)], \]  
(B.11)

\[ \text{doi:10.1088/1742-5468/2010/01/P01010} \]
where

\[
U_1(\lambda) = \lim_{\alpha \to 0^+} \int_0^{\lambda} \frac{dJ}{a^{\alpha+1}} [f(J) + f(-J)] \cos \left( \frac{(\alpha + 1) \arctan(J/a)}{[(J/a)^2 + 1]^{(\alpha+1)/2}} \right),
\]

(B.12)

\[
U_2(\lambda) = \lim_{\alpha \to 0^+} \int_0^{\lambda} \frac{dJ}{a^{\alpha+1}} [f(J) - f(-J)] \sin \left( \frac{(\alpha + 1) \arctan(J/a)}{[(J/a)^2 + 1]^{(\alpha+1)/2}} \right).
\]

(B.13)

The limit \( a \to 0^+ \) has been taken on the right-hand side of equations (B.10) and (B.11). The integrals present in the definition of \( U_1(\lambda) \) and \( U_2(\lambda) \) are computed through a power-series representation of their integrands, yielding the results

\[
U_1(\lambda) = \lim_{\alpha \to 0^+} \sum_{n,l=0}^{\infty} u_{nl} \left( \frac{\lambda}{a} \right)^{2l+\alpha+1} \lambda^{2n-\alpha},
\]

(B.14)

\[
U_2(\lambda) = \lim_{\alpha \to 0^+} \sum_{n,l=0}^{\infty} v_{nl} \left( \frac{\lambda}{a} \right)^{2l+\alpha+2} \lambda^{2n+1-\alpha}.
\]

(B.15)

The explicit forms of the coefficients \( \{u_{nl}\} \) and \( \{v_{nl}\} \) are irrelevant. The analysis of equations (B.10) and (B.11) as \( \alpha \) tends to zero, constrained to the limit \( a \to 0^+ \) in the functions \( U_1(\lambda) \) and \( U_2(\lambda) \), constitutes the final step of the calculation.

One can notice from equation (B.14) that \( U_1(\lambda) \) diverges for \( \lambda \to 0^+ \). However, the transformation of \( f(J) \) according to \( f(J) \to f(J) - f(0) \) removes this divergence and makes \( U_1(\lambda) \) go to zero for \( \lambda \to 0^+ \), provided that \( \alpha < 2 \). This allows us to take the limit \( \lambda \to 0^+ \) in equation (B.10) which leads, after comparison with equation (B.1), to the following result:

\[
\int_{-\infty}^{\infty} \frac{dJ}{2\pi} |J|^\alpha e^{iJ} [f(J) - f(0)] = -\frac{\Gamma(\alpha + 1)}{\pi} \sin \left( \frac{\alpha \pi}{2} \right)
\]

\[
\times \int_0^{\infty} \frac{dJ}{J^{\alpha+1}} [f(J) + f(-J) - 2f(0)], \quad 0 < \alpha < 2.
\]

(B.16)

By integrating the term with \( f(0) \) on the left-hand side of the above equation we get equation (B.3).

The calculation of equations (B.4) and (B.5) proceeds in an analogous way. Depending on the value of \( \alpha \), there are two different situations concerning the behavior of equation (B.15) for \( \lambda \to 0^+ \). For \( \alpha < 1 \), we obtain \( \lim_{\lambda \to 0^+} U_2(\lambda) = 0 \), which allows us to take the limit \( \lambda \to 0^+ \) in equation (B.11). For \( \alpha > 1 \), it is necessary to transform \( f(J) \) according to \( f(J) \to f(J) - f'(0)J \) in order to obtain \( \lim_{\lambda \to 0^+} U_2(\lambda) = 0 \) and to take the limit \( \lambda \to 0^+ \) in equation (B.11).

**References**

[1] Sherrington D and Kirkpatrick S, Solvable model of a spin-glass, 1975 Phys. Rev. Lett. 35 1792

[2] Kirkpatrick S and Sherrington D, Infinite-ranged models of spin-glasses, 1978 Phys. Rev. B 17 4384

[3] Mézard M, Parisi G and Virasoro M A, 1987 Spin Glass Theory and Beyond (World Scientific Lecture Notes in Physics vol 9) (Singapore: World Scientific)

[4] Klein M W, Temperature-dependent internal field distribution and magnetic susceptibility of a dilute Ising spin system, 1968 Phys. Rev. 173 552

doi:10.1088/1742-5468/2010/01/P01010
The phase diagram of Lévy spin glasses

[5] Klein M W, Held C and Zuroff E, Dipole interactions among polar defects: a self-consistent theory with application to OH⁻ impurities in KCL, 1976 Phys. Rev. B 13 3576

[6] Berkov D V, Local-field distribution in systems with dipolar interparticle interaction, 1996 Phys. Rev. B 53 731

[7] Lévy P, 1937 Theory de l'addition de Variables Aléatoires (Paris: Gauthier-Villars)

[8] Gnedenko B V and Kolmogorov A N, 1954 Limit Distributions for Sums of Independent Random Variables (Cambridge: Addison-Wesley)

[9] Biroli G, Bouchaud J P and Potters M, Extreme value problems in random matrix theory and other disordered systems, 2007 J. Stat. Mech. 07019

[10] Cizeau P and Bouchaud J P, Theory of Lévy matrices, 1994 Phys. Rev. E 50 1810

[11] Janzen K, Hartmann A K and Engel A, Replica theory for Lévy spin glasses, 2008 J. Stat. Mech. 04006

[12] Bouchaud J P and Georges A, Anomalous diffusion in random media: statistical mechanisms, models and physical applications, 1990 Phys. Rep. 195 127

[13] Galluccio S, Bouchaud J P and Potters M, Rational decisions, random matrices and spin glasses, 1998 Physica A 259 449

[14] Cizeau P and Bouchaud J P, Mean field theory of dilute spin-glasses with power-law interactions, 1993 J. Phys. A: Math. Gen. 26 L187

[15] Viana L and Bray A J, Phase diagrams for dilute spin glasses, 1985 J. Phys. C: Solid State Phys. 18 3037

[16] Janzen K, Hartmann A K and Engel A, Replica theory for Lévy spin glasses, 2008 J. Stat. Mech. P04006

[17] Mérard M and Parisi G, Mean-field theory of randomly frustrated systems with finite connectivity, 1987 Europhys. Lett. 3 1067

[18] Mérard M and Parisi G, The cavity method at zero temperature, 2003 J. Stat. Phys. 111 1–34

[19] Janzen K, Engel A and Mérard M, The Levy spin glass transition, arXiv:0910.2602 [cond-mat]

[20] Paul W and Baschnagel J, 1999 Stochastic Processes (Berlin: Springer)

[21] Nolan J P, 2007 Stable Distributions: Models for Heavy-Tailed Data (Basle: Birkhauser)

[22] Monasson R, Optimization problems and replica symmetry breaking in finite connectivity spin glasses, 1998 J. Phys. A: Math. Gen. 31 513

[23] Mérard M and Parisi G, The Bethe lattice spin glass revisited, 2001 Eur. Phys. J. B 20 217

[24] Raymond J R and Saad D, Composite systems of dilute and dense couplings, 2008 J. Phys. A: Math. Theor. 41 324014

[25] Raymond J R and Saad D, Equilibrium properties of disordered spin models with two-scale interactions, 2009 Phys. Rev. E 80 031138

[26] de Almeida J R L and Thouless D J, Stability of the Sherrington–Kirkpatrick solution of a spin-glass model, 1978 J. Phys. A: Math. Gen. 11 129

[27] Kwon C and Thouless D J, Spin glass with two replicas on a Bethe lattice, 1991 Phys. Rev. B 10 8379

[28] Aldous D J and Bandyopadhyay A, A survey of max-type recursive distributional equations, 2005 Ann. Appl. Probab. 15 1047

[29] Neri I, Skantzos N S and Bollé D, Gallager error-correcting codes for binary asymmetric channels, 2008 J. Stat. Mech. P10018

[30] de Oliveira V M and Fontanari J F, Replica analysis of the p-spin interaction Ising spin-glass model, 1999 J. Phys. A: Math. Gen. 32 2285

[31] Gradshteyn I S and Ryzhik I M, 2000 Table of Integrals, Series, and Products (San Diego, CA: Academic) p 492 and 493

doi:10.1088/1742-5468/2010/01/P01010