Lattice effective potential of \((\lambda \Phi^4)_4\): 
nature of the phase transition and bounds on the Higgs mass

P. Cea\textsuperscript{1,2}, L. Cosmai\textsuperscript{2}, M. Consoli\textsuperscript{3} and R. Fiore\textsuperscript{4,5}

\textsuperscript{1} INFN - Sezione di Bari - via Amendola 173 - I 70126 - Bari - Italy
\textsuperscript{2} Dipartimento di Fisica - Università di Bari - via Amendola 173 - I 70126 Bari - Italy
\textsuperscript{3} INFN - Sezione di Catania - Corso Italia 57 - I 95129 Catania - Italy
\textsuperscript{4} Dipartimento di Fisica - Università della Calabria - I 87030 Arcavacata di Rende (Cs) - Italy
\textsuperscript{5} INFN - Gruppo Collegato di Cosenza - I 87030 Arcavacata di Rende (Cs) - Italy

Abstract

We present a detailed discussion of Spontaneous Symmetry Breaking (SSB) in \((\lambda \Phi^4)_4\). In the usual approach, inspired by perturbation theory, one predicts a second-order phase transition, the Higgs mass \(m_h\), related to the value of the renormalized 4-point coupling, gets smaller when increasing the ultraviolet cutoff and this leads to the generally quoted upper bounds \(m_h < 700-900\) GeV. On the other hand, by exploring the structure of the effective potential in those approximation consistent with ‘triviality’, where the Higgs mass does not represent a measure of any observable interaction, SSB does not require an ultraviolet cutoff, the phase transition is first-order, such that the massless ‘Coleman-Weinberg’ regime lies in the broken phase, and one gets only \(m_h < 3\) TeV from vacuum stability. To separate out the two alternatives, we present a precise lattice computation of the slope of the effective potential in the region of bare parameters indicated by the Luscher & Weisz and Brahm’s analysis of the critical line. Our lattice data strongly support the latter description of SSB. Indeed, our data cannot be reproduced in perturbation theory, and then they confirm the existence on the lattice of a remarkable phase of \((\lambda \Phi^4)_4\) where SSB is generated through "dimensional transmutation", and show no evidence for residual self-interaction effects of the shifted “Higgs” field \(h(x) = \Phi(x) - \langle \Phi \rangle\), in agreement with “triviality”.

\textsuperscript{a}E-mail: cea@bari.infn.it
\textsuperscript{b}E-mail: cosmai@bari.infn.it
\textsuperscript{c}E-mail: consoli@infnct.infn.it
\textsuperscript{d}E-mail: fiore@cs.infn.it
1 Introduction

Spontaneous Symmetry Breaking (SSB), induced through a self-interacting scalar sector, is the essential ingredient to generate the mass of the intermediate vector bosons in the standard model of electroweak interactions [1]. However, despite of the simplicity and elegance of the Higgs mechanism [2], it is believed that the generally accepted “triviality” [3, 4, 5, 6, 7, 8, 9, 10] of \((\lambda \Phi^4)_4\) implies the scalar sector of the standard model to be just an effective theory, valid up to some cutoff scale. Without a cutoff, the argument goes, there would be no scalar self-interactions, and without them there would be no symmetry breaking. This point of view also leads to upper bounds on the Higgs mass [9, 11, 12].

Recently [13, 14], it has been pointed out, on the basis of very general arguments, that SSB is not incompatible with “triviality”. Indeed, the most general condition for a trivial scattering matrix requires all interaction effects to be reabsorbed into a set of Green’s functions expressible in terms of the first two moments of a Gaussian functional distribution. In this situation, where the simple Hartree-Fock-Bogolubov approximation becomes effectively exact in the continuum limit, one can meaningfully consider a non-zero vacuum expectation value (VEV) of the field, \(\langle \Phi \rangle\), in connection with non-interacting, quasi-particle excitations in the broken symmetry phase. In this sense, a “trivial” theory can still be useful to determine the vacuum of the theory and as such a convenient frame for the gauge fields preserving a perturbatively weak interaction at high energies. This picture, while reconciling the strong evidences for “triviality” with those for a non-trivial effective potential [13, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25], also clarifies the meaning of some explicit studies of “triviality” in the broken symmetry phase (\(\langle \Phi \rangle \neq 0\)) [26], all pointing out the absence of observable interactions. Let us briefly recapitulate the simple logical steps of Ref. [13, 14] leading to this conclusion.

The effective potential determines the zero-momentum 1PI vertices [27] and reflects any non trivial dynamics in the zero-momentum sector of the theory. This remark immediately suggests the relevance of studying the massless version of \((\lambda \Phi^4)_4\) theories. In fact, on one hand, just the situation of a vanishing mass-gap in the symmetric phase, preventing to deduce the uniqueness of the vacuum from the basic results of quantum field theory (see Ref. [8], chaps. 16 and 17), opens the possibility of SSB. On the other hand, since for the massless theory the zero-momentum \((p_\mu = 0)\) is a physical on-shell point, a non-trivial effective potential implies a non-trivial scattering matrix, at least in that limiting and unobservable region of the 4-momentum space, in agreement with the independent evidence for “non-triviality” of Pedersen, Segal and Zhou [28]. As pointed out in Ref. [13, 14], in fact, this (admittedly almost ignored) result does not imply, by itself, the presence of observable scattering processes since all non-trivial interaction effects of the symmetric phase may become unobservable, being fully reabsorbed into a change of the vacuum structure and in the mass of the excitations of the broken symmetry vacuum. In this way, one can reconcile non-trivial SSB with “triviality” and, in the simplest case of the one-component theory with a discrete symmetry \(\Phi \rightarrow -\Phi\), the physical excitation spectrum contains only massive free particles. Furthermore, by relating the equivalent computations in the symmetric phase of
Coleman and Weinberg [27] with those in the broken phase by Jackiw [29] the requirement of “triviality”, i.e. the condition of a free shifted ‘Higgs’ field $h(x) = \Phi(x) - \langle \Phi \rangle$, dictates the form of the effective potential which, close to the continuum limit, has to reduce to the simple expression corresponding to the sum of a classical background and of the zero-point energy of a massive free field. Indeed, massless $(\phi^4)_4$, in all approximations consistent with “triviality”, i.e. when $h(x)$ is effectively governed by a quadratic Hamiltonian (one-loop potential, Gaussian approximation, postgaussian calculations, see in particular Ref. [27], where the Higgs propagator $G(x, y)$ is properly optimized at each value of $\langle \Phi \rangle$, by solving the corresponding non-perturbative gap-equation) provides the same structure for the effective potential and, close to the continuum limit, one finds the simple expression [20, 21, 13, 14] ($\phi_B = \langle \Phi \rangle$ denotes the bare, cutoff-dependent vacuum field and $V_{\text{triv}}$ is a short-hand notation to denote the effective potential in those approximations consistent with “triviality”)

$$V_{\text{triv}}(\phi_B) = \frac{\tilde{\lambda}}{4} \phi_B^4 + \frac{\omega^4(\phi_B)}{64\pi^2} \left( \ln \frac{\omega^2(\phi_B)}{\Lambda^2} - \frac{1}{2} \right) .$$  \hspace{1cm} (1)

In Eq. (1) $\Lambda$ denotes the Euclidean ultraviolet cutoff, $\omega^2(\phi_B) = 3\tilde{\lambda}\phi_B^2$ is the $\phi_B$–dependent mass squared of the shifted field and $\tilde{\lambda}$ is finitely proportional to the bare coupling $\lambda_0$ entering the bare Lagrangian density ($\tilde{\lambda} = \lambda_0$ at one-loop, $\tilde{\lambda} = (2/3)\lambda_0$ in the Gaussian approximation and so on). Eq. (1) provides the most general form of the effective potential in those approximations consistent with “triviality” and leads to SSB. Indeed, the absolute minimum condition occurs at $\phi_B = \pm v_B \neq 0$ where

$$m_h^2 = 3\tilde{\lambda}v_B^2 = \Lambda^2 \exp \left[ -\frac{16\pi^2}{9\lambda} \right]$$  \hspace{1cm} (2)

and one finds

$$W = V_{\text{triv}}(\pm v_B) = -\frac{m_h^4}{128\pi^2} < 0$$  \hspace{1cm} (3)

so that, by using Eq. (2), Eq. (1) can be rewritten as

$$V_{\text{triv}}(\phi_B) = \frac{9\tilde{\lambda}^2v_B^4}{64\pi^2} \left( \ln \frac{\phi_B^2}{v_B^2} - \frac{1}{2} \right) .$$  \hspace{1cm} (4)

As discussed in Ref. [20, 21, 13, 14, 23, 25], a non-perturbative renormalization of the “trivial” effective potential is possible by imposing the requirement of cutoff-independence for the ground state energy density $W$ in Eq. (3) and for the physical correlation length $\xi_h \sim 1/m_h$ in the broken phase. In this case, from

$$\Lambda \frac{dm_h}{d\Lambda} = \left( \Lambda \frac{\partial}{\partial \Lambda} + \beta(\tilde{\lambda}) \frac{\partial}{\partial \lambda} \right) m_h = 0$$  \hspace{1cm} (5)
one deduces

\[ \beta(\tilde{\lambda}) = \Lambda \frac{d \tilde{\lambda}}{d \Lambda} = -\frac{9 \tilde{\lambda}^2}{8 \pi^2} < 0 \]  

which implies that both \( \lambda_0 \) and \( \tilde{\lambda} \) vanish in the continuum limit \( \Lambda \to \infty, m_h = \text{fixed} \) according to

\[ \lambda_0 \sim 3 \tilde{\lambda} = \frac{m_h^2}{v_B^2} = \frac{8 \pi^2}{3 \ln(\Lambda/m_h)} \to 0. \]  

Notice that the non-perturbative \( \beta \)-function obtained from “triviality” is very different from the perturbative \( \beta \)-function

\[ \beta_{\text{pert}}(\lambda_0) = \Lambda \frac{d \lambda_0}{d \Lambda} = \frac{9 \lambda_0^2}{8 \pi^2} - \frac{51 \lambda_0^3}{64 \pi^4} + O(\lambda_0^4) \]  

deduced from the cutoff-independence of the renormalized coupling \( \lambda_R = \lambda_R(\mu^2) \), as computed in a weak coupling expansion \[ \text{in powers of the bare coupling } \lambda_0 \text{ at some non-zero external momenta } p^2 = \mu^2 \]

\[ \Lambda \frac{d \lambda_R}{d \Lambda} = (\Lambda \frac{\partial}{\partial \Lambda} + \beta_{\text{pert}}(\lambda_0) \frac{\partial}{\partial \lambda_0}) \lambda_R = 0. \]  

In particular, by relying on a perturbative evaluation of the higher order effects one would deduce \[ \text{that the one-loop minimum is changed into a false vacuum by the genuine } h \text{-field self-interactions and that SSB does not occur for the massless theory. In this framework, and quite independently of our results, one should realize that in the “trivial” } (\lambda \Phi^4)_4 \text{ theories the validity of the perturbative relations is, at best, unclear} \]. Indeed, perturbation theory, being based on the concept of a cutoff-independent and non-vanishing renormalized coupling at non-zero external momenta, predicts the existence of observable scattering processes that cannot be there if “triviality” is true. In this sense, the unphysical features of the perturbative \( \beta \)-function \[ \text{are just a consequence} \] of the basic assumption behind the perturbative approach – the attempt of defining a continuum limit in the presence of residual interaction effects which, at any finite order, cannot be reabsorbed into the vacuum structure and the particle mass. Therefore, only abandoning from the start the vain attempt of defining an interacting theory, can a meaningful continuum limit be obtained and only approximations to the effective potential consistent with the non-interacting nature of the field \( h(x) \) are reliable (for more details on the general class of these consistent approximations see Ref. \[ \text{Ref.} \]).

As discussed in Ref. \[ \text{Ref.} \], all approximations consistent with the structure in Eqs. \[ \text{Eqs.} \], although providing different \( \tilde{\lambda} \) in their bare forms, are equivalent.
Indeed, they lead to the same form when expressed in terms of the physical renormalized vacuum field $\phi_R$. The underlying rationale for the introduction of $\phi_R$ is the following. Naively, one would plot the effective potential (14) in terms of the bare field $\phi_B$ at fixed $\Lambda$. This attempt, however, becomes more and more difficult when increasing $\Lambda$ since in the continuum limit, when $\Lambda \to \infty$ and $\tilde{\lambda} \to 0$, the slope of the effective potential in terms of $\phi_B$ becomes infinitesimal. Indeed, to produce a finite change in $V_{\text{triv}}$ from 0 to $W$ requires to reach the values $\pm v_B$ and these are located at an infinite distance in units of $m_h$. Thus, in this situation the plot of the effective potential is crucially dependent on the magnitude of the ultraviolet cutoff. However, the effective potential itself is a cutoff-independent quantity and one may naturally consider the question of making this invariance manifest. To do this, let us consider the Renormalization Group (RG) equation for the effective potential and determine the integral curves

$$\tilde{\lambda} = \tilde{\lambda}(\Lambda)$$  \hspace{1cm} (10)

$$\phi_B = \phi_B(\Lambda)$$  \hspace{1cm} (11)

along which $V_{\text{triv}}$ in Eqs. (14) is invariant. This amounts to solve the partial differential equation

$$(\Lambda \frac{\partial}{\partial \Lambda} + \beta(\tilde{\lambda}) \frac{\partial}{\partial \tilde{\lambda}} + \Lambda \frac{d\phi_B}{d\Lambda} \frac{\partial}{\partial \phi_B})V_{\text{triv}}(\Lambda, \tilde{\lambda}, \phi_B) = 0.$$  \hspace{1cm} (12)

By using Eq. (6), obtained from the cutoff-independence of the ground state energy at the minima $\phi_B = \pm v_B$ where the partial derivative with respect to $\phi_B$ in Eq. (12) vanishes identically, we find [17, 19, 20, 13, 23]

$$\Lambda \frac{d\phi_B}{d\Lambda} = \frac{9\tilde{\lambda}}{16\pi^2}\phi_B$$  \hspace{1cm} (13)

which indeed confirms that the product $\tilde{\lambda}\phi_B^2$ is invariant along a given integral curve as dictated by the RG-invariance of Eq. (2). Thus, one finds $\phi_B^2 \sim 1/\tilde{\lambda}$ and the question naturally arises of finding the proper normalization of the vacuum field in units of the natural cutoff-independent scale $m_h$ associated with the absolute minimum of the effective potential. To this end, let us consider the general structure in Eq. (4) and introduce the cutoff-independent combination $\phi_R$ such that

$$\phi_R^2 = \frac{3\tilde{\lambda}\phi_B^2}{8\pi^2X} = \frac{\phi_B^2}{Z_\phi},$$  \hspace{1cm} (14)
\(X\) being an arbitrary, positive \(\lambda\)-independent number. In this way

\[V_{\text{triv}}(\phi_B) = V_{\text{triv}}(\phi_R) = \pi^2 X^2 \phi_R^4 \left( \ln \frac{\phi_R^2}{v_R^2} - \frac{1}{2} \right)\]  

(15)

so that, from the value at \(\phi_B = \pm v_B\),

\[W = V_{\text{triv}}(\pm v_B) = V_{\text{triv}}(\pm v_R) = -\frac{1}{2} \pi^2 X^2 v_R^4 = -\frac{m_h^4}{128\pi^2}\]  

(16)

we find

\[m_h^2 = 8\pi^2 X v_R^2.\]  

(17)

However, when considering the second derivative of the effective potential with respect to \(\phi_R\) at \(\pm v_R\)

\[
\left. \frac{d^2V_{\text{triv}}(\phi_R)}{d\phi_R^2} \right|_{\phi_R=\pm v_R} = 8\pi^2 X^2 v_R^2
\]  

(18)

and depending on the value of \(X\), the quadratic shape in terms of the rescaled field \(\phi_R\) will not agree with the Higgs mass \(m_h\) unless

\[X = 1.\]  

(19)

In this case, namely for

\[V_{\text{triv}}(\phi_R) = \pi^2 \phi_R^4 \left( \ln \frac{\phi_R^2}{v_R^2} - \frac{1}{2} \right),\]  

(20)

\[m_h^2 = 8\pi^2 v_R^2,\]  

(21)

when the effective potential at its minima is locally equivalent to a harmonic potential parametrized in terms of the physical Higgs mass, the continuum theory associated with \(X = 1\) is completely indistinguishable from a free-field theory. Eqs. (20, 21), discovered in all known approximations consistent with “triviality”, should be considered exact (strictly speaking the exact effective potential is the “convex hull” of Eq. (20) [10]).
The only unconventional ingredient of the analysis is the asymmetric rescaling of vacuum field and fluctuation implying $Z_\phi \neq Z_h$, since in any approximation consistent with “triviality” there is no non-trivial rescaling of $h(x)$. However, the introduction of $Z_\phi$ is extremely natural when considering the quantization of the classical Lagrangian

$$L = \frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} r_0 \Phi^2 - \frac{1}{4} \lambda_0 \Phi^4.$$ 

In fact, as it is well known from statistical mechanics, a consistent quantization requires to single out preliminary the zero-momentum mode $\phi_B$

$$\Phi(x) = \phi_B + h(x)$$

(determined from the condition $\int d^4x \ h(x) = 0$), whose essentially classical nature reflects, however, the fundamental quantum phenomenon of Bose condensation. Therefore, beyond perturbation theory, the renormalization of the term containing the field derivatives, defining $Z_h$, is quite unrelated to $Z_\phi$, defined from the scaling properties of the effective potential, and, in those approximations consistent with “triviality”, e.g. preserving $Z_h = 1$ identically as at one-loop or in the Gaussian approximation, one finds $Z_\phi \sim 1/\lambda_0$. The structure $Z_\phi \neq Z_h$, allowed by the Lorentz-invariant meaning of the field decomposition into $p_\mu = 0$ and $p_\mu \neq 0$ components \[13, 14\], is completely consistent with the rigorous indications of quantum field theory. In fact, SSB in the cutoff theory (see Ref. \[10\], Sect.15), while imposing the finiteness of $Z_h$, requires that the bare vacuum field $v_B$ and the Higgs mass $m_h$ do not scale uniformly and one finds

$$\frac{m_h^2}{v_B^2} \to 0$$

in the continuum limit where the lattice spacing $a \sim 1/\Lambda \to 0$ in agreement with Eq. (7). Equation (22), by itself, represents a perfectly acceptable result. Indeed, there is no reason, in principle, why the bare vacuum field, defined at the lattice level, should be finite. One is actually faced with the situation where the vacuum energy $W = V_{\text{eff}}(\pm v_B)$ (related to the critical temperature $T_c$ at which the symmetry is restored \[8\]) is finitely related to $m_h^2$ even though $v_B^2$ itself is not. A familiar example of this situation is gluon condensation in QCD where the vacuum energy, as measured with respect to the perturbative ground state, is finitely related to the particular combination of bare quantities $g_B^2 \langle G_B^2 \rangle$. In the continuum limit where $g_B^2 \to 0$ the bare expectation value $\langle G_B^2 \rangle$ diverges in units of the physical scale of the QCD vacuum as defined, for instance, from the string tension or a glueball mass. The same is true in $(\lambda \Phi^4)_4$ where the bare vacuum field $v_B$ does not remain finite in units of $m_h$, the physical scale setting the correlation length of the spontaneously broken phase. However, despite of this illuminating analogy, pointing out the very general nature of the problem, the trend in Eq. (22) is usually considered as an indication for the inconsistency of SSB in the continuum limit. The reason for this conclusion, most likely, derives from the
operatorial meaning given to the field rescaling. Namely, the perturbative assumption of a single renormalization constant $Z = Z_\phi = Z_h$, to account for the cutoff-dependence both of the vacuum field and of the residue of the shifted field propagator, amounts to introduce a renormalized field operator

$$\Phi(x) = \sqrt{Z} \Phi_R(x),$$

(23)

This relation is a consistent shorthand for expressing the wave function - renormalization in a theory allowing an asymptotic Fock representation, but, beyond perturbation theory, has no justification in the presence of SSB. In fact, in this case, it overlooks that the shifted field $h(x)$, the only allowing for a particle interpretation, is not defined before fixing the vacuum and that the Lehmann spectral-decomposition argument ($0 < Z \leq 1$), valid for a field with vanishing VEV, constrains only the value of $Z = Z_h$. Notice that, quite independently of our results, the possibility of a different rescaling for the vacuum field and the fluctuations is somewhat implicit in the conclusions of the authors of Ref. [10] (see their footnote at page 401: "This is reminiscent of the standard procedure in the central limit theorem for independent random variables with a nonzero mean: we must subtract a mean of order $n$ before applying the rescaling $n^{-1/2}$ to the fluctuation fields").

Equation (22), supported as well by the results of lattice calculations (see Ref. [12] and Sect.3), represents a basic condition to define the continuum limit of SSB in $(\lambda \Phi^4)_4$. At the same time, by requiring $m_h^2$ to be cutoff independent, Eq. (7) implies

$$\lambda_0 \sim \tilde{\lambda} \to 0$$

(24)

for $\Lambda \to \infty$ and, thus, one recovers, with very different techniques, the consistency condition usually denoted as “asymptotic freedom” [4, 9]. Obviously, in the trivial $(\lambda \Phi^4)_4$ theory, where no observable scattering process can survive in the continuum limit, the notion of asymptotic freedom is very different from QCD and has nothing to do with the existence of a renormalized coupling constant $\lambda_R(\mu^2)$ at non vanishing external momenta. Only at zero momentum a non-trivial dynamics is possible and this unobservable effect is fully reabsorbed in the vacuum structure and in the particle mass of the field $h(x)$. Thus “triviality” means that the finite-momentum modes are free-field like. The physics in the SSB vacuum has free particles, and $m_h$ is the mass of any physical (on-shell) particle with any 3-momentum one wishes, including zero. It is true that there is a subtlety about zero 4-momentum, but this affects only off-shell particles, and since the theory is free the experimenter has no way to make off-shell particles.

The remarkable consistency of this theoretical framework suggests to look for additional tests of the structure of the effective potential in Eqs. (14) by comparing with precise lattice simulations of the massless regime. To this end, a few general remarks are needed. While substantial experience has been already gained in the numerical analysis of lattice field theories, the reliability of statements about their continuum limits is still open to questions.
In fact, the equivalence of different procedures to get those limits is not yet controlled by standard theoretical methods. These imply a shift of the problem from the domain of bare computation, including the evaluation of numerical errors and/or approximations, to that of interpretation, exploiting the connection of the numerical results with the formal theory or with suitable models of the continuum limit. For instance, in all lattice simulations performed so far, the perturbative relation $Z_\phi \equiv Z_h$ has been assumed to define, from the average bare field measured on the lattice, a renormalized vacuum field that, in the $O(4)$-symmetric case, is then related to the Fermi constant. In this case, since the lattice data definitely support the trend in Eq. (22) and provide trivially free shifted fields well consistently with $Z_h = 1$ (see Ref. [12]), one frequently deduce upper limits on the Higgs mass which would not exist otherwise. Therefore, a fully model-independent approach is needed to check which relations are actually valid outside the perturbative domain. To this end, the lattice approach to quantum field theories can be very useful. Indeed the lattice offers us the unique opportunity to study a quantum field theory with an ultraviolet cutoff (the inverse of the lattice spacing) by means of non-perturbative methods.

An important progress in this direction has been performed in Ref. [36]. There, the lattice theory defined by the Euclidean action

$$S = \sum_x \left[ \frac{1}{2} \sum_\mu (\Phi(x+\hat{\mu}) - \Phi(x))^2 + \frac{\mu_0}{2} \Phi^2(x) + \frac{\lambda_4}{4} \Phi^4(x) - J \Phi(x) \right]$$

(25)

(x denotes a generic lattice site and, unless otherwise stated, lattice units are understood) was used to compute the VEV of the bare scalar field $\Phi(x)$ in the presence of an “external source” whose strength $J(x) = J$ is $x$-independent

$$\langle \Phi \rangle_J = \left\langle \frac{1}{L^4} \sum_x \Phi(x) \right\rangle_J = \phi_B(J),$$

(26)

where $L$ is the linear dimension of the lattice. Determining $\phi_B(J)$ at several $J$-values is equivalent [37, 38, 39] to inverting the relation

$$J = J(\phi_B) = \frac{dV_{\text{eff}}}{d\phi_B}$$

(27)

and starting from the action in Eq. (25), the effective potential can be rigorously defined for the lattice theory up to an arbitrary integration constant. In this framework, the occurrence of SSB is determined by exploring (for $J \neq 0$) the properties of the function

$$\phi_B(J) = -\phi_B(-J)$$

(28)
in connection with its behaviour in the limit of zero external source

$$\lim_{J \to 0^\pm} \phi_B(J) = \pm v_B \neq 0$$  \hspace{1cm} (29)$$

over a suitable range of the bare parameters \(r_0, \lambda_0\) appearing in Eq. (25). The existence of non-vanishing solutions of Eq. (29) is associated with non-trivial extrema of the effective potential at \(\phi_B = \pm v_B\) whose energy density is lower than (or equal to) the corresponding value in the symmetric phase \(\phi_B = 0\). In fact, through the Legendre transform formalism, the resulting effective potential is convex downward and represents the convex hull \(\Pi\) of the more familiar “double well” effective potential (for more details see Ref. [23, 40]).

The analysis of Ref. [36] was performed for weak bare coupling \(\lambda_0/\pi^2 << 1\), to approach the “triviality” continuum limit in Eq. (24), and exploring the dependence on the bare mass \(r_0\) to provide an operative definition of the massless regime. Here, some remarks are needed. The massless theory is defined [27] from the condition

$$\left. \frac{d^2 V_{\text{eff}}}{d\phi_B^2} \right|_{\phi_B = 0} = 0,$$

(30)

ensuring that the theory has no physical mass scale in its symmetric phase \(\phi_B = 0\). Thus, from Eq. (27), one should explore, in principle, the shape of \(J = J(\phi_B)\) around \(J = 0\). However, in any finite lattice [39], the basic inadequacy of the finite volume calculation shows up in the occurrence of large finite size effects at very small values of \(|J|\). This suggests that a reliable estimate of the massless regime requires to evaluate the response of the system at non-zero \(J\) and then to extrapolate the lattice data toward \(J = 0^\pm\) with some analytic form. To this end, in Ref. [36] Eq. (30) was replaced by a trial form for the source consistent with the “triviality” structure in Eqs. (1,4) but allowing for an explicit scale breaking parameter \(\beta\) [14]:

$$J(\phi_B) = \alpha \phi_B^3 \ln(\phi_B^2) + \beta \phi_B + \gamma \phi_B^3$$

(31)

(for \(\beta = 0\) Eqs. (27,31) imply \(\alpha = (9\tilde{\lambda}^2)/(16\pi^2)\) and \(\gamma = (9\tilde{\lambda}^2)/(16\pi^2) \ln(1/v_B^2)\) in the case of the effective potential in Eqs. (1,4)).

The massless regime at each \(\lambda_0\) was operatively defined from the maximum value of the bare mass \(r_0\) where the 3-parameter fit \((\alpha, \beta, \gamma)\) to the lattice data (for \(|J| \geq 0.05\)) was giving exactly the same \(\chi^2\) of the 2-parameter fit \((\alpha, \beta = 0, \gamma)\). Finally, the numerical results of Ref. [36] were also fitted with analytical forms allowing for the presence of perturbative corrections. As a matter of fact, Eqs. (1,4) agree remarkably well with the lattice results, while the “pro-forma” perturbative leading-log improvement fails to reproduce the Monte Carlo data, thus providing a definite numerical support for the coexistence of SSB and “triviality” proposed in Ref. [13, 14].
A possible objection to this procedure is that the operative definition of the massless theory is not fully model-independent since, by using Eq. (31), one essentially checks the self-consistency of the procedure while a definitive test of the structure of the effective potential requires an a priori estimate of $r_c$ as, for instance, determined from the lattice data of other groups.

After this general introduction, the aim of the present paper is just to provide this more refined lattice test of the “triviality” structure in Eqs. (1,4). Our analysis will be presented in Sect. 2 while Sect. 3 will contain our conclusions and a discussion of the more general consequences of our results.

## 2 Lattice simulation of massless $(\lambda \Phi^4)_4$

The starting point for the analysis of the massless theory is the Schwinger-Dyson equation in the symmetric phase which determines the pole of the scalar propagator from the 1PI self-energy

$$m^2 = r_0 + \Sigma(p^2 = m^2)$$

and leads to the zero-mass condition for the “critical” value of the bare mass

$$r_c = -\Sigma(p^2 = 0).$$

Depending on the adopted regularization scheme, Eq. (33) disposes of the bare mass in the classical lagrangian as a counterterm for the quantum theory and, in this situation, the theory does not possess any physical scale in its symmetric phase $\langle \Phi \rangle = 0$. In dimensional regularization $r_c = 0$ and the scale invariance of the classical theory is preserved up to logarithmic terms. Concerning the lattice theory some remarks are needed.

We stress that the definition of the massless theory is independent on any assumption about the nature of the phase transition in $(\lambda \Phi^4)_4$. More precisely, from the general structure of the effective potential in Eqs. (1,4), we deduce that the system is already in the broken phase at $m^2 = 0$ in contrast with the leading-log improvement of the one loop result [27]. To clarify this point it is worthwhile to recall some rigorous results. In the symmetric phase $\langle \Phi \rangle = 0$, the existence of the $(\lambda \Phi^4)_4$ critical point can be established for the lattice theory (see chapt.17 of Ref. [3]). Namely, for any $\lambda_0 > 0$, a critical value $r_c = r_c(\lambda_0)$ exists, such that the quantity $m(r_0, \lambda_0)$ defined as

$$m(r_0, \lambda_0) = -\lim_{|x-y|\to\infty} \frac{\ln \langle \Phi(x)\Phi(y) \rangle}{|x-y|}$$

(34)
is a continuous, monotonically decreasing, non-negative function for \( r_0 \) approaching \( r_c \) from above and one has \( m(r_0, \lambda_0) = 0 \) for \( r_0 = r_c(\lambda_0) \). For \( r_0 > r_c(\lambda_0) \), \( m(r_0, \lambda_0) \) is the energy of the lowest nonvacuum state in the symmetric phase \( \langle \Phi \rangle = 0 \).

The basic problem for the analysis of the phase transition in the cutoff theory concerns the relation between \( r_c(\lambda_0) \) and the value marking the onset of SSB, say \( r_s(\lambda_0) \), defined as the supremum of the values of \( r_0 \) at which Eq. (29) possesses non-vanishing solutions for a given \( \lambda_0 \). It should be obvious that, in general, \( r_s \) and \( r_c \) correspond to basically different quantities. Indeed, \( r_c \) defines the limiting situation where there is no gap for the first excited state in the symmetric phase \( \langle \Phi \rangle = 0 \), while \( r_s \) can only be determined from a stability analysis after computing the relative magnitude of the energy density in the symmetric and non-symmetric vacua. A widely accepted point of view, based on the picture of a second-order Ginzburg-Landau phase transition with perturbative quantum corrections, is that, indeed, the system is in the broken phase at \( r_0 = r_c - M^2 \) for any \( M^2 > 0 \), thus implying \( r_s = r_c \). In this case, \( -M^2 \) represents the ‘negative renormalized mass squared’ frequently used to describe SSB in pure \( \lambda \Phi^4 \) and related to the second derivative of the effective potential at the origin in field space \( \phi_B = 0 \). However, in this picture the massless theory at \( r_0 = r_c \) still belongs to the symmetric phase thus implying that for

\[
r_0 \to r_c = r_c(\lambda_0)
\]

(from above) one finds a ‘continuum limit’ of the theory at that particular value of the bare coupling. Indeed, by using \( m(r_0, \lambda_0) \) in Eq. (34) to set the physical scale of the theory and approaching the critical point, the correlation length would diverge in units of the lattice spacing.

This conclusion, however, ignores the possibility that

\[
r_c(\lambda_0) < r_s(\lambda_0)
\]

so that the massless regime lies in the broken phase. In this case, there is no continuum limit at finite bare coupling, since the physical correlation length of the theory is not defined through Eq. (34) but has to be determined from the mass of the shifted field after subtracting out the field vacuum expectation value. Thus, understanding the continuum limit of \( (\lambda \Phi^4)_4 \) requires to study the effective potential of the theory by exploring the stability of the symmetric phase for \( r_0 = r_c(\lambda_0) + \epsilon \) at any \( \epsilon > 0 \).

In general, the effective potential is not a finite order polynomial function of the vacuum field \( \phi_B \) and (see chapt.1 of Ref. [10] and chapt.4 of Ref. [41]) one should consider the more general situation

\[
V_{\text{eff}}(\phi_B) = \frac{1}{2} a\phi_B^2 + \frac{1}{4} b\phi_B^4 + \frac{1}{6} c\phi_B^6 + \ldots
\]  

(35)
where \( a, b, c, \ldots \) depend on the bare parameters \((r_0, \lambda_0)\) so that there are several patterns in the phase diagram. In particular, if the coefficient \( b \) can become negative, even though \( a > 0 \), there is a first order phase transition and the system dives in the broken phase passing through a degenerate configuration where \( V_{\text{eff}}(\pm v_B) = V_{\text{eff}}(0) \). A remarkable example of this situation was provided in Ref. [42]. There, the superconducting phase transition was predicted to be (weakly) first order, because of the effects of the intrinsic fluctuating magnetic field which induce a negative fourth order coefficient in the free energy when the coefficient of the quadratic term is still positive. Unfortunately, the predicted effect is too small to be measured but, conceptually, is extremely relevant.

Thus, on general grounds, one may consider the possibility that \( r_c < r_s \) even though for \( r_0 > r_c \) the symmetric phase of the lattice \((\lambda \Phi^4)_4\) has still a mass gap and an exponential decay of the two-point correlation function. The subtlety is that the general theorem 16.1.1 of Ref. [8], concerning the possibility of deducing the uniqueness of the vacuum in the presence of a non-vanishing mass gap in the symmetric phase, holds for the continuum theory. Namely, by introducing the variable \( t = \ln(a_0/a) \), \( a_0 \) being a fixed length scale, and defining the continuum limit as a suitable path in the space of the bare parameters \( r_0 = r_0(t), \lambda_0 = \lambda_0(t) \) (with \( \lambda_0(t) \sim 1/t \) according to Eqs. (7,24)) only paths leading to a vanishing mass gap in the symmetric phase, i.e. for which

\[
\lim_{t \to \infty} m(r_0(t), \lambda_0(t)) = 0
\]

(36)
can consistently account for the occurrence of SSB in quantum field theory. However, at any finite value of the ultraviolet cutoff \( \Lambda \sim 1/a \) there is no reason why the mass gap should vanish before the system being in the broken phase, thus opening the possibility that \( r_s > r_c \). As discussed in Ref. [14, 43], by exploring the \( m \)-dependence of the effective potential of the cutoff theory in those approximations consistent with “triviality”, this indeed occurs and one finds a first order phase transition so that the massless regime lies in the broken phase (see Eqs. (114)). For convenience of the reader, in the case of the Gaussian approximation, we describe in the Appendix the evaluation of the effective potential and its dependence on the bare mass parameter.

At the same time[23, 44, 43], SSB requires \( m \), the mass gap in the symmetric phase, to vanish in units of \( m_h \), the mass gap of the broken phase, in the continuum limit \( t \to \infty \) consistently with Eq. (36) (see Eq. (66) of the Appendix).

As anticipated in the Introduction, a definitive test of the validity of this theoretical framework requires to explore the shape of the effective potential after an \textit{a priori} determination of \( r_c \). To this end we shall use the analysis of lattice data presented by Brahm in Ref. [44]. By considering data from different groups and different lattices, including a parametrization of the finite-size effects and extrapolating the lattice data down to zero mass, Brahm’s analysis provides a rather precise determination of the value of \( r_c \) at \( \lambda_0 = 0.5 \):

\[
r_c = -0.2240 - \frac{1.00 \pm 0.05}{L^2} \pm 0.0010.
\]

(37)
For \( L=16 \), Eq. (37) predicts

\[
  r_c = -0.2279(10) .
\]

In order to check this prediction we have performed a direct numerical calculation of \( r_c \) at \( \lambda_0 = 0.5 \) on a \( 16^4 \) lattice.

An accurate procedure to determine \( r_c(\lambda_0) \) uses the susceptibility \( \chi \):

\[
  \chi = L^4 \left[ \langle |\Phi|^2 \rangle - \langle |\Phi| \rangle^2 \right]
\]

where

\[
  |\Phi| = \frac{1}{L^4} \sum_x |\Phi(x)| .
\]

Indeed it is known that [7] near the critical region \( \chi^{-1} \sim (r_c - r_0) \), modulo logarithmic corrections. However, the logarithmic modifications of the free field scaling law are important only when \( r_0 \) is very close to \( r_c \). Accordingly, we have measured the susceptibility in the symmetric phase \( r_0 < r_c \) and fitted linearly the inverse of the susceptibility. We find that the linear fit is rather good for \( 0.24 \leq r_0 \leq 0.35 \) (see Fig. 1). As a result we obtain

\[
  r_c = -0.22788(138) .
\]

The agreement with Brahm’s prediction is remarkable and, thus, we have two independent indications that the central value \( r_c = -0.2279 \) represents the input definition of the massless regime for a numerical computation of the slope of the effective potential on a \( 16^4 \) lattice with the action Eq. (25) at \( \lambda_0 = 0.5 \). Finally, after obtaining \( J = J(\phi_B) \) in a model-independent way, we shall compare the lattice data with the two alternative descriptions of the phase transition in \((\lambda \Phi^4)_4\).

For our Monte Carlo simulation we used the standard Metropolis algorithm to update the lattice configurations weighted by the action Eq. (25). In order to avoid the trapping into metastable states due to the underlying Ising dynamics we followed the upgrade of the scalar field \( \Phi(x) \) with the upgrade of the sign of \( \Phi(x) \). This is done according to the effective Ising action [47]

\[
  S_{\text{Ising}} = J \sum_x |\Phi(x)| s(x) - \sum_x \sum_{\hat{\mu}} |\Phi(x + \hat{\mu})\Phi(x)| s(x + \hat{\mu}) s(x) ,
\]

where \( s(x) = \text{sign}(\Phi(x)) \). We measured the vacuum expectation value of the scalar field

\[
  \langle \Phi \rangle_J = \frac{1}{N_c} \sum_{i=1}^{N_c} \frac{1}{L^4} \sum_x \Phi^i(x)
\]
where $N_c$ is the number of configurations generated with the action (25), for 16 different values of the external source in the range $0.01 \leq |J| \leq 0.70$. Statistical errors are evaluated using the jackknife algorithm [46] adapted to take into account the correlations between consecutive lattice configurations [47]. Our final results for $\langle \Phi \rangle_J = \phi_B(J)$ are shown in Table 2. We have compared these data with the two alternative approaches motivated by “triviality” or based on perturbation theory. In the former case we have chosen the general form

$$J(\phi_B) = \alpha \phi_B^3 \ln(\phi_B^2) + \beta \phi_B + \gamma \phi_B^3 + \delta \phi_B^3 \ln^2(\phi_B^2)$$ (40)

which, besides including an explicit scale breaking term $\beta$ accounts for possible deviations from the “triviality” structure in Eqs. (1,4). Indeed, the presence of residual, genuine self-interaction effects for the field $h(x)$ may show up in a non vanishing coefficient $\delta$ of the $\ln^2(\phi_B^2)$ term as predicted, for instance, from the structure of the effective potential in a perturbative two-loop calculation. Finally, since we expect to be very close to the massless regime, the effect of a non vanishing mass scale $\pm M^2$ in the argument of the logarithms (apart from those effectively included in the coefficient $\beta$) should be negligible. A consistency check
Table 1: We report the values of $\phi_B(J)$ as obtained with our $16^4$ lattice at $\lambda_0 = 0.5$ and $r_0 = r_c = -0.2279$. Errors are statistical only.

| $J$  | $\phi_B(J)$     | $J$  | $\phi_B(J)$     |
|------|-----------------|------|-----------------|
| -0.010 | -0.288862 (695) | 0.010 | 0.289389 (787)  |
| -0.030 | -0.413565 (321) | 0.030 | 0.414713 (376)  |
| -0.050 | -0.488797 (296) | 0.050 | 0.489132 (249)  |
| -0.075 | -0.557737 (181) | 0.075 | 0.557961 (182)  |
| -0.100 | -0.612497 (169) | 0.100 | 0.612865 (151)  |
| -0.300 | -0.876352 (111) | 0.300 | 0.876518 (95)   |
| -0.500 | -1.03526 (8)    | 0.500 | 1.03532 (7)     |
| -0.700 | -1.15518 (8)    | 0.700 | 1.15528 (7)     |

Table 1: We report the values of $\phi_B(J)$ as obtained with our $16^4$ lattice at $\lambda_0 = 0.5$ and $r_0 = r_c = -0.2279$. Errors are statistical only.

of this assumption can be obtained, apart from the quality of the fit, from the size of the coefficient $\beta$ since, away from criticality, one has $\beta \sim \pm M^2$. The results of the fit to the data in Table I with Eq. (40) is

$$\alpha = (1.535 \pm 0.062) \cdot 10^{-2}$$
$$\beta = (0.3 \pm 6.3) \cdot 10^{-4}$$
$$\gamma = 0.44955 \pm 0.00061$$
$$\delta = (1.3 \pm 6.4) \cdot 10^{-4}$$
$$\frac{\chi^2}{\text{d.o.f}} = \frac{14.4}{16 - 4}$$

It is clear that the model-independent calculation of the slope of the effective potential, is in very good agreement with our predictions based on Eqs. (2). In fact, the fit shows no evidence for non-vanishing coefficients $\beta$ and $\delta$. By constraining $\beta = \delta = 0$ in the fit, we obtain $\alpha = 1.520(18) \times 10^{-2}$, $\gamma = 0.44960(7)$, $\chi^2/\text{d.o.f} = 15.0/(16 - 2)$. In Figure 2 we display our data together with the best fit Eq. (40), with the constraint $\beta = \delta = 0$. Also, the size of the coefficient $\beta$ is so small that the effect of a non vanishing $\pm M^2 \sim \beta$ in the arguments of the logarithms is totally negligible, at least in the explored range of $\phi_B$, $\phi_B^2 > 0.08$. Notice that the only finite size effect included in our analysis is the $1/L^2$ correction to the Brahm’s value for the critical bare mass. Therefore, the excellent $\chi^2$’s of the fits show that, at least
Figure 2: The external current $J$ versus $\phi_B$. The solid line is Eq. (40) with $\beta = \delta = 0$ and $\alpha = 1.520 \times 10^{-2}$, $\gamma = 0.44960$.

for $|J| \geq 0.01$, our $16^4$ lattice behaves as an infinite system.

Let us now compare the lattice data in Table 4 with a perturbative evaluation of the effective potential. Indeed, the small size of the bare coupling $\lambda_0/\pi^2 \sim 0.05$ might suggest that the agreement between the lattice data and Eqs. (1,4) represents just a trivial test of perturbation theory. To this end we have compared with the full two-loop calculation of Ref. [48]. In the dimensional regularization scheme, their expression for the effective potential in the massless regime and for the one component theory is $(m_2^2 = \lambda \phi_B^2/2, \Omega(1) = \frac{3}{4} S - \frac{1}{3} \zeta(2)$ with $S = 1/2^2 + 1/5^2 + 1/8^2 + \ldots$ and $\bar{m}$ includes in the definition of the logarithm additional terms of the MS scheme)

$$V^{2\text{-loop}}(\phi_B) = V_0(\phi_B) + V_1(\phi_B) + V_2(\phi_B)$$

with

$$V_0(\phi_B) = \frac{\lambda}{4!} \phi_B^4$$
\[ V_1(\phi_B) = \frac{1}{64\pi^2} m_2^4 \left[ \ln \frac{m_2^2}{\mu^2} - \frac{3}{2} \right] \]

and

\[ V_2(\phi_B) = \frac{1}{256\pi^4} \frac{\lambda^2 \phi_B^2 m_2^2}{8} \left[ 5 + 8\Omega(1) - 4\ln \frac{m_2^2}{\mu^2} + \ln \phi_B^2 \right] + \frac{1}{256\pi^4} \frac{\lambda m_2^4}{8} \left[ 1 - \ln \frac{m_2^2}{\mu^2} \right]^2 \]

By transforming to our notations, namely

\[
\lambda \equiv 6\lambda_0 \\
\ln \left( \frac{m_2^2}{\mu^2} \right) - \frac{3}{2} \equiv \ln \left( \frac{m_2^2}{\Lambda^2} \right) - \frac{1}{2}
\]

and computing \( J^{2-\text{loop}}(\phi_B) = dV^{2-\text{loop}}/d\phi_B \) we find the final expression

\[
J^{2-\text{loop}}(\phi_B) = \frac{\lambda_0 \phi_B^3}{1 + \frac{9\lambda_0}{16\pi^2} \ln \frac{\Lambda^2}{3\lambda_0 \phi_B^2}} + \frac{\lambda_0^3 \phi_B^3}{256\pi^4} \left[ 27 \ln \frac{\Lambda^2}{3\lambda_0 \phi_B^2} + 54 + 432\Omega(1) \right] \tag{42}
\]

in which we have used the perturbative \( \beta \) function to resum the leading logarithmic terms to all orders. By fitting the lattice data in Table 2 with Eq. (42) for \( \lambda_0 = 0.5 \), and leaving out the scale \( \Lambda \) as a free parameter in the fit, we find the result for the 2-loop, leading-log improved fit

\[
\left( \frac{\chi^2}{\text{d.o.f}} \right)_{2-\text{loop}} = \frac{1162}{16 - 1}
\]

which, indeed, shows that, despite of the small value of the bare coupling, perturbation theory (in one of its more refined versions) is totally unable to describe the lattice data. As anticipated in the Introduction, this result is not unexpected and, therefore, the agreement with Eqs. (14) is not a trivial test of perturbation theory but provides, rather, a non-perturbative test of “triviality”. The crux of the matter has to be found in the qualitative conflict between Eq. (10) (for \( \beta = \delta = 0 \)) and Eq. (12). In the former case, based on a first-order description of the phase transition, the massless regime lies in the broken phase and there are non-trivial minima for the effective potential so that \( J \) vanishes at non zero values \( \phi_B = \pm v_B \). On the other hand, Eq. (12), consistent with a second-order phase transition, can only vanish at \( \phi_B = 0 \) since, within the perturbative approach, one needs a non-vanishing and negative renormalized mass squared \(-M^2\) to obtain SSB. Our lattice data for the response of the system to the external source are precise enough to detect the sizeable difference produced by the two extrapolations towards \( J = 0 \).
Finally, to perform an additional check, we have used the complete form for \( V^{2\text{-loop}}(\phi_B) \) reported in Eq. (41) by allowing for the presence of a negative mass parameter \(-M^2\) as in Ref. [48]. In this case, where the classical potential becomes

\[
V_0(\phi_B) = \frac{\lambda}{4!} \phi_B^4 - \frac{1}{2} M^2 \phi_B^2
\]

and the mass parameter \( m_2^2 \) has to be replaced everywhere by

\[
m_2^2 = \frac{\lambda \phi_B^2}{2} - M^2,
\]

we still obtain an extremely poor fit

\[
\left( \frac{\chi^2}{\text{d.o.f.}} \right)_{2\text{-loop}} = \frac{152}{16 - 2}.
\]

Summarizing: our analysis predicts that, for \( \lambda_0 = 0.5 \), the massless regime of \((\lambda \Phi^4)_4\) corresponds to \( r_0 = r_c = -0.2279\) on a 16\(^4\) lattice. The resulting effective potential, computed in a fully model-independent way, is in excellent agreement with the general “triviality” structure in Eqs. (44) and cannot be reproduced in a perturbative expansion, despite of the small value of the bare coupling \( \lambda_0 / \pi^2 \sim 0.05 \). Our analysis, while confirming the existence on the lattice of a remarkable phase of \((\lambda \Phi^4)_4\) where SSB is generated through “dimensional transmutation” [27], enforces the first numerical evidences of Ref. [36] pointing out the inner contradiction between perturbation theory and “triviality”. The most important consequences of this basic inadequacy will be illustrated in detail in Sect. 3

### 3 Conclusions and outlook

Let us now compare our results with the output of the existing lattice simulations. So far, the theoretical expectations based on “triviality” have been numerically confirmed and there is overwhelming evidence that all observable interaction effects vanish when approaching the continuum limit, i.e. when the physical correlation length of the broken phase becomes large in units of the lattice spacing.

In all interpretations of the lattice simulations performed so far the validity of the perturbative relation (23) has been assumed. Let us refer to the very complete review by C.B.Lang [12]. There, for the O(4) theory, the value of \( Z = Z_h \) is extracted from the large distance decay of the Goldstone boson propagator and used in Eq. (23) to define a renormalized VEV \( u_R \) from the average bare VEV \( v_B \) measured on the lattice. Quite independently of any interpretation, the lattice data provide \( Z_h = 1 \) to very good accuracy (see Tab.II in
Ref. [12]) and completely confirm the trend in Eqs. (7,22) (see Fig.19 in Ref. [12]) so that, in this approach, one finds

\[
\frac{m_h^2}{u_R^2} \sim \frac{m_h^2}{v_B^2} \sim \frac{8\pi^2}{3 \ln(\Lambda/m_h)} \to 0. \tag{43}
\]

The above relation has been interpreted so far on the basis of the leading-log formula for the running coupling constant. Indeed, in perturbation theory, where one relates \(u_R \sim v_B\) to \(m_h\) through the renormalized 4-point function at external momenta comparable to the Higgs mass itself,

\[
\frac{m_h^2}{u_R^2} \sim 3\lambda_R(m_h^2), \tag{44}
\]

in the leading-log approximation

\[
\lambda_R(m_h^2) = \frac{\lambda_0}{1 + \frac{9\lambda_0}{8\pi^2} \ln \frac{\Lambda}{m_h}} \tag{45}
\]

one deduces Eq. (13) for \(\lambda_0 \to \infty\). Therefore, when \(u_R^2 \equiv v_B^2/Z_h \sim v_B^2 \sim 1/G_F\sqrt{2}\), and the result (22) is interpreted within perturbation theory, one concludes that SSB is only possible in the presence of an ultraviolet cutoff (this point of view leads to upper bounds on the Higgs mass [8, 11] \(m_h < 700 \sim 900\) GeV).

As pointed out in the Introduction, however, this interpretation of “triviality” is, at least, suspicious. Indeed, within perturbation theory itself, it is in contradiction with explicit two-loop calculations of \(\beta_{\text{pert}}\). In this case, from Eq. (8) one would deduce that the bare coupling flows toward the ultraviolet fixed point

\[
\lim_{\Lambda \to \infty} \frac{\lambda_0(\Lambda)}{\pi^2} = \frac{24}{17} = \frac{\lambda^*}{\pi^2} \tag{46}
\]

for any \(0 < \lambda_R < \lambda^*\) and, by assuming the validity of Eq. (14), there is no reason why \(\lambda_R\) and \(m_h\) should vanish in the limit \(\Lambda \to \infty\) (\(\lambda^*,\) in any case, disappears at 3-loop but reappears at 4-loops [32]).

Quite independently of this remark, which rather concerns the internal consistency of the perturbative approach to “triviality”, our numerical results of Sect. 2 (and those of Ref. [36]), show that in the response of the lattice theory to the external source \(J\) there is no trace of any residual \(h\)-field self-interaction effect but the lattice effective potential cannot be reproduced in perturbation theory. Therefore, any perturbative interpretation of “triviality”
can hardly be taken as correct. Close to the continuum limit as one can be, we do find a
definite numerical evidence for non trivial minima of the effective potential, and, as such,
there is no reason why SSB should not coexist with “triviality”. However, the Higgs mass
$m_h$ defined from the vacuum energy in Eq. (3), which determines the physical scale of the
broken phase, is quite unrelated to $\lambda_R$, which vanishes, and does not represent a measure of
any interaction.

Before exploring the consequences of our picture of SSB, however, let us briefly discuss the
case of a $(\lambda \Phi^4)_4$ theory in the continuous symmetry O(N) case. The use of radial and angular
fields allows to deduce easily the structure of the effective potential for the O(N) theory. In
fact, as discussed in Ref. [20, 21, 13], one expects Eqs. (20, 21) of the one-component theory,
to be also valid for the radial field in the O(N)-symmetric case. The explanation for this
result is extremely intuitive and originates from Ref. [49] which obtained, for the radial
field, the same effective potential as in the discrete-symmetry case. The Goldstone-boson
fields contribute to the effective potential only through their zero-point energy, that is an
additional constant, since, according to “triviality”, they are free massless fields. Thus, in
the O(2)-symmetric case, one may take the diagram $(V_{\text{eff}}, \phi_B)$ for the one-component theory
and “rotate” it around the $V_{\text{eff}}$ symmetry axis. This generates a three-dimensional diagram
$(V_{\text{eff}}, \phi_1, \phi_2)$ where $V_{\text{eff}}$, owing to the O(2) symmetry, only depends on the bare radial field,

$$\rho_B = \sqrt{\phi_1^2 + \phi_2^2}$$  \hspace{1cm} (47)

in exactly the same way as $V_{\text{eff}}$ depends on $\phi_B$ in the one-component theory; namely

$$(\omega^2(\rho_B^2) = 3\lambda \rho_B^2)$$

$$V_{\text{eff}}(\rho_B) = \frac{\bar{\lambda}}{4} \rho_B^4 + \frac{\omega^4(\rho_B^2)}{64 \pi^2} \left( \ln \frac{\omega^2(\rho_B^2)}{\Lambda^2} - \frac{1}{2} \right).$$  \hspace{1cm} (48)

This has been explicitly checked in Ref. [36] with the Monte Carlo simulation of the O(2)
lattice theory employing the action

$$S = \sum_x \sum_{i=1}^2 \left[ \frac{1}{2} \sum_{\hat{\mu}} (\Phi_i(x + \hat{\mu}) - \Phi_i(x))^2 + \frac{1}{2} r_0 (\Phi_i(x) \Phi_i(x) + \frac{\lambda_0}{4} (\Phi_i(x) \Phi_i(x))^2 - J_i \Phi_i(x) \right]$$  \hspace{1cm} (49)

where $\Phi_1$ and $\Phi_2$ are coupled to two constant external sources $J_1$ and $J_2$. By using $J_1 = J \cos \theta$ and $J_2 = J \sin \theta$ and having defined $\phi_1 = \langle \Phi_1 \rangle_{J_1, J_2}$, $\phi_2 = \langle \Phi_2 \rangle_{J_1, J_2}$, one can compute
the bare radial field Eq. (47)

$$\rho_B = \rho_B(J)$$  \hspace{1cm} (50)
and invert Eq. (50) to obtain the slope of the effective potential. As shown in Ref. [36], the lattice data for $\rho_B = \rho_B(J)$ in the massless regime are remarkably reproduced by

$$J(\rho_B) = \frac{dV_{\text{eff}}(\rho_B)}{d\rho_B} = \frac{9\lambda^2\rho_B^3}{16\pi^2} \ln \frac{\rho_B^2}{v_B^2},$$

(51)

thus providing definite numerical support for the exactness conjecture of Eqs. (20, 21) [13, 14].

Finally, in the post Gaussian calculation of Ref. [25], the numerical solution of the integral equation for the shifted radial field propagator gives for the Higgs mass $m_h = 2.21$ TeV for $N=2$ and $m_h = 2.27$ TeV for $N=4$ to compare with the prediction of Eq. (21) $m_h = 2.19$ TeV for $v_R \sim 246$ GeV if the physical VEV introduced in Sect. 2 is related to the Fermi constant $G_F$ in the usual way.

Therefore, in the most appealing theoretical framework, where SSB is generated from a theory which does not possess any physical scale in its symmetric phase $\langle \Phi \rangle = 0$, we end up with a definite prediction for the Higgs mass, namely $m_h \sim 2.2$ TeV [20, 21, 13, 14]. In general, the Higgs can be lighter or heavier [14], depending on the flow of the bare mass $r_0 = r_0(t)$ in the continuum limit $t \to \infty$. In this case one ends up with the more general result [14]

$$m_h^2 = 8\pi^2\zeta v_R^2$$

(52)

with

$$0 < \zeta \leq 2.$$  

(53)

$\zeta = 1$ corresponds to $r_0(t) = r_c(t)$ and $\zeta = 2$ to $r_0(t) = r_s(t) > r_c(t)$, as defined in Sect. 2 (see also the Appendix). This limiting situation, namely when $m_h/v_R = 4\pi$ and symmetric and broken phases have the same energy, places an upper bound on the Higgs mass $m_h < 3.1$ TeV. The range $0 < \zeta < 1$, corresponding to values $r_0(t) < r_c(t)$, for which the theory cannot be quantized in its symmetric phase $\langle \Phi \rangle = 0$, is allowed by the Renormalization Group analysis of the effective potential [14] and cannot be discarded.

The existence of an upper limit for the ratio $m_h/v_R$ from vacuum stability (and not from ‘triviality’) is a genuine quantum phenomenon which has no counterpart in the semiclassical ‘double well’ picture. It is a direct consequence of the first-order phase transition and is in qualitative agreement with similar estimates based on the Bethe-Salpeter equation. By investigating the condition for a zero-mass bound state in Higgs-Higgs scattering, signaling the instability of the spontaneously broken phase, one finds the result $m_h \lesssim 2.4$ TeV [50] or $m_h \lesssim 3.4$ TeV [51] in different approximations of the Bethe-Salpeter kernel.

Obviously, the above estimates are only valid provided the contribution to the effective potential from the other fields, namely gauge bosons and fermions, is negligible and can be
treated as a small perturbation, as in the original formulation of the Weinberg-Salam theory \cite{1} where SSB is generated in the pure scalar sector (this is certainly possible for $\zeta \sim 1$ where the Higgs mass is large compared to the top quark mass as measured by the CDF and D0 Collaborations, $m_t = 180 \pm 12$ GeV \cite{2}).

However, even though the Higgs mass would turn out to be considerably lighter than our reference value 2.2 TeV, there are substantial implications. In fact, the Higgs phenomenology, on the basis of gauge invariance, depends on the details of the pure scalar sector. For instance, consider the Higgs decay width to $W$ and $Z$ bosons. The conventional calculation would give a huge width, of order $G_F m_h^3 \sim m_h$ for $m_h \sim 1$ TeV. However, in a renormalizable-gauge calculation of the imaginary part of the Higgs self-energy, this result comes from a diagram in which the Higgs supposedly couples strongly to a loop of Goldstone bosons with a physical strength proportional to its mass squared, an effect which, in principle, has nothing to do with the gauge sector but crucially depends on the description of the pure scalar theory at zero gauge coupling. Now, if “triviality” is true, as we believe, all interaction effects of the pure $\lambda \Phi^4$ sector of the standard model have to be reabsorbed into two numbers, namely $m_h$ and $v_R$, and there are no residual interactions. However, if a Higgs particle can decay into two Goldstone bosons there are observable interactions, namely there is a non-trivial scattering matrix for Goldstone-Goldstone scattering with a pole in the complex plane whose real and imaginary parts are related to the Higgs mass and to the Higgs decay width. Beyond perturbation theory, this process cannot be there and, therefore, if “triviality” is true, a heavy Higgs must be a relatively narrow resonance, decaying predominantly to $t\bar{t}$ quarks.

In conclusion, according to our analysis “triviality” implies just the opposite of what is generally believed, namely SSB with elementary scalar fields, and as such the essential ingredient for the Higgs mechanism, poses, by itself, no problems of internal consistency as far as its quantum field theoretical limit is concerned. Truly enough, this is only relevant to the continuum theory and, therefore, we have little to say if the scalar sector of the standard model turns out to be a low-energy effective description of symmetry breaking. In this case, a perturbative approach in terms of a light Higgs mass (say $m_h \lesssim 200$ GeV \cite{53}) is still acceptable, since the scale at which the picture breaks down is exponentially decoupled from the Higgs mass, and physically equivalent to our description in the limit $\zeta \ll 1$. On the other hand, if the Higgs turns out to be very heavy a measure of its decay width will represent a test of our predictions and the experiment will decide between the two descriptions.

\textit{Acknowledgement}

We thank A. Agodi, G. Andronico, P.M. Stevenson, and D. Zappalà for many useful discussions.
Appendix

Let us now address the question of the stability of the symmetric phase in the Gaussian approximation. This is a particularly simple type of calculation and has the advantage, in a ‘trivial’ theory such as \((\lambda \Phi^4)_4\), of being effectively exact, in its renormalized form, as discussed in Sect. 1.

The starting point for our analysis is the effective potential for composite operators introduced by Cornwall, Jackiw and Tomboulis (CJT) \([54]\) which represents a powerful analytic tool to investigate the structure of the effective potential beyond perturbation theory.

The CJT approach is based on the exact relation between the effective potential and the energy functional for a constant bare field configuration and arbitrary equal-time propagator \(G(\vec{x}, \vec{y}) = G(x, y)|_{x_0 = y_0}\)

\[
\int d^3 x \ V_{\text{eff}}(\phi_B) = E[\phi_B, G_0(\phi_B)]
\]

where

\[
E[\phi_B, G] = \min_{\Psi} \langle \Psi | H | \Psi \rangle
\]

with the conditions

\[
\langle \Psi | \Psi \rangle = 1
\]

\[
\langle \Psi | \Phi | \Psi \rangle = \phi_B
\]

\[
\langle \Psi | \Phi(\vec{x})\Phi(\vec{y}) | \Psi \rangle = \phi_B^2 + G(\vec{x}, \vec{y})
\]

and

\[
\left. \frac{\delta E[\phi_B, G]}{\delta G(\vec{x}, \vec{y})} \right|_{G=G_0(\phi_B)} = 0 .
\]

Finally, the absolute minima of \(V_{\text{eff}}\), say \(\pm v_B\), define the ground state energy density \(W\) up to an arbitrary additive constant (for which we shall take the value at \(\phi_B = 0\)) so that

\[
W = V_{\text{eff}}(\pm v_B) - V_{\text{eff}}(0)
\]
and SSB corresponds to the situation \( W < 0 \) for \( v_B \neq 0 \).

In general one may employ different approximation schemes to \( E[\phi_B, G] \) and, therefore, to \( V_{\text{eff}} \). For instance, one may adopt the modified loop expansion discussed in [54] in terms of two-particle irreducible vacuum-vacuum graphs with vertices determined by the shifted interaction Lagrangian and propagators fixed by \( G(x,y) \). This type of ‘loop-expansion’, however, is very different from the usual loop-expansion in powers of \( \hbar \) for the effective potential since, for instance, the inclusion of a single graph in \( E[\phi_B, G] \) produces, through the (in general non-perturbative) solution of Eq. (54), an infinite number of graphs in terms of the 1-loop propagator

\[
D(x, y; \phi_B) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m_h^2 + 3\lambda_0\phi_B^2} \exp[ip(x - y)].
\]  

(55)

We want to emphasize that a systematic expansion in powers of \( \hbar \) is precisely equivalent to a weak coupling expansion in the shifted theory and, as such, unable to provide any meaningful indication for a theory where no observable interaction effect can survive in the continuum limit. In fact, the attempt to renormalize \((\lambda \Phi^4)_4\) in the standard perturbative approach is based on the concept of a cutoff independent and non-vanishing renormalized coupling at non zero external momenta \( \lambda_R(Q^2) \), a concept for which there is no room in a ‘trivial’ theory (in order to fully realize the essentially perturbative nature of the usual loop expansion we address the interested reader to ref. [21] where the meaning of the various contributions and their relation with the perturbative expansion are clearly illustrated).

It is clear from its definition that \( G(x,y) \) is the propagator of the shifted field \( h(x) \) for each given value of the vacuum field \( \phi_B \). Thus, a consistency requirement for SSB in \((\lambda \Phi^4)_4\) is that, in the continuum limit, \( G(x,y) \) has to reduce to a free field propagator in agreement with the basic ‘triviality’ results of Sect. 2. Now, it is well known that in perturbation theory this does not occur. Beyond the lowest order 1-loop approximation in Eq. (55), the shifted field propagator has ultraviolet divergent corrections implying a non trivial cutoff-dependence of \( Z_h \) for the quantum field \( h(x) \) and as such a non-trivial anomalous dimension

\[
\gamma_h(\lambda_0) = -\frac{1}{2} \frac{\partial \ln Z_h}{\partial \ln \Lambda} = \frac{3\lambda_0^2}{256\pi^4} + O(\lambda_0^3) + \cdots
\]

which cannot vanish in the continuum limit \( \Lambda \to \infty \) unless, in the same limit, \( \lambda_0 = \lambda_0(\Lambda) \to 0 \), in contrast with the results obtained from the perturbative \( \beta \)-function.

On the basis of the previous discussion, it should be clear that ‘triviality’ forces to define the continuum theory starting from a regularized version where all shifted field self-interaction effects are neglected or become unobservable, being reabsorbed into its mass. Therefore, only approximations to \( E[\phi_B, G] \) fulfilling the consistency requirement

\[
G(x,y) \to \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m_h^2} \exp[ip(x - y)]
\]
when the ultraviolet regulator is removed, are allowed. A particularly simple class of approximations is provided by the 1-loop and Gaussian effective potential where, by definition, the shifted field $h(x)$ is governed by a quadratic Hamiltonian consistently with “triviality”.

The Gaussian effective potential has been considered as a convenient method to study vacuum stability beyond perturbation theory by many authors and is based on the choice of the Gaussian wave-functionals

$$\Psi_G[\Phi] = (\text{Det } G)^{-1/4} \times \exp\left[-\frac{1}{4} \int d^3x \int d^3y \,(\Phi(\vec{x}) - \phi_B) \, G^{-1}(\vec{x}, \vec{y}) \,(\Phi(\vec{y}) - \phi_B)\right]$$

where

$$G(\vec{x}, \vec{y}) = \int \frac{d^3p}{(2\pi)^3} \frac{\exp[i\vec{p} \cdot (\vec{x} - \vec{y})]}{2\sqrt{p^2 + \omega^2}}$$

$\omega$ being a variational parameter. In this class of states one obtains [5, 53, 13]

$$E_G(\phi_B, \omega) = \int d^3x \, V_G(\phi_B, \omega)$$

with

$$V_G(\phi_B, \omega) = \frac{1}{2} r_0 \phi_B^2 + \frac{1}{4} \lambda_0 \phi_B^4 + I(x, x) + \frac{1}{2} (r_0 - \omega^2 + 3\lambda_0 \phi_B^2) \, G(x, x) + \frac{3\lambda_0}{4} G(x, x)G(x, x)$$

where

$$I(x, y) = \frac{1}{2} G^{-1}(x, y).$$

In Eqs. (56,57) we have used the 4-dimensional notation to express $G(x, x)$ and $I(x, x)$ in terms of the 4-dimensional Euclidean cutoff (appropriate for the lattice theory) as

$$G(x, x) = \frac{1}{16\pi^2}[\lambda^2 + \omega^2 \ln \frac{\omega^2}{\Lambda^2 + \omega^2}]$$

and

$$I(x, x) = \frac{\Lambda^2 \omega^2}{32\pi^2} + \frac{1}{64\pi^2} \omega^4 [\ln \frac{\omega^2}{\Lambda^2 + \omega^2} - \frac{1}{2}] + \frac{\Lambda^4}{64\pi^2} R(q^2) + C$$
where $R$ denotes the ultraviolet finite expression ($q^2 = \frac{\omega^2}{\pi^2}$)

\[ R(q^2) = \ln(1 + q^2) - q^2 + \frac{1}{2} q^4 \]

and $C$ is an $\omega$-independent constant.

In the class of Gaussian states, the integral equation in Eq. (54) becomes a simple algebraic equation in the mass parameter $\omega$ for any given value of $\phi_B$

\[ \omega^2 = r_0 + 3\lambda_0 \phi_B^2 + 3\lambda_0 G(x, x). \]  

(59)

By introducing dimensionless variables

\[ s = \frac{3\lambda_0}{16\pi^2} \]  

(60)

\[ r_0 = -s\Lambda^2 + \epsilon\Lambda^2 \]

\[ \phi_B = f\Lambda \]

one obtains

\[ q^2 = \epsilon + 3\lambda_0 f^2 + sq^2 \ln \frac{q^2}{1 + q^2}. \]

(61)

The Gaussian effective potential at its minimum, where one replaces $q^2 = q^2(f^2)$ from Eq. (61), becomes

\[ V_{\text{eff}}(\phi_B) = V_G(\phi_B, \omega(\phi_B^2)) = \Lambda^4 u_\epsilon(f^2) \]

where $u_\epsilon(f^2)$ can be put in the particularly simple form (up to an uninteresting constant term)

\[ u_\epsilon(f^2) = \frac{q^4(f^2)}{64\pi^2} \left[ \ln \frac{q^2(f^2)}{1 + q^2(f^2)} + \frac{1}{2} \right] + \frac{R(q^2(f^2))}{64\pi^2} + \frac{q^4(f^2)}{12\lambda_0} - \frac{\lambda_0 f^4}{2} \]

(62)

($u_\epsilon(f^2)$ is bounded from below in the limit of large $f^2$). The meaning of the variational parameter $\omega$ as the shifted-field mass is further confirmed by computing the energy of the one-particle states [55]

\[ |1\tilde{p}\rangle = a_\omega^+(\tilde{p})|\Psi\rangle_G \]
where, by using Eq. (59), one finds

\[ E_1(\phi_B, \omega(\phi_B^2); \vec{p}) - E_G(\phi_B, \omega(\phi_B^2)) = \sqrt{\vec{p}^2 + \omega^2(\phi_B^2)} \]

so that, indeed, \( \omega(\phi_B^2) \) is the gap in the energy spectrum of the cutoff theory. In particular, \( \omega(0) \) represents the Gaussian approximation result for \( m(r_0, \lambda_0) \) introduced in Eq. (34) of Sect. 2.

The qualitative behaviour of \( u_\epsilon(f^2) \) is the following. Let us first consider the gap equation at \( f = 0 \). In this case we find

\[ q_2^2(0) = \frac{\epsilon}{1 + s \ln((1 + q^2(0))/(q^2(0)))} < \epsilon \]

which is a positive-definite quantity for any \( \epsilon > 0 \) and one has

\[ \frac{d^2 u_\epsilon}{df^2}\bigg|_{f=0} = q^2(0). \]

At large values of \( \epsilon \) the function \( u_\epsilon(f^2) \) is convex downward and has only one minimum at \( f = 0 \). As \( \epsilon \) becomes smaller (approaching the regime \( \epsilon \sim \exp[-1/s] \)) the function \( u_\epsilon(f^2) \) develops secondary maxima and minima since the fourth-derivative at the origin

\[ (F(q^2) = \ln((1 + q^2)/q^2) - 1/(1 + q^2)) \]

\[ \frac{d^4 u_\epsilon}{df^4}\bigg|_{f=0} = 6\lambda_0 \frac{1 - s F(q^2(0))}{1 + (s/2) F(q^2(0))} \]

becomes negative as anticipated in connection with the sign of the coefficient \( b \) in Eq. (15) of Sect. 2. The secondary minima are located at non zero values of \( f = \pm \bar{f} \) where

\[ q^2(\bar{f}^2) = 2\lambda_0 \bar{f}^2. \]

At

\[ \epsilon = \epsilon_s \sim \frac{0.8}{e} s \exp\left[-\frac{1}{2s}\right] \sim 0.8 \epsilon_{\text{max}} \]

(\( \epsilon_{\text{max}} \) being the value above which the effective potential has no extrema for \( f \neq 0 \)) one obtains

\[ u_{\epsilon_s}(\bar{f}^2) = u_{\epsilon_s}(0). \]
Thus, in the Gaussian approximation, where \( r_c(\lambda_0) = -s\Lambda^2 \) (see Eqs. (58-60) one finds

\[
r_s(\lambda_0) = r_c(\lambda_0) + \epsilon_s \Lambda^2 > r_c(\lambda_0)
\]

as anticipated in Sect. 2. For \( \epsilon < \epsilon_s \) the minima of \( u_\epsilon(f^2) \) at \( \pm \bar{f} \) become deeper than its value at \( f = 0 \) and for \( \epsilon = 0 \) one reaches the massless regime, well within the broken phase. In Fig. 3 we show the shape of \( u_\epsilon(f^2) - u_\epsilon(0) \) for three values of \( \epsilon \) indicative of the overall situation.

The special case \( \epsilon = 0 \) is particularly interesting as discussed in Sect. 1. In this case, we find the simple relation for the effective potential at the absolute minima \( \phi_B = \pm v_B = \pm f\Lambda \)

\[
W = V_{\text{eff}}(\pm v_B) - V_{\text{eff}}(0) = -\frac{m_h^4}{128\pi^2}
\]
with

\[ m_h^2 = \omega^2(v_B^2) = 2\lambda_0 v_B^2 = \Lambda^2 \exp \left[ -\frac{8\pi^2}{3\lambda_0} \right]. \quad (63) \]

To compare with the analogous 1-loop relations, one simply has to replace \( \lambda_0 \rightarrow \frac{3}{2}\lambda_0 \) everywhere in Eq. (63).

In the relevant region of weak bare coupling, where \( s \ll 1, r_s \) and \( r_c \) are numerically so close that a “direct” numerical test of the phase transition is not possible (analogously to the tiny effect predicted in ref.[41]). However, reliable informations can be obtained by comparing the lattice data with various models of the effective potential as shown in Sect. 3.

For \( r_0 \neq r_c \), the parametrization of the effective potential in terms of a renormalized vacuum field \( \phi_R \) such that

\[
\frac{d^2 V_{\text{eff}}}{d\phi_R^2} \bigg|_{\phi_R=\pm v_R} = m_h^2
\]

leads to the more general relations \([14]\) (up to terms which vanish in the limit \( \Lambda \rightarrow \infty, \lambda_0 \rightarrow 0 \) and \( m_h = \text{fixed} \))

\[ V_{\text{eff}}(\phi_R) = \pi^2 \zeta^2 \phi_R^4 \left( \ln \phi_R^2 v_R^2 - \frac{1}{2} \right) + \frac{1}{4}(\zeta - 1)m_h^2 \phi_R^2 \left( 1 - \frac{\phi_R^2}{2v_R^2} \right) \quad (64) \]

\( \zeta \) being defined through

\[ m_h^2 = 8\pi^2 \zeta v_R^2. \quad (65) \]

For all positive values of \( \zeta \) the values \( \phi_R = \pm v_R \) are minima of the effective potential. The minimum has a lower energy than the origin \( \phi_R = 0 \) if \( \zeta < 2 \). At \( \zeta = 2 \), corresponding to the value of the bare mass \( r_0 = r_s \) discussed above, there is a phase transition to the broken symmetry phase. Finally, for \( \zeta = 1 \), corresponding to \( r_0 = r_c \), one finds the Coleman-Weinberg regime \([27]\) and Eqs. \((64,65)\) reduce to Eqs. \((20,21)\) of Sect. 1. The range \( 0 < \zeta < 1 \), corresponding to values \( r_0 < r_c \) for which the theory cannot be quantized in its symmetric phase, is allowed by the RG analysis of the effective potential and cannot be discarded. For \( r_c \leq r_0 \leq r_s \) (corresponding to \( 1 \leq \zeta \leq 2 \)) where SSB coexists with a physical mass gap in the symmetric phase \( \omega(0) \geq 0 \) one finds

\[ \frac{\omega^2(0)}{m_h^2} < s \frac{\zeta - 1}{\zeta} \rightarrow 0 \quad (66) \]
in the bare weak-coupling limit where $\lambda_0$ and $s$ vanish at $m_h = \text{fixed}$. Thus, the mass gap of the symmetric phase becomes infinitesimal in units of $m_h$ in agreement with the general condition for SSB in Eq. (36) of Sect. 2. This result confirms the remarkable consistency of our definition of the continuum limit.

References

[1] S. Weinberg, Phys. Rev. Lett. 19 (1967) 1264; A. Salam, Proc. 8th Nobel Symp., N. Svartholm ed. (Almqvist and Wicksell Stockholm, 1968) 367.

[2] P. W. Higgs, Phys. Lett. 12 (1964) 132); Phys. Rev. Lett. 13 (1964) 508; F. Englert and R. Brout, ibid, 321; G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, ibid, 585; T. W. B. Kibble, Phys. Rev. 155 (1967) 1554.

[3] M. Aizenman, Phys. Rev. Lett. 47 (1981) 1.

[4] J. Fröhlich, Nucl. Phys. B200(FS4) (1982) 281.

[5] A. Sokal, Ann. Inst. H. Poincaré, 37 (1982) 317.

[6] K. G. Wilson and J. Kogut, Phys. Rep. C12 (1974) 75; G. A. Baker and J. M. Kincaid, Phys. Rev. Lett. 42 (1979) 1431; B. Freedman, P. Smolensky and D. Weingarten, Phys. Lett. B113 (1982) 481; D. J. E. Callaway and R. Petronzio, Nucl. Phys. B240 (1984) 577; I. A. Fox and I. G. Halliday, Phys. Lett. B 159 (1985) 148; C. B. Lang, Nucl. Phys. B 265 (1986) 630.

[7] M. Lüscher and P. Weisz, Nucl. Phys. B 290 (1987) 25.

[8] J. Glimm and A. Jaffe, Quantum Physics: A Functional Integral Point of View (Springer, New York, 1981, 2nd Ed. 1987).

[9] D. J. E. Callaway, Phys. Rep. 167 (1988) 241.

[10] R. Fernández, J. Fröhlich, and A. D. Sokal, Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory (Springer-Verlag, Berlin, 1992).

[11] R. Dashen and H. Neuberger, Phys. Rev. Lett., 50 (1983) 1897; J. Kuti, L. Lin, and Y. Shen, Phys. Rev. Lett. 61 (1988) 678; H. Neuberger, U. M. Heller, M. Klomfass, and P. Vranas, in Proceedings of the XXVIth International Conference on High Energy Physics, Dallas, TX, August 1992.

[12] C. B. Lang, Computer Stochastics in Scalar Quantum Field Theory, to appear in Stochastic Analysis and Application in Physics, Proc. of the NATO ASI, Funchal, Madeira, Aug. 1993, ed. L. Streit, Kluwer Acad. Publishers, Dordrecht 1994.
[13] M. Consoli and P. M. Stevenson, *Resolution of the $\lambda\Phi^4$ Puzzle and a 2 TeV Higgs Boson*, Rice University preprint, DE-FG05-92ER40717-5, July 1993, submitted to Annals of Physics. (hep-ph 9303256).

[14] M. Consoli and P. M. Stevenson, Zeit. Phys. C63 (1994) 427.

[15] M. Consoli and A. Ciancitto, Nucl. Phys. B254 (1985) 653.

[16] P. M. Stevenson and R. Tarrach, Phys. Lett. B176 (1986) 436.

[17] P. Castorina and M. Consoli, Phys. Lett. B235 (1989) 302; V. Branchina, P. Castorina, M. Consoli and D. Zappalà, Phys. Rev. D42 (1990) 3587.

[18] R. Muñoz-Tapia and R. Tarrach, Phys. Lett. B256 (1991) 50; R. Tarrach, Phys. Lett. B262 (1991) 294.

[19] V. Branchina, P. Castorina, M. Consoli, and D. Zappalà, Phys. Lett. B274 (1992) 404.

[20] M. Consoli, in “Gauge Theories Past and Future – in Commemoration of the 60th birthday of M. Veltman”, R. Akhoury, B. de Wit, P. van Nieuwenhuizen and H. Veltman Eds., World Scientific 1992, p. 81; M. Consoli, Phys. Lett. B 305 (1993) 78.

[21] V. Branchina, M. Consoli and N. M. Stivala, Zeit. Phys. C57 (1993) 251.

[22] R. Ibañez-Meier and P. M. Stevenson, Phys. Lett. B297 (1992) 144.

[23] U. Ritschel, Phys. Lett. B318 (1993) 617.

[24] L. Polley and U. Ritschel, Phys. Lett. B221 (1989) 44; R. Ibañez-Meier, A. Mattingly, U. Ritschel, and P. M. Stevenson, Phys. Rev. D45 (1992) 2893.

[25] U. Ritschel, Zeit. Phys. C63 345 (1994).

[26] M. G. do Amaral and R. C. Shellard, Phys. Lett. B171 (1986) 285; M. Lüscher and P. Weisz, Nucl. Phys. 295 (1988) 65; J. K. Kim and A. Patrascioiu, “Studying the continuum limit of the Ising model”. University of Arizona preprint AZPH-TH/92-09, July 1992.

[27] S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888.

[28] J. Pedersen, I. E. Segal and Z. Zhou, Nucl. Phys. B376 (1992) 129.

[29] R. Jackiw, Phys. Rev. D9 (1974) 1686.

[30] For instance, according to Fröhlich, in $(\lambda\Phi^4)_4$ “... (renormalized) perturbation theory is slippery ...” (see pag.13 of Ref. [31]) and also “... The traditional view of $\Phi^4_4$ (circa 1950) based on renormalized perturbation theory ... is ruled out ...” (see pag.390 of Ref. [10]).
Eq. (8) predicts an unphysical Landau pole at 1-loop or an ultraviolet fixed point at non-zero bare coupling at 2-loops. This structure reproduces itself in higher orders. Indeed, in odd orders $\beta_1^{1\text{-loop}}, \beta_3^{3\text{-loop}}, \beta_5^{5\text{-loop}}$, each exhibits a Landau singularity whereas in even orders $\beta_2^{2\text{-loop}}, \beta_4^{4\text{-loop}}$, have an ultraviolet fixed point whose magnitude decreases by increasing the perturbative order. In this situation, for a sufficiently high ultraviolet cutoff, the conventional approach based on hierarchy of leading logarithmic, next-to-leading, next-to-next-to-leading, . . . effects looses its meaning and, for any non-vanishing $\lambda_R$, no consistent continuum limit is possible in perturbation theory.

D. Shirkov, *Renormalization Group in different fields of physics*, lectures at KEK, April 1991, KEK-91-13 (Feb. 1992) (unpublished); K. G. Chetyrkin et al., Phys. Lett. B132 (1983) 351; D. I. Kazakov, Phys. Lett. B133 (1983) 406.

M. Consoli and P. M. Stevenson, *‘Triviality’and the perturbative expansion in $\lambda\Phi^4$ theory*, Rice University preprint, November 1994, (revised version) submitted to Physics Letters B.

G. A. Hajj and P. M. Stevenson, Phys. Rev. D 37 (1988) 413; N. M. Stivala, Nuovo Cimento A 106 (1993) 777.

A. Agodi, G. Andronico and M. Consoli, *Spontaneous Symmetry Breaking and the Higgs Mass: a Precision Test of ‘Triviality’ on the Lattice*, Contribution to the Workshop on Electroweak Symmetry Breaking, Budapest, July 9-12 1994, G. Poksik and F. Csikor Eds., World Scientific, in press; A. Agodi, G. Andronico and M. Consoli, *Lattice $(\Phi^4)_4$ effective potential giving spontaneous symmetry breaking and the role of the Higgs mass*, Zeit. Phys. C66 no.3, Apr. 1995, to appear.

K. Huang, E. Manousakis, and J. Polonyi, Phys. Rev. D35 (1987) 3187.

K. Huang, Int. J. Mod. Phys. A4 (1989) 1037; in Proceedings of the DPF Meeting, Storrs, CT, 1988.

D.J.Callaway and D.J.Maloof, Phys. Rev. D27 (1983) 406.

K. Symanzik, Comm. Math. Phys. 16 (1970) 48; T. L. Curtright and C. B. Thorn, Journ. Math. Phys. 25 (1984) 541; A. Dannenberg, Phys. Lett. B202 (1980) 110; V. Branchina, P. Castorina, and D. Zappalà, Phys. Rev. D41 (1990) 1948.

J. C. Tolédano and P. Tolédano, *The Landau Theory of Phase Transitions, Applications to Structural, Incommensurate, Magnetic and Liquid Crystal Systems*, World Scientific 1987.

B. I. Halperin, T. C. Lubenski and S. Ma, Phys. Rev. Lett. 32 (1974) 292.
A. Agodi, M. Consoli, and D. Zappalà, *Spontaneous Symmetry Breaking, ‘Triviality’, and nature of the phase transition in* $\langle \lambda \Phi^4 \rangle_4$, *preprint march 1995, submitted to Annals of Physics.*

D. E. Brahm, *The lattice cutoff for* $\lambda \Phi^4$ *and* $\lambda \Phi^6$, *CMU-HEP94-10 preprint, hep-lat/9403021*, March 1994.

R. C. Brower and P. Tamayo, Phys. Rev. Lett. **62** (1989) 1087.

B. Efron, *Jackknife, the Bootstrap and Other Resampling Plans*, (SIAM Press, Philadelphia, 1982).

See for instance: B. A. Berg and A. H. Billoire, Phys. Rev. **D40** (1989) 550.

C. Ford and D. R. T. Jones, Phys. Lett. **B274** (1992) 409; Erratum, ibidem **B285** (1992) 399.

L. Dolan and R. Jackiw, Phys. Rev. **D9** (1974) 3320.

G. Rupp, Phys. Lett. **B288** (1992) 99.

S. Dilcher, ‘*Multi Higgs Physik’*, Diploma Thesis, University of Freiburg, May 1995.

CDF Collaboration, F. Abe et al., Phys. Rev. Lett. **74** (1995) 2626; D0 Collaboration, S. Abachi et al, ibid. **74** (1995) 2632.

N. Cabibbo, L. Maiani, G. Parisi and R. Petronzio, Nucl. Phys. **B158** (1979) 295.

J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. **D10** (1974) 2428.

P. M. Stevenson, Phys. Rev. **D32** (1985) 1389.