BPS/CFT correspondence V:
BPZ and KZ equations from $qq$-characters

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Abstract

We illustrate the use of the theory of $qq$-characters by deriving the BPZ and KZ-type equations for the partition functions of certain surface defects in quiver $\mathcal{N} = 2$ theories. We generate a surface defect in the linear quiver theory by embedding it into a theory with additional node, with specific masses of the fundamental hypermultiplets at that node. We prove that the supersymmetric partition function of this theory with $SU(2)^r$ gauge group verifies the celebrated Belavin-Polyakov-Zamolodchikov equation of two dimensional Liouville theory. We also study the $SU(n)$ theory with $2n$ fundamental hypermultiplets and the theory with adjoint hypermultiplet. We show that the regular orbifold defect in this theory solves the KZ-like equation of the WZW theory on a four punctured sphere and one-punctured torus, respectively. In the companion paper [66] these equations will be mapped to the Knizhnik-Zamolodchikov equations.

1 Introduction

In the recent paper [53] we attempted to study the novel type of Dyson-Schwinger identities, relating non-perturbative contributions to the correlation functions coming from different topological sectors of the field space. Our main tool is the class of special observables called the $qq$-characters, which are most useful for exploiting these identities. In this paper we shall look at the examples of applications of these identities.

2 Linear quiver theories

In this section we will be dealing with the linear $A$-type quiver $\mathcal{N} = 2$ gauge theories in four dimensions. The gauge group of the corresponding theory is a product of $r$ factors,

$$G_g = U(N) \times \ldots \times U(N)$$

(1)

which we shall label by the vertices $i = 1, \ldots, r$ of the $A_r$ Dynkin diagram. The theory has matter hypermultiplet fields, transforming in the bifundamental representations $(N_i, \bar{N}_{i+1})$, and two pairs of $N$ fundamental hypermultiplets, charged under the $U(N)_1$ and the $U(N)_r$. 

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gauge factors, respectively. The Lagrangian of the theory and the choice of the vacuum is parametrized by the complexified gauge couplings

$$\tau_i = \frac{\delta_i}{2\pi} + \frac{4\pi i q_i}{g_i^2} \in H_+, \quad q_i = \exp 2\pi i \tau_i \in \mathbb{C}^\times_{|\tau| < 1},$$

and the Coulomb moduli $a_i = (a_i, a_i)_{i=1}^{N_1}$, where the range of the index $i$ is extended to $0 \leq i \leq r + 1$ with the convention where $a_0, a$ and $a_{r+1}, a$ encode the masses of the fundamental hypermultiplets, while $q_0 = q_{r+1} = 0$.

In this paper, as in [52] we study vacuum expectation values of the composite operators, the $qq$-characters, which are built out of the $\gamma$-observables. For each node $i$, $i = 0, \ldots, r + 1$

$$\gamma_i(x) \sim \text{Det}(x - \Phi_i)$$

is the regularized characteristic polynomial of the adjoint complex scalar in the vector multiplet of the $i$'th factor of the gauge group (which is actually the flavor group for $i = 0, r + 1$).

The calculations in gauge theory are facilitated by $\Omega$-deformation and the use of equivariant localization [52]. The partition function we are going to study is the sum over the toric instantons, which are in one-to-one correspondence with the collections $\Lambda = (\lambda^{(i,\alpha)})$, $i = 1, \ldots, r$, $\alpha = 1, \ldots, N_2$ of Young diagrams:

$$\lambda^{(i,\alpha)} = \left(\lambda_1^{(i,\alpha)} \geq \lambda_2^{(i,\alpha)} \geq \ldots \geq \lambda_{\ell(\lambda^{(i,\alpha)})}^{(i,\alpha)} > 0\right).$$

The $\gamma$-observables [3] evaluate on $\Lambda$ to

$$\gamma_i(x)|_{\Lambda} = \prod_{a=1}^{N} \left(\frac{(x - a_i, a_i)}{(x - c_\square - \epsilon_1)(x - c_\square - \epsilon_2)}\right)$$

Here $\epsilon = \epsilon_1 + \epsilon_2$, $\epsilon_1, \epsilon_2$ are the two complex parameters of the $\Omega$-background, $\square \equiv (a, b) \in \lambda^{(i,\alpha)} \iff$

$$1 \leq b \leq \lambda^{(i,\alpha)}_a, \quad 1 \leq a \leq \left(\lambda^{(i,\alpha)}_b\right)^\ell$$

and

$$c_\square \equiv a_i + \epsilon_1 (a - 1) + \epsilon_2 (b - 1)$$

The supersymmetric partition function of the theory we discuss is given by the product:

$$Z_N(a, q, \epsilon_1, \epsilon_2) = Z_N(a; q; \epsilon_1, \epsilon_2)^{\text{tree}} Z_N(a; \epsilon_1, \epsilon_2)^{1\text{-loop}} Z_N(a; q; \epsilon_1, \epsilon_2)^{\text{inst}}$$

where

$$Z_N(a; q; \epsilon_1, \epsilon_2)^{\text{tree}} = \prod_{i=1}^{r} \prod_{a=1}^{N} q_i^{-\frac{1}{2\pi i} a_i^2}$$

$$Z_N(a; \epsilon_1, \epsilon_2)^{1\text{-loop}} = \prod_{a, \beta=1}^{N} \prod_{i=1}^{r+1} \frac{\Gamma_2(a_i, a_i - \alpha_i, \beta; \epsilon_1, \epsilon_2)}{\Gamma_2(a_{i-1}, a_i, \beta; \epsilon_1, \epsilon_2)}$$
with \( \Gamma_2(x;\epsilon_1,\epsilon_2) \) the Barnes double gamma function, an entire function of \( x \) with simple zeroes at \( x = m\epsilon_1 + n\epsilon_2, n, m \in \mathbb{Z}_{>0} \). Finally,

\[
Z_N(a, q; \epsilon_1, \epsilon_2)_{\text{inst}} = \sum_{\Lambda} Q_{\Lambda} M_{\Lambda}
\]

where we set \( \lambda^{(0,a)} = \lambda^{(r+1,a)} = \emptyset \),

\[
Q_{\Lambda} = \prod_{i=1}^{r} \prod_{a=1}^{N} q_i^{[\lambda^{(i,a)}]}
\]

and

\[
M_{\Lambda} = \prod_{a,\beta=1}^{N} \left[ \prod_{i=1}^{r a} M_{\epsilon_1, \epsilon_2}(a_{i-1,a} - a_i, \beta; \lambda^{(i,a)}, \lambda^{(i,\beta)}) \right] \prod_{a=1}^{r} \left[ \prod_{i=1}^{N} M_{\epsilon_1, \epsilon_2}(a_i, a_{i-1,a}; \lambda^{(i,a)}, \lambda^{(i,\beta)}) \right]
\]

with

\[
M_{\epsilon_1, \epsilon_2}(x; \lambda, \nu) = \prod_{(a,b) \in \nu} \left( x + \epsilon_1 (a - 1) + \epsilon_2 (\nu_a + 1 - a) \right) \prod_{(a,b) \in \lambda} \left( x + \epsilon_1 (\nu_b + 1 - a) + \epsilon_2 (b - 1) \right)
\]

Note that

\[
M_{\epsilon_1, \epsilon_2}(x; \lambda, \emptyset) = P_{\lambda}(x + \epsilon), \quad M_{\epsilon_1, \epsilon_2}(x; \emptyset, \nu) = (-1)^{|\nu|} P_{\nu}(-x)
\]

where

\[
P_{\lambda}(x) \equiv \prod_{(a,b) \in \lambda} (x + \epsilon_1 (a - 1) + \epsilon_2 (b - 1))
\]

is the so-called content polynomial.

### 2.1 Non-perturbative Dyson-Schwinger identities

The main statement of [53] is that the expectation values \( \langle X_i(x) \rangle_N \) of the \( qq \)-character observables:

\[
X_i(x) = y_i(x) \sum_{J \subseteq [0,r]} \prod_{j \in J} \Xi_j(x + \epsilon h_j)
\]

are entire functions of \( x \), in fact, polynomials of degree \( N \) (cf. (128)):

\[
\langle X_i(x) \rangle = T_i(x) = e_i x^N + \sum_{p=1}^{N} t_{i,p} x^{N-p}
\]

Here

\[
\Xi_i(x) = z_i \frac{y_{i+1}(x + \epsilon)}{y_i(x)}, \quad i = 0, \ldots, r
\]

the parameters \( z_i \) are related to \( q_i \) via:

\[
q_i = z_i / z_{i-1}, \quad i = 0, \ldots, r + 1
\]

so that \( z_{-1} = \infty \), \( z_{r+1} = 0 \) and \( z_0, z_1, \ldots, z_r \) are defined up to an overall rescaling. Finally, the symbol \( \langle O \rangle \), for a function \( O = O_\Lambda \) on the set of \( N \times r \)-tuples of partitions is defined as the complexified statistical average:

\[
\langle O \rangle_N = \frac{1}{Z_N(a, q; \epsilon_1, \epsilon_2)_{\text{inst}}} \sum_{\Lambda} Q_{\Lambda} M_{\Lambda} O_{\Lambda}
\]
2.2 BPZ equation

Consider the \( r + 3 \)-point conformal block

\[
\chi_{r+3}(z) = \left\langle \prod_{i=-1}^{r+1} V_{\Delta_i}(z_i) \right\rangle \text{chiral}
\]  

(22)

of conformal primary operators with dimensions \( \Delta_i, \ i = -1, \ldots, r + 1 \) in the two dimensional conformal Liouville theory with the Virasoro central charge

\[
c = 1 + 6 \left( b + b^{-1} \right)^2
\]  

(23)

When one of the primary dimensions, e.g. \( \Delta_0 \) is equal to either

\[
\Delta_{1,2} = -\frac{1}{2} - \frac{3b^2}{4}, \quad \text{or} \quad \Delta_{1,2} = -\frac{1}{2} - \frac{3}{4b^2},
\]  

(24)

then the corresponding fields \( V_{\Delta_0} \) have null-vectors among their descendants. The decoupling of the null-vectors implies the second-order differential equation for the conformal blocks, the BPZ equation \([5]\):

\[
\left[ \frac{3}{2(2\Delta_0 + 1)} \frac{\partial^2}{\partial z_0^2} + \sum_{i \neq 0} \left( \frac{\Delta_i}{(z_0 - z_i)^2} + \frac{1}{z_0 - z_i} \frac{\partial}{\partial z_i} \right) \right] \chi_{r+3}(z) = 0.
\]  

(25)

where \( i = -1, 0, 1, \ldots, r + 1 \). In addition, (22) obeys the global Virasoro invariance constraints:

\[
\left[ z_0^{-1} V_0^z + \sum_{i \neq 0} z_i^{-1} V_i^z \right] \chi_{r+3}(z) = 0
\]

\[
\left[ V_0^z + \Delta_0 + \sum_{i \neq 0} (V_i^z + \Delta_i) \right] \chi_{r+3}(z) = 0
\]

\[
\left[ z_0 \left( V_0^z + 2\Delta_0 \right) + \sum_{i \neq 0} z_i \left( V_i^z + 2\Delta_i \right) \right] \chi_{r+3}(z) = 0.
\]

(26)

where

\[
V_i^z = z_i \frac{\partial}{\partial z_i}
\]  

(27)

Let us express \( V_{r+1}^z \chi \) and \( V_{r+1}^z \chi \) in terms of \( V_0^z \chi \) and \( V_i^z \chi, \ i = 1, \ldots, r \) using the first and the third equations in (26). The equation (25) and the second equation in (26) now become the partial differential equations in \( r + 1 \) variables \( z_0, z_1, \ldots, z_r \), with \( z_{-1}, z_{r+1} \) playing the role of parameters. Now set

\[
b^2 = \varepsilon_2/\varepsilon_1
\]  

(28)

and

\[
\Delta_0 = -\frac{1}{2} \frac{3\varepsilon_2}{4\varepsilon_1}
\]  

(29)
Without any loss of generality we can now set \( z_{-1} = \infty \) and \( z_{r+1} = 0 \) in (25), with the result:

\[
\begin{align*}
\left[ (\epsilon_1 \nabla^2_0) (\epsilon_1 \nabla^2_0 - \epsilon) + \epsilon_1 \epsilon_2 \left( \Delta_{r+1} + \sum_{i=1}^{r} (\Delta_i u_i^2 - u_i \nabla^2_i) \right) \right] \chi_{r+3} &= 0 \\
\left[ \nabla^2_0 + \Delta_0 - \Delta_{-1} + \Delta_{r+1} + \sum_{i=1}^{r} (\nabla^2_i + \Delta_i) \right] \chi_{r+3} &= 0
\end{align*}
\]

(30)

where

\[
\epsilon_i = \frac{z_0}{z_i - z_0}, \quad i = 1, \ldots, r .
\]

(31)

In this paper we prove that the properly normalized partition function

\[
\frac{Z_2}{Z_1}
\]

(32)

(we fix the normalization of the parameters below) solves the BPZ equation (30) when the parameters \( a_0 \) and \( a_1 \) are in the following relation: For some \( \alpha = 1, \ldots, N \)

\[
a_{0,\beta} = a_{1,\beta}, \quad \beta \neq \alpha \\
a_{0,\alpha} = a_{1,\alpha} + \epsilon_2
\]

(33)

2.3 Remarks

1. In this paper we study the case \( N = 2 \). The \( N > 2 \) generalization is possible, the Liouville theory being replaced by the \( A_{N-1} \) Toda theory.

2. The condition (33) depends on the choice of \( \alpha \). There are, therefore, \( N \) solutions of (30), corresponding to the \( N \) choices of \( \alpha \).

3. The degeneration (33) is not unique. In fact, (33) can be generalized to: For some \( \alpha = 1, \ldots, N \) and \( i = 1, \ldots, r + 1 \)

\[
a_{i-1,\beta} = a_{i,\beta}, \quad \beta \neq \alpha \\
a_{i-1,\alpha} = a_{i,\alpha} + \epsilon_2
\]

(34)

We will not show in this paper that the partition functions which are obtained by the tuning (34) of the Coulomb moduli provide the local solutions to (30) with \( z_0 \leftrightarrow z_i, \Delta_0 \leftrightarrow \Delta_i \), which are analytic in the domain \( |z_{i-2}| > |z_{i-1}| > |z_i| \).

4. The relation (33) can be also generalized to

\[
a_{0,\beta} = a_{1,\beta}, \quad \beta \neq \alpha \\
a_{0,\alpha} = a_{1,\alpha} + (p-1)\epsilon_2 + (q-1)\epsilon_1
\]

(35)

with \( p, q > 1 \). When \( p \) and \( q \) are mutually prime the equation (25) is generalized to the differential equation of the order \( pq \).
3 Derivation

The strategy of our derivation is simple. Expand \( Y_i(x) \), \( \Xi_i(x) \) and \( X_i(x) \) at infinity in \( x \). Specifically, we are interested in the \( x^{-1} \) coefficient \( X_i(-1) \) of the large \( x \) expansion of \( X_i(x) \). The main theorem (18) states that

\[
\langle X_i(-1) \rangle = 0, \quad i = 1, \ldots, r \quad (36)
\]

It is convenient to package (36) into the equation:

\[
\prod_{j=0}^{r} \frac{1}{1 + t z_j} \sum_{i=0}^{r+1} t^i \left\langle X_i(-1) \right\rangle = 0 \quad (37)
\]

Now, from (5) we compute:

\[
\log \Xi_i(x)|_\Lambda = \log(z_i) + \exp \frac{1}{x} A_i^{(1)} + \frac{1}{x^2} \left( \frac{1}{2} A_i^{(2)} + \epsilon_1 \epsilon_2 (k_{i+1}(\Lambda) - k_i(\Lambda)) \right) + \frac{1}{x^3} \left( \frac{1}{3} A_i^{(3)} + \epsilon_1 \epsilon_2 \left( 2(c_{i+1}(\Lambda) - c_i(\Lambda)) - \epsilon (k_{i+1}(\Lambda) + k_i(\Lambda)) \right) \right) + \ldots \quad (38)
\]

where

\[
A_i^{(k)} = \sum_{\alpha=1}^{N} (a_{i,\alpha}^{k} - (a_{i+1,\alpha} - \epsilon)^k)
\]

\[
k_i(\Lambda) = \sum_{\alpha=1}^{N} |\lambda^{(i,\alpha)}|
\]

\[
c_i(\Lambda) = \sum_{\alpha=1}^{N} \sum\sum_{\Box \in \lambda^{(i,\alpha)}} c_{\Box}
\]

3.1 Abelian theory

When \( N = 1 \) the equations (37) read (we skip the index \( \alpha \) in the formulas for \( N = 1 \))

\[
\sum_{j=0}^{r} \frac{t z_j}{1 + t z_j} - a_0 A_j^{(1)} + \epsilon_1 \epsilon_2 \left( k_{j+1} - k_j \right) + \frac{1}{2} \left( A_j^{(2)} + (A_j^{(1)})^2 \right) + \ldots
\]

\[
\sum_{0 \leq i < j \leq r} \frac{t z_i}{1 + t z_j} + \frac{t z_j}{1 + t z_i} (A_j^{(1)} - \epsilon) A_j^{(1)} = 0 \quad (40)
\]

which imply, by taking the residue at \( t = -z_i^{-1} \):

\[
- \epsilon_1 \epsilon_2 \nabla_i^z \log Z_1^{\text{inst}} + (a_i - a_0)(a_i - a_{i+1} + \epsilon) +
\]

\[
(a_i - a_{i+1}) \sum_{j > i} z_j \frac{(a_j - a_{j+1} + \epsilon)}{z_j - z_i} + (a_i - a_{i+1} + \epsilon) \sum_{i > j} \frac{z_j (a_j - a_{j+1})}{z_j - z_i} = 0 \quad (41)
\]
which, in turn, implies:

\[
Z_{\text{inst}}^1 = \prod_{0 \leq i < j \leq r} \left(1 - z_j/z_i\right)^{(z_j/z_{i+1})/z_{i+2}}
\]  

(42)

3.2 Non-abelian theory and surface defects

When \( N = 2 \) the equations (36) (with the help of the appendix) imply the relations

\[
\langle c_i \rangle = \langle \text{polynomials in } k'_j s \rangle = \text{differential operator in } q'_j s \text{ on } Z_2
\]  

(43)

The relations (43) are not very useful, since they express unknown expectation values in terms of the derivatives of the partition function. However, when the parameters are tuned as in (33), the measure (13) forces the partitions \( \lambda(1, \beta) \) to be empty when \( \beta \neq \alpha \), while \( \lambda(1, \alpha) \) is forced to have a linear structure:

\[
\lambda^{(1, \alpha)} = (1^n), \quad n = |\lambda^{(1, \alpha)}|
\]  

(44)

Then:

\[
k_1(\Lambda) = n, \quad c_1(\Lambda) = a_{1, \alpha} n + \frac{1}{2} \varepsilon_1 n(n - 1)
\]  

(45)

Physically, the linear structure of the instantons of the \( U(N) \) gauge factor means that the corresponding gauge field configuration is confined to a two-dimensional plane, where it effectively becomes a vortex. This is happening because the condition (33) makes \( N \) fundamental hypermultiplets (out of \( N^2 \)) nearly massless, thereby opening a Higgs branch. The instantons squeezed into the Abrikosov-Nielsen-Ohlesen strings generate a surface defect in the \( A_{r-1} \) gauge theory. The gauge coupling \( q_1 \) plays the role of the two dimensional Kahler parameter of the supersymmetric sigma model living on the worldsheet of this surface defect.

3.3 Back to the equations

Armed with (45) we express:

\[
\langle c_1 \rangle = (a_{1, \alpha} - \frac{1}{2} \varepsilon_1) q_1 \frac{d}{dq_1} Z_2^{\text{inst}} + \frac{1}{2} \varepsilon_1 \left( q_1 \frac{d}{dq_1} \right)^2 Z_2^{\text{inst}}
\]  

(46)

which makes the residue of (43) at \( t = -z_0^{-1} \) into a differential equation on \( Z_2 \):

\[
\left(\varepsilon_1 \nabla_0^2\right) Z_2^{\text{inst}} + \left( \mu_2 - \mu_1 - \varepsilon_2 - \sum \frac{z_i A_i^{(1)}}{z_i - z_0} \right) (\varepsilon_1 \nabla_0^2) Z_2^{\text{inst}} + \\
\left[ \sum \frac{z_i}{z_i - z_0} \left( -\varepsilon_1 \varepsilon_2 \nabla_0^2 + \frac{1}{2} \left( A_j^{(2)} + (A_j^{(1)})^2 \right) - \mu_2 A_i^{(1)} \right) + \sum \frac{z_i z_j (A_j^{(1)} - \varepsilon) A_j^{(1)}}{(z_i - z_0)(z_j - z_0)} \right] Z_2^{\text{inst}} = 0
\]  

(47)

where \( a_{1, \alpha} = \mu_1, a_{1, \beta} = \mu_2, \beta \in \{1, 2\} \setminus \{\alpha\} \). Now define

\[
\chi(z_0, \ldots, z_r) = z_0^{L_0} \prod_{i=1}^r (z_i/z_0)^{L_i} (1 - z_i/z_0)^{6i} \prod_{0 \leq i < j \leq r} (1 - z_j/z_i)^{T_{ij}} Z_2^{\text{inst}}
\]  

(48)
Since the quiver consists of a single vertex, we omit the subscript encoded in the polynomial

\[ T_{ij} = \frac{2(m_i - \epsilon)m_j}{\epsilon_1 \epsilon_2}, \quad m_i = \frac{A_1(1)}{2} = \epsilon + \sum_{a=1}^{2} \frac{a_{i,a} - a_{i+1,a}}{2}, \quad \delta_i = m_i/\epsilon_1 \]

\[ L_0 = \Delta_1 - \sum_{i=0}^{r+1} \Delta_i, \quad \epsilon_1 \epsilon_2 \Delta_i = m_i(\epsilon - m_i), \quad i = 1, \ldots, r \]

\[ \epsilon_1 \epsilon_2 \Delta_{-1} = \frac{\epsilon^2}{4} - \frac{(\mu_1 + \epsilon_2 - \mu_2)^2}{4} = \frac{\epsilon^2}{4} - \frac{(a_{0,1} - a_{0,2})^2}{4}, \quad \epsilon_1 \epsilon_2 \Delta_{r+1} = \frac{\epsilon^2}{4} - \frac{(a_{r+1,1} - a_{r+1,2})^2}{4}, \quad (49) \]

It is now straightforward to check that \( \chi \) so defined solves the equations (30). The relations \( (49) \) match exactly the conjecture of [1].

4 The higher rank \( U(n) \) theories with orbifold defects

Let us pass to the theories with a single factor gauge group \( U(n) \). We study two examples: the \( A_1 \) theory or the \( \tilde{A}_0 \) theory.

4.1 The \( A_1 \) case.

The \( A_1 \) theory is the \( U(n) \) gauge theory with \( N_f = 2n \) fundamental hypermultiplets. The theory is characterized by the gauge coupling \( q \) and \( 2n \) masses \( m = (m_1, \ldots, m_{2n}) \), which are encoded in the polynomial

\[ P(x) = \prod_{i=1}^{2n} (x - m_i) \]

Since the quiver consists of a single vertex, we omit the subscript \( i \) in \( \chi(x) \) and \( P(x) \).

The fundamental \( A_1 \) \( qq \)-character is equal to

\[ \chi_{1,0}(x) = \chi(x + \epsilon) + q \frac{P(x)}{\chi(x)} \quad (50) \]

The general \( A_1 \) \( qq \)-character depends on a \( w \)-tuple \( \nu \) of complex numbers, \( \nu = (\nu_1, \ldots, \nu_w) \in \mathbb{C}^w \). It is given by:

\[ \chi_{w,\nu}(x) = \sum_{[w] = \{1, \ldots, f\}} q^{[w]} \prod_{i \in l, j \in f} S(v_i, v_j, v_{ij}) \prod_{j \in f} \frac{P(x + v_j)}{\chi(x + v_j)} \prod_{i \in l} \frac{\chi(x + \epsilon + v_i)}{\chi(x + \epsilon + v_i)} \quad (51) \]

It has potential poles in \( \nu \)'s, when \( v_i = v_j \) or \( v_i = v_j + \epsilon \), for \( i \neq j \).

The expression (51) is actually non-singular at the diagonals \( v_i = v_j \). The limit contains, however, the derivatives \( \partial_x \chi \). For example, for \( w = 2, \nu_1 = \nu_2 = 0 \) the \( qq \)-character is equal to:

\[ \chi_{2,0,0}(x) = \chi(x + \epsilon)^2 \left( 1 - q \frac{\epsilon_1 \epsilon_2}{\epsilon} \partial_x \left( \frac{P(x)}{\chi(x) \chi(x + \epsilon)} \right) \right) + 2qP(x) \frac{\chi(x + \epsilon)}{\chi(x)} \left( 1 - \frac{\epsilon_1 \epsilon_2}{\epsilon^2} \right) + q^2 \frac{P(x)^2}{\chi(x)^2} \quad (52) \]
The expression (51) has a first order pole at the hypersurfaces where \( v_i = v_j + \varepsilon \) for some pair \( i \neq j \). The residue of \( \mathcal{X}_{w,\nu} \) is equal to the \( qq \)-character \( \mathcal{X}_{w-2,\nu\setminus\{v_i,v_j\}} \), times the polynomial in \( x \) factor
\[
\prod_{k \neq i,j} S_{1,2}(v_k - v_j)P(x + v_k)
\] (53)

The finite part \( \mathcal{X}_{w,\nu}^{\text{fin}} \) of the expansion of \( \mathcal{X}_{w,\nu} \) in \( v_i \) near \( v_i = v_j + \varepsilon \) is the properly defined \( qq \)-character for the arrangement of weights \( \nu \) near \( v_i = v_j + \varepsilon \). It involves the terms with the derivative \( \partial_x y \). For example
\[
\mathcal{X}_{w,\nu}^{\text{fin}} = y(x + \varepsilon)y(x) + q \left( 1 + \frac{\varepsilon_1 \varepsilon_2}{2\varepsilon^2} \right) P(x - \varepsilon) \frac{y(x + \varepsilon)}{y(x - \varepsilon)} + qP(x) \left( 1 - \frac{\varepsilon_1 \varepsilon_2}{\varepsilon} \right) \frac{\partial_x y(x)}{y(x)} + q^2 \frac{P(x)P(x - \varepsilon)}{y(x)y(x - \varepsilon)}
\] (54)

4.2 The \( \hat{A}_0 \) theory.

The \( \hat{A}_0 \) theory (also known as the \( \mathcal{N} = 2^* \) theory) is characterized by one mass parameter \( m \), the mass of the adjoint hypermultiplet, and the gauge coupling \( q \).

We give the expression for the fundamental character \( \mathcal{X}_1(x) \equiv \mathcal{X}_{1,0}(x) \):
\[
\mathcal{X}_1(x) = \sum_{\lambda} q^{\left| \lambda \right|} \prod_{\square \in \lambda} S_{1,2}(mh_\square + \varepsilon a_\square) \cdot \frac{\prod_{\square \in \partial, \lambda} y(x + \sigma_\square + \varepsilon)}{\prod_{\square \in \partial, \lambda} y(x + \sigma_\square)}
\]
\[
= y(x + \varepsilon) \sum_{\lambda} q^{\left| \lambda \right|} \prod_{\square \in \lambda} S_{1,2}(mh_\square + \varepsilon a_\square) \cdot \prod_{\square \in \lambda} \frac{y(x + \sigma_\square - m) y(x + \sigma_\square + m + \varepsilon)}{y(x + \sigma_\square) y(x + \sigma_\square + \varepsilon)}
\]
\[
= y(x + \varepsilon) + qS_{1,2}(m) \frac{y(x - m) y(x + \varepsilon + m)}{y(x)} + \ldots
\] (55)

Here
\[
\sigma_\square = m(i - j) + \varepsilon(1 - j)
\] (56)

is the content of \( \square \) defined relative to the pair of weights \( (m, -m - \varepsilon) \) (cf. (137)). It is not too difficult to write an expression for the general \( \hat{A}_0 \) \( qq \)-character \( \mathcal{X}_{w,\nu} \), in terms of an infinite sum over the \( w \)-tuples of partitions, but we feel it is not very illuminating.

4.3 Add surface defect

A \( \mathbb{Z}_p \)-type defect in the \( U(n) \) gauge theory \([54]\) (it is the same thing as the \( A_{p-1} \)-type defect \([54]\)) is specified by the choice of the coloring functions: a function \( c : [n] \to \mathbb{Z}_p \), assigning a \( \mathbb{Z}_p \) representation \( R_{c(\alpha)} \) to each color \( \alpha = 1, \ldots, n \); a function \( \sigma : [m] \to \mathbb{Z}_p \), assigning a \( \mathbb{Z}_p \) representation \( R_{\sigma(\ell)} \) to each flavor \( \alpha = 1, \ldots, m = N_f \) of fundamental hypermultiplets (for the \( A_1 \) type theory), or an integer \( s = 0, \ldots, p-1 \), assigning a \( \mathbb{Z}_p \) representation \( R_s \) to the adjoint hypermultiplet (for the \( \hat{A}_0 \) theory).

We shall also find useful to keep track of the multiplicities of colors:
\[
\delta(\omega) = \# \{ \alpha | c(\alpha) = \omega \}
\] (57)
The \( qq \)-characters in the single node gauge theory in the presence of the \( \mathbb{Z}_p \)-defects require, in addition to the “evaluation parameters” \( \nu = (\nu_1, \ldots, \nu_w) \in \mathbb{C}^w \), the \( \mathbb{Z}_p \)-coloring of the evaluation parameters \( \xi = (\xi_1, \ldots, \xi_w) \in (\mathbb{Z}_p)^w \).

### 4.3.1 The \( A_1 \) theory

In the \( U(n) \) theory with \( 2n \) fundamental hypermultiplets, we assign a label \( \omega, \omega = 0, 1, \ldots, p-1 \) to each Coulomb parameter \( a_\alpha \), and each fundamental mass \( m_f \):

\[
P(x) = \prod_{\omega=0}^{p-1} P_\omega(x), \quad P_\omega(x) = \prod_{f \in [2n], \sigma(f) = \omega} (x - m_f)
\]

\[
A(x) = \prod_{\omega=0}^{p-1} A_\omega(x), \quad A_\omega(x) = \prod_{\alpha \in [n], c(\alpha) = \omega} (x - a_\alpha)
\]

where

\[
\text{deg} P_\omega(x) = m_\omega, \quad \text{deg} A_\omega(x) = n_\omega.
\]

The single \( y \)-observable of the bulk theory becomes \( p \) observables \( y_\omega(x) \),

\[
y(x) = \prod_{\omega=0}^{p-1} y_\omega(x)
\]

which are \( p \)-periodic in \( \omega \):

\[
y_{\omega+p}(x) = y_\omega(x)
\]

The fundamental refined \( qq \)-characters \( X_{w,\nu,\xi} \) of the \( A_1 \) theory with the \( \mathbb{Z}_p \)-surface defect \( D_{p,\xi,\sigma} \) are given by:

\[
X_{\omega}(x) = y_{\omega+1}(x + \epsilon) + q_\omega \frac{P_\omega(x)}{y_\omega(x)}
\]

The general refined \( qq \)-characters \( X_{\nu,\xi} \) of the \( A_1 \) theory with the \( \mathbb{Z}_p \)-surface defect \( D_{p,\xi,\sigma} \) are given by:

\[
X_{\nu,\xi}(x) = \sum_{I \subseteq [w]} \prod_{i \in I, j \in [w] \setminus I} S_{i,1,2}(\nu_i - \nu_j) \delta_{\nu_1}(c_i - c_j) S_{-1,1,2}(\nu_i + \nu_j) \delta_{\nu_1}(c_i - c_j+1) \times
\]

\[
\times \prod_{i \in I} y_{c_i+1}(x + \nu_i + \epsilon) \prod_{j \in [w] \setminus I} q_{c_j} y_{c_j}(x + \nu_j)
\]

Let us consider the special case, where \( c_i = \omega \) for all \( i \in [w] \). In this case (61) can be written in the determinant form:

\[
X_{\nu,\xi}(x) = \text{Det} \left[ \Xi_{ij}(x) \right]_{i,j=1}^{w}
\]

with

\[
R_w(x) = \prod_{i=1}^{w} (x - \nu_i)
\]
and
\[
\Xi_{ij}(x) = y_{\omega+1}(x + \nu_i + \epsilon) \delta_{ij} + \frac{R_{\omega}(\nu_i + \epsilon)}{R'_{\omega}(\nu_i)} \frac{P_{\omega}(x + \nu_i)}{y_{\omega}(x + \nu_i)} \frac{q_{\omega}}{v_i - v_j + \epsilon_1} \quad (64)
\]

The formula (62) is useful for computing the $\nu \to 0$ limit of the refined $qq$-character. First, realize the operator $\Xi$ in (62) as the operator acting in the $w$-dimensional space $W$ of polynomials in one variable modulo those which vanish at the points $v_1, \ldots, v_w$: $W = \mathbb{C}[t]/R_{\omega}(t)\mathbb{C}[t]$ as follows
\[
(\Xi(x)f)(t) = \left(y_{\omega+1}(x + t + \epsilon)f(t) + q_{\omega} \sum_{\nu} \prod_{k=0}^{w-1} \frac{(w-i+k)!}{k!(w-i)!} e^{\nu} \left(t^{j-i+k} \frac{P_{\omega}(x + t)}{y_{\omega}(x + t)}\right) \right)_{i,j=1}^{w} \quad (65)
\]
where the contour integral is taken along the large loop $|u| \to \infty$ (the subtraction of $R_{\omega}(t)$ in the numerator makes it obvious that the pole at $u + \epsilon_1 = t$ does not contribute). The matrix elements (64) are written in the basis of “delta”-functions (known as Lagrange interpolating polynomials)
\[
f_j(t) = \frac{R_{\omega}(t)}{(t - v_i)R'_{\omega}(v_i)} \quad (66)
\]
Now pass to the basis $\epsilon_i(t) = t^{i-1}, i = 1, \ldots, w$ and take a limit $R_{\omega}(t) \to t^w$ to get:
\[
\chi_{\omega}^{[w]}(x) = \text{Det} \left( [t^{j-i}] y_{\omega+1}(x + \epsilon) + q_{\omega} \sum_{\nu} \prod_{k=0}^{w-1} \frac{(w-i+k)!}{k!(w-i)!} e^{\nu} \left(t^{j-i+k} \frac{P_{\omega}(x + t)}{y_{\omega}(x + t)}\right) \right)_{i,j=1}^{w} \quad (67)
\]

4.3.2 The $\hat{A}_0$-theory, a.k.a. $N = 2^*$

To write the $qq$-characters for the $U(n)$ $N = 2^*$ theory in the presence of the surface operator $D_{px,q}$ we use the notations (168), (170).

The fundamental $qq$-characters of the $N = 2^*$ theory with the surface operator $D_{px,q}$ are given by (cf. (168)):
\[
\chi_{\omega}(x) = y_{\omega+1}(x + \epsilon) \sum_{\lambda} \mathcal{Q} \left( \omega \begin{array}{c} p \end{array} g \begin{array}{c} \lambda \end{array} \right) \left[ \begin{array}{c} m \begin{array}{c} -m - \epsilon \end{array} \end{array} \begin{array}{c} \begin{array}{c} \epsilon_1 \\ \epsilon_2 \end{array} \end{array} \begin{array}{c} s \begin{array}{c} q \end{array} \end{array} \right] \left[ \begin{array}{c} \lambda \end{array} \right] \times
\times \prod_{\square \in \lambda} \frac{y_{\omega+\kappa_\square + s+1}(x + \sigma_\square + m + \epsilon)}{y_{\omega+\kappa_\square + s+1}(x + \sigma_\square + \epsilon)} \frac{y_{\omega+\kappa_\square}(x + \sigma_\square - m)}{y_{\omega+\kappa_\square}(x + \sigma_\square)} \quad (68)
\]
Here
\[
\sigma_\square = m(i-j) + \epsilon(1-j) \quad (69)
\]
is the content of $\square$ defined relative to the pair of weights $(m, -m - \epsilon)$. For $s = 0$ the expression (68) simplifies to:
\[
\chi_{\omega}(x) = y_{\omega+1}(x + \epsilon) \sum_{\lambda} \mathcal{Q}_{\omega}^\lambda \prod_{\square \in \lambda} \frac{y_{\omega+\kappa_\square + s+1}(x + \sigma_\square + m + \epsilon)}{y_{\omega+\kappa_\square + s+1}(x + \sigma_\square + \epsilon)} \frac{y_{\omega+\kappa_\square}(x + \sigma_\square - m)}{y_{\omega+\kappa_\square}(x + \sigma_\square)} \quad (70)
\]
with $\kappa_\square = 1 - j \mod(p)$.
4.3.3 The *qq*-characters for $U(1)$-defects in $U(n)$ theories

The large $p$-limit of the $\mathbb{Z}_p$ defect is the $U(1)$ defect, which is essentially a boundary condition in the three dimensional gauge theory with eight supercharges. The coloring functions $c, \sigma, \epsilon$ are now integer-valued.

The only change the formula (61) undergoes is the replacement of the $p$-periodic Kronecker symbol by the ordinary one: $\delta(p)(x) \rightarrow \delta(x,0)$.

4.3.4 The *qq*-characters for surface defects in folded $\mathcal{N} = 2^*$ theory

The folded instantons is an example of the gauge origami theory [54, 55, 56] with two plane looks like the

The large in the three dimensional gauge theory with eight supercharges. The coloring functions $c, \sigma, \epsilon$ are now integer-valued.

The only change the formula (61) undergoes is the replacement of the $p$-periodic Kronecker symbol by the ordinary one: $\delta(p)(x) \rightarrow \delta(x,0)$.

$$N_{12} = \sum_{a} e^{a}, \quad N_{23} = \sum_{b} e^{b}$$

With the $\Omega$-deformation parameters $\epsilon_a$, $a \in 4$, the theory on the $\mathbb{C}^2_{12}$ plane looks like the $\mathcal{N} = 2^*$ $U(n)$ theory with the mass $\epsilon_3$ adjoint hypermultiplet, while the theory on the $\mathbb{C}^2_{23}$ plane looks like the $\mathcal{N} = 2^*$ $U(m)$ theory with the mass $\epsilon_1$ adjoint hypermultiplet. We have two $Y$-observables: $y_{12}(x)$ and $y_{23}(x)$:

$$y_{ab}(x)[\lambda] = \prod_{\alpha} (x - a_{ab,\alpha}) \cdot \prod_{(i,j) \in \lambda(ab,\alpha)} \left( 1 + \frac{\epsilon_b}{x - a_{ab,\alpha} - \epsilon_a(i-1) - \epsilon_j} \right)$$

There are two interesting *qq*-characters we shall study, the one corresponding to the rank one theory on $\mathbb{C}^2_{34}$, another corresponding to the rank one theory on $\mathbb{C}^2_{14}$.

Let $N_{34} = e^{\epsilon_3}$. The corresponding *qq*-character has the usual form [55]:

$$\chi_{34}^{\text{fold}}(x) = \hat{y}(x + \epsilon) \sum_{\lambda} q^{[\lambda]} \mu_{1,2}([\lambda], \square) \cdot \prod_{\square \in \lambda} \frac{\hat{y}(x + \sigma_{\square} - \epsilon_3) \hat{y}(x + \sigma_{\square} - \epsilon_4)}{\hat{y}(x + \sigma_{\square}) \hat{y}(x + \sigma_{\square} - \epsilon_3 - \epsilon_4)}$$

where $\sigma_{(i,j)} = \epsilon_3(i-1) + \epsilon_4(j-1)$, and

$$\mu_{a,b}([\lambda]) = \prod_{\square \in \lambda} S_{a,b}(\epsilon_{h(a,b)} h_{\square} + (\epsilon_a + \epsilon_b) a_{\square})$$

where $h(a,b) = \min(4\setminus\{a,b\})$, e.g. $h(1,2) = 3$, $h(2,3) = 1$. However, the novelty is that the $\hat{y}$-function is now

$$\hat{y}(x) = y_{12}(x) \prod_{n=1}^{\infty} \frac{y_{23}(x - \epsilon_1 + \epsilon_3 n)}{y_{23}(x + \epsilon_3 n)}$$

is regularized using the infinite product representation of the $\Gamma$-function. Symmetrically,

$$\chi_{14}^{\text{fold}}(x) = \hat{y}(x + \epsilon_2 + \epsilon_3) \sum_{\lambda} q^{[\lambda]} \mu_{2,3}([\lambda], \square) \cdot \prod_{\square \in \lambda} \frac{\hat{y}(x + \sigma_{\square} - \epsilon_1) \hat{y}(x + \sigma_{\square} - \epsilon_4)}{\hat{y}(x + \sigma_{\square}) \hat{y}(x + \sigma_{\square} - \epsilon_1 - \epsilon_4)}$$
where $\delta_{(i,j)} = \varepsilon_1(i-1) + \varepsilon_4(j-1)$, and

$$
\hat{y}(x) = \hat{y}_{23}(x) \prod_{n=1}^{\infty} \frac{\hat{y}_{12}(x - \varepsilon_3 + \varepsilon_1 n)}{\hat{y}_{12}(x + \varepsilon_1 n)}
$$

is regularized using the infinite product representation of the $\Gamma$-function. Fortunately, in the orbifold case, with the judicial choice of the coloring functions we shall not encounter infinite products.

### 4.3.5 Folding and orbifolding

Let us now add the surface defects stretched both along the $\mathbb{C}^4$ and the $\mathbb{C}^4$ products.

The orbifold case, with the judicial choice of the coloring functions we shall not encounter infinite products.

$$
X_{34,0}^\text{fold}(x) = \hat{y}_{\omega+1}(x + \varepsilon_1 + \varepsilon_2) \prod_{i,j} Q^\omega_{i,j} \mu_{i,j}[\lambda] \prod_{i,j} \hat{y}_{\omega+1}(x + \varepsilon_3 + \varepsilon_1 n) / \hat{y}_{12}(x + \varepsilon_1 n)
$$

\(X_{34,0}^\text{fold}(x) = \hat{y}_{\omega+1}(x + \varepsilon_1 + \varepsilon_2) \prod_{i,j} Q^\omega_{i,j} \mu_{i,j}[\lambda] \prod_{i,j} \hat{y}_{\omega+1}(x + \varepsilon_3 + \varepsilon_1 n) / \hat{y}_{12}(x + \varepsilon_1 n)

$$
X_{14,0}^\text{fold}(x) = \hat{y}_{\omega+1}(x + \varepsilon_2 + \varepsilon_3) \prod_{i,j} Q^\omega_{i,j} \mu_{i,j}[\lambda] \prod_{i,j} \hat{y}_{\omega+1}(x + \varepsilon_3 + \varepsilon_1 n) / \hat{y}_{12}(x + \varepsilon_1 n)
$$

where

$$
Q^\omega_{i,j} = \prod_{i,j} q_{\omega+1-j}
$$

and, for $(a,b) = (1,2)$ or $(2,3)$

$$
\mu_{a,b}[\lambda] = \prod_{i,j} \delta_{a,b}(\varepsilon_h(a,b) n + (\varepsilon_a + \varepsilon_b) \| n \| \delta_{a,b}(\varepsilon_h(a,b) n + (\varepsilon_a + \varepsilon_b) \| n \|

### 4.3.6 More fun with folded theories

Let us present here the $A_1$ analogue of the folded theory. By that we mean the theory which looks like the $A_1$ type theory on the $\mathbb{C}^2$ plane. We get it by performing the $\mathbb{Z}_3$ orbifold of the folded setup $n\mathbb{C}^2_{12} + m\mathbb{C}^2_{23}$ we discussed so far, where the group $\mathbb{Z}_3$ acts in the $\mathbb{C}^2_{34}$ plane: $(z_3,z_4) \mapsto (\omega z_3, \omega^{-1} z_4)$. The theory now has three fractional couplings $q_0, q_1, q_2$. We send $q_1, q_2 \to 0$.

There are six Chan-Paton spaces now, $N_{A,\omega}$, $A = 12, 23$, $\omega = 0, 1, 2$. From the point of view of the $\mathbb{C}^2_{12}$ observer, $N_{12,0}$ is the color space, while $N_{12,1}, N_{12,2}$ are the multiplicities of the fundamental hypermultiplets. This is the content of the $A_1$ theory.

Life is more intricate on the $\mathbb{C}^2_{23}$ plane. Because of the $q_1, q_2 \to 0$ limit the instantons there can only “grow” in the $\mathbb{C}^2_1$ direction, i.e. this theory is effectively two dimensional. Moreover,

$$
S_{23,0} = N_{23,0} - P_{2}K_{23,0}, \ S_{23,1} = N_{23,1} + q_3 P_{2}K_{23,0}, \ S_{23,2} = N_{23,2}
$$

with

$$
P_{2}K_{23,0} = \sum_{\rho} e^{\rho}(1 - q_2^{d_{\rho}})
$$
where $N_{23,0} = \sum_{\beta} e^{b_{\beta}}$ and $d_{\beta}$ are the vortex fluxes.

We can therefore view the combined theory as a surface defect in the $A_1$ type theory. In particular, by choosing $N_{23,0} = e^x$, $N_{23,1} = N_{23,2} = 0$ to be rank one, the resulting operator is identified with the $\varepsilon_1$-deformed $Q(x)$, the “second solution” of the $T - Q$-equation

$$\frac{Q(x + \varepsilon_1 + \varepsilon_2)}{Q(x + \varepsilon_1)} + qP(x)\frac{Q(x - \varepsilon_2)}{Q(x)} = T(x) \quad (83)$$

If, on the other hand, we choose $N_{23,0} = 0 = N_{23,2}$, and $N_{23,1} = e^x$, the resulting surface defect will be the Baxter operator $Q(x)$ itself. See [61, 53, 54, 55] for more details on Baxter operators in gauge theory.

### 4.4 KZ-type equations

In this section we demonstrate that the differential equations from the world of two dimensional conformal field theories, such as the KZ equations [31] and KZB [6] equations are verified by the regular surface defect partition functions of the above supersymmetric gauge theories.

In fact, we shall get the special case of the KZ and KZB equations. For the $A_1$ theory we’ll get the KZ equation for the four-point conformal block, with two generic vertex operators and two minimal ones. For the $\tilde{A}_0$ case we’ll get the KZB equation which is obeyed by the one-point conformal block on the torus of a minimal vertex operator in $SU(n)$ WZW theory, corresponding to the minimal orbit of $SU(n)$. Such equations (in genus one) were studied in [22]. Of course, the unitary WZW theory vertex operators have discrete labels. The level $k$ of the WZW model is also quantized. Our equations have continuous parameters, thus extending the range of natural parameters of the two dimensional CFT into the complex domain.

Throughout this section we use the notation $\varepsilon_3 = m$.

We shall assume the $c$-coloring function of our surface defect to be given by:

$$c(\alpha) = \alpha, \quad \alpha = 1, \ldots, n \quad (84)$$

The $Y_{\omega}(x)$-observable in the presence of such surface operator on the instanton configuration $\lambda$ evaluates to

$$Y_{\omega}(x)[\lambda] = (x - a_{\omega}) \prod_{\beta = 1}^{N} \prod_{\square \in \lambda(\beta)} \left(\frac{x - a_\beta - c_\square - \varepsilon_1}{x - a_\beta - c_\square}\right)^{\delta_\lambda(\beta + j - 1 - \omega)} \frac{\delta_\lambda(\beta + j - \omega)}{\delta_\lambda(\beta + j)} \quad (85)$$

The strategy is to look at the coefficient $[x^{-1}]X_{\omega}(x)$ in the orbifold $qq$-characters $X_{\omega}(x)$, $\omega = 0, \ldots, n - 1$:

$$[x^{-1}] \langle X_{\omega}(x) \rangle = 0 \quad (86)$$

To compute (86), we need to know the large $x$ expansion of $Y_{\omega}(x)$:

$$Y_{\omega}(x) = (x - a_{\omega}) \exp \left(\frac{\varepsilon_1}{x} \nu_{\omega - 1} + \frac{\varepsilon_1 \varepsilon_2}{x^2} k_{\omega - 1} + \frac{\varepsilon_1}{x^2} (\sigma_{\omega - 1} - \sigma_\omega) + \ldots\right) = x - a_{\omega} + \varepsilon_1 \nu_{\omega - 1} + \frac{\varepsilon_1}{x} (\sigma_{\omega - 1} - \sigma_\omega - a_{\omega} \nu_{\omega - 1} + \varepsilon_2 k_{\omega - 1} + \frac{\varepsilon_1}{2} \nu_{\omega - 1}^2) \quad (87)$$
where
\[ k_\omega = \# K_\omega, \quad v_\omega = k_\omega - k_{\omega+1}, \quad \sigma_\omega = \frac{\epsilon_1}{2} k_\omega + \sum_{(\beta, \square) \in K_\omega} (a_\beta + c_\square) \]
\[ K_\omega = \{(\beta, \square) | \beta \in [n], \square = (i, j) \in \lambda(\beta), \beta + j - 1 \equiv \omega \mod(p)\}, \quad (88) \]

4.4.1 The \( A_1 \) case

We take the matter coloring function to be
\[ \sigma(f) = \left\lfloor \frac{(f-1)/2}{2} \right\rfloor + 1, \quad f = 1, \ldots, 2n \quad (89) \]

The \( qq \)-characters are
\[ X_\omega(x) = y_{\omega+1}(x+\epsilon) + q_\omega P_\omega(x) y_{\omega-1}(x), \quad \omega = 0, \ldots, n-1 \quad (90) \]

with
\[ P_\omega(x) = (x+m_{2\omega})(x+m_{2\omega+1}) \quad (91) \]

Using (87) we derive:
\[ \frac{1}{\epsilon_1} [x^{-1}] X_\omega(x) = D_\omega - q_\omega D_{\omega-1} + \frac{\epsilon_1}{2} \left( v_\omega^2 + q_\omega v_{\omega-1}^2 \right) - q_\omega \left( a_\omega + m_{2\omega} + m_{2\omega+1} \right) v_{\omega-1} - a_{\omega+1} v_\omega + q_\omega \frac{P_\omega(a_\omega)}{\epsilon_1} \quad (92) \]

where
\[ D_\omega = \sigma_\omega - \sigma_{\omega+1} + \epsilon_2 k_\omega \quad (93) \]

Thus, we derived a system of equations, relating \( \langle D_\omega \rangle \) to the derivatives of the surface defect partition function \( \Psi \):
\[ \langle D_\omega - q_\omega D_{\omega-1} \rangle = -\frac{1}{2} \epsilon_1 \langle v_\omega^2 \rangle - \frac{1}{2} \epsilon_1 q_\omega \langle v_{\omega-1}^2 \rangle + q_\omega \left( a_\omega + m_{2\omega} + m_{2\omega+1} \right) \langle v_{\omega-1} \rangle + a_{\omega+1} \langle v_\omega \rangle - q_\omega \frac{P_\omega(a_\omega)}{\epsilon_1} \langle 1 \rangle, \quad \omega = 0, \ldots, n-1 \quad (94) \]

which becomes a second order differential equation on \( \Psi \) once we use the obvious relation
\[ \sum_{\omega=0}^{n-1} \langle D_\omega \rangle = \epsilon_2 D^2 \Psi \quad (95) \]

In [66] we map this equation to the KZ equation for the special four-point genus zero conformal blocks of the \( SL(n) \) current algebra.
4.4.2 The regular defects in the $\mathcal{N} = 2^*$ theory

We take $s = 0$. Let us first consider the case of the regular surface defect, i.e. the one for which $\delta(\omega) = 1$ for all $\omega \in \mathbb{Z}_p$.

The large $x$-expansion of the fundamental $\mathbb{Z}_p$-refined $qq$-character $X_\omega(x)$ given by (70) is equal to:

$$
\frac{1}{B_\omega} X_\omega(x) = x + \varepsilon - a_{c-1(\omega+1)} + \varepsilon_1 v_\omega + \frac{1}{x} \left( (\varepsilon_1 v_\omega - a_{c-1(\omega+1)})^2 - \frac{1}{2} a_{c-1(\omega+1)}^2 + \varepsilon_1 D_\omega - m \sum_{\omega' = 0}^{p-1} \left( (m + \varepsilon) \nabla_{\omega'}^q + (\varepsilon_1 v_{\omega'} - a_{c-1(\omega'+1)}) \nabla_{\omega'}^z \right) \log(B_\omega) \right) + \ldots
$$

(96)

It implies the relation between the expectation value of $D_\omega$ and the differential operator acting on the partition function $\hat{\Psi} \equiv \Psi^{\text{inst}}_{A_0, \mathbb{Z}_p}(a; m; \tau; q)$ of the regular surface defect.

$$
0 = \frac{1}{B_\omega} [x^{-1}] \langle X_\omega(x) \rangle = \varepsilon_1 \langle D_\omega \rangle \Psi + \frac{1}{2} (\varepsilon_1 \nabla_{\omega}^z)^2 \Psi - \varepsilon_1 a_{c-1(\omega+1)} \nabla_{\omega}^z \Psi - \varepsilon_1 m \sum_{\omega' = 0}^{p-1} \left( (m + \varepsilon) \nabla_{\omega'}^q + (\varepsilon_1 v_{\omega'} - a_{c-1(\omega'+1)}) \nabla_{\omega'}^z \right) \log(B_\omega)
$$

(97)

Sum over $\omega$ to get the linear differential equation (using some identities from the appendix)

$$
\left( p \varepsilon_1 \varepsilon_2 \nabla^q + \varepsilon_1 (\varepsilon_2 \rho - a_s^+) \cdot \nabla^z + \frac{1}{2} \varepsilon_1^2 \Delta_z + \frac{1}{2} m(m + \varepsilon_1) \Delta_z \log(\Delta) + (m + \varepsilon_1)^2 \hat{u} \right) \Psi = 0
$$

(98)

on the normalized partition function

$$
\hat{\Psi} = \Delta^{1+m/\varepsilon_1} . \Psi
$$

(99)

where $a_s^+ \cdot \nabla^z = \sum_{\omega} a_{c-1(\omega+1)} \nabla_{\omega}^z$ and (cf. (206))

$$
\hat{u} = \Delta^{-1} \left( p \nabla^q + \rho \cdot \nabla^z - \frac{1}{2} \Delta_z \right) \Delta - p \frac{\varepsilon_2}{m + \varepsilon_1} \nabla^q \chi
$$

(100)

actually vanishes, $\hat{u} \equiv 0$, as we now demonstrate using the sub-regular defects.

4.5 The sub-regular defects in the $\mathcal{N} = 2^*$ theory

The sub-regular defect is the one where the multiplicities $\delta(\omega)$ of the $\mathbb{Z}_p$-representations occurring in the coloring of the Chan-Paton space are 0 and 1 with at least one 0.

The fundamental refined $qq$-characters $X_\omega(x)$ behave as $x^{\delta(\omega+1)}$ when $x \to \infty$. We shall have to expand $X_\omega$ with $\delta(\omega + 1) = 0$ up to the $x^{-2}$ terms. The $X_\omega$ with $\delta(\omega + 1) = 1$ are expanded to the $x^{-1}$ order, as in the regular case.

Let us denote by $\hat{\Psi}_\delta$ the instanton part of the partition function of the surface defect characterized by the function $\delta$. Note that when $\delta = \delta_{\omega_0}$ is supported at only one value of $\omega$, i.e. $\delta_{\omega_0}(\omega) = \delta_{\omega_0}$, this partition function coincides with our friend (173)

$$
\hat{\Psi}_{\delta_{\omega_0}} = \tilde{B}_{\omega_0} \equiv \tilde{B}_{\rho, \omega_0}(\varepsilon; q)
$$

(101)
Let us denote by $\nabla^z_\delta, \nabla^q_\delta$ the operators, cf. (204)

$$\nabla^z_\delta = \sum_{\omega=0}^{p-1} \delta(\omega + 1) \nabla^z_\omega, \quad \nabla^q_\delta = \sum_{\omega=0}^{p-1} \delta(\omega + 1) \nabla^q_\omega,$$

(102)

The equations $[x^{-1}] \langle X_\omega(x) \rangle = 0$ give:

$$\varepsilon_1 \nabla^z_\omega \log \tilde{\Psi}_\delta + m \nabla^z_\omega \log B_\omega = 0, \quad \text{when} \quad \delta(\omega + 1) = 0$$

(103)

which imply (202), with the help of (101). Now, using (203), we get:

$$\nabla^z_\omega \tilde{\Psi}_\delta = 0, \quad \delta(\omega + 1) = 0,$$

(104)

where

$$\tilde{\Psi}_\delta = \tilde{B}^{-1} \Psi_\delta, \quad \tilde{B}_\delta = \prod_{\omega} \tilde{B}^\delta_\omega$$

(105)

In turn, the equations $\frac{1}{\varepsilon_2} [x^{\delta(\omega + 1) - 2}] \langle X_\omega(x) \rangle = 0$ produce upon summing over $\omega \in \mathbb{Z}_p$ while using (103):

$$p \varepsilon_1 \varepsilon_2 q \frac{\partial}{\partial q} \tilde{\Psi}_\delta + \left[ \sum_{\omega} \frac{1}{2} (\varepsilon_1 \nabla^z_\omega)^2 + \left( \varepsilon_2 \rho_\omega - \sigma_{c^{-1}(\omega + 1)} \right) \varepsilon_1 \nabla^z_\omega \right] \tilde{\Psi}_\delta + \tilde{U}_\delta(z; q) \tilde{\Psi}_\delta = 0$$

(106)

with

$$\tilde{U}_\delta(z; q) = \left( \varepsilon_1 \varepsilon_2 \left( p \nabla^q + \rho \cdot \nabla^z \right) + \frac{1}{2} \varepsilon_1^2 \Delta_z \right) \log(\tilde{B}_\delta)$$

$$+ \varepsilon_1^2 \sum_{\omega} \nabla^z_\omega \log(\tilde{B}_\delta) \left( \frac{1}{2} \delta_{\omega + 1} \nabla^z_\omega \log(\tilde{B}_\delta) - \nabla^z_\omega \log(\tilde{B}_\delta) \right)$$

$$+ \frac{1}{2} \varepsilon_1 \nabla^z_\delta \left( m \nabla^z_\delta + m + 2 \varepsilon \right) \log(\tilde{B})$$

$$- m(m + \varepsilon) \nabla^q_\delta \log(\tilde{B}) - m \sum_{\omega} \left( \tilde{D}_\omega \right)_\omega$$

(107)

Here $\left( \tilde{D}_\omega \right)_\omega$ is the average of the observable

$$\tilde{D}_\omega = \sum_{\omega'} \delta(\omega' + 1) \left( \varepsilon_{\omega'} - \varepsilon_{\omega + 1} - m \left( \tilde{k}_{\omega'} - \tilde{k}_{\omega + 1} \right) \right)$$

(108)

with respect to the measure:

$$\left( \tilde{D}_\omega \right)_\omega = \frac{1}{\tilde{B}_\omega} \sum_{\lambda} \tilde{D}_\omega \cdot Q^\lambda_\omega \cdot \tilde{B}_\omega \cdot \tilde{B}_\omega$$

(109)

where

$$\tilde{k}_\omega[\lambda] = \# \{ \square = (i, j) | \square \in \lambda, \omega + 1 - j \equiv \omega' \mod p \} ,$$

so that

$$Q^\lambda_\omega = \prod_{\omega'} q_{\tilde{k}_\omega[\lambda]}$$
\[ \tilde{\epsilon}_{\omega'}[\lambda] = \sum_{\square \in \lambda, \omega + 1 = j = \omega' \mod p} \sigma_{\square} \]

It looks like we are not gaining much as the expression (107) looks even worse than (98), (100), since it contains an unknown quantity

\[ \sum_{\omega} \langle \mathcal{D}_{\delta}^{-} \rangle_{\omega'} . \] (110)

However, the summit is close. We know that \( \hat{\Psi}_{\delta_{\omega}} = 1 \) for any \( \omega \in \mathbb{Z}_p \). Therefore,

\[ \tilde{U}_{\delta_{\omega}}(z; q) = 0, \quad \omega \in \mathbb{Z}_p \] (111)

We can use this to simplify (107) quite considerably:

\[ \tilde{U}_{\delta}(z; q) = \tilde{U}_{\delta}(z; q) - \sum_{\omega \in \mathbb{Z}_p} \delta(\omega + 1) \tilde{U}_{\delta_{\omega}}(z; q) \]

which removes all the terms linear in \( \delta \), including (110), leading to

\[ \tilde{U}_{\delta}(z; q) = -\frac{1}{2}m(m + \varepsilon_1) \sum_{\omega' \neq \omega''} \delta_{\omega'+1} \delta_{\omega''+1} \nabla_{\omega'}\nabla_{\omega''} \log(\tilde{\Delta}) \] (112)

Now sum (111) over \( \omega \in \mathbb{Z}_p \) using the expression (107). The terms containing \( \mathcal{D}_{\delta_{\omega}} \) cancel out, leaving us with the identity:

\[ 0 = \left( p \nabla q + \rho \cdot \nabla z \right) \left( \varepsilon_1 \varepsilon_2 \log(\tilde{\mathbb{B}}) - m(m + \varepsilon) \log(\tilde{\mathbb{B}}) \right) + \frac{1}{2} \varepsilon_1 (m + \varepsilon_1) \Delta \log(\tilde{\mathbb{B}}) - \frac{1}{2} \varepsilon_1^2 \sum_{\omega} \left( \nabla_{\omega'} \log(\tilde{\mathbb{B}}) \right)^2 \] (113)

which, upon the substitutions (205), (206) is equivalent to \( \dot{u} = 0 \), where \( \dot{u} \) is given by (100). Use the trigonometric limit (175), (177) as the initial condition in the heat equation to prove the Eq. (209). Now, armed with (209), even without exact knowledge of what the function \( \chi(q) \) is, we conclude:

\[ \tilde{U}_{\delta}(z; q) = -\frac{1}{2}m(m + \varepsilon_1) \sum_{\omega' \neq \omega''} \delta_{\omega'+1} \delta_{\omega''+1} \nabla_{\omega'}\nabla_{\omega''} \log(\theta_{11}(z_{\omega'}/z_{\omega''}; \tau)) \] (114)

This concludes the derivation in the \( \mathcal{N} = 2^* \) case. The equation obeyed by the regular defect \( \delta(\omega) = 1 \), can be now written quite explicitly (we restored the notation \( m = \varepsilon_3 \)):

\[ \left( n \varepsilon_1 \varepsilon_2 \nabla q + \frac{1}{2} \varepsilon_1^2 \Delta z - \varepsilon_3 (\varepsilon_3 + \varepsilon_1) \sum_{\alpha < \beta} \varphi(w_{\alpha}/w_{\beta}; \tau) \right) \Psi = 0 \] (115)

where

\[ \Psi = \prod_{\alpha} w_{\alpha}'^{-1} \Psi \] (116)
and
\[ w_\alpha = z_{c(\alpha)-1}, \quad p_\alpha = \lambda_\alpha - \epsilon_2 p_{c(\alpha)-1} \]  
(117)

The equation (115) is the KZB equation for the special one-point conformal block [22]. This is one of the strongest confirmations of the original conjecture [51].

A couple of remarks are in order.

• The theory in the bulk is actually invariant under the permutation \( \epsilon_3 \leftrightarrow \epsilon_4 \). However, the surface defect is not, since with our choice \( s = 0 \) the transversal \( z_4 \) direction is twisted by the orbifold group \( \mathbb{Z}_p \) while the \( z_3 \) is neutral.

• In the limit to the pure \( N = 2 \) theory, \( m \to \infty \), \( q \to 0 \) with \( m^{2n} q = \Lambda^{2n} \) finite we recover the result of [73] while in the purely perturbative limit \( q \to 0 \) with \( m \) finite we’ll get the results of [10]. Also, our result proves for all \( n \) the conjecture made for \( n = 2 \) in [3].

• It is not much work to extend this calculation in the folded case. We take \( p = n + m \) and choose the coloring function \( c \) so that the \( c(12, \alpha) \neq c(23, \beta) \) for \( \alpha \in [n], \beta \in [m] \). The regular surface defect is solving the generalization of (115):

\[
(m + n)\epsilon_2 \nabla^a \Psi + \frac{1}{2} \epsilon_1 \sum_{\alpha \in [n]} \nabla^2 w_\alpha \Psi + \frac{1}{2} \epsilon_3 \sum_{\beta \in [m]} \nabla^2 u_\beta \Psi -
\]

\[
- (\epsilon_3 + \epsilon_1) \left( \frac{\epsilon_3}{\epsilon_1} \sum_{\alpha < \alpha'} \varphi(w_\alpha/w_\alpha'; \tau) + \frac{\epsilon_1}{\epsilon_3} \sum_{\beta < \beta'} \varphi(u_\beta/u_\beta'; \tau) + \sum_{\alpha, \beta} \varphi(w_\alpha/u_\beta; \tau) \right) \Psi = 0
\]  
(118)

where \( w_\alpha = z_{c(12, \alpha)-1}, u_\beta = z_{c(23, \beta)-1} \).

It would be nice to compare these equations to the KZB equations for the one-point genus one conformal block for the super-Kac-Moody algebra \( \mathfrak{sl}(n|m) \). It appears that we have found the elliptic generalization of the super-Calogero system [73].

5 Bethe/gauge correspondence

In the limit \( \epsilon_2 \to 0 \) our theories become effectively \( N = (2, 2) \) two dimensional with \( \mathbb{C}^2 \) as a worldsheet. In correspondence with the quantum integrable systems [63, 64, 65]. The \( \mathbb{Z}_p \) orbifold creates a point-like singularity in the \( \mathbb{C}^2 \) space, which is the codimension two defect in that effective two dimensional theory. It can be represented by the operator \( \Psi(z) \) in the twisted chiral ring which depends on the additional fractional couplings \( z_\alpha \). As is well-known, the \( \epsilon_2 \to 0 \) limit of (115) becomes the eigenvalue problem:

\[
\Psi(w, z, \alpha, m, \tau; \epsilon_1, \epsilon_2) = e^{\frac{1}{2} \chi(w, z, \alpha, m, \tau; \epsilon_1)} \cdot \chi(w, z, \alpha, m, \tau; \epsilon_1) + \ldots
\]  
(119)

where \( \ldots \) denote the terms which vanish in the \( \epsilon_2 \to 0 \) limit[2] and (we return to the \( m = \epsilon_3 \) notation):

\[
- \frac{1}{2} \epsilon_1^2 \Delta_2 + m(m + \epsilon_1) \sum_{\alpha < \beta} \varphi(w_\alpha/w_\beta; \tau) \chi = E(\alpha, m, \tau; \epsilon_1) \chi
\]  
(120)

[1] With the additional bonus of providing not only an equation, but also its solution
[2] it is important that \( \alpha \) etc. are generic here, see [24] for the first steps in the study of non-generic cases
where
\[ E = q \frac{\partial}{\partial q} \tilde{W}(a, m, \tau; \epsilon_1) \] (121)
and
\[ \tilde{W}(a, m, \tau; \epsilon_1) = \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z(a, m, \tau; \epsilon_1, \epsilon_2) \] (122)
is the effective twisted superpotential [65].

The limit \( \chi(w, a, m, \tau; \epsilon_1) \) is the so-called Jost function of the elliptic Calogero-Moser system.

- In the non-generic case \( m/\epsilon_1 \in \mathbb{Z} \) the paper [14] gives an integral representation for the solution of (115) and a Bethe-ansatz-type formula for the solution of (120). It would be nice to relate our formulas and theirs.

- The folded case would lead to the so far unnoticed two-species generalization of the elliptic Calogero-Moser system:

\[
\hat{H} = -\frac{1}{2} \epsilon_1 \sum_{\alpha \in [n]} \nabla_{w_\alpha}^2 - \frac{1}{2} \epsilon_3 \sum_{\beta \in [m]} \nabla_{u_\beta}^2 + \\
+ (\epsilon_3 + \epsilon_1) \left( \frac{\epsilon_3}{\epsilon_1} \sum_{\alpha' < \alpha''} \varphi(w_{\alpha'}/w_{\alpha''}; \tau) + \frac{\epsilon_1}{\epsilon_3} \sum_{\beta' < \beta''} \varphi(u_{\beta'}/u_{\beta''}; \tau) + \sum_{\alpha, \beta} \varphi(w_{\alpha}/u_{\beta}; \tau) \right) 
\] (123)

whose trigonometric limit was studied in [74].
6 Appendix A. Notations

Here we summarize our notations.

6.1 Roots of unity

\[ i = \sqrt{-1} , \quad (124) \]

and

\[ \omega_p = \exp \frac{2\pi i}{p} \quad (125) \]

so that \( i = \omega_4, -i = \omega_4^3 \).

6.2 Finite sets

We denote by \([a,b]\) the set

\[ [a,b] = \{ n | a \leq n \leq b, n \in \mathbb{Z} \} \quad (126) \]

For the finite set \( J \subset \mathbb{Z} \) we define \( h : J \to [0, \#J - 1] \) the height function:

\[ h_j = \# \{ j' | j' \in J, j' < j \} \quad (127) \]

6.3 Useful sums

Let \((z_i, p_i) \in \mathbb{C}^\times \times \mathbb{C}, i = 0, \ldots, p\). Let \( e_j, j = 0, \ldots, p + 1, e_0 = 1, \) be the elementary symmetric functions of \( z \)'s:

\[ e_j = \sum_{J \subset [0,p], \#J = j} \prod_{j \in J} z_j \quad (128) \]

We have:

\[ \prod_{j=0}^{p} (1 + tz_j) = \sum_{J \subset [0,p]} t^{\#J} \prod_{j \in J} z_j \quad (129) \]

Define:

\[ D^p_k = \prod_{j=0}^{p} \frac{1}{1 + tz_j} \sum_{J \subset [0,p]} t^{\#J} \prod_{j \in J} z_j \sum_{j \in J} h^k_j p_j \quad (130) \]

It is easy to prove by induction, that:

\[ D_0^p = \sum_{j=0}^{p} \frac{t z_j}{1 + t z_j} p_j \quad D_1^p = \sum_{0 \leq i < j \leq p} \frac{t z_i}{1 + t z_i} \frac{t z_j}{1 + t z_j} p_j \]

\[ D_2^p = D_1^p + 2 \sum_{0 \leq i < j < k \leq p} \frac{t z_i}{1 + t z_i} \frac{t z_j}{1 + t z_j} \frac{t z_k}{1 + t z_k} p_k \quad (131) \]
6.4 S-functions

The functions

\[ S_{a,b}(x) = 1 + \frac{\varepsilon_a \varepsilon_b}{x(x + \varepsilon_a + \varepsilon_b)}, \quad S_{+a,b}(x) = 1 + \frac{\varepsilon_a}{x}, \quad S_{-a,b}(x) = 1 - \frac{\varepsilon_a}{x + \varepsilon_a + \varepsilon_b}, \]  

play a prominent rôle. They are related to each other,

\[ S_{-a,b}(x) = S_{+a,b}(-\varepsilon_a - \varepsilon_b - x), \]
\[ S_{a,b}(x) = S_{+a,b}(x)S_{-a,b}(x). \]  

7 Appendix B. Partitions and partition sums

7.1 Partitions

A partition \( \lambda \) is a finite sequence \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0) \) of integers, \( \lambda_i \in \mathbb{Z}_{\geq 0} \), which we sometimes extend to an infinite non-increasing sequence of integers which stabilizes at 0, \( \lambda_i = 0, \ i \gg 0 \). The number of non-zero entries is called the length of the partition \( \lambda \):

\[ \ell(\lambda) = \# \{ i \mid \lambda_i > 0 \} \]

the sum of its elements is called the size

\[ |\lambda| = \sum_{i=1}^{\infty} \lambda_i \]

The square \( \Box \) is the pair \((i,j)\) of integers obeying \( i, j \geq 1 \). The square \( \Box \) belongs to \( \lambda, \Box \in \lambda \), if \( j \leq \lambda_i \) or, equivalently if \( i \leq \lambda_j \). The collection of the squares of the partition \( \lambda \) is its Young diagram (the uppermost square on the left is \((1,1)\)):
7.1.1 The contents

The content $c_{\Box}$ of $\Box$ is defined as:

$$c_{\Box} = \varepsilon_1 (i - 1) + \varepsilon_2 (j - 1).$$

(137)

More precisely, the content (137) is defined relative to the weights $(\varepsilon_1, \varepsilon_2)$. Below, we shall use the contents defined relative to other pairs of weights as well. To avoid any confusion we either denote them by different letters, e.g.

$$\sigma_{\Box} = \varepsilon_3 (i - 1) + \varepsilon_4 (j - 1),$$

(138)

or explicitly specify the pair of the weights, e.g.

$$c_{ab \Box} = \varepsilon_a (i - 1) + \varepsilon_b (j - 1)$$

(139)

Fig. 7

Examples of the contents, with $\lambda_l = 3$

subArms, legs, and hooks

For each $\Box = (i, j) \in \lambda$ one defines the arm-length, the leg-length and the hook-length: $a_{\Box}$, $l_{\Box}$, $h_{\Box}$ by:

$$a_{\Box} = \lambda_i - j, \quad l_{\Box} = \lambda_j^t - i, \quad h_{\Box} = a_{\Box} + l_{\Box} + 1,$$

(140)

Here is the picture for the arm and for the leg of a square $(3, 4)$ in the partition $(15, 9, 9, 9, 7, 4, 1, 1)$:

Fig. 8

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7.1.2 The character

The character \( \chi_\lambda(q_1, q_2) \) of the partition \( \lambda \) is the polynomial in \( q_1, q_2 \)

\[
\chi_\lambda(q_1, q_2) = \sum_{\Box \in \lambda} q_1^{i-1} q_2^{j-1}
\]

(141)

which we can also express as the sum of exponentials of contents using (??):

\[
\chi_\lambda(q_1, q_2) = \sum_{\Box \in \lambda} e^{\beta c_\Box}
\]

(142)

The character contains all information about the partition. In particular,

\[
|\lambda| = \chi_\lambda(1, 1), \quad \ell(\lambda) = \chi_\lambda(1, 0), \quad \lambda_1 = \chi_\lambda(0, 1)
\]

(143)

7.1.3 Dual partition

Fix a partition \( \lambda \). The dual or transposed partition \( \lambda^t \) is defined by the property:

\[
\chi_{\lambda^t}(q_1, q_2) = \chi_\lambda(q_2, q_1)
\]

(144)

In other words,

\[
\lambda^t_j = \#\{ i | \Box = (i, j) \in \lambda \Leftrightarrow j \leq \lambda_i \}
\]

(145)

Obviously:

\[
\ell(\lambda^t) = \lambda_1, \quad |\lambda^t| = |\lambda|
\]

(146)

7.1.4 Bosonic representation

A partition \( \lambda \) can also be identified with a finite sequence of non-negative integers \( (k_1k_2k_3 \ldots k_l \ldots) \)

\[
\lambda \leftrightarrow \left(1^{k_1} 2^{k_2} 3^{k_3} \ldots l^{k_l} \ldots\right),
\]

(147)

where

\[
k_l = \lambda_l^t - \lambda_l^t_{l+1} = \#\{ i | \lambda_i = l \}
\]

(148)

We have the obvious relations:

\[
\ell(\lambda) = \sum_{l=1}^{\infty} k_l, \quad |\lambda| = \sum_{l=1}^{\infty} l k_l
\]

(149)

Since there is no restriction on the \( k_l \)'s apart from them being non-negative, this representation of partition identifies it with a state in the free boson Fock space.

7.1.5 Dual character

is defined by the conjugation \( \beta \mapsto -\beta \) keeping \( \varepsilon_1, \varepsilon_2, a \) intact:

\[
\chi_\lambda^t(q_1, q_2) = \chi_\lambda(q_1^{-1}, q_2^{-1}), \quad \chi_{\Delta}(a, q_1, q_2)^t = \chi_{\Delta}(-a, q_1^{-1}, q_2^{-1})
\]

(150)
7.1.6 Colored partitions

An \( n \)-colored partition \( \lambda \) is the collection \((a_\alpha, \lambda(\alpha))_{\alpha=1}^n\) of \( n \) pairs consisting of a complex number and a partition. An \( n \)-colored square \( \square \) is a pair \((\alpha, \square)\), where \( \alpha \in [n] \) and \( \square \in \lambda(\alpha) \).

The content \( c_\square \) of the \( n \)-colored square is the sum

\[
c_\square = a_\alpha + c_\square
\]  

It is defined relative to the weights \((a, \varepsilon)\). We shall also encounter the contents \( \Sigma_\square \) defined relative to the weights \((\nu, \tilde{\varepsilon})\):

\[
\Sigma_\square = \nu_\alpha + \alpha_\square
\]  

The character \( \chi_\lambda(a, q_1, q_2) \) of the \( n \)-colored partition is the function

\[
\chi_\lambda(a, q_1, q_2) = \sum_{\square \in \lambda} e^{\beta c_\square} = \sum_{\alpha=1}^n e^{\beta a_\alpha} \chi_{\lambda(\alpha)}(q_1, q_2)
\]  

We shall also need

\[
\hat{\chi}_\lambda(\nu, q_3, q_4) = \sum_{\square \in \lambda} e^{\beta \varepsilon_\square} = \sum_{\beta=1}^n e^{\beta \nu_\beta} \chi_{\lambda(\beta)}(q_3, q_4)
\]  

Finally,

\[
\chi^{ab}_\lambda = \sum_{\alpha=1}^{n_{ab}} e^{a_{ab, \alpha}} \chi_{\lambda(\alpha)}(q_a, q_b)
\]  

7.2 Simple partition sums and elliptic functions

7.2.1 Generating functions of lengths and sizes

The following generating functions are computed in an elementary fashion using the bosonic representation \([147]\): that of the number of partitions

\[
\sum_{n=1}^\infty p(n)q^n = \sum_\lambda q^{|\lambda|} = \frac{1}{\phi(q)},
\]  

where

\[
\phi(q) = \prod_{n=1}^\infty (1 - q^n),
\]  

and the more refined generating function

\[
\sum_\lambda f^{(\lambda)} q^{|\lambda|} = \frac{1}{f(t, q)},
\]  

with

\[
f(t, q) = \prod_{n=1}^\infty (1 - tq^n)
\]  

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7.2.2 Elliptic functions

Let us fix our notations for the functions $\eta$, $\theta$, $\wp$. Write

$$q = e^{2\pi i \tau}, \quad \text{Im} \tau > 0$$

Then, define the Dedekind eta function

$$\eta(\tau) = e^{\frac{\pi i}{12} \phi(q)}$$

We shall slightly abuse the notation for the odd theta function

$$\theta_{11}(z; \tau) = i e^{\pi i \tau} z^{\frac{1}{2}} \left( 1 - z^{-1} \right) \phi(q) f(z, q) f(z^{-1}, q),$$

since in most of what follows the sign ambiguity in $i \sqrt{z}$ will cancel out. The series expansion

$$\theta_{11}(z; \tau) = i \sum_{r \in \mathbb{Z} + \frac{1}{2}} (-1)^{r - \frac{1}{2}} z^r e^{\pi i \tau r^2}$$

implies the heat equation:

$$\frac{1}{\pi i} \partial_{\tau} \theta_{11}(z; \tau) = (z \partial_z)^2 \theta_{11}(z; \tau)$$

Weierstrass $\wp$-function:

$$\wp(z; \tau) = - (z \partial_z)^2 \log \theta_{11}(z; \tau) + \frac{1}{\pi i} \partial_{\tau} \log \eta(\tau)$$

is normalized so as to have vanishing $\log(z)$ term in the expansion near $z = 1$.

7.2.3 Rank $p - 1$ theta function

is given by (cf. (200))

$$\Theta_{\Lambda_{p-1}}(z; \tau) = \eta(\tau)^{p-1} \prod_{\alpha > \beta} \frac{\theta_{11}(z_{\alpha}/z_{\beta}; \tau)}{\eta(\tau)}$$

It obeys the heat equation (cf. [192] and [25], [71], [26])

$$p \frac{\partial}{d \tau} \Theta_{\Lambda_{p-1}}(z; \tau) = \pi i \Delta_z \Theta_{\Lambda_{p-1}}(z; \tau)$$

7.3 More partition sums

Fix

$$\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$$

with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 0$.

Fix a non-negative integer $p \in \mathbb{Z}_{\geq 0}$ and $s, \omega \in \mathbb{Z}/p\mathbb{Z}$.

The functions

$$B\left( \begin{array}{c|c} \varepsilon_3 & s \\ \varepsilon_1 & p \end{array} | q \right), \quad Q\left( \begin{array}{c|c} \omega & s \\ \varepsilon_2 & p \end{array} | q \right), \quad \tilde{Q}\left( \begin{array}{c|c} \omega & s \\ \varepsilon_2 & p \end{array} | q \right)$$
are defined on the set of all partitions cf. [132] as follows:

\[
B \left( \begin{array}{c c c c c}
\epsilon_3 & \epsilon_4 & \delta & 0 \\
\epsilon_1 & \epsilon_2 & p & q
\end{array} \right) [\lambda] = \prod_{\Box \in \lambda} S_{+,1,2} (\epsilon_3 h_{\Box} + \epsilon_4 q_\Box) \delta_{\lambda}(s h_{\Box} + a_{\Box}) S_{-,1,2} (\epsilon_3 h_{\Box} + \epsilon_4 q_\Box) \delta_{\lambda}(s h_{\Box} + a_{\Box} + 1),
\]

and

\[
Q \left( \begin{array}{c c c c c}
\omega & \delta & 0 \\
P & p & q
\end{array} \right) [\lambda] = \prod_{\Box \in \lambda} q_{\omega+\kappa_{\Box}}, \quad \tilde{Q} \left( \begin{array}{c c c c c}
\omega & \delta & 0 \\
P & p & q
\end{array} \right) [\lambda] = \prod_{\Box \in \lambda} q_{\omega-\kappa_{\Box}},
\]

where for \( \Box = (i,j) \)

\[
\kappa_{\Box} = s(i-j) + 1 - j \pmod p.
\]

Sometimes we use a shorthand notation:

\[
B^\lambda = B \left( \begin{array}{c c c c c}
\epsilon_3 & \epsilon_4 & 0 \\
\epsilon_1 & \epsilon_2 & p & q
\end{array} \right) [\lambda], \quad \tilde{B}^\lambda = B \left( \begin{array}{c c c c c}
\epsilon_1 & \epsilon_2 & 0 \\
\epsilon_3 & \epsilon_4 & p & q
\end{array} \right) [\lambda]
\]

and

\[
Q^\lambda = Q \left( \begin{array}{c c c c c}
\omega & 0 \\
P & p & q
\end{array} \right) [\lambda] = \prod_{j} q_{\omega+1-j} = \prod_{i} z_{\omega}/z_{\omega-\lambda_i},
\]

\[
\tilde{Q}^\lambda = \tilde{Q} \left( \begin{array}{c c c c c}
\omega & 0 \\
P & p & q
\end{array} \right) [\lambda] = \prod_{j} q_{\omega+j-1} = \prod_{i} z_{\omega+\lambda_i-1}/z_{\omega-1}.
\]

Define the partition sums

\[
B_{\rho,\omega}(\bar{\epsilon};q) = \sum_{\lambda} Q^\lambda \bar{B}^\lambda, \quad \tilde{B}_{\rho,\omega}(\bar{\epsilon};q) = \sum_{\lambda} \tilde{Q}^\lambda \tilde{B}^\lambda.
\]

When there is no confusion about what \( p, \bar{\epsilon}, m \) are, we use the shorter notations

\[
B_{\omega} \equiv B_{\rho,\omega}(\bar{\epsilon}, m, -m - \bar{\epsilon}; q), \quad \tilde{B}_{\omega} \equiv \tilde{B}_{\rho,\omega}(\bar{\epsilon}, m, -m - \bar{\epsilon}; q).
\]

### 7.4 Trigonometric limit

By the trigonometric limit we understand the limit \( q \to 0 \) with \( z \) kept fixed. In this limit the partition sums \( B_{\rho,\omega}(\bar{\epsilon};q), \tilde{B}_{\rho,\omega}(\bar{\epsilon};q) \) are elementary. Indeed, the Eq. [182] implies \( q_0 = 0 \).

Thus, only the partitions with \( \lambda_1 \leq \omega < p \) contribute to the sum for \( B_{\rho,\omega}(\bar{\epsilon};q) \). This restriction implies that for all \( \Box \in \lambda, a_{\Box} < \omega \leq p - 1 \). Therefore only \( \Box = (i, \lambda_i) \) contribute to
the $\mathbb{B}_\omega^\lambda$ measure:

$$\mathbb{B}_{p,\omega}(\vec{\epsilon};\mathbf{Z})_{q \to 0} = \sum_{\lambda,\lambda_1 \leq \omega} \prod_{l=1}^{\ell_1} \frac{z_{\omega - \lambda_i}}{z_{\omega - \lambda_i}} \mathbf{S}_{+;1,2}(\vec{\epsilon}_3 h_{\square}) = \sum_{\lambda,\lambda_1 \leq \omega} \prod_{l=1}^{\ell_1} \left( \frac{z_{\omega}}{z_{\omega - l}} \mathbf{S}_{+;1,2}(\vec{\epsilon}_3 h) \right)$$

$$\sum_{\lambda,\lambda_1 \leq \omega} \prod_{l=1}^{\ell_1} \left( \lambda_i - \lambda_{i+1} + \frac{\epsilon^i_1}{\epsilon^i_2} \right)! \left( \frac{z_{\omega}}{z_{\omega - l}} \right)^{\lambda_i - \lambda_{i+1}}$$

$$\sum_{k_1, k_2, \ldots, k_{\omega - 1} \geq 0} \prod_{l=1}^{\omega - 1} \left( \frac{z_{\omega}}{z_{\omega - 1}} \right)^{k_l} = \prod_{l=1}^{\omega - 1} \left( 1 - \frac{z_{\omega}}{z_{\omega - 1}} \right)^{-1 - \frac{\epsilon^i_1}{\epsilon^i_2}}, \quad (175)$$

where we used the bosonic representation of the partition, cf. (147):

$$\lambda \leftrightarrow \left( \epsilon^1_1, \epsilon^2_1, \ldots, \epsilon^\omega_1 \right) \quad (176)$$

Similarly, only the partitions $\lambda$ with $\lambda_1 \leq p - \omega$ contribute to $\mathbb{B}_{p,\omega}(\vec{\epsilon};\mathbf{Z})$ in the trigonometric limit, thus we can use the bosonic representation (147) to perform the sum explicitly,

$$\mathbb{B}_{p,\omega}(\vec{\epsilon};\mathbf{Z})_{q \to 0} = \prod_{l=1}^{p - \omega} \left( 1 - \frac{z_{\omega + l - 1}}{z_{\omega - 1}} \right)^{-1 - \frac{\epsilon^i_1}{\epsilon^i_2}}, \quad (177)$$

### 7.5 The roots-weights coordinates and moduli spaces

Let $p$ be a positive integer, and

$$\mathbf{q} = (q_0, q_1, \ldots, q_{p-1}) = (q_\omega)_{\omega = 0}^{p-1} \in (\mathbb{C}^\times)^p \quad (178)$$

a collection of non-zero complex numbers. We shall be dealing with generating functions of the form:

$$Z_{A_{p-1}}(\mathbf{q}) = \sum_{\mathbf{k} \in \mathbb{Z}^p_{\geq 0}} \mathbf{q}^\mathbf{k} Z_{A_{p-1}}[\mathbf{k}] \quad (179)$$

where $Z_{A_{p-1}}[k_0, \ldots, k_{p-1}]$ may represent a contribution of $p$ “fractional” instantons of the charges $k_0$, $k_1$, ..., $k_{p-1}$, or a contribution of $p$ instantons belonging to $p$ different gauge groups with an $A$-type quiver interaction between them. The geometric or, physically, weak coupling domain of definition of the generating function (179) is the polydisk $D^{\text{weak}}$: $|q_\omega| < 1$, $\omega = 0, \ldots, p - 1$.

Surprisingly enough, we shall see that an analytic continuation of (179) outside $D^{\text{weak}}$ is often possible. In the fractional case we shall see an emergence of the Weyl symmetry $S_p$ of the $A_{p-1}$-type, even though the original problem didn’t have this symmetry in a manifest way. In the $A$-type quiver case we shall see the emergence of the modular group of the $\mathcal{M}_{0, p+2}$ moduli space of $p + 2$-punctured genus zero curves.

The variables $\mathbf{q}$ are not, however, convenient for exhibiting these symmetries.
7.5.1 Fractional case

Let us extend \( q \) to a \( p \)-periodic function \( \mathbb{Z} \to \mathbb{C}^\times \):

\[
q_{\omega+p} = q_\omega .
\]

Define \( p + 1 \) variables

\[
\mathbf{z} = (z_0, z_1, \ldots, z_{p-1}) = (z_\omega)_{\omega=0}^{p-1} \in (\mathbb{C}^\times)^p ,
\]
and \( q \in \mathbb{C}^\times \) by

\[
q_\omega = z_\omega / z_{\omega-1}, \quad \omega = 1, \ldots, p-1, \quad q_0 = q z_0 / z_{p-1}
\]

Thus,

\[
q = q_0 q_1 \cdots q_{p-1} = q[p] = q[0,p-1] ,
\]

while \( z_\omega \)'s are defined up to an overall rescaling:

\[
\begin{align*}
z_1 &= z_0 q_1, \\
z_2 &= z_0 q_1 q_2, \\
&\vdots \\
z_{p-1} &= z_0 q_1 q_2 \cdots q_{p-1},
\end{align*}
\]

It is natural to extend \( z \) to the quasi-periodic function on \( \mathbb{Z} \):

\[
z_{\omega+p} = q\, z_\omega, \quad \omega \in \mathbb{Z}
\]

The cyclic permutation of the variables \( q_\omega \), i.e.

\[
q_\omega \mapsto q_{\omega+a}, \quad a = 0, 1, \ldots, p-1
\]

acts on \( z_\omega \)'s via:

\[
\begin{align*}
z_\omega &\mapsto z_{\omega+a}, \quad \omega = 0, \ldots, p-1-a \\
z_\omega &\mapsto q z_{\omega+a-p}, \quad \omega = p-a, \ldots, p-1
\end{align*}
\]

It is on the variables \( \mathbf{z} \) that \( S_p \) will act by permutations in the fractional case. Moreover, when the variables \( \mathbf{z} \) are extended as in (185), one gets an action of the affine Weyl group \( S_p \times \mathbb{Z}^p \) by arbitrary finite permutations of \( (z_\omega)_{\omega \in \mathbb{Z}} \). In this way the \( \mathbf{z} \) defined up to the overall rescaling and these extended permutations (keeping \( q \) fixed) parametrize the coarse moduli space \( \text{Bun}_E(PGL_p) \) of holomorphic \( PGL_p \) bundles on the elliptic curve \( E = \mathbb{C}^\times / q\mathbb{Z} \).

7.5.2 Linear quiver case

In the \( A_{p-1} \)-type quiver case, define \( \mathbf{z} \in (\mathbb{C}^\times)^p \) via

\[
\begin{align*}
z_1 &= z_0 q_1, \\
z_2 &= z_0 q_1 q_2, \\
&\vdots \\
z_{p-1} &= z_0 q_1 q_2 \cdots q_{p-1},
\end{align*}
\]

We shall think of \( \mathbf{z} \) as of the coordinates on the cell \( 0 < |z_{p-1}| < |z_{p-2}| < \cdots < |z_1| < |z_0| < \infty \) in the moduli space \( M_{0,p+2} \), which is the space of \( p + 2 \)-tuples of distinct points \( z_{-1}, z_0, \ldots, z_{p-1}, z_p \) on \( \mathbb{CP}^1 \) modulo the overall \( PGL(2) \) action. By fixing the first point \( z_{-1} \) to be \( z_{-1} = \infty \) and the last point \( z_p \) to \( z_p = 0 \) one ends up with the remaining \( \mathbb{C}^\times \) symmetry of the overall rescaling of \( \mathbf{z} \), which can be fixed, e.g. by setting \( z_0 = 1 \).
7.5.3 $A_{p-1}$-quiver case

The formulas (184), without any permutation symmetry of the $z_i$’s, can be interpreted as parameterizing the open cell in the moduli space $M_{1,p}$ of $p$-punctured elliptic curves. The modulus $q = \exp 2\pi i \tau$ defines the elliptic curve $E = \mathbb{C}/2\pi i (\mathbb{Z} \oplus \tau \mathbb{Z}) \cong \mathbb{C}^\times / q^\mathbb{Z}$, while $z_0, z_1, \ldots, z_{p-1}$ determine the location of punctures. This identification is further supported by the modular properties of the partition functions we study in this paper, they transform nicely (albeit in a complex fashion) under the action of $SL_2(\mathbb{Z})$:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad z_i \mapsto \exp \frac{\log(z_i)}{c\tau + d}$$  \hspace{1cm} (189)

with $a, b, c, d \in \mathbb{Z}$ obeying

$$ad - bc = 1.$$  

7.5.4 Differential operators

For $\omega = 0, \ldots, p-1$ let

$$\nabla_\omega^q = q^\omega \frac{\partial}{\partial q^\omega},$$  \hspace{1cm} (190)

$$\nabla_\omega^z = z^\omega \frac{\partial}{\partial z^\omega}. $$  \hspace{1cm} (191)

Let $\Delta_z$ denote the Laplacian in $\log(z_\omega)$’s:

$$\Delta_z = \nabla_z \cdot \nabla_z \equiv \sum_{\omega=0}^{p-1} (\nabla_\omega^z)^2$$  \hspace{1cm} (192)

Let

$$\nabla^q = q \frac{\partial}{\partial q},$$  \hspace{1cm} (193)

when acting on the function of $(z, q)$, i.e. we keep $z$ fixed when differentiating with respect to $q$. The definition (182) implies $\nabla^q = \nabla_0^q$ and:

$$\nabla^q_\omega = \nabla^q_\omega - \nabla^q_{\omega+1},$$  \hspace{1cm} (194)

for $\omega = 0, \ldots, p-1$. Let

$$\mathcal{D}^q = \sum_{\omega=0}^{p-1} \nabla^q_\omega$$  \hspace{1cm} (195)

In terms of $\nabla^z, \nabla^q$ it reads:

$$\mathcal{D}^q = p \nabla^q + \rho \cdot \nabla^z,$$  \hspace{1cm} (196)

where

$$\rho \cdot \nabla^z \equiv \sum_{\omega=0}^{p-1} \rho_\omega \nabla^z_\omega,$$  \hspace{1cm} (197)

and
7.5.5 The rho-vector

Define

\[ \rho = (\rho_0, \rho_1, \ldots, \rho_{p-1}) \in \left( \frac{1}{2} \mathbb{Z} \right)^p \]  

(198)

where

\[ \rho_{\omega} = \omega - \frac{p - 1}{2}, \]  

(199)

\[ \rho_{\rho} = \frac{p(p^2 - 1)}{12} \]  

(200)

We also use the notation

\[ z^\rho = \prod_{\omega=0}^{p-1} z_{\rho,\omega} \]  

(201)

7.6 Differential identities

In the main body of the paper we shall prove the identity

\[ \varepsilon_1 \nabla^z \log(\tilde{B}_{p,\omega}(\varepsilon; q)) + \varepsilon_3 \nabla_{\omega-1}^z \log(B_{p,\omega}(\varepsilon; q)) = 0, \quad \omega'' \neq \omega' \mod p \]  

(202)

which implies (by summing over \( \omega' \), keeping \( \omega'' = \omega \) or by summing over \( \omega'' \), keeping \( \omega' = \omega - 1 \), respectively)

\[ \nabla_{\omega-1}^z \left( \varepsilon_1 \log(\tilde{B}_{\omega}) + m \log(B_{\omega-1}) \right) = m \nabla_{\omega-1}^z \log(\tilde{B}) = \varepsilon_1 \nabla_{\omega-1}^z \log(\tilde{B}) \]  

(203)

where we defined \( B \equiv B_p(\varepsilon, m; q), \tilde{B} \equiv \tilde{B}_p(\varepsilon, m; q) \) via:

\[ B_p(\varepsilon, m; q) \equiv \prod_{\omega=0}^{p-1} B_{p,\omega}(\varepsilon, m - m - \varepsilon; q), \]

\[ \tilde{B}_p(\varepsilon, m; q) \equiv \prod_{\omega=0}^{p-1} \tilde{B}_{p,\omega}(\varepsilon, m - m - \varepsilon; q) \]  

(204)

7.7 \( \Delta \)-functions

Let us define the functions \( \Delta = \Delta_p(\varepsilon, m; q), \tilde{\Delta} = \tilde{\Delta}_p(\varepsilon, m; q) \) by:

\[ \log(\Delta) = -\frac{m}{m + \varepsilon_1} \log(\tilde{B}) + \log(\Delta) = -\frac{\varepsilon_1}{m + \varepsilon_1} \log(\tilde{B}) \]  

(205)

which are equal up to a \( q \)- (and of course \( \varepsilon \))-dependent factor thanks to (203),

\[ \Delta_p(\varepsilon, m; q) = \tilde{\Delta}_p(\varepsilon, m; q) \exp \delta(q; \varepsilon). \]  

(206)

We shall also prove:

\[ \nabla^z_\alpha \nabla^z_\beta \log \Delta = f_{\alpha \beta}(z_\alpha/z_\beta; q) \]  

(207)
for all $\alpha \neq \beta \in \mathbb{Z}_p$, and, finally, the heat equations,

$$\left( p\nabla^q - p\frac{\varepsilon_2}{m + \varepsilon_1}\nabla^q \chi + \rho \cdot \nabla\chi - \frac{1}{2} \Delta z\right) \Delta = 0$$  \hspace{1cm} (208)$$

from which we derive

$$\Delta = z^p q^{-\frac{\rho^2}{2p} \frac{\varepsilon_2}{m + \varepsilon_1} \chi(q)} \Theta_{A_{p-1}}(z; \tau)$$
$$\bar{\Delta} = z^p q^{-\frac{\rho^2}{2p} \frac{\varepsilon_2}{m + \varepsilon_1} \chi(q)} \Theta_{A_{p-1}}(z; \tau)$$ \hspace{1cm} (209)

7.8 Generalization of Macdonald Identities

We conjecture that the stronger statements hold (see below for the list of checks we performed):

$$\mathbb{B}_{p,\omega}(\varepsilon; q) = \left( \phi(q) q^{\frac{\varepsilon_1}{m}} \cdot \Theta_\omega(z; q) \right)^{-\frac{\varepsilon_1}{m}}, \quad \mathbb{\bar{B}}_{p,\omega}(\varepsilon; q) = \left( \phi(q) q^{\frac{\varepsilon_1}{m}} \cdot \bar{\Theta}_\omega(z; q) \right)^{-\frac{\varepsilon_1}{m}}$$  \hspace{1cm} (210)

(These two formulas follow one from another, by exchanging $(\varepsilon_1, \varepsilon_2) \leftrightarrow (\varepsilon_3, \varepsilon_4)$) where, cf. (184):

$$\Theta_\omega(z; q) = \prod_{\alpha=0}^{\omega-1} \left( 1 - z_\omega/z_\alpha \right) \prod_{\alpha=0}^{p-1} f(z_\omega/z_\alpha, q) =$$
$$= \prod_{\ell=0}^\infty \left( 1 - \prod_{j=1}^\ell q_{\omega+1-j} \right) = \prod_{\ell=1}^\infty \left( 1 - z_\omega/z_{\omega-\ell} \right),$$

$$\bar{\Theta}_\omega(z; q) = \prod_{\alpha=\omega}^{p-1} \left( 1 - z_\alpha/z_{\omega-1} \right) \prod_{\alpha=0}^{p-1} f(z_\alpha/z_{\omega-1}, q) =$$
$$= \prod_{\ell=0}^\infty \left( 1 - \prod_{j=1}^\ell q_{\omega+j-1} \right) = \prod_{\ell=1}^\infty \left( 1 - z_{\omega+\ell-1}/z_{\omega-1} \right)$$ \hspace{1cm} (211)

are the “half-theta”-functions, related to each other and the $A_{p-1}$-theta function via

$$\prod_{\omega \in \mathbb{Z}_p} \Theta_\omega(z; q) \sim \prod_{\omega \in \mathbb{Z}_p} \bar{\Theta}_\omega(z; q) \sim \phi(q) z^p q^{-\frac{\rho^2}{2p}} \Theta_{A_{p-1}}(z; \tau)$$  \hspace{1cm} (212)

The symbol “$\sim$” in (212) stands to mean that we are omitting a constant factor like $i^{p(p-1)/2}$, which drops out of the formulas we use in the paper anyway.
The conjecture (210), (211) implies (cf. (166))

$$\chi(q; \bar{\epsilon}) = \left(\frac{\bar{\epsilon}_3}{\bar{\epsilon}_2} - \frac{\bar{\epsilon}_1}{\bar{\epsilon}_4}\right) \log(\phi(q))$$

(213)

7.8.1 Checks of the conjecture

Let us make several comments about (210), (211). First, when \(\bar{\epsilon}_3 \to \infty\), or when \(\bar{\epsilon}_1 \to 0\), the left hand side of (210) becomes:

$$\sum_{\lambda} \prod_{\square \in \lambda} q_{\omega+1-j \mod p} = \sum_{\lambda} \prod_{i=1}^{\infty} \frac{z_{\omega}}{z_{\omega-\lambda_i}} = \prod_{i=1}^{\infty} \left(1 - \frac{z_{\omega}}{z_{\omega-\ell_i}}\right)^{-1} = \Theta_\omega(z; q)^{-1}$$

(214)

by (211). Thus, (210) holds in this limit.

Second, when \(\bar{\epsilon}_1 + \bar{\epsilon}_3 \to 0\), the right hand side of (210) approaches 1. The left hand side does the same, since any \(\lambda \neq \emptyset\) has at least one square \(\square\) with \(a_{\square} = l_{\square} = h_{\square} - 1 = 0\). Such a square contributes the factor \(1 + \bar{\epsilon}_1/\bar{\epsilon}_3\) to (168), which vanishes when \(\bar{\epsilon}_3 = -\bar{\epsilon}_1\).

Thus, (210) holds in this limit too. Moreover, we can compute the limit, as \(\bar{\epsilon}_3 \to -\bar{\epsilon}_1\),

$$\frac{\bar{\epsilon}_1 \log(\bar{\Theta}_\omega)}{\bar{\epsilon}_1 + \bar{\epsilon}_3} \to \sum_{\lambda = wh} \left(\frac{z_{\omega+w-1}}{z_{\omega-1}}\right)^h \times \prod_{l=2}^{h} \left(1 - \frac{1}{l}\right) \times \prod_{l=1}^{h} \left(1 - \frac{1}{l - 1 - \frac{\bar{\epsilon}_1}{\bar{\epsilon}_3}}\right) \times \prod_{k=1}^{\infty} \left(1 - \frac{1}{l - 1 - \frac{\bar{\epsilon}_1 k}{\bar{\epsilon}_3}}\right) = -\log(\bar{\Theta}_\omega(z; q)) + \frac{\bar{\epsilon}_1}{\bar{\epsilon}_3} \log(\phi(q))$$

(215)

where we used the fact that the only partitions \(\lambda\) which contribute to the limit are the ones with exactly one square \(\square\) with \(a_{\square} = l_{\square} = h_{\square} - 1 = 0\), i.e. exactly one south-east corner. The Young diagrams of such partitions are the \(w \times h\) rectangles, with \(\ell_\lambda = h, \ell_{\lambda'} = w, |\lambda| = wh\). In the formula (215) we denote by \([x]_\lambda\) the maximal integer which is strictly less than \(x\), i.e. \([x]_\lambda < x\) for all \(x\), and by \([x]_\lambda\) the usual integer part of \(x\), i.e. \([x]_\lambda\leq x\), and \([x]_\lambda = x\), if \(x \in \mathbb{Z}\).

Finally, when \(p \to \infty\), the only squares with nontrivial contribution to (168) with \(s = 0\) are those with \(a_{\square} = 0\). Now, use the bosonic representation (147). The rows with \(\lambda_i = l\) with
$\lambda^I_{i+1} \leq i \leq \lambda^I_i$ contribute

$$\prod_{h=1}^{\lambda^I_i - \lambda^I_{i+1}} \left( \frac{z_\omega}{z_{\omega-l}} \right)^{\lambda^I_i - \lambda^I_{i+1} - 1} \left( \frac{z_\omega}{z_{\omega-l}} \right)^{\lambda^I_i - \lambda^I_{i+1}} = \left( \frac{z_\omega}{z_{\omega-l}} \right)^{\lambda^I_i - \lambda^I_{i+1}}$$

(216)

to $B_{\lambda^I}^{\lambda^I_0} Q_{\lambda^I}$. Summing over all $\lambda$’s is equivalent to summing over all $k_1, k_2, \ldots$ independently, from 0 to $\infty$, giving rise to:

$$\prod_{l=1}^\infty \left( 1 - \frac{z_{\omega-l}}{z_\omega} \right)^{-1}$$

(217)

which is indeed the $p \to \infty$ limit of the right hand side of the Eq. (210).

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This paper concludes a mini-series of papers [53, 54, 55, 56] dedicated to the BPS/CFT-correspondence. Since we began presenting our findings at various seminars and conferences, e.g. [51, 57, 58] a lot of interesting papers appeared, e.g. [19, 20, 28, 30, 8, 7, 69, 70] where some of our results have been derived as well. Also the papers [24, 23] contain some of the further developments of these ideas.
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