Cosmic acceleration and $f(R)$ theory: perturbed solution in a matter FLRW model

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In the present paper we consider $f(R)$ gravity theories in the metric approach and we derive the equations of motion, focusing also on the boundary condition. In such a way we apply the general equation to a first order perturbed expansion of the Lagrangian. We find that this proceeding can describe the accelerated expansion of the Universe and consequently without needing any dark energy. In particular we find accelerated solutions of the universe’s expansion in a matter dominated universe without any kind of dark energy component.

I. INTRODUCTION

Gravity is supposed to be the only dominant force at large distances during the present epoch. According to this and due to several puzzles concerning the evolution of the Universe (for instance dark energy, dark matter and so on), it is reasonable to consider that we might have not fully understood it on a cosmological scale.

The first attempts to modify the Einstein’s gravity go back in 1919, when Weyl [1] added a quadratic term in the Weyl tensor to the Einstein-Hilbert Lagrangian. Later many authors gave attention to modification of gravity, for example Eddington [2], Lanczos, Bach, Schrodinger and then Buchdahl [3] that analyzed the actions considering singularity free oscillating cosmology. In the 1960’s in the context of quantum gravity the Einstein’s Lagrangian was modified introducing terms containing higher orders of the scalar curvature [4].

Very interesting classes of extended gravity are the so called ”$f(R)$ theories” that coming from a direct generalization of the Einstein’s Lagrangian (for complete reviews see [5–13] and references therein). Among these, $f(R)$ gravity seems to be an interesting model that is relatively simple and it may have many applications in astrophysics, cosmology and in high energy phenomena [14, 15].

The simplest modification of gravity which still preserves all the symmetries of the General Relativity (GR) consists of the extension of Einstein-Hilbert action

$$S_{EH} = -\frac{1}{16\pi G} \int_{\Omega} d^4x \sqrt{-g} R$$

(1)

where instead of $R$, the Ricci scalar curvature, an arbitrary function $f(R)$ is present,

$$S = \int_{\Omega} d^4x \sqrt{-g} f(R)$$

(2)

where $g$ is determinant of the metric $g_{\mu\nu}$. From the technical point of view, this way to proceed directly allows us to write field equations in order to compare them with GR ones. Moreover, they are directly related to scalar-tensor theories by a peculiar conformal transformation of the metric involving a scalar field $\phi$ [16]. This way to proceed gives field equations which are also ghost free [17, 18].

It has been established that our Universe is going through an accelerated phase. Indeed a series of observations based on Supernovae type Ia [19–21] can be explained by the accelerated expansion of the Universe. Within the mathematical framework of the GR and the idea that our cosmo is homogeneous and isotropic and generally speaking, the scientists assume that this acceleration is due to some kind of negative pressure form due to “dark energy”. Cosmologists have proposed many models of dark energy but these ones have many free parameters and they often have constraints from observational data. This discovery has revolutionized modern cosmology. There are many explanations and theoretical models in literature, for example quintessence, k-essence, Chaplygin gas... and so on. The simple explanation for the Universe’s accelerated expansion is a cosmological constant, $\Lambda$, that is to say a nonzero vacuum energy drives the acceleration of the Universe. From a observational viewpoint it is important to say that $\Lambda$CDM model of the Universe
is in agreement with data coming from observations. But this model shows incongruences and is "unnatural", because poses important theoretical questions: what is the reason by which the nonzero vacuum energy should drive the acceleration? Why is the cosmological constant Λ smaller than obtained from quantum field theory by 121 orders of magnitude? This is known as "cosmological constant problem" and it is a fundamental problem in cosmology and physics. In other terms the nature of dark energy as well as its cosmological origin remain unknown and a real mystery.

For a resolution to this problem we invoke the class of \( f(R) \) theories. For sure, this is just a particular class of extended gravity theories. Some different interesting approaches have been studied in \[17\] \[18\]. Nevertheless recent research has shown a plausible alternative to this picture, in fact it has been shown that such cases lead to an effective dark energy \[10, 22–62\]; in other terms these models mimic the accelerated expansion of the Universe by a modification of general relativity that converts the attractive gravity into a repulsive interaction on cosmological scales. If we consider a small correction to the Einstein-Hilbert action, for example, by adding an \( 1/R \) term, we have an acceleration of the Universe because of the \( 1/R \) term which is able to dominate as the Hubble parameter decreases. This theory shows \[10\] that there is an equivalency to a scalar-tensor gravity without scalar kinetic term. It is important to say that this connection to scalar-tensor gravity is provided by a conformal transformation connecting the Einstein frame and the Jordan one and is valid to all extended theories of gravity that have the Einstein-Hilbert action with \( f(R) \), a function of Ricci scalar in which \( f \) is zero, or at least proportional to a total quadri-divergence that doesn’t modify the field equations:

\[
\delta \left( \sqrt{-g} \right) f(R) = 0
\]

so let’s just apply our requirement, i.e.

\[
\delta g_{\mu\nu} S = \int_\Omega d^4x \left[ \delta(\sqrt{-g}) f(R) + \sqrt{-g} \delta(f(R)) \right]
\]

\[
\delta g_{\mu\nu} S = \int_\Omega d^4x \left[ -\frac{1}{2} \sqrt{-g} g_{\mu\nu} f'(R) + \sqrt{-g} f'(R) \delta(g^{\mu\nu} R_{\mu\nu}) \right]
\]

\[
\delta g_{\mu\nu} S = \int_\Omega d^4x \sqrt{-g} \left[ -\frac{1}{2} g_{\mu\nu} f'(R) \delta g^{\mu\nu} + f'(R) \delta g^{\mu\nu} R_{\mu\nu} + f'(R) g^{\mu\nu} \delta R_{\mu\nu} \right]
\]

where we defined \( f'(R) \equiv \frac{df(R)}{dR} \) and we used:

\[
\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}.
\]

The last term in eq. \ref{eq:3} gives rise to the boundary effects due to the fact that \( \delta R_{\mu\nu} \) contains \( (\delta \sqrt{g})_{\partial \Omega} \) that it is nonzero. So let’s see how to treat it. First of all, let’s notice that:

\[
\delta R_{\mu\nu} = \nabla_\alpha \delta \Gamma_{\mu\nu}^\alpha - \nabla_\mu \delta \Gamma_{\alpha\nu}^\alpha.
\]
where $\Gamma^\alpha_{\mu\nu}$ are the usual Christoffel symbols constructed from $g_{\mu\nu}$. Now, let’s rewrite our relation in the local inertial frame where $\Gamma = 0 \Rightarrow \nabla \rightarrow \partial$ and the metricity condition becomes $\partial_\alpha g_{\mu\nu} = 0$. So, we obtain:

$$\delta \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\rho} (\partial_\mu \delta g_{\rho\nu} + \partial_\nu \delta g_{\rho\mu} - \partial_\rho \delta g_{\mu\nu})$$  \hspace{1cm} (6)$$

$$\delta \Gamma_{\alpha\nu} = \frac{1}{2} g^{\alpha\rho} \partial_\rho \delta g_{\alpha\nu}$$  \hspace{1cm} (7)

and

$$(g^{\mu\nu} \delta R_{\mu\nu})_{\Gamma=0} = \partial^\mu \partial^\nu \delta g_{\mu\nu} - g^{\alpha\rho} \partial_\mu \partial^\alpha \delta g_{\alpha\rho}.$$  \hspace{1cm} (8)

At this point, if we release the inertial frame hypothesis, we have to replace $\partial$ with $\nabla$, so that appears as follows:

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla^\mu \nabla^\nu \delta g_{\mu\nu} - \nabla_\mu \nabla^\nu (g^{\alpha\rho} \delta g_{\alpha\rho}) = \nabla_\mu \left[ g_{\alpha\beta} \nabla^\mu \delta g^{\alpha\beta} - \nabla_\nu \delta g^{\mu\nu} \right]$$  \hspace{1cm} (9)

where we used $g^{\alpha\beta} \delta g_{\alpha\beta} = -g_{\alpha\beta} \delta g^{\alpha\beta}$, $\delta g_{\alpha\beta} = -g_{\alpha\rho} \delta g^{\alpha\rho}$ and the metricity condition $\nabla g = 0$. In other word, the last term in (4) becomes:

$$\int d^4x \sqrt{-g} f'(R) \nabla_\mu \left[ g_{\alpha\beta} \nabla^\mu \delta g^{\alpha\beta} - \nabla_\nu \delta g^{\mu\nu} \right]$$

$$= \int d^4x \sqrt{-g} \nabla_\mu \left[ f'(R) (g_{\alpha\beta} \nabla^\mu \delta g^{\alpha\beta} - \nabla_\nu \delta g^{\mu\nu}) \right]$$

$$- \int d^4x \sqrt{-g} \left\{ \nabla_\mu \left[ (\nabla^\mu f'(R)) g_{\alpha\beta} \delta g^{\alpha\beta} \right] - \nabla_\alpha \nabla^\alpha f'(R) g_{\mu\nu} \delta g^{\mu\nu} \right\}$$

$$+ \int d^4x \sqrt{-g} \left\{ \nabla_\nu \left[ (\nabla_\nu f'(R)) \delta g^{\mu\nu} \right] - \nabla_\nu \nabla_\mu f'(R) \delta g^{\mu\nu} \right\}.$$  \hspace{1cm} (10)

It is important to note that second integral and fourth one don’t contribute to the variation; in fact they can be changed in two flux integrals evaluated on the boundary $\partial\Omega$, where $\delta g = 0$. On the other hand, third integral and fifth one give a relevant contribute to the variation made with respect to $g^{\mu\nu}$, that appears as follows:

$$\int d^4x \sqrt{-g} \delta g^{\mu\nu} \left[ R_{\mu\nu} f'(R) - \frac{1}{2} g_{\mu\nu} f(R) + g_{\mu\nu} \Box f'(R) - \nabla_\mu \nabla_\nu f'(R) \right]$$  \hspace{1cm} (11)

where $\Box \equiv \nabla_\alpha \nabla^\alpha$. Here we have to add the variation of the material action made with respect to the metric

$$\int_{\partial\Omega} d^4x \sqrt{-g} \frac{1}{2} T_{\mu\nu} \delta g^{\mu\nu}$$  \hspace{1cm} (12)

in order to obtain all the terms proportional to $\delta g^{\mu\nu}$. There is still another term that contributes to the variation; this one can be rewritten as a flux integral as follows:

$$\int_{\partial\Omega} dS^\mu \sqrt{-g} f'(R) \left[ g_{\alpha\beta} \nabla^\mu \delta g^{\alpha\beta} - \nabla_\nu \delta g^{\mu\nu} \right] \equiv \delta g S_h.$$  \hspace{1cm} (13)

In this way, we obtained a fourth order differential bulk equation that must determine 10 components of the symmetric tensor $g_{\mu\nu}$. In such a way, we are allowed to fix 40 initial conditions, in order to have a well defined solvable problem. We already fixed 20 of them by requiring that $\delta g_{\mu\nu}|_{\partial\Omega} = 0$ so we are still left with 20, that aren’t enough to completely erase the boundary terms. In fact, in $\partial\Omega \partial_\beta \delta g_{\alpha\beta}$ on the boundary appear, that correspond to 80 degrees of freedom. In GR ($f'(R) = 0$) this term is erased by adding the well-know York-Gibbons-Hawking action:

$$S_{YGH} = \int_{\partial\Omega} d^4x \sqrt{-g} \nabla_\mu V^\mu_{YGH} = \int_{\partial\Omega} d^4\xi \sqrt{|h|} \frac{1}{2} K$$  \hspace{1cm} (14)

where $h_{\alpha\beta} \equiv g_{\alpha\beta} + \epsilon_{\alpha\beta\gamma} \partial_\gamma$ and $K \equiv \epsilon^{\mu\nu} K_{\mu\nu} = 2h^{\mu\nu} \nabla_\mu \eta_\nu$. Therefore the generalized boundary term is:

$$S_B = \int_{\partial\Omega} d^4x \sqrt{-g} \nabla_\mu \left[ f'(R) V^\mu_{YGH} \right].$$  \hspace{1cm} (15)
In such a way, by varying with respect to \( g \), we obtain that:

\[
\delta g S_B = \int_{\partial \Omega} d^3 \xi \sqrt{|h|} f'(R) h^\mu{}_{\nu} \partial_{\mu} \delta g_{\nu\alpha} + \int_{\partial \Omega} dS_{\mu} \sqrt{-g} 2K f''(R) g^{\mu\nu} \delta R_{\mu\nu}.
\]  

(16)

The first integral of (16) exactly erases the (13). Otherwise, \( R_{\mu\nu} \) is a symmetric tensor so the requirement that \( \delta R_{\mu\nu} \) must be equal to zero on the boundary corresponds to impose the 20 initial conditions which allows us to completely solve the bulk equations:

\[
R_{\mu\nu} f'(R) - \frac{1}{2} g_{\mu\nu} f(R) + g_{\mu\nu} \Box f'(R) - \nabla_{\mu} \nabla_{\nu} f'(R) = -\frac{1}{2} T_{\mu\nu}.
\]  

(17)

Before concluding this section, let us notice that \((\partial\Omega) \delta R_{\mu\nu} = (\delta R)_{\partial\Omega}\); in other words, the total contributions on the boundary can be rewrite as scalar degree of freedom, using the well-know equivalence between \( f(R) \) gravity and scalar-tensor theory [63].

### III. PERTURBED LAGRANGIAN

In this Section we investigate a perturbed solution of the general relativity in a purely matter dominated Friedmann universe. In particular we’ll obtain the expression of the expansion parameter \( a \).

To this end, let us consider a generic and unknown Lagrangian for a modified theory of gravity which depends only by Ricci scalar \( R = g^{\mu\nu} R_{\mu\nu} \), so we can write \( \mathcal{L} = f(R) \). As we saw in the previous section the field equations are given by (17). Because we are facing to Friedmann-Lemaître-Robertson-Walker (FLRW) models, with only one function to determine, we consider the trace of these equations:

\[
f'(R) - 2f(R) + 3 \Box f'(R) = -\frac{1}{2} T.
\]  

(18)

In general, \( f(R) \) is not specified, therefore it is possible to consider its power series expansion instead of the exact form:

\[
f(R) = \sum_{n=0}^{\infty} c_n R^n.
\]  

(19)

From (19), it directly follows that

\[
f'(R) = \sum_{n=1}^{\infty} n c_n R^{n-1}.
\]  

(20)

In this way, equation (18) can be rewritten as follows:

\[
\left( \sum_{n=1}^{\infty} n c_n R^{n-1} \right) R - 2 \sum_{n=0}^{\infty} c_n R^n + 3 \Box \left( \sum_{n=1}^{\infty} n c_n R^{n-1} \right) = -\frac{1}{2} T
\]  

(21)

so we obtain:

\[
\sum_{n=1}^{\infty} c_n \left[ (n-2)R^n + 3 n \Box R^{n-1} \right] = -\frac{1}{2} T + 2 c_0.
\]  

(22)

At this point we assume that there isn’t a cosmological constant term, i.e. \( c_0 = 0 \), because our purpose is to explain the apparent acceleration of the universe as an effect due to the Lagrangian’s higher order terms without introducing any kind of dark energy. Furthermore, let’s divide all of the equation by \( c_1 = -\frac{1}{2\chi} \) and let’s define \( C_n = \frac{c_n}{c_1} \), so:

\[
-R = \chi T, \quad \text{if} \quad n = 1
\]  

(23)
\[- R + 6 C_2 \Box R = \chi T, \quad \text{if} \quad n = 2\] (24)

and so on. We stop our expansion because we assume that all of the relevant corrections to general relativity can be treated as a first order perturbative correction. Having this consideration in mind, let us consider a flat FLRW metric:

\[ds^2 = dt^2 - a^2(t) \left[ dx^2 + dy^2 + dz^2 \right].\] (25)

In such a way, we obtain \(\Box R = \dddot{R} + 3 \frac{\ddot{a}}{a} \dot{R};\) so eq. \([23]\) becomes:

\[- R = \chi T, \quad \text{if} \quad n = 1\] (26)

\[- R + 6 C_2 \left( \dddot{R} + 3 \frac{\ddot{a}}{a} \dot{R} \right) = \chi T, \quad \text{if} \quad n = 2.\] (27)

Now, because we are interested in finding the perturbative solution of a GR solution of a purely matter dominated Friedmann model, let us expand our function as \(a(t) \approx a_0(t) + C_2 a_1(t),\) considering that \(T = \rho_0 a^{-3} \approx \rho_0 a_0^{-3} \left( 1 - 3 C_2 \frac{a_1}{a_0} \right)\)

\[- R [a_0(t)] = \frac{\chi \rho_0}{a_0^3} \] (28)

\[- R [a(t)] + 6 C_2 \left\{ \dddot{R} [a(t)] + 3 \frac{\ddot{a}(t)}{a(t)} \dot{R} [a(t)] \right\} = \chi T.\] (29)

The solution of eq. \([28]\) is given by \(a_0(t) = \left( 1 + \frac{\sqrt{3} a_0 \chi T}{2 t} \right)^{2/3}.\) Now we want to find the expression of \(a_1\) by expanding eq. \([29]\) as follows:

\[- R [a_0 + C_2 a_1] + 6 C_2 \left\{ \dddot{R} [a_0 + C_2 a_1] + 3 \frac{\ddot{a}_0 + C_2 \dddot{a}_1}{a_0 + C_2 a_1} \dot{R} [a_0 + C_2 a_1] \right\} = \frac{\chi \rho_0}{a_0^3} \left( 1 - 3 C_2 \frac{a_1}{a_0} \right)\] (30)

we obtain

\[- R [a_0] - C_2 a_1 \frac{\partial R [a_0]}{\partial a} + 6 C_2 \left\{ \dddot{R} [a_0] + 3 \frac{\ddot{a}_0}{a_0} \dot{R} [a_0] \right\} = \frac{\chi \rho_0}{a_0^3} \left( 1 - 3 C_2 \frac{a_1}{a_0} \right),\] (31)

that is to say

\[a_1 = \left\{ 6 \dddot{R} [a_0] + 18 \frac{\ddot{a}_0}{a_0} \dot{R} [a_0] \right\} \left( \frac{\partial R [a_0]}{\partial a} - 3 C_2 \frac{\chi \rho_0}{a_0^3} \right)^{-1}\] (32)

where we treated the matter content as relevant only for \(a^{(1)}\). At this step, remembering that \(R(t) = -6 \left\{ (\ddot{a}/a)^2 + \dot{a}/a \right\}\) and inserting the expression for \(a_0\), we finally find:

\[a_1(t) = \frac{3 \chi \rho_0}{2 \left( 1 + \frac{\sqrt{3} a_0 \chi T}{2 t} \right)^{4/3}}.\] (33)

and then:

\[a(t) \approx \left( 1 + \frac{\sqrt{3} \chi \rho_0}{2 t} \right)^{2/3} + C_2 \frac{3 \chi \rho_0}{2 \left( 1 + \frac{\sqrt{3} a_0 \chi T}{2 t} \right)^{4/3}}\]

\[= \left( 1 + \frac{3 \hbar}{2 t} \right)^{2/3} + C_2 \frac{9 \hbar^2}{2 \left( 1 + \frac{3 \hbar}{2 t} \right)^{4/3}}\] (34)
that is the most general expression of the expansion parameter in our model and directly reduces to general relativity solution for $C_2 = 0$, where we defined $h \equiv \sqrt{\frac{3}{2}} \chi_0$.

Differently from the usual GR normalization, the perturbed scale factor as written in (34) is not equal to 1 at the present time. Nevertheless we can recover the usual normalization by redefining $a(t)$ as $a(t) - \frac{9C_2 h^2}{2}$. Indeed we can make this renormalization thanks to eq. (30). In order to convince ourselves, let us notice that all the time derivatives of $a(t)$ are not affected by the subtraction so, the correction appears only in the denominators. More precisely, because the term we subtract is first order, we have to consider it only in the right side and in the zero-th order of the left side in eq. (30). In this way, the additional part appearing from the rescaling in both sides exactly erases each other thanks to the zero-th order equation. From now, we then refer to the following expression for the scale factor:

$$a(t) = \left(1 + \frac{3}{2} \frac{h}{t}\right)^{2/3} + C_2 \frac{9h^2}{2(1 + \frac{3h}{2}t)^{4/3}} - \frac{9C_2 h^2}{2}$$  \hspace{1cm} (35)$$

which is automatically normalized to 1 nowadays ($t = 0$).

In the following sections, we will calculate some fundamental parameters as the Hubble function and the acceleration. In particular, the last section is dedicated to the comparison with supernovae Ia Union2 data, in order to obtain an experimental estimation of $C_2$ by best-fit comparison.

IV. HUBBLE PARAMETER

In the previous section we have found the solution for $a(t)$, that contains two free parameters $h$ and $C_2$. In general relativity $h$ is just the Hubble parameter evaluated today. However, in the perturbed approach, this interpretation is no longer viable: in fact, remembering that the Hubble function is related to the expansion rate by

$$H(t) = \frac{\dot{a}(t)}{a(t)} \approx \frac{\dot{a}_0}{a_0} \left[1 + C_2 \left(\frac{\dot{a}_1}{a_0} - \frac{\dot{a}_1}{a_0}\right)\right]$$  \hspace{1cm} (36)$$

we obtain

$$H(t) = \frac{2h}{2 + 3ht} + \frac{9C_2 h^3}{2} \left[\left(1 + \frac{3}{2}ht\right)^{-5/3} - 3\left(1 + \frac{3}{2}ht\right)^{-3}\right].$$  \hspace{1cm} (37)$$

So the Hubble constant is:

$$H_0 \equiv H(0) = h - 9C_2 h^3.$$  \hspace{1cm} (38)$$

At this point, using the value $H_0 = 67$ km/s Mpc$^{-1}$, we are able to determine a relation among $C_2$ and $h$ by inverting (38), i.e.

$$C_2 = \frac{h - H_0}{9h^3}. \hspace{1cm} (39)$$

This relates $C_2$ to $h$ by $H_0$ and allows us to rewrite (34) as

$$a(t) = \left(1 + \frac{3}{2} \frac{ht}{t}\right)^{2/3} + \frac{h - H_0}{2h} \left[\left(1 + \frac{3}{2}ht\right)^{-4/3} - 1\right].$$  \hspace{1cm} (40)$$

This expression contains the only parameter $h$. Once again, as $C_2$ approaches to 0, $h \to H_0$. By the way we stress once again that this interpretation falls down in the perturbative model and $h$ is just a parameter fixed by the second initial condition $\dot{a}(0) = H_0$.

In the next section, we shall use our solution (40) in order to study the acceleration of the universe.
V. ACCELERATION AND LUMINOSITY DISTANCE

In order to study the accelerating properties of our model, let us study the second derivative of $a(t)$:

$$\ddot{a}(t) = -\frac{h^2}{2 (1 + \frac{3}{2} h t)^{4/3}} + \frac{h - H_0}{2} \frac{7h}{(1 + \frac{3}{2} h t)^{10/3}}. \quad (41)$$

The previous expression, when evaluated at the present time, becomes equal to $(\ddot{a}/a)_{\text{today}}$ because of our choice $a(0) = 1$, so, from (41) we have:

$$\left(\frac{\ddot{a}}{a}\right)_{\today} = \frac{h}{2} (6h - 7H_0). \quad (42)$$

Therefore requiring the acceleration at the present epoch, we have that (42) must be greater than 0 i.e.

$$h < 0 \quad \text{or} \quad h > \frac{7}{6} H_0. \quad (43)$$

However, the values of $h$ next to 0 cannot be considered, because in this region $C_2$ diverges as shown in the eq. (39) and in the fig. 1, where we have rearranged (39) by the definition of $x = h/H_0$. This analysis shows a very important and interesting feature of this perturbative model. Without introducing any kind of dark energy, an accelerated expansion of the Universe is possible even in presence of a purely matter component and in presence of a little correction in the Lagrangian.

At this point, it is very interesting to check our hypothesis with the Supernovae Ia data, in particular, we consider the recent Union2 compilation [65]. To this end, let us evaluate the luminosity distance: it is well known that, in a Friedmann-Lemaître-Robertson-Walker spacetime, the luminosity distance $d_L$ is:

$$d_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')} \quad (44)$$

where $H(z)$ is the Hubble function given in (36) with the constrain (38). The dependence by the redshift is given by consider the expression of $z$ for a stationary and geodesic observer, i.e.

$$1 + z = \frac{1}{a(t)}. \quad (45)$$
In such a way, defining \( x \equiv (1 + \frac{3}{2} h t)^{2/3} \) allows us the rewrite last equation as:

\[
x^3 - A_1(z) x^2 + A_2 = 0
\]

(46)

where

\[
A_1(z) = \frac{h - H_0}{\frac{1}{2} h} + \frac{1}{1+z}
\]

(47)

\[
A_2 = \frac{h - H_0}{\frac{1}{2} h}
\]

(48)

\[
A_3(z) = A_1^2 + \frac{4}{9} \left( \sqrt{81 A_2^2 - 12 A_2 A_1 - 9 A_2} \right)
\]

(49)

Finally, by solving the third order polynomial, we find that

\[
x(z) = \frac{1}{3} \left[ A_1 + A_2 + A_3 \right]
\]

(50)

or equivalently

\[
t(z) = \frac{2}{3 h} \left[ x(z)^{3/2} - 1 \right].
\]

(51)

Once founded this relation, let us insert that one into definition of \( d_L \), and by numerical integration, we are able to evaluate our distance modulus

\[
\mu(z,h) = 5 \log_{10} \left[ \frac{d_L(z,h)}{1 Mpc} \right] + 25.
\]

(52)

We have in principle one free parameter \( h \). We use for the Hubble constant the recent value \( H_0 = 67 \) Km/s Mpc\(^{-1}\) given by Planck data. It is possible to fit experimental data \( \mu^{obs}(z_i) \pm \Delta \mu(z_i) \) with \( \Delta \mu(z_i) \) relative error of the

Figure 2: The Hubble diagram of the Union 2 dataset. The plot illustrates the best-fit result with the only parameter \( h \). We have \( h=13.9 \pm 0.8 \) with \( \chi^2 / \text{d.o.f.} = 1.1 \). We have considered for the Hubble constant the value \( H_0 = 67 \) km/s Mpc\(^{-1}\).
modulus distance with respect to the $i$–th Supernova Ia, by means of a $\chi^2$ analysis with

$$\chi^2 = \sum_{i=1}^{557} \left[ \mu_{\text{obs}}(z_i) - \mu(z_i, h) \frac{\Delta \mu(z_i)}{\Delta \mu(z_i)} \right].$$

In Fig. 2 we plot distance modulus vs redshift in the Hubble diagram. Minimizing the $\chi^2$ expression we have the best-fit value $h = 13.9 \pm 0.8$ with $\chi^2 / \text{d.o.f.} = 1.1$, where we have imposed a constrain on $h$ such that $C_2 < 0.1$. The best-fit red curve is superimposed to the Union 2 data set (with error bars). It is very interesting that we are able to reproduce the corresponding best-fit result for a homogeneous $\Lambda$CDM model in Friedman-Lemaitre-Robertson-Walker metric, without introducing the cosmological constant.

Moreover, the value we found for $C_2$ is equal to $-(2.2 \pm 0.4) \times 10^{-3}$. It is interesting noticing that the best-fit value corresponds to a decelerated expansion.

VI. CONCLUSION

In this research we have investigated a perturbative approach of a FLRW metric in $f(R)$ gravity and in particular the cosmological implications of our model of $f(R)$ gravity has been considered. We have studied a model in order to construct the form of $f(R)$ which in general is responsible for the acceleration of the Universe. This analysis of $f(R)$ model mimics a cosmological evolution consistent with observations.

Many forms of the function $f(R)$ are founded in the literature, here we discuss our model considering a series polynomial expansion of $f(R)$ function in $R^n$. This is interesting if we consider the phenomenological consequence of our study. Capozziello et al. [66, 67] have introduced the action with a term $f(R) \sim R^n$ and they have shown that this lead to an accelerated expansion for $n \approx 3/2$.

We consider a second order expansion of the cosmological parameter $a(t)$. We found an approximate expression given by eq. (34) that contains two parameters. Taking into account the Hubble parameter value it is possible to restrict us at only one parameter obtaining eq. (40). This expression is useful in order to study the permitted values of the parameter in order to have acceleration of the Universe without introducing any exotic component in the energy budget of the Universe. It’s important to underline that we used the Hubble parameter’s value founded by Planck; this could be important in order to reduce the tension of the $H_0$ determinations among the CMB observations and the SNIa ones. By the way this is an intriguing situation because we have not the necessity to introduce a dark energy in order to explain the accelerated expansion of the Universe. On the other hand it is possible to check this hypothesis considering the SN Ia catalog of Union2 data set. In fact fitting the data released by Union 2 we have found that the experimental point in the Hubble diagram can also be accurately described in our model. We have calculated the luminosity distance and the distance modulus $\mu$ only depending by the parameter $h$. In conclusion we want to stress that this work offers a possible explanation of the acceleration of the Universe as a consequence of a dynamical approach in which we consider a perturbative solution of a general relativity solution without introducing cosmological constant and/or dark energy.

Keeping in mind that there are numerous possibility for $f(R)$, we do not forget that our choice is one of the most simple to consider. No doubts the details must be more complicated that the model deputed here. The study if this model may also provide some specific effects that could discriminate between the various models. This will be done in a future work.

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