ASYMPTOTIC GEOMETRY OF TORIC K\"{A}HLER INSTANTONS

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Abstract. The symplectic reduction of a complete toric K\"{a}hler manifold need not be closed or even be a polygon. Sharp differences in behavior occur between those complete toric K\"{a}hler 4-manifolds with closed and with non-closed reductions. This paper establishes geometric criteria for these reductions to be closed, and classifies all asymptotic geometries possible in the case of scalar-flat instantons with closed reductions. Contrasting this, we provide examples of complete instantons with non-closed and non-polygon reductions.

1. Introduction

A K\"{a}hler manifold \((M^4, J, g)\) with a 2-torus action that preserves the symplectic form and the metric is said to be a toric K\"{a}hler 4-manifold. In the usual way, the torus action creates a 2-dimensional leaf-space. This leaf-space is always convex; if the manifold is compact, this polygon is naturally a compact Delzant polygon (see [7], [17]). If the manifold is non-compact, its leaf-space need not be closed, and need not be a polygon. In this paper we lay out the necessary and sufficient conditions that this leaf-space reduction be closed. This should be regarded as a companion paper to [21] and [22].

Before stating our results we must describe this reduction and set some notation. As in [22] we prefer the Arnold-Liouville reduction over the momentum construction; the former requires only two commuting action fields without the necessity of having an action torus. We study toric K\"{a}hler 4-manifolds \((M^4, J_\omega, X^1, X^2)\) where \((M^4, J_\omega)\) is a geodesically complete K\"{a}hler manifold and \(X^1, X^2\) are two symplectomorphic Killing fields that commute; this means \(L_{X^i}\omega = 0, L_{X^j}g = 0,\) and \([X^1, X^2] = 0\). It is convenient to assume \(M^4\) is simply connected; we may pass to the universal cover if not. Because \(L_{X^i}\omega = 0\) we have \(di_{X^i}\omega = 0,\) and therefore functions \(\varphi^1, \varphi^2\) exist with \(\omega(X^i, \cdot) = d\varphi^i.\) These are called the momentum or action coordinates on \(M^4.\) Then \(\nabla\varphi^1 = -JX^1,\) \(\nabla\varphi^2 = -JX^2,\) and the usual Nijenhuis relation implies that also \([\nabla\varphi^1, \nabla\varphi^2] = 0.\) To complete \((\varphi^1, \varphi^2)\) into a coordinate system we first choose a single leaf of the \(\nabla\varphi^1\cdot\nabla\varphi^2\) distribution, assign it coordinates \((\theta_1, \theta_2) = (0, 0),\) and then push \(\theta_1, \theta_2\) along the action created by the \(X^1\cdot X^2\) fields; these are the cyclic or angle coordinates. The coordinates \((\varphi^1, \varphi^2, \theta_1, \theta_2)\) are known as action-angle coordinates; see [4]. The Arnold-Liouville reduction is the map

\[\Phi : M^4 \rightarrow \mathbb{R}^2, \quad \Phi(\varphi^1, \varphi^2, \theta_1, \theta_2) = (\varphi^1, \varphi^2).\]

The locus \(\Sigma^2 = \Phi(M^4)\) in the \(\varphi^1\cdot\varphi^2\) plane is called the reduction of \(M^4.\) If \(M^4\) is compact, it is well-known that \(\Sigma^2\) is a closed convex polygon; see [7]. Because \(\varphi^1\) and \(\varphi^2\) are invariant under the action of the fields, the reduction \(M^4 \rightarrow \Sigma^2\) is generically a Riemannian submersion. Therefore the interior of \(\Sigma^2\) inherits a Riemannian metric, \(g_\Sigma.\) We call \((\Sigma^2, g_\Sigma)\) the metric reduction. In the non-compact...
case, it is generally not true that $\Sigma^2$ need be closed or be a polygon; see Examples 7.1 and 7.2. However if the Kähler metric on $M^4$ is geodesically complete, we recover the fact that $\Sigma^2$ is convex.

**Proposition 1.1** (cf. Proposition 3.1). Assume $(M^4, J, \omega, X^1, X^2)$ is a geodesically complete toric Kähler manifold. Then its reduction $\Sigma^2$ is convex in the $\varphi^1$-$\varphi^2$ plane.

We use the term “boundary point” of $\Sigma^2$ to refer to a boundary point of $\Sigma^2$ in the topology of the coordinate plane (as opposed to the metric topology on $(\Sigma^2, g_\Sigma)$), and we use $\partial \Sigma^2$ to indicate the set of all such points in the $\varphi^1$-$\varphi^2$ plane. We can then make the distinction between “included boundary points,” $(\partial \Sigma^2)_I = \partial \Sigma^2 \cap \Sigma^2$, and “non-included boundary points,” $(\partial \Sigma^2)_N = \partial \Sigma^2 \setminus \Sigma^2$. If $p$ is a point in the $\varphi^1$-$\varphi^2$ coordinate plane and $\epsilon > 0$, we use the notation $D_p(\epsilon)$ to mean the coordinate disk of radius $\epsilon$ around $p$; then $p \in \partial \Sigma^2$ if and only if every $B_p(\epsilon)$ contains both points of $\Sigma^2$ and $\mathbb{R}^2 \setminus \Sigma^2$. The following proposition states that if $p$ is an included boundary point, then nearby the boundary is either an edge or a vertex, so intuitively we retain “polygon-ness” near every included boundary point.

**Proposition 1.2** (Structure of $(\partial \Sigma^2)_I$). The included boundary points $p \in \Sigma^2 \cap \partial \Sigma^2$ are precisely those for which $\Phi^{-1}(p) \subseteq M^4$ consists of points where the distribution $\{X^1, X^2\}$ has rank 1 or 0. In this case, a coordinate disk $D_p(\epsilon)$, $\epsilon > 0$ exists so that, if the rank is 1 at $\Phi^{-1}(p)$ then a linear function $m(\varphi^1, \varphi^2)$ exists so that $\Sigma^2 \cap D_p(\epsilon) = \{m \geq 0\} \cap D_p(\epsilon)$, and if it has rank 0 then two linear functions $m_1$, $m_2$ exist so that $\Sigma^2 \cap D_p(\epsilon) = \{m_1 \geq 0\} \cap \{m_2 \geq 0\} \cap D_p(\epsilon)$.

**Proposition 1.3** (Structure of $(\partial \Sigma^2)_N$). If $p \in \partial \Sigma^2 \setminus \Sigma^2$ is a non-included boundary point, then it is infinitely far away in the sense that if $\{p_i\}$ is any sequence of points with $p_i \to p$ in the coordinate topology, then for any $q \in \Sigma^2$ we have $\text{dist}_{M^4}(\Phi^{-1}(q), \Phi^{-1}(p_i)) \to \infty$.

By Proposition 1.2, the included boundary $(\partial \Sigma^2)_I$ consists of segments, rays, and lines in the $\varphi^1$-$\varphi^2$ plane, joined at vertex points, and given such a segment, ray, or line $l \in (\partial \Sigma^2)_I$, then $\Phi^{-1}(l)$ is the zero-set of a Killing field. We call such a locus a polar locus (this is an analogy with the fact that the zero-set of a rotational Killing field on $S^2$ is the north and south poles of $S^2$). See Figure 2 for a depiction. A consequence of the following proposition is that any such locus is an embedded complex submanifold, so we call such a locus a polar submanifold.

**Proposition 1.4** (Polar submanifolds). Assume $l \subset (\partial \Sigma^2)_I$ is a boundary segment, ray, or line that has one or both of its termini at included vertex points or non-included points. Then $L^2 = \Phi^{-1}(l)$ is a holomorphically embedded, geodesically complete codimension-2 submanifold with a Killing field $\mathcal{X}$.

Any terminus of $l$ at a non-included point corresponds to a manifold end of $L^2$. A terminus of $l$ at an included vertex point corresponds to a zero of the Killing field on $L^2$. If both termini of $l$ are on included vertex points, then the polar submanifold $L^2 = \Phi^{-1}(l)$ is a holomorphically embedded $\mathbb{P}^1$.

We give geometric criteria on $M^4$ that forces $\Sigma^2 \subset M^4$ to be closed. First we remark that the Killing field $\mathcal{X}$ on a polar submanifold $L^2$ gives rise to a momentum function $\varphi$ in the usual way: $d\varphi = i\mathcal{X}\omega$. Then using $\varphi$ we have a standard construction for a distance function $r$ (see [50]), that obeys $dr = |d\varphi|^{-1}d\varphi$. The trajectories of $\nabla r$ are then perpendicular to the Killing field $\mathcal{X}$, and we call such a distance function a radial distance function.
Theorem 1.5 (Criteria for closedness of $\Sigma^2$). Assume that, whenever $L^2 \subset M^4$ is a polar submanifold that has an unbounded radial distance function $r$, one of the following holds:

- The action field $X$ on $L^2$ decays slowly (or not at all): a constant $C_1 > 0$ exist so when $|r|$ is sufficiently large, then $|X| \geq C_1 |r|^{-1}$,
- The negative part of the Gaussian curvature $K$ of $L^2$ decays quickly: when $|r|$ is sufficiently large, then $K \geq -2|r|^{-2}$.

Then the closure of every ray or segment in $\partial \Sigma^2$ consists of included points.

In particular, if every component of $\partial \Sigma^2$ contains at least one point of $(\partial \Sigma^2)_I$, then $\Sigma^2$ is closed.

We remark that, because $\Sigma^2$ is a convex subset of the plane, $\partial \Sigma^2$ is disconnected if and only if it has two components, each being a line. Then $\Sigma^2$ is a closed strip.

Without further conditions on a toric 4-manifold, Theorem 1.5 is as far as we can go. However if $M^4$ is scalar-flat, combining Theorem 1.5 with the classification of [21] allows us to classify all such manifolds whose reductions $\Sigma^2$ are closed. First we show that if the $\Sigma^2$ contains no edges and the scalar curvature of $M^4$ is non-negative, then $M^4$ is flat.

Theorem 1.6 (c.f. Theorem 4.4). Assume $(M^4, J, \omega, X^1, X^2)$ has non-negative scalar curvature $\kappa \geq 0$ and that the distribution $\{X^1, X^2\}$ is always rank 2 (this is the same as $M^4 \to \Sigma^2$ being a Riemannian submersion, which is the same as $\Sigma^2$ being an open subset of the plane).

Then the Riemannian manifold $(M^4, g)$ is flat $\mathbb{R}^4$, the two Killing fields are translational fields, and $(\Sigma^2, g_{\Sigma^2})$ is flat $\mathbb{R}^2$.

The following theorem gives necessary and sufficient conditions on a scalar-flat $(M^4, J, \omega, X^1, X^2)$ for its reduction $\Sigma^2$ to be closed.

Theorem 1.7. Assume $(M^4, J, \omega, X^1, X^2)$ is a simply-connected toric Kähler 4-manifold so that the corresponding Riemannian manifold $(M^4, g)$ is scalar-flat, geodesically complete, and has finite topological type.

Then its reduction $\Sigma^2$ is closed if and only if $M^4$ is one of the following:

1. $(M^4, g)$ is ALE but not flat,
2. $(M^4, g)$ is ALF or ALF-like,
3. $(M^4, g)$ is one-ended and asymptotically equivariant $\mathbb{R}^2 \times S^2$,
4. $(M^4, g)$ is asymptotically exceptional,
5. $(M^4, g)$ is an exceptional half-plane instanton of [21],
6. the reduction $\Sigma^2$ is a closed strip, or
7. $(M^4, g)$ is flat.

Case (3) occurs if and only if the polygon reduction $\Sigma^2$ has parallel rays, but is not a strip; see Figure 1.4 for example. Case (7) includes the case of Theorem 1.6 that distribution $\{X^1, X^2\}$ has rank 2 everywhere. In all cases except (6), $M^4$ is one-ended. In case (6) the manifold $M^4$ is two-ended, and is the only case that was not classified in [21].

Cases (1)-(4) require an explanation of the terms “ALE”, “ALF”, “ALF-like”, “asymptotically exceptional,” and “asymptotically equivariantly $\mathbb{R}^2 \times S^2$.” A manifold end is any connected component of $\Omega = M^4 \setminus B$ where $M^4$ is complete and $B$ is a compact subset. Let $\mathbb{R}^+ \times S^3$ have a flat metric of the form $g_E = dr^2 + r^2 g_{S^3}$ where $g_{S^3}$ is the round metric on $S^3$. A manifold end $\Omega$ is ALE if its universal cover

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Ω is diffeomorphic to \( \mathbb{R}^+ \times S^3 \) by some diffeomorphism \( \pi : \mathbb{R}^+ \times S^3 \to \Omega \) with the property that
\[
|\pi^* g_\Omega - g_E| = o(1) \quad \text{and} \quad |\text{Rm}_\Omega| = o(r^{-2}).
\]

Referring to the ALE-ALF-ALG-ALH schema of [6], [10], an ALF end must also have a diffeomorphism \( \pi : \Omega \to \mathbb{R}^+ \times S^3 \). Letting \( p : S^3 \to S^2 \) be a Riemannian projection generated by a circle-action with fiber \( F = S^1 \), we have the model ALF metric \( g_F = dr^2 + g_f + r^2 p^* g_b \) where \( g_f \) is an \( S^1 \)-invariant metric on the fiber and \( g_b \) is the round metric of Gaussian curvature 4 on the base \( S^2 \). Then the manifold end is ALF if and only if \( \pi \) can be chosen so
\[
|\pi^* g_\Omega - g_F| = o(1) \quad \text{and} \quad |\text{Rm}_\Omega| = o(r^{-2}).
\]

As observed in [23], the ALF model is insufficient for describing scalar-flat manifold ends with cubic volume growth and one collapsing direction. There, it was found that the “chiral Taub-NUTs,” a set of scalar-flat Kähler metrics discovered by Donaldson [9], had strictly quadratic curvature decay \( |\text{Rm}| = O(r^{-2}) \), cubic volume growth, and a Killing field that is asymptotically stable. We adopt the following definition.

**Definition.** A manifold end \((\Omega, g_\Omega)\) is called ALF-like if four conditions hold: a diffeomorphism \( \pi : \mathbb{R}^+ \times S^3 \to \Omega \) exists, curvature decay on \( \Omega \) is quadratic \( |\text{Rm}| = O(r^{-2}) \), volume growth is cubic \( \text{Vol}(B_r(p) \cap \Omega) = O(r^3) \) for any point \( p \in M^4 \), and there is a Killing field \( X \) on \( \Omega \) whose norm is asymptotically stable: along any sequence of points \( p_i \) that diverges to infinity in \( \Omega \), the norm \( |X(p_i)| \) is bounded away from 0 and \( \infty \).

When \( S^2 \) has parallel rays and is closed (but is not the strip) then [21] classifies the metrics on the parent manifolds \((M^4, g)\); these are the model geometries for the asymptotically equivariantly \( \mathbb{R}^2 \times S^2 \) manifolds.

**Definition.** We say that a manifold end \((\Omega, g_\Omega)\) is equivariantly asymptotically \( \mathbb{R}^2 \times S^2 \) provided \( \Omega \) has a pair of commuting Killing fields, and a diffeomorphism exists of the form \( \pi : (\mathbb{R}^2 \setminus B) \times S^2 \to \Omega \), where \( B \subset \mathbb{R}^2 \) is a ball in \( \mathbb{R}^2 \) with the following conditions: under \( \pi \), one of Killing field \( X^1 \) on \( \Omega \) maps to the rotational field on \( S^2 \), and another Killing field \( X^2 \) maps to the rotational field on \( \mathbb{R}^2 \), and on the locus \( \{X^1 = 0\} \), the norm \( |X^2| \) remains bounded away from zero.

Lastly we define “asymptotically exceptional.” As studied in [23], the generalized Taub-NUT metrics are essentially described by a number \( k \in [-1, 1] \) called its “chirality number.” For \( k = 0 \) the metric is the usual ALF Ricci-flat Taub-NUT metric of Gibbons-Hawking. For \( k \neq \pm 1 \) the metrics are ALF-like. At maximum chirality \( k = \pm 1 \) the metric is no longer ALF-like, but has quartic volume growth, and along some unbounded geodesics (but not all) there is no curvature decay. A manifold end modeled in this way we call an “asymptotically exceptional” end.

**Definition.** Assume \((\Omega, g_\Omega)\) is a manifold end, and let \((\mathbb{R}^4, g_{ex})\) be an exceptional Taub-NUT from [21]. Then \( \Omega \) is called an exceptional manifold end if there is a diffeomorphism \( \pi : \mathbb{R}^4 \setminus B \to \bar{\Omega} \), where \( B \subset \mathbb{R}^4 \) is a ball and \( \bar{\Omega} \) is the universal cover of \( \Omega \), and so that
\[
|\pi^* g_\Omega - g_{ex}| = o(1).
\]

Further, we require that \( \Omega \) has a pair of commuting Killing fields, and that the toric structures on \( \Omega \) and the exceptional Taub-NUT are conjugate to each other in the sense that the Killing fields map to each other under the diffeomorphism.
Remark. This paper should be considered a companion to [21] and [22]. The paper [21] classifies possible metrics on scalar-flat toric Kähler 4-manifolds, but begin with hypotheses on Σ² rather than M⁴. The main assumption of [21] was "Hypothesis A," which requires the reduction Σ² be closed. The primary purpose of this paper is to understand geometric conditions on the parent manifold M⁴ that enforces closure on Σ². This way, the geometry of M⁴ can be understood without the necessity of computing its reduction Σ².

2. Preliminaries and Notation

We begin by recalling some of the theory of Toric Kähler manifolds, and by setting notation for the rest of the paper. For reference see for example [13] [1] [8] [9] [2] [3], and the companion paper [21] which has similar notation. Aside from 2.3 this material is well-known, so the presentation here is relatively brief.

2.1. Properties of the reduction. From \( J \nabla \varphi^i = -\frac{\partial}{\partial \varphi^i} \), in action-angle coordinates \((\varphi^1, \varphi^2, \theta_1, \theta_2)\) we can write

\[
g = \begin{pmatrix} G_{ij} & 0 \\ 0 & G^{ij} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -G^{ij} \\ G_{ij} & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}.
\]

where \( G_{ij} \) and its inverse \( G^{ij} \) are the 2 \( \times \) 2 matrices \( G_{ij} = \langle \frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j} \rangle = \langle \nabla \theta_i, \nabla \theta_j \rangle \), \( G^{ij} = \langle \nabla \varphi^i, \nabla \varphi^j \rangle = \langle \frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j} \rangle \). We have \( \mathcal{X}^1 = \frac{\partial}{\partial \varphi^1} \), \( \mathcal{X}^2 = \frac{\partial}{\partial \varphi^2} \), and the generating functions \( \theta_1, \theta_2 \) of the \( \mathcal{X}^1, \mathcal{X}^2 \) action are pluriharmonic, meaning \( d(Jd\theta_i) = 0 \); see Lemma 2.1 of [21]. Therefore pluriharmonic conjugate functions exist, which we denote \( \xi_1, \xi_2 \), given by \( d\xi_i = Jd\theta_i \). This gives a holomorphic chart \( z_1 = \xi_1 + \sqrt{-1} \theta_1 \), \( z_2 = \xi_2 + \sqrt{-1} \theta_2 \), with a coordinate frame and coframe given by

\[
\frac{\partial}{\partial z_1} = \frac{1}{2} (\nabla \varphi^1 - \sqrt{-1} \mathcal{X}^1), \quad \frac{\partial}{\partial z_2} = \frac{1}{2} (\nabla \varphi^2 - \sqrt{-1} \mathcal{X}^2)
\]

\[
dz_1 = d\xi_1 + \sqrt{-1} d\theta_1, \quad dz_2 = d\xi_2 + \sqrt{-1} d\theta_2
\]

which gives Hermitian metric \( h^{ij} = \langle \frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j} \rangle = \frac{1}{2} G^{ij} \). Because the \( \xi_i \) are \( \mathcal{X}^i \)-invariant, we have \( \xi_i = \xi_i(\varphi^1, \varphi^2) \). Consequently the distributions \( \{\nabla \varphi^1, \nabla \varphi^2\} \) and \( \{\nabla \xi_1, \nabla \xi_2\} \) coincide, and both are integrable (because \( [\nabla \varphi^1, \nabla \varphi^2] \) vanishes).

Because the reduction \( \Phi(\varphi^1, \varphi^2, \theta_1, \theta_2) = (\varphi^1, \varphi^2) \) is \( \mathcal{X}^1, \mathcal{X}^2 \) invariant, this is generically a Riemannian submersion, and we have a metric and complex structure on \( \Sigma^2 \). Abbreviating \( \mathcal{V} = \text{Det}(G^{ij}) = |\nabla \varphi^1|^2 |\nabla \varphi^2|^2 - (\nabla \varphi^1, \nabla \varphi^2)^2 \), we have

\[
g_\Sigma = (G_{ij}) \quad \text{and} \quad J_\Sigma = \mathcal{V}^{-\frac{1}{2}} \left( \begin{array}{cc} \langle \mathcal{X}^1, \mathcal{X}^2 \rangle & -|\mathcal{X}^1|^2 \\ |\mathcal{X}^2|^2 & -\langle \mathcal{X}^1, \mathcal{X}^2 \rangle \end{array} \right).
\]

The metric \( g_\Sigma \) is inherited from the parent manifold \((M^4, g)\), but the complex structure \( J_\Sigma \) is not; rather, it is the (dualization of) the Hodge-\( \ast \) on \((\Sigma^2, g)\), and is obtained from \( J_{\Sigma} J_{\Sigma} = (\text{det} g_{ij}) g_{ik} \epsilon^{jk} \) where \( \epsilon \) is the Levi-Civita symbol. Citing Proposition 2.5 of [21], there is elliptic equations for \( \varphi^1, \varphi^2 \) which can be expressed

\[
d(\mathcal{V}^{-\frac{1}{2}} J_\Sigma d\varphi^k) = 0.
\]

From Proposition 2.2 of [21], the Ricci and scalar curvatures on \( M^4 \) are

\[
\rho = -\sqrt{-1} \partial \bar{\partial} \log \mathcal{V}, \quad s = -\triangle \log \mathcal{V}.
\]
The scalar curvature \( s \) on \( M^4 \) can be computed on \( \Sigma^2 \), where
\[
\Delta_{\Sigma} \nu^{\frac{1}{2}} + \frac{1}{2} s \nu^{\frac{1}{2}} = 0.
\]

This is Corollary 2.4 of [21]. Therefore when the \( M^4 \) scalar curvature is zero there exists a natural analytic coordinate \( z = x + \sqrt{-1}y \) on \( \Sigma^2 \) given by \( y = \sqrt{V} \), and \( x \) is an harmonic conjugate of \( y \). Because \( y = \sqrt{V} \) is a volume, we call the isothermal coordinates \((x, y)\) the \textit{volumetric normal coordinates} and the the analytic coordinate \( z = x + \sqrt{-1}y \) the \textit{volumetric normal function}. From Proposition 1.1 of [21], when \( \Sigma^2 \) is closed and has connected boundary, the map to the upper half-space
\[
z : \Sigma^2 \to \{ y \geq 0 \}
\]
is a biholomorphism with \( \partial \Sigma^2 \) mapping bijectively onto \( \{ y = 0 \} \). We remark that in [23], the Ricci form \( \rho \) and the Weyl tensor on \( M^4 \) were also computed from data on \( \Sigma^2 \). The key component of the Weyl tensor is the Gaussian curvature on \( \Sigma \)
\[
\triangle_{\Sigma} \omega = \left| W_{\Sigma} \right|^2.
\]

The scalar curvature \( s \) of \( \Sigma^2 \) can be computed on \( \Sigma^2 \) where
\[
\Delta_{\Sigma} \nu^{\frac{1}{2}} + \frac{1}{2} s \nu^{\frac{1}{2}} = 0.
\]

2.2. \textbf{The Kähler and symplectic potential.} For their uses in Proposition 3.1 below, it is necessary to describe the Kähler and symplectic potentials on \( M^4 \). Let \( P^2 \subset M^4 \) be the union of the polar submanifolds. Then the distribution \( \{ \lambda^1, \lambda^2 \} \) is 2-dimensional on \( M^4 \setminus P^2 \), so \( \Phi : M^4 \setminus P^2 \to \Sigma^2 \) is a Riemannian submersion onto the interior of \( \Sigma^2 \). Any leaf of the \( \{ \nabla \phi^1, \nabla \phi^2 \} \)-distribution maps bijectively onto \( \Sigma^2 \), and \( M^4 \setminus P^2 \) is diffeomorphic to the product of \( \text{Int}(\Sigma^2) \) with any of the \( \lambda^1, \lambda^2 \) leaves (any such leaf is a torus, a cylinder, or a copy of \( \mathbb{R}^2 \)). Because \( \text{Int}(\Sigma^2) \) is contractable, \( M^4 \setminus P^2 \) has no cohomology on either the first or second levels, and therefore \( \omega = \sqrt{-1} \partial \bar{\partial} F \) for some smooth pseudoconvex function \( F = F(\xi^1, \xi^2) \) called the \textit{Kähler potential}. Because \( F \) is pseudoconvex and the distribution \( \{ \nabla \xi^1, \nabla \xi^2 \} \) is Legendrian—meaning \( \omega(\nabla \xi^i, \nabla \xi^j) = 0, i, j = 1, 2 \)—we have that \( F \) is a convex function of \( \xi^1, \xi^2 \). From
\[
d\phi^i = \omega(\frac{\partial}{\partial \theta^i}, \cdot)\quad \text{and} \quad d\xi_i = J d\theta_i,
\]
\[
\phi^1 = \frac{\partial F}{\partial \xi_1}, \quad \phi^2 = \frac{\partial F}{\partial \xi_2}.
\]

See Eq. (4.4) of [13]. These are precisely the holomorphic transforms usually seen in the setting of the Legendre transform, and indeed \( F = F(\xi_1, \xi_2) \) convex so it has a Legendre transform \( G = G(\phi^1, \phi^2) \), called the manifold’s \textit{symplectic potential}. Referring to the \( 2 \times 2 \) matrix \( G_{ij} \) above, it turns out that \( G_{ij} = \frac{\partial^2 F}{\partial \phi^i \partial \phi^j} \); see [13] for a detailed justification of this fact. The Legendre transform links \( F, G \) and the coordinate systems \((\phi^1, \phi^2), (\xi_1, \xi_2)\) by
\[
F(\xi_1, \xi_2) + G(\phi^1, \phi^2) = \xi_1 \phi^1 + \xi_2 \phi^2.
\]

We remark that, because both \( F(\xi_1, \xi_2) \) and \( \phi^1, \phi^2 \) are defined on \( M^4 \setminus P^2 \), also the convex function \( G(\phi^1, \phi^2) \) is defined on the entirety of \( M^4 \setminus P^2 \). On the reduction \( \Sigma^2 \), \( G \) is a smooth convex function on \( \text{Int}(\Sigma^2) \).

2.3. \textbf{A conformal change of the metric.} We compute the Gaussian curvature \( K_{\Sigma} \) of \( \Sigma^2 \) and create a conformal-change formula [19] that will be used in Section 4. We use “s” to indicate the scalar curvature of \((\Sigma^2, g_{\Sigma})\), and “s” to indicate
the scalar curvature on the parent 4-manifold $(M^4, g)$. Of course $K_\Sigma = \frac{1}{2} s_\Sigma$. The metric on $g_\Sigma = (G_{ij})$ on $\Sigma^2$ has the property

\begin{equation}
\frac{\partial G_{ij}}{\partial \phi^k} = \frac{\partial G_{ik}}{\partial \phi^j}
\end{equation}

due to the fact that $G_{ij}$ is a matrix of second partial derivatives, as described above. Using $V = \det(G^{ij}) = |\nabla \phi^1|^2 |\nabla \phi^2|^2 - (\nabla \phi^1, \nabla \phi^2)^2$, the Christoffel symbols are

\begin{equation}
\Gamma^k_{ij} = \frac{1}{2} \frac{\partial G_{ij}}{\partial \phi^k} G^{sk}, \quad \Gamma^k = G^{sj} \Gamma^k_{sj} = -G^{ks} \frac{\partial}{\partial \phi^s} \log V^2.
\end{equation}

The usual formula for scalar curvature in terms of Christoffel symbols is

\begin{equation}
s_\Sigma = G^{ij} \frac{\partial^2}{\partial \phi^i \partial \phi^j} - G^{ij} \frac{\partial G^{sj}}{\partial \phi^i} + G_{st} \Gamma^t - G^{is} G^{jt} G_{st} \Gamma^k_{ik} \Gamma^l_{jl}.
\end{equation}

Applying (14) to find that $\frac{\partial}{\partial \phi^i} (G^{ij} \Gamma^s_{sj}) - \frac{\partial}{\partial \phi^i} (G^{ij} \Gamma^s_{sj}) = 0$, we have

\begin{equation}
s_\Sigma = G^{is} G^{jt} G_{st} \Gamma^k_{ik} \Gamma^l_{jl} - G_{st} \Gamma^t = |\Gamma^k_{ij}|^2 - |\nabla \log V^2|^2.
\end{equation}

Now modify the metric by $g_\Sigma = V^2 g_\Sigma$. The usual conformal-change formula gives

\begin{equation}
s_\Sigma = V^{-\frac{1}{2}} (s_{\Sigma} - \Delta_{\Sigma} \log V^\frac{1}{2})
\end{equation}

\begin{equation}
= V^{-\frac{1}{2}} \left(|\Gamma^k_{ij}|^2 - |\nabla \log V^\frac{1}{2}|^2 - \Delta_{\Sigma} \log V^\frac{1}{2}\right).
\end{equation}

But $\Delta_{\Sigma} \log V^\frac{1}{2} + |\nabla \log V^\frac{1}{2}|^2 = V^{-\frac{1}{2}} \Delta_{\Sigma} V^\frac{1}{2}$, so along with [9] we see the conformally related scalar curvature is simply

\begin{equation}
s_\Sigma = V^{-\frac{1}{2}} \left(|\Gamma^k_{ij}|^2 - V^{-\frac{1}{2}} \Delta_{\Sigma} V^\frac{1}{2}\right) = V^{-\frac{1}{2}} \left(|\Gamma^k_{ij}|^2 + \frac{1}{2} s\right).
\end{equation}

In particular $s \geq 0$ on $M^4$ implies $s_\Sigma \geq 0$. This is the central observation behind the proof of Theorem 1.6 in Section 4.

3. Structure of $\Sigma^2$ near boundary points

Here we prove Propositions 1.2, 1.3 and 1.4.

3.1. Convexity of $\Sigma^2$. Apriori we do not know that the reduction is convex, although it does supports a convex function, the symplectic potential $G$. In the purely symplectic case, if $M^4$ is compact its reduction is a convex polygon [7] but if $M^4$ is non-compact its reduction need not be either convex or polygonal. If $M^4$ has a complete Kähler metric, we recover the fact that its reduction must be convex.

**Proposition 3.1** (cf. Proposition 1.1). Assume the Kähler metric on $M^4$ is complete. Then its reduction $\Sigma^2$ is convex.

**Proof.** Let $\varphi_0$ and $\varphi_1$ be points within $Int(\Sigma^2)$, and let $\gamma(t) = (1 - t)\varphi_0 + t\varphi_1$, $t \in [0, 1]$ be the line segment between them. Then the 1-variable function $G$ along $l$ is a convex function, and

\begin{equation}
g \left( \frac{d}{dt}, \frac{d}{dt} \right) = \frac{d^2 G}{dt^2}, \quad \text{and} \quad \left| \frac{d}{dt} \right| = \sqrt{\frac{d^2 G}{dt^2}}.
\end{equation}

Equation (20) holds when $\gamma$ is a line segment in the coordinate plane but not in other cases, as a result of the fact that $\frac{d}{ds}$ is a constant-coefficient linear combination
of \( \partial_{\tau^2} \) and \( \partial_\tau \). Assuming the entire line segment lies within \( \text{Int}(\Sigma^2) \), we can bound its length is bounded from above using Hölder’s inequality:

\[
\text{Len}(\gamma) = \int_0^1 \left| \frac{d}{dt} \right| dt = \int_0^1 \left( \frac{d^2 G}{dt^2} \right)^{\frac{1}{2}} dt 
\leq \left( \int_0^1 \frac{d^2 G}{dt^2} dt \right)^{\frac{1}{2}} = \left( \frac{dG}{dt} (\phi_0) - \frac{dG}{dt} (\phi_0) \right)^{\frac{1}{2}}.
\]

Varying the endpoints, (21) implies that as long as \( \phi_0, \phi_1 \) remain within in the interior of \( \Sigma^2 \), then the length of \( \gamma \) remains uniformly finite.

Because \( \text{Int}(\Sigma^2) \) is connected we can vary \( \phi_0, \phi_1 \) throughout \( \text{Int}(\Sigma^2) \). If \( \Sigma^2 \) is non-convex then there exists some point \( p \in \partial \Sigma^2 \) that has a supporting segment \( l \) which lies entirely within \( \Sigma^2 \) and its endpoints lie within \( \text{Int}(\Sigma^2) \). By Proposition 3.4 the segment \( l \) must have infinite length. However (21) gives a uniform upper bound on the length of \( l \). This contradiction shows that \( \Sigma^2 \) is convex.

3.2. Structure near boundary points. In the \( \varphi^1, \varphi^2 \) plane it is convenient to refer to a coordinate disk around a point \( p = (p^1, p^2) \), defined simply by

\[
D_p(r) = \left\{ (\varphi^1, \varphi^2) \in \mathbb{R}^2 \mid \sqrt{(\varphi^1 - p^1)^2 + (\varphi^2 - p^2)^2} < r \right\}.
\]

We remark that \( \Phi^{-1}(D_p(r)) \subseteq M^4 \) is a neighborhood of any \( \bar{p} \in \Phi^{-1}(p) \subseteq M^4 \).

We begin with a basic fact.

**Lemma 3.2.** If \( p \in \Sigma^2 \) then the locus \( \Phi^{-1}(p) \subseteq M^4 \) is a connected submanifold. Its tangent bundle is the distribution \( \{X^1, X^2\} \) restricted to \( \Phi^{-1}(p) \). Finally, \( p \in \text{Int}(\Sigma^2) \) if and only if \( \{X^1, X^2\} \) has rank 2 on \( \Phi^{-1}(p) \).

**Proof.** Because \( X^1, X^2 \) are Killing, the orbit of any point \( \bar{p} \in M^4 \) under \( X^1, X^2 \) is a manifold with tangent bundle being the distribution \( \{X^1, X^2\} \) restricted to the orbit. Because \( \Sigma^2 \) is \( M^4 \) modulo the action of these fields, the image under \( \Phi \) of the orbit of any point \( \bar{p} \in M^4 \) is a single point in \( \Sigma^2 \).

The remaining question is whether a point of \( \Sigma^2 \) has connected or disconnected pre-image. Assume a point \( p \in \Sigma^2 \) has two components \( T_1, T_2 \subseteq \Phi^{-1}(\Sigma^2) \). Let \( \gamma(s), s \in [0,1] \) be a geodesic from \( T_1 \) to \( T_2 \) that represents a shortest path between these submanifolds. Consequently the direction \( \dot{\gamma} \) is perpendicular to \( T_1 \) and \( T_2 \) at the two points of contact; in particular \( \langle \dot{\gamma}, X^1 \rangle = 0 \) for both fields \( X^1, X^2 \). Then

\[
\frac{d^2}{ds^2} \langle X, \dot{\gamma} \rangle = \dot{\gamma} \langle X, \dot{\gamma} \rangle + \langle \nabla_\dot{\gamma} X, \dot{\gamma} \rangle = -\langle \text{Rm}(X, \dot{\gamma}) \dot{\gamma}, \dot{\gamma} \rangle = 0
\]

shows that \( \langle \dot{\gamma}, X \rangle = 0 \) along the entirety of \( \gamma \). In particular \( \dot{\gamma} \in \text{span}\{\nabla \varphi^1, \nabla \varphi^2\} \).

As the distribution \( \{\nabla \varphi^1, \nabla \varphi^2\} \) is both integrable and perpendicular to the \( \{X^1, X^2\} \)-distribution, we see that \( \gamma \) remains within an integral leaf of the \( \{\nabla \varphi^1, \nabla \varphi^2\} \)-distribution; we call this leaf \( \Sigma^2 \). Given \( t \in [0,1] \), draw the line segment in \( \Sigma^2 \)

\[
\eta_s(t) = (1-t)\Phi(\gamma(0)) + t\Phi(\gamma(s)).
\]

This now a surface \( (s,t) \mapsto \eta_s(t) \). Because \( \Phi : \Sigma^2 \rightarrow \Sigma^2 \) is a local homeomorphism, we can lift \( (s,t) \mapsto \eta_s(t) \) to \( \Sigma^2 \), which we call \( (s,t) \mapsto \tilde{\eta}_s(t) \). In particular for all \( s \) we have that \( \tilde{\eta}_s(0) = \gamma(0), \tilde{\eta}_s(1) = \gamma(s) \). Then because \( \gamma \) is a minimizing geodesic we always have

\[
\text{Len}(\gamma([0,s])) \leq \text{Len}(\tilde{\eta}_s([0,1]))
\]
However, because \( \eta_\gamma \) is a straight line segment in the \( \varphi^1, \varphi^2 \) coordinates, \((21)\) states
\[
(26) \quad \text{Len}(\gamma([0, s])) \leq \text{Len}(\eta_\gamma([0, 1])) \leq \left( \frac{dG}{dt}(\eta_\gamma(0)) - \frac{dG}{dt}(\eta_\gamma(1)) \right)^\frac{1}{2}.
\]
By assumption \( \Phi(\gamma(0)) = \Phi(\gamma(1)) \)—this is the assumption that the two components \( T_1, T_2 \) map to a common point in \( \Sigma^2 \). Thus \( \eta_\gamma(0) = \eta_\gamma(1) \), so setting \( s = 1 \) in \((26)\) gives \( \text{Len}(\gamma([0, 1])) = 0 \), an impossibility.

The following proposition is well-known in the case of compact symplectic manifolds; see for example [7], [15]. We give a simple geometrically-flavored proof in the Kähler setting.

**Proposition 3.3** (Structure of \( \Sigma^2 \) near included boundary points). Assume \( p = (p^1, p^2) \) is a point of \( (\partial \Sigma^2)_I \). Then on \( \Phi^{-1}(p) \) the distribution \( \{X^1, X^2\} \) has rank 0 or 1, and a coordinate disk \( D_p(\epsilon) \) exists so that one of the following holds.

1. If the distribution \( \{X^1, X^2\} \) has rank 1, then a linear function \( m(\varphi^1, \varphi^2) = \alpha(\varphi^1 - p^1) + \beta(\varphi^2 - p^2) \) exists so that \( D_p(\epsilon) \cap \Sigma^2 = D_p(\epsilon) \cap \{m \geq 0\} \).

2. If the distribution \( \{X^1, X^2\} \) has rank 0, then two linear functions \( m_0(\varphi^1, \varphi^2) = \alpha_0(\varphi^1 - p^1) + \beta_0(\varphi^2 - p^2) \), \( m_1(\varphi^1, \varphi^2) = \alpha_1(\varphi^1 - p^1) + \beta_1(\varphi^2 - p^2) \) exist so that \( D_p(\epsilon) \cap \Sigma^2 = D_p(\epsilon) \cap \{m_0 \geq 0\} \cap \{m_1 \geq 0\} \).

In particular, \( \Sigma^2 \cap D_p(\epsilon) \) is convex.

**Proof.** First, let \( p \in M^4 \) be any point in \( M^4 \) so that \( \{X^1, X^2\} \) has rank 2. Then the distribution \( \{\nabla \varphi^1, \nabla \varphi^2\} \) also has rank 2, so therefore nearby \( q \) the projection \( \Phi : M^4 \to \Sigma^2 \) is a Riemannian submersion that maps neighborhoods of \( p \) to neighborhoods \( \Phi(p) \). Thus if the rank of the distribution \( \{\nabla \varphi^1, \nabla \varphi^2\} \) is 2, then \( \Phi(p) \notin (\partial \Sigma^2)_I \). Therefore \( \Phi(p) \in (\partial \Sigma^2)_I \) implies the rank is 0 or 1.

Now assume the rank of \( \{X^1, X^2\} \) is 1; this means numbers \( \alpha, \beta \) exist so that the Killing field \( X = \alpha \partial^{1} + \beta \partial^{2} \) vanishes at \( p \). As always, the zero-set \( \{X = 0\} \) of any Killing field is a totally geodesic submanifold of even codimension. Because the rank of the distribution at \( p \) is 1, and because the Killing fields commute, a second non-zero Killing field exists that preserves the zero-set of \( X \), so the zero-set \( \{X = 0\} \) through \( p \) has dimension at least 1. The zero-set \( L^2 = \{X = 0\} \) through \( p \) must have dimension 2 because its codimension is even. Because the field \( X \) is holomorphic, we conclude that \( L^2 \) is also \( J \)-invariant. Therefore \( L^2 \) is holomorphic submanifold. Setting \( \varphi = \alpha \varphi^1 + \beta \varphi^2 \), we have that \( \nabla \varphi = JX^1 = 0 \) on \( L^2 \); this means that \( L^2 \) consists of critical points of the function \( \varphi \). To see that \( L^2 \) is either a max or a min of \( \varphi \), consider any short segment perpendicular to \( L^2 \), and rotate it via the field \( X \)—since \( X \) is zero on \( L^2 \), one endpoint of the segment will remain fixed on \( L^2 \) while the other endpoint creates a circle, and the rotated segment will be a disk. Because \( \varphi \) is also \( X \)-invariant, when restricted to this disk the level-sets of \( \varphi \) are the integral curves of \( X \), which are circles. Because its level-sets are compact, \( \varphi \) either has a max or a min at the disk’s center (as opposed to the disk’s center being a saddle of \( \varphi \)). Along \( L^2 \) the function \( \varphi \) remains constant, so now we see that the entire locus \( L^2 \) is max or a min of \( \varphi \), in some neighborhood of \( L^2 \)—replacing our choice of \( \alpha, \beta \) with \( -\alpha, -\beta \) if necessary, we may assume \( \varphi \) obtain a maximum on \( L^2 \). Finally, from \( \varphi^1(p) = p^1 \) and \( \varphi^2(p) = p^2 \), we have the relation \( \alpha(\varphi^1 - p^1) + \beta(\varphi^2 - p^2) \geq 0 \) in a neighborhood of \( L^2 \), with equality if and only if \( (\varphi^1, \varphi^2) \in L^2 \).
In the case that \( \{ X^1, X^2 \} \) has rank 0, then it is the intersection of two totally-geodesic submanifolds on which the distribution has rank 1. Following the above argument twice, once for each submanifold, leads to two linear relations. □

In the case of complete Kähler manifolds, the reduction \( \Sigma^2 \) might not contain all of its boundary points; see the examples in §7. Intuitively, non-included boundary points should represent “points at infinity,” namely places where the manifold extends infinitely far away although the momentum functions remain bounded. The next proposition verifies this intuition.

**Proposition 3.4** (Structure of \( \Sigma^2 \) near non-included boundary points). Assume \((M^4, J, \omega, X^1, X^2)\) is geodesically complete, and let \( p \in (\partial \Sigma^2)_N \) be a non-included boundary point. Then if \( \{ \tilde{p}_i \} \subset M^4 \) is any sequence of points so that \( \Phi(\tilde{p}_i) \to p \) in the coordinate topology on \( \mathbb{R}^2 \), then the sequence \( \tilde{p}_i \) diverges to infinity in the metric topology on \( M^4 \).

**Proof.** First, if \( N \subset M^4 \) is any compact set, then its image \( \Phi(N) \) is compact in the coordinate topology on \( \mathbb{R}^2 \), by continuity of \( \Phi \). Given such a compact set \( N \subset M^4 \) we show that eventually \( \tilde{p}_i \) leaves \( K \). If not, there is a subsequence (still denoted \( \tilde{p}_i \)) that remains within \( N \) but so that \( \Phi(\tilde{p}_i) \) converges to \( p \) in the \( \mathbb{R}^2 \)-topology. By compactness of \( N \), after passing to a subsequence we have \( \tilde{p}_i \to \tilde{p} \in N \). By continuity \( \Phi(\tilde{p}_i) = p_i \to \Phi(\tilde{p}) \in \Sigma^2 \). But this is impossible because we assumed \( p_i \) converges to a non-included point. □

Non-included boundary points need not be given by linear relations, even if \( M^4 \) is complete. For an example, see §7.1 and Figure 3.

4. Reductions with no edges

In this section we prove Theorem 1.6 that if \( M^4 \) has scalar curvature \( s \geq 0 \) and \( \Sigma^2 \) has no edges, then \( M^4 \) is a flat manifold. We do so in stages. First we show that if \( V \) is constant, then \( \tilde{g}_\Sigma \) is flat. Then we use a classical Liouville theorem to show that \( \Delta_{\Sigma^2} \sqrt{V} \leq 0 \) implies \( V \) is constant. To use the classical Liouville theorem, we must determine that the complete manifold \((\Sigma^2, g_{\Sigma})\) is actually biholomorphic to \( \mathbb{C} \), or, equivalently, that it is parabolic in the sense of potential theory [5].

To do this, we show that if \((\Sigma^2, g_{\Sigma})\) is complete, the conformal change from §2.3 gives a metric \( \tilde{g}_\Sigma \) that actually remains complete, and crucially now gives a manifold \((\Sigma^2, \tilde{g}_\Sigma)\) with non-negative Gaussian curvature. Thus \( \Sigma^2 \) is biholomorphic with \( \mathbb{C} \), by the Cheng-Yau criterion for parabolicity. The classical Liouville theorem then forces \( V \) to be constant, and we conclude that \( g_{\Sigma} \) is a flat metric.

**Lemma 4.1.** Assume \((\Sigma^2, g_{\Sigma})\) is geodesically complete. If \( V \) is constant, then \((\Sigma^2, g_{\Sigma})\) is biholomorphic to \( \mathbb{C} \). Further, it is flat, and indeed \( g_{\Sigma} \) has constant coefficients when expressed in \( \varphi^1, \varphi^2 \) coordinates.

**Proof.** With \( V \) constant, then \( d(V^{-1} J_{\Sigma} d\varphi^k) = 0 \) from §8 is just \( d(J_{\Sigma} d\varphi^k) = 0 \), so \( \varphi^1 \) and \( \varphi^2 \) are harmonic. Therefore each determines a holomorphic function which we can write \( z = \varphi^1 + \sqrt{-1} \eta^1 \), \( w = \varphi^2 + \sqrt{-1} \eta^2 \) (where \( d\eta^k = -J_{\Sigma} d\varphi^k \)). Away
from possible critical points these are each a holomorphic coordinate on \( \Sigma^2 \), and

\[
\frac{d}{dz} = \frac{1}{2|\nabla \varphi_1|^2} (\nabla \varphi_1 + \sqrt{-1} J_{\Sigma} \nabla \varphi^1)
\]

\[
\frac{d}{dw} = \frac{1}{2|\nabla \varphi_2|^2} (\nabla \varphi_2 + \sqrt{-1} J_{\Sigma} \nabla \varphi^2)
\]

and we easily compute the transition function:

\[
\frac{dw}{dz} = g \left( \frac{d}{dz}, \nabla w \right) = \frac{\langle \nabla \varphi_1, \nabla \varphi_2 \rangle}{|\nabla \varphi_1|^2} - \sqrt{-1} \frac{\sqrt{V}}{|\nabla \varphi_1|^2}.
\]

But \( \frac{dw}{dz} \) is holomorphic, so its real and imaginary parts are harmonic, so \( |\nabla \varphi_1|^{-2} \) is harmonic. In the \( z \)-coordinate the Hermitian metric is \( h_{\Sigma} = \left| \frac{d}{dz} \right|^2 = \frac{1}{2} |\nabla \varphi_1|^{-2} \), so \( h_{\Sigma} \) is an harmonic function. Thus using \( \triangle h_{\Sigma} = 0 \) the Gaussian curvature is

\[
K_{\Sigma} = -h_{\Sigma}^{-1} \triangle h_{\Sigma} = 8 |\nabla \varphi_1|^2
\]

which is non-negative, forcing the complete manifold \( (\Sigma^2, g_{\Sigma}) \) to be parabolic—this is due to the Cheng-Yau condition for parabolicity; see [3] or the remark below. Equivalently, the complete manifold \( (\Sigma^2, J_{\Sigma}) \) is biholomorphic to \( \mathbb{C} \). But then the classical Liouville theorem implies \( |\nabla \varphi_1|^{-2} \) is actually constant, because it is an harmonic function on \( \mathbb{C} \) bounded from below. The same argument works to prove \( |\nabla \varphi_2|^{-2} \) is constant. Then the fact that \( V \) is constant and \( V = |\nabla \varphi_1|^2 |\nabla \varphi_2|^2 - \langle \nabla \varphi_1, \nabla \varphi_2 \rangle^2 \) also forces \( \langle \nabla \varphi_1, \nabla \varphi_2 \rangle \) to be constant.

Because \( g_{\Sigma}^{ij} = \langle \nabla \varphi_i, \nabla \varphi_j \rangle \) we have that all components of the metric are constants when measured in \( (\varphi^1, \varphi^2) \) coordinates. Thus the Christoffel symbols are zero, so the Gaussian curvature of \( \Sigma^2 \) is zero. \( \square \)

Lemma 4.1 gives flatness of \( g_{\Sigma} \) provided we somehow know \( V \) is a constant. The conditions under which we know this is clarified below in Lemma 4.4. The next lemma gives a hint: if we the reduction is biholomorphic to \( \mathbb{C} \), then \( V \) is constant.

**Lemma 4.2.** Assume \( (\Sigma^2, g_{\Sigma}) \) is geodesically complete, that \( (\Sigma^2, J_{\Sigma}) \) is biholomorphic to \( \mathbb{C} \), and that \( \triangle_{\Sigma} \sqrt{V} \leq 0 \). Then \( (\Sigma^2, g_{\Sigma}) \) is flat, and the metric \( g_{\Sigma} \) has constant coefficients when expressed in \( \varphi^1, \varphi^2 \) coordinates.

**Proof.** Since the function \( V^{\frac{1}{2}} \) is superharmonic and bounded from below on \( \mathbb{C} \), it is constant. From this, Lemma 4.1 provides the result. \( \square \)

Lemmas 4.1 and 4.2 together show \( V = \text{const} \) if and only if \( (\Sigma^2, J_{\Sigma}) \) is biholomorphic to \( \mathbb{C} \), and either condition implies \( (\Sigma^2, g_{\Sigma}) \) is flat. In Theorem 4.3 we prove biholomorphicity to \( \mathbb{C} \), assuming only geodesic completeness and \( \triangle_{\Sigma} \sqrt{V} \leq 0 \). This is done using the conformal change discussed in \( \underline{2.3} \). First we require a fact about this conformal change.

**Lemma 4.3.** Assume \( (\Sigma^2, g_{\Sigma}) \) is geodesically complete and \( \triangle_{\Sigma} \sqrt{V} \leq 0 \). Setting \( \tilde{g}_{\Sigma} = V^{\frac{1}{2}} g_{\Sigma} \), then \( (\Sigma^2, \tilde{g}_{\Sigma}) \) is also geodesically complete.

**Proof.** Let \( \gamma : [0, R] \to \Sigma^2 \) be a unit-speed geodesic in the \( \tilde{g}_{\Sigma} \) metric that gives a shortest \( g_{\Sigma} \)-distance from some internal point \( p \) to \( \partial \Sigma^2 \). For a contradiction, we show that if \( \gamma \) has finite length in \( \tilde{g}_{\Sigma} \), it also has finite length in \( g_{\Sigma} \).

Set \( p = \gamma(0) \) and \( r = \text{dist}(p, \cdot) \). Certainly \( \gamma \) has infinite length in the \( g_{\Sigma} \) metric, due to its inextendability. Therefore \( V \to 0 \) along \( \gamma \), or else \( \gamma \) would remain infinite
in length in the $\tilde{g}_{\Sigma}$ metric. Our main technical step is to create a lower barrier for $\mathcal{V}$ near the $\tilde{g}_{\Sigma}$-boundary of $\Sigma^2$. To do this we use a Bochner-style Laplacian comparison (a technique appearing in [18]) to obtain information on $\Delta r$. This is available because the curvature of $(\Sigma^2, \tilde{g}_{\Sigma})$ is positive: $\tilde{K}_{\Sigma} = \frac{1}{2} \tilde{s}_\Sigma \geq 0$ by (19).

Then comparing the $\tilde{g}_{\Sigma}$ Laplacian $\tilde{\Delta}_\Sigma$ to the Laplacian on flat Euclidean space, the usual comparison theorems provide $\tilde{\Delta}_\Sigma r \leq \Delta_{\text{Eucl}} r = r^{-1}$. Thus $\tilde{\Delta}_\Sigma r^{-1} \geq r^{-3}$.

Now we create our barrier for $\mathcal{V}$. On the outer annulus boundary $\partial B_p(R)$ the function $(r^{-1} - R^{-1})$ equals 0 and so $\mathcal{V}^{\frac{1}{2}} \geq \epsilon (r^{-1} - R^{-1})$ for any $\epsilon$, and on the inner annulus boundary $\partial B_p(R/2)$ because $\mathcal{V}^{\frac{1}{2}} > 0$ there exists some $\epsilon > 0$ so $\mathcal{V}^{\frac{1}{2}} \geq \epsilon (r^{-1} - R^{-1})$ on the inner boundary as well. Thus by harmonicity of $\mathcal{V}^{\frac{1}{2}}$ and Laplacian comparison, we have

\[
\mathcal{V}^{\frac{1}{2}} - \epsilon (r^{-1} - R^{-1}) \geq 0 \quad \text{on } \partial \mathcal{A} \quad \text{and}
\]

\[
\tilde{\Delta}_\Sigma \left( \mathcal{V}^{\frac{1}{2}} - \epsilon (r^{-1} - R^{-1}) \right) \leq -\epsilon r^{-3} < 0.
\]

Then $\mathcal{V}^{\frac{1}{2}} - \epsilon (r^{-1} - R^{-1})$ is superharmonic on $\mathcal{A}$ and non-negative on the boundary, so $\mathcal{V}^{\frac{1}{2}} \geq \epsilon (r^{-1} - R^{-1})$ on $\mathcal{A}$. Because the path $\gamma$ lies within the annulus $\mathcal{A}$ for $t \in [R/2, R]$ we have

\[
\left( (\mathcal{V} \circ \gamma)(t) \right)^{\frac{1}{2}} \geq \epsilon \left( t^{-1} - R^{-1} \right), \quad \text{which is}
\]

\[
\left( (\mathcal{V} \circ \gamma)(t) \right)^{-\frac{1}{4}} < \epsilon^{-\frac{1}{2}} (t^{-1} - R^{-1})^{-\frac{1}{2}} = \epsilon^{-\frac{1}{2}} (tR)^{\frac{3}{2}} (R - t)^{-\frac{1}{2}}.
\]

This allows us to estimate the $g_{\Sigma}$-length of $\gamma$. From $|\gamma|_{g_{\Sigma}} = \mathcal{V}^{-\frac{1}{2}} |\gamma|_{\tilde{g}_{\Sigma}}$, we have

\[
|\gamma|_{g_{\Sigma}} = (\mathcal{V} \circ \gamma)^{-\frac{1}{2}} \cdot |\gamma|_{\tilde{g}_{\Sigma}} \leq \epsilon^{-\frac{1}{2}} (tR)^{\frac{3}{2}} (R - t)^{-\frac{1}{2}} \cdot |\gamma|_{\tilde{g}_{\Sigma}}.
\]

Because also $|\gamma|_{\tilde{g}_{\Sigma}} = 1$, we get

\[
\text{Length}_{g_{\Sigma}}(\gamma) = \int_{R/2}^{R} |\gamma|_{g_{\Sigma}} dt = \int_{R/2}^{R} (\mathcal{V}^{-\frac{1}{2}} \circ \gamma)(t) dt 
\]

\[
\leq \epsilon^{-\frac{1}{2}} R^{\frac{3}{2}} \int_{R/2}^{R} t^2 (R - t)^{-\frac{1}{2}} dt = \epsilon^{\frac{1}{2}} \frac{2 + \pi}{4} R^2.
\]

This computation shows $\gamma$ has finite length in the $g_{\Sigma}$ metric, whereas our assumption that $(\Sigma^2, g_{\Sigma})$ is complete forces it to have infinite length. This contradiction completes the proof. \hfill \Box

**Theorem 4.4** (cf. Theorem 1.6). Assume $(\Sigma^2, g_{\Sigma})$ is geodesically complete and $\Delta_{\Sigma} \sqrt{\mathcal{V}} \leq 0$. Then the metric polygon $(\Sigma^2, g_{\Sigma})$ and the conformally related metric polygon $(\Sigma^2, \tilde{g}_{\Sigma})$ are flat Riemannian manifolds.

**Proof.** By Lemma 4.3 $(\Sigma^2, \tilde{g}_{\Sigma})$ is also complete, and by (19) $\tilde{K}_{\Sigma} = \frac{1}{2} \tilde{s}_{\Sigma} \geq 0$. Invoking the Cheng-Yau criterion for parabolicity we have that $(\Sigma^2, \tilde{g}_{\Sigma})$ is parabolic, therefore its universal cover is biholomorphically $\mathbb{C}$. Then $\mathcal{V}^{\frac{1}{2}}$ is a positive super-harmonic function on $\mathbb{C}$, so it is constant. Lemma 4.1 now gives the conclusion. \hfill \Box

**Corollary 4.5** (cf. Theorem 1.6). Assume $(M^4, J, \omega, X_1, X_2)$ has $s \geq 0$, and assume the distribution $\{X_1, X_2\}$ is always rank 2. Then $M^4$ is flat $\mathbb{R}^4$ and $\Lambda^1, \Lambda^2$ are translational Killing fields.
Proof. Since $V = 0$ if and only if $\{X_1, X_2\}$ is a linearly dependent set, this manifold’s polytope has no edges. The moment map is therefore a Riemannian submersion everywhere, so the metric polytope $(\Sigma^2, g_{\Sigma^2})$ is complete. By equation (19) we have $\triangle_{\Sigma^2} \sqrt{V} \leq 0$. Theorem 4.4 now implies $g_{\Sigma^2}$ is flat; in fact $g_{\Sigma^2}$ is a constant matrix when expressed in $\varphi^1$-$\varphi^2$ coordinates. In the context of the equations (5), we have that $G$ is a constant matrix, so $g$ and $J$ are constant. Thus $M^4$ is flat.

The fact that $M^4 \approx \mathbb{R}^4$ comes from our earlier assumption that $M^4$ is simply connected. The fact that the Killing fields are translational is due to the fact that they are nowhere zero. □

Remark. Crucial to the proofs of Lemma 4.1 and Theorem 4.4 is the fact that a complete, simply connected $\Sigma^2$ with $K_{\Sigma^2} \geq 0$ is biholomorphic to $\mathbb{C}$. This is a simple consequence of volume comparison and the Cheng-Yau criterion for parabolicity: that $\int_1^\infty \frac{t}{\sqrt{t^2 + 1}} dt = \infty$. The assertion that a simply connected, boundaryless, complete Riemann surface is parabolic if and only if it is actually $\mathbb{C}$ is a consequence of the uniformization theorem. The subject of parabolicity has received a great deal of attention; for a tiny sampling of this large subject see [19] [20] [14] [12] [21] and references therein.

5. Asymptotics of known metrics

This section examines the asymptotics of the metrics classified in [21] with two purposes in mind: we show that along polar submanifolds each has Killing fields that do not decay, and we show that when $\Sigma^2$ has parallel rays, then indeed it is “asymptotically equivariantly $\mathbb{R}^2 \times S^2$” in the sense of the definition from the introduction. The analysis here is elementary, although full details would be excessively tedious. Because of this we have skipped some steps and only presented the outcomes of several long, though completely elementary, computations. Letting $r = \sqrt{(x)^2 + (y)^2}$, these computations all revolve around estimating sums of the form $\sum_{i=1}^k C_i \sqrt{(x - x_i)^2 + (y)^2}$, $x_i \in \mathbb{R}$ as $r \to \infty$. The main computation is that

$$(34) \quad \left( \sum_{i=1}^k C_i \sqrt{(x - x_i)^2 + y^2} \right) - \left( \sum_{i=1}^k C_i \right) \sqrt{x^2 + y^2} = -\frac{x}{r} \sum_{i=1}^k C_i x_i + \frac{y^2}{2r^2} \sum_{i=1}^k C_i x_i^2 + O(r^{-2})$$

In this section we prove the following lemmas.

Lemma 5.1. Assume the reduction $(\Sigma^2, g_{\Sigma^2})$ of $M^4$ is a closed polygon with connected non-empty boundary, and $\Sigma^2$ does not have a line on its boundary. Then the parent manifold $M^4$ has two unbounded polar submanifolds, and each has a Killing field which remains bounded away from zero.

Lemma 5.2. Assume the reduction $(\Sigma^2, g_{\Sigma^2})$ of $M^4$ is a closed polygon with parallel rays, but is not the closed strip. Then $M^4$ is asymptotically equivariantly $\mathbb{R}^2 \times S^2$, and the Killing fields on both polar submanifolds are bounded from below.

5.1. Comparing a given polygon metric $g_{\Sigma^2}$ to an asymptotic model. In this section we compute the asymptotic attributes of the scalar-flat metrics on closed polygons, as classified in [21]. By Theorems 1.3 and 1.4 of that paper, as long as
the polygon is not the closed strip or the half-plane, the momentum functions can be expressed in terms of the volumetric normal coordinates in the form

\[ \varphi^1 = A_1 + B_1 x + \sum_{k=1}^{d-1} C_{1,k} \sqrt{(x-x_k)^2 + y^2} + \alpha_1 y^2 \]

\[ \varphi^2 = A_2 + B_2 x + \sum_{k=1}^{d-1} C_{2,k} \sqrt{(x-x_k)^2 + y^2} + \alpha_2 y^2 \]

(35)

where \( A_1, A_2, B_1, B_2, C_{1,k}, C_{2,k}, x_k \) are constants, and \( \alpha_1, \alpha_2 \) are non-negative constants. Leaving out the half-plane and closed-strip cases, there are two types of closed polygons \( \Sigma^2 \): the case where \( \Sigma^2 \) has non-parallel terminal rays, and the case where \( \Sigma^2 \) has parallel terminal rays. In the case of non-parallel terminal rays, through affine transformation of the \( \varphi^1 \)-\( \varphi^2 \) coordinate plane we can assume the terminal rays lie along the \( \varphi^1 \)- and \( \varphi^2 \)-axes, respectively.

The idea is to create a comparison metric that asymptotically matches the metric on \( \Sigma^2 \). To do this, we create comparison functions \( \tilde{\varphi}^1, \tilde{\varphi}^2 \) that maintain the “parametrization speed” along the terminal rays that is obtained by the original momentum functions \( \varphi^1, \varphi^2 \). As noted in [21], the boundary \( \partial \Sigma^2 \) is precisely the locus \( \{ y=0 \} \). Along \( \{ y=0 \} \) the value \( s = \sqrt{(\tilde{\varphi}^1)^2 + (\tilde{\varphi}^2)^2} \) is a constant on rays and segments; see Section 5 of [21]. This is called the “parameterization speed” or the “label” on that segment. On the terminal rays, where \( y=0 \) and \( x \) is very large or very small, we compute

\[ \varphi^1|_{x>>1,y=0} = \left( B_1 + \sum_{k=1}^{d-1} C_{1,k} \right) x + A_1 - \sum_{k=1}^{d-1} C_{1,k} x_k \]

\[ \varphi^1|_{x<<1,y=0} = \left( B_1 - \sum_{k=1}^{d-1} C_{1,k} \right) x + A_1 + \sum_{k=1}^{d-1} C_{1,k} x_k \]

(36)

\[ s|_{x>>1,y=0} = \sqrt{ \left( B_1 + \sum_{k=1}^{d-1} C_{1,k} \right)^2 + \left( B_2 + \sum_{k=1}^{d-1} C_{2,k} \right)^2 } \]

\[ s|_{x<<-1,y=0} = \sqrt{ \left( B_1 - \sum_{k=1}^{d-1} C_{1,k} \right)^2 + \left( B_2 - \sum_{k=1}^{d-1} C_{2,k} \right)^2 } \]

See [21] for a fuller description of “parameterization speed” and its geometric interpretation. We may take an affine recombination of the \( \varphi^1 \)-\( \varphi^2 \) coordinates so

\[ \varphi^1|_{x>>1,y=0} = \sqrt{2} x, \quad \varphi^2|_{x>>1,y=0} = 0 \]

\[ \varphi^1|_{x<<-1,y=0} = 0, \quad \varphi^2|_{x<<-1,y=0} = \sqrt{2} x \]

(37)

\[ s|_{x>>1,y=0} = \sqrt{2}, \quad s|_{x<<-1,y=0} = \sqrt{2} \]

Consequently, for large \( |x| \) the momentum functions have the same restriction to \( \{ y=0 \} \) as the following comparison functions:

\[ \tilde{\varphi}^1 = \frac{1}{\sqrt{2}} \left( x + \sqrt{x^2 + y^2} \right) + \frac{\alpha_1}{2} y^2 \]

\[ \tilde{\varphi}^2 = \frac{1}{\sqrt{2}} \left( -x + \sqrt{x^2 + y^2} \right) + \frac{\alpha_2}{2} y^2 \]

(38)
Next we consider the case that $\Sigma$ has parallel terminal rays. Again we may apply an affine coordinate change to the $\varphi^{1,2}$ plane so that the two terminal rays occur along $\{\varphi_2 = -1\}$ and $\{\varphi_2 = 1\}$ (see Figure 1 for example). Then the comparison functions are

\begin{align}
\varphi^1 &= \frac{1}{2} \left( \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} \right) + \alpha_1 y^2 \\
\varphi^2 &= \frac{1}{2} \left( -\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} \right).
\end{align}

The image $(\tilde{\varphi}^1, \tilde{\varphi}^2) : \{y \geq 0\} \to \mathbb{R}^2$ is the closed half-strip $\{1 \leq \tilde{\varphi}^1, -1 \leq \tilde{\varphi}^2 \leq 1\}$.

Now we will show, in either case, the comparison $\varphi^1, \varphi^2$ to the functions $\tilde{\varphi}^1, \tilde{\varphi}^2$. In both cases we have

\begin{equation}
\tilde{\varphi}^1 - \varphi^1 \big|_{y=0} = 0, \quad \tilde{\varphi}^2 - \varphi^2 \big|_{y=0} = 0
\end{equation}

for $y = 0$ and $|x|$ sufficiently large. When the polygon has non-parallel rays then

\begin{align}
\varphi^1 - \varphi^1 &= \left( -\frac{1}{\sqrt{2}} + B_1 \right) x - \frac{1}{\sqrt{2}} \sqrt{x^2 + y^2} + \sum_{k=1}^{d-1} C_{1,k} \sqrt{(x-x_k)^2 + y^2} \\
\varphi^2 - \varphi^2 &= \left( \frac{1}{\sqrt{2}} + B_1 \right) x - \frac{1}{\sqrt{2}} \sqrt{x^2 + y^2} + \sum_{k=1}^{d-1} C_{2,k} \sqrt{(x-x_k)^2 + y^2}
\end{align}

where, in order that (40) hold we have $B_1 = \frac{1}{\sqrt{2}}$, $B_2 = -\frac{1}{\sqrt{2}}$, $\sum_{k=1}^{d-1} C_{1,k} = \frac{1}{\sqrt{2}}\frac{2d}{2d+1}$, $\sum_{k=1}^{d-1} C_{2,k} = \frac{1}{\sqrt{2}}\frac{d}{2d+1}$. In the case that the polygon has parallel rays, we divide the $C_{2,k}$ into those that are positive $C'_{2,k}$ and those that are positive $C''_{2,k}$.

\begin{align}
\varphi^1 - \varphi^1 &= \frac{1}{2} \left( \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} \right) + \sum_{k=1}^{d-1} C_{1,k} \sqrt{(x-x_k)^2 + y^2} \\
\varphi^2 - \varphi^2 &= \frac{1}{2} \left( \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} \right) - \sum_{k=1}^{d-1} C_{2,k} \sqrt{(x-x_k)^2 + y^2} + \sum_{k=1}^{d-1} C''_{2,k} \sqrt{(x-x_k)^2 + y^2}.
\end{align}

The boundary-matching criterion (40) forces $\sum_{k=1}^{d-1} C'_{2,k} = \sum_{k=1}^{d-1} C''_{2,k} = \frac{1}{2}$ and $\sum_{k=1}^{d-1} C_{1,k} = 1$.

To compare $\varphi^1 - \varphi^1$ asymptotically, we need only compare expressions of the form $C \sqrt{(x-a)^2 + y^2} - \sum_{k=1}^{d-1} C_k \sqrt{(x-x_k)^2 + y^2}$ as $x^2 + y^2$ get large, where $C = \sum_{k=1}^{d-1} C_k$.

Letting $r = \sqrt{x^2 + y^2}$, the asymptotic expansion as $r \to \infty$ is

\begin{equation}
f(x, y) = C \sqrt{x^2 + y^2} - \sum_{k=1}^{d-1} C_k \sqrt{(x-x_k)^2 + y^2} \\
= \left( \sum_{k=1}^{d-1} C_k x_k \right) x - \frac{1}{2r} \left( \sum_{k=1}^{d-1} C_k (x_k)^2 \right) y^2 + O(r^{-2})
\end{equation}

Taking partial derivatives, we find asymptotically that

\begin{align}
\frac{\partial f}{\partial x} &= \frac{1}{r} \left( \sum_{k=1}^{d-1} C_k x_k \right) \left( \frac{y}{r} \right)^2 + O(r^{-2}) \\
\frac{\partial f}{\partial y} &= \frac{1}{r} \left( \sum_{k=1}^{d-1} C_k x_k \right) \frac{xy}{r^2} + O(r^{-2}).
\end{align}
Noting that both \( \xi \) and \( \zeta \) are bounded as \( r \to \infty \), we see that \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = O(r^{-1}) \). Because the differences \( \tilde{\varphi}^i - \varphi^i \) are both of the form of \( f(x, y) \), we conclude that

\[
(45) \quad \frac{\partial}{\partial x} (\tilde{\varphi}^i - \varphi^i) = O(r^{-1}), \quad \text{and} \quad \frac{\partial}{\partial y} (\tilde{\varphi}^i - \varphi^i) = O(r^{-1}).
\]

Now we compare the metric on \( (\Sigma^2, g_\Sigma) \) to the metric \( g_{\text{comp}} \) created by the comparison momentum functions. By (2.14) of [21] the polygon metric is

\[
(46) \quad g_\Sigma = \frac{1}{y} \text{Det} \left( \frac{\partial (\varphi^1, \varphi^2)}{\partial (x, y)} \right) (dx \otimes dx + dy \otimes dy)
\]

where \( \left( \frac{\partial (\varphi^1, \varphi^2)}{\partial (x, y)} \right) \) is the \( 2 \times 2 \) Jacobian for the transition from \( (x, y) \) to \( (\varphi^1, \varphi^2) \) coordinates. Similarly the comparison metric is

\[
(47) \quad g_{\text{comp}} = \frac{1}{y} \text{Det} \left( \frac{\partial (\tilde{\varphi}^1, \tilde{\varphi}^2)}{\partial (x, y)} \right) (dx \otimes dx + dy \otimes dy)
\]

By (45) we have

\[
(48) \quad \text{Det} \left( \frac{\partial (\varphi^1, \varphi^2)}{\partial (x, y)} \right) = \text{Det} \left( \frac{\partial (\tilde{\varphi}^1, \tilde{\varphi}^2)}{\partial (x, y)} \right) + O(r^{-2}).
\]

Therefore when \( |r| >> 1 \) we see \( |g_\Sigma - g_{\text{comp}}| = O(r^{-2}) \). It is not clear what relationship might hold between \( r \) and the distance function on \( g_{\text{comp}} \), except that rays of \( \Sigma^2 \) correspond to unbounded polar submanifolds. This in terms of the distance function, we can only say

\[
(49) \quad |g_\Sigma - g_{\text{comp}}| = o(1).
\]

In particular the norms of the Killing fields \( |\lambda^i| = |d\varphi^i| \) asymptotically approach the norms of the comparison fields \( |d\tilde{\varphi}^i| \). In particular if the norms \( |d\tilde{\varphi}^i| \) remain bounded away from 0, then the norms \( |d\lambda^i| \) remain bounded away from zero.

5.2. The case that \( \Sigma^2 \) does not have parallel rays. By (5.1) the momentum functions \( \varphi^1, \varphi^2 \) are asymptotically close to

\[
(50) \quad \tilde{\varphi}^1 = \frac{1}{\sqrt{2}} \left( x + \sqrt{x^2 + y^2} \right) + \frac{\alpha_1}{2} y^2, \quad \tilde{\varphi}^2 = \frac{1}{\sqrt{2}} \left( -x + \sqrt{x^2 + y^2} \right) + \frac{\alpha_2}{2} y^2
\]

where \( \alpha_1, \alpha_2 \geq 0 \), and by (47) the comparison metric is

\[
(51) \quad g_{\text{comp}} = \frac{1 + 2 M k x + \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} (dx \otimes dx + dy \otimes dy)
\]

where \( M = \frac{a_1 + a_2}{2\sqrt{2}}, \quad k = \frac{a_1 - a_2}{\alpha_1 + \alpha_2} \). See Equation (2-12) of [23] and the discussion in that paper for further details. The case \( \alpha_1 = \alpha_2 = 0 \) gives \( M = 0 \); otherwise we have the ranges \( M \in (0, \infty) \) and \( k \in [-1, 1] \). The case \( k = -1 \) or 1 is the “exceptional” case, the case \( k = 0 \) is the Taub-NUT case, and the cases \( k \in (-1, 0) \cup (0, 1) \) are the ALF-like cases. We compute

\[
(52) \quad d\tilde{\varphi}^1 = \frac{1}{\sqrt{2}} \left( 1 + \frac{x}{\sqrt{x^2 + y^2}} \right) dx + \frac{y}{\sqrt{2}} \left( 1 + \frac{\sqrt{2} \alpha_1 \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} \right) dy
\]

\[
(52) \quad d\tilde{\varphi}^2 = \frac{1}{\sqrt{2}} \left( -1 + \frac{x}{\sqrt{x^2 + y^2}} \right) dx + \frac{y}{\sqrt{2}} \left( 1 + \frac{\sqrt{2} \alpha_2 \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} \right) dy.
\]
The two terminal rays occur when \( \hat{\varphi} = 0 \) and \( \{ y = 0, x < 0 \} \) where \( \hat{\varphi}^1 = 0 \). To compute a quantity such as \( |d\varphi|^2 \) using a metric of the form \( g = N(dx^2 + dy^2) \) where \( N = N(x, y) \) is a function we have \( |d\varphi|^2 = \frac{1}{N} (|\varphi_x|^2 + |\varphi_y|^2) \). Using (51) we find in the first case that

\[
|d\varphi^1|^2 = \frac{2x}{1 + 2(1 + k)Mx}, \quad |d\varphi^2|^2 = 0, \quad \text{where } y = 0, x > 0
\]

and in the second case

\[
|d\varphi^1|^2 = 0, \quad |d\varphi^2|^2 = \frac{2(-x)}{1 + 2(1 - k)M(-x)}, \quad \text{where } y = 0, x < 0
\]

In all cases except \( k = \pm 1 \) we see that that on either polar submanifold, one of the Killing fields is zero and the other is bounded both away from 0 and \( \infty \), as desired. In the case \( k = \pm 1 \), one polar submanifold has a Killing field bounded away from both 0 and \( \infty \), while the other polar submanifold has a Killing field that grows unboundedly.

5.3. The case the polygon has parallel rays. In this case, letting \( \varphi^1, \varphi^2 \) be the momentum functions, then one of these functions is unbounded and the other is bounded; may assume \( \varphi^2 \) is bounded. We may choose for comparison functions

\[
\varphi^1 = \frac{1}{2} \left( \sqrt{(x - 1)^2 + y^2} + \sqrt{(x + 1)^2 + y^2} \right) + \lambda y^2
\]

\[
\varphi^2 = \frac{1}{2} \left( -\sqrt{(x - 1)^2 + y^2} + \sqrt{(x + 1)^2 + y^2} \right)
\]

where \( \lambda \geq 0 \). Then (47) gives comparison metric

\[
g_{\text{comp}} = \frac{1 + \alpha \left( (1 + x)\sqrt{(x - 1)^2 + y^2} + (1 - x)\sqrt{(x + 1)^2 + y^2} \right)}{\sqrt{(x - 1)^2 + y^2} \sqrt{(x + 1)^2 + y^2}} (dx^2 + dy^2)
\]

We compute

\[
d\varphi^1 = \left( \frac{\frac{1}{2}(-1 + x)}{\sqrt{(x - 1)^2 + y^2}} + \frac{\frac{1}{2}(1 + x)}{\sqrt{(x + 1)^2 + y^2}} \right) dx + \frac{y}{2} \left( \frac{1}{\sqrt{(x - 1)^2 + y^2}} + \frac{1}{\sqrt{(x + 1)^2 + y^2}} + 2\lambda \right) dy
\]

\[
d\varphi^2 = \left( \frac{\frac{1}{2}(-1 + x)}{\sqrt{(x - 1)^2 + y^2}} - \frac{\frac{1}{2}(1 + x)}{\sqrt{(x + 1)^2 + y^2}} \right) dx + \frac{y}{2} \left( \frac{1}{\sqrt{(x - 1)^2 + y^2}} - \frac{1}{\sqrt{(x + 1)^2 + y^2}} \right) dy.
\]

The two terminal rays occur when \( \{ y = 0, x > 1 \} \) and \( \{ y = 0, x < -1 \} \). Along both rays we have

\[
|d\varphi^1|^2 = -1 + x^2, \quad |d\varphi^2|^2 = 0, \quad \text{when } y = 0, \text{ and } x > 1 \text{ or } x < -1.
\]

Letting \( \xi^1, \xi^2 \) be the Killing fields on the parent manifold, because \( |\xi^1| = |d\varphi^1| \), we see that the the terminal rays both have an unbounded Killing field.

Lastly, we wish to verify that the comparison metric \( g_{\text{comp}} \) is indeed asymptotically equivariantly \( \mathbb{R}^2 \times S^2 \), as described in the introduction.

First, from (57) we have that \( \varphi^1 = 0 \) precisely on the locus \( E = \{ y = 0, -1 \leq x \leq 1 \} \). Therefore, removing this locus, the field \( \xi^1 = -J\nabla \varphi^1 \) is never zero, so
creates an $S^1$-factor on the parent $M^4 \setminus E$, and the flow-lines of the $\nabla \tilde{\varphi}_1$ field creates an $\mathbb{R}$-factor. Therefore $M^4 \setminus E \approx (\mathbb{R} \times S^1) \times N^2$. To discover the topology of the $N^2$ manifold, we restrict to a level-set $\{ \varphi_1 = c_1 \}$ and quotient by the $X^1$-action. Then $\tilde{\varphi}_2$ varies within the range $[-1, 1]$, and the expression for $d\tilde{\varphi}_2$ can be used to show that $|d\tilde{\varphi}_2| = 0$ precisely when $y = 0$, which occurs at $\tilde{\varphi}_2 = \pm 1$. Therefore also the Killing field $X_2$ reaches 0 at exactly the two points $\tilde{\varphi}_2 = \pm 1$. Then the submanifold $N^2$ is described by $\tilde{\varphi}_2 = \text{const}$, $\tilde{\varphi}_2 \in [-1, 1]$, and the fact that $\tilde{\nabla}X$ generates a circle-action with exactly two zeros. This displays $N^2$ as a copy of $S^2$.

6. The Geometric Classification

Here we prove Theorem 1.5, the general criteria for closure of $\Sigma^2$. From 3.3 every polar submanifold is a totally geodesic, $J$-invariant embedded surface of revolution, with Killing field $X$. From $J$-invariance, we have that $JX = -\nabla \varphi$ is a function whose trajectories are perpendicular to $X$, and there is distance function $r : L^2 \to \mathbb{R}$ defined implicitly by

$$(59) \quad \nabla r = \frac{\nabla \varphi}{|\nabla \varphi|}, \quad \text{which is} \quad dr = |d\varphi|^{-1}d\varphi \quad \text{or} \quad \frac{\partial \varphi}{\partial r} = |\nabla \varphi| = |X|.$$

6.1. Proof of Theorem 1.5

On an unbounded polar submanifold $L^2$ we check the conditions under which its momentum variable remains finite.

Proposition 6.1 (First Boundary Closure Condition). Let $L^2 \subset M^4$ be a polar submanifold with Killing field $X$, corresponding momentum function $\varphi$, and radial distance function $r : L^2 \to \mathbb{R}$. Assume $r$ is unbounded, and a $C_1 < \infty$ exist so $|X| \geq C_1 |r|^{-1}$ for sufficiently large $r$. Then the image $l \subset \partial \Sigma^2$ of $L^2$ is closed.

Consequently, if this hypothesis holds on all unbounded polar submanifolds, then $(\partial \Sigma^2)_l \text{ is closed. In particular if } \partial \Sigma^2 \text{ is connected and } (\partial \Sigma^2)_l \text{ is non-empty, then } \Sigma^2 \text{ is a closed subset of the plane.}$

Proof. We have that $|\nabla \varphi| = |X| > C_1 |r|^{-1}$ implies

$$(60) \quad \frac{\partial \varphi}{\partial r} = |\nabla \varphi| \geq C_1 |r|^{-1}$$
Increasing $r$, $\varphi$ increasing for large $r$. Integrating, $\varphi(r) \geq C_1 \log |r| + C_2$. Thus if $r$ is unbounded, so is the coordinate $\varphi$. Thus if the metric distance $r$ goes to $\infty$ along an edge, then the edge must also extend infinitely in the coordinate plane. Because a non-included point is located infinitely far away in the metric sense, we conclude that no boundary ray or segment can terminate on a non-included point, as claimed. \hfill \Box

**Proposition 6.2 (Second Boundary Closure Condition).** Let $L^2 \subset M^4$ be a polar submanifold with Killing field $\mathcal{X}$ and corresponding momentum function $\varphi$ and distance function $r: L^2 \to \mathbb{R}$. Assume $r$ is unbounded and the Gaussian curvature $K$ on $L^2$ obeys $K \geq -2r^{-2}$, large $r$. Then the image $l \subset \partial \Sigma^2$ of $L^2$ is closed.

Consequently, if this hypothesis holds on all unbounded polar submanifolds, then $(\partial \Sigma^2)_l$ is closed. In particular if $\partial \Sigma^2$ is connected and $(\partial \Sigma^2)_l$ is non-empty, then $\Sigma^2$ is a closed subset of the plane.

**Proof.** The Gaussian curvature of a surface of revolution is $K = -\frac{1}{|\mathcal{X}|} \frac{\partial^2}{\partial r^2}|\mathcal{X}|$. By assumption we have $K > -Cr^{-2}$. For convenience we write $C = k(k-1)$ for some number $k > 1$. Then

\begin{equation}
\frac{\partial^2}{\partial r^2} |\mathcal{X}| = -K \cdot |\mathcal{X}| \leq k(k-1)r^{-2} \cdot |\mathcal{X}|
\end{equation}

Because $|\mathcal{X}| \geq 0$, we can use (61) to bound $|\mathcal{X}|$ from below in the usual way. For a brief explanation of this, notice the operator $\mathcal{L} = \frac{\partial^2}{\partial r^2} - k(k-1)\frac{1}{r^2}$ is a second order elliptic operator in dimension 1, and (61) shows that $\mathcal{L}(|\mathcal{X}|) \leq 0$ so $|\mathcal{X}|$ is a supersolution for this operator. Then letting $f$ be any solution of $\mathcal{L}(f) = 0$ so that at endpoints we have $f(r_0) \leq |\mathcal{X}(r_0)|$, $f(r_1) \leq |\mathcal{X}(r_1)|$ then $f(r) \leq |\mathcal{X}(r)|$ on the interval $r \in [r_0, r_1]$.

The general solution of $\mathcal{L}(f) = 0$ is $f(r) = C_1 r^k + C_2 r^{1-k}$. We can always choose $C_1$, $C_2$ so $f(r_0) = |\mathcal{X}(r_0)|$ and $f(r_1) = 0$; then $f(r) \leq |\mathcal{X}|$. Using the fact that $k > 1$, sending $r_1 \to \infty$ gives $f(r) = r^{k-1} |\mathcal{X}(r_0)| \cdot r^{1-k}$ and we have

\begin{equation}
|\mathcal{X}(r)| \geq \frac{1}{r_0^{1-k}} |\mathcal{X}(r_0)| \cdot r^{1-k} \text{ on } r \in [r_0, \infty).
\end{equation}

When $k \leq 2$ we have $|\mathcal{X}| \geq C r^{-1}$ for large $r$. Because we abbreviated $C = k(k-1)$, the condition $1 < k \leq 2$ are the same as $0 < C \leq 2$. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{polar_submanifold.png}
\caption{Depiction of a polar submanifold $L^2$ over a segment $l$, with the possibility that $l$ terminates on a non-included point.}
\end{figure}
Note that the first equation of (61) can be written $\frac{d^3 \phi}{dr^3} + K \frac{d\phi}{dr} = 0$. Thus for this theorem to be true, it is only necessary that a non-constant solution of this third-order ODE be bounded.

6.2. Proof of Theorem 1.7. The theorem states the reduction $\Sigma^2$ is a closed polygon in the coordinate plane if and only if the parent 4-manifold $M^4$ is one of the cases (1)-(7). The first direction follows from the analytic classification in [21], along with the asymptotic considerations of Section 5 above.

**Proposition 6.3** (c.f. Theorem 1.7). Let $(M^4, J, \omega, \alpha^1, \alpha^2)$ be a complete scalar-flat toric Kähler manifold. If its reduction $\Sigma^2$ is closed, then $M^4$ is one of (1)-(7) of Theorem 1.7.

**Proof.** Because $\partial \Sigma^2$ is closed we have $\partial \Sigma^2 = (\partial \Sigma^2)_I$, and we have three cases: $\partial \Sigma^2 = \emptyset$, $\partial \Sigma^2$ is disconnected, and $\partial \Sigma^2$ has one component.

If $\partial \Sigma^2$ is disconnected then by the convexity of $\Sigma^2$ its boundary must consist of two lines in the $\varphi^1$-$\varphi^2$ plane, so $\Sigma^2$ is the closed strip; this is case (6). In this case nothing further can be said, as this cases remains geometrically unclassified. In the case that $\Sigma^2$ has no boundary then Theorem 3.3 states the distribution $\{\alpha^1, \alpha^2\}$ on the 4-manifold has rank 2 everywhere, so Theorem 1.6 states $M^4$ is flat.

Last is the case $\partial \Sigma^2$ has one component. Then its metric $g_{\Sigma^2}$ is one of those given in the classification of [21]. This classification is achieved first by creating the normal biholomorphism $z : \Sigma^2 \to \{y \geq 0\}$ onto the upper half-plane, then by classifying the possible momentum functions $\varphi^i = \varphi^i(x, y)$ on the upper half-plane. Once the momentum functions are classified, formula (46) gives $g_{\Sigma^2}$, from which we obtain the metric $g$ on $M^4$.

We verify that each of the classified cases in [21] falls into one of the categories (1)-(5). The classification of [21] has three cases: the boundary has two rays that are not parallel, the boundary has two parallel rays, and the boundary is a line.

In the case that $\Sigma^2$ does not have parallel rays (Theorem 1.3 of [21]), the computations of Subsection 5.2 compare the asymptotic structure of the metric to an asymptotic structure determined by the comparison variables $\tilde{\varphi}^1, \tilde{\varphi}^2$ of [23]. The asymptotic structure of this comparison pair was studied extensively in [23], where it was determined that such a metric was flat (in the case $\alpha_1 = \alpha_2 = 0$, ALF (in the case $\alpha_1 = \alpha_2 > 0$), ALF-Like (in the case $\alpha_1, \alpha_2 > 0$) or exceptional (in the case $\alpha_1 = 0$ or $\alpha_2 = 0$, but not both). See Theorems 1.1, 1.2 and 1.3 of [23].

In the case that $\Sigma^2$ has parallel rays, a thorough asymptotic description has yet to be created, but the analysis of Subsection 5.3 gives that any such metric has asymptotic structure determined by the comparison functions $\tilde{\varphi}^1, \tilde{\varphi}^2$ of [55]. In Subsection 5.3 it was verified that the manifold end determined by these functions is asymptotically equivariantly $\mathbb{R}^2 \times S^2$, as defined in the introduction.

The final case is that $\partial \Sigma^2$ is a line then $\Sigma^2$ is a closed half-plane. By Theorem 1.5 of [21] the metric on $M^4$ is either flat $\mathbb{R}^4$ with one rotational and one translational field (case (7)) or else $M^4$ is an “exceptional half-plane instanton” (case (6)).

For the other direction we must show that if $(M^4, J, \omega, \alpha^1, \alpha^2)$ obeys any of the geometric conditions (1)-(7), then its reduction $\Sigma^2$ is closed.

**Proposition 6.4** (c.f. Theorem 1.7). Let $(M^4, J, \omega, \alpha^1, \alpha^2)$ be a complete scalar-flat toric Kähler manifold that satisfies one of (1)-(7) of Theorem 1.7. Then its reduction $\Sigma^2$ is closed.
Proof. Cases (5), (6), and (7) are immediate, as the reduction in case (5) is the closed half-plane (as studied in [23]), the reduction in case (6) is assumed to be the closed strip, and the flat case (7) occurs if the reduction is either \( \mathbb{R}^2 \) (when both Killing fields are translational), or a closed half-plane in \( \mathbb{R}^2 \) (when one Killing fields is translational and the other is rotational), or a closed quarter-plane in \( \mathbb{R}^2 \) (when both Killing fields are rotational).

We examine cases (1)-(4) individually. The strategy is to verify that along polar submanifolds we must have \( |\mathcal{X}| \) bounded from below, in which case Proposition 6.1 states the polygon must be closed. The first case is the ALE case, where the comparison metric is \( g_E = dr^2 + r^2 g_{S^2} \). Because the Killing fields preserve the metric \( g \), which is close to the metric \( g_E \) in the sense that \( |\pi^* g - g_E| = o(1) \) for some diffeomorphism \( \pi \), the push-forward fields on \( g_E \) must asymptotically approach Killing fields on the \( S^3 \) factor. Because the Killing fields commute, their integrated action must form a torus within the symmetry group of \( S^3 \), and a diagonal of that torus produces a Killing field without zeros on \( S^3 \). This completes case (2).

For cases (3) and (4), the definitions of “asymptotically exceptional” and “asymptotically equivariantly \( \mathbb{R}^2 \times S^2 \) includes the requirement that a Killing field exists that is bounded away from zero. Therefore Proposition 6.1 holds in these two cases as well.

7. Examples

We give several examples demonstrating the failure of Theorems 1.3, 1.4, and 1.6 when one or another of the hypotheses are violated.

7.1. A non-polygon example. Consider the function

\[
G(\varphi^1, \varphi^2) = -\frac{1}{2} \log \left( 1 - (\varphi^1)^2 - (\varphi^2)^2 \right)
\]

defined on the open disk \( \Sigma^2 = \{ (\varphi^1)^2 + (\varphi^2)^2 < 1 \} \). Interpreting this as a symplectic potential, we obtain metric \( g_{\Sigma,ij} = G_{ij} \, d\varphi^i \otimes d\varphi^j \) where \( G_{ij} \) is given by \( G_{ij} = \frac{\partial^2 G}{\partial \varphi^i \partial \varphi^j} \). Explicitly,

\[
G_{ij} = \begin{pmatrix}
\frac{1+(\varphi^1)^2-(\varphi^2)^2}{1-(\varphi^1)^2-(\varphi^2)^2} & \frac{2\varphi^1\varphi^2}{1-(\varphi^1)^2-(\varphi^2)^2} \\
\frac{2\varphi^1\varphi^2}{1-(\varphi^1)^2-(\varphi^2)^2} & \frac{1-(\varphi^1)^2-(\varphi^2)^2}{2\varphi^1\varphi^2}
\end{pmatrix}
\]

Then \( (\Sigma^2, g_{\Sigma}) \) is a surface of revolution. Considering a path of the form \( \gamma(t) = (at, bt) \) where \( a^2 + b^2 = 1 \), then \( |\gamma| = \sqrt{2 + \frac{t^2}{at^2}} \) and we see that \( \int_0^1 |\gamma| dt = \infty \), so this surface is complete. Then \( (\Sigma^2, g_{\Sigma}) \) is the Arnold-Liouville reduction of a complete toric Kähler 4-manifold on \( \Sigma^2 \times \mathbb{T}^2 \), whose metric is \( g = G_{ij} d\varphi^i \otimes d\varphi^j + G^{ij} d\theta_i \otimes d\theta_j \).
The corresponding 4-manifold has everywhere rank-2 distribution \( \{X^1, X^2\} \). See Figure 3. The usual formula for the curvature tensor in holomorphic coordinates is

\[
\Omega_j^i = \sqrt{-1} \partial \left( h^{i\bar{j}} \bar{g} \right).
\]

Using the complex coordinates of (6) we have \( h^{i\bar{j}} = \frac{1}{2} G^{i\bar{j}} \) and \( \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \nabla \varphi \), so

\[
\Omega_j^i = \frac{1}{4} \sqrt{-1} \nabla \varphi^k (G^{i\bar{k}} \nabla \varphi^j (G_{\bar{k}s}) \right) dz_k \wedge d\bar{z}_i.
\]

We can compute scalar curvature \( R \) either using the Abreu equation, Equation (10) of [1] which is \( R = -\frac{1}{2} \sqrt{g_{\varphi \varphi}} \), or else using (66) to find that

\[
R = 4 \left( \frac{2}{(1 + (\varphi^2)^2)} \right) \frac{2 - 4(\varphi^2)^2 - 3(\varphi^4) + (\varphi^6)}{(1 + (\varphi^2)^2)^3}
\]

where \( \varphi = \sqrt{(\varphi^1)^2 + (\varphi^2)^2} \) is the radial coordinate. We see \( R \) is not signed; for example \( R = +8 \) at \( \varphi = 0 \) and decreases to \(-3\) at the manifold edge at \( \varphi = 1 \); compare to Theorem 1.6. A tedious but straightforward check shows the norm of curvature is

\[
* (\Omega_j^i \wedge \Omega_l^j) = 32 \left( 3 - 10(\varphi^2)^2 + 29(\varphi^4)^2 + 19(\varphi^6) + 6(\varphi^8) + 12 \right)
\]

and we see that curvature remains bounded on \( M^4 \).

### 7.2. Polygons with scalar-flat and extremal Kähler metrics

Computing metric polygons \( \Sigma^2 \) is especially simple in the case that the 4-manifold has a cohomogeneity-one action of \( U(2) \). Choosing an appropriate radial coordinate, in the Kähler case any such metric is given on some domain within \( \mathbb{R} \times S^3 \) by

\[
g = e^{-r} \left( \frac{1}{4F} (dr)^2 + F \left( \sigma^1 \right)^2 + \left( \sigma^2 \right)^2 + \left( \sigma^3 \right)^2 \right)
\]

where \( F = F(r) \) is any positive function, and \( \{\sigma^1, \sigma^2, \sigma^3\} \) are the left-invariant forms on \( S^3 \) normalized so \( d\sigma^i = -\epsilon^i_{j\ell} \sigma^j \wedge \sigma^k \). The corresponding (1,1) form is

\[
\omega = \frac{1}{2} e^{-r} dr \wedge \sigma^1 + e^{-r} \sigma^2 \wedge \sigma^3
\]

From \( d(\sigma^2 \wedge \sigma^3) = 0 \) and \( d\sigma^1 = -2\sigma^2 \wedge \sigma^3 \), we easily verify \( d\omega = 0 \). The polar coordinates on \( \mathbb{R} \approx C^2 \) are

\[
(r, \psi, \theta, \varphi) \mapsto \left( r \cos(\theta/2) e^{-\frac{i}{2}(\psi + \varphi)}, r \sin(\theta/2) e^{-\frac{i}{2}(\psi - \varphi)} \right)
\]

and \( \psi, \varphi \) parameterize a torus action: the coordinate ranges are \( |\psi - \varphi| < 2\pi \), \( |\varphi| < \pi \) which covers the Clifford torus \( (e^{-\frac{i}{2}(\psi + \varphi)}, e^{-\frac{i}{2}(\psi - \varphi)}) \) exactly once. The coordinate range for \( \theta \) is \( \theta \in [0, \pi] \). The fields \( \frac{\partial}{\partial \psi} \) and \( \frac{\partial}{\partial \varphi} \) are Killing fields. To translate this to the \( (\varphi^1, \varphi^2, \theta_1, \theta_2) \) notation of the Arnold-Liouville reduction, we select angle coordinates \( \theta_1 = \frac{1}{2}(\psi - \varphi), \theta_2 = \psi \). These have ranges \( \theta_1, \theta_2 \in [-\pi, \pi] \), so \( \theta_1, \theta_2 \) parameterize a standard torus. The Killing fields are

\[
X^1 = \frac{\partial}{\partial \theta_1} = 2 \frac{\partial}{\partial \psi}, \quad X^2 = \frac{\partial}{\partial \theta_2} = \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \varphi}.
\]
Before calculating the conjugate momenta, we record the left-invariant forms:

\[ \sigma^1 = \frac{1}{2}(d\psi + \cos(\theta)d\phi), \quad \sigma^2 = \frac{1}{2}(\sin(\psi)d\theta - \cos(\psi)\sin(\theta)d\phi), \]
\[ \sigma^3 = \frac{1}{2}(\cos(\psi)d\theta + \sin(\psi)\sin(\theta)d\phi). \]

One also sees \( \sigma^2 \wedge \sigma^3 = \frac{1}{4}\sin(\theta)(d\theta \wedge d\phi). \) Putting the fields \( X^1, X^2 \) into (70),

\[ \omega \left( \frac{\partial}{\partial \theta^1}, \cdot \right) = \frac{1}{2}e^{-r} dr = d \left( -\frac{1}{2}e^{-r} \right) \]
\[ \omega \left( \frac{\partial}{\partial \theta^2}, \cdot \right) = -\frac{1}{2}e^{-r}(1 + \cos \theta) dr - \frac{1}{4}e^{-r}\sin \theta d\theta = d \left( \frac{1}{2}e^{-r}(1 + \cos \theta) \right) \]

so we may take the momentum functions to be

\[ \varphi^1 = \frac{1}{2}(1 - e^{-r}), \quad \varphi^2 = \frac{1}{2}e^{-r}(1 + \cos \theta). \]

Now we consider some examples from [16]. Referring to (69), the modified Taub-NUT of the first and second kinds and the modified Taub-bolt of the first and second kinds are given respectively by

\[ F(r) = (1 - e^r)^2, \quad r \in (-\infty, 0), \]
\[ F(r) = (1 - e^{-r})^2, \quad r \in (0, \infty), \]
\[ F(r) = 1 + \frac{9}{8}e^{-2r} - \frac{9}{4}e^{-r} + \frac{1}{4}e^r - \frac{1}{8}e^{2r}, \quad r \in (0, \log 3) \]
\[ F(r) = 1 - \frac{1}{8}e^{-2r} + \frac{9}{4}e^{-r} - \frac{9}{4}e^r + \frac{9}{8}e^{2r}, \quad r \in [-\log 3, 0). \]

To create the corresponding polygons, one computes the image of \((\varphi^1, \varphi^2)\) of (75) under any of the four coordinate ranges for \( r \) along with \( \theta \in [0, \pi] \). See Figure 3.

The modified Taub-NUT of the first kind is a scalar-flat 2-ended manifold, with one ALE end and one cusp-like end. In Figure 3, the ALE end corresponds to the unbounded end of the polygon and the cusp-like end corresponds to the dotted line segment. Its underlying complex manifold is \( \mathbb{C}^2 \setminus \{(0,0)\} \). This metric was written down first in [14], then a second time using very different methods in [16]. The modified Taub-NUT of the second kind is extremal Kahler and 1-ended with underlying complex manifold \( \mathbb{C}^2 \). Its end is cusp-like, and its scalar curvature is \( R = 48(1 - e^{-r}) \), which is positive.

The modified Taub-bolt of the first kind is a complete extremal Kahler metric on the total space of \( O(-1) \) (the same total space inhabited by the Burns metric, which is scalar-flat Kahler and asymptotically Euclidean). Its end is cusp-like and its scalar curvature is \( R = 54(1 - e^{-r}) \) which is positive. The modified Taub-bolt of the second kind is a complete, extremal Kahler metric on the total space of \( O(+1) \); its end is cusp-like. Notably it has a holomorphic \( \mathbb{P}^1 \) of positive self-intersection, a rarity for complete manifolds. It has scalar curvature \( R = 6(-1 + e^{-r}) \), which is also positive. Referring to the Figure 3, the fact that the \( \mathbb{P}^1 \)-divisor has positive self-intersection is reflected in the fact that both vertices of the complete segment make acute angles (that self-intersection numbers of the polar submanifolds can be read off from \( \Sigma^2 \) is a fact we shall not explore here). Dashed line segments in the figure correspond to cusp-like ends.

Motivated by these examples, we close the paper with two conjectures.
Figure 3. From left to right: The reduction \((\Sigma^2, g)\) of Example 7.1; the modified Taub-NUT of the first kind on \(\mathbb{C}^2 \setminus (0, 0)\), which is scalar-flat and 2-ended; the modified Taub-NUT of the second kind on \(\mathbb{C}^2\), which is extremal and 1-ended; the modified Taub-bolt of the first kind on \(O(-1)\), and of the second kind on \(O(+1)\), both of which are extremal and 1-ended.

**Conjecture 1.** Assume \((M^4, J, \omega, \lambda^1, \lambda^2)\) is a complete scalar-flat toric Kähler manifold. If \(M^4\) is one-ended, then its reduction \(\Sigma^2\) is closed.

**Conjecture 2.** Assume \((M^4, J, \omega, \lambda^1, \lambda^2)\) is a complete toric Kähler manifold. If \(M^4\) is extremal Kähler, then the closure of its reduction is a polygon.

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