Crowd effects and volatility in a competitive market

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Abstract

We present analytic and numerical results for two models, namely the minority model and the bar-attendance model, which offer simple paradigms for a competitive marketplace. Both models feature heterogeneous agents with bounded rationality who act using inductive reasoning. We find that the effects of crowding are crucial to the understanding of the macroscopic fluctuations, or ‘volatility’, in the resulting dynamics of these systems.

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1. Introduction

There is much attention now being paid in various disciplines to the topic of complex adaptive systems [1–4]. In an economics context, such systems may provide invaluable insight since they can avoid the standard economic assumptions of equilibrium based on rational behaviour by agents. It is now recognised among economists that realistic models of economic markets should include the effects of heterogeneous agents with bounded rationality acting via inductive reasoning. In finance theory, a central problem is to understand the microscopic factors which give rise to macroscopic fluctuations, or ‘volatility’, in financial markets. It would be fascinating if complex adaptive systems could be used to explain such global market properties.

Here we present analytic and numerical results for two models which offer simple paradigms for a competitive marketplace containing competing agents. We show that the effects of crowding are crucial for understanding the macroscopic fluctuations in the resulting dynamics of these systems. For the minority model, introduced by Zhang and co-workers [5,6], we find that an analytic model incorporating ‘crowd’ and ‘anticrowd’ effects can explain the numerically-obtained volatility over a wide range of parameter space. For the bar-attendance model, introduced by Arthur [7], we also find that crowd effects are essential for understanding the volatility.

2. Minority Model

The minority model was introduced by Challet and Zhang [3] and was also discussed by Savit et al. [4] and de Cara et al [8]. The basic model takes the form of a repeated game as follows. Consider an odd number of agents $N$ who must choose whether to be in room ‘0’ or room ‘1’. After every agent has independently chosen a room, the winners are those in the minority room, i.e., the room with fewer agents. The ‘output’ is a single binary digit for each time-step: 0 represents room 0 winning, 1 represents room 1 winning. This output is made available to all agents, and is the only information they can use to decide which room to
choose in subsequent time-steps. Given that the agents are of limited yet similar capabilities, we assign to each agent a ‘brain-size’ \( m \); this is the length of the past history bit-string that an agent can use when making its next decision. Consider \( m = 2 \); the possible \( m = 2 \) history bit-strings are 00, 01, 10, 11. When faced with any one of these \( 2^m = 4 \) histories, the agent can make one of two decisions, 0 or 1. Hence, there are \( 2^{2m} = 16 \) possible strategies which define the decisions in response to all possible \( m = 2 \) history bit-strings. Each strategy can thus be represented by a string of 4 bits \([ijk\ell]\) with \( i, j, k, \ell = 0 \) or 1 corresponding to the decisions based on the histories 00, 01, 10, 11, respectively. For example, strategy \([0000]\) corresponds to deciding to pick room 0 irrespective of the \( m = 2 \) history bit-string. \([1111]\) corresponds to deciding to pick room 1 irrespective of the \( m = 2 \) history bit-string. \([1010]\) corresponds to deciding to pick room 1 given the histories 00 or 10, and pick room 0 given the histories 01 or 11. The agents randomly pick \( s \) strategies at the beginning of the game. After each turn of the game, the agent assigns one (virtual) point to each of his strategies which would have predicted the correct outcome. In addition the agent gets awarded one (real) point if he is successful. At each turn of the game, the agent uses whichever is the most successful strategy among the \( s \) strategies in his possession, i.e., he chooses the one that has gained most (virtual) points.

The important feature of both the minority model, and the bar-attendance model described later, is that the success of any particular strategy is generally short-lived. If all the agents begin to use similar strategies, and hence choose the same room, such a strategy ceases to be profitable and is hence dropped. Hence there is no best strategy for predicting the market for all times. References \[5\] and \[6\] provide various numerical results for the minority model. The main result which emerges from the numerical simulations \[5,6\] of the minority model concerns the volatility (i.e., standard deviation) of the time-series \( x(t) \) corresponding to the number of agents attending a given room, say room 0, at each time step, with each time step taken to be a turn in the game. When the number of strategies per agent \( s \) is small, the volatility \( \sigma \) exhibits a pronounced minimum as a function of the
brain-size $m$. Around this minimum, the volatility $\sigma$ is substantially smaller than the value obtained for the case where each agent makes his decision by tossing a coin [5,6]. A complete analytic theory describing $\sigma$ as a function of $m$ for arbitrary $N$ and $s$ has, however, been lacking.

3. Effects of Crowds and Anticrowds

Here we propose an analytic theory which describes the volatility for arbitrary $m$, $N$ and $s$. We first present the basic idea, before proceeding to explicit expressions. Consider the oversimplified case of $N$ independent agents each deciding whether or not to attend room 0 by tossing a coin. Using standard random-walk results, the total variance $\sigma^2$ of attendance in room 0 is given by the sum of the variances produced by the $N$ independent agents:

$$\sigma^2 = \sum_{i=1}^{N} \sigma_i^2$$

(1)

where $\sigma_i^2 = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$. Hence $\sigma^2 = \frac{N}{4}$. However in reality on any given turn of the minority game, there are a number of agents using the same, or similar, strategies. Consider the subset of agents $n_i$ using a particular strategy $i$. Although there is no information available to a given agent about other individual agents nor is any direct communication allowed between agents, this subset $n_i$ will all act in the same way, i.e., they all go to either room 0 or 1 and hence constitute a crowd. Since the corresponding random-walk ‘step-size’ has become $n_i$, one might think that $\sigma_i^2$ should be given by $\frac{1}{4}n_i^2$. Given that there is no a priori best strategy, however, it is important to realize that there may also be a subset of agents $n_i$ who are using the opposite, or at least very dissimilar, strategies to the first subset $n_i$. We call this second subgroup the anticrowd, and the strategy $\bar{i}$ that they use is anti-correlated to strategy $i$ (e.g. if $i$ is [0000] then $\bar{i}$ is [1111]) [5]. The anticrowd chooses the opposite room to the crowd and hence behaves as a crowd itself. Over the timescale during which the two opposing strategies are being played, the fluctuations of attendance in room 0 are determined only by the net crowd-size $N_i = n_i - n_{\bar{i}}$. Hence $\sigma_i^2$ should instead be given by $\frac{1}{4}N_i^2$. 

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Suppose strategy $i^*$ is the highest scoring at a particular moment in the game: the anti-correlated strategy $\bar{i}^*$ is therefore the lowest scoring at that same moment. In the limit of small brain-size $m$, the size of the strategy space is small. For most values of $s$, it follows that even if an agent picks $\bar{i}^*$ among his $s$ strategies, he will not have to use it since he will most likely have a high scoring strategy in his toolbag. In practice if $m$ is small, the required $s$ can be relatively small since a modest value of $s$ is sufficient for each agent to carry a considerable fraction of all possible strategies. Therefore, many agents will choose to use either $i^*$ itself (if they hold it among their $s$ strategies) or a similar one. Very few agents will have such a poor set of $s$ strategies that they are forced to use a strategy similar to $\bar{i}^*$. In this regime there are practically no anticrowds, hence the crowds dominate. Therefore $N_i \sim N \delta_{i i^*}$ yielding $\sigma_i^2 \sim N^2 \delta_{i i^*}$; the resulting volatility $\sigma^2 \sim N^2$ is larger than the independent agent limit of $N^2$ and is therefore consistent with the numerical simulations \[5,6\]. In the opposite limit of large brain-size $m$, the strategy space is very large, hence agents will have a low chance of holding, and hence playing, the same strategy. In addition, even if an agent has $s$ low-scoring strategies, the probability of his best strategy being strictly anticorrelated to another agent’s best strategy (hence forming a crowd-anticrowd pair) is small. All the crowds and anticrowds are of size 0 or 1 implying that the crowds and anticrowds have effectively disappeared and the agents act independently. We thus have $N_i = 0$ or 1 with $\sum_i N_i \sim N$. The volatility is then given by $\sigma^2 \sim N^2$ which is again consistent with the numerical simulations \[5,6\]. In the intermediate $m$ region where the numerical minimum exists for small $s$, the size of the strategy space is relatively large so that some agents may get stuck with $s$ strategies which are all low scoring. They can hence form anticrowds. The presence of finite-size anticrowds implies that $\sum_i N_i < N$. Considering the extreme case where the crowd and anticrowd are of similar size, we have $N_i \sim 0$ and hence $\sigma_i^2 \sim 0$. The volatility is therefore small ($\sigma \sim 0$) which is again consistent with the numerical results. For a fixed value of $m$, this regime of small volatility will arise for small $s$ since in this case the number of strategies available to each agent is small, hence some of the agents may indeed be forced to use a strategy which
is little better than the poorly-performing $\vec{i}$. In other words, the cancellation effect of the crowd and anticrowd becomes most effective in this intermediate region. It is this interplay between the size of the strategy space and the probabilities of agents forming crowds and anticrowds which gives rise to the rich and non-trivial behaviour of the volatility.

Analytic expressions for the crowd and anticrowd sizes can be obtained in various ways, with varying degrees of accuracy. Here we present one such approach, and show that it yields good quantitative agreement with the numerical results. We need to calculate the number of agents who are using a given strategy $i$ or one which is similar, such that they are acting identically in the majority of turns. We can immediately exploit the results of Ref. [5] concerning the reduced strategy space. In particular, given a strategy $i$, the only strategies that are significantly different from $i$ are the anti-correlated strategy $\vec{i}$ plus the uncorrelated strategies [5]. For example for $m = 2$, given a strategy $i \equiv [0000]$ the strategies which will yield significantly different actions from $i$ are $\vec{i} \equiv [1111]$, plus the uncorrelated strategies [1100], [1010], [1001], [0110], [0101], [0011]. Although the full strategy space contains $2^m = 16$ strategies, the reduced strategy space only contains $2 \cdot 2^m = 8$. Henceforth we only need to consider the reduced strategy space – in agreement with Ref. [5], we have found that the numerical results for the volatility are quantitatively very similar for both the full strategy space and the reduced strategy space [5].

The number of strategies in the reduced strategy space $V_m$ is given by $a = 2 \cdot 2^m$. For $s = 1$ strategy per agent, the probability that a given strategy in $V_m$ will be picked by a given agent at the start of the game is $\frac{1}{a}$. Consider a given moment in the game. We can rank all the strategies as best, 2nd best, 3rd best, etc at that moment. Since agents will always play their best strategy, the probability that the current best strategy is being used by a given agent is given by the probability that the agent actually has this strategy. Similarly, the probability that the 2nd best strategy is being used by a given agent is equal to the probability that the agent has this strategy but does not possess the best strategy. Since repetition of picked strategies is allowed during the initial picking process, it is straightforward to show that for
general s the mean number of agents \( y_r \) using the \( r \)-th best strategy is given by

\[
y_r = N \left( \left[ 1 - \frac{(r-1)}{a} \right]^s - \left[ 1 - \frac{r}{a} \right]^s \right).
\] (2)

Notice that \( \sum_{r=1}^{a} y_r = N \) as required.

Next we consider the probability \( P \) that given a particular strategy is being used, then its anticorrelated strategy is also being used. Consider the list of all \( 2 \cdot 2^m \) strategies in order of the number of virtual points they have collected. The \( i \)-th strategy and the \( [2 \cdot 2^m + 1 - i] \)-th strategy will be anticorrelated. For large values of \( m \) and hence large reduced strategy space, \( P \) will be negligible since it is highly unlikely that if a strategy is used then its anticorrelated strategy is also used. It is only when the total number of strategies in play, which is of order \( N \), greatly exceeds the number of strategies in the reduced strategy space that \( P \) will be of order unity. We can thus approximate \( P \) by \( P(p) = p \) for \( p < 1 \) and \( P(p) = 1 \) for \( p > 1 \), where \( p = N/(2 \cdot 2^m) \). Although the form for \( P(p) \) can be made more accurate, the present expression is reasonable since there are only of order \( N \) strategies out of a possible maximum of \( 2 \cdot 2^m \) which can actually be in play at any one time. Hence, as expected, \( P(p) \) is zero when \( N << 2 \cdot 2^m \) and unity when \( N >> 2 \cdot 2^m \). We have checked that our analytic results for the volatility are fairly insensitive to the precise form of \( P(p) \) as long as \( P(p) \) satisfies the constraints \( P(0) = 0 \) and \( P(p >> 1) = 1 \).

Equation (2) gives \( y_r \) in terms of \( m, s, \) and \( N \). In satisfying \( \sum_{r=1}^{a} y_r = N \), we should in practice take into account the fact that agents exist only as integer values. Hence we should only include \( R \) terms in this sum, subject to the condition that the partial sum equals \( N \) after the quantities \( y_r \) have been rounded to the nearest integer. In addition we choose to round any \( y_r \)'s which are less than one, up to one if \( r \leq R \) such that the first \( R \) terms are all non-zero. There are hence only \( R \) different strategies in play; note that \( R \leq 2 \cdot 2^m \) and \( R \leq N \). The probability \( P(p) \) is, to a reasonable approximation, the probability that the \( [R + 1 - r] \)-th strategy is anticorrelated to the \( r \)-th strategy. The variance can hence be written analytically as
\[ \sigma_{an}^2 = \frac{1}{4} \sum_{r=1}^{\frac{1}{2}(R-g)} [y_r - P(p)y_{R+1-r}]^2 + \frac{1}{4} \sum_{r=1}^{\frac{1}{2}(R-g)} [(1 - P(p))y_{R+1-r}]^2 + \frac{g}{4} [y_{R+1}]^2 \] (3)

where \( g = 0 \) if \( R \) is even and \( g = 1 \) if \( R \) is odd. The first term represents the net effect after pairing off the agents playing anticorrelated strategies. The second term in Eq. (3) reintroduces those agents using strategies that were assumed to be anticorrelated to some more successful strategy, and hence were discarded unnecessarily in the first term. The third term in Eq. (3) is due to the volatility of the group which remains unpaired in the case where the number of different strategies used in the calculation is odd. The third term is usually negligible compared to the first two.

Figure 1 shows the volatility for \( s = 2, s = 4 \) and \( s = 6 \) with \( N = 101 \) agents, as calculated using the analytic results above. Note that since \( m \) is integer, the curves are not smooth. Figure 2 compares these theoretical volatility curves (solid lines) with the numerical simulations (dashed line). The agreement between the analytic and numerical results is good across a wide range of \( m \) and \( s \) values. In particular, the analytic results capture the deepening of the minimum in the volatility as \( s \) decreases. For the range of \( s \) considered, the crowds are much larger than the anticrowds for small \( m \) (i.e., below the minimum). Hence the volatility is large for small \( m \). As \( m \) increases, the crowds and anticrowds begin to compete effectively for small \( s \), hence yielding the minimum as discussed qualitatively earlier in the paper. This minimum disappears with increasing \( s \) since the anticrowds become negligible in size; this is reasonable since for large \( s \) the likelihood of being stuck with a strategy which is essentially anti-correlated to a winning strategy is very small. The agent with such a strategy would typically have several better choices among his \( s \) strategies. For large \( m \) (i.e., above the minimum) the crowds and anticrowds have reduced in size to such a point that the agents act independently. The agreement in Fig. 2 can be improved upon by using a better approximation for \( P(p) \) at the expense of increased analytic complexity. We note that the above arguments also explain the behaviour of the volatility at fixed, small \( m \) shown in Fig. 1: the volatility increases with \( s \) due to the decreasing likelihood of forming substantial anticrowds.
It is interesting to note that the present analytic approach uses an idea which is very common in condensed matter physics: the population of \( N \) agents is treated in terms of clusters. These clusters are chosen such that the dominant correlations in the problem become intra-cluster correlations, i.e., we choose a given cluster \( i \) to contain the \( n_i \) agents using strategy \( i \) and the \( n_i \) agents using strategy \( \bar{i} \). The strong intra-cluster correlations between the crowd \( n_i \) and anticrowd \( n_i \) are treated as accurately as possible, and dominate the weak inter-cluster correlations which can themselves be effectively ignored.

Finally we note that we have also found a new scaling result for the minority model for arbitrary \( N, m \) and \( s \), which goes beyond that reported in Ref. [6]. In particular, if we plot the numerical values of the reduced variable \( \sigma/\sqrt{N} \) as a function of \( (2 \cdot 2^m)/(Ns) \), then the data undergo almost complete data collapse [10]. This collapse becomes increasingly precise for larger \( s \) (e.g. \( s \geq 4 \)).

4. Bar-Attendance Model

Arthur [7] proposed the ‘bar-attendance’ model in which \( N \) adaptive agents, each possessing \( s \) prediction rules or ‘predictors’ chosen randomly from a pool of \( V \), attempt to attend a bar, whose cut-off is \( L \), on a particular night each week. Each week the agents update their best rule for predicting a given week’s attendance based on the past attendance time-series \( x(t) \), which is made known to all agents. As stated in Ref. [5], the minority model seems to be a special case of the bar-attendance model. There are differences however: the output is no longer binary, nor do we restrict all predictors to depend on the same number of past weeks’ data. The predictors in our pool of \( V \) are chosen from a variety of ‘classes’ of rules. For example, one class might comprise rules which take an arithmetic, geometric, or weighted average over the past \( m \) weeks’ attendances; another class might copy the result from week \( m' \); or alternatively, might take the mirror image of \( x(t) \) about \( L \) from week \( m'' \). In Ref. [9], we presented the results of numerical simulations on the bar-attendance model and showed that the volatility of the attendance time-series can exhibit a minimum at small,
but finite, \( s \). Here we are interested in the extent to which crowding effects can explain the dynamics.

Figure 3 shows a plot of the volatility (i.e., standard deviation) \( \sigma \) as a function of the number of agents \( N \) for a predictor pool size \( V = 60 \). When the number of agents is smaller than the cut-off value (i.e. \( N < L \)), the volatility \( \sigma \approx 0 \) as expected. When the number of agents is larger than the cut-off value (i.e. \( N > L \)), the volatility increases with increasing \( N \) \[9\]. As for the minority model above, suppose that each agent decides whether to attend based on a pre-assigned probability. Given the common knowledge of the cut-off \( L \), each agent should attend with a probability \( \frac{L}{N} \) and stay away with a probability \( 1 - \frac{L}{N} \). The volatility can be estimated using the expression for a bounded random walk, i.e., the volatility \( \sigma \sim \sqrt{N \cdot \frac{L}{N} \cdot (1 - \frac{L}{N})} \) which gives \( \sqrt{L(1 - \frac{L}{N})} \). The dashed line in Fig. 3 shows that this is however a poor approximation to the numerical results. The discrepancies imply that we should include correlations between the actions of the \( N \) agents. In particular several agents may be using the same predictor at any one time. Hence, crowd effects will again be important. A crowd model can be developed in a way analogous to that for the minority model. However the model will necessarily be less sophisticated than in Sec. 3 since the ‘predictor-space’ in the bar-attendance problem is less easily described than the strategy space for the (binary) minority model.

Consider the most popular predictor in use at a given stage of the simulation. There will be approximately \( n_1 = N \frac{\hat{p}}{V} \) agents using this predictor. We denote these \( n_1 \) agents as belonging to group 1. All \( n_1 \) agents in group 1 will make the same decision as whether to attend in the subsequent week. Note that in the case that agents, when initially picking predictors from the pool \( V \), are prohibited from picking the same predictor twice, this expression \( n_1 = N \frac{\hat{p}}{V} \) is quite accurate in practice. When agents are allowed to pick the same rule repeatedly, the expression \( \frac{\hat{p}}{V} \) is merely an estimate of \( n_1 \). In what follows, we will disallow repeated-picking of predictors. It is straightforward to show that the \( r \)-th most popular predictor will be used by approximately
\[ n_r = N^s \left[ 1 - \frac{s}{V} \right]^{r-1} \]  

agents. These agents belonging to group \( r \) form a crowd and will all make the same decision as to whether to attend in the subsequent week. As with the minority model, the resulting volatility-squared \( \sigma^2 \) is given by the sum over the volatilities-squared \( \sigma_r^2 \) produced by each group, i.e., \( \sigma^2 = \sum_r \sigma_r^2 \). Now \( \sigma_r^2 \sim n_r^2 \frac{L}{N} (1 - \frac{L}{N}) \), hence \( \sigma \) can be calculated by obtaining \( n_r \) from Eq. (4) above. Note that \( n_r \) is again rounded to the nearest integer due to the discreteness of agent-number. Figure 3 compares the volatility \( \sigma \) obtained using this crowd model (dotted line) to the numerical simulation (solid line). The analytic model is in good quantitative agreement with the simulation results, in stark contrast to the earlier random walk model without crowding (dashed line). Future work will investigate the role of anticrowds in the bar attendance model.

5. Summary

In summary we have presented an analytic analysis of crowding effects in the minority and bar-attendance models. We hope that the present results will stimulate further interest in what is proving to be an exciting field of study for physicists.

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Figure Captions

Figure 1: Analytic results for the minority model volatility (i.e., standard deviation) $\sigma$ for $s = 2$, $s = 4$ and $s = 6$ strategies per agent. Number of agents $N = 101$. Since $m$ can strictly only take integer values, the curves are not smooth.

Figure 2: Comparison between analytic and numerical results for the minority model volatility (i.e., standard deviation) $\sigma$. Results are shown for (a) $s = 2$, (b) $s = 4$ and (c) $s = 6$, where $s$ is the number of strategies per agent. Number of agents $N = 101$. Solid line: analytic result (see text). Dashed line: numerical simulations. Since $m$ is integer, the curves are not smooth.

Figure 3: Bar-attendance model volatility (i.e. standard deviation) $\sigma$ as a function of the number of agents $N$. The number of predictors per agent $s = 3$, the bar cut-off value $L = 60$ and the predictor pool size $V = 60$. Solid line: numerical simulation results. Dashed line: random walk model without crowding (see text). Dotted line: analytic crowd model (see text).
minority model
analytic results

volatility $\sigma$

brain size $m$

- Solid line, $s = 2$
- Dashed line, $s = 4$
- Dotted line, $s = 6$
(a) For $s=2$ and $N=101$, the volatility $\sigma$ decreases as the brain size $m$ increases.

(b) For $s=4$ and $N=101$, the volatility $\sigma$ decreases more sharply as the brain size $m$ increases.

(c) For $s=6$ and $N=101$, the volatility $\sigma$ decreases even more sharply as the brain size $m$ increases.
The graph shows the volatility ($\sigma$) as a function of the number of agents ($N$) in the bar-attendance model. The lines represent different models:

- **Numerical**
- **Random walk**
- **Analytic: crowd model**

The volatility increases with the number of agents, indicating higher variability in attendance as the crowd size grows. The analytic model, represented by the dashed line, shows a smoother increase compared to the numerical and random walk models.