ON THE TITS-KANTOR-KOECHER CONSTRUCTION OF UNITAL JORDAN BIMODULES

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Abstract. In this paper we explore relationship between representations of a Jordan algebra $J$ and the Lie algebra $g$ obtained from $J$ by the Tits-Kantor-Koecher construction. More precisely, we construct two adjoint functors $\text{Lie}: J\text{-mod}_1 \to g\text{-mod}_1$ and $\text{Jor}: g\text{-mod}_1 \to J\text{-mod}_1$, where $J\text{-mod}_1$ is the category of unital $J$-bimodules and $g\text{-mod}_1$ is the category of $g$-modules admitting a short grading. Using these functors we classify $J$ such that its semisimple part is of Clifford type and the category $J\text{-mod}_1$ is tame.

1. INTRODUCTION

The famous Tits-Kantor-Koecher construction relates a unital Jordan (super)algebra $J$ with a Lie (super)algebra $g$ equipped with a short $\mathbb{Z}$-grading. It was introduced independently in [1], [2] and [3] and one of most prominent applications was a classification of simple Jordan superalgebras in [4], [5], [6].

The TKK construction has been proven to be quite efficient and useful in the study of Jordan superalgebras, Jordan superpairs and their superbimodules. Several application of TKK construction in representation theory of semisimple Jordan superalgebras and Jordan superpairs can be found in [7], [8], [9], [10] and [11]. The goal of this paper is to further study and apply this construction to non-semisimple Jordan algebras and their representations.

Recall that a representation of a Jordan algebra $J$ in a vector space $M$ is a linear mapping $\rho$ of $J$ into $\text{End}_k(M)$ such that

$$[\rho(a), \rho(a^2)] = 0, \quad 2\rho(a)\rho(b)\rho(a) + \rho((a \circ a) \circ b) = 2\rho(a)\rho(a \circ b) + \rho(b)\rho(a \circ a).$$

The category of finite dimensional $J$-modules will be denoted by $J\text{-mod}$.

If $J$ is a unital algebra the category of $J$-modules has a decomposition into the direct sum of three subcategories

$$J\text{-mod} = J\text{-mod}_1 \oplus J\text{-mod}_2 \oplus J\text{-mod}_0$$

according to the action of the identity element of $J$. The subcategory $J\text{-mod}_0$ is not interesting since all its objects are trivial modules. The subcategory $J\text{-mod}_2$ consists of modules on which the identity element acts as $\frac{1}{2}$, such modules are called special. The objects of $J\text{-mod}_1$ are called unital modules. One can introduce associative enveloping algebras for $J\text{-mod}$, $J\text{-mod}_1$ and $J\text{-mod}_2$, such that each of these categories is equivalent to the category of modules over the corresponding enveloping algebra.

Recall the classification of simple Jordan algebras over an algebraically closed field $k$ of characteristic zero. With the exception of the case $J = k$, simple Jordan algebras are divided in two groups: Jordan algebras of quadratic form $J(E, q)$, see Section 5 for details, and Jordan algebras of matrix type, see [12]. The latter are called sometimes Hermitian Jordan algebras.

In [13] Jacobson constructed the associative enveloping algebras for $J\text{-mod}_1$ and $J\text{-mod}_2$, when $J$ is finite-dimensional simple, and proved that both categories are semisimple with finitely many simple objects.

The next step is to study non-semisimple Jordan algebras. In this case it is important to classify tame categories $J\text{-mod}_2$ and $J\text{-mod}_1$ (for basics on tame and wild categories see [14]). In [15] the enveloping algebra of $J\text{-mod}_2$ was studied in the case when the semisimple part of $J$ is of matrix type and $\text{rad}^2 J = 0$. Using the coordinatalization theorem for Jordan algebras of matrix type the authors proved that the enveloping algebra and consequently the Ext quiver algebra of $J\text{-mod}_2$
have radical squared equal to zero. Hence they could employ the representation theory of quivers to classify tame \( J\text{-mod}_{\hat{\mathfrak{g}}} \).

In all other cases the above method is not applicable. But it seems likely that we can later deal with the remaining cases using the TKK construction. The main advantage of this approach is the existence of a tensor structure on the category of \( \mathfrak{g}\)-modules and a well developed theory of weights.

In this paper we focus on Jordan algebras, whose semisimple part is a sum of Jordan algebras of quadratic forms. We classify all such algebras with tame \( J\text{-mod}_1 \) without any additional assumptions on the radical, see Theorem \([9,1]\). For this purpose we avoid cases of small dimensions: we start with simple Jordan algebras of dimension greater than 4. It follows from our classification that all such tame Jordan algebras \( J \) satisfy the condition \( \text{rad}^2 J = 0 \). On the other hand, in contrast with \([15]\), the square of the radical of the universal enveloping algebra is not necessarily zero for tame categories. The category \( J\text{-mod}_{\hat{\mathfrak{g}}} \) is studied in a forthcoming paper \([17]\).

In Section 3 we define and study two adjoint functors \( \text{Jor} \) and \( \text{Lie} \) between the category \( J\text{-mod}_1 \) and the category \( \mathfrak{g}\text{-mod}_1 \) of \( \mathfrak{g}\)-modules admitting a short grading. The definition of \( \text{Jor} \) is straightforward. However, not every \( J \)-module can be obtained from a \( \mathfrak{g}\)-module by application of \( \text{Jor} \). To fix this flaw one has to consider the universal central extension \( \hat{\mathfrak{g}} \) of \( \mathfrak{g} \). This problem does not appear in the semisimple case since \( \hat{\mathfrak{g}} = \mathfrak{g} \) but it is already essential for simple Jordan and Lie superalgebras, see \([9]\). Although algebras with non-zero central extensions do not appear in our classification, we formulate statements in full generality for future applications. The second problem worth mentioning here is caused by the fact that the splitting \( J\text{-mod}_1 \oplus J\text{-mod}_0 \) can be lifted to the Lie algebra \( \hat{\mathfrak{g}} \); since some modules can have non-trivial extensions with trivial modules.

That implies, in particular, that left and right adjoint of the functor \( \text{Jor} \) are not isomorphic and the categories \( \hat{\mathfrak{g}}\text{-mod}_1 \) and \( J\text{-mod}_1 \) are not equivalent. Still they are close enough and one can describe projective modules, quivers and relations of \( J\text{-mod}_1 \) in terms of \( \hat{\mathfrak{g}}\text{-mod}_1 \).

In Section 4 we explain how to construct the Ext quivers of \( \hat{\mathfrak{g}}\text{-mod}_1 \) and \( J\text{-mod}_1 \) and compute the radical filtration of projective indecomposable modules.

In Sections 5–9 we classify Jordan algebras with tame categories of unital representations satisfying above mentioned conditions. Our main tool is the representation theory of quivers. All the quiver results we use are collected in Appendix. Although our algebras do not satisfy the condition \( \text{rad}^3 A = 0 \), we use a lot Theorem \([10,4]\) which could be considered as a generalization of this property. Finally, let us mention that all tame associative algebras \( A \) arising from our classification are quadratic and satisfy the conditions \( \text{rad}^3 A = 0 \) and \( A \simeq A^{op} \).

2. Tits-Kantor-Koecher construction for Jordan algebras.

2.1. Jordan algebras and bimodules. Let \( k \) be a field, \( \text{char} k \neq 2 \). A Jordan \( k\)-algebra is a commutative algebra \( J \) such that any \( a, b \in J \) satisfy the Jordan identity:

\[
\begin{align*}
(1) & \quad a \circ b = b \circ a \\
(2) & \quad ((a \circ a) \circ b) \circ a = (a \circ a) \circ (b \circ a).
\end{align*}
\]

For any associative algebra \( A \), one can construct the Jordan algebra \( A^+ \) by introducing on a vector space \( A \) a new multiplication \( a_1 \circ a_2 = \frac{1}{2}(a_1 a_2 + a_2 a_1) \). If \( A \) is Jordan isomorphic to a subalgebra of the algebra \( A^+ \) for a certain associative algebra \( A \) then it is called special, otherwise it is exceptional.

Let \( J \) be a Jordan algebra over \( k \) and \( M \) be a \( k\)-vector space endowed with a pair of linear mappings \( l : J \otimes_k M \to M, (a \otimes m) \mapsto a \cdot m, r : M \otimes_k J \to M, (m \otimes a) \mapsto m \cdot a, a \in J, m \in M. \) Then \( M \) is called a Jordan bimodule over \( J \) if the algebra \( Z = (J \oplus M, \ast) \), where \( \ast \) is a \( k\)-bilinear product

\[
(a_1 + m_1) \ast (a_2 + m_2) = a_1 \circ a_2 + a_1 \cdot m_2 + m_1 \cdot a_2,
\]

for \( a_1, a_2 \in J, m_1, m_2 \in M \), is a Jordan algebra. Observe that \( J \) is a subalgebra in \( Z \) and \( M \) is an ideal with \( M^2 = 0 \). In this case \( Z \) is called the null split extension of \( J \) by the bimodule \( M \). It follows from the Jordan identity \([2]\) that if \( M \) is a Jordan bimodule over \( J \) the corresponding representation \( \rho : J \to \text{End}_k M \) satisfies \([1]\).
2.2. TKK construction. A short grading of an algebra $\mathfrak{g}$ is a $\mathbb{Z}$-grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let $P$ be the commutative bilinear map on $J$ defined by $P(x, y) = x \circ y$. Then we associate to $J$ a Lie algebra $\mathfrak{g} = \text{Lie}(J)$ with short grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ in the following way, see [16]. We set

$$\mathfrak{g}_{-1} = J, \quad \mathfrak{g}_0 = \langle L_a, [L_a, L_b] | a, b \in J \rangle \subset \text{End}_k(J),$$

where $L_a$ denotes the operator of left multiplication in $J$, and

$$\mathfrak{g}_1 = \langle P, [L_a, P] | a \in J \rangle \subset \text{Hom}_k(S^2 J, J)$$

with the following bracket

- $[x, y] = 0$ for $x, y \in \mathfrak{g}_{-1}$ or $x, y \in \mathfrak{g}_1$;
- $[L, x] = L(x)$ for $x \in \mathfrak{g}_{-1}$, $L \in \mathfrak{g}_0$;
- $[B, x](y) = B(x, y)$ for $B \in \mathfrak{g}_1$ and $x, y \in \mathfrak{g}_{-1}$;
- $[L, B](x, y) = L(B(x, y)) - B(L(x), y) + B(x, L(y))$ for any $B \in \mathfrak{g}_1$, $L \in \mathfrak{g}_0$ and $x, y \in \mathfrak{g}_{-1}$.

$\text{Lie}(J)$ is a Lie algebra.

Note that by construction $\text{Lie}(J)$ is generated as a Lie algebra by $\text{Lie}(J)_1 \oplus \text{Lie}(J)_{-1}$.

A short subalgebra of a Lie algebra $\mathfrak{g}$ is an $\mathfrak{sl}_2$ subalgebra spanned by elements $e, h, f$, satisfying $[e, f] = h, [h, e] = -e, [h, f] = f$, such that the eigenspace decomposition of $ad h$ defines a short grading on $\mathfrak{g}$. Consider a Jordan algebra $J$ with a unit element $e$. Then $e, h_J = -L_e$ and $f_J = P$ span a short subalgebra $\alpha_J \subset \text{Lie}(J)$. A $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called minimal if any non-trivial ideal $I$ of $\mathfrak{g}$ intersects $\mathfrak{g}_{-1}$ non-trivially, i.e. $I \cap \mathfrak{g}_{-1}$ is neither 0 nor $\mathfrak{g}_{-1}$.

Lemma 2.1. [16] A Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is minimal if and only if the following conditions hold:

1. if $[a, \mathfrak{g}_{-1}] = 0$ for some $a \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$, then $a = 0$;
2. $[\mathfrak{g}_0, \mathfrak{g}_i] = \mathfrak{g}_i, i = \pm 1$.

Let $\mathcal{J}$ denote the category of unital Jordan algebras in which morphisms are Jordan epimorphisms and let $\mathcal{L}$ denote the category of minimal pairs $(\mathfrak{g}, \alpha)$, where $\mathfrak{g}$ is a Lie algebra, $\alpha$ a short subalgebra of $\mathfrak{g}$, and a morphism $\phi$ from pair $(\mathfrak{g}, \alpha)$ to $(\mathfrak{g}', \alpha')$ is a Lie algebra epimorphism $\phi: \mathfrak{g} \to \mathfrak{g}'$ such that $\phi(\alpha) = \alpha'$. We construct a functor $\mathcal{F}: \mathcal{J} \to \mathcal{L}$ by associating to every unital Jordan algebra $J$ the pair $(\text{Lie}(J), \alpha_J)$ and to every epimorphism $\phi: J \to J'$ of unital Jordan algebras the map $\phi_\mathcal{F}: \text{Lie}(J) \to \text{Lie}(J')$ defined as follows:

$$x \mapsto \phi(x), \quad L_a \mapsto L_{\phi(a)}, \quad P \mapsto P', \quad P'(x, y) = x \circ_J y.$$

Let $\mathfrak{g}$ be a Lie algebra containing an $\mathfrak{sl}_2$-subalgebra $\alpha$ which induces a short grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then $J = (\mathfrak{g}_{-1}, \alpha_{J})$ with $x \circ_J y = [[P, x], y]$ is a Jordan algebra. Moreover, any epimorphism $\phi: (\mathfrak{g}, \alpha) \to (\mathfrak{g}', \alpha')$ in $\mathcal{F}$ defines Jordan algebra epimorphism $\phi_{\mathfrak{g}_{-1}}$. Thus, we have defined a functor $\mathcal{L} \to \mathcal{J}$ which we denote by $\text{Jor}$. The functors $\mathcal{F}$ and $\text{Jor}$ define an equivalence of categories $\mathcal{L}$ and $\mathcal{J}$, see Theorem 5.15, [16].

3. Functors $\text{Lie}$ and $\text{Jor}$ for unital modules

Let $J$ be a unital Jordan algebra and $\mathfrak{g} = \text{Lie}(J)$. By $\hat{\mathfrak{g}}$ we denote the universal central extension of $\mathfrak{g}$. Note that $\hat{\mathfrak{g}}$ contains the $\mathfrak{sl}_2$-subalgebra $\alpha = \langle e, h, f \rangle$, hence the center of $\hat{\mathfrak{g}}$ is in $\mathfrak{g}_0$. It implies that

$$\mathfrak{g}_{\pm 1} = \hat{\mathfrak{g}}_{\pm 1}.$$

Let $\hat{\mathfrak{g}}\text{-mod}_1$ denote the category of $\hat{\mathfrak{g}}$-modules $N$ such that the action of $h \in \alpha$ induces a grading of length 3 on $N$. We will construct two functors

$$\text{Jor}: \hat{\mathfrak{g}}\text{-mod}_1 \to J\text{-mod}_1, \quad \text{Lie}: J\text{-mod}_1 \to \hat{\mathfrak{g}}\text{-mod}_1$$

and show that $\text{Lie}$ is left adjoint to $\text{Jor}$.

To define $\text{Jor}$ let $N \in \hat{\mathfrak{g}}\text{-mod}_1$. Then $N$ has a short grading $N = N_{-1} \oplus N_0 \oplus N_{-1}$. We set $\text{Jor}(N) := N_{-1}$ with action of $J = \mathfrak{g}_{-1} = \hat{\mathfrak{g}}_{-1}$ defined by

$$x(m) = [f, x] m, \quad x \in J = \mathfrak{g}_{-1}, m \in N_{-1}.$$

It is clear that $\text{Jor}$ is an exact functor.
Our next step is to define $\text{Lie} : J\mod_1 \to \hat{\mathfrak{g}}\mod_1$. Let $M \in J\mod_1$. Consider the associated null split extension $J \oplus M$. Let $\mathcal{A} = \text{Lie}(J \oplus M)$. Then by (10) we have an exact sequence of Lie algebras
\[
0 \to N \to \mathcal{A} \xrightarrow{\pi} \mathfrak{g} \to 0,
\]
where $N$ is an abelian Lie algebra and $N_{-1} = M$.

**Lemma 3.1.** Let $\gamma : \hat{\mathfrak{g}} \to \mathfrak{g}$ be the canonical projection. There exists $s : \hat{\mathfrak{g}} \to \mathcal{A}$ such that $\pi \circ s = \gamma$.

**Proof.** Observe that the splitting $\mathcal{A}_{\pm 1} = \mathfrak{g}_{\pm 1} \oplus N_{\pm 1}$ is canonical. Let $\hat{\mathfrak{g}}$ be the Lie subalgebra in $\mathcal{A}$ generated by $\mathfrak{g}_{\mp 1}$. Then we have a surjective homomorphism $\varphi : \hat{\mathfrak{g}} \to \mathfrak{g}$ and $\text{Ker} \varphi \subset \hat{\mathfrak{g}}_0$. We claim that $\text{Ker} \varphi$ lies in the center of $\hat{\mathfrak{g}}$. Indeed, $z \in \text{Ker} \varphi$ implies $[z, \hat{\mathfrak{g}}_{\pm 1}] \subset \hat{\mathfrak{g}}_{\pm 1} \cap \text{Ker} \varphi = 0$ and from $[\hat{\mathfrak{g}}_{-1}, \hat{\mathfrak{g}}_1] = \hat{\mathfrak{g}}_0$ it follows that $[z, \hat{\mathfrak{g}}_0] = 0$. Therefore the map $s : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}} \subset \mathcal{A}$ is as required. □

**Remark 3.2.** For the illustration that $\hat{\mathfrak{g}}$ is essential, see Example [4.7].

The above Lemma implies that $N$ is a $\hat{\mathfrak{g}}$-module. Thus, in particular, we have defined a $\hat{\mathfrak{g}}_0$-module structure on $N_{-1} = M$. Now let $\mathcal{P} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ and we extend the above $\hat{\mathfrak{g}}_0$-module structure on $M$ to a $\mathcal{P}$-module structure by setting $\mathfrak{g}_{-1}M = 0$. Let
\[
\Gamma(M) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathcal{P})} M.
\]
We define $\text{Lie}(M)$ to be the maximal quotient in $\Gamma(M)$ which belongs to $\hat{\mathfrak{g}}\mod_1$. More precisely $\text{Lie}(M) := \Gamma(M)/T$, where $T$ is the submodule in $\Gamma(M)$ generated $\bigoplus_{i \geq 2} \Gamma(M)_i$.

Note that Frobenius reciprocity implies that for any $K \in \hat{\mathfrak{g}}\mod_1$ and any $M \in J\mod_1$
\[
\text{Hom}_{\hat{\mathfrak{g}}}(\text{Lie}(M), K) \simeq \text{Hom}_{\hat{\mathfrak{g}}}(\Gamma(M), K) \simeq \text{Hom}_{\mathcal{P}}(M, K).
\]

On the other hand, we have
\[
\text{Hom}_{\mathcal{P}}(M, K) = \text{Hom}_{\hat{\mathfrak{g}}_0}(M, K_{-1}) = \text{Hom}_{J}(M, \text{Jor}(K)).
\]

**Lemma 3.3.** We have a canonical isomorphism
\[
\text{Hom}_{\hat{\mathfrak{g}}}(\text{Lie}(M), K) \simeq \text{Hom}_{J}(M, \text{Jor}(K)),
\]

**Proof.** Indeed,
\[
\text{Hom}_{\hat{\mathfrak{g}}}(\text{Lie}(M), K) \simeq \text{Hom}_{\mathcal{P}}(M, K) \simeq \text{Hom}_{J}(M, \text{Jor}(K)),
\]
where the first isomorphism is a consequence of (6) and the second follows from (7). □

**Corollary 3.4.** If $P$ is a projective module in $J\mod_1$, then $\text{Lie}(P)$ is a projective module in $\hat{\mathfrak{g}}\mod_1$.

**Proof.** Follows from Lemma 3.3 and exactness of $\text{Jor}$. □

**Lemma 3.5.** $\text{Jor} \circ \text{Lie}$ is isomorphic to the identity functor in $J\mod_1$.

**Proof.** By construction we have $\text{Jor} \circ \text{Lie}(M) = (\text{Lie}(M))_{-1} \simeq M$. □

**Lemma 3.6.** Let $N \in \hat{\mathfrak{g}}\mod_1$. We have an exact sequence of $\hat{\mathfrak{g}}$-modules
\[
0 \to C \to \text{Lie}(\text{Jor}(N)) \to N \to C' \to 0,
\]
where $C, C'$ are some trivial $\hat{\mathfrak{g}}$-modules.

**Proof.** The identity morphism $\text{Jor}(N) \to \text{Jor}(N)$ induces a homomorphism of $\hat{\mathfrak{g}}\mod_1$-modules $\text{Lie}(\text{Jor}(N)) \to N$ by Lemma 3.3. Let $C$ and $C'$ denote the kernel and cokernel of this homomorphism. Then we obtain the sequence (8). Apply $\text{Jor}$ to this sequence. Since $\text{Jor}(\text{Lie}(\text{Jor}(N))) \simeq \text{Jor}(N)$, exactness of $\text{Jor}$ implies $\text{Jor}(C) = \text{Jor}(C') = 0$. Therefore $C$ and $C'$ are trivial $\hat{\mathfrak{g}}$-modules. □
The latter is projectivity of \( Jorb \) which is also exact. Application of \( Jorb \) is injective.

By Lemma 3.7(c) Lie \((\hat{\mathfrak{g}})\) and that gives an isomorphism \( \text{add} \) for details. In this case these functors establish an equivalence of categories, see [17] for details.

\begin{corollary}
(a) Let \( N \in \hat{\mathfrak{g}}\-\text{mod}_1 \) and \( \hat{\mathfrak{g}}N = N \), then the canonical map \( \text{Lie}(Jor(N)) \to N \) is surjective.

(b) Let \( N \in \hat{\mathfrak{g}}\-\text{mod}_1 \) and \( N^\hat{\mathfrak{g}} := \{ x \in N | \hat{\mathfrak{g}}x = x \} = 0 \), then the canonical map \( N \to \text{Lie}(Jor(N)) \) is injective.

(c) If \( M \to L \to 0 \) is exact in \( Jmod_1 \), then \( \text{Lie}(M) \to \text{Lie}(L) \to 0 \) is exact in \( \hat{\mathfrak{g}}\-\text{mod}_1 \);
\end{corollary}

\begin{proof}
Note that (a) and (b) follow from Lemma 3.7 since in (a) we have \( C' = 0 \) and in (b) we have \( C = 0 \). To prove (c) consider the exact sequence \( \text{Lie}(M) \to \text{Lie}(L) \to C \to 0 \), where \( C \) is the cokernel of \( \text{Lie}(M) \to \text{Lie}(L) \) and apply \( Jor \). Then again we have \( \text{Jor}(C) = 0 \). Note that by construction \( \text{Lie}(L) \) is generated by \( L = \text{Lie}(L)_{-1} \) and therefore \( \text{Lie}(L) \) does not have a trivial quotient. Hence \( C = 0 \).
\end{proof}

\begin{lemma}
Let \( P \) be a projective module in \( \hat{\mathfrak{g}}\-\text{mod}_1 \) such that \( \hat{\mathfrak{g}}P = P \). Then \( \text{Jor}(P) \) is projective in \( Jmod_1 \).
\end{lemma}

\begin{proof}
By Corollary 3.7 (a) we have a surjection \( \text{Jor}(\hat{\mathfrak{g}}P) \to P \). However, \( P \) is projective and that gives an isomorphism \( \text{Lie}(\text{Jor}(P)) \simeq P \). Now let \( M \to N \to 0 \) be an exact sequence in \( Jmod_1 \). We rewrite it in the form \( \text{Jor}(\text{Lie}(M)) \to \text{Jor}(\text{Lie}(N)) \to 0 \). Now we use

\[
\text{Hom}_J(\text{Jor}(\hat{\mathfrak{g}}P), \text{Jor}(\text{Lie}(M))) \simeq \text{Hom}_\hat{\mathfrak{g}}(\text{Lie}(\text{Jor}(P)), \text{Lie}(M)),
\]

\[
\text{Hom}_J(\text{Jor}(\hat{\mathfrak{g}}P), \text{Jor}(\text{Lie}(N))) \simeq \text{Hom}_\hat{\mathfrak{g}}(\text{Lie}(\text{Jor}(P)), \text{Lie}(N)).
\]

By Lemma 3.7(c) \( \text{Lie}(M) \to \text{Lie}(N) \to 0 \) is exact, hence we have that

\[
\text{Hom}_\hat{\mathfrak{g}}(P, \text{Lie}(M)) \to \text{Hom}_\hat{\mathfrak{g}}(P, \text{Lie}(N)) \to 0
\]

is also exact. Application of \( \text{Jor} \) implies exactness of

\[
\text{Hom}_J(\text{Jor}(\hat{\mathfrak{g}}P), M) \to \text{Hom}_J(\text{Jor}(\hat{\mathfrak{g}}P), N) \to 0.
\]

The latter is projectivity of \( \text{Jor}(P) \).
\end{proof}

\begin{corollary}
Let \( L \in \hat{\mathfrak{g}}\-\text{mod}_1 \) be simple and non-trivial. Then \( \text{Jor}(L) \) is simple.
\end{corollary}

\begin{corollary}
Let \( M \in \hat{\mathfrak{g}}\-\text{mod}_1 \). If \( (M/\text{rad}M)^\hat{\mathfrak{g}} = 0 \), then \( \text{Jor}(\text{rad}M) = \text{rad}\text{Jor}(M) \).
\end{corollary}

\begin{remark}
One can construct functors \( \text{Lie} \) and \( \text{Jor} \) between the category of special Jordan modules and the category of \( \hat{\mathfrak{g}} \)-modules with grading of length 2. In this case these functors establish an equivalence of categories, see [17] for details.
\end{remark}

4. ON THE CATEGORIES \( \mathfrak{g}\-\text{mod}_1 \) AND \( \mathfrak{g}\-\text{mod}_2 \)

In the rest of the paper we assume that the ground field \( k \) is algebraically closed of characteristic zero. Let \( \mathfrak{g} \) be a finite-dimensional Lie algebra which contains an \( sl_2 \)-subalgebra \( \alpha = \langle e, h, f \rangle \) with short grading \( \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) induced by the action of \( h \). We fix a Levi subalgebra \( \mathfrak{g}_{ss} \subset \mathfrak{g} \) such that \( \alpha \subset \mathfrak{g}_{ss} \) and denote by \( \mathfrak{r} \) the radical of \( \mathfrak{g} \). Then \( \mathfrak{g} \) is a semi-direct sum \( \mathfrak{g}_{ss} \rtimes \mathfrak{r} \). We assume in addition that

\[
\mathfrak{r}^{ss} \subset [\mathfrak{g}, \mathfrak{g}] \cap Z(\mathfrak{g})
\]

and \( \mathfrak{g} \) is generated by \( \mathfrak{g}_1 \) and \( \mathfrak{g}_{-1} \). These assumptions imply that \( \mathfrak{r} \) is a nilpotent Lie algebra. Define a decreasing filtration

\[
\mathfrak{r} = \mathfrak{r}_1 \supset \mathfrak{r}_2 \supset \ldots
\]

by setting \( \mathfrak{r}_i = [\mathfrak{r}, \mathfrak{r}_{i-1}] \) for all \( i > 1 \). Let \( R_i^1 = \mathfrak{r}_i / \mathfrak{r}_{i+1} \) and write \( R_i^1 = \mathfrak{r}_i / \mathfrak{r}_2 \) to simplify notation.

Let \( S \) be the full subcategory of finite-dimensional \( \mathfrak{g} \)-modules consisting of all modules \( M \) such that

\[
M = M_{-1} \oplus M_{-\frac{1}{2}} \oplus M_0 \oplus M_{\frac{1}{2}} \oplus M_1
\]

in the grading induced by the action of \( h \). In this section we prove some general statements about \( S \).

We notice first that \( S = \mathfrak{g}\-\text{mod}_1 \oplus \mathfrak{g}\-\text{mod}_2 \) is a direct sum of categories, and all simple objects in \( S \) are simple as \( \mathfrak{g}_{ss} \)-modules. In what follows we denote by \( tr \) the trivial simple \( \mathfrak{g} \)-module.
Remark 4.1. It is useful to notice that the category $\mathcal{S}$ has a contravariant duality functor which sends $M$ to $M^*$. In particular, $\mathcal{S}$ is equivalent to $\mathcal{S}^{op}$. 

Remark 4.2. We also note that the tensor product of two modules from $\mathfrak{g}\text{-mod}_2$ is a module from $\mathfrak{g}\text{-mod}_1$. So we have a bifunctor $\mathfrak{g}\text{-mod}_2 \times \mathfrak{g}\text{-mod}_2 \to \mathfrak{g}\text{-mod}_1$. In the language of Jordan algebras that corresponds to the well-known bifunctor $J\text{-mod}_2 \times J\text{-mod}_2 \to J\text{-mod}_1$, see for example [12, Section II.10].

4.1. Indecomposable projectives and Ext quivers. Consider the category $(\mathfrak{g}, \mathfrak{g_{ss}})$-mod of all $\mathfrak{g}$-modules $M$ integrable over $\mathfrak{g}_{ss}$. Note that $\mathfrak{g}_{ss}$-integrability implies that the action of $h$ is semisimple and eigenvalues of $h$ are in $\frac{1}{2}\mathbb{Z}$. Clearly, $\mathcal{S}$ is a full abelian subcategory in $(\mathfrak{g}, \mathfrak{g_{ss}})$-mod.

We define a functor

$$s_h: (\mathfrak{g}, \mathfrak{g_{ss}})\text{-mod} \to \mathcal{S}$$

by setting $M^{s_h} = M/N$ where $N$ is the submodule generated by all graded components in $M$ of degree greater or equal than $\frac{h}{2}$. Obviously, we have

$$\text{Hom}_\mathfrak{g}(M^{s_h}, K) = \text{Hom}_\mathfrak{g}(M, K)$$

for any $M \in (\mathfrak{g}, \mathfrak{g_{ss}})\text{-mod}$ and $K \in \mathcal{S}$. In other words, $s_h$ is left adjoint to the embedding functor $\mathcal{S} \to (\mathfrak{g}, \mathfrak{g_{ss}})\text{-mod}$. That implies in particular, that if $P$ is projective in $(\mathfrak{g}, \mathfrak{g_{ss}})$-mod, then $P^{s_h}$ is projective in $\mathcal{S}$.

To construct the projective cover $P(L)$ of a simple module $L \in \mathcal{S}$ consider the induced module

$$I(L) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g_{ss}})} L.$$ 

By Frobenius reciprocity $I(L)$ is the projective cover of $L$ in $(\mathfrak{g}, \mathfrak{g_{ss}})$-mod. Therefore $P(L) = I(L)^{s_h}$ is the projective cover of $L$ in $\mathcal{S}$.

Lemma 4.3. $P(L)$ is finite-dimensional.

Proof. Let $M = P(L)$. Recall that $S(r)$ is the associated graded algebra of the universal enveloping algebra $U(r)$ with respect to the PBW filtration. Let $GrM$ be the corresponding graded $S(r)$-module. Note that $GrM$ inherits the short grading of $M$ and is generated by $L$. Let $I = \text{Ann}_{S(r)} L$. Then $I$ is a $\mathfrak{g}_{ss}$-invariant ideal, and we have $GrM \simeq (S(r)/I) \otimes L$. Consider the $\mathfrak{g}_{ss}$-invariant decomposition $r = r' \oplus Z(\mathfrak{g})$. Let $I' = I \cap S(r')$. It is well know fact of representation theory of semisimple algebraic groups, that if dim $S(r')/I' = \infty$ then there exists a $\mathfrak{g}_{ss}$-highest vector $v \in r'$ such that $v^m \notin I'$ for all $m > 0$. That excludes the possibility dim $S(r')/I' = \infty$ since $v \in r'_1$ and the induced by action of $h$ grading of $S(r')/I'$ must be bounded.

Thus, we have dim $S(r')/I' < \infty$, which implies $I' \supseteq (r')^k$ for some $k > 0$. Therefore we have

$$M = U(Z(\mathfrak{g})) \sum_{i=0}^{k} (r')^i L,$$

where $Z(\mathfrak{g})$ denotes the center of $\mathfrak{g}$.

Since by our assumptions on $\mathfrak{g}$ we have $Z(\mathfrak{g}) \subset [r', r']$ we obtain that for sufficiently large $n$

$$M = \sum_{i=0}^{n} r^i L.$$

Therefore $M$ is finite-dimensional. \qed

Let $\bar{\mathfrak{g}} := \mathfrak{g}/[r, r]$. Then $\bar{\mathfrak{g}} = \mathfrak{g}_{ss} \supseteq R$ where $R = r/[r, r]$ is the abelian radical of $\mathfrak{g}$. We denote by $Q(\mathcal{S})$ the Ext quiver of the category $\mathcal{S}$, by $Q(\bar{\mathfrak{g}})$ the Ext quiver of $\mathfrak{g}\text{-mod}_1$ and by $Q^2(\bar{\mathfrak{g}})$ the Ext quiver of $\mathfrak{g}\text{-mod}_2$. Clearly, $Q(\mathcal{S})$ is the disjoint union of $Q(\bar{\mathfrak{g}})$ and $Q^2(\bar{\mathfrak{g}})$.

Lemma 4.4. Let $L$ and $L'$ be two simple modules in $\mathcal{S}$. Then

$$\text{Ext}^1_{\mathfrak{g}}(L', L) = \text{Ext}^1_{\bar{\mathfrak{g}}}(L', L)$$

and dim $\text{Ext}^1_{\mathfrak{g}}(L', L)$ equals the multiplicity of $L$ in $L' \otimes R$. 

Proof. Consider a non-trivial extension of $L, L' \in S$
\[0 \to L \to M \to L' \to 0,
\]then $rM \subset L$ and $rL = 0$. Therefore $[r, r]M = 0$, and hence $M$ is a $\mathfrak{g}$-module. That implies the first assertion.

Now we use the fact that $M$ splits over $\mathfrak{g}_s, s$, and the action $R \otimes L' \to L$ is a $\mathfrak{g}_{ss}$-invariant map. Therefore $\text{Ext}^1_{\mathfrak{g}}(L', L) \simeq \text{Hom}_{\mathfrak{g}_s}(L' \otimes R, L)$. \(\square\)

**Corollary 4.5.** $Q(\mathfrak{g}) = Q(\bar{\mathfrak{g}})$ and $Q^+(\mathfrak{g}) = Q^+(\bar{\mathfrak{g}})$.

Let $(\mathfrak{g})$ (respectively, $(\bar{\mathfrak{g}})$) be the direct sum of $P(L)$ over all up to isomorphism simple $L$ in $\mathfrak{g}$-$\text{mod}$ (respectively, $\mathfrak{g}$-$\text{mod}_{\bar{\mathfrak{g}}}$), $A(\mathfrak{g}) := \text{End}_{\mathfrak{g}}(P), A^+(\mathfrak{g}) := \text{End}_{\bar{\mathfrak{g}}}(P')$.

By general results usually attributed to Gabriel (see Appendix) we know that $\mathfrak{g}$-$\text{mod}_1$ and $\mathfrak{g}$-$\text{mod}_{\bar{\mathfrak{g}}}$ are equivalent to the categories of finite-dimensional right $A(\mathfrak{g})$-modules and $A^+(\mathfrak{g})$-modules respectively.

**4.2. Radical filtration of indecomposable projectives.** In what follows we will need a description of the first three layers of the radical filtration of an indecomposable projective $P(L)$. To simplify notations we set $P^k(L) := \text{rad}^k P(L)/\text{rad}^{k+1} P(L)$. We have $L = P(L)/\text{rad} P(L)$ by definition and $P^1(L) = (R \otimes L)^{sh}$ by Lemma 14.

Let $U(\mathfrak{r})$ be the universal enveloping algebra of $\mathfrak{r}$ and $\mathfrak{N} = \langle \mathfrak{r} \rangle$ denote the augmentation ideal. We observe first that $I(L) \simeq U(\mathfrak{r}) \otimes L$ as a module over $\mathfrak{r}$. Since the action of $\mathfrak{r}$ is nilpotent on all modules in the category $(\mathfrak{g}, \mathfrak{g}_{ss})$-$\text{mod}$, we obtain that
\[\text{rad}^k I(L) = \mathfrak{N}^k \otimes L.
\]Since $P(L) = I(L)^{sh}$ is a quotient of $I(L)$ we obtain
\[\text{rad}^k P(L) = \mathfrak{N}^k P(L).
\]

We proceed to describing $P^2(L)$. Let $L$ be a simple $\mathfrak{g}_{ss}$-module and
\[\pi : R \otimes R \otimes L \to (R \otimes (R \otimes L)^{sh})^{sh}, \quad p : R^2 \otimes L \to (R^2 \otimes L)^{sh}
\]be the maps induced by the canonical projection $X \to X^{sh}$. Consider also the maps
\[\delta : \Lambda^2 R \to R^2, \quad \delta(x, y) := -[x, y] \mod \mathfrak{r}^3
\]and
\[\text{alt} : \Lambda^2 R \to R \otimes R, \quad \text{alt}(x, y) := x \otimes y - y \otimes x.
\]

**Lemma 4.6.** Let $L$ be a semisimple $\mathfrak{g}_{ss}$-module. Consider the maps
\[(9) \quad \mu : \Lambda^2 R \otimes L \xrightarrow{\text{alt} \otimes 1} R \otimes R \otimes L \xrightarrow{\pi} (R \otimes (R \otimes L)^{sh})^{sh}
\]
and
\[\lambda : \Lambda^2 R \otimes L \xrightarrow{\delta \otimes 1} R^2 \otimes L \xrightarrow{p} (R^2 \otimes L)^{sh}.
\]
Then $P^2(L)$ is isomorphic to the cokernel of $\mu \oplus \lambda$.

**Proof.** The universal enveloping algebra $U(\mathfrak{r})$ is the quotient of the tensor algebra $T(\mathfrak{r})$ by the ideal generated by $x \otimes y - y \otimes x - [x, y]$. In particular, at the second layer of the augmentation filtration we have
\[\mathfrak{N}^2/\mathfrak{N}^3 \simeq (R \otimes R \otimes R^2) / (\text{alt}(x \otimes y) + \delta(x, y))_{x, y \in R}.
\]Therefore $I^2(L) := \text{rad}^2 I(L)/\text{rad}^3 I(L)$ is the cokernel of $\text{alt} \otimes 1 \oplus (\delta \otimes 1)$.

Thus, the statement follows from the commutative diagram
\[(10) \quad \xymatrix{ R \otimes R \otimes L \ar[r] \ar[d] & (R \otimes (R \otimes L)^{sh})^{sh} \ar[d] \\
I^2(L) \ar[r] & P^2(L)}
\] \(\square\)
Example 4.7. Let \( g = \hat{g} = \mathfrak{sl}_2 \) \( \supseteq R \), where \( R = R_1 \oplus R_2 \) is a direct sum of two adjoint representations. Let \( L \) be the standard two dimensional \( \mathfrak{sl}_2 \)-module. Let us calculate \( P^2(L) \) in \( \hat{g}\text{-mod}_2 \).

We observe that

\[
(R \otimes L)^{sh} \simeq L \oplus L, \quad (R \otimes (R \otimes L)^{sh})^{sh} \simeq L \oplus L \oplus L \oplus L.
\]

For any \( (x_1, x_2), (y_1, y_2) \in R \) and \( v \in L \) we have

\[
\pi((x_1, x_2), (y_1, y_2), v) = (x_1 y_1 v, x_1 y_2 v, x_2 y_1 v, x_2 y_2 v),
\]

and

\[
\mu((x_1, x_2), (y_1, y_2), v) = (x_1 y_1 v, x_1 y_2 v - y_1 x_2 v, x_2 y_1 v - y_2 x_1 v, x_2 y_2 v).
\]

One can check that \( \text{Coker } \mu = 0 \) and hence \( P^2(L) = 0 \).

Now consider the universal central extension \( \hat{g} \) of \( g \). Then we have \( \hat{g} = g \oplus k \) as a vector space, \( R^2 = k \) and

\[
\delta((x_1, x_2), (y_1, y_2)) = \text{tr} x_1 y_2 - \text{tr} x_2 y_1.
\]

Then \( R^2 \otimes L = L \) and

\[
\lambda((x_1, x_2), (y_1, y_2), v) = (\text{tr} x_1 y_2 - \text{tr} x_2 y_1)v.
\]

Note that for \( A, B \in \mathfrak{sl}_2 \) and \( v \in V \) we have

\[
ABv + BAv = (2\text{tr} AB)v.
\]

That implies \( P^2(L) = \text{Coker}(\lambda \oplus \mu) = L \).

The above example illustrates that \( g\text{-mod}_2 \) and \( \hat{g}\text{-mod}_2 \) are not equivalent. To construct a similar example for the categories \( g\text{-mod}_1 \) and \( \hat{g}\text{-mod}_1 \), consider the Lie algebra \( g \oplus \mathfrak{sl}_2 \) and \( P(L \otimes V) \), where \( V \) is the standard module over the second copy of \( \mathfrak{sl}_2 \) and \( \otimes \) stands for the exterior tensor product.

Lemma 4.8. Assume that \( [r, r] = 0 \). Then for any \( L \in g\text{-mod}_2 \) we have \( P^2(L) = 0 \).

Proof. We use the fact that \( P^2(L) \) is a \( g_{ss} \)-submodule of \( S^2(R) \otimes L \). Since we have \( (r_1 \otimes L_{\frac{1}{2}})^{sh} = (r_{-1} \otimes L_{-\frac{1}{2}})^{sh} = 0 \), \( P^2(L)_{-\frac{1}{2}} \) is in fact a submodule in

\[
M = (S^2 r_0 \otimes L_{-\frac{1}{2}}) \oplus (r_0 \otimes r_{-1} \otimes L_{\frac{1}{2}}).
\]

By our assumption on \( g \) there are no \( (g_{ss})_{-1} \)-invariant vectors in \( r_0 \). Therefore \( M \) also does not have \( (g_{ss})_{-1} \)-invariant vectors. Hence \( P^2(L) = 0 \). \( \square \)

4.3. Ext quivers \( J\text{-mod}_1 \) and \( \hat{g}\text{-mod}_1 \). Now let \( J := \text{Lie}(g) \). Consider the category \( J\text{-mod}_1 \) and recall the functor \( \text{Jor} : \hat{g}\text{-mod}_1 \to J\text{-mod}_1 \). If \( L \in \hat{g}\text{-mod}_1 \) is simple and not trivial, then \( \text{Jor}(L) \) is simple in \( J\text{-mod}_1 \) and \( \text{Jor}(P(L)) = P(\text{Jor}(L)) \) by Lemma 3.8 and Lemma 3.5.

Lemma 4.9. Let

\[
P(J) = \bigoplus_{L \neq \text{tr}} P(\text{Jor}(L))
\]

and \( A(J) = \text{End}_J(P(J)) \). Then

\[
A(J) = (1 - e_{\text{tr}})A(\hat{g})(1 - e_{\text{tr}}),
\]

where \( e_{\text{tr}} \) is the idempotent corresponding to the projector onto \( P(\text{tr}) \).

Proof. Follows immediately from Lemma 3.8 and the identity

\[
\text{Lie}(P(J)) = (1 - e_{\text{tr}})P,
\]

where \( P \) is the direct sum of all up to isomorphism indecomposable projectives in \( \hat{g}\text{-mod}_1 \). \( \square \)

Corollary 4.10. Let \( Q(J) \) be the Ext quiver of the category \( J\text{-mod}_1 \) and \( Q'(g) \) be the quiver obtained from \( Q(g) \) by removing the vertex corresponding to the trivial representation. Then \( Q'(g) \) is obtained from \( Q(J) \) by removing some edges.
Proof. In notations of the previous proof we have
\[ \text{Jor}(P^1) \subset \text{rad}P(J)/\text{rad}^2P(J). \]
Hence the statement. \hfill \Box

Corollary 4.11. Let \( \hat{g} = g \), the radical \( r = R \) is abelian and simple over \( g_{ss} \). Then \( Q(J) = Q'(g) \).

Proof. We have to check that for any non-trivial simple \( L \in \hat{g}\)-mod \( 1 \) we have
\[ \text{Jor}(P^1(L)) = \text{rad}(\text{Jor}(P(L)))/\text{rad}^2(\text{Jor}(P(L))). \]
As we already mentioned in the previous corollary we have
\[ \text{Jor}(P^1(L)) \subset \text{rad}(\text{Jor}(P(L)))/\text{rad}^2(\text{Jor}(P(L))). \]
If the inclusion is strict, then by Corollary 3.10 \( P^1(L) \) contains a trivial submodule and \( P(L) \) has an indecomposable quotient \( M \) of length 3 such that
\[ M/\text{rad}M = L, \text{rad}M/\text{rad}^2M = tr, \text{rad}^2M = L', \]
where \( L' \) is some irreducible \( g \)-module. Consider the decomposition \( M = L \oplus tr \oplus L' \) over \( g_{ss} \).
Since the action \( R \otimes M \to M \) is \( g_{ss} \)-invariant and \( R(\text{rad}^dM) \subset \text{rad}^{d+1}M \), we have \( L \cong R^n, L' \cong R \) and for any \( x \in R, a \in L, b \in k, c \in L' \)
\[ x(a,b,c) = (0, t_1\langle x, a \rangle, t_2bx) \]
for some \( t_1, t_2 \in k \). By obvious calculation \( xy \neq yx \) if \( x \) and \( y \) are not proportional. Hence there is no such module. \hfill \Box

5. Applying Jor and Lie to the Case of Jordan Algebras of Bilinear Form

Let \( E \) be a finite-dimensional \( k \)-vector space of dimension greater or equal 2 and \( q \) be a symmetric bilinear form on \( E \). Then a Jordan algebra of a bilinear form \( J = J(E, q) \) is a vector space \( E \oplus k \) endowed with a multiplication \( \circ \)
\[ (e_1 + \lambda_1) \circ (e_2 + \lambda_2) = \lambda_1 \lambda_2 + q(e_1, e_2) + \lambda_1 e_2 + \lambda_2 e_1, \]
e_1, e_2 \in E, \lambda_1, \lambda_2 \in k. In what follows we assume that \( q \) is non-degenerate and consequently \( J(E, q) \) is a simple Jordan algebra. It is useful to notice that \( J(E, q) \) is a Jordan subalgebra in the Clifford algebra \( C(E, q) \) generated by \( E \subset C(E, q) \). If \( \dim E \) is even, then \( C(E, q) \cong \text{End}_k(S) \), and \( S \) is a unique up to isomorphism special irreducible \( J \)-module. If \( \dim E \) is odd, then \( C(E, q) \cong \text{End}_k(S^+ \oplus S^-) \), and \( J \) has two simple special modules \( S^+ \) and \( S^- \).
We proceed to describing \( g = \text{Lie}(J) \). Let \( V \) be a \( n \)-dimensional vector space equipped with non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \). The orthogonal Lie algebra \( g = so_n \) is the algebra of endomorphisms \( A : V \to V \) satisfying \( \langle Av, v \rangle + \langle v, Av \rangle = 0 \) for all \( v, w \in V \). If \( L \subset V \) be a subspace of codimension 2 such that \( \langle \cdot, \cdot \rangle \) is non-degenerate on \( F \). Choose a basis \( \xi, \eta \in F^\perp \) such that \( \langle \xi, \eta \rangle = 1, \langle \xi, \xi \rangle = \langle \eta, \eta \rangle = 0 \). Let \( h \in g \) such that \( h\xi = \xi, h\eta = -\eta \), \( h(F) = 0 \). Then \( h \) defines a short \( Z \)-grading of \( g \) such that
\[ g_0 = \{ A \in g \, | \, A(F) \subset F \}, \quad g_1 = \{ A \in g \, | \, A(\xi) \in F, A(F) \subset k\xi \}, \]
\[ g_{-1} = \{ A \in g \, | \, A(\xi) \in F, A(F) \subset k\eta \} \]
Any non-zero element \( f \in g_1 \) defines a Jordan algebra structure on \( g_{-1} \) isomorphic to \( J \). In this way \( n = \dim E + 3 \).

Next we describe simple objects in \( g\)-mod \( 1 \) and \( g\)-mod \( 2 \). This description is slightly different in even and odd case. Let \( n = 2m \) or \( 2m + 1 \), \( \omega_1, \ldots, \omega_m \) denote the fundamental weights. We denote by \( \Gamma \) the spinor representation of \( so_n \) with highest weight \( \omega_m \) for \( n = 2m + 1 \) and by \( \Gamma^\pm \) the spinor representations with highest weights \( \omega_{m-1} \) and \( \omega_m \) for \( n = 2m \), see Section 20.1 in \[18\] for details. Other irreducible fundamental representations of \( so_n \) can be obtained by taking the exterior powers of the standard representation \( V \). If \( n = 2m + 1 \) they are \( \Lambda^nV \) for \( i = 1, \ldots, m-1 \) with highest weights \( \omega_1, \ldots, \omega_{m-1} \) respectively. Note that \( \Lambda^nV \) is irreducible with highest weight \( 2\omega_m \). If \( n = 2m \), then \( \Lambda^nV \) for \( i = 1, \ldots, m-2 \) are fundamental representations, \( \Lambda^{m-1}V \) is irreducible with highest weight \( \omega_{m-1} + \omega_m \). Finally, \( \Lambda^nV \) splits into direct sum of two simple modules \( \Lambda^+V \oplus \Lambda^-V \) with highest weights \( 2\omega_{m-1} \) and \( 2\omega_m \) respectively.
Remark 5.3. An orthogonal Lie algebra

In the case when $\omega$ is its Dynkin diagram. It swaps $L$

Therefore if

The formulas for tensor products are given in [19], table 5, applying Lemma 5.1.

Let $\Gamma$

object in $g$

Next we calculate $(\Lambda r\otimes \Lambda' V)_{sh}$ for simple $M$, $N \in \mathcal{S}$, when $g = \mathfrak{so}_n$.

Lemma 5.4. Let $g = \mathfrak{so}_n$ with $n = 2m$ or $2m + 1$.

(1) For any $1 \leq r \leq q \leq m$,

$(\Lambda^q V \otimes \Lambda' V)_{sh} = \bigoplus_{i=0}^{r} \Lambda^{q-r+2i} V.$

(2) If $n = 2m$, then for any $1 \leq r \leq m - 1$

$(\Lambda^r \otimes \Lambda' V)_{sh} = \begin{cases} 
\Lambda^r V \oplus \Lambda^{m-2} V \oplus \cdots \oplus \Lambda^{m-r} V & \text{if } r \text{ is even,} \\
\Lambda^{m-1} V \oplus \Lambda^{m-3} V \oplus \cdots \oplus \Lambda^{m-r} V & \text{if } r \text{ is odd.}
\end{cases}$

(3) Suppose $n = 2m$, $1 \leq r \leq m$, then

$(\Gamma^r \otimes \Lambda' V)_{sh} = \begin{cases} 
\Gamma_r, & \text{if } r \text{ is even,} \\
\Gamma^r, & \text{if } r \text{ is odd,}
\end{cases}$

(4) If $n = 2m$

$(\Gamma^r \otimes \Gamma^s)_{sh} = (\Gamma^r \otimes \Gamma^s)_{sh} \oplus \bigoplus_{i=1}^{\frac{m}{2}} \Lambda^{m-2i} V,$

$(\Gamma^+ \otimes \Gamma^-)_{sh} = \bigoplus_{i=1}^{\frac{m}{2}} \Lambda^{m-2i+1} V.$

(5) If $n = 2m + 1$,

$(\Gamma \otimes \Gamma)_{sh} = \bigoplus_{i=0}^{m} \Lambda^i V.$

Proof. The formulas for tensor products are given in [19], table 5, applying $\text{sh}$ is straightforward.
6. Admissible quivers

We call the quiver \( Q(\mathfrak{g}) \) admissible if the associative algebra \( kQ'(\mathfrak{g})/\text{rad}^2 \) is tame. That happens exactly when the double quiver of \( Q'(\mathfrak{g}) \) is tame, see Theorem 10.1. Let \( J \) be a unital Jordan algebra and \( \mathfrak{g} = \text{Lie}(J) \). Lemma 4.9 and Corollary 4.10 imply that if \( A(J) \) is tame, then \( Q(\mathfrak{g}) \) is admissible. Therefore the first step towards classification of tame \( A(J) \) is to classify admissible \( Q(\mathfrak{g}) \).

For the rest of this paper \( J \) will be a unital Jordan algebra such that \( J_{ss} \) is a direct sum of Jordan algebras \( J(E,q) \), where \( q \) is non-degenerate and \( \dim E \geq 4 \), \( \mathfrak{g} \) is the Lie algebra obtained from \( J \) by the Tits-Kantor-Koecher construction.

In this section we classify indecomposable Lie algebras \( \mathfrak{g} \) with admissible quivers \( Q(\mathfrak{g}) \) such that \( \mathfrak{g}_{ss} \) is a direct sum of \( \mathfrak{so}_n \) with \( n \geq 7 \). If \( \mathfrak{g}_{ss} = \mathfrak{so}_{n_1} \oplus \mathfrak{so}_{n_2} \), then \( V \) and \( W \) denote the standard representations of \( \mathfrak{so}_{n_1} \) and \( \mathfrak{so}_{n_2} \) respectively.

**Theorem 6.1.** Let \( \mathfrak{g} = \text{Lie}(J) \), where \( J \) is a unital indecomposable Jordan algebra, such that \( J_{ss} \) is a direct sum of Jordan algebras of bilinear form over vector space of dimension greater or equal than 4 and \( \tau \neq 0 \). If \( Q(\mathfrak{g}) \) is admissible, then \( Q(\mathfrak{g}) \) is one of the following quivers:

\[
Q_1^{2m+1} : \quad \begin{array}{cccccccc}
\gamma_0 & \rightarrow & \cdots & \rightarrow & \gamma_m \\
\delta_0 & \rightarrow & \cdots & \rightarrow & \delta_m
\end{array}
\]

\[
Q_1^{2m} : \quad \begin{array}{cccccccc}
\gamma_0 & \rightarrow & \cdots & \rightarrow & \gamma_m \\
\delta_0 & \rightarrow & \cdots & \rightarrow & \delta_m
\end{array}
\]

\[
Q_2 : \quad \begin{array}{cccccccc}
\gamma_0 & \rightarrow & \cdots & \rightarrow & \gamma_m \\
\delta_0 & \rightarrow & \cdots & \rightarrow & \delta_m
\end{array}
\]

\[
Q_3 : \quad \begin{array}{cccccccc}
\gamma_0 & \rightarrow & \cdots & \rightarrow & \gamma_m \\
\delta_0 & \rightarrow & \cdots & \rightarrow & \delta_m
\end{array}
\]
Proof. Suppose so

Further refer to this list as List A

Indecomposable Lie algebras \( g = g = g_{ss} \oplus R \) with admissible \( Q(g) \) are listed below. We will further refer to this list as List A:

1. \( g = \mathfrak{so}_{2m+1} \supset V, m \geq 3, Q(g) = Q_{2m+1}^{1}; \)
2. \( g = \mathfrak{so}_{2m} \supset V, m \geq 4, Q(g) = Q_{2m}^{1}; \)
3. \( g = \mathfrak{so}_8 \supset \Lambda_{\pm}; Q(\mathfrak{so}_8 + \Lambda_{\pm}) = Q_2, \) while \( Q(\mathfrak{so}_8 \supset \Lambda^-) \) is obtained by application of \( \tau \) to \( Q_2, \) see Remark 5.3.
4. \( g = (\mathfrak{so}_8 \oplus \mathfrak{so}_8) \supset \Gamma_1^+ \otimes \Gamma_2^+; Q((\mathfrak{so}_8 \oplus \mathfrak{so}_8) \supset \Gamma_1^+ \otimes \Gamma_2^+) = Q_3, \) the quivers corresponding to \( \Gamma_1^+ \otimes \Gamma_2^+, (\Gamma_1^+ \otimes \Gamma_2^+ \text{ and } \Gamma_1^- \otimes \Gamma_2^-) \) are obtained from \( Q_3 \) applying \( 1 \times \tau \) (respectively \( \tau \times 1 \) and \( \tau \times \tau \)).
5. \( g = (\mathfrak{so}_8 \oplus \mathfrak{so}_{10}) \supset \Gamma_1^+ \otimes \Gamma_2^+; Q((\mathfrak{so}_8 \oplus \mathfrak{so}_{10}) \supset \Gamma_1^+ \otimes \Gamma_2^+) = Q_4, \) while other quivers are obtained by application of \( 1 \times \tau, \tau \times 1 \) and \( \tau \times \tau \) to \( Q_4. \)
6. \( g = (\mathfrak{so}_{10} \oplus \mathfrak{so}_{10}) \supset \Gamma_1^+ \otimes \Gamma_2^+; Q((\mathfrak{so}_{10} \oplus \mathfrak{so}_{10}) \supset \Gamma_1^+ \otimes \Gamma_2^+) = Q_5, \) while other quivers are obtained by application of \( 1 \times \tau, \tau \times 1 \) and \( \tau \times \tau \) to \( Q_5. \)

Proof. Suppose \( J \) satisfies the conditions of the theorem, then \( g = Lie(J) = g_{ss} \supset \tau, \) where \( g_{ss} \) is a direct sum of orthogonal algebras \( \mathfrak{so}_n, n \geq 7. \) Since \( Q(g) = Q(\hat{g}) \) we may assume that \( g = \hat{g} \) and
hence $r = R$. To construct $Q(g)$ we use Lemma 4.4. We start with classifying admissible quivers $Q(g)$ in the case when $R$ is an irreducible faithful $g_{sa}$-module.

Consider first the case $g = \mathfrak{so}_{2m+1} \ni R$, $m \geq 3$. There are $m + 1$ simple modules in the category $\mathfrak{g}$-$\text{mod}_1$, namely $tr$ and $\Lambda^r V$, $r = 1, \ldots, m$, Thus $Q(g)$ has $m + 1$ vertices. Let $R = V$ be the standard representation of $\mathfrak{so}_{2m+1}$. Tensor product formulas in Lemma 5.4(a) imply that the quiver of $\mathfrak{so}_{2m+1} \ni V$ is $Q_{2m+1}^2$. It is admissible by Theorem 10.1(2).

Next we claim that if $R = \Lambda^r V$, $r \geq 2$ then the quiver $Q(g)$ is not admissible. Indeed, Lemma 5.4(b) implies that $\Lambda^m V$ and $\Lambda^{m-1} V$ are simple constituents of $\Lambda^m V \otimes \Lambda^r V$, $\Lambda^m V, \Lambda^{m-1} V, \Lambda^{m-2} V$ are simple constituents of $\Lambda^{m-1} V \otimes \Lambda^r V$. Therefore $Q(g)$ has the following subquiver

\begin{equation}
\Lambda^{m-2} V \quad \Lambda^{m-1} V \quad \Lambda^m V
\end{equation}

The corresponding double quiver is wild by Theorem 10.1(2), hence the double quiver of $Q'(g)$ is wild by Lemma 10.2. Therefore $Q(g)$ is not admissible.

Next, let us consider the case $g = \mathfrak{so}_{2m} \ni R$, with $m \geq 4$. All up to isomorphism simple objects of $\mathfrak{g}$-$\text{mod}_1$ are $tr$, $\Lambda^r V$, $r = 1, \ldots, m-1$ and $\Lambda^+ V, \Lambda^- V$. By Lemma 5.4 $Q(\mathfrak{so}_{2m} \ni V) = Q_{2m}^2$. It is again admissible by Theorem 10.1(2).

Let $R = \Lambda^r V$, $r = 2, \ldots, m - 1$. We will show that $Q(g)$ is not admissible. Let $m$ be even, then $\Lambda^{m-1} V \otimes \Lambda^r V$ contains $\Lambda^{m-1} V$ with multiplicity 2 and $\Lambda^m V$ with multiplicity 1. Hence $Q(g)$ has the subquiver

\begin{equation}
\Lambda^{m-3} V \quad \Lambda^{m-1} V
\end{equation}

Thus, $Q(g)$ is not admissible by Theorem 10.1 and Lemma 10.2.

Similarly, if $m$ is odd, $\Lambda^{m-1} V \otimes \Lambda^r V$ contains $\Lambda^{m-2} V$ with multiplicity 2 and $\Lambda^+ V$ with multiplicity 1. Therefore $Q(g)$ has the wild subquiver

\begin{equation}
\Lambda^{m-2} V \quad \Lambda^{m-1} V \quad \Lambda^+ V
\end{equation}

Thus, $Q(g)$ is not admissible by Lemma 10.2

Let $R = \Lambda^\pm V$. For $g = \mathfrak{so}_8 \ni \Lambda^+ V$ the Ext quiver of $\mathfrak{g}$-$\text{mod}_1$ is $Q_2$, which is admissible. The same applies to $Q(g)$ for $g = \mathfrak{so}_8 \ni \Lambda^- V$, since the involution $\tau$ interchanges the vertices $\Lambda^+ V$ and $\Lambda^- V$ of $Q_2$.

By Lemma 5.4(2) we obtain that $Q'(g)$, where $g = \mathfrak{so}_{10} \ni \Lambda^+ V$, has the subquiver

\begin{equation}
\mathfrak{so}_{10} + \Lambda^+:
\end{equation}

By Theorem 10.1(2) the double quiver of the above quiver is wild. Hence $Q(g)$ is not admissible. The same argument works for $R = \Lambda^- V$.

For $m \geq 6$ one of the following subquivers

\begin{equation}
\Lambda^{m-4} V \quad \Lambda^{m-2} V \quad \Lambda^+ V
\end{equation}

if $m$ is even
is a subquiver of $Q'(\mathfrak{so}_{2n} + \Lambda^+V)$. Both have wild double quivers and hence are not admissible.

Next we move to the case when $\mathfrak{g}_{ss} = \mathfrak{so}_{m_1} \oplus \mathfrak{so}_{m_2}$, $m_1, m_2 \geq 7$, and $R$ is an exterior tensor product of spinor modules $\Gamma$ or $\Gamma^\pm$, depending on the parity of $m_1$ and $m_2$, see Section 5 for details.

First, assume that $m_1 = 2m + 1$ and $m_2 = 2n + 1$ are odd, $m, n \geq 3$. Denote by $\Gamma_i$, $i = 1, 2$ the spinor representations of $\mathfrak{so}_{m_i}$. Let $\mathfrak{g} = (\mathfrak{so}_{2m+1} \oplus \mathfrak{so}_{2n+1}) \vartriangleright \Gamma_1 \boxtimes \Gamma_2$. Lemma 5.4 (5) implies

$$(\Gamma_1 \boxtimes \Gamma_2) \otimes (\Gamma_1 \boxtimes \Gamma_2))^{ss} = \bigoplus_{i=0}^{m} \Lambda^i V \otimes \bigoplus_{i=0}^{n} \Lambda^i W.$$

Therefore $Q'(\mathfrak{g})$ has a wild subquiver

$$(23) \quad \Lambda^2 V \quad \Lambda^3 V \quad \Lambda^2 W \quad W$$

and hence $Q(\mathfrak{g})$ is not admissible.

Next let $m_1 = 2m + 1, m_2 = 2n, n \geq 3, m \geq 4$ and $\mathfrak{g} = (\mathfrak{so}_{2m+1} \oplus \mathfrak{so}_{2n}) \vartriangleright \Gamma_1 \boxtimes \Gamma_2^\pm$. By Lemma 5.4 (4) and (5) we easily obtain that $((\Gamma_1 \boxtimes \Gamma_2^\pm) \otimes (\Gamma_1 \boxtimes (\Gamma_2^\pm)^*))^{ss}$ has at least five simple constituents:

$V, \quad \Lambda^2 V, \quad \Lambda^3 V, \quad \Lambda^2 W, \quad \Lambda^4 W \quad (\Lambda^+W$ if $m = 4$).

Therefore the vertex $\Gamma_1 \boxtimes (\Gamma_2^\pm)^*$ has at least five outgoing arrows in $Q'(\mathfrak{g})$. As in the previous case $Q(\mathfrak{g})$ is not admissible. The case of the algebra $R = \Gamma_1 \boxtimes \Gamma_2^\pm$ can be reduced to the previous one by applying $1 \times \tau$.

Finally, we have to deal with the case when both $m_1 = 2m$ and $m_2 = 2n$ are even, $n \geq m \geq 4$. Using Lemma 5.3 (4), 5 we obtain that quivers $Q(\mathfrak{g}_i)$, $i = 3, 4, 5$ of algebras

$\mathfrak{g}_3 = (\mathfrak{so}_9 \oplus \mathfrak{so}_8) \vartriangleright \Gamma_1^+ \boxtimes \Gamma_2^+, \quad \mathfrak{g}_4 = (\mathfrak{so}_8 \oplus \mathfrak{so}_{10}) \vartriangleright \Gamma_1^+ \boxtimes \Gamma_2^+, \quad \mathfrak{g}_5 = (\mathfrak{so}_{10} \oplus \mathfrak{so}_{10}) \vartriangleright \Gamma_1^+ \boxtimes \Gamma_2^+$

are $Q_3, Q_4$ and $Q_5$ respectively. By direct inspection they are admissible. Furthermore, the same is true for $R = \Gamma_1^+ \boxtimes \Gamma_2^\pm$, by application of $\tau \times 1, 1 \times \tau$ or $\tau \times \tau$.

We claim that if $m \geq 4, n \geq 6$ and $R = \Gamma_1^+ \boxtimes \Gamma_2^\pm$, then $Q(\mathfrak{g})$ is not admissible. Indeed, we use the same argument as before. Lemma 5.3 (4) implies that $((\Gamma_1^+ \boxtimes \Gamma_2^\pm) \otimes (\Gamma_1^+)^* \boxtimes (\Gamma_2^\pm)^*))^{ss}$ has at least five simple constituents:

$\Lambda^2 V, \quad \Lambda^4 V \quad (\Lambda^+V$ if $m = 4), \quad \Lambda^2 W, \quad \Lambda^4 W, \quad \Lambda^6 W \quad (\Lambda^+W$ if $n = 6)$.

There are at least five outgoing arrows in $Q'(\mathfrak{g})$ from the vertex $(\Gamma_1^+)^* \boxtimes (\Gamma_2^\pm)^*$. Theorem 10.1 implies $Q(\mathfrak{g})$ is not admissible.

We have shown that if $R$ is a faithful irreducible module, $\mathfrak{g}$ is indecomposable, then $Q(\mathfrak{g}_{ss} \vartriangleright R)$ is admissible if and only if $\mathfrak{g}$ is one of the algebras from List A. It remains to prove that $Q(\mathfrak{g})$ is not admissible if $R$ is not simple. It follows from the observation that adding an irreducible component to $R$ implies adding at least one outgoing arrow to the vertex corresponding to $\Lambda^2 V$ or $\Lambda^2 W$. We leave it to the reader to check that adding such an arrow to one of the quivers from the list makes the corresponding double quiver wild. \hfill \Box

Remark 6.2. If $\mathfrak{g}$ is from List A, then $\hat{\mathfrak{g}} = \mathfrak{g}$ since $(\Lambda^2 R)^{ss} = 0$. 
7. Relations in the case of abelian radical

Let us assume that \( \mathfrak{g} = \text{Lie}(J) \) is a Lie algebra from **List A**. The goal of this section is to show that for any such \( \mathfrak{g} \) the algebra \( A(J) \) is tame. Recall that by Corollary 5.11, \( A(J) \) is a quotient of the path algebra \( kQ'(g) \) by some ideal \( I \). It turns out that \( I \) is generated by quadratic relations and to describe it it is sufficient to calculate \( P^2(L) \) for simple \( L \) in \( \mathfrak{g} \text{-mod}_1 \). We will see also that \( \text{rad}^2(A(J)) = 0 \).

7.1. The case of simple \( \mathfrak{g}_{ss} \). In this subsection we assume that \( \mathfrak{g}_{ss} \) is simple, i.e. \( \mathfrak{g} \) is a Lie algebra from (1), (2) or (3) of **List A**.

**Proposition 7.1.** Let \( \mathfrak{g} = \mathfrak{so}_n \triangleright V \). Then the first three layers of the radical filtration of indecomposable projectives in \( \mathfrak{g} \text{-mod}_1 \) are as follows

\[
\begin{array}{c|c|c}
\Lambda^r V & \Lambda^{r-1} V \oplus \Lambda^{r+1} V & \Lambda^{r-2} V \oplus \Lambda^r V \oplus \Lambda^{-r} V \\
\hline
\Lambda^r V & \Lambda^{r-1} V & \Lambda^{r-2} V \oplus \Lambda^r V \oplus \Lambda^{-r} V \\
\end{array}
\]

\[(24)\]

if \( n = 2m + 1 \), \( 0 \leq r \leq m \)

if \( n = 2m \), \( 0 \leq r \leq m - 2 \)

**Remark 7.2.** Note that for an odd \( n \) we have \( P^1(\Lambda^m V) = \Lambda^{m-1} V \oplus \Lambda^m V \) due to isomorphism \( \Lambda^m V \cong \Lambda^{m+1} V \). We also assume \( \Lambda^{-1} V = 0 \).

**Proof.** For any \( v \in V \) we introduce the following operators \( m_v, i_v \in \text{End}(\Lambda^* V) \):

\[
\begin{align*}
m_v(x_1 \wedge \cdots \wedge x_r) &= v \wedge x_1 \wedge \cdots \wedge x_r, \\
i_v(x_1 \wedge \cdots \wedge x_r) &= \sum_{i=1}^{r} \langle v, x_i \rangle (-1)^{i-1} x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_r,
\end{align*}
\]

\[x = x_1 \wedge \cdots \wedge x_r \in \Lambda^r V. \]

These operators satisfy the following well-known relations:

\[
\begin{align*}
i_v i_w + i_w i_v &= 0; \\
m_v m_w + m_w m_v &= 0; \\
i_v m_w + m_w i_v &= (v, w).
\end{align*}
\]

Moreover, the action of algebra \( \mathfrak{so}_n \cong \Lambda^2 V \) on \( \Lambda^* V \) can be written as

\[(26)\]

\[
T_{v \wedge w}(x) = m_v i_w(x) - m_w i_v(x), \quad x \in \Lambda^* V, \quad v \wedge w \in \mathfrak{so}_n.
\]

First assume that \( n = 2m + 1 \), then any simple module \( L \) in \( \mathfrak{g} \text{-mod}_1 \) is isomorphic to \( \Lambda^r V \) for some \( 0 \leq r \leq m \).

By Lemma 5.4

\[
P^1(L) = (L \otimes V)^{sh} \cong \Lambda^{r-1} V \oplus \Lambda^{r+1} V.
\]

Note that \( \mathfrak{g}_{ss} \)-invariant maps \( \Lambda^r V \otimes V \to \Lambda^{r+1} V \) and \( \Lambda^r V \otimes V \to \Lambda^{r-1} V \) are given by \( x \otimes v \mapsto m_v(x) \) and \( x \otimes v \mapsto i_v(x) \) respectively. To describe \( P^2(L) \) we use Lemma 4.10 with \( \lambda = 0 \). Indeed,

\[
(R \otimes (R \otimes L)^{sh})^{sh} = \Lambda^{r+2} V \oplus \Lambda^r V \oplus \Lambda^r V \oplus \Lambda^{r-2} V,
\]

and (20), (27) imply

\[
\mu(v, w, x) = (2m_v m_w(x), -T_{v \wedge w}(x), T_{v \wedge w}(x), 2i_v i_w(x)).
\]

That implies \( P^2(L) \cong L \).

Now let \( n = 2m \). In this case the calculation of the radical filtration of \( P(L) \) for a simple \( L = \Lambda^r V \) for \( r \leq m - 1 \) is the same as in the case of odd \( n \). It remains to consider the cases \( L = \Lambda^\pm V \). Then we have \( P^1(L) = (L \otimes V)^{sh} = \Lambda^{m-1} V \). Recall that we have a decomposition \( \Lambda^m V = \Lambda^+ V \oplus \Lambda^- V \). After suitable normalization

\[
\Lambda^\pm V = \{ x \in \Lambda^m V | i_v(x) = \pm \psi m_v(x), \text{ for all } v \in V \},
\]

where \( \psi : \Lambda^{m+1} V \to \Lambda^{m-1} V \) is an isomorphism of simple \( \mathfrak{g}_{ss} \)-modules. Furthermore,

\[
(R \otimes (R \otimes L)^{sh})^{sh} = \Lambda^{m-2} V \oplus \Lambda^{m-1} V,
\]

\[
\mu(v, w, x) = (2i_v i_w(x), T_{v \wedge w}(x)).
\]

The relation (27) imply \( \text{Im} \mu = \Lambda^{m-2} V \oplus L \). Therefore we have \( P^2(\Lambda^\pm V) = \Lambda^\mp V \). \( \square \)
Proposition 7.3. Let \( g = \mathfrak{so}_8 \oplus \Lambda^+ V \). Then \( \Lambda^- V \) is projective and other indecomposable projectives have the following first three layers in the radical filtration

\[
\begin{array}{cccc}
\Lambda^2 V & \Lambda^+ V & \Lambda^3 V & V \\
\Lambda^2 V \oplus \Lambda^+ V & \Lambda^2 V \oplus \Lambda^+ V \oplus \text{tr} & \Lambda^3 V \oplus \text{tr} &
\end{array}
\]

\[(29)\]

Proof. It follows from Lemma \[5, 4\] that \( \Lambda^- V \) is projective in \( g\)-mod\(_1\). To describe the projective covers of \( \Lambda^2 V \) and \( \Lambda^+ V \) we use an automorphism \( \gamma \) of \( \mathfrak{so}_8 \), induced by a rotation of the Dynkin diagram \( D_4 \). Twisting by \( \gamma \) defines the following identifications on simple modules

\[
V \mapsto \Gamma^+, \quad \Lambda^2 V \mapsto \Lambda^2 V, \quad \Lambda^3 V \mapsto (V \otimes \Gamma^-)_0 \quad \Lambda^+ \mapsto S^2 V_0,
\]

where by \( S^2 V_0 \) we denote the traceless part of \( S^2 V \) and \( V \otimes \Gamma^- = (V \otimes \Gamma^-)_0 \oplus \Gamma^+ \).

Let us calculate \( P^2(L) \) for the case \( L = S^2 V_0 \) using Lemma \[4, 3\] with \( \lambda = 0 \). We identify \( S^2 V \) and \( \Lambda^2 V \) with the spaces of symmetric and skew symmetric matrices respectively. We have

\[
(R \otimes (R \otimes L)^{sh})^{sh} = \Lambda^2 V \oplus \Lambda^2 V \oplus S^2 V \oplus S^2 V_0, \quad (R^2 \otimes L)^{sh} = \Lambda^2 V \oplus S^2 V_0,
\]

and

\[
\pi(X \otimes Y \otimes A) = \{(X, [Y, A]), [X, Y, A], [X, (Y, A)], [X, [Y, A]], [X, Y, A])\},
\]

where \( \{C, B\} = CB + BC \) and \( [C, B] = CB - BC \). Next we calculate \( \mu \):

\[
\mu(X \otimes Y \otimes A) = \{[A, [X, Y]], [A, [X, Y]], [A, [X, Y]], [A, [X, Y]], [A, [X, Y]] + 2XAY - 2YAX\},
\]

From this formula we see that cokernel of \( \mu \) is isomorphic to \( S^2 V_0 \oplus \text{tr} \).

Now let \( L = \Lambda^2 V \). Then

\[
(R \otimes (R \otimes L)^{sh})^{sh} = \Lambda^2 V \oplus \Lambda^2 V \oplus S^2 V \oplus S^2 V_0, \quad (R^2 \otimes L)^{sh} = \Lambda^2 V \oplus S^2 V_0,
\]

and

\[
\pi(X \otimes Y \otimes A) = \{(X, [Y, A]), [X, Y, A], [X, [Y, A]], [X, Y, A])\},
\]

\[
\mu(X \otimes Y \otimes A) = \{[A, [X, Y]], [A, [X, Y]], [A, [X, Y]], [A, [X, Y]], [A, [X, Y]] + 2XAY - 2YAX\},
\]

and cokernel of \( \mu \) is isomorphic to \( \Lambda^2 V \).

Next we construct the projective covers of \( V \) and \( \Lambda^3 V \) in \( g\)-mod\(_1\). We will show that both modules have Loewy length two. Let \( \{e_1, e_{-1} | 1 \leq i \leq 4\} \) be the basis of \( V \) such that with respect to the form on \( V \), \( (e_i, e_{-j}) = \delta_{i-j}, \ i, j \in \{1, 2, 3, 4\} \). Then \( \Lambda^4 V \) is spanned by

\[
e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}, \quad i_1 < i_2 < i_3 < i_4,
\]

to check whether given element of \( \Lambda^4 V \) belongs to \( \Lambda^+ V \) we use \[(28)\].

From Lemma \[5, 4\]

\[
P^1(\Lambda^3 V) = (\Lambda^3 V \otimes \Lambda^+ V)^{sh} = V \oplus \Lambda^3 V, \quad P^1(V) = (V \otimes \Lambda^+ V)^{sh} = \Lambda^3 V.
\]

One can check that \( \mathfrak{so}_8\)-invariant maps

\[
\phi_3^3 : \Lambda^3 V \otimes \Lambda^+ V \rightarrow \Lambda^3 V, \quad \phi_1^3 : \Lambda^3 V \otimes \Lambda^+ V \rightarrow V, \quad \phi_0^3 : V \otimes \Lambda^+ V \rightarrow \Lambda^3 V
\]

are given by

\[
(30) \quad \phi_3^3(x_1 \wedge x_2 \wedge x_3 \otimes v_1 \wedge v_2 \wedge v_3 \wedge v_4) = \sum_{1 \leq j, k, l \leq 4 \atop j \neq k \neq l} \text{sgn}(j, k, l)m_{x_j}i_{x_k}i_{x_l}(v_1 \wedge v_2 \wedge v_3 \wedge v_4),
\]

\[
(31) \quad \phi_1^3(x_1 \wedge x_2 \wedge x_3 \otimes v_1 \wedge v_2 \wedge v_3 \wedge v_4) = \sum_{1 \leq j, k, l \leq 4 \atop j \neq k \neq l} \text{sgn}(j, k, l)i_{x_j}i_{x_k}i_{x_l}(v_1 \wedge v_2 \wedge v_3 \wedge v_4),
\]

\[
(32) \quad \phi_0^3(x_1 \otimes v_1 \wedge v_2 \wedge v_3 \wedge v_4) = i_{x_1}(v_1 \wedge v_2 \wedge v_3 \wedge v_4).
\]

Here \( v_1 \wedge v_2 \wedge v_3 \wedge v_4 \in \Lambda^+ V \), \( x_j \in V \), \( 1 \leq j \leq 4 \), \( \text{sgn}(j, k, l) \) is the sign of permutation \( (j, k, l) \in \Sigma_3 \), while \( m_{x_j} \) and \( i_{x_j} \) are given by \[(29)\].
To show that $P^2(V) = 0$ we describe map $\mu$ for $L = V$, see Lemma 4.6. Observe that $\lambda = 0$ therefore $P^2(V) = \text{Coker } \mu$. We have

$$(\Lambda^+V \otimes (\Lambda^+V \otimes V)^{sh}) = V \oplus \Lambda^3V,$$

hence

$$\mu(v, w, x) = (\phi^1_1(x \otimes v) \otimes w) - \phi^1_1(x \otimes (\phi^1_2(x \otimes v) \otimes v), \phi^3_3(x \otimes (\phi^1_2(x \otimes v) \otimes (\phi^1_2(x \otimes v) \otimes v))),$$

where $v, w \in \Lambda^+V$, $x \in V$. Suppose $v = e_1 \land e_2 \land e_3 \land e_4$, $w = e_1 \land e_2 \land e_3 \land e_4$ and $x = e_1$, then

$$\mu(v, w, x) = (e_1 \land e_1 \land e_2 \land e_2 \land e_3 \land e_3 \land e_4).$$

Since $\text{Im } \mu$ is $g_{ss}$-invariant and $\mu(v, w, x)$ generates $V \oplus \Lambda^3V$ as a $g_{ss}$-module, we obtain $P^2(V) = \text{Coker } \mu = 0$.

To check that $P^2(\Lambda^3V) = 0$, note that $(\Lambda^+V \otimes (\Lambda^+V \otimes \Lambda^3V)^{sh}) = V \oplus \Lambda^3V \oplus \Lambda^3V$ and this projection is given by

$$\pi(v, w, y) = (\phi^1_3(x \otimes v) \otimes w), \phi^1_3(x \otimes (\phi^1_2(x \otimes v) \otimes v), \phi^3_3(x \otimes (\phi^1_2(x \otimes v) \otimes (\phi^1_2(x \otimes v) \otimes v))),$$

$v, w, y \in \Lambda^3V$, $x \in \Lambda^3V$ and $\mu(v, w, y) = (alt \otimes 1) \circ \pi$, see Lemma 4.6. Choosing

$$v = e_1 \land e_2 \land e_3 \land e_4 + e_2 \land e_2 \land e_3 \land e_4, \quad w = e_1 \land e_2 \land e_3 \land e_4, \quad y = e_1 \land e_2 \land e_3$$

we obtain that

$$\mu(v, w, y) = (2e_3, e_1 \land e_1 \land e_3 \land e_4 \land e_2 \land e_2 \land e_3 \land e_4, -2e_3) \in \Lambda^3V \oplus \Lambda^3V.$$  

Observe that $g_{ss}$-submodule generated by $\mu(v, w, y)$ coincides with $V \oplus \Lambda^3V \oplus \Lambda^3V$. Hence $P^2(\Lambda^3V) = \text{Coker } \mu = 0$.

**Corollary 7.4.** If $g = so_n \supset V$ or $g = so_3 \supset \Lambda^+V$ and $P^2(L) \neq 0$, then $\text{Jor}(P^2(L))$ is simple and coincides with the socle of $\text{Jor}(P(L)/\text{rad}^3P(L))$.

**Proof.** Follows from direct description of $P(L)$. □

**Theorem 7.5.** (1) If $g = so_{2m+1} \supset V$, then $A(J) = k(Q_1^{2m+1}) / I$, where $I$ is generated by the following relations with $r = 2, \ldots, m - 1$

$$\gamma_{r-1} \gamma_r = \delta_r \delta_{r-1} = 0, \gamma_{r-1} \delta_{r-1} = \delta_r \gamma_r, \gamma_{m-1} \delta_{m-1} = \gamma_m^2.$$

(33)

(2) If $g = so_{2m} \supset V$, then $A(J) = k(Q_1^{2m}) / I$, where $I$ is generated by the following relations with $r = 2, \ldots, m - 2$

$$\gamma_\pm \delta_\pm = \gamma_{r-1} \gamma_r = \delta_r \delta_{r-1} = 0, \gamma_{r-1} \delta_{r-1} = \delta_r \gamma_r, \gamma_{m-2} \delta_{m-2} = \delta^+ \gamma^+ = \delta^- \gamma^-.$$

(34)

(3) If $g = so_3 \supset \Lambda^+V$, then $A(J) = kQ_1^2 / I$, where $I$ is generated by

$$\alpha_1 \beta_1 = \beta_1 \alpha_1 = \beta_1 \gamma_1 = \gamma_1 \alpha_1 = \gamma_1^2 = \alpha_2 \gamma_2 = \gamma_2 \alpha_2 = \beta_2 \gamma_3 = \gamma_3 \beta_2 = \beta_2 \gamma_3 = 0, \gamma^2_2 = \beta_2 \alpha_2, \gamma^2_3 = \alpha_2 \beta_2.$$

(35)

(4) All above algebras are quadratic, satisfy rad$^3A(J) = 0$. Furthermore, in the first two cases $A(J)$ is a Frobenius algebra.

**Proof.** Corollary 7.3 implies that all paths in $Q'(g)$ of length 2 leading from vertex $i$ to vertex $j$ are proportional with non-zero coefficients. Moreover, after suitable normalization one can make them equal.

It is straightforward that the quadratic relations imply rad$^3A(J) = 0$. Finally, in the first two cases $A(J)$ is a Frobenius algebra since $P(L)^* \simeq P(L)$ if $L \neq \Lambda^+V$ and $m$ is odd. In the latter case $P(\Lambda^+V)^* \simeq P(\Lambda^+V)$. □
7.2. Mixed case. Now we will deal with the case when $\mathfrak{g}$ is (4),(5) or (6) from List A. We will prove first some statements about more general situation. Assume that $\mathfrak{g}_{ss} = \mathfrak{g}_l \oplus \mathfrak{g}_r$, where $\mathfrak{g}_l$ and $\mathfrak{g}_r$ are simple Lie algebras and $R = \Gamma_l \boxtimes \Gamma_r$, where $\Gamma_l \in \mathfrak{g}_l \mod \frac{1}{2}$ and $\Gamma_r \in \mathfrak{g}_r \mod \frac{1}{2}$. In our situation $\mathfrak{g}_l$ and $\mathfrak{g}_r$ are orthogonal Lie algebras, hence both $\Gamma_l$ and $\Gamma_r$ are spinor modules. Since spinor modules are minuscule, then its tensor product with any irreducible module is multiplicity free. In particular, $S^2 \Gamma_l$, $\Lambda^2 \Gamma_l$ (respectively, $S^2 \Gamma_r$, $\Lambda^2 \Gamma_r$) are multiplicity free disjoint $\mathfrak{g}_l^*$ (respectively, $\mathfrak{g}_r^*$)-modules.

Note that the $sl_2$-subalgebra $\alpha$ is the diagonal subalgebra in $\alpha_l \oplus \alpha_r$, where $\alpha_l$ and $\alpha_r$ are $sl_2$-subalgebras in $\mathfrak{g}_l$ and $\mathfrak{g}_r$ respectively. Therefore any module $M$ in $\mathfrak{g}$-$mod_1$ is equipped with with $\mathbb{Z}/2 \oplus \mathbb{Z}/2$-grading

$$M = \bigoplus M(1,0) \oplus M(0,1) \oplus M(\frac{1}{2}, \frac{1}{2}) \oplus M(0,0) \oplus M(-1,0) \oplus M(-1,-1) \oplus M(-\frac{1}{2}, \frac{1}{2})$$

such that short grading of $M$ with respect to $\alpha$ is given by

$$M_k = \bigoplus_{i+j=k} M_{i,j}.$$

Lemma 7.6. Let $v$ be a highest weight vector in $L$.

1. $S^k R(v)$ generates $P^k(L)$.

2. If $w \in P^k(L)$ is a highest weight vector, then $w \in S^k R(v)$.

Proof. Both assertions are obvious. \qed

Lemma 7.7. (a) Let $L$ be a simple non-trivial module in $\mathfrak{g}$-$mod_1$. Then $L \in \mathfrak{g}_l \mod_1$, $L \in \mathfrak{g}_r \mod_1$ or $L$ is isomorphic to $A \boxtimes B$ for some simple $A \in \mathfrak{g}_l \mod_\frac{1}{2}$ and $B \in \mathfrak{g}_r \mod_\frac{1}{2}$.

(b) If $L \in \mathfrak{g}_l \mod_1$ (resp., $L \in \mathfrak{g}_r \mod_1$), then $P^3(L) = 0$ and $P^2(L) \in \mathfrak{g}_r \mod_1$ (resp., $\mathfrak{g}_l \mod_1$).

(c) If $L = A \boxtimes B$, then $P^3(L)$ is a trivial $\mathfrak{g}_{ss}$-module.

Proof. (a) Follows easily from the double grading. Indeed, we have the following three possibilities

- $L = L_{1,0} \oplus L_{0,0} \oplus L_{-1,0}$;
- $L = L_{0,1} \oplus L_{0,0} \oplus L_{0,-1}$;
- $L = L_{\frac{1}{2}, \frac{1}{2}} \oplus L_{-\frac{1}{2}, -\frac{1}{2}}$.

(b) Without loss of generality assume that $L = L_{1,0} \oplus L_{0,0} \oplus L_{-1,0}$. Let $v \in L$, $w \in P^3(L)$ be $\mathfrak{g}_{ss}$-highest weight vectors. Then $v \in L_{(1,0)}$, $w \in P^3(L)_{\left(\frac{1}{2}, \frac{1}{2}\right)}$ and by Lemma 7.6 we have

$$w \in \sum_{i,j,k \in \left\{\pm \frac{1}{2}\right\}} R_{(-\frac{1}{2}, i)} R_{(-\frac{1}{2}, j)} R_{\left(\frac{1}{2}, k\right)} v.$$

But $R_{\left(\frac{1}{2}, k\right)} L_{(1,0)} = 0$ by (36). Contradiction.

If we assume that $w \in P^2(L)$ is a highest vector, then by the similar grading consideration we have

$$w \in \sum_{j,k \in \left\{\pm \frac{1}{2}\right\}} R_{\left(-\frac{1}{2}, j\right)} R_{\left(-\frac{1}{2}, k\right)} v.$$ 

This implies that the degree of $w$ is $(0,0)$ or $(0,1)$. Hence $P^2(L) \subseteq \mathfrak{g}_r \mod_1$.

(c) Now let $L = L_{\frac{1}{2}, \frac{1}{2}} \oplus L_{-\frac{1}{2}, -\frac{1}{2}}$, $v \in L$, $w \in P^3(L)$ be $\mathfrak{g}_{ss}$-highest weight vectors. We want to show that the degree of $w$ is $(0,0)$. Indeed, assume without loss of generality that degree of $w$ is $(1,0)$. Then $v \in L_{\frac{1}{2}, \frac{1}{2}}$ and we have

$$w \in \sum_{i,j,k \in \left\{\pm \frac{1}{2}\right\}} R_{\left(-\frac{1}{2}, i\right)} R_{\left(\frac{1}{2}, j\right)} R_{\left(\frac{1}{2}, k\right)} v.$$ 

But $R_{\left(\frac{1}{2}, j\right)} R_{\left(\frac{1}{2}, k\right)} L_{\left(\frac{1}{2}, \frac{1}{2}\right)} = 0$ by (36). Contradiction. \qed

Corollary 7.8. We have $\text{rad}^3 A(J) = 0$. 

Lemma 7.9. Let \( L \) be a simple non-trivial module in \( \mathfrak{g}\text{-}\text{mod}_1 \). Then the length of the radical filtration of the indecomposable projective \( P(L) \) is at most 3. Moreover, it is 3 in one of the following cases

1. \( L \) is a simple submodule in \( S^2\Gamma_l^* \) (respectively, \( \Lambda^2\Gamma_l^* \)). Then \( P^1(L) = \Gamma_l^* \boxtimes \Gamma_r^* \) and \( P^2(L) = S^2\Gamma_r^* \) (respectively, \( \Lambda^2\Gamma_r^* \)).

2. \( L \) is a simple submodule in \( S^2\Gamma_r^* \) (respectively, \( \Lambda^2\Gamma_r^* \)). Then \( P^1(L) = \Gamma_l^* \boxtimes \Gamma_r^* \) and \( P^2(L) = S^2\Gamma_l^* \) (respectively, \( \Lambda^2\Gamma_l^* \)).

3. \( L = \Gamma_l^* \boxtimes \Gamma_r^* \), then \( P(L) \) is self-dual with \( P^2(L) = \Gamma_r^* \boxtimes \Gamma_l^* \) and \( P^1(L) = (\Gamma_l^* \boxtimes \Gamma_r^* \boxtimes \Gamma_r^* \boxtimes \Gamma_l^*)/\text{tr} \).

Proof. We have to consider three cases as in Lemma 7.7.

The first two cases are similar by symmetry. Therefore it is sufficient to consider the case \( L \in \mathfrak{g}_l \text{-}\text{mod}_1 \). Then we have

\[
P^1(L) = (R \otimes L)^{sh} = (\Gamma_l \otimes L)^{sh} \boxtimes \Gamma_r.
\]

Furthermore,

\[
(R \otimes (R \otimes L)^{sh})^{sh} = C \oplus D,
\]

where \( C \in \mathfrak{g}_l \text{-}\text{mod}_1, D \in \mathfrak{g}_r \text{-}\text{mod}_1 \), and we assume in addition that \( C^{sh} = 0 \). Due to Lemma 7.7(b) we know that \( P^2(L) = \text{Coker} \hat{\mu} \), where \( \hat{\mu} \) is the composition of \( \mu \), defined in Lemma 7.6 with the natural projection on \( D \). More precisely, \( D = (\Gamma_l \otimes (\Gamma_l \otimes L)^{sh})^{ \mathfrak{g}_l \boxtimes (\Gamma_l \otimes \Gamma_l)} \). If \( D \neq 0 \), then \( (\Gamma_l \otimes \Gamma_l \otimes L)^{sh} \neq 0 \), which is only possible when \( L \) is a simple submodule in \( (\Gamma_l \otimes \Gamma_l)^* \). In this case the multiplicity of \( L \) in \( (\Gamma_l \otimes \Gamma_l)^* \) is one. Therefore \( D = \Gamma_l \otimes \Gamma_r \).

Let \( L \) be a submodule in \( S^2\Gamma_l^* \). Let \( v_1, v_2 \in \Gamma_l, w_1, w_2 \in \Gamma_r \) and \( x, y \in S^2\Gamma_l^* \). Then

\[
\hat{\mu}(v_1 \boxtimes v_2, w_1 \boxtimes w_2, x, y) = ((x, v_1)\langle y, v_2 \rangle + \langle y, v_1 \rangle\langle x, v_2 \rangle)w_1 \otimes w_2 - \langle x, v_2 \rangle \langle y, v_1 \rangle + \langle y, v_2 \rangle \langle x, v_1 \rangle)w_2 \otimes w_1.
\]

Thus, \( \text{Im} \hat{\mu} = \Lambda^2\Gamma_r \) and hence \( P^2(L) = S^2\Gamma_r^* \).

In the similar way, with the change of sign, one obtains that if \( L \) is a submodule in \( \Lambda^2\Gamma_l^* \), then \( \text{Im} \hat{\mu} = S^2\Gamma_l^* \) and hence \( P^2(L) = S^2\Gamma_l^* \).

We also have an explicit construction of \( P(L) \). Assume for example that \( L \subset S^2\Gamma_l^* \). There exists an indecomposable module \( M \) of length two in \( \mathfrak{g}\text{-}\text{mod}_1 \) with submodule \( \Gamma_r \) and quotient \( \Gamma_l^* \).

Then \( S^2M \in \mathfrak{g}\text{-}\text{mod}_1 \) is indecomposable, with the radical filtration: \( M^0 = S^2\Gamma_l^*, M^1 = \Gamma_l^* \boxtimes \Gamma_r, M^2 = S^2\Gamma_r^* \). One can check that \( L \subset M_0 \) generates a submodule isomorphic to \( P(L) \).

Now assume that \( L = A \boxtimes B \) as in Lemma 7.7(c). If \( A \neq \Gamma_l^* \) and \( B \neq \Gamma_r^* \), then \( P(L)^1 = (L \otimes R)^{sh} = 0 \) and hence \( P(L) = L \). Assume next that \( A \neq \Gamma_l^* \) and \( B = \Gamma_r^* \). Then

\[
P^1(L) = (R \otimes L)^{sh} = \Gamma_l \otimes A
\]

and

\[
(R \otimes (R \otimes L)^{sh})^{sh} = (\Gamma_l \otimes \Gamma_l \otimes A)^{sh} \boxtimes (\Gamma_l \otimes \Gamma_l),
\]

Furthermore, if we denote by \( \pi_l \) the natural projection \( \Gamma_l \otimes \Gamma_l \otimes A \to (\Gamma_l \otimes \Gamma_l \otimes A)^{sh} \), then for all \( v_1, v_2 \in \Gamma_l, w_1, w_2 \in \Gamma_r \) and \( a \in A, b \in B \) we have

\[
\mu(v_1 \boxtimes v_2, w_1 \boxtimes w_2, a \boxtimes b) = (b, w_2)\pi_l(v_1, v_2, a) \boxtimes w_1 - (b, w_1)\pi_l(v_2, v_1, a) \boxtimes w_2.
\]

Since \( \pi_l \) is surjective, we obtain \( \text{Coker} \mu = 0 \) and hence \( P^2(L) = 0 \).

Finally, we assume that \( L = \Gamma_l^* \boxtimes \Gamma_r^* \). In this case

\[
P^1(L) = (R \otimes L)^{sh} = \text{tr} \oplus (\Gamma_l \otimes \Gamma_l^*)_0 \oplus (\Gamma_r \otimes \Gamma_l^*)_0,
\]

where \( (X \otimes X^*)_0 \) denotes the traceless part of \( X \otimes X^* \). Then

\[
(R \otimes (R \otimes L)^{sh})^{sh} = \Gamma_l \boxtimes \Gamma_r^* \oplus (\Gamma_l \otimes (\Gamma_l \otimes \Gamma_l^*)_0)^{sh} \boxtimes \Gamma_r \oplus \Gamma_l \boxtimes (\Gamma_r \otimes (\Gamma_r \otimes \Gamma_r^*)_0)^{sh}.
\]

We notice that \( (\Gamma_l \otimes (\Gamma_l \otimes \Gamma_l^*)_0)^{sh} \simeq \Gamma_l \) with the natural projection \( \pi_l : \Gamma_l \otimes (\Gamma_l \otimes \Gamma_l^*)_0 \to \Gamma_l \) given by the formula

\[
\pi_l(v_1, v_2, x) = \langle x, v_1 \rangle v_2 - \langle x, v_2 \rangle v_1.
\]

We have analogous formula for \( \pi_r : \Gamma_r \otimes (\Gamma_r \otimes \Gamma_r^*)_0 \to \Gamma_r \). Then \( \pi : R \otimes R \otimes L \to (R \otimes (R \otimes L)^{sh})^{sh} \) is defined by

\[
\pi(v_1 \boxtimes w_1, v_2 \boxtimes w_2, x \boxtimes y) = ((x, v_2 \langle y, w_2 \rangle v_3 \boxtimes w_1, \langle y, w_2 \rangle \pi_l(v_1, v_2, x) \boxtimes w_1) + (x, v_2 v_1 \boxtimes \pi_r(w_1, w_2, y)).
\]
By tedious straightforward calculations one obtains that \( \theta : (R \otimes (R \otimes L)^{sh})^h \to \Gamma_1 \otimes \Gamma_r \) defined by \( \theta (x_1, x_2, x_3) = 2 x_1 - x_2 - x_3 \) gives the cokernel of \( \mu \). Therefore, we obtain \( P^2(L) = \Gamma_1 \otimes \Gamma_r \). Finally, let us prove that \( P^3(L) = 0 \). Assume the opposite. Then \( P^3(L) \) is trivial and every submodule \( N \) generated by a simple submodule \( L' \) in \( P^3(L) \) has Loewy length 3 with trivial submodule in \( \text{rad}^3 N \). That contradicts description of \( P^1(L) \) and \( P^2(L) \).

We leave to the reader to check that \( P(L) \simeq M \otimes M^* \) where \( M \) is defined above in this proof. 

We use the last lemma in order to determine \( A(J) \), equivalently the relations in \( Q'(g) \), when \( g \) in List A and \( g_{ss} = s_02m + s_02n, \ m, n \in \{4,5\} \).

**Theorem 7.10.**

1. If \( g = (s_08 + s_08) \ni \Gamma_1^+ \otimes \Gamma_2^+ \), then \( A(J) = kQ'_3/I \), where the ideal \( I \) is generated by

\[
\alpha_i \beta_i = \alpha_j \beta_j, \quad \beta_i \alpha_i = \delta_i \tau_j = \tau_i \delta_j = 0, \quad 1 \leq i, j \leq 4;
\]

\[
\beta_2 \alpha_1 = \beta_4 \alpha_1 = \beta_1 \alpha_2 = \beta_3 \alpha_2 = \beta_2 \alpha_3 = \beta_4 \alpha_3 = \beta_1 \alpha_4 = \beta_3 \alpha_4 = 0.
\]

2. If \( g = (s_08 + s_010) \ni \Gamma_1^+ \otimes \Gamma_2^+ \), then \( A(J) = kQ'_4/I \), where the ideal \( I \) is generated by

\[
\beta_2 \alpha_1 = \beta_j \alpha_1, \quad 1 \leq i, j \leq 4; \quad \alpha_1 \beta_i = \alpha_3 \beta_i = 0, \quad i \neq 4; \quad \alpha_1 \delta_i = 0, \quad i \neq 3; \quad \gamma_i \beta_3 = 0, \quad i = 1, 3;
\]

\[
\gamma_i \beta_1 = 0, \quad i = 1, 2; \quad \gamma_i \beta_2 = \gamma_i \beta_3 = 0, \quad 1 \leq i \leq 3; \quad \delta_i \tau_j = 0, \quad j = 1, 2; \quad \tau_i \rho_j = 0, \quad i = 1, 2.
\]

3. If \( g = (s_010 + s_010) \ni \Gamma_1^+ \otimes \Gamma_2^+ \), then \( A(J) = kQ'_5/I \), where the ideal \( I \) is generated by

\[
\alpha_1 \beta_2 = \alpha_3 \beta_2 = \alpha_2 \beta_1 = \alpha_2 \beta_3 = \gamma_3 \delta_2 = \gamma_2 \delta_3 = 0; \quad \rho_i \tau_i = \rho_j \tau_j, \quad 1 \leq i, j \leq 4.
\]

**Proof.** In order to write down the relations in the path algebra \( kQ'(g) \) it is enough to describe all projective covers of simple non-trivial modules of Loewy length three. (Recall that by Lemma 7.7 all indecomposable projectives have Loewy length at most three).

Let \( g = (s_08 + s_08) \ni \Gamma_1^+ \otimes \Gamma_2^+ \). For \( s_08 \)-spinor module \( \Gamma^+ \) we have

\[
(\Gamma^+)^* = \Gamma^+, \quad S^2(\Gamma^+) = \Lambda^+V \oplus \tau, \quad \Lambda^2(\Gamma^+) = \Lambda^2V.
\]

By Lemma 7.9 we obtain the following indecomposable projectives of Loewy length three.

\[
\begin{align*}
\Lambda^+V & \quad \Lambda^+W & \quad \Lambda^2V & \quad \Lambda^2W & \quad \Gamma_1^+ \otimes \Gamma_2^+ \\
\Gamma_1^+ \otimes \Gamma_2^+ & \quad \Gamma_1^+ \otimes \Gamma_2^+ & \quad \Gamma_1^+ \otimes \Gamma_2^+ & \quad \Gamma_1^+ \otimes \Gamma_2^+ & \quad \Gamma_1^+ \otimes \Gamma_2^+
\end{align*}
\]

The relations in \( A(J) = kQ'_3/I \) follow from (42). They imply \( \text{rad}^3 A(J) = 0 \).

Let \( g = (s_08 + s_010) \ni \Gamma_1^+ \otimes \Gamma_2^+ \). For \( s_010 \)-spinor modules \( \Gamma_2^+ \) we have

\[
(\Gamma_2^+)^* = \Gamma_2^+, \quad S^2(\Gamma_2^+) = \Lambda^3W \oplus W, \quad \Lambda^2(\Gamma_2^+) = \Lambda^3W, \quad \Gamma_2^+ \otimes \Gamma_2^+ = \tau \oplus \Lambda^2W \oplus \Lambda^4W.
\]

By Lemma 7.9 the indecomposable projective modules of Loewy length three in \( g \text{-mod}_1 \) are the following:

\[
\begin{align*}
\Lambda^2V & \quad \Lambda^+V & \quad \Lambda^3W & \quad \Lambda^3W & \quad W & \quad \Gamma_1^+ \otimes \Gamma_2^+
\end{align*}
\]

The relations (39) in \( A(J) \) follow and imply \( \text{rad}^3 A(J) = 0 \).

Finally, if \( g = (s_010 + s_010) \ni \Gamma_1^+ \otimes \Gamma_2^+ \), using (39), we apply Lemma 7.9 to obtain the indecomposable projectives in \( g \text{-mod}_1 \):

\[
\begin{align*}
\Lambda^+V & \quad V & \quad \Lambda^3V & \quad \Lambda^3V & \quad W & \quad \Lambda^3W
\end{align*}
\]

The relations (40) in \( A(J) \) follow and imply \( \text{rad}^3 A(J) = 0 \). 

\[\square\]
7.3. Tameness.

Theorem 7.11. The algebras $A(J)$ described in Theorem 7.5 and Theorem 7.10 are tame.

Proof. First, we deal with the cases $g = so_n \supset V$, $g = so_8 \supset A^\perp V$ and $g = (so_8 \oplus so_8) \supset \Gamma_1^+ \otimes \Gamma_2^+$. We note that $A(J)$ satisfies the conditions of Theorem 10.4. Theorem 10.12 implies that $Jor(so_8 \supset V)$ is of finite type, while $Jor(so_8 \supset A^\perp)$ and $Jor((so_8 \oplus so_8) \supset \Gamma_1^+ \otimes \Gamma_2^+)$ are tame.

Next, let $g = (so_8 \oplus so_10) \supset \Gamma_1^+ \otimes \Gamma_2^+$ then $A(g) = A^1 \oplus A^2 \oplus A^3$ is the direct sum of subalgebras such that $A^1 = k$, $Q(A^2)$ has four vertices and $Q(A^3)$ has ten vertices, see (15). Observe that $rad^2A^2 = 0$ therefore $A^2$ is of finite representation type by Theorem 10.12.

Let us show that $A^3$ is tame. Let $I_1 = \{\langle x_i y_i \rangle \mid \langle x_i y_i \rangle \leq 4 = rad^2A^3e_{\Gamma_1^+ \otimes \Gamma_2^+}$, but $B = A^3/I_1$. Note that the projective $P(\Gamma_1^+ \otimes \Gamma_2^-)$ satisfies the condition of Lemma 10.3. Therefore an indecomposable $A^4$-module non-isomorphic $P(\Gamma_1^+ \otimes \Gamma_2^-)$ is, in fact, a module over the algebra $B$.

Furthermore divide the set of $V(Q(A^4))$ into three subsets, namely $S_1 = \{\Lambda^W, \Gamma_1^+ \otimes \Gamma_2^\perp\}$, $S_2 = \{\Lambda^W, \Gamma_1^\perp \otimes \Gamma_2^\perp\}$ and $T = \{W, \Lambda^W, \Lambda^V, \Lambda^V, \Lambda^V, \Lambda^V\}$ then by Lemma 10.5 any indecomposable $B$-module $M$ is either a $B' = e_{S_1}B e_{S_1}$-module or a $B'' = e_{S_2}B e_{S_2}$-module. The Ext quivers of $B'$ and $B''$ are the following:

Note that $(B')^{op}$-mod is equivalent to $B''^{op}$-mod thus it is sufficient to determine the type of $B'$. Since $Q(B')$ is a tree we determine its representation type by calculating the Tits form, see (47) in Appendix. It can be written in the following form

$$q_{B'}(x) = (x_1 + x_3 + x_6 - x_4)^2 + (x_2 + x_3 + x_7 + x_8 - x_4)^2 + \sum_{i=2,5,7} (x_i - x_{i+1})^2 + x_1^2 + 2x_6(x_2 + x_3).$$

One can see that $q_{B'}$ is weakly non-negative, therefore $B'$ as well as $B''$ are of tame representation type. That implies that $A^3$ is tame.

The last case is $g = (so_10 \oplus so_10) \supset \Gamma_1^\perp \otimes \Gamma_2^\perp$, and $A(g) = A^1 \oplus A^2$, where $Q(A^1)$ has seven vertices while $Q(A^2)$ has ten vertices, see (16). By Theorem 10.4 the algebra $A^1$ is tame. To prove tameness of $A_2$ we split $V(Q(A^2))$ into three subsets, namely $S_1 = \{\Lambda^W, \Lambda^V, \Gamma_1^+ \otimes \Gamma_2^\perp\}$, $S_2 = \{\Lambda^W, \Lambda^V, \Gamma_1^- \otimes \Gamma_2^\perp\}$ and $T = \{W, \Lambda^W, \Lambda^V, V\}$ then by Lemma 10.5 any indecomposable $A^2$-module is either $C = e_{S_1}A^2 e_{S_1}$-module or $C' = e_{S_2}A^2 e_{S_2}$-module, where $Q(C)$ and $Q(C')$ are respectively

Like in the previous case $C^{op}$-mod is equivalent to $C'$-mod therefore it is sufficient to determine the type of $C$. The Tits form corresponding to $C$

$$q_C(x) = (x_1 + x_3 + x_5 + x_7 - x_4)^2 + (x_2 + x_6 - x_4)^2 + (x_1 - x_3)^2 + (x_5 - x_7)^2 + x_2^2 + x_6^2.$$
is weakly non-negative. Therefore, by Theorem 10.6 \( C \) is tame. This finishes the proof. □

8. Algebras with \( [\mathfrak{r}, \mathfrak{r}] \not= 0 \)

In view of Theorem 6.1 our next step is to consider indecomposable Lie algebras \( \mathfrak{g} \) with short grading, irreducible \( R = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}] \) such that \( Q(\mathfrak{g}) \) is admissible and \( [\mathfrak{r}, [\mathfrak{r}, \mathfrak{r}]] = 0 \). Irreducibility of \( R \) implies that \( \mathfrak{r} \) is a nilpotent Lie algebra of nilindex two, and \( [\mathfrak{r}, \mathfrak{r}] \) is a \( \mathfrak{g}_{ss} \)-submodule in \((\Lambda^2 R)^{sh}\).

Therefore we have to consider the following cases

1. \( \mathfrak{g}_{ss} = \mathfrak{so}_m, R = V, R^2 = (\Lambda^2 V)^{sh} = \Lambda^2 V; \)
2. \( \mathfrak{g}_{ss} = \mathfrak{so}_n, R = \Lambda^+ V, R^2 = (\Lambda^2 R)^{sh} = \Lambda^2 V; \)
3. \( \mathfrak{g}_{ss} = \mathfrak{so}_8 \oplus \mathfrak{so}_8, R = \Gamma_1^+ \ltimes \Gamma_2^+ . \) In this case \( [\mathfrak{r}, \mathfrak{r}] = (\Lambda^2 R)^{sh} = \Lambda^2 V \oplus \Lambda^2 W \), hence \( [\mathfrak{r}, \mathfrak{r}] = (\Lambda^2 R)^{sh} \) or \( [\mathfrak{r}, \mathfrak{r}] = \Lambda^2 V \) or \( [\mathfrak{r}, \mathfrak{r}] = \Lambda^2 W \), and the last two cases are clearly isomorphic;
4. \( \mathfrak{g}_{ss} = \mathfrak{so}_6 \oplus \mathfrak{so}_{10}, R = \Gamma_1^+ \ltimes \Gamma_2 . \) In this case \((\Lambda^2 R)^{sh} = \Lambda^3 W\) is irreducible;
5. \( \mathfrak{g}_{ss} = \mathfrak{so}_6 \oplus \mathfrak{so}_{10}, R = \Gamma_1 \ltimes \Gamma_2 . \) In this case \((\Lambda^2 R)^{sh} = 0\), therefore it does not need further consideration.

In what follows we refer to the above list as \textbf{List B}. In this section we will prove that \( A(J) \) is wild for the algebras (1)-(4) in \textbf{List B}.

**Lemma 8.1.** Let \( \mathfrak{g}_{ss} = \mathfrak{so}_m, R = V, [R, R] = \Lambda^2 V = (\Lambda^2 V)^{sh} \). Then \( A(\mathfrak{g}) = kQ_{m}/I_2 \), where \( I_2 \subset \text{rad}^2 kQ_{m} \), i.e. all the relations are of degree 3 or higher.

**Proof.** The proof amounts to showing that for a simple \( L \in \mathfrak{g}-\text{mod}_1 \), we have

\[ P^2(L) = (R \otimes (R \otimes L)^{sh})^{sh}. \]

We use Lemma 4.6 Since \( \delta : \Lambda^2 R \rightarrow R^2 \) is an isomorphism, the map

\[ \mu : \Lambda^2 R \otimes L \rightarrow (R \otimes (R \otimes L)^{sh})^{sh} \]

is the composition

\[ \Lambda^2 R \otimes L \xrightarrow{\delta} (R \otimes L)^{sh} \xrightarrow{\text{alt} \otimes 1} (R \otimes R \otimes L)^{sh} \rightarrow (R \otimes (R \otimes L)^{sh})^{sh}. \]

Hence \( P^2(L) = \text{Coker}(\mu \oplus \lambda) = (R \otimes (R \otimes L)^{sh})^{sh} \). □

**Lemma 8.2.** Let \( \mathfrak{g} \) be as above and \( J = \text{Jor}(\mathfrak{g}) \). Then the Ext quiver of \( J-\text{mod}_1 \) is

\[ m \text{ is odd : } \gamma_0 V \xrightarrow{\delta_1} \Lambda^2 V \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{m-1}} \Lambda^{m-1} V \xrightarrow{\delta_m} \Lambda^m V \xrightarrow{\gamma_m} \Lambda^+ \]

\[ m \text{ is even : } \gamma_0 V \xrightarrow{\delta_1} \Lambda^2 V \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{m-2}} \Lambda^{m-1} V \xrightarrow{\delta_m} \Lambda^{-} \]

and the relations in \( A(J) \) lie in \( \text{rad}^3 A(J) + \gamma_0 \text{rad} A(J) \).

**Proof.** It is a straightforward consequence of Lemma 4.9 The appearance of a new arrow \( \gamma_0 \) follows from Lemma 5.1 Indeed, recall \( Q_{3m}^{3n+1} \) and \( Q_{3m}^{3m} \) of Theorem 6.1 Now \( \gamma_0 = \gamma_0 \delta_0 \) lies in the radical of \( A(J) \) since \( \gamma_0 \delta_0 \) and \( \delta_1 \gamma_1 \) are linearly independent. □

**Lemma 8.3.** Let \( \mathfrak{g} = \mathfrak{so}_8, R = \Lambda^+ V \) and \( R^2 = \Lambda^2 V \). Then the block of \( J-\text{mod}_1 \) containing \( \Lambda^2 V \) and \( \Lambda^+ V \) has the quiver

\[ \gamma_3 \xrightarrow{\alpha_2} \Lambda^2 V \xrightarrow{\beta_2} \Lambda^+ V \xrightarrow{\gamma_3} \gamma_3 \]
and the only relations in this block modulo \( \text{rad}^3 \) are
\[
\alpha_2^2 \gamma_2 = \gamma_3 \alpha_2, \quad \gamma_2 \beta_2 = \beta_2 \gamma_3.
\]

**Proof.** In order to prove this lemma, we have to compute \( P^2(\Lambda^2 V) \) and \( P^2(\Lambda^+ V) \). Using symmetry of the Dynkin diagram of \( \mathfrak{so}_8 \) we identify \( \Lambda^+ V \) with \( S^2 V_0 \) (the latter stands for the traceless part of \( S^2 V \)) as in the proof of Proposition 7.3. If we identify \( \mathfrak{so}_8 \) with the space of skew-symmetric matrices and \( R = S^2 V_0 \) with the space of traceless symmetric matrices, the map \( \delta : S^2 V_0 \times S^2 V_0 \to \Lambda^2 V \) is given by the usual commutator of matrices.

We will do this computation for the case \( L = S^2 V_0 \) leaving the second case to the reader. We have
\[
(R \otimes (R \otimes L)^{sh})^{sh} = \Lambda^2 V \oplus \Lambda^2 V \oplus S^2 V_0 \oplus S^2 V_0, \quad (R^2 \otimes L)^{sh} = \Lambda^2 V \oplus S^2 V_0,
\]
and
\[
\pi(X \otimes Y \otimes A) = ([X, [Y, A]], [X, Y], [X, [Y, A]], [X, [Y, A]]),
\]
where \( \{C, B\} = CB + BC \) and \( [C, B] = CB - BC \).

The canonical projection \( p : R^2 \otimes L \to (R^2 \otimes L)^{sh} \) is given by
\[
p(Z \otimes A) = ([Z, A], [Z, A]),
\]
where \( A \in S^2 V_0, Z \in \Lambda^2 V \). Therefore
\[
\lambda(X, Y, A) = -([X, Y], [X, Y], [X, Y], [X, Y]).
\]

Define the map
\[
\theta : (R \otimes (R \otimes L)^{sh})^{sh} \oplus (R^2 \otimes L)^{sh} \to \Lambda^2 V \oplus S^2 V_0 \oplus S^2 V_0 \oplus \Lambda^2 V \oplus S^2 V_0 \to \Lambda^2 V \oplus S^2 V_0 \to 0
\]
by \( \theta(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 + x_2 - 2x_5, x_3 - x_4, x_4 - x_6) \). We claim that the sequence
\[
\Lambda^2 R \otimes L \xrightarrow{\mu \circ \theta} (R \otimes (R \otimes L)^{sh})^{sh} \oplus (R^2 \otimes L)^{sh} \to \Lambda^2 V \oplus S^2 V_0 \to 0
\]
is exact. Surjectivity of \( \theta \) is trivial. The identity \( \theta(\lambda \oplus \mu) = 0 \) follows from the following identities
\[
\{X, [Y, A]\} + [Y, [X, A]] = 2\{XY, A\},
\]
\[
\{X, [Y, A]\} - [X, [Y, A]] = 2XAY + 2YAX,
\]
\[
[X, [Y, A]] - [Y, [X, A]] = [[X, Y], A].
\]
Balancing the numbers of \( g_{ss} \) components implies that \( \operatorname{Ker} \theta = \operatorname{Im}(\lambda \oplus \mu) \). Hence we obtain
\[
P^2(S^2 V_0) \simeq \Lambda^2 V \oplus S^2 V_0.
\]
Similarly,
\[
P^2(\Lambda^2 V) = \Lambda^2 V \oplus \Lambda^2 V \oplus S^2 V.
\]

The relations modulo \( \text{rad}^3 \) follow by Lemma 4.3. \( \square \)

**Lemma 8.4.** Let \( g_{ss} = \mathfrak{so}_8 \oplus \mathfrak{so}_8 \) or \( \mathfrak{so}_8 \oplus \mathfrak{so}_{10} \), \( R = \Gamma_1^+ \boxtimes \Gamma_2^+ \), \( R^2 = \Lambda^2 \Gamma_2^+ \) and the commutator \( R \otimes R \to R^2 \) be defined by the formula
\[
[v_1 \boxtimes w_1, v_2 \boxtimes w_2] = (v_1, v_2) w_1 \wedge w_2,
\]
where \( v_1, v_2 \in \Gamma_1^+ \), \( w_1, w_2 \in \Gamma_2^+ \) and \( \langle \cdot, \cdot \rangle \) is an invariant scalar product in \( \Gamma_1^+ \). Consider the subcategory of \( g \text{-mod}_1 \) generated by simple submodules in \( (\Gamma_2^+ \boxtimes \Gamma_2^+)^* \) and \( \Gamma_1^+ \boxtimes (\Gamma_2^+)^* \). Let \( \tilde{P}(L) \) denote the projective cover of \( L \) in this subcategory. Then for any simple submodule \( L \) of \( (\Gamma_2^+ \boxtimes \Gamma_2^+)^* \) we have \( \tilde{P}(L) = \Gamma_2^+ \boxtimes (\Gamma_2^+)^* \).

**Proof.** It is clear from the quiver \( Q(\mathfrak{g}) \) that \( \tilde{P}(L) \) is a submodule of \( \Gamma_2^+ \boxtimes (\Gamma_2^+)^* \).

We claim that there exists an indecomposable module \( M \in \mathfrak{g} \text{-mod}_2 \) with the radical filtration
\[
M^0 = (\Gamma_2^+)^*, \quad M^1 = \Gamma_1^+, \quad M^2 = \Gamma_2^+.
\]
To show it, we define the action of \( R \) and \( R^2 \) on \( M \) by the formulas:
\[
v \boxtimes (x_0, x_1, x_2) = (0, \langle w, x_0 \rangle v, (v, x_1) w)
\]
\[
w_1 \wedge w_2 (x_0, x_1, x_2) = (0, 0, \langle w_1, x_0 \rangle w_2 - \langle w_2, x_0 \rangle w_1)
for all \( v \in \Gamma_1^+, w_1, w_2 \in \Gamma_2^+, x_i \in M_i, i = 0, 1, 2 \). Then we have
\[
\begin{align*}
(53) & \quad v_1 \otimes w_1 (v_2 \otimes w_2)(x_0, x_1, x_2) = (0, 0, (w_2, x_0)(v_1, v_2)w_1) \\
& \quad v_2 \otimes w_2 (v_1 \otimes w_1)(x_0, x_1, x_2) = (0, 0, (w_1, x_0)(v_2, v_1)w_2)
\end{align*}
\]
and the reader can see that
\[
[v_1 \otimes w_1, v_2 \otimes w_2] = (v_1, v_2)w_1 \wedge w_2.
\]
Thus \( M \) is a \( g \)-module, it is indecomposable by construction.

Let \( T = M \otimes \langle \Gamma_2^+ \rangle^* \). Then \( T \) is indecomposable with radical filtration
\[
T^0 = \langle \Gamma_2^+ \otimes \Gamma_2^+ \rangle^*, \quad T^1 = \langle \Gamma_1^+ \otimes \Gamma_2^+ \rangle, \quad T^2 = \langle \Gamma_2^+ \otimes \langle \Gamma_2^+ \rangle^* \rangle.
\]
Since \( T^1 \) is simple, any submodule \( L \) of \( T^0 \) generates an indecomposable submodule \( S \) with radical filtration \( S^0 = L, S^1 = T^1, S^2 = T^2 \). But then \( S \) is a quotient of \( P(L) \) and since \( \overline{P}^2(L) \subset T^2 = S^2 \) we have \( S \cong \overline{P}(L) \).

Corollary 8.5. (a) Let \( g_{ss} = so_8 \oplus so_8, R = \Gamma_1^+ \otimes \Gamma_2^+, R^2 = \Lambda^2 V \). Then the quiver of \( A(J) \) is \( Q_3^1 \) and \( \beta_3 \alpha_3, \beta_4 \alpha_4, \beta_3 \alpha_4 \) are linearly independent elements in \( \text{rad}^2 A(J)/\text{rad}^3 A(J) \).

(b) Let \( g_{ss} = so_8 \oplus so_8, R = \Gamma_1^+ \otimes \Gamma_2^+, R^2 = \Lambda^2 W \). Then the quiver of \( A(J) \) is \( Q_4^1 \) and \( \gamma_1 \beta_1, \gamma_1 \beta_2, \gamma_3 \beta_1, \gamma_3 \beta_2, \gamma_3 \beta_3, \) and \( \gamma_3 \beta_4 \) are linearly independent elements in \( \text{rad}^2 A(J)/\text{rad}^3 A(J) \).

Proof. It follows from computation of \( P^2(L) \) for certain simple \( L \). For example, to prove (a) take \( L \) to be a non-trivial simple submodule in \( \Gamma_1^+ \otimes \Gamma_2^+ = \Lambda^2 W \oplus \Lambda^3 W \oplus \text{tr} \). Lemma 5.4 implies \( \Lambda^3 W \oplus \Lambda^4 W \oplus \text{tr} \subset P^2(L) \). On the other hand, it is shown in Lemma 7.10 (1) that \( \Lambda^3 V \subset P^2(\Lambda^2 W) \). Now (a) follows. The proof of (b) is left to the reader.

Theorem 8.6. If \( [x, x] \neq 0 \), then \( A(J) \) is wild.

Proof. One has to show that if \( g \) is a Lie algebra from \((1)-(4)\) List B, then \( A(J) \) is wild. For any simple \( L \in \widehat{g} \)-mod \(_1\) denote by \( e_L \) the idempotent corresponding to the projector onto \( P(L) \). In all cases we use the same method. We consider \( B = eA(J)e \) for some idempotent \( e \in \Lambda(J) \) and show that \( B \) is wild. Then by Lemma 10.2 \( A(J) \) is wild.

Let \( g = so_{2m+1} \cong (V \oplus \Lambda^2 V) \). Put
\[
B = (e_{A_1} + e_{A_{m-1}V})A(J)(e_{A_1} + e_{A_{m-1}V}).
\]
By Lemma 8.2 \( B = kQ/I, \) where
\[
Q : \quad \gamma_{m-2} \delta_{m-2} \gamma_m
\]
\[
\begin{array}{c}
\gamma_{m-2} \delta_{m-2} \\
\Lambda^{m-1} V \\
\gamma_m
\end{array}
\]
and \( I \subset \gamma_{m-2} \delta_{m-2} \text{rad} B + \text{rad}^3 B \). Then \( B \) has a factor algebra isomorphic to \( A_1, \) see (58), and by Lemma 10.7 it is wild.

For \( g = so_{2m} \cong (V \oplus \Lambda^2 V) \) set
\[
B = e_{A_{m-1}V}A(J)e_{A_{m-1}V}.
\]
Then Lemma 8.2 implies that \( \gamma^+ \delta^+, \gamma^- \delta^-, \delta_{m-2} \gamma_{m-2} \) are linearly independent elements in \( \text{rad} B/\text{rad}^2 B \). Thus the quiver of \( B \) is one vertex with at least three loops. From Theorem 10.1.2 it follows that \( B \) is wild.

Let \( g_{ss} = so_8, R = \Lambda^2 V \) and \( R^2 = \Lambda^2 V \). We set
\[
B = (e_{A_1} + e_{A_{2}V})A(J)(e_{A_1} + e_{A_{2}V}).
\]
Lemma 8.3 implies the quiver of \( B \) is (92). Furthermore, \( B \) has a quotient isomorphic to \( A_2, \) see (58), by Lemma 10.7 \( B \) is wild.

Let \( g_{ss} = so_8 \oplus so_8, R = \Gamma_1^+ \otimes \Gamma_2^+, R^2 = \Lambda^2 V \). Set
\[
e_1 = e_{A_{2}V} + e_{A_{2}W} + e_{A_1 W},
\]
\( B = e_1 A(J)e_1 \) and apply Corollary 8.5(a). Then \( B \) has the quiver

\[
\begin{array}{c}
\Lambda^2 V \\ \beta_3 \alpha_1 \\
\Lambda^2 W \\ \beta_4 \alpha_4 \\
\Lambda^4 W \\
\end{array}
\]

(55)

By Theorem 10.1(2) \( B/rad^2 B \) is wild, hence \( B \) is wild.

Finally, we consider \( g_{ss} = so_8 + so_{10} \), \( R = \Gamma_2^+ \otimes \Gamma_2^+ \), \( R^2 = \Lambda^3 W \). Let
\[
e_2 = e_W + e_{\Lambda^2 W} + e_{\Lambda^3 W} + e_{\Lambda^4 W} + e_{\Lambda^5 W}.
\]

\( B = e_2 A(J)e_2 \). Corollary 8.5(b) implies that \( B \) is the path algebra of the quiver

\[
\begin{array}{c}
W \\
\gamma_1 \\
\Lambda^2 V \\
\beta_1 \\
\Lambda^4 W \\
\beta_2 \\
\Lambda^4 W \\
\beta_4 \\
\end{array}
\]

By Theorem 10.1(1) \( B \) is wild.

\( \square \)

**Corollary 8.7.** Let \( g \) be such that \( Jor(g)-mod_1 \) is tame then \( \hat{g} = g \).

### 9. Classification theorem: general case

**Theorem 9.1.** Let \( J \) be a unital Jordan algebra such that \( J_{ss} \) is a direct sum of Jordan algebras of bilinear form over vector space of dimension greater than 4. Then \( J \) is tame if and only if \( \text{Lie}(J) \) is a direct sum of Lie algebras from List A and simple orthogonal algebras.

Let \( g = \bigoplus_{i=1}^r g(i) \) be a direct sum of Lie algebras with short grading. Then the category \( \hat{g}\)-mod_1 has a decomposition in the direct sum

\[
\bigoplus_{i=1}^r g(i) - \text{mod}_1 \oplus \bigoplus_{i<j \leq r} S_{i,j},
\]

where \( S_{i,j} \) is the category of \( g(i) \oplus g(j) \)-modules which have very short grading over \( g(i) \) and \( g(j) \). Simple objects in \( S_{i,j} \) are isomorphic to \( L_1 \otimes L_2 \), where \( L_1 \) is a simple object \( g(i)\)-mod_2 and \( L_2 \) is a simple object \( g(j)\)-mod_2.

**Lemma 9.2.** If \( P(L) \) and \( P(L') \) are projective covers of \( L \) and \( L' \) in \( g(i)\)-mod_2 and \( g(j)\)-mod_2 respectively, then \( P(L) \boxtimes P(L') \) is the projective cover \( L \boxtimes L' \) in \( S_{i,j} \).

**Proof.** As it was explained in Section 1.1

\[
P(L \boxtimes L') = (I(L \boxtimes L'))^{sh}.
\]

Since \( U(g(i) \oplus g(j)) = U(g(i)) \otimes U(g(j)) \) we have

\[
I(L \boxtimes L') = I(L) \boxtimes I(L').
\]

Furthermore,

\[
(I(L) \boxtimes I(L'))^{sh} = (I(L))^{sh} \boxtimes (I(L'))^{sh} = P(L) \boxtimes P(L').
\]

\( \square \)

**Corollary 9.3.** Let \( P \) and \( P' \) are projective generators in \( g(i)\)-mod_2 and \( g(j)\)-mod_2 respectively. Then \( P \boxtimes P' \) is a projective generator in \( S_{i,j} \) and

\[
\text{End}_g(P \boxtimes P') \simeq \text{End}_{g(i)}(P) \otimes \text{End}_{g(j)}(P').
\]

Now we can prove Theorem 9.1.
Proof. It is sufficient to show that if $g(i)$ and $g(j)$ are two algebras from List A then $S_{i,j}$ is tame (if $g(i)$ is simple, this is trivial). First we construct the projective indecomposable modules in $g$-mod$_2$, where $g$ is one of the algebra from List A. By Lemma 4.4 the Ext quiver $Q^\ddagger(g)$ is the following:

$$Q^\ddagger(\mathfrak{s}_0^{2m+1} \ni V) \quad Q^\ddagger(\mathfrak{s}_0^{2m} \ni V) \quad Q^\ddagger(\mathfrak{s}_0 \ni \Lambda^+V) \quad Q^\ddagger((\mathfrak{s}_0 \oplus \mathfrak{s}_8) \ni \Gamma_1 \otimes \Gamma_2^+)$$

Observe that for any algebra $g$ from List A Lemma 4.8 implies that any projective indecomposable module in $g$-mod$_2$ has radical filtration of the length at most two. Therefore it is completely determined by $Q^\ddagger(g)$. Moreover for any simple module $L \in g$-mod$_2$ its projective cover $P(L)$ is either $L$ or $P(L) = \frac{L}{\mathfrak{d}}$, where $P^1(L) = K$ is simple $g$-mod$_2$ module.

Let $P = P(L)$ and $P' = P(L')$ be projective indecomposable modules in $g(i)$-mod$_2$ and $g(j)$-mod$_2$ respectively. Then $P \boxtimes P'$ is one of the following

$$L \boxtimes L' \quad \frac{L \boxtimes L'}{P^1(L) \boxtimes L'} \quad \frac{L \boxtimes L'}{P^1(L) \boxtimes P^1(L')}$$

Since $P^1(L), P^1(L')$ are simple, $P^1(L) \boxtimes P^1(L') \in S_{i,j}$ is also simple. Hence the associative algebra of the category $S_{i,j}$ satisfies the conditions of Theorem 10.4 One can check that if $g(i), g(j)$ are from List A, then the double quiver of the Ext quiver of $S_{i,j}$ is a disjoint union of Dynkin and extended Dynkin diagrams, therefore $S_{i,j}$ is either tame or finite. This finishes the proof.

Example 9.4. The Ext quiver for the category $S_{i,j}$ if $g(i) = \mathfrak{s}_0^{2m+1} \ni V$ and $g(j) = \mathfrak{s}_0^{2m} \ni V$ is

$$\Gamma_1 \otimes \Gamma_j^+ \quad \delta \quad \beta \quad \delta \alpha = \beta \nu$$

all other path of length two are zero

The Ext quiver for the category $S_{i,j}$ if $g(i) = (\mathfrak{s}_0 \oplus \mathfrak{s}_0) \ni \Gamma_1^+ \otimes \Gamma_2^+$ and $g(j) = \mathfrak{s}_0^{2m} \ni V$ is

$$\Gamma_2 \otimes \Gamma_j^+ \quad \alpha_1 \quad \Gamma_1^+ \otimes \Gamma_j^+ \quad \alpha_2 \quad \Gamma_2^+ \otimes \Gamma_j^+ \quad \beta_1 \delta_1 = \delta_{i+1} \alpha_i$$

all other path of length two are zero

$\beta_1 \nu_1 = \nu_{i+1} \alpha_i = 1, 2$;

10. Appendix: Representations of Quivers

In this section we will collect some notions, theorems and methods which will be used to determine the representation type of algebras given as a quiver with relations.

Let $C$ be an abelian category and $P$ be a projective generator in $C$. It is well-known fact (see [20 ex.2 section 2.6] that the functor $\text{Hom}_C(P, M)$ provides an equivalence of $C$ and the category of right modules over the ring $A = \text{Hom}_C(P, P)$. In case when every object in $C$ has the finite length and each simple object has a projective cover, one reduces the problem of classifying indecomposable objects in $C$ to the similar problem for modules over a finite-dimensional algebra (see [21] [22]). If $L_1, \ldots, L_r$ is the set of all up to isomorphism simple objects in $C$ and
$P_1, \ldots, P_r$ are their projective covers, then $A$ is a pointed algebra which is usually realized as the path algebra of a certain quiver $Q$ with relations. The vertices of $Q$ correspond to simple (resp. projective) modules and the number of arrows from vertex $i$ to vertex $j$ equals $\dim \Ext^1_L(L_j, L_i)$ (resp. $\dim \Hom(P_i, \rad P_j / \rad^2 P_j)$).

We apply this approach to the case when $\mathcal{C}$ is $\mathfrak{g}\text{-mod}_1$.

For any quiver $Q$ let $V(Q)$ denote the set of vertices of $Q$ and $Ar(Q)$ the set of its arrows. The quiver double $D(Q)$ of the quiver $Q$ is defined as follows:

$$V(D(Q)) = \{ X^+, X^- \mid X \in V(Q) \}, \text{ } Ar(D(Q)) = \{ \tilde{a} : X^- \to Y^+ \mid \text{if } a : X \to Y \in Ar(Q) \}.$$

The following results are classical.

**Theorem 10.1.**

1. Let $A = k[Q]$ is the path algebra of a quiver $Q$. Then $A$ is of finite (tame) representation type if and only if $Q$ is a disjoint union of oriented Dynkin diagrams (extended Dynkin diagrams).

2. Let $A$ be a finite dimensional associative algebra, such that $\rad^2 A = 0$, $Q$ its quiver. Then $A$ is of finite (tame) representation type if and only if $D(Q)$ is a disjoint union of oriented Dynkin diagrams (extended Dynkin diagrams).

**Lemma 10.2.** [23] 1.4.7 Suppose $A$ is a finite-dimensional algebra.

1. If $e$ is an idempotent of $A$ such that $eAe$ is wild then so is $A$.

2. Let $I$ be an ideal of $A$. If $A/I$ is wild then $A$ is wild as well.

**Lemma 10.3.** Let $A = kQ/I$, $e$ be an indecomposable idempotent and $P = Ae$ is both projective and injective. Assume that $\rad^3 P = 0$, while $\rad^2 P \neq 0$. Then any indecomposable $A$-module $M$ such that $\rad^2 eM \neq 0$ is isomorphic to $P$.

**Proof.** Injectivity of $P$ implies that $\rad^2 P$ is simple and coincides with the socle of $P$. Let $v \in M$ be such that $\rad^2 Ae_v \neq 0$. Then $\rad^2 Ae_v = \rad^2 P$ and therefore $Ae_v \simeq P$. Since $P$ is injective, we obtain that $M = Ae_v \simeq P$. \hfill $\Box$

Next we determine representation type of algebras whose indecomposable projective modules satisfy the condition of the Lemma 10.3.

**Theorem 10.4.** Suppose that $A = kQ/I$ and any indecomposable projective module $P$ such that $\rad^2 P \neq 0$ satisfies the conditions of Lemma 10.3. Then $A$ is of finite (respectively tame) representation type if and only if the double quiver $D(Q)$ is a disjoint union of Dynkin diagrams (respectively extended Dynkin diagrams).

**Proof.** Let $M$ be an indecomposable $A$-module and $\rad^2 M \neq 0$ then by Lemma 10.3 $M$ is projective. Otherwise $M$ is a module over $A/rad^2 A$. The statement follows from Theorem 10.1. \hfill $\Box$

**Lemma 10.5.** Assume that $V(Q)$ is a disjoint union of $S_1, S_r$ and $T$. Assume $Q(T)$ is disjoint, any path from $S_1$ to $S_r$ (or from $S_r$ to $S_1$) contains a vertex from $T$ and any path from $S_r$ to $S_1$ and from $S_1$ to $S_r$ does not contain a vertex from $T$. Let $A = kQ/I$ and any path from $S_1$ to $S_r$ (or from $S_r$ to $S_1$) belongs to $I$. Then for any indecomposable $A$-module $M$ either $e_{S_1} M = 0$ or $e_{S_r} M = 0$, where $e_{S_1} = \sum_{i \in S_1} e_i$ and $e_{S_r} = \sum_{i \in S_r} e_i$.

**Proof.** Let $e_T = \sum_{i \in T} e_i$. One can check that both $e_{S_r} M + e_T Ae_{S_1} M$ and $e_{S_1} M + e_T Ae_{S_r} M$ split as direct summands. Hence one of them is zero. \hfill $\Box$

Recall that for any associative algebra $A = kQ/I$ of finite global dimension the Tits form of $A$ is the quadratic form $q_A : Z^V(Q) \to \mathbb{Z}$ which is defined by

$$q_A(x) = \sum_{i \in V(Q)} x_i^2 - \sum_{i \to j \in E(Q)} x_i x_j + \sum_{i,j \in V(Q)} g(i,j)x_i x_j,$$

where $g(i,j) = |G \cap e_j I e_i|$ for a minimal set $G \subseteq \bigcup_{i,j \in V(Q)} e_j I e_i$ of generators of the ideal $I$.

A quiver $Q$ is called a tree if its underlying graph is a tree (i.e. does not contain cycles), the algebra $A = kQ/I$ is a tree algebra if $Q$ is a tree.
Theorem 10.6. [24, Thm 1.1] Let $A$ be a tree algebra. Then $A$ is tame precisely when the Tits form $q_A$ is weakly non-negative.

In [25] Y. Han has classified tame two-point algebras and minimal wild two-point algebras. We list the following two algebras from Table W, [25], $A_1 = kQ/I_1$ and $A_2 = kQ/I_2$, where

\begin{align}
Q : & \overset{\mu}{\circ} \overset{\nu}{\circ} \overset{\beta}{\circ} \\
I_1 = & \langle \alpha^2 = \mu \alpha = \alpha \nu = \nu \beta = \mu \nu = \beta^2 = 0 \rangle, \\
I_2 = & \langle \mu - \beta \mu = \alpha^2 = \nu \mu = \alpha \nu = \nu \beta = \beta^3 = \beta^2 \mu = 0 \rangle.
\end{align}

Lemma 10.7. [25, Thm 1] Let $B = kQ/I$ be a two-point algebra which has either algebra $A_1$ or algebra $A_2$ as a factor, then $B$ is wild.

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References

[1] J. Tits, Une classe d’algèbres de Lie en relation avec les algèbres de Jordan, Indag. Math. 24, (1962), 530-534.
[2] I. L. Kantor, Classification of irreducible transitively differential groups, Soviet Math. Dokl., 5, (1964), 1404-1407.
[3] M. Koecher, Imbedding of Jordan algebras into Lie algebras I, Amer. J. Math. 89, (1967), 787-816.
[4] V. Kac, Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras, Comm. Algebra, 5, no.13, (1977), 1375-1400.
[5] I. L. Kantor, Jordan and Lie superalgebras determined by a Poisson algebra, Amer. Math. Soc. Transl. Ser. 2, 151, Amer. Math. Soc., Providence, RI, 1992.
[6] V. G. Kac, C. Martínez, E. Zelmanov, Graded simple Jordan superalgebras of growth one, Mem. Amer. Math. Soc. 150, (2001), no. 711.
[7] A. S. Shtern, Representations of an exceptional Jordan superalgebra, Funct. Anal. Appl., 21, no.3. (1987), 253-254.
[8] A. S. Shtern, Representations of finite dimensional Jordan superalgebras of Poisson bracket, Comm. Algebra, 23, (1995), no. 5, 18151823.
[9] C. Martínez, E. Zelmanov, Representation theory of Jordan superalgebras I, Trans. AMS, 362, (2010), no.2, 815846.
[10] S. Krutelevich, Simple Jordan superpairs, Comm. Algebra, 25, no. 8, (1997), 2635-2657.
[11] S. Krutelevich, The Tits-Kantor-Koecher Construction and Birepresentations of the Jordan Superpair $SH(1, n)$, Comm. Algebra, 32, no. 6, (2004), 2117-2148.
[12] N. Jacobson, Structure and representations of Jordan algebras, AMS Colloq. Publ., 39, AMS, Providence 1968.
[13] N. Jacobson, Structure of alternative and Jordan bimodules, Osaka Math. J., 6, (1954), 1-71.
[14] Yu. A. Drozd, Tame and wild matrix problems,, in: “Representations and Quadratic Forms”, Kiev, (1979), 39-74 (AMS Translations 128).
[15] I. Kashuba, S. Ovsienko, I. Shestakov, Representation type of Jordan algebras, Adv. Math. 226, (2011), no. 1, 385-418.
[16] N. Cantarini, V. Kac, Classification of linearly compact Jordan and generalized Poisson superalgebras, J. Algebra, 313, (2007), 100-124.
[17] I. Kashuba, V. Serganova, One-sided representation of Jordan algebras,, in preparation.
[18] W. Fulton, J. Harris, Representation theory, a first course, Graduate Texts in Math., 129, (1991), Springer.
[19] A. L. Onishchik, E. B. Vinberg, Lie Groups and Algebraic Groups, Springer-Verlag, Berlin, (1980).
[20] S. Gelfand, Yu. Manin, Homological Algebra, Springer-Verlag, Berlin, (1999).
[21] P. Gabriel, Indecomposable Representations II, Symposia Mathematica, vol. XI Academic Press, London (1973), 81104.
[22] P. Gabriel, AuslanderReiten sequences and representation-finite algebras, in: Representation Theory, I, Proc. Workshop, Carleton Univ., Ottawa, ON, 1979, in: Lecture Notes in Math., 831, Springer, Berlin, (1980), 171-172.
[23] K. Erdmann, Blocks of Tame Representation Type and Related Algebras, Lecture Notes in Mathematics, 1428, 1990.
[24] T. Brüstle, Tame tree algebras, J. reine angew. Math., 567, (2004), 51-98.
[25] Y. Han, Wild two-point algebras, J. Algebra, 247, (2002), 57-77.