A Burge tree of Virasoro-type polynomial identities

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Abstract

Using a summation formula due to Burge, and a combinatorial identity between partition pairs, we obtain an infinite tree of q-polynomial identities for the Virasoro characters \( \chi_{r,s}^{p,p'} \), dependent on two finite size parameters \( M \) and \( N \), in the cases where:

1. \( p \) and \( p' \) are coprime integers that satisfy \( 0 < p < p' \).
2. If the pair \( (p' : p) \) has a continued fraction \( (c_1, c_2, \ldots, c_{t-1}, c_t + 2) \), where \( t \geq 1 \), then the pair \( (s : r) \) has a continued fraction \( (c_1, c_2, \ldots, c_u-1, d) \), where \( 1 \leq u \leq t \), and \( 1 \leq d \leq c_u \).

The limit \( M \to \infty \), for fixed \( N \), and the limit \( N \to \infty \), for fixed \( M \), lead to two independent boson-fermion-type q-polynomial identities: in one case, the bosonic side has a conventional dependence on the parameters that characterise the corresponding character. In the other, that dependence is not conventional. In each case, the fermionic side can also be cast in either of two different forms.

Taking the remaining finite size parameter to infinity in either of the above identities, so that \( M \to \infty \) and \( N \to \infty \), leads to the same q-series identity for the corresponding character.

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0 Introduction

0.1 A ‘lattice’ of Virasoro characters

Let us consider the set of minimal conformal field theories \( M(p, p') \) of Belavin, Polyakov and Zamolodchikov \(^1\). They are labelled by two coprime integers \( \{p, p'\} \), where \( 0 < p < p' \). If we think of \( \{p, p'\} \) as the coordinates of points on a two-dimensional square lattice, then crudely speaking, one can say that there is a two-fold infinity of these theories \(^2\). For \( p' = p + 1 \), they correspond to the critical limit of the lattice models of Andrews, Baxter and Forrester \(^3\). For \( p' > p + 1 \), they correspond to the critical limit of the models of Forrester and Baxter \(^4\).

The spectrum of a minimal theory \( M(p, p') \) on the cylinder, or equivalently, the set of one-point functions of the corresponding lattice model on the plane, can be written in terms of Virasoro characters \( \chi_{p, p'}^{r, s} \), where \( 1 \leq r < p, 1 \leq s < p' \), and \( \chi_{r, s}^{p, p'} = \chi_{p-r, p'-s}^{p, p'} \). \(^5\) In the sense used above, one can say that there is a four-fold infinity of these characters.

0.2 Q-series identities

The characters \( \chi_{r, s}^{p, p'} \) are \( q \)-series. The Stony Brook group were the first to realise that the form of these \( q \)-series is not unique \(^6\). The different forms arise naturally from the different approaches to computing them. Each approach arises from a specific physical interpretation \(^7\). All we need to know here, is that depending on how they are computed, the characters can be ‘alternating-sign’ series, or ‘constant-sign’ series.

For physical reasons \(^8\), the alternating-sign expressions are called ‘bosonic expressions’. Those of constant-sign are called ‘fermionic expressions’. For all \( p, p', r \) and \( s \), bosonic expressions for \( \chi_{r, s}^{p, p'} \) are given by Rocha-Caridi \(^9\):

\[
\chi_{r, s}^{p, p'} = \frac{1}{(q)_{\infty}} \sum_{k=-\infty}^{\infty} (q^{k^2 p p' + k(p' r - p s)} - q^{k (p r) (p' s)}) \tag{1}
\]

However, fermionic forms for \( \chi_{r, s}^{p, p'} \) are known explicitly only for certain \( p, p', r \) and \( s \). In those cases for which they are known, equating the two forms results in \( q \)-series identities. For example, a fermionic expression for \( \chi_{2, 5}^{2, 5} \) is provided

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1For an introduction to conformal field theory, see [2].
2Two integers that have no common divisor.
3Of course, this is an inaccurate description as one can order all sets with a finite number of elements that take discrete values on the line. However, this description will be useful.
4For an introduction to lattice models, see [3].
5To be precise, the one-point functions are proportional to Virasoro characters [6].
6In the context of affine algebras, a similar observation was made and used in [8].
7For an introduction and a review of the physics behind the different forms, we refer to [7].
by:
\[
\chi_{1,2}^{2.5} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}.
\]  
Equating this expression with the relevant instance of (1) results in one of the celebrated Rogers-Ramanujan (RR) identities. By the same means, if we knew fermionic expressions for all \(\chi_{r,s}^{p,p'}\), then we would end up with a four-fold infinity of generalised RR identities.

0.3 \textit{Q-polynomial identities}

Rather than work in terms of \(q\)-series, it is possible to work in terms of \(q\)-polynomials. The latter arise by taking the size of underlying physical models to be finite and to depend on various \textit{finite size} parameters. In the limit for which the finite size parameters tend to infinity, we recover the original \(q\)-series.

Working in terms of \(q\)-polynomials is particularly suited to the combinatorial approach that we follow in this work: one considers a class of combinatorial objects, \(q\)-counts them in two different ways\footnote{By \textit{q-counting}, we mean counting objects in such a way that one keeps track of a certain statistic that we call the ‘weight’. In the case of partitions, the weight is simply the integer that is partitioned.} and then equates the results. In this work, the combinatorial objects that we consider are pairs of partitions that satisfy certain conditions\cite{10, 11, 12}.

As mentioned above, bosonic expressions are known for all \(\chi_{r,s}^{p,p'}\). It is also straightforward to obtain finite analogues of \(\chi_{r,s}^{p,p'}\) that depend on one finite size parameter\cite{11}. Furthermore, finite analogues that depend on more than one finite size parameter are known\cite{12}, as we will see below.

One approach to obtaining new polynomial RR identities is to explicitly evaluate more generating functions in fermionic form. Another approach is to use summation formulae and \(q\)-series transforms to generate new expressions and identities from known ones.

0.4 \textit{The Bailey transform}

The Bailey transform\cite{13}, with extensions by Andrews\cite{14, 15}, has been used to generate infinite sequences of new identities from known ones\cite{16, 17}. An initial known identity acts as a starting point, or as a ‘seed’ for an infinite sequence of identities. The sequence obtained is one-fold infinite and covers a measure zero subset of the full set of possible identities.

That the Bailey transform allows one to obtain more ‘complicated’ identities (in the sense of larger \(p\) and \(p'\)) from ‘simpler’ ones (in the sense of smaller \(p\) and \(p'\))

\footnote{The original Rogers-Ramanujan identities involve two equalities: an equality between a constant-sign series and an alternating-sign series, and an equality between the latter and a product form. Strictly speaking, it is the latter equality that is challenging from a combinatorial point of view. We do not consider the product form in this work.}
and \( p' \), raises the possibility that one can actually obtain all identities, for all Virasoro characters from a single combinatorially trivial identity. However, it is unclear how to this may be achieved in the context of the Bailey transform, as will be discussed below.

0.5 The Burge transform

In this work, we take a step towards generating the entire set of RSOS character identities from a single simple one. We make use of a transform due to Burge which, in a sense, generalises a restricted version of the Bailey transform: the Bailey transform involves two continuous parameters and the Burge transform generalises the special case of the Bailey transform where both parameters tend to infinity.

0.6 A comparison of two transforms

The Burge transform is ‘stronger’ than the Bailey transform in the following sense: in both cases, one starts from \( q \)-polynomial identities. In the case of the Bailey transform, the polynomials that the transform acts on to generate new identities, depend on a single finite size parameter, say \( L \). The action of the Bailey transform is such that the final result is not a polynomial identity but a \( q \)-series identity.

In the case of the Burge transform, the transform acts on \( q \)-polynomial identities that depend on two finite size parameters, say \( M \) and \( N \). As noticed by Burge [12], working in terms of partition pairs allows more ‘games’ to be played: it allows us to find more transformations under which the generating function of the objects that are counted remains invariant. This extra freedom is what allows us to use the Burge transform to obtain a two-fold infinity of identities rather than a one-fold infinity as in the case of the Bailey transform. It is also what allows us to obtain two independent identities for each character.

Furthermore, the result of applying the Burge transform to a \( q \)-polynomial with two (four) finite size parameters is once again a \( q \)-polynomial with two (four) finite size parameters.

\( \textit{Roughly speaking, this is related to the fact that the objects that are counted are single partitions.} \)

\( \textit{Roughly speaking, this is related to the fact that the objects that are counted are are pairs of partitions.} \)

\( \textit{To be more precise, the Burge transform involves four finite size parameters, say } N, \ M, \ N', \text{ and } M'. \text{ However, the identities that we obtain can all be derived in terms of two parameters only: } M = M', \text{ and } N = N'. \text{ The general case of four parameters is relevant to identities that correspond to the most general RSOS characters. We do not deal with the most general case in this work.} \)
0.7 Outline of results

The multi-parameter polynomials involved in the Burge transform are not precisely finite analogues of Virasoro characters. To obtain the latter, one has to take one of the finite parameters, say $M$ to infinity, and normalise the resulting expressions appropriately. Only then does one obtain a finite analogue of a Virasoro character. Because $p$ can be any positive integer, and $p'$ can be any positive integer that is larger than $p$ and coprime to it, we obtain a two-fold infinite tree of polynomial identities. If one takes the other finite parameter $N$ to infinity, instead of $M$, one obtains another two-fold infinite tree. Taking the finite parameter that is left to infinity, one obtains $q$-series identities for the characters.

We proceed in two steps. Firstly, we use the Burge transform, without any further additions, to obtain an identity for one character of each of the $M(p,p')$ models: $\chi^{p,p'}_{r_0,s_0}$, where $|ps_0 - p'r_0| = 1$.

Secondly, we make use of a simple combinatorial identity, to extend our results to further characters $\chi^{p,p'}_{r_i,s_i}$ as will be described below. What we can say at this point is the following: if $p'/p$ has a continued fraction expansion of the form $(c_1, c_2, \ldots, c_{t-1}, c_t+2)$ with $t \geq 1$, then $s_i/r_i$ has a continued fraction expansion of the form $(c_1, c_2, \ldots, c_{u-1}, d)$, where $1 \leq u \leq t$ and $1 \leq d \leq c_u$.

(As we will see, the case $\{r_0, s_0\}$ is included in this set.)

0.8 Remarks on content

At this point, we wish to emphasise that, mathematically speaking, this work is based entirely on the ideas of Burge, as expressed in [12]. In particular, the idea of generating a tree of polynomial identities that depend on two finite size parameters, and that reduce to two independent polynomial identities by taking one of the two parameters to infinity is contained explicitly in [12]. We believe that even the combinatorial identity that we use to obtain an extended Burge tree, is also known to Burge, although it was not explicitly stated in [12].

The first purpose of this work is to introduce and clarify Burge’s ideas, which are often cited, but seldom read. For this reason we have included derivations of all of the relevant results from [12].

Furthermore, we wish to make an explicit connection with the physically-motivated works on boson-fermion Virasoro character identities. In particular, we wish to use Burge’s results to obtain algorithms to generate the largest

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13 This is the character that has the smallest conformal dimension in the model[18].
14 We can say that the latter correspond to all possible ‘tapered truncations’ of the former.
15 There are two results in [12] that, to the best of our understanding, could only be derived using such an identity.
16 We find that though [12] is ingenious, it is also very succinctly written, and therefore not easy to read. In particular, certain consistency conditions that must be imposed on the partitions pairs are not explicitly stated. This difficulty is further compounded by the fact that it contains, unfortunately, a large number of misprints.
possible set of character identities. The emphasis in this work is on systematic 
and algorithmic methods to generate characters.

0.9 Outline of paper

In §1, we introduce the combinatorial objects that we are interested in: partition 
pairs that obey specific restrictions. In §2, we derive a \( q \)-polynomial summation 
formula due to Burge. We refer to this formula as the ‘Burge transform’. In 
§3, we explain Burge’s algorithm for generating a tree of \( q \)-polynomial identities 
using the Burge transform. The main result here, Theorem 3.8, gives finitised 
character identities for the \( \chi_{p,p',r_0,s_0} \) mentioned above. 

In §4, we introduce a combinatorial identity that enables the algorithm of 
the previous section to further extend the tree of polynomial identities. The 
remaining sections are devoted to our results. In §5, we obtain a tree of poly- 
nomial identities that depend on one finite size parameter and that have a bosonic 
side with a conventional dependence on the parameters that characterise the 
corresponding character. The resulting one-parameter finitised \( q \)-polynomial 
Virasoro characters are unified and presented in Corollary 5.3. 

In §6, we obtain a tree of polynomial identities that depend on one finite size 
parameter and that have a bosonic side with a non-conventional dependence 
on the parameters that characterise the corresponding character. Here, the 
resulting one-parameter finitised \( q \)-polynomial Virasoro characters are presented 
in Corollary 6.3. In §7, we remove both finite size parameters, and obtain a tree 
of character identities. The resulting Virasoro character \((q\text{-series}) \) identities are 
unified and presented in Corollary 7.3. In §8, we include a number of remarks. 

Appendix A details Burge’s proof of his alternating sign generating function 
for the restricted partition pairs. Appendix B provides some details for examples 
given in §6 and §7.

1 Restricted partition pairs

1.1 Combinatorial objects related to Virasoro characters

Although we are ultimately interested in generating identities that express Vira- 
soro characters in two ways with different physical interpretations, the approach 
followed in this work is combinatorial in the following sense: following Burge 
[12], we obtain a tree of \( q \)-polynomial identities by enumerating certain combi-
natorial objects in two different ways, and equating the results.\(^{17}\)

\(^{17}\)To be more precise, we will enumerate these objects only in a special case where the 
conditions they satisfy are so strict that the set of enumerated objects can only be empty. Once 
we obtain an identity that expresses the above fact, we use it as a root of a tree of identities, 
and generate the rest of the tree using the Burge transform, and a simple combinatorial 
identity between partitions.
In previous work, various combinatorial objects have been used in order to obtain finite versions of Virasoro characters:

1. Paths \[19, 20, 21\]
2. Partitions with prescribed hook differences \[16, 19\]
3. Coloured Young diagrams \[22\]
4. Tableaux \[23, 24\]

In this work, we use yet another type of combinatorial object, introduced by Burge \[12\], that are most closely related to the partitions with prescribed hook differences that were introduced in \[11\].

### 1.2 Definitions

A partition \(p = (p_N, p_{N-1}, p_{N-2}, \ldots, p_1)\) in \((N, M)\) is a sequence of \(N\) integer parts \(\{p_N, p_{N-1}, p_{N-2}, \ldots, p_1\}\) such that \(M \geq p_N \geq p_{N-1} \geq \cdots \geq p_1 \geq 0\).

The weight \(\text{wt}(p)\) of \(p\) is the sum of its parts: \(\text{wt}(p) = p_N + p_{N-1} + p_{N-2} + \cdots + p_1\).

Equivalently, \(\text{wt}(p)\) is the number of nodes in the corresponding Young diagram of \(p\).

A partition pair \((q, p)\) in \((N_1, M_1) \times (N_2, M_2)\) is an ordered pair such that \(q\) is a partition that fits in a box of dimensions \((N_1, M_1)\) and \(p\) is a partition that fits in a box of dimensions \((N_2, M_2)\).

A partition pair can be depicted as follows: Firstly, we draw an \(N_2 \times N_1\) ‘Durfee rectangle’. Next, we attach the \(q\) partition to the bottom edge, and attach the \(p\) partition to the right edge, as indicated in the figure below.

(Here \(N' = N - 1\) and \(N'' = N - 2\).) The weight \(\text{wt}((q, p))\) of the partition pair \((q, p)\) is given by \(\text{wt}((q, p)) = \text{wt}(q) + \text{wt}(p)\). In other words, the Durfee rectangle is empty, and does not contribute to the weight of the partition pair.

\[18\] Note that, following Burge, the parts here are labelled unconventionally, with decreasing labels.
Given a set of non-negative integer parameters \( \{N_1, M_1, N_2, M_2, a, b, \alpha, \beta\} \) that satisfy the conditions\(^{19}\)

1. \( a + b > 0 \),
2. \( \alpha + \beta > 0 \),
3. \( -a \leq N_1 - N_2 \leq b \),
4. \( -\beta \leq M_1 - M_2 \leq \alpha \),

we say that a partition pair \((q, p)\) in \((N_1, M_1) \times (N_2, M_2)\) is restricted if

\[
\begin{align*}
    p_i - q_{i+1} - a &\geq 1 - \alpha, \\
    q_i - p_{i+1} - b &\geq 1 - \beta,
\end{align*}
\]

for all meaningful values of the subscripts — that is, for \( \max\{1, a\} \leq i \leq \min\{N_2, N_1 + a - 1\} \) in the first inequality and for \( \max\{1, b\} \leq i \leq \min\{N_1, N_2 + b - 1\} \) in the second — and the following special cases also hold:

- if \( a = 0 \) then \( q_1 \leq \alpha - 1 \);
- if \( b = 0 \) then \( p_1 \leq \beta - 1 \);
- if \( a = N_2 - N_1 \) then \( p_{N_2} \geq M_1 - \alpha + 1 \);
- if \( b = N_1 - N_2 \) then \( q_{N_1} \geq M_2 - \beta + 1 \).

We define \( \mathcal{R}(N_1, M_1, N_2, M_2, a, b, \alpha, \beta) \) to be the set of all such restricted partition pairs. The generating function for these pairs is then defined by:

\[
R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta; q) = \sum_{\mathcal{R}(N_1, M_1, N_2, M_2, a, b, \alpha, \beta)} q^{\text{wt}((q, p))}.
\]

For convenience, we can drop the last parameter in \( \mathcal{R} \), and define

\[
R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta) = R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta; q).
\]

For the purposes of the result given below, we define the \( q \)-shifted factorial \((q)_n\), and the Gaussian polynomial \( \left[ \begin{array}{c} P \\ N \end{array} \right] \) as follows

\[
(q)_n = (1 - q)(1 - q^2)(1 - q^3) \cdots (1 - q^n) \quad (n \geq 0),
\]

\[
\left[ \begin{array}{c} P \\ N \end{array} \right] = \left\{ \begin{array}{ll}
(q)_P & \text{if } P \geq N \geq 0; \\
0 & \text{otherwise.}
\end{array} \right.
\]

Further, we define

\[
g(N_1, M_1, N_2, M_2, x, y) = \left[ \begin{array}{c} N_1 + M_1 + x - y \\ N_1 + x \end{array} \right] \left[ \begin{array}{c} N_2 + M_2 - x + y \\ N_2 - x \end{array} \right].
\]

\(^{19}\)The last two of the following four conditions, and the special cases that follow, are not explicitly stated in \[12\]. However, they are required for consistency\[25\].
1.3 Bosonic generating function

Now that we have defined the combinatorial objects that we are interested in, one can obtain $q$-polynomial identities by counting them in two different ways, and identifying the results. One way of counting is by using ‘inclusion-exclusion’ \[10\]. This was done in \[12\]. We give a detailed proof in Appendix A.

Theorem 1.1 \[12\]

\[
R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta) = \sum_{k=-\infty}^{\infty} q^{k^2(a+b)(\alpha+\beta)+k(\alpha+\beta)(N_1-N_2)+k(\alpha b - \beta a)} \times g(N_1, M_1, N_2, M_2, k(a+b), k(\alpha + \beta)) - \sum_{k=-\infty}^{\infty} q^{k^2(a+b)(\alpha+\beta)+(k(\alpha+\beta)+\alpha)(N_1-N_2)+k(\alpha b + 2a\alpha) + a\alpha} \times g(N_1, M_1, N_2, M_2, k(a+b) + a, k(\alpha + \beta) + \alpha).
\]

Notice the dependence of the generating function $R$ on two Gaussian polynomials. This is a consequence of the fact that $R$ $q$-counts partitions pairs. Also notice its dependence on four finite size parameters. Also notice that (taking $N_1 = N_2$ and $M_1 = M_2$ as we often will), using (1),

\[
\lim_{N \to \infty, M \to \infty} R(N, M, N, M, a, b, \alpha, \beta) = \frac{1}{(q)_\infty} \chi_{a+b, \alpha+\beta}^{a, \alpha},
\]

and thus $R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta)$ is a four-parameter finitised Virasoro character (up to a factor of $(q)_\infty$).

1.4 Comparison of notational conventions

At this point, we wish to compare the different notations used to label the Virasoro characters, the corresponding one-point functions, and their finite size analogues. It will be sufficient for our purposes to compare the notations used in the following four.

BPZ In \[1\], and in the rest of the conformal field theory literature, a Virasoro character is denoted by $\chi_{p, p'}^{r, s}$. This notation is used in \[1\].

FB In \[3\], a polynomial analogue of $\chi_{r, s}^{p, p'}$ that depends on one finite size parameter was obtained (in the process of computing one-point functions). The parameters of Forrester and Baxter are related to those of BPZ as follows:

\[
\{p, p', r, s\} \equiv \{K - \mu, K, b, a\}.
\]
ABBBFV In [11], a polynomial analogue of $\chi_{r,s}^{p,p'}$ that depends on one finite size parameter $L$ was obtained. The parameters of ABBBFV are related to those of BPZ as follows:

$$\{p, p', r, s\} \equiv \{\alpha + \beta, K, \beta, \iota\}.$$ 

Burge In [12], a polynomial analogue of $\chi_{r,s}^{p,p'}$ (up to a factor) that depends on two finite size parameters was obtained. The parameters of Burge, and also of this paper, are related to those of BPZ as follows:

$$\{p, p', r, s\} \equiv \{a + b, \alpha + \beta, a, \alpha\}.$$ 

1.5 The symmetries of the bosonic generating function

Because we are counting pairs of partitions, rather than single ones, the generating function of the partition pairs enjoys a number of symmetries that are not present in the case of single partitions. These symmetries are what allows us to use the Burge transform to obtain more results than in the case of the Bailey transform.

The symmetries of the generating function can be expressed in terms of the following identities between the $R(N_1, M_1, N_2, M_2; a, b, \alpha, \beta)$, and may be proved directly from Theorem 1.1.

**Corollary 1.2** Let $\Delta N = N_1 - N_2$, $\Delta M = M_1 - M_2$ and $A = N_1 M_1 + N_2 M_2$. Then:

1. $R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta)$
   $$= R(N_2, M_2, N_1, M_1, b, a, \beta, \alpha);$$
2. $R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta)$
   $$= R(M_1, N_1, M_2, N_2, \alpha - \Delta M, \beta + \Delta M, a + \Delta N, b - \Delta N);$$
3. $R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta; q)$
   $$= q^A R(N_1, M_1, N_2, M_2, b - \Delta N, a + \Delta N, \beta + \Delta M, \alpha - \Delta M; q^{-1});$$
4. $R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta; q)$
   $$= q^A R(N_2, M_2, N_1, M_1, a + \Delta N, b - \Delta N, \alpha - \Delta M, \beta + \Delta M; q^{-1}).$$

In fact, these identities may alternatively be deduced combinatorially by considering the restricted partition pairs themselves, suitably transforming them, and identifying the restriction on the transformed pairs.

The first identity of Corollary 1.2 follows by interchanging the two partitions that make up the pair. The second follows by taking the conjugate of each partition. The third arises from taking the complements of the two partitions: the first inside a box of dimensions $N_1 \times M_1$ and the second inside a box of dimensions $N_2 \times M_2$. The fourth identity combines the first and third.
2 The Burge transform

Let us suppose that we are able to compute the generating function of a set of partition pairs that obey certain restrictions in two different ways. Equating the results, we obtain a $q$-polynomial identity. Following Burge [12], we can use that identity to obtain a tree of polynomial identities using a summation formula that we refer to as The Burge transform.

In this section, we recall Burge’s derivation of his summation formula in 3 steps.

2.1 A $q$-polynomial form of the $q$-Pfaff-Saalschütz summation formula

The first step is to notice that the $q$-analogue of the Pfaff-Saalschütz summation formula (see eq. (3.3.11) of [10], or eq. (4.2) of [26]) can be written as a $q$-polynomial identity as follows:

\[
\begin{bmatrix}
m_1 + n + B \\
n + A
\end{bmatrix}
\begin{bmatrix}
m_2 + n + A \\
n + B
\end{bmatrix}
= \sum_{i=0}^{n} q^{(i+A)(i+B)} \begin{bmatrix}
m_1 + m_2 + n - i \\
i + A
\end{bmatrix} \begin{bmatrix}
m_1 \\
i + B
\end{bmatrix} \cdot (8)
\]

The second step is to use the above identity to prove the following lemma

Lemma 2.1 Let $\Delta N = N_1 - N_2$. Then

\[g(N_1, M_1, N_2, M_2, x, y) = \sum_{i=0}^{N_2} \left( q^{i^2 + \Delta N - x(i + \Delta N)} \begin{bmatrix}
M_1 + M_2 + N_2 - i \\
N_2 - i
\end{bmatrix} \times g(i + \Delta N, M_1 - i, i, M_2 - \Delta N - i, x, y - x) \right).\]

Proof: On setting $n = N_2$, $A = \Delta N + x$, $B = -x$, $m_1 = M_1 + \Delta N + 2x - y$, and $m_2 = M_2 - \Delta N - 2x + y$ in (8), we obtain:

\[\begin{bmatrix}
N_1 + M_1 + x - y \\
N_1 + x
\end{bmatrix}
\begin{bmatrix}
N_2 + M_2 - x + y \\
N_2 - x
\end{bmatrix}
= \sum_{i=0}^{N_2} \left( q^{(i-x)(i+x+\Delta N)} \begin{bmatrix}
M_1 + M_2 + N_2 - i \\
N_2 - i
\end{bmatrix} \times \begin{bmatrix}
M_1 + \Delta N + 2x - y \\
\Delta N + x + i
\end{bmatrix} \begin{bmatrix}
M_2 - \Delta N - 2x + y \\
i - x
\end{bmatrix} \right),\]

which is the desired result. \qed
2.2 The Burge summation formula

Finally, we are in a position to prove the Burge transform, or equivalently, the Burge summation formula.

**Theorem 2.2** Let \( \Delta N = N_1 - N_2, \Delta M = M_1 - M_2, -a \leq \Delta N \leq b, -\beta \leq \Delta M \leq \alpha \) and \( b - \beta \leq \Delta M + \Delta N \leq \alpha - a \). Then

\[
R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta)
= \sum_{i=0}^{N_2} \left( q^{i^2 + i\Delta N} \left[ \frac{M_1 + M_2 + N_2 - i}{N_2 - i} \right] \right.
\times R(i + \Delta N, M_1 - i, i, M_2 - \Delta N - i, a, b, \alpha - a, \beta - b)
\left. \right).
\]

**Proof:** Following Burge [2], this expression is proved by applying Lemma 2.1 to each term of the generating function given in Theorem 1.1. For the \( k \)th term of the first summation of the expression for \( R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta) \), we take \( x = k(a + b) \) and \( y = k(\alpha + \beta) \) in Lemma 2.1 to obtain:

\[
q^{k(\alpha + \beta)(k(a+b)+\Delta N)+k(a\beta-ba)} g(N_1, M_1, N_2, M_2, k(a+b), k(\alpha + \beta))
= q^{k(\alpha + \beta)(k(a+b)+\Delta N)+k(a\beta-ba)}
\times \sum_{i=0}^{N_2} \left( q^{i^2 + i\Delta N - k(a+b)(k(a+b)+\Delta N)} \left[ \frac{M_1 + M_2 + N_2 - i}{N_2 - i} \right] \right.
\times g(i + \Delta N, M_1 - i, i, M_2 - \Delta N - i, k(a + b), k(\alpha + \beta - a - b))
\left. \right).
\]

Similarly, for the \( k \)th term of the second summation of the expression for \( R(N_1, M_1, N_2, M_2, a, b, \alpha, \beta) \), we take \( x = k(a + b) + a \) and \( y = k(\alpha + \beta) + \alpha \) in Lemma 2.1 to obtain:

\[
q^{k(\alpha + \beta)(k(a+b)+\Delta N)+k(a\beta+ba+2\alpha)+a(a+\Delta N)}
\times g(N_1, M_1, N_2, M_2, k(a+b) + a, k(\alpha + \beta) + \alpha)
= q^{k(\alpha + \beta)(k(a+b)+\Delta N)+k(a\beta+ba+2\alpha)+a(a+\Delta N)}
\times \sum_{i=0}^{N_2} \left( q^{i^2 + i\Delta N - (k(a+b)+a)(k(a+b)+a+\Delta N)} \left[ \frac{M_1 + M_2 + N_2 - i}{N_2 - i} \right] \right.
\times g(i + \Delta N, M_1 - i, i, M_2 - \Delta N - i, k(a + b), k(\alpha + \beta - a - b))
\left. \right).
\]
\[ \times g(i + \Delta N, M_1 - i, i, M_2 - \Delta N - i, \]

\[ k(a + b) + a, k(\alpha \beta - a - b) + \alpha - a) \]

\[ = \sum_{i=0}^{N_2} \left( q^{i^2 + i \Delta N} \left[ \frac{M_1 + M_2 + N_2 - i}{N_2 - i} \right] \right. \]

\[ \times q^{k(\alpha \beta - a - b)(k(a + b) + \Delta N) + k(a(\beta - b) + (\alpha - a) + (a - a)(\alpha + \beta)) + (a - a)(\alpha + \beta)} \]

\[ \times g(i + \Delta N + i, M_1 - i, i, M_2 - \Delta N - i, \]

\[ k(a + b) + a, k(\alpha \beta - a - b) + \alpha - a) \right) , \]

after re-expressing \((k(a + b) + a)(k(a + b) + \Delta N) = k(a + b)(k(a + b) + \Delta N) + k(ab + ba + 2a^2) + (a + \Delta N)\). Summing over all \(k\) for each of these two results, and taking the difference between the two sums, proves the theorem. \(\square\)

### 3 A Burge tree of polynomial identities

In this section, following Burge [12], we obtain an explicit algorithm to generate a tree of polynomial identities that depend on two finite size parameters. We specialise to the case where \(N_1 = N_2\) and \(M_1 = M_2\) and so define \(R(N, M, a, b, \alpha, \beta) = R(N, M, N, M, a, b, \alpha, \beta)\). We generate a tree-like structure giving fermionic expressions for certain \(R(N, M, a, b, \alpha, \beta)\), by expressing these certain \(R(N, M, a, b, \alpha, \beta)\) in terms of various \(R(N', M', 0, 1, 1, 1)\). The latter form the ‘root’ of the tree, and have a particularly simple form which is best obtained via Theorem 1.1.

#### 3.1 The root of the tree

**Lemma 3.1**

\[ \sum_{k=-\infty}^{\infty} q^{k^2 + k} \left( \left[ \begin{array}{c} N + M \\ N + k \end{array} \right] \left[ \begin{array}{c} N + M \\ N - k \end{array} \right] - \left[ \begin{array}{c} N + M - 1 \\ N + k \end{array} \right] \left[ \begin{array}{c} N + M + 1 \\ N - k \end{array} \right] \right) = \delta_{N,0} \delta_{M,0} . \]

**Proof:** In the case where \(P > 0\), one readily obtains

\[ \left[ \begin{array}{c} P \\ N \end{array} \right] = \left[ \begin{array}{c} P - 1 \\ N \end{array} \right] + q^{P-N} \left[ \begin{array}{c} P - 1 \\ N - 1 \end{array} \right] , \]

from (3). Use of this result in the left side of the premise when \(M + N > 0\), yields:

\[ \sum_{k=-\infty}^{\infty} q^{k^2 + k} \left( \left[ \begin{array}{c} N + M \\ N - k \end{array} \right] \left( \left[ \begin{array}{c} N + M - 1 \\ N + k \end{array} \right] + q^{M-k} \left[ \begin{array}{c} N + M - 1 \\ N + k - 1 \end{array} \right] \right) \right) \]
\[\begin{align*}
\sum_{k=-\infty}^{\infty} q^{k^2+M} & \left[ \binom{N + M}{N - k} \binom{N + M - 1}{N + k - 1} \right] \\
& - \sum_{k=-\infty}^{\infty} q^{k^2+2k+1+M} \left[ \binom{N + M}{N - k - 1} \binom{N + M - 1}{N + k} \right] \\
& = \sum_{k=-\infty}^{\infty} q^{k^2+M} \left[ \binom{N + M}{N - k} \binom{N + M - 1}{N + k - 1} \right] \\
& - \sum_{k=-\infty}^{\infty} q^{k^2+M} \left[ \binom{N + M}{N - k} \binom{N + M - 1}{N + k - 1} \right].
\end{align*}\]

having shifted the second summation index \( k \to k - 1 \). The result is manifestly 0. When \( M = N = 0 \), only the \( k = 0 \) term of the first summation is non-zero. Its value is 1 which proves the lemma. \( \square \)

**Corollary 3.2**

1. \( R(N, M, 0, 1, 1, 0) = \delta_{N,0} \delta_{M,0} \);
2. \( R(N, M, 0, 1, 1, 1) = \delta_{M,0} \).

**Proof:** Substituting \( N_1 = N_2 = N \), \( M_1 = M_2 = M \), \( a = \beta = 0 \) and \( b = \alpha = 1 \) into Theorem 1.1 results in the left side of Lemma 3.1 after changing the sign of the first summation index. The first result follows. The second result then also follows on direct application of Theorem 2.2 with \( N_1 = N_2 = N \), \( M_1 = M_2 = M \), \( a = 0 \), \( b = \alpha = \beta = 1 \). \( \square \)

The following similar result will not be required until Section 4.

**Lemma 3.3** If \( b > 0 \) and \( \beta \geq 0 \) then

\[ R(0, M, 0, b, 1, \beta) = \delta_{M,0}. \]

**Proof:** By Theorem 1.1

\[ R(0, M, 0, b, 1, \beta) \]

\[ = \sum_{k=-\infty}^{\infty} q^{k^2(b+1)-kb} \left[ \binom{M + k(b - \beta - 1)}{kb} \binom{M - k(b - \beta - 1)}{-kb} \right] \\
- \sum_{k=-\infty}^{\infty} q^{k^2(b+1)+kb} \left[ \binom{M + k(b - \beta - 1) - 1}{kb} \binom{M - k(b - \beta - 1) + 1}{-kb} \right]. \]
From the definition (6) of the Gaussian, when \( k \neq 0 \) each term here contains a zero factor. Thus

\[
R(0, M, 0, b, 1, \beta) = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} M - 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

The lemma then follows since the first term is always 1, whereas the second term is 0 when \( M = 0 \) and is 1 otherwise. \( \blacksquare \)

3.2 Continued fractions

A binary tree of fermionic expressions for certain \( R(N, M, a, b, \alpha, \beta) \) may now be obtained via Corollary 3.2 using the notion of a continued fraction.

Let \( s \) and \( r \) be positive coprime integers, or \( s = 1 \) and \( r = 0 \). The continued fraction \((c_1, c_2, c_3, \ldots, c_t)\) for the pair \((s:r)\) is defined as follows. The continued fraction for \((1:0)\) is defined to be the sequence () of length zero. Then, for other \((s:r)\), the continued fraction \((c_1, c_2, c_3, \ldots, c_t)\) is defined recursively by setting \( c_1 \) to be the largest integer such that \( c_1 r \leq s \), and taking \((c_2, c_3, c_4, \ldots, c_t)\) to be the continued fraction for \((r:s-c_1 r)\). This recursive procedure clearly terminates. Note that if \( s \) and \( r \) are coprime then so are \( r \) and \( s - c_1 r \) (unless \( r \leq 1 \)). Also note that if \( t > 0 \) then \( c_1 = 0 \) if and only if \( r < s \). Note further that, apart from the cases \((s:r) = (1:1)\) and \((s:r) = (1:0)\), we obtain \( c_t \geq 2 \).

It is useful to permit the continued fraction for which \( t > 1 \) and \( c_t = 1 \) and then to equate \((c_1, \ldots, c_{t-1}, 1)\) and \((c_1, \ldots, c_{t-1} + 1)\).

This definition differs from the usual definition of a continued fraction, but has the advantage of dealing with the useful additional case \((1:0)\). The connection with the usual definition is given in the following lemma.

**Lemma 3.4** Let \( s \) and \( r \) be positive coprime integers and \((c_1, c_2, c_3, \ldots, c_t)\) the continued fraction for \((s:r)\). Then

\[
\frac{s}{r} = c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \frac{1}{\ddots + \frac{1}{c_t}}}}.
\]

**Proof:** Clearly \( t \geq 1 \) and the result holds in the case \( t = 1 \). For the purposes of induction, assume that the result holds for arbitrary \( t \geq 1 \). For \( t \geq 1 \), let \((s:r)\) be such as to have continued fraction \((c_1, c_2, c_3, \ldots, c_{t+1})\). Then, by definition, \((c_2, c_3, \ldots, c_{t+1})\) is the continued fraction for \((r:s-c_1 r)\). The induction hypothesis now implies that the quotient \( r/(s-c_1 r) \) is given by the
right side of the premise once each subscript is increased by one. The result at
\( t+1 \) now follows because \( s/r = c_1 + 1/(r/(s-c_1r)) \). The lemma is then proved
by induction. \( \square \)

### 3.3 A ‘Bailey-type’ algorithm to generate a Burge tree

**Lemma 3.5** For \( c \geq 0 \),

\[
R(N, M, a, b, \alpha + ca, \beta + cb) = \sum_{n \in (c, N)} \left( q^{n_1^2 + n_2^2 + \cdots + n_c^2} \prod_{i=1}^{c} \left[ 2 \left( M - \sum_{j=1}^{i} n_j \right) + n_{i-1} + n_i \right] \right)
\times R(n_c, M - \text{wt}(n), a, b, \alpha, \beta)
\]

the sum being over all partitions \( n = (n_1, n_2, \ldots, n_c) \) in \((c, N)\) for which \( \text{wt}(n) \leq M \).

**Proof:** The result holds trivially when \( c = 0 \). For the purposes of induction, assume that the result holds for a fixed \( c \geq 0 \). Using this followed by Theorem 2.2 (with \( N_1 = N_2 = n_c \) and \( M_1 = M_2 = M - \text{wt}(n) \)) gives:

\[
R(N, M, a, b, \alpha + (c + 1)a, \beta + (c + 1)b) = \sum_{n \in (c, N)} \left( q^{n_1^2 + n_2^2 + \cdots + n_c^2} \prod_{i=1}^{c} \left[ 2 \left( M - \sum_{j=1}^{i} n_j \right) + n_{i-1} + n_i \right] \right)
\times R(n_c, M - \text{wt}(n), a, b, \alpha + a, \beta + b)
\]

\[
= \sum_{n \in (c+1, N)} \left( q^{n_1^2 + n_2^2 + \cdots + n_{c+1}^2} \prod_{i=1}^{c+1} \left[ 2 \left( M - \sum_{j=1}^{i} n_j \right) + n_{i-1} + n_i \right] \right)
\times \sum_{n_{c+1} = 0}^{n_c} q^{n_{c+1}} \left[ 2 \left( M - \sum_{j=1}^{c} n_j \right) + n_c - n_{c+1} \right]
\times R(n_{c+1}, M - \text{wt}(n) - n_{c+1}, a, b, \alpha + a, \beta + b)
\]

\[
= \sum_{n \in (c+1, N)} \left( q^{n_1^2 + n_2^2 + \cdots + n_{c+1}^2} \prod_{i=1}^{c+1} \left[ 2 \left( M - \sum_{j=1}^{i} n_j \right) + n_{i-1} + n_i \right] \right)
\]
unbounded respectively. Also note that either of these values might be specified as \( \infty \pm \alpha \).

Lemma 3.7 \( \text{is the continued fraction for } n^{(1)} \), \( n^{(2)}, \ldots, n^{(x)} \), for \( x \leq t \) (where \( x \) may or may not be fixed) where each partition \( n^{(k)} \) is in \((c_k, n_{k,0})\) for a certain integer \( n_{k,0} \) and \( \text{wt}(n^{(k)}) \leq w_k \) for a certain integer \( w_k \). The values of \( n_{k,0} \) and \( w_k \) will be specified recursively. The parts of \( n^{(k)} \) will be denoted \( n_{k,i} \) for \( 1 \leq k \leq c_k \) and listed (conventionally) in non-increasing order. Thus:

\[
n_{k,0} \geq n_{k,1} \geq n_{k,2} \geq \cdots \geq n_{k,c_k} \geq 0
\]

for \( 1 \leq k \leq x \).

In later results, some of the \( n_{k,0} \) or the \( w_k \) will be specified to be \( \infty \). This will naturally mean that the largest part of \( n^{(k)} \) is unbounded or that \( \text{wt}(n^{(k)}) \) is unbounded respectively. Also note that either of these values might be specified as \( \infty \pm \alpha \) for some integer \( \alpha \). This sum/difference should also be taken to be \( \infty \). \( \square \)

The following rather technical result will be used as a stepping stone to later theorems.

Lemma 3.7 Let \( p \) and \( p' \) be positive coprime integers and let \((c_1, c_2, \ldots, c_t)\) be the continued fraction for \((p' : p)\) satisfying \( t \geq 1 \). For \( 1 \leq u \leq t \) and \( v \geq u - 1 \), let \( r \) and \( s \) be such that \((c_1, c_2, \ldots, c_{u-1}, d_u, d_{u+1}, \ldots, d_v)\) is the continued fraction for \((s : r)\). Now let \( \hat{p} \) and \( \hat{p}' \) be such that \((c_u, c_{u+1}, \ldots, c_t)\) is the continued fraction for \((\hat{p}' : \hat{p})\), and let \( \hat{r} \) and \( \hat{s} \) be such that \((d_u, d_{u+1}, \ldots, d_v)\) is the continued fraction for \((\hat{s} : \hat{r})\). Then \( R(N, M, r, p - r, s, p' - s) \) may be expressed in terms of \( R(N', M', \hat{r}, \hat{p} - \hat{r}, \hat{s}, \hat{p}' - \hat{s}) \), for various \( N' \) and \( M' \), as follows:

\[
R(N, M, r, p - r, s, p' - s) = \sum \left( \sum_{k=1}^{n-1} \sum_{j=1}^{u_k} w_k \prod_{i=1}^{u_k} \frac{2(n_{k,j} + n_{k,j-1} + n_{k,i})}{n_{k,i-1} - n_{k,i}} \right) \times R(n_{u,0}, w_u, \hat{r}, \hat{p} - \hat{r}, \hat{s}, \hat{p}' - \hat{s}),
\]

Thus the result holds at \( c + 1 \), whereupon the theorem follows by induction. \( \square \)

3.4 A ‘Burge-type’ algorithm to generate a Burge tree

Definition 3.6 The following and subsequent theorems give fermionic expressions involving a summation taken over various sequences of partitions that satisfy certain constraints. Typically, given a sequence of integers \((c_1, c_2, \ldots, c_t)\), the sum will be over all sequences of partitions \( n^{(1)}, n^{(2)}, \ldots, n^{(x)} \), for \( x \leq t \) (where \( x \) may or may not be fixed) where each partition \( n^{(k)} \) is in \((c_k, n_{k,0})\) for a certain integer \( n_{k,0} \) and \( \text{wt}(n^{(k)}) \leq w_k \) for a certain integer \( w_k \). The values of \( n_{k,0} \) and \( w_k \) will be specified recursively. The parts of \( n^{(k)} \) will be denoted \( n_{k,i} \) for \( 1 \leq k \leq c_k \) and listed (conventionally) in non-increasing order. Thus:

\[
n_{k,0} \geq n_{k,1} \geq n_{k,2} \geq \cdots \geq n_{k,c_k} \geq 0
\]

for \( 1 \leq k \leq x \).

In later results, some of the \( n_{k,0} \) or the \( w_k \) will be specified to be \( \infty \). This will naturally mean that the largest part of \( n^{(k)} \) is unbounded or that \( \text{wt}(n^{(k)}) \) is unbounded respectively. Also note that either of these values might be specified as \( \infty \pm \alpha \) for some integer \( \alpha \). This sum/difference should also be taken to be \( \infty \). \( \square \)
where the sum is over all sequences \( n^{(1)} \), \( n^{(2)} \), \ldots, \( n^{(u-1)} \) of partitions for which, for \( 1 \leq k < u \), the partition \( n^{(k)} \) is in \( (c_k, n_{k,0}) \) and satisfies \( \text{wt}(n^{(k)}) \leq w_k \), where we define \( w_1 = M, w_k = n_{k-1,c_{k-1}} \) for \( 2 \leq k \leq u \), \( n_{1,0} = N \), and \( n_{k,0} = w_{k-1} - \text{wt}(n^{(k-1)}) \) for \( 2 \leq k \leq u \).

Proof: If \( u = 1 \), \( t > 0 \) and \( v \geq 0 \) then the expression holds trivially. We shall proceed by induction on \( u \) keeping the differences \( t - u \) and \( v - u \) fixed.

For the purposes of induction, assume that the result holds for a fixed \( u \geq 1 \). Now let \((p' : p)\) have continued fraction \((c_1, c_2, \ldots, c_{u+1})\), \((s : r)\) have continued fraction \((c_1, c_2, \ldots, c_u)\), \((\hat{p}' : \hat{p})\) have continued fraction \((c_{u+1}, c_{u+2}, \ldots, c_{2+1})\), and \((\hat{s} : \hat{r})\) have continued fraction \((d_{u+1}, d_{u+2}, \ldots, d_{2+1})\). Then, by definition, \((p : p' - c_1 p)\) has continued fraction \((c_2, c_3, \ldots, c_{t+1})\) and \((r : s - c_1 r)\) has continued fraction \((c_2, c_3, \ldots, c_u)\). Thereupon, on using Lemma 3.3 followed by Corollary 1.2 and then the induction hypothesis,

\[
R(N, M, r, p - r, s, p' - s)
\]

\[
= \sum_{n^{(1)} \in \{c_1, n_{1,0}\}, \text{wt}(n^{(1)}) \leq w_1} \left( q^{n_{1,1}'^2 + n_{1,2}'^2 + \cdots + n_{1,c_1}^2} \prod_{i=1}^{c_1} \left[ 2 \left( w_1 - \sum_{j=1}^{i} n_j \right) + n_{i-1} + n_i \right] \right.
\]

\[
\times R(w_2, n_{2,0}, r, p - r, s - c_1 r, p' - s - c_1 p + c_1 r)
\]

\[
= \sum_{n^{(1)} \in \{c_1, n_{1,0}\}, \text{wt}(n^{(1)}) \leq w_1} \left( q^{n_{1,1}'^2 + n_{1,2}'^2 + \cdots + n_{1,c_1}^2} \prod_{i=1}^{c_1} \left[ 2 \left( w_1 - \sum_{j=1}^{i} n_j \right) + n_{i-1} + n_i \right] \right.
\]

\[
\times R(n_{2,0}, w_2, s - c_1 r, p' - s - c_1 p + c_1 r, r, p - r)
\]

\[
= \sum_{n^{(1)} \in \{c_1, n_{1,0}\}, \text{wt}(n^{(1)}) \leq w_1} \left( q^{n_{1,1}'^2 + n_{1,2}'^2 + \cdots + n_{1,c_1}^2} \prod_{i=1}^{c_1} \left[ 2 \left( w_1 - \sum_{j=1}^{i} n_j \right) + n_{i-1} + n_i \right] \right)
\]

\[
\times \sum_{n^{(2)} \in \{c_2, n_{2,0}\}, \text{wt}(n^{(2)}) \leq w_2} \left( q^{\sum_{k=2}^{u} \sum_{i=1}^{n_{k,1}} n_{k,i}^2} \right)
\]

\[
= \sum_{n^{(3)} \in \{c_3, n_{3,0}\}, \text{wt}(n^{(3)}) \leq w_3} \left( q^{\sum_{k=2}^{u} \sum_{i=1}^{n_{k,1}} n_{k,i}^2} \right)
\]

\[
\vdots
\]

\[
= \sum_{n^{(u)} \in \{c_u, n_{u,0}\}, \text{wt}(n^{(u)}) \leq w_u} \left( q^{\sum_{k=2}^{u} \sum_{i=1}^{n_{k,1}} n_{k,i}^2} \right)
\]

\[
\times \prod_{k=2}^{u} \prod_{i=1}^{c_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right]
\]
\[ \sum_{k=1}^{u} \left( q^{\sum_{k=1}^{u} \sum_{i=1}^{v_k} n_{k,i}^2} \times R(n_{u+1,0}, w_{u+1}, \hat{r}, \hat{s}, \hat{p}', \hat{s}') \right) \]

\[ = \sum_{n^{(1)} \in (c_1, n_{1,0}), \ \text{wt}(n^{(1)}) \leq w_1} \left( q^{\sum_{k=1}^{v_1} n_{k,1}^2} \times R(n_{v_1+1,0}, w_{v+1}, \hat{r}, \hat{s}, \hat{p}, \hat{s}) \right) \]

\[ \leq \frac{1}{w_1} \sum_{n^{(2)} \in (c_2, n_{2,0}), \ \text{wt}(n^{(2)}) \leq w_2} \left( q^{\sum_{k=1}^{v_2} n_{k,2}^2} \times R(n_{v_2+1,0}, w_{v+1}, \hat{r}, \hat{s}, \hat{p}, \hat{s}) \right) \]

\[ \vdots \]

\[ \leq \frac{1}{w_{t-1}} \sum_{n^{(t-1)} \in (c_{t-1}, n_{t-1,0}), \ \text{wt}(n^{(t-1)}) \leq w_{t-1}} \left( q^{\sum_{k=1}^{v_{t-1}} n_{k,t-1}^2} \times R(n_{v_{t-1}+1,0}, w_{v+1}, \hat{r}, \hat{s}, \hat{p}, \hat{s}) \right) \]

where \( w_{u+1} = n_{u,c_\bar{u}} \) and \( n_{u+1,0} = w_u - \text{wt}(n^{(u)}) \). This is the desired result at \( u+1, t+1 \) and \( v+1 \). Hence the lemma is proved by induction. \( \square \)

**Theorem 3.8** Let \( p \) and \( p' \) be positive coprime integers and let \((p' : p)\) have continued fraction \((c_1, c_2, \ldots, c_t, c_t + 2)\). Then let \( r \) and \( s \) be such that \((s : r)\) has continued fraction \((c_1, c_2, \ldots, c_t-1)\). Then \( R(N, M, r, p-r, s, p'-s) \) may be expressed as follows:

\[ R(N, M, r, p-r, s, p'-s) \]

\[ = \sum_{n^{(1)} \in (c_1, n_{1,0}), \ \text{wt}(n^{(1)}) \leq w_1} \left( q^{\sum_{k=1}^{v_1} n_{k,1}^2} \times R(n_{v_1+1,0}, w_{v+1}, \hat{r}, \hat{s}, \hat{p}, \hat{s}) \right) \]

where the sum is over all sequences \( n^{(1)}, n^{(2)}, \ldots, n^{(t)} \) of partitions for which, for \( 1 \leq k \leq t \), the partition \( n^{(k)} \) is in \((c_k, n_{k,0})\) and satisfies \( \text{wt}(n^{(k)}) \leq w_k \), where we define \( w_1 = M, w_k = n_{k-1,c_{k-1}} \) for \( 2 \leq k \leq t \), \( n_{1,0} = N \), and \( n_{k,0} = w_{k-1} - \text{wt}(n^{(k-1)}) \) for \( 2 \leq k \leq t \); and additionally also satisfy \( \text{wt}(n^{(t)}) = w_t \).

**Proof:** On using Lemma 3.7 with \( u = t = v + 1 \), so that \((p' : p)\) has continued fraction \((c_t + 2)\) and \((\hat{s} : \hat{r})\) has continued fraction (), whereupon \( p = 1, p' = c_t + 2, \hat{r} = 0 \) and \( \hat{s} = 1 \), we obtain

\[ R(N, M, r, p-r, s, p'-s) \]

\[ = \sum_{n^{(1)} \in (c_1, n_{1,0}), \ \text{wt}(n^{(1)}) \leq w_1} \left( q^{\sum_{k=1}^{v_1} n_{k,1}^2} \times R(n_{v_1+1,0}, w_{v+1}, \hat{r}, \hat{s}, \hat{p}, \hat{s}) \right) \]

\[ \vdots \]

\[ \sum_{n^{(t-1)} \in (c_{t-1}, n_{t-1,0}), \ \text{wt}(n^{(t-1)}) \leq w_{t-1}} \left( q^{\sum_{k=1}^{v_{t-1}} n_{k,t-1}^2} \times R(n_{v_{t-1}+1,0}, w_{v+1}, \hat{r}, \hat{s}, \hat{p}, \hat{s}) \right) \]

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\[
\times \prod_{k=1}^{i-1} \prod_{i=1}^{c_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right] \\
\times R(n_{t,0}, w_t, 0, 1, 1, c_t + 1).
\]

The use of Lemma 3.2 with \(a = 0\), \(b = \alpha = \beta = 1\), \(c = c_t\), \(N = n_{t,0}\), \(M = w_t\), followed by an application of Corollary 3.2 results in:

\[
R(n_{t,0}, w_t, 0, 1, 1, c_t + 1)
\]

\[
= \sum_{n^{(t)} \text{ in } (c_t, n_{t,0})} \left( q^{n_{t,1}^2 + n_{t,2}^2 + \ldots + n_{t,c_t}^2} \prod_{i=1}^{c_t} \left[ 2 \left( w_t - \sum_{j=1}^{i} n_{t,j} \right) + n_{t,i-1} + n_{t,i} \right] \right) \\
\times R(n_{t,c_t}, w_t - \text{wt}(n^{(t)}), 0, 1, 1, 1)
\]

\[
= \sum_{n^{(t)} \text{ in } (c_t, n_{t,0})} \left( q^{n_{t,1}^2 + n_{t,2}^2 + \ldots + n_{t,c_t}^2} \prod_{i=1}^{c_t} \left[ 2 \left( w_t - \sum_{j=1}^{i} n_{t,j} \right) + n_{t,i-1} + n_{t,i} \right] \right) \\
\times \delta_{w_t, \text{wt}(n^{(t)})}.
\]

which, when substituted in the previous expression, proves the theorem. \(\square\)

In fact, the particular \(r\) and \(s\) specified in Theorem 3.8 have a simple characterisation. We first prove the following:

**Lemma 3.9** Let \(p\) and \(p'\) be positive coprime integers and let \((c_1, c_2, \ldots, c_t)\) be the continued fraction for \((p' : p)\) satisfying \(c_t \geq 2\). Let \(r_0\) and \(s_0\) be such that \((c_1, c_2, \ldots, c_t-1)\) is a continued fraction for \((s_0 : r_0)\). Then:

1. \(p'r_0 - ps_0 = (-1)^t\);
2. \(r_0 \leq p/2\) and \(s_0 \leq p'/2\).
3. \((c_1, c_2, \ldots, c_{t-1}, c_t - 1)\) is a continued fraction for \((p' - s_0 : p - r_0)\).

**Proof:** For each part, the \(t = 1\) case is immediate. For the purposes of induction, assume that each part holds for a fixed \(t \geq 1\). Now let \((p' : p)\) and \((s_0 : r_0)\) have continued fractions \((c_1, c_2, \ldots, c_{t+1})\) and \((c_1, c_2, \ldots, c_t)\) respectively. Then, by definition, \((p : p' - c_1 p)\) and \((r_0 : s_0 - c_1 r_0)\) have continued fractions \((c_2, c_3, \ldots, c_{t+1})\) and \((c_2, c_3, \ldots, c_t)\) respectively. Then, by the induction hypothesis for the first part, \(p(s_0 - c_1 r_0) - (p' - c_1 p)r_0 = (-1)^t\), whereby \(p'r_0 - ps_0 = (-1)^{t+1}\). The first part now follows by induction.
By the induction hypothesis for the second part, \( r_0 \leq p/2 \) and \( s_0 - c_1 r_0 \leq (p' - c_1 p)/2 \) whereupon \( s_0 \leq p'/2 + c_1 (r_0 - p/2) \leq p'/2 \). The second part now also follows by induction.

By the induction hypothesis for the third part, \((p - r_0 ; p' - c_1 p - s_0 + c_1 r_0)\) has continued fraction \((c_2, c_3, \ldots, c_{t + 1} - 1)\). Thereupon \((c_1, c_2, c_3, \ldots, c_{t + 1} - 1)\) is a continued fraction for \((p' - s_0 ; p - r_0)\). The third part now also follows by induction. 

It is now a straightforward task to show that the \( r_0 \) and \( s_0 \) specified in this lemma are the smallest positive integers for which \( |p' r_0 - ps_0| = 1 \). Thus these are the values denoted \( r_{\text{min}} \) and \( s_{\text{min}} \) in [18].

**Note 3.10** By using the first identity of Corollary [12], the fermionic expression for \( R(N, M, p - r, r, p' - s, s) \) is immediately obtained from that of \( R(N, M, r, p - r, s, p' - s) \). In the case of \( r_0 \) and \( s_0 \), we thus also have, by virtue of Theorem [3.8], the fermionic expression for \( R(N, M, r, p - r, s, p' - s) \) where \( r = p - r_0 \) and \( s = p' - s_0 \). Thus for specific \( p \) and \( p' \) having continued fraction \((c_1, c_2, \ldots, c_t)\), we already have fermionic expressions for \( R(N, M, r, p - r, s, p' - s) \) for two different pairs of \( r \) and \( s \): namely those for which \((s,r)\) has continued fractions \((c_1, c_2, \ldots, c_t - 1)\) and \((c_1, c_2, \ldots, c_t - 1)\).

In fact, it is possible to calculate \( R(N, M, p - r_0, r_0, p' - s_0, s_0) \) directly using the technique of this section, but starting (instead) from \( R(N', M', 1, 0, 1, 1) = \delta_{M', 0} \). This also results in precisely the same expression for \( R(N, M, p - r_0, r_0, p' - s_0, s_0) \) as for \( R(N, M, r_0, p - r_0, s_0, p' - s_0) \).

### 4 An extended Burge tree

In this section, we deduce a further relationship between the sets of partition pairs. This is then exploited to provide fermionic expressions for further \( R(N, M, r, p - r, s, p' - s) \). Once again, the continued fraction of \((p' : p)\) determines for which \( r \) and \( s \), the fermionic expression for \( R(N, M, r, p - r, s, p' - s) \) may be determined. However here, for particular \( p \) and \( p' \), this may be achieved for various \( r \) and \( s \).

**Lemma 4.1** Let \( M \), \( N \), \( b \), and \( \beta \) be non-negative integers. Then

1. \( R(N, M + 1, 1, b, 0, \beta + 1) = q^n R(N, M, 1, b, 1, \beta) \);
2. \( R(N + 1, M, 0, b + 1, 1, \beta) = q^m R(N, M, 1, b, 1, \beta) \);
3. \( R(N + 1, M, 0, b + 1, 1, \beta) = q^{m-n} R(N, M + 1, 1, b, 0, \beta + 1) \);
4. \( R(N + 1, M, 0, b + 1, 1, \beta) = q^{m-n} R(M + 1, N, 0, \beta + 1, 1, b) \).

**Proof:** The first expression is proved by setting up a bijection between the sets \( \mathcal{R}(N, M + 1, N, M + 1, 1, b, 0, \beta + 1) \) and \( \mathcal{R}(N, M, N, M, 1, b, 1, \beta) \). Let \((q, p) \in \mathbb{
\[ R(N, M + 1, N, M + 1, 1, b, 0, \beta + 1) \] whence from \([3]\),

\[
\begin{align*}
p_i & \geq q_i + 1 & 1 \leq i \leq N, \\
q_i & \geq p_{i+1-b} - \beta & b \leq i \leq N.
\end{align*}
\]

Since each \( q_i \geq 0 \) and each \( p_i \leq M + 1 \), the first of these inequalities implies that \( p_i \geq 1 \) and \( q_i \leq M \). Thereupon, on setting \( q'_i = q_i \) and \( p'_i = p_i - 1 \) for \( 1 \leq i \leq N \), we have:

\[
\begin{align*}
p'_i & \geq q'_i & 1 \leq i \leq N, \\
q'_i & \geq p'_{i+1-b} + 1 - \beta & b \leq i \leq N.
\end{align*}
\]

This ensures that \((q', p') \in R(N, M, N, M, 1, b, 1, \beta)\). The map \((q, p) \mapsto (q', p')\) clearly defines a bijection between \(R(N, M + 1, N, M + 1, 1, b, 0, \beta + 1)\) and \(R(N, M, N, M, 1, b, 1, \beta)\). Thereupon, since \( \text{wt}((q', p')) = \text{wt}((q, p)) - N \), the first expression is proved. (The first expression may also be proved via the expression \([1,4]\) for the generating function.)

The second expression follows from the first by transforming both sides using Corollary \([3,2]\). Thereupon, the third expression results from combining these two. A further application of Corollary \([3,2]\) then yields the fourth expression.

\[ \square \]

As we shall see, this lemma enables us to determine fermionic expressions for \( R(N, M, r, p-r, s, p'-s) \) where the possible \( r \) and \( s \) are determined by \( p \) and \( p' \) as follows. Let \((p':p)\) have continued fraction \((c_1, c_2, \ldots, c_t)\) with \( t \geq 1 \). Then we may obtain the fermionic expression for \( R(N, M, r, p-r, s, p'-s) \) if \((s:r)\) has continued fraction \((c_1, c_2, \ldots, c_{u-1}, d)\) where \( 1 \leq u \leq t \) and \( 1 \leq d \leq c_u \). (Here we include the possibility that \( u > 1 \) and \( d = 1 \). In such a case, the continued fractions \((c_1, c_2, \ldots, c_{u-1}, 1)\) and \((c_1, c_2, \ldots, c_{u-1} + 1)\) are equated.) This is done in Theorem \([4,4]\). Of course, for these \( r \) and \( s \), Corollary \([1,2]\) then also yields \( R(N, M, p-r, r, p'-s, s) \).

In the following result, we restrict to the \( N > 0 \) case, the \( N = 0 \) case having been already dealt with in Lemma \([3,3]\).

**Theorem 4.2** Let \( p \) and \( p' \) be positive coprime integers and let \((p':p)\) have continued fraction \((c_1, c_2, \ldots, c_t, c_t+2)\) with \( t \geq 1 \). Then \( R(N, M, 0, p, 1, p'-1) \) may be expressed as follows when \( N > 0 \):

\[
\begin{align*}
R(N, M, 0, p, 1, p'-1) & = q^M \sum_{n^{(x)}} q^{\sum_{k=1}^{x} \sum_{i=1}^{c_k} n_{k,i}(n_{k,i}-1)} \prod_{k=1}^{c_k} \prod_{i=1}^{n_{k,i}} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right],
\end{align*}
\]

where the sum is over all sequences \( n^{(1)}, n^{(2)}, \ldots, n^{(x)} \) of partitions with \( 1 \leq x \leq t \) for which, for \( 1 \leq k \leq x \), the partition \( n^{(k)} \) is in \((c_k, n_{k,0})\) and satisfies \( \text{wt}(n^{(k)}) \leq w_k \) with \( \text{wt}(n^{(x)}) = w_x \), where we define \( w_1 = M \), \( w_k = n_{k-1}c_{k-1} - 1 \).
for $2 \leq k \leq x$, $n_{1,0} = N$, and $n_{k,0} = w_{k-1} + 1 - \text{wt}(n^{(k-1)})$ for $2 \leq k \leq x$; and additionally also satisfies $n_{k,c_k} > 0$ for $1 \leq k < x$ and satisfies $n_{x,c_x} = 0$ whenever $x < t$.

Proof: If $t = 1$ then necessarily $p' = c_1 + 2$ and $p = 1$. Then, with $n_{1,0} = N$ and $w_1 = M$, Lemma 3.5 implies that

$$R(N, M, 0, 1, 1, c_t + 1)$$

$$= \sum_{n^{(1)} \in (c_1, n_{1,0})} \left( q^{c_1} \prod_{i=1}^{c_1} n_{1,i}^2 \right) \left[ 2 \left( w_1 - \sum_{j=1}^{i} n_{1,j} \right) + n_{1,i-1} + n_{1,i} \right]$$

$$\times R(n_{1,c_1}, w_1 - \text{wt}(n^{(1)}), 0, 1, 1, 1)$$

$$= \sum_{n^{(1)} \in (c_1, n_{1,0})} \left( q^{c_1} \prod_{i=1}^{c_1} n_{1,i}^2 \right) \left[ 2 \left( w_1 - \sum_{j=1}^{i} n_{1,j} \right) + n_{1,i-1} + n_{1,i} \right]$$

$$\times \delta_{w_1, \text{wt}(n^{(1)})}. \right)$$

on using Corollary 3.3. Therefore, we require those partitions $n^{(1)}$ for which $\text{wt}(n^{(1)}) = w_1 = M$. Furthermore, we then have that $\sum_{i=1}^{c_1} n_{1,i}^2 = M + \sum_{i=1}^{c_1} n_{1,i} (n_{1,i} - 1)$, and therefore the theorem holds in the case $t = 1$, when $x$ can only take the value $x = 1$.

For the purposes of induction, assume that the result holds for a fixed $t \geq 1$. Now let $(p' : p)$ have continued fraction $(c_1, c_2, \ldots, c_t+1 + 2)$. Then, by definition, $(p : p' - c_1)$ has continued fraction $(c_2, c_3, \ldots, c_t+1 + 2)$. Thereupon, on using Lemma 3.3,

$$R(N, M, 0, p, 1, p' - 1)$$

$$= \sum_{n^{(1)} \in (c_1, n_{1,0})} \left( q^{c_1} \prod_{i=1}^{c_1} n_{1,i}^2 \right) \left[ 2 \left( w_1 - \sum_{j=1}^{i} n_{1,j} \right) + n_{1,i-1} + n_{1,i} \right]$$

$$\times R(n_{1,c_1}, w_1 - \text{wt}(n^{(1)}), 0, p, 1, p' - c_1 p - 1).$$

For those cases where $n_{1,c_1} = 0$, by virtue of Lemma 3.3, we obtain:

$$R(N, M, 0, p, 1, p' - 1)$$
These correspond to the terms for which $x = 1$ in the premise.

On the other hand, for those cases where $n_{1,c_1} > 0$, use Lemma 4.1(4) and then the induction hypothesis:

\[
R(N, M, 0, p, 1, p' - 1) = \sum_{n^{(1)} \in (c_1,n_{1,0})} \sum_{\text{wt}(n^{(1)}) = w_1} \left( q^{\sum_{i=1}^{c_1} n_{1,i}^2} \prod_{i=1}^{c_1} \left[ 2 \left( w_1 - \sum_{j=1}^{i} n_{1,j} \right) + n_{1,i-1} + n_{1,i} \right] \right) \\
\times \delta_{w_1, \text{wt}(n^{(1)})} \\
\times R(w_1 + 1 - \text{wt}(n^{(1)}), n_{1,c_1} - 1, 0, p' - c_1 p, 1, p - 1) \\
= \sum_{n^{(1)} \in (c_1,n_{1,0})} \sum_{\text{wt}(n^{(1)}) = w_1} \left( q^{\sum_{i=1}^{c_1} n_{1,i}^2} \prod_{i=1}^{c_1} \left[ 2 \left( w_1 - \sum_{j=1}^{i} n_{1,j} \right) + n_{1,i-1} + n_{1,i} \right] \right) \\
\times q^{w_1+1-n_{1,c_1}-\text{wt}(n^{(1)})} \\
\times \sum_{2 \leq x \leq t+1} \left( q^{\sum_{k=2}^{x} \sum_{i=1}^{c_k} n_{k,i}(n_{k,i-1})} \right) \\
\times R(w_1 + 1 - \text{wt}(n^{(1)})+n_{1,c_1}-1) \\
\times R(w_2, w_2, f_2 > 0) \\
\times R(w_3, w_3, f_3 > 0) \\
\vdots \\
\times R(w_x, w_x, f_x > 0) \\
\times q^{\prod_{k=2}^{x} \prod_{i=1}^{c_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right]}
\]

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\[
q^M \sum_{2 \leq x \leq t+1} \left( q^{\sum_{k=1}^{\infty} \sum_{i=1}^{c_k} n_{k,i}(n_{k,i}-1)} \right)
\]

\[\begin{align*}
&n^{(1)} \text{ in } (c_1,n_{1,0}), \text{ wt } (n^{(1)}) \leq w_1, n_{1,f_1} > 0 \\
&n^{(2)} \text{ in } (c_2,n_{2,0}), \text{ wt } (n^{(2)}) \leq w_2, n_{2,f_2} > 0 \\
&\vdots
\end{align*}\]

\[\begin{align*}
&n^{(x-1)} \text{ in } (c_{x-1},n_{x-1,0}), \text{ wt } (n^{(x-1)}) \leq w_{x-1}, n_{x-1,f_{x-1}} > 0 \\
&n^{(x)} \text{ in } (c_{x},n_{x,0}), \text{ wt } (n^{(x)}) = w_x, n_{x,f_x} = 0 \text{ if } x < t+1
\end{align*}\]

after using \(M = w_1\). Combining this with the previous expression for when \(n_{1,c_1} = 0\), proves the theorem in the case \(t + 1\). Hence, by induction, the theorem holds for all \(t \geq 1\).

\[\square\]

**Example 4.3** We use the above theorem to calculate \(R(6,5,0,6,1,16)\). Via Lemma 3.4, the continued fraction of \((17:6)\) is found to be \((2,1,5)\). Thus \(c_1 = 2\), \(c_2 = 1\) and \(c_3 = 3\). For convenience, we consider separately the sequences of parameters which correspond to the cases \(x = 1\), \(x = 2\) and \(x = 3\).

- \(x = 1\): require a 2-part partition \(n^{(1)} = (n_{1,1}, n_{1,2})\) having \(n_{1,1} \leq 6\), weight \(x = 1\), \(n_{1,1} \leq 6\), weight \(n_{1,2} = 0\). The only possibility is \(n^{(1)} = (5,0)\). Since \(w_1 = 5\) and \(n_{1,0} = 6\), the corresponding summand of Theorem 4.2 is

\[q^{20} \begin{bmatrix} 11 \\ 1 \\ 5 \\ 5 \end{bmatrix}\]

- \(x = 2\): require a 2-part partition \(n^{(1)} = (n_{1,1}, n_{1,2})\) satisfying \(n_{1,1} \leq 6\), \(n_{1,2} \geq 0\), and a 1-part partition \(n^{(2)} = (n_{2,1})\) satisfying \(n_{2,1} = 6 - \text{wt } (n^{(1)})\), \(n^{(2)} = n_{2,1} - 1\) and \(n_{2,1} = 0\). The last two conditions on \(n^{(2)}\) imply that \(n_{1,2} = 1\) and hence the only possible sequences here are \((n_{1,2}, n_{1,2'}; n_{2,1}) = (4, 1; 0), (3, 1; 0), (2, 1; 0)\) or \((1, 1; 0)\). As above, \(w_1 = 5\) and \(n_{1,0} = 6\), and now \(w_2 = 0\) in each case but \(n_{2,0} = 6 - \text{wt } (n^{(1)})\). Thereupon, the corresponding summands of Theorem 4.2 are:

\[q^{12} \begin{bmatrix} 12 \\ 2 \\ 3 \\ 1 \end{bmatrix} + q^6 \begin{bmatrix} 13 \\ 3 \\ 1 \\ 2 \end{bmatrix} + q^2 \begin{bmatrix} 14 \\ 3 \\ 3 \\ 1 \end{bmatrix} + q^2 \begin{bmatrix} 15 \\ 5 \\ 8 \\ 4 \end{bmatrix} + \ldots\]

- \(x = 3\): require a 2-part partition \(n^{(1)} = (n_{1,1}, n_{1,2})\) satisfying \(n_{1,1} \leq 6\), \(n_{1,2} \geq 0\), a 1-part partition \(n^{(2)} = (n_{2,1})\) satisfying \(n_{2,1} = 6 - \text{wt } (n^{(1)})\), \(n^{(2)} = n_{2,1} - 1\) and \(n_{2,1} = 0\), and a 3-part partition \(n^{(3)} = (n_{3,1}, n_{3,2}, n_{3,3})\) satisfying \(n_{3,1} \leq n_{1,2} - \text{wt } (n^{(2)})\) and

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Here, we define: satisfies
\[ n = w \]
Theorem 4.4

Let \( q \) and \( q' = p \) be positive coprime integers and let \((p' : p)\) have continued fraction \((c_1, c_2, \ldots, c_{u-1}, c_u + 2)\) with \( t \geq 1 \). For \( 1 \leq u \leq t \) and \( 1 \leq d \leq c_u \), let \( r \) and \( s \) be such that \((c_1, c_2, \ldots, c_{u-1}, d)\) is the continued fraction for \((s : r)\). Then \( R(N, M, r, p - r, s, p' - s) \) may be expressed as follows:

\[
R(N, M, r, p - r, s, p' - s) = \sum q^{n_u, d + \sum_{k=1}^{u} \sum_{i=1}^{b_k} n_{k,i}^2 + \sum_{k=u+1}^{x} \sum_{i=1}^{c_k} h_{k,i} (n_{k,i} - 1)} \times \prod_{k=1}^{x} \prod_{i=1}^{f_k} 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} + n_{k,i} - n_{k,i-1} \right) .
\]

where, on setting \((f_1, f_2, \ldots, f_x, f_{x+1}) = (c_1, c_2, \ldots, c_{x-1}, d, c_x - d, c_{x+1}, \ldots, c_t)\), the sum is over all sequences \( n^{(1)}, n^{(2)}, \ldots, n^{(x)} \) of partitions with \( u \leq x \leq t + 1 \) for which, for \( 1 \leq k \leq x \), the partition \( n^{(k)} \) is in \((f_k, n_{k,0})\), satisfies \( \text{wt}(n^{(k)}) \leq w_k \) with \( \text{wt}(n^{(x)}) = w_x \), and additionally also satisfies \( \text{wt}(n^{(u)}) < w_u \) if \( u < x \), satisfies \( n_{k,f_k} > 0 \) if \( u < k \leq x \), and satisfies \( n_{x,f_x} = 0 \) whenever \( x < t + 1 \).

Here, we define:

\[
w_1 = M;
\]

\[
w_k = n_{k-1,f_k-1} \quad 2 \leq k \leq u;
\]

\[
w_{u+1} = w_u - 1 - \text{wt}(n^{(u)});
\]

\[
w_k = n_{k-1,f_k-1} - 1 \quad u + 2 \leq k \leq x;
\]

and:

\[
n_{1,0} = N;
\]

\[
n_{k,0} = w_k - 1 - \text{wt}(n^{(k-1)}) \quad 2 \leq k \leq u;
\]

\[
n_{u+1,0} = n_{u,f_u} + 1;
\]

\[
n_{k,0} = w_k + 1 - \text{wt}(n^{(k-1)}) \quad u + 2 \leq k \leq x.
\]
Proof: Let \( \hat{p} \) and \( \hat{p}' \) be such that \((\hat{p}' : \hat{p})\) has continued fraction \((c_u, c_{u+1}, \ldots, c_t + 2)\). Then, on using \(v = u\) and \(d_v = d\) in Theorem 3.7, we obtain:

\[
R(N, M, r, p - r, s, p' - s) = \sum \left( q \sum_{k=1}^{u} \sum_{i=1}^{f_k} n_{k,i}^2 \prod_{k=1}^{u} \prod_{i=1}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right] \right.
\]

where the sum is over all sequences \(n^{(1)}, n^{(2)}, \ldots, n^{(u-1)}\) of partitions for which, for \(1 \leq k < u\), the partition \(n^{(k)}\) is in \((c_k, n_{k,0})\) and satisfies \(\text{wt}(n^{(k)}) \leq w_k\).

Applying first Lemma 3.3 with \(c = d = f_u\) to \(R(n_{u,0}, w_u, 1, \hat{p} - 1, d, \hat{p}' - d)\), and then Corollary 4.2 to the result, yields:

\[
R(N, M, r, p - r, s, p' - s) = \sum \left( q \sum_{k=1}^{u} \sum_{i=1}^{f_k} n_{k,i}^2 \prod_{k=1}^{u} \prod_{i=1}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right] \times R(n_{u,d}, w_u - \text{wt}(n^{(u)}), 1, \hat{p} - 1, 0, \hat{p}' - d\hat{p}) \right)
\]

where, now, the sum is over all sequences \(n^{(1)}, n^{(2)}, \ldots, n^{(u)}\) of partitions for which, for \(1 \leq k \leq u\), the partition \(n^{(k)}\) is in \((f_k, n_{k,0})\) and satisfies \(\text{wt}(n^{(k)}) \leq w_k\).

Now, if \(\text{wt}(n^{(u)}) = w_u\) then \(R(w_u - \text{wt}(n^{(u)}), n_{u,d}, 0, \hat{p}' - d\hat{p}, 1, \hat{p} - 1) = \delta_{n_{u,d}, 0}\) by Lemma 3.3. Otherwise, we may transform this term using Lemma 4.1(4). Thereupon,

\[
R(N, M, r, p - r, s, p' - s) = \sum_{\text{wt}(n^{(u)}) = w_u} q \sum_{k=1}^{u} \sum_{i=1}^{f_k} n_{k,i}^2 \prod_{k=1}^{u} \prod_{i=1}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right]
\]

\[
+ \sum_{\text{wt}(n^{(u)}) < w_u} q \sum_{k=1}^{u} \sum_{i=1}^{f_k} n_{k,i}^2 \prod_{k=1}^{u} \prod_{i=1}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right]
\]

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sequences for \( x \) cases have wt (\( n \)) possible (\( n \)) continued fraction of (5 : 2) is found to be (2

Then Theorem 4.2 yields:

\[
R(n_{u+1}, w_{u+1}, 0, \hat{p}, 1, \hat{p}' - d\hat{p} - 1)
\]

with the sums over all sequences as above.

Now, with \( n_{u+1,0} = n_{u,d} + 1 \) and \( w_{u+1} = w_u - 1 - \text{wt}(n(u)) \), we use the expression for \( R(n_{u+1,0}, w_{u+1}, 0, \hat{p}, 1, \hat{p}' - d\hat{p} - 1) \) given by Theorem 4.2. First note that since \((\hat{p}', \hat{p})\) has continued fraction \((f_{u+1} + d, f_{u+2}, \ldots, f_{t+1} + 2)\), it follows that the continued fraction of \((\hat{p}' - d\hat{p}; \hat{p})\) is \((f_{u+1}, f_{u+2}, \ldots, f_{t+1} + 2)\). Then Theorem 4.2 yields:

\[
R(n_{u+1,0}, w_{u+1}, 0, \hat{p}, 1, \hat{p}' - d\hat{p} - 1)
\]

\[
= q^{w_{u+1}} \sum_{u+1 \leq x \leq t+1} \left( q \sum_{k=u+1}^x \sum_{i=1}^l f_k n_{x,i}(n_{k,i-1}) \right)
\]

\[
n^{(u+1)} \text{ in } (f_{u+1,n_{u+1,0}}, \text{wt}(n^{(u+1)}) \leq w_{u+1}, n_{u+1,f_{u+1}} > 0 \)
\]

\[
n^{(u+2)} \text{ in } (f_{u+1,n_{u+2,0}}, \text{wt}(n^{(u+2)}) \leq w_{u+2}, n_{u+2,f_{u+2}} > 0 \)
\]

\[
\vdots
\]

\[
n^{(x-1)} \text{ in } (f_{x-1,n_{x-1,0}}, \text{wt}(n^{(x-1)}) \leq w_{x-1}, n_{x-1,f_{x-1}} > 0 \)
\]

\[
n^{(x)} \text{ in } (f_{x,n_{x,0}}, \text{wt}(n^{(x)}) = w_x, n_x, f_x = 0 \text{ if } x < t+1 \)
\]

\[
\times \prod_{k=x+1}^u \prod_{i=1}^{f_k} \left( 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right)
\]

with \( n_{k,0} \) and \( w_k \) for \( u < k \leq t+1 \) as in the premise. Combining this with the previous expression, proves the theorem.

\[\square\]

**Example 4.5** We use the above theorem to calculate \( R(3, 4, 2, 14, 5, 32) \). Via Lemma 4.2, the continued fraction of \((37 : 16)\) is found to be \((2, 3, 5)\), and the continued fraction of \((5 : 2)\) is found to be \((2, 2)\). Thus here \( t = 3, u = 2, d = 2, f_1 = 2, f_2 = 2, f_3 = 1 \) and \( f_4 = 3 \). Here, \( x \geq 2 \) and so each partition sequence contains at least the two partitions \( n^{(1)} \) and \( n^{(2)} \). These comprise \( f_1 = 2 \) and \( f_2 = 2 \) parts respectively, for which \( n_{1,1} \leq 3 \) and \( \text{wt}(n^{(1)}) \leq 4, n_{2,1} \leq 4 - \text{wt}(n^{(1)}) \) and \( \text{wt}(n^{(2)}) \leq n_{1,2} \). These constraints give rise to ten possible \((n_{1,1}, n_{1,2}; n_{2,1}, n_{2,2})\). They are:

\[
(3, 0; 0, 0), (2, 1; 1, 0), (2, 0; 0, 0), (1, 1; 1, 0), (1, 0; 0, 0), (0, 0; 0, 0),
\]

\[
(3, 1; 0, 0), (2, 1; 0, 0), (1, 1; 0, 0), (2, 2; 0, 0).
\]

Since \( w_u = n_{1,2} \), the first six cases have \( \text{wt}(n^{(u)}) = w_u \) and the final four cases have \( \text{wt}(n^{(u)}) > w_u \). Thus the first six cases are precisely the partition sequences for \( x = 2 \).
The remaining four cases give rise to those sequences for \( x > 2 \). First we require \( n^{(3)} \) with one part \( n_{3,1} \) for which \( n_{3,1} \leq n_{3,0} \) where \( n_{3,0} = n_{2,2} + 1 \) and \( \text{wt}(n^{(3)}) \leq w_3 \) where \( w_3 = n_{1,2} - 1 - \text{wt}(n^{(2)}) \). In the first three cases, this gives \( w_3 = 0 \) and so \( n_{3,1} = 0 \) in these cases. These are thus the cases for \( x = 3 \).

For the final case, \( n_{1,0} = 1 \) and \( w_3 = 1 \). This cannot give rise to an \( x = 3 \) term since the requirement that \( \text{wt}(n^{(3)}) = w_3 \) forces \( n_{3,1} = 1 \). For an \( x = 4 \) term, we also require \( n_{3,1} = 1 \) here, whereupon \( n_{4,0} = 1 \) and \( w_4 = 0 \), so that only \( n^{(4)} = (0, 0, 0, 0) \) is possible.

The following table summarises these details.

| \((n^{(1)}; n^{(2)}; \ldots; n^{(x)})\) | \(n_{1,0}\) | \(n_{2,0}\) | \(n_{2,1}\) | \(n_{3,0}\) | \(w_3\) | \(n_{4,0}\) | \(w_4\) |
|---|---|---|---|---|---|---|---|
| (3, 0; 0, 0) | 2 | 3 | 4 | 1 | 0 | | |
| (2, 1; 1, 0) | 2 | 3 | 4 | 1 | 1 | | |
| (2, 0; 0, 0) | 2 | 3 | 4 | 2 | 0 | | |
| (1, 1; 1, 0) | 2 | 3 | 4 | 2 | 1 | | |
| (1, 0; 0, 0) | 2 | 3 | 4 | 3 | 0 | | |
| (0, 0; 0, 0) | 2 | 3 | 4 | 4 | 0 | | |
| (3, 1; 0; 0; 0) | 3 | 3 | 4 | 0 | 1 | 1 | 0 |
| (2, 1; 0; 0; 0) | 3 | 3 | 4 | 1 | 1 | 1 | 0 |
| (1, 1; 0; 0; 0) | 3 | 3 | 4 | 2 | 1 | 1 | 0 |
| (2, 2; 0; 0; 1; 0; 0, 0) | 4 | 3 | 4 | 0 | 2 | 1 | 1 | 1 | 0 |

Using this information to evaluate the expression in Theorem 4.4, leads to:

\[
q^3 \left[ \frac{8}{6} \right] \left[ \frac{2}{2} \right] \left[ \frac{1}{1} \right] \left[ \frac{0}{0} \right] + q^6 \left[ \frac{8}{6} \right] \left[ \frac{2}{2} \right] \left[ \frac{1}{1} \right] \left[ \frac{0}{0} \right] + q^4 \left[ \frac{6}{4} \right] \left[ \frac{2}{2} \right] \left[ \frac{1}{1} \right] \left[ \frac{0}{0} \right] + q^3 \left[ \frac{10}{8} \right] \left[ \frac{6}{6} \right] \left[ \frac{4}{4} \right] \left[ \frac{2}{2} \right] \left[ \frac{2}{2} \right] \left[ \frac{1}{1} \right] \left[ \frac{0}{0} \right] + q^2 \left[ \frac{10}{8} \right] \left[ \frac{6}{6} \right] \left[ \frac{4}{4} \right] \left[ \frac{2}{2} \right] \left[ \frac{2}{2} \right] \left[ \frac{1}{1} \right] \left[ \frac{0}{0} \right] + q^3 \left[ \frac{8}{6} \right] \left[ \frac{6}{6} \right] \left[ \frac{4}{4} \right] \left[ \frac{2}{2} \right] \left[ \frac{2}{2} \right] \left[ \frac{1}{1} \right] \left[ \frac{0}{0} \right] \left[ \frac{0}{0} \right].
\]

Evaluating all the Gaussians and summing, results in:

\[
1 + 2q + 5q^2 + 10q^3 + 18q^4 + 28q^5 + \cdots + 114q^{11} + 119q^{12} + 114q^{13} + \cdots + 2q^{23} + q^{24}.
\]

As may be verified, this agrees with the value obtained using Theorem 4.4. (In fact, from the viewpoint of partition pairs, this result is fairly trivial — being the generating function for all pairs of partitions in \((3, 4) \times (3, 4)\).)

Note that, for convenience, the above theorem does not deal with the cases of \((s : r)\) having continued fraction () or \((c_1, c_2, \ldots, c_{t-1}, c_t + 1)\). The former of these, when \( r = 0 \) and \( s = 1 \), is already dealt with in Theorem 4.2, and as explained in Note 3.4, the latter is equal to that resulting from the case \((c_1, c_2, \ldots, c_{t-1})\).
5 Q-polynomial identities with a conventional bosonic side

In the previous section, we obtained q-polynomial identities that depend on two finite size parameters. We can take one of these parameters to infinity, and obtain identities that depend on a single parameter. Depending on which parameter is taken to infinity, we obtain polynomial identities that have a conventional dependence on the rest of the parameters, namely \( \{ p, p', r, s \} \), or a non-conventional dependence. In this section, we present the former. In the next section, we present the latter.

To be more specific, we use the words 'conventional' and 'non-conventional' in the following sense: We compare the dependence of the bosonic side of the q-polynomial identities that we obtain with that of the q-polynomial identities that appear in e.g. [17].

**Theorem 5.1** Let \( p \) and \( p' \) be positive coprime integers and let \((p': p)\) have continued fraction \((c_1, c_2, \ldots, c_{r-1}, c_r + 2)\) with \( t \geq 1 \). For \( 1 \leq u \leq t \) and \( 1 \leq d \leq c_u \), let \( r \) and \( s \) be such that \((c_1, c_2, \ldots, c_{u-1}, d)\) is the continued fraction for \((s: r)\). Then

\[
\lim_{N \to \infty} R(N, M, r, p - r, s, p' - s) = \frac{1}{(q)_{2M}} \sum \left( q^{n_{u,d} + \sum_{k=1}^{u} \sum_{i=1}^{f_k} n_k,i + \sum_{k=u+1}^{t} \sum_{i=1}^{f_k} n_k,i} \right) \prod_{k=1}^{x} \prod_{i=1+\delta_k}^{f_k} \left( 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right),
\]

where, on setting \((f_1, f_2, \ldots, f_t, f_{t+1}) = (c_1, c_2, \ldots, c_{u-1}, d, c_u - d, c_{u+1}, \ldots, c_t)\), the sum is over all sequences \( n^{(1)}, n^{(2)}, \ldots, n^{(x)} \) of partitions with \( u \leq x \leq t + 1 \) for which, for \( 1 \leq k \leq x \), the partition \( n^{(k)} \) is in \((f_k, n_{k,0})\), satisfies \( wt(n^{(k)}) \leq w_k \) with \( wt(n^{(x)}) = w_x \), and additionally also satisfies \( wt(n^{(u)}) < w_u \) if \( u < x \), satisfies \( n_{k,f_k} > 0 \) if \( u < k < x \), and satisfies \( n_{x,f_k} = 0 \) whenever \( x < t + 1 \).

Here, we define:

\[
w_1 = M; \quad w_k = n_{k-1,f_k-1} \quad 2 \leq k \leq u; \quad w_{u+1} = w_u - 1 - wt(n^{(u)}); \quad w_k = n_{k-1,f_k-1} - 1 \quad u + 2 \leq k \leq x;
\]

and:

\[
n_{1,0} = \infty; \quad n_{k,0} = w_{k-1} - wt(n^{(k-1)}) \quad 2 \leq k \leq u; \quad n_{u+1,0} = n_{u,f_u} + 1; \quad n_{k,0} = w_{k-1} + 1 - wt(n^{(k-1)}) \quad u + 2 \leq k \leq x.
\]
Proof: Only the \( i = k = 1 \) Gaussian in Theorem 4.4 involves \( N \) \((N = n_{1,0})\). On using \( M = n_{0,0} \), this term is

\[
\begin{bmatrix}
2M + N - n_{1,1} \\
N - n_{1,1}
\end{bmatrix} = \frac{(q)_{2M+N-n_{1,1}}}{(q)_{2M}(q)_{N-n_{1,1}}}.
\]

In the limit as \( N \to \infty \), this tends termwise to \( 1/(q)_{2M} \). The theorem then follows. \( \square \)

**Example 5.2** To illustrate the above theorem, we extend Example 4.5 to calculate \( \lim_{N \to \infty} R(N, 4, 2, 14, 5, 32) \). As in Example 4.5, the continued fraction of \((37:16)\) is \((2, 3, 5)\), the continued fraction of \((5:2)\) is \((2, 2)\), and \( t = 3, \quad u = 2, \quad d = 2, \quad f_1 = 2, \quad f_2 = 2, \quad f_3 = 1 \) and \( f_4 = 3 \). Now, although the largest part of \( n^{(1)} \) is unbounded, the weight of \( n^{(1)} \) is bounded by \( w_1 = 4 \) as before. Thus, compared with Example 4.5, only one more partition sequence is permitted: it is \((4, 0; 0, 0)\). The full set of valid partition sequences are then summarised in the following table.

| \((n^{(1)}; n^{(2)}; \ldots; n^{(x)})\) | \(x\) | \(n_{1,0}\) | \(w_1\) | \(w_2\) | \(n_{2,0}\) | \(w_2\) | \(n_{3,0}\) | \(w_3\) | \(n_{4,0}\) | \(w_4\) |
|-----------------------------|-----|--------|------|------|--------|------|--------|------|--------|------|
| \((4, 0; 0, 0)\)          | 2   | \(\infty\) | 4    | 0    | 0      |      |        |      |        |      |
| \((3, 0; 0, 0)\)          | 2   | \(\infty\) | 4    | 1    | 0      |      |        |      |        |      |
| \((2, 1; 1, 0)\)          | 2   | \(\infty\) | 4    | 1    | 1      |      |        |      |        |      |
| \((2, 0; 0, 0)\)          | 2   | \(\infty\) | 4    | 2    | 0      |      |        |      |        |      |
| \((1, 1; 1, 0)\)          | 2   | \(\infty\) | 4    | 2    | 1      |      |        |      |        |      |
| \((1, 0; 0, 0)\)          | 2   | \(\infty\) | 4    | 3    | 0      |      |        |      |        |      |
| \((0, 0; 0, 0)\)          | 2   | \(\infty\) | 4    | 4    | 0      |      |        |      |        |      |
| \((3, 1; 0, 0; 0)\)       | 3   | \(\infty\) | 4    | 0    | 1      | 1    | 0      |      |        |      |
| \((2, 1; 0, 0; 0)\)       | 3   | \(\infty\) | 4    | 1    | 1      | 1    | 0      |      |        |      |
| \((1, 1; 0, 0; 0)\)       | 3   | \(\infty\) | 4    | 2    | 1      | 1    | 0      |      |        |      |
| \((2, 2; 0, 0; 1; 0, 0, 0)\) | 4   | \(\infty\) | 4    | 0    | 2      | 1    | 1      | 1    | 1      | 0    |

The calculation of \( \lim_{N \to \infty} R(N, 4, 2, 14, 5, 32) \) using Theorem 5.1 is then similar in detail to that of Example 4.5. We obtain:

\[
\lim_{N \to \infty} R(N, 4, 2, 14, 5, 32) = \frac{1}{(q)_8} (q^{16} [4] [6] [8] + q^9 [5] [1] [8] + q^6 [5] [3] [1] + q^4 [5] [2] [8])
\]

\[
+ q^3 [6] [3] [1] + q^2 [4] [3] [8] + [5] [5] [6] [6] + q^{10} [4] [2] [3] [1]
\]

\[
+ q^5 [5] [3] [1] + q^2 [6] [4] [3] [1] + q^4 [4] [4] [5] [1] [8] + q^8 [4] [4] [4] [3] [1] [8] + q^{16} [4] [4] [3] [1] [8] + q^{15} + q^{16}/(q)_8.
\]

\[
= 1 + q + 2q^2 + 3q^3 + 5q^4 + 5q^5 + 7q^6 + 7q^7 + 8q^8 + \cdots + q^{15} + q^{16}/(q)_8
\]

\[
= 1 + 2q + 5q^2 + 10q^3 + 20q^4 + 34q^5 + 59q^6 + 94q^7 + 149q^8 + 224q^9 + \cdots
\]

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From the above Theorem 5.1 together with Theorem 1.1, one obtains the following conventional \( q \)-polynomial identity for a finite version of the character \( \chi_{p,p'}^{r,s} \), where the finite size parameter is \( N \).

**Corollary 5.3** Let \( \{ p, p', r, s \} \) be as in Theorem 5.1. Then

\[
\sum_{k=-\infty}^{\infty} q^{k^2p'p+(p'pr-ps)} \left[ \frac{2M}{M+p'k} \right] - q^{(kp+r)(kp'+s)} \left[ \frac{2M}{M+p'k+s} \right] = \sum \left( q^{n_{u,d}+\sum_{k=1}^{u} \sum_{i=1}^{f_k} n_{k,i}^2 +\sum_{k=u+1}^{x} \sum_{i=1}^{f_k} n_{k,i}(n_{k,i}-1) \prod_{k=1}^{x} \prod_{i=1+\delta_{1,k}}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right] \right),
\]

where the sum is over the same parameters as that in Theorem 5.1.

**Note 5.4** Notice the dependence of the Gaussian polynomials, on the bosonic side of the above identities, on the parameters \( \{ p, p', r, s \} \): only \( \{ p', s \} \) appear, but not \( \{ p, r \} \). This is what we refer to as a conventional dependence on the parameters. Also notice that the form of the fermionic side is the same as appears in e.g. [17].

### 6 Q-polynomial identities with a non-conventional bosonic side

In this section, we take the finite size parameter, that was left fixed in the previous section, to infinity, and retain the other fixed. We obtain another set of \( q \)-polynomial identities, that depend on a single finite size parameter, and that can be characterised by the fact that their bosonic side has a non-conventional dependence on the parameters \( \{ p, p', r, s \} \) that characterise the model.

**Theorem 6.1** Let \( p \) and \( p' \) be positive coprime integers and let \( (p' : p) \) have continued fraction \( (c_1, c_2, \ldots, c_{t-1}, c_t + 2) \) with \( t \geq 1 \). For \( 1 \leq u \leq t \) and \( 1 \leq d \leq c_u \), let \( r \) and \( s \) be such that \( (c_1, c_2, \ldots, c_{u-1}, d) \) is the continued fraction for \( (s : r) \). Then if \( u > 1 \),

\[
\lim_{M \to \infty} R(N, M, r, p - r, s, p' - s) = \sum \left( q^{n_{u,d}+\sum_{k=1}^{u} \sum_{i=1}^{f_k} n_{k,i}^2 +\sum_{k=u+1}^{x} \sum_{i=1}^{f_k} n_{k,i}(n_{k,i}-1) \prod_{k=1}^{x} \prod_{i=1+\delta_{1,k}}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right] \right),
\]

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Here, we define:

\[ n_1 \]

the only Gaussians involving \( n_1 \).

where, on setting \( k = 0 \),

\[
\lim_{M \to \infty} R(N, M, r, p - r, s, p' - s)
\]

\[
= \sum \left( \frac{q^{n_1,i+n_1,i} + \sum_{i=1}^{f_1} n_1,i + \sum_{s=2}^{r} \sum_{i=1}^{f_k} n_k,i(n_k,i-1)}{\prod_{i=1}^{f_1} (q)_{n_1,i-n_1,i}} \right) \times \prod_{k=2}^{f_k} \prod_{i=1+\delta_{2,k}}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_k,j \right) + n_k,i-1 + n_k,i \right]
\]

otherwise if \( u = 1 \),

\[
\lim_{M \to \infty} R(N, M, r, p - r, s, p' - s)
\]

\[
= \sum \left( \frac{q^{n_1,i+n_1,i} + \sum_{i=1}^{f_1} n_1,i + \sum_{s=2}^{r} \sum_{i=1}^{f_k} n_k,i(n_k,i-1)}{\prod_{i=1}^{f_1} (q)_{n_1,i-n_1,i}} \right) \times \prod_{k=2}^{f_k} \prod_{i=1+\delta_{2,k}}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_k,j \right) + n_k,i-1 + n_k,i \right]
\]

where, on setting \( f_1, f_2, \ldots, f_t, f_{t+1} \) = \( (c_1, c_2, \ldots, c_{u-1}, d, c_u - d, c_u+1, \ldots, c_t) \),

the sum is over all sequences \( n^{(1)}, n^{(2)}, \ldots, n^{(x)} \) of partitions with \( u \leq x \leq t+1 \)

for which, for \( 1 \leq k \leq x \), the partition \( n^{(k)} \) is in \( (f_k, n_k, 0) \), satisfies \( \text{wt}(n^{(k)}) \leq w_k \) with \( \text{wt}(n^{(x)}) = w_x \), and additionally also satisfies \( \text{wt}(n^{(u)}) < w_u \) if \( u < x \),

satisfies \( n_k,f_k > 0 \) if \( u < k < x \), and satisfies \( n_x,f_x = 0 \) whenever \( x < t + 1 \).

Here, we define:

\[ w_1 = \infty; \]

\[ w_k = n_{k-1,f_k-1} \quad 2 \leq k \leq u; \]

\[ w_{u+1} = w_u - 1 - \text{wt}(n^{(u)}); \]

\[ w_k = n_{k-1,f_k-1} - 1 \quad u + 2 \leq k \leq x; \]

and:

\[ n_{1,0} = N; \]

\[ n_{k,0} = w_{k-1} - \text{wt}(n^{(k-1)}) \quad 2 \leq k \leq u; \]

\[ n_{u+1,0} = n_{u,f_u} + 1; \]

\[ n_{k,0} = w_{k-1} + 1 - \text{wt}(n^{(k-1)}) \quad u + 2 \leq k \leq x. \]

**Proof:** Consider the expression given by Theorem 4.1. In the case \( u > 1 \), the only Gaussians involving \( M \) are those factors indexed by \( k = 1 \) and \( i \) for \( 1 \leq i \leq f_1 \) (since \( w_1 = M \)), and that indexed by \( k = 2 \) and \( i = 1 \) (since \( n_{2,0} = M + \sum_{i=1}^{f_1} n_{1,i} \)). All but the last of these terms is:

\[
\left( \frac{q}{2M+n_{1,i-1}+n_{1,i}+2\sum_{j=1}^{i} n_{1,i}} \right) = \frac{(q)_{2M+n_{1,i-1}+n_{1,i}+2\sum_{j=1}^{i} n_{1,i}}}{(q)_{n_{1,i-1}+n_{1,i}+2\sum_{j=1}^{i} n_{1,i}}}
\]

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In the limit as \( M \to \infty \), these tend termwise to \( 1/(q)_{n_{1,i-1}-n_{1,i}} \). The \( k = 2 \), \( i = 1 \) term is:

\[
\begin{aligned}
2w_2 + n_{2,0} - n_{2,1} &= \frac{(q)_{2w_2+n_{2,0}-n_{2,1}}}{(q)_{n_{2,0}-n_{2,1}}}(q)_{2w_2}.
\end{aligned}
\]

As \( M \to \infty \), then \( n_{2,0} \to \infty \) whereupon this term tends termwise to \( 1/(q)_{2w_2} = 1/(q)_{n_{1,f_1}} \).

In the case where \( u = 1 \), we have \( w_1 = M \), \( w_2 = M - \sum_{i=1}^{f_1} n_{1,i} \), \( n_{2,0} = n_{1,f_1} \), and \( n_{3,0} = M - 1 - \sum_{i=1}^{f_1} n_{1,i} - \sum_{i=1}^{f_2} n_{2,i} \). Thus as \( M \to \infty \), all the \( k = 1 \) and \( k = 2 \) terms behave similar to the \( k = 1 \) terms above, and the \( k = 3 \), \( i = 1 \) term behaves as did the \( k = 2 \), \( i = 1 \) term above. The theorem then follows. \( \square \)

**Example 6.2** To illustrate the above theorem, we extend Example 4.5 to calculate \( \lim_{M \to \infty} R(3, M, 2, 14, 5, 32) \). As in Example 4.4, the continued fraction of (37:16) is (2, 3, 5), the continued fraction of (5:2) is (2, 2), and \( t = 3 \), \( u = 2 \), \( d = 2 \), \( f_1 = 2 \), \( f_2 = 2 \), \( f_3 = 1 \) and \( f_4 = 3 \). Now the weight of \( n^{(1)} \) is unbounded. However, the largest part of \( n^{(1)} \) is bounded by \( n_{1,0} = 3 \), thus restricting the possible \( n^{(1)} \) to a finite number. In addition, \( n_{2,0} = \infty \) and \( w_2 \) is finite. Thus the weight of \( n^{(2)} \) is bounded, thereby restricting the possible \( n^{(2)} \) to a finite number. Thus overall, the permitted number of partition sequences is finite. In fact, as may be readily confirmed, there are 21 of them. They are tabulated in Appendix B. Thereupon, as shown in Appendix B, Theorem 6.1 yields:

\[
\begin{aligned}
\lim_{M \to \infty} R(3, M, 2, 14, 5, 32) &= 1 + 2q + 5q^2 + 10q^3 + 18q^4 + 30q^5 + 49q^6 + 74q^7 + 110q^8 + 158q^9 + \cdots \\
\end{aligned}
\]

\( \diamond \)

From the above Theorem 6.1 together with Theorem 1.1, one obtains the following \( q \)-polynomial identity for a finite version of the character \( \chi^{p,p'}_{1,s} \), where the finite size parameter is \( M \). In this case, the expression so obtained has both the bosonic and fermionic sides in a non-conventional form.

**Corollary 6.3** Let \( \{p, p', r, s\} \) be as in Theorem 6.1. If \( u > 1 \) (as defined in Theorem 6.4), then

\[
\begin{aligned}
\sum_{k=-\infty}^{\infty} q^{kp+p'+k(p'-ps)} &\left[ \frac{2N}{N + pk} - q^{(kp+r)(kp'+s)} \right] \left[ \frac{2N}{N + pk + r} \right] \\
&= (q)_{2N} \sum_{k=1}^{f_1} \frac{1}{(q)_{2n_{1,f_1}}} \prod_{i=1}^{f_1} (q)_{n_{1,i-1}-n_{1,i}} \\
&\quad \times \frac{1}{(q)_{2n_{1,f_1}}} \prod_{i=1}^{f_1} (q)_{n_{1,i-1}-n_{1,i}} \\
&= (q)_{2N} \sum_{k=1}^{f_1} \frac{1}{(q)_{2n_{1,f_1}}} \prod_{i=1}^{f_1} (q)_{n_{1,i-1}-n_{1,i}} \\
&\quad \times \frac{1}{(q)_{2n_{1,f_1}}} \prod_{i=1}^{f_1} (q)_{n_{1,i-1}-n_{1,i}} \\
\end{aligned}
\]
\[
\prod_{k=2}^{x} \prod_{i=1+\delta_{2,k}}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right].
\]

otherwise if \( u = 1, \)

\[
\sum_{k=-\infty}^{\infty} q^{k^2 p p' + k(p' r - p s)} \left[ \frac{2N}{N + pk} - q^{(kp+r)(kp'+s)} \left[ \frac{2N}{N + pk + r} \right] \right] = (q)_{2N} \sum \left( q^{n_{1,d} + \sum_{i=1}^{f_1} n_{1,i} + \sum_{k=2}^{x} \sum_{i=1}^{f_k} n_{k,i} (n_{k,i} - 1)} \right)
\]

\[
\times \prod_{k=3}^{x} \prod_{i=1+\delta_{3,k}}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{i} n_{k,j} \right) + n_{k,i-1} + n_{k,i} \right],
\]

where the sums are taken over the same parameters as in Theorem 6.1.

Note 6.4 Notice the dependence of the Gaussian polynomials, on the bosonic side of the above identities, on the parameters \( \{p, p', r, s\} \): only \( \{p, r\} \) appear, but not \( \{p', s\} \). This is what we refer to as a non-conventional dependence on the parameters. Also notice the form of the fermionic side: It is not the same form as appears in e.g. \([17]\). In fact, it is reminiscent of what results from the application of the Bailey transform, see e.g. \([16]\).

7 Q-series identities

In this section, we take both finite size parameters to infinity, and obtain \(q\)-series identities. Notice that, we obtain the same result, irrespectively of the order of removing the finite size parameters.

Theorem 7.1 Let \( p \) and \( p' \) be positive coprime integers and let \((p' : p)\) have continued fraction \((c_1, c_2, \ldots, c_t, c_t + 2)\) with \( t \geq 1 \). For \( 1 \leq u \leq t \) and \( 1 \leq d \leq c_u \), let \( r \) and \( s \) be such that \((c_1, c_2, \ldots, c_u-1, d)\) is the continued fraction for \((s : r)\). Then if \( u > 1, \)

\[
\lim_{\substack{M \to \infty \\ N \to \infty}} R(N, M, r, p-r, s, p'-s) = \frac{1}{(q)_{\infty}} \sum \left( q^{n_{u,d} + \sum_{k=1}^{u} \sum_{i=1}^{f_k} n_{k,i} + \sum_{k=u+1}^{x} \sum_{i=1}^{f_k} n_{k,i} (n_{k,i} - 1)} \right)
\]

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\[ \times \frac{1}{(q)^{2n_1,f_1}} \prod_{i=2}^{f_1} \frac{1}{(q)_{n_1,i-1-n_1,i}} \]
\[ \times \prod_{k=2}^{x} \prod_{i=1+\delta_{2,k}}^{f_k} \left[ 2 \left( \frac{w_k - \sum_{j=1}^{i} n_{k,j}}{n_{k,i-1} - n_{k,i}} \right) \right], \]

otherwise if \( u = 1 \),

\[ \lim_{M \to \infty} \lim_{N \to \infty} R(N, M, r, p + r, s, p' + s) \]
\[ = \frac{1}{(q)_{\infty}} \sum \left( q^{n_{1,d} + \sum_{i=1}^{f_1} n_{1,i} + \sum_{k=2}^{x} \sum_{i=1}^{f_k} n_{k,i}(n_{k,i} - 1)} \right) \]
\[ \times \frac{1}{(q)^{2n_2,f_2}} \prod_{i=2}^{f_1} \frac{1}{(q)_{n_1,i-1-n_1,i}} \prod_{i=1}^{f_2} \frac{1}{(q)_{n_2,i-1-n_2,i}} \]
\[ \times \prod_{k=3}^{x} \prod_{i=1+\delta_{3,k}}^{f_k} \left[ 2 \left( \frac{w_k - \sum_{j=1}^{i} n_{k,j}}{n_{k,i-1} - n_{k,i}} \right) \right], \]

where, on setting \((f_1, f_2, \ldots, f_{t+1}) = (c_1, c_2, \ldots, c_{u-1}, d, c_u - d, c_{u+1}, \ldots, c_t)\),

the sum is over all sequences \( n^{(1)}, n^{(2)}, \ldots, n^{(x)} \) of partitions with \( u \leq x \leq t + 1 \)

for which, for \( 1 \leq k \leq x \), the partition \( n^{(k)} \) is in \((f_k, n_{k,0})\), satisfies \( wt(n^{(k)}) \leq w_k \) with \( wt(n^{(x)}) = w_x \), and additionally also satisfies \( wt(n^{(u)}) < w_u \) if \( u < x \),

satisfies \( n_{k,f_k} > 0 \) if \( u < k < x \), and satisfies \( n_{x,f_x} = 0 \) whenever \( x < t + 1 \).

Here, we define:

\[
w_k = \frac{n_{k-1,f_{k-1}}}{n_{k,0}} = \frac{n_{k-1,f_{k-1}+1}}{n_{k,0}}, \quad 2 \leq k \leq u; \]
\[
w_{u+1} = w_u - 1 - wt(n^{(u)}); \]
\[
w_k = \frac{n_{k-1,f_{k-1}+1}}{n_{k,0}}, \quad u + 1 \leq k \leq x; \]

and:

\[
n_{1,0} = \infty; \]
\[
n_{k,0} = \frac{w_{k-1} - wt(n^{(k-1)})}{n_{k,0} - n_{u,f_u}}, \quad 2 \leq k \leq u; \]
\[
n_{u+1,0} = n_{u,f_u}; \]
\[
n_{k,0} = \frac{w_{k-1} - wt(n^{(k-1)})}{n_{k,0} - n_{u,f_u} + 1}. \]

Proof: In the expression given by Theorem 6.1 (there is no difficulty in taking either limit first), only the term \( 1/(q)_{n_1,0} - n_{1,1} = 1/(q)_{N-n_1,1} \) is affected on taking \( N \to \infty \). This gives rise to \( 1/(q)_{\infty} \), with all other terms being unaffected. \( \square \)
Example 7.2 To illustrate Theorem 7.1, we consider, as in Examples 4.3, 5.2 and 6.2, the case of \( p = 16, p' = 37, r = 2 \) and \( s = 5 \). As in those previous Examples, the continued fraction of \((37:16)\) is \((2,3,5)\), the continued fraction of \((5:2)\) is \((2,2)\), \( t = 3, u = 2, \), \( d = 2, f_1 = 2, f_2 = 2, f_3 = 1 \) and \( f_4 = 3 \).

Unfortunately, unlike the previous examples, the number of partition sequences over which the sum is taken is not finite, since both \( n_{1,0} = \infty \) and \( w_1 = \infty \). Thus \( n^{(1)} \) is any two-part partition \( n^{(1)} = (n_{1,1}, n_{1,2}) \). However, only a finite number of terms are required to guarantee accuracy to any particular order. (In view of the exponent of \( q \) in the expression in Theorem 7.1, if accuracy is required up to the term of order \( z \), then only those partitions for which \( \sum_{i=1}^{\infty} n_{1,i}^2 \leq z \) are required.) In the current case, accuracy up to the term of order 10 is guaranteed by taking those \( n_{1,1} \) and \( n_{1,2} \) lying in the plane, on or within the circle of radius \( \sqrt{10} \), and in the sector for which \( n_{1,1} \geq n_{1,2} \geq 0 \).

There are eight such partitions. They are:

\[(3,1), (3,0), (2,2), (2,1), (2,0), (1,1), (1,0), (0,0).\]

Then, since \( w_2 = n_{1,2} \), the number of possible \( n^{(2)} \) is bounded in each case. Altogether, there are thirteen possible partition sequences with \( n^{(1)} \) as one of the above. They are the sequences marked with an asterisk in the table in Appendix B. Using these in the expression in Theorem 7.1 yields:

\[
\lim_{N \to \infty} \frac{R(N, M, 2, 14, 5, 32)}{N^{2\rho_5}} = \frac{1}{(q)_{\infty}} \left( q^{11} \frac{1}{(q)_2} \frac{1}{(q)_2} \left[ 1 \right] + q^{9} \frac{1}{(q)_2} \frac{1}{(q)_2} \left[ 0 \right] + q^{12} \frac{1}{(q)_4} \frac{1}{(q)_4} \left[ 1 \right] + q^{6} \frac{1}{(q)_4} \frac{1}{(q)_4} \left[ 0 \right]ight.
\]

\[
+ q^{4} \frac{1}{(q)_4} \frac{1}{(q)_2} \left[ 0 \right] + q^{4} \frac{1}{(q)_2} \frac{1}{(q)_2} \left[ 1 \right] + q^{2} \frac{1}{(q)_2} \frac{1}{(q)_2} \left[ 0 \right]
\]

\[
+ q^{10} \frac{1}{(q)_2} \frac{1}{(q)_2} \left[ 0 \right] \left[ 1 \right] + q^{9} \frac{1}{(q)_4} \frac{1}{(q)_4} \left[ 0 \right] \left[ 1 \right] + q^{5} \frac{1}{(q)_2} \frac{1}{(q)_2} \left[ 0 \right] \left[ 1 \right] + q^{2} \frac{1}{(q)_2} \frac{1}{(q)_2} \left[ 0 \right] \left[ 0 \right]
\]

\[
= \frac{1}{(q)_{\infty}} \left( 1 + q + 2q^{2} + 3q^{3} + 5q^{4} + 7q^{5} \right. + 11q^{6} + 15q^{7} + 22q^{8} + 30q^{9} + 41q^{10} + \cdots \right)
\]

\[
= 1 + 2q + 5q^{2} + 10q^{3} + 20q^{4} + 36q^{5} + 65q^{6} + 110q^{7} + 185q^{8} + 300q^{9} + 480q^{10} + \cdots
\]

Note that this generating function differs from that of the unrestricted partition pairs, first at the \( q^{10} \) term. This reflects the fact that the smallest partition pair that does not satisfy the constraints of this example is \((5,5,0,0)\). \(\diamond\)

The above Theorem 7.1 together with (5) and (6) results in the following \( q \)-series identities for the Virasoro character \( \chi_{r,s}^{p,p'} \).
Corollary 7.3 Let \( \{p, p', r, s\} \) be as in Theorem 7.1. If \( u > 1 \) (as defined in Theorem 7.1), then

\[
\frac{1}{(q)^\infty} \left( \sum_{k=-\infty}^{\infty} q^{k^2pp'+k(p'r-ps)} - q^{(kp+r)(kp'+s)} \right)
\]

\[
= \sum \left( q^{n_{u,d} + \sum_{k=1}^{u} \sum_{i=1}^{f_k} n_{k,i}^2 + \sum_{k=u+1}^{x} \sum_{i=1}^{f_k} n_{k,i}(n_{k,i}-1)} \right)
\times \frac{1}{(q)_{2n_{1,f_1}}} \prod_{i=2}^{f_1} (q)_{n_{1,i}-n_{1,i}}
\times \prod_{k=2}^{x} \prod_{i=1+\delta_{2,k}}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{n_{k,j}} \right) + n_{k,i-1} + n_{k,i} \right],
\]

otherwise if \( u = 1 \),

\[
\frac{1}{(q)^\infty} \left( \sum_{k=-\infty}^{\infty} q^{k^2pp'+k(p'r-ps)} - q^{(kp+r)(kp'+s)} \right)
\]

\[
= \sum \left( q^{n_{1,d} + \sum_{k=1}^{1} \sum_{i=1}^{f_k} n_{k,i}^2 + \sum_{k=2}^{x} \sum_{i=1}^{f_k} n_{k,i}(n_{k,i}-1)} \right)
\times \frac{1}{(q)_{2n_{2,f_2}}} \prod_{i=2}^{f_1} (q)_{n_{1,i}-n_{1,i}} \prod_{i=1}^{f_2} (q)_{n_{2,i}-n_{2,i}}
\times \prod_{k=3}^{x} \prod_{i=1+\delta_{3,k}}^{f_k} \left[ 2 \left( w_k - \sum_{j=1}^{n_{k,j}} \right) + n_{k,i-1} + n_{k,i} \right],
\]

where the sums are taken over the same parameters as in Theorem 7.4.

8 Discussion

Q-polynomial identities for the Virasoro characters that we are interested in, were previously studied in [18, 17]. We wish to discuss the similarities and differences between this work and [18, 17].

In this work, we have used purely combinatorial methods: the Burge transform, plus a combinatorial identity. This enabled us to obtain two polynomial identities for each character \( \chi_{p,p',r,s} \) in the subset discussed above. Obtaining identities for more characters would require either an extension of the Burge transform, or further combinatorial identities.

In [18, 17], Bethe Ansatz type methods were used, and an identity for each character in various subsets of \( \chi_{p,p',r,s} \) was obtained. It is our understanding that
the methods of \[18, 17\] are sufficiently general to produce polynomial identities for all characters. However, this task is impeded by computational complexity.

Aside from the fact that our polynomial identities that depend on two finite size parameters have no counterparts in \[18, 17\], there are overlaps between our results and those of \[18, 17\] at the level of identities with one finite size parameter.

Our results of the latter type are of two forms: identities with a conventional bosonic side, and identities with a non-conventional bosonic side. The results of \[18, 17\] are all of the former type. In this case, our results form only a subset of those of \[18, 17\].

For example, it is easy to show that we obtain the same identities as in \[18, 17\] for \(\chi_{p,p}^{r_0,s_0}\), where \(|p_{s_0} - p_{r_0}| = 1\). On the other hand, all identities that we obtain and that have a non-conventional bosonic side are new.

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A Derivation

In this appendix, we derive Theorem 1.1. To do this, we classify each element of \(\mathcal{R}(s, u, t, v, a, b, \alpha, \beta)\) according to its sequence of faults as follows. First note that the conditions (3) and (4) for a partition pair \((q, p)\) in \((s, u) \times (t, v)\) to be restricted may be expressed:

\[
\begin{align*}
p_i - q_{i+1-a} & \geq 1 - \alpha & (a \leq i \leq t) \\
q_i - p_{i+1-b} & \geq 1 - \beta & (b \leq i \leq s),
\end{align*}
\]

where, when required, we take \(q_0 = p_0 = 0\), \(q_{s+1} = u\) and \(p_{t+1} = v\). For \((q, p) \in \mathcal{R}(s, u, t, v, a, b, \alpha, \beta)\), locate the largest \(i\) that violates these conditions. Note that if \(a > 0\) and \(b > 0\) then both conditions cannot be violated simultaneously because if the first is violated so that \(q_{i+1-a} > p_i + \alpha\) then for \(j \geq i + 1 - a\) and \(k \leq i\),

\[
q_j \geq q_{i+1-a} \geq p_i + \alpha \geq p_k + \alpha \geq p_k - \beta + 1
\]

since \(\alpha + \beta \geq 1\), thereby, in the particular case where \(j = i\) and \(k = i + 1 - b\), satisfying the second condition above. The violation at position \(i\) is said to be an \(a\)-fault if it is the first of the two conditions above that is violated. Otherwise it is said to be a \(b\)-fault. Now locate the next largest \(i\) for which a fault of the opposite variety occurs. (Note that if the first fault is an \(a\)-fault at position \(i_1\) then the second fault will be at position \(i_2\) satisfying \(i_2 \leq i_1 - a\), and will be a \(b\)-fault. On the other hand if the first fault is a \(b\) fault at position \(i_1\) then
the second will be an \( a \)-fault at position \( i_2 \leq i_1 - b \).) In this way, produce an alternating sequence of \( a \)s and \( b \)s for each element of \( \mathcal{R}(s, u, t, v, a, b, \alpha, \beta) \).

If either \( a = 0 \) or \( b = 0 \) then it is possible for an \( a \)-fault and a \( b \)-fault to occur at the same \( i \). In this case, both faults must be recorded. If \( a = 0 \), we consider the \( a \)-fault as preceding the \( b \)-fault, whereas if \( b = 0 \), we consider the \( b \)-fault as preceding the \( a \)-fault.

Define \( A_k(s, u, t, v, a, b, \alpha, \beta) \) to be the set that comprises all those elements of \( \mathcal{R}(s, u, t, v, a, b, \alpha, \beta) \) whose sequence of faults contains a subsequence of length \( k \) starting with an \( a \). Likewise, define the set \( B_k(s, u, t, v, a, b, \alpha, \beta) \) to comprise all those elements of \( \mathcal{R}(s, u, t, v, a, b, \alpha, \beta) \) whose sequence of faults contains a subsequence of length \( k \) starting with a \( b \). Note that for each \( k \), \( A_k \subset A_{k-1} \) and \( B_k \subset B_{k-1} \). The generating functions for these sets are defined by:

\[
A_k(s, u, t, v, a, b, \alpha, \beta) = \sum_{\pi \in A_k(s, u, t, v, a, b, \alpha, \beta)} q^{wt(\pi)},
\]

\[
B_k(s, u, t, v, a, b, \alpha, \beta) = \sum_{\pi \in B_k(s, u, t, v, a, b, \alpha, \beta)} q^{wt(\pi)}.
\]

**Lemma A.1**

\[
A_k(s, u, t, v, a, b, \alpha, \beta) = q^{\alpha(s-t+a)}B_{k-1}(t-a, v + \alpha, s + a, u - \alpha, a, b, \alpha, \beta),
\]

\[
B_k(s, u, t, v, a, b, \alpha, \beta) = q^{\beta(t-s+b)}A_{k-1}(t + b, v - \beta, s - b, u + \beta, a, b, \alpha, \beta).
\]

**Proof:** The first of these results is proved by setting up a bijection between the sets \( A_k(s, u, t, v, a, b, \alpha, \beta) \) and \( B_{k-1}(t - a, v + \alpha, s + a, u - \alpha, a, b, \alpha, \beta) \). For each \( (q, p) \in A_k(s, u, t, v, a, b, \alpha, \beta) \), the bijective image is obtained as follows. Let \( i \) be the position of the first \( a \)-fault of \( (q, p) \), so that \( p_i - q_{i-a+1} < 1 - \alpha \), whereupon \( p_i \leq q_{i-a} - 1 \). In addition, since this is the first \( a \)-fault, we have \( p_{i+1} - q_{i-a+2} \geq 1 - \alpha \) whereupon \( p_{i+1} + \alpha > q_{i-a+2} \geq q_{i-a} \). (Actually, care should be taken at the exceptional values. First note that if \( t-s = a \) then \( p_i \geq u - \alpha + 1 = p_{s+1} - \alpha + 1 \) implies that \( p_i - q_{i-a+1} \geq 1 - \alpha \) so that no \( a \)-fault then occurs at \( i = t \). Thus either \( i < t \) or \( t - s < a \). Thus if \( i = t \) then, from \( u-v \leq \alpha \) follows \( q_{i-a} = q_{t-a} \leq q_s \leq u \leq v + \alpha \). Also if \( i = t - 1 \) and \( t - s = a \) then \( q_{i-a} = q_{t-a-1} \leq q_s \leq u \leq p_i + \alpha - 1 < p_{i+1} + \alpha \). Otherwise \( i+1 \leq t \) and \( i - a + 2 \leq s \) and the previous reasoning holds. Also note that if \( a = 0 \) then \( q_1 \leq \alpha - 1 \) implies that \( p_{n} - q_{n-a+1} \geq 1 - \alpha \) so that no \( a \)-fault occurs at \( i = 0 \). Thus \( i \geq \max\{1, a\} \).) These two inequalities ensure that \( p' \) and \( q' \) defined by:

\[
p' = (q_s - \alpha, q_{s-1} - \alpha, \ldots, q_{i+1-a} - \alpha, p_i, p_{i-1}, \ldots, p_1), 
\]

\[
q' = (p_i + \alpha, p_{i-1} + \alpha, \ldots, p_{i+1} + \alpha, q_{i-a}, q_{i-a-1}, \ldots, q_1)
\]

are both partitions. Furthermore, the partition pair \((q', p')\) is in \((t-a, v+\alpha) \times (s+a, u-\alpha)\). Now the faults beyond the \( i \)th position in \((q', p')\) are precisely
as they are in \((q, p)\). This ensures that \((q', p') \in \mathcal{B}_{k-1}(t - a, v + \alpha, s + a, u - \alpha, a, b, \alpha, \beta)\).

The reverse map is defined as follows. Let \((q', p') \in \mathcal{B}_{k-1}(t-a, v+\alpha, s+a, u-\alpha, a, b, \alpha, \beta)\) and let \(i\) be the position of the first \(b\)-fault so that \(q'_{t-b+1} - p'_{t-b+1} < 1 - \beta\). Now locate \(j \geq i + a\) such that \(p'_{j+1} + \alpha \geq q'_{j-a}\) and \(p'_{h+1} + \alpha < q'_{h-a}\) for \(h > j\). If such a \(j\) exists then the second inequality implies that \(p'_{j} \leq p'_{j+2} < q'_{j-a+1} - \alpha\), thereby ensuring that \(p\) and \(q\) defined by

\[
\begin{align*}
p &= (q'_{t-a} - \alpha, q'_{t-a-1} - \alpha, \ldots, q'_{j-a+1} - \alpha, p'_{j}, p'_{j-1}, \ldots, p'_{1}), \\
q &= (p'_{i+a} + \alpha, p'_{i+a-1} + \alpha, \ldots, p'_{i+1} + \alpha, q'_{i-a}, q'_{i-a+1}, \ldots, q'_{1})
\end{align*}
\]

are each partitions. Furthermore \((q, p)\) is in \((s, u) \times (t, v)\). In addition to the faults at and to the right of the \(i\)th position, \((q, p)\) has an \(a\)-fault at position \(j \geq i + a\) since \(p_{j} - q_{j-a+1} = p'_{j} - p'_{j+1} - \alpha \leq -\alpha < 1 - \alpha\). Therefore, in the case where \(j\) can be found, the partition pair \((q, p)\) is an element of \(\mathcal{A}_{k}(s, u, t, v, a, b, \alpha, \beta)\). Also note that if \(k > j\) then \(p_{k} - q_{k-a+1} = q'_{k-a} - \alpha - (p' + \alpha) > q'_{k-a} - \alpha - q'_{k-a} \geq 1 - \alpha\), thus ensuring that the \(a\)-fault of \((q, p)\) at position \(j\) is the first, and therefore that the map described in this paragraph is the inverse of that given in the previous paragraph.

In the case where such a \(j\) cannot be located, so that \(p'_{j+1} + \alpha < q'_{j-a}\) for all \(j \geq i + a\), define

\[
\begin{align*}
p &= (p'_{t}, p'_{t-1}, p'_{t-2}, \ldots, p'_{1}), \\
q &= (p'_{i+a} + \alpha, p'_{i+a-1} + \alpha, \ldots, p'_{i+1} + \alpha, q'_{i-a}, q'_{i-a+1}, \ldots, q'_{1})
\end{align*}
\]

whence an \(a\)-fault has been introduced at position \(t\) since \(p_{t} - q_{t-a+1} = p'_{t} - p'_{t+1} - \alpha \leq -\alpha < 1 - \alpha\). The faults of \((q', p')\) also occur in \((q, p)\) to the right of position \(s-a\). Thus also in this case \((q, p) \in \mathcal{A}_{k}(s, u, t, v, a, b, \alpha, \beta)\). Certainly, the introduced \(a\)-fault is the first fault so that in this case, this map is the inverse of that above.

Thus the two sets \(\mathcal{A}_{k}(s, u, t, v, a, b, \alpha, \beta)\) and \(\mathcal{B}_{k-1}(t-a, v+\alpha, s+a, u-\alpha, a, b, \alpha, \beta)\) are in bijection. The first expression of the lemma then follows after noting that if \((q', p')\) is the bijective image of \((q, p)\) then \(\text{wt}((q, p)) = \text{wt}((q', p')) + (s-t+a)\alpha\).

The second expression is proved in a totally analogous manner.

\[\Box\]

**Lemma A.2**

\[
\begin{align*}
A_{2k}(s, u, t, v, a, b, \alpha, \beta) &= q^{k^2(a+b)(\alpha+\beta)+k(\alpha+\beta)(s-t)+k(\alpha+\beta)\alpha} g(s, u, t, v, k(a+b), k(\alpha+\beta)) \\
B_{2k}(s, u, t, v, a, b, \alpha, \beta) &= q^{k^2(a+b)(\alpha+\beta)-k(\alpha+\beta)(s-t)-k(\alpha+\beta)\alpha} g(t, v, s, u, k(a+b), k(\alpha+\beta))
\end{align*}
\]

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A_{2k+1}(s, t, v, a, b, \alpha, \beta)
= q^{k^2(a+b)(\alpha+\beta)+(k(\alpha+\beta)+\alpha)(s-t)+k(a\beta+ba+2a\alpha)+a\alpha}
\times g(s, u, t, v, k(a+b)+a, k(\alpha+\beta)+\alpha)
B_{2k+1}(s, t, v, a, b, \alpha, \beta)
= q^{k^2(a+b)(\alpha+\beta)-(k(\alpha+\beta)+\beta)(s-t)+k(a\beta+ba+2b\beta)+b\beta}
\times g(t, v, s, u, k(a+b)+b, k(\alpha+\beta)+\beta).

Proof: We proceed by induction. The first two results clearly hold when \( k = 0 \) since the required generating function is that for all partitions, regardless of their faults. Therefore

\[ A_0(s, t, v, a, b, \alpha, \beta) = B_0(s, t, v, a, b, \alpha, \beta) \]
\[ = \left[ \frac{s + u}{s} \right] \left[ \frac{t + v}{t} \right] = g(s, u, t, v, 0, 0) = g(t, v, s, u, 0, 0). \]

Now assume that for a given \( i \), the expressions hold for \( A_i(s, t, v, a, b, \alpha, \beta) \) and \( B_i(s, t, v, a, b, \alpha, \beta) \). We show that this implies that the expressions for \( A_{i+1}(s, t, v, a, b, \alpha, \beta) \) and \( B_{i+1}(s, t, v, a, b, \alpha, \beta) \) hold. In the case where \( i \) is even, let \( i = 2k \) whereupon, on using Lemma [A.1], we obtain:

\[ A_{i+1}(s, t, v, a, b, \alpha, \beta) \]
\[ = q^{a(s-t+a)}B_i(t - a, v + \alpha, s + a, u - \alpha, a, b, \alpha, \beta) \]
\[ = q^{a(s-t+a)+k^2(a+b)(\alpha+\beta)-(k(\alpha+\beta)+\alpha)(t-s-2a)-k(a\beta-\alpha)} \]
\[ \times g(s + a, u - \alpha, t - a, v + \alpha, k(a + b), k(\alpha + \beta)) \]
\[ = q^{a\alpha+k^2(a+b)(\alpha+\beta)+(k(\alpha+\beta)+\alpha)(s-t)+k(a\beta+ba+2a\alpha)} \]
\[ \times g(s, u, t, v, k(a+b)+a, k(\alpha+\beta)+\alpha), \]
where use has been made of

\[ g(s + a, u - \alpha, t - a, v + \alpha, x, y) \]
\[ = \left[ \frac{s + a + u - \alpha + x - y}{s + a + x} \right] \left[ \frac{t - a + v + \alpha - x + y}{t - a - x} \right] \]
\[ = g(s, u, t, v, x + a, y + \alpha). \]

Similarly,

\[ B_{i+1}(s, t, v, a, b, \alpha, \beta) \]
\[ = q^\beta(t-s+b)A_i(t + b, v - \beta, s - b, u + \beta, a, b, \alpha, \beta) \]
\[ = q^{\beta(t-s+b)+k^2(a+b)(\alpha+\beta)+(k(\alpha+\beta)+\alpha)(t-s+2b)+k(a\beta-b\alpha)}. \]
Theorem A.3

generating function

In the case where \( i \) is odd, let \( i = 2k + 1 \) whereupon, on using Lemma A.1.

\[
\begin{align*}
A_{i+1}(s, u, t, v, a, b, \alpha, \beta) &= q^{(s-t+a)}B_i(t - a, v + \alpha, s + a, u - a, a, b, \alpha, \beta) \\
&= q^{(s-t+a)+k^2(a+b)(\alpha+\beta)-(k+1)(\alpha+\beta)}(s-t)+k(a\beta+ba+2\alpha) + b\beta \\
&\quad \times g(s, u, t, v, (k+1)(a+b), (k+1)(\alpha+\beta)) \\
&= q^{(k+1)^2(a+b)(\alpha+\beta)-(k+1)(\alpha+\beta)}(s-t)+k(a\beta+ba+2\alpha) + 2a\beta + a\alpha + b\beta \\
&\quad \times g(t, v, s, u, (k+1)(a+b), (k+1)(\alpha+\beta)).
\end{align*}
\]

Similarly,

\[
\begin{align*}
B_{i+1}(s, u, t, v, a, b, \alpha, \beta) &= q^{(t-s+b)}A_i(t + b, v - \beta, s - b, u + \beta, a, b, \alpha, \beta) \\
&= q^{(t-s+b)+k^2(a+b)(\alpha+\beta)+(k+1)(\alpha+\beta)}(s-t)+k(a\beta+ba+2\alpha) + a\alpha \\
&\quad \times g(t + b, v - \beta, s - b, u + \beta, a + b, a, k(a + \beta) + \alpha) \\
&= q^{k^2(a+b)(\alpha+\beta)-(k+1)(\alpha+\beta)(s-t)+2k(b(a+\beta)+k(a\beta+ba+2\alpha)+2b\alpha+a\alpha+b\beta) \\
&\quad \times g(t, u, v, s, (k+1)(a+b), (k+1)(\alpha+\beta)) \\
&= q^{(k+1)^2(a+b)(\alpha+\beta)-(k+1)(\alpha+\beta)}(s-t)-k(a\beta+ba+2\alpha) + 2a\beta + a\alpha + b\beta \\
&\quad \times g(t, v, s, u, (k+1)(a+b), (k+1)(\alpha+\beta)).
\end{align*}
\]

Thus for both even and odd \( i \), the expressions for \( A_{i+1} \) and \( B_{i+1} \) follow from those for \( A_i \) and \( B_i \). The lemma then follows by induction. \( \square \)

We are now in a position to give a non-constant sign expression for the generating function \( R(s, u, t, v, a, b, \alpha, \beta) \).

Theorem A.3

\[
\begin{align*}
R(s, u, t, v, a, b, \alpha, \beta) &= \sum_{k=-\infty}^{\infty} q^{k^2(a+b)(\alpha+\beta)+k(a+\beta)(s-t)+k(a\beta+ba)} \\
&\quad \times g(s, u, t, v, k(a+b), k(a+\beta)) \\
&\quad - \sum_{k=-\infty}^{\infty} q^{k^2(a+b)(\alpha+\beta)+(k+1)(\alpha+\beta)(s-t)+k(a\beta+ba+2\alpha)} + a\alpha \\
&\quad \times g(s, u, t, v, k(a+b) + a, k(a+\beta) + \alpha).
\end{align*}
\]
Proof: We require the generating function for partition pairs in \((s, u) \times (t, v)\) that have no faults. In this proof for typographical reasons, we drop the arguments \((s, u, t, v, a, b, \alpha, \beta)\) from \(R, A_i, B_i, R, A_i\) and \(B_i\). Note first that \(A_{2k} \subset A_{2k-1}\). Moreover, the set \(A_{2k-1} \setminus A_{2k}\) contains all partition pairs that have a sequence of faults \(abab \cdots ba\) of length \(2k - 1\) or a sequence of faults \(baba \cdots ba\) of length \(2k\). Likewise, the set \(B_{2k-1} \setminus B_{2k}\) contains all partition pairs that have a sequence of faults \(baba \cdots ab\) of length \(2k - 1\) or a sequence of faults \(abab \cdots ab\) of length \(2k\).

Therefore, \((A_{2k-1} \setminus A_{2k}) \cup (B_{2k-1} \setminus B_{2k})\) contains all partition pairs with a sequence of faults of length \(2k\) or \(2k - 1\). The generating function for such sequences is therefore \(A_{2k-1} - A_{2k} + B_{2k-1} - B_{2k}\). Thereupon, the generating function for sequences with no faults is:

\[
R = A_0 - \sum_{k=1}^{\infty} (A_{2k-1} - A_{2k} + B_{2k-1} - B_{2k}).
\]

The theorem then follows from the four expressions given by Lemma A.2 after noting that

\[
g(t, v, s, u, x, y) = \left[ \frac{t + v + x - y}{t + x} \right] \left[ \frac{s + u - x + y}{s - x} \right] = g(s, u, t, v, -x, -y),
\]

and that the fourth expression of Lemma A.2 may be re-expressed

\[
B_{2k-1}(s, u, t, v, a, b, \alpha, \beta) = q^{k^2(a+b)(\alpha+\beta)-(k(\alpha+\beta)-\alpha)(s-t)-k(a\beta+ba+2a\alpha)+a\alpha} \times g(s, u, t, v, -k(a+b)+a, -k(\alpha+\beta)+\alpha).
\]

\[\Box\]

B Details of Examples

Example B.1 Here, we provide the details for Example 6.2. The following table lists the partition sequences used to calculate \(\lim_{M \to \infty} R(3, M, 2, 14, 5, 32)\) by means of Theorem 6.1.
\[
\binom{n(1); n(2); \ldots; n(x)}{x \ n_{1,0} \ n_{2,0} \ n_{3,0} \ n_{4,0} \ n_{4,1}}
\]

| (3, 3, 3, 0) | 2 \ 3 \ \infty \ \infty \ 3 |
| (3, 3, 2, 0) | 2 \ 3 \ \infty \ \infty \ 2 |
| (3, 3, 1, 0) | 2 \ 3 \ \infty \ \infty \ 1 |
| (3, 3, 0, 0) | 2 \ 3 \ \infty \ \infty \ 0 |
| (2, 2, 2, 0) | 2 \ 3 \ \infty \ \infty \ 2 |
| (2, 2, 1, 0) | 2 \ 3 \ \infty \ \infty \ 1 |
| (2, 0, 0, 0) | 2 \ 3 \ \infty \ \infty \ 0 |
| (1, 1, 1, 0) | 2 \ 3 \ \infty \ \infty \ 1 |
| (1, 0, 0, 0) | 2 \ 3 \ \infty \ \infty \ 0 |
| (0, 0, 0, 0) | 2 \ 3 \ \infty \ \infty \ 0 |
| (3, 3, 2, 0, 0) | 3 \ 3 \ \infty \ \infty \ 3 \ 1 \ 0 |
| (3, 3, 1, 1, 0) | 3 \ 3 \ \infty \ \infty \ 3 \ 2 \ 0 |
| (3, 2, 1, 0, 0) | 3 \ 3 \ \infty \ \infty \ 2 \ 1 \ 0 |
| (3, 1, 0, 0, 0) | 3 \ 3 \ \infty \ \infty \ 1 \ 1 \ 0 |
| (2, 2, 1, 0, 0) | 3 \ 3 \ \infty \ \infty \ 2 \ 1 \ 0 |
| (2, 1, 0, 0, 0) | 3 \ 3 \ \infty \ \infty \ 1 \ 1 \ 0 |
| (1, 1, 0, 0, 0) | 3 \ 3 \ \infty \ \infty \ 1 \ 1 \ 0 |
| (3, 3, 1, 0; 1; 0, 0, 0) | 4 \ 3 \ \infty \ \infty \ 3 \ 1 \ 1 \ 1 \ 0 |
| (3, 3, 0, 0; 1; 0, 0, 0) | 4 \ 3 \ \infty \ \infty \ 3 \ 1 \ 2 \ 2 \ 0 |
| (3, 2, 0, 0; 1; 0, 0, 0) | 4 \ 3 \ \infty \ \infty \ 2 \ 1 \ 1 \ 1 \ 0 |
| (2, 2, 0, 0; 1; 0, 0, 0) | 4 \ 3 \ \infty \ \infty \ 2 \ 1 \ 1 \ 1 \ 0 |

(Those sequences marked with an asterisk are also relevant to Example 7.3)

The use of Theorem 7.3 then gives:

\[
\lim_{M \to \infty} R(3, M, 2, 14, 5, 32)
= q^{27} \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{15}} \frac{3}{[3]} + q^{17} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{2}{[2]} + q^{11} \frac{1}{q^{17}} \frac{1}{q^{19}} \frac{1}{q^{21}} \frac{1}{[1]}
+ q^9 \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{[2]} + q^6 \frac{1}{q^{15}} \frac{1}{q^{17}} \frac{1}{q^{19}} \frac{1}{q^{21}} \frac{1}{[1]}
+ q^4 \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{[0]} + q^3 \frac{1}{q^{13}} \frac{1}{q^{15}} \frac{1}{q^{17}} \frac{1}{q^{19}} \frac{1}{[1]} + q^2 \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{20}} \frac{1}{[0]}
+ \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{[2]} + q^{22} \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{[1]} + q^{20} \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{20}} \frac{1}{[2]}
+ q^{14} \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{20}} \frac{1}{[1]} + q^{10} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{20}} \frac{1}{[2]}
+ q^9 \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{3}{[3] + q^5 \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{21}} \frac{2}{[1]} + q^2 \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{3}{[1]}
+ q^9 \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{3}{[3] + q^5 \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{21}} \frac{2}{[1]}
+ q^{18} \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{21}} \frac{3}{[1]} + q^{18} \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{21}} \frac{3}{[1]}
+ q^{13} \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{21}} \frac{1}{q^{23}} \frac{3}{[1]} + q^{13} \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{21}} \frac{1}{q^{23}} \frac{3}{[1]}
+ q^8 \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{21}} \frac{1}{q^{23}} \frac{1}{q^{25}} \frac{3}{[1]}
+ q^8 \frac{1}{q^{10}} \frac{1}{q^{12}} \frac{1}{q^{14}} \frac{1}{q^{16}} \frac{1}{q^{18}} \frac{1}{q^{21}} \frac{1}{q^{23}} \frac{1}{q^{25}} \frac{3}{[1]}
\]

\[= 1 + 2q + 5q^2 + 10q^3 + 18q^4 + 30q^5 + 49q^6 + 74q^7 + 110q^8 + 158q^9 + \cdots\]

\[\diamondsuit\]
References

[1] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. B 241 (1984) 333.

[2] Ph. Di Francesco, P. Mathieu and D. Senechal, Conformal Field Theory, Springer, 1996.

[3] G.E. Andrews, R.J. Baxter and P.J. Forrester, Eight-vertex SOS model and generalized Rogers-Ramanujan-type identities, J. Stat. Phys. 35 (1984), 193.

[4] P.J. Forrester and R.J. Baxter, Further exact solutions of the eight-vertex SOS model and generalizations of the Rogers-Ramanujan identities, J. Stat. Phys. 38 (1985) 435.

[5] R. J. Baxter, Exactly solvable models in statistical mechanics, Academic Press, London, 1982.

[6] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Exactly solvable SOS models: Local height probabilities and theta function identities, Nucl. Phys. B 290 [FS20] (1987) 231.

[7] R. Kedem, B. M. McCoy and E. Melzer, The sums of Rogers, Schur and Ramanujan and the Bose-Fermi correspondence in 1+1 dimensional quantum field theory, hep-th 9304056.

[8] J. Lepowsky and M. Primc, Structure of the standard modules of the affine Lie algebra \( A_{1}^{(1)} \), Contemporary Mathematics, 46 (AMS, Providence, 1985).

[9] A. Rocha-Caridi, Vacuum vector representations of the Virasoro algebra, in Vertex Operators in Mathematics and Physics, eds. J. Lepowsky, S. Mandelstam and I.M. Singer, Springer, 1985.

[10] G.E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and its Applications, 2, Addison-Wesley, 1976.

[11] G.E. Andrews, R.J. Baxter, D.M. Bressoud, W.H. Burge, P.J. Forrester, and G.X. Viennot, Partitions with prescribed hook differences, European J. of Combin. 8 (1987) 4310.

[12] W. H. Burge, Restricted partition pairs, J. of Comb. Th. A 63 (1993) 210.

[13] W. N. Bailey, Proc. London Math. Soc. 50 (1949) 1.
[14] G. E. Andrews, *Multiple series Rogers-Ramanujan type identities*, Pacific J. Math. 114 (1984) 267.

[15] G. E. Andrews, *q-Series: Their development and application in analysis, number theory, combinatorics, physics, and computer algebra*, in CBMS Regional Conf. Ser. in Math., 66, (AMS, Providence, Rhode Island, 1985).

[16] O. Foda and Y.-H. Quano, *Virasoro character identities from the Andrews–Bailey construction*, Int. J. Mod. Phys. A 12 (1997) 1651.

[17] A. Berkovich, B. M. McCoy and A. Schilling, *Rogers–Schur–Ramanujan type identities for the M(p,p') minimal models of conformal field theory*, preprint [q-alg/9607020], to appear in Commun. Math. Phys.

[18] A. Berkovich and B. M. McCoy, *Continued fractions and fermionic representations for characters of M(p,p') minimal models*, Lett. Math. Phys. 37 (1996) 49.

[19] O. Foda and S.O. Warnaar, *A bijection which implies Melzer’s polynomial identities: the \( \chi_{1,1}^{(p,p+1)} \) case*, Lett. Math. Phys. 36 (1996) 145.

[20] S.O. Warnaar, *Fermionic solution of the Andrews-Baxter-Forrester model I. Unification of TBA and CTM methods*, J. Stat. Phys. 82 (1996) 657.

[21] S.O. Warnaar, *Fermionic solution of the Andrews-Baxter-Forrester model. II. Proof of Melzer’s polynomial identities*, J. Stat. Phys. 84 (1997) 49.

[22] O. Foda, M. Okado and S.O. Warnaar, *A proof of polynomial identities of type \( \hat{sl}(n)_1 \otimes \hat{sl}(n)/\hat{sl}(n)_2 \)*, J. Math. Phys. 37 (1996) 965.

[23] S. Kerov, A.N. Kirillovœ and N. Reshetikhin, *J. Sov. Math.* 41 (1988) 916.

[24] S. Dasmahapatra and O. Foda, *Strings, paths and standard tableaux*, [q-alg 9601011], to appear in Int. J. of Mod. Phys. (1997).

[25] Ch. Krattenthaler and I. Gessel, *Cylindric partitions*, Trans. Amer. Math. Soc. 349 (1997), 429-479.

[26] H.W. Gould, *A new symmetrical combinatorial identity*, J. Comb. Theory (A) 13 (1972) 278.