EXAMPLES OF DEGENERATIONS
OF COHEN-MACAULAY MODULES

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Abstract. We study the degeneration problem for maximal Cohen-Macaulay modules and give several examples of such degenerations. It is proved that such degenerations over an even-dimensional simple hypersurface singularity of type \((A_n)\) are given by extensions. We also prove that all extended degenerations of maximal Cohen-Macaulay modules over a Cohen-Macaulay complete local algebra of finite representation type are obtained by iteration of extended degenerations of Auslander-Reiten sequences.

1. Introduction

The degeneration problem of modules has been studied by many authors [2, 5, 7, 8, 12]. For modules over an Artinian algebra, it has been studied by Bongartz [2] in relation with the Auslander-Reiten quiver. In general, but for modules over a commutative Noetherian ring, the second author [7] generalized the theory of Bongartz to maximal Cohen-Macaulay modules and has shown that any extended degenerations of maximal Cohen-Macaulay modules are fundamentally obtained by degenerations of Auslander-Reiten sequences under certain special conditions. For this, several order relations for modules, such as the hom order, the degeneration order, the extension order and the AR order, were introduced, and the connection among them has been studied.

The purpose of this paper is to give several examples of degenerations of maximal Cohen-Macaulay modules and to show how we can describe them. We will be able to give the complete description of degenerations over a ring of even-dimensional simple hypersurface singularity of type \((A_n)\). In fact, we will show that all degenerations of maximal Cohen-Macaulay modules over such a ring are given by extensions (Theorem 3.1). This result depends heavily on the recent work of the second author about the stable analogue of degenerations for Cohen-Macaulay modules over a Gorenstein local algebra [10].

In Section 4 we also investigate the relations among the extended versions of the degeneration order, the extension order and the AR order. As a result we shall show that if \(R\) is of finite Cohen-Macaulay representation type, then all these...
extended orders are identical when restricted on the maximal Cohen-Macaulay modules (Theorem 4.4).

2. Preliminaries and the first examples

In this section, we recall the definition of degeneration and state several known results on degenerations. For the details, we recommend the reader to refer to [8, 10].

Definition 2.1. Let $R$ be a Noetherian algebra over a field $k$, and let $M$ and $N$ be finitely generated left $R$-modules. We say that $M$ degenerates to $N$, or $N$ is a degeneration of $M$, if there is a discrete valuation ring $(V, tV, k)$ that is a $k$-algebra (where $t$ is a prime element) and a finitely generated left $R \otimes_k V$-module $Q$ which satisfies the following conditions:

1. $Q$ is flat as a $V$-module.
2. $Q/tQ \cong N$ as a left $R$-module.
3. $Q[1/t] \cong M \otimes_k V[1/t]$ as a left $R \otimes_k V[1/t]$-module.

The following characterization of degenerations has been proved by the second author [8]. See also [5, 13].

Theorem 2.2 ([8, Theorem 2.2]). The following conditions are equivalent for finitely generated left $R$-modules $M$ and $N$.

1. $M$ degenerates to $N$.
2. There is a short exact sequence of finitely generated left $R$-modules

$$0 \longrightarrow Z \xrightarrow{(\psi)} M \oplus Z \longrightarrow N \longrightarrow 0,$$

such that the endomorphism $\psi$ of $Z$ is nilpotent, i.e. $\psi^n = 0$ for $n \gg 1$.

Remark 2.3. Let $R$ be a Noetherian $k$-algebra.

1. Suppose that a finitely generated $R$-module $M$ degenerates to a finitely generated module $N$. Then as a discrete valuation ring $V$ in Definition 2.1 we can always take the ring $k[t]/(t)$. See [8, Corollary 2.4]. (This is actually a corollary of Theorem 2.2) Thus in the rest of the paper we always take $k[t]/(t)$ as $V$.
2. Assume that there is an exact sequence of finitely generated left $R$-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$ 

Then $M$ degenerates to $L \oplus N$. See [8, Remark 2.5] for the details.
3. Let $M$ and $N$ be finitely generated left $R$-modules and suppose that $M$ degenerates to $N$. Then the modules $M$ and $N$ give the same class in the Grothendieck group, i.e. $[M] = [N]$ as an element of $K_0(mod(R))$, where $mod(R)$ denotes the category of finitely generated $R$-modules and $R$-homomorphisms.

Definition 2.4. Let $R$ be a Noetherian algebra over a field $k$, and let $M$ and $N$ be finitely generated left $R$-modules.

1. We write $M \leq_{\text{deg}} N$ to indicate that $N$ is obtained from $M$ by iterative degenerations; i.e. there is a sequence of finitely generated left $R$-modules $L_0, L_1, \ldots, L_r$ such that $M \cong L_0, N \cong L_r$ and each $L_i$ degenerates to $L_{i+1}$ for $0 \leq i < r$. 

(2) We say that \( M \) degenerates by an extension to \( N \) if there is a short exact sequence \( 0 \to U \to M \to V \to 0 \) of finitely generated left \( R \)-modules such that \( N \cong U \oplus N \).

We write \( M \leq_{ext} N \) to indicate that \( N \) is obtained from \( M \) by iterative degenerations by extensions; i.e. there is a sequence of finitely generated left \( R \)-modules \( L_0, L_1, \ldots, L_r \) such that \( M \cong L_0, N \cong L_r \) and each \( L_i \) degenerates by an extension to \( L_{i+1} \) for \( 0 \leq i < r \).

If \( R \) is a commutative Noetherian local ring, then \( \leq_{deg} \) and \( \leq_{ext} \) are known to be partial orders on the set of isomorphism classes of finitely generated \( R \)-modules, which are called the degeneration order and the extension order respectively. See [7] for the details.

**Remark 2.5.** By virtue of Remark 2.3(2), if \( M \leq_{ext} N \), then \( M \leq_{deg} N \). However the converse is not necessarily true.

For example, consider a ring \( R = k[[x, y]]/(x^2) \). A pair of matrices over \( k[[x, y]] \),

\[
(\varphi, \psi) = \left( \begin{pmatrix} x & y^2 \\ 0 & x \end{pmatrix}, \begin{pmatrix} x & -y^2 \\ 0 & x \end{pmatrix} \right)
\]

is a matrix factorization of the equation \( x^2 \); hence it gives a maximal Cohen-Macaulay \( R \)-module \( N \) that is isomorphic to the ideal \((x, y^2)R\). Actually there is an exact sequence

\[
\cdots \longrightarrow R^2 \xrightarrow{\varphi} R^2 \xrightarrow{\psi} R^2 \xrightarrow{\varphi} R^2 \longrightarrow N \longrightarrow 0.
\]

It is known that \( N \) is indecomposable. See [6, Example (6.5)]. Now we deform the matrices \((\varphi, \psi)\) to

\[
(\Phi, \Psi) = \left( \begin{pmatrix} x + ty & y^2 \\ -t^2 & x - ty \end{pmatrix}, \begin{pmatrix} x - ty & -y^2 \\ t^2 & x + ty \end{pmatrix} \right)
\]

over \( R \otimes_k V \), where \( V = k[t]/(t) \). Then, since \((\Phi, \Psi)\) is still a matrix factorization of \( x^2 \) over the regular ring \( k[[x, y]] \otimes_k V \), we have an exact sequence

\[
\cdots \longrightarrow (R \otimes_k V)^2 \xrightarrow{\Phi} (R \otimes_k V)^2 \xrightarrow{\Psi} (R \otimes_k V)^2 \longrightarrow Q \longrightarrow 0,
\]

where \( Q \) is a maximal Cohen-Macaulay module over \( R \otimes_k V \). In particular \( Q \) is \( V \)-flat. Since \( \Phi \otimes_V V/tV = \varphi \), it follows that \( Q/tQ \cong N \). On the other hand, since \( t^2 \) is a unit in \( R \otimes_k V[1/t] \), after an elementary transformation of matrices, we have \( \Phi \otimes_V V[1/t] \cong \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \); hence \( Q_t \cong R \otimes_k V[1/t] \). As a result we see that \( R \) degenerates to \((x, y^2)R \) in this case, and hence \( R \leq_{deg} (x, y^2)R \).

In general if \( M \leq_{ext} N \) and if \( M \not\cong N \), then \( N \) is a nontrivial direct sum of modules. Since \( N \cong (x, y^2)R \) is indecomposable, we see that \( R \leq_{ext} (x, y^2)R \) can never happen.

We also note that if \( R \) is an Artinian \( k \)-algebra, then the degeneration for finitely generated left modules is known to be transitive by Zwara [11]. Namely, if \( L \) degenerates to \( M \) and if \( M \) degenerates to \( N \), then \( L \) degenerates to \( N \). In general if \( R \) is not necessarily Artinian, we do not know whether this transitivity property holds or not. However, it is rather easy to see the following:

**Remark 2.6.** If \( L \) degenerates to \( M \) and if \( M \) degenerates by an extension to \( N \), then \( L \) degenerates to \( N \).

Now we note that the following lemma holds.
Lemma 2.7. Let $I$ be a two-sided ideal of a Noetherian $k$-algebra $R$, and let $M$ and $N$ be finitely generated left $R/I$-modules. Then $M$ degenerates (resp. degenerates by an extension) to $N$ as a left $R$-module if and only if it does so as a left $R/I$-module.

Proof. Assume $M$ degenerates to $N$ as a left $R/I$-module. Then there is a finitely generated left $R/I \otimes_k V$-module $Q$ satisfying the conditions in Definition 2.1. Regarding $Q$ as a left $R \otimes_k V$-module, we can see that $M$ degenerates to $N$ as a left $R$-module.

Contrarily, assume $M$ degenerates to $N$ as a left $R$-module. Then the $R \otimes_k V$-module $Q$ satisfying the conditions in Definition 2.1 is a left $R/I \otimes_k V$-module. In fact, since $IM = 0$ and since $Q[1/t] \cong M \otimes_k V[1/t]$, we have $IQ[1/t] = 0$. On the other hand, since $Q$ is $V$-flat, the natural mapping $Q \to Q[1/t]$ is injective. Therefore we see that $IQ = 0$, which shows that $Q$ is a module over $R/I \otimes_k V$. Then $Q$ satisfies all the conditions in Definition 2.1 as a left $R/I$-module. Hence $M$ degenerates to $N$ as a left $R/I$-module.

By iterative use of this lemma we have the following corollary.

Corollary 2.8. As in the lemma, let $I$ be a two-sided ideal of a Noetherian $k$-algebra $R$, and let $M$ and $N$ be finitely generated left $R/I$-modules. Then $M \leq_{\text{deg}} N$ (resp. $M \leq_{\text{ext}} N$) as $R$-modules if and only if it does so as left $R/I$-modules.

We make a remark on degenerations for modules over commutative rings that will be used later on.

Remark 2.9. Let $R$ be a commutative Noetherian $k$-algebra. Suppose that a finitely generated $R$-module $M$ degenerates to a finitely generated module $N$. Then the $i$th Fitting ideal of $M$ contains that of $N$ for all $i \geq 0$. Namely, denoting the $i$th Fitting ideal of an $R$-module $M$ by $F^R_i(M)$, we have $F^R_i(M) \supseteq F^R_i(N)$ for all $i \geq 0$. (See [10, Theorem 2.5].)

Now we give an example of modules of finite length for which we can easily describe the degeneration.

Let $R = k[[x]]$ be a formal power series ring over a field $k$ with one variable $x$ and let $M$ be an $R$-module of length $n$. It is easy to see that there is an isomorphism

$$M \cong R/(x^{p_1}) \oplus \cdots \oplus R/(x^{p_n}),$$

where

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i = n.$$

In this case the finite presentation of $M$ is given as follows:

$$0 \longrightarrow R^n \xrightarrow{\begin{pmatrix} x^{p_1} \\ \vdots \\ x^{p_n} \end{pmatrix}} R^n \longrightarrow M \longrightarrow 0.$$
Note that we can easily compute the $i$th Fitting ideal of $M$ from this presentation:
\[ \mathcal{F}^R_i(M) = (x^{p_{i+1} + \cdots + p_n}) \ (i \geq 0). \]

We denote by $p_M$ the sequence $(p_1, p_2, \ldots, p_n)$ of nonnegative integers. Recall that such a sequence satisfying (2.2) is called a partition of $n$.

Conversely, given a partition $p = (p_1, p_2, \ldots, p_n)$ of $n$, we can associate an $R$-module of length $n$ by (2.1), which we denote by $M(p)$. In such a way we see that there is a one-one correspondence between the set of partitions of $n$ and the set of isomorphism classes of $R$-modules of length $n$.

**Definition 2.10.** Let $n$ be a positive integer and let $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ be partitions of $n$. Then we denote $p \preceq q$ if it satisfies $\sum_{i=1}^{j} p_i \geq \sum_{i=1}^{j} q_i$ for all $1 \leq j \leq n$.

We note that the reverse of $\preceq$ is known to be a partial order on the set of partitions of $n$ and is called the dominance order. See [4] page 7 for the definition of the dominance order.

It is known that the degeneration problem for $R$-modules of length $n$ is equivalent to the degeneration problem for Jordan canonical forms of square matrices of size $n$. In the following proposition we show that the degeneration order for $R$-modules of length $n$ coincides with the opposite of the dominance order of corresponding partitions. Note that if $M$ and $N$ are $R$-modules of finite length and if $M$ degenerates to $N$, then the length of $M$ equals the length of $N$, since $[M] = [N]$ in the Grothendieck group.

**Proposition 2.11.** Let $R = k[[x]]$ as above, and let $M, N$ be $R$-modules of length $n$. Then the following conditions are equivalent:

1. $M \preceq_{\text{deg}} N$,
2. $M \preceq_{\text{ext}} N$,
3. $p_M \preceq p_N$.

**Proof.** First of all, we assume $M$ degenerates to $N$, and let $p_M = (p_1, p_2, \ldots, p_n)$ and $p_N = (q_1, q_2, \ldots, q_n)$. Then, by definition, we have the equalities of the Fitting ideals: $\mathcal{F}^R_i(M) = (x^{p_{i+1} + \cdots + p_n})$ and $\mathcal{F}^R_i(M) = (x^{q_{i+1} + \cdots + q_n})$ for all $i \geq 0$. Since $M$ degenerates to $N$, it follows from Remark 2.9 that $\mathcal{F}^R_i(M) \supseteq \mathcal{F}^R_i(N)$ for all $i$. Thus $p_i + 1 + \cdots + p_n \leq q_i + 1 + \cdots + q_n$. Since $\sum_{i=1}^{n} p_i = n = \sum_{i=1}^{n} q_i$, it follows that $p_1 + \cdots + p_i \leq q_1 + \cdots + q_i$ for all $i \geq 0$. Therefore $p_M \preceq p_N$.

Secondly, assume $M \preceq_{\text{deg}} N$. Then there are $R$-modules $L_0, L_1, \ldots, L_r$ such that $M \cong L_0$, $N \cong L_r$, and each $L_i$ degenerates to $L_{i+1}$ for $0 \leq i < r$. It then follows from the above that $p_{L_0} \preceq p_{L_1} \preceq \cdots \preceq p_{L_r}$. Since $\preceq$ is a partial order, we have $p_M \preceq p_N$. Thus we have proved the implication (1) $\Rightarrow$ (3).

Finally we shall prove (3) $\Rightarrow$ (2). To this end let $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ be partitions of $n$. Note that it is enough to prove that the corresponding $R$-module $M(p)$ degenerates by an extension to $M(q)$ whenever $q$ is a successor of $p$ under $\preceq$. (Recall that $q$ is called a successor of $p$ if $p \preceq q$ and there are no partitions $r$ with $p \preceq r \preceq q$ other than $p$ and $q$.)

Assume that $q$ is a successor of $p$ under $\preceq$. Then it is easy to see that there are numbers $1 \leq i < j \leq n$ with $p_i - p_j \geq 2$, $p_i > p_{i+1}$, $p_{j-1} > p_j$ such that the equality $q = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_j, p_{j+1}, \ldots, p_n)$ holds. In this case, setting $L = M((p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{j-1}, p_j, \ldots, p_n))$, we have $M(p) = L \oplus M((p_i, p_j))$ and $M(q) = L \oplus M((p_i - 1, p_j + 1))$. Note that, in general, if $M$ degenerates by
an extension to $N$, then $M \oplus L$ degenerates by an extension to $N \oplus L$, for any $R$-modules $L$. Hence it is enough to show that $M((a, b))$ degenerates by an extension to $M((a - 1, b + 1))$ if $a \geq b + 2$. However there is a short exact sequence of the form:

$$0 \longrightarrow R/(x^{a-1}) \longrightarrow R/(x^a) \oplus R/(x^b) \longrightarrow R/(x^{b+1}) \longrightarrow 0,$$

Thus $M((a, b)) = R/(x^a) \oplus R/(x^b)$ degenerates by an extension to $M((a - 1, b+1)) = R/(x^{a-1}) \oplus R/(x^{b+1})$.

Combining Proposition 2.11 with Corollary 2.8, we have the following corollary, which will be used in the next section.

**Corollary 2.12.** Let $R = k[[x]]/(x^m)$, where $k$ is a field and $m$ is a positive integer, and let $M$, $N$ be finitely generated $R$-modules. Then $M \leq_{\text{deg}} N$ holds if and only if $M \leq_{\text{ext}} N$ holds.

### 3. The second examples

Let $k$ be a field of characteristic 0 and $R = k[[x_0, x_1, x_2, \ldots, x_d]]/(f)$, where $f$ is a polynomial of the form

$$f = x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2 \quad (n \geq 1).$$

Recall that such a ring $R$ is called the ring of simple singularity of type $(A_n)$. Note that $R$ is a Gorenstein complete local ring and has finite Cohen-Macaulay representation type. (Recall that a Cohen-Macaulay $k$-algebra $R$ is said to be of finite Cohen-Macaulay representation type if there are only a finite number of isomorphism classes of objects in $\text{CM}(R)$. See [6].) In this section, we focus on the degeneration problem of maximal Cohen-Macaulay modules over the ring $R$ of simple singularity of type $(A_n)$ and of even dimension. The main result of this section is the following, whose proof will be given in the last part of this section.

**Theorem 3.1.** Let $k$ be an algebraically closed field of characteristic 0 and let $R = k[[x_0, x_1, x_2, \ldots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2)$ as above, where we assume that $d$ is even. For maximal Cohen-Macaulay $R$-modules $M$ and $N$, if $M \leq_{\text{deg}} N$, then $M \leq_{\text{ext}} N$.

To prove the theorem, we need several results concerning the stable degeneration which was introduced by the second author in [10].

Let $A$ be a commutative Gorenstein ring. We denote by $\text{CM}(A)$ the category of all maximal Cohen-Macaulay $A$-modules with all $A$-homomorphisms, and we also denote by $\text{CM}(A)$ the stable category of $\text{CM}(A)$. Recall that the objects of $\text{CM}(A)$ are the same as those of $\text{CM}(A)$, and the morphisms of $\text{CM}(A)$ are elements of $\text{Hom}_A(M, N) = \text{Hom}_A(M, N)/\text{P}(M, N)$ for $M, N \in \text{CM}(A)$, where $\text{P}(M, N)$ denotes the set of morphisms from $M$ to $N$ factoring through projective $A$-modules. For a maximal Cohen-Macaulay module $M$ we denote it by $\overline{M}$ to indicate that it is an object of $\text{CM}(A)$. Since $A$ is Gorenstein, it is known that $\text{CM}(A)$ has a structure of a triangulated category. By definition, $L \to M \to N \to L[1]$ is a triangle in $\text{CM}(A)$ if and only if there is an exact sequence $0 \to L' \to M' \to N' \to 0$ in $\text{CM}(A)$ with $L' \cong L$, $M' \cong M$ and $N' \cong N$ in $\text{CM}(A)$. See [3] Chapter 1, [10] Section 4 for the details.
Let $(R, m, k)$ be a commutative Gorenstein local ring that is a $k$-algebra and let $V = k[t]/(t^i)$ and $K = k(t)$. Note that $R \otimes_k V$ and $R \otimes_k K$ are Gorenstein rings as well. In fact, $R \otimes_k V \cong K^{-1} R[t]$, where $S = \{ f(t) \in k[t] \mid f(0) \neq 0 \}$, and it is well known that a polynomial extension or a localization of a Gorenstein ring is again a Gorenstein ring. Hence, as mentioned above, $\text{CM}(R \otimes_k V)$ and $\text{CM}(R \otimes_k K)$ are triangulated categories. We denote by $\mathcal{L} : \text{CM}(R \otimes_k V) \to \text{CM}(R \otimes_k K)$ (resp. $\mathcal{R} : \text{CM}(R \otimes_k V) \to \text{CM}(R)$) the triangle functor defined by the localization by $t$ (resp. taking $- \otimes V/tV$).

**Definition 3.2 ([10] Definition 4.1).** Let $M, N \in \text{CM}(R)$. We say that $M$ stably degenerates to $N$ if there exists a maximal Cohen-Macaulay module $Q \in \text{CM}(R \otimes_k V)$ such that $\mathcal{L}(Q) \cong M \otimes_k K$ in $\text{CM}(R \otimes_k K)$ and $\mathcal{R}(Q) \cong N$ in $\text{CM}(R)$.

We write $M \leq_{st} N$ if $N$ is obtained from $M$ by iterative stable degenerations, i.e. if there is a sequence of objects $L_0, L_1, \ldots, L_r$ in $\text{CM}(R)$ such that $M \cong L_0$, $N \cong L_r$ and each $L_i$ stably degenerates to $L_{i+1}$ for $0 \leq i < r$.

We also consider the triangle version of the degeneration by an extension.

**Definition 3.3.** We say that $M$ stably degenerates by a triangle to $N$ if there is a triangle of the form $U \to M \to V \to U[1]$ in $\text{CM}(R)$ such that $U \oplus V \cong N$. We write $M \leq_{tri} N$ if there is a finite sequence of modules $L_0, L_1, \ldots, L_r$ in $\text{CM}(R)$ such that $M \cong L_0$, $N \cong L_r$ and each $L_i$ stably degenerates by a triangle to $L_{i+1}$ for $0 \leq i < r$.

**Remark 3.4.** Let $R$ be a commutative Gorenstein local ring that is a $k$-algebra.

1. Let $M, N \in \text{CM}(R)$. If $M$ degenerates to $N$, then $M$ stably degenerates to $N$. Therefore that $M \leq_{deg} N$ forces that $M \leq_{st} N$. (See [10] Lemma 4.2.)

2. Suppose that there is a triangle

$$
\begin{array}{ccc}
L & \longrightarrow & M \\
\downarrow & & \downarrow \\
N & \longrightarrow & L[1]
\end{array}
$$

in $\text{CM}(R)$. Then $M$ stably degenerates to $L \oplus N$; thus $M \leq_{st} L \oplus N$. (See [10] Proposition 4.3.) Therefore $M \leq_{tri} N$ implies that $M \leq_{st} N$.

The following theorem, proved by the second author in [10], shows the relation between stable degenerations and ordinary degenerations.

**Theorem 3.5 ([10] Theorems 5.1, 6.1, 7.1).** Let $(R, m, k)$ be a commutative Gorenstein complete local $k$-algebra, where $k$ is an infinite field. Consider the following four conditions for maximal Cohen-Macaulay $R$-modules $M$ and $N$:

1. $R^m \oplus M$ degenerates to $R^n \oplus N$ for some $m, n \in \mathbb{N}$.

2. There is a triangle

$$
\begin{array}{ccc}
Z & \longrightarrow & M \oplus Z \\
\downarrow & & \downarrow \\
\phi & \longrightarrow & N \longrightarrow Z[1]
\end{array}
$$

in $\text{CM}(R)$ where $\phi$ is a nilpotent element of $\text{End}_R(Z)$.

3. $M$ stably degenerates to $N$.

4. There exists an $X \in \text{CM}(R)$ such that $M \oplus R^m \oplus X$ degenerates to $N \oplus R^n \oplus X$ for some $m, n \in \mathbb{N}$.

Then, in general, the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ hold. If $R$ is an isolated singularity, then $(2)$ and $(3)$ are equivalent. Furthermore, if $R$ is an Artinian ring, then the conditions $(1), (2)$ and $(3)$ are equivalent.
As one of the direct consequences of Theorem 3.5 we have the following corollary.

**Corollary 3.6 ([10] Corollary 6.6).** Let \((R_1, m_1, k)\) and \((R_2, m_2, k)\) be Gorenstein complete local \(k\)-algebras. Assume that both \(R_1\) and \(R_2\) are isolated singularities and that \(k\) is an infinite field. Suppose there is a \(k\)-linear equivalence \(F : \text{CM}(R_1) \rightarrow \text{CM}(R_2)\) of triangulated categories. Then, for \(M, N \in \text{CM}(R_1)\), \(M\) stably degenerates to \(N\) if and only if \(F(M)\) stably degenerates to \(F(N)\).

If \(R\) is a complete (more generally, Henselian) local ring, then it is known that the category \(\text{mod}(R)\) of finitely generated \(R\)-modules and \(R\)-homomorphisms is a Krull-Schmidt category; i.e., every finitely generated \(R\)-module is uniquely a finite direct sum of indecomposable \(R\)-modules. To use this property, all the rings considered below are assumed to be complete local rings.

We need several lemmas to prove Theorem 3.1.

**Lemma 3.7.** Let \(R\) be a Gorenstein complete local ring. If there is a triangle \(L \rightarrow M \rightarrow N \rightarrow L[1]\) in \(\text{CM}(R)\), then there exist nonnegative integers \(m\) and \(n\) such that \(M \oplus R^m \leq_{\text{ext}} L \oplus N \oplus R^n\). In this case, we have \([M \oplus R^m] = [L \oplus N \oplus R^n]\) in \(K_0(\text{mod}(R))\).

**Proof.** If there is a triangle \(L \rightarrow M \rightarrow N \rightarrow L[1]\), then by definition there is a short exact sequence \(0 \rightarrow L' \rightarrow M' \rightarrow N' \rightarrow 0\), where \(L' \cong L\), \(M' \cong M\) and \(N' \cong N\) in \(\text{CM}(R)\). Thus \(M' \leq_{\text{ext}} L' \oplus N'\). Since \(R\) is a complete local ring, we note that \(L'\) (resp. \(M', N'\)) is isomorphic to \(L\) (resp. \(M, N\)) up to free summands, i.e., \(L' \oplus R^a \cong L \oplus R^a\), \(M' \oplus R^b \cong M \oplus R^b\), \(N' \oplus R^c \cong N \oplus R^c\) for integers \(a, a', b, b', c\) and \(c'\). Therefore we have
\[
M \oplus R^{a''+b''+c'} \cong M' \oplus R^{a''+b''+c'} \leq_{\text{ext}} L' \oplus N' \oplus R^{a''+b''+c'} \cong L \oplus N \oplus R^{a''+b''+c'}.
\]

The next lemma will follow easily from the fact that \(\text{Hom}_R(-, R)\) is exact on \(\text{CM}(R)\) if \(R\) is Gorenstein. We leave its proof to the reader.

**Lemma 3.8.** Let \(R\) be a Gorenstein local ring and let \(0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0\) be an exact sequence in \(\text{CM}(R)\). Suppose that \(L\) (resp. \(N\)) contains a free \(R\)-module \(R^n\) as a direct summand. Then there is an exact sequence in \(\text{CM}(R)\) of the form \(0 \rightarrow L' \rightarrow M' \rightarrow N' \rightarrow 0\) (resp. \(0 \rightarrow L \rightarrow M' \rightarrow N' \rightarrow 0\)), where \(L' \oplus R^n \cong L\) (resp. \(N' \oplus R^n \cong N\)) and \(M' \oplus R^n \cong M\).

As a result of this lemma we obtain the following.

**Corollary 3.9.** Let \(R\) be a Gorenstein local \(k\)-algebra and let \(M, N \in \text{CM}(R)\). If \(M \leq_{\text{ext}} N \oplus R^n\) for an integer \(n\), then there is an \(R\)-module \(M' \in \text{CM}(R)\) such that \(M \cong M' \oplus R^n\) and \(M' \leq_{\text{ext}} N\).

Summing up all the results above, we can show the equivalence of the triangle order and the extension order.

**Proposition 3.10.** Let \((R, m, k)\) be a Gorenstein complete local ring and let \(M, N \in \text{CM}(R)\). Assume \([M] = [N]\) in \(K_0(\text{mod}(R))\). Then \(M \leq_{\text{tri}} N\) if and only if \(M \leq_{\text{ext}} N\).

**Proof.** The implication \(M \leq_{\text{ext}} N \Rightarrow M \leq_{\text{tri}} N\) is clear, since if there is an exact sequence \(0 \rightarrow U \rightarrow L \rightarrow V \rightarrow 0\) in \(\text{CM}(R)\), then there is a triangle \(U \rightarrow L \rightarrow V \rightarrow U[1]\) in \(\text{CM}(R)\).
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To prove the other implication, it is enough to show the following:

(*) If there is a triangle $\mathcal{U} \to \mathcal{L} \to \mathcal{V} \to \mathcal{U}[1]$ in $\text{CM}(R)$ and if $[\mathcal{L}] = [\mathcal{U}] + [\mathcal{V}]$ in $K_0(\text{mod}(R))$, then $L \leq_{\text{ext}} U \oplus V$.

Under the assumption of (*), it follows from Lemma 3.7 that $L \oplus R^m \leq_{\text{ext}} U \oplus V \oplus R^n$ for some integers $m$ and $n$. Since there is a degeneration, we have $[L] + [R^m] = [U] + [V] + [R^n]$ in $K_0(\text{mod}(R))$, which forces $[R^m] = [R^n]$ by the assumption. Thus it follows that $m = n$. Therefore, by Corollary 3.9 and by the Krull-Schmidt property, we have $L \leq_{\text{ext}} U \oplus V$, as desired.

As a corollary of the proof of Proposition 3.10, we have the following.

Corollary 3.11. Let $(R, m, k)$ be a Gorenstein complete local ring. Then the relation $\leq_{\text{tri}}$ gives a well-defined partial order on the set of isomorphism classes of objects in $\text{CM}(R)$.

Proof. We have to show that $M \leq_{\text{tri}} N$ and $N \leq_{\text{tri}} M$ implies that $M \cong N$ for $M, N \in \text{CM}(R)$. If $M \leq_{\text{tri}} N$, then it follows from the proof of Proposition 3.10 that $M \oplus R^m \leq_{\text{ext}} N \oplus R^n$ for some integers $m, n$. Likewise if $N \leq_{\text{tri}} M$, then $N \oplus R^{m'} \leq_{\text{ext}} M \oplus R^{n'}$ for some integers $m', n'$. Combining them, we have $M \oplus R^{m+n'} \leq_{\text{ext}} N \oplus R^{m+n'} \leq_{\text{ext}} M \oplus R^{n+m'}$. Since there is a degeneration, all of these modules give the same class in $K_0(\text{mod}(R))$; and as in the same argument as in the proof of Proposition 3.10 we see that $m + n' = n + m'$. Recall that $\leq_{\text{ext}}$ is a partial order on the set of isomorphism classes of objects in $\text{CM}(R)$. (See also the comments after Definition 2.4.) Thus it is concluded that $M \oplus R^{m+n'} \cong N \oplus R^{m+n'}$ in $\text{CM}(R)$, and hence $M \cong N$ in $\text{CM}(R)$.

The following lemma is known as the Knörrer periodicity (cf. [6] Thm. 12.10).

Lemma 3.12. Let $k$ be an algebraically closed field of characteristic 0 and let $S = k[[x_0, x_1, \cdots, x_n]]$ be a formal power series ring. For a nonzero element $f \in (x_0, x_1, \cdots, x_n)S$, we consider the two rings $R = S/(f)$ and $R^f = S[[y, z]]/(f + y^2 + z^2)$. Then the stable categories $\text{CM}(R)$ and $\text{CM}(R^f)$ are equivalent as triangulated categories.

Now we proceed to the proof of Theorem 3.1.

Let $k$ be an algebraically closed field of characteristic 0 and let

$$R = k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2)$$

as in the theorem, where we assume that $d$ is even. Suppose that $M \leq_{\text{deg}} N$ for maximal Cohen-Macaulay $R$-modules $M$ and $N$. We want to show $M \leq_{\text{ext}} N$.

Since $M \leq_{\text{deg}} N$, we have $M \leq_{\text{st}} N$ in $\text{CM}(R)$ and $[M] = [N]$ in $K_0(\text{mod}(R))$, by Remarks 3.3(1) and 2.2(3). Now let us denote $R' = k[[x_0]]/(x_0^{n+1})$, and we note that $\text{CM}(R)$ and $\text{CM}(R')$ are equivalent to each other as triangulated categories. In fact this equivalence is given by using $d/2$-times of Lemma 3.12 since $d$ is even.

Let $\Omega : \text{CM}(R) \to \text{CM}(R')$ be a triangle functor which gives the equivalence. Then, by virtue of Corollary 3.6 we have $\Omega(M) \leq_{\text{st}} \Omega(N)$ in $\text{CM}(R')$. Since $R'$ is an Artinian algebra, the equivalence $(1) \iff (3)$ holds in Theorem 3.5 and thus we have $\tilde{M} \oplus R'^m \leq_{\text{ext}} \tilde{N} \oplus R'^m$, where $\tilde{M}$ (resp. $\tilde{N}$) is a module in $\text{CM}(R')$ with $\tilde{M} \cong \Omega(M)$ (resp. $\tilde{N} \cong \Omega(N)$) and $m, n$ are nonnegative integers. It then follows from Corollary 2.12 that $\tilde{M} \oplus R'^m \leq_{\text{ext}} \tilde{N} \oplus R'^m$. Hence, by Proposition 3.10 we have that $\Omega(M) \leq_{\text{tri}} \Omega(N)$ in $\text{CM}(R')$. Noting that the partial order $\leq_{\text{tri}}$ is preserved under a
triangle functor, we see that $M \leq_{tri} N$ in CM$(R)$. Since $[M] = [N]$ in $K_0(\text{mod}(R))$, applying Proposition 3.10 we finally obtain that $M \leq N$.

**Example 3.13.** Let $R = k[[x_0, x_1, x_2]]/(x_0^2 + x_1^2 + x_2^2)$, where $k$ is an algebraically closed field of characteristic 0. Let $p$ and $q$ be the ideals generated by $(x_0, x_1 - \sqrt{-1} \cdot x_2)$ and $(x_0, x_1 + \sqrt{-1} \cdot x_2)$ respectively. It is known that the set \{R, p, q\} is a complete list of the isomorphism classes of indecomposable maximal Cohen-Macaulay modules over $R$. See [6, Chapter 10]. We see from Theorem 3.1 that all degenerations in CM$(R)$ are given by extensions. By this fact we can easily describe the degenerations in CM$(R)$. For example, the Hasse diagram of degenerations of maximal Cohen-Macaulay $R$-modules of rank 3 is a disjoint union of the following diagrams:

\[
\begin{align*}
\text{R} \oplus \text{p} \oplus \text{q} & \quad \text{R} \oplus \text{q} \\
\text{R}^2 \oplus \text{p} & \quad \text{R}^2 \oplus \text{q}.
\end{align*}
\]

**Remark 3.14.** Theorem 3.1 is expected to hold without the assumption on $d$. Unfortunately, at the moment of writing the manuscript, the authors do not know any appropriate proof for this.

## 4. Extended orders

In the rest of this paper, $R$ denotes a commutative Cohen-Macaulay complete local $k$-algebra, where $k$ is any field.

We shall show that any extended degenerations of maximal Cohen-Macaulay $R$-modules are generated by extended degenerations of Auslander-Reiten (AR) sequences if $R$ is of finite Cohen-Macaulay representation type. For the theory of AR sequences of maximal Cohen-Macaulay modules, we refer to [6]. First of all we recall the definitions of the extended orders generated respectively by degenerations, extensions and AR sequences.

**Definition 4.1 ([7, Definitions 4.11, 4.13]).** The relation $\leq_{DEG}$ on CM$(R)$, which is called the extended degeneration order, is a partial order generated by the following rules:

1. If $M \leq_{deg} N$, then $M \leq_{DEG} N$.
2. If $M \leq_{DEG} N$ and if $M' \leq_{DEG} N'$, then $M \oplus M' \leq_{DEG} N \oplus N'$.
3. If $M \oplus L \leq_{DEG} N \oplus L$ for some $L \in \text{CM}(R)$, then $M \leq_{DEG} N$.
4. If $M^n \leq_{DEG} N^n$ for some natural number $n$, then $M \leq_{DEG} N$.

**Definition 4.2 ([7, Definition 3.6]).** The relation $\leq_{EXT}$ on CM$(R)$, which is called the extended extension order, is a partial order generated by the following rules:

1. If $M \leq_{ext} N$, then $M \leq_{EXT} N$.
2. If $M \leq_{EXT} N$ and if $M' \leq_{EXT} N'$, then $M \oplus M' \leq_{EXT} N \oplus N'$.
3. If $M \oplus L \leq_{EXT} N \oplus L$ for some $L \in \text{CM}(R)$, then $M \leq_{EXT} N$.
4. If $M^n \leq_{EXT} N^n$ for some natural number $n$, then $M \leq_{EXT} N$.
Definition 4.3 (\cite{7} Definition 5.1). The relation $\leq_{AR}$ on $\text{CM}(R)$, which is called the extended AR order, is a partial order generated by the following rules:

1. If $0 \to X \to E \to Y \to 0$ is an AR sequence in $\text{CM}(R)$, then $E \leq_{AR} X \oplus Y$.
2. If $M \leq_{AR} N$ and if $M' \leq_{AR} N'$, then $M \oplus M' \leq_{AR} N \oplus N'$.
3. If $M \oplus L \leq_{AR} N \oplus L$ for some $L \in \text{CM}(R)$, then $M \leq_{AR} N$.
4. If $M^n \leq_{AR} N^n$ for some natural number $n$, then $M \leq_{AR} N$.

The following is the main theorem of this section.

Theorem 4.4. Let $R$ be a Cohen-Macaulay complete local $k$-algebra as above. Adding to this, we assume that $R$ is of finite Cohen-Macaulay representation type. Then the following conditions are equivalent for $M, N \in \text{CM}(R)$:

1. $M \leq_{DEG} N$,
2. $M \leq_{EXT} N$,
3. $M \leq_{AR} N$.

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are clear from the definitions.

To prove (1) $\Rightarrow$ (2), it suffices to show that $M \leq_{EXT} N$ whenever $M$ degenerates to $N$. If $M$ degenerates to $N$, then, by virtue of Theorem \ref{2.2} we have a short exact sequence $0 \to Z \to M \oplus Z \to N \to 0$ with $Z \in \text{CM}(R)$. Thus $M \oplus Z \leq_{ext} N \oplus Z$; hence $M \leq_{EXT} N$.

It remains to prove that (2) $\Rightarrow$ (3), for which we need several preparations.

Under the circumstances of Theorem \ref{4.4} we consider the functor category $\text{Mod}(\text{CM}(R))$ and the Auslander category $\text{mod}(\text{CM}(R))$ of $\text{CM}(R)$. By definition, $\text{Mod}(\text{CM}(R))$ is the category whose objects are contravariant additive functors from $\text{CM}(R)$ to the category of Abelian groups and whose morphisms are natural transformations between functors. Note that $\text{Mod}(\text{CM}(R))$ is an Abelian category. The Auslander category $\text{mod}(\text{CM}(R))$ is a full subcategory of $\text{Mod}(\text{CM}(R))$ consisting of all finitely presented functors. Recall that a functor $F \in \text{Mod}(\text{CM}(R))$ is called finitely presented if there is an exact sequence in $\text{Mod}(\text{CM}(R))$,

$$\text{Hom}_R(\ , M) \to \text{Hom}_R(\ , N) \to F \to 0,$$

for $M, N \in \text{CM}(R)$.

If there is a short exact sequence $0 \to L \to M \to N \to 0$ in $\text{CM}(R)$, then the finitely presented functor $F$ is defined by the exact sequence in $\text{mod}(\text{CM}(R))$:

$$0 \to \text{Hom}_R(\ , L) \to \text{Hom}_R(\ , M) \to \text{Hom}_R(\ , N) \to F \to 0.$$

Such a functor $F \in \text{mod}(\text{CM}(R))$ satisfies $F(R) = 0$. Conversely every element $F \in \text{mod}(\text{CM}(R))$ with the property $F(R) = 0$ is obtained in this way from a short exact sequence in $\text{CM}(R)$.

If $0 \to X \to E \to Y \to 0$ is an AR sequence in $\text{CM}(R)$, then the functor $S$ defined by an exact sequence

$$0 \to \text{Hom}_R(\ , X) \to \text{Hom}_R(\ , E) \to \text{Hom}_R(\ , Y) \to S \to 0$$

is a simple object in $\text{mod}(\text{CM}(R))$ and all the simple objects in $\text{mod}(\text{CM}(R))$ are obtained in this way from AR sequences.

It is proved in \cite{6} (13.7.4) that every object $F$ in $\text{mod}(\text{CM}(R))$ with $F(R) = 0$ has a composition series; i.e., there is a filtration by subobjects $0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F$ such that each $F_i/F_{i-1}$ is a simple object in $\text{mod}(\text{CM}(R))$.
Now we consider a free Abelian group
\[ G(\text{CM}(R)) = \bigoplus \mathbb{Z} \cdot X, \]
where \(X\) runs through all isomorphism classes of indecomposable objects in \(\text{CM}(R)\).

There is a group homomorphism
\[ \gamma : G(\text{CM}(R)) \rightarrow K_0(\text{mod}(\text{CM}(R))), \]
defined by \(\gamma(M) = [\text{Hom}_R(\cdot, M)]\) for \(M \in \text{CM}(R)\).

It is proven in [6, Theorem 13.7] that
\[(4.4) \text{ the group homomorphism } \gamma \text{ is injective.} \]
(See the first paragraph of the proof of Theorem 13.7 in [6].)

The following lemma is essentially due to Auslander and Reiten [1].

**Lemma 4.5.** Under the same assumptions on \(R\) as in Theorem 4.4, let \(0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0\) be a short exact sequence in \(\text{CM}(R)\). Then there are a finite number of AR sequences in \(\text{CM}(R)\):
\[ 0 \rightarrow X_i \rightarrow E_i \rightarrow Y_i \rightarrow 0 \quad (1 \leq i \leq n), \]
such that there is an equality in \(G(\text{CM}(R))\):
\[ L - M + N = \sum_{i=1}^{n} (X_i - E_i + Y_i). \]

**Proof.** Consider the functor \(F \in \text{mod}(\text{CM}(R))\) defined by (4.1). By virtue of (4.3), there is a filtration by subobjects \(0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F\) such that each \(F_i/F_{i-1}\) is a simple object in \(\text{mod}(\text{CM}(R))\). Now take an AR sequence \(0 \rightarrow X_i \rightarrow E_i \rightarrow Y_i \rightarrow 0\) corresponding to the simple functor \(F_i/F_{i-1}\) for each \(i\).

Thus we have equalities in \(K_0(\text{mod}(\text{CM}(R)))\):
\[ [\text{Hom}_R(\cdot, L)] - [\text{Hom}_R(\cdot, M)] + [\text{Hom}_R(\cdot, N)] = \sum_{i=1}^{n} [F_i/F_{i-1}] \]
\[ = \sum_{i=1}^{n} ([\text{Hom}_R(\cdot, X_i)] - [\text{Hom}_R(\cdot, E_i)] + [\text{Hom}_R(\cdot, Y_i)]). \]

This is equivalent to
\[ \gamma(L - M + N) = \sum_{i=1}^{n} \gamma(X_i - E_i + Y_i). \]
Since \(\gamma\) is injective (4.4), we have the equality \(L - M + N = \sum_{i=1}^{n} (X_i - E_i + Y_i)\) in \(G(\text{CM}(R))\). \(\square\)

Now we shall continue the proof of Theorem 4.4. It remains to show that \(M \leq_{\text{EXT}} N\) implies that \(M \leq_{\text{AR}} N\). We have only to prove that if \(M\) degenerates by an extension to \(N\), then \(M \leq_{\text{AR}} N\). Assuming that \(M\) degenerates by an extension to \(N\), we have a short exact sequence \(0 \rightarrow N_1 \rightarrow M \rightarrow N_2 \rightarrow 0\) in \(\text{CM}(R)\) with \(N \cong N_1 \oplus N_2\). Then, by Lemma 4.5, there are a finite number of AR sequences in \(\text{CM}(R)\):
\[ 0 \rightarrow X_i \rightarrow E_i \rightarrow Y_i \rightarrow 0 \quad (1 \leq i \leq n), \]
such that there is an equality in $G(CM(R))$:

$$N_1 - M + N_2 = \sum_{i=1}^{n} (X_i - E_i + Y_i).$$

This equality is equivalent to there being an isomorphism of $R$-modules:

$$M \oplus \sum_{i=1}^{n} (X_i \oplus Y_i) \cong N_1 \oplus N_2 \oplus \sum_{i=1}^{n} E_i.$$ 

Since $E_i \leq_{AR} (X_i \oplus Y_i)$ for all $1 \leq i \leq n$, we have

$$M \oplus \sum_{i=1}^{n} (X_i \oplus Y_i) \cong N_1 \oplus N_2 \oplus \sum_{i=1}^{n} E_i \leq_{AR} N_1 \oplus N_2 \oplus \sum_{i=1}^{n} (X_i \oplus Y_i).$$

Therefore $M \leq_{AR} N_1 \oplus N_2 \cong N$, and the proof is complete. $\square$

**Remark 4.6.** In the paper [7], the second author introduced the order relation $\leq_{\text{hom}}$ as well. Adding to the assumption that $R$ is of finite Cohen-Macaulay representation type, if we assume further conditions on $R$, such as $R$ being an integral domain of dimension 1 or $R$ being of dimension 2, then he showed that $\leq_{\text{hom}}$ is also equal to any of $\leq_{AR}$, $\leq_{\text{EXT}}$ and $\leq_{\text{DEG}}$.

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