THE CONSTRUCTION OF TRIGONOMETRIC
IN VariANTS FOR WEYL GROUPS AND THE
DERIVATION OF CORRESPONDING EXACTLY
SOLVABLE SUTHERLAND MODELS

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Abstract

Trigonometric invariants are defined for each Weyl group orbit on the root lattice. They are real and periodic on the coroot lattice. Their polynomial algebra is spanned by a basis which is calculated by means of an algorithm. The invariants of the basis can be used as coordinates in any cell of the coroot space and lead to an exactly solvable model of Sutherland type. We apply this construction to the $F_4$ case.

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1 Introduction

Integrable models of the Calogero-Moser class and their trigonometric and rational limit models are conventionally described by the simple Lie algebras, i.e. those contained in the classical sequences \(A_n, B_n, C_n, D_n\) or the exceptional set \(G_2, F_4, E_6, E_7, E_8\). We have shown \([1, 2]\) that the Weyl groups underlying these algebras are the essential ingredients in the construction of the trigonometric or rational models. Exact solvability is easily proven by expressing the Schrödinger equation in terms of the Weyl group invariants as coordinates. For the rational models the crystallographic property of the Weyl groups, which guarantees the existence of a root lattice, can be abandoned. For the proper Coxeter groups \(H_3\) and \(H_4\) and the infinite sequence of dihedral groups \([2, 3]\) Calogero type models can be constructed as well.

For the classical Lie algebras and \(G_2\) the Weyl group invariants are not derived but guessed by intuition \([4]\). But this method failed for the exceptional algebras \(F_4\) and \(E_6, E_7, E_8\). Therefore a concept supplying us with all trigonometric polynomial invariants and their algebraic basis is highly desirable. In the case of the rational polynomial invariants the existence of the algebraic basis is guaranteed by Chevalley’s theorem \([5]\). The Jacobian for the transition from cartesian coordinates in root space to the basic invariants as new coordinates can be factorized (“factorization theorem”). Both the Chevalley theorem and the factorization theorem are valid also in the trigonometric case as we shall show here for \(F_4\) (for the classical Lie algebras and \(G_2\) they are known to be valid, too).

Our construction of invariants proceeds as follows. We decompose the root lattice \(\Lambda\) into orbits \(\Omega\) (infinitely many, they can be ordered by the length of vectors which is constant over the orbit). For each orbit we define an invariant trigonometric polynomial

\[
T_\Omega(x) = \sum_{\beta \in \Omega} \exp i(\beta, x) \tag{1.1}
\]

These functions obey fusion rules

\[
T_{\Omega_1}(x)T_{\Omega_2}(x) = \sum_{\Omega_3} C_{\Omega_1\Omega_2}^{\Omega_3} T_{\Omega_3}(x) \tag{1.2}
\]

with “fusion coefficients” \(C_{\Omega_1\Omega_2}^{\Omega_3}\) that are nonnegative integers. In \(1.2\) the null-orbit consisting only of the null-vector in \(\Lambda\) must be included

\[
T_{\Omega_0}(x) = 1 \tag{1.3}
\]

The system of equations \(1.2\) is of triangular shape and can be solved trivially for \(T_{\Omega_{\text{max}}}(x)\). Each pair \(\Omega_1, \Omega_2\) defines in fact a unique \(\Omega_{\text{max}}(\Omega_1, \Omega_2)\) with

\[
C_{\Omega_1\Omega_2}^{\Omega_{\text{max}}(\Omega_1, \Omega_2)} = 1 \tag{1.4}
\]

By recursive substitutions we isolate then an algebraic basis

\[
\{T_{\Omega_1}, T_{\Omega_2}, \ldots T_{\Omega_n}\} \tag{1.5}
\]

\(n = \text{rank (Weyl group)}\)

and for all other orbits we obtain explicitly

\[
T_\Omega(x) = \text{pol}\{T_{\Omega_1}(x), \ldots T_{\Omega_n}(x), T_{\Omega_0}(x)\} \tag{1.6}
\]
This is an explicit and constructive version of Chevalley’s theorem. It is obtained only in a case-by-case study (e.g. for $F_4$). In Section 2 we do this in great detail for $F_4$.

In Section 3 we apply this technique to construct the $F_4$ Sutherland model, using the approach developed in [1, 2]. By the way some minor theorems are proven by explicit calculation (e.g. existence of the $r_{i}^{(a)}$-coefficients as polynomials in the Chevalley basis).

## 2 Trigonometric invariants of Weyl groups.

Weyl groups are generated by reflections along roots $\alpha \in \mathbb{R}_n$ \[ x \in \mathbb{R}_n : s_\alpha x = x - 2 \frac{(\alpha, x)}{(\alpha, \alpha)} \alpha \tag{2.1} \]

Each root $\alpha$ is an integer linear combination of simple roots \[ \{\alpha_1, \alpha_2, ...\alpha_n\} \tag{2.2} \]

These simple roots span an integral lattice $\Lambda \in \mathbb{R}_n$ \[ \beta \in \Lambda : \beta = \sum_{i=1}^{n} m_i \alpha_i, \ m_i \in \mathbb{Z} \tag{2.3} \]

If the Weyl group $W$ acts on a vector $\beta \in \Lambda$ it produces an orbit $\Omega$ \[ \Omega = \{w\beta, \ w \in W\} \tag{2.4} \]

How can such orbit be characterized?

Since $W$ is a discrete subgroup of $O(n)$ acting on $\mathbb{R}_n$ we obtain \[ \|w\beta\|^2 = (w\beta, w\beta) = \|\beta\|^2 \tag{2.5} \]

so that an orbit appears as a discrete set on a sphere of radius $\|\beta\|$ and (see \[(2.3)\]) \[ \|\beta\|^2 = \sum_{i,j} (\alpha_i, \alpha_j)m_i m_j \tag{2.6} \]

Here $(\alpha_i, \alpha_j)$ is contained in the Cartan matrix as \[ A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{\|\alpha_i\|^2} \tag{2.7} \]

Only for simply laced Weyl groups the lengths of the simple roots are all equal.

For the non-simply laced Weyl group $W = F_4$ we use as basis in $\mathbb{R}_4$ \[ \{e_i, \ i \in \{1, 2, 3, 4\}, \ (e_i, e_j) = \delta_{ij}\} \]

Then $W = F_4$ can be generated from the reflections along $e_i$ and $f_i$ (all $i$) where \[ f_1 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \]
\[ f_2 = \frac{1}{2}(e_1 + e_2 - e_3 - e_4) \]
\[ f_3 = \frac{1}{2}(e_1 - e_2 + e_3 - e_4) \]
\[ f_4 = \frac{1}{2}(e_1 - e_2 - e_3 + e_4) \tag{2.8} \]
$(f_i, f_j) = \delta_{ij} \quad (2.9)$

and the permutation group $S_4$ of the basis $\{e_i\}$. The set of roots decomposes into two orbits

$$\Omega_1 = \{\pm e_i, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\} \quad (2.10)$$

$$\Omega_2 = \{\pm e_i \pm e_j, \; i < j\} \quad (2.11)$$

which are characterized by a subscript denoting the length squared of the roots. In addition we need the null-orbit

$$\Omega_0 = \{\text{null-vector}\} \quad (2.12)$$

All other orbits can be characterized by an integral radius squared and a further “degeneracy” label. A list of orbits up to $\|\beta\|^2 = 24$ is given in Table 1.

Trigonometric invariants are defined for each orbit by

$$T_{n,a}(x) = \sum_{\beta \in \Omega_{n,a}} \exp i(\beta, x) \quad (2.13)$$

(i.e. $\|\beta\|^2 = n$), so that

$$T_{n,a}(0) = \#\Omega_{n,a} \quad (2.14)$$

Now we consider pairs of such trigonometric invariants and expand their product as

$$T_{n,a}(x)T_{m,b}(x) = \sum_{k,c} C^{(k,c)}_{(n,a),(m,b)} T_{k,c}(x) \quad (2.15)$$

Applying permutations of $S_4$, reflections along the coordinate axis $e_i$, $i \in \{1, 2, 3, 4\}$ and reflection along $f_1$ to any generating vector yields the whole orbit.

We expand $x \in \mathbb{R}_n$ in a co-root basis

$$x = \sum_i \xi_i \tilde{\alpha}_i \quad (2.16)$$

Then by inversion of (2.13) we obtain

$$(2\pi)^{-n} \int_{\text{cell}} d^n \xi T_{m,a}(x)T_{n,b}(x)T_{k,c}(x) = \#\Omega_{k,c} \cdot C^{(k,c)}_{(m,a),(n,b)} \quad (2.17)$$

Note that the r.h.s. is symmetric in the three orbits. We denote (2.13), (2.17) the “fusion rules” for trigonometric invariants. The “fusion coefficients” $C^{(k,c)}_{(m,a),(n,b)}$ are nonnegative integers.

We expect that a Chevalley theorem of the following type is valid:

Any polynomial in the $\{T_{n,a}(x)\}$ can be expressed as a polynomial in an algebraic basis of invariants including $T_{\Omega_0} = 1$. The number of nontrivial basis elements is rank $W$. 
In the case of $W = F_4$ this algebraic basis is constructed by inversion of (2.13) and consists of

$$T_1, T_2, T_3, T_6^*$$

(2.18)

This inversion is possible by the triangular shape of the fusion rules. Namely inserting (2.13) into (2.15) and using the triangular inequality we obtain

$$C^{(k,c)}_{(n,a)(m,b)} = 0 \quad \text{except possibly for}$$

$$|\sqrt{n} - \sqrt{m}| \leq \sqrt{k} \leq \sqrt{n} + \sqrt{m}$$

(2.19)

Thus there is a maximal $k$ for each $n, m$

$$k_{\text{max}} \leq (\sqrt{n} + \sqrt{m})^2$$

(2.20)

and we solve (2.15) for $T_{k_{\text{max}},c}(x)$ where $c$ is such that the fusion coefficient is one. The result for

$$T_{k,c}, \quad k \notin \{1, 2, 3, 6\}, \quad k \leq 24$$

(2.21)

expressed as a polynomial in $T_1, T_2, T_3, T_6$ is given in Table 2.

The task to introduce the trigonometric invariants as coordinates in each cell of the space $\mathbb{R}_n$ leads to the study of the Jacobian matrix

$$\begin{align*}
\left\{ \frac{\partial T_m}{\partial x_i} \right\}_{m \in \text{basic set of invariants}}
\end{align*}$$

(2.22)

In the program of constructing exactly solvable models we have to compute the (inverse) Riemannian

$$g^{-1}_{mn} = \sum_i \frac{\partial T_m}{\partial x_i} \frac{\partial T_n}{\partial x_i}(x)$$

$$= - \sum_{\beta \in \Omega_m} \sum_{\beta' \in \Omega_n} (\beta, \beta') \exp i(\beta + \beta', x)$$

$$= -\frac{1}{2} \sum_{k,a} (k - m - n)C^{(k,a)}_{(m,n)}T_{k,a}(x)$$

(2.23)

which can obviously be expressed as a polynomial in the basic invariants. For $F_4$ the largest orbit appearing is $\Omega_{24}$ in $g^{-1}_{66}$.

*) We could replace $T_6$ by one of the following invariants: $\{T_7, T_8, T_{9,1}, T_{9,2}, T_{10}\}$ or constant linear combinations thereof.
3 The $F_4$ Sutherland model

From $[2,23]$ a first version of the Riemannian is obtained by insertion of the fusion coefficients

\[
\begin{align*}
g_{11}^{-1} &= -T_1 - T_3 + 4T_1 + 24 \\
g_{12}^{-1} &= -T_5 + 6T_1 \\
g_{13}^{-1} &= -\frac{3}{2}T_7 - 3T_6 - 2T_5 + 3T_3 + 12T_2 + 12T_1 \\
g_{16}^{-1} &= -2T_{11} - 2T_{9,1} + 4T_5 + 6T_3 \\
g_{21}^{-1} &= -2T_8 - 2T_6 + 8T_2 + 48 \\
g_{23}^{-1} &= -2T_{9,1} - 3T_7 + 6T_3 + 24T_1 \\
g_{26}^{-1} &= -3T_{14} - 6T_{12} - 4T_{10} + 6T_6 + 24T_4 + 24T_2 \\
g_{33}^{-1} &= -3T_{12} - 5T_{11} - 8T_{10} - 12T_{9,2} - 6T_{9,1} - 12T_8 - 3T_7 + 4T_5 \\
&\quad + 12T_4 + 21T_3 + 48T_2 + 60T_1 + 288 \\
g_{56}^{-1} &= -4T_{17,1} - 6T_{15} - 6T_{13,2} - 8T_{13,1} - 4T_{11} \\
&\quad + 6T_7 + 16T_5 + 36T_3 + 48T_1 \\
g_{66}^{-1} &= -6T_{24} - 10T_{22} - 16T_{20} - 12T_{18,2} - 24T_{18,1} \\
&\quad - 24T_{16} - 6T_{14} + 8T_{10} + 24T_8 + 42T_6 + 96T_4 + 120T_2 + 576
\end{align*}
\]

(3.1)

A second form is obtained by substitution of the algebraic basis $T_1, T_2, T_3, T_6$ by means of Table 2.

\[
\begin{align*}
g_{11}^{-1} &= -T_1^2 + T_3 + 6T_2 + 12T_1 + 48 \\
g_{12}^{-1} &= -T_1T_2 + 3T_3 + 12T_1 \\
g_{13}^{-1} &= -\frac{3}{2}T_1T_3 + 4T_1T_2 + 12T_1^2 + \frac{3}{2}T_6 - 24T_3 - 42T_2 - 96T_1 - 288 \\
g_{16}^{-1} &= -2T_1T_6 + 2T_2T_3 + 4T_1T_2 - 12T_3 - 48T_1 \\
g_{21}^{-1} &= -2T_2^2 + 12T_1^2 + 2T_6 - 24T_3 - 48T_2 - 96T_1 - 192 \\
g_{23}^{-1} &= -2T_2T_3 + 3T_1T_3 - 4T_1T_2 - 24T_1^2 - 9T_6 + 60T_3 \\
&\quad + 108T_2 + 240T_1 + 576 \\
g_{26}^{-1} &= -4T_2^2T_2 - 24T_3^3 - 3T_2T_6 + 3T_3^2 + 8T_3T_2 - 6T_6T_1 \\
&\quad + 72T_3T_1 + 104T_2T_2 + 12T_2^2 + 96T_6 - 48T_6 + 288T_3 \\
&\quad + 480T_2 + 1536T_1 + 2304 \\
g_{33}^{-1} &= 4T_2T_1^2 + 12T_3^2 - 4T_2T_3 - 3T_3^2 + T_6T_1 - 36T_3T_1 \\
&\quad - 60T_1T_2 - 96T_1^2 + 12T_6 - 48T_3 - 48T_2 - 384T_1 \\
g_{36}^{-1} &= 2T_1T_2T_3 - 16T_2^2T_2 - 4T_3T_5 - 6T_2T_5 - 8T_1T_5 + 72T_2^2 \\
&\quad + 32T_2T_3 - 12T_1T_3 + 144T_1T_2 + 96T_2^2 + 36T_6 \\
&\quad - 240T_3 - 48T_2 - 96T_1 - 2304 \\
g_{66}^{-1} &= -16T_3^3T_2 - 4T_1T_2T_6 - 8T_2^2T_4 + 2T_2T_3^2 \\
&\quad + 48T_1T_2T_3 - 96T_1^2T_2 + 48T_1T_2^2 - 192T_1^3 \\
&\quad - 6T_2^2 + 16T_3T_6 - 24T_2T_6 + 16T_1T_6 + 24T_3^2 \\
&\quad + 512T_2T_3 + 576T_1T_3 + 864T_2^2 + 2880T_1T_2 + 1344T_1^2 \\
&\quad + 96T_6 + 1152T_3 + 6144T_2 + 7680T_1 + 9216
\end{align*}
\]

(3.2)
Since $F_4$ has two orbits in the roots we have according to the factorization theorem

$$\det g^{-1} = \frac{1}{4} P_1 \cdot P_2$$

(3.3)

where $P_1$ corresponds to $\Omega_4$ \[2.10, 2.11\]. We find explicitly

$$P_1 = 110592T_1 + 41472T_2 + 27648T_3 + 110592 - 3456T_6 - 1728T_1T_6$$
$$+ 192T_1T_3 + 432T_3T_6 - 1072T_1T_3T_6 - 784T_2T_6^2$$
$$+ 5184T_2T_3T_6 - 1728T_2^2T_3 + 144T_1T_3^3$$
$$- 48T_1^3T_2 - 648T_2T_6 + 14976T_1T_3 + 4T_6T_3^3$$
$$- T_3^2T_1 + 216T_1T_2T_3 - 18T_6T_1T_3 - 4608T_1^3$$
$$+ 1728T_3^2 + 18432T_1^2 + 3888T_2^2 + 27T_6^2 + 16T_1^6$$
$$+ 4T_3^4$$

(3.4)

$$P_2 = 10616832T_1 + 4423680T_2 + 1769472T_3 + 7077888 - 221184T_6$$
$$- 221184T_1T_6 + 103680T_1T_3^2 - 27648T_5T_6$$
$$+ 34560T_1^3T_3 + 663552T_1^2T_3 + 774144T_2T_3$$
$$+ 4860048T_1T_2 + 1465344T_1^2T_2 - 62208T_1T_6$$
$$- 6912T_1^3T_2 - 78336T_3T_5 + 1990656T_1T_3$$
$$- 1728T_6T_3^3 + 18144T_3^2T_1^2 - T_2^3T_6$$
$$+ 566784T_1T_2T_3 - 17280T_6T_1T_3 - 8T_6T_1^2T_3$$
$$+ 79488T_1^2T_3T_2 - 48384T_6T_1T_2 - 2592T_6T_1T_2^2$$
$$+ 43200T_1T_2^3T_3 + 36T_3^2T_1T_2 - 2592T_3T_2^2$$
$$- 8640T_1T_3T_2 - 4608T_2T_3T_6 - 5184T_2T_1^2T_6$$
$$+ 576T_6T_1^3T_2 - 18T_6^2T_2T_3^2 + 13392T_1T_2^2T_3$$
$$- 144T_2^3T_6 - 1728T_2^2T_3^2 + 96T_3^2T_3T_1$$
$$- 216T_3^2T_1T_2 - 108T_1T_6T_3^2 - 72T_1^2T_2^2$$
$$+ 774144T_1^3 + 138240T_2^2 + 5308416T_1T_2 + 1096704T_2^2$$
$$+ 1728T_2^2 - 20736T_5^2 + 3456T_3^2 - 103680T_1^4 + 129024T_2^3$$
$$+ 6384T_4^2 + 1728T_6^2 + 27T_1^4 + 64T_2^5 + 4T_3^6$$
$$+ 119808T_3T_2^2 + 787968T_1T_2^2 + 36288T_2^2$$
$$+ 88128T_1T_2 + 108T_1T_6 + 45888T_1T_2$$
$$- 18432T_1^3T_2^2 - 9024T_2^2T_6 + 2520T_2^2T_3$$
$$- 192T_6T_3^2 + 864T_1T_3^2 - 432T_6T_3^2 - 328T_3T_6$$
$$- 10368T_1^4T_3 - 32T_3^2T_6^3 + 6592T_3^2T_3 - 2976T_1T_3T_6^2$$
$$+ 432T_4^2T_2 + 1728T_5^2T_2 - 432T_3^2T_3^4 + 1296T_3T_1$$
$$+ 864T_1^4T_6 + 432T_2^2T_3^5 - 34560T_1^2T_2^2 + 32T_4^2$$
$$- 16T_1T_2^4 + 224T_1T_2^4 + 4T_2^3T_3$$
$$- 1296T_6T_1T_2T_3$$

(3.5)
If we reexpress these polynomials in the Chevalley basis \( T_1, T_2, T_3, T_6 \) as functions of Cartesian coordinates we find

\[
P_1 = -2^{24} \prod_{\alpha \in \Omega_1^+} \left[ \sin \frac{1}{2}(\alpha, x) \right]^2 \tag{3.6}
\]

\[
P_2 = -2^{24} \prod_{\alpha \in \Omega_2^+} \left[ \sin \frac{1}{2}(\alpha, x) \right]^2 \tag{3.7}
\]

Here we made use of the fact that each orbit \( \Omega_{k,a} \) can be decomposed into a positive and a negative semiorbit

\[
\Omega_{k,a} = \Omega_{k,a}^+ \cup \Omega_{k,a}^-
\]

by a hypersurface

\[
(\alpha, \xi) = 0
\]

so that

\[
\Omega_{k,a}^\pm = \{ \alpha \in \Omega_{k,a}; (\alpha, \xi) \geq 0 \}
\]

For example, we define

\[
\Omega_{k,a}^+ = \{ \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4 \in \Omega_{k,a}; \mu_1 > 0 \text{ or } \mu_1 = 0, \mu_2 > 0; \text{ or } \mu_1 = \mu_2 = 0, \mu_3 > 0; \text{ or } \mu_1 = \mu_2 = \mu_3 = 0, \mu_4 > 0 \} \tag{3.11}
\]

Then the coefficients \( \xi_n \) of \( \xi \)

\[
\xi = \sum_n \xi_n e_n
\]

must satisfy inequalities, i.e. from \( \Omega_1 \) and \( \Omega_2 \)

\[
\xi_1 > \xi_2 > \xi_3 > \xi_4 > 0 \tag{3.16}
\]

\[
\xi_1 > \xi_2 + \xi_3 + \xi_4 \tag{3.17}
\]

\( \Omega_3 \) implies new inequalities, etc., so that a vector \( \xi \) exists for any finite set of orbits. We assume (3.16),(3.17) to hold throughout.

We consider the asymptotic behaviour along an imaginary direction in \( x \)-space

\[
x = -i \lambda \xi, \quad \lambda \to \infty
\]

In each orbit there exist maximal vectors \( \alpha_{\text{max}}(k,a) \) so that for fixed \( \xi \)

\[
(\alpha_{\text{max}}(k,a), \xi) = \max_{\alpha \in \Omega_{k,a}} (\alpha, \xi)
\]

Then (for unique \( \alpha_{\text{max}}(k,a) \))

\[
T_{k,a} \sim e^{\lambda (\alpha_{\text{max}}(k,a), \xi)} \quad \text{for } \lambda \to \infty
\]
In particular, we have
\[
\begin{align*}
\alpha_{\text{max}}(1) &= e_1 \\
\alpha_{\text{max}}(2) &= e_1 + e_2 \\
\alpha_{\text{max}}(3) &= \frac{1}{2} (3e_1 + e_2 + e_3 + e_4) \\
\alpha_{\text{max}}(4) &= 2e_1 + e_2 + e_3
\end{align*}
\] (3.22)

From (3.18),(3.19) we find as leading term in \( P_1(P_2) \) the unique monomial
\[
- T_1^2 T_3^2 (- T_2^2 T_6^2)
\]
with asymptotic behaviour
\[
P_{1,2} \sim -e^{\lambda(\rho_{1,2},\xi)}
\] (3.26)

where
\[
\begin{align*}
\rho_1 &= 5e_1 + e_2 + e_3 + e_4 \\
\rho_2 &= 6e_1 + 4e_2 + 2e_3
\end{align*}
\] (3.27)

These are special cases of Weyl type vectors
\[
\rho_{k,a} = \sum_{\alpha \in \Omega^+_{k,a}} \alpha
\] (3.28)

The same leading term (3.26) results from (3.6), (3.7). In this fashion the constants in
the factorization formulas (3.6), (3.7) can be controlled.

From (2.23) follows that
\[
det g^{-1} = \frac{1}{4} P_1 P_2 = (\det J)^2
\] (3.29)

where \( J \) is the Jacobian matrix (2.22). Due to (3.6), (3.7) \( \det J \) vanishes in first order
along the hyperplanes
\[
(\alpha, x) = 2n\pi, \ n \in \mathbb{Z}, \alpha \in \Omega_{1,2}
\] (3.30)

The root space \( \mathbb{R}_4 \) is therefore divided into cells bounded by the walls (3.30) where (see
(3.42)) repulsive and impenetrable potentials are positioned.

Next we derive the \( r \)-coefficients from
\[
r_m^{(a)} = \sum_n g_{mn}^{-1} \frac{\partial \log P_a}{\partial T_n}
\] (3.31)

that ought to be polynomials in the \( \{T_n\}_{n \in \{1,2,3,6\}} \). In fact, we find
\[
\begin{align*}
 r^{(1)} &= (-5T_1 - 24, -6T_2 - 6T_1, -9T_3 - 12T_2 - 24T_1, -4T_1 T_2 - 12T_6 - 24T_2 + 24T_1) \\
 r^{(2)} &= (-6T_1, -10T_2 - 48, -12T_3 - 24T_1, -24T_1^2 - 18T_6 + 48T_3 + 96T_2 + 192T_1 + 576)
\end{align*}
\] (3.32)

From (3.2) and (3.32),(3.33) we obtain the algebraic differential operator
\[
D = -\sum_{m,n} \frac{\partial}{\partial T_m} g_{mn}^{-1} \frac{\partial}{\partial T_n} + \sum_a \gamma_a \sum_m r^{(a)}_m \frac{\partial}{\partial T_m}
\] (3.34)
containing two real coupling constants $\gamma_a$ as free parameters. This operator is exactly solvable in terms of polynomial eigenfunctions

$$Dp_\lambda(T_1, T_2, T_3, T_6) = \lambda p_\lambda(T_1, T_2, T_3, T_6) \quad (3.35)$$

One can show that $D$ possesses a flag of invariant polynomial spaces $\{V_N\}$

$$DV_N \subset V_N \quad (3.36)$$

$$V_N = \text{span}\{T_1^{n_1}T_2^{n_2}T_3^{n_3}T_6^{n_6}, n_i \in \mathbb{Z}_+, n_1 + 2n_2 + 3n_3 + 4n_6 \leq N\} \quad (3.37)$$

Thus the eigenfunctions $\{p_\lambda\}$ can be calculated by linear algebra.

On the other hand we know from the general scheme that $D$ corresponds by a ”gauge transformation” [1] to the Schrödinger operator in $\mathbb{R}_4$

$$\mathcal{H} = -\Delta + W \quad (3.39)$$

with a Laplacian $\Delta$ and a potential $W$

$$W = \frac{1}{4} \sum_{a,b} (\gamma_a \gamma_b - \frac{1}{4}) R_{ab} \quad (3.40)$$

$$R_{ab} = \sum_{m,n} g_{mn} \frac{\partial \log P_a}{\partial T_m} \frac{\partial \log P_b}{\partial T_n}$$

$$= \sum_n r_n^{(a)} \frac{\partial}{\partial T_n} \log P_b \quad (3.41)$$

In Cartesian coordinates we can for $F_4$ express

$$R_{ab} = \rho_{ab} \sum_{\alpha \in \Omega^+_a} [\sin \frac{1}{2}(\alpha, x)]^{-2} + C_{ab} \quad (3.42)$$

where

$$\rho_{11} = +1, \quad C_{11} = -28$$
$$\rho_{22} = +2, \quad C_{22} = -56$$
$$\rho_{12} = \rho_{21} = 0, \quad C_{12} = C_{21} = -36 \quad (3.43)$$

can be determined by an asymptotic analysis.

In this fashion we have derived the Sutherland model for $F_4$.

**References**

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A Tables

Table 1: Orbits, generating vectors and cardinals for $F_4$

| $\Omega$ | $\sum n_ie_i$ | $\#\Omega$ |
|----------|---------------|-------------|
| $\Omega_1$ | $e_1$ | 24 |
| $\Omega_2$ | $e_1 + e_2$ | 24 |
| $\Omega_3$ | $e_1 + e_2 + e_3$ | 96 |
| $\Omega_4$ | $2e_1, e_1 + e_2 + e_3 + e_4$ | 24 |
| $\Omega_5$ | $2e_1 + e_2$ | 144 |
| $\Omega_6$ | $2e_1 + e_2 + e_3$ | 96 |
| $\Omega_7$ | $2e_1 + e_2 + e_3 + e_4$ | 192 |
| $\Omega_8$ | $2e_1 + 2e_2$ | 24 |
| $\Omega_{9,1}$ | $2e_1 + 2e_2 + e_3$ | 288 |
| $\Omega_{9,2}$ | $3e_1$ | 24 |
| $\Omega_{10}$ | $3e_1 + e_2, 2e_1 + 2e_2 + e_3 + e_4$ | 144 |
| $\Omega_{11}$ | $3e_1 + e_2 + e_3$ | 288 |
| $\Omega_{12}$ | $2e_1 + 2e_2 + 2e_3, 3e_1 + e_2 + e_3 + e_4$ | 96 |
| $\Omega_{13,1}$ | $3e_1 + 2e_2$ | 144 |
| $\Omega_{13,2}$ | $2e_1 + 2e_2 + 2e_3 + e_4$ | 192 |
| $\Omega_{14}$ | $3e_1 + 2e_2 + e_3$ | 192 |
| $\Omega_{15}$ | $3e_1 + 2e_2 + e_3 + e_4$ | 576 |
| $\Omega_{16}$ | $2e_1 + 2e_2 + 2e_3 + 2e_4, 4e_1$ | 24 |
| $\Omega_{17,1}$ | $3e_1 + 2e_2 + 2e_3$ | 288 |
| $\Omega_{17,2}$ | $4e_1 + e_2$ | 144 |
| $\Omega_{18,1}$ | $3e_1 + 3e_2$ | 24 |
| $\Omega_{18,2}$ | $4e_1 + e_2 + e_3, 3e_1 + 2e_2 + 2e_3 + e_4$ | 288 |
| $\Omega_{19,1}$ | $3e_1 + 3e_2 + e_3$ | 288 |
| $\Omega_{19,2}$ | $4e_1 + e_2 + e_3 + e_4$ | 192 |
| $\Omega_{20}$ | $4e_1 + 2e_2, 3e_1 + 3e_2 + e_3 + e_4$ | 144 |
| $\Omega_{21,1}$ | $4e_1 + 2e_2 + e_3$ | 576 |
| $\Omega_{21,2}$ | $3e_1 + 2e_2 + 2e_3 + 2e_4$ | 192 |
| $\Omega_{22}$ | $3e_1 + 3e_2 + 2e_3, 4e_1 + 2e_2 + e_3 + e_4$ | 288 |
| $\Omega_{23}$ | $3e_1 + 3e_2 + 2e_3 + e_4$ | 576 |
| $\Omega_{24}$ | $4e_1 + 2e_2 + 2e_3$ | 96 |
Table 2: $T_{n,a}$ expressed as polynomial in $T_1, T_2, T_3, T_6$ for $F_4$

\[
T_4 = T_1^2 - 2T_3 - 6T_2 - 8T_1 - 24
\]
\[
T_5 = T_2T_1 - 3T_3 - 6T_1
\]
\[
T_7 = T_3T_1 - 4T_2T_1 - 8T_1^2 - 3T_6 + 22T_3 + 36T_2 + 80T_1 + 192
\]
\[
T_8 = T_2^2 - 6T_1^2 - 2T_6 + 12T_3 + 28T_2 + 48T_1 + 120
\]
\[
T_{9,1} = T_3T_2 - 3T_3T_1 + 8T_2T_1 + 24T_1^2 + 9T_6 - 60T_3 - 108T_2 - 228T_1 - 576
\]
\[
T_{9,2} = T_3^2 - 3T_3T_1 - 3T_2T_1 + 3T_6 - 21T_3 - 36T_2 - 99T_1 - 192
\]
\[
T_{10} = T_2T_1^2 - 2T_3T_2 - 8T_2T_1 - 6T_2^2 - 3T_6 - 30T_2
\]
\[
T_{11} = T_6T_1 - 2T_3T_2 + 3T_3T_1 - 8T_2T_1 - 24T_1^2 - 9T_6 + 63T_3 + 108T_2 + 240T_1 + 576
\]
\[
T_{12} = -4T_3T_1^2 - 8T_1^3 - 2T_6T_1 + 8T_3T_2 + 12T_2^2 + 24T_3T_1 + 56T_2T_1 + T_3^2 + 60T_2^2
+ 40T_3 + 120T_2 + 288T_1 + 288
\]
\[
T_{13,1} = T_2^2T_1 - T_1^3 - 2T_3T_1 - T_3T_2 + 15T_3T_1 + 19T_2T_1 + 24T_1^2 - 9T_6
+ 63T_3 + 108T_2 + 354T_1 + 576
\]
\[
T_{13,2} = T_3T_1^2 - 2T_3^2 - T_6T_1 - 6T_3T_2 - 6T_3T_1 - 8T_2T_1 - 16T_1^2 - 6T_6
+ 20T_3 + 72T_2 + 152T_1 + 384
\]
\[
T_{14} = 8T_2T_1^2 + 24T_3^2 + T_6T_2 + 6T_6T_1 - 72T_3T_1 - 136T_2T_1 - 20T_2^2 - 16T_3T_2 - 3T_3
- 144T_1^2 + 22T_6 - 192T_3 - 400T_2 - 1152T_1 - 1536
\]
\[
T_{15} = T_3T_2T_1 - 3T_3T_1^2 - 4T_3^2T_1 - 8T_2T_1^2 - 3T_2T_1 + 40T_2T_1 + 21T_3T_1
+ 96T_2T_1 + 6T_3^2 + 36T_2^2 + 24T_1^2 + 9T_6 - 6T_3 - 48T_2 - 240T_1 - 576
\]
\[
T_{16} = T_1^4 - 4T_3T_1^2 - 4T_3T_1^2 + 4T_6T_1 + 2T_3^2 + 8T_3T_2 + 6T_2^2 - 16T_3T_1
- 16T_2T_1 - 76T_1^2 + 12T_6 - 40T_3 - 72T_2 - 416T_1 - 552
\]
\[
T_{17,1} = -2T_3T_2T_1 + 3T_3T_1^2 + 4T_2^2T_1 + 16T_2T_1^2 + 12T_1^3 + T_6T_3 + 6T_6T_2 + 5T_6T_1
- 6T_3^2 - 55T_3T_2 - 72T_2^2 - 51T_3T_1 - 200T_2T_1 - 72T_1^2 + 9T_6 - 120T_3
- 492T_2 - 468T_1 - 576
\]
\[
T_{17,2} = -3T_3T_2T_1 + T_2T_1^3 - 3T_3T_1^2 + 3T_2T_2 - T_6T_1 - 36T_2^2 - 19T_3T_2
- 3T_3T_1 - 92T_2T_1 + 24T_1^2 + 9T_6 - 60T_3 - 300T_2 - 234T_1 - 576
\]
\[
T_{18,1} = T_3^2 - 24T_1^3 - 15T_2T_1^2 - 3T_6T_2 - 6T_6T_1 + 3T_3^2 + 30T_3T_2 + 72T_3T_1 + 54T_2^2
+ 192T_2T_1 + 144T_1^2 - 21T_6 - 192T_3 + 549T_2 + 1152T_1 + 1536
\]
\[
T_{18,2} = T_6T_2^2 - 28T_2T_1^2 - 72T_3^3 - 2T_6T_3 - 9T_6T_2 - 26T_6T_1 + 56T_3T_2 + 216T_3T_1 + 84T_2^2
+ 440T_2T_1 + 9T_6^2 + 432T_1^2 - 84T_6 + 576T_3 + 1308T_2 + 3456T_1 + 4608
\]
\[
T_{19,1} = T_3T_2^2 - T_3T_2T_1 + 8T_2T_1^2 + 4T_2^2T_1 - 3T_3T_1^2 - 2T_6T_3 + 3T_6T_2 - 3T_6T_1 - 6T_3^2
- 8T_3T_2 - 36T_2^2 + 18T_3T_1 - 68T_2T_1 + 48T_1^2 + 18T_6 - 69T_3 - 408T_2
- 480T_1 - 1152
\]
\[
T_{19,2} = -8T_3^4 - 4T_2T_3^3 + T_2^2T_1 + 9T_6T_3T_1 + 23T_3T_1^2 + 12T_2^2T_1 - 2T_6T_1^2 + 48T_3T_1^2
+ 48T_3^3 - T_6T_3 - 3T_6T_2 - 7T_6T_1 + 2T_3^2 + 24T_3T_2 + 36T_2^2 + 72T_3T_1 + 244T_2T_1
+ 384T_1^2 + 18T_3 + 192T_2 + 512T_1
\]
\[
T_{20} = -1716T_3 + 48T_6 - 2T_3T_2^2 - 8T_1^2T_2 - 2T_3^2T_6 - 2928T_1 - 636T_3
+ 76T_3T_2 + T_1^2T_2 - 228T_2^2 - 152T_2T_3 + 22T_1T_6 - 264T_1T_3 - 306T_1^2
- 680T_1T_2 + 12T_2T_6 - 27T_3^2 + 24T_1^2T_6 + 120T_1^3 + 4T_3T_6 - 6T_2^3 - 3600 - 6T_1^4
\]
\[
T_{22} = -4608 - 1008T_2 + 111T_6 - 576T_3 - 3456T_1 + 34T_1T_6 + 4T_3T_6 + 8T_3^2 + 16T_2T_3
- 216T_1T_3 - 152T_1T_2 + 88T_1^2T_2 + 56T_1T_2^2 - 2T_1^2T_6 - 8T_1^3T_2 + 15T_2T_6 + T_2T_3^2
- 4T_1^2T_2^2 - 2T_6T_1T_2 + 24T_1T_2T_3 - 432T_1^2 - 9T_3^2 + 36T_2^2 + 72T_1^3 + 12T_2^3
\]

\[
T_{24} = 4320 + 1104T_2 - 152T_6 + 912T_3 + 3648T_1 - 64T_1T_6
- 16T_3T_6 - 8T_3^2 + 96T_2T_3 + 384T_1T_3
+ 128T_1T_2 - 208T_1^2T_2 - 80T_1T_2^2 + 8T_1^2T_6
+ 16T_1^3T_2 - 24T_2T_6 - 2T_2T_3^2 + 4T_2^2T_2
- 48T_1^2T_3 + 4T_6T_1T_2 - 48T_1T_2T_3 + 312T_1^2
+ 48T_3^2 + 12T_2^2 - 192T_1^3 - 8T_2^3 + 12T_1^4 + T_6^2
\]