The mixed Yamabe problem for harmonic foliations

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Abstract

The mixed scalar curvature of a foliated Riemannian manifold, i.e., an averaged mixed sectional curvature, has been considered by several geometers. We explore the Yamabe type problem: to prescribe the constant mixed scalar curvature for a foliation by a conformal change of the metric in normal directions only. For a harmonic foliation, we derive the leafwise elliptic equation and explore the corresponding nonlinear heat type equation. We assume that the leaves are compact submanifolds and the manifold is fibered instead of being foliated, and use spectral parameters of certain Schrödinger operator to find solution, which is attractor of the equation.

Keywords: foliation, Riemannian metric, harmonic, mixed scalar curvature, biconformal, Schrödinger operator, parabolic PDE, attractor

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1 Introduction

Geometrical problems of prescribing curvature-like invariants (e.g. the scalar curvature and the mean curvature) of manifolds and foliations are popular for a long time, see [4] [14] [17]. There are many proofs of positive answer to the Yamabe problem: given a closed Riemannian manifold \((M, g)\) of \(\dim M \geq 3\), find a metric conformal to \(g\) with constant scalar curvature. This geometrical problem is expressed in terms of the existence and multiplicity of solutions of a given elliptic PDE in the Riemannian manifold, see [1] [10]. Several authors developed an analog of the problem for CR manifolds, see [7], and its generalization to contact (real or quaternionic) manifolds. The problem when a Riemannian metric of constant scalar curvature can be produced on a warped product manifolds has been studied in several articles, see [6].

Let a Riemannian manifold \((M, g)\) be endowed with a \(p\)-dimensional foliation \(\mathcal{F}\). Denote by \(\mathcal{D}^\perp (\dim \mathcal{D}^\perp = n)\) the orthogonal distribution (or the normal subbundle) of the tangent bundle \(TM\). In [3], a tensor calculus, adapted to the orthogonal splitting

\[ TM = TF + D^\perp \] (1)

is developed to study the geometry of both the distributions and the ambient manifolds. We have \(g = g_F + g^\perp\), where \(g^\perp (X, Y) := g(X^\perp, Y^\perp)\) and \((\cdot)^\perp\) is the projection of \(TM\) onto \(\mathcal{D}^\perp\). Obviously, biconformal metrics \(\tilde{g} = v^2g_F + u^2g^\perp\) \((u, v > 0)\) preserve [11] and extend the class of conformal metrics (i.e., \(u = v\)). Biconformal metrics (e.g. doubly-twisted products, introduced by Ponge and Reckziegel in [8]) have many applications in differential geometry, relativity, quantum-gravity, etc, see [6]. The \(D^\perp-\) or \(TF-\)conformal metrics correspond to \(v \equiv 1\) or \(u \equiv 1\), see [13] – [16].
The components of the curvature of a foliation can be tangential (or structural), transversal, and mixed. The tangential geometry of a foliation is the geometry infinitesimally modeled by the tangent distribution to the leaves, while the transversal geometry – by the orthogonal distribution $D^\perp$. The transversal scalar curvature is well studied for Riemannian foliations, e.g., the “transversal Yamabe problem”, see [19]. The mixed scalar curvature, $S_{\text{mix}}$, for foliated (sub)manifolds has been considered by several geometers, see [2, 9], but its constancy (so called “mixed Yamabe problem”) is less studied. In [15, 16], we prescribed the sign of $S_{\text{mix}}$ using flows of $D^\perp$-conformal metrics. In this paper we explore the following Yamabe type problem: Given a foliation $\mathcal{F}$ on a Riemannian manifold $(M, g)$, find a $D^\perp$-conformal metric $\tilde{g}$ with leafwise constant mixed scalar curvature.

For a general foliation, the topology of the leaf through a point can change dramatically with the point; this gives many difficulties in studying leafwise parabolic and elliptic equations. Therefore, in the paper (at least in main results) we assume that

\begin{align*}
(a) \quad & \text{the leaves of } \mathcal{F} \text{ are compact and orientable,} \\
(b) \quad & \text{the manifold } M \text{ is fibered (instead of being foliated).}
\end{align*}

The main results of the paper are the following.

**Theorem 1.** Let $\mathcal{F}$ (dim $\mathcal{F} > 1$) be a harmonic and nowhere totally geodesic foliation on a closed Riemannian manifold $(M, g)$ with conditions (2). Then there exists a $D^\perp$-conformal metric $\tilde{g}$ with leafwise constant mixed scalar curvature.

**Theorem 2.** Let $\mathcal{F}$ (dim $\mathcal{F} > 1$) be a totally geodesic foliation on a closed Riemannian manifold $(M, g)$ with conditions (3) and integrable normal distribution. Then there exists a $D^\perp$-conformal metric $\tilde{g}$ with leafwise constant mixed scalar curvature.

Proofs are based on results of Sect. 2 (variation formulae for various geometrical quantities under $D^\perp$-conformal change of a metric), Sect. 3 (Proposition 3), and Sect. 4 (attractor of the nonlinear heat equation on a closed manifold).

2 Preliminaries

**The mixed scalar curvature.** Denote by $R^\nabla(X, Y) = \nabla_Y \nabla_X - \nabla_{\nabla_X Y} + \nabla_{[X, Y]}$ the curvature operator of the Levi-Civita connection $\nabla$. The sectional curvature $K(X, Y) = g(R^\nabla(X, Y)X, Y)$ where $X \in T\mathcal{F}$, $Y \in D^\perp$ are unit vectors, is called *mixed*. It regulates (through the Jacobi equation) the deviation of leaves along the leaf geodesics. Foliations with constant mixed sectional curvature plays important role in differential geometry, but are far from being classified. Examples are $k$-nullity foliations on Riemannian manifolds which are totally geodesic, relative nullity foliations, which determine a ruled structure of submanifolds in space forms, foliations produced by Reeb vector field on Sasakian manifolds, etc. Totally geodesic foliations on complete manifolds with $K_{\text{mix}} \equiv 0$ split. For a $k$-dimensional totally geodesic foliation with $K_{\text{mix}} \equiv 1$ on a closed manifold $M^{n+k}$, we have the Ferus’s inequality $k < \rho(n)$, where $\rho(n) - 1$ is the number of linear independent vector fields on a sphere $S^{n-1}$, see [12].

The *mixed scalar curvature* is an averaged mixed sectional curvature,

$$S_{\text{mix}} = \sum_{j=1}^{n} \sum_{a=1}^{p} K(E_j, E_a),$$

and is independent of the choice of a local orthonormal frame $\{E_j, E_a\}_{j \leq n, a \leq p}$ of $TM$ adapted to $D^\perp$ and $T\mathcal{F}$, see [11] [12] [18]. If either $D^\perp$ or $T\mathcal{F}$ is one-dimensional and tangent to a unit vector field $N$, then $S_{\text{mix}}$ is the Ricci curvature in the $N$-direction.
Let $\mathcal{X}_M$ be the module over $C^\infty(M)$ of all vector fields on $M$, and $\mathcal{X}^\perp$ and $\mathcal{X}^\top$ the modules of all vector fields on $\mathcal{D}^\perp$ and $\mathcal{T}F$, respectively. Extrinsic geometry of a foliation is related to the second fundamental form of the leaves, $h(X,Y) = (\nabla_X Y)^\perp$, where $X, Y \in \mathcal{X}^\top$, and its invariants (e.g., the mean curvature $H = \text{Tr}_g h$). Special classes of foliations such as totally geodesic, $h = 0$ (with the simplest extrinsic geometry); totally umbilical, $h = (H/p) g_F$; and harmonic, $H = 0$, have been studied by many geometers, see survey in [12]. Let $h^\perp$ be the second fundamental form, $H^\perp = \text{Tr}_g h^\perp$ the mean curvature, and $T$ the integrability tensor of $\mathcal{D}^\perp$. We have

\[ 2h^\perp(X,Y) = (\nabla_X Y + \nabla_Y X)^\top, \quad 2T(X,Y) = [X, Y]^\top, \quad X, Y \in \mathcal{X}^\perp. \tag{3} \]

The formula in [18], for foliations reads as

\[ S_{\text{mix}} = \|H^\perp\|^2 - \|h^\perp\|^2 + \|T\|^2 + \|H\|^2 - \|h\|^2 + \text{div}(H^\perp + H). \tag{4} \]

We calculate norms of tensors using local adapted basis as

\[ \|h^\perp\|^2 = \sum_{i,j} \|h^\perp(\mathcal{E}_i, \mathcal{E}_j)\|^2, \quad \|h\|^2 = \sum_{a,b} \|h(E_a, E_b)\|^2, \quad \|T\|^2 = \sum_{i,j} \|T(\mathcal{E}_i, \mathcal{E}_j)\|^2. \]

Next example represents many doubly twisted products with constant $S_{\text{mix}}$.

**Example 1.** (Constant mixed scalar curvature on doubly-twisted products). The **doubly twisted product** of Riemannian manifolds $(B, g_F)$ and $(F, g^\perp)$, is a manifold $M = B \times F$ with the metric $g = v^2 g_F + u^2 g^\perp$ where $v, u \in C^\infty(B \times F)$ are positive functions. It is called the **doubly warped product** of $(B, g_F)$ and $(F, g^\perp)$ if warping functions $v$ and $u$ only depend on the points of $B$ and $F$, respectively.

The leaves $B \times \{y\}$ of a doubly-twisted product $B \times_{(v,u)} F$ and the fibers $\{x\} \times F$ are totally umbilical. We have

\[ h = -(\nabla^\perp \log v) g_F, \quad h^\perp = -(\nabla^\top \log u) g^\perp. \]

By the above, $H = -n \nabla^\perp \log v$, $H^\perp = -p \nabla^\top \log u$, and

\[ \|H\|^2 - \|h\|^2 = (n^2 - n) \|\nabla^\perp v\|^2 / v^2, \quad \|H^\perp\|^2 - \|h^\perp\|^2 = (p^2 - p) \|\nabla^\top u\|^2 / u^2. \]

Next we derive

\[ \text{div } H = -p(\Delta^\top u) / u - (p^2 - p) \|\nabla^\top u\|^2 / u^2, \]

\[ \text{div } H^\perp = -n(\Delta^\perp v) / v - (n^2 - n) \|\nabla^\perp v\|^2 / v^2, \]

where $\Delta^\top$ is the leafwise Laplacian and $\Delta^\perp$ is the fiberwise Laplacian. Substituting in (4) with $T = 0$, we obtain the important formula

\[ S_{\text{mix}} = -n(\Delta^\top u) / u - p(\Delta^\perp v) / v. \]

Let $B$ be a closed manifold. Given a positive function $v \in C^\infty(B \times F)$, define the leafwise Schrödinger operator $\mathcal{H} = -\Delta^\top - \beta \text{ id}$, where $\beta = \frac{2}{n} (\Delta^\perp v) / v$. For any compact leaf, the spectrum of $\mathcal{H}$ is discrete, the least eigenvalue $\lambda_0$ is isolated from other eigenvalues, and the eigenfunction $e_0$ (called the ground state) can be chosen positive. Since $\mathcal{H}(e_0) = \lambda_0 e_0$, we conclude that a doubly-twisted product $B \times_{(v,e_0)} F$ has the leafwise constant mixed scalar curvature equal to $n \lambda_0$. 

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H. Rummel characterized harmonic foliations by existence of an \( F \)-closed differential \( p \)-form that is transverse to \( F \). D. Sullivan’s topological tautness condition is equivalent to the existence of a metric on \( M \) making a foliation harmonic, see [4]. By Lemma 2, \( D^⊥ \)-conformal changes of the metric preserve harmonic foliations.

We focus on the mixed Yamabe problem for harmonic foliations, which amounts to finding a positive solution of the leafwise elliptic equation, see Proposition 2.

\[
-n(\Delta^\perp u + \beta^\perp u) = -2H^\perp(u) + \tilde{S}_{\text{mix}}u + \|h\|^2u^{-1} - \|T\|^2g^{-3},
\]

where \( \beta^\perp = \frac{1}{n}(\|T\|^2 - \|h\|^2 - \tilde{S}_{\text{mix}}) \), and a leafwise constant \( \tilde{S}_{\text{mix}} \) corresponds to a \( D^⊥ \)-conformal metric \( \tilde{g} \). Proposition 1 allows us to reduce (5) to the case of \( H^\perp = 0 \).

**Proposition 1.** Let \( F \) be a foliation on a closed Riemannian manifold \( (M, g) \) with conditions \([2]\). Then there exists a smooth function \( u > 0 \) on \( M \) such that \( H^\perp = 0 \) for the metric \( \tilde{g} = gx^2g^\perp \).

**Proof.** Recall the equality for any \( X, Y \in \mathfrak{X}^\perp \) and \( U, V \in \mathfrak{X}^T \),

\[
g(R^X(U, X)V, Y) = g((\nabla_U C)V - C_V U)(X, Y) + g((\nabla_X A^\perp)Y - A^\perp_X A^\perp_Y)(U, V),
\]

see [12], where the co-nullity operator \( C : TF \times TM \to D^⊥ \) is defined by \( C_U(X) = -(\nabla_X U)^\perp \) for \( U \in \mathfrak{X}^\perp \) and \( X \in \mathfrak{X}_M \). Note that

\[
\sum_j g((\nabla_U C)V(\xi_j), \xi_j) = \sum_j \nabla_U(g(C_V(\xi_j), \xi_j)) = \nabla_U(g(\sum_j h(\xi_j, \xi_j), V)) = g(\nabla_U H^\perp, V).
\]

Thus, tracing (3) over \( D \) and taking the antisymmetric part, we obtain \( d^\perp H^\perp = 0 \), where the 2-form \( d^\perp H^\perp \) is defined by

\[
2d^\perp H^\perp(U, V) = g(\nabla_U H^\perp, V) - g(\nabla_V H^\perp, U) \quad (U, V \in \mathfrak{X}^T).
\]

Then we apply Lemma 1 given below. \(\square\)

**Lemma 1** (see Theorem 1.1 in [14]). Let \( F \) be a foliation on a closed Riemannian manifold \( (M, g) \) with conditions \([2]\), and \( d^\perp H^\perp = 0 \). Then the Cauchy’s problem

\[
\partial_t g = -(2/p)(\text{div}^\perp H^\perp)g^\perp, \quad g_0 = g,
\]

has a unique solution \( g_t \) (\( t \geq 0 \)) that converges as \( t \to \infty \) to a metric with \( H^\perp = 0 \).

**\( D^⊥ \)-conformal change of a metric.** We shall find how various geometrical quantities are transformed under \( D^⊥ \)-conformal change of a metric. The Weingarten operator \( A^\perp_U \) of \( D^⊥ \) and the skew-symmetric operator \( T^\perp_U \), where \( U \in \mathfrak{X}^T \), are given by

\[
g(A^\perp_U(X), Y) = g(h^\perp(X, Y), U), \quad g(T^\perp_U(X), Y) = g(T(X, Y), U).
\]

**Lemma 2.** Given a foliation \( F \) on \( (M, g = gx + g^\perp) \), and \( \phi \in C^1(M) \), define a new metric \( \tilde{g} = gx + e^{2\phi}g^\perp \). Then

\[
\tilde{h}^\perp = e^{-2\phi}h^\perp, \quad \tilde{H}^\perp = e^{-2\phi}H^\perp \quad (7)
\]

\[
\tilde{h}^\perp = e^{2\phi}(h^\perp - (\nabla^\perp \phi)g^\perp), \quad \tilde{H}^\perp = H^\perp - n \nabla^\perp \phi \quad (8)
\]

\[
\tilde{A}^\perp_U = A^\perp_U - U(\phi) \text{id}^\perp, \quad \tilde{T}^\perp_U = e^{-2\phi}T^\perp_U \quad (U \in \mathfrak{X}^T) \quad (9)
\]

Hence, \( D^⊥ \)-conformal variations preserve total umbilicity, harmonicity, and total geodesy of \( F \), and preserve total umbilicity of the normal distribution \( D^⊥ \).
Proof. The Levi-Civita connection $\nabla$ of a metric $g$ is given by the known formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y)$$
$$+ g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) \quad (X, Y, Z \in \mathfrak{x}_M).$$  \hfill (10)

Formula (7) follows from (10):

$$2e^{2\phi}g(\tilde{\nabla}_U V, X) = 2\tilde{g}(\tilde{\nabla}_U V, X)$$
$$= -Xg(U, V) - g([U, X], V) - g([V, X], U) = 2g(\nabla_U V, X).$$

We deduce (7) using $\tilde{H}^\top = e^{-2\phi} \sum_a h(E_a, E_a) = e^{-2\phi} H$. From $\tilde{T} = T$ and

$$g(T^\perp_U^\perp(X, Y)) = e^{-2\phi}\tilde{g}(T^\perp_U^\perp(X, Y)) = e^{-2\phi}\tilde{g}(T(X, Y), U)$$
$$= e^{-2\phi}g(T(X, Y), U) = e^{-2\phi}g(T^\perp_U^\perp(X, Y))$$

formula (6) follows. By (10), for any $X, Y \in \mathfrak{x}^\perp$ and $U \in \mathfrak{x}^\top$ we have

$$g(\tilde{\nabla}_X Y, U) = e^{2\phi}g(\nabla_X Y, U) - e^{2\phi}U(\phi)g(X, Y) - (e^{2\phi} - 1)g(T(X, Y), U).$$  \hfill (11)

From this, skew-symmetry of $T$ and (3), we deduce (11). Then we get (8) using

$$e^{2\phi}g(\tilde{A}^\perp_U(X), Y) = \tilde{g}(\tilde{A}^\perp_U(X), Y) = \tilde{g}(\tilde{h}^\perp(X, Y), U)$$
$$= e^{2\phi}(g(A^\perp_U(X), Y) - U(\phi)g(X, Y)).$$

Similarly, we prove (12):

$$e^{2\phi}g(\tilde{T}^\perp_U^\perp(X), Y) = \tilde{g}(\tilde{T}^\perp_U^\perp(X), Y) = \tilde{g}(\tilde{T}(X, Y), U) = g(T^\perp_U^\perp(X, Y)).$$

The orthonormal bases of $D^\perp$ in both metrics are related by $\tilde{E}_j = e^{-\phi}E_j$. To show this we calculate for any $j \leq n$,

$$1 = \tilde{g}(\tilde{E}_j, \tilde{E}_j) = e^{2\phi}g(e^{-\phi}E_j, e^{-\phi}E_j) = g(E_j, E_j).$$  \hfill (12)

By (3), we have

$$\tilde{h}^\perp(\tilde{E}_j, \tilde{E}_j) = e^{-2\phi}\tilde{h}^\perp(E_j, E_j) = h^\perp(E_j, E_j) - (\nabla^\top X)g(E_j, E_j).$$

From this and definition $H^\perp = \text{Tr}_g h^\perp$, the equality (9) follows.

Remark 1. By Lemma 2 for a leafwise constant $\phi$ we obtain $\tilde{h}^\perp = e^{2\phi}h^\perp$ and $\tilde{H}^\perp = H^\perp$. Hence, $D^\perp$-scalings of $g$ preserve harmonicity and total geodesy of $D^\perp$.

Proposition 2. The mixed scalar curvature of a harmonic foliation $\mathcal{F}$ under $D^\perp$-conformal change of a metric $g = g_{\mathcal{F}} + u^2 g^\perp$, where $u > 0$ is a smooth function, is transformed due to the formula

$$(S_{\text{mix}} - \tilde{S}_{\text{mix}}) u = n \Delta^\top u - 2H^\perp(u) + \|h\|^2_u(u^{-1} - u) - \|T\|^2_u(u^{-3} - u).$$  \hfill (13)

If, in particular, $u$ is leafwise constant (i.e., $\tilde{g}$ is a $D^\perp$-scaling of $g$), then we have

$$\tilde{S}_{\text{mix}} = S_{\text{mix}} - (c^2 - 1)\|h\|^2_u + (c^4 - 1)\|T\|^2_u.$$
Proof. By Lemma \(\ref{lem:estimate} \) we have
\[
\|\tilde{h} \|^2_g = e^{-2\phi} \|h\|^2_g, \quad \|\tilde{T}\|^2_g = e^{-4\phi} \||T||^2_g,
\]
\[
\|\tilde{h} \|^2_g = \|h\|^2_g + n\|\nabla^T \phi\|^2_g - 2 H^1(\phi),
\]
\[
\|\tilde{H} \|^2_g = \|H^1\|^2_g + 2n^2 \|\nabla^T \phi\|^2_g - 2n H^1(\phi), \quad \text{div}^T \tilde{H}^1 = \text{div}^T H^1 - n \Delta^T \phi.
\]
Indeed, formulae for \(\|\tilde{h} \|^2_g\) and \(\|\tilde{T}\|^2_g\) follow from
\[
\|\tilde{h} \|^2_g = \sum_{a,b,i} \tilde{g}(\tilde{h}^T (E_a, E_b), \tilde{E}_i)^2 = e^{-4\phi} \sum_{a,b,i} g(e^{-2\phi} h(E_a, E_b), e^{-4\phi} \tilde{E}_i)^2
\]
\[
= e^{-2\phi} \sum_{a,b,i} g(h(E_a, E_b), \tilde{E}_i)^2 = e^{-2\phi} \|h\|^2_g,
\]
\[
\|\tilde{T}\|^2_g = \sum_{a,i} \tilde{g}(\tilde{T}(\tilde{E}_i), E_a)^2 = \sum_{a,b,i} g(T(e^{-\phi} \tilde{E}_i, e^{-\phi} \tilde{E}_j), E_a)^2
\]
\[
= e^{-4\phi} \sum_{a,b,i} g(T(\tilde{E}_i, \tilde{E}_j), E_a)^2 = e^{-4\phi} \|T\|^2_g.
\]
Formula for \(\text{div}^T \tilde{H}^1\) follows from \(g(\tilde{\nabla}_a U, E_a) = g(\nabla_a U, E_a)\) for \(U \in \mathbb{X}^T\) and
\[
\text{div}^T \tilde{H}^1 = \sum_a \tilde{g}(\tilde{\nabla}_a \tilde{H}^1, E_a) \text{div}^T H^1 - n \text{div}^T (\nabla^T \phi).
\]
From
\[
\|\tilde{h} \|^2_g = \sum_{a,i,j} \tilde{g}(\tilde{h}^1 (\tilde{E}_i, \tilde{E}_j), E_a)^2
\]
\[
= \sum_{a,i,j} (g(h^1 (E_i, E_j) - (\nabla^T \phi) g(E_i, E_j), E_a))^2
\]
\[
= \|h^1\|^2_g - 2g(H^1, \nabla^T \phi) + n\|\nabla^T \phi\|^2_g,
\]
\[
\|\tilde{H} \|^2_g = \|H^1\|^2_g - 2n \|H^1, \nabla^T \phi\) + 2n^2 \|\nabla^T \phi\|^2_g
\]
the formulae for \(\|\tilde{h} \|^2_g\) and \(\|\tilde{H} \|^2_g\) follow. Then, using (14), \(\tilde{E}_i = e^{-\phi} E_i\), and
\[
\tilde{S}_{\text{mix}} = \sum_i \tilde{r}(\tilde{E}_i, \tilde{E}_i) = e^{-2\phi} \sum_i \tilde{r}(E_i, E_i),
\]
we obtain the formula
\[
\tilde{S}_{\text{mix}} = S_{\text{mix}} - n (\Delta^T \phi + \|\nabla^T \phi\|^2_g)
\]
\[
+ 2H^1(\phi) + (e^{-4\phi} - 1)\|T\|^2_g - (e^{-2\phi} - 1)\|h\|^2_g.
\]
Substituting \(\phi = \log u\) and \(\nabla^T \phi = u^{-1}\nabla^T u\), \(\Delta^T \phi = u^{-1} \Delta^T u - u^{-2} \|\nabla^T u\|^2_g\) into (14) yields the required formula (13), which is equivalent to (5).

3 Proof of main results

The mixed Yamabe problem for harmonic foliations amounts to finding a positive solution of (5) for a leafwise constant \(\tilde{S}_{\text{mix}}\). Proposition \(\ref{prop:existence}\) allows us to assume \(H^1 = 0\). In this case, one may associate with (5) the leafwise parabolic equation
\[
\partial_t u - \Delta^T u - (\beta^T + \tilde{S}_{\text{mix}}/n) u = (\|h\|^2_g/n) u^{-1} - (\|T\|^2_g/n) u^{-3},
\]
where $\beta^T = \|T\|^2/n - \|h\|^2/n - S_{\text{mix}}$. We shall study asymptotic behavior of solutions to \((16)\) using spectral parameters of the leafwise Schrödinger operator

$$\mathcal{H}^T = -\Delta^T - \beta^T \text{id}.$$  

The spectrum $\sigma(\mathcal{H}^T)$ on a compact leaf $F$ is an infinite sequence of real eigenvalues $\lambda_0^T \leq \lambda_1^T \leq \ldots \leq \lambda_j^T \leq \ldots$ counting their multiplicities, and $\lim_{j \to \infty} \lambda_j^T = \infty$. One may fix in $L_2(F)$ an orthonormal basis of eigenfunctions $\{e_j\}$, i.e., $\mathcal{H}^T(e_j) = \lambda_j^T e_j$. Under assumption \((2)\), the leafwise constants $\lambda_j^T$ and functions $\{e_j\}$ on $M$ are smooth, the least eigenvalue $\lambda_0^T$ is isolated and obeys inequalities

$$\lambda_0^T \in [-\max_M \beta^T, -\min_M \beta^T],$$

its eigenfunction $e_0$ (called the ground state) may be chosen positive, see \([15, 16]\).

Assume $h \neq 0$ and $\Phi < n \lambda_0^T$, and consider the functions (compare with Sect. 4),

$$\phi_\pm^T(y) = -(n \lambda_0^T - \Phi) y + \min_M (\|h\|^2 e_0^{-2}) y^{-1} - \max_M (\|T\|^2 e_0^{-4}) y^{-3},$$

$$\phi_+^T(y) = -(n \lambda_0^T - \Phi) y + \max_M (\|h\|^2 e_0^{-2}) y^{-1} - \min_M (\|T\|^2 e_0^{-4}) y^{-3}. \quad (17)$$

If their discriminant $D = \min_M (\|h\|^4 e_0^{-4}) - 4 (n \lambda_0^T - \Phi) \max_M (\|h\|^2 e_0^{-4}) > 0$, each of \((17)\) has four real roots (two of them are positive). Their maximal (positive) roots

$$y_-^T = \left(\frac{\min_M (\|h\|^2 e_0^{-2}) + \min_M (\|h\|^4 e_0^{-4}) - 4 (n \lambda_0^T - \Phi) \max_M (\|T\|^2 e_0^{-4})}{2 (n \lambda_0^T - \Phi)}\right)^{1/2},$$

$$y_+^T = \left(\frac{\max_M (\|h\|^2 e_0^{-2}) + \max_M (\|h\|^4 e_0^{-4}) - 4 (n \lambda_0^T - \Phi) \min_M (\|T\|^2 e_0^{-4})}{2 (n \lambda_0^T - \Phi)}\right)^{1/2},$$

obey the inequalities $y_3^- < y_-^T < y_+^T$, where $y_3^T$ is the maximal root of $(\phi_-^T)'(y)$,

$$y_3^T = \left(\frac{\min_M (\|h\|^4 e_0^{-4}) + 12 (n \lambda_0^T - \Phi) \max_M (\|T\|^2 e_0^{-4})}{2 (n \lambda_0^T - \Phi)}\right)^{1/2}. \quad (18)$$

**Proposition 3.** Let $\mathcal{F}$ (dim $\mathcal{F} > 1$) be a harmonic foliation on a Riemannian manifold $(M, g)$ with conditions \((2)\) and $H^\perp = 0$. Then for any leafwise constant $\Phi : M \to \mathbb{R}$ obeying the inequalities

$$n \lambda_0^T - \frac{\min_M (\|h\|^4 e_0^{-4})}{4 \max_M (\|T\|^2 e_0^{-4})} < \Phi < n \lambda_0^T, \quad (19)$$

there exists a unique $u_*$ in the set $\{\tilde{u} \in C^\infty(M) : \tilde{u} > e_0 y_3^T\}$, such that the mixed scalar curvature of the metric $\tilde{g} = g_\mathcal{F} + u_*^2 g^\perp$ is $\Phi$. Moreover, $y_3^T \leq u_* \leq y_+^T$, and $u_* = \lim_{t \to \infty} u(\cdot, t)$, where $u$ solves \((16)\) with $\bar{S}_{\text{mix}} = \Phi$, does not depend on initial value $u(\cdot, 0) = u_0 > e_0 y_3^T$.

**Proof.** Let $\Phi$ obeys \((18)\) and $u_0 > e_0 y_3^T$. Denote by

$$\beta = \beta^T + \Phi/n, \quad \lambda_0 = \lambda_0^T - \Phi/n, \quad \Psi_1 = \|h\|^2/n, \quad \Psi_2 = \|T\|^2/n.$$  

Then \((18)\) becomes \((2a)\), and \((16)\) with $\bar{S}_{\text{mix}} = \Phi$ becomes \((20)\) with $u_0 \in \mathcal{U}_1$. Thus, the claim follows from Theorem 5.
Corollary 1. Let \( F (\dim F > 1) \) be a harmonic foliation on a Riemannian manifold \((M, g)\) with conditions \(\Phi\), integrable normal subbundle \(\mathcal{D}^\perp\), \(H^\perp = 0\) and \(h \neq 0\). Then for any leafwise constant \(\Phi : M \to \mathbb{R}\) such that \(\Phi < n\lambda_0^\perp\) there exists a unique positive function \(u_* \in C^\infty(M)\) such that

\[
\min_M \left( \frac{\|h\|^2 e_0^{-2}}{n \lambda_0^\perp - \Phi} \right) \leq u_* \leq \max_M \left( \frac{\|h\|^2 e_0^{-2}}{n \lambda_0^\perp - \Phi} \right),
\]

and the mixed scalar curvature of the metric \(\tilde{g} = g_F + u_*^2 g^\perp\) is \(\Phi\).

Proof. This is similar to proof of Proposition \(\Phi\). Since \(\Psi_2 \equiv 0\) and \(\lambda_0 > 0\), we conclude that each of \(\phi_- = -\lambda_0 + \Psi_1^\perp y^{-1}\) and \(\phi_+ = -\lambda_0 + \Psi_1^\perp y^{-1}\) has one positive root \(y_1^- = (\Psi_1^\perp / \lambda_0)^{1/2}\) and \(y_1^+ = (\Psi_1^\perp / \lambda_0)^{1/2}\), see also Example \(2(1c)\).

Proof of Theorem \(\Phi\) By Proposition \(\Phi\) there exists a metric \(g_1, \mathcal{D}^\perp\)-conformal to \(g\), for which \(H^\perp = 0\) (the mean curvature of \(\mathcal{D}^\perp\)). By Lemma \(2\) \(H = 0\) is preserved for \(g_1\). By Proposition \(\Phi\) there exists a metric \(\tilde{g}, \mathcal{D}^\perp\)-conformal to \(g_1\), for which \(S_{\text{mix}}\) is leafwise constant; moreover, \(H = 0\) holds.

Proof of Theorem \(\Phi\) By Proposition \(\Phi\) there exists a \(\mathcal{D}^\perp\)-conformal to \(g\) metric \(g_1\), for which \(H^\perp = 0\). Denote by \(\mathcal{H} = -\Delta^\perp - \beta\) id the leafwise Schrödinger operator on \((M, g_1)\) with potential \(\beta = -\frac{1}{n} S_{\text{mix}} (g_1)\). Let \(\epsilon_0 > 0\) be the ground state of \(\mathcal{H}\) with the least eigenvalue \(\lambda_0^\perp\) (leafwise constant). Since \(T = 0 = h\) and \(H^\perp = 0\), equation \(\Phi\) reads as the eigenproblem

\[\mathcal{H}^\perp (u) = \frac{1}{n} \tilde{S}_{\text{mix}} u.\]

Thus, the metric \(\tilde{g} = g_F + \epsilon_0^2 g_1^\perp\) has \(\tilde{S}_{\text{mix}} = n \lambda_0^\perp\); moreover, the equality \(h = 0\) is preserved for \(\tilde{g}\).

4 Auxiliary results for parabolic equations

Let \((F, g)\) be a closed \(p\)-dimensional Riemannian manifold, e.g., a leaf of a compact foliation. Functional spaces over \(F\) will be denoted without writing \((F)\), e.g., \(L_2\) instead of \(L_2(F)\). Let \(H^l\) be the Hilbert space of differentiable by Sobolev real functions on \(F\) with the inner product \((\cdot, \cdot)_l\) and the norm \(\| \cdot \|_l\). In particular, \(H^0 = L_2\) with the product \((\cdot, \cdot)_0\) and the norm \(\| \cdot \|_0\).

If \(B\) is a Banach space, we denote by \(\| \cdot \|_B\) the norm of vectors in this space. Denote by \(\| \cdot \|_{C^k}\) the norm in the Banach space \(C^k\) for \(1 \leq k < \infty\), and \(\| \cdot \|_C\) for \(k = \infty\). In coordinates \((x_1, \ldots, x_p)\) on \(F\), we have \(\|f\|_{C^k} = \max_{|\alpha| \leq k} \max_{|\beta| \leq n} |D^\alpha f|\), where \(\alpha \geq 0\) is the multi-index of order \(|\alpha| = \sum \alpha_i\) and \(D^\alpha\) is the partial derivative.

Proposition 4 (Scalar maximum principle, see \(\Phi\) Theorem 4.4]). Suppose that \(X_t\) and \(g_t\) are smooth families of vector fields and metrics on a closed manifold \(F\), and \(f \in C^\infty(\mathbb{R} \times [0, T))\).

Let \(u : F \times [0, T) \to \mathbb{R}\) be a \(C^\infty\) supersolution to

\[
\partial_t u \geq \Delta_i u - X_i(u) + f(u,t),
\]

and \(y : [0, T] \to \mathbb{R}\) solve the Cauchy’s problem for ODEs:

\[
y'(t) = f(y(t), t), \quad y(0) = C.
\]

If \(u(\cdot, 0) \geq C\), then \(u(\cdot, t) \geq y(t)\) for all \(t \in [0, T)\).

The nonlinear heat equation. We are looking for stable solutions of the elliptic equation, see \(\Phi\) with \(H^\perp = 0\),

\[
\mathcal{H}(u) = \Psi_1(x) u^{-1} - \Psi_2(x) u^{-3},
\]

where \(\mathcal{H}(u) := -\Delta u - \beta u\) is the Schrödinger operator with domain of definition \(H^2\), and \(\Psi_1 > 0, \Psi_2 \geq 0\) and \(\beta\) are smooth functions.
positive stationary solution, see Fig. 1(a), and has no stationary solutions if 4 |
\beta | \Psi_2 < \Psi_1^2 : y_1 \text{ stable}, y_2 \text{ unstable}

Fig. 1: Example 2: the nonlinear heat equation

**Theorem** (Elliptic regularity, see [1]). If 0 \not\in \sigma(\mathcal{H}), then for any nonnegative k \in \mathbb{Z} we have \mathcal{H}^{-1} : H^k \to H^{k+2}.

One can add a real constant to \beta such that \mathcal{H} becomes invertible in L_2 (e.g., \lambda_0 > 0) and \mathcal{H}^{-1} is bounded in L_2. Since by the Elliptic regularity Theorem with k = 0, we have \mathcal{H}^{-1} : L_2 \to H^2, and the embedding of H^2 into L_2 is continuous and compact, see [1], then the operator \mathcal{H}^{-1} : L_2 \to L_2 is compact. Thus, the spectrum of \mathcal{H} is discrete, i.e., consists of an infinite sequence of real eigenvalues \{\lambda_k\}_{k=0} with finite multiplicities, bounded from below and \lim_{j \to \infty} \lambda_j = \infty. The least eigenvalue, \lambda_0 of \mathcal{H} is simple, and corresponding eigenfunction \phi_0(x) (called the ground state) can be chosen positive, see [13]. To solve (19), we shall look for attractor of the Cauchy’s problem for the nonlinear heat equation,

\[ \frac{\partial u}{\partial t} + \mathcal{H}(u) = \Psi_1(x) u^{-1} - \Psi_2(x) u^{-3}, \quad u(x, 0) = u_0(x) > 0. \]  

(20)

Denote by \mathcal{C}_t = F \times [0, t) the cylinder with the base F. By [1] Theorem 4.51, parabolic equation (20) has a unique smooth solution in \mathcal{C}_{t_0} for some t_0 > 0. The following example shows that (20) may have (i) stationary solutions on a closed manifold F; (ii) attractors when \beta < 0, and no attractors when \beta > 0.

**Example 2.** Let \beta and \Psi_1 > 0, \Psi_2 \geq 0 be real constants.

1. The Cauchy’s problem for ODE, corresponding to (21), is

\[ y' = f(y), \quad y(0) = y_0 > 0, \quad f(u) := \beta u + \Psi_1 u^{-1} - \Psi_2 u^{-3}. \]

(21)

(a) Let \beta < 0 and \Psi_2 > 0. Positive stationary solutions of (21) are the roots of a biquadratic equation \[ y^4 + \Psi_1 y^2 - \Psi_2 = 0 \] if \[ 4 |\beta| \Psi_2 < \Psi_1^2 \], then we have two positive solutions \[ y_1, y_2 = \left( \frac{\Psi_1 \pm (\Psi_1^2 - 4 |\beta| \Psi_2)^{1/2}}{2 |\beta|} \right)^{1/2} \] and \[ y_1 > y_2 \]. The linearization of (21) at the point \[ y_k \] (k = 1, 2) is \[ v' = f'(y_k) v \], where \[ f'(y_k) = -|\beta| (y^3 (y^2 - y_1^2) (y^2 - y_2^2)) y = y_k \]. Hence, \[ f'(y_1) < 0 \] and \[ f'(y_2) > 0 \], and \[ y_1 \] is asymptotically stable, but \[ y_2 \] is unstable. If \[ 4 |\beta| \Psi_2 = \Psi_1^2 \], then (21) has one positive stationary solution, see Fig. 1(a), and has no stationary solutions if \[ 4 |\beta| \Psi_2 > \Psi_1^2 \].

(b) Let \beta > 0 and \Psi_2 > 0. Then the biquadratic equation \[ y^4 + \Psi_1 y^2 - \Psi_2 = 0 \] has one positive root \[ y_1 = \left( \frac{-\Psi_1 + (\Psi_1^2 + 4 \beta \Psi_2)^{1/2}}{2 \beta} \right)^{1/2} \]. We find

\[ f'(y_1) = \beta (y^3 (y^2 - y_1^2) (y^2 + \Psi_2 / (\beta y_1^2))) y = y_1 > 0; \]
hence, $y_1$ is unstable. One may also show that in the case $\beta = 0$, (21) has a unique positive stationary solution, which is unstable.

(1c) Let $\Psi_2 = 0$ and $\Psi_1 > 0$. Then $f(y) = \beta y + \Psi_1 y^{-1}$. If $\beta \geq 0$, then there are no positive stationary solutions. If $\beta < 0$, then $f$ has one positive root $y_1 = (\Psi_1/|\beta|)^{1/2}$. The solution $y_1$ is an attractor since $f'(y_1) = -|\beta| (y^{-1}(y-y_1)(y+y_1))' |_{y=y_1} < 0$.

2. Let $F$ be a circle $S^1$ of length $l$. Then (20) is the Cauchy’s problem

\begin{equation}
   u_t = u_{xx} + f(u), \quad u(x, 0) = u_0(x) > 0 \quad (x \in S^1, \; t \geq 0).
\end{equation}

The stationary equation with $u(x)$ for (22) has the form

\begin{equation}
   u'' + f(u) = 0, \quad u(0) = u(l), \quad u'(0) = u'(l), \quad l > 0.
\end{equation}

Rewrite (23) as the dynamical system

\begin{equation}
   u' = v, \quad v' = -f(u) \quad (u > 0).
\end{equation}

Periodic solutions of (25) correspond to solutions of (24) with the same period. The system (24) is Hamiltonian, since $\partial_u v = \partial_v f(u)$, its Hamiltonian $H(u, v)$ (the first integral) solves

\begin{equation}
   \partial_u H(u, v) = f(u), \quad \partial_v H(u, v) = v. \quad \text{Then } H(u, v) = \frac{1}{2}(v^2 + \beta u^2) + \Psi_1 \ln u + \frac{1}{2} \Psi_2 u^{-2}. \quad \text{The trajectories of (24) belong to level lines of } H(u, v). \quad \text{Consider the cases.}
\end{equation}

(2a) Let $\beta < 0$. Then (24) has two fixed points: $(y_i, 0)$ ($i = 1, 2$) with $y_1 > y_2$. To clear up the type of fixed points, we linearize (24) at $(y_i, 0)$,

\begin{equation}
   \eta' = A_i \eta, \quad A_i = \begin{pmatrix} 0 & 1 \\ -f'(y_i) & 0 \end{pmatrix}.
\end{equation}

Since $f'(y_1) < 0$ and $f'(y_2) > 0$, the point $(y_1, 0)$ is a “saddle” and $(y_2, 0)$ is a “center”. The separatrix is $H(u, v) = H(y_1, 0)$, i.e., $v^2 = |\beta|(u^2 - y_1^2) - 2 \Psi_1 \ln(u/y_1) - \Psi_2(u^{-2} - y_1^{-2})$, see Fig. 2(a). The separatrix divides the half-plane $u > 0$ into three simply connected areas. Then $(y_2, 0)$ is a unique minimum point of $H$ in $D = \{(u, v) : H(u, v) < H(y_1, 0), \; 0 < u < y_1\}$. The phase portrait of (24) in $D$ consists of the cycles surrounding the fixed point $(y_2, 0)$, all correspond to non-constant solutions of (23) with various $l$. Other two areas do not contain cycles, since they have no fixed points.

(2b) Let $\beta \geq 0$. Then (24) has one fixed point $(y_1, 0)$ and $f'(y_1) > 0$. Hence, $(y_1, 0)$ is a “center”. Since $(y_1, 0)$ is a unique minimum of $H(u, v)$ in the semiplane $u > 0$, the phase portrait of (24) consists of the cycles surrounding the fixed point $(y_1, 0)$, all correspond to non-constant solutions of (23) with various $l$, see Fig. 2(b).

For $\Psi_2 = 0$ and $\Psi_1 > 0$, the Hamiltonian of (24) is $H(u, v) = \frac{1}{2}(v^2 + \beta u^2) + \Psi_1 \ln u$. Solving $H(u, v) = C$ with respect to $u$ and substituting to (24), we get $u' = \sqrt{-\beta u^2 - 2 \Psi_1 \ln u + 2 C}$.

If $\beta \geq 0$, then (24) has no cycles (since it has no fixed points); hence, (23) has no solutions. If $\beta < 0$, then the separatrix $H(u, v) = H(u_*, 0)$, see (1c), is $v^2 = |\beta|(u^2 - u_*^2) - 2 \Psi_1 \ln(u/u_*)$, (24) has a unique fixed point $(u_*, 0)$ which is a “saddle”. The separatrix divides the half-plane $u > 0$ into four simply connected areas with these lines, see Fig. 1(b). Since each of these areas has no fixed points of (24), the system has no cycles. Hence, $u_*$ is a unique solution of (23).

3. Consider (23) for $\Psi_1 = 0$, $\Psi_2 > 0$ and $l = 2\pi$. Set $p = u'$ and represent $p = p(u)$ as a function of $u$. Then $u'' = dp/du$ and

\begin{equation}
   (p^2)u' = -2\beta u + 2 \Psi_2 u^{-3} \Rightarrow |u'|^2 = C_1 - \beta u^2 - \Psi_2 u^{-2}.
\end{equation}
After separation of variables and integration, we obtain

\[ u = \begin{cases} \sqrt{\frac{C_1}{2\beta} + \frac{1}{2\beta} \sqrt{C_1^2 - 4\beta \Psi_2 \sin(2\sqrt{\beta(x+C_2)})}} & \beta > 0, \\ \sqrt{-\frac{C_1}{2\beta} + \frac{1}{2\beta} \sqrt{C_1^2 + 4\beta \Psi_2 \cosh(2\sqrt{\beta(x+C_2)})}} & \beta < 0. \end{cases} \]

Hence, for \( \beta \leq 0, \) case 2 has no positive solutions, and for \( \beta > 0 \) the solution is \( 2\pi \)-periodic and positive only if
- \( \beta \neq \frac{n\pi}{4} \) (\( n \in \mathbb{N} \)) and \( C_1 = 2(\beta \Psi_2)^{1/2} \); a solution \( u_\ast = (\Psi_2/\beta)^{1/4} \) is unique, or
- \( \beta = \frac{n\pi}{4} \) (\( n \in \mathbb{N} \)); the set of solutions forms a two-dimensional manifold

\[ u_0(C_1, C_2) = \frac{1}{n} \left( 2C_1 + 2(C_1^2 - n^2 \Psi_2)^{1/2} \sin(n(x+C_2)) \right)^{1/2}. \]

**Attractor of the nonlinear heat equation.** Denote by \( \Psi_i^+ = \max_F(\Psi_i e_0^{-2i}) \) and \( \Psi_i^- = \min_F(\Psi_i e_0^{-2i}) \) for \( i = 1, 2 \). Assume that

\[ 0 < \lambda_0 < (\Psi_1^-)^2/(4 \Psi_2^+) \quad (25) \]

and \( \Psi_2^+ > 0 \) (case of \( \Psi_2^- = 0 \) is similar). Each of two functions of variable \( y > 0 \),

\[ \phi_+(y) = -\lambda_0 y + \Psi_1^+ y^{-1} - \Psi_2^- y^{-3}, \quad \phi_-(y) = -\lambda_0 y + \Psi_1^- y^{-1} - \Psi_2^+ y^{-3}, \]

has four real roots, two of which are positive; moreover, \( y_2^- < y_1^- \) and \( y_2^+ < y_1^+ \). Since \( \phi_-(y) < \phi_+(y) \) for \( y > 0 \), we also have \( y_1^- < y_1^+ \).

Denote by \( y_3^- = \left( -\Psi_1^- + \sqrt{(\Psi_1^-)^2 + 12 \Psi_2^+ \lambda_0} \right) / 2\lambda_0 \) a unique positive root of \( \phi'_-(y) \). Clearly, \( y_3^- \in (y_2^-, y_1^-) \). Notice that \( \phi_-(y) > 0 \) for \( y \in (y_2^-, y_1^-) \) and \( \phi_-(y) < 0 \) for \( y \in (0, \infty) \setminus [y_2^-, y_1^-] \); moreover, \( \phi_-(y) \) increases in \( (0, y_3^-) \) and decreases in \( (y_3^-, \infty) \). The line \( z = -\lambda_0 y \) is asymptotic for the graph of \( \phi_-(y) \) when \( y \to \infty \), and \( \lim_{y \to 0} \phi_-(y) = -\infty \). The function \( \phi'_+(y) \) decreases in \( (0, y_4^-) \) and increases in \( (y_4^-, \infty) \), where \( y_4^- := (6 \Psi_2^+ / \Psi_1^-)^{1/2} > y_3^- \), and \( \lim_{y \to -\infty} \phi'_-(y) = -\lambda_0 \), see Fig. 3. Hence,

\[ \mu^-(\sigma) := -\sup_{y \geq y_3^-} \phi_-(y) = \min\{|\phi_-(y_1^- - \sigma)|, \lambda_0\} > 0 \quad (26) \]

for \( \sigma \in (0, y_1^- - y_3^-) \). Similar properties have \( y_3^+, y_4^+ \) and \( \mu^+(\sigma) \) defined for \( \phi'_+(y) \).
Lemma 3. Let \( y(t) \) be the solution of Cauchy’s problem
\[
y' = \phi_-(y), \quad y(0) = y_0^- > 0.
\] (27)

(i) If \( y_0^- > y_2^- \) then \( \lim_{t \to \infty} y(t) = y_1^- \). Furthermore, if \( y_0^- \in (y_2^-, y_1^-) \) then \( y(t) \) is increasing and if \( y_0^- > y_1^- \) then \( y(t) \) is decreasing.

(ii) If \( y_0^- \geq y_1^- - \varepsilon \) for some \( \varepsilon \in (0, y_1^- - y_3^-) \) then
\[
|y(t) - y_1^-| \leq |y_0^- - y_1^-| e^{-\mu^-(\varepsilon) t}.
\] (28)

Similar claims are valid for Cauchy’s problem \( y' = \phi_+(y), \quad y(0) = y_0^+ > 0 \).

Proof. (i) Assume that \( y_0^- \in (y_2^-, y_1^-) \). Since \( \phi_-(y) \) is positive in \( (y_2^-, y_1^-) \), \( y(t) \) is increasing. The graph of \( y(t) \) cannot intersect the graph of the stationary solution \( y_1^- \); hence, the solution \( y(t) \) exists and is continuous on the whole \( [0, \infty) \), and it is bounded there. There exists \( \lim_{t \to \infty} y(t) \), which coincides with \( y_1^- \), since \( y_1^- \) is a unique solution of \( \phi(y) = 0 \) in \( (y_2^-, \infty) \).

The case \( y_0^- > y_1^- \) is treated similarly. Notice that if \( y_0^- \in (y_2^-, y_1^-) \) then \( y(t) \) is increasing, and if \( y_0^- > y_1^- \) then \( y(t) \) is decreasing.

(ii) For \( y_0^- \geq y_1^- - \varepsilon \), where \( \varepsilon \in (0, y_1^- - y_3^-) \), denote \( z(t) = y_1^- - y(t) \). We obtain from (27), using definition of \( \mu^-(\varepsilon) \) and the fact that \( \phi_-(y_1^-) = 0 \),
\[
(z^2)' = 2 z z' = 2z^2 \int_0^1 \phi_-'(y + \tau z) d\tau \leq -2 \mu^-(\varepsilon) z^2.
\]

This differential inequality implies (28). The case \( y_0^- > y_1^- \) is treated similarly. \( \square \)

Under assumption (24), define closed in \( C \) nonempty sets \( U_{2^\varepsilon} \subset U_1 \) with \( \varepsilon \in (0, y_1^- - y_3^-) \) and \( \eta > 0 \) by
\[
U_1^- : = \{ u_0 \in C : u_0/e_0 \geq y_1^- - \varepsilon \},
U_{2^\varepsilon}^- : = \{ u_0 \in C : y_1^- - \varepsilon \leq u_0/e_0 \leq y_1^- + \eta \}.
\]

Then, \( U_1^- \subset U_1 \), where \( U_1 = \{ u_0 \in C : u_0/e_0 > y_3^- \} \) is the open in \( C \) set.

Proposition 5. Let (24) holds. Then Cauchy’s problem (24) with \( u_0 \in U_1^- \) for some \( \varepsilon \in (0, y_1^- - y_3^-) \), admits a unique global solution. Furthermore, the sets \( U_1^- \) and \( U_{2^\varepsilon}^- (\eta > 0) \) are invariant for the corresponding to (24)1 semigroup of operators \( S_t : u_0 \to u(\cdot, t) (t \geq 0) \) in \( C_\infty \).
By the above, the solution \( u(\cdot, t) \) \((t \geq 0)\) solves (20) with \( u_0 \in \mathcal{U}_1^\varepsilon \) for some \( \varepsilon \in (0, y_1^+ - y_3^-) \). Substituting

\[
\bar{u} = e_0 w
\]

and using \( \Delta e_0 + \beta_0 = -\lambda_0 e_0 \), yields the Cauchy’s problem

\[
\partial_t w = \Delta w + (2 \nabla \log e_0, \nabla w) + f(w, \cdot), \quad w(\cdot, 0) = u_0/e_0 \geq y_1^+ - \varepsilon,
\]

for \( w(x, t) \), where

\[
f(w, \cdot) = -\lambda_0 w + (\Psi_1 e_0^{-2}) w^{-1} - (\Psi_2 e_0^{-4}) w^{-3}.
\]

From (20) we obtain the differential inequalities

\[
\phi_-(w) \leq \partial_t w - \Delta w - (2 \nabla \log e_0, \nabla w) \leq \phi_+(w).
\]

By Proposition 4, applied to the left inequality of (31), and Lemma 3 in the maximal domain \( D_M \) of the existence of the solution \( w(x, t) \) of (20) we obtain the inequality

\[
w(\cdot, t) \geq y_1^- - \epsilon e^{-\mu^-(\epsilon)t} \geq y_1^- - \epsilon > 0,
\]

which implies that \( w(x, t) \) cannot “blowdown” to zero. Since \( \phi_+(w) \leq \Psi_1^+ w^{-1} \), from the right inequality of (31), applying Proposition 4 and Lemma 3 to the right inequality of (31), we get

\[
w(\cdot, t) \leq w_+(t) = \left(\left((u_0^+)^2 - \Psi_1^+/\lambda_0\right) e^{-2\lambda_0 t} + \Psi_1^+ / \lambda_0 \right)^{1/2},
\]

where \( w_+(t) \) solves the Cauchy’s problem for ODE

\[
dw_+/dt + \lambda_0 w_+ = \Psi_1^+ w_+^{-1}, \quad w_+(0) = u_0^+ := \max_F (u_0/e_0).
\]

By the above, the solution \( u(x, t) \) of (20) exists for all \((x, t) \in C_\infty \), and the set \( \mathcal{U}_1^\varepsilon \) is invariant for operators \( S_t : u_0 \to u(\cdot, t) \) \((t \geq 0)\). Assuming \( u_0 \in \mathcal{U}_1^\varepsilon \) and applying again Proposition 4 and Lemma 3, we get

\[
w(\cdot, t) \leq y_1^+ + \eta e^{-\mu^+(\sigma)t}, \quad \sigma \in (0, y_1^+ - y_3^+).
\]

Thus, \( u(\cdot, t) \in \mathcal{U}_1^\varepsilon,\eta \) \((t > 0)\). Hence, also the set \( \mathcal{U}_2^\varepsilon,\eta \) is invariant for all \( S_t \).

For a positive function \( f \in C(M) \) define \( \delta_f := (\min_M f) / (\max_M f) \in (0, 1] \).

**Theorem 3.** Let (20) holds. Then (17) has in \( \mathcal{U}_1 \) a unique solution \( u_* \); moreover, \( u_* = \lim_{t \to \infty} u(\cdot, t) \), where \( u \) solves (20) with \( u_0 \in \mathcal{U}_1 \), and \( y_1^- \leq u_* \leq y_1^+ \). Furthermore, for any \( \varepsilon \in (0, y_1^+ - y_3^-) \), the set \( \mathcal{U}_1^\varepsilon \) is attracted by (20), exponentially fast to the point \( u_* \) in C-norm:

\[
\| u(\cdot, t) - u_* \|_{C(F)} \leq \delta_{\varepsilon_0}^{-1} e^{-\mu^-(\varepsilon)t} \| u_0 - u_* \|_{C(F)} \quad (t > 0, \quad u_0 \in \mathcal{U}_1^\varepsilon).
\]

**Proof.** By Proposition 3, the set \( \mathcal{U}_1^\varepsilon \) is invariant for the semigroup of operators \( S_t : u_0 \to u(\cdot, t) \) \((t \geq 0)\) corresponding to (20), i.e., \( S_t (\mathcal{U}_1^\varepsilon) \subseteq \mathcal{U}_1^\varepsilon \) for \( t \geq 0 \). Take \( w^0_0 \in \mathcal{U}_1^\varepsilon \) \((i = 1, 2)\) and denote by

\[
\bar{w}_i(\cdot, t) = S_t (w^0_i), \quad w_1(\cdot, t) = u_1(\cdot, t)/e_0, \quad w^0_i = u^0_i/e_0.
\]

Using (20) and the equalities

\[
2 \bar{w} \Delta \bar{w} = \Delta (\bar{w}^2) - 2 \| \nabla \bar{w} \|^2, \quad \nabla (\bar{w}^2) = 2 \bar{w} \nabla \bar{w}
\]

with \( \bar{w} = w_2 - w_1 \), we obtain

\[
\partial_t ((w_2 - w_1)^2) = 2 (w_2 - w_1) \partial_t (w_2 - w_1) \leq \Delta ((w_2 - w_1)^2) + \langle 2 \nabla \log e_0, \nabla (w_2 - w_1)^2 \rangle + 2 (f(w_2, \cdot) - f(w_1, \cdot))(w_2 - w_1).
\]
We estimate the last term, using \( w_i \geq y_i^- - \varepsilon > y_3^- \) \((i = 1, 2)\), (20) and (30),
\[
(f(w_2, \cdot) - f(w_1, \cdot))(w_2 - w_1) = (w_2 - w_1)^2 \int_0^1 \partial_{w}f(w_1 + \tau(w_2 - w_1), \cdot) \, d\tau \leq -\mu^- (\varepsilon)(w_2 - w_1)^2.
\]
Thus, the function \( v = (w_2 - w_1)^2 \) satisfies the differential inequality
\[
\partial_t v \leq \Delta v + \langle 2 \nabla \log e_0, \nabla v \rangle - 2 \mu^- (\varepsilon) v.
\]
By Proposition 4, \( v(\cdot, t) \leq v_+(t) \), where \( v_+(t) \) solves the Cauchy’s problem for ODE: \( v_+ = -2 \mu^- (\varepsilon) v_+(t), \quad v_+(0) = \|w_0^0 - w_1^0\|^2_C \). Thus,
\[
\|S_t(w_0^0) - S_t(w_1^0)\|_C \leq \|w_2(\cdot, t) - w_1(\cdot, t)\|_C \cdot \max_F e_0 \\
\leq e^{-\mu^- (\varepsilon)t} \|w_2^0 - w_1^0\|_C \cdot \max_F e_0 \leq \delta_0^1 e^{-\mu^- (\varepsilon)t} \|w_2^0 - w_1^0\|_C.
\]
i.e., the operators \( S_t \) \((t \geq 0)\) for (20) satisfy in \( U_1^\varepsilon \) the Lipschitz condition with respect to \( C\)-norm with the Lipschitz constant \( \delta_0^{-1} e^{-\mu^- (\varepsilon)t} \).

By Proposition 5, for any \( t \geq 0 \) the operator \( S_\tau \) for (20) maps the set \( U_1^\varepsilon \), which is closed in \( C \), into itself, and for \( t > \frac{1}{\mu^- (\varepsilon)} \ln \delta_0^{-1} \) it is a contraction there. Since all operators \( S_\tau \) commute one with another, they have a unique common fixed point \( u_* \) in \( U_1^\varepsilon \). Since \( \varepsilon \in (0, y_1^- - y_3^-) \) is arbitrary, \( u_* \) is a unique common fixed point of the operators \( S_t \) in the set \( U_1 \). For any \( w_0 \in U_1^\varepsilon \) and \( t \geq 0 \), (22) holds. Thus, \( u_* \in C \) is a generalized solution of (19). By the Elliptic Regularity Theorem, \( u_* \in C^\infty \) and it is a classical solution. By Proposition 4 \( \mathcal{U}_{3, \eta} \subset U_1^\varepsilon \) is also \( S_t \)-invariant, hence \( u_* \in \mathcal{U}_{3, \eta} \). Since \( \varepsilon \in (0, y_1^- - y_3^-) \) and \( \eta > 0 \) are arbitrary, we get \( y_1^- \leq u_* \leq y_1^- \).

\[
\square
\]

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