The $\beta$-function of the chiral Gross Neveu model at $O(1/N^2)$.

J.A. Gracey,
Department of Applied Mathematics and Theoretical Physics,
University of Liverpool,
P.O. Box 147,
Liverpool,
L69 3BX,
United Kingdom.

Abstract. We compute the $O(1/N^2)$ correction to the critical exponent $2\lambda = -\beta'(g_c)$ for the chiral Gross Neveu model in arbitrary dimensions by substituting the corrections to the asymptotic scaling forms of the propagators into the Schwinger Dyson equations and solving the resulting consistency equations.
1 Introduction.

The large $N$ expansion has proved to be a useful technique in providing an alternative method of analysing renormalizable quantum field theories which possess $N$ fundamental fields. For example, it was demonstrated in [1] that the (perturbatively massless) two dimensional Gross Neveu models with discrete and continuous chiral symmetry exhibit dynamical mass generation as well as being asymptotically free field theories with this technique. This former property cannot be accessed in conventional perturbation theory and such toy models have proved to be important as a step to understanding similar phenomena in complicated four dimensional gauge theories such as quantum chromodynamics. One of the limitations, however, of a conventional large $N$ analysis is that one cannot proceed much beyond the leading order due in part to the appearance of intractable integrals whose treatment is hindered primarily by the presence of the dynamically generated mass.

Methods to overcome this problem have been developed in [2, 3] and applied initially to the $O(N)$ bosonic $\sigma$ model to solve that theory to $O(1/N^2)$. The technique exploits the properties of the field theory at its $d$-dimensional critical point which is defined to be the non-trivial zero of the $\beta$-function. The simplifying feature of studying a model there is the additional symmetry present. As the $\beta$-function vanishes the Green’s function of the fields are conformally invariant and obey simple power law behaviour which, of course, has a massless form. In essence one is using the conformal bootstrap approach to solve the field theory, [4]. As a consequence, one avoids one of the difficulties of the conventional large $N$ approach in that the previously intractable integrals become calculable using techniques developed to compute massless Feynman diagrams, [5]. This technique, known as uniqueness, is in effect a method of conformal integration in $d$-dimensions. Essentially when the sum of the exponents of bosonic propagators at a 3-vertex in coordinate space sum to the dimension of space time, the vertex is replaced by a triangle of propagators whose exponents are related to the original vertex. This represents an integration within the overall Feynman diagram. In essence it is the Yang Baxter star triangle relation, but is more general since it is valid in arbitrary (fixed) space time dimension. Moreover, as one is now dealing with a scale invariant situation the problem of solving the model becomes one of computing the critical indices or exponents of the fields which will have an anomalous portion in addition to the canonical dimension. The anomalous piece is the part which carries the quantum properties of the model and from the universality criterion of statistical mechanics (see, for
example, $[6]$), it will be a function of the spacetime dimension and the basic parameters of the model, such as $N$ in the case of a large $N$ expandable theory, and can be calculated order by order in $1/N$. The equations which one uses to do this are deduced by representing the Dyson equations of the Green’s functions in the critical region of the theory from which one can deduce self consistent equations whose solution yields the exponents, $[3]$. The analytic expressions which result for the anomalous dimensions can be related to the respective renormalization group functions precisely at the critical point, $[4, 5]$. One important function which characterizes a field theory is the $\beta$-function which governs how the coupling constant behaves with the renormalization scale. Ordinarily one calculates it order by order in powers of the perturbative coupling constant, which is assumed to be small. In the large $N$ critical point approach which we concentrate on here it can, however, be determined by computing the critical exponent $2\lambda = -\beta'(g_c)$ where $g_c$ is the critical point. Whilst this may seem to be an indirect way to proceed it is important to note that the $O(1/N^2)$ corrections to the $\beta$-function of the $O(N)$ bosonic $\sigma$ model could only be deduced in this way, $[3]$.

Following the pioneering work of $[2, 3]$ the method has been extended to analyse models with fermions, $[7-11]$. In particular the $O(N)$ Gross Neveu model has been solved at $O(1/N^2)$ with the fermion and auxiliary field anomalous dimensions, $[4, 5]$, and the $\beta$-function exponent $\lambda$, $[8, 10]$, all having been computed in arbitrary dimensions. Now that matter fields have been successfully incorporated within the critical point approach it is possible to examine other models to the same precision. One such model which is currently relevant and related to the $O(N)$ Gross Neveu model is the chiral Gross Neveu model, $[1, 12]$, which possesses a continuous chiral symmetry. This was first discussed in $[12]$ by Nambu and Jona-Lasinio in four dimensions and later in $[13]$ where the connection with hadronic physics was first introduced. Moreover, four fermi interactions have recently been a subject of intense study as it provides a possible alternative to the Higgs mechanism in the standard model. There a fermion fermion bound state plays the role of the Higgs boson, $[14, 15]$ and so it is important to have as complete a picture as possible of the quantum structure of the model. Therefore, the aim of this paper is to compute the exponent $\lambda$ of the chiral Gross Neveu or Nambu–Jona-Lasinio model at $O(1/N^2)$ which relates to the $\beta$-function. This is one order beyond what has been computed before in this model. One property that the chiral Gross Neveu model shares with other models in two dimensions is that it possesses an exact $S$-matrix, $[16]$, from which
an expression for the exact mass gap has recently been deduced, [14]. The
$S$-matrix has been constructed on the assumption that 3-particle $S$-matrix
elements factorize into the product of the constituent 2-particle elements
which is a consequence of the underlying Yang Baxter triangle relation as-
associated with integrable field theories. Whilst the model is asymptotically
free and therefore possesses the property of dimensional transmutation in
two dimensions, it might appear that one has already determined the $\beta$-
function of the 2-dimensional model. However, such a statement overlooks
the implicit nature of the situation. As far as we are aware, the explicit form
of the 2-dimensional $\beta$-function which can in principle be deduced from the
$S$-matrix spectrum, has never been achieved in detail. In providing new in-
formation on the $O(1/N^2)$ structure of the $\beta$-function here we are putting in
place a means of checking any future attempt at that difficult programme.
However, more importantly, our motivation in the present calculation goes
beyond the two dimensional model for two reasons. First, we are determi-
ning the $d$-dimensional structure of the $\beta$-function by exploiting the conformal
nature of the fixed point and applying the $d$-dimensional generalization of
the star-triangle relation, we are able to deduce new information on the three
dimensional model simultaneously. Our result goes beyond anything which
has been calculated before in this area since we are computing at $O(1/N^2)$
and this is relevant for recent lattice simulations of the three dimensional
four fermi interactions for relatively low values of $N$ which have been a topic
of interest lately, [18].

Further, the anomalous dimensions of all the fields of the theory have
been deduced in [13] and by calculating $\lambda$ we will in fact solve the model com-
pletely to second order since the remaining critical exponents characterizing
the model can be deduced through the hyperscaling relations which have
been checked at $O(1/N)$ in [20] although their consistency merely reflects
the renormalizability of the model. Indeed the work we present builds very
much on the calculation of $\lambda$ in the $O(N)$ case, [4]. Although there are sub-
stantially more Feynman graphs to consider due to the presence of another
field it is possible to make use of results obtained for those graphs in the
continuous chirally symmetric case to minimize the work needed to obtain
new results. This is significant when one realises the basic intractability that
a conventional large $N$ analysis would entail. The previous remarks are im-
portant in stating our second motivation on the need to have $d$-dimensional
results. One feature of four fermi theories with various chiral symmetries is
that they are believed to lie in the same universality class as the Yukawa
models with the same chiral structure. Such equivalences, for example, are
established using the technique of the $\epsilon$-expansion. In essence the renormalization group functions, calculated to several orders in perturbation theory, are expanded in powers of $\epsilon$, where $d = 2 + \epsilon$ or $4 - 2\epsilon$ depending upon which dimension the coupling constant of the model is dimensionless in, and compared with the analogous function in a similar expansion in the other model. As the critical exponents calculated within the large $N$ self consistency approach discussed here encode all orders information on perturbation theory, such $d$-dimensional results have been important in confirming such equivalences in the $O(N)$ case, [18].

The paper is organized as follows. In section 2, we review the basic formalism we need for solving the chiral Gross Neveu model and recall the relevant features of the $O(1/N)$ calculation of $\lambda$, which unlike the bosonic $O(N)$ $\sigma$ model calculation at the same order requires the inclusion of several 2-loop Feynman graphs. We devote section 3 to the derivation of the master equation whose solution will yield $\lambda$ at $O(1/N^2)$. This involves considering several three and four loop graphs in the Dyson equations of two of the fields of the model. The equation is solved in section 4 to obtain an arbitrary dimensional expression for the exponent from which we deduce information on the 3-dimensional model as a simple corollary.

2 Preliminaries.

The lagrangian of the model we will analyse is, [1, 12],

$$L = \bar{\psi}i \gamma^5 \psi^i - \frac{1}{2g^2}(\sigma^2 + \pi^2)$$

(2.1)

where $\sigma$ and $\pi$ are auxiliary bosonic fields which when eliminated from (2.1) yields a model with a four fermi interaction which has a $U(1) \times U(1)$ global chiral symmetry. The fermion field $\psi^i$ lies in an $N$-tuplet, $1 \leq i \leq N$, and the parameter $1/N$ will play the role of the coupling constant in our analysis and we use the convention that $\text{tr} 1 = 2$. In the usual approach to solving (2.1) in the large $N$ expansion, [1], one can perform the path integral over the fermion fields in (2.1) as they appear quadratically in the action. To proceed further one uses the saddle point approximation when $N$ is large to deduce from the effective potential that the true vacuum of the theory is not the perturbative one but that where the $\sigma$ and $\pi$ fields become dynamical. Further, in this vacuum a mass is generated for the fermion field although perturbatively it remains massless, [1]. Equipped with this
structure one normally proceeds to solve the model in the large $N$ expansion by renormalizing the various Green’s functions. Consequently, one can deduce the large $N$ approximation of the $\beta$-function, for example. To go beyond the leading order turns out to be impossible due to the appearance of intractable integrals whose divergence structure, handled by some regularization, cannot be extracted. This is chiefly a consequence of the mass present in the propagators of the $\psi$ and $\sigma$ fields where the form of the latter involves a hypergeometric function of the momentum and mass, [1]. (This non-fundamental structure is consistent with its fermion bound state nature.)

As we have already mentioned in the self consistency approach of [2] it is indeed possible to go beyond the leading order by examining the structure of the field theory in the neighbourhood of its $d$-dimensional critical point defined as the non-trivial zero of the $\beta$-function. There the theory possesses a conformal symmetry and the Green’s functions of the fields obey a simple power law behaviour and there is no mass in the problem. By solving for the critical exponents of the power law one can deduce information on the renormalization group functions through a critical point analysis of the renormalization group equation which takes a solvable and simplified form at criticality, [6]. For this paper, taking $1/N$ as the coupling constant allows us to compute the exponent order by order in $1/N$ by solving the appropriate Dyson equations truncated to the order we are interested in.

We will now review the $O(1/N)$ derivation of $2\lambda = -\beta'(g_c)$ in the self consistency formalism to illustrate the general method and to define the notation of the problem. As the $\sigma$ and $\pi$ fields are dynamical, we take, [2, 19],

\[
\psi(x) \sim \frac{A_f}{(x^2)\alpha}[1 + (x^2)^\lambda A']
\]

\[
\sigma(x) \sim \frac{B}{(x^2)^\beta}[1 + (x^2)^\lambda B']
\]

\[
\pi(x) \sim \frac{C}{(x^2)^\gamma}[1 + (x^2)^\lambda C']
\]

as the asymptotic scaling forms of the propagators of the fields in coordinate space in the critical region as $x \to 0$. The quantities $A$, $B$ and $C$ and $A'$, $B'$ and $C'$ are the respective $x$-independent amplitudes, whilst $\alpha$, $\beta$ and $\gamma$ are the full dimensions of the respective fields. We choose to compute in coordinate space in order to implement various integration techniques which
are easier to apply there but note that one can access the momentum space formulation easily via the Fourier transform

\[
\frac{1}{(x^2)^\alpha} = \frac{a(\alpha)}{2^{2\alpha-\mu}} \int_k \frac{e^{ikx}}{(k^2)^{\mu-\alpha}} \tag{2.5}
\]

valid for all \( \alpha \) where \( a(\alpha) = \Gamma(\mu - \alpha)/\Gamma(\alpha) \) and we set the spacetime dimension \( d \) to be \( d = 2\mu \) for convenience. One can deduce the canonical dimensions of \( \alpha, \beta \) and \( \gamma \) from a dimensional analysis of the action but quantum fluctuations through, say, radiative corrections will always adjust these. To allow for this one introduces the anomalous dimensions in the definitions of the exponents via

\[
\alpha = \mu + \frac{1}{2} \eta \quad , \quad \beta = 1 - \eta - \chi_\sigma \quad , \quad \gamma = 1 - \eta - \chi_\pi \tag{2.6}
\]

where \( \eta \) is the fermion anomalous dimension and \( \chi_\sigma \) and \( \chi_\pi \) are the anomalous dimensions of the 3-vertices of (2.1)

In [7, 19] we considered only the leading order terms of the asymptotic scaling forms in the skeleton Dyson equations which allows one to deduce the exponent \( \eta \) at \( O(1/N^2) \). To determine \( \beta \)-function corrections one has to include the higher order corrections as we have in (2.2)-(2.4) which is similar to previous analyses in other models, [3, 7]. Then to deduce the exponents one writes down a set of equations which represent the Dyson equations in the critical region and solves them self consistently. We have illustrated the Dyson equations we need to consider for \( \lambda_1 \), where \( \lambda = \mu - 1 + \sum_{i=1}^{\infty} \lambda_i/N^i \), in fig. 1. Ordinarily to compute, say, \( \eta_1 \) one truncates the equations at one loop, [7]. However, as was noted in [21] for field theories where the basic field is fermionic one has to include the two loop graphs of the \( \sigma \) and \( \pi \) 2-point functions where each has an \((x^2)^\lambda\) insertion from (2.3)-(2.4) on the \( \sigma \) or \( \pi \) internal propagator. The reason for their inclusion is due to a reordering of the powers of \( N \) in the master equation whose solution yields \( \lambda_1 \). The quantities \( \psi^{-1}, \sigma^{-1} \) and \( \pi^{-1} \) in fig. 1 are the 2-point functions and their asymptotic scaling forms can be deduced from (2.2)-(2.4) by inverting each expression in momentum space via (2.5) and they take the \( x \)-space forms, [7].

\[
\psi^{-1}(x) \sim \frac{r(\alpha - 1)^\lambda}{A(x^2)^{2\mu - \alpha + 1}} [1 - A's(\alpha - 1)(x^2)^\lambda] \tag{2.7}
\]

\[
\sigma^{-1}(x) \sim \frac{p(\beta)^\lambda}{B(x^2)^{2\mu - \beta}} [1 - B'q(\beta)(x^2)^\lambda] \tag{2.8}
\]
\[ \pi^{-1}(x) \sim \frac{p(\gamma)}{C(x^2)^{2\mu-\beta}}[1 - C'q(\gamma)(x^2)^\lambda] \]  

where we define

\[
\begin{align*}
p(\alpha) &= \frac{a(\alpha - \mu)}{\pi^{2\mu a(\alpha)}} , & q(\alpha) &= \frac{a(\alpha - \mu + \lambda)a(\alpha - \lambda)}{a(\alpha - \mu)a(\alpha)} \\
r(\alpha) &= \frac{\alpha p(\alpha)}{(\mu - \alpha)} , & s(\alpha) &= \frac{\alpha(\alpha - \mu)q(\alpha)}{(\alpha - \lambda)(\alpha - \mu + \lambda)}
\end{align*}
\]

If we formally denote the values of the 2-loop integrals by \( \Pi_{1A,B,C} \) and \( \Lambda_{1A,B,C} \), where the subscript \( A, B \) or \( C \) denotes an insertion on a \( \psi, \sigma \) or \( \pi \) line respectively, then with (2.2)-(2.4) the Dyson equations of fig. 1 become

\[
\begin{align*}
0 &= r(\alpha - 1)[1 - A's(\alpha - 1)(x^2)^\lambda] + z[1 + (A' + B')(x^2)^\lambda] \\
&+ y[1 + (A' + C')(x^2)^\lambda] \quad (2.11)
\end{align*}
\]

\[
\begin{align*}
0 &= \frac{p(\beta)}{N} [1 - B'q(\beta)(x^2)^\lambda] + 2z[1 + 2A'(x^2)^\lambda] \\
&- z^2[\Pi_1 + (x^2)^\lambda(A'\Pi_{1A} + B'\Pi_{1B})] + zy[\Pi_2 + (x^2)^\lambda(A'\Pi_{2A} + C'\Pi_{2C})] \\
0 &= \frac{p(\gamma)}{N} [1 - C'q(\gamma)(x^2)^\lambda] + 2y[1 + 2A'(x^2)^\lambda] \\
&- y^2[\Lambda_1 + (x^2)^\lambda(A'\Lambda_{1A} + C'\Lambda_{1C})] + zy[\Lambda_2 + (x^2)^\lambda(A'\Lambda_{2A} + B'\Lambda_{2B})] \quad (2.13)
\end{align*}
\]

where we have set \( z = A^2B \) and \( y = A^2C \). Comparing powers of \( x^2 \) the equations decouple into a set which become the consistency equations for \( \eta_1 \) and give, \( [14] \),

\[
\eta_1 = \frac{2\Gamma(2\mu - 1)}{\Gamma(\mu - 1)\Gamma(1 - \mu)\Gamma(\mu + 1)} \quad (2.14)
\]

and a set involving \( A', B' \) and \( C' \). The solution of the latter yield \( \lambda_1 \) where the relevant consistency equation is formed by setting the determinant of the matrix formed by treating \( A', B' \) and \( C' \) as independent basis vectors, to zero. In this matrix one can examine the \( N \) dependence of each element and observe that \( r(\alpha - 1)s(\alpha - 1) = O(1) \) whilst \( p(\beta)q(\beta) = O(1/N) \). Since \( y \) and \( z \) are both \( O(1/N) \) from the \( \eta_1 \) equation it is easy to see the necessity of including the higher order graphs \( \Pi_{1B,C} \) and \( \Lambda_{1B,C} \). Direct calculation, however, reveals that \( \Pi_{1B} = \Pi_{1C} = \Lambda_{1B} = \Lambda_{1C} \equiv \Pi \), where the basic integral, \( \Pi \), has been computed explicitly in the \( O(N) \) model, \( [4, 22] \). It turns out that in manipulating the determinant through elementary row and column
transformations and using, \[19\],

\[ z_1 = y_1 = \frac{\mu \Gamma^2(\mu) \eta_1}{4\pi^2 \mu} \tag{2.15} \]

that one obtains

\[
0 = \det \begin{pmatrix}
    s(\alpha - 1) & 1 & 1 \\
    2 & q(\beta) & z \Pi \\
    0 & 0 & q(\gamma) - z \Pi
\end{pmatrix} \tag{2.16}
\]

where the \( \Pi \)-type contribution has cancelled in the central element through
the additional symmetry present, compared with \[7, 19\]. Consequently, the
consistency equation which results from (2.16),

\[
0 = q(\beta) s(\alpha - 1) - 2 \tag{2.17}
\]

does not require the value of \( \Pi \) and we deduce that

\[
\lambda_1 = -(2\mu - 1)\eta_1 \tag{2.18}
\]

which is in agreement with \[20\] and establishes the correctness of our proce-
dure for this model. This is a non-trivial statement as a second consistency
equation would appear to emerge from (2.16) by setting the lower right el-
lement to zero. We discard this as it does not give a result consistent with
the exponent calculated in the conventional large \( N \) approach. Moreover,
the appearance of such additional solutions is an artefact of the formalism
and has been observed in other models where again it was appropriate to
disregard it, \[22\]. Indeed from the transformations of the determinant we
have made this potentially alternative solution does not take into account
any contribution from the \( \psi \) consistency equation, which again justifies our
stance.

3 Master equation.

To proceed beyond (2.18) and to compute the \( O(1/N^2) \) correction requires
that one expands the quantities \( s(\alpha - 1), q(\beta) \) and \( q(\gamma) \) to the subsequent
order and also includes the higher order corrections to the Dyson equations.
For the former this requires the values of the vertex anomalous dimensions
and \( \eta_2 \). They have been calculated in \[19\] by considering the subsequent
corrections to the Dyson equations which determine the fermion anomalous dimension as

\[ \eta_2 = \eta_1^2 \left[ \Psi(\mu) + \frac{2}{(\mu - 1)} + \frac{1}{2\mu} \right] \]  

(3.1)

\[ \chi_{\sigma 1} = \chi_{\pi 1} = 0 \]  

(3.2)

where \( \Psi(\mu) = \psi(2\mu - 1) - \psi(1) + \psi(2 - \mu) - \psi(\mu) \) and \( \psi(x) \) is the logarithmic derivative of the \( \Gamma \)-function. By considering the scaling behaviour of the 3-vertices at \( O(1/N^2) \) one also finds that

\[ \chi_{\sigma 2} = \chi_{\pi 2} = -\frac{\mu^2(4\mu^2 - 10\mu + 7)\eta_1^2}{2(\mu - 1)^3} \]  

(3.3)

using methods developed from the earlier work of [23, 24].

The main problem in determining \( \lambda_2 \), however, is the inclusion of the higher order corrections. For the fermion equations this is straightforward in that one includes the two loop Feynman graphs of fig. 2, and these together with the two loop graphs of fig. 1 are all that is required to deduce \( \eta_2 \) from the leading order ansatze of (2.2)-(2.4) and (2.7)-(2.9). It is worthwhile detailing the resulting Dyson equation for \( \psi \) explicitly since it will illustrate several important points which arise in the \( \sigma \) and \( \pi \) equations and may be obscured there. We have

\[
0 = r(\alpha - 1)[1 - A's(\alpha - 1)(x^2)^\lambda] + z(x^2)^{\lambda}[1 + (A' + B'')(x^2)^\lambda] \\
+ y(x^2)^{\lambda}[1 + (A' + C'')(x^2)^\lambda] \\
+ z^2(x^2)^{2\lambda}[\Sigma_1 + (x^2)^\lambda(A'\Sigma_{1A} + B'\Sigma_{1B})] \\
- 2yz(x^2)^{\lambda}[\Sigma_2 + (x^2)^\lambda(A'\Sigma_{2A} + B'\Sigma_{2B} + C'\Sigma_{2C})] \\
+ y^2(x^2)^{2\lambda}[\Sigma_3 + (x^2)^\lambda(A'\Sigma_{3A} + C'\Sigma_{3C})] \\

(3.4)
\]

where the mixed two loop graphs have the same value and we have not cancelled the powers of \( x^2 \). The values \( \Sigma \) and \( \Sigma_{A,B,C} \) correspond to the values of the integrals representing the Feynman graphs without symmetry factors which have been displayed explicitly. In analysing similar equations at \( O(1/N^2) \) in other models the analogous higher order two loop graphs are infinite which can be deduced by a detailed computation. To handle such divergences one shifts the exponents of the \( \sigma \) and \( \pi \) fields by an infinitesimal amount \( \Delta \) through \( \chi_\sigma \to \chi_\sigma + \Delta \) and \( \chi_\pi \to \chi_\pi + \Delta \) where \( \Delta \) plays the role of a regularizing parameter. The subsequent simple poles in \( \Delta \) which would occur in the regularized graphs are renormalized by the vertex counterterm.
available from the one loop graphs. For the current equation it turns out that upon adding the divergent graphs together the infinity cancels due to the symmetry induced through the $\gamma^5$ interaction and there is therefore no need to renormalize (3.4) to the order we are working to, $O(1/N^2)$. Thus only the $\Delta$-finite parts of the 2-loop graphs remain in (3.4). As before we decouple the finite equation into that which has already been used to deduce $\eta_2$ and

$$0 = [z + y - r(\alpha - 1)s(\alpha - 1) + z^2\Sigma_{1A} - 2yz\Sigma_{2A} + y^2\Sigma_{3A}]A' + [z + z^2\Sigma_{1B} - 2yz\Sigma_{2B}]B' + [y + y^2\Sigma_{3C} - 2yz\Sigma_{3C}]C' \quad (3.5)$$

where the powers of $x^2$ are absent at this order due to (3.2). The explicit evaluation of the finite parts of $\Sigma_{iB}$ and $\Sigma_{iC}$, however, have been determined in [9] and it turns out that to the order we are working to they are zero. Also if one compares the $N$-dependence of each term of the coefficient of $A'$ in (3.5) it is easy to see that $\Sigma_{iA}$ is $O(1/N^2)$ relative to $r(\alpha - 1)s(\alpha - 1)$ and therefore can be neglected in the correction to the determinant of (2.16). This term would be relevant only for $\lambda_3$. Thus the part of (3.5) which we require for $\lambda_2$ is simply

$$0 = [z + y - r(\alpha - 1)s(\alpha - 1)]A' + zB' + yC' \quad (3.6)$$

where, of course, the $O(1/N^2)$ parts of $z$ and $y$ contribute in the last two terms.

At leading order we had to include the higher order two loop graphs of fig. 1. As was noted in [21] this is a feature of the formalism for models where the fundamental field is fermionic, although in the final equation, (2.17), for the chiral Gross Neveu model the explicit value was not needed. By the same argument one has now to include the representative three and four loop higher order graphs of fig. 3 in the $\sigma$ and $\pi$ equations which we now discuss. We have given only the basic distinct topological structures of the graphs which occur for $\sigma$ where the internal bosonic lines can be either the $\sigma$ or $\pi$ fields. So, for example, there are 4, 4, 4, 8 and 8 respective graphs for each of the basic ones given in fig. 3 for the $\sigma$ equation. Of course, since $\text{tr}\,\gamma^5 = 0$ only half of the four loop graphs survive. Further, when one substitutes the asymptotic scaling forms (2.3) and (2.4) one need only include those graphs in the $\sigma$ consistency equation where there is one $(x^2)^3$ insertion on an internal bosonic line, in a similar way to leading order, [9]. The full consistency equation for $\sigma$ with the graphs of fig. 3 is quite a
long expression and rather than give its complete form we write down only the part relevant for \( \lambda_2 \) as

\[
0 = 4zA' - B' \left[ \frac{p(\beta)q(\beta)}{N} + z^2 \Pi + \Pi_{B2} \right] + C'\left[yz\Pi - \Pi_{C2} \right] \tag{3.7}
\]

where we have set

\[
\Pi_{B2} = 2(\Pi_{2B1} + \Pi_{2B2}) + \Pi_{3B} + \Pi_{4B} - 2(2\Pi_{5B1} + \Pi_{5B2}) - 4(\Pi_{6B1} + 2\Pi_{6B2}) \tag{3.8}
\]
\[
\Pi_{C2} = 2(\Pi_{2C1} + \Pi_{2C2}) + \Pi_{3C} + \Pi_{4C} - 2(2\Pi_{5C1} + \Pi_{5C2}) - 4(\Pi_{6C1} + 2\Pi_{6C2}) \tag{3.9}
\]

and our notation is partially defined in fig. 3. When there is an additional subscript 1 or 2 in (3.8) and (3.9) this corresponds to the two distinct ways of including an insertion on the internal bosonic line and is consistent with definitions given in \[ \text{fig. 3} \]. Using the properties of the \( \gamma \)-matrices and being careful in manipulating factors of \( i \) from the vertex of (2.1) one finds that the contributions from \( \Pi_{2B1}, \Pi_{3B}, \Pi_{2C1} \) and \( \Pi_{3C} \) style graphs each sum to zero through use of (2.15). This is consistent with the fact that one does not require a regularization at this order in \( 1/N \) as the divergent contributions cancel. Subsequently this implies \( \chi_{\sigma 1} = 0 \) which is in agreement with calculations of other Dyson equations. Further, with (2.15) it is possible to simplify (3.8) and (3.9) to

\[
\Pi_{B2} = 4z^3[\Pi^0_{B2} + \Pi^0_{4B} - z_1(2\Pi^0_{5B1} + \Pi^0_{5B2} + 2\Pi^0_{6B1})] \tag{3.10}
\]
\[
\Pi_{C2} = 4z^3[-\Pi^0_{B2} + \Pi^0_{4B} - z_1(2\Pi^0_{5B1} + \Pi^0_{5B2} - 2\Pi^0_{6B1})] \tag{3.11}
\]

using (2.15) where the superscript \( ^0 \) denotes the value of the fundamental graph, with the appropriate insertion, given in fig. 3. Whilst these have all been computed explicitly in the \( O(N) \) case, \[ \text{fig. 3} \], we defer to later the direct substitution into \( \Pi_{B2} \) and \( \Pi_{C2} \).

A similar analysis of the \( \pi \) Dyson equation yields the third of our consistency equations as

\[
0 = 4yA' + B'[yz\Pi - \Pi_{C2}] - C'\left[\frac{p(\gamma)q(\gamma)}{N} + y^2 \Pi + \Pi_{B2} \right] \tag{3.12}
\]

where the same combinations \( \Pi_{B2} \) and \( \Pi_{C2} \) appear as in (3.7). We can now manipulate these formal corrections to (2.11)-(2.13) to deduce the master equation for \( \lambda_2 \). First, though we record that from the \( \eta_2 \) consistency
\[ z_2 = y_2 = \frac{\mu \Gamma^2(\mu) \eta_1^2}{4\pi^2 \mu} \left[ \Psi(\mu) + \frac{2}{(\mu - 1)} \right] \] (3.13)

and following the same transformations of the determinant as we made at leading order with these corrections we are left with the determinant of the \(2 \times 2\) submatrix

\[ 0 = \det \begin{pmatrix} 2z - r(\alpha - 1)s(\alpha - 1) & 2z \\ 4z & -\frac{1}{\lambda} p(\beta) q(\beta) - (\Pi_{B2} + \Pi_{C2}) \end{pmatrix} \] (3.14)

Again the two loop graphs have cancelled though it is important to note that they would in fact have contained \(O(1/N^2)\) information since the integral depends on the exponents \(\eta_1\) and \(\lambda_1\) and they would have contributed to \(\lambda_2\). From (3.14) we deduce that \(\lambda_2\) will emerge from the solution of

\[ 8z^2 = [r(\alpha - 1)s(\alpha - 1) - 2z] \left[ \frac{p(\beta) q(\beta)}{N} + \Pi_{B2} + \Pi_{C2} \right] \] (3.15)

Again we ignore the solution that would come from the lower right element of the full \(3 \times 3\) for the reasons stated earlier. Moreover, we note that the higher order graph contributions appear in a particular combination \(\Pi_{B2} + \Pi_{C2}\) and from (3.10) and (3.11) this simplifies to

\[ \Pi_{B2} + \Pi_{C2} = 8z^3 [\Pi_{1B}^0 - z_1(2\Pi_{5B1}^0 + \Pi_{5B2}^0)] \] (3.16)

Thus we need only consider the contributions from three of the eight basic topologies and insertions of fig. 3 which represents a significant simplification. We note that the graph \(\Pi_{5B1}^0\) corresponds to an \((x^3)^5\) insertion on the top or bottom internal bosonic line.

4 Discussion.

All that remains now is the explicit evaluation of (3.15) at \(O(1/N^2)\). The values of the higher order graphs have been computed in \([9]\) for the Gross Neveu model with a discrete chiral symmetry. This involved extensive use of the technique known as uniqueness, \([5]\), which is applicable to calculating massless Feynman diagrams, and is a method of integration which is based on the conformal properties of the integral in \(d\)-dimensions. Our manipulations for (2.1) have been such that the basic graphs of (3.16) are equivalent
to the values of the graphs of the $O(N)$ case and therefore we record that, \[9\],

\[
\Pi_{oB}^5 = \frac{\pi^4}{(\mu - 1)^2 \Gamma^4(\mu)} \left[ 3\Theta + \frac{1}{(\mu - 1)^2} \right] \tag{4.1}
\]

\[
\Pi_{5B1}^5 = \frac{\pi^6}{2(\mu - 1)^5 (\mu - 2) \Gamma^3(\mu)} \left[ \frac{6\Theta - \Phi - \Psi^2}{(\mu - 1)^2} + \frac{5}{2(\mu - 1)^2} \right] + \frac{1}{(2\mu - 3)(\mu - 1)} \tag{4.2}
\]

\[
\Pi_{5B2}^5 = \frac{\pi^6}{(\mu - 1)^6 \Gamma^3(\mu)} \left[ \frac{6\Theta - \Phi - \Psi^2}{(\mu - 1)^2} + \frac{1}{(\mu - 1)^2} \right] + \frac{2\pi^6}{(\mu - 1)^6 (\mu - 2)^2} \tag{4.3}
\]

where $\Theta(\mu) = \psi'(\mu) - \psi'(1)$ and $\Phi(\mu) = \psi'(2\mu - 1) - \psi'(2 - \mu) - \psi'(\mu) + \psi'(1)$. Thus substituting into (3.16) we have

\[
\Pi_{B2} + \Pi_{C2} = \frac{8\pi^4}{(\mu - 1)^2 \Gamma^4(\mu)} \left[ \frac{2(2\mu - 3)}{(\mu - 2)} (\Phi + \Psi^2) - \frac{3(3\mu - 4)\Theta}{(\mu - 2)} \right] + \frac{2(2\mu - 3)(\mu - 2)^2}{(\mu - 1)(\mu - 2)^2} + \frac{1}{(\mu - 1)^2} \tag{4.4}
\]

A little bit of algebra subsequently leads us to the main result of the paper,

\[
\lambda_2 = \eta_1^2 \left[ \frac{3\mu^2(3\mu - 4)\Theta}{2(\mu - 1)(\mu - 2)} - \frac{\mu^2(2\mu - 3)(\Phi + \Psi^2)}{(\mu - 1)(\mu - 2)} + \frac{4(\mu - 1)}{(\mu - 2)^2 \eta_1} \right] + \frac{2}{(\mu - 1)} + \frac{1}{2\mu} - \frac{\mu^2}{2(\mu - 1)^3 (\mu - 2)^2} \tag{4.5}
\]

in arbitrary dimensions.
As the three dimensional chiral Gross Neveu model has also been the subject of study recently, [18, 20, 25], we can set $\mu = \frac{3}{2}$ in (4.5) to deduce a new result for this case as

$$\lambda = \frac{1}{2} - \frac{16}{3\pi^2 N} + \frac{16[472 + 27\pi^2]}{27\pi^4}$$

(4.6)

This will be important for deducing numerical estimates for this exponent for relatively small values of $N$ since several orders are now known. For example, in [9] an estimate was given for the $O(8)$ Gross Neveu model which agreed with a recent Monte Carlo result, [18].

We conclude with various remarks. First, the chiral Gross Neveu model has now been solved completely at $O(1/N^2)$ and it is important to recognise that this is one order further than had previously been possible. By this we mean that two independent critical exponents $\lambda$ and $\eta$ or $\eta + \chi$ have been determined to this order. The remaining thermodynamic exponents can be deduced through the hyperscaling laws which are valid for renormalizable quantum field theories and which have been checked in [20]. The quantum structure of the model is appreciably simpler than for the $O(N)$ Gross Neveu model in the sense that fewer topologies of Feynman diagrams needed to be considered and this is primarily due to the extra chiral symmetry present in (2.1). Indeed a clear indication of symmetry within the self consistency formalism is the vanishing of the anomalous dimensions of one of the fields or vertices, such as $\chi_\sigma_1 = \chi_{\pi_1} = 0$ here, in all dimensions. A similar phenomena occurs in the supersymmetric $O(N)$ $\sigma$ model where the anomalous dimension of the analogous supermultiplet of auxiliary and Lagrange multiplier fields vanishes at $O(1/N)$ as a consequence of the unbroken supersymmetry present in the model, [26].

References.

[1] D. Gross & A. Neveu, Phys. Rev. D10 (1974), 3235.

[2] A.N. Vasil’ev, Yu.M. Pis’mak & J.R. Honkonen, Theor. Math. Phys. 46 (1981), 104.

[3] A.N. Vasil’ev, Yu.M. Pis’mak & J.R. Honkonen, Theor. Math. Phys. 47 (1981), 465.

[4] A.N. Vasil’ev, Yu.M. Pis’mak & J.R. Honkonen, Theor. Math. Phys. 50 (1982), 127.
[5] M. d’Eramo, L. Peliti & G. Parisi, Lett. Nuovo Cim. 2 (1971), 878.

[6] D.J. Amit, ‘Field Theory, the Renormalization Group and Critical Phenomena’ (McGraw-Hill, New York, 1978).

[7] J.A. Gracey, Int. J. Mod. Phys. A6 (1991), 395; 2755(E).

[8] J.A. Gracey, Phys. Lett. 297B (1992), 293.

[9] J.A. Gracey, Int. J. Mod. Phys. A9 (1994), 567.

[10] S.E. Derkachov, N.A. Kivel, A.S. Stepanenko & A.N. Vasil’ev, ‘On calculation of 1/n expansions of critical exponents in the Gross Neveu model with the conformal technique’ preprint; Theor. Math. Phys. 92 (1992), 1047.

[11] J.A. Gracey, Mod. Phys. Lett. A7 (1992), 1945.

[12] Y. Nambu & J. Lona-Lasinio, Phys. Rev. 122 (1961), 345.

[13] H. Kleinert, Phys. Lett. 59B (1975), 163.

[14] Y. Nambu, in ‘New Theories in Physics’ proceedings of the XI International symposium on Elementary Particle Physics, Kazimierz, Poland, 1988 ed Z. Ajduk, S. Pokorski & A. Trautman (World Scientific, Singapore, 1989); in ‘New Trends in Strong Coupling Gauge Theories’ International workshop, Nagoya, Japan, 1988 ed M. Bando, T. Muta & K. Yamawaki (World Scientific, Singapore, 1989).

[15] W.A. Bardeen, C.T. Hill & M. Lindner, Phys. Rev. D41 (1990), 1647; A. Hasenfratz, P. Hasenfratz, K. Jansen, J. Kuti & Y. Shen, Nucl. Phys. B365 (1991), 79.

[16] B. Berg, M. Karowski, V. Kurak & P. Weisz, Nucl. Phys. B134 (1978), 125; B. Berg & P. Weisz, Nucl. Phys. B146 (1979), 205; E. Abdalla, B. Berg & P. Weisz, Nucl. Phys. B157 (1979), 387.

[17] P. Forgács, S. Naik & F. Niedermayer, Phys. Lett. B283 (1992), 282.

[18] L. Kärkkäinen, Nucl. Phys. B (Proc. Suppl.) 30 (1993), 670; L. Kärkkäinen, R. Lacaze, P. Lacock & B. Petersson, Nucl. Phys. B415 (1994), 781.

[19] J.A. Gracey, Phys. Lett. B308 (1993), 65.
[20] S. Hands, A. Kocić & J.B. Kogut, Ann. Phys. 224 (1993), 29.
[21] J.A. Gracey, Int. J. Mod. Phys. A8 (1993), 2465.
[22] J.A. Gracey, Ann. Phys. 224 (1993), 275.
[23] A.N. Vasil’ev & M.Yu. Nalimov, Theor. Math. Phys. 55 (1983), 423.
[24] A.N. Vasil’ev & M.Yu. Nalimov, Theor. Math. Phys. 56 (1983), 643.
[25] H.J. Pe, Y.P. Kuang, Q. Wang & Y.P. Yi, Phys. Rev. D45 (1992), 4610.
[26] J.A. Gracey, Nucl. Phys. B352 (1991), 183.

**Figure Captions.**

Fig. 1. Leading order Dyson equations to determine $\lambda_1$.

Fig. 2. Higher order graphs for $\psi$ Dyson equation.

Fig. 3. Higher order graphs for $\sigma$ and $\pi$ Dyson equations.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406162v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406162v1