Approximate quantum encryption with even shorter keys

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Abstract

Perfect encryption of a qubit state using the Quantum One-Time Pad (QOTP) requires 2 classical key bits. More generally, perfect QOTP encryption of a $2^n$-dimensional state requires $2^n$ classical bits. However, almost-perfect encryption, with information-theoretic security, can be achieved with only little more than 1 key bit per qubit.

In this paper we slightly improve the key length. We show that key length $n + 2\log \frac{1}{\varepsilon}$ suffices to encrypt $n$ qubits in such a way that the cipherstate has trace distance $\leq \varepsilon$ from the fully mixed state. The previous best result was $n + 2\log \frac{1}{\sqrt{\varepsilon}}$.

Similar to the previous literature, we expand the key to a $2^n$-bit pseudorandom string which is then used as a QOTP key. In this expansion we make use of $2^n$ bits of public randomness which are included as a classical part of the cipherstate. Our key expansion is slightly faster than in previous works.

1 Introduction

1.1 Encryption of quantum states

An encryption is called perfect if the ciphertext reveals no information whatsoever about the message that was encrypted. In the case of classical messages, the length of the key required to achieve perfect encryption is at least the Shannon entropy of the message. The Vernam cipher [Ver18] (also known as One-Time Pad, OTP) performs a bitwise xor of an $n$-bit message and an $n$-bit key; it achieves perfect encryption for any probability distribution of the message.

An equivalent of the Vernam cipher exists for the encryption of quantum states [AMTdW00, BR03, Leu02]. This is known as the quantum Vernam cipher, Quantum One-Time Pad (QOTP) or private quantum channel. In order to perfectly encrypt any $n$-qubit state, the necessary and sufficient key length is $2n$ bits. In its simplest form, QOTP encryption and decryption work by applying to each individual qubit a Pauli operation from the set $\{1, \sigma_x, \sigma_y, \sigma_z\}$. The choice of Pauli operations constitutes the key. For someone who does not know this key, the state after encryption equals the fully mixed state regardless of the original state.

It is possible to get $\varepsilon$-close to the fully mixed state by a randomization process that takes fewer than $2n$ key bits [HLSW04, AS04, DN06, Aub09, ŠdV17]. This is called approximate randomization or almost-perfect encryption. The ‘$\varepsilon$-close’ property can be expressed as a distance with respect to different norms, e.g. the 1-norm (trace norm) or the $\infty$-norm (maximum absolute eigenvalue). In this paper we consider only the 1-norm, since it expresses indistinguishability of states and it is a universally composable measure of security [Can01, BOHL+05, FS09]. Table 1 summarizes known results on approximate randomization, including this work, focusing on the 1-norm.

Work has also been done on quantum encryption with entropic security [Des09, DD10]. However, this security property is not composable and needs assumptions on the adversary’s prior knowledge about the message; we will not consider this topic in the current paper.

Hayden et al. [HLSW04] showed that approximate randomization is possible with a key length of $n + \log n + 2\log \frac{1}{\varepsilon}$ by using sets of random unitaries. Random selection and storage of unitary matrices is very inefficient. Ambainis and Smith [AS04] introduced far more efficient schemes that work with a pseudorandom sequence which selects Pauli operators as in the QOTP. In one of them, they expand the key using small-bias sets and achieve key length $n + 2\log n + 2\log \frac{1}{\sqrt{\varepsilon}}$. This scheme is length-preserving, i.e. the cipherstate consists of $n$ qubits. In another construction, they expand the key by multiplying

\footnote{They also provide a result for the $\infty$-norm, with key length $n + \log n + 2\log \frac{1}{\sqrt{\varepsilon}} + \log 134$; unitaries are drawn from the Haar measure. This was later improved to $n + 2\log \frac{1}{\sqrt{\varepsilon}} + \log 150$ by Aubrun [Aub09].}
we give the details of our construction, and in Section 2.3, Randomization process



Pseudorandom QOTP based on small-bias

Pseudorandom QOTP based on multiplication
in \( GF(2^{2n}) \). Key expansion takes \( O(n \log n) \) time.

Pseudorandom QOTP based on small-bias spaces. Key expansion takes \( O(n^2 \log n) \) time.

Pseudorandom QOTP based on huge Common Reference String.

Pseudorandom QOTP based on affine function in \( GF(2^{\text{key length}}) \). Key expansion takes \( O(n \log n) \) time.

Table 1: Results on almost-perfect encryption of \( n \) qubits, with security definition in terms of the trace distance: \( \|\text{cipherstate} – \text{fully mixed state}\|_1 \leq \varepsilon \).

it with a random binary string of length \( 2n \); this string becomes part of the cipherstate. The key length is reduced to \( n + 2 \log \frac{1}{\varepsilon} \) bits. Dickinson and Nayak [DN06] improved the small-bias based scheme of [AS04] and achieved key length \( n + 2 \log \frac{1}{\varepsilon} + 4 \). Škorić and de Vries [ŠdV17] described a pseudorandom QOTP scheme that has key length \( n + 2 \log \frac{1}{\varepsilon} \), but need an exponentially large common random string to be stored somewhere.

Key length

Ciphertext length

Randomization process

| [HLSW04] | \( n + \log n + 2 \log \frac{1}{\varepsilon} \) | \( n \) qubits | Random units (non-Haar, e.g. Pauli) |
| [AS04] | \( n + 2 \log n + 2 \log \frac{1}{\varepsilon} \) | \( n \) qubits | Pseudorandom QOTP based on small-bias sets. Key expansion takes \( O(n^2) \) time. |
| [AS04] | \( n + 2 \log \frac{1}{\varepsilon} \) | \( n \) qubits + \( 2n \) bits | Pseudorandom QOTP based on multiplication in \( GF(2^{2n}) \). Key expansion takes \( O(n \log n) \) time. |
| [DN06] | \( n + 2 \log \frac{1}{\varepsilon} + 4 \) | \( n \) qubits | Pseudorandom QOTP based on small-bias spaces. Key expansion takes \( O(n^2 \log n) \) time. |
| [ŠdV17] | \( n + 2 \log \frac{1}{\varepsilon} \) | \( n \) qubits | Pseudorandom QOTP based on huge Common Reference String. |
| This paper | \( n + \frac{2}{3} \log \frac{1}{\varepsilon} \) | \( n \) qubits + \( 2n \) bits | Pseudorandom QOTP based on affine function in \( GF(2^{\text{key length}}) \). Key expansion takes \( O(n \log n) \) time. |

1.2 Contribution and outline

We slightly modify the second scheme of Ambainis and Smith [AS04]. Instead of expanding the key by multiplying in \( GF(2^{2n}) \), we append to the key an affine function of the key. The two parameters of the affine function are drawn at random and become part of the cipherstate. The resulting encryption scheme has key length \( \ell = n + \frac{2}{3} \log \frac{1}{\varepsilon} \), which is shorter than previously achieved values. The cipherstate consists of \( n \) qubits and \( 2n \) classical bits. The key expansion is slightly faster than [AS04] because multiplication is done in \( GF(2^\ell) \) instead of \( GF(2^{2n}) \).

The outline of the paper is as follows. In Section 2.1 we introduce notation, and in Section 2.2 the desired \( \varepsilon \)-randomizing security property is specified. The QOTP is briefly recalled in Section 2.3. In Section 3.1 we give the details of our construction, and in Section 3.2 the security proof. We conclude with a brief discussion in Section 4.

2 Preliminaries

2.1 Notation

Expectation over a random variable \( X \) is written as \( \mathbb{E}_X \). We denote the space of density matrices on Hilbert space \( \mathcal{H} \) as \( \mathcal{D}(\mathcal{H}) \). The single-qubit Hilbert space is \( \mathcal{H}_2 \). A bipartite state comprising subsystems ‘A’ and ‘B’ is written as \( \rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B) \). The state of a subsystem is obtained by taking the partial trace over the other subsystem, e.g. \( \rho^A = \text{tr}_B \rho^{AB} \). The identity operator on \( \mathcal{H} \) is denoted by \( 1_{\mathcal{H}} \); we will simply write 1 when the Hilbert space is clear from the context. Similarly we write \( \tau^H \) for the fully mixed state \( 1_{\mathcal{H}} / \text{dim}(\mathcal{H}) \), often omitting the superscript. Let \( M \) be an operator.

\(^2\)We do not see the additional \( 2n \) bits as a problem. Classical storage and transmission are ‘for free’ compared to quantum resources.
with eigenvalues $\lambda_i$. The Schatten 1-norm of $M$ is given by $\|M\|_1 = \text{tr} \sqrt{M^*M} = \sum_i |\lambda_i|$. The induced \textquoteleft trace\textquoteright distance between states $\rho$, $\sigma$ is $\|\rho - \sigma\|_1$.

### 2.2 Security definitions

We use standard definitions of encryption and $\varepsilon$-randomization.

**Definition 2.1 (Encryption scheme)** An encryption scheme with classical key space $\mathcal{K}$, quantum message space $\mathcal{H}$ and quantum ciphertext space $\mathcal{H}'$ consists of a pair $(\text{Enc}, \text{Dec})$. Here $\text{Enc} : \mathcal{K} \times \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H}')$ is a (possibly randomised) algorithm that takes as input a classical key $k \in \mathcal{K}$ and a quantum state $\varphi \in \mathcal{D}(\mathcal{H})$, and outputs a quantum state $\omega = \text{Enc}(k, \varphi) \in \mathcal{D}(\mathcal{H}')$ called the ciphertext. $\text{Dec} : \mathcal{K} \times \mathcal{D}(\mathcal{H}') \rightarrow \mathcal{D}(\mathcal{H})$ is an algorithm that takes as input a key $k \in \mathcal{K}$ and a state $\omega \in \mathcal{D}(\mathcal{H}')$, and outputs a state $\text{Dec}(k, \omega) \in \mathcal{D}(\mathcal{H})$. It must hold that $\forall k \in \mathcal{K}, \varphi \in \mathcal{D}(\mathcal{H}) \text{ Dec}(k, \text{Enc}(k, \varphi)) = \varphi$.

Note that Def. 2.1 allows the ciphertext space to be larger than the plaintext space, $\dim \mathcal{H}' > \dim \mathcal{H}$. We will be working with the special case where the ciphertext consists of a quantum state of the same dimension as the input (n qubits), accompanied by classical information.

The effect of an encryption, with key unknown to the adversary, can be described as a completely positive trace preserving (CPTP) map $R : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H}')$ as follows,

$$ R(\varphi) = \sum_{k \in \mathcal{K}} \frac{1}{|\mathcal{K}|} \text{Enc}(k, \varphi). $$

(1)

**Definition 2.2 ($\varepsilon$-Randomizing)** Let $\varepsilon \geq 0$. A CPTP linear operator $R : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H}')$ is said to be $\varepsilon$-randomizing with respect to a norm $\| \cdot \|$ if

$$ \forall \varphi \in \mathcal{D}(\mathcal{H}) \quad \| R(\varphi) - \tau^{\mathcal{H}'} \| \leq \varepsilon. $$

(2)

We say that $R$ is completely randomizing if $\varepsilon = 0$.

Def. 2.2 is a slight modification of Def. 1.1 in [DN06]; the difference is that we allow $\dim \mathcal{H}' \neq \dim \mathcal{H}$.

It is interesting to note that randomisation as specified in Def. 2.2, with respect to the 1-norm, implies that $R(\varphi)$ is almost independent of any classical information that is correlated with $\varphi$. Let $\rho^{X\Phi}$ denote the quantum-classical state of the classical random variable $X$ and the quantum state $\rho^{X} = \varphi$. We write $\tau = \tau^{\mathcal{H}'}$ and $\rho^{X\Phi} = \sum_x p_x |x\rangle \langle x| \otimes \varphi(x)$, where $p_x$ stands for $\text{Pr}[X = x]$. Furthermore $\Phi'$ denotes the result of the operation $R$ on the $\Phi$ subsystem.

$$\|\rho^{X\Phi'} - \rho^X \otimes \rho^{\Phi'}\|_1 = \|\rho^{X\Phi'} - \rho^X \otimes \tau + \rho^X \otimes \tau - \rho^X \otimes \rho^{\Phi'}\|_1$$

triangle ineq.

$$\leq \|\rho^{X\Phi'} - \rho^X \otimes \tau\|_1 + \|\rho^X \otimes \rho^{\Phi'} - \rho^X \otimes \tau\|_1$$

(3)

$$= \left\| \sum_x p_x |x\rangle \langle x| \otimes [R(\varphi(x)) - \tau] \right\|_1 + \|\rho^{\Phi'} - \tau\|_1$$

(4)

$$= \sum_x p_x \|R(\varphi(x)) - \tau\|_1 + \|R(\varphi) - \tau\|_1$$

(5)

$$\leq 2\varepsilon.$$ (6)

Just as the earlier works [HLSW04, AS04, DN06, ŠdV17] we will use Def. 2.2 (with the 1-norm) as our security definition.

### 2.3 The Quantum One Time Pad (QOTP)

Let $\mathcal{H}_2$ denote the Hilbert space of a qubit. Let $Z$ and $X$ be single-qubit Pauli operators, in the standard basis given by $|0\rangle - |1\rangle$ and $|1\rangle + |0\rangle$. For QOTP encryption of one qubit, the key consists of two bits $s, t \in \{0,1\}$. The encryption of a state $\varphi \in \mathcal{D}(\mathcal{H}_2)$ is given by $X^s Z^t \varphi Z^t X^s$. Decryption is the same operation as encryption. For someone who does not know the key, the state after encryption is $\frac{1}{2} \sum_{s,t \in \{0,1\}} X^s Z^s \varphi Z^t X^* = \mathbb{I}/2$ for any $\varphi$. Hence the QOTP is completely randomizing.
The simplest way to encrypt an \( n \)-qubit state \( \varphi \in \mathcal{D}(\mathcal{H}_2^\otimes n) \) is to encrypt each qubit independently. The key is \( \beta = (\beta_1, \ldots, \beta_n) \in \{0,1\}^{2n} \), with \( \beta_i = (s_i, t_i) \). In the rest of the paper we will use the following shorthand notation for the QOTP cipherstate,
\[
F_\beta(\varphi) = U_\beta \varphi U_\beta^\dagger \quad \text{where} \quad U_\beta = \bigotimes_{i=1}^n X^{s_i} Z^{t_i}.
\]
(8)
It holds that \( 2^{-2n} \sum_{\beta \in \{0,1\}^{2n}} F_\beta(\varphi) = 1/2^n \) for any \( \varphi \in \mathcal{D}(\mathcal{H}_2^\otimes n) \).

3 Our result on approximate randomization with a short key

3.1 The construction

We encrypt an \( n \)-qubit state \( \varphi \in \mathcal{D}(\mathcal{H}_2^\otimes n) \) using a key \( k \in \{0,1\}^\ell \) where \( \ell > n \), and \( \ell \) is an even integer. We construct a pseudorandom sequence \( b \in \{0,1\}^{2n} \) by expanding \( k \in \{0,1\}^\ell \) as follows. Two strings \( u \in \{0,1\}^\ell \), \( v \in \{0,1\}^{2n-\ell} \) are drawn at random. They are interpreted as elements of \( \text{GF}(2^\ell) \). Note that \( 2n - \ell < n \). The string \( b \) is constructed by concatenating \( k \) with an affine function of \( k \),
\[
b(k, u, v) = k \|(uk + v)_{\text{lsb}}.
\]
(9)
The subscript ‘lsb’ (Least Significant Bits) stands for taking the last \( 2n - \ell \) bits of the string; in the finite field representation this corresponds to taking a polynomial in \( x \) modulo \( x^{2n-\ell} \). In (9) the multiplication and addition are operations in \( \text{GF}(2^\ell) \). Instead of \( (uk + v)_{\text{lsb}} \) we can also write \( (uk)_{\text{lsb}} + v \). The cipherstate is given by
\[
\text{Enc}(k, \varphi) = (u, v, F_{b(k,u,v)}(\varphi))
\]
(10)
with \( F \) the QOTP encryption as defined in (8) and \( b(\cdot, \cdot, \cdot) \) as defined by (9). The parameters \( u, v \) are a classical part of the cipherstate.

3.2 Security proof

Eve sees the parameters \( u, v \) but she does not know \( k \). From her point of view the state of the qubits is
\[
R_{uv}(\varphi) \overset{\text{def}}{=} \frac{1}{2^\ell} \sum_{k \in \{0,1\}^\ell} F_{b(k,u,v)}(\varphi).
\]
(11)

Lemma 3.1 It holds that
\[
E_{uv} R_{uv}(\varphi) = \tau.
\]
(12)
Proof: We write \( E_{uv} R_{uv}(\varphi) = E_u [E_v F_{b(k,u,v)}(\varphi)] \). Next, \( E_k E_v F_{b(k,u,v)}(\varphi) = E_{\beta \in \{0,1\}^{2n}} F_\beta(\varphi) = \tau \). The first equality follows from the fact that for any fixed \( u \), the \( k \) and \( v \) together can create any string in \( \{0,1\}^{2n} \) in precisely one way. The second equality is due to the fact that the QOTP is completely randomizing.

Lemma 3.2 Let \( f \) be any (possibly operator valued) function acting on \( \{0,1\}^{2n} \). It holds that
\[
E_{kk'uv} f(b(k, u, v)) f(b(k', u, v)) = 2^{2n-3f} E_\beta f(\beta) f(\beta') + E_{\beta \beta'} f(\beta') f(\beta') - 2^{-f} E_{kgg'} f(k\|g) f(k\|g').
\]
(13)
Here \( \beta, \beta' \in \{0,1\}^{2n} \), and \( E_\beta \) stands for \( 2^{-2n} \sum_{\beta} \). Similarly, \( g, g' \in \{0,1\}^{2n-\ell} \) and \( E_g \) stands for \( 2^{\ell-2n} \sum_{g} \).

Proof: We first look at the summation terms with \( k' \neq k \). We write \( b(k, u, v) = k\|g \) and \( b(k', u, v) = k'\|g' \). Consider \( k, k' \) fixed. For every combination \( (g, g') \) there are exactly \( 2^{2f-2n} \) values of \( (u, v) \) that yield \( (uk_{\text{lsb}} + v) = g + g' \). This allows us to rewrite the summations as
\[
\sum_{kk': k' \neq k} \sum_{uv} f(b(k, u, v)) f(b(k', u, v)) = 2^{2f-2n} \sum_{kk': k' \neq k} \sum_{gg'} f(k\|g) f(k'\|g')
\]
(14)
\[
= 2^{2f-2n} \left( \sum_{kgg'} f(k\|g) f(k'\|g') - \sum_{kgg'} f(k\|g) f(k\|g') \right).
\]
(15)

\(^3\)When the two equations are added the \( v \) disappears and we get \( |u(k+k')|_{\text{lsb}} = g + g' \), which has \( 2^\ell/2^{2n-\ell} \) solutions \( u \). Then, at fixed \( k, k' \), \( g, g' \), \( u \) the solution for \( v \) is unique.

4
Next we look at the \( k' = k \) terms. The summation over \( v \) covers all possible values of \( g \) exactly once. The summation over \( u \) visits every \( g \) exactly \( 2^{2n-\ell} \) times. Hence we can write

\[
\sum_k \sum_{uv} f(b(k, u, v)) f(b(k, u, v)) = 2^{2n-\ell} \sum_k \sum_g f(k|g) f(k|g).
\]  

Combining the \( k' = k \) and \( k' \neq k \) parts we get

\[
\sum_{kk' uv} f(b(k, u, v)) f(b(k', u, v)) = 2^{2n-\ell} \sum_{\beta} f(\beta) f(\beta) + 2^{2\ell-2n} \sum_{\beta' \beta''} f(\beta') f(\beta'') - 2^{2\ell-2n} \sum_{kgg'} f(k|g) f(k|g').
\]  

In order to go from summations to expectations we divide (17) by a factor \((2\ell)^3 2^{2n-\ell} = 2^{2\ell+2n} \). □

**Theorem 3.3** The randomizing map \( R_{uv} : \mathcal{D}(\mathcal{H}_2^{\otimes n}) \rightarrow \mathcal{D}(\mathcal{H}_2^{\otimes n}) \) as described in (11) satisfies

\[
\forall \varphi \in \mathcal{D}(\mathcal{H}_2^{\otimes n}) \quad \mathbb{E}_{uv} \| R_{uv}(\varphi) - \tau \|_1 < \sqrt{2^{3n-3\ell}}.
\]

**Proof:** For any \( \varphi \) we have

\[
\mathbb{E}_{uv} \| R_{uv}(\varphi) - \tau \|_1 \leq \mathbb{E}_{uv} \text{tr} \sqrt{(R_{uv}(\varphi) - \tau)^2}
\]

\[
\leq \text{tr} \sqrt{\mathbb{E}_{uv} (R_{uv}(\varphi) - \tau)^2}
\]

\[
= \text{tr} \sqrt{\mathbb{E}_{uv} E_{kk'} F_{b(k,u,v)}(\varphi) F_{b(k',u,v)}(\varphi) - \tau^2}
\]

\[
= \text{tr} \sqrt{2^{2n-3\ell} \mathbb{E}_{\beta\beta'} F_\beta(\varphi) F_{\beta'}(\varphi) - \mathbb{E}_k [E_g F_{k|g}(\varphi)] [E_{g'} F_{k|g'}(\varphi)]}
\]

In the last step we used \( \mathbb{E}_{\beta\beta'} F_\beta(\varphi) F_{\beta'}(\varphi) = \tau^2 \). Next we use \( F_\beta(\varphi) F_{\beta'}(\varphi) = F_\beta(\varphi^2) \) and \( \mathbb{E}_\beta F_\beta(\varphi^2) = \tau \text{tr} (\varphi^2) \), yielding

\[
\mathbb{E}_{\beta\beta'} F_\beta(\varphi) F_{\beta'}(\varphi) = \tau \text{tr} (\varphi^2).
\]

Furthermore we write \( \varphi = \sum_i p_i \varphi_i^L \otimes \varphi_i^R \), where the index ‘\( L \)’ stands for the first \( \ell/2 \) qubits and ‘\( R \)’ stands for the final \( n - \ell/2 \) qubits. The marginal state of the \( L \)-subsystem is given by \( \varphi_i^L = \text{tr}_R \varphi = \sum_i p_i \varphi_i^L \). We note that \( \mathbb{E}_k F_{k|g}(\varphi) = F_k(\varphi_i^L) \otimes \tau^R \), which gives

\[
\mathbb{E}_k [E_g F_{k|g}(\varphi)] [E_{g'} F_{k|g'}(\varphi)] = E_k F_k(\varphi_i^L) F_k(\varphi_i^L) \otimes (\tau^R)^2 = \tau^L \otimes (\tau^R)^2 \text{tr}_L((\varphi_i^L)^2).
\]

We substitute (24) and (25) into (23). Since the operator under the square root is diagonal, the tr \( \cdots \) is readily computed and gives

\[
\mathbb{E}_{uv} \| R_{uv}(\varphi) - \tau \|_1 \leq \sqrt{2^{3n-3\ell} \text{tr} (\varphi^2 - \mathbb{E}_k (\cdot)^2)^2}
\]

\[
\leq \sqrt{2^{3n-3\ell} - 2^{-\ell}}
\]

\[
< \sqrt{2^{3n-3\ell}}.
\]

In (27) we used \( \text{tr} (\varphi^2) \leq 1 \) and \( \text{tr} (\varphi_i^L)^2 \geq 2^{-\ell/2} \). □

**Theorem 3.4** Our scheme is \( \varepsilon \)-randomizing (Def. 2.2) with respect to the 1-norm when the key length is set to \( \ell = n + \frac{2}{3} \log \frac{1}{\varepsilon} \).

**Proof:** From the adversary’s point of view the cipherstate is \( \mathbb{E}_{uv} |uv\rangle \langle uv| \otimes R_{uv}(\varphi) \). We have to prove that this is \( \varepsilon \)-close to the fully mixed state on the whole output space, i.e. to \( \mathbb{E}_{uv} |uv\rangle \langle uv| \otimes \tau \). We get

\[
\| \mathbb{E}_{uv} |uv\rangle \langle uv| \otimes [R_{uv}(\varphi) - \tau] \|_1 = \mathbb{E}_{uv} \| R_{uv}(\varphi) - \tau \|_1 \leq \sqrt{2^{3n-3\ell}}.
\]

Substituting \( \ell = n + \frac{2}{3} \log \frac{1}{\varepsilon} \) into the final expression yields \( \varepsilon \). □

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4Here we use that \( \ell \) is even.
4 Discussion

It is difficult to pinpoint exactly how the key length is reduced from $n + 2 \log \frac{1}{\varepsilon}$ in previous works to our $n + \frac{2}{3} \log \frac{1}{\varepsilon}$. We suspect that it is due to the fact that our affine function $(uk)_{lsb} + v$ hashes to $\{0, 1\}^{2n-\ell}$ whereas previous constructions map the key to $\{0, 1\}^{2n}$.

It would be interesting to see how our scheme behaves regarding entropic security [Des09, DD10]. We note that our key is shorter than the one obtained in [DD10] for the relevant case where the adversary is not entangled with the message. This question is left for future work.

A small improvement to our scheme could be to draw the parameter $u$ from $\{0, 1\}^{2n-\ell}$ instead of $\{0, 1\}^\ell$.

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