AN INVESTIGATION OF THE COLLATZ CONJECTURE

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January 7, 2019

Abstract: This paper explores special conditions on the starting value of a Collatz sequence which imply that the Collatz conjecture is true. This is the result of the collaboration of a retired mathematics professor (Koelzer) and a retired physics professor (Welling).

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The Collatz conjecture was formulated by L. Collatz in 1937. The conjecture concerns the definition of a sequence of positive integers by means of a simple algorithm. The conjecture states that no matter what the starting value is, the sequence eventually equals 1. For more information on the history of the Collatz problem see [1] and [2].

**The Collatz Conjecture:** Let $n$ be a positive integer. If $n$ is even, divide it by 2 to get $n/2$. If $n$ is odd, multiply it by 3, add 1 and divide the result by 2 to obtain the number $(3n + 1)/2$. Repeat the process indefinitely. The conjecture is that no matter what number you start with, you will always reach 1. Our definition follows that of Terras in [3]. This is equivalent to the original Collatz algorithm but it results in a somewhat shorter sequence. We will assume that our starting value is an odd integer; otherwise we can simply divide by 2 until the number is odd.

**Example:** The Collatz sequence for $n = 19$ is: $19 \rightarrow 29 \rightarrow 44 \rightarrow 22 \rightarrow 11 \rightarrow 17 \rightarrow 26 \rightarrow 13 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$. (14 steps)

In this paper we will present some results concerning the Collatz conjecture. We will assume that the starting integer $N$ in a Collatz sequence is an odd integer, since, if $N$ is even, it can be repeatedly be divided by 2 until the resulting integer is odd.

The following theorem expresses the relation between an odd integer in a Collatz sequence and the next even integer in the sequence.

**Theorem 1:** Let $N$ be an odd integer in a Collatz sequence and define $E$ to be $N + 1$. The successive odd integers in the sequence starting with $N$ are

$$3 \left( \frac{E}{2} \right) - 1, \ 3^2 \left( \frac{E}{2^2} \right) - 1, \ 3^3 \left( \frac{E}{2^3} \right) - 1, \ \cdots, \ 3^{k-1} \left( \frac{E}{2^{k-1}} \right) - 1,$$

where $k$ is the highest power of 2 contained as a factor in $E$. The integer $3^k \left( \frac{E}{2^k} \right) - 1$ is the first even integer after $N$.

**Proof:** Let $m$ be an integer such that $1 \leq m \leq k - 1$. Then $\frac{E}{2^m}$ is an even number and hence $3^m \left( \frac{E}{2^m} \right) - 1$ will be odd. We will show that these integers are the successive odd entries in the Collatz sequence starting with $N$.

**Case 1:** $m = 1$. The next term in the Collatz sequence after $N$ is

$$\frac{3N + 1}{2} = \frac{3}{2} N + \frac{1}{2} = \frac{3}{2} (E - 1) + \frac{1}{2} = \frac{3}{2} E - 1,$$

and so the theorem is proven for $m = 1$.

**Case 2:** $2 \leq m \leq k - 1$. Since $3^{m-1} \left( \frac{E}{2^{m-1}} \right) - 1$ is an odd integer, the next term in the Collatz sequence

$$3 \left( 3^{m-1} \left( \frac{E}{2^{m-1}} \right) - 1 \right) + 1$$

is $\frac{3^{m-1} \left( \frac{E}{2^{m-1}} \right) - 1}{2}$, which simplifies algebraically to $3^m \left( \frac{E}{2^m} \right) - 1$ and the theorem is proved.
in this case. Finally, since \( k \) is the highest power of 2 contained as a factor in \( E \), \( \frac{E}{2^k} \) will be an odd integer. This means that \( 3^k \left( \frac{E}{2^k} \right) - 1 \) is even and it is the first even integer in the Collatz sequence after \( N \).

Given an odd integer \( N \), let \( u \) be the number of steps to the next even integer, say \( M \), and let \( d \) be the number of steps from \( M \) to the next odd integer \( N' \). By Theorem 1 \( u \) is the largest power of 2 in \( N + 1 \) and, since \( M \) is even, \( d \) is the largest power of 2 in \( M \). The relation between \( N \) and \( N' \) is given by the formula

\[
N' = \left( \frac{3}{2} \right)^u (N - 1) \left( \frac{1}{2} \right)^d
\]

Here is a simple corollary to illustrate the theorem:

**Corollary 1:** Suppose \( k \) is a positive integer. Let \( N = 10^k - 1 \) be an odd integer in a Collatz sequence. Then the next even integer in the sequence will be \( 15^k - 1 \).

**Proof:** Using the notation from Theorem 1, \( E = 10^k \) and the largest power of 2 in \( E \) is \( 2^k \). So by the Theorem, the first even integer following \( 10^k - 1 \) is \( 3^k \left( \frac{E}{2^k} \right) - 1 = 15^k - 1 \).

Another result following from Theorem 1 is shown below:

**Theorem 2:** Let \( k, l \) and \( m \) be positive integers with \( m \) odd and let \( N = \left( \frac{2}{3} \right)^k (2^l m + 1) - 1 \). If \( N \) is an odd integer then \( N \) reaches \( m \) in \( l + k \) steps.

**Proof:** Let \( E \) be the even integer \( N + 1 \). Note that \( E \) can be written as \( \frac{2^k (2^l m + 1)}{3^k} \). We claim that \( k \) is the highest power of 2 contained as a factor in \( E \). Suppose \( p \) is the highest power; Since \( 2^k \) is a factor of \( E \), we must have \( k \leq p \).

Furthermore, we can write \( \frac{E}{2^p} \) as \( \frac{2^k \left( \frac{2^l m + 1}{3^k} \right)}{2^p} \). Since \( 2^l m + 1 \) and \( 3^k \) are odd, their quotient is odd and so \( 2^p \) must divide \( 2^k \). This implies \( p \leq k \) and we conclude \( k = p \).

By Theorem 1, \( N \) is followed by \( k - 1 \) odd integers in the Collatz sequence and the even number \( 3^k \left( \frac{E}{2^k} \right) - 1 \). This even number can be rewritten as \( \left( \frac{3}{2} \right)^k \left[ \frac{2^k (2^l m + 1)}{3^k} \right] - 1 \), which algebraically reduces to \( 2^l m \). This means that the next \( l \) terms of the Collatz sequence are all products of \( m \) multiplied by powers of 2 terminating at \( m \) and the total number of steps from \( N \) to \( m \) is \( l + k \).

**Example:** \( k = 3, l = 2, d = 47 \) \( \rightarrow N = (8/27)(4 \cdot 47 + 1) - 1 = 55 \). The Collatz sequence from 55 to 47 is: 55 \( \rightarrow 83 \rightarrow 125 \rightarrow 188 \rightarrow 94 \rightarrow 47 \). (5 steps)

**Remarks:** To apply Theorem 1, it is required to find integers \( k, l \) and \( m \) that make \( N \) odd. Also, it would be nice to show that there is an infinite number of values of \( k, l \) and \( m \) satisfying the hypotheses of the theorem, but that is not essential to the proof.

Here is a special case of Theorem 2:
**Corollary 2:** Let \( k \) and \( l \) be positive integers and let \( N = \left(\frac{2}{3}\right)^k (2^l + 1) - 1 \). If \( N \) is an odd integer then \( N \) reaches 1 in \( l + k \) steps.

**Proof:** Let \( m = 1 \) in Theorem 1.

**Examples:**
- \( k = 1, l = 5 \rightarrow N = \frac{2}{3} (2^5 + 1) - 1 = 21 \). The Collatz sequence for 21 is: 21 \( \rightarrow \) 32 \( \rightarrow \) 16 \( \rightarrow \) 8 \( \rightarrow \) 4 \( \rightarrow \) 2 \( \rightarrow \) 1. (6 steps)
- \( k = 2, l = 9 \rightarrow N = \left(\frac{2}{3}\right)^2 (2^9 + 1) - 1 = 227 \). The Collatz sequence for 227 is: 227 \( \rightarrow \) 341 \( \rightarrow \) 512 \( \rightarrow \) 256 \( \rightarrow \) 32 \( \rightarrow \) 16 \( \rightarrow \) 8 \( \rightarrow \) 4 \( \rightarrow \) 2 \( \rightarrow \) 1. (11 steps)
- \( k = 3, l = 9 \rightarrow N = \left(\frac{2}{3}\right)^3 (2^9 + 1) - 1 = 151 \). The Collatz sequence for 151 is: 151 \( \rightarrow \) 227 \( \rightarrow \) 341 \( \rightarrow \) 512 \( \rightarrow \) 256 \( \rightarrow \) 32 \( \rightarrow \) 16 \( \rightarrow \) 8 \( \rightarrow \) 4 \( \rightarrow \) 2 \( \rightarrow \) 1. (12 steps)

We prove a result that will be useful later in the paper.

**Lemma 1:** Let \( n \) be a positive integer. Then \( 2^{(3^n - 1)} \equiv -1 \mod 3^n \).

**Proof:** We will use mathematical induction \([4]\).

**Case 1:** If \( n = 1 \), \( 2^{(3^1)} = 2 \equiv -1 \mod 3^1 \)

**Case 2:** Assume the lemma is true for \( n = k \); i.e., \( 2^{(3^k - 1)} \equiv -1 \mod 3^k \). We may write \( 2^{(3^k - 1)} = 3^k q - 1 \) for some integer \( q \). To show the equation is true for \( n = k + 1 \), we cube both sides of this equation using the binomial theorem:

\[
2^{3^k} = (2^{3^{k-1}})^3 \\
= (3^k q - 1)^3 \\
= (3^k q)^3 - 3(3^k q)^2 + 3(3^k q) - 1 \\
= q^3 3^{3k} - 3q^2 3^{2k} + 3(3^k q) - 1 \\
= q^3 3^{3(3^k - 1)} - q^2 3^{3k+1} + 3^{k+1} q - 1 \\
= (q^3 3^{3(3^k - 1)} - q^2 3^{3k} + q) 3^{k+1} - 1
\]

So \( 2^{3^k} \) is a multiple of \( 3^{k+1} \) and the formula is verified for \( n = k + 1 \).

**Example:** For \( n = 4 \), \( 2^{(3^4 - 1)} = 2^{27} = 134,217,728 \equiv 80 \equiv -1 \mod 81 \).

**Theorem 3:** Suppose \( k \) is a positive integer and \( l = 3^k - 1 \cdot r \), where \( r \) is an odd integer. If \( N = \left(\frac{2}{3}\right)^k (2^l + 1) - 1 \) then the Collatz sequence beginning with \( N \) terminates at 1 after \( l + k \) steps.

**Proof:** By invoking Corollary 1, it suffices to show that \( N \) is an odd integer under the given hypotheses. By Lemma 2, \( 2^{(3^{k-1})} \equiv -1 \mod 3^k \).

\[
2^{(3^{k-1})} \equiv -1 \mod 3^k \\
\Rightarrow 2^{(3^{k-1}) m} \equiv (-1)^m \mod 3^k \\
\Rightarrow 2^l + 1 \equiv (-1)^m + 1 \equiv 0 \mod 3^k , \text{ since } m \text{ is an odd integer.}
\]

Because \( 2^l + 1 \) is a multiple of \( 3^k \), it follows that \( N \) is an odd integer.
**Example:** \( k = 2, \ l = 3^2 \cdot 5 = 15 \rightarrow N = \left(\frac{2}{3}\right)^2 (2^{15} + 1) - 1 = 14563 \). The Collatz sequence for 14563 is:

14563 \rightarrow 21845 \rightarrow 32768 \rightarrow 16384 \rightarrow 8192 \rightarrow \cdots \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \ (17 \text{ steps})

Corollaries 3 and 4 provide special cases of starting values for Collatz sequences that satisfy the Collatz conjecture and for which we can provide the number of steps required to reach 1.

**Corollary 3:** If \( q \) is an integer and \( N = 4^q - 1 \), then the Collatz sequence starting with \( N \) reaches 1 in \( 2^q \) steps.

**Proof:** Let \( k = 1 \) in Theorem 2 and simplify algebraically.

**Example:** \( q = 4 \rightarrow N = \frac{4^4 - 1}{3} = 85 \).
The Collatz sequence for 85 is: 85 \rightarrow 128 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \. (8 \text{ steps}).

**Corollary 4:** If \( r \) is an odd integer and \( N = \frac{2^{3r+2} - 5}{9} \), then the Collatz sequence starting with \( N \) reaches 1 in \( 3r + 2 \) steps.

**Proof:** Let \( k = 2 \) in Theorem 2 and simplify algebraically.

**Example:** \( r = 3 \rightarrow N = \frac{2^{11} - 5}{9} = 227 \).
The Collatz sequence for 227 is: 227 \rightarrow 341 \rightarrow 512 \rightarrow 256 \rightarrow 128 \rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \. (11 \text{ steps})

We now derive some results which identifies some properties of general Collatz sequences. We begin by categorizing positive odd integers.

**Definitions:** Every odd positive integer \( N \) falls into one of three categories:

(i) \( N = 6r + 1, r = 0, 1, 2, \ldots \), (call this **Type A**) Examples: 1, 7, 13,....
(ii) \( N = 6r + 3, r = 0, 1, 2, \ldots \), (call this **Type B**) Examples: 3, 9, 15,....
(iii) \( N = 6r + 5, r = 0, 1, 2, \ldots \), (call this **Type C**) Examples: 5, 11, 17,....

**Theorem 4:** Every odd integer in a Collatz sequence except possibly the first one is of Type A or C.

**Proof:** Let \( N \) be an odd integer of type B in a Collatz sequence. Then there is an integer \( r \) such that \( N = 6r + 3 \). Suppose the number in the Collatz sequence preceding \( N \) in the sequence is the odd number \( N' \). Then by the Collatz algorithm, \( \frac{3N' + 1}{2} = N = 6r + 3 \). We then have \( 3N' + 1 = 12r + 6 \). Since 3 divides the right-hand side of this equation it must divide the left-hand side, which is impossible. We conclude that \( N \) cannot be immediately preceded by an odd integer.

Therefore we can assume that either \( N \) is at the start of the Collatz sequence or there is an integer \( k \) such that the even numbers \( 2^k N, 2^{k-1} N, \ldots, 2 N \) comprise the sequence before \( N \). This implies that there is an odd number \( N' \) preceding \( 2^k N \). This means that \( \frac{3N' + 1}{2} = 2^k N = 2^k (6r + 3) = 2^k 3(2r + 1) \). So \( 3N' + 1 = 3(2^{k+1})(2r + 1) \). Since 3 divides the right hand side of this equation, it must divide the left side, which is a contradiction and the theorem is proved.

**Remark:** This means that no Collatz sequence contains an odd multiple of three; i.e., Type B except possibly at the start of the sequence.
Example: The Collatz sequence for $N = 9$: $9$ (type B) $\rightarrow$ $14$ $\rightarrow$ $7$ (type A) $\rightarrow$ $11$ (type C) $\rightarrow$ $17$ (type C) $\rightarrow$ $26$ $\rightarrow$ $13$ (type A) $\rightarrow$ $20$ $\rightarrow$ $10$ $\rightarrow$ $5$ (type C) $\rightarrow$ $8$ $\rightarrow$ $4$ $\rightarrow$ $2$ $\rightarrow$ $1$ (type A).

The following is a conjecture that expresses a relationship between an odd integer in a Collatz sequence and the next odd integer.

Conjecture: Let $N$ be an odd integer in a Collatz sequence and let $N' = \frac{3N + 1}{2}$ be the next term in the sequence. If $N'$ factors into $2^l m$ then $m$, the next odd integer in the sequence, is of type A if $l$ is odd and is of type C if $l$ is even.

Examples:

$N = 15 \rightarrow N' = 23 = 2^0 \cdot 23; l = 0$ (even) and $m = 23$ (type C).

$N = 117 \rightarrow N' = 176 = 2^4 \cdot 11; l = 4$ (even) and $m = 11$ (type C).

$N = 49 \rightarrow N' = 74 = 2^1 \cdot 37; l = 1$ (odd) and $m = 37$ (type A).

$N = 133 \rightarrow N' = 200 = 2^3 \cdot 25; l = 3$ (odd) and $m = 25$ (type A).

$N = 341 \rightarrow N' = 512 = 2^9 \cdot 1; l = 9$ (odd) and $m = 1$ (type A).

References

[1] J. C. Lagarias. The 3x + 1 Problem and its Generalizations. *The American Mathematical Monthly*, 92:3–23, 1985.

[2] Eric W. Weisstein. Collatz Problem. from Mathworld–a Wolfram Web Resource. [http://mathworld.wolfram.com/CollatzProblem.html](http://mathworld.wolfram.com/CollatzProblem.html)

[3] R. Terras. A Stopping Time Problem on the Positive Integers. *Acta Arithmetica*, 3(30):241–252, 1976.

[4] Wikipedia Contributors. Mathematical Induction — Wikipedia, The Free Encyclopedia. [https://en.wikipedia.org/w/index.php?title=Mathematical_induction&oldid=87442784](https://en.wikipedia.org/w/index.php?title=Mathematical_induction&oldid=87442784) 2018.