Axion Oscillations in Binary Systems: Angle-action Surgery

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Received 2020 March 31; revised 2020 July 29; accepted 2020 August 11; published 2020 September 24

Abstract

Scalar, tensor waves induce oscillatory perturbations in Keplerian systems that can be probed with measurements of pulsar timing residuals. In this paper, we consider the imprint of coherent oscillations produced by ultralight axion dark matter on the Roemer time delay. We use the angle-action formalism to calculate the time evolution of the observed signal and its dependence on the orbital parameters and the axion phase. We derive exact analytical expressions for arbitrary binary pulsar mass ratio and eccentricity, alleviating the need for long numerical integrations. We emphasize the similarity of the expected signal-to-noise ratio with the response of a harmonic oscillator to an external oscillatory driving. We validate our theoretical predictions with numerical simulations. Our results furnish a useful benchmark for numerical codes and analysis procedures and, hopefully, will motivate the search for such imprints in real data.

Unified Astronomy Thesaurus concepts: Dark matter (353); Binary pulsars (153); Milky Way dark matter halo (1049); Celestial mechanics (211); Perturbation methods (1215)

1. Introduction

Gravitational radiation can resonate with binary systems and produce potentially detectable orbital perturbations (Rudenko 1975; Mashhoon 1978; Turner 1979). When the perturbations arise from a stochastic background of gravitational waves, monitoring the randomness of the orbital elements—through the correlation functions of frequency shifts and timing residuals of pulsars, for instance—can set constraints on the amplitude of such a background (Mashhoon et al. 1981; Mashhoon 1985; Hui et al. 2013). This effect also takes place when a binary system is excited by scalar waves (Annulli et al. 2018), or embedded in a coherent background of very light bosons (Khmelnitsky & Rubakov 2014; Blas et al. 2017; Bošković et al. 2018; Bošković 2019; Rozner et al. 2019).

Light bosons such as QCD axions are interesting dark matter candidates because they can also resolve the strong CP problem in particle physics, as was first proposed by Peccei & Quinn (1977), Wilczek (1978), and Weinberg (1978). Ultralight axions are an extrapolation of the QCD axions—with expected masses in the range \( m_a \sim 10^{-5}\text{–}10^{-3}\text{eV} \) to much smaller masses \( m_a \sim 10^{-25}\text{–}10^{-20}\text{eV} \) (Press et al. 1990; Hu et al. 2000; Peebles 2000). At low redshift, they form a Bose–Einstein condensate that oscillates coherently (unlike a stochastic background) on a timescale \( \propto m_a^{-1} \) (e.g., Sikivie & Yang 2009; Marsh 2016; Hui et al. 2017; Grin et al. 2019; Niemeyer 2019, and references therein). Constraints on their mass \( m_a \) have been set using Ly\( \alpha \) forest measurements (Armengaud et al. 2017; Irič et al. 2017; Kobayashi et al. 2017) and cosmic microwave background lensing (Hložek et al. 2018) on megaparsec scales, dwarf spheroidals (Marsh & Pop 2015; González-Morales et al. 2017; Broadhurst et al. 2019; Marsh & Niemeyer 2019; Safarzadeh & Spergel 2019) and ultradiffuse galaxies (Wasserman et al. 2019) on kiloparsec scales, and galactic core observations on (sub)parsec scales (Bar et al. 2019; Desjacques & Nusser 2019; Davies & Mocz 2020). Pulsar timing offers another avenue to probe the existence of coherent oscillations induced by ultralight scalar

fields (Khmelnitsky & Rubakov 2014; Blas et al. 2017; de Martino et al. 2017). Upper limits on the amplitude of such an oscillating gravitational potential in the Milky Way halo have already been derived from pulsar timing arrays (Porayko & Postnov 2014; Porayko et al. 2018). Cross-correlation of residuals from different pulsars should improve these constraints (Hellings & Downs 1983).

Axion coherent oscillations also resonate with binary pulsars (Blas et al. 2017). While the effect is strongest near resonance, the very small width of the latter (when the coupling is purely gravitational) implies that one shall monitor instantaneous variations (Rozner et al. 2019) or the secular drift of orbital elements (Blas et al. 2019) away from resonances.

Mashhoon (1978) used Lagrange’s planetary equations to develop an approximate theory of the interaction of a weak gravitational wave with a Keplerian binary. In this paper, we use angle-action variables to investigate the instantaneous variations (that is, not averaged over one orbital time) of a Keplerian system produced by an oscillating background of axion dark matter. We refer the reader to Binney & Tremaine (1987) for an overview of the angle-action formalism.

The paper is organized as follows. After a brief presentation of the astrophysical/cosmological context and our numerical implementation in Section 2, we solve for the time evolution of the perturbed binary system using angle-action variables in Section 3. We explore the instantaneous variations of the Roemer time delay as a function of orbital parameters, etc., in Section 4. We conclude in Section 5.

2. Setup

We will use the numerical simulations of Rozner et al. (2019) to validate our theoretical predictions. We consider a binary pulsar system with total mass \( M = m_1 + m_2 \) and reduced mass \( \mu = m_1 m_2/(m_1 + m_2) \). The motion of the binary pulsar is integrated along with the perturbation induced by the coherent axion oscillations using the publicly available framework REBOUND (Rein & Liu 2012) and the fast, adaptive, high-order integrator IAS15 for gravitational dynamics (Rein...
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& Spiegel 2015), accurate to machine precision over a billion orbits. We will be interested in binaries far away from the inspiral phase so that general relativistic corrections can be neglected (see the discussion in Section 4.6). In all the subsequent illustrations, we adopt the same parameters as Rozner et al. (2019), that is,

1. a dark matter density \( \rho_{DM} = 5 \times 10^3 \, M_\odot \, \text{pc}^{-3} \);
2. an axion mass \( m_a = 10^{-30} \, \text{GeV} \);
3. an axion phase \( \alpha = 0 \); and
4. a total binary mass \( M = 2 \, M_\odot \).

The value of \( \rho_{DM} \) is comparable to the density \( \rho_c \) achieved near the hypothetical axion core (of radius \( R_c \sim 1 \, \text{pc} \)) located in the vicinity of the Milky Way halo center when the axion mass is \( 10^{-30} \, \text{GeV} \) (Chavanis 2011). In the solar neighborhood, the dark matter density is smaller by five orders of magnitude, \( \rho_{DM} \sim 0.03 \, M_\odot \, \text{pc}^{-3} \) (Salucci et al. 2010; Read 2014). Furthermore, we conveniently define

\[
\omega_d = 2m_a.
\]

At the fundamental resonance for which \( \Omega = \Omega_0 = \omega_d \), the orbital frequency is \( \Omega_0 \approx 3.062 \times 10^{-6} \), and the semimajor axis is \( a_0 \approx 0.205 \, \text{au} \). Finally, note that the axion phase \( \alpha \equiv \alpha(x) \) generally is a function of the spatial position \( x \) and, thus, actually varies among binary pulsar systems.

Following Rozner et al. (2019), we will focus on the signal imprinted in the Roemer time delay, which can be extracted from measurements of the pulse times of arrival (TOAs) at the detector. The Roemer time delay is the variation of the light-travel time due to perturbations in the distance between the detector and the pulsar. In plain words, axion coherent oscillations induce a perturbation \( \delta r(t) \) to the separation vector of the binary system at a given time \( t \). Ignoring the apparent viewing geometry of the latter for simplicity, this translates into a perturbation \( \Delta T_{\text{TOA}}(t) = \frac{1}{c} |\delta r(t)| \) (\( c \) is the speed of light) in the pulse TOAs. The signal-to-noise ratio (S/N) for this effect can be expressed as

\[
\left( \frac{S}{N} \right)^2 = \frac{1}{\sigma_\Delta^2} \sum_{i=1}^{N} (\Delta T_{\text{TOA}}(t_i))^2,
\]

where \( t_i = i \Delta \) are the times at which a TOA measurement is performed and \( \sigma_\Delta \) is the error on the TOA for a pulse shape averaged over a time interval \( \Delta \). In what follows, we shall adopt \( \sigma_\Delta = 10^{-6} \, \text{s} \) for \( \Delta = 10 \, \text{s} \). The (ideal) number of measurements is \( N = t_{\text{obs}}/\Delta \), where \( t_{\text{obs}} \) is the total time of observations. In practice, TOA measurements will be performed only a fraction \( f_{\text{obs}} \) of the time. We shall hereafter assume \( f_{\text{obs}} = 10^{-3} \).

Equation (2) is the sole observable we shall consider here, as the Roemer delay is the largest effect in magnitude. However, the results presented in Section 3.3 can be used to calculate the signals imprinted in other delays (see, e.g., Edwards et al. 2006, for a detailed overview of timing models).

3. The Axion Perturbation in Angle-action Formalism

To illustrate the power of the angle-action formalism, we will use the Delaunay variables. For simplicity, however, we will focus on the two-dimensional dynamics (justified since the angular momentum vector is conserved also in the perturbed system). Therefore, we can restrict ourselves to the Delaunay angles \( \theta_a = \{ \theta_a, \theta_b \} \) and actions \( J_a = \{ J_a, J_b \} \) (they correspond to the angles \( \{ \theta_2, \theta_3 \} \) and actions \( \{ J_2, J_3 \} \) in Binney & Tremaine 1987). We will designate the polar variables as \( q_a = \{ r, \vartheta \} \) and \( p_a = \{ p_r, p_\vartheta \} \).

3.1. Hamiltonian

The total Hamiltonian of the system is \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \). The unperturbed Hamiltonian

\[
\mathcal{H}_0(J) = -\frac{\mu k^2}{2 \ell^2},
\]

where \( k = GM \mu \), describes the Keplerian motion. In polar coordinates, the perturbation Hamiltonian takes the form

\[
\mathcal{H}_1 = 2\pi G \rho_{DM} \mu \cos(\omega_a t + \alpha) r^2 \equiv \mu \ell \Omega^2 a^2 \cos(\omega_a t + \alpha) \left( \frac{r}{a} \right)^2.
\]

Here \( a \) and \( \Omega \) are the semimajor axis and frequency of the binary, respectively. The amplitude of the perturbation Hamiltonian relative to \( \mathcal{H}_0 \) is quantified by the small parameter

\[
\epsilon = \frac{2\pi \rho_{DM} a^3}{M} = \epsilon_0 \left( \frac{a}{a_0} \right)^3,
\]

where \( \epsilon_0 \) is evaluated at the fundamental resonance. For our fiducial parameters (see Section 2), it is

\[
\epsilon_0 = \frac{2\pi \rho_{DM} a_0^3}{M} \approx 3.073 \times 10^{-14}.
\]

This justifies a perturbative treatment at first order in \( \epsilon \). The \( a^3 \)-scaling reflects the fact that \( |\mathcal{H}_0| \sim \frac{1}{a} \), while

\[
|\mathcal{H}_1| \sim \epsilon \Omega^2 a^2 \sim \epsilon_0 \left( \frac{a}{a_0} \right)^3 a^{-3} a^2 \sim a^2.
\]

As expected, the effect of axion oscillations increases with the physical volume enclosed by the binary motion.

Using the definition of \( \epsilon \), together with Kepler’s third law \( \Omega^2 a^3 = GM \) and the relation \( a = J_r^2 / \mu k \), the overall multiplicative factor in \( \mathcal{H}_1 \) can be conveniently expressed as

\[
\frac{1}{2} \mu \epsilon \Omega^2 a^2 = \frac{1}{2} \epsilon_0 \frac{J_r^4}{a_0^3 \mu^2 k^2}.
\]

Furthermore, the Fourier cosine decomposition of \( (r/a)^2 \) on the unperturbed Keplerian orbit (justified by the smallness of \( \epsilon \), for which \( \theta_a \) equals the mean anomaly \( \mathcal{M} \), reads

\[
\left( \frac{r}{a} \right)^2 = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2} \mathcal{J}_n(n \vartheta) \cos(n \theta_a).
\]

Here and henceforth, \( 0 \leq \epsilon < 1 \) will denote the eccentricity. As a result, the perturbation Hamiltonian in the angle-action
variables can be recast into the form
\[
\mathcal{H}(\theta, J_\theta, J_c; t) = \frac{1}{2} \epsilon_0 \frac{J_c^2}{a_0^2} \frac{1}{\mu^2 k} \left[ \cos(\omega_\theta t + \alpha) \right] \times \left( 5 - 3 \frac{J_c^2}{J_c^2} \right)
\]
\[- \sum_{n=1}^{\infty} \frac{4}{n^2} J_n(ne) \left\{ \cos(\omega_\theta t + n\theta_\theta + \alpha) + \cos(\omega_\theta t - n\theta_\theta + \alpha) \right\},
\]
where it is understood that \( \epsilon \equiv \sqrt{1 - J_c^2/J_c^2}. \) Note also that \( \frac{\partial J_c}{\partial \theta} = \epsilon \Omega. \)

Equation (10) will be useful for the computation of the time evolution of the perturbations (presented in Section 3.3).

### 3.2. Perturbed Displacement

In the variables \((\theta_\theta, \theta_c, J_\theta, J_c)\), the separation vector \(r(t)\) takes the general form
\[
r(t) = r(\theta_\theta(t), \theta_c(t), J_\theta(t), J_c(t)).
\]
The time dependence of the angle-action variables is governed by Hamilton equations:
\[
\dot{\theta}_\theta = -\frac{\partial \mathcal{H}}{\partial J_\theta}, \quad \dot{J}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta_\theta},
\]
\[
\mathcal{H} = \mathcal{H}_0(J_\theta) + \mathcal{H}_1(\theta, J_\theta, J_c; t).
\]
The unperturbed solution given by \(\mathcal{H}_0\) is
\[
r_0(t) = r(\theta^0_\theta(t), \theta^0_c(t), J^0_\theta(t), J^0_c(t)),
\]
with
\[
\theta^0_\theta(t) = 0, \quad \theta^0_c(t) = M(t),
\]
\[
J^0_\theta = J^0_c \sqrt{1 - e^2}, \quad J^0_c = \sqrt{k M a_0}.
\]

Combining the previous expressions, the displacement vector reads
\[
\delta r(t) = r(t) - r_0(t) \approx (\cos \delta, \sin \delta) \delta \dot{r} + r(-\sin \delta, \cos \delta) \delta \dot{\theta}
\]
at first order in the small perturbation \( \epsilon \ll 1 \), with
\[
\delta r = \frac{\partial r}{\partial \theta} \delta \theta + \frac{\partial r}{\partial J_\theta} \delta J_\theta,
\]
\[
\delta \delta = \frac{\partial \delta}{\partial \theta} \delta \theta + \frac{\partial \delta}{\partial J_\theta} \delta J_\theta.
\]

These relations follow from writing the position vector in polar coordinates, \( r = r(\cos \delta, \sin \delta) \). Since the perturbations \( \Delta \theta_\theta(t) \) and \( \Delta J_\theta(t) \) are first order in \( \epsilon \), the partial derivatives of the polar coordinates are computed on the unperturbed orbit. Furthermore, the Einstein summation convention is implied here and throughout this paper.

To calculate the derivatives of the polar coordinates with respect to the angle-action variables, we start from the function generating the canonical transformation to the Delaunay variables. The details and results of the computation can be found in Appendix A.

To conclude this section, one could in principle transform to a new set of angle-action coordinates constructed such that the perturbed Hamiltonian depends on the new action variables solely (up to first order in \( \epsilon \)). This standard procedure is briefly reviewed in, e.g., Annulli et al. (2018). However, it does not give any practical advantage in the computation of the signal we are aiming at. Therefore, we have not implemented it here.

### 3.3. Time Evolution

Next, we compute the perturbations \( \Delta \theta_\theta(t) \) and \( \Delta J_\theta(t) \) to the angle-action variables from the perturbation Hamiltonian Equation (4). We require that the perturbed and unperturbed orbits coincide initially (that is, at the beginning of the observational period), so that \( \Delta \theta_\theta \) and \( \Delta J_\theta \) vanish at \( t = 0 \). We will denote the initial eccentric and mean anomaly as \( \xi_0 \) and \( \mathcal{M}_0 \), respectively. On the unperturbed trajectory, we thus have \( \theta^0_\theta(t) = \mathcal{M}(t) = \Omega t + \mathcal{M}_0 \), with \( \mathcal{M}_0 = \xi_0 - \epsilon \sin \xi_0 \). The offset between the periapsis passages and the peaks of the axion oscillatory forcing evolves with time depending on the initial conditions (\( \alpha, \mathcal{M}_0 \)) and the frequencies (\( \omega_\theta, \Omega \)) unless one sits at a resonance (in which case only \( \alpha \) and \( \mathcal{M}_0 \) matter).

For the angles, Hamilton equations give
\[
\Delta \theta(t) = \int_0^t dt' \frac{\partial \mathcal{H}_1}{\partial J_\theta},
\]
\[
= -3 \epsilon \left\{ \Omega \frac{\omega_\theta}{\omega_\theta} \sqrt{1 - e^2} \left( \sin(\omega_\theta t + \alpha) - \sin \alpha \right) + 2 \epsilon \Omega \frac{\omega_\theta}{\omega_\theta} \sqrt{1 - e^2} \sum_{n=1}^{\infty} \frac{J_n(ne)}{n} \times S(n)_{-1}(\omega_\theta, \Omega, \alpha, \mathcal{M}_0; t)
\]
\[
\Delta \theta_\theta(t) = \int_0^t dt' \frac{\partial \mathcal{H}_1}{\partial J_\theta},
\]
\[
= -6 \epsilon \left\{ \Omega \frac{\omega_\theta}{\omega_\theta} \sum_{n=1}^{\infty} \frac{J_n(ne)}{n} \left[ \left( \Omega \frac{\omega_\theta}{\Omega} \right) S(n)_{-1}(\omega_\theta, \Omega, \alpha, \mathcal{M}_0; t)
\right.
\right.
\]
\[
\left. \left. - \Omega t A_n \left( \Omega \frac{\omega_\theta}{\Omega} \right) \right] + \epsilon \left( \Omega \frac{\omega_\theta}{\omega_\theta} \right) (7 + 3 e^2) \times (\sin(\omega_\theta t + \alpha) - \sin \alpha) - 2 \epsilon \Omega \frac{\omega_\theta}{\omega_\theta} \times S(n)_{+1}(\omega_\theta, \Omega, \alpha, \mathcal{M}_0; t),\right.
\]
\[
\Delta J_\theta(t) = -\int_0^t dt' \frac{\partial \mathcal{H}_1}{\partial \theta},
\]
\[
= 0,
\]

which expresses the conservation of angular momentum, and
\[
\Delta J_\theta(t) = -\int_0^t dt' \frac{\partial \mathcal{H}_1}{\partial \theta},
\]
\[
= 2 \epsilon \left( \Omega \frac{\omega_\theta}{\omega_\theta} \right) \times J_n \sum_{n=1}^{\infty} \frac{J_n(ne)}{n} C(n)_{+1}(\omega_\theta, \Omega, \alpha, \mathcal{M}_0; t).\]
To derive all these expressions, we have substituted the unperturbed solution $\theta_0(t)$ and taken advantage of the relations $J_1 = \mu \Omega a^2$ and $\Omega = \mu k^2 / J_3^3$ to simplify them further. For shorthand convenience, we have also introduced the functions $S_{nq}^{(\pm)}(\omega_i, \Omega, \alpha, M_0; t)$ and $C_{nq}^{(\pm)}(\omega_i, \Omega, \alpha, M_0; t)$, defined as

$$S_{nq}^{(\pm)}(\omega_i, \Omega, \alpha, M_0; t) = \sin(\omega_i t + n\Omega t + \alpha + nM_0) \pm \sin(\omega_i t - n\Omega t + \alpha - nM_0) \quad \left(1 + \frac{n\Omega_i}{\omega_i}\right)^\gamma,$$

$$C_{nq}^{(\pm)}(\omega_i, \Omega, \alpha, M_0; t) = \cos(\omega_i t + n\Omega t + \alpha + nM_0) \pm \cos(\omega_i t - n\Omega t + \alpha - nM_0) \quad \left(1 - \frac{n\Omega_i}{\omega_i}\right)^\gamma.$$

The $(\pm)$ determines their parity under the transformation $n \to -n$. Furthermore, both $S_{nq}^{(\pm)}$ and $C_{nq}^{(\pm)}$ vanish at the initial time $t = 0$.

For $\Delta \theta_\alpha$, the expression is somewhat more involved because one needs to consider two variations:

$$\Delta \theta_\alpha = \Delta \left(\frac{\partial \theta_0}{\partial \Omega} + \frac{\partial \mathcal{H}_\alpha}{\partial \Omega} \right) = \frac{\partial \Omega}{\partial \Omega} \Delta \Omega + \frac{\partial \mathcal{H}_\alpha}{\partial \Omega} \Delta \Omega,$$  \hspace{1cm} (22)

The first term on the right-hand side is the perturbation to the Keplerian frequency. As a result, the angle $\theta_\alpha$ evolves faster (or slower) relative to the unperturbed case. We have

$$\frac{\partial \Omega}{\partial \Omega} \Delta \Omega = -3 \frac{\Omega}{J_3} \Delta \Omega.$$  \hspace{1cm} (23)

This effect vanishes at first order for a perfectly circular orbit ($e = 0$) because the unperturbed orbit sits at the bottom of the effective one-body (radial) potential. As a consequence, any variation in the frequency—or energy—must be second order for the circular case. Using Equation (20) and integrating over time, this becomes

$$-3 \frac{\Omega}{J_3} \int_0^t dt' \Delta \Omega(t') = -6 \left(\frac{\Omega}{\omega_i}\right) \frac{\sum_{n=1}^N J_2(\nu_e)}{n} \times \left[ \left(\frac{\Omega}{\omega_i}\right) S_{nq}^{(\pm)}(\omega_i, \Omega, \alpha; t) \right. \left. - \Omega \mathcal{A}_n \left(\frac{\Omega}{\omega_i}, \alpha, M_0\right) \right].$$  \hspace{1cm} (24)

The last two equalities in Equation (25) assume that the sum over discrete times can be traded for an integral, which is a good approximation when the number of measurements is large.

To validate our analytical results, we numerically evolved the perturbed and unperturbed system in polar coordinates and extracted the time evolution of $\delta r$ and $\delta \theta$, which we compared to our theoretical prediction obtained upon combining Equations (17)–(20) with the partial derivatives Equations (A9) and (A12). In practice, we truncated the series expansion of $\mathcal{H}_\alpha$ at the 20th harmonic. The results are shown in Figure 1 as a function of the eccentric anomaly $\xi$. The initial conditions were set at pericenter passage ($M_0 = \xi_0 = 0$). They assume a highly eccentric orbit with $e \approx 0.866$. As a consequence, the change in orbital frequency is the dominant effect. This translates into a fairly large perturbation in the polar angle $\delta \theta$ (relative to $\delta r$) owing to the term $\frac{\partial^2}{\delta \theta^2} \Delta \theta_\alpha$, which peaks at pericenter passage. Note that $\delta r$ is shown in units of the semimajor axis $a$. We emphasize that our calculation is valid everywhere except in small neighborhoods of size $\sqrt{r}$ centered on the resonances. The near-resonance case is thoroughly discussed in Blas et al. (2017) and Rozner et al. (2019). Note also that the angle perturbations given by Equations (17)–(18) do not depend on the reduced mass $\mu$ of the system, while Equation (20) does through the multiplicative factor of $J_3$. However, the latter cancels out in $\delta r$ and $\delta \theta$. Hence, the perturbation $\delta r$ to the separation vector truly is independent of the reduced mass, as requested by the equivalence principle.

4. Signal-to-noise Ratio for the Roemer Delay

Having solved the equations of motion to first order in $\epsilon$, we will now concentrate on the variations in Roemer time delay and the corresponding S/N as quantified in Section 2.

4.1. General Expression

Taking into account a duty cycle of $f_{obs}$ as advocated above, the S/N Equation (2) for the perturbation to the Roemer time delay can be expressed as

$$\left(\frac{S}{N}\right)^2 = \frac{f_{obs}^2}{\sigma^2 \Delta} \sum_{i=1}^N \Delta \left(\Delta r_{\alpha i}\right)^2(t_i) = \frac{f_{obs}^2}{\sigma^2 \Delta} \int_0^{t_{obs}} dt \frac{\left|\delta r(t)\right|^2}{e^2},$$

$$= \epsilon^2 a^2 \left(\frac{\Omega}{\omega_i}\right)^2 \left(\frac{f_{obs} f_{obs}^{\text{obs}}}{\sigma^2 \Delta e^2} \right) \left\{ \frac{1}{t_{obs}} \int_0^{t_{obs}} dt \left|\delta \mathcal{H}_\alpha(t)\right|^2 \right\},$$  \hspace{1cm} (25)

where, for convenience, we have introduced a dimensionless displacement $\delta \mathcal{H}_\alpha(t)$ defined through the relation

$$\delta r(t) \equiv \epsilon a \left(\frac{\Omega}{\omega_i}\right) \delta \mathcal{H}_\alpha(t).$$  \hspace{1cm} (26)
time variable and recast Equation (25) into

\[
\frac{S}{N}^2 = \left( \frac{\epsilon}{\Omega a} \right)^2 \frac{f_{\text{obs}} f_{\text{obs}}}{\sigma^2 \Delta^2} \times \left\{ \frac{1}{\Omega_{\text{obs}}} \int_{\xi_0}^{\xi_{\text{obs}}+\xi_0} d\xi (1 - \epsilon \cos \xi) |\delta r|^2(\xi) \right\},
\]

(27)

where \(\xi_{\text{obs}}\) denotes the amount of eccentric anomaly elapsed during the observational run. The upper limit \(\xi_{\text{obs}} = \xi_{\text{obs}}(t_{\text{obs}})\) of the integral is determined from \(\Omega t_{\text{obs}} = \mathcal{M}(\xi_{\text{obs}}) - \mathcal{M}(\xi_0)\).

To get insight into the dependence of the S/N on the axion mass and the orbital parameters, consider the limit \(\Omega \gg \omega_a\). In this regime, the perturbation produced by the axion coherent oscillations can be treated as time independent. Therefore, Equation (4) shows that they yield a force of amplitude \(\epsilon \Omega r \sim \epsilon \Omega a\) per unit mass. This implies that the variation \(\delta a\) is

\[
\delta a = \int_0^T dt \delta \dot{r} \sim \epsilon \Omega \int_0^T dt \sim \epsilon \Omega a
\]

(28)

since the orbital period is \(T = 2\pi/\Omega\). For a total observational time \(t_{\text{obs}}\), the change \(\delta a\) thus is

\[
\delta a \sim \int_0^{t_{\text{obs}}} dt \delta \dot{a} \sim \epsilon \Omega a t_{\text{obs}} \sim a^{5/2} t_{\text{obs}}.
\]

(29)

Since the overall amplitude of the S/N is proportional to

\[
\epsilon \cdot \Omega \cdot a \propto a^3 \cdot a^{-3/2} \cdot a \propto a^{5/2},
\]

(30)

Equation (29) suggests that the S/N should behave like \(a^{5/2}\) in the limit \(a \rightarrow 0\), or, equivalently, the term in curly brackets in Equation (27) asymptotes to a constant in the same limit. We will see that this is indeed the case.

4.2. Development at Small Eccentricities

Emission of gravitational waves will eventually circularize the orbit of binary pulsar systems, so that \(e \approx 0\) is a very good approximation in the late stage of the coalescence phase (Peters 1964). Therefore, it is instructive to develop the previous results for low eccentricities. As we shall see now, the S/N can be cast into a simple expression in the limit \(e \rightarrow 0\). Details can be found in Appendix B.
The square $|\delta \mathbf{r}|^2$ of the normalized perturbed displacement can eventually be expressed as

$$
|\delta \mathbf{r}|^2 = |\sin \xi S_{11}^{(+)}/(t) + \cos \xi C_{11}^{(-)}|^2 + 4\left(2(\sin(\omega \alpha + \alpha) - \sin \alpha) + \sin \xi C_{11}^{(-)} - \cos \xi S_{11}^{(+)}/(t)^2 + O(e)\right).
\tag{31}
$$

Integrating this expression from $t = 0$ until $t = t_{\text{obs}}$ returns terms linear in $t_{\text{obs}}$ along with a transient contribution that vanishes in the limit $t_{\text{obs}} \to \infty$. We shall mainly focus on the former in the following discussion since it solely survives for large $\xi_{\text{obs}}$, which is the experimental setup considered here. However, we will also show the full result for the sake of comparison with the data.

### 4.3. The Case $e = 0$

For a circular orbit, the value of the initial mean anomaly $M_0$ and the phase $\alpha$ of the axion field are irrelevant to the S/N. Therefore, we can choose $M_0 = \alpha = 0$ without any restriction. With this simplification, a straightforward calculation shows that

$$
\frac{1}{\xi_{\text{obs}}} \int_{\xi_{\text{obs}}}^{\xi_{\text{obs}}+\xi_0} d\xi |\delta \mathbf{r}|^2(\xi) = R_\infty(\frac{\Omega_0}{\omega_{\text{a}}}) + \text{transient}.
\tag{32}
$$

The transient contribution is of the form

$$
\sum_i c_i \sin(\omega_i\xi_{\text{obs}}),
\tag{33}
$$

where the various amplitudes $c_i$ and frequencies $\omega_i$ (loosely labeled with an index $i$) are functions of the frequencies $m_0$ and $\Omega$. Its explicit expression—which is subdominant in the long-time asymptotic limits—is too long to be given here. The response function $R_\infty(x)$—which dominates in the long-time asymptotic limit—takes the form

$$
R_\infty(x) = 8x^2 \frac{(2 + x^2)}{(1 - x^2)^2}.
\tag{34}
$$

It diverges at the fundamental resonance $x = 1$, and its asymptotic behavior as the argument tends toward zero or infinity is

$$
R_\infty(x) = \begin{cases} 12x^2 + O(x^4) & (x \to 0) \\ 8 + 28x^2 + O(x^4) & (x \to \infty). \end{cases}
\tag{35}
$$

Unsurprisingly, $R_\infty(x)$ is analogous to the response function (or transfer function) of a standard driven harmonic oscillator of frequency $\Omega_0$, in which the axion oscillations $\alpha \sin(\Omega_0/t + \alpha)$ play the role of the external driving force (this can also be seen upon writing the orbit equation for $u(\theta)$, where $u \equiv 1/r$). In the limit $x \to \infty$ (slow driving), this force can be treated as constant, and therefore $R_\infty(x) \to \text{const}$. Since the displacement is independent of frequency. In the limit $x \to 0$ (fast driving), the constant (i.e., frequency-independent) terms in Equation (31) cancel out so that the response function is proportional to $x^2 \sim (\omega_{\text{a}})^{-2}$. All this remains true when $e > 0$ (see Section 4.4).

Substituting the response function into Equation (27), the long-time S/N reads

$$
\left(\frac{S}{N}\right)^2 = \left(\frac{e}{\omega_{\text{a}}}ight)^2 f_{\text{obs}} t_{\text{obs}} \frac{R_\infty(\Omega_{\text{a}})}{\omega_{\text{a}}}.
\tag{36}
$$

Using Equation (30), the S/N expressed as a function of the semimajor axis $a$ scales like

$$
\left(\frac{S}{N}\right) \propto \begin{cases} a^{5/2} & (a \ll a_0) \\ a & (a \gg a_0). \tag{37} \end{cases}
$$

The $a^{5/2}$ behavior in the regime $a \ll a_0$ reflects the scaling $R_\infty(x) \to \text{const}$ of the response function for large orbital frequencies $x \gg 1$.

Our prediction with the response function Equation (34) is shown in Figure 2 as the solid (red) curve. The overlaid dashed (magenta) curve represents the full result (i.e., including the transient contribution). For comparison, the blue data points indicate the simulated S/N for a circular orbit. Note that we have artificially increased $\epsilon_0$ by a factor of $10^5$ in order to reduce the numerical noise.

### 4.4. The Case $0 < e < 1$

When the eccentricity is different from zero, the perturbation to the Keplerian frequency Equation (23) provides the greatest contribution to the signal across the lowest-order resonances in the limit $t_{\text{obs}} \gg \Omega^{-1}$ since the amplitude of this effect grows like $(\Omega t)^2$. The variation of the Keplerian frequency (unlike a simple one-dimensional harmonic oscillator for which the fundamental frequency is fixed) leads to an infinite series of resonances located at $k\Omega = \omega_{\text{a}}$, with $k \in \mathbb{N}$.

#### 4.4.1. Perturbation to the Orbital Frequency

The term proportional to $A_n$ in Equation (23) dominates the perturbation to the orbital frequency. Since a change in the latter affects $t_{\text{obs}}$ solely, its contribution to the S/N of the Roemer time delay is given by

$$
\frac{1}{\xi_{\text{obs}}} \int_{\xi_0}^{\xi_0+\xi_0} d\xi (1 - e \cos \xi) \left[\left(\frac{\partial \delta}{\partial \theta_{\text{c}}}\right)^2 + r^2 \left(\frac{\partial \delta}{\partial \theta_{\text{r}}}\right)^2\right]
\times 36(\Omega \tau)^2 \left\{ \sum_{n=1}^{\infty} \frac{J_n(ne)}{n} A_n \left[\frac{\Omega}{\omega_{\text{a}}}, \alpha, M_0\right] \right\}^2
\tag{38}
$$

For large values of $\xi_{\text{obs}} \gg \xi_0$, the integral over the eccentric anomaly (as emphasized by the curly brackets) asymptotes to $\xi_{\text{obs}}^2$ in the long-time limit, with transient residuals proportional to $\xi_{\text{obs}}$ (in the best case) that can safely be neglected. Therefore, the frequency change yields a contribution

$$
R_\infty \supset 12 \xi_{\text{obs}}^2 \left\{ \sum_{n=1}^{\infty} \frac{J_n(ne)}{n} A_n \left[\frac{\Omega}{\omega_{\text{a}}}, \alpha, M_0\right] \right\}^2.
\tag{39}
$$
to the response function. As we will see shortly, it depends sensitively on the axion phase $\alpha$ and the initial condition $\mathcal{M}_0$. Since, at fixed $t_{\text{obs}}$, the total observed lapse of eccentric anomaly is $\xi_{\text{obs}} \propto \Omega$, the corresponding $S/N$ decays like $\propto a^{-1/2}$ for $a > a_0$ and eventually drops below the $S/N \propto a$ arising from the other perturbations.

### 4.4.2. Instantaneous Perturbations

To calculate the combined effect of these instantaneous perturbations, we must, here again, take into account all the resonances because they lead to the cancellation of the zeroth-order term $\propto \Omega^0$, such that the behavior $S/N \propto a$ is recovered in the regime $\Omega \to 0$. We demonstrate this point in Appendix C. Alternatively, notice that

$$
\delta q_a(t) = \frac{\partial q_a}{\partial \theta_a}(t) \Delta \theta_a(t) + \frac{\partial q_a}{\partial J_a}(t) \Delta J_a(t)
$$

$$
= \int_0^t dt'[q_a(t), \mathcal{H}(t')] \xi_{\text{obs}}(t)
$$

$$
= 4\pi G\rho_{\text{DM}} \int_0^t dt' \cos(\omega_q t' + \alpha) r(t') \xi_{\text{obs}}(t)
$$

$$
= 4\pi G\rho_{\text{DM}} \int_0^t dt' \xi_{\text{obs}}(t)
$$

$$
= 4\pi G\rho_{\text{DM}} \int_0^t dt' \xi_{\text{obs}}(t)
$$

$$
\xi_{\text{obs}}(t) \propto \exp(1 - e \cos \xi')
$$

$$
\times \cos(\omega_q t') \xi'(\xi') + \alpha) r(\xi') q_a(\xi), \ r(\xi') \xi_{\text{obs}}(t).
$$

where $q_a = \{r, \vartheta\}$ are the polar coordinates. We have ignored the frequency change arising from $\mathcal{H}_0$ (since we treat it separately). In the limit $\Omega \to 0$, the variable $t$ evolves independently of the eccentric anomaly, which remains constant and equal to $\xi = \xi_0$ during the entire observational period. As a result, the Poisson brackets converge toward $[\xi_0, r(\xi_0)]$, which must vanish by definition. Therefore, there is no contribution to the $S/N$ proportional to $\epsilon \cdot \Omega \cdot a \sim a^{3/2}$ in the limit $\Omega \to 0$.

To derive the leading nonvanishing contribution, we expand the functions $\mathcal{S}$ and $\mathcal{C}$ that appear in Equations (17)–(20) in the small ratio $\Omega/\omega_0$, as in Equation (C2). Namely, we must retain the argument $n\Omega t$ of the trigonometric functions because $t$ can be arbitrarily large and thus $\Omega t$ is not necessarily small.

Obtaining the exact functional dependence on $e, \alpha$, and $\mathcal{M}_0$ is challenging owing to the presence of a multiplicative factor of $(1 - e \cos \xi)^{-1}$. A rough approximation can be derived upon treating $t$ and $\xi$ as independent variables (an approximation justified by the fact that the frequencies $\Omega$ and $\omega_0$ are vastly different) and setting $\xi = \xi_0$ (which has the advantage of removing factors of $(1 - e \cos \xi)^{-1}$ in the integrand). Successively averaging over $t$ (with $0 \leq t \leq 2\pi/\omega_0$) and $\xi$ (with $0 \leq \xi < 2\pi/\Omega$), we eventually arrive at

$$
\mathcal{R}_\infty \supset 2 \left( \frac{\Omega}{\omega_0} \right)^2 (1 + 33e^2)(2 + \cos 2\alpha).
$$

Although the dependence on $\alpha$ is certainly incorrect (in the limit $e \to 0$, any dependence on $\alpha$ should vanish as outlined in Section 4.3), this shows that the amplitude of this effect mildly increases with the eccentricity.

#### 4.4.3. Response Function for $0 < e < 1$

In analogy with the circular case, the $S/N$ for $0 < e < 1$ can be recast into the form

$$
\left( \frac{S}{N} \right)^2 = \left( \frac{e}{\omega_0} \right) \frac{\mathcal{J}_{\text{obs}} t_{\text{obs}}}{\sigma^2 e^2 \Delta} \mathcal{R}_\infty \left( \mathcal{M}_0; t_{\text{obs}} \right).
$$

in which the response function $\mathcal{R}_\infty$ is the sum of Equations (39) and (41). The $S/N$ behaves like

$$
\left( \frac{S}{N} \right) \propto \begin{cases} a^{3/2} \quad (a \ll a_0) \\ a^{-1/2} \quad (a \approx a_0) \\ a \quad (a \gg a_0), \end{cases}
$$

where $a \sim a_0$ signifies “in the resonant region.” As emphasized earlier, this behavior is consistent with the response of a harmonic oscillator to an external, oscillatory perturbation.

When the $S/N$ is shown as a function of axion mass $m_a$ for a fixed orbital configuration (as would arise from the analysis of a given pulsar timing residuals series), Equation (43) gives the behavior $\propto m_a^{-1}$ for $m_a \ll \Omega$, and $\propto m_a^{-2}$ for $m_a \gg \Omega$. This behavior can be seen in Figure 3, where the sensitivity curve is shown for two different eccentricities.

### 4.5. Validation with Numerical Simulations

In all subsequent illustrations, the theory curve represents the long-time asymptotic result characterized by the response function $\mathcal{R}_\infty$ (that is, we neglect the transient contribution).

Figure 4 demonstrates the impact of the initial mean anomaly on the $S/N$. When $\mathcal{M}_0 = 0$ or $\pi$, the response function
Figure 3. S/N as a function of the axion mass \( m_a \), when the orbital parameters are fixed to their fiducial value (see Section 2). Results are shown for a near-circular \((e = 0.01)\) and highly eccentric \((e = 0.75)\) orbit. For small axion masses \( m_a \ll \Omega_0 \), the S/N scales like \( m_a^{-1} \) as indicated in the figure.

Figure 4. Dependence of the S/N on the initial conditions. The solid and dashed curves show the simulations and theoretical predictions for \( M_0 = 0 \) (blue) and \( M_0 = \pi \) (green). Results are shown for an unperturbed Keplerian orbit with \( e = 0.5 \). An axion phase \( \alpha = 0 \) is assumed throughout.

simplifies to

\[
\mathcal{R}_\infty(x, e, \alpha, M_0; \xi_{\text{obs}}) = 48 \xi_{\text{obs}}^2 \left[ \sum_{n=1}^{\infty} \frac{\chi_d(ne)}{(1 - n^2x^2)} \right]^2
\]

with

\[
\begin{align*}
\sigma(n) &= 1 \quad (M_0 = 0) \\
\sigma(n) &= (-1)^n \quad (M_0 = \pi).
\end{align*}
\]

The resonant patterns vary significantly (the difference can exceed an order of magnitude at a given eccentricity) across the range of values spanned by \( a \). Near the resonances, the discrepancy between the numerical data and the theoretical predictions is due to transients, which are negligible away from resonances provided that \( t_{\text{obs}} \) is larger than a few orbital times.

Figure 5 compares the prediction Equation (42) (dashed curves) to the numerical result (solid curves) for different values of the eccentricity. The model parameters are \( M_0 = \pi \) and \( \alpha = 0 \). The agreement between theory and simulations is excellent for all the configurations considered here.

For the range of semimajor axis values shown here, the contribution Equation (39) dominates the S/N shown in Figures 4 and 5. Notwithstanding, the contribution Equation (41) turns out to be significant for the nearly circular orbit with \( e = 0.01 \) above \( a \gtrsim 0.5 \) au.

In the classical (Newtonian) setting adopted here (see Section 4.6 for a justification), our analytical results are valid so long as the dimensionless parameter

\[
e \simeq 1.43 \times 10^{-15} \left( \frac{\rho_{\text{DM}}}{M_\odot \text{pc}^{-3}} \right) \left( \frac{M}{M_\odot} \right)^{-1} \left( \frac{a}{\text{au}} \right)^3
\]

is significantly smaller than unity, so that a leading-order perturbation treatment is justified. For a realistic binary pulsar mass \( \sim M_\odot \), we have \( e \lesssim 10^{-2} \) so long as the DM mass enclosed within the orbit satisfies

\[
\rho_{\text{DM}} a^3 \lesssim 10^{-3} M_\odot.
\]

For a semimajor axis \( \lesssim 1 \) au, this holds even for DM densities orders of magnitude larger than the solar neighborhood value. Note that \( e \) does not depend on either the axion mass or the orbital eccentricity.

4.6. General Relativistic Corrections

In principle, the smallness of axion perturbations to the orbital motion of a binary system requires the calculation to be carried out within a GR framework (see Hughes 2009; Blanchet 2014, for recent reviews).

GR corrections to \( J_c \) enter only at 2.5PN order, but there are other corrections at lower post-Newtonian orders that induce
variations in $J_a, J_0$. Overall, all these effects are calculable. Furthermore, at leading order in perturbations (relative to the Newtonian solution), most of them cancel out in Equation (25), as the latter involves the difference between the (Newtonian or GR) solution with $\epsilon = 0$ and that with $\epsilon > 0$. Higher-order GR corrections should be negligible unless the binary is about to merge, which is not the situation considered here.

Nevertheless, the loss of energy through the emission of gravitational radiation (or the axion particles themselves if $\Omega > m_a$; see Kumar Poddar et al. 2019) can affect the S/N around the resonances even when the system is far from coalescence. Ignoring the dependence on eccentricity, the power radiated in gravitational waves during one orbital period is (Peters 1964)

$$P_{gw} = \frac{32}{5c^5} \mu^2 G^{1/3} M^{4/3} \Omega^{10/3}.$$ \hfill (47)

By comparison, the power injected by the axion coherent oscillations during one orbital period is

$$P_a \approx \epsilon \mu \Omega^2 a^2 \cdot \Omega = \epsilon \mu G^{2/3} M^2 / \Omega^{5/3}.$$ \hfill (48)

This yields

$$\frac{P_{gw}}{P_a} \approx 4.38 \times 10^{-5} \left(\frac{\rho_{DM}}{M_\odot \text{pc}^{-3}}\right)^{-1} \left(\frac{\Omega}{\Omega_0}\right)^{5/2} \left(\frac{M}{M_\odot}\right)^{11/2} \left(\frac{a}{\text{au}}\right)^{11/2}.$$ \hfill (49)

Upon inserting our fiducial orbital parameters and assuming $\Omega \sim \omega_0$, $\mu = 1 M_\odot$, we find $P_{gw}/P_a \sim 3 \times 10^{-6}$. This ratio is independent of the axion mass and increases with decreasing $\rho_{DM}$. While it is small for the large $\rho_{DM}$ adopted here (see Equation (6), it would be of order unity for a dark matter density comparable to that of the solar neighborhood. In this case, we expect that the damping produced by gravitational wave emission smooths the response function around the resonances (in analogy with a simple one-dimensional damped, driven harmonic oscillator). For reasonable values of $\mu \sim M \sim M_\odot$, this occurs when

$$\left(\frac{\rho_{DM}}{M_\odot \text{pc}^{-3}}\right) \left(\frac{a}{\text{au}}\right)^{11/2} \gtrsim 10^{-5}. \hfill (50)$$

Summarizing, the calculation presented in this paper is accurate so long as the conditions (46) and (50) are simultaneously satisfied and, as explained in Rozner et al. (2019), the system is at least $\sqrt{\epsilon}$ away from resonances.

### 4.7. Resonances

Since the energy loss $\Omega^{-1} P_{gw}$ is very small compared to the binding energy of the system (except for the very last stages of the merger), the orbits shrink adiabatically owing to the emission of gravitational radiation. Using the classical formula Equation (47), the time spent in a semimajor axis interval of width $\Delta a \sim \sqrt{\epsilon}$ is

$$t_{res} \approx \frac{5}{154} \frac{\epsilon^5}{G^3 \mu M^2} a^3 \Delta a.$$ \hfill (51)

Consider now the fundamental resonance centered at $\Omega = \omega_0$. Using the techniques presented in Rozner et al. (2019), the width of the corresponding resonant region (that is, the libration region, which cannot be resolved with our perturbative approach) is

$$\Delta a = \sqrt{\frac{32 c J_0 (\epsilon)}{3 a_0}}.$$ \hfill (52)

Therefore, taking $\epsilon = 0.5$ for illustration, the time spent in the fundamental resonance is

$$t_{res} \approx 0.05 \frac{\epsilon^5}{G^3 \mu M^2} \sqrt{a_0}.$$ \hfill (53)

For our fiducial parameter values, we obtain $t_{res} \approx 6.1 \times 10^7$ yr. This shows that, for the orbital parameters adopted here, the system would stay at resonance for a duration much longer than any realistic observational time. In practice, however, the probability that a binary system will be found at resonance is very small owing to the smallness of $\sqrt{\epsilon}$.

### 5. Conclusions

We investigated the instantaneous variations produced by the coherent oscillations of ultralight axion dark matter of mass $m_a$ on a Keplerian binary. After solving the equations of motion at first order in the (small) perturbations, we focused on the response of the binary separation to this oscillatory driving force, the amplitude of which can be constrained with pulsar timing owing to its impact on the Roemer time delay. The relative amplitude of this effect is proportional to $G \rho_{DM}/M_\odot$ and thus is comparable to the relative imprint of axion oscillations on the gravitational potential (Khmelnitsky & Rubakov 2014) in the resonant region $\Omega \sim m_a$.

We computed the S/N for a measurement of instantaneous variations in the Roemer time delay, providing physical intuition whenever possible. In particular, we emphasized its similarity to the response of a harmonic oscillator to an external oscillatory driving (unsurprisingly, given the duality between the Kepler problem and the two-dimensional harmonic oscillator; see Arnol’d et al. 1990). We outlined the dependence of such a measurement on the orbital parameters, as well as the initial axion and orbital phases. We compared our theoretical predictions to accurate numerical simulations and found excellent agreement for a wide range of eccentricities $0 \leq e < 1$. Although we did not consistently include the back-reaction of the binary system, which can emit energy in the form of gravitational waves, etc. (see, e.g., Annuli et al. 2018, for a recent discussion), we estimate for which parameter values gravitational wave emission becomes relevant. Furthermore, we ignored the orientation of the orbital plane relative to the line of sight to the observer for simplicity, but this can be easily taken into account.

Our exact expressions furnish a useful benchmark for numerical codes and analysis procedures and, hopefully, will motivate the search for such imprints in real data. While we concentrated on dark matter in the form of a Bose–Einstein condensate of ultralight axions (for which the signal induced by oscillations in the gravitational potential is arguably small), our application of the angle-action formalism can, of course, be extended to other dark matter scenarios and/or different couplings (see, e.g., Blas et al. 2017; Nojirő et al. 2019).
V.D. and Y.B.G. acknowledge support by the Israel Science Foundation (grant No. 1395/16).

Appendix A
Generating Function

To calculate the derivatives of the polar coordinates with respect to the angle-action variables, consider the function \( \mathcal{W} \) generating the canonical transformation to the Delaunay variables,

\[
\mathcal{W}(r, \theta, J_\phi, J_\rho) = \int dr \cdot \text{sgn}(\ell) \left( \frac{\mu^2 k^2}{J_c^2} + \frac{2 \mu k}{r} - \frac{J_\rho^2}{r^2} \right) + J_\phi \phi , \tag{A1}
\]

and notice that the first of the two equations of canonical transformations

\[
\frac{\partial \mathcal{W}}{\partial J_\phi} = \theta_b \quad \text{and} \quad \frac{\partial \mathcal{W}}{\partial J_\rho} = \theta_c \tag{A2}
\]

involves the variables \((r, \theta, J_\phi, J_\rho)\), while the second involves only \((r, \theta, J_\phi, J_\rho)\). Therefore, we can write Equation (A2) as \( g_1 = g_2 = 0 \). The auxiliary functions \( g_1 \) and \( g_2 \) are

\[
g_1(r, \theta, J_\phi, J_\rho) = \theta_b - \frac{\partial \mathcal{W}}{\partial J_\phi} , \quad g_2(r, \theta, J_\phi, J_\rho) = \theta_c - \frac{\partial \mathcal{W}}{\partial J_\rho} , \tag{A3}
\]

with the understanding that all the variables should be treated as independent. Next, we can solve \( g_2 = 0 \) for \( r = r(\theta_c, J_\rho, J_\phi) \), which we subsequently substitute into \( g_1 = 0 \) to solve for \( \theta = \theta(\theta_b, \theta_c, J_\rho, J_\phi) \). In differential form, we have

\[
dg_1 = \frac{\partial g_1}{\partial r} dr + \frac{\partial g_1}{\partial \theta} d\theta + \frac{\partial g_1}{\partial J_\phi} dJ_\phi + \frac{\partial g_1}{\partial J_\rho} dJ_\rho ,
\]

\[
dg_2 = \frac{\partial g_2}{\partial r} dr + \frac{\partial g_2}{\partial \theta} d\theta + \frac{\partial g_2}{\partial J_\phi} dJ_\phi + \frac{\partial g_2}{\partial J_\rho} dJ_\rho . \tag{A4}
\]

Setting \( dg_2 = 0 \) implies

\[
\frac{dr}{d\theta_c} = - \left( \frac{\partial g_2}{\partial r} \right)^{-1} \left( \frac{\partial g_2}{\partial \theta_c} d\theta_c + \frac{\partial g_2}{\partial J_\phi} dJ_\phi + \frac{\partial g_2}{\partial J_\rho} dJ_\rho \right). \tag{A5}
\]

Now, since Equation (A2) implies \( \frac{\partial \theta_b}{\partial \theta_c} = 1 \), we find

\[
\frac{dr}{d\theta_c} = - \left( \frac{\partial g_2}{\partial r} \right)^{-1} \frac{\partial g_2}{\partial \theta_c} = - \left( \frac{\partial g_2}{\partial r} \right)^{-1} \frac{\partial}{\partial r} \left( \frac{\partial \mathcal{W}}{\partial \theta_c} \right) . \tag{A6}
\]

Similarly,

\[
\frac{dr}{dJ_\phi} = - \left( \frac{\partial g_2}{\partial r} \right)^{-1} \frac{\partial g_2}{\partial J_\phi} = - \frac{dr}{d\theta_c} \frac{\partial}{\partial J_\phi} \left( \frac{\partial \mathcal{W}}{\partial \theta_c} \right) ,
\]

\[
\frac{dr}{dJ_\rho} = - \left( \frac{\partial g_2}{\partial r} \right)^{-1} \frac{\partial g_2}{\partial J_\rho} = - \frac{dr}{d\theta_c} \frac{\partial}{\partial J_\rho} \left( \frac{\partial \mathcal{W}}{\partial \theta_c} \right) . \tag{A7}
\]

To proceed further, we need

\[
\frac{\partial \mathcal{W}}{\partial J_\rho} = \theta - \text{sgn}(\ell) \arccos \left( \frac{\frac{J_\rho^2}{r^2} - 1}{\sqrt{1 - \frac{\mu k}{r}}} \right) ,
\]

\[
\frac{\partial \mathcal{W}}{\partial J_\phi} = \text{sgn}(\ell) \left[ -\frac{k}{J_c^2} \sqrt{\frac{\mu^2 k^2}{J_c^2} + \frac{2 \mu k}{r} - \frac{J_\rho^2}{r^2}} + \arccos \left( \frac{1 - \frac{\mu k}{r}}{\sqrt{1 - \frac{\mu k}{r}}} \right) \right] . \tag{A8}
\]

In the second equality, the first term on the right-hand side vanishes at the pericenter and apericenter, i.e., \( r = r_p = a(1 \pm e) \). Note also that \( \arccos(x) \) is defined on its main branch \( -\pi \leq x < \pi \). Therefore, these derivatives must be properly incremented (that is, subtract and add Int \((\xi/2\pi + 1/2)\) to the first and second line, respectively) such that the angles \( \theta_b \) and \( \theta_c \) grow monotonically with time. Parameterizing the unperturbed trajectory with the eccentric anomaly \( \xi \), the radial coordinate reads \( r(\xi) = a(1 - e \cos \xi) \) and the partial derivatives of \( r \) reduce to

\[
\frac{\partial r}{\partial \theta_b} = 0 ,
\]

\[
\frac{\partial r}{\partial \theta_c} = \frac{ae \sin \xi}{(1 - e \cos \xi)} ,
\]

\[
\frac{\partial r}{\partial J_\rho} = a \sqrt{1 - e^2} \left( e - \cos \xi \right) ,
\]

\[
\frac{\partial r}{\partial J_\phi} = a \left( -3e + \cos \xi + 3e^2 \cos \xi - e^3 \cos(2\xi) \right) \frac{e \cos \xi - 1}{e \cos \xi - 1} . \tag{A9}
\]

In this derivation, it is essential to take into account the multiplicative factor of \( \text{sgn}(\ell) = \text{sgn}(\sin \xi) \) in the generating function \( \mathcal{W}(r, \theta, J_\phi, J_\rho) \), as it ensures that all the partial derivatives are continuous functions of \( \xi \). The calculation of the derivatives \( \partial \theta_b/\partial \theta_c \) and \( \partial J_\rho/\partial \theta_c \) proceeds analogously. Setting \( dg_1 = 0 \), substituting \( r = r(\theta_c, J_\rho, J_\phi) \), and taking advantage of the fact that \( \frac{\partial g_1}{\partial \theta_c} = -\frac{\partial}{\partial \theta_c} \frac{\partial \mathcal{W}}{\partial J_\phi} = -1 \), we obtain

\[
d\theta = \left[ \frac{dg_1}{d\theta_c} d\theta_c + \frac{dg_1}{dJ_\rho} dJ_\rho \right] + \left[ \frac{dg_1}{dJ_\phi} + \frac{dg_1}{dJ_\rho} \right] dJ_\rho ,
\]

\[
+ \left[ \frac{dg_1}{dJ_\phi} + \frac{dg_1}{d\theta_c} \right] d\theta_c \tag{A10}
\]

For instance, we read off

\[
\frac{\partial \theta}{\partial J_\rho} = \frac{dg_1}{d\theta_c} + \frac{dg_1}{dJ_\rho} = -\frac{dr}{d\theta_c} \frac{\partial}{\partial J_\rho} \left( \frac{\partial \mathcal{W}}{\partial \theta_c} \right) ,
\]

\[
+ \left[ \frac{dg_1}{dJ_\phi} + \frac{dg_1}{d\theta_c} \right] d\theta_c \tag{A11}
\]
After some algebra, we arrive at
\[
\frac{\partial \vartheta}{\partial \theta_b} = 1
\]
\[
\frac{\partial \vartheta}{\partial \theta_c} = \frac{\sqrt{1 - e^2}}{(1 - e \cos \xi)^2}
\]
\[
\frac{\partial \vartheta}{\partial I_b} = \frac{1}{J_c} \left[ e(1 - e \cos \xi)^2 \sin \xi \right]
\]
\[
\frac{\partial \vartheta}{\partial I_c} = \frac{1}{J_c} \left[ \sqrt{1 - e^2} (2 - e^2 - e \cos \xi) \sin \xi \right].
\] (A12)

One can check that the following (equal-time) Poisson bracket vanishes identically for any \( \xi \),
\[
[r, \vartheta]_{\theta_b, I_b} \equiv 0. \quad (A13)
\]
This indicates that the various partial derivatives we computed are consistent with a canonical transformation, as it should be. Note that \([r, \vartheta]\) does not generally vanish when \( r \) and \( \vartheta \) are evaluated at different times on the physical trajectory.

**Appendix B**

**Development at Small Eccentricities**

To proceed, we specialize the partial derivatives of the polar coordinates \((r, \vartheta)\) with respect to the angles and actions (along the unperturbed trajectory, which is now a circular orbit) to the case \( e \to 0 \):
\[
\frac{\partial r}{\partial \theta_b} = 0
\]
\[
\frac{\partial r}{\partial \theta_c} \approx a e \sin \xi + a e^2 \cos \xi \sin \xi
\]
\[
\frac{\partial r}{\partial I_b} \approx \frac{a}{e L_c} \cos \xi - \frac{a}{e L_c} \sin^2 \xi
\]
\[
\quad \quad + \frac{a e}{2 L_c} (\cos 2 \xi - 2) \cos \xi
\]
\[
\frac{\partial r}{\partial I_c} \approx -\frac{a}{e L_c} \cos \xi + \frac{a}{2 L_c} (5 - \cos 2 \xi)
\]
\[
\quad \quad - \frac{ae}{L_c} \cos^3 \xi
\]
\[
\frac{\partial \vartheta}{\partial \theta_b} = 1
\]
\[
\frac{\partial \vartheta}{\partial \theta_c} \approx 1 + 2 e \cos \xi - \left( \frac{1}{2} - 3 \cos^2 \xi \right) e^2
\]
\[
\frac{\partial \vartheta}{\partial I_b} \approx -\frac{2}{e L_c} \sin \xi - \frac{3}{L_c} \sin \xi \cos \xi
\]
\[
\quad \quad - \frac{e}{L_c} (1 + 2 \cos 2 \xi) \sin \xi
\]
\[
\frac{\partial \vartheta}{\partial I_c} \approx \frac{2}{e L_c} \sin \xi + \frac{3}{L_c} \sin \xi \cos \xi
\]
\[
\quad \quad + \frac{e}{L_c} (3 \sin \xi - \sin \xi). \quad (B1)
\]

In each expression, we retained terms up to order \( e \) except for the partial derivatives relative to \( \theta_b \), for which we include terms up to order \( e^2 \) (because the perturbation \( \Delta \theta \) features a contribution of order \( e^{-1} \)). Although the derivatives relative to the actions diverge in the limit \( e \to 0 \), the relation \([r, \vartheta]_{\theta_b, I_b} = 0\) is always satisfied along the physical trajectory (i.e., when \( r \) and \( \vartheta \) are evaluated at a fixed \( \xi \)). We now turn to the expressions for \( \Delta \theta_b \) and \( \Delta \theta_c \) and use the fact that, for small \( ne \ll 1 \), the Bessel functions behave like \( J_n(ne) \sim (ne)^n \). In particular, \( J_n(e) \approx e/2 \). Writing \( J_n'(ne) = J_{n-1}(ne) - \frac{1}{n} J_n(ne) \), Taylor-expanding the Bessel functions in the small argument limit, and retaining terms up to order \( e \) (since the partial derivatives of \( r \) and \( \vartheta \) with respect to the angles \( \theta_b \) and \( \theta_c \) are at best of order \( e^0 \)), we find
\[
\Delta \theta_b = \epsilon \left( \frac{\Omega}{\omega_a} \right) \left\{ S_{11}^{(1)} e + \frac{1}{2} S_{21}^{(1)} e + O(e^2) \right\}
\]
\[
\Delta \theta_c = -\epsilon \left( \frac{\Omega}{\omega_a} \right) \left\{ \frac{S_{11}^{(1)} e}{e} + \frac{S_{21}^{(1)} e}{2} + 7 \sin(\omega_a t + \alpha) + \sin \alpha \right\}
\]
\[
+ \frac{3}{8} \left( 7 S_{11}^{(1)} e + S_{21}^{(1)} e + 3 \left( \frac{\Omega}{\omega_a} \right) S_{12}^{(1)} \right)
\]
\[
- 3 \Omega t A_1 e + O(e^2) \}. \quad (B2)
\]
All the functions \( S_{11}^{(1)} \) and \( C_{12}^{(1)} \) that appear in the previous expressions are evaluated at \( t > 0 \). Furthermore, although the individual deviations \( \Delta \theta_b \) and \( \Delta \theta_c \) diverge in the limit \( e \to 0 \), their contribution to the displacement \( \delta r \) is always well behaved since the latter depends on
\[
\frac{\partial \vartheta}{\partial \theta_b} \Delta \theta_b + \frac{\partial \vartheta}{\partial \theta_c} \Delta \theta_c = \epsilon \left( \frac{\Omega}{\omega_a} \right)
\]
\[
\times (4 \sin(\omega_a t + \alpha) - \sin \alpha) - 3 \cos \xi S_{11}^{(1)} + 14 \sin(\omega_a t + \alpha) - \sin \alpha)
\]
\[
+ 3 \left( \frac{\Omega}{\omega_a} \right) S_{12}^{(1)} \right\}
\]
\[
- 3 \Omega t A_1 e + O(e^2) \}. \quad (B3)
\]
For the action variables, we have \( \Delta I_b = 0 \), while
\[
\Delta I_c = \epsilon I_c \left( \frac{\Omega}{\omega_a} \right) \left[ C_{11}^{(-1)} e + \frac{1}{2} C_{21}^{(-1)} e^2 + O(e^3) \right]. \quad (B4)
\]
The term linear in \( e \) in \( \Delta I_c \) combines with that proportional to \( e^{-1} \) in \( \frac{\partial \vartheta}{\partial I_c} \) to give an \( e^0 \) contribution. More precisely,
\[
\frac{\partial \vartheta}{\partial \theta_b} \Delta I_b + \frac{\partial \vartheta}{\partial \theta_c} \Delta I_c = \epsilon \left( \frac{\Omega}{\omega_a} \right)
\]
\[
\times \left[ 2 \sin \xi C_{11}^{(-1)} e + \sin \xi (3 \cos \xi C_{11}^{(-1)} e + C_{21}^{(-1)} e + O(e^2)) \right]. \quad (B5)\]
Applying the same analysis to the radial coordinate $r$ eventually leads to

$$
\frac{\partial}{\partial \theta_b} \Delta \theta_b + \frac{\partial}{\partial \theta_c} \Delta \theta_c = -e \left\{ \frac{\Omega}{\omega_0} \right\} \\
\times \left\{ \sin \xi \, S^{(+)\xi}_1 + \sin \left[ \cos \xi \, S^{(+)\xi}_1 + \frac{1}{2} \xi C^{(-)\xi}_1 \right] \\
- 3(\sin(\omega_0 t + \alpha) - \sin \alpha) \right\} e + O(e^2) \\
\frac{\partial}{\partial \theta_b} \Delta \theta_b + \frac{\partial}{\partial \theta_c} \Delta \theta_c = -e \left\{ \frac{\Omega}{\omega_0} \right\} \\
\times \left\{ \cos \xi \, C^{(-)\xi}_1 + \left[ \frac{1}{2} \cos \xi C^{(-)0}_1 \right] \\
+ (\cos 2\xi - 3)C^{(-)\xi}_1 - C^{(-)0}_1 \right\} e + O(e^2). \tag{B6}
$$

Putting all this together, the square $|\delta r|^2$ of the normalized perturbed displacement can be recast in the form of Equation (31).

**Appendix C**

**Perturbations in the Limit $\Omega \to 0$**

We consider the regime $\Omega \ll \omega_0$, and expand the functions $S^{(+)1}_1$ and $C^{(-)1}_1$ that appear in Equations (17)–(20) accordingly to obtain

$$
S^{(+)1}_1 \approx 2(\sin(\omega_0 t + \alpha)\cos(nM) - \sin \alpha \cos(nM_0)) \\
- 2n \left( \frac{\Omega}{\omega_0} \right) \cos(\omega_0 t + \alpha) \sin(nM) \\
- \cos \alpha \sin(nM_0),
$$

$$
C^{(-)1}_1 \approx -2(\sin(\omega_0 t + \alpha)\sin(nM) \\
- \sin \alpha \sin(nM_0)) \\
- 2n \left( \frac{\Omega}{\omega_0} \right) \cos(\omega_0 t + \alpha) \cos(nM) \\
- \cos \alpha \cos(nM_0). \tag{C1}
$$

In the limit $\Omega \to 0$, only the first term subsists on the right-hand side of Equation (C2). Furthermore, $M \approx M_0$ since the mean anomaly does not change appreciably during the observational period. Therefore, we find

$$
S^{(+)1}_1 \approx 0 = 2(\sin(\omega_0 t + \alpha) - \sin \alpha) \cos(nM_0) \\
C^{(-)1}_1 \approx -2(\sin(\omega_0 t + \alpha) - \sin \alpha) \sin(nM_0). \tag{C2}
$$

Substituting these expressions into Equations (17)–(20) and taking advantage of the relations

$$
\sum_{n=1}^{\infty} J_n(\pi e) \cos(nM) = \frac{e \cos \xi}{2(1 - e \cos \xi)} \\
\sum_{n=1}^{\infty} J_n(\pi e) \sin(nM) = \frac{e \sin \xi}{2 \sin \xi}, \tag{C3}
$$

the series expansions of $\Delta \theta_b$, $\Delta \theta_c$, and $\Delta \theta_i$ become

$$
\Delta \theta_b \approx 2e \left\{ \frac{\Omega}{\omega_0} \right\} \frac{\sqrt{1 - e^2}}{e} (\cos \xi_0 - e) \\
\times (\sin(\omega_0 t + \alpha) - \sin \alpha) + ... \\
\Delta \theta_c \approx 2e \left\{ \frac{\Omega}{\omega_0} \right\} \left[ 3 - \left( \frac{1}{3} + 3e \right) \cos \xi_0 \\
+ e^2 \cos(2\xi_0) \right] (\sin(\omega_0 t + \alpha) - \sin \alpha) + ... \\
\Delta \theta_i \approx -2e J_i \left\{ \frac{\Omega}{\omega_0} \right\} \left[ \frac{e \sin \xi_0}{2} (\sin(\omega_0 t + \alpha) - \sin \alpha) \right. \\
\times \left. \left( \frac{e \cos \xi_0}{2} \right) \right] \tag{C4}
$$

in the limit $\Omega \to 0$. On setting $\xi = \xi_0$ in Equations (A9) and (A12) (but allowing $t$ to grow freely) and taking into account the leading contribution in Equation (C4) solely, we can check that

$$
\delta r = \frac{\partial}{\partial \theta_b} (\xi_0) \Delta \theta_b + \frac{\partial}{\partial \theta_c} (\xi_0) \Delta \theta_c \\
= e \left\{ \frac{\Omega}{\omega_0} \right\} \left\{ \frac{e \sin \xi_0}{1 - e \cos \xi_0} \right\} \\
\times \left[ 3 - \left( \frac{1}{3} + 3e \right) \cos \xi_0 + e^2 \cos(2\xi_0) \right] \\
- (-3e + \cos \xi_0 + 3e^2 \cos \xi_0 - 3e^3 \cos(2\xi_0)) \\
\times e (e \cos \xi_0 - 1) \\
\times 2e \sin \xi_0 (\sin(\omega_0 t + \alpha) - \sin \alpha) \\
\equiv 0, \tag{C5}
$$

as the term in curly brackets identically vanishes regardless of the value of $\xi_0$. Similar manipulations lead to $\delta r \equiv 0$ under the same assumptions. This implies that the response function must scale like $R_{\infty} \propto (\Omega/\omega_0)^2$ for small $\Omega/\omega_0$.

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**References**

Annulli, L., Bernard, L., Blas, D., & Cardoso, V. 2018, PhRvD, 98, 084001
Armengaud, E., Palanque-Delabrouille, N., Yèche, C., Marsh, D. J. E., & Baur, J. 2017, MNRAS, 471, 4606
Arnoïd, V. L., Kozlov, V. V., & Neishtadt, A.I. 1990, Mathematical aspects of classical and celestial mechanics (Berlin: Springer)
Bar, N., Blum, K., Lacroix, T., & Papi, P. 2019, JCAP, 2019, 045
Binney, J., & Tremaine, S. 1987, Galactic Dynamics (Princeton, NJ: Princeton Univ. Press)
Blanchet, L. 2014, LRR, 17, 2
Blas, D., López Nacir, D., & Sibiryakov, S. 2020, PhRvD, 101, 063016
Blas, D., Nacir, D. L., & Sibiryakov, S. 2017, PhRvL, 118, 261102
Bošković, M. 2019, arXiv:1907.12809
Bošković, M., Duque, F., Ferreira, M. C., Miguel, F. S., & Cardoso, V. 2018, PhRvD, 98, 024037
Broadhurst, T., de Martino, I., Luu, H. N., Smoot, G. F., & Tye, S. H. H. 2020, PhRvD, 101, 083012
Chavanis, P.-H. 2011, PhRvD, 84, 043531
Davies, E. Y., & Mocz, P. 2020, MNRAS, 492, 5721
de Martino, I., Broadhurst, T., Tye, S.-H. H., et al. 2017, PhRvL, 119, 221103
Desjacques, V., & Nusser, A. 2019, MNRAS, 488, 4497
Edwards, R. T., Hobbs, G. B., & Manchester, R. N. 2006, MNRAS, 372, 1549
González-Morales, A. X., Marsh, D. J. E., Peñarrubia, J., & Ureña-López, L. A. 2017, MNRAS, 472, 1346
Grin, D., Amin, M. A., Gruscevic, V., et al. 2019, arXiv:1904.09003
Hellings, R. W., & Downs, G. S. 1983, ApJL, 265, L39
Hlozek, R., Marsh, D. J. E., & Grin, D. 2018, MNRAS, 476, 3063
Hu, W., Barkana, R., & Gruzinov, A. 2000, PhRvL, 85, 1158
Hughes, S. 2009, ARA&A, 47, 107
Hui, L., McWilliams, S. T., & Yang, I.-S. 2013, PhRvD, 87, 084009
Hui, L., Ostriker, J. P., Tremaine, S., & Witten, E. 2017, PhRvD, 95, 043541
Iršič, V., Viel, M., Hahnelt, D. G., Bolton, J. S., & Becker, G. D. 2017, PhRvL, 119, 031302
Khmelnitsky, A., & Rubakov, V. 2014, JCAP, 1402, 019
Kobayashi, T., Murgia, R., de Simone, A., Iršič, V., & Viel, M. 2017, PhRvD, 96, 123514
Kumar Poddar, T., Mohanty, S., & Jana, S. 2019, arXiv:1906.00666
Marsh, D. J., & Niemeyer, J. C. 2019, PhRvL, 123, 051103
Marsh, D. J. E. 2016, PhR, 643, 1
Marsh, D. J. E., & Pop, A.-R. 2015, MNRAS, 451, 2479
Mashhoon, B. 1978, ApJ, 223, 285
Mashhoon, B. 1985, MNRAS, 217, 265
Mashhoon, B., Carr, B. J., & Hu, B. L. 1981, ApJ, 246, 569
Niemeyer, J. C. 2019, arXiv:1912.07064
Nojiri, S., Odintsov, S. D., Oikonomou, V. K., & Popov, A. A. 2019, PhRvD, 100, 084009
Peebles, P. 2000, ApJL, 534, L127
Peters, P. C. 1964, PhRv, 136, B1224
Porayko, N. K., & Postnov, K. A. 2014, PhRvD, 90, 062008
Porayko, N. K., Zhu, X., Levin, Y., et al. 2018, PhRvD, 98, 102002
Press, W. H., Ryden, B. S., & Spergel, D. N. 1990, PhRvL, 64, 1084
Read, J. I. 2014, JPhG, 41, 063101
Rein, H., & Liu, S.-F. 2012, A&A, 537, A128
Rein, H., & Spiegel, D. S. 2015, MNRAS, 446, 1424
Rozner, M., Grishin, E., Ginat, Y. B., Igoshev, A. P., & Desjacques, V. 2019, JCAP, 2020, 061
Rudenko, V. N. 1975, SvA, 19, 270
Safarzadeh, M., & Spergel, D. N. 2019, arXiv:1906.11848
Salucci, P., Nesti, F., Gentile, G., & Frigerio Martins, C. 2010, A&A, 523, A83
Sikivie, P., & Yang, Q. 2009, PhRvL, 103, 111301
Turner, M. S. 1979, ApJ, 233, 685
Wasserman, A., van Dokkum, P., Romanowsky, A.J., et al. 2019, ApJ, 885, 155
Weinberg, S. 1978, PhRvL, 40, 223
Wilczek, F. 1978, PhRvL, 40, 279