Quasi-linear elliptic equations with data in $L^1$ on a compact Riemannian manifold

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Abstract This work is dedicated to the study of quasi-linear elliptic problems with $L^1$ data, the simple model will be the next equation on $(M, g)$ a compact Riemannian manifold.

$$-\Delta_p u = f$$

Where $f \in L^1(M)$. Our goal is to develop the functional framework and tools that are necessary to prove the existence and the uniqueness of the solution for the previous problem. Notice that our argument can be used to deal with a more general class of quasi-linear equations.

Introduction

This article is dedicated to the study of quasi-linear elliptic equations with data in $L^1(M)$, the major difficulty encountered when one is interested in such problems is that the classical theories of existence, either using variational methods or compactness methods, are not applicable. Hence the need to use new techniques to prove the existence and uniqueness of solutions for such problems.

Note that the importance of trying to solve problems with data in $L^1$ is not limited to a purely theoretical framework, but also for applicable reasons, to be convinced it is recommended to the reader various references as for example: [1], [2] and [3] where different examples of equations having an application in physics are pre-
The first significant advance in this direction is due to Stampacchia in [3], where he considers second-order linear elliptic operators with non-regular data of the form

\[ L(u) = f \]

where

\[ L(u) = \sum_{i,j=1}^{N} a_{ij} \frac{\partial^2 u}{\partial x_{ij}} + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} + cu \]

where \( a_{ij}, b_i, c \) are functions with specific hypotheses.

In his famous works, Stampacchia uses the notion of duality to solve these classes of problems. Existence and uniqueness results have been proved in this direction thanks to the linear character and the "regularizing effect" of the operator. Note that in the case where \( L \equiv \Delta \) then the notion of duality coincides with the notion of solution in the sense of distributions. In the linear framework and with the notion of duality, we can even consider data measures.

The extension of Stampacchia’s work to non-linear operators has been done by several mathematicians. The first works were realized by Boccardo, Murat, Gallouet and their collaborators. The main difficulties for non-linear operators linear consists of two points:

1. The sense in which the solution is defined (the meaning of the good solution and the method of its construction).
2. The uniqueness of the "good" solution.

Note that the second question is legitimate given Serrin’s counter example for the non-uniqueness of the solution, see [4].

To go beyond the first difficulty we proceed by approximation by returning In the variational framework, the main step is to demonstrate properties of the solutions for approximate problems that remain conserved by passing to the limit. This passage is feasible by imposing natural conditions on the space of the test functions.

Concerning the second difficulty, we demonstrate partial results, especially for the \( \Delta_p \) operator we are able to demonstrate the uniqueness of the solution. It will be noted that the uniqueness of the solution is usually true.

We organize this work in two sections. In the first section we briefly recall the functional spaces of Sobolev and Marcinkiewicz, on a compact Riemannian manifold, which will be very useful in this paper. In Subsection 1.2.3 we define the notion of the weak solution. Using variational techniques we prove the existence and the uniqueness of the energy solution for the problem:

\[ -\Delta_p u = f, \quad u \in W^{1,p}_0(M) \]

where \( (M, g) \) a Riemannian manifold. This result is a natural extension of the Lax-Milgram Theorem to the non-linear case in a Euclidean space. In Section 1.3 we
present the proof of the Picone inequality in its general version on a compact Riemannian manifold and as a consequence we obtain a comparison principle for quasilinear problems with a “concave” term compared to Laplacian. This result generalizes that of Brezis-Kamin in \([5]\) for the Laplacian, see \([6]\). The second section is dedicated to define the notion of entropy on a compact Riemannian manifold, in which we will study our problem. We begin by defining the functional framework that will be given using the truncation function, ie we analyze the functional properties of \(T_k(u)\) instead of \(u\). Note that \(T_k : \mathbb{R} \rightarrow \mathbb{R}\) defined by:

\[
T_k(s) = \begin{cases} 
  s & \text{if } |s| \leq k \\
  k \text{sign}(s) & \text{if } |s| > k
\end{cases}
\]

After giving the definition of solution in the sense of entropy, we prove the existence and uniqueness of the solution in this context, some properties of the entropy solution in Marcinkiewicz’s spaces on Riemannian manifolds will be deduced. At the end of the section some generalizations for non-homogeneous quasi-linear operators and with second members that may depend on \(u\) will be presented, see \([1]\).

1 Preliminaries

1.1 some definitions

Definition 1 (Equi-integrable functions in \(L^1\)).

Let \(X\) be a set of \(\mathbb{R}^N\). We say that a sequence \(\{f_n\}\) of functions of \(L^1(X)\) is equi-integrable if, for all \(\varepsilon > 0\), there exists \(\delta > 0\) such as \(\text{meas}(E) < \delta\) with \(E \subset X\) will result for all \(n\),

\[
\int_E |f_n(x)| \, dx \leq \varepsilon.
\]

We use often the next result of compactness in \(L^1\).

Lemma 1 (Lemma of Vitali: compactness in \(L^1\)).

Let \(X\) a finite measure set for the Lebesgue measure of \(\mathbb{R}^N\). Let \(\{f_n\}\) a sequence of functions of \(L^1(X)\) which converges everywhere to \(f\), and which is equi-integrable. Then \(f \in L^1(X)\) and \(\{f_n\}\) converges strongly to \(f\) in \(L^1(X)\).

1.2 Functional spaces

1.2.1 Sobolev spaces \(W^{k,p}(M)\)

see \([7]\) and \([8]\)
Let \((M, g)\) a Riemannian manifold, for an integer \(k\) and \(u \in C^\infty(M)\), \(\nabla^k u\) represents the \(k\)-th of the covariant derivative of \(u\) (with the Convention \(\nabla^0 u = u\)), and the norm of \(k\)-th of covariant derivative on a local map is given by the formula:

\[
|\nabla^k u| = g^{i_1 j_1} \ldots g^{i_k j_k} (\nabla^k u)_{i_1 \ldots i_k} (\nabla^k u)_{j_1 \ldots j_k}
\]

where the Einstein summation convention is adopted.

We also recall the notion of Riemannian measure on manifolds, let \(\{U_i, \Phi_i\}\) be any atlas of \(M\). There exists a partition of unity \(\{\eta_i\}\) subordinate to \(\{U_i, \Phi_i\}\),

Given a continuous function \(f: M \to \mathbb{R}\), we define the integral as follows:

\[
\int_M f \, d\sigma_g = \sum_i \int_{\Phi_i(U_i)} (\eta_i \sqrt{\det g \circ \Phi_i}) \, dx.
\]

where \(dx\) is the Lebesgue measure on \(\mathbb{R}^n\).

Let be \(p \geq 1\) a real, and \(k\) a positive integer.

\[
L^p(M) = \{u: M \to \mathbb{R} \text{ measurable} / \int_M |u|^p d\sigma_g < \infty\}
\]

\[
C^p_k(M) = \{u \in C^\infty / \forall j = 0, \ldots, k \int_M |\nabla^j u|^p d\sigma_g < \infty\}
\]

**Definition 2.** The Sobolev space \(W^{k,p}(M)\) is the complete space \(C^p_k(M)\) for the norm

\[
\|u\|_{W^{k,p}(M)} = \sum_{j=0}^k \|\nabla^j u\|_{L^p(M)}
\]

\[
\|u\|_{W^{k,p}(M)} = \|\nabla u\|_p + \|u\|_p
\]

**Definition 3.** We must recall the notion of the geodesic distance for every curve:

\[
\gamma: [a, b] \to M
\]

We define the length of \(\gamma\) by:

\[
l(\gamma) = \int_a^b \sqrt{g(\gamma(t)) \left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)} \, dt
\]

**Remark 1.** For \(x, y \in M\) defining a distance \(d_g\) by:

\[
d_g(x, y) = \inf \{l(\gamma) : \gamma : [0, 1] \to M \quad \gamma(0) = x \quad \gamma(1) = y\}
\]

By the theorem of Hopf-Rinow, we obtain that if \(M\) a Riemannian manifold then compact for all \(x, y \in M\) can be joined by a minimizing curve \(\gamma\) i.e \(l(\gamma) = d_g(x, y)\)
**Proposition 1.** If \( p = 2 \), \( W^{k,2}(M) \) is a Hilbert space for the scalar product

\[
(u,v)_{H^k} = \sum_{j=0}^k \langle \nabla^j u, \nabla^j v \rangle_{L^2}
\]

**Proposition 2.** If \( p > 1 \) then \( W^{k,p}(M) \) is reflexive.

**Proposition 3.** Any reflex normalized space is a Banach space. Then if \( p > 1 \) then \( W^{k,p}(M) \) is Banach.

**Definition 4.** The Sobolev space \( W^{k,p}(M) \) is the closure of \( \mathcal{D}(M) \) in \( W^{k,p}(M) \).

**Theorem 1.** If \( (M,g) \) is complete, then for all \( p \geq 1 \) \( W^{1,p}_0(M) = W^{1,p}(M) \).

**Embeddings of Sobolev:** See [7].

**Lemma 2.** Let \((M,g)\) a complete Riemannian manifold of dimension \(n\). Suppose that inclusion \( W^{1,1}(M) \subset L^{\frac{m}{m-1}}(M) \) is valid. Then, for a whole real number \( 1 \leq q < p \) and an integer \( 0 \leq m < k \) which verify \( \frac{1}{p} = \frac{1}{q} - \frac{(k-m)}{n} \), \( W^{k,p}(M) \subset W^{m,p}(M) \).

**Remark 2.** Note that the proof of the Lemma ?? shows that if \( A \in \mathbb{R} \) is such that \( \forall \ u \in W^{1,1}(M) \),

\[
\left( \int_M |u|^{n/(n-1)} d\sigma_g \right)^{(n-1)/n} \leq A \int_M (|\nabla u| + |u|) d\sigma_g
\]

So, for all \( 1 \leq q < n \) and all \( u \in W^{1,q}(M) \),

\[
\left( \int_M |u|^p d\sigma_g \right)^{1/p} \leq A p (n-1) \left\{ \left( \int_M |\nabla u|^q d\sigma_g \right)^{1/q} + \left( \int_M |u|^q d\sigma_g \right)^{1/q} \right\}
\]

Where \( 1/p = 1/q - 1/n \).

**Theorem 2.** Let \((M,g)\) a compact Riemannian manifold of dimension \(n\). For a real number \( 1 \leq q < p \) and an integer \( 0 \leq m < k \) which verify \( \frac{1}{p} = \frac{1}{q} - \frac{(k-m)}{n} \), \( W^{k,p}(M) \subset W^{m,p}(M) \).

**Theorem 3.** (Rellich-Kondrakov’s Theorem): Let \((M,g)\) a compact Riemannian manifold of \(n\) dimension. \( j \geq 0 \) and \( m \geq 1 \) two integers, \( q \geq 1 \) and \( p \) two real numbers that verify \( 1 \leq p < nq/(n - mq) \), the inclusion \( W^{j+m,q}(M) \subset W^{j,p}(M) \) is compact.

**Corollary 1.** Let \((M,g)\) a compact Riemannian manifold of \(n\) dimension. For everything \( 1 \leq q < n \) and \( p \geq 1 \) such as \( \frac{1}{p} > \frac{1}{q} - \frac{1}{n} \), the inclusion \( W^{1,q}(M) \subset L^p(M) \) is compact.
**Lemma 3. (Inequality of Poincaré):** Let \( D \) a regular domain is bounded in a Riemannian manifold \( M \) and \( 1 \leq p < \infty \). Then there is a constant \( A \) such as:

\[
\left( \int_D |u - u_D|^p d\sigma_g \right)^{\frac{1}{p}} \leq A \left( \int_D |\nabla u|^p d\sigma_g \right)^{\frac{1}{p}},
\]

for everything \( u \in W^{1,p}_{\text{loc}}(M) \), where \( u_D = \frac{1}{\text{vol}(D)} \int_D u d\sigma_g \) is the mean value of \( u \) on \( D \).

By combining this lemma with the Holder inequality, we obtain:

**Corollary 2.** There exists a constant \( c = c_D \) such that

\[
\int_D |u - u_D| d\sigma_g \leq c_D \left( \int_M |\nabla u|^p d\sigma_g \right)^{\frac{1}{p}} \quad \forall u \in W^{1,p}_{\text{loc}}(M)
\]

### 1.2.2 Marcinkiewicz’s spaces.

**Definition 5.** [7] Let \( f : M \to \mathbb{R} \) be a measurable function, its distribution function

\[
\phi_f(k) = \text{meas}\left\{ x \in M : |f(x)| > k \right\}, \quad k > 0,
\]

**Definition 6.** Let \( 0 < q < \infty \) and \((M, g)\) Riemannian manifold, the space Marcinkiewicz \( \mathcal{M}^q(M) \) is the set of functions measurable \( f : M \to \mathbb{R} \) such as

\[
\phi_f(k) \leq Ck^{-q}, \quad C < \infty,
\]

Marcinkiewicz’s space \( \mathcal{M}^q(M) \) is defined the norm

\[
\|f\|_{\mathcal{M}^q(M)} = \inf \left\{ C : \phi_f(k) \leq Ck^{-q}, \quad \text{for all} \quad k > 0 \right\}.
\]

is a Banach space.

Note that if \( f \in L^q(M) \), we have

\[
\int_{\{|f| > k\}} d\sigma_g \leq \int_M \left| f \right|^q d\sigma_g \leq k^{-q} \int_M |f|^q d\sigma_g,
\]

so

\[
\phi_f(k) \leq k^{-q}\|f\|_q^q
\]

and as a conclusion will have \( L^q(M) \subset \mathcal{M}^q(M) \).

For analyze the properties of the spaces \( \mathcal{M}^q(M) \), needs some the next lemma

**Lemma 4.** If \( f \in L^q(M) \) so

\[
\int_M |f(x)|^q dx = \int_0^{+\infty} t^{q-1} \phi_f(t) dt.
\]
Proof.

We start with the case where \( q = 1 \). Let

\[
H(t) = \begin{cases} 
1 & \text{if } t > 0, \\
0 & \text{if } t < 0,
\end{cases}
\]

so

\[
H(|f(x)| - k) = \begin{cases} 
1 & \text{if } |f(x)| > k, \\
0 & \text{if } |f(x)| < k,
\end{cases}
\]

so we have

\[
\int_0^{+\infty} \phi_f(k) dk_g = \int_M \left[ \int_0^{+\infty} H(|f(x)| - k) d\sigma_g \right] dk_g
= \int_M \int_0^{+\infty} H(|f(x)| - k) d\sigma_g, \quad \text{(Fubini)}
= \int_M \int_{|f(x)| > k} 1 d\sigma_g = \int_M \left[ \int_0^{+\infty} 1 dk_g \right] d\sigma_g,
= \int_M |f(x)| d\sigma_g,
\]

so \( \int_0^{+\infty} \phi_f(k) dk_g = \int_M |f(x)| d\sigma_g \) and the result is demonstrated.

We now consider the general case \( q > 1 \). Ask \( g(x) = |f(x)|^q \) so, \( g \in L^1(M) \) and

\[
\phi_g(k) = \text{meas} \left\{ |g| > k \right\} = \text{meas} \left\{ |f|^q > k \right\} = \text{meas} \left\{ |f| > k^{\frac{1}{q}} \right\},
\]

i.e.

\[
\phi_g(k) = \phi_f \left( k^{\frac{1}{q}} \right),
\]

so

\[
\int_M |g(x)| d\sigma_g = \int_0^{+\infty} \phi_f \left( k^{\frac{1}{q}} \right) dk_g,
\]

Ask \( t = k^{\frac{1}{q}} \) so \( k = t^q \) and \( dk = qt^{q-1} \), so that

\[
\int_M |f(x)|^q d\sigma_g = \int_0^{+\infty} t^{q-1} \phi_f(t) dt, \quad \blacksquare
\]

Corollary 3. If \( q \in [1, \infty[ \), so

\[
L^q(M) \subset \mathcal{M}^q(M) \subset L^{q-\varepsilon}(M) \quad \forall \varepsilon > 0,
\]

and for all \( q, \hat{q} \in [1, \infty[ \) we have

\[
\mathcal{M}^q(M) \subset \mathcal{M}^{\hat{q}}(M) \quad \text{if } q \geq \hat{q}.
\]

Proof.
Assuming that $f \in \mathcal{M}^q(M)$ and $\varepsilon > 0$, we have

\[
\int_M |f(x)|^{q-\varepsilon} \, d\sigma_g \leq c_1 + \int_M |g(x)|^{q-\varepsilon} \, d\sigma_g, \\
\leq c_1 + \left( q - \varepsilon \right) \int_0^{\infty} t^{q-\varepsilon-1} \phi_g(t) \, dt, \\
\leq c_1 + c_2 (q - \varepsilon) \int_0^1 t^{q-\varepsilon-1} \, dt + c_3 (q - \varepsilon) \int_1^{\infty} t^{q-\varepsilon-1} \, dt, \\
\leq c_1 + c_2 (q - \varepsilon) + c_3 (q - \varepsilon) \int_1^{\infty} t^{-\varepsilon-1} \, dt, \\
\leq c_4 + (q - \varepsilon) \left[ \frac{1}{\varepsilon} \right]^{\infty}_1, \\
\leq C < \infty,
\]

so $f \in L^{q-\varepsilon}(M)$, and then it results that $\mathcal{M}^q(M) \subset L^{q-\varepsilon}(M)$. As $L^q(M) \subset \mathcal{M}^q(M) \subset L^{q-\varepsilon}(M) \subset \mathcal{M}^{q-\varepsilon}(M)$ for all $q \in [1, \infty]$ and for all $\varepsilon > 0$, so we deduce for all $q, \tilde{q} \in [1, \infty]$ we have

\[\mathcal{M}^{q}(M) \subset \mathcal{M}^\tilde{q}(M) \text{ si } q \geq \tilde{q}.\]

1.2.3 Elliptic problems and the concept of the weak solution.

Let $p > 1$ and $(M,g)$ a compact Riemannian manifold, for $u \in W_0^{1,p}(M)$, we can consider the continuous linear form $-\Delta_p u$ over $W_0^{1,p}(M)$ defined by

\[
\langle -\Delta_p u, v \rangle = \int_M |\nabla u|^{p-2} \nabla u \nabla v \, d\sigma_g.
\]
It’s clear that $-\Delta_p u \in \left(W_0^{1,p}(M)\right)' = W_0^{-1,p'}(M)$ and $\|\Delta_p u\|_{W_0^{-1,p'}(M)} = \|u\|_{W_0^{1,p}(M)}$.

As a consequence, we have the next definition

**Definition 7.** Let $f \in W_0^{-1,p'}(M)$, we say that $u$ is a weak solution of the problem

\[
\begin{cases}
-\Delta_p u = f \text{ in } M, \\
u(x) = 0 \text{ on } \partial M,
\end{cases}
\]

in $W_0^{1,p}(M)$ if and only if

\[
\int_M |\nabla u|^{p-2}\nabla u \nabla \varphi \, d\sigma_g = \left\langle -\Delta_p u, \varphi \right\rangle \quad \forall \varphi \in W_0^{1,p}(M).
\]

The next inequalities will be systematically used in this work.

**Lemma 5.** Let $\xi_1, \xi_2 \in M$, we have

1) If $p \leq 2$,
\[
|\xi_1 + \xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2}\langle \xi_1, \xi_2 \rangle \leq C(p) |\xi_2|^p,
\]
\[
|\xi_1|^p - |\xi_2|^p - p|\xi_2|^{p-2}\langle \xi_2, \xi_1 - \xi_2 \rangle \geq C(p) \frac{|\xi_1 - \xi_2|^2}{(|\xi_2| + |\xi_1|)^{2-p}}.
\]

2) If $p > 2$,
\[
|\xi_1 + \xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2}\langle \xi_1, \xi_2 \rangle \leq \frac{p(p-1)}{2} (|\xi_1| + |\xi_2|)^{p-2}|\xi_2|^2,
\]
\[
|\xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2}\langle \xi_1, \xi_2 - \xi_1 \rangle \geq \frac{C(p)}{2^p-1} |\xi_2 - \xi_1|^p.
\]

For the demonstration, see [9] and [10].

To demonstrate the existence of a weak solution for the previous problem, one often uses variational techniques and arguments of minimization of the convex functional ones. More precisely, we have the next result.

**Theorem 4.** [11] Let $V$ be a reflexive Banach space, $K \subset V$ is a closed non-empty convex, and $J : K \to \mathbb{R} \cup \{+\infty\}$ a semi-continue inferiorly coercive function weakly on $K$.

So $\inf_{u \in K} J(u) < \infty$ and $\exists u_0 \in K$, $J(u_0) = \min_{u \in K} J(u)$.

Moreover, if $J$ is strictly convex, $u_0$ is unique. If $J$ is differentiable in the sense of Gateaux and $K$ then open $J'(u_0) = 0$.

**Theorem 5.** Let $(M, g)$ a Riemannian manifold and $p \in [1, \infty[$. We suppose that $f \in L^q(M)$ with $q = \frac{Np}{N(p-1)+p}$, so there exists a unique solution $u \in W_0^{1,p}(M)$ of the problem
\{-\Delta_p u = f \quad \text{in} \quad M, \\
u = 0 \quad \text{if} \quad \partial M.\}

1.3 Inequality of Picone for the $p-$Laplacian and application.

We begin by formulating the inequality of Picone punctual for the case of $p-$Laplacian.

**Theorem 6.** Let $v > 0, u \geq 0$ two positive class $C^1$ functions, we pose

\[ L(u, v) = |\nabla u|^p + (p - 1)\frac{\mu^p}{\nu^{p-1}}|\nabla v|^p - p \frac{\mu^{p-1}}{\nu^{p-1}}|\nabla v|^{p-2}\nabla v \nabla u. \]

\[ R(u, v) = |\nabla u|^p - \nabla \left( \frac{\mu^p}{\nu^{p-1}} \right) |\nabla v|^{p-2} \nabla v. \]

so $L(u, v) = R(u, v), L(u, v) \geq 0$ and $L(u, v) = 0, almost everywhere$ in $M$ of $u = kv$ in each Connected component of $M$.

The proof of Theorem 6 is simple, it is based on the development of term $\nabla \left( \frac{\mu^p}{\nu^{p-1}} \right) |\nabla v|^{p-2} \nabla v.$

To apply the Picone inequality to nonlinear elliptic equations we need to prove an extension of Theorem 6 in $W^{1, p}_0(M)$, more precisely we have the next lemma

**Lemma 6.** Let $v \in W^{1, p}(M)$ such as $v \geq \delta > 0$ in $M$, so for all $u \in C_0^\infty(M), u \geq 0$

\[ \int_M |\nabla u|^p \geq \int_M \left( \frac{|u|^p}{\nu^{p-1}} \right) (-\Delta_p v). \]

**Proof.**

As $v \in W^{1, p}(M)$ and $v \geq \delta > 0$ in $M$, then it exists a sequence $\{v_n\}$ regular functions such as

\[ \begin{cases} 
    v_n \to v & \text{in} \quad W^{1, p}(M), v_n \in C^1(M) \\
    v_n \to v & a.e, \quad \text{and} \quad v_n > \frac{\delta}{2} \quad \text{in} \quad M. 
\end{cases} \]

As a consequence of the continuity of the operator $-\Delta_p \left( \text{of} \quad W^{1, p}(M) \right)$ in $W^{-1, p'}(M), p' = \frac{p}{p-1}$ we get that $-\Delta_p v_n \to -\Delta_p v$ in $W^{1, p'}(M), p' = \frac{p}{p-1}$. (see[24]). En using the identity of Picone at $v_n$, it results

\[ |\nabla u|^p \geq \nabla \left( \frac{\mu^p}{\nu^{p-1}} \right) |\nabla v_n|^{p-2} \nabla v_n, \]

as
\[
\int_M -\Delta_p v_n \frac{u^p}{v_n^{p-1}} = \int_M |\nabla v_n|^{p-2} \langle \nabla v_n, \nabla \left( \frac{u^p}{v_n^{p-1}} \right) \rangle \\
= p \int_M \frac{u^{p-1}}{v_n^{p-1}} |\nabla v_n|^{p-2} \langle \nabla v_n, \nabla u \rangle - (p - 1) \int_M \frac{u^p}{v_n^p} |\nabla v_n|^p.
\]

Using the hypothesis on the Dominated convergence theorem we conclude
\[
\int_M |\nabla u|^p \geq \int_M \left( -\frac{\Delta_p v}{v^{p-1}} \right) u^p, \quad u \in C_0^\infty(M), u \geq 0.
\]

In a more general context, we have the next result

**Theorem 7.** if \( u \in W_0^{1,p}(M), u \geq 0, v \in W_0^{1,p}(M), -\Delta_p v \geq 0 \) is a measure of Radon bounded, \( v|_{\partial M} = 0, v \geq 0 \), so
\[
\int_M |\nabla u|^p \geq \int_M \left( -\frac{\Delta_p v}{v^{p-1}} \right) u^p.
\]

**Proof.** According to the principle of Maximum strong we have \( v > 0 \) in \( M \). (See [13]). We pose \( v_m(x) = v(x) + \frac{1}{m}, m \in \mathbb{N} \). So \( \Delta_p v_m = \Delta_p v \) and \( \{v_m\} \) converges in \( W^{1,p}(M) \) and a.e to \( v \). Therefore, using Lemma 1.4, on gets the result for all \( \phi \in C_0^\infty(M), \phi \geq 0 \). Now in the general case, by density we deduce the existence of \( u_n \to u \) in \( W_0^{1,p}(M) \), \( u_n \in C_0^\infty(M) \) et \( u_n \geq 0 \), so
\[
\int_M |\nabla u_n|^p \geq \int_M \left( -\frac{\Delta_p v_n}{v_n^{p-1}} \right) u_n^p = \int_M \left( -\frac{\Delta_p v_n}{v_n^{p-1}} \right) u_n^p.
\]

By the hypothesis imposed on \( u \) and according to the Lemma of Fatou we obtain the result. \hfill \Box

1.3.1 Comparison principle.

As application of lemma 6, we demonstrate the next comparison result.

**Lemma 7.** Let \( f \) be a continuous positive function such that \( \frac{f(u)}{u^{p-1}} \downarrow \) with \( 1 < p \). We suppose that \( u, v \in W_0^{1,p}(M) \cap C^1(M) \) are such that
\[
\begin{cases}
-\Delta_p u \geq f(u), u > 0 \text{ in } M \\
-\Delta_p v \leq f(v), v > 0 \text{ in } M
\end{cases}
\]
so \( u \geq v \) in \( M \)

**Proof.** previous inequality implies that
\[-\Delta_p v \frac{\Delta_p u}{u^{p-1}} + \frac{\Delta_p v}{v^{p-1}} \geq \frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}}.\]

Multiply by \( w = (v^p - u^p)^+ \), we get that

\[
\int_M \left( \frac{-\Delta_p u}{u^{p-1}} + \frac{\Delta_p v}{v^{p-1}} \right) (v^p - u^p)^+ \geq \int_M \left( \frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}} \right) (v^p - u^p)^+ 
= \int_{\{v > 1\}} \left( \frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}} \right) (v^p - u^p)^+
\]

By the assumption on \( f \), we conclude that the term on the right in the previous equality is positive. On the other hand as \( w = (v^p - u^p)^+ \), so \( \nabla w = p \left( v^{p-1} \nabla v - u^{p-1} \nabla u \right) \chi_{\{v > u\}} \), so

\[
\int_M \left( \frac{-\Delta_p u}{u^{p-1}} + \frac{\Delta_p v}{v^{p-1}} \right) w = \int_M |\nabla u|^{p-2} \left( \nabla u, \nabla \left( w \frac{\nabla u}{u^{p-1}} \right) \right) - \int_M |\nabla v|^{p-2} \left( \nabla v, \nabla \left( w \frac{\nabla v}{v^{p-1}} \right) \right)
= \int_M |\nabla u|^{p-2} \left( \nabla u, \frac{u^{p-1} \nabla w - (p-1) u^{p-2} \nabla \nabla u}{u^{2(p-1)}} \right) - \int_M |\nabla v|^{p-2} \left( \nabla v, \frac{v^{p-1} \nabla w - (p-1) v^{p-2} \nabla \nabla v}{v^{2(p-1)}} \right)
= \int_{M \cap \{v > u\}} \left[ |\nabla u|^{p-2} \left( \nabla u, \nabla v \right) - (p-1) \frac{u^p}{u^{p-1}} |\nabla u|^p - |\nabla u|^p \right] 
+ \int_{M \cap \{v > u\}} \left[ \frac{v^p}{v^{p-1}} |\nabla v|^{p-2} \left( \nabla v, \nabla u \right) - (p-1) \frac{v^p}{v^{p-1}} |\nabla v|^p - |\nabla v|^p \right]
= \int_{M \cap \{v > u\}} K_1(x) d\sigma_g + \int_{M \cap \{v > u\}} K_2(x) d\sigma_g
\]

and as \( u > 0 \) and \( v > 0 \) in \( m \), using the Picone inequality, \( K_1 \leq 0 \) and \( K_2 \leq 0 \). So

\[
\int_m \left( \frac{-\Delta_p u}{u^{p-1}} + \frac{\Delta_p v}{v^{p-1}} \right) w \leq 0
\]

and Consequently,

\[
\int_{m \cap \{v > u\}} \left( \frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}} \right) (v^p - u^p) \leq 0.
\]

But on the set \( \{v > u\} \), \( \frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}} \geq 0 \), so \(|v > u| = 0\), and we deduce that \( v \leq u \).

Easily demonstrates the extension using Lemma 7.

Lemma 8 (Comparison principle). Let \( u, v \in W_0^{1,p}(M) \cap C^1(M) \) such as
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\[ \begin{align*}
-\Delta_p u &\geq h(x) f(u), \quad u > 0 \text{ in } M \\
-\Delta_p v &\leq h(x) f(v), \quad v > 0 \text{ in } M
\end{align*} \]

where $h$ is a positive function such that $h \neq 0$. So $u \geq v$ in $M$.

Remark 3. The result of Lemma 8 is valid if $h(x) = |x|^{-p}$.

As a direct application of Lemma 8, we obtain the next uniqueness result

**Theorem 8.** The problem

\[ \begin{align*}
-\Delta_p u &= \lambda h(x) u^q \text{ in } M, \quad 0 < q < p - 1 \\
0 &< \lambda \text{ in } M \\
0 &\text{in } M
\end{align*} \]

where $h$ is in the conditions of the preceding theorem, admits a unique solution.

Remark 4. In general, we have the same result of uniqueness if we replace $u^q$ with a function of Carateodory $f(x, u)$ such that \( f(x, u) u^{p-1} \) is decreasing uniformly in $x \in M$.

To demonstrate existence we need to impose more conditions on $f$.

## 2 Theory of existence and uniqueness of solutions for nonlinear elliptic problems with data in $L^1$

### 2.1 Introduction

Consider the problem of form

\[ \begin{align*}
-\Delta_p u &= f \text{ in } M, \\
0 &= \text{on } \partial M.
\end{align*} \]

(1)

where $1 < p < \infty$, $f$ is a measurable function such that $f \in L^1(M)$.

There are three difficulties associated with the study of the equation (1).

1- Find the direction for which the previous equation is well defined.
2- The construction of a solution in the direction obtained.
3- Uniqueness of the solution found.

Note that the most general meaning that can be used is the direction of distribution, ie, $u$ checks

\[ \int_M |\nabla u|^{p-2} \nabla u \nabla \phi d\sigma_s = \int_M f \phi d\sigma_s \quad \forall \phi \in C_0^\infty(M) \]

except that the problem in this context is who we do not have a construction argument (the test function space being too "small"), and the second problem is the
uniqueness of the solution (the operator is nonlinear). Note that for the case \( p = 2 \), the distributional framework is a natural framework for studying equations with a second member in \( L^1 \), because \( \Delta u = 0 \) in the distributions sense implies that \( u \) is harmonic in the classical sense.

To solve the nonlinear problem we need to introduce a new space \( \tau^{1,1}_{loc}(M) \) in which we can make sense of the gradient of \( u \), which in general is not locally integrable. So the idea is to work with the truncations \( T_k(u) \) of the \( u \) solution and expand the space of the test functions to bounded functions with a gradient in a suitable Lebesgue space.

The arguments we will introduce will be applicable to a class of equations general form.

\[-\Delta_p u = F(x,u) \text{ in } D'(M) \quad (1,2)\]

Or \( F \) is a carathodory functions, continuous and decreasing in \( u \) for \( x \) fixed, and measurable in \( x \) for \( u \) fixed. moreover, \( F(x,0) \in L^1(M) \) and \( F(x,c) \in L^1_{loc}(M) \) if \( c \neq 0 \), and if

\[G_c(x) = \sup_{|u| \leq c} |F(x,u)|,\]

so \( G_c \in L^1_{loc}(M) \) for all \( c > 0 \)

### 2.2 Functional Framework

Before discussing the concept of the entropy solution, we will present the functional framework in which the solution is well defined. We start with the introduction of the truncation operator. For a constant \( k > 0 \), we define the function \( T_k : \mathbb{R} \rightarrow \mathbb{R} \) by

\[T_k(s) = \begin{cases} 
    s & \text{if } |s| \leq k, \\
    k \text{sign}(s) & \text{if } |s| > k.
\end{cases}\]

So for a measurable function \( u \) defined in \( M \), \( T_k u \) is defined by \((T_k u)(x) = T_k(u(x))\).

we will use in its subsection Functional spaces :

i) \( \tau^{1,1}_{loc}(M) \) is the set of measurable functions \( u : M \rightarrow \mathbb{R} \) such as for all \( k > 0 \) the truncation function \( T_k(u) \) in \( W^{1,1}_{loc}(M) \).

ii) for \( p \in]1,\infty[ \), \( \tau^{1,p}_{loc}(M) \) is the subset of \( \tau^{1,1}_{loc}(M) \) composed by functions \( u \) such as \( |\nabla (T_k(u))| \in L^p_{loc}(M) \) for all \( k > 0 \).

iii) Of even, \( \tau^{1,p}(M) \) is the subset of \( \tau^{1,1}_{loc}(M) \) composed of functions \( u \), such as, of more \( |\nabla T_k(u)| \in L^p(M) \) for all \( k > 0 \).

iv) Finally, \( \tau^{1,p}_{0}(M) \) is the subset of \( \tau^{1,p}(M) \), composed of functions that can be approximated by class functions \( C^1 \) a compact support in \( M \) in the next sense : a function \( u \in \tau^{1,p}(M) \) in \( \tau^{1,p}_{0}(M) \), if for all \( k > 0 \), it exists a Sequence \( (\phi_n) \subset C^0_0(M) \) such as
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\[ \phi_n \to T_k(u) \quad \text{in} \quad L^1_{\text{loc}}(M) \]
\[ \nabla \phi_n \to \nabla T_k(u) \quad \text{in} \quad L^p(M) \]

This space will play an important role in this work.

We have the next lemma giving some properties of the preceding spaces

**Lemma 9.** for all $p \in [1, \infty]$, we have

1) $W^{1,p}_{\text{loc}}(M) \subset \tau^{1,p}_{\text{loc}}(M)$ et $W^{1,p}_0(M) \subset \tau^{1,p}_0(M)$,

2) $\tau^{1,p}_{\text{loc}}(M) \cap L^\infty_{\text{loc}}(M) = W^{1,p}_{\text{loc}}(M) \cap L^\infty_{\text{loc}}(M)$,

3) $\nabla T_k(u) = \nabla u 1_{\{|u| < k\}}$,

where $1_A$ denotes the characteristic function of a measurable set $A$.

**Proof.**

1) we have

\[ u \in W^{1,p}_{\text{loc}}(M) \Rightarrow u \in W^{1,1}_{\text{loc}}(M) \text{ et } \nabla u \in L^p_{\text{loc}}(M), \]
\[ \Rightarrow T_k(u) \in W^{1,1}_{\text{loc}}(M) \text{ and } \nabla T_k(u) \in L^p_{\text{loc}}(M) \forall k > 0 \]
\[ \Rightarrow u \in \tau^{1,p}_{\text{loc}}(M) \]

so $W^{1,p}_{\text{loc}}(M) \subset \tau^{1,p}_{\text{loc}}(M)$. For the second point, we have

\[ u \in W^{1,p}_0(M) \Rightarrow u \in W^{1,p}(M) \quad \text{and} \quad \exists \{\phi_n\} \subset C_0^\infty(M) \quad \text{such as} \]
\[ \begin{cases} 
\phi_n \to u & \text{in } L^p(M) \\
\nabla \phi_n \to \nabla u & \text{in } L^p(M) 
\end{cases} \]
\[ \Rightarrow u \in \tau^{1,p}(M) \text{ and } \exists \{\phi_n\} \subset C_0^\infty(M) \quad \text{such as} \]
\[ \begin{cases} 
\phi_n \to T_k(u) & \text{in } L^1_{\text{loc}}(M) \\
\nabla \phi_n \to \nabla T_k(u) & \text{in } L^p(M) \forall k > 0, 
\end{cases} \]
\[ \Rightarrow u \in \tau^{1,p}_0(M), \]

so $W^{1,p}_0(M) \subset \tau^{1,p}_0(M)$.

2) as

\[ u \in W^{1,p}_{\text{loc}}(M) \cap L^\infty_{\text{loc}}(M) \Rightarrow u \in W^{1,p}_{\text{loc}}(M) \quad \text{and} \quad u \in L^\infty_{\text{loc}}(M) \]
\[ \Rightarrow u \in \tau^{1,p}_{\text{loc}}(M) \quad \text{and} \quad u \in L^\infty_{\text{loc}}(M) \]
\[ \Rightarrow u \in \tau^{1,p}_{\text{loc}}(M) \cap L^\infty_{\text{loc}}(M) \]

so $W^{1,p}_{\text{loc}}(M) \cap L^\infty_{\text{loc}}(M) \subset \tau^{1,p}_{\text{loc}}(M) \cap L^\infty_{\text{loc}}(M)$. We also have
We have that $u \in \tau_{loc}^{1,p}(M) \cap L^\infty_{loc}(M) \Rightarrow u \in \tau_{loc}^{1,p}(M)$ and $u \in L^\infty_{loc}(M)$
\[ \Rightarrow T_k(u) \in W^{1,1}_{loc}(M) \quad \text{and} \quad \nabla T_k(u) \in L^p_{loc}(M) \quad \text{and} \quad u \in L^\infty_{loc}(M) \]
\[ \Rightarrow u \in L^p_{loc}(M) \quad \text{and} \quad \nabla u \in L^p_{loc}(M) \quad \text{and} \quad u \in L^\infty_{loc}(M) \]
\[ \Rightarrow \quad u \in W^{1,p}_{loc}(M) \quad \text{and} \quad u \in L^\infty_{loc}(M) \]
\[ \Rightarrow \quad u \in W^{1,p}_{loc}(M) \cap L^\infty_{loc}(M) \]

so $\tau_{loc}^{1,p}(M) \cap L^\infty_{loc}(M) \subset W^{1,p}_{loc}(M) \cap L^\infty_{loc}(M)$.
So $\tau_{loc}^{1,p}(M) \cap L^\infty_{loc}(M) = W^{1,p}_{loc}(M) \cap L^\infty_{loc}(M)$.

3) We have
\[ T_k(u) = \begin{cases} u & \text{if } |u| \leq k \\ -k\frac{u}{|u|} & \text{if } |u| > k \end{cases} \]
implies
\[ \nabla T_k(u) = \begin{cases} \nabla u & \text{if } |u| \leq k \\ \nabla u & \text{if } |u| > k \end{cases} \]
so $\nabla T_k(u) = \nabla u 1_{\{|u| < k\}}$.

Note that if $u \in \tau_{loc}^{1,1}(M)$, so $\nabla u$ is not defined even in the sense of distributions, yet we have the next lemma that gives meaning to $\nabla u$.

**Lemma 10.** [1] Let $u \in \tau_{loc}^{1,1}(M)$, it exists a function $v : M \to \mathbb{R}^N$ unique measurable such as
\[ \nabla T_k(u) = v 1_{\{|u| < k\}} \quad \text{a.e.} \]
in others, $u \in W^{1,1}_{loc}(M)$ if and only if $v \in L^1_{loc}(M)$, so $v \equiv \nabla u$ in the usual weak sense.

**Proof.**
We have $\nabla T_k(u) = \nabla u 1_{\{|u| < k\}}$, so for all $u \in \tau_{loc}^{1,1}(M)$ it exists a function $v : M \to \mathbb{R}^N$ measurable such as $v \equiv \nabla u$ a.e. and $v \in L^1_{loc}(M)$.

$v$ is unique in the sense almost everywhere, because:

for all $k, \varepsilon > 0$, we have $T_k(T_k T_k(u) = k(u)$. Therefore, we get in $M_k = \{|u| < k\}$ legality $\nabla T_k + \varepsilon = \nabla T_k$ a.e. hence the result, and so $v$ unique a.e.

It remains to show that $u \in W^{1,1}_{loc}(M)$ if $v \in L^1_{loc}(M)$. Indeed, in this case $\nabla T_k(u) \to v$ in $L^1_{loc}(M)$, so we have to prove that $u \in L^1_{loc}(M)$. By contradiction, if $u \notin L^1_{loc}$, there will be a closed ball $B \subset M$ such as
\[ \int_k = \|T_k(u)\|_{L^1(B)} \to \infty \quad \text{when} \quad k \to \infty, \]
by normalization, $v_k = \frac{T_k(u)}{k}$, so $v_k \to 0$ a.e. $\|v_k\|_{L^1(B)} = 1$ and $\|\nabla v_k\|_{L^1(B)} \to 0$, contradiction with the compactness of the injection of $W^{1,1}(B)$ in $L^1(B)$.
2.3 Solutions in the sense of entropy

In this section we will develop the concept of the solution in the sense of entropy which will allow us to study elliptic equations with second member in $L^1(M)$.

Suppose that $f \in L^1(M)$ and consider the next equation:

$$
\begin{cases}
-\Delta_p u = f(x) & \text{in } M \\
u = 0 & \text{on } \partial M
\end{cases}
$$

(2)

Let $u \in \tau^{1,p}_0(M)$ a solution of the equation (2) in $D'(M)$, so for all $\phi \in C_0^\infty(M)$, we have

$$
\int_M |\nabla u|^{p-2}\nabla u \nabla \phi \, d\sigma_g = \int_M f \phi \, d\sigma_g \quad \forall \phi \in C_0^\infty(M)
$$

Note that $f \in L^1(M)$, so by density and if we posit conditions of the type "Dirichlet homogeneous", so we can take $T_k(u - \phi), k > 0$, as a test function in the previous equation we get

$$
\int_M |\nabla u|^{p-2}\nabla u \nabla T_k(u - \phi) \, d\sigma_g = \int_M T_k(u - \phi) f \, d\sigma_g,
$$

so

$$
\int_{\{u - \phi < k\}} |\nabla u|^{p-2}\nabla u \nabla (u - \phi) \, d\sigma_g = \int_M T_k(u - \phi) f \, d\sigma_g. 
$$

(3)

Note that each term in (3) is well defined: as $\phi \in L^\infty(M)$, so

$$
|u - \phi| < k \Rightarrow |u| - |\phi| < |u - \phi| < k \Rightarrow |u| < k + |\phi| \\
\Rightarrow |u| < k + \|\phi\|_\infty \Rightarrow |u| < \overline{k}
$$

where

$$
\overline{k} = k + \|\phi\|_\infty,
$$
as $|\nabla u|^{p-1} \in L^1(M)$, so
Definition 8 (Solution in the sense of entropy). Let 

We are in a position to give the next definition

Let’s start by demonstrating some properties of the entropy solutions.

So

Since \( T_\xi(u) \in W_0^{1,p}(M) \) if \( u \in \tau_0^{1,p}(M) \) and \( \phi \in L^\infty(M) \cap W_0^{1,p}(M) \), the second member in (4) is bounded, so the first member of (3) is well defined.

We are in a position to give the next definition

Definition 8 (Solution in the sense of entropy). Let \( f \in L^1(M) \), we say that \( u \in \tau_0^{1,p}(M) \) is an entropy solution of the problem (1) if (3) is checked for each \( \phi \in L^\infty(M) \cap W_0^{1,p}(M) \) and for all \( k > 0 \).

Let’s start by demonstrating some properties of the entropy solutions.

Lemma 11. If \( u \in \tau_0^{1,p}(M) \) is an entropy solution of (1) so for all \( k > 0 \)

Therefore, we obtain the next estimate in \( L^p(M) \)

Proof.

As \( u \in \tau_0^{1,p}(M) \Rightarrow T_k(u) \in W_0^{1,p}(M) \Rightarrow T_k(u) \in L^p(M) \). If \( \phi = 0 \) and grace at (3) we will have

Thus we have

Therefore, we have

and we obtain the next estimate in \( L^p(M) \)
2.4 estimates

Before demonstrating the existence of the entropy solution, we will prove some preliminary estimates based on the estimate (5). These estimates will relate to $u$ and $|\nabla u|$ in Marcinkevicius spaces and we can consider them as keys to demonstrate compactness results in $L^q(M)$ spaces with $q$ suitably chosen. The first main result is the next lemma.

**Lemma 12.** Let $1 < p < N$ and $(M, g)$ a Riemannian manifold of dimension $N$ Consider $u \in \mathcal{D}_0^{1,p}(M)$ such as

$$\frac{1}{k} \int_{|u|<k} |\nabla u|^p d\sigma_g \leq \alpha,$$

(6)

for all $k > 0$. So $u \in \mathcal{M}^{p_1}(M)$ with $p_1 = \frac{N(p-1)}{N-p}$. More precisely, there exists $C = C(N, p) > 0$ such as

$$\text{meas} \{|u| > k\} \leq C \alpha^{\frac{N}{N-p}} k^{-p_1}.$$

(7)

**Proof.** Let $1 < p < N$ and $u \in \mathcal{D}_0^{1,p}(M)$, so $T_k(u) \in W_0^{1,p}(M)$ for all $k > 0$, and according to the inequality of Sobolev we have

$$\|T_k(u)\|_{p^*} \leq c(N, p) \|\nabla T_k(u)\|_p,$$

or $p^* = \frac{Np}{N-p}$.

because of (6), we have $\int_M |\nabla T_k(u)|^p d\sigma_g \leq k \alpha$, so $\|\nabla T_k(u)\|_p \leq k\alpha.$

and consequently $\|\nabla T_k(u)\|_{p^*} \leq (k\alpha)^{\frac{1}{p^*}}$, so $\|T_k(u)\|_{p^*} \leq c(N, p)(k\alpha)^{\frac{N}{N-p}}$.

n for $0 < \epsilon \leq k$. We have $\{|u| > \epsilon\} = \{|T_k(u) > \epsilon\}$, so

$$\text{meas} \{|u| > \epsilon\} \leq c_1(N, p)\alpha^{\frac{N}{N-p}} k^{\frac{N}{N-p}} \epsilon^{-\frac{N}{N-p}}.$$

for $\epsilon = k$, $\text{meas} \{|u| > k\} \leq c_1(N, p)\alpha^{\frac{N}{N-p}} k^{\frac{N}{N-p}}$, we obtain

$$\text{meas} \{|u| > k\} \leq C k^{-p_1},$$

or $C = c_1(N, p)\alpha^{\frac{N}{N-p}}$ and $p_1 = \frac{N(p-1)}{N-p}$. So it results than $\phi_u(k) \leq C k^{-p_1}$, and as a conclusion it results that $u \in \mathcal{M}^{p_1}(M)$.

We now prove estimates on the gradient of $u$.

**Lemma 13.** Let $1 < p < N$ and suppose that $u \in \mathcal{D}_0^{1,p}(M)$ satisfied (6) for all $k$, so for all $h > 0$

$$\text{meas} \left\{ |\nabla u| > h \right\} \leq C(N, p)\alpha^{\frac{N}{N-p}} h^{-p_2}, \quad p_2 = \frac{N(p-1)}{N-1}.$$
for \( k, \lambda > 0 \), we pose
\[
\Phi(k, \lambda) = \text{meas}\{ |\nabla u|^p > \lambda, |u| > k \},
\]
according to the Lemma (13) we have
\[
\Phi(k, 0) \leq C(N, p)M^N h^{-p_1}
\]
(8)
As the function \( \lambda \mapsto \Phi(k, \lambda) \) is decreasing, we get for \( k, \lambda > 0 \) and for \( 0 \leq s \leq \lambda \), \( \Phi(0, \lambda) \leq \Phi(0, s) \), so
\[
\Phi(0, \lambda) \leq \Phi(0, s) \Rightarrow \int_0^\lambda \Phi(0, \lambda)ds \leq \int_0^\lambda \Phi(0, s)ds
\]
\[
\Rightarrow \Phi(0, \lambda) \leq \frac{1}{\lambda} \int_0^\lambda \Phi(0, s)ds
\]
and
\[
\frac{1}{\lambda} \int_0^\lambda \Phi(0, s)ds = \frac{1}{\lambda} \int_0^\lambda \Phi(k, s)ds + \frac{1}{\lambda} \int_0^\lambda (\Phi(0, s) - \Phi(k, s))ds
\]
\[
\leq \frac{1}{\lambda} \int_0^\lambda \Phi(k, 0)ds + \frac{1}{\lambda} \int_0^\lambda (\Phi(0, s) - \Phi(k, s))ds
\]
\[
\leq \Phi(k, 0) + \frac{1}{\lambda} \int_0^\lambda (\Phi(0, s) - \Phi(k, s))ds
\]
so
\[
\Phi(0, \lambda) \leq \frac{1}{\lambda} \int_0^\lambda \Phi(0, s)ds \leq \Phi(k, 0) + \int_0^\lambda (\Phi(0, s) - \Phi(k, s))ds
\]
(9)
Note that
\[
\Phi(0, s) - \Phi(k, s) = \text{meas}\{ |u| < k, |\nabla u|^p > s \}
\]
as (7), we will have
\[
\int_0^\infty (\Phi(0, s) - \Phi(k, s))ds = \int_{|u| < k} |\nabla u|^p d\sigma_g \leq k\alpha.
\]
(10)
Finally from (9) and using (8) and (10), we get to
\[
\Phi(0, \lambda) \leq \frac{\alpha k}{\lambda} + C(N, p)\alpha^N h^{-p_1}.
\]
we pose \( P(k) = \frac{\alpha k}{\lambda} + c\alpha^N h^{-p_1} \), so minimizing \( P(k) \), of \( k \), we will have to solve the equation \( P'(k) = 0 \), which implies that
\[
\frac{\alpha}{\lambda} - cp_1\alpha^N k^{-p_1-1} = 0
\]
and so \( k = \left( c\lambda p_1 \alpha^{p} \right)^{\frac{1}{p+1}} \).

Consequently,

\[
\Phi(0, \lambda) \leq k \left[ \frac{\alpha}{\lambda} + c\alpha^{\frac{N}{p+1}} k^{-p_1-1} \right] \leq k \left[ \frac{\alpha}{\lambda} + \frac{\alpha^{N}}{\lambda p_1} \right] \alpha^{\frac{N}{p+1}} k^{-\frac{N}{p+1}}
\]

\[
\leq \frac{\alpha}{\lambda} \left[ 1 + \frac{1}{p_1} \right] \left( c\lambda p_1 \alpha^{\frac{p}{p+1}} \right)^{\frac{N}{p+1}} \leq \frac{\alpha}{\lambda} \left[ 1 + \frac{1}{p_1} \right] (c p_1)^{\frac{N}{p+1}} \lambda^{-\frac{N}{p+1}} \alpha^{\frac{N}{p+1}} k^{-\frac{N}{p+1}}
\]

so

\[
\Phi(0, \lambda) \leq C(N, p) \alpha^{\frac{N}{p+1}} \lambda^{-\frac{N(p-1)}{p+1}} \quad \text{with} \quad C(N, p) = \left[ 1 + \frac{1}{p_1} \right] (c p_1)^{\frac{N}{p+1}} \lambda^{-\frac{N}{p+1}} \alpha^{\frac{N}{p+1}} k^{-\frac{N}{p+1}}
\]

we pose \( \lambda = h^p \), so

\[
\text{meas} \{ |\nabla u|^p > h^p \} \leq C(N, p) \alpha^{\frac{N}{p+1}} h^{-\frac{N(p-1)}{p+1}}
\]

and consequently

\[
\text{meas} \{ |\nabla u| > h \} \leq C(N, p) \alpha^{\frac{N}{p+1}} h^{-p_2} \quad \text{with} \quad p_2 = \frac{N(p-1)}{N-1}
\]

Hence the result.

### 2.5 Existence of the entropy solution

We are in a position to demonstrate the main result of this article, more precisely we have the next theorem

**Theorem 9.** Let \( 1 < p < N \) and let \((M, g)\) a compact Riemannian manifold, so it exists \( u \) an entropy solution of the problem (2) with \( u \in \tau_0^{1,p}(M) \). Furthermore

\[
u \in \mathcal{M}^1(M) \quad \text{and} \quad |\nabla u| \in \mathcal{M}^2(M),
\]

or \( p_1 = \frac{N(p-1)}{N-p} \) and \( p_2 = \frac{N(p-1)}{N-1} \).

In the case \( p > 2 - \frac{1}{N} \) the solution \( u \in W_0^{1,q}(M) \) for all \( q < p_2 \).
Proof.
The main idea of the demonstration is to proceed by approximation.
Step 1.
As \( f \in L^1(M) \) there is a sequence of functions \( \{f_n\} \subset L^\infty(M) \) such that \( f_n \rightarrow f \) in \( L^1(M) \).

for \( f_n \in L^\infty(M) \) it exists \( u_n \in W_0^{1,p}(M) \), the unique weak solution of the problem

\[
\begin{cases}
-\Delta_p u_n = f_n & \text{in } M \\
u_n = 0 & \text{on } \partial M
\end{cases}
\]

(11)

note that \( T_k(u_n) \in L^1(M) \cap L^\infty(M) \) for all \( k > 0 \), so taking \( T_k(u_n) \) as test function in (11) we get that

\[
\int_M |\nabla u_n|^{p-2} \nabla u_n \nabla T_k(u_n) \, d\sigma_g = \int_M f_n T_k(u_n) \, d\sigma_g
\]

so

\[
\int_{\{|u_n|<k\}} |\nabla u_n|^p \, d\sigma_g \leq kc
\]

and so

\[
\frac{1}{k} \int_{\{|u_n|<k\}} |\nabla u_n|^p \, d\sigma_g \leq c
\]

ie,

\[
\int_M |\nabla T_k(u_n)|^p \, d\sigma_g \leq kc
\]

therefore it is concluded that \( \{\nabla T_k(u_n)\} \) is bounded in \( L^p(M) \) for all \( k > 0 \). So it exists \( w_k \) tel que \( T_k(u_n) \rightarrow w_k \) weakly in \( W_0^{1,p}(M) \) for each \( k > 0 \), and \( T_k(u_n) \rightarrow w_k \) a.e in \( M \). we pose \( w_k \equiv T_k(u) \) in the set or \( |w_k| < k \), it’s clear that \( u \) is well defined because \( T_{k+b}(u_n) = T_k(T_h(u_n)) \) and consequently \( T_k(u_n) \rightarrow T_k(u) \) strongly in \( L^p(M) \) for all \( q < p^* \).

According to Lemmas 12 and 13, we have

\[
u_n \in \mathcal{A}^{p_1}(M) \quad \text{et} \quad |\nabla u_n| \in \mathcal{A}^{p_2}(M),
\]

with \( p_1 = \frac{N(p-1)}{N-1} \) and \( p_2 = \frac{N(p-1)}{N-1} \), so

\[
|\nabla u_n|_{p_1(M)} \leq C \quad \text{et} \quad |\nabla u_n|_{p_2(M)} \leq \overline{C},
\]

and as \( L^q(M) \subset \mathcal{A}^q(M) \subset L^{q-\epsilon}(M) \) for all \( q, \epsilon > 0 \), so

\[
|\nabla u_n|_{L^{p_1-\epsilon}(M)} \leq C \quad \text{et} \quad |\nabla u_n|_{L^{p_2-\epsilon}(M)} \leq \overline{C}.
\]

Note that if \( p > 2 - \frac{1}{N} \), so \( p_2 > 1 \) and consequently \( \{u_n\} \) will be bounded in \( W_0^{1,p_2-\epsilon}(M) \) for all \( \epsilon > 0 \) with \( p_2 - \epsilon \geq 1 \), so \( u_n \rightarrow u \) weakly in \( W_0^{1,p_2-\epsilon}(M) \).

So for all \( \phi \in W^{1,\infty}(M) \), we have
\[
\int_M |\nabla u_n|^{p-2} \nabla u_n \nabla \delta \, d\sigma_g = \int_M f_n \delta \, d\sigma_g,
\]
as \( |\nabla u_n|^{p_2-\delta} \equiv \left( |\nabla u_n|^{p-1} \right)^{N-\delta} \) and \( p - 1 < p_2 \), so for \( p - 1 < p_2 - \delta \)
\[
|\nabla u_n|^{p-1} \in L^{N-\delta}(M),
\]
as \( |\nabla \phi| \in L^\infty(M) \) and \( f_n \rightarrow f \) in \( L^1(M) \), so going to the limit when \( n \rightarrow \infty \), we find that
\[
\int_M |\nabla u|^{p-2} \nabla u \nabla \delta \, d\sigma_g = \int_M f \delta \, d\sigma_g.
\]
It’s clear that \( u_n \rightarrow u \) strongly in \( L^\infty(M) \) such as \( 1 \leq \varphi < \varphi' \) with \( \varphi' = \frac{\varphi}{N-\delta} > 1 \)
or \( \varphi' = p_2 - \delta \)

**Step 2.** To analyze the general case \( 1 < p \), we start by demonstrating that \( u \in W^{1,p}_0(M) \).

We pose \( \nabla T_k(u) = \nabla w_k \), is clear that \( \nabla T_k(u) \) is well defined because \( w_k \in W^{1,p}_0(M) \), to go to the limit in \( k \) we will start by show that \( \nabla u_n \) converges to \( \nabla u \) locally in measure. To prove it we show that \( \{\nabla u_n\} \) is a Cauchy sequence in measure.

Let \( \varepsilon > 0 \), so
\[
\{ |\nabla u_n - \nabla u_m| > \varepsilon \} \subset \{ |\nabla u_n| > A \} \cup \{ |\nabla u_m| > A \} \cup \{ |u_n - u_m| > k \}
\]
\[
\cup \{ |u_n - u_m| \leq k, |\nabla u_n| \leq A, |\nabla u_m| \leq A, |\nabla u_n - \nabla u_m| > \varepsilon \}.
\]
(12)

We choose \( A \) big enough as
\[
\text{meas} \{ |\nabla u_n| > A \} \leq \varepsilon \quad \text{for all} \quad n \in \mathbb{N},
\]
(this is possible by Lemma 13).

To estimate the last term in (12), we use the next algebraic inequalities.

for all \( \xi, \eta \in \mathbb{R}^N \), we have
\[
\langle |\xi|^{p-2} \xi - |\eta|^{p-2} \eta, \xi - \eta \rangle \geq 0,
\]
again if \( x \neq \eta \) and \( |\xi| < A, |\eta| < A \), so it exists \( \mu > 0 \) such that
\[
\langle |\xi|^{p-2} \xi - |\eta|^{p-2} \eta, \xi - \eta \rangle \geq \mu.
\]

Knowing that \( -\Delta_p u_n = f_n \) and \( -\Delta_p u_m = f_m \), so by subtracting and using \( T_k(u_n - u_m) \) as a test function, we get
\[
\int_{\{|u_n - u_m| \leq k\}} \left\langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m, \nabla u_n - \nabla u_m \right\rangle d\sigma_g
\]

\[
= \int_M (f_n - f_m) T_k(u_n - u_m) d\sigma_g \leq 2ck.
\]

According to the Lemma 12, we have

\[
\text{meas} \left\{ |u_n - u_m| \leq k, |\nabla u_n| \leq A, |\nabla u_m| \leq A, |\nabla u_n - \nabla u_m| > \epsilon \right\} \\
\leq \text{meas} \left\{ |u_n - u_m| \leq k, \left( |\nabla u_n|^{p-2} u_n - |\nabla u_m|^{p-2} u_m \right), (\nabla u_n - \nabla u_m) \geq \mu \right\} \\
\leq \frac{1}{\mu} \int_{\{|u_n - u_m| \leq k\}} \left\langle |\nabla u_n|^{p-2} u_n - |\nabla u_m|^{p-2} u_m, \nabla u_n - \nabla u_m \right\rangle d\sigma_g \\
\leq \frac{1}{\mu} 2ck \leq \epsilon,
\]

if \( k \) is small enough, as \( k \leq \frac{\mu \epsilon}{2c} \).

So we fix \( A \) and \( k \), if \( n_0 \) big enough, we have to \( n, m \geq n_0 \), \( \text{meas} \left\{ |u_n - u_m| > k \right\} \leq \epsilon \), and so

\[
\text{meas} \{ |\nabla u_n - \nabla u_m| > k \} \leq 2\epsilon.
\]

So \( \{\nabla u_n\} \) converges locally to a \( v \) function and as a consequence a.e. in \( M \). Since \( \{\nabla T_k(u_n)\} \) is bounded in \( L^p(M) \) for all \( k > 0 \) and \( \nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \) weakly in \( L^p(M) \), we deduce that \( v = \nabla u \) a.e. Note that in general \( v \notin (L^1(M))^N \). It is clear that if \( p > 2 - \frac{1}{n} \), then \( v \notin (L^1(M))^N \), and so \( u \in W_0^{1,1}(M) \) and from Lemma 10 we deduce \( \nabla u = v \) a.e.

And consequently \( u \in \tau_{0}^{1,1}(M) \).

To see that \( u \in \tau_0^{1,p}(M) \), we consider \( \phi_n \in C_0^\infty(M) \) such that

\[
\|\nabla \phi_n - \nabla T_k(u_n)\|_{L^p(\Omega)} \leq \frac{1}{n} \quad \text{et} \quad \|\phi_n - T_k(u_n)\|_{L^{p^*}(\Omega)} \leq \frac{1}{n}.
\]

We have then

\[
\nabla \phi_n \rightharpoonup \nabla T_k(u) \quad \text{fortement dans} \quad L^p(M)
\]

and

\[
\phi_n \rightharpoonup T_k(u) \quad \text{fortement in} \quad L_q^{1,loc}(M) \quad \text{for} \quad q < p^*.
\]

As a conclusion we get that \( \phi_n \) converges strongly to \( T_k(u) \) and consequently \( u \in \tau_0^{1,p}(M) \).

**Step 3.** In this step we will demonstrate the strong convergence of truncations in \( W_0^{1,p}(M) \), i.e. for \( k > 0 \) fixed on a \( T_k(u_n) \rightharpoonup T_k(u) \) strongly in \( W_0^{1,p}(M) \).

Note that \( T_k(u_n) \rightharpoonup T_k(u) \) weakly in \( W_0^{1,p}(M) \) for all \( k > 0 \).
Let $k, h > 0$ such that $h > k > 0$, we assume
\[ w_n = T_{2k} (u_n - T_h (u_n) T_k (u_n) - T_k (u)) \]
Taking $w_n$ as a test function in (11), it results
\[ \int_M |\nabla u_n|^{-2} \nabla u_n \nabla w_n d\sigma_g = \int_M f_n w_n d\sigma_g, \]
we pose $I = \int_M |\nabla u_n|^{-2} \nabla u_n \nabla w_n d\sigma_g$, when $k \to \infty$ and $h \to \infty$ we have $\int_M f_n w_n d\sigma_g \to 0$, so $\int_M |\nabla u_n|^{-2} \nabla u_n \nabla w_n d\sigma_g \to 0$.
We pose $\alpha = 4k + h$. if $|u_n| > \alpha$, $\nabla w_n = 0$. So

\[ I = \int_{\{ |u_n| < \alpha \}} |\nabla u_n|^{-2} \nabla u_n \nabla w_n d\sigma_g \]
\[ = \int_{\{ |u_n| < k \}} |\nabla u_n|^{-2} \nabla u_n \nabla w_n d\sigma_g + \int_{\{ k < |u_n| < \alpha \}} |\nabla u_n|^{-2} \nabla u_n \nabla w_n d\sigma_g \]
\[ = \int_M |\nabla T_k (u_n)|^{-2} \nabla T_k (u_n) \nabla (T_k (u_n) - T_k (u)) d\sigma_g \]
\[ + \int_{\{ k < |u_n| < \alpha \}} |\nabla u_n|^{-2} \nabla u_n \nabla w_n d\sigma_g. \]

note that $1$  
if $k < |u_n| \leq h$ so $\nabla w_n = \nabla T_k (u)$  
and  
$2$  
if $h < |u_n| < \alpha$ so $\nabla w_n = \nabla T_k (u)$,  
so

\[ \int_{\{ k < |u_n| < \alpha \}} |\nabla u_n|^{-2} \nabla u_n \nabla w_n d\sigma_g = \int_{\{ |u_n| > k \}} |\nabla T_k (u_n)|^{-2} \nabla T_k (u_n) \nabla T_k (u) d\sigma_g \]
\[ \geq - \int_{\{ |u_n| > k \}} |\nabla T_k (u_n)|^{-1} |\nabla T_k (u)| d\sigma_g. \]

we obtain

\[ I \geq \int_M |\nabla T_k (u_n)|^{-2} \nabla T_k (u_n) \nabla (T_k (u_n) - T_k (u)) d\sigma_g \]
\[ - \int_{\{ |u_n| > k \}} |\nabla T_k (u_n)|^{-1} |\nabla T_k (u)| d\sigma_g. \]

As $\left\{ |\nabla T_k (u_n)|^{-1} \right\}$ is bounded in $L^\infty (M)$, $|\nabla T_k (u)| 1_{\{|u_n| > k\}}$ is bounded in $L^p (M)$ and $|\nabla T_k (u)|^{\rho_{\alpha}} 1_{\{|u_n| > k\}} \to 0$ strongly in $L^p (M)$ when $k \to \infty$, we get that
\[ \int_{\{ |u_n| > k \}} |\nabla T_k (u_n)|^{-1} |\nabla T_k (u)| d\sigma_g \to 0 \quad \text{when} \quad k \to \infty. \]

So
J = \int_M |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \nabla (T_k(u_n) - T_k(u)) \, dx \\
= \int_M \left( |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p-2} \nabla T_k(u) \right) (\nabla T_k(u_n) - \nabla T_k(u)) \, d\sigma_\delta \\
+ \int_M |\nabla T_k(u)|^{p-2} \nabla T_k(u) (\nabla T_k(u_n) - \nabla T_k(u)) \, d\sigma_\delta \\
as \\
\int_M |\nabla T_k(u)|^{p-2} \nabla T_k(u) (\nabla T_k(u_n) - \nabla T_k(u)) \, d\sigma_\delta \rightarrow 0 \quad \text{when} \quad n \rightarrow \infty,

so

J = \int_M \left( |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p-2} \nabla T_k(u) \right) (\nabla T_k(u_n) - \nabla T_k(u)) \, d\sigma_\delta + o(1).

So

I \geq \int_M \left( |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p-2} \nabla T_k(u) \right) (\nabla T_k(u_n) - \nabla T_k(u)) \, d\sigma_\delta + o(1)

\geq c \int_M |\nabla T_k(u_n) - \nabla T_k(u)|^p \, d\sigma_\delta + o(1) \quad \text{if} \quad p \geq 2

and

I \geq C(p) \int_M \frac{|\nabla T_k(u_n) - \nabla T_k(u)|^2}{(|\nabla T_k(u)| + |\nabla T_k(u_n)|)^{2-p}} \, d\sigma_\delta + o(1) \quad \text{if} \quad p < 2.

So

\int_M |\nabla T_k(u_n) - \nabla T_k(u)|^p \, d\sigma_\delta \leq o(1) + \int_M f_n w_n \, d\sigma_\delta \quad \text{if} \quad p \geq 2

and

C(p) \int_M \frac{|\nabla T_k(u_n) - \nabla T_k(u)|^2}{(|\nabla T_k(u)| + |\nabla T_k(u_n)|)^{2-p}} \, d\sigma_\delta \leq o(1) + \int_M f_n w_n \quad \text{if} \quad p < 2.

Consequently

\int_M |\nabla T_k(u_n) - \nabla T_k(u)|^p \, d\sigma_\delta \rightarrow 0 \quad \text{if} \quad p \geq 2,

and

\int_M \frac{|\nabla T_k(u_n) - \nabla T_k(u)|^2}{(|\nabla T_k(u)| + |\nabla T_k(u_n)|)^{2-p}} \, d\sigma_\delta \rightarrow 0 \quad \text{if} \quad p < 2.

For the second case we have
Quasi-linear elliptic equations with data in $L^1$ on a compact Riemannian manifold

\[
\int_M |\nabla T_k(u_n) - \nabla T_k(u)|^p \, d\sigma_g = \int_M \frac{|\nabla T_k(u_n) - \nabla T_k(u)|^p}{(|\nabla T_k(u)| + |\nabla T_k(u_n)|)^{\frac{p}{2}}} \left(\int_M (|\nabla T_k(u)| + |\nabla T_k(u_n)|)^{\frac{p}{2}} \, d\sigma_g\right)^{-\frac{p}{2}}. 
\]

So for $p < 2$,

\[
\int_M |\nabla T_k(u_n) - \nabla T_k(u)|^p \, d\sigma_g \to 0 \text{ for } n \to \infty 
\]

As a conclusion we obtain that $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,p}(M)$ for all $k > 0$.

**Step 4.** To complete the proof it remains to show that $u$ is an entropy solution. Recall that

\[
\begin{cases}
-\Delta_p u_n = f_n & \text{in } M \\
u_n = 0 & \text{on } \partial M
\end{cases}
\]

Let $v \in L^\infty(M) \cap W_0^{1,p}(M)$, for all $k$ fixed $> 0$, we have

\[
\int_M |\nabla u_n|^{p-2} \nabla u_n \nabla T_k(u_n - v) \, d\sigma_g = \int_M f_n T_k(u_n - v) \, d\sigma_g
\]

As $u_n \to u$ a.e. in $M$, and $f_n \to f$ in $L^1(M)$. So \( \int_M f_n T_k(u_n - v) \, d\sigma_g \to \int_M f T_k(u - v) \, d\sigma_g \) for $n \to \infty$.

As $v \in L^\infty(M) \cap W_0^{1,p}(M)$, so it exists a positive constant $c > 0$ such that

\[
\int_M |\nabla u_n|^{p-2} \nabla u_n \nabla T_k(u_n - v) \, d\sigma_g = \int_{|u_n| \leq c} |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n - v) \nabla T_k(u_n - v) \, d\sigma_g
\]

Note that it is sufficient to take $c \geq k + ||v||_\infty$. As $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,p}(M)$, so we conclude that

\[
\int_{|u_n| \leq c} |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \nabla T_k(u_n - v) \, d\sigma_g \to \int_{|u| \leq c} |\nabla T_k(u)|^{p-2} \nabla T_k(u) \nabla T_k(u - v) \, d\sigma_g
\]

for $n \to \infty$. Consequently and for $n \to \infty$ we get that

\[
\int_M |\nabla u|^{p-2} \nabla u \nabla T_k(u - v) \, d\sigma_g = \int_M f T_k(u - v) \, d\sigma_g
\]

So $u$ is an entropy solution of the problem (1, 2).
2.6 Uniqueness of the solution in the sense of entropy

We deal here with the question of the uniqueness of entropy solutions \( u \in \tau_{0}^{1, \rho}(M) \) for the problem (2), note that \( u \) checks (3) for all \( \phi \in L^\infty(M) \cap W_{0}^{1, \rho}(M) \) and for all \( k > 0 \).

The main result of this section is the next theorem

**Theorem 10.** Let \( u_1 \) et \( u_2 \) two functions in \( \tau_{0}^{1, \rho}(M) \), such as \( u_1 \) and \( u_2 \) are entropy solutions to the problem

\[-\Delta_{\rho}u = f(x)\]

so \( u_1 = u_2 \).

**Proof.**

Note that \( f \in L^1(M) \), substitute in the relation (3) with test functions \( T_{h}(u_1) \) and \( T_{h}(u_2) \) and by addition gets that

\[
\begin{align*}
\int_{\{ |u_1 - T_{h}(u_2)| < k \}} |\nabla u_1|^{p-2} \nabla u_1 \nabla (u_1 - T_{h}(u_2)) \, d\sigma_g &= \int_M f T_h(u_1 - T_h(u_2)) \, d\sigma_g, \\
\int_{\{ |u_2 - T_{h}(u_1)| < k \}} |\nabla u_2|^{p-2} \nabla u_2 \nabla (u_2 - T_{h}(u_1)) \, d\sigma_g &= \int_M f T_h(u_2 - T_h(u_1)) \, d\sigma_g.
\end{align*}
\]

By combining the two results we get

\[
\begin{align*}
\int_{\{ |u_1 - T_{h}(u_2)| < k \}} |\nabla u_1|^{p-2} \nabla u_1 \nabla (u_1 - T_{h}(u_2)) \, d\sigma_g \\
+ \int_{\{ |u_2 - T_{h}(u_1)| < k \}} |\nabla u_2|^{p-2} \nabla u_2 \nabla (u_2 - T_{h}(u_1)) \, d\sigma_g &= \int_M f (T_h(u_1 - T_h(u_2)) + T_h(u_2 - T_h(u_1))) \, d\sigma_g. \\
\end{align*}
\]

The conclusion \( u_1 = u_2 \) will be reached after going to the limit \( h \to \infty \) in this formula. Let

\[
I = \int_{\{ |u_1 - T_{h}(u_2)| < k \}} |\nabla u_1|^{p-2} \nabla u_1 \nabla (u_1 - T_{h}(u_2)) \, d\sigma_g \\
+ \int_{\{ |u_2 - T_{h}(u_1)| < k \}} |\nabla u_2|^{p-2} \nabla u_2 \nabla (u_2 - T_{h}(u_1)) \, d\sigma_g.
\]

we pose

\[ A_0 = \{ x \in M : |u_1 - u_2| < k, |u_1| < h, |u_2| < h \}. \]

In \( A_0 \) the first member of (13) is reduced to the next term
$$I_0 = \int_{A_0} \left( |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right) (\nabla u_1 - \nabla u_2) d\sigma_g.$$ 

Let 

$$A_1 = \{ x \in M : |u_1 - T_h(u_2)| < k, |u_2| \geq h \},$$

so 

$$\int_{A_1} |\nabla u_1|^{p-2} \nabla u_1 \nabla (u_1 - T_h(u_2)) d\sigma_g = \int_{A_1} |\nabla u_1|^p d\sigma_g \geq 0,$$

and on set 

$$A_2 = \{ x \in M : |u_1 - T_h(u_2)| < k, |u_1| < h, |u_2| \geq h \},$$

we are getting 

$$\int_{A_2} |\nabla u_1|^{p-2} \nabla u_1 \nabla (u_1 - T_h(u_2)) d\sigma_g = \int_{A_2} |\nabla u_1|^{p-2} \nabla u_1 (\nabla u_1 - \nabla u_2) d\sigma_g \geq - \int_{A_2} |\nabla u_1|^{p-2} \nabla u_1 \nabla u_2 d\sigma_g.$$ 

In the same way, we can define all $A'_1$ and $A'_2$ as 

$$A'_1 = \{ x \in M : |u_2 - T_h(u_1)| < k, |u_1| \geq h \},$$

and 

$$A'_2 = \{ x \in M : |u_2 - T_h(u_1)| < k, |u_1| < h, |u_2| \geq h \}.$$ 

Then the second term of (13) can be written as a sum of 

$$\int_{A'_1} |\nabla u_2|^{p-2} \nabla u_2 (\nabla u_2 - \nabla T_h(u_1)) d\sigma_g = \int_{A'_1} |\nabla u_2|^p d\sigma_g \geq 0$$

and 

$$\int_{A'_2} |\nabla u_2|^{p-2} \nabla u_2 (\nabla u_2 - \nabla T_h(u_1)) d\sigma_g = \int_{A'_2} |\nabla u_2|^{p-2} \nabla u_2 (\nabla u_2 - \nabla u_1) d\sigma_g \geq - \int_{A'_2} |\nabla u_2|^{p-2} \nabla u_2 \nabla u_1 d\sigma_g.$$ 

Therefore we conclude that
\[ I \geq I_0 + \int_{A_1} |\nabla u_1|^p d\sigma_g - \int_{A_2} |\nabla u_1|^{p-2} \nabla u_1 \nabla u_2 d\sigma_g \]
\[ + \int_{A_3} |\nabla u_2|^p d\sigma_g - \int_{A_2} |\nabla u_2|^{p-2} \nabla u_2 \nabla u_1 d\sigma_g \]
\[ \geq I_0 - \left( \int_{A_2} |\nabla u_1|^{p-2} \nabla u_1 \nabla u_2 d\sigma_g + \int_{A_2} |\nabla u_2|^{p-2} \nabla u_2 \nabla u_1 d\sigma_g \right) \]
\[ \geq I_0 - I_3 \]
or
\[ I_3 = \int_{A_2} |\nabla u_1|^{p-2} \nabla u_1 \nabla u_2 d\sigma_g + \int_{A_2} |\nabla u_2|^{p-2} \nabla u_2 \nabla u_1 d\sigma_g \]

The first term of \( I_3 \) can be estimated by

\[ \int_{A_2} |\nabla u_1|^{p-2} \nabla u_1 \nabla u_2 d\sigma_g \leq \int_{A_2} |\nabla u_1|^{p-1} |\nabla u_2| d\sigma_g \]
\[ \leq \left( \int_{A_2} |\nabla u_1|^{p} d\sigma_g \right)^{\frac{1}{p}} \left( \int_{A_2} |\nabla u_2|^{p} d\sigma_g \right)^{\frac{1}{p}} \]
\[ \leq \| \nabla u_1 \|_{L^p(\{|h \leq |u_1| \leq h+k|\})} \| \nabla u_2 \|_{L^p(\{|h-k \leq |u_2| \leq h|\})} . \]

as \( \| \nabla u_1 \|_{L^p(\{|h \leq |u_1| \leq h+k|\})} \| \nabla u_2 \|_{L^p(\{|h-k \leq |u_2| \leq h|\})} \to 0 \) when \( h \to \infty \) for all \( k > 0 \), it results that \( \int_{A_2} |\nabla u_1|^{p-2} \nabla u_1 \nabla u_2 d\sigma_g \) converges to 0 when \( h \to \infty \) for tout \( k > 0 \).

In the same way we obtain the same conclusion for the second term of \( I_3 \).

So we conclude that \( I_3 \) tends to 0 when \( h \to \infty \).

Regarding the second member of (13), knowing that

\[ T_k (u_1 - T_h (u_2)) + T_k (u_2 - T_h (u_1)) \to 0 \text{ a.e. in } M \text{ for } h \to \infty \]
\[ |T_k (u_1 - T_h (u_2)) + T_k (u_2 - T_h (u_1))| \leq 2k \]

and that \( f \in L^1(M) \), so using the dominated Convergence Theorem we get that

\[ \int_{M} f (T_k (u_1 - T_h (u_2)) + T_k (u_2 - T_h (u_1))) d\sigma_g \to 0 \text{ quand } h \to \infty \text{ for all } k > 0 . \]

Combining previous estimates it results that

\[ \int_{A_0(h,k)} \left( |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right) (\nabla u_1 - \nabla u_2) d\sigma_g \leq \epsilon(h) , \]

or \( \epsilon(h) \to 0 \) when \( h \to \infty \) for all \( k \) fixed > 0. Since \( A_0(h,k) \) converges to

\[ \{ x \in M : |u_1 - u_2| < k \} , \]
we conclude that
\[
\int_{|u_1 - u_2| < k} \left( |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right) (\nabla u_1 - \nabla u_2) d\sigma_g \leq 0.
\]
As
\[
\lambda \|\nabla u_1 - \nabla u_2\|_{L^p(|u_1 - u_2| < k)}^p \leq \int_{|u_1 - u_2| < k} \left( |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right) (\nabla u_1 - \nabla u_2) d\sigma_g
\]
if \( p > 2 \) and
\[
\int_{|u_1 - u_2| < k} \frac{|\nabla u_1 - \nabla u_2|^2}{(|\nabla u_1| + |\nabla u_2|)^{2-p}} d\sigma_g \leq
\]
\[
\int_{|u_1 - u_2| < k} \left( |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right) (\nabla u_1 - \nabla u_2) d\sigma_g
\]
for \( p < 2 \), then \( \nabla u_1 - \nabla u_2 = 0 \) a.e. and consequently \( T_k(u_2) = T_k(u_2) \) for all \( k > 0 \). It is clear that \( u_1 - u_2 = c \), using the fact that \( u_1 = u_2 = 0 \) on \( \partial M \), then we concludes that \( u_1 = u_2 \) a.e. Hence the result.

2.7 Some generalizations

The notion of the entropy solution can be defined for a very large class of nonlinear elliptic operators, for example if we consider the next problem
\[
-\text{div}(a(x,u,\nabla u)) = F(x,u)
\]
with \( a(x,s,\xi) : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) is a function of Carathéodory verifying (H1):
\[
|a(x,s,\xi)| \leq c \left( |\xi|^{p-1} + |s|^{p-1} + k(x) \right), \quad p.p., \quad x \in M, (s,\xi) \in \mathbb{R}^N \times \mathbb{R},
\]
(H2): \( a(x,s,\xi)\xi \geq \lambda |\xi|^p, \quad p.p., \quad x \in M, \ (s,\xi) \in \mathbb{R}^N \times \mathbb{R}, \)
(H3): \( F \) is a Caratheodory function, continuous and decreasing in \( u \) for \( x \) fixed and measurable in \( x \) for \( u \) fixed. Furthermore, \( F(x,0) \in L^1(M) \) et \( F(x,c) \in L^1_{\text{loc}}(M) \) if \( c \neq 0 \) and si
\[
G_c(x) = \sup_{|u| \leq c} |F(x,u)|,
\]
\( G_c \in L^1_{\text{loc}}(M) \) for all \( c > 0 \).

So under the conditions (H1), (H2) et (H3) we can define the notion of the solution in the sense of entropy. Regarding the uniqueness of the solution, in general the result is not true but if \( \text{div}(a(x,u,\nabla u)) = \Delta u \), then we can demonstrate the uniqueness of the solution in the sense of entropy.
Keywords: Quasi-linear elliptic equations, variational methods, functional spaces, entropy solution, Riemannian manifold, space Marcinkiewicz.

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