THE PARABOLIC FLOWS FOR COMPLEX QUOTIENT EQUATIONS

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Abstract. We apply the parabolic flow method to solving complex quotient equations on closed Kähler manifolds. We study the parabolic equation and prove the convergence. As a result, we solve the complex quotient equations.

1. Introduction

Let \((M, \omega)\) be a closed Kähler manifold of complex dimension \(n \geq 2\), and \(\chi\) a smooth closed real \((1, 1)\) form in \(\Gamma^k_\omega\), where \(\Gamma^k_\omega\) is the set of all the real \((1, 1)\) forms whose eigenvalue sets with respect to \(\omega\) belong to \(k\)-positive cone in \(\mathbb{R}^n\). In any local coordinate chart, we write

\[
\chi = \frac{\sqrt{-1}}{2} \sum_{i,j} \chi_{ij} dz^i \wedge d\bar{z}^j \quad \text{and} \quad \omega = \frac{\sqrt{-1}}{2} \sum_{i,j} g_{ij} dz^i \wedge d\bar{z}^j.
\]

In this paper, we study the following form of parabolic equations, for \(n \geq k > l \geq 0\)

\[
\frac{\partial u}{\partial t} = \log \frac{\chi^k_u \wedge \omega^{n-k}}{\chi^l_u \wedge \omega^{n-l}} - \log \psi, \quad (1.1)
\]

with initial condition \(u(x, 0) = 0\), where \(\psi \in C^\infty(M)\) is positive and \(\chi_u\) is the abbreviation for \(\chi + \sqrt{-1} \partial \bar{\partial} u\).

The study of the parabolic flows is motivated by complex equations

\[
\chi^k_u \wedge \omega^{n-k} = \psi \chi^l_u \wedge \omega^{n-l}, \quad \chi_u \in \Gamma^k_\omega. \quad (1.2)
\]

When \(\psi\) is constant, it must be \(c\) defined by

\[
c := \frac{\int_M \chi^k \wedge \omega^{n-k}}{\int_M \chi^l \wedge \omega^{n-l}}. \quad (1.3)
\]

These equations include some important geometric equations, which have attracted much attention in mathematics and physics since the breakthrough of Yau [37] (see also Aubin [1]) in Calabi conjecture [4]. The most famous examples are probably the complex Monge-Ampère equation and Donaldson equation [10], which respectively correspond to

\[
\chi^l_u = \psi \omega^n, \quad \chi_u \in \Gamma^n_\omega. \quad (1.4)
\]

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and
\[ \chi_u^n = \frac{\int_M \chi^n \chi_u \wedge \omega^{n-1}}{\int_M \chi \wedge \omega^{n-1}}, \quad \chi_u \in \Gamma^n_\omega. \] (1.5)

Since equation (1.2) is fully nonlinear elliptic, a classical way to solve it is the continuity method. Using this method, the complex Monge-Ampère equation was solved by Yau [37], while Donaldson equation was independently solved by Li, Shi and Yao [20], Collins and Székelyhidi [8] and the author [23]. There have been many extensive studies for equation (1.2) on closed complex manifolds, for example, [7, 9, 16, 17, 27, 28, 29, 38, 39].

Besides the continuity method, the parabolic flow method also have the potential to solve complex quotient equations. The result of Yau [37] was reproduced by Cao [5] through Kähler-Ricci flow. Donaldson equation was actually first solved via the J-flow by Song and Weinkove [22]. There are many results regarding different flows on closed complex manifolds, and we refer readers to [8, 12, 13, 19, 24, 30, 35, 36]. In this paper, we preliminarily explore the parabolic flow equation (1.1) for complex quotient equations, and reprove the solvability of the corresponding complex quotient equations (see [26, 25]).

To solve equation (1.1), some extra condition is in need. We assume that there is a real-valued \( C^2 \) function \( u \) satisfying \( \chi_u \in \Gamma^k_\omega \) and
\[ k\chi_u^{k-1} \wedge \omega^{n-k} > \psi \chi_u^{l-1} \wedge \omega^{n-l}. \]

For convenience, we adopt an equivalent definition of \( u \) due to Székelyhidi [26], which is called \( C \)-subsolution.

**Definition 1.1.** We say that a \( C^2 \) function \( u \) is a \( C \)-subsolution to (1.1) if \( \chi_u \in \Gamma^k_\omega \), and at each point \( x \in M \), the set
\[ \left\{ \tilde{x} \in \Gamma^k_\omega \left| \chi_u^{k-n} \leq \psi \chi_u^{l-n} \wedge \omega^{n-l} \text{ and } \tilde{x} - \chi_u \geq 0 \right. \right\} \]
(1.6)
is bounded.

When \( l = 0 \), a natural \( C \)-subsolution is \( u \equiv 0 \). In other cases, the existence of a \( C \)-subsolution is not easy to verify in applications. In the case \( \psi = c \), Székelyhidi [26] propose a conjecture that the existence of a \( C \)-subsolution is equivalent to that for all \( m \) dimensional subvariety \( V \subset M \), where \( m = n - l, \cdots, n - 1 \),
\[ \frac{k!}{(m-n+k)!} \int_V \chi^{m-n+k} \wedge \omega^{n-k} - \frac{l!c}{(m-n+l)!} \int_V \chi^{m-n+l} \wedge \omega^{n-l} > 0. \] (1.7)

For Donaldson equation, the conjecture was verified in dimensional 2 by Lejmi and Székelyhidi [19] and on toric manifolds by Collins and Székelyhidi [8]. In view of these results, we expect the conjecture to hold in general cases.

The following is the main result of the paper.
Theorem 1.2. Let \((M^n, \omega)\) be a closed Kähler manifold of complex dimension \(n\) and \(\chi\) a smooth closed real \((1,1)\) form in \(\Gamma^k_\omega\). Suppose that there is a \(C\)-subsolution \(u\) and \(\psi \geq c\) for all \(x \in M\), where \(c\) is an invariant as defined in (1.3). Then there exists a long time solution \(u\) to equation (1.1). Moreover, the normalization \(\hat{u}\) of \(u\) is \(C^\infty\) convergent to a smooth function \(\hat{u}_\infty\) where \(\hat{u}\) is defined later in Section 2. Consequently, there is a unique real number \(b\) such that the pair \((\hat{u}_\infty, b)\) solves
\[
\frac{\chi_u \wedge \omega^{n-k}}{\chi_u \wedge \omega^{n-l}} = e^b \psi.
\] (1.8)

Very recently, Phong and Tô [21] were also able to prove the result.

To obtain the \(L^\infty\) estimate, we adapt an approach due to Blocki [3] and Székelyhidi [26]. However the Alexandroff-Bakelman-Pucci maximum principle [32] is dependent on time \(t\), which implies that the \(L^\infty\) estimate for \(u\) probably blows up as time \(t\) approaches \(\infty\). In this case, higher order estimates were also dependent on time, which means we were unable to prove the convergence. To eliminate the influence of time, we apply Alexandroff-Bakelman-Pucci maximum principle locally in time \(t\). The proof of the second order estimates follows the work of Hou, Ma and Wu [17]. In order to apply the work, we need to improve a key lemma in [25]. The gradient estimate thus follows from the blow up argument of Dinew and Kolodziej [9] and Gill [14]. The techniques in this papers can be applied to other flows, while equation (1.1) is the appropriate form to deal with real number \(b\) in Theorem 1.2.

2. Preliminary

In this section, we shall state some notations and prove some preliminary results.

We follow the notations in [15, 25, 24]. Let \(S_k(\lambda)\) denote the \(k\)-th elementary symmetric polynomial of \(\lambda \in \mathbb{R}^n\),
\[
S_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{2.1}
\]

For \(\{i_1, \cdots, i_s\} \subseteq \{1, \cdots, n\}\),
\[
S_{k; i_1 \cdots i_s}(\lambda) = S_k(\lambda|_{\lambda_{i_1} = \cdots = \lambda_{i_s} = 0}). \tag{2.2}
\]

By convention, \(S_0(\lambda) = 0\) and thus \(S_{-1; i_1 \cdots i_s}(\lambda) = S_{-2; i_1 \cdots i_s}(\lambda) = 0\). We express \(X := \chi_u\) and hence in any local coordinate chart,
\[
X_{\bar{i}j} = X \left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) = \chi_{\bar{i}j} + u_{\bar{i}j}. \tag{2.3}
\]

When no confusion occurs in a fixed local coordinate chart, we also abuse \(X\) and \(\chi\) et al. to denote the corresponding Hermitian matrices. We define \(\lambda_s(X)\) as the eigenvalue set of \(X\).
with respect to $\omega$ and thus write $S_k(X) = S_k(\lambda_*(X))$. For simplicity, we use $S_k$ to denote $S_k(X)$. In any local coordinate chart, equation (1.1) can be written as

$$\frac{\partial u}{\partial t} = \log \frac{S_k}{S_l} - \log \Psi,$$

with $\Psi = \frac{C_k^n}{C_k^n} \psi$ and $C_k^n = \frac{n!}{(n-k)!k!}$. For convenience, we use indices to denote covariant derivatives with respect the Chern connection $\nabla$.

Fixing a point $p$, we can choose a normal chart around $p$ such that $g_{ij} = \delta_{ij}$ and $X$ is diagonal. Then there is a natural ordered eigenvalue set

$$\lambda_*(X) = (X_{11}, \cdots, X_{nn}),$$

and we write

$$S_{k;ij\cdots i_s} = S_{k;ij\cdots i_s}(\lambda_*(X)).$$

Moreover, we have the following equalities

$$X_{i\bar{j}j} - X_{j\bar{i}i} = \chi_{i\bar{j}j} - \chi_{j\bar{i}i},$$

and

$$X_{i\bar{j}j} - X_{j\bar{i}i} = R_{j\bar{i}i}X_{\bar{j}i} - R_{i\bar{j}j}X_{\bar{i}j} - G_{i\bar{j}j},$$

where

$$G_{i\bar{j}j} = \chi_{j\bar{i}i} - \chi_{i\bar{j}j} + \sum_{m} R_{j\bar{j}im} \chi_{m\bar{i}} - \sum_{m} R_{i\bar{j}m} \chi_{m\bar{j}}.$$  

For simplicity, we define

$$F(\chi_u) := \log \frac{S_k(\chi_u)}{S_l(\chi_u)}$$

and

$$F_{i\bar{j}} = \frac{\partial F}{\partial u_{i\bar{j}}}.$$  

Then at $p$ under the chosen chart, $\{F_{i\bar{j}}\}$ is diagonal and

$$F_{i\bar{j}} = \frac{S_{k-1;i}}{S_k} - \frac{S_{l-1;i}}{S_l}.$$  

Differentiating equation (2.4) at $p$ under such a normal coordinate chart,

$$\partial_t(\partial_t u) = \sum_i F_{i\bar{i}}(\partial_t u)_{i\bar{i}},$$

$$\partial_t u_m = \sum_i F_{i\bar{i}} X_{i\bar{m}m} - \partial_m \log \Psi,$$

and

$$\partial_t u_{m\bar{m}} \leq \sum_i F_{i\bar{i}} X_{i\bar{m}m} - \sum_{i\neq j} \left( \frac{S_{k-2;i\bar{j}}}{S_k} - \frac{S_{l-2;i\bar{j}}}{S_l} \right) X_{i\bar{j}m} X_{j\bar{i}m} - \partial_m \partial_m \log \Psi.$$
Applying the maximum principle to equation (2.13), we see that \( \partial_t u \) reaches its extremal values at \( t = 0 \). Thanks to the boundedness of \( \partial_t u \), it is easy to see that the flow remains in \( \Gamma^{k}_{\omega} \) at any time.

Fixing a \( C \)-subsolution \( \underline{u} \), the set
\[
\left\{ \tilde{\chi} \in \Gamma^{k}_{\omega}(x) \mid \tilde{\chi}^k \wedge \omega^{n-k} \leq e^\lambda \psi \tilde{\chi}^l \wedge \omega^{n-l} \text{ and } \tilde{\chi} - \chi_{\underline{u}}(x) \geq -\lambda g_{ij} \right\}
\] (2.16)
is also bounded if \( \lambda > 0 \) is small enough. Without loss of generality, we may assume that \( \sup_{M} u = -2\lambda \) in this paper. To obtain the second order estimate, we improve a key lemma in [25] to fit in with the parabolic case.

**Lemma 2.1.** Under the assumptions of Theorem 1.2, there is a constant \( \theta > 0 \) such that
\[
\sum_i F^{\bar{i}}(u_{\bar{i}} - u_{\bar{i}}) \leq F(\chi_u) - \log \Psi - \theta \left( 1 + \sum_i F^{\bar{i}} \right),
\] (2.17)
or
\[
F^{\bar{j}} \geq \theta \left( 1 + \sum_i F^{\bar{i}} \right), \quad \forall j = 1, \cdots, n.
\] (2.18)

**Proof.** Without loss of generality, we may assume that \( X_{1\bar{1}} \geq \cdots \geq X_{n\bar{n}} \). Thus
\[
F^{n\bar{n}} \geq \cdots \geq F^{1\bar{1}}.
\] (2.19)

If \( \lambda > 0 \) is small enough, \( \chi - \lambda \omega \) and \( \underline{u} \) still satisfy Definition 1.1. Since \( M \) is compact, there are uniform constants \( N > 0 \) and \( \sigma > 0 \) such that
\[
F(\chi') > \log \Psi + \sigma,
\] (2.20)
where
\[
\chi' = \chi_u - \lambda g + \begin{pmatrix} N \\ \vdots \\ 0 \end{pmatrix}_{n \times n}.
\] (2.21)

Direct calculation shows that
\[
\sum_i F^{\bar{i}}(u_{\bar{i}} - u_{\bar{i}}) = \sum_i F^{\bar{i}} X_{i\bar{i}} - \sum_i F^{\bar{i}} \chi'_{\bar{i}} + NF^{1\bar{1}} - \lambda \sum_i F^{\bar{i}}
\]
\[
\leq F(\chi_u) - F(\chi') + NF^{1\bar{1}} - \lambda \sum_i F^{\bar{i}}
\]
\[
\leq F(\chi_u) - \log \Psi - \sigma - \lambda \sum_i F^{\bar{i}} + NF^{1\bar{1}}.
\] (2.22)

If
\[
\frac{\min\{\sigma, \lambda\}}{2} \left( 1 + \sum_i F^{\bar{i}} \right) \geq NF^{1\bar{1}},
\] (2.23)
we obtain (2.17); otherwise, inequality (2.18) has to be true.
The argument can be applied to elliptic and parabolic equations with \( C \)-subsolution. For the parabolic equation, we shall use the following corollary. The argument applies to general functions \( F \).

**Corollary 2.2.** Under the assumption of Lemma 2.1 and additionally assuming that \( X_{11} \geq \cdots \geq X_{n1} \), there is a constant \( \theta > 0 \) such that we have either

\[
\sum_i F^i (u_i - u^i) - \partial_t u \leq - \theta \left( 1 + \sum_i F^i \right),
\]

or

\[
F^{11} X_{11} \geq \theta \left( 1 + \sum_i F^i \right).
\]

**Proof.** Note that

\[
X_{11} \geq \frac{S_1}{n} > 0,
\]

and

\[
F \left( \frac{S_1}{n} g \right) \geq F (X) = \partial_t u + \log \Psi
\]

by concavity and symmetry of \( F \). The latter inequality implies that there is a constant \( \sigma > 0 \) such that \( S_1 > \sigma \), and hence the former shows that \( X_{11} > n \sigma \). Combining Lemma 2.1 and the fact that \( X_{11} > n \sigma \), we finish the proof.

We adapt the general \( J \)-functionals\[6, 12\]. Let \( \mathcal{H} \) be the space

\[
\mathcal{H} := \{ u \in C^\infty (M) \mid \chi_u \in \Gamma_k^\omega \}\]

For any curve \( v(s) \in \mathcal{H} \), we define the functional \( J_l \) by

\[
\frac{dJ_l}{ds} = \int_M \frac{\partial v}{\partial s} \chi_v^l \wedge \omega^{n-l}.
\]

Then we have a formula for \( J_l (u) \),

\[
J_l (u) = \int_0^1 \int_M \frac{\partial v}{\partial s} \chi_v^l \wedge \omega^{n-l} ds,
\]

where \( v(s) \) is an arbitrary path in \( \mathcal{H} \) connecting 0 and \( u \). Thanks to the closedness of \( \chi \) and \( \omega \), those functionals are independent from choices of the path. If the integration is over \( v(s) = su \),

\[
J_l (u) = \int_0^1 \int_M u (s \chi_u + (1-s) \chi)^l \wedge \omega^{n-l} ds
\]

\[
= \frac{1}{l+1} \sum_{i=0}^l \int_M u \chi_u^i \wedge \chi^{l-i} \wedge \omega^{n-l}.
\]
Along the solution flow $u(x, t)$ to equation (1.1), we have
\[
\frac{d}{dt} J_l(u) = \int_M \left( \log \frac{\chi^k_u \wedge \omega^{n-k}}{\chi^l_u \wedge \omega^{n-l}} - \log \psi \right) \chi^l_u \wedge \omega^{n-l} \\
\leq \log c \int_M \chi^l_u \wedge \omega^{n-l} - \int_M \log \psi \chi^l_u \wedge \omega^{n-l} \\
\leq 0.
\] (2.32)

Computing $J_l$ on an arbitrary function flow $u(x, t)$ starting from 0 to $T$, it follows that
\[
J_l(u(T)) = \int_0^T \frac{dJ_l}{dt} dt
\] (2.33)

For the solution flow $u(x, t)$ to equation (1.1), let
\[
\hat{u} = u - \frac{J_l(u)}{\int_M \chi^l_u \wedge \omega^{n-l}}.
\] (2.34)

By (2.32), we know that $\partial_t \hat{u} \geq \partial_t u$.

3. The $L^\infty$ estimate

In this section, we shall prove the $L^\infty$ estimate. We follow the approach of Székelyhidi [26] based on the method of Blocki [3].

**Theorem 3.1.** Let $u \in C^2(M \times [0, T))$ be an admissible solution to equation (1.1). Suppose that there is a $C$-subsolution $\bar{u}$ and $\psi \geq c$ for all $x \in M$. Then there exists a uniform constant $C > 0$ such that for any $t \in [0, T)$
\[
\sup_M u(x, t) - \inf_M u(x, t) = \sup_M \hat{u}(x, t) - \inf_M \hat{u}(x, t) < C.
\] (3.1)

To prove Theorem 3.1, we need the following lemma. The argument follows closely those in [36] [24], so we omit the proof.

**Lemma 3.2.**
\[
0 \leq \sup_M \hat{u}(x, t) \leq -C_1 \inf_M \hat{u}(x, t) + C_2 \quad \text{and} \quad \inf_M \hat{u}(x, t) \leq 0.
\] (3.2)

This lemma tells us that it suffices to find a lower bound for $\inf_M (\hat{u} - u)(x, t)$.

**Proof of Theorem 3.1.** We claim that
\[
\inf_M (\hat{u} - u)(x, t) > -2 \sup_{M \times \{0\}} |\partial_t u| - C_0,
\] (3.3)

where $C_0 \geq 0$ is to be determined later.

Since $\partial_t u$ reaches its extremal values at $t = 0$,
\[
|\partial_t \hat{u}| \leq 2 \sup_{M \times \{0\}} |\partial_t u|.
\] (3.4)
Therefore if the lower bound (3.3) does not hold, there must be time $t_0 > 1$ such that
\[
\inf_{M} (\hat{u}_t - \hat{u}) (x,t) = -2 \sup_{M \times \{0\}} |\partial_t u| + 2\lambda. \tag{3.5}
\]

Therefore if the lower bound (3.3) does not hold, there must be time $t_0 > 1$ such that
\[
\inf_{M} (\hat{u}_t - \hat{u}) (x,t_0) = -2 \sup_{M \times \{0\}} |\partial_t u| + C_0. \tag{3.6}
\]

We may assume that $(\hat{u}_t - \hat{u})(x_0, t_0) = \inf_{M} (\hat{u}_t - \hat{u})(x, t_0) < 0$. Following the approach of Székelyhidi\cite{26}, we work in local coordinates around $x_0$, where $x_0$ is the origin and the coordinates are defined for $B_1 = \{z : |z| < 1\}$. Let $v = \hat{u} - \hat{u} - \epsilon |z|^2 - \epsilon(t - t_0) - \inf_{M} (\hat{u}_t - \hat{u})(x, t_0)$ for some small $\epsilon > 0$. We may assume that $\epsilon < \lambda$. It is easy to see that when $t = t_0 - 1$
\[
v = \hat{u} - \hat{u} + \epsilon |z|^2 - \inf_{M} (\hat{u}_t - \hat{u})(x, t_0) \geq 0, \tag{3.7}
\]
and when $|z|^2 = 1$, $t \leq t_0$
\[
v = \hat{u} - \hat{u} - \epsilon(t - t_0) - \inf_{M} (\hat{u}_t - \hat{u})(x, t_0) \geq 0. \tag{3.8}
\]

Moreover,
\[
\inf_{M \times [t_0 - 1, t_0]} v = \inf_{M \times \{t_0\}} v = v(x_0, t_0) = -\lambda. \tag{3.9}
\]

Define the set for $-v$ on $[t_0 - 1, t_0]$,
\[
\Phi(y, t) = \left\{ (p, h) \in \mathbb{R}^{2n+1} \mid -v(x, s) \leq -v(y, t) + p \cdot (x - y), \right. \nonumber
\]
\[
\left. h = -v(y, t) - p \cdot y, \forall x \in B_1, s \in [t_0 - 1, t] \right\}. \tag{3.10}
\]

Then we define the contact set
\[
\Gamma_{-v} = \left\{ (y, t) \in B_1 \times [t_0 - 1, t_0] \mid \Phi(y, t) \neq \emptyset \right\}. \tag{3.11}
\]

In $\Gamma_{-v}$, it must be true that $D_x^2 v \geq 0$ and $\partial v \leq 0$. Therefore,
\[
\{u_{ij}^+\} - \{u_{ij}^-\} \geq -\epsilon I, \tag{3.12}
\]
and
\[
\frac{\chi^k_u \wedge \omega^{n-k}}{\chi^l_u \wedge \omega^{n-l}} \leq \epsilon \psi. \tag{3.13}
\]

If $\epsilon$ is chosen small enough, we obtain an bound $|u_{ij}| < C$ in $\Gamma_{-v}$. By Alexandroff-Bakelman-Pucci maximum principle\cite{32} for parabolic equations, we have
\[
\epsilon \leq C \left[ \int_{\Gamma_{-v} \cap \{v < 0\}} -\partial_t v \det(D_x^2 v) dx dt \right]^\frac{1}{2n+1} \tag{3.14}
\]
\[
\leq C \left[ \int_{\Gamma_{-v} \cap \{v < 0\}} -\partial_t v 2^{2n} (\det(v_{ij}))^2 dx dt \right]^\frac{1}{2n+1}.
\]
Because of the boundedness of $u_{ij}$ and $\partial_t u$, it follows that
\[ \epsilon \leq C |\Gamma_v \cap \{ v < 0 \}|^{\frac{1}{2n+1}}. \] (3.15)

When $v < 0$,
\[ \hat{u} < \underline{u} + \epsilon |z|^2 + \epsilon(t - t_0) + \inf_M (\hat{u} - \underline{u})(x, t_0) < \inf_M (\hat{u} - \underline{u})(x, t_0). \] (3.16)

So
\[ \epsilon^{2n+1} \leq C \left| M \times [t_0 - 1, t_0] \cap \left\{ \hat{u} < \inf_M (\hat{u} - \underline{u})(x, t_0) \right\} \right| \]
\[ \leq C \int_{t_0 - 1}^{t_0} |\inf_M (\hat{u} - \underline{u})(x, t_0)| dt \]
\[ \leq C \int_{t_0 - 1}^{t_0} |\inf_M (\hat{u} - \underline{u})(x, t_0)|^{L^1} dt \]
\[ \leq C \int_{t_0 - 1}^{t_0} \inf_M (\hat{u} - \underline{u})(x, t_0) dt \]
\[ \leq C \int_{t_0 - 1}^{t_0} \inf_M (\hat{u} - \underline{u})(x, t_0) dt \]
\[ \leq C \int_{t_0 - 1}^{t_0} |\inf_M (\hat{u} - \underline{u})(x, t_0)|^{L^1} dt \]

Using the Green’s function of $\omega$, we have a uniform bound for $|\hat{u}(x, t) - \sup_M \hat{u}(x, t)|_{L^1}$. Therefore, there is a uniform constant $C_0 \geq 0$ such that
\[ \inf_{M \times [0, t_0]} (\hat{u} - \underline{u})(x, t) > -C_0, \]
which contradicts the definition of $t_0$. \qed

4. The second order estimate

In this section, we shall prove the second order estimates.

**Theorem 4.1.** There exists a constant $C$ depending on $\sup_{M \times [0, T]} |\hat{u}|$ such that for any $t' \in [0, T]$,
\[ \sup_{M} |\partial \hat{u}| \leq C \left( \sup_{M \times [0, t']} |\nabla u|^2 + 1 \right), \] (4.1)
at any time $t \in [0, t']$.

**Proof.** Following the work of Hou, Ma and Wu [17], we define
\[ H(x, \xi) = \log \left( \sum_{i,j} X_{ij} \xi^i \xi^j \right) + \varphi(|\nabla u|^2) + \rho(\hat{u} - \underline{u}) \] (4.2)
where
\[ \varphi(s) = -\frac{1}{2} \log \left( 1 - \frac{s}{2K} \right), \quad \text{for } 0 \leq s \leq K - 1, \]
\[ \rho(t) = -A \log \left( 1 + \frac{t}{2L} \right), \quad \text{for } -L + 1 \leq t \leq L - 1, \] (4.3)
with

$$K := \sup_{M \times [0,t]} |\nabla u|^2 + \sup_{M \times [0,t']} |\nabla u|^2 + 1,$$

$$L := \sup_{M \times [0,T)} |\hat{u}| + \sup_{M} |u| + 1,$$

$$A := 3L(C_0 + 1)$$

and $C_0$ is to be specified later. Note that

$$\frac{1}{2K} \geq \varphi' \geq \frac{1}{4K} > 0, \quad \varphi'' = 2(\varphi')^2 > 0 \quad (4.4)$$

and

$$\frac{A}{L} \geq -\rho' \geq \frac{A}{3L} = C_0 + 1, \quad \rho'' \geq 2\epsilon (\rho')^2, \quad \text{for all } \epsilon \leq \frac{1}{2A + 1}. \quad (4.5)$$

The function $H$ must achieve its maximum at some point $(p,t_0)$ in some unit direction of $\eta$. Around $p$, we choose a normal chart such that $X_{1\bar{1}} \geq \cdots \geq X_{n\bar{n}}$, and $X_{1\bar{1}} = X_{\eta\bar{\eta}}$ at $p$.

Define

$$H_t := \frac{\partial_t u_{1\bar{1}}}{X_{1\bar{1}}} + \varphi' \partial_t (|\nabla u|^2) + \rho' \partial_t (\hat{u} - \underline{u}), \quad (4.6)$$

$$H_i := \frac{X_{1\bar{i}}}{X_{1\bar{1}}} + \varphi' \partial_i (|\nabla u|^2) + \rho' (u_i - \underline{u}_i), \quad (4.7)$$

and

$$H_{\bar{i}i} := \frac{X_{1\bar{i}\bar{i}}}{X_{1\bar{1}}} - \frac{|X_{1\bar{i}}|^2}{X_{1\bar{1}}^2} + \varphi'' \partial_i (|\nabla u|^2)^2 + \varphi' \partial_i \partial_i (|\nabla u|^2)
+ \rho'' |u_i - \underline{u}_i|^2 + \rho' (u_{\bar{i}\bar{i}} - \underline{u}_{\bar{i}\bar{i}}). \quad (4.8)$$

At $(p,t_0)$, we have $H_t \geq 0$, $H_i = 0$ and $H_{\bar{i}i} \leq 0$. Thus

$$\frac{\partial_t u_{1\bar{1}}}{X_{1\bar{1}}} \geq -\varphi' \partial_t (|\nabla u|^2) - \rho' \partial_t u, \quad (4.9)$$

$$\frac{X_{1\bar{1}}}{X_{1\bar{1}}} = -\varphi' \partial_\bar{i} (|\nabla u|^2) - \rho' (u_i - \underline{u}_i), \quad (4.10)$$

and

$$0 \geq \frac{1}{X_{1\bar{1}}} (X_{1\bar{i}i} + R_{i\bar{i}11} X_{1\bar{1}} - R_{1\bar{i}\bar{i}} X_{1\bar{1}} - G_{1\bar{i}\bar{i}}) - \frac{|X_{1\bar{1}}|^2}{X_{1\bar{1}}^2}
+ \varphi'' \partial_i (|\nabla u|^2)^2 + \varphi' \partial_i \partial_i (|\nabla u|^2) + \rho'' |u_i - \underline{u}_i|^2 + \rho' (u_{\bar{i}\bar{i}} - \underline{u}_{\bar{i}\bar{i}}). \quad (4.11)$$
Multiplying (4.11) by $F^{i\bar{j}}$ and summing it over index $i$,
\begin{align}
0 &\geq \frac{1}{X_{11}} \partial_{\bar{i}} \partial_{i} (\log \Psi) + \frac{1}{X_{11}} \sum_{i \neq j} \left( \frac{S_{k-2;ij}}{S_{k}} - \frac{S_{l-2;ij}}{S_{l}} \right) X_{i\bar{j}1} X_{j\bar{i}1} \\
&\quad + \frac{(k-l)}{X_{11}} \inf_{p} R_{pp11} - \sup_{p} G_{11pp} \sum_{i} F^{i\bar{j}} - \frac{1}{X_{11}} \sum_{i} F^{i\bar{j}} \\
&\quad - \frac{1}{X_{11}^{2}} \sum_{i} F^{i\bar{j}} |X_{11i}|^{2} + \varphi'' \sum_{i} F^{i\bar{j}} |\partial_{i} (|\nabla u|^{2})|^{2} + \varphi' \sum_{i} F^{i\bar{j}} \partial_{i} (|\nabla u|^{2}) \\
&\quad + \rho'' F^{i\bar{j}} |u_{i} - \bar{u}_{i}|^{2} + \rho' \sum_{i} F^{i\bar{j}} (u_{i} - \bar{u}_{i}) \\
&= \frac{1}{X_{11}} \partial_{\bar{i}} \partial_{i} (\log \Psi) + \frac{1}{X_{11}} \sum_{i \neq j} \left( \frac{S_{k-2;ij}}{S_{k}} - \frac{S_{l-2;ij}}{S_{l}} \right) X_{i\bar{j}1} X_{j\bar{i}1} \\
&\quad + \frac{(k-l)}{X_{11}} \inf_{p} R_{pp11} + \inf_{p} R_{pp11} \sum_{i} F^{i\bar{j}} - \sup_{p} G_{11pp} \sum_{i} F^{i\bar{j}} \\
&\quad - \frac{1}{X_{11}^{2}} \sum_{i} F^{i\bar{j}} |X_{11i}|^{2} + \varphi'' \sum_{i} F^{i\bar{j}} |\partial_{i} (|\nabla u|^{2})|^{2} + \varphi' \sum_{i} F^{i\bar{j}} \partial_{i} (|\nabla u|^{2}) \\
&\quad + \rho'' F^{i\bar{j}} |u_{i} - \bar{u}_{i}|^{2} + \rho' \sum_{i} F^{i\bar{j}} (u_{i} - \bar{u}_{i}) - \varphi' \partial_{i} (|\nabla u|^{2}) - \rho' \partial_{i} u. \\
\end{align}

Direct calculation shows that,
\begin{align}
\partial_{i} (|\nabla u|^{2}) &= 2 \sum_{j} \Re \{ \partial_{i} u_{j} u_{j} \}, \\
\partial_{i} (|\nabla u|^{2}) &= \sum_{j} (-u_{j} X_{ij} + u_{ji} u_{j}) + u_{i} X_{i\bar{i}}, 
\end{align}

and
\begin{align}
\tilde{\partial}_{i} \partial_{i} (|\nabla u|^{2}) &\geq -2 \sum_{j} \Re \{ \chi_{i\bar{j}j} u_{j} \} + \sum_{j,k} R_{i\bar{j}j} u_{k} u_{j} + \frac{1}{2} X_{i\bar{i}}^{2} \\
&\quad - 2 \chi_{i\bar{i}}^{2} + 2 \sum_{j} \Re \{ X_{i\bar{j}j} u_{j} \}.
\end{align}

We control some terms in (4.12),
\begin{align}
\varphi' \sum_{i} F^{i\bar{j}} \tilde{\partial}_{i} \partial_{i} (|\nabla u|^{2}) - \varphi' \partial_{i} (|\nabla u|^{2}) \\
&\geq -2 \varphi' \sum_{i,j} F^{i\bar{j}} \Re \{ \chi_{i\bar{j}j} u_{j} \} + \varphi' \sum_{i,j,p} F^{i\bar{j}} R_{i\bar{j}j} p u_{p} u_{j} + \frac{\varphi'}{2} \sum_{i} F^{i\bar{j}} X_{i\bar{i}}^{2} \\
&\quad - 2 \varphi' \sum_{i} F^{i\bar{j}} \chi_{i\bar{i}}^{2} + 2 \varphi' \sum_{j} \Re \{ \partial_{j} (\log \Psi) u_{j} \},
\end{align}
where

\[
- 2\varphi' \sum_{i,j} F^{\bar{i}i} \text{Re}\{\chi_{\bar{i}ij} u_j\} + \varphi' \sum_{i,j,p} F^{\bar{i}i} R_{\bar{i}ijp} u_p u_j + 2\varphi' \sum_{j} \text{Re}\{\partial_j (\log \Psi) u_j\}
\]

\[
\geq - \sup_p (\sum_j |\chi_{p\bar{j}j}| |\nabla u|) \sum_i F^{\bar{i}i} - \sup_{j,p,q} |R_{q\bar{j}j\bar{p}}| |\nabla u| \sum_i F^{\bar{i}i} + \frac{\phi'^{\prime} \sum_i F^{\bar{i}i}}{K} \sum_i \bar{u}^{\bar{i}i} (u_{\bar{i}i} - u_{\bar{i}i}) - \rho \partial_t u.
\]

Combining (4.12)-(4.18),

\[
0 \geq - C_1 + C_2 \sum_i F^{\bar{i}i} + \frac{1}{X_{11}} \sum_{i \neq j} \left( \frac{S_{k-2:ij}}{S_k} - \frac{S_{l-2:ij}}{S_l} \right) X_{ij1} X_{i\bar{j}\bar{i}}
\]

\[
- \frac{1}{X_{11}^2} \sum_i F^{\bar{i}i} |X_{i1i}|^2 + \varphi'' \sum_i F^{\bar{i}i} |\partial_i (|\nabla u|^2)|^2 + \frac{\varphi' \sum_i F^{\bar{i}i} X_{i\bar{i}}^2}{2}
\]

\[
+ \rho'' \sum_i F^{\bar{i}i} |u_{i} - u_{\bar{i}i}|^2 + \rho' \sum_i F^{\bar{i}i} (u_{i\bar{i}} - u_{\bar{i}i}) - \rho' \partial_t u.
\]

Now we can define

\[
\delta = \frac{1}{1 + 2A} = \frac{1}{1 + 6L(C_0 + 1)}
\]

and

\[
C_0 = \frac{C_1 + C_2 + 1}{\theta} + \frac{C_2 + 1}{\lambda}.
\]

**Case 1.** \(X_{n\bar{n}} < -\delta X_{11}\). In this case, \(X_{11}^2 \leq \frac{1}{\rho'} X_{n\bar{n}}^2\) and we just need to bound \(X_{n\bar{n}}^2\).

By (4.10) and (4.4), we have

\[
\frac{1}{X_{11}^2} \sum_i F^{\bar{i}i} |X_{i1i}|^2 = \sum_i F^{\bar{i}i} |\varphi' \partial_i (|\nabla u|^2) + \rho' (u_i - u_{\bar{i}i})|^2
\]

\[
\leq 2(\varphi')^2 \sum_i F^{\bar{i}i} |\partial_i (|\nabla u|^2)|^2 + 2(\rho')^2 \sum_i F^{\bar{i}i} |u_i - u_{\bar{i}i}|^2
\]

\[
\leq \varphi'' \sum_i F^{\bar{i}i} |\partial_i (|\nabla u|^2)|^2 + 36(C_0 + 1)^2 K \sum_i F^{\bar{i}i}.
\]

Substituting (4.22) into (4.19),

\[
C_1 + C_2 \sum_i F^{\bar{i}i} + 36(C_0 + 1)^2 K \sum_i F^{\bar{i}i}
\]

\[
\geq \frac{1}{8K} \sum_i F^{\bar{i}i} X_{i\bar{i}}^2 + \rho' \sum_i F^{\bar{i}i} (u_{i\bar{i}} - u_{\bar{i}i}) - \rho' \partial_t u.
\]
According to Lemma 2.2 there are at most two possibilities. If (2.24) holds true,

\[ C_1 + C_2 \sum_i F^{\bar{i}} + 36(C_0 + 1)^2 K \sum_i F^{\bar{i}} \geq \frac{X_{11}^2}{8nK} \sum_i F^{\bar{i}} + \theta(C_0 + 1) + \theta(C_0 + 1) \sum_i F^{\bar{i}}. \]  

(4.24)

Then

\[ X_{11} < \sqrt{288n(C_0 + 1)} K. \]  

(4.25)

If (2.25) holds true,

\[ C_1 + C_2 \sum_i F^{\bar{i}} + 36(C_0 + 1)^2 K \sum_i F^{\bar{i}} \geq \frac{\theta X_{11}^2}{8K} + \frac{X_{1n}^2}{8nK} \sum_i F^{\bar{i}} + \rho' \sum_i F^{\bar{i}}(u_{\bar{i}} - \omega_{\bar{i}}) - \rho' u_t. \]  

(4.26)

Since

\[ \rho' \sum_i F^{\bar{i}}(u_{\bar{i}} - \omega_{\bar{i}}) - \rho' u_t \geq \rho' \log \frac{S_k}{S_l} - \rho' u_t - \lambda \rho' \sum_i F^{\bar{i}} \]  

(4.27)

it follows that

\[ C_1 + 3(C_0 + 1) \sup_M |\log \Psi| + 36(C_0 + 1)^2 K \sum_i F^{\bar{i}} \geq \frac{\theta X_{11}^2}{8K} + \frac{X_{1n}^2}{8nK} \sum_i F^{\bar{i}}. \]  

(4.28)

Then we either have (4.25) or

\[ X_{11} \leq \frac{8(C_1 + 3(C_0 + 1) \sup_M |\log \Psi|)}{\theta} K. \]  

(4.29)

Case 2. \( X_{1n} \geq -\delta X_{11} \). Define

\[ I = \left\{ i \in \{1, \cdots, n\} \mid F^{\bar{i}} > \delta^{-1} F^{1\bar{1}} \right\}. \]  

(4.30)

Then

\[ \frac{1}{X_{11}} \sum_{i \neq j} \left( \frac{S_{k-2,ij}}{S_k} - \frac{S_{l-2,ij}}{S_l} \right) X_{ij} X_{\bar{i}\bar{j}} \geq \frac{1 - \delta}{1 + \delta} \frac{1}{X_{11}^2} \sum_{i \in I} |X_{11i}|^2 + 2\Re\{X_{11i} \bar{b}_i\}. \]  

(4.31)
where  \( b_i = \chi_{ii1} - \chi_{i11} \). So we have

\[
C_1 + C_2 \sum_i F^{\bar{i}i} + \frac{1}{X_{11}^2} \sum_i F^{\bar{i}i} |X_{11i}|^2 \\
\geq \frac{1 - \delta}{1 + \delta} \sum_{i \in I} F^{\bar{i}i} (|X_{11i}|^2 + 2 \Re \{X_{11i} \bar{b}_i \}) + \varphi'' \sum_{i \in I} F^{\bar{i}i} |\partial_i (|u|^2)|^2 + \rho'' \sum_{i \in I} F^{\bar{i}i} |u_i - \bar{u}_i|^2 \\
+ \frac{1}{8k} \sum_i F^{\bar{i}i} X_{ii}^2 + \rho' \sum_i F^{\bar{i}i} (u_{i\bar{i}} - \bar{u}_{i\bar{i}}) - \rho' u_i.
\]

We need to control the terms in (4.32). By (4.10) and the fact that \( \varphi'' = 2(\varphi')^2 \),

\[
\varphi'' \sum_{i \in I} F^{\bar{i}i} |\partial_i (|u|^2)|^2 \geq 2 \sum_{i \in I} F^{\bar{i}i} \left( \left| \frac{X_{11i}}{X_{11}} \right|^2 - \frac{\delta}{1 - \delta} \rho' (u_i - \bar{u}_i)^2 \right),
\]

and in addition using the fact that \( \rho'' \geq \frac{2\delta}{1 - \delta} (\rho')^2 \) and Schwarz inequality,

\[
\frac{1 - \delta}{1 + \delta} \sum_{i \in I} F^{\bar{i}i} (|X_{11i}|^2 + 2 \Re \{X_{11i} \bar{b}_i \}) - \frac{1}{X_{11}^2} \sum_{i \in I} F^{\bar{i}i} |X_{11i}|^2 \\
+ \rho'' \sum_{i \in I} F^{\bar{i}i} |u_i - \bar{u}_i|^2 + \varphi'' \sum_{i \in I} F^{\bar{i}i} |\partial_i (|u|^2)|^2 \\
\geq \frac{1 - \delta}{1 + \delta} \sum_{i \in I} F^{\bar{i}i} (|X_{11i}|^2 + 2 \Re \{X_{11i} \bar{b}_i \}) - \frac{1}{X_{11}^2} \sum_{i \in I} F^{\bar{i}i} |X_{11i}|^2 \\
+ \rho'' \sum_{i \in I} F^{\bar{i}i} |u_i - \bar{u}_i|^2 + 2 \sum_{i \in I} F^{\bar{i}i} \left( \delta \left| \frac{X_{11i}}{X_{11}} \right|^2 - \frac{\delta}{1 - \delta} \rho' (u_i - \bar{u}_i)^2 \right) \\
\geq \frac{2\delta^2}{1 + \delta} \sum_{i \in I} F^{\bar{i}i} X_{11i}^2 + \frac{2(1 - \delta)}{1 + \delta} \sum_{i \in I} F^{\bar{i}i} \Re \{X_{11i} \bar{b}_i \} \\
\geq \frac{\delta^2}{X_{11}^2} \sum_{i \in I} F^{\bar{i}i} |X_{11i}|^2 - \frac{1 - \delta}{(1 + \delta) \delta^2} \sum_{i \in I} |b_p|^2 \sum_{i \in I} F^{\bar{i}i}.
\]

For the terms without index in \( I \), by (4.22)

\[
\varphi'' \sum_{i \notin I} F^{\bar{i}i} |\partial_i (|u|^2)|^2 - \frac{1}{X_{11}^2} \sum_{i \notin I} F^{\bar{i}i} |X_{11i}|^2 \\
\geq -36(C_0 + 1)^2 K_{\max} \sum_{i \notin I} F^{\bar{i}i} \\
\geq -\frac{36(C_0 + 1)^2 K}{\delta} \sum_{i \notin I} F^{\bar{i}i}.
\]
We may assume that
\[ X_{11}^2 \geq \frac{(1 - \delta)}{(1 + \delta)\delta^2} \sum_p |b_p|^2, \] (4.36)
otherwise, the $C^2$ bound is achieved. Substituting (4.34) and (4.35) into (4.32),
\[ C_1 + (C_2 + 1) \sum_i F^{i\bar{i}} + \frac{36(C_0 + 1)^2 K}{\delta} F^{1\bar{1}} \]
\[ \geq \frac{1}{8K} \sum_i F^{i\bar{i}} X_{i\bar{i}}^2 + \rho' \sum_i F^{i\bar{i}}(u_{i\bar{i}} - u_{\bar{i}i}) - \rho'u_t. \] (4.37)
If (2.24) holds true,
\[ C_1 + (C_2 + 1) \sum_i F^{i\bar{i}} + \frac{36(C_0 + 1)^2 K}{\delta} F^{1\bar{1}} \]
\[ \geq \frac{1}{8K} \sum_i F^{i\bar{i}} X_{i\bar{i}}^2 + \theta(C_0 + 1) + \theta(C_0 + 1) \sum_i F^{i\bar{i}}. \] (4.38)
Then
\[ \frac{36(C_0 + 1)^2 K}{\delta} F^{1\bar{1}} \geq \frac{1}{8K} \sum_i F^{i\bar{i}} X_{i\bar{i}}^2. \] (4.39)
So we have
\[ X_{11} \leq \sqrt{\frac{288}{\delta}} (C_0 + 1)K. \] (4.40)
If (2.25) holds true,
\[ C_1 + (C_2 + 1) \sum_i F^{i\bar{i}} + \frac{36(C_0 + 1)^2 K}{\delta} F^{1\bar{1}} \]
\[ \geq \frac{1}{8K} \sum_i F^{i\bar{i}} X_{i\bar{i}}^2 - 3(C_0 + 1) \sup_M |\log \Psi| + \lambda(C_0 + 1) \sum_i F^{i\bar{i}}. \] (4.41)
Then
\[ C_1 + 3(C_0 + 1) \sup_M |\log \Psi| + \frac{36(C_0 + 1)^2 K}{\delta} F^{1\bar{1}} \geq \frac{\theta X_{11}}{16K} + \frac{1}{16K} F^{i\bar{i}} X_{i\bar{i}}^2. \] (4.42)
So we have either
\[ X_{11} \leq \frac{16(C_1 + 3(C_0 + 1) \sup_M |\log \Psi|)K}{\theta} \] (4.43)
or
\[ X_{11} \leq \frac{24(C_0 + 1)}{\sqrt{\delta}} K. \] (4.44)
5. The gradient estimate

In this section, we shall adapt the blow up argument of Dinew and Kolodziej\[9\] and Gill\[14\].

**Theorem 5.1.** On the maximal time interval \([0, T)\), there is a uniform constant \(C > 0\) such that

\[
\sup_{M \times [0, T)} |\nabla u| \leq C. \tag{5.1}
\]

**Proof.** The argument is very similar to those of Dinew and Kolodziej\[9\] and Gill\[14\], so we just give a brief statement here.

We shall prove the theorem by contradiction and suppose that the gradient estimate (5.1) does not hold. Then there exists a sequence \((x_m, t_m) \in M \times [0, T)\) with \(t_m \to T\) such that \(\lim_{m \to \infty} |\nabla u(x_m, t_m)| \to \infty\) and \(|\nabla u(x_m, t_m)| = \sup_{M \times [0, t_m]} |\nabla u|\). We set \(C_m := |\nabla u(x_m, t_m)|\).

After passing to a subsequence, we may assume that \(x_m \to x \in M\). Fix a normal coordinate chart around \(x\), which we identify with an open set in \(\mathbb{C}^n\) with coordinates \((z_1, \cdots, z^n)\), and such that \(\omega(0) = \beta := \sum_{i,j} \delta_{ij} dz^i \wedge d\bar{z}^j\). Without loss of generality, we may assume that the open set contains \(B_1(0)\). We define, on the ball \(B_{C_m}(0)\) in \(\mathbb{C}^n\),

\[
\tilde{u}_m(z) := u_m\left(\frac{z}{C_m}\right). \tag{5.2}
\]

By passing to a subsequence again, we can find a limit function \(\tilde{u} \in C^{1,\alpha}(\mathbb{C}^n)\). As show in \[9\], it is sufficient to prove that \(\tilde{u}\) is a maximal \(k - sh\) function. Without loss of generality, we may assume that \(\tilde{u}_m\) is \(C^{1,\alpha}\) convergent to \(\tilde{u}\). Then we have

\[
\left[\chi_u\left(\frac{z}{C_m}\right)\right]^k \wedge \left[\omega\left(\frac{z}{C_m}\right)\right]^{n-k} = e^{\partial_t u_m\psi_m\left(\frac{z}{C_m}\right)} \left[\chi_u\left(\frac{z}{C_m}\right)\right]^l \wedge \left[\omega\left(\frac{z}{C_m}\right)\right]^{n-l}. \tag{5.3}
\]

Fixing \(z\), we have

\[
C_m^{2(k-l)} \left[O\left(\frac{1}{C_m^2}\right) \beta + \frac{\sqrt{n-1}}{2} \partial \bar{\partial} \tilde{u}_m(z)\right]^k \wedge \left[\left(1 + O\left(\frac{|z|^2}{C_m^2}\right)\right) \beta\right]^{n-k} = e^{\partial_t u_m\psi_m\left(\frac{z}{C_m}\right)} \left[O\left(\frac{1}{C_m^2}\right) \beta + \frac{\sqrt{n-1}}{2} \partial \bar{\partial} \tilde{u}_m(z)\right]^l \wedge \left[\left(1 + O\left(\frac{|z|^2}{C_m^2}\right)\right) \beta\right]^{n-l}. \tag{5.4}
\]

Since \(\partial_t u\) is bounded,

\[
\left[\frac{\sqrt{n-1}}{2} \partial \bar{\partial} \tilde{u}(z)\right]^k \wedge \beta^{n-k} = 0, \tag{5.5}
\]
which is in the pluripotential sense. Moreover, a similar reasoning tells us that for any \(1 \leq p \leq k\),
\[
\left[ \frac{\sqrt{-1}}{2} \partial \bar{\partial} \tilde{u}(z) \right]^p \wedge \beta^{n-p} \geq 0. \tag{5.6}
\]
By a result of Blocki\[2\], the above (5.5) and (5.6) imply that \(\tilde{u}\) is a maximal \(k - sh\) function in \(\mathbb{C}^n\).

\[\square\]

6. Long time existence and Convergence

If \(T > 0\) is a real number, Theorem 5.1 implies that there is a time-independent gradient estimate on \([0, T]\). By the Evans-Krylov theorem and Schauder estimates, we can obtain \(C^\infty\) estimates on \([0, T]\). Then standard procedure based on implicit function theorem can extend \(u(x, t)\) to \([0, T + \epsilon]\) for some small \(\epsilon > 0\), which contradicts the definition of \(T\). So, \(T\) must be \(\infty\).

Applying Theorem 5.1, the Evans-Krylov theorem and Schauder estimates again, we obtain \(C^\infty\) estimates on \([0, \infty)\). Now we are able to show the convergence of the solution flow. The arguments of Gill\[13\] following Cao\[5\] can be applied verbatim here, and thus the proof is omitted.

From the arguments,
\[
sup_{x \in M} \frac{\partial u}{\partial t}(x, t) - \inf_{x \in M} \frac{\partial u}{\partial t}(x, t) \leq Ce^{-c_0 t}, \tag{6.1}
\]
for some \(c_0 > 0\). Noticing that
\[
\int_M \frac{\partial \hat{u}}{\partial t} \chi_u^l \wedge \omega^{n-l} = 0, \tag{6.2}
\]
for any fixed \(t\) there must be \(y \in M\) such that \(\partial \hat{u}(y, t) = 0\). Therefore
\[
\left| \frac{\partial \hat{u}(x, t)}{\partial t} \right| = \left| \frac{\partial \hat{u}(x, t)}{\partial t} - \frac{\partial \hat{u}(y, t)}{\partial t} \right|
\leq \sup_{x \in M} \frac{\partial u}{\partial t}(x, t) - \inf_{x \in M} \frac{\partial u}{\partial t}(x, t) \leq Ce^{-c_0 t}, \tag{6.3}
\]
and thus
\[
\frac{\partial}{\partial t} \left( \hat{u} + \frac{c}{c_0} e^{-c_0 t} \right) \leq 0. \tag{6.4}
\]
By the \(L^\infty\) estimate, it is easy to see that \(\hat{u} + \frac{c}{c_0} e^{-c_0 t}\) is bounded. By a standard argument, (6.4) implies that \(\hat{u}\) is \(C^\infty\) convergent to a smooth function \(\hat{u}_\infty\).

Rewriting equation (1.1),
\[
\frac{\partial \hat{u}}{\partial t} + \frac{\partial}{\partial t} \int_M \chi^l \wedge \omega^{n-l} = \log \frac{\chi^k \wedge \omega^{n-k}}{\chi^l \wedge \omega^{n-l}} - \log \psi. \tag{6.5}
\]
Letting $t \to \infty$, (6.3) implies that (6.5) converges to a constant $b$.

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