T-matrix approach to the phonon-mediated Casimir interaction

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We develop a theory of the phonon mediated Casimir interaction between two point-like impurities, which is based on the single impurity scattering T-matrix approach. Within this, we show that the Casimir interaction at \( T = 0 \) falls off as a power law with the distance between the impurities. We find that the power in the weak and in the unitary phonon-impurity scattering limits differs, and we relate the power law to the low-energy properties of the single impurity scattering T-matrix. In addition, we consider the Casimir interaction at finite temperature and show that at finite temperatures the Casimir interaction becomes exponential at large distances.

INTRODUCTION

The Casimir effect is one of the most striking manifestations of the so-called zero point motion. It emerges as a consequence of changes in vacuum fluctuations in a confined area of space due to boundaries \([1][2]\). A significant interest to this phenomenon has appeared due to prospective experiments in atomic gases \([7][8]\). From this perspective, low dimensional experiments are the most interesting, since the vacuum fluctuations are predicted to be the strongest in low dimensions \([6][8]\). One of the anticipated consequences of changes in vacuum fluctuations in a confined area is based on the single impurity scattering process \([11][12]\). In our case this approach allows studying the evolution of the Casimir interaction starting from the weak phonon-impurity scattering processes \([13][14]\). In the present work, we adopt the form of the interaction used in Ref. \([7][9][10]\):

\[
\hat{H}_0 = \sum_{k} \left( \varphi(k)\bar{\varphi}(k) + \frac{\omega_k^2}{2}\rho(k)\pi(k) \right),
\]

where the bosonic fields are:

\[
\varphi(x) = \frac{1}{\sqrt{\pi}} \sum_{k} \sqrt{\frac{\omega_k}{2}} \left[ b_{k\omega} e^{i k x} + b_{-k\omega}^\dagger e^{-i k x} \right],
\]

\[
\pi(x) = \frac{i}{\sqrt{\pi}} \sum_{k} \sqrt{\frac{1}{2\omega_k}} \left[ b_{k\omega} e^{i k x} - b_{-k\omega}^\dagger e^{-i k x} \right],
\]

and \( \omega_k = c|k| \).

These bosons interact with two impurities placed at the points \( r/2 \) and \(-r/2\). In the present work, we adopt the form of the interaction used in Ref. \([7][8][10]\):

\[
\hat{H}_{int} = g \left( \varphi(x)\bar{\varphi}(0) \right)_{x=-\frac{r}{2}} + \left. \varphi(x)\bar{\varphi}(x) \right|_{x=\frac{r}{2}}.
\]

(2)

Note that the fields Eq.\([1]\) are defined in such a way that the interaction part has the most simple form. Following the diagrammatic rules for bosons, we define the Green’s functions at the Matsubara axis as \([10]\):

\[
G(x,x',\omega_n) = -\int_0^\beta d\tau e^{-\omega_n \tau} \langle T_r(\varphi(x,\tau)\bar{\varphi}(x',0)) \rangle.
\]

To calculate the Casimir forces, we need the propagators at the coordinates of the impurities \( \pm r \), which are designated as follows:

\[
G_r(\omega_n) = G\left( +\frac{r}{2}, -\frac{r}{2}, \omega_n \right) = G\left( -\frac{r}{2}, +\frac{r}{2}, \omega_n \right),
\]

\[
G_0(\omega_n) = G\left( +\frac{r}{2}, +\frac{r}{2}, \omega_n \right) = G\left( -\frac{r}{2}, -\frac{r}{2}, \omega_n \right).
\]

(3)

The explicit expression for \( G_r(\omega_n) \) and \( G_0(\omega_n) \) for the free phonons can be obtained by the Fourier transform.
of $G(k,\omega_n)$ as

$$G_r(\omega_n) = \int \frac{d^D k}{(2\pi)^D} \frac{\omega_k^2}{\omega_n^2 + \omega_k^2} e^{-ikr}$$

and

$$G_0(\omega_n) = \int \frac{d^D k}{(2\pi)^D} \frac{\omega_k^2}{\omega_n^2 + \omega_k^2}$$
correspondingly. In the continuum case the integrals in Eqs. (4), (5) are divergent, and a cut-off has to be introduced.

**CASIMIR FORCE**

To find the relation between the Casimir force and the impurity scattering amplitude, it is worth to use the familiar relation between the derivative of the thermodynamic potential $\Omega$ with respect to a parameter and the derivative of the total Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{int}$ with respect to the same parameter \[16, 17\]. Using $r$ as a parameter, we find the Casimir force:

$$F(r) = -\frac{\partial \Omega(r)}{\partial r} = -\left\langle \frac{\partial \hat{H}_{int}(r)}{\partial r} \right\rangle.$$  \(6\)

The relation of the resulting Casimir force and the two-impurity $T$-matrix is given by the diagram in Fig. \[1\] \[10\]. The solid line with a tick stands for the derivative of the Green’s function $\frac{\partial G_r(\omega_n)}{\partial \omega_n}$. The two-impurity $T$-matrix $T_2(\omega_n)$, can be deduced from the single impurity scattering matrix $T_1(\omega_n)$ as shown in Fig. \[2\] The corresponding equations read:

$$T_2\left(\frac{r}{2}, \frac{r}{2}, \omega_n\right) = T_1(\omega_n) + T_1(\omega_n)G_r(\omega_n)T_2\left(-\frac{r}{2}, \frac{r}{2}, \omega_n\right)$$

$$T_2\left(-\frac{r}{2}, \frac{r}{2}, \omega_n\right) = T_1(\omega_n)G_r(\omega_n)T_2\left(\frac{r}{2}, \frac{r}{2}, \omega_n\right),$$

where $T_2\left(\pm \frac{r}{2}, \pm \frac{r}{2}, \omega_n\right)$ and $T_1(\omega_n)$ are the two-impurity and the single impurity scattering $T$-matrices, correspondingly. Solving Eqs.\(7\) one gets

$$F(r) = T \sum_{n=-\infty}^{\infty} \frac{(T_1(\omega_n)G_r(\omega_n))^2}{1 - (T_1(\omega_n)G_r(\omega_n))^2} \frac{\partial_r G_r(\omega_n)}{G_r(\omega_n)}$$

**FIG. 1:** Diagrammatic representation of the derivative of the thermodynamic potential with respect distance of the impurities (Casimir force).

**FIG. 2:** Diagrammatic representation of the two-impurity $T$-matrix.

The obtained relation between the Casimir force and the impurity scattering amplitude is the central result of the work. The thermodynamic potential can be restored by the integration over $r$.

**Casimir force at T=0**

First, we consider the Casimir interaction at $T = 0$. The Green’s functions $G_r(\varepsilon)$ can be explicitly calculated for the linear boson spectrum $\omega_k = ck$ from Eq.\(9\). It reads in the dimensions $D = 1.3$ (see for details Appendix A):

$$G_r(\varepsilon) = \left\{ \begin{array}{ll}
\frac{i e^{-\frac{|\varepsilon|}{r}}}{\frac{2}{r}}, & D = 1, \\
\frac{i e^{-\frac{|\varepsilon|}{r}}}{\frac{3}{r}}, & D = 2, \\
\frac{i e^{-\frac{|\varepsilon|}{r}}}{\frac{4\pi r c}{2}}, & D = 3.
\end{array} \right.$$  \(9\)

where $K_0(x)$ is the Bessel function. We would like to note, that for dimensions $D \geq 2$, the large distance scaling of $G_r(\varepsilon)$ is universal and can be expressed in the form

$$G_r(\varepsilon) \sim \varepsilon^{D-1} \frac{D+1}{D-1} e^{-\frac{\pi c}{2}}.$$  \(10\)

Due to the exponential dependence of $G_r(\varepsilon)$ on the energy, the leading contribution to the Casimir effect comes from low energies $\varepsilon \ll \varepsilon^* = 1/r$ at large distances.

**Second order of perturbation theory**

Before proceeding to the calculation of the Casimir interaction in general form with use of the $T$-matrix approximation, we consider the second order of the perturbation theory at $T = 0$. In this approximation, the Casimir force is given by the diagram depicted at Fig. \[8\]

$$F^{(2)}(r) = \partial_r \int_0^\infty \frac{d\varepsilon}{2\pi} \varepsilon g^2(\varepsilon)$$

$G_r(\varepsilon)$ is taken from Eq. \[9\]. Integration over $r$ provides expressions for the Casimir potential energy, which is more compact than the expression for the Casimir force. In general, using the expression for $G_r(\varepsilon)$ form Eq. \[9\] and Eq. \[10\], we get that the Casimir interaction in the second order of the perturbation theory obeys the law
the scaling behavior of the Casimir interaction. At these
matrix in the form (see for details Appendix A):

\[ U(\omega) = \begin{cases} 
-\frac{g^2c}{2\pi r^6}, & D = 1, \\
-\frac{g^2c}{2\pi r^5}, & D = 2, \\
\frac{g^2c}{3\pi r^4}, & D \geq 3.
\end{cases} \]

The expression for \( D = 1 \) agrees with early obtained in [8].

**T-matrix approximation**

**Continuum model**

Keeping only the leading terms in the low energy expansion, one can write out the single impurity \( T_1(\varepsilon) \)-matrix in the form (see for details Appendix A):

\[ T_1(\varepsilon) \approx \begin{cases} 
\frac{1}{a^{D-1+1}}, & \varepsilon_1D = 0 \\
\frac{1}{a^{D-2+2} \log \omega_0}, & \varepsilon_2D = 0 \\
\frac{1}{a^{D-3+3}}, & \varepsilon_3D = 0
\end{cases}, \quad \text{for } D = 1, 2, 3.
\]

Here \( \omega_0 \) is the upper cut-off. It is of the order of the Debye energy.

The analysis of the \( T_1 \)-matrix shows that the coefficient \( A \) does not depend on the coupling constant \( g \) in our case (in general, it may weakly depend on \( g \)), whereas \( a \) shows a strong dependence on \( g \). The scattering amplitude \( a \) continuously grows with increasing of \( g \) and diverges, \( 1/a \rightarrow 0 \), at some critical value \( g_{cr} \). This divergence of \( a \) causes the modification of the Casimir interaction scaling at large distances.

For the finite value of \( a \), the \( T_1 \)-matrix tends to the scattering length \( T_1 \div a \) at small values of \( \varepsilon \ll \varepsilon^* \). As one can see from Eq. (12) the characteristic energy \( \varepsilon^* \) depends on the dimension and for \( D = 1, 3 \) reads:

\[ \varepsilon_1D = c/(aA), \quad \varepsilon_2D = c\sqrt{1/(aA) \log \omega_0} \chi(aA/c)), \quad \varepsilon_3D = c\sqrt{1/(aA)}. \]

Since \( G_1(\varepsilon) \) exponentially depends on \( \epsilon r \), the leading contribution to the Casimir force, as seen from Eq. (8), comes from the energies \( \varepsilon \leq 1/r \). It naturally defines the characteristic length \( r_a \sim c/\varepsilon^* \) of the change of the scaling behavior of the Casimir interaction. At these energies, the \( T_1(\varepsilon) \) matrix can be approximated by the constant \( a \). As a result, at large distances between the impurities \( r \gg r_a \), the scaling of the Casimir interaction is the same as in the second order of the perturbation theory with a renormalized phonon-impurity scattering amplitude \( U_{Cas}(r) \sim T_1^2(\varepsilon = 0)/r^{(2D+1)} \). For \( g = g_{cr} \) the scattering length \( a \rightarrow \infty \) and \( r_a \rightarrow \infty \). Hence, the energy dependence of \( T_1(\varepsilon) \) matrix becomes important. The evaluation of Eq. (8) with \( T_1(\varepsilon) \) from Eq. (12) in the unitary limit shows that the Casimir interaction scaling in the leading order triggers to

\[ U_{Cas}(r) \sim \begin{cases} 
\frac{1}{r}, & D = 1, \\
\frac{1}{r \log r}, & D = 2, \\
\frac{1}{r^{2-\delta}}, & D \geq 3.
\end{cases}\]

The analysis of the intermediate case of large but finite \( r_a \) shows that at small distances \( r \ll r_a \) the scaling in the leading approximation is described by Eq. (13).

**Lattice model**

Now we map the model on a lattice in order to study the general properties of the T-matrix. We analyze an ideal harmonic cubic lattice described by \( \rho_0 = \sum \frac{\delta}{2m} + \frac{m\omega_0^2}{2} \sum_{<i,j>}(\delta - u_i)^2 \), with two embedded impurity atoms with their mass different from the mass of the atoms of the lattice. Here \( \rho \) and \( u \) are the momentum and coordinate operators, \( m \) is the mass of the atoms of the cubic lattice and \( m\omega_0^2 \) is the interaction potential.

The excitations of the ideal harmonic lattice are non-interacting phonons. The Hamiltonian reads \( \rho_0 = \sum_k\omega_k(b_k^{\dagger}b_k + \frac{1}{2}) \). The dispersion of the phonons on a lattice is given as \( \omega_k = \omega_0\sqrt{Z - \sum_{\delta} e^{ik\delta}} \), where the summation is done over the nearest neighbors, and \( Z \) is the number of the nearest neighbors. The effect of the introduced impurity atoms can be considered as a perturbation to the kinetic part of the Hamiltonian: \( V = \frac{g}{2m}(\hat{p}_k^2 + \hat{p}_k^2) \). The coupling constant \( g = (1 - m/M) \) with \( m \) being the mass of atom of the ideal lattice, \( M \) the mass of the impurity atom. The momentum operator \( \hat{p}_k \) is quantized as \( \hat{p}_k = i\sqrt{\frac{m\omega_0}{2}}(\hat{b}_k^{\dagger} - \hat{b}_k) \). This term \( V \) is an equivalent of the phonon-impurity coupling given at [2].

In terms of the bosonic operators \( b_k, b_k^{\dagger} \), it reads

\[ V(r) = g \sum_{k,k'} \sqrt{\omega_k \omega_{k'}}(-\hat{b}_k^{\dagger} b_{k'} \cos \frac{(k - k')r}{2} + \hat{b}_k b_{k'} \cos \frac{(k + k')r}{2} + h.c.), \]

\( r = r_i - r_j \). The value of the coupling constant \( g \) for impurities lighter than the lattice atoms, \( g = 0 \) for \( M = m \), and \( 0 \ll g \ll 1 \) for \( M > m \). In the limit of static impurities, \( M \rightarrow \infty \), the coupling
constant $g \to 1$.

Following the trick of \[10, 16\], we redefine the phononic fields so that $k$-dependence is transferred from the vertex to the Green’s function. The resulting definition corresponds to Eq. \[1\]. Then, the integration in $G_r(\varepsilon)$ and $G_0(\varepsilon)$ is performed over the Brillouin zone, and the integrals in Eqs. \[4\],\[5\] become finite:

\[
G_r(\varepsilon) = V_c \int_{BZ} \frac{d^D k}{(2\pi)^D} \cos kr \frac{\omega_k^2}{\varepsilon^2 + \omega_k^2},
\]
\[
G_0(\varepsilon) = V_c \int_{BZ} \frac{d^D k}{(2\pi)^D} \frac{\omega_k^2}{\varepsilon^2 + \omega_k^2},
\]

$V_c$ is the elementary cell volume.

Now we evaluate $G_r(\varepsilon)$, $G_0(\varepsilon)$, $T_1(\varepsilon)$ and the Casimir interaction for the cubic lattice with the phonon spectrum $\omega_k$:

\[
\omega_k = 2\omega_0 \sqrt{\sum_{i=1}^{D} \sin^2 \frac{k_i \delta}{2}},
\]

where $D$ is the dimension, $\delta$ is the lattice constant. For simplicity, we put $\delta = 1$. Both Green’s functions $G_r(\varepsilon)$ and $G_0(\varepsilon)$ scale with the frequency $\omega_0$:

\[
G_r(\varepsilon) = -\frac{\omega_0}{c} f_D \left( \frac{|\varepsilon|}{2\omega_0} \right),
\]
\[
G_0(\varepsilon) = 1 - \frac{\omega_0}{c} f_D \left( \frac{|\varepsilon|}{2\omega_0} \right),
\]

where the function $f_D(x, r)$ does not contain any divergences and falls off exponentially with energy and distance. It can be analytically calculated in $D = 1$ \[10\] and estimated in the leading approximation for higher dimensions.

The $T_1(\varepsilon)$-matrix can be found exactly by summation of the ladder diagrams:

\[
T_1(\varepsilon) = \frac{1}{1 + \frac{\omega_0}{c} f_D \left( \frac{|\varepsilon|}{2\omega_0} \right)},
\]

The value $g = 1$ corresponds to the unitary limit, at which the scattering amplitude $a = g/(1 - g) \to \infty$. Namely, this limit corresponds to the scattering of bosons on a static impurity considered in \[7\]. Away from this limit the low energy part of the $T$-matrix can be approximated by a constant scattering amplitude.

The function $f_{1D}$ reads:

\[
f_{1D}(x) = \frac{x}{\sqrt{1 + x^2}}(x + \sqrt{1 + x^2})^{-2r}
\]

The expression in the limit of small $\varepsilon$ gives Eq. \[12\], with $a = \frac{\omega_0}{c} f_D \left( \frac{|\varepsilon|}{2\omega_0} \right)$ and $A = \frac{1}{2r}$.

For the dimensions $D = 2, 3$, we evaluated $a$ and $A$ numerically. For a cubic lattice in 2D and 3D $G_r(\varepsilon)$ is identical to \[9\] when $\varepsilon \ll c/\delta$, where $\delta$ is a unitary cell length (we put it to 1 from now on). $G_0(\varepsilon) = 1 - \frac{\omega_0}{c} \log |\omega_0/\varepsilon|$ in 2D. The cut-off value $\omega_0$ depends on the lattice. For the cubic lattice one can approximate $\omega_0 \simeq 28$ in units of energy. The same calculation for the hexagonal lattice leads to the phonon spectrum $\omega_0^2 = \frac{8}{3} \left( \sin^2 \frac{k_x}{2} + \sin^2 \frac{k_y}{2} + \frac{1}{4} \left( \sin^2 \frac{k_x}{2} - \sin^2 \frac{k_y}{2} \right) \right)$, the cutoff $\omega_0 \simeq 32$.

In 3D, in the leading approximation for low energies, $G_0(\varepsilon) = 1 - \frac{1}{4\omega_0^2} \varepsilon^2$.

Then we approximate the Casimir interaction for the linearized spectrum as:

\[
U_{Cas}^{(1D)}(r) = \int_0^\infty \frac{d\varepsilon}{2\pi} \log \left[ 1 - \left( \frac{|\varepsilon|}{2\omega_0} e^{-\frac{|\varepsilon|}{\varepsilon}} - \frac{|\varepsilon|}{(1 - g)/g + |\varepsilon|} \right)^2 \right],
\]
\[
U_{Cas}^{(2D)}(r) = \int_0^\infty \frac{d\varepsilon}{2\pi} \log \left[ 1 - \left( \frac{|\varepsilon|^2}{2\pi c^2} K_0(|\varepsilon|) \frac{|\varepsilon|}{(1 - g)/g + |\varepsilon|} \right) \right],
\]
\[
U_{Cas}^{(3D)}(r) \simeq \int_0^\infty \frac{d\varepsilon}{2\pi} \log \left[ 1 - \left( \frac{|\varepsilon|^2}{14\pi c^2} e^{-\frac{|\varepsilon|}{|\varepsilon|}} - \frac{|\varepsilon|}{(1 - g)/g + |\varepsilon|} \right)^2 \right],
\]

The $r$ dependence of the Casimir interaction Eqs.\[15\] \[17\] for various $g$ and $D = 1..3$ is illustrated in Figs.\[4\] \[5\]
There are two regions, which are determined by the characteristic distance $r_a$. For $r \gg r_a$ one finds the universal scaling $U_D \sim \frac{1}{r^2}$. At distances $r < r_a$ this interaction is not universal. (see Ref. [10] for a detailed discussion of this subject in the one-dimensional case). But at very short distances, $r \ll r_a = \epsilon/\varepsilon^*$, the Casimir interaction can be approximated in the leading order as Eq. (13).

In the unitary limit $g \rightarrow 1$ (the static limit in terms of the lattice model), due to the energy dependence of $T_1(\varepsilon), U_{1D}(r) \sim 1/r, U_{2D}(r) \sim 1/(r \log^2 r)$ and $U_D(r) \sim 1/r^{2D-3}$ for $D \geq 3$ at any $r$, since $r_a \rightarrow \infty$.

**Casimir interaction at finite temperature**

Finite temperatures lead to considerable transformation of the scaling of the Casimir interaction at large distances. To study the temperature effect, we will use the expression for the thermodynamic potential, which can be obtained by integration over $r$ of Eq. (8):

$$\Delta \Omega(r) = T \sum_{\omega_n > 0} \log \left(1 - \left(T_1(\omega_n)G_r(\omega_n)\right)^2\right)$$

We evaluate the thermodynamic potential as the function of the distance between the impurities, substituting Eqs. (9) and (12) in Eq. (13). Since $G_r(\omega_n) \sim e^{-\omega_n r/c}$, there are two limiting cases of $r \gg \lambda_T$ and $r \ll \lambda_T$, where $\lambda_T = c/(2\pi T)$ is the thermal de Broglie wavelength. In the former case one restricts the summation by the first term, in the latter case one has to perform the summation over all Matsubara frequencies.

For the illustration, it is worth to consider the 1D system. In the explicit form, the thermodynamic potential reads:

$$\Delta \Omega^{1D}(r) = T \sum_{n=1}^{\infty} \log \left[1 - \left(\frac{e^{-nr/\lambda_T}}{2\lambda_T(na)^{-1} + 1}\right)^2\right],$$

where we introduced $a = g/(1 - g)$ as the scattering length. Now there are two characteristic lengths: the scattering length $a$ and $\lambda_T$ of the order of the thermal de Broglie length. For $r \gg \lambda_T$ the sum is dominated by the first Matsubara term and we get a universal exponential decay of the interaction:

$$\Delta \Omega^{1D}(r) \approx -T \frac{e^{-2r/\lambda_T}}{(2\lambda_T/a + 1)^2},$$

For $r \ll r_a, r_T$ the Casimir interaction follows $r^{-1}$ law. For the intermediate distances $a \ll r \ll r_T$ the Casimir interaction falls off as $r^{-3}$. A special consideration is required when we are exactly in the unitary limit $a \rightarrow \infty$. In this case the decay of the Casimir interaction for $r \ll r_T$ is precisely $r^{-1}$, and for $r \gg r_T$ further transfers to the exponential behavior Eq. (13) for $r \gg \lambda_T$. These dependencies can be seen at Fig. 4 and Fig. 5.

in which the Casimir interaction at finite temperatures and at $T = 0$ are presented. As a guideline, we depict approximate borders of the change of the Casimir law $r_a$ and $r_T$. The detailed derivation of various limits is provided in Appendix B.

The effect of temperature in 2D and 3D systems can be calculated in the same way. The calculations lead to:

$$\Delta \Omega^{2D} = T \sum_{n=1}^{\infty} \log \left[1 - \left(\frac{2\left(K_0\left(\frac{nr}{\lambda_T}\right)\right)^2}{4\pi^2 \lambda_T^2 + \log \frac{\lambda_T^2 \omega_n^2 + (cn)^2}{(cn)}^2}\right)\right]$$

and

$$\Delta \Omega^{4D} = T \sum_{n=1}^{\infty} \log \left[1 - \left(\frac{\pi T \lambda_T e^{-\frac{\pi r}{\lambda_T}}}{\pi T \lambda_T e^{-\frac{2\pi r}{\lambda_T}} + (4\pi \lambda_T^2 + 1)^2}\right)\right].$$

For $r \gg r_T$ the leading contribution to the thermodynamic potential comes from the first Matsubara frequency:

$$\Delta \Omega^{2D}_{r \gg r_T} \approx -\frac{\pi T \lambda_T}{4\pi} e^{-\frac{2\pi r}{\lambda_T}}$$

and

$$\Delta \Omega^{4D}_{r \gg r_T} \approx -\frac{g T}{r^2} \frac{e^{-\frac{2\pi r}{\lambda_T}}}{(4\pi \lambda_T^2 + 1)^2},$$

 correspondingly.

One can see that at large distances $r \gg r_T$ the decay of the Casimir interaction is exponential in all dimensions. For $r \ll r_T$ and $g < 1$, there is a crossover from the short distance law to the $T = 0$ long range law $r^{- (2D + 1)}$.

**DISCUSSION AND CONCLUSION**

Recently, in Ref. [19], it was predicted that the Casimir interaction has an exponential fall short distances and $r^{-3}$ in a weakly interacting 1D Bose condensate in magnetic traps at $T = 0$. In our model, at short distances the Casimir interaction decays as a power law in the limit of $r \ll r_a$ and becomes exponential at finite temperatures at large distances. We have checked the impact of a weak phonon mediated Casimir interaction, based on the single impurity scatterings $T$-matrix. We show that the energy dependence of the $T$-matrix determines the power law fall of the Casimir interaction at short distances and at large distances in the unitary limit at $T = 0$. For weak
impurity scattering the Casimir interaction is universal at large distances.

At finite temperatures, a new characteristic scale of the order of the thermal de Broglie wavelength appears. For distances much larger the de Broglie wave length, the Casimir interaction decays exponentially with the distance between the impurities.

Acknowledgements. We have benefited from discussions with Alex Kamenev, Mikhail Kiselev, Boris Altschuler, and Ilya Polishchuk. We thank Ulrike Nitzsche for technical assistance.

The work was supported by the German Research Foundation (Deutsche Forschungsgemeinschaft) through the program DFG-Russia, BR4064/5-1. J.v.d.B is also supported by SFB 1143 of the Deutsche Forschungsgemeinschaft.

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Appendix A. Evaluation of $G_\varepsilon(x)$, $G_0(\varepsilon)$ and $T_1(\varepsilon)$

Linear spectrum. Here we evaluate Eq. (4) for any dimension for the linear spectrum $\varepsilon = c|k|$ at large distances. Let us consider an arbitrary number of dimensions $D$. For $r \neq 0$ it reads

$$G_r(\varepsilon) = -\int \frac{dk_\perp}{(2\pi)^D} \frac{\varepsilon^2}{2} k_\perp + c^2 k_\parallel^2 + c^2 k_\parallel^2 e^{-uk_\parallel r}.$$

We represented the vector $k = k_\perp + k_\parallel$ and chose $k_\parallel$ along $r$. After integration over $k_\parallel$, we have

$$- C_D \left( \frac{\varepsilon}{c} \right)^2 \int_0^\infty dk_\perp \frac{k_\perp^{D-2}}{\left( \frac{\varepsilon}{c} \right)^2 + k_\perp^2} e^{-\sqrt{\left( \frac{\varepsilon}{c} \right)^2 + k_\perp^2} r},$$

where $C_D = \frac{\pi}{(2\pi)^D} \int d\Omega_{D-1}$ is a constant containing all angular integrations.

We renormalize the momentum introducing a new dimensionless variable $q$ defined as: $k_\perp \rightarrow q\frac{\varepsilon}{c}$. The contribution of large values of the momentum to the integral is exponentially small, so only small momenta matter here. It turns the integral to

$$- C_D \left( \frac{|\varepsilon|}{c} \right)^D \int_0^\infty dq \frac{q^{D-2}}{1 + q^2} e^{\sqrt{1 + q^2} |\frac{\varepsilon}{c}| r} \approx 0$$

for $r \gg 1$.

The remaining integral can be evaluated exactly and gives

$$2\pi |\frac{\varepsilon}{c}| r - \frac{\varepsilon}{c} \Gamma \left( \frac{D-1}{2} \right).$$

It leads us to Eq. (10):

$$G_0(\varepsilon) \approx C_D 2^{\frac{D-3}{2}} \frac{1}{2} \left( \frac{|\varepsilon|}{c} \right) \frac{\varepsilon}{c} r - \frac{\varepsilon}{c} e^{-\frac{|\varepsilon|}{c} r}.$$

$G_0(\varepsilon)$ diverges on the upper limit. Therefore a cut-off $\omega_0$ of the order of the Debye energy is introduced. Then the integration yields:

$$G_0(\varepsilon) \approx \begin{cases} \frac{\omega_0}{4\pi} - \frac{|\varepsilon|}{2c} \log |\frac{\omega_0}{\varepsilon}|, & D = 1, \\ \frac{\omega_0}{6\pi} - \frac{|\varepsilon|}{2c} \log |\frac{\omega_0}{\varepsilon}|, & D = 2, \\ \frac{\omega_0}{8\pi} - \frac{|\varepsilon|}{2c} \log |\frac{\omega_0}{\varepsilon}|, & D = 3. \end{cases}$$

Since the $T_1(\varepsilon)$-matrix is given by the diagrams Fig. 6 it has the form $T_1(\varepsilon) = \frac{2}{1 - G_0(\varepsilon)}$. Substitution of $G_0(\varepsilon)$ gives in the low energy limit the form of the $T$-matrix given in Eq. (12).

Lattice. The integrals (4) and (5) are convergent on the lattice:

$$G_x(\varepsilon) = V_c \int_{BZ} \frac{d^D k}{(2\pi)^D} \frac{\omega_k^2}{e^{\omega_k^2} + \omega_k^2} e^{i k \cdot x} = V_c \int_{BZ} \frac{d^D k}{(2\pi)^D} e^{i k \cdot x}$$

$$-V_c \int_{BZ} \frac{d^D k}{(2\pi)^D} \frac{\omega_k^2}{e^{\omega_k^2} + \omega_k^2} e^{i k \cdot x} = \delta_{x,0} - \frac{\omega_0}{c} F_D \left( \frac{|\varepsilon|}{2\omega_0}, x \right).$$

For $D = 1$, on a square lattice it results into $G_0(\varepsilon) = 1 - \frac{\omega_0}{c} f_1 \left( \frac{|\varepsilon|}{2\omega_0}, 0 \right)$, with $f_1 (x, r) = \frac{x}{\sqrt{1 + x^2}} \left( x + \sqrt{1 + x^2} \right)^{-2}$. In the small $\varepsilon$ limit, this function turns into $G_0(\varepsilon) = 1 - \frac{\omega_0}{c}$. For higher dimensions, the structure $1 - \frac{\omega_0}{c} F_D \left( \frac{|\varepsilon|}{2\omega_0}, 0 \right)$, with some finite function $F_D$, remains. For $D \geq 3$, the Green function $G_0(\varepsilon)$ can be approximated:

$$G_0(\varepsilon) = 1 - \varepsilon^2 V_c \int_{BZ} \frac{d^D k}{(2\pi)^D} \frac{1}{e^{\varepsilon^2 + \omega_k^2}} \approx 1 - \varepsilon^2 V_c \int_{BZ} \frac{d^D k}{(2\pi)^D} \frac{1}{\omega_k^2}.$$

It means that $G_0(\varepsilon) \approx 1 - A\varepsilon^2$, with constant $A$. For $D = 1$, one gets $G_0(\varepsilon) \sim 1 - \varepsilon$. For $D = 2$ the integral is $G_0(\varepsilon) \sim 1 - \varepsilon^2 \log |\varepsilon|$. 

\[ \begin{array}{c}
\text{FIG. 6: Definition of } T_1(\varepsilon) \\
\end{array} \]
Low temperature corrections in 1D. At low temperatures and short distances, when \( r \ll \lambda_T \), we use the Euler-Maclaurin formula up to the first term to approximate the sum:

\[
U_{Cas}^{1D}(r) \simeq \frac{c}{2\pi r} \int_0^\infty \frac{dx}{2\pi r} \log \left[ 1 - \left( \frac{\frac{g^2}{2\pi} e^{-x}}{1 - g + \frac{g^2}{2\pi} x} \right)^2 \right] - \frac{1}{12} \frac{d}{dT} T \log \left[ 1 - \left( \frac{\frac{g^2\pi n T}{e} - 2\pi^2 n \frac{x}{e}}{1 - g + 2\pi^2 n \frac{x}{e}} \right)^2 \right]_{n=1}.
\]

At \( g = 1 \) it gives

\[
U_{Cas}^{1D}(r) \simeq - \frac{\pi c}{24r} + \frac{2\pi^2 r T}{\lambda_T}, \tag{22}
\]

while for small \( g \) it gives

\[
U_{Cas}^{1D}(r) \simeq - \frac{g^2 c}{32\pi r^3} + \frac{g^2 \pi^6 r T}{\lambda_T^3}. \tag{23}
\]

Low temperature corrections in 2D. In the same way as in the one-dimensional case, there are two limiting cases for \( r \ll \lambda_T \). For \( g \ll 1 \) we have:

\[
U_{Cas}^{2D}(r, T) \simeq - \frac{g^2 c}{128\pi r^3} + \frac{4g^2}{3} \pi^2 T^5 \log^2 \left( \frac{r}{\lambda_T} \right)
\]

At \( g = 1 \) the leading terms are:

\[
U_{Cas}^{2D}(r, T) \simeq - \frac{1}{2} \left( r \log \omega_0 r - \frac{1}{r \log^2 \omega_0 r} \right) - \frac{\pi^2}{6} \left( \frac{\log \left( \frac{r}{\lambda_T} \right)}{\log^2 \omega_0 r} \right).
\]

Low temperature corrections in 3D. For \( r \ll \lambda_T \) we again consider two cases. At \( g = 1 \) we have:

\[
U_{Cas}^{3D} \simeq - c \pi \log \left( \frac{r}{r c^2} \right) + 2T \pi r c,
\]

where \( \log \left( \frac{r}{r c^2} \right) \) is the polylogarithmic function.

Casimir energy at non-zero temperature in the second order of the perturbation theory. Here we evaluate the Casimir energy at non-zero temperature in the second order in relation to the parameter \( g \). The energy is given by:

\[
U^{(2)} = T \sum_{n=1}^{\infty} g^2 G_n(\omega_n),
\]

\( G_n(\omega_n) \) is taken from [9].

In the one-dimensional case the energy reads as:

\[
U_{1D}^{(2)} = T g^2 \sum_{n=1}^{\infty} \left( \frac{\pi n T}{e} \right)^2 e^{-4\pi a T / c} = g^2 T^3 \pi^2 \sinh \left( \frac{\pi T}{\lambda_T} \right).
\]

This expression is in accordance with the result given in the Supplemental Material of [9].

In the three-dimensional case the energy turns into:

\[
U_{3D}^{(2)} = T g^2 \sum_{n=1}^{\infty} \left( \frac{\pi n^2 T^2}{r c^4} \right)^2 e^{-2\pi T / \lambda_T} = g^2 \pi^2 T^5 \frac{\cosh^3 \left( \frac{T}{\lambda_T} \right)}{\sinh \left( \frac{T}{\lambda_T} \right)} + 2 \cosh \left( \frac{T}{\lambda_T} \right) \frac{\cosh \left( \frac{T}{\lambda_T} \right)}{\sinh \left( \frac{T}{\lambda_T} \right)}.
\]