Simplified Parsing Expression Derivatives

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Abstract

This paper presents a new derivative parsing algorithm for parsing expression grammars; this new algorithm is both simpler and faster than the existing parsing expression derivative algorithm presented by Moss[12]. This new algorithm improves on the worst-case space and runtime bounds of the previous algorithm by a linear factor, as well as decreasing runtime by about half in practice. A proof of correctness for the new algorithm is included in this paper, a result not present in earlier work.

1 Introduction

A derivative parsing algorithm for parsing expression grammars (PEGs) was first published by Moss[12]; this paper presents a simplified and improved algorithm, as well as a practical comparison of the two algorithms both to each other and to other PEG parsing methods. This new algorithm preserves or improves the performance bounds of the earlier algorithm, trimming a linear factor off the worst-case time and space bounds, while preserving the linear time and constant space bounds for the broad class of “well-behaved” inputs defined in [12]. As an additional contribution to the theory of parsing expression grammars, this work includes a formal proof of correctness for its algorithm, a result left as conjecture by authors of previous parsing expression derivative algorithms[12, 6]. This paper also presents an extension of the concept of nullability from existing work on derivative parsing[3, 10] to PEGs, proving some useful properties of the given presentation while respecting Ford’s[5] undecidability results.

2 Parsing Expression Grammars

Parsing expression grammars are a language formalism similar in power to the more familiar context-free grammars (CFGs). Ford[5] has shown that any LR(k) language can be represented as a PEG, and that there are also some non-context free languages for which PEGs exist (e.g. $a^n b^n c^n$). It is conjectured, however, that context-free languages exist that cannot be recognized by a PEG. Any such language would need to take advantage of the possible ambiguity in CFG parsing since PEGs are unambiguous by definition, admitting no more than one parse tree for any combination of grammar and input.

PEGs are a formalization of recursive-descent parsing with limited backtracking and infinite lookahead; [1] provides definitions of the fundamental parsing expressions. $a$ is a character literal, matching and consuming a single character of input; $\varepsilon$ is the empty expression which always matches without consuming any input, while $\emptyset$ is the failure expression, which never matches. $A$ is a nonterminal, which is replaced by its corresponding parsing expression $R(A)$ to pro-
vide recursive structure in the formalism. The negative lookahead expression !α provides much of the unique power of PEGs, matching only if its subexpression α does not match, but consuming no input (the positive lookahead expression &α can be expressed as !!α). The sequence expression αβ matches α followed by β, while the alternation expression α/β matches either α or β. Unlike the unordered choice in CFGs, if its first alternative for the unambiguous nature of PEG parsing.

Positive lookahead expression provides much of the formalism. The alternative lookahead expression &expression & can be recursive structure in the formalism. The negative lookahead expression !α provides much of the unique power of PEGs, matching only if its subexpression α does not match, but consuming no input (the positive lookahead expression &α can be expressed as !!α). The sequence expression αβ matches α followed by β, while the alternation expression α/β matches either α or β. Unlike the unordered choice in CFGs, if its first alternative for the unambiguous nature of PEG parsing.

A naïve recursive implementation of \( a(s) \) may run in exponential time and linear space with respect to the input string, though Ford\(^4\) has shown that a memoized packrat implementation runs in linear time and space for all grammars and inputs. This linear time bound also suggests that some context-free languages cannot be recognized by a PEG, as the best known general-purpose CFG parsing algorithms have cubic worst-case time.

Formally, a parsing expression grammar \( G \) is the tuple \( (N, X, \Sigma, R, \sigma) \), where \( N \) is the set of non-terminals, \( X \) is the set of parsing expressions, \( \Sigma \) is the input alphabet, \( R : N \rightarrow X \) maps each nonterminal to its corresponding parsing expression, and \( \sigma \in X \) is the start expression to parse. Each parsing expression \( \varphi \in X \) is a function \( \varphi : \Sigma^* \rightarrow \Sigma^* \cup \{\text{fail}\}, \varphi \notin \Sigma^* \). The language \( L(\varphi) \) accepted by a parsing expression \( \varphi \) is the set of strings matched by that parsing expression; precisely, \( L(\varphi) = \{s \in \Sigma^* : \exists s' \in \Sigma^*, \varphi(s) = s'\} \). \( \varphi \) is said to match \( s \) if \( s \in L(\varphi) \). For some string \( s = s_0 s_{i+1} \cdots s_n \in \Sigma^* \), if \( \varphi(s) = s' \in \Sigma^* \), \( s' \) is always a suffix \( s_j s_{j+1} \cdots s_n \) of \( s \); such a suffix is denoted \( s[j] \). This suffix may be the entire string \( (j = i) \), a strict substring \( (j > i) \), or even the empty string \( \varepsilon \) (distinct from the empty expression \( \varepsilon \)). Note that for some of the new derivative expressions introduced in Section 3, \( j < i \), i.e., \( s \) is a suffix of \( s[j] \).

\begin{equation}
a(s) = \begin{cases} s' & \text{ fail otherwise} \\
\text{fail} & \text{ otherwise}
\end{cases}
\end{equation}

\begin{equation}
\varepsilon(s) = s
\end{equation}

\begin{equation}
\otimes(s) = \text{ fail}
\end{equation}

\begin{equation}
A(s) = (R(A))(s)
\end{equation}

\begin{equation}
!\alpha(s) = \begin{cases} s & \alpha(s) = \text{ fail} \\
\text{fail} & \text{ otherwise}
\end{cases}
\end{equation}

\begin{equation}
\alpha\beta(s) = \begin{cases} s'' & \alpha(s) = s' \land \beta(s') = s'' \\
\text{fail} & \text{ otherwise}
\end{cases}
\end{equation}

\begin{equation}
\alpha/\beta(s) = \begin{cases} s' & \alpha(s) = s' \\
s'' & \alpha(s) = \text{ fail} \land \beta(s) = s'' \\
\text{fail} & \text{ otherwise}
\end{cases}
\end{equation}

A parsing expression \( \varphi \) is nullable if it matches any string; that is, \( L(\varphi) = \Sigma^* \). Since parsing expressions match prefixes of their input, nullable expressions are those that match the empty string \( \varepsilon \) without constraining the rest of the input. Not all expressions that may match without consuming input (i.e., match \( \varepsilon \)) are nullable, though; for instance, \( !a \) matches \( \varepsilon \), but not \( abc \). I define a weakly nullable parsing expression \( \varphi \) as one where \( \varepsilon \in L(\varphi) \). \( \{4\} \) and \( \{3\} \) define a nullability predicate \( \nu \) and a weak nullability predicate \( \lambda \). Both predicates can be computed by iteration to a fixed point.
It is worth noting that if evaluation of \((\mathcal{R}(A))(s)\) includes a left-recursive call to \(A(s)\), parsing expression evaluation will never terminate; for this reason the behaviour of parsing expressions including left-recursion is undefined. Repetition of an expression can be accomplished right-recursively with a nonterminal of the form \(R := α R / ε\), generally written \(α^*\). Fortunately, lack of left-recursion is easily structurally verified. The left-expansion function \(LE\) defined in \(\text{(5)}\) defines the set of immediate sub-expressions possibly left-recursively expanded by evaluation of a parsing expression. The left-expansion of an expression is a subset of its subexpressions \(SUB\), defined in \(\text{(4)}\). The recursive left-expansion and subexpression functions can be computed by iteration to a fixed point as \(LE^+[φ] = LE(φ) \cup γ \in LE(φ) LE^+[γ]\) and \(SUB^+[φ] = SUB(φ) \cup γ \in SUB(φ) SUB^+[γ]\). A parsing expression \(φ\) is well-formed if neither \(φ\) nor any of its subexpressions left-recursively expand themselves; that is, \(∀γ \in \{φ\} \cup SUB^+[φ], γ ∉ LE^+[γ]\). This definition of well-formed is equivalent to the condition for a well-formed grammar introduced by Ford\[5\] § 3.6.

\[
\begin{align*}
SUB(α) &= \{\} \\
SUB(ε) &= \{\} \\
SUB(∅) &= \{\} \\
SUB(A) &= \{\mathcal{R}(A)\} \\
SUB(αα) &= \{α\} \\
SUB(αβ) &= \{α, β\} \\
SUB(α/β) &= \{α, β\}
\end{align*}
\]

\[
\begin{align*}
LE(α) &= \{\} \\
LE(ε) &= \{\} \\
LE(∅) &= \{\} \\
LE(A) &= \{\mathcal{R}(A)\} \\
LE(αα) &= \{α\} \\
LE(αβ) &= \{α, β\}
\end{align*}
\]

The nullability predicate \(ν\) is a conservative underapproximation of which parsing expressions are actually nullable; the main source of imprecision is lookahead expressions. For \(µ\) such that \(L(µ) = ∅\), \(L(∅) = ∅^*\) but \(ν(∅) = ⊥\); however, Ford\[5\] showed that it is undecidable whether the language of an arbitrary parsing expression is empty, precluding a precise description of nullability with respect to lookahead expressions. As shown in Section \(\S\) 3.6, all parsing expressions \(η\) for which \(ν\) holds do match all of \(Σ^*\) (Theorem 1), but for the converse, all that is shown is the weaker statement that \(λ\) holds for any expression which matches the empty string (Theorem 2). The author conjectures that any significantly stronger statement is precluded by Ford’s undecidability results.

It is sometimes useful to assume that all parsing expressions are in a consistent simplified form. None of the simplification rules in Table 1 change the result of the parsing expressions, as can be easily verified by consulting (1), while they have the useful properties of reducing any expression which is structurally incapable of matching to ∅ and of trimming unreachable or redundant subexpressions from sequence and alternation expressions.

| Rule | Simplification |
|------|----------------|
| 1.   | \(αε ≡ α\)    |
| 2.   | \(εβ ≡ β\)    |
| 3.   | \(α∅ ≡ α\)    |
| 4.   | \(∅β ≡ ∅\)    |
| 5.   | \(α/∅ ≡ α\)   |
| 6.   | \(∅/β ≡ β\)   |
| 7.   | \(η/β ≡ η\)   |
| 8.   | \(A ≡ ∅\) if \(A := ∅\) |
| 9.   | \(!η ≡ ∅\)    |
| 10.  | \(!!α ≡ !α\)  |

### 3 Derivative Parsing

The essential idea of derivative parsing, first introduced by Brzozowski\[3\], is to iteratively transform an expression into an expression for the “rest” of the input. For example, given the expression \(γ = foo/bar/baz\), \(d_b(γ) = ar/az\), the suffixes that can follow \(b\) in \(L(γ)\). Once repeated derivatives have been taken for every character in the input string, the resulting expression can be checked to determine whether or not it represents a match. Existing work shows how to compute the derivatives of regular
expressions\cite{3}, context-free grammars\cite{10}, and parsing expression grammars\cite{12 6}. This paper presents a simplified algorithm for parsing expression derivatives, as well as a formal proof of the correctness of this algorithm, an aspect lacking from the earlier presentations.

The chief difficulty in creating a derivative parsing algorithm for PEGs is that backtracking is a fundamental part of the semantics of PEGs. If one alternation branch does not match, the algorithm tries another; similarly, if a lookahead expression matches, the algorithm must attempt to continue parsing its successor from the original point in the input. However, derivative parsing does not backtrack – a derivation is taken in sequence for each character, and all effects of parsing that character must be accounted for in the resulting expression. The earlier formulation by Moss\cite{12} of derivative parsing for PEGs included a system of “backtracking generations” to label possible backtracking options for each expression, as well as a complex mapping algorithm to translate the backtracking generations of parsing expressions to the corresponding generations of their parent expressions. The key observation of the simplified algorithm presented here is that an index into the input string is sufficient to label backtracking choices consistently across all parsing expressions.

\[
\begin{align*}
\varepsilon_j(s[k]) &= s[j+1] \\
\lambda_j\alpha(s[k]) &= \begin{cases} s[j+1] & \text{if } \alpha(s[k]) = \text{fail} \\ \text{fail} & \text{otherwise} \end{cases} \\
\alpha_\beta[\beta_1 \ldots \beta_{i\kappa}](s[k]) &= \alpha_\beta(s[k])
\end{align*}
\]

In typical formulations\cite{3 10 12}, the derivative \(d_c(\varphi)\) is a function from an expression \(\varphi \in \mathcal{X}\) and a character \(c \in \Sigma\) to a derivative expression \(\varphi' \in \mathcal{X}\). Formally, \(\mathcal{L}(d_c(\varphi)) = \{s \in \Sigma^c : c \cdot s \in \mathcal{L}(\varphi)\}\). This paper defines a derivative \(d_{c,i}(\varphi)\), adding an index parameter \(i \in \mathbb{N}\) for the current location in the input string. Additionally, certain parsing expressions are annotated with information about their input position. \(\varepsilon\), which always matches, becomes \(\varepsilon_j\), a match at index \(j\); \(\lambda\alpha\), a lookahead expression which never consumes any characters, becomes \(\lambda_j\alpha\), a lookahead expression at index \(j\). Finally, a sequence expression \(\alpha_\beta\) must track possible indices at which \(\alpha\) may have stopped consuming characters and \(\beta\) began to be parsed; to this end, \(\alpha_\beta\) is annotated with a list of lookahead followers \([\beta_1 \ldots \beta_{i\kappa}]\), where \(\beta_1 \in \mathcal{X}\) is the repeated derivative of \(\beta\) starting at each index \(i_j \in \mathbb{N}\) where \(\alpha\) may have stopped consuming characters. These annotated expressions are formally defined in \(6\); note that these definitions match or fail under the same conditions as those in \(11\), but may consume (or un-consume) a different portion of the input, as shown in Theorem 4. Accompanying extensions of \(\text{SUB}\) and \(\text{LE}\) are defined in \(7\) and \(8\).

\[
\begin{align*}
\text{SUB}(\varepsilon_j) &= \{\} \\
\text{SUB}(\lambda_j\alpha) &= \{\alpha\} \\
\text{SUB}(\alpha_\beta[\beta_1 \ldots \beta_{i\kappa}]) &= \{\alpha, \beta, \beta_1, \ldots, \beta_{i\kappa}\} \\
\text{LE}(\varepsilon_j) &= \{\} \\
\text{LE}(\lambda_j\alpha) &= \{\alpha\} \\
\text{LE}(\alpha_\beta[\beta_1 \ldots \beta_{i\kappa}]) &= \{\alpha, \beta_1, \ldots, \beta_{i\kappa}\}
\end{align*}
\]

To annotate parsing expressions with their indices, \(13\) defines a normalization function \(\langle \bullet \rangle_i\) to annotate parsing expressions; derivative parsing of \(\varphi\) starts by taking \(\langle \varphi \rangle_0\). One useful effect of this normalization function is that all nonterminals in the left-expansion of \(\varphi\) are replaced by their expansion, and thus \(\langle \varphi \rangle_i\) has no recursion in its left-expansion; in particular, structural induction over \(\langle \varphi \rangle_i\) is bounded for well-formed \(\varphi\), per Theorem 3.

\[
\begin{align*}
\langle a \rangle_i &= a \\
\langle \varepsilon \rangle_i &= \varepsilon_i \\
\langle \varnothing \rangle_i &= \varnothing \\
\langle A \rangle_i &= \langle \mathcal{R}(A) \rangle_i \\
\langle \lambda \alpha \rangle_i &= \lambda \langle \alpha \rangle_i \\
\langle \alpha_\beta \rangle_i &= \langle \alpha \rangle_i \beta [\beta_1 = \langle \beta \rangle_i \text{ if } \lambda(\beta)] \\
\langle \alpha / \beta \rangle_i &= \langle \alpha \rangle_i / \langle \beta \rangle_i
\end{align*}
\]

The nullability functions \(\nu\) and \(\lambda\) must also be expanded to deal with the indices added by the normalization process. To this end, I define two functions
match and back from $\mathcal{X}$ to $\mathcal{P}(\mathbb{N})$ (based on definitions in [12]). match and back may be thought of as extended versions of $\nu$ and $\lambda$, respectively, where match (resp. back) is a non-empty set of indices if $\nu$ (resp. $\lambda$) is true; definitions are in (10) and (11) and a proof is included with Theorem 5.

\[
\begin{align*}
\text{back}(a) &= \{\} \\
\text{back}(\varepsilon_i) &= \{i\} \\
\text{back}(\varnothing) &= \{\} \\
\text{back}(\lambda i) &= \{i\} \\
\text{back}(\alpha \beta [\beta_1 \ldots \beta_k]) &= \bigcup_{j \in [i_1 \ldots i_k]} \text{back}(\beta_j) \\
\text{back}(\alpha / \beta) &= \text{back}(\alpha) \cup \text{back}(\beta) \\
\text{match}(a) &= \{\} \\
\text{match}(\varepsilon_i) &= \{i\} \\
\text{match}(\varnothing) &= \{\} \\
\text{match}(\lambda i) &= \{\} \\
\text{match}(\alpha \beta [\beta_1 \ldots \beta_k]) &= \bigcup_{j \in \text{match}(\alpha)} \text{match}(\beta_j) \\
\text{match}(\alpha / \beta) &= \text{match}(\beta)
\end{align*}
\]

Having defined these necessary helper functions, the derivative step function $d_{c,i}$ is defined in (13). To test whether some input string $s = s_1 s_2 \ldots s_n$ augmented with an end-of-string terminal $\# \notin \Sigma$ matches a parsing expression $\varphi$, I define the string derivative (12):

\[
d_{s,i}(\varphi) = (d_{s_{n,i+n-1}} \circ d_{s_{n-1,i+n-2}} \circ \cdots \circ d_{s_1,i+1})(\varphi).
\]

\[
\varphi^{(n)} = d_{\#,n} \circ d_{\#,0}(\langle \varphi \rangle_0)
\]

can be used to recognize $\mathcal{L}(\varphi)$: if $\varphi^{(n)} = \varepsilon_j$, then $\varphi(s) = s[j]$, otherwise $\varphi(s) = \text{fail}$. This assertion is proven in Theorem 3. This process may be short-circuited if some earlier derivative resolves to $\varphi_j$ or $\varnothing$, as Lemma 2 shows these success and failure results are preserved for the rest of the string.

To preserve performance, the derivative step (13) should be memoized, with a fresh memoization table for each derivative step. With such a table, a single instance of $d_{c,i}(\varphi)$ can be used for all derivatives of $\varphi$, changing the tree of derivative expressions into a directed acyclic graph (DAG). The only new expressions added by computation of $d_{c,i}(\varphi)$ are of the form $\langle \beta \rangle_i$, for $\beta$ the successor in some sequence expression $\alpha \beta$. Given that all such $\beta$ must be present in the original grammar, all of these added expressions are of constant size. With memoization, this bounds the increase in size of the derivative expression DAG by a constant factor for each derivative step. back and match are also memoized.

### 4 Proofs

This paper rectifies the absence of formal rigor in the existing literature on parsing expression derivatives: both Moss [12] and the prepublication work of Garnock-Jones et al. [6] leave the correctness of their algorithms and predicates as conjecture with some level of experimental validation. This work, by contrast, includes proofs of correctness for both the nullability predicates and the complete algorithm.

#### 4.1 Nullability

The nullability ($\nu$) and weak nullability ($\lambda$) predicates differ only in their treatment of lookahead expressions, and as such it is useful to observe that $\nu$ is a strictly stronger condition:

**Lemma 1.** For any parsing expression $\varphi$, $\nu(\varphi) \Rightarrow \lambda(\varphi)$.

**Proof.** By cases on definitions of $\nu$ and $\lambda$. \hfill $\square$

The proofs presented in this section rely heavily on the technique of structural induction; nonterminals present some difficulty in this approach, as they may recursively expand themselves, causing the induction to fail to terminate in a base case. The following two lemmas show that this is not an issue for well-formed PEGs:

**Lemma 2.** For any well-formed parsing expression $\varphi$, the set of parsing expressions examined to compute $\lambda(\varphi)$ is precisely $\{\varphi\} \cup \mathcal{L}^+(\varphi)$.
For any well-formed parsing expression \( \varphi \), the set of parsing expressions examined to compute \( \nu(\varphi) \) is a subset of \( \{\varphi\} \cup LE^+(\varphi) \).

Proof. Structural induction as in Lemma 2. Note that for \( \alpha \) the set examined is \( \{\alpha\} \subset \{\alpha\} \cup LE^+(\alpha) \) and that for \( \alpha \beta \), \( \nu(\alpha) \Rightarrow \lambda(\alpha) \) by Lemma 1.

Having demonstrated the feasibility of the structural induction approach, the nullability results claimed in Section 2 can now be shown:

Theorem 1 (Nullability). For any well-formed parsing expression \( \varphi \), \( \nu(\varphi) \Rightarrow L(\varphi) = \Sigma^* \).

Proof. By structural induction on \( \varphi \); by Lemma 3 this induction does terminate. The \( \alpha \), \( \emptyset \), and \( \emptyset \alpha \) cases are trivially satisfied; the \( \varepsilon \) and \( A \) cases follow from definition. For \( \alpha \beta \) where \( \nu(\alpha) \lor \nu(\beta) \), the inductive hypothesis implies that for all \( s \) there exist \( r, t \in \Sigma^*, s = rt \), such that \( \alpha(s) = t \) and \( t \in L(\beta) \). For \( \alpha / \beta \) where \( \nu(\alpha) \lor \nu(\beta) \), if \( s \in \Sigma^* \notin L(\alpha) \), by the inductive hypothesis \( \neg \nu(\alpha) \lor \nu(\beta) \lor s \in L(\beta) \lor s \in L(\alpha / \beta) \).

Theorem 2 (Weak Nullability). For any well-formed parsing expression \( \varphi \), \( \epsilon \in L(\varphi) \Rightarrow \lambda(\varphi) \).

Proof. By structural induction on contrapositive; by Lemma 2 this induction does terminate. The \( \varepsilon \) and \( \alpha \emptyset \) cases are trivially satisfied, the \( \alpha \), \( \emptyset \), and \( A \) cases follow from definition. If \( \neg \lambda(\alpha \beta) \), by the inductive hypothesis \( \epsilon \in L(\alpha) \Rightarrow \epsilon \notin L(\beta) \lor \alpha(\epsilon) = \epsilon \Rightarrow \beta(\epsilon) = \text{fail} \lor \epsilon \notin L(\alpha \beta) \). If \( \neg \lambda(\alpha / \beta) \), \( \neg \lambda(\alpha) \lor \neg \lambda(\beta) \); by the inductive hypothesis \( \epsilon \notin L(\alpha) \lor \epsilon \notin L(\beta) \lor \alpha(\epsilon) = \beta(\epsilon) = \alpha / \beta \) = \text{fail}.

The nullability and weak nullability functions dis-
cussed in this paper are closely related to the → relation defined by Ford\cite[§ 3.5]{ford}. In Ford’s formulation, \( \varphi \to 0 \) means that \( \varphi \) may match while consuming no characters, \( \varphi \to 1 \) means that \( \varphi \) may match while consuming at least one character, and \( \varphi \to f \) means that \( \varphi \) may fail to match. When recursively applied, the simplification rules in Table \ref{table:1} ensure that any expression which is structurally incapable of matching\footnote{Note that this does not include expressions such as \( \langle \alpha \rangle \alpha \), which will never match, but exceed the power of the rules in Table \ref{table:1} to analyze.} is reduced to the equivalent expression \( \varnothing \). Under this transformation, \( \neg \nu(\varphi) \equiv \varphi \to f \), while \( \lambda(\varphi) \equiv \varphi \to 0 \).

### 4.2 Derivatives

As with the nullability predicates, the correctness proof for the derivative parsing algorithm relies heavily on structural induction, and as such must demonstrate the termination of that induction (Theorem \ref{thm:3}). Additionally, it must be shown that the normalization step applied does not meaningfully change the semantics of the parsing expressions (Theorem \ref{thm:4}).

To discuss the effects of the normalization function \( \langle \bullet \rangle_i \), some terminology must be introduced. Since \( \langle \bullet \rangle_i \) applies to the left-expansion of its argument (per Lemma \ref{lem:4}), a normalized parsing expression \( \varphi \) is defined as a well-formed parsing expression with no un-normalized expressions in its left-expansion (i.e., \( \langle \bullet \rangle_i \) and where all sequence expressions \( \alpha \beta^{[\beta_i \ldots \beta_{1k}]} \in \{\varphi\} \cup LE^+(\varphi) \) respect the sequence normalization property \( \langle\rangle \)). The more precise class of \( k \)-normalized parsing expressions are normalized parsing expressions \( \varphi \) with indices no greater than \( k \) (i.e., \( \langle\rangle \)). By contrast, an un-normalized parsing expression \( \varphi \) is one where \( \langle\rangle \).

First I show that normalized parsing expressions have a finite expansion amenable to structural induction:

**Lemma 4.** \( \forall \varphi \in \mathcal{X}, i \in \mathbb{N}, \) the set of parsing expressions expanded by \( \langle \varphi \rangle_i \) is precisely \( LE^+(\varphi) \).

**Proof.** By cases on definitions of \( \langle \bullet \rangle_i \), \( LE \). \( \square \)

**Theorem 3** (Finite Expansion). For any well-formed, un-normalized parsing expression \( \varphi \), \( \langle \varphi \rangle_i \) has a finite expansion.

**Proof.** Follows directly from the definition of well-formed and Lemma \ref{lem:4} \( \square \)

Then I show that normalization does not change the semantics of the parsing expression:

**Theorem 4** (Normalization). For any well-formed, un-normalized parsing expression \( \varphi \), string \( s = s_k s_{k+1} \cdots s_{k+n} \in \Sigma^* \), \( \varphi(s) = \langle \varphi \rangle_k(s) \).

**Proof.** By structural induction on \( \varphi \). By Theorem \ref{thm:3} (Finite Expansion) \( \langle A \rangle_k \) has a finite expansion, thus the structural induction is bounded. \( \langle \varepsilon \rangle_k(s) = \varepsilon(s) \). \( \langle ! \alpha \rangle_k(s) = !_k(\alpha)_k(s) \); by the inductive hypothesis \( \langle \alpha \rangle_k(s) = \alpha(s) \), thus by the definitions in \ref{eq:1} and \ref{eq:2} \( !_k(\alpha)_k(s) = !_k(\alpha)(s) \). The other cases follow directly from the relevant definitions and the inductive hypothesis. \( \square \)

In addition to showing that \( \langle \varphi \rangle_i \) is semantically equivalent to the original expression \( \varphi \), the index sets match and back must be shown to represent equivalent concepts of nullability to \( \nu \) and \( \lambda \), respectively. The following lemmas present some useful properties of match and back; all can be straightforwardly shown by structural induction over \( \varphi \):

**Lemma 5.** For any normalized \( \varphi \in \mathcal{X}, \) \( \text{match}(\varphi) \subseteq \text{back}(\varphi) \).

**Lemma 6.** For any well-formed, un-normalized \( \varphi \in \mathcal{X}, \) \( \text{back}(\langle \varphi \rangle_i) \subseteq \{i\} \).

**Lemma 7.** For any normalized \( \varphi \in \mathcal{X}, \) \( |\text{match}(\varphi)| \leq 1 \).
Given these lemmas, the main result of equivalence to the nullability predicates can be shown, with the corollary that the normalization function \( i \) respects the normalization rule \( \mathcal{L} \):

**Theorem 5 (Nullability Equivalence).** For a well-formed, un-normalized parsing expression \( \varphi \), \( \nu(\varphi) \Rightarrow \text{match}(\langle \varphi \rangle_k) = \{ k \} \) and \( \lambda(\varphi) \Rightarrow \text{back}(\langle \varphi \rangle_k) = \{ k \} \).

**Proof.** By structural induction on \( \varphi \).

— Cases \( a \) and \( \varnothing \) — Vacuously true.

— Case \( \varepsilon \) — Follows from definitions.

— Case \( A \) — \( \langle A \rangle_k \) has a finite expansion [Theorem 3], thus the induction terminates. From there, case follows from inductive hypothesis.

— Case \( !\alpha \) — \( \nu \) statement vacuously true, \( \lambda \) statement follows from definitions.

— Case \( \alpha \beta \) — \( \lambda(\beta) \) implies \( \beta_k \) defined [defn \( \langle \alpha \beta \rangle_k \)] and \( \text{back}(\beta_k) = \{ k \} \) [ind. hyp.]; similarly \( \nu(\beta) \Rightarrow \lambda(\beta) \) [Lemma 1] and \( \nu(\beta) \Rightarrow \text{match}(\beta_k) \) [ind. hyp.]. The rest follows from definitions.

— Case \( \alpha / \beta \) — \( \nu \) statement implicitly requires application of the simplification rules in Table 1; particularly that \( \neg \nu(\alpha) \) [by 7.] ; \( \lambda \) statement follows from definitions.

**Corollary 1.** \( \langle \alpha \beta \rangle_i \) respects \( \mathcal{L} \).

To link up the normalization function with the idea of normalized parsing expressions, I prove the following link:

**Lemma 8.** For any well-formed, un-normalized parsing expression \( \varphi \), \( \langle \varphi \rangle_k \) is \( k \)-normalized.

**Proof.** By definition, \( \sharp \varepsilon_j \) or \( !\alpha \in \text{SUB}^+(\varphi) \); particularly none exist such that \( j > k \). By Lemma 4, \( \langle \varphi \rangle_k \) is applied to all of \( \text{LE}^+(\varphi) \), which by definition of \( \langle \bullet \rangle_k \) replaces all of the \( \varepsilon, A \), and \( \alpha \) with \( \varepsilon_k, \langle \mathcal{R}(A) \rangle_k \) and \( !k \alpha \), respectively, satisfying the definitions of normalized and \( k \)-normalized. Note \( \langle \mathcal{R}(A) \rangle_k \) has a finite expansion [Theorem 3], and \( \mathcal{L} \) is maintained [Corollary 1].

With these initial results in place, I can now move on to proving the primary correctness result for the algorithm, as discussed in Section 3. The first step is to show that success (\( \varepsilon_j \)) and failure (\( \varnothing \)) results persist for the rest of the string. I also show that the derivative step maintains the normalization property \( \mathcal{L} \), and that \( \text{back} \) and \( \text{match} \) have semantics matching their claimed meaning:

**Lemma 9.** Given \( s = s_{k+1}s_{k+2} \cdots s_n \), \( d_{s,k}(\varepsilon_j) = \varepsilon_j \) and \( d_{s,k}(\varnothing) = \varnothing \).

**Proof.** Follows directly from definitions of \( d_{c,i}(\varepsilon_j) \) and \( d_{c,i}(\varnothing) \), applied inductively over \( k \) decreasing from \( n \).

**Corollary 2.** \( d_{c,i}(\alpha \beta [\beta_i \cdots \beta_{ik}]) \) maintains \( \mathcal{L} \).

**Lemma 11.** For a normalized parsing expression \( \varphi \) such that \( \text{match}(\varphi) \neq \{ \} \), and a string \( s = s_{k}s_{k+1} \cdots s_n \), \( d_{s,k}(\varphi) = \varepsilon_{\ell} \) for some \( \ell \leq n + 1 \).

**Proof.** By structural induction on \( \varphi \); by Theorem 3 (Finite Expansion) the expression is finite and thus admits structural induction.

— \( a \), \( \varnothing \), and \( !\alpha \) — vacuously satisfied.

— \( \varepsilon_j \) — Lemma 9

— \( \alpha \beta [\beta_i \cdots \beta_{ik}] \) and \( \alpha / \beta \) — Inductive hypothesis.

With these lemmas established, Theorem 6 shows the main result, that the derivative step possesses the expected semantics, while Theorem 7 demonstrates how the derivative \( d_{\delta,n} \) with respect to the end-of-string terminator fulfills the role typically served by a nullability combinator \( \delta \) in other derivative parsing formulations [3, 11, 6].
**Theorem 6** (Derivative Step). For any well-formed, $k$-normalized parsing expression $\varphi$ and any string $s = s_{k+1}s_{k+2}\cdots s_{k+n}$, $\varphi(s) = d_{s_{k+1}s_{k+2}\cdots s_{k+m},k}(\varphi(s[k+m]))$ for all $m \leq n$.

Proof. By induction on $m$. $m = 0$ is true by identity, and the inductive step is shown by structural induction on $\varphi$. Let $s' = s[k+1]$ and for any parsing expression $\gamma$ let $\gamma' = d_{s_{k+1},k+1}(\gamma)$. Note that by Lemma 9, if $\varphi' \in \{\varepsilon, \varnothing\}$ then $d_{\varepsilon,k} = \varphi'$. Also note that by Lemma 8 and the inductive hypothesis, $\text{match}(\varphi') \neq \emptyset \Rightarrow \varphi'(s') = s[l] \Rightarrow \varphi(s) = s[l]$.

- $\varepsilon$ and $\emptyset$ — By structural induction, $\varphi(s) = \alpha'(s')$.
  - If $\alpha' = \emptyset$, $\varphi'(s') = \alpha'(s') = \emptyset$ and $\varphi(s) = s[l] = \emptyset$.
  - If $\alpha' = \varepsilon$, $\varphi'(s') = \varepsilon_k = \varphi(s) = s[j]$ and also $\alpha(s) = s[j]$.
  - Otherwise $\varphi(s) = \varphi'(s')$ by $\alpha(s) = \alpha'(s')$.

- $\alpha/\beta$ — By similar argument to $!\alpha$
- $\alpha\beta[\beta_1, \ldots, \beta_n]$ — By structural induction, $\alpha(s) = \alpha'(s')$.
  - If $\alpha' = \emptyset$, $\varphi'(s') = \alpha'(s') = \emptyset$ and also $\varphi(s) = \alpha(s) = \emptyset$.
  - If $\alpha' = \varepsilon_k = \varphi'(s') = \beta_k(s')$ [Theorem 4 (Normalization)] and $\alpha(s) = s'[k] = \beta(s')$.
  - If $\alpha' = \varepsilon_j, j \leq k$, $\alpha'(s') = \alpha(s) = s[j]$ and $\varphi(s) = \beta(s[j])$; $\varphi'(s') = \beta_j(s'[j])$ [backward application of case 5 of sequence derivative]. Note that $\text{back}(\varepsilon_j) = \emptyset$ so by repeated application of Lemma 10, $\beta_j$ is defined. $\varphi'(s') = \beta(s[j])$ [Theorem 4 (Normalization)].
  - Otherwise $\varphi(s) = \varphi'(s')$ by structural induction: $\alpha(s) = \alpha'(s')$, $\beta_j(s) = \beta_j(s')$, and $\beta(s) = \beta_k(s')$ by Theorem 4 (Normalization) if applicable.

**Theorem 7** (Derivative Completion). For any string $s = s_1s_2\cdots s_n \in \Sigma^*$ and any $k$-normalized expression $\varphi$, $\varphi(s) = s[l] \iff d_{\varepsilon,k}(\varphi) = \varepsilon_l, l \leq k$ and $\varphi(s) = \emptyset$ if $d_{\varepsilon,k}(\varphi) = \emptyset$.

Proof. By structural induction on $\varphi$.
- Case $\varepsilon$ and $\emptyset$ — By definitions.
- Case $!\alpha$ and $\alpha/\beta$ — Follow directly from the inductive hypothesis and definitions; note $\text{match}(\varepsilon) = \emptyset$.

- Case $\alpha\beta[\beta_1, \ldots, \beta_n]$ —
  - If $\alpha$ does not match $\varepsilon$, then $\alpha\beta(\varepsilon) = \emptyset$ and $d_{\varepsilon,k}(\alpha) = \emptyset$ by the inductive hypothesis and $!\beta$.
  - If $\alpha(\varepsilon) = s[j]$, then $\alpha(\beta(s[j]))$ and $d_{\varepsilon,k}(\alpha) = \varepsilon_j$.
    - If $j = k$, $d_{\varepsilon,k}(\varphi) = d_{\varepsilon,k}(\beta)$; by Theorem 4 (Normalization), $\beta(\varepsilon) = \beta(s[j])$ and by Lemma 10, $\beta(\varepsilon) = \beta(s[j])$ is $k$-normalized, therefore the inductive hypothesis applies.
    - If $j < k$, $d_{\varepsilon,k}(\varphi) = d_{\varepsilon,k}(\beta_j)$; $\beta_j(\varepsilon) = \beta(s[j]) [\text{Theorem 4 (Derivative Step)}]$. 

**Theorem 8** (Derivative Correctness). For any string $s = s_1s_2\cdots s_n \in \Sigma^*$ and well-formed, un-normalized expression $\varphi$, $\varphi(s) = s[l] \iff d_{\varepsilon_n}(\varphi) = \varepsilon_l$ and $\varphi(s) = \emptyset$ if $d_{\varepsilon_n}(\varphi) = \emptyset$, where $d_{\varepsilon_n}(\varphi) = d_{s_0,0}(\varphi)$.

Proof. By Theorem 4 (Normalization) $\varphi(s) = \langle \varphi \rangle_0(s)$ and by Lemma 10, $\langle \varphi \rangle_0$ is $0$-normalized, i.e. $\varphi(s) = d_{s_0,0}(\langle \varphi \rangle_0)$ by Theorem 4 (Derivative Step); the result follows from Theorem 7.
5 Analysis

In [12], Moss demonstrated the polynomial worst-case space and time of his algorithm with an argument based on bounds on the depth and fanout of the DAG formed by his derivative expressions. These bounds, cubic space and quartic time, were improved to constant space and linear time for a broad class of “well-behaved” inputs introduced by Moss in [12].

Given that the bound on \( b \) limits the fanout of the derivative expression DAG, a constant bound on the depth of that DAG implies that the overall size of the DAG is similarly constant-bounded. Intuitively, the bound on the depth of the DAG is a bound on recursive invocations of a nonterminal by itself, applying a sort of “tail-call optimization” for right-recursive invocations such as \( R_{n} := \alpha_{R} / \varepsilon \). The conjunction of both of these bounds defines the class of “well-behaved” PEG inputs introduced by Moss in [12], and by the constant bound on derivative DAG size this algorithm also runs in constant space and linear time on such inputs.

6 Experimental Results

In addition to being easier to explain and implement than the previous derivative parsing algorithm, the simplified parsing expression derivative presented here also has superior runtime performance.

To test this performance, the new simplified parsing expression derivative (SPED) algorithm was compared against the parser-combinator-based recursive descent (Rec.) and packrat (Pack.) parsers used in [12], as well as the parsing expression derivative (PED) implementation from that paper. The same set of XML, JSON, and Java inputs and grammars used in [12] are also used here. Code and test data are available online [13]. All tests were compiled with g++ 6.2.0 and run on a machine with 8GB of RAM, a dual-core 2.6 GHz processor, and SSD main storage.

Figure 1 shows the runtime of all four algorithms on all three data sets, plotted against the input size; Figure 2 shows the memory usage of the same runs, also plotted against the input size, but on a log-log
Contrary to its poor worst-case asymptotic performance, the recursive descent algorithm is actually best in practice, running most quickly on all tests, and using the least memory on all but the largest inputs (where the derivative parsing algorithms' ability to not buffer input gives them an edge). Packrat parsing is consistently slower than recursive descent, while using two orders of magnitude more memory. The two derivative parsing algorithms have significantly slower runtime performance, and memory usage closer to recursive descent than packrat.

Though on these well-behaved inputs all four algorithms run in linear time and space (constant space for the derivative parsing algorithms), the constant factor differs by both algorithm and grammar complexity. The XML and JSON grammars are of similar complexity, with 23 and 24 nonterminals, respectively, and all uses of lookahead expressions $\text{?}a$ and $\text{&?}a$ eliminated by judicious use of the more specialized negative character class, end-of-input, and until expressions described in [12]. It is consequently unsurprising that the parsers have similar runtime performance on those two grammars. By contrast, the Java grammar is significantly more complex, with 178 nonterminals and 54 lookahead expressions, and correspondingly poorer runtime performance.

Both the packrat algorithm and the derivative parsing algorithm presented here trade increased space usage for better runtime. Naturally, this trade-off works more in their favour for more complex grammars, particularly those with more lookahead expressions, as suggested by Moss [12]. Grouping the broadly equivalent XML and JSON tests together and comparing mean speedup, recursive descent is 3.3x as fast as packrat and 18x as fast as SPED on XML and JSON, yet only 1.6x as fast as packrat and 3.7x as fast as SPED for Java. Packrat’s runtime advantage over SPED also decreases from 5.5x to 2.3x between XML/JSON and Java. As can be seen from Figure 3, these speedup numbers are quite consistent across input sizes on a per-grammar basis.

Though the packrat algorithm is a modest constant factor faster than the derivative parsing algorithm across the test suite, it uses as much as 300x as much peak memory on the largest test cases, with the increases scaling linearly in the input size. Derivative parsing, by contrast, maintains a grammar-dependent constant memory usage across all the (well-behaved) inputs tested. This constant
memory usage is within a factor of two on either side of the memory usage of the recursive descent implementation on all the XML and JSON inputs tested, and 3-5x more on the more complex Java grammar. The higher memory usage on Java is likely due to the lookahead expressions, which are handled with runtime backtracking in recursive descent, but extra concurrently-processed expressions in derivative parsing.

Derivative parsing in general is known to have poor runtime performance\cite{10,1}, as these results also demonstrate. However, this new algorithm does provide a significant improvement on the current state of the art for parsing expression derivatives, with a 40% speedup on XML and JSON, a 50% speedup on Java, and an up to 13% decrease in memory usage. This improved performance may be beneficial for use cases that specifically require the derivative computation, such as the modular parsers of Brachthäuser et al.\cite{2} or the sentence generator of Garnock-Jones et al.\cite{6}.

7 Related Work

A number of other recognition algorithms for parsing expression grammars have been presented in the literature. Ford\cite{4} introduced both the PEG formalism and the recursive descent and packrat algorithms. Medeiros and Ierusalimschy\cite{9} have developed a parsing machine for PEGs, similar in concept to a recursive descent parser, but somewhat faster in practice. Mizushima et al.\cite{11} have demonstrated the use of manually-inserted “cut operators” to trim memory usage of packrat parsing to a constant, while maintaining the asymptotic worst-case bounds; Kuramitsu\cite{8} and Redziejowski\cite{14} have built modified packrat parsers which use heuristic table-trimming mechanisms to achieve similar real-world performance without manual grammar modifications, but which sacrifice the polynomial worst-case runtime of the packrat algorithm. Henglein and Rasmussen\cite{7} have proved linear worst-case time and space bounds for their progressive tabular parsing algorithm, with some evidence of constant space usage in practice for a simple JSON grammar, but their lack of empirical comparisons to other algorithms makes it difficult to judge the practical utility of their approach.

Brzozowski\cite{3} introduced derivative parsing for regular expressions; Might et al.\cite{10} extended this to context-free grammars. Adams et al.\cite{1} later improved the CFG derivative of Might et al., proving the same cubic worst-case time bound as shown here for PEG derivatives. Garnock-Jones et al.\cite{6} have posted a preprint of another derivative parsing algorithm for PEGs; their approach elegantly avoids defining new parsing expressions through use of a nullability combinator to represent lookahead followers as later alternatives of an alternation expression. However, unlike this work, their paper lacks a both a proof of correctness and empirical performance results.

Along with defining the PEG formalism, Ford showed a number of fundamental theoretical results, notably that PEGs can represent any $LR(k)$ language\cite{4}, and that it is undecidable in general whether the language of an arbitrary parsing expression is empty\cite{5}, and by corollary that the equivalence of two arbitrary parsing expressions is undecidable. These undecidability results constrain the precision of any nullability function for PEGs, proving to also be a limit on the functions $\nu$ and $\lambda$ de-
fined in this work. Redziejowski has also done significant theoretical work in analysis of parsing expression grammars, notably his adaptation of the FIRST and FOLLOW sets from classical parsing literature to PEGs[15].

8 Conclusion and Future Work

This paper has proven the correctness of a new derivative parsing algorithm for PEGs based on the previously published algorithm in [12]. Its key contributions are simplification of the earlier algorithm through use of global numeric indices for backtracking choices, a proof of algorithmic correctness formerly absent for all published PEG derivative algorithms, and empirical comparison of this new algorithm to previous work. The proof of correctness includes an extension of the concept of nullability from previous literature in derivative parsing[3][10] to PEGs, while respecting existing undecidability results for PEGs[5]. The simplified algorithm also improves the worst-case space and time bounds of the previous algorithm by a linear factor.

While extension of this recognition algorithm to a parsing algorithm remains future work, any such extension may rely on the fact that successfully recognized parsing expressions produce a $e_c$ expression in this algorithm, where $e$ is the index at which the last character was consumed. As one approach, $(\bullet)_b$ might annotate parsing expressions with $b$, the index at which they began to consume characters. By collecting subexpression matches and combining the two indices $b$ and $e$ on a successful match, this algorithm should be able to return a parse tree on match, rather than simply a recognition decision. The parser derivative approach of Might et al.[10] may be useful here, with the added simplification that PEGs, unlike CFGs, have no more than one valid parse tree, and thus do not need to store multiple possible parses in a single node.

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