The recurrence coefficients of semi-classical Laguerre polynomials and the fourth Painlevé equation

Galina Filipuk$^1$, Walter Van Assche$^2$ and Lun Zhang$^2$

$^1$ Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, Warsaw, 02-097, Poland
$^2$ Department of Mathematics, KU Leuven, Celestijnenlaan 200B box 2400, B-3001 Leuven, Belgium

E-mail: filipuk@mimuw.edu.pl, Walter.VanAssche@wis.kuleuven.be and lun.zhang@wis.kuleuven.be

Received 20 January 2012, in final form 23 March 2012
Published 27 April 2012
Online at stacks.iop.org/JPhysA/45/205201

Abstract

We show that the coefficients of the three-term recurrence relation for orthogonal polynomials with respect to a semi-classical extension of the Laguerre weight satisfy the fourth Painlevé equation when viewed as functions of one of the parameters in the weight. We compare different approaches to derive this result, namely, the ladder operators approach, the isomonodromy deformations approach and combining the Toda system for the recurrence coefficients with a discrete equation. We also discuss a relation between the recurrence coefficients for the Freud weight and the semi-classical Laguerre weight and show how it arises from the Bäcklund transformation of the fourth Painlevé equation.

PACS numbers: 02.30.Gp, 02.30.Hq
Mathematics Subject Classification: 34M55

1. Introduction and statement of the results

One of the most important properties of orthogonal polynomials is the three-term recurrence relation. Let $\mu$ be a positive measure on the real line for which all the moments

$$\mu_n = \int x^n \, d\mu(x)$$

exist. It is well known that there exists a sequence $(p_n)_{n \in \mathbb{N}}$ of orthonormal polynomials such that

$$\int p_n(x) p_k(x) \, d\mu(x) = \delta_{n,k},$$

where $\delta_{n,k}$ is the Kronecker delta and the integration is over the support $S \subset \mathbb{R}$ of the measure $\mu$. The three-term recurrence relation then takes the following form:

$$xp_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

where $a_n$ and $b_n$ are the recurrence coefficients.
with the recurrence coefficients given by the following integrals:

\[ a_n = \int x p_n(x) p_{n-1}(x) \, d\mu(x), \quad b_n = \int x^2 p_n^2(x) \, d\mu(x). \]  

(3)

Here it is assumed that \( p_{-1} = 0 \). The recurrence coefficients can also be expressed in terms of determinants containing the moments of the orthogonality measure; we refer to [12, 22, 36] for more information about orthogonal polynomials.

In this paper, we consider polynomials orthogonal on \( \mathbb{R}^+ \) with respect to the so-called semi-classical Laguerre weight, that is, \( d\mu(x) = w(x) \, dx \) and

\[ w(x) = w(x; t) = x^\alpha e^{-x^2 + tx}, \quad x \in \mathbb{R}^+, \]  

(4)

with \( \alpha > -1 \) and \( t \in \mathbb{R} \). Due to the dependence on the parameter \( t \) in the weight function, all the quantities like \( p_n(x) \) and recurrence coefficients \( a_n, b_n \) are actually functions of \( t \). The main result of this paper is as follows:

**Theorem 1.1.** Let \( b_n(t) \) be the recurrence coefficients in (2) associated with the weight function (4); then the function

\[ q_n(z) := 2b_n(2z) - 2z \]  

(5)

satisfies the fourth Painlevé equation

\[ q''_n = \frac{q_n^2}{2q_n} + \frac{3q_n^3}{2} + 4q_n^2 + 2(z^2 - A)q_n + \frac{B}{q_n} \]  

(6)

with parameters

\[ A = 1 + 2n + \alpha, \quad B = -2\alpha^2. \]  

(7)

The fourth Painlevé equation is among the six well-known Painlevé equations, whose solutions are often referred to as nonlinear special functions due to many important applications in mathematics and mathematical physics; cf [13, 20, 31].

We note that for classical orthogonal polynomials (Hermite, Laguerre, Jacobi) one knows these recurrence coefficients explicitly in contrast to non-classical weights. A useful characterization of classical polynomials is the Pearson equation

\[ \frac{d}{dx} (\sigma(x) w(x)) = \tau(x) w(x), \]

where \( \sigma \) and \( \tau \) are polynomials satisfying \( \deg \sigma \leq 2 \) and \( \deg \tau = 1 \). Semi-classical orthogonal polynomials are defined as orthogonal polynomials for which the weight function satisfies a Pearson equation with \( \deg \sigma > 2 \) or \( \deg \tau \neq 1 \). See Hendriksen and van Rossum [21] and Maroni [28]. One can easily check that our weight function defined in (4) corresponds to

\[ \sigma(x) = x, \quad \tau(x) = -2x^2 + tx + 1 + \alpha; \]

hence, we deal with semi-classical orthogonal polynomials.

One of the motivations of our study is the fact that the recurrence coefficients of semi-classical orthogonal polynomials are often related to the solutions of the Painlevé equations \( P_{II} - P_{VI} \). Below we include a few examples of such a connection indicating the weight, the Painlevé equation and the relevant reference:

- the weight \( e^{x^3/3 + tx} \) on \( \{ x : x^3 < 0 \} \) and \( P_{III} \) [27];
- the weight \( x^\alpha e^{-x^2} \) and \( P_{II} \) [10];
- the discrete Charlier weight \( w(k) = a^k/((\beta)k!) \), \( a > 0 \), and \( P_{III} \) (and \( P_{V} \)) [16];
- \( P_{IV} \) appears for the weights \( |x - t|^\nu e^{-x^2} \) in [8], \( x^0 e^{-x^2 + tx} \), \( x > 0 \), and \( |x|^{2\nu + 1} e^{-x^2 + tx} \) (in this paper);
the discrete Meixner weight \((γ)\), \(\frac{c^k}{k!}\), \(c, β, γ > 0\), and \(PV\) \([3, 15]\);

- \(PV\) for the weights \((1-ξθ(x-t))|^x|^r e^{-x}\), where θ is the Heaviside function in \([17]\), \((1+x)^β(1-x)^β e^{-x}\), \(x ∈ (-1, 1)\), in \([1]\);

- \(PV\) for the weights \((1-x)^β(A+Bθ(x-t)), x ∈ [0, 1]\), in \([11]\), \((1-x)^α(t-x)^γ, x ∈ [-1, 1]\), in \([27]\); see also \([18]\) for more examples and applications in random matrix theory.

In this paper, we will apply different approaches to prove theorem 1.1. In particular, we shall use the ladder operators approach, the isomonodromy deformations approach and the Toda system for the recurrence coefficients combined with a discrete equation derived in \([2]\). A similar comparison of the methods is given in \([17]\), where the recurrence coefficients are related to the solutions of the fifth Painlevé equation. Another main objective of the paper is to see how the properties of the orthogonal polynomials are related to the properties of transformations of the Painlevé equation. In particular, by using the Toda system we show that the discrete equation in \([2]\) can be obtained from a Bäcklund transformation of the fourth Painlevé equation. Finally, we shall deal with recurrence coefficients associated with the Freud weight and discuss their connection with the fourth Painlevé equation. We shall also revisit this connection from the point of view of the Bäcklund transformations.

2. Proof of theorem 1.1

To prove theorem 1.1, we need the following two theorems concerning the discrete equations and Toda system for the recurrence coefficients.

2.1. The discrete equations and Toda system

For the semi-classical Laguerre weight given in (4), we have the following discrete equations for the recurrence coefficients.

**Theorem 2.1** (see \([2, theorem 1.1]\)). Let \(a_n\) and \(b_n\) be the recurrence coefficients in (2) associated with the weight (4). Then the quantities

\[
x_n = \frac{\sqrt{2}}{t - 2b_n}, \quad y_n = 2a_n^2 - n - α
\]

satisfy

\[
x_{n-1}x_n = \frac{y_n + n + \frac{α}{2}}{y_n^2 - \frac{α^2}{4}}, \tag{8}
\]

\[
y_n + y_{n+1} = \frac{1}{x_n}\left(\frac{t}{\sqrt{2}} - \frac{1}{x_n}\right). \tag{9}
\]

It is shown in \([2]\) that the system (8)–(9) can be obtained from an asymmetric Painlevé dPIV equation by a limiting process. The proof is based on a Lax pair for the associated orthogonal polynomials. Later on we will show that this system can be obtained from a Bäcklund transformation of \(PV\).

Next, we know that if the positive measure is given by \(\exp(tx) dμ(x)\) on the real line with finite moments for all \(t ∈ \mathbb{R}\), where \(t\) is a real parameter (which is indeed our case), then the coefficients of the orthogonal polynomials depend on \(t\) and satisfy the Toda system \([29, 22, section 2.8, p 41]\) (see also \([3]\) for more details and a direct proof). More precisely,
Theorem 2.2. The recurrence coefficients $a_n(t)$ and $b_n(t)$ of monic polynomials which are orthogonal with respect to $\exp(tx)\,dx\,\mu(x)$ on the real line satisfy the Toda system

\begin{align}
\frac{d}{dt}(a_n^2) &= a_n^2(b_n - b_{n-1}) \\
\frac{d}{dt} b_n &= a_{n+1}^2 - a_n^2.
\end{align}

(10)

(11)

The initial conditions $a_n(0)$ and $b_n(0)$ correspond to the recurrence coefficients of the orthogonal polynomials for the measure $\mu$.

With these two theorems, we are ready to prove theorem 1.1.

2.2. Proof of theorem 1.1

From equation (8) we trivially get

\begin{equation}
x_{n-1} = -\frac{2(2y_n + 2n + \alpha)}{x_n(\alpha^2 - 4y_n^2)}.
\end{equation}

(12)

Inserting the expressions for $a_n^2$ and $b_n$ (in terms of $x_n$ and $y_n$) into equation (11) gives

\begin{equation}
y_{n+1} = \frac{x_n^2y_n + \sqrt{2}x_n^2}{x_n^2}.
\end{equation}

(13)

Substituting equation (13) into (9) gives

\begin{equation}
y_n = \frac{\sqrt{2}x_n - 2\sqrt{2}x_n - 2}{4x_n^2}.
\end{equation}

(14)

Inserting the expressions for $a_n^2$ and $b_n$ into equation (10) gives the following equation for $y_n'$:

\begin{equation}
y_n' = \frac{1}{\sqrt{2}} (y_n + n + \alpha/2) \left( \frac{1}{x_{n-1}} - \frac{1}{x_n} \right).
\end{equation}

(15)

Using (12) we can eliminate $x_{n-1}$ in (15) to get the following expression:

\begin{equation}
y_n' = \frac{-2(2n + \alpha + 2y_n) - x_n^2(\alpha^2 - 4y_n^2)}{4\sqrt{2}x_n}.
\end{equation}

(16)

which involves $y_n'$, $x_n$ and $y_n$. Similarly, equation (14) provides an expression for $x_n'$ in terms of $x_n$ and $y_n$. Differentiating this with respect to $t$ gives an expression for $x_n''$ in terms of $x_n$ and $y_n$ and their derivatives. Using (16) to eliminate $y_n'$, we get $x_n''$ in terms of $x_n$, $x_n'$ and $y_n$. Then using (14) to eliminate $y_n$, we finally get the second-order nonlinear ordinary differential equation for $x_n$ of the form

\begin{equation}
x_n'' = \frac{3}{2} x_n' + \frac{1}{4} \alpha^2 x_n^3 - \frac{x_n}{8} (t^2 - 4 - 8n - 4\alpha) + \frac{t}{\sqrt{2}} - \frac{3}{4x_n}.
\end{equation}

(17)

Finally, substituting

\begin{equation}
x_n(t) = -\frac{\sqrt{2}}{q_n(z)}, \quad t = 2z,
\end{equation}

(18)

into (17), we get the fourth Painlevé equation (6) with parameters (7).
3. Alternative proofs of theorem 1.1

3.1. Ladder operators approach

The ladder operators for orthogonal polynomials have been derived by many authors with a long history; we refer to [4, 6, 5, 9, 35] and the references therein for a quick guide. Nowadays the ladder operators approach has been successfully applied to show the connections of the Painlevé equations and recurrence coefficients of certain orthogonal polynomials; cf [7, 11, 14].

It is the aim of this section to give an alternative proof of theorem 2.1 by using this approach. This, combined with the arguments in section 2.2, will lead to another proof of theorem 1.1.

Preliminaries. Let \( P_n(x) \) be the monic orthogonal polynomials of degree \( n \) in \( x \) with the measure \( d\mu = w(x) \, dx \) on the real line, namely,
\[
P_n(x) = x^n + p_1(n)x^{n-1} + \cdots,
\]
(19)
such that
\[
\int P_m(x)P_n(x) \, d\mu(x) = h_n \delta_{m,n}, \quad h_n > 0, \quad m, n = 0, 1, 2, \ldots.
\]
(20)
The three-term recurrence relation now reads
\[
xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x),
\]
(21)
where
\[
\alpha_n = \frac{1}{h_n} \int xP_n^2(x) \, d\mu(x), \quad \beta_n = \frac{1}{h_{n-1}} \int xP_n(x)P_{n-1}(x) \, d\mu(x),
\]
(22)
and the initial condition is taken to be \( \beta_0 P_{-1} = 0 \).

For later use, we also list the following relations between monic orthogonal polynomials and normalized orthogonal polynomials without proof:
\[
p_n(x) = \frac{1}{\sqrt{h_n}} P_n(x),
\]
(23)
\[
\alpha_n = b_n,
\]
(24)
\[
a_n = \sqrt{\frac{h_n}{h_{n-1}}}, \quad \beta_n = a_n^2 = \frac{h_n}{h_{n-1}}.
\]
(25)
Here, we recall that \( \alpha_n \) and \( b_n \) are the recurrence coefficients associated with normalized orthogonal polynomials as shown in (2).

Assume that the weight function \( w \) vanishes at the endpoints of the orthogonality interval. Following the general set-up (cf [9]), the lowering and raising ladder operators for monic polynomials \( P_n(x) \) in (20) are given by
\[
\left( \frac{d}{dx} + B_n(x) \right) P_n(x) = \beta_n A_n(x) P_{n-1}(x),
\]
(26)
\[
\left( \frac{d}{dx} - B_n(x) - \upsilon(x) \right) P_{n-1}(x) = -A_{n-1}(x) P_n(x),
\]
(27)
with
\[
\upsilon(x) := -\ln w(x)
\]
and
\[ A_n(x) := \frac{1}{h_n} \int \frac{\varphi'(x) - \varphi'(y)}{x - y} [p_n(y)]^2 w(y) \, dy, \]
(28)
\[ B_n(x) := \frac{1}{h_{n-1}} \int \frac{\varphi'(x) - \varphi'(y)}{x - y} p_{n-1}(y) p_n(y) w(y) \, dy. \]
(29)

Note that \( A_n(x) \) and \( B_n(x) \) are not independent, but satisfy the following supplementary conditions [22, lemma 3.2.2 and theorem 3.2.4].

**Proposition 3.1.** The functions \( A_n(x) \) and \( B_n(x) \) defined by (28) and (29) satisfy
\[ B_{n+1}(z) + B_n(z) = (z - \alpha_n) A_n(z) - \varphi'(z), \]  
\( S_1 \)
\[ 1 + (z - \alpha_n)[B_{n+1}(z) - B_n(z)] = \beta_{n+1} A_{n+1}(z) - \beta_n A_{n-1}(z). \]  
\( S_2 \)

From \( S_1 \) and \( S_2 \), we can derive another identity involving \( \sum_{j=0}^{n-1} A_j \) which is often helpful:
\[ \beta_n A_n(x) A_{n-1}(x). \]  
\( S'_2 \)

The conditions \( S_1, S_2 \) and \( S'_2 \) are usually called the compatibility conditions for the ladder operators.

**Analysis of the ladder operators.** Now we shall apply the general set-up of the ladder operators to the polynomials orthogonal with respect to the weight (4).

For the weight function given in (4), we have
\[ \varphi(x) = -\ln w(x) = -\alpha \ln x + x^2 - tx; \]

hence,
\[ \frac{\varphi'(x) - \varphi'(y)}{x - y} = 2 + \frac{\alpha}{xy}. \]

It then follows from (28) and (29) that, if \( \alpha > 0 \),
\[ A_n(x) = 2 + \frac{R_n}{x}, \quad B_n(x) = \frac{r_n}{x}, \]
(30)
where
\[ R_n = \frac{\alpha}{h_n} \int_0^\infty p_n(y)^2 y^{\alpha-1} e^{-y^2+ty} \, dy \]
(31)
and
\[ r_n = \frac{\alpha}{h_{n-1}} \int_0^\infty p_{n-1}(y) p_n(y)^2 y^{\alpha-1} e^{-y^2+ty} \, dy. \]
(32)

Substituting (30) into \( S_1 \) and comparing the coefficients of \( x^0 \) and \( x^{-1} \), we have
\[ R_n - 2\alpha_n + t = 0, \]
(33)
\[ r_n + r_{n+1} = \alpha - \alpha_n R_n. \]
(34)
From \( S_2 \) we similarly get two more conditions:
\[ 1 + r_{n+1} - r_n = 2(\beta_{n+1} - \beta_n). \]
(35)
\[ \alpha_n(r_n - r_{n+1}) = \beta_{n+1}R_{n+1} - \beta_n R_{n-1}. \]  
(36)

Finally, relation (S') gives
\[ r_n + n = 2\beta_n, \]
(37)
\[ \sum_{j=0}^{n-1} R_j - tr_n = 2\beta_n(R_{n-1} + R_n), \]
(38)
\[ r_n^2 - \alpha r_n = \beta_n R_{n-1} R_n. \]
(39)

In particular, it follows from (33) and (37) that
\[ \alpha_n = \frac{R_n + t}{2}, \quad \beta_n = \frac{r_n + n}{2}. \]
(40)

It is clear that (35) is automatically satisfied using (40).

With the above preparations, we are ready to prove theorem 2.1.

**Proof of theorem 2.1.** Recall that \( y_n = 2\alpha_n^2 - n - \alpha/2 \). Using (25), it follows that
\[ y_n = 2\beta_n - n - \alpha/2. \]
(41)

On the other hand, since \( x_n = \frac{\sqrt{\tau}}{r_{2n}}, \) we see from (24) and the first equality in (40) that
\[ x_n = -\frac{\sqrt{\tau}}{R_n}. \]
(42)

Replacing \( x_{n-1}, x_n \) and \( y_n \) in (8) by \( R_{n-1}, R_n \) and \( \beta_n \), respectively, with the help of (42) and (41), it is equivalent to show that
\[ \beta_n R_{n-1} R_n = \left( 2\beta_n - n - \frac{\alpha}{2} \right)^2 - \frac{\alpha^2}{4}. \]
(43)

On account of (39), it is essential to prove
\[ \left( 2\beta_n - n - \frac{\alpha}{2} \right)^2 = r_n^2 - \alpha r_n, \]
(44)

which is immediate by (37).

To show (9), we note that, again with the help of (42) and (41), it suffices to show that
\[ 2(\beta_n + \beta_{n+1}) - 2n - 1 - \alpha = -\frac{1}{2} R_n(t + R_n). \]
(45)

From (37) and (34), it follows that
\[ 2(\beta_n + \beta_{n+1}) - 2n - 1 - \alpha = r_n + r_{n+1} - \alpha = -\alpha_n R_n. \]
(46)

This, together with (33), implies (45).

\[ \square \]

3.2. Isomonodromy deformations approach

Another easy method to show the connection of the recurrence coefficients of the semiclassical Laguerre polynomials with the solutions of the fourth Painlevé equation is by using the isomonodromy deformations approach. The fourth Painlevé equation appears as a result of the compatibility condition \( Y_{xt} = Y_{tx} \) of two linear \( 2 \times 2 \) systems \( Y_x = A(x)Y \) and \( Y_t = B(x)Y \), where the subscript denotes the partial derivative \([23]\). Here

\[
A(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} T & U \\ -Z + \theta_0 & -UV/2 \end{pmatrix},
\]
(47)
where $V = V(T)$, $U = U(T)$, $Z = Z(T)$ and $V(T)$ satisfies the fourth Painlevé equation (6) with

$$A = 2\theta_\infty - 1, \quad B = -8\theta_0^2. \quad (48)$$

Substituting (30) into (27) and (26) (in this order) we get the following linear system:

$$\frac{d}{dx} \begin{pmatrix} p_{n-1}(x) \\ p_n(x) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2\beta_n & -2 \end{pmatrix} x + \begin{pmatrix} -t & -2 \\ R_n\beta_n & -r_n \end{pmatrix} \begin{pmatrix} p_{n-1}(x) \\ p_n(x) \end{pmatrix}. \quad (49)$$

By replacing the vector $(p_{n-1}, p_n)^T$ by $e^{i(x-1)/2}x^{-u/2}(p_{n-1}, p_n)^T$, we get a matrix similar to (47). Hence, we can calculate that

$$U = -2, \quad T = -t/2, \quad V = -R_{n-1}, \quad Z = \theta_0 + \theta_\infty - 2\beta_n, \quad \beta_n = (2\theta_\infty - \alpha + 2r_n)/4, \quad \theta_\infty = (\alpha + 2n)/2.$$ 

Using (33), (37) and (39), we get

$$\lambda = q(\lambda t)$$

is a solution of PIV with parameters $\alpha$, $\beta$ and $\lambda^2 = 1$, then $\lambda^{-1}q(\lambda t)$ is a solution of PIV with parameters $\lambda^2\alpha$ and $\beta$. Hence, the parameters are in agreement with (7) (with $n$ replaced by $n - 1$).

We note that an explicit link between the isomonodromic problem and the Toda system was found in [24].

4. Bäcklund transformations

In this section, we show how to obtain the system (8) and (9) from a Bäcklund transformation of the fourth Painlevé equation. It is known that the fourth Painlevé equation PIV admits a Bäcklund transformation [20]. If $q = q(z)$ is a solution of PIV with parameters $A$ and $B$, then the function

$$\tilde{q} = T_{\epsilon, \mu}q = \frac{q' - \mu q^2 - 2\mu z q - \epsilon \sqrt{-2R}}{2\mu q}$$

is a solution of PIV with new values of the parameters

$$\tilde{A} = \frac{1}{2}(2\mu - 2A + 3\mu \epsilon \sqrt{-2R}), \quad \tilde{B} = -\frac{1}{2}(1 + A\mu + \frac{1}{2}\epsilon \sqrt{-2R})^2,$$

where $\epsilon^2 = \mu^2 = 1$.

It is the aim of this section to prove

Theorem 4.1. The discrete system (8) and (9) for the recurrence coefficients of semi-classical Laguerre polynomials can be obtained from a Bäcklund transformation of the fourth Painlevé equation PIV.

Proof. First we need a nonlinear relation for $x_{n-1}$, $x_n$ and $x_{n+1}$. From the second equation of the system (8) and (9) we get

$$y_{n+1} = \frac{\sqrt{2}t x_n - 2x_n^2 y_n - 2}{2x_n^2}. \quad (50)$$

Using the first equation of this system for the indices $n$ and $n + 1$, we can eliminate $y_n$ by calculating the resultant and obtain a nonlinear relation for $x_{n-1}$, $x_n$ and $x_{n+1}$. Let us denote this cumbersome expression by $E$ for future reference. Clearly, we can also find an expression between $q_n$ and $q_{n+1}$ by using (18).
One can verify directly that, for instance, the compositions \( T_{1,1} \circ T_{1,-1} \circ T_{1,1} \) and \( T_{1,-1} \circ T_{1,1} \circ T_{1,-1} \) give rise to the following transformations. Let \( q = q_n(z) \) be a solution of PIV with (7); then

\[
q_{n+1}(z) = \frac{(2\alpha + 2zq + q^2 - 4q' - q')(2\alpha - 2zq - q^2 + 4q')}{2q(q^2 + 2zq - q' - 4 - 4n - 2\alpha)}
\]

is a solution of PIV with \( A = 3 + 2n + \alpha \) and \( B = -2\alpha^2 \). Similarly,

\[
q_{n-1}(z) = -\frac{(q' + q^2 + 2zq - 2\alpha)(q' + q^2 + 2zq + 2\alpha)}{2q(q' + q^2 + 2zq - 2(2n + \alpha))}
\]

is a solution of PIV with \( A = 2n + \alpha - 1 \) and \( B = -2\alpha^2 \). After substituting these expressions for \( q_{n \pm 1} \) into the nonlinear recurrence relation \( E \), we indeed find that this is identically zero. This completes the proof of our theorem. \( \square \)

**Remark 4.2.** For our purposes it is sufficient to use the standard Bäcklund transformations of the Painlevé transcendents which are given in NIST Digital Library of Mathematical Functions (DLMF project). There are also algebraic aspects of the Painlevé equations. It is known [31, 33] that the Bäcklund transformations of the fourth Painlevé equation form the affine Weyl group of \( A_2^{(1)} \) type. The interested reader can easily re-formulate our transformations within the framework of Noumi–Yamada’s birational representation of \( \tilde{W}(A_2^{(1)}) \) (see [33]).

## 5. Initial conditions of the recurrence coefficients and classical solutions of the fourth Painlevé equation

It is known [20] that the fourth Painlevé equation admits classical solutions as follows. Let us take \( n = 0 \); then PIV with parameters (7) for the function \( q = q_n(z) \) has solutions which satisfy the following Riccati equation:

\[
q' + q^2 + 2zq - 2\alpha = 0.
\]

This equation can further be reduced to the Weber–Hermite equation. Using the change of variables (18) we can calculate that

\[
x_0(t) = \frac{\sqrt{\gamma} \mu_0}{t \mu_0 - 2 \mu_1}
\]

satisfies equation (50). Here \( \mu_k \) is the \( k \)th moment \( \int x^k \, d\mu(x) \). \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Moreover, using (14) we get the initial value \( y_0 = -\alpha/2 \) which coincides with the initial values given in [2]. Thus, we have shown that the initial conditions correspond to classical solutions of the fourth Painlevé equation.

The recurrence coefficients \( a_n^2 \) and \( b_n \) can always be written [12] as ratios of Hankel determinants containing the moments of the orthogonality measure (see also [3]). However, the explicit determinant formulas of classical solutions for all the Painlevé equations are known. We refer the reader to [25, 30, 33, 32] and the references therein for a classification and explicit determinant formulas for classical solutions (classical transcendental and rational) of the fourth Painlevé equation.

## 6. The Freud weight

In this section, we will study the relation between the recurrence coefficients of the semiclassical Laguerre weight and the Freud weight, and show how they are related via the Bäcklund transformation.

3 http://dlmf.nist.gov/32.7
6.1. The recurrence coefficients of the Freud weight

The Freud weight [19, 26, 37] is given by
\[ w_\alpha(x) = |x|^{2\alpha+1} \exp(-x^4 + tx^2), \quad x \in \mathbb{R}, \quad \alpha > -1. \]

The orthonormal polynomials \( \{\tilde{P}_n\} \) with respect to the weight \( w_\alpha \) satisfy
\[ x\tilde{P}_n(x) = A_{n+1}\tilde{P}_{n+1}(x) + A_n\tilde{P}_{n-1}(x). \]

In the case of the monic polynomials \( \{P_n\} \) the recurrence relation is given by
\[ xP_n(x) = \tilde{P}_{n+1}(x) + A_n^2\tilde{P}_{n-1}(x). \]

It is known [19, 26, 37] that the recurrence coefficients satisfy the following second-order nonlinear difference equation,
\[ 4A_n^2(A_{n-1}^2 + A_n^2 + A_{n+1}^2 - t/2) = n + (2\alpha + 1)\Delta_n, \quad \Delta_n = (1 - (-1)^n)/2, \] (51)

which is the first discrete Painlevé equation dP1. The bilinear structure and exact (Casorati determinant) solutions of the (extended) dP1 equation are studied in [34]. The differential equation, which can be derived similarly to the Toda system, is given by
\[ \frac{d}{dt}A_n^2 = A_n^2(A_{n+1}^2 - A_{n-1}^2). \] (52)

**Theorem 6.1.** The function \( f(z) = -2A_n^2(2z) \) satisfies the fourth Painlevé equation (6) with parameters given by
\[ A = -\frac{1}{2}(2 + n + 4\alpha), \quad B = -\frac{n^2}{2}, \]
in case \( n \) is even and
\[ A = \frac{1}{2} - \frac{n}{2} + \alpha, \quad B = -\frac{1}{2}(1 + n + 2\alpha)^2, \]
in case \( n \) is odd.

**Proof.** Although this result is well known (see, e.g., [27]), we briefly show how to derive it in order to be self-contained. For simplicity we introduce the notation \( f_n(t) = A_n^2(t) \). From (51), with \( n \) and \( n - 1 \), we can find \( f_{n-1} \) and \( f_{n-2} \). From (52) we can find \( f_{n+1} \). Substituting these expressions into (52) with \( n - 1 \) we get a second-order differential equation for the function \( f_n(t) \). Introducing a new independent variable \( t = 2z \) and changing \( f(z) = -2f_n(t) \) we get the fourth Painlevé equation for the function \( f(z) \) with the required parameters. \( \square \)

6.2. Connection with the recurrence coefficients of semi-classical Laguerre polynomials and the Bäcklund transformations

For the semi-classical Laguerre weight from theorem 2.1
\[ w_\alpha(x) = x^\alpha e^{-x^2+tx}, \quad x > 0, \quad \alpha > -1, \]

the orthonormal polynomials \( \{p_n^\alpha\} \) satisfy
\[ xp_n^\alpha(x) = a_n^\alpha p_{n+1}^\alpha(x) + b_n^\alpha p_n^\alpha(x) + a_n^\alpha p_{n-1}^\alpha(x). \]

Here we use the notation in [2]. It is known [12] that the polynomials for the Freud weight and the semi-classical Laguerre weight are related by
\[ \tilde{p}_{2n}(x) = p_n^\alpha(x^2), \quad \tilde{p}_{2n+1}(x) = xp_n^\alpha(x^2). \]
and the following relations hold for the recurrence coefficients:

\[
\begin{align*}
da_n' &= A_{2n}a_{2n-1}, \\
b_n' &= A_{2n}^2 + A_{2n+1}^2, \\
da_n^{a+1} &= A_{2n}a_{2n+1}, \\
b_n^{a+1} &= A_{2n+2}^2 + A_{2n+1}^2.
\end{align*}
\] (53)

Since both \(A_{2n}^2\) and \(b_n^a\) in (53) and (54) satisfy the fourth Painlevé equation, we are interested in revisiting these formulas from the point of view of the Bäcklund transformation \(T_{\varepsilon,\mu}\).

Let us consider case (53). On the one hand, as shown before, from (18) we find that

\[
q_n(z) = -2z + 2b_n(t), \quad t = 2z,
\] (55)
satisfies the fourth Painlevé equation with (7). On the other hand, let us look more carefully at the right-hand side of (53). We have

\[
b_n(t) = A_{2n}^2(t) + A_{2n+1}^2(t) = f_2(t) + f_{2n+1}(t).
\] (56)

Denoting

\[
g_1(z) = -2f_{2n}(2z), \quad g_2(z) = -2f_{2n+1}(2z),
\]
we have that \(g_1(z)\) satisfies the fourth Painlevé equation with parameters

\[
A = -1 - n - 2\alpha, \quad B = -2n^2.
\]
The function \(g_2(z)\) is also a solution of the fourth Painlevé equation with parameters

\[
A = -n + \alpha, \quad B = -2(1 + n + \alpha)^2,
\]
and

\[
g_2(z) = T_{1,-1}g_1(z) = \frac{2n - 2zg_1 - g_1' - g_1'^{\prime}}{2g_1}.
\]

This, together with (55) and (56), implies that

\[
q_n(z) = -2z - g_1(z) - g_2(z) = \frac{g_1' - g_1 - 2zg_1 - 2n}{2g_1} = T_{1,1}g_1(z)
\]
satisfies the fourth Painlevé equation with (7). This is in agreement with the previous result. Thus, formula (53) can be viewed as a Bäcklund transformation for the solutions of the fourth Painlevé equation.

By analogy, in case (54) we have that if \(g_1(z)\) satisfies the fourth Painlevé equation with

\[
A = -2 - n - 2\alpha, \quad B = -2(n + 1)^2,
\]
the function

\[
g_2(z) = T_{-1,1}g_1(z) = \frac{2n + 2zg_1 - g_1^2 + g_1'^{\prime}}{2g_1}
\]
satisfies the fourth Painlevé equation with

\[
A = \alpha - n, \quad B = -2(n + 1 + \alpha)^2,
\]
then

\[
q_n(z) = -2z - g_1(z) - g_2(z) = T_{-1,-1}g_1(z)
\]
is a solution of the fourth Painlevé equation with

\[
A = 2 + 2n + \alpha, \quad B = -2(\alpha + 1)^2,
\]
that is, with parameters (7) where \(\alpha\) is replaced by \(\alpha + 1\).
Acknowledgments

Part of this work was carried out while GF was visiting KU Leuven for one month. The financial support of KU Leuven, MIMUW at the University of Warsaw and the hospitality of the Analysis section at KU Leuven is gratefully acknowledged. GF is also partially supported by Polish MNiSzW Grant N N201 397937. WVA is supported by Belgian Interuniversity Attraction Pole P6/02, FWO grant G.0427.09 and KU Leuven Research Grant OT/08/033. LZ is a Postdoctoral Fellow of the Fund for Scientific Research – Flanders (FWO), Belgium. The authors are grateful to one referee for suggesting [24] and, in particular, to the other one for providing a lot of useful and helpful suggestions which substantially improved the presentation of this paper.

References

[1] Basor E, Chen Y and Ehrhardt T 2010 Painlevé V and time-dependent Jacobi polynomials J. Phys. A: Math. Theor. 43 015204
[2] Boelen L and Van Assche W 2010 Discrete Painlevé equations for recurrence coefficients of semiclassical Laguerre polynomials Proc. Am. Math. Soc. 138 1317–31
[3] Boelen L, Filipuk G and Van Assche W 2011 Recurrence coefficients of generalized Meixner polynomials and Painlevé equations J. Phys. A: Math. Theor. 44 035202
[4] Bonan S and Clark D S 1986 Estimates of the orthogonal polynomials with weight $\exp(-s^n)$, $n$ an even positive integer J. Approx. Theory 46 408–10
[5] Bonan S and Nevai P 1984 Orthogonal polynomials and their derivatives: I J. Approx. Theory 40 134–47
[6] Bonan S, Lubinsky D S and Nevai P 1987 Orthogonal polynomials and their derivatives: II SIAM J. Math. Anal. 18 1163–76
[7] Chen Y and Dai D 2010 Painlevé V and a Pollaczek–Jacobi type polynomials J. Approx. Theory 162 2149–67
[8] Chen Y and Feigin M V 2006 Painlevé IV and degenerate Gaussian unitary ensembles J. Phys. A: Math. Gen. 39 12381–93
[9] Chen Y and Ismail M 2005 Jacobi polynomials from compatibility conditions Proc. Am. Math. Soc. 133 465–72
[10] Chen Y and Its A 2010 Painlevé III and a singular linear statistics in Hermitian random matrix ensembles: I J. Approx. Theory 162 270–97
[11] Chen Y and Zhang L 2010 Painlevé VI and the unitary Jacobi ensembles Stud. Appl. Math. 125 91–112
[12] Chihara T S 1978 An Introduction to Orthogonal Polynomials (New York: Gordon and Breach)
[13] Clarkson P A 2006 Painlevé equations—nonlinear special functions Orthogonal Polynomials and Special Functions: Computation and Applications (Lecture Notes in Mathematics vol 1883) (Berlin: Springer) pp 331–411
[14] Dai D and Zhang L 2010 Painlevé VI and Hankel determinants for the generalized Jacobi weight J. Phys. A: Math. Theor. 43 055207
[15] Filipuk G and Van Assche W 2011 Recurrence coefficients of a new generalization of the Meixner polynomials SIGMA 7 068
[16] Filipuk G and Van Assche W 2012 Recurrence coefficients of generalized Charlier polynomials and the fifth Painlevé equation Proc. Am. Math. Soc. at press (arXiv:1106.2959)
[17] Forrester P J and Ormerod C M 2010 Differential equations for deformed Laguerre polynomials J. Approx. Theory 162 653–77
[18] Forrester P J and Witte N S 2006 Random matrix theory and the sixth Painlevé equation J. Phys. A: Math. Gen. 39 12211–33
[19] Freud G 1976 On the coefficients in the recurrence formulae of orthogonal polynomials Proc. R. Irish Acad. Sect. A 76 1–6
[20] Gromak V I, Laine I and Shimomura S 2002 Painlevé Differential Equations in the Complex Plane Studies in Mathematics vol 28 (Berlin: de Gruyter)
[21] Hendriksen E and Van Rossum H 1985 Semi-classical orthogonal polynomials Polynomes Orthogonaux et Applications (Lecture Notes in Mathematics vol 1171) (Berlin: Springer) pp 354–61
[22] Ismail M E H 2005 Classical and quantum orthogonal polynomials in one variable Encyclopedia of Mathematics and its Applications vol 98 (Cambridge: Cambridge University Press)
[23] Jimbo M and Miwa T 1981 Monodromy preserving deformations of linear ODEs with rational coefficients: II Physica D 2 407–48
[24] Joshi N, Kajiwara K and Mazzocco M 2006 Generating function associated with the determinant formula for the solutions of the Painlevé IV equation Funkcial. Ekvac. 49 451–68
[25] Kajiwara K and Ohta Y 1998 Determinant structure of the rational solutions for the Painlevé IV equation J. Phys. A: Math. Gen. 31 2431–46
[26] Magnus A P 1999 Freud’s equations for orthogonal polynomials as discrete Painlevé equations Symmetries and Integrability of Difference Equations (London Mathematical Society Lecture Note Series vol 225) (Cambridge: Cambridge University Press) pp 228–43
[27] Magnus A P 1995 Painlevé type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials J. Comput. Appl. Math. 57 215–37
[28] Maroni P 1987 Prolégomènes à l’étude des polynômes orthogonaux semi-classiques Ann. Mat. Pura Appl. 149 165–84
[29] Moser J 1975 Finitely many mass points on the line under the influence of an exponential potential—an integrable system Dynamical Systems, Theory and Applications (Lecture Notes in Physics vol 38) (Berlin: Springer) pp 469–97
[30] Murata Y 1985 Rational solutions of the second and the fourth Painlevé equations Funkcial. Ekvac. 28 1–32
[31] Noumi M 2004 Painlevé Equations Through Symmetry (Translations of Mathematical Monographs vol 223) (Providence, RI: American Mathematical Society)
[32] Noumi M and Okamoto K 1997 Irreducibility of the second and fourth Painlevé equations Funkcial. Ekvac. 40 139–63
[33] Noumi M and Yamada Y 1999 Symmetries in the fourth Painlevé equation and Okamoto polynomials Nagoya Math. J. 153 53–86
[34] Ohta Y, Kajiwara K and Satsuma J 1996 Bilinear structure and exact solutions of the discrete Painlevé I equation Symmetries and Integrability of Difference Equations (Proceedings and Lecture Notes vol 9) (Providence, RI: American Mathematical Society) pp 265–68
[35] Shohat J 1939 A differential equation for orthogonal polynomials Duke Math. J. 5 401–17
[36] Szegő G 1975 Orthogonal Polynomials 4th edn (AMS Colloquium Publications vol 23) (Providence, RI: American Mathematical Society)
[37] Van Assche W 2007 Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials Discrete Equations, Special Functions and Orthogonal Polynomials ed Elaydi S et al (Singapore: World Scientific) pp 687–725