VIEWS ON LEVEL $\ell$ CURVES, K3 SURFACES AND FANO THREEFOLDS

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Abstract. An analogue of the Mukai map $m_g : \mathcal{P}_g \to \mathcal{M}_g$ is studied for the moduli $\mathcal{R}_{g,\ell}$ of genus $g$ curves $C$ with a level $\ell$ structure. Let $\mathcal{P}^\perp_{g,\ell}$ be the moduli space of 4-tuples $(S,\mathcal{L},C)\subset S$ so that $(S,\mathcal{L})$ is a polarized K3 surface of genus $g$, $\mathcal{E}$ is orthogonal to $\mathcal{L}$ in $\text{Pic}(S)$ and defines a standard degree $\ell$ K3 cyclic cover of $S$, $C\subset |\mathcal{L}|$. We say that $(S,\mathcal{L},\mathcal{E})$ is a level $\ell$ K3 surface. These exist for $\ell \leq 8$ and their families are known. We define a level $\ell$ Mukai map $r_{g,\ell} : \mathcal{P}^\perp_{g,\ell} \to \mathcal{R}_{g,\ell}$, induced by the assignment of $(S,\mathcal{L},\mathcal{E},C)$ to $(C,\mathcal{E}\otimes \mathcal{O}_C)$. We investigate a curious possible analogy between $m_g$ and $r_{g,\ell}$, that is, the failure of the maximal rank of $r_{g,\ell}$ for $g = g_\ell \pm 1$, where $g_\ell$ is the value of $g$ such that $\dim \mathcal{R}^\perp_{g,\ell} = \dim \mathcal{R}_{g,\ell}$. This is proven here for $\ell = 3$. As a related open problem we discuss Fano threefolds whose hyperplane sections are level $\ell$ K3 surfaces and their classification.

1. Introduction

Our aim is to convince the reader, showing a program and new results, of the interest represented by some complex projective varieties whose curvilinear sections are canonical curves $C$ of genus $g$, endowed with a distinguished nonzero $\ell$-torsion element $\eta \in \text{Pic}(C)$. Often one says that $(C,\eta)$ is a level $\ell$ curve of genus $g$, cfr. [7]. Fixing $(g,\ell)$ the moduli space of these pairs is integral, quasi projective and denoted by $\mathcal{R}_{g,\ell}$.

To enter further in the matter let us mention two other names from the title: K3 surface and Fano threefold. The K3 surfaces $S$ we consider are very special: they admit a non split cyclic cover of degree $\ell$, still birational to a K3 surface. This is defined by a line bundle $\mathcal{O}_S(\mathcal{E}) := \mathcal{E}$ such that $h^0(\mathcal{O}_S(\ell\mathcal{E})) = 1$ and $h^0(\mathcal{O}_S(m\mathcal{E})) = 0$ for $m < \ell$. The study of these surfaces stems from Nikulin’s classification of K3 surfaces with an order $\ell$ symplectic automorphism and the classification implies $\ell \leq 8$, [22]. Since then several foundational results, in use here, did follow, cfr. [13] [14] [15] [16] [28].

Now let $\mathcal{L} \subset \text{Pic}(S)$ be a genus $g$ polarization orthogonal to $\mathcal{E}$. Let $\eta := \mathcal{O}_C(\mathcal{E})$, where $C \subset |\mathcal{L}|$ is smooth, then it turns out that $(C,\eta)$ is a level $\ell$ curve. We say that the triple $(S,\mathcal{L},\mathcal{E})$ is a level $\ell$ K3 surface of genus $g$, see definition [31] for some precision. Fixing $\ell$ the moduli of these triples are reducible for infinitely many values of $g$. However a distinguished irreducible component exists for every $g$, namely the moduli space of triples $(S,\mathcal{L},\mathcal{E})$ such that $\text{Pic}(S)$ is the sum of $\mathbb{Z}\mathcal{L}$ and its orthogonal lattice. We denote it by

$$\mathcal{F}^\perp_{g,\ell}.$$ 

Finally we come to the moduli space $\mathcal{P}^\perp_{g,\ell}$ of 4-tuples $(S,\mathcal{L},C)\subset S$ such that $C \subset |\mathcal{L}|$ and $(S,\mathcal{L},\mathcal{E})$ defines a point in $\mathcal{F}^\perp_{g,\ell}$. Such a space is strictly related with the first topic considered in our paper. 

To introduce it let us define the level $\ell$ Mukai map. This is the rational map

$$r_{g,\ell} : \mathcal{P}^\perp_{g,\ell} \to \mathcal{R}_{g,\ell},$$

assigning the moduli point of the 4-tuple $(S,\mathcal{L},\mathcal{E},C)$ to the moduli point of the pair $(C,\eta)$, where $\eta$ is $\mathcal{O}_C(\mathcal{E})$. Let $\mathcal{P}_g$ be the moduli space of triples $(S,\mathcal{L},C)$, where $(S,\mathcal{L})$ is a polarized K3 surface of genus $g$ and $C \subset |\mathcal{L}|$, then the previous name is motivated by the well known Mukai map

$$m_g : \mathcal{P}_g \to \mathcal{M}_g,$$

assigning the moduli point of the triple $(S,\mathcal{L},C)$ to the moduli point of the curve $C$. Some famous connections between canonical curves of genus $g$, K3 surfaces and Fano threefolds are well represented by $m_g$ and, in particular, by a curious variation of its rank. We recall that a rational map $f : X \to Y$ of integral varieties has maximal rank if $\dim f(X) = \min\{\dim X, \dim Y\}$.

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Considering $m_g$ we recall that $\dim \mathcal{P}_g = 19 + g$ and $\dim \mathcal{M}_g = 3g - 3$, therefore $\dim \mathcal{P}_g = \dim \mathcal{M}_g$ iff $g = 11$. Now $m_{11}$ is birational but, curiously, $m_g$ fails to be of maximal rank precisely before and after this transition value, that is, for $g = 11 \pm 1$. For the rest $m_g$ is dominant for $g \leq 9$ and generically injective for $g \geq 13$. As is well known this anomaly is due to the presence behind the scene of some Fano varieties, whose curvilinear sections are general canonical curves of genus $11 \pm 1$, cf. [8, 23, 24, 25].

A task of this paper is to point out the same possible anomalies for the level $\ell$ Mukai maps $r_{g,\ell}$. The case $\ell = 2$ has already been done and it is an experimental origin to this work. If $\ell = 2$ we have $\dim \mathcal{P}_{g,2} = \dim \mathcal{R}_{g,2}$ for $g = 7$. Then $r_{g,2}$ fails to be of maximal rank for $g = 7 \pm 1$ and is birational for $g = 7$, [11, 19, 26]. The 'Fano varieties behind the scene' for $g = 8$ and $g = 6$ are addressed or revisited in section 7.

In section 5 we summarize the question for each $\ell$. Let $g_\ell$ be the unique value of $g$ such that $\dim \mathcal{P}_{g,\ell} = \dim \mathcal{R}_{g,\ell}$, for $l = 2$, 3, 4, 5, 6, 7, 8 we respectively have:

\begin{equation}
(4) 
 g_\ell = 7, 5, 4, 3, 2, 2, 2.
\end{equation}

In this paper we present the following theorem, solving the question for $\ell = 3$.

**Theorem 1.1.** Let $r_{g,3} : \mathcal{P}^+_{g,3} \to \mathcal{R}_{g,3}$ be the level 3 Mukai map then:

1. $r_{4,3}$ has not maximal rank,
2. $r_{5,3}$ is birational,
3. $r_{6,3}$ has not maximal rank.

The image of $r_{4,3}$ is contained in a divisor of $\mathcal{R}_{4,3}$, parametrizing pairs $(C, \eta)$ such that the multiplication map $\mu : H^0(\omega_C \otimes \eta) \otimes H^0(\omega_C \otimes \eta^{-1}) \to H^0(\omega_C^2)$ is not an isomorphism. This case seems interestingly related to the $G_2$-variety, see [23] and section 7.

The proof of (3) is sketched here and it will appear elsewhere. The image of $r_{6,3}$ parametrizes pairs $(C, \eta)$, where $C$ is a curvilinear section of a suitable Gushel-Mukai threefold singular along a rational normal sextic curve, see section 7.

Let $(S, \mathcal{L}, \mathcal{E})$ be a level $\ell$ K3 surface of genus $g$ and $\phi : S \to \mathbb{P}^g$ the morphism defined by $\mathcal{L}$, we assume for simplicity that $\phi$ is birational onto $\mathfrak{S} := \phi(S)$. Then we close this introduction with few lines addressing the classification of Fano threefolds

$$
\mathfrak{X} \subset \mathbb{P}^{g+1}
$$

whose general hyperplane sections are projective models $\mathfrak{X}$ as above. The problem sounds similar to that of classifying threefolds $T \subset \mathbb{P}^g$ whose hyperplane sections are Enriques surfaces, that is, Enriques-Fano threefolds. It seems however quite neglected.

Some examples of threefolds $\mathfrak{X}$ appear in this paper, most are normal and $\text{Sing} \mathfrak{X}$ is a curve. Moreover $\mathfrak{X}$ admits a cyclic cover $\pi : \tilde{X} \to \mathfrak{X}$, branched exactly on $\text{Sing} \mathfrak{X}$. A basic notion of level $\ell$ polarized projective variety $(X, \mathcal{L}, \mathcal{E})$ is introduced in the next section, since it is useful in the cases we want to consider.

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We are happy of contributing to this volume, celebrating Professor Fabrizio Catanese on the occasion of his Seventies. Let us wish to him abundance in mathematics and life as always.

2. **Some preliminaries**

In what follows $X$ is a smooth, irreducible complex projective variety and $\mathcal{L}$ is a big and nef line bundle on $X$, we say that $(X, \mathcal{L})$ is a polarized projective variety. On the other hand we are interested, along this paper, in some families of cyclic coverings

\begin{equation}
(5) 
 \pi : \tilde{X} \to X.
\end{equation}

Then we fix our conventions about, [10, 21] I p.242. By definition $\pi$ is a finite morphism of degree $\ell \geq 2$ and it is the quotient map of the action of an automorphism of order $\ell$ of $\tilde{X}$. We assume that $\tilde{X}$ is normal, up to composing $\pi$ with the normalization map. Hence $\tilde{X}$ is reduced with irreducible
connected components. Starting from \( \pi \), we briefly review the recipe for its construction. Notice that \( \pi_* \mathcal{O}_X \cong \mathcal{A} \), where

\[
\mathcal{A} = \mathcal{O}_X \oplus \mathcal{E}^{-1} \oplus \cdots \oplus \mathcal{E}^{-l+1}
\]

and \( \mathcal{E} \in \text{Pic } X \). Assume \( \tilde{X} \) is connected and hence irreducible. Then \( \pi \) defines the field extension \( \pi^*: k(X) \to k(\tilde{X}) \) and its trace map induces the exact sequence

\[
0 \to \mathcal{E}^{-l} \to \mathcal{O}_X \to \mathcal{O}_B \to 0,
\]

for some \( s \in H^0(\mathcal{E}^l) \). The multiplication by \( s \) defines a structure of \( \mathcal{O}_X \)-Algebra on \( \mathcal{A} \). We have \( \tilde{X} = \text{Spec } \mathcal{A} \), moreover \( \pi \) factors through the projection \( u: \mathbb{P}(\mathcal{A}) \to X \). The branch divisor of \( \pi \) is \( \text{div}(s) \) and will be denoted by \( B \). For \( B \) we fix the notation

\[
B = m_1 B_1 + \cdots + m_r B_r,
\]

where \( B_1, \ldots, B_r \) are prime divisors. Conversely, a pair \((\mathcal{E}, B)\) such that \( B \in |\mathcal{E}^l| \) defines on \( X \) an \( \mathcal{O}_X \)-Algebra structure as above and a cyclic cover \( \pi \). Notice that the condition \( g.c.d.(\ell, m_1, \ldots, m_r) = 1 \) implies the irreducibility of \( \tilde{X} \).

Now let \( C \) be a reduced curve and \( \eta \in \text{Pic } C \) a nontrivial \( \ell \)-torsion element. Then \((C, \eta)\) uniquely defines, using a nonzero vector \( s \in H^0(\eta^l) \), a nonramified cyclic cover

\[
\pi: \tilde{C} \to C,
\]

which is nontrivial. To give a pair \((C, \pi)\) is equivalent to give a singular level \( \ell \) curve \((C, \eta)\). Now recall that a curve \( C \subset X \) is mobile if moves in an irreducible algebraic family covering \( X \), with integral general member. In the Neron-Severi group \( N_1(X) \otimes \mathbb{R} \) the mobile classes of such curves generate an important convex cone, [3] 1.3 (vi), [21] II p. 307. Finally we introduce the following definition.

**Definition 2.1.** Let \( \mathcal{E} \in \text{Pic } X \), the pair \((X, \mathcal{E})\) is a level \( \ell \) structure on \( X \) if:

- \( |\mathcal{E}^l| \neq \emptyset \) and a general \( B \in |\mathcal{E}^l| \) defines an integral cyclic cover,
- there exists a mobile curve \( C \) in \( X \) such that \( CB = 0 \).

Assume \( \dim X = 1 \) then \( X \) is the smooth, integral curve \( C \) and \( \mathcal{E} \) is a line bundle of degree \( 0 \) such that \( \mathcal{E}^l \cong \mathcal{O}_C \). Moreover we are assuming that the cover \( \pi: \tilde{C} \to C \) defined by \( \mathcal{E} \) is integral. Hence \( \mathcal{E} \) is a nontrivial \( \ell \)-torsion element. Then, for curves, the definition is the traditional one. In higher dimension the next property is clear.

**Proposition 2.1.** Let \((X, \mathcal{E})\) be a level \( \ell \) structure on \( X \) and \( C \subset X \) a mobile curve such that \( CE = 0 \), where \( \mathcal{O}_X(CE) \cong \mathcal{E} \). Then \( \mathcal{O}_C(CE) \) is an \( \ell \)-torsion element of \( \text{Pic } C \).

**Proof.** Consider \( D \in |\mathcal{E}^l| \). Since \( C \) is movable we can assume that \( C \) is not a component of \( D \). Then \( C \cap D \) is empty because \( CE = 0 \). This implies that \( \mathcal{E}^l \otimes \mathcal{O}_C \cong \mathcal{O}_C(CE) \cong \mathcal{O}_C \). \( \square \)

**Remark 2.1.** Nevertheless we may have a trivial \( \mathcal{O}_C(CE) \) even when \( \mathcal{E} \) is not, and even generically when \( C \) moves in its family. This is obvious if \( C \) is smooth and rational. Furthermore consider a curve \( F \) and the projection \( p: F \times X \to X \). Then \((F \times X, p^*\mathcal{E})\) is a level \( \ell \)-structure on \( F \times X \) and \( p^*\mathcal{E} \) is trivial on the mobile curve \( p^*(x), x \in X \).

Then, to address the concrete topics of our paper, we turn to polarized pairs \((X, \mathcal{L})\) and we denote by \( d \) the dimension of \( X \). We assume that \( |\mathcal{L}^m| \) is globally generated for \( m >> 0 \) and observe that a general complete intersection of \( d - 1 \) elements of \( |\mathcal{L}^m| \) is a smooth, integral mobile curve, which moves in an irreducible family \( \mathcal{C}_m \) of transversal complete intersections in \( X \).

**Proposition 2.2.** Let \((X, \mathcal{L}, \mathcal{E})\) be as above. Assume \( CE = 0 \), where \( C \in \mathcal{C}_m \) and \( \mathcal{O}_X(CE) \cong \mathcal{E} \). Then \( \mathcal{O}_C(CE) \) is a nontrivial \( \ell \)-torsion element of \( \text{Pic } C \), moreover

\[
h^0(\mathcal{O}_X(kE)) = 0, \quad k \neq 0 \mod \ell.
\]

**Proof.** By induction on \( d = \dim X \). Let \( d = 1 \) then \( X = C \) and \( \{C\} = \mathcal{C}_m \). Since \( \mathcal{E} \) defines an integral cover, the statement follows. Let \( d \geq 2 \) and \( C = D_1 \cdots D_{d-1} \), where \( D_1, \ldots, D_{d-1} \in |\mathcal{L}^m| \), then a general \( D \) in the linear system generated by \( D_1 \cdots D_{d-1} \) is smooth. \( \mathcal{O}_D(D) \) is nef, big and globally generated. Let \( \pi: \tilde{X} \to X \) be the cyclic cover, branched on \( B \), we have, since \( C \) is mobile
and $CB = 0$ we can assume $C \cap B = \emptyset$. Now let $f : X \to \mathbb{P}^n$ be the morphism defined by $|D|$, then $f$ is generically finite onto its image and the same is true for $f \circ \pi : \tilde{X} \to \mathbb{P}^n$. Then $\tilde{C} = \pi^{-1}(C)$ is connected by the connectedness theorem and $\mathcal{O}_C(E)$ is non trivial of $\ell$-torsion in $\text{Pic} C$. Moreover $(D, \mathcal{O}_D(E))$ is a level $\ell$ structure and the second statement follows by induction on $d$. □

Keeping this notation we finally come to the following definition.

**Definition 2.2.** A level $\ell$ polarized variety is a triple $(X, \mathcal{L}, \mathcal{E})$ such that $(X, \mathcal{E})$ is a level structure on $X$ and $C \mathcal{E} = 0$, where $C \in \mathcal{C}_m$.

Actually the triples $(X, \mathcal{L}, \mathcal{E})$ we will consider always satisfy the additional property:

$|\mathcal{L}|$ is base point free and defines a birational morphism onto its image

$$(9) \quad f : X \to \mathbb{P}^n.$$ 

Hence we assume $C = H_1 \cap \cdots \cap H_{d-1} \in \mathcal{C}_1$, where $H_1 \cdots H_{d-1} \in |f^*\mathcal{O}_{\mathbb{P}^n}(1)|$. So $C$ shows the distinguished line bundles $\eta_C := \mathcal{E} \otimes \mathcal{O}_C$ and $\mathcal{L}_C := \mathcal{L} \otimes \mathcal{O}_C$ and these lead us to the varieties we are interested in. For these $\mathcal{L}_C$ is the canonical sheaf $\omega_C$. For the triples considered, we will also have that the restriction $r : H^0(\mathcal{L}) \to H^0(\omega_C)$ is surjective and that $X := f(X)$ is normal. So we are going to deal with projective varieties $X$ whose curvilinear sections are canonical curves $C$, endowed with the étale cover defined by $\eta_C$. This includes K3 surfaces and Fano threefolds with a prescribed level $\ell$ structure.

**3. LEVEL $\ell$ K3 SURFACES**

We begin discussing the families of level $\ell$ polarized K3 surfaces $(S, \mathcal{L}, \mathcal{E})$ and the chances that $C \in |\mathcal{L}|$ be a curve with general moduli. We say that $C^2 = 2g - 2$ is the degree of $(S, \mathcal{L})$ and $g$ its genus. As usual the moduli space of $(S, \mathcal{L})$ is denoted by

$$(10) \quad \mathcal{F}_g,$$

it is an integral quasi projective variety of dimension 19. Let $[S, \mathcal{L}] \in \mathcal{F}_g$ be a general point, we recall that then $\text{Pic} S \cong \mathbb{Z} \mathcal{L}$ and $|\mathcal{L}|$ defines an embedding

$$(11) \quad f : S \to \mathbb{P}^g$$

for $g \geq 3$. Coming to level $\ell$ structures $(S, \mathcal{L}, \mathcal{E})$, these properties are no longer satisfied, as we are going to recall. We fix our notation as follows, the map

$$(12) \quad \pi' : \tilde{S} \to S$$

is the covering morphism defined by $\mathcal{E}$. As already established its branch divisor is

$$B = m_1B_1 + \cdots + m_rB_r,$$

where $B_1, \ldots, B_r$ are the irreducible components of $\text{Supp} B$. Of course, since $\text{Pic} S$ has no torsion, $B$ is not zero. We fix the following convention:

- $r$ is the number of irreducible components of $\text{Supp} B$,
- $t$ is the number of its connected components.

Moreover we set

$$(13) \quad B_1 + \cdots + B_r = B_{\text{red}} = N_1 + \cdots + N_t,$$

where $N_1 \ldots N_t$ denote the connected components of $\text{Supp} B$. Notice that $CB_i = 0$ for $i = 1 \ldots r$. Indeed $C$ is integral and $\text{dim} |C| \geq 1$ so that $CB_i \geq 0$. Since $B \in |\mathcal{E}|$ then $CB = 0$ and this implies $CB_i = 0$. Then, applying the Hodge Index Theorem, $B_i$ is an integral curve on $S$ with $B_i^2 < 0$. Hence $B_i^2 = -2$ and $B_i$ is $\mathbb{P}^1$. The same argument applies to $N_j$ which is a reduced connected curve of arithmetic genus 0. In particular each $N_j$ is contracted by $f$ to a quadratic singularity and $\text{Pic} S$ is not isomorphic to $\mathbb{Z}$.

It is not difficult to see that the Kodaira dimension of $\tilde{S}$ is zero, moreover, with some elaboration, one has the following property, cfr. [15], [22].

**Proposition 3.1.** Either $\tilde{S}$ is birational to a K3 surface or to an abelian surface.

**Definition 3.1.** Let $(S, \mathcal{L}, \mathcal{E})$ be a level $\ell$ K3 surface, we say that:
(1) $(S, \mathcal{L}, E)$ is of K3 type if $\tilde{S}'$ is birational to a K3 surface.
(2) $(S, \mathcal{L}, E)$ is of abelian type if $\tilde{S}'$ is birational to an abelian surface.

Case (2) is scarcely interesting for our purposes. We aim indeed to use the curves $C \in |\mathcal{L}|$ in order to parametrize the moduli space $\mathcal{R}_{g, \ell}$ of level $\ell$ curves in low genus. But in case (2) $C$ has not enough moduli for $g \geq 3$.

We assume since now that $(S, \mathcal{L}, E)$ is a level $\ell$ K3 surface of K3 type. Then, to ameliorate the exposition, we just say with some abuse that $(S, \mathcal{L}, E)$ is a level $\ell$ K3 surface. We say that two triples $(S_n, \mathcal{L}_n, E_n)$, $(n = 1, 2)$, are isomorphic if there exists a birational map $\beta : S_1 \to S_2$ such that $\beta^* \mathcal{L}_2 \cong \mathcal{L}_1$ and $\beta^* E_2 \cong E_1$, $i = 1, 2$.

As mentioned the classification of these triples is due to Nikulin and originates from his paper [22]. The part of interest here is the classification of pairs $(\tilde{S}, G)$, where $\tilde{S}$ is a K3 surface and $G$ is a finite group of symplectic automorphisms of $\tilde{S}$. There exist 14 classes of pairs $(\tilde{S}, G)$ such that $G$ is commutative and $G$ is $\mathbb{Z}/\ell\mathbb{Z}$ exactly for $2 \leq \ell \leq 8$. After the classification, several papers addressed the description of the moduli and the projective models of these K3 surfaces. It is due to mention here [13, 14, 15, 16, 28].

The triple $(S, \mathcal{L}, E)$ determines an associated triple $(\tilde{S}, \tilde{\mathcal{L}}, \gamma)$, where $\gamma \in \text{Aut} \tilde{S}$ is a symplectic automorphisms of order $\ell$ and $(\tilde{S}, \tilde{\mathcal{L}})$ is a polarized K3 surface of degree $\ell(2g - 2)$. We have indeed $B_{\text{red}} = N_1 + \cdots + N_t$, where the summands are the connected components and $-2$-curves. Let $\nu : S \to \overline{S}$ be their contraction morphism, then the Cartesian square

$$
\begin{array}{ccc}
\nu' & \cong & \nu \\
\tilde{S}' & \to & S \\
\downarrow & & \downarrow \\
\tilde{S} & \cong & \overline{S}
\end{array}
$$

is the Stein factorization of $\nu \circ \pi'$. In it $\nu'$ is a birational morphism. Let $G \subset \text{Aut} \tilde{S}'$ be the group whose quotient map is $\pi'$. As we will see $\pi'^* H^3(\tilde{\mathcal{L}}(-E))$ sits in $H^3(\tilde{\mathcal{L}})$ as an eigenspace of the natural representation of $G$ and defines a generator $\gamma$ of $G$. Moreover $\pi$ is the quotient map of the induced action of $G$ on $\tilde{S}$. Conversely, starting from $\pi$ and the minimal desingularization $\nu$, $\pi'$ is reconstructed from the fibre product $\pi \times_{\overline{S}} \nu$.

In order to describe the rational singularities occurring in $\text{Sing} \overline{S}$ we use the notation

$$
T := n_1 T_1 + \cdots + n_s T_s,
$$

where $T_j$ is the singularity type and $n_j$ the number of points of type $T_j$ in $\text{Sing} \overline{S}$.

**Theorem 3.2.** Let $(S, \mathcal{E}, \mathcal{L})$ be a level $\ell$ K3 surface of genus $g$, then one has $2 \leq \ell \leq 8$ and $(S, \mathcal{E})$ satisfies one of the following conditions:

1. $\ell = 2$. One has $t = 8$, $r = 8$ and $T = 8A_1$.
2. $\ell = 3$. One has $t = 6$, $r = 12$ and $T = 6A_2$.
3. $\ell = 4$. One has $t = 6$, $r = 14$ and $T = 4A_3 + 2A_1$.
4. $\ell = 5$. One has $t = 4$, $r = 16$ and $T = 4A_4$.
5. $\ell = 6$. One has $t = 6$, $r = 16$ and $T = 2A_5 + 2A_2 + 2A_1$.
6. $\ell = 7$. One has $t = 3$, $r = 18$ and $T = 3A_6$.
7. $\ell = 8$. One has $t = 4$, $r = 18$ and $T = 2A_7 + A_3 + A_1$.

See [22]. It is also useful to observe that always one has

$$
E^2 = \frac{B_2}{\ell^2} = -4.
$$

Now, in view of the concrete applications in this paper, we mention some relevant properties of the structure of $\text{Pic} S$ and of the moduli of the above triples.

**Definition 3.2.** $\mathcal{F}_{g, \ell}$ is the moduli space of level $\ell$ K3 surfaces of genus $g$. 
As in the case of \((S, \mathcal{L})\), the construction of \(\mathcal{F}_{g, \ell}\) relies on the usual notion of lattice polarized variety, see \([1, 9, 17]\) and \([22]\) for this K3 case. In particular, for every \(g \geq 2\), \(\mathcal{F}_{g, \ell}\) has a standard irreducible component to be constructed as follows. We may have
\[
Z[\mathcal{L}] \oplus M_S \subseteq \text{Pic} \, S,
\]
where the sum is orthogonal. Moreover \(M_S\) has rank \(r\) and it is generated by the classes \([B_1], \ldots, [B_r], [E]\), with \(E \cong \mathcal{O}_S(E)\), so that the relation \(\ell[E] - [B] = 0\) is satisfied in \(\text{Pic} \, S\). We can see the inclusion as the image of a primitive embedding of lattices
\[
v : \mathbb{Z}c \oplus M_\ell \to \text{Pic} \, S,
\]
where \(v(c) := [\mathcal{L}]\) and \(v(M_\ell) = M_S\). The lattice \(M_\ell\) is given with the set of generators \(\{e, b_1, \ldots, b_r\}\) so that \(v(e) = [E], v(b_1) = [B_1], \ldots, v(b_r) = [B_r]\). Notice also that \(c^2 = 2g - 2\), \(e^2 = -4\), \(b_1^2 = \cdots = b_r^2 = -2\), cfr. \([22]\). Fixing these data, the moduli space of triples \((S, \mathcal{L}, \mathcal{E})\) endowed with an embedding \(v\), can be constructed as a moduli space of lattice polarized K3 surfaces \((S, v)\). In our case \(S\) is \(M\)-polarized with \(M := \mathbb{Z}c \oplus M_\ell\) and the induced embedding \(M \subset L := H^2(S, \mathbb{Z})\) is unique up to isometries, \([22]\). Then the moduli space is constructed as quotient of the period domain of these surfaces \(S\). In particular its dimension is \(19 - r\), \([9]\) Section 4.1 and Theorem 1.4.8, \([2]\) Section 2.4 and Proposition 2.6. Moreover a unique irreducible component of it is the closure of the moduli points of pairs \((S, v)\) such that
\[
\text{Pic} \, S = \mathbb{Z}[\mathcal{L}] \oplus M_S.
\]
In this case we will say that \((S, \mathcal{L}, \mathcal{E})\) is a standard triple of genus \(g\) and level \(\ell\). Let us fix our notation:

**Definition 3.3.** \(\mathcal{F}_{g, \ell}^\perp\) is the moduli space of standard triples of genus \(g\) and level \(\ell\).

\(\mathcal{F}_{g, \ell}^\perp\) exists for any \(g \geq 2\) and \(\ell = 2 \ldots 8\). Fixing \(\ell\), \(\mathcal{F}_{g, \ell}^\perp\) is the unique irreducible component of \(\mathcal{F}_{g, \ell}\) along a proper countable set of values \(g \in \mathbb{N}\).

**Remark 3.1.** Let \((S, \mathcal{L}, \mathcal{E})\) be a non standard triple and \(C \subset |\mathcal{L}|\). Then, at least experimentally for \(\ell = 2\), \(C\) is never general in moduli for \(g \geq 4\). This is true even when the parameter count makes that possible in low genus, see \([20]\). The situation is quite different for standard triples. This paper studies indeed the modular properties of \(C\) in this case: standard behavior or peculiarities of \(C\).

4. A STANDARD PROJECTIVE MODEL

Given a standard triple \((S, \mathcal{L}, \mathcal{E})\), let us construct a projective realization of \(S\) useful to our purposes. Consider \(C \subset |\mathcal{L}|\) such that \(C \cap B = \emptyset\) and \(\hat{C}' = \pi''C\). Then the curve \(\hat{C} = \nu_*' \hat{C}'\) is biregular to \(\hat{C}'\) via the contraction \(\nu' : \hat{S}' \to \hat{S}\) and the linear map
\[
\nu_*' : H^0(\mathcal{O}_{\hat{S}'}(\hat{C}')) \to H^0(\mathcal{O}_{\hat{S}}(\hat{C}))
\]
is an isomorphism, we identify the two spaces under it. Then, using \(\hat{C}\), it is easy to remind of the action of the group \(\mathbb{Z}/\ell\mathbb{Z}\) on this space and of its eigenspaces. Let
\[
0 \to \mathcal{O}_{\hat{S}} \to \mathcal{O}_{\hat{S}}(\hat{C}) \to \omega_{\hat{C}} \to 0
\]
be the standard exact sequence, then \(\mathbb{Z}/\ell\mathbb{Z}\) acts on its associated long exact sequence
\[
0 \to H^0(\mathcal{O}_{\hat{S}}) \to H^0(\mathcal{O}_{\hat{S}}(\hat{C})) \to H^0(\omega_{\hat{C}}) \to 0.
\]
As is well known the \(\mathbb{Z}/\ell\mathbb{Z}\)-decomposition of \(H^0(\omega_{\hat{C}})\) is as follows
\[
H^0(\omega_{\hat{C}}) = \bigoplus_{k=1\ldots\ell-1} \pi''^* H^0(\omega_C \otimes \eta^{-k}) \bigoplus \pi''^* H^0(\omega_C).
\]
and this implies that \(H^0(\mathcal{O}_{\hat{S}}(\hat{C}'))\) decomposes as
\[
H^0(\mathcal{O}_{\hat{S}}(\hat{C}')) = \bigoplus_{k=1\ldots\ell-1} \pi''^* H^0(\mathcal{O}_S(H_k)) \bigoplus \pi''^* H^0(\mathcal{O}_S(C)),
\]
where \( O_S(H_1) \ldots O_S(H_{\ell-1}) \in \text{Pic} \, S \) and \( O_C(H_k) \cong \omega_C \otimes \eta^{-k} \), up to reindexing. Since \( \tilde{C} \) has genus \( g = g + (\ell - 1)(g - 1) \) it follows \( \dim H^0(O_S(\tilde{C})) = g + 1 + (\ell - 1)(g - 1) \). In particular the above decomposition immediately implies that

\[
\dim H^0(O_S(H_k)) = \dim H^0(\omega_C \otimes \eta^{-k}) = g - 1, \quad k = 1 \ldots \ell - 1.
\]

In what follows, it is also useful to recall the mentioned fact that \( E^2 = -4 \).

Lemma 4.1. It holds \( h^i(O_S(E)) = h^i(O_S(-E)) = 0 \), for \( i \geq 0 \).

Proof. By assumption \( E \) is not effective. The same is true for \( -E \), since \( \ell E \sim B \) and \( B > 0 \). This implies \( h^0(O_S(E)) = 0 \) and \( h^2(O_S(E)) = h^0(O_S(-E)) = 0 \). Since \( E^2 = -4 \) we have \( \chi(O_S(E)) = 0 \) and then \( h^1(O_S(E)) = 0 \). The same argument applies to \( -E \). \( \square \)

Now we consider the line bundle \( O_S(C - E) \) and the standard exact sequence

\[
0 \to O_S(-E) \to O_S(C - E) \to O_C(C - E) \to 0.
\]

Lemma 4.2. Let \( g \geq 2 \) then the associated long exact sequence is

\[
0 \to H^0(O_S(C - E)) \to H^0(\omega_C \otimes \eta^{-1}) \to 0,
\]

in particular it follows \( \dim [C - E] = g - 2 \) and \( h^i(O_S(C - E)) = 0 \), \( i \geq 1 \).

Proof. By the previous lemma \( h^i(O_S(E)) = h^i(O_S(-E)) = 0 \), for \( i \geq 0 \). Moreover we have \( h^0(\omega_C \otimes \eta^{-1}) = g - 1 \) and \( h^1(\omega_C \otimes \eta^{-1}) = 0 \). Then the statement follows. \( \square \)

Now we observe that the pull-back by \( \pi' \) defines a linear embedding

\[
\pi'^{*} : H^0(O_S(C - E)) \to H^0(O_{\tilde{S}}(\tilde{C}')).
\]

We have indeed \( O_{\tilde{S}}(\tilde{C}') \otimes \pi'^{*}O_S(E - C) \cong O_{\tilde{S}}(\pi'^{*}E) \) and finally

\[
h^0(O_{\tilde{S}}(\pi'^{*}E)) = h^0(\pi'^{*}O_S(\pi'^{*}E)) = h^0(\mathcal{A}(E)) = 1,
\]

with \( \mathcal{A} = O_S \oplus O_S(-E) \oplus \cdots \oplus O_S((1 - \ell)E) \). The equality defines, up to a nonzero constant factor, the linear embedding \( \pi'^{*} \). Then \( \text{Im} \, \pi'^{*} \) is the \( \mathbb{Z}/\ell \mathbb{Z} \)-invariant space

\[
\pi'^{*}H^0(O_S(C - E)).
\]

Proposition 4.3. Let \( g \geq 3 \) and \( \text{Pic} \, S \cong \mathbb{Z}e \oplus M_\ell \), then \( [C - E] \) is base point free.

Proof. Since \( S \) is a K3 surface, it suffices to prove that \( [C - E] \) has no fixed component. Let \( F \) be an integral fixed component of \( [C - E] \), set \( f = F \cdot C \) for a general \( C \). Then \( f \) is a fixed divisor of \( \omega_C \otimes \eta^{-1} \). Applying Riemann-Roch to \( C \) it follows \( \dim [\eta(f)] = \deg f - 1 \). Since \( g \geq 3 \) then \( \deg f \leq 2 \). Hence \( F \) is a line, a conic or \( FC = 0 \). We have \( F \sim xc + \sum y_jB_j + zE \) in Pic \( S \). Assume \( \deg f > 0 \) then \( 0 < CF = (2g - 2)x \leq 2 \) with \( x \in \mathbb{Z} \): a contradiction for \( g \geq 3 \). Let \( CF = 0 \) then \( F^2 = -2 \) by the Hodge Index Theorem and \( F \) is a \( \mathbb{P}^1 \) contracted by \( f|_C : S \to \mathbb{P}^g \). By Lemma 4.2

\[
h^0(C - E) = g - 1 = (C - E)^2/2 + 2.
\]

Let \( M \) be the moving part of the linear system \( [C - E] \), then \( \dim [M] \geq 1 \) and \( MF \geq 0 \). Moreover we have \( C - E \sim M + kF + R \), where \( R \) is a curve not containing \( F \) and \( k \geq 1 \). Let \( G \in [M + F] \) be general then \( G \) contains \( F \): otherwise the curve \( kF \) could not be a component of the element \( G + (k - 1)F + R \in [C - E] \). Hence \( F \) is a fixed component of \( [M + F] \). Now observe that \( MF \geq 0 \) and then consider the standard exact sequence

\[
0 \to O_S(M) \to O_S(M + F) \to O_F(M) \to 0.
\]

We claim that, passing to the associated long exact sequence, it follows

\[
\chi(O_S(M)) = \chi(O_S(M + F))
\]

and \( \chi(O_F(M)) = 0 \). Since \( F = \mathbb{P}^1 \) this implies \( MF < 0 \): a contradiction. To prove the claim consider a smooth \( D \in [M] \). Then either \( D \) is integral of genus \( g - 2 \) and \( h^1(O_S(M)) = 0 \) or \( M \sim (g - 2)N \) and \( N \) is a smooth integral elliptic curve. Via Serre duality we have \( h^2(O_S(M)) = h^2(O_S(M + F)) = 0 \). Moreover \( MF \geq 0 \) implies \( h^1(O_F(M)) = 0 \). Then, in the former case, \( h^1(O_S(M)) = 0 \) implies \( h^1(O_S(M + F)) = 0 \) and the claim follows. In the latter case replace \( M \) by \( N \). Then the equality and the same contradiction follow by the same type of arguments. \( \square \)
Now we introduce a second linear system associated with $E$. At first let us set
\[
(27) \quad B_{\text{red}} := B_1 + \cdots + B_r,
\]
where the summands are the irreducible components of $\text{Supp} B$. Then we recall that
\[
E = \frac{1}{\ell} (m_1 B_1 + \cdots + m_r B_r), \quad \text{with } m_1, \ldots, m_r \in [1 \ldots \ell - 1].
\]

**Definition 4.1.** Set $E = B_{\text{red}} - \ell \cdot m_i B_i + \cdots + \ell \cdot m_r B_r$, where $m_i := \ell - m_i$

Let us denote by $n_i$ the coefficients of the curves $B_i$ in $-\ell E$. Then $n_i \equiv m_i \mod \ell$. More precisely, $E$ is a generator of $\mathbb{Z}/\ell \mathbb{Z} = \langle B_i, E \rangle / \langle B_i \rangle$ and $E$ is its opposite in $\mathbb{Z}/\ell \mathbb{Z}$; in particular it is a different generator of the same group. Hence $E := O_S(E)$ is a level $\ell$ structure, with the same properties of $E$. We notice that $E$ defines a cover $\tilde{\pi}' : \tilde{S}' \to S$ so that $\tilde{\pi}' = \pi' \circ a$ and $a^\ell = id_{S'}$. Then we define
\[
(28) \quad |H| := |C - E|, \quad \hat{H} := |C - E|.
\]
The rational maps associated with these linear systems respectively will be
\[
(29) \quad p : S \to \mathbb{P}, \quad \hat{p} : S \to \hat{\mathbb{P}},
\]
where $\mathbb{P} := |H|^*$ and $\hat{\mathbb{P}} := |\hat{H}|^*$ are the projective space $\mathbb{P}^{g-2}$. Let $i$ be the inclusion
\[
(30) \quad \mathbb{P} \times \hat{\mathbb{P}} \subset \mathbb{P}^{(g-1)^2 - 1}
\]
defined by the Segre embedding, we set $f := i \circ (p \times \hat{p})$ and fix the notation
\[
(31) \quad f : S \to \mathbb{P} \times \hat{\mathbb{P}} \subset \mathbb{P}^{(g-1)^2 - 1}.
\]

**Definition 4.2.** The morphism $f$ is the main projective model of $(S, L, E)$.

The next two remarks are simple but relevant in order to discuss $f$ (the second one follows by a direct computation of $E \cdot E$, where the class $E$ is explicitly given in [22]):
1. $f^* O_{\mathbb{P}^{(g-1)^2 - 1}}(1) \cong O_S(H + \hat{H}) \cong O_S(2C - B_{\text{red}})$,
2. $H \hat{H} = 2g + 2 - t$.

**Proposition 4.4.** The divisors $|H - \hat{H}|$ and $|\hat{H} - H|$ are not effective classes for $\ell \geq 3$ and
\[
(32) \quad h^1(O_S(H - \hat{H})) = h^1(O_S(\hat{H} - H)) = 6 - t.
\]

\textbf{Proof.} We have $H(H - \hat{H}) = \hat{H}(\hat{H} - H) = t - 8$. Since the general elements of $|H|$ and $|\hat{H}|$ are irreducible curves, the first statement follows for $\ell \geq 3$ because then $t \leq 6$. The second statement just follows from Riemann-Roch. \hfill \Box

Now let us consider, for a general $C \in |L|$, the standard exact sequence
\[
(33) \quad 0 \to O_S(C - B_{\text{red}}) \to O_S(2C - B_{\text{red}}) \to O_C(2C - B_{\text{red}}) \to 0.
\]
Since $C$ is smooth and disjoint from $B_{\text{red}}$, then $O_C(-B_{\text{red}})$ is trivial and $|2C - B_{\text{red}}|$ cuts on $C$ a linear system of bicanonical divisors. Moreover we know that both $|H|$ and $|\hat{H}|$ are base point free. Hence the same is true for $|H + \hat{H}| = |2C - B_{\text{red}}|$. Notice that
\[
(2C - B_{\text{red}})^2 = 8(g - 1) - 2t,
\]
which is $\geq 0$ for $g \geq 3$ and any of the prescribed values of $t, \ell$. Actually the zero value is only reached in the known situation $g = 3, \ell = 2$. Hence we assume $g \geq 4$ for $\ell = 2$. Then a general $D \in |H + \hat{H}|$ is a smooth integral curve such that $D^2 > 0$. As is well known, this implies $h^i(O_S(H + \hat{H})) = 0$ for $i \geq 1$ and the next property follows.

**Proposition 4.5.** Let $g$ be as above then $\dim |2C - B_{\text{red}}| = 4g - t - 3$ and the long exact sequence associated with the exact sequence $(33)$ is as follows:
\[
0 \to H^0(O_S(C - B_{\text{red}})) \to H^0(O_S(2C - B_{\text{red}})) \to H^0(\omega^2_C) \to H^1(O_S(C - B_{\text{red}})) \to 0.
\]
The linear system $|C - B_{\text{red}}|$ also deserves some observations. Since we are dealing with a general standard triple $(S, L, E)$, we know that $|C|$ defines a morphism

$$f_{|C|} : S \to \mathbb{P}^g$$

which is the contraction $\nu : S \to \mathbb{S}$, composed with the embedding $\mathbb{S} \subset \mathbb{P}^g$ defined by $[\nu_* C]$. Since a general $C$ is disjoint from $B$, $[\nu_* C]$ is a linear system of Cartier divisors. Let $\mathcal{T}_{\text{Sing}}$ be the ideal sheaf of $\text{Sing} \mathbb{S}$, it is clear that the natural map

$$f_* |C| : H^0 (\mathcal{T}_{\text{Sing}}(1)) \to H^0 (\mathcal{O}_S (C - B_{\text{red}}))$$

is an isomorphism. Then, considering the above exact sequence (33), we have

$$h^0 (\mathcal{O}_S (C - B_{\text{red}})) - h^1 (\mathcal{O}_S (C - B_{\text{red}})) = \chi (\mathcal{O}_S (2C - B_{\text{red}})) - \chi (\omega_C^2) = g + 1 - t.$$  

This implies the next property.

**Proposition 4.6.** It holds $h^1 (\mathcal{O}_S (C - B_{\text{red}})) = 0$ if and only if $h^0 (\mathcal{O}_S (C - B_{\text{red}})) = g + 1 - t$, that is, the points of $\text{Sing} \mathbb{S}$ are linearly independent in $\mathbb{P}^g$.

On the other hand consider the commutative diagram

$$
\begin{array}{cccc}
0 & \to & H^0 (\mathcal{O}_S (C - B_{\text{red}})) & \to \\
\downarrow & & \downarrow & \\
& & H^0 (\mathcal{O}_S (C - B_{\text{red}})) & \\
\downarrow & & \downarrow & \\
& & H^0 (\mathcal{O}_S (H)) \otimes H^0 (\mathcal{O}_S (\tilde{H})) & \\
\rho_H \otimes \rho_R & \downarrow & \mu_S & \to \\
H^0 (\omega_C \otimes \eta^{-1}) \otimes H^0 (\omega_C \otimes \eta) & & H^0 (\omega_C^2) & \\
\rho_C & \downarrow & & \\
& & H^1 (\mathcal{O}_S (C - B_{\text{red}})) & \\
\downarrow & & \downarrow & \\
& & 0 & \\
\end{array}
$$

where $\mu_S$ and $\mu_C$ are the multiplication maps and the vertical arrows are the restriction maps. It follows from Lemma (4.2) that $\rho_H \otimes \rho_R$ is an isomorphism. The next property is clear.

**Proposition 4.7.** If $\mu_C$ is surjective then $h^1 (\mathcal{O}_S (C - B_{\text{red}})) = 0$ i.e. $\rho_C$ is surjective.

Since $\chi (\mathcal{O}_S (C - B_{\text{red}})) = g + 1 - t$ let us point out that $\mu_C$ is not surjective if

$$g < t - 1.$$  

We do not further investigate the diagram, for our applications these results suffice.

5. Views on the Mukai maps in level $\ell$

In this section we only put in large the picture we have outlined in the introduction. This picture concerns the maps in (33) and (34), that is, the Mukai map

$$m_g : \mathcal{P}_g \to \mathcal{M}_g$$

and the level $\ell$ Mukai maps

$$r_{g, \ell} : \mathcal{P}_{g, \ell} \to \mathcal{R}_{g, \ell}.$$  

These maps, and the involved moduli spaces, have been previously considered. We recall that the points of $\mathcal{P}_g$ are the elements $[S, L, C]$ such that $[S, L] \in F_g$ and $C \in |L|$. The Mukai map $m_g$ is the natural forgetful map. We have

1. $m_g$ is dominant for $g \leq 9$. 

(2) $m_g$ is not dominant for $g = 10$.
(3) $m_g$ is birational for $g = 11$.
(4) $m_g$ has 1-dimensional fibre for $g = 12$.
(5) $m_g$ is generically injective for $g \geq 13$.

Thus $m_g$ has not maximal rank for $g = 10, 12$. It is indeed known that a general $[C] \in m_{10}(P_{10})$ is a linear section $C$ of the $G_2$ variety $W \subset \mathbb{P}^{13}$, [23]. Hence the family of 2-dimensional linear sections of $W$ through $C$ is a $\mathbb{P}^3$. It turns out from this fact that the fibre of $m_{10}$ at $[C]$ is 3-dimensional. Then $m_{10}(P_{10})$ has codimension 1. Genus 12 Fano threefolds play a similar role, then a general fibre of $m_{12}$ is a rational curve.

In this perspective, asking about the connections between the moduli space $F_{g,\ell}$, of level $\ell$ K3 surfaces of genus $g$, and $R_{g,\ell}$ is, as observed, natural. For a general point $[S, L, E] \in F_{g,\ell}$ one can ask if $(C, \eta)$, with $C \in |L|$ and $\eta = E \otimes O_C$, defines a general point of $R_{g,\ell}$. More precisely recall that $P_{g,\ell}$ is the moduli space of 4-tuples $(S, L, E, C)$ such that $[S, L, E] \in F_{g,\ell}$ and $C \in |L|$. The level $\ell$ Mukai $r_{g,\ell} : P_{g,\ell} \to R_{g,\ell}$ is the morphism sending $[S, L, E, C] \in P_{g,\ell}$ to the point $[C, \eta_C] \in R_{g,\ell}$, where $\eta$ is $E \otimes O_C$. About the possible dominance of the map $r_{g,\ell}$ we have:

(1) $3g - 3 = \dim R_{g,2} \leq \dim P_{g,2} = 11 + g$ iff $g \leq 7$.
(2) $3g - 3 = \dim R_{g,3} \leq \dim P_{g,3} = 7 + g$ iff $g \leq 5$.
(3) $3g - 3 = \dim R_{g,4} \leq \dim P_{g,4} = 5 + g$ iff $g \leq 4$.
(4) $3g - 3 = \dim R_{g,5} \leq \dim P_{g,5} = 3 + g$ iff $g \leq 3$.
(5) $3g - 3 = \dim R_{g,6} \leq \dim P_{g,6} = 3 + g$ iff $g \leq 3$.
(6) $3g - 3 = \dim R_{g,7} \leq \dim P_{g,7} = 1 + g$ iff $g \leq 2$.
(7) $3g - 3 = \dim R_{g,8} \leq \dim P_{g,8} = 1 + g$ iff $g \leq 2$.

These issues have not been systematically considered but for $\ell = 2$. We close this expository section with a summary on what happens for $\ell = 2, 3$.

5.1. The picture for $\ell = 2$. We have $3g - 3 = \dim M_g \leq \dim P_{g,2} = 11 + g$ iff $g \leq 7$. Again, $r_{g,2}$ behaves unexpectedly near the value of transition, which is now $g = 7$.

(1) $r_{g,2}$ is dominant for $g \leq 5$.
(2) $r_{g,2}$ is not dominant for $g = 6$.
(3) $r_{g,2}$ is birational for $g = 7$.
(4) $r_{g,2}$ has not finite fibres for $g = 8$.
(5) $r_{g,2}$ is generically injective for $g \geq 9$.

These surfaces are known as (standard) Nikulin surfaces. Cases (1), (2), (3) are treated in [11, 12], the remaining ones, (standard and non standard), in [18, 20]. Notice that $r_{g,2}$ is not of maximal rank for $g = 6, 8$. In genus 6 the condition $C \subset S$ implies that the following multiplication map is not an isomorphism as expected:

\[
(37) \quad \mu : \text{Sym}^2 H^0(\omega_C \otimes \eta_C) \to H^0(\omega_C^\otimes 2).
\]

Then $(C, \eta_C)$ does not define a general point of $R_{g,2}$, see [1]. We point out that, studying the two cases where $r_{g,2}$ has not maximal rank, two families of singular Fano threefolds appear. Their hyperplane sections are singular models $\overline{S}$ of general Nikulin surfaces $S$. The existence of these threefolds implies the failure of the maximal rank.
5.2. The picture for $\ell = 3$. We will prove that $r_{g,3}$ behaves unexpectedly near $g = 5$:

1. $r_{g,3}^*$ is dominant for $g \leq 3$,
2. $r_{g,3}^*$ has not maximal rank for $g = 4$,
3. $r_{g,3}^*$ is birational for $g = 5$,
4. $r_{g,3}^*$ has not maximal rank for $g = 6$.

Remark 5.1. The case $g \geq 7$ should be considered for further investigation, addressing the generic injectivity. The (uni)rationality of $R_{g,3}$ is known, or elementary, for $g \leq 5$, cfr. [5 6 27]. We recall that $R_{g,3}$ is of general type for $g \geq 12$ and of Kodaira dimension $\geq 19$ for $g = 11$. [7]. Bruns proved in [4] that $R_{8,3}$ is of general type. The cases $g = 6, 7, 10$, and partially $g = 11$ are open.

6. The Mukai map in level 3

6.1. The case of genus 4. Let $[S, \mathcal{L}, \mathcal{E}, C] \in \mathcal{P}_{g,\ell}^+$ be general and $\ell = 3$, as in section 2, (35) we consider the commutative diagram

$$
\begin{align*}
H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(\tilde{H})) &\xrightarrow{\mu_B} H^0(\mathcal{O}_S(H + \tilde{H})) \\
\mu_H \otimes \mu_{\tilde{H}} &\downarrow \\
H^0(\omega_C \otimes \eta^{-1}) \otimes H^0(\omega_C \otimes \eta) &\xrightarrow{\mu_C} H^0(\omega^{\otimes 2}_C).
\end{align*}
$$

(38)

Since $\ell = 3$ we have $t = 6$ connected components of Supp $B$. Then, by proposition (1.7), $\mu_C$ is not surjective if $g < t - 1 = 5$. This is obvious for $g \leq 3$. For $g = 4$ the dimension count suggests that in $R_{4,3}$ the map $\mu_C$ is not surjective in codimension 1.

Proposition 6.1. Let $[C, \eta] \in R_{4,3}$ be a general point then $\mu_C$ is surjective, moreover the locus of points such that $\mu_C$ is not surjective is an effective Cartier divisor in $R_{4,3}$.

Indeed, for $g = 4$ and $\ell = 3$, this locus turns out to be the locus $D_{g,\ell}$ defined in [7] p. 77. There, for low level $\ell \geq 3$ and for $g \leq 16$, the so defined Torsion bundle conjecture $B$ is proven, which implies that $D_{4,3}$ is an effective Cartier divisor in $R_{4,3}$. Then the next theorem follows. Notice also that, for $g = 4$, theorem 1.7 of [10] implies that $\mu_C$ is an isomorphism for a general $(C, \eta)$.

Theorem 6.2. The map $r_{4,3} : \mathcal{P}_{4,3}^+ \to R_{4,3}$ fails to be dominant.

Remark 6.1. The case $g = 4$ turns out to be of special interest. See the last section for a natural, presently conjectural, geometric interpretation.

6.2. The case of genus 5. Differently from the case $g \leq 4$ the multiplication map

$$
\mu_C : H^0(\omega_C \otimes \eta) \otimes H^0(\omega_C \otimes \eta^{-1}) \to H^0(\omega^{\otimes 2}_C)
$$

can be surjective for $g \geq 5$ and a general point $[C, \eta] \in R_{g,3}$. This property occurs in genus $g = 5$ and makes possible the proof of the next birational theorem.

Theorem 6.3. The Mukai map $r_{5,3} : \mathcal{P}_{5,3}^+ \to R_{5,3}$ is birational.

Before proving it we cannot avoid a long series of preliminaries. We will always assume that $[S, \mathcal{L}, \mathcal{E}, C] \in \mathcal{P}_{5,3}^+$ is a general point, in particular Pic $S \cong \mathbb{Z}_c \oplus M_3$. Let

$$
0 \to \mathcal{O}_S(H + \tilde{H} - C) \to \mathcal{O}_S(H + \tilde{H}) \to \omega^{\otimes 2}_C \to 0.
$$

(39)

be the standard exact sequence, at first we point out the following fact.

Proposition 6.4. The associated long exact sequence is

$$
0 \to H^0(\mathcal{O}_S(H + \tilde{H})) \to H^0(\omega^{\otimes 2}_C) \to 0.
$$

(40)

Since $H + \tilde{H} - C \sim C - B_{\text{red}}$, the next lemma implies the previous statement.

Lemma 6.5. It holds $h^i(\mathcal{O}_S(C - B_{\text{red}})) = 0$ for $i \geq 0$. 

Proof. Since $C(B_{red}-C) < 0$, $h^0(\mathcal{O}_S(B_{red}-C)) = 0$. Hence $h^2(\mathcal{O}_S(C-B_{red}))$ is zero by Serre duality. Since $(C-B_{red})^2 = -4$ then $\chi(\mathcal{O}_S(C-B_{red})) = 0$ and the statement follows if $h^0(\mathcal{O}_S(C-B_{red})) = 0$. Assume $A \in |C-B_{red}|$ then $A$ is not connected. This follows from $\chi(\mathcal{O}_S(A)) = h^0(\mathcal{O}_S(A)) - h^1(\mathcal{O}_S(A)) = 0$ and the standard exact sequence

$$0 \to \mathcal{O}_S(-A) \to \mathcal{O}_S \to \mathcal{O}_A \to 0.$$ 

This implies $A = A_1 + A_2$, where $A_1$ is a connected component and $A_2 = A - A_1$ is a curve. We have $C(A_1 + A_2) = C(C-B_{red}) = 8$ and we can choose $A_1$ so that $CA_1 > 0$. Assume $CA_2 = 0$ then the morphism $\phi : S \to \mathbb{P}^5$, defined by $|C|$, maps birationally $A_1 + A_2 + B_{red}$ onto a degree 8 hyperplane section of $\mathcal{S} = \phi(S)$. This is the curve $\phi_*A_1$, singular at the points of $\phi(B_{red}) = \text{Sing} \mathcal{S}$. These points are the images by $\phi$ of the six connected components of $B_{red}$ and are exactly six. Indeed each fibre of $\phi$ is connected and hence two connected components $V_1, V_2$ of $B_{red}$, contracted to the same point, are connected by an effective divisor $W$ orthogonal to $C$. On the other hand, under our generality assumption, we have $\text{Pic} S \cong \mathbb{Z}C \oplus \mathbb{Z}_3$. Moreover a direct computation shows that, in the negative definite lattice $\mathbb{M}_3$, $\text{Supp} W$ is union of irreducible components of $B_{red}$. Actually one computes that the only classes of irreducible $(-2)$-curves are the classes of $B_1 \ldots B_{12}$. This implies $W = 0$ and $V_1 = V_2$. But then $\phi_*A_1$ is not integral, because it is a hyperplane section of $\phi(S)$ with six singular points. Then there exists an irreducible component $R$ of it such that $0 < CR < 8$. The same is obvious if $CA_2 > 0$. Since $\text{Pic} S \cong \mathbb{Z}C \oplus \mathbb{Z}_3$ we have $[R] = x[C] + [\sum y_i B_i] + z[E]$, with $x, y_i, z \in \mathbb{Z}$. But this implies $0 < CR = 8x < 8$ with $x \notin \mathbb{Z}$: a contradiction.

Proposition 6.6. The linear systems $|H|$ and $|\tilde{H}|$ are not hyperelliptic.

Proof. Let $|H|$ be hyperelliptic, then $|H|$ defines a $2 : 1$ morphism $\psi : S \to \mathbb{P}^3$ onto a quadric surface $Q := \psi(S)$. As is well known the pull-back of a ruling of lines of $Q$ defines a pencil $|F_2|$ of curves such that $F_2^2 = 0$ and $HF_2 = 2$. Moreover $|F_1| := [H - F_2]$ is a pencil of irreducible elliptic curves. The same is true for the moving part of $|F_2|$. Since $H \sim F_2$ and $C \sim H + E$ we have $C(F_1 + F_2) = 8$ and also $CF_i \geq 2$, $i = 1, 2$. Let $|F|$ be the moving part of the pencil $|F_1|$ such that $CF_i$ is minimal, then it follows $2 \leq CF \leq 4$. On the other hand we have $F \sim xC + \sum y_j B_j + zE$ in $\text{Pic} S$. This implies $2 \leq CF = 8x \leq 4$ and $x \notin \mathbb{Z}$: a contradiction. The same argument works for $|\tilde{H}|$.

Lemma 6.7. It holds $h^i(\mathcal{O}_S(2H - \tilde{H})) = h^i(\mathcal{O}_S(2\tilde{H} - H)) = 0$ for $i \geq 0$.

Proof. From $H \sim C - E$ and $\tilde{H} \sim C - \tilde{E}$ we have $2H - \tilde{H} \sim C - 2E + \tilde{E}$, moreover

$$\tilde{H}(\tilde{H} - 2H) = -8 \Rightarrow h^0(\mathcal{O}_S(\tilde{H} - 2H)) = 0 \Rightarrow h^2(\mathcal{O}_S(2H - \tilde{H})) = 0.$$ 

Since $(2H - \tilde{H})^2 = -4$ then $\chi(\mathcal{O}_S(2H - \tilde{H})) = 0$. Hence the statement follows for $2H - \tilde{H}$ if we prove $h^0(\mathcal{O}_S(2H - \tilde{H})) = 0$. For this we observe that the well known descriptions of $E$ and $\tilde{E}$ are as follows. For $i = 1 \ldots 6$ consider $N_i = B_i + B_i'$, that is, the $i$-th connected component of $B_{red} = \sum_{i=1\ldots6} B_i + B_i'$. Then in $\text{Pic} S$ we have

$$(41) \quad [E] = \sum_{i=1\ldots6} \frac{1}{3}[B_i + 2B_i'] \quad [\tilde{E}] = \sum_{i=1\ldots6} \frac{1}{3}[2B_i + B_i']$$ 

up to exchanging $E$ with $\tilde{E}$. Since $2H - \tilde{H} \sim C - 2E + \tilde{E}$, it follows that

$$(42) \quad 2H - \tilde{H} \sim C - \sum_{i=1\ldots6} B_i'.$$ 

This implies that $[2H - \tilde{H}]$ is not an effective class. Indeed let $B' := B_1' + \cdots + B_6'$, observe that $(C - B')B_i = -1$, $i = 1 \ldots 6$. Assume $C - B' \sim F$ where $F$ is an effective divisor. Then $F|B_i = -1$ implies $B_i \subset F$ and $F = F' + B_1 + \cdots + B_6$ where $F'$ is effective. Hence $C - B_{red} \sim F' > 0$: a contradiction to the above lemma.

We will profit of genus 3 curves of the non hyperelliptic linear systems $|H|$ or $|\tilde{H}|$.

Lemma 6.8. It holds $\forall D \in |H|, \quad h^0(\mathcal{O}_D(\tilde{H} - H)) = 0$ and $\forall \tilde{D} \in |\tilde{H}|, \quad h^0(\mathcal{O}_{\tilde{D}}(H - \tilde{H})) = 0$. 

Proof. Let $D \in |H|$, once more consider the standard exact sequence

$$0 \to \mathcal{O}_S(\hat{H} - 2H) \to \mathcal{O}_S(\hat{H} - H) \to \mathcal{O}_D(\hat{H} - H) \to 0$$

and its long exact sequence. We have $h^1(\mathcal{O}_S(\hat{H} - 2H)) = h^1(\mathcal{O}_S(2H - H)) = 0$ by the previous lemma and $h^0(\mathcal{O}_S(\hat{H} - 2H)) = 0$ because $H(\hat{H} - 2H) = -2$. Then it follows $h^0(\mathcal{O}_D(\hat{H} - H)) = h^0(\mathcal{O}_S(\hat{H} - H))$. Finally the latter is zero by Proposition (14).

Let $D \in |H|$ be smooth then $\mathcal{O}_D(\hat{H} - H) \cong \mathcal{O}_D(b)$, where $\deg b = 2$. We fix the notation $b$ for such a divisor and the notation $\mu_D$ for the following multiplication map:

$$\mu_D : H^0(\omega_D) \otimes H^0(\omega_D(b)) \to H^0(\omega_D(b)).$$

Let us also point out that $h^0(\mathcal{O}_D(b)) = 0$ by the above lemma. Moreover we fix the notation

$$\nu_D : H^0(\mathcal{O}_S(H)) \to H^0(\omega_D), \quad \hat{\nu}_D : H^0(\mathcal{O}_S(\hat{H})) \to H^0(\omega_D(b)),$$

$$\rho_D : H^0(\mathcal{O}_S(H + \hat{H})) \to H^0(\omega_D^\otimes 2(b))$$

for the natural restriction maps. Then we consider the commutative diagram:

$$\begin{array}{ccc}
H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(\hat{H})) & \xrightarrow{\mu_S} & H^0(\mathcal{O}_S(H + \hat{H})) \\
H^0(\omega_D) \otimes H^0(\omega_D(b)) & \xrightarrow{\nu_D \otimes \hat{\nu}_D} & H^0(\omega_D^\otimes 2(b)).
\end{array}$$

which is similar to our main diagram.

Proposition 6.9. The vertical arrows and the horizontal arrow $\mu_D$ are surjective.

Proof. Let $p : S \to \mathbb{P}^2$ be the map defined by $|H|$, then $p|D : D \to \mathbb{P}^2 = |\omega_D|^* \, \text{is the canonical map and } |\omega_D(b)| \, \text{is cut on } D$ by $|\mathcal{I}_d|S(3H)|$, where $d$ is any element of $|\omega^\otimes 2(-b)|$, and $\mathcal{I}_d|S(3H)$ is its ideal sheaf. Moreover the map $p^* : |\mathcal{I}_d|\mathbb{P}^2(3)| \to |\omega_D^\otimes 2||$ is an isomorphism and $|\mathcal{I}_d|\mathbb{P}^2(3)| = p^*|\mathcal{I}_d|\mathbb{P}^2(3)$, where $Z = p_*, d$ and $\mathcal{I}_d|\mathbb{P}^2$ is its ideal sheaf. Hence it follows $h^0(\mathcal{I}_d|\mathbb{P}^2(3)) = h^0(\omega_D^\otimes 2(-b)) = h^0(\mathcal{O}_D(b)) = 0$ and $h^1(\mathcal{I}_d|\mathbb{P}^2(3)) = h^0(\mathcal{O}_D(b)) = 0$. This easily implies $h^i(\mathcal{I}_d|\mathbb{P}^2(3 - i)) = 0$ for $i > 0$, that is, $\mathcal{I}_d|\mathbb{P}^2$ is 3-regular. Hence, by Castelnuovo-Mumford regularity theorem, the multiplication map

$$\mu : H^0(\mathcal{I}_d|\mathbb{P}^2(1)) \otimes H^0(\mathcal{I}_d|\mathbb{P}^2(3)) \to H^0(\mathcal{I}_d|\mathbb{P}^2(4))$$

is surjective. Now consider the standard exact sequence of ideal sheaves

$$0 \to \mathcal{I}_d|\mathbb{P}^2(4) \to \mathcal{I}_d|\mathbb{P}^2(4) \to \mathcal{I}_d|\mathbb{P}^2(4) \to 0$$

and its associated long exact sequence. Since $\mathcal{I}_d|\mathbb{P}^2(4) \cong \mathcal{O}_{\mathbb{P}^2}$ it follows that

$$h^0(\rho) : H^0(\mathcal{I}_d|\mathbb{P}^2(4)) \to H^0(\omega_D^\otimes 2(b))$$

is surjective. On the other hand we have $\mu_D \circ \lambda = h^0(\rho) \circ \mu$, where $\lambda$ is the tensor product

$$\lambda_1 \otimes \lambda_2 : H^0(\mathcal{O}_S(1)) \otimes H^0(\mathcal{I}_d|\mathbb{P}^2(3)) \to H^0(\omega_D) \otimes H^0(\omega_D^\otimes 2(b))$$

of the natural isomorphisms $\lambda_1 : H^0(\mathcal{O}_S(1)) \to H^0(\omega_D)$ and $\lambda_2 : H^0(\mathcal{I}_d|\mathbb{P}^2(3)) \to H^0(\omega_D^\otimes 2(b))$. Since $\lambda$ is an isomorphism and $h^0(\rho)$ and $\mu$ are surjective, then $\mu_D$ is surjective. The surjectivity of $\rho_D$ follows from the vanishing of $h^1(\mathcal{O}_S(H))$ and the standard exact sequence

$$0 \to \mathcal{O}_S(H) \to \mathcal{O}_S(H + \hat{H}) \to \omega_D^\otimes 2(b) \to 0.$$ 

Since $\omega_D^\otimes 2(b)$ is $O_D(H + \hat{H})$, the surjectivity of $\nu_D$ follows from the above exact sequence twisted by $-\hat{H}$. Finally the exact sequence

$$0 \to \mathcal{O}_S(\hat{H} - H) \to \mathcal{O}_S(\hat{H}) \to \omega_D(b) \to 0$$

implies that $\hat{\nu}_D$ is an isomorphism. Indeed we have $h^0(\mathcal{O}_S(\hat{H} - H)) = h^1(\mathcal{O}_S(\hat{H} - H)) = 0$ in its long exact sequence by (32). Hence $\nu_D \otimes \hat{\nu}_D$ is surjective too.

Proposition 6.10. The map $\mu_S : H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(\hat{H})) \to H^0(\mathcal{O}_S(H + \hat{H}))$ is surjective.
Proof. Let us consider again the commutative diagram (45), that is,
\[ H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(\hat{H})) \xrightarrow{\mu_S} H^0(\mathcal{O}_S(H+\hat{H})) \]
\[ \Downarrow \rho_D \otimes \rho_D \quad \Downarrow \rho_D \]
\[ H^0(\omega_D) \otimes H^0(\omega_D(b)) \xrightarrow{\mu_D} H^0(\omega_D^2(b)). \]

Counting dimensions we have \( \dim \ker \mu_S \geq 4 \), hence it suffices to show that the equality holds. Now we know that \( \mu_D \) and \( \nu_D \otimes \nu_D \) are surjective. Let \( K \) be the Kernel of \( \mu_D \circ (\nu_D \otimes \nu_D) \), then the dimension count gives \( \dim K = 8 \) and, of course, we have \( \ker \mu_S \subseteq K \). Therefore, to prove \( \dim \ker \mu_S = 4 \), it suffices to produce a 4-dimensional subspace \( V \subseteq K \) such that \( V \cap \ker \mu_S = 0 \).

To this purpose consider the space of decomposable vectors \( \mathcal{V} \) and the same is true for \( \mathcal{V}^* \).

Now we go back, in genus 5, to our usual diagram (45) in section 2. This is
\[ H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(\hat{H})) \xrightarrow{\mu_S} H^0(\mathcal{O}_S(H+\hat{H})) \]
\[ \Downarrow \rho_H \otimes \rho_H \quad \Downarrow \rho_C \]
\[ H^0(\mathcal{O}_C(\eta) \otimes H^0(\mathcal{O}_C(\eta)^{-1}) \xrightarrow{\mu_C} H^0(\mathcal{O}_C^2). \]

Proposition 6.11. \( \mu_C : H^0(\mathcal{O}_C(\eta) \otimes H^0(\mathcal{O}_C(\eta)^{-1}) \rightarrow H^0(\mathcal{O}_C^2) \) is surjective.

Proof. We have already shown that \( \mu_S \) and \( \mu_H \otimes \rho_H \) are surjective. By (40) and its related lemma the same is true for \( \rho_C \). Hence the surjectivity of \( \mu_C \) follows. \( \square \)

Let \( \mathbb{P}^{15} := \mathbb{P}(H^0(\mathcal{O}_S(H))^* \otimes H^0(\mathcal{O}_S(\hat{H}))^*) \) and let \( \mathbb{P}^3 \times \mathbb{P}^3 := \mathbb{P}(|H^* \times |\hat{H}|^* \rightarrow \mathbb{P}^3 \times \mathbb{P}^3 \cap \mathbb{P}^{15} \).

In other words \( f \) is just the morphism defined by the complete linear system \(|H + \hat{H}|\) composed with the linear embedding \( \mathbb{P}^{11} \subseteq \mathbb{P}^{15} \).

Proposition 6.12. The map \( p \times \hat{p} \) is an embedding for a general point \([S,L,E] \in \mathcal{F}_{5,3}^1\).

Proof. The linear systems \(|H|\) and \(|\hat{H}|\) are non hyperelliptic. Hence \( p, \hat{p} \) are generically injective and the same is true for \( f \). In particular \( f : S \rightarrow f(S) \) is biregular over \( f(S) - \text{Sing} f(S) \) and \( \text{Sing} f(S) \) is a finite set of rational double points. Let \( R \subseteq S \) be an integral curve contracted by \( f \) then \( R \) is biregular to \( \mathbb{P}^1 \) but it is not \( B_i \). Indeed \( R \) is contracted by \( p \) and \( \hat{p} \) while \( B_i \) is not, as one can directly compute. Notice also that \( C \simeq \frac{1}{2}(H + \hat{H} + B_{\text{red}}) \). Therefore, since \( R C \geq 0 \), it follows
\[ RC = \frac{1}{2} \sum_{i=1 \ldots 12} RB_i \geq 0 \]
with \( RB_i \geq 0 \). Assume \( RB_i = 0 \) for each \( i \), then \( RC = 0 \). Since the Picard group of \( S \) is \( \mathbb{Z}[L] \oplus M_3 \), \( R \) is necessarily contained in \( M_3 = \mathbb{Z}[L] \). By (15) the unique \((-2)\)-curves contained in \( M_3 \) are the \( B_i \)'s, which contradicts the fact that \( R \) cannot be a \( B_i \). Now assume that \( RB_i \geq 2 \) for some \( B_i \) and consider, among the maps \( p \) and \( \hat{p} \), the one not contracting \( B_i \), say \( p \). Then \( p \) embeds \( B_i \) as a line. On the other hand \( p \) contracts \( R \cdot B_i \), which is a divisor of degree \( \geq 2 \) in \( B_i \) : a contradiction. This implies \( RB_i = 1 \) for each \( i \).

Finally consider two distinct curves as above, say \( B_1 \) and \( B_2 \), which are contracted by \( p \). Let us also claim that \( p(B_1) \) and \( p(B_2) \) are distinct points for a general \((S,L,E)\). Since \( RB_1 = RB_2 = 1 \) then \( p(R) \) is not a point: a contradiction.
We now prove that \( p(B_1) \neq p(B_2) \) for a general \((S, L, \mathcal{E})\). If two curves are contracted by a map \( p \) to the same point, there is a tree of \((-2)\)-curves connecting these curves which is contracted by \( p \). Since \( p \) is defined by \([H]\), the \((-2)\)-curves contracted by \( p \) are orthogonal to \( H \) in \( \mathbb{Z}[L] \oplus M_3 \), which is the Picard group of a general \( S \). By a direct computation one observes that the negative defined lattice orthogonal to \( H \) contains exactly 12 \((-2)\)-classes, which are \( \pm B_i \) for \( i = 1, \ldots, 6 \). Since \( B_iB_j = 0 \) if \( i, j \in \{1, \ldots, 6\} \) and \( i \neq j \), \( p(B_1) \neq p(B_2) \).

At this point the special geometry determined by \( \mu_S \) appears, we have

\[
\text{Ker} \mu_S = H^0(\mathcal{I}(1,1)),
\]

where \( \mathcal{I} \) is the ideal sheaf of \( \mathbb{P}^1 \cdot (\mathbb{P}^3 \times \mathbb{P}^3) \) in \( \mathbb{P}^3 \times \mathbb{P}^3 \) and \( \dim \text{Ker} \mu_S = 4 \). Let

\[
\Sigma := \mathbb{P}^1 \cdot (\mathbb{P}^3 \times \mathbb{P}^3),
\]

then \( f(S) \) sits in \( \mathbb{P}^1 \) as a K3 surface of degree 20 and \( f(S) \subseteq \Sigma \). Now assume that the intersection scheme \( \Sigma \) is proper, then \( \Sigma \) is a K3 surface of degree 20 and hence

\[
f(S) = \Sigma.
\]

Postponing its proof, we therefore assume the following claim.

**Claim** For a general triple \((S, L, \mathcal{E})\) the intersection scheme \( \Sigma \) is proper.

Then we prove the birationality of the Mukai map \( r_{5,3} : \mathcal{P}_{5,3}^+ \to \mathcal{R}_{5,3} \).

**Proof of the birationality.** Since \( \mathcal{P}_{5,3}^+ \) and \( \mathcal{R}_{5,3} \) are irreducible of the same dimension, it suffices to show that \( r_{5,3} \) is birational onto \( \mathcal{M} := r_{5,3}(\mathcal{P}_{5,3}^+) \). Let \( x = [S, L, \mathcal{E}, C] \) be general in \( \mathcal{P}_{5,3}^+ \) and \( y = r_{5,3}(x) \), then \( y = [C, \eta] \) with \( \eta := \mathcal{E} \otimes \mathcal{O}_C \). Let \( y \in \mathcal{M} \) be general, we prove that a unique \( x = [S, L, \mathcal{E}, C] \) exists so that \([C, \mathcal{E} \otimes \mathcal{O}_C] = y\). We already know, for a general \( y = [C, \eta] \in \mathcal{M} \), the surjectivity of the multiplication map

\[
\mu_C : H^0(\omega_C \otimes \eta) \otimes H^0(\omega_C \otimes \eta^{-1}) \to H^0(\omega_C^{[2]}),
\]

because this condition is open and non empty on \( \mathcal{M} \). Then, applying to \( \mu_C \) the same construction applied to \( \mu_S \), one obtains

\[
C \subseteq \Sigma := \mathbb{P}^1 \cdot (\mathbb{P}^3 \times \mathbb{P}^3) \subseteq \mathbb{P}^1.
\]

Let \( V = H^0(\omega_C \otimes \eta)^* \) and \( \tilde{V} = H^0(\omega_C \otimes \eta^{-1})^* \), here \( C \) is bicanonically embedded in \( \mathbb{P}^1 := \mathbb{P}(\text{Im} \mu_C)^* \) and the inclusion is the Segre embedding \( \mathbb{P}(V) \times \mathbb{P}(\tilde{V}) \subset \mathbb{P}(V \otimes \tilde{V}) \). Now the properness of \( \Sigma \) is an open condition on \( \mathcal{M} \), not empty under our claim. Then \((\Sigma, \mathcal{O}_\Sigma(1))\) is a polarized K3 surface as above. Since \( y = r_{5,3}(x) \) for some \( x = [S, L, \mathcal{E}, C] \), the commutative diagram \((17)\) implies that \([\Sigma, \mathcal{O}_\Sigma(1)] = [S, L] \). Therefore \( \mu_C \) defines a rational map, sending \( y = [C, \eta] \in \mathcal{M} \) to \( x \in \mathcal{P}_{5,3}^+ \), which is inverse to \( r_{5,3} \).\( \square \)

**Proof of the claim.** Since each component of \( \Sigma \) has dimension \( \geq 2 \), it suffices to construct one \( \mathbb{D} \in (\mathcal{O}_{\mathbb{P}^3}(\mathbb{P}^3(1,1))) \) so that \( \mathbb{D} \cdot \Sigma = \mathbb{D} \cdot S \). We choose the hyperplane section

\[
\mathbb{D} = (P \times P^3) + (P \times \tilde{P}),
\]

where \( P \) and \( \tilde{P} \) are general planes. Then we have \( \mathbb{D} \cdot S = D + \tilde{D} \), where \( D \in [H] \) and \( \tilde{D} \in [\tilde{H}] \) are smooth, non hyperelliptic curves of genus \( 3 \). We show, only for \( D \), that

\[
D = \mathbb{P}^1 \cdot (P \times \mathbb{P}^3), \quad \tilde{D} = \mathbb{P}^1 \cdot (P \times \tilde{P}).
\]

The map \( p : D \to P \) is the canonical map; we fix on \( P \) coordinates \((x) = (x_1 : x_2 : x_3)\). The map \( \tilde{p} : D \to \mathbb{P}^3 \) is defined by \([\omega_D(b)]\), where \( \deg b = 2 \) and \( h^0(\mathcal{O}_D(b)) = 0 \). This implies that \( \omega_D(b) \) is very ample, we fix coordinates \((y) = (y_1 : \cdots : y_4) \) on \( \mathbb{P}^3 \). The resolution of \( \mathcal{O}_{\tilde{p}(D)}(1) \cong \omega_D(b) \) is definitely ample known, [15]. We have the exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^3(-1)} \otimes A \to \mathcal{O}_{\mathbb{P}^3} \otimes \omega_D(b) \to 0,
\]

\( A = (a_{ij}) \) being a \( 4 \times 3 \) matrix of linear forms in \((y)\). Then \( \tilde{p}(D) \) is a determinantal curve defined by the cubic minors of \( A \). In particular \( A \) has rank \( 3 \) on \( \mathbb{P}^3 = \tilde{p}(D) \) and, since \( \tilde{p} : D \to \tilde{p}(D) \) is birational and \( \tilde{p}(D) \) is smooth, it also follows that \( \tilde{p}(D) \) is the set of points \( y \in \mathbb{P}^3 \) such that \( A \) has exactly rank \( 2 \). This implies that the equations \( a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 = 0 \), \( i = 1 \ldots 4 \), define a complete
intersection \( \hat{D} \subset P \times \mathbb{P}^3 \) such that \( \text{Supp} \hat{D} = D \). Finally one easily computes that \( \hat{D} \) and \( D \) have the same degree 10 with respect to \( \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1,1) \). This implies \( \hat{D} = D \) and the claim follows. \( \square \)

6.3. The case of genus 6.

**Theorem 6.13.** The Mukai map \( r_{6,3} : \mathcal{P}_{6,3}^+ \to \mathcal{R}_{6,3} \) has not maximal rank.

In this paper we only sketch the proof of this theorem and its geometric motivation: see section 7 and also [27]. We postpone some details to further investigation on \( \mathcal{R}_{6,3} \). We conclude that the mentioned analogies are confirmed for \( \ell = 3 \): the Mukai maps

\[
m_{11 \pm 1}, r_{7 \pm 1,2}, r_{5 \pm 1,3}
\]

have not maximal rank, while they are birational for \( g = 11, 7, 5 \). These maps are not dominant for \( g = 10, 6, 4 \) and they have positive dimensional fibre for \( g = 12, 8, 6 \).

7. Views on Fano threefolds with sections of level 2 or 3

We close this paper discussing some families of Fano threefolds \( \overline{X} \subset \mathbb{P}^{g+1} \), whose general hyperplane sections are singular K3 surfaces \( S \) of the considered types. Then \( S \) is endowed with a degree \( \ell \) cyclic cover \( \pi : \tilde{S} \to S \) with branch locus \( \text{Sing} \; S \). Moreover its minimal desingularization \( \nu : S \to \bar{S} \) fits in a standard level \( \ell \) K3 surface \( (S, \mathcal{L}, \mathcal{E}) \), so that \( \mathcal{L} \cong \nu^*\mathcal{O}_S(1) \) and \( \mathcal{E} \) induces \( \pi : \tilde{S} \to \bar{S} \). We have \( \ell = 2, 3 \).

For some families a natural cyclic cover \( \pi_{\overline{X}} : \tilde{X} \to \overline{X} \) is visible, with branch locus the curve \( \text{Sing} \; \overline{X} \). However we do not address it here. The existence of these families implies that \( r_{g,\ell} \) has not maximal rank. They correspond to the peculiar values

\[
(g, \ell) = (6,3), (6,2), (8,2), (4,3).
\]

For \( \ell = 2 \) these families are known. [11] [19] [20]. The case \( (6,2) \) is revisited here with emphasis on a singular quadratic complex of the Grassmannian \( G(2,5) \). This implies that \( r_{6,2} \) is not of maximal rank. For \( (6,3) \) we introduce a family of Gushel - Mukai threefolds singular along a rational normal sextic curve. This is responsible for the failure of the maximal rank of \( r_{6,3} \). The case \( (8,2) \) is similar and not treated here, [20]. Finally we point out the plausible relation of the case \( (4,3) \) to the \( G_2 \)-variety.

7.1. A singular Gushel - Mukai threefold: \( \ell = 3 \) and \( g = 6 \). We sketch the geometric construction implying theorem [6,13]. Let \( g = 6 \) and \( \ell = 3 \), keeping our notation we consider \( p \times \tilde{p} : S \to \mathbb{P}^4 \times \mathbb{P}^4 \). Then \( p \) is defined by the linear system

\[
|H| = |C - \sum_{i=1}^{6} (B_i + 2B'_i)|,
\]

where \( B_i + B'_i \), are the connected components of \( B_{\text{red}} \). Let \( x_0 := [S, \mathcal{L}, \mathcal{E}, C] \in \mathcal{P}_{6,3}^+ \) be a general point, then a standard analysis shows that \( p : S \to p(S) \) is the contraction of \( \sum B_i \) to six points and that \( p(B'_i) \) is a line. Moreover we have

\[
p(S) = F_0 \cap Q,
\]

where \( F_0 \) is a cubic and \( Q \) a smooth quadric. Notice that \( p(C) \) is the embedding defined by \( \omega_C \otimes \eta^{-1} \), since \( CB_i = 0 \) then \( p(C) \cap \text{Sing} \; p(S) = \emptyset \). Let \( C' := p(C) \) and let

\[
0 \to I_{p(S)}(3) \to I_{p(C)}(3) \to I_{C'|p(S)}(3) \to 0
\]

be the standard exact sequence of ideal sheaves of \( Q \), we notice the isomorphisms \( I_{p(S)}(3) \cong \mathcal{O}_Q \) and \( p_* : H^0(\mathcal{O}_S(3H - C)) \to H^0(I_{p(C)}|p(S)(3)) \). This implies that

\[
0 \to H^0(\mathcal{O}_Q) \to H^0(\mathcal{I}_C(3)) \to H^0(\mathcal{O}_S(3H - C)) \to 0
\]

is its associated long exact sequence. It easily follows that \( C' \) is projectively normal. A second standard step is the remark that \( \mathcal{O}_S(3H - C) \) is a genus 3 polarization of \( S \). Now let \( M \in |3H - C| \), then \( p_*(C + M) \in |I_{p(C)}|p(S)(3)| \) and it is cut on \( p(S) \) by a cubic hypersurface. Therefore we have in \( Q \) the complete intersection scheme

\[
p_*(C + M) = F_0 \cap F_{\infty} \cap Q,
\]
where \( F_0, F_\infty \) are cubics. Let \( S'_0 = F_0 \cdot Q \) and \( S'_\infty = F_\infty \cdot Q \). We consider the pencil
\[
P_M = \{ S'_t, t \in \mathbb{P}^1 \},
\]
of cubic sections of \( Q \) generated by \( S'_0 \) and \( S'_\infty \). We can assume \( p(S) = S'_0 \), notice that a general \( S'_t \) is a possibly singular K3 surface, smooth along \( C' \). Let \( \sigma_t : S_t \to S'_t \) be its minimal desingularization and \( C_t := \sigma_t^* C' \), then \( S_t \) is endowed with the line bundles
\[
\mathcal{H}_t := \sigma_t^* \mathcal{O}_Q(1), \quad \mathcal{L}_t := \mathcal{O}_{S_t}(C_t), \quad \mathcal{E}_t := \mathcal{L}_t \otimes \mathcal{H}_t^{-1}.
\]
For \( t = 0 \) the fourtuple \((S_t, \mathcal{L}_t, \mathcal{E}_t, C_t)\) defines the point \( x_0 = [S, \mathcal{L}, \mathcal{E}, C] \) of \( \mathcal{P}^1_{6,3} \). For \( t \neq 0 \) we have constantly \( C_t = C \). Now consider the family of fourtuples
\[
\{ (S_t, \mathcal{L}_t, \mathcal{E}_t, C_t), t \in \mathbb{P}^1 \},
\]
then the assignment \( t \mapsto [S_t, \mathcal{L}_t] \in \mathcal{F}_0 \) defines a non constant rational map \( m : \mathbb{P}^1 \to \mathcal{F}_0 \). Assume \((S_t, \mathcal{L}_t, \mathcal{E}_t)\) is a K3 surface of level 3 for a general \( t \). Then \( m \) lifts to a map \( \tilde{m} : \mathbb{P}^1 \to \mathcal{P}^1_{6,3} \), sending \( t \) to \([S_t, \mathcal{L}_t, \mathcal{E}_t, C_t]\), and the next statement immediately follows.

**Proposition 7.1.** If \((S_t, \mathcal{L}_t, \mathcal{E}_t)\) is a K3 surface of level 3 for a general \( t \), the curve \( \tilde{m}(\mathbb{P}^1) \) is in the fibre at the point \([C, \eta] \) of the Mukai map \( r_{6,3} \), which is therefore not of maximal rank.

The assumption mentioned in the statement depends on the choice of the element \( M \) in \( |3H - C| \) and in general it is not satisfied. However the assumption is satisfied choosing in \(|M|\) the very special element
\[
M_0 := 2A + \sum_{i=1,...,6} B_i,
\]
where \( A \) is the unique element of \(|C - \sum_{i=1,...,6}(B_i + B'_i)|\). The curve \( A \) is biregular to \( \mathbb{P}^1 \) and \( p|A \) embeds it as a rational normal quartic curve. Let \( A' = p(A) \), then the base scheme of \( P_{M_0} \) is a non reduced, complete intersection curve and its 1-cycle is
\[
p_*(M_0 + C) = 2A' + C'.
\]
In other words the surfaces \( S'_t \) intersect along a contact curve \( A' \) of multiplicity two and along \( C' \). It turns out that a general \( \text{Sing} S'_t \) consists of six nodes moving in \( A' \) and each node belongs to a line in \( S'_t \). This can be shown using the special property that \( \eta \cong \omega_{C'}(-1) \in \text{Pic} C \) is of 3-torsion. Omitting further details of this construction, let us just say that \( M_0 \) defines a pencil of level 3 and genus 6 K3 surfaces as required.

To close geometrically this sketch let \( A \) be the non reduced component, supported on \( A' \), of the base curve of \( P_{M_0} \) and \( \mathcal{I}_{A|Q} \) its ideal sheaf. Consider the rational map
\[
\phi : Q \to \mathbb{P}^7
\]
defined by the linear system \(|\mathcal{I}_{A|Q}(3)|\). Let us notice the following property.

**Proposition 7.2.** The map \( \phi \) is birational onto its image \( W \), which is a singular Gushel - Mukai threefold whose general hyperplane sections are singular K3 surfaces \( S \) as above.

Therefore \( W \) is a complete intersection of type \((1,1,2)\) in the Grassmannian \( G(2,5) \). We notice that \( \text{Sing} W \) is a rational normal sextic curve. This completes our sketch.

7.2. The tangential quadratic complex of \( \mathbb{P}^4 \): \( \ell = 2 \) and \( g = 6 \). Let \( G_n \) be the Plücker embedding of the Grassmannian of lines of \( \mathbb{P}^n \), a quadratic complex is just a quadratic section of \( G_n \). Let \( Q \subset \mathbb{P}^n \) be a quadric, then the family \( T \) of tangent lines to \( Q \) is a quadratic complex, named sometimes the tangential quadratic complex. We assume \( Q \) is smooth, then \( T \) is a Fano variety. Notice that \( \text{Sing} T \) is the Hilbert scheme of lines of \( Q \), of codimension and multiplicity 2 in \( T \).

Now we assume \( n \) is even. Then \( T \) has a unique nontrivial quasi étale 2:1 cover
\[
\pi : \tilde{T} \to T,
\]
whose branch locus is \( \text{Sing} T \). Let us describe the known map \( \pi \) in the case \( n = 4 \), since it is linked to the Mukai map \( r_{6,2} : \mathcal{P}^1_{6,2} \to \mathcal{R}_6 \) and its behavior. This is treated in [11]. For \( n = 4 \) the Hilbert scheme of lines of \( Q \) is the 2-Veronese embedding of \( \mathbb{P}^3 \), say
\[
V \subset G_4 \subset \mathbb{P}^9.
\]
Let \( t \in T \), consider the pencil \( \{ H_p, p \in t \} \), where \( H_p \) is the polar hyperplane to \( Q \) at \( p \). Its base locus is a plane \( P_t \) and \( Q_t := P_t \cdot Q \) is a conic. Since \( t \) is tangent to \( Q \), a standard exercise shows that \( \text{Sing} \; Q_t = t \cap Q \). This defines a smooth, integral correspondence
\[
\tilde{T} := \{(t, r) \in T \times V \mid r \subset Q_t\}.
\]
Notice that its projection onto \( T \) is a quasi étale 2 : 1 cover branched on \( V \), say
\[
\pi : \tilde{T} \to T.
\]
Indeed the fibre \( \zeta_t := \pi^*(t) \) is the Hilbert scheme of lines of \( Q_t \) and is finite of length 2. Then \( \zeta_t \) is smooth iff rank \( Q_t = 2 \) iff \( t \notin V \) and \( \zeta_t \) has multiplicity 2 iff rank \( Q_t = 1 \) iff \( t \in V \).

Now it is well known that a general 2-dimensional linear section \( \tilde{S} = T \cap \mathbb{P}^6 \) is the model defined by \( |C| \) of \( S \), where \([S, L, E] \in \mathcal{F}_6 \) is general. In particular \( \text{Sing} \; \tilde{S} = V \cap \mathbb{P}^6 \) is an even set of 8 nodes, defining \( \pi|\tilde{S} \) with \( \tilde{S} = \pi^{-1}(S) \), cfr. [11][19][20]. For \( \ell = 2 \) and \([S, L, E] \in \mathcal{F}_g \), the surface \( S \), or its model \( \tilde{S} \), is known as a standard Nikulin surface of genus \( g \). Therefore we can say that a general 3-dimensional linear section of \( T \) is a Fano threefold whose hyperplane sections are standard Nikulin surfaces of genus 6. Let us denote such a section by
\[
X = T \cap \mathbb{P}^7,
\]
notice that \( \text{Sing} \; X \) is a curvilinear section of \( V \), hence an elliptic curve of degree 8.

Finally let \( C \) and \( \tilde{S} \) respectively be the family of general curvilinear sections \( C \) and that of general 2-dimensional linear sections \( \tilde{S} \) of \( T \). Consider the family of pairs
\[
\mathcal{P} := \{(C, \tilde{S}) \in C \times \tilde{S} \mid C \subset \tilde{S}\}.
\]
Let \((C, \tilde{S}) \in \mathcal{P} \) then \( C \) is a canonical curve and \( C \in |\mathcal{O}(1)| \). Let \( \nu : S \to \tilde{S} \) be the desingularization then \( \nu^*C \in |L| \) and \( \eta := E \otimes \mathcal{O}_{\nu^*C} \) defines \( \pi|\tilde{C} \), where \( \tilde{C} = \pi^{-1}(C) \). Then the assignment of \((C, \tilde{S})\) to \([S, L, E, \nu^*C] \) defines a dominant rational map
\[
m : \mathcal{P} \to \mathcal{P}^+.
\]

We already know that the Mukai map \( r_{6,2} \) fails to be of maximal rank. However we can now see this fact from a geometric perspective: the existence of the Fano variety \( T \) and its quasi finite 2 : 1 cover \( \pi \). Indeed this implies that \( C \subset \tilde{C} \) is contained in a higher dimension family of sections \( \tilde{S} \) of \( T \), so that \( C \) cannot have general moduli.

More precisely the parameter space \( C \) is open in the Grassmannian \( G(5,9) \), hence \( \dim C = 24 \). Moreover \( \text{Aut} \; Q \subset \text{Aut} \; \mathbb{P}^4 \) has dimension 10 and acts faithfully on \( C \). Then we have \( \dim \mathcal{C} // \text{Aut} \; Q = 14 < \dim r_{6,2} = 15 \). Hence \( r_{6,2} \) cannot be dominant.

**Remark 7.1.** Let \( C \subset \mathcal{C} \) then \( \tilde{C} = \pi^{-1}(C) \) is a smooth, integral curve of genus 11. We have \( \tilde{C} \subset \tilde{S} \subset \tilde{X} \subset \mathbb{P}^{12} \), where \( \tilde{X} = \pi^{-1}(X) \) is a non prime Fano threefold of genus 11. We just mention that \( \tilde{C} \) is the base locus of a pencil of hyperplane sections of \( \tilde{X} \) and that the birational Mukai map \( m_{11} : \mathcal{P}_{11} \to \mathcal{M}_{11} \) is not invertible at \( \tilde{C} \).

7.3. The \( G_2 \)-variety: \( \ell = 3 \) and \( g = 4 \). A geometric interpretation seems plausible and it is possibly postponed to future work. It relates to the failure of the Mukai map in genus 10. As in [14] let \( \pi : \tilde{S} \to \tilde{S} \) be the cover induced by \( E \) and \( \nu : S \to \tilde{S} \) the desingularization map. For a general \( C \) the map \( \nu : C \to \tilde{S} \setminus \text{Sing} \; \tilde{S} \) is an embedding, then we set \( \tilde{C} := \nu(C) \). Let \( \tilde{C} = \pi^{-1}(C) \) then \( (\tilde{S}, \mathcal{O}_E(C)) \) is a K3 surface of genus 10. This suggests that \( \tilde{S} \) embeds in the \( G_2 \)-variety \( W \subset \mathbb{P}^{13} \) as a linear section, [23]. Now a general curvilinear section of \( W \) is not general as a genus 10 curve. In the same way, if it is a triple cover of a genus 4 curve, it seems not a general genus 4 triple cover.

**References**

[1] A. Beauville *Applications aux espaces de modules*, in *Géométrie des surfaces K3: modules et périodes*, Exp. 13 Astérisque 126 (1985) 141-152
[2] A. Beauville *Fano threefolds and K3 surfaces*, in *The Fano Conference*, Proceedings edited by A. Collino, A. Conte, M. Marchisio, Dip. di Matematica Univ. di Torino (2004) 175-184
[3] S. Boucksom, J.P. Demailly, M. Păun, T. Peternell *The pseudoeffective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Alg. Geom. 22 (2013) 201-24
[4] G. Bruns *Twists of Mukai bundles and the geometry of the level 3 modular variety over \( \mathcal{M}_8 \)*, Trans. Amer. Math. Soc. (2017) 8359-8376
[5] I. Bauer, F. Catanese The Rationality of Certain Moduli Spaces of Curves of Genus 3, in Cohomological and Geometric Approaches to Rationality Problems Progress in Math. J. 282 Birkhäuser, Basel (2009) 1-16
[6] I. Bauer, A. Verra The rationality of the moduli space of genus 4 curves endowed with an order 3 subgroup of their Jacobian, Michigan Math. J. 59 (2010) 483-504
[7] A. Chiodo, D. Eisenbud, G. Farkas, F.O. Schreyer Syzygies of torsion bundles and the geometry of the level 1 modular variety over \( \text{M}_g \), Inventiones Mathematicae 194 (2013), 73-118.
[8] C. Ciliberto, A.F. Lopez, R. Miranda Projective degenerations of K3 surfaces, Gaussian maps, and Fano threefolds, Inventiones mathematicae 114, (1993) 641-667
[9] I Dolgachev Integral quadratic forms: applications to algebraic geometry (after V. Nikulin), S` em. Bourbaki 1982/83, Exp. 611, Ast` erisque 105-106, SMF, Paris (1983) 251-278
[10] H Esnault, E. Viehweg Lectures on vanishing theorems, DMV Seminar, 20 Birkhäuser, Basel, (1992)
[11] G. Farkas, A. Verra, Moduli of theta characteristics via Nikulin surfaces, Math. Annalen 354 (2012), 465–496.
[12] G. Farkas, A. Verra, Prym varieties and moduli of polarized Nikulin surfaces Adv. Math. 290 (2016), 314-328.
[13] A. Garbagnati Y. Prieto Monta˜ nez Order 3 symplectic automorphisms on K3 surfaces ArXiv math.AG 2102.01207 (2021)
[14] A. Garbagnati, A. Sarti, Symplectic automorphisms of prime order on K3 surfaces J. Algebra 318 (2007) 323-350
[15] A. Garbagnati, On K3 surfaces quotients of K3 or abelian surfaces, Canad. J. Math.69 (2017) 338-372
[16] A. Garbagnati, A. Sarti, Projective models of K3 surfaces with an even set, Adv. Geometry 8 (2008), 413-440.
[17] D. Huybrechts Lectures on K3 Surfaces Cambridge studies in advanced mathematics 158 Cambridge UP (2016)
[18] M. Homma On projective normality and defining equations of a projective curve of genus three embedded by a complete linear system, Tsukuba J. Math. 4 (1980), 269-279.
[19] A. Knutsen, A. Lelli Chiesa, A. Verra, Half Nikulin surfaces and moduli of Prym curves J. Inst. Math. Jussieu (2019), 1-38
[20] A. Knutsen, A. Lelli Chiesa, A. Verra, Moduli of non-standard Nikulin surfaces in low genus Annali Sc. Norm. Sup. Pisa 21 (2020) 361-384
[21] R. Lazarsfeld Positivity in Algebraic Geometry, volumes I and II, Ergebnisse der Mathematik 48 and 49 Springer, Basel (2004)
[22] V. Nikulin, Finite automorphism groups of Kähler K3 surfaces Trans. Moscow Math. Soc. 38 (1980), 71-135.
[23] S. Mukai Curves, K3 Surfaces and Fano 3-folds of Genus \( \leq 10 \), in Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata I Kinokunya, Tokio (1987) 357-377
[24] S. Mukai, Curves and K3 surfaces of genus eleven , in Moduli of Vector Bundles, Lecture Notes in Pure and Applied Mathematics, 179, Dekker, New York, (1996) 189-197
[25] F.O. Schreyer Geometry and Algebra of Prime Fano 3-folds of Genus 12 Compositio Mathematica 127 (1999)
[26] A. Verra Geometry of Nikulin surfaces of genus 8 and rationality of their moduli, in K3 Surfaces and their Moduli Progress in Math. 315 (2016) 345-364
[27] A. Verra K3 surfaces and moduli of étale cyclic covers of curves, Slides of a talk, Workshop on Complex Algebraic Geometry, Barcelona (2018)
[28] B. van Geemen, A. Sarti, Nikulin involutions on K3 surfaces, Math. Z. 255 (2007), 731–753.

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