An 8-dimensional non-formal simply connected symplectic manifold

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Abstract

We answer in the affirmative the question posed by Babenko and Taimanov [3] on the existence of non-formal simply connected compact symplectic manifolds of dimension 8.

1 Introduction

Simply connected compact manifolds of dimension less than or equal to 6 are formal [18, 11], and there are simply connected compact manifolds of dimension greater than or equal to 7 which are non-formal [20, 10, 9, 6, 12]. If we are treating the symplectic case, the story is not so straightforward. Lupton and Oprea [15] conjectured that any simply connected compact symplectic manifold is formal. Babenko and Taimanov [2, 3] disproved this conjecture giving examples of non-formal simply connected compact symplectic manifolds of any dimension bigger than or equal to 10, by using the symplectic blow-up [16]. They raise the question of the existence of non-formal simply connected compact symplectic manifolds of dimension 8. The techniques of construction of symplectic manifolds used so far [1, 3, 6, 7, 11, 14, 21, 22] have not proved fruitful when addressing this problem. In this note, we answer the question in the affirmative by proving the following.

Theorem 1.1 There is a simply connected compact symplectic manifold of dimension 8 which is non-formal.

To construct such a manifold, we introduce a new technique to produce symplectic manifolds, which we hope can be useful for obtaining examples with interesting properties. We consider a non-formal compact symplectic 8-dimensional manifold with a symplectic non-free action of a finite group such that the quotient space is a non-formal orbifold which is simply connected. Then we resolve symplectically the singularities to produce a smooth symplectic 8-manifold satisfying the required properties. The origin of the idea stems from our study of Guan’s examples [13] of compact holomorphic symplectic manifolds which are not Kähler.

2 A simply-connected symplectic 8-manifold

Consider the complex Heisenberg group $H_{\mathbb{C}}$, that is, the complex nilpotent Lie group of complex matrices of the form

$$\begin{pmatrix}
1 & u_2 & u_3 \\
0 & 1 & u_1 \\
0 & 0 & 1
\end{pmatrix},$$

1
and let \( G = H_C \times C \), where \( C \) is the additive group of complex numbers. We denote by \( u_4 \) the coordinate function corresponding to this extra factor. In terms of the natural (complex) coordinate functions \((u_1, u_2, u_3, u_4)\) on \( G \), we have that the complex 1-forms \( \mu = du_1, \nu = du_2, \theta = du_3 - u_2 du_1 \) and \( \eta = du_4 \) are left invariant, and

\[
d\mu = d\nu = d\eta = 0, \quad d\theta = \mu \wedge \nu.
\]

Let \( \Lambda \subset C \) be the lattice generated by 1 and \( \zeta = e^{2\pi i/3} \), and consider the discrete subgroup \( \Gamma \subset G \) formed by the matrices in which \( u_1, u_2, u_3, u_4 \in \Lambda \). We define the compact (parallelizable) nilmanifold

\[
M = \Gamma \backslash G.
\]

We can describe \( M \) as a principal torus bundle

\[
T^2 = C/\Lambda \hookrightarrow M \rightarrow T^6 = (C/\Lambda)^3,
\]

by the projection \((u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_4)\).

Now introduce the following action of the finite group \( \mathbb{Z}_3 \)

\[
\rho : G \to G
\]

\[
(u_1, u_2, u_3, u_4) \mapsto (\zeta u_1, \zeta u_2, \zeta^2 u_3, \zeta u_4).
\]

This action satisfies that \( \rho(p \cdot q) = \rho(p) \cdot \rho(q) \), for \( p, q \in G \), where the dot denotes the natural group structure of \( G \). The map \( \rho \) is a particular case of a homothetic transformation (by \( \zeta \) in this case) which is well defined for all nilpotent simply connected Lie groups with graded Lie algebra. Moreover \( \rho(\Gamma) = \Gamma \), therefore \( \rho \) induces an action on the quotient \( M = \Gamma \backslash G \). The action on the forms is given by

\[
\rho^* \mu = \zeta \mu, \quad \rho^* \nu = \zeta \nu, \quad \rho^* \theta = \zeta^2 \theta, \quad \rho^* \eta = \zeta \eta.
\]

The complex 2-form

\[
\omega = i \mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta} + i \eta \wedge \bar{\eta}
\]

is actually a real form which is clearly closed and which satisfies \( \omega^4 \neq 0 \). Thus \( \omega \) is a symplectic form on \( M \). Moreover, \( \omega \) is \( \mathbb{Z}_3 \)-invariant. Hence the space

\[
\tilde{M} = M/\mathbb{Z}_3
\]

is a symplectic orbifold, with the symplectic form \( \tilde{\omega} \) induced by \( \omega \). Our next step is to find a smooth symplectic manifold \( \tilde{M} \) that desingularises \( \tilde{M} \).

**Proposition 2.1** There exists a smooth compact symplectic manifold \( (\tilde{M}, \tilde{\omega}) \) which is isomorphic to \((\tilde{M}, \tilde{\omega})\) outside the singular points.

**Proof** : Let \( p \in M \) be a fixed point of the \( \mathbb{Z}_3 \)-action. Translating by a group element \( g \in G \) taking \( p \) to the origin, we may suppose that \( p = (0, 0, 0, 0) \) in our coordinates. At \( p \), the symplectic form is

\[
\omega_0 = i du_1 \wedge d\bar{u}_1 + du_2 \wedge du_3 + d\bar{u}_2 \wedge d\bar{u}_3 + i du_4 \wedge d\bar{u}_4.
\]

Take now \( \mathbb{Z}_3 \)-equivariant Darboux coordinates around \( p, \Phi : (B, \omega) \to (B_{C^1}(0, \epsilon), \omega_0) \), for some \( \epsilon > 0 \). This means that \( \Phi^* \omega_0 = \omega \) and \( \Phi \circ \rho = d\rho_p \circ \Phi \), where we interpret \((B_{C^1}(0, \epsilon), \omega_0) \subset \)
(\mathcal{T}_p M, \omega_0) \cong (\mathbb{C}^4, \omega_0)$ in the natural way. (The proof of the existence of usual Darboux coordinates in \cite{17} pp. 91–93 carry over to this case, only being careful that all the objects constructed should be $\mathbb{Z}_3$-equivariant.) We denote the new coordinates given by $\Phi$ as $(u_1, u_2, u_3, u_4)$ again (although they are not the same coordinates as before).

Now introduce the new set of coordinates:

$$ (w_1, w_2, w_3, w_4) = \left( u_1, \frac{1}{\sqrt{2}}(u_2 + \bar{u}_3), \frac{i}{\sqrt{2}}(u_3 - \bar{u}_2), u_4 \right). $$

Then the symplectic form $\omega$ can be expressed as

$$ \omega = i (dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2 + dw_3 \wedge d\bar{w}_3 + dw_4 \wedge d\bar{w}_4). $$

Moreover, with respect to these coordinates, the $\mathbb{Z}_3$-action $\rho$ is given as

$$ \rho(w_1, w_2, w_3, w_4) = (\zeta w_1, \zeta w_2, \zeta^2 w_3, \zeta w_4). $$

With this Kähler model for a neighbourhood $B$ of $p$, we may resolve the singularity of $B/\mathbb{Z}_3$ with a non-singular Kähler model. Basically, blow up $B$ at $p$ to get $\tilde{B}$. This replaces the point with a complex projective space $\mathbb{P}^3$ in which $\mathbb{Z}_3$ acts as

$$ [w_1, w_2, w_3, w_4] \mapsto [\zeta w_1, \zeta w_2, \zeta^2 w_3, \zeta w_4] = [w_1, w_2, w_3, w_4]. $$

Therefore there are two components of the fix-point locus of the $\mathbb{Z}_3$-action on $\tilde{B}$, namely the point $q = [0, 0, 1, 0]$ and the complex projective plane $H = \{[w_1, w_2, 0, w_4]\} \subset F = \mathbb{P}^3$. Next blow up $\tilde{B}$ at $q$ and at $H$ to get $\tilde{\mathbb{Z}_3}$. The point $q$ is substituted by a projective space $H_1 = \mathbb{P}^3$. The normal bundle of $H \subset \tilde{B}$ is the sum of the normal bundle of $H \subset F$, which is $\mathcal{O}_{\mathbb{P}^2}(1)$, and the restriction of the normal bundle of $F \subset \tilde{B}$ to $H$, which is $\mathcal{O}_{\mathbb{P}^3(-1)}|_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-1)$. Therefore the second blow-up replaces the plane $H$ by the $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ defined as $H_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. The strict transform of $F \subset \tilde{B}$ under the second blow-up is the blow up $\tilde{F}$ of $F = \mathbb{P}^3$ at $q$, which is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, actually $\tilde{F} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. See Figure 1 below.

![Figure 1: Desingularisation process](image)

The fix-point locus of the $\mathbb{Z}_3$-action on $\tilde{B}$ are exactly the two disjoint divisors $H_1$ and $H_2$. Therefore the quotient $\tilde{B}/\mathbb{Z}_3$ is a smooth Kähler manifold \cite{4} page 82). This provides a symplectic resolution of the singularity $B/\mathbb{Z}_3$. To glue this Kähler model to the symplectic form in the complement of the singular point we use Lemma 2.2 below. We do this at every fixed point to get a smooth symplectic resolution of $\tilde{M}$.

\[ \boxed{\text{QED}} \]
Lemma 2.2 Let \((B, \omega_0)\) be the standard Kähler ball in \(\mathbb{C}^n, n > 1\), and let \(\Pi\) be a finite group acting linearly (by complex isometries) on \(B\) whose only fixed point is the origin. Let \(\phi : (\tilde{B}, \omega_1) \to (B/\Pi, \omega_0)\) be a Kähler resolution of the singularity of the quotient. Then there is a symplectic form \(\Omega\) on \(\tilde{B}\) which coincides with \(\omega_0\) near the boundary, and with a positive multiple of \(\omega_1\) near the exceptional divisor \(E = \phi^{-1}(0)\). Moreover \(\Omega\) is tamed by the complex structure.

Proof: Since \(\phi : (\tilde{B}, \omega_1) \to (B/\Pi, \omega_0)\) is holomorphic, \(\omega_0\) and \(\omega_1\) are Kähler forms in \(\tilde{B} - E = B - \{0\}\) with respect to the same complex structure \(J\). Therefore \((1 - t)\omega_0 + t\omega_1\) is a Kähler form on \(\tilde{B}\), for any number \(0 < t < 1\). (Note that \(\omega_0|_E = 0\), where we denote again by \(\omega_0\) the pull-back to \(\tilde{B}\).

Fix \(\delta > 0\) small and let \(A = \{z \in B | \delta < |z| < 2\delta\} \subset B\). Since \(A\) is simply connected, we may write \(\omega_1 - \omega_0 = d\alpha\), with \(\alpha \in \Omega^1(A)\), which we can furthermore suppose \(\Pi\)-invariant.

Let \(\rho : [0, \infty) \to [0, 1]\) be a smooth function whose value is \(1\) for \(r \leq 1.1\delta\) and \(0\) for \(r \geq 1.9\delta\). Define

\[\Omega = \omega_0 + \epsilon \, d(\rho(|z|)\alpha).\]

This equals \(\omega_0\) for \(|z| \geq 1.9\delta\), and \(\omega_0 + \epsilon (\omega_1 - \omega_0) = (1 - \epsilon)\omega_0 + \epsilon \omega_1\) for \(|z| \leq 1.1\delta\). For \(1.1\delta \leq |z| \leq 1.9\delta\), let \(C > 0\) be a bound of \(d(\rho\alpha)(u,Ju)\), for any \(u\) unitary tangent vector (with respect to the Kähler form \(\omega_0\)). Choose \(0 < \epsilon < \min\{1, C^{-1}\}\). Then \(\Omega(u,Ju) > 0\) for any non-zero \(u\).

Proposition 2.3 The manifold \(\tilde{M}\) is simply connected.

Proof: Fix the base points: let \(p_0 \in M = \Gamma \backslash G\) be the image of \((0,0,0,0) \in G\) and let \(\tilde{p}_0 \in \tilde{M}\) be the image of \(p_0\) under the projection \(M \to \tilde{M}\). There is an epimorphism of fundamental groups

\[\Gamma = \pi_1(M) \to \pi_1(\tilde{M}),\]

since the \(\mathbb{Z}_3\)-action has a fixed point \([5, \text{Corollary 6.3}]\). Now the nilmanifold \(M\) is a principal 2-torus bundle over the 6-torus \(T^6\), so we have an exact sequence

\[\mathbb{Z}^2 \hookrightarrow \Gamma \to \mathbb{Z}^6.\]

Let \(\tilde{p}_0 = \pi(p_0)\), where \(\pi : M \to T^6\) denotes the projection of the torus bundle. Clearly, \(\mathbb{Z}_3\) acts on \(\pi^{-1}(\tilde{p}_0) \cong T^2 = \mathbb{C}/\Lambda\) with 3 fixed points, and the quotient space \(T^2/\mathbb{Z}_3\) is a 2-sphere \(S^2\). So the restriction to \(\mathbb{Z}^2 = \pi_1(T^2) \subset \pi_1(M) = \Gamma\) of the map \(\Gamma \to \pi_1(\tilde{M})\) factors through \(\pi_1(T^2/\mathbb{Z}_3) = \{1\}\), hence it is trivial. Thus the map \(\Gamma \to \pi_1(\tilde{M})\) factors through the quotient \(\mathbb{Z}^0 \to \pi_1(\tilde{M})\). But \(M\) contains three \(\mathbb{Z}_3\)-invariant 2-tori, \(T_1, T_2\), and \(T_3\) (which are the images of \(\{(u_1,0,0,0)\}, \{(0,u_2,0,0)\}\) and \(\{(0,0,0,u_4)\}\), respectively) such that \(\pi_1(\tilde{M})\) is generated by the images of \(\pi_1(T_1), \pi_1(T_2)\), and \(\pi_1(T_3)\). Again, each quotient \(T_i/\mathbb{Z}_3\) is a 2-sphere, hence \(\pi_1(\tilde{M})\) is generated by \(\pi_1(T_1/\mathbb{Z}_3) = \{1\}\), which proves that \(\pi_1(\tilde{M}) = \{1\}\).

Finally, the resolution \(\tilde{M} \to M\) consists of substituting, for each singular point \(p\), a neighbourhood \(B/\mathbb{Z}_3\) of it by a non-singular model \(\tilde{B}/\mathbb{Z}_3\). The fiber over the origin of \(\tilde{B}/\mathbb{Z}_3 \to B/\mathbb{Z}_3\) is simply connected: it consists of the union of the three divisors \(H_1 = \mathbb{P}^3, H_2 = \mathbb{P}(O_{\mathbb{P}^3}(-1) \oplus O_{\mathbb{P}^3}(1))\) and \(\tilde{F}/\mathbb{Z}_3 = \mathbb{P}(O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(3))\), all of them are simply connected spaces, and their intersection pattern forms no cycles (see Figure 1). Therefore, a simple Seifert-Van Kampen argument proves that \(\tilde{M}\) is simply connected.

Lemma 2.4 The odd degree Betti numbers of \(\tilde{M}\) are \(b_1(\tilde{M}) = b_3(\tilde{M}) = b_5(\tilde{M}) = b_7(\tilde{M}) = 0.\)
Proof: As \( \widetilde{M} \) is simply connected, then \( b_1(\widetilde{M}) = 0 \). Next, using Nomizu’s theorem \([19]\) to compute the cohomology of the nilmanifold \( M \), we easily find that \( H^3(M) = W \oplus \overline{W} \), where

\[
W = \langle [\mu \wedge \bar{\mu} \wedge \eta], [\nu \wedge \bar{\nu} \wedge \eta], [\mu \wedge \nu \wedge \eta], [\mu \wedge \bar{\nu} \wedge \eta], [\nu \wedge \eta \wedge \bar{\eta}], [\mu \wedge \nu \wedge \theta],
\]

\[
\quad [\mu \wedge \bar{\nu} \wedge \bar{\theta}], [\bar{\mu} \wedge \nu \wedge \bar{\theta}], [\mu \wedge \bar{\mu} \wedge \bar{\theta}], [\nu \wedge \bar{\nu} \wedge \theta], [\mu \wedge \eta \wedge \bar{\theta}], [\mu \wedge \eta \wedge \theta], [\nu \wedge \eta \wedge \bar{\theta}], [\nu \wedge \eta \wedge \theta], [\bar{\mu} \wedge \bar{\nu} \wedge \theta], [\bar{\mu} \wedge \bar{\nu} \wedge \bar{\theta}] \rangle
\]

and \( \overline{W} \) is its complex conjugate. (Here \( H^*(X) \) denotes cohomology with complex coefficients.) Clearly \( \rho \) acts as multiplication by \( \zeta \) on \( W \) and as multiplication by \( \zeta^2 = \bar{\zeta} \) on \( \overline{W} \). Therefore \( H^3(\overline{M}) = H^3(M)^{\overline{\mathbb{Z}}} = 0 \).

The desingularisation process of Proposition \(2.1\) consists on removing contractible neighborhoods of the form \( B_i/\mathbb{Z}_3 \), \( B_i \cong B_{\mathbb{C}^3}(0, \epsilon) \), around each fixed point \( p_i \), and inserting a non-singular Kähler model \( \widetilde{B}_i/\mathbb{Z}_3 \) which retracts to the “exceptional divisor” \( E_i = \phi^{-1}(0) \), \( \phi : \widetilde{B}_i/\mathbb{Z}_3 \to B_i/\mathbb{Z}_3 \). We glue along the region \( A/\mathbb{Z}_3 \) which retracts into \( S^7/\mathbb{Z}_3 \), a rational homology 7-sphere. An easy Mayer-Vietoris argument then shows that \( H^j(\widetilde{M}) = H^j(M) \oplus (\bigoplus H^j(E_i)) \) for \( 0 < j < 7 \). All the \( E_i \) are diffeomorphic to the 6-dimensional complex manifold depicted in Figure 1, which consists of the union of \( H_1 = \mathbb{P}^3 \), \( H_2 = \mathbb{P}(O_{\mathbb{P}^2}(-1) \oplus O_{\mathbb{P}^2}(1)) \) (a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^3 \)) and \( F/\mathbb{Z}_3 = \mathbb{P}(O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(3)) \) (another \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^2 \)), intersecting in copies of the complex projective plane. So \( H^3(E_i) = 0 \) and hence \( H^3(\overline{M}) = 0 \).

The statement \( b_5(\overline{M}) = b_7(\overline{M}) = 0 \) follows from Poincaré duality. \( \Box \)

3 Non-formality of the constructed manifold

Formality for a simply connected manifold \( M \) means that its rational homotopy type is determined by its cohomology algebra. Let us recall its definition (see \([8, 22]\) for more details). Let \( X \) be a simply connected smooth manifold and consider its algebra of differential forms \( (\Omega^*(X), d) \). Let \( \psi : (\bigwedge V, d) \to (\Omega^*(X), d) \) be a minimal model for this algebra \([8]\). Then \( X \) is formal if there is a quasi-isomorphism \( \psi' : (\bigwedge V, d) \to (H^*(X), d) = 0 \), i.e. a morphism of differential algebras, inducing the identity on cohomology.

Lemma 3.1 Let \( X \) be a simply connected smooth manifold with \( H^3(X) = 0 \), and let \( a, x_1, x_2, x_3 \in H^2(X) \) be cohomology classes satisfying that \( a \cup x_i = 0 \), \( i = 1, 2, 3 \). Choose forms \( \alpha, \beta_i \in \Omega^2(X) \) and \( \xi_i \in \Omega^3(X) \), with \( a = [\alpha] \), \( x_i = [\beta_i] \) and \( \alpha \wedge \beta_i = d\xi_i \), \( i = 1, 2, 3 \). If the cohomology class

\[
[\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2] \in H^5(X)
\]

is non-zero, then \( X \) is non-formal.

Proof: First, notice that

\[
d(\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2) = \alpha \wedge \beta_1 \wedge \xi_2 \wedge \beta_3 - \xi_1 \wedge \alpha \wedge \beta_2 \wedge \beta_3 +
\]

\[
+ \alpha \wedge \beta_2 \wedge \xi_3 \wedge \beta_1 - \xi_2 \wedge \alpha \wedge \beta_3 \wedge \beta_1 + \alpha \wedge \beta_3 \wedge \xi_1 \wedge \beta_2 - \xi_3 \wedge \alpha \wedge \beta_1 \wedge \beta_2 = 0,
\]

so \(1\) is a well-defined cohomology class.

Second, let us see that the cohomology class \(1\) does not depend on the particular forms \( \alpha, \beta_i \in \Omega^2(X) \) and \( \xi_i \in \Omega^3(X) \) chosen. If we write \( a = [\alpha + df] \), with \( f \in \Omega^1(X) \), then \( (\alpha + df) \wedge \beta_i = d(\xi_i + f \wedge \beta_i) \) and

\[
(\xi_1 + f \wedge \beta_1) \wedge (\xi_2 + f \wedge \beta_2) \wedge \beta_3 + (\xi_2 + f \wedge \beta_2) \wedge (\xi_3 + f \wedge \beta_3) \wedge \beta_1 + (\xi_3 + f \wedge \beta_3) \wedge (\xi_1 + f \wedge \beta_1) \wedge \beta_2 = \]
so the cohomology class (1) does not change by changing the representative of \( a \). If we change the representatives of \( x_i \), say for instance \( x_1 = [\beta_1 + df] \), \( f \in \Omega^1(X) \), then \( \alpha \wedge (\beta_1 + df) = d(\xi_1 + \alpha \wedge f) \) and

\[
(\xi_1 + \alpha \wedge f) \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge (\beta_1 + df) + \xi_3 \wedge (\xi_1 + \alpha \wedge f) \wedge \beta_2 = \\
= \xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2 + d(f \wedge \xi_2 \wedge \xi_3),
\]

so the cohomology class (1) does not change again. Finally, if we change the form \( \psi \) and \( \hat{\psi} \), balls into the fixed points. It induces a map \( \hat{\psi} : \Omega^3(X) \) closed, hence exact since \( H^3(X) = 0 \). Also in this case the cohomology class (1) does not change.

To see that \( X \) is non-formal, consider the minimal model \( \psi : (\wedge V, d) \to (\Omega^*(X), d) \) for \( X \). Then there are closed elements \( \hat{a} \), \( \hat{x}_i \in (\wedge V)^3 \) whose images are 2-forms \( \alpha, \beta_i \) representing \( a, x_i \). Since \( [\hat{a} \cdot \hat{x}_i] = 0 \), there are elements \( \hat{\xi}_i \in (\wedge V)^3 \) such that \( d\hat{\xi}_i = \hat{a} \cdot \hat{x}_i \). Let \( \hat{\xi}_i = \psi(\hat{\xi}_i) \in \Omega^3(X) \). So \( d\hat{\xi}_i = \alpha \wedge \beta_i, i = 1, 2, 3 \).

If \( X \) is formal, then there exists a quasi-isomorphism \( \psi' : (\wedge V, d) \to (H^*(X), 0) \). Note that \( \psi'(\hat{\xi}_i) = 0 \) since \( H^3(X) = 0 \). Then

\[
[\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2] = \psi'(\hat{\xi}_1 \wedge \hat{\xi}_2 \wedge \hat{x}_3 + \hat{\xi}_2 \wedge \hat{\xi}_3 \wedge \hat{x}_1 + \hat{\xi}_3 \wedge \hat{\xi}_1 \wedge \hat{x}_2) = 0,
\]

contradicting our assumption. This proves that \( X \) is non-formal.

**Theorem 3.2** The manifold \( \tilde{M} \) is non-formal.

**Proof:** We start by considering the nilmanifold \( M \). Consider the closed forms:

\[
\alpha = \mu \wedge \tilde{\mu}, \quad \beta_1 = \nu \wedge \tilde{\nu}, \quad \beta_2 = \nu \wedge \tilde{\eta}, \quad \beta_3 = \tilde{\nu} \wedge \eta.
\]

Then

\[
\alpha \wedge \beta_1 = d(-\theta \wedge \mu \wedge \nu), \quad \alpha \wedge \beta_2 = d(-\theta \wedge \mu \wedge \tilde{\eta}), \quad \alpha \wedge \beta_3 = d(\theta \wedge \mu \wedge \eta).
\]

All the forms \( \alpha, \beta_1, \beta_2, \beta_3 \), \( \xi_1 = -\theta \wedge \mu \wedge \nu \), \( \xi_2 = -\theta \wedge \mu \wedge \tilde{\eta} \) and \( \xi_3 = \tilde{\theta} \wedge \mu \wedge \eta \) are \( \mathbb{Z}_3 \)-invariant. Hence they descend to the quotient \( \tilde{M} = M / \mathbb{Z}_3 \). Let \( q : M \to \tilde{M} \) denote the projection, and define \( \hat{\alpha} = q_* \alpha \), \( \hat{\beta}_i = q_* \beta_i \), \( \hat{\xi}_i = q_* \xi_i \), \( i = 1, 2, 3 \). Now take a \( \mathbb{Z}_3 \)-equivariant map \( \varphi : M \to M \) which is the identity outside some small balls around the fixed points, and contracts some smaller balls into the fixed points. It induces a map \( \hat{\varphi} : \tilde{M} \to \tilde{M} \) such that \( \varphi \circ q = q \circ \varphi \). The forms \( \hat{\alpha} = \hat{\varphi}^* \hat{\alpha}, \hat{\beta}_i = \hat{\varphi}^* \hat{\beta}_i, \hat{\xi}_i = \hat{\varphi}^* \hat{\xi}_i, i = 1, 2, 3 \), are zero in a neighbourhood of the fixed points, therefore they define forms on \( \tilde{M} \), by extending them by zero along the exceptional divisors. Note that \( \hat{\alpha}, \hat{\beta}_i \in \Omega^2(\tilde{M}) \) are closed forms and \( \hat{\xi}_i \in \Omega^3(\tilde{M}) \) satisfies \( d\hat{\xi}_i = \hat{\alpha} \wedge \hat{\beta}_i, i = 1, 2, 3 \).

By Lemma 2.3, \( H^2(\tilde{M}) = 0 \), so we may apply Lemma 3.1 to the cohomology classes \( a = [\alpha], b_i = [\beta_i] \in H^2(\tilde{M}), i = 1, 2, 3 \). The cohomology class

\[
[\hat{\xi}_1 \wedge \hat{\xi}_2 \wedge \hat{\beta}_3 + \hat{\xi}_2 \wedge \hat{\xi}_3 \wedge \hat{\beta}_1 + \hat{\xi}_3 \wedge \hat{\xi}_1 \wedge \hat{\beta}_2] = [\hat{\varphi}^* q_* (\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2)] = \hat{\varphi}^* q_* (2[\theta \wedge \mu \wedge \nu \wedge \eta \wedge \theta \wedge \mu \wedge \tilde{\nu} \wedge \tilde{\eta}]) \neq 0,
\]

QED
since its integral is
\[
\int_{\tilde{M}} \varphi^* q_*(2[\theta \wedge \mu \wedge \nu \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\eta}]) = \int_{\tilde{M}} \varphi^* q_*(2[\theta \wedge \mu \wedge \nu \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\eta}]) \\
= 3 \int_{\tilde{M}} \varphi^*(2[\theta \wedge \mu \wedge \nu \wedge \eta \wedge \theta \wedge \mu \wedge \nu \wedge \eta]) \\
= 6 \int_{\tilde{M}} [\theta \wedge \mu \wedge \nu \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\eta}] \neq 0.
\]

By Lemma 3.3, \(\tilde{M}\) is non-formal.

**Remark 3.3** The symplectic manifold \((\tilde{M}, \tilde{\omega})\) is not hard-Lefschetz. The \(\mathbb{Z}_3\)-invariant form \(\nu \wedge \bar{\nu}\) on \(M\) is not exact, but \(\omega^2 \wedge \nu \wedge \bar{\nu} = 2d(\theta \wedge \mu \wedge \eta \wedge \eta \wedge \bar{\theta} \wedge \bar{\mu})\). This form descends to the quotient \(\tilde{M}\) and can be extended to \(\tilde{M}\) via the process done at the end of the proof of the previous theorem. Therefore the map \([\omega]^2: H^2(\tilde{M}) \to H^6(\tilde{M})\) is not injective.

Cavalcanti [7] gave the first examples of simply connected compact symplectic manifolds of dimension \(\geq 10\) which are hard Lefschetz and non-formal. Yet examples of non-formal simply connected compact symplectic 8-manifolds satisfying the hard Lefschetz property have not been constructed.

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