An optimal result for global existence and boundedness in a three-dimensional Keller-Segel(-Navier)-Stokes system (involving a tensor-valued sensitivity with saturation)

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Abstract

This paper is concerned with the following Keller-Segel(-Navier)-Stokes system with (rotational flux)

\[
\begin{align*}
  n_t + u \cdot \nabla n & = \Delta n - \nabla \cdot \left( nS(x, n, c) \nabla c \right), \quad x \in \Omega, t > 0, \\
  c_t + u \cdot \nabla c & = \Delta c - c + n, \quad x \in \Omega, t > 0, \\
  u_t + \kappa (u \cdot \nabla) u + \nabla P & = \Delta u + n \nabla \phi, \quad x \in \Omega, t > 0, \\
  \nabla \cdot u & = 0, \quad x \in \Omega, t > 0
\end{align*}
\]

(KSNF)

in a bounded domain \( \Omega \subset \mathbb{R}^3 \) with smooth boundary, where \( \kappa \in \mathbb{R} \) is given constant, \( \phi \in W^{1,\infty}(\Omega) \), \( |S(x, n, c)| \leq C_S (1 + n)^{-\alpha} \) and the parameter \( \alpha \geq 0 \). If \( \alpha > \frac{1}{3} \), then for all reasonably regular initial data, a corresponding initial-boundary value problem for (KSNF) possesses a globally defined weak solution. This result improves

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the result of Wang (Math. Models Methods Appl. Sci., 27(14):2745–2780, 2017), where the global very weak solution for system \((KSNF)\) is obtained. Moreover, if \(\kappa = 0\) and \(S(x, n, c) = CS(1 + n)^{-\alpha}\), then the system \((KSF)\) exists at least one global classical solution which is bounded in \(\Omega \times (0, \infty)\). In comparison to the result for the corresponding fluid-free system, the optimal condition on the parameter \(\alpha\) for both global (weak) existence and boundedness are obtained. Our proofs rely on Maximal Sobolev regularity techniques and a variant of the natural gradient-like energy functional.

**Key words:** Navier-Stokes system; Keller-Segel model; Global existence; Boundedness; Tensor-valued sensitivity

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1 Introduction

Chemotaxis, the biased movement of cells (or organisms) in response to chemical gradients, plays an important role coordinating cell migration in many biological phenomena (see Hillen and Painter [10]). Let $n$ denote the density of the cells and $c$ present the concentration of the chemical signal. In 1970s, Keller and Segel ([14]) proposed a mathematical system for chemotaxis phenomena through a system of parabolic equations. The mathematical model reads as

$$
\begin{align*}
    n_t &= \Delta n - \nabla \cdot (nS(x, n, c)\nabla c), \quad x \in \Omega, \quad t > 0, \\
    c_t &= \Delta c - c + n, \quad x \in \Omega, \quad t > 0,
\end{align*}
$$

where $S$ is a given chemotactic sensitivity function, which can either be a scalar function, or more general a tensor valued function (see e.g. Xue and Othmer [40]). During the past four decades, the Keller-Segel models (1.1) and its variants have attracted extensive attentions, where the main issue of the investigation was whether the solutions of the models are bounded or blow-up (see Winkler et al. [1], Hillen and Painter [10], Horstmann [11]). For instance, if $S := S(n)$ is a scalar function satisfying $S(s) \leq C(1 + s)^{-\alpha}$ for all $s \geq 1$ and some $\alpha > 1 - \frac{2}{N}$ and $C > 0$, then all solutions to the corresponding Neumann problem are global and uniformly bounded (see Horstmann and Winkler [12]). While, if $N \geq 2$, $\Omega \subset \mathbb{R}^N$ is a ball and $S(s) > cs^{-\alpha}$ for some $\alpha < 1 - \frac{2}{N}$ and $c > 0$, then the solution to problem (1.1) may blow up (see Horstmann and Winkler [12]). Therefore,

$$\alpha = 1 - \frac{2}{N} \quad (1.2)$$

is the critical blow-up exponent, which is related to the presence of a so-called volume-filling effect. For the more related works in this direction, we mention that a corresponding quasilinear version, the logistic damping or the signal is consumed by the cells has been deeply investigated by Cieślak and Stinner [5, 6], Tao and Winkler [22, 32, 39] and Zheng et al. [42, 43, 50, 45, 51].

As in the classical Keller-Segel model where the chemoattractant is produced by bacteria, the corresponding chemotaxis-fluid model is then Keller-Segel(-Navier)-Stokes system of the
form
\[
\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (nS(x,n,c) \nabla c), & x \in \Omega, t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - n - c, & x \in \Omega, t > 0, \\
  u_t + \kappa (u \cdot \nabla) u + \nabla P &= \Delta u + n \nabla \phi, & x \in \Omega, t > 0, \\
  \nabla \cdot u &= 0, & x \in \Omega, t > 0,
\end{align*}
\]
where \( n \) and \( c \) are defined as before, \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with smooth boundary. Here \( u, P, \phi \) and \( \kappa \in \mathbb{R} \) denote, respectively, the velocity field, the associated pressure of the fluid, the potential of the gravitational field and the strength of nonlinear fluid convection. And \( S(x,n,c) \) is a chemotactic sensitivity tensor satisfying
\[
S \in C^2(\bar{\Omega} \times [0, \infty)^2; \mathbb{R}^{3 \times 3})
\] (1.4)
and
\[
|S(x,n,c)| \leq C_S (1 + n)^{-\alpha} \quad \text{for all } (x,n,c) \in \Omega \times [0, \infty)^2
\] (1.5)
with some \( C_S > 0 \) and \( \alpha > 0 \). Problem (1.3) is proposed to describe chemotaxis–fluid interaction in cases when the evolution of the chemoattractant is essentially dominated by production through cells (see Winkler et al. [1], Hillen and Painter [10]).

Before going into our mathematical analysis, we recall some important progresses on system (1.3) and its variants. The following chemotaxis–fluid model was proposed by Tuval et al. [26], which is a closely related variant of (1.3)
\[
\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (nS(x,n,c) \nabla c), & x \in \Omega, t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - nf(c), & x \in \Omega, t > 0, \\
  u_t + \kappa (u \cdot \nabla) u + \nabla P &= \Delta u + n \nabla \phi, & x \in \Omega, t > 0, \\
  \nabla \cdot u &= 0, & x \in \Omega, t > 0,
\end{align*}
\] (1.6)
where \( f(c) \) is the consumption rate of the oxygen by the cells. In the last few years, by making use of energy-type functionals, system (1.6) and its variants have attracted extensive attentions (see e.g. Chae et. al. [4], Duan et. al. [7], Liu and Lorz [15, 17], Tao and Winkler [25, 34, 35, 38], Zhang and Zheng [41] and references therein). For example, Winkler
(38) established the global existence of weak solution in a three-dimensional domain when \( S(x, n, c) \equiv 1 \) and \( \kappa \neq 0 \). For more literatures related to this model, we can refer to Tao and Winkler [23, 24] and the reference therein.

If the chemotactic sensitivity \( S(x, n, c) \) is regarded as a tensor rather than a scalar one (see Xue and Othmer [40]), (1.6) turns into a chemotaxis(-Navier)-Stokes system with rotational flux. Due to the presence of the tensor-valued sensitivity, the corresponding chemotaxis-Stokes system loses some energy structure, which plays a key role in previous studies for the scalar sensitivity case (see Cao [3], Winkler [37]). Therefore, only very few results appear to be available on chemotaxis-Stokes system with such tensor-valued sensitivities (see e.g. Ishida [13], Wang et al. [28, 29] and Winkler [37]). In fact, when assuming \( f(c) = c \) and (1.4)–(1.5) holds, Ishida [13] proved that (1.6) admits a bounded global weak solution in 2-dimensions with nonlinear diffusion. While, in 3-dimensions, Winkler (see Winkler [37]) showed that the chemotaxis-Stokes system (\( \kappa = 0 \) in the first equation of (1.6)) with the nonlinear diffusion (the coefficient of diffusion satisfies \( m > \frac{7}{6} \)) possesses at least one bounded weak solution which stabilizes to the spatially homogeneous equilibrium \( (\frac{1}{|\Omega|} \int_\Omega n_0, 0, 0) \).

In contrast to the large number of the existed results on (1.6), the mathematical analysis of (1.3) regarding global and bounded solutions is far from trivial, since, on the one hand its Navier-Stokes subsystem lacks complete existence theory (see Wiegner [31]) and on the other hand the previously mentioned properties for Keller-Segel system can still emerge (see Wang, Xiang et. al. [19, 29, 30, 27], Zheng [48, 49]). In fact, in 2-dimensional, if \( S = S(x, n, c) \) is a tensor-valued sensitivity fulfilling (1.4) and (1.5), Wang and Xiang [29] proved that Stokes-version (\( \kappa = 0 \) in the first equation of (1.3)) of system (1.3) admits a unique global classical solution which is bounded. These condition for \( \alpha \) is optimal according to (1.2). And similar results are also valid for the three-dimensional Stokes-version (\( \kappa = 0 \) in the first equation of (1.3)) of system (1.3) with \( \alpha > \frac{1}{2} \) (see Wang and Xiang [30]). While if 3-dimensional, Wang and Liu [16] showed that Keller-Segel-Navier-Stokes (\( \kappa \neq 0 \) in the first equation of (1.3)) system (1.3) admits a global weak solutions for tensor-valued sensitivity \( S(x, n, c) \) satisfying (1.4) and (1.5) with \( \alpha > \frac{3}{7} \). Recently, due to the lack of enough regularity and compactness properties for the first equation, by using the idea originating from Winkler (see Winkler [37])
Wang (see Wang [27]) obtained the global very weak solutions system (1.3) under the assumption that $S$ satisfies (1.4) and (1.5) with $\alpha > \frac{1}{3}$, which in light of the known results for the fluid-free system mentioned above is an optimal restriction on $\alpha$ (see (1.2)). However, for the global (strongly than the result of [27]) weak solutions is still open. In this paper, we try to obtain the enough regularity and compactness properties (see Lemmas 3.4, 5.1 and 5.2), then show that system (1.3) possesses a globally defined weak solution (see Definition 2.1), which improves the result of [27]. Moreover, with the help of Maximal Sobolev regularity and some carefully analysis, if $S := S(n) = C_S(1 + n)^{-\alpha}$ is a scalar function which satisfies that $\alpha > \frac{1}{3}$, the boundedness of solution to Keller-Segel-Stokes ($\kappa = 0$ in the first equation of (1.3)) system (1.3) is also obtained. Recalling the condition (1.2) for global existence in the fluid-free setting, as implied by the previously mentioned studied (see Horstmann and Winkler [12]), this result appears to be optimal with respect to $\alpha$.

We sketch here the main ideas and methods used in this article. One novelty of this paper is that we use the Maximal Sobolev regularity (see Hieber and Prüss [9]) approach to show the existence of bounded solutions. The Maximal Sobolev regularity approach has been widely used to obtain the existence of bounded solutions of the quasilinear parabolic–parabolic Keller–Segel system with logistic source (see e.g. Cao [2] and Zheng [47]). However, it seems that no one used such method to obtain the existence of bounded solutions to Keller–Segel-Stokes system. We should pointed that the idea of this paper can also be used to deal with Keller–Segel-Stokes system with nonlinear diffusion (see Zheng [49]). In fact, by using the idea of this paper, one can prove that if the coefficient of diffusion satisfies $m > \frac{4}{3}$, then Keller–Segel-Stokes system (with nonlinear diffusion) exists at least one global weak solution which is bounded in $\Omega \times (0, \infty)$. The conditions $m > \frac{4}{3}$ is also optimal due to the fact that the 3D fluid-free system admits a global bounded classical solution for $m > \frac{4}{3}$ (see the Introduction of Tao and Winkler [22]).

Throughout this paper, we assume that

$$\phi \in W^{1,\infty}(\Omega) \quad (1.7)$$
and the initial data \((n_0, c_0, u_0)\) fulfills

\[
\begin{cases}
  n_0 \in C^\kappa(\bar{\Omega}) & \text{for certain } \kappa > 0 \text{ with } n_0 \geq 0 \text{ in } \Omega, \\
  c_0 \in W^{1,\infty}(\Omega) & \text{with } c_0 \geq 0 \text{ in } \bar{\Omega}, \\
  u_0 \in D(\mathcal{A}_r) & \text{for some } \gamma \in (\frac{3}{4}, 1) \text{ and any } r \in (1, \infty),
\end{cases}
\]

where \(\mathcal{A}_r\) denotes the Stokes operator with domain \(D(\mathcal{A}_r) := W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega) \cap L^r_0(\Omega)\), and \(L^r_0(\Omega) := \{\varphi \in L^r(\Omega)|\nabla \cdot \varphi = 0\}\) for \(r \in (1, \infty)\) ([21]).

In the context of these assumptions, the first of our main results asserts global weak existence of a solution in the following sense.

**Theorem 1.1.** Let \(\Omega \subset \mathbb{R}^3\) be a bounded domain with smooth boundary, (1.7) and (1.8) hold, and suppose that \(S\) satisfies (1.4) and (1.5) with some \(\alpha > \frac{1}{3}\).

Then the problem (1.3) possesses at least one global weak solution \((n, c, u, P)\) in the sense of Definition 2.1.

**Remark 1.1.** (i) From Theorem 1.1, we conclude that if the algebraic saturation with \(\alpha > \frac{1}{3}\) is sufficient to guarantee the existence of global (weak) solutions. Compared to the results (1.2), we know such a restriction on \(\alpha\) is optimal.

(ii) Obviously, \(\frac{3}{7} > \frac{1}{2}\), Theorem 1.1 improves the results of Liu and Wang ([16]), who showed the global weak existence of solutions in the cases \(S(x, n, c)\) satisfying (1.4) and (1.5) with \(\alpha > \frac{3}{7}\).

Moreover, if in addition we assume that \(\kappa = 0\) and \(S(x, n, c) = C_S(1 + n)^{-\alpha}\), then the solutions will actually be bounded:

**Theorem 1.2.** Let \(\Omega \subset \mathbb{R}^3\) be a bounded domain with smooth boundary, (1.7) and (1.8) hold. Moreover, assume that \(\kappa = 0\) and \(S(x, n, c) = C_S(1 + n)^{-\alpha}\), then for any choice of \(n_0, c_0\) and \(u_0\) fulfilling (1.8), the problem (1.3) possesses a global classical solution \((n, c, u, P)\) for which \(n, c\) and \(u\) are bounded in \(\Omega \times (0, \infty)\) in the sense that there exists \(C > 0\) fulfilling

\[
\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t > 0.
\]
Remark 1.2. (i) If $u \equiv 0$, Theorem 1.2 is coincides with Theorem 4.1 of [12], which is optimal according to the fact that the 3D fluid-free system (1.1) admits a global bounded classical solution for $\alpha > \frac{1}{3}$ as mentioned before.

(ii) The condition of $S(x, n, c) = C_S(1+n)^{-\alpha}$ can be replaced by $S := S(n)$ which satisfies $S(n) \leq C_S(1+n)^{-\alpha}$.

This paper is organized as follows. In Section 2, we firstly give the definition of weak solutions to (1.3), the regularized problems of (1.3) and some preliminary properties. Section 3 and Section 4 will be devoted to an analysis of regularized problems of (1.3). Next, on the basis of the compactness properties thereby implied, in Section 5 and Section 6 we can pass to the limit along an adequate sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$ and thereby verify the Theorem 1.1. In Section 7, in view of the Maximal Sobolev regularity techniques, we will show Theorem 1.2 by applying the standard Alikakos-Moser iteration. Indeed, by using the Maximal Sobolev regularity techniques, we firstly, establish an energy-type inequality which will play a key role in the derivation of further estimates. Then, we develop some $L^p$-estimate techniques to raise the a priori estimate of solutions from $L^1(\Omega) \to L^{q_0}(\Omega)(q_0 > \frac{3}{2})$, and then use the standard Alikakos-Moser iteration and the standard parabolic regularity arguments to show Theorem 1.2.

2 Preliminaries

In light of the strongly nonlinear term $(u \cdot \nabla)u$, the problem (1.3) has no classical solutions in general, and thus we consider its weak solutions. The concept of (global) weak solution for (1.3) we shall pursue in this sequel will be given in the follows.

Definition 2.1. Let $T > 0$ and $(n_0, c_0, u_0)$ fulfills (1.8). Then a triple of functions $(n, c, u)$ is called a weak solution of (1.3) if the following conditions are satisfied

$$
\begin{align*}
    n &\in L^1_{loc}(\bar{\Omega} \times [0, T)), \\
    c &\in L^1_{loc}([0, T); W^{1,1}(\Omega)), \\
    u &\in L^1_{loc}([0, T); W^{1,1}(\Omega)),
\end{align*}
$$

(2.1)
where $n \geq 0$ and $c \geq 0$ in $\Omega \times (0,T)$ as well as $\nabla \cdot u = 0$ in the distributional sense in $\Omega \times (0,T)$, moreover,

$$u \otimes u \in L^1_{loc}(\bar{\Omega} \times [0,\infty); \mathbb{R}^{3 \times 3}) \text{ and } n \text{ belong to } L^1_{loc}(\bar{\Omega} \times [0,\infty)),$$

\hspace{1cm} \text{(2.2)}

and

$$-\int_0^T \int_{\Omega} n \varphi_t - \int_{\Omega} n_0 \varphi(\cdot,0) = -\int_0^T \int_{\Omega} \nabla n \cdot \nabla \varphi + \int_0^T \int_{\Omega} n S(x,n,c) \nabla c \cdot \nabla \varphi + \int_0^T \int_{\Omega} n u \cdot \nabla \varphi$$

\hspace{1cm} \text{(2.3)}

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0,T))$ satisfying $\frac{\partial \varphi}{\partial n} = 0$ on $\partial \Omega \times (0,T)$ as well as

$$-\int_0^T \int_{\Omega} c \varphi_t - \int_{\Omega} c_0 \varphi(\cdot,0) = -\int_0^T \int_{\Omega} \nabla c \cdot \nabla \varphi - \int_0^T \int_{\Omega} c \varphi + \int_0^T \int_{\Omega} n \varphi + \int_0^T \int_{\Omega} c u \cdot \nabla \varphi$$

\hspace{1cm} \text{(2.4)}

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0,T))$ and

$$-\int_0^T \int_{\Omega} u \varphi_t - \int_{\Omega} u_0 \varphi(\cdot,0) - \kappa \int_0^T \int_{\Omega} u \otimes u \cdot \nabla \varphi = -\int_0^T \int_{\Omega} \nabla u \cdot \nabla \varphi - \int_0^T \int_{\Omega} n \nabla \phi \cdot \varphi$$

\hspace{1cm} \text{(2.5)}

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0,T); \mathbb{R}^3)$ fulfilling $\nabla \varphi \equiv 0$ in $\Omega \times (0,T)$. If $\Omega \times (0,\infty) \rightarrow \mathbb{R}^3$ is a weak solution of (1.3) in $\Omega \times (0,T)$ for all $T > 0$, then we call $(n,c,u)$ a global weak solution of (1.3).

Our goal is to construct solutions of (1.3) as limits of solutions to appropriately regularized problems. To achieve this, in order to deal with the strongly nonlinear term $(u \cdot \nabla) u$, we introduce the following approximating equation of (1.3):

\begin{align*}
\begin{cases}
    n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon &= \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x,n_\varepsilon,c_\varepsilon) \nabla c_\varepsilon), \quad x \in \Omega, t > 0, \\
c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon &= \Delta c_\varepsilon - c_\varepsilon + F_\varepsilon(n_\varepsilon), \quad x \in \Omega, t > 0, \\
u_{\varepsilon t} + \nabla P_\varepsilon &= \Delta u_\varepsilon - \kappa (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon + n_\varepsilon \nabla \phi, \quad x \in \Omega, t > 0, \\
\nabla \cdot u_\varepsilon &= 0, \quad x \in \Omega, t > 0, \\
\nabla n_\varepsilon \cdot \nu = \nabla c_\varepsilon \cdot \nu &= 0, \quad u_\varepsilon = 0, \quad x \in \partial \Omega, t > 0, \\
n_\varepsilon(x,0) = n_0(x), c_\varepsilon(x,0) = c_0(x), u_\varepsilon(x,0) = u_0(x), & x \in \Omega,
\end{cases}
\end{align*}

(2.6)

where
\[ F_\varepsilon(s) = \frac{1}{\varepsilon} \ln(1 + \varepsilon s) \] for all \( s \geq 0 \) and \( \varepsilon > 0 \) (2.7)

as well as
\[ S_\varepsilon(x, n, c) = \rho_\varepsilon(x) S(x, n, c), \quad x \in \bar{\Omega}, \quad n \geq 0, \quad c \geq 0 \] (2.8)
and
\[ Y_\varepsilon w := (1 + \varepsilon A)^{-1} w \] for all \( w \in L^2_\sigma(\Omega) \) (2.9)
is the standard Yosida approximation. Here \((\rho_\varepsilon)_{\varepsilon \in (0, 1)} \in C_0^\infty(\Omega)\) be a family of standard cut-off functions satisfying \( 0 \leq \rho_\varepsilon \leq 1 \) in \( \Omega \) and \( \rho_\varepsilon \to 1 \) in \( \Omega \) as \( \varepsilon \to 0 \).

The local solvability of (2.6) can be derived by a suitable extensibility criterion and a slight modification of the well-established fixed point arguments in Lemma 2.1 of [38] (see also [37], Lemma 2.1 of [18]), so here we omit the proof.

**Lemma 2.1.** Assume that \( \varepsilon \in (0, 1) \). Then there exist \( T_{\max, \varepsilon} \in (0, \infty) \) and a classical solution \((n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)\) of (2.6) in \( \Omega \times (0, T_{\max, \varepsilon}) \) such that

\[
\begin{cases}
  n_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\max, \varepsilon})) \cap C^{2,1}(\bar{\Omega} \times [0, T_{\max, \varepsilon})), \\
  c_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\max, \varepsilon})) \cap C^{2,1}(\bar{\Omega} \times [0, T_{\max, \varepsilon})), \\
  u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\max, \varepsilon})) \cap C^{2,1}(\bar{\Omega} \times [0, T_{\max, \varepsilon})), \\
  P_\varepsilon \in C^{1,0}(\bar{\Omega} \times [0, T_{\max, \varepsilon})),
\end{cases}
\] (2.10)
classically solving (2.6) in \( \Omega \times [0, T_{\max, \varepsilon}) \). Moreover, \( n_\varepsilon \) and \( c_\varepsilon \) are nonnegative in \( \Omega \times (0, T_{\max, \varepsilon}) \), and

\[
\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \to \infty \quad \text{as} \quad t \to T_{\max, \varepsilon},
\] (2.11)
where \( \gamma \) is given by (1.8).

**Lemma 2.2.** ([33, 46]) Let \((e^{\tau \Delta})_{\tau \geq 0}\) be the Neumann heat semigroup in \( \Omega \) and \( p > 3 \). Then there exist positive constants \( c_1 := c_1(\Omega), c_2 := c_2(\Omega) \) and \( c_3 := c_3(\Omega) \) such that

\[
\|\nabla e^{\tau \Delta} \varphi\|_{L^p(\Omega)} \leq c_1(\Omega)\|\nabla \varphi\|_{L^p(\Omega)} \quad \text{for all} \quad \tau > 0 \quad \text{and any} \quad \varphi \in W^{1,p}(\Omega)
\] (2.12)
and

\[
\|\nabla e^{\tau \Delta} \varphi\|_{L^p(\Omega)} \leq c_2(1 + \tau^{-\frac{1}{2}})\|\varphi\|_{L^\infty(\Omega)} \quad \text{for all} \quad \tau > 0 \quad \text{and each} \quad \varphi \in L^\infty(\Omega)
\] (2.13)
as well as
\[
\|e^{\tau \Delta} \nabla \cdot \varphi \|_{L^\infty(\Omega)} \leq c_3(1 + \tau^{-\frac{1}{p}}) \|\varphi\|_{L^p(\Omega)} \text{ for all } \tau > 0 \text{ and all } \varphi \in C^1(\bar{\Omega}; \mathbb{R}^N) \text{ fulfilling } \varphi \cdot \nu = 0 \text{ on } \partial \Omega.
\]
(2.14)

3 A priori estimates for the regularized problem (2.6) which is independent of \( \varepsilon \)

In this section, we are going to establish an iteration step to develop the main ingredient of our result. The iteration depends on a series of a priori estimate. In order to proceed, firstly, we recall some properties for \( F_\varepsilon \) and \( F'_\varepsilon \), which play important role in showing Theorem 1.1.

Lemma 3.1. Assume \( F_\varepsilon \) is given by (2.7). Then
\[
0 \leq F'_\varepsilon(s) = \frac{1}{1 + \varepsilon s} \leq 1 \text{ for all } s \geq 0 \text{ and } \varepsilon > 0
\]
(3.1)
as well as
\[
\lim_{\varepsilon \to 0^+} F_\varepsilon(s) = s, \quad \lim_{\varepsilon \to 0^+} F'_\varepsilon(s) = 1 \text{ for all } s \geq 0
\]
(3.2)
and
\[
0 \leq F_\varepsilon(s) \leq s \text{ for all } s \geq 0.
\]
(3.3)

Proof. Recalling (2.7), by tedious but simple calculations, we can derive (3.1)–(3.3). □

The proof of this lemma is very similar to that of Lemmas 2.2 and 2.6 of [25] (see also Lemma 3.2 of [27]), so we omit its proof here.

Lemma 3.2. There exists \( \lambda > 0 \) independent of \( \varepsilon \) such that the solution of (2.6) satisfies
\[
\int_\Omega n_\varepsilon + \int_\Omega c_\varepsilon \leq \lambda \text{ for all } t \in (0, T_{max,\varepsilon}).
\]
(3.4)

Lemma 3.3. Let \( \alpha > \frac{1}{3} \). Then there exists \( C > 0 \) independent of \( \varepsilon \) such that the solution of (2.6) satisfies
\[
\int_\Omega n_\varepsilon^{2\alpha} + \int_\Omega c_\varepsilon^2 + \int_\Omega |u_\varepsilon|^2 \leq C \text{ for all } t \in (0, T_{max,\varepsilon}).
\]
(3.5)
Moreover, for $T \in (0, T_{\text{max}, \varepsilon})$, it holds that one can find a constant $C > 0$ independent of $\varepsilon$ such that

$$
\int_0^T \int_\Omega \left[ n_\varepsilon^{2\alpha-2} |\nabla n_\varepsilon|^2 + |\nabla c_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right] \leq C. \tag{3.6}
$$

**Proof.** The proof consists two cases.

Case $2\alpha \neq 1$ : We first obtain from $\nabla \cdot u_\varepsilon = 0$ in $\Omega \times (0, T_{\text{max}, \varepsilon})$ and straightforward calculations that

$$
\begin{align*}
\text{sign}(2\alpha - 1) & \frac{1}{2\alpha} \frac{d}{dt} \| n_\varepsilon \|^2_{L^{2\alpha}(\Omega)} \\
+ & \text{sign}(2\alpha - 1)(2\alpha - 1) \int_\Omega n_\varepsilon^{2\alpha-2} |\nabla n_\varepsilon|^2 \\
= & - \int_\Omega \text{sign}(2\alpha - 1)(2\alpha - 1) n_\varepsilon^{2\alpha-2} \nabla \cdot (n_\varepsilon F'_\varepsilon(n_\varepsilon) S_S(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon) \\
\leq & \text{sign}(2\alpha - 1)(2\alpha - 1) \int_\Omega n_\varepsilon^{2\alpha-2} n_\varepsilon F'_\varepsilon(n_\varepsilon) |S_S(x, n_\varepsilon, c_\varepsilon)| \| \nabla n_\varepsilon \| \| \nabla c_\varepsilon \|
\end{align*}
$$

for all $t \in (0, T_{\text{max}, \varepsilon})$. Therefore, due to (3.1), in light of (1.5) and (2.7), with the help of the Young inequality, we can estimate the right of (3.7) by following

$$
\begin{align*}
\text{sign}(2\alpha - 1)(2\alpha - 1) & \int_\Omega n_\varepsilon^{2\alpha-2} n_\varepsilon F'_\varepsilon(n_\varepsilon) |S_S(x, n_\varepsilon, c_\varepsilon)| \| \nabla n_\varepsilon \| \| \nabla c_\varepsilon \| \\
\leq & \text{sign}(2\alpha - 1)(2\alpha - 1) \int_\Omega n_\varepsilon^{2\alpha-2} n_\varepsilon C_S(1 + n_\varepsilon)^{-\alpha} \| \nabla n_\varepsilon \| \| \nabla c_\varepsilon \| \\
\leq & \text{sign}(2\alpha - 1) \frac{2\alpha - 1}{2} \int_\Omega n_\varepsilon^{2\alpha-2} |\nabla n_\varepsilon|^2 \\
+ & \frac{2\alpha - 1}{2} C_S^2 \int_\Omega n_\varepsilon^{2\alpha-2} n_\varepsilon^2 (1 + n_\varepsilon)^{-2\alpha} |\nabla c_\varepsilon|^2 \\
\leq & \text{sign}(2\alpha - 1) \frac{2\alpha - 1}{2} \int_\Omega n_\varepsilon^{2\alpha-2} |\nabla n_\varepsilon|^2 \\
+ & \frac{2\alpha - 1}{2} C_S^2 \int_\Omega |\nabla c_\varepsilon|^2 \\
\text{for all } t \in (0, T_{\text{max}, \varepsilon}),
\end{align*}
$$

where in the last inequality we have used the fact that $n_\varepsilon^{2\alpha-2} n_\varepsilon^2 (1 + n_\varepsilon)^{-2\alpha} \leq 1$ for all $\varepsilon \geq 0$, $n_\varepsilon$ and $\alpha \geq 0$. Inserting (3.8) into (3.7), we conclude that

$$
\begin{align*}
\text{sign}(2\alpha - 1) & \frac{1}{2\alpha} \frac{d}{dt} \| n_\varepsilon \|^2_{L^{2\alpha}(\Omega)} + \text{sign}(2\alpha - 1) \frac{2\alpha - 1}{2} \int_\Omega n_\varepsilon^{2\alpha-2} |\nabla n_\varepsilon|^2 \\
\leq & \frac{2\alpha - 1}{2} C_S^2 \int_\Omega |\nabla c_\varepsilon|^2 \\
\text{for all } t \in (0, T_{\text{max}, \varepsilon}).
\end{align*}
$$

To track the time evolution of $c_\varepsilon$, taking $c_\varepsilon$ as the test function for the second equation of
(2.6), using $\nabla \cdot u_\varepsilon = 0$ and (3.3), with the help of the Hölder inequality yields that

$$\frac{1}{2} \frac{d}{dt} \| c_\varepsilon \|_{L^2(\Omega)}^2 + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |c_\varepsilon|^2 = \int_\Omega F_\varepsilon(n_\varepsilon) c_\varepsilon \leq \int_\Omega n_\varepsilon c_\varepsilon \leq \|n_\varepsilon\|_{L^\frac{6}{5}(\Omega)} \|c_\varepsilon\|_{L^6(\Omega)}$$

for all $t \in (0, T_{\max, \varepsilon})$.

An application of the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ in the three-dimensional setting, in view of (3.4), there exist positive constants $C_1$ and $C_2$ such that

$$\|c_\varepsilon\|_{L^6(\Omega)} \leq C_1 \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + C_1 \|c_\varepsilon\|_{L^1(\Omega)}^2 \leq C_1 \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + C_2 \text{ for all } t \in (0, T_{\max, \varepsilon}).$$

Thus by means of the Young inequality and (3.11), we proceed to estimate

$$\frac{1}{2} \frac{d}{dt} \| c_\varepsilon \|_{L^2(\Omega)}^2 + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |c_\varepsilon|^2 \leq \frac{1}{2C_1} \|c_\varepsilon\|_{L^6(\Omega)}^2 + \frac{C_1}{2} \|n_\varepsilon\|_{L^\frac{6}{5}(\Omega)}^2 \leq \frac{1}{2} \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + \frac{C_1}{2} \|n_\varepsilon\|_{L^\frac{6}{5}(\Omega)}^2 + C_3 \text{ for all } t \in (0, T_{\max, \varepsilon})$$

and some positive constant $C_3$ independent of $\varepsilon$. Therefore,

$$\frac{1}{2} \frac{d}{dt} \| c_\varepsilon \|_{L^2(\Omega)}^2 + \frac{1}{2} \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |c_\varepsilon|^2 \leq \frac{C_1}{2} \|n_\varepsilon\|_{L^\frac{6}{5}(\Omega)}^2 + C_3 \text{ for all } t \in (0, T_{\max, \varepsilon}).$$

(3.12)

To estimate $\|n_\varepsilon\|_{L^\frac{6}{5}(\Omega)}$ for all $t \in (0, T_{\max, \varepsilon})$, we should notice that $\alpha > \frac{1}{3}$ which ensures that $\frac{2}{6\alpha - 1} < 2$, in light of (3.4), and hence the Gagliardo–Nirenberg and the Young inequalities allow us to estimate that for any $\delta_1 > 0$,

$$\|n_\varepsilon\|_{L^\frac{6}{5}(\Omega)}^2 = \|n_\varepsilon^\alpha\|_{L^{\frac{6}{5\alpha}}(\Omega)}^2 \leq C_4 (\|\nabla n_\varepsilon^\alpha\|_{L^2(\Omega)}^2 \|n_\varepsilon^\alpha\|_{L^{\frac{6}{5\alpha}}(\Omega)}^\frac{\alpha - \frac{2}{6\alpha - 1}}{2} + \|n_\varepsilon^\alpha\|_{L^{\frac{6}{5\alpha}}(\Omega)}^\frac{\alpha}{2}) \leq \delta_1 \|\nabla n_\varepsilon^\alpha\|_{L^2(\Omega)}^2 + C_5 \text{ for all } t \in (0, T_{\max, \varepsilon})$$

with some positive constants $C_4$ and $C_5$ independent of $\varepsilon$. This together with (3.13) contributes to

$$\frac{1}{2} \frac{d}{dt} \| c_\varepsilon \|_{L^2(\Omega)}^2 + \frac{1}{2} \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |c_\varepsilon|^2 \leq \frac{C_1}{2} \delta_1 \|\nabla n_\varepsilon^\alpha\|_{L^2(\Omega)}^2 + C_6 \text{ for all } t \in (0, T_{\max, \varepsilon})$$

(3.15)
and some positive constant $C_6$. Taking an evident linear combination of the inequalities provided by (3.9) and (3.15), one can obtain that

$$\begin{align*}
sign(2\alpha - 1)\frac{1}{2\alpha} \frac{d}{dt}\|n\|_{L^{2\alpha}(\Omega)}^{2\alpha} + |2\alpha - 1|C_S^{2 \alpha} \frac{d}{dt}\|c\|_{L^{2\alpha}(\Omega)}^{2\alpha} \\
+ \frac{|2\alpha - 1|}{2} C_S^{2 \alpha} \int_{\Omega} |\nabla c|^2 + 2|2\alpha - 1|C_S^{2 \alpha} \int_{\Omega} |c|^2 \\
+ (\sign(2\alpha - 1) - C_1\delta_1^{\alpha - 2}|2\alpha - 1|C_S^{2 \alpha}) \int_{\Omega} n^{\alpha - 2} |\nabla n|^2 \\
\leq C_7 \text{ for all } t \in (0, T_{\max, \varepsilon})
\end{align*}$$

(3.16)

and some positive constant $C_7$. Since $\sign(2\alpha - 1)\frac{2\alpha - 1}{2} = \frac{|2\alpha - 1|}{2}$, we may choose $\delta = \frac{1}{4} \frac{2\alpha - 1}{C_1\delta_1^{\alpha - 2} C_S^{2 \alpha}}$ in (3.16) then implies that

$$\begin{align*}
sign(2\alpha - 1)\frac{1}{2\alpha} \frac{d}{dt}\|n\|_{L^{2\alpha}(\Omega)}^{2\alpha} + |2\alpha - 1|C_S^{2 \alpha} \frac{d}{dt}\|c\|_{L^{2\alpha}(\Omega)}^{2\alpha} \\
+ \frac{|2\alpha - 1|}{2} C_S^{2 \alpha} \int_{\Omega} |\nabla c|^2 + 2|2\alpha - 1|C_S^{2 \alpha} \int_{\Omega} |c|^2 \\
+ \frac{|2\alpha - 1|}{4} \int_{\Omega} n^{\alpha - 2} |\nabla n|^2 \\
\leq C_7 \text{ for all } t \in (0, T_{\max, \varepsilon}).
\end{align*}$$

(3.17)

If $2\alpha > 1$, then $\sign(2\alpha - 1) = 1 > 0$, thus, integrating (3.17) in time, we can obtain

$$\int_{\Omega} n^{\alpha} + \int_{\Omega} c^{\alpha} \leq C_7 \text{ for all } t \in (0, T_{\max, \varepsilon})$$

(3.18)

and

$$\int_0^T \int_{\Omega} [n^{\alpha - 2} |\nabla n|^2 + |\nabla c|^2] \leq C_7 \text{ for all } T \in (0, T_{\max, \varepsilon})$$

(3.19)

and some positive constant $C_7$. While if $2\alpha < 1$, then $\sign(2\alpha - 1) = -1 < 0$, hence, in view of (3.4), integrating (3.17) in time and employing the Hölder inequality, we conclude that there exists a positive constant $C_8$ such that

$$\int_{\Omega} n^{\alpha} + \int_{\Omega} c^{\alpha} \leq C_8 \text{ for all } t \in (0, T_{\max, \varepsilon})$$

(3.20)

and

$$\int_0^T \int_{\Omega} [n^{\alpha - 2} |\nabla n|^2 + |\nabla c|^2] \leq C_8 \text{ for all } T \in (0, T_{\max, \varepsilon}).$$

(3.21)

Case $2\alpha = 1$: Using the first equation of (2.6) and (2.7), from integration by parts and
applying (1.5) and using (3.1), we obtain

\[
\frac{d}{dt} \int_\Omega n_\varepsilon \ln n_\varepsilon = \int_\Omega \nabla n_\varepsilon \cdot (n_\varepsilon F'(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon) - \nabla c_\varepsilon) 
\leq -\int_\Omega |\nabla n_\varepsilon|^2 + \int_\Omega C_S(1 + n_\varepsilon)^{\alpha} \frac{n_\varepsilon}{n_\varepsilon} |\nabla n_\varepsilon||\nabla c_\varepsilon| 
\text{ for all } t \in (0,T_{\max,\varepsilon}),
\]  

(3.22)

which combined with the Young inequality and \(2\alpha = 1\) implies that

\[
\frac{d}{dt} \int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \leq \frac{1}{2} C_S^2 \int_\Omega |\nabla c_\varepsilon|^2 
\text{ for all } t \in (0,T_{\max,\varepsilon}).
\]  

(3.23)

On the other hand, due to \(2\alpha = 1\) yields to \(\alpha > \frac{1}{3}\), employing almost exactly the same arguments as in the proof of (3.10)–(3.16) (the minor necessary changes are left as an easy exercise to the reader), we conclude the estimate

\[
\int_\Omega n_\varepsilon \ln n_\varepsilon + \int_\Omega c_\varepsilon^2 \leq C_9 \text{ for all } t \in (0,T_{\max,\varepsilon})
\]  

(3.24)

and

\[
\int_0^T \int_\Omega \left[ \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + |\nabla c_\varepsilon|^2 \right] \leq C_9 \text{ for all } T \in (0,T_{\max,\varepsilon}).
\]  

(3.25)

Now, multiplying the third equation of (2.6) by \(u_\varepsilon\), integrating by parts and using \(\nabla \cdot u_\varepsilon = 0\)

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi \text{ for all } t \in (0,T_{\max,\varepsilon}).
\]  

(3.26)

Here we use the Hölder inequality, the Young inequality, (1.7) and the continuity of the embedding \(W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)\) and to find \(C_{10}\) and \(C_{11} > 0\) such that

\[
\int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi \leq \|\nabla \phi\|_{L^\infty(\Omega)} \|n_\varepsilon\|_{L^6(\Omega)} \|u_\varepsilon\|_{L^6(\Omega)} 
\leq C_{10} \|\nabla \phi\|_{L^\infty(\Omega)} \|n_\varepsilon\|_{L^6(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)} 
\leq \frac{1}{2} \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 + C_{11} \|n_\varepsilon\|_{L^6(\Omega)}^2 \text{ for all } t \in (0,T_{\max,\varepsilon}).
\]  

(3.27)

Next, observing that (3.3), in view of \(\alpha > \frac{1}{3}\), by (3.14) and using the Young inequality and the Gagliardo–Nirenberg inequality yields

\[
\int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi \leq \frac{1}{2} \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 + C_S \|\nabla n_\varepsilon^\alpha\|_{L^2(\Omega)}^{\frac{2}{\alpha(\alpha - 1)}} \|n_\varepsilon^\alpha\|_{L^2(\Omega)}^{\frac{2}{\alpha(\alpha - 1)}} 
\leq \frac{1}{2} \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 + \|\nabla n_\varepsilon^\alpha\|_{L^2(\Omega)}^2 + C_{12} \text{ for all } t \in (0,T_{\max,\varepsilon})
\]  

(3.28)
and some positive constant $C_{12}$. Now, inserting (3.27) and (3.28) into (3.27) and using (3.21) and (3.25), one have
\[
\int_{\Omega} |u_\varepsilon|^2 \leq C_{13} \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \tag{3.29}
\]
and
\[
\int_0^T \int_{\Omega} |\nabla u_\varepsilon|^2 \leq C_{13} \text{ for all } T \in (0, T_{\text{max}, \varepsilon}) \tag{3.30}
\]
and some positive constant $C_{14}$. Finally, collecting (3.20)–(3.21), (3.24)–(3.25) and (3.29)–(3.30), we can get (3.5)–(3.6).

With the help of Lemma 3.3 in light of the Gagliardo–Nirenberg inequality and an application of well-known arguments from parabolic regularity theory, we can derive the following Lemma:

**Lemma 3.4.** Let $\alpha > \frac{1}{3}$. Then for each $T \in (0, T_{\text{max}, \varepsilon})$, there exists $C > 0$ independent of $\varepsilon$ such that the solution of (2.6) satisfies
\[
\int_0^T \int_{\Omega} \left[ |\nabla n_\varepsilon|^{\frac{10}{3} + 2\alpha} + n_\varepsilon^6 \right] \leq C(T + 1) \text{ if } \frac{1}{3} < \alpha \leq \frac{1}{2}, \tag{3.31}
\]
\[
\int_0^T \int_{\Omega} \left[ |\nabla n_\varepsilon|^{10\alpha} + n_\varepsilon^{10\alpha} \right] \leq C(T + 1) \text{ if } \frac{1}{2} < \alpha < 1 \tag{3.32}
\]
as well as
\[
\int_0^T \int_{\Omega} \left[ |\nabla n_\varepsilon|^2 + n_\varepsilon^{\frac{10}{3}} \right] \leq C(T + 1) \text{ if } \alpha \geq 1 \tag{3.33}
\]
and
\[
\int_0^T \int_{\Omega} \left[ c_\varepsilon^{\frac{4}{3}} + |u_\varepsilon|^{\frac{4}{3}} \right] \leq C(T + 1). \tag{3.34}
\]

**Proof.** Case $\frac{1}{3} < \alpha \leq \frac{1}{2}$: Due to (3.4), (3.5) and (3.6), in light of the Gagliardo–Nirenberg inequality, for some $C_1$ and $C_2 > 0$ which are independent of $\varepsilon$, one may verify that
\[
\int_0^T \int_{\Omega} \frac{n_\varepsilon^{6\alpha + 2}}{3^{\alpha + 2}} \leq \int_0^T \int_{\Omega} \left( \|n_\varepsilon^{\alpha}\|_{L^\infty(\Omega)}^{\frac{6\alpha + 2}{3^{\alpha + 2}}} \right)^2 \leq C_1 \int_0^T \left( \|n_\varepsilon^{\alpha}\|_{L^2(\Omega)}^2 + \|n_\varepsilon\|_{L^\infty(\Omega)} \right) \leq C_2(T + 1) \text{ for all } T > 0. \tag{3.35}
\]
Therefore, employing the Hölder inequality (with two exponents $\frac{4}{3\alpha+1}$ and $\frac{4}{3-3\alpha}$), we conclude that there exists a positive constant $C_3$ such that

$$
\int_0^T \int_\Omega |\nabla n_\varepsilon|^\frac{3\alpha+1}{3\alpha+2} \leq \left[ \int_0^T \int_\Omega n_\varepsilon^{2\alpha-2} |\nabla n_\varepsilon|^2 \right]^\frac{3\alpha+1}{4} \left[ \int_0^T \int_\Omega n_\varepsilon^{\frac{6\alpha+2}{3\alpha+2}} \right]^\frac{3-3\alpha}{4} 
$$

(3.36)

Case $\frac{1}{2} < \alpha < 1$ : Again by (3.34), (3.35) and (3.36) and the Gagliardo–Nirenberg inequality the Hölder inequality (with two exponents $\frac{\frac{3+2\alpha}{5\alpha}}{3+\frac{2\alpha}{3\alpha}}$ and $\frac{\frac{3+2\alpha}{3-3\alpha}}{3+\frac{2\alpha}{3\alpha}}$), we derive that there exist positive constants $C_4, C_5$ and $C_6$ such that

$$
\int_0^T \int_\Omega n_\varepsilon^{\frac{10\alpha}{3}} = \left[ \int_0^T \int_\Omega n_\varepsilon^{\frac{3+2\alpha}{2}} n_\varepsilon^{\frac{\frac{3+2\alpha}{3\alpha}}{3+\frac{2\alpha}{3\alpha}}} + n_\varepsilon^{\frac{\frac{3+2\alpha}{3-3\alpha}}{3+\frac{2\alpha}{3\alpha}}} \right] \leq C_5(T+1) \text{ for all } T > 0.
$$

(3.37)

and

$$
\int_0^T \int_\Omega |\nabla n_\varepsilon|^\frac{10\alpha}{3+3\alpha} \leq \left[ \int_0^T \int_\Omega n_\varepsilon^{2\alpha-2} |\nabla n_\varepsilon|^2 \right]^\frac{3\alpha}{3+\frac{2\alpha}{3\alpha}} \left[ \int_0^T \int_\Omega n_\varepsilon^{\frac{6\alpha+2}{3\alpha+2}} \right]^\frac{3-3\alpha}{3+\frac{2\alpha}{3\alpha}} 
$$

(3.38)

Case $\alpha \geq 1$ : Multiply the first equation in (2.6) by $n_\varepsilon$, in view of (2.7) and using $\nabla \cdot u_\varepsilon = 0$, we derive

$$
\frac{1}{2} \frac{d}{dt} \|n_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\nabla n_\varepsilon|^2 = -\int_\Omega n_\varepsilon \nabla \cdot (n_\varepsilon F'_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon) 
$$

(3.39)

$$
\leq \int_\Omega n_\varepsilon F'_\varepsilon(n_\varepsilon) |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)||\nabla n_\varepsilon||\nabla c_\varepsilon| \text{ for all } t \in (0, T_{\text{max},\varepsilon}).
$$

Recalling (1.5) and (2.7) and using $\alpha \geq 1$, by Young inequality, we derive that

$$
\int_\Omega n_\varepsilon F'_\varepsilon(n_\varepsilon) |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)||\nabla n_\varepsilon||\nabla c_\varepsilon| \leq C_S \int_\Omega |\nabla n_\varepsilon||\nabla c_\varepsilon| 
$$

(3.40)

$$
\leq \frac{1}{2} \int_\Omega |\nabla n_\varepsilon|^2 + \frac{C_2^2}{2} \int_\Omega |\nabla c_\varepsilon|^2 \text{ for all } t \in (0, T_{\text{max},\varepsilon}).
$$

Here we have use the fact that

$$
n_\varepsilon F'_\varepsilon(n_\varepsilon) |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \leq C_S n_\varepsilon (1 + n_\varepsilon)^{-1} \leq C_S.
$$
Therefore, collecting (3.39) and (3.40) and using (3.6), we conclude that

\[ \int_\Omega n_\varepsilon^2 \leq C_7 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}) \]  (3.41)

and

\[ \int_0^T \int_\Omega |\nabla n_\varepsilon|^2 \leq C_7 (T + 1). \]  (3.42)

Hence, due to (3.41)–(3.42), (3.5) and (3.6), in light of the Gagliardo–Nirenberg inequality, we derive that there exist positive constants \( C_8, C_9, C_{10}, C_{11}, C_{12} \) and \( C_{14} \) such that

\[ \int_0^T \int_\Omega n_\varepsilon^{10} \leq C_8 \int_0^T \left( \|\nabla n_\varepsilon\|^2_{L^2(\Omega)} \|n_\varepsilon\|^\frac{4}{3}_{L^2(\Omega)} + \|n_\varepsilon\|^\frac{10}{3}_{L^2(\Omega)} \right) \leq C_9 (T + 1) \quad \text{for all } T > 0 \]  (3.43)

as well as

\[ \int_0^T \int_\Omega c_\varepsilon^{10} \leq C_{10} \int_0^T \left( \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} \|c_\varepsilon\|^\frac{4}{3}_{L^2(\Omega)} + \|c_\varepsilon\|^\frac{10}{3}_{L^2(\Omega)} \right) \leq C_{11} (T + 1) \quad \text{for all } T > 0 \]  (3.44)

and

\[ \int_0^T \int_\Omega |u_\varepsilon|^{10} \leq C_{12} \int_0^T \left( \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|u_\varepsilon\|^\frac{4}{3}_{L^2(\Omega)} + \|u_\varepsilon\|^\frac{10}{3}_{L^2(\Omega)} \right) \leq C_{14} (T + 1) \quad \text{for all } T > 0. \]  (3.45)

Finally, combined with (3.35)–(3.38) and (3.42)–(3.45), we can get the results.

\[ \square \]

### 4 The global solvability of regularized problem (2.6)

The main task of this section is to prove the global solvability of regularized problem (2.6). To this end, we firstly, need to establish some \( \varepsilon \)-dependent estimates for \( n_\varepsilon, c_\varepsilon \) and \( u_\varepsilon \).

#### 4.1 A priori estimates for the regularized problem (2.6) which depends on \( \varepsilon \)

In this subsection, on the basis of Lemma 3.3, we thereby obtain some regularity properties for \( n_\varepsilon, c_\varepsilon \) and \( u_\varepsilon \) in the following form.
Lemma 4.1. Let $\alpha > \frac{1}{3}$. Then there exists $C := C(\varepsilon) > 0$ depending on $\varepsilon$ such that the solution of (2.6) satisfies

$$\int_{\Omega} n_\varepsilon^{2\alpha+2} + \int_{\Omega} |\nabla u_\varepsilon|^2 \leq C \quad \text{for all } t \in (0, T_{\max,\varepsilon}).$$

(4.1)

In addition, for each $T \in (0, T_{\max,\varepsilon})$, one can find a constant $C > 0$ depends on $\varepsilon$ such that

$$\int_{0}^{T} \int_{\Omega} n_\varepsilon^{2\alpha} |\nabla n_\varepsilon|^2 + |\Delta u_\varepsilon|^2 \leq C.$$

(4.2)

Proof. Multiply the first equation in (2.6) by $n_\varepsilon^{1+2\alpha}$, in view of (2.7) and using $\nabla \cdot u_\varepsilon = 0$, we derive

$$\frac{1}{2} \frac{d}{dt} \|n_\varepsilon\|_{L^{2\alpha+2}(\Omega)}^{2\alpha+2} + (1 + 2\alpha) \int_{\Omega} n_\varepsilon^{2\alpha} |\nabla n_\varepsilon|^2$$

$$= - \int_{\Omega} n_\varepsilon^{1+2\alpha} \nabla \cdot (n_\varepsilon F'_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon)$$

$$\leq (1 + 2\alpha) \int_{\Omega} n_\varepsilon^{2\alpha} n_\varepsilon F'_\varepsilon(n_\varepsilon) |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)||\nabla n_\varepsilon||\nabla c_\varepsilon| \quad \text{for all } t \in (0, T_{\max,\varepsilon}).$$

(4.3)

Recalling (1.5) and (2.7), by Young inequality, one can see that

$$\int_{\Omega} n_\varepsilon^{2\alpha} n_\varepsilon F'_\varepsilon(n_\varepsilon) |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)||\nabla n_\varepsilon||\nabla c_\varepsilon|$$

$$\leq C \varepsilon (1 + 2\alpha) \int_{\Omega} n_\varepsilon^{2\alpha} (1 + n_\varepsilon)^{1-\alpha} |\nabla n_\varepsilon||\nabla c_\varepsilon|$$

$$\leq C \varepsilon (1 + 2\alpha) \int_{\Omega} n_\varepsilon^{2\alpha} |\nabla n_\varepsilon||\nabla c_\varepsilon|$$

$$\leq \frac{(1 + 2\alpha)}{2} \int_{\Omega} n_\varepsilon^{2\alpha} |\nabla n_\varepsilon|^2 + C_1 \int_{\Omega} |\nabla c_\varepsilon|^2 \quad \text{for all } t \in (0, T_{\max,\varepsilon}),$$

(4.4)

where $C_1$ is a positive constant, as all subsequently appearing constants $C_2, C_3, \ldots$ possibly depend on $\varepsilon$. Here we have used the fact that $F'_\varepsilon(n_\varepsilon) \leq \frac{1}{\varepsilon n_\varepsilon}$. Inserting (4.4) into (4.3) and using (3.6), we derive that

$$\int_{\Omega} n_\varepsilon^{2\alpha+2} \leq C_2 \quad \text{for all } t \in (0, T_{\max,\varepsilon})$$

(4.5)

and

$$\int_{0}^{T} \int_{\Omega} n_\varepsilon^{2\alpha} |\nabla n_\varepsilon|^2 \leq C_2 \quad \text{for all } T < T_{\max,\varepsilon}.$$

(4.6)

Now, due to $D(1 + \varepsilon A) := W^{2,2}(\Omega) \cap W^{0,2}_{0,\sigma}(\Omega) \hookrightarrow L^\infty(\Omega)$, by (3.5), we derive that for some $C_3 > 0$ and $C_4 > 0$,

$$\|Y_\varepsilon u_\varepsilon\|_{L^\infty(\Omega)} = \|(I + \varepsilon A)^{-1} u_\varepsilon\|_{L^\infty(\Omega)} \leq C_3 \|u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C_4 \quad \text{for all } t \in (0, T_{\max,\varepsilon}).$$

(4.7)
Next, testing the projected Stokes equation $u_{\varepsilon t} + Au_{\varepsilon} = P[-\kappa(Y_{\varepsilon}u_{\varepsilon} \cdot \nabla)u_{\varepsilon} + n_{\varepsilon}\nabla \phi]$ by $Au_{\varepsilon}$, we derive

$$
\frac{1}{2} \frac{d}{dt} \|A^{1/2}u_{\varepsilon}\|^2_{L^2(\Omega)} + \int_{\Omega} |Au_{\varepsilon}|^2 \leq \int_{\Omega} Au_{\varepsilon}P(-\kappa(Y_{\varepsilon}u_{\varepsilon} \cdot \nabla)u_{\varepsilon}) + \int_{\Omega} P(n_{\varepsilon}\nabla \phi)Au_{\varepsilon} \leq \frac{1}{2} \int_{\Omega} |Au_{\varepsilon}|^2 + \kappa^2 \int_{\Omega} |(Y_{\varepsilon}u_{\varepsilon} \cdot \nabla)u_{\varepsilon}|^2 + \|\nabla \phi\|^2_{L^\infty(\Omega)} \int_{\Omega} n_{\varepsilon}^2 \text{ for all } t \in (0, T_{\text{max},\varepsilon}).
$$

On the other hand, in light of the Gagliardo–Nirenberg inequality, the Young inequality and (4.7), there exists a positive constant $C_5$ such that

$$
\kappa^2 \int_{\Omega} |(Y_{\varepsilon}u_{\varepsilon} \cdot \nabla)u_{\varepsilon}|^2 \leq \kappa^2 \|Y_{\varepsilon}u_{\varepsilon}\|^2_{L^\infty(\Omega)} \int_{\Omega} |\nabla u_{\varepsilon}|^2 \leq \kappa^2 \|Y_{\varepsilon}u_{\varepsilon}\|^2_{L^\infty(\Omega)} \int_{\Omega} |\nabla u_{\varepsilon}|^2 \leq C_5 \int_{\Omega} |\nabla u_{\varepsilon}|^2 \text{ for all } t \in (0, T_{\text{max},\varepsilon}).
$$

Here we have the well-known fact that $\|A(\cdot)\|_{L^2(\Omega)}$ defines a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ on $D(A)$ (see Theorem 2.1.1 of [21]). Now, recalling that $\|A^{1/2}u_{\varepsilon}\|^2_{L^2(\Omega)} = \|\nabla u_{\varepsilon}\|^2_{L^2(\Omega)}$, therefore, substituting (4.9) into (4.8) yields

$$
\frac{1}{2} \frac{d}{dt} \|\nabla u_{\varepsilon}\|^2_{L^2(\Omega)} + \int_{\Omega} |\Delta u_{\varepsilon}|^2 \leq C_6 \int_{\Omega} |\nabla u_{\varepsilon}|^2 + \|\nabla \phi\|^2_{L^\infty(\Omega)} \int_{\Omega} n_{\varepsilon}^2 \text{ for all } t \in (0, T_{\text{max},\varepsilon}).
$$

(4.10)

In view of $\alpha > \frac{1}{3}$ yields to $2\alpha + 2 > \frac{8}{3} > 2$, thus, collecting (4.5) and (4.10) and applying some basic calculation, we can get the results.

**Lemma 4.2.** Under the assumptions of Theorem 1.1, it holds that there exists $C := C(\varepsilon) > 0$ depends on $\varepsilon$ such that

$$
\int_{\Omega} |\nabla c_{\varepsilon}(\cdot, t)|^2 \leq C \text{ for all } t \in (0, T_{\text{max},\varepsilon})
$$

(4.11)

and

$$
\int_{0}^{T} \int_{\Omega} |\Delta c_{\varepsilon}|^2 \leq C \text{ for all } T \in (0, T_{\text{max},\varepsilon}).
$$

(4.12)

**Proof.** Firstly, testing the second equation in (2.6) against $-\Delta c_{\varepsilon}$, employing the Young
inequality and using (3.3) yields

\[ \frac{1}{2} \frac{d}{dt} \| \nabla c_\varepsilon \|^2_{L^2(\Omega)} = \int_\Omega -\Delta c_\varepsilon (\Delta c_\varepsilon - c_\varepsilon + F_\varepsilon(n_\varepsilon) - u_\varepsilon \cdot \nabla c_\varepsilon) \]

\[ = -\frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 - \int_\Omega \| \nabla c_\varepsilon \|^2 - \int_\Omega F_\varepsilon(n_\varepsilon) \Delta c_\varepsilon - \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon) \Delta c_\varepsilon \]  

(4.13)

for all \( t \in (0, T_{\text{max}, \varepsilon}) \). Next, one needs to estimate the last term on the right-hand side of (4.13). Indeed, in view of the Sobolev embedding theorem \( (W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)) \), then applying (4.1) and (3.5), we derive from the Hölder inequality, the Gagliardo–Nirenberg inequality and the Young inequality that there exist positive constants \( C_1, C_2, C_3 \) and \( C_4 \) such that

\[ \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon| |\Delta c_\varepsilon| \leq \| u_\varepsilon \|_{L^6(\Omega)} \| \nabla c_\varepsilon \|_{L^3(\Omega)} \| \Delta c_\varepsilon \|_{L^2(\Omega)} \]

\[ \leq C_1 \| \nabla c_\varepsilon \|_{L^3(\Omega)} \| \Delta c_\varepsilon \|_{L^2(\Omega)} \]

\[ \leq C_2 (\| \Delta c_\varepsilon \|^\frac{1}{2} \| L^2(\Omega) \| \| c_\varepsilon \|^\frac{1}{2} \| L^2(\Omega) \| + \| c_\varepsilon \|^\frac{2}{3} \| L^2(\Omega) \| \| \Delta c_\varepsilon \|_{L^2(\Omega)}) \]

\[ \leq C_3 (\| \Delta c_\varepsilon \|^\frac{7}{2} \| L^2(\Omega) \| + \| \Delta c_\varepsilon \|_{L^2(\Omega)}) \]

\[ \leq \frac{1}{4} \| \Delta c_\varepsilon \|^2_{L^2(\Omega)} + C_4 \text{ for all } t \in (0, T_{\text{max}, \varepsilon}). \]

Inserting (4.11) into (4.13) and using (4.1), one obtains (4.11) and (4.12). This completes the proof of Lemma 4.2.

Lemma 4.3. Let \( \alpha > \frac{1}{3} \). Assume the hypothesis of Theorem 1.1 holds. Then there exists a positive constant \( C := C(\varepsilon) \) depends on \( \varepsilon \) such that the solution of (2.6) from Lemma 2.1 satisfies

\[ \| A^\gamma u_\varepsilon(\cdot, t) \|_{L^2(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \]  

(4.15)

as well as

\[ \| u_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \]  

(4.16)

and

\[ \| \nabla c_\varepsilon(\cdot, t) \|_{L^q(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \]  

(4.17)

and some \( 3 < q < 6 \).
Proof. Let $h_{\varepsilon}(x, t) = \mathcal{P}[n_{\varepsilon} \Delta \phi - \kappa(Y_{\varepsilon}u_{\varepsilon} \cdot \nabla)u_{\varepsilon}]$. Due to $\alpha > \frac{1}{3}$, then along with (4.11), (1.7) and (4.7), there exist positive constants $q_0 > \frac{3}{2}$ and $C_1$ such that

$$\|n_{\varepsilon}(\cdot, t)\|_{L^{q_0}(\Omega)} \leq C_1 \text{ for all } t \in (0, T_{\text{max}, \varepsilon})$$  \hspace{1cm} (4.18)

and

$$\|h_{\varepsilon}(\cdot, t)\|_{L^{q_0}(\Omega)} \leq C_1 \text{ for all } t \in (0, T_{\text{max}, \varepsilon}).$$  \hspace{1cm} (4.19)

Hence, by $q_0 > \frac{3}{2}$, we pick an arbitrary $\gamma \in (\frac{3}{4}, 1)$ and therefore, $-\gamma - \frac{3}{2}(\frac{1}{q_0} - \frac{1}{2}) > -1$. Then in view of the smoothing properties of the Stokes semigroup ([8]), we derive that for some $C_2 > 0$ and $C_3 > 0$, we have

$$\|A^\gamma u_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} \leq \|A^\gamma e^{-tA}u_0\|_{L^2(\Omega)} + \int_0^t \|A^\gamma e^{-\tau A}h_{\varepsilon}(\cdot, \tau)\|_{L^2(\Omega)} d\tau$$

$$\leq \|A^\gamma u_0\|_{L^2(\Omega)} + C_2 \int_0^t (t - \tau)^{-\gamma - \frac{3}{2}(\frac{1}{q_0} - \frac{1}{2})} e^{-\lambda(t-\tau)} \|h_{\varepsilon}(\cdot, \tau)\|_{L^{q_0}(\Omega)} d\tau$$

$$\leq C_3 \text{ for all } t \in (0, T_{\text{max}, \varepsilon}).$$  \hspace{1cm} (4.20)

Observe that $\gamma > \frac{3}{4}$, $D(A^\gamma)$ is continuously embedded into $L^\infty(\Omega)$, therefore, due to (4.20), we derive that there exists a positive constant $C_4$ such that

$$\|u_{\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} \leq C_4 \text{ for all } t \in (0, T_{\text{max}, \varepsilon}).$$  \hspace{1cm} (4.21)

On the there hand, observing that (4.11), with the help of the Sobolev imbedding theorem, we derive for any $l < 6$, there exists a positive constant $C_5$ such that

$$\|c_{\varepsilon}(\cdot, t)\|_{L^l(\Omega)} \leq C_5 \text{ for all } t \in (0, T_{\text{max}, \varepsilon}),$$  \hspace{1cm} (4.22)

which together with the Hölder inequality implies that for any fixed $\tilde{q} \in (3, 6)$

$$\|c_{\varepsilon}(\cdot, t)\|_{L^{\tilde{q}}(\Omega)} \leq C_6 \text{ for all } t \in (0, T_{\text{max}, \varepsilon}).$$  \hspace{1cm} (4.23)

Now, involving the variation-of-constants formula for $c_{\varepsilon}$ and applying $\nabla \cdot u_{\varepsilon} = 0$ in $x \in \Omega, t > 0$, we have

$$c_{\varepsilon}(t) = e^{t(\Delta - 1)}c_0 + \int_0^t e^{(t-s)(\Delta - 1)}(F_{\varepsilon}(n_{\varepsilon}(s)) + \nabla \cdot (u_{\varepsilon}(s)c_{\varepsilon}(s)))ds, \text{ } t \in (0, T_{\text{max}, \varepsilon}),$$  \hspace{1cm} (4.24)
which implies that
\[
\|\nabla e^{(\Delta-1)}c_0\|_{L^q(\Omega)} \leq \|\nabla e^{(t-s)(\Delta-1)}F_\varepsilon(n_\varepsilon(s))\|_{L^q(\Omega)} + \int_0^t \|\nabla e^{(t-s)(\Delta-1)} \cdot (u_\varepsilon(s)c_\varepsilon(s))\|_{L^q(\Omega)} ds,
\]
where \(3 < q < \min\{\frac{3q_0}{3-q_0}, q\}\). To deal with the right-hand side of (4.25), in view of (1.8), we first use Lemma 2.2 to get that
\[
\|\nabla e^{(t-s)(\Delta-1)}c_0\|_{L^q(\Omega)} \leq C_7 \quad \text{for all } t \in (0, T_{max,\varepsilon}).
\]
Since (4.18) and (4.23) yields to
\[
-\frac{1}{2} - \frac{3}{2}(\frac{1}{q_0} - \frac{1}{q}) > -1,
\]
then we derive from Lemma 2.2 and (4.23) and (4.21) that there exist constants \(C_{10}, C_{11}, C_{12}\) and \(C_{13}\) such that
\[
\int_0^t \|\nabla e^{(t-s)(\Delta-1)} \cdot (u_\varepsilon(s)c_\varepsilon(s))\|_{L^{\tilde{q}}(\Omega)} ds \leq C_9 \quad \text{for all } t \in (0, T_{max,\varepsilon}).
\]
Finally we will deal with the third term on the right-hand side of (4.25). Indeed, we choose \(0 < \iota < \frac{1}{2}\) satisfying \(\frac{1}{2} + \frac{3}{2}(\frac{1}{q_0} - \frac{1}{q}) < \iota\) and \(\tilde{\kappa} \in (0, \frac{1}{2} - \iota)\). In view of the Hölder inequality, then we derive from Lemma 2.2 and (4.23) and (4.21) that there exist constants \(C_{10}, C_{11}, C_{12}\) and \(C_{13}\) such that
\[
\int_0^t \|\nabla e^{(t-s)(\Delta-1)} \cdot (u_\varepsilon(s)c_\varepsilon(s))\|_{L^{\tilde{q}}(\Omega)} ds \leq C_{13} \quad \text{for all } t \in (0, T_{max,\varepsilon}).
\]
Here we have used the fact that
\[
\int_0^t (t-s)^{-\iota - \frac{1}{2} - \tilde{\kappa}} e^{-\lambda(t-s)} ds \leq \int_0^\infty \sigma^{-\iota - \frac{1}{2} - \tilde{\kappa}} e^{-\lambda\sigma} d\sigma < +\infty.
\]
Finally, collecting (4.25)–(4.28), we can obtain there exists a positive constant $C_{14}$ such that
\[
\int_{\Omega} |\nabla c_\varepsilon(t)|^q \leq C_{14} \text{ for all } t \in (0, T_{\max, \varepsilon}) \text{ and some } q \in (3, \min\{\frac{3q_0}{3-q_0^+}, \bar{q}\}). \quad (4.29)
\]
The proof Lemma 4.3 is complete.

Then we shall establish global existence in approximate problem (2.6) by using Lemmas 4.1, 4.2.

**Lemma 4.4.** Let $\alpha > \frac{1}{3}$. Then for all $\varepsilon \in (0, 1)$, the solution of (2.6) is global in time.

**Proof.** Assuming that $T_{\max, \varepsilon}$ be finite for some $\varepsilon \in (0, 1)$. Fix $T \in (0, T_{\max, \varepsilon})$. Let $M(T) := \sup_{t \in (0, T)} \|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}$ and $\tilde{h}_\varepsilon := F'_\varepsilon(n_\varepsilon)S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon + u_\varepsilon$. Then by Lemma 4.3, (1.5) and (3.1), there exists $C_1 > 0$ such that
\[
\|\tilde{h}_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \leq C_1 \text{ for all } t \in (0, T_{\max, \varepsilon}) \text{ and some } 3 < q < 6. \quad (4.30)
\]
Hence, due to the fact that $\nabla \cdot u_\varepsilon = 0$, again, by means of an associate variation-of-constants formula for $v$, we can derive
\[
n_\varepsilon(t) = e^{(t-t_0)\Delta}n_\varepsilon(\cdot, t_0) - \int_{t_0}^{t} e^{(t-s)\Delta}\nabla \cdot (n_\varepsilon(\cdot, s)\tilde{h}_\varepsilon(\cdot, s))ds, \quad t \in (t_0, T), \quad (4.31)
\]
where $t_0 := (t - 1)_+$. If $t \in (0, 1]$, by virtue of the maximum principle, we derive that
\[
\|e^{(t-t_0)\Delta}n_\varepsilon(\cdot, t_0)\|_{L^\infty(\Omega)} \leq \|n_0\|_{L^\infty(\Omega)}, \quad (4.32)
\]
while if $t > 1$ then with the help of the $L^p$-$L^q$ estimates for the Neumann heat semigroup and Lemma 3.2, we conclude that
\[
\|e^{(t-t_0)\Delta}n_\varepsilon(\cdot, t_0)\|_{L^\infty(\Omega)} \leq C_2(t - t_0)^{-\frac{\alpha}{2}}\|n_\varepsilon(\cdot, t_0)\|_{L^1(\Omega)} \leq C_3. \quad (4.33)
\]
Finally, we fix an arbitrary $p \in (3, q)$ and then once more invoke known smoothing properties of the Stokes semigroup and the Hölder inequality to find $C_4 > 0$ such that
\[
\int_{t_0}^{t} \|e^{(t-s)\Delta}\nabla \cdot (n_\varepsilon(\cdot, s)\tilde{h}_\varepsilon(\cdot, s))\|_{L^\infty(\Omega)}ds \\
\leq C_4 \int_{t_0}^{t} (t - s)^{-\frac{1}{2} - \frac{\alpha}{2p}}\|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}\|\tilde{h}_\varepsilon(\cdot, s)\|_{L^q(\Omega)}ds \\
\leq C_4 \int_{t_0}^{t} (t - s)^{-\frac{1}{2} - \frac{\alpha}{2p}}\|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}\|\tilde{h}_\varepsilon(\cdot, s)\|_{L^q(\Omega)}ds \\
\leq C_4 \int_{t_0}^{t} (t - s)^{-\frac{1}{2} - \frac{\alpha}{2p}}\|u_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)}\|u_\varepsilon(\cdot, s)\|_{L^p(\Omega)}\|\tilde{h}_\varepsilon(\cdot, s)\|_{L^q(\Omega)}ds \\
\leq C_5 M^b(T) \text{ for all } t \in (0, T), \quad (4.34)
\]
where 
\[ b := \frac{pq - q + p}{pq} \in (0, 1) \] and

\[ C_5 := C_4 C_1^{2 - b} \int_0^1 \sigma^{-\frac{1}{2} - \frac{3}{2p}} d\sigma. \]

Since \( p > 3 \), we conclude that \(-\frac{1}{2} - \frac{3}{2p} > -1\). In combination with (4.31)–(4.34) and using the definition of \( M(T) \) we obtain \( C_6 > 0 \) such that

\[ M(T) \leq C_6 + C_6 M^b(T) \quad \text{for all } T \in (0, T_{\max, \varepsilon}). \] (4.35)

Hence, in view of \( b < 1 \), with some basic calculation, in light of \( T \in (0, T_{\max, \varepsilon}) \) was arbitrary, we can get

\[ \| n_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} \leq C_7 \quad \text{for all } t \in (0, T_{\max, \varepsilon}). \] (4.36)

In order to prove the boundedness of \( \| \nabla c_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} \), we rewrite the variation-of-constants formula for \( c_\varepsilon \) in the form

\[ c_\varepsilon(\cdot, t) = e^{(\Delta - 1)t} c_0 + \int_0^t e^{(t-s)(\Delta - 1)} [F_\varepsilon(n_\varepsilon)(s) - u_\varepsilon(s) \cdot \nabla c_\varepsilon(s)] ds \quad \text{for all } t \in (0, T_{\max, \varepsilon}). \]

Now, we choose \( \theta \in (\frac{1}{2} + \frac{3}{2q}, 1) \), where \( 3 < q < 6 \) (see (4.29)), then the domain of the fractional power \( D((\Delta + 1)^\theta) \rightarrow W^{1,\infty}(\Omega) \). Hence, in view of \( L^p-L^q \) estimates associated heat semigroup, (4.16), (4.17) and (3.3), we derive that there exist positive constants \( \lambda, C_8, C_9, C_{10} \) and \( C_{11} \) such that

\[ \| c_\varepsilon(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq C_8 \| (-\Delta + 1)^\theta c_\varepsilon(\cdot, t) \|_{L^q(\Omega)} \]

\[ \leq C_9 t^{-\theta} e^{-\lambda t} \| c_0 \|_{L^q(\Omega)} + C_9 \int_0^t (t-s)^{-\theta} e^{-\lambda(t-s)} \| (F_\varepsilon(n_\varepsilon)(s) - u_\varepsilon(s) \cdot \nabla c_\varepsilon(s)) \|_{L^q(\Omega)} ds \]

\[ \leq C_{10} + C_{10} \int_0^t (t-s)^{-\theta} e^{-\lambda(t-s)} [\| n_\varepsilon(s) \|_{L^q(\Omega)} + \| u_\varepsilon(s) \|_{L^\infty(\Omega)} \| \nabla c_\varepsilon(s) \|_{L^q(\Omega)}] ds \]

\[ \leq C_{11} \quad \text{for all } t \in (0, T_{\max, \varepsilon}). \] (4.37)

Here we have used the Hölder inequality as well as

\[ \int_0^t (t-s)^{-\theta} e^{-\lambda(t-s)} \leq \int_0^\infty \sigma^{-\theta} e^{-\lambda\sigma} d\sigma < +\infty. \]

In view of (4.15), (4.37) and (4.36), we apply Lemma 2.1 to reach a contradiction.
5 Regularity properties of time derivatives

In order to prove the limit functions $n, c$ and $u$ gained below (see Section 6), we will rely on an additional regularity estimate for $n_\varepsilon F'_\varepsilon(n_\varepsilon)S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon$, $u_\varepsilon \cdot \nabla c_\varepsilon$, $n_\varepsilon u_\varepsilon$ and $c_\varepsilon u_\varepsilon$.

**Lemma 5.1.** Let $\alpha > \frac{1}{3}$, (1.7) and (1.8) hold. Then for any $T > 0$, one can find $C > 0$ independent of $\varepsilon$ such that

$$
\int_0^T \int_\Omega \left[ |n_\varepsilon F'_\varepsilon(n_\varepsilon)S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon|^{\frac{3\alpha+1}{2}} + |n_\varepsilon u_\varepsilon|^{\frac{10(3\alpha+1)}{9(\alpha+2)}} \right] \leq C(T+1) \quad \text{if} \quad \frac{1}{3} < \alpha \leq \frac{1}{2},
$$

(5.1)

$$
\int_0^T \int_\Omega \left[ |n_\varepsilon F'_\varepsilon(n_\varepsilon)S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon|^{\frac{10\alpha}{3+\alpha}} + |n_\varepsilon u_\varepsilon|^{\frac{10\alpha}{3(\alpha+1)}} \right] \leq C(T+1) \quad \text{if} \quad \frac{1}{2} < \alpha < 1
$$

(5.2)

as well as

$$
\int_0^T \int_\Omega \left[ |n_\varepsilon F'_\varepsilon(n_\varepsilon)S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon|^2 + |n_\varepsilon u_\varepsilon|^\frac{4}{3} \right] \leq C(T+1) \quad \text{if} \quad \alpha \geq 1
$$

(5.3)

and

$$
\int_0^T \int_\Omega \left[ |u_\varepsilon \cdot \nabla c_\varepsilon|^\frac{5}{4} + |c_\varepsilon u_\varepsilon|^\frac{5}{3} \right] \leq C(T+1).
$$

(5.4)

**Proof.** Firstly, by (1.5), (3.1) and (2.8), we derive that

$$n_\varepsilon F'_\varepsilon(n_\varepsilon)S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \leq C_{S}n_\varepsilon^{(1-\alpha)+}
$$

with $(1-\alpha)^+ = \max\{0, 1-\alpha\}$. Case $\frac{1}{3} < \alpha \leq \frac{1}{2}$: It is not difficult to verify that

$$
\frac{2}{3\alpha+1} = \frac{1}{2} + \frac{3}{6\alpha+2}(1-\alpha)
$$

and

$$
\frac{9(\alpha+2)}{10(3\alpha+1)} = \frac{3}{10} + \frac{3}{6\alpha+2}.
$$

From this and by (3.31), and recalling (3.45) and the Hölder inequality, we can obtain (5.1). Other cases, can be proved very similarly. Therefore, we omit it. 

To prepare our subsequent compactness properties of $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ by means of the Aubin-Lions lemma (see Simon [20]), we use Lemmas 3.2, 3.4 to obtain the following regularity property with respect to the time variable.

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Lemma 5.2. Let $\alpha > \frac{1}{3}$, (1.7) and (1.8) hold. Then for any $T > 0$, one can find $C > 0$ independent if $\varepsilon$ such that

$$\int_0^T \|\partial_t n_\varepsilon(\cdot, t)\|_{(W^{2,4}(\Omega))^*} dt \leq C(T + 1) \quad \text{if} \quad \frac{1}{3} < \alpha \leq \frac{8}{21}, \quad (5.5)$$

$$\int_0^T \|\partial_t n_\varepsilon(\cdot, t)\|_{(W^{1,10(\alpha+1)}(\Omega))^*} dt \leq C(T + 1) \quad \text{if} \quad \frac{8}{21} < \alpha \leq \frac{1}{2}, \quad (5.6)$$

$$\int_0^T \|\partial_t n_\varepsilon(\cdot, t)\|_{(W^{1,7/\alpha-3}(\Omega))^*} dt \leq C(T + 1) \quad \text{if} \quad \frac{1}{2} < \alpha < 1, \quad (5.7)$$

$$\int_0^T \|\partial_t n_\varepsilon(\cdot, t)\|_{(W^{1,7/2}(\Omega))^*} dt \leq C(T + 1) \quad \text{if} \quad \alpha \geq 1 \quad (5.8)$$

as well as

$$\int_0^T \|\partial_t c_\varepsilon(\cdot, t)\|_{(W^{1,5}(\Omega))^*} dt \leq C(T + 1) \quad (5.9)$$

and

$$\int_0^T \|\partial_t u_\varepsilon(\cdot, t)\|_{(W^{1,5}(\Omega))^*} dt \leq C(T + 1). \quad (5.10)$$

Proof. In the proof, we only prove the case $\frac{8}{21} < \alpha \leq \frac{1}{2}$, since, other case can be proved similarly. Firstly, an elementary calculation ensures that

$$\frac{3\alpha + 1}{2} > \frac{10(3\alpha + 1)}{9(\alpha + 2)} \quad \text{and} \quad 1 = \frac{9(\alpha + 2)}{10(3\alpha + 1)} + \frac{21\alpha - 8}{10(3\alpha + 1)}. \quad (5.11)$$

Next, testing the first equation of (2.6) by certain $\varphi \in C^\infty(\Omega)$, we have

$$\left\{ \begin{array}{l}
\int_\Omega (n_\varepsilon, t) \varphi \\
\int_\Omega \left[ \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon F_\varepsilon'(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon) - u_\varepsilon \cdot \nabla n_\varepsilon \right] \varphi \\
\int_\Omega \left[ -\nabla n_\varepsilon \cdot \nabla \varphi + n_\varepsilon F_\varepsilon'(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon \cdot \nabla \varphi + n_\varepsilon u_\varepsilon \cdot \nabla \varphi \right] \\
\leq \left\{ \begin{array}{c}
\|\nabla n_\varepsilon\|_{L^{10(3\alpha+1)}(\Omega)}^{10(3\alpha+1)} + \|n_\varepsilon F_\varepsilon'(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon\|_{L^{10(3\alpha+1)}(\Omega)}^{10(3\alpha+1)} + \|n_\varepsilon u_\varepsilon\|_{L^{9(\alpha+2)}(\Omega)}^{10(3\alpha+1)} \end{array} \right\} \|\varphi\|_{W^{1,10\alpha+3}\Omega} \quad (5.12)
\end{array} \right.$$
where $C_1$ is a positive constant independent of $\varepsilon$. Finally, (5.5) is a consequence of (3.31), (5.1), (5.11) and the Hölder inequality.

\[ \square \]

6 Passing to the limit. Proof of Theorem 1.1

Based on above lemmas and by extracting suitable subsequences in a standard way, we could see the solution of (1.3) is indeed globally solvable.

Lemma 6.1. Let (1.4), (1.5) and (1.7) and (1.8) hold, and suppose that $\alpha > \frac{1}{3}$. There exists $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and such that as $\varepsilon := \varepsilon_j \searrow 0$ we have

\[ n_\varepsilon \to n \quad \text{a.e. in } \Omega \times (0, \infty) \quad \text{and in } L^r_{loc}(\overline{\Omega} \times [0, \infty)) \quad \text{with } \ r = \begin{cases} \frac{3\alpha + 1}{2} & \text{if } \frac{1}{3} < \alpha \leq \frac{1}{2}, \\ \frac{10\alpha}{3 + 2\alpha} & \text{if } \frac{1}{2} < \alpha < 1, \\ 2 & \text{if } \alpha \geq 1, \end{cases} \]

(6.1)

\[ \nabla n_\varepsilon \rightharpoonup \nabla n \quad \text{in } \Omega \times (0, \infty) \quad \text{and in } L^r_{loc}(\overline{\Omega} \times [0, \infty)) \quad \text{with } \ r = \begin{cases} \frac{3\alpha + 1}{2} & \text{if } \frac{1}{3} < \alpha \leq \frac{1}{2}, \\ \frac{10\alpha}{3 + 2\alpha} & \text{if } \frac{1}{2} < \alpha < 1, \\ 2 & \text{if } \alpha \geq 1, \end{cases} \]

(6.2)

\[ c_\varepsilon \to c \quad \text{in } L^2_{loc}(\overline{\Omega} \times [0, \infty)) \quad \text{and a.e. in } \Omega \times (0, \infty), \]

(6.3)

\[ \nabla c_\varepsilon \rightharpoonup \nabla c \quad \text{a.e. in } \Omega \times (0, \infty), \]

(6.4)

\[ u_\varepsilon \to u \quad \text{in } L^2_{loc}(\overline{\Omega} \times [0, \infty)) \quad \text{and a.e. in } \Omega \times (0, \infty), \]

(6.5)

as well as

\[ \nabla c_\varepsilon \rightharpoonup \nabla c \quad \text{in } L^2_{loc}(\overline{\Omega} \times [0, \infty)) \]

(6.6)

and

\[ \nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^2_{loc}(\overline{\Omega} \times [0, \infty)) \]

(6.7)

and

\[ u_\varepsilon \rightharpoonup u \quad \text{in } L^{10\alpha}_{loc}(\Omega \times [0, \infty)) \]

(6.8)

with some triple $(n, c, u)$ which is a global weak solution of (1.3) in the sense of Definition 2.1.
Proof. From Lemmas 3.3, 3.4, 5.1, 5.2 and the Aubin–Lions lemma ([20]), we can derive (6.1)–(6.3) and (6.5)–(6.8) holds. Next, let \( g_\varepsilon(x, t) := -c_\varepsilon + F_\varepsilon(n_\varepsilon) - u_\varepsilon \cdot \nabla c_\varepsilon \). With the notation, the second equation of (2.6) can be rewritten in the component form as

\[
c_{\varepsilon t} - \Delta c_\varepsilon = g_\varepsilon. \quad (6.9)
\]

Case \( \frac{1}{3} < \alpha \leq \frac{1}{2} \): Observing that

\[
\frac{5}{4} < \frac{4}{3} < \min\left\{ \frac{6\alpha + 2}{3}, \frac{10}{3} \right\} \text{ for } \frac{1}{3} < \alpha \leq \frac{1}{2},
\]

thus, recalling (3.31), (3.34) and (5.4) and applying the Hölder inequality, we conclude that \( g_\varepsilon \) is bounded in \( L^{\frac{5}{4}}(\Omega \times (0, T)) \) for any \( \varepsilon \in (0, 1) \), we may invoke the standard parabolic regularity theory to (6.9) and infer that \( (c_\varepsilon)_{\varepsilon \in (0, 1)} \) is bounded in \( L^{\frac{5}{4}}((0, T) ; W^{2,\frac{5}{4}}(\Omega)) \). Thus, by virtue of (5.9) and the Aubin–Lions lemma we derive that the relative compactness of \( (c_\varepsilon)_{\varepsilon \in (0, 1)} \) in \( L^{\frac{5}{4}}((0, T) ; W^{1,\frac{5}{4}}(\Omega)) \). We can pick an appropriate subsequence which is still written as \( (\varepsilon_j)_{j \in \mathbb{N}} \) such that \( \nabla c_{\varepsilon_j} \to z_1 \) in \( L^{\frac{5}{4}}(\Omega \times (0, T)) \) for all \( T \in (0, \infty) \) and some \( z_1 \in L^{\frac{5}{4}}(\Omega \times (0, T)) \) as \( j \to \infty \), hence \( \nabla c_{\varepsilon_j} \to z_1 \) a.e. in \( \Omega \times (0, \infty) \) as \( j \to \infty \). In view of (6.6) and the Egorov theorem we conclude that \( z_1 = \nabla c \), and whence (6.4) holds. Next, we pay our attention to the case \( \frac{1}{2} < \alpha < 1 \): By straightforward calculations, and using relation \( \frac{1}{2} < \alpha < 1 \), one has

\[
\frac{5}{4} < \frac{5}{3} < \min\left\{ \frac{10\alpha}{3}, \frac{10}{3} \right\}
\]

Therefore, noticing that (3.32), (3.34), and using (5.4), it follows from the Hölder inequality that

\[
g_\varepsilon \text{ is bounded in } L^{\frac{5}{4}}(\Omega \times (0, T)) \text{ for any } \varepsilon \in (0, 1). \quad (6.10)
\]

Employing almost exactly the same arguments as in the proof of Case \( \frac{1}{3} < \alpha \leq \frac{1}{2} \), and taking advantage of (6.10), we conclude the estimate (6.6). Case \( \alpha \geq 1 \) is similar to case \( \frac{1}{3} < \alpha \leq \frac{1}{2} \), we omit it.

In the following, we shall prove \( (n, c, u) \) is a weak solution of problem (1.3) in Definition 2.1. In fact, \( \alpha > \frac{1}{3} \) yields to

\[
r > 1,
\]
where \( r \) is given by (6.1). Therefore, with the help of (6.1)–(6.3), (6.5)–(6.7), we can derive (2.1). Now, by the nonnegativity of \( n_\varepsilon \) and \( c_\varepsilon \), we derive \( n \geq 0 \) and \( c \geq 0 \). Next, due to (6.7) and \( \nabla \cdot u_\varepsilon = 0 \), we conclude that \( \nabla \cdot u = 0 \) a.e. in \( \Omega \times (0, \infty) \). On the other hand, in view of (5.1), (5.2) and (5.3), we conclude that

\[
n_\varepsilon F'(n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon)) \nabla c_\varepsilon \rightharpoonup (n S(x, n, c)) \nabla c \text{ a.e. in } \Omega \times (0, \infty) \text{ as } \varepsilon := \varepsilon_j \searrow 0 \text{ for each } T \in (0, \infty),
\]

(6.11)

where \( r \) is given by (6.1). On the other hand, it follows from (1.4), (2.8), (3.2), (6.1), (6.3) and (6.4) that

\[
n_\varepsilon F'(n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon)) \nabla c_\varepsilon \rightarrow nS(x, n, c) \nabla c \text{ a.e. in } \Omega \times (0, \infty) \text{ as } \varepsilon := \varepsilon_j \searrow 0.
\]

(6.12)

Again by the Egorov theorem, we gain \( z_2 = nS(x, n, c) \nabla c \), and hence (6.11) can be rewritten as

\[
n_\varepsilon F'(n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon)) \nabla c_\varepsilon \rightharpoonup nS(x, n, c) \nabla c \text{ in } L^r(\Omega \times (0, T)) \text{ as } \varepsilon := \varepsilon_j \searrow 0 \text{ for each } T \in (0, \infty),
\]

(6.13)

which together with \( r > 1 \) implies the integrability of \( nS(x, n, c) \nabla c \) in (2.2) as well. It is not hard to check that

\[
\frac{10(3\alpha + 1)}{9(\alpha + 2)} > 1 \text{ if } \frac{1}{3} < \alpha \leq \frac{1}{2} \text{ and } \frac{10\alpha}{3(\alpha + 1)} > 1 \text{ if } \frac{1}{2} < \alpha < 1.
\]

Thereupon, recalling (5.1), (5.2) and (5.3), we infer that for each \( T \in (0, \infty) \)

\[
n_\varepsilon u_\varepsilon \rightharpoonup z_3 \text{ in } L^{\tilde{r}}(\Omega \times (0, T)) \text{ with } \tilde{r} = \begin{cases} \frac{10(3\alpha + 1)}{9(\alpha + 2)} & \text{if } \frac{1}{3} \leq \alpha \leq \frac{1}{2}, \\ \frac{10\alpha}{3(\alpha + 1)} & \text{if } \frac{1}{2} < \alpha < 1, \\ \frac{5}{3} & \text{if } \alpha \geq 1. \end{cases} \text{ as } \varepsilon := \varepsilon_j \searrow 0.
\]

(6.14)

This, together with (6.1), and (6.5), implies

\[
n_\varepsilon u_\varepsilon \rightarrow nu \text{ a.e. in } \Omega \times (0, \infty) \text{ as } \varepsilon := \varepsilon_j \searrow 0.
\]

(6.15)

Along with (6.14) and (6.15), the Egorov theorem guarantees that \( z_3 = nu \), whereupon we derive from (6.14) that

\[
n_\varepsilon u_\varepsilon \rightarrow nu \text{ in } L^{\tilde{r}}(\Omega \times (0, T)) \text{ with } \tilde{r} = \begin{cases} \frac{10(3\alpha + 1)}{9(\alpha + 2)} & \text{if } \frac{1}{3} \leq \alpha \leq \frac{1}{2}, \\ \frac{10\alpha}{3(\alpha + 1)} & \text{if } \frac{1}{2} < \alpha < 1, \\ \frac{5}{3} & \text{if } \alpha \geq 1. \end{cases} \text{ as } \varepsilon := \varepsilon_j \searrow 0
\]

(6.16)
for each $T \in (0, \infty)$.

As a straightforward consequence of $(6.3)$ and $(6.5)$, it holds that

$$c_{\varepsilon}u_{\varepsilon} \to cu \quad \text{in} \quad L^1_{\text{loc}}(\bar{\Omega} \times (0, \infty)) \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0. \quad (6.17)$$

Thus, the integrability of $nu$ and $cu$ in $(2.2)$ is verified by $(6.3)$ and $(6.5)$.

Next, by $(6.5)$ and using the fact that $\|Y_{\varepsilon}\varphi\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(\Omega)}(\varphi \in L^2_{\sigma}(\Omega))$ and $Y_{\varepsilon}\varphi \to \varphi$ in $L^2(\Omega)$ as $\varepsilon \searrow 0$, we derive that there exists a positive constant $C_1$ such that

$$\|Y_{\varepsilon}u_{\varepsilon}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \leq \|Y_{\varepsilon}[u_{\varepsilon}(\cdot, t) - u(\cdot, t)]\|_{L^2(\Omega)} + \|Y_{\varepsilon}u(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}$$

$$\leq \|u_{\varepsilon}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} + \|Y_{\varepsilon}u(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}$$

$$\to 0 \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0 \quad (6.18)$$

and

$$\|Y_{\varepsilon}u_{\varepsilon}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}^2 \leq \left( \|Y_{\varepsilon}u_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} + \|u(\cdot, t)\|_{L^2(\Omega)} \right)^2$$

$$\leq \left( \|u_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} + \|u(\cdot, t)\|_{L^2(\Omega)} \right)^2$$

$$\leq C_1 \quad \text{for all} \quad t \in (0, \infty) \quad \text{and} \quad \varepsilon \in (0, 1). \quad (6.19)$$

Now, thus, by $(6.3)$, $(6.18)$ and $(6.19)$ and the dominated convergence theorem, we derive that

$$\int_0^T \|Y_{\varepsilon}u_{\varepsilon}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}^2 \, dt \to 0 \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0 \quad \text{for all} \quad T > 0, \quad (6.20)$$

which implies that

$$Y_{\varepsilon}u_{\varepsilon} \to u \quad \text{in} \quad L^2_{\text{loc}}([0, \infty); L^2(\Omega)). \quad (6.21)$$

Now, combining $(6.3)$ with $(6.21)$, we derive

$$Y_{\varepsilon}u_{\varepsilon} \otimes u_{\varepsilon} \to u \otimes u \quad \text{in} \quad L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)) \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0. \quad (6.22)$$

Therefore, by $(6.13)$, $(6.16)$–$(6.17)$ and $(6.22)$ we conclude that the integrability of $nS(x, n, c)\nabla c, nu$ and $cu, u \otimes u$ in $(2.2)$. Finally, for any fixed $T \in (0, \infty)$, applying $(6.1)$, we can derive

$$\int_0^T \|F_{\varepsilon}(n_{\varepsilon}(\cdot, t)) - n(\cdot, t)\|_{L^r(\Omega)} \, dt$$

$$\leq \int_0^T \|F_{\varepsilon}(n_{\varepsilon}(\cdot, t)) - F_{\varepsilon}(n(\cdot, t))\|_{L^r(\Omega)} \, dt + \int_0^T \|F_{\varepsilon}(n(\cdot, t)) - n(\cdot, t)\|_{L^r(\Omega)} \, dt$$

$$\leq \|F_{\varepsilon}\|_{L^\infty(\Omega \times (0, \infty))} \int_0^T \|n_{\varepsilon}(\cdot, t) - n(\cdot, t)\|_{L^r(\Omega)} \, dt + \int_0^T \|F_{\varepsilon}(n(\cdot, t)) - n(\cdot, t)\|_{L^r(\Omega)} \, dt, \quad (6.23)$$

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where \( r \) is the same as (6.1). Besides that, we also deduce from (3.3) and \( r > 1 \) that
\[
\|F_\varepsilon(n(\cdot, t)) - n(\cdot, t)\|_{L^r(\Omega \times (0,T))}^r \leq 2^r \|n(\cdot, t)\|^r
\] (6.24)
for each \( t \in (0, T) \), which together with (6.1) shows the integrability of \( \|F_\varepsilon(n(\cdot, t)) - n(\cdot, t)\|_{L^r(\Omega \times (0,T))} \) on \( (0, T) \). Thereupon, by virtue of (3.2), we infer from the dominated convergence theorem that
\[
\int_0^T \|F_\varepsilon(n) - n\|_{L^r(\Omega)}^r \, dt \to 0 \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0 \quad (6.25)
\]
for each \( T \in (0, \infty) \). Inserting (6.25) into (6.23) and using (6.1) and (3.1), we can see clearly that
\[
F_\varepsilon(n) \to n \quad \text{in} \quad L^r_{\text{loc}}(\bar{\Omega} \times [0, \infty)) \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0. \quad (6.26)
\]
Finally, according to (6.1)–(6.3), (6.5), (6.7), (6.6), (6.13), (6.16), (6.17), (6.21), (6.22) and (6.26), we may pass to the limit in the respective weak formulations associated with the the regularized system (2.6) and get the integral identities (2.3)–(2.5). \( \square \)

7 A priori estimates for the problem (1.3)

By a straightforward adaptation of the reasoning in Lemma 2.1 of [38], one can derive the following basic statement on local solvability and extensibility of solutions to (1.3).

**Lemma 7.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary and the initial data \((n_0, c_0, u_0)\) fulfills (1.8). Then there exist \( T_{\text{max}} \in (0, \infty) \) and a classical solution \((n, c, u, P)\) of (1.3) in \( \Omega \times (0, T_{\text{max}}) \) such that
\[
\begin{align*}
\text{n} & \in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})), \\
c & \in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})), \\
u & \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})), \\
P & \in C^{1,0}(\bar{\Omega} \times (0, T_{\text{max}})),
\end{align*}
\] (7.1)
classically solving (1.3) in \( \Omega \times [0, T_{\text{max}}) \). Moreover, \( n \) and \( c \) are nonnegative in \( \Omega \times (0, T_{\text{max}}) \), and
\[
\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|A^\gamma u(\cdot, t)\|_{L^2(\Omega)} \to \infty \quad \text{as} \quad t \nearrow T_{\text{max}}, \quad (7.2)
\]
where \( \gamma \) is given by (1.8).
In order to discuss the boundedness and classical solution of (1.3), in light of Lemma 7.1, we can pick any $s_0 \in (0, T_{\text{max}})$ and $s_0 \leq 1$, there exists $\beta > 0$ such that

$$\|n(\tau)\|_{L^\infty(\Omega)} \leq \beta \quad \|u(\tau)\|_{W^{1,\infty}(\Omega)} \leq \beta \quad \text{and} \quad \|c(\tau)\|_{W^{2,\infty}(\Omega)} \leq \beta \quad \text{for all} \quad \tau \in [0, s_0]. \quad (7.3)$$

**Lemma 7.2.** ([37, 44]) Let $l \in [1, +\infty)$ and $r \in [1, +\infty]$ be such that

$$\begin{cases} l < \frac{3r}{3-r} & \text{if} \quad r \leq 3, \\ l \leq \infty & \text{if} \quad r > 3. \end{cases} \quad (7.4)$$

If $\kappa = 0$ and for all $K > 0$ there exists $C = C(l, r, K)$ such that

$$\|n(\cdot, t)\|_{L^r(\Omega)} \leq K \quad \text{for all} \quad t \in (0, T_{\text{max}}), \quad (7.5)$$

then

$$\|Du(\cdot, t)\|_{L^l(\Omega)} \leq C \quad \text{for all} \quad t \in (0, T_{\text{max}}), \quad (7.6)$$

where $(n, c, u, P)$ is a solution of (1.3).

**Lemma 7.3.** ([9, 46, 47]) Suppose $\gamma \in (1, +\infty)$, $g \in L^\gamma((0, T); L^\gamma(\Omega))$ and $v_0 \in W^{2,\gamma}(\Omega)$ such that $\frac{\partial v_0}{\partial \nu} = 0$. Let $v$ be a solution of the following initial boundary value

$$\begin{cases} v_t - \Delta v + v = g, & (x, t) \in \Omega \times (0, T), \\ \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial \Omega \times (0, T), \\ v(x, 0) = v_0(x), & (x, t) \in \Omega. \end{cases} \quad (7.7)$$

Then there exists a positive constant $C_\gamma := C_{\gamma, \infty}$ such that if $s_0 \in (0, T)$, $v(\cdot, s_0) \in W^{2,\gamma}(\Omega)(\gamma > N)$ with $\frac{\partial v(\cdot, s_0)}{\partial \nu} = 0$, then

$$\int_{s_0}^T e^{\gamma s} (\|v(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma + \|\Delta v(\cdot, t)\|_{L^\gamma(\Omega)}^\gamma) \, ds \leq C_\gamma \left( \int_{s_0}^T e^{\gamma s} \|g(\cdot, s)\|_{L^\gamma(\Omega)}^\gamma \, ds + e^{\gamma s} (\|v_0(\cdot, s)\|_{L^\gamma(\Omega)}^\gamma + \|\Delta v_0(\cdot, s)\|_{L^\gamma(\Omega)}^\gamma) \right). \quad (7.8)$$

The proof of the following lemma is very similar to that of Lemmas 3.2–3.3, so we omit its proof here.

**Lemma 7.4.** There exists $\tilde{\lambda} > 0$ such that the solution of (1.3) satisfies

$$\int_\Omega n + \int_\Omega c \leq \tilde{\lambda} \quad \text{for all} \quad t \in (0, T_{\text{max}}). \quad (7.9)$$
Lemma 7.5. Let $\alpha > \frac{1}{3}$, $S(x, n, c) = C_S(1 + n)^{-\alpha}$ and $\kappa = 0$. Then there exists $C > 0$ such that the solution of (1.3) satisfies

$$\int_{\Omega} n^{2\alpha} + \int_{\Omega} c^2 + \int_{\Omega} |u|^2 \leq C \quad \text{for all} \quad t \in (0, T_{\text{max}}).$$

(7.10)

Moreover, for $T \in (0, T_{\text{max}})$, it holds that one can find a constant $C > 0$ such that

$$\int_{0}^{T} \int_{\Omega} [n^{2\alpha - 2} |\nabla n|^2 + |\nabla c|^2 + |\nabla u|^2] \leq C.$$  

(7.11)

Lemma 7.6. Let $p = \frac{13}{8}$, $\alpha \in \left(\frac{1}{3}, \frac{3}{4}\right]$ and $\theta = \frac{3}{2}$. Then there exists a positive constant $\tilde{l}_0 \in (\frac{59}{20}, 3)$ such that

$$\frac{5}{6} \cdot \frac{1}{\theta(p+1-\alpha)} + \frac{1}{l_0} - \frac{1}{l_0 + \frac{2}{3} - \frac{1}{p+1-\alpha}} < 1,$$  

(7.12)

where $\theta' = \frac{\theta}{\theta - 1} = 3$.

Proof. It is easy to verify that

$$\frac{24}{55} < \frac{1}{p + 1 - \alpha} < \frac{8}{15}$$

and

$$1 - \frac{1}{l_0} - \frac{1}{\theta(p+1-\alpha)} = \frac{2}{3} - \frac{1}{l_0 + \frac{2}{3} - \frac{1}{p+1-\alpha}}.$$

These together with some basic calculation yield to (7.12). \qed

Now, let us derive the following a priori bounded for the solutions of model (1.3), which plays a key role in obtaining the main results.

Lemma 7.7. Let

$$S(x, n, c) = C_S(1 + n)^{-\alpha}$$  

(7.13)

and $\kappa = 0$. If

$$\frac{1}{3} < \alpha \leq \frac{3}{4},$$  

(7.14)

then there exists a positive constant $p_0 > \frac{3}{2}$ such that the solution of (1.3) from Lemma 7.1 satisfies

$$\int_{\Omega} n^{p_0}(x, t)dx \leq C \quad \text{for all} \quad t \in (0, T_{\text{max}}).$$  

(7.15)
Proof. Let \( p = \frac{13}{8} \). Taking \( n^{p-1} \) as the test function for the first equation of (1.3) and combining with the second equation and using \( \nabla \cdot u = 0 \), we obtain
\[
\frac{1}{p} \frac{d}{dt} \|n\|_{L^p(\Omega)}^p + (p - 1) \int_\Omega n^{p-2} |\nabla n|^2 = - \int_\Omega n^{p-1} \nabla \cdot (nCS(1 + n)^{-\alpha} \nabla c) = (p - 1) \int_\Omega n^{p-1} C_S(1 + n)^{-\alpha} \nabla n \cdot \nabla c \quad \text{for all } t \in (0, T_{\max}),
\]
which derives,
\[
\frac{1}{p} \frac{d}{dt} \|n\|_{L^p(\Omega)}^p + (p - 1) \int_\Omega n^{p-2} |\nabla n|^2 \leq - \frac{p + 1 - \alpha}{p} \int \Omega n^p + (p - 1) \int_\Omega n^{p-1} C_S(1 + n)^{-\alpha} \nabla n \cdot \nabla c + \frac{p + 1 - \alpha}{p} \int_\Omega n^p \quad \text{for all } t \in (0, T_{\max}).
\]

Here, for any \( \varepsilon_1 > 0 \), we invoke the Young inequality to find that
\[
\frac{p + 1 - \alpha}{p} \int_\Omega n^p \leq \varepsilon_1 \int_\Omega n^{p+\frac{2}{3}} + C_1(\varepsilon_1, p),
\]
where
\[
C_1(\varepsilon_1, p) = \frac{2}{p + \frac{2}{3}} \left( \varepsilon_1 \frac{p + \frac{2}{3}}{p} \right)^{-\frac{3}{4}} \left( \frac{p + 1 - \alpha}{p} \right)^{\frac{p+\frac{2}{3}}{4}} |\Omega|.
\]

Once more integrating by parts, in view of (7.13), we also find that
\[
(p - 1) \int_\Omega n^{p-1} C_S(1 + n)^{-\alpha} \nabla n \cdot \nabla c = (p - 1) \int_\Omega \nabla \int_0^\tau n^{p-1} C_S(1 + \tau)^{-\alpha} d\tau \cdot \nabla c = - (p - 1) \int_\Omega \int_0^\tau n^{p-1} C_S(1 + \tau)^{-\alpha} d\tau \Delta c \leq C_S(p-1) \int_\Omega n^{p-\alpha} |\Delta c|,
\]
so that the Young inequality implies
\[
\frac{C_S(p-1)}{p - \alpha} \int_\Omega n^{p-\alpha} |\Delta c| \leq \int_\Omega n^{p+1-\alpha} + \frac{1}{p + 1 - \alpha} \left[ \frac{p + 1 - \alpha}{p - \alpha} \right]^{-(p-\alpha)} \left( \frac{C_S(p-1)}{p - \alpha} \right)^{p+1-\alpha} \int_\Omega |\Delta c|^{p+1-\alpha}
\]
\[
= \int_\Omega n^{p+1-\alpha} + A_1 \int_\Omega |\Delta c|^{p+1-\alpha},
\]
where
\[
A_1 := \frac{1}{p + 1 - \alpha} \left[ \frac{p + 1 - \alpha}{p - \alpha} \right]^{-(p-\alpha)} \left( \frac{C_S(p-1)}{p - \alpha} \right)^{p+1-\alpha}.
\]
Thus, inserting (7.18) and (7.20) into (7.17), we get
\[
\frac{1}{p} \frac{d}{dt} \|n\|^p_{L^p(\Omega)} + (p - 1) \int_\Omega n^{p-2} |\nabla n|^2 \leq \varepsilon_1 \int_\Omega n^{p+\frac{2}{3}} + \int_\Omega n^{p+1-\alpha} - \frac{p+1-\alpha}{p} \int_\Omega n^p + A_1 \int_\Omega |\Delta c|^{p+1-\alpha} + C_1(\varepsilon_1, p) \quad \text{for all } t \in (0, T_{max}).
\]
Since, \( \alpha > \frac{1}{3} \), yields to \( p + 1 - \alpha < p + \frac{2}{3} \), therefore, by the Young inequality, we conclude that
\[
\frac{1}{p} \frac{d}{dt} \|n\|^p_{L^p(\Omega)} + \frac{4(p-1)}{p^2} \|\nabla n\|^2_{L^2(\Omega)} \leq 2\varepsilon_1 \int_\Omega n^{p+\frac{2}{3}} - \frac{p+1-\alpha}{p} \int_\Omega n^p + A_1 \int_\Omega |\Delta c|^{p+1-\alpha} + C_2(\varepsilon_1, p),
\]
where
\[
C_2(\varepsilon_1, p) = \frac{\alpha - \frac{1}{3}}{p + \frac{2}{3}} \left( \varepsilon_1 \frac{p + \frac{2}{3}}{p + 1 - \alpha} \right)^{-\frac{p+1-\alpha}{p}} \left( \frac{p+1-\alpha}{\alpha} \right)^{\frac{p+1}{p}} |\Omega|.
\]
On the other hand, by the Gagliardo–Nirenberg inequality and (3.4), one can get there exist positive constants \( \mu_0 \) and \( \mu_1 \) such that
\[
\int_\Omega n^{p+\frac{2}{3}} = \|n^{\frac{2}{3}}\|_{L^{2p/(p+2)}(\Omega)}^2 \leq \mu_0 (\|\nabla n\|^2_{L^2(\Omega)} n^{\frac{2}{3}}_{L^{2p/(p+2)}(\Omega)} + \|n\|^2_{L^p(\Omega)})^{\frac{2(p+\frac{2}{3})}{p}} \leq \mu_1 (\|\nabla n\|^2_{L^2(\Omega)} + 1).
\]
Collecting (7.21) and (7.22), we derive that
\[
\frac{1}{p} \frac{d}{dt} \|n\|^p_{L^p(\Omega)} \leq (2\varepsilon_1 - \frac{4(p-1)}{p^2} \frac{1}{\mu_1}) \int_\Omega n^{p+\frac{2}{3}} - \frac{p+1-\alpha}{p} \int_\Omega n^p + A_1 \int_\Omega |\Delta c|^{p+1-\alpha} + C_3(\varepsilon_1, p) \quad \text{for all } t \in (0, T_{max}),
\]
where
\[
C_3(\varepsilon_1, p) = C_2(\varepsilon_1, p) + \frac{4(p-1)}{p^2}.
\]
For any \( t \in (s_0, T_{max}) \), employing the variation-of-constants formula to (7.23), we obtain
\[
\frac{1}{p} \|n(t)\|^p_{L^p(\Omega)} \leq \frac{1}{p} e^{-(p+1-\alpha)(t-s_0)} \|n(s_0)\|^p_{L^p(\Omega)} + (2\varepsilon_1 - \frac{4(p-1)}{p^2} \frac{1}{\mu_1}) \int_{s_0}^t e^{-(p+1-\alpha)(t-s)} \int_\Omega n^{p+\frac{2}{3}} ds + A_1 \int_{s_0}^t |\Delta c|^{p+1-\alpha} dx ds + C_3(\varepsilon_1, p) \int_{s_0}^t e^{-(p+1-\alpha)(t-s)} ds \leq (2\varepsilon_1 - \frac{4(p-1)}{p^2} \frac{1}{\mu_1}) \int_{s_0}^t e^{-(p+1-\alpha)(t-s)} \int_\Omega n^{p+\frac{2}{3}} ds + A_1 \int_{s_0}^t e^{-(p+1-\alpha)(t-s)} \int_\Omega |\Delta c|^{p+1-\alpha} dx ds + C_4(\varepsilon_1, p)
\]
We derive from the Young inequality that for any \( \delta > 0 \) with some constants \( C \),

\[
C_4 := C_4(\varepsilon, p) = \frac{1}{p} e^{-(p+1-\alpha)(t-s_0)} \| \eta(s_0) \|_{L^p(\Omega)}^p + C_3(\varepsilon, p) \int_{s_0}^t e^{-(p+1-\alpha)(t-s)} \, ds.
\]

Due to (7.9), employing Lemma 7.2, we derive that

\[
\| D u(\cdot, t) \|_{L^1(\Omega)} \leq C_5 \text{ for all } t \in (0, T_{\text{max}}) \text{ and for any } l < \frac{3}{2},
\]

so that the Sobolev imbedding theorem implies that

\[
\| u(\cdot, t) \|_{L^{5/3}(\Omega)} \leq C_6 \text{ for all } t \in (0, T_{\text{max}}) \text{ and for any } l_0 < 3.
\]

Now, due to Lemma 7.3 and the second equation of (1.3) and using the Hölder inequality, we have

\[
\begin{align*}
A_1 & = \frac{1}{\rho} \int_{s_0}^t e^{-(p+1-\alpha)(t-s)} \int_{\Omega} |\Delta c|^{p+1-\alpha} \, ds \\
& = A_1 e^{-(p+1-\alpha)t} \int_{s_0}^t e^{(p+1-\alpha)s} \int_{\Omega} |\Delta c|^{p+1-\alpha} \, ds \\
& \leq 2^{p+1-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} \left( \int_{s_0}^t \int_{\Omega} e^{(p+1-\alpha)s} (|u \cdot \nabla c|^{p+1-\alpha} + n^{p+1-\alpha}) \, ds \right) + C_{p+1-\alpha} \| c(s_0, t) \|_{W^{2, p+1-\alpha}}^{p+1-\alpha} \\
& \leq 2^{p+1-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} \left( \int_{s_0}^t \int_{\Omega} e^{(p+1-\alpha)s} (\| u \|_{L^{\theta(p+1-\alpha)}(\Omega)}^{p+1-\alpha} + n^{p+1-\alpha}) \, ds \right) + C_7
\end{align*}
\]

for all \( t \in (s_0, T_{\text{max}}) \), where \( \theta = \frac{3}{2}, \theta' = \frac{\theta}{\theta-1} = 3 \),

\[
C_7 = 2^{p+1-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} e^{(p+1-\alpha)s_0} \| c(s_0, t) \|_{W^{2, p+1-\alpha}}^{p+1-\alpha}.
\]

Next, with the help of the Gagliardo–Nirenberg inequality and (3.5), we derive that

\[
\begin{align*}
\| \nabla c \|_{L^{\theta(p+1-\alpha)}(\Omega)}^{p+1-\alpha} & \leq C_8 \| \Delta c \|_{L^{(p+1-\alpha)}(\Omega)}^{(1-a)(p+1-\alpha)} + C_9 \| c \|_{L^2(\Omega)}^{p+1-\alpha} \\
& \leq C_9 \| \Delta c \|_{L^{(p+1-\alpha)}(\Omega)}^{(1-a)(p+1-\alpha)} + C_9
\end{align*}
\]

with some constants \( C_8 > 0 \) and \( C_9 > 0 \), where

\[
a = \frac{\frac{5}{6} - \frac{1}{\theta(p+1-\alpha)}}{\frac{5}{6} - \frac{1}{p+1-\alpha}} \in (0, 1).
\]

We derive from the Young inequality that for any \( \delta \in (0, 1) \),

\[
\begin{align*}
\| u \|_{L^{\theta(p+1-\alpha)}(\Omega)}^{p+1-\alpha} & \leq C_9 \| \Delta c \|_{L^{(p+1-\alpha)}(\Omega)}^{(1-a)(p+1-\alpha)} + C_9 \| u \|_{L^{\theta(p+1-\alpha)}(\Omega)}^{p+1-\alpha} \\
& \leq \delta \| \Delta c \|_{L^{(p+1-\alpha)}(\Omega)}^{(1-a)(p+1-\alpha)} + C_{10} \| u \|_{L^{\theta(p+1-\alpha)}(\Omega)}^{p+1-\alpha} + C_9 \| u \|_{L^{\theta(p+1-\alpha)}(\Omega)}^{p+1-\alpha}.
\end{align*}
\]
where $C_{10} = (1 - a) \left( \delta \times \frac{1}{a} \right)^{-\frac{1}{p} - \alpha} C_9^{1 - \alpha}$.

Inserting $(7.29)$ into $(7.27)$, we conclude that
$$A_1 \int_{s_0}^{t} e^{-(p+1-\alpha)(t-s)} \int_{\Omega} |\Delta c|^{p+1-\alpha} ds \leq 2^{p+1-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} \int_{s_0}^{t} e^{(p+1-\alpha)s} \|\Delta c\|^{p+1-\alpha}_{L^\theta(p+1-\alpha)(\Omega)} ds + 2^{p+1-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} C_{10} \int_{s_0}^{t} e^{(p+1-\alpha)s} \|u\|^{p+1-\alpha}_{L^\theta(p+1-\alpha)(\Omega)} ds + 2^{p+1-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} \int_{s_0}^{t} e^{(p+1-\alpha)s} n^{p+1-\alpha} ds + C_7$$

for all $t \in (s_0, T_{max})$. Therefore, choosing $\delta = \frac{1}{2} 2^{p+1-\alpha} C_{p+1-\alpha}$ yields to
$$A_1 \int_{s_0}^{t} e^{-(p+1-\alpha)(t-s)} \int_{\Omega} |\Delta c|^{p+1-\alpha} ds \leq 2^{p+2-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} C_{10} \int_{s_0}^{t} e^{(p+1-\alpha)s} \|u\|^{p+1-\alpha}_{L^\theta(p+1-\alpha)(\Omega)} ds + 2^{p+2-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} C_{9} \int_{s_0}^{t} \|u\|^{p+1-\alpha}_{L^\theta(p+1-\alpha)(\Omega)} ds + 2^{p+1-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} \int_{s_0}^{t} e^{(p+1-\alpha)s} n^{p+1-\alpha} ds + C_7.$$  \hfill (7.30)

On the other hand, by Lemma $7.6$ we may choose $\frac{50}{20} < \tilde{l}_0 < 3$ such that
$$\frac{5}{6} - \frac{1}{(p+1-\alpha)} + \frac{1}{l_0} - \frac{1}{\theta(p+1-\alpha)} - \frac{1}{l_0} + \frac{2}{3} - \frac{1}{p+1-\alpha} < 1. \hfill (7.32)$$

Therefore, it follows from the Gagliardo–Nirenberg inequality, $(7.26)$ and the Young inequality that there exist constants $C_{11} = C_{11}(p) > 0$ and $C_{12} = C_{12}(p) > 0$ such that
$$\|u\|^{p+1-\alpha}_{L^\theta(p+1-\alpha)(\Omega)} \leq \|Au\|^{p+1-\alpha}_{L^\theta(p+1-\alpha)(\Omega)} \|u\|^{p+1-\alpha}_{L^0(\Omega)} \leq \|Au\|^{p+1-\alpha}_{L^\theta(p+1-\alpha)(\Omega)} C_{11} \leq \|Au\|^{p+1-\alpha}_{L^\theta(p+1-\alpha)(\Omega)} + C_{12} \hfill (7.33)$$

with
$$\tilde{a} = \frac{1}{l_0} - \frac{1}{\theta(p+1-\alpha)} \in (0, 1).$$

Here we have use the fact that $\frac{p+1-\alpha}{l_0} - \frac{1}{\theta(p+1-\alpha)} \in (0, 1)$. In light of $\frac{1}{1-\alpha} > 1$, similarly, we derive that
$$\|u\|^{p+1-\alpha}_{L^\theta(p+1-\alpha)(\Omega)} \leq \|Au\|^{p+1-\alpha}_{L^\theta(p+1-\alpha)(\Omega)} + C_{13}. \hfill (7.34)$$
Collecting (7.31), (7.33) and (7.34), we derive that

\begin{equation}
A_1 \int_{s_0}^t e^{-(p+1-\alpha)(t-s)} \int_{\Omega} |\Delta c|^{p+1-\alpha} ds \\
\leq 2^{p+2-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} C_{10} \int_{s_0}^t e^{(p+1-\alpha)s} \|Au\|_{L^{p+1-\alpha}(\Omega)} ds \\
+ 2^{p+2-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} C_9 \int_{s_0}^t e^{(p+1-\alpha)s} \|Au\|_{L^{p+1-\alpha}(\Omega)} ds \\
+ 2^{p+2-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} \int_{s_0}^t \int_{\Omega} e^{(p+1-\alpha)s} n^{p+1-\alpha} ds + 2C_7
\end{equation}

(7.35)

where \( C_{14} = 2[2^{p+1-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha}(C_{10}C_{12} + C_9C_{13}) + C_7] \). Putting \( \tilde{u}(\cdot, s) := e^s u(\cdot, s) \), \( s \in (s_0, t) \), we obtain from the third equation in (1.3) that

\begin{equation}
\tilde{u}_s = \Delta \tilde{u} + \tilde{u} + e^s n \nabla \phi + e^s \nabla P,
\end{equation}

(7.36)

which implies that

\begin{equation}
\tilde{u}_s + A\tilde{u} = \mathcal{P}(\tilde{u} + e^s n \nabla \phi + e^s \nabla P),
\end{equation}

(7.37)

where \( \mathcal{P} \) denotes the Helmholtz projection mapping \( L^2(\Omega) \) onto its subspace \( L^2_\sigma(\Omega) \) of all solenoidal vector field. Thus by \( p < 2 \) and (7.26), we derive from Lemma 7.3 (see also Theorem 2.7 of [8]) that there exist positive constants \( C_{15}, C_{16}, C_{17} \) and \( C_{18} \) such that

\begin{equation}
\begin{align*}
\int_{s_0}^t e^{(p+1-\alpha)s} \|Au(\cdot, t)\|_{L^{p+1-\alpha}(\Omega)} ds \\
\leq C_{15} \left( \int_{s_0}^t e^{(p+1-\alpha)s} \|u(\cdot, s)\|_{L^{p+1-\alpha}(\Omega)} + \|n(\cdot, s)\|_{L^{p+1-\alpha}(\Omega)} ds + e^{(p+1-\alpha)t} + 1 \right) \\
\leq C_{16} \left( \int_{s_0}^t e^{(p+1-\alpha)s} \|u(\cdot, s)\|_{L^{p+1-\alpha}(\Omega)} \frac{l_0 - p + 1 + \alpha}{l_0} + \|n(\cdot, s)\|_{L^{p+1-\alpha}(\Omega)} ds + e^{(p+1-\alpha)t} + 1 \right) \\
\leq C_{17} \int_{s_0}^t e^{(p+1-\alpha)s} \|n(\cdot, s)\|_{L^{p+1-\alpha}(\Omega)} ds + (1 + C_{18}) e^{(p+1-\alpha)t}.
\end{align*}
\end{equation}

(7.38)

Here we have used the fact that

\[ \frac{15}{8} \leq p + 1 - \alpha < \frac{55}{24} < \frac{59}{20} < \tilde{l}_0. \]
Inserting (7.38) into (7.35), we derive that
\[ A_1 \int_{s_0}^{t} e^{-(p+1-\alpha)(t-s)} \int_{\Omega} |\Delta c|^{p+1-\alpha} ds \leq 2^{p+2-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} [C_{10} + C_9] \int_{s_0}^{t} e^{(p+1-\alpha)s} ||n(\cdot, s)||_{L^{p+1-\alpha}(\Omega)} ds + (1 + C_{18}) e^{(p+1-\alpha)t} \]
\[ + 2^{p+2-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} \int_{s_0}^{t} \int_{\Omega} e^{(p+1-\alpha)s} n^{p+1-\alpha} ds + C_{14} \]
\[ \leq C_{19} \int_{s_0}^{t} e^{(p+1-\alpha)s} ||n(\cdot, s)||_{L^{p+1-\alpha}(\Omega)} ds + C_{20}, \]
(7.39)

where \( C_{19} = 2^{p+1-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} [C_{10} + C_9] C_{17} + 2^{p+1-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} \) and
\[ C_{20} := C_{20}(p) = 2^{p+2-\alpha} A_1 e^{-(p+1-\alpha)t} C_{p+1-\alpha} [C_{10} + C_9] (1 + C_{18}) e^{(p+1-\alpha)t} + C_{14}. \]

Collecting (7.24) and (7.39), applying Lemma 7.6 and the Young inequality, we derive that
\[ \frac{1}{p} ||n(t)||_{L^p(\Omega)}^p \leq (2 \varepsilon_1 - \frac{4(p-1)}{p^2} \frac{1}{\mu_1}) \int_{s_0}^{t} e^{-(p+1-\alpha)(t-s)} \int_{\Omega} n^{p+\frac{2}{3}} ds \]
\[ + C_{19} \int_{s_0}^{t} e^{(p+1-\alpha)(t-s)} \int_{\Omega} n^{p+1-\alpha} ds + C_{21} \]
\[ \leq (3 \varepsilon_1 - \frac{4(p-1)}{p^2} \frac{1}{\mu_1}) \int_{s_0}^{t} e^{-(p+1-\alpha)(t-s)} \int_{\Omega} n^{p+\frac{2}{3}} ds + C_{22} \]
(7.40)

with \( C_{21} = C_{20} + C_4(\varepsilon_1, p) \) and \( C_{22} = \frac{\alpha-\frac{2}{3}}{\alpha+\frac{2}{3}} \left( \varepsilon_1 \left( p+\frac{2}{3} \right) - \frac{p+1-\alpha}{\alpha+\frac{2}{3}} \right) (C_{19})^{\frac{p+\frac{2}{3}}{\alpha+\frac{2}{3}}} + C_{21} \). Thus, choosing \( \delta \) and \( \varepsilon_1 \) small enough (e.g. \( \varepsilon_1 < \frac{(p-1)}{p^2} \frac{1}{\mu_1} \)) in (7.40), using (7.34) and the Hölder inequality, we derive that there exits a positive constant \( p_0 > \frac{3}{2} \) such that
\[ \int_{\Omega} n^{p_0}(x,t) dx \leq C_{23} \quad \text{for all} \quad t \in (0, T_{\text{max}}). \]
(7.41)

The proof of Lemma 7.7 is completed. \( \square \)

If we can find parameters that allow for an application of Lemmas 7.1 and 7.3 at the same time, we can conclude boundedness of \( n \). This is the goal we pursue in the following lemma:

**Lemma 7.8.** Let \( \alpha > \frac{1}{3} \), \( S(x, n, c) = C_S(1 + n)^{-\alpha} \) and \( \kappa = 0 \). Then there exists a positive constant \( q_0 > \frac{3}{2} \) such that the solution of (1.3) from Lemma 7.1 satisfies
\[ \int_{\Omega} n^{q_0}(x,t) dx \leq C \quad \text{for all} \quad t \in (0, T_{\text{max}}). \]
(7.42)
Proof. Let
\[ q_0 = \begin{cases} p_0 & \text{if } \frac{1}{3} < \alpha \leq \frac{2}{3}, \\ 2\alpha & \text{if } \alpha > \frac{3}{4}, \end{cases} \]  
where \( p_0 \) is the same as Lemma 7.7. Then obviously, \( q_0 > \frac{3}{2} \), hence, in view of Lemmas 7.7 and 7.5 yields to (7.42). The proof of Lemma 7.8 is completed.

With the help of Lemma 7.8 in light of the Gagliardo–Nirenberg inequality and an application of well-known arguments from parabolic regularity theory, we can derive the following Lemma:

**Lemma 7.9.** Assume the hypothesis of Theorem 1.2 holds. Then for \( p > 2 \), one can find a constant \( C > 0 \) such that the solution of (1.3) satisfies
\[ \int_\Omega c^p \leq C \quad \text{for all } t \in (0, T_{\text{max}}). \]  
(7.44)

Proof. Firstly, for any \( p > 2 \), taking \( c^{p-1} \) as the test function for the second equation of (1.3) and using \( \nabla \cdot u = 0 \), the Hölder inequality and (7.42) yields that
\[ \begin{align*}
\frac{1}{p} \frac{d}{dt} \|c\|^p_{L^p(\Omega)} + (p-1) \int_\Omega c^{p-2} |\nabla c|^2 + \int_\Omega c^p \\
= \int_\Omega nc^{p-1} \\
\leq \left( \int_\Omega n^\frac{2}{3} \right)^\frac{1}{2} \left( \int_\Omega c^3(p-1) \right)^\frac{1}{3} \\
\leq C_1 \left( \int_\Omega c^{3(p-1)} \right)^\frac{1}{3} \quad \text{for all } t \in (0, T_{\text{max}}),
\end{align*} \]  
(7.45)
where in the last inequality we have used the fact that (7.42) and the Hölder inequality. Now, due to (7.10), in light of the Gagliardo–Nirenberg inequality, we derive that there exist positive constants \( C_2 \) and \( C_3 \) such that
\[ \left( \int_\Omega c^{3(p-1)} \right)^\frac{1}{3} \leq \|c\|_{L^2(\Omega)} \left( \frac{2(p-1)}{\mu_1(p-1) + 1} \right)^\frac{2(p-1)}{p} + \|c\|_{L^2(\Omega)} \leq C_2 \left( \|\nabla c\|_{L^2(\Omega)}^{\mu_1(p-1) + 1} + \|c\|_{L^2(\Omega)} \right)^{\frac{2(p-1)}{p}} \]  
(7.46)
with some positive constants $C_2$ and $C_3$ and
\[
\mu_1 = \frac{\frac{3p}{4} - \frac{3p}{6(p-1)}}{\frac{1}{2} + \frac{3p}{4}} = \frac{\frac{3}{4} - \frac{3}{6(p-1)}}{\frac{1}{2} + \frac{3p}{4}} \in (0, 1).
\]

Inserting (7.46) into (7.45), in view of the fact that $2 \frac{3p-5}{3p-2} < 2$, therefore, by using the Young inequality, we derive that
\[
\frac{1}{p} \frac{d}{dt} \|c\|_{L^p(\Omega)}^p + \frac{p-1}{2} \int_\Omega c^{p-2} |\nabla c|^2 + \int_\Omega c^p \leq C_4 \quad \text{for all } t \in (0, T_{\text{max}}),
\]
which combined with an ODE comparison argument entails (7.44). \hfill \Box

Underlying the estimates established above (Lemmas 7.4–7.5), we can derive the following boundedness results by invoking a Moser-type iteration and the standard parabolic regularity arguments (see the proof of Lemmas 4.3 and 4.4).

**Lemma 7.10.** Let $\alpha > \frac{1}{3}$ and $\gamma$ be as in (1.8). Then one can find a positive constant $C$ such that
\[
\|n(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}}) \tag{7.48}
\]
and
\[
\|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}}) \tag{7.49}
\]
as well as
\[
\|u(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}}). \tag{7.50}
\]
Moreover, we also have
\[
\|A^\gamma u(\cdot, t)\|_{L^2(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}}). \tag{7.51}
\]

**Proof.** Employing the same arguments as in the proof of Lemmas 7.4–7.5, and taking advantage of (7.42) and (7.44), we conclude the estimates (7.48)–(7.51). The proof of Lemma 7.10 is completed. \hfill \Box

Combining Lemma 7.1 and Lemma 7.10, we readily prove Theorem 1.2.

**Proof of Theorem 1.2.** In view of Lemma 7.10 $\|u(\cdot, t)\|_{L^\infty(\Omega)}$, $\|c(\cdot, t)\|_{W^{1,\infty}(\Omega)}$ and $\|A^\gamma u(\cdot, t)\|_{L^2(\Omega)}$ are bounded uniformly with respect to $t \in (0, T_{\text{max}})$. Thereupon the assertion of Theorem 1.2 is immediately obtained from Lemma 7.1. 

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