IMPROVEMENT OF NUMERICAL RADIUS INEQUALITIES

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Abstract. We develop upper and lower bounds for the numerical radius of $2 \times 2$ off-diagonal operator matrices, which generalize and improve on the existing ones. We also show that if $A$ is a bounded linear operator on a complex Hilbert space and $|A|$ stands for the positive square root of $A$, i.e., $|A| = (A^*A)^{1/2}$, then for all $r \geq 1$, $w^{2r}(A) \leq \frac{1}{4}\|A^{2r} + |A|^2r\| + \frac{1}{2}\min \{\|\Re(|A|^r|A|^r\|, w^r(A^2)\}$ where $w(A)$, $\|A\|$ and $\Re(A)$, respectively, stand for the numerical radius, the operator norm and the real part of $A$. This (for $r = 1$) improves on existing well-known numerical radius inequalities.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$ induced by the inner product. Let $\mathfrak{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$ with the identity operator $I$ and the zero operator $O$. Let $A \in \mathfrak{B}(\mathcal{H})$. We denote by $|A| = (A^*A)^{1/2}$ the positive square root of $A$, and $\Re(A) = \frac{1}{2}(A + A^*)$ and $\Im(A) = \frac{1}{2i}(A - A^*)$, respectively, stand for the real and imaginary part of $A$. The numerical range of $A$, denoted by $W(A)$, is defined as $W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$. We denote by $\|A\|$ and $w(A)$ the operator norm and the numerical radius of $A$, respectively, and are defined as

$$\|A\| = \sup \{\|Ax\| : x \in \mathcal{H}, \|x\| = 1\}$$

and

$$w(A) = \sup \{\|\langle Ax, x \rangle\| : x \in \mathcal{H}, \|x\| = 1\}.$$ 

It is well-known that the numerical radius $w(\cdot)$ defines a norm on $\mathfrak{B}(\mathcal{H})$ and is equivalent to the operator norm $\| \cdot \|$. In fact, the following double inequality holds:

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \quad (1.1)$$
The inequalities in (1.1) are sharp. The first inequality becomes equality if \( A^2 = O \), and the second one turns into equality if \( A \) is normal. For various refinements of (1.1), we refer the reader to [3, 4, 5, 6, 9, 17] and references therein. In particular, Kittaneh [13] improved the inequalities in (1.1) by establishing that

\[
\frac{1}{4} \|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2} \|A^*A + AA^*\|. \tag{1.2}
\]

Kittaneh [14] also improved the upper bound of \( w(T) \) in (1.1) to show that

\[
w(A) \leq \frac{1}{2} \left( \|A\| + \|A^2\|^{1/2} \right). \tag{1.3}
\]

Further, Abu-Omar and Kittaneh [1] obtained the following inequality which refines both the upper bounds in (1.2) and (1.3):

\[
w^2(A) \leq \frac{1}{4} \|A^*A + AA^*\| + \frac{1}{2} w(A^2). \tag{1.4}
\]

Recently, Bhunia and Paul [2] also improved both the upper bounds in (1.2) and (1.3) by developing that

\[
w^2(A) \leq \frac{1}{4} \|A^*A + AA^*\| + \frac{1}{2} w(|A||A^*|). \tag{1.5}
\]

In this paper, we derive inequalities for the bounds of the numerical radius which generalize and improve on both in (1.4) and (1.5). Further, we obtain a lower bound for the numerical radius which generalizes and improves on the existing ones. Applications of the obtained inequalities are also given.

2. Main Results

We begin our work with noting that for \( A, B \in \mathbb{B}(\mathcal{H}) \), the \( 2 \times 2 \) off-diagonal operator matrix

\[
\begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
\]

\( \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}) \) and the numerical radius of the matrix is denoted by \( w \left( \begin{pmatrix} O & A \\ B & 0 \end{pmatrix} \right) \). To achieve our aim in this paper we need the following four lemmas. First lemma follows easily.

**Lemma 2.1.** If \( A, B \in \mathbb{B}(\mathcal{H}) \), then

\[
w \left( \begin{pmatrix} A & O \\ O & B \end{pmatrix} \right) = \max \left\{ w(A), w(B) \right\}.
\]

Second lemma deals with positive operators, and is known as McCarthy inequality.
Lemma 2.2. ([15, p. 20]). If $A \in \mathbb{B}(\mathcal{H})$ is positive, then for $r \geq 1$

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle,$$

for all $x \in \mathcal{H}$ with $\|x\| = 1$.

Third lemma deals with vectors in $\mathcal{H}$, and is known as Buzano’s inequality.

Lemma 2.3. ([10]) If $x, y, e \in \mathcal{H}$ with $\|e\| = 1$, then

$$|\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} \left( \|x\| \|y\| + |\langle x, y \rangle| \right).$$

Fourth lemma is known as mixed Schwarz inequality.

Lemma 2.4. ([12]) If $A \in \mathbb{B}(\mathcal{H})$, then

$$|\langle Ax, y \rangle| \leq \langle |A|x, x \rangle^{1/2} \langle |A^*|y, y \rangle^{1/2},$$

for all $x, y \in \mathcal{H}$.

We are now in a position to prove our first desired inequality.

Theorem 2.5. If $A, B \in \mathbb{B}(\mathcal{H})$, then

$$w^{2r} \begin{pmatrix} O & A \\ B & O \end{pmatrix}$$

$$\leq \frac{1}{4} \max \left\{ \| |B|^{2r} + |A^*|^{2r} \|, \| |A|^{2r} + |B^*|^{2r} \| \right\} + \frac{1 - \alpha}{2} \max \left\{ w^{r}(AB), w^{r}(BA) \right\}$$

$$+ \frac{\alpha}{2} \max \left\{ \| \Re(|B|^{r} |A^*|^{r}) \|, \| \Re(|A|^{r} |B^*|^{r}) \| \right\},$$

for all $\alpha \in [0, 1]$ and for all $r \geq 1$. 
Proof. Let \( T = \begin{pmatrix} O & A \\ B & O \end{pmatrix} \) and let \( x \in \mathcal{H} \oplus \mathcal{H} \) with \( \|x\| = 1 \). Then we have,

\[
|\langle Tx, x \rangle|^{2r}
\]
\[
= \alpha|\langle Tx, x \rangle|^{2r} + (1 - \alpha)|\langle Tx, x \rangle|^{2r}
\]
\[
\leq \alpha \left( \langle |T|x, x \rangle \right)^{r/2} \left( \langle |T^*|x, x \rangle \right)^{r/2} + (1 - \alpha)|\langle Tx, x \rangle|^{2r} \quad \text{(by Lemma 2.4)}
\]
\[
= \alpha \left( \langle |T|x, x \rangle \right)^{r/2} \left( \langle |T^*|x, x \rangle \right)^{r/2} + (1 - \alpha)|\langle Tx, x \rangle|^{2r} \quad \text{(by Lemma 2.2)}
\]
\[
\leq \alpha \left( \langle |T|r, x \rangle + \langle |T^*|r, x \rangle \right)^{2} + (1 - \alpha)|\langle Tx, x \rangle|^{2r} \quad \text{(by Lemma 2.2)}
\]
\[
\leq \frac{\alpha}{4} \left( \langle |T|r + |T^*|r \rangle \right)^{2} + (1 - \alpha)|\langle Tx, x \rangle|^{2r} \quad \text{(by Lemma 2.3)}
\]
\[
\leq \frac{\alpha}{4} \left( \langle |T|r + |T^*|r \rangle \right)^{2} + \frac{1 - \alpha}{2} \left( \langle |T|x \rangle \| |T^*|x \rangle + |\langle Tx, x \rangle| r \right)
\]
\[
\leq \frac{\alpha}{4} \left( \langle |T|^2r + |T^*|2r \rangle \right)^{2} + \frac{1 - \alpha}{2} \left( \langle \frac{2}{2} + |\langle T2x, x \rangle| r \right)
\]
\[
= \frac{\alpha}{4} \left( \langle |T|^2r + |T^*|2r \rangle \right)^{2} + \frac{1 - \alpha}{2} \left( \langle \frac{2}{2} + |\langle T2x, x \rangle| r \right)
\]
\[
= \frac{1}{4} \left( \langle |T|^2r + |T^*|2r \rangle \right)^{2} + \frac{1 - \alpha}{2} \left( \langle \frac{2}{2} + |\langle T2x, x \rangle| r \right)
\]
\[
\leq \frac{1}{4} \left( \langle |T|^2r + |T^*|2r \rangle \right)^{2} + \frac{1 - \alpha}{2} \left( \langle \frac{2}{2} + |\langle T2x, x \rangle| r \right)
\]
\[
\leq \frac{1}{4} \left( \langle |T|^2r + |T^*|2r \rangle \right)^{2} + \frac{1 - \alpha}{2} \left( \langle \frac{2}{2} + |\langle T2x, x \rangle| r \right)
\]

Thus, taking supremum over \( \|x\| = 1 \), we have

\[
w^{2r}(T) \leq \frac{1}{4} \left( \langle |T|^2r + |T^*|2r \rangle \right)^{2} + \frac{1 - \alpha}{2} \left( \langle \frac{2}{2} + |\langle T2x, x \rangle| r \right)
\]
That is,
\[
\begin{bmatrix}
O & A \\
B & O
\end{bmatrix} \leq \frac{1}{4} \left( \begin{bmatrix}
|B|^2r + |A^*|^2r & O \\
O & |A|^2r + |B^*|^2r
\end{bmatrix} + \frac{\alpha}{2} \left( \begin{bmatrix}
\Re(|B|^r |A^*|^r) & O \\
O & \Re(|A|^r |B^*|^r)
\end{bmatrix} + \frac{1 - \alpha}{2} w^r \begin{bmatrix}
AB & O \\
O & BA
\end{bmatrix} \right) \right),
\]
Therefore, the required inequality follows from Lemma 2.1.

**Remark 2.6.** In particular, considering \( \alpha = 1 \) and \( r = 1 \) in the above theorem we get that
\[
w^2 \left( \begin{bmatrix}
O & A \\
B & O
\end{bmatrix} \right) \leq \frac{1}{4} \max \left\{ \|B\|^2 + |A^*|^2 \|, \|A\|^2 + |B^*|^2 \| \right\} + \frac{1}{2} \max \left\{ \|\Re(|A|^r |B^*|^r)\|, \|\Re(|A^*|^r |B|^r)\| \right\},
\]
which refines the existing one [8, Th. 2.10], namely
\[
w^2 \left( \begin{bmatrix}
O & A \\
B & O
\end{bmatrix} \right) \leq \frac{1}{4} \max \left\{ \|B\|^2 + |A^*|^2 \|, \|A\|^2 + |B^*|^2 \| \right\} + \frac{1}{2} \max \left\{ w(|A^*|), w(|A||B^*|) \right\}.
\]

To obtain our next result we need the following lemma, which can be found in [11, Lemma 2.1].

**Lemma 2.7.** If \( A, B \in \mathcal{B}(\mathcal{H}) \), then
\[
w \left( \begin{bmatrix}
A & B \\
B & A
\end{bmatrix} \right) = \max \left\{ w(A + B), w(A - B) \right\}.
\]
In particular, \( w \left( \begin{bmatrix}
O & B \\
B & O
\end{bmatrix} \right) = w(B). \)

**Corollary 2.8.** If \( A \in \mathcal{B}(\mathcal{H}) \), then
\[
w^2r(A) \leq \frac{1}{4} \| |A|^2r + |A^*|^2r \| + \frac{1}{2} \min \left\{ \|\Re(|A|^r |A^*|^r)\|, w^r(A^2) \right\},
\]
for all \( r \geq 1 \).

**Proof.** Considering \( A = B \) in Theorem 2.5, and then using Lemma 2.7 we infer that
\[
w^2r(A) \leq \frac{1}{4} \| |A|^2r + |A^*|^2r \| + \frac{1}{2} \left\{ \alpha \|\Re(|A|^r |A^*|^r)\| + (1 - \alpha)w^r(A^2) \right\}, \tag{2.2}
\]
for all \( \alpha \in [0, 1] \) and for all \( r \geq 1 \). This implies the desired inequality.
\[ \square \]
Remark 2.9. (i) Since $\|\Re(|A||A^*|)\| \leq w(|A||A^*|)$, so we would like to remark that Corollary 2.8 (for $r = 1$) gives stronger inequality than that in both (1.4) and (1.5).

(ii) If for norm one sequences $\{x_n\}$ in $\mathcal{H}$ with $|\langle \Re(|T||T^*|)x_n, x_n\rangle| \to \|\Re(|T||T^*|)\|$ and $|\langle \Im(|T||T^*|)x_n, x_n\rangle| \to \lambda (\neq 0)$, then Corollary 2.8 (for $r = 1$) gives strictly stronger inequality than that in (1.5).

We next prove the following theorem.

Theorem 2.10. Let $A \in \mathfrak{B} (\mathcal{H})$ be such that $\Re(|A||A^*|) = O$. Then $W(A)$ is a circular disk with center at the origin and radius $\frac{1}{2} \sqrt{\|A^*A + AA^*\|}$.

Proof. From Corollary 2.8, we get for the case $r = 1$,

$$w^2(A) \leq \frac{1}{4} \|A^2 + |A^*|^2\| + \frac{1}{2} \|\Re(|A||A^*|)\|.$$  \tag{2.3}

The first inequality in (1.2) together with (2.3) we infer that $w(A) = \frac{1}{2} \sqrt{\|A^2 + |A^*|^2\|} = \frac{1}{2} \sqrt{\|A^*A + AA^*\|}$ if $\Re(|A||A^*|) = O$. Therefore, from [7, Th. 2.14] it follows that $W(A)$ is a circular disk with center at the origin and radius $\frac{1}{2} \sqrt{\|A^*A + AA^*\|}$. \hfill \Box

We observe that the converse of the above result is, in general, not true. There exists an operator $A \in \mathfrak{B} (\mathcal{H})$ for which $w(A) = \frac{1}{2} \sqrt{\|A^2 + |A^*|^2\|}$, but $\Re(|A||A^*|) \neq O$.

Consider $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then one can verify that $w(A) = \frac{1}{2} \sqrt{\|A^2 + |A^*|^2\|} = \frac{1}{\sqrt{2}}$, but $\Re(|A||A^*|) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Next we obtain an inequality for the numerical radius of the sum of $n$ operators which generalizes Theorem 2.5. For this, we need the following Bohr’s inequality which deals with positive real numbers.

Lemma 2.11. ([16]) If $a_i \geq 0$ for each $i = 1, 2, \ldots, n$, then

$$\left( \sum_{i=1}^{n} a_i \right)^r \leq n^{r-1} \sum_{i=1}^{n} a_i^r,$$

for all $r \geq 1$. 

Theorem 2.12. If $A_i \in \mathbb{B}(\mathcal{H})$ for $i = 1, 2, \ldots, n$, then
\[
w^{2r} \left( \sum_{i=1}^{n} A_i \right) \leq \frac{n^{2r-1}}{4} \left\| \sum_{i=1}^{n} (|A_i|^{2r} + |A_i^*|^{2r}) \right\|
+ \frac{n^{2r-1}}{2} \left( \alpha \left\| \sum_{i=1}^{n} \Re (|A_i|^r |A_i^*|^r) \right\| + (1 - \alpha) \sum_{i=1}^{n} w^r (A_i^2) \right),
\]
for all $\alpha \in [0, 1]$ and for all $r \geq 1$.

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then by using Lemma 2.11 we infer that
\[
\left\| \left( \sum_{i=1}^{n} A_i \right) x, x \right\|^{2r} \leq \left( \sum_{i=1}^{n} | \langle A_i x, x \rangle | \right)^{2r}
\leq n^{2r-1} \left( \sum_{i=1}^{n} | \langle A_i x, x \rangle |^{2r} \right)
= n^{2r-1} \left( \alpha \sum_{i=1}^{n} | \langle A_i x, x \rangle |^{2r} + (1 - \alpha) \sum_{i=1}^{n} | \langle A_i x, x \rangle |^{2r} \right).
\]

Now proceeding similarly as in Theorem 2.5 we have the desired inequality. \hfill \Box

Next we obtain a lower bound for the numerical radius of the $2 \times 2$ off-diagonal operator matrix \( \begin{pmatrix} O & A \\ B & O \end{pmatrix} \). By considering the unitary operator matrix \( \begin{pmatrix} O & I \\ I & O \end{pmatrix} \), the weak unitary invariance property for the numerical radius gives that
\[
w \left( \begin{pmatrix} O & A \\ B & O \end{pmatrix} \right) = w \left( \begin{pmatrix} O & B \\ A & O \end{pmatrix} \right).
\]

Theorem 2.13. If $A, B \in \mathbb{B}(\mathcal{H})$, then
\[
w \left( \begin{pmatrix} O & A \\ B & O \end{pmatrix} \right) \geq \frac{1}{2} \left\| \Re(A) + i \Im(B) \right\| + \frac{1}{4} \left\| A + B^* \right\| - \left\| A - B^* \right\|, \tag{2.4}
\]
and
\[
w \left( \begin{pmatrix} O & A \\ B & O \end{pmatrix} \right) \geq \frac{1}{2} \left\| \Re(B) + i \Im(A) \right\| + \frac{1}{4} \left\| A + B^* \right\| - \left\| A - B^* \right\|. \tag{2.5}
\]

Proof. It is well-known that
\[
w \left( \begin{pmatrix} O & A \\ B & O \end{pmatrix} \right) \geq \left\| \Re \left( \begin{pmatrix} O & A \\ B & O \end{pmatrix} \right) \right\| = \frac{1}{2} \left\| A + B^* \right\|
\]
and 
\[ w \begin{pmatrix} O & A \\ B & O \end{pmatrix} \geq \| \Im \begin{pmatrix} O & A \\ B & O \end{pmatrix} \| = \frac{1}{2} \| A - B^* \|. \]

Therefore, we have
\[ w \begin{pmatrix} O & A \\ B & O \end{pmatrix} \geq \frac{1}{2} \max \left\{ \| A + B^* \|, \| A - B^* \| \right\} \]
\[ = \frac{1}{2} \left( \frac{\| A + B^* \| + \| A - B^* \|}{2} + \frac{\| A + B^* \| - \| A - B^* \|}{2} \right) \]
\[ = \frac{1}{2} \left( \frac{\| (A + B^*) + (A^* - B) \|}{2} + \frac{\| A + B^* \| - \| A - B^* \|}{2} \right) \]
\[ = \frac{1}{2} \left( \| \Re(A) - i \Im(B) \| + \frac{\| A + B^* \| - \| A - B^* \|}{2} \right) \]
\[ = \frac{1}{2} \| \Re(A) - i \Im(B) \| + \frac{1}{4} \| A + B^* \| - \| A - B^* \| \]. \]

This implies the inequality (2.4). Interchanging \( A \) and \( B \) in (2.4) we get the inequality (2.5).

As a consequence of Theorem 2.13 we get the following corollaries.

**Corollary 2.14.** Let \( A \in \mathcal{B}(\mathcal{H}) \). Then
\[ w(A) \geq \frac{1}{2} \| A \| + \frac{1}{2} \left\| \Re(A) \right\| - \left\| \Im(A) \right\| \]. \] (2.6)

**Proof.** This follows clearly from Theorem 2.13 by considering \( A = B \). \( \square \)

**Corollary 2.15.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \). Then
\[ w \begin{pmatrix} O & A \\ B & O \end{pmatrix} \geq \frac{1}{2} \max \left\{ \| A \|, \| B \| \right\} + \frac{1}{4} \| A + B^* \| - \| A - B^* \| \]. \] (2.7)

**Proof.** Considering the operator \( \begin{pmatrix} O & A \\ B & O \end{pmatrix} \) and applying the inequality (2.6) we get the desired inequality (2.7).

**Remark 2.16.** (i) Consider the matrix \( A = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 2 + 3i & 0 \\ 0 & 0 \end{pmatrix} \). Then, \( \max \left\{ \| A \|, \| B \| \right\} = \sqrt{13} \) and \( \| \Re(A) + i \Im(B) \| = \sqrt{18} \). Again, if we assume that \( A = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 0 \\ 0 & 2 + 3i \end{pmatrix} \), then \( \max \left\{ \| A \|, \| B \| \right\} = \sqrt{13} \) and \( \| \Re(A) + i \Im(B) \| = \sqrt{18} \).
Thus, we would like to remark that the inequalities obtained in Theorem 2.13 and Corollary 2.15 are, in general, not comparable.

(ii) We observe that the inequalities in Theorem 2.13 and Corollary 2.15 are sharp.

Next, we study the following necessary conditions for the equalities of

\[ w \left( \begin{array}{cc} O & A \\ B & O \end{array} \right) = \frac{1}{2} \| \Re(A) + i \Im(B) \| \]

and

\[ w \left( \begin{array}{cc} O & A \\ B & O \end{array} \right) = \frac{1}{2} \| \Re(B) + i \Im(A) \|. \]

**Proposition 2.17.** Let \( A, B \in \mathbb{B} \left( \mathcal{H} \right) \). Then the following results hold.

(i) If \( w \left( \begin{array}{cc} O & A \\ B & O \end{array} \right) = \frac{1}{2} \| \Re(A) + i \Im(B) \| \), then \( \| A + B^* \| = \| A - B^* \| = \| \Re(A) + i \Im(B) \| \).

(ii) If \( w \left( \begin{array}{cc} O & A \\ B & O \end{array} \right) = \frac{1}{2} \| \Re(B) + i \Im(A) \| \), then \( \| A + B^* \| = \| A - B^* \| = \| \Re(B) + i \Im(A) \| \).

**Proof.** Suppose \( w \left( \begin{array}{cc} O & A \\ B & O \end{array} \right) = \frac{1}{2} \| \Re(A) + i \Im(B) \| \). Then from the inequality (2.4) it follows that \( \| A + B^* \| = \| A - B^* \| \). Therefore,

\[
\frac{1}{2} \| A + B^* \| \leq w \left( \begin{array}{cc} O & A \\ B & O \end{array} \right) = \frac{1}{2} \| \Re(A) + i \Im(B) \|
\]

\[
= \frac{1}{2} \left\| \frac{A + A^*}{2} + i \frac{B - B^*}{2i} \right\|
\]

\[
\leq \frac{1}{2} \left( \frac{1}{2} \| A - B^* \| + \frac{1}{2} \| A + B^* \| \right)
\]

\[
= \frac{1}{2} \| A^* \| \quad \text{(since } \| A + B^* \| = \| A - B^* \|). \]

This completes the proof of (i). The proof of (ii) follows from (i) by interchanging \( A \) and \( B \). \( \square \)

Finally, as a consequence of Theorem 2.13 we obtain the following inequality.

**Corollary 2.18.** If \( A, B \in \mathbb{B} \left( \mathcal{H} \right) \), then

\[
w \left( \begin{array}{cc} O & A \\ B & O \end{array} \right) \geq \frac{1}{4} \| A + B \| + \frac{1}{4} | a - b | + \frac{1}{4} \| A + B^* \| - \| A - B^* \|,
\]

where \( a = \| \Re(A) + i \Im(B) \|, b = \| \Re(B) + i \Im(A) \| \). This inequality is sharp.
Proof. It follows from (2.4) and (2.5) that
\[ w \left( \begin{bmatrix} O & A \\ B & O \end{bmatrix} \right) \geq \frac{1}{2} \max\{a, b\} + \frac{1}{4} \left\| A + B^* \right\| - \left\| A - B^* \right\|. \] (2.8)

Now,
\[ \max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2} |a - b| \geq \frac{1}{2} \left\| A + B \right\| + \frac{1}{2} |a - b|. \] (2.9)

Therefore, combining (2.9) with (2.8) we infer that the desired inequality. The proof of sharpness is trivial. \(\square\)

We end this section with the following result.

**Proposition 2.19.** Let \( A, B \in \mathbb{B}(\mathcal{H}) \). If \( w \left( \begin{bmatrix} O & A \\ B & O \end{bmatrix} \right) = \frac{1}{4} \left\| A + B \right\| \), then the following results hold:
\begin{enumerate}
  \item \( \left\| A \right\| = \left\| B \right\|. \)
  \item \( \left\| A + B^* \right\| = \left\| A - B^* \right\|. \)
  \item \( \left\| \Re(A) + i \Im(B) \right\| = \left\| \Re(B) + i \Im(A) \right\|. \)
\end{enumerate}

Proof. The proof of (i) follows from (2.7). The proofs of (ii) and (iii) follow from Corollary 2.18. \(\square\)

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