Nonasymptotic Guarantees for Low-Rank Matrix Recovery with Generative Priors

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Abstract

Many problems in statistics and machine learning require the reconstruction of a low-rank signal matrix from noisy data. Enforcing additional prior information on the low-rank component is often key to guaranteeing good recovery performance. One such prior on the low-rank component is sparsity, giving rise to the sparse principal component analysis problem. Unfortunately, this problem suffers from a computational-to-statistical gap, which may be fundamental. In this work, we study an alternative prior where the low-rank component is in the range of a trained generative network. We provide a non-asymptotic analysis with optimal sample complexity, up to logarithmic factors, for low-rank matrix recovery under an expansive-Gaussian network prior. Specifically, we establish a favorable global optimization landscape for a mean squared error optimization, provided the number of samples is on the order of the dimensionality of the input to the generative model. As a result, we establish that generative priors have no computational-to-statistical gap for structured low-rank matrix recovery in the finite data, nonasymptotic regime. We present this analysis in the case of both the Wishart and Wigner spiked matrix models.

1 Introduction

In this paper we study the problem of estimating a spike vector \( y_\star \in \mathbb{R}^n \) from data \( Y \) consisting of a rank-1 matrix perturbed with random noise. In particular, the following random models for \( Y \) will be considered.

- **The Spiked Wishart Model** in which \( Y \in \mathbb{R}^{N \times n} \) is given by:
  \[ Y = uu_\star^\top + \sigma Z, \]
  where \( \sigma > 0, u \sim \mathcal{N}(0, I_n) \) and \( Z \) are independent and \( Z_{ij} \) are i.i.d. from \( \mathcal{N}(0, 1) \).

- **The Spiked Wigner Model** in which \( Y \in \mathbb{R}^{n \times n} \) is given by:
  \[ Y = y_\star y_\star^\top + \nu \mathcal{H}, \]
  where \( \nu > 0, \mathcal{H} \in \mathbb{R}^{n \times n} \) is drawn from a Gaussian Orthogonal Ensemble \( \text{GOE}(n) \), i.e. \( \mathcal{H}_{ii} \sim \mathcal{N}(0, 2/n) \) for all \( 1 \leq i \leq n \) and \( \mathcal{H}_{ij} = \mathcal{H}_{ji} \sim \mathcal{N}(0, 1/n) \) for \( 1 \leq j < i \leq n \).

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Spiked random matrices have been extensively studied in recent years as they serve as a mathematical model for many statistical inverse problems such as PCA [33, 2, 21, 56], synchronization over graphs [1, 8, 32] and community detection [42, 20, 46]. They are, moreover, connected to the rank-1 case of other linear inverse problems such as matrix sensing and matrix completion under RIP-like assumptions on the measurements operator [12, 63].

In high-dimensional/low signal-to-noise ratio regimes, it is fundamental to leverage additional prior information on the low-rank component in order to obtain consistent estimates of \( y^* \). Recent works, however, have discovered that some priors give rise to gaps between what is statistically-theoretically optimal and can be achieved with unbounded computational resources and what instead can be achieved with polynomial-time algorithms. A prominent example is represented by the Sparse PCA problem in which the vector \( y^* \) in (1) is taken to be sparse (see next section and [9, 36] for surveys of recent approaches).

In this paper we study the spiked random matrix models (1) and (2), where the prior information on the planted signal \( y^* \) comes from a learned generative network. In particular, we assume that a generative neural network \( G : \mathbb{R}^k \to \mathbb{R}^n \) with \( k < n \), has been trained on a data set of spikes, and the unknown spike \( y^* \in \mathbb{R}^n \) lies on the range of \( G \), i.e. we can write \( y^* = G(x^*) \) for some \( x^* \in \mathbb{R}^k \).

As a mathematical model for the trained \( G \), we consider a \( d \)-layer feed forward network of the form:

\[
G(x) = \text{relu}(W_d \cdots \text{relu}(W_2 \text{relu}(W_1 x)) \cdots)
\]

with weight matrices \( W_i \in \mathbb{R}^{n_i \times n_{i-1}} \) and \( \text{relu}(x) = \max(x, 0) \) is applied entrywise. We furthermore assume that the network is expansive and the weights are Gaussian. This modeling assumption was introduced by [29], and additionally it and its variants were used in [30, 26, 40, 25, 55]. See Section 1.1 for justifications of this model.

Generative priors have been shown to close a computational-to-statistical gap in the Compressive Phase Retrieval problem. With a sparsity prior, the information-theoretically optimal sample complexity is proportional to the sparsity level \( s \) of the signal, on the other hand the best known algorithms (convex methods [28, 38, 47], iterative thresholding [15, 58, 62], etc.) require a sample complexity proportional to \( s^2 \) for stable recovery, a barrier which might not be resolvable by polynomial-time algorithms [11, 10]. Under the generative prior (3), [26] has shown that, compressive phase retrieval is possible via gradient descent over a nonlinear objective with sample complexity proportional (up to log factors) to the underlying signal dimensionality, \( k \). This result suggests that it may be possible to use generative priors to close other computational-to-statistical gaps such as for models (1) and (2). Indeed, recently [7] considered these low-rank models and the generative network prior (3) and shown that in the asymptotic limit \( k, n, N \to \infty \) with \( n/k = O(1) \) and \( N/n = O(1) \), an Approximate-Message Passing algorithm achieves the statistical information-theoretic lower bound and no computational-to-statistical gap is present.

This paper analyzes the low-rank matrix models (1) and (2) under the generative network prior (3).

The contributions of this paper are as follows. We demonstrate that low-rank matrix recovery in the non-asymptotic finite-data regime does not have computational-to-statistical gaps when enforcing a generative prior. This provides a second problem for which generative priors have closed such gaps in a non-asymptotic case. We further corroborate these findings by proposing a (sub)gradient algorithm which, as shown by our numerical experiments, is able to recover the sought spike with optimal sample complexity. This paper, therefore, strengthens the case for generative networks as priors for statistical inverse problems, not only because of their ability to learn natural signal priors, but also because of their capacity to lead to statistically optimal polynomial-time algorithms and zero computational-to-statistical gaps.
1.1 Problem formulation and main results

We consider the rank-one matrix recovery problem under a deep generative prior. We assume that the signal spike $y_\star = G(x_\star)$. To estimate $y_\star$, we propose to first find an estimate $\hat{x}$ of the latent variable $x_\star$ and then apply $G(\hat{x}) \approx y_\star$. We thus consider the following minimization problem:

$$\min_{x \in \mathbb{R}^k} f(x) := \frac{1}{4}\|G(x)G(x)^\top - M\|_F^2. \tag{4}$$

where:

- for the **Wishart model** (1) we take $M = \Sigma_N - \sigma^2 I_n$,
- for the **Wigner model** (2) we take $M = Y$.

Even though the objective function (4) is nonconvex, we show that it enjoys a favorable global optimization geometry for Gaussian weight matrices $\{W_i\}_{i=1}^d$. The informal version of our main results for the two spiked models is given below.

**Theorem 1** (Informal). Let $y_\star = G(x_\star)$ for a given a generative network $G : \mathbb{R}^k \to \mathbb{R}^n$ as in (3). Assume that each layer is sufficiently expansive $n_{i+1} = \Omega(n_i \log n_i)$ and the weights are Gaussian. Consider the minimization problem (4) and assume that up to factors dependent on the number of layers $d$:

- for the **Wishart model**: $\sqrt{k \log n / N} = O(1)$,
- for the **Wigner model**: $\nu \sqrt{k \log n / n} = O(1)$.

With high probability:

A. for any nonzero point $x \in \mathbb{R}^k$ outside two small neighborhoods of $x_\star$ and $-\rho_d x_\star$, there is a direction of descent;

B. the objective function values near $-\rho_d x_\star$ are larger than those near $x_\star$;

C. for any point $x$ in the small neighborhood around of $x_\star$, up polynomials in $d$:

- for the **Wishart model**:
  $$\|G(x) - y_\star\|_2 \lesssim \sqrt{\frac{k \log n}{N}}, \tag{5}$$
- for the **Wigner model**:
  $$\|G(x) - y_\star\|_2 \lesssim \nu \sqrt{\frac{k \log n}{n}}. \tag{6}$$

This theorem demonstrates the favorable optimization geometry of the problem (4) and implies that rank-1 matrix recovery with a deep (random) generative network prior can be solved rate-optimally by simply minimizing over the range of the network via simple and computationally tractable algorithms such as gradient descent methods. The iterates of these methods would converge to one of the two neighborhoods where the gradients are small (Theorem 1A), and avoiding the bad neighborhood of $-\rho_d x_\star$ can be done by exploiting the knowledge of the properties of the loss function (Theorem 1B) as done in Algorithm 1 below and shown in the numerical experiments.

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\[1\] Under the deterministic conditions on the generative network (see below for details), it was shown in [29] that $G$ is invertible and therefore there exists a unique $x_\star$ that satisfies $y_\star = G(x)$. 
Finally for any point $x$ in the “benign neighborhood” of $x_\star$, the reconstruction error $\|G(x) - y_\star\|$ will have information-theoretically optimal rates (5) and (6) (Theorem 1C), corresponding (up to log factors) to the best achievable even in the simple linear case.

Regarding the Gaussian weight assumption, we observe that there is empirical evidence that the distribution of the weights of deep neural networks have properties consistent with those of Gaussian matrices [4]. Moreover, these observations have been used in advancing the theoretical understanding of deep network trained in supervised setting and in particular their ability to preserve the metric structure of the data [24]. The randomness assumption has been further used by [3] to show that autoencoders with random weights can be learned polynomially. More recently, an avalanche of works (see for example [39, 23, 49, 43, 17]) have been dedicated to theoretical guarantees for training deep neural networks in the close-to-random regime of the Neural Tangent Kernel [31]. Finally as for the case of compressed sensing in which the analysis of the random setting has led to considerable understanding of the problem as well as tangible practical innovations, we hope that the analysis of the random setting for deep generative networks will provide insights and generate novel developments in the field of statistical inverse problems.

2 Related work

Sparse PCA and other computational-to-statistical gaps.

Given a large number of samples data $\{y_i\}_{i=1}^N \in \mathbb{R}^n$ the important statistical task of finding the directions that explain most of the variance (principal components) is classically solved by PCA. Insights on the statistical performance of this algorithm can be gained by studying spiked covariance models [33]. Under this model it is assumed that the data are of the form:

$$y_i = u_i y_\star + \sigma z_i$$  \hfill (7)

where $\sigma > 0$, $u_i \sim \mathcal{N}(0, 1)$ and $z_i \sim \mathcal{N}(0, I_n)$ are independent and identically distributed, and $y_\star$ is the unit norm planted spike. Note that a matrix $Y \in \mathbb{R}^{N \times n}$ with rows $\{y_i\}_i$ can be written as (1), and the $y_i$s are i.i.d. samples of $\mathcal{N}(0, \Sigma)$ where the population covariance matrix is $\Sigma = y_\star y_\star^\top + \sigma^2 I_n$.

Principal Component Analysis, then, estimates $y_\star$ via the leading eigenvector $\hat{y}$ of the empirical covariance matrix $\Sigma_N = \frac{1}{N} \sum_{i=1}^N y_i y_i^\top$. Standard techniques of high dimensional probability then show that as long as

$$\min_{\epsilon = \pm} \|\epsilon \hat{y} - y_\star\|_2 \lesssim \sqrt{\frac{n}{N}}, \hfill \hfill (8)$$

However, in the modern high dimensional data regime, it is not uncommon to consider cases where the ambient dimension of the data $n$ is larger, or of the order, of the number of samples $N$. In this case, bounds of the form (8) become meaningless. Even worse, in the asymptotic regime $n/N \to c > 0$ and for $\sigma^2$ large enough, the spike $y_\star$ and the estimate $\hat{y}$ becomes orthogonal [34]. Moreover, minimax techniques can be used to show that in this regime no other estimators can achieve better overlap with $y_\star$ [57].

These negative results motivated the use of additional structural prior on the spike $y_\star$, aimed at reducing the sample complexity of the problem. In recent years various priors has been studied such as positivity [45], cone constraints [22] and in particular sparsity [34], [64]. In the latter case $y_\star$

We write $f(n) \gtrsim g(n)$ if $f(n) \geq Cn$ for some constant $C > 0$ that might depend $\sigma$ and $\|y_\star\|^2$. Similarly for $f(n) \lesssim g(n)$.
is assumed to be $s$-sparse, and it can be shown that for $N \gtrsim s \log n$ and $n \gtrsim s$, then the $s$-sparse largest eigenvector $\hat{y}_s$ of $\Sigma_N$:

$$\hat{y}_s = \arg\max_{y \in \mathbb{S}_2^{n-1}, \|y\|_0 \leq s} y^T \Sigma_N y$$

satisfies the condition:

$$\min_{\epsilon = \pm} \|\epsilon \hat{y}_s - y_*\|_2 \lesssim \sqrt{\frac{s \log n}{N}}.$$ 

In particular the number of samples must scale linearly with the intrinsic dimension $s$ of the signal. These rates are also minimax optimal, see for example [56] for the mean squared error and [2] for the support recovery. Despite these encouraging results no known polynomial time algorithm is known that achieves such performances and for example the covariance thresholding algorithm of [35] requires $N \gtrsim s^2$ samples in order to obtain exact support recovery or estimation rate:

$$\min_{\epsilon = \pm} \|\epsilon \hat{y}_s - y_*\|_2 \lesssim \sqrt{\frac{s^2 \log n}{N}},$$

as shown in [21]. In summary, only computationally intractable algorithms are known to reach the statistical limit $N = \Omega(s)$ for Sparse PCA, while polynomial time methods are only sub-optimal requiring $N = \Omega(s^2)$. The study of this computational-to-statistical gap was initiated by [11] who investigated the detection problem via a reduction to the planted clique problem which is conjectured to be computationally hard.

The hardness of sparse PCA has been further suggested in a series of recent works [16, 41, 37, 14]. These works fit in the growing and important body of literature on computational-to-statistical gaps, which have also been found and studied in a variety of other contexts such as tensor principal component analysis [52], community detection [19] and synchronization over groups [50]. Many of these problems can be phrased as recovery a spike from a spiked random matrix models, and the hardness can then be viewed as arising from imposing simultaneously both low-rankness and additional prior information on the signal (sparsity in case of Sparse PCA). This difficulty can be found in sparse phase retrieval as well, where [38] has shown that for an $s$-sparse signal of dimension $n$ lifted to a rank-one matrix, $m = O(s \log n)$ number of quadratic measurements are enough to ensure well-posedness, while $m \geq O(s^2 / \log^2 n)$ measurements are necessary for the success of natural convex relaxations of the problem. Similarly [48] studies the recovery of simultaneously low-rank and sparse matrices, and show the existence of a gap between what can be achieved with convex and tractable relaxations and nonconvex and intractable methods.

### Recovery with a generative network prior

Recently, in the wake of successes of deep learning, deep generative networks have gained popularity as a novel approach to encoding and enforcing priors. They have been successfully used as a prior for various statistical estimation problems such as compressed sensing [13], blind deconvolution [5], inpainting [61], and many more [54, 60, 51, 59], etc.

Parallel to these empirical successes, a recent line of works have investigated theoretical guarantees for various statistical estimation tasks with generative network priors. Following the work of [13], [29] have given global guarantees for compressed sensing, followed then by many others for various inverse problems [53, 44, 25, 6, 51]. In particular [27] have shown that $m = \Omega(k \log n)$ number of measurements are sufficient to recover a signal from random phaseless observations, assuming that the signal is the output of a trained generative network with latent space of dimension $k$. Note that, contrary to the sparse phase retrieval problem, generative priors for phase retrieval allow optimal
sample complexity, up to logarithmic factors, with respect to the intrinsic dimension of the signal. Further, when modeled by generative priors, that dimensionality could be much smaller than the sparsity level \( s \) under a sparsity prior and an appropriate basis.

Recently \cite{7} has shown that when \( y_\star \) is in the range of an expansive-Gaussian generative network with Relu activation functions, then low-rank matrix recovery does not have a computational-to-statistical gap, in the asymptotic limit \( k, n, N \to \infty \) with \( n/k = \mathcal{O}(1) \) and \( N/n = \mathcal{O}(1) \). They also provide a spectral algorithm and demonstrate that it is able to match asymptotically the information-theoretical optimal. These methods were then extended to the phase-retrieval problem in \cite{6}.

### 3 Low-rank matrix recovery under a generative network prior

We are now ready to formulate our main theoretical result for the spiked random models. Its analysis will be based on the following assumptions on the weights of the network.

**Assumption 1.** The generative network \( G \) defined in (3), has weights \( W_i \in \mathbb{R}^{n_i \times n_i} \) with i.i.d. entries from \( \mathcal{N}(0, 1/n_i) \) and satisfying the expansivity condition with constant \( \epsilon > 0 \):

\[
n_{i+1} \geq c \epsilon^{-2} \log(1/\epsilon) n_i \log n_i
\]

for all \( i \) and a universal constant \( c > 0 \).

Due to the non-smoothness of the Relu activation functions, the generative network \( G \) and the loss function \( f \) are not differentiable everywhere. Therefore, following \cite{30}, we resort to some concepts from nonsmooth analysis \(^3\). Since \( f \) is continuous and piecewise smooth, at every point \( x \in \mathbb{R}^k \), \( f \) has a Clarke subdifferential given by:

\[
\partial f(x) = \text{conv}\{v_1, v_2, \ldots, v_T\}
\]

where \( \text{conv} \) denotes the convex hull of the vectors \( v_1, \ldots, v_T \), gradients of the \( T \) smooth functions adjoint at \( x \). In particular at a point where \( f \) is differentiable \( \partial f(x) = \{\tilde{v}_x\} \). where \( \tilde{v}_x \) will denote the gradient of \( f \) when it exists.

**Theorem 2** (Global Landscape Analysis). Assume Assumption 1 is satisfied with \( \epsilon \leq K_1 d^{-90} \), consider the minimization problem (4) and assume that the noise variance \( \omega \) satisfies \( \omega \leq K_2\|x_\star\|_2^2 2^{-d}/d^{12} \)

where:

- for the **Spiked Wishart Model** (1) take \( M = \Sigma_N - \sigma^2 I_n \), and:

\[
\omega := \sqrt{\frac{1}{n} \log(n) (3 n_1^d n_2^{d-1} \ldots n_{d-1}^2)}
\]

- for the **Spiked Wigner Model** (2) take \( M = Y \) and:

\[
\omega := \sqrt{\frac{1}{n} \log(n) (3 n_1^d n_2^{d-1} \ldots n_{d-1}^2)}
\]

\(^3\)The reader is referred to \cite{18} for more details.
Then for some absolute constants $C_1, C_2 > 0$ with probability at least
\[
1 - 2e^{-k \log n} - \sum_{i=2}^{d} 8n_i \exp(-C_1 n - 2 - 8n_1 e^{-C_2 \epsilon^2 \log(1/\epsilon)k})
\]
the following holds.

For all $x \in \mathbb{R}^k$:
- if $x \notin B(x_*, r_+) \cup B(-\rho_d x_*, r_-) \cup \{0\}$ and $v_x \in \partial f(x)$:
  \[
  D_{-v_x} f(x) < 0
  \]
where
\[
r_+ = K_3(d^4 \epsilon^{1/2} + 2d \omega \|x_*\|^{-2})d^{10}\|x_*\|_2,
\]
and
\[
r_- = K_4(d^2 \epsilon^{1/4} + 2d^{3/2} \omega^{1/2}\|x_*\|^{-1})d^{10}\|x_*\|
\]
- if $x \in B(0, \|x_*\|_2/16\pi) \setminus \{0\}$ and $v_x \in \partial f(x)$ then
  \[
  \langle x, v_x \rangle < 0
  \]
while if $x = 0$ and $v \in S^{k-1}$ then
\[
D_{-v} f(0) = 0
\]
Here $\rho_d$ is a positive number that converges to 1 as $d \to \infty$, $K_1, \ldots, K_4$ are universal constants and $B(x, r)$ denotes the Euclidean ball of radius $r$ around $x$.

Note that the quantity $2^d$ in the hypothesis and conclusions of the theorem, is an artifact of the scaling of the network and it should not be taken as requiring exponentially small noise. Indeed under the assumptions on the weights specified above, these matrices have spectral norm approximately 1, while the application of the ReLU function zeros out approximately half of the entries of its argument leading to an “effective” operator norm of approximately $1/2$.

This theorem shows that while $x = 0$ is a critical point, the gradient in its vicinity points in its direction, that is it is a local max. Moreover the theorem gives existence of descent directions outside two neighborhoods of $x_*$ and $-\rho_d x_*$. 

The next proposition will describe the local properties of these two neighborhoods.

**Proposition 1.** Assume that the assumptions of Theorem 2 are satisfied.

A. If $x \in B(x_*, r_+)$ and $y \in B(\rho_d x_*, r_-)$ it holds that:
\[
f(y) > f(x)
\]

B. In addition, for $K_6$ and $K_7$ positive absolute constants:

- for the **Spiked Wishart Model**, for all $x \in B(x_*, r_+)$:
  \[
  \|G(x) - G(x_*)\| \leq K_6 \left( d^4 \epsilon^{1/2} + \frac{\omega}{\|y_*\|^2} \right) d^{10}\|G(x_*)\|_2,
  \]

- for the **Spiked Wigner Model**, for all $x \in B(x_*, r_+)$:
  \[
  \|G(x) - G(x_*)\| \leq K_7 \left( d^4 \epsilon^{1/2} + \frac{\omega}{\|y_*\|^2} \right) d^{10}\|G(x_*)\|_2,
  \]

The previous results imply that, for $\epsilon$ small enough and under the assumptions on the noise level $\omega$, any point $x$ in the benign neighborhood $B(x_*, r_+)$ has reconstruction error $\|G(x) - G(x_*)\|$ which scales optimally according to (5) or (6).
3.1 Proofs outline and techniques

The bulk of the analysis will be based on deterministic conditions on the weights of the network. In particular we leverage a set of technical results recently introduced by [29].

For \( W \in \mathbb{R}^{n \times k} \) and \( x \in \mathbb{R}^k \), define the operator \( W_{+,x} := \text{diag}(Wx > 0)W \) such that \( \text{relu}(Wx) = W_{+,x}x \). Moreover let \( W_{i,+,x} = (W_1)_{+,x} = \text{diag}(W_1x > 0)W_1 \), and for \( 2 \leq i \leq d \):

\[
W_{i,+,x} = \text{diag}(W_i, \Pi_{j=1}^{i-1} W_{j,+,x} > 0)W_i,
\]

where \( \Pi_{j=1}^{i-1} W_i = W_d W_{d-1} \ldots W_1 \). Finally we let \( \Lambda_x = \Pi_{j=1}^{i} W_{j,+,x} \) and note that \( G(x) = \Lambda_x x \).

**Definition 1 (Weight Distribution Condition [29]).** We say that \( W \in \mathbb{R}^{n \times k} \) satisfies the **Weight Distribution Condition (WDC)** with constant \( \epsilon > 0 \) if for all \( x_1, x_2 \in \mathbb{R}^k \):

\[
||W_{+,x_1}^\top W_{+,x_2} - Q_{x_1,x_2}||_2 \leq \epsilon,
\]

where

\[
Q_{x_1,x_2} = \frac{\pi - \theta_{x_1,x_2}}{2\pi} I_k + \frac{\sin \theta_{x_1,x_2}}{2\pi} M_{\hat{x}_1 \leftrightarrow \hat{x}_2}
\]

and \( \theta_{x_1,x_2} = \angle(x_1, x_2) \), \( \hat{x}_1 = x_1/||x_1||_2 \), \( \hat{x}_2 = x_2/||x_2||_2 \), \( I_k \) is the \( k \times k \) identity matrix and \( M_{\hat{x}_1 \leftrightarrow \hat{x}_2} \) is the matrix that sends \( \hat{x}_1 \mapsto \hat{x}_2 \), \( \hat{x}_2 \mapsto \hat{x}_1 \), and with kernel span(\( \{x_1, x_2\} \))

Note that \( Q_{x_1,x_2} \) is the expected value of \( W_{+,x_1}^\top W_{+,x_2} \) when \( W \) has rows \( w_i \sim \mathcal{N}(0, I_k/n) \) and if \( x_1 = x_2 \) then \( Q_{x_1,y_2} \) is an isometry up to the scaling factor \( 1/2 \). Under the Assumption 1, then [29] shows that, the WDC holds with high probability for all layers of the generative network \( G \). This condition ensures that the angle between two vectors in the latent space is approximately preserved at the output layer and in turn guarantees the invertibility of the network.

Next we observe that at a differentiable point the gradient of \( f \), defined in (4), is given by:

\[
\tilde{v}_x := \Lambda_x^\top [\Lambda_x x x^\top \Lambda_x^\top - M] \Lambda_x x.
\]

Using the WDC, then, we demonstrate that \( \tilde{v}_x \) concentrates up to the noise level around a direction \( h_x \in \mathbb{R}^k \) which is a continuous function of nonzero \( x \) and \( \hat{x}_* \). Furthermore using the characterization (9) of the Clarke subdifferential, we show that this concentration extends also to non-differentiable points for subgradients.

A direct analysis then shows that any directional derivative of \( f \) at zero is zero and that \( h_x \) is small in a neighborhood of \( x_* \) and its negative multiple \( -\rho_d x_* \). This in turn guarantees the existence of a descent direction in the complement of these sets.

Similarly we use the WDC to show that up to noise level, the loss function \( f \) concentrates around:

\[
f_E(x) = \frac{1}{4} \left( \frac{1}{2\sigma_d} ||x||_2^4 + \frac{1}{2\sigma_d} ||x_*||_2^4 - 2\langle x, h_x \rangle^2 \right).
\]

where \( \tilde{h}_{x,x_*} \) is continuous for nonzero \( x \) and \( x_* \). Directly analyzing the properties of \( f_E \) in a neighborhood of \( x_* \) and \( -\rho_d x_* \) allows to derive the first point of Proposition 1. The last part of Proposition 1 follows by noticing that the generator \( G \) is locally Lipschitz.

Finally we extend a technique of [30] to control the noise: in our case a Gaussian Orthogonal matrix \( H \) for the spiked Wigner model (2), and \( \Sigma_N - \Sigma \) for the spiked Wishart model. The analysis is based on a counting argument on the subspaces spanned by a depth \( d \) generative networks, which leads to the sought rate-optimal bounds.
4 A subgradient method and numerical experiments

Informed by the analysis in the previous sections, we propose a gradient method for the solution of (4) and verify empirically its optimal properties.

Recall that the main properties of the landscape of the minimization problem (4) were the global minimum in a neighborhood of the true latent vector $x^\star$ and a flat region in correspondence of $-\rho d x^\star$. Moreover the latter region has larger loss function values than those in the vicinity of $x^\star$. In order to overcome the non-convexity and avoid this bad region we run gradient descent from a random non-zero initialization and its negation. We then pick the iterate which has smaller final loss value.

Algorithm 1 Gradient method for the minimization problem (4)

1: **Input:** Weights $W_i$, observation matrix $M$, step size $\alpha > 0$, number of iterations $T$
2: Choose $\hat{x}_0 \in \mathbb{R}^k \setminus \{0\}$ arbitrary
3: Let $x_{1,0} = \hat{x}_0$ and $x_{2,0} = -\hat{x}_0$
4: for $i = 1, 2$ do
5: for $j = 0, 1, \ldots, T - 1$ do
6: Compute $v_{x_{i,j}} \in \partial f(x_{i,j})$
7: $x_{i,j+1} \leftarrow x_{i,j} - \alpha v_{x_{i,j}}$
8: if $f(x_{1,T}) < f(x_{2,T})$ then
9: Return: $x_{1,T}, G(x_{1,T})$
10: else
11: Return: $x_{2,T}, G(x_{2,T})$

Note that it will be highly unlikely that the iterates will be at a non-differentiable point, therefore in practice we can consider Algorithm 1 with descent direction $\tilde{v}_x = \nabla f(x)$.

We verify our theoretical claims on synthetic generative priors. We consider 2-layer generative networks with Relu activation functions, hidden layer of dimension $n_1 = 250$ and output dimension $n = 1700$ and varying number of latent dimension $k \in [10, 30, 70]$. We randomly sample the weights of the matrix independently from $\mathcal{N}(0, 2/n_i)$, which removes that $2^d$ dependence in Theorem 2. We then consider data $Y$ according to spiked models (1) and (2), where $x^\star \in \mathbb{R}^k$ is chosen so that $y^\star = G(x^\star)$ has unit norm. For the Wishart model we vary the samples $N$ while for the Wigner model we vary the noise level $\nu$ so that the following quantities remain constant for the different networks (latent dimension $k$):

$$\theta_{WS} := \sqrt{k \log(n_1^2 n)/N}, \quad \theta_{WG} := \nu \sqrt{k \log(n_1^2 n)/n}$$

We then plot the MSE given by $\|G(x) - G(x^\star)\|$ against $\theta_{WS} \approx \sqrt{k/N}$ and $\theta_{WG} \approx \nu \sqrt{n}$. As predicted by Theorem 2 the errors scale linearly with respect to these control parameters, and moreover all the plots overlap confirming that these rates are tight with respect to the order of $k$. 


Figure 1: Mean Squared Error for the recovery of a spike $y_\star = G(x_\star)$ in Wishart and Wigner model with random generative network priors. Average over 50 random drawing of the network weights and samples. These plots demonstrate that the reconstruction error follow closely our theory.

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A Global landscape analysis under deterministic conditions

As mentioned, the proof of Theorem 2 and Proposition 1, will be based on deterministic conditions on the weights of the network and the noise matrix. In particular we will consider the minimization problem (4) with:

\[ M = G(x_*)^T G(x_*) + H, \]

for an unknown symmetric matrix \( H \in \mathbb{R}^{n \times n} \) and nonzero \( x_* \in \mathbb{R}^k \).

Recall the definition 1 of the WDC. Below we will say that a \( d \)-layer generative network \( G \) of the form (3), satisfies the WDC with constant \( \epsilon > 0 \) if every weight matrix \( W_i \) has the WDC with constant \( \epsilon \) for all \( i = 1, \ldots d \).

We can now describe the landscape of the minimization problem (4) in a deterministic setting.

**Theorem 3.** Consider a generative network \( G : \mathbb{R}^k \rightarrow \mathbb{R}^n \) as in (3) and the minimization problem (4) with unknown nonzero \( x_* \) and symmetric \( H \). Fix \( \epsilon > 0 \) such that \( K_1 d^{16} \sqrt{\epsilon} \leq 1 \) and let \( d \geq 2 \).

Suppose that \( G \) satisfies the WDC with constant \( \epsilon \) and assume that:

\[ \| \Lambda^T H \Lambda \|_2 \leq \frac{\omega}{2d} \]

for all \( x \in \mathbb{R}^k \),

with \( 2^d d^{12} w \leq K_2 \| x_* \|^2 \) and \( K_2 < 1 \).

Then for all \( x \in \mathbb{R}^k \):

- if \( x \notin B(x_*, r_+) \cup B(-\rho_d x_*, r_-) \cup \{0\} \) and \( v_x \in \partial f(x) \):
  \[ D_{-v_x} f(x) < 0 \]

where

\[ r_+ = K_3 (d^{4 \epsilon^{1/2}} + 2^d \omega \| x_* \|^{-2}) d^{10} \| x_* \|_2, \]

and

\[ r_- = K_4 (d^{2 \epsilon^{1/4}} + d^{4/2} \omega^{1/2} \| x_* \|^{-1}) d^{10} \| x_* \| \]

- if \( x \in B(0, \| x_* \|/16\pi) \setminus \{0\} \) and \( v_x \in \partial f(x) \) then
  \[ \langle x, v_x \rangle < 0 \]

while if \( x = 0 \) and \( v \in S^{k-1} \) then

\[ D_{-v} f(0) = 0 \]

Here \( \rho_d \) is a positive number that converges to 1 as \( d \rightarrow \infty \) and \( K_1, \ldots, K_4 \) are universal constants.

Similarly, below we give the deterministic version of Proposition 1.

**Proposition 2.** Under the assumptions of Theorem 3, for any \( \phi_d \in [\rho_d, 1] \), it holds that:

\[ f(x) < f(y) \]

for all \( x \in B(\phi_d x_*, \phi \| x_* \| d^{-12}) \) and \( y \in B(-\phi_d x_*, \phi \| x_* \| d^{-12}) \) where \( \phi < 1 \) is a universal constant.

The rest of the paper is organized as follows. After summarizing the notation used throughout the paper in Section A.1 and deriving concentration results for the subgradients from the WDC in Section A.2, we give the proof of Theorem 3 in Section A.3. In Section A.4 we prove Proposition 2, while Section A.5 contains the proofs of supplementary lemmas needed in the main results. Finally in Section B, we derive the main Theorem 2 and Proposition 2 from the corresponding deterministic ones by controlling the noise terms and recalling a result of [29] which show that the WDC holds with high probability.
A.1 Notation

We now collect the notation that is used throughout the paper. For any real number $a$, let $\text{relu}(a) = \max(a, 0)$ and for any vector $v \in \mathbb{R}^n$, denote the entrywise application of relu as $\text{relu}(v)$.

Let $\text{diag}(Wx > 0)$ be the diagonal matrix with $i$-th diagonal element equal to 1 if $(Wx)_i > 0$ and 0 otherwise. For any vector $x$ we denote with $\|x\|$ its Euclidean norm and for any matrix $A$ we denote with $\|A\|$ its spectral norm and with $\|A\|_F$ its Frobenius norm. The euclidean inner product between two vectors $a$ and $b$ is $\langle a, b \rangle$, while for two matrices $A$ and $B$ their Frobenius inner product will be denoted by $\langle A, B \rangle_F$. For any nonzero vector $x \in \mathbb{R}^n$, let $\hat{x} = x/\|x\|$.

For a set $S$ we will write $|S|$ for its cardinality and $S^c$ for its complement. Let $B(x, r)$ be the Euclidean ball of radius $r$ centered at $x$, and $S_{k-1}$ be the unit sphere in $\mathbb{R}^k$. We will write $\gamma = \Omega(\delta)$ to mean that there exists a positive constant $C$ such that $\gamma \geq C\delta$ and similarly $\gamma = O(\delta)$ if $\gamma \leq C\delta$. Additionally we will use $a = b + O(\delta)$ when $\|a - b\| \leq \delta$, where the norm is understood to be the absolute value for scalars, the Euclidean norm for vectors and the spectral norm for matrices.

A.2 Preliminaries

Recall that at a differentiable point the gradient of $f$ is denoted by $\tilde{v}_x$ and given in (10). By the WDC $\tilde{v}_x$ concentrates up to the noise level around the direction $h_x \in \mathbb{R}^k$:

$$h_x := \left[\frac{1}{2d}xx^T - \tilde{h}_{x,x^*} \tilde{h}_{x,x^*}^T\right]x,$$

where $\tilde{h}_{x,x^*}$ is defined below and is a continuous function of $x$ and $x^*$. The vector field $\tilde{h}_{x,x^*}$ depends on a function that controls how the angles are contracted by the deep network, and defined as:

$$g(\theta) := \cos^{-1}\left(\frac{(\pi - \theta) \cos \theta + \sin \theta}{\pi}\right)$$

With this definition we let $\tilde{h}_{x,x^*}$ be:

$$\tilde{h}_{x,x^*} := \frac{1}{2d} \left[\prod_{i=0}^{d-1} \frac{\pi - \theta_i}{\pi} x^* + \prod_{i=1}^{d-1} \sin \frac{\theta_i}{\pi}\left(\prod_{j=i+1}^{d-1} \frac{\pi - \theta_j}{\pi}\right) \|x^*\| \tilde{h}_{x,x^*}^T \right]$$

where $\theta_i := g(\theta_{i-1})$ for $g$ given by (14) and $\theta_0 = \angle(x,y)$. For brevity of notation below we will use $\tilde{h}_x = \tilde{h}_{x,x^*}$. For later convenience we also will define the following vectors:

$$p_x := \Lambda_x^T \Lambda_x x;$$
$$q_x := \Lambda_x^T \Lambda_{x^*} x^*;$$
$$\tilde{v}_x := [p_x p_x^T - q_x q_x^T] x;$$
$$\eta_x := \Lambda_x^T H \Lambda_{x} x.$$

and note that when $f$ is differentiable at $x$, then $\tilde{v}_x := \nabla f(x) = \tilde{v}_x - \eta_x$, in particular for zero noise $\tilde{v}_x = \tilde{v}_x$.

We now observe the following facts.
Lemma 1 (Lemma 8 in [29]). Suppose that $d \geq 2$ and the WDC holds with $\epsilon < 1/(16\pi d^2)^2$, then for all nonzero $x, x_* \in \mathbb{R}^k$,

$$\langle \Lambda_x x, \Lambda_x x_* \rangle \geq \frac{1}{4\pi} \frac{1}{2d} \|x\|_2^2 \|x_*\|_2, \quad (15)$$

$$\|\Lambda_x^T \Lambda_x x_* - \tilde{h}_{x,x_*}\| \leq 24 \frac{d^2 \sqrt{\epsilon}}{2d} \|x_*\|, \text{ and } (16)$$

$$\|\Lambda_x\|^2 \leq \frac{1}{2d} (1 + 2\epsilon)^d \leq \frac{1 + 4\epsilon d}{2d} \leq \frac{13}{12} \frac{1}{2d}. \quad (17)$$

**Proof.** The first two bounds can be found in [29, Lemma 8]. The third bound follows noticing that the WDC implies:

$$\|\Lambda_x\|^2 \leq \Pi_{i=d}^1 \|W_{i,+} x\|^2 \leq \frac{1}{2d} (1 + 2\epsilon)^d \leq \frac{1 + 4\epsilon d}{2d} \leq \frac{13}{12} \frac{1}{2d}$$

where we used $\log(1+z) \leq z$ and $e^z \leq (1 + 2z)$ for all $0 \leq z \leq 1$.

The next lemma shows that the noiseless gradient $\bar{\nabla}_x$, concentrates around $h_x$.

Lemma 2. Suppose $d \geq 2$ and the WDC holds with $\epsilon < 1/(16\pi d^2)^2$, then for all nonzero $x, x_* \in \mathbb{R}^k$:

$$\|\bar{\nabla}_x - h_x\| \leq 86 \frac{d^4 \sqrt{\epsilon}}{2d} \max(\|x_*\|^2, \|x\|^2) \|x\| \quad (18)$$

We now use the characterization of the Clarke subdifferential given in (9), to derive a bound on the concentration of $v_x \in \partial f(x)$ around $h_x$ up to the noise level.

Lemma 3. Under the assumption of Lemma 2, and with $H$ satisfying (11), for any $v_x \in \partial f(x)$:

$$\|v_x - h_x\| \leq 86 \frac{d^4 \sqrt{\epsilon}}{2d} \max(\|x_*\|^2, \|x\|^2) \|x\| + \frac{\omega}{2d} \|x\| \quad (19)$$

### A.3 Proof of Theorem 3

We define the set $S_\beta$ outside which we can lower bound the gradient as:

$$S_\beta := \{ x \in \mathbb{R}^k | \|h_x\| \leq \frac{\beta}{2d} \max(\|x\|^2, \|x_*\|^2) \|x\| \}$$

with:

$$\beta = 5 \cdot (86d^4 \sqrt{\epsilon} + 2d \omega \|x_*\|^{-2}) \quad (18)$$

Outside the set $S_\beta$ the gradient is bounded below and the landscape has favorable optimization geometry. Due to the continuity and piecewise smoothness of the generator $G$ and in turn of the loss function $f$, for any $x, y \neq 0$ there exists a sequence of $\{x_n\} \to x$ such that $f$ is differentiable at each $x_n$ and $D_y f(x) = \lim_{n \to \infty} \nabla f(x_n) \cdot y$. It follows that:

$$D_{-v_x} f(x) = -\lim_{n \to \infty} \tilde{v}_{x_n} \cdot \frac{v_x}{\|v_x\|}$$
as $\nabla f(x_n) = \tilde{v}_{x_n}$. Regarding the right hand side of the above, observe that:

$$\langle \tilde{v}_{x_n} \cdot v_x \rangle = h_{x_n} \cdot h_x + (v_{x_n} - h_{x_n}) \cdot h_x + h_{x_n} \cdot (v_x - h_x) + (\tilde{v}_{x_n} - h_{x_n}) \cdot (v_x - h_x)$$

$$\geq h_{x_n} \cdot h_x - \|\tilde{v}_{x_n} - h_{x_n}\| \|h_x\| - \|h_{x_n}\| \|v_x - h_x\| - \|\tilde{v}_{x_n} - h_{x_n}\| \|v_x - h_x\|,$$

$$\geq h_{x_n} \cdot h_x - \frac{86d^4\sqrt{\epsilon} + 2d^4\omega \|x_*\|^2}{2d^2} \max(\|x_n\|^2, \|x_*\|^2) \|x_x\| + \max(\|x\|^2, \|x_*\|^2) \|x_x\| \|h_{x_n}\|$$

$$- \left( \frac{86d^4\sqrt{\epsilon} + 2d^4\omega \|x_*\|^2}{2d^2} \right)^2 \max(\|x\|^2, \|x_*\|^2) \max(\|x_n\|^2, \|x_*\|^2) \|x_n\||x_x||x_x|$$

where the second inequality follows from Lemma 3. Moreover as $h_x$ is continuous in $x$ for all nonzero $x$:

$$\lim_{n \to \infty} \tilde{v}_{x_n} \cdot v_x \geq \|h_x\|^2 - 2\frac{86d^4\sqrt{\epsilon} + 2d^4\omega \|x_*\|^2}{2d^2} \max(\|x\|^2, \|x_*\|^2) \|x_x\|$$

$$- \left( \frac{86d^4\sqrt{\epsilon} + 2d^4\omega \|x_*\|^2}{2d^2} \right)^2 \max(\|x\|^2, \|x_*\|^2) \|x\|^2$$

$$\geq \|h_x\|^2 \left[ \|h_x\| - \frac{4\epsilon^2}{2d^2} \right] \max(\|x\|^2, \|x_*\|^2) \|x_x\||x_x|$$

$$\geq \frac{1}{2} \left( \|h_x\|^2 - 2 \left( \frac{86d^4\sqrt{\epsilon} + 2d^4\omega \|x_*\|^2}{2d^2} \right)^2 \max(\|x\|^2, \|x_*\|^2) \|x\|^2 \right)$$

By our choice of $\beta$ in (18) it follows that for any $x \in S_{\beta}^\delta \setminus \{0\}$:

$$\|h_x\| - 4\frac{86d^4\sqrt{\epsilon} + 2d^4\omega \|x_*\|^2}{2d^2} \max(\|x\|^2, \|x_*\|^2) \|x_x\| \geq \frac{\max(\|x\|^2, \|x_*\|^2)}{2d^2} \|x\| \left( \beta - 4\frac{86d^4\sqrt{\epsilon} + 2d^4\omega \|x_*\|^2}{2d^2} \right),$$

so that:

$$\lim_{n \to \infty} v_{x_n} \cdot v_x \geq \frac{\|h_x\|}{2} \max(\|x\|^2, \|x_*\|^2) \frac{86d^4\sqrt{\epsilon}}{2d^2} \|x\| > 0.$$
A.3.1 Control of the zeros of $h_x$

In this section we show that $h_x$ is nonzero outside two neighborhoods of $x_*$ and $-\rho_d x_*$. 

Lemma 4. Suppose $8\pi d^6 \sqrt{\beta} \leq 1$. Define:

$$\rho_d := \sum_{i=0}^{d-1} \frac{\sin \tilde{\theta}_i}{\pi} \left( \prod_{j=i+1}^{d-1} \frac{\pi - \tilde{\theta}_j}{\pi} \right)$$

where $\tilde{\theta}_0 = \pi$ and $\tilde{\theta}_i = g(\tilde{\theta}_{i-1})$. If $x \in S_\beta$, then we have either:

$$|\tilde{\theta}_0| \leq 32d^4 \pi \beta \quad \text{and} \quad ||x||^2 - ||x_*||^2 \leq 258\pi \beta d^6 ||x_*||$$

or

$$|\tilde{\theta}_0 - \pi| \leq 8\pi d^4 \sqrt{\beta} \quad \text{and} \quad ||x||^2 - \rho_d^2 ||x_*||^2 \leq 281\pi^2 \sqrt{\beta} d^{10} ||x_*||.$$ 

In particular, we have:

$$S_\beta \subset B(x_*, R_1 \beta d^{10} ||x_*||) \cup B(-\rho_d x_*, R_2 \sqrt{\beta} d^{10} ||x_*||)$$

where $R_1, R_2$ are numerical constants and $\rho_d \to 1$ as $d \to \infty$.

Proof. Without loss of generality, let $x_* = e_1$ and $x = r \cos \bar{\theta}_0 \cdot e_1 + r \sin \bar{\theta}_0 \cdot e_2$, for some $\bar{\theta}_0 \in [0, \pi]$, and $r \geq 0$. Recall that we call $\hat{x} = x/||x||$ and $\hat{x}_* = x_*/||x_*||$. We then introduce the following notation:

$$\xi = \prod_{i=0}^{d-1} \frac{\pi - \tilde{\theta}_i}{\pi}, \quad \zeta = \sum_{i=0}^{d-1} \frac{\sin \tilde{\theta}_i}{\pi} \prod_{j=i+1}^{d-1} \frac{\pi - \tilde{\theta}_j}{\pi}, \quad r = ||x||, \quad R = \max(r^2, 1),$$

(19)

where $\theta_i = g(\tilde{\theta}_{i-1})$ with $g$ as in (14), and observe that $2^d h_x = (\xi \hat{x}_* + \zeta \hat{x})$. Let $\alpha := 2^d \langle \hat{h}_x, \hat{x} \rangle$, then we can write:

$$h_x = \left[ \frac{\langle x, x \rangle}{2^{2d} r^2} x - \langle \hat{h}_x, x \rangle \hat{h}_x \right] = \frac{r}{2^{2d}} [r^2 \hat{x} - \alpha (\xi \hat{x}_* + \zeta \hat{x})].$$

Using the definition of $\hat{x}$ and $\hat{x}_*$ we obtain:

$$\frac{2^{2d} h_x}{r} = [r^2 - \alpha \zeta \cos \bar{\theta}_0 - \alpha \xi] \cdot e_1 + [r^2 - \alpha \zeta] \sin \bar{\theta}_0 \cdot e_2,$$

and conclude that since $x \in S_\beta$, then:

$$||r^2 - \alpha \zeta \cos \bar{\theta}_0 - \alpha \xi|| \leq \beta R \quad \text{(20)}$$

$$||r^2 - \alpha \zeta \sin \bar{\theta}_0|| \leq \beta R. \quad \text{(21)}$$
We now list some bounds that will be useful in the subsequent analysis. We have:

\[
\begin{align*}
\bar{\theta}_i & \leq \bar{\theta}_{i-1} \text{ for } i \geq 1 \quad (22) \\
\bar{\theta}_i & \leq \cos^{-1}(1/\pi) \text{ for } i \geq 2 \quad (23) \\
|\xi| & \leq 1 \quad (24) \\
|\zeta| & \leq \frac{d}{\pi \sin \bar{\theta}_0} \quad (25) \\
\zeta & \geq \frac{\pi - \bar{\theta}_0}{\pi} d^{-3} \quad (26) \\
\bar{\theta}_i & \leq \frac{3\pi}{i + 3} \text{ for } i \geq 0 \quad (27) \\
\bar{\theta}_i & \geq \frac{\pi}{i + 1} \text{ for } i \geq 0 \quad (28) \\
\bar{\theta}_0 = \pi + O_1(\delta) & \Rightarrow |\xi| \leq \frac{\delta}{\pi} \quad (29) \\
\bar{\theta}_0 = \pi + O_1(\delta) & \Rightarrow \zeta = \rho_d + O_1(3d^3\delta) \text{ if } \frac{d^2\delta}{\pi} \leq 1 \quad (30) \\
1/\pi & \leq \alpha \leq 1. \quad (31)
\end{align*}
\]

The identities (22) through (30) can be found in Lemma 16 of [30], while the identity (31) follows by noticing that \( \alpha = \xi \cos \bar{\theta}_0 + \zeta = \cos \theta_d \) and using (23) together with \( d \geq 2 \).

**Bound on \( R \).** We now show that if \( x \in S_\beta \), then \( r^2 \leq 4d \) and therefore \( R \leq 4d \).

If \( r^2 \leq 1 \), then the claim is trivial. Take \( r^2 > 1 \), then note that either \( |\sin \bar{\theta}_0| \geq 1/\sqrt{2} \) or \( |\cos \bar{\theta}_0| \geq 1/\sqrt{2} \) (or both) must hold. If \( |\sin \bar{\theta}_0| \geq 1/\sqrt{2} \) then from (21) it follows that \( r^2 - \alpha \zeta \leq \sqrt{2} \beta R = \sqrt{2} \beta r^2 \) which implies:

\[
\begin{align*}
\frac{\alpha \zeta}{1 - \sqrt{2} \beta} & \leq \frac{1}{d} \frac{d}{2} \\
\frac{\alpha \zeta + \sqrt{2} \zeta}{1 - \sqrt{2} \beta} & \leq 4d
\end{align*}
\]

using (25) and (31) in the second inequality and \( \beta < 1/4 \) in the third. Next take \( |\cos \bar{\theta}_0| \geq 1/\sqrt{2} \), then (20) implies \( |r^2 - \alpha \zeta| \leq \sqrt{2}(\beta r^2 + \alpha \xi) \) which in turn results in:

\[
\begin{align*}
r^2 & \leq \frac{\alpha (\zeta + \sqrt{2} \xi)}{1 - \sqrt{2} \beta} \leq 4d
\end{align*}
\]

using (24), (25), (31) and \( \beta < 1/4 \). In conclusion if \( x \in S_\beta \) then \( r^2 \leq 4d \Rightarrow R \leq 4d \).

**Bounds on \( \bar{\theta}_0 \).** We now show we only have to analyze the small angle case \( \bar{\theta}_0 \approx 0 \) and the large angle case \( \bar{\theta}_0 \approx \pi \).

At least one of the following three cases must hold:

1. \( \sin \bar{\theta}_0 \leq 16 \beta \pi d^4 \): Then we have \( \bar{\theta} = O_1(32 \beta \pi d^4) \) or \( \bar{\theta} = \pi + O_1(32 \beta \pi d^4) \) as \( 32 \beta \pi d^4 < 1 \).

2. \( |r^2 - \alpha \zeta| \leq \sqrt{2} R \): Then (20), (31) and \( \beta < 1 \) yield \( |\xi| \leq 2 \sqrt{2} \beta R \). Using (26), we then get \( \bar{\theta} = \pi + O_1(2 \sqrt{2} \beta^2 d^3 R) \).

3. \( \sin \bar{\theta}_0 > 16 \beta \pi d^4 \) and \( |r^2 - \alpha \zeta| \geq \sqrt{2} R \): Then (21) gives \( |r^2 - \alpha \zeta| \leq \beta M/\sin \bar{\theta}_0 \) which used with (20) leads to:

\[
|\alpha \xi| \leq \beta R + |r^2 - \alpha \zeta| \leq \beta R + \frac{\beta R}{\sin \bar{\theta}_0} \leq 2 \frac{\beta R}{\sin \bar{\theta}_0}.
\]
Then using (31), the assumption on \( \sin \tilde{\theta}_0 \) and \( R \leq 4d \) we obtain \( \xi \leq d^{-3}/2 \). The latter together with (26) leads to \( \tilde{\theta}_0 \geq \pi/2 \). Finally as \( |r^2 - \alpha \zeta| \geq \sqrt{3}R \) then (21) leads to \( |\sin \tilde{\theta}_0| \leq \sqrt{3} \). Therefore as \( \tilde{\theta}_0 \geq \pi/2 \) and \( \beta < 1 \), we can conclude that \( \tilde{\theta}_0 = \pi + O_1(2\sqrt{3}) \).

Inspecting the three cases, and recalling that \( R \leq 4d \), we can see that it suffices to analyze the small angle case \( \tilde{\theta}_0 = O_1(32d^4\pi \beta) \) and the large angle case \( \tilde{\theta} = \pi + O_1(8\sqrt{3}\pi^2 d^4) \).

**Small angle case.** We assume \( \tilde{\theta}_0 = O_1(\delta) \) with \( \delta = 32d^4\pi \beta \) and show that \( \|x\|^2 \approx \|x_\star\|^2 \).

We begin collecting some bounds. Since \( \tilde{\theta}_0 \leq \theta_0 \leq \delta \), then \( 1 \geq \xi \geq (1 - \delta/\pi) \) \( \geq 1 + O_1(2d\delta/\pi) \) assuming \( d\delta/\pi \leq 1/2 \), which holds true since \( 64d^5\beta < 1 \). Moreover from (25) we have \( \zeta = O_1(d\delta/\pi) \).

Finally observe that \( \cos \tilde{\theta}_0 = 1 + O_1(\tilde{\theta}_0^2/2) = 1 + O_1(\delta/2) \) for \( \delta < 1 \). We then have \( \alpha = 1 + O_1(2d\delta) \) so that \( \alpha \zeta = O_1(d^2\delta) \) and \( \alpha \xi = 1 + O_1(4d^2\delta) \). We can therefore rewrite (20) as:

\[
(r^2 + O_1(d^2\delta))(1 + O_1(\delta/2)) - (1 + O_1(4d^2\delta)) = O_1(\beta R).
\]

Using the bound \( r^2 \leq R \leq 4d \) and the definition of \( \delta \), we obtain:

\[
\begin{align*}
r^2 - 1 &= O_1 \left( \frac{\delta r^2}{2} + d^2\delta + \frac{d^2\delta^2}{2} + 4d^2\delta + 4d\beta \right) \\
&= O_1(8d^2\delta + 4d\beta) \\
&= O_1(258\pi d^6 \beta)
\end{align*}
\]

**Large angle case.** Here we assume \( \tilde{\theta} = \pi + O_1(\delta) \) with \( \delta = 8\sqrt{3}\pi^2 d^4 \) and show that it must be \( \|x\|^2 \approx \|x_\star\|^2 \).

From (29) we know that \( \xi = O_1(\delta/\pi) \), while from (30) we know that \( \zeta = \rho_d + O_1(3d^3\delta) \) as long as \( 8\sqrt{3}\pi d^6 \leq 1 \). Moreover for large angles and \( \delta < 1 \), it holds \( \cos \tilde{\theta}_0 = -1 + O_1((\tilde{\theta}_0 - \pi)^2/2) = -1 + O_1(\delta^2/2) \). These bounds lead to:

\[
\alpha = \xi \cos \tilde{\theta}_0 + \zeta
= \rho_d + O_1 \left( \frac{\delta}{\pi} + \frac{\delta^3}{2\pi} + 3d^3\delta \right)
= \rho_d + O_1(4d^3\delta),
\]

and using \( \rho_d \leq d \):

\[
\begin{align*}
\alpha \zeta &= \rho_d^2 + O_1(4d^5\delta \rho_d + 3d^3\delta \rho_d + 12d^6\delta) = \rho_d^2 + O_1(20d^6\delta), \\
\alpha \xi &= O_1 \left( \frac{\delta}{\pi} \rho_d + 4 \frac{d^3\delta^2}{\pi} \right) = O_1(2d^3\delta).
\end{align*}
\]

Then recall that (20) is equivalent to \( (r^2 - \alpha \zeta) \cos \tilde{\theta}_0 - \alpha \xi = O_1(4\beta d) \), that is:

\[
(r^2 - \rho_d^2 + O_1(20d^6\delta))(1 + O_1(\delta^2/2)) + O_1(2d^3\delta) = O_1(4\beta d)
\]

and in particular:

\[
\begin{align*}
r^2 - \rho_d^2 &= O_1 \left( 20d^6\delta + 10d^6\delta^2 + \frac{\rho_d \delta^2}{2} + \frac{r^2}{2} + 2d^3\delta + 4\beta d \right) \\
&= O_1(35d^6\delta + 4\beta d) \\
&= O_1(281\pi^2 \sqrt{3} d^{10})
\end{align*}
\]
where we used $\rho_d \leq d$, the definition of $\delta$ and $\delta < 1$.

**Controlling the distance.** We have shown that it is either \( \tilde{\theta}_0 \approx 0 \) and \( \|x\|^2 \approx \|x_*\|^2 \) or \( \tilde{\theta}_0 \approx \pi \) and \( \|x\|^2 \approx \rho_d^2 \|x_*\|^2 \). We can therefore conclude that it must be either \( x \approx x_* \) or \( x \approx -\rho_d x_* \).

Observe that if a two dimensional point is known to have magnitude within \( \Delta r \) of some \( r \) and is known to be within an angle \( \Delta \theta \) from 0, then its Euclidean distance to the point of coordinates \( (r,0) \) is no more that \( \Delta r + (r + \Delta r) \Delta \theta \). Similarly we can write:

\[
\|x - x_*\| \leq \|x\| - \|x_*\| + (\|x_*\| + \|x\| - \|x_*\|)\tilde{\theta}_0. \quad (34)
\]

In the small angle case, by (32), (34), and \( \|x_*\| \|x\| - \|x_*\| \leq \|\|x\|^2 - \|x_*\|^2\| \), we have:

\[
\|x - x_*\| \leq 258 \pi d^6 \beta + (1 + 258 \pi d^6 \beta) 32 d^4 \pi \beta \leq 550 \pi d^{10} \beta.
\]

Next we notice that $\rho_2 = 1/\pi$ and $\rho_d \geq \rho_{d-1}$ as follows from the definition and (27), (28). Then considering the large angle case and using (33) we have:

\[
\|x\| - \rho_d \leq \frac{281 \pi^2 \sqrt{\beta} d^{10}}{\|x\| + \rho_d} \leq 281 \pi^3 \sqrt{\beta} d^{10}.
\]

The latter, together with (34), yields:

\[
\|x + \rho_d x_*\| \leq \|x\| - \rho_d + (\rho_d + \|x\| - \rho_d)(\pi - \tilde{\theta}_0) \\
\leq 281 \pi^3 \sqrt{\beta} d^{10} + (d + 281 \pi^3 \sqrt{\beta} d^{10}) 8 \sqrt{\beta} \pi^2 d^4 \\
\leq 284 \pi^3 \sqrt{\beta} d^{10}
\]

where in the second inequality we have used $\rho_d \leq d$ and in the third $8 \sqrt{\beta} \pi d^6 \leq 1$.

We conclude by noticing that $\rho_d \to 1$ as $d \to 1$ as shown in [30, Lemma 16].

**A.4 Proof of Proposition 2**

Recall that $f(x) := 1/4 \|G(x)G(x)^T - G(x_*)G(x_*)^T - H\|_F^2$, we next define the following loss functions:

\[
f_0(x) := \frac{1}{4} \|G(x)G(x)^T - G(x_*)G(x_*)^T\|_F^2, \\
f_H(x) := f_0(x) - \frac{1}{2} \langle G(x)G(x)^T - G(x_*)G(x_*)^T, H \rangle_F, \\
f_E(x) := \frac{1}{4} \left( \frac{1}{2d^4} \|x\|^4 + \frac{1}{2d^4} \|x_*\|^4 - 2\langle x, \tilde{h}_x \rangle \right). 
\]

In particular notice that $f(x) = f_H(x) + 1/4 \|H\|_F^2$. Below we show that assuming the WDC is satisfied $f_0(x)$ concentrates around $f_E(x)$.

**Lemma 5.** Suppose that $d \geq 2$ and the WDC holds with $\epsilon < 1/(16 \pi d^2)$, then for all nonzero $x, x_* \in \mathbb{R}^k$

\[
|f_0(x) - f_E(x)| \leq \frac{16}{d^2\sqrt{\epsilon}} (\|x\|^4 + \|x_*\|^4) d^4 \sqrt{\epsilon}
\]

We next consider the loss $f_E$ and show that in a neighborhood $-\rho_d x_*$, this loss function has larger values than in a neighborhood of $x_*$. 

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Lemma 6. Fix $0 < a \leq 1/(2\pi^3 d^3)$ and $\phi_d \in [\rho_d, 1]$ then:

$$f_E(x) \leq \frac{1}{2d + 2} \|x_*\|^4 + \frac{1}{2d + 2} [(a + \phi_d)^4 - 4\phi_d^2 + 2a2\pi da \|x_*\|^4 \quad \forall x \in B(\phi_d x_*, a\|x_*\|)) \quad \text{and}$$

$$f_E(x) \geq \frac{1}{2d + 2} \|x_*\|^4 + \frac{1}{2d + 2} [(a - \phi_d)^4 - 2\phi_d^2 - 40\pi d^4 a \|x_*\|^4 \quad \forall x \in B(-\phi_d x_*, a\|x_*\|)).$$

The above two lemmas are now used to prove Proposition 2.

Proof of Proposition 2. Let $x \in B(\pm \phi_d x_*, \varphi \|x_*\|)$ for a $0 < \varphi < 1$ that will be specified below, and observe that by the assumptions on the noise:

$$\|(G(x)G(x)^T - G(x_*G(x_*^T), H)_F) \leq |G(x)^T H G(x)| + |G(x_*)^T H G(x_*)| \leq \frac{\omega}{2d} (\|x\|^2 + \|x_*\|^2) \leq \frac{\omega}{2d} (\|\phi_d + \varphi\|^2 + 1) \|x_*\|^2,$$

and therefore by Lemma 5:

$$|f_0(x) - f_E(x)| + \frac{1}{2} (|G(x)G(x)^T - G(x_*G(x_*^T), H)_F) \leq \frac{16}{2d} (|\phi_d + \varphi|^4 + 1) \|x_*\|^4 d^4 \sqrt{\epsilon} + \frac{\omega}{2d} (|\phi_d + \varphi|^2 + 1) \|x_*\|^2 \leq \frac{272}{2d} \|x_*\|^4 d^4 \sqrt{\epsilon} + \frac{\omega}{27} (|\phi_d + \varphi|^2 + 1) \|x_*\|^2.$$ We next take $\varphi = \epsilon$ and $x \in B(\phi_d x_*, \varphi \|x_*\|)$, so that by Lemma 6 and the assumption $2d^4 d^{12}w \leq K_2 \|x_*\|^2$, we have:

$$f_H(x) \leq f_E(x) + |f_0(x) - f_E(x)| + \frac{1}{2} (|G(x)G(x)^T - G(x_*G(x_*^T), H)_F) \leq \frac{1}{2d + 2} [1 + (\epsilon + \phi_d)^4 - 2\phi_d^2 + 2\pi de \|x_*\|^4 + 272d^4 \sqrt{\epsilon} \|x_*\|^4 + \frac{\omega}{2d + 2}(2 + 2\epsilon + \epsilon^2) \|x_*\|^2$$

$$\leq \frac{1}{2d + 2} [1 - 2\phi_d^2 + (\epsilon + \phi_d)^4] \|x_*\|^4 + \frac{1}{2d} (\frac{3}{2} \rho_d^2 \|x_*\|^4 - 2(2d + 272d^4) \sqrt{\epsilon} \|x_*\|^4 + (\frac{\omega}{2d}) \|x_*\|^2$$

$$\leq \frac{1}{2d + 2} [1 - 2\rho_d^2 + (\epsilon + \phi_d)^4] \|x_*\|^4 + \frac{1}{2d} (\frac{3}{2} K_d \|x_*\|^4 - (\frac{\omega}{2d}) \|x_*\|^2$$

Similarly if $y \in B(-\phi_d x_*, \varphi \|x_*\|)$, and $\varphi = \epsilon$ we obtain:

$$f_H(y) \geq f_E(y) - |f_0(y) - f_E(y)| - \frac{1}{2} (|G(y)G(y)^T - G(x_*G(x_*^T), H)|$$

$$\geq \frac{1}{2d + 2} [1 - 2\phi_d^2 \rho_d^2 + (\epsilon - \phi_d)^4] \|x_*\|^4 - \frac{1}{2d} (\frac{3}{2} \rho_d^2 \|x_*\|^4 - 2(2d + 272d^4) \sqrt{\epsilon} \|x_*\|^4 - \frac{\omega}{2d} \|x_*\|^2$$

$$\geq \frac{1}{2d + 2} [1 - 2\rho_d^2 \phi_d^2 + (\epsilon - \phi_d)^4] \|x_*\|^4 - \frac{1}{2d} (\frac{3}{2} K_d \|x_*\|^4 - (\frac{\omega}{2d}) \|x_*\|^2 - K_2 \|x_*\|^4 d^{-12}.$$ In order to guarantee that $f(y) > f(x)$, it suffices to have:

$$2(1 - \rho_d^2) \phi_d^2 - 8K_d d^{-12} > 4C_d \sqrt{\epsilon}$$

with $C_d := (544d^4 + 10\pi d^3 \pi + 3K_d d^{-12} + \pi d/2 + 1/100)$, that is to require:

$$\varphi = \epsilon < \left(\frac{(1 - \rho_d^2) \phi_d^2 - 4K_d d^{-12}}{2C_d}\right)^2.$$ Finally notice that by Lemma 17 in [30] it holds that $1 - \rho_d \geq (K(d + 2))^{-2}$ for some numerical constant $K$, we therefore choose $\epsilon = \varphi / d^{12}$ for some $\varphi > 0$ small enough.
A.5 Supplementary proofs

Below we prove Lemma 2 on the concentration of the gradient of \( f \) at a differentiable point.

**Proof of Lemma 2.** We begin by noticing that:

\[
\bar{v}_x - h_x = \left[ \langle p_x, x \rangle p_x - \langle x, x \rangle \frac{x}{2d} \right] + \left[ \langle \tilde{h}_x, x \rangle x - \langle q_x, x \rangle x \right].
\]

Below we show that:

\[
\| \langle p_x, x \rangle p_x - \langle x, x \rangle \frac{x}{2d} \| \leq \frac{50}{2d} d^3 \sqrt{r} \max \{ \| x \|^2, \| x_\ast \|^2 \} \| x \|. \tag{35}
\]

and

\[
\| \langle q_x, x \rangle p_x - \langle \tilde{h}_x, x \rangle \tilde{h}_x \| \leq \frac{36}{2d} d^4 \sqrt{r} \max \{ \| x \|^2, \| x_\ast \|^2 \} \| x \|. \tag{36}
\]

from which the thesis follows.

Regarding equation (35) observe that:

\[
\| \langle p_x, x \rangle p_x - \langle x, x \rangle \frac{x}{2d} \| = \| \langle p_x, x \rangle [p_x - \frac{x}{2d}] + \langle p_x - \frac{x}{2d}, \frac{x}{2d} \rangle x \|
\leq \left( \| \Lambda_x x \|^2 + \frac{\| x \|^2}{2d} \right) \| p_x - \frac{x}{2d} \|
\leq \frac{50}{2d} d^3 \sqrt{r} \| x \|^3
\]

where in the first inequality we used \( \langle p_x, x \rangle = \| \Lambda_x x \|^2 \) and in the second we used equations (16) and (17) of Lemma 1.

Next note that:

\[
\| \langle q_x, x \rangle q_x - \langle \tilde{h}_x, x \rangle \tilde{h}_x \| = \| \langle q_x, x \rangle (q_x - \tilde{h}_x) + \langle q_x - \tilde{h}_x, x \rangle \tilde{h}_x \|
\leq (\| q_x \| + \| \tilde{h}_x \|) \| x \| \| q_x - \tilde{h}_x \|
\leq \left( \frac{13}{12} + \frac{d}{\pi} \frac{\| x \| \| x_\ast \|}{2d} \right) \| q_x - \tilde{h}_x \|
\leq \frac{3}{2} d \frac{\| x \| \| x_\ast \|}{2d} \| q_x - \tilde{h}_x \|
\]

where in the second inequality we have the bound (17) and the definition of \( \tilde{h}_x \). Equation (36) is then found by appealing to equation (16) in Lemma 1.

The previous lemma is now used to control the concentration of the subgradients \( v_x \) of \( f \) around \( h_x \).

**Proof of Lemma 3.** When \( f \) is differentiable at \( x \), \( \nabla f(x) = \bar{v}_x = \bar{v}_x + \eta_x \), so that by Lemma 2 and the assumption on the noise:

\[
\| v_x - h_x \| \leq \| \bar{v}_x - h_x \| + \| \eta_x \|
\leq 86 \frac{d^4 \sqrt{r}}{2d} \max \{ \| x_\ast \|^2, \| x \|^2 \} \| x \| + \frac{\omega}{2d} \| x \|. \tag{37}
\]

Observe, now, that by \( (9) \), for any \( x \in \mathbb{R}^k \), \( v_x \in \partial f(x) = \text{conv}(v_1, \ldots, v_l) \), and therefore \( v_x = a_1 v_1 + \cdots + a_T v_T \) for some \( a_1, \ldots, a_T \geq 0, \sum_i a_i = 1 \). Moreover for each \( v_i \) there exist a \( w_i \) such that
We can then conclude that:

\[ \| v_x - h_x \| \leq \sum_{i=1}^{T} a_i \| v_i - h_x \| \]

\[ \leq \sum_{i=1}^{T} a_i \lim_{\delta \to 0} \| \tilde{v}_{x+\delta w_i} - h_{x+\delta w_i} \| \]

\[ \leq 86 \frac{d^4 \sqrt{\epsilon}}{2d^4} \max(\|x_*\|^2, \|x\|^2) \|x\| + \frac{\omega}{2d} \|x\|. \]

We now prove Lemma 5 on the concentration of the noiseless objective function.

**Proof of Lemma 5.** Observe that:

\[ |f_0(x) - f_E(x)| \leq \frac{1}{4} \|G(x)\|^4 - \frac{1}{2d^3} \|x\|^4 \]

\[ + \frac{1}{4} \|G(x_*)\|^4 - \frac{1}{2d^3} \|x_*\|^4 \]

\[ + \frac{1}{2} \langle G(x), G(x_*) \rangle^2 - \langle x, \tilde{h}_x \rangle \].

We analyze each term separately. The first term can be bounded as:

\[ \frac{1}{4} \|G(x)\|^4 - \frac{1}{2d^3} \|x\|^4 = \frac{1}{4} \|G(x)\|^2 + \frac{1}{2d^3} \|x\|^2 \]

\[ \leq \frac{1}{4} \left( \frac{13}{12} + 1 \right) \|x\|^2 \|G(x)\|^2 - \frac{1}{2d^3} \|x\|^2 \]

\[ \leq \frac{1}{4} \left( \frac{13}{12} + 1 \right) \|x\|^2 \frac{d^3 \sqrt{\epsilon}}{2d^3} \|x\|^2 \]

\[ \leq \frac{1}{2d^3} \|x\|^4 \]

where in the first inequality we used (17) and in the second inequality (16) . Similarly we can bound the second term:

\[ \frac{1}{4} \|G(x_*)\|^4 - \frac{1}{2d^3} \|x_*\|^4 \leq \frac{1}{2d^3} 13d^3 \sqrt{\epsilon} \|x_*\|^4 \].

We next note that \( \|\tilde{h}_x\| \leq 2^{-d}(1 + d/\pi)\|x_*\| \) and therefore from (17) and \( d \geq 2 \) we obtain:

\[ \|G(x)\| \|G(x_*)\| + \|x\| \|\tilde{h}_x\| \leq \frac{1}{2d^3} \left( \frac{13}{12} + 1 + \frac{d}{\pi} \right) \|x\| \|x_*\| \leq \frac{1}{2d^3} \frac{3}{2} \|x\| \|x_*\| \]

We can then conclude that:

\[ \frac{1}{2} \langle G(x), G(x_*) \rangle^2 - \langle x, \tilde{h}_x \rangle^2 \leq \frac{1}{2} \langle x, A_{x_*) A_{x} x - \tilde{h}_x \rangle \]

\[ \|G(x)\| \|G(x_*)\| + \|x\| \|\tilde{h}_x\| \]

\[ \leq \frac{1}{2} \|x\| 24 \frac{d^3 \sqrt{\epsilon}}{2d^3} \|x_*\| \]

\[ \leq \frac{1}{2} \|x\| 24 \frac{d^3 \sqrt{\epsilon}}{2d^3} \|x_*\| \]

\[ \leq \frac{9}{2d^3} d^3 \sqrt{\epsilon} (\|x_*\|^4 + \|x\|^4) \]

\[ \square \]
Below we prove lower and upper bound on the loss $f_E$ as in Lemma 6.

**Proof of Lemma 6.** Let $x \in B(\phi_d x_* , a \|x_*\|)$ then observe that $0 \leq \tilde{\theta} \leq \bar{\theta}_0 \leq \pi a / 2 \phi_d$ and $(\phi_d - a)\|x_*\| \leq \|x\| \leq (a + \phi_d)\|x_*\|$.

Then observe that:

$$
\langle x, \tilde{h}_d \rangle = \frac{1}{2d}(\prod_{i=0}^{d-1} \frac{\pi - \theta_i}{\pi}) \|x_*\| \|x\| \cos \tilde{\theta}_0 + \frac{1}{2d} \sum_{i=0}^{d-1} \sin \theta_i \prod_{j=i+1}^{d-1} \frac{\pi - \theta_j}{\pi} \|x_*\| \|x\|
\geq \frac{1}{2d}(\prod_{i=0}^{d-1} \frac{\pi - \pi a / 2 \phi_d}{\pi}) (\phi_d - a)\|x_*\|^2 (1 - \frac{\pi^2 a^2}{8 \phi_d^2})
\geq \frac{1}{2d}(1 - \frac{da}{\phi_d}) (\phi_d - a) (1 - \frac{\pi^2 a^2}{8 \phi_d^2}) \|x_*\|^2.
$$

using $\cos \theta \geq 1 - \theta^2 / 2$ and $(1 - x)^d \geq (1 - 2dx)$ as long as $0 \leq x \leq 1$. We can therefore write:

$$
f_E(x) - \frac{\|x_*\|^4}{2^{d+2}} \leq \frac{1}{2^{d+2}} \|x\|^4 - \frac{1}{2^{d+1}} (1 - \frac{da}{\phi_d})^2 (\phi_d - a)^2 (1 - \frac{\pi^2 a^2}{8 \phi_d^2}) \|x_*\|^4
\leq \frac{1}{2^{d+2}} \left[ (\phi_d + a)^4 - 2(1 - 2\frac{da}{\phi_d}) (\phi_d - a)^2 (1 - \frac{\pi^2 a^2}{4 \phi_d^2}) \right] \|x_*\|^4
$$

where in the second inequality we used $(1 - x)^2 \geq 1 - 2x$ for all $x \in \mathbb{R}$. We then observe that:

$$(1 - 2\frac{da}{\phi_d}) (\phi_d - a)^2 (1 - \frac{\pi^2 a^2}{4 \phi_d^2}) \geq (1 - \frac{\pi^2 a^2}{4 \phi_d^2} - 2\phi_d - a)(\phi_d - a)(1 - \frac{\pi^2 a^2}{4 \phi_d^2})
\geq \frac{\phi_d^2}{a} - a(\frac{1}{2\phi_d^2} + 2\phi_d) + a(a - 2\phi_d)(1 - \frac{\pi^2 a^2}{4 \phi_d^2})
\geq \frac{\phi_d^2}{a} - a(\frac{1}{2\phi_d^2} + 2\phi_d + 2\phi_d)
\geq \frac{\phi_d^2}{a} - \pi da,
$$

where in the second inequality we have used $\pi^2 d^3 a \leq 2$ and in the last one $d \geq 2$ and $\phi_d \leq 1$. We can then conclude that:

$$
f_E(x) - \frac{\|x_*\|^4}{2^{d+2}} \leq \frac{1}{2^{d+2}} \left[ (\phi_d + a)^4 - 2(\phi_d^2 - \pi da) \right] \|x_*\|^4
$$

We next take $x \in B(-\phi_d x_* , a \|x_*\|)$ which implies $0 \leq \pi - \bar{\theta}_0 \leq \pi^2 a / 2 =: \delta$ and $\|x\| \leq (a + \phi_d)\|x_*\|$.

We then note that for $\xi$ and $\zeta$ as defined in (19) we have:

$$
\|2^d x^\top \tilde{h}_x\|^2 \leq (|\xi| + |\zeta|)^2 (a + \phi_d)^2 \|x_*\|^4
\leq (\frac{\delta}{\pi} + 3d^3 \delta + \rho_d)^2 (a + \phi_d)^2 \|x_*\|^4
\leq (\frac{\pi^2}{2} a + \rho_d)^2 (a + \phi_d)^2 \|x_*\|^4
\leq (2\pi^2 d^3 a + \rho_d^2)(a + \phi_d)^2 \|x_*\|^4
\leq 20\pi d^3 a + \rho_d^2 \phi_d^2
$$

where the second inequality is due to (29) and (30), the rest from $d \geq 2$, $\rho_d \leq \phi_d \leq 1$ and $2\pi^3 d^3 a \leq 1$.

Finally using $(\phi_d - a)\|x_*\| \leq \|x\|$, we can then conclude that:

$$
f_E(x) - \frac{\|x_*\|^4}{2^{d+2}} \geq \frac{1}{2^{d+2}} \left[ (\phi_d - a)^4 - 2(20\pi d^3 a + \rho_d^2 \phi_d^2) \right] \|x_*\|^4.
$$
B Proofs for the random spiked and generative models

We are now ready to prove our main results for random spiked models and generative networks with random weights. We begin by recalling the following fact on the WDC of a single Gaussian layer.

**Lemma 7** (Lemma 11 in [29]). Fix $0 < \epsilon < 1$ and suppose $W \in \mathbb{R}^{n \times k}$ has i.i.d. $\mathcal{N}(0, 1/n)$ entries. Then if $n \geq C_2 k \log k$, then with probability at least $1 - 8n \exp(-\gamma_0)$, $W$ satisfies the WDC with constant $\epsilon$. Here $C_\epsilon$ and $\gamma_0$ depend polynomially on $\epsilon^{-1}$.

By a union bound over all layers, using the above result we can conclude that the WDC holds simultaneously for all layers of the network with probability at least:

$$1 - \sum_{i=2}^{d} 8n_i \exp(-C_1 n_i^{-2} - 8n_1 e^{-C_2 \epsilon^2 \log(1/\epsilon)^k}).$$

Note in particular that this argument does not requires the independence of the layers.

By Lemma 7, with high probability the random generative network $G$ satisfies the WDC. Therefore if we can guarantee the assumptions on the noise term, then the proof of the main Theorem 2 follows from the deterministic Theorem 3 and the previous lemma.

Before turning to the bounds of the noise terms in the spiked models, we recall the following lemma which bounds the number of possible regions defined by a deep Relu network.

**Lemma 8** (Proof of Lemma 8 in [30]). Consider a network $G$ as defined in (3) with $d \geq 2$, weight matrices $W_i \in \mathbb{R}^{n_i \times n_{i-1}}$ with i.i.d. entries $\mathcal{N}(0, 1/n_i)$ and $\log(10) \leq k/4 \log(n_1)$. Then, with probability one, for any $x \neq 0$ the number of different matrices $\Lambda_x$ is:

$$|\{\Lambda_x : x \neq 0\}| \leq 10^{d^2} (n_1^{d-1} \ldots n_d)^k \leq (n_1^{d-1} \ldots n_d)^{2k}.$$

In the next section we use this lemma to control the noise term $\Lambda_x^H \mathcal{H} \Lambda_x$ where:

- in the Spiked Wishart Model $H = \Sigma_N - \Sigma$;
- in the Spiked Wigner Model $H = \mathcal{H}$.

We then conclude in section B.3 with the proof of Proposition 1.B.

B.1 Spike Wigner Model

Recall that in the Wigner model $Y = G(x_*)G(x_*)^T + \mathcal{H}$ and the symmetric noise matrix $\mathcal{H}$ follows a Gaussian Orthogonal Ensemble GOE($\nu$, $n$), that is $\mathcal{H}_{ij} \sim \mathcal{N}(0, 2\nu/n)$ for all $1 \leq i \leq n$ and $\mathcal{H}_{ij} \sim \mathcal{N}(0, \nu/n)$ for $1 \leq j < i \leq n$. Our goal is to bound $\|\Lambda_x^H \mathcal{H} \Lambda_x\|$ uniformly over $x$ with high probability.

Fix $x \in \mathbb{R}^k$, and let $\mathcal{N}_{1/4}$ be a $1/4$-net on the sphere $S^{k-1}$ such that $|\mathcal{N}_{1/4}| \leq 9^k$ and:

$$\|\Lambda_x^H \mathcal{H} \Lambda_x\| \leq 2 \max_{z \in \mathcal{N}_{1/4}} |\langle \Lambda_x^H \mathcal{H} \Lambda_x z, z \rangle|.$$

For any $z \in \mathcal{N}_{1/4}$ let $\ell_{x,z} := \Lambda_x z \in \mathbb{R}^n$ and note that by the assumption on the entries of $\mathcal{H}$ it holds that $\ell_{x,z}^H \mathcal{H} \ell_{x,z} \sim \mathcal{N}(0, \nu^2 \|\ell_{x,z}\|^4/n)$. In particular by Lemma 1, the quadratic form $\ell_{x,z}^H \mathcal{H} \ell_{x,z}$ is sub-Gaussian with parameter $\gamma_2$ given by:

$$\gamma_2 := \frac{\nu^2}{n} \left(\frac{13}{12}\right)^2 \frac{1}{2^{2d}}.$$
Then for fixed $x \in \mathbb{R}^k$, standard sub-Gaussian tail bounds and a union bound over $\mathcal{N}_{1/4}$ give:

$$
P[\|\Lambda_x^T \mathcal{H} x\| \geq 2u] \leq \mathbb{P} \left[ \max_{z \in \mathcal{N}_{1/4}} \|\ell_x^T \mathcal{H}^2 x\| \geq u \right] \leq \sum_{z \in \mathcal{N}_{1/4}} \mathbb{P}[\|\ell_x^T \mathcal{H}^2 x\| \geq u] \leq 2 \cdot 9^k e^{-\frac{u^2}{2}}.
$$

Lemma 8, then ensures that the number of possible $\Lambda_x$ is at most $(n_1^d n_2^{d-1} \cdots n_d)^{2k}$, so a union bound over this set allows to conclude that:

$$
P[\|\Lambda_x^T \mathcal{H} x\| \geq 2u] \leq \nu \sqrt{\frac{30k \log(3n^d n_2^{d-1} \cdots n_d)}{n}}, \text{ for all } x \geq 1 - 2e^{-k \log(n)}.
$$

### B.2 Spike Wishart Model

Recall that the data $\{y_i\}_{i=1}^N$ are i.i.d. samples from $\mathcal{N}(0, \Sigma)$ where $\Sigma = G(x^*) G(x^*)^T + \sigma^2 I_n$. In the minimization problem (4) we take $\mathcal{H} = \mathcal{N} - \sigma^2 I_n$, where $\mathcal{N}$ is the empirical covariance matrix. The symmetric noise matrix $H$ is then given by $H = \mathcal{N} - \Sigma$ and by the Law of Large Numbers $H \to 0$ as $N \to \infty$. We bound $\|\Lambda_x^T \mathcal{H} x\|$ with high probability uniformly over $x \in \mathbb{R}^k$.

Fix $x \in \mathbb{R}^k$, let $\mathcal{N}_{1/4}$ be a $1/4$-net on the sphere $S^{k-1}$ such that $|\mathcal{N}_{1/4}| \leq 9^k$, and notice that:

$$
\|\Lambda_x^T \mathcal{H} x\| \leq 2 \max_{z \in \mathcal{N}_{1/4}} |z^T \Lambda_x^T \mathcal{H} x z|.
$$

By a union bound on $\mathcal{N}_{1/4}$ we obtain for any fixed $z \in \mathcal{N}_{1/4}$:

$$
P[\|\Lambda_x^T \mathcal{H} x\| \geq 2u] \leq 9^k \mathbb{P}[|z^T \Lambda_x^T \mathcal{H} x z| \geq u].
$$

Let $\ell_x := \Lambda_x z$ and note that:

$$z^T \Lambda_x^T \mathcal{H} x z = \frac{1}{N} \sum_{i=1}^N (\ell_x^T y_i)^2 - \mathbb{P}[|\ell_x^T y_i|^2]
$$

Since $\nu_i := \ell_x^T y_i \sim \mathcal{N}(0, \gamma^2)$ where $\gamma^2 = \ell_x^T \Sigma \ell_x$, then for $u \leq (n_1^d n_2^{d-1} \cdots n_d)^{2k}$, then proceeding as for the Wigner case by a union bound over all possible $\Lambda_x$:

$$
P[\|\Lambda_x^T \mathcal{H} x\| \geq 2u] \leq 9^k \mathbb{P}[\left| \frac{1}{N} \sum_{i=1}^N (\nu_i / \gamma)^2 - 1 \right| \geq \frac{u}{\gamma^2}] \leq 2 \exp \left[ k \log 9 - \frac{N u^2}{8 \gamma^4} \right].
$$

Recall now that $|\{\Lambda_x | x \neq 0\}| \leq (n_1^d n_2^{d-1} \cdots n_d)^k$, then proceeding as for the Wigner case by a union bound over all possible $\Lambda_x$:

$$
P[\|\Lambda_x^T \mathcal{H} x\| \geq \sqrt{\frac{24k \log(3n^d n_2^{d-1} \cdots n_d)}{n}} \gamma^2, \text{ for all } x \geq 1 - 2e^{-k \log(3n)}
$$

Finally observe that $\|\Lambda_x\| \leq \frac{13}{12} \frac{1}{2^t}$. Combining this inequality with the bound on the spectral norm of $\Sigma$, gives the final result.
B.3 Proof of Proposition 1.B

We recall now the following fact on the local Lipschitz property of the generative network.

Lemma 9 (Lemma 21 in [30]). Suppose \( x \in B(x_\star, d\sqrt{\epsilon\|x\|}) \), and the WDC holds with \( \epsilon < 1/(200)^{4/6} \). Then it holds that:

\[
\|G(x) - G(x_\star)\| \leq \frac{1.2}{2^{d/2}}\|x - x_\star\|.
\]

The proof of Proposition 1.B follows now from the above Lemma together with the bounds (15) and (17), the assumptions on \( \epsilon \) and the noise term.