RIGIDITY OF NEWTON DYNAMICS

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Abstract. We study rigidity of rational maps that come from Newton’s root finding method for polynomials of arbitrary degrees. We establish dynamical rigidity of these maps: each point in the Julia set of a Newton map is either rigid (i.e. its orbit can be distinguished in combinatorial terms from all other orbits), or the orbit of this point eventually lands in the filled-in Julia set of a polynomial-like restriction of the original map. In particular, the Julia set is locally connected everywhere except possibly where it is renormalizable. As a corollary we show that for arbitrary Newton maps the boundary of every component of the basin of a root is locally connected.

In the parameter space of Newton maps of arbitrary degree we obtain the following rigidity result: any two combinatorially equivalent Newton maps are quasiconformally conjugate in a neighborhood of their Julia sets provided that they either non-renormalizable, or they are both renormalizable “in the same way”.

Our main tool is the concept of complex box mappings due to Kozlovski, Shen, van Strien; we also extend a dynamical rigidity result for such mappings so as to include irrationally indifferent or renormalizable situations.

1. Introduction and main results

1.1. Local connectivity, topological models, and rigidity. We investigate the fine structure in the dynamical systems formed by iteration of Newton maps of polynomials: the goal is to show that any two points within any given dynamical system can be distinguished in combinatorial terms (“dynamical rigidity”), and similarly that any two Newton dynamical systems can be distinguished combinatorially as well (“parameter rigidity”). Analogous rigidity results are known to be false for polynomial dynamics, and our main result is that they hold for the Newton dynamics everywhere except when embedded polynomial dynamics interferes (both in the dynamical plane and in parameter space). These results are strongest possible: embedded non-rigidity of polynomial dynamics makes rigidity in the Newton dynamics impossible.

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This research connects to and builds upon a deep body of research on polynomial dynamics, initiated by Douady and Hubbard in their seminal Orsay Notes [DH1] and extended in celebrated work by Yoccoz [H], Lyubich and coauthors (see e.g. [L1, DL]), Kozlovski–van Strien [KvS1], and numerous others. The goal in much of this work is often phrased as showing that polynomial Julia sets are locally connected (many of these are, but not all; see for instance Milnor [M1]). The importance of local connectivity of Julia set comes from several closely connected aspects:

- when Julia sets are locally connected, they have simple and satisfactory topological models, for instance in terms of Thurston laminations [T, S5] or Douady’s pinched disks [Do];
- any two points in the Julia sets can be distinguished in terms of symbolic dynamics, for instance in the complement of pairs of dynamic rays that land at common periodic or preperiodic points.

For instance, Yoccoz’ theorem on quadratic polynomials can be phrased as saying that all quadratic polynomials that are non-renormalizable and for which both fixed points are repelling have locally connected Julia sets, or equivalently that any two points in the Julia set can be distinguished in terms of their itineraries with respect to the single non-dividing fixed point (usually called the $\alpha$ fixed point).

Meanwhile, it is known that a polynomial Julia set is locally connected, and all its points can be distinguished in terms of symbolic dynamics, when the dynamics is not infinitely renormalizable and no periodic points are irrationally indifferent [KvS1]. In many cases with irrationally indifferent periodic points, or in the infinitely renormalizable setting, the Julia sets are locally connected anyway (compare e.g. [DR]); however, there are explicitly known examples when local connectivity fails, especially in the presence of Cremer points [M2, § 18] and in certain infinitely renormalizable cases [M1].

The research on local connectivity of polynomial Julia sets is among the deepest in all of dynamical systems. It has often been thought that the dynamics of rational maps must be even more complicated because polynomials have a basin of infinity that provides a simple and good coordinate system for the study of the dynamics, in particular through dynamic rays and their landing properties. In this paper, we propose a rather opposite point of view, at least for the dynamics of rational maps that are Newton maps of polynomials, that we phrase as the following principle:

**Rational Rigidity Principle** (dynamical version). *In the dynamics of any polynomial Newton map, the Julia set at any given point $z$ is locally connected, and the dynamics of $z$ can be distinguished by symbolic dynamics from any other point $z'$, unless the Newton dynamics is renormalizable and admits an embedded polynomial Julia set that fails to be locally connected or fails to be combinatorially rigid, and that contains the point $z$ (as well as $z'$).*

Here we say that a polynomial Julia set is embedded in the Newton dynamics when the latter is renormalizable and a domain of renormalization has a Julia set that is quasiconformally conjugate to the given polynomial Julia set.
There is a parallel discussion in parameter space that has also started with the work by Douady and Hubbard [DH1] on the Mandelbrot set:

- if the Mandelbrot set is locally connected, then it has a simple and satisfactory topological model, for instance in terms of Thurston laminations [T, S5] and Douady’s pinched disks [Do];
- if the Mandelbrot set is locally connected, then any two parameters in its boundary (the bifurcation locus) have Julia sets that can be distinguished in terms of symbolic dynamics;
- if the Mandelbrot set is locally connected, then hyperbolic dynamics is open and dense in the space of quadratic polynomials.

For spaces of polynomial maps beyond quadratic polynomials, local connectivity is not the right concept (see a discussion below); instead the goal is to establish rigidity for instance in the form that any two polynomials for which the Julia sets are combinatorially indistinguishable are already quasiconformally conjugate. This rigidity conjecture is false in general, but it holds for instance when the polynomial dynamics is not renormalizable. Again, the study of parameter spaces of rational maps seems harder than for polynomials, but still we propose an analogous rigidity principle also in parameter space:

**Rational Rigidity Principle** (parameter space version). *Any two polynomial Newton maps that are combinatorially equivalent are quasiconformally equivalent, provided these Newton maps are either non-renormalizable, or they are both renormalizable “in the same way”: the little Julia sets are hybrid equivalent and embedded into the Newton dynamics in combinatorially the same way.*

Both versions of our rational rigidity principle, in the dynamical plane and in parameter space, can be interpreted as saying that “the Newton dynamics behaves well unless embedded polynomial dynamics interferes”, so contrary to frequent belief the dynamics of rational maps does not exhibit any additional complications beyond those known from polynomials, once a good combinatorial structure is established — at least in the case of polynomial Newton maps, which are the first family of rational maps for which a good combinatorial structure has been established [LMS1, LMS2].

1.2. **Statement of results on rigidity.** After this overview, we now provide a more precise statement of results. The Newton map of a polynomial $p: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined to be the rational map $N_p(z) := z - p(z)/p'(z)$; we call such a map a polynomial Newton map.

Our goal is to distinguish all orbits of $N_p$ in combinatorial terms; more precisely, in terms of symbolic dynamics. For polynomials, this is issue is closely related to the topology of the Julia set, in the sense that in many cases the distinction of all orbits is possible when the Julia set is locally connected. In analogy to [S1, S3], we define the fiber of a point $z \in \hat{\mathbb{C}}$ as the set of points whose orbits are combinatorially indistinguishable from that of $z$; this is a compact connected set (see Definition 2.3 for a precise definition in general, and Section 4 specifically for Newton maps). We say that the fiber of $z$ is trivial if it consists of $z$ alone. Providing sufficient conditions for trivial fibers is one of the chief goals of this paper.
The purpose of Newton’s method is to find roots of \( p \): every root is an attracting fixed point of \( N_p \). The points that converge to these roots are known as the basins of the roots. They can be classified in terms of Newton graphs that connect the connected components of these basins to the roots; see [LMS1, LMS2]. From the dynamics point of view, the other points are more interesting: these points are either in the Julia set \( J(N_p) \) or are contained in Fatou components that eventually have period 2 or higher (note that for every polynomial Newton map every Fatou component of period 1 is a basin of a root because every fixed point is either attracting or the repelling fixed point at \( \infty \), and there cannot be Herman rings either [Sh]).

Our first main theorem (Theorem A) says that for every polynomial Newton map every point that is not attracted to a root can fail to have trivial fiber only if it belongs to (or is mapped to) an embedded quasiconformal copy of the filled Julia set of an actual polynomial mapping. This is the dynamical version of the Rational Rigidity Principle for Newton maps: rational Newton maps are dynamically rigid (have trivial fibers) except where polynomial dynamics interferes.

**Theorem A** (Dynamical Rigidity for Newton maps). Let \( N_p \) be a polynomial Newton map of degree \( d \geq 2 \). Then for every point \( z \in \hat{\mathbb{C}} \) at least one of the following possibilities holds true:

(B) \( z \) belongs to the Basin of attraction of a root of \( p \);

(T) \( z \) has Trivial fiber;

(R) \( z \) belongs, or is mapped by some finite iterate, to the filled Julia set of Renormalizable dynamics (a polynomial-like restriction of \( N_p \) with connected Julia set).

Theorem A immediately implies the following corollary.

**Corollary 1.1** (Local connectivity or renormalizable dynamics). Let \( N_p \) be a polynomial Newton map and denote its Julia set by \( J(N_p) \). Then for every \( z \in J(N_p) \), the set \( J(N_p) \) is locally connected at \( z \), except possibly when \( z \) belongs to, or is mapped to, the Julia set of some renormalizable polynomial-like restriction of \( N_p \). \( \square \)

**Remark.** Many polynomial Julia sets are known to be locally connected, and even have all their fibers trivial, so the corresponding results can be imported to the Newton dynamics: if \( z \) belongs to some renormalizable polynomial-like restriction of \( N_p \) for which the polynomial Julia set has trivial fiber at the point corresponding to \( z \), then the Julia set of \( N_p \) has trivial fiber at \( z \), and in particular is locally connected at \( z \). The idea of proof for this statement is that two points \( w, w' \) in a polynomial Julia set are in different fibers if and only if there is a pair of (pre)periodic dynamic rays landing at the same point in the polynomial Julia set that separates \( w \) from \( w' \), and within the dynamics of \( N_p \) these rays can be replaced by “bubble rays” consisting of sequences of components of basins that converge to the same landing point with the same separation properties. (Note, however, that it is not clear that if a polynomial Julia set is locally connected at some point, then its fiber must be trivial, and this point can be separated from every other point in the Julia set; compare [S1, S3].)

There are polynomial Julia sets that are known not to be locally connected (for instance those having Cremer points, as well as some infinitely renormalizable polynomials). If such
polynomial Julia sets happen to be ‘embedded’ in the Newton dynamical plane, the Julia set of the Newton map may or may not be locally connected at the corresponding points. Roesch [R] has shown that cubic Newton maps are locally connected in many cases even when they are renormalizable and the corresponding (quadratic) Julia sets are not.

**Remark.** The degree of a Newton map $N_p$ is always equal to the number of distinct roots of $p$ (ignoring multiplicities). It is well known that if $N_p$ has degree $d = 2$, then the Julia set $J(N_p)$ is a quasi-circle through $\infty$ (in particular, it is a straight line if $p$ has degree 2 as well), and the two complementary domains are the basins of the two roots. This case is trivial, and the case $d = 1$ is even more trivial, so they are excluded from our discussions and we assume $d \geq 3$.

Another corollary of the dynamical rigidity of Newton maps (Theorem A) is the following result; a proof can be found in Section 5.

**Corollary 1.2 (Boundary of the root basin locally connected).** For every Newton map, every component in the basin of any root has locally connected boundary.

**Remark.** Note that in general the boundaries of the components of the basins of roots are not simple curves: they are pinched when the corresponding immediate basins have more than one access to $\infty$. This may happen for all degrees $d \geq 3$.

Our second main result (Theorem B) is a parameter space counterpart to Theorem A. We say that two Newton maps are *combinatorially equivalent* if their Newton graphs coincide (see Definition 6.1 for details); an equivalent way of saying this is that the all the components of the basins of the roots are connected to each other in the same way (some examples are shown in Figure 1).

**Theorem B (Parameter rigidity for Newton maps).** If two polynomial Newton maps are combinatorially equivalent, then they are quasiconformally conjugate in a neighborhood of the Julia set provided

1. either they are both non-renormalizable,
2. or they are both renormalizable, and there is a bijection between their domains of renormalization that respects hybrid equivalence between the little Julia sets as well as their combinatorial position.

The domain of this quasiconformal conjugation can be chosen to include all Fatou components not in the basin of the roots, and its antiholomorphic derivative vanishes on those Fatou components as well as on the entire Julia set.

Moreover, if these Newton maps are normalized so that they attracting-critically-finite, then they are even affine conjugate.

The conditions in the renormalizable case mean the following: the renormalizable “little Julia sets” should correspond to the same polynomial dynamics (up to a quasiconformal conjugation that is conformal on the filled-in Julia set of the polynomials), and they should be connected to the Newton graph (i.e. the graph consisting of all components of the basins of the roots) at the same combinatorial position. This will be made precise in Section 6.
Finally, a Newton map is called *attracting-critically-finite* if the orbit of every critical point in the basin of a root is eventually fixed; this can be accomplished by a routine quasiconformal surgery on a compact subset of the basins of the roots (see Section 4).

Figure 1. The dynamical planes of various degree four Newton maps; the basins of different roots are shown in different colors. Different Newton maps can often be distinguished combinatorially in terms of the combinatorial structure of touching components of the basins of the roots. Renormalizable parts of the dynamical plane are shown in black.

1.3. Box mappings. One of our key tools are *complex box mappings* as introduced by Kozlovski and van Strien [KvS1]. These maps are natural generalizations of polynomial-like maps to the case when the domains of definition and the ranges are disconnected. For any point $z$ in such a box mapping, the fiber $\text{fib}(z)$ is the set of points that have the same symbolic dynamics as $z$ with respect to the connected components for its domain of definition: that is, the set of points with the same itinerary through these connected components. Again, a precise definition will come later (Section 3). Kozlovski and van Strien show that certain box mappings have all their fibers trivial (in different language; see [KvS1, Theorem 1.4 (1)] and also [KvS2]). Our third main result (Theorem C) is an upgrade to their theorem: our result applies to all box mappings and provides sufficient conditions for most individual fibers to be trivial. Similarly as for polynomial-like maps, some points can only be iterated finitely many times; we say that such points *escape* (from the box mapping).

**Theorem C** (Generalized rigidity for complex box mappings). Consider an arbitrary box mapping and an arbitrary non-escaping point $z$. Then at least one of the following cases occurs:

- (T) $z$ has **Trivial fiber**;
- (R) $z$ belongs, or is mapped by some finite iterate, to the filled Julia set of **Renormalizable dynamics** (a polynomial-like restriction of the given box mapping with connected Julia set);
the domain of the box mapping contains a periodic component that maps onto itself by some iterate of the box map, and \( z \) eventually maps to such a component (so the fiber of \( z \) is equal to the closure of its component); \( (NE) \)

(CB) the orbit of \( z \) converges the boundary of the domain of definition of the box mapping.

Observe that the first two possibilities here exactly match the two possibilities in Theorem A for points not in basins of the roots. The last two are “pathological” cases that are admitted by the fairly general definition of box mappings (see Definition 3.1), but do not naturally arise in many cases where box mappings are extracted from dynamical systems on \( \hat{\mathbb{C}} \) (see also \( [KvS2] \)). This will be exactly the case in the proof of Theorem A. The letters \( (NE) \) denote an orbit that lands in a periodic component with no escape: all points on this component remain there forever. A periodic component without escaping points will be called an \( (NE) \) component.

The concept of renormalization (in the sense of Douady–Hubbard, as well as of Kozlovski–van Strien) is discussed in Section 3.

Earlier work on Newton’s method. Newton’s method as a dynamical system has been studied by various people for a long time, in many cases with a focus on the cubic case: in this case there is a single free critical point and the parameter space is complex one-dimensional, like the well-studied case of the dynamics of quadratic polynomials and the Mandelbrot set. In particular, we would like to mention the classical work by Tan Lei \([TL]\) with a combinatorial study of the Newton parameter space, with a recent refinement by Roesch, Wang, and Yin in \([RWY]\). In \([R]\), Roesch has shown that the Newton map of a cubic polynomial has locally connected Julia set in many cases, even when it is renormalizable and the embedded polynomial Julia set is not locally connected.

More recently, there are two manuscripts that study Newton’s method of arbitrary degrees in a similar spirit as we do here. The main result of Wang, Yin, and Zeng \([WYZ]\) is the result that immediate basins have locally connected boundary, similar to our Corollary 1.2. Roesch, Yin, and Zeng show in \([RYZ]\) that all non-renormalizable Newton maps are rigid (in parameter space); this corresponds to our Theorem B in the special case of non-renormalizability.

There is also recent work on Newton’s method as an efficient root finder. Among the early results are a paper by Przytycki \([Prz]\) that shows that immediate basins are always simply connected, and one by Manning \([Ma]\) that shows where to start the Newton iteration to find some “exposed roots”. A sufficient small set of starting points that always finds all roots was constructed in \([HSS]\) with a refinement in \([BLS]\). Estimates on the necessary number of iterations were given in \([S2, S6, BAS]\); see also the overview in \([S4]\). Finally, experiments that highlight the efficiency of Newton’s method for certain polynomials of degrees exceeding one billion were described in \([SS1, RSS]\).

Notation. In order to lighten notation, we write \( f^k \) for the \( k \)-fold iterate of a map \( f \), that is \( f^k := f \circ f \ldots \circ f \), \( k \) times.
We will also write $\text{Crit}(f)$ for the set of critical points of a map $f$, and put $\text{orb}(z) := \{f^k(z) : k \geq 0\}$ for the orbit of a point $z$ under the dynamics of $f$.

2. ON GENERAL PUZZLES

In this preparatory section we start with some general discussion and fix terminology concerning puzzles. In the end of the section we will prove triviality of fibers in some fairly general cases. These auxiliary results will be used later in the proof of the generalized rigidity principle — Theorem C.

Let $g: U \to V$ be a holomorphic map between two open sets $U \subseteq V \subset \hat{\mathbb{C}}$ so that connected components of $U$ resp. $V$ have disjoint closures; we do not require that $U$ or $V$ be simply connected. We describe a setting of puzzles in the spirit of the well known Yoccoz puzzles, adapted to the needs of our Newton dynamics. Suppose that there exists a nested sequence $(S_n)_{n=0}^\infty$ of open sets such that $U = S_0 \supset S_1 \supset S_2 \ldots$, every component of $S_{n+1}$ is either compactly contained in or coincide with the corresponding component of $S_n$ and for every $n \geq 0$ the restriction $g: S_{n+1} \to S_n$ is a proper map. Further assume that the closure of each $S_n$ can be represented as a (not necessarily finite) union of closed topological disks $P_i^n$ ($i$ runs over some finite or countable index set $I_n$) that can only intersect along their boundaries. We call each $P_i^n$ a puzzle piece of depth $n$; the union of all puzzle pieces of depth $n$ comprises the puzzle partition (of $S_n$) of depth $n$. We will also call the topological graph $\Gamma_n := \bigcup_{i \in I_n} \partial P_i^n$ the puzzle boundary of depth $n$: vertices of this graph are either points on $\partial S_n$ where at least two puzzle pieces meet, or points in $S_n$ where at least three puzzle pieces meet (note here that $\partial S_n \subset \Gamma_n$ for every $n$); an edge of $\Gamma_n$ is a vertex-free set homeomorphic to an interval that connects two vertices. For simplicity we assume that all edges in all $\Gamma_n$ are smooth and the boundary of each puzzle piece contains finitely many vertices. (In the special case that $Y$ is a component of $S_n$ such that $Y = P_i^n$ for some $i$, this definition of edges and vertices does not apply; in this case, we choose an arbitrary point on $\partial Y$ as a vertex and let the rest of $\partial Y$ be an edge that connects the vertex to itself.)

**Remark.** The previous paragraph describes, in fairly large generality, a construction that includes not only the setting of the well known Yoccoz puzzles (where each $S_n$ consists of finitely many puzzle pieces), but it also caters for two settings that will be specified in the upcoming sections. First, we will be interested in puzzles for complex box mappings (see Definition 3.1). In that case, $g$ will be a box mapping, where each $S_n$ will be a (possibly infinite) union of open topological disks with disjoint closures; the closure of each of the disks will serve as a puzzle piece of depth $n$. In other words, for a box mapping and for any given $n$ the set of puzzle pieces of depth $n$ equals the set of closures of the connected components of $S_n$; see Definition 3.2 for details. For the second time the construction in the previous paragraph will be specified for polynomial Newton maps $N_p$ (see Section 4). There $g$ will stand for a particularly chosen iterate of the Newton map, while $S_n$ will be the Riemann sphere minus finitely many suitably chosen closed topological disks bounded by equipotentials in the respective basins of roots of $p$; each of these removed disks is a neighborhood of either a root of $p$ or an iterated preimage of such a root for a bounded number of iterations (see Definition 4.3 for details).
Definition 2.1 (Markov property). The union of all puzzle pieces of all depths has the Markov property if:

1. any two puzzle pieces are either nested or have disjoint interiors; in the former case the puzzle piece of bigger depth is contained in the puzzle piece of a smaller depth;
2. the image of each puzzle piece \( P^i_n \) of depth \( n > 0 \) is a puzzle piece \( P^j_{n-1} \) of depth \( n - 1 \), and the restriction \( g: P^i_n \to P^j_{n-1} \) is a branched covering.

Equivalently, the Markov property can be stated in terms of puzzles: the union of all puzzle pieces of all depths has the Markov property if

\[ g(\Gamma_n \cap S_{n+1}) \subset \Gamma_n \]
for all \( n \geq 0 \). (Note that for puzzles coming from box mappings this condition is automatically satisfied since \( \Gamma_n \cap S_{n+1} = \emptyset \), see Definition 3.2).

We will say that \( g \) is a holomorphic map with well-defined Markov partition if \( g: U \to V \), with \( U \subseteq V \subseteq \hat{\mathbb{C}} \), is a holomorphic map as described at the beginning of the section and for which there exists a nested sequence of open sets \( U = S_0 \supset S_1 \supset \ldots \) with a well-defined puzzle partition into puzzle pieces the union of which has the Markov property.

Definition 2.2 (Puzzle piece centered at a point). Given a point \( x \in S_n \), define \( P_n(x) \) to be the union of all puzzle pieces of depth \( n \) containing \( x \).

From the definition above it is clear that if \( x \) is not on the boundary of any puzzle piece of depth \( n \) (equivalently, if \( x \notin \Gamma_n \)), then \( P_n(x) \) is the unique puzzle piece of depth \( n \) containing the point \( x \). Otherwise, \( P_n(x) \) is a union of puzzle pieces with \( x \) in their common boundary. Note that these sets do not form a Markov partition: it may be that \( P_n(x) \) and \( P_n(y) \) are different with intersecting interiors if \( x \) or \( y \) are in \( \Gamma_n \). However, it is still true that the restriction \( g: P_n(x) \to P_{n-1}(g(x)) \) is a branched covering.

Let us spell out an elementary argument that will be used several times below without explicit mention. If \( x \) is a point that does not belong to the puzzle boundary of any depth, then every point on the orbit of \( x \) also does not belong to the puzzle boundary of any depth, and hence \( P_n(g^k(x)) \) is the puzzle piece (of depth \( n \)) for all \( n \) and \( k \) (as long as \( x \) can be iterated \( k \) times).

We say that a point \( x \in U \) escapes if \( x \in S_n \setminus \overline{S}_{n+1} \) for some \( n \geq 0 \). Thus the set of non-escaping points of \( g \) (the non-escaping set of \( g \)) is precisely \( \bigcap_{n=0}^{\infty} \overline{S}_n \); this is the set of points that can be iterated infinitely often.

Definition 2.3 (Fiber, trivial fiber). For a non-escaping point \( x \), the set

\[ \text{fib}(x) := \bigcap_{n \geq 0} P_n(x) \]

is called the fiber of \( x \) (with respect to the partition of \( S_n \)). We say that \( x \) has trivial fiber if \( \text{fib}(x) = \{x\} \).
The Markov property of puzzle partitions is a powerful combinatorial property allowing us to study maps from the point of view of symbolic dynamics. We define the itinerary at level \( n \) of a point \( x \) as the sequence \( (P_n(g^i(x)))_{i=0}^{\infty} \). Two points have the same itinerary if their itineraries at all levels coincide. In particular, the fiber \( \text{fib}(x) \) consists of all points that have the same itineraries at all levels. In other words, \( \text{fib}(x) \) consists of all points that are dynamically indistinguishable with respect to our puzzle partition. Hence, if the fiber is trivial, then the dynamics of \( g \) at \( x \) is rigid: there is no point other than \( x \) with the same itinerary as \( x \).

We will also say that the fiber \( \text{fib}(x) \) is periodic if \( x \) has periodic itinerary; this property is independent of a particular choice of a point in the fiber.

A point \( x \) is called combinatorially recurrent if \( x \) does not belong to the puzzle boundary of any depth and the orbit of \( g(x) \) under \( g \) intersects \( \bar{P}_n(x) \) for every \( n \). This implies that the orbit of \( g(x) \) intersects every \( \bar{P}_n(x) \) infinitely often: if this was not true, then for some \( n \) there was a largest \( k \) with \( g^k(x) \in \bar{P}_n(x) \), so \( g^k(x) \in \bar{P}_m(x) \) for all \( m \geq n \); hence \( g^k(x) \in \text{fib}(x) \). But then \( g^j(x) \) and \( g^{k+j}(x) \) are in the same fiber for every \( j \geq 0 \), so \( g^j(x) \in \text{fib}(x) \) for all \( j \).

We start by a pair of fairly standard technical lemmas, and then proceed by proving some of the simplest cases when fibers are trivial.

**Lemma 2.4** (Annulus pull-back under branched covering). Let \( f: Y' \to Y \) be a branched covering of degree at most \( D \) between two closed topological disks. Suppose \( A \subset Y \) is an open annulus with \( \text{mod}(A) = \mu \), and assume that an annulus \( A' \subset Y' \) is a component of \( f^{-1}(A) \). Then \( \text{mod}(A') \geq \mu/D^2 \).

**Proof.** The branched cover \( f: Y' \to Y \) has at most \( D-1 \) critical points. Hence the annulus \( A \) has a parallel sub-annulus \( B \) of modulus \( \mu/D \) that avoids all critical values (recall that \( B \) is a parallel sub-annulus of an annulus \( A \) if a biholomorphic map that uniformizes \( A \) to a round annulus \( A_0 \) sends \( B \) to a concentric round sub-annulus \( B_0 \) of \( A_0 \)). Then all \( f \)-preimages of \( B \) are annuli that map to \( B \) by unbranched covering maps of degrees at most \( D \). One of them, say \( B' \), is an essential sub-annulus in \( A' \), and thus \( \text{mod}(A') \geq \text{mod}(B') \geq \mu/D^2 \). \( \square \)

**Remark.** In fact, in the previous lemma one can prove the stronger bound \( \text{mod}(A') \geq \mu/D \), see [KL2, Lemma 4.5].

**Lemma 2.5** (First entry maps have uniformly bounded degrees). Let \( g \) be a holomorphic map with a well-defined Markov partition. Then there exists a constant \( D \in \mathbb{N} \) with the following property: for every puzzle piece \( Y \) of any depth \( n \) and for every point \( z \) that does not belong to the puzzle boundary of any depth, if \( k \geq 1 \) is the least index so that \( g^k(z) \in Y \), then the map \( g^k: P_{n+k}(z) \to P_n(g^k(z)) = Y \) has degree bounded by \( D \) (independently of \( n \) and \( k \)).

**Proof.** Consider the sequence of puzzle pieces \( (P_{n+k-i}(g^i(z)))_{i=0}^{k-1} \). We claim that these \( k \) puzzle pieces have disjoint interiors. If not, then by the Markov property (Definition 2.1) we have \( P_{n+k-i}(g^i(z)) \subset P_{n+k-j}(g^j(z)) \) for some \( i < j < k \) and \( g^{k-j}(P_{n+k-j}(g^j(z))) = \)
$P_n(g^k(z)) = Y$, hence $g^{k-j}(P_{n+k-i}(g^j(z))) \subset Y$, so $g^{k-j+i}(z) \in Y$ in contradiction to minimality of $k$ (see Figure 2).

**Figure 2.** An illustration on how to conclude contradiction to minimality of $k$ in the proof of Lemma 2.5.

In particular, each critical point of $g$ can lie in the interior of at most one puzzle piece in this sequence. Therefore, the claim follows with $D$ equal to the product of the degrees of all critical points of $g$. \hfill \qed

The following lemma, in a similar manner as Lemma 2.5, gives us control over the degree of the first entry map to the union of puzzle pieces that contains all critical puzzle pieces. We will say that a fiber is critical if it contains a critical point.

**Lemma 2.6** (First entry to union of critical puzzle pieces has uniformly bounded degree). Let $(x_i)_{i \in I}$ be a finite set of points with distinct fibers which includes all critical fibers of $g$. Suppose that there exists a depth $m \geq 0$ so that all puzzle pieces $P_m(x_i)$ of depth $m$ are pairwise disjoint, and an integer $s > 0$ so that all $(P_m(x_i) \setminus P_{m+s}(x_i))_{i \in I}$ are non-degenerate annuli. Then there is a constant $\mu > 0$ with the following property: for every $y \in U$ for which there exists a $k \geq 0$ so that $g^k(y) \in \bigcup_i P_{m+s}(x_i)$, let $k = k(y)$ be minimal with this property; then there exists an essential open annulus $A \subseteq \hat{P}_{m+k}(y) \setminus P_{m+s+k}(y)$ such that $\text{mod}(A) \geq \mu$.

**Proof.** Consider an arbitrary $y' \in U$ for which there exists a $k' \leq s$ so that $g^{k'}(y') \in \bigcup_i P_{m+s}(x_i)$, and suppose again that $k'$ is minimal with this property, and that $y'$ is not on the boundary of a puzzle at any depth. To fix notation, suppose that $x_0$ is a point in $(x_i)_{i \in I}$ with $g^{k'}(y') \in P_{m+s}(x_0)$.

We claim that then the set $\hat{P}_{m+k'}(y') \setminus P_{m+s+k'}(y')$ contains an annulus that separates $P_{m+s+k'}(y')$ from $\partial P_{m+k'}(y')$ and that has modulus bounded below.

We have $P_{m+s}(g^{k'}(y')) = P_{m+s}(x_0)$ and hence $\hat{P}_m(g^{k'}(y')) \setminus P_{m+s}(g^{k'}(y')) = \hat{P}_m(x_0) \setminus P_{m+s}(x_0)$, and by hypothesis this is a non-degenerate annulus of some modulus, say $\mu(x_0) > 0$.

Now we take a preimage of this annulus under $g^{k'}$. The map $g^{k'}$ sends $\hat{P}_{m+k'}(y')$ to the puzzle piece $\hat{P}_m(g^{k'}(y')) = \hat{P}_m(x_0)$ at depth $m$, and this is a branched cover of degree bounded in terms of $m$ and $k' \leq s$ and the degrees of the critical points of $g$. 


Therefore, 
\[ g^{-k'} \left( \hat{P}_m(g^{k'}(y')) \cup P_{m+s}(g^{k'}(y')) \right) \cap \hat{P}_{m+k'}(y') \]
will in general not be an annulus, but an open disk with several closed disks removed. However, it does contain an annulus that separates \( P_{m+s+k'}(y') \) from \( \partial P_{m+k'}(y') \), and that is an essential annulus with modulus bounded below in terms of \( \mu(x_0), m, s \) and the degrees of the critical points of \( g \). Since there are only finitely many \( x_i \), this modulus is bounded below by a number \( \mu > 0 \) that depends on \( g, s, m \) and the set \( \{x_i\} \), but not on \( y' \).

Now consider a point \( y \) for which there exists a minimal \( k = k(y) \) as required in the lemma, and so that \( y \) is not on the puzzle boundary at any depth. If \( k \leq s \), then \( y \) is one of the \( y' \) discussed earlier.

The proof for the case \( k > s \) is similar to Lemma 2.5. Again, consider the “orbit of puzzle pieces” \( \left( P_{m+k-t}(g^t(y)) \right)^k_{t=0} \). For \( t < k \), the point \( g^t(y) \) does not visit any critical puzzle piece of depth \( m + s \). Since for \( t < k - s \), the depth of the surrounding pieces \( P_{m+k-t}(g^t(y)) \) exceeds \( m + s \), the entire pieces \( P_{m+k-t}(g^t(y)) \) are non-critical. Therefore, the map 
\[ g^{k-s} : \hat{P}_{m+k}(y) \to \hat{P}_{m+s}(g^{k-s}(y)) \]
is biholomorphic. In particular, \( \hat{P}_{m+k}(y) \setminus P_{m+s+k}(y) \) is conformally equivalent to 
\[ \hat{P}_{m+s}(g^{k-s}(y)) \setminus P_{m+2s}(g^{k-s}(y)) \] .
The claim now follows from the first part, applied to \( y' = g^{k-s}(y) \) and \( k' = s \).

The \( \omega \)-limit set of a point \( z \) is the set
\[ \omega(z) := \bigcap_{n \in \mathbb{N}} \{ g^k(z) : k > n \} . \]

Note that \( \overline{\text{orb}(z)} = \text{orb}(z) \cup \omega(z) \).

**Lemma 2.7** (Accumulation on periodic fiber implies trivial fiber). *Let \( g \) be a holomorphic map with a well-defined Markov partition. Suppose that \( z \) is a non-escaping point of \( g \) so that the \( \omega \)-limit set of \( z \) intersects the fiber \( \text{fib}(y) \) of some periodic point \( y \) but the orbit of \( z \) is disjoint from \( \text{fib}(y) \). Assume additionally that \( \text{fib}(y) \), as well as all those critical fibers of \( g \) that intersect \( \omega(z) \), are compactly contained in the corresponding puzzle pieces of any depth. Then \( \text{fib}(z) = \{z\} \).*

**Proof.** Our proof goes along the lines of the proof of [RY, Lemma 3], except for the final step where Lemma 2.6 will provide us with the suitable annuli to run the pull-back argument.

The proof itself may look a bit technical in notation, but the underlying idea is simple: as long as the orbit of \( z \) stays sufficiently close to \( \text{fib}(y) \), that is in some fixed puzzle piece \( P_{n_0}(y) \) that contains no further critical points other than those are already in \( \text{fib}(y) \), the puzzle pieces along this orbit are mapped forward injectively. When the orbit leaves \( P_{n_0}(y) \) and later returns back (\( z \) accumulates on \( \text{fib}(y) \) by hypothesis), it does so with uniformly bounded degree by Lemma 2.5. This allows us, by pulling back suitable annuli (given by Lemma 2.6), to conclude that \( \text{fib}(z) = \{z\} \), whether or not the fiber of \( y \) is trivial.
Up to passing to an iterate of $g$, assume that $y$ is a fixed point, and let us adopt the notation $f$ for this iterate of $g$; thus $f^{k}(P_{n+k}(y)) = P_{n}(y)$ for all $n$ and $k$.

Let $(c_{f})_{i \in I}$ be the set of all critical fibers of $f$, different from fib($y$), that intersect $\omega(z)$ (if the fiber of $y$ is not critical, then this is just the set of all critical fibers of $f$ that intersect $\omega(z)$; here $I$ is some finite index set, which in the simplest case might be empty). For every critical fiber $c_{f}$ pick a critical point $c_{i} \in c_{f}$ representing this fiber (this choice might not be unique). Let us choose $n_{0}$ so that $P_{n_{0}}(y) \cap \text{Crit}(f) \subset \text{fib}(y)$ and $P_{n_{0}}(c_{i}) \cap \text{Crit}(f) \subset c_{f}$ for every $i \in I$; this is possible by definition of a fiber. By increasing $n_{0}$ if necessary, we also assume that orb($z$) does not intersect any critical puzzle piece of depth $n_{0}$ except those around $c_{i}$, $i \in I$ and, possibly, $y$. Up to an index shift, assume $n_{0} = 0$. Further on, fix a depth $s > 0$ such that all the annuli $P_{0}(y) \setminus P_{s}(y)$ and $P_{0}(c_{i}) \setminus P_{s}(c_{i})$ are non-degenerate. The depth $s$ exists by the assumption of the lemma: all the fibers fib($y$) and $c_{f}$ are compactly contained in the corresponding puzzle pieces of any depth. Define $P := P_{s}(y) \cup \bigcup_{i \in I} P_{s}(c_{i})$; this is the union of the puzzle pieces of depth $s$ containing $y$ and all critical fibers on which the orbit of $z$ accumulates.

For given $n$, let $k_{n}$ be the smallest integer such that $f^{k_{n}}(z) \in P_{n}(y)$; such an index exists because $z$ accumulates on fib($y$). However, since the orbit of $z$ never enters fib($y$) by hypothesis, there exists a smallest integer $m_{n} > n$ such that $f^{k_{n}}(z) \notin P_{m_{n}}(y)$, hence $f^{k_{n}}(z) \in P_{m_{n}-1}(y) \setminus P_{m_{n}}(y)$. Finally, let $l_{n} \geq 0$ be minimal so that $f^{k_{n}+m_{n}+l_{n}}(z) \in P$; again, such an index exists because $z$ accumulates on fib($y$), critical fibers fib($c_{i}$), $i \in I$, and by the choice of what we call the zero depth puzzle pieces. However, it might happen that $f^{k_{n}+m_{n}+l_{n}}(z)$ lands not in $P_{s}(y)$ but in a critical puzzle piece in $P$; denote by $c = c(n)$ the point from the set $\{y\} \cup \bigcup_{i \in I} \{c_{i}\}$ such that $f^{k_{n}+m_{n}+l_{n}}(z) \in P_{s}(c)$ (see Figure 3 for a schematic drawing of the puzzle pieces involved).

We claim that there exists an essential open sub-annulus

$$A_{n} \subset P_{k_{n}+m_{n}+l_{n}}(z) \setminus P_{k_{n}+m_{n}+l_{n}+s}(z)$$

such that

$$\text{mod}(A_{n}) \geq \frac{\mu}{D^{2}},$$

where $D$ is given by Lemma 2.5 and $\mu$ is given by Lemma 2.6, and hence the factor $\mu/D^{2}$ is independent of $z$ and $n$. We will do this in three steps; since we are pulling back, they come in reverse order. The third step is $f^{k_{n}}: P_{k_{n}+m_{n}+l_{n}}(z) \to P_{m_{n}+l_{n}}(f^{k_{n}}(z))$ controlled by Lemma 2.5; the second step is a sequence of $m_{n}$ conformal iterates to $P_{l_{n}}(f^{k_{n}+m_{n}}(z))$; and in the first step this puzzle piece is sent by $f^{l_{n}}$ to $P_{0}(f^{k_{n}+m_{n}+l_{n}}(z))$, controlled by Lemma 2.6 again. These three steps are illustrated in Figure 3 (left, center, and right); the annuli we are pulling back are contained in the shaded rings.

**Step 1.** Since $l_{n}$ was chosen to be minimal so that $f^{k_{n}+m_{n}+l_{n}}(z) \in P$, and $c$ is such that $f^{k_{n}+m_{n}+l_{n}}(z) \in P_{s}(c) \subset P$, Lemma 2.6 guarantees that there exists an essential open sub-annulus

$$A''_{n} \subset P_{l_{n}}(f^{k_{n}+m_{n}}(z)) \setminus P_{l_{n}+s}(f^{k_{n}+m_{n}}(z))$$
Figure 3. Puzzle pieces of various depths involved in the proof of Lemma 2.7. Observe that the fiber \( \text{fib}(y) \) may or may not belong to some of the puzzle pieces in the sequence \( (P_{m_n-i} f^{k_n+i}(z))_{i=1}^{m_n-1} \); the picture shows the case when it never happens. Moreover, the first landing of the orbit of \( z \) after the time \( k_n + m_n \) to the union \( \mathcal{P} = P_s(z) \cup \bigcup_j P_s(c_j) \) may or may not be in the puzzle piece \( P_s(z) \); the picture shows the case when the orbit of \( z \) lands in some other puzzle piece \( P_s(c) \in \mathcal{P} \).

such that

\[
\text{(2.2) } \quad \text{mod}(A''_n) \geq \mu,
\]

where \( \mu \) does not depend on \( z \) and \( n \). (Strictly speaking, in order to apply Lemma 2.6, we have to enlarge \( \mathcal{P} \) so that it would contain all critical puzzle pieces of depth \( s \); but since, by construction, the orbit of \( z \) visits only those critical puzzles already in \( \mathcal{P} \), this enlargement of \( \mathcal{P} \) does not alter the conclusion.)
Step 2. We claim that there exists an open essential sub-annulus
\[ A'_n \subset \tilde{P}_{m_n + l_n} \left( f^{k_n}(z) \right) \setminus P_{m_n + l_n + s} \left( f^{k_n}(z) \right) \]
such that $A'_n$ is a conformal copy of $A''_n$, and hence
\[ \mod(A'_n) = \mod(A''_n). \]

We argue as follows. The puzzle pieces around $y$ are, as always, nested like $P_0(y) \supset P_1(y) \supset P_2(y) \supset \ldots$, and since $y$ is a fixed point, each one is the image of the next one under $f$. Since $f^{k_n}(z) \in P_{m_n - 1}(y) \setminus P_{m_n}(y)$, all the points $f^{k_n}(z), f^{k_n+1}(z), \ldots, f^{k_n+m_n}(z)$ are in $P_0(y) \setminus P_{m_n}(y)$. But by construction all critical points in $P_0(y)$ are already in fib$(y)$ and hence in $P_{m_n}(y)$. Therefore, for $i \in \{0, 1, \ldots, m_n - 1\}$, the puzzle pieces of depth $m_n - i$ around $f^{k_n+i}(z)$ do not contain critical points. Together, this shows that the map $f^{m_n} : P_{m_n} \left( f^{k_n}(z) \right) \to P_0 \left( f^{k_n+m_n}(z) \right)$ has degree 1, and the same is true for its restriction $f^{m_n} : P_{m_n + l_n} \left( f^{k_n}(z) \right) \to P_{l_n} \left( f^{k_n+m_n}(z) \right)$, and hence the claim in Step 2 follows with $A'_n$ as the conformal pull-back of $A''_n$ under this restricted map.

Step 3. Similarly to Step 1, since $k_n$ is the first iterate so that $f^{k_n}(z) \in P_n(y)$, the map $f^{k_n} : P_{k_n+l_n}(z) \to P_n(y)$ has degree at most $D$ by Lemma 2.5. The same is then true for its restriction
\[ f^{k_n} : P_{k_n+m_n+l_n}(z) \to P_{m_n+l_n} \left( f^{k_n}(z) \right). \]

Let $A_n$ be a preimage of the annulus $A'_n$ under this restricted map chosen in such a way that $A_n$ is an essential sub-annulus in $\tilde{P}_{k_n+m_n+l_n}(z) \setminus P_{k_n+m_n+l_n+s}(z)$. By Lemma 2.4,
\[ \mod(A_n) \geq \frac{\mod(A'_n)}{D^2}. \]

Combining (2.2), (2.3) and (2.4) we obtain (2.1).

This argument can be carried out for infinitely many $n$: we choose a sequence $n_j$ so that once $A_{n_j-1}$ is fixed, the value of $n_j$ is chosen so that $P_{k_{n_j}+m_{n_j}+l_{n_j}}(z)$ is contained in the bounded component of $\mathbb{C} \setminus A_{n_j-1}$. This way, we obtain infinitely many disjoint annuli with moduli bounded below that all separate $z$ from all previous annuli, and using the standard Grötzsch inequality this implies that the fiber of $z$ is trivial. \(\square\)

We say that a puzzle piece $P_n$ of depth $n$ is \textit{weakly protected} if there exists a puzzle piece $P_m$ of depth $m < n$ such that $P_n$ is compactly contained in $P_m$. If $m = n - 1$, then $P_n$ is \textit{protected}. The following lemma guarantees compact containment of pullbacks of certain weakly protected puzzle pieces.

**Lemma 2.8** (First return to weakly protected puzzle piece, recurrent case). Let $g$ be a holomorphic map with a well-defined Markov partition. Suppose that $z$ is a point that does not belong to the puzzle boundary of any depth and for which there exists a $k \geq 1$ with $g^k(z) \in P_n(z)$, and suppose further that the minimal such $k$ has the property that $P_{n+k}(z) \subset P_n(z)$; in particular, $P_{n+k}(z)$ is weakly protected. Then every component of the set \( \{ z' \in P_{n+k}(z) \colon \exists m \geq 1 : g^m(z') \in P_{n+k}(z) \} \) is compactly contained in $P_{n+k}(z)$. 

Proof. By the Markov property, the closure of any component of the set \( \{ z' \in P_{n+k}(z) : \exists m \geq 1 : g^m(z) \in P_{n+k}(z) \} \) is a puzzle piece of depth at least \( n+k+1 \). Let \( Y \subset P_{n+k}(z) \) be one of these puzzle pieces. If \( n+k+l \) with \( l \geq 1 \) is the depth of \( Y \), then \( g^l : Y \to P_{n+k}(z) \) is a branched covering. Since \( k \) is the first return time of the orbit of \( g(z) \) back to \( P_n(z) \), the puzzle pieces \( (P_{n+k-l}(g^j(z)))_{j=0}^{k-1} \) have disjoint interiors. Therefore, \( l \geq k \).

Suppose that \( Y \) is not compactly contained in \( P_{n+k}(z) \). Then \( g^k(Y) \) is not compactly contained in \( g^k(P_{n+k}(z)) = P_n(g^k(z)) = P_n(z) \), and there exists a point \( w \in \partial(g^k(Y)) \cap \partial P_n(z) \). But since puzzle pieces are nice sets, i.e. \( g^m(\partial P_n(z)) \cap P_n(z) = \emptyset \) for all \( m \geq 0 \), we have in particular \( g^{-k}(w) \notin P_n(z) \). But we must have \( g^{-k}(w) \in \partial g^{-k}(g^k(Y)) = \partial g^l(Y) = \partial P_{n+k}(z) \cap P_n(z) \), a contradiction.

Lemma 2.9 (First return to weakly protected puzzle piece, non-recurrent case). Let \( g \) be a holomorphic map with a well-defined Markov partition. Suppose that \( z \) is a combinatorially non-recurrent point that does not belong to the puzzle boundary of any depth. Let \( n > 0 \) be a depth such that the orbit of \( z \) never returns to \( P_n(z) \). If there exists \( k > 0 \) such that \( P_{n+k}(z) \) is weakly protected by \( P_n(z) \), then every component of the set \( \{ z' \in P_{n+k}(z) : \exists m \geq 1 : g^m(z') \in P_{n+k}(z) \} \) is compactly contained in \( P_{n+k}(z) \).

Proof. Similarly to the proof of Lemma 2.8, the closure of every component in the set \( \{ z' \in P_{n+k}(z) : \exists m \geq 1 : g^m(z') \in P_{n+k}(z) \} \) is a puzzle piece of depth at least \( n+k+1 \); let \( Y \) be such a puzzle piece, and \( n+k+l \) with \( l \geq 1 \) be its depth. Since the orbit of \( z \) never returns to \( P_n(z) \), it follows that for every \( s \in \{1, \ldots, k\} \) the puzzle piece \( P_{n+k-s} \circ g^s(z) = g^s(P_{n+k}(z)) \) is disjoint from \( P_n(z) \). The same is true for \( g^k(Y) \) because \( Y \subset P_{n+k}(z) \); hence \( l \geq k+1 \). Finally, since \( P_{n+k}(z) \) is weakly protected by \( P_n(z) \), and \( g^l(Y) = P_{n+k}(z) \), by pulling back \( P_n(z) \) we conclude that the puzzle piece \( Y \) is weakly protected by the puzzle piece of depth \( n+l \). Since \( l \geq k+1 \), this puzzle piece lies in \( P_{n+k}(z) \). Therefore, \( Y \) is compactly contained in \( P_{n+k}(z) \).

Remark. It is possible to show that if \( P_n \) is protected, then every component of the first return domain to \( P_n \) is compactly contained in \( P_n \), see [L2, §31].

3. Complex box mappings and rigidity (Theorem C)

In this section we review the notion of complex box mappings introduced in [KSS, KvS1] (with some further clarification in [KvS2]) and prove a generalized version of trivial fibers for such mappings (Theorem C). This result is of interest in its own right, and it is a key ingredient in the proof of our Rational Rigidity Principle (Theorem A).

Definition 3.1 (Complex box mapping [KvS1, KvS2]). A holomorphic map \( F : \mathcal{U} \to \mathcal{V} \) between two open sets \( \mathcal{U} \subset \mathcal{V} \subset \hat{\mathbb{C}} \) is a complex box mapping if the following holds:

1. \( F \) has finitely many critical points;
2. \( \mathcal{V} \) is the union of finitely many open Jordan disks with disjoint closures;
3. every component \( V \) of \( \mathcal{V} \) is either a component of \( \mathcal{U} \), or \( V \cap \mathcal{U} \) is a union of Jordan disks with pairwise disjoint closures, each of which is compactly contained in \( V \);
(4) for every component $U$ of $\mathcal{U}$ the image $F(U)$ is a component of $\mathcal{V}$, and the restriction $F: U \to F(U)$ is a proper map.

Following [DH2], a proper holomorphic map $f: U \to V$ of degree $d \geq 2$ between two open topological disks $U$ and $V$ with $\overline{U} \subset V \subset \hat{\mathbb{C}}$ is called a polynomial-like map (in the sense of Douady–Hubbard [DH2]). By the straightening theorem, such a polynomial-like map is hybrid equivalent to a polynomial of degree $d$, and this polynomial is unique (up to affine conjugation) if the filled Julia set $K(f) := \bigcap_{n \geq 1} f^{-n}(V)$ is connected. Moreover, connectivity of $K(f)$ is equivalent to the condition that all critical points of $f$ are contained in $K(f)$.

When $V$ is connected and $U$ has only finitely many components, and all of these are compactly contained in $V$, then the corresponding box mapping may be regarded as a polynomial-like map in the sense of Douady–Hubbard (generalized to several components of $U$). For general box mappings, however, $U$ is allowed to have infinitely many components, and in many applications this is important. Such generality in the definition of a box mapping results in phenomena that do not occur for polynomial-like maps. For example, a box mapping might wandering domains, or it might have a filled Julia set that is all of $U$; see below, as well as [KvS2].

**Definition 3.2** (Puzzle piece of box mapping). For a box mapping $F: \mathcal{U} \to \mathcal{V}$, we define a puzzle piece of depth $n$ to be a component of $F^{-n}(V)$. A puzzle piece is called critical if it contains at least one critical point.

The set $K(F) := \{z \in \mathcal{U}: F^n(z) \in \mathcal{U} \text{ for all } n \geq 0\}$ is the filled Julia set of the box mapping $F: \mathcal{U} \to \mathcal{V}$; this is the set of non-escaping points. Similarly, the Julia set is defined as $J(F) := \partial K(F)$.

**Definition 3.3** (Box renormalizable box mappings). We call a complex box mapping $F: \mathcal{U} \to \mathcal{V}$ box renormalizable around a critical point $c \in \mathcal{U}$ if there exists a puzzle piece $W$ at some depth containing $c$, and an integer $s > 1$ (called the period of the renormalization) such that $F^{sk}(c') \in \hat{W}$ for every critical point $c' \in W$ and every $k \geq 0$, and $s$ is minimal with this property. The filled Julia set of this box renormalization is defined analogously as $\{z \in \mathcal{U}: F^{sk}(z) \in \hat{W} \text{ for all } k \geq 0\}$. In this context we call $c$ a box renormalizable critical point.

A complex box mapping $F: \mathcal{U} \to \mathcal{V}$ is called box renormalizable if it is box renormalizable around at least one critical point in $\mathcal{U}$, and non-box renormalizable otherwise.

**Remark.** If we denote by $Y$ the component of $F^{-s}(W)$ containing $c$, then either $Y \subset \hat{W}$, or $Y = W$.

If $Y$ is compactly contained in $W$ (that is $Y \subset \hat{W}$), then the restriction $F^s: \hat{Y} \to \hat{W}$ is a polynomial-like map in the sense of Douady–Hubbard, and we simply say that $F$ is renormalizable. Moreover, the filled Julia set of the renormalization around $c$ is connected, by the standard theory of polynomial-like maps mentioned above.

In the case that a puzzle piece $W_0$ contains several critical points among which some have their entire $F^s$-orbits in $W_0$ and others do not, then one can shrink $W_0$ to a puzzle
piece $W$ of greater depth that contains only those critical points that do not escape, and then $F$ is renormalizable around these critical points.

If $Y = W$, then $F^s: \hat{Y} \to \hat{W}$ is a proper self-map of a disk without escaping points, and hence the filled Julia set of $F^s$, restricted to $Y$, is equal to $Y$. This is a “pathological case” that may occur for box renormalizable maps, but it is not included in our definition of renormalization; it does not occur in a number of interesting cases arising from dynamics on $\hat{\mathbb{C}}$. Our definition of box renormalizability of a box map coincides with the notion of renormalizability as used in [KvS1, Definition 1.3]).

**Remark.** If $c$ is a renormalizable critical point, and $\varrho := F^s: \hat{Y} \to \hat{W}$ is a corresponding to $c$ polynomial-like map, then the filled Julia set of $\varrho$ is, in fact, equal to $\text{fib}(c)$ (the set of all points with the same periodic itinerary). To see this, first observe that the fiber of any non-escaping point in $\hat{Y}$ under $\varrho$ is contained in the filled Julia set of $\varrho$. Moreover, each of such fibers is compactly contained in every puzzle piece of any depth. For a given non-escaping point $z$, if $\text{fib}(z) = \text{fib}(c)$, then the claim follows. Otherwise, there exists a pair of separating puzzle pieces $P \subset \hat{Y}$ and $P' \subset \hat{Y}$ of the same depth with $\text{fib}(z) \subset P$ and $\text{fib}(c) \subset P'$ constructed as $\varrho$-pullbacks of $W$. Since the filled Julia set of $\varrho$ is connected and contains both $\text{fib}(z)$ and $\text{fib}(c)$, it should then intersect the boundaries of $P$ and $P'$ escape under $\varrho$, a contradiction.

We will be using the following theorem by Kozlovski and van Strien [KvS1, Theorem 1.4] (with some clarification in [KvS2]); this theorem plays a crucial role in their study of rigidity for multicritical complex and real polynomials (see [KSS, KvS1]; see also [KL2] for the original proof of the Kahn–Lyubich Covering Lemma, a crucial technical ingredient used to obtain the all-important complex bounds).

**Theorem 3.4 (Rigidity for complex box mappings [KvS1, KvS2]).** Assume that $F: U \to V$ is a non-renormalizable complex box mapping for which all periodic points are repelling. Then each point in its Julia set has trivial fiber or converges to the boundary of $U$. □

(Here we rephrased the statement in [KvS1, KvS2] using the language of fibers.)

**Lemma 3.5 (Fibers compactly contained in puzzle pieces).** Consider a box mapping $F: U \to V$ and a non-escaping point $z$. If the orbit of $z$ does not eventually land in a cycle of periodic components without escaping points (NE), then the fiber of $z$ is compactly contained in any puzzle piece it is contained in.

**Proof.** It suffices to prove that every puzzle piece around $z$ compactly contains another puzzle piece around $z$ at greater depth. To do this, let $W_n$ be the puzzle piece of any depth $n$ containing $z$ and for $k \leq n$ denote by $W_{n-k} = F^k(W_n)$ the puzzle piece around $F^k(z)$ at depth $n - k$, so that $W_0$ is a component of $U$.

If any $W_{n-k}$ contains a puzzle piece around $F^k(z)$ at greater depth than $n - k$ that is a proper subset, then this proper subset must be compactly contained, and the claim follows by pull-back to $W_n$. Otherwise, in particular $W_0$ is not only a component of $U$ but also a component of $V$. As we iterate forward, we cannot keep visiting components of $U$ that are also components of $V$ (by finiteness of $V$ this would yield a cycle of (NE)-components
which is excluded by hypothesis), so we must reach a component of $\mathcal{U}$ that is compactly contained in its component of $\mathcal{V}$, and the claim follows.

Proof of the generalized rigidity principle (Theorem C). Let $F: \mathcal{U} \to \mathcal{V}$ be a complex box mapping, and let $z \in K(F)$ be a non-escaping point. The claim of the theorem is that then at least one of the following holds: $z$ has (T) trivial fiber, or it has (R) renormalizable dynamics, or it has “pathological dynamics” in the sense that (NE) it eventually maps to a periodic component without escaping points, or (CB) the orbit converges to $\partial \mathcal{U}$.

It thus suffices to consider a non-escaping point $z$ to which neither (NE) nor (CB) apply. We need to show that $z$ satisfies either (T) or (R). By Lemma 3.5, the fiber $\text{fib}(z)$ is compactly contained in puzzle pieces of any depth, and absence of (CB) implies that $\omega(z) \cap \mathcal{U} \neq \emptyset$.

Denote by $\mathcal{R} := \{\text{pf}_j\}$ be the set of all periodic fibers of $F$ that are renormalizable in the sense that they are contained in a puzzle piece $W_i$ of level $n_i$ and so that $F^{n_i}: W_i \to F^{n_i}(W_i)$ is a polynomial-like map with connected Julia set and of degree at least 2. These puzzle pieces can be chosen to be disjoint for different renormalizable fibers, and each of them contains at least one critical point of $F$. Therefore, $\mathcal{R}$ contains only finitely many fibers. All their Julia sets contain at least one fixed point of $F^{n_i}$.

For the given point $z \in K(F)$, we have the following three possibilities:

1. there exists an $i$ such that $\text{orb}(z) \cap \text{pf}_i \neq \emptyset$;
2. $\text{orb}(z) \cap \text{pf}_i = \emptyset$ for all $i$, but there exists an index $j$ such that $\omega(z) \cap \text{pf}_j \neq \emptyset$;
3. $\overline{\text{orb}(z)}$ does not intersect the fibers in $\mathcal{R}$.

Let us make several remarks concerning this case distinction. Since the components of $\mathcal{U}$ have disjoint closures and are compactly contained in the respective components of $\mathcal{V}$ (unless they coincide), all fibers of $F$ are disjoint. Therefore, in the first case above the index $i$ defines a unique cycle of fibers. However, in the second case the index $j$ might be not unique; for us it is enough to have at least one such index. Note that it is impossible that the orbit of $z$ intersects one fiber in $\mathcal{R}$ and accumulates on another fiber since the fibers in $\mathcal{R}$ are periodic and disjoint.

Case (1) is exactly possibility (R) in the statement of Theorem C, so in this case we are done. It remains to show that in the other two cases, the fiber of $z$ is trivial.

Let us now treat case (2). The fiber $\text{pf}_j$ contains a periodic point, but the orbit of $z$ never lands in a periodic fiber. This is the situation of Lemma 2.7; we explain why the assumptions in this lemma are satisfied. The fiber $\text{pf}_j$ is compactly contained in all its puzzle pieces by the condition on renormalizability. Moreover, since the orbit of $z$ never lands in an (NE) component by the hypothesis, each critical fiber on which it can further accumulate is compactly contained in any surrounding puzzle piece of any depth by Lemma 3.5 (note that, by the definition of a fiber, all such fibers must contain a non-escaping critical point; moreover, the orbit of this non-escaping critical point never lands in an (NE) component). Therefore, the assumptions of Lemma 2.7 are satisfied, and thus the fiber of $z$ is trivial.

In order to tackle case (3), we will show that the orbit of $z$ is eventually contained in the Julia set of a non-renormalizable box mapping $F': \mathcal{U}' \to \mathcal{V}'$ with only repelling periodic
points. The map \( F' \) will be obtained as a suitable restriction of (iterates of) \( F \), and the conclusion will then follow from Theorem 3.4.

Let \( \text{Crit}_R := \{ c \in \text{Crit}(F) : \exists i : \text{orb}(c) \cap \text{pf}_i \neq \emptyset \} \) be the set of all critical points the orbits of which either land in, or accumulate at, one of the periodic fibers in \( R \). Also define \( \text{Crit}_{N_E} \) to be the set of all critical points of \( F \) that eventually map to components of \( \mathcal{U} \) of type (NE) (periodic components without escaping points); by construction, \( \text{Crit}_R \cap \text{Crit}_{N_E} = \emptyset \). Periodic critical points in \( \text{Crit}_{N_E} \) are box renormalizable but not renormalizable in our (i.e. the Douady–Hubbard) sense.

Finally, write \( \text{Crit}_{nR} := \text{Crit}(F) \setminus (\text{Crit}_R \cup \text{Crit}_{N_E}) \). The critical points in \( \text{Crit}_{nR} \) are not box renormalizable. Consider the union \( V' \) of the interiors \( V' \) of critical puzzle pieces of the same sufficiently large depth such that:

- (a) each \( V' \) contains at least one point in \( \text{Crit}_{nR} \);
- (b) \( V' \cap \text{Crit}(F) = \text{Crit}_{nR} \);
- (c) if \( c \in \text{Crit}_{nR} \) is not combinatorially recurrent, and \( V' \) is the component of \( V' \) containing \( c \), then the puzzle piece \( V' \) is chosen of sufficiently large depth so that \( c \) never re-enters this puzzle piece.

Critical points in \( \text{Crit}_{nR} \) are in different fibers than those in \( \text{Crit}_R \) or in \( \text{Crit}_{N_E} \). Condition (b) can be satisfied by choosing sufficiently large depth of the puzzle pieces, and condition (c) can be satisfied by possibly increasing this depth; since there are only finitely many critical points, this yields finite depth. Condition (a) is clear.

Finally, set \( \mathcal{U}' := \{ z' \in V' : \exists n : F^n(z') \in V' \} \) to be the set of points in \( V' \) that eventually return to \( V' \). The set \( \mathcal{U}' \) is open. Indeed, let \( k > 0 \) the depth of all (interiors of) puzzle pieces in \( V' \). For every \( z' \in \mathcal{U}' \), if \( n \) is minimal such that \( F^n(z') \in V' \), then \( P_{n+k}(z') \subset \mathcal{U}' \). Moreover, \( \partial P_{n+k}(z') \cap \mathcal{U}' = \emptyset \). Hence, every component of \( \mathcal{U}' \) is the interior of a puzzle piece of the original box mapping \( F : \mathcal{U} \to V \). In this way we obtain a well-defined map \( F' : \mathcal{U}' \to V' \) (choosing the least \( n \) for every \( z' \)): \( F' \) restricted to a component \( U' \) of \( \mathcal{U}' \) is a proper map \( F^n : U' \to V' \) from \( U' \) to a component \( V' \) of \( V' \).

By construction, \( F' \) is a box mapping (in the sense of Definition 3.1) whose periodic points are all repelling and all critical points are non-renormalizable, so the box mapping \( F' \) is non-renormalizable as well. Moreover, \( F' \) has no (NE) components, again by construction. By Theorem 3.4, every point in \( J(F') \) either has trivial \( F' \)-fiber, or converges to the boundary of \( \mathcal{U}' \). We argue that trivial \( F' \)-fiber implies trivial fiber with respect to the original box mapping \( F \). Indeed, since \( F' \) restricted to any component \( U' \) of \( \mathcal{U}' \) is a proper map between interiors of two puzzle pieces of \( F \), and the map itself is an iterate of \( F \) (see above), we conclude that the set of puzzle pieces defined for the box mapping \( F' \) is a subset of the set of puzzle pieces defined for \( F \). Hence, if \( y \in J(F') \) has trivial \( F' \)-fiber, i.e. there exist arbitrary small puzzle neighborhoods around \( y \) constructed with respect to \( F' \), then these are also small neighborhoods of \( y \) for \( F \) as well, so \( y \) has trivial fiber.

Let us come back to case (3) in the case distinction above. We know that \( \overline{\text{orb}(z)} \) is disjoint from all fibers in \( R \). Our last case distinction is whether or not \( \overline{\text{orb}(z)} \) intersects \( J(F') \). (Note that here and everywhere below \( \text{orb}(y) \) stands for the orbit of \( y \) under \( F' \).)
Suppose first that $\text{orb}(z)$ intersects $J(F')$. This implies that the orbit of $z$ intersects $\mathcal{V}'$ in an infinite set. Therefore, $\text{orb}(z) \cap \mathcal{V}' \subset \mathcal{U}'$, by construction of $\mathcal{U}'$ as the first return domain to $\mathcal{V}'$ (‘first’ here refers to the choice of minimal $n$ for each $z'$ in the definition of $F': \mathcal{U}' \to \mathcal{V}'$). And since $F'$ restricted to any component of $\mathcal{U}'$ is an iterate of $F$, none of the points in $\text{orb}(z) \cap \mathcal{V}'$ can escape under $F'$ (equivalently, none can land after some number of iterations of $F'$ into $\mathcal{V}' \setminus \mathcal{U}'$). Therefore, $\text{orb}(z) \cap \mathcal{V}'$ belongs to the non-escaping set $K(F') = J(F')$ (all periodic points of $F'$ are repelling and it has no (NE) components). Since the orbit of $z$ lands in $J(F')$, and the fibers of the points in $J(F')$ are trivial, we conclude that $\text{fib}(z) = \{z\}$. This is exactly case (T) in Theorem C.

Finally, let us show that if the orbit of $z$ does not satisfy (NE) and (CB), and $\text{orb}(z)$ avoids $J(F')$ and the elements in $\mathcal{R}$, then $\text{fib}(z) = \{z\}$. Let $w \in \omega(z)$ be an accumulation point of $\text{orb}(z)$ that does not lie on the boundary of a puzzle piece of any depth; such point exists since $z$ does not satisfy (CB). Moreover, since the orbit of $z$ does not land in an (NE) component, the fiber of $w$ is compactly contained in puzzle pieces of any depth (Lemma 3.5). Finally, by the last assumptions on $z$, there exists a depth $n$ large enough so that the orbit of $z$ is disjoint from the set $\bigcup_{c \in \text{Crit}(F)} P_n(c)$. Consider a non-degenerate annulus $A := \hat{P}_n(w) \setminus P_{n+m}(w)$ for some large $m$. Since the orbit of $z$ accumulates on $w$, there exists an increasing sequence $(n_i)$ with $F^{n_i}(z) \in P_{n+m}(w)$. By the choice of $n$, the annuli $A_i := \hat{P}_{n+n_i}(z) \setminus P_{n+n_i+m}(z)$ are conformal copies of $A$ under $F^{n_i}$. Therefore, by possibly passing to a subsequence of $(n_i)$, we obtain a sequence $(A_i)$ of nested annuli of equal moduli. By the Grötzsch inequality, this immediately yields $\text{fib}(z) = \{z\}$, which is case (T) in Theorem C. This concludes case (3) from the case distinction above, and finishes the proof of Theorem C.

4. PUZZLES FOR NEWTON MAPS

The proof of the Rational Rigidity Principle (Theorem A) will rely on the puzzle construction for Newton maps introduced in [DLSS]. We will review the key steps of this construction, together with some of its properties that we will use in the further sections.

In the polynomial case, puzzles were constructed by Branner–Hubbard and Yoccoz (with much further work since then) starting with neighborhoods $S_n$ of the filled-in Julia set that are bounded by suitably chosen equipotential curves. These are subdivided by finitely many pairs of dynamic rays landing at common repelling periodic or preperiodic points (for quadratic polynomials, usually at the $\alpha$-fixed point and its iterated preimages). The closures of complementary components of $\overline{S_n}$ minus the ray pairs are called puzzle pieces (of depth $n$), and they form a Markov partition.

However, for rational maps it is not at all clear how to carry over such a construction; in particular, there are no obvious substitutes for the basin of infinity and the Böttcher coordinates available in the polynomial case that give rise to good Markov partitions.

A notable exception are Newton maps of polynomials. For these, puzzles with similar properties as in the polynomial case have recently been constructed in [DLSS, Theorem B]. This result will be one of the main ingredients in the construction leading to our Theorem A.
Theorem 4.1 (Puzzles for polynomial Newton maps). Every polynomial Newton map $N_p$, possibly after a natural quasiconformal surgery in the basins of the roots of $p$, has an iterate $g := N_p^M$ for which there exists a forward invariant graph $\Gamma \subset \hat{\mathbb{C}}$ such that for every $n \geq 0$ the complementary components of $g^{-n}(\Gamma)$ in $\hat{\mathbb{C}}$ have the Markov property under $g$ and are topological disks with Jordan boundary (subject to a suitable truncation in the basins of the roots of $p$). The latter components define a (Newton) puzzle partition of depth $n$. □

We will unwrap some of the steps in Theorem 4.1: we will explain the necessary quasiconformal surgery, how we choose $\Gamma$, and what we mean by a suitable truncation in the basins of roots.

Let us start with the quasiconformal surgery. By [DLSS, Proposition 2.10], for every polynomial $p$ with Newton map $N_p$ there exists a polynomial $\tilde{p}$ with Newton map $N_{\tilde{p}}$ and a quasiconformal homeomorphism $\tau: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ with $\tau(\infty) = \infty$ such that

1. $\deg p \geq \deg N_p = \deg N_{\tilde{p}} = \deg \tilde{p}$;
2. all the critical points of the Newton map $N_{\tilde{p}}$ that are attracted to the roots of $\tilde{p}$ are, in fact, mapped to the corresponding roots by some finite iterate of $N_{\tilde{p}}$;
3. $\tau$ conjugates $N_p$ and $N_{\tilde{p}}$ in neighborhoods of the Julia sets of $N_p$ and $N_{\tilde{p}}$ union all Fatou components (if any) that do not belong to the basins of roots; moreover, $\tau$ has non-zero dilatation only in the basin of the roots, but not on the Julia set (if it has positive measure) or on Fatou components away from the root basins.

The immediate basin $U_\xi$ of a root $\xi$ is the component of the basin of $\xi$ that contains $\xi$. For every $\xi$, the restriction of the Newton map to $U_\xi$ yields a proper self-map of $U_\xi$ of some degree $k = k(\xi) \geq 2$. Every immediate basin is simply connected [Prz], and the quasiconformal surgery consists of replacing a disk neighborhood of the root in its immediate basin by a disk with dynamics $z \mapsto z^k$; the degree of the self-maps on $U_\xi$ is unchanged by this procedure, but afterwards there is a single critical point in $U_\xi$, and it is a fixed point (a simple root of $\tilde{p}$). The degree of $N_p$ will drop if $\xi$ was a multiple root of $p$. Moreover, the dynamics in the preimage components of $U_\xi$ is adjusted so as to make sure that all critical points in any preimage component coincide and land exactly on the root. All this can be accomplished by a quasiconformal surgery within a compact subset of finitely many components of the basin. The procedure is standard and described in [DMRS].

The key issue is condition (2); it assures that the dynamics of $N_{\tilde{p}}$ restricted to the basins of the roots is postcritically finite, while keeping the dynamics elsewhere unchanged (there may for instance still be critical points with dense orbits in the Julia set). Following [DMRS], we call a polynomial with this property attracting-critically-finite.

Condition (2) implies that the roots of $\tilde{p}$ are simple, so they are critical fixed points of $N_{\tilde{p}}$, and these are the only critical points in the immediate basins (possibly of higher multiplicities as critical points, but not as roots). Condition (3) implies that the dynamics of fibers of points in the Julia sets are the same, up to quasiconformal conjugation; in particular, triviality of fibers is preserved.
It is known that for every immediate basin $U_\xi$ the boundary point $\infty$ is always accessible through one or several accesses that are invariant up to homotopy; this number of invariant accesses equals $k(\xi) - 1$ is one less than the degree of $N_p$ as a self-map of the immediate basin of the root [HSS, Proposition 6]. For the modified Newton map $N_p$, the accesses to $\infty$ are in fact invariant as curves, without need for a homotopy; this is the point of the surgery. Since $\tau$ must map basins and in particular immediate basins to basins and immediate basins, and it is a conjugation in a neighborhood of $\infty$, it must respect accesses to $\infty$ up to homotopy within immediate basins, and in particular the circular order at $\infty$ of these accesses (quasiconformal homeomorphisms preserve orientation).

From now on we will assume that our Newton maps are attracting-critically-finite. This property allows us to define the basic combinatorial object associated to a Newton map: the channel diagram (and eventually fibers). By definition (see [HSS, DMRS] for a detailed discussion), the channel diagram is a finite topological graph $\Delta$ such that its vertex set consists of all fixed points of $N_p$ (that is, $\infty$ and the roots of $p$), and each edge of $\Delta$ is invariant under the dynamics and connects $\infty$ to some root within the respective immediate basin. The union of the interiors of all such fixed rays across all immediate basins gives the edge set of $\Delta$.

By construction, the graph $\Delta$ is invariant (as a set) under $N_p$; it encodes the mutual locations of the immediate basins. The Newton graph at level $n$ (denoted by $\Delta_n$) is the connected component of $N_p^{-n}(\Delta)$ containing $\infty$. Each edge of the Newton graph is an iterated preimage of a fixed ray in an immediate basin, and hence the Newton graph intersects the Julia set of $N_p$ only at $\infty$, poles and prepoles, and intersects the Fatou set of $N_p$ along basins of roots.

The Newton graph at level $n$ is the foundation for the definition of the puzzle of depth $n$. The construction rests on the following theorem, which is a compilation of two results, [DMRS, Theorem 3.4] and [DLSS, Proposition 3.1].

**Theorem 4.2 (Connectivity properties of Newton graphs).** Let $N_p$ be an attracting-critically-finite Newton map; then

1. there exists a least integer $N$ so that $\Delta_N$ contains all poles of $N_p$;
2. there exists a least integer $K > N$ so that for every component $V$ of $\hat{\mathbb{C}} \setminus \Delta$ there exists a topological circle $X_V \subset \Delta_K \cap V$ that passes through all finite fixed points in $\partial V$ and separates $\infty$ from all critical values of $N_p$ in $V$ (see Figure 4). □

Finally, we are ready to complete the unwrapping Theorem 4.1 and present the construction of puzzles that was carried out in [DLSS, Section 3]. Our puzzles will be defined for a suitable iterate of the Newton map $N_p$. Let $X := \bigcup_V X_V$, where the union is taken over all components $V$ of $\hat{\mathbb{C}} \setminus \Delta$ of the circles described in Theorem 4.2 (2). It is not hard to see that that $K$ in Theorem 4.2 (2) is the smallest integer so that $N_p^K(X) = \Delta$. Define $M := N - 1 + K$. It follows from Theorem 4.2 (1) that $M$ is the smallest index so that $\Delta_M$ contains all prepoles of level $K$ (where a point $z$ is called a (pre-)pole of level $n > 0$ if $n$ is minimal such that $N_p^n(z) = \infty$; with this definition, $K$ equals to the largest level of (pre-)poles in $X$). Indeed, since $\infty \in \Delta_N$, every component of $N_p^{-1}(\Delta_N)$ contains a
pole. Hence, $\Delta_{N+1} = N^{-1}_p(\Delta_N)$ because $\Delta_N \subset \Delta_{N+1}$ and $\Delta_N$ contains all the poles. Therefore, $\Delta_{N+1}$ contains all the prepoles of level 2. Proceeding inductively, we see that $\Delta_M$ contains all prepoles of level $K$, and $M$ is the smallest with such property (as $N$ was also the smallest). This property of $\Delta_M$ guarantees $X \subset \Delta_M$. Write $g := N^M_p$ for the $M$-th iterate of $N_p$. This is the iterate for which we will construct puzzles.
In order to define Newton puzzles at any depth $n \geq 0$, the main ingredient is the Newton graph at level $n$. In particular, the components of $\hat{C} \setminus \Delta_n$ have the Markov property (Definition 2.1); this is discussed in [DLSS, Section 3]. However, and this is the main technical difficulty, these components do not necessarily have Jordan boundaries (the main difficulty are (pre-)poles on the boundary that can be accessible in more than one way, like the point $\infty$ for some immediate basins). It turns out that this problem can be remedied by adding $X$ to $\Delta$ as follows. Define $\Delta^+_n := \Delta \cup X$, and (in analogy to $\Delta_n$) let $\Delta^+_n$ be the component of $N_p^{-n}(\Delta^+_0)$ containing $\infty$, for $n > 0$. By [DLSS, Lemma 3.5 (3)], we have

\[
 g^{-1}(\Delta^+_n) = N_p^{-M}(\Delta^+_n) = \Delta^+_{(n+1)M} \quad \text{for all } n \geq 2.
\]

Note here that $n$ should be at least two for the following reasons. Since $\infty \in \Delta_n \subset \Delta^+_n$ for every $n \geq 2$, each component of $g^{-1}(\Delta^+_n)$ contains a pole of level $M$. In order for (4.1) to hold, $n$ should be such that $\Delta^+_n$ would contain all poles of level $M$. We know that $\Delta_M$ contains all poles of level $K < M$, and hence the same is true for $\Delta^+_M \supset \Delta_M$; this level might not be good enough. But then $\Delta_{2M}$ contains all poles of level $M + K > M$, and this is already sufficient. Therefore, (4.1) holds for all $n \geq 2$; for precise details see [DLSS, Lemma 3.5].

We set $\Gamma'_0 := \Delta^+_2$, and similarly $\Gamma'_n := \Delta^+_{(n+2)M}$ for all $n > 0$. In this notation, Property (4.1) transforms to

\[
 g^{-1}(\Gamma'_n) = \Gamma'_{n+1} \supset \Gamma'_n \quad \text{for all } n \geq 0
\]

where the last inclusion follows from [DLSS, Lemma 3.5 (2)].

In order to construct puzzles, we start by picking an equipotential curve in each of the immediate basins of roots (with respect to the coordinates where the immediate basin is represented by $D$ with the root at the center, this is a concentric circle, for instance at radius $1/2$). Consider the pull-backs of these equipotentials under $g$ to the components of the basins that intersect $\Gamma'_0$ (see Figure 4). The union of all these equipotentials will give us the set $\partial S_0$. The set $S_0$ is then defined to be the unique unbounded component of $\hat{C} \setminus \partial S_0$. For a given $n > 1$ define inductively $S_n$ to be the unbounded component of $g^{-1}(S_{n-1})$. Since the roots of $p$ are superattracting fixed points of $g$, we have $S_0 \supset S_1 \supset \ldots \supset S_n \supset \ldots$. Note that the intersection of the $S_n$ is the complement of the root basins, and hence it contains those and only those critical points of $g$ that are not mapped to the roots of $p$ by some iterate of $g$ (or equivalently, of $N_p$).

**Definition 4.3** (Newton puzzles). For given $n \geq 0$, the Newton puzzle of depth $n$ is the graph $\Gamma_n := \partial S_n \cup (\Gamma'_n \cap S_n)$. The closure of a connected component of $S_n \setminus \Gamma_n$ is a Newton puzzle piece of depth $n$.

Newton puzzle pieces (of depth $n$) provide a puzzle partition of $S_n$. Their union forms a neighborhood of the Julia set of $N_p$. Theorem 4.1 guarantees that these puzzle pieces are closed topological disks with Jordan boundaries, and that they have the Markov property in the sense of Definition 2.1: essentially, this follows from (4.2)). (See also [DLSS, Theorem 3.9] for a stronger result.)
4.1. Properties of Newton puzzles. Let us review the properties of Newton puzzles proven in [DLSS]. As mentioned above, we will be working with the iterate $g$ of an attracting-critically-finite Newton map $N_p$ for which we have well-defined puzzle pieces. For simplicity, we will keep calling $g$ a Newton map, even though it is an iterate of a Newton map.

It follows directly from the construction that for every puzzle piece $P_n$ of depth $n$ the “Julia boundary” $\partial P_n \cap J(g)$ is a finite set consisting of (pre)poles of level at most $K + (n + 2)M$, whereas $\partial P_n$ intersects the Fatou set of $g$ within the basins of the roots, and $\partial P_n$ intersects any particular Fatou component along two pre-fixed internal rays with common endpoint an iterated preimage of some root (or not at all).

From Definition 2.2 it is clear that if $x \in J(g)$ is neither $\infty$ nor a (pre)pole, then $P_n(x)$ is the unique puzzle piece of depth $n$ (as defined at the beginning of Section 4) that contains $x$. Otherwise, $P_n(x)$ is a finite union of puzzle pieces with $x$ as their common boundary point.

The following two lemmas are [DLSS, Theorem 3.9 (4) and (5)].

**Lemma 4.4** (Infinity and (pre)poles have trivial fibers). If $x$ is $\infty$, a pole or a prepole, then $\text{fib}(x) = \{x\}$. □

**Lemma 4.5** (Fibers are compactly contained). Every $x \in \hat\mathbb{C}$ that is not in the basin of a root has the property that its fiber is contained in $\hat P_n(x)$ for every depth $n \geq 0$. Stronger yet, for every $n \geq 0$ there is an $m > n$ so that $P_m(x) \subset \hat P_n(x)$. □

Here comes a helpful result saying that the only possible obstruction to triviality of a fiber is when the orbit accumulates at critical fibers.

**Lemma 4.6** (Avoiding critical points that are not (pre)poles implies trivial fiber). Let $z \in \mathbb{C}$ be a point, not in the basin of any root, such that its orbit does not accumulate at any critical fiber except for, possibly, critical (pre)poles. Then $\text{fib}(z) = \{z\}$.

**Proof.** Since the orbit of $z$ does not accumulate on critical fibers, except, possibly, at critical (pre)poles, there exists $n > 0$ such that $\text{orb}(g(z))$ is disjoint from the interiors of all puzzle pieces of depth $n$ that contain critical values in their interiors. Furthermore, since the fiber of $\infty$ is trivial (Lemma 4.4), we can enlarge $n$ if necessary so that $\hat P_n(\infty)$ contains no values.

We first consider the case that there exists a point $w \in \omega(z)$ that does not belong to the puzzle boundary of any depth. In particular, $w$ is neither $\infty$ nor a (pre)pole; and $w$ cannot be in any root basin since $z$ is not. By Lemma 4.5, every $k > n$ has an $l > 0$ so that $A := \hat P_k(w) \setminus \hat P_{k+l}(w)$ is a non-degenerate annulus. Since the orbit of $z$ accumulates at $w$, there exists an increasing sequence $(k_i)$ with $g^{k_i}(z) \in \hat P_{k+l}(w)$.

We claim that for all $i$, the annulus $A_i := \hat P_{k+k_i}(z) \setminus \hat P_{k+l+k_i}(z)$ is a conformal copy of $A$. Indeed, we can pull back $\hat P_k(w) \ni g^{k_i}(z)$ univalently for $k_i$ iterates along the orbit from $z$ to $g^{k_i}(z)$ and obtain $\hat P_{k+k_i}(z)$: the only possible obstacle would be a critical value in $\hat P_{k+k_i-j}(g^j(z))$ for $j \in \{1,2,\ldots,k_i\}$, but this would mean that $g^j(z)$ was in a puzzle piece of depth $k + k_i - j \geq k > n$ with a critical value in its interior, which is excluded
by hypothesis. We thus have \( \text{mod}(A_i) = \text{mod}(A) \), and the \( A_i \) are nested (possibly after passing to a subsequence of \((k_i)\)); hence, by the Grötzsch inequality, \( \text{fib}(z) = \{z\} \).

We are thus left with the case when \( \omega(z) \) consists only of points in the puzzle boundary, but not in the basin of any root. By construction of puzzles, this means that \( \omega(z) \) consists only of \( (\text{pre})\)-poles, and hence contains \( \infty \). Since the fiber of \( \infty \) is trivial (Lemma 4.4), we can exclude the case that \( z \) is a \( (\text{pre})\)-pole itself.

Since the orbit of \( z \) accumulates at \( \infty \), and \( \infty \) is a repelling fixed point, there exist infinitely many points \( y \in \text{orb}(z) \) such that \( y \in \hat{P}_{n+2}(\infty) \) and \( g(y) \in \hat{P}_{n+1}(\infty) \setminus P_{n+2}(\infty) \). Let \( w \in P_{n+1}(\infty) \setminus \hat{P}_{n+2}(\infty) \) be an accumulation point of such \( g(y) \)'s. By our assumption on \( \omega(z) \), the point \( w \) must be a \( (\text{pre})\)-pole. Let \( Y \subset P_{n+1}(\infty) \setminus \hat{P}_{n+2}(\infty) \) be a puzzle piece of depth \( n + 2 \) containing \( w \) and such that \( \text{orb}(z) \) intersects \( Y \) in an infinite set.

We claim that \( B := P_n(w) \setminus Y \) is a non-degenerate annulus. To see this, let \( W \) be the puzzle piece of depth \( n + 1 \) such that \( Y \subset W \) and \( \infty \in \partial W \). From our choice of \( n \) it follows that \( P_{m+1}(\infty) \) is compactly contained in \( P_m(\infty) \) for every \( m \geq n \): the point \( \infty \) is a repelling fixed point, and the set \( \hat{P}_n(\infty) \) lies in the linearizing neighborhood around \( \infty \) as it contains no critical values. Therefore, the set \( P_n(w) \) is a puzzle piece of depth \( n \) (rather than a union of puzzle pieces), \( \infty \in \partial P_n(w) \), and we have the inclusion \( Y \subset W \subset P_n(w) \). Furthermore, \( \partial W \cap \partial P_n(w) \) consists of \( \infty \) and two pieces of fixed rays meeting at \( \infty \). But since \( \partial Y \) is disjoint from \( \infty \), and hence cannot contain pieces of fixed rays, it does not intersect \( \partial W \cap \partial P_n(w) \). Therefore, \( \hat{Y} \subset P_n(w) \), and hence the annulus \( B \) is non-degenerate. The rest of the proof for \( B \) is the same as for the annulus \( A \).

We now turn to a discussion of the more interesting cases, when critical points are renormalizable or at least combinatorially recurrent.

A critical point \( c \) of a Newton map \( g \) is called \textit{combinatorially periodic} if there exists a puzzle piece \( W \) at some depth containing \( c \), and an integer \( s > 1 \) such that \( g^s(c') \in W \) for every critical point \( c' \in W \) and every \( k \geq 0 \). Combinatorially periodic critical points are never mapped to \( \infty \) and do not belong to the basins of roots. Moreover, the following lemma shows that all combinatorially periodic critical points are, in fact, renormalizable in the sense of Douady–Hubbard: each of them lies in the non-escaping set of a suitable polynomial-like restriction (with domain being the interior of a puzzle piece). This lemma is a slight modification of [DLSS, Proposition 3.16], and essentially follows from Lemma 4.5.

**Lemma 4.7** (Polynomial-like renormalization at combinatorially periodic points). If \( x \) is a combinatorially periodic critical point, then there exists \( n > 0 \) and a least \( k = k(n) > 0 \) so that the map \( g^k : \hat{P}_{n+k}(x) \to \hat{P}_n(x) \) is a polynomial-like map of degree \( d \geq 2 \) with connected filled Julia set equal to \( \text{fib}(x) \). Moreover, for sufficiently large \( n \) and for any two combinatorially periodic points \( x \) and \( x' \) as above, either \( \text{fib}(x) = \text{fib}(x') \) or \( P_n(x) \cap P_n(x') = \emptyset \).

We obtain that in the Newton setting being combinatorially periodic is equivalent to being renormalizable: the inclusion in one direction is given by Lemma 4.7; inclusion in the other direction follows by definition. (Note here that the renormalization period
$k = k(n)$ from Lemma 4.7 can be larger than the least period coming from combinatorial periodicity.) In the sequel, we will use these terms interchangeably.

As defined in Section 2, a point $x$ is combinatorially recurrent if the orbit of $g(x)$ under $g$ intersects every puzzle piece at $x$ (we had shown in Section 2 that this implies that the orbit visits every such puzzle piece infinitely often). In this case, we can define a strictly increasing sequence $(n_i)$ of return times as follows, starting at arbitrary $n_0 \geq 0$: given $n_i$, let $k_i$ be minimal so that $g^{k_i}(x) \in P_{n_i}(x)$ and set $n_{i+1} := n_i + k_i$. Then $g^{k_i}$ sends $P_{n_{i+1}}(x)$ to $P_{n_i}(g^{k_i}(x)) = P_{n_i}(x)$.

Every combinatorially periodic (and hence renormalizable, see Lemma 4.7) critical point is combinatorially recurrent. In this case, the sequence $k_i = n_{i+1} - n_i$ is eventually constant. There is a converse to this observation, as follows:

**Lemma 4.8** (First return times for the pullback nest). For a given $n_0 \geq 0$, the sequence $k_i = n_{i+1} - n_i$ associated to a combinatorially recurrent critical point $x$ via the pullback construction above is monotonically increasing. It is bounded (and hence eventually constant) if and only if $x$ is renormalizable.

**Proof.** Monotonicity of $k_i$ follows immediately from the definition that $k_i$ is minimal so that $g^{k_i}(x) \in P_{n_i}(x)$: a larger value of $i$ means a smaller set $P_{n_i}(x)$, and hence it can take only longer to return into the smaller set.

If the sequence $(k_i)$ is bounded, and hence eventually constant, then $x$ is combinatorially periodic with respect to $P_{n_i}(x)$ for $i$ large enough. Thus $x$ is renormalizable by Lemma 4.7. Conversely, if $x$ is renormalizable, then $x$ is periodic with some period $k$ (which is equal to the renormalization period). This implies that eventually $k_i \leq k$; hence the sequence is bounded. \hfill \Box

For combinatorially recurrent critical points that are not renormalizable we can assure that the boundaries of the puzzle pieces in the pullback nest constructed above are disjoint for all sufficiently large depths, due to the following lemma.

**Lemma 4.9** (Non-renormalizable recurrent points are well inside). For every $n_0 \geq 0$, if $x$ is a combinatorially recurrent critical point that is non-renormalizable, and $(P_{n_i}(x))_{i \geq 0}$ is the nest obtained by pulling back $P_{n_0}(x)$ along the orbit of $x$, then there exists $j$ large enough so that $P_{n_{i+1}}(x) \subset P_{n_i}(x)$ for all $i \geq j$.

**Proof.** The proof is similar to the proof of [DMRS, Proposition 3.10]. First of all, possibly by increasing $n_0$ we can assume that $\partial P_{n_0}(x)$ is disjoint from $\infty$: this is possible since $\infty$ has trivial fiber (Lemma 4.4), and hence the puzzle pieces containing $\infty$ cannot intersect all elements in the nest $(P_{n_i}(x))_{i \geq 0}$. Hence we can assume that the boundaries of the puzzle pieces in the nest are disjoint from periodic points of $g$.

Since $\partial P_{n_0}(x)$ contains finitely many (pre-)poles and no periodic points, there exists $k$ large enough so that for all (pre-)poles $w \in \partial P_{n_0}(x)$ and all $r \geq k$ we have $g^r(w) \notin \partial P_{n_0}(x)$. By Lemma 4.8, the sequence $(k_i)_{i \geq 0}$ of first return times tends to infinity. Therefore, there is a minimal $j$ so that $k_j \geq k$. Let us show that the lemma holds for this $j$. If not, then there exists $i \geq j$ with $\partial P_{n_{i+1}}(x) \cap \partial P_{n_i}(x) \neq \emptyset$. This intersection, by construction of
puzzle pieces (see the beginning of Subsection 4.1), must contain a (pre-)pole, say $z$. Hence $\partial P_{n_i}(x)$ contains the (pre-)poles $z$ and $g^{k_i}(z)$, and they are distinct (there are no periodic points on the boundary). Mapping these two points forward, it follows that $\partial P_{n_0}$ must contain a pair of distinct (pre-)poles $w$ and $g^{k_i}(w)$. But $k_i \geq k_j \geq k$, and this contradicts our choice of $k$. \hfill \Box

5. Proof of dynamical rigidity for Newton maps (Theorem A)

We will prove Theorem A by applying the following strategy. In the dynamical plane of the Newton map $N_p$ (or its iterate) we will extract a box mapping that contains all critical points that do not belong to the Newton graph at any level, and hence will be able to conclude rigidity-or-polynomial-like dynamics using Theorem C. The remaining cases will be treated by means of Lemma 4.4 and Lemma 4.6.

We start implementing our strategy by making the following case distinction of the critical points

\begin{equation}
\text{Crit}(g) = \text{Crit}_B \sqcup \text{Crit}_I \sqcup \text{Crit}_{nR} \sqcup \text{Crit}_{RnR} \sqcup \text{Crit}_R
\end{equation}

as follows:

- (B) we say that $c \in \text{Crit}_B$ if the orbit of $c$ converges to one of the roots of the polynomial $p$ (critical points in the Basin of a root);
- (I) we say that $c \in \text{Crit}_I$ if the orbit of $c$ lands at $\infty$ (critical points landing at Infinity);
- (nR) we say that $c \in \text{Crit}_{nR} \setminus (\text{Crit}_I \sqcup \text{Crit}_B)$ if $c$ is a combinatorially non-Recurrent critical point;
- (RnR) we say that $c \in \text{Crit}_{RnR}$ if $c$ is a combinatorially Recurrent non-Renormalizable critical point;
- (R) we say that $c \in \text{Crit}_R$ if $c$ is a (combinatorially recurrent) Renormalizable critical point.

By construction, the last three classes of critical points are disjoint (note that the labels allow a unique interpretation if we want to make a complete case distinction between combinatorially non-recurrent, combinatorially recurrent non-renormalizable, and renormalizable (thus necessarily combinatorially recurrent) critical points).

In what follows, if not explicitly mentioned otherwise, every orbit will be understood as an orbit under iterations of $g$. Similarly, puzzle pieces, fibers and triviality of fibers will be understood for the iterate $g$ of the Newton map $N_p$.

The following two lemmas (Lemma 5.1 and 5.2) guarantee that out of the Newton dynamical plane we can extract a box mapping that captures the dynamics of all non-renormalizable critical points of $g$.

Lemma 5.1 (Newton box mappings for recurrent non-renormalizable critical points). Suppose $\text{Crit}_{RnR} \neq \emptyset$. Then there exists a box mapping $F: U \to V$ such that $\text{Crit}_{RnR} \subset V$. Moreover, this box mapping has the property that it has no (NE) components and for every point $z \in \mathbb{C}$ whose orbit intersects $V$ in an infinite set it follows that $\text{orb}(z) \cap V$ lies in the non-escaping set of $F$. 
Proof. The explicit construction of such a box mapping is similar to [KvS1, Corollary 2.1]. The details are as follows.

Since all fibers of points in \( \text{Crit}(g) \setminus (\text{Crit}_B \sqcup \text{Crit}_I) \) are compactly contained in the corresponding critical puzzle pieces of any depth (by Lemma 4.5), there exists a depth \( s > 0 \) so that \( P_s(c) \cap \text{Crit}(g) \subseteq \text{fib}(c) \) for every \( c \in \text{Crit}(g) \setminus (\text{Crit}_B \sqcup \text{Crit}_I) \), and two critical puzzle pieces of depth \( s \) centered at points in \( \text{Crit}(g) \setminus (\text{Crit}_B \sqcup \text{Crit}_I) \) are either disjoint, or coincide. Everywhere below we assume that the critical puzzle pieces are chosen of depth at least \( s \). In particular, this assumption guarantees that all critical puzzle pieces are chosen at sufficiently large depth so that they do not intersect \( \text{Crit}_B \sqcup \text{Crit}_I \).

We argue by induction over the set \( \text{Crit}_{RnR} \).

**Step 1: Base of induction.** Let \( c_1 \in \text{Crit}_{\mathbb{R}_nR} \) be a combinatorially recurrent non-renormalizable critical point. Consider a puzzle piece \( W_1 \) with \( c_1 \in W_1 \) and set \( \hat{W}_1 := \{ z \in \hat{C} : \exists n: g^n(z) \in W_1 \} \) (the set of points that eventually map to the interior of \( W_1 \)). Let \( W'_1 \) be the union of those components in \( \hat{W}_1 \) that contain a critical point. We want to choose \( W_1 \) so that \( W_1 \cap W'_1 \) is compactly contained in \( W_1 \) (note that \emph{a priori} \( W_1 \cap W'_1 \) and \( W_1 \) can have intersecting boundaries). This is done by choosing \( W_1 \) to be a puzzle piece of sufficiently large depth, see Lemma 4.9. Since, by our standing assumption, \( W_1 \) is chosen of depth at least \( s \), it follows that \( W_1 \cap W'_1 \) consists of a single critical puzzle piece, say \( W_1^* \), and the intersection of \( W_1^* \) with \( \text{Crit}(g) \) is contained in \( \text{fib}(c_1) \). (In particular, this means that \( \text{fib}(c_1) \) does not contain non-recurrent critical points.) Set \( W_1 := W_1^* \).

**Step 2: Inductive step.** Assume \( W_k \) and \( c_1, \ldots, c_k \) are chosen (for some \( k \geq 1 \)). If all combinatorially recurrent non-renormalizable critical points of \( g \) eventually map to \( W_k \), then the induction for \( \text{Crit}_{\mathbb{R}_nR} \) is over. Otherwise, choose another combinatorially recurrent non-renormalizable critical point \( c_{k+1} \in \text{Crit}_{\mathbb{R}_nR} \) that is never mapped to \( W_k \). Find a puzzle piece \( W_{k+1} \) with \( c_{k+1} \in W_{k+1} \) having the same properties as \( W_k \), and of depth at least the depths of all puzzle pieces in \( W_k \), and set \( W_{k+1} := W_k \cup W^*_{k+1} \), with \( W^*_{k+1} \) defined similarly to \( W_1^* \).

Suppose that the induction for \( \text{Crit}_{\mathbb{R}_nR} \) ended up with the set \( W_m \). By construction, any point in \( \text{Crit}_{\mathbb{R}_nR} \) either lies in, or is mapped to \( W_m \). Using \( W_m \), we can now associate to the set of all recurrent non-renormalizable critical points a box mapping \( F : \mathcal{U} \to \mathcal{V} \) with \( \text{Crit}_{\mathbb{R}_nR} \subseteq \mathcal{V} \) as follows.

Set \( V := \left\{ z \in \hat{C} : \exists n: g^n(z) \in W_m \right\} \). Since \( W_m \) is a finite union of interiors of puzzle pieces, it follows that each component of \( V \) has a constant number of iterations until the orbit reaches \( W_m \), and hence \( V \) is a (possibly infinite) union of interiors of puzzle pieces. Define \( \mathcal{V} \) to be the union of \( W_m \) and all components in \( V \setminus W_m \) that contain critical points. Again, since all puzzle pieces in question are at least of depth \( s \), each critical component in \( \mathcal{V} \) contains a single critical fiber. Furthermore, by construction of \( W_m \), we have guaranteed \( \text{Crit}_{\mathbb{R}_nR} \subseteq \mathcal{V} \).

We then define \( \mathcal{U} := \left\{ z \in V : \exists n: g^n(z) \in V \right\} \) to be the set of points in \( \mathcal{V} \) that eventually return to \( \mathcal{V} \). By the Markov property (see Definition 2.1), all connected components of \( \mathcal{U} \) are (interiors of) puzzle pieces; hence, we have a well-defined map \( F : \mathcal{U} \to \mathcal{V} \), where \( F \) restricted to a component \( U \) of \( \mathcal{U} \) is a proper map \( g^n : U \to \mathcal{V} \) between \( U \) and a component
$V$ of $\mathcal{V}$. The set $\mathcal{U}$ is the first return domain for $\mathcal{V}$ in the following sense: for each proper map $g^n: U \to V$ the sets in the sequence $(g^i(U))_{i=1}^{n-1}$ are disjoint from $\mathcal{V}$. (In particular, this guarantees that the number of critical points of $F$ is finite, and in fact is a subset of $\text{Crit}(g)$ containing $\text{Crit}_{\mathcal{R}_n\mathcal{R}}$; see the explanation in the next paragraph.)

The map $F: \mathcal{U} \to \mathcal{V}$ is a box mapping in the sense of Definition 3.1. Indeed, the critical components of $V \setminus \mathcal{W}_m$ are simultaneously components of $\mathcal{U}$ and $\mathcal{V}$, and hence there only finitely many of those; moreover, only for finitely many components $U$ in $\mathcal{W}_m \cap \mathcal{U}$ the map $F|_{U}$ will be of degree greater than one: these will be the components containing respectful critical fibers $\text{fib}\left(\mathcal{C}_k\right)$, and there will be $m$ of such components in total. Therefore, $F$ has only finitely many critical points (property (1) of Definition 3.1). Properties (2) and (4) are satisfied by construction. Finally, by Lemma 2.8, since each of the puzzle pieces $\mathcal{W}_k$ is weakly protected by $\mathcal{W}_k$, we conclude that the first return domain for $\mathcal{W}_m$ is compactly contained in $\mathcal{W}_m$. This guarantees property (3) of Definition 3.1.

Let us now show that the box mapping $F: \mathcal{U} \to \mathcal{V}$ has no (NE) components, and that every orbit that intersects $\mathcal{V}$ in an infinite set, in fact, lands in the non-escaping set of $F$.

Absence of periodic components without escaping points ((NE) components) follows by construction: any cycle of (NE) complements of $\mathcal{U}$ should pass through the set $\mathcal{W}_m \subset \mathcal{V}$. But this is impossible since all components in $\mathcal{W}_m \cap \mathcal{U}$ are compactly contained in $\mathcal{W}_m$ by the way how we defined the box mapping.

Finally, since $\mathcal{U}$ was constructed as a first return domain to $\mathcal{V}$, the orbit of any point $z$ intersecting $\mathcal{V}$ in an infinite set must, in fact, satisfy $\text{orb}(z) \cap \mathcal{V} \subset \mathcal{U}$. Hence, the set $\text{orb}(z) \cap \mathcal{V}$ lies in the non-escaping set of $F$ (see the end of the proof of Theorem C for the similar argument). In particular, $\text{Crit}_{\mathcal{R}_n\mathcal{R}}$ is in the non-escaping set of $F$. The lemma is proven.

\begin{lemma}[Newton box mappings for non-recurrent critical points] Let $\text{Crit}'_{\mathcal{R}_n\mathcal{R}}$ be the subset of $\text{Crit}_{\mathcal{R}_n\mathcal{R}}$ consisting of all non-recurrent critical points that do not accumulate on or are mapped to any of $\text{fib}(c)$ with $c \in \text{Crit}_{\mathcal{R}_n\mathcal{R}} \cup \text{Crit}_R$. If $\text{Crit}'_{\mathcal{R}_n\mathcal{R}} \neq \emptyset$, then there exists a box mapping $F: \mathcal{U} \to \mathcal{V}$ such that $\mathcal{V} \cap \text{Crit}(g) = \text{Crit}'_{\mathcal{R}_n\mathcal{R}}$. Moreover, $F$ has the property that it has no (NE) components and for every point and for every point $z \in \mathbb{C}$ whose orbit intersects $\mathcal{V}$ in an infinite set it follows that $\text{orb}(z) \cap \mathcal{V}$ lies in the non-escaping set of $F$.
\end{lemma}

\begin{proof}
Similarly to Lemma 5.1, we assume that all puzzle pieces are chosen of depth at least $s$. By increasing $s$ if necessary, we may assume that for every $c \in \text{Crit}'_{\mathcal{R}_n\mathcal{R}}$ the orbit of $c$ never re-enters $P_s(c)$.

Since the points in $\text{Crit}'_{\mathcal{R}_n\mathcal{R}}$ never mapped to or accumulate at the fibers of points in $\text{Crit}_{\mathcal{R}_n\mathcal{R}} \cup \text{Crit}_R$, their combinatorial mapping behavior can be easily understood as follows. For a pair of points $x$ and $y$, we say that $x > y$ if $\text{orb}(x) \cap \text{fib}(y) \neq \emptyset$, but $\text{orb}(y) \cap \text{fib}(x) = \emptyset$. Since all critical points in $\text{Crit}'_{\mathcal{R}_n\mathcal{R}}$ are combinatorially non-recurrent, it is impossible for $c, c' \in \text{Crit}'_{\mathcal{R}_n\mathcal{R}}$ to have $\text{orb}(c) \cap \text{fib}(c') \neq \emptyset$ and $\text{orb}(c') \cap \text{fib}(c) \neq \emptyset$. Therefore, the set $\text{Crit}'_{\mathcal{R}_n\mathcal{R}}$ with the strict order $<$ is a partially (strictly) ordered set. Let $c^*$ be a minimal element in $\text{Crit}'_{\mathcal{R}_n\mathcal{R}}$ with respect to $<$. The orbit of $c^*$ cannot land in or accumulate on the fibers of points from $\text{Crit}_{\mathcal{R}_n\mathcal{R}}$ (since $c^*$ is minimal); moreover, it cannot intersect or accumulate
on the fibers of points from \( \text{Crit}_{R_n R} \sqcup \text{Crit}_R \) by the definition of \( \text{Crit}_{n R}' \). Lemma 4.6 then guarantees \( \text{fib}(c^*) = \{ c^* \} \).

Define \( W := \bigcup_{c^* \in \text{Crit}_{n R}'} \tilde{P}_m(c^*) \), where the union is taken over all minimal elements in \( \text{Crit}_{n R}' \), and the depth \( m \) is chosen to be so that the (interiors of) puzzle pieces in \( W \) have disjoint closures and \( P_m(c^*) \) is compactly contained in \( P_{s}(c^*) \) for every minimal element \( c^* \). Such \( m \) exists because \( \text{fib}(c^*) = \{ c^* \} \).

As in Lemma 5.1, set \( V := \{ z \in \hat{C} : \exists n : g^n(z) \in \hat{W} \} \). By construction of \( W \), any component \( Y \) of \( V \setminus W \) that contains a critical point (note that these critical points can be only from \( \text{Crit}_{n R}' \)) is an interior of a critical puzzle piece, and there exists \( k \geq 1 \) and a minimal element \( c^* \) such that \( g^k : Y \to \tilde{P}_m(c^*) \) is a branched covering. Possibly by increasing \( m \), we can assume that \( Y \cap W = \emptyset \) for every such \( Y \). Finally, define \( V \) to be the union of \( W \) and all \( Y \) components in \( V \setminus W \).

Similarly as in the proof of Lemma 5.1, define \( U := \{ z \in V : \exists n : g^n(z) \in V \} \) to be the first return domain to \( V \). Again, we have a well-defined map \( F : U \to V \) which is a box mapping in the sense of Definition 3.1: all \( Y \) components of \( V \setminus W \) will be simultaneously components of \( U \) and \( V \), and for every \( c^* \) each component of \( U \cap \tilde{P}_n(c^*) \) is compactly contained in \( P_n(c^*) \), by Lemma 2.9.

Finally, \( F \) has no periodic components without escaping points ((NE) components) since every component of \( U \) is eventually mapped to \( W \), and each of the components in \( W \) has an escaping point (the respective \( c^* \)). The orbit property for \( F \) follows similarly as in Lemma 5.1. \( \square \)

**Proof of Theorem A.** The statement of Theorem A is invariant under passing to an iterate of \( N_p \) and is unaffected by the quasiconformal surgery as defined in Section 4. Therefore, we can assume that \( N_p \) is attracting-critically-finite, and consider the iterate \( g = N_p^M \) with the well-defined puzzle partition.

Let \( z \in \hat{C} \) be given. Consider the following cases (listed in the order in which we will deal with them later):

1. \( \overline{\text{orb}(z)} \cap \text{Crit}_B \neq \emptyset \), that is \( z \) lies in the basins of roots and is being attracted to one of them;
2. \( \text{orb}(z) \cap \text{Crit}_I \neq \emptyset \), that is the orbit of \( z \) lands at a (pre-)pole which is critical.
3. \( \overline{\text{orb}(z)} \cap (\text{Crit}(g) \setminus \text{Crit}_I) = \emptyset \) and \( \text{orb}(z) \cap \text{Crit}_I = \emptyset \); that is \( z \) is not a (pre-)pole and the \( \omega \)-limit set of \( \text{orb}(z) \) does not intersect the critical set of \( g \), except, perhaps, critical (pre-)poles.
4. \( \text{orb}(z) \cap \text{Crit}_{R_n R} \neq \emptyset \), that is the orbit of \( z \) contains or accumulates on some combinatorially recurrent non-renormalizable critical point;
5. \( \text{orb}(z) \cap \text{Crit}_R \neq \emptyset \), but \( \text{orb}(z) \cap \text{Crit}_{R_n R} = \emptyset \); in other words, the orbit of \( z \) contains or accumulates on some renormalizable critical point and does not accumulate on or intersect the set of recurrent non-renormalizable critical points;
orb($z$) $\cap$ Crit$_{nR} \neq \emptyset$, but orb($z$) $\cap$ (Crit$_{RnR}$ $\cup$ Crit$_R$) = $\emptyset$; that is, the orbit of $z$ contains or accumulates on a non-recurrent critical point and does not accumulate on or intersect any of the recurrent critical points;

By construction, all the cases above are disjoint. Let us show in each of the cases which of the alternatives (B), (T), or (R) in Theorem A holds. This will finish the proof of the theorem.

1. This is precisely case (B) of Theorem A.
2. If orb($z$) $\cap$ Crit$_I \neq \emptyset$, then fib($z$) = {$z$} by Lemma 4.4. Triviality of the fiber implies that we are in case (T) of Theorem A.
3. Suppose orb($z$) does not intersect the critical set of $g$, except, perhaps, at poles or pre-poles which are critical. But then we are in position of applying Lemma 4.6, from which we conclude that $z$ satisfies property (T) of Theorem A.
4. By Lemma 5.1, there exists a box mapping $F: U \rightarrow V$ such that $V \cap$ Crit($g$) $\supset$ Crit$_{RnR}$. Therefore, if the orbit of $z$ accumulates on Crit$_{RnR}$, then it also intersects the filled-in Julia set $K(F)$ of $F$, again by Lemma 5.1. Since the box mapping $F$ contains no (NE) components (still Lemma 5.1), and the orbit of $z$ is not of (CB) type (those are taken care of in case (3)), the conclusion of Theorem A follows from the generalized rigidity principle (Theorem C).
5. Suppose $c \in$ Crit$_R$ is a point the orbit of $z$ accumulates at. Since $c$ is renormalizable, by Lemma 4.7 there exists a polynomial-like mapping $\varrho: U \rightarrow V$ with the filled Julia set equal to fib($c$). Therefore, if the orbit of $z$ lands in fib($c$), then we are in case (R) of Theorem A. Otherwise, since the Julia set fib($c$) necessarily contains a repelling periodic point, we conclude fib($z$) = {$z$} by Lemma 2.7. Note that the critical fibers that are not fibers of pre-poles, are compactly contained in puzzle pieces of any depth by Lemma 4.5; hence, Lemma 2.7 can be applied for the point $z$. Therefore, $z$ satisfies property (T).
6. Define Crit$_{nR}'$ to be the maximal subset in Crit$_{nR}$ of points whose orbits do not intersect or accumulate on Crit$_{RnR}$ $\cup$ Crit$_R$. It follows that orb($z$) $\cap$ Crit$_{nR}'$ $\neq$ $\emptyset$. By Lemma 5.2, there exists a box map $F: U \rightarrow V$ with $V \cap$ Crit($g$) = Crit$_{nR}'$. By the generalized rigidity principle (Theorem C), all points in the non-escaping set of $F$ have trivial fibers (observe that $F$ has no box renormalizable critical points). As in case (4), using the result of Lemma 5.2, we conclude fib($z$) = {$z$}. This argument finishes the proof of the dynamical rigidity for Newton maps.

Using Theorem A, we can now prove that for every Newton map, each of its immediate basins has locally connected boundary (Corollary 1.2). This result clearly extends to all (non-immediate) components of the basins.

Proof of Corollary 1.2. We do not lose generality by considering a polynomial Newton map $N_p$ that is attracting-critically-finite because the relevant surgery leaves the claim unchanged. Moreover, every rational map has the same Julia set as any of its iterates, so we may replace $N_p$ by an iterate $g := N_p^M$ and thus work in the setting of Section 4.

Let $\xi$ be a root of $p$ with immediate basin $U_\xi$. By Theorem A, every $z \in \partial U_\xi$ either has trivial fiber or it maps after finitely many iterations to the little filled-in Julia set, say $K$. 

of a polynomial-like restriction of \(g\) with connected Julia set. In the first case, \(U_\xi\) is locally connected at \(z\) and we are done, so (after replacing \(z\) by a point on its orbit) it suffices to assume that \(z \in K\).

The little Julia set comes with a polynomial-like restriction \(g^k: \hat{P}_{n+k}(z) \to \hat{P}_{n}(z)\) for some \(n \geq 0\) and \(k \geq 1\), and all critical points of \(g^k\) on \(\hat{P}_{n}(z)\) are already in \(K\). We claim that \(z\) is fixed by \(g^k\), that it is the only point in \(K \cap \partial U_\xi\), and that its fiber intersects \(\partial U_\xi\) only in \(\{z\}\). A single periodic dynamic ray within \(U_\xi\) lands at \(z\), and the impression of this ray consists of the point \(z\) alone. This establishes local connectivity of \(U_\xi\) at \(z\) as well.

For fixed \(n\) so that \(P_n(z)\) does not contain any critical points that are not in \(K\), define the annulus \(A := P_n(z) \setminus K\). For every \(w_0 \in A\), every \(w_1 \in (g^k)^{-1}(w_0)\) and every simple curve \(\gamma_0 \subset A\) connecting \(w_0\) to \(w_1\), one can construct a curve \(\gamma\) that starts at \(w_0\) and converges to \(\partial K\) by connecting a preimage component \(\gamma_1\) of \(\gamma_0\) to \(w_1\), then connecting another preimage component of \(\gamma_1\) to the end of \(\gamma_1\), and so on. A standard hyperbolic contraction argument shows that this curve must converge to a point \(p' \in \partial K\) that is fixed by \(g^k\) (after straightening, this curve is homotopy to a dynamic ray landing at the straightened image of \(p'\) in the homotopy class (relative to the filled Julia set) of a fixed dynamic ray).

There are only finitely many fixed points of \(g^k\) on \(\partial K\) and only finitely many homotopy classes of fixed rays, so many choices of \(w_0, w_1\) and \(\gamma_0\) will lead to homotopic curves. We will show that one such invariant curve lands at \(z\) that is an internal ray of \(U_\xi\) fixed by \(g^k\), and there are two further curves that land at \(z\) that together separate \(K \setminus \{z\}\) from \(U_\xi\).

Indeed, since \(N_p\) is attracting-critically-finite, the dynamics of \(N_p\) on \(U_\xi\) is conformally conjugate to \(z \mapsto z^m\) on \(\mathbb{D}\) for some \(m \geq 2\), so dynamic rays on \(U_\xi\) are well defined together with their usual mapping properties, and periodic rays land at periodic points.

By construction of puzzles, the puzzle piece \(P_n(z)\) intersects \(U_\xi\) in a domain, say \(D_n\), bounded by two pre-fixed dynamic rays, say \(R_n\) and \(R_n'\), together with some equipotential in \(U_\xi\). We clearly have \(D_{n+1} \subset D_n\) for all \(n\), with common boundary only on \(\partial U_\xi\). The rays \(R_n\) and \(R_n'\) must converge from both sides to a single ray, say \(R\), that is fixed by \(g^k\). This ray \(R\) must land at a point \(p \in \partial U_\xi\) that is also fixed by \(g^k\).

One particular example of a curve \(\gamma \subset A\) that lands at a fixed point in \(\partial K\) can be constructed by choosing \(w_0, w_1\) and \(\gamma_0\) on \(R\); then the entire curve \(\gamma\) is a subset of \(R\) and lands at \(p\).

Construct another curve \(\Gamma\) as follows: connect the landing points of the rays \(R_{n+1}\) and \(R_{n+2}\) within \(A\) by a curve \(\Gamma_0 \subset A\) that avoids the ray \(R\) (this fixes the homotopy class of this curve in \(A\)) and extend as before. The two curves \(\Gamma_0\) and \(\gamma_0\) (between \(w_0\) and \(w_1\)) have finite hyperbolic distance within \(A\), and this distance is preserved by taking preimages (with respect to the hyperbolic distance of preimage domains of \(A\); hence the distance is contracted with respect to \(A\)), so both curves must land at the same point \(p\). A third curve \(\Gamma'\) that lands at \(p\) can be constructed analogously starting from the landing points of the rays \(R'_{n+1}\) and \(R'_{n+2}\). The hyperbolic distance argument shows that all three curves land through the same access to \(p\) relative to \(K\). Therefore, the union \(\Gamma \cup \{p\} \cup \Gamma'\) disconnects
$P_{n+1}(z)$, but it does not disconnect $K$. Therefore, it separates $K \setminus \{p\}$ from $U_\xi$ within $P_{n+1}(z)$.

The conclusions now follow: the only point in $K \cap \partial U_\xi$ is $p = z$, so $z$ is fixed by $g^k$. Therefore the fiber of $z$, which is $K$, intersects $U_\xi$ only in $\{z\}$. This shows that $U_\xi$ is locally connected at $z$ (and moreover that the impression of the ray $R$ consists of the point $z$ alone). □

6. Proof of parameter rigidity for Newton maps (Theorem B)

In this section we prove parameter rigidity for Newton maps (Theorem B). This will be accomplished by combining the rigidity results of Kozlovski–van Strien [KvS1, KvS2] together with our results from Section 5.

Let $N_p: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be an attracting-critically-finite Newton map, and $\Delta_n$ be the Newton graph of level $n \geq 0$ for $N_p$. Let $L > 0$ be the smallest level such that $\Delta_L$ contains all the critical points that eventually land on the Newton graph (either at $\infty$ or at a root), as well as all poles of $N_p$; such a level exists by Theorem 4.2. Similarly, let $N_{\tilde{p}}: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be another attracting-critically-finite Newton map with Newton graph $\tilde{\Delta}_n$ at level $n \geq 0$, and level $\tilde{L}$ analogous to $L$.

**Definition 6.1** (Combinatorial equivalence of Newton maps). Two attracting-critically-finite Newton maps $N_p$ and $N_{\tilde{p}}$ are combinatorially equivalent if

1. $L = \tilde{L}$, and the Newton graphs $\Delta_L$ and $\tilde{\Delta}_L$ are homeomorphic and topologically conjugate, respecting vertices;
2. there is a bijection between critical points of $N_p$ on $\hat{\mathbb{C}} \setminus \Delta_L$ and of $N_{\tilde{p}}$ on $\hat{\mathbb{C}} \setminus \Delta_L$ that respects degrees and itineraries with respect to (complementary components of) the Newton graphs.

Two Newton maps (not necessarily attracting-critically-finite) are combinatorially equivalent if the quasiconformal surgery in the basins of roots (described in Section 4) turns them into combinatorially equivalent (attracting-critically-finite) Newton maps.

We will deduce Theorem B from the corresponding rigidity results for box mappings. In order to state those, we need to define a notion of combinatorially equivalent box mappings. For a given box mapping $F$, let $PC(F)$ be the postcritical set of $F$.

**Definition 6.2** (Itinerary of puzzle pieces relative to curve family, [KvS1]). Let $F: U \to V$ be a box mapping without (NE) components, and $X \subset \partial V$ a finite set with one point on each component of $\partial V$. Moreover, let $\Gamma$ be a collection of simple curves in $V \setminus (U \cup PC(F))$, one for each $y \in F^{-1}(X)$, that connects $y$ to a point in $X$. Then for every $n \geq 0$ and for each component $U'$ of $F^{-n}(U)$ there exists a simple curve connecting $\partial U'$ to $X$ of the form $\gamma_0 \ldots \gamma_n$ where $F^k(\gamma_k) \in \Gamma$. The word $(\gamma_0, F(\gamma_1), \ldots, F^n(\gamma_n))$ is called the $\Gamma$-itinerary of $U'$.

Note that the $\Gamma$-itinerary of $U'$ is not uniquely defined (there is a unique finite word for every $y' \in F^{-n}(x) \cap \partial U'$); however, different components of $F^{-n}(U)$ have different $\Gamma$-itineraries.
Definition 6.3 (Combinatorial equivalence of non-renormalizable box mappings, [KvS1]).
Two non-renormalizable complex box mappings \( F: \mathcal{U} \to \mathcal{V} \) and \( \tilde{F}: \tilde{\mathcal{U}} \to \tilde{\mathcal{V}} \), both without (NE) components, are called combinatorially equivalent w.r.t. some homeomorphism \( H: \mathcal{V} \to \tilde{\mathcal{V}} \) with \( H(\mathcal{U}) = \tilde{\mathcal{U}} \) and with \( H(PC(F) \setminus \mathcal{U}) = PC(\tilde{F}) \setminus \tilde{\mathcal{U}} \) if there exists a curve family \( \Gamma \) as in Definition 6.2 so that each critical point \( c \in \text{Crit}(F) \) is mapped to a critical point \( \bar{c} \in \text{Crit}(\tilde{F}) \) with the property that for every integer \( k \geq 0 \) and \( n \geq 0 \), the \( \Gamma \)-itineraries of the puzzle piece \( P_n(F^k(c)) \) coincide with the \( \Gamma \)-itineraries of \( P_n(\tilde{F}^k(c)) \) (where \( \tilde{\Gamma} = H(\Gamma) \)).

(\text{Note that every critical point of } F \text{ may have several itineraries, and they should all coincide with the corresponding critical point of } \tilde{F}.)

Let us now quote the main parameter rigidity results for non-renormalizable box mappings due to Kozlovski–van Strien (essentially proven in [KvS1], with some clarifying remarks given in [KvS2]). For a given box mapping \( F: \mathcal{U} \to \mathcal{V} \) and a point \( x \in \mathcal{U} \), define \( r_F(x) := \text{mod}(P_0(x) \setminus \mathcal{P}_1(x)) \), where, by definition of puzzle pieces, \( P_0(x) \), resp. \( \mathcal{P}_1(x) \), is the component of \( \mathcal{V} \), resp. \( \mathcal{U} \), containing \( x \). Finally, for a given \( \delta > 0 \) write \( K_{\delta}(F) := \{ x \in K(F) : \limsup_{k \to \infty} r_F(F^k(x)) > \delta \} \). The set \( K_{\delta}(F) \) is a subset of the non-escaping set of \( F \) consisting of points that visit only those components of \( \mathcal{U} \) that are “well inside” the respective components of \( \mathcal{V} \) (the “wellness” is measured by \( \delta \)). Clearly, for every \( \delta > 0 \) the set \( K_{\delta}(F) \) contains no points that converge to \( \partial \mathcal{V} \) (property (CB)) or that belong to an (NE) component.

Theorem 6.4 ([KvS1]). The Julia set of any non-renormalizable complex box mapping with only repelling periodic points carries no measurable invariant line fields. \( \square \)

Theorem 6.5 ([KvS1, KvS2]). Let \( F: \mathcal{U} \to \mathcal{V} \) be a non-renormalizable complex box mapping whose periodic points are all repelling and that contains no (NE) components. Assume that \( \tilde{F}: \tilde{\mathcal{U}} \to \tilde{\mathcal{V}} \) is another complex box mapping without (NE) components for which there exists a quasiconformal homeomorphism \( H: \mathcal{V} \to \tilde{\mathcal{V}} \) so that

1. \( H(\mathcal{U}) = \tilde{\mathcal{U}} \);
2. \( \tilde{F} \circ H = H \circ F \) on \( \partial \mathcal{U} \);
3. \( \tilde{F} \) is combinatorially equivalent to \( F \) w.r.t. \( H \).

Moreover, assume that \( K(F) = K_{\delta}(F) \) and \( K(\tilde{F}) = K_{\delta}(\tilde{F}) \) for some \( \delta > 0 \), and that the boundary of each component of \( \mathcal{U}, \mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}} \) consists of piecewise smooth arcs. Then \( F \) and \( \tilde{F} \) are quasiconformally conjugate, and this conjugation agrees with \( H \) on the boundary of \( \mathcal{V} \). \( \square \)

Note that it is not necessary to assume that \( \tilde{F} \) is non-renormalizable or has only repelling periodic points (see the remark after the statement of Theorem 1.4 in [KvS1]).

Suppose \( N_p \) and \( \tilde{N}_p \) are combinatorially equivalent Newton maps, and let \( g \), resp. \( \tilde{g} \), be the iterates of \( N_p \), resp. \( \tilde{N}_p \), for which we have well-defined Markov partitions as in Section 4. Let \( Y_n \) be a puzzle piece of depth \( n \geq 0 \) defined for \( g \), and \( \tilde{Y}_n \) be the corresponding puzzle piece defined for \( \tilde{g} \). We will say that a homeomorphism \( h: \tilde{Y}_n \to \tilde{Y}_n \) respects the
standard boundary marking if \( h \) extends to a continuous map \( \overline{h} : \partial \mathcal{Y}_n \to \partial \overline{\mathcal{Y}}_n \), and this extension agrees with the map induced by the homeomorphism between \( \Delta_L \) and \( \overline{\Delta}_L \).

**Proof of Theorem B.** Consider two Newton maps \( N_p \) and \( N_{\overline{p}} \) that are combinatorially finite. For simplicity, we first consider the case that they are both attracting-critically-finite. In Section 4, we constructed a unique minimal iterate \( M \) so that \( g := N_p^M \) has a well-defined Markov partition. By combinatorial equivalence, the same holds for \( \overline{g} := N_{\overline{p}}^M \) with the same iterate \( M \).

Since the Newton maps in question are combinatorially equivalent, the itineraries of the corresponding critical points for \( g \) and \( \overline{g} \) with respect to the puzzle partitions will be equal (possibly by increasing the starting depth of the partitions in order to guarantee that all critical prepoles are on the puzzle boundary of zero depth).

Let \( n \geq 0 \) be a large enough depth of the puzzle so that for \( g \) and \( \overline{g} \) if a pair of critical puzzle pieces of depth \( n \) coincide, i.e. \( P_n(c_1) = P_n(c_2) \) and \( \overline{P}_n(\overline{c}_1) = \overline{P}_n(\overline{c}_2) \), then the \( g \)-fibers of \( c_1 \) and \( c_2 \) are equal, as well as the \( \overline{g} \)-fibers of \( \overline{c}_1 \) and \( \overline{c}_2 \).

Let \( P_n \), resp. \( \overline{P}_n \), be the union of the puzzle pieces of \( g \), resp. \( \overline{g} \), of depth \( n \), and let \( U_0 \), resp. \( \overline{U}_0 \), be the union of the interiors of the critical puzzle pieces in \( P_n \), resp. \( \overline{P}_n \). Construct a quasiconformal homeomorphism \( h_0 : U_0 \to \overline{U}_0 \) such that:

- for every \( P_n \in P_n \) with \( \overline{P}_n \subset U_0 \), if \( \overline{P}_n \in \overline{P}_n \) is the corresponding puzzle piece, then \( h_0(P_n) = \overline{P}_n \);
- \( h_0 \) respects the standard boundary marking;
- if \( g \), and hence \( \overline{g} \), is renormalizable, \( g^k : \overline{P}_{n+k} \to \overline{P}_n \) is a polynomial-like restriction of \( g \), and \( \overline{g}^k : \overline{P}_{n+k} \to \overline{P}_n \) is the corresponding polynomial-like restriction of \( \overline{g} \), and \( \varphi : \overline{P}_n \to \overline{P}_n \) is a hybrid equivalence between these restrictions, then \( h_0|_{\overline{P}_n} = \varphi \).

Note that the last property of \( h_0 \) can be guaranteed due to the hypothesis that there exists a bijection between domains of renormalization of the Newton maps that respects hybrid equivalence and combinatorial positions. More precisely this means that for a given polynomial-like map \( g^k : \overline{P}_{n+k} \to \overline{P}_n \) (see Lemma 4.7), the map \( g^k : \overline{P}_{n+k} \to \overline{P}_n \) between the corresponding to \( P_{n+k} \) and \( P_n \) puzzle pieces \( \overline{P}_{n+k} \) and \( \overline{P}_n \) is a polynomial-like mapping, and it is hybrid equivalent to \( g^k : P_{n+k} \to P_n \).

Let \( V \) be the union of all connected components of \( U_0 \) containing all non-renormalizable critical fibers of \( g \). Set \( U'_0 := U_0 \setminus V \); this will be the union of the components containing all renormalizable critical fibers. Define \( \overline{V} \) and \( \overline{U}'_0 \) for \( \overline{U}_0 \) in an analogous way. Construct non-renormalizable box mappings \( F : U \to V \) and \( \overline{F} : \overline{U} \to \overline{V} \) by considering first return domains to \( V \), resp. \( \overline{V} \), under \( g \), resp. \( \overline{g} \) (see Lemmas 5.1 and 5.2 in Section 5). These box maps satisfy conditions (1)–(3) of Theorem 6.5 with the homeomorphism \( h_0 \) (see [KvS1, Proposition 4.1]). Moreover, as it was shown in [KvS2], the construction as in Lemmas 5.1 and 5.2 produces a box mapping for which the filled Julia set equals \( K_\delta \) for some \( \delta > 0 \). Hence, by Theorem 6.5, there exists a quasiconformal homeomorphism \( h_1 : V \to \overline{V} \) that conjugates \( F \) to \( \overline{F} \) and respects the standard boundary marking on the boundary of \( U \).
Let $U_1$, resp. $\tilde{U}_1$, be the set of all critical components of $U$, resp. $\tilde{U}$. By construction, $h_1$ conjugates (the iterates of) $g$ to $\tilde{g}$ on $U_1$. Let $h_2: U_1 \sqcup U'_0 \to \tilde{U}_1 \sqcup \tilde{U}'_0$ be the quasiconformal conjugation between $g$ and $\tilde{g}$ on the respective sets such that $h_2|_{U_1} = h_1$ and $h_2|_{U'_0} = h_0$. By the pullback argument, this map can be globalized to a quasiconformal conjugacy between $g$ and $\tilde{g}$ in some neighborhood of their Julia sets. Moreover, this neighborhood can be chosen to include all the Fatou components not in the basins of the roots. Since the small filled Julia sets of $g$ and $\tilde{g}$ given by the renormalizable dynamics of case (R) in Theorem A are not only quasiconformally conjugate, but also hybrid equivalent (by hypothesis), and furthermore, the remaining points of the Julia sets that fall into case (T) of Theorem A do not carry measurable invariant line fields (by Theorem 6.4), the constructed conjugation has vanishing antiholomorphic derivative on the complement to the basin of the roots. This finishes the proof for $g$ and $\tilde{g}$. The result for $N_p$ and $\tilde{N}_p$ follows by a standard argument because the construction of $g$ and $\tilde{g}$ using the same canonically defined iterate $M$ is natural.

Finally, if $N_p$ and $\tilde{N}_p$ were attracting-critically-finite from the beginning, then they are conformally conjugate on the basins of the roots by definition, and this conjugation extends the quasiconformal conjugation just constructed to an affine conjugation. □

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