THE PSEUDO-LINDLEY ALPHA POWER TRANSFORMED DISTRIBUTION, MATHEMATICAL CHARACTERIZATIONS AND ASYMPTOTIC PROPERTIES

MODOU NGOM †, MOUMOUNI DIALLO ††, ADJA MBARKA FALL †††, AND GANE SAMB LO ††††

Abstract. We introduce a new generalization of the Pseudo-Lindley distribution by applying alpha power transformation. The obtained distribution is referred as the Pseudo-Lindley alpha power transformed distribution (PL-APT). Some tractable mathematical properties of the PL-APT distribution as reliability, hazard rate, order statistics and entropies are provided. The maximum likelihood method is used to obtain the parameters' estimation of the PL-APT distribution. The asymptotic properties of the proposed distribution are discussed. Also, a simulation study is performed to compare the modeling capability and flexibility of PL-APT with Lindley and Pseudo-Lindley distributions. The PL-APT provides a good fit as the Lindley and the Pseudo-Lindley distribution. The extremal domain of attraction of PL-APT is found and its quantile and extremal quantile functions studied. Finally, the extremal value index is estimated by the double-indexed Hill’s estimator (Ngom and Lo, 2016) and related asymptotic statistical tests are provided and characterized.

Keywords. alpha power Transformation of distribution functions; Lindley’s distribution; pseudo-Lindley distribution; extreme value theory; Doubly indexed Hill’s estimator; reliability; hazard rate; maximum likelihood method; quantile function; extreme quantile function; asymptotic laws; Lambert function.

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1. Introduction

In the last decades, the Lindley distribution (see Lindley (1958, 1965)), with parameter $\theta > 0$, has been a center of interests of many research activities. The family of Lindley distribution of one parameter $\theta > 0$ has the following cumulative distribution function (cdf)

\[
F_L(x) = \left(1 - \left(1 + \frac{\theta x}{1 + \theta}\right) \exp(-\theta x)\right) 1_{(x \geq 0)}.
\]

The corresponding probability distribution function (pdf) of (1.1) is given by

\[
f_L(x) = \left(\frac{\theta^2}{1 + \theta} (1 + x) \exp(-\theta x)\right) 1_{(x \geq 0)}.
\]

Lindley’s statistical distribution, which has been proposed as an alternative model to fit data with non-monotone hazard rate, and its different generalizations have attracted a great attention from researchers. This is justified by the importance of such distribution in many areas as reliability for example. Let us cite a few number of important examples. A new bounded domain probability density feature in view of a generalized Lindley distribution is considered in Ghitany et al. (2008). The Lindley distribution has been used for modeling competing risks in lifetime data in Mazucheli and Achcar (2011). A statistical inference on the parameter in its progressive Type-II censoring scheme is provided in Krishna and Kumar (2011). Gomez et al. (2014) use the Log-Lindley distribution in the application of strength systems reliability in the field of insurance and inventory management. A comparison study of the adequacy of exponential and Lindley distributions on modeling of lifetime data is studied by Shanker et al. (2015). A study is carried out by Hafez et al. (2020), using the accelerated life tests under censored sample and the importance of the distribution is introduced applying an experimental application.

However, the Lindley distribution has an increasing failure rate and this makes it not flexible in lifetime data modeling. Because of the above mentioned importance, a significant number of generalizations has been introduced to improve its ability to analyze various types of lifetime data with a high degree of skewness and kurtosis. The later generalizations continue
themselves to be extended. The idea is to correct this flaw by increasing the number of parameters. Indeed, most of the generalizations introduced other parameters in hope of capturing the complexity data in lifetime data. One of these direct generalizations is developed in Zeghdoudi and Nedjar (2016) and is named as the Pseudo-Lindley distribution with two parameters \( \theta > 0 \) and \( \beta > 1 \). The cdf and pdf of the Pseudo-Lindley distribution are defined respectively by

\[
F_{PL}(x) = \left(1 - \beta^{-1} (\beta + \theta x) \exp(-\theta x)\right)1_{(x \geq 0)}, \tag{1.3}
\]

\[
f_{PL}(x) = \left(\frac{\theta (\beta - 1 + \theta x) \exp(-\theta x)}{\beta}\right)1_{(x \geq 0)} \tag{1.4}.\]

Further statistical studies on the Pseudo-Lindley distribution are available in Lo et al. (2020, 2019) which focused on the asymptotic theory of moments estimators, the extreme values characterization and estimations with among other topics. Also, a discrete version of the Pseudo-Lindley distribution is proposed by Irshad et al. (2021) with a stress on the mathematical properties.

On another side, Mahdavi and Kundu (2017) introduced a powerful method of creating new statistical distributions named the alpha power transformed (APT). For any cdf \( F \) with respective lower and upper endpoints

\[\text{lep}(F) = \inf\{x \in \mathbb{R}, F(x) > 0\}, \quad \text{uep}(F) = \sup\{x \in \mathbb{R}, F(x) > 0\},\]

its APT \( G_\alpha \) is defined, for \( \alpha \in ]0, +\infty[\setminus\{1\} \) as

\[
G_\alpha(x) = \frac{1 - \alpha^{F(x)}}{1 - \alpha}, \quad x \in [\text{lep}(F), \text{uep}(F)]. \tag{1.5}
\]

We may see that \( G_\alpha \) is a cdf by considering the two cases \( 0 < \alpha < 1 \) and \( \alpha > 1 \). In fact, by using the non-decreasingness of \( F \), we have, for \( \alpha > 1 \), that \( \alpha^{F(x)} = \exp(F(x) \log \alpha) \) is non-decreasing and hence \( 1 - \alpha^{F(x)} \) non-increasing. Since the denominator \( 1 - \alpha \) is negative, we get that \( G_\alpha \) is non-decreasing. A similar method shows that \( G_\alpha \) is still non-decreasing for \( 0 < \alpha < 1 \). Besides, \( G_\alpha \) is right-continuous and \( \lim_{x \to \text{lep}(F)} G_\alpha(x) = 0 \) and \( \lim_{x \to \text{uep}(F)} G_\alpha(x) = 1 \).
Furthermore, we have \( \text{lep}(F) = \text{lep}(G_\alpha) \) and \( \text{uep}(F) = \text{uep}(G_\alpha) \). The definition may be extended to \( \alpha = 1 \) by taking \( G_1 = F \).

Whenever \( F \) has a probability density function \( (pdf) \) \( f \), the \( APT \) \( G_\alpha \), has the \( pdf \) defined as follows

\[
g_\alpha(x) = \frac{\log(\alpha)}{\alpha - 1} f(x) \alpha^{F(x)}, \quad x \in [\text{lep}(G_\alpha), \text{uep}(G_\alpha)],
\]

for \( \alpha \in ]0, +\infty[\setminus\{1\} \) and \( g_1 = f \).

One of the reasons of appealing to the \( APT \) is its ability to make distribution more flexible to fit correctly and adequately some lifetime data. For example the following uses of the \( APT \) have been made: Aldahlan (2020) for log-logistic distributions, ZeinEldin et al. (2021) for the Inverse Lomax distribution, Eghwerido (2021) for the Teissier distribution and Ijaz et al. (2021) for exponential distribution, to cite a few.

The alpha power transformed quasi Lindley distribution \( (APTQL) \) has been studied in Unyime and Ette (2021) using the quasi Lindley distribution introduced by Shanker and Mishra (2013). The \( cdf \) of the \( APTQL \) is defined for \( \beta > -1 \) and \( \theta > 0 \) by

\[
F_\alpha(x) = \frac{1}{1 - \alpha} \left( 1 - \alpha^{1 - \left( \frac{\beta + 1 + \theta x}{\beta + 1} \right) \exp(-\theta x)} \right), \quad x \geq 0,
\]

if \( \alpha \in ]0, +\infty[\setminus\{1\} \) and

\[
F_1(x) = 1 - \left( \frac{\beta + 1 + \theta x}{\beta + 1} \right) \exp(-\theta x), \quad x \geq 0.
\]

The aim of this paper is contributing to the current trend by precisely applying the \( APT \) to the Pseudo-Lindley distribution (see Zeghdoudi and Nedjar (2016) ), already cited above. We will show that the new class of distributions, called the Pseudo-Lindley Alpha Power Transformed distribution \( (PL\text{-APT}) \), can be used to improve the flexibility of continuous real lifetime data over the Pseudo-Lindley distribution. We study the statistical properties which will be summarized in a reserved paragraph 7.

The rest of the paper is organized as follows. In Section 2, we present the model of the \( PL\text{-APT} \) such as the \( cfd \) and the \( pdf \). We precise the type of
distribution when the parameters take some particular values. Section 3 is devoted to some statistical properties of the PL-APT related reliability and hazard rate functions, order statistic and entropies and likelihood method of parameters estimations. The asymptotic properties of the new family are presented in Section 4. Its quantile function and the related extreme expansions are studied in Subsection 4.1 and the extremal behaviors in Subsection 4.2, where the double-indexed Ngom and Lo (2016)'s statistic is used to estimate the extreme index value and its asymptotic law is expanded. The proof of the expressions of its quantile function and its expansions are stated in an appendix from page 25. Section 5 concludes the paper and gives perspectives.

2. The model

Let $X$ be a random variable following a PL-APT with parameters $\alpha$, $\beta$ and $\theta$ denoted by $X \sim \text{PL-APT}(\alpha, \beta, \theta)$. According to equations (1.3) and (1.6) the cdf of the PL-APT is defined as follows: if $\alpha \notin [0, +\infty[\setminus\{1\}$,

\begin{equation}
G_\alpha(x) = \frac{1}{1 - \alpha} \left(1 - \alpha^{1 - \beta^{-1} (\beta + \theta x) \exp(-\theta x)}\right) 1_{(x \geq 0)}, \tag{2.1}
\end{equation}

and

\begin{equation}
G_1(x) = (1 - \beta^{-1} (\beta + \theta x) \exp (-\theta x)) 1_{(x \geq 0)}. \tag{2.2}
\end{equation}

From equations (2.1), (2.2) and (1.6), we find the pdf of the PL-APT defined by

\begin{equation}
g_\alpha(x) = \left(\frac{\theta \log (\alpha) (\beta - 1 + \theta x) \exp (-\theta x)}{\beta (\alpha - 1)} \alpha^{1 - \beta^{-1} (\beta + \theta x) \exp(-\theta x)}\right) 1_{(x \geq 0)}, \tag{2.3}
\end{equation}

if $\alpha \notin [0, +\infty[\setminus\{1\}$, and

\begin{equation}
g_1(x) = \left(\frac{\theta (\beta - 1 + \theta x) \exp (-\theta x)}{\beta}\right) 1_{(x \geq 0)}. \tag{2.4}
\end{equation}
(1) If the parameter $\alpha = 1$ then the PL-APT distribution corresponds to the Pseudo-Lindley distribution developed by Lo et al. (2020). The figure 1 shows the graphs of the cdf and the pdf with several values of the parameters $\alpha$, $\beta$ and $\theta$.

(2) If the parameter $\alpha = 1$, $\beta = 1 + \theta$ and $\theta > 0$ then the PL-APT distribution corresponds to the Lindley distribution developed by Lindley in the two papers Lindley (1958, 1965). The figure 2 shows the graphs of the cdf and the pdf with several values of the parameters $\alpha$, $\beta$ and $\theta$.

(3) If the parameter $\alpha \in ]0, +\infty \setminus \{1\}$ then the cdf and the pdf of the PL-APT distributions are represented by the graphs of figure 3 below for several values of the parameters $\alpha$, $\beta$ and $\theta$.

3. Mathematical properties

Some basic statistical properties of the PL-APT distribution with parameters $\alpha$, $\theta > 0$ and $\beta > 1$ are derived and established in this section.
Figure 2. Graphs of the cdf (left) and pdf (right) for the Lindley distribution with several values of parameters $\alpha$, $\beta$ and $\theta$.

Figure 3. Graphs of the cdf (left) and pdf (right) for the PL-APT distribution with several values of parameters $\alpha$, $\beta$ and $\theta$. 
3.1. **Reliability.** The reliability function of the PL-APT distribution is expressed as follows for \( \alpha \in ]0, +\infty[ \setminus \{1\} \),

\[
R_\alpha(x) = 1 - G_\alpha(x) \\
= 1 - \frac{1 - \alpha^{1-\beta^{-1}(\beta+\theta x) \exp(-\theta x)}}{1 - \alpha} \\
= \frac{\alpha}{\alpha - 1} \left( 1 - \alpha^{-\beta^{-1}(\beta+\theta x) \exp(-\theta x)} \right),
\]

\( x \in [\text{lep}(F), \text{uep}(F)] \) and for \( x \in [\text{lep}(F_1), \text{uep}(F_1)] \),

\[
R_1(x) = 1 - G_1(x) \\
= \beta^{-1}(\beta + \theta x) \exp(-\theta x).
\]

3.2. **Hazard rate function.** The mathematical formula for hazard function which is otherwise called failure rate is defined as follows:

if \( \alpha \in ]0, +\infty[ \setminus \{1\} \),

\[
h_\alpha(x) = \frac{g_\alpha(x)}{1 - G_\alpha(x)}, \ x \in [\text{lep}(F), \text{uep}(F)]
\]

and

\[
h_1(x) = \frac{f(x)}{1 - F(x)}, \ x \in [\text{lep}(F), \text{uep}(F)].
\]

So, we obtain the hazard function of the PL-APT distribution with parameters \( \alpha, \theta > 0 \) and \( \beta > 1 \) as follows:

if \( \alpha \in ]0, +\infty[ \setminus \{1\} \),

\[
(3.1) \quad h_\alpha(x) = \frac{\beta^{-1}\theta \log(\alpha) (\beta - 1 + \theta x) \exp(-\theta x) \alpha^{1-\beta^{-1}(\beta+\theta x) \exp(-\theta x)}}{\alpha - \alpha^{1-\beta^{-1}(\beta+\theta x) \exp(-\theta x)}},
\]

\( x \in [\text{lep}(F), \text{uep}(F)] \) and for \( x \in [\text{lep}(F_1), \text{uep}(F_1)] \),
\[ h_1(x) = \frac{\theta (\beta - 1 + \theta x)}{\beta + \theta x}. \]

\[ (3.2) \]

3.3. \textbf{Order statistics and Entropies.} Let \( X_1, X_2, \ldots, X_n \) be a sample of random variables of size \( n \) following a \textit{PL-APT} distribution with parameters \( \alpha, \theta > 0 \) and \( \beta > 1 \) and \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) the order statistics of the processes. Then, the \textit{pdf} of the \textit{k}th order statistic \( X_{(k)} \) denoted by \( g_k(x) \) is defined as follows:
(1): If $\alpha \in ]0, +\infty[ \backslash \{1\}$ then

\[
g_k(x) = \begin{cases} 
\frac{n(n-1)!}{(n-k)!(k-1)!} \left( \frac{1-\alpha^{1-\beta^{-1}}(\beta+\theta x) \exp(-\theta x)}{1-\alpha} \right)^{k-1} \\
\times \left( 1 - \frac{1-\alpha^{1-\beta^{-1}}(\beta+\theta x) \exp(-\theta x)}{1-\alpha} \right)^{n-k} \\
\times \frac{\theta \log(\alpha)(\beta-1+\theta x) \exp(-\theta x)}{\beta(\alpha-1)} \alpha^{1-\beta^{-1}}(\beta+\theta x) \exp(-\theta x) 
\end{cases}
\]

(3.3)

(2): If $\alpha = 1$ then

\[
g_k(x) = \begin{cases} 
\frac{n(n-1)!}{(n-k)!(k-1)!} (1 - \beta^{-1}(\beta + \theta x) \exp(\theta x))^{k-1} \\
\times \left( \beta^{-1}(\beta + \theta x) \exp(\theta x) \right)^{n-k} \\
\times \frac{\theta (\beta-1+\theta x) \exp(\theta x)}{\beta} 
\end{cases}
\]

(3.4)

We obtain the minimum and the maximum order statistics respectively when $k = 1$ and $k = n$. If the size $n$ of the sample is an odd number then there exists an integer number $m$ such that $n = 2m + 1$ and $k = m + 1$, and the distribution of the median is defined by:

(1): If $\alpha \in ]0, +\infty[ \backslash \{1\}$ then

\[
g_{m+1}(x) = \begin{cases} 
\frac{2m(2m+1)!}{(m!)^2} \left( \frac{1-\alpha^{1-\beta^{-1}}(\beta+\theta x) \exp(-\theta x)}{1-\alpha} \right)^m \\
\times \left( 1 - \frac{1-\alpha^{1-\beta^{-1}}(\beta+\theta x) \exp(-\theta x)}{1-\alpha} \right)^m \\
\times \frac{\theta \log(\alpha)(\beta-1+\theta x) \exp(-\theta x)}{\beta(\alpha-1)} \alpha^{1-\beta^{-1}}(\beta+\theta x) \exp(-\theta x) 
\end{cases}
\]

(2): If $\alpha = 1$ then

\[
g_{m+1}(x) = \begin{cases} 
\frac{2m(2m+1)!}{(m!)^2} (1 - \beta^{-1}(\beta + \theta x) \exp(-\theta x))^m \\
\times \left( \beta^{-1}(\beta + \theta x) \exp(-\theta x) \right)^m \\
\times \frac{\theta (\beta-1+\theta x) \exp(-\theta x)}{\beta} 
\end{cases}
\]
3.4. **Parameters estimation.** We estimate the parameters of the PL-APT distribution by the maximum likelihood method. Let \( X_1, X_2, ..., X_n \) be a random sample from the PL-APT distribution. Then, the log-likelihood function of the PL-APT distribution \( l_n = \log l(\beta; \theta; x_1, x_2, ..., x_n) \) is given by

\[
(3.5) \quad l_n = \sum_{i=1}^{n} \log \left( \frac{\theta \log (\alpha) (\beta - 1 + \theta x_i) \exp (-\theta x_i)}{\beta (\alpha - 1)} \alpha^{1-\beta^{-1}(\beta+\theta x_i)e^{-\theta x_i}} \right)
\]

\[
= n (\log \theta + \log \log \alpha - \log \beta - \log (\alpha - 1)) + \sum_{i=1}^{n} \log (\beta - 1 + \theta x_i)
\]

\[
- \theta \sum_{i=1}^{n} x_i + \log \alpha \sum_{i=1}^{n} (1 - \beta^{-1} (\beta + \theta x_i) \exp (-\theta x_i)) .
\]

The estimate values of \( \theta \) and \( \beta \), points in which the log-likelihood function attains its maximum, are the solutions of likelihood equations (3.6) and (3.7) obtained by using the partial derivative for each parameter on equation (3.5) and equating to zero. We have

\[
(3.6) \quad \frac{\partial l_n}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{x_i}{\beta - 1 + \theta x_i} - \left(1 + \frac{1}{\beta}\right) \log \alpha \sum_{i=1}^{n} x_i e^{-\theta x_i}
\]

\[
+ \log \alpha \sum_{i=1}^{n} \left(\frac{\beta + \theta x_i}{\beta}\right) x_i e^{-\theta x_i} = 0,
\]

and

\[
(3.7) \quad \frac{\partial l_n}{\partial \beta} = -n\beta^{-1} + \sum_{i=1}^{n} \frac{1}{\beta - 1 + \theta x_i} + \log \alpha \sum_{i=1}^{n} \beta^{-2} \theta x_i \exp (-\theta x_i) = 0.
\]

The Likelihood equations (3.6) and (3.7) can not be solved explicitly since they are nonlinear functions of parameters \( \theta \) and \( \beta \). Therefore, iterative methods such as Newton-Raphson algorithm (NR) should be utilized to obtain the solution of these equations simultaneously.
4. Asymptotic properties

In this part, we study the asymptotic properties specially the quantile function, the extremal quantile function and the extremal index estimation of the PL-APT distribution.

4.1. Quantile and extremal quantile functions.

4.1.1. Quantile function. The quantile function for the PL-APT distribution is obtained by solving for $x$ the non-linear equation $G_\alpha (x) = u$. The quantile function of the PL-APT distribution, for any value of $\alpha > 0$, as follows:

If $\alpha = 1$

(4.1) \[ x(u) = -\frac{\beta}{\theta} - \frac{1}{\theta} W_{-1} [\beta (u - 1) \exp (-\beta)] , \]

and if $\alpha \in ]0, +\infty[\backslash\{1\}$ then

(4.2) \[ x(u) = -\frac{\beta}{\theta} - \frac{1}{\theta} W_{-1} \left\{ -\beta \exp (-\beta) + \frac{\beta \exp (-\beta)}{\log \alpha} \log [1 - (1 - \alpha) u] \right\} . \]

The proof of equation (4.2) is presented in Appendix (A2), page 29.

Obviously, we defined the first quartile, the median and the third quartile of the PL-APT distribution by $Q_1$, $Q_2$ and $Q_3$ respectively. For several values of the parameters, the Tables 1 and 2 comprise values of the quantile specially the first quartile, the median and the third quartile of the PL-APT distribution.

For $\alpha \in ]0, +\infty[\backslash\{1\}$, we have the table 1 and for $\alpha = 1$ we have the table 2.

We can remark from Table 1 and Table 2 that:

1) if $\alpha$ increases, $\beta$ increases and $\theta$ constant then the value of each of $Q_1$, $Q_2$ and $Q_3$ increases,

2) if $\alpha = \beta$ and $\theta$ increases then the value of each of $Q_1$, $Q_2$ and $Q_3$ decreases.
### Table 1. First Quartile (Q1), Median (Q2) and Third Quartile (Q3) for Selected Values of the Parameters of the PL-APT Distribution.

| θ | α | β | Q1       | Q2       | Q3       |
|---|---|---|----------|----------|----------|
| 0.5 | 1.1 | 0.7026004 | 1.2219140 | 1.7944430 |
| 0.6 | 1.5 | 0.2253118 | 0.4127261 | 0.5760581 |
| 1.5 | 2.5 | 0.4515757 | 0.8640310 | 1.2577340 |
| 1.5 | 1.1 | 0.28104020 | 0.4887654 | 0.7177772 |
| 2 | 1.5 | 0.09012474 | 0.1650904 | 0.2304232 |
| 2 | 2.5 | 0.18063030 | 0.3456124 | 0.5030937 |
| 3 | 1.5 | 0.14052010 | 0.24438270 | 0.3588886 |
| 3 | 2.5 | 0.04506237 | 0.08254522 | 0.1152116 |
| 5.2 | 1.5 | 0.05210488 | 0.0969589 | 0.14512320 |
| 5.2 | 2.5 | 0.02599752 | 0.04762224 | 0.06646824 |

4.1.2. **Extremal quantile function.** To find the extremal quantile function of the PL-APT distribution we solve the equation $G_\alpha(x) = 1-u$, with $u \in (0; 1)$. If $\alpha = 1$ then $G_\alpha(x) = F(x)$, is the cdf of the Pseudo-Lindley distribution. Its asymptotic properties are developed by Lo et al. (2020). Now, we focus on the condition that $\alpha \in [0, +\infty \setminus \{1\}$ and have the extremal quantile defined as follows:

\[
G^{-1}_\alpha (1-u) = C_0 + \theta^{-1} \log \left( \frac{1}{u} \right) + \theta^{-1} \log \left( \log \left( \frac{1}{u} \right) \right) + \frac{\theta^{-1} \log (-C(\alpha, \beta))}{\log \left( \frac{1}{u} \right)} + \theta^{-1} K(u),
\]

(4.3)

where $K(u) \rightarrow 0$ as $u \rightarrow 0$. 
4.2. **Extremes.** In this part, we present the extremal properties of the PL-APT distribution. We establish its domain of attraction, the expansion of the maximum values and finish with the study of the extremal value index estimation.

4.2.1. **Domain of attraction.** For any positive $\lambda$, and $\alpha \in [0, +\infty \setminus \{1\}$ we have the following limit to determine the domain of attraction of the PL-APT distribution,

$$
\lim_{u \to 0} \left( G_{\alpha}^{-1}(1 - \lambda u) - G_{\alpha}^{-1}(1 - u) \right) = L(\lambda),
$$

where

$$
s(u) = -u \left( G_{\alpha}^{-1}(1 - u) \right)', \quad 0 < u < 1.
$$
We have
\[
L(\lambda) = \lim_{u \to 0} \left( \frac{\theta^{-1} \log \left( \frac{1}{\lambda} \right) + \frac{1}{\theta} \log \left( 1 + \frac{\log 1/\lambda}{\log 1/u} \right) + \theta^{-1} K(u)}{-u (G^{-1}(1-u))'} \right)
\]
\[
= \lim_{u \to 0} \left( \frac{\theta^{-1} \log \left( \frac{1}{\lambda} \right) + \frac{1}{\theta} \log \left( 1 + \frac{\log 1/\lambda}{\log 1/u} \right) + \theta^{-1} K(u)}{-u \left( \frac{-1}{\theta} \times \frac{1}{u} \right)} \right)
\]
\[
= \lim_{u \to 0} \left( \log \left( \frac{1}{\lambda} \right) + \log \left( 1 + \frac{\log 1/\lambda}{\log 1/u} \right) + \theta^{-1} K(u) \right)
\]
\[
= - \log \lambda.
\]

Since
\[
\lim_{u \to 0} \left( \frac{G^{-1}_\alpha(1-\lambda u) - G^{-1}_\alpha(1-u)}{s(u)} \right) = - \log (\lambda),
\]
by the $\pi$–variation criteria developed in Lo et al. (2021-2016) (see Proposition 11, page 88), we conclude that $G_\alpha$ belongs to the Gumbel domain denoted by $G_\alpha \in D(\Lambda)$.

4.2.2. Expansion of the maximum values. Let $Z_n = - \log(nU_{1,n})$. By using the Renyi representation, we have that, if $\alpha \in ]0, +\infty[ \setminus \{1\}$
\[
M_1 = X_{n,n} - G^{-1}_\alpha(1 - 1/n) = G^{-1}_\alpha(1 - U_{1,n}) - G^{-1}_\alpha(1 - 1/n)
\]
\[
M_1 = C_0 + \theta^{-1} \log \left( \frac{1}{U_{1,n}} \right) + \theta^{-1} \log \left( \log \left( \frac{1}{U_{1,n}} \right) \right) + \theta^{-1} \log \left( -C(\alpha, \beta) \right) + \theta^{-1} K(U_{1,n})
\]
\[
- C_0 - \theta^{-1} \log(n) - \theta^{-1} \log \left( \log (n) \right) - \frac{\theta^{-1} \log \left( -C(\alpha, \beta) \right)}{\log(n)} - \theta^{-1} K(1/n).
\]
\[
= - \theta^{-1} \log(nU_{1,n}) + \theta^{-1} \log \left( 1 + \frac{Z_n}{\log n} \right) - \theta^{-1} \log \left( -C(\alpha, \beta) \right) \left( \frac{1}{\log(U_{1,n})} + \frac{1}{\log(n)} \right)
\]
\[
+ \theta^{-1} (K(U_{1,n}) - K(1/n)),
\]
and hence
\[
\frac{X_{n,n} - G^{-1}_\alpha (1 - 1/n)}{(1/\theta)} = -\log (nU_{1,n}) + \log \left(1 + \frac{-\log (nU_{1,n})}{\log n}\right) + \frac{-\log (nU_{1,n}) \log (-C(\alpha, \beta))}{(\log n)(\log U_{1,n})} \\
+ (K(U_{1,n}) - K(1/n)) \\
= Z_n + \log \left(1 + \frac{Z_n}{\log n}\right) + \frac{Z_n \log (-C(\alpha, \beta))}{(\log n)(\log U_{1,n})} + (K(U_{1,n}) - K(1/n)).
\]

Since \(\log \left(1 + \frac{Z_n}{\log n}\right)\) and \(\frac{Z_n \log (-C(\alpha, \beta))}{(\log n)(\log U_{1,n})}\) converge both to 0 as \(u \to 0\) then we have

\[
\frac{X_{n,n} - G^{-1}_\alpha (1 - 1/n)}{(1/\theta)} = Z_n + O_P(1).
\]

So we have \(X_{n,n}\) converge to a Gumbel law \(\Lambda\) with cdf

\[
\Lambda(x) = \exp \left(-\exp(-x)\right), x \in \mathbb{R}.
\]

Likewise, for \(k = k(n) \to +\infty\) such that \(k(n)/n \to 0\), we have

\[
M_k = X_{n-k,n} - G^{-1}_\alpha (1 - k/n) \\
= G^{-1}_\alpha (1 - U_{k+1,n}) - G^{-1}_\alpha (1 - k/n) \\
= \log \left(\frac{1}{U_{k+1,n}}\right) + \log \left(\frac{1}{U_{k+1,n}}\right) + \frac{\log (-C(\alpha, \beta))}{\log \left(\frac{1}{U_{k+1,n}}\right)} + K(U_{k+1,n}) \\
- \log (n/k) - \log (\log(n/k)) - \frac{\log (-C(\alpha, \beta))}{\log(n/k)} - K(k/n).
\]

Hence,
\[ M_k = -\log \left( nU_{k+1,n}/k \right) + \log \left( 1 + \frac{-\log \left( nU_{k+1,n}/k \right)}{\log \left( n/k \right)} \right) + \log \left( -C(\alpha,\beta) \right) \left( \frac{1}{\log (U_{k+1,n})} + \frac{1}{\log (n/k)} \right) + K(U_{k+1,n}) - K(k/n) \]

\[ = T_n + \log \left( 1 + \frac{T_n}{\log q_n} \right) + \log \left( -C(\alpha,\beta) \right) \left( \frac{\log (kU_{k+1,n}/n)}{\log (U_{k+1,n}) \log (n/k)} \right) + O_{\mathbb{P}} \left( (\log q_n)^{-2} \right), \]

where \( T_n = -\log \left( nU_{k+1,n}/k \right) \) and \( q_n = n/k \) which goes to \( +\infty \) as \( n \to +\infty \).

So, we have

\[ \frac{X_{n-k,n} - G^{-1}_\alpha (1-k/n)}{1/\theta} = T_n + \log \left( 1 + \frac{T_n}{\log q_n} \right) + \log \left( -C(\alpha,\beta) \right) \left( \frac{\log (kU_{k+1,n}/n)}{\log (U_{k+1,n}) \log (n/k)} \right) + O_{\mathbb{P}} \left( (\log q_n)^{-2} \right). \]

### 4.2.3. Estimation of the extreme value index

In Ngom and Lo (2016) create a new class of estimators of the extreme value index built around the statistic defined by

\[ T_n(f,s) = \sum_{j=1}^{k(n)} f(j) [\log X_{n-j+1,n} - \log X_{n-j,n}]^s, \]

where \( f \) is a positive measurable mapping defined from \( \mathbb{N} - \{0\} \) to \( \mathbb{R} - \{0\} \), and \( s \) is a positive real number. To estimate the extreme value index, it is necessary to define the following expressions. We have

\[ a_n(f,s) = \Gamma(s+1) \sum_{j=1}^{k(n)} f(j) j^{-s}, \]

\[ s_n^2(f,s) = \left\{ \Gamma(2s+1) - \Gamma^2(s+1) \right\} \sum_{j=1}^{k(n)} f^2(j) j^{-2s} \]

and
\[ B_n(f, s) = \max \left\{ \frac{f(j)}{s_n(f, s)} j^{-s}, 1 \leq j \leq k \right\}. \]

The Ngom and Lo (2016) estimator, called the double Hill estimator is defined by the expression below

\[ M_n(f, s) = \left( \frac{T_n(f, s)}{a_n(f, s)} \right)^{1/s}. \]

We remark that, if \( f(j) = j \) and \( s = 1 \) then \( M_n(f, s) = M_n(j, 1) = H_n \) is the Hill (1975) estimator and if \( f(j) = j^\tau; s > 0 \) and \( s = 1 \) then \( M_n(f, s) = M_n(j^\tau, 1) = T_n(\tau, 1) \) is the Deme et al. (2012) estimator.

**Theorem 1.** We have

(a) If the following conditions \( \frac{a_n(f, s)}{s_n(f, s)} \to 0 \) and \( B_n(f, s) \to 0 \) hold as \( n \to +\infty \), then

\[ \frac{a_n(f, s)}{s_n(f, s)} [M_n(f, s) - \gamma^s] \to \mathcal{N}(0, \gamma^{2s}). \]

(b) Furthermore, if

\[ \frac{a_n(f, s)}{s_n(f, s)} \to +\infty \]

then

\[ \frac{a_n(f, s)}{s_n(f, s)} [M_n(f, s) - \gamma^s] \to \mathcal{N}(0, \gamma^{2s} C^2(s)). \]

**Proof.** Here, we establish the proof of Theorem 1.

As the cdf \( G_\alpha \) of the PL-APT belongs to the attraction domain of Gumbel, it is known that to estimate the extremal index it is equivalent to use in the calculus the \( G_\alpha^{-1}(1 - u) \) or \( \log G_\alpha^{-1}(1 - u) \).
Let
\[
R_{j,n} = f(j) \left( \log X_{n-j+1,n} - \log X_{n-j,n} \right)^s
\]
\[
= f(j) \left( G^{-1}_\alpha (1 - U_{j,n}) - G^{-1}_\alpha (1 - U_{j+1,n}) \right)^s
\]
\[
= f(j) \left( (1/\theta) \log \left( \frac{U_{j+1,n}}{U_{j,n}} \right) + C_2 A_n + O_{\mathbb{P}}(B_n) \right)^s
\]
\[
= f(j) \left( (1/\theta) j^{-1} E_{j,n} + C_2 A_n + O_{\mathbb{P}}(B_n) \right)^s
\]

with
\[
A_n = \max \{ U_{j,n}^2, U_{j+1,n}^2 \} \leq -2 \log \left( \frac{U_{j+1,n}}{U_{j,n}} \right)
\]

and
\[
O_{\mathbb{P}}(B_n) = O_{\mathbb{P}}\left( (\log n)^{-2} \right).
\]

By the mean value theorem and \( j \in \{1, \ldots, k(n)\} \), \( s > 1 \), we get

\[
R_{j,n} - (1/\theta)^s f(j) j^{-s} E_{j,n}^s
\]
\[
\leq s f(j) |C_2 A_n + O_{\mathbb{P}}(B_n)| \left( (1/\theta) j^{-1} E_{j,n} + |C_2 A_n| + |O_{\mathbb{P}}(B_n)| \right)^{s-1}
\]
\[
\leq (1/\theta) s f(j) j^{-1} E_{j,n} \left( (1/\theta) j^{-1} E_{j,n} + |C_2 A_n| + |O_{\mathbb{P}}(B_n)| \right)^{s-1}
\]
\[
\leq (1/\theta) s f(j) j^{-1} E_{j,n} \left( D_s (1/\theta)^{s-1} j^{s-1} E_{j,n}^{s-1} + (1/\theta) D_s^2 j^{-1} E_{j,n} + O_{\mathbb{P}}\left( \frac{D_s^2}{(\log n)^{-2}} \right) \right).
\]

Applying the sum, we get

\[
\sum_{j=1}^{k(n)} \left( R_{j,n} - (1/\theta)^s f(j) j^{-s} E_{j,n}^s \right)
\]
\[
\leq (1/\theta) s \sum_{j=1}^{k(n)} f(j) j^{-1} E_{j,n} \left( D_s (1/\theta)^{s-1} j^{s-1} E_{j,n}^{s-1} + (1/\theta) D_s^2 j^{-1} E_{j,n} + O_{\mathbb{P}}\left( \frac{D_s^2}{(\log n)^{-2}} \right) \right)
\]
where

\[ S_n (f, s) = \sum_{j=1}^{k(n)} f(j) j^{-s} E_{j,n}^s \]

and

\[ T_n (f, s) = \sum_{j=1}^{k(n)} f(j) (\log X_{n-j+1,n} - \log X_{n-j,n})^s. \]

The random variable \( S_n (f, s) \) is a sequence of partial sum of random real values and independent random variables indexed by \( j \in \{1, \ldots, k(n)\} \) with first and second moments

\[ \mu_1 = \Gamma (s + 1) f (j) j^{-s} \text{ and } \mu_2 = (\Gamma (2s + 1) - \Gamma (s + 1)^2) f (j) j^{-s}. \]

Its asymptotic normality is given as follows by using the Levy-Feller-Linderberg (see Lo (2018), Theorem 20),

\[
\left( \frac{1}{s_n (f, s)} \sum_{j=1}^{k(n)} (f(j) j^{-s} (E_{j,n}^s - \Gamma (s + 1))) \right) \sim \mathcal{N} (0, 1)
\]
and $B_n(f, s) \longrightarrow 0$ as $n \longrightarrow +\infty$,

where

$$B_n(f, s) = \frac{1}{C(s)} \left\{ \frac{\text{Var} (f(j)j^{-s}(E^s_{j,n} - \Gamma (s + 1)))}{\sum_{j=1}^{k(n)} \text{Var} (f(j)j^{-s}(E^s_{j,n} - \Gamma (s + 1)))} \right\}.$$

By combining the Lindeberg condition, the Cauchy-Schwarz inequality and the central limit theorem, we get for $S_n(f, s)$ the result below

$$S_n(f, s) - (1/\theta)^s a_n(f, s) \quad \frac{s_n(f, s)}{S_n(f, s)} \sim N(0, 1).$$

The continuation of the inequality (IL) implies

$$\frac{T_n(f, s) - (1/\theta)^s S_n(f, s)}{s_n(f, s)} - \frac{S_n(f, s) - (1/\theta)^s a_n(f, s)}{s_n(f, s)} \leq O_P \left( \frac{S_n(f, 1)}{S_n(f, s) \log n} \right).$$

The right hand side of the inequality (IL1) above tends to zero in probability if and only if

$$\left( \frac{S_n(f, 1)}{S_n(f, s) \log n} \right) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty.$$

Let

$$M_n(f, s) = \left( \frac{T_n(f, s)}{a_n(f, s)} \right)^{1/s}.$$

By combining the results above for the left hand side of the inequality (IL1) above, we arrive at

$$\frac{T_n(f, s) - (1/\theta)^s S_n(f, s)}{s_n(f, s)} = a_n(f, s) \frac{T_n(f, s)}{s_n(f, s)} \left[ \frac{T_n(f, s)}{a_n(f, s)} - (1/\theta)^s \right].$$

So, we have
\[
\frac{a_n(f, s)}{s_n(f, s)} \left[ T_n(f, s) - \frac{1}{\theta}s \right] = Z_n + O_P(1).
\]

If \( a_n(f, s)/s_n(f, s) \rightarrow +\infty \), then we have by using the delta method applied to \( g(t) = t^{1/s} \),

\[
\frac{a_n(f, s)}{s_n(f, s)} \left[ T_n(f, s) - \frac{1}{\theta}s \right] \sim \mathcal{N} \left( 0, \left( \frac{1}{\theta} \right)^{2s} C^2(s) \right).
\]

\[\square\]

5. Conclusion

In this paper, the PL-APT distribution, which is flexible for modeling lifetime data, is presented. This study is motivated by the extensive use of the Pseudo-Lindley distribution in Statistics and Economics. The PL-APT distribution provides more flexibility than the Lindley and the Pseudo-Lindley distributions to analyze lifetime data. The PL-APT distribution has several new and known properties as its mathematical properties and asymptotic convergence of the extreme value index. Its parameters are estimated by the maximum likelihood method. The possibility of expanding Pseudo-Lindley into other areas can be achieved with the new quantile distribution and the extremal quantile distribution of the PL-APT. In a next article, we face to study the simulations with application of real lifetime data.

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Appendix (A1): Extremal quantile of the PL-APT distribution.

We solve for \( x \) the equation \( G_\alpha(x) = 1 - u, \ \alpha \in ]0, +\infty[\setminus\{1\} \). Thus, we have

\[
1 - \frac{\alpha^{1-\beta^{-1}(\beta+\theta x)\exp(-\theta x)}}{1-\alpha} = 1 - u
\]

\[
1 - \alpha^{1-\beta^{-1}(\beta+\theta x)\exp(-\theta x)} = (1 - u)(1 - \alpha)
\]

\[
\alpha^{1-\beta^{-1}(\beta+\theta x)\exp(-\theta x)} = \alpha + u(1 - \alpha)
\]

\[
1 - \beta^{-1}(\beta + \theta x)\exp(-\theta x) = \log(\alpha + u(1 - \alpha)) \times (\log \alpha)^{-1}
\]

\[
-\beta^{-1}(\beta + \theta x)\exp(-\theta x) = -1 + \log(\alpha + u(1 - \alpha)) \times (\log \alpha)^{-1}
\]

\[
-(\beta + \theta x)\exp(-\theta x) = -\beta + \frac{\beta}{\log \alpha} \log(\alpha + u(1 - \alpha)) .
\]

Multiplying both sides by \( \exp(-\beta) \), we have

\[
-(\beta + \theta x)\exp(-\theta x)\exp(-\beta) = \exp(-\beta) \left( -\beta + \frac{\beta}{\log \alpha} \log(\alpha + u(1 - \alpha)) \right)
\]

(5.1) \( (-\beta - \theta x)\exp(-\beta - \theta x) = -\beta\exp(-\beta) + \frac{\beta\exp(-\beta)}{\log \alpha} \log(\alpha + u(1 - \alpha)) \).

The right hand side of the equation (5.1) satisfies the negative branch \( W_{-1}(\cdot) \) of the Lambert \( W \) function. Let \( W(x) = -\beta - \theta x \). Herein, we have

\[
W(x)\exp(W(x)) = -\beta\exp(-\beta) + \frac{\beta\exp(-\beta)}{\log \alpha} \log(\alpha + u(1 - \alpha)) ,
\]

and
\[-\beta - \theta x = W_{-1} \left( -\beta \exp(-\beta) + \frac{\beta \exp(-\beta)}{\log \alpha} \log (\alpha + u (1 - \alpha)) \right).\]

Hence,

\[-\beta - \theta x = W_{-1} \left( -\beta \exp(-\beta) + \frac{\beta \exp(-\beta)}{\log \alpha} \log (\alpha + u (1 - \alpha)) \right),\]

\[-\theta x = \beta + W_{-1} \left( -\beta \exp(-\beta) + \frac{\beta \exp(-\beta)}{\log \alpha} \log (\alpha + u (1 - \alpha)) \right),\]

\[x = -\frac{\beta}{\theta} - \frac{1}{\theta} W_{-1} \left( -\beta \exp(-\beta) + \frac{\beta \exp(-\beta)}{\log \alpha} \log (\alpha + u (1 - \alpha)) \right),\]

\[x = -\beta \frac{1}{\theta} - \frac{1}{\theta} W_{-1} \left( \frac{\beta}{\log \alpha} \log (\alpha + u (1 - \alpha)) \right),\]

where

\[(5.2) \quad (A(\alpha, \beta, u)) = -\beta \exp(-\beta) + \frac{\beta \exp(-\beta)}{\log \alpha} \log (\alpha + u (1 - \alpha)).\]

Now we deal with the expression \(A(\alpha; \beta; u)\) below

\[A(\alpha; \beta; u) = -\beta \exp(-\beta) + \frac{\beta \exp(-\beta)}{\log \alpha} \log (\alpha + u (1 - \alpha))\]

\[= -\beta \exp(-\beta) + \frac{\beta \exp(-\beta)}{\log \alpha} \left( \log \alpha + \log \left( 1 + u \left( \frac{1 - \alpha}{\alpha} \right) \right) \right)\]

\[= \frac{\beta \exp(-\beta)}{\log \alpha} \log \left( 1 + u \left( \frac{1 - \alpha}{\alpha} \right) \right)\]

\[= \frac{\beta \exp(-\beta)}{\log \alpha} \left[ u \left( \frac{1 - \alpha}{\alpha} \right) + O(u^2) \right]\]

\[= u \left( \frac{1 - \alpha}{\alpha} \right) \left( \frac{\beta \exp(-\beta)}{\log \alpha} \right) + O(u^2)\]

\[= u C(\alpha, \beta) + O(u^2),\]

where
\[ C(\alpha, \beta) = \frac{(1 - \alpha) \beta \exp(-\beta)}{\alpha \log \alpha}, \]

\( C(\alpha, \beta) \) is a negative number for \( \alpha \in ]0, +\infty[ \setminus \{1\} \) and \( \beta > 0 \).

Now, the inverse Lambert function has the following expansion at the neighborhood of zero \( (x = 0^-) \):

\[ W_{-1}(x) = \log(-x) - \log(-\log(-x)) + O\left(\frac{\log(-\log(-x))}{\log(-x)}\right). \]

By applying the \( A(\alpha; \beta; u) \) on it, we have

\[ W_{-1}(A(\alpha; \beta; u)) = \log(-A(\alpha; \beta; u)) - \log(-\log(-A(\alpha; \beta; u))) \\
+ O\left(\frac{\log(-\log(-A(\alpha; \beta; u)))}{\log(-A(\alpha; \beta; u))}\right). \]

Since the expansion of \( A(\alpha; \beta; u) \) gives \( A(\alpha; \beta; u) = uC(\alpha, \beta) + O(u^2) \), we have

\[ -A(\alpha; \beta; u) = -uC(\alpha, \beta) + O(u^2). \]

Applying the logarithm both sides, we have

\[
\log(-A(\alpha; \beta; u)) = \log\left[-uC(\alpha, \beta) + O(u^2)\right] \\
= \log\left[-uC(\alpha, \beta) (1 + O(u))\right] \\
= \log(u) + \log\left(-C(\alpha, \beta)\right) + O(u). \\
\rightarrow -\infty.
\]

And, by re-applying the logarithm both sides, we have
\[- \log (-A(\alpha; \beta; u)) = - \log (u) - \log (-C(\alpha, \beta)) + O(u)\]

\[- \log [- \log (-A(\alpha; \beta; u))] = - \log [- \log (u) - \log (-C(\alpha, \beta)) + O(u)]\]

\[
= - \log \left[ - \log (u) \left( 1 + \frac{\log (-C(\alpha, \beta))}{\log (u)} + O\left( \frac{-u}{\log (u)} \right) \right) \right]
\]

\[
= - \log (- \log (u)) - \log \left( 1 + \frac{\log (-C(\alpha, \beta))}{\log (\frac{1}{u})} + O\left( \frac{-u}{\log (\frac{1}{u})} \right) \right)
\]

\[
= - \log \left( \log \left( \frac{1}{u} \right) \right) + \log \left( \frac{-C(\alpha, \beta)}{\log \left( \frac{1}{u} \right)} \right) + O\left( \frac{u}{\log \left( \frac{1}{u} \right)} \right).
\]

\[\rightarrow -\infty.\]

Hence,

\[
\frac{\log (- \log (-A(\alpha; \beta; u)))}{\log (-A(\alpha; \beta; u))} = \frac{- \log \left( \log \left( \frac{1}{u} \right) \right) + \frac{\log(-C(\alpha, \beta))}{\log \left( \frac{1}{u} \right)} + O\left( \frac{u}{\log \left( \frac{1}{u} \right)} \right)}{\log (u) + \log (-C'(\alpha, \beta)) + O(u)}
\]

\[
= \frac{- \log \left( \log \left( \frac{1}{u} \right) \right) \left[ 1 + \frac{\log(-C(\alpha, \beta))}{- \log \left( \frac{1}{u} \right) \log \left( \frac{1}{u} \right)} + O\left( \frac{u}{\log \left( \frac{1}{u} \right) \log \left( \frac{1}{u} \right)} \right) \right]}{\log (u) \left[ 1 + \frac{\log(-C(\alpha, \beta))}{\log (u)} + O\left( \frac{u}{\log (u)} \right) \right]}
\]

\[
= \frac{\log \left( \log \left( \frac{1}{u} \right) \right)}{\log \left( \frac{1}{u} \right)} \left[ 1 + O\left( \frac{u}{\log (u)} \right) \right],
\]

which tends to 0 as \(u \rightarrow 0\).

Therefore,

\[
W_{-1}(A(\alpha; \beta; u)) = \log (u) - \log \left( \log \left( \frac{1}{u} \right) \right)
\]

\[
+ \log (-C(\alpha, \beta)) + \frac{\log C(\alpha, \beta)}{\log \left( \frac{1}{u} \right)} + O\left( \frac{u}{\log \left( \frac{1}{u} \right)} \right).
\]
Hence,

\[
x = -\frac{\beta}{\theta} - \frac{1}{\theta} [W_{-1}(A(\alpha; \beta; u))]
\]

\[
= -\frac{\beta}{\theta} + \frac{1}{\theta} \log \left( \frac{1}{u} \right) + \frac{1}{\theta} \log \left( \log \left( \frac{1}{u} \right) \right) - \frac{1}{\theta} \log (-C(\alpha, \beta))
\]

\[
+ \frac{1}{\theta} \times \frac{\log (-C(\alpha, \beta))}{\log \left( \frac{1}{u} \right)} + O \left( \frac{u}{\log \left( \frac{1}{u} \right)} \right)
\]

\[
= -\beta \theta^{-1} - \theta^{-1} \log (-C(\alpha, \beta)) + \theta^{-1} \log \left( \frac{1}{u} \right) + \theta^{-1} \log \left( \log \left( \frac{1}{u} \right) \right)
\]

\[
+ \frac{\theta^{-1} \log (-C(\alpha, \beta))}{\log \left( \frac{1}{u} \right)} + \theta^{-1} K(u)
\]

\[
= C_0 + \theta^{-1} \log \left( \frac{1}{u} \right) + \theta^{-1} \log \left( \log \left( \frac{1}{u} \right) \right) + \frac{\theta^{-1} \log (-C(\alpha, \beta))}{\log \left( \frac{1}{u} \right)} + \theta^{-1} K(u).
\]

Where \( K(u) = O \left( \frac{u}{\log \left( \frac{1}{u} \right)} \right) \) and \( C_0 = -\beta \theta^{-1} - \theta^{-1} \log (-C(\alpha, \beta)) \).

Therefore, if \( \alpha \in ]0, +\infty[ \backslash \{1\} \), we have

(5.3)

\[
G_{\alpha}^{-1}(1-u) = C_0 + \theta^{-1} \log \left( \frac{1}{u} \right) + \theta^{-1} \log \left( \log \left( \frac{1}{u} \right) \right) + \frac{\theta^{-1} \log (-C(\alpha, \beta))}{\log \left( \frac{1}{u} \right)} + \theta^{-1} K(u).
\]

**Appendix (A2) : Quantile of the PL-APT distribution.**

For \( \alpha = 1 \), the quantile function of the PL-APT is the quantile of Pseudo-Lindley and it is solution of the equation \( F(x) = u \). We have
Thus, the quantile function of Pseudo-Lindley distribution is

\[ x(u) = -\frac{\beta}{\theta} - \frac{1}{\theta} W_{-1}(\beta(u - 1) \exp(-\beta)). \] (5.4)

For \( \alpha \in ]0, +\infty[ \setminus \{1\} \), the quantile function of the \textit{PL-APT} distribution is the solution of the equation \( G_{\alpha}(x) = u \). we have

\[
1 - \alpha^{1 - \beta^{-1}(\beta + \theta x) \exp(-\theta x)} = u \\
\frac{1 - \alpha^{1 - \beta^{-1}(\beta + \theta x) \exp(-\theta x)}}{1 - \alpha} = u (1 - \alpha) \\
\alpha^{1 - \beta^{-1}(\beta + \theta x) \exp(-\theta x)} = 1 - u (1 - \alpha) \\
[1 - \beta^{-1}(\beta + \theta x) \exp(-\theta x)] \log \alpha = \log [1 - u (1 - \alpha)] \\
1 - \beta^{-1}(\beta + \theta x) \exp(-\theta x) = (\log \alpha)^{-1} \log [1 - u (1 - \alpha)] \\
\beta^{-1}(\beta + \theta x) \exp(-\theta x) = 1 - (\log \alpha)^{-1} \log [1 - u (1 - \alpha)] \\
(\beta + \theta x) \exp(-\theta x) = \beta - \beta (\log \alpha)^{-1} \log [1 - u (1 - \alpha)] \\
(\beta + \theta x) \exp(-\beta - \theta x) = \beta \exp(-\beta) - \beta \exp(-\beta) (\log \alpha)^{-1} \log [1 - u (1 - \alpha)].
\]

We apply the negative Lambert function on both sides to obtain
\[-\beta - \theta x = W_{-1} \left\{ \beta \exp(-\beta) - \beta \exp(-\beta) (\log \alpha)^{-1} \log \left[ 1 - u (1 - \alpha) \right] \right\} \]
\[-\theta x = \beta + W_{-1} \left\{ \beta \exp(-\beta) - \beta \exp(-\beta) (\log \alpha)^{-1} \log \left[ 1 - u (1 - \alpha) \right] \right\} \]
\[x = -\frac{\beta}{\theta} - \frac{1}{\theta} W_{-1} \left\{ \beta \exp(-\beta) - \beta \exp(-\beta) (\log \alpha)^{-1} \log \left[ 1 - u (1 - \alpha) \right] \right\}.\]

Finally, we have
\[x(u) = -\frac{\beta}{\theta} - \frac{1}{\theta} W_{-1} \left\{ \beta \exp(-\beta) - \beta \exp(-\beta) (\log \alpha)^{-1} \log \left[ 1 - u (1 - \alpha) \right] \right\}.\]