NNLO QCD CORRECTIONS TO $\bar{B} \rightarrow X_s \gamma$

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Current status of the NNLO QCD corrections to $\bar{B} \rightarrow X_s \gamma$ is reviewed. The calculations include three-loop matching conditions, four-loop anomalous dimensions as well as two- and three-loop on-shell amplitudes. Certain parts of the three-loop matrix elements are found by interpolation in the charm quark mass between the large-$\beta_0$ approximation in the $m_c = 0$ case and the complete result in the $m_c \gg m_b/2$ case.

1 Introduction

The weak radiative $\bar{B}$-meson decay is known to be a sensitive probe of new physics. Thus, it is essential to calculate the Standard Model value of its branching ratio as precisely as possible. In order to do so, one writes

$$B(\bar{B} \rightarrow X_s \gamma)_{E_{\gamma} > 1.6 \text{ GeV}} = B(\bar{B} \rightarrow X_c e \bar{\nu})^{\exp} \left[ \frac{\Gamma(\bar{b} \rightarrow s \gamma)}{\Gamma(\bar{b} \rightarrow c e \bar{\nu})} \right]_{\text{LO EW}} f \left( \frac{\alpha_s(M_W)}{\alpha_s(m_b)} \right) \times$$

$$\times \left\{ 1 + \mathcal{O}(\alpha_s) + \mathcal{O}(\alpha_s^2) + \mathcal{O}(\alpha_{\text{em}}) + \mathcal{O} \left( \frac{\Lambda^2}{m_b^2} \right) + \mathcal{O} \left( \frac{\Lambda^2}{m_c^2} \right) + \mathcal{O} \left( \frac{\alpha_s \Lambda}{m_b} \right) \right\},$$

where $B(\bar{B} \rightarrow X_c e \bar{\nu})^{\exp}$ is the measured semileptonic branching ratio. The $b$-quark radiative and semileptonic decay widths $\Gamma(\bar{b} \rightarrow s \gamma)$ and $\Gamma(\bar{b} \rightarrow c e \bar{\nu})$ are calculated perturbatively at the leading order in electroweak interactions and neglecting QCD effects. Normalization to the semileptonic rate is introduced to eliminate uncertainties from the CKM angles and overall factors of $m_b^5$. The Leading Order (LO) QCD correction factor is denoted by $f \left( \alpha_s(M_W)/\alpha_s(m_b) \right)$.

Higher-order perturbative and non-perturbative corrections are listed in the second line of Eq. (1). Their approximate sizes are indicated. The non-perturbative effects arise only as corrections, thanks to the heaviness of the $b$-quark ($m_b \gg \Lambda \equiv \Lambda_{\text{QCD}}$) and to the inclusive character
of the considered decay. The $O(\alpha_s)$, $O(\alpha_{em})$, $O(\Lambda^2/m_b^2)$ and $O(\Lambda^2/m_t^2)$ contributions are known since many years. On the other hand, the indicated sizes of the $O(\alpha_s^2)$ and $O(\alpha_s\Lambda/m_b)$ corrections are only estimates that were made before the actual calculation of the Next-to-Next-to-Leading (NNLO) QCD corrections ($O(\alpha_s^2)$)\(^6\).

The current experimental world average for the branching ratio reads\(^\text{II}\)

$$B(\bar{B} \to X_s\gamma)_{E_{T}>1.6\,\text{GeV}}^{\exp} = \left(3.55 \pm 0.24^{+0.09}_{-0.10} \pm 0.03\right) \times 10^{-4},$$

where the combined error is approximately equal to the expected size of the $O(\alpha_s^2)$ effects. Thus, the currently reported NNLO calculation is well-motivated. It is hoped that the $O(\alpha_s\Lambda/m_b)$ uncertainty can be reduced in the future by performing a dedicated analysis.

2 **The effective Lagrangian**

Resummation of $(\alpha_s \ln M_W^2/m_b^2)^n$ at each order in $\alpha_s$ is most conveniently performed in the framework of an effective theory that arises from the SM after decoupling of the heavy electroweak bosons and the top quark. The Lagrangian of such a theory reads

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCDxQED}}(u,d,c,b) + \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu) Q_i.$$   

The operators $Q_i$ and numerical values of their Wilson coefficients at $\mu \simeq m_b$ are as follows:

$$Q_i = \begin{cases} 
(\bar{s}\Gamma_i c)(\bar{c}\Gamma'_i b), & i = 1, 2, \quad |C_i(m_b)| \sim 1, \\
(\bar{s}\Gamma_i b)\Sigma_q(\bar{q}\Gamma'_i q), & i = 3, 4, 5, 6, \quad |C_i(m_b)| < 0.07, \\
\frac{em_b}{16\pi^2} \bar{s}_L \sigma^{\mu\nu} b_R F_{\mu\nu}, & i = 7, \quad C_7(m_b) \sim -0.3, \\
\frac{gm_b}{16\pi^2} \bar{s}_L \sigma^{\mu\nu} T^a b_R G_{\mu\nu}, & i = 8, \quad C_8(m_b) \sim -0.15. 
\end{cases}$$

Here, $\Gamma$ and $\Gamma'$ stand for various products of the Dirac and color matrices.\(^6\) The perturbative calculations are performed in three steps: (i) **Matching**: Evaluating $C_i(\mu_0)$ at $\mu_0 \sim M_W$ by requiring equality of the SM and effective theory Green’s functions at the leading order in (external momenta)/$M_W$. (ii) **Mixing**: Deriving the effective theory Renormalization Group Equations (RGE) and evolving $C_i(\mu)$ from $\mu_0$ down to $\mu_b \sim m_b$. (iii) **Matrix elements**: Evaluating the on-shell $b \to X_s^{\text{parton}}\gamma$ amplitudes at $\mu_b \sim m_b$.  

![Figure 1: Sample 1-loop diagram.](image1.png) ![Figure 2. Sample 2-loop diagram.](image2.png)
3 Current status of the NNLO calculation

The 3-loop matching for $Q_7$ and $Q_8$ was evaluated more than two years ago. The 3-loop mixings in the $Q_1$-$Q_6$ and $Q_7$-$Q_8$ sectors were found more recently. The yet unpublished results on the 4-loop mixing of $Q_1$-$Q_6$ into $Q_7$ will be used below. The effect of the unknown 4-loop mixing of $Q_1$-$Q_6$ into $Q_8$ is expected to be small. Nevertheless, its calculation is underway.

As far as the matrix elements are concerned, contributions to the decay rate that are proportional to $|C_7(\mu_b)|^2$ are completely known at the NNLO. These two-loop results have recently been confirmed by independent groups.

Two- and three-loop matrix elements in the so-called large-$\beta_0$ approximation were found as an expansion in $m_c/m_b$ that is convergent for $m_c < m_b/2$, i.e. in the physical domain. The remaining (“beyond-large-$\beta_0$”) contributions to the matrix elements were calculated in the limit $m_c \gg m_b/2$ and then interpolated to smaller values of $m_c$ (see the next section).

In order to relate our result with $E_{\text{cut}} = 1.6 \text{GeV}$ to the measurements with cuts at 1.8 GeV (Belle) and 1.9 GeV (BaBar), one needs to evaluate ratios of the decay rates with different cuts. Such a calculation at the NNLO has recently been completed. However, the final numerical results are not yet available, and the average in Eq. does not include them.

4 Interpolation in $m_c$

Let us parametrize the NNLO correction to the branching ratio in terms of three quantities $\delta_i$:

$$B(\bar{B} \to X_s \gamma)_{E_{\gamma}>1.6 \text{GeV}} \equiv B_{\text{NNLO}}(r) = B_{\text{NLO}}(r) + B_{\text{NNLO}}(0.262) \left( \delta_1 + \delta_2(r) + \delta_3(r) \right),$$

where $r = m_c / m_b^S$ and $B_{\text{NLO}}(0.262) \approx 3.38 \times 10^{-4}$. The quantities $\delta_i$ contain terms depending on different contributions to the Wilson coefficient perturbative expansion

$$C_i(\mu_b) = C_i^{(0)}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C_i^{(1)}(\mu_b) + \left( \frac{\alpha_s(\mu_b)}{4\pi} \right)^2 C_i^{(2)}(\mu_b) + \ldots.$$ 

In particular, $\delta_1$ contains terms proportional to $C_i^{(0)} C_j^{(2)}$ and $C_i^{(1)} C_j^{(1)}$, $\delta_2$ – terms proportional to $C_i^{(0)} C_j^{(0)}$, and $\delta_3$ – terms proportional to $C_i^{(0)} C_j^{(1)}$. The so-called large-$\beta_0$ part of $\delta_2$ is found from the fermionic contributions to this quantity: $\delta_2 = A n_f + B = \delta_2^{\text{rem}} + \delta_2^{\text{em}}$, where $\delta_2^{\text{rem}} = -\frac{3}{2} (11 - 2/3 n_f) A$ and $\delta_2^{\text{em}} = \frac{33}{2} A + B$. Here, $n_f = 5$ is the number of active flavors in the effective theory. While $\delta_2^{\text{rem}}$ is known for all $r$, $\delta_2^{\text{rem}}$ has been calculated only for $r \gg \frac{1}{2}$ and needs to be interpolated to lower values of $r$.

Preliminary results of this interpolation are summarized in Fig. 3 (for $\mu_b = m_b$). The two vertical lines mark the experimental bounds on $r$. Dashed curves show the calculated asymptotic
behavior of each $\delta_i$ for $r \gg \frac{1}{2}$ at the leading order in $1/(4r^2)$, i.e. including only the constant and logarithmic terms. Solid lines show either the known exact dependence on $r$ (for $\delta_1$ and $\delta_3$), the known small-$r$ expansion (for $\delta_2^{\beta}$) or the interpolation (for $\delta_2^{\text{em}}$). The interpolation can be performed approximating $\delta_2^{\text{em}}$ by a linear combination of four functions:

$$\delta_2^{\text{em}}(r) = a B_{\text{NLO}}(r) + b r \frac{d}{dr} B_{\text{NLO}}(r) + c \delta_2^{\beta}(r) + d$$

(7)

The coefficients $a$, $b$, $c$ and $d$ are determined in a unique manner from the asymptotic behavior at large $r$ and from the requirement that either $\delta_2^{\text{em}}(0) = 0$ (lower curves) or $\delta_1 + \delta_2^{\text{em}}(0) + \delta_3(0) = 0$ (upper curves). The assumed functional dependence of $\delta_2^{\text{em}}$ on $r$ is motivated by the $r$-dependence of renormalization-induced effects. In many explicitly calculated examples, such effects have been found to dominate over other terms of the same order.

5 Conclusion

The two ways of interpolation lead to two values of the NNLO branching ratio $B_{\text{NNLO}} = 3.06 \times 10^{-4}$ or $B_{\text{NNLO}} = 3.24 \times 10^{-4}$, for $E_\gamma > 1.6$ GeV. These values are around 6% apart from each other and around 7% below the NLO result $B_{\text{NLO}} = 3.38 \times 10^{-4}$. Their average $3.15 \times 10^{-4}$ is around $1.5\sigma_{\text{exp}}$ below the experimental result in Eq. 2. Consequently, extensions of the SM predicting a suppression of the $b \to s\gamma$ amplitude are going to be more constrained than previously.

6 Acknowledgments

I am greatly indebted to the contributors to Refs. for their participation in the NNLO enterprise. I am grateful to the organizers of the 40th Rencontres de Moriond for their invitation and care. I acknowledge support from the Polish Committee for Scientific Research under the grant 2 P03B 078 26 and from the European Community’s Human Potential Programme under the contract HPRN-CT-2002-00311, EURIDICE.

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