ON THE ZARISKI COMPACTNESS OF MINIMAL SPECTRUM AND FLAT COMPACTNESS OF MAXIMAL SPECTRUM

ABOLFAZL TARIZADEH

Abstract. In this article, a few interesting and non-trivial results are obtained. Specially, the Zariski compactness of the minimal spectrum and flat compactness of the maximal spectrum are characterized.

1. Introduction

The minimal spectrum of a commutative ring specially its compactness has been the main topic of many articles in the literature over the years, see e.g. [1], [3], [5], [6], [7], [8], [10], [11], [12], [13]. Amongst them, the well-known result of Quentel [12, Proposition 1] can be considered one of the most important results in this context. But his proof, as presented there, is sketchy. In fact, it is merely a plan of the proof, not the proof itself. In the present article, i.e. Theorem 4.9 and Corollary 4.11, a new and purely algebraic proof is given for this non-trivial result. Dually, a new result is also given for the compactness of the maximal spectrum w.r.t. the flat topology, see Theorem 4.5.

In Theorem 3.1 the patch closures are computed in a certain way. This result plays a major role in proving Theorem 4.9. The noetherianess of the prime spectrum w.r.t. the Zariski topology is also characterized, see Theorem 5.1. It is worthy to mention that in the literature, as far as the author knows, there is no known non-trivial characterization for the noetherianess of the Zariski topology. In this article, all of the rings are commutative.

---

0 2010 Mathematics Subject Classification: 13A99, 13B10, 13C11.
Key words and phrases: minimal spectrum; maximal spectrum; maximal flat epimorphic extension; flat topology; patch topology; absolutely flat ring.
2. Preliminaries

Here we briefly recall some material which are needed in the sequel.

Every ring map $\varphi : A \rightarrow B$ induces a map $\varphi^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ between the corresponding spectra which maps each prime ideal $p$ of $B$ into $\varphi^{-1}(p)$.

Already in [2] and more recently and independently in [14] we have rediscovered the Hochster’s inverse topology (see [4, Prop. 8]) on the prime spectrum by a new and algebraic method. We call it the flat topology (it is worthy to mention that during the writing [14] we were not aware of [2] and Hochster’s work). Therefore the flat topology and Hochster’s inverse topology are exactly the same things. In fact, for a given ring $R$, then the collection of subsets $V(f) = \{ p \in \text{Spec}(R) : f \in p \}$ with $f \in R$ forms a sub-basis for the opens of the flat topology. It behaves as the dual of the Zariski topology, for more details see [14].

Recall that a ring $R$ is said to be absolutely flat (or, von-Neumann regular) if each $R$–module is $R$–flat. This is equivalent to the statement that each element $a \in R$ can be written as $a = a^2b$ for some $b \in R$. Clearly absolutely flat rings are stable under taking quotients and localizations. Every prime ideal of an absolutely flat ring is maximal since an absolutely flat domain is a field.

To see the definition of “the maximal flat epimorphic extension of a ring” please consider [9, Prop 3.4].

3. Patch Closures

Recall that for a given ring $R$ then the collections of subsets of the form $D(f) \cap V(I)$ with $f \in R$ and $I$ is a f.g. ideal of $R$ is a basis for the opens of the patch topology on $\text{Spec}(R)$, for more details see [14] §2. In the following result, which we shall use it later, the patch closures are computed in a certain way:

**Theorem 3.1.** Let $R$ be a ring and let $D$ be a subset of $\text{Spec}(R)$. Consider the patch topology on $\text{Spec}(R)$ and also consider the canonical ring map $\pi : R \rightarrow \prod_{p \in D} \kappa(p)$ where $\kappa(p)$ is the residue field of $R$ at $p$. Then $\gamma(D) = \text{Im} \pi^*$ where $\gamma(D)$ denotes the closure of $D$ in $\text{Spec}(R)$.
Proof. For each prime ideal \( p \in D \), the canonical map \( R \to \kappa(p) \) factors as \( R \xrightarrow{\pi} A \xrightarrow{s_p} \kappa(p) \) where \( A := \prod_{q \in D} \kappa(q) \) and \( s_p \) is the canonical projection. Let \( J_p := s_p^{-1}(0) \). Then clearly \( p = \pi^{-1}(J_q) \). This implies that \( \gamma(D) \subseteq \text{Im } \pi^* \). To prove the reverse inclusion we act as follows. First we show that \( \gamma(D') = \text{Spec}(A) \) where \( D' = \{ J_p : p \in D \} \).

Clearly the ring \( A \) is absolutely flat. Therefore by [15, Theorem 4.7], the Zariski and patch topologies over \( \text{Spec}(A) \) are the same. Let \( f = (f_p)_{p \in D} \) be a non-zero element of \( A \). Thus there exists some \( p \in D \) such that \( f_p \neq 0 \). This implies that \( J_p \in D(f) \). Therefore every non-empty standard Zariski-open of \( \text{Spec}(A) \) meets \( D' \). This means that \( \gamma(D') = \text{Spec}(A) \). We have \( \pi^*(D') = D \subseteq \gamma(D) \). It follows that \( D' \subseteq (\pi^*)^{-1}(\gamma(D)) \).

Therefore \( \text{Im } \pi^* = \pi^*(\text{Spec}(A)) = \pi^*(\gamma(D')) \subseteq \gamma(D) \). □

The above Theorem suggests us the following natural questions which are unknown in general:

1. Does it true that the Zariski closure of \( D \) in \( \text{Spec}(R) \) is equal to \( \text{Im } \pi^* \) where \( \pi : R \to \prod_{p \in D} R/p \) is the canonical map?

2. Does it true that the flat closure of \( D \) in \( \text{Spec}(R) \) is equal to \( \text{Im } \pi^* \) where \( \pi : R \to \prod_{p \in D} R_p \) is the canonical map?

4. Minimal and maximal spectra

Lemma 4.1. Let \( p \) be a minimal prime of a ring \( R \). If there is a Zariski quasi-compact open subset \( U \) of \( \text{Spec}(R) \) such that \( p \notin U \) then there exists an element \( f \in R \setminus p \) such that \( D(f) \cap U = \emptyset \).

Proof. There are finitely many elements \( g_1, \ldots, g_n \in R \) such that \( U = \bigcup_{i=1}^n D(g_i) \). We have \( g_i \in p \) for all \( i \). It follows that the image of \( g_i \) under the canonical map \( R \to R_p \) is nilpotent. Thus there is an element \( f \in R \setminus p \) and a natural number \( N \) such that \( f g_i^N = 0 \) for all \( i \). Then clearly \( D(f) \cap U = \emptyset \). □
Proposition 4.2. The minimal spectrum of a ring $R$ w.r.t. the Zariski topology is Hausdorff and totally disconnected. Moreover, $\text{Min}(R) \cap D(f)$ is a clopen subset of $\text{Min}(R)$ for all $f \in R$.

Proof. It implies from the above Lemma. □

Lemma 4.3. Let $m$ be a maximal ideal of a ring $R$. If there is a flat quasi-compact open subset $U$ of $\text{Spec}(R)$ such that $m \not\in U$ then there exists an element $f \in m$ such that $V(f) \cap U = \emptyset$.

Proof. There is an element $g \in R$ such that $U \subseteq V(g)$ and $g \not\in m$. Thus there are elements $a \in R$ and $f \in m$ such that $ag + f = 1$. Clearly $V(f) \cap V(g) = \emptyset$. □

As the dual of Proposition 4.2 we have:

Proposition 4.4. The maximal spectrum of a ring $R$ w.r.t. the flat topology is Hausdorff and totally disconnected. Moreover, $\text{Max}(R) \cap V(f)$ is a clopen subset of $\text{Max}(R)$ for all $f \in R$.

Proof. It is an immediate consequence of the above Lemma. □

It is well-known that the maximal spectrum of a ring is quasi-compact w.r.t. the Zariski topology. Dually, its minimal spectrum is quasi-compact w.r.t. the flat topology. But the minimal (resp. maximal) spectrum of a ring is not necessarily quasi-compact w.r.t. the Zariski (resp. flat) topology. As a specific example, $\text{Max}(\mathbb{Z})$ is not quasi-compact w.r.t. the flat topology since the set of prime numbers is infinite. By [14, Theorem 3.20], there is a ring $A$ whose prime ideals have precisely the reverse order of the primes of $\mathbb{Z}$ and so $\text{Min}(A)$ is not Zariski quasi-compact. Concerning to the these we have the following interesting results:

Theorem 4.5. The maximal spectrum of a ring $R$ is compact w.r.t. the flat topology if and only if $R/J(R)$ is absolutely flat where $J(R)$ is the radical Jacobson of $R$.

Proof. If $A := R/J(R)$ is absolutely flat then $\text{Max}(A) = \text{Spec}(A)$. Therefore the image of the map $\text{Spec}(A) \to \text{Spec}(R)$ induced by the
canonical map $R \rightarrow A$ is equal to $\text{Max}(R)$. Hence, by [14, Proposition 3.4], $\text{Max}(R)$ is quasi-compact. Conversely, let $f \in R \setminus J(R)$. Let $m$ be a maximal ideal of $R$ such that $f \notin m$. Then there are elements $a \in R$ and $c \in m$ such that $1 = af + c$. Now by applying the quasi-compactness of $\text{Max}(R)$ then we may find a finite number of elements $a_1, ..., a_n$ and $c_1, ..., c_n$ of $R$ such that $\text{Max}(R) \subseteq V(f) \cup \left( \bigcup_{i=1}^{n} V(c_i) \right)$ and $1 = a_if + c_i$ for all $i$. This implies that $f_{c_1...c_n} \in J(R)$ and $1 = bf + c_1...c_n$ for some $b \in R$. It follows that $f - bf^2 \in J(R)$. Therefore $R/J(R)$ is absolutely flat. □

In order to establish the dual of Theorem 4.5, the following results are required:

**Lemma 4.6.** Let $R \subseteq S$ be an extension of rings with $R$ absolutely flat then $S$ is $R$–faithfully flat.

**Proof.** Suppose $S \otimes_R M = 0$ for some $R$–module $M$. From the following exact sequence of $R$–modules $0 \rightarrow R \rightarrow S \xrightarrow{\pi} S/R \rightarrow 0$ we obtain the following long exact sequence of $R$–modules

$$\cdots \rightarrow \text{Tor}_1^R(S/R, M) \rightarrow R \otimes_R M \rightarrow S \otimes_R M \rightarrow S/R \otimes_R M \rightarrow 0.$$

But $\text{Tor}_1^R(S/R, M) = 0$ since $S/R$ is $R$–flat. It follows that $M \simeq R \otimes_R M = 0$. □

**Lemma 4.7.** Every injective flat ring map $\varphi : R \rightarrow A$ with $R$ reduced and $A$ absolutely flat then induces an injective flat epimorphism $\psi : R \rightarrow B$ such that $B$ is absolutely flat.

**Proof.** Consider the following commutative diagram

$$\begin{array}{ccc}
R & \xrightarrow{\varphi} & A \\
\downarrow{\eta} & & \downarrow{\eta'} \\
R^{(-1)}R & \xrightarrow{\varphi'} & A^{(-1)}A
\end{array}$$

where $R^{(-1)}R$ and $A^{(-1)}A$ are the point-wise localizations of $R$ and $A$, respectively, see [15, §3]. By [15, Lemma 4.6], $\eta'$ is an isomorphism. Let $\theta := (\eta')^{-1} \circ \varphi' : R^{(-1)}R \rightarrow A$ also let $B := \text{Im}(\theta)$ which is an absolutely flat ring since, by [15, Theorem 3.8], $R^{(-1)}R$ is absolutely flat.
flat. The map $\theta$ factors as $i \circ \theta'$ where $\theta' : R^{(-1)} \rightarrow B$ is the canonical surjection which is induced by $\theta$ and $i : B \rightarrow A$ is the canonical injection. Let $\psi := \theta' \circ \eta : R \rightarrow B$ which is an epimorphism. By lemma 4.6, $i$ is faithfully flat. Therefore $\psi$ is injective and flat since $\varphi = i \circ \psi$. 

Lemma 4.8. Let $\varphi : R \rightarrow A$ be a ring map such that for each prime $q$ of $A$, $A_q$ is $R_p$-flat where $p := \varphi^{-1}(q)$. Then $A$ is $R$-flat.

Proof. It is an easy exercise. 

Theorem 4.9. If the minimal spectrum of a reduced ring $R$ is Zariski compact then $\mathcal{M}(R)$, the maximal flat epimorphic extension of $R$, is absolutely flat.

Proof. The minimal spectrum of $R$ is a patch closed subset of $\text{Spec}(R)$. Because suppose there is some $q \in \gamma(\text{Min}(R))$ which is not in $\text{Min}(R)$. Then for each $p \in \text{Min}(R)$ there exists some $f \in q$ which is not in $p$. By applying the quasi-compactness of $\text{Min}(R)$ then we may find a finitely many elements $f_1, ..., f_n \in q$ such that $\text{Min}(R) \subseteq \bigcup_{i=1}^n D(f_i)$. Clearly $\bigcap_{i=1}^n V(f_i)$ is a patch open neighbourhood of $q$. Therefore it meets $\text{Min}(R)$. But this is a contradiction. Thus $\gamma(\text{Min}(R)) = \text{Min}(R)$. Let $A := \bigprod_{p \in \text{Min}(R)} R_p$. It is absolutely flat since $R_p = \kappa(p)$ for all $p \in \text{Min}(R)$. By Theorem 3.1, $\text{Min}(R) = \text{Im} \pi^*$ where $\pi : R \rightarrow A$ is the canonical map. Hence by Lemma 4.8, $\pi$ is flat. It is also injective. Thus by Lemma 4.7, there exists an injective flat epimorphism $\psi : R \rightarrow B$ such that $B$ is absolutely flat. By [9, Prop 3.4], there is a (unique injective) ring map $h : B \rightarrow S$ such that $\eta = h \circ \psi$ where $S := \mathcal{M}(R)$ and $\eta : R \rightarrow S$ is the canonical map. It follows that for each prime $q$ of $S$ then $p := \eta^{-1}(q)$ is a minimal prime of $R$ because $B$ is absolutely flat and every flat ring map has the going-down property. The canonical map $\kappa(p) = R_p \rightarrow R_p \otimes_R S_q \simeq S_q$ is a flat epimorphism since flat maps and epimorphisms are stable under the base change. It is also faithfully flat since $pS \neq S$. Every faithfully flat epimorphism is an isomorphism. Therefore, by [15, Prop 2.4], $S$ is absolutely flat. 

Lemma 4.10. Let $\varphi : R \to A$ be an injective flat ring map. If $\text{Min}(A)$ is Zariski compact then $\text{Min}(R)$ is as well.

**Proof.** By [14, Lemma 3.9], $\text{Min}(R) \subseteq \varphi^*(\text{Min}(A))$. Moreover the inverse image of every minimal prime of $A$ under $\varphi$ is a minimal prime of $R$ since every flat ring map has the going-down property. Therefore $\varphi^*(\text{Min}(A)) = \text{Min}(R)$. Hence $\text{Min}(R)$ is quasi-compact. □

Now as the dual of Theorem 4.9 we have:

**Corollary 4.11.** The minimal spectrum of a ring $R$ is Zariski compact if and only if $\mathcal{M}(R/\mathfrak{N})$ is absolutely flat where $\mathfrak{N}$ is the nil-radical of $R$.

**Proof.** The canonical map $R \to R/\mathfrak{N}$ induces a homeomorphism between the corresponding spectra which maps $\text{Min}(R/\mathfrak{N})$ onto $\text{Min}(R)$. Then apply Theorem 4.9 for the implication “⇒” and Lemma 4.10 for the reverse. □

**Corollary 4.12.** The minimal spectrum of a reduced ring $R$ is Zariski compact if and only if the total ring of fractions of the polynomial ring $R[x]$ is absolutely flat.

**Proof.** Clearly $R[x]$ is reduced and every minimal prime of it is of the form $p[x]$ where $p$ is a minimal prime of $R$. Suppose $\text{Min}(R[x]) \subseteq \bigcup D(f_\alpha)$ where $f_\alpha \in R[x]$ for all $\alpha$. Then $\text{Min}(R) \subseteq \bigcup D(c_\alpha)$ where $c_\alpha$ is some coefficient of $f_\alpha$. Thus compactness of the minimal spectrum of $R$ is equivalent to that of $R[x]$. Then apply Corollary 4.11 and the fact that the total ring of fractions of the polynomial ring $R[x]$ is canonically isomorphic to $\mathcal{M}(R[x])$. □

5. **Noetherian property**

Here we give a characterization for noetherianness of the Zariski topology in terms of the flat topology:

**Theorem 5.1.** The prime spectrum of a ring $R$ is noetherian w.r.t. the Zariski topology if and only if the flat opens of $\text{Spec}(R)$ are stable under the arbitrary intersections.
Proof. First assume that \( \text{Spec}(R) \) is Zariski noetherian. If \( I \) is an arbitrary ideal of \( R \) then \( V(I) \) is a flat open. Because \( U = \text{Spec}(R) \setminus V(I) \) is Zariski quasi-compact and hence there is a finite set \( \{ f_1, ..., f_n \} \) of elements of \( R \) such that \( U = \bigcup_{i=1}^{n} D(f_i) \). It follows that \( V(I) = \bigcap_{i=1}^{n} V(f_i) \).

To prove the assertion it suffices to show that the intersection of every family of basis flat opens is flat open. By [14, Theorem 3.2], the basis flat opens are of the form \( V(I) \) where \( I \) is a f.g. ideal of \( R \). Let \( \{ I_\alpha \} \) be a family of ideals of \( R \). If \( p \in \bigcap_\alpha V(I_\alpha) \) then clearly \( V(p) \subseteq \bigcap_\alpha V(I_\alpha) \).

Therefore \( \bigcap_\alpha V(I_\alpha) \) is flat open. To prove the converse, it suffices to show that every Zariski open \( U = \text{Spec}(R) \setminus V(I) \) of \( \text{Spec}(R) \) is quasi-compact where \( I \) is an ideal of \( R \). But \( V(I) \) is flat open since \( V(I) = \bigcap_{f \in I} V(f) \). It is also quasi-compact, apply [14, Proposition 3.4] to the canonical map \( R \to R/I \). Thus there is a finite set \( \{ I_1, ..., I_s \} \) of f.g. ideals of \( R \) such that \( V(I) = \bigcup_{k=1}^{s} V(I_k) = V(I_1I_2...I_s) \). Note that \( I_1I_2...I_s = \langle f_1, ..., f_n \rangle \) is a f.g. ideal of \( R \). Therefore \( \sqrt{I} \) is equal to the radical of a f.g. ideal of \( R \). We have then \( U = \text{Spec}(R) \setminus V(\sqrt{I}) = \text{Spec}(R) \setminus \bigcap_{i=1}^{n} V(f_i) = \bigcup_{i=1}^{n} D(f_i) \).

\[ \square \]

As the dual of Theorem 5.1 please consider [14, Theorem 4.2].

Acknowledgements. The author would like to give sincere thanks to Professor Irena Swanson for valuable comments about the manuscript.

References

[1] Artico, Giuliano. and Marconi, Umberto. On the compactness of minimal spectrum, Rendiconti del Seminario Matematico della Università di Padova, 56, 79-84 (1976).
[2] Dobbs D., Fontana M. and Papick I. On the flat spectral topology, Rendiconti di Mathematica. (4) 1981, Vol. 1, Serie VII.
[3] Henriksen, Melvin. and Jerison, Meyer. The space of minimal prime ideals of a commutative ring, Trans. of the AMS, 115, 110-130 (1965).
[4] Melvin. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 142 (1969), 43-60.
[5] Hochster, Melvin. The minimal prime spectrum of a commutative ring, Can. J. Math., Vol. XXIII, No. 5, 1971, pp. 749-758.
[6] Hong, C. Y. et al. The minimal prime spectrum of rings with annihilator conditions, Journal of Pure and Applied Algebra, 213, 1478-1488 (2009).
[7] Kist, Joseph. Compact spaces of minimal prime ideals, Mathematische Zeitschrift, 1969, Vol. 111, Issue 2, pp. 151-158.

[8] Knox, M. et al. Generalizations of complemented rings with applications to rings of continuous functions. J. Algebra. Appl. 7, 124 (2008).

[9] Lazard, Daniel. Épimorphismes plats, Séminaire Samuel. Algèbre commutative, tomme 2 (1967-1968).

[10] Matlis, Eben. The minimal prime spectrum of a reduced ring, Illinois Journal of Mathematics, Vol. 27, Num. 3, Fall 1983.

[11] Mewborn, A. C. Some conditions on commutative semiprime rings, J. Algebra, 13, 422-431 (1969).

[12] Quentel, Yann. Sur la compacité du spectre minimal d’un anneau, Bulletin de la S. M. F, 99 (1971), p. 265-272.

[13] Schwartz, Niels. and Tressl, Marcus. Elementary properties of minimal and maximal points in Zariski spectra, J. Algebra, 323, 698-728 (2010).

[14] Tarizadeh, A. Flat topology and its dual aspects, preprint, arXiv:1503.04299v10 [math.AC].

[15] Tarizadeh, A. On the separability of prime spectra, preprint, arXiv:1608.05835v5 [math.AC].

Department of Mathematics, Faculty of Basic Sciences, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran.

E-mail address: ebulfez1978@gmail.com