THE DIFFERENTIAL EQUATIONS ASSOCIATED WITH
CALOGERO-MOSER-SUTHERLAND PARTICLE MODELS IN
THE FREEZING REGIME

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Abstract. Multivariate Bessel processes describe Calogero-Moser-Sutherland
particle models and are related with $\beta$-Hermite and $\beta$-Laguerre ensembles.
They depend on a root system and a multiplicity $k$. Recently, several limit
theorems for $k \to \infty$ were derived where the limits depend on the solutions
of associated ODEs in these freezing regimes. In this paper we study the solutions
of these ODEs which are are singular on the boundaries of their domains. In
particular we prove that for a start in arbitrary boundary points, the ODEs
always admit unique solutions in their domains for $t > 0$.

1. Introduction

Calogero-Moser-Sutherland particle systems on $\mathbb{R}$ with $N$ particles can be
described as multivariate Bessel processes on closed Weyl chambers in $\mathbb{R}^N$. These
processes are time-homogeneous diffusions with well-known generators of the transi-
tion semigroups, and they are solution of the associated stochastic differential
equations (SDEs); see [A, CDGRVY, DV, R1, R2, RV] for the background. These
Bessel processes $(X_{t,k})_{t \geq 0}$ are classified via root systems, a possibly multidi-
sensional multiplicity parameter (often called a coupling constant) $k$ and by their start-
ing points. Moreover, these processes are related to the $\beta$-Hermite and $\beta$-Laguerre
ensembles of Dumitriu and Edelman [DE1, DE2]; see e.g. [AV1].

We briefly recapitulate the generators of $(X_{t,k})_{t \geq 0}$ for the most important cases,
the root systems $A_{N-1}$ and $B_N$. For $A_{N-1}$, we have $k \in ]0, \infty[$, the processes live
on the closed Weyl chamber

$$C_{N}^{A} := \{ x \in \mathbb{R}^N : x_1 \geq x_2 \geq \ldots \geq x_N \},$$

and the generators of the transition semigroups are

$$L_{A}f := \frac{1}{2} \Delta f + k \sum_{i=1}^{N} \left( \sum_{j \neq i} \frac{1}{x_i - x_j} \right) \frac{\partial}{\partial x_i} f,$$

(1.1)

where we assume reflecting boundaries. For $B_N$, we have $k = (k_1, k_2) \in ]0, \infty[^2$, the
processes live on

$$C_{N}^{B} := \{ x \in \mathbb{R}^N : x_1 \geq x_2 \geq \ldots \geq x_N \geq 0 \},$$

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and the generators of the transition semigroups are

\[ Lf := \frac{1}{2} \Delta f + k_2 \sum_{i=1}^{N} \sum_{j \neq i} \left( \frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \right) \frac{\partial}{\partial x_i} f + k_1 \sum_{i=1}^{N} \frac{1}{x_i} \frac{\partial}{\partial x_i} f \]  

(1.2)

with reflecting boundaries. If we define the weight functions \( w_k \) of the form

\[ w_k^A(x) := \prod_{i<j} (x_i - x_j)^{2k}, \quad w_k^B(x) := \prod_{i<j} (x_i^2 - x_j^2)^{2k_2} \cdot \prod_{i=1}^{N} x_i^{2k_1}, \]  

(1.3)

the generators may be written in a unified way as

\[ Lf := \frac{1}{2} \Delta f + \frac{1}{2} \nabla \ln w_k \cdot \nabla f. \]  

(1.4)

The associated SDEs then have the form

\[ dX_{t,k} = dB_t + \frac{1}{2} \nabla (\ln w_k)(X_{t,k}) \, dt \]  

(1.5)

on the interior of the associated Weyl chamber \( C_N \) with an \( N \)-dimensional Brownian motion \( (B_t)_{t \geq 0} \). It is known from [CDGRVY, Sch, GM] that for each starting point in the closed Weyl chamber \( C_N \), \( (1.5) \) has a unique strong solution \( (X_{t,k})_{t \geq 0} \) whenever all components of \( k \) are positive. Moreover, if all components of \( k \) are at least 1/2, we have the following behaviour: If the process starts in the interior of \( C_N \), then \( (X_{t,k})_{t \geq 0} \) does not hit the boundary \( \partial C_N \) of \( C_N \) almost surely. If it starts at the boundary, it leaves the boundary immediately.

Assume now that \( k = \beta > 0 \) in the A-case and \( k = (k_1, k_2) = (\nu \beta, \beta) \) with \( \nu > 0 \) fixed and \( \beta > 0 \) in the B-case. Then in both cases, the renormalized processes \( (\tilde{X}_{t,k} := \frac{1}{\beta} X_{t,k})_{t \geq 0} \) satisfy \( d\tilde{X}_{t,k} = \frac{1}{\nu^2} dB_t + \frac{1}{2} \nabla (\ln w)(\tilde{X}_{t,k}) \, dt \) with \( w \) of the form

\[ w^A(x) := \prod_{i<j} (x_i - x_j)^2, \quad w_k^B(x) := \prod_{i<j} (x_i^2 - x_j^2)^2 \cdot \prod_{i=1}^{N} x_i^{2\nu}. \]  

(1.6)

In [AKM1, AKM2, AV1, V, VW], several limit theorems like laws of large numbers and central limit theorems were derived for \( \tilde{X}_{t,k} \) and \( X_{t,k} \) for \( \beta \to \infty \), where the limits are described in terms of the solutions of the associated ordinary differential equations (ODEs)

\[ \frac{dx(t)}{dt} = \frac{1}{2} \nabla (\ln w)(x(t)) \]  

(1.7)

in the interior of \( C_N \). Unfortunately, only for particular starting points \( x_0 \), the gradient systems \( (1.7) \) can be solved explicitly. For instance, by [AV1], one has solutions of the form \( x(t) = \sqrt{c + t} \cdot z \) for \( c > 0 \) where the coordinates of \( z \in \mathbb{R}^N \) are the ordered zeros of classical orthogonal polynomials of order \( N \). Formally, these solutions also exist for \( c = 0 \) where we start in the singular point \( x(0) = 0 \in \partial C_N \).

The aim of this paper is to study several properties of the solutions of \( (1.7) \) in a systematic way for the most interesting root systems of types A, B, and D; see Section 4 for the details the the root systems \( D_n \). In particular, we show that all solutions of \( (1.7) \) can be determined explicitly up to determining the zeros of polynomial of order \( N \) via transformations of \( (1.7) \) using elementary symmetric polynomials. We also obtain from this approach that for all starting points \( x_0 \in \partial C_N \) there is a unique continuous solution \( x \) of \( (1.7) \) with \( x(t) \) in the interior of \( C_N \) for \( t > 0 \). The arguments will be similar for root systems of types A, B, and D; these three cases will be studied in the next three sections.
2. The root system $A_{N-1}$

For the root system $A_{N-1}$, the ODE (1.7) has the form
\[
\frac{dx(t)}{dt} = H(x(t)) \quad \text{with} \quad H(x) := \left(\sum_{j \neq 1} \frac{1}{x_1 - x_j}, \ldots, \sum_{j \neq N} \frac{1}{x_N - x_j}\right) \quad (2.1)
\]
on the interior $W_N^A$ of the Weyl chamber $C_N^A$. By classical results on ODEs the following is straightforward; see Lemma 2.1 of [AV1]:

**Lemma 2.1.** For $\epsilon > 0$ consider the open subset $U_\epsilon := \{ x \in C_N^A : d(x, \partial C_N^A) > \epsilon \}$ with the Euclidean distance $d$ on $\mathbb{R}^N$. Then $H$ is Lipschitz-continuous on each $U_\epsilon$, and for each $x_0 \in U_\epsilon$, the ODE (2.1) with $x(0) = x_0$ admits a unique solution. This solution exists for all $t > 0$ with $x(t) \in U_\epsilon$.

We next recapitulate from [AV1] that (2.1) has the following particular solutions:

**Lemma 2.2.** Let $(H_N)_{N \geq 0}$ be the Hermite polynomials which are orthogonal w.r.t. the density $e^{-x^2}$ as in the monograph [S]. Let $z := (z_1, \ldots, z_N) \in W_N^A$ be the vector consisting of the ordered zeros of the Hermite polynomial $H_N$. Then for each $c > 0$,

\[
x(t) := \sqrt{2t + c^2} \cdot z \text{ is the solution of (2.1) with } x(0) = cz.
\]

The growth of these particular solutions w.r.t. the Euclidean norm on $\mathbb{R}^N$ is typical; see [VW]:

**Lemma 2.3.** For each solution $x$ of (2.1) with start in $W_N^A$,
\[
\|x(t)\|^2 = N(N-1)t + \|x(0)\|^2. \quad (2.2)
\]

Lemma 2.3 motivates a decomposition of solutions into an easy radial part and a spherical part on the unit sphere in $\mathbb{R}^N$. If doing so, the spherical parts of the special solutions in 2.2 correspond to a stationary solution. This stationary solution satisfies the following stability result:

**Lemma 2.4.** For each initial value $x_0 \in W_N^A$, the solution $x$ of (2.1) has the form
\[
x(t) = \sqrt{N(N-1)t + \|x_0\|^2} \cdot \phi(t) \quad (t \geq 0)
\]
where $\phi$ satisfies
\[
\|\phi(t)\| = 1 \quad \text{and} \quad \lim_{t \to \infty} \phi(t) = \sqrt{\frac{2}{N(N-1)}} \cdot z.
\]

**Proof.** Using (2.2), we define
\[
\phi(t) := (\phi_1(t), \ldots, \phi_N(t)) := \frac{1}{\sqrt{N(N-1)t + \|x_0\|^2}} \cdot x(t) = \frac{x(t)}{\|x(t)\|} \quad (2.3)
\]
with $\|\phi(t)\| = 1$. The ODE (2.1) implies that
\[
\frac{d}{dt}(\phi_i(t)) = \frac{\dot{x}_i(t)}{\sqrt{N(N-1)t + \|x_0\|^2}} - \frac{N(N-1) \cdot \phi_i(t)}{2(N(N-1)t + \|x_0\|^2)^{3/2}}
\]
\[
= \frac{1}{N(N-1)t + \|x_0\|^2} \left( \sum_{j \neq i} \frac{\sqrt{N(N-1)t + \|x_0\|^2}}{x(t) - x_j(t)} - \frac{N(N-1)}{2} \phi_i(t) \right)
\]
\[
= \frac{1}{N(N-1)t + \|x_0\|^2} \left( \sum_{j \neq i} \phi_i(t) - \phi_j(t) - \frac{N(N-1)}{2} \phi_i(t) \right).
\]
Therefore,
\[ \psi(t) := \phi\left( \frac{N(N-1)}{2} t^2 + \|x_0\|^2 t \right) \quad (t \geq 0) \]
satisfies
\[ \dot{\psi}_i(t) = \sum_{j \neq i} \frac{1}{\psi_i(t) - \psi_j(t)} - \frac{N(N-1)}{2} \psi_i(t) \quad (i = 1, \ldots, N) \quad (2.4) \]
with \( \psi(0) = \phi(0) = x_0/\|x_0\| \). The ODE (2.4) is a gradient system \( \dot{\psi} = (\nabla u)(\psi) \) with
\[ u(y) := 2 \sum_{i,j=1,\ldots,N, i<j} \ln(y_i - y_j) - \|y\|^2 \cdot N(N-1)/4. \]
It now follows from Lemma 2.2 of [AV1] (or see [AKM1] or Section 6.7 of [S]) that \( u \) admits a unique local maximum on \( C^A \), that this maximum is a global one, and that it located at
\[ \sqrt{\frac{2}{N(N-1)} \cdot z} \]
where, by (D.22) of [AKM1], this vector has \( \| \cdot \|_2 \)-norm 1. We conclude from Section 9.4 of [HS] on gradient systems that this point is an asymptotically stable equilibrium point of the ODE (2.3). This and (2.3) now lead to the claim. □

We cannot determine \( x \) explicitly for arbitrary starting points and \( N \). On the other hand, Lemma 2.5 is a special case of the observation that for each symmetric polynomial \( p \) in \( N \) variables, \( t \mapsto p(x(t)) \) is a polynomial in \( t \) which can be computed explicitly. In this context we shall now employ the elementary symmetric polynomials \( e_k := e_k^N \) \( (k = 0, \ldots, N) \) in \( N \) variables which are characterized by
\[ \prod_{k=1}^N (z - x_k) = \sum_{k=0}^N (-1)^{N-k} e_{N-k}(x) z^k \quad (z \in \mathbb{C}, x = (x_1, \ldots, x_N)). \quad (2.5) \]
In particular, \( e_0 = 1 \), \( e_1(x) = \sum_{k=1}^N x_k, \ldots, e_N(x) = \prod_{k=1}^N x_k \). As each symmetric polynomial in \( N \) variables is a polynomial in \( e_1, \ldots, e_N \) by a classical result, the following lemma shows that all symmetric polynomials of \( x \) are polynomials in \( t \).

**Lemma 2.5.** For each initial value \( x_0 = (x_{0,1}, \ldots, x_{0,N}) \in W^A_N \), let \( x \) be the solution of (2.1). Then, for \( k = 0, \ldots, N, t \mapsto e_k(x(t)) \) is a polynomial in \( t \) of degree \( (at most) \lceil \frac{k}{2} \rceil \) where the leading coefficient of order \( \lceil \frac{k}{2} \rceil \) is given by
\[ \frac{(-1)^k \cdot N!}{2^l \cdot l!(N-2l)!} \quad (k = 2l \leq N) \quad \text{and} \quad \frac{(-1)^k \cdot (N-1)!}{2^l \cdot l!(N-2l-1)!} \sum_{j=1}^N x_{0,j} \quad (k = 2l+1 \leq N). \]

**Proof.** This is trivial for \( k = 0 \) and can be easily verified for \( k = 1 \). For \( k \geq 2 \) we use induction on \( k \) and use the following notion: For a non-empty set \( S \subset \{1, \ldots, N\} \), the vector \( x \in \mathbb{R}^{|S|} \) is the vector with the coordinates \( x_i \) for \( i \in S \) in the natural ordering on \( S \). We then have for \( k \geq 2 \) that
\[ \frac{d}{dt} e_k(x(t)) = \sum_{j=1}^N \frac{dx_j(t)}{dt} \cdot e_{k-1}^{N-1}(x_{(1, \ldots, N) \setminus \{j\}}(t)). \]
Therefore, by the differential equation,
\[
\frac{d}{dt}e_k(x(t)) = \sum_{j=1}^{N} \sum_{i:j \neq j} e_{k-1}^{N-1}(x_{\{1,\ldots,N\} \setminus \{j\}}(t)) \frac{x_j(t) - x_i(t)}{x_j(t) - x_i(t)}
\]
\[= \frac{1}{2} \sum_{i,j=1,\ldots,n; i \neq j} e_{k-1}^{N-1}(x_{\{1,\ldots,N\} \setminus \{j\}}(t)) - e_{k-1}^{N-1}(x_{\{1,\ldots,N\} \setminus \{i\}}(t)) \frac{x_j(t) - x_i(t)}{x_j(t) - x_i(t)}.\]  

Moreover, simple combinatorial computations yield for \(i \neq j\)
\[
eq_k^{N-1}(x_{\{1,\ldots,N\} \setminus \{j\}}(t)) - e_k^{N-1}(x_{\{1,\ldots,N\} \setminus \{i\}}(t)) = (x_i(t) - x_j(t))e_{k-2}^{N-2}(x_{\{1,\ldots,N\} \setminus \{i,j\}}(t))
\]
and
\[
\sum_{i,j=1,\ldots,n; i \neq j} e_{k-2}^{N-2}(x_{\{1,\ldots,N\} \setminus \{i,j\}}(t)) = (N - k + 2)(N - k + 1)e_{k-2}^{N}(x(t)).
\]

Therefore, by (2.6) and (2.8),
\[
\frac{d}{dt}e_k(x(t)) = -\frac{1}{2} (N - k + 2)(N - k + 1)e_{k-2}^{N}(x(t)).
\]  

This recurrence relation and the cases \(k = 0, 1\) now lead to the claim. \(\square\)

**Remark 2.6.** For each \(r \in \mathbb{R}\) and each solution \(x\) of (2.1), the function \(x^r(t) := x(t) + r \cdot (1, \ldots, 1)\) is also a solution of (2.1). This implies that we may assume \(\sum_{j=1}^{N} x_{0,j} = 0\) without loss of generality for our initial conditions. If we do so, the degrees of the polynomials \(t \mapsto e_k(x(t))\) for odd \(k\) can be chosen to be even smaller.

Lemma 2.8 and Eq. (2.4) can be used to compute all solutions of (2.1) on \(W^A_N\). First, one has to determine the polynomials \(e_k(x(t))\) \(k = 1, \ldots, N\). In a second step, one has to determine the ordered, different zeros of the polynomials in (2.4) from the coefficients of the polynomials. This relation corresponds to some diffeomorphism. We present an example.

**Example 2.7.** Let \(N = 3\). Let \(x_0 \in C^A_3\) with \(x_{0,1} + x_{0,2} + x_{0,3} = 0\) according to Remark 2.6. We here obtain
\[e_1(x(t)) = 0, \quad e_1(x(t)) = -3t + e_2(x_0), \quad e_3(x(t)) = e_3(x_0),\]
and thus
\[\prod_{k=1}^{3}(z - x_k(t)) = z^3 + (e_2(x_0) - 3t)z - e_3(x_0).\]  

We now apply Cardano’s formula in the cases irreducible. We first observe that the existence of 3 real zeros implies \(3t - e_2(x_0) > 0\); we have the solutions
\[x_k(t) = \sqrt[3]{4t - \frac{4}{3}e_2(x_0)} \cdot \cos(\frac{1}{3} \arccos(\frac{\sqrt{27}}{2} \cdot \frac{e_3(x_0)}{(3t - e_2(x_0))^{3/2}}) + \frac{2}{3}(1 - k)\pi),\]
for \(k = 1, 2, 3\). The correct ordering \(x_1(t) \geq x_2(t) \geq x_3(t)\) here follows easily from the case \(t \to \infty\) in which case
\[x(t) = \sqrt[3]{4t - \frac{4}{3}e_2(x_0)} \cdot ((\sqrt{3/4}, 0), -\sqrt{3/4}) + O(t^{-3/2})\]
\[= \sqrt{6}t - 2e_2(x_0) \cdot (\sqrt{1/2}, 0, -\sqrt{1/2}) + O(t^{-1}) = \ddot{x}(t) + O(t^{-1}).\]
with a solution $\hat{x}$ of our differential equation of the type of Lemma 2.7. This improves Lemma 2.4 in a quantitative way for $N = 3$.

We also remark that the discriminant of the polynomial (2.10) is

$$\Delta := e_3(x_0)^2/4 + (e_2(x_0) - 27)^3/27.$$ 

By Cardano’s formula, $\Delta = 0$ holds if and only if we have multiple (real) zeros in (2.10), i.e., a point in $\partial C_A^3$. Hence, if we formally start our solution at some $x_0 \in \partial C_A^3$ with $x_{0,1} + x_{0,2} + x_{0,3} = 0$, then $x(t) \in W_3^A$ exists for all $t > 0$. By Remark 2.6, this observation also holds for arbitrary starting points in $C_A^3$.

The existence of solutions with a start on the boundary in Example 2.7 holds for arbitrary dimensions:

**Theorem 2.8.** Let $N \geq 2$. Then, for each starting point $x_0 \in C_A^N$, the ODE (2.7) has a unique solution for all $t \geq 0$ in the following sense: For each $x_0 \in C_A^N$ there is a unique continuous function $x : [0, \infty) \rightarrow C_A^N$ with $x(0) = x_0$ such that for $t \in [0, \infty)$, $x(t) \in W_3^N$ holds, and $x (\cdot) : [0, \infty) \rightarrow W_3^N$ satisfies (2.7).

**Proof.** By Lemma 2.7, it suffices to assume $x_0 \in \partial C_A^N$. We transform (2.7) via elementary symmetric polynomials as in Lemma 2.5. Then, for $t > 0$ and $k = 2, \ldots, N$,

$$\frac{d}{dt} e_k^N(x(t)) = -\frac{1}{2} (N - k + 2) (N - k + 1) e_{k-2}^N(x(t)) \quad (2.11)$$

where in particular,

$$\frac{d}{dt} e_2^N(x(t)) = -\frac{1}{2} N(N - 1).$$

Moreover, for $k = 1$,

$$\frac{d}{dt} e_1^N(x(t)) = \sum_{i,j=1,\ldots,N:i\neq j} \frac{1}{x_i(t) - x_j(t)} = 0. \quad (2.12)$$

These equations form an ODE whose solutions $\hat{e}(t) := (e_1^N(x(t)), \ldots, e_N^N(x(t)))$ are polynomials in $t$ by Lemma 2.5. As the mapping $e : C_A^N \rightarrow E_N := \{y = (y_1, \ldots, y_N) \in \mathbb{R}^N : \text{the polynomial}

$$P(z) := \sum_{j=0}^{N} (-1)^{N-j} y_{N-j} z^j \text{ with } y_0 = 1 \text{ has only real zeros}\}$,$

$$x \mapsto (e_1(x), \ldots, e_N(x)) \quad (2.13)$$

is continuous and injective, the ODE (2.7) with start in $x_0$ has at most one solution.

For the existence of a solution we notice that the restriction $e : W_3^N \rightarrow e(W_3^N)$ of the mapping $e$ is a diffeomorphism while the extension to the closure $e : C_A^N \rightarrow e(C_A^N) = e(W_3^N) \subset E_N$ is a homeomorphism. We claim that the inverse mapping of $e$ transforms the existent polynomial solutions of the ODEs (2.12) and (2.11) back into solutions of the ODE in (2.7) in the sense of the theorem. For this we prove that for any starting point $x_0 \in \partial C_A^N$ in (2.7) and its image $e(x_0) \in e(W_3^N)$, the solution $\hat{e}(t)$ ($t \geq 0$) of the ODEs (2.12) and (2.11) with $\hat{e}(0) = e(x_0)$ satisfies $\hat{e}(t) \in e(W_3^N)$ for $t > 0$ sufficiently small. If this is shown it follows that the preimage of $(\hat{e}(t))_{t \geq 0}$ under $e$ is a solution of (2.7) in the sense of the theorem.
To prove this statement, we recapitulate that for each starting point in \( e(W_N^A) \), the solution of the ODEs (2.12) and (2.11) satisfies \( \hat{e}(t) \in e(W_N^A) \) for \( t \geq 0 \), and that for all fixed \( t \geq 0 \), the solutions \( \hat{e}(t) \) depend continuously from arbitrary starting points in \( \mathbb{R}^N \) by a classical theorem on ODEs. We thus conclude that for each starting point in \( \hat{e}(0) \in e(W_N^A) \) we have \( \hat{e}(t) \in e(W_N^A) \) for \( t \geq 0 \).

We next observe that for each solution \( x \) of (2.11) with start in \( x(0) \in \partial C_N^A \), and for each \( c > 0 \), the function \( t \mapsto \frac{1}{\sqrt{c}} x(ct) \) also solves the ODE in (2.11) with start in \( x(0)/\sqrt{c} \). Moreover, for each solution \( \hat{e}(t) \) of the ODEs (2.12) and (2.11) with start in \( e(x(0)) \), and each \( c > 0 \), the function
\[
t \mapsto \hat{e}_c(t) := (c^{-1/2} e_1( ct ), c^{-2/2} e_2( ct ), \ldots, c^{-N/2} e_N( ct ))
\]
is also a solution of the ODEs (2.12) and (2.11) with start in
\[
(c^{-1/2} e_1( x(0) ), \ldots, c^{-N/2} e_N( x(0) )).
\]

Assume now that there is a starting point \( x(0) \in \partial C_N^A \) and some \( t_0 > 0 \), such that the solution \( (e(t))_{t \geq 0} \) of (2.12) and (2.11) with start in \( e(x(0)) \) satisfies \( \hat{e}(t) \notin e(W_N^A) \) for \( t \in [0,t_0] \). This means that for \( t \in [0,t_0] \),
\[
\hat{e}(t) \notin e(W_N^A) \subset Y := \{ y \in \mathbb{R}^N : \hat{D}(y) = 0 \}
\]
for the discriminant mapping \( \hat{D} : \mathbb{R}^N \to \mathbb{R} \) which is defined as follows: For \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \) we form the discriminant
\[
D(x) := \prod_{1 \leq i < j \leq N} (x_i - x_j)^2.
\]

\( D \) is a symmetric polynomial in \( x_1, \ldots, x_N \) and thus, by a classical result on elementary symmetric polynomials, a polynomial \( \hat{D} \) in \( e_1(x), \ldots, e_N(x) \). This is the discriminant mapping used above. In summary we obtain that \( \hat{D}(\hat{e}(t)) = 0 \) for \( t \in [0,t_0] \) with some polynomial \( D \) which yields that \( \hat{D}(\hat{e}(t)) = 0 \) for all \( t \geq 0 \). As \( Y \cap e(W_N^A) = \emptyset \), we conclude that \( \hat{e}(t) \notin e(W_N^A) \) holds for all \( t \geq 0 \). Therefore, for all \( c > 0 \), the modified solutions from (2.14) satisfy \( \hat{e}_c(t) \notin e(W_N^A) \) for \( t \geq 0 \). For \( c \to \infty \), the starting points \( \hat{e}_c(0) \) of these solutions tend to the vector \( 0 \in \mathbb{R}^N \).

On the other hand, if \( z_1 > z_2 > \ldots > z_N \) are the ordered zeros of the Hermite polynomial \( H_N \), then \( x_0(t) := \sqrt{2t} \cdot (z_1, \ldots, z_N) \) solves (2.11) with start in \( 0 \) in the sense of our theorem by [AV1]. This implies that \( e(0) \) is the solution of (2.12) and (2.11) with start in \( 0 \) where for this solution \( \hat{e}_0(t) \in e(W_N^A) \) holds for all \( t > 0 \). We thus obtain a contradiction to the continuous dependence of the solutions of the ODEs (2.12) and (2.11) from the starting points. We thus obtain that there is no starting point \( x(0) \in \partial C_N^A \) and no \( t_0 > 0 \) with \( \hat{e}(t) \notin e(W_N^A) \) for \( t \in [0,t_0] \). This completes the proof.

Theorem 2.8 can be supplemented by the following observation:

**Lemma 2.9.** Let \( (x(t))_{t \geq 0} \subset W_N^A \) be a solution of the ODE (2.1). Then there is a unique \( t_0 < 0 \) such that \( (x(t))_{t \geq t_0} \) is a solution of (2.1) with \( x(t_0) \in \partial C_N^A \) in the sense of Theorem 2.8 i.e., \( (x(t))_{t > t_0} \subset W_N^A \) solves (2.7).

**Proof.** By Lemma 2.1 the RHS of (2.1) is locally Lipschitz continuous on \( W_N^A \). We thus have a maximal solution \( (x(t))_{t \geq t_0} \subset W_N^A \) with some \( t_0 \in [-\infty,0] \). On the other hand, by Lemma 2.8 this solution must satisfy \( \| x(t) \|^2 = N(N-1) t + \| x(0) \|^2 \) for \( t > t_0 \) which implies that \( t_0 \geq -\| x(0) \|^2/(N(N-1)) \) and that \( x(t) \) remains in
exists, we conclude that \( \lim t \to t_0 \) from the preceding proof. As by the results of the preceding proof \( \lim_{t \to t_0} e(x(t)) \) exists, we conclude that \( \lim_{t \to t_0} x(t) =: x(t_0) \in \partial C_N^A \) exists. This yields the claim. □

3. The root system \( B_N \)

For the root systems \( B_N \) we fix some constant \( \nu > 0 \) as described in the introduction. The ODE (3.1) here has the form

\[
\frac{dx(t)}{dt} = H_\nu(x(t)) \quad \text{with} \quad H_\nu(x) := \left( \sum_{j \neq i} \left( \frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \right) + \frac{\nu}{x_i} \right)
\]

on the interior \( W_N^B \) of the closed Weyl chamber \( C_N^B \). We recapitulate from [AV1]:

Lemma 3.1. Let \( \nu > 0 \). For \( \epsilon > 0 \) consider the open subset

\[
U_\epsilon := \{ x \in C_N^B : x_N > \frac{\epsilon^\nu}{N-1}, \quad \text{and} \quad x_i - x_{i+1} > \epsilon \quad \text{for} \quad i = 1, \ldots, N-1 \}
\]

of \( C_N^B \). Then for each \( x_0 \in U_\epsilon \), the ODE (3.1) with \( x(0) = x_0 \) admits a unique solution. This solution exists for all \( t > 0 \) with \( x(t) \in U_\epsilon \).

We next recapitulate from [AV1] that (3.1) has some particular solutions. For this we recall that for \( \alpha > 0 \) the Laguerre polynomials \( L_\alpha^{(n)} \) are orthogonal w.r.t. the density \( e^{-x} \cdot x^\alpha \); see the monograph [S] for details. We know from [AV1]:

Lemma 3.2. Let \( \nu > 0 \). Let \( z^{(\nu-1)}_1 > \ldots > z^{(\nu-1)}_N > 0 \) be the ordered zeros of \( L_N^{(\nu-1)} \), and \( y := (y_1, \ldots, y_N) \in W_N^B \) with

\[
2(z^{(\nu-1)}_1, \ldots, z^{(\nu-1)}_N) = (y_1^2, \ldots, y_N^2).
\]

Then for each \( c > 0 \), \( x(t) := \sqrt{t+c^2} \cdot y \) is a solution of (3.1).

Again, the growth of these particular solutions is typical; see [VW]:

Lemma 3.3. For each solution \( x \) of (3.1) with start in \( W_N^B \),

\[
\|x(t)\|^2 = 2N(N+\nu-1)t + \|x(0)\|^2.
\]

We again decompose solutions of (3.1) into an easy radial part and and a spherical part where the spherical parts of the solutions in (2.2) correspond to a stationary solution. This stationary solution satisfies the following stability result:

Lemma 3.4. For each starting point \( x_0 \in W_N^B \), the solution \( x \) of (3.1) has the form

\[
x(t) = \sqrt{2N(N+\nu-1)t + \|x_0\|^2} \cdot \phi(t) \quad (t \geq 0)
\]

with

\[
\|\phi(t)\| = 1 \quad \text{and} \quad \lim_{t \to \infty} \phi(t) = \frac{2}{N(N-1)} y
\]

and the vector \( y \) from Lemma 3.2.
Proof. The proof is similar to that of Lemma 2.4. Using (3.3), we define
\[
\phi(t) := \frac{1}{\sqrt{2N(N + \nu - 1)t + \|x_0\|^2}} \cdot \frac{x(t)}{\|x(t)\|}
\]
with \(\|\phi(t)\| = 1\). (3.1) implies that
\[
\frac{d}{dt}(\phi_i(t)) = \frac{\dot{x}_i(t)}{\sqrt{2N(N + \nu - 1)t + \|x_0\|^2}} = \frac{N(N + \nu - 1) \cdot x_i(t)}{(2N(N + \nu - 1)t + \|x_0\|^2)^{3/2}}
\]
\[
= \frac{1}{2N(N + \nu - 1)t + \|x_0\|^2} \left( \sum_{j \neq i} \frac{1}{\phi_i(t) - \phi_j(t)} + \sum_{j \neq i} \frac{1}{\phi_i(t) + \phi_j(t)} \right) + \frac{\nu}{\phi_i(t)} - N(N + \nu - 1) \cdot \psi_i(t)
\]
Hence, \(\psi(t) := \phi \left( N(N + \nu - 1)^2 + \|x_0\|^2 t \right)\) for \(t \geq 0\) satisfies
\[
\frac{\dot{\psi}_i(t)}{\psi_i(t)} = \sum_{j \neq i} \frac{\psi_i(t) - \psi_j(t)}{\psi_i(t) + \psi_j(t)} + \sum_{j \neq i} \frac{1}{\psi_i(t) + \psi_j(t)} + \frac{\nu}{\psi_i(t)} - N(N + \nu - 1) \cdot \psi_i(t)
\]
for \(i = 1, \ldots, N\) with \(\psi(0) = \phi_0(0) = x_0/\|x_0\|\). The ODE (3.5) is a gradient system \(\dot{\psi} = (\nabla u)(\psi)\) with
\[
u(y) := \sum_{i, j = 1, \ldots, N, i < j} \left( \ln(y_i - y_j) + \ln(y_i + y_j) \right) + \nu \sum_{i = 1}^{N} \ln y_i - \frac{N(N + \nu - 1)}{2} \|y\|^2.
\]
It now follows from Lemma 3.2 of [AV1] (or see [AKM2] or Section 6.7 of [S]) that \(u\) admits a unique local maximum on \(C_N^R\), that this maximum is a global one, and that it located at \(y\) with
\[
(y_1^2, \ldots, y_N^2) = \frac{1}{N(N + \nu - 1)}(z_1^{(\nu-1)}, \ldots, z_N^{(\nu-1)}).
\]
We notice that \(\|y\| = 1\) holds; this follows either from Lemmas 3.2 and 3.3 or from (C.10) in [AKM2]. These observations and (3.4) now lead to the claim as in the proof of Lemma 2.4. \(\square\)

In order to describe the general solutions \(x(t)\) of (3.1) we again use the elementary symmetric polynomials \(e_k^N\) in \(N\) variables and put
\[
\hat{e}_k(x) := \hat{e}_k^N(x) := e_k^N(x_1^2, \ldots, x_N^2) \quad (k = 0, \ldots, N).
\]

Lemma 3.5. For each \(x_0 \in W_N^R\), consider the solution \(x(t)\) of (3.1). Then, for \(k = 0, \ldots, N\), \(t \mapsto \hat{e}_k(x(t))\) is a polynomial in \(t\) of degree \(k\) with leading coefficient
\[
2^k(N + \nu - 1)(N + \nu - 2) \cdots (N + \nu - k) \cdot \binom{N}{k} \quad (k \leq N).
\]

Proof. The statement is trivial for \(k = 0\) and follows from (3.3) for \(k = 1\). For \(k \geq 2\) we use induction on \(k\). We use the notations \(x_S(t)\) from Section 2 and put...
\( \tilde{e}_k^R(x) := e_k^R(x_1^2, \ldots, x_R^2) \) for \( R = 1, \ldots, N \). Then for \( k \geq 2 \),
\[
\frac{d}{dt}\tilde{e}_k(x(t)) = 2 \sum_{j=1}^{N} \frac{dx_j(t)}{dt} \cdot x_j(t) \cdot \tilde{e}_{k-1}^{N-1}(x_{\{1,\ldots, N\} \setminus \{j\}}(t))
\]
Therefore, by (3.1),
\[
\frac{d}{dt}\tilde{e}_k(x(t)) = 2 \sum_{j=1}^{N} \left( 2 \sum_{i \neq j} x_j(t)^2 \sum_{i \neq j} x_j(t)^2 - x_i(t)^2 \right) \tilde{e}_{k-1}^{N-1}(x_{\{1,\ldots, N\} \setminus \{j\}}(t))
\]
\[
= 2 \sum_{i,j=1,\ldots, N; i \neq j} \frac{x_j(t)^2 \tilde{e}_{k-1}^{N-1}(x_{\{1,\ldots, N\} \setminus \{j\}}(t)) - x_i(t)^2 \tilde{e}_{k-1}^{N-1}(x_{\{1,\ldots, N\} \setminus \{i\}}(t))}{x_j(t)^2 - x_i(t)^2} + 2\nu \sum_{j=1}^{N} \tilde{e}_{k-1}^{N-1}(x_{\{1,\ldots, N\} \setminus \{j\}}(t))
\]  
(3.6)
Simple combinatorial computations show that for \( k \leq N - 1 \),
\[
x_j(t)^2 \tilde{e}_{k-1}^{N-1}(x_{\{1,\ldots, N\} \setminus \{j\}}(t)) - x_i(t)^2 \tilde{e}_{k-1}^{N-1}(x_{\{1,\ldots, N\} \setminus \{i\}}(t)) = (x_j(t)^2 - x_i(t)^2)\tilde{e}_{k-1}^{N-2}(x_{\{1,\ldots, N\} \setminus \{i,j\}}(t))
\]  
(3.7)
and
\[
\sum_{i,j=1,\ldots, N; i \neq j} \tilde{e}_{k-1}^{N-2}(x_{\{1,\ldots, N\} \setminus \{i,j\}}(t)) = (N - k + 1)(N - k)\tilde{e}_{k-1}^{N}(x(t))
\]  
(3.8)
Moreover,
\[
x_j(t)^2 \tilde{e}_{k-1}^{N-1}(x_{\{1,\ldots, N\} \setminus \{j\}}(t)) - x_i(t)^2 \tilde{e}_{k-1}^{N-1}(x_{\{1,\ldots, N\} \setminus \{i\}}(t)) = 0
\]  
(3.9)
Furthermore,
\[
\sum_{j=1}^{N} \tilde{e}_{k-1}^{N-1}(x_{\{1,\ldots, N\} \setminus \{j\}}(t)) = (N - k + 1)\tilde{e}_{k-1}^{N}(x(t))
\]  
(3.10)
Therefore, by (3.10 - 3.11), for \( k \leq N \),
\[
\frac{d}{dt}\tilde{e}_k(x(t)) = 2(N - k + 1)(N - k + \nu)\tilde{e}_{k-1}^{N}(x(t))
\]  
(3.11)
This recurrence relation and the known cases \( k = 0, 1 \) lead easily to the claim. \( \Box \)

**Example 3.6.** Let \( \nu > 0 \) and \( N = 2 \). Assume that we start in \( x_0 = (x_{0,1}, x_{0,2}) \in C^2 \). Then, (3.11) and Lemma 5.3 imply that
\[
(z - x_1(t)^2)(z - x_2(t)^2) = z^2 - (x_1(t)^2 + x_2(t)^2)z + x_1(t)^2x_2(t)^2
\]
with
\[
x_1(t)^2 + x_2(t)^2 = 4(1 + \nu)t + \|x_0\|^2; \quad x_1(t)^2x_2(t)^2 = 4\nu(1 + \nu)t^2 + 2\nu\|x_0\|^2t + x_{0,1}^2x_{0,2}^2.
\]
Since the components of \( x \) are non-negative, this yields that
\[
x_1(t) = \left( \frac{1}{2} \left( 4(1 + \nu)t + \|x_0\|^2 + \sqrt{16(1 + \nu)t^2 + 8\|x_0\|^2t + (x_{0,1}^2 - x_{0,2}^2)^2} \right) \right)^{1/2},
\]
\[
x_2(t) = \left( \frac{1}{2} \left( 4(1 + \nu)t + \|x_0\|^2 - \sqrt{16(1 + \nu)t^2 + 8\|x_0\|^2t + (x_{0,1}^2 - x_{0,2}^2)^2} \right) \right)^{1/2}
\]
This implies in particular that for \( t \to \infty \),
\[
x_1(t) = \left( \frac{1}{2} (4(1+\nu) t + \|x_0\|^2) \right)^{1/2} \left( 1 + \sqrt{\frac{1}{1 + \nu} \left( \frac{1}{x_0\|^2} - 4x_0^2 \right)} \right)^{1/2} \]
\[
= \left( \frac{1}{2} (4(1+\nu) t + \|x_0\|^2) \right)^{1/2} \left( 1 - \sqrt{\frac{1}{1 + \nu}} \right)^{1/2} + O(t^{-1/2})
\]
and in the same way
\[
x_2(t) = \left( \frac{1}{2} (4(1+\nu) t + \|x_0\|^2) \right)^{1/2} \left( 1 - \sqrt{\frac{1}{1 + \nu}} \right)^{1/2} + O(t^{-1/2}), \tag{3.12}
\]
which may be seen as a quantitative version of Lemma 3.4 for \( N = 2 \). We also observe that our solutions \( x \) exist when we start at any point \( x_0 \in \partial C_2^B \) and that for these solutions, \( x(t) \in W_2^B \) holds for all \( t > 0 \).

This last observation holds for all \( N \geq 2 \):

**Theorem 3.7.** Let \( N \geq 2 \) and \( \nu > 0 \). Then, for each starting point \( x_0 \in C_2^B \), the ODE \((3.7)\) has a unique solution for all \( t \geq 0 \) in the sense of Theorem 2.8.

**Proof.** The proof is similar to that of Theorem 2.8. We keep the notations from there and describe the modifications. We again transform solutions \( x \) of the ODE \((3.1)\). Using the notation \( \hat{e}_k(x) := e_k(x_1^2, \ldots, x_k^2) \) as above, we conclude from the proof of Lemma 3.5 that for \( t > 0 \) and \( k = 2, \ldots, N \),
\[
\frac{d}{dt}(\hat{e}_k^N(x(t))) = 2(N - k + 1)(N - k + \nu)\hat{e}_{k-1}^N(x(t)). \tag{3.13}
\]
Moreover, for \( k = 1 \),
\[
\frac{d}{dt}(\hat{e}_1^N(x(t))) = 2N(N + \nu - 1). \tag{3.14}
\]
\((3.14)\) and \((3.13)\) form an ODE, whose solutions \( \hat{e}(t) := (\hat{e}_1^N(x(t)), \ldots, \hat{e}_N^N(x(t))) \) are polynomials in \( t \) where for \( k = 1, \ldots, N \), the \( k \)-th component \( \hat{e}_k(t) \) is a polynomial of maximal order \( k \). As the mapping \( e : C_2^B \to E_N \) with \( E_N \) as in \((2.13)\) and \( e(x) := (\hat{e}_1^N(x), \ldots, \hat{e}_N^N(x)) \) is continuous and injective, \((3.11)\) has at most one solution.

For the existence of a solution we proceed as in the proof of Theorem 2.8 and show that for any starting point \( x_0 \in \partial C_2^B \) and its image \( \hat{e}(x_0) \in e(W_2^B) \), the solution \( \hat{e}(t) \) \((t \geq 0)\) of the ODEs \((3.14)\) and \((3.13)\) with \( \hat{e}(0) = e(x_0) \) satisfies \( \hat{e}(t) \in e(W_2^B) \) for \( t > 0 \) sufficiently small. To prove this, we first observe that by the same reasons as in the proof of Theorem 2.8 for each starting point in \( \hat{e}(0) \in e(W_2^B) \) we have \( \hat{e}(t) \in e(W_2^B) \) for all \( t \geq 0 \). Moreover, for each prospective solution \( (x(t))_{t \geq 0} \) of \((3.1)\) with start in \( x(0) \in \partial C_N^A \), and for each \( c > 0 \), the function \( t \mapsto \sqrt{c} x(ct) \) also is also a prospective solution of \((3.1)\) with start in \( x(0)/\sqrt{c} \). This observation corresponds with the fact that for each solution \( \hat{e}(t) \) of the ODEs \((3.14)\) and \((3.13)\) with start in \( e(x(0)) \), and each \( c > 0 \), the function
\[
t \mapsto (c^{-1} \hat{e}_1(ct), c^{-2} \hat{e}_2(ct), \ldots, c^{-N} \hat{e}_N(ct)) \tag{3.15}
\]
is also a solution of \((3.14)\) and \((3.13)\) with start in \( (c^{-1} \hat{e}_1(x(0)), \ldots, c^{-N} \hat{e}_N(x(0))) \).

Assume now that there is a starting point \( x(0) \in \partial C_N^B \) and some \( t_0 > 0 \), such that the solution \( \hat{e}(t) \) \((t \geq 0)\) of \((3.14)\) and \((3.13)\) with start in \( e(x(0)) \) satisfies \( \hat{e}(t) \not\in e(W_2^B) \)
for \( t \in [0, t_0] \). We notice that \( \partial C_N^D \) consists of two (overlapping) parts, namely points \( x \) with \( x_N = 0 \) and points, where at least two coordinates are equal. Assume now that \( x(0) = (x(0)_1, \ldots, x(0)_N) \) satisfies \( x(0)_N = \ldots = x(0)_{N-l+1} = 0 \) and \( x(0)_{N-l} > 0 \) with some \( l = 1, \ldots, N \). Then \( \tilde{e}_N(x(0)) = \ldots = \tilde{e}_{N-l+1}(x(0)) = 0 \) and \( \tilde{e}_{N-l}(x(0)) > 0 \) by the form of the elementary symmetric polynomials. This, \( \nu > 0 \), \( 3.7 \), and the Taylor expansion now imply that \( \tilde{e}_N(x(t)) > 0 \) for \( t \in [0, t_0] \) and a suitable \( t_0 > 0 \). This means that our assumption \( \tilde{c}(t) \notin \varepsilon(W_N^D) \) for \( t \in [0, t_0] \) is caused by the fact that at least two coordinates of \( x(t) \) are equal for \( t \in [0, t_0] \).\( ^{\dagger} \) From this we conclude as in the proof of Theorem 2.8 that \( \tilde{c}(t) \notin \varepsilon(W_N^D) \) holds for all \( t \geq 0 \). With this observation the proof can be completed precisely as the proof of Theorem 2.8 where one has to use the fact that the solution of \( 4.1 \) with start in 0 is given by \( x(t) := \sqrt{2t} \cdot y \) for the vector \( y \) from Lemma 3.2.  

A slightly more complicated variant of Theorem 3.7 can be stated for the case \( \nu = 0 \) under some restriction. This result will be a consequence of the study of the root system \( D_N \) in the end of the next section. We finally observe that Theorem 3.7 can be supplemented by an analogue of Lemma 2.9 with the same proof.

4. The root system \( D_N \)

For the root systems \( D_N \), we consider the associated Weyl chamber

\[
C_N^D := \{ x \in \mathbb{R}^N : \quad x_1 \geq \ldots \geq x_{N-1} \geq |x_N| \} \subset \mathbb{R}^N
\]
as well as its interior \( W_N^D \). The ODE \( 1.7 \) has in this case the form

\[
\frac{dx(t)}{dt} = H_D(x(t)) \quad \text{with} \quad H_D(x) := \left( \frac{\sum_{j \neq 1} \left( \frac{1}{x_j-x_1} + \frac{1}{x_j+x_1} \right)}{\sum_{j \neq N} \left( \frac{1}{x_j-x_N} + \frac{1}{x_j+x_N} \right)} \right)
\]
on \( W_N^D \). Similar to the preceding cases, we have by Lemma 4.1 of [AV1]:

**Lemma 4.1.** For \( \epsilon > 0 \) consider the open subset \( U_\epsilon := \{ x \in C_N^D : d(x, \partial C_N^D) > \epsilon \} \) of \( C_N^D \). Then for each starting point \( x_0 \in U_\epsilon \), the ODE \( 1.7 \) with \( x(0) = x_0 \) admits a unique solution. This solution exists for all \( t > 0 \) with \( x(t) \in U_\epsilon \).

We next recapitulate some facts on Laguerre polynomials and proceed as in Section 4 of [AV1]. Using the representation

\[
L_N^{(\alpha)}(x) := \sum_{k=0}^{N} \binom{N+\alpha}{N-k} (-x)^k k! \quad (\alpha \in \mathbb{R}, \ N \in \mathbb{N})
\]
of the Laguerre polynomials according to (5.1.6) of Szegö [S], we form the polynomial \( L_N^{(-1)} \) of order \( N \geq 1 \) where, by (5.2.1) of [S],

\[
L_N^{(-1)}(x) = -\frac{x}{N} L_N^{(1)}(x).
\]

We denote the \( N \) ordered zeros of \( L_N^{(-1)} \) by \( z_1 \geq \ldots \geq z_{N-1} \geq z_N = 0 \). Similar to the preceding cases, we obtain from Section 4 of [AV1]:

**Lemma 4.2.** Let \( y \in C_N^D \) be the vector with

\[
2 \cdot (z_1, \ldots, z_{N-1}, z_N = 0) = (y_1^2, \ldots, y_N^2).
\]

Then for each \( c > 0 \), \( x(t) := \sqrt{t+c^2} \cdot y \) is a solution of \( 4.1 \).

\[\Box\]
Again, the growth of these particular solutions is typical; see [VW]:

**Lemma 4.3.** For each solution \( x \) of \((4.1)\) with start in \( x(0) \in W^D_N \),
\[
\|x(t)\|^2 = 2N(N-1)t + \|x(0)\|^2.
\]

We again have a stability result for the special solutions in Lemma 4.2. Its proof is completely analog to that of Lemma 3.4, we omit the proof.

**Lemma 4.4.** For each \( x_0 \in W^D_N \), the solution \( x \) of \((4.1)\) has the form
\[
x(t) = \sqrt{2N(N-1)}t + \|x_0\|^2 \cdot \phi(t) \quad (t \geq 0)
\]
where \( \phi \) satisfies
\[
\|\phi(t)\| = 1 \quad \text{and} \quad \lim_{t \to \infty} \phi(t) = \frac{2}{N(N-1)}y
\]
with the vector \( y \) from Lemma 4.4.

Beside the particular solutions in Lemma 4.2, we here have the following observation. This result fits with Eq. (4.2) for \( L^N_{-1} \).

**Lemma 4.5.** Let \( x_0 = (x_0,1,\ldots,x_{0,N}) \in W^D_N \) with \( x_{0,N} = 0 \). Then the associated solution \( x \) of \((4.1)\) satisfies \( x(t)_N = 0 \) for all \( t \), and the first \( N-1 \) components \( (x(t)_1,\ldots,x(t)_{N-1}) \) solve the ODE \((4.7)\) with dimension \( N-1 \) and \( \nu = 2 \).

Moreover, if \( x_{0,N} > 0 \) or \( < 0 \), then for all \( t \), \( x(t)_N > 0 \) or \( < 0 \) respectively.

**Proof.** If \( x_{0,N} = 0 \), then by \((4.1)\), \( \frac{d}{dt}x(t)_N = 0 \). This shows the first statements. This and the fact that the curves \( (x(t))_\nu \) are either equal or do not intersect then show the second statement. \( \square \)

It is possible also to derive a result which is analog to Lemma 4.2. We skip this statement and proceed directly to the following analogue of Theorem 3.7.

**Theorem 4.6.** Let \( N \geq 2 \). Then, for each starting point \( x_0 \in C^D_N \), the ODE \((4.1)\) has a unique solution for all \( t \geq 0 \) in the sense of Theorem 2.8.

**Proof.** The proof is similar to that of Theorem 3.7 and we keep the notations from Theorem 2.8. For \( x_0 \in W^D_N \), the assertion is clear. Now let \( x_0 = (x_1(0),\ldots,x_N(0)) \in \partial C^D_N \). We transform \((4.1)\) as in Theorem 3.7 for \( \nu = 0 \) and obtain that for a solution \( (x(t))_{t \geq 0} \) of \((4.1)\), the function \( \hat{\epsilon}(t) := (\hat{\epsilon}^N_1(x(t)),\ldots,\hat{\epsilon}^N_k(x(t))) \) satisfies
\[
\frac{d}{dt}(\hat{\epsilon}^N_k(x(t))) = 2(N-k+1)(N-k)\hat{\epsilon}^N_{k-1}(x(t)) \quad (k \geq 2) \quad (4.5)
\]
and
\[
\frac{d}{dt}(\hat{\epsilon}^N_1(x(t))) = 2N(N-1) . \quad (4.6)
\]
In particular, for \( k = N \),
\[
\frac{d}{dt}(\hat{\epsilon}^N_N(x(t))) = 0 . \quad (4.7)
\]
We now consider different cases. If \( x_N(0) = 0 \), then we obtain from \((4.7)\) that \( \hat{\epsilon}^N_N(x(t)) = 0 \) and thus \( x_N(t) = 0 \) for all \( t \geq 0 \). If we insert the trivial component \( x_N(t) = 0 \) for \( t \geq 0 \) into our ODE \((4.1)\), we get an ODE of the form \((3.1)\) of type B in \( N-1 \) dimensions with some \( \nu > 0 \). Therefore, in this case, Theorem 4.6 follows from Theorem 3.7. Assume now that \( x_N(0) \neq 0 \). Here, the theorem can be proved in the same way as Theorem 2.8 where one has to use the fact that the solution of \((4.1)\) with start in 0 is given by \( x(t) := \sqrt{2t} \cdot (\sqrt{y_1},\ldots,\sqrt{y}_{N-1},0) \). \( \square \)
There is also an analogue of Lemma 2.9 for the case D with the same proof.

Clearly, the ODE (4.1) of type D is closely related with the ODE (3.1) of type B for \( \nu = 0 \) by Lemma 4.5. In particular, we obtain with Theorem 4.6:

**Corollary 4.7.** Let \( N \geq 2 \) and \( \nu = 0 \) for the root system \( B_N \). Then, for each starting point \( x_0 = (x_1(0), \ldots, x_N(0)) \in C_B^R \) with \( x_N(0) > 0 \), the ODE (4.1) has a unique solution \( x \) for all \( t \geq 0 \) in the sense of Theorem 2.8 with \( x(t) \in W_{B_N}^R \) for \( t > 0 \). On the other hand, if \( x_N(0) = 0 \), then there is no solution \( x \) of (3.1) with \( x(t) \in W_{B_N}^R \) for \( t > 0 \).

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