INVERSE BOUNDARY VALUE PROBLEM OF DETERMINING UP TO A SECOND ORDER TENSOR APPEAR IN THE LOWER ORDER PERTURBATION OF A POLYHARMONIC OPERATOR

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Abstract. We consider the following perturbed polyharmonic operator \( L(x, D) \) of order \( 2m \) defined in a bounded domain \( \Omega \subset \mathbb{R}^n, n \geq 3 \) with smooth boundary, as
\[
L(x, D) \equiv (-\Delta)^m + \sum_{j,k=1}^{n} A_{jk} D_j D_k + \sum_{j=1}^{n} B_j(x) D_j + q(x),
\]
where \( A \) is a symmetric 2-tensor field, \( B \) and \( q \) are vector field and scalar potential respectively. We show that the coefficients \( A = [A_{jk}] \), \( B = (B_j) \) and \( q \) can be recovered from the associated Dirichlet-to-Neumann data on the boundary. Note that, this result shows an example of determining higher order (2nd order) symmetric tensor field in the class of inverse boundary value problem.

1. Introduction and statement of the main results

Let \( \Omega \subset \mathbb{R}^n, n \geq 3 \) be a bounded domain with smooth connected boundary. Let us consider the following perturbed polyharmonic operator \( L(x, D) \) of order \( 2m \), with perturbations up to second order, of the following form:
\[
L(x, D) \equiv (-\Delta)^m + \sum_{j,k=1}^{n} A_{jk}(x) D_j D_k + \sum_{j=1}^{n} B_j(x) D_j + q(x); \quad m \geq 2,
\]
where \( D_j = \frac{1}{i} \partial_{x_j}, A \in W^{3,\infty}(\Omega, \mathbb{C}^n), B \in W^{2,\infty}(\Omega, \mathbb{C}^n) \) and \( q \in L^{\infty}(\Omega, \mathbb{C}) \). For simplicity, we call \( A_\alpha : \Omega \mapsto \mathbb{C}^{|\alpha|} \), as \( A_\alpha = A \) whenever \( |\alpha| = 2 \), and \( A_\alpha = B \) whenever \( |\alpha| = 1 \).

Consider the domain of this operator to be
\[
D(L(x, D)) = \left\{ u \in H^{2m}(\Omega) \mid u|_{\partial\Omega} = (-\Delta)u|_{\partial\Omega} = \cdots = (-\Delta)^{m-1}u|_{\partial\Omega} = 0 \right\}.
\]
The operator \( L(x, D) \) with the domain \( D(L(x, D)) \) is an unbounded closed operator on \( L^2(\Omega) \) with a purely discrete spectrum [15]. We make the assumption that 0 is not an eigenvalue of the operator \( L(x, D) : D(L(x, D)) \to L^2(\Omega) \). Let us denote
\[
\gamma u = \left( u|_{\partial\Omega}, \cdots, (-\Delta)^k u|_{\partial\Omega}, \cdots, (-\Delta)^{m-1} u|_{\partial\Omega} \right),
\]
then for any \( f = (f_0, f_1, \ldots, f_{m-1}) \in \prod_{i=0}^{m-1} H^{2m-2i-\frac{3}{2}}(\partial\Omega) \), the boundary value problem,
\[
\begin{align*}
L(x, D)u &= 0 \quad \text{in } \Omega \\
\gamma u &= f \quad \text{on } \partial\Omega
\end{align*}
\]
has a unique solution \( u \in H^{2m}(\Omega) \).

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Let us define the corresponding Neumann trace operator $\gamma^#_m$ by

$$\gamma^#_m u = \left( \partial_{\nu} u|_{\partial \Omega}, \ldots, \partial_{\nu}(-\Delta)^k u|_{\partial \Omega}, \ldots, \partial_{\nu}(-\Delta)^{m-1} u|_{\partial \Omega} \right)$$

where $\nu$ is the outer unit normal to the boundary $\partial \Omega$, and the corresponding Dirichlet-to-Neumann map (D-N map) is given by,

$$\mathcal{N} : \prod_{i=0}^{m-1} H^{2m-2i-\frac{1}{2}}(\partial \Omega) \to \prod_{i=0}^{m-1} H^{2m-2i-\frac{3}{2}}(\partial \Omega)$$

$$\mathcal{N}(f) = \gamma^#_m u = \left( \partial_{\nu} u|_{\partial \Omega}, \ldots, \partial_{\nu}(-\Delta)^k u|_{\partial \Omega}, \ldots, \partial_{\nu}(-\Delta)^{m-1} u|_{\partial \Omega} \right)$$

where $u \in H^{2m}(\Omega)$ is the solution of (1.2). We also define the Cauchy data set as the graph of the D-N map

$$\mathcal{C}^N = \left( u|_{\partial \Omega}, \ldots, (-\Delta)^{m-1} u|_{\partial \Omega}, \partial_{\nu} u|_{\partial \Omega}, \ldots, \partial_{\nu}(-\Delta)^m u|_{\partial \Omega} \right)$$

Equations of this kind has its own resemblance in the study of continuum mechanics of buckling problems [2] and in the related field of elasticity [12].

Inverse problems concerning higher order operators is an active field of research in recent days. It addresses the recovery of the coefficient from measurements taken on full or on a part of the boundary $\partial \Omega$, and the corresponding Dirichlet-to-Neumann data. In this paper we are interested in the recovery of the coefficients in (1.1) from the boundary Dirichlet-Neumann data. To be specific, we are interested in the recovery of the second order tensor or matrix $A_\alpha$ for $|\alpha| = 2$ associated with the second order perturbed term in the operator $\mathcal{L}(x,D)$. Recovery of lower order perturbations mainly zeroth order and first order, of biharmonic and polyharmonic operators has been considered in prior works [16, 17, 19, 20, 13, 14, 5, 6], lower regularity of the coefficients [21, 3, 4], stability issues [10, 9] from the boundary Dirichlet-Neumann data. In this paper we are interested in the recovery of the coefficients in (1.1) from the boundary Dirichlet-Neumann data. To be specific, we are interested in the recovery of the second order tensor or matrix $A_\alpha$ for $|\alpha| = 2$ associated with the second order perturbed term in the operator $\mathcal{L}(x,D)$. Recovery of lower order perturbations mainly zeroth order and first order, of biharmonic and polyharmonic operators has been considered in prior works [16, 17, 19, 20, 13, 14, 5, 6]. In [16, 17], recovery of the zeroth order perturbations of the biharmonic operator was studied, and recently in [20, 19] recovery of the first and zeroth order perturbations of the biharmonic operator with partial boundary Neumann data and of the polyharmonic operator with full boundary Neumann data were considered. Similar work on non polyharmonic higher order elliptic operator ($\sum_{i=1}^{n} D_{x_i}^{2m}$, $m \geq 2$) has been also considered in [14]. A natural question to ask is whether higher order perturbations of the polyharmonic operator can be recovered from boundary Dirichlet-Neumann map. Such attempted was initiated in [13], where the authors had considered the higher order perturbations (upto $m^{th}$ order) of polyharmonic operator $(\Delta)^m$, but the coefficients attached to those lower order perturbed terms were restricted among the class of first order and zeroth order tensors accordingly to odd and even order perturbations. So, the question becomes more interesting when one involves higher order tensor with the higher order perturbations. In this paper, we show that if we consider a perturbation of the polyharmonic operator $(\Delta)^m$ of the form (1.1), with $m > 2$ then all the coefficients $A,B,q$ can be recovered from the boundary Dirichlet-Neumann data. For $m = 2$, we recover $A$ as an isotropic matrix along with $B$ and $q$, this can be seen in [13]. In other words, this article provides an example of recovery of matrix in this class of inverse boundary value problems.

For the biharmonic operator that for being $m = 2$, one may have gauge invariance in the recovery of the matrix coefficient. For instance, let us consider the following fourth order generalization of the inverse problem associated with the magnetic Schrödinger operator [26], as $u \in H^4(\Omega)$ solves

$$\sum_{k,l=1}^{n} (D_k D_l + A_{kl})^2 u + qu = 0 \text{ in } \Omega,$$

where $\nu$ is the outer unit normal to the boundary $\partial \Omega$.
with the given Navier boundary condition \((u|_{\partial \Omega}, (-\Delta u)|_{\partial \Omega}) \in H^\frac{\nu}{2}(\partial \Omega) \times H^\frac{\nu}{2}(\partial \Omega)\). Note that, (1.5) can be written in the form of (1.1) as

\[
(-\Delta)^2 u + \sum_{k,l} A'_{kl} D_k D_l u + \sum_j B'_j D_j u + q'u = 0
\]

for some \(A', B'\) and \(q'\) as a function of \(A, q\).

Then for any \(\phi \in C^4(\Omega)\) with \(\text{supp } \phi \subset \subset \Omega\), we observe \(ue^\phi \in H^4(\Omega)\) solves

\[
\sum_{k,l=1}^n \left( D_k D_l + A_{kl} + \frac{\partial^2 \phi}{\partial x_k \partial x_l} + \frac{\partial \phi}{\partial x_k} \frac{\partial \phi}{\partial x_l}\right)^2 u + qu = 0 \text{ in } \Omega. \tag{1.6}
\]

Since \(\text{supp } (\phi) \subset \Omega\), then \((ue^\phi, (-\Delta(ue^\phi)))\) and the normal derivative \((\partial_n(ue^\phi), \partial_n(-\Delta(ue^\phi)))\) carries the same boundary value as \((u, (-\Delta)u)\) does. Therefore the DN map (1.3) remains unchanged, i.e. \(N_{A,q} = N_{\tilde{A},q}\), where \(\tilde{A}\) equals to its gauge invariance

\[
\tilde{A}_{kl} = A_{kl} + \frac{\partial^2 \phi}{\partial x_k \partial x_l} + \frac{\partial \phi}{\partial x_k} \frac{\partial \phi}{\partial x_l}. \tag{1.7}
\]

This stands as an obstacle to the uniqueness of \(A\). However, if we consider \(A_{ij}\) to be an isotropic matrix, i.e. \(A_{ij}(x) = a(x)\delta_{ij}\) for some scalar function \(a\) and \(\delta_{jk}\) denotes the Kronecker delta functions, then full recovery of that isotropic matrix is possible and was first explored in [13]. We also note that the gauge invariance \((\tilde{A} - A)\) in (1.7) is not governed by the 2 form \(d_s V\) where \(V\) is a 1-tensor field and \(d_s V\) is the symmetrized inner derivative defined as

\[
(d_s V)_{jk} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} V_k + \frac{\partial}{\partial x_k} V_j \right).
\]

This follows from the fact that \((\tilde{A} - A)\) does not belong to the kernel of the corresponding Saint Venant operator \(W\) (see [24, Theorem 2.2.1]) defined as

\[
W_{ijkl}(F) = \frac{\partial^2 F_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 F_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 F_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 F_{jk}}{\partial x_i \partial x_l},
\]

where \(F\) is a symmetric 2 tensor field. In our case one can get \(\phi \in C^4_c(\mathbb{R}^n)\) such that

\[
W(\tilde{A} - A) = W(\frac{\partial^2 \phi}{\partial x_k \partial x_l} + \frac{\partial \phi}{\partial x_k} \frac{\partial \phi}{\partial x_l}) \neq 0. \tag{1.8}
\]

Whereas, in the case of the magnetic Schrödinger operator \((D_k + A_k)^2 + q\), the gauge invariance of the vector field \((\tilde{A} - A)\) with the same boundary data is governed by the 1 form \(dV\) where \(V\) is a scalar field, see [26, 22].

This article is a natural extension of the results obtained in [13, 20, 19] which considered the recovery of only zeroth and first order tensor from the lower order perturbations. To the best of the authors knowledge, inverse problems involving upto higher order perturbations of polyharmonic operator which are essentially determined by higher order tensors has not been investigated in previous studies. The novelty of this work provides a fresh look in order to determine the tensors of order more than one, which we don’t see often.

Here we take the opportunity to mention that inverse problems with boundary information arise naturally in several imaging applications including seismic and medical imaging, electrical impedance tomography to name a few. The techniques we rely on to prove our main result are
based on the pioneering works done for inverse boundary value problems involving Schrödinger operators [8, 27, 7, 18, 11].

Now we state the main result of this article.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n, n \geq 3 \) be a bounded domain with smooth connected boundary. Let \( \mathcal{L}(x, D) \) and \( \mathcal{\tilde{L}}(x, D) \) be two operators defined as in (1.1) having \( m \geq 2 \) with the coefficients \( A_\alpha, \tilde{A}_\alpha \in W^{1+|\alpha|, \infty}(\mathbb{R}^n, \mathbb{C}^{|\alpha|}) \cap \mathcal{L}'(\mathcal{E}(\Omega)), |\alpha| = 1, 2 \) and \( q, \tilde{q} \in L^\infty(\Omega, \mathbb{C}) \). We assume that for \( m = 2, A_\alpha, \tilde{A}_\alpha \) with \( |\alpha| = 2 \) are isotropic matrix. Let us assume that 0 is not an eigenvalue of the operators \( \mathcal{L}, \mathcal{\tilde{L}} \) and \( \mathcal{N}, \mathcal{\tilde{N}} \) are the corresponding Dirichlet-to-Neumann maps respectively. If

\[
\mathcal{N}(f)|_{\partial \Omega} = \mathcal{\tilde{N}}(f)|_{\partial \Omega} \quad \text{for all } f \in \prod_{i=0}^{m-1} H^{2m-2i-\frac{1}{2}}(\partial \Omega),
\]

then

\[
A_\alpha = \tilde{A}_\alpha, \quad \text{for } |\alpha| = 1, 2; \quad \text{and } q = \tilde{q}, \quad \text{in } \Omega.
\]

**Remark 1.2.** The result for \( m = 2 \) in this Theorem 1.1 with the assumption the second order perturbations are governed by isotropic matrices, follows from [13, Theorem 1.1].

Finally let us consider the same problem (1.2) given with the Dirichlet boundary conditions instead of Navier boundary conditions. Let us denote

\[
\gamma_D u = \left( u|_{\partial \Omega}, \partial_\nu u|_{\partial \Omega}, \cdots, \partial_\nu^k u|_{\partial \Omega}, \cdots, \partial_\nu^{m-1} u|_{\partial \Omega} \right)
\]

Then for \( f = (f_0, f_1, \cdots, f_{m-1}) \in \prod_{i=0}^{m-1} H^{2m-i-\frac{1}{2}}(\partial \Omega) \) we consider the boundary value problem

\[
\mathcal{L}(x, D)u = 0 \quad \text{in } \Omega
\]

\[
\gamma_D u = f \quad \text{on } \partial \Omega.
\]

(1.9)

The corresponding Neumann trace is

\[
\gamma_D^\# = \left( \partial_\nu^m u|_{\partial \Omega}, \cdots, \partial_\nu^{m-1} u|_{\partial \Omega} \right) \in \prod_{i=m}^{2m-1} H^{2m-i-\frac{1}{2}}(\partial \Omega)
\]

where \( u \in H^{2m}(\Omega) \) is the solution to the Dirichlet problem (1.9). See [1, 15] for the wellposedness of the forward problem (1.9). We introduce the set of Cauchy data for the operator \( \mathcal{L}(x, D) \) with the Dirichlet boundary condition by

\[
\mathcal{C}^D = \left( u|_{\partial \Omega}, \partial_\nu u|_{\partial \Omega}, \partial_\nu^m u|_{\partial \Omega}, \cdots, \partial_\nu^{2m-2} u|_{\partial \Omega}, \partial_\nu^{2m-1} u|_{\partial \Omega} \right)
\]

where \( u \in H^{2m}(\Omega) \) solving (1.9). We have this following result:

**Corollary 1.3.** We assume \( m \geq 2 \) and \( \alpha = 1, 2, A_\alpha, \tilde{A}_\alpha \) and \( q, \tilde{q} \) satisfy the same conditions as in Theorem 1.1. Then \( \mathcal{C}^D = \mathcal{\tilde{C}}^D \) implies that for each \( |\alpha| = 1, 2, A_\alpha = \tilde{A}_\alpha \) and \( q = \tilde{q} \) in \( \Omega \). Proceeding in a similar way as of Theorem 1.1, in this case we end up with an integral identity same as (3.6). Then following the same analysis one can show uniqueness of the lower order perturbations in \( \Omega \).

This article is organized as follows. Section 2 is devoted to the interior Carleman estimate, which is used to prove the existence of a complex geometric optics (C.G.O.) type solution for the equation (1.2). Finally, in Section 3 we derive an integral identity involving the perturbations and then prove the main result using Fourier analysis techniques.
In this section we derive Complex Geometric Optics (CGO) type solutions for the operator $L$ and its formal $L^2$ adjoint $L^*$ based on Carleman estimate.

2.1. Interior Carleman estimates. Let us assume that

$$L(x, D) = (-\Delta)^m + \sum_{|\alpha|=1} A_{\alpha}(x) D^\alpha + q$$

where $\alpha$ is a multi-index, $A_{\alpha} \in W^{1+|\alpha|, \infty}(\Omega, \mathbb{C}^{n\mid \alpha\mid})$ and $q \in L^\infty(\Omega)$. Note that $L^*$ the the $L^2$ adjoint of $L$, also has the same form as that of $L$ with different $A_{\alpha}$, and $q$. We will derive an interior Carleman estimate for the conjugated semiclassical version of the operator $L$ as well as its adjoint operator.

First we consider the principal part of the conjugated semiclassical version of the perturbed operator $L(x, hD)$, which is given as $(-\Delta)^m$. Then by adding the lower order terms to it finally we derive the required Carleman estimate for the conjugated semiclassical version of the operator $L(x, D)$.

We start by recalling the definition of a limiting Carleman weight for the semiclassical Laplacian $-h^2\Delta$. Let $\tilde{\Omega} \subset \subset \Omega$ be a bounded domain with smooth boundary. Let $\Omega$ be an open set in $\mathbb{R}^n$ such that $\Omega \subset \subset \tilde{\Omega}$ and $\varphi \in C^\infty(\tilde{\Omega}, \mathbb{R})$. Consider the conjugated operator $P_{0, \varphi} = e^{\tilde{x}\hat{\varphi}}(-h^2\Delta)e^{-\tilde{x}\hat{\varphi}}$ with its semiclassical symbol $p_{0, \varphi}(x, \xi)$.

**Definition 2.1 ([18])**. We say that $\varphi$ is a limiting Carleman weight for $(-h^2\Delta)$ in $\tilde{\Omega}$ if $\nabla \varphi \neq 0$ in $\tilde{\Omega}$ and the Poisson bracket of Re($p_{0, \varphi}$) and Im($p_{0, \varphi}$) satisfying

$$\{\text{Re}(p_{0, \varphi}), \text{Im}(p_{0, \varphi})\}(x, \xi) = 0$$

whenever $p_{0, \varphi}(x, \xi) = 0$ for $(x, \xi) \in (\tilde{\Omega} \times \mathbb{R}^n)$.

An example of such $\varphi$ is the linear weight defined as $\varphi(x) = \alpha \cdot x$, where $\alpha \in \mathbb{R}^n$ with $|\alpha| = 1$ or logarithmic weights $\varphi(x) = \log|x - x_0|$ with $x_0 \notin \tilde{\Omega}$. Throughout this article we consider the limiting Carleman weight to be $\varphi(x) = (\alpha \cdot x)$ where $\alpha \in \mathbb{R}^n$ with $|\alpha| = 1$.

As the principal symbol of the semiclassical conjugated operator $e^{\tilde{x}\hat{\varphi}}(-h^2\Delta)^m e^{-\tilde{x}\hat{\varphi}}$ is given by $p_{0, \varphi}^m$ which is not of principal type, the idea of Carleman weight for polyharmonic operator is irrelevant. One always has the Poisson bracket of Re($p_{\varphi}^m$) and Im($p_{\varphi}^m$) is zero when $p_{\varphi}(x, \xi) = 0$, $(x, \xi) \in (\tilde{\Omega} \times \mathbb{R}^n)$. In order to get the Carleman estimate for the polyharmonic operator we iterate the Carleman estimate for the semiclassical Laplacian.

We use the semiclassical Sobolev spaces $H_{scl}^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$ equipped with the norm

$$\|u\|_{H_{scl}^s(\mathbb{R}^n)} = \|\langle hD\rangle^s u\|_{L^2(\mathbb{R}^n)}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

With these notations we now prove the following proposition.

**Proposition 2.2.** Let for each $\alpha$ with $|\alpha| = 1, 2$, $A_{\alpha} \in W^{1+|\alpha|, \infty}(\Omega, \mathbb{C}^{n\mid \alpha\mid})$, $q \in L^\infty(\Omega, \mathbb{C})$ and $\varphi$ be a limiting Carleman weight for the semiclassical Laplacian on $\tilde{\Omega}$. Then for $0 < h \ll 1$ and $-2m \leq s \leq 0$, we have

$$h^m \|u\|_{H_{scl}^{s+2m}} \leq C \|h^{2m} e^{\tilde{x}\hat{\varphi}} L(x, D) e^{-\tilde{x}\hat{\varphi}} u\|_{H_{scl}^s}, \text{ for all } u \in C^\infty_0(\Omega).$$

The constant $C = C_{s, \Omega, A_{\alpha}, q}$ is independent of $h$. 


Proof. Let us consider the convexified Carleman weight as in [18] defined as

\[ \varphi_\epsilon = \varphi + \frac{h}{2\epsilon} \varphi^2 \] on \( \tilde{\Omega} \).

We begin with the Carleman estimate for the semiclassical Laplacian with a gain of two derivatives proved in [18]:

\[ \frac{h}{\sqrt{\epsilon}} \|u\|_{H^{s+2}_{scl}} \leq C \left\| e^{\frac{\varphi_\epsilon}{h}} (-h^2 \Delta) e^{-\frac{\varphi_\epsilon}{h}} u \right\|_{H^s_{scl}}, \text{ for all } u \in C_0^\infty(\Omega). \] (2.3)

Let \(-2m \leq s \leq 0\), then iterating the above estimate (2.3) \(m\) times, we get the following estimate:

\[ \left( \frac{h}{\sqrt{\epsilon}} \right)^m \|u\|_{H^{s+2m}_{scl}} \leq C \left\| e^{\varphi_\epsilon} (-h^2 \Delta)^m e^{-\varphi_\epsilon} u \right\|_{H^s_{scl}}, \text{ for all } u \in C_0^\infty(\Omega). \] (2.4)

Next to get the required Carleman estimate for \( \mathcal{L}(x, D) \), we add the lower order perturbations (given in (2.1)) to the above estimate. Similar literature on convexified Carleman estimates can be found in [11] in the context of a magnetic Schrödinger operator.

We first add the zeroth order term \((h^{2m} q)\), where \(q \in L^\infty(\Omega, \mathbb{C})\) in the above estimate. We get

\[ \|h^{2m} qu\|_{H^s_{scl}} \leq h^{2m} \|q\|_{L^\infty} \|u\|_{L^2} \leq h^{2m} \|q\|_{L^\infty} \|u\|_{H^{s+2m}_{scl}}. \]

Next we consider the first order term \(h^{2m} A_\alpha D^{\alpha} u\), with \(|\alpha| = 1\) (here \(A_\alpha \in W^{2,\infty}(\Omega, \mathbb{C}^n)\)). We observe

\[ h^{2m-1} e^{\frac{\varphi_\epsilon}{h}} \sum_{|\alpha|=1} A_\alpha(hD)^\alpha e^{-\frac{\varphi_\epsilon}{h}} u = h^{2m-1} \sum_{|\alpha|=1} A_\alpha(-D^{\alpha} \varphi_\epsilon + hD^{\alpha}) u. \] (2.5)

The first term in the right hand side of the above expression can be estimated as

\[ \|(A_\alpha D^{\alpha} \varphi_\epsilon) u\|_{H^s_{scl}} \leq \|A_\alpha D^{\alpha} \varphi_\epsilon\|_{L^\infty} \|u\|_{H^{s+2m}_{scl}} \]

Now as \( \varphi_\epsilon = \varphi + \frac{h}{\epsilon} \varphi^2 \) and \(0 < h \ll \epsilon \ll 1\), that is, \(0 < \frac{h}{\epsilon} < 1\). Hence \(\|D^{\alpha} \varphi_\epsilon\|_{L^\infty} = \mathcal{O}(1)\) for any \(\alpha\) and consequently we get

\[ \|(A_\alpha D^{\alpha} \varphi_\epsilon) u\|_{H^s_{scl}} \leq \mathcal{O}(1) \|u\|_{H^{s+2m}_{scl}}. \]

For the second term in the right hand side of (2.5) we observe that

\[ \|A_\alpha(hD)^\alpha u\|_{H^s_{scl}} \leq \|h D^{\alpha}(A_\alpha u)\|_{H^s_{scl}} + h \|(D^{\alpha} A_\alpha) u\|_{H^s_{scl}} \leq \mathcal{O}(1) \|A_\alpha u\|_{H^{s+1}_{scl}} + \mathcal{O}(h) \|u\|_{H^{s+2m}_{scl}} \leq \mathcal{O}(1) \|u\|_{H^{s+2m}_{scl}}. \]

The last inequality follows from the fact that the operator given as multiplication by \(A_\alpha\) is continuous from \(H^{s+2m}_{scl}\) to \(H^{s+1}_{scl}\) where \(A_\alpha \in W^{1+|\alpha|, \infty}, |\alpha| \geq 1\). Therefore for \(|\alpha| = 1\) we have

\[ \|h^{2m-1} e^{\frac{\varphi_\epsilon}{h}} (A_\alpha hD^{\alpha}) e^{-\frac{\varphi_\epsilon}{h}} u\|_{H^s_{scl}} \leq \mathcal{O}(h^{2m-1}) \|u\|_{H^{s+2m}_{scl}}. \] (2.6)
Now consider the term $h^{2m} A_\alpha \delta^\alpha e^{\frac{2\pi i}{h} u}$, for $|\alpha| = 2$.

\begin{align}
&h^{2m-2} e^{\frac{2\pi i}{h} u} \sum_{|\alpha|=2} A_\alpha h^2 \delta^\alpha e^{\frac{-2\pi i}{h} u} \\
= & h^{2m-2} e^{\frac{2\pi i}{h} u} \sum_{\alpha_1+\alpha_2 = \alpha, |\alpha|=|\alpha_1|=|\alpha_2|=1} A_\alpha h^2 \delta^\alpha e^{\frac{-2\pi i}{h} u} \\
= & h^{2m-2} \sum_{|\alpha|=|\alpha_2|=1} A_\alpha (D^{\alpha_1} \varphi \delta D^{\alpha_2} \varphi - h D^{\alpha_1+\alpha_2} \varphi + h D^{\alpha_1} \varphi \delta D^{\alpha_2} + D^{\alpha_2} \varphi \delta h D^{\alpha_1} + h^2 D^{\alpha_1+\alpha_2}) u.
\end{align}

(2.7)

For the first two terms in the right hand side of (2.7), we get

$$\|A_\alpha (D^{\alpha_1} \varphi \delta D^{\alpha_2} \varphi - h D^{\alpha_1+\alpha_2} \varphi) u\|_{H^s_{\text{sc}}} \leq C \|A_\alpha (D^{\alpha_1} \varphi \delta D^{\alpha_2} \varphi - h D^{\alpha_1+\alpha_2} \varphi)\|_{L^\infty} \|u\|_{H^s_{\text{sc}}} \leq O(1) \|u\|_{H^{s+2m}_{\text{sc}}}.$$  

(2.8)

Analyzing the third term in (2.7), we see

$$\|A_\alpha D^{\alpha_1} \varphi \delta h D^{\alpha_2} u\|_{H^s_{\text{sc}}} \leq C \|h D^{\alpha_2} (A_\alpha u \delta D^{\alpha_1} \varphi)\|_{H^s_{\text{sc}}} + Ch \|D^{\alpha_2} (A_\alpha D^{\alpha_1} \varphi)\|_{L^\infty} \|u\|_{H^{s+2m}_{\text{sc}}} \leq O(1) \|(A_\alpha D^{\alpha_1} \varphi) u\|_{H^{s+1}_{\text{sc}}} + Ch \|u\|_{H^{s+2m}_{\text{sc}}} \leq O(1) \|u\|_{H^{s+2m}_{\text{sc}}}.$$  

where, for the first term we use the continuity of the multiplication operator $A_\alpha : H^{s+2m}_{\text{sc}} \to H^{s+1}_{\text{sc}}$ whenever $A_\alpha \in W_{1+|\alpha|, \infty}$, $|\alpha| \geq 1$.

Now, consider the last term of the expression on the right hand side of (2.7), we get

$$\|A_\alpha h^2 \delta^\alpha u\|_{H^s_{\text{sc}}} \leq \|h^2 \delta^\alpha (A_\alpha u)\|_{H^s_{\text{sc}}} + 2 \|h^2 \delta D^{\alpha_1} (A_\alpha D^{\alpha_2} u)\|_{H^s_{\text{sc}}} + \|h^2 \delta^\alpha (A_\alpha u)\|_{H^s_{\text{sc}}} \leq O(1) \|A_\alpha u\|_{H^{s+2}_{\text{sc}}} + O(h) \|u\|_{H^{s+2m}_{\text{sc}}} + O(h^2) \|u\|_{H^{s+2m}_{\text{sc}}} \leq O(1) \|u\|_{H^{s+2m}_{\text{sc}}} + O(h) \|u\|_{H^{s+2m}_{\text{sc}}} + O(h^2) \|u\|_{H^{s+2m}_{\text{sc}}}.$$  

Here in the first term we use the continuity of the multiplication operator $A_\alpha : H^{s+2m}_{\text{sc}} \to H^{s+2}_{\text{sc}}$ whenever $A_\alpha \in W_{1+|\alpha|, \infty}$, $|\alpha| \geq 2$. To prove the continuity, it suffices to consider the complex interpolation for the cases $s = 0$ and $s = -2m$. The inequality on the second term on the right hand side follows as before by using the continuity of the multiplication operator $A_\alpha : H^{s+2m}_{\text{sc}} \to H^{s+1}_{\text{sc}}$ whenever $A_\alpha \in W_{1+|\alpha|, \infty}$, $|\alpha| \geq 1$.

Then adding all the lower order terms up to order 2 in (2.4) and choosing $h \ll \epsilon \ll 1$ small enough and using the standard bounds i.e. $1 \leq e^{\frac{2\pi i}{h} u} \leq C$, $\frac{1}{2} \leq 1 + \frac{h}{\epsilon} \varphi \leq \frac{3}{2}$ we get our desired estimate (2.2).

Let us denote

\[ L_\varphi(x, D) = h^{2m} e^{\frac{2\pi i}{h} \varphi} L(x, D) e^{-\frac{2\pi i}{h} \varphi}. \]

The formal $L^2$ adjoint of $L_\varphi(x, D)$ would be $L^*_\varphi(x, D) = h^{2m} e^{-\frac{2\pi i}{h} \varphi} L^*(x, D) e^{\frac{2\pi i}{h} \varphi}$, where $L^*(x, D)$ is the formal $L^2$-adjoint of the operator $L(x, D)$. As $L^*(x, D)$ has the similar form as $L(x, D)$ with the same regularity of the coefficients, since $-\varphi$ is also a limiting Carleman weight if $\varphi$ is, the Carleman estimate derived in Proposition 2.2 holds for $L^*_\varphi(x, D)$ as well. The following proposition establishes an existence result for an inhomogeneous equation analogous to the results in [19, 20].
Proposition 2.3. Let \( A_\alpha \in W^{1,|\alpha|,\infty}(\Omega, \mathbb{C}^{n*}) \), \( |\alpha| = 1, 2 \) and \( q \in L^\infty(\Omega, \mathbb{C}) \) and \( \varphi \) be any limiting Carleman weight for the semiclassical Laplacian on \( \tilde{\Omega} \). For \( 0 < h \ll 1 \) sufficiently small, the equation

\[
\mathcal{L}_\varphi(x, D)u = v \quad \text{in} \quad \Omega,
\]

has a solution \( u \in H^2_{scl}(\Omega) \), for \( v \in H^2_{scl} \) satisfying,

\[
h^m\|u\|_{H^2_{scl}} \leq C\|v\|_{H^2_{scl}}.
\]

The constant \( C > 0 \) is independent of \( h \) and depends only on \( A_\alpha, \alpha = 1, 2 \) and \( q \).

Here we skip the proof of the above proposition as it is standard in the literature of Calderón type inverse problems (see [19, 20]).

2.2. Construction of C.G.O. solutions. Now we construct complex geometric optics type solutions of the equation \( \mathcal{L}(x, D)u = 0 \) based on Proposition 2.3. We propose a solution of the form

\[
u = e^{\frac{(\varphi + i\psi)}{h}}(a_0(x) + ha_1(x) + r(x; h)),
\]

where \( 0 < h \ll 1 \), \( \varphi(x) \) is a limiting Carleman weight for the semiclassical Laplacian. The real valued phase function \( \psi \) is chosen such that \( \psi \) is smooth near \( \overline{\Omega} \) and solves the following Eikonal equation \( p_{0, \varphi}(x, \nabla \psi) = 0 \) in \( \overline{\Omega} \). The functions \( a_0, a_1 \) are complex amplitudes which solve certain transport equations, which we will define later. The function \( r \) is the correction term which satisfies the following estimate \( \|r\|_{H^2_{scl}} = O(h^2) \).

We consider \( \varphi \) and \( \psi \) to be

\[
\varphi(x) = \omega \cdot x, \quad \psi(x) = \tilde{\omega} \cdot x,
\]

where \( \omega, \tilde{\omega} \in \mathbb{R}^n \) are such that \( \omega \cdot \tilde{\omega} = 0 \) and \( |\tilde{\omega}| = |\omega| \). Observe that \( \varphi \) and \( \psi \) solves the Eikonal equation \( p_{0, \varphi}(x, \nabla \psi) = 0 \) in \( \tilde{\Omega} \), that is \( |\nabla \varphi| = |\nabla \psi| \) and \( \nabla \varphi \cdot \nabla \psi = 0 \).

Proposition 2.4. Let us consider the equation

\[
\mathcal{L}(x, D)u = (-\Delta)^m u + \sum_{|\alpha|=1,2} A_\alpha D^\alpha u + qu = 0,
\]

where \( A_\alpha \in W^{1,|\alpha|,\infty}(\Omega, \mathbb{C}^{n|\alpha|}) \), \( |\alpha| = 1, 2 \) and \( q \in L^\infty(\Omega, \mathbb{C}) \). We assume that for \( m = 2 \), \( A_\alpha \) with \( |\alpha| = 2 \) is an isotropic matrix. Then for all \( 0 < h \ll 1 \), there exists a solution \( u \in H^2(\Omega) \) of (2.11) of the form

\[
u(x, h) = e^{\frac{(\varphi + i\psi)}{h}}(a_0(x) + ha_1(x) + r(x; h))
\]

where \( \varphi \) and \( \psi \) are as in (2.10). Here \( a_0, a_1 \) are complex amplitudes satisfying certain transport equations and \( r \in H^2(\Omega) \) satisfies the estimate \( \|r\|_{H^2_{scl}} = O(h^2) \).

Proof. Let us write \( T = [(\nabla \varphi + i\nabla \psi) \cdot \nabla] \) and consider the conjugated operator

\[
e^{-\frac{(\varphi + i\psi)}{h}} h^{2m} \mathcal{L}(x, D) e^{\frac{(\varphi + i\psi)}{h}} u
\]

\[
e^{\frac{(-\varphi + i\psi)}{h}} (\varphi + i\psi)^m u
\]

\[
+ \sum_{\alpha=\alpha_1+\alpha_2, \ |\alpha_1|=|\alpha_2|=1} h^{2m-2}A_\alpha (D^\alpha_1(\varphi + i\psi)D^\alpha_2(\varphi + i\psi) + 2hD^\alpha_1(\varphi + i\psi)D^\alpha_2 + h^2D^\alpha) u
\]

\[
+ \sum_{|\alpha|=1} h^{2m-1}A_\alpha (D^\alpha(\varphi + i\psi) + hD^\alpha) u + h^{2m}qu.
\]
Let us substitute the form of \( u \) in (2.9) in the Equation 2.13. We would like to get
\[
e^{-\frac{(\varphi + i\psi)}{h}} h^{2m} \mathcal{L}(x, D) \left( e^{\frac{(\varphi + i\psi)}{h}} r(x, h) \right) \text{ is of } O(h^{m+2})
\]
In order to obtain that it is enough to consider the coefficient of \( h^m \) and \( h^{m+1} \) in terms of the complex amplitudes \( a_0, a_1 \) in the above expression (2.13) and equate them to zero, which will give two transport equations. Let us begin with considering the coefficient of \( h^m \). We get the following transport equation for \( a_0 \) as
\[
T^m a_0 = 0 \quad \text{in } \Omega, \quad \text{whenever } m > 2,
\]
and
\[
\left( -2T \right)^m + \sum_{\alpha = \alpha_1 + \alpha_2, \quad |\alpha| = |\alpha_2| = 1} A_\alpha D^{\alpha_1}(\varphi + i\psi) \cdot D^{\alpha_2}(\varphi + i\psi) \right) a_0 = 0 \quad \text{in } \Omega, \quad \text{whenever } m = 2.
\]
As we have assumed that \( A_\alpha \) with \(|\alpha| = 2\) is an isotropic matrix for \( m = 2 \), so the term
\[
\sum_{\alpha = \alpha_1 + \alpha_2, \quad |\alpha| = |\alpha_2| = 1} A_\alpha D^{\alpha_1}(\varphi + i\psi) \cdot D^{\alpha_2}(\varphi + i\psi)
\]
becomes zero in (2.15). It follows from the fact that if \( A_\alpha = mI \) then
\[
\sum_{\alpha = \alpha_1 + \alpha_2, \quad |\alpha| = |\alpha_2| = 1} A_\alpha D^{\alpha_1}(\varphi + i\psi) \cdot D^{\alpha_2}(\varphi + i\psi) = m(\omega + i\bar{\omega}) \cdot (\omega + i\bar{\omega}) = 0.
\]
Hence, we get (2.14) as the transport equation of \( a_0 \) for all \( m \geq 2 \). The solution of this homogeneous transport equation \( T^m a_0 = 0 \) in \( \Omega \) for \( m \geq 2 \) always exists and can be taken in \( C^\infty(\Omega) \) class, see [19, 20].

Next we consider the coefficient of \( h^{m+1} \) and obtain the following transport equation of \( a_1 \) as
\[
(-2T)^m a_1 = -\sum_{k=0}^{m-1} ((-2T)^k \circ (-\Delta) \circ (-2T)^{m-1-k}) a_0
\]
\[
+ \sum_{\alpha = \alpha_1 + \alpha_2, \quad |\alpha| = |\alpha_2| = 1} A_\alpha D^{\alpha_1}(\varphi + i\psi) D^{\alpha_2}(\varphi + i\psi) a_0 \quad \text{in } \Omega, \quad \text{whenever } m > 2
\]
and for \( m = 2 \),
\[
(-2T)^m a_1 = -\sum_{k=0}^{m-1} ((-2T)^k \circ (-\Delta) \circ (-2T)^{m-1-k}) a_0
\]
\[
+ 2 \sum_{\alpha = \alpha_1 + \alpha_2, \quad |\alpha| = |\alpha_2| = 1} A_\alpha D^{\alpha_1}(\varphi + i\psi) D^{\alpha_2} a_0 + 2 \sum_{|\alpha| = 1} A_\alpha D^{\alpha}(\varphi + i\psi) a_0 \quad \text{in } \Omega.
\]
The above inhomogeneous transport equations are solvable and we solve them in \( W^{2,\infty}(\Omega) \) to ensure \( u \) in (2.12) is indeed in \( H^2(\Omega) \). In order to show the solvability of (2.16), (2.17) in \( W^{2,\infty}(\Omega) \) one can break it into a system of \( m \geq 2 \) linear equations as follows.

Given \( f \in W^{2,\infty}(\Omega) \), find \( v_1 \in W^{2,\infty}(\Omega) \) solving \( Tv_1 = f \) in \( \Omega \);

Given \( v_1 \in W^{2,\infty}(\Omega) \), find \( v_2 \in W^{2,\infty}(\Omega) \) solving \( Tv_2 = v_1 \) in \( \Omega \);

Proceeding as before, given \( v_{m-1} \in W^{2,\infty}(\Omega) \) find \( a_1 \in W^{2,\infty}(\Omega) \) solving \( Ta_1 = v_{m-1} \) in \( \Omega \).

The solvability of the linear inhomogeneous equation
\[
Tg = (\omega + i\bar{\omega}) \cdot \nabla g = f \quad \text{in } \Omega
\]
for \( \omega \perp \tilde{\omega} = 0 \) and \( |\omega| = |\tilde{\omega}| = 1 \) is well known, see [22]. Let \( f \in W^{k,\infty}(\Omega, \mathbb{C}) \), \( k \geq 0 \) then we get a solution \( g \in W^{k,\infty}(\Omega, \mathbb{C}) \) as

\[
g(x) = \int_{\mathbb{R}^2} \tilde{f}(x - \omega y_1 - \tilde{\omega} y_2) dy_1 dy_2
\]

where \( \tilde{f} \in W^{k,\infty}(\mathbb{R}^n; \mathbb{C}) \) with \( \text{supp}(\tilde{f}) \) is a compact set containing \( \Omega \) and \( \tilde{f} = f \in \Omega \).

We considered \( a_0 \in C^\infty(\overline{\Omega}) \) and for \( a_1 \) to be in at least in \( W^{2,\infty}(\Omega) \) we require \( A_\alpha \in W^{2,\infty}(\Omega) \) for \( |\alpha| = 1, 2 \) whenever \( m \geq 2 \). As we have assumed \( A_\alpha \in W^{1+|\alpha|,\infty}(\Omega) \) for \( \alpha = 1, 2 \) hence it justifies \( a_1 \in W^{2,\infty}(\Omega) \) and hence it is in \( H^2(\Omega) \).

The remainder term \( r(x, h) \) satisfies

\[
e^{-\frac{(\omega+\omega \cdot h)}{h}} h^{2m} \mathcal{L}(x, D) \left( e^{-\frac{(\omega+\omega \cdot h)}{h}} r(x, h) \right) = -e^{-\frac{(\omega+\omega \cdot h)}{h}} h^{2m} \mathcal{L}(x, D) \left( e^{\frac{(\omega+\omega \cdot h)}{h}} (a_0 + h a_1) \right) = O(h^{m+2}).
\]

By choosing \( a_0 \in C^\infty(\overline{\Omega}) \) and \( a_1 \in W^{2,\infty}(\Omega) \) the reminder term \( r \) solves

\[
e^{-\frac{\tilde{\omega}}{h}} h^{2m} \mathcal{L}(x, D) e^{\frac{\tilde{\omega}}{h}} r(x; h) = O(h^{m+2}) \quad \text{in} \quad H^{2m}(\Omega),
\]

which is solvable thanks to the Proposition 2.3. We get a solution \( r \in H^2(\Omega) \) satisfying the estimate \( \|r\|_{H^2} = O(h^2) \). \qed

3. Determination of the coefficients

Our first step is the standard method of extending the problem to a larger simply connected domain. Similar to [20, 27] we have the following result.

**Proposition 3.1.** Let \( \Omega \subset \subset \tilde{\Omega} \) be two bounded domains in \( \mathbb{R}^n \) with smooth boundaries, and let \( A_\alpha, \tilde{A}_\alpha \in W^{1+|\alpha|,\infty}(\tilde{\Omega}, \mathbb{C}^{n|\alpha|}) \) for \( |\alpha| = 1, 2 \), and \( q, \tilde{q} \in L^\infty(\Omega, \mathbb{C}) \) satisfy \( A_\alpha = \tilde{A}_\alpha \) for \( |\alpha| = 1, 2 \) and \( q = \tilde{q} \) in \( \Omega \). If the Cauchy data set (cf. (1.4)) \( \mathcal{C}_{A_\alpha, q}(\Omega) = \mathcal{C}_{\tilde{A}_\alpha, \tilde{q}}(\Omega) \), then \( \mathcal{C}_{A_\alpha, q}(\tilde{\Omega}) = \mathcal{C}_{\tilde{A}_\alpha, \tilde{q}}(\tilde{\Omega}) \).

The proof of the above proposition follows from a standard analysis technique, which is given in [20, 27].

We recall that \( A_\alpha, \tilde{A}_\alpha \in W^{1+|\alpha|,\infty}(\mathbb{R}^n, \mathbb{C}^{n|\alpha|}) \cap \mathcal{E}'(\overline{\Omega}) \) for \( |\alpha| = 1, 2 \) and \( q, \tilde{q} \in L^\infty(\Omega) \). Let \( \tilde{\Omega} \) be a smooth simply connected domain in \( \mathbb{R}^n \) such that \( \Omega \subset \subset \tilde{\Omega} \). We extend all the lower order perturbed coefficients \( A_\alpha, \tilde{A}_\alpha, q, \tilde{q} \) by zero to \( \Omega \setminus \Omega \), and denote these extensions by the same letters. Now using the Proposition 3.1 we get \( \mathcal{C}_{A_\alpha, q}(\tilde{\Omega}) = \mathcal{C}_{\tilde{A}_\alpha, \tilde{q}}(\tilde{\Omega}) \). Therefore now we can study the problem in a bigger domain \( \tilde{\Omega} \) which is simply connected.

3.1. Integral identity involving the coefficients \( A_\alpha, q \). We recall that

\[
\mathcal{L}(x, D) \equiv (-\Delta)^m + \sum_{|\alpha|=1}^2 A_\alpha(x) D^\alpha + q(x),
\]

where \( A_\alpha \in W^{1+|\alpha|,\infty}(\tilde{\Omega}, \mathbb{C}^{n|\alpha|}) \) for \( |\alpha| = 1, 2 \) and \( q \in L^\infty(\tilde{\Omega}, \mathbb{C}) \). We write the formal \( L^2 \) adjoint of this operator, \( \mathcal{L}^*(x, D) \) is of the form

\[
\mathcal{L}^*(x, D) \equiv (-\Delta)^m + \sum_{|\alpha|=1}^2 A_\alpha^2(x) D^\alpha + q^2(x),
\]
We have the following integral identity

$$
\int_\Omega (L(x, D)u) v dx - \int_\Omega u L^*(x, D)v dx = 0, \quad \forall u \in H^m_0(\Omega), v \in H^m(\Omega).
$$

(3.2)

where $H^m_0(\Omega)$ is the closure of $C^\infty(\Omega)$ functions in $H^m(\Omega)$ norm.

Let $m \geq 2$ and $u, \tilde{u} \in H^m(\Omega)$ solves

$$
\begin{align*}
L(x, D)u &= 0 \quad \text{in } \Omega, \\
\tilde{L}(x, D)\tilde{u} &= 0 \quad \text{in } \Omega,
\end{align*}
$$

(3.3)

with $(-\Delta)^l u|_{\partial \Omega} = (-\Delta)^l \tilde{u}|_{\partial \Omega}$, for $l = 0, 1, \ldots, (m - 1)$.

From the assumption of the Theorem 1.1 and Proposition 3.1 on $\partial \Omega$ we now get

$$
\partial_v (-\Delta)^l u = \partial_v (-\Delta)^l \tilde{u}, \quad \text{for } l = 0, 2, \ldots, (m - 1).
$$

(3.4)

So we have $(u - \tilde{u}) \in H^m_0(\Omega)$. Therefore

$$
L(x, D)(u - \tilde{u}) = \sum_{1 \leq |\alpha| \leq 2} (A_\alpha - \tilde{A}_\alpha) D^\alpha \tilde{u} + (q - \tilde{q}) \tilde{u}.
$$

(3.5)

Let $v \in H^m(\Omega)$ satisfies $L^*(x, D)v = 0$ in $\Omega$, then from the integral identity (3.2) we get

$$
\int_\Omega \left( \sum_{1 \leq |\alpha| \leq 2} (A_\alpha - \tilde{A}_\alpha) D^\alpha \tilde{u} + (q - \tilde{q}) \tilde{u} \right) v dx = 0
$$

(3.6)

Next we choose $\tilde{u}$ and $v$ to be the CGO type solutions constructed in the previous subsection 2.2. We choose $\varphi = \omega \cdot x$ and $\psi = \tilde{\omega} \cdot x$ for $\tilde{u}$ and $\varphi = -\omega \cdot x$ and $\psi = \tilde{\omega} \cdot x$ for $v$, where $\omega, \tilde{\omega} \in \mathbb{R}^n$ satisfying $|\omega| = |	ilde{\omega}| = 1$ and $\omega \cdot \tilde{\omega} = 0$. For $h > 0$ small enough, we set the solutions are of the form

$$
\begin{align*}
\tilde{u}(x) &= e^{\frac{\omega \cdot x + i\tilde{\omega} \cdot x}{h}} (\tilde{a}_0(x, \omega + i\tilde{\omega}) + h\tilde{a}_1(x, \omega + i\tilde{\omega}) + \tilde{r}(x, \omega + i\tilde{\omega}; h)) \quad \text{in } \Omega, \\
v(x) &= e^{\frac{-\omega \cdot x + i\tilde{\omega} \cdot x}{h}} (a_0(x, -\omega + i\tilde{\omega}) + ha_1(x, -\omega + i\tilde{\omega}) + r(x, -\omega + i\tilde{\omega}; h)) \quad \text{in } \tilde{\Omega}.
\end{align*}
$$

(3.7)

The amplitudes $\tilde{a}_0(\cdot, \omega + i\tilde{\omega}), a_0(\cdot, -\omega + i\tilde{\omega}) \in C^\infty(\Omega)$ satisfy the transport equations

$$
\begin{align*}
((\omega + i\tilde{\omega}) \cdot \nabla)^m \tilde{a}_0(x, \omega + i\tilde{\omega}) &= 0 \quad \text{in } \Omega, \\
(-\omega + i\tilde{\omega}) \cdot \nabla)^m a_0(x, -\omega + i\tilde{\omega}) &= 0 \quad \text{in } \tilde{\Omega}, \quad m \geq 2
\end{align*}
$$

(3.8)

and

$$
\|\tilde{r}\|_{H^m_{\tilde{\Omega}}} \leq C(h^2).
$$

(3.9)
Now substituting (3.7) in (3.6) we get

\[
0 = \sum_{\alpha = \alpha_1 + \alpha_2, \atop |\alpha_1| = |\alpha_2| = 1} \int_{\tilde{\Omega}} (A_\alpha - \tilde{A}_\alpha) \left( \frac{(\omega + i\tilde{\omega})_\alpha}{h} \right) (a_0 + h\tilde{a}_1 + \tilde{r}) (a_0 + ha_1 + r) \, dx \\
+ \sum_{\alpha = \alpha_1 + \alpha_2, \atop |\alpha_1| = |\alpha_2| = 1} \int_{\tilde{\Omega}} (A_\alpha - \tilde{A}_\alpha) \left( \frac{(\omega + i\tilde{\omega})_\alpha}{h} \right) D^{\alpha_1} (a_0 + h\tilde{a}_1 + \tilde{r}) (a_0 + ha_1 + r) \, dx \\
+ \sum_{|\alpha| = 2} \int_{\tilde{\Omega}} (A_\alpha - \tilde{A}_\alpha) D^\alpha (a_0 + h\tilde{a}_1 + \tilde{r}) (a_0 + ha_1 + r) \, dx \\
+ \sum_{|\alpha| = 1} \int_{\tilde{\Omega}} (A_\alpha - \tilde{A}_\alpha) D^{\alpha_1} (a_0 + h\tilde{a}_1 + \tilde{r}) (a_0 + ha_1 + r) \, dx \\
+ \int_{\tilde{\Omega}} (q - \tilde{q}) (a_0 + h\tilde{a}_1 + \tilde{r}) (a_0 + ha_1 + r) \, dx.
\]  

(3.10)

Now we consider following two cases for the different forms of the second order perturbation.

**Case(1):** Let us assume that \( m > 2 \) and \((A_\alpha - \tilde{A}_\alpha)\) for \(|\alpha| = 2\) is not an isotropic matrix. Then multiplying (3.10) by \( h^2 \) and letting \( h \to 0 \) we get

\[
\sum_{\alpha = \alpha_1 + \alpha_2, \atop |\alpha_1| = |\alpha_2| = 1} \int_{\tilde{\Omega}} (A_\alpha - \tilde{A}_\alpha) (\omega + i\tilde{\omega})_\alpha (\omega + i\tilde{\omega})_{\alpha_1} \tilde{a}_0 \tilde{a}_0 \, dx = 0.
\]

(3.11)

This follows from the fact that \( A_\alpha, \tilde{A}_\alpha \in L^\infty(\tilde{\Omega}), \tilde{a}_0, a_0 \in C^\infty(\tilde{\Omega}) \) and \( a_1, \tilde{a}_1 \in W^{2,\infty}(\tilde{\Omega}) \). Note that here we have crucially used the fact that \( \|\tilde{r}\|_{L^2}, \|r\|_{L^2} = O(h^2), \|D^2\tilde{r}\|_{L^2} = O(h), \) for \(|\beta| = 1 \) and \( \|D^\alpha\tilde{r}\|_{L^2} = O(1) \), for \(|\alpha| = 2 \).

**Case(2):** Let us assume that \((A_\alpha - \tilde{A}_\alpha)\) for \(|\alpha| = 2\) is isotropic. Observe that the first term in (3.10) is always zero. Hence, multiplying (3.10) by \( h \) and letting \( h \to 0 \) we get

\[
\sum_{\alpha = \alpha_1 + \alpha_2, \atop |\alpha_1| = |\alpha_2| = 1} \int_{\tilde{\Omega}} (A_\alpha - \tilde{A}_\alpha) (\omega + i\tilde{\omega})_\alpha D^{\alpha_1} \tilde{a}_0 \tilde{a}_0 \, dx + \sum_{|\alpha| = 1} \int_{\tilde{\Omega}} (A_\alpha - \tilde{A}_\alpha) (\omega + i\tilde{\omega})_\alpha \tilde{a}_0 \tilde{a}_0 \, dx = 0.
\]

(3.12)

It follows from the fact that \( A_\alpha, \tilde{A}_\alpha \in L^\infty(\tilde{\Omega}), \tilde{a}_0, a_0 \in C^\infty(\tilde{\Omega}) \) and \( a_1, \tilde{a}_1 \in W^{2,\infty}(\tilde{\Omega}) \). Note that here we use \( \|\tilde{r}\|_{L^2}, \|r\|_{L^2} = O(1), \|hD^2\tilde{r}\|_{L^2} = O(h) \) with \(|\beta| = 1 \), and \( \|h^jD^\alpha\tilde{r}\|_{L^2} = O(h^j) \) with \(|\alpha| = 2, j = 1, 2 \), specially in the third term of (3.10).

### 3.2. Determination of the coefficients

In this section we will establish the equality of \( A_\alpha = \tilde{A}_\alpha \) for \(|\alpha| = 1, 2 \) and \( q = \tilde{q} \) in \( \Omega \) from the integral identities (3.11) and (3.12).

**Proof of Theorem 1.1.** Let us begin with the integral identity (3.11) in the Case (1) mentioned above.

In [25], it is proved that a symmetric 2-form \((A_\alpha - \tilde{A}_\alpha) \in W^{3,\infty}(\tilde{\Omega}, \mathbb{C}^n)\) can be uniquely decomposed in an 1-form \( V \in W^{4,\infty}(\tilde{\Omega}, \mathbb{C}^n) \) with \( V = 0 \) on \( \partial\tilde{\Omega} \) and a symmetric 2-form
\[ F \in W^{3,\infty}(\tilde{\Omega}, \mathbb{C}^n) \] so that

\[ (A_{\alpha} - \tilde{A}_{\alpha}) = F + d_s V, \quad \text{in } \tilde{\Omega}. \]  

(3.13)

Here \( F \) is divergence free, i.e.

\[ (\delta F)_j = \sum_{k=1}^n \frac{\partial F_{jk}}{\partial x_k} = 0, \quad \text{in } \tilde{\Omega}, \quad j = 1, 2, \ldots, n \]

and \( d_s \) is the symmetrized differentiation defined as

\[ (d_s V)_{jk} = \frac{1}{2} \left( \frac{\partial V_j}{\partial x_k} + \frac{\partial V_k}{\partial x_j} \right), \quad \text{in } \tilde{\Omega}. \]

The above expression (3.13) can be realized in the following manner. Let us take any vector field \( B \in W^{k,\infty}(\tilde{\Omega}), \ k \geq 1 \), and consider the following Laplace equation \(-\Delta \varphi = \text{div} B\) in \( \tilde{\Omega} \). We can get a unique solution \( \varphi \in W^{k+1,\infty}(\tilde{\Omega}) \) subject to suitable Dirichlet or Neumann boundary condition. For example, we get a unique solution \( \varphi \in W^{3,\infty}(\tilde{\Omega}) \) by solving

\[ (-\Delta) \varphi = \text{div} B \quad \text{in } \tilde{\Omega}, \quad \text{with } \varphi = 0 \text{ on } \partial \tilde{\Omega}. \]

Then we can write \( B = \tilde{B} + \nabla \varphi \) in \( \tilde{\Omega} \), where \( \tilde{B} = (B - \nabla \varphi) \) with \( \text{div} \tilde{B} = 0 \) in \( \tilde{\Omega} \), and \( \varphi = 0 \) on \( \partial \tilde{\Omega} \). Applying the above technique for each row (or column) vector of the matrix \((A_{\alpha} - \tilde{A}_{\alpha})\) and using symmetrization one can obtain the decomposition in (3.13).

Using the decomposition (3.13) in (3.11) we get

\[ \sum_{\alpha=\alpha_1+\alpha_2, \ |\alpha_1|=|\alpha_2|=1} \int_{\tilde{\Omega}} (F + d_s V)_\alpha (\omega + i\tilde{\omega})_{\alpha_1} (\omega + i\tilde{\omega})_{\alpha_2} a_0 \bar{a}_0 \, dx = 0. \]  

(3.14)

We choose \( a_0 = 1, \bar{a}_0 = e^{-ix\xi} \), where \( \xi \in \mathbb{R}^n \) is perpendicular to \( \omega \) and \( \tilde{\omega} \). Then \( a_0, \bar{a}_0 \) are indeed solutions of the transport equations (3.8) and we get

\[ \sum_{\alpha=\alpha_1+\alpha_2, \ |\alpha_1|=|\alpha_2|=1} \int_{\tilde{\Omega}} (F + d_s V)_\alpha (\omega + i\tilde{\omega})_{\alpha_1} (\omega + i\tilde{\omega})_{\alpha_2} e^{-ix\xi} \, dx = 0. \]

Since, \( V|_{\partial \tilde{\Omega}} = 0 \) and \( \xi \cdot \omega = \xi \cdot \tilde{\omega} = 0 \) we get

\[ \int_{\tilde{\Omega}} (d_s V)_\alpha (\omega + i\tilde{\omega})_{\alpha_1} (\omega + i\tilde{\omega})_{\alpha_2} e^{-ix\xi} \, dx = (\omega + i\tilde{\omega}) \cdot (\omega + i\tilde{\omega}) \int_{\tilde{\Omega}} (V \cdot (\omega + i\tilde{\omega})) \, e^{-ix\xi} \, dx = 0. \]

Therefore,

\[ \sum_{\alpha=\alpha_1+\alpha_2, \ |\alpha_1|=|\alpha_2|=1} \int_{\tilde{\Omega}} F_\alpha (\omega + i\tilde{\omega})_{\alpha_1} (\omega + i\tilde{\omega})_{\alpha_2} e^{-ix\xi} \, dx = 0. \]  

(3.15)

Now extending \( F \) by 0 outside \( \tilde{\Omega} \) and denoting the extended function by the same notation \( F \in L^2(\mathbb{R}^n) \) we get

\[ \sum_{j,k=1}^n \tilde{F}_{jk}(\xi) (\omega + i\tilde{\omega})_j (\omega + i\tilde{\omega})_k = 0. \]  

(3.16)

Let us now fix \( \xi \in \mathbb{R}^n \setminus \{0\} \). Consider an orthonormal basis \( \{\mu_1, \mu_2, \ldots, \mu_n\} \) of \( \mathbb{R}^n \) so that \( \mu_n = \frac{\xi}{|\xi|} \). Choose \( \omega = \mu_p \) and \( \tilde{\omega} = \mu_q \) for \( p \neq q \) and \( 1 \leq p, q \leq (n-1) \). Observe that the construction of the
solutions \( \tilde{u} \) and \( v \) allows us to make these choices of \( \omega \) and \( \tilde{\omega} \). In fact, without loss of generality, the relation in (3.16) is true whenever \( \xi, \omega, \tilde{\omega} \in \mathbb{R}^n \) such that \{\( \xi, \omega, \tilde{\omega} \)\} are mutually perpendicular and \(|\omega| = |\tilde{\omega}| = 1\).

Now, equation (3.16) implies

\[
\sum_{j,k=1}^{n} \hat{F}_{jk}(\xi) ((\mu_p)_j((\mu_1)_k - (\mu_2)_k) + 2i((\mu_p)_j(\mu_q)_k)) = 0. \tag{3.17}
\]

Choosing \(-\mu_q\) in place of \(\mu_q\) we get an another orthonormal basis of \(\mathbb{R}^n\) and consequently we find

\[
\sum_{j,k=1}^{n} \hat{F}_{jk}(\xi) (((\mu_p)_j(\mu_q)_k - (\mu_q)_j(\mu_q)_k) - 2i((\mu_p)_j(\mu_q)_k)) = 0. \tag{3.18}
\]

Adding and subtracting equations (3.17) and (3.18) for \(1 \leq p, q \leq n - 1\), \(p \neq q\) we obtain

\[
\langle \hat{F}(\xi)\mu_p, \mu_p \rangle = \langle \hat{F}(\xi)\mu_q, \mu_q \rangle \quad \text{and} \quad \langle \hat{F}(\xi)\mu_p, \mu_q \rangle = 0 = \langle \hat{F}(\xi)\mu_q, \mu_p \rangle. \tag{3.19}
\]

Since \(F\) is divergence free, we also have

\[
\sum_{k=1}^{n} \hat{F}_{jk}(\xi)\xi_k = 0 \quad \text{for each} \quad j = 1, \ldots, n. \tag{3.20}
\]

Using the symmetry of \(\hat{F}(\xi)\) we obtain \(\langle \hat{F}(\xi)\mu_p, \xi \rangle = 0\) for any \(p = 1, \ldots, n - 1\). This leads to have the following representation of the element \(\hat{F}(\xi)\mu_p\) with respect to orthonormal basis \{\(\mu_1, \ldots, \mu_{n-1}; \xi\)\} of \(\mathbb{R}^n\) as

\[
\hat{F}(\xi)\mu_p = \sum_{j=1}^{n-1} d_j^{(p)}(\xi)\mu_j, \quad p = 1, \ldots, n - 1,
\]

for some \(d_j^{(p)}(\xi) \in \mathbb{C}, \quad j = 1, \ldots, n - 1\). Using the second relation in (3.19) we see \(d_j^{(p)}(\xi) = 0\) whenever \(j \neq p\). Hence,

\[
\hat{F}(\xi)\mu_p = d_p^{(p)}(\xi)\mu_p, \quad p = 1, \ldots, n - 1.
\]

Now, the first relation in (3.19) implies

\[
d_p^{(p)}(\xi) = \langle \hat{F}(\xi)\mu_p, \mu_p \rangle = \langle \hat{F}(\xi)\mu_q, \mu_q \rangle = d_q^{(q)}(\xi), \quad \forall p, q = 1, \ldots, (n - 1).
\]

Hence, we have \(\hat{F}(\xi)\mu_p = d(\xi)\mu_p\), where

\[
d(\xi) = \int_{\mathbb{R}^n} \langle F(x)\mu_p, \mu_p \rangle e^{-ix\cdot\xi} dx, \quad \forall 1 \leq p \leq n - 1. \tag{3.21}
\]

Therefore, \(\hat{F}(\xi)\) has eigenvalue \(d(\xi)\) of multiplicity \((n - 1)\) corresponding to the eigenvectors \(\mu_1, \mu_2, \ldots, \mu_{n-1}\) and eigenvalue 0 corresponding to the eigenvector \(\xi\). Let us now define the orthonormal matrix

\[
P = \begin{pmatrix} \mu_1 & \mu_2 & \ldots & \mu_{n-1} & \xi \\ \frac{\xi}{|\xi|} \end{pmatrix}
\]

and observe that \(\hat{F} = P^tDP\), where \(P^t\) is the transpose of the matrix \(P\) and \(D(\xi)\) is the diagonal matrix given as \(D(\xi) = \text{diag}(d(\xi), \ldots, d(\xi), 0)\).
The above expression shows that \( F \) where we get
\[
\hat{F}(\xi) = d(\xi) \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right), \quad \xi \in \mathbb{R}^n \setminus \{0\}
\]
and \( \hat{F}(0) \) can be taken as \( \int_{\Omega} F(x) \, dx \).

Consequently, in the physical space \( x \in \mathbb{R}^n \) we have
\[
F_{jk}(x) = d_\#(x) \delta_{jk} + R_j R_k (d_\#(x)),
\]
where \( d_\# \in L^2(\mathbb{R}^n) \) with \( \hat{d}_\#(\xi) = d(\xi) \) and \( R_j \) is the classical Riesz transformation defined as \( \hat{R}_j f(\xi) = \frac{\xi_j}{|\xi|^2} \hat{f}(\xi) \), for \( f \in L^2(\mathbb{R}^n) \).

Since \( F_{jk} \in L^2(\mathbb{R}^n) \) supported inside \( \widetilde{\Omega} \), therefore trace \( [F(x)] = 0 \) in \( \mathbb{R}^n \setminus \widetilde{\Omega} \). As \( \sum_{j=1}^n R_j^2 = -I \), from (3.23) we get trace \( [F(x)] = (n - 1) d_\#(x) \), therefore we derive \( d_\#(x) = 0 \) in \( \mathbb{R}^n \setminus \widetilde{\Omega} \).

Having \( d_\# \in L^2(\mathbb{R}^n) \) with its compact support, let us consider \( \tilde{d} \in H^2(\mathbb{R}^n) \) solving uniquely
\[
- \Delta \tilde{d} = d_\# \quad \text{in} \quad \mathbb{R}^n.
\]

Then by using a standard property of the Riesz transform given as \( R_j R_k (\Delta) = \frac{\partial^2}{\partial x_j \partial x_k} \), from (3.23) we get
\[
F_{jk}(x) = d_\#(x) \delta_{jk} + \frac{1}{2} \left[ \frac{\partial}{\partial x_j} \left( \frac{\partial \tilde{d}}{\partial x_k} \right) + \frac{\partial}{\partial x_k} \left( \frac{\partial \tilde{d}}{\partial x_j} \right) \right] \quad \text{in} \quad \mathbb{R}^n.
\]

The above expression shows that \( F_{jk} \in L^2(\mathbb{R}^n) \) can be written as a sum of two \( L^2(\mathbb{R}^n) \) functions.

As a next step, we show \( \tilde{d} = 0 \) in \( \mathbb{R}^n \setminus \widetilde{\Omega} \). Since we got \( d_\#(x) = 0 \) in \( \mathbb{R}^n \setminus \widetilde{\Omega} \). Then from (3.24) we get \( \frac{\partial^2}{\partial x_j \partial x_k} \tilde{d} = 0 \) in \( \mathbb{R}^n \setminus \widetilde{\Omega} \) for all \( j, k = 1, \ldots, n \). This implies \( \tilde{d} \) is some linear function in \( x \) in the domain \( \mathbb{R}^n \setminus \widetilde{\Omega} \). However, as \( \tilde{d} \in H^2(\mathbb{R}^n) \) i.e. having \( L^2(\mathbb{R}^n \setminus \widetilde{\Omega}) \)-integrability of \( \tilde{d} \) in the unbounded (in all direction) domain \( \mathbb{R}^n \setminus \widetilde{\Omega} \) that linear function in \( x \) in \( \mathbb{R}^n \setminus \widetilde{\Omega} \) could only be zero function, i.e. \( \tilde{d} = 0 \) in \( \mathbb{R}^n \setminus \widetilde{\Omega} \). Therefore, \( d_\# \in H^2_0(\widetilde{\Omega}) \).

So our decomposition in (3.13) is now modified into
\[
\left( A - \hat{A} \right)_{jk} = (F + d_s V)_{jk} = d_\#(x) \delta_{jk} + \frac{1}{2} \left[ \frac{\partial}{\partial x_j} \left( \frac{\partial \tilde{d}}{\partial x_k} + V_k \right) + \frac{\partial}{\partial x_k} \left( \frac{\partial \tilde{d}}{\partial x_j} + V_j \right) \right] \quad \text{in} \quad \widetilde{\Omega}. \quad (3.25)
\]

We denote \( \tilde{V} = \left( \nabla \tilde{d} + V \right) \in H^1(\widetilde{\Omega}) \) and observe that \( \tilde{V} \mid_{\partial \widetilde{\Omega}} = 0 \), since \( \tilde{d} \in H^2_0(\widetilde{\Omega}) \) and \( V \mid_{\partial \widetilde{\Omega}} = 0 \).

Next we show that \( \tilde{V} = 0 \) in \( \widetilde{\Omega} \). For that we consider (3.14) with the form of \( (F + d_s V) \) given in (3.25) and obtain
\[
\sum_{j,k=1}^n \int_{\widetilde{\Omega}} \frac{1}{2} \left( \frac{\partial \tilde{V}_j}{\partial x_k} + \frac{\partial \tilde{V}_k}{\partial x_j} \right) (\omega + i \tilde{\omega})_j (\omega + i \tilde{\omega})_k a_0 a_0 \, dx = 0. \quad (3.26)
\]

We consider \( a_0 = e^{-ix \cdot \xi} \) and \( a_0 = (\omega \cdot x) \) in \( \widetilde{\Omega} \), with \( \omega \cdot \omega = 1 \) and \( \omega \cdot \tilde{\omega} = \omega \cdot \xi = 0 = \tilde{\omega} \cdot \xi \), which are indeed the solutions of the transport equations (3.8). By integration by parts with using \( \tilde{V} = 0 \) on
Then following [26, 20], from (3.27) one can directly conclude that
\[ \partial_j \tilde{V}_k - \partial_k \tilde{V}_j = 0 \quad \text{in } \tilde{\Omega}, \; \forall j, k = 1, \ldots, n. \]

Since \( \tilde{\Omega} \) is simply connected, thus \( \tilde{V} = \nabla p \) for some scalar function \( p \in H^2(\tilde{\Omega}) \).

Since \( \tilde{V} = 0 \) on \( \partial \tilde{\Omega} \), we infer \( \nabla_{\text{tan}} p|_{\partial \tilde{\Omega}} = (\nabla p - (\partial_{\text{c}} p)\nu)|_{\partial \tilde{\Omega}} = 0 \) and consequently get that \( p \) is constant \( (p|_{\partial \tilde{\Omega}} = c \in \mathbb{C}) \) on the boundary. Here \( \nabla_{\text{tan}} \) denotes the tangential component of \( \nabla \) along the boundary \( \partial \tilde{\Omega} \). Hence, by considering \( (p - c) \) in place of \( p \), we assume that \( p = 0 \) on \( \partial \tilde{\Omega} \).

Therefore, from (3.26) we get \( p = \partial_{\nu} p = 0 \) on \( \partial \tilde{\Omega} \) and consequently we see
\[
0 = \sum_{j,k=1}^{2} \frac{\partial^2 p}{\partial x_j \partial x_k} (\omega + i \tilde{\omega})_j (\omega + i \tilde{\omega})_k \tilde{\alpha}_0 \tilde{\alpha}_0 \, dx = \int_{\tilde{\Omega}} p(x) (\omega \cdot \nabla)^2 (\tilde{\alpha}_0 \tilde{\alpha}_0) \, dx. \tag{3.28}
\]

Consider \( \tilde{\alpha}_0 = e^{-ix \cdot \xi} \) and \( a_0 = (\omega \cdot x)^2 \) in \( \tilde{\Omega} \) with \( \omega \cdot \omega = 1 \) and \( \omega \cdot \tilde{\omega} = \omega \cdot \xi = 0 \), which are indeed solutions of the transport equations (3.8) for \( m > 2 \). Therefore we get
\[
\int_{\tilde{\Omega}} p(x) e^{-ix \cdot \xi} \, dx = 0.
\]

Hence, \( p = 0 \) in \( \tilde{\Omega} \) and so, \( \tilde{V} = 0 \) in \( \tilde{\Omega} \).

This shows that \( (A_{\alpha} - \tilde{A}_{\alpha}) \) (see (3.25)) is an isotropic matrix given as
\[
(A - \tilde{A})_{\alpha} = d_{\#} I \quad \text{in } \tilde{\Omega} \text{ for } |\alpha| = 2.
\]

Hence we fall into the regime of Case (2).

Next, we turn back to the Case (2) when \( (A_{\alpha} - \tilde{A}_{\alpha}) \) is isotropic for \( |\alpha| = 2 \). In that case we have the integral identity (3.12). First we show that \( (A_{\alpha} - \tilde{A}_{\alpha}) = 0 \) in \( \Omega \) for \( |\alpha| = 1 \) and then we move into proving \( (A_{\alpha} - \tilde{A}_{\alpha}) = 0 \) in \( \tilde{\Omega} \) for \( |\alpha| = 2 \).

Let us consider \( \tilde{a}_0 = 1 \) and \( a_0 = e^{ix \cdot \xi} \) in \( \tilde{\Omega} \), which are indeed the solutions of the transport equations (3.8). Plugging in the functions \( a_0 \) and \( \tilde{a}_0 \) in (3.12) we get
\[
\sum_{|\alpha| = 1} \int_{\tilde{\Omega}} (A_{\alpha} - \tilde{A}_{\alpha})(\omega + i \tilde{\omega})_\alpha e^{-ix \cdot \xi} \, dx = 0. \tag{3.29}
\]

This implies \( (A_{\alpha} - \tilde{A}_{\alpha}) = \nabla \tilde{p} \), \( |\alpha| = 1 \), for some scalar function \( \tilde{p} \in W^{3,\infty}(\tilde{\Omega}) \). Moreover, as \( (A_{\alpha} - \tilde{A}_{\alpha}) = 0 \) in a neighborhood of \( \partial \tilde{\Omega} \), hence we conclude that \( \tilde{p} \) is a constant \( (\tilde{p}|_{\partial \tilde{\Omega}} = \tilde{c} \in \mathbb{C}) \) on \( \partial \tilde{\Omega} \). Hence by considering \( (\tilde{p} - \tilde{c}) \), we assume that \( \tilde{p} = 0 \) on \( \partial \tilde{\Omega} \). Next by putting this in (3.12) we get
\[
\int_{\tilde{\Omega}} \nabla \tilde{p} \cdot (\omega + i \tilde{\omega}) \tilde{a}_0 \tilde{a}_0 \, dx = 0. \tag{3.30}
\]

Now in (3.30) we consider \( \tilde{a}_0 = 1 \) and \( a = e^{ix \cdot \xi}(\omega \cdot x) \) in \( \tilde{\Omega} \), which are the solution of the transport equations (3.8). Using integration by parts with the fact that \( \tilde{p} = 0 \) on \( \partial \tilde{\Omega} \), we obtain
\[
\int_{\tilde{\Omega}} \tilde{p}(x) e^{-ix \cdot \xi} \, dx = 0.
\]

Therefore \( \tilde{p} = 0 \) in \( \tilde{\Omega} \) and hence \( (A_{\alpha} - \tilde{A}_{\alpha}) = 0 \) in \( \Omega \), for \( |\alpha| = 1 \).
Next in order to show the same for $|\alpha| = 2$ with $A_{\alpha}$ and $\tilde{A}_{\alpha}$ being isotropic matrices, we go back to (3.12) and put $A_{\alpha} = \tilde{A}_{\alpha}$ for $|\alpha| = 1$ in $\tilde{\Omega}$. Taking $\tilde{a}_0 = (\omega \cdot x)$ and $a_0 = e^{ix \cdot \xi}$ in $\tilde{\Omega}$ we get
\[
\int_{\tilde{\Omega}} m(x) e^{-ix \cdot \xi} \, dx = 0,
\]
where $A_{\alpha} - \tilde{A}_{\alpha} = m(x)I$. Thus $m = 0$ in $\tilde{\Omega}$, consequently $(A_{\alpha} - \tilde{A}_{\alpha}) = 0$ for $|\alpha| = 2$ in $\tilde{\Omega}$. So in the Case (2) we have shown $A_{\alpha} = \tilde{A}_{\alpha}$ for $\alpha = 1, 2$ in $\Omega$.

Now we go back to Case (1) again and by the above analysis we get $A_{\alpha} = \tilde{A}_{\alpha}$ for $|\alpha| = 2$ in $\tilde{\Omega}$. Now we will show that for Case (1), $A_{\alpha} = \tilde{A}_{\alpha}$ for $|\alpha| = 1$ in $\tilde{\Omega}$. In order to do that, we go back to identity (3.10) and put $A_{\alpha} = \tilde{A}_{\alpha}$ for $|\alpha| = 2$ in $\tilde{\Omega}$. By doing that we get first three terms in the identity becomes zero. Now by multiplying both sides of the identity (3.10) by $h$ and letting $h \to 0$ we get
\[
\sum_{|\alpha|=1} \int_{\tilde{\Omega}} (A_{\alpha} - \tilde{A}_{\alpha})(\omega + i\tilde{\omega})_{\alpha} \tilde{a}_0 \overline{a_0} \, dx = 0.
\]
Here we consider $\tilde{a}_0 = e^{-ix \cdot \xi}$ and $a_0 = 1$ in $\tilde{\Omega}$ and end up with having (3.29). Hence we get $A_{\alpha} = \tilde{A}_{\alpha}$ for $|\alpha| = 1$ in $\tilde{\Omega}$.

Hence, in both cases we have $A_{\alpha} = \tilde{A}_{\alpha}$ for $|\alpha| = 1, 2$ in $\tilde{\Omega}$. The remaining part is to show the uniqueness of the zeroth order perturbation, that is $(q - \tilde{q}) = 0$ in $\tilde{\Omega}$. Putting $A_{\alpha} = \tilde{A}_{\alpha}$ for $|\alpha| = 1, 2$ in $\tilde{\Omega}$ in the identity (3.10) we get the first five terms are zero. Next, by letting $h \to 0$ we obtain
\[
\int_{\tilde{\Omega}} (q - \tilde{q}) \tilde{a}_0 \overline{a_0} \, dx = 0.
\]
By considering $\tilde{a}_0 = e^{-ix \cdot \xi}$ and $a_0 = 1$ in $\tilde{\Omega}$, we end up with having
\[
\int_{\tilde{\Omega}} (q - \tilde{q}) e^{-ix \cdot \xi} \, dx = 0,
\]
which proves $q = \tilde{q}$ in $\tilde{\Omega}$. This completes the proof of the Theorem 1.1. \qed

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