Stable, metastable and unstable states in the mean-field random-field Ising model at $T = 0$

M L Rosinberg$^1$, G Tarjus$^1$ and F J Pérez-Reche$^2$

$^1$ Laboratoire de Physique Théorique de la Matière Condensée, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 05, France
$^2$ Department of Chemistry, University of Cambridge, Cambridge CB2 1EW, UK
E-mail: mlr@lptmc.jussieu.fr, tarjus@lptmc.jussieu.fr and fjp23@cam.ac.uk

Received 18 July 2008
Accepted 10 September 2008
Published 7 October 2008

Online at stacks.iop.org/JSTAT/2008/P10004
doi:10.1088/1742-5468/2008/10/P10004

Abstract. We compute the probability distribution of the number of metastable states at a given applied field in the mean-field random-field Ising model at $T = 0$. Remarkably, there is a non-zero probability in the thermodynamic limit of observing metastable states on the so-called ‘unstable’ branch of the magnetization curve. This implies that the branch can be reached when the magnetization is controlled instead of the magnetic field, in contrast to the situation for the pure system.

Keywords: classical phase transitions (theory), disordered systems (theory), energy landscapes (theory), metastable states
Stable, metastable and unstable states in the mean-field RFIM at $T = 0$

Contents

1. Introduction 2
2. The $T = 0$ RFIM on a fully connected lattice 2
3. Metastable states 5
4. Conclusion 9
References 10

1. Introduction

As is well known, the main approximation underlying mean-field theories of phase transitions consists in neglecting any spatial dependence of the order parameter. In simple systems such as fluids or ferromagnets this leads to an equation of state (e.g. the van der Waals equation) that exhibits a continuous loop below the critical temperature $T_c$. In this framework, one can distinguish between stable (equilibrium), metastable, and unstable states. The intermediate, unstable branch of the loop is associated with maxima of the free energy and has a negative slope (whence a negative susceptibility), and the unstable and metastable regions of the phase diagram are separated by a spinodal line where the free-energy barrier vanishes. Whereas the concept of metastability can be in some sense extended to systems with short-range interaction in finite dimensions where it becomes a matter of time scales, the unstable branch is a complete artifact of the mean-field approximation and the homogeneous states along this branch do not represent any real physical situation.

Things are different in the presence of disorder since inhomogeneities are induced even at a microscopic scale. As a result, configurations formed by a multitude of small domains of the two phases can become metastable over an extended range of the ‘external’ field (e.g. the fluid pressure or the magnetic field). Interestingly, this can already occur at the mean-field level and the purpose of this paper is to emphasize that metastable states are present all along the so-called unstable branch in the mean-field random-field Ising model (RFIM) at $T = 0$. This feature is in fact a precursor of what happens in the RFIM in finite dimensions, and it has consequences for experiments in actual random-field systems at low temperature.

2. The $T = 0$ RFIM on a fully connected lattice

We consider a collection of $N$ Ising spins ($s_i = \pm 1$) interacting via the Hamiltonian

$$\mathcal{H} = -\frac{J}{2N} \sum_{i \neq j} s_i s_j - \sum_i (h_i + H)s_i,$$

(1)

where $J > 0$, $H$ is a uniform external field, and $\{h_i\}$ is a collection of random fields drawn identically and independently from some probability distribution $\mathcal{P}(h)$. (In the following, we consider a Gaussian distribution with zero mean and standard deviation $\Delta$.) This
mean-field model can be obtained from the usual short-range RFIM by placing the spins on a fully connected lattice (the connectivity is then equal to $N-1$ and the exchange interaction is rescaled by $N$ to ensure a proper thermodynamic limit). Equation (1) can be also rewritten as

$$H = -\sum_i f_i s_i,$$

where $f_i = J(m - s_i/N) + h_i + H$ is the effective field acting on each spin $i$ and $m = (\sum_i s_i)/N$ is the magnetization per spin.

It is straightforward to compute the equilibrium properties of this model by the replica method, without all the complications that plague the case of random exchange. In particular, the average magnetization in the thermodynamic limit $N \to \infty$ is the solution of the self-consistent equation [1]

$$m(H) = \int P(h) \tanh[\beta(Jm(H) + H + h)] \, dh,$$

(3)

where $\beta = 1/k_B T$. (In [1], only the case $H = 0$ is considered, but the generalization to $H \neq 0$ is straightforward.) At $T = 0$, the equation becomes

$$m_0(H) = \int P(h) \text{sgn}(Jm_0(H) + H + h) \, dh$$

$$= 2p(m_0(H)) - 1,$$

(4)

where $\text{sgn}(x) = x/|x|$ and $p(m) = \int_{-\infty}^{\infty} P(h) \, dh$. This yields $m_0(H) = \text{erf}((H + Jm_0(H))/\Delta\sqrt{2})$ in the case of the Gaussian distribution, where $\text{erf}(x)$ is the error function. Since the magnetization per spin is a self-averaging quantity, $m_0(H)$ is also the expected value of the magnetization for a very large sample.

Below the critical disorder $\Delta_0^c$ ($\Delta_0^c = \sqrt{2/\pi} J$ for the Gaussian distribution), equation (4) has three solutions in a certain range of the field and the curve $m_0(H)$ exhibits a characteristic ‘van der Waals’ loop with an intermediate branch along which $\partial m/\partial H < 0$ (see figure 1; we take $J = 1$ in all figures). As usual, this behavior is associated with the non-convexity of the (free) energy. At a given field $H$, the ground state corresponds to the solution with the lowest overall energy whereas the intermediate branch has the largest energy. In the space of replica magnetizations [1], such a branch corresponds to an absolute maximum. The result from the replica method can be shown to be exact [1], and it seems natural to describe the intermediate branch as ‘unstable’, naively transposing the situation found in the pure system (i.e. in the absence of random field). However, this is misleading. Indeed, since equation (4) results from an average over disorder (which restores translational invariance) when $N \to \infty$, it says nothing about the orientation of the spins in a given (finite $N$) sample. It turns out that some of the states along the intermediate branch are metastable (i.e. local minima of the energy) and not unstable. Indeed, equation (4) is trivially verified if

$$s_i = \text{sgn}(f_i), \quad i = 1, \ldots, N,$$

(5)

which is the definition of the so-called one-spin-flip stable states. There is nothing new here: it is known that equation (4) also describes the non-equilibrium (hysteretic) behavior of the mean-field RFIM at $T = 0$ with the single-spin-flip (Glauber) dynamics [2]. As

doi:10.1088/1742-5468/2008/10/P10004
Stable, metastable and unstable states in the mean-field RFIM at $T = 0$.

Figure 1. Metastable states in the mean-field Gaussian RFIM for a single disorder realization of size $N = 5000$ with $\Delta = 0.5$ (the increment in the field is $\Delta H = 25 \times 10^{-4}$). The dashed curve represents the solution of equation (4). The inset shows that there may exist a few states with different magnetizations at the same field.

$H$ is slowly varied from $\pm \infty$, this dynamics imposes at any field that the spins with $h_i < -Jm - H$ point down whereas the other ones point up, and the self-consistent equation for the average magnetization, $m(H) = \int \mathcal{P}(h) s_i \, dh$, is just equation (4). (Note incidentally that this corresponds to an ‘annealed’ average, but it yields the correct result in this case.) The non-equilibrium RFIM at $T = 0$ has been the subject of extensive studies in recent years [3] and the mean-field model, despite some peculiar features (the equilibrium and non-equilibrium critical disorders coincide and there is no hysteresis for $\Delta > \Delta_c$), has the advantage that many interesting properties can be computed exactly (for instance the size and duration of avalanches) [2, 4]. For $\Delta < \Delta_c$, the non-equilibrium system explores the lower (or upper) branch of the loop until $H$ reaches one of the two branching fields where $dm/dH$ diverges and the magnetization jumps discontinuously. This ‘infinite avalanche’ [2] occurs when each flipping spin triggers one other spin on average, which corresponds to the condition $2J\mathcal{P}(-Jm_0(H) - H) = 1$. Hereafter, we shall use the shorthand $\mathcal{P}_0^*$ for the quantity $\mathcal{P}(-Jm_0(H) - H)$.

The point that we want to emphasize here is that a configuration where each spin satisfies equation (5) is a metastable state by definition, whatever the value taken by $m_0(H)$ (of course, a state on the intermediate branch has a larger energy than the corresponding ground state and it cannot be reached by controlling the field). This is illustrated in figure 1 that shows the magnetizations of the metastable states in a single sample of size $N = 5000$ with $\Delta = 0.5$ as a function of $H$.

3 This results from an enumeration of all metastable states in the sample at the field $H$. The enumeration can be done very easily in the mean-field model: for each value of $M$ (from $-N$ to $+N$), one first aligns each spin with its local field $f_i = J(M - s_i)/N + h_i + H$, trying successively the two possible orientations $s_i = \pm 1$; one then computes $\sum_i s_i$. The configuration is metastable if the sum is equal to $M$. 
are stable states in the intermediate region and that they gather along a curve that will become the so-called “unstable” branch in the thermodynamic limit. At a given field \( H \), the number of these states is very small, obviously not exponentially growing with system size (so the probability of a randomly chosen state being stable decreases very fast with \( N \)). On the other hand, there is a non-zero probability in the thermodynamic limit that a sample has metastable states along this branch, as we now show.

3. Metastable states

For a disorder realization of size \( N \), consider the ensemble of metastable configurations at the field \( H \) with exactly \( P \) spins up (and thus an overall magnetization \( M = 2P - N \)). The number of such configurations is

\[
\mathcal{N}(M, H) = \text{Tr}_{\{s_i\}} \prod_i \Theta(s_i H) \delta_K(\sum_i s_i - M) ,
\]

(6)

where \( \Theta(x) \) is the Heaviside step function and \( \delta_K \) is the Kronecker \( \delta \). It is easy to see that there cannot be more than one metastable state with magnetization \( M \) in a given realization. Therefore, \( \mathcal{N}(M, H) \) is a random variable that only takes the values 0 or 1. Its average over disorder is

\[
\overline{\mathcal{N}(M, H)} = \binom{N}{P} p(m - 1/N)^P [1 - p(m + 1/N)]^{N-P} ,
\]

(7)

where the two terms in the right-hand side represent the probabilities of having \( P \) spins up and \( N - P \) spins down at the field \( H \), respectively. Using the Stirling approximation for the factorial and expanding \( p(m \pm 1/N) \) to first order in \( 1/N \), we find

\[
\overline{\mathcal{N}(M, H)} \sim \sqrt{\frac{2}{\pi N}} \frac{e^{\phi(m)}}{\sqrt{1 - m^2}} e^{-JP(\delta - Jm)((1+m)(1-m)/2p(m)+(1-m)/2[1-p(m)])} ,
\]

(8)

with

\[
\phi(m) = \frac{1 + m}{2} \ln \frac{2p(m)}{1 + m} + \frac{1 - m}{2} \ln \frac{2[1 - p(m)]}{1 - m} .
\]

(9)

This number is exponentially small (\( \phi(m) < 0 \)) except when \( m = 2p(m) - 1 \equiv m_0(H) \), i.e. when \( m \) is solution of equation (4): \( \phi(m) \) is then maximum and equal to 0. Expanding \( \phi(m) \) to second order close to \( m = m_0(H) \), we find that the average number of metastable states at the field \( H \) is finite when \( N \to \infty \) and is given by

\[
\overline{\mathcal{N}(H)} = \sum_M \overline{\mathcal{N}(M, H)} \sim \frac{N}{2} \int dm \overline{\mathcal{N}(M, H)} \to \frac{e^{-2JP_0^*}}{[1 - 2JP_0^*]} .
\]

(10)

This result is valid both above and below \( \Delta_0^2 \), and for \( \Delta < \Delta_0^2 \) the three branches of \( m_0(H) \) must be considered separately (with \( 2JP_0^* \geq 1 \) along the intermediate branch). The behavior of \( \overline{\mathcal{N}(H)} \) as a function of \( m_0(H) \) is shown in figure 2 for \( \Delta/J = 0.5 \). It is worth noting that \( \overline{\mathcal{N}(H)} \) diverges at the two branching fields (i.e. at the spinodal endpoints) where \( 2p'(m) = 2JP_0^* = 1 \) and \( \phi''(m_0) = -[1 - 2JP_0^*]^2/(1 - m_0^2) = 0 \) (from now on, the dependence of \( m_0 \) on \( H \) will not be indicated for brevity). In this case, the large-deviation function \( \phi(m) \) must be expanded to fourth order about \( m = m_0 \) and \( \overline{\mathcal{N}(H)} \).
Figure 2. Average number of one-spin-flip (solid line) and two-spin-flip (dashed line) stable states along the curve \( m_0(H) \) for \( \Delta = 0.5 \). Both quantities diverge at the spinodal endpoints.

scales like \( N^{1/4} \). At the critical point (\( \Delta = \Delta_c^0 \) and \( H = m_0 = 0 \)), \( \phi(m) \) must be expanded to the sixth order and \( \mathcal{N}(H) \) scales like \( N^{1/3} \). The special form of \( \phi(m) \) is responsible for these unusual mean-field exponents.

Note that it is crucial to take into account the contributions of order \( 1/N \) in the local fields \( f_i \) in order to obtain the correct result. This is due to the fact that the function \( \phi(m) \) is zero at the saddle point, so the associated complexity is zero. (This is different from the situation found in the mean-field spin glass model discussed in [5].) On the other hand, the subdominant terms play no role in determining the fraction of metastable states with magnetization \( m \) at the field \( H \), which is only given as usual by the fluctuations around the saddle point:

\[
\frac{\mathcal{N}(m, H)}{\mathcal{N}(H)} \sim \sqrt{\frac{-N \phi''(m_0)}{2\pi}} e^{\frac{N \phi''(m_0)}{2} (m - m_0)^2}. \tag{11}
\]

This becomes a \( \delta \)-distribution when \( N \to \infty \) if \( \phi''(m_0) \neq 0 \).

More generally, we can compute all the moments \( \mathcal{N}(H)^n \) and use this information to obtain the full probability distribution \( P(q) = \frac{\delta_k(\mathcal{N}(H) - q)}{q^n} \) in the limit \( N \to \infty \) (with \( \mathcal{N}(H)^n \equiv q^n = \sum q^n P(q) \)). This amounts to counting all possible ways of ordering \( n \) magnetizations \( M_1, M_2, \ldots, M_n \), given that \( \mathcal{N}(M_i, H)\mathcal{N}(M_j, H) = \mathcal{N}(M_i, H) \) when \( M_i = M_j \) since \( \mathcal{N}(M, H) \) is just 0 or 1. This yields

\[
\mathcal{N}(H)^n = \sum_{r=1}^{n} \sum_{n_1, n_2, \ldots, n_r \geq 1} (n_1, n_2 \ldots n_r)! \sum_{M_1, M_2, \ldots, M_r} > \mathcal{N}(M_1, H)\mathcal{N}(M_2, H), \ldots, \mathcal{N}(M_r, H), \tag{12}
\]

where \((n_1, n_2, \ldots, n_r)! = n!/(n_1!n_2! \ldots, n_r!)\) is a multinomial coefficient and the sum runs over all \((n_1, n_2, \ldots, n_r)\) such that \( \sum_{i=1}^{r} n_i = n \) and \( n_1, n_2, \ldots, n_r \geq 1 \). The notation \( \sum > \)
indicates that the sum over the magnetizations $M_1, M_2, \ldots, M_r$ is restricted to a specific order, say $M_r > M_{r-1} > \cdots > M_2 > M_1$. We then define $M_i = 2P_i - N$, where $P_i$ is the number of spins up, and introduce the (strictly) positive quantities $Q_i = P_i - P_{i-1}$ ($i = 2, \ldots, r$). It is easy to see that equation (7) generalizes to

$$\frac{\mathcal{N}(M_1, H)\mathcal{N}(M_2, H)\cdots\mathcal{N}(M_r, H)}{N!} = \frac{p(m_1 - 1/N)^{P_1}}{P_1!} \times \prod_{i=2}^{r} \frac{[p(m_i - 1/N) - p(m_{i-1} + 1/N)]^{Q_i}}{Q_i!} \times \left[ 1 - p(m_r + 1/N) \right]^{N - P_1 - \sum_{i=2}^{r} Q_i}.$$  \hfill (13)

Since only the magnetizations in the close neighborhood of $m_0$ contribute to the sum $\Sigma^r$ when $N \gg 1$ (with the $Q_i$ being at most of the order $N^\alpha$ with $0 < \alpha < 1$), we can expand $p(m_i \pm 1/N)$ around $p(m_0)$ to first order in $1/N$ and use $1 - P_i/N \sim (1 - m_0)/2 = 1 - p(m_0)$. After some straightforward manipulations, we then obtain

$$\lim_{N \to \infty} \frac{\mathcal{N}(M_1, H)\mathcal{N}(M_2, H)\cdots\mathcal{N}(M_r, H)}{\mathcal{N}(H)} = \left[ \frac{\sum_{Q \geq 1} (Q - 1)^Q}{Q!} (2JP_0^* e^{-2JP_0^*})^Q \right]^{r-1},$$

and, after inserting this expression into equation (12),

$$\lim_{N \to \infty} \frac{\mathcal{N}(H)}{\mathcal{N}(H)} = \sum_{n=1}^{\infty} \sum_{n_1, n_2, \ldots, n_r \geq 1} (n_1, n_2, \ldots, n_r)! [a(m_0) - 1]^{r-1},$$  \hfill (15)

where

$$a(m_0) = \sum_{k \geq 0} \frac{(k - 1)^k}{k!} (2JP_0^* e^{-2JP_0^*})^k. $$  \hfill (16)

We recognize in (16) the series expansion of $z/[W(z)(1+W(z))]$ near the origin, where $W(z)$ is the so-called Lambert function, defined as the root of the equation $W(z)e^{W(z)} = z$ [6]. This series converges for $|z| < 1/e$, which is always true in equation (16) where $z = -2JP_0^* e^{-2JP_0^*}$. (The series (16) only refers to the principal branch $W_0(z)$ which takes on values between $-1$ to $+\infty$ for $z \geq -1/e$ and is analytic at $z = 0$.) As a result,

$$a(m_0) = \frac{-2JP_0^* e^{-2JP_0^*}}{W_0(-2JP_0^* e^{-2JP_0^*})[1 + W_0(-2JP_0^* e^{-2JP_0^*})]},$$

which yields

$$a(m_0) = \begin{cases} 
\frac{e^{-2JP_0^*}}{1 - 2JP_0^*} = \frac{\mathcal{N}(H)}{\mathcal{N}(H)} & \text{if } 2JP_0^* < 1, \\
\frac{e^{W_0(-2JP_0^* e^{-2JP_0^*})}}{1 + W_0(-2JP_0^* e^{-2JP_0^*})} & \text{if } 2JP_0^* > 1.
\end{cases}$$  \hfill (18)
Knowing from equation (15) all the moments $q$ of the probability distribution $P(q)$, we can build the generating function

$$\sum_{q \geq 0} e^{-\lambda q} P(q) = 1 + \sum_{n \geq 1} (-\lambda)^n q^n$$

(19)

to obtain

$$\sum_{q \geq 0} e^{-\lambda q} P(q) = \frac{1 + (a - \overline{q})(e^\lambda - 1)}{1 + a(e^\lambda - 1)},$$

(20)

with $\overline{q} \equiv N(H)$. This equation can be inverted, showing that $P(q)$ decreases exponentially for $q \geq 1$. More precisely, we have

$$P(0) = 1 - \frac{\overline{q}}{a},$$

$$P(q) = \frac{\overline{q}}{a(a - 1)} \left( \frac{a - 1}{a} \right)^q$$

for $q \geq 1$,

(21)

where both $\overline{q}$ and $a$ are functions of $m_0(H)$.

For $\Delta > \Delta_0$, $a = \overline{q}$, so $P(0) = 0$ and the most probable value of $N(H)$ is $q = 1$, as could be expected. (What is perhaps less expected is that $P(q) \neq 0$ for $q > 1$ and that $\overline{q} = N(H) > 1$.) For $\Delta < \Delta_0$, the most probable value of $N(H)$ on the intermediate branch is $q = 0$ except very close to the spinodal endpoints where it is again $q = 1$ (for $1 < 2JP_0 < 1.0073$, $P(1) = \overline{q}/a^2 > P(0)$). Note also that $P(q)$ decreases more and more slowly when approaching the spinodals, as the inverse characteristic scale $\xi^{-1} = \ln(1 - 1/a) \to 0$. In all cases, there is a non-zero probability of observing a finite number of metastable states at a given field, as illustrated in figure 3, that results from an exact enumeration of all metastable states in 5000 disorder realizations of size $N = 20000$ at $H = 0$ for $\Delta = 1$ and 0.5 (in the latter case, only the states in the vicinity of the intermediate branch, i.e. around $m_0 = 0$, are counted). One has $a = \overline{q} \approx 2.12$ for $\Delta = 1$, and $\overline{q} \approx 0.340$, $a \approx 1.324$ for $\Delta = 0.5$. The numerical data shown in the figure are in very good agreement with the predictions of equation (21).

The above calculations can be generalized to $2, 3, \ldots, k$-spin-flip stable states, i.e. to spin configurations whose energy cannot be lowered by the flip of any subset of $1, 2, \ldots, k$ spins [8]. It is easy to see that a configuration with $P$ spins up and $N - P$ spins down is $k$-stable if it is $(k - 1)$-stable and if the random fields on the $P$ spins up satisfy $\sum_{\alpha=1}^k h_{i_{\alpha}} > -k(Jm + H) + k^2J/N$ whereas the fields on the $N - P$ spins down satisfy $\sum_{\alpha=1}^k h_{i_{\alpha}} < -k(Jm + H) - k^2J/N$ for any subset $\{i_1, i_2, \ldots, i_k\}$. From this, one can for instance compute the average number of two-spin-flip stable states at the field $H$ and find that

$$\overline{N}^{(2)}(H) \to \frac{[2e^{-2JP_0} - e^{-3JP_0}]^2}{|1 - 2JP_0|}$$

(22)

\[\text{\footnotesize The present calculations show that there is not a unique metastable configuration in the thermodynamic limit, contrary to what is suggested by [7]. Of course, this has no influence on the macroscopic properties discussed by these authors.}\]

\[\text{\footnotesize In the non-equilibrium evolution, however, the one-spin-flip dynamics imposes a well-defined path among these states that depends on the field history.}\]

\[\text{\footnotesize doi:10.1088/1742-5468/2008/10/P10004}\]
Stable, metastable and unstable states in the mean-field RFIM at $T = 0$

Figure 3. Probability of finding $q$ metastable states in zero field for $\Delta = 1$ (blue circles) and $\Delta = 0.5$ (red squares) in the mean-field RFIM (the statistics is over 5000 disorder realizations of size $N = 20000$). For $\Delta = 0.5$, only the states in the vicinity of the intermediate branch (around $m = 0$) are counted. The dashed lines are guides for the eye.

when $N \to \infty$. The comparison with $\mathcal{N}^{(1)}(H) \equiv \overline{\mathcal{N}(H)}$ is shown in figure 2. Of course, one has $\mathcal{N}^{(k)}(H) \leq \mathcal{N}^{(k-1)}(H) \cdots \leq \mathcal{N}^{(1)}(H)$. We have not investigated the behavior for $k \sim \sqrt{N}$ [9]. In any case, in order to go from a metastable state on the intermediate branch to the ground state on the lower or upper branch, one needs to flip a number of spins of order $N$ (which corresponds to taking into account another solution of the saddle-point equation).

4. Conclusion

In this paper, we have computed the probability distribution of the number of metastable states along the so-called ‘unstable’ branch of the mean-field RFIM at $T = 0$ and shown that the average number remains finite in the thermodynamic limit.

The presence of a few metastable states along the intermediate part of the curve $m_0(H)$ for $\Delta < \Delta_c$ can be considered as a precursor of the phenomenology observed in the $T = 0$ RFIM with finite-range exchange interaction. This will be investigated in a forthcoming paper [10] dealing with random graphs of large but finite connectivity $z$. Preliminary results indicate that, as soon as $z$ is finite, a strip of finite width develops around the curve $m_0(H)$ in the field–magnetization plane, a strip in which the density of the typical metastable states scales exponentially with the system size. This occurs both above and below $\Delta_c$. As the connectivity decreases, one expects the strip to widen but to remain distinct from the actual hysteresis loop in the regime of weak disorder. In fact, as suggested in [11,12], one may associate the discontinuity in the hysteresis loop below $\Delta_c$ with the existence of a gap in the magnetization of the metastable states beyond a
Stable, metastable and unstable states in the mean-field RFIM at $T = 0$

certain value of the field. Of course, all this is strictly valid only at $T = 0$. However, as is well known, free-energy barriers are very large in random-field systems and thermally activated processes are not expected to play a significant role on experimental time scales, at least at low temperature. Therefore, the above picture is expected to be relevant to real situations. In particular, the presence of metastable states (and not simply unstable ones as in pure systems) in the central part of the hysteresis loop means that this region could be experimentally accessible, for instance by controlling the magnetization instead of the magnetic field (and more generally the extensive variable conjugated to the external field). This can be put in relation with the re-entrant hysteresis loops that are observed in some magnetic systems [13] or in shape-memory alloys [14] (see also the discussion in [15]).

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