Shor’s algorithm with fewer (pure) qubits

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Abstract

In this note we consider optimised circuits for implementing Shor’s quantum factoring algorithm. First I give a circuit for which none of the about $2^n$ qubits need to be initialised (though we still have to make the usual $2^n$ measurements later on). Then I show how the modular additions in the algorithm can be carried out with a superposition of an arithmetic sequence. This makes parallelisation of Shor’s algorithm easier. Finally I show how one can factor with only about $1.5n$, and maybe even fewer.

1 Introduction

Over the years people have looked into how Shor’s quantum factoring algorithm could actually be implemented on a quantum computer. In particular people have tried to optimise the algorithm (the quantum part of it). An early work was by César Miquel in 1996 [1]. Later, people were mostly trying to reduce the size of a quantum computer needed for factoring (thus the number of qubits). The circuit of Stéphane Beauregard from 2003 [2] uses only $2n + 3$ qubits to factor an $n$-bit number.

An important observation has always been that much of the computation can be done classically. A lot of quantities can be “precomputed” conventionally. For the most part the quantum circuits are just classical reversible circuits to compute a modular exponentiation, which is reduced to many (actually some $4n^2$) modular additions of $n$-bit numbers. A non-classical circuit is Tom Draper’s addition technique from 2000 [3] using a quantum Fourier transform which allows to do without the additional $n$ work qubits which are usually needed to hold the carry bits.

The first observation in this context which I present is more of theoretical than of practical interest. Parker and Plenio (4, 2000) have pointed out that $n$ of the $2n$ qubits used in these implementations need not be initialised (set to $|0\rangle$) at the beginning. (But in the title and abstract of their paper they seem to claim even more than that). I now show how the algorithm can be modified so that actually none of the about $2n$ qubits need to be initialised. Still, in mine as in their algorithm we have to do some $2n$ measurements in
the course of the algorithm. As measuring a qubit also resets it, it would therefore not be correct to say that we need no pure qubits at all. The technique works by modifying the (controlled) modular multiplications. Instead of the usual modular multiplication $|\alpha, 0\rangle \rightarrow |a \cdot \alpha, 0\rangle$, (where $a$ is a classically known number) I show how to do $|\alpha, \beta\rangle \rightarrow |a \cdot \alpha, a^{-1} \cdot \beta\rangle$ without need for further work qubits. Instead of the usual 2 modular multiplications for this step (one by $a$, the other by $a^{-1}$, everything mod $N$), I now need 3 of them. In a further observation I show how the slowdown caused by this can be compensated by using a “trinary” quantum Fourier transform.

The second observation is of a different kind. It looks at the modular additions (modulo the number $N$ to be factored) out of which the algorithm essentially consists. The fact that these are not usual additions, but modular ones causes some complications (effectively several additions have to be made). In particular when using Draper’s “Fourier-addition” we need to Fourier transform back to the usual basis after each addition, which is costly. I now propose an approximate way to compute modular additions with a quantum method (as opposed to a classical reversible circuit). We need to make the $n$-qubit registers larger by some $O(\log n)$ qubits. Then instead of a number $|b\rangle$ with $0 \leq b < N$, we consider an equal (“uniform”) superposition $\sum_x |b + xN\rangle$. Using this technique allows to simplify the overall outline of the algorithm and may allow a considerable speedup, mainly because it allows a lot of parallelisation. The depth of the overall circuit for Shor’s algorithm can then be reduced to $O(n^2)$.

The third result gives a sizable reduction in the number of qubits needed in Shor’s algorithm. Namely from the present number of about $2n$ qubits (Beau-regard) to about $1.5n$. I show two things. One is how to replace modular multiplication by an $n$-bit factor with two multiplications by factors half this length. Then I show how such “short” multiplications can be done with accordingly fewer qubits. (If somehow we manage to further factor these multipliers, further reductions down to closer to $1.0n$ would be possible.) The presentation, especially for this third result, is sketchy.

## 2 Review of circuits for Shor’s algorithm

The number $N$ we want to factor is an $n$-bit number, thus $n = \lceil \log_2 N \rceil$ (log_2 N rounded up). $N$ may e.g. be the product of two unknown large prime numbers, thus $N = p \cdot q$. The problem of factorisation can be reduced to finding the period of the periodic function $f(x) = a^x \mod N$, where $a$ is essentially an arbitrary integer (really $0 < a < N$ and $a$ should be coprime with $N$). The “order finding” quantum algorithm to do this can be described as follows. We have a $2n$-qubit register and an $n$-qubit register:

$$
\sum_{x=0}^{2^n-1} |x\rangle|0\rangle \rightarrow \sum_{x=0}^{2^n-1} |x\rangle|a^x \mod N\rangle \rightarrow \sum_{x=0}^{2^n-1} FT(|x\rangle) |a^x \mod N\rangle
$$

After Fourier transforming the first ($2n$-qubit) register, we measure it. (From the result, the period of $a^x \mod N$ can be computed classically with good
success probability.) The modular exponentiation can be computed by noting (everything is understood to be mod \(N\)):

\[
a^x = x_0 + 2x_1 + \ldots + 2^{2n-1} x_{2n-1} = a^{x_0} \cdot \left(a^{2}\right)^{x_1} \cdot \left(a^{2^2}\right)^{x_2} \cdot \ldots \cdot \left(a^{2^{2n-1}}\right)^{x_{2n-1}}
\]

where \(x_0, x_1, \ldots\) are the bits of \(x\). The numbers \(a^{x_i}\) can be computed through repeated squaring (at every step reducing mod \(N\)). Thus in the end we really only have to make a sequence of (modular) multiplications, each conditioned on a different bit of \(x\). Now look at the subsequent Fourier transform and measurement of the \(x\)-register. Fourier transform and measurement can be combined (“Fourier sampling”, “semiclassical Fourier transform”) so that we first measure the high bits of the \(x\)-register (one after the other, starting from the highest value one). Indeed the whole procedure can be described as simply measuring one qubit after another, each in a basis depending on the previous measurement outcomes (of the higher qubits). Given that the \(x\)-register is initially in a product state, we see that we need not keep these \(2n\) qubits all at the same time. For each \(x\)-qubit we can prepare it in state \(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\), then use it to control a modular multiplication and then measure it in the appropriate basis. (Note that for this to work, we have to carry out the multiplications with \(a^{2^i}\) in “reversed order”, thus \(i = 2n - 1, \ldots, 1, 0\).) Thus instead of the whole \(2n\)-qubit \(x\)-register, now we need only 1 qubit.

Each modular multiplication is of the form \(|\alpha\rangle \rightarrow |a \cdot \alpha\rangle\), where \(\alpha\) is the number initially in the register, and we simply use “\(a\)” for any number we want to multiply with (really it’s a power of \(a\)). Parker and Plenio [4] have pointed out that if we don’t initialise the register, we effectively compute \(\alpha \cdot a^x\ mod\ N\) instead of \(a^x\ mod\ N\), where \(\alpha\) is the number that happens to be initially in the register. (Why we can think of “not initialised” in this way is explained later.) For most \(\alpha\) this function \((x)\) has the same periodicity as the original one (namely for all \(\alpha\) that are coprime with \(N\)). So Parker and Plenio said that none of the initial qubits needed to be initialised. For this they assumed that they had a unitary operator that would modularly multiply a register with any given \(a\) coprime with \(N\) (indeed this operation is reversible and thus unitary): \(U_a : |\alpha\rangle \rightarrow |a \cdot \alpha\rangle\). But we don’t know how to do this efficiently without a supply of another \(O(n)\) properly initialised qubits. In the usual technique we use an auxiliary \(n\)-qubit register and make two steps whereby we add a multiple of one register to the other register (everything mod\(N\)):

\[
|\alpha, 0\rangle \rightarrow |\alpha, a \cdot \alpha\rangle \rightarrow |\alpha - a^{-1} \cdot (a \cdot \alpha), a \cdot \alpha\rangle = |0, a \cdot \alpha\rangle
\]

where in the second step we use (modular) multiplication with \(a^{-1}(\mod N)\) (which we can precompute classically using Euclid’s algorithm).

### 3 Not initialising any qubits

The above sequence of operations doesn’t give the desired result if the auxiliary register is not initialised to \(|0\rangle\), but this can easily be fixed. We start by applying
the usual two steps to registers in any initial state but add an appropriate third step:

$$|\alpha, \beta\rangle \rightarrow |\alpha, \beta + a\alpha\rangle \rightarrow |\alpha - a^{-1}(\beta + a\alpha)\rangle = -a^{-1}\beta, \beta + a\alpha\rangle \rightarrow |-a^{-1}\beta, a\alpha\rangle$$

Thus by adding a SWAP and multiplication by $-1$ (which is rather easy), we can do $|\alpha, \beta\rangle \rightarrow |a \cdot \alpha, a^{-1} \cdot \beta\rangle$. This is just as good for order finding, as the function $(a^x, a^{-x} \beta)$ has still the same period which we were looking for (at least for most numbers $\alpha, \beta$). We can also write the sequence of operations as $2 \times 2$ matrices (acting on a pair of numbers in $Z_N$):

$$
\begin{pmatrix}
-1 & 1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & a \\
1 & a
\end{pmatrix}
\begin{pmatrix}
1 & -a^{-1} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha & \beta \\
\alpha & a^{-1}\beta
\end{pmatrix}
= 
\begin{pmatrix}
a \cdot \alpha & a \cdot \beta \\
a^{-1}\beta & \beta
\end{pmatrix}
$$

The technique which was described here for modular multiplication actually works in any group setting and can thus also be used for Shor’s discrete logarithm (dlog) algorithm.

Finally a remark on what it means to not initialise some qubits. Clearly the algorithm may fail if e.g. someone has intentionally prepared the qubits in some “bad” state. But we can always ensure that they are in the maximally mixed state, namely by randomly applying one of the four Pauli operators to each qubit. I thus have simply assumed that “uninitialised registers” are in this maximally mixed state, which is equivalent to thinking that the register is at random in one of the “computational” basis states $|\alpha\rangle$.

Actually so far we have only shown that the two main registers involved in Shor’s algorithm need not be initialised, but there are a few more qubits, e.g. in Beauregards circuit we need 3 more qubits. As this is a constant (and small) number, we still get a reasonable success probability for Shor’s algorithm if we don’t initialise these qubits, too. Thus one may claim that none of the initial qubits need to be initialised.

3.1 Saving time with a trinary quantum Fourier transform

The multiplication steps are conditioned on bits of $x$, thus either we multiply (modularly) with $a$, or we do nothing. But if we can do $|\alpha, \beta\rangle \rightarrow |aa, a^{-1} \cdot \beta\rangle$, it is clear that by swapping the 2 registers before and after the operation, we can also do the inverse: $|\alpha, \beta\rangle \rightarrow |a^{-1} \alpha, a \beta\rangle$. Note that such controlled swaps are cheap compared to the multiplication steps. Thus effectively we can multiply (say the first register) with either $a$, 1, or with $a^{-1}$, depending on some control qubits. (Note that this “control” hardly increases the cost as the individual modular additions, out of which the multiplications consist, are anyways conditional.) Thus is makes sense to take advantage of all three possibilities. Multiplication with $a$, 1, or $a^{-1}$ is very similar to multiplication with $a^2$, $a$, or 1, it only shifts the final (periodic) function by a constant amount: $f(x) \rightarrow f(x - \text{const.})$. Thus we now imagine that $x$ is given in “trinary” form $x = x_0 + 3x_1 + 3^2x_2 + \ldots + x_{2n'-1}3^{2n'-1}$, where $n'$ is smaller than $n$ by about a factor of $\log_2 3$. Note that
it is not important whether we store the bits of \( x \) in actual physical “qutrits” or whether we e.g. use 2 qubits.

The final Fourier transform in Shor’s algorithm now has to be replaced with a trinary one (of order \( 3^{2n'} \)). Everything works just as well with such a trinary quantum Fourier transform, in particular the “semiclassical” version is analogous and we can essentially proceed as before: Prepare a “qutrit” (possibly realised with 2 qubits) in an equal superposition. Use it to control a modular multiplication (now 3 possibilities!). Then measure the “qutrit” in a basis determined by the previous measurement outcomes.

For deriving the circuit for a trinary (or any “\( p \)-ary” of order \( p^n \)) quantum Fourier transform and seeing that this works, I only give some hints. I find it useful to consider the transform of a basis state \( |b\rangle \rightarrow 1/\sqrt{\ldots\sum_x \omega^{bx} |x\rangle} \). It turns out that this is again a product state, where the state of a given “qudit” depends only on some of the original digits of \( b \) (from \( b_0 \) up to a maximum).

Start by seeing how the one qudit which depends on all original digits can be obtained (namely by a generalised Hadamard transform, followed by controlled phases). From there on down, each qudit can be obtained in a similar way. Thus the circuit has the same structure for any “\( p \)-ary” quantum Fourier transform.

The dominant cost in the controlled multiplications are the individual multiplication steps. In the modified algorithm we have 3 instead of 2 of these. But this slowdown is more than compensated through the use of the trinary Fourier transform. Thus we now even use a bit less time by a factor \( 3/2 \cdot \log_3 2 \approx 0.946 \ldots \)

4 Modular addition with equal “coset superpositions”

Most of Shor’s algorithm actually consists of (conditional) modular additions of a fixed (classically known) number to a quantum register: \( |\alpha\rangle \rightarrow |\alpha + a \mod N\rangle \). This operation is of course reversible (just subtract \( a \) modulo \( N \)), but it can be a bit cumbersome to implement it. If we first simply add and only then reduce modulo \( N \) (thus possibly subtracting \( N \)), then this last step would not be reversible. What one can do is to first make a comparison (of the quantum register with a suitable “classical” number) which determines whether \( N \) will have to be subtracted. Then with the final result in the quantum register, one can make another suitable comparison to “uncompute” this control qubit. Each of these comparisons essentially amounts to an addition (or rather a subtraction). And, what is worse, when using Draper’s Fourier-addition technique, each time one has to Fourier transform back to the usual basis for reading out the result of a comparison.

I now propose a modular addition technique which is faster, although it uses a few more qubits. Instead of representing a number in \( \mathbb{Z}_N \) simply by a number \( b \) in the range \( 0 \ldots N-1 \), we prepare an equal superposition of many terms of the arithmetic sequence \( b + x \cdot N \) with \( x = 0, 1, \ldots \). Thus we do \( |b\rangle \rightarrow \sum_x | b + xN \rangle \).
It is enough if the new register is larger by some $O(\log n)$ qubits (actually some $2\log n + 10$ qubits should be more than enough for Shor’s algorithm) and the range of $x$ will be accordingly.

Now we simply add (non-modularly) the number $a$ to this register:

$$\sum_{x=0}^{x_{\text{max}}-1} |b + xN\rangle \rightarrow \sum_{x=0}^{x_{\text{max}}-1} |a + b + xN\rangle$$

The point is that the outcome is close to the (desired) outcome $\sum_{x} |(a + b) \mod N + xN\rangle$. Thus $\sum_{x} (a + b + xN) \approx \sum_{x} ((a + b) \mod N + xN)$, in the sense that the fidelity (overlap) is close to 1. The “ladder” of peaks in the superposition simply may get shifted by one period. So the loss of fidelity per addition is on the order of $O(1/x_{\text{max}})$ and thus can be made very small.

### 4.1 Converting to the “coset representation”

First let’s see how we can transform back and forth between the usual “representative element” representation (of a number in $\mathbb{Z}_N$) $|b\rangle$ and the “coset superposition” representation $\sum_{x} |b + xN\rangle$. (Here the coset is given by the elements $b + xN$ of the arithmetic sequence, the subgroup being the multiples of $N$ and the overall group are the integers $\mathbb{Z}$.) Given $|b\rangle$, we need another small register prepared in an equal (“uniform”, “flat”) superposition $1/\sqrt{x_{\text{max}} \sum_{x} |x\rangle}$. It is easiest to prepare this superposition if $x_{\text{max}}$ is a power of 2. (It may be advantageous to choose a different $x_{\text{max}}$, and in this case, too it is not difficult to prepare the equal superposition.) Then we apply a simple (classical reversible) operation to these 2 registers. Thus:

$$\text{prepare } \frac{1}{\sqrt{x_{\text{max}}}} \sum_{x=0}^{x_{\text{max}}-1} |x\rangle, \quad \text{then do } |b\rangle|x\rangle \rightarrow |b + xN\rangle$$

This last step is not hard to carry out. Imagine that we first extend the range of the first (“$b$”) register (by adding a few qubits in state $|0\rangle$ at the top end of the register). Then we go through the bits of $x$ from least significant to most significant. Conditioned on bit $x_i$ we add $2^i N$ to the first register. Then we uncompute $x_i$ by checking whether the content of the first register is $\geq 2^i N$.

### 4.2 Use in Shor’s algorithm

First a quick review of how modular multiplications are decomposed into $n$ modular additions. (Actually this is similar to the decomposition of the modular exponentiation into modular multiplications.) The individual modular multiplication steps are of the form $|\alpha\rangle|\beta\rangle \rightarrow |\alpha\rangle|\alpha a + \beta\rangle$ (again “$a$” stand for any power of $a$). We write (everything mod $N$):

$$\alpha a = (\alpha_0 + 2\alpha_1 + \ldots) a = \alpha_0 a + \alpha_1 (2a) + \alpha_2 (2^2 a) + \ldots$$
The numbers $2^j a$ (really we will have $2^j a^2$) can again be precomputed and reduced mod $N$. Thus everything is decomposed into $n$ controlled modular additions of precomputed integers in the range $0 \ldots N - 1$. Each addition is conditioned on a different qubit of the first (“α-”) register.

We can leave the “accumulation register” in the coset representation during all these $n$ additions. Under the plausible assumptions that the precomputed numbers on average have a size of about $N/2$ and that on average we add up only about half of these numbers, the sum will be around $n/2 \cdot N/2$. I think that by subtracting (from the accumulation register) a multiple of $N$ close to this, we can improve the fidelity (thus reduce the error of the modular addition technique).

So for a modular multiplication step $|\alpha, \beta\rangle \rightarrow |\alpha, \beta + a\alpha\rangle$ we would leave the second register through all $n$ modular additions in the “coset” representation. Usually we would imagine that the first (“control”) register would be in the usual representation, but actually we can also leave it in the coset representation. Essentially this means that instead of multiplying with $\alpha$ we may multiply with some $\alpha + x N$, but modulo $N$ this doesn’t make any difference. A disadvantage is that we now would have to carry out some $O(\log n)$ more modular addition steps (per modular multiplication) and also that the first register needs this many more qubits.

Thus in practice we may prefer to switch the first register back to the usual representation while it acts as a control register. But if we don’t, the layout of Shor’s algorithm becomes quite simple. We have to convert the 2 registers to the “coset representation” only at the beginning, and later no switching back will be needed. Each multiplication step simply consists of a sequence of $n + O(\log n)$ regular additions. The only thing that doesn’t look simple are the “strange” precomputed numbers we have to add... :-)

Finally note that while modular multiplication can be carried out with both registers in the “coset representation”, this wouldn’t work as well for an addition of the form $|\alpha, \beta\rangle \rightarrow |\alpha, \beta + \alpha\rangle$ as then the fidelity loss would be large, at least for the present scheme.

Note that the additions we do, naturally come out to be modulo the size $2^d$ of the register, independently of whether we use a classical reversible method or Draper’s Fourier-addition. This is no problem for the approximative modular addition described here. We can choose the parameters such that a modular reduction never occurs, but even if it does, the error per modular addition is still small.

If we use Draper’s Fourier addition, we can leave the second (“target”) register Fourier transformed for a whole sequence of additions. This allows for a lot of parallelisation, as each addition can be carried out in a single time step. (I think approximative classical reversible techniques exist which allow a similar parallelisation of addition and also don’t need auxiliary work qubits. Here “approximative” would mean that the circuit works correctly for all but a few inputs, which should be good enough for Shor’s algorithm. For such techniques see e.g. my 1998 work on implementing Shor’s algorithm.)
4.3 Error estimate

The question is how large we have to make the “equal coset superposition” \( \sum_x |b + xN\rangle \) (with \( x = 0 \ldots x_{\text{max}} - 1 \)) to get a good enough approximation for Shor’s algorithm. Roughly we can argue that in each modular addition we lose about \( 1/x_{\text{max}} \) in fidelity. In order to keep the overall fidelity loss of the \( 4n^2 \) additions at, say, below 1%, we see that \( x_{\text{max}} \approx 1000n^2 \) should be more that enough. This corresponds to using some \( 2 \log n + 10 \) additional qubits for the coset representation of a register. Note that this “analysis” is not rigorous as it assumes that losses in fidelity simply add up. While in general errors can behave worse than that, I think that in the present case the assumption is correct. (A rather easy “worst case” analysis shows that an \( x_{\text{max}} \) on the order of \( n^4 \) is provably enough.)

4.4 Related work

In 2000 Hales and Hallgren [5] have published an improved approximate technique to carry out the quantum Fourier transform for any order, e.g. for a large prime. Their technique is simpler, faster and uses fewer qubits than Kitaev’s earlier method. One might think of using their technique to carry out a quantum Fourier transform of order \( N \) on the register to which we have to add numbers modulo \( N \). Then Draper’s Fourier addition technique could be used to directly do additions modulo \( N \). Actually the “coset superposition” technique which I propose is very similar to this. Hales and Hallgren also first carry out the conversion \( |b\rangle \rightarrow \sum_x |b + xN\rangle \) and then Fourier transform the whole register (thus modulo a power of 2). Because they have Fourier transformed a “periodic” state, they get peaks which they then “extract” to get the final result. If in conjunction with my “coset superpositions” we also use Draper’s Fourier addition (modulo some \( 2^n \)), we thus essentially do the same as Hales and Hallgren, except their last step, the “extraction” of the Fourier peaks, which fortunately turns out to not being necessary.

Also I understand that John Watrous [6] has been using uniform superpositions of subgroups (and cosets) in his work on quantum algorithms for solvable groups. Thus he also used coset superpositions to represent elements of the factor group (and probably also to carry out factor group operations on them). In our case the overall group are the integers, the (normal) subgroup are the multiples of \( N \). The factor group who’s elements we want to represent is \( Z_N \equiv Z/(NZ) \). We now represent these elements by superpositions over the cosets of the form \( b + xN \). A problem in our case is that we can do things only approximatively as the integers and the cosets are infinite sets.

4.5 Some wishful thinking...

As I have pointed out it has already been shown that much of the computation in Shor’s algorithm can be carried out classically (as pre- and post-processing). It would be nice if the quantum part of Shor’s algorithm could be further reduced
(at the expense of a “reasonable” amount of additional classical computation).

E.g. it is not clear whether maybe a modular multiplication step could not be
simplified. It is not even clear whether we could not carry it out with only $O(n)$ quantum gates and maybe also with only one $n$-qubit register, although I doubt that this is possible.

It would also be nice to find better quantum techniques for other modular arithmetic operations. Namely a while ago I have unsuccessfully tried to think about how “modular inversion” $x \rightarrow x^{-1}(\mod N)$ could be done more elegantly (maybe using Fourier transforms or the like) than the classical (and “classic” :- ) technique. In 2003 with John Proos in a work on elliptic curves we used a cumbersome reversible implementation of Euclid’s algorithm to do that.

Also note that further simplifications of Shor’s algorithm might lead to some insights (into quantum computation). E.g. if the quantum part could be reduced to some more natural operations. Also nice would be if we could do without any pure qubits and projective measurements, like the trace estimation problem of Knill and Laflamme in their “power of one qubit” paper. Another possibly practically useful advance would be to “break up” Shor’s algorithm into several smaller (quantumly) independent quantum parts, but again I don’t see how this could be achieved. One can also investigate the possibility to replace the modular multiplications by numbers of the form $a^2 \mod N$ by multiplications with other powers of $a$ or with any other suitably chosen numbers.

5 Shor’s algorithm with $1.5n$ qubits (maybe less)

5.1 Modular multiplication with smaller factors

Usually we have to multiply (mod $N$) a quantum register with a fixed $n$-bit number. Here I show that if this classical factor is shorter, we can accordingly save qubits. Later I will show how we can do Shor’s algorithm with such shorter factors, namely how we can replace a single $n$-bit factor by two $n/2$-bit factors.

So say the classical factor “$a$” we want to multiply with has only $n’$ bits with $n’ < n$, e.g. $n’ = n/2$. We want to do $|\alpha\rangle \rightarrow |a\cdot \alpha \mod N\rangle$ while using fewer work qubits than usually. I propose the following sequence of three steps:

$$\alpha \ (n \text{ bits}) \rightarrow a\cdot \alpha \ (n + n’ \text{ bits}) \rightarrow a\cdot \alpha \mod N(= a\cdot \alpha - q\cdot N) , q \ (n + n’ \text{ bits}) \rightarrow a\cdot \alpha \mod N \ (n \text{ bits})$$

The first step is a usual multiplication (not modular). In the second step we divide by $N$, getting the remainder and the (integer) quotient $q$. In the last step we “uncompute” the quotient.

The first step (normal multiplication) is rather straight-forward. It is a sequence of controlled additions, conditioned on bits of $\alpha$ after each of which the controlling bit is uncomputed. Note that this way the total number of qubits is always at most $n + n’$. We can go through the bits of $\alpha$ both ways but it’s a bit easier if we go from most significant to least significant as then uncomputing
the bits of $\alpha$ is easiest. (E.g. when $a$ is odd the lowest bit of the evolving sum is simply equal to the bit we need to uncompute.)

The second step is the reduction modulo $N$. It is a usual division consisting of a sequence of subtractions of $2^i \cdot N$ for $i = n', n' - 1, \ldots, 0$. For each subtraction one bit of the quotient $q$ is computed, from most to least significant, each bit indicating whether the corresponding subtraction was done or not.

The third step (uncomputing of $q$) is less straightforward. I show how the reverse can be done, thus computing $q$ from the remainder $a\alpha \mod N = a\alpha - qN$. First consider computing the remainder of this modulo $a$: $(a\alpha - qN) \mod a = -qN \mod a$. Computing the remainder of a quantum register modulo a fixed number is indeed not hard to do. We need an “accumulation” register into which we can add numbers modulo $a$. For each qubit of the original quantum register we then add, conditioned on this qubit, the (classically precomputed) number $2^i \mod a$. To directly obtain $q$ (instead of $-qN \mod a$) we now simply replace these numbers with $2^i(-N)^{-1} \mod a$.

Thus in total we need only $n + n'$ qubits for modular multiplication with factors of size $n'$ (with $n' < n$).

### 5.2 Getting multiplication factors of size $n/2$

Usually in Shor’s algorithm we have to multiply by $n$-bit factors, namely by the numbers $a^2 \mod N$. If we could factor these numbers into smaller numbers, we could accordingly save qubits. Here I show how we can write these numbers as quotients (mod $N$) of two half size numbers. (Note that dividing by such a number is simply the reverse of multiplying.) Say the original factor was $a$. Now consider the extended Euclidean algorithm one would use to compute $a^{-1} \mod N$. In this algorithm we get a sequence of integer linear combinations of $a$ and $N$ of the form $r \cdot a + k \cdot N = r'$. In the course of the algorithm the integer $r$ gets larger while $r'$ gets smaller, while always $|r| \cdot |r'| < N$. By stopping the algorithm in the middle we get two numbers $r, r'$ both of size about $\sqrt{N}$, with $a = r' / r \mod N$. (Note that the integer coefficients $a$ and $k$ can be negative, but $r'$ is usually taken to remain positive.)

Actually it’s not hard to show that there appears always a linear combination (and sometimes two) in Euclid’s algorithm where (the absolute values of) both $r$ and $r'$ are smaller than $\sqrt{N}$.

In summary this gives us a factoring algorithm with about $1.5n$ qubits. In a more careful count I arrive at $1.5n + 2$ qubits (when $n$ is odd, round up). For this I used a modular addition circuit which uses only 1 work qubit and of course also Draper’s Fourier addition technique to save qubits. (By the way, I wonder whether there is maybe a general way to get rid of a single work qubit, e.g. at the expense of increasing the number of gates by some factor, say 8 ...?)

#### 5.2.1 Even smaller factors?

If we could somehow write our $n$-bit factors $a$ as a product (or quotient) of even smaller factors we could further save qubits. One idea might be to try to
(further) factor the \( n/2 \)-bit numbers obtained above. A full factorisation would be hard. (But still less hard than factoring \( n \)-bit numbers... actually one idea is to use the quantum algorithm to do this. This wouldn’t increase the number of qubits, but would increase the quantum running time quite a bit). Easier would be to just look for small primefactors. At any rate for each step we would need to have a choice of numbers which we could try to (partially) factor. Indeed given the linear combinations in the middle of Euclid’s algorithm one can find many \( r, r' \) pairs with sizes of about \( n/2 \) bits. Also the \( n \)-bit factors \( a \) need not be the numbers usually taken. (E.g. one can think of a scheme whereby one is allowed to use essentially arbitrary products of the numbers \( a^2 \mod N \) for small ranges of \( i \).) Note that we can’t hope to find enough fully “smooth” numbers (numbers with only small primefactors), at least not for \( n \)-bit numbers, as otherwise one could use this for efficient classical factoring... but I think that maybe (classical) factorisation into small enough factors is possible to be able to run Shor’s algorithm with only, say, \( 1.1n \) qubits.

6 Remarks

6.1 Useful for future real implementations?

Note that here we haven’t tried to find a “best” (most efficient) circuit for future actual implementations. The tradeoffs one would like to make for physical realisations are not clear, e.g. how many more qubits one is willing to use to speed the algorithm up. Also in reality one probably needs many more qubits for error correction and the spatial arrangement and connections between qubits are an issue. Still, maybe some of the techniques worked out for abstract circuits may one day be useful...

6.2 Optimisation of reversible and quantum circuits

There seem to be quite some opportunities for the optimisation of the quantum part of a circuit (at the expense of a “reasonable” amount of classical computation). If we restrict just to classical reversible circuits the task can be formulated as finding (efficiently computable) short circuits. Note that the possibility to shift work to the classical (fixed) part gives us extra opportunities for optimisation. Also the possibility of doing things only approximately (e.g. wrong for a small fraction of the computational basis states) allows for extra possibilities.

6.3 Conclusions

We have sketched three results. One is that even fewer qubits need to be initialised for factoring than known so far. Then a non-classical modular addition technique with “coset superpositions” was proposed which may be of use for parallelising computations. Finally I have shown how instead of the usual \( 2n + \ldots \) qubits, we can run Shor’s algorithm with only \([1.5n] + 2\) qubits. It seems likely that this can be further reduced, at least somewhat.
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