Ubiquity and a general logarithm law
for geodesics.

Victor Beresnevich∗  Sanju Velani
York York

In memory of Bill Parry

Abstract

There are two fundamental results in the classical theory of metric Diophantine approximation: Khintchine’s theorem and Jarník’s theorem. The former relates the size of the set of well approximable numbers, expressed in terms of Lebesgue measure, to the behavior of a certain volume sum. The latter is a Hausdorff measure version of the former. We start by discussing these theorems and show that they are both in fact a simple consequence of the notion of ‘local ubiquity’. The local ubiquity framework introduced here is a much simplified and more transparent version of that in [4]. Furthermore, it leads to a single local ubiquity theorem that unifies the Lebesgue and Hausdorff theories. As an application of our framework we consider the theory of metric Diophantine approximation on limit sets of Kleinian groups. In particular, we obtain a general Hausdorff measure version of Sullivan’s logarithm law for geodesics – an aspect overlooked in [4].

1 Introduction

1.1 Background: the classical theory

To set the scene, we follow the opening discussion of [4] and introduce a basic lim sup set whose study has played a central role in the development of the classical theory of metric Diophantine approximation. Given a real, positive decreasing function \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \), let

\[
W(\psi) := \{ x \in [0,1] : |x - p/q| < \psi(q) \text{ for i.m. rationals } p/q \ (q > 0) \},
\]

where ‘i.m.’ means ‘infinitely many’. This is the classical set of \( \psi \)-well approximable numbers in the theory of Diophantine approximation. The fact that we have restricted

∗EPSRC Advanced Research Fellow, EP/C54076X/1
our attention to the unit interval rather than the real line is purely for convenience. It is natural to refer to the function $\psi$ as the *approximating function*. It governs the ‘rate’ at which points in the unit interval must be approximated by rationals in order to lie in $W(\psi)$. It is not difficult to see that $W(\psi)$ is a lim sup set. For $n \in \mathbb{N}$, let

$$W(\psi, n) := \bigcup_{k^{n-1} < q \leq k^n} \bigcup_{0 \leq p \leq q} B(p/q, \psi(q)) \cap [0, 1]$$

where $k > 1$ is fixed and $B(c, r)$ is the open interval centred at $c$ of radius $r$. The set $W(\psi)$ consists precisely of points in the unit interval that lie in infinitely many $W(\psi, n)$; that is

$$W(\psi) = \limsup_{n \to \infty} W(\psi, n) := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} W(\psi, n).$$

Investigating the measure theoretic properties of the set $W(\psi)$ underpins the classical theory of metric Diophantine approximation. We begin by considering the ‘size’ of $W(\psi)$ expressed in terms of the ambient measure $m$; i.e. one-dimensional Lebesgue measure. On exploiting the lim sup nature of $W(\psi)$, a straightforward application of the convergence part of the Borel–Cantelli lemma from probability theory yields that

$$m(W(\psi)) = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} k^{2n} \psi(k^n) < \infty.$$ 

Notice that since $\psi$ is monotonic, the convergence/divergence property of the above sum is equivalent to that of $\sum_{r=1}^{\infty} r \psi(r)$.

A natural problem now arises. Under what conditions is $m(W(\psi)) > 0$? The following fundamental result provides a beautiful and simple criteria for the ‘size’ of the set $W(\psi)$ expressed in terms of Lebesgue measure.

**Khintchine’s Theorem (1924)**  
Let $\psi$ be a real, positive decreasing function. Then

$$m(W(\psi)) = \begin{cases} 
0 & \text{if} \quad \sum_{r=1}^{\infty} r \psi(r) < \infty, \\
1 & \text{if} \quad \sum_{r=1}^{\infty} r \psi(r) = \infty.
\end{cases}$$

Thus, in the divergence case, which constitutes the main substance of Khintchine’s theorem, not only do we have positive Lebesgue measure but full Lebesgue measure. To the best of our knowledge, this turns out to be the case for all naturally occurring lim sup sets – not just within the number theoretic setup. Usually, there is a standard argument which allows one to deduce full measure from positive measure – such as the invariance of the lim sup set or some related set, under an ergodic transformation. In any case, we shall prove a general result which directly implies the above full measure statement.
It is worth mentioning that in Khintchine’s original statement the stronger hypothesis that \( r^2\psi(r) \) is decreasing was assumed. The fact that this additional hypothesis is unnecessary has been known for sometime.

Returning to the convergence case, we cannot obtain any further information regarding the ‘size’ of \( W(\psi) \) in terms of Lebesgue measure — it is always zero. Intuitively, the ‘size’ of \( W(\psi) \) should decrease as the rate of approximation governed by the function \( \psi \) increases. In short, we require a more delicate notion of ‘size’ than simply Lebesgue measure. The appropriate notion of ‘size’ best suited for describing the finer measure theoretic structures of \( W(\psi) \) is that of generalized Hausdorff measures. The Hausdorff \( f \)-measure with respect to a dimension function \( f \) is a natural generalization of Lebesgue measure. A dimension function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is an increasing, continuous function such that \( f(r) \to 0 \) as \( r \to 0 \). The Hausdorff \( f \)-measure with respect to the dimension function \( f \) will be denoted throughout by \( \mathcal{H}_f \) and is defined as follows. Suppose \( F \) is a non-empty subset of a metric space \( (\Omega, d) \). For \( \rho > 0 \), a countable collection \( \{B_i\} \) of balls in \( \Omega \) with radii \( r_i \leq \rho \) for each \( i \) such that \( F \subset \bigcup_i B_i \) is called a \( \rho \)-cover for \( F \). For a dimension function \( f \) define

\[
\mathcal{H}_\rho^f(F) = \inf \left\{ \sum_i f(r_i) : \{B_i\} \text{ is a } \rho-\text{cover of } F \right\},
\]

where the infimum is over all \( \rho \)-covers. The Hausdorff \( f \)-measure \( \mathcal{H}_f(F) \) of \( F \) with respect to the dimension function \( f \) is defined by

\[
\mathcal{H}_f(F) := \lim_{\rho \to 0} \mathcal{H}_\rho^f(F) = \sup_{\rho > 0} \mathcal{H}_\rho^f(F).
\]

In the case that \( f(r) = r^s (s \geq 0) \), the measure \( \mathcal{H}_f \) is the usual \( s \)-dimensional Hausdorff measure \( \mathcal{H}^s \) and the Hausdorff dimension \( \dim F \) of a set \( F \) is defined by \( \dim F := \inf \{s : \mathcal{H}_f^s(F) = 0\} = \sup \{s : \mathcal{H}_f^s(F) = \infty\} \). In particular when \( s \) is an integer \( \mathcal{H}^s \) is a constant multiple of \( s \)-dimensional Lebesgue measure. For further details see \[12,14,17\].

Again on exploiting the \( \lim \sup \) nature of \( W(\psi) \), a straightforward covering argument provides a simple convergence condition under which \( \mathcal{H}_f(W(\psi)) = 0 \). Thus, in view of the development of the Lebesgue theory it is natural to ask for conditions under which \( \mathcal{H}_f(W(\psi)) \) is strictly positive.

The following fundamental result provides a beautiful and simple criteria for the ‘size’ of the set \( W(\psi) \) expressed in terms of Hausdorff measures.

**Jarník’s Theorem (1931)**  Let \( f \) be a dimension function such that \( r^{-1}f(r) \to \infty \) as \( r \to 0 \) and \( r^{-1}f(r) \) is decreasing. Let \( \psi \) be a real, positive decreasing function. Then

\[
\mathcal{H}_f(W(\psi)) = \begin{cases} 
0 & \text{if } \sum_{r=1}^{\infty} r f(\psi(r)) < \infty, \\
\infty & \text{if } \sum_{r=1}^{\infty} r f(\psi(r)) = \infty.
\end{cases}
\]

Clearly the above theorem can be regarded as the Hausdorff measure version of Khintchine’s theorem. As with the latter, the divergence part constitutes the main
substance. Notice, that the case when \( H^f \) is comparable to one–dimensional Lebesgue measure \( m \) (i.e. \( f(r) = r \)) is excluded by the condition \( r^{-1} f(r) \to \infty \) as \( r \to 0 \). Analogous to Khintchine’s original statement, in Jarník’s original statement the additional hypotheses that \( r^2 \psi(r) \) is decreasing, \( r^2 \psi(r) \to 0 \) as \( r \to \infty \) and that \( r^2 f(\psi(r)) \) is decreasing were assumed. Thus, even in the simple case when \( f(r) = r^s \) \( (s \geq 0) \) and the approximating function is given by \( \psi(r) = r^{-\tau} \log r \) \( (\tau > 2) \), Jarník’s original statement gives no information regarding the \( s \)–dimensional Hausdorff measure of \( W(\psi) \) at the critical exponent \( s = 2/\tau \) – see below. That this is the case is due to the fact that \( r^2 f(\psi(r)) \) is not decreasing. However, as we shall see these additional hypotheses are unnecessary. Furthermore, with the theorems of Khintchine and Jarník as stated above it is possible to combine them to obtain a singe unifying statement (see \[\text{[23]}\]) that provides a complete measure theoretic description of \( W(\psi) \).

Returning to Jarník’s theorem, note that in the case when \( H^f \) is the standard \( s \)–dimensional Hausdorff measure \( H^s \) (i.e. \( f(r) = r^s \)), it follows from the definition of Hausdorff dimension that

\[
\dim W(\psi) = \inf \{ s : \sum_{r=1}^{\infty} r^s \psi(r) < \infty \} .
\]

Previously, Jarník (1929) and independently Besicovitch (1934) had determined the Hausdorff dimension of the set \( W(r \mapsto r^{-\tau}) \), usually denoted by \( W(\tau) \), of \( \tau \)–well approximable numbers. They proved that for \( \tau > 2 \), \( \dim W(\tau) = 2/\tau \). Thus, as the ‘rate’ of approximation increases (i.e. as \( \tau \) increases) the ‘size’ of the set \( W(\tau) \) expressed in terms of Hausdorff dimension decreases. As discussed earlier, this is in precise keeping with one’s intuition. Obviously, the dimension result implies that

\[
\mathcal{H}_s(W(\tau)) = \begin{cases} 0 & \text{if } s > 2/\tau \\ \infty & \text{if } s < 2/\tau \end{cases}
\]

but gives no information regarding the \( s \)–dimensional Hausdorff measure of \( W(\tau) \) at the critical value \( s = \dim W(\tau) \). Clearly, Jarník’s zero–infinity law implies the dimension result and that for \( \tau > 2 \)

\[
\mathcal{H}^{2/\tau}(W(\tau)) = \infty .
\]

Furthermore, the ‘zero–infinity’ law allows us to discriminate between sets with the same dimension and even the same \( s \)–dimensional Hausdorff measure. For example, with \( \tau \geq 2 \) and \( 0 < \epsilon_1 < \epsilon_2 \) consider the approximating functions

\[
\psi_{\epsilon_i}(r) := r^{-\tau} (\log r)^{-\frac{\tau}{2}(1+\epsilon_i)} \quad (i = 1, 2) .
\]

It is easily verified that for any \( \epsilon_i > 0 \),

\[
m(W(\psi_{\epsilon_i})) = 0 , \quad \dim W(\psi_{\epsilon_i}) = 2/\tau \quad \text{and} \quad \mathcal{H}^{2/\tau}(W(\psi_{\epsilon_i})) = 0 .
\]

However, consider the dimension function \( f \) given by \( f(r) = r^{2/\tau}(\log r^{-1/\tau})^{\epsilon_1} \). Then \( \sum_{r=1}^{\infty} r f(\psi_{\epsilon_i}(r)) \asymp \sum_{r=1}^{\infty} (r \log r)^{1+\epsilon_1-\epsilon_1} \), where as usual the symbol \( \asymp \) denotes
comparability (the quotient of the associated quantities is bounded from above and below by positive, finite constants). Hence, Jarník’s zero–infinity law implies that

$$\mathcal{H}^f(W(\psi_1)) = \infty \quad \text{whilst} \quad \mathcal{H}^f(W(\psi_2)) = 0.$$ 

Thus the Hausdorff measure $\mathcal{H}^f$ does make a distinction between the ‘sizes’ of the sets under consideration; unlike $s$–dimensional Hausdorff measure.

Within this classical setup, it is apparent that Khintchine’s theorem together with Jarník’s zero–infinity law provide a complete measure theoretic description of $W(\psi)$ – see [2,3] for a single unifying statement. In short, our central aim is to establish analogues of the divergence parts of these classical results within a general framework. Recall, that the divergence parts constitute the main substance of the classical statements.

1.2 The general setup and fundamental problems

The setup described below is a much simplified version of that considered in [4]. In particular, we make no attempt to incorporate the linear forms theory of metric Diophantine approximation. However this does have the advantage of making the exposition more transparent and also leads to cleaner statements which are more than adequate for the application we have in mind.

Let $(\Omega, d)$ be a compact metric space equipped with a non-atomic, probability measure $m$. Let $\mathcal{R} = \{R_\alpha \subset \Omega : \alpha \in J\}$ be a family of points $R_\alpha$ of $\Omega$ indexed by an infinite, countable set $J$. The points $R_\alpha$ will be referred to as resonant points for reasons which will become apparent later. Next, let $\beta : J \to \mathbb{R}^+: \alpha \mapsto \beta_\alpha$ be a positive function on $J$. Thus, the function $\beta$ attaches a ‘weight’ $\beta_\alpha$ to the resonant point $R_\alpha$. To avoid pathological situations within our framework, we shall assume that the number of $\alpha$ in $J$ with $\beta_\alpha$ bounded above is always finite.

Given a decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ let

$$\Lambda(\psi) = \{x \in \Omega : x \in B(R_\alpha, \psi(\beta_\alpha)) \text{ for infinitely many } \alpha \in J\}.$$ 

The set $\Lambda(\psi)$ is a ‘$\limsup$’ set; it consists of points in $\Omega$ which lie in infinitely many of the balls $B(R_\alpha, \psi(\beta_\alpha))$ centred at resonant points. Clearly, even in this abstract setup it is natural to refer to the function $\psi$ as the *approximating function*. It governs the ‘rate’ at which points in $\Omega$ must be approximated by resonant sets in order to lie in $\Lambda(\psi)$.

Before continuing our discussion, we rewrite $\Lambda(\psi)$ in a fashion which brings its ‘$\limsup$’ nature to the forefront. For $n \in \mathbb{N}$, let

$$\Delta(\psi, n) := \bigcup_{\alpha \in J : k^{n-1} < \beta_\alpha \leq k^n} B(R_\alpha, \psi(\beta_\alpha)) \quad \text{where } k > 1 \text{ is fixed.}$$
By assumption the number of $\alpha$ in $J$ with $k^{n-1} < \beta_\alpha \leq k^n$ is finite regardless of the value of $k$. Thus, $\Lambda(\psi)$ is precisely the set of points in $\Omega$ which lie in infinitely many $\Delta(\psi, n)$; that is

$$\Lambda(\psi) = \limsup_{n \to \infty} \Delta(\psi, n) := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \Delta(\psi, n).$$

The main line of our investigation is motivated by the following pair of fundamental problems regarding the measure theoretic structure of $\Lambda(\psi)$. In turn the fundamental problems are motivated by the classical theory described in the previous section. It is reasonably straightforward to determine conditions under which $m(\Lambda(\psi)) = 0$. In fact, this is implied by the convergence part of the Borel–Cantelli lemma from probability theory whenever

$$\sum_{n=1}^{\infty} m(\Delta(\psi, n)) < \infty. \tag{1}$$

In view of this it is natural to consider:

**Problem 1** Under what conditions is $m(\Lambda(\psi))$ strictly positive?

Under a ‘local ubiquity’ hypothesis and a ‘$m$-volume’ divergent sum condition, our first theorem provides a complete solution to this problem; namely that $\Lambda(\psi)$ has full $m$–measure. This statement can be viewed as the analogue of Khintchine’s theorem.

Reiterating the above measure zero statement, if the approximating function $\psi$ decreases sufficiently quickly so that (1) is satisfied, the corresponding lim sup set $\Lambda(\psi)$ is of zero $m$–measure. As with the classical setup of (1), in this case we cannot obtain any further information regarding the ‘size’ of $\Lambda(\psi)$ in terms of the ambient measure $m$ — it is always zero. In short, we require a more delicate notion of ‘size’ than simply the given $m$-measure. In keeping with the classical development, we investigate the ‘size’ of $\Lambda(\psi)$ with respect to the Hausdorff measures $\mathcal{H}^f$ where $f$ is a dimension function. Again, provided a certain ‘$f$-volume’ sum converges, it is reasonably simple to determine conditions under which $\mathcal{H}^f(\Lambda(\psi)) = 0$. Naturally, we consider:

**Problem 2** Under what conditions is $\mathcal{H}^f(\Lambda(\psi))$ strictly positive?

This problem turns out to be far more subtle than the previous one regarding $m$-measure. However, under a ‘local ubiquity’ hypothesis and an ‘$f$-volume’ divergent sum condition, together with mild conditions on the dimension function, our second theorem shows that $\mathcal{H}^f(\Lambda(\psi)) = \infty$. Thus, $\mathcal{H}^f(\Lambda(\psi))$ satisfies an elegant ‘zero–infinity’ law whenever the convergence of the ‘$f$-volume’ sum implies $\mathcal{H}^f(\Lambda(\psi)) = 0$ as is often the case. In particular, this latter statement is true for the standard $s$-dimensional Hausdorff measure $\mathcal{H}^s$. Thus, in the language of geometric measure theory the sets $\Lambda(\psi)$ are not $s$-sets. Furthermore, from such zero–infinity laws it is easy to deduce the Hausdorff dimension of $\Lambda(\psi)$.
In order to illustrate and clarify the above setup and our line of investigation, we return to the basic lim sup set of $W(\psi)$. The classical set $W(\psi)$ of $\psi$-well approximable numbers in the theory of one dimensional Diophantine approximation can clearly be expressed in the form $\Lambda(\psi)$ with

$$\Omega := [0, 1], \quad J := \{(p, q) \in \mathbb{N} \times \mathbb{N} : 0 \leq p \leq q\}, \quad \alpha := (p, q) \in J,$$

$$\beta_\alpha := q, \quad R_\alpha := p/q \quad \text{and} \quad \Delta(R_\alpha, \psi(\beta_\alpha)) := B(p/q, \psi(q)).$$

The metric $d$ is of course the standard Euclidean metric; $d(x, y) := |x - y|$. Thus in this basic example, the resonant points $R_\alpha$ are simply rational points $p/q$. Furthermore,

$$\Delta(\psi, n) := \bigcup_{k^{n-1} < q \leq k^n} \bigcup_{p=0}^q B(p/q, \psi(q))$$

and $W(\psi) = \limsup \Delta(\psi, n)$ as $n \to \infty$.

For this basic example, the solution to our first fundamental problem is given by Khintchine’s theorem and the solution to the second by Jarník’s theorem. Together, these theorems provide a complete measure theoretic description of $W(\psi)$. In the case of the general framework, analogues of these results should be regarded as the ultimate pair of results describing the metric structure of the lim sup sets $\Lambda(\psi)$. Alternatively, they provide extremely satisfactory solutions to the fundamental problems. Analogues of the convergence parts of the classical results usually follow by adapting the ‘natural cover’

$$\{\Delta(\psi, n) : n = m, m + 1, \ldots \} \quad (m \in \mathbb{N})$$

of $\Lambda(\psi)$. Our key aim is to establish analogues of the divergence parts of the classical results for $\Lambda(\psi)$.

## 2 Ubiquity

In order to make any reasonable progress with the fundamental problems we impose various conditions on the compact metric measure space $(\Omega, d, m)$. Moreover, we require the notion of a ‘local’ ubiquitous system which will underpin our line of investigation.

Throughout, a ball centred at a point $x$ and radius $r$ is defined to be the set \( \{y \in \Omega : d(x, y) < r\} \) or \( \{y \in \Omega : d(x, y) \leq r\} \) depending on whether it is open or closed. In general, we do not specify whether a certain ball is open or close since it will be irrelevant. Notice, that by definition any ball is automatically a subset of $\Omega$. We shall impose the following regularity condition on the measure of balls.

**(M)** There exist positive constants $\delta$ and $r_\circ$ such that for any $x \in \Omega$ and $r \leq r_\circ$, $a r^\delta \leq m(B(x, r)) \leq b r^\delta$. 

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The constants $a$ and $b$ are independent of the ball and without loss of generality we assume that $0 < a < 1 < b$. Notice that the above condition implies that $\dim \Omega = \delta$ and furthermore that $\mathcal{H}^\delta(\Omega)$ is strictly positive and finite. Indeed, $m$ is a comparable to $\delta$–dimensional Hausdorff measure $\mathcal{H}^\delta$.

2.1 The ubiquitous system

The following ‘system’ contain the key measure theoretic structure necessary for our attack on the fundamental problems. Recall that $\mathcal{R}$ denotes the family of resonant sets $R_\alpha$ and that the function $\beta$ attaches a ‘weight’ $\beta_\alpha$ to each resonant set $R_\alpha \in \mathcal{R}$.

Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a function with $\rho(r) \to 0$ as $r \to \infty$ and let

$$\Delta(\rho, n) := \bigcup_{\alpha \in J(n)} B(R_\alpha, \rho(k^n)),$$

where $k > 1$ is a fixed real number and

$$J(n) := \{\alpha \in J : \beta_\alpha \leq k^n\}.$$

Definition (Local $m$–ubiquity) Let $B = B(x, r)$ be an arbitrary ball with centre $x$ in $\Omega$ and radius $r \leq r_0$. Suppose there exists a function $\rho$ and absolute constants $\kappa > 0$ and $k > 1$ such that

$$m(B \cap \Delta(\rho, n)) \geq \kappa m(B) \quad \text{for } n \geq n_\rho(B). \quad (2)$$

Then the pair $(\mathcal{R}, \beta)$ is said to be a local $m$-ubiquitous system relative to $(\rho, k)$.

Loosely speaking, the definition of local ubiquity says that the set $\Delta(\rho, n)$ locally approximates the underlying space $\Omega$ in terms of the measure $m$. By ‘locally’ we mean balls centred at points in $\Omega$. The function $\rho$, will be referred to as the ubiquitous function. The actual values of the constants $\kappa$ and $k$ in the above definition are irrelevant – it is their existence that is important. In practice, the local $m$–ubiquity of a system can be established using standard arguments concerning the distribution of the resonant sets in $\Omega$, from which the function $\rho$ arises naturally. To illustrate this, we return to the classical lim sup set of §1.1.

The set $W(\psi)$ of $\psi$–well approximable numbers has already been shown to fit within our general lim sup setup – see §1.2. Now let $m$ be one–dimensional Lebesgue measure. Clearly $m$ satisfies the measure condition (M) with $\delta = 1$. With this in mind, we have the following statement concerning local ubiquity within the classical setup.

Lemma 1 There is a constant $k > 1$ such that the pair $(\mathcal{R}, \beta)$ is a local $m$-ubiquitous system relative to $(\rho, k)$ where $\rho : r \mapsto \text{constant} \times r^{-2}$. 

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Proof. Let \( I = [a, b] \subset [0, 1] \). By Dirichlet’s theorem, for any \( x \in I \) there are coprime integers \( p, q \) with \( 1 \leq q \leq k^n \) satisfying \( |x - p/q| < (qk^n)^{-1} \). Clearly, \( aq - 1 \leq p \leq bq + 1 \). Thus, for a fixed \( q \) there are at most \( m(I)q + 3 \) possible values of \( p \). Trivially, for \( n \) large

\[
m \left( I \cap \bigcup_{q \leq k^{n-1}} \bigcup_p B \left( \frac{p}{q}, \frac{1}{qk^n} \right) \right) \leq 2 \sum_{q \leq k^{n-1}} \frac{k}{qk^n} (m(I)q + 3) \leq \frac{2}{k} m(I).
\]

It follows that for \( k \geq 6 \),

\[
m \left( I \cap \bigcup_{q \leq k^n} \bigcup_p B \left( \frac{p}{q}, \frac{k}{k^{2n}} \right) \right) \geq m \left( I \cap \bigcup_{k^{n-1} < q \leq k^n} \bigcup_p B \left( \frac{p}{q}, \frac{k}{k^{2n}} \right) \right) \\
\geq m(I) - \frac{3}{k} m(I) \geq \frac{1}{2} m(I).
\]

\[\triangleleft\]

It will be evident from our ‘ubiquity’ theorems, that Lemma 1 is sufficient for directly establishing the divergence part of both Khintchine’s theorem and Jarník’s zero–infinity law – see §2.3.

A remark on related systems. In the case that \( \Omega \) is a bounded subset of \( \mathbb{R}^n \) and \( m \) is \( n \)-dimensional Lebesgue measure, the notion of ubiquity was originally formulated by Dodson, Rynne & Vickers\[11\] to obtain lower bounds for the Hausdorff dimension of the sets \( \Lambda(\psi) \). Their notion of ubiquity is closely related to our notion of a ‘local \( m \)-ubiquitous’ system and furthermore coincides with the ‘regular systems’ of Baker & Schmidt\[1\]. Both these systems have proved very useful in obtaining lower bounds for the Hausdorff dimension of lim sup sets. However, both\[1\] and\[11\] fail to shed any light on the problems considered in this paper. For further details regarding regular systems and the original formulation of ubiquitous systems see\[4, 7\]. Recently and independently, in\[8\] the notion of an optimal regular system introduced in\[2\] has been re-formulated to obtain divergent type Hausdorff measures results for subsets of \( \mathbb{R}^n \). This re-formulated notion is essentially equivalent to our notion of local \( m \)-ubiquity in which \( m \) is \( n \)-dimensional Lebesgue measure and the ubiquity function is comparable to \( \rho : r \to r^{-1/n} \). Furthermore, even with these restrictions our notion of local \( m \)-ubiquity is not equivalent to that of an optimal regular system since we make no assumption on the growth of \( \# J(n) \).

2.2 The ubiquity statements

Recall, that an approximating function \( \psi \) is a real, positive decreasing function and that a ubiquity function \( \rho \) is a real, positive function such that \( \rho(r) \to 0 \) as \( r \to \infty \). Before
stating our main results we introduce one last notion. Given a real number \( k > 1 \), a function \( h \) will be said to be \( k\)-regular if there exists a strictly positive constant \( \lambda < 1 \) such that for \( n \) sufficiently large

\[
h(k^{n+1}) \leq \lambda h(k^n) .
\]  

(3)

The constant \( \lambda \) is independent of \( n \) but may depend on \( k \). A consequence of local ubiquity are the following pair of theorems. They constitute the main theorems appearing in [4] tailored to the setup considered here.

**Theorem BDV1**  
Let \( (\Omega, d) \) be a compact metric space equipped with a measure \( m \) satisfying condition (M) such that any open subset of \( \Omega \) is \( m \)-measurable. Suppose that \( (\mathcal{R}, \beta) \) is a local \( m \)-ubiquitous system relative to \( (\rho, k) \) and that \( \psi \) is an approximating function. Furthermore, suppose that either \( \psi \) or \( \rho \) is \( k \)-regular and that

\[
\sum_{n=1}^{\infty} \left( \frac{\psi(k^n)}{\rho(k^n)} \right)^{\delta} = \infty .
\]  

(4)

Then

\[
m(\Lambda(\psi)) = m(\Omega) .
\]

**Theorem BDV2**  
Let \( (\Omega, d) \) be a compact metric space equipped with a measure \( m \) satisfying condition (M). Suppose that \( (\mathcal{R}, \beta) \) is a locally \( m \)-ubiquitous system relative to \( (\rho, k) \) and that \( \psi \) is an approximation function. Let \( f \) be a dimension function such that \( r^{-\delta} f(r) \to \infty \) as \( r \to 0 \) and \( r^{-\delta} f(r) \) is decreasing. Let \( g \) be the real, positive function given by

\[
g(r) := f(\psi(r))\rho(r)^{-\delta} \quad \text{and let} \quad G := \limsup_{n \to \infty} g(k^n) .
\]  

(5)

(i) Suppose that \( G = 0 \) and that \( \rho \) is \( k \)-regular. Then,

\[
\mathcal{H}^{f}(\Lambda(\psi)) = \infty \iff \sum_{n=1}^{\infty} g(k^n) = \infty .
\]  

(6)

(ii) Suppose that \( 0 < G \leq \infty \). Then, \( \mathcal{H}^{f}(\Lambda(\psi)) = \infty . \)

Clearly, the assumption that the function \( 0 < G \leq \infty \) in part (ii) implies the divergent sum condition in part (i). The case when the dimension function \( f \) is \( \delta \)-dimensional Hausdorff measure \( \mathcal{H}^{\delta} \) is excluded from the statement of Theorem BDV2 by the condition that \( r^{-\delta} f(r) \to \infty \) as \( r \to 0 \). This is natural since otherwise Theorem BDV1 implies that \( m(\Lambda(\psi)) > 0 \) which in turn implies that \( \mathcal{H}^{\delta}(\Lambda(\psi)) \) is positive and finite. In other words \( \mathcal{H}^{\delta}(\Lambda(\psi)) \) is never infinite. However, given that the measure \( m \) is comparable to \( \mathcal{H}^{\delta} \) – a simple consequence of condition (M) – we are able to combine the above statements and obtain a single unifying theorem.
**Theorem 1** Let \((\Omega, d)\) be a compact metric space equipped with a measure \(m\) satisfying condition (M) such that any open subset of \(\Omega\) is \(m\)-measurable. Suppose that \((\mathcal{R}, \beta)\) is a locally \(m\)-ubiquitous system relative to \((\rho, k)\) and that \(\psi\) is an approximation function. Let \(f\) be a dimension function such that \(r^{-\delta} f(r)\) is monotonic. Furthermore, suppose that \(\rho\) is \(k\)-regular and that

\[
\sum_{n=1}^{\infty} \frac{f(\psi(k^n))}{\rho(k^n)^\delta} = \infty. \tag{7}
\]

Then,

\[
\mathcal{H}^f(\Lambda(\psi)) = \mathcal{H}^f(\Omega). \tag{8}
\]

The condition that \(r^{-\delta} f(r)\) is monotonic is a natural condition which is not particularly restrictive. Note that if the dimension function \(f\) is such that \(r^{-\delta} f(r) \to \infty\) as \(r \to 0\) then \(\mathcal{H}^f(\Omega) = \infty\) and Theorem 1 leads to the same conclusion as Theorem BDV2. Here we make use of the following fact: if \(f\) and \(g\) are two dimension functions such that the ratio \(f(r)/g(r)\to 0\) as \(r \to 0\), then \(\mathcal{H}^f(F) = 0\) whenever \(\mathcal{H}^g(F) < \infty\). On the other hand, Theorem 1 with \(f(r) = r^\delta\) implies that \(\mathcal{H}^\delta(\Lambda(\psi)) = \mathcal{H}^\delta(\Omega)\). This together with the fact that the measure \(m\) is comparable to \(\mathcal{H}^\delta\) implies that \(m(\Lambda(\psi)) = m(\Omega)\) – the conclusion of Theorem BDV1.

### 2.3 The classical results

For the classical set \(W(\psi)\) of \(\psi\)-well approximable numbers, Lemma 1 in §2 establishes local \(m\)-ubiquity. Clearly, the ubiquity function \(\rho\) satisfies (3) (i.e. \(\rho\) is \(u\)-regular) and so Theorem BDV1 establishes the divergent part of Khintchine’s Theorem. On the other hand, Theorem BDV2 establishes the divergent part of Jarník’s Theorem. By making use of the ‘natural cover’ of \(W(\psi)\), the convergent parts of these classical results are easily established.

In the above discussion we have opted to establish the classical results of Khintchine and Jarník separately. In the past these results have always been thought of as separate entities with Jarník’s Theorem being regarded as a refinement of Khintchine’s Theorem – but not containing Khintchine’s Theorem. However, it is easily seen that Lemma 1 together with Theorem 1 leads to the following unification of the fundamental classical results.

**Theorem (Khintchine–Jarník)** Let \(f\) be a dimension function such that \(r^{-1} f(r)\) is monotonic. Let \(\psi\) be a real, positive decreasing function. Then

\[
\mathcal{H}^f(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} r f(\psi(r)) < \infty, \\ \mathcal{H}^f([0,1]) & \text{if } \sum_{r=1}^{\infty} r f(\psi(r)) = \infty. \end{cases}
\]
An important observation. It is worth standing back a little and think about what we have actually used in establishing the classical results – namely local ubiquity. Within the classical setup, local ubiquity is a simple measure theoretic statement concerning the distribution of rational points with respect to Lebesgue measure – the natural measure on the unit interval. From this we are able to obtain the divergent parts of both Khintchine’s Theorem (a Lebesgue measure statement) and Jarník’s Theorem (a Hausdorff measure statement). In other words, the Lebesgue measure statement of local ubiquity seems to underpin the general Hausdorff measure theory of the lim sup set $W(\psi)$. That this is the case is by no means a coincidence – see [5, 6]. In fact, in view of the Mass Transference Principle introduced in [5] one actually has that

$$\text{Khintchine’s Theorem} \implies \text{Jarník’s Theorem}.$$ 

Thus, the Lebesgue theory of $W(\psi)$ underpins the general Hausdorff theory. This at first glance is rather surprising in that the Hausdorff theory had previously been thought to have been a subtle refinement of the Lebesgue theory. However, given that the Lebesgue statement of local ubiquity implies the general Hausdorff theory we should not be too surprised.

### 2.4 Where to go from here with ubiquity?

Let $\psi$ be a real, positive decreasing function. For $x$ in the unit interval and $N \in \mathbb{N}$, let

$$R(x, N) := \# \{ 1 \leq q \leq N : |x - p/q| < \psi(q) \text{ for some } p \in \mathbb{Z} \}.$$ 

In view of Khintchine’s Theorem, if $\sum q \psi(q)$ diverges then for almost all $x$ we have that $R(x, N) \to \infty$ as $N \to \infty$. An obvious question now arises: can we saying anything more precise about the behavior of the counting function $R(x, N)$? Within the classical theory of Diophantine approximation we have the following remarkable quantitative statement of Khintchine’s Theorem.

**Schmidt’s Theorem (1964).** Suppose that $2q \psi(q) < 1$ and that $\sum_{q=1}^{\infty} q \psi(q) = \infty$. Then, for almost all $x$

$$R(x, N) \sim 2 \sum_{q=1}^{N} q \psi(q).$$

Schmidt actually proves the above asymptotic statement with an error term. Note that the condition $2q \psi(q) < 1$ simply means that for any fixed $q$ there is at most one $p \in \mathbb{Z}$ such that $|x - p/q| < \psi(q)$ – this avoids counting multiplicities.

In view of above discussion, in particular the work of Schmidt, a gaping inadequacy with the ubiquity framework is exposed. In describing the $m$-measure theoretic structure of a lim sup, the analogue of Khintchine’s Theorem should be regarded as the first
step. The ultimate aim should be to obtain a quantitative version of such a result; i.e. the analogue of Schmidt’s Theorem. Thus we ask the following question. Is there a natural ‘stronger’ form of local ubiquity which would enable us to obtain a quantitative form of Theorem BDV1 analogous to Schmidt’s Theorem? Obviously, it would be highly desirable to establish such a form. Even a ubiquity framework that would yield a comparable rather than asymptotic analogue of Schmidt’s Theorem would be desirable; i.e. a framework which within the classical setup implies that

$$\mathcal{R}(x, N) \asymp \sum_{q=1}^{N} q \psi(q).$$

3 Diophantine approximation and Kleinian Groups

The classical results of Diophantine approximation, in particular those from the one dimensional theory, have natural counterparts and extensions in the hyperbolic space setting. In this setting, instead of approximating real numbers by rationals, one approximates limit points of a fixed Kleinian group $G$ by points in the orbit (under the group) of a certain distinguished limit point $y$. Beardon and Maskit have shown that the geometry of the group is reflected in the approximation properties of points in the limit set. The elements of $G$ are orientation preserving Möbius transformations of the $(n + 1)$–dimensional unit ball $B_{n+1}$. Let $\Lambda$ denote the limit set of $G$ and let $\delta$ denote the Hausdorff dimension of $\Lambda$. For any element $g$ in $G$ we shall use the notation $L_g := |g'(0)|^{-1}$, where $|g'(0)|$ is the (Euclidean) conformal dilation of $g$ at the origin.

Let $\psi$ be an approximating function and let

$$W_g(\psi) := \{\xi \in \Lambda : |\xi - g(y)| < \psi(L_g) \text{ for i.m. } g \in G\}.$$ 

This is the set of points in the limit set $\Lambda$ which are ‘close’ to infinitely many (‘i.m.’) images of a ‘distinguished’ point $y$. The ‘closeness’ is of course governed by the approximating function $\psi$. The limit point $y$ is taken to be a parabolic fixed point if the group has parabolic elements and a hyperbolic fixed point otherwise.

Geometrically finite groups with parabolics: Let us assume that the geometrically finite group has parabolic elements so it is not convex co-compact. Thus our distinguished limit point $y$ is a parabolic fixed point, say $p$. Associated with $p$ is a geometrically motivated set $T_p$ of coset representatives of $G_p \backslash G := \{g G_p : g \in G\}$; so chosen that if $g \in T_p$ then the orbit point $g(0)$ of the origin lies within a bounded hyperbolic distance from the top of the standard horoball $H_{g(p)}$. The latter, is an $(n + 1)$–dimensional Euclidean ball contained in $B_{n+1}$ such that its boundary touches the unit ball $S^n$ at the point $g(p)$. Let $R_g$ denote the Euclidean radius of $H_{g(p)}$. As a consequence of the definition of $T_p$, it follows that

$$\frac{1}{CL_g} \leq R_g \leq \frac{C}{L_g}$$
where $C > 1$ is an absolute constant. Also, it is worth mentioning that the balls in the standard set of horoballs $\{ H_{g(p)} : g \in T_p \}$ corresponding to the parabolic fixed point $p$ are pairwise disjoint. For further details and references regarding the above notions and statements see any of the papers [15, 19, 23]. With reference to our general framework, let $\Omega := \Lambda$, $J := \{ g : g \in T_p \}$, $\alpha := g \in J$, $\beta_{\alpha} := CR_{g^{-1}}$ and $R_{\alpha} := g(p)$. Thus, the family $R$ of resonant sets $R_{\alpha}$ consists of orbit points $g(p)$ with $g \in T_p$. Furthermore, $B(R_{\alpha}, \psi(\beta_{\alpha})) := B(g(p), \psi(CR_{g^{-1}}))$ and

$$\Delta(\psi, n) := \bigcup_{g \in T_p : k^{n-1} < CR_{g^{-1}} \leq k^n} B(g(p), \psi(CR_{g^{-1}})) .$$

Here $k > 1$ is a constant. Then

$$W_p(\psi) \supset \Lambda(\psi) := \limsup_{n \to \infty} \Delta(\psi, n) .$$

Now, let $m$ be Patterson measure and $\delta = \dim \Lambda$. Thus $m$ is a non-atomic, $\delta$–conformal probability measure supported on $\Lambda$. We are assuming that the group has parabolic elements, thus in general $m$ does not satisfy condition (M) and so our ubiquity statements are not applicable. However, if we restrict our attention to groups of the first kind then $\Lambda = S^n$ and $m$ is simply $n$–dimensional Lebesgue measure on unit sphere $S^n$. Also note that $\delta = n$ in this case. Thus for groups of the first kind, $m$ clearly satisfies condition (M) and we have the following statement concerning local ubiquity.

**Proposition 1** Let $k \geq k_o$ – a positive group constant. Then then pair $(R, \beta)$ is a local $m$–ubiquitous system relative to $(\rho, k)$ where $\rho : r \rightarrow \rho(r) := \text{constant} \times r^{-1}$.

The proposition follows from the following two facts which can be found in [15, 19, 21]. They are valid in general, but for groups of the first kind they are particularly easy to establish.

- **Local Horoball Counting Result:** Let $B$ be an arbitrary Euclidean ball in $S^n$ centred at a limit point. For $\lambda \in (0, 1)$ and $r \in \mathbb{R}^+$ define

$$A_{\lambda}(B, R) := \{ g \in T_p : g(p) \in B \text{ and } \lambda R \leq R_g < R \} .$$

There exists a positive group constant $\lambda_o$ such that if $\lambda \leq \lambda_o$ and $R < R_o(B)$, then

$$k_1^{-1} R^{-\delta} m(B) \leq \#A_{\lambda}(B, R) \leq k_1 R^{-\delta} m(B) ,$$

where $k_1$ is a positive constant independent of $B$ and $R_o(B)$ is a sufficiently small positive constant which does depend on $B$.

- **Disjointness Lemma:** For distinct elements $g, h \in T_p$ with $\lambda < R_g/R_h < \lambda^{-1}$, one has $B(g(p), \lambda R_g) \cap B(h(p), \lambda R_h) = \emptyset$.
Proof of Proposition \footnote{1}. To prove the proposition, let \( \rho(r) := C(kr)^{-1} \) where \( k := 1/\lambda > 1/\lambda_0 \) and \( B \) be an arbitrary ball centred at a limit point. Then for \( n \) sufficiently large

\[
m( B \cap \bigcup_{g \in J(n)} B(g(p), \rho(k^n))) = m( B \cap \bigcup_{g \in \mathcal{J}_n: C R_g^{-1} \leq k^n} B(g(p), \rho(k^n))) \geq m( \bigcup_{g \in \mathcal{J}_n: \rho(p) \in \frac{1}{2} B, k^{n-1} < C R_g^{-1} \leq k^n} B(g(p), \rho(k^n))) \gg m(\frac{1}{2} B) \gg m(B).
\]

Thus, in view of Proposition \footnote{1} and the fact that the measure \( m \) is of type (M) and that \( \rho \) is \( k \)-regular, Theorem 1 yields the divergent part of the following statement. The convergent part is easy – just use the ‘natural cover’ given by the lim sup set \( W_p(\psi) \) under consideration. Also we make use of the following simple fact. Suppose that \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) is a real, positive monotonic function, \( \alpha \in \mathbb{R} \) and \( k > 1 \). Then the divergence and convergence properties of the sums

\[
\sum_{n=1}^{\infty} k^n \alpha h(k^n) \quad \text{and} \quad \sum_{r=1}^{\infty} r^{\alpha-1} h(r)
\]

coincide.

\[\begin{align*}
\mathcal{H}^f(W_p(\psi)) &= \left\{ \begin{array}{ll}
0 & \text{if } \sum_{r=1}^{\infty} f(\psi(r)) r^{n-1} < \infty, \\
\mathcal{H}^f(S^n) & \text{if } \sum_{r=1}^{\infty} f(\psi(r)) r^{n-1} = \infty.
\end{array} \right.
\end{align*}\]

In the above theorem, on taking \( f(r) = r^n \) we obtain the analogue of Khintchine’s theorem with respect to the measure \( m \) supported on the limit set; i.e. \( n \)-dimensional
Lebesgue measure on $S^n$. The theorem is this case, under a certain regularity condition on $\psi$, has previously been established in [21, 23, 24]. Indeed, in [23] the analogue of Khintchine’s theorem with respect to Patterson measure is established without the condition that the group is of the first kind. It is worth mentioning that the more general local ubiquity framework of [4] also yields this statement even though Patterson measure does not generically satisfy condition (M). However, the condition (M) on the measure is essential even in [4] for establishing general Hausdorff measure ‘divergent’ results and the full analogue of Theorem 2 without the ‘first kind’ restriction is currently out of reach – precise Hausdorff dimension statements are known [15].

When interpreted on the upper half plane model $\mathbb{H}^2$ of hyperbolic space and applied to the modular group $\operatorname{SL}(2, \mathbb{Z})$, Theorem 2 implies the classical result associated with the lim sup set $W(\psi)$ as stated in §2.3.

**Convex co-compact groups:** These are geometrically finite Kleinian groups without parabolic elements. Thus, the distinguished limit point $y$ is a hyperbolic fixed point. For convex co-compact groups, Patterson measure $m$ satisfies condition (M) and the situation becomes much more satisfactory – we don’t not have to assume that the group is of the first kind.

Let $L$ be the axis of the conjugate pair of hyperbolic fixed points $y$ and $y'$, and let $G_{yy'}$ denote the stabilizer of $y$ (or equivalently $y'$). Then there is a geometrically motivated set $T_{yy'}$ of coset representatives of $G_{yy'} \backslash G$; so chosen that if $g \in T_{yy'}$ then the orbit point $g(0)$ of the origin lies within a bounded hyperbolic distance from the summit $s_g$ of $g(L)$ – the axis of the hyperbolic fixed pair $g(y)$ and $g(y')$. The summit $s_g$ is simply the point on $g(L)$ ‘closest’ to the origin. For $g \in T_{yy'}$, let $H_{g(y)}$ be the horoball with base point at $g(y)$ and radius $R_g := 1 - |s_g|$. Then the top of $H_{g(y)}$ lies within a bounded hyperbolic distance of $g(0)$. Furthermore, as a consequence of the definition of $T_{yy'}$, it follows that $C^{-1} \leq R_g L_g \leq C$ where $C > 1$ is an absolute constant. We are now able to define the subset $\Lambda(\psi)$ of $W_y(\psi)$ in exactly the same way as in the parabolic case with $y$ replacing $p$ and $T_{yy'}$ replacing $T_p$.

Essentially the arguments given in [19], can easily be modified to obtain the analogue of the local horoball counting result stated above for the parabolic case. We leave the details to the reader. In turn, this enables one to establish Proposition [1] for convex co-compact groups – the statement remains unchanged. Since $m$ is of type (M) and $\rho$ is $k$–regular for any $k > 1$, Theorem [1] yields the divergent part of the following statement. The convergent part is straightforward to establish.

**Theorem 3** Let $G$ be a convex co-compact Kleinian group and $y$ be a hyperbolic fixed point. Let $f$ be a dimension function such that $r^{-\delta} f(r)$ is monotonic. Let $\psi$ be a real,
positive decreasing function. Then

\[ \mathcal{H}^f(W_y(\psi)) = \begin{cases} 
0 & \text{if } \sum_{r=1}^{\infty} f(\psi(r)) r^{\delta-1} < \infty, \\
\mathcal{H}^f(\Lambda) & \text{if } \sum_{r=1}^{\infty} f(\psi(r)) r^{\delta-1} = \infty.
\end{cases} \]

In the above theorem, on taking \( f(r) = r^\delta \) we obtain the convex co-compact analogue of Khintchine’s theorem with respect to the measure \( m \) supported on the limit set; i.e. Patterson measure on \( \Lambda \). This Khintchine analogue, under a certain regularity condition on \( \psi \), has been known for sometime – see for example [10]. Regarding the general Hausdorff measure aspect of the above theorem, previously only dimension statements were known – see [25]. Theorem 3 not only implies these dimension statements but also gives the \( s \)-dimensional Hausdorff measure \( \mathcal{H}^s \) of \( W_y(\psi) \) at the critical exponent \( s = \dim W_y(\psi) \).

### 3.1 Consequences of Theorem 2

Throughout, \( G \) is a geometrically finite group of the first kind with parabolic elements. In this section we bring into play the real strength of Theorem 2. Let \( \tau \geq 1 \) and \( \epsilon > 0 \) be arbitrary. Consider the approximating functions

\[ \psi(r) := r^{-\tau} (\log r)^{-\frac{s}{\tau}} \quad \text{and} \quad \psi_\epsilon(r) := r^{-\tau} (\log r)^{-\frac{s}{\tau}(1+\epsilon)}. \]

Let

\[ E_p(\tau) := W_p(\psi) \setminus W_p(\psi_\epsilon), \]

where \( p \) is our ‘distinguished’ parabolic fixed point of \( G \). Thus, a limit point \( \xi \) is in the set \( E(\tau) \) if

\[ |\xi - g(y)| < \psi(L_g) \quad \text{for infinitely many } g \text{ in } G, \]

and for any \( \epsilon > 0 \)

\[ |\xi - g(y)| \geq \psi_\epsilon(L_g) \quad \text{for all but finitely many } g \text{ in } G. \]

In other words, the approximation properties of \( \xi \) by the orbit of the parabolic fixed point is ‘sandwiched’ between the approximating functions \( \psi \) and \( \psi_\epsilon \). Now consider the dimension function

\[ f : r \to f(r) := r^{\frac{s}{\tau}}. \]

Hence, \( \mathcal{H}^f \) is simply \( n/\tau \)-dimensional Hausdorff measure \( \mathcal{H}^{n/\tau} \). A straightforward application of Theorem 2 yields that

\[ \mathcal{H}^{\frac{n}{\tau}}(W_p(\psi)) = \mathcal{H}^{\frac{n}{\tau}}(S^n) \quad \text{and} \quad \mathcal{H}^{\frac{n}{\tau}}(W_p(\psi_\epsilon)) = 0. \]

Now, \( \mathcal{H}^{n/\tau}(S^n) > 0 \) (in fact it is equivalent to the \( n \)-dimensional Lebesgue measure of the unit sphere \( S^n \) when \( \tau = 1 \) and is infinite if \( \tau > 1 \)) and so we obtain the following statement.
Lemma 2 For $\tau \geq 1$, $\dim E_p(\tau) = n/\tau$ and furthermore

$$\mathcal{H}^\tau (E_p(\tau)) = \mathcal{H}^\tau (S^n)$$

The main observation used in extracting Lemma 2 from Theorem 2 is the following: if we have two sets $A$ and $B$ with $m(A) > 0$ and $m(B) = 0$ then $m(A \setminus B) = m(A) > 0$. This simple observation can be implemented to obtain the analogue of the lemma for general exact order sets – see (11, 13, 14) for a discussion of this notion within the classical framework of Diophantine approximation. Briefly, given two approximating functions $\varphi$ and $\psi$ with $\varphi$ in some sense ‘smaller’ than $\psi$, consider the set $E_p(\psi, \varphi) := W_p(\psi) \setminus W_p(\varphi)$. Thus the approximation properties of limit points $\xi$ in $E_p(\psi, \varphi)$ are ‘sandwiched’ between the functions $\varphi$ and $\psi$. Under suitable conditions on the ‘smallness’ of $\varphi$ compared to $\psi$ it is possible to obtain the analogue of Lemma 2 for the set $E_p(\psi, \varphi)$ – see [3] for the classical statements. In view of the above observation, the key is to construct an appropriate dimension function $f$ for which $\mathcal{H}^f (W_p(\psi)) = \mathcal{H}^f (S^n)$ and $\mathcal{H}^f (W_p(\varphi)) = 0$.

In the case that $\tau = 1$, Lemma 2 has a well known dynamical interpretation in terms of the ‘rate’ of excursions by geodesics into a cuspidal end of the associated hyperbolic manifold $\mathcal{M} = B^{n+1}/G$; namely Sullivan’s logarithm law for geodesics [24]. We are now in the position to naturally place this law within the general Hausdorff measure setting. First some notation. Let $P$ denote a complete set of parabolic fixed points inequivalent under $G$. Clearly the orbit $G(P)$ of points in $P$ under $G$ is the complete set of parabolic fixed points of $G$. Since $G$ is geometrically finite of the first kind with parabolic elements, the associated hyperbolic manifold $\mathcal{M}$ consists of a compact part $X_0$ with a finite number of attachments:

$$\mathcal{M} = X_0 \cup \bigcup_{p \in P} Y_p$$

where each $p$ in $P$ determines an exponentially ‘thinning’ end $Y_p$ – usually referred to as a cuspidal end – attached to $X_0$.

We shall write 0 for the projection of the origin in $B^{n+1}$ to the quotient space $\mathcal{M}$. Let $S^n$ be the unit sphere of the tangent space to $\mathcal{M}$ at 0, and for every vector $v$ in $S^n$ let $\gamma_v$ be the geodesic emanating from 0 in the direction $v$. Furthermore, for $t$ in $\mathbb{R}^+$, let $\gamma_v(t)$ denote the point achieved after travelling time $t$ along $\gamma_v$. Now fix a $p \in P$. We define a function

$$\text{pen}_p : \mathcal{M} \to \mathbb{R}^+$$

$$x \mapsto \begin{cases} 0 & x \notin Y_p \\ \text{dist} (x, 0) & x \in Y_p. \end{cases}$$

where dist is the induced metric on $\mathcal{M}$. This is the penetration of $x$ into the cuspidal end $Y_p$. A relatively simple argument (see [18, 24, 25]) shows that the excursion pattern
of a random geodesic into a cuspidal end $Y_p$ is equivalent to the approximation of a random limit point of $G$ by the base points of standard horoballs in $\{H_{g(p)} : g \in T_p\}$.

In particular, for any $\alpha$ in $[0,1]$, consider the set $S_p(\alpha)$ of directions $v$ in $S^n$ such that

$$\limsup_{t \to \infty} \frac{\text{pen}_p(\gamma_v(t)) - \alpha t}{\log t} = \frac{1}{n}.$$ 

Then the problem of determining the measure theoretic structure of $S_p(\alpha)$ is equivalent to determining the measure theoretic structure of $E_p(\tau)$ with $\tau = 1/(1 - \alpha)$. In view of this, the following result can be regarded as a dynamical interpretation of Lemma 2 in terms of the geodesic excursions into the cuspidal ends of $M$.

**Theorem 4 (A general logarithm law for geodesics)** Let $G$ be a geometrically finite group of the first kind with parabolic elements. For $\alpha \in [0,1)$, we have that

$$\mathcal{H}^{n(1-\alpha)}(S_p(\alpha)) = \mathcal{H}^{n(1-\alpha)}(S^n).$$

In the case $\alpha = 0$, so that $\mathcal{H}^{n(1-\alpha)}$ is equivalent to $n$-dimensional Lebesgue measure, the theorem reduces to Sullivan’s famous logarithm law for geodesics. The theorem simple says that Sullivan’s logarithm law survives for $\alpha > 0$ if we appropriately ‘rescale’ $n$-dimensional Lebesgue measure.

**Remark.** In this section we have chosen to demonstrate the power of Theorem 2. We could just as easily have picked on Theorem 3 and established analogues statements to Lemma 2 and Theorem 4 for convex co-compact groups. The latter would be a statement along the lines suggested by the dynamical interpretation of the Diophantine approximation results in [10].

We end our discussion by studying limit points which are ‘extremely’ well approximable by the orbit of a parabolic fixed point. In view of the above discussion, they correspond to geodesics which exhibit an ‘extremely’ rapid excursion pattern into a cuspidal end of $M$. For $\omega > 0$, let us say that a limit point $\xi$ is $\omega$–Liouville if

$$|\xi - g(y)| < \exp(-L^\omega_g)$$ for infinitely many $g$ in $G$.

Let $L_p(\omega)$ denote the set of $\omega$–Liouville limit points. Note that if $\xi \in L_p(\omega)$, then for any real number $\tau$ we have that $|\xi - g(y)| < L^{-\tau}_g$ for infinitely many $g$ in $G$ – hence the reference to Liouville since in the classical framework, a real number $x$ is said to be Liouville if $|x - p/q| < q^{-\tau}$ for infinitely many rationals $p/q$, irrespective of the value of $\tau$. It is easy to see that for any $s > 0$

$$\sum_{r=1}^{\infty} r^{n-1} (\exp(-r^\omega))^s < \infty,$$

of a random geodesic into a cuspidal end $Y_p$ is equivalent to the approximation of a random limit point of $G$ by the base points of standard horoballs in $\{H_{g(p)} : g \in T_p\}$. In particular, for any $\alpha$ in $[0,1]$, consider the set $S_p(\alpha)$ of directions $v$ in $S^n$ such that

$$\limsup_{t \to \infty} \frac{\text{pen}_p(\gamma_v(t)) - \alpha t}{\log t} = \frac{1}{n}.$$ 

Then the problem of determining the measure theoretic structure of $S_p(\alpha)$ is equivalent to determining the measure theoretic structure of $E_p(\tau)$ with $\tau = 1/(1 - \alpha)$. In view of this, the following result can be regarded as a dynamical interpretation of Lemma 2 in terms of the geodesic excursions into the cuspidal ends of $M$.

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In the case $\alpha = 0$, so that $\mathcal{H}^{n(1-\alpha)}$ is equivalent to $n$-dimensional Lebesgue measure, the theorem reduces to Sullivan’s famous logarithm law for geodesics. The theorem simple says that Sullivan’s logarithm law survives for $\alpha > 0$ if we appropriately ‘rescale’ $n$-dimensional Lebesgue measure.

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$$|\xi - g(y)| < \exp(-L^\omega_g)$$ for infinitely many $g$ in $G$.

Let $L_p(\omega)$ denote the set of $\omega$–Liouville limit points. Note that if $\xi \in L_p(\omega)$, then for any real number $\tau$ we have that $|\xi - g(y)| < L^{-\tau}_g$ for infinitely many $g$ in $G$ – hence the reference to Liouville since in the classical framework, a real number $x$ is said to be Liouville if $|x - p/q| < q^{-\tau}$ for infinitely many rationals $p/q$, irrespective of the value of $\tau$. It is easy to see that for any $s > 0$

$$\sum_{r=1}^{\infty} r^{n-1} (\exp(-r^\omega))^s < \infty,$$
regardless of \( \omega \) and so the sets \( L_p(\omega) \) are of zero dimension. However, given \( \epsilon \geq 0 \), let \( f_\epsilon \) be the dimension function given by

\[
f_\epsilon(r) := \left( \log \frac{1}{r} \right)^{\frac{1}{\omega}} \times \left( \log \log \frac{1}{r} \right)^{-(1+\epsilon)}.
\]

On applying Theorem 2, we obtain the following statement.

**Lemma 3** Let \( G \) be a geometrically finite group of the first kind with parabolic elements. For \( \omega > 0 \),

\[
\mathcal{H}^{f_\epsilon}(L_p(\omega)) = \begin{cases} 
0 & \text{if } \epsilon > 0, \\
\infty & \text{if } \epsilon = 0.
\end{cases}
\]

In terms of the dimension theory, when we are confronted with sets of dimension zero it is natural to change the usual ‘\( r^s \)-scale’ in the definition of Hausdorff dimension to a logarithmic scale. For \( s > 0 \), let \( f_s \) be the dimension function given by \( f_s(r) := (- \log r)^s \). The **logarithmic Hausdorff dimension** of a set \( F \) is defined by \( \dim \log F := \inf \{ s : \mathcal{H}^{f_s}(F) = 0 \} = \sup \{ s : \mathcal{H}^{f_s}(F) = \infty \} \). It is easily verified that if \( \dim F > 0 \) then \( \dim \log F = \infty \) – precisely as one should expect. The following statement is a simple consequence of Lemma 3.

**Corollary 1** Let \( G \) be a geometrically finite group of the first kind with parabolic elements. For \( \omega > 0 \),

\[
\dim \log L_p(\omega) = \frac{n}{\omega}.
\]

Furthermore, \( \mathcal{H}^{f_s}(L_p(\omega)) = \infty \) at the critical exponent \( s = n/\omega \).

**Remark.** Theorem 3 yields the analogues statements to Lemma 3 and Corollary 1 for convex co-compact groups. Apart from replacing \( n \) by \( \delta := \dim \Lambda \), the statements are identical to those above.

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Victor V. Beresnevich: Department of Mathematics, University of York,
Heslington, York, YO10 5DD, England.
e-mail: vb8@york.ac.uk

Sanju L. Velani: Department of Mathematics, University of York,
Heslington, York, YO10 5DD, England.
e-mail: slv3@york.ac.uk