DoCoM-SGT: Doubly Compressed Momentum-assisted Stochastic Gradient Tracking Algorithm for Communication Efficient Decentralized Learning

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Abstract

This paper proposes the Doubly Compressed Momentum-assisted Stochastic Gradient Tracking algorithm (DoCoM-SGT) for communication efficient decentralized learning. DoCoM-SGT utilizes two compression steps per communication round as the algorithm tracks simultaneously the averaged iterate and stochastic gradient. Furthermore, DoCoM-SGT incorporates a momentum based technique for reducing variances in the gradient estimates. We show that DoCoM-SGT finds a solution $\bar{\theta}$ in $T$ iterations satisfying $E[\|\nabla f(\bar{\theta})\|^2] = O(1/T^{2/3})$ for non-convex objective functions; and we provide competitive convergence rate guarantees for other function classes. Numerical experiments on synthetic and real datasets validate the efficacy of our algorithm.

1 Introduction

Decentralized algorithms tackle an optimization problem with inter-connected agents/workers possessing local data without relying on a central server. For many scenarios relevant to large-scale machine learning, these algorithms improve computational scalability and preserve data privacy. Owing to these reasons, decentralized algorithms have become the critical enabler for applications such as sensor networks (Schizas et al., 2007), federated learning (Konečný et al., 2016; Wang et al., 2021), etc.

This paper concentrates on the communication efficiency issue with decentralized algorithms, which is a key bottleneck as the latter rely heavily on the bandwidth limited inter-agent communication links (Wang et al., 2021). An inefficient design may lead to significant overhead and slow down to the application. Several approaches have been studied to tame with this issue. The first approach is to consider the optimal algorithm design. Scaman et al. (2019); Uribe et al. (2021) studied algorithms with an optimal iteration complexity, Sun & Hong (2019); Sun et al. (2020); Lu & De Sa (2021) focused on non-convex problems and studied lower bounds on the number of communications needed; also see (Gorbunov et al., 2019). We remark that a common algorithm design to achieve optimal rates is to balance between computation and communication by performing several computation (i.e., gradient) steps before communication.

Perhaps a more direct approach to improve communication efficiency is to apply compression in every communication step of algorithms. This idea was first studied in the context of distributed optimization where workers/agents communicate to a central server. A number of algorithms have been studied for the distributed setting with compression strategies such as sparsification (Stich et al., 2018; Alistarh et al., 2018; Wangni et al., 2018), quantization (Alistarh et al., 2017; Bernstein et al., 2018; Reisizadeh et al., 2020), low-rank approximation (Vogels et al., 2019), etc., often used in combination with an error compensation technique (Mishchenko et al., 2019; Tang et al., 2019). See a recent study via a unified framework in (Richtárik et al., 2021).

For decentralized optimization where a central server is not employed, the design of compression-enabled algorithm is more challenging. Tang et al. (2018a) proposed an extrapolation compression method, Koloskova et al. (2019b,a) proposed the CHOCO-SGD algorithm which combines decentralized SGD (Lian et al., 2017) with error compensation. Despite the simplicity and reasonable practical performance demonstrated, algorithms such as CHOCO-SGD suffer from a sub-optimal iteration complexity, and their analysis show that the performance

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Concentrating on the stochastic optimization setting, this paper addresses the aforementioned issues by developing a communication efficient algorithm with three ingredients: (A) compression, (B) gradient tracking, and (C) momentum-based variance reduction. Our contributions are:

- We derive the **Doubly Compressed Momentum-assisted Stochastic Gradient Tracking** (DoCoM-SGT) algorithm which utilizes two levels of error-compensated compressions for tackling stochastic optimization problem in a communication efficient manner. Incorporated with gradient tracking, our algorithm is able to find a stationary solution without relying on additional conditions such as bounded similarity between data distributions.

- We provide a unified convergence analysis for DoCoM-SGT. Let \( f(\theta) \) be the averaged objective function across the network, to be defined in (1), our main result shows that DoCoM-SGT finds a solution, \( \tilde{\theta} \), in \( T \) iterations and communications rounds with \( \mathbb{E}[\|\nabla f(\tilde{\theta})\|^2] = \mathcal{O}(1/T^{2/3}) \) for general smooth (possibly non-convex) objective functions in Theorem 4.1, and with \( \mathbb{E}[f(\tilde{\theta}) - f^*] = \mathcal{O}(\log T/T) \) for objective functions satisfying the Polyak-Łojasiewicz condition in Corollary 4.1. For the latter case, we further show that if deterministic gradients are available, then DoCoM-SGT converges linearly in terms of the optimality gap. These convergence rates are comparable to state-of-the-art algorithms.

- We empirically evaluate the performance of DoCoM-SGT on training linear models and deep learning models using synthetic and real data, on non-convex losses.

Our analysis relies on the construction of a new Lyapunov function that handles the coupled terms between the errors of compression, gradient tracking and average iterates; see Lemma 4.5. We emphasize that obtaining the \( \mathcal{O}(1/T^{2/3}) \) bound requires a number of subtle modifications to be demonstrated in the proof outline of Sec. 4.1. Lastly, we compare the iteration complexities of state-of-the-art algorithms in Table 1. As seen, DoCoM-SGT is the only algorithm with compression and \( \mathcal{O}(\epsilon^{-3}) \) complexity.

**Notations.** \( \| \cdot \| \), \( \| \cdot \|_F \) denote Euclidean norm, Frobenius norm, respectively. The subscript-less operator \( \mathbb{E}[\cdot] \) is total expectation taken over all randomizations in operand.

### 1.1 Related Works

**Decentralized Optimization.** Algorithms for decentralized optimization have been first studied in (Nedic & Ozdaglar, 2009). The main idea is to mix communication (i.e., consensus) with optimization (i.e., gradient) steps. It has been extended to the stochastic setting (a.k.a. DSGD) in (Ram et al., 2010), and to directed graphs (Tsianos et al., 2012; Assran et al., 2019). Notably, Qu & Li (2017) proposed a gradient tracking technique where agents communicate local gradients to accelerate convergence.

In the stochastic non-convex optimization setting, we note that Lian et al. (2017) provided a performance analysis of DSGD; Lu et al. (2019) proposed GNSD which combines gradient tracking with stochastic gradient (also see (Tang et al., 2018b)); Lu & De Sa (2021) proposed DeTAG with optimal computation-communication tradeoff; Xin et al. (2021) proposed GT-HSGD which extended GNSD with momentum-based variance reduction, and a similar algorithm is in (Zhang et al., 2021). Note that the latter idea was proposed in Tran-Dinh et al. (2021); Cutkosky & Orabona (2019) to achieve optimal sampling complexity for centralized SGD. See a recent survey in (Chang et al., 2020).

**Communication Efficient Algorithm.** Methods for reducing communication burden in decentralized al-

### Table 1: Comparison of decentralized stochastic optimization algorithms for smooth non-convex objective functions

| Algorithms       | Iteration Complexity | Compress. | Remarks |
|------------------|----------------------|-----------|---------|
| DSGD             | \( \mathcal{O}(\max\left\{ \frac{\sigma_n^2}{n}\epsilon^{-4}, \frac{n(\sigma_n^2 + \sigma^2)}{\rho^2\epsilon^2}\right\}) \) | ×         | \( \epsilon^2 = \sup_{i,\theta} \|\nabla f_i(\theta) - \nabla f(\theta)\|^2 \)
| GNSD             | \( \mathcal{O}\left(\frac{1}{\rho^2\epsilon^2}\epsilon^{-4}\right) \) | ×         | \( C_0, C_1 \) are not explicitly defined, see (Lu et al., 2019). |
| DeTAG            | \( \mathcal{O}(\max\left\{ \frac{\sigma_n^2}{n\rho^2}\epsilon^{-4}, \log(\frac{n+\alpha_n\epsilon^{-1}}{\rho^2\epsilon^2}) \right\}) \) | ×         | \( \sigma_0 \) is variance of stoc. gradient at init., \( B \) is rounds of comm. per iteration. |
| GT-HSGD          | \( \mathcal{O}(\max\left\{ \frac{\sigma_n^2}{n}\epsilon^{-3}, \frac{C_n}{\sigma_n^2}\epsilon^{-4}, \frac{\sigma^2}{\rho^2\epsilon^2} \right\}) \) | ×         | \( G = \sup_{i,\theta} \mathbb{E}_{Z_i}[\|\nabla f_i(\theta; Z_i)\|^2] \)
| CHOCO-SGD        | \( \mathcal{O}(\max\left\{ \frac{\sigma_n^2}{n}\epsilon^{-3}, \frac{C_n}{\sigma_n^2}\epsilon^{-4}, \frac{1}{\rho^2\epsilon^2} \right\}) \) | ✓         | See Theorem 4.1 |
| DoCoM-SGT        | \( \mathcal{O}(\max\left\{ \frac{\sigma_n^2}{n}\epsilon^{-3}, \frac{C_n}{\sigma_n^2}\epsilon^{-4}, \frac{1}{\rho^2\epsilon^2} \right\}) \) | ✓         | |

depends on the data similarity across agents which is not ideal in light of applications such as federated learning (Konečný et al., 2016).
algorithms have been developed. For instance, (Aysal et al., 2008; Kashyap et al., 2007; Reisizadeh et al., 2019) studied quantization for average consensus protocol which is a main building block for decentralized algorithms. Notably, recent works (Liu et al., 2020; Liao et al., 2021; Song et al., 2021) showed that combining compression with gradient tracking technique lead to algorithms that converge linearly to an optimal solution. We remark that these algorithms bear similar structure to DoCoM-SGT, yet they focus on strongly convex objectives and consider deterministic gradients; see (Kovalev et al., 2021) with extension to stochastic setting.

2 Problem Setup & Background

We consider a weighted and undirected graph \( G = (\mathcal{N}, \mathcal{E}, \mathbf{W}) \) with the node set \( \mathcal{N} = \{1, \ldots, n\} \) representing a set of \( n \) agents, the edge set \( \mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \) representing the communication links between agents, and \( \mathbf{W} \in \mathbb{R}^{n \times n} \) is a weighted adjacency matrix. Note that self-loops are included such that \( \{i, i\} \in \mathcal{E} \) for all \( i \).

Our goal is to tackle the following stochastic optimization problem in a decentralized manner by the \( n \) agents on \( G \):

\[
\min_{\theta \in \mathbb{R}^d} f(\theta) := \frac{1}{n} \sum_{i=1}^{n} f_i(\theta),
\]

where \( f_i : \mathbb{R}^d \to \mathbb{R} \) is a continuously differentiable (possibly non-convex) objective function known to the \( i \)th agent. In particular, the objective function can be expressed as the expectation \( f_i(\theta) = \mathbb{E}_{\zeta \sim \mu_i}[f_i(\theta; \zeta)] \) such that \( \mu_i \) denotes the data distribution available at agent \( i \). Throughout this paper, we assume:

**Assumption 1.** There exists \( L \geq 0 \) such that for any \( i = 1, \ldots, n \), the gradient of \( f_i(\cdot; \zeta) \) is \( L \)-Lipschitz, i.e.,

\[
\|\nabla f_i(\theta; \zeta) - \nabla f_i(\theta'; \zeta)\| \leq L\|\theta - \theta'\|,
\]

for any \( \theta, \theta' \in \mathbb{R}^d \) and \( \zeta \in \text{supp}(\mu_i) \).

**Assumption 2.** The adjacency matrix \( \mathbf{W} \in \mathbb{R}^{n \times n} \) satisfies:

1. (graph topology) \( W_{ij} = 0 \) if \( \{i, j\} \notin \mathcal{E} \),
2. (doubly stochastic) \( \mathbf{W} \mathbf{1}_n = \mathbf{W}^\top \mathbf{1}_n = \mathbf{1}_n \),
3. (mixing) let \( \mathbf{U} \in \mathbb{R}^{n \times (n-1)} \) be a matrix with orthogonal columns satisfying \( \mathbf{I}_n - (1/n)\mathbf{1}_n\mathbf{1}_n^\top = \mathbf{U}\mathbf{U}^\top \), then there exists \( \rho \in (0, 1) \) such that \( \|
\mathbf{U}^\top \mathbf{W} \mathbf{W}^\top \mathbf{U}\| \leq 1 - \rho \).
4. (bounded eigenvalue) there exists \( \bar{\omega} \in (0, 2] \) such that \( \|
\mathbf{W} - \mathbf{I}_n\| \leq \bar{\omega} \).

The above conditions are standard. Assumption 1 requires the objective function to be smooth\(^1\), and there exists \( \mathbf{W} \) such that Assumption 2 is satisfied when \( G \) is a connected graph; see Boyd et al. (2004). Moreover, the gradient of \( f_i \) can be estimated as \( \nabla f_i(\theta; \zeta) \) satisfying:

**Assumption 3.** There exists \( \sigma \geq 0 \) such that for any \( \theta \in \mathbb{R}^d \), \( i = 1, \ldots, n \), the gradient estimate \( \nabla f_i(\theta; \zeta) \) with \( \zeta \sim \mu_i \) is unbiased with bounded second order moment, i.e.,

\[
\mathbb{E}[\nabla f_i(\theta; \zeta)] = \nabla f_i(\theta), \quad \mathbb{E}[\|
\nabla f_i(\theta; \zeta) - \nabla f_i(\theta)\|^2] \leq \sigma^2,
\]

where the expectations are taken w.r.t. \( \zeta \sim \mu_i \).

**DSGD and CHOCO-SGD Algorithms.** Equipped with Assumption 2, 3, a common practice for tackling (1) in a decentralized manner is to utilize \( \mathbf{W} \) as a mixing matrix. To illustrate the basic idea, we observe the decentralized stochastic gradient (DSGD) algorithm (Ram et al., 2010; Lian et al., 2017): at iteration \( t \),

\[
\theta_i^{t+1} = \sum_{j=1}^{n} W_{ij} \theta_j^t - \eta \nabla f_i^t, \quad \forall i,
\]

where \( \eta > 0 \) is the step size, \( \nabla f_i^t = \mathbb{E}[\nabla f_i(\theta_i^t; \zeta_t)] \) is the unbiased stochastic gradient with the data \( \zeta_t^i \sim \mu_i \) drawn independently upon fixing \( \theta_i^t \) and satisfying Assumption 3. For agent \( i \), the consensus step \( \sum_{j=1}^{n} W_{ij} \theta_j^t \) can be computed with a local average among the neighbors of \( i \).

A drawback of (3) is that agents are required to transmit \( d \) real numbers on \( G \) to their neighbors at every iteration. In practice, the communication links between agents are bandwidth limited and such algorithm may be undesirable when \( d \gg 1 \). To this end, a remedy is to apply compression to messages transmitted on \( G \).

Formally, we consider a stochastic compression operator \( Q : \mathbb{R}^d \to \mathbb{R}^d \) satisfying the condition:

\[\text{Our analysis is extensible to a slightly relaxed condition replacing (2) with } \mathbb{E}[\|
\nabla f_i(\theta; \zeta) - \nabla f_i(\theta'; \zeta)\|^2] \leq L^2\|\theta - \theta'\|^2.\]
Assumption 4. For any $x \in \mathbb{R}^d$, the compressor output $Q(x)$ is the random vector $\hat{Q}(x; \xi)$ with $\xi \sim \pi_x$ such that there exists $\delta \in (0, 1]$ satisfying

$$
E \left[ \|x - Q(x)\|^2 \right] = E \left[ \|x - \hat{Q}(x; \xi)\|^2 \right] \leq (1 - \delta) \|x\|^2.
$$

The above is a general condition on compressors as discussed in (Koloskova et al., 2019b). It is satisfied by a number of common designs. For instance, with $k \leq d$, the top-$k$ (resp. random-$k$) sparsifier given by

$$[Q(x)]_i = \begin{cases} x_i, & \text{if } i \in I_x, \\ 0, & \text{otherwise,} \end{cases}$$

where $I_x \subseteq \{1, \ldots, d\}$ with $|I_x| = k$ is the set of the coordinates of $x$ with the largest $k$ magnitudes (resp. uniformly selected at random), satisfies Assumption 4 with $\delta = \frac{k}{d}$. Other compressors such as random quantization can also satisfy Assumption 4; see (Alistarh et al., 2017; Stich et al., 2018; Alistarh et al., 2018). Note that sending $Q(x)$ in (4) over a communication channel requires only $k$ real number transmission. This achieves a $\frac{k}{d}$ compression ratio.

However, applying $Q(\cdot)$ to the consensus step in (3) directly does not lead to a convergent algorithm as the compression error will accumulate with $t \to \infty$. The CHOCO-SGD algorithm (Koloskova et al., 2019b) resolves the issue by incorporating an error feedback step: at iteration $t$,

$$
\hat{\theta}^{t+1} = \hat{\theta}^t + Q(\theta^t - \eta \nabla \hat{f}^t - \hat{\theta}^t), \quad (5a)
$$

$$
\theta^{t+1} = \theta^t - \eta \nabla \hat{f}^t + \gamma \sum_{j=1}^n W_{ij}(\theta^{t+1} - \hat{\theta}^{t+1}), \quad (5b)
$$

for all $i$, where $\gamma > 0$ is the consensus step size, and $\eta, \nabla \hat{f}^t$ were defined in (3). Instead of transmitting a compressed version of $\theta^t - \eta \nabla \hat{f}^t$ directly, CHOCO-SGD maintains an auxiliary variable $\hat{\theta}^t$ that accumulates the compressed difference $Q(\theta^t - \eta \nabla \hat{f}^t - \hat{\theta}^t)$. Subsequently the main variable $\theta^t$ is updated through a consensus step with this auxiliary variable. Koloskova et al. (2019a) proved that in $T$ iterations, CHOCO-SGD finds a near-stationary solution of (1), $(\theta^t)_{i=1}^\tau$ with $\tau \in \{0, \ldots, T-1\}$, satisfying $E[\|\nabla f((n^{-1} \sum_{i=1}^n \theta^t_i))\|^2] = \mathcal{O}(1/\sqrt{T})$.

A drawback of CHOCO-SGD is that the convergence of the algorithm requires the stochastic gradient $E[\|\nabla \hat{f}^t\|^2]$ to be bounded for any $i, t$, see Table 1 and (Koloskova et al., 2019b,a). The latter condition can be relaxed into requiring that the data similarity $\sup_{\theta \in \mathbb{R}^d} \|\nabla f_i(\theta) - \nabla f(\theta)\| \neq 0$, where the local objective functions have to be close to each other. Nevertheless, these quantities are not easy to control for applications such as federated learning Konečný et al. (2016) as the local data are non-i.i.d.

3 Proposed DoCoM-SGT Algorithm

Taking a closer look at CHOCO-SGD (5) reveals that when the data available at the agents are heterogeneous (a.k.a. non-i.i.d.), i.e., $\sup_{\theta \in \mathbb{R}^d} \|\nabla f_i(\theta) - \nabla f(\theta)\| \neq 0$, the algorithm can only utilize local gradient estimates $\nabla \hat{f}^t_i \approx \nabla f_i(\hat{\theta}^t_i)$. This estimate can be large even when the solution $\theta^t_i$ is close to a stationary point of (1). As a result, the algorithm needs to incorporate a small step size $\eta$ (or vanishing step size as $t \to \infty$) to compensate for the accumulated error.

We propose the Doubly Compressed Momentum-assisted Stochastic Gradient Tracking (DoCoM-SGT) algorithm which offers improved convergence properties over CHOCO-SGD. Let $\eta > 0$ be step size, $\gamma, \beta \in (0, 1]$, the DoCoM-SGT algorithm at iteration $t \in \mathbb{N}$ reads

$$
\theta^{t+1} = \theta^t - \eta g^t + \gamma \sum_{j=1}^n W_{ij}(\theta^{t+1} - \hat{\theta}^{t+1})
$$

$$
\hat{\theta}^{t+1} = \hat{\theta}^t + Q(\theta^t - \eta g^t - \hat{\theta}^t),
$$

$$
v^{t+1} = \beta \nabla \hat{f}^{t+1} + (1 - \beta) [v^t + \nabla \hat{f}^{t+1} - \nabla \hat{f}^t],
$$

$$
g^{t+1} = g^t + v^{t+1} - v^t + \gamma \sum_{j=1}^n W_{ij}(\theta^{t+1} - \hat{\theta}^{t+1})
$$

$$
\hat{g}^{t+1} = \hat{g}^t + Q(g^t + v^{t+1} - v^t - g^t),
$$

where we draw the sample $\zeta^{t+1} \sim \mu_i$ at agent $i$ (or a minibatch of samples) and define $\nabla \hat{f}^{t+1} \equiv \nabla f_i(\theta^{t+1}; \zeta^{t+1})$, $\nabla \hat{f}^t_i \equiv \nabla f_i(\theta^t_i; \zeta^{t+1})$ such that the stochastic gradients in (6c) are formed using the same data batch. In
Lastly, we notice that Cutkosky & Orabona (2019); Tran-Dinh et al. (2021). This section analyzes the expected convergence rate of the DoCoM-SGT algorithm in seeking a (near-)stationary solution of (1). We demonstrate that it achieves state-of-the-art performance for decentralized optimization.

Algorithm 1 DoCoM-SGT Algorithm

1. **Input**: mixing matrix $W$; step sizes $\eta, \gamma$; momentum $\beta$; initial batch number $b_0$; initial iterate $\theta^0 \in \mathbb{R}^d$. 
2. Initialize $\theta_i^0 = \theta^0, \forall i \in [n], \theta_{i,j}^0 = \theta^0, \forall (i, j) \in \mathcal{E}$. 
3. Initialize stochastic gradient estimate 
   
   \[
   g_i^0 = \frac{1}{b_0} \sum_{r=1}^{b_0} \nabla f_i(\theta_i^r; \xi_i^0); \quad \{\xi_i^0\}_{r=1}^{b_0} \sim \mu_i
   \]

4. for $i$ in $0, \ldots, T-1$ do

5. \quad $\forall i : \theta_i^{t+\frac{1}{2}} = \theta_i^t - \eta g_i^t$

6. \quad for $(i, j) \in \mathcal{E}$ (notice $(i, i) \in \mathcal{E}$) do

7. \quad \quad Agent $j$ receive $Q(\theta_i^{t+\frac{1}{2}} - \hat{\theta}_{ij}^t)$ from agent $i$

8. \quad \quad Set $\hat{\theta}_{ij}^{t+1} = \hat{\theta}_{ij}^t + \frac{\beta}{Q} (\theta_i^{t+\frac{1}{2}} - \hat{\theta}_{ij}^t)$

9. \quad for $(i, j) \in \mathcal{E}$ (notice $(i, i) \in \mathcal{E}$) do

10. \quad \quad Agent $j$ receive $Q(g_i^{t+\frac{1}{2}} - \hat{\theta}_{ij}^t)$ from agent $i$

11. \quad \quad Set $\hat{g}_{ij}^{t+1} = \hat{g}_{ij}^t + Q(g_i^{t+\frac{1}{2}} - \hat{g}_{ij}^t)$

12. \quad \quad Agent $j$ receive $Q(g_i^{t+\frac{1}{2}} - \hat{g}_{ij}^t)$ from agent $i$

13. \quad Agent $j$ receive $Q(g_i^{t+\frac{1}{2}} - \hat{g}_{ij}^t)$ from agent $i$

14. Agent $j$ receive $Q(g_i^{t+\frac{1}{2}} - \hat{g}_{ij}^t)$ from agent $i$

15. Agent $j$ receive $Q(g_i^{t+\frac{1}{2}} - \hat{g}_{ij}^t)$ from agent $i$

16. Agent $j$ receive $Q(g_i^{t+\frac{1}{2}} - \hat{g}_{ij}^t)$ from agent $i$

17. Agent $j$ receive $Q(g_i^{t+\frac{1}{2}} - \hat{g}_{ij}^t)$ from agent $i$

18. Agent $j$ receive $Q(g_i^{t+\frac{1}{2}} - \hat{g}_{ij}^t)$ from agent $i$

19. **Output**: pick the $T$th iterate $\theta_i^T$, where $T$ is uniformly selected from $\{0, \ldots, T-1\}$; or the last iterate $\theta_i^T$.

Algorithm 1 provides the pseudo-code of DoCoM-SGT which provides details on the initialization and implementation.

The DoCoM-SGT algorithm features two ingredients: (A) a gradient tracking step with compression where each agent maintains an estimate of the averaged gradient $n^{-1} \sum_{i=1}^{n} \nabla \hat{f}_i$; (B) a momentum-based variance reduction step to improve convergence rate, where our update form is similar to that of GT-HSGD (Xin et al., 2021).

We observe that the compressed consensus step on $\{\theta_i^t\}_{t=1}^{\infty}$ (6a), (6b) resembles that of CHOCO-SGD (5) except the local update is computed along the direction $\hat{g}_i^t$; the latter can be updated according to another compressed consensus steps in (6d), (6e) which aims at tracking the dynamically updated average gradient estimator $g_i^t \approx n^{-1} \sum_{j=1}^{n} v_{ij}$. Moreover, (6c) uses a variance reduced estimate of the gradient with a recursive step similar to Cutkosky & Orabona (2019); Tran-Dinh et al. (2021).

Lastly, we notice that DoCoM-SGT shares a similar communication and computation cost per iteration as CHOCO-SGD, except that an extra communication step (with compression) is needed for the tracking of $n^{-1} \sum_{i=1}^{n} v_i^t$ and an extra computation step is needed for computing $\nabla \hat{f}_i$ in (6d), (6e). Similar to CHOCO-SGD, the DoCoM-SGT algorithm requires each agent to store the auxiliary variables $\{\hat{\theta}_j, \hat{g}_j\}_{j \in \mathcal{N}_i}$ of its neighbors to apply error compensation. As we will show later, the above shortcomings can be overcome as DoCoM-SGT has a better convergence rate.

### 4 Convergence Analysis

This section analyzes the expected convergence rate of the DoCoM-SGT algorithm in seeking a (near-)stationary solution of (1). We demonstrate that it achieves state-of-the-art performance for decentralized optimization.

Let $\bar{\theta} := n^{-1} \sum_{i=1}^{n} \theta_i^t$ be the averaged iterate, $\overline{\nabla} := n^{-1} \sum_{i=1}^{n} \|g_i^0\|^2$ be the initial expected gradient norm, $f^* := \min_{\theta} f(\theta)$ be the optimal objective value. We first summarize the convergence results under the mentioned assumptions where (1) is possibly non-convex:

**Theorem 4.1.** Under Assumption 1, 2, 3, 4. Suppose that the step sizes satisfies

\[
\eta \leq \min \left\{ \eta_{\infty}, \sqrt{\frac{3n}{(8C_\beta)}} \right\}, \quad \gamma \leq \gamma_{\infty},
\]

(7)
where $\gamma_\infty$, $\eta_\infty$ are defined in (18). We set $\beta \in (0,1)$, $\bar{\beta} = \min\{\frac{\rho}{\beta}, \frac{\delta}{\beta}, \beta\}$. Then, for any $T \geq 1$, it holds

\[
\frac{1}{T} \sum_{t=0}^{T-1} E \left[ \frac{1}{2} \|\nabla f(\theta_t^t)\|^2 + \frac{L^2}{n} \sum_{i=1}^{n} \|\theta_i^t - \bar{\theta}_t^\ast\|^2 \right] \leq \frac{f(\bar{\theta}_0^t) - f^\ast}{\eta T/2} + C_\sigma \frac{2\beta^2 \sigma^2}{\beta n} + \frac{4\sigma^2}{b_0 \beta T n} + \frac{\gamma^2}{\beta T} \frac{384 L^2 \bar{C}_0}{(1-\gamma)^4},
\]

where

\[
C_\sigma = 4 + \frac{\eta^2}{\gamma^3} \frac{672 L^2 n}{\rho^3} + \frac{\eta^2}{\gamma^2} \frac{6L^2 n \rho \delta}{25 \omega^2} + \frac{\gamma^2}{\gamma^2} \frac{4L^2}{\rho \omega^2},
\]

\[
C_\rho = 8(1-\beta)^2 L^2 (1-\rho \gamma)^2 + \frac{L^2 n}{\rho \gamma} \left( 96 + \frac{141 \rho^2}{400 \omega^2} \right).
\]

### Convergence Rate

Setting the step sizes and parameters as $\beta = \Theta(\frac{n^{1/3}}{T^{2/3}})$, $\eta = \Theta(\frac{n^{2/3}}{T^{3/2}})$, $\gamma = \gamma_\infty$, $b_0 = \Omega(T^{1/3})$. Further, we select the $T$th iterate such that $T$ is independently and uniformly selected from $\{0,\ldots,T-1\}$ [cf. the output of Algorithm 1], similar to (Ghadimi & Lan, 2013). For a sufficiently large $T$, it can be shown that

\[
n^{-1} \sum_{i=1}^{n} E \left[ \|\nabla f(\theta^T_i)\|^2 \right] = O \left( \frac{L(f(\bar{\theta}_0^t) - f^\ast)}{(nT)^{2/3}} + \frac{\sigma^2}{(nT)^{2/3}} \frac{n \bar{C}_0}{\delta^2 \rho T^3} + \frac{\sigma^2 \gamma^{5/3}}{\delta^3 \rho T^{3/4}} \right),
\]

where we have used (8) and the Lipschitz continuity of $\nabla f_i(\cdot)$ [cf. Assumption 1] to derive a bound on the gradient of individual iterate $\theta^T_i$. For any agent $i = 1, \ldots, n$, the iterate $\theta^T_i$ at the output of Algorithm 1 is guaranteed to be $O(1/T^{2/3})$-stationary to (1). Notice that this is a state-of-the-art convergence rate for first order stochastic optimization even in the centralized setting; see (Cutkosky & Orabona, 2019; Tran-Dinh et al., 2021). Our rate is comparable to or faster than a number of decentralized algorithms with or without compression; see Table 1.

### Impacts of Network Topology and Compressor

Eq. (10) indicates the impacts of network topology (due to $\rho$) and compressor (due to $\delta$) vanish as $T \rightarrow \infty$. This can be observed by recognizing that the last two terms in (10) are in the order of $O(1/T), O(1/T^{4/3})$. In Appendix A.10, we demonstrate with a similar set of step sizes, for any $T \geq T_{\text{trans}} = \Omega(n^3 \bar{C}_0/(\sigma^6 \delta^6 \rho^{12}))$, DoCoM-SGT enjoys a matching convergence behavior as a centralized SGD algorithm employing a momentum-based variance reduced gradient estimator with a batch size of $n$, e.g., (Tran-Dinh et al., 2021). In the latter case, we have $n^{-1} \sum_{i=1}^{n} E \left[ \|\nabla f(\theta^T_i)\|^2 \right] = O(\sigma^2/nT^{2/3})$. Here, the constant $T_{\text{trans}}$ is also known as the transient time of the decentralized algorithm (Pu et al., 2020).

Besides, we remark that our result does not require any assumption on the data heterogeneity level nor the boundedness of gradient as in CHOCO-SGD (Koloskova et al., 2019a) or DSGD (Lian et al., 2017). As mentioned before, this is a consequence of the gradient tracking procedure applied. In Appendix B, we provide a separate analysis for the case of $\beta = 1$, i.e., when no momentum is applied in the algorithm (6c). Interestingly, in the latter case, the fastest convergence rate achievable in our analysis is only $O(1/\sqrt{T})$ [cf. (52)], indicating that the momentum term may be crucial in accelerating DoCoM-SGT.

### PL Condition

Finally, we show that the convergence rate can be improved when the objective function satisfies the Polyak-Lojasiewicz (PL) condition:

**Assumption 5.** For any $\theta \in \mathbb{R}^d$, it holds that

\[
\|\nabla f(\theta)\|^2 \geq 2\mu [f(\theta) - f^\ast].
\]

Notice that the PL condition is satisfied by strongly convex functions as well as a number of non-convex functions; see Karimi et al. (2016). We obtain:

**Corollary 4.1.** Under Assumption 1, 2, 3, 4, 5. Suppose that the step size condition (7) holds and $\beta \in (0,1)$. Then, for any $t \geq 1$, it holds

\[
\Delta^t + \frac{L^2}{\rho n} \sum_{i=1}^{n} E \left[ \|\theta^T_i - \bar{\theta}^\ast\|^2 \right] \leq \left( 1 - \bar{\beta} \right) \Delta^0 + \frac{2\sigma^2}{\beta n} \psi^0 + \frac{\eta^2}{\beta \sigma} \frac{2C\sigma^2}{n},
\]

where $\bar{\beta} := \min \{\eta \mu, \bar{\beta}/2\}$, $\Delta^t := E [f(\theta^T)] - f^\ast$ is the expected optimality gap and the constant $C_\sigma$ is defined in (9). Notice that $\psi^0$ can be upper bounded with (22).
Setting the step sizes and parameters as $\beta = \Theta(\log T/T)$, $\eta = \Theta(\log T/T)$, $\gamma = \gamma_\infty$, $b_0 = \Omega(1)$. For sufficiently large $T$, it can be shown that
\[
E[f(\bar{\theta}^T)] - f^* = \mathcal{O}(\log T/T), \tag{13}
\]
\[
\frac{1}{n} \sum_{n=1}^n E[\|\bar{\theta}_n^T - \bar{\theta}^T\|_2^2] = \mathcal{O}(\log T/T), \tag{14}
\]
see Appendix A.11. Moreover, in the deterministic gradient case with $\sigma^2 = 0$, we can select $\beta = \Theta(1)$, $\eta = \Theta(1)$. Then, (12) shows that DoCoM-SGT converges linearly to an optimal solution such that $E[f(\bar{\theta}^T)] - f^* = \mathcal{O}(1 - \bar{\beta}^T)$. The latter matches the performance of recently proposed algorithms with compression Liu et al. (2020); Liao et al. (2021); Song et al. (2021); Kovalev et al. (2021).

4.1 Proof of Theorem 4.1

We preface the proof by defining the following notations for the variables in DoCoM-SGT. For any $t \geq 0$:
\[
\Theta_t = \left( (\theta_1^t)^\top \right), \quad V_t = \left( (v_1^t)^\top \right), \quad G_t = \left( (g_1^t)^\top \right),
\]
which are $n \times d$ matrices. Similarly, we define the matrices $\hat{\Theta}_t^t$, $\hat{G}_t^t$ based on $\{\hat{\theta}_1^t\}_{i=1}^n$, $\{\hat{g}_1^t\}_{i=1}^n$, and the matrices $\nabla \hat{F}_t$, $\nabla \hat{F}_t$, $\nabla F$ based on $\{\nabla \hat{F}_1^t\}_{i=1}^n$, $\{\nabla \hat{F}_1^t\}_{i=1}^n$, $\{\nabla F_i\}_{i=1}^n$.

The norm of the matrix $\Theta_0^t = U^\top \Theta^t$, i.e., $\|\Theta_0^t\|_F$, measures the consensus error of the iterate $\Theta^t$ since $\Theta_0^t = U^\top (I - (1/n)11^\top) \Theta^t$; similarly, we denote $G_0^t = U^\top G^t$ such that $G_0^t$ measures the consensus error of $G^t$.

Denote the average variables $\bar{\theta}^t = n^{-1}1^\top \Theta_t^0$, $\bar{v}^t = n^{-1}1^\top V_t^0$, $\bar{g}^t = n^{-1}1^\top G_t^0$, $\nabla \bar{F}^t = n^{-1}1^\top \nabla F^t$. We have the following observation regarding the $\bar{\theta}$-update:

Lemma 4.2. Under Assumption 1 and the step size condition $\eta \leq \frac{1}{\pi}$. Then, for any $t \geq 0$, it holds
\[
f(\bar{\theta}^{t+1}) \leq f(\bar{\theta}^t) - \frac{\eta}{2} \|\nabla f(\bar{\theta}^t)\|^2 + \frac{L^2 \eta}{n} \|\Theta_0^t\|^2_F \tag{15}
+ \eta \left\| \bar{v}^t - \nabla \bar{F}^t \right\|^2 - \frac{\eta}{4} \|\bar{g}^t\|^2.
\]

The proof is relegated to Appendix A.2 and is established using the relation $\bar{\theta}^{t+1} = \bar{\theta}^t - \eta \bar{g}^t$. We remark that the above lemma utilizes just Assumption 1 and results in a deterministic bound of $f(\bar{\theta}^{t+1})$.

From Lemma 4.2, we observe that controlling $\|\nabla f(\bar{\theta}^t)\|^2$ requires bounding $\|\Theta_0^t\|^2_F$ and $\|\bar{v}^t - \nabla \bar{F}^t\|^2$. We have a set of coupled recursion formulas as

Lemma 4.3. Under Assumption 2, 4. Then, for any $t \geq 0$, it holds
\[
E[\|\Theta_0^{t+1}\|^2_F] \leq \left(1 - \frac{\rho \gamma}{2}\right) E[\|\Theta_0^t\|^2_F] + \frac{2}{\rho} \frac{\eta^2}{\gamma} E[\|G^t\|^2_F]
+ \frac{2 \eta^2}{\rho \gamma} E\left[\left\| \Theta^t = \eta G^t - \hat{\Theta}^t \right\|^2_F\right]. \tag{16}
\]

Lemma 4.4. Under Assumption 1, 2, 3, 4 and let $\beta \in [0, 1)$. Then, for any $t \geq 0$, it holds
\[
(1 - \beta)^2 E\left[\left\| \bar{\theta}^{t+1} - \nabla \bar{F}^{t+1} \right\|^2\right]
\leq E\left[\left\| \bar{v}^t - \nabla \bar{F}^t \right\|^2\right] + \frac{2 \beta^2}{(1 - \beta)^2} \frac{\sigma^2}{n}
+ \frac{8 L^2}{n^2} \eta^2 (1 - \rho \gamma)^2 E\left[\left\| G^t \right\|^2_F + \frac{n}{2} \left\| \bar{g}^t \right\|^2\right]
+ \frac{8 L^2}{n^2} \omega^2 \gamma^2 E\left[\left\| \Theta^t = \eta G^t - \hat{\Theta}^t \right\|^2_F + \|\Theta_0^t\|^2_F\right]. \tag{17}
\]
The proofs are relegated to Appendix A.3, A.4: the latter also provides a bound on the difference matrix \( \mathbb{E} \left[ \left\| \nabla^t - \nabla F^t \right\|^2 \right] \). Moreover, in Appendix A.1, we show that \( \mathbb{E}[\|G^t\|^2_F], \mathbb{E}[\|\Theta^t - \bar{\Theta}^t\|^2_F], \mathbb{E}[\|\hat{G}^t - \hat{G}^t\|^2_F] \) can be bounded with a similar set of coupled recursions.

Together, let us define the Lyapunov function:

\[
\mathcal{V}(t) = \mathbb{E} \left[ L \frac{\|\Theta^t\|^2}{F} + n \left\| \nabla^t - \nabla \mathcal{F}^t \right\|^2 + \frac{1}{n} \left\| \mathcal{V}^t - \nabla \mathcal{F}^t \right\|^2 \right]
\]

where \( a, b, c > 0 \) are determined below. We observe

**Lemma 4.5.** Under Assumption 1, 2, 3, 4 and let \( \beta \in (0, 1) \). Suppose that the step sizes satisfy:

\[
\begin{align*}
\gamma &\leq \min \left\{ \frac{1}{4p}, \frac{\rho m}{64\omega^2}, \frac{\delta_0}{10\omega}, \frac{\rho \sqrt{1 - \gamma}}{250\omega^2} \right\} =: \gamma_{\infty}, \\
\eta &\leq \frac{\gamma}{L} \min \left\{ \sqrt{\frac{1 - \beta}{\beta n}}, \frac{\rho^2}{45}, \frac{\rho^2}{240\omega} \right\} =: \eta_{\infty}.
\end{align*}
\]

Then, for any \( t \geq 0 \), it holds

\[
\mathcal{V}^{t+1} \leq (1 - \beta)\mathcal{V}^t + \beta^2 C_\sigma \sigma^2 + \eta^2 C_\rho \mathbb{E} \left[ \left\| \varphi^t \right\|^2 \right],
\]

where we have set \( a = \frac{96\lambda^2}{\rho^2 \gamma} \gamma^2, b = \frac{\eta^2}{\rho (1 - \gamma)}, c = \frac{\gamma^2}{1 - \gamma} \frac{48\lambda^2 \omega^2}{\delta \rho} \) in the definition of \( \mathcal{V}^t \), and \( C_\sigma, C_\rho, \beta \) were defined in Theorem 4.1.

The proof is relegated to Appendix A.8 where we demonstrated how to derive a set of tight parameters for \( a, b, c \).

Equipped with (19) and define \( \Delta^t := \mathbb{E}[f(\hat{\theta}^t)] - f^* \). From Lemma 4.2, we can deduce that

\[
\begin{align*}
\Delta^{t+1} + \frac{2\eta}{n_\beta} \varphi^{t+1} &\leq \Delta^t + \frac{2\eta}{n_\beta} \varphi^t + \frac{2\eta}{n_\beta} \beta^2 C_\sigma \sigma^2 \\
&- \eta \mathbb{E} \left[ \frac{1}{2} \left\| \nabla f(\hat{\theta}^t) \right\|^2 + \frac{L^2}{n} \left\| \Theta^t_0 \right\|^2_F \right] \\
&+ (n_\beta)^{-1} 2\eta^3 C_\rho - 4^{-1} \eta \mathbb{E} \left[ \left\| \varphi^t \right\|^2 \right]
\end{align*}
\]

Setting \( \eta \leq \sqrt{\frac{2n_\beta}{\rho^2 \gamma}} \) as in (7) shows that the last term in the r.h.s. of the above can be upper bounded by zero. Summing up both sides of (20) from \( t = 0 \) to \( t = T - 1 \) yields

\[
\begin{align*}
\eta \sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{1}{2} \left\| \nabla f(\hat{\theta}^t) \right\|^2 + \frac{L^2}{n} \left\| \Theta^t_0 \right\|^2_F \right] \\
&\leq \Delta^0 + \frac{2\eta}{n_\beta} \varphi^0 + \frac{2\eta T}{n_\beta} \beta^2 C_\sigma \sigma^2
\end{align*}
\]

Furthermore, with the initialization, choice of \( a, b, c \) and the step size \( \gamma \leq \gamma_{\infty} \), it can be shown that

\[
\varphi^0 \leq \frac{2a^2}{b_0} + \frac{192L^2 n}{\rho^2 \gamma^2 (1 - \gamma)} \frac{\rho \gamma}{\rho^2 \gamma^2} \gamma^2.
\]

Dividing both sides of the inequality (21) by \( \eta T \) and observing that \( \left\| \Theta^t_0 \right\|^2_F \geq \left\| (I - (1/n)11^T) \Theta^t \right\|^2_F \) concludes the proof of Theorem 4.1.

**Proof of Corollary 4.1.** Applying the PL condition of Assumption 5 to the inequality (15) shows that

\[
\begin{align*}
\Delta^{t+1} &\leq (1 - \eta \mu) \Delta^t + \eta \mathbb{E} \left[ \frac{L^2}{n} \left\| \Theta^t_0 \right\|^2_F + \left\| \varphi^t - \nabla \mathcal{F}^t \right\|^2_F \right] \\
&- \frac{\eta}{4} \mathbb{E} \left[ \left\| \varphi^t \right\|^2 \right].
\end{align*}
\]

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Combining with Lemma 4.5 shows that

$$\Delta^{t+1} + \frac{2n}{\beta n} \gamma^{t+1} \leq \left(1 - \beta\right) \left[ \Delta^t + \frac{2n}{\beta n} \gamma^t \right] + \frac{\eta^2}{\beta n} 2C_g \sigma^2$$

$$+ \left( \frac{\eta^2}{\beta n} 2C_g - \frac{2}{\gamma} \right) \mathbb{E} \left[ \|g^t\|^2 \right],$$

where we have used $1 - \beta + \frac{\pi}{2n} \frac{n}{n} \leq 1 - \beta$. Setting $\eta^2 \leq \frac{\pi}{4\epsilon\alpha}$ and telescoping the above relation concludes the proof.

5 Numerical Experiments

Setup. In all experiments, we compare DoCoM-SGT to decentralized stochastic gradient algorithms including GNSD (Lu et al., 2019), DeTAG (Lu & De Sa, 2021), GT-HSGD (Xin et al., 2021), and compressed algorithms including CHOCO-SGD (Koloskova et al., 2019b). Our experiments are performed by running the optimization algorithms (i.e., training) on a 40 cores Intel(R) Xeon(R) Gold 6148 CPU @ 2.40GHz server with MPI-enabled PyTorch and evaluating the performance of trained models on a Tesla K80 GPU server. To simulate the scenario of heterogeneous data distribution, each agent has a disjoint set of training samples, while we evaluate each model by its performance on all of the training/testing data in the network.

Hyperparameter Tuning. For all algorithms we choose the learning rate $\eta$ from $\{0.1, 0.01, 0.001\}$, and fix the regularization parameter as $\lambda = 10^{-4}$ [cf. (24)]. For compressed algorithms, we adopt the top-$k$ compressor and we select the consensus step size $\gamma$ starting from the compression ratio $k/d$, and then increment at steps of 0.01 until divergence. For DeTAG, we adopt the parameters from (Lu & De Sa, 2021). For DoCoM-SGT and GT-HSGD, we choose the best momentum parameter $\beta$ in $\{0.0001, 0.001, 0.01, 0.1, 0.5, 0.9\}$ and fix the initial batch number as $b_{0,i} = m_i$. We choose the batch sizes such that all algorithms spend the same amount of computation on stochastic gradient per iteration. The tuned parameters and additional results can be found in Appendix C.

Synthetic Data with Linear Model. We consider a set of synthetic data generated with the leaf benchmarking framework (Caldas et al., 2019). The task is to train a linear classifier for a set of 1000-dimensional features with $n = 1443$ samples partitioned into $n = 25$ non-i.i.d. portions, each held by an agent that is connected to the others on a ring graph with uniform edge weights. Each feature vector is labeled into one of 5 classes. Altogether, the local dataset for the $i$th agent is given by $\{x_j, \{f_{j,k} \}_{k=1}^{5} m_i \}_{j=1}^{m_i}$, where $m = \sum_{i=1}^{25} m_i$,
\( x_j^i \in \mathbb{R}^{1000} \) denotes the \( j \)th feature, and \( \{\ell^i_{j,k}\}_{k=1}^5 \in \{0, 1\}^5 \) is the label such that \( \ell^i_{j,k} = 1 \) if the \( j \)th feature has label \( k \in \{1, ..., 5\} \).

To train a linear classifier \( \theta = (\theta_1, ..., \theta_5) \in \mathbb{R}^{5000} \) in a decentralized manner, we consider (1) with the following non-convex objective function that models a modified logistic regression problem with sigmoid loss and \( \ell_2 \) regularization:

\[
f_i(\theta) = \frac{1}{m} \sum_{j=1}^{m} \sum_{k=1}^{5} \phi(\ell^i_{j,k} \langle x_j^i, \theta_k \rangle) + \frac{\lambda}{2} \|\theta\|_2^2,
\]

where \( \phi(z) = (1 + e^{-z})^{-1} \) and \( \lambda = 10^{-4} \) is the regularization parameter. Notice that \( f_i(\theta) \) is a non-convex function in \( \theta \), and we estimate its gradient stochastically by sampling a mini-batch of data. Our numerical experiment results are presented in Fig. 1 as we compare the worst agent’s loss values \( \max_i f(\theta^i) \) and consensus gap \( \max_i \|\theta^i - \theta^j\|_2^2 \) against the communication cost and iteration number.

With the same number of iterations, DoCoM-SGT performs slightly slower than the other algorithms such as GT-HSGD, DeTAG in terms of the loss value, yet DoCoM-SGT achieves the fastest convergence in terms of the communication cost (number of floats transmitted). This is a main advantage since DoCoM-SGT incorporates compression for every message exchanged. Furthermore, we observe that DoCoM-SGT outperforms CHOCO-SGD significantly in the due to the use of gradient tracking and momentum. Lastly, DoCoM-SGT finds a solution with the lowest consensus gap (that is 35 times lower than CHOCO-SGD) given the same communication budget.

**FEMNIST Data with LeNet-5.** We consider training a LeNet-5 (with \( d = 60850 \) parameters) neural network on the FEMNIST dataset. The dataset contains \( m = 805263 \) samples of \( 28 \times 28 \) hand-written character images, each belongs to one of the 62 classes. The samples are partitioned into \( n = 36 \) agents according to the groups specified in (Caldas et al., 2019). These agents are arranged according to a ring topology with uniform edge weights. We tackle (1) with \( f_i(\theta) \) taken as the cross entropy loss function of the local dataset and an \( \ell_2 \) regularization is applied with the parameter of \( \lambda = 10^{-4} \). We use a decreasing learning rate at \( \eta^0, \eta^0/10, \eta^0/100 \) during the 0-15th epoch, 16-30th epoch, 30-50th epoch of training; see Table 3.

Fig. 2 compares the worst-agent’s loss function, \( \max_i f(\theta^i) \), against the communication cost and iteration number. We observe that the communication efficiency gap between the compressed and uncompressed algorithms has widened, in which DoCoM-SGT and CHOCO-SGD can achieve the same level of loss values with 10-20x less communication cost. Moreover, DoCoM-SGT has similar performance as CHOCO-SGD in terms of the training loss, and it yields a better consensus gap than CHOCO-SGD with the same communication budget. Notice that in this experiment, we selected a compression ratio \( k/d \) of 0.05 and 0.1 for DoCoM-SGT and CHOCO-SGD.

**Conclusions.** We have proposed the DoCoM-SGT algorithm for communication efficient decentralized learning and shown that the algorithm achieves a state-of-the-art \( O(\epsilon^{-3}) \) iteration complexity. Future works include investigating the effect of reducing the frequency of (compressed) communication, as done in (near-)optimal algorithms such as Sun et al. (2020); Lu & De Sa (2021).
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A  Missing Proofs from Section 4.1

Using the matrix notations defined in the preface of Section 4.1, we observe that DoCoM-SGT (6) can be expressed conveniently as

\begin{align*}
\Theta^{t+1} &= \Theta^t - \eta G^t + \gamma(W - I)\hat{G}^{t+1} \\
\hat{G}^{t+1} &= \hat{G}^t + Q(\Theta^t - \eta G^t - \hat{G}^t) \\
V^{t+1} &= \beta \nabla F^{t+1} + (1 - \beta)(V^t + \nabla \hat{F}^{t+1} - \nabla \hat{F}^t) \quad \text{where} \quad Q(X) = \begin{pmatrix} Q(x_1)^T \\ \vdots \\ Q(x_n)^T \end{pmatrix}, \forall X \in \mathbb{R}^{n \times d}.
\end{align*}

The above simplified expression of the algorithm will be useful for developing our subsequent analysis.

A.1  Bounds on \(\|G_o^t\|_F^2, \|\Theta^t - \eta G^t - \hat{G}^t\|_F^2\) and \(\|G^t - \hat{G}^t\|_F^2\)

**Lemma A.1.** Under Assumption 1, 2, 3, 4 and the step size conditions \(\eta \leq \frac{\rho \gamma}{10L(1-\rho \gamma)\sqrt{1+\gamma^2 \omega^2}} \), \(\gamma \leq \frac{1}{8\omega} \). For any \(t \geq 0\), it holds

\[
\mathbb{E} \left[ \|G_o^{t+1}\|_F^2 \right] \leq \left(1 - \frac{\rho \gamma}{4}\right) \mathbb{E} \left[ \|G_o^t\|_F^2 \right] + \gamma \frac{2\omega^2}{\rho} \mathbb{E} \left[ \|G^t - \hat{G}^t\|_F^2 \right] + \gamma \frac{2L^2\omega^2}{\rho} \mathbb{E} \left[ \|\Theta^t\|_F^2 \right]
\]

\[
\quad + \gamma \frac{13L^2}{\rho} \omega^2 (1 - \delta) \mathbb{E} \left[ \|\Theta^t - \eta G^t - \hat{G}^t\|_F^2 \right] + \gamma \frac{\rho \gamma}{5} \mathbb{E} \left[ \|\hat{g}^t\|_F^2 \right] + \frac{7}{\rho \gamma} \beta^2 \mathbb{E} \left[ \|V^t - \nabla F^t\|_F^2 \right] + \frac{7n}{\rho \gamma} \beta^2 \sigma^2.
\]

**Lemma A.2.** Under Assumption 1, 2, 3, 4 and the step size conditions \(\eta \leq \min\{\frac{\rho \gamma}{16L\gamma(1-\delta)(1+3\gamma^2 L^2)/(1+2/\delta)}, \frac{1}{4\delta}, \frac{1}{124\omega^2}\} \), \(\eta^2 \gamma \leq \frac{\rho \gamma}{1248\omega^2} \). For any \(t \geq 0\), it holds

\[
\mathbb{E} \left[ \|\Theta^{t+1} - \eta G^t - \hat{G}^{t+1}\|_F^2 \right] \leq \left(1 - \frac{\delta}{8}\right) \mathbb{E} \left[ \|\Theta^t - \eta G^t - \hat{G}^t\|_F^2 \right] + \eta \frac{50}{\delta} \mathbb{E} \left[ \|G_o^t\|_F^2 \right] + \eta \frac{3\omega^2}{\rho} \mathbb{E} \left[ \|G^t - \hat{G}^t\|_F^2 \right]
\]

\[
\quad + \frac{29}{\delta} \omega^2 \gamma^2 + \frac{38L^2\omega^2}{\rho} \mathbb{E} \left[ \|\Theta^t\|_F^2 \right] + \frac{18\rho^2}{\delta} n \mathbb{E} \left[ \|\hat{g}^t\|_F^2 \right] + \left(18 + \frac{84}{\rho \gamma} \right) \beta^2 \frac{\eta^2 \gamma^2}{\delta} \frac{n \alpha^2}{\delta} (25)
\]

**Lemma A.3.** Under Assumption 1, 2, 3, 4 and the step size conditions \(\gamma \leq \frac{\delta}{8\omega} \), \(\eta \leq \frac{\rho \gamma}{10L(1-\rho \gamma)\sqrt{1+\gamma^2 \omega^2}} \). For any \(t \geq 0\), it holds

\[
\mathbb{E} \left[ \|G^{t+1} - \hat{G}^{t+1}\|_F^2 \right] \leq \left(1 - \frac{\delta}{8}\right) \mathbb{E} \left[ \|G^t - \hat{G}^t\|_F^2 \right] + \frac{10}{\delta} \gamma^2 \left(\omega^2 + \frac{\rho^2}{8}\right) \mathbb{E} \left[ \|G_o^t\|_F^2 \right]
\]

\[
\quad + \gamma^2 \frac{122L^2\omega^2}{\delta} \mathbb{E} \left[ \|\Theta^t\|_F^2 \right] + \gamma^2 \frac{60L^2\omega^2}{\delta} \mathbb{E} \left[ \|\Theta^t - \eta G^t - \hat{G}^t\|_F^2 \right] + \gamma^2 \frac{3\rho^2}{5\delta} n \mathbb{E} \left[ \|\hat{g}^t\|_F^2 \right] + \frac{31}{\delta} \beta^2 \mathbb{E} \left[ \|V^t - \nabla F^t\|_F^2 \right] + \frac{31}{\delta} \beta^2 \sigma^2.
\]

The proofs can be found in Appendix A.5, A.6, A.7.

A.2  Proof of Lemma 4.2

Using the L-smoothness of \(f\) [cf. Assumption 1], we obtain:

\[
f(\hat{g}^{t+1}) \leq f(\hat{g}^t) + \langle \nabla f(\hat{g}^t) | \hat{g}^{t+1} - \hat{g}^t \rangle + \frac{L}{2} \|\hat{g}^{t+1} - \hat{g}^t\|^2
\]
= f(\theta') - \eta \langle \nabla f(\theta') \mid \tilde{g}' \rangle + \frac{L\eta^2}{2} \|g'\|^2

= f(\theta') - \frac{\eta}{2} \left( \|g'\|^2 + \|\nabla f(\theta')\|^2 - \|g' - \nabla f(\theta')\|^2 \right) + \frac{L\eta^2}{2} \|g'\|^2

\leq f(\theta') - \frac{\eta}{4} \|g'\|^2 + \frac{\eta}{2} \|\nabla f(\theta')\|^2 + \frac{\eta}{2} \|g' - \nabla f(\theta')\|^2

\leq f(\theta') - \frac{\eta}{4} \|g'\|^2 - \frac{\eta}{2} \|\nabla f(\theta')\|^2 + \eta \left( \|g' - \nabla f(\theta')\|^2 + \|\nabla f(\theta')\|^2 \right)

\leq f(\theta') - \frac{\eta}{4} \|g'\|^2 - \frac{\eta}{2} \|\nabla f(\theta')\|^2 + \eta \|g' - \nabla f(\theta')\|^2 + \frac{L^2\eta^2}{n} \|\Theta' - \Theta\|^2_F

(26)

where (a) is due to \( \eta \leq \frac{1}{2L} \). We remark that \( \langle x \mid y \rangle = x^T y \) denotes the inner product between the vectors \( x, y \).

Note that by construction and the initialization \( v_0 = g_0 \), we have \( \tilde{g}' = \tilde{v}' \) for any \( t \geq 0 \); see (6d). Applying the upper bound

\[ \|\Theta' - \Theta\|^2_F = \|\Theta'\|^2_F \leq \|\Theta\|^2_F \]

leads to Lemma 4.2.

A.3 Proof of Lemma 4.3

Observe that

\[ \Theta_{o}^{t+1} = U^T (\Theta' - \eta G' + \gamma (W - I) \Theta') \]

\[ = U^T \left\{ \Theta' - \eta G' + \gamma (W - I) \left[ \hat{\Theta}' + Q(\Theta' - \eta G' - \hat{\Theta}') - \Theta' + \eta G' + \Theta' - \eta G' \right] \right\} \]

\[ = U^T \left\{ [I + \gamma (W - I)] (\Theta' - \eta G') + \gamma (W - I) \left[ Q(\Theta' - \eta G' - \hat{\Theta}') - (\Theta' - \eta G' - \hat{\Theta}') \right] \right\} \]

(28)

Notice that it holds \( U^T (I + \gamma (W - I)) = U^T (I + \gamma (W - I)) UU^T \) and \( \|UU^T (I + \gamma (W - I)) U\| \leq 1 - \rho \gamma \). Taking the Frobenius norm on (28) and the conditional expectation \( \mathbb{E}_t[\cdot] \) on the randomness in DoCoM-SGT up to the \( t \)th iteration:

\[ \mathbb{E}_t[\|\Theta_{o}^{t+1}\|^2_F] \]

\[ = \mathbb{E}_t \left[ \left\| U^T (I + \gamma (W - I)) UU^T (\Theta' - \eta G') + \gamma U^T (W - I) \left[ Q(\Theta' - \eta G' - \hat{\Theta}') - (\Theta' - \eta G' - \hat{\Theta}') \right] \right\|^2_F \right] \]

\[ \leq (1 + \alpha) \left\| U^T (I + \gamma (W - I)) U (\Theta_{o}^t - \eta G_{o}^t) \right\|^2_F \]

\[ + (1 + \alpha^{-1}) \mathbb{E}_t \left[ \left\| U^T (W - I) \left[ Q(\Theta' - \eta G' - \hat{\Theta}') - (\Theta' - \eta G' - \hat{\Theta}') \right] \right\|^2_F \right] \]

\[ \leq (1 + \alpha) \left\| U^T (I + \gamma (W - I)) U \right\|^2 \|\Theta_{o}^t - \eta G_{o}^t\|^2_F \]

\[ + (1 + \alpha^{-1}) \gamma^2 \|U^T (W - I)\|^2 \mathbb{E}_t \left[ \left\| Q(\Theta' - \eta G' - \hat{\Theta}') - (\Theta' - \eta G' - \hat{\Theta}') \right\|^2_F \right] \]

\[ \leq (1 + \alpha) \left( (1 - \rho \gamma)^2 \left\| \Theta_{o}^t - \eta G_{o}^t \right\|^2_F + (1 + \alpha^{-1}) \bar{\omega}^2 \gamma^2 (1 - \delta) \left\| \Theta' - \eta G' - \hat{\Theta}' \right\|^2_F \right) \]

\[ \leq (1 + \alpha) \left( (1 - \rho \gamma)^2 (1 + \beta) \right) \left\| \Theta_{o}^t \right\|^2_F + \left( 1 + \alpha \right) (1 - \rho \gamma) (1 + \beta^{-1}) \eta^2 \|G_{o}^t\|^2_F \]

\[ + \bar{\omega}^2 \gamma^2 (1 + \alpha^{-1}) (1 - \delta) \left\| \Theta' - \eta G' - \hat{\Theta}' \right\|^2_F \]

(29)

\[ \leq (1 - \frac{\rho \gamma^2}{2}) \left\| \Theta_{o}^t \right\|^2_F + \frac{2}{\rho \gamma^2} \eta^2 \|G_{o}^t\|^2_F + \frac{\bar{\omega}^2 \gamma^2}{\rho} \left\| \Theta' - \eta G' - \hat{\Theta}' \right\|^2_F \]

where (a) is due to \( \|W - I\| \leq \bar{\omega} \) and (b) is due to the choices \( \alpha = \frac{\rho \gamma}{1 - \rho \gamma^2}, \beta = \frac{\bar{\omega}^2 \gamma}{2} \). The proof is completed.

A.4 Proof of Lemma 4.4

Defining \( \nabla \hat{F}' = n^{-1}1^T \nabla \hat{F}', \nabla \hat{F}' = n^{-1}1^T \nabla \hat{F}' \), we get

\[ \tilde{v}'^{t+1} - \nabla \hat{F}'^{t+1} = \nabla \hat{F}'^{t+1} + (1 - \beta)(\tilde{v}' - \nabla \hat{F}') - \nabla \hat{F}'^{t+1} \]

(30)
\[(1 - \beta)(\hat{v}^t - \nabla \hat{F}^t) - \beta(\nabla F^{t+1} - \nabla \hat{F}^{t+1})
+ (1 - \beta)(\nabla F^t - \nabla \hat{F}^t - (\nabla F^{t+1} - \nabla \hat{F}^{t+1}))\]

It follows that
\[
E \left[ \|\hat{v}^{t+1} - \nabla \hat{F}^{t+1}\|^2 \right] \leq (1 - \beta)^2 E \left[ \|v^t - \nabla F^t\|^2 \right] + 2\beta^2 E \left[ \|\nabla F^{t+1} - \nabla \hat{F}^{t+1}\|^2 \right]
+ 2(1 - \beta)^2 E \left[ \|\nabla F^t - \nabla \hat{F}^t - (\nabla F^{t+1} - \nabla \hat{F}^{t+1})\|^2 \right]
\leq (1 - \beta)^2 E \left[ \|v^t - \nabla F^t\|^2 \right] + 2\beta^2 \frac{\sigma^2}{n} + 2(1 - \beta)^2 \frac{L^2}{n^2} E \left[ \|\Theta^{t+1} - \Theta^t\|^2 \right]
\]

Furthermore, applying Lemma A.4 leads to
\[
E \left[ \|\hat{v}^{t+1} - \nabla \hat{F}^{t+1}\|^2 \right] \leq (1 - \beta)^2 E \left[ \|v^t - \nabla F^t\|^2 \right] + 2\beta^2 \frac{\sigma^2}{n}
+ 8(1 - \beta)^2 \frac{L^2}{n^2} \eta^2 (1 - \rho \gamma)^2 E \left[ \|G_o^t\|_F^2 \right]
+ 4n(1 - \beta)^2 \frac{L^2}{n^2} \eta^2 (1 - \rho \gamma)^2 E \left[ \|g^t\|^2 \right]
+ 8(1 - \beta)^2 \frac{L^2}{n^2} \omega^2 \gamma^2 E \left[ \|\Theta^t\|^2 \right]
+ 4(1 - \beta)^2 \frac{L^2}{n^2} \omega^2 \gamma^2 (1 - \delta) E \left[ \|\Theta^t - \eta G^t - \hat{\Theta}^t\|^2 \right]
\] (31)

This concludes our proof for the stated lemma.

**Bound on the Matrix Form** Observe that
\[
V^{t+1} - \nabla F^{t+1} = \nabla \hat{F}^{t+1} + (1 - \beta)(V^t - \nabla \hat{F}^t) - \nabla F^{t+1}
= (1 - \beta)(V^t - \nabla F^t) - \beta(\nabla F^{t+1} - \nabla \hat{F}^{t+1})
+ (1 - \beta)\left(\nabla F^t - \nabla \hat{F}^t - (\nabla F^{t+1} - \nabla \hat{F}^{t+1})\right)
\] (32)

Taking the full expectation yields
\[
E \left[ \|V^{t+1} - \nabla F^{t+1}\|^2 \right] \leq (1 - \beta)^2 E \left[ \|V^t - \nabla F^t\|^2 \right] + 2\beta^2 n\sigma^2 + 2(1 - \beta)^2 L^2 \frac{n^2}{n^2} E \left[ \|\Theta^{t+1} - \Theta^t\|^2 \right]
\]

Again, applying Lemma A.4 leads to
\[
E \left[ \|V^{t+1} - \nabla F^{t+1}\|^2 \right] \leq (1 - \beta)^2 E \left[ \|V^t - \nabla F^t\|^2 \right] + 2\beta^2 n\sigma^2
+ 8(1 - \beta)^2 L^2 \eta^2 (1 - \rho \gamma)^2 E \left[ \|G_o^t\|_F^2 \right]
+ 4n(1 - \beta)^2 L^2 \eta^2 (1 - \rho \gamma)^2 E \left[ \|g^t\|^2 \right]
+ 8(1 - \beta)^2 L^2 \omega^2 \gamma^2 E \left[ \|\Theta^t\|^2 \right]
+ 4(1 - \beta)^2 L^2 \omega^2 \gamma^2 (1 - \delta) E \left[ \|\Theta^t - \eta G^t - \hat{\Theta}^t\|^2 \right]
\] (33)

This concludes our proof.

**A.5 Proof of Lemma A.1**

We begin by observing the update for $G_o^{t+1}$ as:
\[
G_o^{t+1} = U^T \left[ G^t + V^{t+1} - V^t + \gamma (W - I) \hat{G}^{t+1} \right]
= U^T \left[ G^t + V^{t+1} - V^t + \gamma (W - I) \left( \hat{G}^t + Q(G^t + V^{t+1} - V^t - \hat{G}^t) \right) \right]
\]
Taking the full expectation and applying Lemma A.5 give

Furthermore, applying Lemma A.4 yields

The above implies that

Taking \( \alpha_0 = \frac{\rho \gamma}{1 - \rho \gamma} \), \( \alpha_1 = \frac{\rho \gamma}{2} \) gives

Taking the full expectation and applying Lemma A.5 give

Furthermore, applying Lemma A.4 yields

The step size condition

implies that

This concludes our proof.

A.6 Proof of Lemma A.2

Observe that

\[ E_t \left[ \left\| \Theta^{t+1} - \eta G^{t+1} - \hat{G}^{t+1} \right\|_F^2 \right] = E_t \left[ \left\| \Theta^{t+1} - \eta G^{t+1} - (\hat{G}^t + Q(\Theta^t - \eta G^t - \hat{G}^t)) \right\|_F^2 \right] \]
Using the step size condition:

\[ E \left[ \| \theta^{t+1} - \eta G^{t+1} - (\Theta^t - \eta G^t) + \eta (\Theta^t - \eta G^t - \hat{\Theta}^t) \|_F^2 \right] \]

\[ \leq (1 + \frac{2}{\delta}) E \left[ \| \theta^{t+1} - \Theta^t - \eta (G^{t+1} - G^t) \|_F^2 + (1 + \delta)(1 - \delta) \| \Theta^t - \eta G^t - \hat{\Theta}^t \|_F^2 \right] \]

\[ \leq 2(1 + \frac{2}{\delta}) E \left[ \| \theta^{t+1} - \Theta^t \|_F^2 \right] + 2\eta^2(1 + \frac{2}{\delta}) E \left[ \| G^{t+1} - G^t \|_F^2 \right] + (1 - \frac{\delta}{2}) \| \Theta^t - \eta G^t - \hat{\Theta}^t \|_F^2 \]

(35)

Note that as

\[ G^{t+1} - G^t = G^{t+1} - 1(\hat{g}^{t+1})^T + 1(\hat{g}^{t+1})^T - [G^t - 1(\hat{g}^t)^T + 1(\hat{g}^t)^T] \]

we obtain the bound

\[ E \left[ \| G^{t+1} - G^t \|_F^2 \right] \leq \frac{1}{n} E \left[ \| V^t (V^t - V^t) \|_F^2 \right] \]

\[ + 2 E \left[ \| G^{t+1} \|_F^2 \right] \]

\[ + 2 E \left[ \| G^t \|_F^2 \right] \]

With Lemma A.5, we substitute back into (63) and obtain

\[ E \left[ \| \theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^t \|_F^2 \right] \leq (1 - \frac{\delta}{2}) E \left[ \| \theta^{t+1} - \eta G^t - \hat{\Theta}^t \|_F^2 \right] + 4\eta^2(1 + \frac{2}{\delta}) E \left[ \| G^{t+1} \|_F^2 \right]

\[ + 2(1 + \frac{2}{\delta})(3\eta^2 L^2 + 1) E \left[ \| \theta^{t+1} - \Theta^t \|_F^2 \right] + 6\beta^2 \eta^2(1 + \frac{2}{\delta}) E \left[ \| V^t - \nabla F^t \|_F^2 \right] \]

\[ + 6\beta^2 \eta^2(1 + \frac{2}{\delta}) n \sigma^2 \]

(36)

We further apply Lemma A.4 to obtain

\[ E \left[ \| \theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^t \|_F^2 \right] \leq (1 - \frac{\delta}{4}) E \left[ \| \theta^{t+1} - \eta G^t - \hat{\Theta}^t \|_F^2 \right] + \frac{12}{\delta} \eta^2 E \left[ \| G^{t+1} \|_F^2 \right]

\[ + 18 \beta^2 \eta^2 n \sigma^2 + \frac{41}{\delta} \eta^2 E \left[ \| G^t \|_F^2 \right] \]

\[ + \frac{29}{\delta} \omega^2 \gamma^2 E \left[ \| \theta^t \|_F^2 \right] + \frac{15}{\delta} \eta^2 n E \left[ \| g^t \|_F^2 \right] \]

(37)

Using the step size condition:

\[ \gamma^2 \leq \frac{1}{16} \frac{\omega^2}{\delta} \frac{(1 + \delta)^2 (1 + 3\eta^2 L^2)(1 + 1/4L)}{\delta} \]

and we recall that \( \eta \leq 1/(4L) \), the upper bound in (65) can be simplified as

\[ E \left[ \| \theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^t \|_F^2 \right] \leq (1 - \frac{\delta}{4}) E \left[ \| \theta^{t+1} - \eta G^t - \hat{\Theta}^t \|_F^2 \right] + \frac{12}{\delta} \eta^2 E \left[ \| G^{t+1} \|_F^2 \right]

\[ + \frac{18}{\delta} \beta^2 \eta^2 n \sigma^2 + \frac{41}{\delta} \eta^2 E \left[ \| G^t \|_F^2 \right] \]

\[ + \frac{29}{\delta} \omega^2 \gamma^2 E \left[ \| \theta^t \|_F^2 \right] + \frac{15}{\delta} \eta^2 n E \left[ \| g^t \|_F^2 \right] \]

The above bound can be combined with Lemma A.1 and \( \gamma \leq 1 \) to give

\[ E \left[ \| \theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^t \|_F^2 \right] \leq \left( 1 - \frac{\delta}{4} + \frac{156}{\rho \delta} \frac{L^2 (1 - \delta)}{\rho \delta} \right) E \left[ \| \theta^t - \eta G^t - \hat{\Theta}^t \|_F^2 \right] \]

\[ + \frac{12}{\delta} \eta^2 E \left[ \| G^{t+1} \|_F^2 \right] \]

\[ + \frac{18}{\delta} \beta^2 \eta^2 n \sigma^2 \]

\[ + \frac{41}{\delta} \eta^2 E \left[ \| G^t \|_F^2 \right] \]

\[ + \frac{29}{\delta} \omega^2 \gamma^2 E \left[ \| \theta^t \|_F^2 \right] \]

\[ + \left( 18 + \frac{84}{\rho \gamma} \right) \frac{2 \beta^2 \eta^2 n \sigma^2}{\delta} \]

Taking \( \eta^2 \gamma \leq \frac{\delta^2 \rho}{1248(1 - \delta) \omega^2 L^2} \) and \( \gamma \leq \frac{\delta}{8d} \) simplifies the bound into

\[ E \left[ \| \theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^t \|_F^2 \right] \leq \left( 1 - \frac{\delta}{8} \right) E \left[ \| \theta^t - \eta G^t - \hat{\Theta}^t \|_F^2 \right] \]
Using the step size condition from (59), i.e., $\eta \leq \frac{50 \delta}{\delta \omega} E \left[ \| G^t \|^2_F \right] + \eta \frac{3\delta}{\delta} E \left[ \| G^t - \hat{G}^t \|^2_F \right]$

+ $\frac{29}{\delta} \omega^2 \gamma^2 + \frac{38L^2 \omega^2}{\delta} \left[ E \left[ \| \Theta^t \|^2_F \right] + \frac{18 \eta^2}{\delta} n \ E \left[ \| \theta' \|^2 \right] \right]$

+ $\left( 18 + \frac{8 \eta}{\rho\gamma} \right) \frac{\beta^2 \eta}{\delta} E \left[ \| V^t - \nabla F^t \|^2_F \right] + \left( 18 + \frac{8 \eta}{\rho\gamma} \right) \frac{\beta^2 \eta^2 \sigma^2}{\delta}$

This concludes the proof.

### A.7 Proof of Lemma A.3

We begin by observing the following recursion for $G^t - \hat{G}^t$:

$$G^{t+1} - \hat{G}^{t+1} = G^t + V^{t+1} - V^t + (\gamma (W-I) - I) \hat{G}^{t+1}$$

$$= \gamma (W-I) (G^t + V^{t+1} - V^t)$$

$$+ (\gamma (W-I) - I) \left[ \mathcal{Q}(G^t + V^{t+1} - V^t) - (G^t + V^{t+1} - V^t - \hat{G}^t) \right].$$

This implies

$$E \left[ \| G^{t+1} - \hat{G}^{t+1} \|^2_F \right] \leq (1 + \alpha_0) (1 + \gamma \omega^2) (1 - \delta) E \left[ (1 + \alpha_1) \| G^t - \hat{G}^t \|^2_F + (1 + \alpha_1^{-1}) \| V^{t+1} - V^t \|^2_F \right]$$

+ $2(1 + \alpha_0^{-1}) \gamma \omega^2 E \left[ \| G^t \|^2_F + \| V^{t+1} - V^t \|^2_F \right]$

Taking $\alpha_0 = \frac{4}{7}, \alpha_1 = \frac{6}{5}$ and the step size condition

$$\gamma \leq \frac{\delta}{8\omega} \leq \frac{\delta}{6\omega}$$

give

$$E \left[ \| G^{t+1} - \hat{G}^{t+1} \|^2_F \right] \leq \left( 1 - \frac{\delta}{8} \right) E \left[ \| G^t - \hat{G}^t \|^2_F \right] + \frac{10 \gamma \omega^2}{\delta} E \left[ \| G^t \|^2_F \right]$$

+ $\left( 1 - \frac{\delta}{4} \right) (1 + \frac{8}{\delta}) + 2 \gamma \omega^2 (1 + \frac{4}{\delta}) E \left[ \| V^{t+1} - V^t \|^2_F \right]$

$$\leq \left( 1 - \frac{\delta}{8} \right) E \left[ \| G^t - \hat{G}^t \|^2_F \right] + \frac{10 \gamma \omega^2}{\delta} E \left[ \| G^t \|^2_F \right] + \frac{10(1 + \gamma \omega^2)}{\delta} E \left[ \| V^{t+1} - V^t \|^2_F \right]$$

Applying Lemma A.5 and Lemma A.4 gives

$$E \left[ \| G^{t+1} - \hat{G}^{t+1} \|^2_F \right] \leq \left( 1 - \frac{\delta}{8} \right) E \left[ \| G^t - \hat{G}^t \|^2_F \right] + \frac{10 \gamma \omega^2}{\delta} E \left[ \| G^t \|^2_F \right] + \frac{31}{\delta} \beta^2 \left( E \left[ \| V^t - \nabla F^t \|^2_F \right] + n \sigma^2 \right)$$

+ $30(1 + \gamma \omega^2) L^2 E \left[ \| \Theta^{t+1} - \Theta^t \|^2_F \right]$

$$\leq \left( 1 - \frac{\delta}{8} \right) E \left[ \| G^t - \hat{G}^t \|^2_F \right] + \frac{10 \delta}{\delta} \left( \gamma \omega^2 + 12 \gamma \omega^2 (1 + \gamma \omega^2)(1 - \rho \gamma)^2 \right) E \left[ \| G^t \|^2_F \right]$$

+ $\frac{120(1 + \gamma \omega^2) L^2 \omega^2}{\delta} E \left[ \| \Theta^t \|^2_F \right]$

+ $\frac{60(1 + \gamma \omega^2) L^2 \omega^2 \gamma(1 - \delta)}{\delta} E \left[ \| \Theta^t - \gamma G^t - \hat{G}^t \|^2_F \right]$

+ $\frac{60(1 + \gamma \omega^2) L^2 \omega^2 \gamma(1 - \rho \gamma)^2}{\delta} E \left[ \| \theta' \|^2 \right]$

+ $\frac{31}{\delta} \beta^2 \left( E \left[ \| V^t - \nabla F^t \|^2_F \right] + n \sigma^2 \right)$

Using the step size condition from (59), i.e., $\eta L^2(1 - \rho \gamma)^2(1 + \gamma \omega^2) \leq \frac{\delta^2}{100}$, and $\gamma \leq \frac{\delta}{8\omega}$ simplifies the above to

$$E \left[ \| G^{t+1} - \hat{G}^{t+1} \|^2_F \right] \leq \left( 1 - \frac{\delta}{8} \right) E \left[ \| G^t - \hat{G}^t \|^2_F \right] + \frac{10 \delta}{\delta} \gamma \omega^2 \left( \omega^2 + \frac{\rho^2}{\delta} \right) E \left[ \| G^t \|^2_F \right]$$
\[
+ \gamma^2 \frac{122 L^2 \omega^2}{\delta} E \left[ \| \Theta^t \|_F^2 \right] + \gamma^2 \frac{60 L^2 \omega^2}{\delta} E \left[ \| \Theta^t - \eta G^t - \hat{G}^t \|_F^2 \right] \\
+ \gamma^2 \frac{3 \beta^2 96}{5 \delta} n E \left[ \| \tilde{g}^t \|_2^2 \right] + \frac{31}{8} \beta^2 E \left[ \| V^t - \nabla F^t \|_2^2 \right] + \frac{31}{8} \beta^2 n \sigma^2.
\]

This concludes our proof.

A.8 Proof of Lemma 4.5

Combining Lemma 4.3, 4.4, A.1, A.3, A.2 and (33) yields

\[
E \left[ L^2 \| \Theta^{t+1} \|_2^2 + n \| \tilde{g}^{t+1} - \nabla F^{t+1} \|_2^2 + \frac{1}{\delta} \| \tilde{V}^{t+1} - \nabla F^{t+1} \|_F^2 + a \| G^{t+1} \|_2^2 + b \| G^{t+1} - \Theta^{t+1} \|_F^2 + c \| \Theta^{t+1} - \eta G^{t+1} - \hat{G}^{t+1} \|_F^2 \right] \\
\leq \left( 1 - \frac{\rho \gamma}{8} + \frac{16}{n} (1 - \beta)^2 \omega^2 \gamma^2 + a \gamma \frac{25 \omega^2}{\rho} + b \gamma^2 \frac{122 \omega^2}{\delta} + c \gamma^2 \frac{29 \omega^2}{\delta L^2} + \eta^2 \frac{38 \omega}{\rho} \right) E \left[ L^2 \| \Theta^t \|_2^2 \right] \\
+ (1 - \beta)^2 n E \left[ \| \tilde{g}^t - \nabla F^t \|_2^2 \right] + \left( (1 - \beta)^2 + a b^2 \frac{7 n}{\rho \gamma} + b \beta^2 \frac{31 n}{\delta} + c b^2 \eta^2 (18 + \frac{84}{\rho \gamma}) \right) \frac{1}{n} E \left[ \| V^t - \nabla F^t \|_2^2 \right] \\
+ a \left( 1 - \frac{\rho \gamma}{8} + \frac{1}{a} \eta^2 \frac{2 L^2}{\rho} + \frac{116}{n} \frac{6 \eta^2 (1 - \beta)^2 (1 - \rho \gamma)^2 L^2}{\delta} + \frac{b \gamma}{a} \frac{10}{\delta} \omega^2 + \frac{\rho^2}{8} + \frac{c \gamma^2}{\delta} \frac{50}{\delta} \right) E \left[ \| G^t \|_2^2 \right] \\
+ b \left( 1 - \frac{\rho \gamma}{8} + \frac{1}{b} \eta^2 \frac{2 L^2}{\rho} + \frac{c \gamma^2}{\delta} \frac{2 \omega^2}{\rho} \right) E \left[ \| G^t - \Theta^t \|_2^2 \right] \\
+ c \left( 1 - \frac{\rho \gamma}{8} + \frac{1}{c} \eta^2 \frac{2 L^2}{\rho} + \frac{18}{n} \frac{6 \gamma^2 (1 - \beta)^2 L^2 \omega^2 (1 - \delta) + \frac{a}{c} \gamma \frac{13 L^2}{\rho} \omega^2 (1 - \delta) + \frac{b \gamma}{c} \frac{60 L^2 \omega^2}{\delta}}{8} \right) E \left[ \| \Theta^t - \eta G^t - \hat{G}^t \|_2^2 \right] \\
+ \left( \frac{4}{n} \beta^2 + a b^2 \frac{7 \gamma}{\rho \gamma} + b \beta^2 \frac{31 \gamma}{\delta} + b c \beta^2 \eta^2 (18 + \frac{84}{\rho \gamma}) \right) n \sigma^2 \\
+ \left( \frac{8}{n} (1 - \beta)^2 L^2 \eta^2 (1 - \rho \gamma)^2 + a \gamma \frac{5}{8} + b \gamma^2 \frac{3 \rho^2}{5 \delta} + c \gamma \frac{18}{\delta} \right) \frac{1}{n} E \left[ \| \tilde{g}^t \|_2^2 \right]
\]

Our goal is to find conditions on step sizes and the choices of \( a, b, c \) such that

\[
1 - \frac{\rho \gamma}{8} + \frac{16}{n} (1 - \beta)^2 \omega^2 \gamma^2 + a \gamma \frac{25 \omega^2}{\rho} + b \gamma^2 \frac{122 \omega^2}{\delta} + c \gamma^2 \frac{29 \omega^2}{\delta L^2} + \eta^2 \frac{38 \omega}{\rho} \leq 1 - \frac{\rho \gamma}{8} \\
(1 - \beta)^2 + a b^2 \frac{7 n}{\rho \gamma} + b \beta^2 \frac{31 n}{\delta} + c b^2 \eta^2 (18 + \frac{84}{\rho \gamma}) \leq (1 - \beta) \\
1 - \frac{\rho \gamma}{4} + \frac{1}{a} \eta^2 \frac{2 L^2}{\rho} + \frac{116}{n} \frac{6 \eta^2 (1 - \beta)^2 (1 - \rho \gamma)^2 L^2}{\delta} + \frac{b \gamma}{a} \frac{10}{\delta} \omega^2 + \frac{\rho^2}{8} + \frac{c \gamma^2}{\delta} \frac{50}{\delta} \leq 1 - \frac{\rho \gamma}{8} \\
1 - \frac{\delta}{8} + \frac{1}{b} \gamma \frac{2 \omega^2}{\rho} + \frac{c \gamma^2}{\delta} \frac{2 \omega^2}{\rho} \leq 1 - \frac{\delta}{8} \\
1 - \frac{\delta}{8} + \frac{1}{c} \gamma \frac{2 \omega^2}{\rho} + \frac{18}{n} \frac{6 \gamma^2 (1 - \beta)^2 L^2 \omega^2 (1 - \delta) + \frac{a}{c} \gamma \frac{13 L^2}{\rho} \omega^2 (1 - \delta) + \frac{b \gamma}{c} \frac{60 L^2 \omega^2}{\delta}}{8} \leq 1 - \frac{\delta}{8}.
\]

To this end, with the step size condition

\[
\gamma \leq \min \left\{ \frac{1}{4 \rho} \cdot \frac{\rho \gamma}{64 (1 - \beta)^2 \omega^2}, \frac{n}{8 (1 - \beta)^2 (1 - \rho \gamma)} \right\},
\]

the above set of inequalities can be guaranteed if \( a, b, c \) satisfy

\[
\frac{96 L^2}{\rho \gamma^2} \eta^2 \leq a \leq \min \left\{ \frac{(1 - \beta) \gamma \rho}{21 n}, \frac{2}{13 (1 - \delta)}, \frac{\rho^2}{600 \omega^2} \right\} \\
\max \left\{ a \frac{\gamma}{1 - \gamma} \frac{32 \omega^2}{\rho \delta}, c \frac{\eta^2}{1 - \gamma} \frac{48 \omega^2}{\rho \delta} \right\} \leq b \leq \min \left\{ \frac{\eta^2}{\gamma^3 \frac{5 \rho (\omega^2 + \frac{\rho^2}{\delta})}{30}}, \frac{\delta}{3 \rho \gamma}, \frac{933 \gamma}{2928 \gamma^2} \right\} \\
\frac{\gamma}{1 - \gamma} \frac{48 L^2 \omega^2}{\delta \rho} \leq c \leq \min \left\{ \frac{\delta \rho L^2}{1392 \gamma^2 \omega^2}, \frac{2 \delta L^2}{\beta \rho \eta^2 n (18 + \frac{84}{\rho \gamma})}, \frac{\gamma}{\eta^2} \frac{\rho^2}{1824 \omega^2} \right\}
\]

Notice that the step size conditions:

\[
\eta^2 \leq \min \left\{ \frac{(1 - \beta) \gamma^3 \rho^3}{2016 \beta n L^2}, \frac{\gamma^2 \rho^4}{624 (1 - \delta) L^2}, \frac{\gamma^2 \rho^4}{57600 \omega^2 L^2} \right\}
\]

(S2)
guarantees the existence of $a$ which satisfies (38). In particular, we take $a = \frac{96L^2}{\rho^2} \eta^2$.

At the same time, with the step size conditions:

$$\eta^2 \leq \min \left\{ \frac{(1 - \gamma)\gamma}{\beta n} \frac{464\gamma}{(18 + \frac{8}{\gamma})\rho L^2} \gamma^2 \frac{29\rho\bar{\omega}}{388L^2}, \quad \frac{\gamma^2}{1 - \gamma} \leq \frac{\delta^2\rho^2}{66816\omega^4} \right\}, \quad (S3)$$

we guarantee the existence of $c$ which satisfies (40). In particular, we take $c = \frac{\gamma^2}{1 - \gamma} \frac{48L^2\omega^2}{\delta \rho}$. This simplifies (39) into

$$\max \left\{ \frac{\eta^2}{\gamma(1 - \gamma) - \delta \rho}, \frac{\eta^2\gamma}{(1 - \gamma)^2 - \delta \rho^2} \right\} \leq b \leq \min \left\{ \frac{\eta^2}{\gamma^3} \frac{2\delta L^2}{\beta n}, \frac{3\rho^2}{30\rho \gamma}, \frac{29\gamma^2 \omega^2}{93\beta n}, \frac{29\gamma^2 \omega^2}{2928\gamma^2 \omega^2} \right\} \quad (41)$$

Combining with the step size conditions:

$$\eta^2 \leq \min \left\{ \frac{\gamma^2(\omega^2 + \frac{\gamma^2}{8} \eta^2 \gamma^2)}{12L^2}, \frac{(1 - \beta)\gamma^3 \frac{5\rho(\omega^2 + \frac{\gamma^2}{8})}{186L^2}, \frac{\gamma^2}{1 - \gamma} \frac{5\rho(\omega^2 + \frac{\gamma^2}{8})}{5856\omega^2 L^2} \right\}, \quad (S4)$$

guarantees the existence of $b$ which satisfies (41). Finally, we take $b = \frac{\eta^2}{\gamma(1 - \gamma)} \frac{3072\omega^2 L^2}{\delta \rho}$. Using the upper bound on $\gamma^2/(1 - \gamma)$ from (S4) and the above choices of $a, b, c$ yield:

$$\Psi^{t+1} \leq \left( \min \left\{ \frac{\rho^2}{8}, \frac{\delta \gamma}{\beta \gamma}, (1 - \beta) \frac{\gamma}{\beta \gamma}, \frac{\gamma}{\beta \gamma}, \frac{\gamma}{\beta \gamma}, \frac{\gamma}{\beta \gamma}, \frac{\gamma}{\beta \gamma}, \frac{\gamma}{\beta \gamma} \right\} \right)^2 \Psi^t + \beta^2 \left[ 4 + \frac{\eta^2 672L^2 \rho^2}{\gamma 25\omega^2} + \frac{\eta^2 6L^2 \rho^2 \delta}{\gamma 8\omega^2} + \frac{\eta^2 4L^2 n}{\gamma 2\omega^2} \right] \sigma^2$$

$$+ \eta^2 \left[ 6(1 - \beta)^2 2L^2 (1 - \rho n)^2 + \frac{L^2 n}{\rho \gamma} \left( 96 + \frac{141 \rho^2}{400 \omega^2} \right) \right] E \left[ \left\| \theta^t \right\|^2 \right]$$

Furthermore, we observe that the above steps require step size conditions (S1), (S2), (S3), (S4). Together with the requirements in Lemma A.4.4, A.1, A.3, A.2, we need

$$\eta^2 \leq \min \left\{ \frac{(1 - \beta)\gamma^3 \rho^4}{2016\beta n L^2}, \frac{(1 - \beta)\gamma^3 \rho^4}{18(1 + \frac{8}{\gamma})\rho L^2}, \frac{29\rho \bar{\omega}}{388L^2}, \frac{\gamma^2(\omega^2 + \frac{\gamma^2}{8} \eta^2)}{12L^2}, \frac{(1 - \beta)\gamma^3}{\beta n} \frac{5\rho(\omega^2 + \frac{\gamma^2}{8})}{186L^2}, \frac{5\gamma^2 \rho^2(\omega^2 + \frac{\gamma^2}{8})}{5856\omega^2 L^2}, \frac{\delta^2 \rho^2}{624(1 - \delta) L^2}, \frac{57600\bar{\omega}^2 L^2}{57600\bar{\omega}^2 L^2}, \frac{100L^2(1 - \rho n)^2(1 + \gamma^2 \omega^2)}{1248\omega^2 L^2 \gamma}, \frac{\delta^2 \rho^2}{66816\omega^3}, \frac{67800(\omega^2 + \frac{\gamma^2}{8} \eta^2)}{7680\omega^2}, \frac{\delta^2 \rho^2}{3\omega^2 \rho} \right\}$$

$$\gamma \leq \min \left\{ \frac{1}{4\rho}, \frac{64(1 - \beta)^2 \omega^2}{8(1 - \beta)^2 (1 - \delta \rho)^4}, \frac{\delta}{8\omega}, \frac{4\omega \sqrt{(1 - \delta)(1 + 3\eta^2 L^2)(1 + 2/\delta)}}{4\omega \sqrt{(1 - \delta)(1 + 3\eta^2 L^2)(1 + 2/\delta)}} \right\}, \quad \frac{\gamma^2}{1 - \gamma} \leq \min \left\{ \frac{\delta^2 \rho^2}{66816\omega^3}, \frac{67800(\omega^2 + \frac{\gamma^2}{8} \eta^2)}{7680\omega^2}, \frac{\delta^2 \rho^2}{3\omega^2 \rho} \right\}$$

Taking the restriction that $\bar{\omega} \in [1, 2]$, the above can be simplified and implied by

$$\gamma \leq \min \left\{ \frac{1}{4\rho}, \frac{64(1 - \beta)^2 \omega^2}{10\omega}, \frac{\delta}{259\omega^2} \right\} =: \gamma_\infty,$$

$$\eta \leq \frac{\gamma}{L} \min \left\{ \frac{1 - \beta}{\beta n}, \frac{\rho^2}{45 \cdot 240\omega} \right\} =: \eta_\infty.$$ 

This concludes the proof.

### A.9 Auxiliary Lemmas

**Lemma A.4.** Under Assumption 2, 4. For any $t \geq 0$, it holds

$$\mathbb{E} \left[ \left\| \theta^{t+1} - \Theta^t \right\|^2 \right] \leq 4\eta^2 (1 - \rho \gamma)^2 \mathbb{E} \left[ \left\| G_t \right\|^2 \right] + 2n \eta^2 (1 - \rho \gamma)^2 \mathbb{E} \left[ \left\| \dot{\theta}^t \right\|^2 \right]$$

$$+ 4\omega^2 \gamma^2 \mathbb{E} \left[ \left\| \dot{\theta}^t \right\|^2 \right] + 2\omega^2 \gamma^2 (1 - \delta) \mathbb{E} \left[ \left\| \theta^t - \eta G^t - \dot{\theta}^t \right\|^2 \right].$$
Proof. We observe that:

\[
\|\Theta^{t+1} - \Theta^t\|^2_F = \|\eta G^t - \gamma (W - I) \hat{\Theta}^t\|^2_F \\
= \|(I + \gamma(W - I))(-\eta G^t + \gamma(W - I)\Theta^t + \gamma(W - I) \left[ Q(\Theta^t - \gamma G^t - \hat{\Theta}^t) - (\Theta^t - \gamma G^t - \hat{\Theta}^t) \right])\|^2_F \\
\leq 2 \|(I + \gamma(W - I))(-\eta G^t) + \gamma(W - I)\Theta^t\|^2_F \\
+ 2 \|(\gamma(W - I))(Q(\Theta^t - \gamma G^t - \hat{\Theta}^t) - (\Theta^t - \gamma G^t - \hat{\Theta}^t))\|^2_F \\
\leq 2 \left[ \gamma^2 (\|I + \gamma(W - I))G^t\|^2_F + \gamma^2 (\|W - I)\Theta^t\|^2_F + 2 \langle (I + \gamma(W - I))(-\eta G^t) \mid \gamma(W - I)\Theta^t \rangle \right] \\
+ 2\gamma^2 \gamma \|Q(\Theta^t - \gamma G^t - \hat{\Theta}^t) - (\Theta^t - \gamma G^t - \hat{\Theta}^t)\|^2_F
\]

(43)

Observe that \((I + \gamma(W - I))(-\eta G^t) \mid \gamma(W - I)\Theta^t) = (I + \gamma(W - I))(-\eta U G^o) \mid \gamma(W - I)\Theta^t)\), the above leads to:

\[
\mathbb{E}_t \left[ \|\Theta^{t+1} - \Theta^t\|^2_F \right] \leq 2\gamma^2(1 - \epsilon)^2 \|G^t\|^2_F + 2\gamma^2(1 - \epsilon)^2 \|G^o\|^2_F + 4\gamma^2 \|W - I)\Theta^t\|^2_F \\
+ 2\gamma^2 \gamma^2 (1 - \delta) \|\Theta^t - \gamma G^t - \hat{\Theta}^t\|^2_F
\]

(44)

Notice that:

\[
\mathbb{E} \left[ \|G^t\|^2_F \right] = \mathbb{E} \left[ \|(1/n)11^t + UU^t\)G^t\|^2_F \right] = \mathbb{E} \left[ \|UU^t G^t\|^2_F + \|(1/n)11^t G^t\|^2_F \right] \\
\leq \mathbb{E} \left[ \|G^t\|^2_F + n \|g^t\|^2 \right]
\]

(45)

By combining (61), (62), and the fact \(\|W - I)\Theta^t\|^2_F \leq \omega^2 \|\Theta^o\|^2_F\), we have:

\[
\mathbb{E} \left[ \|\Theta^{t+1} - \Theta^t\|^2_F \right] \leq 4\gamma^2(1 - \epsilon)^2 \mathbb{E} \left[ \|G^t\|^2_F + 2n\gamma^2(1 - \epsilon)^2 \|g^t\|^2 \right] \\
+ 4\omega^2 \gamma^2 (1 - \delta) \mathbb{E} \left[ \|\Theta^t - \gamma G^t - \hat{\Theta}^t\|^2_F \right]
\]

This concludes the proof. □

Lemma A.5. Under Assumption 1, 3. For any \(t \geq 0\), it holds:

\[
\mathbb{E} \left[ \|V^{t+1} - V^t\|^2_F \right] \leq 3L^2 \mathbb{E} \left[ \|\Theta^{t+1} - \Theta^t\|^2_F \right] + 3\beta^2 \mathbb{E} \left[ \|V^t - \nabla F^t\|^2_F \right] + 3n \beta^2 \sigma^2
\]

Proof. Observe that:

\[
V^{t+1} - V^t = \nabla \hat{F}^{t+1} + (1 - \beta)(V^t - \nabla \hat{F}^t) - V^t \\
= \nabla \hat{F}^{t+1} - \nabla \hat{F}^t - \beta(V^t - \nabla F^t) + \beta(\nabla \hat{F}^t - \nabla F^t)
\]

It holds that:

\[
\mathbb{E} \left[ \|V^{t+1} - V^t\|^2_F \right] \leq 3L^2 \mathbb{E} \left[ \|\Theta^{t+1} - \Theta^t\|^2_F \right] + 3\beta^2 \mathbb{E} \left[ \|V^t - \nabla F^t\|^2_F \right] + 3\beta^2 \mathbb{E} \left[ \|\nabla \hat{F}^t - \nabla F^t\|^2_F \right] \\
\leq 3L^2 \mathbb{E} \left[ \|\Theta^{t+1} - \Theta^t\|^2_F \right] + 3\beta^2 \mathbb{E} \left[ \|V^t - \nabla F^t\|^2_F \right] + 3n \beta^2 \sigma^2
\]

This concludes the proof. □

A.10 Transient Time of DoCoM-SGT

We follow a similar argument as in (10). Particularly, consider setting the step sizes and parameters as \(\beta = \Theta(\frac{1}{T^2}), \eta = \Theta(\frac{1}{T}), \gamma = \gamma_\infty, b_0 = \Omega(T^{1/3})\). Then for sufficiently large \(T\), we obtain:

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \|\nabla f(\hat{\Theta}^t_i)\|^2 \right] = O \left( \frac{L \left( f(\hat{\theta}^0) - f^* \right)}{T^{2/3}} + \frac{\sigma^2}{\kappa T^{3/2}} + \frac{C_0}{\delta^2 \rho^3 T} + \frac{\sigma^2}{\delta^3 \beta^3 T^{4/3}} \right)
\]

(46)
The transient time can be calculated by bounding $T$ such that the second term dominates over the last two terms. We get

$$T_{\text{trans}} := \Omega \left( \max \left\{ \frac{n^2 \bar{G}_0^3}{\sigma^3 \delta^3 b^2}, \frac{n^{1.5}}{\delta^{1.5} \gamma b^2} \right\} \right)$$  \hspace{1cm} (47)

Taking $\sigma \leq 1$ guarantees that $T_{\text{trans}} = \Omega(\frac{n^2 \bar{G}_0^3}{\sigma^3 \delta^3 b^2})$.

### A.11 Sublinear bound of $O(\log T/T)$ under PL Condition

We consider setting $\beta = \eta = \log T/(\mu T)$, $\gamma = \gamma_\infty$, $b_0 = 1$. Notice that for a sufficiently large $T$, these step sizes will satisfy (7). Furthermore, setting $t = T$, the upper bound in (12) is given by:

$$\left(1 - \frac{\log T}{T}\right)^T \left( \Delta^0 + \frac{2}{n} \left( 2\sigma^2 + \frac{192L^2 n}{\rho^2 \gamma^2 (1 - \gamma)} \bar{G}_0 \frac{(\log T)^2}{T} \right) \right) + \frac{2C_n \sigma^2 \log T}{n} \frac{\mu T}{\rho^T}$$  \hspace{1cm} (48)

We observe that

$$\left(1 - \frac{\log T}{T}\right)^T \leq e^{-\log T} = \frac{1}{T}.$$  

Thus the expression in (48) can be further upper bounded by $O(\log T/T)$. This implies (13), (14).

### B Convergence Analysis of DoCoM-SGT with $\beta = 1$

This section provides the convergence analysis of (6) for the special case when the momentum parameter is $\beta = 1$, i.e., there is no momentum applied. Observe that the DoCoM-SGT algorithm can be simplified as

$$\Theta^{t+1} = \Theta^t - \eta G^t + \gamma (W - I) \hat{G}^{t+1},$$

$$\hat{G}^{t+1} = \hat{G}^t + \alpha (\Theta^t - \eta G^t - \hat{G}^t),$$

$$G^{t+1} = G^t + \nabla \hat{F}^{t+1} - \nabla \hat{F}^t + \gamma (W - I) \hat{G}^{t+1},$$

$$\hat{G}^{t+1} = \hat{G}^t + \alpha (G^t + \nabla \hat{F}^{t+1} - \nabla \hat{F}^t - \hat{G}^t),$$

where we have eliminated the use of $V^t$ since $V^t = \nabla \hat{F}^t$ for any $t \geq 0$. Additionally, we define $\Theta^t = (1/n) \mathbf{1}^\top \Theta^t$.

In such setting, the convergence analysis has to follow a different path from the case of $\beta < 1$. We begin by analyzing the 1-iteration progress with

**Lemma B.1.** Under Assumption 1, 2, 3, and the step size satisfies $\eta \leq 1/(4L)$. Then, for any $t \geq 0$, it holds

$$E_t[ \| \omega^{t+1} \|_F^2 ] \leq \frac{3L^2 \eta}{4n} \| \Theta^t \|_F^2 + \frac{L \eta^2 \sigma^2}{2n}. \hspace{1cm} (49)$$

The proof is relegated to Appendix B.1. Notice that the above lemma departs from Lemma 4.2 as it results in a bound that depends only on the consensus error. Our next endeavor is to bound $\| \Theta^t \|_F^2$, which can be conveniently controlled by Lemma 4.3 as quoted below:

$$E_t[ \| \Theta^t \|_F^2 ] \leq \left(1 - \frac{\rho T}{4} \right) E_t[ \| \Theta^t \|_F^2 ] + \frac{2}{\rho} \frac{\sigma^2}{\gamma} E_t[ \| G^t \|_F^2 ] + \frac{\sigma^2}{\rho} \gamma E_t[ \| \Theta^t - \eta G^t - \hat{G}^t \|_F^2 ]. \hspace{1cm} (50)$$

Moreover,

**Lemma B.2.** Under Assumption 1, 2, 3, 4. Suppose the step size satisfies $\eta \leq \frac{\rho^3}{8L(1-\rho^3) \sqrt{1+\gamma^2}}$, $\gamma \leq \frac{1}{\sqrt{2}}$. Then, for any $t \geq 0$, the consensus error of $G^{t+1}$ is bounded as follows:

$$E_t[ \| G^{t+1} \|_F^2 ] \leq \left(1 - \frac{\rho T}{4} \right) E_t[ \| G^t \|_F^2 ] + \frac{2 \sigma^2}{\rho} E_t[ \| G^t - \hat{G}^t \|_F^2 ] + \gamma \left( \frac{\rho}{4} + \frac{18L^2 \omega^2}{\rho} \right) E_t[ \| \theta^t \|_F^2 ] + \gamma \frac{9L^2 \bar{G}_0^3}{\rho} \omega^2 E_t[ \| \Theta^t - \eta G^t - \hat{G}^t \|_F^2 ] + \gamma \frac{\rho m}{4} E_t[ \| \nabla f(\hat{\theta}^t) \|_2^2 ] + \left( \frac{7n}{\rho^2} + \frac{\rho^3}{8} \right) \sigma^2.$$

The proof is relegated to Appendix B.2. We also bound the subsequent terms by
Lemma B.3. Under Assumption 1, 2, 3, 4. Suppose the step size satisfies \( \eta \leq \frac{\rho \gamma}{8L(1-\rho \gamma)\sqrt{1+\gamma^2\omega^2}} + \frac{\delta}{8\varepsilon} \). Then, for any \( t \geq 0 \), it holds:

\[
E \left[ \left\| G^{t+1} - \hat{G}^{t+1} \right\|_F^2 \right] \leq \left( 1 - \frac{\delta}{8} \right) E \left[ \left\| G^t - \hat{G}^t \right\|_F^2 \right] + \frac{10}{\delta} \gamma^2 \left( \omega^2 + \frac{\rho^2}{8} \right) E \left[ \left\| G^t_\omega \right\|_F^2 \right] + \gamma^2 \frac{5}{4\delta} \left( \rho^2 + 72L^2\omega^2 \right) E \left[ \left\| \Theta^t_\omega \right\|_F^2 \right] + \gamma^2 \frac{40L^2\omega^2}{\delta} E \left[ \left\| \Theta^t - \eta G^t - \hat{G}^t \right\|_F^2 \right] + \gamma^2 \frac{5\rho^2}{4\delta} \eta E \left[ \left\| \nabla f(\hat{\theta}^t) \right\|^2 \right] + \frac{5}{\delta} \left( \frac{\rho^2}{8} + 7n \right) \sigma^2 .
\]

Lemma B.4. Under Assumption 1, 2, 3, 4. Suppose the step size satisfies \( \eta \leq \frac{\sqrt{\gamma}}{4\omega \sqrt{(1-\delta)(1+\eta^2L^2)}(1+2/\delta)} \). Then, for any \( t \geq 0 \), it holds

\[
E \left[ \left\| \Theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^{t+1} \right\|_F^2 \right] \leq \left( 1 - \frac{\delta}{4} \right) E \left[ \left\| \Theta^t - \eta G^t - \hat{\Theta}^t \right\|_F^2 \right] + \eta^2 \frac{12}{\delta} \frac{\rho^2}{8} E \left[ \left\| G^t_\omega \right\|_F^2 \right] + \frac{24}{\delta} \eta^2 \sigma^2 + \frac{38}{\delta} \gamma^2 \eta \eta E \left[ \left\| \Theta^t_\omega \right\|_F^2 \right] + \frac{26}{\delta} \left( \gamma^2 \omega^2 + \omega^2 \gamma^2 \right) E \left[ \left\| \Theta^t_\omega \right\|_F^2 \right] + \eta^2 \frac{29n}{\delta} E \left[ \left\| \nabla f(\hat{\theta}^t) \right\|^2 \right] + \eta^2 \frac{24\sigma^2}{\delta} \left( 1 + \frac{4n}{\rho \gamma} \right) \eta\gamma \leq \frac{\delta\rho}{864\omega^2 L^2} .
\]

Furthermore, if the step size satisfies \( \eta^2 \gamma \leq \frac{\delta\rho}{864\omega^2 L^2} \), then

\[
E \left[ \left\| \Theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^{t+1} \right\|_F^2 \right] \leq \left( 1 - \frac{\delta}{8} \right) E \left[ \left\| \Theta^t - \eta G^t - \hat{\Theta}^t \right\|_F^2 \right] + \eta^2 \frac{50}{\delta} \frac{\rho^2}{8} E \left[ \left\| G^t_\omega \right\|_F^2 \right] + \frac{38}{\delta} \gamma^2 \eta \eta E \left[ \left\| G^t_\omega \right\|_F^2 \right] + \left( \frac{\eta^2}{\delta} \frac{29}{\delta} \frac{L^2}{\rho} \left( 1 + \frac{\omega}{\rho} \right) + \frac{26}{\delta} \gamma^2 \omega^2 \right) E \left[ \left\| \Theta^t_\omega \right\|_F^2 \right] + \eta^2 \frac{29n}{\delta} E \left[ \left\| \nabla f(\hat{\theta}^t) \right\|^2 \right] + \eta^2 \frac{24\sigma^2}{\delta} \left( 1 + \frac{4n}{\rho \gamma} \right) \eta\gamma \leq \frac{\delta\rho}{864\omega^2 L^2} .
\]

The proofs are relegated to Appendix B.3, B.4, respectively. Combining the above lemmas and optimizing the bounds for step sizes lead to

Lemma B.5. Under Assumption 1, 2, 3, 4 and the step size conditions

\[
\eta \leq \frac{1}{L} \min \left\{ \frac{4}{8(1-\rho \gamma)\sqrt{1+\gamma^2\omega^2}}, \frac{\rho^2\gamma}{105\omega}, \frac{\rho^2\delta\sqrt{1-\gamma}}{1303\omega}, \frac{\rho^2\delta\sqrt{1-\gamma}}{1303\omega^{3/2}}, \frac{\sqrt{3}}{2} \frac{2(1-\gamma)}{8(1-\rho \gamma)\sqrt{1+\gamma^2\omega^2}}, \frac{320\omega^2}{\rho^3\delta^2\gamma^3} + \frac{15\omega^2}{\rho^2\delta^2\gamma^3} \right\}, \quad \gamma \leq \min \left\{ \frac{8}{\delta}, \frac{1}{\rho^2 \sqrt{8(1-\delta)(1+\eta^2L^2)(1+2/\delta)}}, \frac{\rho \delta \sqrt{1-\delta}/(8\omega^2)}{88\omega^2 \delta \omega^2 + \rho^2}, \frac{\rho \delta \sqrt{1-\delta}/(8\omega^2)}{123\omega^2} \right\} , \quad \eta^2 \gamma \leq \frac{\delta\rho}{864\omega^2 L^2} ,
\]

Define the constants:

\[
C_{\sigma}^{NM} = \frac{192}{1-\gamma} \left[ \frac{2(1-\gamma)}{\rho^3\gamma^3} + \frac{320\omega^2}{\rho^3\delta^2\gamma^3} + \frac{15\omega^2}{\rho^2\delta^2\gamma^3} \right], \quad C_{\sigma F}^{NM} = \frac{12}{1-\gamma} \left[ \frac{1-\gamma}{\rho^3} + \frac{256\omega^2}{\rho^3\gamma} + \frac{58\omega^2}{\rho^3} \right] .
\]

Then, for any \( t \geq 0 \), it holds

\[
E \left[ \left\| \Theta^{t+1}_\omega \right\|_F^2 + a \left\| G^{t+1}_\omega \right\|_F^2 + b \left\| G^{t+1} - \hat{G}^{t+1} \right\|_F^2 + c \left\| \Theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^{t+1} \right\|_F^2 \right] \leq \left( 1 - \frac{\min\{\rho, \delta\}}{8} \right) \frac{\gamma}{\rho^3} E \left[ \left\| \Theta^t_\omega \right\|_F^2 + a \left\| G^t_\omega \right\|_F^2 + b \left\| G^t - \hat{G}^t \right\|_F^2 + c \left\| \Theta^t - \eta G^t - \hat{\Theta}^t \right\|_F^2 \right] + \frac{5}{\delta} \eta^2 C_{\sigma} \sigma n \sigma^2 + \frac{\eta^2 C_{\sigma F}^{NM} n}{\delta} E \left[ \left\| \nabla f(\hat{\theta}^t) \right\|^2 \right]
\]

where \( a = \frac{48\omega^2}{\rho^3\gamma^3}, b = \frac{1536\omega^2}_3^3}{\rho^3\gamma^3(1-\gamma)}, c = \frac{24\omega^2}{\rho\delta(1-\gamma)} \).

The proof is relegated to Appendix B.5.

To simplify notations, we let \( \rho^{NM} := \min\{\rho, \delta\}/8 \). Using \( E[\left\| G^t_\omega \right\|_F^2] = n\sigma^2 \) and Lemma B.5 imply that

\[
E \left[ \left\| \Theta^{t}_\omega \right\|_F^2 \right] \leq a \left( 1 - \rho^{NM} \gamma \right) t n \sigma^2 + \eta^2 \sum_{s=0}^{t-1} \left( 1 - \rho^{NM} \gamma \right)^{-s-1} \left\{ C_{\sigma} n \sigma^2 + C_{\sigma F}^{NM} n E \left[ \left\| \nabla f(\hat{\theta}^s) \right\|^2 \right] \right\}
\]
Using the B.1 Proof of Lemma B.1 that similar result to Corollary 4.1 can be established under the PL condition Assumption 5.

Setting \( \eta \), where we denote \( \bar{\Theta} \). Therefore, under the additional step size condition \( \eta \leq \sqrt{\frac{\rho_{\text{NM}, \gamma}}{6C_f \bar{L}^2 \bar{T}}} \), it holds

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \nabla f(\bar{\theta}^t) \|^2 \right] \leq \frac{8}{\eta} \mathbb{E} \left[ \frac{f(\bar{\theta}^0)}{T} - f(\bar{\theta}^T) \right] + \frac{4a^2 L}{n} \left( 1 + C_{\text{NM}, \gamma} \frac{3L_n}{2\rho_{\text{NM}, \gamma}} \eta \right)
\]

Setting \( \eta = O(1/\sqrt{T}) \) shows the expected convergence rate of \( \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \nabla f(\bar{\theta}^t) \|^2 \right] = O(1/\sqrt{T}) \). We remark that similar result to Corollary 4.1 can be established under the PL condition Assumption 5.

### B.1 Proof of Lemma B.1

Using the \( L \)-smoothness of \( f \), we obtain:

\[
f(\bar{\theta}^{t+1}) \leq f(\bar{\theta}^t) + \langle \nabla f(\bar{\theta}^t) \mid \bar{\theta}^{t+1} - \bar{\theta}^t \rangle + \frac{L}{2} \| \bar{\theta}^{t+1} - \bar{\theta}^t \|^2
\]

\[
= f(\bar{\theta}^t) - \eta \| \nabla f(\bar{\theta}^t) \|^2 + \eta \langle \nabla f(\bar{\theta}^t) \mid \bar{g}^t - \nabla f(\bar{\theta}^t) \rangle + \frac{L\eta^2}{2} \| \bar{g}^t \|^2
\]

Taking the conditional expectation on the inner product:

\[
- \eta \mathbb{E}_t \langle \nabla f(\bar{\theta}^t) \mid \bar{g}^t - \nabla f(\bar{\theta}^t) \rangle
\]

\[
= - \eta \mathbb{E}_t \left( \nabla f(\bar{\theta}^t) \right) \left[ \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\bar{\theta}^t_i) - \nabla f(\bar{\theta}^t) \right]
\]

\[
\leq \frac{\eta}{2} \| \nabla f(\bar{\theta}^t) \|^2 + \frac{\eta}{2} \left\| \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\bar{\theta}^t_i) - \nabla f(\bar{\theta}^t) \right\|^2
\]

\[
\leq \frac{\eta}{2} \| \nabla f(\bar{\theta}^t) \|^2 + \frac{\eta}{2n} \sum_{i=1}^{n} \| \nabla f_i(\bar{\theta}^t_i) - \nabla f_i(\bar{\theta}^t) \|^2
\]

\[
\leq \frac{\eta}{2} \| \nabla f(\bar{\theta}^t) \|^2 + \frac{\eta L^2}{2n} \| \bar{\theta}^t - \bar{\theta}^t \|^2
\]

\[
= \frac{\eta}{2} \| \nabla f(\bar{\theta}^t) \|^2 + \frac{\eta L^2}{2n} \| \Theta^t - \Theta^t \|^2
\]

where we denote \( \Theta^t = 1(\bar{\theta}^t)^\top = \frac{1}{n} \mathbf{1} \mathbf{1}^\top \Theta^t \).

Putting back into (53) yields

\[
\mathbb{E}_t[f(\bar{\theta}^{t+1})] \leq f(\bar{\theta}^t) - \frac{\eta}{2} \| \nabla f(\bar{\theta}^t) \|^2 + \frac{L\eta^2}{2} \mathbb{E}_t \| \bar{g}^t \|^2 + \frac{\eta L^2}{2n} \| \Theta^t - \Theta^t \|^2.
\]
Observe that as $\mathbb{E}_t \|\hat{g}^i\|^2 = \frac{1}{n} \mathbb{E}_t \|1^T G^i\|^2$, we have
\[
\mathbb{E}_t \|1^T G^i\|^2_F = \mathbb{E}_t \left\| \sum_{i=1}^n \nabla f_i(\hat{\theta}^t; \zeta^{t+1}) \right\|^2 = \mathbb{E}_t \left\| \sum_{i=1}^n \nabla f_i(\hat{\theta}^t; \zeta^{t+1}) - \sum_{i=1}^n \nabla f_i(\hat{\theta}^t) \right\|^2 + \| \sum_{i=1}^n \nabla f_i(\hat{\theta}^t) \|^2 \\
\leq n\sigma^2 + 2 \left\| \sum_{i=1}^n \{ \nabla f_i(\hat{\theta}^t) - \nabla f_i(\hat{\theta}^t) \} \right\|^2 + 2 \left\| \sum_{i=1}^n \nabla f_i(\hat{\theta}^t) \right\|^2 \\
\leq n\sigma^2 + 2nL^2 \|\Theta^t - \hat{\Theta}^t\|^2_F + 2n^2 \|\nabla f(\hat{\theta}^t)\|^2 .
\]
Putting back into (54), and assuming that $-(\frac{1}{2} - L\eta) \leq -\frac{1}{4} \Leftrightarrow \eta \leq \frac{1}{4}$,
\[
\mathbb{E}_t[f(\hat{\Theta}^{t+1})] \leq f(\hat{\theta}^t) - \frac{\eta}{2} \|\nabla f(\hat{\theta}^t)\|^2 + \frac{Ln^2}{2n} \left( n\sigma^2 + 2nL^2 \|\Theta^t - \hat{\Theta}^t\|^2_F + 2n^2 \|\nabla f(\hat{\theta}^t)\|^2 \right) + \frac{\eta L^2}{2n} \|\Theta^t - \hat{\Theta}^t\|^2_F \\
= f(\hat{\theta}^t) - \eta \left(\frac{1}{2} - L\eta\right) \|\nabla f(\hat{\theta}^t)\|^2 + \left(\frac{L^2\eta^2}{n} + \frac{L^2\eta}{2n}\right) \|\Theta^t - \hat{\Theta}^t\|^2_F + \frac{L\eta^2\sigma^2}{2n} \\
\leq f(\hat{\theta}^t) - \frac{\eta}{4} \|\nabla f(\hat{\theta}^t)\|^2 + \frac{3L^2\eta^2}{4n} \|\Theta^t - \hat{\Theta}^t\|^2_F + \frac{L\eta^2\sigma^2}{2n} .
\]
The proof is completed.

\section*{B.2 Proof of Lemma B.2}
We preface the proof by stating two lemmas that will be instrumental to the proof of Lemma B.2. Their proofs can be found in the later part of this subsection.

\begin{lemma}
Under Assumption 3. For any $t \geq 0$, it holds:
\[
\mathbb{E} \left[ \|\nabla \hat{F}^{t+1} - \nabla \hat{F}^t\|^2_F \right] \leq 2L^2 \mathbb{E} \left[ \|\Theta^{t+1} - \Theta^t\|^2_F \right] + 3n\sigma^2 .
\]
\end{lemma}

\begin{lemma}
Under Assumption 1, 2, 3, 4. For any $t \geq 0$, it holds
\[
\mathbb{E}_t \left[ \|\Theta^{t+1} - \Theta^t\|^2_F \right] \leq 4\eta^2 (1 - \rho)^2 \|G^t\|_F^2 + 4 (\eta^2 (1 - \rho)^2 L^2 + \bar{\omega}^2 \gamma^2) \|\Theta^t\|^2_F + 2\omega^2 \gamma^2 (1 - \delta) \|\Theta^t - \eta G^t - \hat{\Theta}^t\|^2_F + 2\eta^2 (1 - \rho)^2 \sigma^2 + 4\eta^2 (1 - \rho)^2 \|\nabla f(\hat{\theta}^t)\|^2 .
\]
\end{lemma}

\textbf{Proof of Lemma B.2.} We begin by observing the update for $G^{t+1}$ as:
\[
G^{t+1} = U^T [G^t + \nabla \hat{F}^{t+1} - \nabla \hat{F}^t + \gamma (W-I) \hat{\Theta}^{t+1}] \\
= U^T \left[ G^t + \nabla \hat{F}^{t+1} - \nabla \hat{F}^t + \gamma (W-I)(G^t + \mathcal{Q}(G^t + \nabla \hat{F}^{t+1} - \nabla \hat{F}^t - \hat{G}^t)) \right] \\
= U^T \left[ (I + \gamma (W-I))(G^t + \nabla \hat{F}^{t+1} - \nabla \hat{F}^t) \right] \\
+ \gamma U^T (W-I) \left[ \mathcal{Q}(G^t + \nabla \hat{F}^{t+1} - \nabla \hat{F}^t - \hat{G}^t) - (G^t + \nabla \hat{F}^{t+1} - \nabla \hat{F}^t - \hat{G}^t) \right]
\]
The above implies that
\[
\mathbb{E}_t \left[ \|G^{t+1}\|_F^2 \right] \leq (1 + \alpha_0)(1 - \rho)^2 \mathbb{E}_t \left[ \|G^t + U^T (\nabla \hat{F}^{t+1} - \nabla \hat{F}^t)\|^2_F \right] \\
+ (1 + \alpha_0^{-1})\gamma^2 \omega^2 (1 - \delta) \mathbb{E}_t \left[ \|\nabla \hat{F}^{t+1} - \nabla \hat{F}^t - \hat{G}^t\|^2_F \right] \\
\leq (1 + \alpha_0)(1 - \rho)^2 \mathbb{E}_t \left[ (1 + \alpha_1) \|G^t\|^2_F + (1 + \alpha_1^{-1}) \|\nabla \hat{F}^{t+1} - \nabla \hat{F}^t\|^2_F \right] \\
+ 2(1 + \alpha_0^{-1})\gamma^2 \omega^2 (1 - \delta) \mathbb{E}_t \left[ \|G^t - \hat{G}^t\|^2_F + \|\nabla \hat{F}^{t+1} - \nabla \hat{F}^t\|^2_F \right]
\]
Taking $\alpha_0 = \frac{\rho^2}{1 - \rho^2}, \alpha_1 = \frac{\rho^2}{2}$ gives
\[
\mathbb{E}_t \left[ \|G^{t+1}\|_F^2 \right] \leq \left( 1 - \frac{\rho^4}{2} \right) \|G^t\|^2_F + \frac{2\omega^2}{\rho} \|G^t - \hat{G}^t\|^2_F
\]

\(26\)
Taking the full expectation and applying Lemma B.6 give

\[ E \left[ \|G_o^{t+1}\|^2_F \right] \leq \left( 1 - \frac{\rho\gamma}{2} \right) E \left[ \|G_o^t\|^2_F \right] + \gamma \frac{2\omega^2}{\rho} E \left[ \|G^t - \hat{G}^t\|^2_F \right] + \frac{2}{\rho\gamma} \left( 1 + \gamma^2 \omega^2 \right) \left( 3n\sigma^2 + 2L^2 \| \left( \Theta^{t+1} - \Theta^t \right) \|^2_F \right) \]

Furthermore, applying Lemma B.7 yields

\[ E \left[ \|G_o^{t+1}\|^2_F \right] \leq \left( 1 - \frac{\rho\gamma}{2} + \eta^2 \frac{16L^2(1 + \gamma^2 \omega^2)(1 - \rho\gamma)^2}{\rho\gamma} \right) E \left[ \|G_o^t\|^2_F \right] + \gamma \frac{2\omega^2}{\rho} E \left[ \|G^t - \hat{G}^t\|^2_F \right] + \frac{16L^2(1 + \gamma^2 \omega^2)}{\rho\gamma} \left( \eta^2 L^2(1 - \rho\gamma)^2 + \omega^2 \gamma^2 \right) E \left[ \|\Theta_o^t\|^2_F \right] + \frac{8L^2(1 + \gamma^2 \omega^2)}{\rho\gamma} \omega^2 \gamma^2 \left( 1 - \delta \right) E \left[ \|\Theta^t - \eta G^t - \hat{\Theta}^t\|^2_F \right] + \frac{16L^2(1 + \gamma^2 \omega^2)}{\rho\gamma} \left( 1 - \rho\gamma \right)^2 n \eta^2 E \left[ \|\nabla f(\bar{b}^t)\|^2 \right] + \left( \frac{6}{\rho\gamma} \left( 1 + \gamma^2 \omega^2 \right) + \frac{8}{\rho\gamma} \eta^2 \left( 1 + \gamma^2 \omega^2 \right) L^2 \left( 1 - \rho\gamma \right)^2 \right) \sigma^2 \]

Using the step size condition

\[ \eta \leq \frac{\rho\gamma}{8L(1 - \rho\gamma)\sqrt{1 + \gamma^2 \omega^2}}, \quad \gamma \leq \frac{1}{8\omega} \tag{59} \]

implies that

\[ E \left[ \|G_o^{t+1}\|^2_F \right] \leq \left( 1 - \frac{\rho\gamma}{4} \right) E \left[ \|G_o^t\|^2_F \right] + \gamma \frac{2\omega^2}{\rho} E \left[ \|G^t - \hat{G}^t\|^2_F \right] + \gamma \left( \frac{\rho}{4} + \frac{18L^2\omega^2}{\rho\gamma} \right) E \left[ \|\Theta_o^t\|^2_F \right] + \gamma \frac{9L^2}{\rho} \omega^2 \left( 1 - \delta \right) E \left[ \|\Theta^t - \eta G^t - \hat{\Theta}^t\|^2_F \right] + \gamma \frac{3n}{4} E \left[ \|\nabla f(\bar{b}^t)\|^2 \right] + \left( \frac{7n}{\rho\gamma} + \frac{\rho\gamma}{8} \right) \sigma^2 \]

This concludes our proof. \( \square \)

**B.2.1 Proofs for the Auxiliary Lemmas**

**Proof of Lemma B.6** Observe that:

\[ E \left[ \left\| \hat{\nabla} F^{t+1} - \nabla \hat{F}^t \right\|^2_F \right] = E \left[ \left\| (\hat{\nabla} F^{t+1} - \nabla F^{t+1}) - (\nabla \hat{F}^t - \nabla F^t) \right\|^2_F + \left\| \nabla F^{t+1} - \nabla F^t \right\|^2_F \]

\[ + 2E \left[ \left\langle (\hat{\nabla} F^{t+1} - \nabla F^{t+1}) - (\nabla \hat{F}^t - \nabla F^t) \left| \nabla F^{t+1} - \nabla F^t \right. \right. \right] \]

Notice that

\[ E \left[ \left\langle (\hat{\nabla} F^{t+1} - \nabla F^{t+1}) - (\nabla \hat{F}^t - \nabla F^t) \left| \nabla F^{t+1} - \nabla F^t \right. \right. \right] = -E \left[ \left\langle \nabla \hat{F}^t - \nabla F^t \left| \nabla F^{t+1} - \nabla F^t \right. \right. \right] \]

As such,

\[ E \left[ \left\| \hat{\nabla} F^{t+1} - \nabla \hat{F}^t \right\|^2_F \right] \leq 3n\sigma^2 + 2L^2 E \left[ \| \Theta^{t+1} - \Theta^t \|^2_F \right]. \]

This concludes the proof. \( \square \)

**Proof of Lemma B.7** Observe that:

\[ \| \Theta^{t+1} - \Theta^t \|^2_F = \| \eta G^t - \gamma (W - I) \hat{b}^{t+1} \|^2_F \]

\[ = \| (I + \gamma (W - I))(-\eta G^t) + \gamma (W - I) \Theta^t + \gamma (W - I) \left[ Q(\Theta^t - \eta G^t - \hat{\Theta}^t) - (\Theta^t - \eta G^t - \hat{\Theta}^t) \right] \|^2_F \]
\begin{align*}
\leq & \ 2 \left\| (I + \gamma (W - I)) (-\eta G^t) + \gamma (W - I) \Theta^t \right\|_F^2 \\
& + 2 \left\| (W - I) \left[ Q (\Theta^t - \eta G^t - \Theta^t) - (\Theta^t - \eta G^t - \Theta^t) \right] \right\|_F^2 \\
\leq & \ 2 \left\| (I + \gamma (W - I)) G^t \right\|_F^2 + \gamma^2 \left\| (W - I) \Theta^t \right\|_F^2 + 2 \left\langle (I + \gamma (W - I)) (-\eta G^t) \ | \ (W - I) \Theta^t \right\rangle \\
& + 2 \omega^2 \gamma^2 \left\| Q (\Theta^t - \eta G^t - \Theta^t) - (\Theta^t - \eta G^t - \Theta^t) \right\|_F^2,
\end{align*}
\tag{60}
Observe that \( \left\langle (I + \gamma (W - I)) (-\eta G^t) \ | \ (W - I) \Theta^t \right\rangle = (I + \gamma (W - I)) (-\eta G^t) \left\| (W - I) \Theta^t \right\|_F^2 \), the above leads to
\begin{align*}
\mathbb{E}_t \left[ \left\| \Theta^{t+1} - \Theta^t \right\|_F^2 \right] & \leq 2 \eta^2 (1 - \rho \gamma)^2 \left\| G^t \right\|_F^2 + 2 \eta^2 (1 - \rho \gamma)^2 \left\| G^t \right\|_F^2 + 4 \eta^2 \left\| (W - I) \Theta^t \right\|_F^2 \\
& + 2 \omega^2 \gamma^2 (1 - \delta) \left\| \Theta^t - \eta G^t - \Theta^t \right\|_F^2,
\end{align*}
\tag{61}
Notice that using (55), we obtain
\begin{align*}
\mathbb{E}_t \left[ \left\| G^t \right\|_F^2 \right] = \mathbb{E}_t \left[ \left\| ((1/n)11^T + UU^T) G^t \right\|_F^2 \right] = \mathbb{E}_t \left[ \left\| UU^T G^t \right\|_F^2 + \left\| (1/n)11^T G^t \right\|_F^2 \right] \\
& \leq \left\| G^t \right\|_F^2 + 2 \omega^2 \left\| \Theta^t \right\|_F^2 + 2 n \left\| \nabla f(\hat{\theta}) \right\|^2 + \sigma^2
\end{align*}
\tag{62}
By combining (61), (62), and the fact \( \left\| (W - I) \Theta^t \right\|_F^2 \leq \omega^2 \left\| \Theta^t \right\|_F^2 \), we have
\begin{align*}
\mathbb{E}_t \left[ \left\| \Theta^{t+1} - \Theta^t \right\|_F^2 \right] & \leq 4 \eta^2 (1 - \rho \gamma)^2 \left\| G^t \right\|_F^2 + 4 \eta^2 (1 - \rho \gamma)^2 L^2 \left\| \Theta^t \right\|_F^2 + 4 \eta^2 (1 - \rho \gamma)^2 \left\| \nabla f(\hat{\theta}) \right\|^2 \\
& + 2 \omega^2 (1 - \rho \gamma)^2 \sigma^2 + 4 \omega^2 \gamma^2 \left\| \Theta^t \right\|_F^2 + 2 \omega^2 \gamma^2 (1 - \delta) \left\| \Theta^t - \eta G^t - \Theta^t \right\|_F^2
\end{align*}
This concludes the proof.

\section*{B.3 Proof of Lemma B.3}
We begin by observing the following recursion for \( G^t - \hat{G}^t \):
\begin{align*}
G^{t+1} - \hat{G}^{t+1} &= G^t + \nabla \hat{F}^{t+1} - \nabla \hat{F}^t + (\gamma (W - I) - I) \hat{G}^{t+1} \\
&= \gamma (W - I) (G^t - \hat{G}^t) + (\gamma (W - I) - I) \left[ Q (G^t - \hat{G}^t) - (G^t - \hat{G}^t) \right].
\end{align*}
This implies
\begin{align*}
\mathbb{E} \left[ \left\| G^{t+1} - \hat{G}^{t+1} \right\|_F^2 \right] & \leq \left( 1 + \alpha_0 \right) (1 + \gamma \omega)^2 (1 - \delta) \mathbb{E} \left[ \left( 1 + \alpha_1 \right) \left\| G^t - \hat{G}^t \right\|_F^2 + \left( 1 + \alpha_1 \right) \left\| \nabla \hat{F}^{t+1} - \nabla \hat{F}^t \right\|_F^2 \right] \\
& + 2 \left( 1 + \alpha_0 \right) \gamma \omega^2 \mathbb{E} \left[ \left\| G^t \right\|_F^2 + \left\| \nabla \hat{F}^{t+1} - \nabla \hat{F}^t \right\|_F^2 \right]
\end{align*}
Taking \( \alpha_0 = \frac{4}{7} \), \( \alpha_1 = \frac{6}{8} \) and the step size condition
\[ \gamma \leq \frac{\delta}{8 \omega} \leq \frac{\delta}{6 \omega} \]
give
\begin{align*}
\mathbb{E} \left[ \left\| G^{t+1} - \hat{G}^{t+1} \right\|_F^2 \right] & \leq \left( 1 - \frac{\delta}{8} \right) \mathbb{E} \left[ \left\| G^t - \hat{G}^t \right\|_F^2 \right] + \frac{10 \gamma \omega^2}{\delta} \mathbb{E} \left[ \left\| G^t \right\|_F^2 \right] \\
& + \left( 1 - \frac{\delta}{4} \right) (1 + \frac{8}{\delta}) + 2 \gamma \omega^2 (1 + \frac{4}{\delta}) \mathbb{E} \left[ \left\| \nabla \hat{F}^{t+1} - \nabla \hat{F}^t \right\|_F^2 \right] \\
& \leq \left( 1 - \frac{\delta}{8} \right) \mathbb{E} \left[ \left\| G^t - \hat{G}^t \right\|_F^2 \right] + \frac{10 \gamma \omega^2}{\delta} \mathbb{E} \left[ \left\| G^t \right\|_F^2 \right] + \frac{10 (1 + \gamma \omega^2)}{\delta} \mathbb{E} \left[ \left\| \nabla \hat{F}^{t+1} - \nabla \hat{F}^t \right\|_F^2 \right]
\end{align*}
Applying Lemma B.6 and Lemma B.7 gives
\begin{align*}
\mathbb{E} \left[ \left\| G^{t+1} - \hat{G}^{t+1} \right\|_F^2 \right] & \leq \left( 1 - \frac{\delta}{8} \right) \mathbb{E} \left[ \left\| G^t - \hat{G}^t \right\|_F^2 \right] + \frac{10 \gamma \omega^2}{\delta} \mathbb{E} \left[ \left\| G^t \right\|_F^2 \right] + \frac{30 (1 + \gamma \omega^2)}{28} \sigma^2
\end{align*}
\[
\begin{align*}
&+ \frac{20(1 + \gamma^2 \omega^2)L^2}{\delta} \mathbb{E} \left[ \| \Theta^{t+1} - \Theta^t \|^2_F \right] \\
&\leq \left( 1 - \frac{\delta}{8} \right) \mathbb{E} \left[ \| G^t - \hat{G}^t \|^2_F \right] + \frac{10}{\delta} \left( \gamma^2 \omega^2 + 8\eta^2 L^2 (1 + \gamma^2 \omega^2) (1 - \rho \gamma)^2 \right) \mathbb{E} \left[ \| G^t_o \|^2_F \right] \\
&+ \frac{80(1 + \gamma^2 \omega^2)L^2}{\delta} \left( \eta^2 L^2 (1 - \rho \gamma)^2 + \omega^2 \gamma^2 \right) \mathbb{E} \left[ \| \Theta^t_o \|^2_F \right] \\
&+ \frac{40(1 + \gamma^2 \omega^2)L^2}{\delta} \omega^2 \gamma^2 (1 - \delta) \mathbb{E} \left[ \| \Theta^t - \eta G^t - \hat{G}^t \|^2_F \right] \\
&+ 80(1 + \gamma^2 \omega^2)L^2 n\eta^2 (1 - \rho \gamma)^2 \mathbb{E} \left[ \| \nabla f(\bar{\theta}^t) \|^2 \right] \\
&+ \frac{10(1 + \gamma^2 \omega^2)}{\delta} \left( 4L^2 \eta^2 (1 - \rho \gamma)^2 \sigma^2 + 3n \sigma^2 \right)
\end{align*}
\]

Using the step size condition from (59), i.e., \( \eta^2 L^2 (1 - \rho \gamma)^2 (1 + \gamma^2 \omega^2) \leq \frac{\rho^2 \gamma^2}{8} \), simplifies the above to

\[
\mathbb{E} \left[ \| G^{t+1} - \hat{G}^{t+1} \|^2_F \right] \leq \left( 1 - \frac{\delta}{8} \right) \mathbb{E} \left[ \| G^t - \hat{G}^t \|^2_F \right] + \frac{10}{\delta} \gamma^2 \left( \omega^2 + \frac{\rho^2 \gamma^2}{8} \right) \mathbb{E} \left[ \| G^t_o \|^2_F \right] \\
+ \gamma^2 \frac{5}{48} \left( \rho^2 + 72L^2 \omega^2 \right) \mathbb{E} \left[ \| \Theta^t_o \|^2_F \right] + \gamma^2 \frac{40L^2 \omega^2}{\delta} \mathbb{E} \left[ \| \Theta^t - \eta G^t - \hat{G}^t \|^2_F \right] \\
+ \gamma^2 \frac{5\rho^2}{4\delta} n \mathbb{E} \left[ \| \nabla f(\bar{\theta}^t) \|^2 \right] + \frac{5}{\delta} \left( \frac{\rho^2 \gamma^2}{8} + 7n \right) \sigma^2
\]

This concludes our proof.

**B.4 Proof of Lemma B.4**

Observe that

\[
\mathbb{E}_t \left[ \| \Theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^{t+1} \|^2_F \right] = \mathbb{E}_t \left[ \| \Theta^{t+1} - \eta G^{t+1} - (\hat{\Theta}^t + \mathcal{Q}(\Theta^t - \eta G^t - \hat{\Theta}^t)) \|^2_F \right] \\
= \mathbb{E}_t \left[ \| \Theta^{t+1} - \eta G^{t+1} - (\Theta^t - \eta G^t) + (\Theta^t - \eta G^t - \hat{\Theta}^t) - \mathcal{Q}(\Theta^t - \eta G^t - \hat{\Theta}^t) \|^2_F \right] \\
\leq (1 + 2 \frac{2}{\delta}) \mathbb{E}_t \left[ \| \Theta^{t+1} - \Theta^t - \eta (G^{t+1} - G^t) \|^2_F \right] + (1 + \frac{\delta}{2}) (1 - \delta) \left\| \Theta^t - \eta G^t - \hat{\Theta}^t \right\|^2_F \\
\leq 2(1 + 2 \frac{2}{\delta}) \mathbb{E}_t \left[ \| \Theta^{t+1} - \Theta^t \|^2_F \right] + 2\eta^2 (1 + \frac{2}{\delta}) \mathbb{E}_t \left[ \| G^{t+1} - G^t \|^2_F \right] + (1 - \frac{\delta}{2}) \left\| \Theta^t - \eta G^t - \hat{\Theta}^t \right\|^2_F \tag{63}
\]

We can bound the second term as

\[
\mathbb{E} \left[ \| G^{t+1} - G^t \|^2_F \right] = \frac{1}{n} \mathbb{E} \left[ \| 1^T (\nabla \hat{F}^{t+1} - \nabla \hat{F}^t) \|^2 \right] + 2 \mathbb{E} \left[ \| G^{t+1}_o \|^2_F \right] + 2 \mathbb{E} \left[ \| G^t_o \|^2_F \right]
\]

Observe that

\[
\mathbb{E} \left[ \| 1^T (\nabla \hat{F}^{t+1} - \nabla \hat{F}^t) \|^2 \right] \leq \mathbb{E} \left[ \| 1^T (\nabla \hat{F}^{t+1} - \nabla F^{t+1}) \|^2 \right] + 2 \mathbb{E} \left[ \| 1^T (\nabla \hat{F}^t - \nabla F^t) \|^2 \right] \\
+ 2 \mathbb{E} \left[ \| 1^T (\nabla F^{t+1} - \nabla F^t) \|^2 \right] \\
\leq 3n \sigma^2 + 2n L^2 \mathbb{E} \left[ \| \Theta^{t+1} - \Theta^t \|^2_F \right]
\]

Substituting back into (63) yields

\[
\mathbb{E} \left[ \| \Theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^{t+1} \|^2_F \right] \leq (1 - \frac{\delta}{2}) \mathbb{E} \left[ \| \Theta^t - \eta G^t - \hat{\Theta}^t \|^2_F \right] + 4\eta^2 (1 + \frac{2}{\delta}) \mathbb{E} \left[ \| G^{t+1}_o \|^2_F + \| G^t_o \|^2_F \right] \\
+ 2(1 + \frac{2}{\delta})(\eta^2 L^2 + 1) \mathbb{E} \left[ \| \Theta^{t+1} - \Theta^t \|^2_F \right] + 6\sigma^2 (1 + \frac{2}{\delta}) \eta^2 \tag{64}
\]

We further apply Lemma B.7 to obtain

\[
\mathbb{E} \left[ \| \Theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^{t+1} \|^2_F \right] \leq (1 - \frac{\delta}{2}) \mathbb{E} \left[ \| \Theta^t - \eta G^t - \hat{\Theta}^t \|^2_F \right] + 4\eta^2 (1 + \frac{2}{\delta}) \mathbb{E} \left[ \| G^{t+1}_o \|^2_F + \| G^t_o \|^2_F \right]
\]
\[ + 6\sigma^2(1 + \frac{2}{\delta})\eta^2 + 4\bar{\omega}^2(1 + \eta^2L^2)(1 + \frac{2}{\delta})\gamma^2(1 - \delta)E \left[ \left\| \Theta' - \eta G^t - \hat{\Theta}' \right\| _F^2 \right] \\
+ 8(1 + \eta^2L^2)(1 + \frac{2}{\delta})\eta^2(1 - \rho\gamma)^2E \left[ \left\| G''_o \right\| _F^2 + L^2 \left\| \Theta''_o \right\| _F^2 + n \left\| \nabla f(\hat{\theta}') \right\|^2 + \sigma'^2/2 \right] \\
+ 8(1 + \eta^2L^2)(1 + \frac{2}{\delta})\omega^2\gamma^2E \left[ \left\| \Theta''_o \right\| _F^2 \right] \] (65)

Using the step size condition:
\[ \gamma^2 \leq \frac{\delta}{16\bar{\omega}^2(1 - \delta)(1 + \eta^2L^2)(1 + 2/\delta)} \]
and we recall that \( \eta \leq 1/(4L) \), the upper bound in (65) can be simplified as
\[ E \left[ \left\| \Theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^{t+1} \right\| _F^2 \right] \leq (1 - \frac{\delta}{4})E \left[ \left\| \Theta' - \eta G^t - \hat{\Theta}' \right\| _F^2 \right] + \eta^2 \frac{12}{\delta} E \left[ \left\| G''_o \right\| _F^2 \right] \\
+ \frac{24}{\delta}\eta^2\sigma^2 + \frac{38}{\delta}\eta^2 E \left[ \left\| G''_o \right\| _F^2 \right] + \frac{26}{\delta} \left( \eta^2 L^2 + \omega^2\gamma^2 \right) \left\| \Theta''_o \right\| _F + \eta^2 n \left\| \nabla f(\hat{\theta}') \right\|^2 \]

This concludes the proof of the first part.

The above bound can be combined with Lemma B.2 and \( \gamma\rho \leq 1 \) to give
\[ E \left[ \left\| \Theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^{t+1} \right\| _F^2 \right] \leq \left( 1 - \frac{\delta}{8} \right) E \left[ \left\| \Theta' - \eta G^t - \hat{\Theta}' \right\| _F^2 \right] \\
+ \eta^2 \frac{50}{\delta} E \left[ \left\| G''_o \right\| _F^2 \right] + \eta^2 \frac{3\omega^2}{\rho} E \left[ \left\| G^t - \hat{G}' \right\| _F^2 \right] \\
+ \eta^2 \frac{50L^4}{\delta} \left( 1 + \frac{\omega}{\rho} \right) + \frac{26}{\delta}\omega\gamma^2 E \left[ \left\| \Theta''_o \right\| _F^2 \right] \\
+ \eta^2 \frac{29n}{\delta} E \left[ \left\| \nabla f(\hat{\theta}') \right\|^2 \right] + \eta^2 \frac{24\sigma^2}{\delta} \left( 1 + \frac{4n}{\rho\gamma} \right) \]

Taking \( \eta^2\gamma \leq \frac{\bar{\omega}^2\rho}{500L^2\delta} \) simplifies the bound into
\[ E \left[ \left\| \Theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^{t+1} \right\| _F^2 \right] \leq \left( 1 - \frac{\delta}{8} \right) E \left[ \left\| \Theta' - \eta G^t - \hat{\Theta}' \right\| _F^2 \right] \\
+ \eta^2 \frac{50}{\delta} E \left[ \left\| G''_o \right\| _F^2 \right] + \eta^2 \frac{3\omega^2}{\rho} E \left[ \left\| G^t - \hat{G}' \right\| _F^2 \right] \\
+ \eta^2 \frac{29n}{\delta} E \left[ \left\| \nabla f(\hat{\theta}') \right\|^2 \right] + \eta^2 \frac{24\sigma^2}{\delta} \left( 1 + \frac{4n}{\rho\gamma} \right) \]

This concludes the proof.

**B.5 Proof of Lemma B.5**

Let \( a, b, c > 0 \) be some constants to be determined later, combining Lemma 4.3, B.2, B.3, B.4 yields
\[ E \left[ \left\| \Theta''_o \right\| _F^2 + a \left\| G''_o \right\| _F^2 + b \left\| G^{t+1} - \hat{G}'^t \right\| _F^2 + c \left\| \Theta^{t+1} - \eta G^{t+1} - \hat{\Theta}^{t+1} \right\| _F^2 \right] \]
\[ \leq \left( 1 - \frac{\rho\gamma}{2} + a\gamma(\frac{\rho}{4} + 18L^2\omega^2 + \frac{18L^2\omega^2}{\rho}) + b\gamma^2(\frac{2}{3}(\rho^2 + 45L^2\omega^2) + \frac{c}{3} \left[ 29L^4(1 + \frac{\omega}{\rho})\gamma^2 + 26\omega^2\gamma^2 \right]) E \left[ \left\| \Theta''_o \right\| _F^2 \right] \\
+ a \left[ 1 - \frac{\rho\gamma}{4} + \frac{1}{a}\eta^2 \frac{2\omega^2}{\rho\gamma} + \frac{b}{a}\gamma^2 \frac{10}{3}(\delta\omega^2 + \rho^2) + \frac{c}{a}\eta^2 \frac{50}{\delta} \right] E \left[ \left\| G''_o \right\| _F^2 \right] \\
+ b \left[ 1 - \frac{\delta}{8} + \frac{a}{b}\eta^2 \frac{2\omega^2}{\rho\gamma} + \frac{c}{b}\eta^2 \frac{3\omega^2}{\rho\gamma} \right] E \left[ \left\| G^t - \hat{G}' \right\| _F^2 \right] \\
+ c \left[ 1 - \frac{\delta}{8} + \frac{1}{c}\frac{\omega^2}{\rho\gamma} + \frac{a}{c}\gamma \frac{9L^2\omega^2}{\rho \gamma} + \frac{b}{c}\gamma \frac{40L^2\omega^2}{\rho \delta} \right] E \left[ \left\| \Theta' - \eta G^t - \hat{\Theta}' \right\| _F^2 \right] \]
Observe that with this choice of $\rho$, the bound on $\gamma$ can be implied by:

$$\gamma^2 \leq \frac{\rho^2 \delta^3 (1 - \delta/8\omega)}{7680 \omega^2 (\delta \omega^2 + \rho^2)}$$

Observe that with this choice of $b$ and the step size condition, we have $40L^2 \gamma^2 b \leq \gamma$. Finally, the condition on $c$ is simplified to

$$\frac{24\omega^2}{\rho \delta (1 - \gamma)} \leq c \leq \min \left\{ \frac{\rho \delta \gamma}{12 (\frac{58L^4}{\rho} \eta^2 + 26\omega^2 \gamma^2)}, \frac{\delta}{25 \rho \gamma}, \frac{32}{\rho^2 \gamma} \right\}$$

The above condition is feasible if

$$\frac{\gamma^2}{1 - \gamma} \leq \min \left\{ \frac{4 \delta}{3 \rho \omega^2}, \frac{\delta^2}{600 \omega^2}, \frac{\rho^2 \delta^2}{14976 \omega^2}, \frac{\delta}{16704 L^4 \omega^3} \right\}$$

Notice that the step size condition:

$$\eta^2 \leq \frac{\rho^4 \gamma^2}{10944 L^2 \omega^2}$$

guarantees the existence of $a$ which satisfies (66). In particular, we take $a = \frac{48}{\rho^2 \omega^2} \eta^2$. This simplifies (67), (68) into

$$\frac{48 \omega^2}{\rho \delta (1 - \gamma)} \eta^2 \leq b \leq \min \left\{ \frac{1}{\gamma} \frac{1104 L^2 \omega^2}{\gamma^3 \omega^2 (\delta \omega^2 + \rho^2)}, \frac{\rho \delta \gamma}{12 (\frac{58L^4}{\rho} \eta^2 + 26\omega^2 \gamma^2)} \right\}$$

we impose the extra condition $c \leq \frac{32}{\rho^2 \gamma}$. We obtain the simplification:

$$\frac{1536 \omega^2}{\rho \delta (1 - \gamma)} \eta^2 \leq b \leq \min \left\{ \frac{1}{\gamma} \frac{1104 L^2 \omega^2}{\gamma^3 \omega^2 (\delta \omega^2 + \rho^2)}, \frac{\rho \delta \gamma}{12 (\frac{58L^4}{\rho} \eta^2 + 26\omega^2 \gamma^2)} \right\}$$

Again, the condition on $b$ is feasible if

$$\eta^2 \leq \frac{\rho^4 \delta^3 (1 - \gamma)}{1536 \times 1104 \times L^2 \omega^4}, \quad \frac{\gamma^2}{1 - \gamma} \leq \frac{\rho^2 \delta^3}{7680 \omega^2 (\delta \omega^2 + \rho^2)}$$

and we take $b = \frac{1536 \omega^2}{\rho \delta (1 - \gamma)} \eta^2$. Note that as $\gamma \leq \delta/8\omega$, the bound on $\gamma$ can be implied by:

$$\gamma^2 \leq \frac{\rho^2 \delta^3 (1 - \delta/8\omega)}{7680 \omega^2 (\delta \omega^2 + \rho^2)}$$

The above set of inequalities can be guaranteed if $a, b, c$ satisfy

$$\frac{48}{\rho^2 \omega^2}\eta^2 \leq a \leq \frac{\rho^2}{228L^2 \omega^2}$$

and we impose the extra condition $c \leq \frac{32}{\rho^2 \gamma}$. We obtain the simplification:

$$\frac{1536 \omega^2}{\rho \delta (1 - \gamma)} \eta^2 \leq b \leq \min \left\{ \frac{1}{\gamma} \frac{1104 L^2 \omega^2}{\gamma^3 \omega^2 (\delta \omega^2 + \rho^2)}, \frac{\rho \delta \gamma}{12 (\frac{58L^4}{\rho} \eta^2 + 26\omega^2 \gamma^2)} \right\}$$

we impose the extra condition $c \leq \frac{32}{\rho^2 \gamma}$. We obtain the simplification:

$$\frac{1536 \omega^2}{\rho \delta (1 - \gamma)} \eta^2 \leq b \leq \min \left\{ \frac{1}{\gamma} \frac{1104 L^2 \omega^2}{\gamma^3 \omega^2 (\delta \omega^2 + \rho^2)}, \frac{\rho \delta \gamma}{12 (\frac{58L^4}{\rho} \eta^2 + 26\omega^2 \gamma^2)} \right\}$$

The above condition is feasible if

$$\frac{\gamma^2}{1 - \gamma} \leq \min \left\{ \frac{4 \delta}{3 \rho \omega^2}, \frac{\delta^2}{600 \omega^2}, \frac{\rho^2 \delta^2}{14976 \omega^2}, \frac{\delta}{16704 L^4 \omega^3} \right\}$$

The above condition is feasible if

$$\frac{\gamma^2}{1 - \gamma} \leq \min \left\{ \frac{4 \delta}{3 \rho \omega^2}, \frac{\delta^2}{600 \omega^2}, \frac{\rho^2 \delta^2}{14976 \omega^2}, \frac{\delta}{16704 L^4 \omega^3} \right\}$$

The above condition is feasible if

$$\frac{\gamma^2}{1 - \gamma} \leq \min \left\{ \frac{4 \delta}{3 \rho \omega^2}, \frac{\delta^2}{600 \omega^2}, \frac{\rho^2 \delta^2}{14976 \omega^2}, \frac{\delta}{16704 L^4 \omega^3} \right\}$$
This is stronger than the typical Lipschitz continuity on the Assumption 1, i.e., we require that this is due to the stronger requirements on Lipschitz continuity of the gradient of objective function in \( \beta \)

a small momentum parameter (e.g., reduced algorithms (including GT-HSGD \( d = 60850 \) parameters. Additionally, we observe that when training LeNet-5 using momentum-based variance

Table 3 summarizes the tuned hyper parameters used by the experiment in Fig. 2. We conduct a similar

C.2 FEMNIST Dataset

In Fig. 3, we provide additional numerical results on the trajectories of gradient norm, training/testing accuracy

\[ \begin{array}{cccccc}
\text{Algorithms} & \text{Learning rate \( \eta \)} & \text{Consensus step size \( \gamma \)} & \text{Momentum param. \( \beta \)} & \text{Compress. ratio \( k/d \)} & \text{Batch size} \\
GNSD & 0.01 & - & - & - & 4 \\
DeTAG (\( R = 3 \)) & 0.01 & - & - & - & 1 \\
GT-HSGD & 0.01 & - & 0.01 & - & 2 \\
CHOCO-SGD & 0.01 & 0.32 & - & 0.1 & 4 \\
DoCoM-SGT & 0.01 & 0.15 & 0.01 & 0.05 & 2 \\
\end{array} \]

and we take \( c = \frac{24\omega^2}{\rho^3(1-\gamma)} \).

The above choice of \( a, b, c \) ensures that

\[
\begin{align*}
\mathbb{E} & \left( \left\| \Theta_{o}^{t+1} \right\|_F^2 + a \left\| G_{o}^{t+1} \right\|_F^2 + b \left\| G_{t}^{t+1} - \hat{G}_{t}^{t+1} \right\|_F^2 + c \left\| \Theta_{t}^{t+1} - \eta G_{t}^{t+1} - \hat{\Theta}_{t}^{t+1} \right\|_F^2 \right) \\
\leq & \left( 1 - \min \left\{ \frac{\rho}{\sqrt{2}}, \frac{\delta}{\sqrt{2}} \right\} \right) \gamma \mathbb{E} \left( \left\| \Theta_{o}^{t} \right\|_F^2 + a \left\| G_{o}^{t} \right\|_F^2 + b \left\| G_{t}^{t} - \hat{G}_{t}^{t} \right\|_F^2 + c \left\| \Theta_{t}^{t} - \eta G_{t}^{t} - \hat{\Theta}_{t}^{t} \right\|_F^2 \right) \\
+ & \frac{192\rho}{1-\gamma} \left[ \frac{2(1-\gamma)}{\rho^3\gamma^3} + \frac{320\omega^2}{\rho^3\delta^2\gamma} + \frac{15\omega^2}{\rho^3\delta^2\gamma} \right] \sigma^2 \eta^2 + \frac{12\rho}{1-\gamma} \left[ \frac{1-\gamma}{\rho^4\gamma} + \frac{256\omega^2}{\rho^4\delta^2\gamma} + \frac{58\omega^2}{\rho^4\delta^2} \right] \eta^2 \mathbb{E} \left( \left\| \nabla f(\hat{\theta}) \right\|_2^2 \right)
\end{align*}
\]

The proof is completed.

C Additional Numerical Results

In this section we provide additional plots and the tuned hyper-parameters of our simulation.

C.1 Synthetic Dataset

This dataset is generated from the benchmark framework leaf (Caldas et al., 2019). The number of data points possessed by each agent is different. In particular, we have the distribution \( \{ m_i \}_{i=1}^{25} \) which follows \[ 470, 403, 91, 84, 79, 51, 51, 38, 31, 25, 24, 19, 14, 10, 9, 6, 6, 5, 5, 4, 4, 4, 3, 3 \]. Table 2 provides the tuned parameters used for the experiment in Fig. 1.

In Fig. 3, we provide additional numerical results on the trajectories of gradient norm, training/testing accuracy of the algorithms. Similar comparisons between DoCoM-SGT and existing algorithms are observed. Notably, we see that DoCoM-SGT achieves the best gradient stationary solution in limited communication budget and recovers the same level of gradient stationary solution as the uncompressed GT-HSGD at the last iteration. Also, we observe that DoCoM-SGT is the first to achieve the best accuracy with the least network cost.

C.2 FEMNIST Dataset

Table 3 summarizes the tuned hyper parameters used by the experiment in Fig. 2. We conduct a similar benchmark experiment as (Lu & De Sa, 2021) and consider training a LeNet-5 neural network which has \( d = 60850 \) parameters. Additionally, we observe that when training LeNet-5 using momentum-based variance reduced algorithms (including GT-HSGD and DoCoM-SGT), we found that the algorithms can be unstable when a small momentum parameter (e.g., \( \beta = 0.1 \)) is adopted, unlike the experiments on synthetic data. We suspect that this is due to the stronger requirements on Lipschitz continuity of the gradient of objective function in Assumption 1, i.e., we require

\[ \sup_{\zeta \in \text{supp}(D_1)} \left\| \nabla f_i(\theta; \zeta) - \nabla f_i(\theta'; \zeta) \right\| \leq L \left\| \theta - \theta' \right\|, \forall \theta, \theta' \in \mathbb{R}^d. \]

This is stronger than the typical Lipschitz continuity on the expected gradient which only demands \( \left\| \mathbb{E}[\nabla f_i(\theta; \zeta)] - \mathbb{E}[\nabla f_i(\theta'; \zeta)] \right\| \leq L \left\| \theta - \theta' \right\| \). Particularly, the convergence of these momentum-based algorithms depend on the less smooth loss landscape from the neural network model and data distribution.

\footnotemark[2]

\footnotetext[2]{or as mentioned in Footnote 1, the condition can be slightly relaxed into \( \mathbb{E}_\zeta[\left\| \nabla f_i(\theta; \zeta) - \nabla f_i(\theta'; \zeta) \right\|^2] \leq L^2 \left\| \theta - \theta' \right\|^2 \).}
Figure 3: **Additional Results on Synthetic Data and Linear Model.** Worst-agent’s gradient norm, training/testing accuracy against the 32-bit floats transmitted (top) and iteration no. (bottom).

Table 3: Tuned hyper-parameters for LeNet-5 on FEMNIST. †As suggested in (Lu & De Sa, 2021), DeTAG is more stable only when using relatively smaller learning rate.

| Algorithms     | Learning rate $\eta$ | Consensus step size $\gamma$ | Momentum param. $\beta$ | Compress. ratio $k/d$ | Batch size |
|----------------|----------------------|------------------------------|-------------------------|-----------------------|------------|
| GNSD           | 0.1                  | -                            | -                       | -                     | 128        |
| DeTAG ($R = 3$) | 0.01†                | -                            | -                       | -                     | 43         |
| GT-HSGD        | 0.1                  | -                            | 0.99                    | -                     | 64         |
| CHOCO-SGD      | 0.1                  | 0.4                          | -                       | 0.1                   | 128        |
| DoCoM-SGT      | 0.1                  | 0.4                          | 1                       | 0.05                  | 64         |

In Fig. 4, Fig. 5, we provide additional numerical results on the trajectories of gradient norm, training/testing accuracy of the algorithms. Notice that one epoch consists of 196 iterations. Similar comparisons between DoCoM-SGT and existing algorithms are observed.
Figure 4: Additional Results on FEMNIST Data with LeNet-5. Worst-agent’s loss value and consensus gap against iteration number.

Figure 5: Additional Results on FEMNIST Data with LeNet-5. Worst-agent’s gradient norm, training/testing accuracy against the 32-bit floats transmitted (top) and iteration no. (bottom).