GEOMETRY OF COISOTROPIC SUBMANIFOLDS IN SYMPLECTIC MANIFOLDS AND KÄHLER MANIFOLDS

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Abstract. The first purpose of this paper is to generalize the well-known Maslov indices of maps of open Riemann surfaces with boundary lying on Lagrangian submanifolds to maps with boundary lying on coisotropic submanifolds in symplectic manifolds. For this purpose, we first define the notion of Maslov loops of coisotropic Grassmanians and their indices. Then we introduce the notions of transverse Maslov bundle of coisotropic submanifolds, and gradable coisotropic submanifolds. We then define graded coisotropic submanifolds and the coisotropic Maslov index of the maps with boundary lying on such graded coisotropic submanifolds, which reduces to the standard Maslov index of disc maps for the case of Lagrangian submanifolds. The second purpose is to study the geometry of coisotropic submanifolds in Kähler manifolds. We introduce the notion of the leafwise mean curvature form and transverse canonical bundle of coisotropic submanifolds and study various geometric properties thereof. Finally we combine all these to define the notion of special coisotropic submanifolds for the case of Calabi-Yau manifolds, and prove various consequences on their properties of the coisotropic Maslov indices.

§1. Introduction

A triple \((X, \omega, J)\) is called an almost Kähler manifold where \((X, \omega)\) is symplectic and \(J\) is compatible to \(\omega\) in that linear form defined by

\[
g(X, Y) := \omega(X, JY)
\]

is positive definite. The mean curvature one form \(\alpha_L\) of the Lagrangian submanifold \(L \subset (X, g)\) is then defined by

\[
\alpha_L := (H|_\omega)|_{TL}
\]

(1.1)

Morvan [Mo] and Dazord [Da] proved that the mean curvature one form, of the Lagrangian submanifold \(L \subset (X, \omega, J)\) becomes closed when \((X, \omega, J)\) is Einstein-Kähler, i.e., \(J\) is integrable and its Ricci form \(\rho\) satisfies

\[
\rho = \lambda \omega, \quad \text{for} \quad \lambda \in \mathbb{R}.
\]

(1.2)

More precisely, Dazord [Da] proved the identity

\[
d\alpha_L = i^* \rho
\]

(1.3)

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on general Kähler manifolds from which together with (1.2) follows closedness of \( \alpha_L \) in the Einstein-Kähler case. We will call the mean curvature class the real one dimensional cohomology class \( [\alpha_L] \in H^1(L; \mathbb{R}) \) of the Lagrangian submanifold \( L \subset (X, \omega, J) \). In fact, Morvan (in the case of \( \mathbb{C}^n \)) and Dazord (in the Calabi-Yau, i.e., in the Ricci-flat case) proved that the mean curvature one form \( \frac{1}{2} \alpha_L \) is the one form whose de Rham cohomology class is an integral class which represents the well-known Maslov class \([Ar]\) of Lagrangian submanifolds in symplectic geometry. The latter measures the rotation index of the angle of the tangent plane of the Lagrangian submanifolds which illustrates an interesting interplay between symplectic and Riemannian geometry of Lagrangian submanifolds in Einstein-Kähler manifolds. In this respect, the author [Oh2] also proved that the mean curvature class \( \alpha_L \) is invariant under the Hamiltonian isotopy of \( (X, \omega) \) (see [Oh2] for its precise meaning).

One purpose of the present paper is to generalize all of the above facts on Lagrangian submanifolds on Kähler manifolds to coisotropic submanifolds \( Y \subset (X, \omega) \):

A submanifold \( Y \) is called coisotropic if

\[
(TY)^\omega \subset TY
\]  

where \( (TY)^\omega \) is the symplectic orthogonal complement to \( TY \) defined by

\[
(T_Y Y)^\omega = \{ v \in T_y X \mid \omega(v, \cdot) \equiv 0 \}.
\]

Coisotropic submanifolds have played some important role in symplectic geometry in relation to generalizing calculus of Lagrangian submanifolds to some corresponding calculus in Poisson manifolds in relation to the geometric quantization (see e.g., [We]). Recently they attracted some physicist’s attention [KaOr] in an attempt to correct and complete Kontsevich’s homological mirror symmetry proposal in which inclusion of coisotropic \( D \)-branes plays an important role. For the case of Kähler manifolds, we introduce the notion of leafwise mean curvature vector of a coisotropic submanifold \( Y \subset (X, \omega, J) \) and attempt to prove all the analogs in this coisotropic case to the above mentioned relationships between the Maslov index and the mean curvature class that are valid for Lagrangian submanifolds. It turns out that in a suitable foliated or leafwise context, all the coisotropic analogs to the above mentioned geometric properties of Lagrangian submanifolds can be proved.

We also introduce the notion of Maslov loops of coisotropic Grassmanians

\[
\Gamma_k(\mathbb{R}^{2n}, \omega_0) = \{ C \in Gr_{n+k}(\mathbb{R}^{2n}) \mid C^\omega_0 \subset C, \dim \ker \omega_0|_C = n-k \}
\]

and their indices. Note that when \( k = 0 \), \( C \) become Lagrangian and \( k = n \), it becomes the full space \( \mathbb{R}^{2n} \). This class is not an homotopy invariant unlike the Lagrangian case but enjoys certain symplectic invariance property (see section 2 and 3). Then we introduce the notions of gradable coisotropic submanifolds and graded coisotropic submanifolds in general symplectic manifolds, for which we also define an index for any (disc) map \( w : (D, \partial D) \to (X, Y) \).

In section 7, we will also introduce the notion of special coisotropic submanifolds in the Calabi-Yau case which generalize the notion of special Lagrangian submanifolds. And we will derive various consequence on the coisotropic Maslov index of disc maps with boundary lying on special coisotropic submanifolds.
Finally in section 8, we analyze the symplectic II-transverse curvature $F_{\Pi}$ introduced in [OP1] with respect to the orthogonal splitting $\Pi : TY = TF \oplus N_JF$ in the case of Kähler manifolds $(X, \omega, J)$ and relate the curvature with the classical Levi form for the case of hypersurfaces $Y \subset (X, \omega, J)$.

The present work is a by-product of the joint works [OP1,2] with Jae-Suk Park. We would like to thank Jae-Suk Park for exciting collaboration on the coisotropic $D$-branes. We also thank Wei-Dong Ruan for his interest on this work and for pointing out a couple of imprecise points in our calculations in the precious version of this paper.

§2. Maslov loops of coisotropic Grassmanians and their indices

In this sub-section, we introduce the notion of Maslov loops of coisotropic subspaces and their associated indices.

Denote the set of coisotropic subspace of rank $2k$ in the symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$ by

$$\Gamma_k(\mathbb{R}^{2n}, \omega_0) = \{ C \in Gr_{n+k}(\mathbb{R}^{2n}) \mid C^\omega \subset C, \dim C^\omega = n-k \}.$$

From the definition, for any coisotropic subspace we have the canonical flag,

$$0 \subset C^\omega \subset C \subset \mathbb{R}^{2n}.$$ 

Combining this with the standard complex structure $j$ on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, we have the splitting

$$C = H_C \oplus C^\omega$$ 

where $H_C$ is the complex subspace of $C$. The following proposition is not difficult check, whose proof we omit and leave to [OP1].

**Proposition 2.1.** Let $0 \leq k \leq n$ be fixed. The unitary group $U(n)$ acts transitively on $\Gamma_k$. The corresponding homogeneous space is given by

$$\Gamma_k(\mathbb{R}^{2n}, \omega_0) \cong U(n)/U(k) \times O(n-k)$$

where $U(k) \times O(n-k) \subset U(n)$ is the isotropy group of the coisotropic subspace $\mathbb{C}^k \oplus \mathbb{R}^{n-k} \subset \mathbb{C}^n$. In particular we have

$$\dim \Gamma_k(\mathbb{R}^{2n}, \omega_0) = \frac{(n+3k+1)(n-k)}{2}.$$ 

Now we have the following commutative diagram of short exact sequences

\[
\begin{array}{ccc}
0 & \rightarrow & H \\
\downarrow & & \downarrow \\
0 & \rightarrow & H \\
\end{array}
\]

Suppose $\gamma : S^1 \rightarrow \Gamma_k(\mathbb{R}^{2n}, \omega_0)$ be a continuous loop and consider the flag

$$0 \subset \gamma(\theta)^\omega \subset \gamma(\theta) \subset \mathbb{R}^{2n}.$$
Considering
\[ S_\gamma := C_\gamma^\omega \oplus jC_\gamma^\omega, \quad C_\gamma^\omega(t) := \gamma(\theta(t))^{-} \]
we associate to the each coisotropic loop \( \gamma \) the pair \((S_\gamma, L_\gamma)\) of loops of symplectic vector space \( S_\gamma(\theta) \) of dimension \( 2(n-k) \) and its Lagrangian subspace \( L_\gamma(\theta) = C_\gamma^\omega \)
for \( \theta \in S^1 \). We denote by \((C^\omega)^- \subset C^*\) the set of annihilators of \( C^\omega \) in the dual space \( C^*\) of \( C\).

We will need the following general discussion on the coisotropic subbundle of symplectic vector bundles. Let \((E, \sigma, J) \to N\) be a symplectic vector bundle with a compatible complex structure \( J \) on the fiber. Let \( C \subset E \) be a coisotropic subbundle and \( C^\sigma \) the kernel bundle of \( C \). Denote by \( C_\sigma = C/C^\sigma \) the quotient symplectic vector bundle and \( J_\sigma \) the induced complex structure. We then identify \( C_\sigma \) with the orthogonal complement of \( C_\sigma \) in \( C \).

We have the splitting
\[ C_\sigma \otimes J_\sigma C = C_{1,0}^{\sigma, J} \oplus C_{0,1}^{\sigma, J}. \]
with respect the complex structure \( J \). We recall that as a complex vector bundle \((C_\sigma, J_\sigma)\) and \( C_{0,1}^{\sigma, J} = C_{1,0}^{\sigma, J}\) are canonically isomorphic. We suppress the \( J_\sigma \) from notations and write the splitting
\[ C_\sigma \otimes C = \Lambda^{1,0}(C_\sigma^*) + \Lambda^{0,1}(C_\sigma^*) \]
associated to \( C_\sigma^*\). Note that \( C_\sigma^*\) is naturally isomorphic to \((C^\sigma)^- \subset C^*,\) the set of annihilators of \( C^\sigma\).

**Definition 2.2.**

1. We denote the determinant line bundle of \( \Lambda^{1,0}(C_\sigma^*)\) by
\[ K_C = K_{(C,E,\Omega,J)} := \Lambda^{k,0}(C_\sigma^*) = \Lambda^k(\Lambda^{1,0}(C_\sigma^*)) \]
and call it the transverse canonical bundle of \( C \). We also denote by \( \Omega^{k,0}(C_\sigma^*)\) the set of smooth sections thereof. We call any section \( \zeta \in \Omega^{k,0}(C_\sigma^*)\) of unit norm a transverse complex volume form of \( C \).

2. We call the square bundle \( K_C^\otimes 2 \) the transverse Maslov line bundle of \( C \subset (E, \Omega, J)\).

We note that since the real line bundle \((\det_R(C^\sigma))^\otimes 2\) is trivial and \( \det_R E \) is always trivial. In general, the transverse Maslov bundle \( K_C^\otimes 2 \) is not trivial as a complex vector bundle.

**Definition 2.3.** Let \( C \subset (E, \Omega) \) be a coisotropic subbundle of nullity \( n-k \) or of rank \( 2k \).

1. We call the coisotropic subbundle \( C \subset (E, \sigma) \) gradable if \( K_C^\otimes 2 \) is trivial, for a (and so any) compatible almost complex structure.

2. For a given compatible almost complex structure \( J \) on \( (E, \Omega) \), we call a section \( \zeta \in \Gamma(K_C^\otimes 2)\) a transverse Maslov section of \( C \). We call the pair \((C, \sigma)\) a graded coisotropic subbundle of \( E \).
Examples 2.4.

1. Any Lagrangian subbundle is gradable in this sense since in that case the transverse Maslov bundle is just a scalar as the transverse space is trivial.
2. Any coisotropic subbundle over the base \( N = S^1 \) is gradable since any complex line bundle over \( S^1 \) is trivial.
3. Consider \((X, \omega, J)\) is Calabi-Yau, or more generally any symplectic manifold \((X, \omega)\) with \(2c_1(X, \omega) = 0\). Then for any coisotropic submanifold \(Y \subset (X, \omega)\), the coisotropic subbundle \(T_Y \subset (TX|_Y, \omega)\) is gradable (see section 7).

Now we go back to the case \(E = S^1 \times (\mathbb{R}^{2n}, \omega_0)\) the trivial symplectic vector bundle over \(S^1\) with the standard complex structure \(j\), with a map \(\gamma: S^1 \to \Gamma_k(\mathbb{R}^{2n}, \omega_0)\) which corresponds to a coisotropic subbundle of \(E\). This is a special case where \(N = S^1\). In this case, the real bundle \((C^\sigma)^{\otimes 2} \to S^1\) is always trivial.

Since any symplectic vector bundle is trivial over \(S^1\), both \(C^\sigma\) and \((\det_C C^\sigma)^{\otimes 2}\) are also trivial (\(C^\sigma\) is Calabi-Yau).

Definition 2.5.

1. We denote
\[
K_\gamma = \Lambda^{k,0}(S_\gamma)
\]
and call the transverse canonical bundle of \(\gamma\) (with respect to the complex structure induced from the standard complex structure of \(\mathbb{C}^n\)). We call any section \(\zeta\) of \(K_\gamma^{\otimes 2}\) a transverse Maslov section of \(\gamma\).
2. We call the pair \((\gamma; \zeta)\) a Maslov loop.

Now we will associate an integer to each given Maslov loop \((\gamma; \zeta)\). We fix a global section \((\Omega^k_R)^{\otimes 2}(C^\sigma_\gamma)\) which we write as
\[
(e_{k+1} \wedge \cdots \wedge e_n)^{\otimes 2}
\]
for any local orthonormal frame
\[
\{e_{k+1}, \cdots, e_n\}
\]
of \(C^\sigma_\gamma\). It is obvious that (2.2) is independent of the choice of orthonormal frame. However \(e_{k+1} \wedge \cdots \wedge e_n\) is not globally defined in general. In case \(C^\sigma_\gamma\) is oriented, one can make this globally defined considering only the oriented frames. We denote by \(\{e_1, \cdots, e_{k+1}, \cdots, e_n, f_1, \cdots, f_{k+1}, \cdots, f_n\}\) any Darboux frame of \(\mathbb{C}^n\), that extends the frame \(\{e_{k+1}, \cdots, e_n\}\) and respects the orthogonal splitting
\[
\gamma(\theta) = S_\gamma(\theta) \oplus C^\sigma_\gamma(\theta).
\]

More precisely, \(\{e_1, \cdots, e_{k+1}, f_1, \cdots, f_n\}\) defines a Darboux orthonormal frame of \(S_\gamma\) and \(\{e_{k+1}, \cdots, e_n, f_{k+1}, \cdots, f_n\}\) one for \(C^\sigma_\gamma \oplus jC^\sigma_\gamma\). Denote \(L_\gamma = C^\sigma_\gamma \subset S_\gamma\) which is a Lagrangian subbundle of \(S_\gamma\).

Noting that \(S_\gamma \cong L_\gamma \otimes \mathbb{C} \cong (S_\gamma)^{1,0}\), we choose the unitary frame \(\{u_1, \cdots, u_k\}\) of \((S_\gamma)^{1,0}\) given as
\[
u_j = e_j + if_j
\]
associated to the Darboux frame \( \{ e_1, \cdots, e_k, f_1, \cdots, f_k \} \). We denote by \( \{ \theta^1, \cdots, \theta^n \} \) the unitary frame dual to \( \{ u_1, \cdots, u_n \} \).

We know that the square of
\[
 u_{k+1} \wedge \cdots \wedge u_n
\]
defines a global section of \( K_\gamma^\otimes 2 \). Pairing this with the standard complex volume form
\[
 dz = dz_1 \wedge \cdots \wedge dz_n,
\]
it gives rise to a natural transverse Maslov section of \( \gamma \)
\[
 \zeta_{(\gamma; dz)} := \left( (u_{k+1} \wedge \cdots \wedge u_n) dz \right)^\otimes 2 . \tag{2.4}
\]

The following is easy to check whose proof we omit.

**Lemma 2.6.** (2.4) does not depend on the choice of the orthonormal frame \( \{ e_{k+1}, \cdots, e_n \} \)
of \( C_\gamma^\omega \) or its extended orthonormal Darboux frame \( \{ e_1, \cdots, e_n, f_1, \cdots, f_n \} \), but depends only on \( \gamma \).

Now let \( \zeta \) be any given transverse Maslov section of \( \gamma \). Then we can write
\[
 \zeta(\theta) = g(\theta)\zeta_{(\gamma; dz)}(\theta) \tag{2.5}
\]
for a well-defined function \( g = g_{(\gamma; \zeta)} : S^1 \to S^1 \).

**Definition 2.7.** We define the index \( \mu(\gamma; \zeta) \) of the Maslov loop \( (\gamma; \zeta) \) by
\[
 \mu(\gamma; \zeta) := \deg(g_{(\gamma; \zeta)}). \tag{2.6}
\]

**Remark 2.8.**

1. Note that in the Lagrangian case there is no transverse direction and so the bundle \( K_\gamma \) just becomes a scalar. Furthermore \( \zeta_{(\gamma; dz)} \) becomes the standard angle function for the Lagrangian loops in \( \mathbb{C}^n \), and hence Definition 2.6 reduces to the standard Maslov index of the loop of Lagrangian Grassmanians.

2. In fact, the above discussion does not depend on the particular choice of the standard complex structure \( j \) on \( \mathbb{R}^{2n} \) but can be repeated verbatim for the complex structure on \( E_\gamma = \mathbb{C}^n \) induced from any almost complex structure \( J \) compatible to the standard symplectic structure \( \omega_0 \). Using the fact that the set of compatible almost complex structure is contractible and the degree is an homotopy invariant, the coisotropic Maslov index is an invariant depending only on the homotopy class of the complex line bundle \( K_{(J, \gamma)} \). We note that this homotopy class is an invariant depending only on the coisotropic loop \( \gamma \). In this sense, the index \( \mu(\gamma; \zeta) \) defined above is a symplectic invariant.

3. A complete parallel discussion can be carried out in the case of Calabi-Yau manifolds as we will discuss in section 7 later.
From the definition, it is clear that to discuss invariance property of the index $\mu(\gamma; \zeta)$, one needs more than just a homotopy of the loop $\gamma$ but also to dictate a proper condition for the section $\zeta$ as well. This deviates from the homotopy invariance of the classical Maslov index of Lagrangian loops [Ar].

However it has the property of symplectic invariance which we now describe. Let $\gamma \in \Gamma_k(\mathbb{R}^{2n}, \omega_0)$ and $\zeta \in \Gamma(K_\gamma)$ be a Maslov section. For any loop $A : S^1 \to Sp(2n)$, it naturally induces the push-forward complex structure $A_*j$ on $\mathbb{R}^{2n}$ and a push-forward coisotropic loop $A \cdot \gamma : S^1 \to \Gamma(\mathbb{R}^{2n}, \omega_0)$. It induces a pair $(A \cdot \gamma; A_*\zeta)$ of $A_*\zeta \in K(A_*j, A \cdot \gamma)$.

**Theorem 2.9.** Let $\gamma \in \Gamma_k(\mathbb{R}^{2n}, \omega_0)$ and $\zeta \in \Gamma(K_{(j, \gamma)}).$ Let $A : S^1 \to Sp(2n)$ be any loop of symplectic matrices. Then we have

$$\mu(A \cdot \gamma; A_*\zeta) = \mu(\gamma; \zeta).$$

(2.7)

§3. Graded coisotropic submanifolds, and the coisotropic Maslov index

In this section, we will introduce the notion of grading on coisotropic submanifolds and define an index of maps carrying a transverse Maslov section with it.

First we denote by $F$ the null foliation of the coisotropic submanifold $Y \subset (X, \omega)$ and consider the leafwise normal bundle $N_F$ and its dual $N_F^*$ respectively.

We note that $J$ preserves both $N_F \subset T_X|_Y$ and $N_F^* \cong (T_F \oplus NY)^\perp \subset T^*X|_Y$ and so induces decomposition of the complexifications

$$N_F \otimes \mathbb{C} = N^{1,0}F \oplus N^{0,1}F$$
$$N^*F \otimes \mathbb{C} = (N^*)^{1,0}F \oplus (N^*)^{0,1}F.$$

By definition, the transverse canonical bundle is defined by

$$K_Y \to Y = \det(\Lambda^{1,0}(N^*F)).$$

In general $K_Y^{\otimes 2} \to Y$ is not trivial. However it is trivial if $Y$ is either Lagrangian or if the total space $X$ satisfies $2c_1(X, \omega_0) = 0$.

This leads us to the following notion.

**Definition 3.1.**

1. We call a coisotropic submanifold $Y \subset (X, \omega)$ gradable if its transverse Maslov bundle $K_Y^{\otimes 2}$ is trivial.

2. We call a pair $(Y, [\Delta])$ a graded coisotropic submanifold where $\Delta \in \Gamma(K_Y^{\otimes 2})$ and $[\Delta]$ its homotopy class. Then we call the $\Delta$ transverse Maslov charge and its homotopy class $[\Delta]$ a grading of $Y$.

Here we note that our definition of graded coisotropic submanifolds is manufactured so that an index for a map with boundary lying on the given coisotropic submanifold can be defined which is always the case for Lagrangian submanifolds. We would like to emphasize that it is not a generalization of the graded Lagrangian submanifold used in [Ko] or [Se]. For example, according to our definition, any Lagrangian submanifold is gradable and canonically graded.
Obviously for given compatible almost complex structure $J$ on $(X, \omega)$, the space of transverse Maslov charges of $Y$ is a principal homogeneous space of $C^\infty(Y, S^1)$. In other words, we have

$$\Delta - \Delta' \in C^\infty(Y, S^1).$$

(3.1)

We are now ready to define the coisotropic Maslov index of a map $w : (\Sigma, \partial \Sigma) \to (X, Y)$ in the presence of transverse Maslov charge $\Delta$ of $Y$ which we denote by $\mu_{(Y, \Delta)}(w)$. We will suppress $\Delta$ from the notation, whenever there is no danger of confusion.

Consider the pull-back $w^*TX$ and its symplectic trivialization

$$\Phi : w^*TX \to \Sigma \times \mathbb{C}^n.$$ Such a trivialization, even the unitary one, always exists as long as $\partial \Sigma \neq \emptyset$. Furthermore it is unique up to homotopy when $\Sigma$ is a disc. We will restrict to the case of discs in this paper and postpone more general discussions of the higher genus cases elsewhere. From now on, we assume $\Sigma$ is a disc $D = D^2$. We will denote by

$$\{u_1, \cdots, u_n\}$$

the corresponding unitary frame such that

$$\text{span}_{\mathbb{C}}\{u_1, \cdots, u_k\} = (TY^\omega \oplus JTY^\omega)_{-J}.$$  

(3.2)

In the presence of transverse Maslov charge $\Delta$ on $Y$, we require that the push-forward grading

$$[\Delta_\Phi] := [(\Phi_\partial D)_*(\Delta)]$$

on the loop $\alpha_\Phi \in \Gamma_k(\mathbb{R}^{2n}, \omega_0)$ coincides with the canonical grading defined in section 2. In terms of the above unitary frame, this means that

$$[(\gamma)^*\Delta] = [(\gamma)^*(\theta^1 \wedge \cdots \wedge \theta^k)]$$  

(3.3)

where $\gamma = \partial w : \partial D \to Y$. More precisely, we have

**Lemma 3.2.**

1. We can always choose a unitary frame of $\gamma^*(TX) \{u_1, \cdots, u_n\}$ over $\partial D$ that satisfies (3.2) and (3.3).

2. We can then extend the above frame over $\partial D$ to that of $w^*TX$ over $D$.

**Proof.** Recall that any complex vector bundle over $\partial D \cong S^1$ is trivial. This proves (1). The statement (2) follows from the fact that $\pi_2(U(n), U(k) \otimes \{id\}) = 0$ when $n \geq 2$ and $0 \leq k \leq n$.  

We will call such a trivialization $\Phi$ an admissible trivialization of $w^*TX$ with respect to $(Y, \Delta)$. We will sometimes denote by $\Phi_\Delta$ if necessary to explicitly state this choice.

Now we are ready to define the coisotropic Maslov index for the map $w : (D, \partial D) \to (X, Y)$ with respect to the transverse Maslov charge $\Delta$ on $Y$. 

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**Definition 3.3.** Let \( w \) be as above, and let \( \Phi : w^*TX \to \Sigma \times \mathbb{C}^n \) be an admissible trivialization. Then we define the index

\[
\mu_{(Y, \Delta)}(w) = [\Delta] - [\Phi^*(\zeta_{(\alpha; dz)})] \in \mathbb{Z}
\]

and call it the coisotropic Maslov index of the map \( w \) with respect to \( \Delta \).

It is easy to check the above definition does not depend on the choice of admissible trivializations \( \Phi_\Delta \).

In general, the coisotropic Maslov index will not be invariant under the homotopy of maps \( w_t : (D, \partial D) \to (X, Y) \) for \( t \in [0, 1] \). However this is so when the transverse Maslov charge \( \Delta \) is also parallel. We will prove the following theorem in the next section.

**Theorem 3.4.** Suppose that the transverse Maslov charge \( \Delta \) on \( Y \) is parallel with respect to the connection induced from the canonical connection from \((X, \omega, J)\). Let \( \{(w_t)\}_{0 \leq t \leq 1} \) be a smooth family of maps with boundary lying on \( Y \). Then we have

\[
\mu_{(Y, \Delta)}(w_0) = \mu_{(Y, \Delta)}(w_1).
\]

An obvious necessary condition for the existence of such a parallel section is vanishing of the curvature of \( K_Y \). This theorem leads us to the following definition.

**Definition 3.5.** We call a coisotropic submanifold \( Y \) a transversally Ricci-flat if \( K_Y \) is flat.

We will study various geometric properties of transversally Ricci-flat coisotropic submanifolds in Kähler manifolds later in section 5-7.

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**§4. Coisotropic submanifolds in almost Kähler manifolds**

In this section we recall the basic facts on the connection on the almost Kähler manifolds \((X, \omega, J)\) following [Kb].

We now consider an almost complex structure \( J \) that is compatible to \( \omega \), i.e., the triple \((X, \omega, J)\) defines an almost Kähler manifold with the standard convention

\[
g = \omega(\cdot, J \cdot).
\]

We will choose a connection \( \nabla \) on \( TX \) which preserves both \( J \) and \( \omega \). Such a connection always exist and in addition becomes unique if we impose the condition that

\[
\nabla \omega = 0 = \nabla J
\]

and the corresponding torsion form \( \Theta \) is of type \((0,2)\).

**Definition 4.1.** A connection \( \nabla \) of the almost Kähler manifold \((X, \omega, J)\) is called the canonical connection (see [Kb]), if its torsion form is of type \((0,2)\).
**Theorem 4.2 [Theorem 5.1, Kb]**. Every almost Kähler manifold \((X, \omega, J)\) carries the unique canonical connection. This connection also satisfies

\[
\sum \Theta^i \wedge \bar{\theta}^i = 0. \tag{4.1}
\]

When we say that \((X, \omega, J)\) is an almost Kähler manifold, we will always assume that it carries the canonical connection. Now we need to do some basic calculations involving the moving frame adapted to the given coisotropic submanifold \(Y \subset X\).

We choose an orthonormal frame of \(X\)

\[
\{e_1, e_2, \cdots, e_n, f_1, \cdots, f_n\} \tag{4.2}
\]

such that

\[
\text{span}_\mathbb{R} \{e_1, \cdots, e_k, e_{k+1} \cdots, e_n, f_1, \cdots, f_k\} = TY \subset TX
\]

and

\[
\text{span}_\mathbb{R} \{e_{k+1}, \cdots, e_n\} = (TY)^\omega := E \subset TY.
\]

The vectors

\[
u_j = \frac{1}{2}(e_j - if_j), \quad j = 1, \cdots, n
\]

form a unitary frame of \(T^{(1,0)}X\) and the vectors

\[
u_j = \frac{1}{2}(e_j + if_j), \quad j = 1, \cdots, n
\]

form a unitary frame of \(T^{(0,1)}X\). Let

\[
\{e_1^*, \cdots, e_n^*, f_1^*, \cdots, f_n^*\}
\]

be the dual frame of (4.2). The complex valued one forms

\[
\theta^j = \alpha^j + i\beta^j, \quad j = 1, \cdots, n
\]

form a unitary frame of \(TX\) which are dual to the unitary frame \(\{u_1, \cdots, u_n\}\).

In terms of the above mentioned canonical connection \(\nabla\), we have

\[
\nabla u_j = \sum_i \omega_j^i \otimes u_i + \sum \tau_j^\ell \otimes \bar{\nu}_\ell \tag{4.3}
\]

where \(\omega_j^i\) is the connection one form

\[
\omega_j^i = \sum \omega_{jk}^{ij} \theta^k + \omega_{jk}^{ij} \bar{\theta}^k.
\]

The first structure equation with respect to the frame \(\{\theta^1, \cdots, \theta^n\}\) becomes

\[
d\theta^j = -\sum_i \omega_j^i \wedge \theta^i + \Theta^j \tag{4.4}
\]
where the torsion form $\Theta^j$ is of the type $(0,2)$

$$\Theta^j = \sum_i \tau^j_i \wedge \bar{\theta}^i = \sum_{i,\ell} N^j_{\ell i} \bar{\theta}^\ell \wedge \theta^i.$$  \hfill (4.5)

Furthermore if we set $N^j_{\ell i} = N_{ij\ell}$, then (4.1) implies

$$N_{j\ell i} = N_{i\ell j} = N_{ij\ell}$$

(see [Kb]), or equivalently

$$\tau^j_i = \tau^j_i.$$  \hfill (4.6)

The 2-nd structure equation of $\{\theta_1, \ldots, \theta_n\}$ is

$$d\omega = -\omega \wedge \omega + K$$

(4.7)

where $K$ is the curvature 2-form.

The following proposition will play an important role in the proof Theorem 3.4 for the maps $w : (D^2, \partial D^2) \to (X,Y)$ and also in our calculation of covariant derivative of the transverse Maslov section later.

**Proposition 4.3.** Suppose that $J$ and $\nabla$ are as above. Let $\gamma : [0,1] \to Y \subset X$ be a smooth curve on $Y$ and let $\Pi_\nabla$ be the parallel translation from $T_{\gamma(0)}X$ to $T_{\gamma(1)}X$ in $X$. Then $\Pi_\gamma$ maps $(E \oplus JE)|_{\gamma(0)} \subset T_{\gamma(0)}X$ to $(E \oplus JE)|_{\gamma(1)} \subset T_{\gamma(1)}X$. In particular, the subbundle $H_E := E^{1,J} \subset TY$ is also invariant under the parallel translation along such curves.

**Proof.** Note that in terms of the metric $g$, it is enough to prove that for any vector field $\eta$ on $Y$ such that $\langle \eta(\gamma(0)), \xi_0 \rangle_g = 0$, the parallel translation $\Pi_\gamma(\xi_0)$ also satisfies

$$\langle \eta(\gamma(1)), \Pi_\gamma(\xi_0) \rangle = 0$$

(4.8)

when $\xi_0 \in E \oplus JE$. We choose an orthonormal frame of $X$

$$\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$$

(4.9)

adapted to $Y$. The vectors

$$u_j = \frac{1}{2}(e_j - if_j), \quad j = 1, \cdots, n$$

form a unitary frame of $T^{(1,0)}X$ and the vectors

$$\bar{u}_j = \frac{1}{2}(e_j + if_j), \quad j = 1, \cdots, n$$

form a unitary frame of $T^{(0,1)}X$. Let

$$\{\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n\}$$

be the dual frame of (4.9) The complex valued one forms

$$\theta^j = \alpha^j + i\beta^j, \quad j = 1, \cdots, n$$
form a unitary frame of $TX$ which is dual to the unitary frame $\{u_1, \ldots, u_n\}$.
Substituting (4.5) and

$$\omega^j_i = \sum_{\ell} \omega^j_{i\ell} \theta^\ell + \sum_{\ell} \omega^j_{i\ell} \bar{\theta}^\ell$$

into (4.4), we can write

$$d\theta^j = -\sum_{i, \ell} \omega^j_{i\ell} \theta^\ell \wedge \theta^i - \sum_{i, \ell} \omega^j_{i\ell} \bar{\theta}^\ell \wedge \theta^i + \sum_{i, \ell} N^j_{i\ell} \theta^\ell \wedge \bar{\theta}^i.$$

It is easy to see that for $d\theta^j$ with $j = 1, \ldots, k$ to be in the ideal generated by $\{\theta^1, \ldots, \theta^k, \bar{\theta}^1, \ldots, \theta^k\}$ on $Y$, we must have

$$\omega^j_{i\ell} = \omega^j_{i\ell} \bar{\theta}^\ell = N^j_{i\ell} = 0,$$

for $i, \ell \geq k + 1$. (4.10)

Let $\xi : [0, 1] \to T^{(1,0)}X$ be the unique solution for

$$\begin{cases} \nabla_{\gamma'(t)} \xi = 0 \\ \xi(0) = \xi_0 \in E \oplus JE|_{\gamma(0)} \end{cases}$$

for given $\xi_0$. By definition of the parallel translation, we have $\Pi_\gamma(\xi_0) = \xi(1)$. We compute

$$\frac{d}{dt} \langle u_j(\gamma(t)), \xi(t) \rangle = \langle \nabla_t u_j, \xi \rangle + \langle u_j, \nabla_t \xi \rangle = \langle \nabla_t u_j, \xi \rangle.$$

On the other hand we derive from (4.3) and (4.10),

$$\nabla_t u_j = \sum_i k \omega^j_i(\gamma(t)) u_i + \sum_{\ell} k \tau^j_{\ell}(\gamma(t)) \bar{\pi}_\ell$$

and hence

$$\frac{d}{dt} \langle u_j(\gamma(t)), \xi(t) \rangle = \sum_i k \omega^j_i(\gamma(t)) \langle u_i, \xi \rangle + \sum_{\ell} k \tau^j_{\ell}(\gamma(t)) \langle \bar{\pi}_\ell, \xi \rangle$$

for $1 \leq i, j \leq k$. Similarly we also derive the equation for $\langle \bar{\pi}_i, \xi \rangle$ with $1 \leq i, j \leq k$. Together we have a system of linear first order ODE for $\langle u_i, \xi \rangle$ and $\langle \bar{\pi}_i, \xi \rangle$ for $1 \leq i \leq k$ with the initial condition

$$\langle u_i(\gamma(0)), \xi(0) \rangle = \langle \bar{\pi}_i(\gamma(0)), \xi(0) \rangle = 0.$$

This proves that $\langle u_i(\gamma(t)), \xi(t) \rangle = \langle \bar{\pi}_i(\gamma(t)), \xi(t) \rangle \equiv 0$ for all $t \in [0, 1]$ and in particular at $t = 1$. This finishes the proof. □

**Proof of Theorem 3.4.** Because of the symplectic invariance of the index $\mu(\gamma; \zeta)$, we may choose any trivialization of $w_t^*TX$ for the definition of $\mu(Y, \Delta)(w_t)$. We denote the parameterized map

$$W : [0, 1] \times D \to X; \quad W(t, z) = w_t(z)$$
and fix a trivialization of \( W^*(TX) \)

\[
\Phi : W^*(TX) \to [0,1] \times D \times \mathbb{C}^n
\]

Under this trivialization, \((\partial w_t)^*TY\) gives rise to a loop of coisotropic subspaces \( \alpha_{w_t} : S^1 \to \Gamma_k(T_{w(0)}X) \cong \Gamma_k(\mathbb{C}^n) \) for each \( t \in [0,1] \). Now we consider the parallel translations

\[
\Pi^0_t : (\partial w_t)^*TX \to (\partial w_0)^*TX
\]

along the paths

\[
t \mapsto w_t(\theta)
\]

for each \( \theta \in \partial D \). The pairs \((S_{\alpha_{w_t}}, L_{\alpha_{w_t}})\) corresponding to the loop \( t \mapsto \Pi^0_t \cdot \alpha_{w_t} \in \Gamma_k(\mathbb{C}^n) \) are mapped to a \( t \)-parameter family of the pairs

\[
(\Pi^0_t(S_{\alpha_{w_t}}), \Pi^0_t(L_{\alpha_{w_t}}))
\]

in \( \mathbb{C}^n \) and Proposition 4.3 implies that

\[
\Pi^0_t(S_{\alpha_{w_t}}) \equiv S_{\alpha_{w_0}} \subset \mathbb{C}^n.
\] (4.11)

for all \( t \in [0,1] \). Furthermore since we assume that \( \Delta \) is parallel, \( \Delta \) is invariant under the parallel translation \( \Pi^0_t \), i.e., we have

\[
\Pi^0_t(\Delta_t) = \Delta_0
\] (4.12)

where \( \Delta_t = (\partial w_t)^* \Delta \). Then \( \mu\{Y,\Delta\}(w_0) = \mu\{Y,\Delta\}(w_1) \) follows from (4.11), (4.12), Proposition 4.3 and homotopy invariance of the usual Maslov index of the Lagrangian Grassmanians. \( \square \)

§5. Leafwise mean curvature vector

From now on, we use indices \( i, j, \ldots \) from 1 to \( n \), \( a, b, \cdots \) from 1 to \( k \) and \( \alpha, \beta, \cdots \) from \( k+1, \cdots, n \) and use the summation convention for the repeated indices. We restrict the 1-st structure equation (4.4) to \( Y \). Since the distribution \( E \) on \( Y \) is integrable and \( f^*_a = 0 \) on \( Y \), both \( d\theta^a \) and \( d\bar{\theta}^a \) should be contained in the ideal generated by \( \{\theta^1, \cdots, \theta^k, \bar{\theta}^1, \cdots, \bar{\theta}^k\} \). The 1-st structure equation provides

\[
d\theta^a = -\omega_b^a \wedge \theta^b + \Theta^a = -\omega_b^a \wedge \theta^b - \omega_c^a \wedge \theta^c + \Theta^a
\]

on \( Y \). We will come back to the non-integrable case in the appendix and restrict our attention to the integrable case for the rest of the paper.

From now on, we will assume \((X, \omega, J)\) is Kähler, i.e., \( \Theta \equiv 0 \). In this case, the above first structure equation becomes

\[
d\theta^a = -\omega_i^a \wedge \theta^i = -\omega_b^a \wedge \theta^b - \omega^a_\alpha \wedge \theta^\alpha
\] (5.1)

Since this is contained in the ideal generated by \( \{\theta^1, \cdots, \theta^k, \bar{\theta}^1, \cdots, \bar{\theta}^k\} \), and

\[
\omega^a_\alpha \wedge \theta^\alpha = \omega^a_{\alpha j} \theta^j \wedge \theta^\alpha + \omega^a_{\alpha \bar{j}} \bar{\theta}^j \wedge \theta^\alpha
\]
we derive
\[ \omega^{\alpha}_{a \beta} \theta^\beta \wedge \theta^\alpha + \omega^*_{a \beta} \bar{\theta}^\beta \wedge \bar{\theta}^\alpha = 0 \]
and so
\[ \omega^\alpha_{a \beta} = \omega^\alpha_{b \alpha}, \quad \omega^*_{a \beta} = 0. \] (5.2)
Now we need to incorporate some real aspect of the submanifold \( Y \subset (X, \omega) \). For this, we write the connection one form \( \omega^i_j \) as
\[ \omega^i_j = \alpha^{i}_{j} + \beta^{i}_{j} \]
where \( \alpha^{i}_{j} \) and \( \beta^{i}_{j} \) are real one forms. The unitarity of connection, i.e., the skew-Hermitian property of \( (\omega^i_j) \) implies
\[ \alpha^{i}_{j} = -\alpha^{j}_{i}, \quad \beta^{i}_{j} = \beta^{j}_{i}. \] (5.3)
The first structure equation \( d\theta^\ell = -\omega^\ell_j \wedge \theta^j \) becomes
\[ \begin{cases} d\theta^\ell = -\omega^\ell_j \wedge \theta^j \\ df^*_k = -\beta^k_j \wedge \theta^j - \alpha^k_j \wedge \theta^j \end{cases} \] (5.4)
For \( k + 1 \leq \alpha \leq n \), since \( f^*_\alpha = 0 \) on \( Y \), we derive
\[ 0 = df^*_\alpha = -\beta^\alpha_j \wedge \theta^j - \alpha^\alpha_j \wedge \theta^j \] (5.5)
on \( Y \). By Cartan’s lemma, we conclude
\[ \beta^\alpha_j = A^\alpha_{j\ell} e^*_\ell + B^\alpha_{j\beta} f^*_\beta \\
\alpha^\alpha_j = C^\alpha_{j\beta} e^*_\beta + D^\alpha_{j\beta} f^*_\beta \] (5.6)
where the coefficients satisfy
\[ A^\alpha_{j\ell} = A^\alpha_{j\ell}, \quad B^\alpha_{j\beta} = C^\alpha_{j\beta}, \quad D^\alpha_{j\beta} = D^\alpha_{j\beta}. \] (5.7)
Therefore the second fundamental form \( S = \sum_{\alpha} S^\alpha f^*_\alpha \) of \( Y \) is given by the symmetric matrix
\[ S^\alpha = \begin{pmatrix} A^\alpha_{j\ell} & C^\alpha_{j\beta} \\ B^\alpha_{j\beta} & D^\alpha_{j\beta} \end{pmatrix}. \]
Now we are ready to define the leafwise mean curvature vector.

**Definition 5.1.** Let \( Y \subset X \) be a coisotropic submanifold and let \( S = S^\alpha f^*_\alpha \) be the second fundamental form of \( Y \subset (X, g) \). The leafwise mean curvature vector of \( Y \) is the partial trace of \( S \)
\[ \vec{H}^\parallel := S(e_\beta, e_\beta) = S^\alpha(e_\beta, e_\beta)f^*_\alpha. \] (5.8)
The leafwise mean curvature one form of \( Y \) is defined by
\[ \alpha^\parallel_Y := \vec{H}^\parallel|\omega. \] (5.9)
It is easy to check that the right hand side of (5.8) is independent of the choice of the frame adapted to \( Y \) and so the leafwise mean curvature vector and so the leafwise mean curvature one form are well-defined.

In the above moving frame, the leafwise mean curvature vector is given by
\[ \vec{H}^\parallel = A^\alpha_{\beta\beta} f^*_\alpha \]
Recalling the Kähler form is given by \( \omega = \frac{i}{2} \theta^j \wedge \bar{\theta}^j = e^*_k \wedge f^*_k \), we prove that the mean curvature one form becomes
\[ \vec{H}^\parallel|\omega = -A^\alpha_{\beta\beta} e^*_\alpha \] (5.10)
We summarize the above calculations into the following proposition
Proposition 5.2. Let \( Y \subset (X, \omega) \) be coisotropic and \((X, \omega, J)\) be Kähler. Then the leafwise mean curvature one form has the formula
\[
\alpha^\parallel := \bar{H}^\parallel \omega|_{TY} = -A_{\alpha\alpha}^\beta e^\alpha_\beta = -A_{\beta\alpha}^\alpha e^\beta_\alpha.
\] (5.11)

Now we will derive leafwise derivatives of various leafwise differential forms, including \(\alpha^\parallel_Y\). To carry out this derivation in a coherent manner, we need to provide a brief review of the concept of \(E\)-de Rham complex \([NT]\) associated to the structure of Lie algebroid \([Mac]\).

Definition 5.3. Let \( M \) be a smooth manifold. A Lie algebroid on \( M \) is a triple \((E, \rho, [\cdot, \cdot])\), where \( E \) is a vector bundle on \( M \), \([\cdot, \cdot]\) is a Lie algebra structure on the sheaf of sections of \( E \), and \( \rho \) is a bundle map
\[
\rho: E \to TM
\] (5.12)
such that the induced map
\[
\Gamma(\rho): \Gamma(M; E) \to \Gamma(TM)
\] (5.13)
is a Lie algebra homomorphism and, for any sections \( \sigma \) and \( \tau \) of \( E \) and a smooth function \( f \) on \( M \), the identity
\[
[\sigma, f\tau] = \rho(\sigma)[f] \cdot \tau + f \cdot [\sigma, \tau].
\] (5.14)

Definition 5.4 [Definition 2.2, NT]. Let \((E, \rho, [\cdot, \cdot])\) be a Lie algebroid on \( M \). The \(E\)-de Rham complex \([E\Omega^\bullet(M), E d]\) is defined by
\[
E\Omega^\bullet(E^*) = \Gamma(\Lambda^\bullet(E^*))
\]
\[
E d\omega(\sigma_1, \cdots, \sigma_{k+1}) =
\]
\[
= \sum_i (-1)^i \rho(\sigma_i) \omega(\sigma_1, \cdots, \tilde{\sigma}_i, \cdots, \sigma_{k+1})
\]
\[
+ \sum_{i<j} (-1)^{i+j-1} \omega([\sigma_i, \sigma_j], \sigma_1, \cdots, \tilde{\sigma}_i, \cdots, \tilde{\sigma}_j, \cdots, \sigma_{k+1}).
\] (5.15)

The cohomology of this complex will be denoted by \(E^H^*(M)\) and called the \(E\)-de Rham cohomology of \( M \). An \(E\)-connection on a vector bundle \( F \) on \( M \) is a linear map
\[
\nabla: \Gamma(F \otimes \Lambda^\bullet(E^*)) \to \Gamma(F \otimes \Lambda^{\bullet+1}(E^*))
\] (5.16)
satisfying the Leibnitz rule:
\[
\nabla(f\sigma) = E d(f) \cdot \sigma + f \cdot \nabla \sigma.
\]

Similarly we can define the notion of \(E\)-curvature \(E^K\) in \(\Gamma(\Lambda^2(E^*) \otimes \text{End}(F))\) in an obvious way.

We would like to emphasize that \(E^H^*(M)\) is not a topological invariant of \( Y \) but an invariant of the Lie algebroid \( E = TY^\omega \) or of the null foliation \( F \) of \( Y \). For
example, the class is not invariant under the general homotopy but invariant only under the homotopy tangential to the foliation in an obvious sense. We can also denote $TY^\omega = TF$ the tangent bundle of the null foliation $F$ and by $E^* = T^*F$ the cotangent bundle of $F$.

In our case, $M = Y$ and $E := TY^\omega$ and the anchor map $\rho : E \to TY$ is nothing but the inclusion map $i : TY^\omega \to TY$. The integrability of $TY^\omega$ implies that the restriction of the Lie bracket on $\Gamma(TY)$ to $\Gamma(TY^\omega)$ defines the Lie bracket $[,]$ on $\Gamma(E)$. Therefore the triple

$$(E = TY^\omega, \rho = i, [ , ])$$

defines the structure of Lie algebroid and hence the $E$-differential and $E$-connections.

In our case, the corresponding $E$-differential is nothing but $d_F$ the exterior derivative along the null foliation $F$. and its cohomology, denoted by $H^* (F, \omega)$, the cohomology $H^* (F)$ of the foliation $F$. We denote the $E$-connection of a vector bundle $F$ over $Y$ of this Lie algebroid by $\nabla^\omega$ in general.

Now we go back to further study the leafwise mean curvature vector. The the Ricci form $i\rho$ of the metric $g = \omega(\cdot, J \cdot)$ represents the curvature of the canonical line bundle

$$K := \Lambda^{n,0} (T^* X \otimes \mathbb{C}).$$

We first recall one geometric fact about Lagrangian submanifolds on the general Kähler manifolds. For a given Lagrangian embedding $i : L \hookrightarrow (X, \omega)$, we consider the pull-back bundle $F = i^* K \to L$. If $L$ is orientable, the line bundle $i^* K$ is always trivial. In general, the square $i^* K^2$ is trivial. In fact, for the orientable Lagrangian submanifold, there exists a canonical section which we denote by $\Omega_L$:

we choose a positively oriented Darboux frame

$$\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$$

so that $\{e_1, \cdots, e_n\}$ spans $TL$ and the associated holomorphic frame $\{\theta_1, \cdots, \theta_n\}$ of $\Lambda^{n,0} (TX \otimes \mathbb{C})$ adapted to $L$. In particular, the $n$-form

$$e_1^* \wedge \cdots \wedge e_n^*$$

provides the volume form of $Y$ with respect to the induced metric. Then the complex $n$-form

$$\Omega_L = \theta_1 \wedge \cdots \wedge \theta_n$$

where $\theta_j = e_j^* + i f_j^*$ provides the canonical section.

Now we study the general coisotropic cases. The discussion below will be parallel with the case in $(\mathbb{R}^{2n}, \omega_0)$ considered in section 2. Recalling that the direct sum

$$E := TF \oplus N_j Y \subset TX|_Y$$

is invariant under the action of $J : TX|_Y \to TX|_Y$, we consider the complexification

$$E \otimes \mathbb{C} = T^{1,0}E \oplus T^{0,1}E$$

where the splitting is with respect to the complex structure $J : TF \oplus N_j Y \to TF \oplus N_j Y$. One can also consider the dual bundle

$$E^* = N^*F \oplus T^*F \cong N_j Y \oplus TF = E.$$  (5.17)
Then we consider the top exterior power of $E^{1,0}$, $\det E^{1,0}$, as a complex line vector bundle.

Note that the complex line bundle $\det E^{1,0}) \otimes^2$ has a canonical section given by

$$(u_{k+1} \wedge \cdots \wedge u_n) \otimes^2$$

for any adapted frame $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ and its associated Hermitian frame. It is easy to check that this form does not depend on the choice of the adapted frame (4.9) and so globally well-defined.

We will now compute the covariant derivative

$$\nabla (u_{k+1} \wedge \cdots \wedge u_n)$$

with respect to the induced connection on $(\det E^{1,0}) \otimes^2$ from that of $(T_F \oplus J(T_F)) \otimes \mathbb{C}$. More precisely we have the following

**Lemma 5.5.** The canonical connection on $TX|_Y$ naturally induces a connection on $T_F \otimes J(T_F) \subset TX|_Y$ by restriction.

**Proof.** This is an immediate consequence of Proposition 4.3.

We will also compute the $E$-covariant derivative

$$\nabla^E (u_{k+1} \wedge \cdots \wedge u_n)$$

where the $E$-connection $\nabla^E$ is nothing but the restriction of the induced connection on $F = \det E^{1,0}$ from the connection $\nabla$ on $T_F \otimes J(T_F)$ defined by Lemma 5.5. One can easily check that $\nabla^E$ really satisfies the defining property $E$-connection in Definition 5.4.

**Proposition 5.6.** Let $(X, \omega, J)$ be Kähler and $i : Y \hookrightarrow (X, \omega)$ be a coisotropic embedding. We denote by $\nabla$ the connection on $\det E^{1,0} \rightarrow Y$ defined in Lemma 5.5. Then we have

$$\nabla (u_{k+1} \wedge \cdots \wedge u_n) = i^* (\omega^n) \cdot (u_{k+1} \wedge \cdots \wedge u_n) \quad (5.18)$$

and

$$\nabla^E (u_{k+1} \wedge \cdots \wedge u_n) = -i\alpha^\parallel_Y \cdot (u_{k+1} \wedge \cdots \wedge u_n) \quad (5.19)$$

In particular, we have

$$\text{curv}(\det(E^{1,0})) = i^*(d\omega^n) \quad (5.20)$$

for the curvature of the line bundle $\det(E^{1,0})$ is given by and its $E$-curvature is given by

$$E\text{curv}(\det(E^{1,0})) = -id_Y (\alpha^\parallel_Y). \quad (5.21)$$

As in the Lagrangian case [Oh2], the following is an immediate corollary of Proposition 5.6 whose proof we omitted.
Corollary 5.7 (Compare with [Corollary 3.3, Oh2]). Let \( i : Y \to X \) be any coisotropic embedding. Then the holonomy of the bundle \( \det(\mathcal{E}^{1,0}) \) over a loop \( \gamma \subset Y \) with respect to the induced connection is given by

\[
\exp(-i \int_{\gamma} \alpha_{\parallel}^Y)
\]

provided that \( \gamma \) is tangent to the null foliation of \( Y \), i.e., \( \gamma'(t) \in T Y^\omega \) for all \( t \).

We first state the following symmetry property of the second fundamental form \( B \) of coisotropic submanifolds \( Y \) in a Kähler manifolds, which is the analog of the symmetry property for the Lagrangian submanifolds [Lemma 3.10, Oh1].

Lemma 5.8. Let \( B \) be the second fundamental form of the coisotropic submanifold \( Y \subset X \). Consider the tri-linear form on \( TF \) defined by

\[
(X, Y, Z) \mapsto \langle B(X, Y), JZ \rangle.
\]

Then we have

\[
\langle B(X, Y), JZ \rangle = \langle B(X, Z), JY \rangle
\]

for all \( X, Y, Z \in TF \).

Proof. This is a re-statement of the property that \( A_{\alpha \beta \gamma}^Y \) is fully symmetric over \( \alpha, \beta \) and \( \gamma \) which follows from (5.3) and (5.7). \( \square \)

Proof of Proposition 5.6. For any \( X \in TY \), we compute \( \nabla_X (u^{k+1} \wedge \cdots \wedge u^n) \) with \( u^{k+1} \wedge \cdots \wedge u^n \) taken as a unit frame of \( \det(\mathcal{E}^{1,0}) \). Then we have

\[
\nabla_X (u^{k+1} \wedge \cdots \wedge u^n) = \sum_{\alpha=k+1}^{n} u^{k+1} \wedge \cdots \wedge \nabla_X u_{\alpha} \wedge \cdots \wedge u^n. \tag{5.22}
\]

From the way how the connection \( \nabla \) on \( TF \oplus J(TF) \) is defined in Lemma 5.5, we have On the other hand Proposition 4.3 together with the structure equations (4.1) restricted to \( Y \) implies that the covariant derivative becomes

\[
\nabla_X u_{\alpha} = \omega_{\alpha}^\gamma(X)u_{\gamma}
\]

and hence

\[
\nabla_X (u^{k+1} \wedge \cdots \wedge u^n) = \omega_{\alpha}^\gamma(X)(u^{k+1} \wedge \cdots \wedge u^n). \tag{5.23}
\]

This proves (5.20). On the other hand by restricting to \( TF \) and using the fact that \( \alpha_{\parallel}^Y = 0 \) we have

\[
\nabla_X (u^{k+1} \wedge \cdots \wedge u^n) = i \beta_{\alpha}^\gamma(X)(u^{k+1} \wedge \cdots \wedge u^n). \tag{5.24}
\]

for \( X \in TF \). On the other hand, we have already shown that \( \beta_{\alpha}^\gamma = A_{\gamma}^\alpha e_{\alpha}^\gamma = -\alpha_{\parallel}^\gamma \) in (5.6), (5.11) and the symmetry property of \( A_{\alpha \beta \gamma}^Y \). This finishes the proof (5.21). \( \square \)
§6. Pre-Kähler manifolds and its transverse canonical bundle

The beginning discussion in this section is intrinsic in that it depends only on the corresponding pre-symplectic structure $\omega_Y$ on $Y$, while the discussions in the previous sections are extrinsic in that it describes the property of the coisotropic embedding into $(X, \omega)$. In the end of the section, we will derive a compatibility condition between them.

For this purpose, it seems to useful to define another intrinsic notion

**Definition 6.1.** Let $(Y, \omega_Y)$ be a pre-symplectic manifold and fix a projection $\Pi : TY \to TY$ and its associated splitting $TY = TF \oplus G_{\Pi}$. Let $J$ be an almost complex structure on the normal bundle $NF \subset TY$ compatible to the pre-symplectic structure $\omega_Y$ in the sense that

$$g|_{NF} = \omega_Y(\cdot, J\cdot)|_{NF}$$

We call the triple $(Y, \omega_Y, J)$ a pre-Kähler manifold. We denote by $\text{curv}(K_Y)$ and $E\text{curv}(K_Y)$ the curvature and $E$-curvature of the transverse canonical bundle $K_Y$ of $Y$, respectively.

Obviously any coisotropic submanifold $Y$ in a Kähler manifold $(X, \omega, J)$ carries the induced pre-Kähler structure.

Considering $\theta^1 \land \cdots \land \theta^k$ as a local frame of $K_Y$, a straightforward computation shows that the covariant derivative

$$\nabla_X(\theta^1 \land \cdots \land \theta^k) = -\omega^a(X)(\theta^1 \land \cdots \land \theta^k)$$

(6.1)

for any $X \in TY$. Here again we use Proposition 4.3 to define the natural connection on the bundle $HE := (TF)^\perp_{J}$ induced from $TX|_Y$. The calculation is done with this natural connection.

Therefore the curvature of $K_Y$ is given by

$$\text{curv}(K_Y) = -i^*(d\omega^a).$$

(6.2)

On the other hand if we denote by

$$i^*_F(\omega^a)$$

the pull-back of the form $\omega^a$ to the leaves of the null-foliation $F$, the the $E$-curvature of the bundle $K_Y$ is given by the leafwise differential

$$E\text{curv}(K_Y) = -d_F(i^*_F \omega^a).$$

(6.3)

These forms do not depend on the choice of frames adapted to $Y$ but depends only on the triple $(Y, \omega_Y, J_Y)$.

We now derive the following which is the coisotropic analog of the result by Morvan [Mo] and Dazord [D], which relates the intrinsic curvatures $(Y, \omega_Y, J_Y$ and the extrinsic curvatures of $Y \subset (X, \omega, J)$ and the ambient Ricci-curvature of $(X, \omega, J)$.
Theorem 6.2. Let $(X, \omega, J)$ Kähler and $i: Y \subset (X, \omega, J)$ be any coisotropic submanifold. Let $K = (K^j_i)$ be the curvature two form and $i\rho = K^j_i$ the Ricci form of the metric. Then we have the formula
\[ -\text{curv}(K_Y) = -\text{curv}(\det \mathcal{E}^{1,0}) + ii^*(\rho) \] (6.4)
\[ -E\text{curv}(K_Y) = \text{id}_F(\alpha^\parallel_Y) + ii^*_F(\rho) \] (6.5)
where the two form $i^*_F(\rho)$ is the restriction to $TY^\omega = TF \subset TY$ of the Ricci form $\rho$ of $X$. Equivalently, we have
\[ -\text{curv}(K_Y) + \text{curv}(\det \mathcal{E}^{1,0}) = ii^*(\rho) \] (6.6)
\[ -E\text{curv}(K_Y) - \text{id}_F(\alpha^\parallel_Y) = ii^*_F(\rho) \] (6.7)

Proof. We will follow the above used notations in section 4. We first note that
\[ d\beta = \text{Im}(d\omega) \] (6.8)
Now we note that if we restrict $\beta^\omega_\gamma$ in (5.6) to the leaves of $E$, we have
\[ i^*_F(\beta^\omega_\gamma) = A^\alpha_{\gamma\mu}e^\mu. \] (6.9)
On the other hand, taking the trace of the 2-nd structure equation we have
\[ d\omega = -\omega_i^j \wedge \omega^j_i + K^j_j = i\rho. \] (6.10)
We decompose the left hand side of (6.9) and rewrite
\[ d\omega^\alpha_a + d\omega^a_\alpha = i\rho. \]
This immediately proves (6.6). Restricting this to the leaves of $F$, we have
\[ d_F(i^*_F(\omega^\alpha_a)) + d_F(i^*_F(\omega^a_\alpha)) = ii^*_F(\rho). \] (6.11)
On the other hand, we derive from (6.8)
\[ d_F(i^*_F(\beta^\alpha_\beta)) = \text{id}_F(i^*_F(\beta^\beta_\alpha)) = -\text{id}_F(\alpha^\parallel_Y) \] (6.12)
and
\[ -E\text{curv}(K_Y) = d_F(i^*_F(\omega^\alpha_a)) \]
from the definition (6.3) of the $E$-curvature of $K_Y$. This finishes the proof. \qed

Corollary 6.3. If $(X, \omega, J)$ is Einstein-Kähler, i.e., $\rho = \lambda \omega$, then we have
\[ \text{curv}(K_Y) - \text{curv}(\det \mathcal{E}^{1,0}) = 0. \] (6.13)
\[ E\text{curv}(K_Y) + \text{id}_F(\alpha^\parallel_Y) = 0. \] (6.14)
This is the coisotropic analog to the well-known fact that the mean curvature one form is always closed for the Lagrangian submanifolds in Einstein-Kähler manifolds.

Now we restrict to the case when $Y$ carries a parallel transverse Maslov charge $\Delta$. An obvious necessary condition for $Y$ to carry such a parallel transverse Maslov charge is the vanishing of its ordinary curvature
\[ \text{curv}(K_Y) = 0 \] (6.15)
i.e., $Y$ must be transversally Ricci-flat. Therefore we have the following

Proposition 6.4. Suppose that $Y$ satisfies (6.15), i.e., is transversally Ricci-flat. Then we have
\[ -\text{id}_F(\alpha^\parallel_Y) = i^*_F(\rho). \]
In particular, for the Einstein-Kähler case, the mean leafwise mean curvature form is leafwise closed and so define an infinitesimal deformation of $Y$ as a coisotropic submanifold.
§7. Special coisotropic submanifolds in Calabi-Yau manifolds

In this section, we restrict to the case when \((X, \omega, J)\) is Calabi-Yau and study special features of geometry of coisotropic submanifolds thereof. A coisotropic submanifold will have an induced pre-Kähler structure \((Y, \omega_Y, J_Y)\) from \(X\).

Let \(\Omega\) be a holomorphic volume form of \(X\) with constant length one or equivalently a non zero holomorphic section of the canonical line bundle \(K\) with length one.

For any given coisotropic submanifold \(Y \subset (X, \omega)\), taking the restriction of \(\Omega\) to the leaves of \(\mathcal{F}\), we define a bundle map
\[
\tilde{\Omega}_Y : (\det(T^{1,0}E))^{\otimes 2} \to (K_Y)^{\otimes 2}
\]
by
\[
(\xi)^{\otimes 2} \mapsto (\xi | \Omega)^{\otimes 2}.
\]
In particular, this pushes forward the global section \((u_{k+1} \wedge \cdots \wedge u_n)^{\otimes 2} \in \Gamma(E^{1,0})\) to a global section on \(K_Y^{\otimes 2}\). Therefore we have

**Definition & Proposition 7.1.** Any coisotropic submanifold \(Y\) in Calabi-Yau manifolds is canonically graded by the section induced by \(\Omega\). We denote this canonical transverse Maslov charge by \(\Delta_{(\Omega; Y)}\) and call it the canonical transverse Maslov charge of \(Y\) and the corresponding grading the canonical grading.

This canonical grading induced by \(\Omega\) enables us to define the coisotropic Maslov index for the maps \(w : (D, \partial D) \to (X, Y)\) for all coisotropic submanifolds \(Y\) in \(X\) simultaneously, which we denote by \(\mu_{(Y; \Omega)}\).

Furthermore for the given frame adapted to \(Y\), we can also write
\[
\Omega = f \theta^1 \wedge \cdots \wedge \theta^n
\]
for a locally defined function \(f\) with values in \(S^1\). Since \(\Omega\) is parallel, we have
\[
0 = \nabla_X (\Omega) = \nabla_X (f \theta^1 \wedge \cdots \wedge \theta^n) = (df(X) - f \omega^j(X)) \theta^1 \wedge \cdots \wedge \theta^n
\]
and hence
\[
df(X) - f \omega^j(X) = 0
\]
for any \(X \in TY\). In the same frame, we have
\[
(u_{k+1} \wedge \cdots \wedge u_n) | \Omega = f (\theta^1 \wedge \cdots \wedge \theta^k).
\]

Now we compute the covariant derivatives of both sides of (7.2) separately. From the right hand side, we derive
\[
\nabla_X (f \theta^1 \wedge \cdots \wedge \theta^k) = (df(X) - f \omega^j(X)) (\theta^1 \wedge \cdots \wedge \theta^k)
\]
for \(X \in TY\). For the left hand side, using (5.18) and the fact that \(\Omega\) is parallel, we derive
\[
\nabla_X (u_{k+1} \wedge \cdots \wedge u_n) | \Omega = \omega^\alpha_j(X)(u_{k+1} \wedge \cdots \wedge u_n) | \Omega).
\]
Comparing (7.3) and (7.4) with the identity (7.2), we derive
\[ f\omega^a_\alpha(X) = df(X) - f\omega^a_\alpha(X) \]
for all \( X \in TY \), i.e., we have
\[ 0 = d\gamma - i^*(\omega^a_\alpha) - i^*(\omega^a_\alpha) = d\gamma - i^*(\omega^j_\beta) \] (7.5)
where \( \gamma = \log f \). Restricting to \( TF \subset TY \), we also have
\[ 0 = d_F\gamma - i^F_*(\omega^a_\alpha) + i\alpha^a_\beta \] (7.6)
Note that the obvious integrability conditions for the equation (7.5) and (7.6) are
\[ d(i^*(\omega^a_\alpha) + i^*(\omega^a_\alpha)) = 0 \] (7.7)
and
\[ d_F(i^F_*(\omega^a_\alpha) - i\alpha^a_\beta) = 0 \] (7.8)
respectively, which, we have already shown in section 6, holds on any Einstein-Kähler manifold and so does on Calabi-Yau manifold.

**Definition 7.2.** Let \((X, \omega, J)\) be Calabi-Yau. We call \( Y \subset (X, \omega, J) \) special coisotropic submanifold (respectively leafwise special coisotropic submanifold) if the canonical section \( \Delta_{(Y, \Omega)} \) is parallel (respectively leafwise parallel), or equivalently if
\[ d\gamma - i^*(\omega^a_\alpha) = 0 \] (7.9)
and
\[ d_F\gamma - i^F_*(\omega^a_\alpha) = 0 \] (7.10)
with respect to the given frame, respectively.

Obviously one necessary condition for the embedding \( Y \subset (X, \omega, J) \) is special coisotropic is that the pre-Kähler structure \((Y, \omega_Y, J_Y)\) must be transversely Ricci-flat.

We recall that for the Lagrangian case, \( NF = \{0\} \) and
\[ (i^\ast\Omega)^\otimes 2 = g(\Omega_L)^\otimes 2 \]
for \( g : L \to S^1 \) which is a globally well-defined function and called the angle function of the Lagrangian submanifold. Therefore in this case, the above condition means
\[ g = (u_1 \wedge \cdots \wedge u_n|\Omega)^\otimes 2 \]
satisfies \( dg = 0 \) on \( L \), i.e., \( f \) is constant. Therefore Definition 7.2 reduces to the usual special Lagrangian condition for the Lagrangian case.

Furthermore for the Lagrangian case, special Lagrangian condition implies minimality of Lagrangian submanifolds and vice versa (at least for orientable Lagrangian submanifolds).

The following theorem is the coisotropic analog to this fact.
Theorem 7.3. Let \((X, \omega, J)\) be Calabi-Yau and \(Y \subset (X, \omega, J)\) be a leafwise special coisotropic submanifold. Then \(Y\) is leafwise special coisotropic if and only if it satisfies \(Y\) is leafwise minimal, i.e.,

\[ \alpha^\parallel_Y = 0 \]

Proof. This immediately follows from (7.5). □

Now we relate the above geometric study of coisotropic submanifolds in Calabi-Yau with the coisotropic Maslov index of the maps \(w: (D, \partial D) \to (X, Y)\).

For given \(w\), we choose our unitary frame \(\{v_1, \cdots, v_n\}\) on \(w^*TX\) so that it is admissible to the transverse Maslov charge \(\Delta_{(Y, \Omega)}\). We denote by \(\{\phi^1, \cdots, \phi^n\}\) its dual frame. The admissibility means

\[ (\partial w)^{*}(\Delta_{(Y, \Omega)}) = (\phi^1 \wedge \cdots \wedge \phi^k)^{\otimes 2}. \quad (7.11) \]

Now we write

\[ (u_{k+1} \wedge \cdots \wedge v_n)^{\otimes 2} = g \cdot (e_{k+1} \wedge \cdots \wedge e_n)^{\otimes 2} \]

for \(g: \partial D \to S^1\), where we recall \(TF = \text{span}_R \{e_{k+1}, \cdots, e_n\}\).

The Maslov index \(\mu_{(Y, \Omega)}(w)\) is computed by the degree of the map

\[ g: \partial D \to S^1 \]

\[ \mu_{(Y, \Omega)}(w) = \frac{1}{2\pi} \int_{\partial D} g^*d\theta \quad (7.12) \]

where \(d\theta\) is the canonical angular form on \(S^1\).

On the other hand, we can choose our frame \(\{v'_1, \cdots, v'_n\}\) of \(w^*TX\), which is not necessarily admissible, so that

\[ v'_j = u_j \quad \text{on} \quad (\partial w)^*TY \quad (7.13) \]

where \(\{u_1, \cdots, u_n\}\) the restriction of a frame of \((\partial w)^*TX\) adapted to \((\partial w)^*TY\) over \(\partial D\). We write

\[ \Omega = f \cdot (\phi')^1 \wedge \cdots \wedge (\phi')^n \]

for some \(f: \partial D \to S^1\). We write

\[ (v'_{k+1} \wedge \cdots \wedge v'_n)|\Omega^{\otimes 2} = f^2((\phi')^1 \wedge \cdots \wedge (\phi')^k)^{\otimes 2} \quad (7.14) \]

In particular, recalling Definition 3.3 of \(\mu_{(Y, \Omega)}(w)\), (7.14) implies

\[ \mu_{(Y, \Omega)}(w) = \text{deg}(f^2). \]

Noting that the calculation in the beginning of this section is purely on \(Y\), we will have exactly the same equation as in (7.5) with the connection one form \(\omega^i_j\) replaced by \((\omega')^i_j\) with respect to the frame \(\{v'_1, \cdots, v'_n\}\). However when restricted to \(Y\), we have

\[ (\omega')^i_j = \omega^i_j. \]

This proves the following local index formula
Theorem 7.4. For any coisotropic submanifold $Y$ and a frame $\{u_1, \ldots, u_n\}$ of $(\partial w)^*TX$ adapted to $(\partial w)^*TY$, we have

$$
\mu_{(Y;\Omega)}(w) = \frac{i}{\pi} \int_{\partial w} i^*(\omega_j^k) = \frac{i}{\pi} \int_{\partial w} (i^*(\omega_a^k) + i^*(\omega_a^k))
$$

(7.15)

for all $w : (D, \partial D) \rightarrow (X, Y)$. In particular, coisotropic Maslov index depends only on the boundary map $\partial w$.

Corollary 7.5. Let $Y \subset (X, \omega)$ be a coisotropic embedding. If the unitary frame $\{u_1, \ldots, u_n\}$ of $TX|_Y$ adapted to $Y$ extends over the neighborhood of the image of $w$, then $\mu_{(Y;\Omega)}(w) = 0$.

Proof. In this case, we note that the integral (7.15) becomes

$$
\frac{i}{\pi} \int_{\partial w} i^*(\omega_j^k) = \frac{i}{\pi} \int_{w} i*(d\omega_j^k) = \frac{i}{\pi} \int_{w} \rho = 0
$$

since $\rho \equiv 0$ for the Calabi-Yau metric. □

Corollary 7.6. Suppose that $Y$ is a special coisotropic submanifold. Then we have

$$
\mu_{(Y;\Omega)}(w) = \frac{i}{\pi} \int_{\partial w} i^*(\omega_a^k).
$$

(7.16)

Note that (7.16) for special Lagrangian submanifolds reduces to the well-known fact that the Maslov index vanishes since in that case $K_Y^{\otimes 2}$ is just a scalar $\mathbb{C}$.

§8. The transverse symplectic curvature in Kähler manifolds

In [OP1], we have introduced the notion of $\Pi$-transverse symplectic curvature $F_\Pi \in \Gamma(\Lambda^2 (N^*F) \otimes TF)$, which measures non-integrability of the complementary subbundle $G_\Pi$ of $TY$ in a given splitting

$$
TY = TF \oplus G_\Pi.
$$

Then the complementary subbundle $G_\Pi$ is integrable, if and only if $F_\Pi = 0$.

In the (almost)-Kähler case, we have the canonical Riemannian splitting

$$
TY = TF \oplus (TF)^{\perp, J}.
$$

We denote the corresponding curvature as $F = F_Y$. We first recall the definition

Definition 8.1 [Definition 4.1, OP1]. The transverse symplectic curvature of $Y$ is a $TF$-valued two form on $N_JF$ or a section of $\Lambda^2 (N^*F) \otimes TF$ defined as follows: Let $\pi : TY \rightarrow N_JF$ be the orthogonal projection. For any given $v, u \in N_JF|_y$, we define

$$
F(v, u) := [X, Y]|^\parallel(y)
$$

where $X, Y$ be any vector field on $Y$ with $X(y) = v, Y(y) = u$ that is normal to the foliation and $(\cdot)\parallel$ the component of $(\cdot)$ tangential to the null-foliation.
It was proven in [OP1] that this is well-defined as an element
\[ F \in \Gamma(\Lambda^2(N^*F) \otimes T^*F). \]

We will compute the components of the intrinsic curvature \( F \) with respect to those of the second fundamental form of the embedding \( Y \subset (X, g) \) where \( g \) is the associated Kähler metric of \( X \).

Let \( \{e_1, \cdots, e_n, f_1, \cdots, f_n\} \) be an orthonormal frame of \( TX \) adapted to \( Y \) and \( \omega^i_j = \alpha^i_j + i\beta^i_j \) be the associated Hermitian frame. The first structure equation is
\[ d\theta^i = -\omega^i_j \wedge \theta^j \]
or equivalently
\[
\begin{cases}
  de^*_i = -\alpha_j^i \wedge e^*_j + \beta_j^i \wedge f^*_j \\
  df^*_i = -\beta_j^i \wedge e^*_j - \alpha_j^i \wedge f^*_j.
\end{cases}
\]

As in section 5, we have
\[
\begin{align*}
\beta_j^i &= A^i_{jk} e^*_k + B^i_{jk} f^*_k \\
\alpha_j^i &= C^i_{jk} e^*_k + D^i_{jk} f^*_k
\end{align*}
\]
where the second fundamental form of \( Y \subset (X, g) \) is given by \( S = S^\alpha f_\alpha \) with
\[
S^\alpha = \left( A^\alpha_{ij}, C^\alpha_{jk}, B^\alpha_{ij}, D^\alpha_{jk} \right).
\]

Now we compute \( F \). A straightforward calculation leads to
\[
\begin{align*}
F(e_\alpha, e_\beta) &= [e_\alpha, e_\beta] = e^*_\alpha([e_\alpha, e_\beta])e_\alpha = -de^*_\alpha(e_\alpha, e_\beta)e_\alpha \\
&= (\alpha_j^\alpha(e_\alpha)\delta_j^\beta + \alpha_j^\beta(e_\beta)\delta_j^\alpha)e_\alpha \\
&= (\alpha_j^\alpha(e_\alpha) + \alpha_j^\beta(e_\beta))e_\alpha \\
&= (C^\alpha_{\alpha\beta} + C^\beta_{\alpha\beta})e_\alpha
\end{align*}
\]
and similarly
\[
F(f_\alpha, f_\beta) = (-A^\alpha_{\alpha\beta} - D^\alpha_{\alpha\beta})e_\alpha
\]
\[
F(f_\alpha, e_\beta) = (-B^\alpha_{\alpha\beta} + B^\beta_{\alpha\beta})e_\alpha.
\]

This proves the following formula
\[
F = \left( \frac{1}{2}(C^\alpha_{\alpha\beta} - C^\beta_{\alpha\beta})e_\alpha \right) e_\alpha \wedge e_\beta + \left( \frac{1}{2}(-B^\alpha_{\alpha\beta} + B^\beta_{\alpha\beta})e_\alpha \right) f^*_\alpha \wedge f^*_\beta
\]
\[
+ \left( -D^\alpha_{\alpha\beta} - A^\alpha_{\alpha\beta} \right) e_\alpha \wedge f^*_\beta.
\]

In terms of the unitary frame, we can also write
\[
F = F^{2.0} + F^{1.1} + F^{0.2}
\]
where
\[
F^{2.0} = \left( \frac{1}{4}(C^\alpha_{\alpha\beta} - C^\beta_{\alpha\beta})e_\alpha \right) \theta^\alpha \wedge \theta^\beta
\]
\[
F^{1.1} = \left( \frac{i}{2} \left( (-D^\alpha_{\alpha\beta} - A^\alpha_{\alpha\beta})e_\alpha \right) \theta^\alpha \wedge \bar{\theta}^\beta
\]
\[
F^{0.2} = \left( \frac{1}{4}(C^\alpha_{\alpha\beta} - C^\beta_{\alpha\beta})e_\alpha \right) \bar{\theta}^\alpha \wedge \bar{\theta}^\beta.
\]

In particular, we have obtained the formula for the symplectic transverse mean curvature \( \rho_Y^{\text{trans}} \) defined in [OP1]
\[
\rho_Y^{\text{trans}} = (A^\alpha_{\alpha\alpha} - D^\alpha_{\alpha\alpha})e_\alpha.
\]
Theorem 8.2.

(1) $F$ is of type $(1,1)$ if and only if

$$C^\alpha_{ab} = C^\alpha_{ba}$$

in addition to (5.7), in which case we have

$$F = F^{1,1} = \left( \frac{i}{2} ( -D^\alpha_{ab} - A^\alpha_{ba} ) e^a \right) \theta^a \wedge \overline{\theta}^b .$$

(2) We have $F = 0$ if and only if the second fundamental form $S$ satisfies

$$C^\alpha_{ab} = C^\alpha_{ba}, \quad A^\alpha_{ba} = -D^\alpha_{ba}$$

in addition to (5.7).

Now we consider the hypersurface case in detail.

Example 8.3. Let $(X, \omega, J)$ be any almost Kähler manifold and consider the hypersurface $Y \subset X$ defined by

$$Y = \{ x \in M \mid \rho(x) = 1 \}$$

(8.9)

for a smooth function $\rho : X \to \mathbb{R}$ such that

$$|\nabla \rho(x)|_g \equiv 1$$

(8.10)

for any $x \in Y$, where $\nabla \rho$ is the gradient of $\rho$ with respect to the associated metric $g = \omega(\cdot, J\cdot)$. We denote by $X_\rho = J\nabla \rho$ is the Hamiltonian vector field of $\rho$. It follows that $X_\rho$ is tangent to $Y$ and

$$TF = \text{span}_\mathbb{R} \{ X_\rho \}$$

(8.10)

and $N_J F \subset TY$ is nothing but the distribution of the maximal complex or $J$-invariant subspace of $TY$.

By the definition of $F$, we have

$$F(X, Y) = [X, Y]_{\mathbb{R}} = ([X, Y], X_\rho)_g X_\rho$$

$$= -\omega(J[X, Y], X_\rho) X_\rho = d\rho \circ J([X, Y]) X_\rho$$

(8.11)

for any $X, Y \in N_J F$. If we denote

$$d^c \rho = -d\rho \circ J$$

(8.12)

as usual, we have

$$N_J F = \ker d\rho \cap \ker d^c \rho.$$  

(8.13)

It then follows that

$$-d^c \rho([X, Y]) = dd^c (X, Y)$$

(8.14)

for any $X, Y \in N_J F$. Combining (8.11) and (8.14), we have proved

$$F(X, Y) = dd^c \rho(X, Y) X_\rho.$$  

(8.15)
In general, the two forms $dd^c$ on the complex vector bundle $N_J F \otimes \mathbb{C}$ will not be of type $(1,1)$, unless the almost complex structure is integrable. On the other hand, in the Kähler case, the following identity is well-known

$$dd^c \rho = 2i \partial \bar{\partial} \rho$$

whose restriction to $N_J F$ is the well-known Levi form on the hypersurface. Therefore in the integrable Kähler case, the transverse symplectic curvature $F$ is automatically of type $(1,1)$ and the two form

$$\langle F, X \varphi \rangle |_{N_J F}$$

reduces to the Levi form of the hypersurface. The case $F = 0$ corresponds to the hypersurface that is Levi-flat and the associated foliation of $N_J F$ in that case is nothing but the foliation by the maximally complex submanifolds in $Y$.

Motivated by this example, we now introduce the following definition

**Definition 8.4.** We say that a pre-Kähler manifold $(Y, \omega_Y, J_Y)$ is integrable, if the associated transverse curvature $F$ is of type $(1,1)$, or equivalently the second fundamental $S = S^\alpha f_\alpha$ in any adapted frame satisfies

$$S^\alpha|_{N_J F} = \begin{pmatrix} A^\alpha, C^\alpha \\ C^\alpha, D^\alpha \end{pmatrix}$$

where all $A^\alpha$, $C^\alpha$ and $D^\alpha$ are symmetric $k \times k$ matrices.

We will further study geometry of such structures elsewhere in the future.

**Appendix**

**A.1. Calculation for the almost Kähler case**

In this appendix, we generalize the formula in Theorem 4.3 in the (non-integrable) almost Kähler manifold as in the spirit of the calculation carried out by Schoen and Wolfson [SW] for the Lagrangian submanifolds in the almost Kähler case.

For this, we need to compare the canonical connection used in the main part of this paper and the Levi-Civita connection of the metric $g = \omega(\cdot, J \cdot)$.

The corresponding first structure equation with respect to the oriented orthonormal frame

$$\{ e_1, \cdots, e_n, f_1, \cdots, f_n \}$$

can be obtained by decomposing (3.3) into the real and the imaginary parts: Writing

$$\omega^j_\ell = \alpha^j_\ell + i \beta^j_\ell$$

$$\tau^j_\ell = \gamma^j_\ell + i \delta^j_\ell$$

turns (3.3) into

$$\begin{align*}
    de^*_j &= -(\alpha^j_\ell + \beta^j_\ell) \wedge e^*_\ell + (\beta^j_\ell - \delta^j_\ell) \wedge f^*_\ell & \text{(A.1)} \\
    df^*_j &= -(\beta^j_\ell + \delta^j_\ell) \wedge e^*_\ell + (\alpha^j_\ell - \gamma^j_\ell) \wedge f^*_\ell & \text{(A.2)}
\end{align*}$$

The symmetry properties of $\omega_{ij}^l$ and $\tau_{ij}^l$

$$\omega_{ij}^l = -\omega_{ji}^l, \quad \tau_{ij}^l = \tau_{ji}^l,$$

coming from the unitarity of $\omega$ and from (3.6), immediately imply that (A.1-2) defines a torsion free Riemannian connection of $g$ which is nothing but the unique Levi-Civita connection of $g$.

Now we apply the same kind of analysis as in section 4 noting that the definitions of $E$-differential and $E$-connections depend only on the symplectic structure and the given coisotropic submanifold $Y \subset (X, \omega)$, but not on the almost complex structure $J$.

In the non-integrable case, there will be torsion terms appearing in various places. First, (4.4) is replaced by (A.1-2) and (4.5) by

$$0 = df^*_\alpha = -(\beta^\alpha_j + \delta^\alpha_j) \wedge e^*_j - (\alpha^\alpha_b - \gamma^\alpha_b) \wedge f^*_j \tag{A.3}$$

on $Y$. Again by Cartan’s lemma, (4.6) is replaced by

$$\beta^\alpha_j + \delta^\alpha_j = A_j^\alpha e^*_j + B_j^\alpha f^*_j$$

$$\alpha^\alpha_b - \gamma^\alpha_b = C_j^\alpha e^*_j + D_j^\alpha f^*_j \tag{A.4}$$

By the same reasoning as in section 4 (see [SW] for a similar calculation for the Lagrangian case), we still obtain the same formula

$$\alpha^\parallel Y = A_{\beta\alpha}^\alpha e^*_\beta \tag{A.5}$$
as (4.11).

On the other hand, the formula (4.18) is replaced by

$$\beta^\gamma_j + \delta^\gamma_j = A_j^\gamma e^*_j$$
when restricted to $TF$ and so (4.21) replaced by

$$i\rho = d\omega^\beta + d\omega^b = d(\beta^\beta_j - \delta^\beta_j) + E \text{ curv}(KY). \tag{A.7}$$

Restricting this to $TF$ and combining with (A.5-7), we have obtained the following coisotropic analog to [Proposition A.1, SW].

**Theorem A.1.** Let $(X, \omega, J)$ be almost Kähler with the canonical connection. Let $K = (K^l_j)$ be the curvature two form and $i\rho = K^l_j$ be the Ricci form of the connection. Then we have the formula

$$id_{F\alpha}^\parallel Y - E \text{ curv}(KY) = i_F^\tau(\rho) + d_F(\tau)$$

where $\tau \in \Gamma(T^*F)$ is a one form canonically pulled-back via the embedding $i : Y \to X$ from the torsion form of the canonical connection of $X$.

**A.2. Criterion for the minimality of the null foliation**

In this section, we study the question when the null foliation becomes a minimal foliation in the sense that each leaf is a minimal submanifold of $Y$.

Note that the form

$$\nu_Y := e^*_{k+1} \wedge \cdots \wedge e^*_n \in \Lambda^{n-k}(E^*) = \Omega^{n-k}(F)$$
is independent choice of the frames $\{e_1, \ldots, e_n, f_1, \ldots, f_k\}$ adapted to $Y$ and so defines a globally well-defined $n-k$ form on $Y$ which restricts to the volume form on each leaf. We recall from [Ru, Su, Ha] that existence of such form is a necessary and sufficient condition for the manifold $(Y, g_Y)$ is foliated by minimal leaves of dimension $n-k$. In our case, the following theorem shows that the form $\nu_Y$ plays the role of such form.
Theorem A.2. A leaf \( \Sigma \) of the null foliation is a minimal submanifold if and only if the from \( \nu_Y \) is relatively closed on \( \Sigma \): namely,

\[
d\nu_Y(X_1, \ldots, X_{n-k+1}) = 0
\]  

on \( \Sigma \) if the first \( n-k \) vector fields \( X_i \) are tangent to the leaves.

Proof. With respect to the frame \( \{ e_1, \ldots, d_n, f_1, \ldots, f_k \} \) on \( Y \), the first structure equation becomes

\[
de e^*_i = -\alpha^i \wedge e^*_a + \beta^i_c \wedge f^*_c
\]  

\[
df^*_a = -\beta^a_j \wedge e^*_j - \alpha^a_b \wedge f^*_b.
\]  

On each leaf \( \Sigma \) of the null foliation, provided by \( e^*_a = 0 = f^*_b \), we have

\[
0 = de^*_a = -\alpha^a \wedge e^*_a
\]  

\[
0 = df^*_a = -\beta^a \wedge e^*_a.
\]  

By applying Cartan’s lemma and comparing with (4.6), we have

\[
\alpha^a = -\alpha^a_c = C^a_{b\beta} e^*_\beta, \quad \beta^a = \beta^a_a = A^a_{\alpha\gamma} e^*_\gamma.
\]  

Therefore the second fundamental form \( S_\Sigma \) of \( \Sigma \) in \( Y \) is given by the matrix

\[
S_\Sigma = (C^a_{b\beta} e^*_\beta \otimes e^*_\gamma) e_b + (A^a_{\alpha\gamma} e^*_\alpha \otimes e^*_\gamma) f_a
\]  

and its mean curvature vector by

\[
H_\Sigma = C^a_{b\alpha} e_b + A^a_{\alpha\gamma} f_a.
\]  

Hence the leaf \( \Sigma \) is minimal in \( Y \) if and only if

\[
C^a_{b\alpha} = 0 = A^a_{\alpha\gamma}(A.13)
\]  

for all \( a, b = 1, \ldots, k \).

Now we compute the differential \( d\nu_Y \)

\[
d\nu_Y = d(e^*_a \wedge \cdots \wedge e^*_n)
\]  

\[
= \sum_j (-1)^j e^*_a \wedge \cdots \wedge de^*_a \wedge \cdots \wedge e^*_n
\]  

The first structure equation (A.9) and (4.6) can be written as

\[
de e^*_a = -\alpha^a \wedge e^*_a - \alpha^a \wedge e^*_a
\]  

\[
= -(C^a_{b\beta} e^*_j + D^a_{b\gamma} f^*_c) \wedge e^*_b - \alpha^a \wedge e^*_a (A^a_{b\gamma} e^*_\gamma + B^a_{b\gamma} f^*_c) \wedge f^*_b.
\]  

It is then straightforward to derive, by substituting this into (A.14) and using \( \alpha^a = 0 \),

\[
d\nu_Y = \sum_{k+1 \leq \alpha \leq n} (-1)^j e^*_a \wedge \cdots \wedge (-C^a_{b\beta} e^*_j \wedge e^*_b) \wedge \cdots \wedge e^*_n
\]  

\[
= \sum_{k+1 \leq \alpha \leq n} (-1)^j e^*_a \wedge \cdots \wedge (A^a_{b\gamma} e^*_\gamma + f^*_c) \wedge f^*_b
\]  

\[
= (C^a_{b\alpha} e^*_b - A^a_{\alpha\gamma} f^*_c) \wedge \nu_Y
\]  

for the \((n-k+1)\)-tuple \((X_1, \ldots, X_{n-k+1})\) with the first \(n-k \) vector fields \( X_i \) are tangent to the leaves. This finishes the proof. \( \square \)
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