THE ASYMPTOTIC BEHAVIOR OF THE CODIMENSION SEQUENCE OF AFFINE G-GRADED ALGEBRAS

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ABSTRACT. Let $W$ be an affine PI algebra over a field of characteristic zero graded by a finite group $G$. We show that there exist $\alpha_1, \alpha_2 \in \mathbb{R}, \beta \in \frac{1}{2}\mathbb{Z}$, and $l \in \mathbb{N}$ such that $\alpha_1 n^{2l^n} \leq c_n^G(W) \leq \alpha_2 n^{2l^n}$. Furthermore, if $W$ has a unit then the asymptotic behavior of $c_n^G(W)$ is $\alpha n^{2l^n}$ where $\alpha \in \mathbb{R}, \beta \in \frac{1}{2}\mathbb{Z}, l \in \mathbb{N}$.

INTRODUCTION

Throughout this article $F$ is an algebraically closed field of characteristic zero and $W$ is an affine associative $F$- algebra graded by a finite group $G$. We also assume that $W$ is a PI-algebra; i.e., it satisfies an ordinary polynomial identity. In this article we study $G$-graded polynomial identities of $W$, and in particular the corresponding $G$-graded codimension sequence. Let us briefly recall the basic setup. Let $X^G$ be a countable set of variables $\{x_{i,g} : g \in G; i \in \mathbb{N}\}$ and let $F \langle X^G \rangle$ be the free algebra on the set $X^G$. Given a polynomial in $F \langle X^G \rangle$ we say that it is a $G$-graded identity of $W$ if it vanishes upon any admissible evaluation on $W$. That is, a variable $x_{i,g}$ assumes values only from the corresponding homogeneous component $W_g$. The set of all $G$-graded identities is an ideal in the free algebra $F \langle X^G \rangle$ which we denote by $Id^G(W)$. Moreover, $Id^G(W)$ is a $G$-graded $T$-ideal, namely, it is closed under $G$-graded endomorphisms of $F \langle X^G \rangle$.

Let $R^G(W)$ denote the relatively free algebra $F \langle X^G \rangle/Id^G(W)$, and $C^G_n(W)$ denote the space $P^G_n/P^G_n \cap Id^G(A)$ where $P^G_n$ is the $|G|^n \cdot n!$ dimensional $F$-space spanned by all permutations of the (multilinear) monomials $x_{1,g_1}, x_{2,g_2}, \ldots, x_{n,g_n}$ where $g_i \in G$. We also define the $n$-th coefficient $c^G_n(W)$ of the codimension sequence of $W$ by $c^G_n(W) = \dim_F(C^G_n(W))$. In [AB2] Aljadeff and Belov showed that for every affine $G$-graded algebra there exists a finite dimensional $G$-graded algebra with the same ideal of graded identities, thus with the same codimension sequence. We denote such algebra by $A_W$.

In the last few years several papers have been written which generalize results from the theory of ordinary polynomial identities to the $G$-graded case (e.g. see [Sv, AB1, AB2]). One of these papers was by E.Aljadeff, A.Giambruno and D.La Matina who showed, in different collaborations (see [GL, AG, AGL]), that, as in the nongraded case, $\lim_{n \to \infty} \sqrt[n]{c^G_n(W)}$ exists, and it is a nonnegative integer called the $G$-graded exponent of $W$, and denoted by $exp^G(W)$. In this paper we expand this result and prove that,

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in addition to the “exponential part”, there is a “polynomial part” to the asymptotics of \( c_n^G(W) \). More precisely, we prove:

**Theorem (A).** Let \( G \) be a finite group, and \( W \) an affine \( G \)-graded \( F \)-algebra where \( F \) is a field of characteristic 0. Suppose that \( W \) satisfies an ordinary polynomial identity. Then there exist \( \alpha_1, \alpha_2 > 0, \beta \in \frac{1}{2}\mathbb{Z} \), and \( l \in \mathbb{N} \) such that

\[
\alpha_1 n^\beta l^n \leq c_n^G(W) \leq \alpha_2 n^\beta l^n.
\]

A conclusion of this theorem is that \( \lim_{n \to \infty} \log_n \left( \frac{c_n(W)}{\exp(W)} \right) \) (the power of the “polynomial part”) exists, and is an integer or a half-integer.

Furthermore, if \( W \) has a unit, we have found the structure of the codimension sequence’s asymptotics.

**Theorem (B).** Let \( G \) be a finite group, and \( W \) an unitary affine \( G \)-graded \( F \)-algebra where \( F \) is a field of characteristic 0. Suppose that \( W \) satisfies an ordinary polynomial identity. Then there exist \( \beta \in \frac{1}{2}\mathbb{Z}, l \in \mathbb{N}, \) and \( \alpha \in \mathbb{R} \) such that

\[
c_n^G(W) \sim \alpha n^\beta l^n.
\]

Theorems [A] and [B] generalize results from the theory of ordinary polynomial identities. Indeed, in [BR] Berele and Regev proved the nongraded version of theorems [A] and [B] for PI algebras satisfying a Capelli identity (e.g. affine algebras, see [GZ2]). However, in Theorem A (for ungraded algebras) Berele and Regev made an additional assumption, namely, that the codimension sequence of the PI algebra is eventually non-decreasing. Recently, in [GZ3], Giambruno and Zaicev showed that the codimension sequence of any PI algebra is eventually nondecreasing and therefore the additional assumption mentioned above can be removed. Here we prove a \( G \)-graded version of Giambruno and Zaicev’s result and so the assumption on the monotonicity of the \( G \)-graded codimension sequence is also unnecessary.

We begin our article (Section 1) by mentioning some notations from the representation theory of \( GL_n(F) \) and \( S_n \). Section 2 is dealing with the case of affine (not necessarily unitary) \( G \)-graded algebras (Theorem A). And Section 3 is about the case of unitary affine \( G \)-graded algebras (Theorem B).

### 1. Preliminaries

In this section we recall some notations and definitions from the representation theory of \( GL_n(F) \) and \( S_n \).

**Definition 1.1.** A *partition* is a finite sequence of integers \( \lambda = (\lambda_1, \ldots, \lambda_k) \) such that \( \lambda_1 \geq \cdots \geq \lambda_k > 0 \). The integer \( k \) denoted by \( h(\lambda) \), and called the *height* of \( \lambda \). The set of all partitions is denoted by \( \Lambda \), and the set of all partitions of height less or equal to \( k \) is denoted by \( \Lambda^k \).
We say that \( \lambda \) is a partition of \( n \in \mathbb{N} \) if \( \sum_{i=1}^{k} \lambda_i = n \). In this case we write \( \lambda \vdash n \) or \( |\lambda| = n \).

It is known (e.g. see [Bru]) that the irreducible \( S_n \) representations are indexed by partitions of \( n \). Denote by \( D_\lambda \) the \( S_n \) irreducible representation indexed by \( \lambda \), and by \( \chi_\lambda \) the corresponding character. By Maschke’s theorem every \( S_n \)-representation \( V \) is completely reducible, so we can write \( V = \bigoplus_{\lambda \vdash n} m_\lambda D_\lambda \), and its character

\[
\chi(V) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda.
\]

A \( GL_n(F) \)-representation \( Y = SpF\{y_1, y_2, \ldots \} \) is called a polynomial if there is a set of polynomials \( \{f_{i,j}(z_{s,t})|i, j \in \mathbb{N}\} \subset F[z_{s,t}|1 \leq s, t \leq n] \) such that for every \( i \) only finite number of \( f_{i,j}(z_{s,t})'s \) are nonzero, and the action of \( GL_n(F) \) on \( Y \) is given by:

\[
P \cdot y_i = \sum_j f_{i,j}(p_{s,t})y_j
\]

for every \( P = (p_{s,t}) \in GL_n(F) \). We say that \( Y \) is \( q \)-homogeneous if all the nonzero \( f_{i,j} \)'s are homogeneous polynomials of (total) degree \( q \). For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), one defines the weight space of \( Y \) associated to \( \alpha \) by

\[
Y^\alpha = \{y \in Y| \text{diag}(z_1, \ldots, z_n) \cdot y = z_1^{\alpha_1} \cdots z_n^{\alpha_n}y\}.
\]

It is known that

\[
Y = \bigoplus_{\alpha} Y^\alpha.
\]

The series \( H_Y(t_1, \ldots, t_n) = \sum_{\alpha} (\text{dim}_F Y^\alpha) t_1^{\alpha_1} \cdots t_n^{\alpha_n} \) is called the Hilbert (or Poincare) series of \( Y \). The Hilbert series is known to be symmetric in \( t_1, \ldots, t_n \), so we recall an important basis of the space of symmetric series, namely the Schur functions. For convenience we use the following combinatorial definition: (Note that although this definition is not the classical one, it is equivalent to it e.g. see [Bru]).

Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \) be a partition. The Young diagram associated with \( \lambda \) is the finite subset of \( \mathbb{Z} \times \mathbb{Z} \) defined as \( D_\lambda = \{(i, j) \in \mathbb{Z} \times \mathbb{Z}|i = 1, \ldots, k, \; j = 1, \ldots, \lambda_i\} \). We may regard \( D_\lambda \) as \( k \) arrays of boxes where the top one is of length \( \lambda_1 \), the second of length \( \lambda_2 \), etc. For example

\[
D_{(4,3,3,1)} =
\]

A Schur function \( s_\lambda \in \mathbb{Z}[t_1, \ldots, t_n]^{S_n} \) is a polynomial such that the coefficient of \( t_1^{\alpha_1} \cdots t_n^{\alpha_n} \) is equal to the number of ways to insert \( a_1 \) ones, \( a_2 \) twos, \ldots, and \( a_n \) \( n \)'s in \( D_\lambda \) such that in every row the numbers are non-decreasing, and in any column the numbers are strictly increasing. Note that \( s_\lambda \) is homogenous of degree \( |\lambda| \).
Example.

(1) If \( \lambda \) is a partition of height one, i.e. \( \lambda = (\lambda_1) \), then the corresponding Schur function is
\[
s_{(\lambda_1)}(t_1, \ldots, t_n) = \sum_{a_1 + \cdots + a_n = \lambda_1} t_1^{a_1} \cdots t_n^{a_n}.
\]

(2) If \( \lambda = (2, 1) \) and \( n = 2 \) we have
\[
s_{(2,1)}(t_1, t_2) = t_1^2 t_2 + t_1 t_2^2
\]
since the only two ways to set ones and twos in \( \mathcal{D}_{(2,1)} \) is \( \begin{bmatrix} 1 & 1 \\ 2 \end{bmatrix} \) and \( \begin{bmatrix} 1 & 2 \\ 2 \end{bmatrix} \).

The next lemma is easily implied from the definition of the Schur functions.

Lemma 1.2.

(1) \( s_\lambda(t_1, \ldots, t_n) = 0 \) if and only if \( h(\lambda) > n \).

(2) \( s_\lambda(t_1, \ldots, t_n, 0, \ldots, 0) = s_\lambda(t_1, \ldots, t_n) \).

As \( \{ s_\lambda(t_1, \ldots, t_n) \}_{\lambda \in \Lambda^n} \) inform a basis of the space of symmetric series in \( n \) variables there exist integers \( \{ m_\lambda \}_{\lambda \in \Lambda^n} \) such that
\[
H_Y(t_1, \ldots, t_n) = \sum_{\lambda \in \Lambda^n} m_\lambda s_\lambda(t_1, \ldots, t_n)
\]
On the other hand, the group \( S_n \) is embed in \( GL_n(F) \) via permutation matrices, hence \( Y \) is also an \( S_n \) representation. Note that if \( P \) is a permutation matrix there is \( \sigma \in S_n \) such that \( \text{diag}(z_1, \ldots, z_n) \cdot P = P \cdot \text{diag}(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) \). Hence for every \( y \) in the weight space \( Y^{(1^n)} \) (\( 1^n \) means \( 1, \ldots, 1 \) \( n \) times) we have
\[
\text{diag}(z_1, \ldots, z_n) \cdot Py = P \cdot \text{diag}(z_{\sigma(1)}, \ldots, z_{\sigma(n)})y = z_1 \cdots z_n Py
\]
so \( Py \in Y^{(1^n)} \), and we obtain that \( Y^{(1^n)} \) is a sub \( S_n \) representation of \( Y \). Moreover, by the representation theory of \( GL_n(F) \) the \( S_n \) character of \( Y^{(1^n)} \) is \( \sum_{\chi \vdash n} m_{\chi} \chi_\lambda \) with the same coefficients \( m_\lambda \) as in the Hilbert series of \( Y \).

Next we recall some terminology and facts about \( G \)-graded relatively free algebras. Denote \( X_n^G = \{ x_{i,g} \mid g \in G, 1 \leq i \leq n \} \subset X^G \), and consider the free algebra \( F \langle X_n^G \rangle \) which is a subalgebra of \( F \langle X^G \rangle \). We equip \( F \langle X_n^G \rangle \) with a \( G \)-grading by setting the homogenous degree of a monomial \( x_{i_1,g_1} \cdots x_{i_l,g_l} \) to be \( g_1 \cdots g_l \in G \). If we suppose that \( I \) is a \( G \)-graded \( T \)-ideal of the free algebra \( F \langle X_n^G \rangle \), then it is easy to see that the grading on \( F \langle X_n^G \rangle \) induces (naturally) a grading on \( F \langle X_n^G \rangle/I \), giving us the corresponding \( T \)-ideal of \( G \)-graded identities \( \text{Id}^G \left( F \langle X_n^G \rangle/I \right) \subset F \langle X^G \rangle \).

We say that \( I \) is PI if \( \text{Id}^G \left( F \langle X_n^G \rangle/I \right) \neq \{0\} \) i.e. \( F \langle X_n^G \rangle/I \) is a PI algebra. Note that the \( T \)-ideal \( F \langle X_n^G \rangle \cap \text{Id}^G \left( F \langle X_n^G \rangle/I \right) \) is exactly \( I \).

For every \( g \in G \) the group \( GL_n(F) \) acts linearly on the \( n \)-dimensional vector space \( S p_F \{ x_{1,g}, \ldots, x_{n,g} \} \). Given these actions, we define a polynomial \( GL_n(F) \) representation
structure on $F \langle X_n^G \rangle$ via $P(x_{i_1,g_1} \cdots x_{i_l,g_l}) = P(x_{i_1,g_1}) \cdots P(x_{i_l,g_l})$ for every $P \in GL_n(F)$. This structure induces (naturally) a structure of $GL_n(F)$ representation on the relatively free algebra $F \langle X_n^G \rangle/\mathcal{J}$, giving us the corresponding Hilbert series $H_{F \langle X_n^G \rangle/\mathcal{J}}(t_1, ..., t_n)$.

We recall from [AB1] (lemma 2.10 and 2.11) the following properties of that kind of relatively free algebra $F \langle X_n^G \rangle/\mathcal{J}$.

This structure induces (naturally) a structure of $GL_n(F)$ representation on the relatively free algebra $F \langle X_n^G \rangle/\mathcal{J}$, giving us the corresponding Hilbert series $H_{F \langle X_n^G \rangle/\mathcal{J}}(t_1, ..., t_n)$.

We recall from [AB1] (lemma 2.10 and 2.11) the following properties of that kind of relatively free algebra $F \langle X_n^G \rangle/\mathcal{J}$.

Let $\mathcal{J}_1, \mathcal{J}_2$ be $G$-graded $T$-ideals of $F \langle X_n^G \rangle$, then

1. $H_{F \langle X_n^G \rangle/\mathcal{J}_1 \cap \mathcal{J}_2} = H_{F \langle X_n^G \rangle/\mathcal{J}_1} + H_{F \langle X_n^G \rangle/\mathcal{J}_2} - H_{F \langle X_n^G \rangle/\mathcal{J}_1 + \mathcal{J}_2}$.
2. If $\mathcal{J}_1 \subset \mathcal{J}_2$ then

$$H_{F \langle X_n^G \rangle/\mathcal{J}_2} = H_{F \langle X_n^G \rangle/\mathcal{J}_1} + H_{\mathcal{J}_2/\mathcal{J}_1}.$$

In our case recall that $Id_n^G(W) := F \langle X_n^G \rangle \cap Id^G(W)$ is $G$-graded $T$-ideal which is PI for every $n \in \mathbb{N}$, and let

$$\left\{ R_n^G(W) = F \langle X_n^G \rangle/Id_n^G(W) \right\}_{n \in \mathbb{N}}$$

be a collection of affine relatively free $G$-graded algebras of $W$, and consider the corresponding Hilbert series $H_{R_n^G(W)}(t_1, ..., t_n) = \sum_{\lambda \in \Lambda} m_\lambda^{(n)} s_\lambda(t_1, ..., t_n)$. We can show that we need only one set of coefficients $\{m_\lambda\}_{\lambda \in \Lambda}$ to describe $H_{R_n^G(W)}(t_1, ..., t_n)$ for every $n$. Indeed, it is easy to see that by definition $\text{diag}(z_1, ..., z_n) \cdot x_{i_1,g_1} \cdots x_{i_l,g_l} = z_{i_1} \cdots z_{i_l} x_{i_1,g_1} \cdots x_{i_l,g_l}$, hence the weight space $R_n^G(W)^\alpha$ is the span of all the monomials in $R_n^G(W)$ with $\alpha_i$ appearances of variables from the set $\{x_{i,g}|g \in G\}$ for every $i = 1, ..., n$. Therefore, we can write $R_n^G(W) = \bigoplus (R_m^G(W))^\alpha$ for every $r < m$, where the summing is over all the $\alpha \in \mathbb{N}^m$ such that $\alpha = (\alpha_1, ..., \alpha_r, 0, ..., 0)$. Thus, we get the Hilbert series of $R_n^G(W)$ by putting $t_{r+1} = \cdots = t_m = 0$ in the Hilbert series of $R_m^G(W)$, i.e.

$$H_{R_n^G(W)}(t_1, ..., t_r) = \sum_{\lambda \in \Lambda^r} m_\lambda^{(r)} s_\lambda(t_1, ..., t_r) = \sum_{\lambda \in \Lambda^m} m_\lambda^{(m)} s_\lambda(t_1, ..., t_r, 0, ..., 0).$$

Moreover, by Lemma 1.2 we get

$$\sum_{\lambda \in \Lambda^r} m_\lambda^{(r)} s_\lambda(t_1, ..., t_r) = \sum_{\lambda \in \Lambda^r} m_\lambda^{(m)} s_\lambda(t_1, ..., t_r) + \sum_{\lambda \in \Lambda^{m \setminus \Lambda^r}} m_\lambda^{(m)} \cdot 0,$$

hence for every $r < m$ the coefficients $m_\lambda^{(r)}$ and $m_\lambda^{(m)}$ are the same for every $\lambda \in \Lambda^r$. Making it possible to denote $m_\lambda^{(n)} = m_\lambda$, and write for every $n \in \mathbb{N}$

$$H_{R_n^G(W)}(t_1, ..., t_n) = \sum_{\lambda \in \Lambda^n} m_\lambda s_\lambda(t_1, ..., t_n).$$

Remember that $A_W$ is a finite dimensional $G$-graded algebra such that $Id^G(A_W) = Id^G(W)$, thus with the same $m_\lambda$. By [Go] (Lemma 1 and Lemma 7) $m_\lambda = 0$ if $\lambda \notin \Lambda^k$ where $k = dim_F(A_W)$, so all the nonzero coefficients $m_\lambda$ appear in the series $H_{R_n^G(W)}(t_1, ..., t_k)$. Notice that for every $n$ the weight space $R_n^G(W)^{(n)}$ is exactly $C_n^G(W)$, thus $c_n^G(W) = \sum_{\lambda \vdash n} m_\lambda d_\lambda$ where $d_\lambda = dim_F D_\lambda$. The conclusion is that the $G$-graded
codimension sequence of $W$ is determined by a single Hilbert series $H_{R_k^G}(W)(t_1, \ldots, t_k)$. We denote this series by $H_W(t_1, \ldots, t_k)$.

As mention in the introduction, Berele and Regev proved versions of Theorems \[A\] and \[B\] for (nongraded) affine PI algebras. Their proofs rely on a result about symmetric series which we will also use. Before stating this result we need a definition.

**Definition 1.4.** A series $H(t_1, \ldots, t_k)$ is called rational function if we can write it as $f(t_1, \ldots, t_k)/g(t_1, \ldots, t_k)$ where $f, g$ are polynomials, moreover a rational function $f(t_1, \ldots, t_k)/g(t_1, \ldots, t_k)$ is called nice if $g(t_1, \ldots, t_k) = \prod_{\alpha \in \Delta} (1 - t_1^{a_1} \cdots t_k^{a_k})$ where $\Delta$ is some subset of $\mathbb{N}^k$.

**Theorem 1.5 (\[BR\].)** Let $H(t_1, \ldots, t_k) = \sum_{\lambda \in \Lambda} a_\lambda s_\lambda$ be a series, and denote $y_n^H = \sum_{\lambda} a_\lambda d_\lambda$ where $d_\lambda$ is the dimension of the $\lambda$’s irreducible representation of $S_n$, and $a_\lambda \in \mathbb{N}$. We assume that

1. There is an integer $r$ such that $a_\lambda = 0$ for every partition $\lambda$ satisfying $\lambda_{r+1} \geq M$ where $M$ is a constant which depends only on $r$.
2. $H(t_1, \ldots, t_k)$ is nice rational function.

Then there is an integer $d$ such that for every $0 \leq m \leq d - 1$ there are constants $\beta_m \in \frac{1}{2} \mathbb{Z}$, and $\alpha_m \geq 0$, such that

$$y_{n,m,t}^H \sim \alpha_m \beta_m^m \gamma_{m,t}^{m,t}; \quad t \to \infty$$

where $n_{m,t} = m + td$, and $l$ is the minimal integer satisfying condition 1.

2. **Affine $G$-graded Algebras**

In this section we prove Theorem A. Our first step is to show that $H_W(t_1, \ldots, t_k) = \sum_{\lambda \in \Lambda} m_\lambda s_\lambda$ satisfies the conditions of Theorem \[1.5\] hence $y_n^H = c_n^G(W)$ satisfies the conclusion of the theorem. Recall that according to the first condition there is an integer $r$ such that $m_\lambda = 0$ for every partition $\lambda$ satisfying $\lambda_{r+1} \geq M$ where $M$ is a constant which depends only on $r$. In \[Go\] A.S Gordienko (see Lemma 1 and Lemma 7) showed that this condition holds for finite dimensional $G$-graded PI $F$-algebras, however we know (see the introduction) that there is a finite dimensional $G$-graded algebra $A_W$ such that $H_W = H_{R_k^G(A_W)}$, so indeed $H_W$ satisfies the first condition of Theorem \[1.5\]. Let us show that the second condition of Theorem \[1.5\] holds, namely that $H_W$ is nice rational. In \[AB\] Aljadeff and Belov proved that the Hilbert series of $F\langle X_k^G \rangle/\mathcal{I}$ is rational for every $G$ - graded $T$ - ideal $\mathcal{I}$ of $F\langle X_k^G \rangle$ which is PI. In fact, with slight changes in their proof, we can prove that the Hilbert series is also nice.

**Theorem 2.1.** Let $\mathcal{I}$ be a $G$ - graded $T$ - ideal of $F\langle X_k^G \rangle$ which is PI then the Hilbert series of $F\langle X_k^G \rangle/\mathcal{I}$ is a nice rational function.

**Proof.** Assume by contradiction that $H_{F\langle X_k^G \rangle/\mathcal{I}}(t_1, \ldots, t_k)$ is not nice rational. Then the set of all PI $G$ - graded $T$-ideal of $F\langle X_k^G \rangle$ whose Hilbert series is not nice rational, is
idalgebras which we will present later. Now, from 2.1 we get that the definition of basic algebra. For our purpose, we just have to use one property of basic $A$-ideals. This of course contradicts the Specht property of $A$ where (2.1) $\langle \text{id} \rangle$ is not empty. Furthermore, we may assume that there exists a maximal element in this set, because otherwise (by Zorn’s lemma) there is an infinite ascending sequence in this set. This sequence does not stabilize, and hence its union is a nonfinite generated $T$-ideal. This of course contradicts the Specht property of $G$-graded $T$-ideals (see [AB2] section 12).

So let $J$ be a maximal PI $G$-graded $T$-ideal of $F\langle x_k^G \rangle$ such that $H_{F\langle x_k^G \rangle/J}$ is not nice rational. From [AB2] (Theorem 11.2) we know that

$$Id^G(F\langle x_k^G \rangle/J) = Id^G(A_1 \oplus \cdots \oplus A_l)$$

where $A_i$ is a $G$-graded basic $F$-algebra for every $i$, and $id^G(A_i) \nsubseteq id^G(A_j)$ for any $1 \leq i, j \leq m$ with $i \neq j$. We refer the reader to [AB2] for more details - in particular, the definition of basic algebra. For our purpose, we just have to use one property of basic algebras which we will present later. Now, from [2.1] we get that $J = Id_k^G(F\langle x_k^G \rangle/J) = Id_k^G(A_1) \cap Id_k^G(A_2 \oplus \cdots \oplus A_l)$. Therefore, if we denote $J_1 = Id_k^G(A_1)$ and $J_2 = Id_k^G(A_2 \oplus \cdots \oplus A_l)$ then we get from Lemma 1.3

$$H_{F\langle x_k^G \rangle/J} = H_{F\langle x_k^G \rangle/J_1} + H_{F\langle x_k^G \rangle/J_2} - H_{F\langle x_k^G \rangle/J_1+J_2}.$$ 

Now, if $l > 1$ then $J \nsubseteq J_1, J_2$ from the maximality of $J$ the series $H_{F\langle x_k^G \rangle/J_1}, H_{F\langle x_k^G \rangle/J_2}$ and $H_{F\langle x_k^G \rangle/J_1+J_2}$ are all nice rational functions. Therefore, $H_{F\langle x_k^G \rangle/J}$ is nice rational which is a contradiction. Hence, $l = 1$, and

$$Id^G(F\langle x_k^G \rangle/J) = Id^G(A)$$

where $A$ is a basic $F$-algebra.

We recall from [AB1] (see section 2) that for any basic algebra $A$ there is a $G$-graded $T$-ideal $K$ strictly containing $J = Id_k^G(A)$ such that $K/J$ is a finite module over some affine commutative $F$-algebra, hence by [St] (Chapter 1 Theorem 2.3) $H(K/J)$ is a nice rational function. Now, by the maximality of $J$ the Hilbert series of $F\langle x_k^G \rangle/K$ is nice rational. Moreover, by Lemma 1.3 we get

$$H_{F\langle x_k^G \rangle/J} = H_{F\langle x_k^G \rangle/K} + H_{K/J}$$

hence $H_{F\langle x_k^G \rangle/J}$ is nice rational which contradicts our assumption. \hfill \Box

So indeed we can apply Theorem 1.5 on $H_W$, and we arrive at the following corollary for affine $G$-graded algebra with eventually non-decreasing codimension sequence (that is $e_n^G(W) \leq e_{n+1}^G(W)$ for every $n > N$ for some $N \in \mathbb{N}$).

**Corollary 2.2.** If the sequence $e_n^G(W)$ is eventually non-decreasing then there exists $\alpha_1, \alpha_2 > 0$, $\beta \in \frac{1}{2}\mathbb{Z}$, and $l \in \mathbb{N}$ such that

$$\alpha_1 n^\beta l^n \leq e_n^G(W) \leq \alpha_2 n^\beta l^n.$$
Proof. We need to show that all the $\beta_m$’s form Theorem 1.5 are equal. Suppose that there are $0 \leq m, l \leq d - 1$ such that $\beta_m < \beta_l$, and denote $s = l - m \mod d$. Note that $s = m - l + qd$ thus

$$n_m, t + s = m + td + s = m + td + l - m + qd = l + (t + q)d = n_{l, t + q}$$

therefore

$$\lim_{t \to \infty} \frac{c_{n_{m, t} + s}(W)}{c_{n_{m, t}}(W)} = \lim_{t \to \infty} \frac{c_{n_l, t + q}(W)}{c_{n_{l, t}}(W)} = \lim_{t \to \infty} \frac{\alpha_l n_{l, t} \beta_t n_{l, t}}{\alpha_m n_{m, t} \beta_t n_{m, t}} = 0$$

which is a contradiction since $c_n(W)$ is eventually non-decreasing. \qed

The final stage of Theorem A’s proof is removing the non-decreasing assumption from Proposition 2.2.

**Proposition 2.3.** Let $G$ be a finite group, and $W$ an affine $G$-graded $F$-algebra where $F$ is a field of characteristic 0. Suppose that $W$ satisfying an ordinary polynomial identity. Then the $G$-graded codimension sequence of $W$ is eventually non-decreasing.

Proof. Recall that there exists a finite dimensional algebra $A_W$ having the same codimension sequence as $W$, so we may prove the proposition for $A_W$. Let $A_W = S + J$ be the Wedderburn-Malcev decomposition of $A_W$, and we may assume that the semi-simple part $S$, and the Jacobson radical $J$ are $G$-graded. It is known (see [BR0]) that $J$ is nilpotent, i.e. there is an integer $t$ such that $J^t = 0$. We claim that $c_n(A_W) \leq c_{n+1}(A_W)$ for every $n > t$.

If $S = 0$ then $A_W$ is nipotent of degree $t$, thus $c^G_n(A_W) = 0$ for every $n > t$. So assume that $S \neq 0$, and let $1_S$ be its unit element $1_S$. Because $1_S^2 = 1_S$, the linear operator $T : A_W \to A_W$; $T(x) = x \cdot 1_S$ has eigenvalues 1 or 0, so we have the decomposition $A_W = A_0 \oplus A_1$ where $A_0, A_1$ are the eigenspaces of 0 and 1 respectively. Now, let $B = B_0 \cup B_1$ be a basis of $A_W$ where $B_0 \subset A_0$ and $B_1 \subset A_1$. We may assume that all the elements in the basis are $G$-homogenous. Since $F$ is of characteristic 0, it is enough to substitute the variables in a polynomial $f \in F \langle X^G \rangle$ by element form $B$ in order to determine if $f$ is an identity or not.

Denote $G = \{g_1, ..., g_s = e\}$, and for any $h = (h_1, ..., h_n) \in G^n$ denote $P_h = S p_F \{x_{h_1, 1}, ..., x_{h_n, n}\}$ and $C_h = P_h / p_{h}^{td(A_W)}$. Note that $C_n^G(A_W) = \bigoplus_{h \in G^n} C_h$, and $C_h \cong C_k$ if the vector $k$ is a permutation of the vector $h$. Therefore, if we denote $P_{(g_1^{n_1}, ..., g_s^{n_s})} = P_{n_1, ..., n_s}$ and $C_{(g_1^{n_1}, ..., g_s^{n_s})} = C_{n_1, ..., n_s}$ ( $g_i^{n_i}$ means $g_i, ..., g_i, n_i$ times), we can write

$$c_n^G(A_W) = \sum_{n_1 + \cdots + n_s = n} \binom{n}{n_1, ..., n_s} c_{n_1, ..., n_s}$$

where $\binom{n}{n_1, ..., n_s} = \frac{n!}{n_1! \cdots n_s!}$ is the generalized binomial coefficient and

$$c_{n_1, ..., n_s} = \text{dim}_F C_{n_1, ..., n_s}.$$
Given \( t < n = n_1 + \ldots + n_s \), write \( c_{n_1, \ldots, n_s} = k \) and let
\[ f_1(x_{1, h_1}, \ldots, x_{n, h_n}), \ldots, f_k(x_{1, h_1}, \ldots, x_{n, h_n}) \]
be a set of linear independent polynomials in \( P_{n_1, \ldots, n_s}/P_{n_1, \ldots, n_s} \cap \text{Id}^G(W) \) (note that \( h_1 = \cdots = h_{n_1} = g_1, h_{n_1+1} = \cdots = h_{n_1+n_2} = g_2 \) and so on). For every \( i = 1, \ldots, k \) we define the following polynomials in \( P_{n_1, \ldots, n_s+1} \) (recall that \( g_s = e \), so \( P_{n_1, \ldots, n_s+1} = P_{(h_1, \ldots, h_n, e)} \)).

\[
p_i(x_{1, h_1}, \ldots, x_{n, h_n}, x_{n+1, e}) = \sum_{j=1}^{k} f_i(x_{1, h_1}, \ldots, x_{j, h_j} x_{n+1, e}, \ldots, x_{n, h_n}).
\]

We want to show that \( p_1, \ldots, p_k \subseteq P_{n_1, \ldots, n_s+1}/P_{n_1, \ldots, n_s+1} \cap \text{Id}^G(W) \) are linearly independent, so let \( \sum_{i=1}^{k} \alpha_i p_i \) be a nontrivial linear combination of the \( p_i \)'s. Since \( f_1, \ldots, f_k \) are linear independent, there are \( b_j, h_j \in B \cap (A_W)_{h_j} \) such that \( \sum_{i=1}^{k} \alpha_i f_i(b_1, h_1, \ldots, b_{n, h_n}) \neq 0 \). Since \( n \) is greater than the nilpotency index of \( J \), one of the \( b_j, h_j \)'s is necessarily in \( S \), and \( S \subseteq A_1 \) so \( q = |B_1 \cap \{b_1, h_1, \ldots, b_{n, h_n}\}| \) is a nonzero integer. Let us substitute \( b_1, h_1, \ldots, b_{n, h_n}, 1_S \) in \( p_i \). Note that \( b_j, h_j \cdot 1_S = b_j, h_j \) if \( b_j, h_j \in B_1 \), and \( b_j, h_j \cdot 1_S = 0 \) if \( b_j, h_j \notin B_0 \), therefore

\[
p_i(b_1, h_1, \ldots, b_{n, h_n}, 1_S) = q \cdot f_i(b_1, h_1, \ldots, b_{n, h_n})
\]

and

\[
\sum_{i=1}^{k} \alpha_i p_i(b_1, h_1, \ldots, b_{n, h_n}, 1_S) = q \cdot \sum_{i=1}^{k} \alpha_i f_i(b_1, h_1, \ldots, b_{n, h_n}) \neq 0.
\]

Thus the polynomial \( \sum_{i=1}^{k} \alpha_i p_i(x_{1, h_1}, \ldots, x_{n, h_n}, x_{n+1, e}) \) is not an identity, so

\[
c_{n_1, \ldots, n_s} = k \leq c_{n_1, \ldots, n_s+1}^G
\]

for every \( n_1, \ldots, n_s \). Hence by 2.2

\[
c_n^G(W) = \sum_{n_1 + \ldots + n_s = n} c_{n_1, \ldots, n_s}^G \leq \sum_{n_1 + \ldots + n_s = n} \left( \begin{array}{c} n+1 \\ n_1, \ldots, n_s+1 \end{array} \right) c_{n_1, \ldots, n_s+1}^G \leq c_{n+1}^G(W)
\]

where \( \left( \begin{array}{c} n+1 \\ n_1, \ldots, n_s+1 \end{array} \right) \) is due to the combinatorial identity

\[
\left( \begin{array}{c} n+1 \\ n_1, \ldots, n_s+1 \end{array} \right) = \left( \begin{array}{c} n \\ n_1, \ldots, n_s \end{array} \right) + \sum_{i=1}^{s-1} \left( \begin{array}{c} n \\ n_1, \ldots, n_s - i, 1, \ldots, n_s \end{array} \right).
\]

\[\square\]

From 2.2 and 2.3 we conclude:

**Theorem (A).** Let \( G \) be a finite group, and \( W \) an affine \( G \)-graded \( F \)-algebra where \( F \) is a field of characteristic 0. Suppose \( W \) satisfies an ordinary polynomial identity.
Then there exists $\alpha_1, \alpha_2 > 0$, $\beta \in \frac{1}{2}\mathbb{Z}$, and $l \in \mathbb{N}$ such that
\[
\alpha_1 n^\beta l^n \leq c_n^G(W) \leq \alpha_2 n^\beta l^n.
\]

A conclusion drawn from theorem A is that the codimension sequence’s asymptotics has a “polynomial part” of degree $\beta \in \frac{1}{2}\mathbb{Z}$. More precisely,

**Corollary 2.4.** The following limit exists and it is a half integer.

\[
\lim_{n \to \infty} \log_n \left( \frac{c_n^G(W)}{\exp^G(W)^n} \right)
\]

### 3. Unitary Affine $G$-graded Algebras

In this section we assume that $W$ has a unit. We show that, as in the nongraded case, the asymptotics of the codimension sequence of $W$ is a constant times polynomial part times exponential part (Theorem B).

In [D] V.Drensky showed that in the non-graded case (or in another words for $G = \{e\}$) there is a subspace $B_n$ of $R_n^{(e)}(W)$ such that $R_n^{(e)}(W) = F[z_1, \ldots, z_n] \otimes B_n$ for every $n \in \mathbb{N}$. We start this section by proving this result for any finite group $G$. In order to keep the notations as light as possible, whenever there is an element $f + I$ in some quotient space $A/I$ where $A \subset F \langle X_n^G \rangle$ we omit the $I$ and leave only the polynomial $f$ when the convention is that all the operations on such polynomials are modulo $I$. So let $B_n$ be the subspace of $R_n^G(W)$ spanned by all the polynomials $x_{i_1, h_1} \cdots x_{i_r, h_r} [x_{i_{r+1}, g_1}, \ldots, x_{i_{r+g_1}, g_1}] \cdots [\ldots, x_{i_r, g_r}]$ where $h_j \neq e$ for $i = 1, \ldots, r$. Note that $B_n$ is clearly a sub polynomial $GL_n(F)$ representation of $R_n^G(W)$, so the Hilbert series $H_{B_n}(t_1, \ldots, t_n) = \sum_{\lambda \in \Lambda^n} a^{(n)}_{\lambda} s_\lambda(t_1, \ldots, t_n)$ is well defined. Moreover, we can write $B_r = \bigoplus B_n^\alpha$ for every $r < m$, where the sum is over all the $\alpha \in \mathbb{N}^m$ such that $\alpha = (\alpha_1, \ldots, \alpha_r, 0, \ldots, 0)$. Thus, we can get the Hilbert series of $B_r$ by putting $t_{r+1} = \cdots = t_m = 0$ in the Hilbert series of $B_m$, i.e.

\[
H_{B_r}(t_1, \ldots, t_r) = \sum_{\lambda \in \Lambda^r} a^{(r)}_{\lambda} s_\lambda(t_1, \ldots, t_r) = \sum_{\lambda \in \Lambda^m} a^{(m)}_{\lambda} s_\lambda(t_1, \ldots, t_r, 0, \ldots 0).
\]

Moreover, from Lemma 1.2

\[
\sum_{\lambda \in \Lambda^r} a^{(r)}_{\lambda} s_\lambda(t_1, \ldots, t_r) = \sum_{\lambda \in \Lambda^r} a^{(m)}_{\lambda} s_\lambda(t_1, \ldots, t_r) + \sum_{\lambda \in \Lambda^m \setminus \Lambda^r} a^{(m)}_{\lambda} \cdot 0.
\]

Hence, for every $r < m$ the coefficients $a^{(r)}_{\lambda}$ and $a^{(m)}_{\lambda}$ are the same for every $\lambda \in \Lambda^r$. Making it possible to denote $a^{(n)}_{\lambda} = a_{\lambda}$, and write for every $n \in \mathbb{N}$

\[
H_{B_n} = \sum_{\lambda \in \Lambda^n} a_{\lambda} s_\lambda(t_1, \ldots, t_n).
\]

We wish to show that, as in $R_n^G(W)$, a single Hilbert series determines all the coefficients $\{a_{\lambda}\}_{\lambda \in \Lambda}$, and that Theorem 1.5 can be applied to this series. Fix an integer $t$, 

\[10\]
and recall that for every polynomial $GL_n(F)$ representation $Y$ the $t$ homogenous sub representation of $Y$ is $Y^{(t)} = \bigoplus_{|\alpha|=t} Y^\alpha$. Let us define for every $s = 0, \ldots, t$ the spaces $Q_s = R^{(t-s)}_{n,e} D^{(s)}_n \subset ( R^G_n(W) )^{(t)}$ where $R_{n,e} = F(x_{1,e}, \ldots, x_{n,e})/Id^G(W) \cap F(x_{1,e}, \ldots, x_{n,e})$, and the spaces $M_p = \sum_{s=p}^t Q_s$ for $p = 0, \ldots, t$ and $M_{t+1} = \{0\}$. It is easy to see that $Q_s$ and $M_p$ are both polynomial $GL_n(F)$ representations for every $s$ and $p$.

**Lemma 3.1.**

(1) $( R^G_n(W) )^{(t)} = M_0$

(2) $( R^G_n(W) )^{(t)} \cong \bigoplus_{p=0}^t M_p/M_{p+1}$ as $GL_n(F)$ modules.

**Proof.** It is clear that $M_0 \subset ( R^G_n(W) )^{(t)}$. For the other inclusion, note that the free algebra $F \langle X^G_n \rangle$ is a universal enveloping of the free Lie algebra on $X^G_n$. Hence, according to Poincare-Birkhoff-Witt theorem, if $l_1, l_2, \ldots$ is a totally ordered basis of the free Lie algebra, then $F \langle X^G_n \rangle$ has a basis $\{ l_1^{\beta_1} \ldots l_r^{\beta_r} | \beta_i \geq 0 \}$. In our case we choose $l_i = x_{i,e}^* t_{i}$ for $i = 1, \ldots, n$, and the other basis elements are arranged in an order such that $\deg(l_j) \leq \deg(l_{j+1})$ for every $j > n$. We conclude that there is a basis $K$ of $F \langle X^G_n \rangle$ consisting of polynomials of the form $x_{1,e}^{\alpha_1} \ldots x_{n,e}^{\alpha_n} t_{1}^{h_1} \cdots t_{r}^{h_r} [x_{r+1,g_1}, \ldots] \cdots [x_{s,g_s}]$ which is in $R^{(s)}_{n,e} D^{(s)}_n$ since $h_i \neq e$ for every $i = 1, \ldots, r$. Moreover, the quotient map is a $G$-graded homomorphism from $F \langle X^G_n \rangle^{(t)}$ onto $( R^G_n(W) )^{(t)}$, thus $( R^G_n(W) )^{(t)}$ is spanned by the $t$-homogenous elements of $K$ (modulo the identities).

For the second part recall that every $GL_n(F)$ representation is completely reducible, i.e. $( R^G_n(W) )^{(t)} = \bigoplus_{i \in I} V_i$ where $I$ is a finite set of indexes and $V_i$ is irreducible $GL_n(F)$ representation for every $i \in I$. Moreover, $Q_s = \bigoplus_{i \in I_s} V_i$ where $I_s$ is a subset of $I$, and $M_p = \bigoplus_{i \in \cup_{s=p}^{t} I_s} V_i$. Therefore, $M_p/M_{p+1} \cong \bigoplus_{i \in I_p} V_i$ where $I_p = \bigcup_{s=p}^{t} I_s - \bigcup_{s=p+1}^{t} I_s$. So to prove the second part we need to show that $I$ is a disjoint union of $I_p$ where $p = 0, \ldots, t$. The inclusion $\bigcup_{p=0}^{t} J_p \subset I$ is trivial. For the other direction, recall that by part one $I = \bigcup_{s=0}^{t} I_s$, so for every $i \in I$ there is a minimal $p$ such that $i \in I_p$. Hence $i$ belongs to $\bigcup_{s=p}^{t} I_s$ and not to $\bigcup_{s=p+1}^{t} I_s$. By definition, $i \in J_p$, and we conclude that $I = \bigcup_{p=0}^{t} J_p$. To show that the $J_p$’s are disjoint suppose that there is an index $i$ in $J_p \cap J_{p'}$ for $p > p'$. The set $J_p$ is a subset of $I_p$, and $p \geq p' + 1$, thus $i \in \bigcup_{s=p'+1}^{t} I_s$. Hence, by definition, $i \notin J_{p'}$, which is a contradiction. \[\square\]

The next step is to find a basis of $M_p/M_{p+1}$ for every $p \leq t$. In order to achieve this, we need some preparations:

**Lemma 3.2.** Let $\{y_1, \ldots, y_k\}$ be a set of non commutative variables where $k \geq 2$, then we have the following identity

\[(3.1) \quad y_1 \cdots y_k = y_2 y_1 y_3 \cdots y_k + \sum_{i=0}^{k-2} y_{i+1} \cdots y_k [y_1, \ldots, y_{k-i}] \]
where the inner sum is over some subset of

\[ \{(j_1, \ldots, j_i, l_1, \ldots, l_{k-i}) \in \mathbb{N}^k \mid (j_1, \ldots, j_i, l_1, \ldots, l_{k-1}) = \{1, \ldots, k\}\}. \]

**Proof.** By induction on \( k \). The base of the induction, \( k = 2 \), is true by the definition of the commutator \( y_1y_2 = y_2y_1 + [y_1, y_2] \). For the induction step let us assume that the claim is true for \( k \). So, by multiplying the identity \[3.1\] from the right by \( y_{k+1} \) we get:

\[
y_1 \cdots y_{k+1} = y_2y_1y_3 \cdots y_{k+1} + \sum_{i=0}^{k-2} \sum_{j_i} y_{l_1} \cdots y_{l_{k-i}} [y_{l_1}, \ldots, y_{l_{k-i}}] y_{k+1}.
\]

We complete the proof by using the commutator definition \( zw = wz + [z, w] \) with \( z = [y_1, \ldots, y_{k-1}] \) and \( w = y_{k+1} \) for every element in the sum. \( \square \)

**Definition 3.3.** We say that \( f \in F\langle X_n^G \rangle \) is \( g \) multi-homogenous of degree \( \gamma = (\gamma_1, \ldots, \gamma_n) \), if substituting the variables \( x_{i,g} \) by \( t_i x_{i,g} \) in \( f \) gives

\[
\left( \prod_{i=1}^n t_i^{\gamma_i} \right) f.
\]

In other words for every \( i = 1, \ldots, n \) the variable \( x_{i,g} \) appears \( \gamma_i \) times in every monomial of \( f \). We denote the \( g \) multi-homogenous degree of polynomial \( f \) by \( \text{deg}_g(f) \).

It is known that for any \( g \in G \) if \( f = \sum f_{\gamma} \) is a \( G \)-graded identity, where \( f_{\gamma} \) is \( g \) multi-homogenous of degree \( \gamma \), then \( f_{\gamma} \) is also \( G \)-graded identity for every \( \gamma \). Therefore, the decomposition \( B_n^{(p)} = \bigoplus_{\gamma} \left( B_n^{(p)} \right)_{\gamma} \) where \( \left( B_n^{(p)} \right)_{\gamma} \) is the subspace of \( B_n^{(p)} \) consisting of \( g \) multi-homogenous polynomials of degree \( \gamma \). We may therefore assume that for every \( g \in G \) there is a basis \( \{ f_{j}^{(p)}(x_{l,h}) \mid j = 1, \ldots, \dim_F B_n^{(p)} \} \) of \( B_n^{(p)} \) consisting of \( g \) multi-homogenous polynomials. Now we are ready to present the basis of \( M_{p}/M_{p+1} \).

**Proposition 3.4.** For every \( p \leq t \) let \( \{ f_{j}^{(p)}(x_{l,h}) \mid j = 1, \ldots, \dim_F B_n^{(p)} \} \) be a basis of \( B_n^{(p)} \) consisting of \( e \) multi-homogenous polynomials. Then:

1. For every \( 1 \leq j \leq \dim_F B_n^{(p)} \) and \( \sigma \in S_{t-p} \)
   
   \[
x_{i_1,e} \cdots x_{i_{t-p},e} f_{j}^{(p)}(x_{l,h}) \equiv x_{i_1,\sigma(1),e} \cdots x_{i_{t-p},\sigma(t-p),e} f_{j}^{(p)}(x_{l,h}) \pmod{M_{p+1}}
   \]

2. The set
   
   \[
   U_p = \left\{ x_{i_1,e}^{\alpha_1} \cdots x_{n,e}^{\alpha_n} f_{j}^{(p)}(x_{l,h}) \mid \left| \alpha \right| = t-p, j = 1, \ldots, \dim_F B_n^{(p)} \right\}
   \]
   is a basis of \( M_{p}/M_{p+1} \).

**Proof.** To prove the first part it is enough to show that for every \( 1 \leq r \leq t-p-1 \)

\[
x_{i_1,e} \cdots x_{i_{t-r},e} f_{j}^{(p)}(x_{l,h}) \equiv x_{i_1,e} \cdots x_{r+1,e} x_{r,e} \cdots x_{i_{t-p},e} f_{j}^{(p)}(x_{l,h}) \pmod{M_{p+1}}.
\]

Recall that \( f_{j}^{(p)} \) is a linear combination of polynomials of the form

\[
x_{i_1,h_1} \cdots x_{i_q,h_q} [x_{l_1,q_1}, \ldots, l_{p-q,q}] \text{ where } h_i \neq e.
\]

Hence, by using Lemma \[3.2\] on
\( x_{r,e} \cdots x_{i_t,e} x_{i_{t-1},e} \cdots x_{i_q,h_q} \) for every such monomial, we obtain
\[
x_{i_1,e} \cdots x_{i_{t-1},e} f_j^{(p)}(x_{l,h}) = x_{i_1,e} \cdots x_{i_{r-1},e} x_{i_{r},e} \cdots x_{i_{t-1},e} f_j^{(p)}(x_{l,h}) + P(x_{l,h})
\]
where \( P(x_{l,h}) \) is a linear combination of polynomials of the form
\[
x_{j_1,e}, \ldots, x_{j_{t-q-1},e} x_{i_{1},h_1} \cdots x_{i_{q},h_q} [x_{i_{1},g_1}, \ldots, [\ldots, x_{l,k,g_k}]] \text{ with } k + q > p. \text{ Thus } P(x_{l,h}) \in M_{p+1},
\]
and the first part is proven.

Obviously \( U_p \) spans \( M_p/M_{p+1} \). It remains to be show that \( U_p \) is linearly independent, so let
\[
T = \sum_{j, \alpha} \beta_{j, \alpha} x_{1,e}^{\alpha_1} \cdots x_{n,e}^{\alpha_n} f_j^{(p)}(x_{l,h})
\]
be a nontrivial linear combination of the elements in \( U_p \). Assume by contradiction that \( T \) is an identity of \( W \). Therefore \( T_\gamma \), the \( e \) multi-homogenenous component of degree \( \gamma \) of \( T \), is also an identity for every \( \gamma \in \mathbb{N}^n \). We choose \( \gamma \) such that \( T_\gamma \) is a nontrivial linear combination, and write
\[
T_\gamma = \sum_j \sum_{\alpha = \gamma - \deg(f_j^{(p)})} \beta_{j, \alpha} x_{1,e}^{\alpha_1} \cdots x_{n,e}^{\alpha_n} f_j^{(p)}
\]
where the first sum is over every \( 1 \leq j \leq \dim_F B_n^{(p)} \) such that \( \deg(f_j^{(p)}) \leq \gamma_i \) for every \( i = 1, \ldots, n \) (note that every index \( j \) is appearing at most once in the sum).

Now, let substitute \( x_{i,e} \) by \( x_{i,e} + 1 \) in \( T_\gamma \) and denote the obtained identity by \( S \). Notice that since \( [y + 1, z] = [y, z] \) the polynomials \( f_j^{(p)} \) are stay the same under this substitution (in the part which is not commutator products in \( f_j^{(p)} \) there are no \( e \)-homogenenous variables). Therefore, the \( p \)-homogenous component of \( S \) is
\[
\sum_j \beta_{j, \alpha} f_j^{(p)}
\]
which is also an identity of \( W \). But \( f_j^{(p)} \) are linear independent (modulo the identities) and not all the \( \beta_{j, \alpha} \) are zero, and this is a contradiction. \( \square \)

**Corollary 3.5.**

1. \( R_n^G(W) \cong F[z_1, \ldots, z_n] \otimes B_n \) as \( GL_n(F) \) representation.
2. \( c_n^G(W) = \sum_{s=0}^{n} \binom{n}{s} \delta_s \) where \( \delta_s = \dim_F B_s^{(1^s)} \).

**Proof.** By 3.4 the mapping from \( M_p/M_{p+1} \) to \( F[z_1, \ldots, z_n]^{(t-p)} \otimes B_n^{(p)} \) taking \((\prod_{i=1}^{n} x_{i,e}^{\alpha_i}) f_j^{(p)}\) to \((\prod_{i=1}^{n} z_i^{\alpha_i}) \otimes f_j^{(p)}\) is a \( GL_n(F) \) representation isomorphism, thus by Lemma 3.1 \( R_n^G(W)^{(t)} \cong (F[z_1, \ldots, z_n] \otimes B_n)^{(t)} \) for every \( t \).

For the proof of the second part, note that
\[
C_n^G(W) \cong (F[z_1, \ldots, z_n] \otimes B_n)^{(1^n)} = \bigoplus_{\beta \in \{0,1\}^n} F[z_1, \ldots, z_n]^{(1^n)-\beta} \otimes B_n^{\beta}
\]
The dimension of $F[z_1, \ldots, z_n]^{1^\beta}$ is 1 for every $\beta \in \{0, 1\}^n$. Furthermore, it is easy to see that $B_n^{(\beta)} \cong B_n^{(\beta)}$ for every $\beta \in \{0, 1\}^n$, and there are $\binom{n}{s}$ vectors $\beta \in \{0, 1\}^n$ such that $|\beta| = s$. Thus $\text{dim}_F C_n^G(W) = c_n^G(W) = \sum_{s=0}^{n} \binom{n}{s} \delta_s$. □

With this corollary we can write $H_{R_n^G(W)} = H_{F[z_1, \ldots, z_n]} \cdot H_{B_n}$.

Recall that $H_{R_n^G(W)}(t_1, \ldots, t_n) = \sum_{\lambda} m_{\lambda} s_{\lambda}(t_1, \ldots, t_n)$ and $H_{B_n}(t_1, \ldots, t_n) = \sum_{\lambda} a_{\lambda} s_{\lambda}(t_1, \ldots, t_n)$.

Moreover, the Hilbert series of $F[z_1, \ldots, z_n]$ is known to be

$$\sum_{\alpha \in \mathbb{N}^n} t_1^{\alpha_1} \cdots t_n^{\alpha_n} = \prod_{i=1}^{n} (1 - t_i)^{-1} = \sum_{k \in \mathbb{N}} s(k)(t_1, \ldots, t_n),$$

thus we have the identity

$$\sum_{\lambda} m_{\lambda} s_{\lambda} = \sum_{\mu \in L_{\lambda}} \sum_{k \in \mathbb{N}} a_{\mu} s_{\mu} s(k).$$

This identity gives us the following arithmetic connection between $m_{\lambda}$ and $a_{\mu}$.

**Proposition 3.6.** For every partition $\lambda$

$$m_{\lambda} = \sum_{\mu \in L_{\lambda}} a_{\mu},$$

where $L_{\lambda}$ is the set of all partition $\mu$ such that $\lambda_{i+1} \leq \mu_i \leq \lambda_i$ for every $i = 1, \ldots, h(\mu)$.

**Proof.** By Pieri’s formula (see [M]) $s_{\mu} s(k) = \sum_{\nu} s_{\nu}$ where the sum is over all partitions $\nu$ obtained from $\mu$ by adding $k$ boxes to its Young’s diagram, no two of them in the same column. We have to show that $s_{\lambda}$ appearing in the sum on right hand side of $s_{\mu} s(\lambda-\mu) = \sum_{\nu} s_{\nu}$ if and only if $\lambda_{i+1} \leq \mu_i \leq \lambda_i$ for every $i$.

Suppose that $s_{\lambda}$ appearing in the sum on the right hand side of $s_{\mu} s(\lambda-\mu) = \sum_{\nu} s_{\nu}$. It is obvious that $\mu_i \leq \lambda_i$ for every $i$, since we are adding boxes to $\mu$ in order to get $\lambda$.

Let us assume that there is an index $i$ such that $\mu_i < \lambda_{i+1} \leq \lambda_i$. In the $\mu_i + 1$ column, then, we necessarily added two boxes which is a contradiction. (see Figure 3.1)

![Figure 3.1](image)

For the other direction, let $\lambda$ and $\mu$ be partitions such that $\lambda_{i+1} \leq \mu_i \leq \lambda_i$ for every $i$. Suppose that in the $j$’s column we added two boxes in the $k$’s and $i$’s row where $k > i$. This implies that $\mu_i < j \leq \lambda_k \leq \lambda_{i+1}$ and this is a contradiction. □
The conclusion of the above proposition is that the coefficients \( a_\lambda \) inherit two properties from the coefficients \( m_\lambda \). Here is the first one.

**Corollary 3.7.** There exists an integer \( q \) such that \( a_\lambda = 0 \) if \( \lambda \notin \Lambda^q \).

**Proof.** Recall from Section 1 that \( m_\lambda = 0 \) if \( \lambda \notin \Lambda^{dim_F(A_W)} \) where \( A_W \) is the finite dimensional algebra with the same \( G \)-graded identities as \( W \).

Let \( \mu \) be a partition such that \( h(\mu) > dim_F(A_W) - 1 \) (i.e. \( \mu \notin \Lambda^{dim_F(A_W)-1} \)), and define \( \lambda = (\mu_1, ..., \mu_{h(\mu)}; 1) \). Since \( h(\lambda) = h(\mu) + 1 > dim_F(A_W) \) we know that \( m_\lambda = \sum_{\nu \in L_\lambda} a_\nu = 0 \), so \( a_\nu = 0 \) for any \( \nu \in L_\lambda \). Moreover, it is easy to see that \( \lambda_{i+1} \leq \mu_i \leq \lambda_i \) for every \( i \) thus \( \mu \in L_\lambda \). \( \square \)

By this corollary, and since \( H_{B_n}(t_1, ..., t_n) = \sum_{\lambda \in \Lambda^n} a_\lambda s_\lambda(t_1, ..., t_n) \) for every \( n \), we conclude that there is a single Hilbert series (here it is \( H_{B_q}(t_1, ..., t_q) \)) which determines all the coefficients \( a_\lambda \). The second corollary of 3.6 is that, as in \( R_n^G(W) \) (see Section 1), the series \( H_{B_q}(t_1, ..., t_q) \) satisfies the first assumption of Theorem 1.5.

**Corollary 3.8.** Let \( H_{B_q}(t_1, ..., t_q) = \sum \alpha_\lambda s_\lambda(t_1, ..., t_q) \). Then there is an integer \( r \) such that \( a_\lambda = 0 \) for every partition \( \lambda \) satisfying \( \lambda_{r+1} \geq M \) where \( M \) is a constant depend only on \( r \).

Moreover the integer \( r = \exp^G(W) - 1 \) is the minimal satisfying this condition.

**Proof.** From \([Go]\) there exists an integer \( M \) such that \( m_\lambda = 0 \) for every partition \( \lambda \) satisfies \( \lambda_{l+1} \geq M \) where \( l = \exp^G(W) \). Suppose that \( \mu \) is a partition such that \( \mu_l \geq M \). We define a partition \( \lambda = (\mu_1, ..., \mu_l, \mu_{l+2}, ..., \mu_{h(\mu)}) \). Obviously, \( \lambda_{l+1} = \mu_l \geq M \), so \( m_\lambda = \sum_{\nu \in L_\lambda} a_\nu = 0 \), so \( a_\nu = 0 \) for any \( \nu \in L_\lambda \). Moreover, one can check that \( \lambda_{i+1} \leq \mu_i \leq \lambda_i \) for every \( i \) so \( \mu \in L_\lambda \).

To show the minimality of \( l - 1 \), we have to show that for every integer \( p \) there is a partition \( \mu \) with \( \mu_{l-1} \geq p \) such that \( a_\mu \neq 0 \). We know from \([Go]\) that for every integer \( p \) there is a partition \( \mu \) with \( \lambda_l \geq p \) such that \( m_\lambda = \sum_{\nu \in L_\lambda} a_\nu \neq 0 \). Hence, there is \( \mu \in L_\lambda \), which satisfies \( \mu_{l-1} \geq \lambda_l \geq p \), such that \( a_\mu \neq 0 \). \( \square \)

The Hilbert series \( H_{B_q} \) is also nice rational because \( H_{R_n^G(W)} \) is a nice rational function, and since 3.2 imply that

\[ H_{B_q}(t_1, ..., t_q) = \prod_{i=1}^{q} (1-t_i)H_{R_n^G(W)}(t_1, ..., t_q). \]

We get that the conclusion of Theorem 1.5 holds for the series \( \{\delta_s\}_{s \in \mathbb{N}} \), and together with 3.5

\[ c_n^G(W) \sim \sum_{m=0}^{d-1} \sum_{0 \leq s \leq n} \binom{n}{s} \alpha_m s^\beta m (l-1)^s \]

\( s = m \mod d \)
where $\beta_m \in \frac{1}{2}\mathbb{Z}$ for every $m$. To estimate this sum, let $\omega$ be a primitive $d^{th}$ root of unit and note that $\sum_{t=0}^{d-1} \omega^{(s-m)t}$ is zero if $s \neq m (mod\ d)$, and is $d$ if $s = m (mod\ d)$. Thus,

$$c_n^G(W) \sim \sum_{m=0}^{d-1} \sum_{0 \leq s \leq n} \binom{n}{s} \alpha_m s^\beta_m (l-1)^s$$

where $s = m (mod\ d)$

$$= \frac{1}{d} \sum_{m=0}^{d-1} \sum_{s=0}^{n} \sum_{t=0}^{d-1} \omega^{(s-m)t} \binom{n}{s} \alpha_m s^\beta_m (l-1)^s$$

$$= \sum_{m=0}^{d-1} \sum_{t=0}^{d-1} \frac{\alpha_m}{d} \omega^{-mt} \sum_{s=0}^{n} \binom{n}{s} s^\beta_m (\omega^t(l-1))^s$$

By lemma 1.1 in [BeR], for every $m$ and $t$ the expression $\sum_{s=0}^{n} \binom{n}{s} s^\beta_m (\omega^t(l-1))^s$ asymptotically equal to:

$$\mu_{m,t} n^\beta_m (\omega^t(l-1) + 1)^n$$

where $\mu_{m,t}$ is a constant which does not depend on $n$. Moreover, the absolute value of the expression $\omega^t(l-1) + 1$ is maximal when $t = 0$, so

$$c_n^G(W) \sim \alpha n^\beta l^n$$

where $\beta = \max(\beta_m) \in \frac{1}{2}\mathbb{Z}$. And theorem B is proven.

REFERENCES

[AB1] E. Aljadeff and A. Kanel-Belov, Hilbert series of PI relatively free G-graded algebras are rational functions. Bull. Lond. Math. Soc. 44 (2012), no. 3, 520–532.

[AB2] E. Aljadeff and A. Kanel-Belov, ‘Representability and Specht problem for G-graded algebras’, Adv. Math. 225 (2010) 2391–2428

[AG] E. Aljadeff and A. Giambruno, Multialternating graded polynomials and growth of polynomial identities, to appear in Proc. Amer. Math. Soc.

[AGL] E. Aljadeff, A. Giambruno and D. La Mattina, Graded polynomial identities and exponential growth, J. Reine Angew. Math. 650 (2011), 83–100.

[BeR] W.Beckner and A.Regev, Asymptotic estimates using probability. Adv. Math. 138 (1998), no. 1, 1–14.

[BRo] A. Kanel-Belov and L.H. Rowen, Computational aspects of polynomial identities. Research Notes in Mathematics, 9. A K Peters, Ltd., Wellesley, MA, 2005.

[BR] A.Berele, and A.Regev, Asymptotic behaviour of codimensions of p. i. algebras satisfying Capelli identities. Trans. Amer. Math. Soc. 360 (2008), no. 10, 5155–5172.

[Bru] Bruce .E. Sagan, The Symmetric Group: Representations, Combinatorial Algorithms and Symmetric Functions, Section 4.4, 155-157

[D] V. Drensky, Codimensions of T-ideals and Hilbert series of relatively free algebras, J. of Alg. 89 (1984), 178–223. MR765766 (86b:16010)

[GL] A. Giambruno and D. La Mattina, Graded polynomial identities and codimensions: computing the exponential growth. Adv. Math. 225 (2010), no. 2, 859–881.
[GZ1] A. Giambruno and M. V. Zaicev, On codimension growth of finitely generated associative algebras, Adv. Math. 140 (1998), 145–155.

[GZ2] Giambruno, Antonio; Zaicev, Mikhail, Polynomial identities and asymptotic methods. Mathematical Surveys and Monographs, 122. American Mathematical Society, Providence, RI, 2005.

[GZ3] A. Giambruno; M. V. Zaicev, Growth of polynomial identities: Is the sequence of codimensions eventually nondecreasing? Bull. Lond. Math. Soc. 46 (2014), no. 4, 771–778.

[Go] A. S. Gordienko, Amitsur’s conjecture for associative algebras with a generalized Hopf action. J. Pure Appl. Algebra 217 (2013), no. 8, 1395–1411.

[M] I. G. Macdonald, (1995), Symmetric functions and Hall polynomials, Oxford Mathematical Monographs (2nd ed.), The Clarendon Press Oxford University Press

[St] R. Stanley, Combinatorics and Commutative Algebra, Birkhäuser, Boston, 1983.

[Stu] B. Sturmfels, On vector partition functions, J. Combin. Theory Ser. A 72 (1995) 302–308.

[Sv] I. Sviridova, Identities of pi-algebras graded by a finite abelian group. (English summary) Comm. Algebra 39 (2011), no. 9, 3462–3490.

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