The Chromatic Quasisymmetric Class Function of a Digraph

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Abstract. We introduce a quasisymmetric class function associated with a group acting on a double poset or on a directed graph. The latter is a generalization of the chromatic quasisymmetric function of a digraph introduced by Ellzey, while the former is a generalization of a quasisymmetric function introduced by Grinberg. We prove representation-theoretic analogues of classical and recent results, including $F$-positivity, and combinatorial reciprocity theorems. We deduce results for orbital quasisymmetric functions, and study a generalization of the notion of strongly flawless sequences.

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1. Introduction

Given a graph $G$, let $\mathcal{G}$ be a subgroup of the automorphism group of $G$. Then, $\mathcal{G}$ acts on the set of $k$-colorings of $G$. If we let $\chi_{\mathcal{G}}(G, k)$ denote the number of orbits of this action, then the resulting function is a polynomial in $k$, called the orbital chromatic polynomial and studied by Cameron and Kayibi [5]. Jochemko [13] found a combinatorial reciprocity theorem by giving a combinatorial interpretation to $(-1)^n \chi_{\mathcal{G}}(G, -k)$, where $n$ is the number of vertices of $G$.

Similarly, given a poset $P$, a subgroup $\mathcal{G}$ of the automorphism group of $P$ acts on the set of order-preserving maps $\varphi : P \to \{1, \ldots, k\}$. If we let $\Omega_{\mathcal{G}}(P, k)$ denote the number of orbits of this action, we obtain the orbital order polynomial that was introduced by Jochemko [13], who proved a combinatorial reciprocity theorem for $\Omega_{\mathcal{G}}(P, k)$. These results were later generalized to quasisymmetric functions associated with a double poset by Grinberg [10]. One of
our primary interests is proving combinatorial reciprocity theorems for orbital polynomial invariants associated with combinatorial objects.

Stapledon [19] studied the equivariant Ehrhart quasipolynomial of a polytope. Let $\mathfrak{G}$ be a finite group acting linearly on a lattice $M'$ of rank $n$, and let $P$ be a $d$-dimensional $\mathfrak{G}$-invariant lattice polytope. Let $M$ be a translation of the intersection of the affine span of $P$ and $M'$ to the origin, and consider the induced representation $\rho : \mathfrak{G} \to GL(M)$. If $\chi(m)$ is the permutation character associated with the action of $G$ on the lattice of points in the $m$th dilate of $P$, then $\chi(m)$ is a quasipolynomial in $m$ whose coefficients are elements of $R(\mathfrak{G})$, the ring of virtual characters of $\mathfrak{G}$. Stapledon proved several results concerning the equivariant Ehrhart quasipolynomial.

Motivated by these past results, we study quasisymmetric class functions. These are class functions associated with the symmetry group $\mathfrak{G}$ of a combinatorial object, whose values are quasisymmetric functions. Equivalently, they are quasisymmetric functions whose coefficients are class functions. If we let $\mathfrak{G}$ be the trivial group, then we obtain ordinary quasisymmetric functions (and should re-derive classical results). In fact, we can always obtain the ordinary quasisymmetric function by evaluating all characters at the identity element. We obtain corresponding orbital quasisymmetric functions, and various polynomial specializations.

The goal of this paper is to study a quasisymmetric class function generalization of the $D$-partition enumerator of a double poset and of the chromatic polynomial of a directed graph. The former is a class function generalization of an invariant introduced by Grinberg, which in turn is a generalization of the labeled $P$-partition enumerator studied by Gessel [9]. We define double posets and related terminology in Sect. 3. Given a double poset $D$ on a set $N$, let $\mathfrak{G}$ be a subset of the automorphism group of $D$. Then, $\mathfrak{G}$ acts on the set of $D$-partitions. For $g \in \mathfrak{G}$, we define

$$\Omega(D, \mathfrak{G}, x; g) = \sum_{\sigma : g\sigma = \sigma} \prod_{v \in N} x_{\sigma(v)},$$

where we are summing over $D$-partitions fixed by $g$. Then, $\Omega(D, \mathfrak{G}, x)$ is a QSYM-valued class function.

Stanley introduced the chromatic symmetric function [18], a symmetric function generalization of the chromatic polynomial. This has been generalized to a chromatic quasisymmetric function by Shareshian and Wachs [16] and to directed graphs by Ellzey [7]. We will study a class function generalization of Ellzey’s invariant, defined more explicitly in Sect. 4. Much like the generalization of Shareshian and Wachs, our invariant has an extra variable $t$: our invariant is a class function that takes on values in the ring of quasisymmetric functions over the field $\mathbb{C}(t)$. Given a digraph $G$ on a set $N$, let $\mathfrak{H}$ be a subset of the automorphism group of $G$. Then, $\mathfrak{H}$ acts on the set of proper colorings of the underlying undirected graph. Ellzey defines a statistic $\text{asc}(f)$ for a coloring. We show that this statistic is $\mathfrak{H}$-invariant. For $g \in \mathfrak{H}$, we define

$$\chi(G, \mathfrak{H}, t, x; g) = \sum_{f : g\cdot f = f} t^{\text{asc}(f)} \prod_{v \in N} x_{f(v)},$$
where we sum over proper colorings of $G$. Much like in the case of double posets, the resulting invariant is a class function whose values are quasisymmetric functions over $\mathbb{C}[t]$.

Our primary interest is to study generalizations of $F$-positivity results, inequalities, and combinatorial reciprocity theorems. Let $C(G, \text{QSYM})$ be the set of class functions with values and coefficients in QSYM. If we take an element $\chi(x)$ of $C(G, \text{QSYM})$ and a given basis $B$ for quasisymmetric functions, we say that $\chi(x)$ is $B$-effective if $\chi(x)$ can be expressed in the basis $B$ with coefficients that are characters of representations of $G$, then $R = \{\chi_i B_\alpha : i \in [k], B_\alpha \in B\}$ forms a basis for $C(G, \text{QSYM})$. If $\chi(x)$ is $B$-effective, then $\chi(x)$ can be expressed as a linear combination of $\chi_i B_\alpha$ with non-negative integer coefficients.

We prove that the $D$-partition quasisymmetric class function for locally special posets is $F$-effective in Theorem 5. The notion of locally special was first introduced by Grinberg, under the name tertispecial. He suggests locally special as an alternative name. Our results specialize to both known and new results in the literature. We give a proof that the corresponding orbital $D$-partition enumerator is $F$-positive. This implies that locally special double posets have $F$-positive $D$-partition enumerators, which appears to be new. We prove in Theorem 6 that $\chi(G, \mathfrak{H}, t, x)$ is $F$-effective.

We study polynomial invariants $\Omega(D, \mathfrak{G}, x)$ and $\chi(G, \mathfrak{F}, t, x)$ as well. There are lots of results about inequalities for coefficients of chromatic polynomials of graphs with respect to different bases, including recent work that the coefficients of $(-1)^n \chi(G, -x)$ are unimodal [12] and strongly flawless [14]. Given a sequence $(f_0, \ldots, f_d)$, we say the sequence is strongly flawless if the following inequalities are satisfied:

1. For $0 \leq i \leq \frac{d-1}{2}$, we have $f_i \leq f_{i+1}$.
2. For $0 \leq i \leq \frac{d}{2}$, we have $f_i \leq f_{d-i}$.

For this paper, we are focused on the sequence of coefficients for a polynomial $p(x)$ with respect to the basis $(x^k)$. We refer to these coefficients as the $f$-vector, and say $p(x)$ is strongly flawless if the $f$-vector is strongly flawless and non-negative. We introduce a representation-theoretic generalization: we require $f_i$ to be effective characters, and we interpret inequalities of the form $f_i \leq f_k$ as saying that $f_k - f_i$ is also a character. We refer to such a sequence of characters as effectively flawless. We show that $\Omega(D, \mathfrak{G}, x)$ and $\chi(G, \mathfrak{F}, t, x)$ are effectively flawless in Sect. 6.

We discuss combinatorial reciprocity theorems. In [17], Stanley defines a combinatorial reciprocity theorem as ‘a result which establishes a kind of duality between two enumeration problems’. The book by Beck and Sanyal [4] is full of many examples of such results. In general, we suppose that we have a vector subspace $V$ of a ring of formal power series, and that $V$ comes equipped with an involution $\omega$. We have a subset $C \subset V$ that consists of combinatorial generating functions. Given two generating functions $f, g \in C$, a combinatorial reciprocity theorem is the statement that $f = \omega g$. This is more general than

the examples that appear in Beck and Sanyal’s book, but still fits the general notion Stanley originally proposed.

In this paper, $V$ is usually a vector space of class functions from a group $G$ to quasisymmetric functions of a fixed degree $d$, and $\omega = (-1)^d S \text{sgn}$, where $S$ is the antipode for the Hopf algebra of quasisymmetric functions and sgn is the character of the sign representation. We say a quasisymmetric class function $p(x)$ is $M$-realizable if the coefficients in the $M$ basis are permutation characters. We let $C$ be the set of $M$-realizable quasisymmetric class functions. Hence, a combinatorial reciprocity theorem for a quasisymmetric class function $p(x) \in C$ consists of showing that the coefficients of $(-1)^d S \text{sgn} p(x)$ in the $M$ basis are permutation characters. The sgn character arises naturally for similar combinatorial reciprocity statements in the works of Stapledon, Grinberg, and Jochemko. We are able to deduce combinatorial reciprocity theorems for corresponding orbital invariants, and for polynomial invariants as well.

We prove a combinatorial reciprocity theorem for double posets in Theorem 2, which involves taking duals of partial orders, and a combinatorial reciprocity theorem for digraphs in Theorem 3, which involves group actions on pairs $(O, f)$, where $O$ is an acyclic orientation and $f$ is a compatible coloring.

The paper is organized as follows. In Sect. 2, we define quasisymmetric functions, review some representation theory, and discuss set compositions. We discuss polynomials, and quasisymmetric class functions. In Sect. 3, we define double posets, $D$-partitions, and the corresponding $D$-partition quasisymmetric class function. Then, we prove some basic facts about $\Omega(D, G, x)$. We discuss some properties about locally special double posets that we need for later proofs. In Sect. 4, we define the chromatic quasisymmetric class function, and provide a formula expressing $\chi(G, f, t, x)$ in terms of quasisymmetric class functions related to double posets coming from acyclic orientations of $G$. In Sect. 5, we prove our combinatorial reciprocity theorems for $\Omega(D, G, x)$, and $\chi(G, f, t, x)$. In Sect. 6, we show that our polynomial invariants are effectively flawless. We show other properties about the quasisymmetric functions, and study some examples to show how these properties fail for $h$-vectors. In Sect. 7, we prove $F$-effectiveness for $\Omega(D, G, x)$ and $\chi(G, f, t, x)$. We establish the corresponding $h$-effectiveness for the related polynomial invariants, and deduce some $F$-positivity results as corollaries. In Sect. 8, we define our orbital quasisymmetric functions, and deduce facts about these invariants from the results we have obtained about the quasisymmetric class functions. Finally, in Sect. 9, we discuss some open problems.

2. Preliminaries

Given a basis $B$ for a vector space $V$ over $\mathbb{C}$, and $\vec{\beta} \in B, \vec{v} \in V$, we let $[\vec{\beta}]\vec{v}$ denote the coefficient of $\vec{\beta}$ when we expand $\vec{v}$ in the basis $B$.

Let $x = x_1, x_2, \ldots$ be a sequence of commuting indeterminates. Recall that an integer composition $\alpha$ of a positive integer $n$ is a sequence $(\alpha_1, \ldots, \alpha_k)$ of positive integers, such that $\alpha_1 + \cdots + \alpha_k = n$. We write $\ell(\alpha) = k$, and
α | n. Let n ∈ N and let $f(x) ∈ \mathbb{C}[[x]]$ be a homogeneous formal power series in x, where the degree of every monomial in $f(x)$ is n. Then, $f(x)$ is a quasisymmetric function if it satisfies the following property: for every $S = \{i_1, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$, and every integer composition $α \models n$ with $ℓ(α) = k$, we have $[\prod_{j=1}^{k} x_{i_j}^{α_j}]f(x) = [\prod_{j=1}^{k} x_{i_j}^{α_j}]f(x)$. Often, we will define quasisymmetric functions that are generating functions over functions. Given a function $w : S → N$, we define

$$x^w = \prod_{v ∈ S} x_{w(v)}.$$

For example, the chromatic symmetric function of a graph $G$ is defined as $\sum_{f : V → N} x^f$ where the sum is over all proper colorings of $G$.

Given an integer composition $α = (α_1, α_2, \ldots, α_k)$ of n, we let

$$M_α = \sum_{i_1 < \cdots < i_k} \prod_{j=1}^{k} x_{i_j}^{α_j}.$$n

These are the monomial quasisymmetric functions, which form a basis for the ring of quasisymmetric functions.

The second basis we focus on is the basis of fundamental quasisymmetric functions, first introduced by Gessel [9]. The set of integer compositions is partially ordered by refinement. We write $α ≤ β$ if $β$ is a refinement of $α$. Note that many authors use the opposite partial order. With respect to this partial order, the set of integer compositions forms a lattice. The fundamental quasisymmetric functions $F_α$ are defined by

$$F_α = \sum_{β ≥ α} M_β.$$n

There is an important linear transformation on quasisymmetric functions called the antipode

$$S(M_α) = (-1)^{ℓ(α)} \sum_{β ≤ α} M_β^\leftarrow,$n

where $β^\leftarrow$ is the composition given by reversing the order of $β$. Antipodes exist for any graded connected bialgebra, and are analogous to inversion for groups.

Our proofs rely a lot on working with set compositions, and quasisymmetric functions related to set compositions. Given a finite set $N$, a set composition is a sequence $(S_1, \ldots, S_k)$ of disjoint non-empty subsets whose union is $N$. We denote set compositions as $S_1|S_2|\cdots|S_k$, and refer to the sets $S_i$ as blocks. We use $C \models N$ to denote that $C$ is a set composition of $N$, and let $ℓ(C) = k$ be the length of the composition. Given $C$, the associated integer composition is

$$α(C) = (|C_1|, |C_2|, \ldots, |C_k|).$$

We refer to $α(C)$ as the type of $C$. We partially order set compositions by refinement. We write $B ≤ C$ if $C$ is a refinement of $B$. Finally, given a set composition $C$ of type $β$ and $α ≤ β$, let $C_α(C)$ be the unique set composition of type $α$, such that $C_α(C) ≤ C$. 
2.1. Group Actions and Class Functions

Given a group action \( \mathcal{G} \) on a set \( X \), we let \( X/\mathcal{G} \) denote the set of orbits. For \( x \in X \), \( \mathcal{G}_x \) is the stabilizer subgroup, and \( \mathcal{G}(x) \) is the orbit of \( x \). Also, a transversal is a subset \( T \subseteq X \), such that \( |T \cap O| = 1 \) for every orbit \( O \) of \( X \). Finally, for \( g \in \mathcal{G} \), we let \( \text{Fix}_g(X) = \{ x \in X : gx = x \} \).

There is an action of \( \mathcal{G}_N \) on the collection of all set compositions of \( N \). Given a permutation \( g \in \mathcal{G}_N \), and a set composition \( C \sqsupseteq N \), we let

\[
gC = g(C_1)|g(C_2)| \cdots |g(C_k),
\]

where \( g(C_i) = \{ gx : x \in C_i \} \).

We assume familiarity with the theory of complex representations of finite groups—see [8] for basic definitions. Recall that, given any group action of \( \mathcal{G} \) on a finite set \( X \), there is a group action on \( \mathbb{C}^X \) as well, which gives rise to a representation. The resulting representations are called permutation representations. Let \( R \) be a \( \mathbb{C} \)-algebra. Then, an \( R \)-valued class function is a function \( \chi : \mathcal{G} \rightarrow R \), such that, for every \( \mathbf{g}, \mathbf{h} \in \mathcal{G} \), and \( \chi \in C(\mathcal{G}, R) \), we have \( \chi(\mathbf{hgh}^{-1}) = \chi(\mathbf{g}) \). Let \( C(\mathcal{G}, R) \) be the set of \( R \)-valued class functions from \( \mathcal{G} \) to \( R \). For our paper, \( R \) is usually QSYM or \( \mathbb{C}[x] \). We refer to \( \chi \in C(\mathcal{G}, \mathbb{C}) \) as class functions when no confusion arises.

There is an orthonormal basis of \( C(\mathcal{G}, \mathbb{C}) \) given by the characters of the irreducible representations of \( \mathcal{G} \). We refer to elements \( \chi \in C(\mathcal{G}, \mathbb{C}) \) that are integer combinations of irreducible characters as virtual characters, and elements that are non-negative integer linear combinations as effective characters. Finally, we say that \( \chi \) is a permutation character if it is the character of a permutation representation. We partially order \( C(\mathcal{G}, \mathbb{C}) \) by saying \( \chi \leq \mathcal{G} \psi \) if \( \psi - \chi \) is an effective character.

Let \( \mathbf{B} \) be a basis for \( R \). For \( b \in \mathbf{B} \), \( g \in \mathcal{G} \), and \( \chi \in C(\mathcal{G}, R) \), let \( \chi_b(g) = |b|\chi(g) \). Then, \( \chi_b \) is a \( \mathcal{C} \)-valued class function. Thus, we can write \( \chi = \sum_{b \in \mathbf{B}} \chi_b b \). Conversely, given a family \( \chi_b \) of \( \mathcal{C} \)-valued class functions, one for each \( b \in \mathbf{B} \) the function \( \chi \) defined by \( \chi(g) = \sum_{b \in \mathbf{B}} \chi_b(g)b \) is an \( R \)-valued class function in \( C(\mathcal{G}, R) \).

Let \( \chi \) be an \( R \)-valued class function. We say that \( \chi \) is \( \mathbf{B} \)-effective if \( \chi_b \) is an effective character for all \( b \in \mathbf{B} \). We say that \( \chi \) is \( \mathbf{B} \)-realizable if \( \chi_b \) is a permutation character for all \( b \). If \( \mathbf{B} \) has a partial order on it, then we say \( \chi \) is \( \mathbf{B} \)-increasing if, for all \( b \leq c \in \mathbf{B} \), we have \( \chi_b \leq \mathcal{G} \chi_c \). Assuming \( \chi \) is \( \mathbf{B} \)-effective, this is equivalent to saying that \( \chi_c \) is the character for the representation of \( \mathcal{G} \) on some module \( V \), and \( \chi_b \) is the character of a representation of a submodule of \( V \).

A quasisymmetric class function is a QSYM-valued class function. Given a quasisymmetric class function \( f(\mathcal{G}, \mathbf{x}) \), we can write \( f(\mathcal{G}, \mathbf{x}) = \sum_{\alpha \leq n} f_{\mathcal{G}, \alpha} M_{\alpha} \), where the \( f_{\mathcal{G}, \alpha} \in C(\mathcal{G}, \mathbb{C}) \). Thus, quasisymmetric class functions may also be viewed as quasisymmetric functions whose coefficients are class functions.

**Proposition 1.** Let \( \chi \in C(\mathcal{G}, \text{QSYM}) \) have degree \( d \). If \( \chi \) if \( F \)-effective, then \( \chi \) is \( M \)-increasing.
Proof. Let $\chi = \sum_{\alpha | d} \psi_\alpha F_\alpha$. Then, there exists $\mathcal{G}$-modules $W_\alpha$, such that $\psi_\alpha$ is the character of the representation of $\mathcal{G}$ on $W_\alpha$. If we let $V_\alpha = \bigoplus_{\beta \leq \alpha} W_\beta$, then $V_\alpha$ has character $\sum_{\beta \leq \alpha} \psi_\beta = [M_\alpha] \chi$.

Let $\alpha \leq \gamma$. Then, we see that $V_\alpha$ is a submodule of $V_\gamma$. Hence, $\chi_\gamma - \chi_\alpha$ is the character of the complement of $V_\alpha$ in $V_\gamma$. Thus, $\chi$ is $M$-increasing. \hfill $\square$

Given a subgroup $H$ of $\mathcal{G}$, and an $R$-valued class function class function $\chi \in C(H, R)$, we define the induced class function $\chi \uparrow H \in C(\mathcal{G}, R)$ by

$$\chi \uparrow H \mathrm{(}g\mathrm{)} = \frac{1}{|H|} \sum_{t \in H} \chi(t g^{-1}).$$

Finally, we define a function $\langle \cdot, \cdot \rangle : C(\mathcal{G}, R) \times C(\mathcal{G}, R) \to R$ by $\langle \chi, \psi \rangle = \frac{1}{|\mathcal{G}|} \chi(g) \psi(g)$ where $\pi$ is the complex conjugate. In the case where $R = \mathbb{C}$, this is the usual inner product on class functions.

Proposition 2. Let $\mathcal{G}$ be a finite group, and let $R$ be a $\mathbb{C}$-algebra with basis $B$. Fix $\chi \in C(\mathcal{G}, R)$.

1. For $b \in B$, we have $[b] \langle \chi \uparrow_{\mathcal{G}} \rangle = ([b] \chi) \uparrow_{\mathcal{G}}$.
2. Given an irreducible character $\psi$, we have $\langle \chi, \psi \rangle = \sum_{b,c \in B} \langle \chi_b, \psi_c \rangle b \cdot c$.
3. If $\chi$ is $B$-effective, and $\psi$ is an irreducible character, then $\langle \psi, \chi \rangle$ is $B$-positive.
4. Suppose $B$ is partially ordered. Let $\psi \in C(B, \mathbb{C})$. If $\chi$ is $B$-increasing, then for all $b \leq c$ in $B$, we have $[b] \langle \psi, \chi \rangle \leq [c] \langle \psi, \chi \rangle$.

Proof. Let $g \in \mathcal{G}$. Then

$$\chi \uparrow_{\mathcal{G}} \mathrm{(}g\mathrm{)} = \frac{1}{|\mathcal{G}|} \sum_{t \in \mathcal{G}} \chi(t g^{-1})$$

$$= \frac{1}{|\mathcal{G}|} \sum_{t \in \mathcal{G}} \sum_{b \in B} \chi_b(t g^{-1}) b$$

$$= \sum_{b \in B} \left( \frac{1}{|\mathcal{G}|} \sum_{t \in \mathcal{G}} \chi_b(t g^{-1}) \right) b$$

Thus, we see that the first result follows from comparing the coefficient of $b$ on both sides.

Let $\psi \in C(\mathcal{G}, R)$. Then

$$\langle \chi, \psi \rangle = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \chi(g) \psi(g)$$

$$= \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \left( \sum_{b \in B} \chi_b(g) b \right) \left( \sum_{c \in B} \psi_c(g) c \right)$$

$$= \sum_{b \in B} \sum_{c \in B} \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \chi_b(g) \psi_c(g) b \cdot c.$$
For the third claim, let $\psi$ be an irreducible character. Let $b \in B$. Using the second claim, we have $\langle b \rangle \langle \psi, \chi \rangle = \langle \psi, [b] \chi \rangle \geq 0$. Hence, $\langle \psi, \chi \rangle$ is $B$-positive.

Now, suppose that $\chi$ is $B$-increasing. Let $b \leq c \in B$. Then, there is a representation of $\mathfrak{S}$ whose character is $\rho := [c] \chi - [b] \chi$. Using the second claim, we see that

$$[c] \langle \psi, \chi \rangle - [b] \langle \psi, \chi \rangle = \langle \psi, [c] \chi - [b] \chi \rangle = \langle \psi, \rho \rangle \geq 0.$$  

Hence, $[c] \langle \psi, \chi \rangle \geq [b] \langle \psi, \chi \rangle$.

\begin{proof}
\end{proof}

2.2. Principal Specialization

Given a polynomial $p(x)$ of degree $d$, define $h(t) = (1 - t)^{d+1} \sum_{m \geq 0} p(m) t^m$. The sequence of coefficients of $h(t)$ is the $h$-vector of $p(x)$. We define the $f$-vector $(f_0, \ldots, f_d)$ via $p(x) = \sum_{i=0}^d f_i \binom{x}{i}$. We say that $p(x)$ is strongly flawless if the following inequalities are satisfied:

1. for $0 \leq i \leq \frac{d-1}{2}$, we have $f_i \leq f_{i+1}$.
2. For $0 \leq i \leq \frac{d}{2}$, we have $f_i \leq f_{d-i}$.

There is a lot of interest in log-concave and unimodal sequences in combinatorics. We consider strongly flawless sequences to be interesting, as strongly flawless unimodal sequences can be seen as a generalization of symmetric unimodal sequences. Examples of results with strongly flawless sequences include the work of Hibi [11] and Juhnke-Kubitzke and Van Le [14].

Given a quasisymmetric function $F(x)$ of degree $d$, there is an associated polynomial $ps(F)(x)$ given by principal specialization. For $x \in \mathbb{N}$, we set

$$x_i = \begin{cases} 1 & i \leq x \\ 0 & i > x. \end{cases}$$

The resulting sequence is a polynomial function in $x$ of degree $d$, which we denote by $ps(F)(x)$. We let $(f_0, \ldots, f_d)$ and $(h_0, \ldots, h_d)$ be the $f$-vector and $h$-vector of $ps(F)(x)$. If we write $F(x) = \sum_{\alpha \vdash d} c_\alpha M_\alpha$, then $f_i = \sum_{\alpha \vdash d: |\alpha| = i} c_\alpha$. Similarly, if we write $F(x) = \sum_{\alpha \vdash n} d_\alpha F_\alpha$, then $h_i = \sum_{\alpha \vdash d: |\alpha| = i} d_\alpha$.

The set $\mathbb{C}[x]$ is a Hopf algebra, with antipode given by $Sp(x) = p(-x)$. Also, $\varphi : \text{QSYM} \to \mathbb{C}[x]$ given by $\varphi(F(x)) = ps(F)(x)$ is a Hopf algebra homomorphism and $\varphi(SF(x)) = ps(F)(-x)$.

Let $F(x)$ be a quasisymmetric class function of degree $d$, and $g \in \mathfrak{S}$. Define $ps(F) \in C(\mathfrak{S}, \mathbb{C}[x])$ by $ps(F)(x; g) = ps(F(x; g))(x)$. We refer to $ps(F)$ as the principal specialization, which results in an polynomial class function. If we write $ps(F) = \sum_{i=0}^d f_i \binom{x}{i}$, then $(f_0, \ldots, f_d)$ is the equivariant $f$-vector of $ps(F)$, which consists of permutation characters. If we write $\sum_{m \geq 0} ps(F)t^m = \frac{h(t)}{(1-t)^{d+1}}$, then the coefficients of $h(t)$ are the equivariant $h$-vector of $ps(F)$. Note that the entries of the equivariant $h$-vector are virtual characters. We say that $ps(F)$ is $h$-effective if the entries are effective characters. We say that $ps(F)$ is effectively flawless if we have the following system of inequalities:

1. for $0 \leq i \leq \frac{d-1}{2}$, we have $f_i \leq \varnothing f_{i+1}$.
2. For $0 \leq i \leq \frac{d}{2}$, we have $f_i \leq \varnothing f_{d-i}$.
Proposition 3. Let $F(x)$ be a quasisymmetric class function be of degree $d$, and $g \in \mathcal{G}$.

1. If we write $F(x) = \sum_{\alpha \models d} \chi_\alpha M_\alpha$, then $\text{ps}(F) = \sum_{i=0}^{d} \chi_\alpha \left( \frac{x}{\ell(\alpha)} \right)$.
2. If we write $F(x) = \sum_{\alpha \models d} \psi_\alpha F_\alpha$, then $h(t) = \sum_{\alpha \models d} \psi_\alpha t^{\ell(\alpha)}$. If $F(x)$ is $F$-effective, then $\text{ps}(F)$ is $h$-effective.
3. If $G(x) \in C(\mathcal{G},\text{QSYM})$ with $(-1)^d \text{sgn} SF(x) = G(x)$, where $\text{sgn}$ is the character of the sign representation, then
   $$(-1)^d \text{sgn} \text{ps}(F)(-x) = \text{ps}(G)(x).$$
4. If $F(x)$ is $M$-realizable and $M$-increasing, then $\text{ps}(F)$ is effectively flawless.
5. Let $\psi$ be an irreducible character. If $F(x)$ is $F$-effective, then $\langle \psi, \text{ps}(F) \rangle$ is $h$-positive. If $F(x)$ is $M$-increasing, then $\langle \psi, \text{ps}(F) \rangle$ is strongly flawless.

Proof. The first three results are proven in a similar manner. Let $g \in \mathcal{G}$. Then,
$$\text{ps}(F)(x; g) = \text{ps}(F(x; g))(x).$$
Since $F(x; g) = \sum_{\alpha \models d} \chi_\alpha (g) M_\alpha$, we have
$$\text{ps}(F(x; g)) = \sum_{\alpha \models d} \chi_\alpha (g) \left( \frac{x}{\ell(\alpha)} \right).$$

The result follows.

For the fourth result, let $d$ be the degree of $F(x)$. For each $\alpha \models d$, let $V_\alpha$ be a $\mathcal{G}$-module with character $[M_\alpha]F(x)$. Since $F(x)$ is $M$-increasing, we know there exists injective $\mathcal{G}$-invariant functions $\theta_{\alpha,\beta} : V_\alpha \to V_\beta$ for every $\alpha \leq \beta \models d$. We let $V_i = \bigoplus_{\alpha \models d : \ell(\alpha) = i} V_\alpha$. Then, the character of $V_i$ is $f_i$. To show the inequalities, it suffices to find $\mathcal{G}$-invariant injections between $V_i$ and $V_j$. Then, $f_j - f_i$ is the character of the complement of $V_i$ in $V_j$.

We need to recall that the Boolean lattice, and hence the lattice of integer compositions, has a symmetric chain decomposition, a result due to DeBruijn [6]. Let $C(d)$ be the set of integer compositions of $d$. A symmetric chain decomposition is a partition of $C(d)$ into saturated chains $c_1, \ldots, c_m$ with the property that, for each chain $c_i$, the sum of the ranks of the first and last element of $c_i$ is $d$.

Fix a symmetric chain decomposition $D$. Fix integers $i$ and $j$ such that $1 \leq i < j \leq d - i$. Consider an integer composition $\alpha$ with $\ell(\alpha) = i$. Then, there exists a chain $x_1 < x_2 < \cdots < x_k$ in $D$ with $x_{i+\ell(x_1)+1} = \alpha$. Define $\varphi_{i,j}(\alpha) = x_j-\ell(x_1)+1$. We see that the following two facts are true:

1. If $i \leq \frac{d-1}{2}$, then $\varphi_{i,i+1}$ is injective.
2. If $i > \frac{d}{2}$, then $\varphi_{i,d-i}$ is a bijection.

Let $1 \leq i \leq \frac{d-1}{2}$, and let $\alpha \models d$ with $\ell(\alpha) = i$. We define $\theta_{i,j} : V_i \to V_j$ for requiring $\theta_{i,j}|_{V_\alpha} = \theta_{\alpha,\varphi_{i,j}(\alpha)}$. Then, $\theta_{i,j}(V_\alpha) \subseteq V_{\varphi_{i,j}(\alpha)}$.

Thus, $\theta_{i,j}$ is an injective $\mathcal{G}$-invariant map. Thus, $V_i$ is isomorphic to a submodule of $V_{i+1}$, and we have $f_i \leq \mathcal{G} f_{i+1}$. 

We can obtain results about polynomial class functions from the corresponding quasisymmetric class functions.
Now, let $i \leq \frac{d}{2}$. Let $\theta_i : V_i \to V_{d-i}$ be given by $\theta_i|_{V_{\alpha}} = \theta_{\alpha, \varphi, i, d-i}$. By a similar argument, $\theta_i$ is injective and $\mathfrak{S}$-invariant. Hence, $V_i$ is isomorphic to a submodule of $V_{d-i}$, and $f_i \leq \emptyset f_{d-i}$.

For the last result, let $\psi$ be an irreducible character. A simple calculation shows that $\langle \psi, \text{ps}(F) \rangle = \text{ps}(\langle \psi, F \rangle)$. If $F(x)$ is $F$-effective, then $\langle \psi, F \rangle$ is $F$-positive. Since the entries of the $h$-vector are non-negative sums of coefficients in the $F$-basis, the $h$-vector of $\text{ps}(\langle \psi, F \rangle)$ is non-negative.

Finally, suppose that $F(x)$ is $M$-increasing. Let $\alpha \leq \beta$ be integer compositions. We see that

$$[M_\alpha]\langle \psi, F(x) \rangle = \langle \psi, [M_\alpha]F(x) \rangle \leq \langle \psi, [M_\beta]F(x) \rangle = [M_\beta]\langle \psi, F(x) \rangle.$$

If we define a quasisymmetric class function $G : \{e\} \to \text{QSYM}$ by $G(x; e) = \langle \psi, F(x) \rangle$, then we see that $G$ is $M$-increasing. Hence, $\text{ps}(G(x))$ is strongly flawless. We conclude that $\langle \psi, \text{ps}(F) \rangle$ is strongly flawless. \hfill $\Box$

### 3. Double Posets

Now, we will discuss double posets. The Hopf algebra of double posets was introduced by Malvenuto and Reutenauer [15]. Grinberg associated a quasisymmetric function to any double poset, which is a generalization of Gessel’s $P$-partition enumerator. This quasisymmetric function is studied extensively by Grinberg [10], who proved a combinatorial reciprocity theorem.

Given a finite set $N$, a double poset on $N$ is a triple $(N, \leq_1, \leq_2)$ where $\leq_1$ and $\leq_2$ are both partial orders on $N$. Often, for standard poset terminology, we will use $\leq_i$ as a prefix to specify which of the two partial orders is being referred to. For instance, a $\leq_1$-order ideal is a subset that is an order ideal with respect to the first partial order, and a $\leq_1$-covering relation refers to a pair $(x, y)$, such that $x <_1 y$.

Let $D$ be a double poset on a finite set $N$, and let $f : N \to \mathbb{N}$. Then, $f$ is a $D$-partition if and only if it satisfies the following two properties:

1. For $i \leq_1 j$ in $D$, we have $f(i) \leq f(j)$.
2. For $i \leq_1 j$ and $j \leq_2 i$ in $D$, we have $f(i) < f(j)$.

Let $P_D$ be the set of $D$-partitions. We define the $D$-partition enumerator by

$$\Omega(D, x) = \sum_{f \in P_D} \prod_{v \in N} x_{f(v)}. \tag{3}$$

Given a double poset $D$, a pair $(m, m') \in M$ is an inversion if $m <_1 m'$ and $m' <_2 m$. Given a set composition $C \models N$, we say that $C$ is a $D$-set composition if it satisfies the following two properties:

1. For every $i$, $C_1 \cup C_2 \cup \cdots \cup C_i$ is a $\leq_1$-order ideal.
2. For every $i$, there are no inversions in $C_i$.

Let $X_D$ be the set of $D$-set compositions.

**Proposition 4.** Let $D$ be a double poset. Then, $\Omega(D, x) = \sum_{C \in X_D} M_{\alpha(C)}$ with $\alpha(C)$ defined as in Eq. (2).
Proof. Let $f \in P_D$. Let $i_1 < i_2 < \cdots < i_k$ be the natural numbers for which $f^{-1}(i_j) \neq \emptyset$. Define $C(f) = f^{-1}(i_1)|f^{-1}(i_2)| \cdots |f^{-1}(i_k)$. This is the composition associated with $f$. Moreover, we see that $C(f) \in X_D$. Given $C \in X_D$, we see that $M_\alpha(C) = \sum_{f \in P_D: C(f) = C} x^f$. Thus, we obtain

$$
\sum_{C \in X_D} M_\alpha(C) = \sum_{C \in X_D} \sum_{f \in P_D: C(f) = C} x^f = \sum_{f \in P_D} x^f.
$$

Given a double poset $D$, and a permutation $g \in \mathfrak{S}_N$, we define a new double poset $gD$ on $N$ as follows: for $i, j \in N$ and $k \in \{1, 2\}$, we say $i \leq_k j$ in $gD$ if and only if $g^{-1}(i) \leq_k g^{-1}(j)$ in $D$. A permutation $g$ is an automorphism of $D$ if $gD = D$. We let $\text{Aut}(D)$ be the automorphism group of $D$. For instance, for the double poset in Fig. 1, the permutation $(ac)(bd)$ is the only nontrivial automorphism. Similarly, the only nontrivial automorphism of the double poset in Fig. 2 is the permutation $(a)(bd)(c)$.

Let $\mathfrak{G} \subseteq \mathfrak{S}_N$, where $N$ is the vertex set of $D$. For $g \in \mathfrak{G}$ and $f : N \rightarrow N$, let $g \cdot f$ be defined by $(g \cdot f)(v) = f(g^{-1}(v))$ for all $v \in N$. This defines an action of $\mathfrak{G}$ on $\mathbb{N}^N$. Moreover, we see that $x^f = x^{g \cdot f}$, where $x^f$ is defined by Eq. (1). If $f \in P_D$ for some double poset $D$, then we observe that $g \cdot f \in P_{g^{-1}D}$.

For a double poset $D$ on $N$, $\mathfrak{G} \subseteq \text{Aut}(D)$, and $g \in \mathfrak{G}$, let

$$
\Omega(D, \mathfrak{G}, x; g) = \sum_{f \in P_D: g \cdot f = f} x^f.
$$

We call $\Omega(D, \mathfrak{G}, x)$ the $D$-partition quasisymmetric class function. After Lemma 1, we show that $\Omega(D, \mathfrak{G}, x)$ is a class function whose values are formal power series. The fact that those values are quasisymmetric functions follows from any of the identities in Theorem 1.

**Lemma 1.** Let $D$ be a double poset on $N$, $\mathfrak{G} \subseteq \mathfrak{S}_N$, and let $g, h \in \mathfrak{S}_N$. Given $g, h \in \mathfrak{G}$, such that $hgh^{-1} \in \mathfrak{G}$, we have

$$
\Omega(D, \mathfrak{G}, x; hgh^{-1}) = \Omega(h^{-1}D, h^{-1}\mathfrak{G}h, x; g).
$$
Figure 2. A double poset, with $\leq_1$ on the left, and $\leq_2$ on the right

Proof. For $g, h \in \mathcal{G}$, we see that

$$
\Omega(D, \mathcal{G}, x; hgh^{-1}) = \sum_{f \in P_D : \text{hgh}^{-1}f = f} x^f
$$

$$
= \sum_{f' \in P_D : \text{gf} = f'} x^{g \cdot f'}
$$

$$
= \sum_{f' \in P_{\text{hgh}^{-1}D} : \text{gf} = f'} x^{f'}
$$

$$
= \sum_{f' \in P_{\text{h}^{-1}D} : \text{gf} = f'} x^{f'}
$$

$$
= \Omega(h^{-1}D, h^{-1}\mathcal{G}h, x; g),
$$

where the second equality comes from setting $f' = h^{-1}f$, and reindexing the sum over $f'$ instead. The last equality follows from the observation that $hgh^{-1} \in \mathcal{G}$ if and only if $g \in h^{-1}\mathcal{G}h$. Moreover, $h^{-1}\mathcal{G}h \subseteq \text{Aut}(h^{-1}D)$. □

Let $\mathcal{G} \subseteq \text{Aut}(D)$. Then, $P_D$ is $\mathcal{G}$-invariant. For $h, g \in \mathcal{G}$, applying Lemma 1, we have $\Omega(D, \mathcal{G}, x; hgh^{-1}) = \Omega(h^{-1}D, h^{-1}\mathcal{G}h, x; g) = \Omega(D, \mathcal{G}, x; g)$. Hence, $\Omega(D, \mathcal{G}, x)$ is a class function whose values are formal power series in countably many variables.

Naturally, there is an order polynomial class function as well: given a positive integer $n$, we let $X_n,D$ be the set of $D$-partitions $\sigma : D \to [n]$. Then, $\mathcal{G}$ acts on $X_n,D$ and we let $\Omega(D, \mathcal{G}, n)$ be the resulting character.

We give two alternative formulas for $\Omega(D, \mathcal{G}, x; g)$, and another formula for the order polynomial class function. Let $X_{\alpha,D}$ be the set of $D$-set compositions of type $\alpha$. Then, $\mathcal{G}$ acts on $X_{\alpha,D}$. Let $\chi_{\alpha,D}$ be the resulting character.

Theorem 1. Let $D$ be a double poset on a finite set $N$ and let $\mathcal{G} \subseteq \text{Aut}(D)$. Then, we have the following identities:

1.

$$
\Omega(D, \mathcal{G}, x; g) = \sum_{C \in \text{Fix}_g(X_D)} M_{\alpha(C)}.
$$

2.

$$
\Omega(D, \mathcal{G}, x) = \sum_{\alpha = |N|} \chi_{\alpha,D} M_{\alpha}.
$$
3. \[ \Omega(D, \mathfrak{G}, x) = \sum_{\alpha \vdash |N|} \chi_{\alpha, D} \left( \frac{x}{\ell(\alpha)} \right). \]

Proof. Fix a double poset \( D \) on a finite set \( N \).

For the first formula, fix \( g \) and let \( f \in \text{Fix}_g(P_D) \). Define \( C(f) \) as in the proof of Proposition 4. We see that \( g f = f \) if and only if \( gC(f) = C(f) \). Thus, we obtain
\[
\sum_{C \in \text{Fix}_g(X_D)} M_{\alpha(C)} = \sum_{C \in \text{Fix}_g(X_D)} \sum_{f \in \text{Fix}_g(P_D) : C(f) = f} x^f = \sum_{f \in \text{Fix}_g(P_D)} x^f.
\]
The first identity follows from the definition of \( \Omega(D, \mathfrak{G}, x; g) \).

To prove the second identity, we see that
\[
\sum_{C \in \text{Fix}_g(X_D)} M_{\alpha(C)} = \sum_{\alpha \vdash |N|} \sum_{C \in \text{Fix}_g(X_{\alpha, D})} M_\alpha = \sum_{\alpha \vdash |N|} \chi_{\alpha, D}(g) M_\alpha,
\]
and so, by the first formula, we conclude that \( \Omega(D, \mathfrak{G}, x) = \sum_{\alpha \vdash |N|} \chi_{\alpha, D} M_\alpha \).

The third formula follows immediately from principal specialization, and the fact that \( ps M_\alpha = \left( \frac{x}{\ell(\alpha)} \right) \).

As an example, consider the double poset \( D \) in Fig. 1, and let \( \mathfrak{G} = \text{Aut}(D) \). Let \( \rho \) denote the regular representation. Then
\[
\Omega(D, \mathfrak{G}, x) = M_{2,2} + \rho(M_{2,1,1} + M_{1,2,1} + M_{1,1,2} + 2M_{1,1,1,1})
= F_{2,2} + \text{sgn}(F_{2,1,1} + F_{1,1,2} - F_{1,1,1,1}) + \rho F_{1,2,1}.
\]

As another example, consider the double poset \( D \) in Fig. 2, and let \( \mathfrak{G} = \text{Aut}(D) \). Let \( \rho \) denote the regular representation. Then
\[
\Omega(D, \mathfrak{G}, x) = M_{1,3} + M_{1,2,1} + \rho(M_{1,1,2} + M_{1,1,1,1})
= F_{1,3} + \text{sgn} F_{1,1,2}.
\]

3.1. Properties of Double Posets

A double poset is \textit{locally special} if whenever \( y \leq_1 \)-covers \( x \), then \( x \) and \( y \) are \( \leq_2 \)-comparable. The double poset in Fig. 2 is locally special, while the double poset in Fig. 1 is not. Grinberg [10] gives several examples of locally special posets, including double posets coming from skew shapes and labeled posets.

We say that an inversion pair \( (x, y) \) is a \textit{descent pair} if \( x \prec_1 y \).

Lemma 2. Let \( D \) be a locally special double poset, and let \( I \subseteq J \) be \( \leq_1 \)-order ideals. If \( D \) has an inversion pair \( (x, y) \) with \( x, y \in J \setminus I \), then \( D \) has a descent pair \( (w, z) \) with \( w, z \in J \setminus I \).

Proof. We prove the result by induction on \( |J \setminus I| \). Suppose that \( (x, y) \) is an inversion pair with \( x, y \in J \setminus I \). If \( |J \setminus I| = 2 \), then the result is immediate. Therefore, suppose \( |J \setminus I| > 2 \).

Choose \( t \), such that \( x \prec_1 t \leq_1 y \). Since \( D \) is locally special, we have \( x \leq_2 t \) or \( t \leq_2 x \). In the latter case, we have found a descent pair \( (x, t) \). In the former
case, the pair \((t, y)\) forms an inversion pair. We observe that then interval \([t, y]\)

is equal to \((y) \setminus (t)\), where \((a)\) is the principal order ideal generated by \(a\). Since

\(|[t, y]| < |J \setminus I|\), by induction, there is a descent pair \((w, z)\) in \([t, y]\), and we have \(x \leq w \prec z \leq y\).

For any set \(S \subseteq N\), and a partial order \(P\) on \(N\), we let \(P|_S\) denote the

induced poset on \(S\). The same notation is also used for linear orders (which are a special case of partial orders), and for double posets.

We say that a linear order \(\ell\) of \(N\) is \(D\)-compatible if for all pairs \((I, J)\) of

\(\leq_1\)-order ideals with \(I \subseteq J\), the following biconditional statement holds: the

linear order \(\ell|_{J \setminus I}\) is a \(\leq_1\)-linear extension of \(D|_{J \setminus I}\) if and only if \(D|_{J \setminus I}\) contains

no inversion pairs.

**Lemma 3.** Let \(D\) be a double poset on a finite set \(N\). If \(D\) is locally special, then there exists a \(D\)-compatible linear order.

**Proof.** Let \(D\) be a locally special double poset. Let \(G(D)\) be the directed graph obtained by taking the directed edges of the Hasse diagram of \(\leq_1\), and reversing the direction on edges \(x \prec y\) if \(x \geq y\). We claim that \(G(D)\) is acyclic. Suppose that we have a directed cycle \(C\) in \(G(D)\). Let \(C\) have vertices \(x_0, x_1, \ldots, x_k\) in order. Note that this means that \(x_i \prec x_{i+1}\) or \(x_{i+1} \prec x_i\)

for all \(i\). Since \(D\) is locally special, we have \(x_i \leq x_{i+1}\) for all \(i\), which is a contradiction. Thus, there is no directed cycle.

We say \(x \leq_P y\) if there is a directed path from \(y\) to \(x\) in \(G(D)\). Let \(\ell\) be a linear extension of \(P\). We claim that \(\ell\) is \(D\)-compatible.

Let \(I \subseteq J\) be \(\leq_1\)-order ideals. Suppose that there are no inversions in \(J \setminus I\). Then, we see that \(P|_{J \setminus I} = D|_{J \setminus I}\), and thus, \(\ell|_{J \setminus I}\) is a linear extension of \(D|_{J \setminus I}\). Suppose, instead, there is an inversion pair \((x, y)\) in \(J \setminus I\). By Lemma 2, we can choose \((x, y)\) to be a descent pair. Since \(x \geq y\), we see that there is a directed edge in \(G(D)\) from \(y\) to \(x\), and thus, \(x \geq_P y\). Since \(\ell\) is a linear extension of \(P\), we have \(x \succ_P y\). Therefore, \(\ell|_{J \setminus I}\) is not a linear extension of \(D|_{J \setminus I}\).

Given a \(D\)-compatible linear order \(\ell\), we can lexicographically order the collection of linear orders of \(N\): given two linear orders \(\pi\) and \(\sigma\), consider the first \(i\) where \(\pi_i \neq \sigma_i\). Then, we say \(\pi <_\ell \sigma\) if \(\pi_i <_\ell \sigma_i\). Let \(C\) be a \(D\)-set composition for \(D\). We let \(\ell(C)\) be the lexicographically first total refinement of \(C\). Finally, we can totally order \(X_{\alpha, D}\), the set of \(D\)-set compositions of type \(\alpha\), given \(C, C' \in X_{\alpha, D}\), we say \(C \leq_\ell C'\) if and only if \(\ell(C) \leq_\ell \ell(C')\).

We see that a \(D\)-set composition with only singleton blocks is a \(\leq_1\)-linear extension. We prove a proposition regarding when such linear extensions are increasing with respect to \(\ell\). We say that a \(\leq_1\)-linear extension \(\pi\) is increasing if \(\pi_1 <_\ell \pi_2 <_\ell \cdots <_\ell \pi_n\).

**Proposition 5.** Let \(D\) be a locally special double poset, and let \(\ell\) be a \(D\)-compatible linear order. Let \(I \subseteq J\) be \(\leq_1\)-order ideals of \(D\). Then, \(D|_{J \setminus I}\) has an increasing \(\leq_1\)-linear extension \(\sigma\) if and only if \(D\) has no inversions in \(J \setminus I\). In that case, \(\ell|_{J \setminus I} = \sigma\), and \(\sigma\) is lexicographically least.
Proof. We prove the result by induction on \( k = |J \setminus I| \). Suppose that \( D|_{J \setminus I} \) has an increasing \( \leq 1 \)-linear extension \( \sigma \). Then, \( D|_{J \setminus (I \cup \{\sigma_1\})} \) has a \( \leq 1 \)-increasing linear extension. By induction, we see that \( \sigma|_{J \setminus \{\sigma_1\}} = \ell|_{J \setminus I}. \) Similarly, if we let \( J' = J \setminus \{\sigma_k\} \) and we see that \( D|_{J' \setminus I} \) has an increasing \( \leq 1 \)-linear extension, and thus, \( \sigma|_{\{\sigma_1, \ldots, \sigma_{k-1}\}} = \ell|_{J \setminus I}. \) Therefore, we have \( \ell|_{J \setminus I} = \sigma. \) Hence, \( \ell|_{J \setminus I} \) is a linear extension of \( D|_{J \setminus I} \), and by definition of \( D \)-compatible order, this means that \( J \setminus I \) does not contain any inversions.

Now, we suppose that \( J \setminus I \) has no inversions. Then, \( \ell \) restricted to \( J \setminus I \) is a linear extension of \( D|_{J \setminus I} \) with respect to \( \leq 1 \). Moreover, \( \ell|_{J \setminus I} \) is increasing.

Let \( \sigma = \ell|_{J \setminus I}. \) Let \( \tau \) be another increasing \( \leq 1 \)-linear extension. Suppose \( \tau_1 \neq \sigma_1. \) Then, \( \tau_k = \sigma_1 \) for some \( k > 1 \). However, then \( \tau_1 > \tau_k \), and hence, \( \tau \) is not increasing. Thus \( \tau_1 = \sigma_1. \) By induction, we have \( \tau|_{J \setminus \{\tau_1\}} \) and \( \sigma|_{J \setminus \{\sigma_1\}} \) are both increasing \( \leq 1 \)-linear extensions of \( D|_{J \setminus (I \cup \{\tau_1\})} \), and hence are equal by induction. Thus, \( \sigma = \tau. \)

\[ \square \]

4. Digraph Coloring

We refer to directed graphs as digraphs. We require that there is at most one directed edge between any two vertices. An example appears in Fig. 3.

Given a digraph \( G \) on \( N \), a coloring is a function \( f : N \rightarrow \mathbb{N} \) which satisfies

1. for every edge \( (u, v) \), \( f(u) \neq f(v). \)

An edge \( (u, v) \) is an \( f \)-ascent if \( f(u) < f(v). \) We let \( \text{asc}(f) \) denote the number of \( f \)-ascents. An example of a coloring in Fig. 3 is given by \( f(A) = 1, f(B) = 2, f(C) = 3, \) and \( f(D) = 4. \) This coloring has exactly three ascents, from \( A \) to \( B, \) from \( B \) to \( C, \) and from \( C \) to \( D. \) Let \( C_G \) denote the set of all colorings, and we let \( C_{n,G} = \{ f \in C_G : f(N) \subseteq [n]\}. \) Finally, we let \( C_{k,n,G} = \{ f \in C_{n,G} : \text{asc}(f) = k\}. \)
Definition 1. The chromatic quasisymmetric function is
\[ \chi(G, t, x) = \sum_{f \in C_G} t^{\text{asc}(f)} x^f. \]

For example, for the digraph $G$ in Fig. 3, we have $\chi(G, x) = 2t^2 M_{2,2} + 4t^2 M_{2,1,1} + 4t^2 M_{1,2,1} + 4t^2 M_{1,1,2} + (4t^3 + 16t^2 + 4t) M_{1,1,1,1}$. For instance, $2M_{2,2}$ comes from colorings $f$ where $f(A) = f(C)$ and $f(B) = f(D)$. In all such cases, there ends up being two ascents.

Now, we define the automorphism group of a digraph. Given a digraph $G$ on a finite set $N$, a bijection $g : N \to N$ is an automorphism if for every $u, v \in N$, we have $(u, v) \in E(G)$ if and only if $(g(u), g(v)) \in E(G)$. Let $\text{Aut}(G)$ be the set of automorphisms of $G$, which forms a group. For the digraph $G$ appearing in Fig. 3, the automorphism group is isomorphic to $C_4$, the cyclic group of order 4, acting by rotations.

Now, we define the chromatic quasisymmetric class function. Let $\mathcal{H} \subseteq \text{Aut}(G)$.

\[ \chi(G, \mathcal{H}, t, x; g) = \sum_{f \in \text{Fix}_g(C_G)} t^{\text{asc}(f)} x^f, \]

where the sum is over all proper colorings of $G$. This defines a class function on $\mathcal{H}$ whose values are quasisymmetric functions over $\mathbb{Q}[t]$. This is the chromatic quasisymmetric class function associated with $G$. The identities in Lemma 4 show that $\chi(G, \mathcal{H}, t, x)$ is a class function whose values are quasisymmetric functions over $\mathbb{C}[t]$.

Likewise, for $n \in \mathbb{N}$, define the chromatic polynomial class function to be
\[ \chi(G, \mathcal{H}, t, n; g) = \sum_{f \in \text{Fix}_g(C_{n,G})} t^{\text{asc}(f)}, \]

where we sum over all proper colorings $f : N \to \mathbb{N}$, such that $f(N) \subseteq \{1, \ldots, n\}$.

As an example, consider the digraph in Fig. 3, and let $\mathcal{H} = \mathbb{Z}/4\mathbb{Z}$ act via rotation. Let $\rho$ denote the regular representation. Then $\chi(G, \mathcal{H}, t, x) = (1 + \text{sgn})t^2 M_{2,2} + \rho t(M_{2,1,1} + t M_{1,2,1} + t M_{1,1,2}) + (1 + 4t + t^2) M_{1,1,1,1}$.

We let $\chi_i : \mathbb{Z}/4\mathbb{Z} \to \mathbb{C}$ be given by $\chi_i(j) = i^j$, and $\chi_{-i} : \mathbb{Z}/4\mathbb{Z} \to \mathbb{C}$ be given by $\chi_{-i}(j) = (-i)^j$. Then
\[ \chi(G, \mathcal{H}, t, x) = (1 + \text{sgn})t^2(F_{2,2} + F_{1,1,1,1}) + i^2(\chi_i + \chi_{-i})(F_{2,1,1} + F_{1,1,2}) + \rho(i^2 F_{2,1,2} + (i + i^2 + i^3) F_{1,1,1,1}). \]

We detail some formulas relating the chromatic quasisymmetric class function of a digraph $G$ to $D$-partition quasisymmetric class functions. The key concept for proving identities is an acyclic orientation. For a directed graph $G$, an acyclic orientation is another digraph $O$ on the same vertex set, with no directed cycles, such that, for every $u, v \in N$, we have $(u, v) \in E(O)$ if and only if $(u, v) \in E(O)$ or $(v, u) \in E(O)$. An $O$-ascent is an edge $(u, v) \in E(O)$ where $(u, v) \in E(O)$. We let $\text{asc}(O)$ be the number of $O$-ascents. We let $\mathcal{A}(G)$
Lemma 4. Let $G$ be a directed graph and let $\mathfrak{A} \subseteq \text{Aut}(G)$.

1. For $h \in \mathfrak{A}$, we have

$$\chi(G, h, t, x; h) = \sum_{O \in \text{Fix}_h(A(G))} t^{\text{asc}(O)} \Omega(P_O, hO, x; h).$$

2. We have

$$\chi(G, h, t, x) = \sum_{O \in A(G)} \frac{t^{\text{asc}(O)}}{|\mathfrak{A}(O)|} \Omega(P_O, hO, x) \uparrow_{\mathfrak{A}O}.$$

3. Let $T$ be a transversal for the group action of $\mathfrak{A}$ on $A(G)$. We have

$$\chi(G, h, t, x) = \sum_{O \in T} t^{\text{asc}(O)} \Omega(P_O, hO, x) \uparrow_{\mathfrak{A}O}.$$

Proof. Let $g \in \mathfrak{A}$, and let $f$ be a proper coloring of $G$ fixed by $g$. Let $O_f$ be the orientation of $G$ given by directing $v$ to $u$ if $f(v) > f(u)$. Then, $O_f$ is an acyclic orientation. We see that $gO_f = O_f$, and that $\text{asc}(f) = \text{asc}(O_f)$. Then, $f$ is a $P_{O_f}$-partition. Thus

$$\chi(G, h, t, x; g) = \sum_{f \in \text{Fix}_g(C_G)} t^{\text{asc}(f)} x^f$$

$$= \sum_{O \in \text{Fix}_g(A(G))} \sum_{f \in \text{Fix}_g(A(G)) : O_f = O} t^{\text{asc}(O)} x^f$$

$$= \sum_{O \in \text{Fix}_g(A(G))} t^{\text{asc}(O)} \Omega(P_O, hO, x; g).$$

To prove our second formula, we have

$$\sum_{O \in A(G)} \frac{t^{\text{asc}(O)}}{|\mathfrak{A}(O)|} \Omega(P_O, hO, x) \uparrow_{\mathfrak{A}O} (g)$$

$$= \sum_{O \in A(G)} \frac{t^{\text{asc}(O)}}{|\mathfrak{A}(O)|} \frac{1}{|\mathfrak{A}O|} \sum_{h \in \mathfrak{A}O} \sum_{hgh^{-1} \in \mathfrak{A}O} \Omega(P_O, hO, x; hgh^{-1})$$

$$= \frac{1}{|\mathfrak{A}|} \sum_{h \in \mathfrak{A}} \sum_{O \in A(G)} t^{\text{asc}(O)} \Omega(P_{O^{-1}O}, h^{-1}O, x; gh^{-1})$$

$$= \frac{1}{|\mathfrak{A}|} \sum_{h \in \mathfrak{A}} \sum_{O \in A(G)} t^{\text{asc}(O)} \Omega(P_{O^{-1}O}, h^{-1}O, x; g)$$

$$= \frac{1}{|\mathfrak{A}|} \sum_{h \in \mathfrak{A}} \sum_{O \in A(G)} t^{\text{asc}(O)} \Omega(P_O, h^{-1}O, x; g).$$
\[
\sum_{O \in \text{Fix}_G(A(G))} t^{\text{asc}(O)} \Omega(P_O, \mathcal{H}_O, \mathbf{x}; g).
\]

The first equality is a formula for computing induced characters. The second equality comes from the Orbit–Stabilizer Theorem, and changing the order of summation. The third equality is due to Lemma 1. Finally, we can reindex our summation by replacing \( O \) with \( hO \). Since \( A(G) \) is invariant under \( H \), we end up with the same number of terms in the sum. Since \( \text{asc}(O) = \text{asc}(hO) \), we obtain the fourth equality. The last equality is then immediate.

Let \( T \) be a transversal for \( H \) acting on \( A(G) \). For \( O \in T \) and \( U \in H(O) \), we claim that

\[
\Omega(P_U, \mathcal{H}_U, \mathbf{x}) \uparrow_{\mathcal{H}_U}^g = \Omega(P_{tO}, \mathcal{H}_{tO}, \mathbf{x}) \uparrow_{\mathcal{H}_{tO}}^g.
\]

(4)

Then, we have

\[
\sum_{O \in A(G)} \frac{t^{\text{asc}(O)} \Omega(P_O, \mathcal{H}_O, \mathbf{x}) \uparrow_{\mathcal{H}_O}^g}{|\mathcal{H}_O|} = \sum_{U \in \mathcal{T}} \sum_{O \in \mathcal{H}(U)} \frac{t^{\text{asc}(U)} \Omega(P_O, \mathcal{H}_O, \mathbf{x}) \uparrow_{\mathcal{H}_O}^g}{|\mathcal{H}(U)|}
= \sum_{U \in \mathcal{T}} \sum_{O \in \mathcal{H}(U)} \frac{t^{\text{asc}(U)} \Omega(P_U, \mathcal{H}_U, \mathbf{x}) \uparrow_{\mathcal{H}_U}^g}{|\mathcal{H}(U)|}
= \sum_{U \in \mathcal{T}} t^{\text{asc}(U)} \Omega(P_U, \mathcal{H}_U, \mathbf{x}) \uparrow_{\mathcal{H}_U}^g,
\]

where the second equality follows from Eq. (4).

5. Combinatorial Reciprocity Theorem

As stated in the introduction, we consider the setting of a combinatorial reciprocity theorem to consist of a subspace \( V \) of a ring of formal power series in some number of variables, a subset \( C \subseteq V \), and an involution \( \omega \) on \( V \). Then, given two generating functions \( f, g \in C \), a combinatorial reciprocity theorem is the claim that \( f = \omega g \).
For example, let $V$ be the vector space of polynomial functions of degree $d$. Given a polynomial $p(x)$, we can consider the formal power series $\sum_{n \geq 0} p(n)t^n$. In this way, $V$ is a subspace of the ring of formal power series in $t$. We let $C$ be the subset of $V$ corresponding to polynomials where $p(n) \geq 0$ for all $n$. We know that $V$ is part of a Hopf algebra, and so $\omega = (-1)^d S$ is an involution on $V$. This is the setting for many combinatorial reciprocity theorems in the literature.

Similarly, one can let $V$ be the vector space of (quasi)symmetric functions of degree $d$, let $C$ be the set of (quasi)symmetric functions with non-negative expansions in the monomial basis, and let $\omega = (-1)^d S$, where $S$ is the antipode. There are several examples of combinatorial reciprocity theorems involving QSYM or sym, including Theorem 4.2 of Stanley [18] or Theorem 4.2 of Grinberg [10].

Our combinatorial reciprocity theorems involve letting $V$ be the vector space of quasisymmetric class functions, $C$ be the set of $M$-realizable quasi-symmetric class functions, and $\omega = (-1)^d S \text{sgn}$.

5.1. Combinatorial Reciprocity for Double Posets

Given a double poset $D$, the dual poset $D^*$ is obtained by reversing the first partial order $\leq_1$ of $D$. We prove a combinatorial reciprocity theorem for locally special double posets. The orbital version of the combinatorial reciprocity theorem was previously obtained by Grinberg [10]. We also prove a combinatorial reciprocity theorem for the chromatic quasisymmetric class function of a digraph.

**Theorem 2.** Let $D$ be a locally special double poset on a finite set $N$, and let $G \subseteq \text{Aut}(D)$. Then,

$$(-1)^{|N|} s\text{gn} \Omega(D, G, x) = \Omega(D^*, G, x).$$

Also,

$$(-1)^{|N|} s\text{gn} \Omega(D, G, -x) = \Omega(D^*, G, x).$$

The second result follows from the first via principal specialization. Our proof relies a lot on the work of Grinberg [10].

First, we discuss a weighted generalization of $\Omega(D, G, x)$. Given a weight function $w: N \to \mathbb{N}$, a double poset $D$ on $N$, and a $D$-partition $\sigma$, define

$$x^{w, \sigma} = \prod_{i \in N} x^{w(i)}_{\sigma(i)}.$$ 

We let $\Omega(D, w, x) = \sum_\sigma x^{w, \sigma}$ where the sum is over all $D$-partitions. The following is Theorem 4.2 of Grinberg [10].

**Proposition 6.** Let $D$ be a double poset on a finite set $N$, and let $w: N \to \mathbb{N}$. If $D$ is locally special, then

$$(-1)^{|N|} s\text{gn} \Omega(D, w, x) = \Omega(D^*, w, x).$$

We let $\text{Aut}(D, w)$ be the set of automorphisms $g$ of $D$ with the property that $w \circ g = w$. Let $\mathcal{G}$ be a subgroup of $\text{Aut}(D, w)$. Then, $\mathcal{G}$ acts on the set of $D$-partitions. We let $\Omega(D, w, \mathcal{G}, x; g) = \sum_\sigma x^{w, \sigma}$, where the sum is over all
Given $g \in \mathcal{G}$, and $i \in D$, we let $g(i) = \{g^m \cdot i : m \in \mathbb{Z}\}$ be the cycle of $g$ containing $i$. Let $\text{Cyc}(g) = \{g(i) : i \in D\}$ be the set of cycles of $g$. We can define a new weight function $w/g : \text{Cyc}(g) \to \mathbb{N}$ by $w/g(C) = \sum_{x \in C} w(x)$. Given two cycles $C_1$ and $C_2$ of $\text{Cyc}(g)$, and $i \in \{1, 2\}$, we say $C_1 \leq_i C_2$ if there exists $x \in C_1$ and $y \in C_2$, such that $x \leq_i y$. This turns $\text{Cyc}(g)$ into a double poset that we denote by $D/g$. The following is a combination of parts of Propositions 7.5 and 7.6 of Grinberg [10].

**Lemma 5.** Let $D$ be a double poset on a finite set $N$, let $w : N \to \mathbb{N}$, and let $\mathcal{G} \subseteq \text{Aut}(D, w)$. Given $g \in \mathcal{G}$, we have

$$\Omega(D, w, \mathcal{G}, x; g) = \Omega(D/g, w/g).$$

Moreover, if $D$ is locally special, then so is $D/g$.

Finally, we let $\mathbf{1} : N \to \mathbb{N}$ be the function defined by $\mathbf{1}(n) = 1$ for all $n \in N$.

**Proof of Theorem 2.** We have

$$(-1)^{|N|} \text{sgn } (g) \Omega(D, w, \mathcal{G}, x; g) = (-1)^{|N|}(-1)^{|N| - \text{cyc}(g)} \Omega(D/g, 1/g)$$

$$= \Omega(D/g^*, 1/g)$$

$$= \Omega(D^*/g, 1/g)$$

$$= \Omega(D^*, w, \mathcal{G}, x; g),$$

where the first and last equalities are due to Lemma 5, and the second equality is due to Proposition 6. \qed

### 5.2. Combinatorial Reciprocity Theorem for Digraphs

Now, we discuss a combinatorial reciprocity theorem for directed graphs. Given a digraph $G$ on $N$, an acyclic coloring is a pair $(O, f)$ satisfying the following:

1. $O$ is an acyclic orientation of $G$.
2. For every edge $(u, v) \in E(O)$, $f(u) \leq f(v)$.

We modify the definition of descent. An $O$-descent is an edge $(u, v) \in E(G)$ where $(v, u) \in E(O)$. Let $A_G$ be the acyclic colorings. If $\mathfrak{H} \subseteq \text{Aut}(G)$, then $\mathfrak{H}$ acts on $A_G$. For $g \in \mathfrak{H}$, define

$$\chi(G, \mathfrak{H}, t, x; g) = \sum_{(O, f) \in \text{Fix}_g(A_G)} t^{\text{des}(f)} x^f.$$

Then, $\chi(G, \mathfrak{H}, t, x)$ is a class function whose values are quasisymmetric functions with coefficients in $\mathbb{Q}[t]$.

**Theorem 3.** Let $G$ be a digraph and let $\mathfrak{H} \subseteq \text{Aut}(G)$. Then

$$(-1)^{|N|} \text{sgn } \chi(G, \mathfrak{H}, t, x) = \chi(G, \mathfrak{H}, t, x)$$

and

$$(-1)^{|N|} \text{sgn } \chi(G, \mathfrak{H}, t, -x) = \chi(G, \mathfrak{H}, t, x).$$
Proof. Let \( g \in \mathcal{H} \). Then
\[
(-1)^{|N|} S \operatorname{sgn} \chi(G, \mathcal{H}, t, x; g) = \sum_{O \in \text{Fix}_g(A(G))} t^{\operatorname{asc}(O)} (-1)^{|N|} S \operatorname{sgn} \Omega(P_O, \mathcal{H}_O, x; g)
\]
\[
= \sum_{O \in \text{Fix}_g(A(G))} t^{\operatorname{asc}(O)} \sum_{f:(\overrightarrow{O}, f) \in \text{Fix}_g(A_G)} x^f
\]
\[
= \sum_{(O, f) \in \text{Fix}_g(A_G)} t^{\operatorname{des}(O)} x^f = \chi(G, \mathcal{H}, t, x; g),
\]
where the first equality follows from Lemma 4, and the second equality is due to Theorem 2. The third equality comes from observing that \( f \in P_{P_O} \) if and only if \( (\overrightarrow{O}, f) \) is an acyclic coloring. The fourth equality comes from observing that \( \operatorname{asc}(O) = \operatorname{des}(\overrightarrow{O}) \), and reindexing the summation by substituting \( O \) with \( \overrightarrow{O} \). □

6. Flawlessness

In this section, we study the property of being \( M \)-increasing or effectively flawless. Let \( D \) be a double poset on a finite set \( N \), and let \( \mathcal{G} \subseteq \text{Aut}(D) \). For \( \alpha \models |N| \), let \( V_{\alpha, D} \) be the vector space with basis \( X_{\alpha, D} \). Then, \( V_{\alpha, D} \) is a \( \mathcal{G} \)-module.

Given \( \alpha \leq \beta \models |N| \), we define maps \( \theta_{\alpha, \beta} : V_{\alpha, D} \to V_{\beta, D} \). Given a \( D \)-set composition \( C \) of type \( \alpha \), we let
\[
\theta_{\alpha, \beta}(C) = \sum_{C' \in X_{\beta, D} : C' \geq C} C'.
\]

Proposition 7. Let \( D \) be a double poset on a finite set \( N \). Let \( \alpha \leq \beta \leq \gamma \) be integer compositions of \( |N| \). Then, \( \theta_{\alpha, \beta} \) is an injective \( \mathcal{G} \)-invariant map. Moreover, we have \( \theta_{\beta, \gamma} \circ \theta_{\alpha, \beta} = \theta_{\alpha, \gamma} \).

Proof. We see that \( \theta_{\alpha, \beta} \) is \( \mathcal{G} \)-invariant. To see that the map is injective, it is enough to note that the map \( f : V_{\beta, D} \to V_{\alpha, D} \) given by \( f(C) = C_{\alpha}(C) \) is a section for \( \theta_{\alpha, \beta} \). Hence, \( \theta_{\alpha, \beta} \) is injective and \( f \) is surjective.

Let \( \alpha \leq \beta \leq \gamma \), and let \( C \in X_{\alpha, D} \). Then
\[
\theta_{\beta, \gamma} \circ \theta_{\alpha, \beta}(C) = \sum_{C' \in X_{\beta, D} : C' \geq C} \theta_{\beta, \gamma}(C')
\]
\[
= \sum_{C' \in X_{\beta, D} : C' \geq C} \sum_{C'' \in X_{\gamma, D} : C'' \geq C'} C''
\]
\[
= \sum_{C'' \in X_{\gamma, D} : C'' \geq C} \sum_{C' \in X_{\beta, D} : C' \geq C'} C''
\]
Table 1. The character table for $\mathbb{Z}_3 \times \mathbb{Z}_2$

|   |  $\{e\}$ |  $\{\sigma\}$ |  $\{\sigma^2\}$ |  $\{\tau\}$ |  $\{\tau\sigma\}$ |  $\{\tau\sigma^2\}$ |
|---|-----------|----------------|------------------|--------------|-----------------|------------------|
| $\chi_1$ | 1         | 1              | 1                | 1            | 1               | 1                |
| $\chi_2$ | 1         | $\omega$      | $\overline{\omega}$ | 1            | $\omega$       | $\overline{\omega}$ |
| $\chi_3$ | 1         | $\overline{\omega}$ | $\omega$ | 1            | $\overline{\omega}$ | $\omega$ |
| $\chi_4$ | 1         | 1              | 1                | $-1$         | $-1$           | $-1$             |
| $\chi_5$ | 1         | $\omega$      | $\overline{\omega}$ | $-1$         | $-\omega$      | $-\overline{\omega}$ |
| $\chi_6$ | 1         | $\overline{\omega}$ | $\omega$ | $-1$         | $-\omega$      | $-\overline{\omega}$ |

$$= \sum_{C'' \in X_{\gamma,D}: C'' \geq C} C'' = \theta_{\alpha,\gamma}(C).$$

The penultimate equality follows from the fact that there is exactly one set composition $C'$ of type $\beta$ with the property that $C \leq C' \leq C''$. Moreover, if $C$ and $C''$ are $D$-set compositions, then $C'$ is as well. Hence, the inner summation only involves one term.

**Theorem 4.** Let $D$ be a double poset on a finite set $N$, and let $\mathfrak{G} \subseteq \text{Aut}(D)$. Then, $\Omega(D, \mathfrak{G}, x)$ is $M$-increasing and $\Omega(D, \mathfrak{G}, x)$ is effectively flawless.

**Proof.** Let $D$ be a double poset on a finite set $N$, and let $\mathfrak{G} \subseteq \text{Aut}(D)$. For $\alpha \models |N|$, we see that $[M_\alpha] \Omega(D, \mathfrak{G}, x)$ is the character of the representation of $\mathfrak{G}$ on $V_{\alpha,D}$. Let $\alpha \leq \beta \models |N|$. Since $\theta_{\alpha,\beta}$ is injective and $\mathfrak{G}$-invariant, $V_{\alpha,D}$ is isomorphic to a submodule of $V_{\beta,D}$. Thus, $[M_\beta] \Omega(D, \mathfrak{G}, x) - [M_\alpha] \Omega(D, \mathfrak{G}, x)$ is the character corresponding to the complement of $\theta_{\alpha,\beta}(V_{\alpha,D})$ in $V_{\beta,D}$. Hence, $\Omega(D, \mathfrak{G}, x)$ is $M$-increasing.

Since $\Omega(D, \mathfrak{G}, x)$ is $M$-increasing and $\Omega(D, \mathfrak{G}, x)$ is the principal specialization of $\Omega(D, \mathfrak{G}, x)$, it follows from Proposition 3 (4) that $\Omega(D, \mathfrak{G}, x)$ is effectively flawless. □

Now, we discuss a counterexample to extending Theorem 4 to studying coefficients in the $F$ basis and to studying the $h$-vector. Let $D$ be the poset in Fig. 4. We define $\leq_2$ to be the opposite partial order of $\leq_1$. Hence, $\Omega(D, x)$ counts strict $D$-partitions. We let $\mathfrak{G} = \mathbb{Z}_2 \times \mathbb{Z}_3$, viewed as the group $\langle (abc), (de) \rangle$. Let $\omega$ be a cube root of unity. If we let $\sigma = (abc)$ and $\tau = (de)$, then Table 1 is the character table for $\mathfrak{G}$. 

\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node (a) at (0,0) [shape=circle,draw] {a};
    \node (b) at (1,0) [shape=circle,draw] {b};
    \node (c) at (2,0) [shape=circle,draw] {c};
    \node (d) at (1,1) [shape=circle,draw] {d};
    \node (e) at (2,1) [shape=circle,draw] {e};

    \draw[->] (a) -- (b);
    \draw[->] (a) -- (c);
    \draw[->] (b) -- (d);
    \draw[->] (b) -- (e);
    \draw[->] (c) -- (d);
    \draw[->] (c) -- (e);
\end{tikzpicture}
\caption{A Hasse diagram}
\end{figure}
Then
\[
\Omega(D, \mathfrak{G}, x) = \chi_1 F_{3,2} + (\chi_2 + \chi_3)(F_{1,2,2} + F_{2,1,2}) + \chi_4 F_{3,1,1} \\
+ (\chi_5 + \chi_6)(F_{1,2,1,1} + F_{2,1,1,1}) + \chi_1 F_{1,1,1,2} + \chi_4 F_{1,1,1,1,1}.
\]
Thus, \(\Omega(D, \mathfrak{G}, x)\) is not \(F\)-increasing. We see that
\[
\sum_{m \geq 0} \Omega(D, \mathfrak{G}, m) t^m = \frac{\chi_1 t^2 + (2\chi_2 + 2\chi_3 + \chi_4)t^3 + (2\chi_5 + 2\chi_6 + \chi_1)t^4 + \chi_4 t^5}{(1-t)^6}.
\]
Then, the \(h\)-vector is \((0, 0, \chi_1, 2\chi_2 + 2\chi_3 + \chi_4, 2\chi_5 + 2\chi_6 + \chi_1, \chi_4)\). We see that \(h_2\) and \(h_3\) are not comparable, so the \(h\)-vector is not strongly flawless.

Similarly, we can view the Hasse diagram in Fig. 4 as a digraph \(G\) with \(\mathfrak{H} = \mathbb{Z}_3 \times \mathbb{Z}_2\). Then, \([t^0]\chi(G, \mathfrak{H}, t, x) = \Omega(D, \mathfrak{H}, x)\). Thus, we have an example of a directed graph where the chromatic quasisymmetric class function is not \(F\)-increasing, and where the \(h\)-vector of the chromatic polynomial class function is not effectively flawless.

7. F-Effectiveness and \(h\)-Effectiveness

In this section, we state and prove several effectiveness theorems.

**Theorem 5.** Let \(D\) be a locally special double poset on a finite set \(N\), and let \(\mathfrak{G} \subseteq \text{Aut}(D)\). Then, \(\Omega(D, \mathfrak{G}, x)\) is \(F\)-effective.

We prove Theorem 5 in the next subsection. First, we focus on corollaries and other related theorems. First, we obtain the follow result from Proposition 2 (3).

**Corollary 1.** Let \(D\) be a locally special double poset on a finite set \(N\), and let \(\mathfrak{G} \subseteq \text{Aut}(D)\). Given an irreducible character \(\psi\), we have \(\langle \psi, \Omega(D, \mathfrak{G}, x) \rangle\) is \(F\)-positive.

We obtain the following theorem for the chromatic quasisymmetric class function of a digraph.

**Theorem 6.** Let \(G\) be a digraph, and let \(\mathfrak{H} \subseteq \text{Aut}(G)\). For any \(k \in \mathbb{N}\), we have \([t^k]\chi(G, \mathfrak{H}, t, x)\) is \(F\)-effective.

For any irreducible character \(\psi\) of \(\mathfrak{H}\), we have \(\langle \psi, [t^k]\chi(G, \mathfrak{H}, t, x) \rangle\) is \(F\)-positive.

The second statement follows from applying Proposition 2 (3) to the first statement, so we focus on proving the first statement.

**Proof.** Let \(T\) be a transversal for the action of \(\mathfrak{H}\) on \(\mathcal{A}(G)\). For each acyclic orientation \(O \in T\), we write \(\Omega(P_O, \mathfrak{H}_O, x) = \sum_{\alpha = |N|} \psi_{\alpha, P_O} F_{\alpha}\) where the characters \(\psi_{\alpha, P_O}\) are effective. Using Lemma 4, we have
\[
\chi(G, \mathfrak{H}, t, x) = \sum_{O \in T} t^{\text{asc}(O)} \Omega(P_O, \mathfrak{H}_O, x) t^{\mathfrak{H}_O}
\]
\[ \times \sum_{O \in \mathcal{T}} t^{\text{asc}(O)} \left( \sum_{\alpha \models |N|} \psi_{\alpha,P,O} F_{\alpha} \right) \uparrow_{\mathfrak{S}_O}^{\mathfrak{S}} = \sum_{\alpha \models |N|} \left( \sum_{O \in \mathcal{T}} t^{\text{asc}(O)} \psi_{\alpha,P,O} \uparrow_{\mathfrak{S}_O}^{\mathfrak{S}} \right) F_{\alpha}. \]

We deduce results about \( h \)-effectiveness by applying Proposition 3.3.

**Corollary 2.** Let \( D \) be a locally special double poset on a finite set \( N \), and let \( \mathfrak{S} \subseteq \text{Aut}(D) \). Then, \( \Omega(D, \mathfrak{S}, x) \) is \( h \)-effective.

Let \( G \) be a directed graph on a finite set \( N \) with \( \mathfrak{S} \subseteq \text{Aut}(G) \). Then, \( \chi(G, \mathfrak{S}, t, x) \) is \( h \)-effective.

**Theorem 7.** Let \( G \) be a digraph, and let \( \mathfrak{S} \subseteq \text{Aut}(G) \). For \( k \in \mathbb{N} \), we have \( [t^k]\chi(G, \mathfrak{S}, t, x) \) is \( M \)-increasing and \( [t^k]\chi(G, \mathfrak{S}, t, x) \) is effectively flawless.

**Proof.** Let \( G \) be a directed graph, and let \( \mathfrak{S} \subseteq \text{Aut}(G) \). By Theorem 6, we see that \( [t^k]\chi(G, \mathfrak{S}, t, x) \) is \( F \)-effective. Thus, \( [t^k]\chi(G, \mathfrak{S}, t, x) \) is \( M \)-increasing. Since \( \text{ps}(\chi(G, \mathfrak{S}, t, x)) = \chi(G, \mathfrak{S}, t, x) \), it follows that \( [t^k]\chi(G, \mathfrak{S}, t, x) \) is effectively flawless.

### 7.1. Proof of Theorem 5

For \( \alpha \leq \beta \models |N| \), define \( \theta_{\alpha,\beta} \) by Eq. (5). Our first step is to prove the following proposition.

**Proposition 8.** Let \( D \) be a double poset on a finite set \( N \). Let \( \alpha \leq \beta \models |N| \) and \( \gamma \leq \beta \). Then, we have \( \theta_{\alpha,\beta}(V_{\alpha,D}) \cap \theta_{\gamma,\beta}(V_{\gamma,D}) = \theta_{\alpha \wedge \gamma,\beta}(V_{\alpha \wedge \gamma,D}) \), where \( \alpha \wedge \gamma \) is an integer composition that coarsens \( \alpha \) and \( \gamma \).

Before we prove the proposition, we define a function on set compositions that we use in the proof of Proposition 8. Fix a \( D \)-compatible linear order \( \ell \), which exists by Lemma 3. Let \( \alpha \leq \beta \) and \( \gamma \leq \beta \). Fix \( \bar{v} \in \theta_{\alpha,\beta}(V_{\alpha,D}) \cap \theta_{\gamma,\beta}(V_{\gamma,D}) \). Consider \( C \in X_{\beta,D} \cap \theta_{\alpha,\beta}(V_{\alpha,D}) \). Then, \( C_{\alpha}(C) \in X_{\alpha,D} \). Let \( C^1 = C_{\beta}(\ell(C_{\alpha}(C))) \). We see that \( C_{\alpha}(C) \leq C^1 \), so if \( C^1 \) contained any inversions, then so does \( C_{\alpha}(C) \). Since \( C^1 \leq \ell(C_{\alpha}(C)) \), we know that the union of the first several blocks of \( C^1 \) is the union of the first several blocks of \( \ell(C_{\alpha}(C)) \). Thus, \( C^1 \) is a \( D \)-set composition.

Since \( C_{\alpha}(C) \leq C \) and \( C^1 \leq \ell(C_{\alpha}(C)) \), we have \( \ell(C_1) \leq \ell \ell(C_{\alpha}(C)) < \ell \ell(C) \). Thus, \( C^1 \leq \ell C \). Moreover, since \( \bar{v} \in \theta_{\alpha,\beta}(V_{\alpha,D}) \), we know \( \bar{v} = \theta_{\alpha,\beta}(\bar{w}) \) for some \( \bar{w} \in V_{\alpha,D} \). Since \( C_{\alpha}(C) \leq C^1 \), it follows from the definition of \( \theta_{\alpha,\beta} \) that \( [C^1]\bar{v} = [C_{\alpha}(C)]\bar{w} \). Similarly, since \( C_{\alpha}(C) \leq C \), we have \( [C]\bar{v} = [C_{\alpha}(C)]\bar{w} \). Thus, \( [C^1]\bar{v} = [C]\bar{v} \).

Let \( C^2 = C_{\beta}(\ell(C_{\gamma}(C^1))) \). By similar reasoning, \( C^2 \leq \ell C^1 \) and \( [C^2]\bar{v} = [C^1]\bar{v} = [C]\bar{v} \). For every integer \( k \), we define

\[
C^{k+1} = \begin{cases} C_{\beta}(\ell(C_{\alpha}(C^k))) & \text{if } k \text{ is even} \\ C_{\beta}(\ell(C_{\gamma}(C^k))) & \text{if } k \text{ is odd}. \end{cases}
\]
Thus, we obtain a sequence of compositions $C^k$, such that $C^{k+1} \leq \ell C^k$ and $[C^k]v = [C]v$ for all $k$.

Let $m$ be the first integer where $C^{m+1} = C^m$. Define $f(C) = C_\alpha \wedge \gamma(\ell(C^m))$.

**Lemma 6.** Let $v \in \theta_{\alpha,\beta}(V_{\alpha,D}) \cap \theta_{\gamma,\beta}(V_{\gamma,D})$. Consider $C \in X_{\beta,D}$. Fix $f(C)$ as defined above. Then, $f(C) \in X_{\alpha \wedge \gamma,D}$.

Moreover, given $B$, such that $B \geq f(C)$, we have $f(B) = f(C)$.

Before we give the proof, we define another operation on set compositions. Given $C \models N$, and $S \subset N$, the set composition $C \cap S$ is defined by taking $(C_1 \cap S, C_2 \cap S, \ldots, C_\ell \cap S)$, and then deleting any coordinates corresponding to empty intersections. For instance, $123|456|789 \cap \{1, 7, 8\} = 1|78$.

**Proof.** We show that $f(C)$ is a $D$-set composition. Write $f(C) = C'_1 \cdot \cdots C'_r$. Since $f(C) \leq C^m$, the set composition $C'_1 \cdot \cdots C'_r$ is a union of blocks of $C^m$, and hence is a $\leq \ell$-order ideal. Thus, it suffices to show that that $C'_i$ does not contain any inversions. Define $D' = C_\alpha(C^m) \cap C'_i$, which is a set composition of $C'_i$. Let $\ell'$ be the restriction of $\ell(C^m)$ to $C'_i$. Then, $\ell'$ is lexicographically least in $C'_i$; if there exists $\tau < \ell'$, then we could modify $\ell(C^m)$ by replacing $\ell(C^m)|_{C'_i}$ with $\tau$ and obtain a new refinement of $C^m$ that is lexicographically smaller than $\ell(C^m)$. Since each block $D'_l$ of $D'$ contains no inversions, by Proposition 5, $\ell'|_{D'_l}$ is strictly increasing. If we let $E' = C_\gamma(C^m) \cap C'_i$, then by similar reasoning $\ell'$ is strictly increasing when restricted to each block of $E'$. We see that the finest common coarsening of $D'$ and $E'$ is the set composition $(C_i)$. Thus, $\ell(C^m)$ is strictly increasing on all of $C'_i$. By Proposition 5, it follows that $D$ has no inversions in $C'_i$. Therefore, $C'$ is a $D$-set composition. We see that $C^m = C_\beta(\ell(f(C)))$.

Now, let $B \geq f(C)$. For every integer $k$, we define

$$B^{k+1} = \begin{cases} C_\beta(\ell(C_\alpha(B^k))) & k \text{ is even} \\ C_\beta(\ell(C_\gamma(B^k))) & k \text{ is odd.} \end{cases}$$

Thus, we obtain a sequence of compositions $B^k$, such that $B^{k+1} \leq \ell B^k$ and $[B^k]v = [B]v$ for all $k$. We observe that $B^{k+1}$ involves permuting the elements from blocks of $B^k$ that belong to the same block of $C_\alpha(B^k)$ (or $C_\gamma(B^k)$). Hence, $C_{\alpha \wedge \gamma}(B^{k+1}) = C_{\alpha \wedge \gamma}(B^k) = f(C)$ for all $k$. Thus, $B^{k+1} \geq f(C)$ for all $k$.

If we let $m$ be the first integer for which $B^{m+1} = B^m$, then we have $f(B) = C_{\alpha \wedge \gamma}(B^m) = f(C)$.

**Proof of Proposition 8.** First, we see that

$$\theta_{\alpha \wedge \gamma, \beta}(V_{\alpha \wedge \gamma,D}) = (\theta_{\alpha, \beta} \circ \theta_{\alpha \wedge \gamma, \alpha})(V_{\alpha \wedge \gamma,D})$$

$$= \theta_{\alpha, \beta}(\theta_{\alpha \wedge \gamma, \alpha}(V_{\alpha \wedge \gamma,D}))$$

$$\subseteq \theta_{\alpha, \beta}(V_{\alpha,D}).$$

By a similar argument, $\theta_{\alpha \wedge \gamma, \beta}(V_{\alpha \wedge \gamma,D}) \subseteq \theta_{\gamma, \beta}(V_{\gamma,D})$. Hence

$$\theta_{\alpha \wedge \gamma, \beta}(V_{\alpha \wedge \gamma,D}) \subseteq \theta_{\alpha, \beta}(V_{\alpha,D}) \cap \theta_{\gamma, \beta}(V_{\gamma,D}).$$
\[v \in \theta_{\alpha,\beta}(V_{\alpha,D}) \cap \theta_{\gamma,\beta}(V_{\gamma,D}). \] For any \( C \in X_{\beta,D} \), let \( m(C) = C_{\beta}(\ell(f(C))) \). Then, we have \( |C|v = |m(C)|v \). Moreover, for any \( C' \geq f(C) \), by Lemma 6, we have \( f(C) = f(C') \). Hence, \( m(C') = m(C) \) and \( |C|v = |C'|v \).

Also, for any \( C \in X_{\beta,D} \), we know that \( f(C) \in X_{\alpha \land \gamma,D} \) by Lemma 6.

We define \( \bar{w} \in \theta_{\alpha \land \gamma,\beta}(W_{\alpha \land \gamma,D}) \) as follows. Given \( B \in X_{\alpha \land \gamma,D} \), we define \( [B]\bar{w} = [C]v \) for any \( C \in X_{\beta,D} \), such that \( f(C) = B \). By what we have just observed, \( [B]\bar{w} \) does not depend upon which \( C \) with \( f(C) = B \) we choose. Thus, \( \theta_{\alpha \land \gamma,\beta}(\bar{w}) = \bar{v} \). Hence, \( \bar{v} \in \theta_{\alpha \land \gamma,\beta}(V_{\alpha \land \gamma,D}) \). \( \square \)

For \( \alpha \models |N| \), we define \( U_{\alpha,D} \) to be the module
\[ U_{\alpha,D} = \text{span} \bigcup_{\beta < \alpha} \theta_{\beta,\alpha}(X_{\beta,D}). \]

Then \( U_{\alpha,D} \) is a \( \mathcal{G} \)-module. Hence there exists another \( \mathcal{G} \)-module \( W_{\alpha,D} \) such that \( V_{\alpha,D} = U_{\alpha,D} \oplus W_{\alpha,D} \).

**Proposition 9.** Let \( D \) be a locally special double poset on a finite set \( N \). Let \( \alpha \leq \beta \models |N| \) and \( \gamma \leq \beta \). Then, we have \( \theta_{\alpha,\beta}(W_{\alpha,D}) \cap \theta_{\gamma,\beta}(W_{\gamma,D}) = \{0\} \).

**Proof.** Let \( \bar{y} \in \theta_{\alpha,\beta}(W_{\alpha,D}) \cap \theta_{\gamma,\beta}(W_{\gamma,D}) \). Then, \( \bar{v} = \theta_{\alpha,\beta}(\bar{y}) \) for some \( \bar{y} \in W_{\alpha,D} \). By Propositions 8 and 7, we have \( \bar{v} \in \theta_{\alpha \land \gamma,\beta}(V_{\alpha \land \gamma,D}) = \theta_{\alpha,\beta} \circ \theta_{\alpha \land \gamma,\alpha}(V_{\alpha \land \gamma,D}) \). Thus, \( \bar{v} = \theta_{\alpha,\beta}(\bar{z}) \) for some \( \bar{z} \in \theta_{\alpha \land \gamma,\alpha}(V_{\alpha \land \gamma,D}) \). Since \( \theta_{\alpha,\beta} \) is injective, we have \( \bar{y} = \bar{z} \). However, by the definition of \( W_{\alpha,D} \), it follows that \( \bar{y} = \bar{z} = 0 \), and thus, \( \bar{v} = 0 \). Hence, \( \theta_{\alpha,\beta}(W_{\alpha,D}) \cap \theta_{\gamma,\beta}(W_{\gamma,D}) = \{0\} \). \( \square \)

**Lemma 7.** Let \( D \) be a locally special poset on \( N \) with \( \mathcal{G} \subseteq \text{Aut}(D) \). Then
\[ V_{\alpha,D} = \bigoplus_{\beta \leq \alpha} \theta_{\beta,\alpha}(W_{\beta,D}) \tag{6} \]
as \( \mathcal{G} \)-modules for all \( \alpha \models |N| \), where \( \bigoplus \) denotes the internal direct sum.

**Proof.** We prove Eq. (6) by induction on \( \ell(\alpha) \). If \( \ell(\alpha) = 1 \), then \( \alpha = |N| \). By definition, \( U_{\alpha,D} = \{0\} \), and \( W_{\alpha,D} = V_{\alpha,D} \). Thus, Eq. (6) follows, as there is only one term in the direct sum, and it is equal to \( V_{\alpha,D} \).

Now, suppose that \( \ell(\alpha) > 1 \). Since \( V_{\alpha,D} = U_{\alpha,D} \oplus W_{\alpha,D} \)
\[ \bigoplus_{\beta \leq \alpha} \theta_{\beta,\alpha}(W_{\beta,D}) = \theta_{\alpha,\alpha}(W_{\alpha,D}) \oplus \bigoplus_{\beta < \alpha} \theta_{\beta,\alpha}(W_{\beta,D}) \]
and \( \theta_{\alpha,\alpha} \) is the identity map, it suffices to show that
\[ U_{\alpha,D} = \bigoplus_{\beta < \alpha} \theta_{\beta,\alpha}(W_{\beta,D}). \tag{7} \]

Let \( \beta < \alpha \). Since \( W_{\beta,D} \subseteq V_{\beta,D} \) as \( \mathcal{G} \)-submodules, and since \( \theta_{\beta,\alpha} \) is injective and \( \mathcal{G} \)-invariant, we see that \( \theta_{\beta,\alpha}(W_{\beta,D}) \subseteq \theta_{\beta,\alpha}(V_{\beta,D}) \subseteq \text{span} \theta_{\beta,\alpha}(X_{\beta,D}) \subseteq U_{\alpha,D} \) as \( \mathcal{G} \)-submodules. Thus
\[ \bigoplus_{\beta < \alpha} \theta_{\beta,\alpha}W_{\beta,D} \subseteq U_{\alpha,D} \]
as \( \mathcal{G} \)-submodules.
Let $\beta < \alpha$. For the other direction, we see that
\[
\theta_{\beta, \alpha}(X_{\beta, D}) \subseteq \theta_{\beta, \alpha}(V_{\beta, D})
\]
\[
= \theta_{\beta, \alpha} \left( \bigoplus_{\gamma \leq \beta} \theta_{\gamma, \beta}(W_{\gamma, D}) \right)
\]
\[
= \bigoplus_{\gamma \leq \beta} \theta_{\beta, \alpha} \circ \theta_{\gamma, \beta}(W_{\gamma, D})
\]
\[
= \bigoplus_{\gamma \leq \beta} \theta_{\gamma, \alpha}(W_{\gamma, D})
\]
\[
\subseteq \bigoplus_{\gamma < \alpha} \theta_{\gamma, \alpha}(W_{\gamma, D}),
\]
where all the inclusions and equalities are as $\mathcal{G}$-sets. The first inclusion follows
from the fact that $X_{\beta, D} \subseteq V_{\beta, D}$, and $\theta_{\beta, \alpha}$ is injective and $\mathcal{G}$-invariant. We
use the induction hypothesis to obtain the first equality. The second equality
follows from the fact that $\theta_{\beta, \alpha}$ is injective and $\mathcal{G}$-invariant. The third equality
follows from the fact that $\theta_{\beta, \alpha} \circ \theta_{\gamma, \beta} = \theta_{\gamma, \alpha}$. The last inclusion is immediate.

Thus
\[
\bigcup_{\beta < \alpha} \theta_{\beta, \alpha}(X_{\beta, D}) \subseteq \bigoplus_{\beta < \alpha} \theta_{\beta, \alpha}(W_{\beta, D})
\]
as $\mathcal{G}$-sets. If we take the linear span on the left-hand side of the inclusion, then
we have proven Eq. (7). Therefore, Eq. (6) holds. \hfill \Box

Proof of Theorem 5. Let $D$ be a locally special poset on $N$ with $\mathcal{G} \subseteq \text{Aut}(D)$. Let us write $\Omega(D, \mathcal{G}, x) = \sum_{\alpha = \mid N \mid} \psi_{\alpha, D} F_\alpha$ where the $\psi_{\alpha, D}$ are virtual characters. Since $[M_\alpha] \Omega(D, \mathcal{G}, x) = \chi_\alpha(D, \mathcal{G})$, we have $\chi_\alpha(D, \mathcal{G}) = \sum_{\beta \leq \alpha} \psi_{\beta, D}$. Since $\theta_{\beta, \alpha}$ is injective and $\mathcal{G}$-invariant, we see that $\theta_{\beta, \alpha}(W_{\beta, D}) \simeq W_{\beta, D}$ for all $\beta \leq \alpha$. Applied to Eq. (6), we obtain that
\[
V_{\alpha, D} \simeq \bigoplus_{\beta \leq \alpha} W_{\beta, D}
\]
as $\mathcal{G}$-modules. If we let $\xi_{\alpha, D}$ be the character of $W_{\alpha, D}$, then
\[
\sum_{\beta \leq \alpha} \psi_{\beta, D} = \chi_{\alpha, D} = \sum_{\beta \leq \alpha} \xi_{\beta, D}.
\]
Since $\psi_{\mid N \mid, D} = \chi_{\mid N \mid, D} = \xi_{\mid N \mid, D}$, it follows from induction on $\ell(\alpha)$ that $\psi_{\alpha, D} = \xi_{\alpha, D}$. Hence, $\psi_{\alpha, D}$ is an effective character for all $\alpha = \mid N \mid$. \hfill \Box

8. Orbital Invariants

In this section, we define orbital quasisymmetric function invariants. In the case of double posets, the resulting invariant was already studied by Grinberg [10]. These are quasisymmetric functions whose coefficients count the number of orbits of a group action. Due to Burnside’s Lemma, these invariants can be computed as $\langle 1, \chi \rangle$, where 1 is the trivial character, and $\chi$ is the quasisymmetric
class function. As a result, we see that we derive many results for our orbital invariants from the class functions.

Let $D$ be a double poset on $N$, and let $\mathcal{G} \subseteq \text{Aut}(D)$. We define the orbital quasisymmetric $D$-partition enumerator by

$$\Omega^O(D, \mathcal{G}, x) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \Omega(D, \mathcal{G}, x; g)$$

and the orbital order polynomial by

$$\Omega^O(D, \mathcal{G}, x) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \Omega(D, \mathcal{G}, x; g).$$

Let $G$ be a digraph on a finite set $N$, and let $\mathcal{H} \subseteq \text{Aut}(G)$. We define the orbital chromatic quasisymmetric function of a digraph $G$ by

$$\chi^O(G, \mathcal{H}, t, x) = \frac{1}{|\mathcal{H}|} \sum_{g \in \mathcal{H}} \chi(G, \mathcal{H}, t, x; g)$$

and the orbital chromatic polynomial by

$$\chi^O(G, \mathcal{H}, t, x) = \frac{1}{|\mathcal{H}|} \sum_{g \in \mathcal{H}} \chi(G, \mathcal{H}, t, x; g).$$

The following results follow from Burnside’s Lemma. Property 2 is Grinberg’s definition of the orbital $D$-partition enumerator. Our results involve $\mathcal{G}$-sets, so we use terminology and notation from Sect. 2.1.

**Proposition 10.** Let $D$ be a double poset on a finite set $N$ and let $\mathcal{G} \subseteq \text{Aut}(D)$. Let $G$ be a digraph on $N$, and let $\mathcal{H} \subseteq \text{Aut}(G)$.

1. Given $\alpha \models |N|$, we see that $X_{\alpha,D}$ is a $\mathcal{G}$-set. We have $[M_\alpha] \Omega^O(D, \mathcal{G}, x) = |X_{\alpha,D}/\mathcal{G}|$, the number of orbits of $X_{\alpha,D}$ under the action of $\mathcal{G}$.
2. Let $T$ be a transversal for $\mathcal{G}$ acting on $P_D$. Then, $\Omega^O(D, \mathcal{G}, x) = \sum_{f \in T} x^f$.
3. For $n \in \mathbb{N}$, we have that $X_{n,D}$ is a $\mathcal{G}$-set. Moreover, $\Omega^O(D, \mathcal{G}, n) = |X_{n,D}/\mathcal{G}|$.
4. Let $T$ be a transversal for $\mathcal{H}$ acting on $C_G$. Then, $\Omega^O(G, \mathcal{H}, x) = \sum_{f \in T} x^f$.
5. For $n \in \mathbb{N}$, and $k > 0$, we have that $C_{k,n,G}$ is a $\mathcal{H}$-set. Moreover, $[t^k] \chi^O(G, \mathcal{H}, t, n) = |C_{k,n,G}/\mathcal{H}|$.

**Proof.** We will prove the first and second claim. The rest are similar. For the first, we observe that

$$[M_\alpha] \Omega^O(D, \mathcal{G}, x) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} [M_\alpha] \Omega(D, \mathcal{G}, x; g) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} |\text{Fix}_g(X_{\alpha,D})|.$$ 

For the second identity, we see that

$$\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \Omega(D, \mathcal{G}, x; g) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \sum_{f \in \text{Fix}_g(X_D)} x^f.$$
\[ \frac{1}{|\mathcal{G}|} \sum_{f \in X_D} x^f \sum_{g \in \mathcal{G}} 1 \]
\[ = \sum_{f \in X_D} \frac{1}{|\mathcal{G}(f)|} x^f \]
\[ = \sum_{h \in \mathcal{T}} x^h \sum_{f \in \mathcal{G}(h)} \frac{1}{|\mathcal{G}(f)|}. \]

The second equality is just a rearrangement of terms. The third equality follows from the Orbit-Stabilizer Theorem. The last equality involves splitting the summation into a double sum, and recognize that \( x^f \) is constant on orbits.

Finally, the inner summation simplifies to 1. \( \square \)

We obtain several facts about the coefficients of the orbital invariants with respect to various bases.

**Corollary 3.** Let \( D \) be a double poset on \( N \). Given \( \mathcal{G} \subseteq \text{Aut}(D) \), we have \( \Omega^O(D, \mathcal{G}, x) \) is \( M \)-increasing, and \( \Omega^O(D, \mathcal{G}, x) \) is strongly flawless. If \( D \) is locally special, then \( \Omega^O(D, \mathcal{G}, x) \) is \( F \)-positive, and \( \Omega^O(D, \mathcal{G}, x) \) is \( h \)-positive.

Let \( G \) be a directed graph on \( N \), and let \( H \subseteq \text{Aut}(G) \). For \( k \in \mathbb{N} \), we see that \( t^k \chi^O(G, H, x) \) is \( F \)-positive and \( M \)-increasing. Moreover, \( \chi^O(G, H, x) \) is \( h \)-positive and strongly flawless.

**Proof.** Let \( D \) be a double poset on \( N \), and let \( \mathcal{G} \subseteq \text{Aut}(D) \). Since \( \Omega^O(D, \mathcal{G}, x) \) is \( M \)-increasing, then if we take the inner product with the trivial character, Proposition 2 (4) implies that \( \Omega^O(D, \mathcal{G}, x) \) is \( M \)-increasing. Also, Proposition 3 (5) implies that \( \Omega^O(D, \mathcal{G}, x) \) is strongly flawless.

Suppose that \( D \) is locally special. Then, by Corollary 1, applied with \( \psi \) being the trivial character, we see that \( \Omega^O(D, \mathcal{G}, x) \) is \( F \)-positive. Since \( \Omega^O(D, \mathcal{G}, x) = \text{ps}(\Omega^O(D, \mathcal{G}, x)) \), it follows from Proposition 3 (5) that \( \Omega^O(D, \mathcal{G}, x) \) is \( h \)-positive.

Let \( G \) be a directed graph, and let \( H \subseteq \text{Aut}(G) \). Fix \( k \in \mathbb{N} \). By Theorem 6, we see that \( [t^k] \chi^O(G, H, x) \) is \( F \)-positive. Hence, it is also \( M \)-increasing. By principal specialization and Proposition 3 (5), we see that \( [t^k] \chi^O(G, H, x) \) is \( h \)-positive, and \( [t^k] \chi^O(G, H, x) \) is strongly flawless. \( \square \)

### 8.1. Orbital Combinatorial Reciprocity Results

We can obtain a combinatorial reciprocity for the orbital \( D \)-partition enumerator, although it involves the notion of coeven \( D \)-partition. Given a group \( \mathcal{G} \subseteq \mathcal{G}_N \), and an action of \( \mathcal{G} \) on a set \( X \), let we say that an element \( x \in X \) is \( \mathcal{G} \)-coeven if the stabilizer subgroup \( \mathcal{G}_x \) is a subgroup of the alternating group \( \mathfrak{A}_N \). Let \( X^+ \) be the set of \( \mathcal{G} \)-coeven elements. Then, \( \mathcal{G} \) acts on \( X^+ \).

The following Lemma is essentially a restatement of Lemma 7.7 in Grinberg [10], although our proof is slightly different.

**Lemma 8.** Let \( \mathcal{G} \subseteq \mathcal{G}_N \) be a group acting on a finite set \( X \). Then

\[ |X^+ / \mathcal{G}| = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \text{sgn}(g)|\text{Fix}_g(X)|. \]
Proof. We have
\[
\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \text{sgn}(g)|\text{Fix}_g(X)| = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \sum_{x \in X: gx = x} \text{sgn}(g)
\]
\[
= \sum_{x \in X} \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}_x} \text{sgn}(g).
\]
Suppose that \( \mathcal{G}_x \not\subseteq \mathcal{A}_N \). Then, \( \mathcal{H} = \mathcal{A}_N \cap \mathcal{G} \) is a normal subgroup of \( \mathcal{G} \) of index 2. Thus, half the elements \( g \) of \( \mathcal{G}_x \) are even, and half are odd, and these elements have opposite signs under \( \text{sgn} \). Hence, the inner sum is zero in that case.

Hence, we are left with those \( x \) for which \( \mathcal{G}_x \subseteq \mathcal{A}_N \). Then, we obtain
\[
\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \text{sgn}(g)|\text{Fix}_g(X)| = \sum_{x \in X^+} \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}_x} \text{sgn}(g)
\]
\[
= \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \sum_{x \in \text{Fix}_g(X^+)} 1
\]
\[
= |X^+/\mathcal{G}|.
\]
\( \square \)

We define the \( \mathcal{G} \)-coeven quasisymmetric function by
\[
\Omega_+^+(D, \mathcal{G}, x) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \text{sgn}(g)\Omega(D, \mathcal{G}, x; g)
\]
and the \( \mathcal{G} \)-coeven orbital polynomial by
\[
\Omega_+^+(D, \mathcal{G}, x) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \text{sgn}(g)\Omega(D, \mathcal{G}, x; g).
\]

Using Lemma 8, we obtain the following results.

**Proposition 11.** Let \( D \) be a double poset on a finite set \( N \) and let \( \mathcal{G} \subseteq \text{Aut}(D) \). Let \( \mathcal{G} \) be a digraph on \( N \), and let \( \mathcal{H} \subseteq \text{Aut}(\mathcal{G}) \).

1. We have \( [M_\alpha]\Omega^+(D, \mathcal{G}, x) = |X_{\mathcal{G},D}^+| \).
2. Let \( T \) be a transversal for \( \mathcal{G} \) acting on \( P_\mathcal{G}^+ \). Then, \( \Omega^+(D, \mathcal{G}, x) = \sum_{f \in T} x^f \).
3. For \( n \in \mathbb{N} \), we have \( \Omega^+(D, \mathcal{G}, n) = |X_{n,D}^+/\mathcal{G}| \).
4. If \( T \) is a transversal for \( \mathcal{H} \) acting on \( C_\mathcal{G}^+ \), then \( \chi^+(G, \mathcal{H}, t, n) = \sum_{f \in T} t^{\asc(f)} x^f \).
5. For \( n \in \mathbb{N} \), and \( k > 0 \), we have \( [t^k]\chi^+(G, \mathcal{H}, t, n) = |C_{k,n,G}^+/\mathcal{H}| \).

Now, we discuss the Combinatorial Reciprocity Theorem for orbital quasisymmetric functions. The first result is Theorem 4.7 of Grinberg [10].

**Theorem 8.** Let \( D \) be a locally special double poset on a finite set \( N \), and let \( \mathcal{G} \subseteq \text{Aut}(D) \). Then, we have the following identities.

1. \( (-1)^{|N|} S \Omega^O(D, \mathcal{G}, x) = \Omega^+(D^*, \mathcal{G}, x) \).
2. \((-1)^{|N|}S^{\Omega^+}(D, \mathcal{G}, x) = \Omega^O(D^*, \mathcal{G}, x)\).
3. \((-1)^{|N|}\Omega^O(D, \mathcal{G}, -x) = \Omega^+(D^*, \mathcal{G}, x)\).
4. \((-1)^{|N|}\Omega^+(D, \mathcal{G}, -x) = \Omega^O(D^*, \mathcal{G}, x)\).

Now, we discuss the Combinatorial Reciprocity Theorem for orbital chromatic quasisymmetric functions.

**Theorem 9.** Let \(G\) be a digraph on a finite set \(N\), and let \(\mathcal{H} \subseteq \text{Aut}(G)\). Then, we have the following identities:
1. \((-1)^{|N|}S\chi^O(G, \mathcal{H}, t, x) = \chi^+(G, \mathcal{H}, t, x)\).
2. \((-1)^{|N|}S\chi^+(G, \mathcal{H}, t, x) = \chi^O(G, \mathcal{H}, t, x)\).
3. \((-1)^{|N|}\chi^O(G, \mathcal{H}, t, -x) = \chi^+(G, \mathcal{H}, t, x)\).
4. \((-1)^{|N|}\chi^+(G, \mathcal{H}, t, -x) = \chi^O(G, \mathcal{H}, t, x)\).

**Proof.** All formulas are proven in a similar manner, so we only prove the first formula. Let \(G\) be a digraph on a finite set \(N\), and let \(\mathcal{H} \subseteq \text{Aut}(G)\). By Theorem 3, we have
\[
(-1)^{|N|}S\chi^O(G, \mathcal{H}, t, x) = \frac{(-1)^{|N|}}{|\mathcal{H}|} \sum_{g \in \mathcal{H}} S\chi(G, \mathcal{H}, t, x; g)\]
\[
= \frac{1}{|\mathcal{H}|} \sum_{g \in \mathcal{H}} \text{sgn}(g)\chi(G, \mathcal{H}, t, x; g)\]
\[
= \chi^+(G, \mathcal{H}, t, x).\]
\[
\square
\]

**9. Future Directions**
First, we note that Stapledon [19] defines a different generalization of the \(h^*\)-vector. His work involves a group \(\mathcal{G}\) acting on the lattice points of a polytope. Given a quasipolynomial \(p(x)\), whose coefficients are characters, he defines
\[
\sum_{n \geq 0} p(n)t^n = \frac{h^*(t)}{(1 - t) \det[I - t\rho]}.
\]
The \(h^*\)-vector is defined for any class function that takes on values in the ring of quasipolynomials, while our \(h\)-vector is only defined for class functions in the ring of polynomials, so the \(h^*\)-vector is a more general invariant. When we restrict to polynomials, the \(h^*\)-vector is different than the \(h\)-vector. Is the order polynomial class function of a double poset \(h^*\)-effective? Is the chromatic polynomial of a digraph \(h^*\)-effective?

There are other bases of quasisymmetric functions. A very recent basis is the basis of quasisymmetric power sums \(\Psi_\alpha\), introduced in [3]. It has been shown that the \(P\)-partition enumerator and the chromatic quasisymmetric function of a digraph are both \(\Psi\)-positive [2]. Is the \(D\)-partition quasisymmetric class function \(\Psi\)-effective? Is the chromatic quasisymmetric class function \(\Psi\)-effective?
We would like to have a better description of $\Omega^O(D, \mathcal{G}, x)$ in the $F$ basis. We know that the coefficients are positive. What do they count?

On a similar note, is the $f$-vector $\Omega^O(D, \mathcal{G}, x)$ effectively unimodal? This would mean that there exists an $i$, such that $f_j \leq G f_k$ for $k \leq i$, and $f_j \geq G f_k$ for $i \leq j$. This is still an open question even for trivial group actions. Our example in Fig. 4 shows that the $h$-vector for a locally special double poset can fail to be effectively unimodal. We can obtain the $h$-vector of $\Omega^O(D, \mathcal{G}, x)$ by taking the coefficients of $\chi_1$ in the $h$-vector of $\Omega(D, \mathcal{G}, x)$. Doing so results in the sequence $(0, 0, 1, 0, 1, 0)$ which fails to be unimodal.

Finally, we could consider proving our results using the theory of combinatorial Hopf monoids, as studied by Aguiar and Mahajan [1]. We will pursue this idea in a subsequent paper, using the theory of Hopf monoids to prove $F$-effectiveness and combinatorial reciprocity results for quasisymmetric class function invariants associated with the other combinatorial Hopf monoids.

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