Lax representations for separable systems from Benenti class

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Abstract

In this paper we construct Lax pairs for Stäckel systems with separation curves from so-called Benenti class. For each system of considered family we present an infinite family of Lax representations, parameterized by smooth functions of spectral parameter.

Keywords and phrases: Lax representation, Stäckel system, Benenti system, Hamiltonian mechanics

1 Introduction

Classical Stäckel systems belong to important class of integrable and separable Hamiltonian ODE’s. The constants of motion of these systems are quadratic in momenta and describe many physical systems of classical mechanics. The Stäckel systems are defined by separation relations, i.e. relations involving canonical variables $(\lambda_i, \mu_i)_{i=1,...,n}$ in which the Hamilton-Jacobi equations separate to a system of decoupled ordinary differential equations. The separation relations of classical Stäckel system are represented by $n$ algebraic equations of the form

$$\sigma_i(\lambda_i) + \sum_{k=1}^{n} H_k S_{ik}(\lambda_i) = \frac{1}{2} f_i(\lambda_i) \mu_i^2, \quad i = 1, 2, \ldots, n,$$  \hspace{1cm} (1.1)

where $n$ is the number of degrees of freedom of the system, i.e. $2n$ is the dimension of the corresponding phase space on which the system is defined, $H_1, H_2, \ldots, H_n$ are $n$ Hamiltonians, $f$, $\sigma$ and $S$ are arbitrary smooth functions. Solving the system (1.1), under assumption that $\det S \neq 0$, with respect to all $H_i$ we get $n$ Hamiltonians expressed in variables $(\lambda_i, \mu_i)_{i=1,...,n}$, which from construction will be in involution, i.e. their Poisson brackets vanish $\{H_i, H_j\} = 0$, and which Hamilton-Jacobi equations separate. In other words the system (1.1) describes a Liouville-integrable and separable Hamiltonian system.

The special, but particularly important class of Stäckel systems is the Benenti class [1, 2]. This class is described by the following separation relations

$$\sigma_i(\lambda_i) + H_1 \lambda_i^{n-1} + H_2 \lambda_i^{n-2} + \cdots + H_n = \frac{1}{2} f_i(\lambda_i) \mu_i^2, \quad i = 1, 2, \ldots, n.$$  \hspace{1cm} (1.2)

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There is an extended literature on systems from class (1.2), however less can be found on their Lax representation. A Lax representation of a Liouville integrable Hamiltonian system is a set of matrices $L$, $U_k$ ($k = 1, 2, \ldots, n$) which satisfy the system of Lax equations
\[
\frac{dL}{dt_k} = [U_k, L], \quad k = 1, 2, \ldots, n,
\]
under the assumption that the time evolution with respect to $t_k$ is governed by the Hamiltonian vector field $X_{H_k}$.

In the paper we find an infinite family of Lax representations for an arbitrary Stäckel system from the Benenti class generated by the following canonical form of separation curves
\[
X_k = E_k(\lambda, \mu) + V_k(\lambda), \quad X_k(\lambda, \mu) = \pi dH_k(\lambda, \mu)
\]
where separation relations (1.2) are reconstructed by $n$ copies of (1.4) with $(\lambda_i, \mu_i) = 1, \ldots, n$. In literature, the reader can find Lax representation for systems related to various subcases of separation curves from the family (1.4). First constructions of Lax representation for separable systems with separation curves of particular hyperelliptic form was constructed by Mumford [3]. For $\sigma(\lambda)$ in polynomial form and $f(\lambda) = \lambda^{2r}$, $r \in \mathbb{N}$, Lax equations were constructed explicitly in [4] and analyzed in particular coordinate frames. In [5], using different technique, Lax representation was constructed for the subclass of separation curves (1.4) with $f(\lambda)$ being polynomials of order $n + 1$, $n$ and $n - 1$, respectively, with distinct roots. Yet another subcases of family (1.4) together with the construction of Lax representations by various techniques, the reader can find in [6] and [7].

Here we present the explicit form of infinite family of admissible Lax representations, for systems generated by (1.3) with $\sigma(\lambda)$ and $f(\lambda)$ being arbitrary smooth functions, in separation coordinates, so called Vieté coordinates and flat coordinates, if such exist. The paper is organized as follows. In Section 2 we present basic facts about Benenti systems. In Section 3 we present in explicit form the infinite family (1.3) in separation coordinates and state that the characteristic equation of each Lax matrix corresponds to the separation curve (1.4) of Benenti system. All results of this section we gather in Theorem 2, which we prove in Section 4. Section 5 contains the particular Lax representations in Vieté coordinates and their relation with flat coordinates. Finally, Section 6 contains several examples illustrating the theory.

2 Preliminaries

Let us consider separable systems generated by separation curves for canonical coordinates in the form (1.4). Actually, solving $n$ copies of (1.4) with $(\lambda_i, \mu_i) = 1, \ldots, n$ we get $n$ Hamiltonians $H_k(\lambda, \mu)$ and $n$ related vector fields $X_k(\lambda, \mu)$, $k = 1, \ldots, n$
\[
H_k(\lambda, \mu) = E_k(\lambda, \mu) + V_k(\lambda), \quad X_k(\lambda, \mu) = \pi dH_k(\lambda, \mu)
\]
such that
\[
\{\lambda_i, \mu_j\}_\pi = \delta_{ij}, \quad \{\lambda_i, \lambda_j\}_\pi = \{\mu_i, \mu_j\}_\pi = 0, \quad \{H_i, H_j\}_\pi = \pi (dH_i, dH_j) = 0, \quad [X_i, X_j] = 0.
\]
Geodesic parts $E_k$, defined by
\[
\sum_{j=1}^{n} E_j \lambda^{n-j} = \frac{1}{2} f(\lambda)\mu^2,
\]
in canonical coordinates $(\lambda_i, \mu_i) = 1, \ldots, n$ take the form
\[
E_j(\lambda, \mu) = -\frac{1}{2} \sum_{i=1}^{n} \frac{\partial \rho_j f(\lambda_i) \mu^2_i}{\Delta_i} = \frac{1}{2} \sum_{i=1}^{n} (K_j G_i) \mu^2_i,
\]
where

\[ \rho_k = (-1)^k \sigma_k, \quad \Delta_i = \prod_{k \neq i} (\lambda_i - \lambda_k), \]  

(2.4)

\( \sigma_k \) are elementary symmetric polynomials, \( G \) is contravariant metric tensor defined by \( E_1 \) and \( K_j \) are Killing tensors of \( G \), where

\[ G^{rs} = \frac{f(\lambda_r)}{\Delta_r} \delta^{rs}, \quad (K_j)^{rs}_r = -\frac{\partial \rho_j}{\partial \lambda_r} \delta^r_s. \]

Notice that for

\[ u(\lambda) = \prod_{k=1}^{n} (\lambda - \lambda_k) = \sum_{k=0}^{n} \rho_k \lambda^{n-k}, \quad \rho_0 \equiv 1 \]  

(2.5)

we have

\[ \Delta_i = u_i(\lambda), \quad u_i(\lambda) := -\frac{\partial u(\lambda)}{\partial \lambda_i} = \prod_{k \neq i} (\lambda_i - \lambda_k) = -\sum_{k=1}^{n} \frac{\partial \rho_k}{\partial \lambda_i} \lambda^{n-k}. \]  

(2.6)

Basic potentials \( V^{(\gamma)}_k \), corresponding to monomials \( \sigma(\lambda) = \lambda^\gamma, \gamma \in \mathbb{Z} \), and defined by

\[ \lambda^\gamma + \sum_{k=1}^{n} V^{(\gamma)}_k \lambda^{n-k} = 0, \quad \gamma \in \mathbb{Z} \]  

(2.7)

are constructed by the formula

\[ V^{(\gamma)} = R^\gamma V^{(0)}, \quad V^{(\gamma)} = (V^{(\gamma)}_1, \ldots, V^{(\gamma)}_n)^T, \]  

(2.8)

where

\[ R = \begin{pmatrix} -\rho_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -\rho_n & 0 & 0 & 0 \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\rho_n} \\ 0 & 0 & 0 & \vdots \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -\frac{\rho_{n-1}}{\rho_n} \end{pmatrix} \]

(2.9)

\( V^{(0)} = (0, \ldots, 0, -1)^T \). Notice that for \( \gamma = 0, \ldots, n-1 \)

\[ V^{(\gamma)}_k = -\delta_{k,n-\gamma} \]  

(2.10)

that is

\[ V^{(0)} = (0, \ldots, 0, -1)^T, \ldots, V^{(n-1)} = (-1, 0, \ldots, 0)^T. \]  

(2.11)

The first nontrivial positive potential is

\[ V^{(n)} = (\rho_1, \ldots, \rho_n)^T \]  

(2.12)

and the negative one

\[ V^{(-1)} = \left( \frac{1}{\rho_n}, \ldots, \frac{\rho_{n-1}}{\rho_n} \right)^T. \]  

(2.13)

### 3 Lax representation in separation coordinates

In order to describe the Lax representation let us introduce the following notation for the division of polynomial by polynomial and the division of pure Laurent polynomial by polynomial. Let

\[ f(\lambda) = \sum_{k=0}^{m} f_k \lambda^k, \quad g(\lambda^{-1}) = \sum_{k=1}^{m} g_k \lambda^{-k}, \quad a(\lambda) = \sum_{k=s}^{n} a_k \lambda^k. \]  

(3.1)
For polynomial \( f(\lambda) \) of order \( m \) and \( a(\lambda) \) of order \( n \), such that \( m \geq n \), \( \left[ \frac{f(\lambda)}{a(\lambda)} \right]_+ \) is a polynomial part of \( \frac{f(\lambda)}{a(\lambda)} \) of order \( (m - n) \) and \( f(\lambda) \) modulo \( a(\lambda) \) denotes the reminder of \( \frac{f(\lambda)}{a(\lambda)} \), i.e.

\[
f(\lambda) \mod a(\lambda) = \text{rem} \left[ \frac{f(\lambda)}{a(\lambda)} \right], \quad (3.2)
\]

being a polynomial of order less than \( n \). In the division algorithm we divide \( f(\lambda) \) by the highest order term of \( a(\lambda) \). For pure Laurent polynomial \( g(\lambda^{-1}) \) of order \( m \) and polynomial \( a(\lambda) \) of order \( n \) and \( 0 \leq s \leq n \) being the lowest order term, \( \left[ \frac{g(\lambda^{-1})}{a(\lambda)} \right]_+ \) is a pure Laurent polynomial of order \( m + s \) and \( g(\lambda^{-1}) \) modulo \( a(\lambda) \) denotes the reminder of \( \frac{g(\lambda^{-1})}{a(\lambda)} \) being again a polynomial of degree less than \( n \). In the division algorithm we divide \( g(\lambda^{-1}) \) by the lowest order term of \( a(\lambda) \). In particular

\[
\frac{f(\lambda)}{a(\lambda)} = \left[ \frac{f(\lambda)}{a(\lambda)} \right]_+ + \frac{1}{a(\lambda)} \text{rem} \left[ \frac{f(\lambda)}{a(\lambda)} \right], \quad (3.3)
\]

or

\[
f(\lambda) = f(\lambda) \mod a(\lambda) + a(\lambda) \left[ \frac{f(\lambda)}{a(\lambda)} \right]_+, \quad (3.4)
\]

and similarly

\[
\frac{g(\lambda^{-1})}{a(\lambda)} = \left[ \frac{g(\lambda^{-1})}{a(\lambda)} \right]_+ + \frac{1}{a(\lambda)} \text{rem} \left[ \frac{g(\lambda^{-1})}{a(\lambda)} \right], \quad (3.5)
\]

\[
g(\lambda^{-1}) = g(\lambda^{-1}) \mod a(\lambda) + a(\lambda) \left[ \frac{g(\lambda^{-1})}{a(\lambda)} \right]_+. \quad (3.6)
\]

In the case of arbitrary Laurent polynomial \( F(\lambda, \lambda^{-1}) = f(\lambda) + g(\lambda^{-1}) \), the division by polynomial \( a(\lambda) \) splits onto two parts described above.

**Remark 1.** The results of this section can be generalized to arbitrary smooth functions \( f(\lambda) \) and \( \sigma(\lambda) \) defined on an open subset \( \mathcal{U} \subset \mathbb{R} \) such that for any phase space point \((\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)\) each \( \lambda_i \in \mathcal{U}, \quad i = 1, 2, \ldots, n. \) For this we have to note that if \( g(\lambda) \) is some smooth function on \( \mathcal{U} \) and

\[
a(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)
\]

is a polynomial of order \( n \) whose roots \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{U} \), then the fraction \( \frac{a(\lambda)}{a(\lambda)} \) can be uniquely written as

\[
\frac{g(\lambda)}{a(\lambda)} = h(\lambda) + \frac{r(\lambda)}{a(\lambda)}, \quad (3.7)
\]

where \( h(\lambda) \) is a smooth function on \( \mathcal{U} \) and \( r(\lambda) \) is a polynomial of order less than \( n \). Indeed, we calculate that

\[
\frac{g(\lambda)}{a(\lambda)} = \frac{h_1(\lambda)}{(\lambda - \lambda_2)(\lambda - \lambda_3) \cdots (\lambda - \lambda_n)} + \frac{h_0(\lambda_1)}{a(\lambda)}
= \frac{h_2(\lambda)}{(\lambda - \lambda_3)(\lambda - \lambda_4) \cdots (\lambda - \lambda_n)} + \frac{h_0(\lambda_1) + h_1(\lambda_2)(\lambda - \lambda_1)}{a(\lambda)} = \ldots
= h_n(\lambda) + \frac{h_0(\lambda_1) + h_1(\lambda_2)(\lambda - \lambda_1) + \cdots + h_{n-1}(\lambda_n)(\lambda - \lambda_1) \cdots (\lambda - \lambda_{n-1})}{a(\lambda)},
\]

where

\[
h_0(\lambda) = g(\lambda), \quad h_i(\lambda) = \frac{h_{i-1}(\lambda) - h_{i-1}(\lambda)}{\lambda - \lambda_i}, \quad i = 1, 2, \ldots, n.
\]

From the Taylor theorem we can see that each function \( h_i(\lambda) \) is smooth on \( \mathcal{U} \). In particular, we have that

\[
\lim_{\lambda \to \lambda_i} \frac{d^k}{d\lambda^k} h_i(\lambda) = \frac{1}{k + 1} \frac{d^{k+1}}{d\lambda^{k+1}} h_{i-1}(\lambda_i).
\]
Putting \( h(\lambda) = h_n(\lambda) \) and \( r(\lambda) = h_0(\lambda) + h_1(\lambda_2)(\lambda - \lambda_1) + \cdots + h_{n-1}(\lambda)(\lambda - \lambda_1) \cdots (\lambda - \lambda_{n-1}) \) we get \((3.7)\). The uniqueness of decomposition \((3.7)\) follows from the fact that if

\[
g(\lambda) = \frac{h_1(\lambda)}{a(\lambda)} + \frac{r_1(\lambda)}{a(\lambda)} = h_2(\lambda) + \frac{r_2(\lambda)}{a(\lambda)},
\]

then

\[
h_1(\lambda) - h_2(\lambda) = \frac{r_2(\lambda) - r_1(\lambda)}{a(\lambda)},
\]

where the left hand side is a smooth function on \( U \) and the right hand side is a rational function with singularities at \( \lambda_1, \lambda_2, \ldots, \lambda_n \in U \). Therefore, \( h_1(\lambda) - h_2(\lambda) = 0 \) and \( r_2(\lambda) - r_1(\lambda) = 0 \). Denoting \( h(\lambda) \) by \( \left[ \frac{g(\lambda)}{a(\lambda)} \right]_+ \) and \( r(\lambda) \) by \( g(\lambda) \mod a(\lambda) \) we can reformulate the results of this section in terms of smooth functions \( f(\lambda) \) and \( \sigma(\lambda) \).

We will consider infinitely many Lax matrices \( L \in \mathfrak{sl}(2, \mathbb{R}) \) parameterized by smooth everywhere non-zero functions \( g(\lambda) \) defined on the same domain as functions \( f(\lambda) \) and \( \sigma(\lambda) \). For simplicity we can take \( f(\lambda) \) and \( \sigma(\lambda) \) as Laurent polynomials and \( g(\lambda) = \lambda^r \) for \( r \in \mathbb{Z} \). The Lax matrices in the canonical representation \((\lambda, \mu)\), parameterized by \( g(\lambda) \), are taken in the form

\[
L(\lambda) = \begin{pmatrix} v(\lambda) & u(\lambda) \\ w(\lambda) & -v(\lambda) \end{pmatrix},
\]

where \( u(\lambda) \) is given by \((2.8)\) and

\[
v(\lambda) = \sum_{i=1}^{n} \frac{g(\lambda_i) \mu_i}{\lambda - \lambda_i} = \sum_{i=1}^{n} \frac{u(\lambda)}{\lambda - \lambda_i} \mu_i \left[ \frac{g(\lambda_i) \mu_i}{\Delta_i} \right] = -\sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial \rho_k}{\partial \lambda_i} \frac{g(\lambda_i) \mu_i}{\Delta_i} \lambda^{n-k}.
\]

Notice that \( u(\lambda_i) = 0 \) and \( v(\lambda_i) = g(\lambda_i) \mu_i \). Moreover,

\[
w(\lambda) = \lambda^{\frac{n}{2}} f(\lambda) \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+, \tag{3.10}
\]

where \( F(x, y) = \frac{1}{2} f(x) y^2 - \sigma(x) \). The function \( w(\lambda) \) splits onto kinetic part \( w_E(\lambda) \) and potential part \( w_V(\lambda) \) respectively:

\[
w(\lambda) = w_E(\lambda) + w_V(\lambda) = \left( \frac{g^2(\lambda)}{f(\lambda)} \frac{f(\lambda)v^2(\lambda)/g^2(\lambda)}{u(\lambda)} \right)_+ + \left( \frac{g^2(\lambda)}{f(\lambda)} \frac{\sigma(\lambda)}{u(\lambda)} \right)_+. \tag{3.11}
\]

The main result we state in the following theorem.

**Theorem 2.** For arbitrary \( g(\lambda) \), separation curve \((1.4)\) is reconstructed as follows

\[
\det [L(\lambda) - g(\lambda) I] = 0 \iff \sigma(\lambda) + \sum_{k=1}^{n} H_k \lambda^{n-1} = \frac{1}{2} f(\lambda) \mu^2. \tag{3.12}
\]

Lax equations for systems generated by separation curve \((1.4)\) take the form

\[
\frac{d}{dt_k} L(\lambda) = [U_k(\lambda), L(\lambda)], \tag{3.13}
\]

where the Lax matrix \( L(\lambda) \) is defined by \((3.8)\) and \((3.10)\) and

\[
U_k(\lambda) = \left[ \frac{B_k(\lambda)}{u(\lambda)} \right]_+, \quad B_k(\lambda) = \left( \frac{f(\lambda)}{g(\lambda)} \right) \left[ \frac{u(\lambda)}{\lambda^{n-k+1}} \right]_+ L(\lambda). \tag{3.14}
\]
The proof of the above theorem is involved so we present it in the separate section.

Let us notice that for a given Lax representation \((L, U)\), there exist infinitely many gauge equivalent Lax representations \((L', U')\). Actually, let \(\Omega\) be a 2 \times 2 invertible matrix, with matrix elements dependent on phase space coordinates but independent on spectral parameter \(\lambda\). Then, for

\[ L' = \Omega L \Omega^{-1}, \quad U' = \Omega U \Omega^{-1} + \Omega \Omega^{-1} \]

one can show that

\[ L_t = [U, L] \iff L'_t = [U', L'] \]

and

\[ \det(L - g(\lambda)\mu I) = \det(L' - g(\lambda)\mu I) = 0. \]

Hence, from the construction, such class of equivalent Lax representations has the same \(\lambda\)-structure.

4 Proof of Theorem 2

First, let us prove the following Lemma.

**Lemma 3.** The following equality holds

\[ \sum_{k=1}^{n} H_k \lambda^{n-k} = F(\lambda, v(\lambda)/g(\lambda)) \mod u(\lambda) \quad (4.1) \]

for \(F(x, y) = \frac{1}{2}f(x)y^2 - \sigma(x)\) and \(H_k\) defined by the linear system \((1.2)\).

**Proof.** In the proof we will use the property that a polynomial of order less than \(n\) is uniquely specified by its values at \(n\) distinct points. The functions \(H_k = H_k(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)\) satisfy the equations \((1.2)\)

\[ \sum_{k=1}^{n} H_k(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)\lambda_i^{n-k} = F(\lambda_i, \mu_i), \quad i = 1, 2, \ldots, n. \]

For fixed \(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n\) such that \(\lambda_i \neq \lambda_j\) for \(i \neq j\) the expression

\[ \sum_{k=1}^{n} H_k(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)\lambda^{n-k} \]

is a polynomial in \(\lambda\) of order \(n - 1\), which takes values \(F(\lambda_i, \mu_i)\) at \(\lambda = \lambda_i\). On the other hand

\[ F(\lambda, v(\lambda)/g(\lambda)) \mod u(\lambda) = F(\lambda, v(\lambda)/g(\lambda)) - u(\lambda) \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ \]

is also a polynomial in \(\lambda\) of order \(n - 1\), which takes the same values \(F(\lambda_i, \mu_i)\) at \(\lambda = \lambda_i\), since \(u(\lambda_i) = 0\) and \(v(\lambda_i) = g(\lambda_i)\mu_i\). This proves the equality \((4.1)\).

Now we can pass to the proof of formula \((3.12)\).
Proof of (3.12). We calculate that

\[
\det [L(\lambda) - g(\lambda)\mu I] = \det \begin{pmatrix} u(\lambda) - g(\lambda)\mu & u(\lambda) \\ v(\lambda) - g(\lambda)\mu & w(\lambda) \end{pmatrix} \\
= -(v(\lambda) - g(\lambda)\mu)(v(\lambda) + g(\lambda)\mu) - u(\lambda)w(\lambda) \\
= -v^2(\lambda) + g^2(\lambda)\mu^2 + 2g(\lambda)f(\lambda)u(\lambda) \left( \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right) \\
+ \frac{1}{2}f(\lambda)v(\lambda)/g(\lambda) - \sigma(\lambda) - u(\lambda) \left( \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right) \\
- \frac{1}{2}g^2(\lambda)f(\lambda)\sigma(\lambda) + g^2(\lambda)\mu^2 \\
= -2g^2(\lambda)f(\lambda) \left( \frac{1}{2}f(\lambda)v(\lambda)/g(\lambda) - \sigma(\lambda) - u(\lambda) \left( \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right) \right) \\
+ \frac{1}{2}g^2(\lambda)f(\lambda)\sigma(\lambda) + g^2(\lambda)\mu^2 \\
= 2g^2(\lambda)f(\lambda) \left( -\sum_{k=1}^{n} H_k(\lambda^n - k) + \frac{1}{2}f(\lambda)\mu^2 - \sigma(\lambda) \right),
\]

where in the last equality we used Lemma 4. This proves (3.12).

Now we will show that the Lax equations (3.13) hold. The proof is based on the following lemmas.

Lemma 4. The Poisson bracket of \( u(\lambda) \) and \( v(\lambda) \) is equal

\[
\{u(\lambda), u(\lambda')\} = 0, \quad \{v(\lambda), v(\lambda')\} = 0, \quad (4.2a)
\]

\[
\{u(\lambda), v(\lambda')\} = \{u(\lambda'), v(\lambda)\} = -\sum_{k=1}^{n} \left( g(\lambda) \left[ \frac{u(\lambda)}{\lambda^{n-k+1}} \right] \mod u(\lambda) \right) \lambda^{n-k}. \quad (4.2b)
\]

Proof. In the proof we will use the property that a polynomial of order less than \( n \) is uniquely specified by its values at \( n \) distinct points. The first equality in (4.2a) is straightforward. The second equality follows from

\[
\{v(\lambda_i), v(\lambda_j)\} = \{g(\lambda_i)\mu_i, g(\lambda_j)\mu_j\} = 0 \quad \text{for } i, j = 1, 2, \ldots, n.
\]

For the proof of (4.2b) note that

\[
\{\rho_k, v(\lambda_j)\} = \{\rho_k, g(\lambda_j)\mu_j\} = (-1)^k \sum_{1 \leq l_1 < l_2 < \cdots < l_k \leq n} \lambda_{l_1} \lambda_{l_2} \cdots \lambda_{l_k} \{g(\lambda_j)\mu_j\}
\]

\[
= -g(\lambda_j) \sum_{m=0}^{k-1} \lambda^m \rho_{k-m-1} \{\lambda_{l_1} \cdots \lambda_{l_k}\} = -g(\lambda_j) \sum_{m=0}^{k-1} \lambda^m \rho_{k-m-1}
\]

is a value of the polynomial \((-g(\lambda) \sum_{m=0}^{k-1} \lambda^m \rho_{k-m-1}) \mod u(\lambda)\) at \( \lambda = \lambda_j \), since \( u(\lambda_j) = 0 \). Because this polynomial is of order less than \( n \) we can write

\[
\{\rho_k, v(\lambda)\} = \left( -g(\lambda) \sum_{m=0}^{k-1} \lambda^m \rho_{k-m-1} \right) \mod u(\lambda) = -g(\lambda) \left[ \frac{u(\lambda)}{\lambda^{n-k+1}} \right] \mod u(\lambda).
\]

Thus

\[
\{u(\lambda'), v(\lambda)\} = \sum_{k=1}^{n} \{\rho_k, v(\lambda)\} \lambda^{n-k} = -\sum_{k=1}^{n} \left( g(\lambda) \left[ \frac{u(\lambda)}{\lambda^{n-k+1}} \right] \mod u(\lambda) \right) \lambda^{n-k},
\]

which proves the second equality in (4.2b). For the proof of the first equality in (4.2b) we calculate that

\[
\{u(\lambda'), v(\lambda)\} = \left\{ \prod_{i=1}^{n} (\lambda' - \lambda_i), \sum_{i=1}^{n} g(\lambda_i)\mu_i \prod_{j \neq i} \frac{\lambda - \lambda_j}{\lambda_i - \lambda_j} \right\} = \sum_{i=1}^{n} g(\lambda_i) \left\{ \prod_{i=1}^{n} (\lambda' - \lambda_i), \mu_i \right\} \prod_{j \neq i} \frac{\lambda - \lambda_j}{\lambda_i - \lambda_j}
\]

\[
= -\sum_{i=1}^{n} g(\lambda_i) \{\lambda_i, \mu_i\} \prod_{j \neq i} \frac{(\lambda' - \lambda_j)(\lambda - \lambda_j)}{\lambda_i - \lambda_j} = \{u(\lambda), v(\lambda')\}.
\]
Lemma 5. Let $q(\lambda)$ be a polynomial of order $n$ and $p(\lambda)$ a smooth function defined on a domain containing all roots of $q(\lambda)$, then
\[
\sum_{k=1}^{n} \lambda^{n-k} p(\lambda) \left[ \lambda^{-n+k-1} q(\lambda) \right]_+ \mod q(\lambda) = \sum_{k=1}^{n} \lambda^{n-k} p(\lambda) \left[ \lambda'^{-n+k-1} q(\lambda') \right]_+ \mod q(\lambda').
\] (4.3)

Proof. We have that
\[
p(\lambda) \left[ \lambda^{-n+k-1} q(\lambda) \right]_+ \mod q(\lambda) = r(\lambda) \left[ \lambda^{-n+k-1} q(\lambda) \right]_+ \mod q(\lambda),
\]
where $r(\lambda) = p(\lambda) \mod q(\lambda)$. Since $\frac{r(\lambda)}{q(\lambda)} = 0$ it farther follows that
\[
p(\lambda) \left[ \lambda^{-n+k-1} q(\lambda) \right]_+ \mod q(\lambda) = r(\lambda) \left[ \lambda^{-n+k-1} q(\lambda) \right]_+ - q(\lambda) \left[ \frac{r(\lambda)}{q(\lambda)} \left[ \lambda^{-n+k-1} q(\lambda) \right]_+ \right]_+.
\]
Using this equality we get
\[
\sum_{k=1}^{n} \lambda^{n-k} p(\lambda) \left[ \lambda^{-n+k-1} q(\lambda) \right]_+ \mod q(\lambda) = \sum_{k=1}^{n} \lambda^{n-k} \left( r(\lambda) \left[ \lambda^{-n+k-1} q(\lambda) \right]_+ - q(\lambda) \left[ \lambda^{-n+k-1} r(\lambda) \right]_+ \right)
\]
\[
= \sum_{k=1}^{n} \lambda^{n-k} \left( r(\lambda') \left[ \lambda'^{-n+k-1} q(\lambda') \right]_+ - q(\lambda') \left[ \lambda'^{-n+k-1} r(\lambda') \right]_+ \right)
\]
\[
= \sum_{k=1}^{n} \lambda^{n-k} p(\lambda') \left[ \lambda'^{-n+k-1} q(\lambda') \right]_+ \mod q(\lambda'),
\]
where the second equality is easily proven by expanding the polynomials. \[\square\]

Lemma 6. The action of Hamiltonian vector fields $X_k = \{ H_k, \cdot \}$ on $u(\lambda)$ and $v(\lambda)$ is equal
\[
X_k u(\lambda) = \frac{\partial F}{\partial y} (\lambda, v(\lambda)/g(\lambda)) \left[ \frac{u(\lambda)}{\chi^{n-k+1}} \right]_+ \mod u(\lambda),
\] (4.4a)
\[
X_k v(\lambda) = g(\lambda) \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ \left[ \frac{u(\lambda)}{\chi^{n-k+1}} \right]_+ \mod u(\lambda),
\] (4.4b)
where $F(x, y) = \frac{1}{2} f(x) y^2 - \sigma(x)$.

Proof. Using Lemmas 5, 4, and 3 we calculate that
\[
\sum_{k=1}^{n} \{ u(\lambda'), H_k \} \chi^{n-k} = \{ u(\lambda'), F(\lambda, v(\lambda)/g(\lambda)) \} \mod u(\lambda)
\]
\[
= \sum_{i=1}^{n} \frac{\partial u(\lambda')}{\partial \lambda_i} \frac{\partial F}{\partial \mu_i} (\lambda, v(\lambda)/g(\lambda)) \frac{1}{g(\lambda)} \frac{\partial v(\lambda)}{\partial \mu_i} \mod u(\lambda)
\]
\[
= \{ u(\lambda'), v(\lambda) \} \frac{1}{g(\lambda)} \frac{\partial F}{\partial \mu} (\lambda, v(\lambda)/g(\lambda)) \mod u(\lambda)
\]
\[
= -\sum_{k=1}^{n} \left( \frac{\partial F}{\partial y} (\lambda, v(\lambda)/g(\lambda)) \left[ \frac{u(\lambda)}{\chi^{n-k+1}} \right]_+ \mod u(\lambda) \right) \chi^{n-k}
\]
\[
= -\sum_{k=1}^{n} \left( \frac{\partial F}{\partial y} (\lambda', v(\lambda')/g(\lambda')) \left[ \frac{u(\lambda')}{\chi^{n-k+1}} \right]_+ \mod u(\lambda') \right) \chi^{n-k}.
\]
By comparing the coefficients of $\lambda^{n-k}$ on the left and right hand side of the above equality we get equation (4.4a). For the proof of (4.4b) we first calculate

$$\frac{\partial}{\partial \lambda_i} \left( F(\lambda, v(\lambda)/g(\lambda)) \mod u(\lambda) \right) = \frac{\partial}{\partial \lambda_i} \left( F(\lambda, v(\lambda)/g(\lambda)) - u(\lambda) \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ \right)$$

$$= \frac{1}{g(\lambda)} \frac{\partial F}{\partial y}(\lambda, v(\lambda)/g(\lambda)) \frac{\partial v(\lambda)}{\partial \lambda_i} - \frac{\partial u(\lambda)}{\partial \lambda_i} \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ - u(\lambda) \left[ \frac{1}{g(\lambda)u(\lambda)} \frac{\partial F}{\partial y}(\lambda, v(\lambda)/g(\lambda)) \frac{\partial v(\lambda)}{\partial \lambda_i} - \frac{F(\lambda, v(\lambda)/g(\lambda))}{u^2(\lambda)} \frac{\partial u(\lambda)}{\partial \lambda_i} \right]_+$$

$$+ u(\lambda) \left[ \frac{F(\lambda, v(\lambda)/g(\lambda)) \partial u(\lambda)}{u^2(\lambda)} \right]_+$$

Since $\left[ \frac{1}{u(\lambda)} \frac{\partial u(\lambda)}{\partial \lambda_i} \right]_+ = 0$ we can write

$$\left[ \frac{1}{u(\lambda)} \frac{\partial u(\lambda)}{\partial \lambda_i} \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ = \left[ \frac{1}{u(\lambda)} \frac{\partial u(\lambda)}{\partial \lambda_i} \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ \right]_+$$

$$= \frac{\partial u(\lambda)}{\partial \lambda_i} \left[ \frac{1}{u(\lambda)} \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ \right]$$

and we get

$$\frac{\partial}{\partial \lambda_i} \left( F(\lambda, v(\lambda)/g(\lambda)) \mod u(\lambda) \right) = \frac{1}{g(\lambda)} \frac{\partial v(\lambda)}{\partial \mu_i} \frac{\partial F}{\partial y}(\lambda, v(\lambda)/g(\lambda)) \mod u(\lambda) - \frac{\partial u(\lambda)}{\partial \lambda_i} \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ \mod u(\lambda).$$

Using this equality and Lemmas 3, 4 and 5 we have

$$\sum_{k=1}^{n} \left\{ v(\lambda'), H_k \right\} \lambda^{n-k} = \left\{ v(\lambda'), F(\lambda, v(\lambda)/g(\lambda)) \mod u(\lambda) \right\}$$

$$= \sum_{i=1}^{n} \left( \frac{\partial v(\lambda')}{\partial \lambda_i} \frac{\partial v(\lambda)}{\partial \mu_i} \frac{1}{g(\lambda)} \frac{\partial F}{\partial y}(\lambda, v(\lambda)/g(\lambda)) \mod u(\lambda) - \frac{\partial v(\lambda')}{\partial \mu_i} \frac{\partial v(\lambda)}{\partial \lambda_i} \frac{1}{g(\lambda)} \frac{\partial F}{\partial y}(\lambda, v(\lambda)/g(\lambda)) \mod u(\lambda) \right.$$

$$+ \left. \frac{\partial v(\lambda')}{\partial \mu_i} \frac{\partial u(\lambda)}{\partial \lambda_i} \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ \mod u(\lambda) \right)$$

$$= \left\{ v(\lambda'), v(\lambda') \right\} \frac{1}{g(\lambda)} \frac{\partial F}{\partial y}(\lambda, v(\lambda)/g(\lambda)) \mod u(\lambda) + \left\{ \lambda(\lambda'), v(\lambda') \right\} \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ \mod u(\lambda)$$

$$= - \sum_{k=1}^{n} \left( g(\lambda) \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ \left[ \frac{u(\lambda)}{\lambda^{n-k+1}} \right]_+ \mod u(\lambda) \right) \lambda^{n-k}$$

$$= - \sum_{k=1}^{n} \left( g(\lambda') \left[ \frac{F(\lambda', v(\lambda')/g(\lambda'))}{u(\lambda') \lambda^{n-k+1}} \right]_+ \mod u(\lambda') \right) \lambda^{n-k}.$$
We can now compute \( X_k u(\lambda) \) and \( X_k v(\lambda) \) for simplicity, we will consider the particular class of systems where \( f(\lambda) = \lambda^m, g(\lambda) = \lambda^r \), \( m, r \in \mathbb{Z} \).

Let us express considered systems and their Lax representation in so-called Vieté coordinates

5 Lax representation in Vieté coordinates and flat coordinates

Let us express considered systems and their Lax representation in so-called Vieté coordinates

5 Lax representation in Vieté coordinates and flat coordinates

Proof of (3.13). The equations (4.4) can be rewritten in the form

\[
X_k u(\lambda) = \frac{f(\lambda)}{g(\lambda)} v(\lambda) \left[ \frac{u(\lambda)}{\lambda^{n+k+1}} \right]_+ - u(\lambda) \left[ \frac{f(\lambda)}{g(\lambda)} v(\lambda) \left[ \frac{u(\lambda)}{\lambda^{n+k+1}} \right]_+ \right], \tag{4.5a}
\]

\[
X_k v(\lambda) = \frac{1}{2} \frac{f(\lambda)}{g(\lambda)} w(\lambda) \left[ \frac{u(\lambda)}{\lambda^{n+k+1}} \right]_+ + \frac{1}{2} u(\lambda) \left[ \frac{f(\lambda)}{g(\lambda)} w(\lambda) \left[ \frac{u(\lambda)}{\lambda^{n+k+1}} \right]_+ \right]. \tag{4.5b}
\]

We can now compute \( X_k w(\lambda) \)

\[
X_k w(\lambda) = -2 \frac{g^2(\lambda)}{f(\lambda)} X_k \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+.
\]

Since \( \left[ \frac{X_k u(\lambda)}{u(\lambda)} \right]_+ = 0 \) we can write

\[
\left[ \frac{F(\lambda, v(\lambda)/g(\lambda)) X_k u(\lambda)}{u(\lambda)} \right]_+ = \left[ \frac{F(\lambda, v(\lambda)/g(\lambda))}{u(\lambda)} \right]_+ \left[ \frac{X_k w(\lambda)}{u(\lambda)} \right]_+ = -\frac{1}{2} \left[ \frac{f(\lambda)}{g(\lambda)} \right]_+ \left[ \frac{X_k u(\lambda)}{u(\lambda)} \right]_+. \tag{4.6}
\]

By (4.5) and (4.6) we have

\[
X_k w(\lambda) = -2 \frac{g^2(\lambda)}{f(\lambda)} \left[ \frac{f(\lambda)}{g^2(\lambda)} v(\lambda) X_k v(\lambda) \right]_+ - \frac{g^2(\lambda)}{f(\lambda)} \left[ \frac{f(\lambda)}{g^2(\lambda)} w(\lambda) X_k u(\lambda) \right]_+.
\]

\[
= -\frac{g^2(\lambda)}{f(\lambda)} \left[ \frac{f(\lambda)}{g^2(\lambda)} v(\lambda) \left[ \frac{f(\lambda)}{g(\lambda)} \left[ \frac{u(\lambda)}{\lambda^{n+k+1}} \right]_+ \right]_+ \right]_+ + \frac{g^2(\lambda)}{f(\lambda)} \left[ \frac{f(\lambda)}{g^2(\lambda)} w(\lambda) \left[ \frac{f(\lambda)}{g(\lambda)} \left[ \frac{u(\lambda)}{\lambda^{n+k+1}} \right]_+ \right]_+ \right]_+.
\]

\[
= w(\lambda) \left[ \frac{f(\lambda)}{g(\lambda)} \left[ \frac{u(\lambda)}{\lambda^{n+k+1}} \right]_+ \right]_+ - v(\lambda) \left[ \frac{f(\lambda)}{g(\lambda)} \left[ \frac{u(\lambda)}{\lambda^{n+k+1}} \right]_+ \right]_. \tag{4.7}
\]

From (4.5) and (4.7) we get

\[
\frac{d}{dt} L(\lambda) = X_k L(\lambda) = [U_k(\lambda), L(\lambda)].
\]

\[
\Box
\]

5 Lax representation in Vieté coordinates and flat coordinates

Let us express considered systems and their Lax representations in so called Vieté coordinates

\[
q_i = \rho_i(\lambda),
\]

\[
p_i = -\sum_{k=1}^{n} \frac{\mu_{ik}}{\Delta_k}, \quad i = 1, \ldots, n. \tag{5.1}
\]

Here, for simplicity, we will consider the particular class of systems where \( f(\lambda) = \lambda^m, g(\lambda) = \lambda^r, m, r \in \mathbb{Z} \).

The Hamiltonians (2.3) take the form

\[
H_j = \sum_{i,k} (K_j G_m)^{ik} q_j p_k + V^{(\gamma)}(q),
\]
\[
\left( G_m \right)^{ik} = -\sum_{l=0}^{k-1} q_{k-l-1} V_{i}^{(m+l)}(q), \quad \left( K_j \right)^{ik} = -\sum_{l=0}^{j-1} q_{j-l-1} V_{i}^{(n+l-k)}(q)
\]

and basic potentials \( V^{(\gamma)} \) in \( q \) coordinates are generated by the recursion matrix

\[
R = \begin{pmatrix}
-q_1 & 1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
-q_n & 0 & 0 & 0 \\
\end{pmatrix}, \quad R^{-1} = \begin{pmatrix}
0 & 0 & \frac{-1}{q_n} \\
0 & \ddots & \ddots & \vdots \\
0 & \ddots & 0 & \frac{-1}{q_n} \\
0 & \ddots & 0 & 1 \\
\end{pmatrix}, \quad (5.2)
\]

For Lax representation, we get immediately

\[
u(\lambda; q) = \sum_{k=0}^{n} q_k \lambda^{n-k}, \quad q_0 \equiv 1. \quad (5.3)
\]

From (2.7) and (5.1) it follows that

\[
\sum_{k=1}^{n} \frac{\lambda_k^{n+r-i} \mu_k}{\Delta_k} = -\sum_{j=1}^{n} V_j^{(n+r-i)} \sum_{k=1}^{n} \frac{\lambda_k^{n-j} \mu_k}{\Delta_k} = \sum_{j=1}^{n} V_j^{(n+r-i)} p_j.
\]

So,

\[
v(\lambda; q, p) = -\sum_{k=1}^{n} \left( \sum_{i=1}^{n} \frac{\partial \rho_k}{\partial \lambda_i} \frac{\lambda_i^r \mu_i}{\Delta_i} \right) \lambda^{n-k}
\]

\[
= \sum_{k=1}^{n} \left[ \sum_{i=1}^{n} \left( \frac{k-1}{\Delta_i} \sum_{s=0}^{k-s} \rho_s \lambda_i^{r+k-s-1} \mu_i \right) \right] \lambda^{n-k}
\]

\[
= \sum_{k=1}^{n} \left[ \sum_{s=0}^{k-1} q_s \left( \sum_{j=1}^{n} V_j^{(r+k-s-1)} p_j \right) \right] \lambda^{n-k}, \quad (5.4)
\]

where we used the identity

\[
\frac{\partial \rho_k}{\partial \lambda_i} = -\frac{k-1}{\Delta_i} \rho_s \lambda_i^{k-s-1}. \quad (5.5)
\]

Notice that in particular for \( r = 0 \)

\[
v(\lambda; q, p) = -\sum_{k=1}^{n} \left[ \sum_{j=0}^{k-1} q_{k-j-1} p_{n-j} \right] \lambda^{n-k}. \quad (5.6)
\]

Thus, the substitutions (5.3), (5.4), (3.11) and (3.14) in \( L(\lambda; q, p) \) and \( U_k(\lambda; q, p) \) lead to Lax equations (3.13), for \( f(\lambda) = \lambda^m, \ g(\lambda) = \lambda^r \), written in canonical \((q, p)\) coordinates.

For particular cases, when \( m = 0, 1, \ldots, n \) the contravariant metric tensor

\[
G = \text{diag} \left( \frac{\lambda^m}{\Delta_1}, \ldots, \frac{\lambda^m}{\Delta_n} \right)
\]

defined by \( E_1 \) in (2.7) is flat, so one can use various flat coordinates. In the next section, for particular
Our first example is a system described by a separation curve of the canonical form constructed in \[8\].

### 6 Examples

Our second example is a system described by a separation curve of the canonical form

\[
\lambda^5 + H_1 \lambda^2 + H_2 \lambda + H_3 = \frac{1}{2} \mu^2,
\]

i.e. \( n = 3 \) and \( f(\lambda) = 1 \). This is the case for which there exist flat coordinates \([5,7]\), related to Vieté coordinates by

\[
q_1 = x_1, \quad q_2 = x_2 + \frac{1}{4} x_1^2, \quad q_3 = x_3 + \frac{1}{2} x_1 x_2,
\]

\[
p_1 = y_1 - \frac{1}{2} x_1 y_2 + \left( \frac{1}{4} x_1^2 - \frac{1}{2} x_2^2 \right) y_3, \quad p_2 = y_2 - \frac{1}{2} x_1 y_3, \quad p_3 = y_3.
\]

In flat coordinates Hamiltonians are

\[
H_1 = \frac{1}{2} y_2^2 + y_1 y_3 + \frac{1}{2} x_1^3 - \frac{3}{2} x_1 x_2 + x_3,
\]

\[
H_2 = y_1 y_2 + \frac{1}{2} x_1 y_2^2 - \frac{1}{2} x_3 y_2^2 + \frac{1}{2} x_1 y_1 y_3 - \frac{1}{2} x_2 y_2 y_3 + \frac{3}{16} x_1^4 - x_1 x_3 - x_2^2,
\]

\[
H_3 = \frac{1}{2} y_1^2 + \frac{1}{8} x_1^2 y_2^2 + \frac{1}{8} x_2 y_3^2 + \frac{1}{2} x_1 y_1 y_2 + \frac{1}{2} x_2 y_1 y_3 - \left( \frac{1}{4} x_1 x_2 + x_3 \right) y_2 y_3 + \frac{3}{4} x_1^2 x_3 + \frac{3}{8} x_1^2 x_2 - x_2 x_3 - \frac{1}{2} x_1 x_2^2
\]

and Lax representation for \( g(\lambda) = 1 \) takes the form

\[
L = \begin{pmatrix}
-y_1 \lambda^2 - \left( y_2 + \frac{1}{2} x_1 y_3 \right) \lambda - y_1 - \frac{1}{2} x_1 y_2 - \frac{1}{2} x_2 y_3 & \lambda^3 + x_1 \lambda^2 + (\frac{1}{4} x_1^2 + x_2) \lambda + x_3 + \frac{1}{2} x_1 x_2 \\
2 \lambda^2 - (y_3^2 + 2 x_1) \lambda - 2 y_1 y_3 + \frac{3}{2} x_1^2 - 2 x_2 & y_3 \lambda^2 + \left( y_2 + \frac{1}{2} x_1 y_3 \right) \lambda + y_1 + \frac{3}{2} x_1 y_2 + \frac{1}{2} x_2 y_3
\end{pmatrix},
\]

\[
U_1 = \begin{pmatrix}
0 & \frac{1}{2} \\
0 & 0
\end{pmatrix}, \quad U_2 = \begin{pmatrix}
\frac{1}{2} y_3 & \frac{1}{2} \lambda + \frac{1}{2} x_1 \\
0 & \frac{1}{2} y_3
\end{pmatrix},
\]

\[
U_3 = \begin{pmatrix}
\frac{1}{2} y_3 \lambda - \frac{1}{2} y_2 - \frac{1}{2} x_1 y_3 & \frac{1}{2} \lambda^2 + \frac{1}{2} x_1 \lambda + \frac{1}{2} x_1^2 + \frac{1}{2} x_2 \\
\lambda - \frac{1}{2} y_3^2 - x_1 & \frac{1}{2} y_3 \lambda + \frac{1}{2} y_2 + \frac{1}{2} x_1 y_3
\end{pmatrix}.
\]

Our second example is a system described by a separation curve of the canonical form

\[
H_1 \lambda + H_2 = \frac{1}{2} \lambda \mu^2 + \lambda^4,
\]
i.e. \( n = 2 \) and \( f(\lambda) = \lambda \). This is one of the integrable cases of the Henon-Heiles system. Actually, in Cartesian coordinates, related to Viete coordinates by

\[
q_1 = -x_1, \quad q_2 = -\frac{1}{4}x_2^2, \quad p_1 = -y_1, \quad p_2 = -\frac{2y_2}{x_2},
\]

both Hamiltonians are

\[
H_1 = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + x_1^3 + \frac{1}{2}x_1^2x_2^2,
\]

\[
H_2 = \frac{1}{2}x_2y_1y_2 - \frac{1}{2}x_1y_1^2 + \frac{1}{4}x_1^2x_2^2 + \frac{1}{16}x_2^4.
\]

Lax representation for \( g(\lambda) = 1 \) takes the form

\[
L = \begin{pmatrix}
\frac{2y_2}{x_2} \lambda + y_1 - \frac{x_1y_2}{x_2} & \lambda^2 - x_1\lambda - \frac{1}{4}x_2^2 \\
-2\lambda - \left( \frac{4y_1^2}{x_2^2} + 2x_1 \right) + \left( \frac{4y_2x_1}{x_2^2} - \frac{4y_1y_2}{x_2} - 2x_1^2 - \frac{1}{2}x_2^2 \right) & \lambda^{-1} - \frac{2y_2}{x_2} \lambda - y_1 + \frac{2y_1y_2}{x_2}
\end{pmatrix},
\]

\[
U_1 = \begin{pmatrix}
y_2 & \frac{1}{2} \lambda \\
-1 & -\frac{y_2}{x_2}
\end{pmatrix}, \quad U_2 = \begin{pmatrix}
y_2 \lambda - \frac{x_1y_2}{x_2} + \frac{1}{2}y_1 & \frac{1}{2} \lambda^2 - \frac{1}{2}x_1 \lambda \\
-\lambda - \frac{2y_2}{x_2} - x_1 + \frac{y_2}{x_2} + \frac{x_1y_2}{x_2} - \frac{1}{2}y_1
\end{pmatrix}.
\]

Lax representation for \( g(\lambda) = \lambda \) is

\[
L = \begin{pmatrix}
y_1 \lambda + \frac{1}{2}x_2y_2 & \lambda^2 - x_1\lambda - \frac{1}{4}x_2^2 \\
-2\lambda^3 - 2x_1\lambda^2 - (2x_1^2 + \frac{1}{2}x_2^2) \lambda + y_2^2 & -y_1 \lambda - \frac{1}{4}x_2y_2
\end{pmatrix},
\]

\[
U_1 = \begin{pmatrix}
0 & \frac{1}{2} \\
-\lambda - 2x_1 & 0
\end{pmatrix}, \quad U_2 = \begin{pmatrix}
\frac{1}{2}y_1 & \frac{1}{2} \lambda - \frac{1}{2}x_1 \\
-\lambda^2 - x_1 \lambda - x_1^2 - \frac{1}{2}x_2^2 & -\frac{1}{2}y_1
\end{pmatrix},
\]

while Lax representation for \( g(\lambda) = \lambda^2 \) is of the form

\[
L = \begin{pmatrix}
(x_1y_1 + \frac{1}{2}x_2y_2)\lambda + \frac{1}{4}x_2^2y_1 & \lambda^2 - x_1 \lambda - \frac{1}{4}x_2^2 \\
-2\lambda^5 - 2x_1\lambda^4 - (2x_1^2 + \frac{1}{2}x_2^2) \lambda^3 + (y_1^2 + y_2^2)\lambda^2 + y_1(x_1y_1 + x_2y_2)\lambda + \frac{1}{2}x_1y_1^2 & -L_{11}
\end{pmatrix},
\]

\[
U_1 = \begin{pmatrix}
-\frac{1}{2}y_1\lambda^{-1} & \frac{1}{2} \lambda^{-1} \\
-\lambda^2 - 2x_1 \lambda - (3x_1^2 + \frac{1}{2}x_2^2) - \frac{1}{2}y_1^2 \lambda^{-1} & \frac{1}{2}y_1 \lambda^{-1}
\end{pmatrix},
\]

\[
U_2 = \begin{pmatrix}
\frac{1}{2}x_1y_1\lambda^{-1} & \frac{1}{2} - \frac{1}{2}x_1 \lambda^{-1} \\
\lambda^3 - x_1 \lambda^2 - (x_1^2 + \frac{1}{2}x_2^2) \lambda + \frac{1}{2}(y_1^2 + y_2^2 - x_1x_2^2) + \frac{1}{2}x_1y_1^2 \lambda^{-1} & -\frac{1}{2}x_1y_1 \lambda^{-1}
\end{pmatrix}.
\]

Lax representations for \( g(\lambda) = 1 \) and \( g(\lambda) = \lambda^2 \) are new one, at least to the knowledge of authors, while that for \( g(\lambda) = \lambda \) is well known (see for example [3] or [7]).

Our last example is a system described by a separation curve of the canonical form

\[
\lambda^{-2} + H_1 \lambda + H_2 = \frac{1}{2} \lambda^{-1} \mu^2,
\]

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i.e. \( n = 2 \), and \( f(\lambda) = \lambda^{-1} \). Contrary to the previous cases the metric defined by \( H_1 \) is non-flat. In Vieté coordinates

\[
H_1 = -\frac{1}{2} p_1^2 - \frac{q_1 p_1 p_2^2}{q_2} + \frac{1}{2} \left ( 1 - \frac{q_1^2}{q_2} \right ) p_2^2 - \frac{q_1}{q_2}, \quad H_2 = -\frac{1}{2} q_1 p_2^2 + \left ( 1 - \frac{q_1^3}{q_2} \right ) p_1 p_2 + \left ( 1 - \frac{1}{2} q_2^2 \right ) p_2^2 + \frac{1}{2} q_2 - \frac{q_1^2}{q_2}.
\]

Then, the Lax representation for \( g(\lambda) = 1 \) takes the form

\[
L = \begin{pmatrix}
-p_2 \lambda - p_1 - q_1 p_2 & \lambda^2 + q_1 \lambda + q_2 \\
-p_2 \lambda - p_1 - q_1 p_2 & \lambda^2 + q_1 \lambda + q_2
\end{pmatrix},
\]

\[
U_1 = \begin{pmatrix}
-\frac{1}{2} p_1 + q_1 p_2 & \lambda^{-1} \\
-\frac{1}{2} p_1 + q_1 p_2 & \lambda^{-1}
\end{pmatrix},
\]

\[
U_2 = \begin{pmatrix}
-\frac{1}{2} q_1 (p_1 + q_1 p_2) & \lambda^{-1} \\
-\frac{1}{2} q_1 (p_1 + q_1 p_2) & \lambda^{-1}
\end{pmatrix},
\]

for \( g(\lambda) = \lambda^{-1} \) we have

\[
L = \begin{pmatrix}
\frac{p_1 + q_1 p_2}{q_2} \lambda + \frac{q_1 (p_1 + q_1 p_2)}{q_2} - p_2 & \lambda^2 + q_1 \lambda + q_2 \\
-(\frac{p_1 + q_1 p_2}{q_2})^2 - (\frac{q_1 (p_1 + q_1 p_2)}{q_2})^2 & -(\frac{p_1 + q_1 p_2}{q_2})^2 - (\frac{q_1 (p_1 + q_1 p_2)}{q_2})^2
\end{pmatrix} - \frac{p_1 + q_1 p_2}{q_2} - \frac{q_1 (p_1 + q_1 p_2)}{q_2} + p_2,
\]

\[
U_1 = \begin{pmatrix}
0 & \lambda^{-1} + \frac{1}{2} \lambda^{-2} \\
\frac{p_1 + q_1 p_2}{q_2} - \frac{q_1 (p_1 + q_1 p_2)}{q_2} + 2 \frac{q_1^2}{q_2} & \lambda^{-1} - \frac{q_1}{q_2} \lambda^{-2} + \frac{1}{2} \lambda^{-3}
\end{pmatrix},
\]

\[
U_2 = \begin{pmatrix}
\frac{1}{2} p_1 + q_1 \lambda & \lambda^{-1} + \frac{1}{2} \lambda^{-2} \\
\frac{q_1 (p_1 + q_1 p_2)}{q_2} - \frac{1}{2} q_1 (p_1 + q_1 p_2)^2 + 2 \frac{q_1^2}{q_2} & \lambda^{-1} + \frac{1}{2} \lambda^{-2} + \frac{q_1^2}{q_2} \lambda^{-3}
\end{pmatrix},
\]

and for \( g(\lambda) = \lambda \)

\[
L = \begin{pmatrix}
-p_1 \lambda + q_2 p_2 & \lambda^2 + q_1 \lambda + q_2 \\
-(\frac{p_1 + q_1 p_2}{q_2})^2 - p_2^2 + 2 \frac{q_1 p_2}{q_2} & \lambda^2 + 2 q_1 p_2 + \frac{1}{2} \lambda^2 - q_2 p_2^2 + p_1 \lambda - q_2 p_2
\end{pmatrix},
\]

\[
U_1 = \begin{pmatrix}
\frac{1}{2} p_2 \lambda^{-2} - \frac{1}{2} q_1 p_1 p_2 & \lambda^{-1} + \frac{1}{2} \lambda^{-2} \\
\frac{p_1 p_2 + q_1 p_2^2}{q_2} + \frac{1}{2} \lambda p_2 & \lambda^{-1} - \frac{q_1}{q_2} \lambda^{-2} - \frac{1}{2} p_1 p_2 + q_1 p_2^2 + \frac{1}{2} \lambda \lambda^{-1}
\end{pmatrix},
\]

\[
U_2 = \begin{pmatrix}
\frac{1}{2} q_2 p_2 \lambda^{-2} - \frac{1}{2} q_1 p_1 p_2 & \lambda^{-1} + \frac{1}{2} \lambda^{-2} \\
\frac{1}{2} q_2 p_2 \lambda^{-2} + \frac{1}{2} q_1 p_1 p_2 - q_2 p_2^2 + q_1 p_2 & \lambda^{-1} - \frac{q_1}{q_2} \lambda^{-2} - \frac{1}{2} p_1 p_2 + q_1 p_2^2 + \frac{1}{2} \lambda \lambda^{-1}
\end{pmatrix}.
\]
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