SEGRE INVARIANT AND A STRATIFICATION OF THE MODULI SPACE OF COHERENT SYSTEMS

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Abstract. The aim of this paper is to generalize the \( m \)-Segre invariant for vector bundles to coherent systems. Let \( X \) be a non-singular irreducible complex projective curve of genus \( g \) over \( \mathbb{C} \) and \((E, V)\) be a coherent system on \( X \) of type \((n, d, k)\). For any pair of integers \( m, t, 0 < m < n, 0 \leq t \leq k \) we define the \((m, t)\)-Segre invariant, denoted by \( S_{m,t}^\alpha \) and show that \( S_{m,t}^\alpha \) induces a semicontinuous function on the families of coherent systems. Thus, \( S_{m,t}^\alpha \) gives a stratification of the moduli space \( G(\alpha; n, d, k) \) of \( \alpha \)-stable coherent systems of type \((n, d, k)\) on \( X \) into locally closed subvarieties \( G(\alpha; n, d, k; m, t; s) \) according to the value \( s \) of \( S_{m,t}^\alpha \). We study the stratification, determine conditions under which the different strata are non-empty and compute their dimension.

1. Introduction

Let \( X \) be a non-singular irreducible complex projective curve of genus \( g \) over \( \mathbb{C} \). The \( m \)-Segre invariant for vector bundles was first introduced by Lange and Narasimhan in [15] for vector bundles of rank \( n = 2 \). Then, it was generalized by Brambila-Paz and Lange in [7] for vector bundles of rank \( n \geq 2 \) (see also [26]). The \( m \)-Segre invariant was used to give a stratification of the moduli space \( M(n, d) \) of stable vector bundles of rank \( n \) and degree \( d \) in order to determine topological and geometric properties of \( M(n, d) \) (see for instance [23], [24], [16]). Moreover, it has been generalized to study other moduli spaces (see [3], [8]).

The aim of this paper is to generalize the \( m \)-Segre invariant for vector bundles to coherent systems. A coherent system \((E, V)\) on \( X \) consists of a holomorphic vector bundle \( E \) on \( X \) and a subspace \( V \) of the space of sections \( H^0(X, E) \). Associated with the coherent systems, there is a notion of stability which depends on a real parameter \( \alpha \). This notion allows the construction of the moduli space \( G(\alpha; n, d, k) \) of \( \alpha \)-stable coherent systems of type \((n, d, k)\) and thus leads to a family of moduli spaces. For a further treatment of the subject see [22], [5] and [4].

Let \( \alpha > 0 \) and \((E, V)\) be a coherent system of type \((n, d, k)\). For any pair of integers \( m, t, 0 < m < n, 0 \leq t \leq k \) we define the \((m, t)\)-Segre invariant (see Definition 3.3) as,

\[
S_{m,t}^\alpha(E, V) := (mn) \min \{ \mu_\alpha(E, V) - \mu_\alpha(F, W) \}
\]

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where the minimum is taken over all principal subsystems (see Remark 2.2) \((F,W)\) of type \((m,d_F,t)\) of \((E,V)\), and \(\mu_\alpha\) denote the \(\alpha\)-slope of coherent systems.

Using similar techniques as Maruyama in [19], [20] and [21] we show that the \((m,t)\)-Segre invariant induces a function (called the \((m,t)\)-Segre function) on the families of coherent systems and prove our first result (see Theorem 3.5),

**Theorem 1.1.** The \((m,t)\)-Segre function is lower semicontinuous.

As consequence of Theorem 1.1, the \((m,t)\)-Segre invariant yields a stratification of the moduli space \(G(\alpha;n,d,k)\) into locally closed subvarieties which we denote as

\[
G(\alpha;n,d,k;m,t;s) := \{(E,V) \in G(\alpha;n,d,k) : S^{\alpha}_{m,t}(E,V) = s\}
\]

according to the value \(s\) of \(S^{\alpha}_{m,t}\). We show a bound for the possible values that can take \(s\) (see Proposition 3.4), determine certain values of \(m, t\) and \(s\) under which the stratum \(G(\alpha;n,d,k;m,t;s)\) is non-empty (see Theorem 4.2) and compute a bound of its dimension,

**Theorem 1.2.** Let \(\alpha = \frac{p}{q} \in \mathbb{Q}^+\) and \(n \geq 2\), \(d > 0\), \(k \geq 1\) be integer numbers. Suppose that there exist \(n_1, n_2, d_1, d_2 > 0\), \(t_1, t_2 \geq 0\) integer numbers such that

\[
n_1n_2(g - 1) - d_1n_2 + d_2n_1 + t_2n_1(g - 1) - t_1t_2 > 0,
\]

\[
q(n_1d_2 - n_2d_1) + p(n_1t_2 - n_2t_1) = 1.
\]

and the moduli spaces \(G(\alpha;n_1,d_1,t_1)\) and \(G(\alpha;n_2,d_2,t_2)\) are non-empty. Then, the stratum

\[
G(\alpha;n,d,k;n_1,t_1;1/q)
\]

is non-empty, where \(n = n_1 + n_2\), \(d = d_1 + d_2\) and \(k = t_1 + t_2\).

The paper is organized as follows, Section 2 contains a brief summary of the \(m\)-Segre invariant for vector bundles and relevant material on coherent systems. In Section 3 we define the \((m,t)\)-Segre invariant for coherent systems and coherent systems of subtype \((a)\), show some technical results which allows to prove Theorem 1.1. In Section 4 we study the stratification of \(G(\alpha;n,d,k)\), determine conditions under which the different strata are non-empty and compute a bound of their dimension. In section 5 we consider coherent systems of type \((2,13,4)\) on a general curve of genus 6.

Notation: For a vector bundle \(E\) we shall denote by \(rk_E\) the rank and by \(d_E\) the degree. We will denote by \(\omega_X\) the canonical sheaf on \(X\).

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2. Review of Segre Invariant and Coherent Systems

Let $X$ be a non-singular irreducible complex projective curve of genus $g$ over $\mathbb{C}$. This section contains a brief summary on the $m$–Segre invariant for vector bundles, for more details see [15] and [7]. We recall the main results that we will use on coherent systems for a further treatment of the subject see [4] and [5].

2.1. The $m$–Segre invariant for vector bundles. Let $E$ be a vector bundle of rank $n$ and degree $d$ on $X$. For any integer $m$, $1 \leq m \leq n - 1$ the $m$–Segre invariant is defined as

$$S_m(E) := md - n \max\{d_F\}$$

where the maximum is taken over all subbundles $F$ of rank $m$ of $E$. The $m$–Segre invariant was first introduced by Lange and Narasimhan in [15] for vector bundles of rank $n = 2$, then it was generalized by Brambila-Paz and Lange in [7] and Russo and Teixidor in [26] for vector bundles of rank $n \geq 2$. Recall that the slope of a vector bundle $E$ denoted by $\mu(E)$, is the quotient

$$\mu(E) := \frac{d}{n}.$$ 

So, the $m$–Segre invariant can be written as

$$S_m(E) = (nm) \min_{\text{rank } F = m} \{\mu(E) - \mu(F)\}.$$ 

Note that for a suitable vector bundle $E$, $S_k(E)$ may take arbitrarily negative values (for instance $E$ to be a suitable direct sum of line bundles). However, Hirschowitz in [11] gives an upper bound

$$S_k(E) \leq m(n - m)(g - 1) + (n - 1).$$

The following theorem studies the behavior of $S_m$ over families of vector bundles.

**Theorem 2.1.** [13, Lemma 1.2] Let $Y$ be a variety and $\mathcal{E}$ be a family of vector bundles of rank $n$ and degree $d$ parametrized by $Y$. For any integer $m$, $1 \leq m \leq n - 1$ the $m$–Segre invariant defines functions

$$S_m : Y \to \mathbb{Z}$$

$$y \mapsto S_m(\mathcal{E}_y).$$

The function $S_m$ is lower semicontinuous.

Denote by $M(n, d)$ the moduli space of stable vector bundles of rank $n$ and degree $d$ on $X$. By Theorem 2.1 the function $S_m : M(n, d) \to \mathbb{Z}$ gives a stratification of $M(n, d)$ into locally closed subvarieties

$$M(n, d; m; s) := \{E \in M(n, d) : S_m(E) = s\}$$

according to the value $s$ of $S_m$. 
The stratification of $M(2, d)$ was studied in [15]. There was shown that for $s > 0$, $s \equiv d \mod 2$ the algebraic variety $M(2, d; s)$ is non-empty, irreducible and of dimension

$$\dim M(2, d; s) = \begin{cases} 3g + s - 2, & \text{if } s \leq g - 2 \\ 4g - 3, & \text{if } s \geq g - 1. \end{cases}$$

For the stratification of $M(n, d)$, it is shown in [7] that for $g \geq n + 1$ and $0 < s \leq m(n - m)(g - 1) + (n + 1)$, $s \equiv md \mod n$ the variety $M(n, d; m; s)$ is non-empty and it has an irreducible component $M^0(n, d; m; s)$ of dimension

$$\dim M^0(n, d; m; s) = \begin{cases} (n^2 + m^2 - nm)(g - 1) + s - 1, & \text{if } s \leq m(n - m)(g - 1) \\ n^2(g - 1) + 1, & \text{if } s \geq m(n - m)(g - 1). \end{cases}$$

The main difficulty in the study of the stratification of $M(n, d)$ is to show that the different varieties $M(n, d; m; s)$ are non-empty. This was shown in [14] and [11] for $s \geq m(n - m)(g - 1)$ for the generic case and for $s \leq \frac{m(n - m)(g - 1)}{\max\{m, n - m\}}$ in [2]. Special cases were considered by Russo and Teixidor in [26].

Such stratification has been used by many authors to get topological and geometric properties of $M(n, d)$ and the $m$–Segre invariant has been generalized to study other moduli spaces. For instance, Popa in [23] determined a bound for the $m$–Segre invariant, this result is applied to the study of generalized theta line bundles on the moduli space $M(n, d)$. Bhosle and Biswas in [3] defined an analogue of the $m$–Segre invariant for parabolic bundles in order to study the moduli space $M(n, d)$ of stable parabolic bundles of rank $n$ and degree $d$ with fixed parabolic structure at a finite set of distinct closed points of $X$. Similarly, Choe and Insong in [8] studied the moduli space $M_{2n}$ of semistable symplectic bundles of rank $2n$ on $X$.

2.2. Coherent systems. A coherent system or Brill-Noether pair on $X$ of type $(n, d, k)$ is a pair $(E, V)$ where $E$ is a holomorphic vector bundle on $X$ of rank $n$ and degree $d$ and $V \subseteq H^0(X, E)$ is a subspace of dimension $k$.

Remark 2.2.

- A coherent subsystem of $(E, V)$ is a coherent system $(F, W)$ such that $F$ is a subbundle of $E$ and $W \subseteq V \cap H^0(X, F)$. The subsystem $(F, W)$ is called principal if $W = V \cap H^0(X, F)$.
- A quotient coherent system of $(E, V)$ is a coherent system $(G, Z)$ together with a homomorphism $\phi : (E, V) \rightarrow (G, Z)$ such that $E \rightarrow G$ and $V \rightarrow Z$ are surjective.

Remark 2.3. In general, a subsystem does not define a quotient system. However, any principal subsystem $(F, W)$ of $(E, V)$ defines a corresponding quotient system $(G, Z)$ which fit in the exact sequence

$$0 \rightarrow (F, W) \rightarrow (E, V) \rightarrow (G, Z) \rightarrow 0.$$
Definition 2.4. A family of coherent systems of type \((n, d, k)\) on \(X\) parametrized by a variety \(T\) consists of a pair \((\mathcal{E}, \mathcal{V})\) where:

- \(\mathcal{E}\) is a family of vector bundles on \(X\) parametrized by \(T\) such that \(\mathcal{E}_t = \mathcal{E}_{X \times \{t\}}\) has degree \(d\) and rank \(n\) for all \(t \in T\).
- \(\mathcal{V}\) is a locally free subsheaf of \(p_T^* \mathcal{E}\) of rank \(k\) such that the fibers \(\mathcal{V}_t\) map injectively to \(H^0(X, \mathcal{E}_t)\) for all \(t \in T\) where \(p_T\) denotes the canonical projection of \(X \times T\) on \(T\).

By Serre-Grothendieck duality Theorem and by [17, Lemme 4.9.] the vector spaces \(\text{Ext}^1_{p_T}(\mathcal{E}, \omega_{X \times T/T}_t)\) and \(H^0(X, \mathcal{E}_t)\) are duals, hence it implies another way of stating the Definition 2.4.

Definition 2.5. A family of coherent systems of type \((n, d, k)\) on \(X\) parametrized by a variety \(T\) consists of a pair \((\mathcal{E}, \Gamma)\) where:

- \(\mathcal{E}\) is a family of vector bundles on \(X\) parametrized by \(T\) such that \(\mathcal{E}_t = \mathcal{E}_{X \times \{t\}}\) has degree \(d\) and rank \(n\) for all \(t \in T\).
- \(\Gamma\) is a locally free quotient sheaf of \(\text{Ext}^1_{p_T}(\mathcal{E}, \omega_{X \times T/T})\) of rank \(k\) on \(T\) where \(p_T\) denotes the canonical projection of \(X \times T\) on \(T\) and \(\omega_{X \times T/T} = p_T^* \omega_X\) the relative canonical sheaf by the projection \(p_T\).

Associated to the coherent systems there is a notion of stability which depends on a real parameter \(\alpha\). For a real number \(\alpha\), the \(\alpha\)–slope of a coherent system \((E, V)\) of type \((n, d, k)\) is defined by

\[
\mu_\alpha(E, V) := \frac{d}{n} + \frac{k}{n} \alpha.
\]

We say, \((E, V)\) is \(\alpha\)–stable (resp. \(\alpha\)–semistable) if for all proper subsystems \((F, W)\),

\[
\mu_\alpha(F, W) < \mu_\alpha(E, V), \text{ (resp. } \leq).\]

The moduli space of \(\alpha\)–stable coherent systems of fixed type was constructed by Le Potier in [17], by King and Newstead in [12] and by Ragavendra and Vishwanath in [25] by the methods of Geometric Invariant Theory. We shall denote the moduli space of \(\alpha\)–stable coherent systems of type \((n, d, k)\) by \(G(\alpha; n, d, k)\).

Necessary conditions for non-emptiness of \(G(\alpha; n, d, k)\) are \(d > 0, \alpha > 0, (n - k)d < \alpha\), for a discussion of recent progress on the non-emptiness problem we refer the reader to [22]. Now, we describe some well known facts about the moduli space of coherent systems.

Definition 2.6. We say that \(\alpha > 0\) is a virtual critical value if it is numerically possible to have a proper subsystem \((F, W)\) of type \((m, d_F, t)\) such that \(\frac{k}{m} \neq \frac{k}{n}\) but \(\mu_\alpha(E, V) = \mu_\alpha(F, W)\). If there is a coherent system \((E, V)\) and a subsystem \((F, W)\) such that this actually holds, we say \(\alpha\) is a critical value. We say \(\alpha = 0\) is a critical value.
For numerical reasons for any \((n, d, k)\) there are finitely many critical values
\[
0 = \alpha_0 < \alpha_1 < \ldots < \alpha_L < \begin{cases} 
\frac{d}{n-k}, & \text{if } k < n \\
\infty, & \text{if } k \geq n.
\end{cases}
\]
These induce a partition of the \(\alpha\)-range into a set of open intervals such that within the interval \((\alpha_i, \alpha_{i+1})\) the property of \(\alpha\)-stability is independent of \(\alpha\). If \(k \geq n\) the moduli spaces coincide for any two different values of \(\alpha\) in the range \((\alpha_L, \infty)\), (see [5, Proposition 4.6]). We will denote by \(G_i = G_i(n, d, k)\) the moduli space in the interval \((\alpha_i, \alpha_{i+1})\) and by \(G_L := G_L(n, d, k)\) for \(\alpha > \alpha_L\).

Coherent systems form an abelian category and the functors \(\text{Hom}((E, V), -)\) are left exact. Hence their derived functors denoted by \(\text{Ext}^i((E, V), -)\) are well defined. For a more detailed treatment we refer the reader to [10].

Given two coherent systems \((E_1, V_1), (E_2, V_2)\) of type \((n_1, d_1, k_1), (n_2, d_2, k_2)\), respectively one defines the groups
\[
\mathbb{H}^0_{21} := \text{Hom}((E_2, V_2), (E_1, V_1)),
\]
\[
\mathbb{H}^i_{21} := \text{Ext}^i((E_2, V_2), (E_1, V_1)), \text{ for } i > 0.
\]
and consider the long exact sequence
\[
0 \longrightarrow \text{Hom}((E_2, V_2), (E_1, V_1)) \longrightarrow \text{Hom}(E_2, E_1) \longrightarrow \text{Hom}(V_2, H^0(X, E_1)/V_1) \\
\longrightarrow \text{Ext}^1((E_2, V_2), (E_1, V_1)) \longrightarrow \text{Ext}^1(E_2, E_1) \longrightarrow \text{Hom}(V_2, H^1(X, E_1)) \\
\longrightarrow \text{Ext}^2((E_2, V_2), (E_1, V_1)) \longrightarrow 0.
\]
From [5, Proposition 3.2] follows
\[
\dim \text{Ext}^1((E_2, V_2), (E_1, V_1)) = C_{21} + \dim \mathbb{H}^0_{21} + \dim \mathbb{H}^2_{21},
\]
where
\[
C_{21} = n_1n_2(g-1) - d_1n_2 + d_2n_1 + k_2d_1 - k_2n_1(g-1) - k_1k_2.
\]

**Proposition 2.7.** The space of equivalence classes of extensions
\[
(2.1) \quad 0 \longrightarrow (E_1, V_1) \longrightarrow (E, V) \longrightarrow (E_2, V_2) \longrightarrow 0
\]
is isomorphic to \(\text{Ext}^1((E_2, V_2), (E_1, V_1))\). Moreover, the quotient of the space of non-trivial extensions by the natural action of \(\mathbb{C}^*\) can be identified with the projective space
\[
\mathbb{P}\text{Ext}^1((E_2, V_2), (E_1, V_1)).
\]

Note that if \(\text{Aut}(E_i, V_i) = \mathbb{C}^*\), then the isomorphism classes of \((E, V)\) appearing in the middle of (2.1) will be parametrized by \(\mathbb{P}\text{Ext}^1((E_2, V_2), (E_1, V_1))\).

**Definition 2.8.** For any \((n, d, k)\), the Brill-Noether number \(\beta(n, d, k)\) is defined by
\[
\beta(n, d, k) = n^2(g-1) + 1 - k(k - d + n(g-1)).
\]
### 3. Segre Invariant for Coherent Systems

In this section, we introduce the \((m, t)\)-Segre invariant for coherent systems and show some properties which will be used later. We show that the \((m, t)\)-Segre invariant induces a semicontinuous function on the families of coherent systems, the proof proceeds as Maruyama in [19], [20] and [21] for the \(m\)-Segre invariant for vector bundles. Unless otherwise stated we assume that \(\alpha > 0\) is a rational number.

Let \((E, V)\) be a coherent system of type \((n, d, k)\). For any pair of integers \(m, t\), \(0 < m < n\), \(0 \leq t \leq k\) consider the set

\[
P_{m,t}(E, V) := \{(F, W) \subset (E, V) : \text{rk}_F = m, \dim W = t \text{ and } (F, W) \text{ principal}\}.
\]

By Remark 2.3, the set \(P_{m,t}(E, V)\) consists of all coherent subsystems \((F, W)\) of type \((m, d_F, t)\) of \((E, V)\) for which there exists an exact sequence of coherent systems

\[
0 \to (F, W) \to (E, V) \to (G, Z) \to 0.
\]

**Remark 3.1.** Let \((E, V)\) be a coherent system of type \((n, d, k)\).

- The set \(P_{m,0}(E, V)\) is not in correspondence with the set of all subbundles of rank \(m\) of \(E\). There exists subbundles \(F \subset E\) of rank \(m\) for which \(\dim V \cap H^0(X, F) > 0\).
- The set \(P_{n,t}(E, V)\) is empty for all \(t \neq k\) and \(P_{n,k}(E, V) = \{(E, V)\}\).

**Remark 3.2.** Since the degrees of subbundles of \(E\) are bounded above (see [18, Lemma 5.4.1]), we have that for a fixed value of \(\alpha\) the coherent system \((E, V)\) does not admit subsystems of \(\alpha\)-slope high arbitrary, i.e, the set

\[
\{\mu_\alpha(F, W) : (F, W) \subset (E, V)\}
\]

is bounded above.

**Definition 3.3.** Let \(\alpha > 0\) and \((E, V)\) be a coherent system of type \((n, d, k)\). The \((m, t)\)-Segre invariant for coherent systems denoted by \(S^\alpha_{m,t}\) is defined by

\[
S^\alpha_{m,t}(E, V) := (mn) \min_{(F, W) \in P_{m,t}(E, V)} \{\mu_\alpha(E, V) - \mu_\alpha(F, W)\},
\]

provided that \(P_{m,t}(E, V) \neq \emptyset\). If \(P_{m,t}(E, V) = \emptyset\), we define \(S^\alpha_{m,t}(E, V) = \infty\).

By Remark 3.2, \(S^\alpha_{m,t}(E, V)\) is a rational number well defined depending only on \((E, V)\), \(m\) and \(t\). Since any \((F, W) \in P_{m,t}(E, V)\) defines an exact sequence of coherent systems

\[
0 \to (F, W) \to (E, V) \to (G, Z) \to 0,
\]

Definition 3.3 is equivalent to

\[
S^\alpha_{m,t}(E, V) = n(n - m) \min \{\mu_\alpha(G, Z) - \mu_\alpha(E, V)\}
\]

where the minimum is taken over any quotient coherent system of \((E, V)\) of type \((n - m, d - d_F, k - t)\).

Here are some basic properties of this concept.
Proposition 3.4. Let $\alpha > 0$ and $(E, V)$ be a coherent system of type $(n, d, k)$, then

1. $(E, V)$ is $\alpha$–stable (resp. $\alpha$–semistable), if and only if
$$S_{m,t}^\alpha(E, V) > 0,$$ (resp. $\geq 0$)
for any pair of integers $m, t$, $0 < m < n$, $0 \leq t \leq k$.

2. If $P_{m,t}(E, V) \neq \emptyset$ then
$$S_{m,t}^\alpha(E, V) \leq m(n - m)(g - 1) + (n - 1) + \alpha(mk - nt).$$

Proof. The proof of (1) follows directly of the definition of $\alpha$–stability for coherent systems. For (2), Hirschowitz in [11] showed that $S_m(E) \leq m(n - m)(g - 1) + (n - 1)$, then for $(F, W) \in P_{m,t}(E, V)$ we would have $S_{m,t}^\alpha(E, V) \leq m(n - m)(g - 1) + (n - 1) + \alpha(mk - nt)$ as claimed.

A coherent system $(F, W) \in P_{m,t}(E, V)$ is called maximal if and only if
$$S_{m,t}^\alpha(E, V) = (mn)(\mu_\alpha(E, V) - \mu_\alpha(F, W)).$$

In the following section we define the $(m, t)$–Segre function and prove that it is lower semicontinuous.

3.1. Semicontinuity of the Segre function. Let $Y$ be a variety and $(\mathcal{E}, \mathcal{V})$ be a family of coherent systems on $X$ of type $(n, d, k)$ parametrized by $Y$. For any pair of integers $m$, $t$, $0 < m < n$, $0 \leq t \leq k$ the $(m, t)$–Segre function is defined as
$$S_{m,t}^\alpha : Y \rightarrow \mathbb{R} \cup \{\infty\}$$
$$y \mapsto S_{m,t}^\alpha(\mathcal{E}, \mathcal{V})_y.$$ The aim of this section is to prove the following theorem.

Theorem 3.5. The $(m, t)$–Segre function is lower semicontinuous.

The proof of the theorem consists in showing that for any $b \in \mathbb{R}$ the set
$$A_b := \{y \in Y : b < (nm)(\mu_\alpha(\mathcal{E}, \mathcal{V})_y - \mu_\alpha(F, W)) \text{ for any } (F, W) \in (\mathcal{E}, \mathcal{V})_y\}$$
is open in $Y$.

To show that $A_b$ is open in $Y$ we proceed as Maruyama in [19], [20] and [21] for the $m$–Segre invariant for vector bundles. We define coherent systems of subtype $(a)$ and show that it is an open condition.

3.1.1. Coherent Systems of Subtype $(a)$. Let $n \geq 1$, $k \geq 0$ be integer numbers. We will denote by $(a)$ a sequence of $(n - 1)(k + 1)$ rational numbers, that is
$$(a) := (a_{ij} : 0 < i < n, \ 0 \leq t \leq k) \in \mathbb{Q}^{(n-1)(k+1)}.$$
The sequence $(a) := (a_{ij} = 0,$ for all pair $i, j) will be denoted by $(0).$ Let $(a), (b) \in \mathbb{Q}^{(n-1)(k+1)}$ we say $(a)$ is greater than or equal than $(b)$ denoted by $(a) \geq (b),$ if $a_{ij} \geq b_{ij}$ for any pair $i, j.$ Hence, the set $\mathbb{Q}^{(n-1)(k+1)}$ is a partial order set.

The following definition extends the usual notion of $\alpha-$stability of coherent systems.

**Definition 3.6.** Let $\alpha > 0,$ $(E, V)$ be a coherent system of type $(n, d, k)$ and $(a) = (a_{ij}) \in \mathbb{Q}^{(n-1)(k+1)}$.

- $(E, V)$ is of $\alpha-$subtype $(a)$ if
  \[ \mu_\alpha(F, W) < \mu_\alpha(E, V) + a_{m,t} \]
  for all coherent subsystems $(F, W) \in P_{m,t}(E, V)$.
- $(E, V)$ is of $\alpha-$cotype $(a)$ if
  \[ \mu_\alpha(E, V) - a_{m',t'} < \mu_\alpha(G, Z) \]
  for all quotient coherent system $(G, Z)$ of $(E, V)$ of type $(m', d_G, t').$

Unless otherwise stated we assume that $\alpha > 0$ is a fixed value. For simplicity of notation we write subtype $(a)$ (resp. cotype) instead $\alpha-$subtype $(a)$ (resp. $\alpha-$cotype).

Here are some elementary properties of these concepts.

**Remark 3.7.** Let $(E, V)$ be a coherent system of type $(n, d, k)$ and $(a), (b) \in \mathbb{Q}^{(n-1)(k+1)}$.

1. $(E, V)$ is $\alpha-$stable if and only if it is of subtype $(0)$.
2. If $(E, V)$ is of subtype $(a) := (a_{ij})$ with $a_{ij} < 0$ for any $i, j,$ then $(E, V)$ is $\alpha-$stable.
3. If $(E, V)$ is of cotype $(a) := (a_{ij})$ with $a_{ij} < 0$ for any $i, j,$ then $(E, V)$ is $\alpha-$stable.
4. If $(E, V)$ is of subtype $(a),$ then it is of subtype $(b)$ for all $(a) \leq (b)$.
5. If $(E, V)$ is of cotype $(a),$ then it is of cotype $(b)$ for all $(a) \leq (b)$.

The following proposition establishes a relationship between coherent systems of subtype $(a)$ and cotype $(b)$.

**Proposition 3.8.** Let $(E, V)$ be a coherent system of type $(n, d, k).$ The coherent system $(E, V)$ is of subtype $(a),$ if and only if it is of cotype $(b)$ where $(b)$ is the sequence defined by

\[ (b) := (b_{ij} = a_{n-i,k-j} \frac{n-i}{i} : 0 < i < n, 0 \leq j \leq k). \]

**Proof.** Let $(a) := (a_{ij}), (b) := (b_{ij} = a_{n-i,k-j} \frac{n-i}{i} : 0 < i < n, 0 \leq j \leq k) \in \mathbb{Q}^{(n-1)(k+1)}$ and $(E, V)$ be a coherent system of type $(n, d, k)$ of subtype $(a).$ Let $(G, Z)$ be a quotient coherent system of $(E, V)$ of type $(m', d_G, t')$ and $(F, W) \in P_{m,t}(E, V),$ which fit into the following exact sequence

\[ 0 \rightarrow (F, W) \rightarrow (E, V) \rightarrow (G, Z) \rightarrow 0. \]

Since $(E, V)$ is of subtype $(a)$ the following inequality holds

\[ \mu_\alpha(F, W) < \mu_\alpha(E, V) + a_{m,t} \]
that is
\[ n(d_F + \alpha t) < m(d + \alpha k) + (mn)a_{m,t}, \]
which is equivalent to
\[ n[d - d_G + \alpha(k - t')] < (n - m')(d + \alpha k) + n(n - m')a_{n-m',k-t'} \]
and this gives
\[ \mu_\alpha(E, V) - a_{n-m',k-t'} \frac{n - m'}{m'} < \mu_\alpha(G, Z). \]
That is
\[ \mu_\alpha(E, V) - b_{m',t'} < \mu_\alpha(G, Z). \]
Therefore, \((E, V)\) is of cotype \((b)\). In the same way, we can see that if \((E, V)\) is of cotype \((b)\), then it is of subtype \((a)\) which completes the proof. \(\square\)

Let \((a) \in \mathbb{Q}^{(a-1)(k+1)}\) and \(a_{m,t}\) be the \((m, t)\)--member of the sequence \((a)\). For \(1 < m < n\), we denoted by \((a - a_{m,t})\) the sequence defined as
\[ (a - a_{m,t}) := (a_{i,j} - a_{m,t} : 0 < i < m, 0 < j < t) \in \mathbb{Q}^{(n-m)(k-t)}. \]

**Example 3.9.** Let \(n = 3, k = 2\) and \((a) = (a_{1,0}, a_{1,1}, a_{1,2}, a_{2,0}, a_{2,1}, a_{2,2}) \in \mathbb{Q}^6\). The sequence \((a - a_{21})\) is defined by
\[ (a - a_{21}) := (a_{10} - a_{21}, a_{11} - a_{21}) \in \mathbb{Q}^2. \]

The following lemma yields information about coherent systems that are not of subtype \((a)\).

**Lemma 3.10.** Let \(\alpha > 0\) and \((E, V)\) be a coherent system of type \((n, d, k)\). If \((E, V)\) is not of subtype \((a)\), then there exist a pair of integers \(m, t, 0 < m < n, 0 \leq t \leq k\) and a coherent system in \(P_{m,t}(E, V)\) such that it is of subtype \((a - a_{m,t})\) and it has \(\alpha\)--slope greater than or equal to \(\frac{d + \alpha k}{n} + a_{m,t}\).

**Proof.** Since \((E, V)\) is not of subtype \((a)\) there exist a pair of integers \(m', t', 0 < m' < n, 0 \leq t' \leq k\) and a coherent system \((F', W') \in P_{m',t'}(E, V)\) such that
\[ \mu_\alpha(F', W') \geq \mu_\alpha(E, V) + a_{m',t'}. \]
Suppose that \((F', W')\) is not of subtype \((a - a_{m',t'})\). By induction on rank we obtain a pair of integers \(m, t, 0 < m < m', 0 \leq t \leq t'\) and a coherent system \((F, W) \in P_{m,t}(F', W') \subset P_{m,t}(E, V)\) which satisfies
\[ \mu_\alpha(F, W) \geq \mu_\alpha(F', W') + a_{m,t} - a_{m',t'}. \]
Replacing (3.1) in (3.2) follows that
\[ \mu_\alpha(F, W) \geq \mu_\alpha(E, V) + a_{m',t'} + a_{m,t} - a_{m',t'} \]
\[ = \mu_\alpha(E, V) + a_{m,t} \]
\[ = \frac{d + \alpha k}{n} + a_{m,t}. \]
Therefore \((F, W)\) is one of the systems in \(P_{m,t}(E, V)\) which satisfies the lemma. \(\square\)

Let us denote by \(A(n, d, k)\) the set of isomorphism classes of coherent systems on \(X\) of type \((n, d, k)\). For fix \(\alpha > 0\) and \((a) \in \mathbb{Q}^{(n-1)(k+1)}\) let

\[
B((a)) := \{(E, V) \in A(n, d, k) : (E, V) \text{ is of subtype } (a)\}.
\]

The following lemma show that the set \(B((a))\) is bounded. The proof proceeds as Le Potier in [17, Théoreme 4.11.] and use the following result.

**Proposition 3.11.** [12, Proposition 2.6.] For fixed \(n, d, b\), there is a bounded family containing all torsion-free sheaves \(E\) on \(X\) with \(rk_E = n\), \(d_E = d\) and such that all non-zero subsheaves \(F\) of \(E\) have slope \(\mu(F) \leq b\).

**Lemma 3.12.** The set \(B((a))\) is bounded.

**Proof.** Let \((a) \in \mathbb{Q}^{(n-1)(k+1)}\) and \(\bar{a} := \max_{a_{i,j} \in (a)} a_{i,j}\). Taking \(b = \frac{d + \alpha k}{n} + \bar{a}\) in Proposition 3.11 it follows that the set of isomorphism classes of vector bundles occurring in \(B((a))\) is bounded. Let \(E\) be the family of vector bundles on \(X\) parametrized by \(S\) such that for each \((E, V) \in B((a)), E\) is isomorphic to \(E_s\) for some \(s \in S\).

Let \(p_X\) and \(p_S\) the canonical projections of \(X \times S\) on \(X\) and \(S\), respectively. We will denote by \(\omega_{X \times S/S} = p_X^*(\omega_X)\) the relative canonical sheaf by the projection \(p_S\). Consider the sheaf \(Ext^1_{p_S}(\mathcal{E}, \omega_{X \times S/S})\) on \(S\). By [17, Lemme 4.9] the fiber \(Ext^1_{p_S}(\mathcal{E}, \omega_{X \times S/S})_s\) is isomorphic to \(Ext^1(\mathcal{E}_s, \omega_X)\) and by change-base Theorem

\[
Ext^1(\mathcal{E}_s, \omega_X) \cong H^1(X, \mathcal{E}_s^* \otimes \omega_X).
\]

Let us denote by

\[
\pi : \mathcal{G} = Grass(Ext^1_{p_S}(\mathcal{E}, \omega_{X \times S/S}), k) \to S
\]

the Grassmannian of quotient coherent locally free sheaves of rank \(k\), equipped with the universal family

\[
\pi^*(Ext^1_{p_S}(\mathcal{E}, \omega_{X \times S/S})) \to \mathcal{Y} \to 0.
\]

By change-base Theorem

\[
\pi^*(Ext^1_{p_S}(\mathcal{E}, \omega_{X \times S/S})) \cong Ext^1_{p_S}(\mathcal{E}', \omega_{X \times G/G})
\]

where \(\mathcal{E}' = (\pi \times id_X)^* \mathcal{E}\).

Hence the pair \((\mathcal{E}', \mathcal{Y})\) defines a family of coherent systems on \(X\) parametrized by \(\mathcal{G}\) in the sense of the Definition 2.5. Note that by the way in which we build the family \((\mathcal{E}', \mathcal{Y})\) we have that every member of \(B((a))\) is isomorphic to one of \(\{(\mathcal{E}', \mathcal{Y})_y : y \in Y\}\) which proves the lemma. \(\square\)

In the remainder of this section we show some local properties of the coherent systems of subtype \((a)\), without loss of generality we assume that \(Y\) is the spectrum of a discrete valuation ring. Denote by \(y\) (resp. \(y_0\)) the generic point (resp. closed point) of \(Y\).
Lemma 3.13. Let \((\mathcal{E}, \mathcal{V})\) be a family of coherent systems on \(X\) parametrized by \(Y\). Let \((G, Z)_y\) be a quotient coherent system of \((\mathcal{E}, \mathcal{V})_y\). Then there exists a unique quotient family of coherent systems \((G, Z)\) of \((\mathcal{E}, \mathcal{V})\) such that the restriction on \(X \times \{y\}\) is \((G, Z)_y\).

Proof. Let \((\mathcal{E}, \mathcal{V})\) be a family of coherent systems on \(X\) parametrized by \(Y\) and \((G, Z)_y\) be a quotient coherent system of \((\mathcal{E}, \mathcal{V})_y\). As \(G_y\) is a quotient bundle of \(\mathcal{E}_y\), from [9, Lemme 3.7.] follows that there exists a unique quotient bundle \(G\) of \(\mathcal{E}\) on \(X \times Y\), flat over \(Y\) such that \(G_y\) is precisely \(G_y\). Moreover, as the morphism \(V_y \rightarrow Z_y\) is surjective from [9, Lemme 3.7.] there is a unique quotient bundle \(Z\) of \(V\), flat over \(Y\) such that \(Z_y\) is \(Z_y\). Therefore, \((G, Z)\) is a family of coherent systems on \(X\) parametrized by \(Y\) that satisfies the lemma. \(\square\)

The following lemma shows that the properties subtype and cotype are stable under specialization.

Lemma 3.14. Let \(\alpha > 0\) and \((\mathcal{E}, \mathcal{V})\) be a family of coherent systems of type \((n, d, k)\) parametrized by \(Y\).

1. If \((\mathcal{E}, \mathcal{V})_y\) is of cotype \((a)\), then \((\mathcal{E}, \mathcal{V})_y\) is of cotype \((a)\).
2. If \((\mathcal{E}, \mathcal{V})_y\) is of subtype \((a)\), then \((\mathcal{E}, \mathcal{V})_y\) is of subtype \((a)\).

Proof. Let \(\alpha > 0\) and \((\mathcal{E}, \mathcal{V})\) be a family of coherent systems of type \((n, d, k)\) parametrized by \(Y\).

1. Suppose \((\mathcal{E}, \mathcal{V})_y\) is not of cotype \((a)\), then there exist a pair of integers \(m', t'\), \(0 < m' < n\), \(0 \leq t' \leq k\) and a quotient coherent system \((G, Z)_y\) of \((\mathcal{E}, \mathcal{V})_y\) of type \((m', d_G, t')\) which satisfies

\[
\mu_\alpha(G, Z)_y \leq \mu_\alpha(\mathcal{E}, \mathcal{V})_y - a_{m', t'}.
\]

By Lemma 3.13 there exists a family of coherent systems \((G, Z)\) on \(X\) parametrized by \(Y\) such that the restriction on \(X \times \{y\}\) is \((G, Z)_y\). Since the degree, the rank and the dimension are invariants in the family, it follows that

\[
\mu_\alpha(G, Z)_y \leq \mu_\alpha(\mathcal{E}, \mathcal{V})_y - a_{m', t'}.
\]

Hence, \((\mathcal{E}, \mathcal{V})_y\) is not of cotype \((a)\).

2. Suppose \((\mathcal{E}, \mathcal{V})_y\) is not of subtype \((a)\), by Proposition 3.8 the coherent system \((\mathcal{E}, \mathcal{V})_y\) is not of cotype \((b)\), where

\[
(b) = (b_{i,j} = a_{n-i,k-j} \frac{n-i}{i} : 0 < i < n, 0 \leq j \leq k),
\]

by (1), we have \((\mathcal{E}, \mathcal{V})_y\) is not of cotype \((b)\). Repeated application of the Proposition 3.8 we conclude that \((\mathcal{E}, \mathcal{V})_y\) is not of subtype \((a)\) as we desired. \(\square\)

Now, we are ready to prove that subtype is an open property for coherent systems.
Theorem 3.15. Let \( \alpha > 0, n \geq 1, k \geq 0 \) integer numbers, \((a) \in \mathbb{Q}^{(n-1)(k+1)}\) and \(Y\) be a variety. If \((E, V)\) is a family of coherent systems on \(X\) of type \((n, d, k)\) parametrized by \(Y\) then:

1. The set
   \[ Y_n((b)) = \{ y \in Y : (E, V)_y \text{ is not of cotype } (b) \} \]
   is a closed set in \(Y\) where \((b)\) is the sequence defined by
   \[ (b) := (b_{ij} = a_{n-i,k-j} \frac{n-i}{k} : 0 < i < n, 0 \leq j \leq k). \]

2. The set
   \[ Y((a)) = \{ y \in Y : (E, V)_y \text{ is of subtype } (a) \} \]
   is an open set in \(Y\).

Proof. (1) Let \((a) := (a_{i,j}), (b) := (b_{ij} = a_{n-i,k-j} \frac{n-i}{k} : 0 < i < n, 0 \leq j \leq k) \in \mathbb{Q}^{(n-1)(k+1)}\) and \((E, V)\) be a family of coherent systems on \(X\) of type \((n, d, k)\) parametrized by \(Y\). For any triple of integers \(m', d', t'\), \(0 < m' < n, 0 \leq t' \leq k\) let us consider the set \(\Delta(m', d', t')\) consisting of isomorphism classes of coherent systems \((G, Z)\) such that

   \(a) (G, Z)\) is of type \((m', d', t')\),

   \(b) (E, V)_y \to (G, Z) \to 0\) for some \(y \in Y\),

   \(c) \mu_\alpha(G, Z) \leq \frac{d_m a_k}{n} - b_{m', t'}\),

   \(d) (G, Z)\) is of cotype \((b - b_{m', t'})\).

Let \(Quot_{(m', d', t')}(E, V)\) denote the Quot-scheme which parametrizes all quotient coherent systems of \((E, V)\) of type \((m', d', t')\) (see [10, Section 1.6.]). Denote by \(p_{i,j}\) the canonical projection on \(X \times Y \times Quot_{(m', d', t')}(E, V)\) for \(i, j = 0, 1, 2\). We denote by

\[ p_{1,2}^*(E, V) \to (G, Z) \to 0 \]

the universal quotient coherent system on \(X \times Y \times Quot_{(m', d', t')}(E, V)\). By Proposition 3.8, Lemma 3.12 and property \((d)\) the set \(\Delta(m', d', t')\) is bounded. In particular, note that every member of \(\Delta(m', d', t')\) is isomorphic to one \(q \in Quot_{(m', d', t')}(E, V)\). Consider the sheaf

\[ \mathcal{H}om_{p_{2,3}}(p_{1,2}^*(E, V), (G, Z)) \]

on \(X \times Quot_{(m', d', t')}(E, V)\) and denote by \(\Gamma(m', d', t')\) the support of \((3.3)\), that is

\[ \Gamma(m', d', t') := \text{Supp}(\mathcal{H}om_{p_{2,3}}(p_{1,2}^*(E, V), (G, Z))) \]

\[ = \{ (y, q) \in X \times Quot_{(m', d', t')}(E, V) : \mathcal{H}om((E, V), (G, Z))_{(y, q)} \neq 0 \}. \]

We denote by \(\pi_Y(\Gamma(m', d', t'))\) the image of \(\Gamma(m', d', t')\) under the canonical projection \(\pi_Y : X \times Quot_{(m', d', t')}(E, V) \to Y\). Note that the set \(\pi_Y(\Gamma(m', d', t'))\) is constructible. Define the set \(H\) by

\[ H := \bigcup_{(m', d', t')} \pi_Y(\Gamma(m', d', t')) \]
which is constructible since it is a finite union of constructible sets. We claim that
\[ H = Y_n((b)). \]

In fact, if \( y \in Y_n((b)) \) the Lemma 3.10 and the Proposition 3.8 implies that there exist integers \( m', t', 0 < m' < n, 0 \leq t' \leq k \) and a quotient coherent system \((G, Z)\) of \((E, V)\) such that \((G, Z)\) is of cotype \((b - m', t')\) and
\[
\mu_\alpha(G, Z) \leq \frac{d + \alpha k}{n} - b_{m', t'}.
\]
It follows that there exists \( q \in Quot_{(m', d', t')}(E, V) \) such that \((G, Z)\) is isomorphic to the element corresponding to \( q \), then \((y, q) \in \Gamma((m', d', t')) \). Therefore \( y \in \pi_Y(\Gamma(m', d', t')) \subset Y \) as we required.

Conversely, if \( y \in H \), there exist integers \( m', d' \) and \( t' \) and a point \( q \in Quot_{(m', d', t')}(E, V) \) such that \((y, q) \in \Gamma(m', d', t') \subset Y \times Quot_{(m', d', t')}(E, V) \). By properties \((b)\) and \((c)\) we have \((E, V)_y\) is not of cotype \((b)\). Hence \( y \in Y_n((b)) \) and \( Y_n((b)) \subset H \). Therefore, \( H = Y_n((b)) \).

Since \( Y_n((b)) \) is a constructible set and by Proposition 3.14 it is stable under specialization, we conclude that \( Y_n((b)) \) is closed in \( Y \) as we required.

(2) Let \((E, V)\) be a family of coherent systems of type \((n, d, k)\) parametrized by \( Y \). By (1) the set
\[
Y_n((b)) = \{ y \in Y : (E, V)_y \text{ is not of cotype } (b) \}
\]
is a closed set in \( Y \), where \((b)\) is the sequence
\[
(b) := (b_{ij} = a_{n-i,k-j} \frac{n-i}{k} : 0 < i < n, 0 \leq j \leq k).
\]
By Proposition 3.8 the set \( Y_n((b)) \) is equivalent to the set
\[
Y^n((a)) := \{ y \in Y : (E, V)_y \text{ is not of subtype } (a) \}.
\]
Therefore,
\[
Y((a)) = \{ y \in Y : (E, V)_y \text{ is of subtype } (a) \}
\]
is an open set in \( Y \) as we required.

Now we are able to prove that the \((m, t)\)-Segre function is lower semicontinuous, (see Theorem 3.5).

**Proof of the Theorem 3.5.** Let \((E, V)\) be a family of coherent systems of type \((n, d, k)\) parametrized by \( Y \). In order to prove that the \((m, t)\)-Segre function
\[
S_{m,t}^\alpha: Y \rightarrow \mathbb{R} \cup \{\infty\}
\]
\[
y \mapsto S_{m,t}^\alpha(E, V)_y
\]
is lower semicontinuous we need to show that the set
\[
S(b) := \{ y \in Y : S_{m,t}^\alpha(E, V)_y > b \}
\]
is open in $Y$ for any $b \in \mathbb{R}$. Note that the set $S(b)$ is equivalent to the set
\[ S(b) = \{ y \in Y : \mu_\alpha(F, W) < \mu_\alpha(E, V) - \frac{b}{nm}, \text{ for any } (F, W) \in P_{m,t}(E, V) \}. \]

Let $A(b)$ denote the set
\[ A(b) := \{ (a) := (a_{i,j}) \in \mathbb{Q}^{(n-1)(k+1)} : a_{m,t} = -\frac{b}{nm}, \ 0 < m < n, 0 \leq t \leq k \}. \]

From Theorem 3.15, for any $(a) \in A(b)$ the set
\[ Y((a)) = \{ y \in Y : (E, V)_y \text{ is of subtype } (a) \} \]
is an open set in $Y$. Note that
\[ \bigcup_{(a) \in A(b)} Y(a) = S(b). \]

Hence $S(b)$ is an open set in $Y$, since it is an arbitrary union of open sets. Therefore, we conclude that Segre function $S^\alpha_{m,t}$ is lower semicontinuous as we required.

4. Stratification of $G(\alpha; n, d, k)$ according to the invariant $S^\alpha_{m,t}$

In this section we use the $(m, t)$-Segre invariant to induce a stratification of the moduli space $G(\alpha; n, d, k)$ of $\alpha$-stable coherent systems on $X$ of type $(n, d, k)$. If $\text{GCD}(n, d, k) = 1$ by [6, Proposition A.8.] there exists a universal family $(E, V)$ parametrized by $G(\alpha; n, d, k)$. If $\text{GCD}(n, d, k) \neq 1$ working locally in the étale topology we can assume that there is a family $(E, V)$ parametrized by $G(\alpha; n, d, k)$.

Let $(E, V)$ be a family of coherent systems on $X$ parametrized by $G(\alpha; n, d, k)$. From Theorem 3.5 the $(m, t)$-Segre function is lower semicontinuous, hence it induces a stratification of the moduli space $G(\alpha; n, d, k)$ into locally closed subvarieties
\[ G(\alpha; n, d, k; m, t; s) := \{ (E, V) \in G(\alpha; n, d, k) : S^\alpha_{m,t}(E, V) = s \} \]
according to the value $s$ of $S^\alpha_{m,t}$. Note that every pair of integers $m, t, 0 < m < n$, $0 \leq t \leq k$ define a stratification of $G(\alpha; n, d, k)$. That means we would have $(n-1)(k+1)$ different stratifications for the same moduli space $G(\alpha; n, d, k)$. Moreover, if $\alpha_i, \alpha_{i+1}$ are consecutive critical values, then the stratification is independent of $\alpha$ within the interval $(\alpha_i, \alpha_{i+1})$. It can be used to analyze the differences between the consecutive moduli spaces in the family $\{G_0, G_1, \ldots, G_L\}$.

One of the main difficulties in the study of the stratification of $G(\alpha; n, d, k)$ is to show that the different strata are non-empty. Our strategy consists in to construct extensions of coherent systems
\[ 0 \rightarrow (F, W) \rightarrow (E, V) \rightarrow (G, Z) \rightarrow 0 \]
in which $(E, V)$ is $\alpha$-stable and $(F, W) \in P_{m,t}(E, V)$ is maximal. In this section we determine values of $\alpha$, $m$, $t$ and $s$ under which the different strata are non-empty and determine their dimension.
Let $\alpha = \frac{p}{q} \in \mathbb{Q}^+$. Suppose that the stratum
\begin{equation}
G(\alpha; n, d, k; m, t; 1/q)
\end{equation}
is non-empty. If $(E, V) \in G(\alpha; n, d, k; m, t; 1/q)$, then there exists $(F, W) \in P_{m,t}(E, V)$ such that
\begin{equation}
S_{m,t}^\alpha(E, V) = (mn)(\mu_\alpha(E, V) - \mu_\alpha(F, W)) = \frac{1}{q},
\end{equation}
and an exact sequence
\begin{equation*}
0 \to (F, W) \to (E, V) \to (G, Z) \to 0
\end{equation*}
where the quotient system $(G, Z)$ is of type $(m', d_G, t')$. Note that (4.2) implies
\begin{equation*}
q(md - nd_F) + p(mk - nt) = 1
\end{equation*}
that is equivalent to
\begin{equation*}
q(md_G - m'd_F) + p(mt' - m't) = 1.
\end{equation*}

**Remark 4.1.** Let $\alpha = \frac{p}{q} \in \mathbb{Q}^+$. If $(E, V)$ is $\alpha-$stable, then for any coherent subsystem $(F, W) \subset (E, V)$ of type $(m, d, t)$ we have
\begin{equation*}
0 < \frac{1}{qnm} \leq \mu_\alpha(E, V) - \mu_\alpha(F, W).
\end{equation*}

The following theorem establishes conditions under which the stratum (4.1) is non-empty.

**Theorem 4.2.** Let $\alpha = \frac{p}{q} \in \mathbb{Q}^+ and \ n \geq 2, d > 0, k \geq 1$ be integer numbers. Suppose that there exist $n_1, n_2, d_1, d_2 > 0, t_1, t_2 \geq 0$ integer numbers such that
\begin{equation}
n_1n_2(g - 1) - d_1n_2 + d_2n_1 + t_2n_1(g - 1) - t_1t_2 > 0,
\end{equation}
\begin{equation}
q(n_1d_2 - n_2d_1) + p(n_1t_2 - n_2t_1) = 1.
\end{equation}
and the moduli spaces $G(\alpha; n_1, d_1, t_1)$ and $G(\alpha; n_2, d_2, t_2)$ are non-empty. Then, the stratum
\begin{equation*}
G(\alpha; n, d, k; n_1, t_1; 1/q)
\end{equation*}
is non-empty, where $n = n_1 + n_2, d = d_1 + d_2 and k = t_1 + t_2$.

The proof of the theorem makes use of the following results.

**Theorem 4.3.** Let $\alpha = \frac{p}{q} \in \mathbb{Q}^+ and (E, V) \in G(\alpha; n, d, k)$. If $(F, W) \in P_{m,t}(E, V)$ and
\begin{equation}
q(md - nd_F) + p(mk - nt) = 1
\end{equation}
then, $(F, W)$ is $\alpha-$stable and maximal. Moreover, the quotient system $(G, Z)$ defined by $(F, W)$ is $\alpha-$stable.
Proof. Let $\alpha = \frac{p}{q} \in \mathbb{Q}^+$, $(E,V) \in G(\alpha;n,d,k)$ and $(F,W) \in P_{m,t}(E,V)$ which satisfy (4.5). Let $(G,Z)$ the quotient coherent system of $(E,V)$ of type $(m',d',t')$ that fit in the exact sequence,

$$0 \rightarrow (F,W) \rightarrow (E,V) \rightarrow (G,Z) \rightarrow 0.$$ 

We claim that $(F,W)$ and $(G,Z)$ are $\alpha-$stable coherent systems.

(1) Let $(F_1,W_1)$ be a subsystem of $(F,W)$ of type $(n_1,d_{F_1},k_1)$, $n_1 \leq m$. If $n_1 = m$ we would have $F \cong F_1$ and $\text{dim } W_1 < \text{dim } W$, hence $\mu_{\alpha}(F_1,W_1) < \mu_{\alpha}(F,W)$. If $n_1 < m$ since $(E,V)$ is $\alpha-$stable by Remark 4.1 we have

\begin{equation}
\mu_{\alpha}(F_1,W_1) \leq \mu_{\alpha}(E,V) - \frac{1}{qnn_1}, \tag{4.6}
\end{equation}

and by (4.5),

\begin{equation}
\mu_{\alpha}(E,V) = \mu_{\alpha}(F,W) + \frac{1}{qnm}. \tag{4.7}
\end{equation}

Replacing (4.7) in (4.6) we obtain

$$\mu_{\alpha}(F_1,W_1) \leq \mu_{\alpha}(F,W) + \frac{1}{qnm} - \frac{1}{qnn_1} = \mu_{\alpha}(F,W) - \frac{1}{qn} \left(\frac{m-n_1}{mn_1}\right), \quad 0 < m-n_1$$

and

$$< \mu_{\alpha}(F,W),$$

Hence, $(F,W)$ is $\alpha-$stable as we required.

(2) In order to prove that $(G,Z)$ is $\alpha-$stable, note by [10, Section 1.2.] that for any coherent system $(G_1,Z_1)$ the functor $\text{Hom}((G_1,Z_1),-) \text{ is left exact. Applying}$ this functor to the extension

$$0 \rightarrow (F,W) \rightarrow (E,V) \rightarrow (G,Z) \rightarrow 0$$

we have the induced homomorphism

\begin{equation}
\text{Hom}((G_1,Z_1),(G,Z)) \rightarrow \text{Ext}^1((G_1,Z_1),(F,W)). \tag{4.8}
\end{equation}

Suppose that $(G_1,Z_1)$ is a subsystem of $(G,Z)$ of type $(n_1',d_1',k_1')$ and consider the exact sequence

$$0 \rightarrow (F,W) \rightarrow (E_1,V_1) \rightarrow (G_1,Z_1) \rightarrow 0$$

which by (4.8) defines the following diagram

$$\begin{array}{cccccc}
0 & \rightarrow & (F,W) & \rightarrow & (E_1,V_1) & \rightarrow & (G_1,Z_1) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & (F,W) & \rightarrow & (E,V) & \rightarrow & (G,Z) & \rightarrow & 0
\end{array}$$
where \((E_1, V_1)\) is of type \((n', d_{E_1}, k')\) and

\[
\mu_\alpha(G_1, Z_1) = \mu_\alpha(E_1, V_1) \frac{n'}{n_1'} - \mu_\alpha(F, W) \frac{m}{n_1'}. \tag{4.9}
\]

Since \((E, V)\) is \(\alpha\)-stable by Remark 4.1 we have

\[
\mu_\alpha(E_1, V_1) \leq \mu_\alpha(E, V) - \frac{1}{qnn'} \tag{4.10}
\]

and

\[
\mu_\alpha(F, W) \leq \mu_\alpha(E, V) - \frac{1}{qnm} \tag{4.11}
\]

Replacing (4.10) and (4.11) in (4.9) we obtain

\[
\mu_\alpha(G_1, Z_1) \leq \left(\mu_\alpha(E, V) - \frac{1}{qnn'}\right) \frac{n'}{n_1'} - \left(\mu_\alpha(E, V) - \frac{1}{qnm}\right) \frac{m}{n_1'}, \quad n' + m = n_1'.
\]

As \((E, V)\) is \(\alpha\)-stable we conclude that

\[
\mu_\alpha(G_1, Z_1) \leq \mu_\alpha(E, V) < \mu_\alpha(G, Z).
\]

Therefore \((G, Z)\) is \(\alpha\)-stable as we required.

Finally, we prove that \((F, W)\) is maximal, suppose \((F', W')\) is not maximal then there exists \((F', W')\) such that \(\mu_\alpha(F, W) < \mu_\alpha(F', W')\). Thus

\[
0 < qnm(\mu_\alpha(E, V) - \mu_\alpha(F', W')) < qnm(\mu_\alpha(E, V) - \mu_\alpha(F, W)) = q(md - nd_F) + p(mk - nt) = 1
\]

which is a contradiction, because \(qnm(\mu_\alpha(E, V) - \mu_\alpha(F', W'))\) is an integer number. Hence, \((F, W)\) is maximal which completes the proof. \(\square\)

**Theorem 4.4.** Let \(\alpha = \frac{p}{q} \in \mathbb{Q}^+\) and \((F, W), (G, Z)\) be \(\alpha\)-stable coherent systems of type \((m, d_F, t)\) and \((m', d_G, t')\), respectively which satisfies

\[
q(md_G - m'd_F) + p(mt' - m't) = 1. \tag{4.12}
\]

If

\[
0 \to (F, W) \to (E, V) \to (G, Z) \to 0
\]

is a non-trivial extension of coherent systems, then the coherent system \((E, V)\) is \(\alpha\)-stable and

\[
S_{m,t}^\alpha(E, V) = (nm)(\mu_\alpha(E, V) - \mu_\alpha(F, W)) = \frac{1}{q}.
\]

**Proof.** Let \(\alpha = \frac{p}{q} \in \mathbb{Q}^+\) and \((F, W), (G, Z)\) be \(\alpha\)-stable coherent systems of type \((m, d_F, t), (m', d_G, t')\), respectively which satisfies (4.12). Suppose that there exists a non-trivial extension

\[
0 \to (F, W) \to (E, V) \to (G, Z) \to 0 \tag{4.13}
\]
where \((E, V)\) is of type \((n, d, k)\). Let \((E', V')\) a subsystem of \((E, V)\) of type \((n', d', k')\) and \((G_1, Z_1), (F_1, W_1)\) be the image and the kernel of the morphism \((E', V') \to (G, Z)\), respectively. We have the following diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (F_1, W_1) & \rightarrow & (E', V') & \rightarrow & (G_1, Z_1) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (F, W) & \rightarrow & (E, V) & \rightarrow & (G, Z) & \rightarrow & 0 \\
\end{array}
\]

where \((F_1, W_1)\) and \((G_1, Z_1)\) are coherent systems of type \((n_1, d_{F_1}, t_1)\) and \((n'_1, d_{G_1}, t'_1)\), respectively. Since \((F, W)\) and \((G, Z)\) are \(\alpha\)-stable by Remark 4.1, we have

\[
\mu_\alpha(F_1, W_1) \leq \mu_\alpha(F, W) - \frac{1}{qm_1}
\]

and

\[
\mu_\alpha(G_1, Z_1) \leq \mu_\alpha(G, Z) - \frac{1}{qm'n'_1}.
\]

Also, by (4.12) we have

\[
\mu_\alpha(F, W) = \mu_\alpha(E, V) - \frac{1}{qn}\]

and

\[
\mu_\alpha(G, Z) = \mu_\alpha(E, V) + \frac{1}{qm'}.
\]

To show that \((E, V)\) is \(\alpha\)-stable we consider all possibilities of \((F_1, W_1)\) and \((G_1, Z_1)\). We claim \(\mu_\alpha(E', V') < \mu_\alpha(E, V)\);

(i) Assume \((F_1, W_1) = 0\). Hence \((E', V') \cong (G_1, Z_1)\). From (4.15) and (4.17) follows that

\[
\mu_\alpha(E', V') = \mu_\alpha(G_1, Z_1) \\
\leq \mu_\alpha(G, Z) - \frac{1}{qm'n'_1} \\
= \mu_\alpha(E, V) - \frac{1}{qm'}\left(\frac{1}{n'_1} - \frac{1}{n}\right), \ 0 < n - n'_1 \\
< \mu_\alpha(E, V).
\]

(ii) Assume \((G_1, Z_1) = 0\). We have \((E', V') \cong (F_1, W_1)\). From (4.16) and since \((F, W)\) is \(\alpha\)-stable follows that

\[
\mu_\alpha(E', V') = \mu_\alpha(F_1, W_1) < \mu_\alpha(F, W) < \mu_\alpha(E, V).
\]

(iii) Assume \((G_1, Z_1) = (G, Z)\). Hence \((F_1, W_1) \neq (F, W)\) and

\[
\mu_\alpha(E', V') = \mu_\alpha(F_1, W_1)\frac{n_1}{n'} + \mu_\alpha(G, Z)\frac{m'}{n'}.
\]
Replacing (4.14) in (4.18) we obtain
\begin{equation}
\mu_\alpha(E', V') \leq \left( \mu_\alpha(F, W) - \frac{1}{qmn_1} \right) \frac{n_1}{n'} + \mu_\alpha(G, Z) \frac{m'}{n'}.
\end{equation}

Replacing (4.16) and (4.17) in (4.19) follows that
\begin{equation}
\mu_\alpha(E', V') \leq \left( \mu_\alpha(E, V) - \frac{1}{qmn} - \frac{1}{qmn_1} \right) \frac{n_1}{n'} + \left( \mu_\alpha(E, V) + \frac{1}{qmn'} \right) \frac{m'}{n'}
\end{equation}
\begin{equation}
= \mu_\alpha(E, V) - \frac{n_1}{qmn} \left( \frac{1}{n_1} + \frac{1}{qmn'} \right) \frac{m'}{n'} + \left( \mu_\alpha(E, V) + \frac{1}{qmn'} \right) \frac{m'}{n'}
\end{equation}
\begin{equation}
< \mu_\alpha(E, V),
\end{equation}

(iv) Assume \((F, W_1) = (F, W)\). Hence \((G_1, Z_1) \neq (G, Z)\) and
\begin{equation}
\mu_\alpha(E', V') = \mu_\alpha(F, W) \frac{m}{n'} + \mu_\alpha(G_1, Z_1) \frac{n_1'}{n'}.
\end{equation}

We proceed as in (iii) and conclude that \(\mu_\alpha(E', V') < \mu_\alpha(E, V)\).

(v) Assume \((F, W_1) \neq (F, W)\) and \((G_1, Z_1) \neq (G, Z)\). It follows that
\begin{equation}
\mu_\alpha(E', V') = \mu_\alpha(F, W_1) \frac{n_1}{n'} + \mu_\alpha(G_1, Z_1) \frac{n_1'}{n'}.
\end{equation}

Replacing (4.14) and (4.15) in (4.20) we have
\begin{equation}
\mu_\alpha(E', V') \leq \left( \mu_\alpha(F, W) - \frac{1}{qmn_1} \right) \frac{n_1}{n'} + \left( \mu_\alpha(G, Z) - \frac{1}{qmn_1} \right) \frac{n_1'}{n'}.
\end{equation}

Replacing (4.16) and (4.17) in (4.21), it follows that
\begin{equation}
\mu_\alpha(E', V') \leq \left( \mu_\alpha(E, V) - \frac{1}{qmn} - \frac{1}{qmn_1} \right) \frac{n_1}{n'} + \left( \mu_\alpha(E, V) + \frac{1}{qmn'} \right) \frac{m'}{n'}
\end{equation}
\begin{equation}
= \mu_\alpha(E, V) - \frac{n_1}{qmn} \left( \frac{1}{n_1} + \frac{1}{qmn'} \right) \frac{m'}{n'} - \frac{n_1'}{qmn'} \left( \frac{n - n_1'}{nn_1} \right), 0 < n - n_1'
\end{equation}
\begin{equation}
< \mu_\alpha(E, V).
\end{equation}

From (i), (ii), (iii), (iv) and (v), it follows that \(\mu_\alpha(E', V') < \mu_\alpha(E, V)\). Therefore, \((E, V)\) is \(\alpha\)-stable as we required.

Since \((E, V)\) is \(\alpha\)-stable and \((F, W)\) fit into the sequence (4.13) by Theorem 4.3, the system \((F, W) \in P_{m,t}(E, V)\) is maximal and
\begin{equation}
S^\alpha_{m,t}(E, V) = (mn)(\mu_\alpha(E, V) - \mu_\alpha(F, W))
\end{equation}
which completes the proof. \(\square\)

**Proof of the Theorem 4.2.** Let \(\alpha = \frac{p}{q}\) and \(n_1, n_2, d_1, d_2, t_1, t_2\) be integer numbers that satisfy the hypothesis of the theorem. Let \((F, W) \in G(\alpha; n_1, d_1, t_1)\) and \((G, Z) \in G(\alpha; n_2, d_2, t_2)\). From (4.3) and [5, Proposition 3.2.],
\begin{equation}
\dim \text{Ext}^1((G, Z), (F, W)) > 0,
\end{equation}
hence there exists a non-trivial extension
\[ 0 \to (F, W) \to (E, V) \to (G, Z) \to 0. \]

As \((F, W), (G, Z)\) are \(\alpha\)-stable and these satisfy (4.4) from Theorems 4.3 and 4.4, we conclude that \((E, V)\) is \(\alpha\)-stable and \((F, W) \in P_{m,t}(E, V)\) is maximal. Therefore \(G(\alpha; n, d, k; m, t; 1/q)\) is non-empty which completes the proof.

The following theorem gives us the dimension of the stratum \(G(\alpha; n, d, k; m, t; 1/q)\).

**Theorem 4.5.** Under the hypothesis of Theorem 4.2. Suppose that
\[ \dim \text{Ext}^2((F_2, W_2), (F_1, W_1)) = \text{cte} \]
for any \((F_i, W_i) \in G(\alpha; n_i, d_i, t_i)\) for \(i = 1, 2\). Then, the dimension of the stratum \(G(\alpha; n, d, k; n_1, t_1; 1/q)\) is bounded above by
\[ \dim G(\alpha; n_1, d_1, t_1) + \dim G(\alpha; n_2, d_2, t_2) + C_{21} - 1, \]
where \(C_{21} = n_1n_2(g - 1) - d_1n_2 + d_2n_1 + t_2d_1 - t_2n_1(g - 1) - t_1t_2.\)

**Proof.** Let \(\alpha = \frac{p}{q} \in \mathbb{Q}^+\) and \(n_1, n_2, d_1, d_2 > 0, t_1, t_2 \geq 0\) be integer numbers that satisfy the hypothesis of the Theorem 4.2. Working locally in the étale topology if necessary, we can assume without loss of generality that there exists a family \((\mathcal{F}_i, \mathcal{W}_i)\) of coherent systems on \(X\) of type \((n_i, d_i, t_i)\) parametrized by \(G_i := G(\alpha; n_i, d_i, t_i), i = 1, 2.\) Let \(p_{i,j}\) denote the canonical projections of \(X \times G_1 \times G_2\) for \(i, j = 0, 1, 2.\) By Theorem 4.3 follows that in the exact sequence
\begin{equation}
(4.22) \quad 0 \longrightarrow (F_1, W_1) \longrightarrow (E, V) \longrightarrow (F_2, W_2) \longrightarrow 0
\end{equation}
\((E, V)\) is \(\alpha\)-stable, it implies that \(\dim \text{Hom}((F_2, W_2), (F_1, W_1)) = 0\) (see [12, Corollary 2.5.1]). Suppose that \(\dim \text{Ext}^2((F_2, W_2), (F_1, W_1)) = \text{cte},\) then by [5, Proposition 3.2.] we have
\[ \dim \text{Ext}^1((F_2, W_2), (F_1, W_1)) = C_{21} + \text{cte}. \]

By [10, Corollaire 1.20], there is a vector bundle
\[ \Gamma := \text{Ext}^1(p_{0,2}^*(\mathcal{F}_2, \mathcal{W}_2), p_{0,1}^*(\mathcal{F}_1, \mathcal{W}_1)) \]
over \(G_1 \times G_2\) whose fibre over \(((F_1, W_1), (F_2, W_2))\) is \(\text{Ext}^1((F_2, W_2), (F_1, W_1))\) for all \((F_i, W_i), i = 1, 2.\) Note that \(\mathbb{P}\Gamma\) parametrizes the non-trivial extensions (4.22) up to scalar multiples.

By [6, Lemma A.10], there exists a universal extension
\begin{equation}
(4.23) \quad 0 \longrightarrow (id \times \pi)^*p_{0,1}^*(\mathcal{F}_1, \mathcal{W}_1) \otimes O_{\mathbb{P}\Gamma}(1) \longrightarrow (id \times \pi)^*(\mathcal{E}, \mathcal{V}) \longrightarrow (id \times \pi)^*p_{0,2}^*(\mathcal{F}_2, \mathcal{W}_2) \longrightarrow 0
\end{equation}
on \(X \times \mathbb{P}\Gamma.\)

Define the set
\[ U := \{ p \in \mathbb{P}\Gamma : (\mathcal{E}, \mathcal{V})_p \text{ is } \alpha\text{-stable and } S_{m,t}^\alpha(\mathcal{E}, \mathcal{V})_p = 1/q \}. \]
From the lower semicontinuous of the function $S_{m,t}^\alpha$, Theorem 4.2 and $\alpha$–stability being an open condition we conclude that the set $U$ is non-empty and open in $\mathbb{P}\Gamma$. Restricting the sequence (4.23) on $X \times U$ from the universal property of the moduli space $G(\alpha; n, d, k)$ we have a morphism

$$f : U \longrightarrow G(\alpha; n, d, k; n_1, t_1; 1/q) \subset G(\alpha; n, d, k).$$

Note that if $p \in U$, then $f(p)$ is precisely the point of $G(\alpha; n, d, k)$ representing to $(E, V)$. We now determine the dimension of the stratum

$$\dim G(\alpha; n, d, k; m, t; 1/q) = \dim U - \dim f^{-1}(E, V)$$

$$= \dim \mathbb{P}\Gamma - \dim f^{-1}(E, V)$$

$$= \dim G_1 + \dim G_2 + C_{21} - 1 - \dim f^{-1}(E, V)$$

$$\leq \dim G_1 + \dim G_2 + C_{21} - 1,$$

where $(F_i, W_i) \in G_i$ for $i = 1, 2$. This proves the theorem. □

In general, it is not easy to compute the dimension of $Ext^2((F_2, W_2), (F_1, W_1))$. However, to establish a better bound of the dimension of the stratum $G(\alpha; n, d, k; n_1, t_1; 1/q)$ in the Theorem 4.5 we could define a stratification $\{S_t\}$ of $G(\alpha; n_1, d_1, t_1) \times G(\alpha; n_2, d_2, t_2)$ such that on each $S_t$ we have that

(4.24) \quad \dim Ext^2((F_2, W_2), (F_1, W_1)) := a.

Hence

$$\dim Ext^1((F_2, W_2), (F_1, W_1)) := C_{21} + a$$

will be constant on each $S_t$. By [5, Lemma 3.3.] the quantity (4.24) is bounded on $G(\alpha; n_1, d_1, t_1) \times G(\alpha; n_2, d_2, t_2)$. Taking the maximum of these dimensions we have a better bound of $\dim G(\alpha; n, d, k; n_1, t_1; 1/q)$.

5. Applications to cross critical values

In this section we apply the previous result for a particular case. Let $X$ be a general curve of genus 6 and $(n, d, k) = (2, 13, 4)$. For $(2, 13, 4)$ the non-zero virtual critical values belong to

$$\left\{ \frac{2d' - 13}{4 - 2k'} : 0 \leq k' \leq 4 \right\} \cap (0, \infty).$$

Here $1/4$ and $1/2$ are the first virtual critical values. Note that $\alpha = 1/4$ is not a critical value, because there is no coherent system $(E, V)$ and a subsystem $(L_1, W_1)$ of type $(1, 6, 4)$ on a general curve of genus 6 (see [1]). The first critical value is $\alpha = 1/2$ since the coherent system $(E, V) := (L_1 \oplus L_2, W_1 \oplus W_2)$ satisfies $\mu_\alpha(E, V) = \mu_\alpha(L_1, W_1)$ where $(L_1, W_1)$ and $(L_2, W_2)$ are coherent systems of type $(1, 6, 3)$ and $(1, 7, 1)$, respectively. Denote by $\alpha_0 = 0$, $\alpha_1 = 1/2$ the first critical values.

Note that for any $p \in \mathbb{Z}^+$, $\alpha_p := \frac{p}{2p+1}$ is in the interval $(\alpha_0, \alpha_1)$, hence $G_0(2, 13, 4) = G(\alpha_p; 2, 13, 4)$. Since $n = 2$ and $k = 4$, we have 5 different stratifications for the moduli
space $G_0(2, 13, 4)$ each one induced by the functions $S^\alpha_{1,0}, S^\alpha_{1,1}, S^\alpha_{1,2}, S^\alpha_{1,3}$ and $S^\alpha_{1,4}$, respectively.

For instance, for $S^\alpha_{1,3} : G_0(2, 13, 4) \to \mathbb{R} \cup \{\infty\}$, we have that $S^\alpha_{1,3}(E, V) \leq 10 - 2\alpha_p$ provided that $P_{1,3}(E, V) \neq \emptyset$. Moreover, from $\alpha$-stability and by classical Brill-Noether theory (see [1]), it follows that

$$S^\alpha_{1,3}(E, V) = \begin{cases} s_1 := \frac{1}{2\alpha_p + 1}, & \text{if } P_{1,3}(E, V) \neq \emptyset \\ \infty, & \text{if } P_{1,3}(E, V) = \emptyset. \end{cases}$$

Hence the function $S^\alpha_{1,3}$ induces the stratification

$$G_0(2, 13, 4) := G_0(2, 13, 4; 1, 3; s_1) \sqcup G_0(2, 13, 4; 1, 3; \infty).$$

From Theorem 4.2 it follows that the stratum $G_0(2, 13, 4; 1, 3; s_1)$ is non-empty. Moreover by Theorem 4.5 and [5, Lemma 3.3.], we get $\dim G_0(2, 13, 4; 1, 3; s_1) = 11$. Since every irreducible component $G$ of $G_0(2, 13, 4)$ has dimension $G \geq \beta(2, 13, 4) = 17$ (see [10, Corollaire 3.14.]) we conclude that the stratum $G_0(2, 13, 4; 1, 3; \infty)$ is non-empty.

This stratification yields information about of how to change the moduli space $G_0(2, 13, 4)$ when it crosses the critical value $\alpha_1$.

**Theorem 5.1.** Let $X$ be a general curve of genus 6 and $G_0(2, 13, 4)$ be the moduli space in the interval $(\alpha_0, \alpha_1)$. Then, the stratum $G_0(2, 13, 4; 1, 3; s_1)$ induced by $S^\alpha_{1,3}$ is not a subset of $G_1(2, 13, 4)$.

**Proof.** Note that any $(E, V) \in G_0(2, 13, 4; 1, 3; s_1) \subset G_0(2, 13, 4)$ can be written in an exact sequence

$$0 \to (L_1, W_1) \to (E, V) \to (L_2, W_2) \to 0$$

where $(L_1, W_1)$ and $(L_2, W_2)$ are coherent systems of type $(n_1, d_1, t_1) = (1, 6, 3)$ and $(n_2, d_2, t_2) = (1, 7, 2)$, respectively and

$$S^\alpha_{1,3}(E, V) = 2(\mu_{\alpha_p}(E, V) - \mu_{\alpha_p}(L_1, W_1)) = s_1.$$ 

For the critical value $\alpha_1 = 1/2$, we get

$$\mu_{\alpha_1}(E, V) = \frac{13 + 2\alpha_1}{2} = \mu_{\alpha_1}(L_1, W_1) = 6 + 3\alpha_1.$$ 

Therefore by [5, Lemma 6.2.], $(E, V)$ is unstable for all $\alpha > \alpha_1$. Hence

$G(\alpha; 2, 13, 4; 1, 3; s_1) \not\subset G_1(2, 13, 4)$ which proves the theorem. \qed

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