UNO-schemes of the second order of accuracy for calculating waves in an elastic-plastic body

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Abstract. The possibility of increasing the efficiency of calculating waves in an elastic-plastic body by the Godunov method due to its UNO-modification is considered. The efficiency is estimated by comparing the results of computations by the Godunov method and its modification of the one-dimensional and two-dimensional problems on the propagation of waves in an elastic-plastic body, arising under the action on its free surface. It is shown that in the problems under consideration the UNO-modification is much more effective. In particular, it allows one to obtain numerical solutions of nearly the same accuracy by using significantly (four times) coarser grids with consuming less computer time by an order of magnitude.

1. Introduction

The classical Godunov method [1, 2] is stable and monotone and is widely used for studying dynamics of various perturbations in a body [3]. However, in computing the propagating waves with acceptable accuracy, it may require very fine computational grids, i.e., large computer costs, due to its first order of accuracy.

It is shown in [4-6] that the efficiency of computations of linear waves in an elastic body by the Godunov method can be increased using its TVD- and UNO-modifications of the second order of accuracy [7] (TVD – Total Variation Diminishing; UNO – Uniformly Non Oscillatory). As is known, in TVD- and UNO-schemes, the condition of monotonicity of a numerical solution is replaced by the condition of its total variation diminishing. The TVD-schemes satisfy that condition strictly, whereas the UNO-schemes meet it approximately (at the level of approximation errors).

The present paper illustrates the possibility of improving the efficiency of wave computation in an elastic-plastic body by the classical Godunov method by applying its UNO-modification proposed earlier by the authors for calculating elastic waves [4, 5].

2. Problem statement

The dynamics of a body is governed by the following equations

\[
\frac{\partial u}{\partial t} = \frac{\partial (S_{xx} - P)}{\partial x} + \frac{\partial S_{xy}}{\partial y}, \quad \frac{\partial v}{\partial t} = \frac{\partial S_{xy}}{\partial x} + \frac{\partial (S_{yy} - P)}{\partial y},
\]

where \(u\) and \(v\) are the displacements in the \(x\) and \(y\) directions, \(S_{xx}\) and \(S_{yy}\) are the components of the stress tensor, \(S_{xy}\) is the shear stress, \(P\) is the pressure, and \(\rho\) is the density of the material.

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\[ \frac{\partial S_{xx}}{\partial t} = \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right), \quad \frac{\partial S_{xy}}{\partial t} = -\frac{2}{3} \mu \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right), \quad \frac{\partial P}{\partial t} = -K \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]

where \( t \) is time; \( x, y \) are the Cartesian coordinates; \( \rho \) is the density; \( u, v \) are the components of the velocity vector; \( S_{xx}, S_{yy}, S_{xy} \) are components of the deviator \( S \) of the stress tensor \( \sigma \). \( P \) is the all-round (hydrostatic) pressure; \( \sigma = S - PE \); \( E \) is the tensor unit, \( K = \lambda + (2/3)\mu \) is the coefficient of volumetric expansion; \( \lambda = \rho (c_1^2 - 2c_2^2), \mu = \rho c_2^2 \) are the Lame parameters, \( c_1 \) and \( c_2 \) are the longitudinal and shear velocities of sound.

In the plastic zones the von Mises condition \( \sigma_i = Y_0 \) [8] is posed where \( Y_0 \) is the yield stress of the body material, \( \sigma_i \) is the stress intensity,

\[ \sigma_i = \sqrt{S_{xx}^2 + S_{yy}^2 + S_{xy}^2}. \]

At the infinite distance from the place of the load application, the body remains undisturbed. Initially the body is non-deformed and unstressed.

The possibility of increasing the efficiency of computation of waves in an elastic-plastic body by using a UNO-modification of the classical Godunov method in comparison with applying the Godunov method itself is investigated. In the both techniques, at each time step, the stresses in the body are first computed by solving equations (1) without taking the plasticity into account. Further, if at a point the yield condition is violated, i.e., if the stress intensity at that point is greater than the yield limit, then the tensor \( S \) at that point is corrected by the formula \( S_{corr} = \left( Y_0/\sigma_0 \right) S \).

3. A UNO-modification of the Godunov method

In presenting the UNO-modification of the Godunov method, it is convenient to represent system (1) in the form

\[ \mathbf{q}_t + \mathbf{f}_x + \mathbf{g}_y = 0, \]

where

\[ \mathbf{q} = \left( \rho, \rho v, S_{xx}, S_{yy}, S_{xy}, P \right)^T, \]

\[ \mathbf{f} = \left( -S_{xx} - P, S_{yy} - \frac{4}{3} \mu u_x, \frac{2}{3} \mu u_y, \mu v, -Ku \right)^T, \]

\[ \mathbf{g} = -\left( S_{xy} - P, -S_{yy} - P, \frac{2}{3} \mu v, \frac{4}{3} \mu u, \mu u, -Kv \right)^T. \]

The UNO-modification algorithm consists of three steps.

Step 1. Computation of the values of the unknown vector \( \mathbf{q} \) in the computational grid cells on a half-integer time layer

\[ q_{i+k}^{n+1/2} = q_{i+k}^n + (q_{i+k})^n \frac{\Delta t}{2}, \]

\( i, k \) is the cell number; \( \Delta t \) is the time step. The time derivative of \( \mathbf{q} \) is evaluated by the following expression resulted from (2)

\[ \mathbf{q}_t = -A \mathbf{q}_x - B \mathbf{q}_y, \]

where \( A = \mathbf{f}, B = \mathbf{g} \). Thus \( (q_{i+k})^n = -A_{i+k}^n (q_{i+k})^n - B_{i+k}^n (q_{i+k})^n \). Here \( A_{i+k}^n = A(q_{i+k})^n, B_{i+k}^n = B(q_{i+k})^n \), whereas \( (q_{i+k})^n \) is calculated by

\[ (q_{i+k})^n = \frac{1}{h_x} \min \text{mod} \left( \Lambda_{i+1/2,k}^1, \frac{1}{2} \Lambda_{i+1/2,k}^2, \Lambda_{i-1/2,k}^1, \frac{1}{2} \Lambda_{i-1/2,k}^2 \right), \]

where \( h_x \) is the grid step in \( x \) direction.
\[
\min \text{mod}(a,b) = \frac{\text{sgn}(a) + \text{sgn}(b)}{2} \min(|a|, |b|),
\]
\[
\Delta_1^{l+1/2,k} = q_{i+1,k}^n - q_{i,k}^n, \quad \Delta_1^{l-1/2,k} = q_{i-1,k}^n - q_{i,k}^n,
\]
\[
\Delta_2^{l+1/2,k} = \min \text{mod}(q_{i+1,k}^n - 2q_{i,k}^n + q_{i-1,k}^n, q_{i+2,k}^n - 2q_{i+1,k}^n + q_{i,k}^n),
\]
\[
\Delta_2^{l-1/2,k} = \min \text{mod}(q_{i,k}^n - 2q_{i-1,k}^n + q_{i-2,k}^n, q_{i+1,k}^n - 2q_{i,k}^n + q_{i-1,k}^n).
\]

The grid function \((q_y)^n_k\) is computed similarly.

Step 2. Computation of numerical flows through the faces of cells
\[
f_{i+1/2}^{n+1/2} = f(q_{i+1/2,k}^{n+1/2}, g_{i+1/2}^{n+1/2} = g(q_{i+1/2,k}^{n+1/2}),
\]
where \(q_{i+1/2,k}^{n+1/2} = R(q_{i+1/2,k}^{n+1/2}, q_{i+2,k}^{n+1/2}), \quad q_{i,k+1/2}^{n+1/2} = R(q_{i,k+1/2,k}^{n+1/2}, q_{i+1,k+1/2,k}^{n+1/2}), \quad R(a, b) \) is the Riemann problem solution,
\[
q_{i+1/2,k}^{n+1/2} = s_{i+1,k}^n h_{y/2}, \quad q_{i+1/2,k}^{n+1/2} = s_{i,k+1}^n h_{x/2}.\]

Step 3. Computation of the values of the unknown vector on a new time layer
\[
\frac{q_{i+1}^n - q_{i}^n}{\Delta t} + \frac{f_{i+1/2}^{n+1/2} - f_{i+1/2}^{n+1/2}}{h_{y}} + \frac{g_{i+k/2}^{n+1/2} - g_{i+k/2}^{n+1/2}}{h_{x}} = 0 .
\]

The time step is computed as
\[
\Delta t = 0.95 \left( \frac{c_1}{h_x} + \frac{1}{h_y} \right)^{-1}.
\]

4. Efficiency of the UNO-modification of Godunov’s method

The efficiency of the considered UNO-modification is demonstrated by computing the propagation of one- and two-dimensional waves in an unstressed elastic-plastic body simulated by a half-space \(y \leq 0\) with a free surface \(y = 0\). The waves in the body arise as a result of its free surface pressure variation. The free surface pressure \(p_f(t)\) is first linearly increased (in \(0 < t < t^*/2\)), then (in \(t^*/2 < t < t^*\)) is linearly removed, and after that it remains at the initial zero level (figure 1).

![Figure 1. The free surface pressure variation with time. Dots 1 and 2 indicate the moments at which numerical solutions are analyzed](image)

The material of the body is considered to be a nickel alloy with the following mechanical characteristics: Young modulus \(E = 196 \text{ GPa}\), Poisson ratio \(\nu = 0.3\), density \(\rho = 8000 \text{ kg} / \text{m}^3\), yield stress \(Y_0 = 125 \text{ MPa}\).

4.1. The propagation of one-dimensional waves

One-dimensional waves are considered, arising in the body under the pressure variation of figure 1 on the whole free surface \(y = 0\). The computation domain is a segment \([-d, 0]\) where \(d = 12L, L = c_1 t^*\). The right end of the segment is the free surface with the pressure \(p_f(t)\), the left one is an artificial boundary.
with non-reflective conditions. The features of propagation of a pulse excited in the elastic-plastic body are illustrated in figure 2a by the reference solution computed on a very fine grid. It is seen that the leading front of the pulse splits into an elastic precursor that propagates into the body at a velocity \( c_1 \), and a plastic wave moving in the same direction with velocity \( c_p = \sqrt{K/\rho} < c_1 \) (moment 1). The trailing edge is an elastic wave with the profile determined by the boundary pressure \( p_R(t) \). Over time, the elastic precursor moves away from the plastic wave, whereas the rear elastic wave catches up with the plastic one and interacts with it. As a result, the amplitude of the plastic wave decreases and becomes comparable with that of the elastic wave (moment 2).

![Figure 2](image)

**Figure 2.** Propagation of a one-dimensional pulse in an elastic-plastic body (a, dots correspond to the reference solution obtained on a very fine uniform grid with a step \( h = 0.0025L \) and propagation of a similar pulse without allowing for the plasticity effect (b, dots correspond to the analytical solution). The dashed and solid curves are respectively computed by the Godunov method and its UNO-modification on a uniform grid with a step \( h = 0.01L \). Time-moments 1, 2 are shown in figure 1.

It can be seen in figure 2a that the UNO-scheme gives much more accurate results than the Godunov scheme. Its solution visually coincides with the reference one at both time moments. At moment 1, the UNO-scheme error in resolving the maximum value is 0.8%, whereas the Godunov scheme error is 6.4% (8 times more). In addition, the Godunov scheme somewhat smears the fronts of the elastic precursor and the plastic wave. At moment 2 there is no pronounced "peak" in the solution, therefore the maximum value is well resolved by the both schemes. At the same time, the Godunov scheme noticeably smooths the wave fronts.

Figure 2b shows the analytical and numerical solutions of this problem in its elastic approximation (i.e., without allowance for plasticity) to estimate the plasticity effect. It can be seen that this approximation is characteristic of a pronounced "peak". The UNO-scheme solution here visually coincides with the exact one, while the Godunov method results noticeably deviate from it. In particular, the UNO-scheme solution errors in resolving the maximum value are 2% at moment 1 and 4% at moment 2, whereas the corresponding Godunov method errors are 10% and 24%.

4.2. The propagation of two-dimensional waves

The considered two-dimensional waves arise in the body under the action of the load presented in figure 1 in the limited area \(-R \leq x \leq R\) of the free surface \( y = 0 \). The computational domain is a square \([0, d] \times [0, -d]\), \( d = 4R \). On the upper side of this square, the pressure is equal to \( p_R(t) \) on the segment \( 0 \leq x \leq R \) and is zero outside it. The left side of this square is a motionless rigid wall (its normal velocity is zero), the right and the bottom sides are artificial boundaries.

Figure 3 presents the stress intensity fields at two moments shown in figure 1, computed by the Godunov method and its UNO-modification on grids resulting in close numerical solutions. Over time, the perturbed area in the body increases along both axes. A plasticity zone (where \( \sigma_1 = Y_0 \)) is formed in the vicinity of the loaded part of the body surface, initially as a thin layer adjacent to the surface. Then it becomes thicker, moves away from the surface, deforms (moment 1), and after that decreases in size and disappears (by moment 2).
Figure 3. Stress intensity contours in an elastic-plastic body at moments 1 (a) and 2 (b) indicated in figure 1. The dashed lines are the Godunov method solution on a uniform grid with a step \( h = R/512 \), the solid lines are the UNO-modification solution on a uniform grid with a step \( h = R/128 \). The shaded area in figure (a) is the plasticity zone.

It can be seen in figures 3 and 4 that at moment 1 the results of the Godunov method and its UNO-modification are very close to one another, whereas at moment 2 the UNO scheme somewhat more accurately resolves the solution extrema and less smears the wave fronts.

Figure 4. Stress intensity distributions along the y-axis. The solid curves correspond to the UNO-modification \((h = R/128)\), the dashed curves to the Godunov method \((h = R/512)\). The dotted curves represent the corresponding one-dimensional problem solution obtained by the UNO-modification \((h = R/128)\).

Figure 4 makes it possible to estimate the effect of the two-dimensionality of the problem. It can be seen that at moment 1, when the influence of the unloaded part of the surface is still weak in the vicinity of the y axis, the solutions of the one- and two-dimensional problems differ insignificantly. At moment 2, the pulses of the one- and two-dimensional problems in the region \( y < -R \) are in fact coincident, whereas behind their trailing fronts the wave patterns differ significantly.

5. Conclusion
The efficiency of wave computation in an elastic-plastic body by a second-order accurate UNO-modification of the Godunov method is numerically estimated using the efficiency of the classical first-order accurate version of the Godunov method for comparison. One- and two-dimensional problems are considered, in which the pressure on the free surface of a large body first linearly increases for a certain time, then also linearly decreases, and after that remains constant at the initial zero level. It is shown that the UNO-modification is much more efficient than the Godunov method.
particular, in the two-dimensional problem considered its computer time costs are less than those of the Godunov method by more than seventy times (its operating memory needs are 3.5 times less).

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