We addressed the problem of generalizing the extensive postulates of the standard thermodynamics in order to extend it to the study of nonextensive systems. We did it in analogy with the traditional analysis, starting from the microcanonical ensemble, but this time, considering its equivalence with some generalized canonical ensemble in the thermodynamic limit by means of the scaling properties of the fundamental physical observables.

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I. INTRODUCTION

Traditionally, the Statistical Mechanics and the standard Thermodynamics have been formulated to be applied to the study of extensive systems, in which the consideration of the additivity and homogeneity conditions, and the realization of the thermodynamic limit are indispensable ingredients for their good performance. In spite of its great success, this theory seems to be inadequate to deal with many physical contexts. Nowadays it is known that the above conditions can be considered as reasonable approximations for those systems containing a huge number of particles interacting by means of short range forces [1–3] when these systems are not suffering a phase transition [4,5].

In the last years considerable efforts have been devoted to the study of nonextensive systems. The available and increasingly experimental evidences on anomalies presented in the dynamical and macroscopic behavior in plasma and turbulent fluids [6,7], astrophysical systems [8,9], nuclear and atomic clusters [10,11], granular matter [12], glasses [13,14] and complex systems [15] constitute a real motivation for the generalization of the Thermodynamics.

One of the most important tendencies in the resolution of this problem is the generalization of Boltzmann-Gibbs' formalism through the consideration of a more general entropy form than the Shannon-Boltzmann-Gibbs' :

\[ S_{SBG} = - \sum_k p_k \ln p_k. \] (1)

This is the case of the Renyi [22] and Sharma-Mittal [23] entropies and the so popular Tsallis non-extensive entropy [24], to mention some of them. It has been said that in the second half of last century close to 30 different entropic forms have been proposed! This number is increasing continuously through the years.

The main advantage of all these Probabilistic Thermodynamic Formalisms (PThF), those based on the consideration of the concept of information entropy, is the versatility of applications, be already well inside the physical sceneries, in the study of the behavior of systems during the equilibrium or dynamical processes, as well as other interdisciplinary branches of knowledge. However, in our opinion, this kind of theory undergoes two remarkable defects. The first, the non-uniqueness of information entropy concept, which is evident. The second, it is not possible to determine the application domain a priori for each specific entropic form. For example, in the case of the Tsallis’ nonextensive statistics we can refer the determination of the entropic index \( q \). That is why we consider that the statistical description of nonextensive systems must start from their microscopic characteristics.

Ordinarily, the macroscopic description of systems in thermodynamic equilibrium through its microscopic properties has the starting point in considering the microcanonical ensemble. Although there is no way to justify this selection, it is generally accepted its great significance since from this ensemble the Thermostatistics could be deduced without invoking anything outside the mechanics.

Boltzmann [25], Gibbs [26], Einstein [27,28] and Ehrenfests [29] recognized the hierarchical primacy of the microcanonical ensemble with respect to the canonical and Gran-canonical ensembles. The last ones can be derived from the first by considering certain particular conditions: the extensivity postulates. These are the essential ideas to generalize the traditional results for the justification of any generalized Boltzmann-Gibbs’ formalism,
to put these on the level of the microscopic description, in the ground of Mechanics.

The present paper will be devoted to the analysis of the necessary conditions to be taken into account in order to perform this derivation. The same will be done in analogy with the traditional case, starting from the microcanonical ensemble, but this time generalizing the traditional postulates to cover systems with long-range interactions.

II. MICROcanonical Ensemble: Its Geometric Aspects and Scaling Properties.

Let us show now that a reasonable way to generalize the traditional Thermodynamic is assuming the scaling postulates: The equivalence of the microcanonical description with a generalized canonical one during the realization of the Thermodynamic Limit by means of the self-similarity scaling properties of the fundamental physical observables.

A. The thermodynamic limit.

Analyzing the three postulates that sustain the validity of the standard thermodynamic it is easy to see that the realization of the thermodynamic limit possesses a most universal character. In fact, the additivity and homogeneity conditions are associated more indissolubly to the short-range nature of the interactions in comparison with the characteristic linear dimension of a sufficient large system.

However, in the real world we can find examples of systems where the thermodynamic limit does not take place, since they are not composed by a huge number of particles. As examples of these, it can be mentioned the molecular and atomic clusters. A very interesting problem is the extension of the macroscopic description for this kind of systems as well as the question about when it could be considered that these systems have reached the thermodynamic limit.

An important step toward the resolution of these problems was accomplished by D.H.E. Gross with the development of the Microcanonical Thermostatistics [15,29,30], a formalism based on the consideration of the microcanonical ensemble. In this approach is possible to accomplish the description of some finite systems, extending in this way the study of phase transitions to them. On the other hand, its possibilities are very attractive when exploring the behavior of the macroscopic observables with the increase of the number of particles and analyzing the convergence in some cases in an ordinary extensive system, allowing us in this way to give a valuation about when the thermodynamic limit is established in the system.

A remarkable aspect of this approach is that the justification of the canonical ensemble is not supported by the consideration of the Gibbs’ argument: a closed system composed by a subsystem with a weak interaction with a thermal bath, but by means of equivalence between the microcanonical and canonical ensemble in the thermodynamic limit for this kind of system. The Gibbs’ argument is sustained by the possibility of the separation of this subsystem from the whole system, that is, in the independence or weak correlation of this subsystem with the remaining part. It is easy to understand that this argument can not be applied for many nonextensive systems, since in them there are long-range correlations due to the presence of long-range interactions.

Thus, the equivalence of the microcanonical ensemble with the canonical ensemble during the thermodynamic limit could be easily extended for the nonextensive systems since this argument has an universal character stronger than the Gibbs’. We have to point out that this equivalence also supports the thermodynamic formalism based on the Legendre Transformation between the thermodynamic potentials, which is usually valid without restriction for the probabilistic interpretation of the thermodynamics [29].

The thermodynamic formalism of the Gross’ theory have been defined in a way that be equivalent in the thermodynamic limit with the ordinary form of the Thermodynamic for the case of the extensive systems. That is why this theory is appropriate to study systems that in the thermodynamic limit become in ordinary extensive system (see for example in ref. [31,32]). However it is not trivial its application to systems with long-range interactions because we do not know their asymptotical properties as well as the corresponding generalized canonical ensemble.

B. Geometrical aspects of the PDF.

A feature of the PThF is that the description of the macroscopic state of the systems is performed by means of the intensive parameters of the generalized canonical distributions, β, its means, they are supported by the validity of some generalized zero principle of the Thermodynamics. This section will be devoted to the analysis of the prescriptions for this kind of description.

Let us pay special attention to the geometrical aspects that posses the Probabilistic Distribution Functions (PDF) of the ensembles. For the case of the microcanonical ensemble its PDF is given by:

$$\omega_M (X; I, N, a) = \frac{1}{\Omega (I, N, a)} \delta [I - I_N (X; a)],$$  \(2)$$

where \(\mathcal{R}_I = \{I_N (X; a)\}\) is the set of movement integrals of the distribution, and \(\Omega (I, N, a)\) is the accessible states density of the microscopic state. Any function of
the above movement integrals is itself a movement integral. If we have a complete independent set of such a functions this set of functions will be equivalent to the set $\mathcal{R}_I$, that is, both of them represent the same macroscopic state. That is why it is more exactly to speak about the movement integrals abstract space of the microcanonical distribution, $\mathfrak{S}_N$. Therefore, every physical quantity or behavior has to be equally reproduced by any representation of the space of macroscopic state, $(\mathfrak{S}_N; a)$.

It is easy to show that the microcanonical distribution is invariant under the transformation group of local reparametrizations or Diffeomorfism of the movement integrals space, $\text{Diff} (\mathfrak{S}_N)$. Let $\mathcal{R}_I$ and $\mathcal{R}_\phi$ be two representations of $\mathfrak{S}_N$, where:

$$I \equiv \{I^1, I^2 \ldots I^n\} \text{ and } \phi \equiv \{\phi^1 (I), \phi^2 (I) \ldots \phi^n (I)\} .$$

By the property of the $\delta$-function we have:

$$\delta [\phi (I) - \phi (I_N (X; a))] = \left| \frac{\partial \phi}{\partial I} \right|^{-1} \delta [I - I_N (X; a)] ,$$

hence:

$$\tilde{\Omega} (\phi, N, a) = \left| \frac{\partial \phi}{\partial I} \right|^{-1} \Omega (I, N, a) ,$$

and therefore:

$$\omega_M (X; \phi, N, a) = \omega_M (X; I, N, a) \equiv \omega_M (X; \mathfrak{S}_N, a) .$$

From here, two interesting remarks are derived. The first: This local reparametrization invariance is the maximal symmetry that a geometrical theory could possess, and it is usually associated to the local properties of a curved general space. This property is in complete contradiction with the ergodic chauvinism of the traditional distributions. This is translated as the preference of the energy over all the integrals of movement corresponding to the system. In the microcanonical ensemble this preference is not justified since it does not depend on the representation of the movement integrals space. The second: The movement integrals are defined by the commutativity relation with the Hamiltonian of the system. In the case of closed systems, the hamiltonian is the total energy, a conserved quantity. In the microcanonical distribution it represents one of the coordinates of the point belonging to $\mathfrak{S}_N$, in an specific representation. When we change the representation, the energy loses its identity. Since we can not fix the commutativity of the energy with the other integrals, we must impose that all movement integrals commute between them in order to preserve these conditions. As we see, even from the classical point of view, the reparametrization invariance suggests that the set of movement integrals have to be simultaneously measurable. In the quantum case, this is an indispensable request for the correct definition of the statistical distribution. This property is landed to the classical distribution by the correspondence between the Quantum and the Classic Physics.

On the other hand, all the PDF derived from the PThF must depend on the movement integrals in a lineal combination with the canonical intensive parameters:

$$p (X; \beta, N) = F (\beta, N; \beta \cdot I_N (X)) ,$$

therefore, the most general symmetry group that preserves this functional form is the local unitary lineal group, $\text{SL}_c (\mathbb{R}^n)$ ($n$ is the dimension of $\mathfrak{S}_N$), which is related with the euclidean vectorial spaces. It is supposed that an spontaneous symmetry breaking takes place during the thermodynamic limit, from $\text{Diff} (\mathfrak{S}_N)$ to $\text{SL}_c (\mathbb{B}^n)$, where $\mathbb{B}^n$ is the space of the generalized canonical parameters. The way in which this hypothetical symmetry breaking happens will determine the specific form of the PDF in the thermodynamic limit.

What do we understand as an spontaneous symmetry breaking? Ordinarily, the microcanonical ensemble could be described equivalently by any representation of the movement integrals space. However, with the increasing of the system degrees of freedom, some specific representations will be more adequate to describe the macroscopic state because it shows in a better way the general properties of the system (i.e., in the case of the traditional systems, the extensivity).

C. The scaling symmetry of the fundamental physical observables.

In complete analogy with the traditional analysis, we identify the cause of this spontaneous symmetry breaking with the scaling properties of the fundamental macroscopic observables during the thermodynamic limit: the behavior of the movement integrals, the external parameters and the accessible states density with the increasing of the system degrees of freedom.

As it was pointed out previously, these symmetry properties will be just expressed for certain set of representations, $\mathcal{M} = \{\mathcal{R}_I\}$. Only in the frame of this set of admissible representations will be valid the equivalence between the microcanonical description and the generalized canonical one. The remaining symmetry among representations belonging to $\mathcal{M}$ constitutes the local group $\text{SL}_c (\mathbb{R}^n)$ of the canonical description.

In order to access to the scaling behavior of the movement integrals, the asymptotical dependence of the states density for large values of the number of particles, $W_{\text{asym}} (I^*, N, a)$, must be obtained:

$$W (I^*, N, a) \Rightarrow W_{\text{asym}} (I^*, N, a) ,$$

where $\mathcal{R}_{I^*} \in \mathcal{M}$. Let $W_o$ be defined as:

$$W_o = W_{\text{asym}} (I_o, N_o, a_o) ,$$

where $I_o$ is a set of independent movement integrals.
where $N_o$ could be considered to be in the asymptotical region $N \gg 1$. The scaling behavior of the macroscopic observables is dictated by the symmetry of the asymptotical states density under the self-similar scaling transformations:

$$
\begin{align*}
N_o \to N &= \alpha N_o \\
I_o^* \to I^* &= \alpha^{\sigma} I_o^* \\
a_o \to a &= \alpha^{\gamma} a_o
\end{align*}
\Rightarrow W_{asym} (I^*, N, \alpha) = \mathcal{F} [W_o, \alpha].
$$

(10)

In the above definition, $\alpha$ is the scaling parameter, $\sigma$ and $\pi_a$ are real constants characterizing the scaling transformations. As we see, all the movement integrals are transformed by the same scaling law in order to satisfy the $SL(\mathbb{R}^n)$ invariance of the PDF in the thermodynamic limit. Finally, $\mathcal{F}$ is a functional dependence of $W_o$ and $\alpha$ satisfying the self-similarity condition:

$$
2.- \quad \mathcal{F} [\mathcal{F} [x, \alpha_1], \alpha_2] = \mathcal{F} [x, \alpha_1 \alpha_2].
$$

(11)

The above request could be satisfied by the following generic form:

$$
\mathcal{F} [x, \alpha] = f_{\infty} [\alpha^\sigma f_{\infty}^{-1} (x)],
$$

(12)

where $f_{\infty} (x)$ is an increasing function of $x$ with a monotonic first derivative in the asymptotical region $x \gg 1$, and $\sigma$, a real constant. The function $f_{\infty} (x)$ defines the specific scaling law of the system. Let us introduce the function $f (x)$, which is an increasing function of $x$ with a monotonic first derivative for all the positive real values of $x$ satisfying the following conditions:

$$
f (0) = 1 \quad \text{(Normalization)},
$$

(13)

$$
f (x) \xrightarrow[x \to 0]{} f_{\infty} (x) \quad \text{(Convergency)}.
$$

(14)

This function allows us to define the function $\gamma (I, N; a)$:

$$
\gamma (I^*, N; a) = \frac{1}{N^\sigma} f_{\infty}^{-1} [W (I^*, N; a)],
$$

(15)

which is scaling invariant during the realization of the thermodynamic limit. It is not difficult to understand that the objective of the canonical description is the knowledge of the set of admissible representations, $\mathcal{M}$, the scaling laws of the fundamental macroscopic observables (integrals of movement and external parameters) as well as the determination of the scaling invariant function $\gamma_{\infty} (I^*, N; a)$:

$$
\gamma_{\infty} (I^*, N; a) = \lim_{\alpha \to \infty} \gamma (I^*, N; a) = \frac{1}{N^\sigma} f_{\infty}^{-1} [W (I_o^*, N_o; a_o)].
$$

(16)

Although $\gamma_{\infty} (I^*, N; a)$ is not a physical observable, from this function could be derived other quantities which are experimentally measurable, allowing in this way to obtain information about the ordering of the system. This fact aims at the following conclusion: In the generalized canonical description, the function:

$$
S_f = f^{-1} [W (I^*, N; a)],
$$

(17)

plays the role of the generalized entropy. From now on we will refer the definition of the Eq.(17) as the generalized Boltzmann´s Principle. The function $f^{-1} (x)$ defines the so called counting rule for the entropy. Our proposition suggests the relative significance of the entropy concept. In the thermodynamic limit the generalized Boltzmann´s entropy may possess a probabilistic interpretation with the same style of the Shannon-Boltzmann-Gibbs or the Tsallis nonextensive entropy. The specific form of the thermodynamic formalism could be determined by means of the conditions for the equivalence of the ensembles, maybe, through some generalization of the ordinary Laplace’s Transformation [2].

A final remark: Ordinarily, the validity of the equivalence between the microcanonical and the canonical ensemble must satisfy a stability condition, which is ordinarily improved by the exigency of the concavity of the entropy function. This mathematical demand is intimately bounded to the homogeneity condition, the non occurrence of phase transitions in the system. In the case of the finite system this exigency leads to the generalization of the microcanonical thermostatistic of D.H.E. Gross.

### III. Conclusions

We have analyzed a possible way to generalize the extensive postulates in order to extend the Thermodynamics to the study of some nonextensive systems. We showed the superiority of the argument of the equivalence of the microcanonical ensemble with some generalized canonical ensemble with regard to the Gibbs’ one. The ergodic chauvinism of the classical thermodynamics is not justified in the microcanonical ensemble since this description is reparametrization invariant. This property has a relevant role in the passage from one description to the other, with the occurrence of a hypothetical spontaneous breaking of this symmetry in the thermodynamic limit by means of the scaling properties of the fundamental physical observables. The generalized Boltzmann´s Principle, the definition of the generalized Boltzmann´s entropy [the Eq.(17)], have been defined in order to deal the study of systems with an arbitrary scaling laws [ i.e., exponentials, potentials (fractals), etc.].

The previous arguments can be applied to the study of some nonextensive systems, allowing in this way to justify a thermodynamic probabilistic formalism analogue to the traditional one without the necessity of appealing to additional postulates.
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