Circular coloring and fractional coloring in planar graphs

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Abstract
We study the following Steinberg-type problem on circular coloring: for an odd integer $k \geq 3$, what is the smallest number $f(k)$ such that every planar graph of girth $k$ without cycles of length from $k + 1$ to $f(k)$ admits a homomorphism to the odd cycle $C_k$ (or equivalently, is circular $(k, \frac{k-1}{2})$-colorable). Known results and counterexamples on Steinberg’s Conjecture indicate that $f(3) \in \{6, 7\}$. In this paper, we show that $f(k)$ exists if and only if $k$ is an odd prime. Moreover, we prove that for any prime $p \geq 5$,

$$p^2 - \frac{5}{2}p + \frac{3}{2} \leq f(p) \leq 2p^2 + 2p - 5.$$  

We conjecture that $f(p) \leq p^2 - 2p$, and observe that the truth of this conjecture implies Jaeger’s conjecture that every planar graph of girth $2p - 2$ has a homomorphism to $C_p$ for any prime $p \geq 5$. Supporting this conjecture, we prove a related fractional coloring result that every planar graph of girth $k$ without cycles of length from $k + 1$ to $\left\lceil \frac{22k}{3} \right\rceil$ is fractional $(k : \frac{k-1}{2})$-colorable for any odd integer $k \geq 5$.

Keywords
planar graphs, circular coloring, cycle length, fractional coloring, girth
INTRODUCTION

The circular chromatic number of a graph is a natural generalization of the chromatic number of a graph, introduced by Vince [28]. For two positive integers $k$ and $d$ with $k \geq 2d$, a circular $(k, d)$-coloring of a graph $G$ is a mapping $\varphi : V(G) \rightarrow \{0, 1, ..., k - 1\}$ such that $d \leq |\varphi(u) - \varphi(v)| \leq k - d$ whenever $uv \in E(G)$. The circular chromatic number $\chi_c(G)$ of $G$ is defined as the infimum of rational numbers $\frac{k}{d}$ for which $G$ has a circular $(k, d)$-coloring. Notice that a circular $(k, 1)$-coloring of a graph $G$ is just an ordinary proper $k$-coloring of $G$. We call $\chi_c(G)$ a refined measure of coloring because $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ for every graph $G$, as proved in [4, 28], where $\chi(G)$ is the chromatic number of $G$. Perhaps one of the most intriguing problems concerning circular coloring of planar graphs is the following conjecture, motivated from the dual of Jaeger’s circular flow conjecture [19].

**Conjecture 1.1.** For any planar graph $G$ of girth at least $2t$, $\chi_c(G) \leq 2 + \frac{2}{t}$.

The $t = 1$ case of this conjecture is the celebrated Four Color Theorem proved by Appel and Haken [2] in 1976; the $t = 2$ case is the classical Grötzsch’s theorem [15] from 1959 that every triangle-free planar graph is 3-colorable. Conjecture 1.1 remains open for each $t \geq 3$. A result of Hell and Zhu [16] shows that Conjecture 1.1 is true for $K_4$-minor-free graphs, a subclass of planar graphs.

When $t = 2s$ is even, it is not hard to observe that a graph $G$ is circular $(2t + 2, t)$-colorable if and only if $G$ admits a homomorphism to the odd cycle $C_{2s+1}$. Indeed, $\chi_c(C_{2s+1}) = \frac{2s+1}{s}$ and each color class contains exactly one vertex of $C_{2s+1}$ under a circular $(2s + 1, s)$-coloring; thus a circular $(2s + 1, s)$-coloring is also called a $C_{2s+1}$-coloring for convenience. For partial results of Conjecture 1.1, Dvořák and Postle [13] showed that every planar graph of girth at least 10 is $C_5$-colorable. In [11], by duality from flow results, a simpler proof of Dvořák and Postle’s result was obtained, and it was extended to the next case that every planar graph of girth at least 16 is $C_7$-colorable. Independently, Postle and Smith-Roberge [23] also proved that every planar graph of girth at least 16 is $C_7$-colorable through the density of $C_7$-critical graphs. The current best general result was due to Lovász, Thomassen, Wu, and Zhang [21], from the dual of their more general flow results, that for each even $t$, $\chi_c(G) \leq \frac{2t+2}{t}$ for every planar graph $G$ of girth at least $3t$. For odd $t$, a recent flow results in [20] also showed that $\chi_c(G) \leq \frac{2t+2}{t}$ for every planar graph $G$ of girth at least $3t + 1$.

Another influential coloring problem on planar graphs is Steinberg’s Conjecture (see [26]) from 1976, which asserts that every planar graph without cycles of length 4 or 5 is $C_5$-colorable. We ask the following generalization on $C_k$-coloring.

**Question 1.2.** For any integer $k \geq 3$, what is the smallest number $f(k)$ such that every planar graph of girth $k$ without cycles of length from $k + 1$ to $f(k)$ is $C_k$-colorable?

As an approach to Steinberg’s Conjecture, Erdős (see [26]) asked to bound and determine $f(3)$. Abbott and Zhou [1] first established that $f(3) \leq 11$. The bounds are progressively improved to $f(3) \leq 9$ by Borodin [5] and by Sanders and Zhao [24] independently, and to $f(3) \leq 7$ by Borodin, Glebov, Raspau, and Salavatipour [6], that is, every planar graph without cycles of length from 4 to 7 is 3-colorable. However, Steinberg’s Conjecture has been disproved by Cohen-Addad, Hebdige, Kráľ’, Li, and Salgado [10], that is, there exists a planar graph without cycles of length 4 or 5 that is not 3-colorable. Those results imply that $f(3) \in \{6, 7\}$.

Our first main result of this paper describes the existence of $f(k)$ for all $k$. 
Theorem 1.3. The value $f(k)$ exists as a finite number if and only if $k$ is an odd prime. Moreover, for any prime $p \geq 5$,

$$p^2 - \frac{5p - 3}{2} \leq f(p) \leq 2p^2 + 2p - 5.$$ 

We suspect that the lower bound in Theorem 1.3 is close to the exact value of $f(p)$, and propose the following conjecture for upper bound.

Conjecture 1.4. For any prime $p \geq 5$, $f(p) \leq p(p - 2)$. That is, every planar graph of girth $p$ without cycles of length from $p + 1$ to $p(p - 2)$ is $C_p$-colorable.

The following connection between Conjectures 1.1 and 1.4 is observed.

Proposition 1.5. Let $p \geq 5$ be a prime. The truth of Conjecture 1.4 implies the validity of Conjecture 1.1 for $t = p - 1$. That is, Conjecture 1.4 implies that every planar graph of girth at least $2p - 2$ is $C_p$-colorable.

Proposition 1.5 indicates that proving Conjecture 1.4 may be difficult. But on the other hand, it also suggests that Conjecture 1.4 may provide a possible new approach to solve Conjecture 1.1 for $t = p - 1$ with odd prime $p$. Particularly, the $p = 5$ case of Conjecture 1.4 not only implies that every planar graph of girth 8 is $C_5$-colorable, but also implies the Five Coloring Theorem as shown in Observation 2.6.

The fractional chromatic number of a graph is another well-known variation of the chromatic number. For positive integers $a$ and $b$ with $a \geq b$, a fractional $(a : b)$-coloring $\varphi$ of a graph $G$ is a set coloring such that each vertex assigns a $b$-element subset of $\{1, \ldots, a\}$ satisfying $\varphi(u) \cap \varphi(v) = \emptyset$ whenever $uv \in E(G)$. The fractional chromatic number of $G$, denoted by $\chi_f(G)$, is the infimum of the fractions $\frac{a}{b}$ such that $G$ admits a fractional $(a : b)$-coloring. Notice that a fractional $(a : 1)$-coloring of a graph $G$ coincides with an ordinary proper $a$-coloring of $G$. The fractional coloring was first introduced by Hilton, Rado, and Scott [17] in 1973 to seek for a proof of the Four Color Problem. Since then, it has been the focus of many intensive research efforts, see [25]. For a graph $G$, let $\omega(G)$ and $\alpha(G)$ denote the clique number and the independence number of $G$, respectively. It is well known (cf. [29,30]) that

$$\max \left\{ \omega(G), \frac{|V(G)|}{\alpha(G)} \right\} \leq \chi_f(G) \leq \chi_c(G) \leq \lceil \chi_c(G) \rceil = \chi(G).$$

One may also consider the fractional coloring variations of Conjecture 1.1 and Question 1.2. Analogous to Conjecture 1.1, Naserasr [22] conjectured that every planar graph of girth at least $2s + 2$ is fractional $(2s + 1 : s)$-colorable. It is proved for $K_4$-minor-free graphs in [3,14] that every $K_4$-minor-free graph of girth at least $2s$ is fractional $(2s + 1 : s)$-colorable. Our second main result provides a fractional coloring result of Question 1.2, which particularly confirms the fractional coloring version of Conjecture 1.4 for prime $p \geq 11$ in a strong sense.

Theorem 1.6. For any odd integer $k \geq 5$, every planar graph of girth $k$ without cycles of length from $k + 1$ to $\left\lceil \frac{2k}{3} \right\rceil$ is fractional $(k : \frac{k-1}{2})$-colorable.
In a follow-up work [18], we also prove the remaining cases \((p = 5, 7)\) of the fractional coloring version of Conjecture 1.4 with some refined arguments and additional efforts.

The rest of this paper is organized as follows. We introduce some preliminaries and prove Proposition 1.5 in Section 2. The proof of Theorem 1.3 is presented in Section 3 and the proof of Theorem 1.6 is completed in Section 4. We end this paper with a few remarks in Section 5.

2 | PRELIMINARIES

We start with some basic notation and terminologies. Let \(G = (V(G), E(G))\) be a simple finite graph. For a vertex \(v \in V(G)\), the neighborhood \(N_G(v)\) of a vertex \(v\) is the set of vertices adjacent to \(v\), and denotes \(d_G(v) = |N_G(v)|\). The distance between two vertices \(u\) and \(v\), denoted by \(d_G(u, v)\), is the length of a shortest path from \(u\) to \(v\) in \(G\). The subscript \(G\) is often omitted if the graph \(G\) is clear from the context. For \(S \subseteq V(G)\), \(G - S\) denotes the graph obtained from \(G\) by deleting all the vertices of \(S\) together with all the edges incident to at least one vertex in \(S\). For a positive integer \(i\), let \([i] = \{1, 2, ..., i\}\). We use \(i^+\) to denote a number equal or greater than \(i\). An \(i\)-vertex \((i^-\text{-vertex, resp.)} is a vertex of degree \(i\) (at least \(i\), resp.). Similarly, in a plane graph, an \(i\)-face \((i^-\text{-face, resp.)} is a face of degree \(i\) (or at least \(i\), resp.). In the rest of this paper, we usually assume \(k \geq 5\) is an odd integer and \(p \geq 5\) is a prime implicitly.

A common method in graph coloring is to study certain coloring properties of typical graphs under given precoloring. This usually provides some reducible subgraphs and facilitates a discharging proof. We shall define precoloring properties for circular coloring and fractional coloring, respectively. Let \(H\) be a graph with a vertex subset \(S \subseteq V(H)\). A precoloring \(\omega\) assigns colors in \([k]\) to vertices in \(S\) such that \(H[S]\) is properly \(C_k\)-colored. The graph \(H\) is called \(\omega\)-colorable if the precoloring \(\omega\) of \(S\) can be extended to \(V(H)\) to obtain a \(C_k\)-coloring of \(H\).

Similarly, a precoloring \(\varphi\) of \(S\) assigns colors in \(\varphi\) to vertices in \(S\) such that \(H[S]\) is properly fractional \((k : \frac{k-1}{2})\)-colored. We say that \(H\) is \(\varphi\)-colorable if the precoloring \(\varphi\) of \(S\) can be extended to all vertices of \(H\) to obtain a fractional \((k : \frac{k-1}{2})\)-coloring.

We first observe the following fact on precoloring of \(k\)-cycle for \(C_k\)-coloring, which will be useful.

Lemma 2.1. Let \(G = v_0v_1 ... v_{k-1}v_0\) be an odd cycle of length \(k\). Let \(\omega\) be a precoloring of \(\{v_i, v_j\} \subseteq V(G)\). Then \(G\) is \((\omega, \{v_i, v_j\})\)-colorable if and only if

\[
\omega(v_i) - \omega(v_j) \equiv \frac{k-1}{2} \cdot (i - j) \mod k. \tag{1}
\]

Proof. If \(\omega\) can be extended to a \(C_k\)-coloring \(\tilde{\omega}\) of \(G\), then the \(C_k\)-coloring \(\tilde{\omega}: V(G) \mapsto \{0, 1, ..., k - 1\}\) provides a coloring of \(G\) such that

either \(\tilde{\omega}(v_i) \equiv \frac{k-1}{2} \cdot t + \tilde{\omega}(v_0) \mod k\) for each \(0 \leq t \leq k - 1\),

or \(\tilde{\omega}(v_i) \equiv \frac{k+1}{2} \cdot t + \tilde{\omega}(v_0) \mod k\) for each \(0 \leq t \leq k - 1\).
Hence, for \( v_i, v_j \in V(G) \) we have Equation (1).

Conversely, if Equation (1) holds, then we can properly define a \( C_k \)-coloring of \( G \) as above. This proves the lemma. \( \square \)

In a graph \( G \), a \( d-C_k \)-replacement operation on a given edge \( e = xy \in E(G) \) is to replace the edge \( e \) with a \( k \)-cycle \( C_k = v_0v_1 ... v_{k-1}v_0 \) by identifying \( x \) with \( v_0 \) and identifying \( y \) with \( v_d \).

When \( d \) is not explicitly stated, we just call it a \( C_k \)-replacement operation on the edge \( e \in E(G) \).

Lemma 2.1 implies the following relation between \( C_k \)-coloring and \( d-C_k \)-replacement operation.

**Proposition 2.2.** Let \( G \) be a graph, and let \( G(d, k) \) be a graph obtained from \( G \) by applying \( d-C_k \)-replacement operation on each edge of \( G \). Assume that \( d \) and \( k \) are coprime, that is, \( \gcd(d, k) = 1 \). Then \( G \) is \( C_k \)-colorable if and only if \( G(d, k) \) is \( C_k \)-colorable.

**Proof.** Let \( \varphi \) be a \( C_k \)-coloring of \( G \). Define a precoloring \( \omega \) of \( G(d, k) \) by coloring each vertex \( u \in V(G) \subset V(G(d, k)) \) with \( \omega(u) \equiv d\varphi(u) \pmod{k} \). Since \( \varphi(u) - \varphi(v) \in \left\{ \frac{k-1}{2}, \frac{k+1}{2} \right\} \)

for each edge \( uv \in E(G) \), we have, in the graph \( G(d, k) \),

\[
\omega(u) - \omega(v) \equiv d\varphi(u) - d\varphi(v) \equiv \frac{k-1}{2} \cdot d \quad \text{or} \quad \frac{k+1}{2} \cdot d \pmod{k}.
\]

It follows from Lemma 2.1 that \( \omega \) can be extended to a \( C_k \)-coloring of \( G(d, k) \) by coloring each \( k \)-cycle of \( G(d, k) \) properly.

Conversely, assume that \( G(d, k) \) admits a \( C_k \)-coloring \( \omega \). Then for each edge \( uv \in E(G) \), we have \( \omega(u) - \omega(v) \in \left\{ \frac{k-1}{2} \cdot d, \frac{k+1}{2} \cdot d \right\} \pmod{k} \) by Lemma 2.1.

Define \( \varphi = d^{-1}\omega \pmod{k} \). (Note that \( d^{-1} \) exists in \( \mathbb{Z}_k \) since \( \gcd(d, k) = 1 \).) Then \( \varphi(u) - \varphi(v) \in \left\{ \frac{k-1}{2}, \frac{k+1}{2} \right\} \) for each edge \( uv \in E(G) \). That is, \( \varphi \) restricted to \( V(G) \) provides a proper \( C_k \)-coloring of \( G \). \( \square \)

Applying Lemma 2.1, we also show that \( f(k) \) does not exist for nonprime \( k \) by construction using \( d-C_k \)-replacement operations.

**Proposition 2.3.** Let \( k > 0 \) be an odd nonprime integer. Then \( f(k) \) does not exist. That is, for any integer \( m > k \) there exist planar graphs of girth \( k \) without cycles of length from \( k + 1 \) to \( m \) admitting no \( C_k \)-coloring.

**Proof.** Denote \( k = st \), where \( s, t \) are positive integers with \( t \geq s > 1 \). Take an \( (m + 1) \)-cycle \( z_0z_1z_2 ... z_mz_0 \). For each \( 0 \leq i \leq m - 1 \), apply \( s-C_k \)-replacement operation on the edge \( z_iz_{i+1} \). Let \( G \) be the resulting graph. Then \( G \) is a planar graph of girth \( k \) without cycles of length from \( k + 1 \) to \( ms \). See Figure 1A for the construction of \( G \) when \( k = 9 \) and \( m = 13 \).

It is routine to check that \( G \) is not \( C_k \)-colorable. To see this, suppose for a contradiction that \( \omega : V(G) \mapsto \{0, 1, ..., k - 1\} \) is a \( C_k \)-coloring of \( G \). By Lemma 2.1, for each \( 0 \leq i \leq m - 1 \), we have
Thus \( \omega(z_{i+1}) - \omega(z_i) \) is a multiple of \( s \) since \( k = st \). This implies that
\[
\omega(z_m) - \omega(z_0) = \sum_{i=0}^{m-1} (\omega(z_{i+1}) - \omega(z_i)) \quad \text{is a multiple of } s.
\]

On the other hand, as \( z_mz_0 \) is an edge in \( E(G) \), we must have
\[
|\omega(z_m) - \omega(z_0)| \in \left\{ \frac{k-1}{2}, \frac{k+1}{2} \right\}.
\]
But as \( k = st \), neither \( \frac{k-1}{2} \) nor \( \frac{k+1}{2} \) is a multiple of \( s \), a contradiction. This completes the proof.

In contrast, we will show below in Theorem 3.4 that \( f(p) \) exists as a quadratic function of \( p \) for odd prime \( p \). Now we give a low bound of \( f(p) \) with similar arguments as Proposition 2.3.

**Proposition 2.4.** For any prime \( p \geq 5 \), there exist planar graphs of girth \( p \) without cycles of length from \( p + 1 \) to \( p^2 - \frac{5p-1}{2} \) admitting no \( C_p \)-coloring. That is, \( f(p) \geq p^2 - \frac{5}{2}p + \frac{3}{2} \).

**Proof.** Construct a graph \( W_p \) from a \((2p - 3)\)-cycle \( z_0z_1 \ldots z_{2p-4}z_0 \) by adding a new center vertex \( x \) connecting each \( z_i \) with a new path of length \( p - 2 \) for \( 0 \leq i \leq 2p - 4 \). This graph \( W_p \) was constructed by DeVos (see [8]) to show the tightness of Conjecture 1.1, that is, \( W_p \) is a planar graph of girth \( 2p - 3 \) without \( C_p \)-coloring. To see that \( W_p \) is not \( C_p \)-colorable, we prove by contradiction. Suppose to the contrary that \( \omega \) is a \( C_p \)-coloring of \( W_p \). If \( \omega(x) = \omega(z_i) \) for some \( i \), then after identifying \( x \) and \( z_i \) in the path of length \( p - 2 \) between \( x \) and \( z_i \), we obtain a \( C_p \)-coloring of \((p - 2)\)-cycle, a contradiction. So \( \omega(x) \neq \omega(z_i) \) for each \( 0 \leq i \leq 2p - 4 \). Hence the \((2p - 3)\)-cycle \( z_0z_1 \ldots z_{2p-4}z_0 \) admits a
Construct a graph $H_p$ from $W_p$ by applying $\left(\frac{p-1}{2}\right)$-replacement operation on each edge of $W_p$. See Figure 1B for the construction of $H_5$. Since $W_p$ is not $C_p$-colorable, we obtain that $H_p$ is not $C_p$-colorable by Proposition 2.2. As $W_p$ has girth $p^2 - 2$, $H_p$ is of girth $p$ and without cycles of length from $p + 1$ to $(2p - 3)\frac{p-1}{2} - 1$.

Next, we shall prove Proposition 1.5 using analogous approaches.

**Proposition 2.5** (Restatement of Proposition 1.5). Let $p \geq 3$ be a prime. If $f(p) \leq (p - 2)$, then every planar graph of girth at least $2p - 2$ is $C_p$-colorable.

*Proof.* Assume that $f(p) \leq p(p - 2)$. That is, every planar graph of girth $p$ without cycles of length from $p + 1$ to $p(p - 2)$ is $C_p$-colorable. Let $G$ be a planar graph of girth at least $2p - 2$. Apply the $\left(\frac{p-1}{2}\right)$-replacement operation on each edge of $G$ to obtain a graph $G\left(\frac{p-1}{2}, p\right)$. Then $G\left(\frac{p-1}{2}, p\right)$ is a planar graph of girth $p$ without cycles of length from $p + 1$ to $p(p - 2)$. Since $f(p) \leq p(p - 2)$, we know that $G\left(\frac{p-1}{2}, p\right)$ is $C_p$-colorable. Hence $G$ is $C_p$-colorable as well by Proposition 2.2.

Similar arguments also show that the $p = 5$ case of Conjecture 1.4 is stronger than the Five Color Theorem.

**Observation 2.6.** The truth of $f(5) \leq 17$ implies that every planar graph is 5-colorable.

*Proof.* Assume that $f(5) \leq 17$, that is, every planar graph of girth 5 without cycles of length from 6 to 17 is $C_5$-colorable. Let $G$ be a planar graph, and $H$ be the graph obtained from $G$ by replacing each edge with a path of length 3. Let $F$ be the graph obtained from $H$ by applying $2$-$C_5$-replacement operation on each edge of $H$. Then by construction $F$ is a planar graph of girth 5 without cycles of length from 6 to 17, and hence $F$ is $C_5$-colorable by $f(5) \leq 17$. Now, by Proposition 2.2 and Lemma 2.1, the $C_5$-coloring $\omega$ of $F$ induces a proper 5-coloring of $G$, since $\omega(u) \neq \omega(v)$ whenever $uv \in E(G)$.

At the end of this section, we define some graphs, serving for reducible configurations in later proofs.

**Definition 2.7.** Let $G$ be a graph.

(i) A *thread* in $G$ is a path whose internal vertices are 2-vertices in $G$. The end vertices of the path are called the end vertices of the thread. A thread with end vertices $x, y$ is also called an $(x, y)$-thread, denoted by $T(x, y)$. An $s$-thread is a thread with $s$ internal vertices. A $(k_1, k_2, ..., k_t)$-thread $T_x$ in $G$ is a subgraph consisting of distinct $k_1$-thread, $k_2$-thread, ..., $k_t$-thread which share a common end vertex $x$, where $t \geq 3$. 

The common end vertex $x$ is called a $(k_1, k_2, ..., k_t)$-vertex. Let $y_i$ be the other end vertex of the $k_i$-thread, and define $\{y_1, y_2, ..., y_t\}$ to be the end vertices of $T_x$. If $z$ is a 2-vertex of an $(x, y)$-thread, then we say $x$ and $z$ are weakly adjacent.

(ii) An $s$-necklace in $G$ is a subgraph obtained from an $s$-thread by applying $C_k$-replacement operations on some edges. A vertex $z$ is an end vertex of the $s$-necklace if and only if $z$ is an end vertex of the $s$-thread. A necklace with end vertices $x, y$ is also called an $(x, y)$-necklace, denoted by $N(x, y)$. A $(k_1, k_2, ..., k_t)$-necklace $N_x$ is a subgraph obtained from a $(k_1, k_2, ..., k_t)$-thread $T_x$ by applying $C_k$-replacement operations on some edges. The vertex $x$ is called the center vertex of $N_x$. A vertex $z$ is an end vertex of the $(k_1, k_2, ..., k_t)$-necklace if and only if $z$ is an end vertex of the $(k_1, k_2, ..., k_t)$-thread. A $(k_1, k_2, k_3)$-bull-necklace is a subgraph obtained from a $(k_1, k_2, k_3)$-thread by applying $C_k$-replacement operations on some edges of the $k_3$-thread. A $(k_1, k_2, ..., k_t)$-crown-necklace is obtained from a $(k_1, k_2, ..., k_t)$-necklace by replacing the center vertex with a $k$-cycle. A vertex $z$ is an end vertex of the $(k_1, k_2, ..., k_t)$-crown-necklace if and only if $z$ is an end vertex of the $(k_1, k_2, ..., k_t)$-necklace. See Figure 2 for some examples.

3 | THE $C_p$-COLORING FOR PRIME $p$

This section is aiming to show $f(p) \leq 2p^2 + 2p - 5$ in Theorem 1.3. We first present some reducible configurations under precoloring in Section 3.1, and then complete the proof in Section 3.2 by a discharging method. Unlike some standard discharging arguments, our method mainly analyzes certain modified graphs obtained from the original graph, which benefits in handling some structures involving $p$-cycles.

**Figure 2** Examples for Definition 2.7
3.1 Precoloring and reducible subgraphs for $C_p$-coloring

Let $H$ be a thread, or a necklace, or a $(k_1, k_2, ..., k_t)$-thread, or a $(k_1, k_2, ..., k_t)$-necklace, or a $(k_1, k_2, ..., k_t)$-crown-necklace with $W$ being the end vertex set of $H$. The graph $H$ is called reducible if in any graph $G$ containing $H$ as a subgraph, any $C_p$-coloring of $G - (V(H) \setminus W)$ can be extended to a $C_p$-coloring of $G$. In other words, it is equivalent to say that $H$ is $(\omega, W)$-colorable for any precoloring $\omega$ of $W$. It is known from $[8,23,29]$ that some threads and certain $(k_1, k_2, ..., k_t)$-threads are reducible configurations for $C_k$-coloring. Our main reducible configurations in this section are certain necklaces and crowns, generalizing from threads, for prime $p$.

We need the following well-known Cauchy–Davenport Theorem over prime field. For two sets $A, B$, define $A + B = \{a + b : a \in A, b \in B\}$.

**Theorem 3.1** (Cauchy–Davenport Theorem, $[9,12]$). Let $p$ be a prime. If $A$ and $B$ are two nonempty subsets of $\mathbb{Z}_p$, then we have

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$ 

**Lemma 3.2.** Let $N(x_0, x_{s+1})$ be an $s$-necklace, where for each $0 \leq i \leq s$, there is either an edge $x_i x_{i+1}$ or a $p$-cycle between $x_i$ and $x_{i+1}$ consisting of a $k_i$-thread and a $(p - 2 - k_i)$-thread. Let $\omega$ be a precoloring of $x_0$, and let $B(x_i)$ be the set of available colors of $x_i$ from a coloring of $x_{i-1}$ for each $i \in [s + 1]$, where $B(x_0) = \{\omega(x_0)\}$. Then each of the following holds.

(i) We have $|B(x_i)| \geq |i + 1, p|$ for each $i \in [s + 1]$.

(ii) If $s \geq p - 2$, then an $s$-necklace is reducible for $C_p$-coloring.

**Proof.** (i) For any $s \geq i \geq 0$, we shall count the number of colors $\omega(x_{i+1})$ that can be extended from a color $\omega(x_i)$ of $x_i$. Note that $x_0$ receives a fixed coloring $\omega(x_0)$. If $x_0 x_i$ is an edge in $G$, then we have $\omega(x_i) \in \{\omega(x_0) + \frac{p-1}{2}, \omega(x_0) - \frac{p-1}{2}\}$, that is, $B(x_i) = \{\omega(x_0) + \frac{p-1}{2}, \omega(x_0) - \frac{p-1}{2}\}$. The arithmetic operations here and below are taken modulo $p$. If there is a $p$-cycle between $x_0$ and $x_i$ which consists of a $k_0$-thread and a $(p - 2 - k_0)$-thread, then by Lemma 2.1 we have $\omega(x_i) \in \{\omega(x_0) + \frac{p-1}{2}(k_0 + 1), \omega(x_0) - \frac{p-1}{2}(k_0 + 1)\}$, which gives $B(x_i) = \{\omega(x_0) + \frac{p-1}{2}(k_0 + 1), \omega(x_0) - \frac{p-1}{2}(k_0 + 1)\}$. Hence, in any case, we have $|B(x_i)| = 2$.

Below we shall apply induction to show $|B(x_i)| \geq \min\{i + 1, p\}$ for each $i \in [s + 1]$. The basic case $i = 1$ is proved above. Assume the statement $|B(x_i)| \geq \min\{i + 1, p\}$ holds for any integer at most $i$. For the case $i + 1$, we shall show that $|B(x_{i+1})| \geq \min\{i + 2, p\}$. Similar as before, if $x_i x_{i+1}$ is an edge of $G$, then

$$B(x_{i+1}) = \left\{b + \frac{p-1}{2}, b - \frac{p-1}{2} : b \in B(x_i)\right\};$$
if there is a $p$-cycle between $x_i$ and $x_{i+1}$ consisting of a $k_i$-thread and a $(p - 2 - k_i)$-thread, then by Lemma 2.1 we have

$$B(x_{i+1}) = \left\{ b + \frac{p - 1}{2}(k_i + 1), b - \frac{p - 1}{2}(k_i + 1) : b \in B(x_i) \right\}. $$

Using the notation in Theorem 3.1, we have

$$\text{either} B(x_{i+1}) = B(x_i) + \left\{ \frac{p - 1}{2}, -\frac{p - 1}{2} \right\} \text{ or } B(x_{i+1}) = B(x_i) + \left\{ \frac{p - 1}{2}(k_i + 1), -\frac{p - 1}{2}(k_i + 1) \right\}. $$

By Theorem 3.1, we obtain that $|B(x_{i+1})| \geq \min\{|B(x_i)| + 1, p\} \geq \min\{i + 2, p\}$. This proves the claim that $|B(x_i)| \geq \min\{|i + 1, p\}$ for each $i \in [s + 1]$.

(ii) Fix a $C_p$-coloring $\omega$ of $G - (V(N(x_0, x_{s+1})) \setminus \{x_0, x_{s+1}\})$. We show that $\omega$ can be extended to a $C_p$-coloring of $G$. We still let $B(x_i)$ be the set of available colors of $x_i$ from a coloring of $x_{i-1}$ for each $1 \leq i \leq s$, where $B(x_0) = \{\omega(x_0)\}$. By (i), we particularly have that $|B(x_i)| \geq p - 1$ for each $p - 2 \leq j \leq s$. For the $s$-necklace $N(x_0, x_{s+1})$, $\omega(x_{s+1})$ is a fixed color, and so its restriction requires that $\omega(x_i) \in \{\omega(x_{s+1}) + \frac{p - 1}{2}, \omega(x_{s+1}) - \frac{p - 1}{2}\}$

when $x_i x_{i+1}$ is an edge, and $\omega(x_i) \in \{\omega(x_{s+1}) + \frac{p - 1}{2}(k_i + 1), \omega(x_{s+1}) - \frac{p - 1}{2}(k_i + 1)\}$

when there is a $p$-cycle between $x_i$ and $x_{s+1}$ consisting of $k_i$-thread and a $(p - 2 - k_i)$-thread. Since $|B(x_i)| \geq p - 1$, we have both $B(x_i) \cap \{\omega(x_{s+1}) + \frac{p - 1}{2}, \omega(x_{s+1}) - \frac{p - 1}{2}\} \neq \emptyset$ and $B(x_i) \cap \{\omega(x_{s+1}) + \frac{p - 1}{2}(k_i + 1), \omega(x_{s+1}) - \frac{p - 1}{2}(k_i + 1)\} \neq \emptyset$. Therefore, there exists an available color for the choice of $x_i$ in $B(x_i)$, and so $\omega$ can be extended to a $C_p$-coloring of $G$ by appropriately coloring each of $x_1, x_2, ..., x_s$ and by Lemma 2.1.

**Lemma 3.3.** For a $(k_1, k_2, ..., k_t)$-necklace or a $(k_1, k_2, ..., k_t)$-crown-necklace, if it holds that

$$\max_{1 \leq i \leq t} \{k_i\} \leq p - 2$$

and

$$\sum_{i=1}^{t} k_i \geq (p - 2)t - p + 1,$$

then it is reducible for $C_p$-coloring.

**Proof.** Let $H$ be a $(k_1, k_2, ..., k_t)$-necklace or a $(k_1, k_2, ..., k_t)$-crown-necklace with end vertex set $W$. For each $i \in [t]$, let $x_i, y_i$ be the end vertices of the $k_i$-necklace in $H$, where $x_i \in W$. If $H$ is a $(k_1, k_2, ..., k_t)$-necklace, then $y_1 = y_2 = \cdots = y_t$ is a common vertex. If $H$ is a $(k_1, k_2, ..., k_t)$-crown-necklace, then $y_1, y_2, ..., y_t$ (may or may not be identical) are lying in a common $p$-cycle. In the later case, suppose that we color a selected vertex $y_i$ of the common $p$-cycle with color $b$. Then denote the color of $y_i$ by $\phi(y_i) = b + \frac{p - 1}{2}d_i$, where $d_i$ is the distance from $y_0$ to $y_i$ in the cyclic order for each $1 \leq i \leq t$. In the former case, we apply the same notation and set that $y_0 = y_1 = y_2 = \cdots = y_t$, $\omega(y_i) = b$, and $d_i = 0$ for each $i$. 
Fix a precoloring $\omega$ of $W$. We show that $\omega$ can be extended to a $C_p$-coloring of $H$ by selecting an appropriate value of $b$ with application of Lemma 3.2(i).

For each $1 \leq i \leq t$, let $B_i$ be the set of available colors of $y_i$ such that the coloring $\omega(x_i)$ and $\omega(y_i) \in B_i$ can be extended to a $C_p$-coloring of the $k_i$-necklace. By Lemma 3.2(i), we have $|B_i| \geq k_i + 2$. Let $D_i = \{\beta : \beta = \alpha - \frac{p-1}{2}d_i, \alpha \in B_i\}$. Clearly, $|D_i| = |B_i| \geq k_i + 2$.

Thus we have

$$
\left| \bigcap_{i=1}^{t} D_i \right| \geq \sum_{i=1}^{t} |D_i| - (t - 1) \left| \bigcup_{i=1}^{t} D_i \right|
\geq \sum_{i=1}^{t} (k_i + 2) - (t - 1)p
= \sum_{i=1}^{t} k_i - (p - 2)t + p \geq 1.
$$

Hence $\bigcap_{i=1}^{t} D_i \neq \emptyset$ holds. Then we can select an element $b \in \bigcap_{i=1}^{t} D_i$ and color $y_i$ with $\omega(y_i) = \varphi(y_i) = b + \frac{p-1}{2}d_i$ for each $i \in [t]$. By definition, the coloring $\omega(x_i)$ and $\omega(y_i)$ can be extended to a $C_p$-coloring of the $k_i$-necklace for each $i \in [t]$. Therefore, $H$ is reducible for $C_p$-coloring.

3.2 The proof of Theorem 1.3

By Proposition 2.3, $f(k)$ does not exist if $k > 0$ is not an odd prime integer. Proposition 2.4 indicates $p^2 - \frac{5}{2}p + \frac{3}{2} \leq f(p)$ for a prime $p \geq 5$. To complete the proof of Theorem 1.3, it suffices to show that every planar graph of girth $p$ without cycles of length from $p + 1$ to $2(p - 1)(p + 2) - 1$ is $C_p$-colorable. In fact, we show the following mild stronger theorem.

**Theorem 3.4.** Let $G$ be a plane graph of girth $p$ without cycles of length from $p + 1$ to $2(p - 1)(p + 2) - 1$, and let $\omega$ be a precoloring of a $p$-cycle $C$ of $G$. Then $G$ is $(\omega, V(C))$-colorable.

**Proof:** Suppose to the contrary that $G$ is a counterexample with $|E(G) \setminus E(C)|$ minimized. Clearly, we have $E(G) \setminus E(C) \neq \emptyset$ and $|V(G)| > p$.

**Claim 1.**

(i) $G$ is 2-connected. In particular, $\delta(G) \geq 2$.
(ii) Every $p$-cycle in $G$ bounds a face. In particular, $C$ is a facial $p$-cycle of $G$.

**Proof of Claim 1.**

(i) If $G$ is not 2-connected, then there exist proper induced subgraphs $G_1$ and $G_2$ of $G$ and a vertex $v \in V(G_2)$ such that $E(G) = E(G_1) \cup E(G_2)$, $V(G_1) \cap V(G_2) \subseteq \{v\}$, and $V(C) \subseteq V(G_1)$. By the minimality of the counterexample, $\omega$ can be extended to a $C_p$-coloring $\tilde{\omega}$ of $G_1$. Take an edge $uv \in E(G_2)$. If $uv$ is in a $p$-cycle, then we let $C'$ be a
p-cycle containing $uv$ and let $G'_2 = G_2$. Otherwise, we construct a new graph $G'_2$ from $G_2$ by adding a new $(p - 2)$-thread between $u$ and $v$ to form a new $p$-cycle $C'$. Note that $G'_2$ contains no cycles of length from $p + 1$ to $2(p - 1)(p + 2) - 1$ in any case. Let $\omega'$ be a precoloring of $C'$ such that $\omega'(v) = \tilde{\omega}(v)$ (if $v \notin V(G_1)$), then $G$ is not connected and we take $\omega'(v)$ to be an arbitrary color). Since $E(G'_2) \setminus E(C') < |E(G) \setminus E(C)|$ and by the minimality of the counterexample, $\omega'$ can be extended to a $C_p$-coloring $\tilde{\omega}'$ of $G'_2$. So $\omega'$ and $\tilde{\omega}$ combine to provide a $C_p$-coloring of $G$, which is a contradiction.

(ii) Suppose for a contradiction that a $p$-cycle $K$ of $G$ does not bound a face. Let $G_1$ be the subgraph of $G$ drawn outside (and including) $K$, and let $G_2$ be the subgraph of $G$ drawn inside (and including) $K$. We may, without loss of generality, assume that $V(C) \subset V(G_1)$. By the minimality of the counterexample, $\omega$ can be extended to a $C_p$-coloring $\tilde{\omega}$ of $G_1$. Let $\omega'$ be the restriction of $\tilde{\omega}$ on $V(K)$. Then $\omega'$ can be extended to a $C_p$-coloring $\tilde{\omega}'$ of $G_2$ by the minimality of $G$. The union of $\tilde{\omega}$ and $\tilde{\omega}'$ is a $C_p$-coloring of $G$ extending $\omega$, which is a contradiction.

By Claim 1(ii), $C$ must be a facial cycle of $G$. Re-embedding $G$ on the plane if needed, we can assume that the face bounded by $C$ is the outer face of $G$, denoted by $f_0$. Let $H$ be a thread, or a necklace, or a $(k_1, k_2, ..., k_t)$-thread, or a $(k_1, k_2, ..., k_t)$-crown-necklace of $G$ with end vertex set $W$. If $V(H) \setminus W \subseteq V(G) \setminus V(C)$, then we say $H$ is valid in $G$.

Claim 2. Each of the following holds.

(i) $G$ contains no valid $(p - 2)^+$-thread.

(ii) $G$ contains no valid $(p - 2)^*$-necklace.

(iii) $G$ contains neither a valid $(k_1, k_2, ..., k_t)$-necklace nor a valid $(k_1, k_2, ..., k_t)$-crown-necklace with $\sum_{i=1}^{t} k_i \geq (p - 2)t - p + 1$, where $t \geq 3$.

Proof of Claim 2. Note that an $s$-thread is a special $s$-necklace without performing $C_p$-replacement operation, and so Claim 2(i) follows from Claim 2(ii). Suppose, for a contradiction, that $G$ has a valid $(p - 2)^+$-necklace, or a valid $(k_1, k_2, ..., k_t)$-necklace, or a valid $(k_1, k_2, ..., k_t)$-crown-necklace described above, denoted by $H$ with $W$ being its end vertex set. By the minimality of $G$, $\omega$ can be extended to a $C_p$-coloring $\tilde{\omega}$ of $G - (V(H) \setminus W)$. By Lemmas 3.2(ii) and 3.3, $\tilde{\omega}$ can be extended to a $C_p$-coloring of $G$, that is, $\omega$ can be extended to a $C_p$-coloring of $G$, a contradiction.

By Claim 1(ii), any $p$-cycle in $G$ is a facial $p$-cycle (the boundary of a $p$-face). Since $G$ contains no cycles of length from $p + 1$ to $2(p - 1)(p + 2) - 1$, any two $p$-cycles have no common edges. A vertex $v$ of a facial $p$-cycle $K$ with $d_G(v) \geq 3$ is called an attachment-vertex of $K$. Since $G$ is 2-connected by Claim 1(i), every $p$-cycle contains at least two attachment-vertices.

Next, we construct two graphs $G'$ and $G''$ modified from $G$ for later proof. Let $G'$ be the graph obtained from $G$ by replacing $K$ with an edge $uv$ for any facial $p$-cycle $K$ other than $C$ with exactly two attachment-vertices $u, v$. Note that $d_G(u, v) \geq 2$ by Claim 2(i). By construction, each edge of $G'$ is corresponding to either an edge of $G$ or a $p$-cycle consisting of a $t$-thread and a $(p - 2 - t)$-thread with $p - 2 \geq t \geq 2$. Note that the shorter one of $t$-thread and $(p - 2 - t)$-thread has length at most $\frac{p - 1}{2}$.
Let $K$ be a facial $p$-cycle of $G'$ other than $C$ with attachment-vertices $v_1, v_2, ..., v_r$. Then $r \geq 3$ by the construction of $G'$. To stick $K$, we mean to delete all the vertices of $V(K) \setminus \{v_1, v_2, ..., v_r\}$ and add a new vertex $v_k^*$ inside face $K$ to join each vertex of \{v_1, v_2, ..., v_r\}. The vertex $v_k^*$ is called a sticking vertex, where the degree of $v_k^*$ is at least 3. Let $G'$ be the graph obtained from $G'$ by sticking all the facial $p$-cycles of $G'$ except $C$.

By the construction of $G'$, we immediately observe the following: $G'$ is a plane graph with outer face $f_0$ bounded by $C$; the minimal degree $\delta(G') \geq 2$; for any $uv \in E(G')$, at most one vertex of $\{u, v\}$ is in $V(G') \setminus V(G)$; each 2-vertex $v$ of $G'$ is either a 2-vertex of $G$ or an attachment-vertex of a facial $p$-cycle of $G$; each vertex $v \in V(G') \setminus V(G)$ is a sticking vertex with $d_{G'}(v) \geq 3$. These facts will be used implicitly in the rest of the proof.

We further obtain the claim below concerning cycles of $G'$.

**Claim 3.** The new constructed graph $G'$ is a plane graph of girth $p$ without cycles of length from $p + 1$ to $4(p + 2) - 1$. Furthermore, $C$ is the only one $p$-cycle of $G'$.

**Proof of Claim 3.** Recall that each $p$-cycle in $G$ is a facial $p$-cycle by Claim 1(ii). By the construction of $G'$, $C$ is the only one $p$-cycle of $G'$. Let $Q = x_0x_1 \ldots x_mx_0$ be a cycle of $G'$ other than $C$. If $x_i$ is a sticking vertex, then $x_i$ corresponds to a facial $p$-cycle $K_i$ of $G$, and $x_{i-1}$ and $x_{i+1}$ are two attachment-vertices of $K_i$, thus the two edges $x_{i-1}x_i, x_ix_{i+1}$ together correspond to a segment of $K_i$ whose length is at most $p - 2$ as $K_i$ has at least three attachment-vertices. If both $x_j$ and $x_{j+1}$ are not sticking vertices, then $x_jx_{j+1}$ corresponds to either an edge of $G$ or a $p$-cycle of $G$ consisting of two threads, where the shorter one has length at most $\frac{p - 1}{2}$. It is also clear that any two sticking vertices are not adjacent. Hence, for each sticking vertex in the cycle $Q$ its two incident edges together correspond to a path of length at most $p - 2$ in $G$, and for each edge in $Q$ not incident to sticking vertex it corresponds to a thread of length at most $\frac{p - 1}{2}$. Hence the cycle $Q$ corresponds to a cycle of length at most $\frac{p - 1}{2}m$ in $G$. So we have $\frac{p - 1}{2}m \geq 2(p - 1)(p + 2)$, which gives $m \geq 4(p + 2)$. Therefore, each cycle of $G'$ except $C$ has length at least $4(p + 2)$.

In the graph $G'$, a thread or a $(k_1, k_2, ..., k_r)$-thread $H$ with end vertex set $W$ is called valid if $V(H) \setminus W \subseteq V(G') \setminus V(C)$.

**Claim 4.** Each valid $(s + 2)$-thread of $G'$ corresponds to a valid $s$-necklace of $G$. In particular, $G'$ contains no valid $p^+$-thread by Claim 2(ii).

**Proof of Claim 4.** Let $P = x_0x_1 \ldots x_{s+2}x_{s+3}$ be a valid $(s + 2)$-thread of $G'$. For any $i \in [s + 2]$, $x_i \in V(G') \setminus V(C)$ by definition. Noting that $x_i$ is a 2-vertex and by the construction of $G'$, we have $x_i \in V(G)$. Hence for each $i \in [s + 1]$, the edge $x_ix_{i+1}$ in $G'$ corresponds to either an edge of $G - V(C)$ or a $p$-cycle of $G - V(C)$ consisting of two threads by its construction. Therefore, $x_ix_{s+3}$ corresponds to a valid $s$-necklace of $G$. In particular, a valid $p^+$-thread of $G'$ corresponds to a valid $(p - 2)^+$-necklace of $G$, and thus $G'$ contains no valid $p^+$-thread by Claim 2(ii).

**Claim 5.** $G'$ contains no valid $(k_1, k_2, ..., k_t)$-thread with $\sum_{i=1}^t k_i \geq pt - p + 1$ and $t \geq 3$. 

Proof of Claim 5. Suppose to the contrary that \( G^* \) has a valid \((k_1, k_2, \ldots, k_t)\)-thread \( T_v \) such that \( \sum_{i=1}^{t} k_i \geq pt - p + 1 \) and \( t = d_{G^*}(v) \geq 3 \). For each \( j \in [t] \), let \( x_j \) be the end vertex (other than \( v \)) of the \( k_j \)-thread in the \((k_1, k_2, \ldots, k_t)\)-thread, let \( y_i \) be the neighbor of \( x_i \) on the \( k_i \)-thread, and let \( z_i \) be the neighbor of \( v \) on the \( k_i \)-thread. Then \( V(T_v) \setminus \{y_1, y_2, \ldots, y_t\} \subseteq V(G^*) \setminus V(C) \). By Claim 4, we have \( k_i \leq p - 1 \) for any \( i \in [t] \).

Note that for each \( i \in [t] \), \( y_i \) is not a sticking vertex since it is a 2-vertex in \( G^* \). If \( x \) is a vertex in \( V(G) \), then the \((x, y_i)\)-thread from \( x \) to \( y_i \) corresponds to a \((k_i - 1)\)-necklace \( N(x, y_i) \) in \( G - V(C) \) for each \( i \in [t] \). Hence \( G \) contains a valid \((k_1 - 1, k_2 - 1, \ldots, k_t - 1)\)-necklace \( N_v \) with end vertices \( y_1, y_2, \ldots, y_t \). Since \( \sum_{i=1}^{t} k_i \geq pt - p + 1 \), we have \( \sum_{i=1}^{t} (k_i - 1) \geq (p - 1)t - p + 1 \), which contradicts Claim 2(iii).

Assume instead that \( x \) is not a vertex in \( V(G) \). Then \( x \) is a sticking vertex in \( G^* \), which corresponds to a \( p \)-cycle \( K_x \) in \( G - V(C) \). Thus, \( z_i \) is an attachment-vertex of \( K_x \) for each \( i \in [t] \). Hence the \((z_i, y_i)\)-thread corresponds to a \((k_i - 2)\)-necklace \( N(z_i, y_i) \) in \( G - V(C) \) for each \( i \in [t] \). Thus \( G \) contains a valid \((k_1 - 2, k_2 - 2, \ldots, k_t - 2)\)-crown-necklace with end vertices \( y_1, y_2, \ldots, y_t \). Similarly, we have \( \sum_{i=1}^{t} (k_i - 2) \geq (p - 2)t - p + 1 \) by \( \sum_{i=1}^{t} k_i \geq pt - p + 1 \), which contradicts Claim 2(iii) again.

Now we shall complete the proof by discharging method on \( G^* \). Any face other than \( f_0 \) is called an internal face of \( G^* \). The vertices of \( V(G^*) \setminus V(C) \) are called internal vertices of \( G^* \). The degree \( d_{G^*}(f) \) of a face \( f \) is the number of edges in its boundary, cut edges being counted twice. Let \( F(G^*) \) be the set of faces of \( G^* \). From Euler Formula, we have

\[
\sum_{v \in V(G^*)} \left( \frac{p}{2} d_{G^*}(v) - (p + 2) \right) + \sum_{f \in F(G^*)} (d_{G^*}(f) - (p + 2)) = -2(p + 2),
\]

which implies

\[
\sum_{v \in V(G^*)} \left( \frac{p}{2} d_{G^*}(v) - (p + 2) \right) + (d_{G^*}(f_0) + p) + \sum_{f \in F(G^*) \setminus \{f_0\}} (d_{G^*}(f) - (p + 2)) = -2. \tag{2}
\]

Assign an initial charge \( ch_0(v) = \frac{p}{2} d_{G^*}(v) - (p + 2) \) for each \( v \in V(G^*) \), \( ch_0(f_0) = 2p \) and \( ch_0(f) = d_{G^*}(f) - (p + 2) \) for each \( f \in F(G^*) \setminus \{f_0\} \). Hence the total charge is \(-2\) by Equation (2).

We redistribute the charges according to the following rules.

(RI) Every \((4p + 8)^t\)-face of \( G^* \) gives charge \( \frac{3}{4} \) to each of its incident internal vertices.

(RII) Every \(3^t\)-vertex of \( G^* \) gives charge \( \frac{1}{4} \) to each of its weakly adjacent internal 2-vertices.

(RIII) The outer face \( f_0 \) gives charge 2 to each of its incident vertices.

Note that in (RII) each \(3^t\)-vertex of \( G^* \) gives no charge to its weakly adjacent 2-vertices in \( V(C) \), since each vertex in \( V(C) \) is not internal by definition.

Let \( ch \) denote the charge assignment after performing the charge redistribution using the rules (RI), (RII), and (RIII).

Claim 6. \( ch(f) \geq 0 \) for each \( f \in F(G^*) \).

Proof of Claim 6. By Claim 3, \( G^* \) is a plane graph of girth \( p \) without cycles of length from \( p + 1 \) to \( 4(p + 2) - 1 \). If \( d_{G^*}(f) = p \), then \( f \) must be the outer face \( f_0 \), and thus
\(ch_0(f_0) = 2p\). By (RIII), \(f\) sends charge 2 to each of its incident vertices, and hence \(ch(f) = ch_0(f_0) - 2p = 0\). Now assume \(d_G^*(f) \geq 4(p + 2)\). Then \(f\) sends charge \(\frac{3}{4}\) to each incident internal vertices by (RI), and so \(ch(f) \geq ch_0(f) - \frac{3}{4}d_G^*(f) = (d_G^*(f) - (p + 2)) - \frac{3}{4}d_G^*(f) = \frac{1}{4}(d_G^*(f) - 4(p + 2)) \geq 0\). \(\square\)

**Claim 7.** \(ch(v) \geq 0\) for each \(v \in V(G')\).

**Proof of Claim 7.** First we assume \(d_G^*(v) = 2\). Then \(ch_0(v) = -2\). If \(v \in V(C)\), then \(v\) receives charge 2 from \(f_0\) by (RIII). Thus \(ch(v) = ch_0(v) + 2 = 0\). For an internal 2-vertex \(v\), by Claims 1 and 4, \(v\) is weakly adjacent to two \(3^+\)-vertices, and thus \(v\) receives charge \(\frac{1}{4} \times 2\) by (RII). By (RI), \(v\) receives charge \(\frac{3}{4} \times 2\) from its incident faces. Hence \(ch(v) = -2 + \frac{1}{2} + \frac{3}{2} = 0\).

Now we assume \(d_G^*(v) \geq 3\). Let \(t(v)\) be the number of internal 2-vertices weakly adjacent to \(v\). By (RII), \(v\) sends charge \(\frac{1}{4}t(v)\) to its weakly adjacent internal 2-vertices. If \(v \in V(C)\), then \(t(v) \leq (p - 1)(d_G^*(v) - 2)\) as each thread in \(G^*\) contains at most \((p - 1)\) internal 2-vertices by Claim 4. Note that \(v\) receives charge \(\frac{3}{4}d_G^*(v) - 1\) from its incident \((4p + 8)^+\)-faces by (RI), and receives charge 2 from \(f_0\) by (RIII). Then

\[
ch(v) = \frac{1}{4}t(v) + \frac{3}{4}d_G^*(v) - 1 + 2 \\
\geq \left( \frac{p}{2}d_G^*(v) - (p + 2) \right) - \frac{1}{4}(p - 1)(d_G^*(v) - 2) + \frac{3}{4}(d_G^*(v) - 1) + 2 \\
= \frac{p + 4}{4}d_G^*(v) - \frac{1}{2}p - \frac{5}{4} \\
\geq \frac{p + 4}{4} \cdot 3 - \frac{1}{2}p - \frac{5}{4} \\
= \frac{p + 7}{4} > 0.
\]

Assume instead that \(v\) is an internal vertex. By Claims 4 and 5, \(t(v) \leq pd_G^*(v) - p\). By (RI), \(v\) receives charge \(\frac{p}{4}d_G^*(v)\) from its incident faces. Hence

\[
ch(v) = \left( \frac{p}{2}d_G^*(v) - (p + 2) \right) + \frac{3}{4}d_G^*(v) - \frac{1}{4}t(v) \\
\geq \frac{p}{2}d_G^*(v) - p - 2 + \frac{3}{4}d_G^*(v) - \frac{1}{4}(pd_G^*(v) - p) \\
= \frac{p + 3}{4}d_G^*(v) - \frac{3}{4}p - 2 \\
\geq \frac{p + 3}{4} \cdot 3 - \frac{3}{4}p - 2 \\
= \frac{1}{4} > 0.
\]

By Equation (2) and Claims 6 and 7, we have
\[ -2 = \sum_{v \in V(G^*)} \phi_0(v) + \sum_{f \in E(G^*)} \phi_0(f) = \sum_{v \in V(G^*)} \phi(v) + \sum_{f \in E(G^*)} \phi(f) \geq 0, \]

a contradiction. This contradiction completes the proof of Theorem 3.4. \qed

4 | THE FRACTIONAL COLORING

This section is devoted to prove Theorem 1.6. We first study some graphs with precoloring extensions in Section 4.1, serving for reducible configurations, and then present the proof of Theorem 1.6 in Section 4.2 by a discharging method.

4.1 | Precoloring graphs for fractional \( \left( k : \frac{k-1}{2}\right) \)-coloring

We start with the following property on coloring of paths.

**Lemma 4.1.** Let \( P = v_1v_2 \ldots v_t \) be a path with \( 2 \leq t \leq k \), and let \( \phi \) be a fractional \( \left( k : \frac{k-1}{2}\right) \)-coloring of \( P \). Then \( |\phi(v_i) \cap \phi(v_{i+1})| \geq \frac{k-t}{2} \) if \( t \) is odd, and \( |\phi(v_i) \cap \phi(v_{i+1})| \leq \frac{t-2}{2} \) if \( t \) is even.

**Proof.** We prove by induction. Since \( \phi \) is a fractional \( \left( k : \frac{k-1}{2}\right) \)-coloring of \( P \), we have \( \phi(v_i) \cap \phi(v_{i+1}) = \emptyset \) for each \( i \in [t - 1] \). Thus Lemma 4.1 holds for \( t = 2 \). If \( t = 3 \), noting that \( (\phi(v_1) \cup \phi(v_2)) \subseteq [k] \setminus \phi(v_2) \), then \( |\phi(v_1) \cup \phi(v_2)| \leq k - |\phi(v_2)| \), and thus \( |\phi(v_1) \cap \phi(v_2)| = |\phi(v_1)| + |\phi(v_2)| - |\phi(v_1) \cup \phi(v_2)| \geq \frac{k-1}{2} + \frac{k-1}{2} - \left( k - \frac{k-1}{2} \right) = \frac{k-3}{2} \).

That is, Lemma 4.1 holds for \( t = 3 \). Assume Lemma 4.1 holds for any value smaller than \( t \). Now we consider \( \phi(v_1) \cap \phi(v_t) \). First we assume \( t \) is even. Then \( t - 1 \) is odd, and \( |\phi(v_1) \cap \phi(v_{t-1})| \geq \frac{k-t+1}{2} \) by induction hypothesis. Since \( \phi(v_1) \cap \phi(v_{t-1}) = \emptyset \), we have \( \phi(v_1) \cap \phi(v_t) \subseteq [k] \setminus (\phi(v_{t-1}) \cup \phi(v_1)) \), which implies \( |\phi(v_1) \cap \phi(v_t)| \leq k - |\phi(v_{t-1})| - |\phi(v_1)| \cap \phi(v_{t-1})| \leq \frac{k-1}{2} - \frac{k-t+1}{2} = \frac{-t+2}{2} \). Now we assume \( t \) is odd. Then \( t - 1 \) is even, and \( |\phi(v_1) \cap \phi(v_{t-1})| \leq \frac{t-3}{2} \) by induction hypothesis. As \( \phi(v_1) \cap \phi(v_{t-1}) = \emptyset \), we have \( (\phi(v_1) \cup \phi(v_t)) \subseteq [k] \setminus (\phi(v_{t-1}) \cup \phi(v_1)) \), which implies \( |\phi(v_1) \cup \phi(v_t)| \leq k - |\phi(v_{t-1})| + |\phi(v_t) \cup \phi(v_{t-1})| \).

Thus \( |\phi(v_1) \cap \phi(v_t)| = |\phi(v_1)| + |\phi(v_t)| - |\phi(v_1) \cup \phi(v_t)| \geq \frac{k-1}{2} + \frac{k-1}{2} - k + \frac{k-1}{2} - |\phi(v_1) \cup \phi(v_{t-1})| \geq \frac{k-1}{2} + \frac{k-1}{2} - k + \frac{k-1}{2} - \frac{t-3}{2} = \frac{k-t}{2} \). Therefore, Lemma 4.1 holds by induction. \qed

Recall that, for \( S \subset V(H) \), \( H \) is \( \phi_S \)-*colorable* if the precoloring \( \phi \) of \( S \) can be extended to a fractional \( \left( k : \frac{k-1}{2}\right) \)-coloring of \( H \). Note that the number \( \frac{k-2}{4} + (-1)^{d(x,y)} \cdot \frac{k-2d(x,y)}{4} \) is always an integer; in fact it is \( \frac{k-1-d(x,y)}{2} \) if \( d(x,y) \) is even, and \( \frac{d(x,y)-1}{2} \) if \( d(x,y) \) is odd.
Lemma 4.2. Let \( C \) be a cycle of length \( k \). Let \( \varphi \) be a precoloring of \( \{x, y\} \subseteq V(C) \). Then \( C \) is \( \varphi_{\{x,y\}} \)-colorable if and only if

\[
|\varphi(x) \cap \varphi(y)| = \frac{k - 2}{4} + (-1)^t \cdot \frac{k - 2d(x,y)}{4}.
\]

Proof. Denote \( C = x_0x_1 \ldots x_{k-1}x_0 \), where \( x_0 = x, x_t = y \), and \( d(x,y) = t \leq \frac{k-1}{2} \).

Assume that \( C \) is \( \varphi_{\{x,y\}} \)-colorable, and let \( \tilde{\varphi} \) be a fractional \((k : \frac{k-1}{2})\)-coloring of \( C \) extended by \( \varphi \). Denote \( P_1 = x_0x_1 \ldots x_t \) and \( P_2 = x_0x_{k-t-1} \ldots x_t \). Then \( P_1 \) is a path of order \( t + 1 \) and \( P_2 \) is a path of order \( k - t + 1 \). Note that \( \tilde{\varphi} \) also provides a fractional \((k : \frac{k-1}{2})\)-coloring of \( P_1 \) and of \( P_2 \). If \( t \) is even, then by Lemma 4.1, we have

\[
|\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| \geq \frac{k-t-1}{2} \text{ as } |V(P_t)| = t + 1 \text{ is odd and } |\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| \leq \frac{k-t-1}{2} \text{ as } |V(P_2)| = k - t + 1 \text{ is even.}
\]

Thus \( |\varphi(x_0) \cap \varphi(x_t)| = |\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| = \frac{k-t-1}{2} \). If \( t \) is odd, then by Lemma 4.1, we have

\[
|\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| \leq \frac{t-1}{2} \text{ as } |V(P_t)| = t + 1 \text{ is even and } |\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| \geq \frac{t-1}{2} \text{ as } |V(P_2)| = k - t + 1 \text{ is odd.}
\]

Hence \( |\varphi(x_0) \cap \varphi(x_t)| = |\tilde{\varphi}(x_0) \cap \tilde{\varphi}(x_t)| = \frac{t-1}{2} \).

Conversely, assume that \( a = |\varphi(x) \cap \varphi(y)| = \frac{k-2}{4} + (-1)^t \cdot \frac{k-2d(x,y)}{4} \). Without loss of generality, we may assume \( \varphi(x_0) = \{1, 2, \ldots, \frac{k-1}{2}\} \). If \( t \) is even, we assume \( \varphi(x_0) \cap \varphi(x_t) = \{1, 2, \ldots, a\} \) and \( \varphi(x_t) \setminus \varphi(x_0) = \{\frac{k+1}{2} + a, 1, \frac{k+1}{2} + a + 2, \ldots, k\} \). If \( t \) is odd, we assume \( \varphi(x_0) \cap \varphi(x_t) = \{\frac{k-1}{2} - a, 1, \frac{k-1}{2} - a + 2, \ldots, \frac{k-1}{2}\} \) and \( \varphi(x_t) \setminus \varphi(x_0) = \{\frac{k-1}{2} + 1, \frac{k-1}{2} + 2, \ldots, k - a - 1\} \). We define a coloring by setting \( \varphi(x_0) = \{1, 2, \ldots, \frac{k-1}{2} - i\} \cup \{k - i + 1, k - i + 2, \ldots, k\} \) and \( \varphi(x_0) = \{\frac{k+1}{2} - i, \frac{k+1}{2} - i + 1, \ldots, k - i - 1\} \) for \( 0 \leq i \leq \frac{k-1}{2} \). It is routine to check that \( \varphi \) is a fractional \((k : \frac{k-1}{2})\)-coloring of \( C \).

\( \square \)

Lemma 4.3. Let \( N(x,y) \) be a necklace with a precoloring \( \varphi \) of \( \{x,y\} \). Suppose that the distance between \( x \) and \( y \) is \( d(x,y) = t \leq \frac{k+1}{2} \). If

\[
|\varphi(x) \cap \varphi(y)| = \frac{k - 2}{4} + (-1)^t \cdot \frac{k - 2t}{4},
\]

then \( N(x,y) \) is \( \varphi_{\{x,y\}} \)-colorable.

Proof. We prove by induction. The statement holds for \( t = 0, 1 \). Assume that it holds for any value smaller than \( t \). If \( x \) and \( y \) are in the same \( k \)-cycle, then the statement holds from Lemma 4.2. Otherwise, we can always find a vertex \( u \) in the shortest \((x,y)\)-path \( xz_{t-1} \ldots z_t y \) which divides the necklace into two separated necklaces that one is from \( x \) to \( u \) and the other is from \( u \) to \( y \). More precisely, if \( xz_t \) is not contained in a \( k \)-cycle, then we choose \( u = z_t \); otherwise, we choose \( u = z_j \) where \( j \) is the largest index such that \( z_{j-1}z_j \) is in the \( k \)-cycle containing \( xz_t \). Note that \( u \) is a cut vertex of \( H \) that divides the necklace \( H \) into two separated necklaces. Now we shall try to provide a coloring \( \varphi(u) \) of \( u \) and then
apply induction on the \((x, u)\)-necklace and on the \((u, y)\)-necklace. This can be achieved if we can find \(a\) colors from \(\varphi(x)\setminus \varphi(y)\), \(b\) colors from \(\varphi(x) \cap \varphi(y)\), \(c\) colors from \(\varphi(y) \setminus \varphi(x)\), and the rest colors from \([k] \setminus (\varphi(x) \cup \varphi(y))\) to formulate \(\varphi(u)\) satisfying the induction hypothesis.

Let \(d(x, u) = s\). Then \(d(u, y) = t - s\). Formally, we need to find a nonnegative integer solution \((a, b, c)\) of the following system of inequalities:

\[
\begin{align*}
0 &\le a \le |\varphi(x) \setminus \varphi(y)| = \frac{k}{4} - (-1)^t \cdot \frac{k - 2t}{4}, \\
0 &\le b \le |\varphi(x) \cap \varphi(y)| = \frac{k - 2}{4} + (-1)^t \cdot \frac{k - 2t}{4}, \\
0 &\le c \le |\varphi(y) \setminus \varphi(x)| = \frac{k}{4} - (-1)^t \cdot \frac{k - 2t}{4}, \\
0 &\le \frac{k - 1}{2} - a - b - c \le |[k] \setminus (\varphi(x) \cup \varphi(y))| = 1 + \frac{k - 2}{4} + (-1)^t \cdot \frac{k - 2t}{4}, \\
a + b &\le \frac{k - 2}{4} + (-1)^s \cdot \frac{k - 2s}{4}, \\
b + c &\le \frac{k - 2}{4} + (-1)^{t-s} \cdot \frac{k - 2(t - s)}{4}.
\end{align*}
\]

Let

\[
\alpha = (-1)^t \cdot \frac{k - 2t}{4}, \quad \beta = (-1)^t \cdot \frac{k - 2s}{4}, \quad \text{and} \quad \gamma = (-1)^{t-s} \cdot \frac{k - 2(t - s)}{4}.
\]

Plugging \(a = -b + \frac{k - 2}{4} + \beta\) and \(c = -b + \frac{k - 2}{4} + \gamma\) into the above system of inequalities, we have

\[
\begin{align*}
\alpha + \beta - \frac{1}{2} &\le b \le \frac{k - 2}{4} + \beta, \\
0 &\le b \le \frac{k - 2}{4} + \alpha, \\
\alpha + \gamma - \frac{1}{2} &\le b \le \frac{k - 2}{4} + \gamma, \\
\beta + \gamma - \frac{1}{2} &\le b \le \frac{k}{4} + \alpha + \beta + \gamma.
\end{align*}
\]

Let

\[
M = \max \left\{ \alpha + \beta - \frac{1}{2}, 0, \alpha + \gamma - \frac{1}{2}, \beta + \gamma - \frac{1}{2} \right\} \quad \text{and} \quad N = \min \left\{ \frac{k - 2}{4} + \beta, \frac{k - 2}{4} + \alpha, \frac{k - 2}{4} + \gamma, \frac{k}{4} + \alpha + \beta + \gamma \right\}.
\]

We can actually show that \(0 \le M \le N\) by a one-by-one compression, and then setting \(b = M\) provides a valid solution of the above system of inequalities. This method will be applied in a similar but more complicated Lemma 4.5 below.

Here an alternative way to do so is to check case by case on the parity as follows.
• If \( t \) is odd and \( s \) is odd, then set \( b = M = N = \frac{s - 1}{2} \).
• If \( t \) is odd and \( s \) is even, then set \( b = M = N = \frac{t - s - 1}{2} \).
• If \( t \) is even and \( s \) is odd, then set \( b = M = N = 0 \).
• If \( t \) is even and \( s \) is even, then set \( b = M = N = \frac{k - 1 - t}{2} \).

Then this solution \((a, b, c)\) provides a coloring \( \varphi(u) \) as desired. \(\square\)

We present this version of the proof of Lemma 4.3 to provide an overview of the more complicated Lemma 4.5 below when \( d(x, y) \) is relatively large. We also need the following technical inequality.

**Proposition 4.4.** Let \( s, t \) be integers with \( 1 \leq s \leq \frac{k-1}{2} \) and \( \frac{k+1}{2} \leq t \leq s + \frac{k+1}{2} \). Denote

\[
\beta = (-1)^t \cdot \frac{k - 2s}{4} \quad \text{and} \quad \gamma = (-1)^{t-s} \cdot \frac{k - 2(t - s)}{4}.
\]

Let \( \ell \) be a fixed integer with \( \frac{k - t}{2} - \frac{(-1)^y + 1}{4} \leq \ell \leq \frac{t - 1}{2} - \frac{(-1)^{y+1} + 1}{4} \). Define

\[
M = \max \left\{ \beta + \ell - \frac{k}{4}, 0, \gamma + \ell - \frac{k}{4}, \beta + \gamma - \frac{1}{2} \right\} \quad \text{and} \quad N = \min \left\{ \frac{k - 2}{4} + \beta, \ell, \frac{k - 2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2} \right\}.
\]

Then \( M \) and \( N \) are integers satisfying

\[ 0 \leq M \leq N. \]

**Proof.** It is routine to check that each term in \( M \) and in \( N \) is an integer by discussing the parity of \( t \) and \( s \). To show that \( M \leq N \), it suffices to check 16 inequalities one by one.

- \( \beta + \ell - \frac{k}{4} \leq N = \min \left\{ \frac{k - 2}{4} + \beta, \ell, \frac{k - 2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2} \right\} \);
- \( 0 \leq N = \min \left\{ \frac{k - 2}{4} + \beta, \ell, \frac{k - 2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2} \right\} \);
- \( \gamma + \ell - \frac{k}{4} \leq N = \min \left\{ \frac{k - 2}{4} + \beta, \ell, \frac{k - 2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2} \right\} \);
- \( \beta + \gamma - \frac{1}{2} \leq N = \min \left\{ \frac{k - 2}{4} + \beta, \ell, \frac{k - 2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2} \right\} \).

It turns out to become the following:

- \( \ell \leq \frac{k - 1}{2}, \beta \leq \frac{k}{4}, \beta - \gamma \leq \frac{k - 1}{2} - \ell, -\gamma \leq \frac{k + 2}{4} \).
• \(-\beta \leq \frac{k-2}{4}, 0 \leq \ell, -\gamma \leq \frac{k-2}{4}, -\gamma - \beta \leq \ell + \frac{1}{2}\),

• \(\gamma - \beta \leq \frac{k-1}{2} - \ell, \gamma \leq \frac{k}{4}, \ell \leq \frac{k-1}{2}, -\beta \leq \frac{k+2}{4}\),

• \(\gamma \leq \frac{k}{4}, \beta + \gamma \leq \ell + \frac{1}{2}, \beta \leq \frac{k}{4}, 0 \leq \ell + 1\).

Except some trivial ones that \(|\beta| \leq \frac{k}{4}, |\gamma| \leq \frac{k}{4}, 0 \leq \ell \leq \frac{k-1}{2}\), this reduces to the following:

• \(-\gamma - \beta \leq \ell + \frac{1}{2}, \beta + \gamma \leq \ell + \frac{1}{2}, \beta - \gamma \leq \frac{k-1}{2} - \ell, \text{ and } \gamma - \beta \leq \frac{k-1}{2} - \ell\).

Those inequalities above are all true since

• \(\gamma + \beta \leq |\gamma| + |\beta| \leq \frac{k-2s}{4} + \frac{k-2(t-s)}{4} = \frac{k-1}{2} \leq \ell + \frac{1}{2}\) and

• \(|\beta| - |\gamma| + \ell \leq |\beta| + |\gamma| + \ell \leq \frac{k-2s}{4} + \frac{k-2(t-s)}{4} + \frac{t-1}{2} = \frac{k-1}{2}\).

This proves that \(0 \leq M \leq N\). \(\square\)

**Lemma 4.5.** Let \(N(x, y)\) be a necklace with a precoloring \(\varphi\) of \(\{x, y\}\). Suppose that the distance between \(x\) and \(y\) satisfies \(d(x, y) = t \geq \frac{k+1}{2}\). If

\[
\frac{k - t}{2} - \frac{(-1)^t + 1}{4} \leq |\varphi(x) \cap \varphi(y)| \leq \frac{t - 1}{2} - \frac{(-1)^t + 1}{4},
\]

then \(H\) is \(\varphi_{[x, y]}\)-colorable.

**Proof.** The basic case \(t = \frac{k+1}{2}\) has already been handled in Lemma 4.3. We shall prove Lemma 4.5 by induction. Similarly, there exists a cut vertex \(u\) of \(H\) in the shortest \((x, y)\)-path that divides the necklace into two parts (two separated necklaces), one is from \(x\) to \(u\) and the other is from \(u\) to \(y\). We choose such cut vertex \(u\) with the smallest distance from \(x\). So either \(xu\) is an edge or \(x\) and \(u\) are in the same \(k\)-cycle, and hence we have \(d(x, u) = s \leq \frac{k-1}{2}\). We shall divide the discussion into two cases depending on the value of \(d(u, y) = t - s\).

**Case 1.** \(d(u, y) = t - s \leq \frac{k+1}{2}\).

Note that in this case \(t \leq s + \frac{k+1}{2} \leq k\). Now we shall try to find \(a\) colors from \(\varphi(x) \setminus \varphi(y)\), \(b\) colors from \(\varphi(x) \cap \varphi(y)\), \(c\) colors from \(\varphi(y) \setminus \varphi(x)\), and the rest colors from \([k] \setminus (\varphi(x) \cup \varphi(y))\) to formulate \(\varphi(u)\) satisfying the induction hypothesis. Formally, similar to the proof of Lemma 4.3, we need to find a nonnegative integer solution \((a, b, c)\) of the following system of inequalities:
\[
\begin{align*}
0 \leq a & \leq |\varphi(x) \setminus \varphi(y)|, \\
0 \leq b & \leq |\varphi(x) \cap \varphi(y)|, \\
0 \leq c & \leq |\varphi(y) \setminus \varphi(x)|, \\
0 \leq \frac{k-1}{2} - a - b - c & \leq |[k] \setminus (\varphi(x) \cup \varphi(y))|, \\
a + b & = \frac{k-2}{4} + (-1)^s \cdot \frac{k-2s}{4}, \\
b + c & = \frac{k-2}{4} + (-1)^{t-s} \cdot \frac{k-2(t-s)}{4}.
\end{align*}
\]

Let \( \ell = |\varphi(x) \cap \varphi(y)| \) be a fixed number with \( \frac{k-1}{2} - \frac{(-1)^s + 1}{4} \leq \ell \leq \frac{k-1}{2} - \frac{(-1)^s + 1}{4} \).

Denote
\[
\beta = (-1)^s \cdot \frac{k-2s}{4} \quad \text{and} \quad \gamma = (-1)^{t-s} \cdot \frac{k-2(t-s)}{4}.
\]

Then by plugging \( a \) and \( c \) into the above system of inequalities, it reduces to the following:

\[
\begin{align*}
\beta + \ell - \frac{k}{4} \leq b & \leq \frac{k-2}{4} + \beta, \\
0 \leq b & \leq \ell, \\
\gamma + \ell - \frac{k}{4} \leq b & \leq \frac{k-2}{4} + \gamma, \\
\beta + \gamma - \frac{1}{2} \leq b & \leq \beta + \gamma + \ell + \frac{1}{2}.
\end{align*}
\]

Let
\[
M = \max\left\{ \beta + \ell - \frac{k}{4}, 0, \gamma + \ell - \frac{k}{4}, \beta + \gamma - \frac{1}{2} \right\}
\quad \text{and} \quad
N = \min\left\{ \frac{k-2}{4} + \beta, \ell, \frac{k-2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2} \right\}.
\]

By Proposition 4.4, \( M \) and \( N \) are integers satisfying \( 0 \leq M \leq N \). Therefore, we choose
\[
b = M, \quad a = \frac{k-2}{4} + (-1)^s \cdot \frac{k-2s}{4} - M, \quad \text{and} \quad
\]
\[
c = \frac{k-2}{4} + (-1)^{t-s} \cdot \frac{k-2(t-s)}{4} - M,
\]

providing a desired nonnegative integer solution \((a, b, c)\).

**Case 2.** \( d(u, y) = t - s \geq \frac{k+3}{2} \).
We are still trying to find \( a \) colors from \( \varphi(x) \setminus \varphi(y) \), \( b \) colors from \( \varphi(x) \cap \varphi(y) \), \( c \) colors from \( \varphi(y) \setminus \varphi(x) \), and the rest colors from \( [k] \setminus (\varphi(x) \cup \varphi(y)) \) to form \( \varphi(u) \) satisfying the induction hypothesis. This formulates similar system of inequalities as follows:

\[
\begin{cases}
0 \leq a \leq |\varphi(x) \setminus \varphi(y)|, \\
0 \leq b \leq |\varphi(x) \cap \varphi(y)|, \\
0 \leq c \leq |\varphi(y) \setminus \varphi(x)|, \\
0 \leq \frac{k-1}{2} - a - b - c \leq |[k] \setminus (\varphi(x) \cup \varphi(y))|, \\
\alpha + \beta = \frac{k-2}{4} + (-1)^s \cdot \frac{k-2s}{4}, \\
\frac{k-t+s}{2} - \frac{(-1)^{t-s} + 1}{4} \leq b + c \leq \frac{t-s-1}{2} - \frac{(-1)^{t-s} + 1}{4}.
\end{cases}
\]

Notice that \( \frac{k-2 - (-1)^{\frac{k+1}{2}}}{4} \) is an integer (this value comes from the case \( d(x, y) = \frac{k+1}{2} \)), and we have

\[
\frac{k-t+s}{2} - \frac{(-1)^{t-s} + 1}{4} \leq \frac{k-2 - (-1)^{\frac{k+1}{2}}}{4} \leq \frac{t-s-1}{2} - \frac{(-1)^{t-s} + 1}{4}.
\]

So it is enough to find a solution \((a, b, c)\) with the last inequality replaced by

\[
b + c = \frac{k-2 - (-1)^{\frac{k+1}{2}}}{4}.
\]

Let \( \ell = |\varphi(x) \cap \varphi(y)| \) be a fixed number with \( \frac{k-t}{2} - \frac{(-1)^{t+1} + 1}{4} \leq \ell \leq \frac{t-1}{2} - \frac{(-1)^{t+1} + 1}{4} \).

Note that \( 0 \leq \ell \leq \frac{k-1}{2} \).

Denote

\[
\beta = (-1)^s \cdot \frac{k-2s}{4} \quad \text{and} \quad \gamma = (-1)^{\frac{k+1}{2}} \cdot \frac{1}{4}.
\]

Then with similar calculation, by plugging \( a \) and \( c \) into the above system of inequalities, it becomes the following:

\[
\begin{cases}
\beta + \ell - \frac{k}{4} \leq b \leq \frac{k-2}{4} + \beta, \\
0 \leq b \leq \ell, \\
\gamma + \ell - \frac{k}{4} \leq b \leq \frac{k-2}{4} + \gamma, \\
\beta + \gamma - \frac{1}{2} \leq b \leq \beta + \gamma + \ell + \frac{1}{2}.
\end{cases}
\]

We still let
\[
M = \max \left\{ \beta + \ell - \frac{k}{4}, 0, \gamma + \ell - \frac{k}{4}, \beta + \gamma - \frac{1}{2} \right\} \quad \text{and} \\
N = \min \left\{ \frac{k - 2}{4} + \beta, \frac{k - 2}{4} + \gamma, \beta + \gamma + \ell + \frac{1}{2} \right\}.
\]

By Proposition 4.4 with \( t - s = \frac{k + 1}{2} \), \( M \) and \( N \) are integers satisfying \( 0 \leq M \leq N \). Therefore, we can choose \( b = M \) and corresponding \( a \) and \( c \) to form a desired solution \((a, b, c)\). This completes the proof. \( \square \)

By Lemma 4.5, we have the following corollary.

**Corollary 4.6.** Let \( N(x, y) \) be a necklace. If \( d(x, y) \geq k \), then \( N(x, y) \) is \( \varphi_{\{x, y\}} \)-colorable for any precoloring \( \varphi \) of \( \{x, y\} \).

Recall Definition 2.7 that a \((k_1, k_2; k_3)\)-bull-necklace is a subgraph obtained from a \((k_1, k_2, k_3)\)-thread by applying \( C_k \)-replacement operation on some edges of the \( k_3 \)-thread. For \( 1 \leq t \leq \frac{k - 1}{2} \), let \( B_v(t, s) \) be a \((t - 1, t - 1; r)\)-bull-necklace \( N_v \) with end vertices \( x, y, z \) and \( d(v, z) = s \).

**Lemma 4.7.** For a bull-necklace \( B_v(t, s) \) with end vertices \( x, y, z \), if \( t \leq \frac{k - 1}{2} \) and \( t + s \geq k \), then \( B_v(t, s) \) is \( \varphi_{\{x, y, z\}} \)-colorable for any precoloring \( \varphi \) of \( \{x, y, z\} \) satisfying \( |\varphi(x) \cap \varphi(y)| = \frac{k - 1 - 2t}{2} \).

**Proof.** Let \( \varphi \) be a precoloring of \( \{x, y, z\} \) such that \( |\varphi(x) \cap \varphi(y)| = \frac{k - 1 - 2t}{2} \). Denote \( A = \varphi(x) \setminus \varphi(y) \), \( B = \varphi(x) \cap \varphi(y) \), \( C = \varphi(y) \setminus \varphi(x) \), and \( D = [k] \setminus (\varphi(x) \cup \varphi(y)) \). Then \( |A| = |B| = \frac{k - 1 - 2t}{2} \), and \( |D| = \frac{k + 1 + 2t}{2} \). Let \( S \) be a subset of \([k]\) such that \( S \subseteq B \) if \( t \) is even and \( S \subseteq D \) if \( t \) is odd. Then \( |S| = \frac{k + 1 + 2t}{2} - \frac{1}{2} \frac{(-1)^y + 1}{2} \). Denote \( S_1 = S \setminus \varphi(z) \) and \( S_2 = S \cap \varphi(z) \). \( \square \)

We first make the following claim.

**Claim 1.** Each of the following holds:

(i) either \( |A \cap \varphi(z)| \geq - \frac{t + 1}{2} - \frac{(-1)^y + 1}{4} - |S_2| \) or \( C \cap \varphi(z) \geq \frac{t + 1}{2} - \frac{(-1)^y + 1}{4} - |S_2| \);

(ii) either \( |A \setminus \varphi(z)| \geq \frac{t + 1}{2} - \frac{(-1)^y + 1}{4} - |S_1| \) or \( C \setminus \varphi(z) \geq \frac{t + 1}{2} - \frac{(-1)^y + 1}{4} - |S_1| \).

**Proof of Claim 1.**

(i) Notice that
\[ |A \cap \varphi(z)| + |C \cap \varphi(z)| = |(A \cup C \cup S) \cap \varphi(z)| - |S| \]
\[ = |A \cup C \cup S| + |\varphi(z)| - |(A \cup C \cup S) \cup \varphi(z)| - |S| \]
\[ \geq \left[ t + t + \frac{k + 1 - 2t}{2} - \frac{(-1)^t + 1}{2} \right] + \frac{k - 1}{2} - |S| \]
\[ = t - \frac{(-1)^t + 1}{2} - |S| \]
\[ \geq 2\left( \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S| \right) - 1. \]

Hence (i) holds.

(ii) Similarly, notice that
\[ |A \setminus \varphi(z)| + |C \setminus \varphi(z)| = |(A \cup C \cup S) \setminus \varphi(z)| - |S| \]
\[ \geq \left( 2t + \frac{k + 1 - 2t}{2} - \frac{(-1)^t + 1}{2} \right) - \frac{k - 1}{2} - |S| \]
\[ = t + 1 - \frac{(-1)^t + 1}{2} - |S| \]
\[ \geq 2\left( \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S| \right). \]

Thus (ii) holds. \( \square \)

Next, we show that there are certain subsets of \( A \) and \( C \) of large size for candidates of \( \varphi(v) \).

**Claim 2.** There exist \( A_1 \subseteq A \setminus \varphi(z) \), \( A_2 \subseteq A \cap \varphi(z) \), \( C_1 \subseteq C \setminus \varphi(z) \), and \( C_2 \subseteq C \cap \varphi(z) \) such that \( |A_1| + |A_2| = \frac{t - 1}{2} + \frac{(-1)^t + 1}{4} \), \( |C_1| + |C_2| = \frac{t - 1}{2} + \frac{(-1)^t + 1}{4} \), \( |A_1| + |S| + |C_1| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} \), and \( |A_2| + |S| + |C_2| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} \).

**Proof of Claim 2.** By Claim 1(i), we may assume without loss of generality that \( |A \cap \varphi(z)| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S| \).

If \( |C \setminus \varphi(z)| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S| \), then we can choose \( C_1 = C \setminus \varphi(z) \) and \( C_2 \subseteq C \cap \varphi(z) \) such that \( |C_1| + |C_2| = \frac{t - 1}{2} + \frac{(-1)^t + 1}{4} \) and \( |C_1| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S| \).

This is feasible since \( |C_1| = t \geq \frac{t - 1}{2} + \frac{(-1)^t + 1}{4} \). By \( |A \cap \varphi(z)| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S| \), we can also choose \( A_1 = A \setminus \varphi(z) \) and \( A_2 \subseteq A \cap \varphi(z) \) such that \( |A_1| + |A_2| = \frac{t - 1}{2} + \frac{(-1)^t + 1}{4} \) and \( |A_1| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S| \).

Hence we have \( |A_1| + |S| + |C_1| \geq |S| + |C_1| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} \) and \( |A_2| + |S| + |C_2| \geq |A_2| + |S| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} \).

Assume instead that \( |C \setminus \varphi(z)| < \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S| \). Notice that
\[ |C \cap \varphi(z)| = |Cl - C\varphi(z)| \]
\[ \geq t - \left( \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S_1| \right) \]
\[ = t - \frac{t + 1}{2} + \frac{(-1)^t + 1}{4} + (|S_1| - |S_2|) \]
\[ = \frac{t - 1}{2} + \frac{(-1)^t + 1}{4} + \left( \frac{k + 1 - 2t}{2} - \frac{(-1)^t + 1}{2} \right) - |S_2| \]
\[ = \frac{k - t}{2} - \frac{(-1)^t + 1}{4} - |S_2| \]
\[ \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S_2|. \]

Hence we can choose \( C_1 = C\varphi(z) \) and \( C_2 \subseteq C \cap \varphi(z) \) such that \( |C_1| + |C_2| = \frac{t - 1}{2} + \frac{(-1)^t + 1}{4} \) and \( C_2 \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S_1| \). By Claim 1(ii) and as \( \varphi(z) \), we have \( |C\varphi(z)| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S_1| \). Thus we can select \( A_1 = A\varphi(z) \) and \( A_2 \subseteq A \cap \varphi(z) \) such that \( |A_1| + |A_2| = \frac{t - 1}{2} + \frac{(-1)^t + 1}{4} \) and \( |A_1| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} - |S_1| \). Therefore, we have \( |A_1| + |S_1| + |C_1| \geq |A_1| + |S_1| \geq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} \) as desired.

Now we choose such \( A_1 \subseteq A\varphi(z) \), \( A_2 \subseteq A \cap \varphi(z) \), \( C_1 \subseteq C\varphi(z) \), and \( C_2 \subseteq C \cap \varphi(z) \) as in Claim 2. Let \( \varphi(v) = A_1 \cup A_2 \cup S_1 \cup S_2 \cup C_1 \cup C_2 \). Then
\[ |\varphi(v)| = \left( \frac{t - 1}{2} + \frac{(-1)^t + 1}{4} \right) + \left( \frac{k + 1 - 2t}{2} - \frac{(-1)^t + 1}{2} \right) + \left( \frac{t - 1}{2} + \frac{(-1)^t + 1}{4} \right) \]
\[ = \frac{k - t}{2}. \]

Moreover, \( |\varphi(v) \cap \varphi(x)| = |A_1| + |A_2| + |S_1| + |S_2| = \frac{k - 1 - t}{2} \) if \( t \) is even and \( |\varphi(v) \cap \varphi(x)| = |A_1| + |A_2| = \frac{t - 1}{2} \) if \( t \) is odd; \( \varphi(v) \cap \varphi(y) = |C_1| + |C_2| \)
\[ \frac{k - 1 - t}{2} \] if \( t \) is even and \( \varphi(v) \cap \varphi(x) = |C_1| + |C_2| = \frac{t - 1}{2} \) if \( t \) is odd.

Notice that \( (A_1 \cup S_1 \cup C_1) \subseteq [k]\varphi(z) \) and \( (A_2 \cup S_2 \cup C_2) \subseteq \varphi(z) \). Hence by Claim 2 we have
\[ \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} \leq |\varphi(v) \cap \varphi(z)| \leq \frac{k - 1}{2} - \left( \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} \right). \]

Since \( s + t \geq k \), we have
\[ \frac{k - s}{2} - \frac{(-1)^s + 1}{4} \leq \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} \]
and \( \frac{k - 1}{2} - \left( \frac{t + 1}{2} - \frac{(-1)^t + 1}{4} \right) \)
\[ \leq \frac{s - 1}{2} - \frac{(-1)^s + 1}{4}, \]

which implies
Thus $B_v(t, s)$ is $\varphi_{[x,y,z,v]}$-colorable by Lemmas 4.3 and 4.5.

\[ \frac{k - s}{2} - \frac{(-1)^s + 1}{4} \leq |\varphi(v) \cap \varphi(z)| \leq \frac{s - 1}{2} - \frac{(-1)^s + 1}{4}. \]

4.2 | The proof of Theorem 1.6

Now we are ready to prove Theorem 1.6 restated below in terms of plane graph.

**Theorem 1.6.** For any odd integer $k \geq 5$, every plane graph of girth at least $k$ without cycles of length from $k + 1$ to $\left\lfloor \frac{2k}{3} \right\rfloor$ is fractional $\left( k : \frac{k-1}{2} \right)$-colorable.

**Proof.** Suppose, for a contradiction, that $G$ is a counterexample with $|V(G)| + |E(G)|$ minimized. □

**Claim 1.** $G$ is 2-connected. In particular, $\delta(G) \geq 2$.

**Proof of Claim 1.** Clearly, $G$ is connected. If $G$ is not 2-connected, then there exist proper induced subgraphs $G_1$ and $G_2$ of $G$ and a vertex $v \in V(G_2)$ such that $E(G) = E(G_1) \cup E(G_2)$ and $V(G_1) \cap V(G_2) = \{v\}$. By the minimality of $G$, $G_1$ has a fractional $\left( k : \frac{k-1}{2} \right)$-coloring $\varphi_1$ and $G_2$ has a fractional $\left( k : \frac{k-1}{2} \right)$-coloring $\varphi_2$. Exchange the colors if needed such that $\varphi_1(v) = \varphi_2(v)$, then $\varphi_1$ and $\varphi_2$ combine to become a $\left( k : \frac{k-1}{2} \right)$-coloring of $G$, which is a contradiction. □

For $1 \leq t \leq \frac{k-1}{2}$, let $F_v(t, s)$ be a graph obtained from a bull-necklace $B_v(t, s)$ with end vertices $x, y, z$ by joining a new $(x, y)$-path of length $k - 2t$ connecting $x$ and $y$, where the vertices in the new $(x, y)$-path may have arbitrary degrees in $G$. That is, $F_v(t, s)$ consists of a $k$-cycle $C_v$ and a necklace $N(v, z)$ with a common vertex $v$, where in the $k$-cycle $C_v$ there exist two $(t - 1)$-threads, one is from $x$ to $v$ and the other is from $y$ to $v$.

**Claim 2.** $G$ contains no $F_v(t, s)$ with $1 \leq t \leq \frac{k-1}{2}$ and $t + s \geq k$, where $s = d(v, z)$.

**Proof of Claim 2.** Suppose to the contrary that $G$ contains an $F_v(t, s)$ with $1 \leq t \leq \frac{k-1}{2}$ and $t + s \geq k$. By the minimality of $G$, $G - (V(N(v, z)) \setminus \{v, z\})$ has a $\left( k : \frac{k-1}{2} \right)$-coloring $\varphi$. If $2t \leq \frac{k-1}{2}$, then $d_G(x, y) = 2t$, and if $2t \geq \frac{k+1}{2}$, then $d_G(x, y) = k - 2t$. By Lemma 4.2, we always have $|\varphi(x) \cap \varphi(y)| = \frac{k-1-2t}{2}$. Let $\varphi'$ be the restriction of $\varphi$ to $G - (V(B_v(t, s)) \setminus \{x, y, z\})$. As $|\varphi'(x) \cap \varphi'(y)| = \frac{k-1-2t}{2}$, $B_v(t, s)$ is $\varphi_{[x,y,z]}$-colorable by Lemma 4.7. That is, $\varphi'$ can be extended to a $\left( k : \frac{k-1}{2} \right)$-coloring of $G$, which is a contradiction. □
From $G$, we obtain a subgraph $G'$ as follows: for each facial $k$-cycle $C$ of $G$, if there exists a 2-vertex in $C$, then we delete all the 2-vertices of a longest thread of $C$. Clearly, the obtained graph $G'$ is a plane graph of girth at least $k$, and contains no cycles of length from $k + 1$ to $\left\lceil \frac{2k}{3} \right\rceil$. Furthermore, each facial $k$-cycle of $G'$ contains no 2-vertices. It is easy to see that $G'$ has minimal degree at least 2 by its construction.

Let $T(v, x)$ be a $(v, x)$-thread of $G'$ and let $u = N_{G'}(v) \cap V(T(v, x))$. If there exists $w \in N_{G'}(v) \setminus \{u\}$ such that $uv$ and $vw$ are in a common $k$-cycle of $G$, then we say $v$ is a bad end vertex of $T(v, x)$; otherwise, $v$ is called a good end vertex of $T(v, x)$.

**Claim 3.** Let $T(v, x)$ be a $(v, x)$-thread of $G'$ with a good end vertex $v$. Then $d_G(v, x) \leq k - 1$.

**Proof of Claim 3.** Suppose to the contrary that $d_G(v, x) \geq k$. If $x$ is also a good end vertex of $T(v, x)$, then the thread $T(v, x)$ in $G'$ corresponds to a necklace $H$ with end vertices $v, x$ in $G$. By the minimality of $G$, $G - (V(H) \setminus \{v, x\})$ has a $(k : \frac{k - 1}{2})$-coloring $\varphi$. Since $d_G(v, x) = d_G(x, v) \geq k$ by construction, $H$ is $\varphi_{\{v, x\}}$-colorable by Corollary 4.6. That is, $\varphi$ can be extended to a $(k : \frac{k - 1}{2})$-coloring of $G$, which is a contradiction.

Therefore we assume that $x$ is a bad end vertex of $T(v, x)$. By definition, let $y = N_{G'}(x) \cap V(T(v, x))$ such that there exists a $k$-cycle $C_x$ of $G$ containing both $xy$ and $xz$ for some $z \in N_{G'}(x) \setminus \{y\}$. Let $w \in V(C_x) \cap V(T(v, x))$ such that $d_G(x, w)$ as large as possible. By the construction of $G'$, we obtain that the $(x, w)$-thread in $G$ satisfies $d(x, w) \leq \frac{k - 1}{2}$, and that there is a deleted thread from $w$ to some vertex, say $(w, u)$-thread, in the $k$-cycle $C_x$ such that $d(u, w) \geq d(x, w)$. Thus $G$ contains a bull $B_w(d(w, x), d(w, v))$, which provides an $F_w(d(w, x), d(w, v))$ in $G$, contradicting to Claim 2.

**Claim 4.** $G'$ contains no $(\frac{3k - 3}{2})^+$-thread.

**Proof of Claim 4.** Suppose to the contrary that $G'$ has a $(\frac{3k - 3}{2})^+$-thread $T(v, x)$. Then $d_G(v, x) = d_G(x, v) \geq \frac{3k - 1}{2}$. By Claim 3, $v$ and $x$ are both bad end vertices of $T(v, x)$. Let $u$ be the neighbor of $v$ in $T(v, x)$. Then there exists a $k$-cycle $C_v$ of $G$ containing both $vu$ and $vw$ for some $w \in N_{G'}(v) \setminus \{u\}$. Let $y \in V(C_v) \cap V(T(v, x))$ such that $d_G(v, y)$ is as large as possible. By the construction of $G'$, we have $d(v, y) \leq \frac{k - 1}{2}$, and so $d(x, y) = d(v, x) - d(v, y) \geq k$. Now $T(v, x) - (V(T(v, x)) \cap V(C_v) \setminus \{y\})$ is an $(x, y)$-thread from $x$ to $y$ in $G'$ with $y$ being a good end vertex, which is a contradiction to Claim 3.

**Claim 5.** $G'$ contains no $(k_1, k_2, k_3)$-thread such that $k_1 + k_2 + k_3 \geq \frac{11k - 17}{3}$.

**Proof of Claim 5.** Suppose to the contrary that $G'$ has a $(k_1, k_2, k_3)$-vertex $v$ such that $k_1 + k_2 + k_3 \geq \frac{11k - 17}{3}$ with end vertices $x, y, z$. Then $d_G(v, x) + d_G(v, y) + d_G(v, z) \geq \frac{11k - 8}{3}$. 
If there are no two edges incident to \( v \) in \( G' \) lying in a common \( k \)-cycle of \( G \), then we may assume, without loss of generality, that \( d_{G'}(v, x) + d_{G'}(v, y) + d_{G'}(v, z) > k \). Now the \((x, v)\)-thread from \( x \) to \( v \) has length at least \( k \) with \( v \) as a good end vertex, which is a contradiction to Claim 3.

If there exist two edges incident to \( v \) in \( G' \) containing in a \( k \)-cycle \( C_v \) of \( G \), then we may suppose that \( C_v \) has no common vertex other than \( v \) with the \((v, z)\)-thread \( T(v, z) \). Thus \( v \) is a good end vertex of the \((v, z)\)-thread \( T(v, z) \), and so \( d_{G'}(v, z) \leq k - 1 \) by Claim 3. Let \( u \) be the common vertex of \( C_v \) and the \((v, x)\)-thread \( T(v, x) \) such that \( d_{G'}(v, u) \) as large as possible, and let \( w \) be the common vertex of \( C_v \) and the \((v, y)\)-thread \( T(v, y) \) such that \( d_{G'}(v, w) \) as large as possible. By the construction of \( G' \), we have \( d_{G'}(v, u) + d_{G'}(v, w) \leq \frac{2k}{3} \), since the deleted \((u, w)\)-thread is a longest thread in \( C_v \). Now we have

\[
d_{G'}(x, u) + d_{G'}(y, w) = d_{G'}(v, x) + d_{G'}(v, y) + d_{G'}(v, z) - (d_{G'}(v, u) + d_{G'}(v, w))
\geq \frac{11k - 8}{3} - \frac{2k}{3} - (k - 1) = 2k - \frac{5}{3}.
\]

Thus \( \max\{d_{G'}(x, u), d_{G'}(y, w)\} \geq k \), say \( d_{G'}(x, u) \geq k \). Hence the \((x, u)\)-thread \( T(x, u) \) is of length at least \( k \) with \( u \) as a good end vertex, a contradiction to Claim 3. □

Now we complete the proof by a discharging method on \( G' \).
Let \( F(G') \) be the set of faces of \( G' \). From Euler Formula, we have

\[
\sum_{v \in V(G')} \left( \frac{k - 2}{2} d_{G'}(v) - k \right) + \sum_{f \in F(G')} (d_{G'}(f) - k) = -2k. \tag{3}
\]

Assign an initial charge \( ch_0(v) = \frac{k - 2}{2} d_{G'}(v) - k \) for each \( v \in V(G') \), and \( ch_0(f) = d_{G'}(f) - k \) for each \( f \in F(G') \). Hence the total charge is \(-2k\) by Equation (3).

We redistribute the charges according to the following rules.
(R1) Every \( \left\lceil \frac{22k}{3} \right\rceil^+ \)-face of \( G' \) gives charge \( \frac{19}{22} \) to each of its incident vertices.
(R2) Every \( 3^+ \)-vertex of \( G' \) gives charge \( \frac{3}{22} \) to each of its weakly adjacent 2-vertices.
Let \( ch \) denote the charge assignment after performing the charge redistribution using rules (R1) and (R2).

Claim 6. \( ch(f) \geq 0 \) for \( f \in F(G') \).

**Proof of Claim 6.** Clearly, each \( k \)-face \( f \) has charge \( ch(f) = ch_0(f) = 0 \). Each \( \left\lceil \frac{22k}{3} \right\rceil^+ \)-face \( f \) sends charge \( \frac{19}{22} \) to each incident vertices by (R1). So \( ch(f) = ch_0(f) - \frac{19}{22} d_{G'}(f) = (d_{G'}(f) - k) - \frac{19}{22} d_{G'}(f) = \frac{3}{22} d_{G'}(f) - k \geq 0 \) as \( d_{G'}(f) \geq \left\lceil \frac{22k}{3} \right\rceil \). □
Claim 7. \( ch(v) \geq 0 \) for \( v \in V(G') \).

**Proof of Claim 7.** Let \( v \) be a vertex of \( G' \). Then \( d_{G'}(v) \geq 2 \) by Claim 1 and the construction of \( G' \).

First we assume \( d_{G'}(v) = 2 \). Then \( ch_0(v) = -2 \). By Claims 1 and 4, \( v \) is weakly adjacent to two \( 3^+ \)-vertex, and thus \( v \) receives charge \( \frac{3}{22} \times 2 \) by (R2). By (R1), \( v \) receives charge \( \frac{19}{22} \times 2 \) from its two incident faces. Hence \( ch(v) = -2 + \frac{3}{22} \times 2 + \frac{19}{22} \times 2 = 0 \).

Now we assume \( d_{G'}(v) \geq 3 \). Let \( t(v) \) be the number of \( 2 \)-vertices weakly adjacent to \( v \). Suppose \( v \) is adjacent to \( r(v) \) facial \( k \)-cycles. Since \( G' \) contains no cycles of length from \( k+1 \) to \( 2k \), any two \( k \)-cycles of \( G' \) have no edges in common, and thus \( r(v) \leq \frac{d_{G'}(v)}{2} - 1 \). By Claim 4 and by the construction of \( G' \), each thread incident to \( v \) contains at most \( \frac{3k-3}{2} - 1 \) \( 2 \)-vertices and each \( k \)-cycle contains no \( 2 \)-vertices, and so we have \( t(v) \leq \frac{3k-5}{2} (d_{G'}(v) - 2r(v)) \). By (R1), \( v \) receives \( \frac{19}{22} (d_{G'}(v) - r(v)) \) from its incident faces. By (R2), \( v \) sends \( \frac{3}{22} \) to each of its weakly adjacent \( 2 \)-vertices. Therefore, we have

\[
ch(v) = \left( \frac{k-2}{2} d_{G'}(v) - k \right) + \frac{19}{22} (d_{G'}(v) - r(v)) - \frac{3}{22} t(v). \tag{4}
\]

Assume that \( d_{G'}(v) \geq 4 \). By Equation (4), it follows from \( t(v) \leq \frac{3k-5}{2} (d_{G'}(v) - 2r(v)) \) that

\[
ch(v) \geq \frac{k-2}{2} d_{G'}(v) - k + \frac{19}{22} (d_{G'}(v) - r(v)) - \frac{3}{22} \times \frac{3k-5}{2} (d_{G'}(v) - 2r(v))
\]

\[
= \frac{13k + 9}{44} d_{G'}(v) - k + \frac{9k - 34}{22} r(v)
\]

\[
\geq \frac{13k + 9}{44} d_{G'}(v) - k
\]

\[
\geq \frac{13k + 9}{44} \cdot 4 - k
\]

\[
= \frac{2k + 9}{11} > 0.
\]

Assume instead that \( d_{G'}(v) = 3 \). Then \( ch_0(v) = \frac{k-6}{2} \) and \( r(v) \leq 1 \). If \( r(v) = 1 \), then \( t(v) \leq \frac{3k-5}{2} \) by Claim 4. Thus by Equation (4) we have \( ch(v) = \frac{k-6}{2} + \frac{19}{22} \times 2 - \frac{3}{22} \times \frac{3k-5}{2} = \frac{13k - 41}{44} \geq \frac{6}{11} \). If \( r(v) = 0 \), then \( t(v) \leq \frac{1}{3} (11k - 17) \) by Claim 5. Thus by Equation (4) we have \( ch(v) = \frac{k-6}{2} + \frac{19}{22} \times 3 - \frac{3}{22} \times \frac{11k-17}{3} = \frac{4}{11} > 0 \). This proves Claim 7.

Combining Equation (3), Claims 6 and 7, we have

\[
-2k = \sum_{v \in V(G')} ch_0(v) + \sum_{f \in F(G')} ch_0(f) = \sum_{v \in V(G')} ch(v) + \sum_{f \in F(G')} ch(f) \geq 0,
\]

a contradiction. This contradiction finishes the proof of Theorem 1.6. \( \square \)
**CONCLUDING REMARKS**

In this paper, we obtain two Steinberg-type results on circular coloring and fractional coloring as Theorems 1.3 and 1.6. Improving the bound to \( f(p) \leq p(p - 2) \) would provide solutions to Conjecture 1.1 for \( t = p - 1 \) when \( p \geq 5 \) is a prime, and completely determining the value \( f(p) \) seems to be more challenging. Theorem 1.6 confirms the fractional coloring version of Conjecture 1.4 for \( p \geq 11 \), since \( \frac{2p}{3} \leq p(p - 2) \) when \( p \geq 11 \). In a follow-up paper [18], we also verify the remaining cases \( (p = 5, 7) \) of the fractional coloring version of Conjecture 1.4 with refined arguments and additional configurations. Those results provide evidence to Conjectures 1.1 and 1.4.

A nature question is to consider variations of Question 1.2 concerning odd cycles. However, naive odd cycle versions of Theorems 1.3 and 1.6 are false, that is, for any \( t > \frac{k - 1}{2} \), there exist planar graphs \( G \) of odd girth \( k \) without odd cycles of length from \( k + 2 \) to \( 2t + 1 \) satisfying \( \chi_c(G) \geq \chi_f(G) > \frac{2k}{k - 1} \). To see this, we construct a graph \( G \) by taking \( 2t \) disjoint copies of \( k \)-cycle, where each \( k \)-cycle contains two distinguished edges \( x_i y_i, y_i z_i \) for each \( i \in [2t] \), adding edges \( x_i y_{i+1}, z_i y_{i+1} \) for each \( i \in [2t - 1] \), and adding a new vertex \( v \) to connect edges \( v_{2t}, v_{2t}, v_1 \). See Figure 3 for the construction of \( G \). We claim that \( \chi_f(G) > \frac{2k}{k - 1} \). In fact, if \( \varphi \) is a fractional \( \left( ka, \frac{k - 1}{2}a \right) \)-coloring of \( G \), then it is easy to show, by an argument similar to Lemmas 4.1 and 4.2, that \( \varphi(x_i) \cap \varphi(z_i) = \frac{k - 1}{2}a - a \). This implies \( \varphi(y_i) = \varphi(y_{i+1}) \) for each \( i \in [2t - 1] \) and \( \varphi(y_{2t}) = \varphi(v) \), which indicates \( \varphi(y_1) = \varphi(v) \). But there is an edge \( y_1 v \) between \( y_1 \) and \( v \), a contradiction. Hence \( \chi_c(G) \geq \chi_f(G) > \frac{2k}{k - 1} \).

Two \( k \)-cycles are called adjacent if they share at least one common edge. Notice that the above-constructed graph \( G \) contains adjacent \( k \)-cycles. It would be possible to consider the following modified odd cycle versions without adjacent \( k \)-cycles.

**Question 5.1.** Does there exist a smallest number \( g(p) \) for each prime \( p \geq 3 \) such that every planar graph of odd girth \( p \) without adjacent \( p \)-cycles and without odd cycles of length from \( p + 2 \) to \( g(p) \) is \( C_p \)-colorable?

The results from [7,10,27] imply that \( g(3) = 7 \). It would be interesting to show the existence of \( g(p) \) for every prime \( p \geq 5 \). Furthermore, is it true that \( g(p) \leq f(p) + 1 \)?

A similar question arises for fractional coloring.

**FIGURE 3** Construction of \( G \) when \( t = 3 \) and \( k = 7 \)
**Question 5.2.** Does there exist a smallest number $h(k)$ for each odd integer $k \geq 3$ such that every planar graph of odd girth $k$ without adjacent $k$-cycles and without odd cycles of length from $k + 2$ to $h(k)$ is fractional $(k : \frac{k-1}{2})$-colorable?

From Theorem 1.6, it is plausible that $h(k)$ exists as a linear function of $k$.

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**REFERENCES**

1. H. L. Abbott and B. Zhou, *On small faces in 4-critical graphs*, Ars Combin. 32 (1991), 203–207.
2. K. Appel and W. Haken, *Every planar map is four colorable, I, discharging*, Illinois J. Math. 21 (1977), 429–490.
3. L. Beaudou, F. Foucaud, and R. Naserasr, *Homomorphism bounds and edge-colourings of $K_4$-minor-free graphs*, J. Combin. Theory Ser. B 124 (2017), 128–164.
4. J. A. Bondy and P. Hell, *A note on the star chromatic number*, J. Graph Theory. 14 (1990), 479–482.
5. O. V. Borodin, *Structural properties of plane graphs without adjacent triangles and an application to 3-colorings*, J. Graph Theory. 21 (1996), 183–186.
6. O. V. Borodin, A. N. Glebov, A. Raspaud, and M. R. Salavatipour, *Planar graphs without cycles of length from 4 to 7 are 3-colorable*, J. Combin. Theory Ser. B. 93 (2005), 668–673.
7. O. V. Borodin, A. N. Glebov, M. Montassier, and A. Raspaud, *Planar graphs without 5- and 7-cycles and without adjacent triangles are 3-colorable*, J. Combin. Theory Ser. B. 99 (2009), 303–311.
8. O. V. Borodin, S.-J. Kim, A. V. Kostochka, and D. B. West, *Homomorphisms from sparse graphs with large girth*, J. Combin. Theory Ser. B. 90 (2004), 147–159.
9. A. L. Cauchy, *Recherches sur les nombres*, J. Éc. Polytech. 9 (1813), 99–123.
10. V. Cohen-Addad, M. Hebdige, D. Král’, Z. Li, and E. Salgado, *Steinberg’s conjecture is false*, J. Combin. Theory Ser. B. 122 (2017), 452–456.
11. D. Cranston and J. Li, *Circular flows in planar graphs*, SIAM J. Discrete Math. 34 (2020), 497–519.
12. H. Davenport, *On the addition of residue classes*, J. Lond. Math. Soc. 10 (1935), 30–32.
13. Z. Dvořák and L. Postle, *Density of 5/2-critical graphs*, Combinatorica. 37 (2017), 863–886.
14. W. Goddard and H. Xu, *Fractional, circular, and defective coloring of series-parallel graphs*, J. Graph Theory. 81 (2016), 146–153.
15. H. Grötzsch, *Ein dreifarbensatz für dreikreisfreie netze auf der kugel*, Math. Natur. Reihe. 8 (1959), 109–120.
16. P. Hell and X. Zhu, *The circular chromatic number of series-parallel graphs*, J. Graph Theory. 33 (2000), 14–24.
17. A. Hilton, R. Rado, and S. Scott, *A (<5)-colour theorem for planar graphs*, Bull. Lond. Math. Soc. 5 (1973), 302–306.
18. X. Hu and J. Li, *Fractional coloring planar graphs under Steinberg-type conditions*, submitted.
19. F. Jaeger, *On circular flows in graphs*, Finite and Infinite Sets (Eger, 1981) Colloquia Mathematica Societatis, J. Bolyai. 37 (1984), 391–402.
20. J. Li, Y. Wu, and C.-Q. Zhang, *Circular flows via extended Tutte orientations*, J. Combin. Theory Ser. B. 145 (2020), 307–322.
21. L. M. Lovász, C. Thomassen, Y. Wu, and C.-Q. Zhang, *Nowhere-zero 3-flows and modulo k-orientations*, J. Combin. Theory Ser. B. 103 (2013), 587–598.
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