Magnetic Catalysis in Quantum Electrodynamics

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Abstract

We derive the (Wilsonian) low energy effective Lagrangian for Quantum Electrodynamics under external constant magnetic field by integrating out all electrons except those in the lowest Landau level. We find the one-loop effective Lagrangian contains a marginal four-Fermi interaction with anomalous dimension, \( \frac{(\ln 2)^2}{2\pi^2} e^4 \). Renormalization group analysis shows that the four-Fermi interaction will break chiral symmetry in QED if the external magnetic field is extremely strong, \( B > 10^{42} \) gauss, or if the Landau gap, \( \sqrt{|eB|} > 6.5 \times 10^{10} \) GeV.

11.30.Rd, 12.20.-m, 11.10.Gh, 12.20.Ds
Recently, it has been shown that external (constant) magnetic field acts as catalysis of dynamical symmetry breaking in quantum electrodynamics with or without a non-renormalizable four-Fermi interaction [1–3]. It is then generalized to QCD under external chromo-magnetic field [4]. The essence of this magnetic catalysis is that electrons of energy much less than the Landau gap ($E \ll \sqrt{|eB|}$) are effectively 1 + 1 dimensional, since the quantum fluctuations perpendicular to the external magnetic field are suppressed by $\frac{E}{\sqrt{|eB|}}$.

Due to this dimensional reduction, the critical coupling for dynamical symmetry breaking becomes zero. Namely, dynamical symmetry breaking occurs for any arbitrarily weak attraction. This has been shown either by calculating the vacuum energy or by solving the Schwinger-Dyson equations for the fermion two-point function.

On the other hand, dynamical symmetry breaking is believed to occur when particles interact strongly as in QCD. Indeed, it is found that four-Fermi interaction of electrons in the lowest Landau level (LLL) are marginal and the $\beta$-function of four-Fermi coupling is negative for attractive interaction, leading to strong attraction at low energy [3]. Thus, the result of Schwinger-Dyson analysis can be understood in terms of renormalization group (RG). But, it was unclear how the weak electromagnetic interaction leads to dynamical symmetry breaking when the four-Fermi interaction is absent, as shown in the Schwinger-Dyson analysis [1–3], since the Coulomb interaction of electrons remains weak at low energy [4]. In this paper, we attempt to understand the dynamical symmetry breaking in pure QED under external magnetic field in terms of RG analysis. In this attempt, we find that, if one integrates out electrons in the higher Landau levels, there will be a new low-energy effective operator for electron-photon coupling at tree level, among others, and this operator will generate a four-Fermi interaction at one-loop, which becomes strong in infrared region.

Magnetic catalysis is a very interesting phenomenon and has potential applications in astrophysics such as the cooling process of neutron stars or particle interactions in early universe under premodial magnetic field [3]. In order for the magnetic catalysis to operate, the external magnetic field has to be strong enough so that the average energy of charged particles is much smaller than the Landau gap. Namely, the gap has to be bigger than the rest mass energy, $\sqrt{|eB|} \gg m$, or, in the early universe, the temperature if the particles are relativistic, $\sqrt{|eB|} \gg T$. Therefore the electrons will be catalyzed only when the external magnetic fields are stronger than a critical field, $B > B_c$, where $B_c = \frac{m^2}{|e|} \simeq 10^{14}$ gauss (G). If electrons are massless, $B_c = 0$ and therefore for any weak external magnetic field they will be catalyzed to get a dynamical mass, $m_{dy} \simeq \sqrt{|eB|}e^{-1/g}$, where $g$ is the coupling at the cut-off scale, $\sqrt{|eB|}$. But, the effect is relevant only at distance larger than $1/m_{dy}$, which is enormously large for weak field. In this paper, we consider electrons of energy larger than the rest mass energy but smaller than the Landau gap, $m < E < \sqrt{|eB|}$, and neglect the electron mass.

As derived by Schwinger [6], the electron propagator in a constant external magnetic field is given as

$$S(x, y) = \tilde{S}(x - y) \exp \left[ \frac{ie}{2} (x - y)^\mu A^\mu_{\text{ext}}(x + y) \right], \quad (1)$$

with the Fourier transform of $\tilde{S}$,
\[ \mathcal{S}(k) = i e^{-k_\perp^2/|eB|} \sum_{n=0}^{\infty} (-1)^n \frac{D_n(eB, k)}{k^2 - 2|eB|n}, \]  

(2)

where \( k_\perp \) is the 3-momentum perpendicular to the direction of the external magnetic field, \( k_\parallel = k - k_\perp \), and

\[
D_n(eB, k) = 2k_\parallel \left[ P_- L_n \left( \frac{2k_\perp^2}{|eB|} \right) - P_+ L_{n-1} \left( \frac{2k_\perp^2}{|eB|} \right) \right] + 4k_\perp L_{n-1}^1 \left( \frac{2k_\perp^2}{|eB|} \right).
\]

(3)

\( L_n^\alpha \) are the associate Laguerre polynomials and \( P_+ (P_-) \) is the projection operator which projects out the electrons of spin (anti-) parallel to the magnetic field direction. For \( \vec{B} = B \hat{z} \), \( 2P_\pm = 1 \pm i \gamma^1 \gamma^2 \text{sign}(eB) \).

To find the Landau level, we need to solve the following eigenvalue equation,

\[
[\vec{\alpha} \cdot (\vec{p} - e\vec{A}^{\text{ext}})] \Psi = E \Psi.
\]

(4)

We take \( \vec{A}^{\text{ext}} = (-\frac{B}{2} y, \frac{B}{2} x, 0) \). The eigenvalues are indexed by collective index \( A = (\alpha, \beta, n, k_\parallel) \) and given by

\[
E_A = \alpha \sqrt{k_\parallel^2 + 2|eB|n}
\]

(5)

where \( \alpha = \pm \) denotes the sign of the energy, \( \beta = \pm \frac{1}{2} \) is the spin component along the magnetic field, and the quantum number \( n \) is given by

\[
2n = 2n_r + 1 + |m_\parallel| - \text{sign}(eB)(m_\parallel + 2\beta).
\]

(6)

\( n \) is a nonnegative integer that labels the Landau level. Here \( n_r \) is the number of nodes of radial eigenfunction, \( m_\parallel \) is the angular momentum of the eigenfunction. The eigenfunction is

\[
U_A = N_A e^{ik_\parallel z} e^{im_\parallel \phi} r_\perp^{m_\parallel} L_{m_\parallel}(|eB|r_\perp^2/2) \exp \left(-|eB|r_\perp^2/4\right) u_{\alpha,\beta},
\]

(7)

where \( N_A \) is the normalization and the radial distance \( r_\perp = \sqrt{x^2 + y^2} \). \( L_n^m(x) \) is the associated Laguerre polynomial and \( u_{\alpha,\beta} = \chi_\alpha \otimes \eta_\beta \) is the eigenvector of \( \sigma_3 \otimes \sigma_3 \) where two \( \sigma_3 \) correspond to the energy and the spin, respectively.

Since the eigenfunctions form an orthonormal basis, one can expand the electron field as

\[
\Psi(x) = \int \sum_A \psi_A(t) U_A(x).
\]

(8)

Now, we divide the electron field into two groups

\[
\Psi = \psi + \Psi_{n \neq 0},
\]

(9)

where \( \psi \) contains only the lowest Landau level \( (n = 0) \) in the mode expansion in Eq. (8) and \( \Psi_{n \neq 0} \) contains the rest.

The low energy effective Lagrangian is obtained by integrating out the higher modes,

\[
\exp \left( i \int d^4x \mathcal{L}_{\text{eff}}(\psi, A_\mu) \right) = \int \mathcal{D}\Psi_{n \neq 0} \exp \left( i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^2 + \bar{\Psi} i \mathcal{D} \Psi \right] \right),
\]

(10)
where the covariant derivative \( D_\mu = \partial_\mu + ieA_\mu + ieA^\text{ext}_\mu \). In practice, one derives the effective Lagrangian by matching all the “one light particle irreducible” (1LPI) diagrams systematically in a loop expansion.

At tree level, \( \Psi_{n\neq 0} \) exchange generates a new photon-electron vertex in the effective theory (Fig. 1):

\[
\Gamma_n^0(p_1, p_2, p_1', p_2') = ie^2\bar{\psi}(p_1)A(p_2)e^{-k^2/eB} \sum_{n=1}^{\infty} (-1)^n \frac{D_n(eB, k)}{k^2 - 2|eB|n} A(p_1')\psi(p_2') \tag{11}
\]

\[
= i\frac{e^2}{|eB|}\bar{\psi} A \left[ k||i\gamma_1\gamma_2 \ln 2 \text{sign}(eB) + k_\perp \right] A\psi + O \left( \frac{k^2}{|eB|} \right), \tag{12}
\]

where \( k = p_1 + p_2 = p_1' + p_2' \). In the second line, we have expanded the electron propagator in powers of momentum and used \( L_n - 1(0) = 1, L_1 - 1(0) = n \). The tree-level effective Lagrangian is then

\[
\mathcal{L}_{0\text{eff}} = -\frac{1}{4} F^2_{\mu\nu} + \bar{\psi}(iD)\psi - \frac{e^2}{2|eB|}\bar{\psi}\gamma^\alpha A_\alpha \tilde{\gamma}^\mu i \partial_\mu \gamma^\beta \psi A_\beta + \cdots, \tag{13}
\]

The ellipsis refers to operators containing more photons and higher power of derivatives, and

\[
\tilde{\gamma}^\mu = \begin{cases} 
  i\gamma^\mu \gamma^1 \gamma^2 \ln 2 \text{sign}(eB) & \text{if } \mu = 0, 3 \\
  \gamma^\mu & \text{if } \mu = 1, 2. 
\end{cases} \tag{14}
\]

To match at one-loop (see Fig. 2), we have to consider graphs with two, three, four, and more external fields to get

\[
\mathcal{L}_{1\text{eff}}^0 = -\frac{1}{4}(1 + a_1) F^2_{\mu\nu} + (1 + b_1)\bar{\psi}(iD)\psi - (1 + c_1)\frac{ie^2}{|eB|}\bar{\psi}A^\gamma_{\mu} \tilde{\gamma}^\mu \partial_\parallel A\psi
\]

\[
- \frac{ie^2}{|eB|}\bar{\psi}A\tilde{\gamma}^\mu \partial_\perp A\psi + \frac{g_1}{|eB|} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right] + \cdots, \tag{15}
\]

where the coefficients \( a_1, b_1, c_1, g_1 \) are dimensionless and of order \( e^2 \). The ellipsis denotes terms with more external fields and derivatives. Note that the new electron-photon effective couplings get different corrections, depending on the component of the total momentum carried by electron and photon.

We find that the four-Fermi operator arises, in a chirally invariant form, at one-loop when we match four-Fermion amplitudes both in the full theory and in the effective theory. The one-loop amplitude for four-Fermi interaction in the full theory is

\[
A_4 = i\frac{3f e^4}{32\pi^2|eB|} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right]. \tag{16}\]

The amplitude should converge since the Feynman diagram is finite and, in general, the dimensionless number \( f \) will be of order one.

In the effective theory, the one-loop amplitude is

\[
i\frac{g_1}{4|eB|} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right], \tag{17}\]
with
\[ g_1 = \frac{e^4 (\ln 2)^2}{4\pi^2} \left[ -\frac{1}{\epsilon} - 1.34 + \gamma - \frac{\pi^2}{8} + \ln \frac{|eB|}{\mu^2} \right] + \delta g_1, \] (18)

where \( \delta g_1 \) is the one-loop counter term for the four-Fermi amplitude to remove the pole divergence and to ensure the matching at the cutoff scale, \( \mu = \sqrt{|eB|} \), which is
\[ g_1(\mu = \sqrt{|eB|} - 0) = \frac{3f e^4}{8\pi^2}, \] (19)

After matching the operators in the effective theory with those in the full theory at \( \sqrt{|eB|} \), we run the operators down to the scale of interest. We find that \( a_1, b_1 \) do not scale as we expect from the Ward-Takahashi identity because the electric charge does not run at low energy due to the dimensional reduction. But, \( c(\mu) = 1 + c_1 + \cdots \) and \( g(\mu) = g_1 + \cdots \) do. To get the right value for the strength of the four-Fermi operator, we have to rescale the fields to get a conventionally normalized kinetic term,
\[ \psi(x) \rightarrow \psi'(x) = \sqrt{1 + b_1}\psi(x), \] (20)

and take into account of the scale dependence of \( c_1(\mu) \). But, since these are higher-order corrections to \( g_1 \), the coefficient of the four-Fermi operator at \( \mu = \sqrt{|eB|} \) is just \( \frac{3f e^4}{8\pi^2} \) in the leading order. As shown in [3], this four-Fermi operator is marginal: Under the scale transformation, \( E \rightarrow sE \), the electron should transform as \( \psi \rightarrow s^{1/2}\psi \), since the radial dependence of \( \psi \) is fixed as \( \exp(-r_{\perp}^2|eB|/4) \) and therefore the coordinate \( x, y \) should not scale. The dimensional parameter, \( |eB| \), in front of the operator will cancel out upon the momentum integration in perturbation theory, leaving the dimensionless parameter \( g = g_1 + O(\hbar^2) \) as the expansion parameter.

Since the energy dispersion of photon is \( E = \pm \sqrt{k_{\|}^2 + k_{\perp}^2} \), photon with momentum parallel to the magnetic field \( (k_{\perp} = 0) \) has the scaling dimension 0 under the scale transformation, \( k_{\|} \rightarrow sk_{\|} \) and \( k_{\perp} \rightarrow k_{\perp} \), while photon with nonvanishing \( k_{\perp} \) has the scaling dimension 1. Therefore, the new photon-electron coupling is marginal if the photon carries no perpendicular momentum component. But, it turns out that its anomalous dimension is negative, \( \mu \frac{d}{d\mu} c_1(\mu) = e^2 \ln 2/\pi^2 > 0 \). Therefore, the coupling becomes irrelevant if we include the one-loop correction. On the other hand, the four-Fermi interaction becomes relevant at quantum level. Below the cutoff scale, the renormalization group equation for \( g_1 \),
\[ \mu \frac{d}{d\mu} g_1 = -\frac{(\ln 2)^2}{2\pi^2} e^4, \] (21)

which is very small. Hence, \( g_1 \) remains almost constant as the scale changes. Integrating this equation from \( \sqrt{|eB|} \) to \( m_e \) gives
\[ g_1(m_e) = g_1(\sqrt{|eB|}) + 4\alpha^2(\ln 2)^2 \ln \frac{|eB|}{m_e^2}, \] (22)

where \( \alpha = e^2/4\pi \) is the fine-structure constant at scale \( \sqrt{|eB|} \).

In this picture, the QED coupling runs down till the Landau gap scale. Below the scale it does not run, but a new marginal four-Fermi coupling appears due to quantum effects.
of the higher level modes and gets enhanced at low energy. In order for the four-Fermi coupling to break chiral symmetry, the coupling has to run extremely long and become of order 1 at \( \mu \approx m_e \). To find the lower bound for the Landau gap that allows sufficient evolution for the four-Fermi interaction to break chiral symmetry, we need to know the value of the fine-structure constant, \( \alpha \), at the Landau gap scale, which is given for \( \sqrt{|eB|} > M_Z \) as [4,5]

\[
\frac{1}{\alpha(\sqrt{|eB|})} = \frac{1}{\alpha(M_Z)} - \frac{1}{2\pi \cdot \frac{14}{15} \ln \left( \frac{\sqrt{|eB|}}{M_Z} \right)},
\]

(23)

where \( \alpha(M_Z) = 0.00782 \), the Z mass, \( M_Z = 91.175 \pm 0.021 \text{ GeV} \) and we assume the standard model particle content. Combining Eq.’s (22) and (23), we find that \( \alpha(\sqrt{|eB|}) \approx 1/124.8 \) and the low energy effective four-Fermi coupling becomes of order one if \( |eB| > 1.6 \times 10^{28} m_e^2 \approx 4.2 \times 10^{21} \text{ GeV}^2 \), or \( B > 10^{42} \text{ G} \). (Note that in the effective theory the value of electric charge is given by the value at \( \mu = \sqrt{|eB|} \).)

Therefore, the effective four-Fermi interaction will break the chiral symmetry of QED if \( B > 10^{42} \text{ G} \), which is much stronger than any known magnetic field in nature. The dynamical mass will be then

\[
m_{\text{dyn}} \approx \sqrt{|eB|} \exp \left( -\frac{2}{(\ln 2)^2 \alpha^2} \right).
\]

(24)

Note that this result is different from the one obtained from Schwinger-Dyson analysis, \( m_{\text{dyn}} \approx \sqrt{|eB|} \exp \left( -\frac{x^2}{\sqrt{2|e|}} \right) \). This is expected, since, according to RG analysis, the relevant operator needed for infrared instability occurs by quantum effects of electrons in the higher Landau levels, which is higher order in coupling constant expansion, while the Schwinger-Dyson analysis shows dynamical mass is generated solely by the electrons in the lowest Landau level. But, in the Schwinger-Dyson analysis, one has to integrate the loop momentum over all ranges. Hence, the effective four-Fermi interaction may not be negligible if one includes the running effect of the coupling, going beyond the ladder approximation. Therefore, it will be interesting to solve the Schwinger-Dyson equations in the effective Lagrangian derived in this paper.

In conclusion, we derive the (one-loop) low energy effective Lagrangian for Quantum Electrodynamics under a constant external magnetic field by integrating out the higher Landau level modes \( (n \neq 0) \). We find that the effective Lagrangian includes a new electron-photon coupling which generates a marginal four-Fermi interaction at one-loop level. Since the anomalous dimension of the four-Fermi interaction is positive though very small, about \( 2.2 \times 10^{-4} \), it gets enhanced at low energy. We find that the effective four-Fermi interaction will break chiral symmetry if the external magnetic field is extremely strong, \( B > 1.6 \times 10^{28} m_e^2 / |e| \approx 10^{42} \text{ G} \).

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FIGURES

\[ \begin{array}{c}
\text{FIG. 1. Tree level matching condition. Diagram on the left is in the full theory, while the one on the right is in the effective theory. The heavy line corresponds to electron in the higher Landau levels, light lines LLL electrons, and wiggly lines photons; number beneath the vertex counts the loop order.} \\
\end{array} \]
FIG. 2. One-loop matching conditions