Totally Nondegenerate Models and Standard Manifolds in CR Dimension One

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Abstract
It is shown that two Levi–Tanaka and infinitesimal CR automorphism algebras, associated with a totally nondegenerate model of CR dimension one are isomorphic. As a result, the model surfaces are maximally homogeneous and standard. This gives an affirmative answer in CR dimension one to a certain question formulated by Beloshapka.

Keywords CR manifolds · Total nondegeneracy · Standard manifolds · Tanaka prolongation

Mathematics Subject Classification 32V40 · 22F30

1 Introduction

The concept of standard manifolds is initiated for the first time by the Japanese mathematician Tanaka in Ref. [13] in the framework of differential systems. Roughly speaking, let $\mathfrak{m} := \bigoplus_{i < 0} \mathfrak{m}_i$ be a fundamental graded algebra (see below for definitions) and let $M(\mathfrak{m})$ be the connected and simply connected Lie group associated with it. Then, the subspace $\mathfrak{m}_{-1}$ of $\mathfrak{m}$ defines a left invariant regular differential system on this Lie group which Tanaka called it by the standard differential system of type $\mathfrak{m}$. Moreover, he called $M(\mathfrak{m})$ by the standard manifold of this type.

When $\mathfrak{m}_{-1} := T^c M$ is the distribution associated with the regular CR structure of a finite type CR manifold $M$, then it is possible to extend it by successive Lie brackets to a

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certain finite graded fundamental algebra \( m := \bigoplus_{\mu \leq i \leq -1} m_i \). As is quite customary in CR geometry, one may ask about the structure of the Lie algebra \( \text{aut}_{CR}(M) \) of infinitesimal CR automorphisms of \( M \). Tanaka showed that (see [13, Proposition 10.7]) if \( M \) is a standard CR manifold of the type \( m \), then the desired Lie algebra \( \text{aut}_{CR}(M) \) is actually the Levi–Tanaka prolongation of \( m \) which, roughly speaking, is the maximal transitive extension of \( m \) containing—as its zero component—of all derivations \( d : m \to m \) with respect to the associated complex structure map.

In Ref. [11], Naruki studied the hulls of holomorphy of standard real manifolds of the second type, namely those of the type \( m := m_{-1} \oplus m_{-2} \). He also considered the holomorphic extension problem for these manifolds. Moreover, Medori and Nacinovich [6,7] considered standard homogeneous CR manifolds, where their types are certain Levi–Tanaka prolongation \( \mathcal{G} \) of some pseudocomplex graded Lie algebras \( m \). In particular, they showed that [7, Theorem 4.3] if one considers \( M \) as the homogeneous manifold obtained by the quotient of the associated Lie group \( \text{Lie}(\mathcal{G}) \) by \( \text{Lie}(\mathcal{G}_+) \), where \( \mathcal{G}_+ \) is the extended part of \( m \) in \( \mathcal{G} \), then it is possible to equip \( M \) with a natural complex structure so that \( \text{Lie}(\mathcal{G}) \) acts on \( M \) as its group of CR automorphisms and moreover \( M \) is a standard CR manifold of the type \( \mathcal{G} \).

Medori and Nacinovich also found in Ref. [6] a significant relation between maximal homogeneity and standardness in the class of nondegenerate CR manifolds. Indeed, they showed that these two notions are actually equivalent. More precisely, they show that every maximally homogeneous nondegenerate CR manifold, regular with the symbol algebra \( m \), is standard for the type of the Levi–Tanaka prolongation of \( m \).

On the other hand, Beloshapka [3] introduced his interesting model surfaces of totally nondegenerate CR manifolds. His approach in the construction of such models was quite analytical. The nice features of these models, gathered in Theorem 14 of this paper, may motivate one to seek about the standardness (and equivalently maximal homogeneity) of these models. Beloshapka, himself, formulated an open question about the relation between his models and standard manifolds established by Tanaka (cf. [3, Question 2]). Due to the fact that Tanaka proceeded along a geometric approach in his theory of transitive prolongations, then finding the desired relationship may help one to discover some more geometric features of Beloshapka’s totally nondegenerate models. Indeed, seeking the standardness of Beloshapka’s models can be regarded as finding some analytic-geometric interactions of these models.

Recently in Ref. [12], we considered biholomorphic equivalence problem between Beloshapka’s models in the specific CR dimension one. Our main approach in that work was actually the classical one of Cartan. We solved the mentioned equivalence problem and discovered as a result the precise structure of the infinitesimal CR automorphisms of the models under question. Our main goal in this paper is to employ the achieved results in Ref. [12] to show that in CR dimension one, all Beloshapka’s models are indeed standard—and hence maximally homogeneous, as well.

2 Preliminary Definitions and Results

For an arbitrary smooth real manifold \( M \), an even rank subbundle \( T^e M \subset TM \) is called an almost CR structure if it is equipped with a fiber preserving complex structure map.
$J : T^c M \rightarrow T^c M$ satisfying $J \circ J = -id$. In this case, $M$ is called an almost CR manifold of CR dimension $n := \frac{1}{2} \cdot (\text{rank} \, T^c M)$ and codimension $k := \dim M - 2n$. According to the principles in CR geometry [2,9], the complexified bundle $\mathbb{C} \otimes T^c M$ decomposes as

$$
\mathbb{C} \otimes T^c M := T^{1,0} M \oplus T^{0,1} M,
$$

where

$$
T^{1,0} M := \{ X - i J(X) : X \in T^c M \},
$$

and $T^{0,1} M = \overline{T^{1,0} M}$. By definition, $M$ is called a CR manifold with the CR structure $T^c M$ if $T^{1,0} M$ is involutive in the sense of Frobenius. Such CR manifold is called a generic submanifold of $\mathbb{C}^{n+k}$ if it can be represented locally as the graph of some $k$ defining functions $f_1, \ldots, f_k$ with $\partial f_1 \wedge \cdots \wedge \partial f_k \neq 0$ (cf. [2]).

### 2.1 Totally Nondegenerate CR Manifolds of CR Dimension One

Let $M \subset \mathbb{C}^{1+k}$ be a real analytic generic submanifold of CR dimension one, codimension $k$ and, hence, of real dimension $2 + k$. As is known [2,8,9], the holomorphic subbundle $T^{1,0} M \subset \mathbb{C} \otimes T M$ can be generated by some single holomorphic vector fields $\mathcal{L}$. Set $D_1 := T^{1,0} M + T^{0,1} M$ and also define successively $D_j = D_{j-1} + [D_1, D_{j-1}]$ for $j > 1$. The iterated Lie brackets between the generators $\mathcal{L}$ and $\overline{\mathcal{L}}$ of $D_1$ induce a filtration:

$$
D_1 \subset D_2 \subset D_3 \subset \cdots
$$

on the complexified tangent bundle $\mathbb{C} \otimes T M$. Our distribution $D_1$ is minimal or bracket generating if for each $p \in M$, there exists some (minimal) integer $\rho(p)$ satisfying $D_{\rho(p)}(p) = \mathbb{C} \otimes T_p M$. Moreover, it is regular, if the already mentioned function $\rho$ is constant. In this case, the number $\rho := \rho(p)$ is called the degree of nonholonomy of the distribution $D_1$.

**Definition 2.1** (see [12, Definition 1.1]). An arbitrary (local) real analytic CR generic submanifold $M \subset \mathbb{C}^{1+k}$ of CR dimension one and codimension $k$ is totally nondegenerate of the length $\rho$ whenever the distribution $D_1 = T^{1,0} M + T^{0,1} M$ is regular with the minimum possible degree of nonholonomy $\rho$. In this case, we have the induced filtration:

$$
D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_\rho = \mathbb{C} \otimes T M,
$$

(2.1)

of the minimum possible length.

The notion of total nondegeneracy has a close connection with the theory of complex free Lie algebras (see [8,12,14]). Clearly, in order to achieve the minimum length $\rho$ of the above filtration, each subbundle $D_\ell \setminus D_{\ell-1}$ must contain maximum possible
number of independent iterated Lie brackets between \( L \) and \( \overline{L} \) of the length \( \ell \), taking it into account that respecting the skew-symmetry and Jacobi identity is unavoidable. This number, that we denote it by \( n_\ell \), is computable by means of the well-known Witt’s (inductive) formula (cf. [8, Theorem 2.6]). By a careful inspection on the above definition, one verifies that the length \( \rho \) of our \( k \)-codimensional totally nondegenerate CR manifold \( M \) is in fact the smallest integer \( \ell \) satisfying

\[
\text{rank}_\mathbb{R}(\mathbb{C} \otimes TM) = 2 + k \leq n_\ell. \tag{2.2}
\]

Moreover—as is the case with the free Lie algebras—no linear relation exists between the iterated brackets of \( L \) and \( \overline{L} \) in the lengths \( \leq \rho - 1 \), except those generated by skew symmetry and Jacobi identity. Notice that this rule entry into force until the length \( \rho - 1 \) and, by contrast, one may encounter unpredictable treatments of iterated brackets in the lengths \( \geq \rho \).

Thus, as a frame for the complexified bundle \( \mathbb{C} \otimes TM \), the distribution \( D_\rho \) is generated by the iterated brackets between \( L \) and its conjugation \( \overline{L} \) up to the length \( \rho \). Following [12], let us show this frame by

\[
\{ L_{1,1}, L_{1,2}, L_{2,3}, \ldots, L_{\rho,2+k} \}, \tag{2.3}
\]

where \( L_{1,1} := L, L_{1,2} := \overline{L} \) and \( L_{\ell,i} \) is the \( i \)th appearing independent vector field obtained as an iterated bracket of the length \( \ell \).

In Ref. [3], Beloshapka showed that after appropriate weight assignment to the complex coordinates \((z, w_1, \ldots, w_k)\), every \( k \)-codimensional totally nondegenerate submanifold of \( \mathbb{C}^{1+k} \) can be represented as the graph of some \( k \) real analytic defining functions (cf. [12, Theorem 1.1]):

\[
\begin{align*}
  w_1 &= \Phi_1(z, \overline{z}) + O([w_1]), \\
  &\vdots \\
  w_k &= \Phi_k(z, \overline{z}, \overline{w}) + O([w_k]),
\end{align*} \tag{2.4}
\]

where \([w_j]\) is the assigned weight to \( w_j \) and where \( \Phi_j \) is a weighted homogeneous complex-valued polynomial in terms of \( z, \overline{z}, w_j \) and other complex variables \( w_\bullet \) of the weights \([w_\bullet]\) < \([w_j]\). Moreover, \( O(t) \) denotes some certain sum of monomials of the weights \( > t \). In this case, the weight \([w_k]\) of the last variable \( w_k \) is equal to the length \( \rho \) of such CR manifold. Beloshapka also introduced:

\[
M := \left\{ \begin{array}{l}
  w_1 = \Phi_1(z, \overline{z}), \\
  \vdots \\
  w_k = \Phi_k(z, \overline{z}, \overline{w}),
\end{array} \right. \tag{2.5}
\]

as a model of all \( k \)-codimensional totally nondegenerate manifolds, represented by (2.4). He also established a practical way to construct the associated defining polynomials \( \Phi_\bullet \) (see [3] or [12, §2]). These models are all homogeneous, of finite type and enjoy several other nice properties [3, Theorem 14] that exhibit their significance.
Convention 2.2 We stress that throughout this paper, we only deal with Beloshapka’s totally nondegenerate CR models and, for the sake of brevity, we call them by “CR models” or “models”. From now on, we fix the notation $M$ for a certain totally nondegenerate model of CR dimension one, codimension $k$ and length $\rho$. Moreover, since we mainly utilize the results of [12] and as is the case with that paper, we also assume that $\rho \geq 3$.

2.2 Symbol Algebra

As we saw, the regular distribution $D_1 = T^{1,0}M + T^{0,1}M$ of $M$ induces the filtration (2.1) of the minimum length $\rho$. Set $g_{-1} := D_1$ and $g_{-\ell} := D_{\ell} \setminus D_{\ell-1}$ for $\ell > 1$. By definition, $g_{-\ell}$ is actually the vector space generated by all iterated Lie brackets between $L_i$ and $L_i$ of the precise length $\ell$. As is shown in Ref. [12, Proposition 3.2] (see also Remark 3.3 of that paper) and independent of the choice of the points $p \in M$ that the above subdistributions $D_j$ are taken on them, the vector space:

$$g_- := g_{-\rho} \oplus \cdots \oplus g_{-1}$$

equipped with the standard Lie bracket of vector fields is essentially a unique $\rho$-th kind graded Lie algebra of dimension $2 + k$, satisfying $[g_{-i}, g_{-j}] = g_{-(i+j)}$. This algebra is fundamental, that is: it can be generated by means of the Lie brackets between the elements of $g_{-1}$. In this case, the regular distribution $D_1$ is called of constant type $g_-$ and, moreover, $g_-$ is called by the symbol algebra of $M$. We emphasize that the Lie algebra $g_-$ and the distribution $D_\rho$ are actually two equal spaces of which the former is regarded as a Lie algebra while the latter is regarded as a frame for the complexified bundle $\mathbb{C} \otimes T M$.

2.3 Lie Algebras of Infinitesimal CR Automorphisms

Let $\{\Gamma_{1,1}, \Gamma_{1,2}, \ldots, \Gamma_{\rho,2+k}\}$ to be the (lifted) coframe, dual to (2.3), of an arbitrary totally nondegenerate CR manifold of codimension $k$ which is biholomorphic (or CR-diffeomorphic) to $M$. Recently in Ref. [12], we have studied—by means of Élie Cartan’s classical approach—the problem of biholomorphic equivalence to $M$ and found its associated constant type structure equations as:

$$
\begin{align*}
&d \Gamma_{\ell,i} = (n_i \alpha + \tilde{n}_i \tilde{\alpha}) \wedge \Gamma_{\ell,i} + \sum_{\ell_1 + \ell_2 = \ell} c^i_{j,n} \Gamma_{\ell_1,j} \wedge \Gamma_{\ell_2,n} \quad (\ell = 1, \ldots, \rho, \ i = 1, \ldots, 2+k), \\
&d \alpha = 0, \\
&d \tilde{\alpha} = 0,
\end{align*}
$$

(2.6)

where $c^i_{j,n}$s are some constant integers and where $n_i$ and $\tilde{n}_i$, visible among the expression of $d\Gamma_{\ell,i}$, are respectively the number of appearing $L_{1,i}$ and $L_{1,i}$ in constructing $L_{\ell,i}$ as an iterated bracket of them. Moreover, $\alpha$ and $\tilde{\alpha}$ are two certain Maurer–Cartan 1-forms added after prolongation steps of the method. Occasionally, it is possible to have these forms as real, i.e. $\alpha = \tilde{\alpha}$, depending on the CR model $M$, under study [12].
As is known [9,12], if the final structure equations of an equivalence problem to a certain \( r \)-dimensional smooth manifold \( M \) equipped with some lifted coframe \( \{ \gamma^1, \ldots, \gamma^r \} \) are of the constant type:

\[
d\gamma^k = \sum_{1 \leq i < j \leq r} c^k_{ij} \gamma^i \wedge \gamma^j \quad (k = 1 \ldots r),
\]

then \( M \) is (locally) diffeomorphic to an \( r \)-dimensional Lie group \( G \), where its corresponding Lie algebra \( g \) has the basis elements \( \{ v_1, \ldots, v_r \} \)—corresponding to \( \{ \gamma^1, \ldots, \gamma^r \} \)—with the structure constants:

\[
[v_i, v_j] = -\sum_{k=1}^r c_{ij}^k v_k \quad (1 \leq i < j \leq r).
\]

We discovered in Ref. [12] that the Lie algebra \( g \) associated with the final structure equations (2.6) is actually the desired Lie algebra \( \text{aut}_{CR}(M) \) of infinitesimal CR automorphisms of \( M \). To realize the structure of this algebra through the above discussion, let us associate \( v_\ell, i \) with the 1-form \( \Gamma_{\ell i} \) for \( \ell = 1, \ldots, \rho \) and \( i = 1, \ldots, 2 + k \) and also associate \( v_0 \) and \( v_0 \) with \( \alpha \) and \( \alpha \) as the basis elements of \( \text{aut}_{CR}(M) \). Clearly in the case that \( \alpha = \alpha \), we will have \( v_0 = v_0 \).

**Proposition 2.3** (cf. [12, Proposition 6.1]) The Lie algebra \( \text{aut}_{CR}(M) \) is graded of the form:

\[
\text{aut}_{CR}(M) := g_{-\rho} \oplus \cdots \oplus g_{-1} \oplus g_0,
\]

satisfying \( [g_{-i}, g_{-j}] = g_{-(i+j)} \) for \( i, j = 0, \ldots, \rho \), where \( g_- \) is (isomorphic to) the \((2 + k)\)-dimensional symbol algebra of \( M \), where each homogeneous component \( g_{-\ell} \) is constructed by the basis elements \( v_{\ell, i} \) and where \( g_0 \) is an Abelian Lie subalgebra of dimension either 1 or 2, generated by \( v_0 \) and \( v_0 \). The Lie brackets between these basis elements are determined by the constant type structure equations (2.6).

**Definition 2.4** A graded Lie algebra \( g := \bigoplus_{i \in \mathbb{Z}} g_i \) is transitive whenever \( [x_i, g_-] \neq 0 \) for each nonzero element \( x_i \in g_i \) with \( i \geq 0 \). Also, it is nondegenerate if \( [x_{-1}, g_{-1}] \neq 0 \) for each nonzero element \( x_{-1} \in g_{-1} \).

**Proposition 2.5** The Lie algebra \( \text{aut}_{CR}(M) = g_- \oplus g_0 \) is nondegenerate and transitive.

**Proof** As is known (cf. [7, p. 201]), every prolongation of the symbol algebra associated with a Levi nondegenerate CR manifold is nondegenerate. Total nondegeneracy of \( M \) implies its Levi nondegeneracy and hence \( \text{aut}_{CR}(M) \), as a prolongation of the symbol algebra \( g_- \), is nondegenerate. To prove that it is transitive, we have to check possible Lie brackets between the generators \( v_{1,1}, v_{1,2} \) of \( g_{-1} \) and \( v_0, v_0 \) of \( g_0 \) by looking for the wedge products between \( \Gamma_{1,1}, \Gamma_{1,2} \) and \( \alpha, \alpha \) throughout the structure equations.
Such products exist only in the structure equations $d\Gamma_{1,1}$ and $d\Gamma_{1,2}$. First let us consider the case $\alpha \neq \bar{\alpha}$, where we have

$$d\Gamma_{1,1} = \alpha \wedge \Gamma_{1,1} \quad \text{and} \quad d\Gamma_{1,2} = \bar{\alpha} \wedge \Gamma_{1,2}.$$ 

This implies that

$$[v_0, v_{1,1}] = -v_{1,1}, \quad [v_0, v_{1,2}] = 0,$$

$$[v_\bar{0}, v_{1,1}] = 0, \quad [v_\bar{0}, v_{1,2}] = -v_{1,2}. \quad (2.7)$$

For the case $\alpha = \bar{\alpha}$, the subalgebra $g_0$ is generated by the single element $v_0$ and the above equations give

$$[v_0, v_{1,1}] = -v_{1,1}, \quad [v_0, v_{1,2}] = 0,$$

$$[v_0, v_{1,1}] = 0, \quad [v_0, v_{1,2}] = -v_{1,2}. \quad (2.8)$$

In any case, one observes that the Lie algebra $\text{aut}_{CR}(M)$ is transitive, as was expected. $\square$

### 2.4 Tanaka Prolongation and Standard Manifolds

In Ref. [13], Noboru Tanaka showed that associated with each finite dimensional fundamental graded algebra $m := \bigoplus_{-\mu \leq i \leq -1} m_i$, there exists a unique, up to isomorphism, Lie algebra $g(m) := \bigoplus_{i \geq -\mu} g^i(m)$, satisfying:

(i) $g^i(m) = m_i$, for each $i = -\mu, \ldots, -1$.

(ii) $g(m)$ is transitive.

(iii) $g(m)$ is the maximal Lie algebra with the above two properties.

This algebra is known as the (full) Tanaka prolongation of $m$. He also established a practical method to construct successively the components $g^i(m)$ (cf. [1,7,10,11,13]). In particular, the zero component $g^0(m)$ is the collection of all derivations $d : m \to m$ that preserve the gradation, i.e. $d(m_{-1}) \subset m_{-1}$.

**Definition 2.6** A graded Lie algebra $m := \bigoplus_{i < 0} m_i$ is said to be *pseudocomplex* (or CR) if there exists some complex structure map $J : m_{-1} \to m_{-1}$ satisfying $J \circ J = -id$ and:

$$[x_{-1}, y_{-1}] = [J(x_{-1}), J(y_{-1})], \quad \text{for each } x_{-1}, y_{-1} \in m_{-1}. \quad (2.9)$$

In the case that the graded fundamental algebra $m := \bigoplus_{-\mu \leq i \leq -1} m_i$ is pseudocomplex, one defines as follows the so-called *Levi–Tanaka* prolongation $\mathcal{G}(m) := \bigoplus_{i \geq -\mu} \mathcal{G}^i(m)$ of $m$, essentially as a transitive subalgebra of $g(m)$: first, for each $i \leq -1$, set $\mathcal{G}^i(m) := m_i$. By definition, the zero component $\mathcal{G}^0(m)$ is the collection of all derivations $d \in g^0(m)$ with respect to the associated complex structure map $J$, i.e.

$$d(J(x_{-1})) = J(d(x_{-1})), \quad \text{for each } x_{-1} \in m_{-1}. \quad (2.9)$$
The Lie bracket between two elements $d \in \mathcal{G}^0(m)$ and $x \in m_i$ is defined as $[d, x] := d(x)$. Assuming that the components $\mathcal{G}^{l'}(m)$ are already constructed for any $l' \leq l - 1$; the $l$th component $\mathcal{G}^l(m)$ of the prolongation consists of $l$-shifted graded linear morphisms $m \to m \oplus \mathcal{G}^0(m) \oplus \mathcal{G}^1(m) \oplus \cdots \oplus \mathcal{G}^{l-1}(m)$ that are derivations, namely

$$\mathcal{G}^l(m) = \left\{ d \in \bigoplus_{k \leq -1} \text{Lin}(\mathcal{G}^k(m), \mathcal{G}^{k+l}(m)) : d([y, z]) = [d(y), z] + [y, d(z)], \ \forall y, z \in m \right\}.$$

Now, for $d \in \mathcal{G}^k(m)$ and $e \in \mathcal{G}^l(m)$, by induction on the integer $k + l \geq 0$, one defines the bracket $[d, e] \in \mathcal{G}^{k+l}(m) \otimes m^*$ by:

$$[d, e](x) = [[d, x], e] + [d, [e, x]] \quad \text{for} \ x \in m.$$

In the case that $m := g_-$ is the symbol algebra of a certain manifold $M$, then $\mathcal{G}(m)$ is called the Levi–Tanaka algebra of $M$.

For the Lie algebra $\text{aut}_{CR}(M) = g_- \oplus g_0$, associated with our fixed CR model $M$, the Lie brackets between the basis elements $v_{1,1}$, $v_{1,2}$ of $g_{-1}$ and $v_0$, $v_{\overline{\alpha}}$ of $g_0$ are presented in (2.7) and (2.8) in two possible cases of $\alpha \neq \overline{\alpha}$ and $\alpha = \overline{\alpha}$. For some technical reasons, we substitute these basis elements with

$$\begin{align*}
\text{for} \ g_{-1} : \ x := v_{1,1} + v_{1,2}, \ y := i(v_{1,1} - v_{1,2}), \\
\text{for} \ g_0 : \ \begin{cases} 
\ d := v_0 + v_{\overline{\alpha}}, \\
\ r := i(v_0 - v_{\overline{\alpha}}), \ \text{where} \ \alpha \neq \overline{\alpha}, \\
\ d := v_0, \ \text{where} \ \alpha = \overline{\alpha}.
\end{cases}
\end{align*}$$

Then, according to (2.7) and (2.8), we have

$$\begin{cases} 
[d, x] = -x, \ [d, y] = -y, \ [r, x] = -y, \ [r, y] = x \ \text{where} \ \alpha \neq \overline{\alpha}, \\
[d, x] = -x, \ [d, y] = -y \ \text{where} \ \alpha = \overline{\alpha}.
\end{cases}$$

Furthermore, let us define the complex structure map $J : g_{-1} \to g_{-1}$ by $J(x) = y$ and $J(y) = -x$.

**Proposition 2.7** Assume as before that $\text{aut}_{CR}(M) = g_- \oplus g_0$. By the already defined complex structure map $J$, the symbol algebra $g_-$ of $M$ is pseudocomplex. Moreover, we have $g_0 \subseteq \mathcal{G}^0(g_-)$.

**Proof** By the above definition of $J$, one readily verifies that for two basis elements $x, y$ of $g_{-1}$ we have $[x, y] = [J(x), J(y)]$ which implies that $g_-$ is pseudocomplex. For the second part of the assertion, first notice that according to Proposition 2.5, $\text{aut}_{CR}(M)$ is transitive and hence, by definition, it is a subalgebra of the Tanaka prolongation $g(g_-) = \bigoplus_{i \geq 0} \mathcal{G}^i(g_-)$. Consequently, we have $g_0 \subseteq \mathcal{G}^0(g_-)$. Then, it suffices to show that the elements of $g_0$ respect the above complex structure map $J$. In other
words, we have to show that \( J([D, X]) = [D, J(X)] \) for \( D = d, r \) and \( X = x, y \) (cf. (2.9)). It needs just some elementary computations that we leave them to the reader.

\[ \square \]

### 2.5 Main Result

After providing preliminary definitions and results, concerning the subject, now we are ready to explain precisely the main aim of this paper. First, we need the following two crucial definitions:

**Definition 2.8** (cf. [7]). Let \( \mathcal{G}(m) := \bigoplus_{i \geq -\mu} \mathcal{G}_i^i(m) \) be the Levi–Tanaka prolongation of a pseudocomplex fundamental Lie algebra \( m \). Assume that \( G \) is the connected and simply connected Lie group with the Lie algebra \( \mathcal{G}(m) \) and also \( G_+ \) is a closed analytic Lie subgroup of \( G \) with \( \mathcal{G}_+(m) := \bigoplus_{i \geq 0} \mathcal{G}_i^i(m) \) as its Lie algebra. Then, the simply connected \( G \)-homogeneous space \( S(\mathcal{G}(m)) := G/G_+ \) is called the standard manifold associated with the Levi–Tanaka prolongation \( \mathcal{G}(m) \).

**Remark 2.9** The above homogeneous manifold \( S(\mathcal{G}(m)) = G/G_+ \) is actually CR. To introduce its associated CR structure, consider first the natural projection \( \pi : G \to S(\mathcal{G}(m)) \) and let \( e \) and \( o \) be the identity elements of the groups \( G \) and \( G/G_+ \). The CR structure \( T_o S(\mathcal{G}(m)) \) of \( S(\mathcal{G}(m)) \) is defined at the identity as:

\[ T_o S(\mathcal{G}(m)) := \pi_*(m_{-1}). \]

Now, at each arbitrary point \( \pi(g) \) of \( S(\mathcal{G}(m)) \), the desired CR structure is defined as the translation of the above fiber \( T_o S(\mathcal{G}(m)) \), from the identity \( o \) to it, through the left multiplication map \( L_g \), i.e.

\[ T_{\pi(g)} S(\mathcal{G}(m)) := L_g T_o S(\mathcal{G}(m)). \]

Roughly speaking, this CR structure is indeed the extension of \( m_{-1} \) to arbitrary points of \( S(\mathcal{G}(m)) \). It is invariant by the action of \( G \) on \( S(\mathcal{G}(m)) \). For more details, we refer the reader to [7, §4].

An arbitrary CR manifold is standard if it is biholomorphic to a certain standard CR manifold. We have also the following—seemingly different but completely relevant—definition

**Definition 2.10** (cf. [6]). An arbitrary Levi nondegenerate CR manifold \( M \) with the symbol algebra \( m \) is maximally homogeneous if \( \dim \text{aut}_{CR}(M) = \dim \mathcal{G}(m) \).

In [6], Medori and Nacinovich showed that a nondegenerate CR manifold \( M \), regular of type \( m \), is maximally homogeneous if and only if it is biholomorphic to the associated standard manifold \( S(\mathcal{G}(m)) \).

In 2004, Beloshapka formulated the question of whether his CR models are standard or not (see [3, Question 2]). Our main aim in this paper is to answer this question affirmatively in CR dimension one. More precisely, the main result of this paper is as follows:
Theorem 2.11 For each Beloshapka’s totally nondegenerate model of CR dimension one, two associated Levi–Tanaka and infinitesimal CR automorphism algebras are isomorphic. As a result, such models are maximally homogeneous and standard.

This result not only answers in part Beloshapka’s question but also provides infinitely many examples of standard manifolds with the certain known geometric-algebraic structures. We prove this theorem at the next section for CR models of the lengths $\rho \geq 3$ (cf. Convention 2.2). By this upcoming proof, it remains only one model of which the correctness of the theorem should be proved in its case. It is nothing but the length two model $H \subset \mathbb{C}^2$ of codimension one which is known as the Heisenberg sphere and is defined in coordinates $(z, w)$ of $\mathbb{C}^2$ as the graph of the following single polynomial equation:

$$w - \bar{w} = 2i z \bar{z}.$$ 

Both the Lie algebra $\text{aut}_{CR}(H)$ and the Levi–Tanaka prolongation $\mathcal{G}(g_-)$ associated with this exceptional model are computed explicitly in [10, §§2, 3], where they are found as two isomorphic 8-dimensional graded algebras. Accordingly, the equality $\text{aut}_{CR}(H) = \mathcal{G}(g_-)$ is plainly satisfied in this case.

Remark 2.12 It is worth to emphasize that according to [3, Proposition 3], Beloshapka has proved (at least implicitly) that every totally nondegenerate CR model $M$ is diffeomorphic to the standard model associated with its symbol algebra, as two smooth manifolds. But, anyway, to conclude that $M$ is a standard manifold we need to have the already mentioned diffeomorphism to be CR. Unfortunately, Beloshapka’s result does not bring directly such desired feature and our main objective in the next section is actually to prove it.

3 Proof of Theorem 2.11

For our length $\rho \geq 3$ CR model $M$, Proposition 2.7 indicates that $\text{aut}_{CR}(M) = g_- \oplus g_0$ is a subalgebra of the Levi–Tanaka algebra $\mathcal{G}(g_-)$. In this section, we prove the reverse inclusion $\text{aut}_{CR}(M) \supseteq \mathcal{G}(g_-)$.

In this case that the $(-1)$-component $g_{-1}$ of the symbol algebra $g_- = \bigoplus_{-\rho \leq i \leq -1} g_i$ is of dimension two and according to last subsection 5.6 of [7], $\mathcal{G}^j(g_0)$ is trivial for all $j \geq 1$. Thus, the Levi–Tanaka algebra $\mathcal{G}(g_-)$ associated with our CR model $M$ is of the short form:

$$\mathcal{G}(g_-) = g_- \oplus \mathcal{G}^0(g_-).$$

Consequently, our problem reduces to prove the inclusion $\text{aut}_{CR}(M) \supseteq \mathcal{G}^0(g_-)$.

Let $\text{Aut}_{CR}(M)$ be the connected and simply connected Lie group of all CR automorphisms of $M$, namely the collection of all automorphisms $h : M \to M$ satisfying $h_* (T^c M) = T^c M$. The associated Lie algebra with this finite dimensional group is $\text{aut}_{CR}(M)$. We also denote by $\text{Aut}_0(M)$ the connected isotropy subgroup of $\text{Aut}_{CR}(M)$ at the origin. All automorphisms belonging to this subgroup are linear.
and its associated Lie algebra is $g_0$. Finally, let $G_-$ be the connected and simply connected Lie subgroup associated with the symbol algebra $g_-$ of $M$. According to [3, Proposition 3], our CR model $M$ is an $\text{Aut}_\text{CR}(M)$-homogeneous space and there exists a certain diffeomorphism:

$$\Gamma : M \rightarrow \frac{\text{Aut}_\text{CR}(M)}{\text{Aut}_\text{t}(M)} = G_- \quad \text{with} \quad \Gamma(0) = e,$$

(3.1)

where $e$ is actually the identity element of $G_-$. Let us denote by $\text{Aut}_J(g_-)$ the Lie group of all automorphisms of $g_-$, preserving the gradation and respecting the complex structure map $J$ defined on the pseudocomplex algebra $g_-$. According to Corollary 3, page 76 of [13] (see also [5, Proposition 1.120]), the associated Lie algebra with this group is the zero component $\mathcal{G}^0(g_-)$ of the Levi–Tanaka algebra of $M$. On the other hand, in this case that $G_-$ is connected and simply connected, two automorphism Lie groups $\text{Aut}(G_-)$ and $\text{Aut}(g_-)$ are isomorphic through the map:

$$\Phi : \text{Aut}(G_-) \rightarrow \text{Aut}(g_-)$$

$$f \mapsto f_* e,$$

where $f_* e$ is the differentiation of $f : G_- \rightarrow G_-$ at the identity element $e$ (cf. [4]). Let $\text{Aut}_J(G_-) \subset \text{Aut}(G_-)$ contains all automorphisms $f$ with $f_* e \in \text{Aut}_J(g_-)$. Then, clearly we have:

**Lemma 3.1** Through the above isomorphism $\Phi$, two Lie groups $\text{Aut}_J(g_-)$ and $\text{Aut}_J(G_-)$ are isomorphic. As a result, the Lie algebra associated with $\text{Aut}_J(G_-)$ is $\mathcal{G}^0(g_-)$.

We aim to show that the Lie group $\text{Aut}_J(G_-)$ can be regarded as a subgroup of $\text{Aut}_\text{CR}(M)$. As a result of this claim, we have

$$\text{Lie}(\text{Aut}_J(G_-)) \subseteq \text{Lie}(\text{Aut}_\text{CR}(M)) \quad \text{or equivalently} \quad \mathcal{G}^0(g_-) \subseteq \text{aut}_\text{CR}(M),$$

(3.2)

as was desired. For this purpose and denoting by $\text{Aut}(M)$ the collection of all (not necessarily CR) automorphisms from $M$ to itself, we define

$$\Psi : \text{Aut}_J(G_-) \rightarrow \text{Aut}(M)$$

$$f \mapsto \Gamma^{-1} \circ f \circ \Gamma =: F$$

(3.3)

where $\Gamma$ is the above diffeomorphism (3.1). Having $f$, as an automorphism of $G_-$, and $\Gamma$ as two certain diffeomorphisms then $\Psi(f)$, that we denote it by $F$ henceforth, is an automorphism. We claim that

**Claim 3.2** $F$ is a CR automorphism and hence belongs to $\text{Aut}_\text{CR}(M)$.

To prove this claim, we need first the following auxiliary lemma:
Lemma 3.3 There exist two independent basis vector fields $L_1$ and $L_2$ of $T^c M$ such that both $X_1 := \Gamma^*_x(L_1)$ and $X_2 := \Gamma^*_x(L_2)$, as two vector fields defined on $G_-$ are left invariant.

Proof Let $L_1$ and $L_2$ be two arbitrary generators of $T^c M$ and consider $X_1 := \Gamma^*_x(L_1)$ and $X_2 := \Gamma^*_x(L_2)$ as two, not necessarily left invariant, vector fields defined on $G_-$. As is known [3, Proposition 4(c)], the Lie subalgebra $g_{-1}$ of $\text{aut}_C G(M)$ can be generated by $x_1 := X_1(e)$ and $x_2 := X_2(e)$. Now, assume that $\tilde{X}_1$ and $\tilde{X}_2$ are the (unique) independent left invariant vector fields associated with $x_1$ and $x_2$, respectively. Let $D_1$ be the distribution on $G_-$ generated by $\tilde{X}_1$ and $\tilde{X}_2$ and define successively $D_j := D_{j-1} + [D_1, D_{j-1}]$ for $j > 1$. Since $\tilde{X}_1$ and $\tilde{X}_2$ are left invariant, the value of each iterated bracket between them at an arbitrary point $g \in G_-$ is actually the translation of this value at $e$ through the differential map $L_{g^*}$ of the left multiplication by $g$, i.e.

$$[\tilde{X}_{i_1}, [\tilde{X}_{i_2}, [\tilde{X}_{i_3}, [... , [\tilde{X}_{i_{\ell-1}}, \tilde{X}_{i_\ell}]]]]]_g = L_{g^*}([x_{i_1}, [x_{i_2}, [x_{i_3}, [... , [x_{i_{\ell-1}}, x_{i_\ell}]]]])], \quad (i_j = 1, 2).$$

This indicates that at each arbitrary point of $G_-$, the filtration $D_1 \subset D_2 \subset D_3 \subset \ldots$ has exactly similar treatment as that at the identity element $e$. Consequently, at each arbitrary point $g \in G_-$, the above filtration induces the same fundamental symbol algebra $g_{-1}$ generated by $x_1$ and $x_2$. Set $L_i := \Gamma^*_{g^{-1}}(\tilde{X}_i)$ for $i = 1, 2$. The subbundle $\tilde{T}^c M$ generated by these two independent vector fields can be regarded as a certain CR structure for $M$ via the complex structure map $J : \tilde{T}^c M \to \tilde{T}^c M$ defined by $J(L_1) := L_2$ and $J(L_2) := -L_1$. According to Theorem 3, page 109 of [2], determining a basis for the CR structure of an smooth generic submanifold depends only upon its defining functions and hence $\tilde{T}^c M$ and $T^c M$ are two equivalent CR structures for $M$. Therefore, we can consider $\tilde{L}_1$ and $\tilde{L}_2$ as the basis elements of the CR structure of $M$ where their image vector fields $\tilde{X}_1 = \Gamma^*_x(\tilde{L}_1)$ and $\tilde{X}_2 = \Gamma^*_x(\tilde{L}_2)$ are left invariant on $G_-$. 

Proof of Claim 3.2 Since $F = \Psi(f)$ is a diffeomorphism, then it suffices to show that $F_x(T^c M) = T^c M$. Let $L_1$ and $L_2$ be two basis vector fields of $T^c M$ such that $X_1 = \Gamma^*_x(L_1)$ and $X_2 = \Gamma^*_x(L_2)$ are left invariant, with $X_1(e) = x_1$ and $X_2(e) = x_2$ as the basis elements of $g_{-1}$. By definition, since $f \in \text{Aut}_f(G_-)$ then its differentiation $f_{se}$ preserves the gradation and hence we have

$$f_{se}(x_1) := a_1 x_1 + a_2 x_2, \quad \text{and} \quad f_{se}(x_2) := b_1 x_1 + b_2 x_2,$$

for some constant integers $a_\bullet$ and $b_\bullet$. Now, since $X_1$ and $X_2$ are left invariant, then it yields that

$$f_x(X_1) := a_1 X_1 + a_2 X_2, \quad \text{and} \quad f_x(X_2) := b_1 X_1 + b_2 X_2.$$
at each arbitrary point \( g \in G_- \). Consequently, \( f_* \) preserves as well the subdistribution \( \mathcal{D}_1 := \langle X_1, X_2 \rangle \), i.e.

\[
f_*(\mathcal{D}_1(g)) = \mathcal{D}_1(h), \quad \text{with} \quad h := f(g).
\]

Let \( p \in M \), with \( \Gamma(p) = g \) and \( f(g) = h \). Since \( \Gamma_*(T_c^c M) = \mathcal{D}_1 \), then we have

\[
F_*(T_p^c M) = \Gamma_{*h}^{-1} \circ f_{*g} \circ \Gamma_{*p}(T_p^c M)
= \Gamma_{*h}^{-1} \circ f_{*g}(\mathcal{D}_1(g)) = \Gamma_{*h}^{-1}(\mathcal{D}_1(h))
= T_{F(p)}^c M.
\]

This completes the proof. \( \square \)

Consequently, we can take the map \( \Psi \), introduced in (3.3) as

\[
\Psi : \text{Aut}_J(G_-) \longrightarrow \text{Aut}_{CR}(M).
\]

One also readily verifies that this map is actually an injective Lie group homomorphism. Therefore, we can regard \( \text{Aut}_J(G_-) \) as a Lie subgroup of \( \text{Aut}_{CR}(M) \). Then, according to (3.2), we have \( \mathcal{G}^0(g_-) \subseteq \text{aut}_{CR}(M) \), as was desired. This completes the proof of the main Theorem 2.11. \( \square \)

**Remark 3.4** It may be worth to notice that since all the automorphisms \( f \in \text{Aut}_J(G_-) \) fix the identity element \( e \) of \( G_- \), then its associated CR automorphism \( F \in \text{Aut}_{CR}(M) \) preserves the origin:

\[
F(0) = \Gamma_0^{-1} \circ f \circ \Gamma(0) = 0.
\]

Therefore, the induced function \( F \) belongs to the isotropy subgroup \( \text{Aut}_0(M) \) of \( \text{Aut}_{CR}(M) \) at the origin. Then, in a more precise manner, we can regard \( \text{Aut}_J(G_-) \) as a subgroup of the isotropy group \( \text{Aut}_0(M) \). This implies the inclusion \( \mathcal{G}^0(g_-) \subseteq g_0 \) and, consequently, we have \( \mathcal{G}^0(g_-) = g_0 \) as was expected.

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