FRACTIONAL SCHröDINGER-POISSON SYSTEMS WITH A GENERAL SUBCRITICAL OR CRITICAL NONLINEARITY

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Abstract. We consider a fractional Schrödinger-Poisson system with a general nonlinearity in subcritical and critical case. The Ambrosetti-Rabinowitz condition is not required. By using a perturbation approach, we prove the existence of positive solutions. Moreover, we study the asymptotics of solutions for a vanishing parameter.

1. Introduction and main result

We are concerned with the fractional nonlinear Schrödinger-Poisson system

\begin{equation}
\begin{aligned}
(-\Delta)^s u + \lambda \phi u &= g(u) \quad \text{in } \mathbb{R}^3, \\
(-\Delta)^t \phi &= \lambda u^2 \quad \text{in } \mathbb{R}^3,
\end{aligned}
\end{equation}

where \( \lambda > 0 \), \((-\Delta)^\alpha\) is the fractional Laplacian operator for \( \alpha = s, t \in (0, 1) \). The fractional Schrödinger equation was introduced by Laskin [28] and arose in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes. The operator \((-\Delta)^\alpha\) can be seen as the infinitesimal generators of Lévy stable diffusion processes [2]. If \( \lambda = 0 \), the system (1.1) reduces to the nonlinear fractional scalar field equation

\begin{equation}
(-\Delta)^s u = g(u) \quad \text{in } \mathbb{R}^3.
\end{equation}

This equation is related to the standing waves for fractional scalar field equation

\begin{equation}
i\phi_t - (-\Delta)^s \phi + g(\phi) = 0 \quad \text{in } \mathbb{R}^3,
\end{equation}

which is a physically relevant generalization of the classical NLS. For power type nonlinearities the fractional Schrödinger equation was derived by Laskin [28] by replacing the Brownian motion in the path integral approach with the so called Lévy flights, see e.g. [29]. So, the equation we want to study presents first of all as a perturbation of a physically meaningful equation. Also, in [21, 22] the author obtained deep results about uniqueness and non-degeneracy of ground states for (1.2) in case \( g(u) = |u|^{p-2}u - u \) for subcritical \( p \). See also [33] where the soliton dynamics for (1.3) with an external potential was investigated. In [24], the author studies the evolution equation associated with the one dimensional system

\begin{equation}
\begin{aligned}
-\Delta u + \lambda \phi u &= g(u) \quad \text{in } \mathbb{R}, \\
(-\Delta)^t \phi &= \lambda u^2 \quad \text{in } \mathbb{R}.
\end{aligned}
\end{equation}

In this case the diffusion is fractional only in the Poisson equation. Our system is more general and contain this as a particular case. If \( K_{\alpha}(x) = |x|^\alpha - N \), in [20] the following equation is studied

\( \sqrt{-\Delta} u + u = (K_2 * |u|^2)u, \quad u \in H^{1/2}(\mathbb{R}^3), \quad u > 0 \).

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and in [19] it is shown that the dynamical evolution of boson stars is described by the nonlinear evolution equation

\[ i \partial_t \psi = \sqrt{-\Delta + m^2} \psi - (K_2 * |\psi|^2)\psi \quad (m \geq 0) \]

for a field \( \psi : [0, T) \times \mathbb{R}^3 \to \mathbb{C} \) (see also [23]). The square root of the Laplacian also appears in the semi-relativistic Schrödinger-Poisson-Slater systems, [6]. See also the model studied in [15]. Observe that, taking formally \( s = t = 1 \), then system (1.1) reduces to the classical Schrödinger-Poisson system

\[
(1.5) \quad \begin{cases}
-\Delta u + \lambda \phi u = g(u) & \text{in } \mathbb{R}^3, \\
-\Delta \phi = \lambda u^2 & \text{in } \mathbb{R}^3.
\end{cases}
\]

It describes systems of identically charged particles interacting each other in the case where magnetic effects can be neglected [7]. In recent years, the Schrödinger-Poisson system (1.5) has been widely studied by many researchers. Here we would like to cite some related results, for example, positive solutions [10], ground state solutions [3], semi-classical states [16], sign-changing solutions [25]. See also [1] and the references therein. In [4], Azzollini, d’Avenia and Pomponio were concerned with (1.5) under the Berestycki-Lions conditions \((H2)-(H4)\) with \( s = 1 \). The authors proved that (1.5) admits a positive radial solution if \( \lambda > 0 \) small enough. For the critical case, we refer to [36] and a recent work [38] of the authors of the present work.

1.1. Main results. In the present paper, we are mainly concerned with the positive solutions of (1.1). First, we consider the subcritical case with the Berestycki-Lions conditions. Precisely, we assume the following hypotheses on \( g \):

\[(H1) \ g \in C^1(\mathbb{R}, \mathbb{R}); \]
\[(H2) \ -\infty < \liminf_{\tau \to 0} \frac{g(\tau)}{\tau} \leq \limsup_{\tau \to 0} \frac{g(\tau)}{\tau} = -m < 0; \]
\[(H3) \ \limsup_{\tau \to \infty} \frac{g(\tau)}{\tau^{2s-1}} \leq 0, \text{ where } 2s = \frac{6}{3-2s}; \]
\[(H4) \text{ there exists } \xi > 0 \text{ such that } G(\xi) := \int_0^\xi g(\tau) d\tau > 0. \]

Our first result can read as

**Theorem 1.1.** Suppose that \( g \) satisfies (H1)-(H4) and \( 2t + 4s \geq 3 \).

(i) There exists \( \lambda_0 > 0 \) such that, for every \( \lambda \in (0, \lambda_0) \), system (1.1) admits a nontrivial positive radial solution \((u_\lambda, \phi_\lambda)\).

(ii) Along a subsequence, \((u_\lambda, \phi_\lambda)\) converges to \((u, 0)\) in \( H^s(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \) as \( \lambda \to 0 \), where \( u \) is a radial ground state solution of (1.2).

**Remark 1.2.** The hypotheses (H2)-(H4) are the so-called Berestycki-Lions conditions, which were introduced in [8] to get the ground state of (1.2) with \( s = 1 \). Under (H1)-(H4), X. Chang and Z.-Q. Wang [13] proved the existence of ground state solutions to (1.2) for \( s \in (0, 1) \). The hypothesis (H1) is only used to get the better regularity of solutions to (1.2), which can guarantee the Pohožaev’s identity. By the Pohožaev’s identity, (H4) is necessary.

**Remark 1.3.** The hypothesis \( 2t + 4s \geq 3 \) is just used to guarantee that the Poisson equation \((-\Delta)^s \phi = \lambda u^2\) make sense, due to \( \mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^2_t(\mathbb{R}^3) \). For the details, see Section 2 below.

In [38], without the Ambrosetti-Rabinowitz condition, the authors of the present work considered the existence and concentration of positive solutions to (1.1) in the critical case for \( s = t = 1 \). It is natural to wonder if similar results can hold for the critical fractional case. This is just our second goal of the present paper. In the critical case, we assume the following hypotheses on \( g \):
\[(H2)' \lim_{\tau \to 0} \frac{g(\tau)}{\tau} = -a < 0; \]
\[(H3)' \lim_{\tau \to \infty} \frac{g(\tau)}{\tau^{2s-1}} = b > 0; \]
\[(H4)' \text{ there exists } \mu > 0 \text{ and } q < 2^* \text{ such that } g(\tau) - b\tau^{2s-1} + a\tau \geq \mu \tau^{q-1} \text{ for all } \tau > 0. \]

Our second result can read as follows.

**Theorem 1.4.** Suppose that \(g\) satisfies (H1), \((H2)'-(H4)'.

(i) The limit problem (1.2) admits a ground state solution if \(\max\{2^*_s - 2, 2\} < q < 2^*_s\).
(ii) Let \(2t + 4s \geq 3\), then there exists \(\lambda_0 > 0\) such that, for every \(\lambda \in (0, \lambda_0)\), system (1.1) admits a nontrivial positive radial solution \((u_\lambda, \phi_\lambda)\) if \(\max\{2^*_s - 2, 2\} < q < 2^*_s\).
(iii) Along a subsequence, \((u_\lambda, \phi_\lambda)\) converges to \((u, 0)\) in \(H^s(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) as \(\lambda \to 0\), where \(u\) is a radial ground state solution of (1.2).

**Remark 1.5.** In the case \(s = 1\), the hypotheses \((H2)'-(H4)'\) were introduced in [37] (see also [5]) to obtain the ground state of the scalar field equation \(-\Delta u = g(u)\) in \(\mathbb{R}^N\). In [34], X. Shang and J. Zhang considered the fractional problem (1.2) in the critical case (see also [35]). With the help of the monotonicity of \(\tau \mapsto g(\tau)/\tau\), the ground state solutions were obtained by using the Ambrosetti-Rabinowitz condition and the monotonicity of \(g(\tau)/\tau\). Theorem 1.4 seems to be the first result in this direction.

**Remark 1.6.** Without loss generality, from now on, we assume that \(a = b = \mu = 1\).

In the rest of the paper, we use the perturbation approach to prove Theorem 1.1 and 1.4. Similar argument also can be found in [38].

The paper is organized as follows.
In Section 2 we introduce the functional framework and some preliminary results.
In Section 3 we construct the min-max level.
In Section 4, we use a perturbation argument to complete the proof of Theorem 1.1.
In Section 5, we give the proof of Theorem 1.4.

**Notations.**
- \(\|u\|_p := (\int_{\mathbb{R}^3} |u|^p \, dx)^{1/p}\) for \(p \in [1, \infty)\).
- \(2^*_s := \frac{6}{3-2\alpha}\) for any \(\alpha \in (0, 1)\).
- \(\hat{u} = \mathcal{F}(u)\) is the Fourier transform of \(u\).

## 2. Preliminaries and functional setting

### 2.1. Fractional order Sobolev spaces.

The fractional Laplacian \((-\Delta)^\alpha\) with \(\alpha \in (0, 1)\) of a function \(\phi : \mathbb{R}^3 \to \mathbb{R}\) is defined by
\[
\mathcal{F}((-\Delta)^\alpha \phi)(\xi) = |\xi|^{2\alpha} \mathcal{F}(\phi)(\xi), \quad \xi \in \mathbb{R}^3,
\]
where \(\mathcal{F}\) is the Fourier transform, i.e.,
\[
\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \exp(-2\pi i \xi \cdot x) \phi(x) \, dx,
\]
i is the image unit. If \(\phi\) is smooth enough, it can be computed by the following singular integral
\[
(-\Delta)^\alpha \phi(x) = c_\alpha \text{ P.V.} \int_{\mathbb{R}^3} \frac{\phi(x) - \phi(y)}{|x - y|^{3+2\alpha}} \, dy, \quad x \in \mathbb{R}^3.
\]
where $c_\alpha$ is a normalization constant and P.V. stands the principal value. For any $\alpha \in (0, 1)$, we consider the fractional order Sobolev space

$$H^\alpha(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2\alpha}|\hat{u}|^2 \, d\xi < \infty \right\},$$

endowed with the norm

$$\|u\|_\alpha = \left( \int_{\mathbb{R}^3} (1 + |\xi|^{2\alpha})|\hat{u}|^2 \, d\xi \right)^{1/2}, \quad u \in H^\alpha(\mathbb{R}^3),$$

and the inner product

$$(u, v)_\alpha = \int_{\mathbb{R}^3} (1 + |\xi|^{2\alpha})\hat{u}\hat{v} \, d\xi, \quad u, v \in H^\alpha(\mathbb{R}^3).$$

It is easy to know the inner products on $H^\alpha(\mathbb{R}^3)$

$$u, v \mapsto \int_{\mathbb{R}^3} (1 + |\xi|^{2\alpha})\hat{u}\hat{v} \, d\xi,$$

$$u, v \mapsto \int_{\mathbb{R}^3} (uv + (-\Delta)^{\alpha/2}u(-\Delta)^{\alpha/2}v) \, dx,$$

are equivalent (see [34]). The homogeneous Sobolev space $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ is defined by

$$\mathcal{D}^{\alpha,2}(\mathbb{R}^3) = \{ u \in L^{2\alpha}_\infty(\mathbb{R}^3) : |\xi|^{\alpha}\hat{u} \in L^2(\mathbb{R}^3) \},$$

which is the completion of $C^\infty_0(\mathbb{R}^3)$ under the norm

$$\|u\|_{\mathcal{D}^{\alpha,2}}^2 = \|(-\Delta)^{\alpha/2}u\|_2^2 = \int_{\mathbb{R}^3} |\xi|^{2\alpha}|\hat{u}|^2 \, d\xi, \quad u \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3),$$

and the inner product

$$(u, v)_{\mathcal{D}^{\alpha,2}} = \int_{\mathbb{R}^3} (-\Delta)^{\alpha/2}u(-\Delta)^{\alpha/2}v \, dx, \quad u, v \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3).$$

For the further introduction on the Fractional order Sobolev space, we refer to [17]. Let

$$H^{s}_0(\mathbb{R}^3) = \{ u \in H^3(\mathbb{R}^3) : u(x) = u(|x|) \}.$$

Now, we introduce the following Sobolev embedding theorems.

**Lemma 2.1** (see [27]). For any $\alpha \in (0, 1)$, $H^\alpha(\mathbb{R}^3)$ is continuously embedded into $L^q(\mathbb{R}^3)$ for $q \in [2, 2^*_\alpha]$ and compactly embedded into $L^q_{\text{loc}}(\mathbb{R}^3)$ for $q \in [1, 2^*_\alpha)$. Moreover, $H^\alpha_0(\mathbb{R}^3)$ is compactly embedded into $L^q(\mathbb{R}^3)$ for $q \in (2, 2^*_\alpha)$.

**Lemma 2.2** (see [12, 17]). For any $\alpha \in (0, 1)$, $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*_\alpha}(\mathbb{R}^3)$, i.e., there exists $S_\alpha > 0$ such that

$$\left( \int_{\mathbb{R}^3} |u|^{2^*_\alpha} \, dx \right)^{2/2^*_\alpha} \leq S_\alpha \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}u|^2 \, dx, \quad u \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3).$$

### 2.2. The variational setting

Now, we study the variational setting of (1.1). By Lemma 2.1, $H^s(\mathbb{R}^3) \hookrightarrow L^{12/(3+2t)}(\mathbb{R}^3)$ if $2t + 4s \geq 3$. Then, for $u \in H^s(\mathbb{R}^3)$, by Lemma 2.2 the linear operator $P : \mathcal{D}^{1,2}(\mathbb{R}^3) \rightarrow \mathbb{R}$ defined by

$$P(v) = \int_{\mathbb{R}^3} u^2v \leq \|u\|_{12/(3+2t)}^2\|v\|_{2^*_t} \leq C\|u\|_s^2\|v\|_{\mathcal{D}^{1,2}},$$
The proof is similar as that in (2.1)

\[ \phi_u^t(x) := \lambda c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|^{3-2t}} \, dy, \]

where we have set

\[ c_t = \frac{\Gamma(\frac{3}{2} - 2t)}{\pi^{\frac{3}{2}} 2^{2t} \Gamma(t)}. \]

Formula (2.1) is called the \( t \)-Riesz potential. Substituting (2.1) into (1.1), we can rewrite (1.1) in the following equivalent form

\[ (-\Delta)^s u + \lambda \phi_u^t u = g(u), \quad u \in H^s(\mathbb{R}^3). \]

We define the energy functional \( \Gamma_\lambda : H^s(\mathbb{R}^3) \to \mathbb{R} \) by

\[ \Gamma_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 \, dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 \, dx - \int_{\mathbb{R}^3} G(u) \, dx, \]

with \( G(\tau) = \int_0^\tau g(\zeta) \, d\zeta \). Obviously, the critical points of \( \Gamma_\lambda \) are the weak solutions of (2.2).

**Definition 2.3.**

1. We call \( (u, \phi) \in H^s(\mathbb{R}^3) \times D^{s,2}(\mathbb{R}^3) \) is a weak solution of (1.1) if \( u \) is a weak solution of (2.2).
2. We call \( u \in H^s(\mathbb{R}^3) \) is a weak solution of (2.2) if

\[ \int_{\mathbb{R}^3} ((-\Delta)^{s/2} u (-\Delta)^{s/2} v + \lambda \phi_u^t uv) \, dx = \int_{\mathbb{R}^3} g(v) v \, dx, \quad \forall v \in H^s(\mathbb{R}^3). \]

Setting

\[ T(u) := \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 \, dx, \]

then we summarize some properties of \( \phi_u^t, T(u) \), which will be used later.

**Lemma 2.4.** If \( t, s \in (0, 1) \) and \( 2t + 4s \geq 3 \), then for any \( u \in H^s(\mathbb{R}^3) \), we have

1. \( u \mapsto \phi_u^t : H^s(\mathbb{R}^3) \to D^{s,2}(\mathbb{R}^3) \) is continuous and maps bounded sets into bounded sets.
2. \( \phi_u^t(x) \geq 0, x \in \mathbb{R}^3 \) and \( T(u) \leq c\|u\|_{H^s}^4 \) for some \( c > 0 \).
3. \( T(u(\tau)) = \tau^{3+2t} T(u) \) for any \( \tau > 0 \) and \( u \in H^s(\mathbb{R}^3) \).
4. If \( u_n \to u \) weakly in \( H^s(\mathbb{R}^3) \), then \( \phi_{u_n} \to \phi_u \) weakly in \( D^{s,2}(\mathbb{R}^3) \).
5. If \( u_n \to u \) weakly in \( H^s(\mathbb{R}^3) \), then \( T(u_n) = T(u) + T(u_n - u) + o(1) \).
6. If \( u \) is a radial function, so is \( \phi_u^t \).

**Proof.** The proof is similar as that in [31], so we omit the details here. \( \square \)

### 3. The subcritical case

#### 3.1. The modified problem

It follows from Lemma 2.4 that \( \Gamma_\lambda \) is well defined on \( H^s(\mathbb{R}^3) \) and is of class \( C^1 \). Since we are concerned with the positive solutions of (2.2), similar as that in [8] (see also [13]), we modify our problem first. Without loss generality, we assume that

\[ 0 < \xi = \inf\{ \tau \in (0, \infty) : G(\tau) > 0 \}, \]

where \( \xi \) is given in (H4). Let \( \tau_0 = \inf\{ \tau > \xi : g(\tau) = 0 \} \in [\xi, \infty], \) define \( \tilde{g} : \mathbb{R} \to \mathbb{R}, \)

\[ \tilde{g}(\tau) = \begin{cases} g(\tau) & \text{if } \tau \in [0, \tau_0] \\ 0 & \text{if } \tau \geq \tau_0, \end{cases} \]
and \( \tilde{g}(\tau) = 0 \) for \( \tau \leq 0 \). If \( u \in H^s(\mathbb{R}^3) \) is a solution of (2.2) where \( g \) is replaced by \( \tilde{g} \), then by the maximum principle [11] we get that \( u \) is positive and \( u(x) \leq \tau_0 \) for any \( x \in \mathbb{R}^3 \), i.e., \( u \) is a solution of the original problem (2.2) with \( g \). Thus, from now on, we can replace \( g \) by \( \tilde{g} \), but still use the same notation \( g \). In addition, for \( \tau > 0 \), let
\[
g_1(\tau) = \max\{g(\tau) + m\tau, 0\}, \quad g_2(\tau) = g_1(\tau) - g(\tau),
\]
then \( g_2(\tau) \geq m\tau \) for \( \tau \geq 0 \),
\[
\lim_{\tau \to 0} \frac{g_1(\tau)}{\tau} = 0, \quad \lim_{\tau \to +\infty} \frac{g_1(\tau)}{\tau^{2^*_s - 1}} = 0,
\]
and for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that
\[
g_1(\tau) \leq \varepsilon g_2(\tau) + C_\varepsilon \tau^{2^*_s - 1}, \quad \tau \geq 0.
\]
Let \( G_i(\tau) = \int_0^\tau g_i(\tau) \, d\tau, \ i = 1, 2 \), then by (3.1) and (3.2) for any \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that
\[
G_1(\tau) \leq \varepsilon G_2(\tau) + C_\varepsilon |\tau|^{2^*_s}, \quad \tau \in \mathbb{R}.
\]

3.2. The limit problem. In the following, we will find the solutions of (2.2) by seeking the critical points of \( \Gamma_\lambda \). If \( \lambda = 0 \), problem (2.2) becomes
\[
(\Delta)^s u = g(u), \quad u \in H^s(\mathbb{R}^3),
\]
which is referred as the limit problem of (2.2). We define an energy functional for the limiting problems (3.4) by
\[
L(u) = \frac{1}{2} \int_{\mathbb{R}^3} |(\Delta)^{s/2} u|^2 \, dx - \int_{\mathbb{R}^3} G(u) \, dx, \quad u \in H^s(\mathbb{R}^3).
\]
In [13], X. Chang and Z.-Q. Wang proved that, with the same assumptions on \( g \) in Theorem 1.1, there exists a positive ground state solution \( U \in H^s_r(\mathbb{R}^3) \) of (3.4). Moreover, each such solution \( U \) of (3.4) satisfies the Pohozaev identity
\[
\frac{3 - 2s}{2} \int_{\mathbb{R}^3} |(\Delta)^{s/2} U|^2 \, dx = 3 \int_{\mathbb{R}^3} G(U) \, dx.
\]
Let \( S \) be the set of positive radial ground state solutions \( U \) of (3.4), then \( S \neq \emptyset \) and we have the following compactness result, which plays a crucial role in the proof of Theorem 1.1.

**Proposition 3.1.** Under the assumptions in Theorem 1.1, \( S \) is compact in \( H^s_r(\mathbb{R}^3) \).

As shown in [9], for general \( s \in (0, 1) \), we do not have a similar radial lemma in \( H^s_r(\mathbb{R}^3) \). So the Strauss’s compactness lemma(see [8]) is not applicable here. Before we prove Proposition 3.1, we start with the following compactness lemma, which is a special case of [13, Lemma 2.4].

**Lemma 3.2 (see [13]).** Assume \( Q \in C(\mathbb{R}, \mathbb{R}) \) satisfy
\[
\lim_{\tau \to 0} \frac{Q(\tau)}{\tau^2} = \lim_{|\tau| \to +\infty} \frac{Q(\tau)}{|\tau|^{2^*_s}} = 0,
\]
and there exists a bounded sequence \( \{u_n\}_{n=1}^\infty \subset H^s_r(\mathbb{R}^3) \) for some \( v \in L^1(\mathbb{R}^3) \) with
\[
\lim_{n \to \infty} Q(u_n(x)) = v(x), \quad a.e. \ x \in \mathbb{R}^3.
\]
Then, up to a subsequence, we have \( Q(u_n) \to v \) strongly in \( L^1(\mathbb{R}^3) \) as \( n \to \infty \).
Proof of Proposition 3.1 Let \( \{u_n\}_{n=1}^{\infty} \subset S \) and denote by \( E \) the least energy of (3.4), then for any \( n, u_n \) satisfies \( L(u_n) = E \) and the Pohožaev identity (4.5), which implies that

\[
E = \frac{s}{3} \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \, dx \quad \text{and} \quad \int_{\mathbb{R}^3} G(u_n) \, dx = \frac{3 - 2s}{2s}E.
\]

Obviously, \( \{\|(-\Delta)^{s/2}u_n\|_2\} \) is bounded. It follows from Lemma 2.2 that \( \{\|u_n\|_2\} \) is bounded. By (3.3), as we can see in [8], \( \{\|u_n\|_2\} \) is bounded, which yields that \( \{u_n\} \) is bounded in \( H^s_\tau(\mathbb{R}^3) \). Without loss generality, we can assume that there exists \( u_0 \in H^s_\tau(\mathbb{R}^3) \) such that \( u_n \rightarrow u_0 \) weakly in \( H^s_\tau(\mathbb{R}^3) \), strongly in \( L^q(\mathbb{R}^3) \) for \( q \in (2, 2^*_s) \) and \( u_n(x) \rightarrow u_0(x) \) a.e. \( x \in \mathbb{R}^3 \).

In the following, we adopt some ideas in [8] to prove that \( u_n \rightarrow u_0 \) strongly in \( H^s_\tau(\mathbb{R}^3) \). For \( u \in H^s(\mathbb{R}^3) \), let

\[
J(u) = \frac{s}{3} \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \, dx \quad \text{and} \quad V(u) = \int_{\mathbb{R}^3} G(u) \, dx,
\]

then, we know \( u_n \) is a minimizer of the following constrained minimizing problem

\[
\inf \left\{ J(u) : u \in H^s_\tau(\mathbb{R}^3), V(u) = \frac{3 - 2s}{2s}E \right\}.
\]

By (3.1) and Lemma 3.2, we get that

\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} G_1(u_n) = \int_{\mathbb{R}^3} G_1(u_0).
\]

Then by the Fatou’s Lemma, \( V(u_0) \geq \frac{3 - 2s}{2s}E \), which implies that \( u_0 \not\equiv 0 \). Meanwhile, it is easy to know that \( J(u_0) \leq E \). Similar as that in [8], we know that \( u_0 \) satisfies \( J(u_0) = E \) and \( V(u_0) = \frac{3 - 2s}{2s}E \). It yields that

\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} G_2(u_n) = \int_{\mathbb{R}^3} G_2(u_0).
\]

By the Fatou’s Lemma, we know \( \|u_n\|_2 \rightarrow \|u_0\|_2 \) as \( n \rightarrow \infty \). Thus, \( u_n \rightarrow u_0 \) strongly in \( H^s_\tau(\mathbb{R}^3) \). The proof is completed. \( \square \)

3.3. The minimax level. Take \( U \in S \), let

\[
U_\tau(x) = U(\frac{x}{\tau}), \tau > 0,
\]

then by the definition of \( \hat{U} = F(U) \), we know \( \hat{U}(\frac{x}{\tau}) = \tau^3 \hat{U}(t) \). Then

\[
\int_{\mathbb{R}^3} |(-\Delta)^{s/2}U_\tau|^2 \, dx = \int_{\mathbb{R}^3} |\xi|^2 |\hat{U}(\frac{\xi}{\tau})|^2 = \tau^{3 - 2s} \int_{\mathbb{R}^3} |(-\Delta)^{s/2}U|^2 \, dx.
\]

By the Pohožaev’s identity,

\[
L(U_\tau) = \left( \frac{\tau^{3 - 2s}}{2} - \frac{3 - 2s}{6} \tau^3 \right) \int_{\mathbb{R}^3} |(-\Delta)^{s/2}U|^2.
\]

Thus, there exists \( \tau_0 > 1 \) such that \( L(U_\tau) < -2 \) for \( \tau \geq \tau_0 \). Set

\[
D_\lambda \equiv \max_{\tau \in [0, \tau_0]} \Gamma_\lambda(U_\tau).
\]

By virtue of Lemma 2.4, \( \Gamma_\lambda(U_\tau) = L(U_\tau) + O(\lambda) \). Note that \( \max_{\tau \in [0, \tau_0]} L(U_\tau) = E \), we get that \( D_\lambda \rightarrow E \), as \( \lambda \rightarrow 0^+ \).

Moreover, similar to [38], we can prove the following lemma, which is crucial to define the uniformly bounded set of the mountain pathes(see below).
Lemma 3.3. There exist $\lambda_1 > 0$ and $C_0 > 0$, such that for any $0 < \lambda < \lambda_1$ there hold

$$\Gamma_{\lambda}(U_{\tau_0}) < -2, \quad \|U_{\tau}\|_s \leq C_0, \quad \forall \tau \in (0, \tau_0], \quad \|u\|_s \leq C_0, \quad \forall u \in S.$$ 

Now, for any $\lambda \in (0, \lambda_1)$, we define a min-max value $C_{\lambda}$:

$$C_{\lambda} = \inf_{\gamma \in \Upsilon_{\lambda}} \max_{\tau \in [0, \tau_0]} \Gamma_{\lambda}(\gamma(\tau)),$$

where

$$\Upsilon_{\lambda} = \left\{ \gamma \in C([0, \tau_0], H^s_r(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(\tau_0) = U_{\tau_0}, \|\gamma(\tau)\|_s \leq C_0 + 1, \tau \in [0, \tau_0] \right\}.$$ 

Obviously, for $\tau > 0$,

$$\|U_{\tau}\|_s^2 = \tau^{3-2s}\|(-\Delta)^{s/2}U\|_2^2 + \tau^3\|U\|_2^2.$$ 

Then we can define $U_0 \equiv 0$, so $U_{\tau} \in \Upsilon_{\lambda}$. Moreover,

$$\limsup_{\lambda \to 0^+} C_{\lambda} \leq \lim_{\lambda \to 0^+} D_{\lambda} = E.$$ 

Proposition 3.4. $\lim_{\lambda \to 0^+} C_{\lambda} = E$.

Proof. It suffices to prove that

$$\liminf_{\lambda \to 0^+} C_{\lambda} \geq E.$$ 

Now, we give the following mountain pass value

$$b = \inf_{\gamma \in \Upsilon} \max_{\tau \in [0,1]} L(\gamma(\tau)),$$

where

$$\Upsilon = \left\{ \gamma \in C([0,1], H^s_r(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) < 0 \right\}.$$ 

It follows from [13, Lemma 3.2] that $L$ satisfies the mountain pass geometry. As we can see in [26], $b$ agrees with the least energy level of (3.4), i.e., $b = E$. Note that $\phi(\gamma) \geq 0, x \in \mathbb{R}^3$, then $\gamma(\cdot) = \gamma(\tau_0) \in \Upsilon$ for any $\gamma \in \Upsilon_{\lambda}$. It follows that $C_{\lambda} \geq b$, concluding the proof.

3.4. Proof of Theorem 1.1. Now for $\alpha, d > 0$, define

$$\Gamma_{\lambda}^\alpha := \left\{ u \in H^s_r(\mathbb{R}^3) : \Gamma_{\lambda}(u) \leq \alpha \right\}$$

and

$$S^d = \left\{ u \in H^s_r(\mathbb{R}^3) : \inf_{v \in S} \|u - v\|_s \leq d \right\}.$$ 

In the following, we will find a solution $u \in S^d$ of problem (2.2) for sufficiently small $\lambda > 0$ and some $0 < d < 1$. The following proposition is crucial to obtain a suitable (PS)-sequence for $\Gamma_{\lambda}$ and plays a key role in our proof.

Proposition 3.5. Let $\{\lambda_i\}_{i=1}^\infty$ be such that $\lim_{i \to \infty} \lambda_i = 0$ and $\{u_{\lambda_i}\} \subset S^d$ with

$$\lim_{i \to \infty} \Gamma_{\lambda_i}(u_{\lambda_i}) \leq E \quad \text{and} \quad \lim_{i \to \infty} \Gamma'_{\lambda_i}(u_{\lambda_i}) = 0.$$ 

Then for $d$ small enough, there is $u_0 \in S$, up to a subsequence, such that $u_{\lambda_i} \to u_0$ in $H^s_r(\mathbb{R}^3)$.

Proof. For convenience, we write $\lambda$ for $\lambda_i$. Since $u_\lambda \in S^d$ and $S$ is compact, we know $\{u_\lambda\}$ is bounded in $H^s_r(\mathbb{R}^3)$. Then by Lemma 2.4 we see that

$$\lim_{i \to \infty} L(u_{\lambda}) \leq E \quad \text{and} \quad \lim_{i \to \infty} L'(u_{\lambda}) = 0.$$ 

It follows from [13, Lemma 3.3] that, there is $u_0 \in H^s_r(\mathbb{R}^3)$, up to a subsequence, such that $u_\lambda \to u_0$ strongly in $H^s_r(\mathbb{R}^3)$. Obviously, $0 \not\in S^d$ for $d$ small. This implies that $u_0 \neq 0, L(u_0) \leq E$ and $L'(u_0) = 0$. Thus, $L(u_0) = E$, i.e., $u_0 \in S$. The proof is completed. □
By Proposition 3.5, for small $d \in (0, 1)$ such that there exist $\omega > 0, \lambda_0 > 0$ such that
\begin{equation}
\|\Gamma_\lambda'(u)\|_s \geq \omega \text{ for } u \in \Gamma^{D, \lambda}_\lambda \cap (S^d \setminus S^{d}_{\frac{3}{2}}) \text{ and } \lambda \in (0, \lambda_0).
\end{equation}
Similar to [38], we have

**Proposition 3.6.** There exists $\alpha > 0$ such that for small $\lambda > 0$,
\[ \Gamma_\lambda(\gamma(\tau)) \geq C_\lambda - \alpha \implies \gamma(\tau) \in S^{d}_{\frac{3}{2}}, \]
where $\gamma(\tau) = U(\frac{\tau}{\gamma}), \tau \in (0, \tau_0]$.

**Proof.** From Lemma 2.4 and the Pohožaev’s identity,
\[ \Gamma_\lambda(\gamma(\tau)) = \left(\frac{\tau^{3-2s}}{2} - \frac{3-2s}{6} \tau^3\right) \int_{\mathbb{R}^d} \left|(-\Delta)^{s/2}U\right|^2 + \lambda\tau^{3+2s}T(U). \]
Then
\[ \lim_{\lambda \to 0^+} \max_{\tau \in [0, \tau_0]} \Gamma_\lambda(\gamma(\tau)) = \max_{\tau \in [0, \tau_0]} \left(\frac{\tau^{3-2s}}{2} - \frac{3-2s}{6} \tau^3\right) \int_{\mathbb{R}^d} \left|(-\Delta)^{s/2}U\right|^2 = E. \]
The conclusion follows. \(\qed\)

Similarly as that in [38], thanks to (3.6) and Proposition 3.6, we can prove the following proposition, which assures the existence of a bounded Palais-Smale sequence for $\Gamma_\lambda$.

**Proposition 3.7.** For $\lambda > 0$ small enough, there exists $\{u_n\}_n \subset \Gamma^{D, \lambda}_\lambda \cap S^d$ such that $\Gamma'_\lambda(u_n) \to 0$ as $n \to \infty$.

**Proof of Theorem 1.1 concluded.** It follows from Proposition 3.7 that there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, there exists $\{u_n\}_n \in \Gamma^{D, \lambda}_\lambda \cap S^d$ with $\Gamma'_\lambda(u_n) \to 0$ as $n \to \infty$. Noting that $S$ is compact in $H^s_2(\mathbb{R}^3)$, we get that $\{u_n\}_n$ is bounded in $H^s_2(\mathbb{R}^3)$. Assume that $u_n \rightharpoonup u_\lambda$ weakly in $H^s_2(\mathbb{R}^3)$, then $\Gamma'_\lambda(u_\lambda) = 0$. It follows from the compactness of $S$ that $u_\lambda \in S^d$ and $\|u_n - u_\lambda\|_s \leq 3d$ for $n$ large. So $u_\lambda \neq 0$ for small $d > 0$. By Lemma 2.4,
\[ \Gamma_\lambda(u_n) = \Gamma_\lambda(u_\lambda) + \Gamma_\lambda(u_n - u_\lambda) + o(1). \]
Noting that $G_2(\tau) \geq \frac{m}{2}\tau^2$ for any $\tau \in \mathbb{R}$, it follows from (3.3) that for some $C > 0$,
\[ \Gamma_\lambda(u_n - u_\lambda) \geq \frac{1}{2} \int_{\mathbb{R}^3} \left|(-\Delta)^{s/2}(u_n - u_\lambda)\right|^2 + \frac{m}{4} |u_n - u_\lambda|^2 \ dx \]
\[ - C \int_{\mathbb{R}^3} |u_n - u_\lambda|^{2^*_s} \ dx. \]
Then by Lemma 2.2, for small $d > 0$, it is easy to verify that $\Gamma_\lambda(u_n - u_\lambda) \geq 0$ for large $n$. So $u_\lambda \in \Gamma^{D, \lambda}_\lambda \cap S^d$ with $\Gamma'_\lambda(u_\lambda) = 0$. Thus $u_\lambda$ is a nontrivial solution of (2.2). Finally, by Proposition 3.5 we can get the asymptotic behavior of $u_\lambda$ as $\lambda \to 0^+$. The proof is completed. \(\qed\)

4. The critical case
In this section, we consider the Schrödinger-Possion system (1.1) in the critical case. First, we establish the existence of ground state solutions to the fractional scalar field equation (1.2) with a general critical nonlinear term. Then by the perturbation argument, we seek the solutions of (1.1) in some neighborhood of the ground states to (1.2).
4.1. The limit problem. In this subsection, we use the constraint variational approach to seek the ground state solutions of (1.2). Similar argument also can be found in [8,18,37]. Let

\[ T(u) = \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \, dx, \quad V(u) = \int_{\mathbb{R}^3} G(u) \, dx. \]

We recall that \( U \) is said to be a ground state solution of (1.2) if and only if \( I(U) = m_0 \), where \( m_0 := \inf \{ I(u) : u \in H^s(\mathbb{R}^3) \setminus \{0\} \} \) is a solution of (1.2) and

\[ I(u) = T(u) - V(u). \]

The existence of ground state is reduced to look at the constraint minimization problem

(4.1) \( M := \inf \{ T(u) : V(u) = 1, u \in H^s(\mathbb{R}^3) \} \)

and eventually to remove the Lagrange multiplier by some appropriate scaling. Now, we state the main result in this subsection.

**Theorem 4.1.** Let \( s \in (0,1) \) and assume that \((H2)'- (H4)' \) and

(\(H0\)) \( g \in C(\mathbb{R}, \mathbb{R}) \) and \( g \) is odd, i.e., \( g(-\tau) = -g(\tau) \) for \( \tau \in \mathbb{R} \).

Then (1.2) admits a positive ground state solution.

**Remark 4.2.** Since we are concerned the positive solution of (1.2), \((H0)\) can be replaced by

(\(H0)' : g \in C(\mathbb{R}^+, \mathbb{R}) \). Moreover, similar to Theorem 4.1, a similar result in \( \mathbb{R}^N(N > 2s) \) also can be obtained.

**Proof of Theorem 4.1.** The proof follows the lines of that in [37]. For the completeness, we give the details here.

**Step 1.** Let \( M \) be given by (4.1) and \( S_s \) be the Sobolev best constant in Lemma 2.2 for \( s \in (0,1) \), then we claim that

\[ 0 < M < \frac{1}{2} (2_s^*)^{\frac{3-2s}{s}} S_s. \]

First, we prove that \( \{ u \in H^s(\mathbb{R}^3) : V(u) = 1 \} \neq \emptyset \). By [14,32], \( S_s \) can be achieved by

\[ U_\varepsilon(x) = \kappa \varepsilon^{-\frac{3-2s}{2s}} \left( \mu^2 + \frac{x}{\varepsilon S_{2s}^3} \right)^{\frac{3-2s}{s}}, \]

for any \( \varepsilon > 0 \), where \( \kappa \in \mathbb{R}, \mu > 0 \) are fixed constants. Let \( \varphi \in C^\infty_0(\mathbb{R}^3) \) is a cut-off function with support \( B_2 \) such that \( \varphi \equiv 1 \) on \( B_1 \) and \( 0 \leq \varphi \leq 1 \) on \( B_2 \), where \( B_r := \{ x \in \mathbb{R}^3 : |x| < r \} \). Let \( \psi_\varepsilon(x) = \varphi(x)U_\varepsilon(x) \), it follows from [32] that

(4.2) \[ \int_{\mathbb{R}^3} |\psi_\varepsilon|^2 = S_s^{3/(2s)} + O(\varepsilon^3), \quad \int_{\mathbb{R}^3} |(-\Delta)^{s/2}\psi_\varepsilon|^2 = S_s^{3/(2s)} + O(\varepsilon^{3-2s}). \]

Let

\[ v_\varepsilon = \frac{\psi_\varepsilon}{\| \psi_\varepsilon \|_{2s}}, \]

then \( \| (-\Delta)^{s/2} v_\varepsilon \|_2^2 \leq S_s + O(\varepsilon^{3-2s}) \). Let

\[ \Gamma_\varepsilon := \frac{1}{q} \| v_\varepsilon \|_q^q - \frac{1}{2} \| v_\varepsilon \|_2^2, \]

then by \((H4)'\) we have \( V(v_\varepsilon) \geq \frac{1}{2} \| v_\varepsilon \|_2^2 + \Gamma_\varepsilon \). In the following, we will show that

(4.3) \[ \lim_{\varepsilon \to 0} \frac{\Gamma_\varepsilon}{\varepsilon^{3-2s}} = +\infty. \]
By $\max\{2^*_s - 2, 2\} < q < 2^*_s$, we know that $(3 - 2s)q > 3$. Then it is easy to know there exist $C_1(s), C_2(s) > 0$ such that

$$
\|v_\varepsilon\|_q^q \geq \frac{1}{\|\psi_\varepsilon\|_2^q} \int_{B_1} |U_\varepsilon|^q \geq C_1(s)\varepsilon^{3 - \frac{3 - 2s}{2}} \int_0^{\varepsilon S_s^{2/(1-s)}} \frac{r^2}{(\mu^2 + r^2)^{\frac{3 - 2s}{2}}}dr
$$

$$
= O(\varepsilon^{3 - \frac{(3 - 2s)\mu}{2}}),
$$

$$
\|v_\varepsilon\|_2^2 \leq \frac{1}{\|\psi_\varepsilon\|_2^2} \int_{B_2} |U_\varepsilon|^2 \leq C_2(s)\varepsilon^{2s} \int_0^{\varepsilon S_s^{2/(1-s)}} \frac{r^2}{(\mu^2 + r^2)^{3 - 2s}}dr
$$

$$
= \begin{cases} 
O(\varepsilon^{2s}), & \text{if } s < \frac{3}{4}; \\
O(\varepsilon^{2s}\ln \frac{1}{\varepsilon}), & \text{if } s = \frac{3}{4}; \\
O(\varepsilon^{3 - 2s}), & \text{if } s > \frac{3}{4}.
\end{cases}
$$

Then, we obtain

$$
\Gamma_\varepsilon \geq O(\varepsilon^{3 - \frac{(3 - 2s)\mu}{2}}), \quad \text{if } s \in (0, 1).
$$

Noting that $\max\{2^*_s - 2, 2\} < q < 2^*_s$, it is easy to verify that (4.3) is true. Thus, it follows that $V(v_\varepsilon) > 0$ for small $\varepsilon > 0$. By a scaling, we get that $\{u \in H^s(\mathbb{R}^3) : V(u) = 1\} \neq \emptyset$.

Next, obviously, $M \in (0, +\infty)$. For small $\varepsilon > 0$, $V(v_\varepsilon) > 0$, then

$$
M \leq \frac{T(v_\varepsilon)}{(V(v_\varepsilon))^{\frac{2}{2s}}} \leq \frac{1}{2} \frac{\|(-\Delta)^{s/2} v_\varepsilon\|_2^2}{\left(\frac{2}{2s} + \Gamma_\varepsilon\right)} \leq \frac{1}{2} (2^*_s)^{\frac{2}{2s}} S_s \frac{1 + O(\varepsilon^{N - 2s})}{(1 + 2^*_s \Gamma_\varepsilon)^{\frac{2}{2s}}}.
$$

If $p \geq 1$, then $(1 + t)^p \leq 1 + p(1 + t)^{1+p}t$, for all $t \geq -1$. It follows from (4.3) that

$$
(1 + O(\varepsilon^{N - 2s}))^{\frac{2}{2s}} - 1 \leq \frac{2^*_s}{2} (1 + O(\varepsilon^{N - 2s}))^{1 + \frac{2s}{2s}} O(\varepsilon^{N - 2s}) < 2^*_s \Gamma_\varepsilon,
$$

for small $\varepsilon > 0$, which yields $1 + O(\varepsilon^{N - 2s}) < (1 + 2^*_s \Gamma_\varepsilon)^{2/2s}$. Then $M < \frac{1}{2} (2^*_s)^{\frac{3 - 2s}{2}} S_s$.

**Step 2.** $M$ can be achieved. Noting that $g$ is odd and the Fractional Polya-Szegö inequality [30], without loss of generality, we can assume that there exists a positive minimizing sequence $\{u_n\} \subset H^s(\mathbb{R}^3)$ such that $V(u_n) = 1$ and $T(u_n) \to M$ as $n \to \infty$. By Lemma 2.2, it is easy to know that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. By Lemma 2.1, we can assume that $u_n \to u_0$ weakly in $H^s(\mathbb{R}^3)$, strongly in $L^q(\mathbb{R}^3)$ and a.e. in $\mathbb{R}^3$. Set $v_n = u_n - u_0$, then $T(u_n) = T(v_n) + T(u_0) + o(1)$, and

$$
\|u_n\|_2^2 = \|v_n\|_2^2 + \|u_0\|_2^2 + o(1), \quad \|u_n\|_2^2 = \|v_n\|_2^2 + \|u_0\|_2^2 + o(1),
$$

where $o(1) \to 0$ as $n \to \infty$. Let $f(s) = g(s) - s^{2^*_s - 1} + s$, then it follows from Lemma 3.2 that

$$
\int_{\mathbb{R}^3} F(u_n) = \int_{\mathbb{R}^3} F(u_0) + \int_{\mathbb{R}^3} F(v_n) + o(1).
$$

So, $V(u_n) = V(v_n) + V(u_0) + o(1)$.

Next, we prove $u_0$ is the minimizer for $M$. Set $S_n = T(v_n), S_0 = T(u_0), V(v_n) = \lambda_n, V(u_0) = \lambda_0$, we have $\lambda_n = 1 - \lambda_0 + o(1)$ and $S_n = M - S_0 + o(1)$. Under a scale change, we get that

$$
(4.4) \quad T(u) \geq M(V(u))^{\frac{3 - 2s}{2}},
$$
for all \( u \in H^s(\mathbb{R}^3) \) and \( V(u) \geq 0 \). By (4.4) we have \( \lambda_0 \in [0, 1] \). If \( \lambda_0 \in (0, 1) \), then by (4.4) again,

\[
M = \lim_{n \to \infty} (S_0 + S_n) \\
\geq \lim_{n \to \infty} M \left( (\lambda_0)^{\frac{3-2s}{3}} + (\lambda_n)^{\frac{3-2s}{3}} \right) \\
= M \left( (\lambda_0)^{\frac{3-2s}{3}} + (1 - \lambda_0)^{\frac{3-2s}{3}} \right) \\
> M(\lambda_0 + 1 - \lambda_0) = M,
\]

which is a contradiction. On the other hand, if \( \lambda_0 = 0 \), then \( S_0 = 0 \), which implies \( u_0 = 0 \). Then

\[
\limsup_{n \to \infty} \|v_n\|_{2}^2 \geq (2_\ast^s)^{\frac{3-2s}{3}}
\]

and

\[
M = \frac{1}{2} \lim_{n \to \infty} \|(-\Delta)^{s/2}v_n\|_2^2 \geq \frac{1}{2}(2_\ast^s)^{\frac{3-2s}{3}} \lim_{n \to \infty} \inf \frac{\|(-\Delta)^{s/2}v_n\|_2^2}{\|v_n\|_{2}^2} \geq \frac{1}{2}(2_\ast^s)^{\frac{3-2s}{3}} S_\ast,
\]

which is again a contradiction. Then we conclude \( \lambda_0 = 1 \), i.e., \( M \) is achieved by \( u_0 \).

Finally, let \( U(\cdot) = u_0(\cdot/\sigma_0) \), where \( \sigma_0 = (\frac{3-2s}{3}M)^{1/2} \), then \( U \) is a ground state solution of (1.2). The proof is completed.

**Remark 4.3.** Furthermore, similar as that in [13], if we assume that \( g \in C^1(\mathbb{R}, \mathbb{R}) \) additionally, \( U \) satisfies the Pohožaev identity

\[
(4.5) \quad \frac{3-2s}{2} \int_{\mathbb{R}^3} \|(-\Delta)^{s/2}U\|^2 \, dx = 3 \int_{\mathbb{R}^3} G(U) \, dx.
\]

Similar as that in [26,37], \( U \) is also a mountain pass solution.

Let \( S_1 \) be the set of positive radial ground state solutions \( U \) of (1.2), then similar to Step 2 in the proof of Theorem 4.1, we have the following compactness result.

**Proposition 4.4.** Under the assumptions in Theorem 4.1, \( S_1 \) is compact in \( H^s(\mathbb{R}^3) \).

4.2. Proof of Theorem 1.4. In the following, we are ready to prove Theorem 1.4. Similar to Section 3, take \( U \in S_1 \), let

\[ \quad U_\tau(x) = U(\frac{x}{\tau}), \quad \tau > 0, \]

then there exists \( \tau_1 > 1 \) such that \( I(U_\tau) < -2 \) for \( \tau \geq \tau_1 \). Set

\[ \quad D^1_\lambda \equiv \max_{\tau \in [0, \tau_1]} \Gamma_\lambda(U_\tau), \]

there exist \( \lambda_2 > 0 \) and \( C_1 > 0 \) such that for any \( 0 < \lambda < \lambda_2 \),

\[ \phi \neq \Upsilon_\lambda = \{ \gamma \in C([0, \tau_1], H^s(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(\tau_1) = U_{\tau_1}, \|\gamma(\tau)\|_s \leq C_1 + 1, \tau \in [0, \tau_1] \}. \]

Then for any \( \lambda \in (0, \lambda_1) \), we define a min-max value \( C^1_\lambda \):

\[ \quad C^1_\lambda = \inf_{\gamma \in \Upsilon_\lambda} \max_{\tau \in [0, \tau_1]} \Gamma_\lambda(\gamma(\tau)). \]

Similar to Section 3, we have

**Proposition 4.5.** \( \lim_{\lambda \to 0^+} C^1_\lambda = \lim_{\lambda \to 0^+} D^1_\lambda = m \), where \( m \) is the least energy of (1.2).
Now for $\alpha, d > 0$, define

$$\Gamma_{\lambda}^{\alpha} := \{ u \in H_{s}^{\alpha}(\mathbb{R}^{3}) : \Gamma_{\lambda}(u) \leq \alpha \}$$

and

$$S_{1}^{d} = \left\{ u \in H_{s}^{\alpha}(\mathbb{R}^{3}) : \inf_{v \in S_{1}} \| u - v \|_{s} \leq d \right\}.$$

Similar as that in Section 3, for small $\lambda > 0$ and some $0 < d < 1$, we will find a solution $u \in S_{1}^{d}$ of (2.2) in the critical case. Similar to [38], we can get the following compactness result, which can yield the gradient estimate of $\Gamma_{\lambda}$.

**Proposition 4.6.** Let $\{\lambda_{i}\}_{i=1}^{\infty}$ be such that $\lim_{i \to \infty} \lambda_{i} = 0$ and $\{u_{\lambda_{i}}\} \subset S_{1}^{d}$ with

$$\lim_{i \to \infty} \Gamma_{\lambda_{i}}(u_{\lambda_{i}}) \leq m \quad \text{and} \quad \lim_{i \to \infty} \Gamma'_{\lambda_{i}}(u_{\lambda_{i}}) = 0.$$

Then for $d$ small enough, there is $u_{1} \in S_{1}$, up to a subsequence, such that $u_{\lambda_{i}} \to u_{1}$ in $H_{s}^{\alpha}(\mathbb{R}^{3})$.

**Proof.** For convenience, we write $\lambda$ for $\lambda_{i}$. Since $\lambda \in S_{1}^{d}$ and $S_{1}$ is compact, we know $\{u_{\lambda}\}$ is bounded in $H_{s}^{\alpha}(\mathbb{R}^{3})$. Moreover, up to a subsequence, there exists $u_{1} \in S_{1}^{d}$ such that $u_{\lambda} \to u_{1}$ weakly in $H_{s}(\mathbb{R}^{3})$, a.e. in $\mathbb{R}^{3}$ and $\|u_{\lambda} - u_{1}\|_{s} \leq 3d$ for $i$ large. Then by Lemma 2.4 we see that

$$\lim_{i \to \infty} I(u_{\lambda}) \leq m \quad \text{and} \quad \lim_{i \to \infty} I'(u_{\lambda}) = 0.$$

Then $I'(u_{1}) = 0$. Obviously, $u_{0} \neq 0$ if $d$ small. So $I(u_{1}) \geq m$. Meanwhile, thanks to Lemma 3.2, $I(u_{\lambda}) = I(u_{1}) + I(u_{\lambda} - u_{1}) + o(1)$, then we have

$$I(u_{\lambda} - u_{1}) = \frac{1}{2} \| u_{\lambda} - u_{1} \|_{s}^{2} - \frac{1}{2\alpha} \| u_{\lambda} - u_{1} \|_{2^*}^{2^*} + o(1) \leq o(1).$$

Then by Lemma 2.2, for $d$ small enough, $u_{\lambda} \to u_{1}$ strongly in $H_{s}^{\alpha}(\mathbb{R}^{3})$. The proof is completed. \[\square\]

By Proposition 4.6, for small $d \in (0, 1)$ there exist $\omega_{1} > 0, \lambda_{2} \in (0, \lambda_{1})$ such that

$$\|\Gamma'_{\lambda}(u)\|_{s} \geq \omega_{1} \text{ for } u \in \Gamma_{\lambda}^{D_{1}^{\alpha}} \cap (S_{1}^{d} \setminus S_{1}^{d}) \text{ and } \lambda \in (0, \lambda_{2}).$$

Similar to Section 3, we have

**Proposition 4.7.** There exists $\alpha_{1} > 0$ such that for small $\lambda > 0$,

$$\Gamma_{\lambda}(\gamma(\tau)) \geq C_{\lambda}^{1} - \alpha_{1} \text{ implies that } \gamma(\tau) \in S_{1}^{d},$$

where $\gamma(\tau) = U(\tau), \tau \in (0, \tau_{1}]$.

**Proof of Theorem 1.4 concluded.** With the help of (4.6) and Proposition 4.7, similarly as that in [38], for $\lambda > 0$ small enough, there exists $\{u_{n}\}_{n} \subset \Gamma_{\lambda}^{D_{1}^{\alpha}} \cap S_{1}^{d}$ such that $\Gamma'_{\lambda}(u_{n}) \to 0$ as $n \to \infty$. Similar as above, there exists $u_{\lambda} \in S_{1}^{d}$ with $u_{\lambda} \neq 0$ for small $d > 0$. Moreover, up to a subsequence, $u_{n} \to u_{\lambda}$ weakly in $H_{s}^{\alpha}(\mathbb{R}^{3})$ and a.e. in $\mathbb{R}^{3}$, and $\|u_{n} - u_{\lambda}\|_{s} \leq 3d$ for $n$ large. Furthermore, $\Gamma'_{\lambda}(u_{\lambda}) = 0$. By Lemma 2.4,

$$\Gamma_{\lambda}(u_{n}) = \Gamma_{\lambda}(u_{\lambda}) + \Gamma_{\lambda}(u_{n} - u_{\lambda}) + o(1).$$

By (H2)'-(H3)', for some $C > 0$,

$$\Gamma_{\lambda}(u_{n} - u_{\lambda}) \geq \frac{1}{2} \int_{\mathbb{R}^{3}} \|(-\Delta)^{s/2}(u_{n} - u_{\lambda})\|^{2} + \frac{1}{2} \|u_{n} - u_{\lambda}\|^{2} \, dx$$

$$- C \int_{\mathbb{R}^{3}} |u_{n} - u_{\lambda}|^{2} \, dx.$$
Then by Lemma 2.2, \( \liminf_{\lambda \to \infty} \Gamma_\lambda(u_\lambda - u_\lambda) \geq 0 \) for small \( d > 0 \). So \( u_\lambda \in \Gamma_\lambda^{D_1} \cap S_1^d \) with \( \Gamma_\lambda'(u_\lambda) = 0 \). Thus \( u_\lambda \) is a nontrivial solution of (2.2). The asymptotic behavior of \( u_\lambda \) follows from Proposition 4.6. The proof is completed. \( \square \)

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