A quiver approach to affine Hecke algebras

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Abstract

We give a presentation of (localized) affine Hecke algebras in terms of weights of the polynomial subalgebra and varied Demazure-BGG type operators. This paper extends the result [Rou, Theorem 3.18], where the affine and degenerate affine Hecke algebras for $GL_n$ were considered.

Introduction

We give a presentation of (localized) affine Hecke algebras in terms of weights of the polynomial subalgebra and varied Demazure-BGG type operators. This paper extends the result [Rou, Theorem 3.18], where the affine and degenerate affine Hecke algebras for $GL_n$ were considered.

In the first chapter we discuss the Bernstein presentation of the affine Hecke algebra,

$$\mathcal{H} \cong \mathcal{H}^f \otimes_{\mathcal{A}} \mathcal{A}.$$ 

Here, $\mathcal{A} \cong \mathbb{C}[X]$, is the group ring of a lattice, hence may be considered as the ring of regular functions on a complex torus, $\mathcal{T} \cong Y \otimes_{\mathbb{Z}} \mathbb{C}^*$. 

Before going further, we define the interpolating Hecke algebra $\mathcal{H}^h$, a Rees algebra. This algebra over $\mathbb{C}[h]$, when specialized to $h = 0$ yields the degenerate affine Hecke algebra, and for $h \neq 0$ yields the affine Hecke algebra. It is interesting that many analogous formulas appearing in the theory of $\mathcal{H}$ and $\mathbb{H}$ may be interpolated to give one formula for $\mathcal{H}^h$, which specializes appropriately.

Next, we consider the non-unital algebra,

$$\mathcal{A}^h = \bigoplus_{\lambda \in \text{Hom}_{\text{alg}}(\mathcal{A}^h, \mathcal{C})} \mathcal{A}^h_{\lambda},$$
which is the direct sum over all localizations of $A^h$ at points of $\mathfrak{T}^h$. By the generalized eigenspace decomposition, this ring has the same finite unital representations as $A^h$. We extend this ring, defining the localized affine Hecke algebra, $\dot{H}^h$.

Though the localized algebras $\dot{H}^h$ essentially appear in [Lus], the advantage of this paper is a presentation of this algebra with generators, as well as a basis, which preserve the weight space decomposition of finite representations. This is the content of chapter 3. As in [Rou], we consider more general algebras, $\dot{H}^h(G)$, depending on a datum $G$ which is analogous to the polynomial Cartan matrix $(Q_{i,j}(u, u'))$. In section 2.6 it is shown that the data $G$ is indeed determined by an integral matrix. It is unclear at this time where else the more general algebras $\dot{H}^h(G)$ appear, but the considerable applications of the quiver Hecke algebras defined in [Rou] give cause for their study.

The basis constructed in Theorem 2.5.4 is well suited to the algebraic study of characters of finite representations. In the last chapter we provide an algebraic construction of all irreducible representations which have a non-zero eigenspace $V_\lambda$ with $\lambda$ a standard parabolic weight, in both the equal and unequal parameters case. This is accomplished by studying the so-called weight Hecke algebras, $\lambda H^h$, which turn out to be matrix rings for $\lambda$ standard parabolic. In the equal parameter case this may be shown geometrically, by noting that $\lambda H^h$ is isomorphic the the algebraic $K$-theory of the product of two flag varieties for the centralizer of $\lambda$ in the complex group $G$ associated to the root data (c.f. [Gi]). This chapter ends with an example computation of the character of such a representation.

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1 Affine Hecke algebras

1.1 Bernstein’s presentation

We recall the definition of affine Hecke algebras, following [Lus]. Let \((X, Y, R, \hat{R}, \Pi)\) be a simply connected, reduced root datum. Thus, \(X, Y\) are finitely generated free abelian groups in perfect pairing we denote by \(\langle \cdot, \cdot \rangle\). Further, the finite subsets \(R \subset X\), \(\hat{R} \subset Y\) are in a given bijection \(\alpha \mapsto \hat{\alpha}\). The set \(R\) is invariant under the simple reflections, \(s_{\alpha}\), which are given by the following explicit formula:

\[
s_{\alpha}(x) = x - \langle x, \hat{\alpha} \rangle \alpha.
\]

Similarly, it is required that \(\hat{R}\) be invariant under \(s_{\hat{\alpha}}\),

\[
s_{\hat{\alpha}}(y) = y - \langle y, \alpha \rangle \hat{\alpha}.
\]

Recall that \(W\) denotes the finite Weyl group of the system, and \(\Pi \subset R\) is a root basis.

Lastly, the root datum being simply connected means \(Y\) is the \(\mathbb{Z}\)-span of the coroots, \(\hat{R}\). Thus, \(\{\hat{\alpha} \mid \alpha \in \Pi\}\) forms a basis of \(Y\). This simplifies a number of the formulas in [Lus]. In particular \(\hat{\alpha} \notin 2Y\) for any \(\alpha \in \Pi\).

Finally, we also fix a parameter set, which for this paper is given by a collection \(q_{\alpha} = q_{\alpha} \in \mathbb{C}^{*}, \alpha \in \Pi\) such that \(q_{\alpha} = q_{\beta}\) whenever the order of \(s_{\alpha}s_{\beta}\) in \(W\) is odd.

With this data we associate the affine Hecke algebra, \(\mathcal{H}\), which appears naturally in the complex, admissible representation theory of the associated algebraic group over \(p\)-adic fields.

For \(\alpha, \beta \in \Pi, \alpha \neq \beta\) let \(m_{\alpha, \beta}\) be the order of \(s_{\alpha}s_{\beta}\) in the Weyl group. As convention, we also put \(m_{\alpha, \alpha} = 2\). We let \(\mathcal{H}^{f}\) denote the finite Hecke algebra, which is the \(\mathbb{C}\)-algebra generated by symbols \(T_{\alpha} = T_{s_{\alpha}}\), indexed by \(\Pi\), with the relations:

i. \(\cdots T_{\beta}T_{\alpha} = \cdots T_{\alpha}T_{\beta}\), \(\alpha \neq \beta\), with \(m_{\alpha, \beta}\) terms on both sides,
ii. \((T_\alpha + 1)(T_\alpha - q_\alpha) = 0, \quad \alpha \in \Pi.\)

Let \(\mathcal{A}\) be the group ring of \(X\),

\[
\mathcal{A} = \mathbb{C}[e^x, x \in X],
\]

\[
e^x e^{x'} = e^{x+x'},
\]

which we remark is a domain.

We denote by \(\mathcal{H}\) the affine Hecke algebra of the root system. Recall that as an additive group,

\[
\mathcal{H} = \mathcal{H}^f \otimes_{\mathbb{C}} \mathcal{A}.
\]

As for the ring structure, recall that \(\mathcal{H}^f\) and \(\mathcal{A}\) are subrings which obey the following commutativity relation,

\[
T_\alpha f - s_\alpha(f)T_\alpha = \frac{f - s_\alpha(f)}{1 - e^{-\alpha}}.
\]

While \((1 - e^{-\alpha})^{-1} \notin \mathcal{A}\), the right side of the above commutativity formula is in \(\mathcal{A}\). For example, when \(\langle x, \alpha \rangle > 0\),

\[
\frac{e^x - e^{s_\alpha(x)}}{1 - e^{-\alpha}} = e^x \cdot \frac{1 - e^{-(x,\check{\alpha})\alpha}}{1 - e^{-\alpha}}
\]

\[
= e^x \cdot (1 + e^{-\alpha} + \cdots e^{\langle(x,\check{\alpha})-1\rangle(-\alpha)})
\]

\[
= e^x + e^{x-\alpha} + \cdots + e^{s_\alpha(x)+\alpha},
\]

and when \(\langle x, \alpha \rangle < 0\),

\[
\frac{e^x - e^{s_\alpha(x)}}{1 - e^{-\alpha}} = e^x \cdot \frac{1 - e^{-(x,\check{\alpha})\alpha}}{1 - e^{-\alpha}}
\]

\[
= -e^{x+\alpha} \cdot \frac{1 - e^{-\langle(x,\check{\alpha})\rangle\alpha}}{1 - e^{-\alpha}}
\]

\[
= -e^{x+\alpha} \cdot (1 + e^{\alpha} + \cdots e^{\langle(x,\check{\alpha})-1\rangle\alpha}),
\]

\[
= -(e^{x+\alpha} + e^{x+2\alpha} + \cdots + e^{s_\alpha(x)}).
\]

We will discuss the fraction above, and the formulas for it in the following sections.
1.2 Graded and interpolating affine Hecke algebras

The degenerate affine Hecke algebra is introduced in this section, along with an interpolating Hecke algebra. We show how the quiver Hecke algebra approach may be applied to the interpolating Hecke algebra.

Let \((X, Y, R, \tilde{R}, \Pi)\) be a simply connected root datum. A set of parameters for the degenerate affine Hecke algebra, \(H\), is a collection, \(c_{\alpha} \in \mathbb{C}^*\), such that \(c_{\alpha} = c_{\beta}\) when \(m_{\alpha, \beta}\) is odd. Let \(A = S(X)\) be the symmetric algebra of \(X\), a polynomial algebra over \(\mathbb{C}\) with variables given by a basis of \(X\). As an additive group, let \(H = \mathbb{C}[W] \otimes_{\mathbb{C}} A\), where \(\mathbb{C}[W]\) denotes the group ring of the Weyl group, generated by the simple reflections \(s_{\alpha} \in W, \alpha \in \Pi\). Let \(\mathbb{C}[W]\) and \(A\) be subrings of \(H\), and impose the following commutativity relation between \(s_{\alpha} \in \mathbb{C}[W]\) and \(x \in X \subset A\):

\[
s_{\alpha} \cdot x - x \cdot s_{\alpha} = \frac{x - s_{\alpha}(x)}{\alpha}.
\]

The interpolating Hecke algebra, \(H^h\), is an algebra defined using a parameter \(h\) such that when \(h = 0\), \(H^h \cong H\) and for \(h \neq 0\), \(H^h \cong \mathcal{H}\). The parameters \(q_s, c_s\) for \(H\) and \(H^h\) must also satisfy \(q_s = 1 + h c_s\).

Consider the polynomial ring, \(\mathcal{A}^h\), over \(\mathbb{C}[h]\) with generators \(\{P_x \mid x \in X\}\). The symmetric algebra, \(A = S(X)\), is the quotient of \(\mathcal{A}^h\) by the relation \(P_x + P_y = P_{x+y}, h = 0\). Let \(\mathcal{A}^h\) be the quotient of \(\mathcal{A}^h\) by the relations

\[
P_x + P_y + h P_x P_y = P_{x+y},
\]

\[
P_0 = 0.
\]

It is clear that for \(h = 0\), \(\mathcal{A}^h \cong S(X)\). Let,

\[
U_x = 1 + h P_x.
\]

Notice that,

\[
U_x U_y = h^2 P_x P_y + h(P_x + P_y) + 1, \\
= h P_{x+y} + 1, \\
= U_{x+y}.
\]

It follows that \(\mathcal{A}^h\) is isomorphic to the \(\mathbb{C}[h]\)-subalgebra of \(\mathbb{C}[h^{\pm 1}] \otimes_{\mathbb{C}} \mathcal{A}\) generated by \(\{P_x = h^{-1}(e^x - 1)\}_{x \in X}\). Hence, \(\mathcal{A}^h[h^{-1}]\) is isomorphic to the group ring of \(X\) over \(\mathbb{C}[h^{\pm 1}]\). Note that \(W\) acts on \(\mathcal{A}^h\) and this action is compatible with the actions of \(W\).
on $A$ and $\mathcal{A}$. This gives an action of $W$ on $\mathcal{F}^h := \text{Hom}_{\mathbb{C}}(\mathcal{A}^h, \mathbb{C})$. Moreover, as the root datum $(X, Y, R, \tilde{R}, \Pi)$ is simply connected, we find for $\alpha \in \Pi$, that $s_\alpha(\lambda) = \lambda$ if and only if $\lambda(P_{-\alpha}) = 0$.

As an additive group let $\mathcal{H}^h = \mathcal{H}^f \otimes_{\mathbb{C}} \mathcal{A}^h$ be the tensor product of the finite Hecke algebra, $\mathcal{H}^f$, associated to the Weyl group $W$ with parameter set $q_s$, and $\mathcal{A}^h$.

Let $\mathcal{A}^h$ and $\mathcal{H}^f$ be subalgebras, and give $\mathcal{H}^h$ the following commutativity relation:

$$T_\alpha P_x - P_{s_\alpha(x)}T_\alpha = \begin{cases} 0 & \text{if } \langle x, \tilde{\alpha} \rangle = 0, \\ c_\alpha \left( \langle x, \tilde{\alpha} \rangle + h(P_x + P_{x-\alpha} + \cdots + P_{s_\alpha(x)+\alpha}) \right) & \text{if } \langle x, \tilde{\alpha} \rangle > 0, \\ c_\alpha \left( \langle x, \tilde{\alpha} \rangle - h(P_{x+\alpha} + P_{x+2\alpha} + \cdots + P_{s_\alpha(x)}) \right) & \text{if } \langle x, \tilde{\alpha} \rangle < 0. \end{cases}$$

From these relations we find $\mathcal{H}^h \cong \mathbb{H}$ for $h = 0$. For $h \neq 0$, we note that for $\langle x, \tilde{\alpha} \rangle > 0$, the number of terms in the sum $P_x + P_{x-\alpha} + \cdots + P_{s_\alpha(x)+\alpha}$ is precisely $\langle x, \tilde{\alpha} \rangle$. Similarly, for $\langle x, \tilde{\alpha} \rangle < 0$ the number of terms in $P_{x+\alpha} + P_{x+2\alpha} + \cdots + P_{s_\alpha(x)}$ is precisely $-\langle x, \tilde{\alpha} \rangle$. Using the fact that $U_y = 1 + hP_y$, as well as $q_\alpha - 1 = hc_\alpha$ we see,

$$T_\alpha U_x - U_{s_\alpha(x)}T_\alpha = \begin{cases} 0 & \text{if } \langle x, \tilde{\alpha} \rangle = 0, \\ (q_\alpha - 1) \left( U_x + U_{x-\alpha} + \cdots + U_{s_\alpha(x)+\alpha} \right) & \text{if } \langle x, \tilde{\alpha} \rangle > 0, \\ -(q_\alpha - 1) \left( U_{x+\alpha} + U_{x+2\alpha} + \cdots + U_{s_\alpha(x)} \right) & \text{if } \langle x, \tilde{\alpha} \rangle < 0. \end{cases}$$

These are nothing more than the commutativity relations for the affine Hecke algebra $\mathcal{H}$.

### 1.3 BGG and Demazure operators

Now we define BGG operators and a variant of Demazure operators to discuss the representations of Hecke algebras on their commutative subalgebras. Define $\Delta_\alpha : S(X) \to S(X)$ by the following formula:

$$\Delta_\alpha(f) = \frac{f - s_\alpha(f)}{\alpha}.$$ 

As usual, the right side of the formula actually lies in $S(X)$. Note that the commutativity relation for $\mathbb{H}$ may be written

$$s_\alpha \cdot f - s_\alpha(f) \cdot s_\alpha = c_{s_\alpha} \Delta_\alpha(f),$$

where $f \in A$. 

Define $D_{\alpha} : \mathcal{A} \rightarrow \mathcal{A}$ by the following formula:

$$D_{\alpha}(f) = \frac{f - s_{\alpha}(f)}{1 - e^{-\alpha}}.$$ 

Note that the commutativity relation for $\mathcal{H}$ may be written

$$T_{\alpha}f - s_{\alpha}(f)T_{\alpha} = (q_{\alpha} - 1)D_{\alpha}(f).$$

Recall that the algebra $\mathcal{A}^h$ is a subalgebra of the localization $\mathbb{C}[X][h, h^{-1}]$ gotten by mapping $P_x$ to $h^{-1}(e^x - 1)$. We compute:

$$hD_{\alpha}(P_x) = D_{\alpha}(e^x - 1) = D_{\alpha}(e^x) = \begin{cases} 0 & \text{if } \langle x, \check{\alpha} \rangle = 0, \\ \langle x, \check{\alpha} \rangle + h(P_x + P_{x-\alpha} + \cdots P_{s_{\alpha}(x)+\alpha}) & \text{if } \langle x, \check{\alpha} \rangle > 0, \\ \langle x, \check{\alpha} \rangle - h(P_{x+\alpha} + P_{x+2\alpha} + \cdots P_{s_{\alpha}(x)}) & \text{if } \langle x, \check{\alpha} \rangle < 0, \end{cases}$$

to see that $D_{\alpha} : \mathcal{A}^h \rightarrow h^{-1}\mathcal{A}^h$. To put it informally, $D_{\alpha}$ is singular at $h = 0$. Nonetheless we have a well defined operator $hD_{\alpha} : \mathcal{A}^h \rightarrow \mathcal{A}^h$.

We make a few remarks about the operators we have just defined. The first is that the classical Demazure operators, $\tilde{D}_{\alpha} : \mathcal{A} \rightarrow \mathcal{A}$, given by

$$\tilde{D}_{\alpha} : f \mapsto \frac{f - e^{-\alpha}s_{\alpha}(f)}{1 - e^{-\alpha}},$$

may be expressed in terms of $D_{\alpha}$. Let $\rho \in X$ be defined by $\langle \rho, \check{\alpha} \rangle = 1, \alpha \in \Pi$, and recall that $\hat{e}^{\rho} : \mathcal{A} \rightarrow \mathcal{A}$ is the invertible operator given by multiplication by $e^{\rho}$. We claim that,

$$\tilde{D}_{\alpha} = \hat{e}^{-\rho} \circ D_{\alpha} \circ \hat{e}^{\rho}.$$ 

Indeed, $s_{\alpha}(e^{\rho}) = e^{-\alpha}e^{\rho}$, hence,

$$\hat{e}^{-\rho} \circ D_{\alpha} \circ \hat{e}^{\rho}(f) = e^{-\rho} \cdot \frac{e^{\rho}f - s_{\alpha}(e^{\rho}f)}{1 - e^{-\alpha}} = \frac{f - e^{-\alpha}s_{\alpha}(f)}{1 - e^{-\alpha}}.$$ 

Thus, it would be appropriate to refer to $D_{\alpha}$ as a twisted Demazure operator. As the classical Demazure operators satisfy the braid relations, so do the $D_{\alpha}$

$$\cdots D_{\beta}D_{\alpha} = \cdots D_{\alpha}D_{\beta}, \quad m_{\alpha, \beta} \text{ terms.}$$
It is also important to note that $D_\alpha$ does not specialize to $\Delta_\alpha$ when $h = 0$, but rather $hD_\alpha$ specializes to $\Delta_\alpha$. This proves that the $\Delta_\alpha$ also satisfy the braid relations. Further, the quadratic relation $D_\alpha^2 = D_\alpha$ gives $(hD_\alpha)^2 = h(hD_\alpha)$, which specializes to 0 when $h \to 0$, showing $\Delta_\alpha^2 = 0$.

### 1.4 Demazure-Lusztig representations of Hecke algebras

Consider the representation of the finite Hecke algebra $\mathcal{H}^f$ on $\mathbb{C}$ given by sending $T_\alpha \mapsto q_\alpha$. There is a representation of $\mathcal{H}^h$ on $\mathcal{H}^h \otimes \mathcal{H}^f \mathbb{C} \cong \mathcal{A}^h$. We write $T_\alpha \mapsto \hat{q}_\alpha$, $\ldots : \mathcal{A}^h \to \mathcal{A}^h$ for the action of $T_\alpha, q_\alpha, \ldots$ as operators on $\mathcal{A}^h$. We claim,

$$T_\alpha - q_\alpha : P_\omega \mapsto (c_\alpha + q_\alpha P_\omega - hD_\alpha(P_\omega)). \quad (2)$$

The claim is obvious for $\langle x, \hat{\alpha} \rangle = 0$. We show the case $\langle x, \hat{\alpha} \rangle > 0$, the other case being nearly identical. By the commutativity relation for $\mathcal{H}^h$ we find:

$$\begin{align*}
T_\alpha - q_\alpha(P_\omega) &= P_{s_\alpha(\omega)}T_\alpha \otimes 1 + (q_\alpha - 1)D_\alpha(P_\omega) - q_\alpha P_\omega, \\
&= q_\alpha P_{s_\alpha(\omega)} - q_\alpha P_\omega + c_\alpha hD_\alpha(P_\omega) \\
&= q_\alpha (P_{s_\alpha(\omega)} - P_\omega) + (q_\alpha - 1)(h^{-1}\langle x, \hat{\alpha} \rangle + P_\omega + \cdots + P_{s_\alpha(\omega) + \alpha}) \\
&= q_\alpha (h^{-1}\langle x, \hat{\alpha} \rangle + P_{x - \alpha} + \cdots + P_{s_\alpha(\omega)}) - (h^{-1}\langle x, \hat{\alpha} \rangle + P_\omega + \cdots + P_{s_\alpha(\omega) + \alpha}) \\
&= (q_\alpha (1 + hP_{-\alpha}) - 1)D_\alpha(P_\omega) \\
&= (q_\alpha - 1 + q_\alpha hP_{-\alpha})D_\alpha(P_\omega) \\
&= (c_\alpha + q_\alpha P_{-\alpha})hD_\alpha(P_\omega) \\
&= (c_\alpha + P_{-\alpha} + hc_\alpha P_{-\alpha})hD_\alpha(P_\omega).
\end{align*}$$

Here we have used that $D_\alpha(P_\omega) = h^{-1}D_\alpha(U_x)$, and that $1 + hP_{-\alpha} = U_{-\alpha}$ which satisfies the relation, $U_{-\alpha} U_y = U_{y - \alpha}$.

### 1.5 Weight spaces of $\mathcal{A}^h$-modules

Let $h \in \mathbb{C}$ and define $\mathcal{T}^h = \text{Hom}_{\mathcal{A}^h}(\mathcal{A}^h, \mathbb{C})$. As $\mathcal{A}^h$ is isomorphic to the symmetric algebra of $X$ for $h = 0$ and the group ring of $X$ for $h \neq 0$, we find that $\mathcal{T}^h \cong \mathfrak{h}$ for $h = 0$ and $\mathcal{T}^h \cong \mathcal{T}$ for $h \neq 0$. Motivated by the following paragraph we will call $\mathcal{T}^h$ the space of weights for the algebra $\mathcal{A}^h$.

Let $V$ be a finite dimensional $\mathcal{A}^h$-module. There is canonical generalized eigenspace (weight space) decomposition:

$$V = \bigoplus_{\lambda \in \Omega} V_\lambda,$$
where for $\lambda \in \mathcal{T}^h$, $V_\lambda = \{ v \in V \mid (f - \lambda(f))^n v = 0 \text{ for all } f \in \mathcal{A}^h, n \gg 0 \}$ and $\Omega = \Omega(V) \subset \mathcal{T}^h$ is the finite subset of $\lambda$ such that $V_\lambda \neq 0$.

In discussing the weight spaces, $V_\lambda$, it is convenient to introduce a non-unital localization $\mathcal{A}^h$ of $\mathcal{A}^h$. We set $\mathcal{A}^h = \bigoplus_{\lambda \in \mathcal{T}^h} \mathcal{A}^h$, where $\mathcal{A}^h = \mathcal{H}^h[f^{-1} \mid \lambda(f) \neq 0]$ with the unit element denoted by $1_\lambda$. We have an isomorphism from the category of finite dimensional $\mathcal{A}^h$-modules to the category of unital (with $1_\Lambda$ acting by the identity on $V_\lambda$) finite dimensional $\mathcal{A}^h$-modules, sending $V \mapsto \bigoplus_{\lambda \in \Omega} V_\lambda$, on which $1_\lambda$ acts as the projection onto $V_\lambda$.

As we will see, if $V$ is a finite dimensional representation of $\mathcal{H}^h$ and $\lambda \in \Omega(V)$ is a weight of the subalgebra $\mathcal{A}^h \subset \mathcal{H}^h$ which is not invariant under $s_\alpha$, then $T_\alpha$ does not preserve the weight space $V_\lambda$, nor does it permute the weight spaces. In fact $T_\alpha(V_\lambda) \subset V_\lambda \oplus V_{s_\alpha(\lambda)}$. There is, however, a relation $1_\lambda T_\alpha 1_\lambda = f_\alpha 1_\lambda$, in $\text{End}_C(V)$, where $f_\alpha \in \mathcal{A}^h$ will be made explicit. In this work we describe generators and relations of $\mathcal{H}^h$ which permute the weight spaces of finite representations.

To that end we introduce the non-unital localized Hecke algebra $\hat{\mathcal{H}}^h$. As an additive group, $\hat{\mathcal{H}}^h = \mathcal{H}^h \otimes \mathcal{A}^h \cong \bigoplus_{\lambda \in \mathcal{T}^h} \mathcal{H}_\lambda \otimes \mathcal{A}_\lambda$.

The algebra $\mathcal{A}^h$ is a subalgebra of $\hat{\mathcal{H}}^h$. For $\Lambda \in \mathcal{T}^h/W$, put $1_\Lambda = \sum_{\lambda \in \Lambda} 1_\lambda$ and $T_\alpha^\Lambda = T_\alpha \otimes 1_\Lambda \in \hat{\mathcal{H}}^h$. There is an inclusion of the finite Hecke algebra,

$$
\mathcal{H}_f \hookrightarrow \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda \otimes \mathcal{A}_\lambda \mathcal{A}^h, \\
T_\alpha \mapsto T_\alpha^\Lambda.
$$

Moreover there is an orthogonal decomposition,

$$
\hat{\mathcal{H}}^h = \bigoplus_{\Lambda \in \mathcal{T}^h/W} \mathcal{H}_1^\Lambda.
$$

Recall the commutativity relation in $\mathcal{H}^h$ may be written

$$
T_\alpha f - s_\alpha(f) T_\alpha = c_\alpha hD_\alpha(f).
$$

Note that $(-P_\alpha) \cdot hD_\alpha(f) = f - s_\alpha(f)$. This relation may be used to extend $hD_\alpha$ to an operator on $\mathcal{A}^h$ as follows. Let $f \in \mathcal{A}^h$ be given. If $s_\alpha(\lambda) \neq \lambda$, then $P_\alpha$ is invertible in both $\mathcal{A}^h_\lambda$ and $\mathcal{A}^h_{s_\alpha(\lambda)}$, and we set,

$$
hD_\alpha(f) = (-P_\alpha)^{-1} f - (-P_\alpha)^{-1} s_\alpha(f) \in \mathcal{A}^h_\lambda \oplus \mathcal{A}^h_{s_\alpha(\lambda)}.
$$
If $s_\alpha(\lambda) = \lambda$, we appeal to the description of $\mathcal{A}^h$ as one of $\mathcal{A}$ or $dA$, where the action of $hD_\alpha(f)$ may be written as a fraction lying in $\mathcal{A}^h$.

Finally, the commutativity relation for $\mathcal{H}^h 1_\Lambda$ may be written exactly as before:

$$T_\alpha^A f - s_\alpha(f)T_\alpha^A = c_\alpha hD_\alpha(f),$$

where $f \in \mathcal{A}^h 1_\Lambda$. This gives a complete description of the algebra structure of $\mathcal{H}^h$. A finite dimensional unital representation of $\mathcal{H}^h$ may be viewed as a finite dimensional representation of $\mathcal{H}^h$ on which $1_\Lambda$ acts as projection onto the $\lambda$ weight space.

We show that a finite dimensional representation of $\mathcal{H}^h$ indeed gives rise to a representation of $\mathcal{H}^h$, where $1_\Lambda$ acts as the projection onto the $\lambda$ weight space.

**Lemma 1.5.1.** Let $V$ be a finite dimensional representation of $\mathcal{H}$. Let $1_\Lambda \in \text{End}_C(V)$ be the projection onto $V_\Lambda$. If $\alpha \in \Pi$ with $s_\alpha(\lambda) = \lambda$ then $T_\alpha(\lambda) \subset V_\Lambda$, hence,

$$T_\alpha 1_\Lambda = 1_\Lambda T_\alpha = 1_\Lambda T_\alpha + c_\alpha hD_\alpha(1_\Lambda).$$

Moreover, if $s_\alpha(\lambda) \neq \lambda$, then $(P_{-\alpha})^{-1} \in \mathcal{A}_\lambda^h, \mathcal{A}_{s_\alpha(\lambda)}^h$, and may be considered as an operator on $V_\Lambda, V_{s_\alpha(\lambda)}$. Also, $T_\alpha(V_\Lambda) \subset V_\Lambda \oplus V_{s_\alpha(\lambda)}$ and the following commutativity relation holds,

$$T_\alpha 1_\Lambda - 1_\Lambda s_\alpha(\lambda) T_\alpha = c_\alpha (-P_{-\alpha})^{-1}(1_\Lambda - 1_{s_\alpha(\lambda)}),$$

$$= c_\alpha hD_\alpha(1_\Lambda).$$

**Proof.** Let $x \in X$ and $z \in \mathbb{C}$. With the simple identity, $a^N - b^N = (a - b) \sum_{0 \leq i \leq N-1} a^i b^{N-i-1}$ we find

$$(-P_{-\alpha})hD_\alpha((P_x - z)^N) = (P_x - z)^N - (P_{s_\alpha(x)} - z)^N,$$

$$= (P_x - P_{s_\alpha(x)}) \sum_{0 \leq i \leq N-1} (P_x - z)^i (P_{s_\alpha(x)} - z)^{N-i-1},$$

$$= ((-P_{-\alpha})hD_\alpha(P_x)) \cdot \sum_{0 \leq i \leq N-1} (P_x - z)^i (P_{s_\alpha(x)} - z)^{N-i-1}.$$

As $\mathcal{A}^h$ is a domain,

$$(P_x - z)^N T_\alpha = T_\alpha(P_{s_\alpha(x)} - z)^N + hD_\alpha(P_x) \sum_{0 \leq i \leq N-1} (P_x - z)^i (P_{s_\alpha(x)} - z)^{N-i-1}. $$
Let $\lambda \in \mathcal{T}^h$ with $s_\alpha(\lambda) = \lambda$ and suppose $N, z$ are such that $(P_x - z)^{[N/2]}(V_\lambda) = (P_{s_\alpha(x)} - z)^{[N/2]}(V_\lambda) = 0$. In this case, the above expression shows that $(P_x - z)^N T_\alpha 1_\lambda v = 0$ for $v \in V$. Thus, $T_\alpha(V_\lambda) \subset V_\lambda$ and $T_\alpha 1_\lambda = 1_\lambda T_\alpha$.

Now, suppose $s_\alpha(\lambda) \neq \lambda$ so that $\lambda(P_{-\alpha}) \neq 0$. Thus, $(P_{-\alpha})^{-1} \in \mathscr{A}_\lambda^h, \mathscr{A}_{s_\alpha(\lambda)}^h$ and $(P_{-\alpha})^{-1}$ may be considered as an operator on $V_\lambda, V_{s_\alpha(\lambda)}$ (as the operator $P_{-\alpha}$ has a lone eigenvalue which is non-zero). We also suppose that $N, z$ are picked so that $(P_x - z)^N(V_\lambda) = 0$. Then,

$$(P_{s_\alpha(x)} - z)^N (T_\alpha 1_\lambda - c_\alpha(-P_{-\alpha})^{-1}1_\lambda)v = T_\alpha(P_{s_\alpha(x)} - z)^N 1_\lambda v$$

$$+ c_\alpha(-P_{-\alpha})^{-1}(P_{s_\alpha(x)} - z)^N 1_\lambda v,$$

$$= 0.$$

In particular, it follows that $T_\alpha(V_\lambda) \subset V_\lambda \oplus V_{s_\alpha(\lambda)},$ and moreover,

$$T_\alpha 1_\lambda - 1_{s_\alpha(\lambda)}T_\alpha = c_\alpha(-P_{-\alpha})^{-1}(1_\lambda - 1_{s_\alpha(\lambda)}),$$

$$= c_\alpha hD_\alpha(1_\lambda).$$

in $\text{End}_\C(V)$.

To show that a finite dimensional representation of $\mathcal{H}^h$ lifts to a finite dimensional representation of $\mathcal{H}^h$, we note that the ring of $W$-invariants, $(\mathcal{A}^h)^W$, is the center of $\mathcal{H}^h$ and each finite representation splits into a direct sum of generalized eigenspaces $V_\Lambda$ of the center of $\mathcal{H}^h$ according to the central characters $\Lambda \in \mathcal{T}^h/W$:

$$V = \sum_{\Lambda \in \mathcal{T}^h/W} V_\Lambda.$$ 

We can decompose the operators $T_\alpha = \sum_\Lambda T_\alpha^\Lambda, f = \sum_\Lambda f_\Lambda$. The above lemma shows that these operators give an action of $\mathcal{H}^h$.

### 2 Quiver Hecke Algebra

#### 2.1 A few Weyl group lemma’s

We will need the following lemmas for the definition of the quiver Hecke algebra. Let $W$ be the Weyl group of the reduced root datum, $(X, Y, R, \check{R}, \Pi)$. Recall that $\Pi \subset R$ defines a length function $\ell$ on $W$. 
Definition. Let $W' \subset W$ be a subgroup. We say that $W'$ is a standard parabolic subgroup if there is a subset $\Pi^\lambda \subset \Pi$ so that $W'$ is the subgroup generated by $\{s_\alpha \in W \mid \alpha \in \Pi^\lambda\}$. We call a subgroup parabolic if it is $W$-conjugate to a standard parabolic subgroup.

Let $\lambda \in \mathcal{T}$. We say that $\lambda$ is standard parabolic, if there is a subset $\Pi^\lambda \subset \Pi$ so that the stabilizer of $\lambda$ in the Weyl group $W$ is the standard parabolic subgroup generated by $\{s_\alpha \in W \mid \alpha \in \Pi^\lambda\}$. We call a weight parabolic (resp. parabolic with respect to $W'$) if it is in the $W$-orbit (resp. $W'$-orbit) of a standard parabolic weight (resp. standard parabolic with respect to $W'$).

Lemma 2.1.1. Let $w = s_{\alpha_0} \cdots s_{\alpha_i} = s_{\beta_0} \cdots s_{\beta_j}$ be two reduced expressions. Then there is a permutation $p$ of $\{1, \ldots, n\}$ so that whenever $p(i) = j$,

\[
s_{\alpha_0} \cdots s_{\alpha_{i-1}}(\alpha_i) = s_{\beta_1} \cdots s_{\beta_{j-1}}(\beta_j),
\]

\[
s_{\alpha_0} \cdots s_{\alpha_{i+1}}(\alpha_i) = s_{\beta_n} \cdots s_{\beta_{j+1}}(\beta_j).
\]

Proof. This is simply a restatement of the following standard theorem, see [Hu, Section 1,7].

Let $w = s_{\alpha_0} \cdots s_{\alpha_n}$ be a reduced decomposition. Put $\gamma_i = s_{\alpha_0} \cdots s_{\alpha_{i+1}}(\alpha_i)$. Then the roots $\gamma_1, \ldots, \gamma_n$ are all distinct and the set $\{\gamma_1, \ldots, \gamma_n\}$ equals $R^+ \cap w^{-1}R^-$, which is the set of $\gamma \in R^+$ such that $w(\gamma) \in R^-$. \hfill \square

Lemma 2.1.2. Every left coset of a parabolic subgroup has an element of minimal length.

More precisely, let $\Pi^P \subset \Pi$ be a subset of the simple roots, and $W^P$ the subgroup of $W$ generated by the $s_\alpha, \alpha \in \Pi^P$. Fix some $w_0 \in W$. Every left $w_0W^Pw_0^{-1}$-coset $u \cdot w_0W^Pw_0^{-1}$ has a unique element $u_0$ of minimal length. Moreover, if $w_0$ is of minimal length in its left $W^P$-coset, the set, $(w_0W^Pw_0^{-1})^\perp$ of Weyl group elements which are of minimal length in their left $w_0W^Pw_0^{-1}$-coset is given by,

\[
(w_0W^Pw_0^{-1})^\perp = \{u \in W \mid \text{for all } \alpha \in \Pi^P, \ell(us_{w_0(\alpha)}) > \ell(u)\}.
\]

Proof. The case of standard parabolic subgroups, meaning $w_0 = e$, follows from the previous lemma.

Using that same claim, if $\gamma \in R^+$, and $w(\gamma) \in R^-$, we can find an index $i$, with, $\gamma = s_{\alpha_0} \cdots s_{\alpha_i}(\alpha_i)$, which implies

\[
\begin{align*}
s_\gamma &= (s_n \cdots s_{\alpha_{i+1}})s_{\alpha_i} (s_{\alpha_{i+1}} \cdots s_{\alpha_n}), \\
s_{\alpha_{i+1}} \cdots s_{\alpha_n}s_\gamma &= s_{\alpha_i}(s_{\alpha_{i+1}} \cdots s_{\alpha_n}), \\
w_\gamma &= s_{\alpha_1} \cdots s_{\alpha_{i-1}}s_{\alpha_{i+1}} \cdots s_{\alpha_n}.
\end{align*}
\]
Using $\ell(w) = n$, we see that $\ell(ws_{\gamma}) < \ell(w)$. Reversing the roles of $w$ and $ws_{\gamma}$, if $w(\gamma) \in R^+$, then $\ell(w) < \ell(ws_{\gamma})$.

For $\delta \in R^+$, $\ell(s_\delta)$ is odd, so the following is also true: $\ell(ws_\delta) > \ell(w)$ if and only if $w(\delta) \in R^+$.

Assume that $w_0$ is the unique element of minimal length in its left $W^P$-coset. Thus, for all $\alpha \in \Pi^P$, $w_0(\alpha) \in R^+$. Consider the map $u \mapsto uw_0$. It sends left $w_0W^Pw_0^{-1}$-cosets to left $W^P$-cosets. By the above remarks, it also gives a bijection,

$$\{u \in W \mid \text{for all } \alpha \in \Pi^P, \ell(us_{w_0(\alpha)}) > \ell(u)\} \rightarrow \{v \in W \mid \text{for all } \alpha \in \Pi^P, \ell(vs_\alpha) > \ell(v)\}.$$  

As mentioned, the latter set has a unique element in each left $W^P$-coset, and the second claim follows. The first claim follows from the fact that the coset $uw_0W^Pw_0^{-1}$ does not depend on which left coset representative of $w_0$ is chosen.

\[\square\]

Remark 2.1.3. It is clear that the stabilizer $W^\lambda$ of a parabolic weight is a parabolic subgroup, and the left cosets of $W^\lambda$ are in bijection with the $W$-orbit of $\lambda$. Thus, the above lemma implies that for any $\lambda$ parabolic, and for any $\lambda' \neq \lambda$ there is a unique $w_0 \in W$ of minimal length with the property that $w(\lambda) = \lambda'$.

2.2 Weyl group orbits on complex tori

Let $(X, Y, R, \hat{R}, \Pi)$ be a simply connected root data associated to the complex semisimple algebraic group $G$. Associated to this data is a maximal torus, $T = Y \otimes_{\mathbb{Z}} \mathbb{C}^\ast$ and a dual torus, $T^\ast = X \otimes_{\mathbb{Z}} \mathbb{C}^\ast$, both with actions of the Weyl group $W$. The purpose of this section is to compute the stabilizers of elements of $T$ in the Weyl group $W$.

To compute stabilizers of weights $\lambda \in T$, start with the Euclidean space $E = X \otimes_{\mathbb{Z}} \mathbb{R}$, which has the Killing form, $(\cdot | \cdot)$ as a standard inner product. For $\alpha \in R$ let $H_\alpha = \{x \in E \mid (x | \alpha) = 0\}$. Set $C = \{x \in E \mid (x | \alpha) > 0, \alpha \in \Pi\}$, the Weyl chamber associated to the set of simple roots $\Pi$, which is a connected component of $E - \cup_{\alpha \in \Pi} H_\alpha$. Recall that $W$ acts simply transitively on the set of connected components of $E - \cup_{\alpha \in R} H_\alpha$. Thus, every $x \in E$ is conjugate under $W$ to an element in the closure, $\overline{C}$ of $C$. The stabilizer of an element in $\overline{C}$ is the subgroup of $W$ generated by those $s_\alpha, \alpha \in \Pi$ for which $(x, | \alpha) = 0$, and so in particular is a standard parabolic subgroup. It follows that the stabilizer of any $x \in E$ is a parabolic subgroup of $W$.

This result may be extended to the associated Cartan subalgebra, $\mathfrak{h} = X \otimes_{\mathbb{Z}} \mathbb{C}$, by noting that $\mathfrak{h} = E \oplus iE$, where $i \in \mathbb{C}$ satisfies $i^2 = -1$. Given $x, y \in \mathfrak{h}$ we have $w(x + iy) = w(x) + iw(y)$, so that the stabilizer of $x + iy \in \mathfrak{h}$ is the intersection of
the stabilizers of \( x \) and \( y \), which are two parabolic subgroups of \( W \). It is a theorem of Steinberg that the intersection of two parabolic subgroups is again a parabolic subgroup.

To compute the stabilizers of elements of \( \mathcal{T} = Y \otimes \mathbb{Z} \mathbb{C}^* \) requires a little more work. Using the exponential map \( z \mapsto exp(2\pi iz) \) we have an isomorphism \( \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^* \), and hence an isomorphism \( \mathcal{T} \cong \mathfrak{h}^*/Y \). We have a polar decomposition, \( \mathcal{T} \cong ((Y \otimes \mathbb{Z} \mathbb{R})/Y) \oplus (iY \otimes \mathbb{Z} \mathbb{R}) \). We have already studied stabilizers on the latter summand.

Consider \( Y \) embedded into \( E = X \otimes \mathbb{Z} \mathbb{R} \) via the Killing form. The image in \( E/Y \) of an element \( x \in E \) is stable under the action of \( w \in W \) exactly when \( w(x) - x \in Y \), which is to say, when there exists an element \( a \in Y \) for which \( w(x) + a = x \). This reasoning leads us naturally to consider the action of the extended affine Weyl group \( \hat{W} = Y \rtimes W \), a reflection subgroup of the affine transformation group of \( E \).

For simplicity, consider now the case that \( G \) is simple and simply connected. There is a unique highest root \( \alpha_0 \in \mathfrak{R}^+ \), and the extended affine Weyl group \( \hat{W} = Y \rtimes W \) is a Coxeter group generated by the reflections \( s_\alpha, \alpha \in \Pi \) fixing the origin, and one reflection \( s_{\alpha_0,1} \) which fixes a hyperplane \( H_{\alpha_0,1} = \{ x \in E \mid (x | \alpha_0) = 1 \} \), not passing through the origin. We may describe a fundamental domain for the action of \( \hat{W} \) on \( E \). Let \( A \subset E \) be the set \( \{ x \in E \mid (x | \alpha) > 0, \alpha \in \Pi, (x|\alpha_0) < 1 \} \). This is a connected component of \( E - \cap_{k \in \mathbb{Z}, \alpha \in \Pi} H_{\alpha,k} \) where \( H_{\alpha,k} = \{ x \in E \mid (x | \alpha) = k \} \). It is a fact that \( \hat{W} \) acts simply transitively on the set of connected components above, \([Hu]\). It follows that every \( x \in E \) is conjugate under \( \hat{W} \) to an element of the closure \( \overline{A} \) of \( A \). It follows that the stabilizer of an element \( x \in \overline{A} \) is either the standard parabolic subgroup of \( W \) fixing \( x \) in the case \( (x | \alpha_0) \neq 1 \), or the group generated by same standard parabolic subgroup of \( W \) and the reflection \( s_{\alpha_0,1} \) when \( (x | \alpha_0) = 1 \).

Returning to the action of \( W \) on \( E/Y \), we find that \( x \) is conjugate to an element whose stabilizer is either a standard parabolic, or generated by a standard parabolic and the reflection \( s_{\alpha_0} \). The general case of the stabilizer of an element \( \lambda \in \mathcal{T} \) yields the following.

**Lemma 2.2.1.** Assume that \( G \) is a simple, simply connected complex reductive algebraic group. Define an augmented standard parabolic subgroup of its Weyl group \( W \) to be one generated by a standard parabolic subgroup of \( W \) and the reflection \( s_{\alpha_0} \). Then every \( \lambda \in \mathcal{T} \) is \( W \)-conjugate to an element whose stabilizer is either a parabolic subgroup or the intersection of a parabolic subgroup and an augmented standard parabolic subgroup. These groups and their conjugates are exactly the subgroups that can occur as stabilizers of elements of \( \mathcal{T} \).

In rank 2, the only augmented parabolic subgroups which aren’t parabolic subgroups have index two in the Weyl group.
Proof. Using the polar decomposition, we see the stabilizer of a point is the intersection of the stabilizers of its components. One of these will always be parabolic and the other one is either parabolic, or conjugate to an augmented standard parabolic. The intersection of two parabolic subgroups is still a parabolic subgroup.

In type $GL_n$ we find that the augmented standard parabolic subgroups are in fact parabolic subgroups. Thus every stabilizer is parabolic.

Consider the rank two root systems. The augmented standard parabolic subgroups in $A_1 \times A_1$ and $A_2$ are parabolic subgroups already.

In type $B_2$ let $\alpha \in \Pi$ be the short root and $\beta \in \Pi$ the long root. The highest root is then $\alpha_0 = s_\alpha(\beta) = \beta + 2\alpha$. The only non-trivial augmented parabolic subgroup is,

$$\langle s_\beta, s_\alpha = s_\alpha s_\beta s_\alpha \rangle,$$

which may be seen as the Weyl group of the sub-root system given by the long roots, which has type $A_1 \times A_1$, thus these two elements commute. The coroot lattice is generated by $\check{\alpha}, \check{\beta}$, and we let $(a, b) \in \mathbb{C}^* \times \mathbb{C}^*$ denote the element $(a \otimes \check{\alpha}) \cdot (b \otimes \check{\beta})$ in $\mathcal{T} = \mathbb{C}^* \otimes_{\mathbb{Z}} Y$. We find $s_{\check{\alpha}}(\check{\beta}) = \check{\beta} + \check{\alpha}$ and $s_{\check{\beta}}(\check{\alpha}) = \check{\alpha} + 2\check{\beta}$, so the action of $W$ on $(a, b)$ is determined as follows,

$$s_{\check{\alpha}} : (a, b) \mapsto (a^{-1}b, b),$$

$$s_{\check{\beta}} : (a, b) \mapsto (a, a^2b^{-1}).$$

It follows that $(a, b)$ is $s_{\check{\beta}}, s_{\alpha_0}$-invariant when $a^2 = b^2 = 1$. Thus, there is a unique $W$-orbit with stabilizer which is not parabolic, given by the two elements $(-1, -1)$ and $(1, -1)$. We note that pairing with the element $e^\alpha$ viewed as a function on the torus $e^\alpha \in \mathbb{C}[\mathcal{T}]$ gives,

$$\langle e^\alpha, (\pm 1 \otimes \check{\alpha}) \cdot (-1 \otimes \check{\beta}) \rangle = \pm 1.$$

Now consider type $G_2$. Again, let $\alpha \in \Pi$ be the short root, and $\beta \in \Pi$ be the long root. The highest root is $\alpha_0 = s_\alpha(\beta) = \beta + 3\alpha$. The only non-trivial augmented parabolic subgroup is,

$$\langle s_\beta, s_\alpha = s_\alpha s_\beta s_\alpha \rangle,$$

which similarly may be seen as the sub-root system, of type $A_2$, given by the long roots. Hence, the two elements above satisfy the braid relation,

$$s_\beta s_{\alpha_0} s_\beta = s_{\alpha_0} s_\beta s_{\alpha_0}.$$
The coroot lattice is generated by \( \check{\alpha}, \check{\beta} \) and we let \((a, b) \in \mathbb{C}^* \times \mathbb{C}^* \) denote the element 
\((a \otimes \check{\alpha}) \cdot (b \otimes \check{\beta}) \) in \( \mathcal{T} = \mathbb{C}^* \otimes_{\mathbb{Z}} Y \). We find, 
\( s_{\check{\alpha}}(\check{\beta}) = \check{\beta} + \check{\alpha}, \ s_{\check{\beta}}(\check{\alpha}) = \check{\alpha} + 3\check{\beta} \). Thus 
the action of \( W \) on \( \mathcal{T} \) is given by,
\[
\begin{align*}
s_{\check{\alpha}} : (a, b) &\mapsto (a^{-1}b, b), \\
s_{\check{\beta}} : (a, b) &\mapsto (a, a^3b^{-1}).
\end{align*}
\]
To be \( s_{\check{\beta}}, s_{\check{\alpha}} \)-invariant is equivalent to \( b = 1, a^3 = 1 \). Thus there is only one \( W \)-orbit 
of elements which have a non-parabolic stabilizer, and they are given by the two 
elements \((\omega, 1), (\omega^{-1}, 1)\) where \( \omega \) is a non-trivial cubic root of unity. We note that 
pairing with the element \( e^\alpha \) viewed as a function on the dual torus \( e^\alpha \in \mathbb{C}[\mathcal{T}] \) gives,
\[
\langle e^\alpha, (\omega \otimes \check{\alpha}) \cdot (1 \otimes \check{\beta}) \rangle = \omega^2.
\]

2.3 The datum \( G \) and its conditions

Remark 2.3.1. We now define an abstract set of datum on which our definition of quiver Hecke algebra depends. The definition of quiver Hecke algebra we give extends that of [Rou], where actually two quivers are given. In that context, one quiver \( \Gamma \) is fixed. Its vertices are identified with a subset of the base field \( k \). This quiver corresponds to the eigenvalues that the standard coweights, \( e_i \in \mathbb{Z}^n \) take, where \( \mathbb{Z}^n \) is the X-lattice for the root datum for \( GL_n \). Next, an integer \( n \) is fixed, and a second quiver is constructed depending on \( n \) and \( \Gamma \). It does not seem possible to generalize the first quiver to a general root system, but the second quiver, and the resulting ‘quiver Hecke algebra’ construction may still be carried out. We replace the ‘Cartan datum’ \( Q_{i,j}(u, v) \) from [Rou] with a set \( G \) of rational functions in \( \mathcal{A}^h \), and determine what properties the set of datum must satisfy to form an algebra which satisfies the PBW property, much like in [Rou].

Fix \( h \in \mathbb{C} \) and regard \( \mathcal{A}^h \) as the specialization of the interpolating ring at \( h \). Thus, either \( h = 0 \) in which case \( \mathcal{A}^h \) is isomorphic to the symmetric algebra, \( \cong S(X) \), or \( h \neq 0 \) in which case \( \mathcal{A}^h \) is isomorphic to the group ring \( \mathbb{C}[X] \). Set \( \mathcal{T}^h = \text{Hom}_{\mathbb{C}-\text{alg}}(\mathcal{A}^h, \mathbb{C}) \) be the set of \( \mathbb{C} \)-algebra homomorphisms from \( \mathcal{A}^h \) to \( \mathbb{C} \), which is either isomorphic to \( \mathfrak{h} = Y \otimes_{\mathbb{Z}} \mathbb{C} \) for \( h = 0 \), or isomorphic to \( \mathcal{T} = Y \otimes_{\mathbb{Z}} \mathbb{C}^* \), for \( h \neq 0 \). We put
\[
\mathcal{A}^h = \bigoplus_{\lambda \in \mathcal{T}^h} \mathcal{A}^h_{\lambda},
\]
the localized, non-unital algebra associated to $\mathscr{A}^h$.

Let $G = \langle G^\lambda_\alpha \rangle_{\lambda \in \mathcal{P}, \alpha \in \Pi}$ be a collection of non-zero rational functions, $G^\lambda_\alpha \in \mathscr{A}^h_\lambda \setminus \{0\} = \mathscr{A}^h_1 \setminus \{0\}$. We list now a few conditions that this data is required to satisfy.

For notational purposes, we need the following lemma, and definition

**Lemma 2.3.2.** Let $\alpha, \beta \in \Pi$. We put $W^{\alpha,\beta} = \langle s_\alpha, s_\beta \rangle$, the subgroup of $W$ generated by $s_\alpha, s_\beta$. We also put $m = m^{\alpha,\beta}$ as the order of $s_\alpha s_\beta$ when $\alpha \neq \beta$ and 2 otherwise, and finally we write $w^{\alpha,\beta}$ for the longest element in $W^{\alpha,\beta}$. We have,

$$w^{\alpha,\beta}(\alpha) = \begin{cases} -\alpha & \text{if } m \text{ even}, \\ -\beta & \text{if } m \text{ odd}. \end{cases}$$

Instead of using cases, we will simply write $w^{\alpha,\beta}s_\alpha(\alpha)$ which is equal to $\alpha$ for $m^{\alpha,\beta}$ even, and $\beta$ for $m^{\alpha,\beta}$ odd.

**Definition.** Let $\lambda \in \mathcal{P}$ $\alpha \in \Pi$. The weight $\lambda$ is said to be $\alpha$-exceptional if there exists $\beta \in \Pi$ with $m^{\alpha,\beta} = 4, 6$, $\lambda$ is not parabolic with respect to $W^{\alpha,\beta}$ and $s_\alpha(\lambda) \neq \lambda$.

Assume that $G$ satisfies the following conditions.

1. For $\lambda, \alpha$ as above,

$$s_\alpha(G^\lambda_\alpha) = G^{s_\alpha(\lambda)}_\alpha.$$

We shall refer to this as the *associative relation* on $G$.

2. If $s_\alpha(\lambda) = \lambda$ then $G_\alpha = 1$.

3. For any $\alpha, \beta \in \Pi$,

$$w^{\alpha,\beta}s_\alpha(G^\lambda_\alpha) = G^{w^{\alpha,\beta}s_\alpha(\lambda)}_{w^{\alpha,\beta}s_\alpha(\alpha)}$$

where, again, $w^{\alpha,\beta} \in W^{\alpha,\beta}$ is the longest element. We will refer to this relation as the *braid relation* on $G$. Note that in the case $\alpha = \beta$, we have that $w_{\alpha,\alpha} = s_\alpha$ and the condition is vacuous.

4. If $\lambda$ is $\alpha$-exceptional then $G^\lambda_\alpha = 1$. 
2.4 The quiver Hecke algebra

Let $G = (G_\alpha^\lambda)_{\lambda \in T^h, \alpha \in \Pi}$ be a collection satisfying the conditions in section 2.3. We define the quiver Hecke algebra $H^h(G)$ associated to this choice in analogy with a quiver algebra with relations over the ring $A$, see [Rou]. Underlying this construction of a quiver algebra with relations is the quiver with vertices $T^h$, and arrows $f: \lambda \rightarrow \lambda, r_\lambda^\alpha: \lambda \rightarrow s_\alpha(\lambda)$, whenever $f \in A^h_\lambda, \alpha \in \Pi$. We remark that the arrows $r_\lambda^\alpha$ give precisely the Coxeter graph of the action of $\{s_\alpha | \alpha \in \Pi\}$ on $T^h$.

First, define $\tilde{H}^h(G)$ as the non-unitary algebra given by adjoining generators $r_\lambda^\alpha$ to $A^h$ which satisfy the following relations.

- $r_\lambda^\alpha 1_\nu = 1_{s_\alpha(\nu)}r_\lambda^\alpha = \delta_{\lambda,\nu}r_\lambda^\alpha$.
- $r_\lambda^\alpha s_\alpha(\lambda)r_\lambda^\alpha = \begin{cases} G_\alpha^\lambda & \text{if } s_\alpha(\lambda) \neq \lambda, \\ hD_\alpha(f) & \text{if } s_\alpha(\lambda) = \lambda. \end{cases}$
- For $f \in A_\lambda$,
  $$r_\lambda^\alpha f - s_\alpha(f)r_\lambda^\alpha = \begin{cases} 0 & \text{if } s_\alpha(\lambda) \neq \lambda, \\ hD_\alpha(f) & \text{if } s_\alpha(\lambda) = \lambda. \end{cases}$$
- For $\alpha, \beta \in \Pi$ distinct and $\lambda \in T^h$ a standard parabolic weight with respect to $\{\alpha, \beta\}$, or $\lambda$ which is not parabolic with respect to $W^{\alpha,\beta}$:
  $$r_\alpha^\lambda \cdots r_{\alpha_1}^\lambda = r_\beta^\mu \cdots r_{\beta_1}^\mu,$$
  where $m = m_{\alpha,\beta}$ is the order of $s_\alpha s_\beta$ in $W$,

$$\alpha_i = \begin{cases} \alpha & \text{if } i \text{ odd}, \\ \beta & \text{if } i \text{ even}, \end{cases}$$

$$\beta_i = \begin{cases} \beta & \text{if } i \text{ odd}, \\ \alpha & \text{if } i \text{ even}, \end{cases}$$

and $\lambda_i = s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\lambda), \mu_i = s_{\beta_{i-1}} \cdots s_{\beta_1}(\mu)$. This is known as the braid relation.
Finally, let $\mathcal{I}_\lambda$ be the right $\mathcal{A}_h$-module consisting of elements $I \in \mathcal{H}_h^h(G)_1$ such that there is $f \in \mathcal{A}_h^h \setminus \{0\}$ with $I \cdot f = 0$. We define $\mathcal{H}_h^h(G) = \mathcal{H}_h^h(G)/\bigoplus I_\lambda$. Thus, there is no right polynomial torsion in $\mathcal{H}_h^h(G)$.

Given a degenerate affine Hecke algebra $\mathcal{H}$, we will produce in section 2.7 a family $G$ and an isomorphism $\mathcal{H}_h^h(G) \to \mathcal{H}_h^h$.

## 2.5 The PBW property and a faithful representation

This section analyzes the structure of $\mathcal{H}_h^h(G)$. We start by defining a filtration $(\mathcal{F}^n)$ on $\mathcal{H}_h^h(G)$, letting $\mathcal{F}^n \subset \mathcal{H}_h^h(G)$ be the $\mathcal{A}_h$-linear span of all products $r_{\alpha_1}^\lambda \cdots r_{\alpha_i}^\lambda$ with up to $n$ terms in them. We see $\mathcal{F}^n \cdot \mathcal{F}^m \subset \mathcal{F}^{n+m}$.

**Lemma 2.5.1.** Fix some $\lambda \in \mathcal{F}_h^h$ and let $\mathcal{B}_1 = (\alpha_n, \ldots, \alpha_1)$, $\mathcal{B}_2 = (\beta_n, \ldots, \beta_1)$, $\alpha_i, \beta_i \in \Pi$, be two ordered collections of simple roots with the same cardinality such that $s_{\alpha_n} \cdots s_{\alpha_1} = s_{\beta_n} \cdots s_{\beta_1}$. Then,

$$r_{\alpha_n}^\lambda \cdots r_{\alpha_1}^\lambda - r_{\beta_n}^\mu \cdots r_{\beta_1}^\mu \in \mathcal{F}^{n-1},$$

where, $\lambda_i = s_{\alpha_i-1} \cdots s_{\alpha_1} (\lambda)$, $\mu_i = s_{\beta_i-1} \cdots s_{\beta_1} (\lambda)$.

**Proof.** We prove the assertion by induction on $n$. The cases of $n = 0, 1$ are trivial.

First, put $w = s_{\alpha_n} \cdots s_{\alpha_1}$. Suppose that $\ell(w) < n$. By the deletion condition, there exists, $1 \leq i < j \leq n$ with,

$$s_{\alpha_j} \cdots s_{\alpha_i+1} = s_{\alpha_j-1} \cdots s_{\alpha_i}.$$

By induction, we may assume

$$r_{\alpha_j}^\lambda \cdots r_{\alpha_i+1}^\lambda - r_{\alpha_j-1}^\lambda \cdots r_{\alpha_i+1}^\lambda \in \mathcal{F}^{j-i-1},$$

with the appropriately chosen $\lambda'_k = \lambda_k$, $1 \leq k \leq i + 1, j + 1 \leq k \leq n$. Thus,

$$r_{\alpha_n}^\lambda \cdots r_{\alpha_{j+1}}^\lambda \left( r_{\alpha_j}^\lambda \cdots r_{\alpha_{i+1}}^\lambda \right) r_{\alpha_i}^\lambda \cdots r_{\alpha_1}^\lambda - r_{\alpha_n}^\lambda \cdots r_{\alpha_{j+1}}^\lambda \left( r_{\alpha_j}^\lambda \cdots r_{\alpha_{i+1}}^\lambda \right) r_{\alpha_i}^\lambda \cdots r_{\alpha_1}^\lambda \in \mathcal{F}^{n-1}.$$

Because $r_{\alpha_i}^\lambda r_{\alpha_i}^\lambda \in \mathcal{F}^1$, the second term is in $\mathcal{F}^{n-1}$, hence $r_{\alpha_n}^\lambda \cdots r_{\alpha_1}^\lambda \in \mathcal{F}^{n-1}$. The claim now follows for non-reduced expressions.

Now we show that the assertion is true in the case of a braid relation. Let $\alpha, \beta \in \Pi$ be distinct, and let $W_{\alpha, \beta}$ be the dihedral subgroup of $W$ generated by $s_{\alpha}, s_{\beta}$.
2 QUIVER HECKE ALGEBRA

$m = m_{\alpha, \beta}$ be the order of $s_\alpha s_\beta$, and set

\[
\alpha_i = \begin{cases} 
\alpha, & \text{if } i \text{ odd,} \\
\beta, & \text{if } i \text{ even.}
\end{cases}
\]

\[
\beta_i = \begin{cases} 
\beta, & \text{if } i \text{ odd,} \\
\alpha, & \text{if } i \text{ even.}
\end{cases}
\]

We will show,

\[
r_{\alpha_{m}}^{\lambda_{m}} \cdots r_{\alpha_{1}}^{\lambda_{1}} - r_{\beta_{m}}^{\mu_{m}} \cdots r_{\beta_{1}}^{\mu_{1}} \in \mathcal{F}^{m-1}.
\] (3)

First, suppose $\lambda$ is not a parabolic weight. By analyzing the four simply connected semisimple groups of rank 2 in lemma 2.2.1, we see the only such $\lambda$ have $m_{\alpha, \beta} = 4, 6$ and the $W^{\alpha, \beta}$ orbit of $\lambda$ has order two. In either case, the braid relation in the definition of $\mathcal{H}^{h}(G)$ shows that the difference in (3) is zero.

Now, assume $\lambda$ is parabolic with respect to $W^{\alpha, \beta}$. If the stabilizer of $\lambda$ has 1 element, or is $W^{\alpha, \beta}$ itself, then $\lambda$ was a standard parabolic weight with respect to $\{\alpha, \beta\}$ and we are done, as the braid relation shows that the difference in (3) is zero.

Thus, assume that $\lambda$ is a parabolic weight, but not a standard parabolic weight. Then there is a unique $1 \leq t < m$ so that $\lambda_{t+1} = s_{\alpha_{t}} \cdots s_{\alpha_{1}}(\lambda)$ is a standard parabolic weight with $s_{\alpha_{t+1}}(\lambda_{t+1}) = \lambda_{t+1}$. We will swap $\alpha, \beta$ if it happens that $t \geq \frac{m}{2}$, which has the effect of changing $t$ to $m - t - 1$. From now on, we assume $t < \frac{m}{2}$.

Define $\lambda_i = \lambda_{i+1}, \mu_i = \mu_{i+1}$, and multiply the difference in (3) on the right by $r_{\alpha_{1}}^{\lambda_{1}} \cdots r_{\alpha_{t}}^{\lambda_{t}}$. The two terms that appear are grouped as follows:

\[
r_{\alpha_{m}}^{\lambda_{m}} \cdots r_{\alpha_{t+1}}^{\lambda_{t+1}}(r_{\alpha_{t}}^{\lambda_{t}} \cdots r_{\alpha_{1}}^{\lambda_{1}} \cdots r_{\alpha_{t}}^{\lambda_{t-1}}) - \\
r_{\beta_{m}}^{\mu_{m}} \cdots r_{\beta_{t+1}}^{\mu_{t+1}}(r_{\beta_{t+1}}^{\mu_{t+1}} \cdots r_{\beta_{1}}^{\mu_{1}} \cdots r_{\alpha_{t}}^{\lambda_{t-1}}).
\]

As $\beta_{1} \neq \alpha_{1}$, the last $m$ entries of the second term alternate between $\alpha$ and $\beta$, and start at the parabolic weight $\lambda_{-t} = \lambda_{t+1}$. Thus, they may be switched using the braid relation to the following:

\[
r_{\alpha_{m}}^{\lambda_{m}} \cdots r_{\alpha_{t+1}}^{\lambda_{t+1}}(r_{\alpha_{t}}^{\lambda_{t}} \cdots r_{\alpha_{1}}^{\lambda_{1}} \cdots r_{\alpha_{t}}^{\lambda_{t-1}}) - \\
r_{\beta_{m}}^{\mu_{m}} \cdots r_{\beta_{t+1}}^{\mu_{t+1}}(r_{\beta_{m-t}}^{\mu_{m-t}} \cdots r_{\beta_{1}}^{\mu_{1}} \cdots r_{\alpha_{m}}^{\lambda_{m}} \cdots r_{\alpha_{t+1}}^{\lambda_{t+1}}).
\]

We combine the last $2t$ entries in the first term and the first $2t$ entries in the second term to simplify this expression,

\[
r_{\alpha_{m}}^{\lambda_{m}} \cdots r_{\alpha_{t+1}}^{\lambda_{t+1}} P - \\
P' r_{\alpha_{m}}^{\lambda_{m}} \cdots r_{\alpha_{t+1}}^{\lambda_{t+1}}.
\]
where,

\[ P = \prod_{i=1}^{t} s_{\alpha_i} \cdots s_{\alpha_{t+1}}(G_{\alpha_i}^{\lambda_{-i}}), \]

\[ P' = \prod_{j=m-t+1}^{m} s_{\beta_m} \cdots s_{\beta_{j+1}}(G_{\beta_j}^{\mu{-j}}). \]

Using the commutativity relation between \( r_{\alpha}^{\lambda} \) and elements of \( \mathcal{H}^h \) we find that the above expression is equal to,

\[ s_{\alpha_m} \cdots s_{\alpha_{t+1}} (P) \cdot r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_{t+1}}^{\lambda_{t+1}} + \]

\[ s_{\alpha_m} \cdots s_{\alpha_{t+1}} (hD_\alpha (P)) \cdot r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_{t+2}}^{\lambda_{t+2}} - \]

\[ P' \cdot r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_{t+1}}^{\lambda_{t+1}}. \]

We claim that \( s_{\alpha_m} \cdots s_{\alpha_{t+1}} (P) = P' \). We can use the permutation \( p \) from lemma \( 2.1.1 \) to show that the \( i \)-th term in the product expression for \( s_{\alpha_m} \cdots s_{\alpha_{t+1}} (P) \) is the same as the \( j \)-th term in the expression for \( P' \), where \( j = p(i) \). In fact, let \( j = m - i + 1 = p(i) \). Then the corresponding terms are exactly,

\[ s_{\alpha_m} \cdots s_{\alpha_{t+1}} G_{\alpha_i}^{\lambda_{-i}} ; \]

\[ s_{\beta_m} \cdots s_{\beta_{j+1}} G_{\beta_j}^{\mu{-j}}. \]

The braid relation for \( G \) axiomatizes the above equality.

Now, \( \mu_j = w_\ell(\lambda_i) \) and \( \beta_j = w_\ell(\alpha_i) \). All in all, the difference in \( \mathfrak{3} \) simplifies to,

\[ s_{\alpha_m} \cdots s_{\alpha_{t+2}} (hD_\alpha (P)) \cdot r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_{t+2}}^{\lambda_{t+2}}. \]

The above expression has \( m - t - 1 \) terms of the form \( r_{\alpha}^{\lambda} \), and we can combine \( t \) of them (using the assumption that \( t \leq m/2 \)) after we multiply on the right by \( r_{\alpha_t}^{\lambda_{-t}} \cdots r_{\alpha_1}^{\lambda_{-1}} \).

To summarize, let’s define the following non-zero element of \( \mathcal{H}^h \):

\[ R = r_{\alpha_1}^{\lambda_{-1}} \cdots r_{\alpha_t}^{\lambda_{-t}} r_{\alpha_t}^{\lambda_{-1}} \cdots r_{\alpha_1}^{\lambda_{-1}}. \]

We have shown that the following relation holds in \( \mathcal{H}^h(G) \):

\[ (r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_1}^{\lambda_1} - r_{\beta_m}^{\mu_m} \cdots r_{\beta_1}^{\mu_1}) R = s_{\alpha_m} \cdots s_{\alpha_{t+2}} (hD_\alpha (P)) \cdot r_{\alpha_m}^{\lambda_m} \cdots r_{\alpha_{t+2}}^{\lambda_{t+2}}. R. \]
The relation that $H^h(G)$ has no right $\mathcal{A}^h$-torsion implies:

$$r_{\alpha_n}^\lambda \cdots r_{\alpha_1}^\lambda - r_{\beta_n}^\mu \cdots r_{\beta_1}^\mu = s_{\alpha_m} \cdots s_{\alpha_{t+2}}(hD_{\alpha_{t+1}}(P)) \cdot r_{\alpha_m}^\lambda \cdots r_{\alpha_{2t+2}}^\lambda,$$

where again,

$$P = \prod_{i=1}^t s_{\alpha_t} \cdots s_{\alpha_{t+1}}(G^\lambda_{\alpha_i}).$$

Finally, we show the assertion for reduced expressions. Let $w \in W$ with $\ell(w) = n$, and take two expressions $w = s_{\alpha_n} \cdots s_{\alpha_1} = s_{\beta_n} \cdots s_{\beta_1}$ of minimal length. We show:

$$r_{\alpha_n}^\lambda \cdots r_{\alpha_1}^\lambda - r_{\beta_n}^\mu \cdots r_{\beta_1}^\mu \in \mathcal{F}^{n-1},$$

by reducing to a smaller length case, or by using a braid relation. We need to apply the following lemma, which follows directly from lemma 2.1.1, possibly many times.

**Lemma 2.5.2.** Let $u \in W$ with $\ell(u) = m$ and consider two reduced expressions $u = s_{\delta_m} \cdots s_{\delta_1} = s_{\gamma_m} \cdots s_{\gamma_1}$ in $W$. Then there is a unique $1 \leq i_0 \leq m$ with

$$\delta_1 = s_{\gamma_1} \cdots s_{\gamma_{i_0-1}}(G_{i_0}),$$

$$s_{\gamma_{i_0-1}} \cdots s_{\gamma_1} s_{\delta_1} = s_{\gamma_{i_0}} \cdots s_{\gamma_1}.$$

Applying the lemma directly to the two expressions we have for $w$, we see if $i_0 < n$, then by induction we have

$$r_{\beta_1} \cdots r_{\beta_1} - r_{\beta_{i_0-1}} \cdots r_{\beta_1} r_{\alpha_1} \in \mathcal{F}^{i_0-1}.$$

Though we drop the weights $\mu_i, \lambda_i$ the reader may check this does no harm.

By the inductive hypothesis,

$$r_{\alpha_n} \cdots r_{\alpha_2} - r_{\beta_n} \cdots r_{\beta_{i_0-1}} r_{\alpha_{i_0}}^\lambda \cdots r_{\beta_1} \in \mathcal{F}^{n-2}.$$

Thus, we get

$$\left(r_{\alpha_n} \cdots r_{\alpha_1} - r_{\beta_n} \cdots r_{\beta_{i_0-1}} r_{\beta_{i_0}^\lambda} \cdots r_{\beta_1}^\mu \right) +$$

$$\left(r_{\beta_n} \cdots r_{\beta_{i_0-1}} r_{\beta_{i_0}^\mu} \cdots r_{\beta_1}^\mu r_{\alpha_1} - r_{\beta_n} \cdots r_{\beta_1}^\mu \right) \in \mathcal{F}^{n-1}.$$
We assume $i_0 = n$, or
\[
\alpha_1 = s_{\beta_1} \cdots s_{\beta_{n-1}}(\beta_n),
\]
\[
s_{\beta_{n-1}} \cdots s_{\beta_1} s_{\alpha_1} = s_{\beta_n} \cdots s_{\beta_1} = w.
\]

Similar to the argument above, we have by induction,
\[
r_{\beta_{n-1}} \cdots r_{\beta_1} - r_{\alpha_{n}} \cdots r_{\alpha_2} \in \mathcal{F}^{n-2}.
\]

Thus, the following two assertions are equivalent,
\[
r_{\beta_{n-1}} \cdots r_{\beta_1} - r_{\alpha_{n}} \cdots r_{\alpha_1} \in \mathcal{F}^{n-1},
\]
\[
r_{\beta_{n-1}} \cdots r_{\beta_1} - r_{\beta_{n-1}} \cdots r_{\beta_1} r_{\alpha_1} \in \mathcal{F}^{n-1}.
\]

We now apply the lemma above to the second expression, finding an $i_0$ with,
\[
\beta_1 = s_{\alpha_1} s_{\beta_1} \cdots s_{\beta_{i_0-2}}(\beta_{i_0-1}).
\]

Again, either $i_0 < n$ in which case we apply the induction to show the claim, or we show the following two assertions are equivalent,
\[
r_{\beta_{n-1}} \cdots r_{\beta_1} - r_{\beta_{n-1}} \cdots r_{\beta_1} r_{\alpha_1} \in \mathcal{F}^{n-1}
\]
\[
r_{\beta_{n-1}} \cdots r_{\beta_1} r_{\alpha_1} - r_{\beta_{n-2}} \cdots r_{\beta_1} r_{\alpha_1} r_{\beta_1} \in \mathcal{F}^{n-1}.
\]

Using the same trick we show either the second claim or the equivalence of the following two assertions,
\[
r_{\beta_{n-1}} \cdots r_{\beta_1} r_{\alpha_1} - r_{\beta_{n-2}} \cdots r_{\beta_1} r_{\alpha_1} r_{\beta_1} \in \mathcal{F}^{n-1},
\]
\[
r_{\beta_{n-3}} \cdots r_{\beta_1} r_{\alpha_1} r_{\beta_1} - r_{\beta_{n-2}} \cdots r_{\beta_1} r_{\alpha_1} r_{\beta_1} \in \mathcal{F}^{n-1}.
\]

At this point, if $m_{\alpha,\beta} = 3$ we are done due to the above proof for the braid relation. If $m_{\alpha,\beta}$ is larger, we keep applying this algorithm to eventually find a braid relation.

This finishes the proof of the lemma. $\square$

**Corollary 2.5.3.** Let $B$ be a set of reduced expressions $r_{\alpha_n} \cdots r_{\alpha_1}^\lambda$ so that every $w \in W$ is represented exactly once. Then $B$ generates $\mathcal{H}^h(G)_{1\lambda}$ as a right $\mathcal{A}_\lambda^h$-module.

Let $gr\mathcal{H}^h(G)$ be the graded algebra associated to the filtration $(\mathcal{F}^n)$. We wish to describe the structure of $gr\mathcal{H}^h(G)$. 


Let $^0\mathcal{H}^I$ be the finite nil-Hecke algebra. This is the algebra with generators $r_{s_\alpha}, \alpha \in \Pi,$ satisfying:
\[
\begin{align*}
r_{s_\alpha}^2 &= 0, \\
\cdots r_{s_\beta} r_{s_\alpha} &= \cdots r_{s_\alpha} r_{s_\beta}, & \text{with } m_{\alpha,\beta} \text{ terms.}
\end{align*}
\]

We form the wreath product algebra
\[
\mathcal{A}^h \ltimes ^0\mathcal{H}^I,
\]
which as a $\mathbb{C}$ vector space is given by the tensor product, $\mathcal{A}^h \otimes_{\mathbb{C}} ^0\mathcal{H}^I.$ We give the multiplication by setting,
\[
1 \otimes r_{s_\alpha} \cdot f \otimes 1 = s_\alpha(f) \otimes r_{s_\alpha}.
\]

There is a natural surjective morphism
\[
\mathcal{A}^h \ltimes ^0\mathcal{H}^I \to \text{gr}\mathcal{H}^h(G).
\]
We say that $\mathcal{H}^h(G)$ has the PBW property if this morphism is an isomorphism.

**Theorem 2.5.4.** The following assertions hold:

- $\mathcal{H}^h(G)$ satisfies the PBW property.
- For every $\lambda \in \mathcal{T}^h,$ $\mathcal{H}^h(G) 1_\lambda$ is a free $\mathcal{A}^h_\lambda$-module with basis $B.$

**Proof.** The first two assertions are equivalent thanks to the generating family $B$ mentioned in the above corollary.

**Lemma 2.5.5.** Given a family $G$ satisfying the conditions of section 2.3, there exists a splitting family $F = (F_\alpha^\lambda), F_\alpha \in \mathcal{A}^h_\lambda,$ which satisfy the following conditions:

1. One of $F_\alpha^\lambda$ or $F_{s_\alpha(\lambda)}^\lambda$ is equal to 1.
2. $F_\alpha^\lambda \cdot s_\alpha(F_{s_\alpha(\lambda)}^\lambda) = G_\lambda^\lambda.$

**Remark 2.5.6.** This lemma takes the place of the splitting $Q_{i,j}(u,u') = P_{i,j}(u,u')P_{j,i}(u,u)$ in [Rou].
Proof. Fix $\lambda \in \mathcal{T}$, $\alpha, \beta \in \Pi$ distinct. If $s_\alpha(\lambda) = \lambda$, we put $F^\lambda_\alpha = F^s_\alpha(\lambda) = 1$. Note, in this case, $G^\lambda_\alpha = 1$. We see that $w_{\alpha,\beta}s_\alpha(\lambda) = w_{\alpha,\beta}(\lambda)$, and because $s_{w_{\alpha,\beta}s_\alpha(\alpha)}w_{\alpha,\beta} = w_{\alpha,\beta}s_\alpha$, we have $F^{w_{\alpha,\beta}s_\alpha(\alpha)}_{w_{\alpha,\beta}s_\alpha(\alpha)} = F^{w_{\alpha,\beta}(\lambda)}_{w_{\alpha,\beta}s_\alpha(\alpha)}$, so this choice is consistent with the braid relation.

Assume $s_\alpha(\lambda) \neq \lambda$, and set $F^\lambda_\alpha = G^\lambda_\alpha, F^{s_\alpha(\lambda)}_\alpha = 1$. Consider the set,

$$\{\lambda, s_\alpha(\lambda), w_{\alpha,\beta}s_\alpha(\lambda), w_{\alpha,\beta}(\lambda)\}. \quad (4)$$

As $s_\alpha(\lambda) \neq \lambda$, we have $w_{\alpha,\beta}s_\alpha(\lambda) \neq w_{\alpha,\beta}(\lambda)$. In accordance with the braid relations, we set

$$F^{w_{\alpha,\beta}s_\alpha(\lambda)}_{w_{\alpha,\beta}s_\alpha(\alpha)} := w_{\alpha,\beta}s_\alpha(F^\lambda_\alpha),$$
$$F^{w_{\alpha,\beta}(\lambda)}_{w_{\alpha,\beta}s_\alpha(\alpha)} := w_{\alpha,\beta}s_\alpha(F^{s_\alpha(\lambda)}_\alpha). \quad (5)$$

If $m_{\alpha,\beta}$ is odd, the four pairs

$$\{(\lambda, \alpha), (s_\alpha(\lambda), \alpha), (w_{\alpha,\beta}(\lambda), w_{\alpha,\beta}s_\alpha(\lambda)), (w_{\alpha,\beta}s_\alpha(\lambda), w_{\alpha,\beta}s_\alpha(\alpha))\}, \quad (6)$$

are distinct, so we have not defined any element of $F$ twice. If $m_{\alpha,\beta}$ is even and $\lambda = w_{\alpha,\beta}s_\alpha(\lambda)$, then the two sides of (5) are already equal. Thus, we have defined the two elements, $F^\lambda_\alpha, F^{s_\alpha(\lambda)}_\alpha$ twice, but with the same values each time. If $m_{\alpha,\beta}$ is even and $\lambda = w_{\alpha,\beta}(\lambda)$, then $\lambda$ is $\alpha$-exceptional so all four values in (5) are 1.

Now, let $\gamma \in \Pi$ be distinct from $\alpha, \beta$ and define $F^{w_{\alpha,\gamma}s_\alpha(\lambda)}_{w_{\alpha,\gamma}s_\alpha(\alpha)}, F^{w_{\alpha,\gamma}(\lambda)}_{w_{\alpha,\gamma}s_\alpha(\alpha)}$ as above. To show that no contradiction forms, it is enough to consider the case

$$w_{\alpha,\gamma}(\lambda) = w_{\alpha,\beta}s_\alpha(\beta), \quad (7)$$
$$w_{\alpha,\gamma}s_\alpha(\alpha) = w_{\alpha,\beta}s_\alpha(\alpha). \quad (8)$$

As $\alpha, \beta, \gamma$ are distinct and

$$w_{\alpha,\beta}s_\alpha(\alpha) = \begin{cases} \alpha \quad \text{if } m_{\alpha,\beta} \text{ even,} \\ \beta \quad \text{if } m_{\alpha,\beta} \text{ odd,} \end{cases}$$

the above two equalities may only occur when $m_{\alpha,\beta}$ and $m_{\alpha,\gamma}$ are both odd. Considering the rank 3 root systems, this may only occur in the two cases $m_{\alpha,\beta} = 2, m_{\alpha,\gamma} = 4, 6$ and $m_{\alpha,\beta} = 4, 6, m_{\alpha,\gamma} = 2$. We show the case of irreducible rank 3 root system, the other cases being similar.

Consider the simply connected root datum associated to $B_3$. 

[Note: The text is a proof of a mathematical proposition involving quiver Hecke algebra, discussing the definition and properties of elements $F_\alpha, F_\beta, F_\gamma$, and considering special cases for odd and even indices, along with conditions for $\alpha, \beta, \gamma$ being distinct. The proof involves defining specific elements and their relations, considering their values under certain conditions, and concluding with a case analysis for odd and even indices.]
Let $\Pi = \{\alpha, \beta, \gamma\}$, where $\alpha$ is the short root, $m_{\alpha,\beta} = 4$, and $m_{\alpha,\gamma} = 2$. By an explicit calculation with the element $\lambda = (x, y, z) \in \mathbb{C}^3$ corresponding to $(\alpha \otimes x) \cdot (\beta \otimes y) \cdot (\gamma \otimes z) \in \mathcal{P}^h \otimes \mathbb{C}^*$, we find only four elements $\lambda$ with $w_{\alpha,\gamma}(\lambda) = w_{\alpha,\beta}s_\alpha(\lambda)$. All four of these elements have the property $s_\alpha(\lambda) = \lambda$, hence all four of the values occurring in (5) are 1.

Now let $\Pi = \{\alpha, \beta, \gamma\}$ with $\alpha$ the short root, $m_{\alpha,\beta} = 2$, $m_{\alpha,\gamma} = 4$. In this case there are only four elements $\lambda$ with $w_{\alpha,\gamma}(\lambda) = w_{\alpha,\beta}s_\alpha(\lambda)$. Two of these elements are $s_\alpha$-invariant, and the other two are $\alpha$-exceptional. Thus all values occurring in (5) are 1.

A similar phenomenon happens with the root datum for $C_3$, but all the roots $\lambda$ with $w_{\alpha,\gamma}(\lambda) = w_{\alpha,\beta}s_\alpha(\lambda)$ are in fact $s_\alpha$-invariant.

This shows that we may define $F_\alpha^{\lambda}$ consistently. □

Now take a splitting family $F_\alpha^{\lambda} \in \mathcal{A}^h$ for $G$.

For $s_\alpha(\lambda) = \lambda$ we let $r_\alpha^{\lambda}$ act as $hD_\alpha 1_\lambda$. Otherwise we let $r_\alpha^{\lambda}$ act as $s_\alpha F_\alpha^{\lambda} 1_\lambda$. To show this representation is well defined we only need to check the relations.

The only difficult relation is the braid relation in the case where $\lambda$ is a parabolic weight with respect to $W_{\alpha,\beta}$, but is fixed by only one of the weights.

For this case, suppose $\lambda$ is $s_\alpha$ invariant and not $s_\beta$ invariant. We set

$$
\alpha_i = \begin{cases} 
\alpha, & \text{if } i \text{ odd}, \\
\beta, & \text{if } i \text{ even}
\end{cases}
$$

$$
\beta_i = \begin{cases} 
\alpha, & \text{if } i \text{ even}, \\
\beta, & \text{if } i \text{ odd}
\end{cases}
$$

The relevant relation we must show is equivalent to

$$
s_{\alpha_m}s_{\alpha_{m-1}} \cdots s_{\alpha_2}D_\alpha 1 = D_{\beta_m}s_{\beta_{m-1}} \cdots s_{\beta_1}.
$$

If we consider $D_\alpha$ as given by the fraction, $\frac{1 - s_\alpha}{1 - e^{-\alpha}}$, then the relation

$$
D_{\alpha_m} = (w_\ell s_\alpha)D_{\alpha_1}(w_\ell s_\alpha)^{-1}
$$

makes the desired relation above obvious.

The above morphism defines a faithful representation of $\mathcal{H}^h(G)$ on $\mathcal{A}^h$.

The image of the set $B \subset \mathcal{H}^h(G) 1_\lambda$ gets mapped to $(\mathcal{A}^h \wr W) 1_\lambda$, and is linearly independent over $1_\lambda \mathcal{A}^h 1_\lambda$. □
2.6 Isomorphism class of $\mathcal{H}^h(G)$

The main result of this section shows that the isomorphism class of $\mathcal{H}^h(G)$ depends only on the valuations of $G^\lambda_\alpha$ in $\mathcal{A}^h_\lambda$ for $s_\alpha$.

**Theorem 2.6.1.** Let $G = (G^\lambda_\alpha)$ and $H = (H^\lambda_\alpha)$ be datum satisfying the conditions from section 2.3. Suppose $(g^\lambda_\alpha)_{\lambda \in \mathcal{A}^h_\alpha, \alpha \in \Pi}$ is the set of functions $g^\lambda_\alpha = H^\lambda_\alpha/G^\lambda_\alpha$, and suppose that the $g^\lambda_\alpha$ are invertible rational functions, $g^\lambda_\alpha \in (\mathcal{A}^h_\lambda)^*$. Then there is an isomorphism,

$$\mathcal{H}^h(H) \rightarrow \mathcal{H}^h(G)$$

**Proof.** Suppose $G$ and $H$ are sets of datum satisfying the conditions from section 2.3. Suppose, further, that $g^\lambda_\alpha = H^\lambda_\alpha/G^\lambda_\alpha$ is a unit in $\mathcal{A}^h_\lambda$. By the splitting lemma, 2.5.5, there exists a splitting family $(F) = (F^\lambda_\alpha)$ for $(g^\lambda_\alpha)$. As each $F^\lambda_\alpha$ is either 1 or $g^\lambda_\alpha$, we find that $F^\lambda_\alpha$ is also invertible in $\mathcal{A}^h_\lambda$. Consider the elements,

$$r^\lambda_\alpha = r^\lambda_\alpha F^\lambda_\alpha \in \mathcal{H}^h(G).$$

From the proof of Theorem 2.5.4 we find the elements $r^\lambda_\alpha$ satisfy the same relations as $r^\lambda_\alpha \in \mathcal{H}^h(H)$. Moreover they generate, along with $\mathcal{A}^h$ the algebra $\mathcal{H}^h(G)$. This proves our claim. \[\square\]

2.7 Affine Hecke algebras as quiver Hecke algebras

Given a root datum $(X, Y, R, \tilde{R}, \Pi)$ and set of parameters $c_\alpha \in \mathbb{C}^*$, $\alpha \in \Pi$, with a fixed $h \in \mathbb{C}$, we construct datum $G$ satisfying the properties above, and an isomorphism $\mathcal{H}^h(G) \rightarrow \mathcal{H}^h$.

If $s_\alpha(\lambda) = \lambda$ or $\lambda$ is $\alpha$-exceptional, let $G^\lambda_\alpha = 1$. Otherwise, for $\lambda$ with $s_\alpha(\lambda) \neq \lambda$ let

$$G^\lambda_\alpha = (c_\alpha + q_\alpha P_{-\alpha})(P_{-\alpha} - c_\alpha)(-P_{-\alpha})^{-2}$$

**Theorem 2.7.1.** The data $G$ constructed above satisfies the conditions of section 2.3 so $\mathcal{H}^h(G)$ is well defined. Consider the map $\mathcal{H}^h(G) \rightarrow \mathcal{H}^h$ which is the identity on $\mathcal{A}^h$, and on generators is given by:

$$r^\lambda_\alpha \mapsto \begin{cases} (c_\alpha + q_\alpha P_{-\alpha})^{-1}(T_{s_\alpha} - q_{s_\alpha})1_\lambda, & \text{if } s_\alpha(\lambda) = \lambda, \\ \frac{P_{-\alpha}}{c_\alpha + P_{-\alpha} + hc_\alpha P_{-\alpha}} 1_{s_\alpha(\lambda)}T_{s_\alpha} 1_\lambda, & \text{if } \lambda \text{ is } \alpha\text{-exceptional,} \\ 1_{s_\alpha(\lambda)}T_{s_\alpha} 1_\lambda, & \text{else}. \end{cases}$$

This map is well defined and it is an isomorphism.
Proof. We easily see that $G^\lambda_\alpha$ satisfies the associative property, and the braid relation follows from the Weyl group lemmas. It follows that $\mathcal{H}^h(G)$ is well defined. To check that the above map is well defined we must check the 4 relations from section 2.4 on the generators, and confirm that there is no right $\mathcal{H}^h_\lambda$-torsion in $\mathcal{H}^h_1\lambda$.

Abusing notation, we use $r^\lambda_\alpha$ for its image in $\mathcal{H}^h$. From the definition of $\mathcal{H}^h$ we have that $1_{s_\alpha(\nu)}r^\lambda_\alpha = r^\lambda_\alpha 1_\nu = \delta_{\lambda,\nu} r^\lambda_\alpha$.

We now check the quadratic relation,

$$ r^{s_\alpha(\lambda)}_\alpha r^\lambda_\alpha = \begin{cases} G^\lambda_\alpha & \text{if } s_\alpha(\lambda) \neq \lambda, \\ r^\lambda_\alpha & \text{if } s_\alpha(\lambda) = \lambda. \end{cases} $$

First, suppose $s_\alpha(\lambda) \neq \lambda$. We have the quadratic relation $T^2_\alpha = (q_{s_\alpha} - 1)T_\alpha + q_{s_\alpha}$.

We multiply on the left and right by $1_\lambda$ to obtain,

$$ 1_\lambda T^2_\alpha 1_\lambda = (q_{s_\alpha} - 1)1_\lambda T_\alpha 1_\lambda + q_{s_\alpha} 1_\lambda, $$

$$ = (q_{s_\alpha} - 1)c_\alpha (-P^-_\alpha)^{-1}1_\lambda + q_{s_\alpha}. $$

On the other hand, we have:

$$ 1_\lambda T^2_\alpha 1_\lambda = 1_\lambda T_\alpha (1_\lambda + 1_{s_\alpha(\lambda)})T_\alpha 1_\lambda, $$

$$ = 1_\lambda T_\alpha 1_{s_\alpha(\lambda)}T_\alpha 1_\lambda + c^2_\alpha (-P^-_\alpha)^{-2}. $$

Equating the two expressions yields the equality:

$$ 1_\lambda T_{s_\alpha} 1_{s_\alpha(\lambda)} T_{s_\alpha} 1_\lambda = (c_\alpha + q_\alpha P_-^\alpha)(P^-_\alpha - c_\alpha)(-P^-_\alpha)^{-2}. $$

Consequently, we find that $1_{s_\alpha(\lambda)} T_{s_\alpha} 1_\lambda$ is invertible when $\lambda(P^-_\alpha) \neq c_\alpha, -q^{-1}c_\alpha$.

Next, we verify the commutativity relation:

$$ r^\lambda_\alpha f - s_\alpha(f)r^\lambda_\alpha = \begin{cases} 0 & \text{if } s_\alpha(\lambda) \neq \lambda, \\ hD_\alpha(f) & \text{if } s_\alpha(\lambda) = \lambda. \end{cases} $$

First, suppose $s_\alpha(\lambda) \neq \lambda$. We simply multiply the original commutativity relation, $T_\alpha f - s_\alpha(f)T_\alpha = c_\alpha hD(f)$, on the left by $1_{s_\alpha(\lambda)}$ and on the right by $1_\lambda$. Since $1_\lambda 1_{s_\alpha(\lambda)} = 0$, the claim follows.

Now suppose $s_\alpha(\lambda) = \lambda$. We check directly:

$$ (T_{\alpha} - q_\alpha)f - s_\alpha(f)(T_{\alpha} - q_\alpha) = c_\alpha hD_\alpha(f) - q_\alpha (f - s_\alpha(f)), $$

$$ = (c_\alpha + q_\alpha P^-_\alpha) hD_\alpha(f). $$
One could also expand the expression,

\[(c_\alpha + q_\alpha P_\alpha)^{-1} (T_\alpha - q_\alpha)(c_\alpha + q_\alpha P_\alpha)^{-1} (T_\alpha - q_\alpha)\]

and verify the quadratic relation, \((r^\lambda_\alpha)^2 = h^2 r^\lambda_\alpha\), but we will use the induced representation of \(\mathcal{H}^h\) on \(\mathcal{A}^h\) for this and the braid relations.

Finally we verify the braid relations. The only standard parabolic subgroups of the Coxeter group \((W^{\alpha,\beta}, \{s_\alpha, s_\beta\})\) are \(W^{\alpha,\beta}, \langle e \rangle, \langle s_\alpha \rangle, \langle s_\beta \rangle\).

Suppose that the stabilizer of \(\lambda\) is the trivial group \(\langle e \rangle\). The element \(r^\lambda_\alpha\) is given by \(1 s_\alpha (\lambda) T s_\alpha 1 \lambda\). In this case, with \(\lambda' = \cdots s_\alpha s_\beta (\lambda)\), we have

\[1 \lambda' \cdots T s_\alpha T s_\beta \cdots 1 \lambda = 1 \lambda' \cdots T s_\alpha 1 s_\alpha s_\beta (\lambda) T s_\beta 1 s_\alpha (\lambda) T s_\alpha 1 \lambda,\]

and similarly for \(\cdots T s_\beta T s_\alpha T s_\beta\). Thus the braid relation for \(T s_\alpha, T s_\beta\) yields the braid relation between \(r^\alpha_\alpha, r^\beta_\beta\).

Consider, now, the case where the stabilizer of \(\lambda\) is \(s_\alpha\). In this case, we also have

\[1 \lambda' \cdots T s_\alpha T s_\beta \cdots 1 \lambda = 1 \lambda' \cdots T s_\alpha 1 s_\alpha s_\beta (\lambda) T s_\beta 1 s_\alpha (\lambda) T s_\alpha 1 \lambda.\]

Replacing the rightmost \(T s_\alpha\) with \((c_\alpha + q_\alpha P_\alpha)^{-1}(T s_\alpha - q_\alpha)\) and using the commutativity relation yields the desired result.

Finally, suppose that \(\text{stab}_{W^{\alpha,\beta}}(\lambda) = W^{\alpha,\beta}\). We will use the Demazure-Lusztig representation of \(\mathcal{H}^h\) from section 1.4.

Recall equation (2), which gives the formula for the action of \(\widehat{T}_\alpha - q_\alpha\) on \(\mathcal{A}^h\):

\[\widehat{T}_\alpha - q_\alpha : f \mapsto (c_\alpha + q_\alpha P_\alpha) hD_\alpha (f).\]

We extend the action of \(\mathcal{H}^h\) on \(\mathcal{A}^h\) to an action of \(\mathcal{H}^h 1 \lambda\) on \(\mathcal{A}^h\), and find \(r^\lambda_\alpha = hD_\alpha\). As the Demazure-Lusztig representation is faithful this shows the braid relation between \(r^\lambda_\alpha, r^\beta_\beta\), as well as the quadratic relation \(r^\lambda_\alpha r^\lambda_\alpha = h r^\lambda_\alpha\).

From the structure theory of \(\mathcal{H}^h\) we see it has no polynomial torsion, and the same PBW basis, by the same Demazure-Lusztig representation, thus the map in question is an isomorphism.

\[\square\]

3 Weight Induction

3.1 A pair of adjoint functors

Fix \(\lambda \in \mathcal{T}^h\) and consider the algebra \(\lambda \mathcal{H}^h := 1_\lambda \mathcal{H}^h 1_\lambda\), which we refer to as the weight Hecke algebra. There is a functor on finite dimensional representations which
we will call the $\lambda$-weight restriction functor,
\[
w{\text{Res}}_\lambda : \mathcal{H}^h - \text{mod} \longrightarrow \lambda \mathcal{H}^h_\lambda - \text{mod},
V \mapsto V_\lambda.
\]

The weight restriction functor admits a left adjoint, which we will call $w{\text{Ind}}_\lambda$, or the $\lambda$-weight induction functor,
\[
w{\text{Ind}}_\lambda : \lambda \mathcal{H}^h_\lambda - \text{mod} \longrightarrow \mathcal{H}^h - \text{mod},
V_\lambda \mapsto \mathcal{H}^h 1_\lambda \otimes_{\lambda \mathcal{H}^h_\lambda} V_\lambda.
\]

**Proposition 3.1.1.** Let $\lambda \in \mathcal{P}^h$ be a weight. The following gives a construction of all irreducible representations $V$ of $\mathcal{H}^h$ for which $V_\lambda \neq 0$, in terms of irreducible representations of $\lambda \mathcal{H}^h_\lambda$.

1. Let $V_\lambda$ be a non-zero, irreducible $\lambda \mathcal{H}^h_\lambda$-module. Then, $w{\text{Ind}}_\lambda(V_\lambda)$ has a unique irreducible quotient, $L(w{\text{Ind}}_\lambda(V_\lambda))$.

2. Conversely, suppose that $V$ is an irreducible representation of $\mathcal{H}^h$ with $V_\lambda \neq 0$. Then, $V_\lambda$ is an irreducible $\lambda \mathcal{H}^h_\lambda$-module, and there is a canonical, non-zero map
\[
w{\text{Ind}}_\lambda(V_\lambda) \to V,
\]
which arises from the adjunction applied to the identity map, $V_\lambda \to V_\lambda$, and identifies $V$ with $L(w{\text{Ind}}_\lambda(V_\lambda))$.

3. Finally, the kernel of the above map is the largest $\mathcal{H}^h$-submodule $U$ of $w{\text{Ind}}_\lambda(V_\lambda)$ for which $U_\lambda = 0$. This kernel may be computed in terms of the $\lambda \mathcal{H}^h_\lambda$-module structure on $V_\lambda$.

**Proof.** For the first claim, we let $U \subseteq I_\lambda(V_\lambda)$ be a proper submodule. We claim that $U_\lambda = 0$. If not, then since $V_\lambda$ is an irreducible $\lambda \mathcal{H}^h_\lambda$-module, $U_\lambda = V_\lambda$. But, $V_\lambda$ generates $\mathcal{H}^h 1_\lambda \otimes_{\lambda \mathcal{H}^h_\lambda} V_\lambda$ as an $\mathcal{H}$-module, so we would have $U = w{\text{Ind}}_\lambda(V_\lambda)$, a contradiction.

Now we note that the interior sum of two submodules $U, U' \subseteq w{\text{Ind}}_\lambda(V_\lambda)$ with $U_\lambda = U'_\lambda = 0$ is a submodule, $U + U'$, with $(U + U')_\lambda = 0$. So there is a unique maximal proper submodule, categorized as the sum of all $\mathcal{H}^h$-submodules $U$ with $U_\lambda = 0$.

For the second claim, we notice that $w{\text{Ind}}_\lambda(V_\lambda)_\lambda = V_\lambda$. If there were a non-trivial submodule $U_\lambda \subset V_\lambda$, then the image $U'$ of the map
\[
w{\text{Ind}}_\lambda(U) \to V
\]
3 WEIGHT INDUCTION

would be a submodule of $V$ with $0 \subsetneq U' \subsetneq V$. Thus, $U''$ would be a non-trivial $\mathcal{H}^h$-submodule of $V$.

The last claim follows from the proof of the first claim. We show how to describe the maximal proper submodule.

Let $U'_{\lambda}$ be the left $\mathcal{A}^h$-span of the elements of the form $r \otimes v \in 1_{\lambda'} \mathcal{H}^h 1_{\lambda} \otimes V_{\lambda}$ for which $1_{\lambda'} \mathcal{H}^h 1_{\lambda}(r \otimes v) = 0$. Then $U'_{\lambda}$ is clearly the $\lambda'$-weight space of the maximal proper submodule. We may describe this set as $1_{\lambda'} \mathcal{H}^h 1_{\lambda} \otimes V_{\lambda}^{\lambda'}$, where $V_{\lambda}^{\lambda'}$ is the kernel of the action of $1_{\lambda'} \mathcal{H}^h 1_{\lambda}, \mathcal{H}^h 1_{\lambda} \subset_{\lambda} \mathcal{H}^h$ on $V_{\lambda}$.

3.2 The Demazure algebra

For simply connected root datum, $(X, Y, R, \check{R}, \Pi)$, we may identify the algebra, $\text{End}_{(\mathcal{A}^h)^W} (\mathcal{A}^h)$ with the Demazure algebra, $^h \mathcal{H}$, an interpolating version of the affine nil-Hecke algebra $^h H$ of [Rou]. As a vector space this algebra is equal to a tensor product

$\mathcal{A}^h \otimes_{\mathbb{C}} ^h \mathcal{H}^f$

of $\mathcal{A}^h$ with the finite Demazure algebra, $^h \mathcal{H}$ of $W$. The latter algebra is the algebra with generators $\tau_\alpha, \alpha \in \Pi$, which satisfy the braid relation between $\tau_\alpha, \tau_\beta$, as well as the quadratic relation

$\tau_\alpha^2 = h \tau_\alpha$.

The algebra structure of $^h \mathcal{H}$ is given by letting $\mathcal{A}^h$ and $^h \mathcal{H}^f$ be subalgebras, and giving the commutativity relation,

$\tau_\alpha f - s_\alpha(f) \tau_\alpha = hD_\alpha(f)$.

The following theorem is an algebraic link between the Demazure algebra and the weight Hecke algebra, $^\lambda \mathcal{H}^h$, for certain $\lambda \in \mathcal{T}^h$.

It is clear that for a weight, $\lambda \in \mathcal{T}^h$, the subalgebra of $^\lambda \mathcal{H}^h$ generated by $\mathcal{A}^h_\lambda$ and $r_\alpha^\lambda$ with $s_\alpha(\lambda) = \lambda$ is isomorphic to a Demazure algebra with possibly smaller root datum, $(X, Y, R^\lambda, \check{R}^\lambda, \Pi^\lambda), \Pi^\lambda = \{ \alpha \in \Pi \mid s_\alpha(\lambda) = \lambda \}$. The question is; when is this subalgebra the entirety of $1_{\lambda} \mathcal{H}^h 1_{\lambda}$?

We can use simple Weyl group lemmas and the structure theorem of the quiver Hecke algebra $\mathcal{H}(G)$ to give the solution to this question.

**Theorem 3.2.1.** Suppose $\lambda \in \mathcal{T}^h$ is a standard parabolic weight, i.e. the stabilizer of $\lambda$ in $W$ is a standard parabolic subgroup of $W$. Then the weight-Hecke algebra $^\lambda \mathcal{H}^h$ is isomorphic to the Demazure algebra associated to the root data, $(X, Y, R^\lambda, \check{R}^\lambda, \Pi^\lambda)$. 
Corollary 3.2.2. If $\lambda \in \mathcal{P}$ is a parabolic weight, then there is, up to isomorphism, only one irreducible representation $V$ of $\mathcal{H}^h$ with $V_\lambda \neq 0$.

Proof. By corollary 2.5.3, we see that $\lambda \mathcal{H}^h_\lambda$ is spanned by products $r_{\alpha_n} \cdots r_{\alpha_1}$ with $w = s_{\alpha_n} \cdots s_{\alpha_1}$ a reduced expression for $w \in W$, a Weyl group element which stabilizes $\lambda$. By assumption the stabilizer is generated by $s_{\alpha}, \alpha \in \Pi$ fixing $\lambda$, and a reduced expression will use only these terms $s_{\alpha}, \alpha \in \Pi^\lambda$.

Now, for $(X, Y, R, \hat{R}, \Pi)$ simply connected, a parabolic subgroup corresponding to $\Pi^\lambda$ will also be simply connected. Thus, the subalgebra $\lambda \mathcal{H}^h_\lambda \cong h \mathcal{H}$ will be a matrix algebra over $\mathcal{A}^W$. Modulo the kernel of the central character corresponding to the irreducible representation, the algebra is a matrix algebra over $\mathbb{C}$. Thus, the weight Hecke algebra $\lambda \mathcal{H}^h_\lambda$ has only one irreducible representation with a non-zero weight $\lambda$. In fact, it’s dimension is $\#W^\lambda$, the cardinality of the stabilizer of $\lambda$. \qed

### 3.3 Example computation

We compute an irreducible representation $L(V_\lambda)$ of the affine Hecke algebra associated to $SL_3$ with parameter $q$, where $\lambda$ is a parabolic weight, and $V_\lambda$ is the unique irreducible representation of the weight Hecke algebra $\lambda \mathcal{H}_\lambda$.

Let us take $(X, Y, R, \hat{R}, \Pi)$ the standard root datum for $SL_3$. We have the fundamental weights, $\omega_\alpha = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}), \omega_\beta = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}) \in \mathbb{R}^3$, which generate the free abelian group $X$. We have $\mathcal{A} = \mathbb{Z}[e^{\pm \omega_\alpha}, e^{\pm \omega_\beta}]$.

Pick a parabolic weight $\lambda = (q, q, 1) \in \text{Hom}_{alg}(\mathcal{A}, \mathbb{C})$. The parabolic braid relation shows that $1_{\lambda} \mathcal{H} 1_{\lambda}$ is the affine nil-Hecke algebra for the root datum, $(X, Y, \{\alpha\}, \{\hat{\alpha}\}, \{\alpha\})$. In particular it has only 1 irreducible representation $V_\lambda$. We study here the weight induced module, $w\text{Ind}_\lambda(V_\lambda)$, and the kernel $U$ of the map $w\text{Ind}_\lambda(V_\lambda) \to L(w\text{Ind}_\lambda(V_\lambda))$.

We may describe $V_\lambda$ as a representation on $\mathcal{A}/\mathcal{I}(\lambda)$, where $\mathcal{I}(\lambda) = \ker(\lambda : \mathcal{A}^{(\epsilon_\alpha)} \to \mathbb{C})$. Using the basis $1, e^{\pm \omega_\alpha}$, of this space, we may write the operators $e^{\pm \omega_\alpha}, e^{\pm \omega_\beta} \in \mathcal{A}$ as,

$$\beta \mapsto \begin{bmatrix} 4 \cdot \frac{2}{3} & 1 \cdot \frac{2}{3} \\ -1 & \frac{2}{3} \end{bmatrix},$$

$$\alpha \mapsto \begin{bmatrix} -2 \cdot \frac{2}{3} & -2 \cdot \frac{2}{3} \\ 2 & 2 \cdot \frac{2}{3} \end{bmatrix}.$$

For parabolic $\lambda$, we actually have a surjection, $r_{\lambda'w^{\lambda}} : 1_{\lambda'} \mathcal{H} 1_{\lambda} \to 1_{\lambda} \mathcal{H} 1_{\lambda}$, where $\lambda'w^{\lambda}$ is a unique shortest element sending $\lambda$ to $\lambda'$. Thus

$$U(q,1,q) = \iota^q_{\beta}(q_{\beta,1}) \otimes \ker((q_{\beta,1})_{\iota^q_{\beta}} : V_\lambda \to V_\lambda),$$

$$U(1,q,q) = \iota^q_{\alpha}(q_{\alpha,1}) \otimes \ker((q_{\alpha,1})_{\iota^q_{\alpha}} : V_\lambda \to V_\lambda).$$
It follows that the weight dimensions of the irreducible $H$-module $L = L(\text{Ind}_\lambda(V_\lambda))$ are, in the order $L(q,q,1), L(q,1,q), L(1,q,q)$, given by 2,1,0.

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