Non-signaling boxes and quantum logics

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Abstract

Using a quantum logic approach we analyze the structure of the so-called non-signaling theories respecting relativistic causality, but allowing correlations violating bounds imposed by quantum mechanics such as CHSH inequality. We discuss the relations among such theories, quantum mechanics, and classical physics. Our main result is the construction of a probability theory adequate for the simplest instance of a non-signaling theory—the two non-signaling boxes world—in which we exhibit its differences in comparison with classical and quantum probabilities. We show that the question of whether such a theory can be treated as a kind of ‘generalization’ of the quantum theory of the two-qubit system cannot be answered positively. Some of its features put it closer to the quantum world—on the one hand, for example, the measurements are destructive, though on the other hand the Heisenberg uncertainty relations are not satisfied. Another interesting property contrasting it from quantum mechanics is that the subset of ‘classically correlated states’, i.e. the states with only classical correlations, does not reproduce the classical world of the two two-state systems. Our results establish a new link between quantum information theory and the well-developed theory of quantum logics.

Keywords: quantum logics, non-signaling theories, generalized probability

(Some figures may appear in colour only in the online journal)

1. Introduction

The theory of so-called non-signaling boxes has recently become popular in quantum information theory. The application of non-signaling theories ranges from discussing the fundamental properties of nature [1, 2] to proving the security of ciphering protocols [3] or finding bounds on communication complexity [4].
In this work we will focus on the simplest example of non-signaling boxes—a system defined in [5] (see figure 1, see also [6])—composed of two spatially separated boxes, each with one binary input (with values denoted by \{x, y\}) and one binary output (with values denoted by \{0, 1\}). The model is supposed to describe the most elementary system composed of two separated subsystems. We can think of inputs as observables that we choose to measure, and outputs as the results of measurements. Performing multiple measurements we will obtain a sequence of outcomes allowing us to determine the relative frequency \(P(ab)\) of getting any pair of outputs \(0, 0\) \(0, 1\) \(1, 0\) \(1, 1\) given any pair of inputs \(a\) \(b\) \(x\) \(y\) that can be explained when \(P\) is treated as a probability distribution determined by the actual state of the system.

Let us thus define the state of a system as \(P(x, y)\): \(0, 0\) \(0, 1\) \(1, 0\) \(1, 1\) fulfilling

\[
P_1 \quad 0 \leq P(\alpha, \beta|ab) \leq 1 \quad \text{(positivity)}
\]

\[
P_2 \quad \sum_{\alpha, \beta} P(\alpha, \beta|ab) = 1 \quad \text{(normalization)}
\]

\[
P_3 \quad \sum_{\alpha} P(\alpha, \beta|ab) = \sum_{\alpha} P(\alpha, \beta|cb) \quad \text{and} \quad \sum_{\beta} P(\alpha, \beta|ab) = \sum_{\beta} P(\alpha, \beta|ac) \quad \text{(non-signaling condition)},
\]

where we always sum over the whole domain, and all unbound variables are universally quantified. The last property, non-signaling, is supposed to encode the principle of relativistic causality, i.e. ‘what happens in one box does not influence the other’ [1] obeyed by spatially separated subsystems. In the sequel, we will refer to this particular example of non-signaling boxes as the (2, 2)-box world.

Our aim is to investigate whether we can actually call \(P\) a probability. If yes we can denote by:

\[
\langle ab \rangle = \sum_{\alpha, \beta \in \{0, 1\}} (-1)^{\alpha \beta} P(\alpha, \beta|ab),
\]

a quantity that we can interpret as the mean value of the ‘observable \((1 - 2a)(1 - 2b)\)’. It is called in literature a correlation. Then the following CHSH [7]-like inequality holds,

\[
|\langle xx \rangle + \langle xy \rangle + \langle yx \rangle - \langle yy \rangle| \leq 4,
\]

since the absolute value of each term on the left hand side does not exceed one. In particular for the state described by
the maximal value of four is obtained. We recall that if in (2) one substitutes \( (ab) \) for the expectation value given by quantum mechanics (i.e. \( \operatorname{Tr} \rho ab \) for some state \( \rho \)), then the right hand side of the inequality is \( 2\sqrt{2} \) (we will refer to this number as Tsirelson’s bound). When \( (ab) \) is an expectation value in classical physics (i.e. \( \int a(x)b(x)p(x)dx \) for some probability measure \( p(x)dx \) on the phase space \( \Gamma \)’) then the upper bound of the left hand side of (2) equals two\(^1\). The fact that the left hand side of (2) can be greater than \( 2\sqrt{2} \) leads to the conclusion [1, 5] that there are theories respecting relativistic causality, but still exhibiting ‘stronger correlations’ than quantum mechanics\(^2\).

In classical and quantum physics the interpretation of a state in terms of probabilities is well established and does not pose any conceptual problems. In the case of non-signaling boxes, however, a more careful analysis is needed. Namely, the state \( P \) is in fact only a function that satisfies P1–P3. We will call such a function a PR-box state\(^3\). At this stage, the only reason to call it probability is motivated by a thought experiment in which one imagines that by having such boxes, performing measurements and writing down the results one will obtain a sequence of outcomes that can be explained when \( P \) is treated as a probability distribution. However, there are no physical non-signaling boxes, and in our opinion the interpretation of mathematical objects used in the theory cannot be based on thought experiments. If the interpretation of \( P \) as a probability is not justified, then neither is drawing any conclusions from (2). The meaning of non-signaling is not clear either.

In the following, exploiting the framework of quantum logics (see e.g. [14]) we were able to show by construction that the logic of propositions in the (2, 2)-box world is an ortho-modular poset (all relevant definitions are given in the next section). This is sufficient to interpret \( P \) as a generalized probability [15]. Moreover, by showing that propositions that can be identified with propositions about only one of the boxes are pairwise compatible, we prove that the (2,2)-box world can indeed describe spatially separated subsystems (it seems that in general the non-signalling condition is a weaker restriction than compatibility).

The paper is organized in the following way: for the reader’s convenience, we begin with a short introduction to the framework of quantum logics in section 2. In section 3 we construct the logic of the (2, 2)-box world and analyze its basic structural properties.

In sections 4.1 and 4.2 we compare the obtained probability theory with classical and quantum theory. We make the interesting observation that although from a mathematical

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\(^{1}\) Although we mention CHSH-like inequalities (and, in particular, violation of Tsirelson’s bound) several times, being acquainted with them is not essential for understanding this paper. Thus, for the sake of compactness, we do not provide an introduction to this topic. Readers who are unfamiliar with this notion can easily find a variety of references. Among the multitude of choices we recommend a very interesting book by Streater [8], or a shorter paper by Griffiths [9].

\(^{2}\) It is worth pointing out that the problem of violating the \( 2\sqrt{2} \) bound in (2) was already discussed in the algebraic setting by Landau in [10, 11] in 1992. In a very general setting of Segal’s axiomatization [12], Landau showed that the violation of Tsirelson’s bound implies non-distributivity of the algebra of observables. The only known examples of non-distributive Segal’s algebras [13] are not suitable for the description of two causally separated systems. Thus, it seems that in the algebraic setting, relativistic causality is enough to single out quantum mechanics among other reasonable theories. The only possible exception could be provided by some yet unknown non-distributive Segal algebras.

\(^{3}\) PR stands for Popescu and Rohrlich who introduced the concept [1].
point of view a quantum logic is a more general structure than an orthomodular lattice (the structure that is behind quantum probability) or Boolean algebra (that describes classical probability), the simple statement that the theory of non-signaling boxes is more general than quantum mechanics is not justified. This stems from the fact that the logic of even the simplest quantum model (a qubit), has an infinite number of propositions, contrary to the 82 propositions of the (2, 2)-box world. From this point of view, the logic of the (2, 2)-box world is much more similar to the logic of an analogous classical system. In fact, we show in section 4.2 that (2, 2)-box world logic can be represented by wisely chosen subsets of the phase space of a classical two box system. This has profound consequences. In particular, an analog of Heisenberg uncertainty relations does not hold in the (2, 2)-box world.

In parallel, we study the properties of the (2, 2)-box world with a set of states restricted to the so-called ‘classically correlated boxes’. We show that the resulting logical structure, contrary to intuition, is exactly the same as the logical structure of the unrestricted (2, 2)-box world. Consequently, ‘classically correlated boxes’ are not ‘classical’.

In section 4.3 we show how the so-called ‘non-local’ states are responsible for the peculiarities of (2,2)-box world logic. Final remarks can be found in section 5.

All computations were performed using Wolfram Mathematica. For reference, the notebook can be found in a git repository, see [16]. We emphasize that we rely on the exact symbolic methods provided by Mathematica. The presented results are not based on any numerical calculations; instead, we used symbolic manipulation systems if a lot of combinatorics or exhaustive checks over a large but finite set were needed.

2. Framework of quantum logics

For reference, we provide here a very concise introduction to quantum logics. We recall only those definitions and facts that will be used in the sequel. The interested reader can find much more detailed expositions in [14] or in [15].

The primary object of our interest will be a partially ordered set (poset), i.e. a set equipped with a reflexive, antisymmetric and transitive relation.

**Definition 1** (see definition 1.1.1 in [14]). A quantum logic is a partially ordered set $\mathcal{L}$ with a map $\alpha: \mathcal{L} \to \mathcal{L}$ such that

L1 there exists the greatest (denoted by 1) and the least (denoted by 0) element in $\mathcal{L}$,
L2 map $a \mapsto a' \ast$ is order reversing, i.e. $a \leq b \ast$ implies that $b' \leq a'$,
L3 map $a \mapsto a'$ is idempotent, i.e. $(a')' = a$,
L4 for a countable family $\{a_i\}$, such that $a_i \leq a_j'$ for $i \neq j$, the supremum $\bigvee \{a_i\}$ exists,
L5 if $a \leq b$ then $a \vee (b \wedge a')$ exists and $b = a \vee (b \wedge a')$ (orthomodular law),

where $a \vee b$ is the least upper bound and $a \wedge b$ the greatest lower bound of $a$ and $b$.

**Definition 2** (see definition 2.4.1 in [14]). An element $a \in \mathcal{L}$ is called an atom if $a \neq 0$ and for every $b \leq a$ either $b = 0$ or $b = a$. A logic $\mathcal{L}$ is called atomic if for every $b \in \mathcal{L}$ there exists an atom $a \in \mathcal{L}$ such that $a \leq b$. A logic $\mathcal{L}$ is called atomistic whenever every $b \in \mathcal{L}$ can be written as a supremum of all atoms that are less than or equal to $b$. 


Definition 3 (see definition 2.1.1 in [14]). Let \( \mathcal{L} \) be a quantum logic. Function \( \mu : \mathcal{L} \to [0, 1] \) is called a probability measure or state on \( \mathcal{L} \) if and only if

i. \( \mu(0) = 0 \), \( \mu(1) = 1 \),

ii. \( \mu(a_1 \lor a_2 \lor \ldots) = \sum_{k=1}^{\infty} \mu(a_k) \), whenever \( a_i \leq a_j \) for \( i \neq j \).

Definition 4 (see definition 4.1.1 in [14]). Let \( \mathcal{L} \) be a quantum logic. Function \( X : B(\mathbb{R}) \to \mathcal{L} \), where \( B(\mathbb{R}) \) is a family of Borel sets on \( \mathbb{R} \), is called an \( \mathcal{L} \)-valued measure on \( B(\mathbb{R}) \) or an observable if

i. \( X(\mathbb{R}) = 1 \),

ii. \( X(\mathbb{R} \setminus A) = X(A)^c \),

iii. \( X(A_1 \cup A_2 \ldots) = X(A_1) \lor X(A_2) \lor \ldots \) for any countable family \( \{A_i\} \subset B(\mathbb{R}) \) of pairwise disjoint sets.

Example 5. Let \( \mathcal{H} \) be a separable Hilbert space. Denote by \( \mathcal{P}(\mathcal{H}) \) the set of orthogonal projections on \( \mathcal{H} \), ordered by the subspace inclusion. Define \( P' = 1 - P \), where \( 1 \) is the identity operator. Then \( \mathcal{P}(\mathcal{H}) \) is a quantum logic. Observe that for any pair \( P, Q \in \mathcal{P}(\mathcal{H}) \), \( P \lor Q \) and \( P \land Q \) exist. A quantum logic with such properties is called an orthomodular lattice.

\( \mathcal{L} \)-valued measures correspond via the spectral theorem to the self-adjoint operators acting on the Hilbert space \( \mathcal{H} \).

Similarly, a probability measure on \( \mathcal{P}(\mathcal{H}) \) defines a quantum mechanical state described by a density matrix. When the measure is supported by only one rank-1 projector, then the corresponding state is pure.

Example 6. Let \( \Gamma \) be a phase space of some classical model with some measure \( \nu \). Denote by \( B(\Gamma) \) a family of \( \nu \)-measurable subsets of \( \Gamma \), ordered by set inclusion. For any \( A \in B(\Gamma) \) we set \( A' = \Gamma \setminus A \). Then \( B(\Gamma) \) is a quantum logic. Since \( A \lor B = A \cup B \) and \( A \land B = A \cap B \), it is also an orthomodular lattice. Moreover, for any triple \( A, B, C \in B(\Gamma) \) distributivity law is satisfied:

\[
A \lor (B \land C) = (A \lor B) \land (A \lor C).
\]

The orthomodular lattice for which the distributivity law holds is called a Boolean algebra.

A strong connection between the structure of a Boolean algebra and Kolmogorov’s axiomatization of probability motivates the use of an orthomodular lattice or an orthomodular poset as an axiomatic definition of generalized probability [15]. Elements of a quantum logic \( \mathcal{L} \) are interpreted as events. In physical terms, these correspond to two-valued measurements, called propositions. Outcomes of a proposition can be conveniently labeled as ‘yes’ and ‘no’. For a given state \( \mu \) on \( \mathcal{L} \) and \( a \in \mathcal{L} \) value \( \mu(a) \) is interpreted as a probability of obtaining the answer ‘yes’ for proposition \( a \). If \( a \leq b^c \) we say that \( a \) and \( b \) are disjoint.

Quantum mechanics has taught us that it is important to distinguish subsets of propositions that can be described by a classical probability model. This motivates the following definition:

Definition 7 (see definition 1.2.1 in [14]). Let \( \mathcal{L} \) be a quantum logic. Elements \( a, b \in \mathcal{L} \) are said to be compatible, what will be denoted by \( a \leftrightarrow b \), whenever there exist mutually disjoint propositions \( a_i, b_1, c \) such that \( a = a_i \lor c \), \( b = b_1 \lor c \).
Definition 8 (see definition 1.2.2 in [14]). A subset $\mathcal{K} \subset \mathcal{L}$ of quantum logic $\mathcal{L}$ is called a sublogic of $\mathcal{L}$ whenever:

i. $0 \in \mathcal{K}$,

ii. $a \in \mathcal{K}$ implies $a' \in \mathcal{K}$,

iii. For a countable family $\{a_i\} \subset \mathcal{K}$ of pairwise disjoint elements, the supremum $\bigvee \{a_i\}$ taken in $\mathcal{L}$ belongs to $\mathcal{K}$.

If $\mathcal{K}$ is a maximal Boolean sublogic of $\mathcal{L}$, then it is called a block.

A compatible pair, $a, b$, of propositions from $\mathcal{L}$ can be simultaneously measured, i.e. there exists a Boolean sublogic of $\mathcal{L}$ that contains $a$ and $b$ (theorem 1.3.23 of [14]). But it is important to remember that, in general, if $A \subset \mathcal{L}$ is a subset of the orthomodular poset $\mathcal{L}$ such that all elements of $A$ are pairwise compatible, then there might not exist any Boolean sublogic $B \subset \mathcal{L}$ such that $A \subset B$. For that we need a stronger definition of compatibility of sets. In this paper we will not consider anything more than a compatible pair, so for the sake of brevity we will omit the details (see section 1.3 of [14]).

Following class of quantum logics will be of profound importance for us:

Definition 9 (see section 1.1 in [14]). Let $\Omega$ be a set and $\Delta$ be a family of subsets of $\Omega$ such that:

i. $\emptyset \in \Delta$,

ii. if $A \in \Delta$ then $\Omega \setminus A \in \Delta$,

iii. for any countable family $\{A_i\}_{i \in I} \subset \Delta$ of pairwise disjoint sets $\bigcup_{i \in I} A_i \in \Delta$.

We say that $(\Omega, \Delta)$ is a concrete logic with partial order induced by set inclusion.

Example 10. Let $\Omega = \{1, \ldots, 2n\}$ and $\Delta$ be a family of subsets of $\Omega$ with an even number of elements. Then $(\Omega, \Delta)$ is a concrete logic which is a Boolean algebra for $n = 1$, an orthomodular lattice for $n = 2$, and a quantum logic for $n \geq 3$.

In order to determine whether a logic is concrete or not, we need a few more notions and results, namely:

Definition 11 (see definition 44 in [15]). Let $\mathcal{L}$ be a quantum logic. The set of states $\mathcal{S}$ is said to be rich if for any disjoint pair of propositions $p, q$ there exists a state $\sigma \in \mathcal{S}$ such that $\sigma(p) = 1$ and $\sigma(q) > 0$.

We say that $\mathcal{L}$ is rich whenever it has a rich subset of states. We say that $\mathcal{L}$ is 2-rich whenever it has a rich set of two-valued states (i.e. states with a property that $\forall q \in \mathcal{L}$, $\sigma(q) = 1$ or $\sigma(q) = 0$).

Theorem 12 (see theorem 48 in [15]). A quantum logic $\mathcal{L}$ is set-representable, i.e. there exists an order-preserving isomorphism between $\mathcal{L}$ and some concrete logic $(\Omega, \Delta)$ if and only if $\mathcal{L}$ is 2-rich.

Concrete logics exhibit some classical properties. Let us firstly summon a reformulation of Heisenberg’s uncertainty relation in the language of quantum logics. Following [15], let $X$ be a real observable on the quantum logic $\mathcal{L}$. Then for any state $\mu$ on $\mathcal{L}$ we define the
expected value of $X$ as
\[ \mu(X) := \int_{\mathbb{R}} t \mu(X(dt)), \]
whenever the integral exists. Similarly, we define a variance of $X$ in state $\mu$
\[ \Delta_\mu X := \int_{\mathbb{R}} (t - \mu(X))^2 \mu(X(dt)). \]
Then for a given pair of real observables $X, Y$ either
\[ \forall \varepsilon > 0 \exists \mu \text{ with finite variance for } X \text{ and } Y, \quad \left( \Delta_\mu X \right) \left( \Delta_\mu Y \right) < \varepsilon \tag{4} \]
or
\[ \exists \varepsilon > 0 \forall \mu \text{ with finite variance for } X \text{ and } Y, \quad \left( \Delta_\mu X \right) \left( \Delta_\mu Y \right) \geq \varepsilon. \tag{5} \]
In the former case we say that the Heisenberg uncertainty relations are not satisfied, while in the latter we say that the Heisenberg uncertainty relations are satisfied. We have:

**Theorem 13** (see theorems 50 and 129 in [15]). If $\mathcal{L}$ is a concrete logic, then Heisenberg’s uncertainty relations are not satisfied.

### 3. Propositional system of the (2, 2)-box world

In this section we construct the propositional system of the (2, 2)-box world described in the introduction (see [5, 6]). To avoid confusion with notions introduced in the previous section we will reserve the notion of a state for a measure on an orthomodular poset, and keep the previously introduced name PR-box state for a function fulfilling P1–P3 from section 1. In fact, our aim is to prove that such a function is a state on $\mathcal{L}$.

The construction is a rather standard procedure, involving arguments similar to the ones used by Mackey in his axiomatic approach to quantum mechanics [17] (we remark that the framework we use is more general than either Mackey’s axioms or Piron’s axioms, [18]). We emphasize that the procedure does not add anything new to the original definition of the (2, 2)-box world. We only explore the structure that it already has.

We start by making a few observations. Firstly, the logic $\mathcal{L}$ of the (2, 2)-box world, if it exists, must contain propositions corresponding to the most elementary questions in the (2, 2)-box world, i.e. questions of the form ‘does measuring $a$ on the first subsystem and $b$ on the second yield the result $\alpha$ on the first subsystem and $\beta$ on the second?’ We will denote the corresponding proposition by $[ab,\alpha\beta]$.

Secondly, any PR-box state on a (2, 2)-box world should correspond to some state on $\mathcal{L}$. Consequently, if $P$ is a PR-box state, then the state $\rho_P$ should satisfy
\[ \rho_P([ab,\alpha\beta]) = P(\alpha\beta|ab). \]
By the definition and properties of the (2, 2)-box world (see theorem 2 in [6]), propositions from the set
\[ A = \{ [ab,\alpha\beta] \mid a, b \in \{x, y\}, \alpha, \beta \in \{0, 1\} \} \tag{6} \]
are sufficient to describe completely any measurement in the (2, 2)-box world, so any other propositions must be built from the elements of $A$. Moreover, $\mathcal{L}$ must contain two trivial propositions, which we denote by $\mathbb{1}$ (trivial ‘yes’) and $0$ (trivial ‘no’).
Given a question corresponding to the proposition \([ab, \alpha\beta]\) we can always ask the negated question (i.e. interchange answer ‘yes’ with ‘no’). We will denote the proposition described by such a question with \([ab, \alpha\beta]'\). Because we expect that the set of the PR-box states is rich enough to distinguish different observables and determine their ordering, we can formally define \([ab, \alpha\beta]'\) by

\[ \rho_p([ab, \alpha\beta]) = 1 - \rho_p([ab, \alpha\beta]) \quad \forall \text{ PR-box states } P, \]  

and \(r \leq q\) if and only if

\[ \rho_p(r) \leq \rho_p(q), \quad \forall \text{ PR-box states } \rho. \]  

If for some fixed pair of propositions \(r \leq q'\), then

\[ \rho_p(p) + \rho_p(q) \leq 1, \quad \forall \text{ PR-box states } P, \]  

and so \(r\) and \(q\) cannot both be true. Thus in principle, the question: ‘is \(r\) or \(q\) true?’ should make sense. We denote the proposition corresponding to such questions by \(r \oplus q\) and define it by

\[ \rho_p(p \oplus q) = \rho_p(p) + \rho_p(q). \]  

We could proceed without this assumption, but from the operational point of view it is well justified and this seems to be in line with the general idea of non-signaling theories. Moreover, this construction is essential for identifying propositions about a single subsystem and can be used to express complementary propositions, e.g.

\([xx, 00]' = [xx, 01] \oplus [xx, 10] \oplus [xx, 11]\].

By construction, the operation \(\oplus\) is commutative and associative.

In order to generate all possible propositions in the \((2, 2)\)-box world, we define recursively a sequence of sets:

\[ \mathcal{L}_0 = \mathcal{A} \cup \{0, 1\}, \]

\[ \mathcal{L}_{i+1} = \mathcal{L}_i \cup \mathcal{L}_{i+1}^c \cup \mathcal{L}_{i+1}^p \]

where,

\[ \mathcal{L}_{i+1}^c = \{q'; q \in \mathcal{L}_i\}, \]

\[ \mathcal{L}_{i+1}^p = \{r \oplus q; r, q \in \mathcal{L}_i, \text{ s. t. } \forall \text{ PR-box states } P \rho_p(r) + \rho_p(q) \leq 1\}. \]

Computing \(\mathcal{L}_{i+1}^p\) requires maximizing the left hand side of equation (9) with respect to the PR-box states. If we represent a PR-box state \(P\) as a matrix (table):

\[
\rho_p = (\rho_p) = \begin{pmatrix}
P(00|xx) & P(00|xy) & P(00|yx) & P(00|yy) \\
P(01|xx) & P(01|xy) & P(01|yx) & P(01|yy) \\
P(10|xx) & P(10|xy) & P(10|yx) & P(10|yy) \\
P(11|xx) & P(11|xy) & P(11|yx) & P(11|yy)
\end{pmatrix},
\]

then the properties P1, P2, P3 are just linear constraints on \(\rho_p\). The non-signaling conditions become,
The normalization conditions are given by

\[ \rho_{11} + \rho_{31} = \rho_{13} + \rho_{33}, \quad \rho_{21} + \rho_{41} = \rho_{23} + \rho_{43}, \]
\[ \rho_{12} + \rho_{32} = \rho_{14} + \rho_{34}, \quad \rho_{22} + \rho_{42} = \rho_{24} + \rho_{44}, \]
\[ \rho_{11} + \rho_{21} = \rho_{12} + \rho_{22}, \quad \rho_{31} + \rho_{41} = \rho_{32} + \rho_{42}, \]
\[ \rho_{13} + \rho_{23} = \rho_{14} + \rho_{24}, \quad \rho_{33} + \rho_{43} = \rho_{34} + \rho_{44}. \]  

(12)

The normalization conditions are given by

\[ \rho_{11} + \rho_{21} + \rho_{31} + \rho_{41} = 1, \]
\[ \rho_{12} + \rho_{22} + \rho_{32} + \rho_{42} = 1, \]
\[ \rho_{13} + \rho_{23} + \rho_{33} + \rho_{43} = 1, \]
\[ \rho_{14} + \rho_{24} + \rho_{34} + \rho_{44} = 1. \]  

(13)

and positivity means that \( \rho_{ij} \geq 0 \). Thus, testing whether \( r \oplus q \) exists is a linear programming problem that can be solved exactly. We used Wolfram Mathematica [19] for this purpose. Similarly, determining whether \( r \leq q \) is also a linear programming problem with the objective function now given by

\[ \rho_p(r) - \rho_p(q) \leq 0, \]

(in fact, \( r \oplus q \) is defined whenever \( r \leq q' \), so the former problem is a special case of the latter). Let us also observe that due to non-signaling constraints, for some \( r, q \) we have

\[ \rho_p(r) = \rho_p(q), \]

i.e. \( r \leq q \) and \( q \leq r \). In this case, we will write that \( r \sim q \) (e.g. [xx, 00] \( \oplus \) [xx, 01] \( \sim \) [xy, 00] \( \oplus \) [xy, 01]).
Finally, let us remark that for the \((2, 2)\)-box world \(L_i = L_{i+1}\) for \(i \geq 4\). It follows from the observation that subsets of \(A\) with pairwise orthogonal elements can have a cardinality of at most \(4\). Then we define \(L\) as a quotient of \(L_4\) with respect to the equivalence relation \(\sim\), i.e. \(L = L_4/\sim\).

The partially ordered set \(L\) has 82 elements. Any further property of \(L\) can be obtained by analyzing the directed graph representing the partial ordering of \(L\) (see figure 2 for its Greechie diagram). Again, we used Mathematica to obtain all the results reported in the sequel. We want to emphasize that the results are exact as they follow from traversing the mentioned graph. We sum up the basic properties of \(L\) in the following proposition:

**Theorem 14.** Let \(L\) be an above-constructed set of propositions about the \((2, 2)\)-box world. Then

1. \(L\) is an atomistic quantum logic,
2. let
   
   \[
   x_{ij} = [xx, \alpha 0] \oplus [xx, \alpha 1], \quad y_{ij} = [xy, \alpha 0] \oplus [xy, 1\alpha],
   \]
   then all the pairs of propositions \((x_{ij}, y_{ij})\), \((x_{ij}, \mathbb{I}), (y_{ij}, \mathbb{I})\), \((y_{ij}, y_{ij})\) are compatible, while pairs \((x_{ij}, y_{ij}), (\mathbb{I}, \mathbb{I})\) are not,
3. the set of all states on \(L\) coincide with the set of all PR-box states on \(L\),
4. the logic \(L\) is set-representable,
5. there are pairs of blocks of \(L\) that have two atoms in common, (i.e. the blocks do not form an almost disjoint system, see definition 2.4.2 in [14]).

**Proof.** To show (i) we directly check the requirements of definition 1 (we perform an exhaustive check [16]). It is interesting to note that \(L\) is not a lattice, as 32 pairs of propositions do not have a unique least upper bound. For example, the minimal elements of the upper bound of \([xx, 00]\) and \([yy, 00]\) are \([xy, 11]'\) and \([yx, 11]'\).

Similarly, (ii) can also be checked directly using the definition. For example, for \(x_1\), \(I_1\) and

\[
\begin{align*}
    a &= [xx, 10], \\
    b &= [xx, 01], \\
    c &= [xx, 11],
\end{align*}
\]

we have that \(x_1 = a \lor c, I_1 = b \lor c\), and all of \(a, b, c\) are mutually disjoint. To observe the non-compatibility of the remaining pairs, e.g. \((x_1, I_1)\) it is enough to observe that \((x_1) = x_0\) and

\[
x_1 \lor (x_0 \land y_1) \neq (x_1 \lor x_0) \land (x_1 \lor y_1),
\]

thus the distributivity law does not hold and, consequently, \(x_1, y_1\) do not span a Boolean algebra.

To show (iii) we use the fact that \(L\) is, by construction, an atomistic logic and any state is determined by the value on its atoms. The set of atoms of \(L\) is exactly the set \(A\), so we have obvious mapping between states on \(L\) and PR-box states on the \((2, 2)\)-box world. We need to show that the restrictions imposed on states by the order structure on \(L\) (see definition 3) are not weaker than P1–P3. Again, through an exhaustive check, we observe that the former and the latter linearly depend on each other. Note that this is not a trivial property, because as we observe in section 4.2, we obtain the same order structure on \(L\) with a much more restricted set of PR-box states.
For (iv) we show again through an exhaustive check of all $2^{16}$ matrices with binary entries that the set of states on $\mathcal{L}$ is 2-rich (there are exactly 16 two-valued states which are, in fact, extreme states of the so-called classically correlated boxes, discussed in more detail in section 4.2). Consequently, by theorem 12, $\mathcal{L}$ is set-representable. Concrete representation will be given in section 4.2.

Finally, for (v) we can compute all blocks of $\mathcal{L}$. They are spanned by the following subsets of atoms of $\mathcal{L}$:

\[
\begin{align*}
\{&[xx, 00], [xx, 01], [xx, 10], [xx, 11]\}, &\{&[xy, 00], [xy, 01], [xx, 10], [xx, 11]\}, \\
\{&[xx, 00], [xx, 01], [xy, 10], [xy, 11]\}, &\{&[xy, 00], [xy, 01], [xx, 10], [xx, 11]\}, \\
\{&[yx, 00], [yx, 01], [yx, 10], [xx, 11]\}, &\{&[xx, 00], [yx, 01], [yx, 10], [xy, 11]\}, \\
\{&[yy, 00], [yx, 01], [yx, 10], [yx, 11]\}, &\{&[yy, 00], [yy, 01], [yx, 10], [xy, 11]\}, \\
\{&[yx, 00], [yy, 01], [yy, 10], [xy, 11]\}, &\{&[yy, 00], [yy, 01], [yy, 10], [yy, 11]\}.
\end{align*}
\] 

Thus we see that it is actually justified to call a function $P$ from the definition of the $(2, 2)$-box world a probability, but we should expect that it will have some properties that are not possessed by quantum probability. Clearly, equation (2) express one of them.

Moreover, propositions that we can identify with propositions about the left or right box, i.e. $x, x_{11}, x_{10},$ etc, are compatible and thus simultaneously measurable, which is required by the principle of relativistic causality. Note that it seems that compatibility is, in general, more restrictive than non-signaling, and the problem of compatibility in non-signaling boxes with higher numbers of inputs and outputs should thus be addressed in the future.

An interesting property (iii) will further be discussed in light of the results presented in section 4.2. Moreover, it allows us to drop the distinction between states and PR-box states in the sequel.

4. Properties of the $(2, 2)$-box world

In the previous section we constructed the logic of the $(2, 2)$-box world and showed that it has a structure that is consistent with naïve interpretations of non-signalling theories. Here we will explore its properties more deeply.

4.1. The $(2, 2)$-box world versus the two-qubit system

The most profound difference between the $(2, 2)$-box world and the two-qubit system stems from the fact that the former, being a concrete logic, does not satisfy Heisenberg uncertainty relations (see theorem 13). So from this perspective, the $(2, 2)$-box world is more classical than the two-qubit system.

When it comes to the violation of Tsirelson’s bound, in light of the results summarized in theorem 14, it is clear that we can expect qualitative differences when the orthomodular lattice structure (the logic of quantum models) is replaced by the more general structure of quantum logic. In a similar way the classical bound of two for CHSH-type inequalities is violated when the Boolean algebra is replaced by a more general orthomodular lattice structure.

Another essential difference between the $(2, 2)$-box world and the two-qubit system lies in the fact that the logic of even the simplest quantum model has an infinite number of
propositions, while the \((2, 2)\)-box world has only 82 of them. In particular, we cannot claim that quantum mechanics is a special case of the theory of non-signaling boxes. To see how the number of propositions is important for the logical structure let us consider the following example.

Consider the two-qubit system described by \( \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \). Let \( P \otimes I, Q \otimes I \in \mathcal{B}(\mathcal{H}) \) be a pair of non-commuting projectors. Analogously, choose two non-commuting projectors \( I \otimes R, I \otimes S \) representing measurements on the second subsystem. Now the join of \( P \otimes S \) and \( Q \otimes R \) exists and by definition is equal to the projector onto the smallest subspace that contains both images of \( P \otimes S \) and \( Q \otimes R \). However, this projector cannot be expressed by a linear combination of the projectors that we have used so far.

In other words, if we denote projector \( P \otimes S \) by \( \text{pr}_{xy} \), \( Q \otimes R \) by \( \text{pr}_{yz} \), etc, then \( \text{pr}_{xy} \lor \text{pr}_{yz} \) in quantum mechanics exists, but it is not a linear combination of \([ab] \). On the other hand, in the \((2, 2)\)-box world there is no equivalent of the above quantum mechanical proposition. Consequently, the more general structure of the \((2, 2)\)-box world is a result of a depleted set of propositions when compared to quantum mechanics. We elaborate on this statement in section 4.3.

To sum up, despite the obvious fact that a quantum logic is a more general object than an orthomodular lattice, the simple statement that the \((2, 2)\)-box world is a generalization of some quantum model (e.g. two qubits) is not justified.

### 4.2. Restriction to the classical case

We would like to analyze the logical structure of the \((2, 2)\)-box world when we restrict ourselves to states which are convex combinations of the following 16 PR-box states [5]:

\[
P_{\text{multi}}(\alpha \beta | ab) = \begin{cases} 1 & \text{if} \alpha = ma + n \text{ mod } 2, \beta = lb + k \text{ mod } 2, \\ 0 & \text{otherwise}, \end{cases}
\]  

(15)

where \( m, l, n, k \in \{0, 1\} \) and input values \( x, y \) are treated as 0, 1 respectively. These states are called classically correlated boxes or local boxes [5]. We repeat the construction of the logic from section 3 but with respect to this restricted set of states. It is quite amazing that we get exactly the same logic \( \mathcal{L} \). Consequently, these states are not ‘classical’ in the sense of forming a closed set to which we can apply classical probability rules, but indeed only ‘classically correlated’, i.e. chosen in such way that they do not violate CHSH-like inequalities. Let us thus examine the relation of the ‘classically correlated’ boxes to the truly classical boxes, i.e. boxes implemented by the classical system.

In classical physics a system is described by its phase space. For a system that can be fully described by two dichotomic observables, the phase space is a set of four points. The phase space of a compound system is a Cartesian product of phase spaces of the components, thus the phase space of the true classical \((2, 2)\)-box world is a 16 element set. We can think of it as

\[
\Gamma = \{ (a, b, c, d) \mid a, b, c, d \in \{0, 1\} \},
\]

where each 4-tuple represents the values of four dichotomic observables in that point of \( \Gamma \) (i.e. a pure state). We assume that \( a, b \) are the values of \( x \) and \( y \), respectively on the first subsystem and \( c, d \) are values of \( x \) and \( y \) on the second subsystem.
Any probability measure on this set can be represented by a point within the 15-simplex:

\((p_1, p_2, \ldots, p_{15}), \quad \text{where } p_i \geq 0, \sum_i p_i \leq 1.\)

It is important to note that any point of a simplex can be uniquely represented as a convex combination of extreme points. In other words a mixed state ‘remembers’ how it was made. This constitutes a remarkable property of classical theories.

To any proposition of type \([xy, \alpha \beta]\) there corresponds a subset \(E([xy, \alpha \beta])\) of \(\Gamma\) defined in the following way:

\[
E\left( [z_1z_2, \alpha \beta] \right) = \begin{cases} 
(\{\alpha, b, d\} \in \Gamma \mid b, d \in \{0, 1\}) & \text{for } z_1 = x, z_2 = x \\
(\{\alpha, b, c, \beta\} \in \Gamma \mid b, c \in \{0, 1\}) & \text{for } z_1 = x, z_2 = y \\
(\{\alpha, c, \beta\} \in \Gamma \mid a, \beta \in \{0, 1\}) & \text{for } z_1 = y, z_2 = x \\
(\{\alpha, c, \beta\} \in \Gamma \mid a, c \in \{0, 1\}) & \text{for } z_1 = y, z_2 = y 
\end{cases}
\]  

(16)

We can define a mapping \(\varphi\) from the simplex of classical states on \(\Gamma\) into the set of classically correlated boxes on the (2, 2)-box world. Using the matrix representation of (2, 2)-box world states one expresses it explicitly

\[
\varphi(\mu) = \rho_{\mu} = \begin{bmatrix}
\mu(E[xx, 00]) & \mu(E[xy, 00]) & \mu(E[yx, 00]) & \mu(E[yy, 00]) \\
\mu(E[xx, 01]) & \mu(E[xy, 01]) & \mu(E[yx, 01]) & \mu(E[yy, 01]) \\
\mu(E[xx, 10]) & \mu(E[xy, 10]) & \mu(E[yx, 10]) & \mu(E[yy, 10]) \\
\mu(E[xx, 11]) & \mu(E[xy, 11]) & \mu(E[yx, 11]) & \mu(E[yy, 11])
\end{bmatrix},
\]  

(17)

where \(\mu\) is a probability measure on \(\Gamma\). It is easy to see that this map maps pure classical states onto extreme classically correlated states \((15)\) and thus onto the set of all classically correlated states. But \(\varphi\) is not injective. As an example consider the following two probability measures:

\[
\mu_1(\{u\}) = \begin{cases} 
1/2 & \text{if } u \in \{(1, 0, 1, 1), (1, 1, 1, 0)\} \\
0 & \text{otherwise}
\end{cases}
\]

\[
\mu_2(\{u\}) = \begin{cases} 
1/4 & \text{if } u \in \{(1, 0, 1, 1), (1, 1, 1, 0), (1, 1, 0, 1)\} \\
0 & \text{otherwise}
\end{cases}
\]

Image of both of them under mapping \((17)\) equals

\[
\varphi(\mu_1) = \varphi(\mu_2) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

Consequently this is not an affine isomorphism and the set of classically correlated states is not ‘classically shaped’, as the image of \(\varphi\) is not a simplex. This means that classically correlated states do not decompose in a unique way into a convex combination of extremal states. As a result, the measurements in the (2, 2)-box world are destructive, even when we restrict PR-box states to classically correlated states. Let us consider this in more detail.

Assume that we are given ‘sources’ of different (2, 2)-box world boxes and a device that allows them to be mixed. Denote by
Now we set up our mixing device to output \( \rho_1 \) and \( \rho_2 \) with probability \( 1/2 \). The resulting state encoding our uncertainty would be \( \rho = 1/2 \rho_1 + 1/2 \rho_2 \). Now we ask the question \([xy, 11]\). Because we mix states \( \rho_1 \) and \( \rho_2 \), in the classical world we should get the answer ‘yes’. But in the \((2, 2)\)-box world we can obtain the same state \( \rho \) by mixing completely different states: \( 1/2 \rho_3 + 1/2 \rho_4 \). In the latter case, for the second question we should always get the answer ‘no’. To overcome this ambiguity we must assume that the measurements are destructive, i.e. that either each box can be measured only once, or that the box undergoes state transformation under measurement. But then we need to postulate how the state is changed (e.g. after the positive answer to question \([xy, ab]\) it transforms to a uniform mixture of all the extreme states in which \([xy, ab]\) is certain).

Finally, the correspondence described in (15) between questions in the \((2, 2)\)-box world and subsets of \( \Gamma \) is an order preserving isomorphism between the logic of the \((2, 2)\)-box world and subsets of \( \Gamma \). Thus \((\Gamma, E(\mathcal{L}))\) is a concrete logic corresponding to \( \mathcal{L} \). Although this corollary seems obvious, we verified it “by Mathematica” independently.

### 4.3. Embedding the \((2, 2)\)-box world into an orthomodular lattice

The map \( \varphi \) defined in the previous section induces an embedding

\[
\begin{array}{|c|c|}
\hline
\text{classically correlated} & \text{classical model} \\
\text{\((2, 2)\)-box world boxes} & \text{\((\text{Boolean algebra on } \Gamma)\)} \\
\hline
\end{array}
\]

which is understood in the following sense: a propositional system is embedded into a larger structure and states are extended to states on this larger structure (the map is not injective, but we can take an arbitrary element in the pre-image of \( \varphi \) for each classically correlated \((2, 2)\)-box world state). A similar construction can be performed for ‘quantumly correlated states’. Using the analogy discussed in section 4.1 it is clear that we can define a map \( \psi \)

\[
\psi(\sigma) = \begin{pmatrix}
\text{Tr}(\sigma P' \otimes R') & \text{Tr}(\sigma P' \otimes S') & \text{Tr}(\sigma Q' \otimes R') & \text{Tr}(\sigma Q' \otimes S') \\
\text{Tr}(\sigma P' \otimes R) & \text{Tr}(\sigma P' \otimes S) & \text{Tr}(\sigma Q' \otimes R) & \text{Tr}(\sigma Q' \otimes S) \\
\text{Tr}(\sigma P \otimes R') & \text{Tr}(\sigma P \otimes S') & \text{Tr}(\sigma Q \otimes R') & \text{Tr}(\sigma Q \otimes S') \\
\text{Tr}(\sigma P \otimes R) & \text{Tr}(\sigma P \otimes S) & \text{Tr}(\sigma Q \otimes R) & \text{Tr}(\sigma Q \otimes S)
\end{pmatrix},
\]

where \( \sigma \) is a density matrix of the two-qubit system, and \( P' = 1 - P \), etc. Clearly \( \psi \) maps quantum states into a subset of states of the \((2, 2)\)-box world, which we would call ‘quantumly correlated’, in the same manner as \( \varphi \) does for classical states. Consequently we also have an embedding.
quantumly correlated (2, 2)-box world boxes quantum model (projection lattice)

Now it is natural to ask if we can embed the whole (2, 2)-box world structure in some larger orthomodular lattice, i.e. we ask if the following embedding exists

(2, 2)-box world boxes ↦ orthomodular lattice

Intuition suggests that the answer to this question is negative because the orthomodular lattice is a propositional system of quantum mechanics and due to a violation of Tsirelson’s bound this embedding should not be possible. Nevertheless, the authors are not aware of any proof of Tsirelson’s bound that relies solely on the structure of the orthomodular lattice. Moreover, there are orthomodular lattices which are not lattices of projections of some von Neumann algebra, so in principle, violation of Tsirelson’s bound for an orthomodular lattice is possible. As orthomodular lattices seem to possess a nicer physical interpretation than the more general orthomodular posets (for details see [18]), the question that has arisen in this paragraph becomes interesting.

Theorem 15. The logic of the (2, 2)-box world cannot be embedded into an orthomodular lattice in a way that preserves all the (2, 2)-box world states.

Proof. We will show this by contradiction. Let

\[ q_1 = [xx, 11], \quad q_2 = [yy, 11]. \]

The minimal upper bound of \( q_1 \) and \( q_2 \) consists of elements (one can use the graph of partial order to track this):

\[ r_1 = [xy, 00], \quad r_2 = [yx, 00]. \]

Assume that the unique element \( q_1 \lor q_2 \) exists. By definition, for any state \( \rho \)

\[
\rho(q_1) \leq \rho(q_1 \lor q_2) \leq \rho(r_1), \tag{18}
\]

\[
\rho(q_2) \leq \rho(q_1 \lor q_2) \leq \rho(r_2). \tag{19}
\]

Consider the following states

\[
\rho_1 = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}, \quad \rho_0 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

and convex combination \( \rho_3 = \lambda \rho_1 + (1 - \lambda) \rho_0 \). It follows that \( \rho_1(q_1 \lor q_2) = 1/2 \) and \( 1/2 \leq \rho_0(q_1 \lor q_2) \leq 1 \). For \( 1/4 \leq \lambda \leq 3/4 \) the state \( \rho_3 \) is a classically correlated state. For \( \lambda = 3/4 \) it can be equivalently written as a convex combination of the following eight classically correlated states, each with weight equal to 1/8:
We check that due to (18) we have \( \sigma_i(q_i \lor q_2) = 1 \) for \( i = 1, 2, 3 \) and \( \sigma_i(q_1 \lor q_2) = 0 \) for the remaining five states. Thus \( \rho_{3,4}(q_1 \lor q_2) = 3/8 \). On the other hand
\[
3/4 \rho_i(q_1 \lor q_2) + 1/4 \rho_1(q_1 \lor q_2) = 3/8,
\]
which immediately implies that \( \rho_0(q_1 \lor q_2) = 0 \) in contradiction to the previously obtained bounds on the value of \( \rho_0(q_1 \lor q_2) \). Consequently, it is not possible to define a unique join if we want to allow all (2, 2)-box world states to extend to valid states on the larger structure. □

5. Outlook

The presented analysis shows how the (2, 2)-box world emerges. We start with classical logic over a 16-element phase space. Then we carefully select 82 propositions in a way that will allow us to interpret the resulting structure in terms of a system composed of two subsystems. In particular, we need to preserve compatibility between questions that we want to assign to different components (in the sense that questions are compatible whenever they span a Boolean algebra). As a result, we obtain an orthomodular poset \( \mathcal{L} \) (the order is induced by the order of classical logic). Finally, we take all possible probability measures as admissible states. In this way, due to the link between probability and logic, we obtain a generalized probability theory. One of the features of this theory is a violation of Tsirelson’s bound of the quantum probability theory.

This perspective ‘hides’ the non-signaling condition (P3) in the appropriate selection of 82 propositions from classical logic. This can possibly help to define non-signaling systems that consist of more than two boxes, as our main concern is now the compatibility of certain questions: the notion which has clear meaning in any orthomodular poset.

Moreover, the presented link between non-signaling theories and quantum logics can shed new light on the problem of describing composite systems in the language of quantum logics (as far as the authors know, there is no unique or canonical way to build the logic of composite systems from the logics of components, see [14, 20]). The first step in this direction would be the identification of how (2, 2)-box world logic arises from the very simple logics of separate boxes.

The authors are not aware of any prior results related to the violation of Tsirelson’s bound in the framework of quantum logics. The presented analysis suggests that other
examples of quantum logics studied in the literature can exhibit the violation of Tsirelson’s bound. These could provide new and interesting models for quantum information theory. It is also interesting to examine how compliance with Tsirelson’s bound for an orthomodular lattice is related to the property of being the projection lattice of some von Neumann algebra.

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