Chern-Simons Theory, Knot Invariants, Vertex Models and Three-manifold Invariants*

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Abstract

Chern-Simons theories, which are topological quantum field theories, provide a field theoretic framework for the study of knots and links in three dimensions. These are rare examples of quantum field theories which can be exactly and explicitly solved. Expectation values of Wilson link operators yield a class of link invariants, the simplest of them is the famous Jones polynomial. Other invariants are more powerful than that of Jones. These new invariants are sensitive to the chirality of all knots at least upto ten crossing number unlike those of Jones which are blind to the chirality of some of them. However, all these invariants are still not good enough to distinguish a class of knots called mutants. These link invariants can be alternately obtained from two dimensional vertex models. The \( R \)-matrix of such a model in a particular limit of the spectral parameter provides a representation of the braid group. This in turn is used to construct the link invariants. Exploiting theorems of Lickorish and Wallace and also those of Kirby, Fenn and Rourke which relate three-manifolds to surgeries on framed links, these link invariants in \( S^3 \) can also be used to construct three-manifold invariants.

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1 Introduction

Topological field theories provide a bridge between quantum field theories and topology of low dimensional manifolds. A topological field theory is independent of the metric \( g_{\mu \nu} \) of the manifold on which it is defined. This means that expectation value of the energy-momentum tensor, which is given by the functional variation of the partition function with respect to the metric \( g_{\mu \nu} \) is zero, \( \langle T_{\mu \nu} \rangle = \frac{\delta Z}{\delta g_{\mu \nu}} = 0 \). The topological operators \( W \) of such a theory are metric independent, \( \frac{\delta}{\delta g_{\mu \nu}} \langle W \rangle = 0 \).

An example of a topological field theory is Chern-Simons gauge theory on a three-manifold. This theory provides a field theoretic framework for the study of knots and links in a given three manifold \([1]-[10]\). It was Schwarz who first conjectured \([2]\) that the now famous Jones polynomial \([3]\) may be related to Chern-Simons theory. Witten in his pioneering paper \([4]\) set up the general framework to study knots and links through Chern-Simons field theories. Wilson loop operators are the topological operators of this theory. Expectation value of these operators are the topological invariants for knots and links. The simplest of these invariants is that of Jones which is associated with spin half representation in an \( SU(2) \) Chern-Simons theory \([4]\). Other representations and other semi-simple gauge groups yield other knot invariants. These invariants are also intimately related to the integrable vertex models in two dimensions \([11, 13]\). Representation theory of quantum groups provided yet another direct framework in which these invariants can be studied \([14]\). A mathematically important development is that these link invariants provide a method of obtaining topological invariants for three-manifold \([15]-[16]\). In the following, we shall review these developments.

2 Chern-Simons field theory and link invariants

For a matrix valued connection one-form \( A \) of the gauge group \( G \), the Chern-Simons action \( S \) is given by

\[
kS = \frac{k}{4\pi} \int_{M^3} tr(AdA + \frac{2}{3} A^3)
\]

The coupling constant \( k \) takes integer values in the quantum theory. We shall, except when stated otherwise, take the gauge group \( G \) to be \( SU(2) \) and the three-manifold \( M^3 \) to be \( S^3 \) for definiteness. Clearly action (1) does not have any metric of \( M^3 \) in it. The topological operators are the Wilson loop (knot) operators defined as

\[
W_j[C] = tr_j Pexp \oint_C A_j
\]
for an oriented knot $C$ carrying spin $j$ representation. A few simple knots are:

\[
\begin{array}{c}
\text{KNOTS} \\
\begin{array}{c}
\text{j} \\
\text{0}_1 \\
j \\
\text{3}_1 \\
j \\
\text{4}_1 \\
j \\
\text{5}_1 \\
j \\
\text{5}_2
\end{array}
\end{array}
\]

The Wilson operators are independent of the metric of the three-manifold. For a link $L$ made up of oriented component knots $C_1, C_2, \ldots C_r$ carrying spin $j_1, j_2, \ldots j_r$ representations respectively, we have the Wilson link operator defined as

\[
W_{j_1j_2\ldots j_r}[L] = \prod_{\ell=1}^{r} W_{j_\ell}[C_\ell] \tag{3}
\]

A few two-component links are:

\[
\begin{array}{c}
\begin{array}{c}
\text{j}_1 \\
\text{j}_2 \\
\text{0}_1
\end{array}
\quad
\begin{array}{c}
\text{j}_1 \\
\text{j}_2 \\
\text{2}_1
\end{array}
\quad
\begin{array}{c}
\text{j}_1 \\
\text{j}_2 \\
\text{4}_1
\end{array}
\end{array}
\]

We are interested in the functional averages of the link operators:

\[
V_{j_1j_2\ldots j_r}[L] = Z^{-1} \int_{S^3} [dA] W_{j_1j_2\ldots j_r}[L] e^{ikS}, \quad \text{where} \quad Z = \int_{S^3} [dA] e^{ikS} \tag{4}
\]

Here the integrands in the functional integrals are metric independent. So is the measure $\mathcal{F}$. Therefore, these expectation values depend only on the isotopy type of the oriented link $L$ and the set of representations $j_1, j_2, \ldots j_r$ associated with the component knots.

The expectation values of Wilson link operators (4) can be determined exactly in Chern-Simons theory. For this purpose two ingredients, one from quantum field theory and other from mathematics of braids, are used [8]:

\[\text{(i) Field theoretic input:}\]

The first ingredient is that the Chern-Simons theory on a three-manifold with boundary is essentially characterized by a corresponding Wess-Zumino conformal field theory on that boundary[4]:

\[
\begin{array}{c}
\text{CS} \\
\text{SU(2) CS theory with coupling k on M} \\
\Sigma \quad \Theta M = \Sigma
\end{array} \quad \Longrightarrow \quad \begin{array}{c}
\text{WZ} \\
\text{SU(2) k WZ theory on } \Sigma
\end{array}
\]
And Chern-Simons functional average for Wilson lines ending at \( n \) points in the boundary is described by the associated Wess-Zumino theory on the boundary with \( n \) punctures carrying the representations of the free Wilson lines:

\[
\sum_{j_1} \sum_{j_2} \cdots \sum_{j_n}.
\]

The Chern-Simons functional integral can be represented by a vector in the Hilbert space \( \mathcal{H} \) associated with the space of \( n \)-point correlator of the Wess-Zumino conformal field theory on the boundary \( \Sigma \). In fact, these correlators provide a basis for this boundary Hilbert space. There are more than one possible basis. These different bases are related by duality of the correlators of the conformal field theory.

(ii) Mathematical input: The second input we shall need is the close connection knots and links have with braids. An \( n \)-braid is a collection of non-intersecting strands connecting \( n \) points on a horizontal plane to \( n \) points on another horizontal plane directly below the first set of \( n \) points. The strands are not allowed to go back upwards at any point in their travel. The braid may be projected onto a plane with two horizontal planes collapsing to two parallel rigid rods. The over-crossings and under-crossings of the strands are to be clearly marked. When all the strands are identical, we have ordinary braids. The theory of such braids, first developed by Artin, is well studied. These braids form a group. However, for our purpose here we need to orient the individual strands and further distinguish them by putting different colours on them. We shall represent different colours by different \( SU(2) \) spins. These braids, unlike braids made from unoriented identical strands, have a more general structure than a group. These instead form a groupoid. The necessary aspects of the theory of such braids have been presented in ref.8.

One way of relating the braids to knots and links is through closure of braids. We obtain the closure of a braid by connecting the ends of the first, second, third, .... strands from above to the ends of the respective first, second, third ..... strands from below as shown in (A):
There is a theorem by Alexander\cite{19} which states that any knot or link can be obtained as closure of a braid. This construction of a knot or link is not unique.

There is another construction associated with braids which relates them to knots and links. This is called plating. Consider a $2m$-braid, with pairwise adjacent strands carrying the same colour and opposite orientations. Then connect the $(2i - 1)$th strand with $(2i)$th from above as well as below. This yields the plat of the given braid as shown in (B) above. Then there is a theorem due to Birman\cite{20} which relates plats to links. This states that a coloured-oriented link can be represented (though not uniquely) by the plat of an oriented-coloured $2m$-braid.

Use of these two inputs, namely relation of Chern-Simons theory to the boundary Wess-Zumino theory and presentation of knots and links as closures or platts of braids leads to an explicit and complete solution of the Chern-Simons theory. For this purpose, consider a manifold $S^3$ from which two non-intersecting three-balls are removed. This manifold has two boundaries, each an $S^2$. We place $2n$ Wilson line-integrals over lines connecting these two boundaries through a weaving pattern $B$ as shown in the Figure (a) below:

This is a $2n$-braid placed in this manifold. The strands are specified on the upper boundary by giving $2n$ assignments $(\hat{j}_1^*, \hat{j}_1, \hat{j}_2^*, \hat{j}_2, \ldots, \hat{j}_n^*, \hat{j}_n)$. Here $\hat{j} = (j, \epsilon)$ represents the spin $j$ and orientation $\epsilon$ ($\epsilon = \pm 1$ for a strand going into or away from the boundary) and conjugate assignment $\hat{j}^* = (j, -\epsilon)$ indicates reversal of the orientation. Similar specifications are done with respect to the lower boundary by the spin-orientation assignments $(\hat{\ell}_1^*, \hat{\ell}_1, \hat{\ell}_2^*, \hat{\ell}_2, \ldots, \hat{\ell}_n^*, \hat{\ell}_n)$. Then the assignments $\{\hat{\ell}_i\}$ are just a permutation of $\{\hat{j}_i\}$. Chern-Simons functional integral over this manifold is a
state in the tensor product of the Hilbert spaces associated with the two boundaries, \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). This state can be expanded in terms of some convenient basis. These bases are given by the conformal blocks for 2n-point correlators of the \( SU(2)_k \) Wess-Zumino conformal field theory.

An arbitrary braid can be generated by a sequence of elementary braiding. The eigenvalues of these elementary braids are given by conformal field theory. A knot is given with some framing. Standard framing is such that the self-linking number of the knot (linking number of the knot and its frame) is zero. The braiding eigenvalues depend on the framing. For standard framing, eigenvalues for an elementary braiding two strands carrying spins \( j, j' \) and with orientations \( \epsilon, \epsilon' \) are:

\[
\lambda_t(j,j') = \begin{cases} 
\lambda_t^{(+)}(j,j') = (-)^{j+j'-t} q^{(C_j+C_j')/2+C_{\min(j,j')}-C_t/2} & \text{if } \epsilon \epsilon' = +1 \\
(\lambda_t^{(-)}(j,j'))^{-1} = (-)^{|j-j'|-t} q^{(C_j-C_j')/2-C_t/2} & \text{if } \epsilon \epsilon' = -1
\end{cases}
\]

where \( q = \exp\left(\frac{2\pi i}{k+2}\right) \) and \( C_j \) is the quadratic Casimir invariant of the spin \( j \) representation, \( C_j = j(j+1) \) and \( t \) takes the values allowed in the product of representations of spin \( j \) and \( j' \) by the fusion rules of \( SU(2)_k \) Wess-Zumino conformal field theory, \( t = |j-j'|, |j-j'|+1, \ldots \min(j+j', k-j-j') \). When \( \epsilon \epsilon' = +1 \), the two strands have the same orientation and the braid generator introduces a right-handed half-twist. On the other hand for \( \epsilon \epsilon' = -1 \), two strands are anti-parallel and braid generator introduces a left-handed half-twist. Thus \( \lambda_t^{(+)}(j,j') \) and \( \lambda_t^{(-)}(j,j') \) are the eigen-values for elementary braiding introducing right-handed half-twists in parallelly and anti-parallelly oriented strands respectively.

Writing the weaving pattern \( B \) in Figure (a) above in terms of the elementary braids, the Chern-Simons functional integral over this manifold is given by a matrix \( B(\{j_i\}, \{\ell_i\}) \) in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \).

To plat this braid, we consider two balls with Wilson lines as shown in Figures (b) and (c) above. We glue these respectively from above and below onto the manifold of Figure (a). This yields a link in \( S^3 \).

The Chern-Simons functional integral over the ball (c) is given by a vector in the Hilbert space associated with the \( S^2 \) boundary. This vector \( \langle \psi(\{\ell_i\}) | \) can again be written in terms of a convenient basis of this Hilbert space. Similarly, the functional integral over the ball of Figure (b) above is a vector \( \langle \psi(\{j_i\}) | \) in the associated dual Hilbert space. Gluing these two balls on to each other gives \( n \) disjoint unknots carrying spins \( j_1, j_2, \ldots, j_n \) in \( S^3 \). Their invariant factorizes into the invariants for \( n \) individual unknots. Thus the inner product of the vectors representing the functional integrals over
manifolds (b) and (c) is given by
\[
\langle \psi(\{j_i\}) | \psi(\{j_i\}) \rangle = \prod_{i=1}^{n} [2j_i + 1]
\]
where \([2j + 1]\) is the invariant for an unknot carrying spin \(j\) and the square brackets represent the \(q\)-numbers:
\[
[x] = (q^{\frac{x}{2}} - q^{-\frac{x}{2}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}}).
\]

Next gluing the two balls (b) and (c) on to the manifold of Figure (a), is just taking the matrix element of the matrix \(B\) between these two vectors. This would thus yield a link-invariant [8]:

**Proposition:** Expectation value of a Wilson operator for an arbitrary \(2n\) coloured-oriented link with a plat representation in terms of a coloured-oriented braid \(B(\{j_i\}, \{\ell_i\})\) generated as a word in terms of the braid generators is given by
\[
V[L] = \langle \psi(\{j_i\}) | B(\{j_i\}, \{\ell_i\}) | \psi(\{\ell_i\}) \rangle
\]

This proposition along with the earlier stated result of Birman, allows us to evaluate these invariants for any arbitrary link. Jones polynomials are obtained by placing spin 1/2 representations on all the component knots. When spin 1 representations are placed on them, we have Akutsu-Wadati/Kauffman invariant[22, 11]. For higher spins, we have new invariants.

These invariants are generally sensitive to the chirality of many knots. For example, for left- and right-handed trefoils \(T_L, T_R\), these invariants are different: \(V_j(T_L) \neq V_j(T_R)\) for \(j = \frac{1}{2}, 1, \frac{3}{2} \ldots\). The invariants for the mirror reflected knots are give by simple complex conjugation. These invariants do not detect chirality of all knots. For example, upto ten crossing numbers, there are six chiral knots, \(9_{42}, 10_{48}, 10_{71}, 10_{91}, 10_{104}\) and \(10_{125}\) (as listed in the knot tables of Rolfsen’s book[21]) which are not distinguished from their mirror images by Jones (spin 1/2) polynomials:

\[
\begin{align*}
9_{42} & \quad 10_{48} & \quad 10_{71} \\
10_{91} & \quad 10_{104} & \quad 10_{125}
\end{align*}
\]

SIX CHIRAL KNOTS
Kauffman/Akutsu-Wadati (spin one) polynomials do detect the chirality of four of them, namely \(10_{48}, 10_{91}, 10_{104} \) and \(10_{125}\). But for \(9_{42}\) and \(10_{71}\) both Jones and Kauffman polynomials are not changed under chirality transformation \((q \rightarrow q^{-1})\). However, the new spin 3/2 invariants are powerful enough to distinguish these knots from their mirror images\[9\].

While it does appear that these new invariants are more powerful as we go up in spin, \(j = 1/2, 1, 3/2 \ldots\), it is not true that chirality of all knots can be detected by these invariants. We shall give below an example where none of the invariants obtained from Chern-Simons theories detect the chirality of a 16 crossing knot.

## 3 Mutants and their Chern-Simons invariants

Besides chirality, there is another interesting property of knots and links which we would like to be detected by knot invariants. This is the “mutation” of knots and links. To study this, consider a link \(L_1\) obtained from two rooms \(S\) and \(R\) with two strands going in and two leaving in each of them as shown in the Figure \(L_1\):

The mutant links are obtained in the following way: (i) Remove one of the rooms, say \(S\), from \(L_1\) and rotate it through \(\pi\) about any one of the three orthogonal axes \((\gamma_i)\) as shown in the Figure:
Clearly only two of these rotations are independent: \( \gamma_3 = \gamma_1 \ast \gamma_2 \). (ii) Change the orientations of the lines inside the rotated room \( \mathcal{R} \) to match with the fixed orientations of the external legs of the original room \( \mathcal{S} \). (iii) Then, replace this room back in \( L_1 \).

This yields for \( \gamma_1 \) and \( \gamma_2 \) mutations of \( \mathcal{R} \), mutant links \( L_2 \) and \( L_3 \) as shown above.

We shall argue that no invariants obtained from Chern-Simons theory distinguish these mutants. In the following discussion in this Section we shall take the gauge group \( G \) of the Chern-Simons theory to be arbitrary and shall not specialize to the \( SU(2) \) gauge group.

Next observe that the link \( L_1 \) in \( S^3 \) can be obtained by gluing a three-ball containing room \( \mathcal{S} \) as shown in Figure (i) below with another three-ball with oppositely oriented boundary \( S^2 \) containing room \( \mathcal{R} \) as shown in Figure (ii). Similarly, gluing Figures (iii) and (iv) on to Figure (i) will give corresponding mutant links \( L_2 \) and \( L_3 \) respectively:

\[ \langle \chi(S; r, r, r, r) \rangle \]  
\[ \langle \chi(R; r, r, r, r) \rangle \]  
\[ \langle \chi(\gamma_1 R; r, \bar{r}, \bar{r}, r) \rangle \]  
\[ \langle \chi(\gamma_2 R; r, \bar{r}, \bar{r}, r) \rangle \]

Notice, the diagrams (a) and (b) in each of the Figures (iii) and (iv) are equivalent; these can be changed into each other by simple isotopic moves of the strands. The functional integrals over these balls can be represented by vectors in the Hilbert space associated with the four-punctured boundary \( S^2 \) of each of them. For example, for the three-ball in Figure (ii), we may write the functional integral in a convenient basis \( |\phi^\text{side}_l(r, \bar{r}, \bar{r}, r)\rangle \) characterized by the four-point correlators of the associated Wess-Zumino conformal field theory on \( S^2 \) as:

\[ |\chi(R; r, \bar{r}, \bar{r}, r)\rangle = \sum_l \nu_l(R) |\phi^\text{side}_l(r, \bar{r}, \bar{r}, r)\rangle \]

where \( \nu_l(R) \) are coefficients characterizing the entanglements in the room \( \mathcal{S} \) and the basis with superscript “side” refer to an eigen-basis for the elementary braiding generators \( b_1 \) and \( b_3 \) introducing left-handed half-twists in the first two and last two
anti-parallel strands respectively:

\begin{align*}
\text{b}_1 |\phi^\text{side}_{l}(r, \bar{r}, \bar{r}, r)\rangle &= (\lambda^\text{(-)}_l(r, \bar{r}))^{-1} |\phi^\text{side}_{l}(\bar{r}, r, \bar{r}, r)\rangle \\
\text{b}_3 |\phi^\text{side}_{l}(r, \bar{r}, \bar{r}, r)\rangle &= (\lambda^\text{(-)}_l(r, \bar{r}))^{-1} |\phi^\text{side}_{l}(r, \bar{r}, r, \bar{r})\rangle
\end{align*}

(5)

Here the twisted side two strands are antiparallel and carry representations \(r\) and \(\bar{r}\), the index \(l\) runs over all the irreducible representations in the fusion rule of \(r \otimes \bar{r}\) of the corresponding Wess-Zumino model. An equivalent basis \(|\phi^\text{cent}_m\rangle\) is one where braid generator \(b_2\), which introduces right-handed half-twists in the central two strands, is diagonal:

\begin{align*}
\text{b}_2 |\phi^\text{cent}_m(\bar{r}, r, r, \bar{r})\rangle &= \lambda^\text{(+)m}(r, r) |\phi^\text{cent}_m(\bar{r}, r, r, \bar{r})\rangle .
\end{align*}

(6)

Since this refers to twisting of parallel strands both carrying representation \(r\), the index \(m\) refers to the allowed irreducible representations in the fusion rule \(r \otimes r\) of the corresponding Wess-Zumino model. The two bases are related by \(q\)-Racah coefficient of the quantum group \(G_q\): 

\begin{align*}
|\phi^\text{side}_{l}(\bar{r}, r, r, \bar{r})\rangle &= \sum_m a_{lm} \begin{pmatrix} \bar{r} & r \\ r & \bar{r} \end{pmatrix} |\phi^\text{cent}_m(\bar{r}, r, r, \bar{r})\rangle .
\end{align*}

The eigenvalues \(\lambda^\text{(-)}_l(r, \bar{r})\) and \(\lambda^\text{(+)m}(r, r)\) for right-handed half-twists in two anti-parallel and parallel strands are respectively [4]:

\begin{align*}
\lambda^\text{(-)}_l(r, \bar{r}) &= (-1)^{l} q^{C_l/2} ; \\
\lambda^\text{(+)m}(r, r) &= (-1)^{m} q^{2C_r - C_m/2} ,
\end{align*}

(7)

where \(C_r\), \(C_m\) and \(C_l\) are the quadratic Casimir invariants in the representations \(r\), \(m\) and \(l\) respectively. Depending upon the representation \(l (m)\) occurring symmetrically or antisymmetrically in the tensor product \(r \otimes \bar{r}\) \((r \otimes r)\), \(\epsilon = \pm 1\). Further \(q = \exp 2\pi i/(k + C_v)\), where \(C_v\) is the quadratic Casimir invariant in the adjoint representation and \(k\) is the Chern-Simons coupling.

Now notice that the diagram (iiiib) can be generated by applying the braid generators \(b_1\) and \(b_3^{-1}\) on the diagram in Figure (ii) with interchanged orientation and representation assignments on the first and second, third and fourth strands ending on the boundary. Therefore, we can relate the vectors representing the functional integral over these manifolds as

\begin{align*}
|\chi(\gamma_1 R; r, \bar{r}, \bar{r}, r)\rangle &= b_1 b_3^{-1} |\chi(R; \bar{r}, r, r, \bar{r})\rangle
\end{align*}

Since \(b_1\) and \(b_3\) commute and are diagonal in the same basis with same eigenvalues [3], \(b_1 b_3^{-1} |\phi^\text{side}_{l}(\bar{r}, r, r, \bar{r})\rangle = |\phi^\text{side}_{l}(r, \bar{r}, \bar{r}, r)\rangle\). Thus the Chern-Simons functional integral for
the manifold of Figure (iii) is same as that for the manifold in Figure (ii):

\[ |\chi(\gamma_1; r, \bar{r}, \bar{r}, r) \rangle = |\chi(R; r, \bar{r}, \bar{r}, r)\rangle. \]  

(8)

Such statements will not hold if we increase the number of Wilson lines in these manifolds.

In order to obtain the action of a $\gamma_2$-mutation, let us now consider the Chern-Simons functional integral over the three-ball shown in Figure (iv). Notice that this diagram can be obtained from that in Figure (ii) by applying $b_1b_2b_3b_2b_1$ on it:

\[ |\chi(\gamma_2; r, \bar{r}, \bar{r}, r) \rangle = b_1b_2b_3b_2b_1 |\chi(R; r, \bar{r}, \bar{r}, r)\rangle. \]

(9)

Next we use the fact (Eqns. (5) and (7)) that $b_1 = b_3$ in the Hilbert space associated with four-punctured $S^2$ carrying representations $(\bar{r}, \bar{r}, r, r)$:

\[ b_1 |\phi_{side}(\bar{r}, \bar{r}, r, r)\rangle = b_3 |\phi_{side}(\bar{r}, \bar{r}, r, r)\rangle \]

Further, for an $n$-strand braid on $S^2$, there is an identity $b_1b_2...b_{n-2}b_{n-1}b_{n-2}...b_2b_1 = 1$. This in our case $n = 4$, reduces to $b_1b_2b_3b_2b_1 = 1$. This makes the functional integral over the three-ball of Figure (iv) equal to that of Figure (ii):

\[ |\chi(\gamma_2; r, \bar{r}, \bar{r}, r) \rangle = |\chi(R; r, \bar{r}, \bar{r}, r)\rangle. \]

(10)

Now the Chern-Simons functional integrals over $S^3$ containing links $L_1$, $L_2$ and $L_3$, $V_r[L_1]$, $V_r[L_2]$ and $V_r[L_3]$, are given by the products of the dual vector $|\chi(S; \bar{r}, r, \bar{r}, r)\rangle$ representing the functional integral over the manifold shown in Figure (i) containing room $S^3$ with $|\chi(R; r, \bar{r}, \bar{r}, r)\rangle$, $|\chi(\gamma_1; r, \bar{r}, \bar{r}, r)\rangle$ and $|\chi(\gamma_2; r, \bar{r}, \bar{r}, r)\rangle$ representing respectively the functional integrals over three-balls in Figures (ii), (iii) and (iv):

\[ V_r(L_1) = \langle \chi(S; \bar{r}, r, \bar{r}, r) |\chi(R; r, \bar{r}, \bar{r}, r)\rangle, \]
\[ V_r(L_2) = \langle \chi(S; \bar{r}, r, \bar{r}, r) |\chi(\gamma_1; r, \bar{r}, \bar{r}, r)\rangle, \]
\[ V_r(L_3) = \langle \chi(S; \bar{r}, r, \bar{r}, r) |\chi(\gamma_2; r, \bar{r}, \bar{r}, r)\rangle. \]

Equations (8) and (9) then imply:

\[ V_r[L_1] = V_r[L_2] = V_r[L_3]. \]

(10)

Thus we have shown that invariants of a link and its mutants are identical for every representation $r$ of a compact semi-simple gauge group, placed on all the Wilson lines constituting the links.
The well-known invariants viz., Jones\cite{Jones}, HOMFLY \cite{HOMFLY} and Kauffman \cite{Kauffman} polynomials are obtained from $SU(2)$, $SU(N)$ and $SO(N)$ Chern-Simons theories respectively. Also Akutsu-Wadati polynomials \cite{Akutsu-Wadati} obtained from $N$ state vertex models correspond to $SU(2)$ with spin $N/2$ representation being placed on the knot/link. Hence the fact that all these polynomials do not distinguish mutants is a special case of the above result.

Now let us give an example of a pair of sixteen crossing mutant knots:

![A 16 CROSSING MUTANT PAIR](image)

The two knots are related by mutation of the room indicated by dashed enclosure. One of them is chiral, other is not. Since their invariants from Chern-Simons theories are the same, here is an example of a chiral knot whose chirality can not be detected by any of these invariants.

### 4 Exactly solvable vertex models

The knot invariants obtained from the Chern-Simons field theories can also be obtained from statistical mechanical models in two dimensions\cite{11,13,12}. In these models the variables live on the bonds of a square lattice. Their properties are described by the so called $R$- matrix: $R_{m_1 m_2}(u)$, where $u$ is the spectral parameter. This matrix satisfies the Yang-Baxter equation. Sometimes another $R$-matrix related to this by a permutation is used: $\tilde{R} = \sigma R$ where the operation $\sigma$ interchanges the lower two indices:

$$\tilde{R}_{m_1, m_2} = R_{m_2, m_1}.$$ 

The simplest model of interest is the six-vertex model of Lieb and Wu \cite{24}. The $R$-matrix of this model is a $4 \times 4$ matrix with six non-zero entries. Its elements $R_{m_1, m_2}(u)$ are explicitly given by:
\[
\begin{array}{c|ccccc}
\langle (m_1' \ m_2') \rangle & (\frac{1}{2} \ \frac{1}{2}) & (\frac{1}{2} \ -\frac{1}{2}) & (-\frac{1}{2} \ \frac{1}{2}) & (-\frac{1}{2} \ -\frac{1}{2}) \\
\hline
(\frac{1}{2} \ \frac{1}{2}) & \sinh(\mu - u) & 0 & 0 & 0 \\
(\frac{1}{2} \ -\frac{1}{2}) & 0 & -\sinh u \ e^{-u}\sinh\mu & 0 \\
(-\frac{1}{2} \ \frac{1}{2}) & 0 & e^{u}\sinh\mu & -\sinh u & 0 \\
(-\frac{1}{2} \ -\frac{1}{2}) & 0 & 0 & 0 & \sinh(\mu - u)
\end{array}
\]

This may be compactly rewritten as:

\[
R_{m_1 m_2}^{m_1' m_2'}(u) = \sum_j \left[ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & j \\ m_2 & m_1 & m \end{array} \right] \lambda_j(u) \left[ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & j \\ m_1' & m_2' & m \end{array} \right]
\]

where the square brackets here are the \(SU(2)\) quantum Clebsch-Gordon coefficients with deformation parameter identified as \(q = e^{2\mu}\); spin \(j\) is to be summed over values 0, 1 and \(m\) takes the associated \(m\) values 0 and 0, \(\pm 1\) for the two values of \(j\) respectively; and further

\[
\lambda_0(u) = \sinh(\mu + u), \quad \lambda_1(u) = \sinh(\mu - u).
\]

The \(\hat{R}\)-matrix for this case then reads (same as above with interchange of \(m_1\) and \(m_2\) in the right hand side):

\[
\hat{R}_{m_1 m_2}^{m_1' m_2'}(u) = \sum_j \left[ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & j \\ m_1 & m_2 & m \end{array} \right] \lambda_j(u) \left[ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & j \\ m_1' & m_2' & m \end{array} \right].
\]

\(\lambda_0\) and \(\lambda_1\) are the two independent eigenvalues of \(\hat{R}\). As the spectral parameter \(u\) is taken to infinity, these two eigenvalues are seen to be proportional to the braiding eigenvalues \(\lambda_j(\frac{1}{2}, \frac{1}{2})\) for strands carrying spin half representations of the \(SU(2)_k\) Wess-Zumino conformal field theory with the identification \(q = e^{\frac{2\pi i}{k+2}}\). It is this relation of the \(R\)-matrix in the limit \(u \to \infty\) with braiding matrix which allows construction of knot invariants from the vertex model [11]. On the other hand, corresponding to the general braiding matrices with braiding eigenvalues \(\lambda_j(j_1, j_2)\) associated with two strands carrying arbitrary \(SU(2)\) representations of spins \(j_1\) and \(j_2\) obtained from the conformal field theory, there should be general solutions of the Yang-Baxter solutions whose eigenvalues in the limit of large \(u\) reduce to these braiding eigen-values. We propose a generalization of the formula [11]:

\[
\left( R_{j_1 j_2}^{j_1' j_2'} \right)^{m_1 m_2}_{m_1' m_2'}(u) = \sum_j \left[ \begin{array}{ccc} j_2 & j_1 & j \\ m_2 & m_1 & m \end{array} \right] \lambda_j(u) \left[ \begin{array}{ccc} j_1' & j_2' & j \\ m_1' & m_2' & m \end{array} \right]
\]
and \[
\left( R^{j_1j_2} \right)_{m_1m_2}^{m_1'm_2'} (u) = \sum_{j \in m} \left[ \begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right] \lambda_j(u) \left[ \begin{array}{ccc} j_1 & j_2 & j \\ m_1' & m_2' & m \end{array} \right].
\]

Here the \( SU(2) \) spin \( j \) runs over the values allowed for the irreducible representations in the product of representations with spins \( j_1 \) and \( j_2 \): \( j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2| \)
and \( m = -j, -j + 1, \ldots, j - 1, j; \) \( m_1, m_1' = -j_1, -j_1 + 1, \ldots, j_1 \) and \( m_2, m_2' = -j_2, -j_2 + 1, \ldots, j_2 \).
Only non-zero matrix elements are those with \( m = m_1 + m_2 = m_1' + m_2' \). The generalized eigen-values are now given by:

\[
\lambda_j(u) = \left( \prod_{\ell = |j_1 - j_2| + 1, |j_1 - j_2| + 2, \ldots, j} \sinh(\ell \mu - u) \right)^{j_1} \left( \prod_{k = j+1, j+2, \ldots, j_1+j_2} \sinh(k \mu + u) \right)^{j_2}. \tag{14}\]

In the limit \( u \to \infty \), these eigen-values do indeed, up to a proportionality constant, reduce to the braiding eigenvalues for two strands carrying spins \( j_1 \) and \( j_2 \) in the \( SU(2) \) conformal field theory. The generalized \( R \)-matrix satisfies the Yang-Baxter equation:

\[
\sum_{m_1', m_2', m_3'} \left( R^{j_1j_2} \right)_{m_1m_2}^{m_1'm_2'} (u) \left( R^{j_3j_4} \right)_{m_3}^{m_3'} (u + v) \left( R^{j_5j_6} \right)_{m_4}^{m_4'} (v) = \sum_{m_1', m_2', m_3'} \left( R^{j_5j_6} \right)_{m_2m_3}^{m_2'm_3'} (v) \left( R^{j_3j_4} \right)_{m_1m_3}^{m_1'm_3'} (u + v) \left( R^{j_1j_2} \right)_{m_1'm_2'}^{m_1m_2} (v). \tag{15}\]

For \( j_1 = j_2 = \frac{1}{2} \), this \( R \)-matrix is the same as that of 6-vertex model above. For other low values of \( (j_1, j_2) \), these solutions of the Yang-Baxter equation correspond to other well known vertex models:

- \( (R^1\frac{1}{2})_{m_1m_2}^{m_1'm_2'} (u) \): 6-vertex model of Lieb and Wu\[24\]
- \( (R^{1\frac{1}{2}})_{m_1m_2}^{m_1'm_2'} (u) \): 19-vertex model of Zamolodchikov and Fateev\[25\]
- \( (R^{\frac{1}{2}2})_{m_1m_2}^{m_1'm_2'} (u) \): 44-vertex model
- \( (R^{2\frac{1}{2}})_{m_1m_2}^{m_1'm_2'} (u) \): 85-vertex model

Thus, an alternate route to the same knot invariants as those emerge from the Chern-Simons theory is to obtain the braid representations from the \( \hat{R} \)-matrix by taking the limit \( u \to \infty \):

\[
(\text{Vertex model}) \xrightarrow{\hat{R}(u) \text{ matrix}} \xrightarrow{u \to \infty} \text{Braid representations (from CS theory)} \xrightarrow{\text{Yang-Baxterization}} \text{CS knot/link invariant}.
\]
5 Three-manifold invariants

The invariants of knots and links in $S^3$ obtained from the Chern-Simons theory can be used to construct three-manifold invariants [4, 15, 16]. This provides an important tool to study topological properties of three-manifolds. Starting step in this construction is a theorem due to Lickorish and Wallace [26, 21]:

**Fundamental theorem of Lickorish and Wallace:** Every closed, orientable, connected three-manifold, $M^3$ can be obtained by surgery on an unoriented framed knot or link $[L, f]$ in $S^3$.

The framing $f$ of a link $L$ is defined by associating with every component knot $K_s$ of the link an accompanying closed curve $K_{sf}$ parallel to the knot and winding $n(s)$ times in the right-handed direction. That is the linking number $lk(K_s, K_{sf})$ of the component knot and its frame is $n(s)$. A particular framing is the so called vertical framing where the frame is thought to be just vertically above the two dimensional projection of the knot as shown below. We may indicate this sometimes by putting $n(s)$ writhes in the strand making the knot or even by just simply writing the integer $n(s)$ next to the knot as shown below:

Next the surgery on a framed link $[L, f]$ made of component knots $K_1, K_2, ..., K_r$ with framing $f = (n(1), n(2), ..., n(r))$ in $S^3$ is performed in the following manner. Remove a small open solid torus neighbourhood $N_s$ of each component knot $K_s$, disjoint from all other such open tubular neighbourhoods associated with other component knots. In the manifold left behind $S^3 - (N_1 \cup N_2 \cup ..., N_r)$, there are $r$ toral boundaries. On each such boundary, consider a simple closed curve (the frame) going $n(s)$ times along the meridian and once along the longitude of the associated knot $K_s$. Now do a modular transformation on such a toral boundary such that the framing curve bounds a disc. Glue back the solid tori into the gaps. This yields a new manifold $M^3$. The theorem of Lickorish and Wallace assures us that every closed, orientable, connected three-manifold can be constructed in this way.
This construction of three-manifolds by surgery is not unique: surgery on more than one framed link can yield homeomorphic manifolds. But the rules of equivalence of framed links in \( S^3 \) which yield the same three-manifold on surgery are known. These rules are known as Kirby moves.

**Kirby calculus on framed links in** \( S^3 \): Following two elementary moves (and their inverses) generate Kirby calculus[27]:

**Move I.** For a number of unlinked strands belonging to the component knots \( K_s \) with framing \( n(s) \) going through an unknotted circle \( C \) with framing +1, the unknotted circle can be removed after making a complete clockwise twist from below in the disc enclosed by the circle \( C \):

\[
\begin{align*}
C & \\
\text{n(s)} & \quad \rightarrow \\
L & \\
\text{n'(s)} = n(s) - (\text{lk}(K_s, C))^2
\end{align*}
\]

In the process, in addition to introducing new crossings, the framing of the various resultant component knots, \( K'_s \) to which the affected strands belong, change from \( n(s) \) to \( n'(s) = n(s) - (\text{lk}(K_s, C))^2 \).

**Move II.** Drop a disjoint unknotted circle with framing \(-1\) without any change in the rest of the link:

\[
\begin{align*}
\text{X} & \\
C & \quad \rightarrow \quad \text{X} \quad \quad \\
\text{I} & \\
\text{L} & \\
\text{n'(s)} = n(s) - (\text{lk}(K_s, C))^2
\end{align*}
\]

Two Kirby moves (I) and (II) and their inverses generate the conjugate moves[16]:

**Move \( \bar{I} \).** Here a circle \( C \) with framing \(-1\) and enclosing a number strands can be removed after making a complete anti-clockwise twist from below in the disc bounded by the curve \( C \):
Again, this changes the framing of the resultant knots $K'_s$ to which the enclosed strands belong from $n(s)$ to $n'(s) = n(s) + (lk(K_s, C))^2$.

**Move II.** A disjoint unknotted circle with framing $+1$ can be dropped without affecting the rest of the kink:

Thus Lickorish-Wallace theorem and equivalence of surgery under Kirby moves reduces the theory of closed, orientable, connected three-manifolds to the theory of framed unoriented links via a one-to-one correspondence:

$\left( \begin{array}{c} \text{Framed links in } S^3 \text{ modulo} \\ \text{equivalence under Kirby moves} \end{array} \right) \leftrightarrow \left( \begin{array}{c} \text{Closed, orientable, connected three-} \\ \text{manifolds modulo homeomorphisms} \end{array} \right)$

This consequently allows us to characterize three-manifolds by the invariants of the associated unoriented framed knots and links obtained from the Chern-Simons theory in $S^3$. This can be done by constructing an appropriate combination of the invariants of the framed links which is unchanged under Kirby moves:

$\left( \begin{array}{c} \text{Invariants of a framed unoriented link} \\ \text{which do not change under Kirby moves} \end{array} \right) = \left( \begin{array}{c} \text{Invariants of associated} \\ \text{three-manifold} \end{array} \right)$

We shall now construct one such invariant from the link invariants of $SU(2)$ Chern-Simons theory.

**Invariants for unoriented knots and links from $SU(2)$ Chern-Simons theory:** The knot/link invariants we discussed earlier were constructed for oriented links with standard framing. The braiding eigenvalues given in Sec. 2 reflect this property. For our present purpose, we need to have invariants for unoriented links in vertical framing. This is achieved by taking the eigen-values for the braid matrix introducing
right-handed or left-handed \((R/L)\) half-twist in two parallel strands carrying spins \(j, j'\) as:

\[
\begin{align*}
R_{j \rightarrow j'}: & \quad \lambda_{\ell,R}(j, j') = \lambda_{\ell}(j, j') = (-1)^{|j-j'|-\ell} q^{-\ell(C_j+C_{j'}-C_{\ell})/2}, \\
L_{j \rightarrow j'}: & \quad \lambda_{\ell,L}(j, j') = (\lambda_{\ell}(j, j'))^{-1} = (-1)^{|j-j'|-\ell} q^{\ell(C_j+C_{j'}-C_{\ell})/2},
\end{align*}
\]

and for anti-parallel strands:

\[
\begin{align*}
R_{j \rightarrow j'}: & \quad \lambda_{\ell,R}(j, j') = (\lambda_{\ell}(j, j'))^{-1}, \\
L_{j \rightarrow j'}: & \quad \lambda_{\ell,L}(j, j') = \lambda_{\ell}(j, j').
\end{align*}
\]

Clearly these eigenvalues do not see the orientations on the strands; these are sensitive only to over-crossing and under-crossing.

In standard framing, a writhe can be stretched without affecting the link. In vertical framing this is not so. In this case the invariants of knots get changed by a phase when a writhe is smoothed out as:

\[
R_{j \rightarrow j'} = (\lambda_0(j, j))^{-1} \int j = q^{C_j} \int j, \quad \text{and} \quad L_{j \rightarrow j'} = \lambda_0(j, j) \int j = q^{-C_j} \int j.
\]

Thus, invariant for an unknot with framing +1 and −1 is related to the invariant for an unknot with zero framing as:

\[
V \left[ R_{j \rightarrow j} \right] = q^{C_j} V \left[ \int j \right] = q^{C_j} [2j + 1], \quad \text{and} \quad V \left[ L_{j \rightarrow j} \right] = q^{-C_j} V \left[ \int j \right] = q^{-C_j} [2j + 1]
\]

The invariant for a Hopf link carrying spins \(j_1\) and \(j_2\) on the component knots and with vertical framing can be obtained in two ways using the braiding and inverse braiding:

\[
V \left[ j_1 \bigcirc j_2 \right] = \sum_{\ell} [2\ell + 1] (\lambda_{\ell}(j_1, j_2))^2 = q^{-C_{j_1}-C_{j_2}} \sum_{\ell} [2\ell + 1] q^{C_{\ell}}, \quad \text{and} \quad V \left[ L_{j_1 \rightarrow j_2} \right] = \sum_{\ell} [2\ell + 1] (\lambda_{\ell}(j_1, j_2))^{-2} = q^{C_{j_1}+C_{j_2}} \sum_{\ell} [2\ell + 1] q^{-C_{\ell}}.
\]

These are equal, as they should be, due to the identity:

\[
q^{-C_{j_1}-C_{j_2}} \sum_{\ell} [2\ell + 1] q^{C_{\ell}} = q^{C_{j_1}+C_{j_2}} \sum_{\ell} [2\ell + 1] q^{-C_{\ell}} = [(2j_1 + 1) (2j_2 + 1)].
\]
Consider next the Hopf link $H(j_1, j_2)$ with framing $+1$ for each of its component knots:

\[
\begin{array}{c}
\includegraphics{HopfLink} \\
H(j_1, j_2)
\end{array}
\]

The invariant for this link is given by

\[
V[H(j_1, j_2)] = q^{C_{j_1} + C_{j_2}} V[j_1 \otimes j_2] = q^{C_{j_1} + C_{j_2}} [(2j_1 + 1) (2j_2 + 1)].
\] (16)

Next we wish to construct a combination of these invariants which would be unchanged under Kirby move I:

\[
\begin{array}{c}
\includegraphics{KirbyMoveI} \\
\text{Kirby move I}
\end{array}
\]

That is, we solve the following equation for $\mu_\ell$ and $\alpha$:

\[
\sum_{\ell=0,1/2,1,...,k/2} \mu_\ell V[H(j, \ell)] = \alpha [2j + 1],
\] (17)

where $[2j + 1]$ is the invariant for an unknot carrying spin $j$ representation. This solution is given by

\[
\mu_\ell = S_{0\ell}, \quad \alpha = e^{\pi i c/4}, \quad c = \frac{3k}{k + 2},
\] (18)

where

\[
S_{j\ell} = \sqrt{\frac{2}{k + 2}} \sin \frac{\pi (2j + 1) (2\ell + 1)}{k + 2} = [(2j + 1) (2\ell + 1)] S_{00}.
\]

This can be easily verified by using the identity:

\[
\sum_\ell S_{j\ell} q^{C_\ell} S_{\ell m} = e^{\pi i c/4} q^{-C_j - C_m} S_{jm},
\]

which follows readily by noticing that the matrices $S_{j\ell}$ and $T_{j\ell} = q^{C_j} e^{-\pi i c/12} \delta_{j\ell}$ are the generators of the modular transformations $\tau \to -1/\tau$ and $\tau \to \tau + 1$ on the characters of the Wess-Zumino $SU(2)_k$ conformal field theory and hence satisfy relations $S^2 = 1$, and $(ST)^3 = 1$. This last relation implies $STS = T^* S T^*$ which is the identity above.

Now let us consider the following two links $H(X; j, \ell)$ and $U(X; j)$:
where $X$ as an arbitrary entanglement inside the box. The link $H(X; j, \ell)$ is the connected sum of the link $U(X; j)$ and a framed Hopf link $H(j, \ell)$. Factorization properties of invariants of such a connected sum of links yields:

$$[2j + 1] V[H(X; j, \ell)] = V[U(X; j)] V[H(j, \ell)].$$

This further implies:

$$\sum \mu_\ell V[H(X; j, \ell)] = \alpha V[U(X; j)].$$

It is possible to generalize this relation for the following links $H(X; j_1, j_2, \ldots, j_n; \ell)$ and $U(X; j_1, j_2, \ldots, j_n)$:

$$\sum \mu_\ell V[H(X; j_1, j_2, \ldots, j_n; \ell)] = \alpha V[U(X; j_1, j_2, \ldots, j_n)].$$

This relation then reads:

$$\sum \mu_\ell V[H(X; j_1, j_2, \ldots, j_n; \ell)] = \alpha V[U(X; j_1, j_2, \ldots, j_n)].$$

Also, for a link containing a disjoint unknot with framing $-1$:

$$\sum \mu_\ell V \left[ \begin{array}{c} j \ \times \ \ell \end{array} \right] = \alpha V \left[ \begin{array}{c} j \ \times \ \ell \end{array} \right]$$

This follows readily due to the exact factorizations of invariants of disjoint links into those of the individual links and use of the identity: $\sum_\ell S_{0\ell} q^{-C_\ell} S_{00} = e^{-\pi i c/4} S_{00}$. 

20
Clearly the Eqns. (19) and (20) respectively correspond to the two generators of the Kirby calculus. Therefore, this allows us to state the following result:

*For a framed unoriented link* \([L, f]\) *with component knots* \(K_1, K_2, ..., K_r\) *and framing* \(f = (n_1, n_2, ..., n_r)\), *the following is an invariant of the associated closed, connected, orientable three-manifold obtained by surgery on the link (upto possible changes of the framing of the manifold):*

\[
F[L, f] = \sum_{j_i=0,1/2,1,...,k/2} \mu_{j_1} \mu_{j_2} .... \mu_{j_r} V[L; n_1, n_2, ..., n_r; j_1, j_2, ..., j_r]
\]  

(21)

Under Kirby moves \(F[L, f]\) changes only by possible phase factors (i.e., powers of \(\alpha\) or \(\alpha^*\)) associated with the framing of the manifold. Particularly, under the two Kirby generators \(F[L; f]\) changes as:

![Diagrams](https://example.com/diagram.png)

Here we have depicted the manifold invariant \(F\) by the affected portion of the diagram of the framed link on which surgery is to be performed.

The frame dependence of the manifold invariant \(F\) can be compensated by noticing the following property of the linking matrix. For a framed link \([L, f]\) whose component knots \(K_1, K_2, ..., K_r\) have framings (self-linking numbers) as \(n_1, n_2, ..., n_r\) respectively, the linking matrix is defined as

\[
W[L, f] = \begin{pmatrix}
  n_1 & lk(K_1, K_2) & lk(K_1, K_3) & ... & lk(K_1, K_r) \\
  lk(K_2, K_1) & n_2 & lk(K_2, K_3) & ... & lk(K_2, K_r) \\
  .. & .. & n_3 & ... & .. \\
  .. & .. & .. & ... & .. \\
  lk(K_r, K_1) & .. & .. & .. & n_r
\end{pmatrix}
\]

where \(lk(K_i, K_j)\) is the linking number of knots \(K_i\) and \(K_j\). The signature of the linking matrix is given by

\[
\sigma[L, f] = (\text{no. of } +\text{ve eigenvalues of } W) - (\text{no. of } -\text{ve eigenvalues of } W)
\]

Then this signature for the framed link \([L, f]\) and those for the links \([L', f']\) obtained by transformation by the two elementary generators of the Kirby calculus are related in
a simple fashion:

Kirby move I: $\sigma[L, f] = \sigma[L', f'] + 1$;  
Kirby move II: $\sigma[L, f] = \sigma[L', f'] - 1$.

Now, if we define a new three-manifold invariant by  
$$\hat{F}[L, f] = \alpha^{-\sigma[L, f]} F[L, f],$$
then this invariant will not see the changes of manifold framings under Kirby moves; it would be exactly unchanged by the Kirby moves. Thus we may state the following important result:

**Proposition:** For a framed link $[L, f]$ with component knots, $K_1$, $K_2$, ..., $K_r$ and their framings respectively as $n_1$, $n_2$, ..., $n_r$, the quantity

$$\hat{F}[L, f] = \alpha^{-\sigma[L, f]} \sum_{\{j_i\}} \mu_{j_1} \mu_{j_2} \ldots \mu_{j_r} V[L; n_1, n_2, \ldots, n_r; j_1, j_2, \ldots, j_r]$$

constructed from invariants $V$ of the unoriented framed link, is an invariant of the associated three-manifold obtained by surgery on that link.

**Explicit examples:** Now let us give the value of this invariant for some simple three-manifolds. The surgery descriptions of manifolds $S^3$, $S^2 \times S^1$ and $RP^3$ are given by an unknotted with framing +1, 0 and +2 respectively. The above invariant for these manifolds is:

$$\hat{F}[S^3] = 1, \quad \hat{F}[S^2 \times S^1] = \frac{1}{S_{00}}, \quad \hat{F}[RP^3] = \alpha \sum_{j=0, \frac{1}{2}, 1, \frac{3}{2}, \ldots} \frac{S_{0j} q^{-2j}}{S_{00}}.$$

A more general example is the whole class of Lens spaces $L(p, q)$; above three manifolds are special cases of this class of manifolds. These are obtained by surgery on a framed link made of successively linked unknots with framing given by integers $a_1, a_2, \ldots, a_n$:

$$[L, f] = \begin{array}{c}
\begin{array}{c}
\text{a}_1 \\
\text{a}_2 \\
\vdots \\
\text{a}_n
\end{array}
\end{array}$$

where these framing integers provide a continued fraction representation for the ratio of two integers $p, q$:

$$\frac{p}{q} = a_n - \frac{1}{a_{n-1} - \frac{1}{\ldots - a_2 a_1}}.$$
The invariant for these manifolds can readily be evaluated and is given by the formula:

\[
\hat{F} [\mathcal{L}(p, q)] = \alpha^{-\sigma[L, f]} \alpha^{(\sum a_i)/3} \frac{(S M^{(p, q)})_{00}}{S_{00}},
\]

where matrix \(M^{(p, q)}\) is given in terms of the modular matrices \(S\) and \(T\):

\[
M^{(p, q)} = T^{an} S T^{a_{n-1}} S \ldots T^{a_2} S T^{a_1} S.
\]

Another example we take up is the Poincare manifold \(P^3\) (dodecahedral space). It is given \[21\] by surgery on a right-handed trefoil knot with framing +1:

\[
\begin{array}{c}
\begin{array}{c}
\text{R}\\ \text{R}\\ \text{L}
\end{array}
\end{array}
\]

Notice, each right-handed crossing of the trefoil introduces +1 linking number between the knot and its vertical framing, and each of the two left-handed writhes contributes −1 so that the total frame number of this knot is +1. Using the proposition above, the invariant for this manifold can be calculated. It turns out to be:

\[
\hat{F}[P^3] = \alpha^{-1} \sum_{m, \ell, j=0, \frac{1}{2}, \ldots, \frac{3}{2}} (-)^j S_{0\ell} S_{0j} S_{\ell m} S_{0\ell} S_{jm} q^{-5C_\ell + \frac{1}{2}C_j}.
\]

It is of interest that this invariant by inspection turns out to be related in a simple way to the partition functions \(Z[M^3]\) of the Chern-Simons theory in all those three-manifolds \(M^3\) for which these have been calculated: \(\hat{F}[M^3] = \frac{Z[M^3]}{S_{00}}\). This thus provides an alternative method of calculating the Chern-Simons partition function.

The three-manifold invariants presented here use link invariants from \(SU(2)\) Chern-Simons theory. It is clear that a similar construction can be done with link invariants from Chern-Simons gauge theories based on other semi-simple groups. These would yield new three-manifold invariants.

Next question we may ask is: Is this three-manifold invariant complete? Two manifolds \(M\) and \(M'\) for which the invariants \(\hat{F}[M]\) and \(\hat{F}[M']\) are different can not be homeomorphic to each other. But the converse is not always true; for two arbitrary manifold, the invariants need not be always different. Recall the invariants obtained from Chern-Simons theory for mutant knots are not distinct. Hence, manifold obtained by surgery on mutant knots can not be distinguished by this three-manifold invariant.
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