7-EQUATION ON \((p, q)\)-FORMS ON CONIC NEIGHBOURHOODS OF 1-CONVEX MANIFOLDS

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1. INTRODUCTION AND THE MAIN THEOREM

Let \(\pi : Z \to X\) be a submersion from a complex manifold \(Z\) to a 1-convex manifold \(X\) with the exceptional set \(S\), which is also a manifold. The motivation for this work was to construct solutions of the \(\partial\) -equation with at most polynomial poles at \(\pi^{-1}(S)\) in a particular geometric situation (see Figure 1), namely on a conic neighbourhood \(V\) of sections of \(\pi : Z \to X\). The results of this paper may provide one step in the proof of such a claim. It turned out that it is possible to construct solutions of the \(\partial\) -equation such that their \(L^2\)-norms on \(V_\delta := \{z \in V, d(z, \pi^{-1}(S)) > \delta\}\) with respect to some ambient Hermitian metric \(h_Z\) on \(Z\) and \(h_E\) on \(E\) grow at most polynomially as \(\delta \to 0\). To this end weights that give rise to positive Nakano curvature and have polynomial behaviour on \(\pi^{-1}(S)\) are constructed. Sup-norm estimates are given in the case \(q = 1\). The main theorem of the present paper is the following:

**Theorem 1.1** (Nakano positive curvature tensor in bidegree \((p, q)\)). Let \(X\) be an \(n\)-dimensional complex manifold, \(X\) a 1-convex manifold, \(S \subset X\) its exceptional set, which is also a manifold, \(\pi : Z \to X\) a holomorphic submersion with \(r_0\)-dimensional fibres, \(\sigma : E \to Z\) a holomorphic vector bundle and \(a : X \to Z\) a holomorphic section. Let \(\varphi : X \to [0, \infty)\) be a plurisubharmonic exhaustion function, strictly plurisubharmonic on \(X \setminus S\) and \(\varphi^{-1}(0) = S\). Let \(U = \varphi^{-1}([0, c])\) for some \(c > 0\) be a given holomorphically convex set and let \(s \in \{1, \ldots, n\}\). Then there exist an open neighbourhood \(V\) of \(a(U \setminus S)\) conic along \(a(S)\), a Nakano positive Hermitian metric \(h\) on \(E|_V\) with at most polynomial poles on \(\pi^{-1}(S)\) and such that
the Chern curvature tensor $i\Theta(E \otimes \Lambda^*TZ)$ restricted to $V$ is Nakano positive and has at most polynomial poles and zeroes on $\pi^{-1}(S)$.

For standard techniques for solving the $\bar{\partial}$-equation we refer to Demailly’s book *Complex analytic and algebraic geometry* [Dem]. Our main tool follows.

**Theorem 1.2** (Theorem VIII-4.5, [Dem]). If $(W,\omega)$ is complete and $A_{E,\omega} \geq 0$ in bidegree $(p,q)$, then for any $\bar{\partial}$-closed form $u \in L^2_{p,q}(W, E)$ with

$$\int_W \langle A_{E,\omega}^{-1}u, u \rangle dV < \infty$$

there exists $v \in L^2_{p,q-1}(W, E)$ such that $\bar{\partial}v = u$ and

$$\|v\|^2 \leq \int_W \langle A_{E,\omega}^{-1}u, u \rangle dV.$$

It seems difficult to handle the commutator $A_{E,\omega}$ (see Sect. VII-7, [Dem] for computations) in the case of $(p,q)$-forms because it has terms of mixed signs for $p < n$. Therefore, we view the $(p,q)$-form $u$ as a $(n,q)$-form $u_1$ with coefficients in $E_1 = E \otimes \Lambda^{n-p}TZ$ by invoking the isomorphism $\Lambda^pT^*Z \simeq \Lambda^{n-p}TZ \otimes \Lambda^nT^*Z$. If $u$ is closed so is $u_1$. In addition, in bidegree $(n,q)$, there is an analog of Theorem 1.2 for a noncomplete Kähler metric provided that the manifold possesses a complete one (Theorem VIII-6.1, [Dem]). Moreover, the positivity of $A_{E_1,\omega}$ follows from the positivity of the Chern curvature tensor $i\Theta(E_1) = i\Theta(E) + i\Theta(\Lambda^{n-p}TZ)$.

To use both theorems we first need a Nakano positive Hermitian metric on $E|_V$ and a Kähler metric on $V$, both with polynomial zeroes or poles on $\pi^{-1}(S)$. The Kähler metric $\omega = i\partial\bar{\partial}\Phi$ constructed in Section 2 and the Nakano positive Hermitian metric given by Theorem 1.1 in [Pre1] have the desired properties. The space $(V, \omega)$ is not complete but if we take a smaller neighbourhood conic along $a(N(g))$, $N(g) := g^{-1}(0)$, for some holomorphic function $g : X \to \mathbb{C}$ with $g(S) = 0$, it contains a conic Stein neighbourhood $V'$ which is complete Kähler (Figure 2). With the above notation we have

**Corollary 1.3** ($\bar{\partial}$-equation in bidegree $(p,q)$). Let $g : X \to \mathbb{C}$ be holomorphic with $N(g) = g^{-1}(0) \supset S$. Let $V'$ be an open Stein neighbourhood of $a(U \setminus N(g))$, conic along $a(N(g))$, let $u$ be a closed $(p,q)$-form on $V'$, $u_1$ the corresponding $(n,q)$-form with coefficients in $E_1 = E \otimes \Lambda^{n-p}TZ$ and let $h$ be the metric on $E$ from Theorem
with \( s = n - p \) and \( h_1 \) the induced metric on \( E_1 \). Denote by \( A = A_{E \otimes \Lambda^{n-p} T_Z, \omega} \) the commutator and assume that

\[
\int_{V'} \langle A^{-1} u_1, u_1 \rangle_{h_1} dV_\omega < \infty.
\]

Then there exist an \((n, q - 1)\)-form \( v_1 \) with

\[
\| v_1 \|^2 = \int_{V'} \langle v_1, v_1 \rangle_{h_1} dV_\omega \leq \int_{V'} \langle A^{-1} u_1, u_1 \rangle_{h_1} dV_\omega.
\]

**Corollary 1.4.** If \( u \) is smooth and \( v_1 \) is the minimal norm solution, then \( v_1 \) and the associated \((p, q - 1)\)-form \( v \) are smooth. The \( L^2 \)-norms of \( v_1 \) on the sets \( V'_\delta := \{ z \in V', d(z, \pi^{-1}(S)) > \delta \} \) with respect to \( h_Z \) and \( h_E \) grow at most polynomially with respect to \( \delta \) as \( \delta \to 0 \) and the same holds for the \( L^2 \)-norms of the corresponding \((p, q - 1)\)-form \( v \).

If \( q = 1 \) we can use Lemma 4.5 in [Pre1] which is an adaptation of Lemma 3.2 in [FL] to get sup-norm estimates from the Bochner-Martinelli-Koppelman formula if we take a slightly smaller Stein neighbourhood \( V'' \subset V' \), conic along \( a(N(g)) \) (Figure 2).

**Corollary 1.5.** If \( q = 0 \) and the initial form \( u \) is smooth on a neighbourhood of \( a(U \setminus S) \), the form \( v \) has at most polynomial poles on \( \pi^{-1}(S) \).

**Notation.** The notation from Theorem 1.1 is fixed throughout the paper. Let \( h_Z \) be a Hermitian metric defined on the manifold \( Z \) and let \( \sigma : E \to Z \) be a holomorphic vector bundle of rank \( r \) equipped with a Hermitian metric \( h_E \). The local coordinate system in a neighbourhood \( V_{z_0} \subset Z \) of a point \( z_0 \in a(U) \) is \((z, w)\), where \( z \) denotes the horizontal and \( w \) the vertical (or fibre) direction and \( z_0 = (0, 0) \). More precisely, every point in \( a(U) \) has \( w = 0 \) and points in the same fibre have the same first coordinate. If the point \( z_0 \) is in \( a(S) \) we write the \( z \)-coordinate as \( z = (z_1, z_2) \), where \( a(S) \cap V_{z_0} = \{ z_2 = 0, w = 0 \} \cap V_{z_0} \). The manifold \( Z \) is \( n \)-dimensional and the dimension of the fibres \( Z_{z_0} \) is constant, \( r_0 = \dim Z_{z_0} \). The notation \( \zeta_1, \ldots, \zeta_n \) is sometimes used for local coordinates in \( Z \).

2. **Construction of the Kähler metric \( \omega \)**

The Kähler metric \( \omega \) will be obtained from the Kähler potential \( \Phi = \varphi_0 + \varphi_1 \) and is similar to the one constructed in Subsection 2.1 in [Pre1]. The only difference
is that we choose a specific plurisubharmonic function \( \varphi_0 \) instead of the given \( \varphi \) in order to be able to study the curvature properties of \( \omega \). The construction is explained below.

Since the Remmert reduction \( p : X \to \hat{X} \) of \( X \) is a Stein space with finitely many isolated singular points it has a proper holomorphic embedding \( f : \hat{X} \to \mathbb{C}^M \) for some large \( M \). Consequently, the holomorphic functions \( f_1 \circ p, \ldots, f_M \circ p \) generate the cotangent space \( T^* X \setminus S \) and the function \( \hat{\varphi}_0 := \sum |f_i \circ p|^2 \) is a plurisubharmonic exhaustion function of \( X \), strictly plurisubharmonic on \( X \setminus S \). The functions \( f_i := f_i \circ p \circ \pi \) are defined on \( Z \) and they generate the horizontal cotangent space on \( a(X \setminus S) \). Define \( \varphi_0 := \hat{\varphi}_0 \circ \pi = \sum |f_i|^2 \) and denote by \( k_0 \) the maximal order of degeneracy of \( i\partial\bar{\partial}\varphi_0 \) at \( \pi^{-1}(S) \).

For given \( l > 2 \) the construction in Subsection 2.1 in [Pre1] with the function \( \varphi \) replaced by \( \varphi_0 \) yields almost holomorphic functions \( f_{M+1}, \ldots, f_N \), holomorphic to a degree \( l \) with zeroes of order at least \( k_0 \) on \( \pi^{-1}(S) \). To be precise, for every sufficiently large \( k \) by Theorem A there exist sections \( F_{M+1}, \ldots, F_N \in \Gamma(U', \mathcal{J}(a(S))^k(\mathcal{J}(a(U'))/\mathcal{J}^{l+1}(a(U')))) \), \( U \subseteq U' \) which locally generate the sheaf on \( U \). We further assume that \( k > k_0 \). The functions \( f_{M+1}, \ldots, f_N \), are obtained by patching together particular local lifts of these sections using the partition of unity \( \{\chi_j, U_j\} \) which we (can) choose to depend on the horizontal variables only and so \( f_{M+1}, \ldots, f_N \), are holomorphic in vertical directions. In local coordinates \( (z, w) \) near \( a(S) \) we thus have

\[
f_i(z, w) = \sum_{|\alpha| = k, 0 < |\beta| \leq l} c_{\alpha\beta} z_\alpha^\beta w^\beta + \sum_{i,j,l} \chi_j(z) f_{ijl}(z, w)
\]

with \( f_{ijl} \in O(||z_2||^k||w||^{l+1}) \) holomorphic on open sets \( U_j \). The functions \( f_i \) satisfy the estimate \( \partial f_i(z, w) \approx ||z_2||^k||w||^{l+1} \). Moreover, we can express \( z_2^\alpha w_j = \sum g_{i\alpha}(z)f_i(z, w) + O(||z_2||^k||w||^{l+1}) \) with \( g_{i\alpha} \) holomorphic and from this we infer that \( \partial w f_i, 1 \leq j \leq r_0, M + 1 \leq i \leq N \) generate the vertical cotangent bundle on a neighbourhood \( V_U \) of \( a(U) \) in \( Z \) except on \( \pi^{-1}(S) \). Consequently, the matrix corresponding to \( \partial w \partial w \sum |f_i|^2 \) is of the form \( ||z_2||^{2k}G \), with \( G \) invertible. Define

\[
\Phi := \sum f_i \bar{f}_i \quad \text{and} \quad \omega := i\partial\bar{\partial}\Phi.
\]

We claim that the function \( \Phi \) is a Kähler potential on a conic neighbourhood of \( a(U \setminus S) \) and \( \omega \) a Kähler metric.

Write local coordinates as \( (\zeta_1, \ldots, \zeta_n) = (z, w) \) and represent the Levi form

\[
i\partial\bar{\partial}\Phi = i \sum h_{jk} d\zeta_j \wedge d\bar{\zeta}_k
\]

by a matrix \( H = \{h_{jk}\} \). In local coordinates \( (z, w) \) the nonnegative part of \( \omega \), \( \omega_+ = i \sum \partial f_i \wedge \bar{\partial} f_i \), represented in the matrix form as \( H_+ \), can be estimated from below by

\[
H_+(z, w) \geq \begin{bmatrix} ||z_2||^{2k_0} + ||w||^2 ||z_2||^{2k-2} & ||w|| ||z_2||^{2k-1} \\ ||w|| ||z_2||^{2k-1} & ||z_2||^{2k} \end{bmatrix},
\]

where we have estimated the decay of \( \varphi_0 \) by \( ||z_2||^{2k_0} \) from below. The possibly negative part \( \omega_- = i \sum \partial\bar{\partial} f_i f_i + f_i \partial\bar{\partial} f_i + \partial f_i \wedge \bar{\partial} f_i \) degenerates at least as \( ||w||^2 ||z_2||^{k-1} \). It is clear that for \( ||z_2|| > \delta \) and small \( ||w|| \) or \( ||z_2|| \leq \delta \) and \( ||w|| \leq ||z_2||^2 \) the matrix \( H \) is strictly positive definite and thus \( \omega \) is a Kähler metric on a neighbourhood of \( a(U \setminus S) \), conic along \( a(S) \). Let \( (z, w) \) be local coordinates near \( a(S) \) and define
$H_0(z) := H(z, 0) = H_+(z, 0)$, $H_1 = H - H_0$. It follows that $H_1 = \mathcal{O}(\|w\| z_2^{2k-1})$ and that $H_0$ decreases polynomially (in some directions) as we approach $\pi^{-1}(S)$ and its degeneracy is bounded from below by $\|z_2\|^{2k}$,

$$H_0(z) \gtrsim \begin{bmatrix} \|z_2\|^{2k_0} & 0 \\ 0 & \|z_2\|^{2k} \end{bmatrix}.$$  

Notice that $H_0$ is strictly positive on a neighbourhood of $a(\mathcal{U})$ except on $\pi^{-1}(S)$ (and therefore invertible) and $\|H_0^{-1}\|$ degenerates in the worst case as $\|z_2\|^{-\kappa}$ for some $\kappa \geq 0$. Because $S$ is compact, there exists one $\kappa$ for all points in $a(S)$. Write $H = H_0(I + H_0^{-1} H_1)$, $H^{-1} = (I + H_0^{-1} H_1)^{-1} H_0^{-1}$, then

$$\|H_0^{-1} H_1\| \approx \|z_2\|^{2k-1-\kappa}\|w\|$$

in the worst case and this term is small, $\|H_0^{-1} H_1\| < \|z_2\|^{3\kappa + k_1}$ on conic neighbourhoods of the form $\|w\| \leq \|z_2\|^{4\kappa + k_1}$, and so

$$(2.2) \quad H^{-1} = H_0^{-1} + H_0^{-1} H_1 \sum_0^\infty (H_0^{-1} H_1)^n H_0^{-1} = H_0^{-1} + N, \|N\| \leq \|z_2\|^{2\kappa + k_1}.$$  

Inside this cone the degeneracy of the inverse $H^{-1}$ is governed by $H_0^{-1}$.

### 3. Theorems on curvatures

#### 3.1. Basic theorems on curvatures.

Before proceeding to the proof we recall some formulae from Demailly’s *Complex analytic and algebraic geometry* [Dem].

Let $(X, \omega)$ be a Kähler manifold, $E \to X$ a rank $r$ vector bundle equipped with a Hermitian metric $h$. The matrix $H$ that corresponds to $h$ in local coordinates is given by $\langle u, v \rangle_h = \sum h_{\lambda\mu} u_\lambda v_\mu = u^T H v$. Let $i\Theta(E)$ be the Chern curvature form of the metric and $\Lambda$ the adjoint of the operator $u \to u \wedge \omega$, defined on $(p, q)$-forms. Denote by $L^2_{p,q}(X, E)$ the space of $(p, q)$-forms with bounded $L^2$-norms with respect to the $h$ and let $A_{E, \omega} = [i\Theta(E), \Lambda]$ be the commutator.

In bidegree $(n, q)$ the positivity of $A_{E, \omega}$ follows from Nakano positivity of $E$. Let $e_1, \ldots, e_r$ be a local frame of $E$. If the metric is locally represented by a matrix $H$ then

$$(3.1) \quad i\Theta(E) = i\partial\bar{\partial} (P^{-1} \partial \bar{\partial}) = i \sum c_{j\kappa\lambda\mu} dz_j \wedge \bar{dz}_k \otimes e_\lambda^* \otimes e_\mu.$$  

If $e_1, \ldots, e_r$ is an orthonormal frame, then the Hermitian form $\theta_E$ defined on $TX \otimes E$, which is associated to $i\Theta(E)$, takes the form

$$(3.2) \quad \theta_E = \sum c_{j\kappa\lambda\mu} (dz_j \otimes e_\lambda^*) \otimes (\bar{dz}_k \otimes e_\mu).$$

The curvature tensor (3.1) is *Griffiths positive* if the form (3.2) is positive on decomposable tensors $\tau = \xi \otimes v$, $\xi \in TX$, $v \in E$, $\theta_E(\tau, \tau) = \sum c_{j\kappa\lambda\mu} \xi_j \bar{\xi}_k v_\lambda \bar{v}_\mu$ and *Nakano positive* if it is positive on $\tau = \sum \tau_{j\lambda}(\partial/\partial z_j) \otimes e_\lambda$, $\theta_E(\tau, \tau) = \sum c_{j\kappa\lambda\mu} \tau_{j\lambda} \bar{\tau}_{k\mu}$. In a non-orthonormal frame we have $\theta_E(\tau, \tau) = \sum c_{j\kappa\lambda\mu} \tau_{j\lambda} \bar{\tau}_{k\mu} h_{\mu\nu}$. Proposition VII-9.1, [Dem] states that

$$(3.3) \quad \text{if } \theta(E) > \text{Griff} \text{ then } r \text{ Tr}_E(\theta(E)) \otimes h - \theta(E) > \text{Nak} 0.$$  

The metric $h$ on $E$ induces the metric $h^s$ on $\Lambda^s E$. Let $L$ be an $s$-tuple of (not necessarily ordered) indices $L = (\lambda_1, \ldots, \lambda_s)$ and denote $e_L := e_{\lambda_1} \wedge \ldots \wedge e_{\lambda_s}$. If $\sigma$ is a
permutation, then \( e_{\sigma(L)} = \text{sign}(\sigma)e_L \). Let \( L, M \in \{(\lambda_1, \ldots, \lambda_s), 1 \leq \lambda_1 < \ldots < \lambda_s \leq r \} =: \mathcal{L}, L = (\lambda_1, \ldots, \lambda_s), M = (\mu_1, \ldots, \mu_s) \). The coefficient \( H_{LM} = \langle e_L, e_M \rangle_{h^s} \) in the matrix \( H^s \) representing the induced metric \( h^s \) is

\[
H^s_{LM} = \det H(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_s),
\]

where \( H(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_s) \) is a submatrix of \( H \) generated by rows \( \lambda_1, \ldots, \lambda_s \) and columns \( \mu_1, \ldots, \mu_s \) of the matrix \( H \). If \( e_1, \ldots, e_r \) are orthonormal at \( z \) so are their wedge products \( \{e_L, L \in \mathcal{L} \} \).

The induced Chern curvature tensor on \( \Lambda^s(E) \), \( i\Theta(\Lambda^s(E)) = \sum_{j,k} i\Theta(\Lambda^s(E))_{jk} dz_j \wedge d\bar{z}_k \) is defined by formula V-(4.5'), [Dem],

\[
i\Theta(\Lambda^s(E))_{jk}(e_L) = i \sum_{1 \leq l \leq s} e_{l_1} \wedge \ldots \wedge \Theta(E)_{jk} e_{l_s} \wedge \ldots \wedge e_{l_s}.
\]

It is known that \( E \geq_{\text{Nak}} 0 \) implies \( \Lambda^s(E) \geq_{\text{Nak}} 0 \). The following lemma gives an explicit formula for the curvature \( i\Theta(\Lambda^s(E)) \) in terms of the curvature \( i\Theta(E) \) and shows that if the associate Hermitian form \( \theta(E) \) has at most polynomial poles on \( \pi^{-1}(S) \), so does \( \theta(\Lambda^s(E)) \).

**Lemma 3.1.** If \( i\Theta(E) \) is Nakano nonpositive (nonnegative), then \( i\Theta(\Lambda^s(E)), 1 \leq s \leq r, \) is also Nakano nonpositive (nonnegative).

**Proof.** By formula (3.4) we have

\[
i\Theta(\Lambda^s(E))(e_L)_{jk} = i \sum_{l,\mu} (-1)^{j-1} c_{jk\lambda\mu} e_\mu \wedge e_{L'_l},
\]

where \( L'_l \) is obtained from \( L \) by removing the \( l \)-th index. Let \( L(\lambda, \mu) \) denote the (not ordered) multi-index obtained by replacing the index \( \lambda \) in the multi-index \( L \) by \( \mu \). We define that \( e_\lambda(\lambda, \mu) = 0 \) if and only if \( \lambda \notin L \) or \( \mu \in L \setminus \{\lambda\} \). If \( \lambda_l = \lambda \), then \( e_\mu L'_l = (-1)^{j-1} e_{L(\lambda, \mu)} \) and

\[
i\Theta(\Lambda^s(E)) = i \sum_{j, k, L, M} c_{jkLM}^s dz_j \wedge d\bar{z}_k \otimes e_L \otimes e_M
\]

\[
\sum_{\lambda \in L, \mu \notin L'} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda \otimes e_\mu L'.
\]

Here the bijection between the sets \( \{L : |L| = s\} \) and \( \{\lambda L' : |L'| = s - 1, \lambda \notin L'\} \) is used, where we compare multi-indices as sets. Let \( \tau = \sum_{j,L} \tau_{j,L}(\partial/\partial z_j) \otimes e_L \) with additional properties \( \tau_{j,\sigma(L)} := \text{sign}(\sigma)\tau_{j,L} \) for any permutation \( \sigma \) and \( \tau_{j,L} = 0 \) if there are at least two equal indices in \( L \). The bundle \( \Lambda^s(E) \) is Nakano positive if the bilinear form

\[
\theta_{\Lambda^s(E)}(\tau, \tau) = \sum_{j, k, |S| = s} c_{jkLM}^s \tau_{j,L} \bar{T}_k, S \langle e_M, e_S \rangle_{h^s}
\]

\[
= \sum_{j, k, |S| = s, \lambda \notin L'} c_{jk\lambda\mu} \tau_{j,\lambda L'} \bar{T}_k, S \langle e_\mu L', e_S \rangle_{h^s}
\]
is positive. Assume that the local frame \( e_1, \ldots, e_r \) is orthonormal. Then \( \{e_L, L \in \mathcal{L} \} \) are orthonormal and
\[
\theta_{\Lambda^*}(E)(\tau, \tau) = \sum_{j,k,L',\lambda,\mu} c_{jk\lambda\mu} \tau_j \tau_{jL'} \tau_{k\lambda L'} = \sum_{L'} \theta_E(\tau_{L'}, \tau_{L'}),
\]
where the form \( \tau_{L'} \) is defined by \( \tau_{L'} = \sum_{j, \lambda} \tau_j \tau_{jL'} (\partial/\partial z_j) \otimes \epsilon_\lambda \) for any multi-index \( L' \) of length \( s - 1 \). Hence, if \( \theta_E \) is Nakano nonpositive (nonnegative), so is \( \theta_{\Lambda^*} \).

### 3.2. Almost nonpositivity of \( \omega \)

In this section we study properties of the form \( \omega \) constructed in Section 2. Let \( V \subseteq Z \) be a neighbourhood of \( a(U \setminus S) \), conic along \( a(S) \). The metric \( \omega \) on a vector bundle \( E \rightarrow V \) is *almost Nakano nonpositive* if the curvature tensor \( i\Theta(E) \) has a decomposition \( i\Theta(E) = i\Theta_0(E) + i\Theta_1(E) \), where \( i\Theta_0(E) \) is nonpositive and \( i\Theta_1(E) \) is locally of the form \( i\Theta_1(E)(z, w) = \mathcal{O}(||w||^l ||z||^k) \) near points in \( a(S) \) and \( i\Theta_1(E)(z, w) = \mathcal{O}(||w||^l) \) near points in \( a(U \setminus S) \), for some \( l \in \mathbb{N}, k \in \mathbb{Z} \).

The main theorem in this subsection is

**Theorem 3.2.** (Almost Nakano nonpositive Kähler metric) Let \( Z, X, S, a, U \) and \( \omega \) be as in Theorem 1.1. There exist a neighbourhood \( V \) of \( a(U \setminus S) \) conic along \( a(S) \) such that the metric \( \omega \) on \( TZ|_V \) is almost Nakano nonpositive.

**Corollary 3.3.** Let \( h_\omega \) be the metric on \( TZ|_V \) induced by \( \omega \). Then \( h_\omega \phi^* \) is Nakano negative on a smaller neighbourhood—which we again denote by \( V \) —of \( a(U \setminus S) \), conic along \( a(S) \).

**Proof of Theorem 3.2** Write \( f = (f_1, \ldots, f_N)^T \) and \( \sum |f_i|^2 = f^T \bar{f} \) and let \( H \) denote the matrix corresponding to the metric. If \( D \) denotes the holomorphic and \( \bar{D} \) the antiholomorphic derivative with respect to \((z, w)\),
\[
Df = \begin{bmatrix}
f_{1, z_1} & f_{1, z_2} & \cdots & f_{1, w_{r_0}} \\
f_{2, z_1} & f_{2, z_2} & \cdots & f_{2w_{r_0}} \\
\vdots & \vdots & \ddots & \vdots \\
f_{N, z_1} & f_{N, z_2} & \cdots & f_{N, w_{r_0}}
\end{bmatrix},
\]
then
\[
H = D\bar{D}(f^T \bar{f}) = (Df)^T \bar{Df} + D\bar{f}^T Df + f^T \bar{L} f + L f^T \bar{f}.
\]

The Levi form \( LF \) of the vector is calculated as \( LF = (Lf_1, \ldots, Lf_N) \). With the notation defined prior to (2.1) we have \( H_+ = (DF)^T \bar{Df} \) and \( H_- = H - H_+ = (DF)^T \bar{Df} + f^T L f + L f^T \bar{f} \). Since \( f_i \) are holomorphic or almost holomorphic to the degree \( l \), we have \( \bar{D} f = \mathcal{O}(||w||^{l+1}) \), terms \( \bar{D} Df, Lf, H_- \) are of the form \( \mathcal{O}(||w||^l) \) and because of the holomorphicity of \( f_i \) in \( w \)-directions the terms \( \bar{D} H_-, \bar{D} H_-, \partial \bar{D} f, \partial \bar{D} \bar{H} \) are of the form \( \mathcal{O}(||w||^{l-1}) \). The term \( H_+ \) gives
\[
\partial \bar{D} H_+ = \partial \bar{D} ((DF)^* Df) = -\bar{D} (DF)^* Df = -\bar{D} (DF)^* Df + (DF)^*(\partial Df)) = -(\partial Df)^* \land (\partial Df) + (DF)^* \land \bar{D} f + (\bar{D} (DF)^*) Df + (DF)^* \partial \bar{D} \bar{D} f
\]
and so
\[
(\partial Df)^* \land (\partial Df) + (DF)^* (\partial Df) = \mathcal{O}(||w||^{l-1}).
\]
By the above estimates we conclude that
\[
\partial H = \partial (H_+ + H_-) = (DF)^*(\partial Df) + \mathcal{O}(||w||^{l-1})
\]
in the conic domain.
and similarly
\[ \overline{\partial} H = (\partial D f)^* D f + \mathcal{O}(\|w\|^{l-1}). \]
Since the metric is Kähler, we may assume that the coordinates \( \zeta_1, \ldots, \zeta_n \) near the point \( z_0 \in a(U \setminus S) \) are such that \( H(z_0) = I \) to the second order. The curvature form equals
\[
i \Theta(TZ) = i \overline{\partial} (H^{-1} \partial \overline{H}) = -i \overline{H}^{-1} \overline{\partial} H \overline{H}^{-1} \wedge \partial \overline{H} + i \overline{H}^{-1} \overline{\partial} \partial \overline{H} \]
\[
= -i \overline{\partial} \overline{H} = i(\partial D f)^* \wedge (\partial D f) + \mathcal{O}(\|w\|^{l-1})
\]
by the above assumptions. We claim that \( i(\partial D f)^* \wedge (\partial D f) \) is Nakano nonpositive. Denote the dual tangent vectors by \( e_\lambda := \partial/\partial \zeta_\lambda \) and let
\[
i \Theta_0 = i(\partial D f)^* \wedge (\partial D f) = i \sum_{j,k,\lambda,\mu} c_{j,k,\lambda,\mu} d\zeta_j \wedge d\overline{\zeta}_k \otimes e_\lambda \otimes e_\mu.
\]
Then we have
\[
c_{j,k,\lambda,\mu} = -\sum_{i=1}^N \frac{\partial^2 f_i}{\partial \zeta_j \partial \zeta_\lambda} \frac{\partial^2 f_i}{\partial \zeta_k \partial \zeta_\mu}
\]
and so
\[
\theta_0(\tau, \tau) = -\sum_i \left| \sum_{j,\lambda} \frac{\partial^2 f_i}{\partial \zeta_j \partial \zeta_\lambda} \tau_{j,\lambda} \right|^2 \leq 0.
\]

Remark 3.4. If the functions \( f_i \) were holomorphic, then the Nakano nonpositivity of \( i \Theta(VT) \) on \( V \) could be inferred from the fact that the metric on \( TZ|_V \) is the metric induced on the subbundle \( F \leq V \times \mathbb{C}^N \) by the standard metric on \( \mathbb{C}^N \) via the holomorphic vector bundle isomorphism \( TZ|_V \to F, TZ \ni v \mapsto (df_1(v), \ldots, df_N(v)) \in \mathbb{C}^N \). It is known that the Nakano curvature decreases in subbundles by VII-(6.10), [Dem]. Because \( f_i \) are almost holomorphic, we get an ‘error’ term, denoted by \( i \Theta_1 \) in the sequel and we will show that it decreases arbitrarily fast on conic neighbourhoods. On the section \( a(U \setminus S) \) the ‘holomorphic’ part of the curvature tensor is exactly \( i \Theta_0 \).

The curvature form \( i \Theta(TZ) \) is then almost Nakano nonpositive on an open neighbourhood of \( a(U \setminus S) \). If we want to prove that the neighbourhood is conic we have to show that the part of the curvature which contains \( w \)-variables decreases polynomially sufficiently fast in some conic neighbourhood. Let \( (z, w) \) be local coordinates at \( (z, 0) \in a(S) \). We cannot assume that \( H = I \) to the second order at \( (z, 0) \) because \( H(z, 0) \) is degenerate on \( a(S) \).

We have to split the curvature tensor into the part depending only on \( z \)-variables—we have just proved that it is nonpositive—and the rest, which we want to be small on conic neighbourhoods. By estimates (2.2), (3.6), (3.7), (3.8) we have for \( \|w\| \leq \|z_2\|^{4k+k_1} \),
\[
\overline{H}^{-1} \overline{\partial} \overline{H} = \overline{H}^{-1} ((\partial D f)^* \wedge (\partial D f) + \mathcal{O}(\|w\|^{l-1}))
\]
\[
= \overline{H}_0^{-1} (\partial D f)^* \wedge (\partial D f) + \mathcal{O}(\|w\|^{l-1} \|z_2\|^{-n})
\]
\[
= \overline{H}_0^{-1} (\partial D f)^* \wedge (\partial D f) + \mathcal{O}(\|z_2\|^{2k+k_1}) + \mathcal{O}(\|w\|^{l-1} \|z_2\|^{-n}),
\]
The curvature tensor \( H^{-1} \partial H \) on a conic neighbourhood of \( a(U \setminus S) \), conic along \( a(S) \). The ‘error term’ \( i\Theta_1(TZ) \) is in the worst case

\[
O(\|w\|)O(\|z_2\|^{-k} + \|z_2\|^{-2k}) + O(\|z_2\|^{2k+1}) + O(\|w\|^{-1}\|z_2\|^{-k}) + O(\|z_2\|^{-2k})O(\|w\|^{-1})
\]

and it decreases at least as \( \|z_2\|^{-k} \) on conic neighbourhoods \( \|w\| \leq \|z_2\|^{4k+1} \).

**Proof of Corollary 3.3** The part \( H_0 \) of the Levi form of \( \Phi \) near \( (z,0) \in a(S) \) that dominates in the matrix \( H \) on a cone is bounded from below by \( \|z_2\|^{2k}I \). The potentially positive part of the Nakano curvature, \( i\Theta_1 \), is of the form

\[
O(\|z_2\|^{k_1})
\]

on \( \|w\| \leq \|z_2\|^{4k+1} \) and can be compensated by \(-\partial \Phi / 2\) for \( k_1 > 2k \) thus making the curvature tensor

\[
(i\Theta_0(z,0) - i\partial \Phi(z,w)) + (i\Theta_1(z,w) - i\partial \Phi(z,w))
\]

strictly Nakano negative.

4. **Proof of the main theorem**

By Theorem 1.1 in [Pre1] the bundle \( E \) can be endowed with a Nakano positive Hermitian metric \( h_0 \) on a conic neighbourhood of \( a(U \setminus S) \) with polynomial poles on \( \pi^{-1}(S) \). Let \( \omega = i\partial \Phi \) be the given metric. In order to solve the \( \partial \Phi \)-equation in bidegree \((p,q)\) we have to show that the curvature tensor

\[
i\Theta(E \otimes \Lambda^sTZ) + iL\Psi = i\Theta(E) + i\Theta(\Lambda^sTZ) + iL\Psi
\]

is positive (or at least nonnegative) for some strictly plurisubharmonic weight \( \Psi \) and \( s = n - p \).

The Kähler metric \( h \) induced by the Kähler form \( \omega \) has almost nonpositive Nakano curvature. Let \( h^s \) be the metric on \( \Lambda^sTZ \) induced by \( h \) and \( h_1 \) the metric
induced by the form $\omega_1 = \omega e^\Phi$. The latter has strictly Nakano negative curvature tensor

$$i\Theta(TZ)_{\omega_1} = i\Theta(TZ)_\omega - i\bar{\partial}\partial\Phi$$

by Corollary 3.3. If the original metric on $TZ$ is represented by $H$, then the new one is $H_1 = He^\Phi$ and the induced metric $h_1^s$ on $\Lambda^sTZ$ is represented by the matrix

$$H_1^s = H^s e^s\Phi.$$

Since the Chern curvature tensor of the Hermitian metric $\omega_1$ on $TZ$ is Nakano negative the induced curvature tensor on $\Lambda^sTZ$ is also Nakano negative by Lemma 3.1 and equals

$$i\Theta(\Lambda^sTZ)_{h_1^s} = i\Theta(\Lambda^sTZ)_{h^s} - is\bar{\partial}\partial\Phi <_{\text{Nak}} 0.$$

Write $F := \Lambda^sTZ$ and let $\theta_F$ be the bilinear form on $TZ \otimes F$ associated to $i\Theta(F)$. Since the rank of the bundle $F$ is $(n)_s$, formula (3.3) gives

$$\theta_1 = -\left(\frac{n}{s}\right) \text{Tr}_F(\theta_F) h_1^s \otimes h_1^s + \theta_{F_h^s} >_{\text{Nak}} 0.$$ We observe that $\theta_1$ is the curvature form associated to the Chern curvature tensor of the bundle $F \otimes (\det F^*)^{(\nu)}$ with the metric induced by $h_1$. The induced metric on $\det F^*$ equals

$$(4.1) \quad h_{\det F^*} = (\det H_1)^{-\binom{n-1}{s-1}} = (\det H)^{-\binom{n-1}{s-1}} e^{-\binom{n-1}{s-1}\Phi}$$

by the identity

$$(4.2) \quad \det F = \det \Lambda^sTZ = (\det TZ)^{-\binom{n-1}{s-1}}$$

and so

$$i\Theta(\det F^*)_{h_1} = i\Theta(\det F^*)_h + \left(\frac{n-1}{s-1}\right) i\bar{\partial}\partial\Phi.$$ Then

$$\theta_1 = \theta((\det F^*)^{(\nu)}_h \otimes h_1^s + e^{s\Phi} \theta(F)_h + \left(\frac{n}{s}\right) \left(\frac{n-1}{s-1} - s\right) \partial\bar{\partial}\Phi \otimes h_1^s >_{\text{Nak}} 0$$

or

$$(4.3) \quad \theta = \theta((\det F^*)^{(\nu)}_h \otimes h^s + \theta(F)_h + \left(\frac{n}{s}\right) \left(\frac{n-1}{s-1} - s\right) \partial\bar{\partial}\Phi \otimes h^s >_{\text{Nak}} 0.$$ To complete the proof we view this expression as part of the curvature of the metric $e^{-\Phi} h^s$ on $\Lambda^sTZ$. Let $\Phi_1 := (\binom{n}{s}) (\binom{n-1}{s-1} - s)\Phi$. Then the weight $e^{-\Phi_1}$ gives the last term of $\theta$. Observe that

$$i\Theta(\det F^*_h) = i\bar{\partial}\partial\log(\det H)^{-\binom{n-1}{s-1}} = i\bar{\partial}\partial\log(\det H)^{-\binom{n-1}{s-1}}.$$ Write

$$F = F \otimes (\det F^*)^{(\nu)} \otimes (\det F)^{(\nu)} = F \otimes (\det F^*)^{(\nu)} \otimes (\det TZ)^{-\binom{n-1}{s-1}}$$

by invoking (4.2) and define the Nakano positive metric on $(\det TZ)^{-\binom{n-1}{s-1}}(\nu)$ in the following way. Let $v_i$ be smooth sections of $\det TZ$, given by Proposition 2.1 in [Pre1], which are holomorphic to the degree $l_2 > 2$ with zeroes of order $k_2$ on $\pi^{-1}(S)$ and such that they generate the bundle $\det TZ$ on a neighbourhood of $a(U \setminus S)$, conic along $a(S)$. Let $v$ be a local holomorphic section, $v_i = \alpha_i v$ and define

$$\Phi_2 := \log \sum \langle v_i, v_i \rangle_H = \log \langle v, v \rangle_H + \log \sum |\alpha_i|^2.$$
Then the metric $e^{-\Psi} h^s$ for $\Psi = \Phi + \Phi_1 + \binom{n}{s} \binom{n-1}{s-1} \Phi_2$ has
\[
i \binom{n}{s} \binom{n-1}{s-1} \partial \overline{\partial} \log \det H + i \Theta(\Lambda^s TZ) \omega + i \binom{n}{s} \binom{n-1}{s-1} \partial \overline{\partial} \Phi
\]
\[+ i \partial \overline{\partial} \Phi + i \binom{n}{s} \binom{n-1}{s-1} \partial \overline{\partial} \log \sum |\alpha_i|^2\]
as a curvature tensor. The first three terms give a Nakano positive curvature by (4.3) and the last is also Nakano positive in a suitable conic neighbourhood because the negative part of $\partial \overline{\partial} \log \sum |\alpha_i|^2$ is of the form
\[C_1 \frac{||w||^{l_2}}{||z_2||} + C_2 ||w||^{2l_2} + C_3 ||w||^{l_2} + C_4 ||w||^{l_2-1}\]
and its modulus decreases at least as $||z_2||^{k_1}$ on conic neighbourhoods $||w|| \leq ||z_2||^{k_1/2}$ (see [Pre1], p. 14 for details). Because $i \partial \overline{\partial} \Phi$ is strictly plurisubharmonic with the rate of degeneracy at most $||z_2||^{2k}$ (independent of the shape of the cone if it is sharp enough) it compensates the negativity of $\partial \overline{\partial} \log \sum |\alpha_i|^2$ provided $k_1 > 2k$. □

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