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Integral Equations of Non-Integer Orders and Discrete Maps with Memory

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Abstract: In this paper, we use integral equations of non-integer orders to derive discrete maps with memory. Note that discrete maps with memory were not previously derived from fractional integral equations of non-integer orders. Such a derivation of discrete maps with memory is proposed for the first time in this work. In this paper, we derived discrete maps with nonlocality in time and memory from exact solutions of fractional integral equations with the Riemann–Liouville and Hadamard type fractional integrals of non-integer orders and periodic sequence of kicks that are described by Dirac delta-functions. The suggested discrete maps with nonlocality in time are derived from these fractional integral equations without any approximation and can be considered as exact discrete analogs of these equations. The discrete maps with memory, which are derived from integral equations with the Hadamard type fractional integrals, do not depend on the period of kicks.

Keywords: fractional integral equation; fractional calculus; fractional dynamics; discrete map with memory; processes with memory; Riemann–Liouville fractional integral; Hadamard type fractional integral

MSC: 26A33 Fractional derivatives and integrals; 34A08 Fractional differential equations; 45G10: Other nonlinear integral equations

1. Introduction

The first mathematical model of processes with memory has been proposed by Ludwig Boltzmann in 1874 and 1876 [1–3] for isotropic viscoelastic media. Then the processes with memory were described in a book by Vito Volterra in 1930 [4,5], where the integral equations were used to take into account fading memory. The presence of memory in a process means that this process depends on the history of changes of the process in the past during a finite time interval. Obviously, such processes cannot be described by differential equations containing only derivatives of integer order with respect to time. To describe processes with memory we should use integral equations [6] or integro-differential equations. Among the integral and integro-differential operators, there are operators that form a calculus. These operators are called fractional derivatives and integrals, and the calculus of these operators is called the fractional calculus.

In mathematics, the differential and integral equations of non-integer order, fractional derivatives and integrals are well known (for example, see books [7–11] and handbooks [12,13]) and has a long history from 1695 [14–18]. Fractional differentiation and fractional integration go back to many great mathematicians such as Leibniz, Liouville, Riemann, Abel, Weyl, Kober, Erdelyi, Hadamard, Riesz, and others. Fractional differential and integral equations of non-integer orders with respect to time are a powerful tool for describing processes with non-locality in time and memory in physics [19,20], economics [21,22], biology [23], and other sciences.

An important approach to description of nonlinear dynamics is discrete-time maps (for example, see [24–29]). Discrete maps with memory are considered in the papers [30–35].
Memory in discrete maps means that the present step depends on all past steps. In works [30–35], the form of the maps with memory was postulated and not derived from any equations. It should be emphasized that all these discrete maps with memory were not derived from any differential or integral equations. Therefore it is important to derive discrete maps with memory from differential or integral equations that describe nonlinear dynamical systems with memory.

For the first time, discrete maps with memory were obtained from fractional differential equations in works [36–38] (see also Chapter 18 in book ([29], pp. 409–453) and [22,39,40]). These discrete maps with memory were derived from exact solutions of the fractional differential equations with periodic kicks without any approximations (for details, see [37,38] and ([29], pp. 409–453)). This approach has been applied in works [41–52], where the existence of new kinds of attractors and new types of chaotic behavior were proved by computer simulations.

Note that discrete maps with memory were not previously derived from fractional integral equations of non-integer orders. Such a derivation of discrete maps with memory is proposed for the first time in this work. In the proposed paper, discrete maps with non-locality in time and memory are derived from fractional integral equations with Riemann–Liouville fractional integrals [7–11] and Hadamard type fractional integrals [53–61]. We should note that these integral equations are nonlocal equations. In this work, we derive exact solutions of fractional integral equations with periodic kicks. These solutions are obtained for arbitrary positive order of integral equations. These maps with nonlocality in time and memory are obtained from exact solutions to these fractional integral equations for discrete time points. The proposed discrete maps describe discrete-time dynamics of systems with nonlocality in time, and periodic kicks.

2. Riemann–Liouville Fractional Integral and Derivative

Let us define the Riemann–Liouville fractional integral [7,10].

**Definition 1.** The left-sided Riemann–Liouville fractional integral of the order \( \alpha > 0 \) is defined by the equation

\[
\left( I_{RL, a+}^\alpha X \right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} X(\tau) \, d\tau,
\]

where \( \Gamma(\alpha) \) is the gamma function, and it is assumed that \( X(t) \in L_1(a, b) \).

**Remark 1.** The Riemann–Liouville integral (1) is a generalization of the standard integration [7]. Note that the Riemann–Liouville integration (1) of the order \( \alpha = 1 \) gives the standard integration of the first order:

\[
\left( I_{RL, a+}^1 X \right)(t) = \int_a^t X(\tau) \, d\tau.
\]

For \( \alpha = 2 \), Equation (1) gives the standard integration of the second order,

\[
\left( I_{RL, a+}^2 X \right)(t) = \left( \int_a^t X(\tau_2) \, d\tau_2 \right) \, d\tau_1 = \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 X(\tau_2).
\]

For \( \alpha = N \in \mathbb{N} \), Equation (1) gives the standard integration of the integer order \( N \) in the form

\[
\left( I_{RL, a+}^N X \right)(t) = \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \ldots \int_a^{\tau_2} d\tau_{N-1} X(\tau_N) = \frac{1}{(N-1)!} \int_a^t (t-\tau)^{N-1} X(\tau) \, d\tau.
\]

Let us define the Riemann–Liouville fractional derivatives [7,10].
Definition 2. The left-sided Riemann–Liouville fractional derivative of the order \( \alpha > 0 \) is defined by the equation

\[
(D^\alpha_{RL,a^+}X)(t) = \frac{d^N}{dt^N} \left( I^{N-a}_{RL,a^+}X \right)(t) = \frac{1}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \int_a^t (t-\tau)^{N-\alpha-1} X(\tau) \, d\tau,
\]

where \( N - 1 \leq \alpha < N \), and \( \Gamma(\alpha) \) is the gamma function. A sufficient condition of the existence of fractional derivatives (5) is \( X(t) \in AC^N[a, b] \). The space \( AC^N[a, b] \) consists of functions \( X(t) \), which have continuous derivatives up to order \( N - 1 \) on \( [a, b] \) and function \( X^{(N-1)}(t) \) is absolutely continuous on the interval \( [a, b] \).

We note that the Riemann–Liouville derivatives of orders \( \alpha = 1 \) and \( \alpha = 0 \) give the Equations \( D^1_{RL,a^+}X(t) = X^{(1)}(t) \) and \( D^0_{RL,a^+}X(t) = X(t) \), respectively ([10], p. 70).

Let us give the fundamental theorems of fractional calculus for the Riemann–Liouville operators.

Let \( D^\alpha_{RL,a^+} \) and \( I^\alpha_{RL,a^+} \) be the left-sided Riemann–Liouville fractional derivative and integral of the order \( \alpha > 0 \), respectively [7,10]. The equation

\[
(D^\alpha_{RL,a^+}I^\alpha_{RL,a^+}X)(t) = X(t)
\]

holds for \( X(t) \in L_1(a, b) \) and \( X(t) \in L_p(a, b) \) with \( p \geq 1 \), and \( t \in (a, b) \). Equation (6) was proved in [7,10] (see Theorem 2.4 of ([7], pp. 44–45), and Lemma 2.4 of ([10], p. 74)). Equation (6) represents the first fundamental theorem of fractional calculus for Riemann–Liouville operators.

In general, the reverse order of the sequence of actions of the operators leads to another result

\[
(I^\alpha_{RL,a^+}D^\alpha_{RL,a^+}X)(t) = X(t) - \sum_{k=0}^{N-1} p^\alpha_k (a+) \Gamma(\alpha-k) (t-a)^{\alpha-k-1},
\]

where

\[
p^\alpha_k (a+) = \lim_{t \to a^+} \left( \frac{d^{N-k-1}}{dt^{N-k-1}} I^{N-a}_{RL,a^+}X \right)(t).
\]

Equation (7) holds for the function \( X(t) \in L_1(a, b) \) has a summable fractional derivative \( D^\alpha_{RL,a^+}X(t) \), i.e., \( I^{N-a}_{RL,a^+}X(t) \in AC^N[a, b] \). This statement was proved in [7,10] (see Theorem 2.4 of ([7], pp. 44–45), and Lemma 2.5 of ([10], pp. 74–75)). Equation (7) represents the second fundamental theorem of fractional calculus for Riemann–Liouville operators.

3. Discrete Maps from Fractional Integral Equations with Riemann–Liouville Fractional Integral

Let us consider the nonlinear fractional integral equation

\[
(I^\alpha_{RL,a^+}X)(t) + KG[X(t)] \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) = 0,
\]

in which perturbation is a periodic sequence of delta-function-type kicks, \( T \) is a period, \( K \) is an amplitude of the kicks, \( G[X] \) is some real-valued function, \( I^{\alpha}_{RL,a^+} \) is the Riemann–Liouville fractional integral of the order \( N - 1 < \alpha < N \), where \( N \in \mathbb{N} \).
functions and the functions $G[X(t)]$ is meaningful, if the function $G[X(t)]$ is continuous at the points $t = kT$.

We can use $G[X(t - \epsilon)]$ with $0 < \epsilon < T$ ($\epsilon \to 0+$) instead of $G[X(t)]$ to make a sense of the left side of Equation (9), when $X(kT - 0) \neq X(kT + 0)$, [48–50].

To derive discrete maps with memory from fractional integral Equation (9), we will use the first fundamental theorem of fractional calculus for the Riemann–Liouville fractional operators in the form of Equation (6), we obtain

\begin{equation}
\label{eq:10}
X(t) = -\frac{K}{\Gamma(-\alpha)} \sum_{k=1}^{n} (t - kT)^{-\alpha-1} G[X(kT)].
\end{equation}

**Proof.** Applying the Riemann–Liouville fractional derivative of the order $\alpha$ to Equation (10) and the first fundamental theorem of fractional calculus for the Riemann–Liouville fractional operators in the form of Equation (6), we obtain

\begin{equation}
\label{eq:12}
X(t) = -K \left(D^{n}_{RL,0+} G[X(\tau)] \sum_{k=1}^{\infty} \delta\left(\frac{\tau}{T} - k\right)\right)(t).
\end{equation}

Using Equation (5), we get

\begin{equation}
\label{eq:13}
X(t) = -\frac{K}{\Gamma(N-\alpha)} \sum_{k=1}^{n} \frac{d^{N}}{dt^{N}} \int_{0}^{t} (t - \tau)^{N-\alpha-1} G[X(\tau)] \sum_{\epsilon=1}^{\infty} \delta\left(\frac{\tau}{T} - k\right) d\tau.
\end{equation}

For $nT < t < (n+1)T$, Equation (13) can be represented in the form

\begin{equation}
\label{eq:14}
X(t) = -\frac{K}{\Gamma(N-\alpha)} \sum_{k=1}^{n} \frac{d^{N}}{dt^{N}} \int_{0}^{t} (t - \tau)^{N-\alpha-1} G[X(\tau)] \delta\left(\frac{\tau}{T} - k\right) d\tau.
\end{equation}

Using the property of the Dirac delta-function

\begin{equation}
\label{eq:15}
\int_{0}^{t} f(\tau) \delta\left(\frac{\tau}{T} - k\right) d\tau = T f(kT),
\end{equation}

which is satisfied if $f(\tau)$ is continuous function at $\tau = kT$ and $0 < kT < t$, Equation (14) with $nT < t < (n+1)T$ takes the form

\begin{equation}
\label{eq:16}
X(t) = -\frac{KT}{\Gamma(N-\alpha)} \sum_{k=1}^{n} \left(\frac{d^{N}}{dt^{N}} (t - kT)^{N-\alpha-1}\right) G[X(kT)],
\end{equation}

\begin{equation}
\label{eq:17}
X(t) = -\frac{KT}{\Gamma(N-\alpha)} \sum_{k=1}^{n} \left(\Gamma(N-\alpha) \frac{\Gamma(N-\alpha)}{\Gamma(-\alpha)} (t - kT)^{-\alpha-1}\right) G[X(kT)].
\end{equation}

As a result, we get

\begin{equation}
\label{eq:18}
X(t) = -\frac{KT}{\Gamma(-\alpha)} \sum_{k=1}^{n} (t - kT)^{-\alpha-1} G[X(kT)],
\end{equation}

where $N - 1 < \alpha < N$.

This ends the proof. □
Using Theorem 1, we can prove the following theorem.

**Theorem 2.** Let $X(t) \in L_1[a, b]$, and $G[X(t)] \in AC^N[a, b]$, where $a > 0$. The fractional integral Equation (10) with $N - 1 < \alpha < N$ has solution for the left side of the kicks ($t = kT - \varepsilon$, $\varepsilon > 0$), in the form of the discrete maps

$$X_{n+1} - X_n = -\frac{K T^{-\alpha}}{\Gamma(-\alpha)} G[X_n] - \frac{K T^{-\alpha}}{\Gamma(-\alpha)} \sum_{k=1}^{n-1} V_{-\alpha}(n-k) G[X_k],$$

where

$$X_k = \lim_{\varepsilon \to 0^+} X(kT - \varepsilon)$$

with $k = 1, \ldots, n, n + 1,$ and

$$V_{-\alpha}(z) = (z + 1)^{-\alpha-1} - z^{-\alpha-1}, \ (z > 0).$$

**Proof.** For the left side of the $(n+1)$th and $n$th kicks ($t = (n+1)T - \varepsilon$ and $t = nT - \varepsilon$, $\varepsilon > 0$), where

$$X_k = \lim_{\varepsilon \to 0^+} X(kT - \varepsilon), \ (k = n, n + 1)$$

Equation (11) is represented by the equations

$$X_{n+1} = -\frac{K T^{-\alpha}}{\Gamma(-\alpha)} \sum_{k=1}^{n} (n + 1 - k)^{-\alpha-1} G[X_k],$$

$$X_n = -\frac{K T^{-\alpha}}{\Gamma(-\alpha)} \sum_{k=1}^{n-1} (n - k)^{-\alpha-1} G[X_k].$$

Subtracting Equation (24) from Equation (23), we obtain the discrete map

$$X_{n+1} - X_n = -\frac{K T^{-\alpha}}{\Gamma(-\alpha)} G[X_n] - \frac{K T^{-\alpha}}{\Gamma(-\alpha)} \sum_{k=1}^{n-1} V_{-\alpha}(n-k) G[X_k],$$

where we use the function $V_{-\alpha}(z)$ in the form (21).

This ends the proof. ☐

**Remark 2.** For $T = 1$ and $N - 1 < \alpha < N$, we can use the Euler reflection formula to represent the proposed discrete map in the form

$$X_{n+1} - X_n = K \sin(\pi\alpha) \Gamma(\alpha + 1) G[X_n] + K \sin(\pi\alpha) \Gamma(\alpha + 1) \sum_{k=1}^{n-1} V_{-\alpha}(n-k) G[X_k],$$

where $V_{-\alpha}(z)$ is defined by Equation (21).

**Remark 3.** Equation (19) can be called the universal map with Riemann–Liouville type memory. If $G[X] = -X$, then Equation (19) gives a generalization of the Anosov-type system, where Riemann–Liouville type memory is taken into account.

If $-K G[X] = (r - 1) X - r X^2$, then Equation (19) gives a generalization of the logistic map, where Riemann–Liouville type memory is taken into account.

For $G[X] = \sin(X)$, Equation (19) is a generalization of the standard or Chirikov-Taylor map [26], where Riemann–Liouville type memory is taken into account.
4. Hadamard Type Fractional Integral and Derivative Operators and Its Properties

Let us give the definitions and some properties of the Hadamard type fractional integral and derivative. These fractional operators and its properties are described in ([10], pp. 110–120) (see also Sections 18.3 and 23.1 in book [7,53]).

Let us consider the fractional integrals of the order \( \alpha > 0 \) that are given by the following definition ([10], pp. 110–120).

**Definition 3.** The left-sided Hadamard type fractional integral of the order \( \alpha > 0 \) is defined by the equations

\[
\Bigl( J_{a+}^{\alpha} X \Bigr)(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left( \frac{\tau}{t} \right)^{\alpha-1} X(\tau) \frac{d\tau}{\tau},
\]

where \( \mu \in \mathbb{R}, t \in (a,b) \).

**Remark 4.** Equation (27) with \( \mu = 0 \) is called the Hadamard fractional integral [7,10], which was proposed in work [54]. Equation (27) was proposed in paper [55] (see also [56–61]), where this integral operator is called the Hadamard type fractional integral, but it can be called the Butzer-Kilbas-Trujillo fractional integral.

**Remark 5.** The operator \( J_{a+}^{\alpha} \) is bounded in the space \( X_{\ell}^{p}(a,b) \), where \( \mu \in \mathbb{R}, \mu > c, p \geq 1, a > 0 \) (see Theorem 2.1 in [56], p. 1194). The space \( X_{\ell}^{p}(a,b) \) with \( c \in \mathbb{R} \) and \( p \geq 1 \) is the weighted \( L_{p} \)-space consists of those complex-valued Lebesgue measurable functions \( X(t) \) on \( (a,b) \) for which

\[
||X||_{X_{\ell}^{p}} = \left( \int_{a}^{b} ||\tau^{\mu} X(t)||_{p} \frac{dt}{\tau} \right)^{1/p} < \infty.
\]

In particular, when \( c = 1/p \), the space \( X_{\ell}^{p}(a,b) \) coincides with the space \( L_{p}(a,b) \), i.e., \( X_{\ell}^{p}(a,b) = L_{p}(a,b) \).

For integer values of \( \alpha = m \), where \( m \in \mathbb{N} \), fractional Equation (27) has ([56], p. 1191) the form

\[
\Bigl( J_{a+}^{m} X \Bigr)(t) = t^{-m} \int_{a}^{t} d\tau \int_{a}^{\tau} \frac{dt_{1}}{t_{1}} \int_{a}^{t_{1}} \frac{dt_{2}}{t_{2}} \ldots \int_{a}^{t_{m-1}} t_{m}^{\mu} X(t_{m}) \frac{dt_{m}}{t_{m}}
\]

\[
= \frac{1}{(m-1)!} \int_{a}^{t} \left( \frac{\tau}{t} \right)^{m-1} \left( \ln \frac{\tau}{t} \right)^{\mu} X(\tau) \frac{dt}{\tau}.
\]

Let us give the definition of the Hadamard type fractional derivative (for example, see Section 18.3 in [7] and ([10], pp. 111–122)).

**Definition 4.** The left-sided Hadamard type fractional derivative of the order \( N - 1 < \alpha < N, \ N \in \mathbb{N}, \) is defined by the Equation

\[
\Bigl( D_{a+}^{\alpha} X \Bigr)(t) = t^{-\mu} \left( \frac{d}{dt} \right)^{N} \frac{1}{\Gamma(N-\alpha)} t^{\mu} \left( J_{a+}^{N-\alpha} X \right)(t) = t^{-\mu} \frac{1}{\Gamma(N-\alpha)} t^{\mu} \int_{a}^{t} \left( \frac{\tau}{t} \right)^{\mu} \left( \ln \frac{\tau}{t} \right)^{N-\alpha-1} X(\tau) \frac{dt}{\tau},
\]

where \( J_{a+}^{N-\alpha} \) is the Hadamard type fractional integral of the order \( N - \alpha \in (0,1) \) and \( \tau > a > 0 \).

For \( \alpha = m \in \mathbb{N} \), the Hadamard type fractional derivative (30) is the standard differential operator ([10], p. 112) of the integer orders in the form

\[
\Bigl( D_{a+}^{m} X \Bigr)(t) = t^{-\mu} \left( \frac{d}{dt} \right)^{m} \left( \tau^{\mu} X(t) \right).
\]
Remark 6. The Hadamard type fractional derivative $D_{a+}^\alpha$ exists almost everywhere on the space $AC_{\xi,\mu}^N[a,b]$. This statement is proved as Theorem 3.2 in ([56], p. 1198). The space $AC_{\xi,\mu}^N[a,b]$ consists of functions $X(t)$ on $[a,b]$ that have the derivatives $(t \, d/dt)^k (t\mu \, X(t))$ for $k = 1, \ldots, N - 1$ and
\[
\left( t \frac{d}{dt} \right)^{N-1} (t\mu \, X(t)) \in AC[a,b],
\]
where $\delta$ denotes the operator $\delta = t \, d/dt$.

Remark 7. The fractional integro-differential operators (30) can be called the Butzer-Kilbas-Trujillo fractional derivatives, since these type of operators were first suggested in articles [55,56]. The properties of the Hadamard type fractional derivatives and integrals are described in papers [55–61]. The fundamental theorems of fractional calculus for the Hadamard type fractional derivatives, since these type of operators were first suggested in articles [55,56]. The second fundamental theorem of fractional calculus, which holds for $X(t) \in X^\mu_p(a,b)$, is described by Lemmas 3 and 5 in paper ([60], p. 735) (see also [57,61]).

The fundamental theorems of fractional calculus for the Hadamard type fractional derivatives are described by Lemmas 3 and 5 in paper ([60], p. 735) (see also [57,61]).

The first fundamental theorem of fractional calculus for the Hadamard type fractional operators is described by Equation (19) and Lemma 3 in paper ([60], p. 735) (see also Theorem 4.7 of ([56], p. 1203) and [57,61]). This theorem states the following. If $a > 0$, $a > 0$, $\mu \geq c$, and $p \geq 1$, then the equation
\[
\left( D_{a+}^\alpha \left( J_{a+}^\alpha X \right) \right)(t) = X(t)
\]
holds for $X(t) \in X^\mu_p(a,b)$. In particular, if $\mu > 1/p$, then (32) is valid for $X(t) \in L^p(a,b)$ ([56], p. 1203).

Equation (32), which describes the first fundamental theorem of fractional calculus, will be used to derive discrete maps from fractional integral equations with Hadamard type fractional integrals. The second fundamental theorem of fractional calculus, which is described by Lemma 5 in paper ([60], p. 735), will not be used in our work to obtain discrete maps with nonlocality in time and memory.

5. Discrete Map from Fractional Integral Equation with Hadamard Type Fractional Integral

Let us consider the fractional integral equation with the Hadamard type fractional derivative
\[
\left( J_{a+}^\alpha X \right)(t) + K \, G[X(t)] \sum_{k=1}^\infty \delta \left( \frac{t}{T} - k \right) = 0,
\]
where $J_{a+}^\alpha$ is the left-sided Hadamard type fractional integral, $N - 1 < \alpha \leq N$, and $0 < a < T$. Here, $T$ is period of the periodic sequence of kicks, $K$ is an amplitude of the kicks, $G[X]$ is some real-valued function.

Fractional integral Equation (33) contains the Dirac delta-functions, which are the generalized functions [62,63]. The generalized functions are treated as functionals on a space of test functions. Therefore, Equation (33) should be considered in a generalized sense, i.e., on the space of test functions, which are continuous. In Equation (33), the product of the delta-functions and the functions $G[X(t)]$ is meaningful if the function $G[X(t)]$ is continuous at the points $t = kT$.

We can use $G[X(t - \epsilon)]$ with $0 < \epsilon < T (\epsilon \to 0^+)$ instead of $G[X(t)]$ to make a sense of the right side of Equation (33), when $X(kT - 0) \neq X(kT + 0)$, [48–50].

To derive discrete maps with memory from fractional integral Equation (33), we can use the first fundamental theorem of fractional calculus for the Hadamard type fractional operators.

Let us prove the following theorem for the integral equation with the Hadamard type fractional integral.
Theorem 3. Let \( X(t) \in X^p_c(a,b), \) where \( \mu \geq c, \) and \( p \geq 1, \) and \( G[X(t)] \in AC^N_{\alpha,\mu}(a,b). \) The fractional integral Equation (33) with \( N - 1 < \alpha < N \) and \( T > a > 0 \) has the solution for \( t \in (nT,(n+1)T), \) \( N \in \mathbb{N} \) in the form

\[
X(t) = -\frac{K}{\Gamma(-\alpha)} \sum_{k=1}^{\infty} \left( \frac{kT}{t} \right)^\mu \frac{1}{k} \left( \ln \frac{t}{kT} \right)^{-\alpha-1} G[X(kT)] ,
\]

(34)

where \( 0 < a < T. \)

Proof. The action of the Hadamard type fractional derivative on Equation (33) gives

\[
\left( D^N_{a+\mu} f_{a+\mu} X \right)(t) + K \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) G[X(t)] = 0.
\]

(35)

Using the first fundamental theorem of fractional calculus in the form of Equation (32), Equation (35) takes the form

\[
X(t) = -K \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) G[X(t)].
\]

(36)

Using definition of the Hadamard type fractional derivative in form (30), we get

\[
X(t) = -K \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) X(kT).
\]

(37)

where

\[
\left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) X(kT) = \frac{1}{\Gamma(N-\alpha)} \int_{a}^{t} \left( \frac{\tau}{T} \right)^{\mu} \left( \ln \frac{t}{\ln \tau} \right)^{N-\alpha-1} X(\tau) \, d\tau ,
\]

(38)

where \( 0 < N - \alpha < 1. \)

Substitution of (38) into Equation (37) gives

\[
X(t) = -K \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) X(kT).
\]

(39)

where \( N - 1 < \alpha < N \) and \( 0 < a < T. \)

For \( nT < (n+1)T, \) Equation (39) can be written as

\[
X(t) = -K \Gamma(N-\alpha) \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) X(kT).
\]

(40)

Using the property of the Dirac delta-function

\[
\int_{0}^{t} f(\tau) \, d\tau = T f(kT),
\]

(41)

which is satisfied if \( f(\tau) \) is continuous function at \( \tau = kT \) and \( 0 < kT < t, \) Equation (40) with \( nT < t < (n+1)T \) takes the form

\[
X(t) = -K \Gamma(N-\alpha) \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) \left( \sum_{k=1}^{\infty} \delta \left( \frac{t}{T} - k \right) \right) X(kT).
\]

(42)

Then we use Equation 2.7.16 in Property 2.24 of ([10] p. 112) in the form

\[
\left( D^N_{a+\mu} \left( \ln \frac{t}{\alpha} \right)^{\beta-1} \right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left( \ln \frac{t}{\alpha} \right)^{\beta-\alpha-1}.
\]

(43)
For $\alpha = N, \beta = N - \alpha, a = kT$ and Equation (31), Equation (43) has the form

$$t^{-\mu} \left( \frac{d}{dt} \right)^N \left( t^\mu \left( \ln \frac{\tau}{kT} \right)^{N-\alpha-1} \right) = \frac{\Gamma(N-\alpha)}{\Gamma(-\alpha)} \left( \ln \frac{t}{\tau} \right)^{-\alpha-1}. \tag{44}$$

Using Equation (44), Equation (42) is represented in the form

$$X(t) = -\frac{K}{\Gamma(-\alpha)} \sum_{k=1}^{n-1} \left( W_{-\alpha,\mu}(n+1,k) - W_{-\alpha,\mu}(n,k) \right) G[X], \quad (45)$$

Equation (45) describes the solution of the fractional integral equation for $t \in (nT, (n+1)T), N \in \mathbb{N}$.

This ends the proof. \(\square\)

Using Theorem 3, we can prove the following theorem for discrete maps.

**Theorem 4.** Let $X(t) \in X^\mu_{\alpha,\mu}(a,b)$, where $\mu \geq c$ and $p \geq 1$, and $G[X(t)] \in AC^{\mu}_{\alpha,\mu}(a,b)$. The fractional integral Equation (33) with $N - 1 < \alpha < N$ and $T > a > 0$ has solution for the left side of the kicks ($t = kT - \varepsilon, \varepsilon > 0$), in the form of the discrete maps

$$X_{n+1} - X_n = -\frac{K}{\Gamma(-\alpha)} W_{-\alpha,\mu}(n+1,n)G[X_n] -$$

$$\frac{K}{\Gamma(-\alpha)} \sum_{k=1}^{n-1} \left( W_{-\alpha,\mu}(n+1,k) - W_{-\alpha,\mu}(n,k) \right) G[X], \tag{46}$$

where

$$X_k = \lim_{\varepsilon \to 0+} X(kT - \varepsilon) \tag{47}$$

with $k = 1, \ldots, n, n + 1$, and the function $W_{\alpha,\mu}(x,y)$ is defined by the equation

$$W_{\alpha,\mu}(x,y) = \left( \frac{x}{y} \right)^{\mu} \left( \frac{\ln \frac{x}{y}}{y} \right)^{\alpha-1}, (x > y > 0). \tag{48}$$

**Proof.** For the left side of the $(n+1)$th and $n$th kicks ($t = (n+1)T - \varepsilon$ and $t = nT - \varepsilon$), $\varepsilon > 0$, where

$$X_k = \lim_{\varepsilon \to 0+} X(kT - \varepsilon), (k = n, n + 1) \tag{49}$$

Equation (45) is represented by the equations

$$X_{n+1} = -\frac{K}{\Gamma(-\alpha)} \sum_{k=1}^{n} \left( \frac{k}{n+1} \right)^{\mu} \left( \frac{n+1}{k} \right)^{-\alpha-1} G[X_k], \tag{50}$$

$$X_n = -\frac{K}{\Gamma(-\alpha)} \sum_{k=1}^{n-1} \left( \frac{k}{n} \right)^{\mu} \left( \frac{n}{k} \right)^{-\alpha-1} G[X_k]. \tag{51}$$

Subtracting Equation (51) from Equation (50), we obtain the discrete map

$$X_{n+1} - X_n = -\frac{K}{\Gamma(-\alpha)} W_{-\alpha,\mu}(n+1,n)G[X] -$$

$$\frac{K}{\Gamma(-\alpha)} \sum_{k=1}^{n-1} \left( W_{-\alpha,\mu}(n+1,k) - W_{-\alpha,\mu}(n,k) \right) G[X], \quad (52)$$

where the function $W_{\alpha,\mu}(x,y)$ in Equation (48) for $x > y > 0$ is used.

This ends the proof. \(\square\)
Remark 8. We should emphasize an important property of the proposed discrete map with memory (46), namely the fact that this map does not depend on the period of kicks $T > 0$.

Remark 9. Equation (46) can be called the universal map with Hadamard type memory.

If $G[X] = -X$, then Equation (46) gives a generalization of the Anosov-type system, where Hadamard type memory is taken into account.

If $-K G[X] = (r - 1) X - r X^2$, then Equation (46) gives a generalization of the logistic map, where Hadamard type memory is taken into account.

For $G[X] = \sin(X)$, Equation (46) is a generalization of the standard or Chirikov-Taylor map [26], where Hadamard type memory is taken into account.

6. Conclusions

In this paper, new discrete maps with nonlocality in time and memory are derived from fractional integral equations with Riemann–Liouville and Hadamard type fractional integrals of non-integer and periodic kicks. We get exact solution of the proposed nonlinear fractional integral equations with kicks. The proposed discrete maps with nonlocality in time and memory are derived from these solutions without any approximations. Note that discrete maps with nonlocality in time and memory have not previously been derived from fractional integral equations of non-integer order. This derivation is proposed for the first time in this paper.

The proposed maps are derived from nonlinear integral equations of non-integer order for general form of non-locality. Therefore the proposed maps are nonlinear maps characterized by a high degree of universality for the considered types of nonlocality. Nonlocal nonlinear mappings of this type, obtained from fractional differential equations, are used to describe processes in economics [22,40], quantum physics [64], and biology [65].

We should also note that the discrete maps with memory, which are derived from integral equations with the Hadamard type fractional integrals, do not depend on the period of kicks $T$.

We assume that fractional integral equations with kicks can be important for the study of nonlinear and chaotic dynamics with memory. The proposed discrete maps with memory, which are derived from fractional integral equations with the Riemann–Liouville and Hadamard type fractional integrals, can be used in computer simulations. We hope that the proposed discrete maps with memory can simplify simulations of the chaotic behavior of dynamics with nonlocality in time and memory in computer simulations. However, this modeling remains an open question, and hopefully it will be solved in future research.

Note that in this paper we propose discrete mappings for only two types of nonlocalities in time. As further possible research, it seems interesting to obtain discrete maps from other types of fractional integral equations for describing dynamics with other types of nonlocalities and memory. In future studies, it is also important to investigate the dependence of the conditions for the onset of chaotic behavior on the type of nonlocality. In this regard, the work [66] is of great importance, in which for the first time methods of such studies are proposed for nonlocalities described by translation invariant kernels, and discrete maps that are represented in the form of a discrete convolution.

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