Compact Polygons

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Einleitung

Verallgemeinerte Polygone sind natürliche Verallgemeinerungen projektiver Ebenen; an die Stelle des Axioms, daß zwei Punkte sich durch eine eindeutig bestimmte Gerade verbinden lassen, tritt die Forderung, daß zwei Elemente (Punkte oder Geraden) sich durch einen eindeutig bestimmten Polygonzug einer gewissen Länge verbinden lassen. Projektive Ebenen sind verallgemeinerte Dreiecke. "Erfunden" wurden verallgemeinerte Polygone (oder sphärische Gebäude vom Rang 2) von Tits, um eine geometrische Interpretation für gewisse einfache (Ausnahme-) Liegruppen zu finden; sphärische Gebäude liefern eine einheitliche geometrische Interpretation für alle einfachen algebraischen Gruppen. Nun tritt bei sphärischen Gebäuden das gleiche Phänomen wie in der projektiven Geometrie auf: die sphärischen Gebäude vom Rang $\geq 3$ sind in einem gewissen Sinne klassisch, das heißt, sie stammen von klassischen oder von einfachen algebraischen Gruppen. Verallgemeinerte Polygone sind dagegen (wie projektive Ebenen) im allgemeinen nicht klassisch, und gerade deswegen interessant.

Will man substantielle Aussagen über verallgemeinerte Polygone machen, so sind zusätzliche Strukturannahmen wie etwa Endlichkeit oder die Existenz gewisser Automorphismen notwendig. Eine andere Möglichkeit ist, das Polygon zusätzlich mit einer topologischen Struktur zu versehen und zu verlangen, daß der oben erwähnte Polygonzug stetig von seinen Enden abhängt. Im wesentlichen ist das die Definition eines topologischen Polygons (in dieser Dissertation wird etwas weniger Stetigkeit verlangt). Bereits dieser recht allgemeine Ansatz liefert Einschränkungen für die Topologie: der unterliegende topologische Raum wird regulär und ist entweder zusammenhängend oder total unzusammenhängend. Ist das Polygon lokalkompakt, so wird die Topologie des Polygons metrisierbar und abzählbar.

Wesentlich stärkere Aussagen über die globale topologische Struktur ergeben sich mit Hilfe der algebraischen Topologie, wenn man voraussetzt, daß die Topologie auf dem Punktraum des verallgemeinerten $n$-Eckes lokalkompakt und endlichdimensional ist. In diesem Falle ist $n \in \{3, 4, 6\}$, das heißt, es gibt unter diesen topologischen Voraussetzungen neben den projektiven Ebenen überhaupt nur verallgemeinerte Vier- und Sechsecke (rein inzidenzgeometrisch lassen sich mit Hilfe freier Konstruktionen verallgemeinerte $n$-Ecke für jedes $n$ konstruieren).

Eine unmittelbare Anwendung dieser Ergebnisse über die algebraische Topologie
verallgemeinerter Polygone ist die Klassifikation punkthomogener kompakter Polygone mit gleichen topologischen Parametern. Dieses Resultat beinhaltet als Spezialfall die Klassifikation der punkthomogenen kompakten projektiven Ebenen, die in den siebziger und achtziger Jahren von Salzmann und Löwen abgeschlossen wurde (im Prinzip läßt es sich auch als Klassifikation der homogenen Fokalmannigfaltigkeiten isoparametrischer Hyperflächen mit gleichen Multiplizitäten deuten). Im Folgenden sollen die Methoden etwas genauer beschrieben werden.

Um die Topologie auf dem Punktraum \( P \) eines topologischen \( n \)-Eckes zu verstehen, wird zunächst die induzierte Topologie auf den Punktreihen betrachtet. Sticht man einen beliebigen Punkt \( \infty_L \) aus der Punktreihe \( L \) heraus, dann läßt sich auf dem Komplement \( L - \{\infty_L\} \) eine Addition einführen, die es erlaubt, Gleichungen (von einer Seite) stetig zu lösen. Zusammen mit der zweifachen Homogenität von \( L \) unter der Projektivitätengruppe folgt daraus (ganz ähnlich wie bei topologischen Gruppen) die Regularität von \( L \).

Weiter kann man eine (sehr schwache) Multiplikation einführen; wenn \( L \) wegzusammenhängend ist, dann läßt sich die affine Punktreihe \( L - \{\infty_L\} \) mit Hilfe dieser Multiplikation gleichmäßig auf das Neutralelement der Addition zusammenziehen.

Wählt man eine Fahne \((p, \ell)\), so erhält man eine Filtrierung des Punktraumes

\[
\{p\} \subseteq L \subseteq p^\perp \subseteq \ldots \subseteq P
\]

durch \( n \) Mengen von Punkten, die sich ungefähr wie folgt beschreiben lassen: die Elemente der \( k \)-ten Menge lassen sich durch einen Polygonzug der Länge \( \leq k \) mit der Fahne \((p, \ell)\) verbinden. Die Mengen dieser Filtrierung sind die Schubertvarietäten des \( n \)-Eckes. Jede Schubertvarietät ist eine abgeschlossene Teilmenge des Punktraumes, und die mengentheoretische Differenz zweier aufeinanderfolgender Schubertvarietäten (die zugehörige Schubertzelle) ist ein Produkt affiner Punktreihen und affiner Geradenbüscheln. Insbesondere ist der Punktraum \( P \) lokal ein Produkt aus Punktreihen und Geradenbüscheln. Alle Eigenschaften der mengentheoretischen Topologie von \( P \) ergeben sich aus diesem Resultat.

Für die algebraische Topologie von \( P \) ist das folgende Theorem von Löwen von zentraler Bedeutung: ist \( X \) ein endlichdimensionaler, kompakter ANR (absoluter Umgebungsretrakt) mit der Eigenschaft, daß das Komplement jedes Elementes \( x \in X \) kontrahierbar ist, dann ist \( X \) eine verallgemeinerte Mannigfaltigkeit und homotopieäquivalent zu einer Sphäre.

Dieses Kriterium läßt sich auf die Punktreihen und Geradenbüschel eines verallgemeinerten \( n \)-Eckes anwenden. Damit wird zunächst jede Schubertzelle eine verallgemeinerte Mannigfaltigkeit, deren Ein-Punkt-Kompaktifizierung eine Homotopie-sphäre ist. Wenn \( n > 3 \) ist, dann sind die Schubertvarietäten im allgemeinen keine verallgemeinerten Mannigfaltigkeiten mehr; der Punkstern \( p^\perp \) etwa, also die Menge aller Punkte, die mit \( p \) durch eine Gerade verbindbar sind, hat in seinem Mittelpunkt
p im allgemeinen eine Singularität. Diese Singularitäten lassen sich "glätten", indem man die Schubertvarietät $p^\perp$ durch den Raum der zugehörigen Galerien ersetzt;

$$\text{Gall}_2(\ell, p) \cong \{(h, q) \mid h \text{ geht durch } p \text{ und } q \text{ liegt auf } h\}$$

ist eine verallgemeinerte Mannigfaltigkeit, und $p^\perp \cong \text{Gall}_2(\ell, p)/ (\mathcal{L}_p \times \{p\})$ ist ein Quotient diese Raumes. Diese Beobachtung, verbunden mit der Tatsache, daß die Menge der $k$-Galerien ein $k$-fach iteriertes Bündel ist, erlaubt die Berechnung der Homologiegruppen der Schubertvarietäten; es ergibt sich, daß jede Schubertvarietät im Punktraum einen Erzeuger der Homologiegruppen repräsentiert, ein Resultat, das für projektive Räume (oder Graßmann-Mannigfaltigkeiten) wohlbekannt ist.

Schärfere Aussagen über die algebraische Topologie erhält man mit Knarrs topologischer Veroneseeinbettung: der doppelte Abbildungszylinder über dem Fahnenraum ist eine Homotopiesphäre. Mit einem Theorem von Münzner folgt, daß die Coxetergruppe eines lokalkompakten, endlichdimensionalen n-Eckes kristallographisch ist (die Coxetergruppe eines verallgemeinerten $n$-Eckes ist eine Diedergruppe der Ordnung $2n$). Es gibt unter diesen topologischen Voraussetzungen also nur projektive Ebenen, Vierecke und Sechsecke. Dieses Resultat (das topologische Pendant zu den Sätzen von Tits-Weiss und Feit-Higman) wurde zuerst von Knarr unter der Zusatzannahme bewiesen, daß das $n$-Eck eine topologische Mannigfaltigkeit ist.

Setzt man voraus, daß die Punktreihen und Geradenbüschel lokal euklidisch sind, dann wird die Schubertzellularzerlegung eine CW-Zerlegung. Für projektive Ebenen wurde das erstmals von Breitsprecher bewiesen; in kleinen Dimensionen lassen sich damit gewisse Schubertvarietäten topologisch klassifizieren. Für projektive Ebenen wurde das bereits von Salzmann, Breitsprecher und Buchanan gezeigt.

Eine Anwendung dieser Ergebnisse ist die geometrische Klassifikation gewisser homogener $n$-Ecke. Gegeben sei ein lokalkompaktes, zusammenhängendes verallgemeinertes $n$-Eck mit punkttransitiver Automorphismengruppe. Zunächst folgt, daß die Automorphismengruppe eine Liegruppe ist, und daß $n = 3, 4, 6$ ist. Für $n = 3, 6$ ist der Punktraum entweder $(n - 1)$-dimensional, oder er hat positive Eulercharakteristik $n$. Für Vierecke ist diese Bedingung nicht notwendig erfüllt, weswegen wir sie für $n = 4$ zusätzlich fordern. Unter diesen Voraussetzungen ist das $n$-Eck klassisch (das heißt Moufangsch) und – via Cartans Klassifikation der einfachen Liegruppen – explizit bekannt. Der Beweis zerfällt in zwei Teile: wenn die Dimension des Punktraumes $n - 1$ ist, so folgt, daß die auftretenden Liegruppen klein und dadurch eindeutig bestimmt sind; der andere Fall ergibt sich mit Hilfe der Borel-De Siebenthal Klassifikation maximaler Untergruppen von maximalem Rang in kompakten Liegruppen.

Verallgemeinerte Vierecke nehmen hier eine Sonderstellung ein; es gibt nicht-Moufangsche, punkthomogene verallgemeinerte Vierecke (mit Eulercharakteristik 0). Auch die Erfahrung, daß die nichtklassischen kompakten Ebenen durch die klassischen Ebenen dominiert werden, hat für Vierecke keinerlei Entsprechung. Kompakte Sechsecke zeigen wieder ein ähnliches Verhalten wie kompakte projektive Ebe-
nen; allerdings kennt man hier gegenwärtig überhaupt keine nicht-Moufangschen Beispiele.

Über die kompakten Vierecke mit Eulercharakteristik 0 ist gegenwärtig noch wenig bekannt. Ferus-Karcher-Münzner und Thorbergsson haben gezeigt, daß jede Darstellung einer reellen Cliffordalgebra ein solches verallgemeinertes Viereck liefert. Diese Konstruktion umfaßt alle kompakten zusammenhängenden hermiteschen und reell-orthogonalen Moufang-Vierecke, darüber hinaus aber sehr viele nicht-Moufangsche Beispiele. Einige dieser Vierecke haben punkttransitive Automorphismengruppen. In dieser Richtung gibt es, auch in der Verbindung zur Differentialgeometrie, noch interessante offene Fragen.

Im ersten Kapitel werden alle benötigten Fakten über verallgemeinerte $n$-Ecke zusammengestellt. Das zweite Kapitel beschäftigt sich mit der mengentheoretischen Topologie verallgemeinerner Polygone. Einige der Resultate sind inzwischen Folklore, aber eine einheitliche Darstellung (für beliebiges $n$) lag bisher nicht vor.

Das dritte Kapitel ist das Zentrum dieser Dissertation. Es behandelt die algebraische Topologie endlichdimensionaler Polygone. Als ’Richtschnur’ für dieses Kapitel haben vor allem die Arbeiten von Breitsprecher und Löwen über die algebraische Topologie projektiver Ebenen gedient. Im vierten Kapitel werden lokal euklidische Polygone untersucht. Diese Polygone sind in natürlicher Weise CW-Komplexe. In Verbindung mit einem solchen Polygon treten verschiedene Vektorbündel auf, die sich (in kleinen Dimensionen) zur Klassifikation bestimmter Schubertvarietäten verwenden lassen. Im fünften Kapitel werden homogene Polygone untersucht. Mit Hilfe der Ergebnisse aus Kapitel 3 werden die punkthomogenen kompakten zusammenhängenden Polygone mit gleichen topologischen Parametern vollständig klassifiziert. Im letzten Kapitel werden verschiedene topologische Hilfsmittel bereitgestellt.

Mathematik lebt vom Austausch und von Diskussionen; für beides möchte ich mich bei meinen MitdoktorandInnen (und Habilitanden) Richard Bödi, Martina Jäger, Michael Joswig, Bernhard Mühlherr und Markus Stroppel bedanken. Norbert Knarr und Stephan Stolz haben mir viele gute Anregungen gegeben, ohne die die Kapitel 3, 4 und 6 sicher kürzer geworden wären. Für das schöne Thema und die vorzügliche Betreuung dieser Dissertation möchte ich mich bei Theo Grundhöfer sehr herzlich bedanken. Das Arbeiten in der AG Geometrie und Topologie war für mich sowohl mathematisch als auch persönlich immer sehr angenehm; dafür gebührt nicht zuletzt Herrn Salzmann mein Dank.

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Notation:

⊆ subset
A − B set-theoretic difference \{a ∈ A \mid a \notin B\}
N natural numbers \{0, 1, 2, 3, \ldots\}
Z integers
Z_k integers mod k
Q rational numbers
R real numbers
C complex numbers
H quaternions
O octonions (Cayley numbers)
I unit interval
S^k k-dimensional unit sphere
A \hookrightarrow B topological imbedding (homeomorphic onto its image)
X/A quotient space obtained by collapsing A \subseteq X to a point
\cong isomorphism in the appropriate category
(e.g. homeomorphism of topological spaces)
\simeq homotopy equivalence
[X;Y] set of all homotopy classes of maps from X to Y
[X;Y]^0 set of all base-point preserving homotopy classes of maps
(between the pointed spaces X, Y)

We use singular homology and cohomology, unless indicated otherwise.
Chapter 1

Generalized polygons

The first six sections of this chapter contain the basic definitions concerning incidence structures and generalized polygons. The most important notions are the space of galleries $\text{Gall}_k(u, v)$ based on a flag, and the corresponding Schubert cells and Schubert varieties. Our approach to the topology of generalized polygons is wholly based on these spaces.

In Section 7 it is shown that each Schubert cell is a product of punctured pencils of lines and punctured point rows; this decomposition is natural with respect to automorphisms, once an ordinary $n$-gon is chosen (the Schubert cells are precisely the preimages of the vertices of this ordinary $n$-gon with respect to the associated retraction).

In the last section we define the addition and multiplication on the point rows and pencils of lines. It is well-known that in the case of a projective plane, these algebraic operations belong to a ternary field, and that there are strong analogies to ordinary fields. For $n > 3$ the algebraic properties of these binary operations are much weaker; the addition yields only a right loop, i.e. equations may be solved only from the right-hand side. The multiplication has even weaker properties. However, it turns out that these algebraic operations still have all the properties which are needed to carry over most of the concepts used in the theory of topological projective planes to generalized polygons.

Generalized polygons were introduced by Tits in [Tits59]. See also [Tits74], [Bro89], [Ron89].

1.1 Incidence structures

An incidence structure is a triple $\mathcal{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$, consisting of a point space $\mathcal{P}$, a line space $\mathcal{L}$, and a flag space $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{L}$. We assume that $\mathcal{P}$ and $\mathcal{L}$ are nonempty, disjoint sets. The union $\mathcal{V} = \mathcal{P} \cup \mathcal{L}$ is called the set of vertices. Two vertices $x, y \in \mathcal{V}$
are called incident, if \((x, y)\) or \((y, x)\) is a flag; we denote the corresponding flag by \(\text{fl}(x, y)\). If \((p, \ell)\) is a flag, we say that the point \(p\) lies on the line \(\ell\), and that the line \(\ell\) passes through the point \(p\).

Given a set of vertices \(A \subseteq V\), we let \(V_A\) denote the set of all vertices that are incident with some member of \(A\); we put \(\mathcal{P}_A = V_A \cap \mathcal{P}\), \(\mathcal{L}_A = V_A \cap \mathcal{L}\), and \(\mathcal{F}_A = (V_A \times V \cup V \times V_A) \cap \mathcal{F}\). In the special case that \(A\) consists of a single point \(p\), we call \(\mathcal{L}_p\) the pencil of lines through \(p\); similarly, if \(A\) consists of a single line \(\ell\), then \(L = \mathcal{P}_\ell\) is called the point row consisting of all points lying on the line \(\ell\).

It is customary to put \(x^+ = \{y \in V \mid y \text{ is incident with some } z \in V_x\}\); this set is sometimes called the star or perp of \(x\).

The incidence structure \(\mathfrak{P}\) is called thick, if every point row and every pencil of lines contains at least 3 elements.

A \(k\)-chain joining \(x_0\) and \(x_k\) is a sequence \(x = (x_0, x_1, \ldots, x_k)\) of vertices with the property that \(x_{i-1}\) is incident with \(x_i\) for \(1 \leq i \leq k\). A \(k\)-chain stammers, if \(x_i = x_{i-2}\) for some \(1 < i \leq k\). An ordinary \(k\)-gon is a \(2k\)-chain \((x_0, x_1, x_2, \ldots, x_{2k-1}, x_0)\) with the property that \(x_i \neq x_j\) for \(0 \leq i < j < 2k\). If two vertices \(x, y\) can be joined by a \(k\)-chain, but not by any \(k'\)-chain for \(k' < k\), we say that the distance \(d(x, y)\) is \(k\). If there is no chain joining \(x\) and \(y\), then we put \(d(x, y) = \infty\). If the distance between any pair of vertices is finite, the incidence structure \(\mathfrak{P}\) is called connected (in the graph-theoretic sense). The diameter of an incidence structure is the supremum of all the distances between vertices in \(V\). If \(\mathcal{P}\) has finite diameter \(n\), then we call two vertices \(x, y\) opposite, if \(d(x, y) = n\). We put \(\text{opp } x = \{y \in V \mid d(x, y) = n\}\). We also put \(V^{(d \geq k)} = \{(x, y) \in V \times V \mid d(x, y) \geq k\}\), and \(V^{(d = k)} = \{(x, y) \in V \times V \mid d(x, y) = k\}\).

### 1.2 Galleries

A \((k + 1)\)-chain is called a gallery of length \(k\). Given a flag \((p, \ell)\), let \(\text{Gall}_k(p, \ell)\) denote the set of all \((k + 1)\)-chains of the form \((x_0 = p, x_1 = \ell, x_2, \ldots, x_{k+1})\), and let \(\text{Gall}_k(\ell, p)\) denote the set of all \((k + 1)\)-chains of the form \((x_0 = \ell, x_1 = p, x_2, \ldots, x_{k+1})\).

Put \(\{u, v\} = \{p, \ell\}\). We let \(\text{StamGall}_k(u, v)\) denote the set of all stammering galleries in \(\text{Gall}_k(u, v)\). The non-stammering galleries \(\text{PropGall}_k(u, v) = \text{Gall}_k(u, v) - \text{StamGall}_k(u, v)\) are called proper galleries. Note that \(\text{PropGall}_k(u, v)\) is nonempty, provided that there are at least two points on every line and two lines through every point.

The gallery \(x = (u, v, x_2, \ldots, x_{k+1})\) ends at the flag \(\text{fl}(x_k, x_{k+1}) \in \mathcal{F}\). We say also that the gallery \(x\) ends at the vertex \(x_{k+1}\).

There is a canonical projection

\[
\text{pr} : \text{Gall}_{k+1}(u, v) \rightarrow \text{Gall}_k(u, v)
\]

\[
(u, v, \ldots, x_{k+1}, x_{k+2}) \mapsto (u, v, \ldots, x_{k+1})
\]
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and a section

\[ s : \text{Gall}_k(u, v) \rightarrow \text{StamGall}_{k+1}(u, v) \]
\[ (u, v, \ldots, x_k, x_{k+1}) \mapsto (u, v, \ldots, x_k, x_{k+1}, x_k). \]

Clearly \( pr \circ s = \text{id}. \)

There is also an injection

\[ \text{Gall}_k(v, u) \rightarrow \text{StamGall}_{k+1}(u, v) \]
\[ (v, u, x_2, \ldots, x_{k+1}) \mapsto (u, v, u, x_2, \ldots, x_{k+1}). \]

1.3 **Automorphisms**

An *automorphism* of an incidence structure is a bijection of the set of vertices that maps points to points, lines to lines, and flags to flags. It is clear that automorphisms map \( k \)-chains to \( k \)-chains, and that they preserve the distance \( d \).

Homomorphisms between generalized polygons are investigated in [BK 94b].

1.4 **Coset geometries**

Suppose we are given a group \( G \) with subgroups \( A, B \). Then we may form the *coset geometry* \( (G/A, G/B, G/(A \cap B)) \). The points are the cosets of \( A \), the lines are the cosets of \( B \), and two cosets are incident, if their intersection is nonempty (there is a slight problem if \( A = B \), but this case is not interesting anyway). The flag space of this incidence structure can be identified with \( G/(A \cap B) \). The coset \( g(A \cap B) \) represents the flag \((gA, gB)\).

Note that if \( G \) acts as a flag-transitive automorphism group on an incidence structure \( \mathcal{P} \), and if \((p, \ell)\) is a flag, then the coset geometry \((G/G_p, G/G_\ell, G/G_p,\ell)\) is isomorphic to \( \mathcal{P} \). The existence of ordinary \( k \)-gons in \( \mathcal{P} \) can be translated into a group-theoretic property, see [GKK94, 3.1].

1.5 **Generalized polygons**

A thick incidence structure \( \mathcal{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F}) \) is called a *generalized \( n \)-gon*, if it satisfies the following two conditions:

(i) \( \mathcal{P} \) contains no ordinary \( k \)-gons for \( 2 \leq k < n \).

(ii) Any two vertices \( x, y \) are contained in an ordinary \( n \)-gon.
It is readily verified that the generalized digons are exactly the trivial thick incidence structures \((P, L, P \times L)\). So we assume from now on that \(n > 2\).

An incidence structure is called a partial \(n\)-gon, if it satisfies condition (i). This condition guarantees that for any two vertices \(x, y\) with \(d(x, y) = k < n\), the \(k\)-chain \((x, x_1, \ldots, x_{k-1}, y)\) joining \(x\) and \(y\) is unique. So we may define a map \(f_k(x, y) = x_{k-1}\) on the set \(V_k^{(d=k)}\) of pairs of vertices at distance \(k\). We are particularly interested in the map \(f_{n-1}\).

Note that for every non-stammering \(k\)-chain \((x_0, \ldots, x_k)\), we have \(d(x_0, x_k) = k\), provided that \(k \leq n\).

The following lemma will be required later.

1.5.1 Lemma (cp. [Tits74, 3.30]) Let \(x, y\) be vertices in a generalized \(n\)-gon. If \(x\) and \(y\) have the same type, then there exists a vertex \(z \in \text{opp } x \cap \text{opp } y\). If \(x\) and \(y\) have different types, then there exists a vertex \(z \in \text{opp } x\) with \(d(z, y) = n - 1\).

**Proof.** Let \(z\) be a vertex in \(\text{opp } x\) that has maximal distance \(k\) to \(y\), and suppose that \(k \neq n, n - 1\). Choose a vertex \(b \in V_z - \{f_k(y, z)\}\). We have \(d(b, x) = n - 1\) and \(d(b, y) = k + 1\), hence there is an element \(z' \in V_b - \{f_{n-1}(x, b), f_{k+1}(y, b)\}\). But now \(d(z', y) = k + 2\) and \(d(z', x) = n\), a contradiction. (Here we needed the fact that the incidence structure \(P\) is thick.)

1.6 Projectivities

Let \(P = (P, L, \mathcal{F})\) be a generalized \(n\)-gon, and let \(x, y\) be vertices at distance \(n\). We may define a map \([y, x]\) from \(V_x\) to \(V_y\) by \(z \mapsto f_{n-1}(z, y)\), with inverse \([x, y]\). These maps are called perspectivities. A concatenation of perspectivities is called a projectivity, and we write \([z, y][y, x] = [z, y, x]\) etc.

1.6.1 Lemma Given any two vertices \(x, y\) of the same type, there exists a projectivity between \(V_x\) and \(V_y\); if \(n\) is odd, then such a projectivity exists also if \(x\) and \(y\) have different types.

**Proof.** Let \(x, y \in V\) be vertices. By Lemma [1.5.1], there exists a vertex \(z \in \text{opp } x \cap \text{opp } y\). Now \([y, z, x]\) is a projectivity as required.

The last statement is clear, because opposite vertices have different types if \(n\) is odd.

The set of projectivities of \(P\) forms a groupoid; the group \(\Pi(x)\) of all projectivities that start and end at a given vertex \(x\) is called the group of projectivities of \(P\) at \(x\); by the lemma above, its equivalence type as a permutation group depends only on the type of \(x\). It will be shown below in [1.8.2] that \(V_x\) is doubly homogeneous under the action of \(\Pi(x)\).
1.7 Schubert cells

Let \((p, \ell)\) be a flag in a projective plane, and let \(L\) be the point row corresponding to \(\ell\). Then the point space \(\mathcal{P}\) can be decomposed as \(\{p\} \cup (L - \{p\}) \cup (\mathcal{P} - L)\). Now choose points \(q \in L - \{p\}\), \(o \in \mathcal{P} - L\), and let \(H\) be the point row corresponding to \(h = o \lor q\). We may introduce coordinates as follows:

\[
\begin{align*}
L - \{p\} &\to \mathcal{L}_o - \{o \lor p\} : x \mapsto x \lor o \\
\mathcal{P} - L &\to (H - \{q\}) \times (\mathcal{L}_q - \{p \lor q\}) : x \mapsto (x \lor q, (p \lor x) \land h).
\end{align*}
\]

There is a similar decomposition of the flag space and of the line space.

We want to generalize these concepts to arbitrary polygons.

1.7.1 Definition Let \(\mathfrak{P}\) be a generalized \(n\)-gon. Let \((p, \ell)\) be a flag, and let \(\{u, v\} = \{p, \ell\}\). For \(0 \leq k \leq n\) we put

\[
\mathcal{F}_k(u, v) = \{(q, h) \in \mathcal{F} | \text{there is an } x \in \text{PropGall}_k(u, v) \text{ that ends at } (q, h)\}.
\]

The set \(\mathcal{F}_k(u, v)\) is called a Schubert cell. For \(0 < k < n\), Schubert cells of different types have no flags in common, whereas \(\mathcal{F}_k(u, v) = \mathcal{F}_k(v, u)\) for \(k = 0, n\). Hence there are \(2n\) Schubert cells that cover the flag space. If we choose any ordinary \(n\)-gon \(x\) containing \((p, \ell)\), then the Schubert cells are precisely the preimages of the retraction \((\mathcal{F}, (p, \ell)) \to (x, (p, \ell))\), see [Tits74, 3.3].

The sets

\[
\text{Cl}\mathcal{F}_k(u, v) = \mathcal{F}_k(u, v) \cup \bigcup_{j<k}(\mathcal{F}_j(u, v) \cup \mathcal{F}_j(v, u))
\]

are called the closed Schubert cells or Schubert varieties. It is immediate that the Schubert variety \(\text{Cl}\mathcal{F}_k(u, v)\) consists precisely of all the flags that occur as ends of galleries in \({\text{Gall}}_k(u, v)\).

Similarly, we put

\[
\begin{align*}
\mathcal{P}_{2k}(\ell, p) &= \{q \in \mathcal{P} | \text{there is an } x \in \text{PropGall}_{2k}(\ell, p) \text{ that ends at } q\} \\
\mathcal{P}_{2k+1}(p, \ell) &= \{q \in \mathcal{P} | \text{there is an } x \in \text{PropGall}_{2k+1}(p, \ell) \text{ that ends at } q\}
\end{align*}
\]

for \(0 \leq 2k < n\) and \(1 \leq 2k + 1 < n\), respectively. In the same way we define

\[
\begin{align*}
\mathcal{L}_{2k}(p, \ell) &= \{h \in \mathcal{L} | \text{there is an } x \in \text{PropGall}_{2k}(p, \ell) \text{ that ends at } h\} \\
\mathcal{L}_{2k+1}(\ell, p) &= \{h \in \mathcal{L} | \text{there is an } x \in \text{PropGall}_{2k+1}(\ell, p) \text{ that ends at } h\}
\end{align*}
\]

for \(0 \leq 2k < n\) and \(1 \leq 2k + 1 < n\), respectively. The \(2n\) sets of this type are also called Schubert cells; they form a partition of \(\mathcal{P}\) and \(\mathcal{L}\), respectively. We also introduce Schubert varieties by

\[
\text{Cl}\mathcal{P}_k(u, v) = \mathcal{P}_0(\ell, p) \cup \mathcal{P}_1(p, \ell) \cup \mathcal{P}_2(\ell, p) \cup \ldots \cup \mathcal{P}_k(u, v),
\]
and
\[ \text{Cl} \mathcal{L}_k(v, u) = \mathcal{L}_0(p, \ell) \cup \mathcal{L}_1(\ell, p) \cup \mathcal{L}_2(p, \ell) \cup \ldots \cup \mathcal{L}_k(v, u); \]
these are the vertices that can be reached by (possibly stammering) galleries of the corresponding type. Note also that
\[ \text{Cl} \mathcal{P}_k(u, v) = \{ p \in \mathcal{P} \mid d(v, p) \leq k \} \]
is precisely the set of all points that have distance \( \leq k \) to \( v \) and thus does not depend on \( u \). Similarly,
\[ \text{Cl} \mathcal{L}_k(v, u) = \{ \ell \in \mathcal{L} \mid d(u, \ell) \leq k \}. \]

This decomposition is called the Schubert cell decomposition of the \( n \)-gon \( \mathcal{P} \) (with respect to the flag \((p, \ell)\)). The sets \( \mathcal{P}_{n-1}(u, v) \), \( \mathcal{L}_{n-1}(v, u) \), and \( \mathcal{F}_n(u, v) \) are called the big cells.

Returning to the example of the projective plane, we have
\[
\begin{align*}
\mathcal{P}_0(\ell, p) &= \{ p \} \\
\mathcal{P}_1(p, \ell) &= L - \{ p \} \\
\mathcal{P}_2(\ell, p) &= \mathcal{P} - L.
\end{align*}
\]
In order to introduce coordinates in the generalized \( n \)-gon \( \mathcal{P} \), we choose an ordinary \( n \)-gon \( x = (x_0 = p, x_1 = \ell, x_2, \ldots, x_{2n-1}, x_0) \).

Let \( q \in \mathcal{P}_{2k+1}(p, \ell) \), and let \( y = (p, \ell, y_2, \ldots, y_{2k+2} = q) \) be the corresponding gallery. Note that \( d(y_j, x_{n+j}) = n \), because the \( n \)-chain \((y_j, y_{j-1}, \ldots, y_0, x_{2n-1}, \ldots, x_{n+j})\) does not stammer. Hence we have the relation
\[ y_{j-1} = f_{n-1}(x_{n+j+1}, y_j), \]
and therefore, the bijection \( \mathcal{P}_{2k+1}(p, \ell) \to \text{PropGall}_{2k+1}(p, \ell) \) can be expressed in terms of the map \( f_{n-1} \) and the ordinary \( n \)-gon \( x \).

Now we attach to \( q \) the \( 2k + 1 \) coordinates
\[ q_{j-1} = f_{n-1}(y_j, x_{n+j-1}) \in V_{x_{n+j-1}} - \{ x_{n+j} \}, \]
where \( 2 \leq j \leq 2k + 2 \). The point \( q \) can be recovered from these coordinates in terms of the function \( f_{n-1} \), because
\[ y_j = f_{n-1}(q_{j-1}, y_{j-1}). \]
Thus there is a bijection
\[ \mathcal{P}_{2k+1}(p, \ell) \to (\mathcal{V}_{x_{n+1}} - \{x_{n+2}\}) \times \cdots \times (\mathcal{V}_{x_{n+2k+1}} - \{x_{n+2k+2}\}) \]
that can in both directions be expressed in terms of the function \( f_{n-1} \).

Observe also that \( \text{pr}_1 \) maps \( \mathcal{F}_{2k+1}(p, \ell) \) bijectively onto \( \mathcal{P}_{2k+1}(p, \ell) \) for \( 2k + 1 < n \), with inverse \( q \mapsto (q, f_{n-1}(x_{n+2k+3}, q)) \).

The other Schubert cells can be treated in a similar way, and we get the following result:

**1.7.2 Proposition** Let \((p, \ell)\) be a flag in the generalized \( n \)-gon \( \mathfrak{P} \), and let \( x = (x_0 = p, x_1 = \ell, x_2, \ldots, x_{2n-1}, x_0) \) be an ordinary \( n \)-gon. The maps

\[
\begin{align*}
\text{PropGall}_{2k}(\ell, p) &\to \mathcal{F}_{2k}(\ell, p) & \xrightarrow{\text{pr}_1} & \mathcal{P}_{2k}(\ell, p) & (2k < n) \\
\text{PropGall}_{2k+1}(\ell, p) &\to \mathcal{F}_{2k+1}(\ell, p) & \xrightarrow{\text{pr}_1} & \mathcal{P}_{2k+1}(\ell, p) & (2k + 1 < n) \\
\text{PropGall}_{2k}(\ell, p) &\to \mathcal{L}_{2k}(p, \ell) & \xrightarrow{\text{pr}_1} & \mathcal{L}_{2k+1}(\ell, p) & (2k < n) \\
\text{PropGall}_{n}(\ell, p) &\to \mathcal{F}_{n}(p, \ell) \\
\text{PropGall}_{n}(\ell, p) &\to \mathcal{F}_{n}(\ell, p)
\end{align*}
\]

are bijections, and their inverses can be expressed in terms of the function \( f_{n-1} \) and the ordinary \( n \)-gon \( x \).

Moreover, there are bijections

\[
\begin{align*}
\mathcal{P}_{2k}(\ell, p) &\to (\mathcal{V}_{x_n} - \{x_{n-1}\}) \times \cdots \times (\mathcal{V}_{x_{n+2k}} - \{x_{n+2k+1}\}) \\
\mathcal{P}_{2k+1}(\ell, p) &\to (\mathcal{V}_{x_{n+1}} - \{x_{n+2}\}) \times \cdots \times (\mathcal{V}_{x_{n+2k+1}} - \{x_{n+2k+2}\}) \\
\mathcal{L}_{2k}(\ell, p) &\to (\mathcal{V}_{x_{n+1}} - \{x_{n+2}\}) \times \cdots \times (\mathcal{V}_{x_{n+2k}} - \{x_{n+2k+1}\}) \\
\mathcal{L}_{2k+1}(\ell, p) &\to (\mathcal{V}_{x_{n+1}} - \{x_{n+2}\}) \times \cdots \times (\mathcal{V}_{x_{n+2k}} - \{x_{n+2k+1}\}) \\
\mathcal{F}_{k}(\ell, p) &\to (\mathcal{V}_{x_{n+1}} - \{x_{n+2}\}) \times \cdots \times (\mathcal{V}_{x_{n+k+1}} - \{x_{n+k+1}\}) \\
\mathcal{F}_{k}(\ell, p) &\to (\mathcal{V}_{x_{n+1}} - \{x_{n+2}\}) \times \cdots \times (\mathcal{V}_{x_{n+k+1}} - \{x_{n+k+1}\}) \\
\mathcal{F}_{k}(\ell, p) &\to (\mathcal{V}_{x_{n+1}} - \{x_{n+2}\}) \times \cdots \times (\mathcal{V}_{x_{n+k+1}} - \{x_{n+k+1}\}).
\end{align*}
\]
that can (in both directions) be expressed in terms of the function $f_{n-1}$ and the ordinary $n$-gon $\mathbf{x}$.

All of these maps are natural with respect to automorphisms, i.e. it does not matter if the automorphism is applied to the coordinates, or if the coordinates are taken with respect to the image of the ordinary $n$-gon $\mathbf{x}$ under the automorphism. In particular, an automorphism is completely determined by its action on the sets $V_{x_0}, \ldots, V_{x_{2n-1}}$. 

The following fact will be required later.

1.7.3 Proposition Let $q \in P_{n-1}(u, v)$. Let $C \subseteq P$ be a Schubert cell (with respect to $(u, v)$) different from $P_{n-1}(u, v)$.

There exists an incident pair $(u', v')$ with the following properties:

(i) $q \in P_{n-1}(u', v')$

(ii) $C \subseteq P_{n-1}(u', v')$

(iii) The map $x \mapsto f_{n-1}(x, v')$ has constant value $z$ on $C$, and $f_{n-1}(q, v') \neq z$.

Proof. We treat only case that $n-k$ is odd (the case that $n-k$ is even is similar); so $C = P_k(u, v)$ for some $k < n-2$.

Inductively, we can find a non-stammering chain $(x_0 = v, x_1 = u, x_2, \ldots, x_{n-k})$ with the property that $d(x_j, q) \geq n-1$ for $0 \leq j \leq n-k$. Thus for $o \in P_k(u, v)$, we have $d(o, x_{n-k-1}) = d(q, x_{n-k-1}) = n-1$, and $f_{n-1}(o, x_{n-k-1}) = x_{n-k-2}$. On the other hand, $f_{n-1}(q, x_{n-k-1}) \neq x_{n-k-2}$, because $d(x_{n-k-2}, q) = n$. Thus we may put $(u', v') = (x_{n-k}, x_{n-k-1})$. 

\[ \square \]
1.8 The algebraic operations

Let $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a generalized $n$-gon, and let $(x_0, x_1, \ldots, x_{2n-1}, x_0)$ be an ordinary $n$-gon in $\mathcal{G}$. Let $a \in \mathcal{V}_{x_{n-1}} - \{x_{n-2}, x_n\}$. Since $d(a, x_0) = d(a, x_{2n-2}) = n$, we may define a projectivity $\mu_a = [x_{2n-2}, a, x_0]$. This projectivity maps $x_1$ to $x_{2n-3}$ and fixes $x_{2n-1}$. Choose an element $1_L \in \mathcal{V}_{x_{n-1}} - \{x_{n-2}, x_n\}$, and consider the map

$$\mu_a^{-1} \mu_{1_L}(x) = f_{n-1}(f_{n-1}(f_{n-1}(x, 1_L), x_{2n-2}), a), x_0).$$

The right-hand side makes sense also for $a = x_{n-2}$, provided that $x \neq x_{2n-1}$, and $f_{n-1}(f_{n-1}(f_{n-1}(x, 1_L), x_{2n-2}), x_{2n-2}, x_0) = x_1$. Similarly, if $x \neq x_1$, then $f_{n-1}(f_{n-1}(f_{n-1}(x, 1_L), x_{2n-2}), x_1, x_0) = x_{2n-1}$.

This leads to the following definition:

1.8.1 Definition (cp. [GKK94, 1.1]) Let $(x_0, x_1, \ldots, x_{2n-1}, x_0)$ be an ordinary $n$-gon in the generalized $n$-gon $\mathcal{G}$. Put $0_K = x_1$, $\infty_K = x_{2n-1}$, $0_L = x_{n-2}$, and $\infty_L = x_n$. Set $K = \mathcal{V}_{x_0} - \{\infty_K\}$, $L = \mathcal{V}_{x_{n-1}} - \{\infty_L\}$, and choose an element $1_L \in L - \{0_L\}$. We may define maps

\[
\begin{align*}
K \times L & \to K \\
(x, y) & \mapsto x \bullet y = f_{n-1}(f_{n-1}(x, 1_L), y, x_0) \\
K \times (L - \{0_L\}) & \to K \\
(x, y) & \mapsto x/y = f_{n-1}(f_{n-1}(x, y), 1_L, x_0)
\end{align*}
\]

These maps have the following properties:

(i) The relations $x \bullet 0_L = 0_K \bullet y = 0_K$ and $x \bullet 1_L = x$ hold for all $(x, y) \in K \times L$.

(ii) The relation $(x \bullet y)/y = (x/y) \bullet y = x$ holds for all $(x, y) \in K \times L - \{0_L\}$.

The map $\bullet$ is called the multiplication with respect to $(x_0, x_1, \ldots, x_{2n-1}, 1_L)$.

We shall also need the map

$$\rho_b : \quad L \cup \{\infty_L\} \to K \cup \{\infty_K\},$$

$$y \mapsto f_{n-1}(f_{n-1}(f_{n-1}(b, 1_L), x_{2n-2}), y), x_0),$$

which is defined for $b \in K - \{0_k\}$. Note that $\rho_b(0_L) = 0_K$, and $\rho_b(1_L) = b$.

Next, choose an element $e \in \mathcal{V}_{x_{n+1}} - \{x_n, x_{n+2}\}$. For $y \in K$, consider the projectivity $\pi_y = [x_0, x_n, f_{n-1}(e, y), x_{n+2}, f_{n-1}(e, x_1), x_n, x_0] \in \Pi(x_0)$. Note that $\pi_y(\infty_K) = \infty_K$, $\pi_y(0_K) = y$, and $\pi_{0_K} = \text{id}_{\mathcal{V}_{x_0}}$. This leads to the following definition.
1.8.2 Definition (cp. [GKK94, 1.4]) For \( x, y \in K \), we define maps \( \pm : K \times K \rightarrow K \) by \( x + y = \pi_y(x) \) and \( x - y = \pi_y^{-1}(x) \). These maps have the following properties:

(i) \( x + 0_K = 0_K + x = x \) for all \( x \in K \).

(ii) \( (x + y) - y = (x - y) + y = x \) for all \( x, y \in K \).

Thus \((K, 0_K, +, -)\) is a right loop in the sense of [BK94a].

In particular, the group of all projectivities is doubly-transitive on \( K \cup \{\infty_K\} \).
Chapter 2

Topological polygons

This chapter deals with the set-theoretic topology of topological generalized polygons. The definition of a topological polygon is rather general; we require only that the geometric operation \( f_{n-1} \) be continuous on its domain, and that there exists at least one non-trivial open set in \( \mathcal{P} \cup \mathcal{L} \) (2.0.1).

The algebraic operations defined on the point rows and the pencils of lines imply that every Schubert cell is homogeneous and regular (2.1.2). Moreover, the big cells are open (2.1.3), hence the point space, the line space, and the flag space are Hausdorff spaces (in fact even regular, as Jäger proved (2.1.13)). The flag space \( \mathcal{F} \) is closed in the product \( \mathcal{P} \times \mathcal{L} \) (2.1.12), and the maps \( \mathcal{F} \to \mathcal{P}, \mathcal{F} \to \mathcal{L} \) are locally trivial bundles (2.1.8).

A topological polygon is either connected or totally disconnected (2.2.3). If it is path connected, then it is locally path-connected and locally contractible (2.2.5). The point rows are discrete if and only if the pencils of lines are discrete (2.2.6). The Schubert varieties are the topological closures of the corresponding Schubert cells, provided that the topology is not discrete (2.2.8). In particular, the point rows and the sets \( p^\perp \) are closed.

If the topology of the polygon is locally compact, then it is second countable and metrizable [GKK94] (2.4.3). If the polygon is in addition connected, then it is path-connected (and thus locally contractible) and compact.

The topological triangles are precisely the topological projective planes, see [Sal57, Sal67]. For topological quadrangles see also [For81, GK90, Sch92].

2.0.1 Definition A generalized \( n \)-gon \( \mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F}) \) is called a topological \( n \)-gon, if the point space \( \mathcal{P} \) and the line space \( \mathcal{L} \) carry topologies such that the map \( f_{n-1} \) is continuous on its domain \( \mathcal{V}^{(d=n-1)} = \{ (x, y) \in \mathcal{V} \times \mathcal{V} | d(x, y) = n - 1 \} \) (we endow the set of vertices \( \mathcal{V} = \mathcal{P} \cup \mathcal{L} \) with the sum topology).

In order to avoid trivialities, we shall always assume that there exists an open set in \( \mathcal{V} \) besides \( \emptyset \), \( \mathcal{P} \), \( \mathcal{L} \), and \( \mathcal{V} \).
CHAPTER 2. TOPOLOGICAL POLYGONS

The following observations are simple, but important:

2.0.2 Proposition Ever projectivity is a homeomorphism. All point rows are homeomorphic, doubly homogeneous spaces, and similarly, all pencils of lines are homeomorphic, doubly homogeneous spaces. If $n$ is odd, then all point rows and all pencils of lines are homeomorphic via projectivities.

The coordinate functions defined in 1.7.2 are homeomorphisms. The algebraic operations $\bullet$, $/$, and $\pm$ defined in 1.8.1, 1.8.2 are continuous. \(\square\)

2.0.3 Proposition ([Hof58]) Let $(G, 0, \pm)$ be a right loop in the sense of [BK94a], see 1.8.2. Suppose that $G$ is a topological $T_1$-space, and that the maps $\pm$ are continuous. Then $G$ is a regular (i.e. a $T_3$-) space.

Proof. Let $U$ be a neighborhood of 0. Choose a neighborhood $V$ of 0 with $V + V \subseteq U$. We claim that $V \subseteq V + V$. For every element $x \in V$, there is a net $(x_\nu) \subseteq V$ converging to $x$. Hence the net $(x - x_\nu)$ converges to 0. Thus there is an element $y \in V$ with $x - y \in V$, and therefore we have $x = (x - y) + y \in V + V$.

Since $G$ is homogeneous, it is a regular space. \(\square\)

2.1 The topology of Schubert cells

Let $\mathcal{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a topological $n$-gon.

2.1.1 Lemma Every point row and every pencil of lines is a $T_1$-space.

Proof. We may assume that the point space $\mathcal{P}$ contains a proper, open subset $U$, with $\emptyset \neq U \neq \mathcal{P}$. Thus there exists a point row $L$ with $L - U \neq L \neq L \cap U$. Since $L$ is doubly homogeneous, every point is closed in $L$.

If $n$ is odd, then all point rows and all pencils of lines are homeomorphic via projectivities, so we are done.

If $n$ is even, then the map $\rho_b$ of 1.8.1 provides a nonconstant map of some pencil of lines $\mathcal{L}_p$ into some point row. Thus $\mathcal{L}_p$ contains proper, nonempty open subsets, and the claim follows for the pencils of lines by duality. \(\square\)

2.1.2 Corollary Every Schubert cell is homogeneous and regular. \(\square\)

2.1.3 Theorem Put $(u, v) = (p, \ell)$ if $n$ is even, and $(u, v) = (\ell, p)$ if $n$ is odd. The big cells $\mathcal{P}_{n-1}(u, v)$, $\mathcal{L}_{n-1}(v, u)$, and $\mathcal{F}_{n}(u, v)$ are open.

Proof. Let $q \in \mathcal{P}_{n-1}(u, v)$, and let $C \subseteq \mathcal{P}$ be a Schubert cell different from $\mathcal{P}_{n-1}(u, v)$. By 1.7.3 there exists an incident pair $(u', v')$ with $C \cup \{q\} \subseteq \mathcal{P}_{n-1}(u', v')$,
2.1. THE TOPOLOGY OF SCHUBERT CELLS

and such that the map \( x \mapsto f_{n-1}(x, v') \) has constant value \( z \neq f_{n-1}(q, v') \) on \( C \).

Therefore there exists an open neighborhood \( U \) of \( q \) that does not meet \( C \). We may do this for each of the \( n-1 \) Schubert cells in \( \mathcal{P} \) different from \( \mathcal{P}_{n-1}(u, v) \); taking the intersection of the corresponding \( n-1 \) open neighborhoods of \( q \), we obtain an open neighborhood of \( q \) contained in \( \mathcal{P}_{n-1}(u, v) \).

The same applies to \( \mathcal{L}_{n-1}(v, u) \), and finally \( \mathcal{F}_n(u, v) = \text{pr}^{-1}_1(\mathcal{P}_{n-1}(u, v)) \cap \text{pr}^{-1}_2(\mathcal{L}_{n-1}(v, u)) \).

\[ \blacksquare \]

2.1.4 Corollary The point space \( \mathcal{P} \), the line space \( \mathcal{L} \), and the flag space \( \mathcal{F} \) are Hausdorff spaces.

\[ \blacksquare \]

2.1.5 Lemma Let \( x \) be a vertex, and let \( V^{(k)}_x \) denote the set of all vertices that have distance \( k \) to \( x \). Then for \( k < n \), the minimal \( k \)-chain \( (x = x_0, x_1, \ldots, x_{k-1}, x_k = y) \) that joins \( y \in V^{(k)}_x \) and \( x \) depends continuously on \( y \).

Proof. Clearly, it suffices to show that \( x_{k-1} \) depends continuously on \( y \). So let \( y \in V^{(k)}_x \). We may choose a non-stammering chain \( (x_{-n+k}, x_0, \ldots, x_k) \). Thus \( y \) is contained in the open cell determined by the pair \( (x_{-n+k}, x_{-n+k+1}) \), and \( x_{k-1} = f_{n-1}(x_{-n+k+1}, x_k) \).

\[ \blacksquare \]

2.1.6 Lemma Let \( p \in \mathcal{P} \) be a point and consider the map \( p^\perp - \{p\} \to \mathcal{L}_p, \ q \mapsto f_2(p, q) \). This defines a locally trivial bundle with the punctured point rows through \( p \) as fibers. Let \( x \) be a vertex opposite to \( p \). Then the map \( \ell \mapsto f_{n-1}(x, \ell) \) is a section of this bundle. In particular, there is an imbedding \( \mathcal{L}_p \hookrightarrow p^\perp \).

Proof. Let \( \ell \in \mathcal{L}_p \) be a line, and put \( y = f_{n-1}(\ell, x), \ z = f_{n-1}(p, u) \). The map \( \mathcal{L}_p - \{\ell\} \times V_y - \{z\} \to p^\perp - \{p\}, \ (h, w) \mapsto f_{n-1}(w, h) \) is a trivialization of \( p^\perp - \{p\} \to \mathcal{L}_p \) over the open set \( \mathcal{L}_p - \{\ell\} \).

This may be put in a slightly more general form:

2.1.7 Lemma Let \( p \) be a point, and let \( V^{(k)}_p \) denote the set of all vertices \( x \) with \( d(x, p) = k \). For \( k < n \), let \( x_j \) be the \( j \)-th vertex in the minimal \( k \)-chain from \( x_k \in V^{(k)}_p \) to \( p \). The map \( x_k \mapsto x_j \) defines for \( 0 < j < k < n \) a locally trivial bundle whose fibers are Schubert cells.

For example, we get bundles \( p^\perp - \{p\} \to \mathcal{L}_p \) (for \( n \geq 3 \)), and \( \mathcal{P} - L \to L \) (for \( n = 4 \)).

Proof. This is clear from the coordinatization with respect to \( (p, \ell) \), where \( \ell \) is a line through \( p \).

\[ \blacksquare \]
2.1.8 Proposition The maps \( \text{pr}_1 : F \to P \) and \( \text{pr}_2 : F \to L \) are locally trivial bundles, and hence open.

Proof. Let \( p \in P \) be a point. Choose a big cell \( P_{n-1}(u, v) \) containing \( p \). For every \( q \in P_{n-1}(u, v) \), we have \( d(u, q) = n \), thus we may consider the map \( \mathcal{V}_u \times P_{n-1}(u, v) \to F \), \((q, z) \mapsto (q, f_{n-1}(z, q))\). This is clearly a trivialization of \( \text{pr}_1 \) over the open set \( P_{n-1}(u, v) \).

\[ \blacksquare \]

2.1.9 Corollary If \( U \) is an open set of points, then the set \( L_U \) of all lines that meet \( U \) is open (and dually).

\[ \blacksquare \]

2.1.10 Corollary The map \( \text{pr} : \text{Gall}_k(u, v) \to \text{Gall}_{k-1}(u, v) \) is a locally trivial bundle, with a section \( s \) as defined in 1.2.

Proof. This bundle is just the pullback of one of the bundles \( F \to P \) or \( F \to L \) by the map that sends a gallery to the vertex where it ends.

\[ \blacksquare \]

2.1.11 Proposition The Schubert varieties are closed subspaces in \( P, L, \) and \( F \), respectively. In particular, every point row, every pencil of lines, and every star \( x^\perp \) is closed.

Proof. The Schubert variety \( \text{Cl} P_{n-2}(v, u) \) is closed, because it is the complement of the big cell \( P_{n-1}(u, v) \). We proceed by induction on \( n - k \) and may assume that \( \text{Cl} L_{k+1}(u, v) \) is closed. Put \( U = L - \text{Cl} L_{k+1}(u, v) \). Now \( P_U = P - \text{Cl} P_k(u, v) \) is open.

\[ \blacksquare \]

2.1.12 Proposition The flag space \( F \) is closed in \( P \times L \).

Proof. Let \((p_\nu, \ell_\nu)\) be a net of flags, converging to \((p, \ell)\). By Lemma 1.5.1 there is a vertex \( u \in \text{opp} p \) with \( d(u, \ell) = n - 1 \). Choose a vertex \( v \in \mathcal{V}_u \). Then the point \( p \) is contained in the big cell \( P_{n-1}(u, v) \), therefore we may assume that all points \( p_\nu \) are contained in \( P_{n-1}(u, v) \), and thus \( d(u, \ell_\nu) = d(u, \ell) = n - 1 \). Now we have the relation \( \ell_\nu = f_{n-1}(f_{n-1}(\ell_\nu, u), p_\nu) \). Passing to the limit we obtain \( \ell = f_{n-1}(f_{n-1}(\ell, u), p) \in L_p \).

\[ \blacksquare \]

The following theorem is due to Jäger.

2.1.13 Theorem (Jäger) Every Schubert variety is regular.

Proof. It suffices to show that \( P \) and \( L \) are regular. Let \( q \in P_{n-1}(u, v) \), and let \( A \subseteq P - \{x\} \) be closed. For every Schubert cell \( C \subseteq P \), there exists an open set \( U_C \) containing \( A \cap C \), and an open set \( W_C \) containing \( q \) with \( U_C \cap W_C = \emptyset \) by 1.7.3 and 2.1.2. Thus, the union of the open sets \( U_C \) and the intersection of the open sets \( W_C \) are disjoint open neighborhoods of \( A \) and \( q \), respectively.

\[ \blacksquare \]
2.2 Connectivity properties

2.2.1 Lemma If the (path-) component of some point \( p \) in a point row \( L \) is non-trivial, then the point row \( L \) is (path-) connected.

Proof. This is clear from the fact that \( L \) is doubly homogeneous. \( \square \)

2.2.2 Proposition Let \( \mathcal{P} \) be a topological \( n \)-gon. Then the following assertions hold:

(i) If the point rows are (path-) connected, then the pencils of lines are (path-) connected.

(ii) If the point rows are discrete, then the pencils of lines are discrete.

Proof. This is clear if \( n \) is odd.

If \( n \) is even, and if the point rows are (path-) connected, then the map \( \rho_b \) of 1.8.1 provides a nonconstant map from some point row into some pencil of lines. Therefore the pencils of lines are (path-) connected by 2.2.1.

Next, assume that the point rows are not discrete. Consider the multiplication \( K \times L \rightarrow K \), where \( H = L \cup \{\infty_L\} \) is a point row, and \( K \cup \{\infty_K\} \) is a pencil of lines. Since \( H \) is homogeneous and not discrete, there is a net \( (x_\nu) \subseteq L - \{0_L\} \) converging to \( 0_L \). Choose \( b \in K - \{0_K\} \). Now \( y_\nu = b \cdot x_\nu \) converges to \( b \cdot 0_L = 0_K \), but \( b \cdot x_\nu \neq 0_K \), hence 0\( _K \) is not isolated in \( K \). \( \square \)

2.2.3 Proposition The following are equivalent:

(i) Some point row is (path-) connected.

(ii) Some pencil of lines is (path-) connected.

(iii) The point space is (path-) connected.

(iv) The line space is (path-) connected.

(v) The flag space is (path-) connected.

If one of these equivalent conditions is violated, then the (path-) component of every point, line, or flag is trivial.

Proof. The conditions (i), (ii) are equivalent by 2.2.2. If the point rows and the pencils of lines are (path-) connected, then the flag space is (path-) connected, because for any flag \( (p, \ell) \), the space \( \{p\} \times L \cup (L_p \times \{\ell\}) \) of all flags that have a point or a line in common with \( (p, \ell) \) is (path-) connected. Thus \( \mathcal{P} \) and \( L \) are (path-) connected.
Now suppose that the point rows and the pencils of lines are not (path-) connected. Then the (path-) component of every point in a point row is trivial, and thus the (path-) component of every member of a Schubert cell is trivial. Thus, if the point rows are not connected, then the big cells are totally disconnected. Since any two points (any two lines, any two flags) are contained in a big cell, the spaces \( \mathcal{P}, \mathcal{L} \text{ and } \mathcal{F} \) are totally disconnected.

If the point rows and pencils of lines are not path-connected, then every map of the unit interval into a Schubert cell is constant; hence the path-component of every point in the point space is trivial.

\[ \text{✷} \]

2.2.4 Definition We call a topological polygon (path-) connected, if it satisfies one of the five equivalent conditions of 2.2.3. If it is not connected, we call it totally disconnected.

2.2.5 Proposition Let \( \mathfrak{P} \) be a path-connected polygon. Then \( \mathcal{P}, \mathcal{L}, \mathcal{F}, \) every point row, and every pencil of lines is locally contractible. Every Schubert cell is pseudo-isotopically contractible.

Proof. Consider the multiplication defined in 1.8.1. Let \( \gamma : \mathbb{I} \to L \) be a path with \( \gamma(0) = 0_L \) and \( \gamma(1) = 1_L \). Let \( U \) be a neighborhood of \( 0_K \). Every element \( (0_K, t) \in U \times \mathbb{I} \) has a neighborhood \( V_t \times I_t \subseteq U \times \mathbb{I} \) with \( V_t \bullet \gamma(I_t) \subseteq U \). Since the unit interval \( \mathbb{I} \) is compact, there is a finite subcovering \( I_{t_1}, \ldots, I_{t_k} \) of \( \mathbb{I} \). We put \( V = \bigcap_{i=1}^{k} V_{t_i} \). Now \( V \bullet \gamma(\mathbb{I}) \subseteq U \), hence \( V \) can be contracted to \( 0_K \) within \( U \).

Thus every point row and every pencil of lines is locally contractible. This implies that every Schubert cell is locally contractible, and the claim follows.

The space \( K \) is pseudo-isotopically contractible by means of the map \( (x, t) \mapsto x \bullet \gamma(t) \) (we may assume that \( \gamma(t) \neq 0_K \) for \( t > 0 \)).

\[ \text{✷} \]

2.2.6 Proposition Let \( \mathfrak{P} \) be a topological polygon. The following are equivalent:

(i) Some point row is discrete.

(ii) Some pencil of lines is discrete.

(iii) The point space is discrete.

(iv) The line space is discrete.

(v) The flag space is discrete.

(vi) The point space has an isolated element.
(vii) The line space has an isolated element.

(viii) The flag space has an isolated element.

Proof. Conditions (i) and (ii) are equivalent by 2.2.2. If the point rows and pencils of lines are discrete, then the big cells are discrete, and hence $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ are discrete. The other implications are clear. □

2.2.7 Definition We call a topological polygon discrete, if it satisfies one of the eight equivalent conditions of 2.2.6.

2.2.8 Corollary If the polygon $\mathfrak{P}$ is not discrete, then the Schubert varieties are the topological closures of the corresponding Schubert cells.

Proof. If $\mathfrak{P}$ is not discrete, then the point rows are not discrete by 2.2.6. Because point rows are homogeneous, no point on a point row is isolated. Now let $\mathcal{P}_k(u,v)$ be a Schubert cell, and let $q \in \mathcal{P}_{k-1}(v,u)$. Choose $\ell \in \mathcal{L}_q - \{f_{k-1}(u,q)\}$. The corresponding punctured point row $L - \{q\}$ is contained in $\mathcal{P}_k(u,v)$, hence $q$ is contained in the closure of $\mathcal{P}_k(u,v)$. Therefore, the closure of $\mathcal{P}_k(u,v)$ contains $\mathcal{P}_{k-1}(v,u)$, and the claim follows by induction on $k$.

The proofs for the other kinds of Schubert cells are similar. □

2.3 Automorphisms

2.3.1 Proposition Let $\phi$ be an automorphism of a topological $n$-gon. If there is a point row $L$ and a pencil of lines $\mathcal{L}_p$, such that the restrictions $L \xrightarrow{\phi|L} \phi(L)$ and $\mathcal{L}_p \xrightarrow{\phi|\mathcal{L}_p} \phi(\mathcal{L}_p)$ are continuous, then $\phi$ is continuous on $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$. If $n$ is odd, then it suffices that the restriction $L \xrightarrow{\phi|L} \phi(L)$ is continuous for some point row $L$.

In particular, every root collineation is a homeomorphism.

Proof. The restriction of $\phi$ to any point row or pencil of lines is continuous. For if $h$ is a line with point row $H$, then there exists a vertex $z \in \text{opp} \ell \cap \text{opp} h$. Now $\phi[H = [\phi(h), \phi(z), \phi(\ell)] \circ (\phi|L) \circ [\ell, z, h]$ is continuous.

By 1.7 the restriction of $\phi$ to any Schubert cell is continuous, hence $\phi$ is continuous. □

This proposition is generalized to homomorphisms of topological polygons in [BK94b].
2.4 Locally compact polygons

2.4.1 Definition A topological polygon $\mathcal{P}$ is called \textit{locally compact}, if the point rows and the pencils of lines are locally compact.

2.4.2 Proposition Let $\mathcal{P}$ be a topological polygon. The following are equivalent:

(i) The polygon $\mathcal{P}$ is locally compact.

(ii) The point space $\mathcal{P}$ is locally compact.

(iii) The line space $\mathcal{L}$ is locally compact.

(iv) The flag space $\mathcal{F}$ is locally compact.

Proof. If $\mathcal{P}$ is locally compact, then every Schubert cell is locally compact, hence the spaces $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ are locally compact. Conversely, if $\mathcal{P}$ is locally compact, then the big cell in $\mathcal{P}$ is locally compact, hence every punctured point row and every punctured pencil of lines is locally compact. □

The following theorem is proved in [GKK94]. There, it is stated for compact polygons, but the compactness is never really used in the proof.

2.4.3 Theorem [GKK94, 1.5,1.6] Let $\mathcal{P}$ be a locally compact, non-discrete polygon. Then the point space, the line space, the flag space, every point row, and every pencil of lines is second countable. In particular, each of these spaces is metrizable (and thus paracompact) and separable.

If $\mathcal{P}$ is in addition connected, then it is path connected and hence locally contractible. □

2.4.4 Corollary Let $\mathcal{P}$ be a locally compact, connected polygon. Fix a point $p \in \mathcal{P}$ and consider the bundle map $\mathcal{V}_p^{(k)} \to \mathcal{V}_p^{(j)}$, $x_k \mapsto x_j$ defined in 2.1.7 for $0 < j < k < n$. This bundle map admits a section and is a homotopy equivalence. For example in a generalized quadrangle, we get homotopy equivalences $\mathcal{P} - L \simeq \ell^\perp - \{\ell\} \simeq L$.

Proof. The bundle has contractible fibers, thus it admits a section and is shrinkable, see [Dol63]. □
2.5 Compact polygons

2.5.1 Definition A topological polygon $\mathfrak{P}$ is called compact, if each point row and each pencil of lines is compact.

2.5.2 Proposition Let $\mathfrak{P}$ be a topological polygon. The following are equivalent:

(i) The polygon $\mathfrak{P}$ is compact.

(ii) The point space and the line space are compact.

(iii) The flag space is compact.

Proof. Suppose $\mathfrak{P}$ is compact. If the base and the fiber of a locally trivial bundle are compact Hausdorff spaces, then the total space is also a compact Hausdorff space.

Let $(p, \ell)$ be a flag. The iterated bundle $\text{Gall}_n(p, \ell)$ is compact. Thus the flag space $F$ is compact, because it is the image of $\text{Gall}_n(p, \ell)$ under the map $(x_0, \ldots, x_{n+1}) \mapsto \text{fl}(x_n, x_{n+1})$.

The other implications are trivial. $\blacksquare$

I do not know if the compactness of the point space implies the compactness of the line space in case that $n$ is even. It does, provided that $\mathfrak{P}$ is connected, see 2.5.5.

2.5.3 Corollary (cp. [GKK94, 1.2]) Let $\mathfrak{P}$ be a compact polygon. If the point rows are finite, then the pencils of lines are finite. $\blacksquare$

2.5.4 Proposition (cp. [GvM90, 2.1(a)]) Let $\mathfrak{P}$ be a generalized polygon. Suppose that $\mathcal{P}$ and $\mathcal{L}$ are compact Hausdorff spaces, and that the subspace $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{L}$ is closed. Then $\mathfrak{P}$ is a compact polygon.

Proof. Clearly, the space of all $(n-1)$-chains $C^{(n-1)}$ and the space of all stammering $(n-1)$-chains $S^{(n-1)}$ are compact. Let $A = \{(x, y) \in \mathcal{V} \times \mathcal{V} | d(x, y) < n-1\}$. Then $A$ is compact, because it is the image of $S^{(n-1)}$ under the map $(x_0, \ldots, x_{n-1}) \mapsto (x_0, x_{n-1})$.

We want to show that the graph $G$ of the function $f_{n-1}$ is closed in $X = ((\mathcal{V} \times \mathcal{V}) - A) \times \mathcal{V}$. It is given by

$$G = \{(x_0, x_{n-1}, x_{n-2}) \mid (x_0, \ldots, x_{n-1}) \in C^{(n-1)} - S^{(n-1)}\}.$$ 

The set $Y = \{(x_0, x_{n-1}, x_{n-2}) \in \mathcal{V} \times \mathcal{V} \times \mathcal{V} \mid (x_0, \ldots, x_{n-1}) \in C^{(n-1)}\}$ is compact and hence closed in $\mathcal{V} \times \mathcal{V} \times \mathcal{V}$, and thus $G = X \cap Y$ is closed in $X$. Therefore, the map $f_{n-1}$ is continuous. $\blacksquare$
2.5.5 Proposition Let $\mathcal{P}$ be a connected, locally compact polygon. Then $\mathcal{P}$ is compact.

Proof. By duality, it suffices to show that some (and hence every) pencil of lines is compact. Pick a point $q \in \mathcal{P}_{n-1}(u, v)$, and let $U \subseteq \mathcal{P}_{n-1}(u, v)$ be an open neighborhood of $q$ with compact closure. Since $\mathcal{P}$ is connected, the boundary $\partial U$ of $U$ is compact and nonempty. Every point row $L$ through $p$ meets $\mathcal{P}_{n-2}(v, u)$, thus $L \cap \partial U$ is nonempty. Now the map $\partial U \rightarrow \mathcal{L}_q, x \mapsto f_2(x, q)$ is a continuous surjection.

For compact, non-connected polygons, we get the following result:

2.5.6 Proposition Let $\mathcal{P}$ be a compact polygon. If $\mathcal{P}$ is not connected, then the point space, the line space, the flag space, every point row, and every pencil of lines is homeomorphic to the Cantor set.

Proof. Being totally disconnected, these spaces are zero-dimensional, and the result follows from [HY61, 2-98].

The following problem seems to be open: If the point rows are (locally) compact, does it follow that the pencils of lines are (locally) compact? Note that it is an open question if there exist generalized polygons with finite point rows and infinite pencils of lines.
Chapter 3

Finite-dimensional polygons

The main theme of this chapter is the algebraic topology of a finite-dimensional $n$-gon $\mathcal{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$, that is, a locally compact $n$-gon with the property that the dimensions $m, m'$ of the point rows and of the pencils of lines are finite and positive (3.1.2).

Let $C$ be an $r$-dimensional Schubert cell (in $\mathcal{P}$, say), let $X = \bar{C}$ be the corresponding Schubert variety, and put $A = X - C$. Thus $A$ is a smaller Schubert variety. In the first section it is shown that $A \subseteq X$ is a cofibration, that $X, C$, and $A$ are ANRs, and that $C$ is a generalized manifold. It follows that $\mathcal{P}, \mathcal{L},$ and $\mathcal{F}$ are generalized manifolds.

The question is then how the homology of $X$ is related to the homology of $A$. It turns out that the pair $(X, A)$ has the same homology as an $r$-sphere; in fact, the one-point compactification $X/A$ of the Schubert cell $C$ is a homotopy $r$-sphere (3.4.3). Hence $X$ and $A$ have the same homology, except possibly in dimensions $r, r - 1$. We show that for a suitable choice of the coefficient domain $R$, the exact sequence

$$0 \to \mathcal{H}_r(A; R) \to \mathcal{H}_r(X; R) \to \mathcal{H}_r(X, A; R) \to \mathcal{H}_{r-1}(A; R) \to \mathcal{H}_{r-1}(X; R) \to 0$$

breaks up to

$$0 \to \mathcal{H}_r(A; R) \to \mathcal{H}_r(A; R) \oplus R \xrightarrow{\mathcal{d}} R \to 0$$

$$0 \to \mathcal{H}_{r-1}(A; R) \xrightarrow{\mathcal{d}} \mathcal{H}_{r-1}(X; R) \to 0.$$

Thus $\mathcal{H}_*(X; R) = R^k$, where $k$ is the number of the Schubert cells contained in $X$. In particular, $\mathcal{H}_*(\mathcal{P}; R) = R^n$. The main idea is to use the gallery space to construct a linking cycle, a method developed by Palais, Terng, and Thorbergsson for isoparametric submanifolds. Similar results hold for the cohomology of $(X, A)$ (3.2.3).
Now suppose that $X$ is a Schubert variety in the flag space. In the next section, we consider the double mapping cylinder $DX$ of

$$\text{pr}_1(X) \leftarrow X \rightarrow \text{pr}_2(X).$$

It turns out that $DX$ is contractible if $X \neq \mathcal{F}$, and that $DF$ is a homotopy $(\dim \mathcal{F} + 1)$-sphere. It follows from a theorem of Münzner that $n \in \{3, 4, 6\}$, i.e. a finite-dimensional polygon is a projective plane, a quadrangle, or a hexagon \((3.3.6)\). This is one of our main results; it was first proved by Knarr \([\text{Kna90}]\) under the additional assumption that the Schubert cells are locally euclidean. For polygons with equal parameters $m = m'$ we may calculate the Steenrod squares; as an application, we show that the point space does not factor as a product \((3.3.8)\). It is also shown that for a quadrangle with parameters $(m, 1)$ the flag space $F$ is a product $F = P \times S^1$, provided that $P$ is orientable \((3.3.9)\).

The connectivity properties of $DX$ are useful in order to determine the homotopy properties of the inclusion $A \subseteq X$. Essentially, it turns out that the pair $(X, A)$ is $(\dim X - 1)$-connected \((3.4.1)\), and thus the Schubert cell decomposition has similar properties as a CW decomposition (there is a difficulty with the point and the flag space of $(1, m')$-quadrangles for $m' > 2$, though, due to the fact that the fundamental groups of these spaces are non-trivial). The section closes with a table of the fundamental groups of $P$, $L$, and $F$ \((3.4.11)\).

In the last section we prove that the bundle map $F \rightarrow P$ does not admit a section, provided that $n \neq 4$ or that $m = m'$ \((3.5.1)\) with additional assumptions for the case of $(1, 1)$-quadrangles). This fact is useful in connection with automorphism groups, see Chapter \([3]\) and it leads also to a non-existence result for closed ovoids in certain polygons. It is proved in \((3.3.9)\) that for $n = 4$, the bundle $F \rightarrow P$ is sometimes trivial, hence quadrangles may behave quite differently.

This seems to be a general phenomenon: There are strong analogies between projective planes, quadrangles with equal parameters $m = m'$, and hexagons, and one might conjecture that for these geometries the dimension determines the homotopy type (or even the homeomorphism type) of $P$, $L$, and $F$, although a proof of this conjecture even for $n = 3$ seems to run into some subtle and difficult questions about characterizations of manifolds and on the homotopy fiber $\text{TOP}(k)/\text{O}(k)$ for $k = 4, 8$ (but see \([\text{Kra94}]\) for the smooth case).

In any case, this conjecture fails for quadrangles with $m \neq m'$ \([\text{Wng88}, \text{Tho92}]\). So far, little is known about these quadrangles, their topology, and their automorphism groups.

### 3.1 Dimension of Schubert cells

We will use the following facts from dimension theory.
3.1. DIMENSION OF SCHUBERT CELLS

3.1.1 Proposition Let $A$ be a subspace of a second countable, metrizable space $X$. Then the covering dimension $\dim A$, the large inductive dimension $\text{Ind} A$, and the small inductive dimension $\text{ind} A$ of $A$ coincide $[\text{Eng89, 7.3.3}]$, $[\text{Pea75, 4 5.9}]$. Moreover, we have the inequality

$$\dim A \leq \dim X \tag{1}$$

cp. $[\text{Eng89, 7.3.4}]$, $[\text{Pea75, 4 6.4}]$. If $A$ is closed in $X$, then

$$\dim X = \max\{\dim A, \dim (X - A)\}; \tag{2}$$

cp. $[\text{Pea75, 3 5.8, 4 4.2, 4 4.8}]$, combined with (1). If $Z$ is a second countable, metrizable space, then the inequality

$$\dim (X \times Z) \leq \dim X + \dim Z \tag{3}$$

holds $[\text{Eng89, 7.3.17}]$, $[\text{Pea75, 9 3.3}]$.

Let $R$ be a principal ideal domain. We denote the sheaf-theoretic dimension by $\dim_R$, cp. $[\text{Bre67}]$. If $X$ is paracompact and if the covering dimension of $X$ is finite, then

$$\dim_Z X = \dim X, \tag{4}$$

see $[\text{Löw83, 4.1}]$. If $X$ is a locally compact Hausdorff space, then it follows readily from the universal coefficient theorem that $\dim_R X \leq \dim_Z X \tag{Bre67, II 15.5, 18.3}$.

\[\square\]

3.1.2 Definition A locally compact $n$-gon is called finite-dimensional, if the covering dimension $m = \dim L$ of the point rows and the covering dimension $m' = \dim L_p$ of the pencils of lines are finite and positive (the case of zero-dimensional compact polygons is covered by $[2.5.6]$). We call $(m, m')$ the topological parameters of the $n$-gon $\mathfrak{P}$, and we put $m_k(p, \ell) = m + m' + m + \ldots$ ($k$ summands), and $m_k(\ell, p) = m' + m + m' + \ldots$ ($k$ summands).

Note that a finite-dimensional polygon $\mathfrak{P}$ is second countable, metrizable, path-connected, locally contractible, and compact by $[2.4.3, 2.5.5]$.

3.1.3 Proposition Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a locally compact polygon. The following are equivalent:

(i) The polygon $\mathfrak{P}$ is finite-dimensional.

(ii) The point space $\mathcal{P}$ has finite and positive covering dimension.

(iii) The line space $\mathcal{L}$ has finite and positive covering dimension.
(iv) The flag space $F$ has finite and positive covering dimension.

Proof. If $\mathcal{P}$ is finite-dimensional, then every Schubert cell is finite-dimensional (because of the product formula 3.1.1 (3)), and thus every Schubert variety is finite-dimensional by 3.1.1 (2) (because it is the union of a Schubert cell and a smaller Schubert variety). Therefore $\mathcal{P}$, $\mathcal{L}$, and $F$ are finite-dimensional.

Conversely, if $\mathcal{P}$ is finite-dimensional, then the point rows and the pencils of lines are finite-dimensional by 3.1.1 (1), because they may be imbedded into $p^\perp \subseteq \mathcal{P}$ by 2.1.6. $\blacksquare$

From now on, we assume that $\mathcal{P}$ is a finite-dimensional $n$-gon with parameters $(m, m')$.

3.1.4 Proposition Every (punctured) point row and every (punctured) pencil of lines of $\mathcal{P}$ is an ANR (i.e. an absolute neighborhood retract for the class of all metrizable spaces, see 6.1.8). Thus every Schubert cell is an ANR. The point space, the line space, and the flag space are ANRs.

Proof. By 2.4.3, every (punctured) point row is locally contractible. Being a finite-dimensional, locally contractible space, it is an ANR [Hu65, V 7.1]. A finite product of ANRs is again an ANR [Hu65, III 7.6]. A space which is locally an ANR is an ANR [Hu65, III 8.1]. $\blacksquare$

3.1.5 Theorem Every point row of $\mathcal{P}$ is a generalized $m$-manifold (see 6.3), homotopy equivalent to an $m$-sphere. Similarly, every pencil of lines is a generalized $m'$-manifold, homotopy equivalent to an $m'$-sphere.

If $m, m' \leq 2$, then the point rows and the pencils of lines are homeomorphic to spheres, and every Schubert cell is homeomorphic to some euclidean space. In particular, $\mathcal{P}$ is a manifold in the sense of Chapter 4.

Every Schubert cell is a generalized manifold. Hence

\[
\begin{align*}
\dim P_k(u, v) &= \dim \text{Cl}P_k(u, v) = m_k(u, v), \\
\dim L_k(v, u) &= \dim \text{Cl}L_k(v, u) = m_k(v, u), \\
\dim F_k(u, v) &= \dim \text{Cl}F_k(u, v) = m_k(u, v).
\end{align*}
\]

The point space, the line space, and the flag space are generalized manifolds of dimension $\dim \mathcal{P} = n^\frac{m+m'}{2} - m'$, $\dim \mathcal{L} = n^\frac{m+m'}{2} - m$, and $\dim \mathcal{F} = n^\frac{m+m'}{2}$, respectively.

Proof. The point rows and the pencils of lines are generalized manifolds and homotopy spheres by Löwen’s Theorem 3.3.6. Being a generalized manifold is a local property, hence every Schubert cell is a generalized manifold by 6.3.4. $\blacksquare$
3.1. DIMENSION OF SCHUBERT CELLS

3.1.6 Theorem Let $\mathfrak{P}$ be a finite-dimensional polygon. Then every Schubert variety is an ANR (and thus locally contractible).

For every $k \geq 0$ the gallery space $\text{Gall}_k(u, v)$, as well as the subset of stammering galleries $\text{StamGall}_k(u, v)$, is an ANR. Moreover, $\text{Gall}_k(u, v)$ is a generalized $m_k(u, v)$-manifold. If $m, m' > 1$, then $\text{Gall}_k(u, v)$ is simply connected and hence orientable.

Each of the following spaces is a compact ANR, and hence the inclusions

$$\text{StamGall}_k(u, v) \subseteq \text{Gall}_k(u, v)$$
$$\text{ClP}_k(u, v) \subseteq \text{ClP}_{k+1}(v, u)$$
$$\text{ClL}_k(v, u) \subseteq \text{ClL}_{k+1}(u, v)$$
$$\text{ClF}_k(u, v) \cup \text{ClF}_k(u, v) \subseteq \text{ClF}_{k+1}(u, v)$$
$$\text{ClL}_{k+1}(v, u) \subseteq \text{ClL}_{k+1}(u, v) \cup \text{ClF}_{k+1}(v, u)$$

are cofibrations by 6.1.8.

Proof. Being an ANR is a local property [Hu65, III 8.1], hence the iterated bundles $\text{Gall}_k(u, v)$ are ANRs by induction on $k$. By 3.3.4, the space $\text{Gall}_k(u, v)$ is a generalized $m_k(u, v)$-manifold. If $m, m' > 1$, then it follows inductively from the homotopy exact sequence for bundles that the total space of the bundle $\text{Gall}_k(u, v) \rightarrow \text{Gall}_{k-1}(u, v)$ is simply connected.

Next, we want to show that the set $\text{StamGall}_k(u, v)$ is an ANR. This is trivial for $k = 0$, so we proceed again by induction. Consider the maps

$$\text{Gall}_{k-1}(u, v) \xrightarrow{\Delta} \text{Gall}_k(u, v) \xrightarrow{\text{pr}} \text{Gall}_{k-1}(u, v).$$

The set $A = \text{pr}^{-1}(\text{StamGall}_{k-1}(u, v))$ is an ANR, because it is the total space of the bundle $\text{pr}^{-1}(\text{StamGall}_{k-1}(u, v)) \rightarrow \text{StamGall}_{k-1}(u, v)$, and because $\text{StamGall}_{k-1}(u, v)$ is an ANR. Clearly, the set $B = s(\text{Gall}_{k-1}(u, v))$, as well as the intersection $A \cap B = s(\text{StamGall}_{k-1}(u, v))$ is an ANR. Thus the union $A \cup B = \text{StamGall}_k(u, v)$ is an ANR by 6.1.8.

Finally, we want to show that the Schubert varieties are ANRs. This follows again by induction. Consider the relative homeomorphism

$$(\text{Gall}_k(u, v), \text{StamGall}_k(u, v)) \rightarrow (\text{ClP}_k(u, v), \text{ClP}_{k-1}(v, u)).$$

By 6.1.8, the space $\text{ClP}_k(u, v)$ is an ANR. The proof for the other types of Schubert varieties is similar. The fact that $\text{ClF}_k(u, v) \cup \text{ClF}_k(v, u)$ is an ANR follows again from 6.1.8. $\square$
3.2 Homological properties of finite-dimensional polygons; first results

3.2.1 Definition Throughout this section, we let \((X, A)\) and \((Y, B)\) denote one of the following pairs.

\[
\begin{align*}
(X, A) & \quad (Y, B) \\
(\text{Cl}P_{k+1}(v, u), \text{Cl}P_k(u, v)) & \quad (\text{Gall}_{k+1}(v, u), \text{StamGall}_{k+1}(v, u)) \\
(\text{Cl}L_{k+1}(u, v), \text{Cl}L_k(u, v)) & \quad (\text{Gall}_{k+1}(u, v), \text{StamGall}_{k+1}(u, v)) \\
(\text{Cl}F_{k+1}(u, v), \text{Cl}F_k(u, v) \cup \text{Cl}F_k(v, u)) & \quad (\text{Gall}_{k+1}(u, v), \text{StamGall}_{k+1}(u, v)) \\
(\text{Cl}F_{k+1}(u, v) \cup \text{Cl}F_{k+1}(v, u), \text{Cl}F_k(u, v)) & \quad (\text{Gall}_{k+1}(v, u), \text{StamGall}_{k+1}(v, u))
\end{align*}
\]

Consider the canonical map \((Y, B) \to (X, A)\) that sends a gallery to its end. The restriction \(Y - B \to X - A\) is a homeomorphism, and we get a homeomorphism \((Y/B, \ast) \cong (X/A, \ast)\). We put \(r = \dim(X - A)\).

3.2.2 Lemma The pairs \((X, A)\) and \((Y, B)\) have the same (co-) homology as \((\mathbb{S}^r, \ast)\). The quotient space \(Y/B\) is a generalized \(r\)-manifold and a (co-) homology \(r\)-sphere. The collapsing map \(Y \to Y/B\) has \(R\)-degree 1, i.e. it induces isomorphisms in \(r\)-dimensional (co-) homology with coefficients in \(R\), where \(R = \mathbb{Z}\) for \(m, m' > 1\), and \(R = \mathbb{Z}_2\) else.

Proof. The space \(Y/B = Y - B \cup \{\ast\}\) has finite covering dimension by \([3.1.1]\) (2), thus its sheaf-theoretic dimension is finite as well. Now \(H_\bullet(Y/B, Y - B) \cong H_\bullet(Y/B)\), because \(Y - B\) is contractible. The quotient \(Y/B\) is a wedge of point rows and pencils of lines, thus it has the same (co-) homology as an \(r\)-sphere by \([3.1.3]\).

Thus the space \(Y/B\) has the ‘right’ local homology groups, and by \([1.3.7]\) it is a generalized \(r\)-manifold.

Now pick \(y \in Y - B\), and consider the excision maps

\[
(Y, Y - \{y\}) \hookrightarrow (Y - B, Y - (B \cup \{y\})) \to (Y/B, Y/B - \{y\}).
\]

It follows that the collapsing map induces an isomorphism

\[
H_\bullet(Y, Y - \{y\}; R) \to H_\bullet(Y/B, Y/B - \{y\}; R),
\]

and hence

\[
H_\bullet(Y; R) \xrightarrow{\sim} H_\bullet(X; R)
\]

is an isomorphism, since \(X\) and \(Y\) are \(R\)-orientable. \(\square\)
### 3.3. THE VERONESE IMBEDDING

#### 3.2.3 Theorem
The maps $(A, \emptyset) \to (X, \emptyset) \to (X, A)$ induce short exact sequences

$$0 \to H_\ast(A; R) \to H_\ast(X; R) \to H_\ast(X, A; R) \to 0$$

which are split (note that $H_\ast(X, A; R) \cong R \cong H^\ast(X, A; R)$ by 3.2.2). Again, the coefficient ring $R$ is $\mathbb{Z}_2$ if $1 \in \{m, m'\}$, and $\mathbb{Z}$ else. Thus the attaching of the Schubert cell $X - A$ to the Schubert variety $A$ corresponds to adding an $r$-dimensional (co-)homology class to the (co-)homology of $A$.

In particular, every Schubert variety $X$ represents a (co-)homology class in dimension $r = \dim(X - A)$, and the (co-)homology of a Schubert variety (and in particular of $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$) is (additively) a free $R$-module over the Schubert varieties contained in it.

The point space is orientable if $m > 1$ (see also 4.2.2).

**Proof.** Consider the long exact sequence of the pair $(X, A)$. Since $X/A$ has the same (co-)homology as an $r$-sphere, we have only to show that $H_r(X; R) \to H_r(X, A; R)$ is surjective (that $H_r(X; R) \leftarrow H^r(X, A; R)$ is injective).

But this as well as the splitting follows from the diagram

$$\begin{align*}
Y & \to (Y, B) \to (Y/B, *) \\
\downarrow & \downarrow \downarrow \cong \\
X & \to (X, A) \to (X/A, *)
\end{align*}$$

and the fact that the composite $Y \to (Y, B) \to (X, A)$ induces an isomorphism in $r$-dimensional (co-)homology by 3.2.2, 6.1.4. \hfill \Box

#### 3.2.4 Lemma
If $m' > 1$, then the inclusions $L \hookrightarrow p^\perp \hookrightarrow \ldots \hookrightarrow \mathcal{P}$ induce isomorphisms in 1-dimensional homology.

**Proof.** By 3.2.3, we have $H_i(\mathcal{P}_{k+1}(v, u), \mathcal{P}_k(u, v)) = 0$ for $k \geq 1$ and $i < m' + m$. Therefore we get isomorphisms $H_1(L) \xrightarrow{\cong} H_1(p^\perp) \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_1(\mathcal{P})$. \hfill \Box

### 3.3 The Veronese imbedding

#### 3.3.1 Definition
Let $A \subseteq \mathcal{F}$ be a subset of the flag space. The double mapping cylinder $DA$ is defined as follows: on $A \times I$ we introduce an equivalence relation by putting

- $(a, 0) \sim (a', 0)$ if $pr_1(a) = pr_1(a')$
- $(a, 1) \sim (a', 1)$ if $pr_2(a) = pr_2(a')$
- $(a, s) \sim (a, s)$ for $0 \leq s \leq 1$.
The quotient \( DA = (A \times I)/\sim \) is called the double mapping cylinder of \( A \) \([DKP70, 1.29]\). Note that the double mapping cylinder is obtained by pasting the (unreduced) mapping cylinders of \( A \to \text{pr}_1(A) \) and \( A \to \text{pr}_2(A) \) together along \( A \). Thus, if \( A \) as well as \( \text{pr}_1(A), \text{pr}_2(A) \) are ANRs, then \( DA \) is also an ANR \([Hu63, VI 1.2]\).

We may consider \( \mathcal{P}, \mathcal{L}, \) and \( \mathcal{F} \) as subspaces of \( DF \). These imbeddings are called the topological Veronese imbeddings of \( \mathcal{P}, \mathcal{L}, \) and \( \mathcal{F} \).

Note that up to homotopy, we may replace the maps \( \text{pr}_1(A) \leftarrow A \to \text{pr}_2(A) \) by the inclusions \( DA - \mathcal{L} \leftarrow DA - (\mathcal{P} \cup \mathcal{L}) \to DA - \mathcal{P} \).

Our next aim is the following result.

3.3.2 Theorem The double mapping cylinder \( DF \) of the flag space is homotopy equivalent to an \((n(m + m')/2 + 1)\)-sphere.

We fix a flag \((p, \ell) \in \mathcal{F}\), and we put \( X_k = \text{Cl}\mathcal{F}_k(p, \ell), Y_k = \text{Cl}\mathcal{F}_k(\ell, p) \). The one-point compactification of a space \( Z \) is denoted by \( Z^+ \).

As an intermediate step, we prove the following result.

3.3.3 Proposition For \( 0 \leq k < n \), the double mapping cylinders \( DX_k, DY_k \) and \( D(X_k \cup Y_k) \) are contractible.

Proof. This is certainly true for \( DX_0 = DY_0 = \{(p, \ell)\} \times I \). Now we assume that the double mapping cylinders \( DX_{k-1}, DY_{k-1} \) and \( D(X_{k-1} \cup Y_{k-1}) \) are contractible. Then the collapsing map

\[
DX_k \mapsto DX_k/D(X_{k-1} \cup Y_{k-1})
\]

is a homotopy equivalence by \([6.1.3]\). Now

\[
DX_k - D(X_{k-1} \cup Y_{k-1}) = \mathcal{F}_k(p, \ell) \times (0, 1]
\]

for \( k < n \). The one-point compactification of this space is just the reduced cone \( C_{\mathcal{F}_k(p, \ell)^+} \) of the one-point compactification of \( \mathcal{F}_k(p, \ell) \) and thus contractible. Hence we find that

\[
DX_k/D(X_{k-1} \cup Y_{k-1}) = C_{\mathcal{F}_k(p, \ell)^+}
\]

is contractible. Similarly, \( DY_k/D(X_{k-1} \cup Y_{k-1}) = C_{\mathcal{F}_k(\ell, p)^+} \) and \( D(X_k \cup Y_k)/D(X_{k-1} \cup Y_{k-1}) = C_{\mathcal{F}_k(p, \ell)^+} \vee C_{\mathcal{F}_k(\ell, p)^+} \) are contractible. \( \square \)

Proof of the theorem. Since \( D(X_{n-1} \cup Y_{n-1}) \) is contractible, \( DF \) is homotopy equivalent to the one-point compactification of \( \mathcal{F}_n(p, \ell) \times (0, 1) \), and this is precisely the reduced suspension of \( \mathcal{F}_n(p, \ell)^+ \).

Now \( \pi_1(S(\mathcal{F}_n(p, \ell)^+)) = 0 \), because \( \mathcal{F}_n(p, \ell)^+ = \mathcal{F}/(X_{n-1} \cup Y_{n-1}) \) is path connected. Thus \( DF \) is a simply connected homology \((n(m + m')/2 + 1)\)-sphere by \([3.2.2]\). Being a compact ANR, the space \( DF \) is homotopy equivalent to a CW complex, and hence to an \((n(m + m')/2 + 1)\)-sphere by \([6.5.1]\). \( \square \)
3.3.4 Theorem The double mapping cylinder $DF$ is a generalized $(n(m+m')/2+1)$-manifold.

Proof. See §3.8.

3.3.5 Corollary For any coefficient domain $R$, we have the relations

$$H_{\dim \mathcal{F}}(\mathcal{F}; R) \cong R \cong H_{\dim \mathcal{F}}(\mathcal{F}; R);$$

in particular, the flag space is orientable. The maps $pr_{1}, pr_{2}$ induce isomorphisms $H_{j}(\mathcal{F}; R) \cong H_{j}(\mathcal{P}; R) \oplus H_{j}(\mathcal{L}; R)$ and $H^{j}(\mathcal{F}; R) \cong H^{j}(\mathcal{P}; R) \oplus H^{j}(\mathcal{L}; R)$ in dimension $j$, for $0 < j < \dim \mathcal{F}$.

Proof. This follows from the Mayer-Vietoris sequence of $(DF; DF - \mathcal{P}, DF - \mathcal{L})$.

The following theorem was first proved by Knarr [Kna90] under the assumption that the polygon $\mathcal{P}$ is a manifold (see Chapter 6).

3.3.6 Theorem Let $\mathcal{P}$ be a finite-dimensional $n$-gon with parameters $(m, m')$. Then $n \in \{3, 4, 6\}$. If $n = 3$, then $m = m' \in \{1, 2, 4, 8\}$. If $n = 6$, then $m = m' \in \{1, 2, 4\}$. If $n = 4$, and if $m, m' > 1$, then either $m = m' \in \{2, 4\}$, or $m + m'$ is odd.

The cohomology rings of $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ are listed in section 6.4. Some of the Steenrod squares are calculated in §3.7.

Proof. This follows from Münzner’s Theorem, see §6.4.1.

In case that $n \neq 4$ or that $m = m'$, it is easy to calculate the Steenrod squares. We use the notation of §7.4 for the $\mathbb{Z}_{2}$-cohomology.

3.3.7 Lemma Let $\mathcal{P}$ be a finite-dimensional $n$-gon. If $n \neq 4$, or if $m = m'$, then the Steenrod squares are as follows:

- If $n = 3, 6$, then $Sq x_{m} = x_{m} + x_{m}^{2}$, and $Sq y_{m} = y_{m} + y_{m}^{2}$. If $n = 6$, then $x_{3m} = x_{3m} + x_{m}x_{3m}$, and $y_{3m} = y_{3m} + y_{m}y_{3m}$.
- If $n = 4$, then $x_{m} = x_{m} + x_{m}^{2}$, $y_{m} = y_{m}$, and $y_{2m} = y_{2m} + y_{m}y_{2m}$.

Proof. The operation of $Sq$ on $x_{m}, y_{m}$ follows readily from the properties of $Sq$, see eg. [Bre93, VI 15.]. Now let $n = 4$. Then $x_{2m} = x_{2m}^{2} + x_{m}y_{m}$, hence $y_{2m} = x_{2m}^{2} + x_{m}y_{m} + x_{m}^{2}y_{m} = y_{2m} + y_{m}y_{2m}$. For $n = 6$ consider $x_{m}y_{m} = x_{3m} + y_{3m}$. Applying $Sq$ to both sides yields $x_{3m}^{2} + x_{m}y_{m}^{2} = Sq x_{3m} + Sq y_{3m}$. Thus $x_{3m} = x_{3m} + x_{m}x_{3m}$ and $y_{3m} = y_{3m} + y_{m}y_{3m}$. □
3.3.8 Theorem Let $\mathfrak{P}$ be a finite-dimensional $n$-gon. If $n \neq 4$, or if $m = m'$, then neither $\mathcal{P}$ nor $\mathcal{L}$ is homeomorphic to a product of two topological spaces (containing more than one point, of course).

Proof. Recall that the $\mathbb{Z}_2$-Poincaré polynomial of a space $X$ is defined by $P_X(t) = \sum_{i \geq 0} \beta_i t^i$, where $\beta_i = \dim_{\mathbb{Z}_2} H^i(X; \mathbb{Z}_2)$.

A generalized $k$-manifold $M$ factors as a product $M = X \times Y$ only if $X$ and $Y$ are generalized $i$- and $(k - i)$-manifolds [Bre67, V 15.8]. We have the relation $p_X(t)p_Y(t) = p_M(t)$ for the $\mathbb{Z}_2$-Poincaré polynomials (taken with respect to sheaf-theoretic $\mathbb{Z}_2$-cohomology). There is a natural isomorphism $H^\bullet(\mathfrak{P}; \mathbb{Z}_2) \cong H^\bullet(\mathfrak{P}; \mathbb{Z}_2)$, because $\mathfrak{P}$ is compact and $HLC$ (6.3.2).

Next, note that the $\mathbb{Z}_2$-cohomology of a space with $\mathbb{Z}_2$-Poincaré polynomial $1 + t^k$ is necessarily of the form $\mathbb{Z}_2[2k]/(x_k^2)$, and the $\mathbb{Z}_2$-cohomology of a generalized manifold with with $1 + t^k + t^{2k}$ as $\mathbb{Z}_2$-Poincaré polynomial is of the form $\mathbb{Z}_2[2k]/(x_k^2)$ by [Bre67, V 10.6] (since every generalized manifold is $\mathbb{Z}_2$-orientable).

If $n = 3$, then the $\mathbb{Z}_2$-Poincaré polynomial of $\mathfrak{P}$ is given by $1 + t^m + t^{2m}$, with $m \in \{1, 2, 4, 8\}$. These polynomials do not factor over $\mathbb{N}$.

If $n = 4$, then the $\mathbb{Z}_2$-Poincaré polynomial is $1 + t^m + t^{2m} + t^{3m} = (1 + t^m)(1 + t^{2m})$, and $m \in \{1, 2, 4\}$. Now

$$H^\bullet(\mathfrak{P}; \mathbb{Z}_2) = \mathbb{Z}_2[x_m]/(x_4^2),$$

and

$$H^\bullet(\mathcal{L}; \mathbb{Z}_2) = \mathbb{Z}_2[y_m, y_{2m}]/(y_m^2, y_{2m}^2).$$

Clearly, the $\mathbb{Z}_2$-cohomology of $\mathfrak{P}$ is not isomorphic to

$$[\mathbb{Z}_2[u_m]/(u_m^2)]^\bullet \otimes_{\mathbb{Z}_2} [\mathbb{Z}_2[v_{2m}]/(v_{2m}^2)]^\bullet = \mathbb{Z}_2[u_m, v_{2m}]/(u_m^2, v_{2m}^2),$$

hence $\mathfrak{P}$ does not factor. The $\mathbb{Z}_2$-cohomology ring of $\mathcal{L}$ is of this form; however, we have the relation $S^m_1 y_{2m} = y_{2m} y_m \notin \langle 1, y_{2m} \rangle$. Since the Steenrod squares commute with maps, we conclude that $\mathcal{L}$ does not factor.

If $n = 6$, then the $\mathbb{Z}_2$-Poincaré polynomial of $\mathfrak{P}$ and $\mathcal{L}$ is $1 + t^m + t^{2m} + t^{3m} + t^{4m} + t^{5m}$, for $m \in \{1, 2, 4\}$. It factors as $(1 + t^m)(1 + t^{2m} + t^{4m}) = (1 + t^m)(1 + t^{2m})(1 + t^{3m})$ over $\mathbb{N}$. The first factorization is excluded by the structure of the $\mathbb{Z}_2$-cohomology ring, and the second possibility is excluded again by the fact that $S^m_1 x_{3m} = x_{3m} x_m \notin \langle 1, x_{3m} \rangle$.

Note that for quadrangles with $m \neq m'$, the point space can be a product of two spheres [FKM81, Wng88, Tho92].

3.3.9 Theorem Let $\mathfrak{P}$ be a finite-dimensional quadrangle with $m' = 1$. If $m = 1$, assume in addition that $\mathfrak{P}$ is orientable, see [4.2.2. Then the circle bundle $\mathcal{F} \to \mathfrak{P}$ is trivial, i.e. $\mathcal{F} = S^1 \times \mathfrak{P}$. 
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Proof. By Kneser's result [6.2.7], there is an orthogonal 2-plane bundle \( \xi \) over \( \mathcal{P} \) that has \( \mathcal{F} \rightarrow \mathcal{P} \) as its circle bundle, and by assumption \( \xi \) is orientable. We may identify the total space of \( \xi \) with \( D\mathcal{F} - L \). Consider the diagram

\[
(\mathbb{R}^2, \mathbb{R}^2 - \{0\}) \rightarrow (D\mathcal{F} - L, D\mathcal{F} - (\mathcal{P} \cup L)) \leftarrow D\mathcal{F} - L \rightarrow \mathcal{P}
\]

\[
\downarrow \quad \downarrow \\
(D\mathcal{F}, D\mathcal{F} - \mathcal{P}) \leftarrow D\mathcal{F}.
\]

Since \( H^2(D\mathcal{F}) = 0 \), the Euler class of \( \xi \), i.e. the image of the orientation class \( U_\xi \in H^2(D\mathcal{F} - L, D\mathcal{F} - (\mathcal{P} \cup L)) \) in \( H^2(\mathcal{P}) \), vanishes. Thus the bundle \( \xi \) is trivial by 6.2.4. \( \Box \)

3.4 Homotopy properties

We may use the double mapping cylinder to calculate some homotopy groups of the Schubert cell decomposition. We use the notation of the previous sections.

3.4.1 Theorem Suppose that \( m, m' > 1 \). Let \((X, A)\) denote one of the pairs of 3.2.1, and put \( r = \dim(X - A) \). Then the pair \((X, A)\) is \((r - 1)\)-connected, and \( \pi_r(X, A) = \mathbb{Z} \). In particular, every Schubert variety is \((\min\{m, m'\} - 1)\)-connected.

Proof. By 3.1.3 the Schubert varieties \( L = \text{Cl}\mathcal{P}_1(p, \ell) \) and \( \mathcal{L}_p = \text{Cl}\mathcal{L}_1(\ell, p) \) are simply connected. Now suppose that \( \text{Cl}\mathcal{P}_k(u, v) \) is simply connected. It follows from the homotopy sequence for bundles that the total space \( \mathcal{F}_{\text{Cl}\mathcal{P}_k(u, v)} = \text{Cl}\mathcal{F}_{k+1}(u, v) \) is simply connected. Up to homotopy, we may replace the diagram

\[
\text{Cl}\mathcal{P}_k(u, v) \leftarrow \text{Cl}\mathcal{F}_{k+1}(u, v) \rightarrow \text{Cl}\mathcal{L}_{k+1}(u, v)
\]

by the inclusions

\[
D\text{Cl}\mathcal{F}_{k+1}(u, v) - \mathcal{L} \leftarrow D\text{Cl}\mathcal{F}_{k+1}(u, v) - (\mathcal{L} \cup \mathcal{P}) \rightarrow D\text{Cl}\mathcal{F}_{k+1}(u, v) - \mathcal{P}.
\]

Since the double mapping cylinder \( D\text{Cl}\mathcal{F}_{k+1}(u, v) \) is simply connected by 3.3.2, we conclude from the Seifert-Van Kampen Theorem that \( \text{Cl}\mathcal{L}_{k+1}(u, v) \) is simply connected. Thus every Schubert variety is simply connected. By the Seifert-Van Kampen Theorem, the union \( \text{Cl}\mathcal{F}_k(u, v) \cup \text{Cl}\mathcal{F}_k(v, u) \) is also simply connected, see [Whi78, II 2.5 and p.94]. From the relative Hurewicz isomorphism Theorem [Spa60, 7.5.4] we infer that \( \pi_s(X, A) \cong H_s(X, A) = 0 \) for \( s < r \), and that \( \pi_r(X, A) \cong H_r(X, A) = \mathbb{Z} \). \( \Box \)

3.4.2 Corollary If \( m, m' > 1 \), then \( \mathcal{P}_k(u, v) \) is homotopy equivalent to a CW complex \( X \) with \( k + 1 \) cells

\[
X = e^0 \cup e^m \cup e^{m+m'} \cup \ldots \cup e^{m_k(u, v)}.
\]
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A similar statement holds for the other kinds of Schubert varieties.

Proof. The space $P_k(u,v)$ is simply connected by [3.4.1], and $H_\bullet(P_k(u,v)) \cong \mathbb{Z}^k$, with $\mathbb{Z}$-Poincaré polynomial $1 + t^m + t^{m+m'} + \cdots + t^{m_k(u,v)}$ by [3.2.3]. Hence the claim follows from [Wal65, 4.1].

3.4.3 Lemma The quotient $X/A$ is homotopy equivalent to an $r$-sphere.

Proof. By [3.1.5] and by induction, it suffices to consider the following two cases:

(1) $1 \in \{m, m'\}$. Then the Schubert cell $X - A$ is homeomorphic to $\mathbb{R} \times K$, where the one-point compactification $K^+$ is a homotopy sphere. The one-point compactification of $\mathbb{R} \times K$ is the reduced suspension of $K^+$, and hence simply connected.

(2) $m, m' > 1$. Then the Schubert cell $X - A$ is homeomorphic to $K \times L$, where $K^+$ and $L^+$ are simply connected homotopy spheres. Thus $\pi_1(K^+ \times L^+) = 0 = \pi_1(K^+ \vee L^+)$, cp. [Dck91, II 5.9]. From the relative Hurewicz isomorphism, we find that $\pi_1(K^+ \times L^+, K^+ \vee L^+) \cong H_1(K^+ \times L^+, K^+ \vee L^+) = 0$. It follows from [6.1.7] that $\pi_1(K \times L)^+ \cong \pi_1(K^+ \vee L^+) = 0$.

Hence in each case the quotient $X/A$ is simply connected, and the claim follows from 3.2.2 and 6.5.1, since every ANR is homotopy equivalent to a CW complex [Web68, p.218].

The following corollary may be used to calculate some homotopy groups of the Schubert varieties.

3.4.4 Corollary If $m, m' > 1$, then $\pi_s(X, A) \cong \pi_s(S^r)$ for $0 \leq s \leq r + \min\{m, m'\} - 2$, where $r = \dim(X - A)$.

Proof. This follows from [6.1.7].

We will see in [4.1.3] that the Schubert cell decomposition is indeed a CW decomposition, provided that $m, m' \leq 2$, or if the point rows and pencils of lines are locally euclidean. Hence the pair $(X, A)$ is also $(r - 1)$-connected in these cases.

It remains to investigate quadrangles with parameters $(1, m')$ and $m' > 2$. In this case we do not get a complete result. However, we may calculate the fundamental groups of the Schubert varieties.

3.4.5 Proposition Suppose that $\mathcal{P}$ is a finite-dimensional quadrangle with parameters $m = 1$ and $m' > 1$. Then $L$ is homeomorphic to $\mathbb{S}^1$, and $p^+$ is the one-point compactification of $L_p \times \mathbb{R}$. The inclusions $L \subseteq p^+ \subseteq \mathcal{P}$ induce isomorphisms $\mathbb{Z} = \pi_1 L \cong \pi_1 p^+ \cong \pi_1 \mathcal{P}$. The Schubert varieties $L_p, \ell^+$ and $L$ are $(m' - 1)$-connected.

Proof. We use the same ideas as in the proof of [3.4.1]. The bundle map $\mathcal{F}_L \rightarrow L$ induces an isomorphism on the fundamental groups, because the fiber is $(m' - 1)$-connected. Again, we conclude from the Seifert-Van Kampen Theorem, applied to
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[87x708]D \mathcal{F}_L, that \ell^\perp is simply connected. Similarly, it follows from the diagram \( p^\perp \leftarrow \mathcal{F}_{p^\perp} \rightarrow \mathcal{L} \) that \( \mathcal{L} \) is simply connected. The line bundle \( p^\perp \rightarrow \{p\} \rightarrow \mathcal{L}_p \) is trivial, because \( \mathcal{L}_p \) is a cohomology \( m' \)-sphere, see 5.2.4 (using this fact, it is not difficult to see that the pair \( (p^\perp, \mathcal{L}) \) is in fact \( m' \)-connected). In 3.4.11 we prove that \( \pi_1 \mathcal{P} \cong \mathbb{Z} \). Thus, the fundamental group of every Schubert variety in \( \mathcal{P} \) is abelian, and the claim about the inclusions \( \mathcal{L} \subseteq p^\perp \subseteq \mathcal{P} \) follows from 3.2.4.

\[ \boxed{\text{3.4.11 Proposition}} \]

Let \( \mathcal{P} \) be a finite-dimensional polygon. The fundamental group of \( \mathcal{F} \) is generated by the images of \( \pi_1(L \times \{\ell\}) \cong \pi_1 S^m \) and \( \pi_1(\{p\} \times \mathcal{L}_p) \cong \pi_1 S^{m'} \) in \( \pi_1 \mathcal{F} \).

The fundamental group of the point space is generated by the image of \( \pi_1 \mathcal{L} \) in \( \pi_1 \mathcal{P} \) (and dually, the fundamental group of the line space is generated by the image of \( \pi_1 \mathcal{L}_p \)).

Proof. (cp. [GH87, 3.5]) It follows from the exact sequence of a fibration that the maps \( \pi_1 \mathcal{F} \rightarrow \pi_1 \mathcal{P}, \pi_1 \mathcal{F} \rightarrow \pi_1 \mathcal{L} \) are surjective, and the kernels of these maps are precisely the images of \( \pi_1(\{p\} \times \mathcal{L}_p) \) and \( \pi_1(L \times \{\ell\}) \), respectively. Now we may replace \( \mathcal{P} \leftarrow \mathcal{F} \rightarrow \mathcal{L} \) by \( D \mathcal{F} \leftarrow \mathcal{L} \leftarrow D \mathcal{F} \leftarrow \mathcal{P}, \mathcal{L} \leftarrow \mathcal{P} \leftarrow D \mathcal{F} \), and the claim follows from the Seifert-Van Kampen Theorem.

\[ \boxed{\text{3.4.7 Corollary}} \]

[GH87, 3.5] The fundamental group \( \pi_1 \mathcal{F} \) acts trivially on the homotopy groups \( \pi_s \mathcal{F} \) for \( s > 1 \).

Proof. Suppose \( m' = 1 \). It follows from the homotopy exact sequence of the circle bundle \( \mathcal{F} \rightarrow \mathcal{P} \) that \( \pi_s \mathcal{F} \rightarrow \pi_s \mathcal{P} \) is an injection for \( s = 2 \), and an isomorphism for \( s > 2 \), and the projection \( \mathcal{F} \rightarrow \mathcal{P} \) kills the image of \( \pi_1(\{p\} \times \mathcal{L}_p) \). Thus \( \pi_1(\{p\} \times \mathcal{L}_p) \) operates trivially on \( \pi_s \mathcal{F} \).

\[ \boxed{\text{3.4.8 Corollary}} \]

(cp. [GH87, 4.8]) If \( \mathcal{F} \rightarrow \mathcal{P} \) is non-orientable, then the image of \( \pi_{m'}(\{p\} \times \mathcal{L}_p) \) in \( \pi_{m'} \mathcal{F} \) is cyclic of order at most 2.

Proof. If \( \mathcal{F} \rightarrow \mathcal{P} \) is non-orientable, then some \( a \in \pi_1 \mathcal{F} \) acts (through the epimorphism \( \pi_1 \mathcal{F} \rightarrow \pi_1 \mathcal{P} \)) as \(-1\) on \( \pi_{m'}(\{p\} \times \mathcal{L}_p) = \mathbb{Z} \). On the other hand, \( a \) acts trivially on \( \pi_{m'} \mathcal{F} \), hence every element in the image of \( \pi_{m'}(\{p\} \times \mathcal{L}_p) \cong \mathbb{Z} \) is of order at most 2.

\[ \boxed{\text{3.4.9 Corollary}} \]

(cp. [GH87, 4.8]) Let \( \mathcal{P} \) be a finite-dimensional quadrangle with parameters \((1, m')\), for some \( m' > 1 \). Then \( \mathcal{P} \) and \( \mathcal{F} \rightarrow \mathcal{P} \) are orientable if and only if \( m' \) is even.

Proof. If \( \mathcal{F} \rightarrow \mathcal{P} \) is orientable, then \( 1 + m' \) is odd by [Mim81, Satz 7].
Suppose that $\mathcal{F} \to \mathcal{P}$ is non-orientable. Then there is a two-fold covering $\tilde{\mathcal{P}} \to \mathcal{P}$ such that the induced bundle $\tilde{\mathcal{F}} \to \tilde{\mathcal{P}}$ is orientable. By the corollary above, the image of $\pi_{m'}(\{p\} \times \mathcal{L}_p)$ in $\pi_{m'}\tilde{\mathcal{F}}$ is finite. From the diagram

$$
\begin{array}{ccc}
\pi_{m'}(\{p\} \times \mathcal{L}_p) & \to & \pi_{m'}\tilde{\mathcal{F}} \\
\downarrow \cong & & \downarrow \\
H_{m'}(\{p\} \times \mathcal{L}_p) & \to & H_{m'}(\tilde{\mathcal{F}})
\end{array}
$$

we conclude that $H_{m'}(\{p\} \times \mathcal{L}_p; \mathbb{Q}) \to H_{m'}(\mathcal{F}; \mathbb{Q})$ is trivial, and the claim follows from 6.2.11.

3.4.10 Corollary The integral cohomology of a quadrangle with parameters $(1,2k)$ is as given in 6.4 4_3.

3.4.11 Theorem Let $\mathcal{P}$ be a finite-dimensional $n$-gon with parameters $(m,m')$. Up to duality, the fundamental groups of $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ are as follows:

| $m, m'$ > 1 | $\pi_1 \mathcal{P}$ | $\pi_1 \mathcal{L}$ | $\pi_1 \mathcal{F}$ |
|-------------|-----------------|-----------------|-----------------|
| $m = 1$, $m' > 1$ | $\mathbb{Z}$ | $0$ | $\mathbb{Z}$ |
| $m = m' = 1$, $n = 3,6$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Q}_8$ |
| $m = m' = 1$, $n = 4$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z} + \mathbb{Z}_2$ |

Here, $\mathbb{Q}_8$ denotes the quaternion group of order 8. In the case $n = 4$, $m = m' = 1$, the point space $\mathcal{P}$ is orientable, and $\mathcal{L}$ is not, see 4.2.3. (A simply connected generalized manifold is orientable.)

Proof. Up to homotopy, we may replace the maps $\mathcal{P} \xrightarrow{pr_1} \mathcal{F} \xrightarrow{pr_2} \mathcal{L}$ by the inclusions $D\mathcal{F} - \mathcal{L} \hookrightarrow D\mathcal{F} - (\mathcal{P} \cup \mathcal{L}) \hookrightarrow D\mathcal{F} - \mathcal{P}$. Assume that $m' > 1$. Since $pr_1$ is a locally trivial fibration whose fiber is a homotopy $m'$-sphere, the induced map $\pi_1\mathcal{F} \to \pi_1\mathcal{P}$ is an isomorphism. The Seifert-Van Kampen Theorem, applied to $(D\mathcal{F} - \mathcal{L}, D\mathcal{F} - \mathcal{P}, D\mathcal{F} - (\mathcal{P} \cup \mathcal{L}))$ yields $\pi_1\mathcal{L} = 0$. From the homotopy exact sequence of $S^m \to \mathcal{F} \to \mathcal{L}$, we deduce that $\pi_1\mathcal{F}$ is cyclic, and hence isomorphic to $H_1(\mathcal{F})$.

The case $m = m' = 1$ will be treated separately in 4.3.3.

3.5 Sections and ovoids

From the knowledge of the cohomology of finite-dimensional polygons, we get the following result which was first obtained by Breitsprecher [Brs71, 2.3.2] for projective planes.
3.5. SECTIONS AND OVOIDS

3.5.1 Theorem Let $\mathfrak{P}$ be a finite-dimensional $n$-gon with parameters $(m, m')$. If $n \neq 4$, or if $m = m' > 1$, then the bundles $F \to P$ and $F \to L$ admit no sections.

If $\mathfrak{P}$ is a quadrangle with parameters $(1, 1)$, and if the point space is orientable (see 4.2.2), then the bundle $F \to L$ admits no section (the bundle $F \to P$ is trivial by 3.3.4).

Proof. Assume that there is a section $s : P \to F$. We put $R = \mathbb{Z}_2$ for $n = 3, 6$, and $R = \mathbb{Q}$ for $n = 4$, $m = m' > 1$. Let $x_1, x_2, y_1, y_2$ be generators of $H^m(P; R)$, $H^{2m}(P; R)$, $H^m(L; R)$, and $H^{2m}(L; R)$, respectively. By 3.3.5 we may identify $H^m(P; R) \oplus H^m(L; R)$ with $H^m(F; R)$, and $H^{2m}(P; R) \oplus H^{2m}(L; R)$ with $H^{2m}(F; R)$. We have the relations $x_1^2 = \alpha x_2 y_2^2 = \beta y_2$, and $x_1 y_1 = x_2 + y_2$, with $\alpha, \beta \in R$. Note that $s^* \pr_1^* = \text{id}$.

For $n = 3, 6$, we get $(\alpha, \beta) = (1, 1)$, and for $n = 4$ we get $(\alpha, \beta) \in \{(1, 2), (2, 1)\}$, according to 3.4. Now let $ax_1 + y_1$ be a generator of $\ker(s^* : H^m(F; R) \to H^m(P; R))$.

Since $s^*$ is a ring homomorphisms, we have

$$0 = s^* ((ax_1 + y_1)(y_1 - (\beta + \alpha)x_1)) = s^* (-(a^2 \alpha + a \alpha \beta + \beta)x_2) = -(a^2 \alpha + a \alpha \beta + \beta)x_2.$$  

But the polynomial $f(a) = a^2 \alpha + a \alpha \beta + \beta$ has no roots in $R$, hence we get a contradiction.

In the case of an $(1, 1)$-quadrangle, let $ay_1 + x_1$ be in the kernel of

$$s^* : \mathbb{Z}_2[x_1, y_1, y_2]/(x_1^4, y_1^2, y_2^2, x_1^2 + y_2 + x_1 y_1) \to \mathbb{Z}_2[y_1, y_2]/(y_1^2, y_2^2).$$

Then $0 = s^* (((1 + a)y_1 + x_1)(ay_1 + x_1)) = s^* (y_1 x_1 + x_1^2) = y_2$, a contradiction. \qed

3.5.2 Corollary (Breitsprecher) Every continuous collineation $\phi$ of a finite-dimensional projective plane has a fixed point.

Proof. Otherwise, the map $p \mapsto (p, p \vee \phi(p))$ would be a section. \qed

Generalized quadrangles with $m \neq m'$ may have sections; in fact, the bundle $F \to P$ is sometimes trivial, see 3.3.9. Every compact connected Moufang quadrangle or hexagon admits an involution with no fixed point and no fixed line (because every isoparametric foliation is invariant under the antipodal map of the ambient sphere).

An ovoid in a generalized $n$-gon (for $n \geq 4$) is a set of points $O \subseteq P$ with the property that every point row meets $O$ in exactly one point. Cameron has proved that the point space of every generalized $n$-gon with infinite point rows and infinite pencils of lines of the same cardinality may be partitioned into ovoids [Cam94].
3.5.3 Corollary Let $\mathcal{P}$ be a finite-dimensional $n$-gon, for some $n \geq 4$. If $n = 4$, assume in addition that $m = m'$, and that $\mathcal{P}$ is orientable. Then $\mathcal{P}$ contains no closed ovoid.

Proof. Let $O \subseteq \mathcal{P}$ be a closed and therefore compact ovoid. Then $\mathcal{F}_O \to \mathcal{L}_O = \mathcal{L}$ is a continuous bijection and hence a homeomorphism. Thus the bundle $\mathcal{F} \to \mathcal{L}$ has a section. \qed
Chapter 4

Polygons which are manifolds

In this section we investigate polygons which are locally euclidean. In this case the Schubert cells are homeomorphic to some euclidean spaces, and the point rows and the pencils of lines are spheres (more generally the one-point compactification of every Schubert cell is a sphere). This fact is due to the contractibility properties of the Schubert cells. The Schubert cell decomposition of $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ is a CW decomposition (1.1.3), a fact that was proved for projective planes by Breitsprecher [Br71]. The proof becomes surprisingly simple, if one constructs first a CW decomposition of the gallery spaces (which is easy).

The double mapping cylinder is a manifold; it follows from the proof of the generalized Poincaré conjecture that it is indeed a sphere [K90]. Therefore $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ may be imbedded into $S^{\dim F+1}$ (with normal disk bundles) (1.2). We calculate the Stiefel-Whitney classes of these spaces, as well as the Stiefel-Whitney classes of the normal bundles, provided that $n \neq 4$ or that $m = m'$ (4.2).

In the next section we obtain a complete topological classification of the bundles $p^\perp - \{p\} \to \mathcal{L}_p$, provided that $m = m' \leq 2$ (4.3.1). For $n = 3$, this is of course Salzmann’s and Breitsprecher’s topological classification of $\mathcal{P}$. For $n = 3$ and $m = m' \leq 2$, Knarr’s Veronese imbedding leads to a simple proof of the topological classification of the flag space (4.3.2).

The last section gives a topological criterion for a partial $n$-gon $\mathfrak{P}$ to be an $n$-gon: ‘if the dimension is right’, then $\mathfrak{P}$ is a compact $n$-gon.

4.1 A CW decomposition for $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$

4.1.1 Definition A topological polygon $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a manifold, if the point rows and the pencils of lines are locally homeomorphic to $\mathbb{R}^m$ and $\mathbb{R}^{m'}$, respectively, for some numbers $m, m' > 0$. Thus the parameters of $\mathfrak{P}$ are $(m, m')$. Note that $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ are second countable and metrizable by 2.4.3.

A finite-dimensional polygon with parameters $m, m' \leq 2$ is a manifold by 3.1.3.
4.1.2 Proposition Suppose the topological \(n\)-gon \(\mathcal{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})\) is a manifold with parameters \((m, m')\). Then every point row is homeomorphic to an \(m\)-sphere, and every pencil of lines is homeomorphic to an \(m'\)-sphere. The point space \(\mathcal{P}\), the line space \(\mathcal{L}\), and the flag space \(\mathcal{F}\) are compact connected manifolds of dimension \(n(m + m')/2 - m', n(m + m')/2 - m\) and \(n(m + m')/2\), respectively. The maps \(\mathcal{F} \to \mathcal{P}\) and \(\mathcal{F} \to \mathcal{L}\) are locally trivial \(m'\)- and \(m\)-sphere bundles, respectively.

Proof. Every punctured point row and every punctured pencil of lines is a pseudo-isotopically contractible manifold. By the result of Harrold [Har65], it is homeomorphic to some euclidean space. \(\square\)

4.1.3 Theorem Suppose the topological \(n\)-gon \(\mathcal{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})\) is a manifold with parameters \((m, m')\). Fix a flag \((p, \ell)\). The Schubert cell decomposition of \(\mathcal{P}\) with respect to \((p, \ell)\) is a CW decomposition. Thus, the point space \(\mathcal{P}\) consists of \(n\) cells of dimension \(0, m, m + m', m + m' + m, \ldots\) Similarly, the line space \(\mathcal{L}\) consists of \(n\) cells, and the flag space consists of \(2n\) cells.

In order to prove the theorem, we need the following lemma.

4.1.4 Lemma The space \(\text{Gall}_k(u, v)\) is a manifold. It is obtained from the subspace \(\text{StamGall}_k(u, v)\) by attaching an \(m_k(u, v)\)-cell,
\[
\text{Gall}_k(u, v) = \text{StamGall}_k(u, v) \cup e^{m_k(u, v)}.
\]

Proof. We proceed by induction on \(k\). For \(k = 0\), there is nothing to show. Now we consider the sphere bundle \(pr : \text{Gall}_{k+1}(u, v) \to \text{Gall}_k(u, v)\), with the section \(s\) defined in 1.2. The proper galleries in \(\text{Gall}_k(u, v)\) are contained in an \(m_k(u, v)\)-cell. Hence by 5.2.12, the proper galleries \(\text{PropGall}_{k+1}(u, v) = pr^{-1}(\text{PropGall}_k(u, v)) - s(\text{PropGall}_k(u, v))\) are contained in an \(m_{k+1}(u, v)\)-cell
\[
(e^{m_{k+1}(u, v)}, e^{m_{k+1}(u, v)}) \to (\text{Gall}_{k+1}(u, v), \text{StamGall}_{k+1}(u, v)).
\]

Proof of the theorem. By the lemma above, there is an attaching map
\[
(e^{m_k(u, v)}, e^{m_k(u, v)}) \rightarrow (\text{Gall}_k(u, v), \text{StamGall}_k(u, v)) \downarrow (\text{ClF}_k(u, v), \text{ClF}_{k-1}(u, v) \cup \text{ClF}_{k-1}(v, u)).
\]
A similar construction works for the point space and the line space. \(\square\)
4.2 The Veronese imbedding, revisited

The following fact was first proved by Knarr [Kna90].

4.2.1 Proposition If the $n$-gon $\mathcal{P}$ is a manifold (with parameters $(m, m')$), then the double mapping cylinder is a $(\dim \mathcal{F} + 1)$-sphere. Hence $\mathcal{F}$ can be imbedded in $\mathbb{R}^{\dim \mathcal{F}}$ as a topological hypersurface with trivial normal bundle. The point space and the line space can be imbedded in $\mathbb{R}^{\dim \mathcal{F}}$ as topological submanifolds with normal disk bundles of dimension $m' + 1$, $m + 1$, respectively.

Proof. Put $E = D\mathcal{F} - \mathcal{L}$. Then $E \to \mathcal{P}$ is a locally trivial open disk bundle, and thus $E$ is a topological manifold. Being a homotopy sphere of dimension $> 3$, the double mapping cylinder is a sphere, see 6.5.3.

4.2.2 Proposition Let $\mathcal{P}$ be an $n$-gon which is a manifold. Suppose that $n \neq 4$ or that $m = m'$. We get the following table for the total Stiefel-Whitney classes $w$ of $\mathcal{P}, \mathcal{L}, \mathcal{F},$ and of the sphere bundles $\mathcal{F} \to \mathcal{P}, \mathcal{F} \to \mathcal{L}$ (cp. [Brs71] for $n = 3$).

| $n$   | $w(\mathcal{P})$ | $w(\mathcal{L})$ | $w(\mathcal{F})$ | $w(\mathcal{F} \to \mathcal{P})$ | $w(\mathcal{F} \to \mathcal{L})$ |
|-------|------------------|------------------|------------------|-------------------------------|-------------------------------|
| $n = 3, 6$ | $1 + x_m + x_m^2$ | $1 + y_m + y_m^2$ | $1$ | $1 + x_m$ | $1 + y_m$ |
| $n = 4$ | $1$ | $1 + y_m$ | $1$ | $1$ | $1 + y_m$ |

Note that the orientability of the manifolds and their normal disk bundles is determined by the corresponding first Stiefel-Whitney classes $w_1$. Thus, the point space of an $(1, 1)$-quadrangle is orientable, but the line space is not (up to exchanging $\mathcal{P}$ and $\mathcal{L}$, of course).

In each case the number $m + 1$ is the minimal codimension for an imbedding of $\mathcal{L}$ into some euclidean space [Spa66, 6 10.23].

Proof. The Stiefel-Whitney classes of $\mathcal{P}$ and $\mathcal{L}$ can be easily calculated from 3.3.7 and the Wu formula [Bre93, VI 17]. Since the sums of the topological tangent bundles and the normal bundles of $\mathcal{P}, \mathcal{L}, \mathcal{F}$ are trivial, the total Stiefel-Whitney classes of these bundles are inverses of each other.

4.3 Classification in small dimensions

It was shown in 3.1.7 that the point rows and pencils of lines of a finite-dimensional polygon with parameters $m, m' \leq 2$ are spheres. Next, we want to classify the small Schubert varieties $\text{Cl}_{\mathcal{P}}(p, \ell) = p^\perp$ and $\text{Cl}_{\mathcal{L}}(p, \ell) = \ell^\perp$. This generalizes results of Salzmann [Sal67] and Breitsprecher [Brs71] about projective planes of small dimension.
4.3.1 Theorem Let $\mathcal{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a compact, connected, finite dimensional $n$-gon with equal parameters $m = m' \leq 2$. This implies by 3.1.3 that $\mathcal{P}$ is a manifold. Let $p$ be a point and let $\ell$ be a line. There exist orthogonal $m$-plane bundles $\eta, \eta'$ over $L$ and over $L_p$, such that the sphere bundles $S(\eta \oplus \varepsilon)$ and $S(\eta' \oplus \varepsilon')$ are isomorphic to the sphere bundles $\mathcal{F}_L \to L$, and $\mathcal{F}_{L_p} \to L_p$, respectively. The trivial line bundles $\varepsilon, \varepsilon'$ correspond to the sections $q \to (q, \ell)$ and $h \to (p, h)$, respectively. Moreover, there are homeomorphisms $E(\eta) \to p^\perp - \{p\}, E(\eta') \to \ell^\perp - \{\ell\}$ that take the fibers of the vector bundles onto the punctured point rows through $p$ and the punctured pencils of lines through $\ell$, respectively. Note that $p^\perp$ and $\mathcal{F}_{L_p}$ are completely determined, once the vector bundle $\eta$ is known.

Let $\eta_\mathbb{R}$ and $\eta_\mathbb{C}$ denote the real and the complex Hopf vector bundle over the $1$- and the $2$-sphere, respectively. The Thom spaces of these bundles are of course the real and the complex projective plane.

Up to duality, i.e. up to exchanging $\mathcal{P}$ and $\mathcal{L}$, precisely one of the following cases occurs:

(i) $\mathcal{P}$ is a $2$-dimensional projective plane (thus $m = m' = 1$), and the bundles $\eta \cong \eta' \cong \eta_\mathbb{R}$ are isomorphic to the real Hopf vector bundle. Thus $\mathcal{F}_L \cong \mathcal{F}_{L_p}$ is the Klein bottle.

(ii) $\mathcal{P}$ is a $4$-dimensional projective plane (thus $m = m' = 2$), and the bundles $\eta \cong \eta' \cong \eta_\mathbb{C}$ are isomorphic to the complex Hopf vector bundle.

(iii) $\mathcal{P}$ is a $3$-dimensional quadrangle (thus $m = m' = 1$). Then $\eta \cong \eta_\mathbb{R}$ is isomorphic to the real Hopf vector bundle, and $\eta'$ is the trivial line bundle over $S^1$. Hence $\mathcal{F}_{L_p}$ is the Klein bottle, and $\mathcal{F}_L$ is a $2$-torus.

(iv) $\mathcal{P}$ is a $6$-dimensional quadrangle (thus $m = m' = 2$). Then $\eta \cong \eta_\mathbb{C}$ is isomorphic to the complex Hopf vector bundle, and $\eta'$ is isomorphic to the tangent bundle of the $2$-sphere.

(v) $\mathcal{P}$ is a $5$-dimensional hexagon (thus $m = m' = 1$). Then the bundles $\eta \cong \eta' \cong \eta_\mathbb{R}$ are isomorphic to the real Hopf vector bundle. Thus $\mathcal{F}_L \cong \mathcal{F}_{L_p}$ is the Klein bottle.

(vi) $\mathcal{P}$ is a $10$-dimensional hexagon (thus $m = m' = 2$). Then $\eta \cong \eta_\mathbb{C}$ is isomorphic to the complex Hopf vector bundle, and $\eta' \cong \eta_\mathbb{C} \otimes \mathbb{C}^3$ is isomorphic to the unique $2$-plane bundle over $S^2$ that is obtained by pulling back the complex Hopf vector bundle by a map $S^2 \to S^2$ of degree $3$.

Proof. Consider the locally trivial $S^m$-bundle $\mathcal{F}_{L_p} \to L_p$. By 6.2.4, the bundle $\mathcal{F}_{L_p} \to L_p$ is the sphere bundle of an orthogonal vector bundle $\xi$. The map $\ell \mapsto (p, \ell)$ provides a section. Let $\varepsilon$ denote the corresponding trivial line bundle, and put $\eta = \varepsilon^\perp$. 
Now the total space of \( \eta \) is homeomorphic to \( \mathcal{F}_{\eta^{'}} - (\{p\} \times L_p) \) (by a stereographic projection in the fibers), which in turn is homeomorphic to \( p^{'-} - \{p\} \). Thus the Thom space of \( \eta \) has the same cohomology as \( p^{'-} \), which is known by \( \ref{3.2.3} \) and \( \ref{6.3} \).

In the case \( m = m' = 1 \), there are just two line bundles over \( S^2 \), namely the trivial bundle and the Möbius bundle. For \( n = 3, 6 \), we have \( H^*(\ell^1; \mathbb{Z}_2) \cong H^*(p^1; \mathbb{Z}_2) = \mathbb{Z}_2[x_1]/(x_1^2) \), whereas for \( n = 4 \), we have \( H^*(p^1; \mathbb{Z}_2) = \mathbb{Z}_2[x_1]/(x_1^3) \), and \( H^*(\ell^1; \mathbb{Z}_2) = \mathbb{Z}_2[x_1, x_2]/(x_1^2, x_2, x_1x_2) \). This completes the case \( m = m' = 1 \).

The oriented 2-plane bundles over \( S^2 \) are classified by their Euler class, cp. \( \ref{6.2.4} \). For \( n = 3 \), we have \( H^*(\mathcal{L}) \cong H^*(\mathcal{P}) = \mathbb{Z}[x_2]/(x_2^2) \). For \( n = 4 \), we have \( H^*(p^1) =\mathbb{Z}[x_2]/(x_2^3) \), and \( H^*(\ell^1; \mathbb{Z}) = \mathbb{Z}[x_2, x_4]/(x_1^2, x_4 - 2x_2^2, x_2x_4) \). For \( n = 6 \), we have \( H^*(p^1) = \mathbb{Z}[x_2]/(x_2^3) \), and \( H^*(\ell^1; \mathbb{Z}) = \mathbb{Z}[x_2, x_4]/(x_1^2, x_4 - 3x_2^2, x_2x_4) \).

Now we may apply \( \ref{6.2.5} \) to obtain the Euler class of the bundles \( \eta, \eta' \). \( \square \)

The flag spaces of low-dimensional projective planes have been classified by Breitsprecher \( \cite{Bre72} \) and Buchanan \( \cite{Buc78} \). With the aid of Knarr’s imbedding theorem, the proof becomes considerably simpler.

\subsection{4.3.2 Theorem}

Let \( \mathcal{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F}) \) be a compact connected \( 2m \)-dimensional projective plane. If \( m \in \{1, 2\} \), then the \( m \)-sphere bundle \( \mathcal{F} \to \mathcal{P} \) is isomorphic to its classical counterpart.

Proof. By \( \ref{6.2.7} \), there exists an orthogonal \((m + 1)\)-plane bundle \( \xi \) over \( \mathcal{P} \) that has \( \mathcal{F} \to \mathcal{P} \) as its sphere bundle.

If \( m = 1 \), the total Stiefel-Whitney class \( \omega \) of \( \xi \) is given by \( 1 + x_1 \), cp. \( \ref{4.2.2} \). The bundle \( \xi \) has no section by \( \ref{3.5.1} \), and \( \xi \) is not a sum of two copies of the tautological bundle \( \eta \) over \( \mathcal{P} \cong \mathcal{P}_2 \mathbb{R} \), because \( \omega(\eta \oplus \eta) = 1 + x_1^2 \), cp. \( \cite{MS74} \), p.43. Thus \( \xi \) does not split, see \( \ref{6.2.6} \). Consider the two-fold covering \( f : \mathbb{S}^2 \to \mathcal{P} \). It follows from \( \ref{6.2.6} \) that \( \xi \) is uniquely determined by its pullback \( f^*\xi \), and hence by the fundamental group of the sphere bundle \( S(f^*\xi) \), see \( \cite{Ste51} \), 26.2. Since \( \pi_1\mathcal{F} = Q_8 \), we conclude that \( \pi_1 S(f^*\xi) = \mathbb{Z}_4 \).

Now suppose that \( m = 2 \). According to \( \cite{DW59} \), 3., we have only to calculate the first Pontrjagin class and the second Stiefel-Whitney class of \( \xi \) in order to classify \( \mathcal{F} \). The total Pontrjagin class of \( \mathcal{P}_2 \mathbb{C} \) is \( 1 + 3x_2 \), \( \cite{MS74} \), p.178. By \( \ref{6.2.10} \), we get \( p(\xi) = 1 - 3x_2 \). The total Stiefel-Whitney class of \( \xi \) is \( 1 + x_m \) by \( \ref{4.2.3} \).

Finally, we want to calculate the fundamental groups for the case \( m = m' = 1 \).

\subsection{4.3.3 Proposition}

Let \( \mathcal{P} \) be a finite-dimensional polygon with parameters \((1, 1)\). Then the fundamental groups of \( \mathcal{P}, \mathcal{L} \), and \( \mathcal{F} \) are as given in \( \ref{3.4.1} \).

Proof. Since \( m, m' \leq 2 \), the polygon is a manifold, and we may use the CW decomposition of \( \ref{4.1.3} \). The 2-skeleton of the point space is \( p^{'-} \), and the 2-skeleton of the line space is \( \ell^{'-} \). Thus we get the fundamental groups of \( \mathcal{P} \) and \( \mathcal{L} \) by \( \ref{4.3.1} \).
The 1-skeleton of the flag space is \( \{p\} \times \mathcal{L}_p \cup L \times \{\ell\} \cong S^1 \vee S^1 \). The 2-skeleton of \( \mathcal{F} \) is \( \mathcal{F}_{\mathcal{L}_p} \cup \mathcal{F}_L \), and \( \mathcal{F}_{\mathcal{L}_p} \cap \mathcal{F}_L = \{p\} \times \mathcal{L}_p \cup L \times \{\ell\} \sim S^1 \vee S^1 \). Consider the diagram

\[
\begin{array}{ccc}
\{p\} \times \mathcal{L}_p \cup L \times \{\ell\} & \overset{\sim}{\longrightarrow} & S^1 \vee S^1 \\
\downarrow & & \downarrow \\
\mathcal{F}_{\mathcal{L}_p} & & \mathcal{F}_L.
\end{array}
\]

Passing to the fundamental groups, we get a diagram

\[
\langle a, b \rangle \overset{\sim}{\longrightarrow} \langle a, b \mid R_1 \rangle \overset{\sim}{\longrightarrow} \langle a, b \mid R_2 \rangle
\]

where \( R_1 = aba^{-1}b \) and \( R_2 = bab^{-1}a \) for \( n = 3, 6 \), and \( R_1 = aba^{-1}b \) and \( R_2 = bab^{-1}a^{-1} \) for \( n = 4 \) by \([4.3.1]\).

By the Seifert-Van Kampen Theorem we get the result stated in \([3.4.1]\). \(\square\)

### 4.4 A Lemma on partial polygons

Suppose we are given an incidence structure \( \mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F}) \) which is a partial \( n \)-gon. We want to show that if \( \mathcal{P} \) and \( \mathcal{L} \) carry nice topologies such that \( (\mathcal{P}, \mathcal{L}, \mathcal{F}) \) 'looks like a compact \( n \)-gon', then it is indeed a compact \( n \)-gon. The next lemma makes this more precise.

#### 4.4.1 Lemma

Let \((\mathcal{P}, \mathcal{L}, \mathcal{F})\) be a thick partial \( n \)-gon. Suppose that \( \mathcal{P}, \mathcal{L}, \) and \( \mathcal{F} \) are compact connected manifolds, and that \( \text{pr}_1 \) and \( \text{pr}_2 \) are locally trivial bundle maps, with \( m' \)- and \( m \)-manifolds as fibers. If the dimension of \( \mathfrak{P} \) is right, that is, if

\[
\dim \mathcal{F} = \frac{n(m + m')}{2} = m_n(p, \ell) = m_n(\ell, p),
\]

then \( \mathfrak{P} \) is a compact connected \( n \)-gon, and in fact a manifold.

**Proof.** Let \((p, \ell) \in \mathcal{F}\) be an arbitrary flag. Since \( \text{pr}_1, \text{pr}_2 \) are locally trivial bundles, the iterated bundle \( \text{Gall}_n(p, \ell) \) is a compact \( n(m + m')/2 \)-dimensional manifold. The subset \( \text{PropGall}_n(p, \ell) \) is nonempty and open, so we may choose a connected component \( M \) of \( \text{Gall}_n(p, \ell) \) that contains an open nonempty set \( U \) of non-stammering galleries. Since \( \mathfrak{P} \) is a partial \( n \)-gon, the map \( (x_0, \ldots, x_{n+1}) \mapsto \text{fl}(x_n, x_{n+1}) \) maps \( U \) injectively onto its image in \( \mathcal{F} \), and the preimage of the image of \( U \) is precisely \( U \). By \([5.5.2]\), the map is surjective. Thus \( d(\ell, z) \leq n \) for every vertex \( z \in \mathcal{V} \). Similarly, \( d(p, z) \leq n \) for every \( z \in \mathcal{V} \). Since \((p, \ell)\) was an arbitrary flag, the diameter of \( \mathfrak{P} \) is \( n \), and thus \( \mathfrak{P} \) is a generalized \( n \)-gon. By \([2.5.4]\), it is a compact \( n \)-gon, and hence a manifold. \(\square\)

The topological assumptions of this lemma are certainly satisfied, if \( \mathfrak{P} \) is a compact coset geometry of the right dimension of a connected Lie group, or if \( \mathfrak{P} \) comes...
from an isoparametric hypersurface. Thus, the main property that has to be checked for this kind of geometries in order to show that they are compact $n$-gons is the non-existence of ordinary $k$-gons for $2 \leq k < n$. 
CHAPTER 4. POLYGONS WHICH ARE MANIFOLDS
Chapter 5

Point homogeneous polygons

In this chapter we consider compact polygons with point transitive automorphism groups. Point homogeneous compact projective planes of dimension at most 16 have been classified by Salzmann; it turns out that these planes are precisely the four classical compact connected Moufang planes [Sal75]. Later, Löwen showed that the dimension of a point homogenous (and thus finite-dimensional) compact projective plane is at most 16 [Löw83]. Löwen gave also a second proof for the classification (for $m > 1$) which made strong use of the Borel-De Siebenthal classification of maximal subgroups of maximal rank in compact simple Lie groups and the fact that the Euler characteristic of a projective plane is 3 [Löw81]. His proof uses some facts about involutive automorphisms of projective planes; however, it turns out that the differential-geometric properties of the group action are already ‘good enough’ to reconstruct the plane uniquely from the group.

Generalized hexagons with $m > 1$ have Euler characteristic 6, and generalized quadrangles with $m = m' > 1$ have Euler characteristic 4 (generalized quadrangles with $m \neq m'$ have Euler characteristic 0). Thus it seems reasonable to try to classify the point homogenous polygons of positive Euler characteristic.

On the other hand, if both parameters $m = m' = 1$, then $n$ is an upper bound for the dimension of compact automorphism groups, and this fact leads to a classification of the point homogeneous polygons with parameters $(1, 1)$.

The main result of this chapter is Theorem 5.2.3: If a compact connected $n$-gon $\mathcal{P}$ admits a point transitive group of automorphisms $\Gamma$, and if $n \neq 4$ or if $m = m' > 1$, then $\Gamma$ is transitive on the flag space $\mathcal{F}$. By the main result of [GKK92], this implies that $\mathcal{P}$ is classical. In the case of quadrangles with parameters $m = m' = 1$, a point transitive group need not be flag transitive; however, the existence of a point transitive group of automorphisms implies that the quadrangle is the real symplectic quadrangle (5.2.7). Thus we obtain a complete classification of point homogeneous $n$-gons, provided that $n \neq 4$ or that $m = m'$ (5.2.8).

The notation for compact Lie groups is adapted from [GKK92]; thus $SO_k\mathbb{R}$ is
5.1 Compact transformation groups

5.1.1 Lemma Suppose that $U_k\mathbb{C}$ acts on $S^2$. If the action is not transitive, then there is a fixed point. If $k > 2$, then there is always a fixed point.

Proof. If the action is not transitive, then the orbits are circles or points. Thus, $SU_k\mathbb{C} < U_k\mathbb{C}$ is contained in the kernel of the action, and we have in fact an $S^1$-action on $S^2$. By [Spa66, 4.7.12], this action has a fixed point. For $k > 2$, there is no subgroup of codimension 2 and rank $k-1$ in $U_k\mathbb{C}$. □

5.1.2 Lemma Every action of $SO_5\mathbb{R} \times SO_2\mathbb{R}$ on a 2-sphere has a fixed point.

Proof. The action cannot be transitive, since otherwise the isotropy group would have the same rank as $SO_5\mathbb{R} \times SO_2\mathbb{R}$. But the largest subgroup of rank 3 is clearly $SO_4\mathbb{R} \times SO_2\mathbb{R}$. Hence $SO_5\mathbb{R}$ is contained in the kernel of the action, and we get an $SO_2\mathbb{R}$ action on $S^2$. Thus there is a fixed point by [Spa66, 4.7.12]. □

5.1.3 Theorem (Szenthe) [Sze74], [GKK94, 2.2] If a locally compact, connected, second countable group $\Gamma$ acts as an effective and transitive transformation group on a connected, locally contractible space $X$, then $\Gamma$ is a Lie transformation group, and $X \cong \Gamma/\Gamma_x$ is a (smooth) manifold. □

5.1.4 Theorem [Mon50] Let $X = \Gamma/\Gamma_x$ be a compact connected homogeneous space of a Lie group $\Gamma$. If the fundamental group of $X$ is finite, then every maximal compact connected subgroup of $\Gamma$ acts transitively on $X$. □

5.1.5 Theorem Let $X$ be a simply connected, compact manifold, with positive Euler characteristic $\chi(X) = n \leq 6$. Let $\Delta$ be a compact, connected, effective and transitive Lie transformation group on $X$, with isotropy group $\Delta_x$. The fact that $\chi(X) > 0$ implies that $\Delta_x$ has the same rank as $\Delta$, and that $\Delta_x$ is connected. Moreover, the group $\Delta$ is a product of centerless and simple compact Lie groups, and $\Delta_x$ factors accordingly [Wan49].

There are precisely the following possibilities (we list only the Lie algebras of the groups):

*(n = 2)* The group $\Delta$ is centerless and simple, and $\Delta_x$ is a maximal connected subgroup. There are only the following possibilities:

| $\Delta$ | $\Delta_x$ | dim $X$ |
|---|---|---|
| $\mathfrak{g}_2$ | $\mathfrak{a}_2$ | 6 | $X = S^6$ |
| $\mathfrak{g}_2$ | $\mathfrak{a}_k$ | $2k$ | $X = S^{2k}$ |
(n = 3) The group $\Delta$ is centerless and simple, and $\Delta_x$ is a maximal connected subgroup. There are only the following possibilities:

| $\Delta$ | $\Delta_x$ | $\text{dim } X$ |
|----------|------------|----------------|
| $a_2$ | $a_1 + \mathbb{R}$ | 4 | $X = P_2\mathbb{C}$ |
| $c_3$ | $c_1 + c_2$ | 8 | $X = P_2\mathbb{H}$ |
| $f_4$ | $b_4$ | 16 | $X = P_2\mathbb{O}$ |
| $g_2$ | $a_1 + a_1$ | 8 | |

(n = 4) If $\Delta$ is not simple, then we have a product of two transformation groups listed in $(n = 2)$.

Otherwise, $\Delta$ is centerless and simple. If $\Delta_x$ is a maximal connected subgroup, then there are only the following possibilities:

| $\Delta$ | $\Delta_x$ | $\text{dim } X$ |
|----------|------------|----------------|
| $a_3$ | $a_2 + \mathbb{R}$ | 6 | $X = P_3\mathbb{C}$ |
| $c_2$ | $a_1 + \mathbb{R}$ | 6 | |
| $c_4$ | $c_1 + c_3$ | 12 | $X = P_3\mathbb{H}$ |

Otherwise, there is a maximal connected subgroup $\Phi$ with $\Delta > \Phi > \Delta_x$, and $\Delta/\Phi$, $\Phi/\Delta_x$ are even-dimensional spheres. There is only the following possibility:

| $\Delta$ | $\Phi$ | $\Delta_x$ | $\text{dim } X$ |
|----------|--------|------------|----------------|
| $c_2$ | $a_1 + a_1$ | $a_1 + \mathbb{R}$ | 6 |

(n = 5) The group $\Delta$ is centerless and simple, and $\Delta_x$ is a maximal connected subgroup. There are only the following possibilities:

| $\Delta$ | $\Delta_x$ | $\text{dim } X$ |
|----------|------------|----------------|
| $a_4$ | $a_3 + \mathbb{R}$ | 8 | $X = P_4\mathbb{C}$ |
| $c_5$ | $c_1 + c_4$ | 16 | $X = P_4\mathbb{H}$ |

(n = 6) If $\Delta$ is not simple, then we have a product of a transformation group listed in $(n = 2)$ with a transformation group listed in $(n = 3)$.

Otherwise, $\Delta$ is centerless and simple. If $\Delta_x$ is a maximal connected subgroup, then there are only the following possibilities:

| $\Delta$ | $\Delta_x$ | $\text{dim } X$ |
|----------|------------|----------------|
| $a_3$ | $a_1 + a_1 + \mathbb{R}$ | 8 | |
| $a_5$ | $a_4 + \mathbb{R}$ | 10 | $X = P_5\mathbb{C}$ |
| $b_3$ | $b_1 + \mathbb{O}$ | 12 | |
| $b_3$ | $b_2 + \mathbb{R}$ | 10 | |
| $c_4$ | $c_2 + c_2$ | 24 | |
| $c_6$ | $c_1 + c_5$ | 20 | $X = P_5\mathbb{H}$ |
Otherwise, there is a maximal connected subgroup $\Phi$ with $\Delta > \Phi > \Delta_x$, and $\Delta/\Phi$, $\Phi/\Delta_x$ are among the spaces listed in $(n = 2)$ and $(n = 3)$ (note, however, that $\Phi$ need not be effective on $\Phi/\Delta_x$). There are only the following possibilities:

\[
\begin{array}{|c|c|c|c|}
\hline
\Delta & \Phi & \Delta_x & \text{dim } X \\
\hline
\alpha_2 & \alpha_1 + \mathbb{R} & \mathbb{R} + \mathbb{R} & 6 \\
\gamma_3 & \gamma_1 + \gamma_2 & \gamma_1 + \gamma_1 + \gamma_1 & 12 \\
\gamma_3 & \gamma_1 + \gamma_2 & \mathbb{R} + \gamma_2 & 10 \\
\gamma_4 & \gamma_4 & \gamma_4 & 24 \\
\beta_2 & \beta_2 & \alpha_1 + \mathbb{R} & 10 \\
\beta_2 & \alpha_1 + \alpha_1 & \alpha_1 + \mathbb{R} & 10 \\
\beta_2 & \alpha_1 + \alpha_1 & \mathbb{R} + \alpha_1 & 10 \\
\hline
\end{array}
\]

\textbf{Proof.} This follows from the Borel-De Siebenthal classification of maximal subgroups of maximal rank. See [Wan49], [BDS49], [Wol84, 8.10], and [GKK94, 2.4 and Table 1].

\section{5.2 The classification}

\textbf{5.2.1 Theorem (Burns-Spatzier) [BS87, 2.1]} Let $\mathcal{P}$ be a compact connected polygon. The group of all continuous automorphisms of $\mathcal{P}$, endowed with the compact-open topology, is a locally compact, second countable topological transformation group.

\textbf{5.2.2 Lemma [GKK94, 3.2]} Let $\mathcal{P}$ be a finite-dimensional polygon with parameters $m = m' = 1$. Suppose a compact Lie group $\Delta$ acts as an effective automorphism group on $\mathcal{P}$. Then the isotropy group $\Delta_{p,\ell}$ of every flag $(p, \ell)$ is finite; hence $\dim \Delta \leq n$, and if $\dim \Delta = n$, then $\Delta$ is flag transitive.

It follows that the isotropy group $\Delta_p$ of a point (or a line) is at most one-dimensional, and if $\Delta_p$ is one-dimensional, then it is transitive on the pencil of lines $\mathcal{L}_p$.

\textbf{Proof.} Let $(p, \ell)$ be a flag. The connected component $(\Delta_{p,\ell})^1$ has a fixed point on $L \times \{\ell\} \cong S^1$ and on $\{p\} \times \mathcal{L}_p \cong S^1$; thus it fixes every flag on $\{p\} \times \mathcal{L}_p \cup L \times \{\ell\}$. But this implies that $(\Delta_{p,\ell})^1$ fixes every flag.

\textbf{5.2.3 Theorem} Let $\mathcal{P}$ be a compact, connected $n$-gon. Suppose that a closed subgroup $\Gamma$ of the automorphism group of $\mathcal{P}$ acts transitively on the point space $\mathcal{P}$. This implies that $\mathcal{P}$ is finite dimensional, so the topological parameters $(m, m')$ of $\mathcal{P}$ are defined.
5.2. THE CLASSIFICATION

If \( n \neq 4 \), or if \( m = m' > 1 \), then the group \( \Gamma \) is transitive on \( \mathcal{F} \). By the main result (Theorem 3.8) of [GKK94], the polygon \( \mathcal{P} \) is a compact connected Moufang polygon. (The point transitive groups are listed explicitly in [GKK94].)

Proof. (1) By 5.1.3, the group \( \Gamma \) is a Lie group, and \( \mathcal{P} \) is a homogeneous space. Hence \( \mathcal{P} \) is finite-dimensional by 3.1.3.

By 3.4.11, the point space \( \mathcal{P} \) has a finite fundamental group in the cases that we consider. Thus, a compact connected subgroup \( \Delta \) of \( \Gamma \) is still transitive on \( \mathcal{P} \) by 5.1.4.

(2) Let \( p \) be a point. The isotropy group \( \Delta_p \) fixes no line \( \ell \in \mathcal{L}_p \), because otherwise the map \( \delta p \mapsto (\delta p, \delta \ell) \), with \( \delta \in \Delta \), would provide a section of the bundle \( \mathcal{F} \to \mathcal{P} \), contradicting 3.5.1.

(3) Now suppose that \( m = m' = 1 \). Then \( n = 3, 6 \), and \( \dim \mathcal{P} = n - 1 \leq \dim \Delta \).

If \( \dim \Delta = n - 1 \in \{2, 5\} \), then \( \Delta \) has an infinite fundamental group, and \( \Delta \to \Delta / \Delta_p \) is a covering. But the fundamental group of \( \mathcal{P} \) is \( \mathbb{Z}_2 \). Thus \( \dim \Delta = n \), and by 5.2.2, \( \Delta \) is transitive on \( \mathcal{F} \).

(4) The remaining cases are \( n = 3, 4, 6 \), and \( m = m' > 1 \). Thus \( \mathcal{P} \) is simply connected, and we may apply 5.1.3, since the Euler characteristic of \( \mathcal{P} \) is \( n = 3, 4, 6 \). Let \( p \) be a point, and let \( \Delta_p \) be the stabilizer of \( p \).

If \( n = 3 \), then \( (\Delta, \Delta_p) \in \{ (\text{PSU}_3 \mathbb{C}, \text{U}_3 \mathbb{C}), (\text{PU}_3 \mathbb{H}, \text{PU}_1 \mathbb{H} \times \text{U}_2 \mathbb{H}), (\text{F}_4, \text{Spin}_9), (\text{G}_2, \text{SO}_4 \mathbb{R}) \} \). Since \( \pi_2 G_2 = 0 = \pi_1 \mathcal{P} \), it follows that \( \pi_2 (G_2 / \text{SO}_4 \mathbb{R}) \cong \pi_1 \text{SO}_4 \mathbb{R} = \mathbb{Z}_2 \), and this excludes this space as a candidate for an 8-dimensional projective plane. Thus the point space \( \mathcal{P} \) is homeomorphic to the point space of one of the four classical compact connected planes. It is readily seen that in each of the three cases, \( \Delta \) contains no proper closed subgroups of codimension \( \leq 2m \) besides the conjugates of \( \Delta_p \). Thus, \( \Delta \) is also transitive on the line space, and \( \Delta_p \) fixes some line \( h \). But \( \Delta_p \) has only one orbit \( X \) of dimension \( \leq m \) in \( \mathcal{P} - \{p\} \), namely the cut locus of \( p \) (because \( \mathcal{P} \) is a compact symmetric space of rank 1 under the action of \( \Delta \)), and this set is precisely the polar line of \( p \) with respect to the elliptic polarity. Hence \( H = X \) is a classical point row, and therefore each point row of \( \mathcal{P} \) is a point row of the classical plane. It is well known that the elliptic motion groups of the four classical compact connected Moufang planes are flag transitive.

Now let \( n = 4 \). It follows from the structure of \( H^*(\mathcal{P}) \) that \( \mathcal{P} \) is not a product of two even-dimensional spheres (see also 3.3.8). Thus we have to consider the simple groups listed in 5.1.3. The group \( \Delta_p \) acts without fixed points on the homology \( m \)-sphere \( \mathcal{L}_p \) by (2). The point space \( \mathcal{P} \) and the line space \( \mathcal{L} \) are 3m-dimensional, and \( m \in \{2, 4\} \).

The pair \( (\text{PSU}_4 \mathbb{C}, \text{U}_3 \mathbb{C}) \) is excluded by 5.1.1 and (2).

The group \( \text{PU}_4 \mathbb{H} \) has (up to conjugation) only one subgroup of codimension \( \leq 12 \), as is easily seen. Thus, if \( \text{PU}_4 \mathbb{H} \) is point transitive, then it is line transitive.
as well, and $\mathcal{P}$ and $\mathcal{L}$ are homeomorphic. But the cohomology rings of $\mathcal{P}$ and $\mathcal{L}$ are not isomorphic, and this excludes the pair $(\text{PU}_4 \mathbb{H}, (\text{U}_3 \mathbb{H} \times \text{U}_1 \mathbb{H})/\pm 1)$.

In the two remaining cases, $\Delta_p = \text{U}_2 \mathbb{C}$. It follows from (2) and [5.1.1] that $\Delta_p$ is transitive on $\mathcal{L}_p$, and hence $\Delta = \text{U}_2 \mathbb{H}$ is transitive on $\mathcal{F}$. (There are two possibilities for the imbedding $\Delta_p \subseteq \Delta$, corresponding to the complex symplectic quadrangle and its dual.)

Finally, let $n = 6$. Then $\dim \mathcal{P} = \dim \mathcal{L} = 5m \in \{10, 20\}$. It is clear from the structure of $H^\bullet(\mathcal{P})$ that the action of $\Delta$ does not factor as a product (see also [3.3.8]), so we have again only to consider the simple groups listed in [5.2.1].

The groups $\text{U}_6 \mathbb{C}$ and $\text{SO}_5 \mathbb{R} \times \text{SO}_2 \mathbb{R}$ are excluded as candidates for $\Delta_p$ by (2) and [5.1.1].

The groups $\text{PU}_3 \mathbb{H}$ and $\text{PU}_6 \mathbb{H}$ have only one conjugacy class of subgroups of codimension $\leq 10$ and $\leq 20$, respectively. Hence, if one of these groups is point transitive, then it is line transitive, and $\mathcal{P}$ and $\mathcal{L}$ are homeomorphic. But the cohomology rings of $\mathcal{P}$ and $\mathcal{L}$ are not isomorphic.

Thus, $\Delta = \text{G}_2$ and $\Delta_p = \text{U}_2 \mathbb{C}$. It follows as in the case $(n = 4)$ from (2) and [5.1.1] that $\Delta$ is flag transitive (there are again two imbeddings $\text{U}_2 \mathbb{C} \subseteq \text{G}_2$, due to the fact that the complex hexagon is not self-dual).

The group $\text{SO}_3 \mathbb{R}$ acts as a sharply point transitive group on the real symplectic quadrangle. Thus we have to modify our assumptions for $(1,1)$-quadrangles.

5.2.4 Theorem Let $\Psi$ be a finite-dimensional quadrangle with parameters $(1,1)$. If its automorphism group contains a semi-simple compact subgroup $\Delta$ of positive dimension, then $\Psi$ is (up to duality) the real symplectic quadrangle.

Proof. (cp. the proof of [GKK94, 3.4] due to Knarr) By [5.2.2] we have $\dim \Delta \leq 4$, and by assumption $\dim \Delta \geq 3$. Thus we may assume that $\Delta$ is connected and of type $a_1$. Passing to the dual quadrangle, if necessary, we may assume moreover that $\pi_1 \mathcal{P} = \mathbb{Z}_2$, and that $\pi_1 \mathcal{L} = \mathbb{Z}$.

Let $\ell \in \mathcal{L}$ be a line. Since $\Delta$ cannot be transitive on $\mathcal{L}$, the isotropy group $\Delta_\ell$ has positive dimension. By [5.2.3] the isotropy group $\Delta_\ell$ is transitive on the point row $L$. Since this is true for every point row, $\Delta$ is transitive on $\mathcal{P}$, and thus $\Delta \cong \text{SO}_3 \mathbb{R}$, and $\mathcal{P} \cong \mathcal{P}_3 \mathbb{R} \cong \text{SO}_3 \mathbb{R}$. Every point row in $\mathcal{P}$ is a translate of a 1-parameter group in $\text{SO}_3 \mathbb{R}$ and thus a classical point row in $\mathcal{P}_3 \mathbb{R} \cong \text{SO}_3 \mathbb{R}$. By the result of Dienst [Die80] on quadrangles which are imbeddable in projective spaces, $\Psi$ is the real symplectic quadrangle. \hfill \square

5.2.5 Lemma Let $\Gamma$ be a connected Lie group, and let $M = \Gamma/\Gamma_x$ be a homogeneous space of $\Gamma$, with fundamental group $\pi$. Let $\Delta$ be a maximal compact subgroup of $\Gamma$. If $\Delta$ contains no almost-simple factor (i.e. if $\Delta$ is trivial or a torus), then $M$ is
an Eilenberg-MacLane space $K(\pi, 1)$, and thus the manifold $M$ has the same (co-) homology as the group $\pi$.

**Proof.** Assume that $\Delta$ is a torus. Passing to the universal covering group, we may assume that $\Gamma$, as well as every closed connected subgroup of $\Gamma$, is contractible \cite[V 3.1]{Hoc63}. It follows from the homotopy sequence of $\Gamma_x \to \Gamma \to M$ that the homotopy groups of $M$ vanish in dimension $\geq 2$. \hfill $\blacksquare$

### 5.2.6 Corollary

Let $\mathfrak{P}$ be a compact connected polygon. Let $\Gamma$ be a Lie group that acts as a point transitive automorphism group on $\mathfrak{P}$. Then $\Gamma$ contains a compact semi-simple subgroup of positive dimension.

**Proof.** Since $\mathcal{P}$ is homogeneous, $\mathfrak{P}$ is finite-dimensional, and $\pi_1\mathcal{P}$ is cyclic. The homology of the point space is not isomorphic to the homology of a cyclic group, cp. \cite[V 7.6]{Whi78}. \hfill $\blacksquare$

### 5.2.7 Corollary

Let $\mathfrak{P}$ be a compact connected quadrangle with a point transitive automorphism group. If $\mathfrak{P}$ has equal parameters $m = m' = 1$, then $\mathfrak{P}$ is the real symplectic quadrangle.

**Proof.** Since $\mathcal{P}$ is homogeneous, $\mathfrak{P}$ is finite-dimensional, and $\pi_1\mathcal{P}$ is cyclic. The homology of the point space is not isomorphic to the homology of a cyclic group, cp. \cite[V 7.6]{Whi78}. \hfill $\blacksquare$

### 5.2.8 Corollary

Let $\mathfrak{P}$ be a compact connected $n$-gon. Suppose that a group of continuous automorphisms acts transitively on the point space $\mathcal{P}$. Then $\mathfrak{P}$ is finite dimensional. If $n \neq 4$, or if $m = m'$, then $\mathfrak{P}$ is (up to duality) one of the following polygons:

3$\mathbb{R}$ The real projective plane.
3$\mathbb{C}$ The complex projective plane.
3$\mathbb{H}$ The quaternion projective plane.
3$\mathbb{O}$ The octonion (Cayley) projective plane.
4$\mathbb{R}$ The real symplectic quadrangle.
4$\mathbb{C}$ The complex symplectic quadrangle.
6$\mathbb{R}$ The real Moufang hexagon (associated with $G_{2(3)}$).
6$\mathbb{C}$ The complex Moufang hexagon (associated with $G_{2}^{\mathbb{C}}$).

Ferus-Karcher-Münzner have constructed point homogeneous quadrangles (with $m \neq m'$) which are not Moufang \cite{FKM81, Tho92}. \hfill $\blacksquare$
Chapter 6

Miscellanea

What I tell you three times is true.
L. Carroll, Hunting of the snark

In this chapter we collect some topological results which are needed in various proofs, but which are not directly connected with the geometry of a topological polygon.

6.1 Cofibrations and ANRs

6.1.1 Definition Let \((X, A)\) be a topological pair. Recall that the inclusion \(A \subseteq X\) is called a cofibration, if the extension problem

\[
A \times \mathbb{I} \cup X \times \{0\} \to Y
\]

\[
\downarrow
\]

\[
X \times \mathbb{I}
\]

has a solution for every space \(Y\) [DKP70, I], [Whi78, I 5], [Dek91, II 3], [Bre93, VII 1]. The cofibration is called closed, if \(A\) is a closed subspace. If \(X\) is a Hausdorff space, then \(A\) is necessarily closed [DKP70, 1.17].

A pointed space \((X, \ast)\) is called well-pointed, if \(\{\ast\} \subseteq X\) is a cofibration.

An inclusion \(A \subseteq X\) is a cofibration if and only if \(A \times \mathbb{I} \cup X \times \{0\}\) is a retract of \(X \times \mathbb{I}\) [DKP70, 1.22].

6.1.2 Proposition Consider the cocartesian diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
X & \to & X \cup_f B.
\end{array}
\]
CHAPTER 6. MISCELLANIES

If $A \subseteq X$ is a cofibration, then the pushout $B \to X \cup_f B$ is also a cofibration \([\text{DKP70}, 7.36]\).

\[ \text{\textbf{6.1.3 Proposition}} \]

If $A \subseteq X$ is a closed cofibration, then $(X/A, *)$ is well-pointed. If $A$ is contractible, then $(X, A) \to (X/A, *)$ is a homotopy equivalence \([\text{DKP70}, 2.36]\).

\[ \text{\textbf{6.1.4 Proposition}} \]

If $A \subseteq X$ is a closed cofibration, then the collapsing map $(X, A) \to (X/A, *)$ induces an isomorphism in homology and cohomology \([\text{Bre93}, \text{VII 1.7}]\).

In particular, if $(X, x)$ and $(Y, y)$ are well-pointed disjoint spaces, then the map $(X \cup Y, \{x, y\}) \to (X \vee Y, *)$ induces an isomorphism in (co-)homology, i.e. the (co-)homology of the one-point union $(X \vee Y, *)$ is the direct sum of the (co-)homology of $(X, x)$ and $(Y, y)$.

\[ \text{\textbf{6.1.5 Corollary}} \]

If $(X_1, x_1)$, $(X_2, x_2)$ are well-pointed homology and cohomology spheres of dimension $m$ and $m'$, respectively, then $(X \wedge Y, *)$ is a homology and cohomology sphere of dimension $m + m'$.

Proof. The subspaces $X_1 \times \{x_2\}, \{x_1\} \times X_2$ form an excisive couple, since the inclusion $(X_1 \times \{x_2\}, \{x_1, x_2\}) \subseteq (X_1 \vee X_2, \{x_1\} \times X_2)$ induces an isomorphism in singular homology by 6.1.4, see \([\text{Spa66}, 4.6.4] \). By the Künneth Theorem \([\text{Spa66}, 5.3.10] \), we get an isomorphism $[H_*(X_1, x_1) \otimes H_*(X_2, x_2)] \cong H_*(X_1 \times X_2, X_1 \vee X_2)$. The inclusion $X_1 \vee X_2 \subseteq X_1 \times X_2$ is a cofibration by \([\text{DKP70}, 3.20] \), hence we get an isomorphism $H_*(X_1 \times X_2, X_1 \vee X_2) \cong H_*(X_1 \wedge X_2, *)$. Thus $X_1 \wedge X_2$ is a homology $(m + m')$-sphere. It follows from the universal coefficient theorem \([\text{Spa66}, 5.5.3] \) that $X_1 \wedge X_2$ is also a cohomology $(m + m')$-sphere.

\[ \text{\textbf{6.1.6 Theorem (Homotopy Excision Theorem)}} \]

Suppose $X = A \cup B$ is the union of two closed subspaces $A, B$. If the intersection $A \cap B$ is a strong deformation retract of some neighborhood in $A$, and if the pairs $(A, A \cap B)$ and $(B, A \cap B)$ are $n$- and $m$-connected, respectively, where $m, n \geq 0$, then the map

$$\pi_r(B, A \cap B) \to \pi_r(X, A)$$

is an isomorphism for $0 \leq r \leq n + m - 1$, and an epimorphism for $r = m + n$. \(\square\)
6.1.7 Corollary Let $A \subseteq X$ be a closed cofibration. If the pair $(X, A)$ is $m$-connected, and if $A$ is $n$-connected, for $m, n \geq 0$, then the collapsing map induces isomorphisms

$$\pi_r(X, A) \to \pi_r(X/A)$$

for $0 \leq r \leq m + n$, and an epimorphism for $r = m + n + 1$. In particular, $X/A$ is $m$-connected.

Proof. Let $C'_A$ denote the unreduced cone over $A$. We may apply the Homotopy Excision Theorem 6.1.6 to the space $X \cup C'_A$. Since $C'_A$ is contractible, the pair $(C'_A, A)$ is $(n + 1)$-connected. Thus the inclusion $(X, A) \subseteq (X \cup C'_A, C'_A)$ induces an isomorphism for the homotopy groups in dimension $\leq n + m$, and an epimorphism in dimension $m + n + 1$. The collapsing map $(X \cup C'_A, C'_A) \to (X/A, *)$ is a homotopy equivalence by 6.1.3. \qed

6.1.8 Definition A metrizable space $X$ is called an absolute neighborhood retract or ANR for short, if for every imbedding $X \subseteq M$ of $X$ as a closed subspace into a metrizable space $M$, there exists an open neighborhood $U$ of $X$ such that $X$ is a retract of $U$ \cite[III 6]{Hu65}.

Every locally contractible metrizable space of finite covering dimension is an ANR \cite[V 7.1]{Hu65}.

A closed subspace $A$ of an ANR $X$ is an ANR if and only if the inclusion $A \subseteq X$ is a cofibration \cite[IV 3.2]{Hu65}.

Suppose that $X, A, B$ are ANR’s, that $A \subseteq X$ is closed, and that $f : A \to B$ is continuous. If the pushout $X \cup_f B$ is metrizable, then it is an ANR \cite[VI 1.2]{Hu65}.

In particular, suppose that $Z = A \cup B$ is the union of two closed subsets. If $Z$ is metrizable, and if $A, B$, and $A \cap B$ are ANR’s, then $Z$ is an ANR.

Any ANR is homotopy equivalent to a CW complex \cite[p.218]{Web68}.

\qed

6.2 Bundles and homotopy

6.2.1 Proposition Let $X, Y$ be pointed spaces, and let $[X; Y]^0$ denote the set of all base-point preserving homotopy classes of maps from $X$ to $Y$. If $X$ is well-pointed, then there is a natural action of $\pi_1 Y$ on $[X; Y]^0$ \cite[7 3.4]{Spa66}. If $Y$ is path-connected, then the space $[X; Y]$ of all free (i.e. not necessarily base-point preserving) homotopy classes is the orbit space $[X; Y]^0/\pi_1 Y$ \cite[III 1.11]{Whi78} \cite[5.10.8]{Dek97}.

The pointed space $Y$ is called simple, if it is path-connected, and if the $\pi_1 Y$-action on $[X; Y]^0$ is trivial for all well-pointed spaces $X$. Thus we have $[X; Y]^0 \cong [X; Y]$ for simple spaces $Y$. Every path-connected $H$-space (in particular every Eilenberg-MacLane space) is simple \cite[7 3.5]{Spa66} \cite[III 4.18]{Whi78}.

\qed
6.2.2 Proposition Suppose \(Y, Y'\) are well-pointed spaces. Let \(f : Y \to Y'\) be a base-point preserving map. If \(f\) induces an isomorphism between all homotopy groups of \(Y\) and \(Y'\), then \(f\) induces an isomorphism \([X; Y]^0 \cong [X; Y']^0\) for every well-pointed CW complex \(X\). \[\text{Spal\text{\texttt{d}}, 7.14, 7.15}\].

6.2.3 Proposition Given a topological group \(G\), let \(BG\) denote its classifying space (unique up to homotopy equivalence). If \(X\) is a paracompact space, then the set of homotopy classes \([X; BG]\) is in one-to-one correspondence with the set of all isomorphism classes of principal \(G\)-bundles over \(X\). The total space of the universal principal \(G\)-bundle is contractible, thus \(\pi_j G \cong \pi_{j+1} BG\). \[\text{Hus94, I.4.12}, [Dol63].\]

6.2.4 Proposition Let \(X\) be a paracompact space having the homotopy type of a CW complex. The map that assigns to a line bundle over \(X\) its first Stiefel-Whitney class \(w_1\) is a bijection between the isomorphism classes of line bundles over \(X\) and \(H^1(X; \mathbb{Z}_2)\). The map that assigns to an oriented 2-plane bundle over \(X\) its Euler class \(e\) is a bijection between the isomorphism classes of oriented 2-plane bundles over \(X\) and \(H^2(X)\).

Proof. The classifying spaces \(BO(1)\) and \(BSO(2)\) are Eilenberg-MacLane spaces of type \(K(\mathbb{Z}_2, 1)\) and \(K(\mathbb{Z}, 2)\), respectively. We have to check that the maps

\[\begin{align*}
[X; BO(1)] & \xrightarrow{\cong} [X; K(\mathbb{Z}_2, 1)] \\
[X; BSO(2)] & \xrightarrow{\cong} [X; K(\mathbb{Z}, 2)]
\end{align*}\]

are bijections.

This is true if \(X\) is a sphere, see eg. \[\text{Dck91, IV 4.10}.\] In case that \(X\) is a CW complex, we get by the above remarks a commutative diagram

\[
\begin{array}{ccc}
[X; BSO(2)] & \xleftarrow{\cong} & [X; BSO(2)]^0 \\
\downarrow e & & \downarrow \cong \\
[X; K(\mathbb{Z}, 2)] & \xleftarrow{\cong} & [X; K(\mathbb{Z}, 2)]^0,
\end{array}
\]

and a similar diagram for \(BO(1)\). \[\text{Spal\text{\texttt{d}}, 7.14, 7.15}\].

6.2.5 Proposition Let \(\xi\) be an oriented 2k-plane bundle over a 2k-sphere \(S^{2k}\). Let \(x\) denote a generator of \(H^{2k}(S^{2k})\), and let \(e_\xi = ax\) denote the Euler class of \(\xi\). Then the cohomology ring of the Thom space \(T_\xi\) of \(\xi\) is given by

\[H^*(T_\xi) = \mathbb{Z}[u, v]/(v^2, u^2 - av, uv),\]
where \( u \) and \( v \) generate \( H^{2k}(T_\xi) \) and \( H^{4k}(T_\xi) \), respectively.

**Proof.** Let \( U_\xi \in H^{2k}(E, E_0) \) be the orientation class of \( \xi \), and let \( \theta : H^\bullet(S^{2k}) \to H^\bullet(E, E_0) \) denote the Thom isomorphism. We have the relation \( \theta(1) = U_\xi \), and \( \theta(e_\xi) = U^{2k}_\xi \) \([\text{MS74}, \text{p. 99}]\). On the other hand, \( H^{2k}(E, E_0) \) is generated by \( \theta(1) \), and \( H^{4k}(E, E_0) \) is generated by \( \theta(x) \). Finally, the reduced cohomology of \( T_\xi \) is naturally isomorphic to that of \( (E, E_0) \).

\( \square \)

### 6.2.6 Proposition \([\text{Lev63}], [\text{Dck91}, 5.13] \)

Let \( \xi \) be a 2-plane bundle over the real projective plane \( \mathbb{P}^2 \mathbb{R} \). Let \( \eta \) denote the tautological line bundle over \( \mathbb{P}^2 \mathbb{R} \), and let \( f : S^2 \to \mathbb{P}^2 \mathbb{R} \) denote the two-fold covering of \( \mathbb{P}^2 \mathbb{R} \).

If \( \xi \) splits as a sum of two line bundles, then \( \xi \) is isomorphic to one of the bundles \( \varepsilon^2, \varepsilon \oplus \eta, \eta \oplus \eta \). (This follows from \( 6.2.4 \) and the fact that \( H^1(\mathbb{P}^2 \mathbb{R}; \mathbb{Z}) = \mathbb{Z}_2 \).)

Otherwise, \( \xi \) is uniquely determined by its pullback \( f^*\xi \in \text{Vect}^2(S^2) \), and hence by the fundamental group of the sphere bundle \( S(f^*\xi) \), cp. \([\text{Ste51}, 26.2]\). \( \square \)

### 6.2.7 Theorem (Kneser) \([\text{Kne26}], [\text{Fri73}], [\text{KS77, V § 5}] \)

Let \( \text{Top} \left( S^m \right) \) denote the group of all homeomorphisms of the \( m \)-sphere. If \( m \leq 2 \), then the inclusion \( \text{O}(m + 1) \to \text{Top} \left( S^m \right) \) is a homotopy equivalence. Therefore the structure group of any locally trivial \( m \)-sphere bundle can be reduced to \( \text{O}(m + 1) \) in these dimensions, that is, there exists always an orthogonal \((m + 1)\)-plane bundle that has this sphere bundle as its unit sphere bundle.

\( \square \)

### 6.2.8 Theorem

Let \( \xi, \xi' \) be two \( k \)-plane bundles over a paracompact space \( X \). If the underlying topological \( \mathbb{R}^k \)-bundles are isomorphic, then the rational Pontrjagin classes of \( \xi \) and \( \xi' \) are the same.

**Proof.** The rational cohomology of \( \text{BO}(k) \) is a polynomial ring generated by the Pontrjagin classes of the universal \( k \)-plane bundle \([\text{MS74}, \text{p.182}]\). Since the Pontrjagin classes are stable \([\text{MS74}, 15.2]\), the cohomology of \( \text{BO} \) is a polynomial ring generated by the universal Pontrjagin classes, and \( H^\bullet(\text{BO}(k); \mathbb{Q}) \leftarrow H^\bullet(\text{BO}; \mathbb{Q}) \) is an epimorphism.

The homotopy fiber \( \text{TOP}/\text{O} \) is simply connected and has finite homotopy groups in every dimension \([\text{KS77, V § 5}]\). Thus \( H^\bullet(\text{TOP}/\text{O}; \mathbb{Q}) = 0 \) by \([\text{Spa64}, 9.6.15, 5.2.8, 5.5.3]\). Hence the spectral sequence of the fibration \( \text{BO} \to \text{BTOP} \) collapses, and we get an isomorphism \( H^\bullet(\text{BO}; \mathbb{Q}) \cong H^\bullet(\text{BTOP}; \mathbb{Q}) \), cp. also \([\text{MS74}, \text{Epilogue}]\).

Let \( f, f' \) be classifying maps for \( \xi \) and \( \xi' \), respectively. The classifying maps of the underlying \( \mathbb{R}^m \) bundles \( X \xrightarrow{f} \text{BO} \to \text{BTOP}(k) \) and \( X \xrightarrow{f'} \text{BO} \to \text{BTOP}(k) \) are homotopic, and the claim follows from the homotopy-commutative diagram

\[
\begin{array}{ccc}
X & \to & \text{BO}(k) & \to & \text{BO} \\
\downarrow & & \downarrow \\
 & \text{BTOP}(k) & \to & \text{BTOP}.
\end{array}
\]
6.2.9 Corollary (Novikov’s Theorem) Let $M$ be a smooth $k$-manifold. The rational Pontrjagin classes of $M$ do not depend on the differentiable structure, but only on the topological structure of $M$.

Proof. By Kister’s result [Kis64], the underlying $\mathbb{R}^k$ bundle of the tangent bundle of $M$ is independent of the differentiable structure of $M$. \qed

6.2.10 Corollary Let $\xi$ be a vector bundle over a smooth manifold $M$. If the total space of $\xi$ is homeomorphic to an open subset of some euclidean space, then the total rational Pontrjagin classes of $M$ and of $\xi$ are inverse to each other, i.e.

$$p(M)p(\xi) = 1.$$  

Proof. We may as well assume that $\xi$ is smooth [Hir76, 4.3.5]. The total Pontrjagin class of the total space $E(\xi)$ of $\xi$ vanishes by 6.2.9. Since $\xi$ is a normal bundle of $M \hookrightarrow E(\xi)$, it follows that $1 = p(TM \oplus \xi) = p(M)p(\xi)$, cp. [MS74, 15.3]. \qed

6.2.11 Proposition Let $E \rightarrow B$ be a locally trivial, orientable bundle. Assume that the fiber $F$ is a rational homology $n$-sphere. If one of the maps $H_n(F; \mathbb{Q}) \rightarrow H_n(E; \mathbb{Q})$ or $H_n(F; \mathbb{Q}) \leftarrow H_n(E; \mathbb{Q})$ is trivial, then $n$ is odd.

Proof. It follows from the universal coefficient theorem for field coefficients that $F$ is a rational cohomology $n$-sphere, and that the map $H^n(F; \mathbb{Q}) \leftarrow H^n(E; \mathbb{Q})$, being the adjoint of $H_n(F; \mathbb{Q}) \rightarrow H_n(E; \mathbb{Q})$, is trivial, if the latter map is trivial [Mun84, §53].

Suppose it is. Consider the cohomology spectral sequence of $E \rightarrow B$ [Spa66, 9.5]. The map $H^n(E; \mathbb{Q}) \rightarrow H^n(F; \mathbb{Q})$ is the composite

$$H^n(E; \mathbb{Q}) = F^0H^n(E; \mathbb{Q}) \rightarrow F^0H^n(E; \mathbb{Q})/F^1H^n(E; \mathbb{Q}) = E_{\infty,0}^0 \rightarrow E_{\infty,1}^0 = H^0(B; H^n(F; \mathbb{Q})) \cong H^n(F; \mathbb{Q})$$

[Spa66, 9.5], hence the map $E_{\infty,0}^0 \rightarrow E_{\infty,1}^0$ is trivial.

Now $E_{2,t}^s = H^s(B; H^t(F; \mathbb{Q})) = 0$ unless $t \in \{0, n\}$, since the fiber $F$ is a rational cohomology $n$-sphere. Therefore, all differentials of the spectral sequence except for $E_{2,n}^{s,n} \xrightarrow{d_{s,n}} E_{2,s+n+1,0}$ vanish. Since $E_{n+2} = E_{\infty}$ is the cohomology of this cochain complex, there is an exact sequence

$$0 \rightarrow E_{\infty,0}^0 \rightarrow E_{2,0}^0 \rightarrow E_{2,1}^{n+1,0} \rightarrow E_{\infty,1}^{n+1,0} \rightarrow 0.$$
It follows that the horizontal arrows in the diagram
\[
\begin{array}{ccc}
E_{2,n}^0 & \rightarrow & E_{2,n+1,0}^0 \\
\downarrow \cong & & \downarrow \cong \\
H^0(B; \mathbb{Q}) & \cup \Omega & \rightarrow H^{n+1}(B; \mathbb{Q})
\end{array}
\]
are monomorphisms, and thus \( \Omega \neq 0 \neq 2\Omega \); hence \( n \) is odd [Spa66, 9.5.2]. \( \square \)

**6.2.12 Lemma** Let \( B \) be a finite CW complex, and let \( p : E \rightarrow B \) be a locally trivial \( m \)-sphere bundle with a section \( s : B \rightarrow E \). Then there is the structure of a CW complex on \( E \), with \( s(B) \) as a subcomplex. For each cell \( \iota : e^k \rightarrow B \), the set \( E_\iota(e^k) \) consists of an \((m+k)\)-cell.

**Proof.** We proceed by induction on \( k = \dim B \). If \( B \) consists of a finite number of points, then there is nothing to show. Now assume that the claim holds for sphere bundles over \( k \)-dimensional CW complexes. Let \( B^{(k)} \) denote the \( k \)-skeleton of \( B \). We may assume that the restriction \( E_{B^{(k)}} \rightarrow B^{(k)} \) has the claimed property. Let \((e^{k+1}, \dot{e}^{k+1}) \rightarrow (B, B^{(k)})\) be a cell, and consider the induced bundle
\[
\begin{array}{ccc}
E' & \rightarrow & E \\
\downarrow \uparrow s' & & \downarrow \uparrow s \\
e^{k+1} & \rightarrow & B.
\end{array}
\]
Since \( e^{k+1} \) is contractible, the bundle \( E' \xrightarrow{s'} e^{k+1} \) is isomorphic to the trivial bundle \( e^{k+1} \times S^m \rightarrow e^{k+1} \). Thus we find a map
\[
(e^{k+m+1}, \dot{e}^{k+m+1}) \rightarrow (E', E'_e \cup s'(e^{k+1})) \rightarrow (E, E_{B^{(k)}} \cup s(B^{(k)})).
\]
\( \square \)

**6.3 Generalized manifolds**

Let \( R \) be a principal ideal domain. We denote sheaf-theoretic cohomology by \( \check{H}^* \), and Borel-Moore homology by \( \check{H}_* \). Compact supports are indicated by the letter \( c \). Borel-Moore homology is defined only for locally compact Hausdorff spaces, and it has some unusual features: it is not clear that \( \check{H}_{-1}(X; R) = 0 \), and the group \( \check{H}_c(X; R) \) depends on whether \( R \) is viewed as an \( \mathbb{Z} \)- or an \( R \)-module. However, these atrocities disappear for 'good' spaces [Bre67, V].

The notation follows essentially Bredon’s book [Bre67].
6.3.1 Lemma Let \( S : U \mapsto S(U) \) be a presheaf on a space \( X \), and let \( S \) denote the sheaf that it generates. Let \( W \subseteq X \) be open. The restriction \( S|W \) is canonically isomorphic to the sheaf generated by the restriction of \( S \) to \( W \).

Proof. This is clear from the construction of \( S \), see [Bre67, I 1.3]. \( \square \)

6.3.2 Proposition Let \((X, A)\) be a pair. If \( X \) and \( A \) are paracompact and HLC, then there is a natural isomorphism

\[ \tilde{H}^\bullet(X, A; R) \cong \tilde{H}^\bullet_c(X, A; R) \]

[Bre67, III 2.]. If \( X \) and \( A \) are locally compact and HLC, then there is a natural isomorphism

\[ H_\bullet(X, A; R) \cong \tilde{H}_\bullet_c(X, A; R) \]

[Bre67, V 12.6]. Note that \( \tilde{H}_\bullet(X, X - \{x\}; R) \cong \tilde{H}_\bullet(X, X - \{x\}; R) \) by [Bre67, V 5.8]. \( \square \)

6.3.3 Definition A locally compact Hausdorff space \( X \) is called a generalized \( n \)-manifold over \( R \), if it has the following properties:

(i) the sheaf-theoretic dimension \( \dim_R X \) is finite [Bre67, II 15.].

(ii) \( X \) is cohomological locally connected (\( clc_R \)), i.e. for every point \( x \in X \) and every open neighborhood \( U \) of \( x \), there exists an open neighborhood \( V \subseteq U \) of \( x \), such that the induced map \( \tilde{H}^\bullet(V; R) \leftarrow \tilde{H}^\bullet(U; R) \) is trivial.

(iii) for each \( x \in X \),

\[ \tilde{H}_\bullet(X, X - \{x\}; R) = \begin{cases} R & \text{for } i = n \\ 0 & \text{for } i \neq n. \end{cases} \]

(iv) The orientation sheaf, i.e. the sheaf generated by the presheaf \( V \mapsto \tilde{H}_\bullet(X, X - V; R) \) is locally constant.

Thus, a generalized \( n \)-manifold over \( R \) is what Bredon [Bre67] calls an \( n \)-cm\( R \).

We call \( X \) a generalized \( n \)-manifold, if it is a generalized \( n \)-manifold over every principal ideal domain \( R \). We are mainly interested in the cases \( R = \mathbb{Z}, \mathbb{Z}_p, \mathbb{Q} \).
6.3.4 Proposition Let $X$ be a locally compact Hausdorff space. If every point $x \in X$ has an open neighborhood $U$ such that $U$ is a generalized $n$-manifold, then $X$ is a generalized $n$-manifold. Conversely, every open subset of a generalized $n$-manifold is again a generalized $n$-manifold.

A product of a generalized $n$-manifold and a generalized $m$-manifold is a generalized $(m+n)$-manifold \cite{Bre67, V 15.8}.

Proof. If every point has a neighborhood which has finite sheaf-theoretic dimension, then $\dim R X$ is finite as well \cite{Bre67, II 15.8}. The orientation sheaf is locally constant by 6.3.1. \hfill \blacksquare

6.3.5 Theorem Let $X$ be a locally compact ANR of finite covering dimension. Then the following are equivalent:

(i) There exists a number $n$ such that for every $x \in X$

$$H_i(X, X - \{x\}) = \begin{cases} 
  \mathbb{Z} & \text{for } i = n \\
  0 & \text{for } i \neq n.
\end{cases}$$

(ii) For all $x, y \in X$, the groups $H_i(X, X - \{x\})$ and $H_i(X, X - \{y\})$ are finitely generated and isomorphic.

(iii) The space $X$ is a generalized $n$-manifold over $\mathbb{Z}$.

(iv) The space $X$ is a generalized $n$-manifold.

If one of these equivalent conditions is satisfied, then $\dim_R X = \dim_Z X = \dim X = n$.

Proof. Being a generalized manifold is a local property, hence we may assume that $X$ is second countable by passing to a relatively compact open subset.

Since the covering dimension of $X$ is finite, its sheaf-theoretic dimension is finite as well. The space $X$ is locally contractible and hence $HLC$ and $clc_R$. Now $\tilde{H}_\ast(X, X - \{x\}) = H_\ast(X, X - \{x\})$ by 6.3.2. By \cite{Bre69b, 3.2}, $X$ is a generalized $n$-manifold over $\mathbb{Z}$. This establishes the equivalence of (i), (ii), and (iii).

Now suppose that (i) holds, and let $R$ be a principal ideal domain. It follows from the universal coefficient theorem that

$$H_\ast(X, X - \{x\}) \otimes R \cong H_\ast(X, X - \{x\}; R) \cong \tilde{H}_\ast(X, X - \{x\}; R)$$

and thus we infer from \cite{Bre69a} that $X$ is a generalized manifold over $R$. \hfill \blacksquare

Now L"owen’s Theorem is an immediate consequence:
6.3.6 Theorem (Löwen) \[\text{[Löw83]}\] Let \(X\) be a compact ANR of finite covering dimension \(\dim X = n\). Suppose that for every \(x \in X\), the complement \(X - \{x\}\) is acyclic (i.e. \(\tilde{H}_*(X - \{x\}) = 0\)). Then \(X\) is an (orientable) generalized \(n\)-manifold, and a homology \(n\)-sphere.

If \(n \leq 2\), then \(X\) is an \(n\)-sphere.

If \(\pi_1(X - \{x\}) = 0 = \pi_1(X - \{y\})\) for two elements \(x, y \in X\), then \(X\) is homotopy equivalent to an \(n\)-sphere.

Proof. Since \(X\) is locally contractible and compact, its homology is finitely generated in each dimension \[\text{[Spa66, 6.9.11]}\]. The complement of each \(x \in X\) is acyclic, and therefore all local homology groups in dimension \(i\) are isomorphic to \(H_i(X)\). By 6.3.4, \(X\) is a generalized \(n\)-manifold. Thus \(\tilde{H}_*(X; R) \cong H_*(X, X - \{x\}; R) \cong \tilde{H}_*(S^n; R)\), and therefore \(X\) is \(R\)-orientable.

If \(\dim X \leq 2\), then \(X\) is locally euclidean \[\text{[Wil49, p.272]}\], and the claim follows from the classification of the compact orientable 1- and 2-manifolds.

If \(n > 1\), then the Mayer-Vietoris sequence yields \(H_0(X - \{x, y\}) = \mathbb{Z}\), and from the Seifert-Van Kampen Theorem we conclude that \(\pi_1 X = 0\). Thus \(X\) is homotopy equivalent to a sphere by 6.5.1. \[\square\]

6.3.7 Lemma Let \(X\) be a locally compact, finite-dimensional ANR. The (unreduced) open cone \(M = C'_X - X\) is a generalized \((n + 1)\)-manifold if and only if \(X\) is a generalized \(n\)-manifold with \(H_*(X) \cong H_*(S^n)\).

Proof. Clearly, \(C'_X - X\) is a finite-dimensional ANR. Let \(x\) denote the tip of the cone. Now \(H_*(C'_X - (X \cup \{x\})) = H_*(X \times (0, 1)) \cong H_*(X)\), and the claim follows from 6.3.5. \[\square\]

6.3.8 Corollary Let \(E \to B\) be a locally trivial bundle with fiber \(F\). Suppose that \(E, B, F\) are finite-dimensional ANRs. If \(E\) is a generalized manifold, then the fiber \(F\) and the base \(B\) are generalized manifolds. The (unreduced) open mapping cylinder is a generalized manifold if and only if \(H_*(F) = H_*(S^\dim F)\).

Proof. This follows from the previous lemma and \[\text{[Bre67, V 15.8]}\]. \[\square\]

6.4 Münzner’s Theorem

6.4.1 Theorem (Münzner) Let \(\mathcal{F}^{pr}_1 \mathcal{P}\) and \(\mathcal{F}^{pr}_2 \mathcal{L}\) be locally trivial bundles with homotopy \(m'\)- and \(m\)-spheres as fibers.

Suppose that the fibers and the bases of these bundles are locally compact, finite dimensional ANRs.

Assume moreover that \(\mathcal{P}, \mathcal{L}\) and \(\mathcal{F}\) are generalized manifolds of dimension \(r - m', r - m\) and \(r\), respectively.
Consider the double mapping cylinder $D\mathcal{F}$ over $\text{pr}_1, \text{pr}_2$. If $D\mathcal{F}$ is a homology $(r + 1)$-sphere, then

$$
\begin{align*}
H^\bullet(\mathcal{F}; \mathbb{Z}_2) & \cong \mathbb{Z}_2^{2n} \\
H^\bullet(\mathcal{P}; \mathbb{Z}_2) & \cong \mathbb{Z}_2^n \\
H^\bullet(\mathcal{L}; \mathbb{Z}_2) & \cong \mathbb{Z}_2^n
\end{align*}
$$

where $n \in \{1, 2, 3, 4, 6\}$. If $m, m' > 1$, then

$$
\begin{align*}
H^\bullet(\mathcal{F}) & \cong \mathbb{Z}^{2n} \\
H^\bullet(\mathcal{P}) & \cong \mathbb{Z}^n \\
H^\bullet(\mathcal{L}) & \cong \mathbb{Z}^n.
\end{align*}
$$

More specifically, the cohomology rings of $\mathcal{P}, \mathcal{L}$ and $\mathcal{F}$ is as follows. Here, the natural inclusions between the listed rings correspond to the monomorphisms

$$H^\bullet(\mathcal{P}; R) \xrightarrow{\text{pr}_1^*} H^\bullet(\mathcal{F}; R) \xrightarrow{\text{pr}_2^*} H^\bullet(\mathcal{L}; R).$$

Compare also Hebda [Heb88] for the $\mathbb{Z}_2$-cohomology. The subscripts indicate the degree of the cohomology classes.

1. $n = 1$ and $m = m'$. The cohomology of $\mathcal{P} \leftarrow \mathcal{F} \rightarrow \mathcal{L}$ is the same as that of $*$ \leftarrow $S^m \rightarrow *$

2. $n = 2$, $m, m'$ arbitrary. The cohomology of $\mathcal{P} \leftarrow \mathcal{F} \rightarrow \mathcal{L}$ is the same as that of $S^m \leftarrow S^m \times S^{m'} \rightarrow S^{m'}$

3. $n = 3$ and $m = m' = 1$.

$$
\begin{align*}
H^\bullet(\mathcal{P}; \mathbb{Z}_2) & = \mathbb{Z}_2[x_1]/(x_1^3) \\
H^\bullet(\mathcal{L}; \mathbb{Z}_2) & = \mathbb{Z}_2[y_1]/(y_1^3) \\
H^\bullet(\mathcal{F}; \mathbb{Z}_2) & = \mathbb{Z}_2[x_1, y_1]/(x_1^3, y_1^3, x_1^2 + x_1 y_1 + y_1^2).
\end{align*}
$$

3. $n = 3$ and $m = m' \in \{2, 4, 8\}$

$$
\begin{align*}
H^\bullet(\mathcal{P}) & = \mathbb{Z}[x_m]/(x_m^3) \\
H^\bullet(\mathcal{L}) & = \mathbb{Z}[y_m]/(y_m^3) \\
H^\bullet(\mathcal{F}) & = \mathbb{Z}[x_m, y_m]/(x_m^3, y_m^3, x_m^2 - x_m y_m + y_m^2).
\end{align*}
$$
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41 $n = 4$ and $m = m' = 1$

\[
\begin{align*}
H^\bullet(\mathcal{P}; \mathbb{Z}_2) &= \mathbb{Z}_2[x_1]/(x_1^4) \\
H^\bullet(\mathcal{L}; \mathbb{Z}_2) &= \mathbb{Z}_2[y_1, y_2]/(y_1^2, y_2^2) \\
H^\bullet(\mathcal{F}; \mathbb{Z}_2) &= \mathbb{Z}_2[x_1, y_1, y_2]/(x_1^4, y_1^2, y_2 + x_1^2 + x_1y_1)
\end{align*}
\]

42 $n = 4$, $m = 1$ and $m' > 1$.

The $\mathbb{Z}_2$-cohomology rings of $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ are graded $\mathbb{Z}_2$-algebras generated by homogeneous elements $(x_m, x_{m+m'})$, $(y_m, y_{m'+m})$, and $(x_m, y_{m' + m}, x_{m+m'}, y_{m'+m})$, respectively, subject to the relations $x_m^2 = x_{m+m'} = 0$, $y_m^2 = y_{m'+m} = 0$ and $x_m^2 = y_m^2 = x_{m+m'} = y_{m'+m} = y_{m'+m} + x_{m+m'} + x_my_{m'} = 0$ (the subscripts indicate the degrees).

43 $n = 4$, $m, m' > 1$, and $m + m'$ odd.

The integral cohomology rings of $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{F}$ are anticommutative graded $\mathbb{Z}$-algebras generated by homogeneous elements $(x_m, x_{m+m'})$, $(y_m, y_{m'+m})$, and $(x_m, y_{m' + m}, x_{m+m'}, y_{m'+m})$, respectively, subject to the relations $x_m^2 = x_{m+m'} = 0$, $y_m^2 = y_{m'+m} = 0$ and $x_m^2 = y_m^2 = x_{m+m'} = y_{m'+m} = y_{m'+m} + x_{m+m'} - x_my_{m'} = 0$ (the subscripts indicate the degrees).

The restriction $m, m' > 1$ is not really necessary, cp. [3.4.11].

44 $n = 4$ and $m = m' \in \{2, 4\}$.

\[
\begin{align*}
H^\bullet(\mathcal{P}) &= \mathbb{Z}[x_m]/(x_m^4) \\
H^\bullet(\mathcal{L}) &= \mathbb{Z}[y_m, y_{2m}]/(y_{2m}^2, y_m^2 - 2y_m) \\
H^\bullet(\mathcal{F}) &= \mathbb{Z}[x_m, y_{2m}]/(x_m^4, y_{2m}^2, y_m^2 - 2y_{2m}, y_{2m} + x_m^2 - y_mx_m)
\end{align*}
\]

61 $n = 6$ and $m = m' = 1$.

\[
\begin{align*}
H^\bullet(\mathcal{P}; \mathbb{Z}_2) &= \mathbb{Z}_2[x_1, x_3]/(x_1^3, x_3^3) \\
H^\bullet(\mathcal{L}; \mathbb{Z}_2) &= \mathbb{Z}_2[y_1, y_3]/(y_1^3, y_3^3) \\
H^\bullet(\mathcal{F}; \mathbb{Z}_2) &= \mathbb{Z}_2[x_1, x_3, y_1, y_3]/(x_1^3, y_1^3, y_3^3, x_1^2 + y_1, x_1 + y_3 + x_3^2 + x_1y_1)
\end{align*}
\]

62 $n = 6$ and $m' = 2, 4$.

\[
H^\bullet(\mathcal{P}) = \mathbb{Z}[x_m, x_{3m}]/(x_{3m}^2, x_m^2 - 2x_{3m})
\]

The rings $H^\bullet(\mathcal{L})$ and $H^\bullet(\mathcal{F})$ have bases $\{y_m, y_{2m}, y_{3m}, y_{4m}, y_my_{4m} \}$ and $\{x_m, y_m, x_m^2, y_{2m}, x_{3m}, y_{3m}, x_mx_{3m}, y_{4m}, x_m^2x_{3m}, y_{4m}y_{4m}, y_{3m}y_{3m}, x_{3m}y_{3m} \}$ respectively. The missing products can easily be calculated from the equations $y_m^2 = 3y_{2m}$, $y_{2m}y_{2m} = 2y_{3m}$, $y_{2m}y_{3m} = 3y_{4m}$, $y_m^2y_{3m} = 0$, $x_my_{3m} = x_m^2 + y_{2m}$, and the fact that the integral cohomology modules are torsion-free. For example $3y_{2m}y_{3m} = y_m^2y_{3m} = 3y_{m}y_{4m}$, hence $y_{2m}y_{3m} = y_{m}y_{4m}$. 

Proof. Münzner uses only standard properties of manifolds, like Poincaré and Alexander duality, and the coefficient sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$. His proof is purely algebraic, hence his arguments are also valid for generalized manifolds. The restrictions on the dimension in the cases $3_2, 4_4, 6_2$ are not given in [Mün81], but they follow from the structure of the cohomology rings and 6.5.4. 

6.5 Odds and ends

6.5.1 Proposition Suppose $X$ is a simply connected CW complex and a homology $m$-sphere. Then $X$ is homotopy equivalent to an $m$-sphere.

Proof. Consider the Hurewicz isomorphism $\pi_m(X) \to H_m(X)$, $h \mapsto h_\ast[S^m]$. Let $f : S^m \to X$ denote the generator of $\pi_m(X)$. The map $f$ induces an isomorphism $H_\ast(S^m) \to H_\ast(X)$ (because $f_\ast[S^m]$ is a generator of $H_m(X)$), and therefore $f$ is a homotopy equivalence [Spa66, 7.6.24] [Bre93, VII 11.15].

6.5.2 Proposition Let $M, N$ be compact connected $k$-manifolds, and let $f : M \to N$ be a continuous map. Suppose that there exists an element $x \in M$ with $\{x\} = f^{-1}(f(x))$, and a neighborhood $U$ of $x$ such that the restriction $f|U$ is injective. Then $f$ is surjective.

Proof. By invariance of domain, $f|U : U \to f(U)$ is a homeomorphism. Consider the diagram

\[
\begin{array}{ccc}
H_k(M; \mathbb{Z}_2) & \to & H_k(M - \{x\}; \mathbb{Z}_2) \\
\downarrow & & \downarrow \\
H_k(N; \mathbb{Z}_2) & \to & H_k(N - \{f(x)\}; \mathbb{Z}_2)
\end{array}
\]

The vertical arrow at the right is an isomorphism. The horizontal arrows on the left-hand side are isomorphisms, because $M, N$ are $\mathbb{Z}_2$-orientable, and the horizontal arrows on the right-hand side are excision isomorphisms. Thus $f_\ast : H_k(M; \mathbb{Z}_2) \to H_k(N; \mathbb{Z}_2)$ is an isomorphism. Suppose there is an element $y \in N - f(M)$. Then $f$ factors as $M \to N - \{y\} \to N$. But $H_k(N - \{y\}; \mathbb{Z}_2) = 0$, a contradiction.

6.5.3 Theorem (Generalized Poincaré conjecture) Let $M$ be a simply connected $n$-manifold. If $M$ has the same homology as an $n$-sphere, and if $n \geq 4$, then $M$ is homeomorphic to a sphere.

Proof. For $n = 4$, this is included in Freedman’s classification of 4-manifolds [Fre82].

For $n \geq 5$, this follows from Smale’s proof [Sma61] of the combinatorial generalized Poincaré conjecture and the fact that the Kirby-Siebenmann obstruction [KS77] vanishes, since $H^4(M; \mathbb{Z}_2) = 0$. There is also a direct proof by Newman [New66].
6.5.4 Theorem (Adams-Atiyah) \cite{AA66} Let $X$ be a space having the homotopy type of a finite CW complex. Assume that the integral cohomology of $X$ has no torsion, and that the integral cohomology groups of $X$ vanish in all dimensions different from $0, 2m, 4m, 8m, \ldots$

If there is an $x \in H^{2m}(X; \mathbb{Z}_2)$ with $x^2 \neq 0$, then $2m \in \{2, 4, 8\}$.

If there is an $x \in H^{2m}(X; \mathbb{Z}_3)$ with $x^3 \neq 0$, then $2m \in \{2, 4\}$. \qed
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