Twisted Supersymmetry, Fermion-Boson Mixing and Removal of UV-IR Mixing

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Abstract

Exact supersymmetry can be maintained on non-anticommutative superspace with a twisted coproduct on the supergroup. We show that the usual exchange statistics for the superfields is not compatible with the twisted action of the superpoincaré group and find a statistics which is consistent with the twisted coproduct and imply interesting phenomena such as mixing of fermions and bosons under particle exchange. We also show that with the new statistics, the $S$-matrix becomes completely independent of the deformation parameter.

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1 Introduction

Recently it has been noted that certain spacetime symmetries which were thought to be explicitly broken on noncommutative spaces, do have a well defined action on these noncommutative spaces with a twisted coproduct. In fact this result was already well formulated in the theory of quantum groups. For review see[1, 2]. A natural generalization of noncommutative spaces are non-anticommutative superspaces, where the algebraic relations between superspace coordinates are deformed. Such spaces have also been shown to arise in certain limits of string theory[4]. Just like noncommutative spaces, these deformed superspaces break supersymmetry, but again full supersymmetry can be implemented by defining a twisted action of supersymmetry generators on the product of two fields[5, 6].

It was shown in [7], in the context of Poincaré group and Moyal plane, that one can not consistently impose the usual symmetrization or anti-symmetrization on the tensor product of fields compatibly with the twisted Poincaré symmetry.

In this paper we briefly review the Drinfel’d twist for deformed superspace and then show that in this case again, one can not impose the usual statistics. The new statistics gives the novel phenomenon of fermion-boson mixing under particle exchange. In the end we show that with the new statistics, the $S$-operator becomes completely independent of the deformation parameter.

2 Twisted Superspace and Twisted Supersymmetry

A simple deformation of superspace consists of the following superspace algebra

\[
\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}
\]

\[
\{\bar{\theta}^\dot{\alpha}, \bar{\theta}^\dot{\beta}\} = \{\bar{\theta}^\dot{\alpha}, \theta^\beta\} = [y^\mu, \theta^\alpha] = [y^\mu, \bar{\theta}^\dot{\alpha}] = 0
\] (1)

where $y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha}$ is the chiral coordinate.

This deformation of superspace has been considered by Seiberg and others and has been extensively studied over the past years.

The above deformation can be implemented on the usual superfunctions by the star product

\[ f \star g = m_o \cdot F^{-1} (f \otimes g). \] (2)

Where $f$ and $g$ are functions on superspace, $m_o$ is the usual multiplication map of the superspace algebra and

\[ F = exp(-\frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta). \] (3)

$F$ is called “the twist element”.

2
At first sight the above deformation does not seem to preserve supersymmetry, e.g.,

$$[y^\mu, y^\nu] = 0$$

is not invariant under translations generated by $\overline{Q}$, hence this space is sometimes said to have $N = 1/2$ supersymmetry. But one can define a new action of the superpoincaré generators on the product of fields (a deformed Leibniz’s rule) so that all the defining relations are preserved under supersymmetry. The new action is given in terms of a deformed coproduct of the superpoincaré algebra. The deformed coproduct is given by

$$\Delta_\theta(g) = F \Delta_\circ(g) F^{-1} \quad (4)$$

where $g$ is a superpoincaré algebra element and $\Delta_\circ$ is the usual undeformed coproduct

$$\Delta_\circ(g) = 1 \otimes g + g \otimes 1$$

The action of superalgebra on product of two superfunctions through the deformed coproduct is given by

$$g \triangleright (f \ast g) = m_\circ F^{-1} \Delta_\theta(g) (f \otimes g) \quad (5)$$

This deformation of the superspace algebra along with the deformation of the coproduct on superpoincaré group (or algebra) is what is known as Drinfel’d Twist.

### 3 Twisted Statistics

Following[7], where it was shown that the usual statistics is incompatible with the twisted coproduct of the poincaré group, we investigate the statistics on deformed superspace.

Consider the tensor product of two chiral scalar superfields. Usually we take the tensor product to be symmetric at space-like separations.

$$\Phi \otimes \Phi(y, \theta ; y', \theta') = \Phi \otimes \Phi(y', \theta' ; y, \theta) \quad (6)$$

We will show that the above relation on the tensor product of two scalar superfields is not compatible with the twisted action of the superpoincaré group.

Let us expand the field $\Phi$ into a Fourier expansion

$$\Phi(y, \theta) = \int d^4k d^2\kappa a_{k,\kappa} e^{iky} e^{i\kappa \theta}$$

then the above tensor product can be written as

$$\Phi \otimes \Phi(y, \theta ; y', \theta') = \int d^4k d^4k' d^2\kappa d^2\kappa' a_{k,\kappa} a_{k',\kappa'} e^{iky} e^{i\kappa \theta} \otimes e^{ik'y} e^{i\kappa' \theta'} \quad (7)$$

Now under a $\overline{Q}$ translation, according to the twisted coproduct rule, this goes to
\[ \Delta_{\theta}(g\xi(\zeta)) \Phi \otimes \Phi(y, \theta ; y', \theta') \]
\[ = \int d^1k \, d^4k' \, d^2\kappa \, d^2\kappa' e^{C_{\alpha\beta} k_{\mu} k'_{\nu} \sigma_{\alpha\beta} \sigma_{\mu\nu} \xi_{\mu} \xi'_{\nu}} e^{i k_{\mu}(y^\mu + 2i \theta^\nu \sigma^\nu \xi)} e^{i k'_{\mu}(y'^\mu + 2i \theta'^\nu \sigma^\nu \xi')} e^{\kappa \theta} \otimes e^{i k'_{\mu}(y'^\mu + 2i \theta'^\nu \sigma^\nu \xi')} e^{\kappa' \theta} \]  
(8)

Now under the permutation of the two fields we get

\[ \sigma \Delta_{\theta}(g\xi(\zeta)) \Phi \otimes \Phi(y, \theta ; y', \theta') \]
\[ = \int d^1k \, d^4k' \, d^2\kappa \, d^2\kappa' e^{C_{\alpha\beta} k_{\mu} k'_{\nu} \sigma_{\alpha\beta} \sigma_{\mu\nu} \xi_{\mu} \xi'_{\nu}} e^{i k'_{\mu}(y'^\mu + 2i \theta'^\nu \sigma^\nu \xi')} e^{i k_{\mu}(y^\mu + 2i \theta^\nu \sigma^\nu \xi)} e^{\kappa' \theta} \otimes e^{i k_{\mu}(y^\mu + 2i \theta^\nu \sigma^\nu \xi)} e^{\kappa \theta} \]  
(9)

where \( \sigma \) is the permutation which interchanges the position of the two fields in the tensor product.

On the other hand if we first apply the permutation and then perform the super transformation, using \( \xi \) we get

\[ \Delta_{\theta}(g\xi(\zeta)) \sigma \Phi \otimes \Phi(y, \theta ; y', \theta') \]
\[ = \int d^1k \, d^4k' \, d^2\kappa \, d^2\kappa' e^{C_{\alpha\beta} k_{\mu} k'_{\nu} \sigma_{\alpha\beta} \sigma_{\mu\nu} \xi_{\mu} \xi'_{\nu}} e^{i k'_{\mu}(y'^\mu + 2i \theta'^\nu \sigma^\nu \xi')} e^{i k_{\mu}(y^\mu + 2i \theta^\nu \sigma^\nu \xi)} e^{\kappa' \theta} \otimes e^{i k_{\mu}(y^\mu + 2i \theta^\nu \sigma^\nu \xi)} e^{\kappa \theta} \]  
(10)

where in the last line we have used the fact that \( C_{\alpha\beta} \) is symmetric in \( \alpha, \beta \).

From (9) and (10) we have that

\[ \sigma \Delta(g) \Phi \otimes \Phi \neq \Delta(g) \sigma \Phi \otimes \Phi. \]  
(11)

This means that the usual statistics is incompatible with the twisted action of supersymmetry, and we cannot impose symmetric statistics on the tensor product of fields

But we can impose

\[ \Phi \otimes \Phi(y, \theta ; y', \theta') = \mathcal{F}^{-2} \Phi \otimes \Phi(y', \theta' ; y, \theta) \]  
(12)

consistently with the twisted action of superpoincaré group. These relations reduce to standard symmetrization when the deformation parameter \( C_{\alpha\beta} \) goes to zero.

One can check that with this deformed permutation, which we will call \( \sigma_{\theta} \), we have

\[ \sigma_{\theta} \Delta_{\theta}(g) \Phi \otimes \Phi(y, \theta ; y', \theta') \]
\[ = \int d^1k \, d^4k' \, d^2\kappa \, d^2\kappa' e^{-C_{\alpha\beta} k_{\mu} k'_{\nu} \sigma_{\alpha\beta} \sigma_{\mu\nu} \xi_{\mu} \xi'_{\nu}} e^{i k'_{\mu}(y'^\mu + 2i \theta'^\nu \sigma^\nu \xi')} e^{i k_{\mu}(y^\mu + 2i \theta^\nu \sigma^\nu \xi)} e^{\kappa' \theta} \otimes e^{i k_{\mu}(y^\mu + 2i \theta^\nu \sigma^\nu \xi)} e^{\kappa \theta} \]

and exactly the same expression for \( \Delta_{\theta}(g) \sigma_{\theta} \Phi \otimes \Phi \). Same statements can be proved for the transformations generated by other elements of the superpoincaré algebra.
4 Fermion-Boson Mixing

The above relations on the tensor product of two chiral fields imply interesting statistics for the component fields. The new statistics mixes the bosons with fermions under an exchange of particle. Expanding the both sides of (12) into component fields, we find the relations

\[ A(y)A(y') = A(y')A(y) - C^{\alpha\beta}\psi_\alpha(y')\psi_\beta(y) + \frac{1}{2}C^{\alpha\beta}C_{\alpha\beta}F(y')F(y) \]

\[ \psi_\alpha(y)\psi_\beta(y') = -\psi_\beta(y')\psi_\alpha(y) + C_{\alpha\beta}F(y')F(y) \]

\[ F(y)F(y') = F(y')F(y) \] (13)

5 Removal of UVIR mixing

As an example of the use of twisted statistics, we show that the S-matrix, in this formalism becomes completely independent of the deformation parameter \( C^{\alpha\beta} \).

Let’s take \( \Phi \) to be a free chiral scalar superfield, and consider an interaction Hamiltonian of the form

\[ H_I(x_o) = \lambda \int d^3x \ d^2\theta \ d^2\bar{\theta} \delta(\bar{\theta}) \Phi^* \Phi \] (14)

The S-matrix is

\[ S_\theta = T \exp \left( - \int dx_0 H_I(x_0) \right) = T \exp \left( - \int d^4x d^2\theta d^2\bar{\theta} \delta(\bar{\theta}) \Phi(y, \theta)^* \Phi(y, \theta) \right) \]

here \( \theta \) in \( S_\theta \) just reminds us that our theory is on deformed superspace.

The fields \( \Phi \) obey the twisted statistics

\[ \Phi(y, \theta) \Phi(y', \theta') = e^{-\frac{1}{4}C^{\alpha\beta}\partial_\alpha \partial_{\theta'}^\beta} \Phi(y', \theta') \Phi(y, \theta) \]

We can take care of this statistics by writing the fields as

\[ \Phi = \Phi_o e^{-\frac{1}{4}C^{\alpha\beta}\overline{\partial}_\alpha \overline{\partial}_{\beta}} \] (15)

where \( \Phi_o \) has the usual statistics, the differential to the left acts only on the field and the differential to the right acts on every thing to the right of it.
For example, consider the product of the fields (from now on we will suppress the y dependence of the fields)

\[
\Phi(\theta)\Phi(\theta') = \Phi_o(\theta) e^{-\frac{1}{2} C^{\alpha\beta} \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta} \Phi_o(\theta') e^{-\frac{1}{2} C^{\alpha\beta} \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta} = e^{-\frac{1}{2} C^{\alpha\beta} \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta} \Phi_o(\theta)\Phi_o(\theta') e^{-\frac{1}{2} C^{\alpha\beta} \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta} \Phi_o(\theta')\Phi_o(\theta)
\]

showing that the field \( \Phi \) has the correct statistics.

Now let us consider the \( O(\lambda^2) \) in the \( S \)-matrix.

\[
S^{(2)}_\theta \sim T \int d^4x d^4x' d^2\theta d^2\theta' d^2\overrightarrow{\theta} d^2\overrightarrow{\theta'} \delta(\overrightarrow{\theta}) \delta(\overrightarrow{\theta'}) \Phi \Phi \Phi(\theta) \Phi \Phi \Phi(\theta')
\]

Consider

\[
\Phi \Phi \Phi(\theta) = \Phi e^{\frac{1}{2} C^{\alpha\beta} \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta} \Phi e^{\frac{1}{2} C^{\alpha\beta} \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta} \Phi(\theta)
\]

Substituting the expression for \( \Phi \) from (15), we have

\[
\Phi \Phi \Phi(\theta) = \Phi_o \Phi_o \Phi_o(\theta) e^{-\frac{1}{2} C^{\alpha\beta} \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta} \Phi_o \Phi_o \Phi_o(\theta')
\]

Then the relevant integral becomes

\[
\int d^2\theta d^2\theta' \Phi \Phi \Phi(\theta) \Phi \Phi \Phi(\theta')
= \int d^2\theta d^2\theta' e^{-\frac{1}{2} C^{\alpha\beta} \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta} \Phi_o \Phi_o \Phi_o(\theta) \Phi_o \Phi_o \Phi_o(\theta') e^{-\frac{1}{2} C^{\alpha\beta} \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta}
= \int d^2\theta d^2\theta' \Phi_o \Phi_o \Phi_o(\theta) \Phi_o \Phi_o \Phi_o(\theta')
\]

where in the last line we have used

\[
\int d\theta \overrightarrow{\partial}_\theta f(\theta) = 0
\]

But then we have removed all dependence on \( C^{\alpha\beta} \) from the \( O(\lambda^2) \) term and reduced it to the undeformed form i.e.,

\[
S^{(2)}_\theta = S^{(2)}_o
\]

This proof generalizes straight-forwardly to higher orders in \( \lambda \) and the proof to first order is even simpler. Hence we have proved, to all orders in perturbation theory

\[
S_\theta = S_o
\]
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