Impossibility of Local State Transformation via Hypercontractivity

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Abstract: Local state transformation is the problem of transforming an arbitrary number of copies of a bipartite resource state to a bipartite target state under local operations. That is, given two bipartite states, is it possible to transform an arbitrary number of copies of one of them into one copy of the other state under local operations only? This problem is a hard one in general since we assume that the number of copies of the resource state is arbitrarily large. In this paper we prove some bounds on this problem using the hypercontractivity properties of some super-operators corresponding to bipartite states. We measure hypercontractivity in terms of both the usual super-operator norms as well as completely bounded norms.

1. Introduction

Local state transformation is the problem of transforming a given bipartite resource state $\rho_{AB}$ to another bipartite target state $\sigma_{A'B'}$ under local operations only; i.e., one raises the question whether there exist completely-positive trace preserving (CPTP) super-operators $\Phi_{A\rightarrow A'}$ and $\Psi_{B\rightarrow B'}$ such that $\Phi \otimes \Psi (\rho_{AB}) = \sigma_{A'B'}$. Solving this problem by brute-force search on the space of CPTP maps is not feasible when the dimensions of $\rho_{AB}$ and $\sigma_{A'B'}$ are large, in which case imposing necessary conditions can be useful.

Local operations do not generate entanglement, so if $\sigma_{A'B'}$ is more entangled than $\rho_{AB}$ then such $\Phi, \Psi$ do not exist. So measures of entanglement provide us with bounds on the problem of local state transformation. Likewise, to attack this problem for classical states, we may use measures of correlation. For instance if the mutual information $I(A; B)$ of $\rho_{AB}$ is less than that of $\sigma_{A'B'}$, then the latter cannot be generated from the former under local operations.

These bounds, however, usually fail when infinitely many copies of the resource state are available and we need to generate only one copy of the target state, i.e., when we want to transform $\rho_{AB}^{\otimes n}$ for a sufficiently large $n$, into $\sigma_{A'B'}$ under local operations. The point is that most measures of entanglement and correlation (such as mutual information,
entanglement of formation, squashed entanglement, etc.) tend to infinity on \( \rho_{AB}^{\otimes n} \) as \( n \) gets larger and larger if \( \rho_{AB} \) is not uncorrelated or unentangled. Thus the following question arises naturally: is there a measure of correlation or entanglement that takes on the same value on all \( \rho_{AB}^{\otimes n} \) for all \( n \)?

1.1. Maximal correlation. There is a measure of correlation for bipartite classical states (distributions) called maximal correlation. This measure was first introduced by Hirschfeld [1] and Gebelein [2] and then studied by Rényi [3, 4]. Among other properties, maximal correlation satisfies the data processing inequality. Namely, it does not increase under local operations. More importantly, maximal correlation of \( n \) independent copies of a bipartite distribution is equal to the maximal correlation of only one copy. Given these two properties, maximal correlation gives a bound on the problem of local state transformation.

Maximal correlation has recently been defined for quantum states [5]. For a bipartite state \( \rho_{AB} \), its maximal correlation \( \mu(\rho_{AB}) \) is defined by

\[
\mu(\rho_{AB}) = \max \left| \operatorname{tr} \left( \rho_{AB} X_A \otimes Y_B^\dagger \right) \right| \text{subject to } \operatorname{tr}(\rho_A X_A) = \operatorname{tr}(\rho_B Y_B) = 0, \quad \operatorname{tr}(\rho_A X_A X_A^\dagger) = \operatorname{tr}(\rho_B Y_B Y_B^\dagger) = 1. \tag{1}
\]

This definition is reduced to the classical maximal correlation when \( \rho_{AB} \) is classical. It is shown in [5] that maximal correlation satisfies the following two important properties:

(i) \( \mu(\rho_{A_1B_1} \otimes \rho'_{A_2B_2}) = \max\{\mu(\rho_{A_1B_1}), \mu(\rho'_{A_2B_2})\} \).
(ii) If \( \sigma_{A'B'} = \Phi \otimes \Psi(\rho_{AB}) \) then \( \mu(\rho_{AB}) \geq \mu(\sigma_{A'B'}) \).

As a result, using maximal correlation we may prove the impossibility of local state transformation in some cases even if infinitely many copies of the resource state are available. For example if we define

\[
\rho_{AB}^{(\alpha)} = (1 - \alpha) \frac{I_{AB}}{4} + \alpha |\psi\rangle\langle\psi|_{AB}, \tag{2}
\]

where \( |\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \) and \( I_{AB}/4 \) is the maximally mixed state, then \( \mu(\rho_{AB}^{(\alpha)}) = \alpha \).

This means that, if \( \beta > \alpha \), having even infinitely many copies of \( \rho_{AB}^{(\alpha)} \) we cannot generate a single copy of \( \rho_{AB}^{(\beta)} \) under local operations.

Let us give another example. Let \( \zeta_{UV} \) be the bipartite distribution over two bits defined by

\[
\zeta_{00} = \zeta_{01} = \zeta_{10} = 1/3 \quad \text{and} \quad \zeta_{11} = 0. \tag{3}
\]

Then as pointed out in [6] we have \( \mu(\zeta_{UV}) = 1/2 \). As a result, two parties who have shared infinitely many copies of \( \rho_{AB}^{(\alpha)} \) cannot generate the bipartite correlation \( \zeta_{UV} \) by local measurements if \( \alpha < 1/2 \).

Maximal correlation characterizes all states from which (perfect) shared randomness can be distilled under local operations [5, 7]. Nevertheless, as one expects, it does not solve the problem of local state transformation in general. In the above example we see that maximal correlation does not rule out the possibility of locally transforming \( n \) copies of \( \rho_{AB}^{(\alpha)} \) to \( \zeta_{UV} \) when \( \alpha \geq 1/2 \). The possibility of such a transformation for \( \alpha = 1 \) is easily verified, but we do not know the answer for \( 1/2 < \alpha < 1 \).
1.2. Hypercontractivity. Another idea to attack the problem of local state transformation is hypercontractivity. This idea is due to Ahlswede and Gács [8], and has recently been revisited by Kamath and Anantharam [6] and Anantharam et al. [9] (see also [10] by Raginsky). Here is a rough description of the hypercontractivity method.

 Via Choi-Jamiołkowski isomorphism a bipartite state $\rho_{AB}$ corresponds to a completely-positive super-operator $\Omega_\rho$ from the space of register $A$ to that of $B$. Suppose that for a CPTP map $\Psi_{B \rightarrow B'}$ we have $\sigma_{AB'} = I \otimes \Psi(\rho_{AB})$, where $I$ is the identity super-operator. This equation in terms of the corresponding super-operators gives $\Omega_\sigma = \Psi \circ \Omega_\rho$.

Recall that for every $1 \leq p \leq \infty$ we may define a $p$-norm $\| \cdot \|_p$ (also called the Schatten norm). Then for every $1 \leq p, q \leq \infty$ we may consider the super-operator norm

$$\| \Omega_\rho \|_{p \rightarrow q} = \sup_{X \neq 0} \frac{\| \Omega_\rho(X) \|_q}{\| X \|_p}.$$ 

From the definition of this norm and $\Omega_\sigma = \Psi \circ \Omega_\rho$ we obtain

$$\| \Omega_\sigma \|_{p \rightarrow q} \leq \| \Psi \|_{q \rightarrow q} \| \Omega_\rho \|_{p \rightarrow q},$$

which puts restrictions on $\sigma$ in terms of super-operator norms.

This restriction however is not very strong since $\| \Psi \|_{q \rightarrow q}$ could be very large. To overcome this problem, instead of $\rho_{AB}$ we may consider the super-operator corresponding to some normalization of $\rho_{AB}$ which we denote by $\tilde{\rho}_{AB}$. If the normalization is done properly, $\sigma_{AB'} = I \otimes \Psi(\rho_{AB})$ will give $\Omega_{\tilde{\sigma}} = \tilde{\Psi} \circ \tilde{\Omega}_\rho$, where again $\tilde{\Psi}$ is some normalization of $\Psi$. As a result,

$$\| \Omega_{\tilde{\sigma}} \|_{p \rightarrow q} \leq \| \tilde{\Psi} \|_{q \rightarrow q} \| \tilde{\Omega}_\rho \|_{p \rightarrow q}.$$ 

To get rid of the dependency on $\Psi$ in the above equation, the normalization is defined in such a way that

$$\| \tilde{\Psi} \|_{q \rightarrow q} \leq 1, \quad (4)$$

for all values of $q$ and all CPTP maps $\Psi$. Putting these two inequalities together, we arrive at

$$\| \Omega_{\tilde{\sigma}} \|_{p \rightarrow q} \leq \| \Omega_{\tilde{\rho}} \|_{p \rightarrow q}.$$ 

Yet this is not the final step since in the problem of local state transformation we assume that infinitely many copies of the resource state is available. Indeed we should compare the maximum of $\| \tilde{\Omega}_{\rho^n} \|_{p \rightarrow q}$ over all $n$, to $\| \Omega_{\tilde{\sigma}} \|_{p \rightarrow q}$. We have $\Omega_{\tilde{\rho}^n} = \tilde{\Omega}_{\rho}^{\otimes n}$ and by the definition of super-operator norm

$$\| \Omega_{\tilde{\rho}^n} \|_{p \rightarrow q} \geq \| \Omega_{\tilde{\rho}} \|_{p \rightarrow q}^{n}.$$ 

(5)

Thus $\| \tilde{\Omega}_{\rho^n} \|_{p \rightarrow q}$ tends to infinity as $n \rightarrow \infty$ if $\| \tilde{\Omega}_{\rho} \|_{p \rightarrow q} > 1$, in which case comparing to $\| \Omega_{\tilde{\sigma}} \|_{p \rightarrow q}$ gives no bound.

This observation suggests considering the set of all pairs $(p, q)$ such that

$$\| \Omega_{\tilde{\rho}^n} \|_{p \rightarrow q} \leq 1,$$

for all $n$. This set is called the hypercontractivity ribbon [6]. Putting everything together we conclude that if $\sigma_{A'B'}$ can be locally generated from copies of $\rho_{AB}$, then the hypercontractivity ribbon of $\sigma_{A'B'}$ is a subset of that of $\rho_{AB}$.
1.3. Quantum hypercontractivity ribbon. The main contribution of this paper is to generalize the idea of hypercontractivity in the classical case [6,8,9] to the quantum setting. This idea is presented here based on the notation of the quantum theory, so this generalization may seem straightforward. Nevertheless there are some difficulties. The first one is proving the upper bound on the super-operator norm of for all CPTP maps , i.e., Eq. (4). This inequality in the classical case is a simple consequence of Hölder’s inequality, but is highly non-trivial in the quantum case. Here we prove (4) based on the theory of complex interpolation and the Riesz–Thorin theorem. Another important difference is that inequality (5) is indeed an equality in the classical case. This equality simplifies very much the analysis of hypercontractivity. In the quantum case however the super-operator norm is known to not be multiplicative even on completely-positive (CP) maps. We thus suggest that, in addition to the usual super-operator norm, one should consider the completely bounded norm. The latter one has the property of being multiplicative.

By generalizing the idea of hypercontractivity to the quantum setting, we prove the impossibility of transforming copies of defined in (2) to defined by (3) under local operations if . This result does not seem to be reproducible by other methods in quantum information theory. We also study the relation between the hypercontractivity ribbon and some other measures of correlation. In particular we show that maximal correlation gives a bound on the hypercontractivity ribbon of . This result in the classical case is due to Ahlswede and Gács [8].

The rest of this paper is organized as follows. In the following section we review the required tools including Hölder’s inequalities, completely-bounded norms, the Riesz–Thorin theorem and Choi–Jamiołkowski isomorphism. Section 3 includes the main definitions and main results of this paper. In particular the hypercontractivity ribbon is defined in this section and a data processing type property is proved. Some properties of the hypercontractivity ribbon, in particular its relation to the maximal correlation are discussed in this section. In Sect. 4 we compute the hypercontractivity ribbon for some examples, and explain how log-Sobolev inequalities can be used to compute the ribbon. Concluding remarks come in Sect. 5.

2. Preliminaries

In this paper we assume that the reader is familiar with basic notions of quantum information theory [11] such as Hilbert spaces, density matrices, CPTP maps etc. Here we just fix some notations.

We denote the Hilbert space corresponding to quantum register by , which throughout this paper is assumed to be finite dimensional. is the space of linear operators acting on . The identity operator acting on is denoted by .

In this paper we use Dirac’s notation as follows. A vector in is denoted by a ‘ket’ . Similarly a vector in the dual space is denoted by a ‘bra’ , i.e., is the functional corresponding to the vector . We may think of as a column vector. Then is a row vector corresponding to the conjugate transpose of . As a result the inner product of is equal to . Similarly is a linear operator acting on by . We also use the notation for a linear operator by which we mean .

Throughout this paper we fix an orthonormal basis for the Hilbert space (the computational basis), where . The transpose of
\( X \in \mathbf{L}(\mathcal{H}_A) \) with respect to this basis is denoted by \( X^T \), and \( X^* = (X^\dagger)^T \) where \( X^\dagger \) is the adjoint of \( X \). Note also that, \( \{|i\rangle\langle j| : 0 \leq i, j \leq d_A - 1 \} \) forms a basis for \( \mathbf{L}(\mathcal{H}_A) \).

For a hermitian operator \( X \in \mathbf{L}(\mathcal{H}_A) \) we let \( X^{-1} \) to be the inverse of \( X \) restricted to the support of \( X \), i.e., \( X^{-1}X = XX^{-1} \) is the hermitian projection on the span of eigenvectors of \( X \) corresponding to non-zero eigenvalues. Furthermore, by \( X \geq Y \) (and \( Y \leq X \)) we mean that \( X - Y \) is positive semi-definite.

2.1. Schatten norms. For \( p \geq 1 \) the Schatten \( p \)-norm of \( X \in \mathbf{L}(\mathcal{H}_A) \) is defined by

\[
\|X\|_p = \text{tr} \left( |X|^p \right)^{1/p},
\]

where \( |X| := (X^\dagger X)^{1/2} \). For \( p = \infty \) we let

\[
\|X\|_\infty = \lim_{p \to \infty} \|X\|_p,
\]

which is equal to the usual operator norm of \( X \):

\[
\|X\|_\infty = \sup\{\|X|v\| : |v\rangle \in \mathcal{H}_A, \||v\rangle\| = 1\}.
\]

\( \| \cdot \|_p \) satisfies triangle inequality and is a norm on \( \mathbf{L}(\mathcal{H}_A) \). We clearly have \( \|X^T\|_p = \|X^*\|_p = \|X^\dagger\|_p = \|X\|_p \). Moreover \( \|UXV\|_p = \|X\|_p \) for all unitary operators \( U, V \).

Hölder’s inequality states that for every \( 1 \leq p \leq \infty \),

\[
\|XY\|_1 \leq \|X\|_p \|Y\|_{p'},
\]

where \( 1 \leq p' \leq \infty \) is the Hölder conjugate of \( p \), i.e.,

\[
\frac{1}{p} + \frac{1}{p'} = 1.
\]

Moreover, \( \| \cdot \|_{p'} \) is the dual norm of \( \| \cdot \|_p \), i.e., for every \( X \),

\[
\|X\|_p = \sup_{\|Y\|_{p'} = 1} \left| \text{tr} (XY) \right|.
\]

A generalization of Hölder’s inequality (see for example [12]) states that for every \( r, p, q > 0 \) with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \),

\[
\|XY\|_r \leq \|X\|_p \|Y\|_q.
\]

Then by a simple induction we have

\[
\|X_1 \ldots X_k\|_r \leq \|X_1\|_{p_1} \ldots \|X_k\|_{p_k},
\]

for all \( r, p_1, \ldots, p_k > 0 \) with \( \frac{1}{r} = \frac{1}{p_1} + \ldots + \frac{1}{p_k} \).

For \( 1 \leq p, q \leq \infty \) the super-operator norm of \( \Phi : \mathbf{L}(\mathcal{H}_A) \to \mathbf{L}(\mathcal{H}_B) \) is defined by

\[
\|\Phi\|_{p \to q} := \sup_{X \neq 0} \frac{\|\Phi(X)\|_q}{\|X\|_p}.
\]

When \( \Phi \) is CP, the supremum could be taken over positive semi-definite \( X \) [13–15] (see also [16]), i.e.
\[
\|\Phi\|_{p \rightarrow q} = \sup_{X > 0} \frac{\|\Phi(X)\|_q}{\|X\|_p}.
\] (7)

Observe that by definition
\[
\|\Phi(X)\|_q \leq \|\Phi\|_{p \rightarrow q} \|X\|_p,
\]
for all \(X \in \text{L}(\mathcal{H}_A)\). Moreover,
\[
\|\Phi \circ \Psi\|_{p \rightarrow q} \leq \|\Psi\|_{p \rightarrow r} \|\Phi\|_{r \rightarrow q}.
\]

The super-operator norm is not multiplicative. That is although the inequality
\[
\|\Phi \otimes \Psi\|_{p \rightarrow q} \geq \|\Phi\|_{p \rightarrow q} \|\Psi\|_{p \rightarrow q},
\]
is easily verified, the reverse inequality does not hold in general. It is indeed well-known in quantum information theory that the super-operator norm is not multiplicative even in the case of \(p = q = 1\). To obtain a multiplicative super-operator norm we may consider the completely bounded norms.

2.2. Completely bounded norms. In this section we review completely bounded norms. For details we refer the reader to [16] and references there including [17,18].

As mentioned above the \(1 \rightarrow 1\) super-operator norm is not multiplicative. To make it multiplicative we usually define the completely bounded norm as
\[
\|\Phi\|_{\text{CB}, 1 \rightarrow 1} = \sup_d \|\mathcal{I}_d \otimes \Phi\|_{1 \rightarrow 1} = \sup_d \sup_{X \neq 0} \frac{\|\mathcal{I}_d \otimes \Phi(X)\|_1}{\|X\|_1},
\]
where by \(\mathcal{I}_d\) we mean the identity super-operator acting on a \(d\)-dimensional Hilbert space. This norm is also called the diamond norm [19].

The completely bounded norm \(p \rightarrow q\) can be defined similarly when \(p = q\). Nevertheless, when \(p \neq q\) this definition does not make sense since the supremum may not even exist. The point is that the norm \(\|\mathcal{I}_d\|_{p \rightarrow q}\) of the identity operator is not equal to 1.

To overcome this difficulty the completely bounded norm can be defined by
\[
\|\Phi\|_{\text{CB}, p \rightarrow q} := \sup_d \|\mathcal{I}_d \otimes \Phi\|_{(t,p) \rightarrow (t,q)} = \sup_d \sup_{X} \frac{\|\mathcal{I}_d \otimes \Phi(X)\|_{(t,q)}}{\|X\|_{(t,p)}}.
\] (9)

Here the \(d\)-dimensional auxiliary space on which \(\mathcal{I}_d\) acts, is equipped with the \(t\)-Schatten norm, while the input and output spaces of \(\Phi\) are equipped with \(p\) and \(q\) norms respectively. In fact, for every \(X_{AB} \in \text{L}(\mathcal{H}_A) \otimes \text{L}(\mathcal{H}_B)\) its \((t,p)\)-norm can be defined via the theory of non-commutative vector valued \(L_p\) spaces [17]. If \(X_{AB} = Y_A \otimes Z_B\) then
\[
\|Y_A \otimes Z_B\|_{(t,p)} = \|Y_A\|_t \|Z_B\|_p.
\] (10)

But the definition of the \((t,p)\)-norm for a general \(X_{AB}\) is complicated and is deferred to Appendix A since we do not require the exact form here. We instead review some basic properties of completely bounded norms.

We know that when \(p = q = 1\), the supremum over \(d\) in (9) is indeed a maximum and is attained by letting \(d\) to be equal to the dimension of the input space of \(\Phi\) [19]. In general, however, the supremum over \(d\) is necessary.
In (9) the choice of $1 \leq t \leq \infty$ is arbitrary; For all values of $t$ we get the same number. In fact the completely bounded norm had been first defined for $t = \infty$, but then Pisier [17] showed that all values of $t$ result in the same operator norm.

From the definition of completely bounded norm (9) we have

$$\|\Phi \circ \Psi\|_{\text{CB}, p \to q} \leq \|\Psi\|_{\text{CB}, p \to r} \|\Phi\|_{\text{CB}, r \to q}.$$  

Also by considering $d = 1$ in the definition we find that

$$\|\Phi\|_{p \to q} \leq \|\Phi\|_{\text{CB}, p \to q}.$$  

In this paper we use the following important theorem proved in [16] (Theorems 11, 12, 13 and Corollary 14 therein).

**Theorem 1** [16].

(a) For CP super-operators $\Phi$, $\Psi$ and $1 \leq p, q \leq \infty$ we have

$$\|\Phi \otimes \Psi\|_{\text{CB}, p \to q} = \|\Phi\|_{\text{CB}, p \to q} \|\Psi\|_{\text{CB}, p \to q}.$$  

(b) If $\Phi$ is CP, the supremum in (9) for every $d$ is achieved at a positive semi-definite $X$.

(c) If $1 \leq q \leq p \leq \infty$ and $\Phi$ is CP then

$$\|\Phi\|_{\text{CB}, p \to q} = \|\Phi\|_{p \to q}.$$  

As a result, by (a) the super-operator norm $\|\cdot\|_{p \to q}$ is multiplicative on CP maps when $q \leq p$.

Part (b) of the above theorem is a generalization of the fact that the supremum in operator norm $\|\Phi\|_{p \to q}$ is achieved at a positive semi-definite $X$ when $\Phi$ is completely positive [13–15].

2.3. Riesz–Thorin theorem. Most of our results are based on the theory of complex interpolation. Here we do not need this theory in detail, so we just give a brief review. For more details we refer the reader to [20, 21].

Let $\mathcal{V}_0$ and $\mathcal{V}_1$ be two (complex) Banach spaces, i.e., two normed vector spaces that are complete under their norms. Moreover suppose that $\mathcal{V}_0$, $\mathcal{V}_1$ both can be embedded into one and the same larger vector space, i.e. $\mathcal{V}_0 \subset \mathcal{V}$ and $\mathcal{V}_1 \subset \mathcal{V}$ for some vector space $\mathcal{V}$. In this case $(\mathcal{V}_0, \mathcal{V}_1)$ is called an interpolation couple. Then in the theory of complex interpolation for every $0 \leq \theta \leq 1$, a Banach space is constructed that is somehow an intermediate space in between $\mathcal{V}_0$ and $\mathcal{V}_1$. These spaces are denoted by

$$[\mathcal{V}_0, \mathcal{V}_1]_\theta.$$  

The main example of interpolation spaces is Schatten classes. Let $L_p(\mathcal{H})$ be the space of linear operators on the Hilbert space $\mathcal{H}$ equipped with the $p$-norm. Then it is well-known that for $1 \leq p_0 \leq p_1 \leq \infty$ we have

$$[L_{p_0}(\mathcal{H}), L_{p_1}(\mathcal{H})]_\theta = L_{p_\theta}(\mathcal{H}),$$  \hspace{1cm} (11)

where $p_\theta$ is given by

$$\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$  \hspace{1cm} (12)
The other important example is the interpolation of \((p, q)\)-norms mentioned in the previous section. If we let \(L_{(p,q)}(\mathcal{H}_{AB})\) to be the space \(L(\mathcal{H}_{AB}) = L(\mathcal{H}_A) \otimes L(\mathcal{H}_B)\) equipped with the \((p, q)\)-norm then we have

\[
[L_{(p_0,q_0)}(\mathcal{H}_{AB}), L_{(p_1,q_1)}(\mathcal{H}_{AB})]_\theta = L_{(p_\theta,q_\theta)}(\mathcal{H}_{AB}),
\]

where again \(p_\theta\) and similarly \(q_\theta\) are defined by (12).

The main result that we require from the theory of interpolation is the following variant of the Riesz-Thorin theorem taken from [21]. Here for ease of notation we state the theory only in the case where the corresponding Banach spaces (as sets) are subsets of finite dimensional matrices.

Let

\[
S := \{z \in \mathbb{C} : 0 \leq \text{Re}z \leq 1\}.
\]

A map \(f : S \to L(\mathcal{H})\) is called holomorphic (bounded, continuous) if the corresponding maps to matrix entries are holomorphic (bounded, continuous).

**Theorem 2.** Let \(\mathcal{V}_0, \mathcal{V}_1 \subseteq L(\mathcal{H}_A)\) and \(\mathcal{W}_0, \mathcal{W}_1 \subseteq L(\mathcal{H}_B)\) be interpolations couples. Suppose that for every \(z \in S\) we have a super-operator \(T_z : L(\mathcal{H}_A) \to L(\mathcal{H}_B)\) such that for every \(X \in L(\mathcal{H}_A)\) the map \(z \mapsto T_z(X)\) is holomorphic and bounded in the interior of \(S\) and continuous on the boundary. Then we may consider \(T_z\) as a map from \(\mathcal{V}_k\) to \(\mathcal{W}_k\) \((k = 0, 1)\) and consider its super-operator norm

\[
\|T_z\|_{\mathcal{V}_k \to \mathcal{W}_k} = \sup_X \frac{\|T_z(X)\|_{\mathcal{W}_k}}{\|X\|_{\mathcal{V}_k}}.
\]

Define

\[
M_0 = \sup_{t \in \mathbb{R}} \|T_{it}\|_{\mathcal{V}_0 \to \mathcal{W}_0}, \quad M_1 = \sup_{t \in \mathbb{R}} \|T_{1+it}\|_{\mathcal{V}_1 \to \mathcal{W}_1}.
\]

Then for \(0 \leq \theta \leq 1\) we have

\[
\|T_\theta\|_{\mathcal{V}_0 \to \mathcal{W}_0} \leq M_0^{1-\theta} M_1^\theta,
\]

where \(\mathcal{V}_\theta = [\mathcal{V}_0, \mathcal{V}_1]_\theta\) and \(\mathcal{W}_\theta = [\mathcal{W}_0, \mathcal{W}_1]_\theta\).

Using (11) this theorem in particular gives the following.

**Theorem 3.** For every \(z \in S\), assume that \(T_z : L(\mathcal{H}_A) \to L(\mathcal{H}_B)\) is a linear operator such that for a fixed \(X\), \(z \mapsto T_z(X)\) is holomorphic and bounded in the interior of \(S\) and continuous on the boundary. Then for \(0 \leq \theta \leq 1\) we have

\[
\|T_\theta\|_{p_\theta \to q_\theta} \leq \left(\sup_{t \in \mathbb{R}} \|T_{it}\|_{p_0 \to q_0}\right)^{1-\theta} \left(\sup_{t \in \mathbb{R}} \|T_{1+it}\|_{p_1 \to q_1}\right)^\theta,
\]

where \(p_\theta, q_\theta\) are defined by (12).

In Appendix B we give a proof of the Riesz–Thorin theorem in the above special case. The proof in this case however captures the main ideas behind the Riesz–Thorin theorem in the general case.
2.4. Choi–Jamiołkowski isomorphism. We denote the unnormalized maximally entangled state by

\[ |\chi\rangle_{AA'} = \sum_{i=0}^{d_A-1} |i\rangle_A |i\rangle_{A'}, \]

where \{|0\rangle, |1\rangle, \ldots, |d_A - 1\rangle\} is a fixed orthonormal basis for \(\mathcal{H}_A\), and \(\mathcal{H}_{A'}\) is a copy of \(\mathcal{H}_A\). For a given \(\eta_{AB} \in \mathbf{L}(\mathcal{H}_A) \otimes \mathbf{L}(\mathcal{H}_B)\) we may consider the corresponding superoperator \(\Omega_\eta : \mathbf{L}(\mathcal{H}_A) \rightarrow \mathbf{L}(\mathcal{H}_B)\) via the Choi–Jamiołkowski isomorphism:

\[ \eta_{AB} = \mathcal{I}_A \otimes \Omega_\eta (|\chi\rangle\langle\chi|_{AA'}) = \sum_{i,j=0}^{d_A-1} |i\rangle\langle j| \otimes \Omega_\eta(|i\rangle\langle j|). \]

Then for every \(X \in \mathbf{L}(\mathcal{H}_A)\) we have

\[ \Omega_\eta(X) = \text{tr}_A \left( \eta_{AB}(X^T \otimes I_B) \right), \quad (13) \]

where \(\text{tr}_A\) denotes the partial trace with respect to subsystem \(A\). This in particular implies that

\[ \text{tr} \left( Y \Omega_\eta(X) \right) = \text{tr} \left( \eta_{AB}(X^T \otimes Y) \right). \quad (14) \]

Moreover \(\Omega_\eta\) is CP if and only if \(\eta\) is positive semi-definite. Also observe that \(\Omega_\eta \otimes \Omega_{\eta'} = \Omega_{\eta \otimes \eta'}\).

Using (14) and Hölder’s duality we have

\[ \|\Omega_\eta\|_{p \rightarrow q} = \sup_{\|X\|_p = \|Y\|_{q'}} |\text{tr} (\eta_{AB}(X \otimes Y))|. \quad (15) \]

Similarly the completely bounded norm is computed as

\[ \|\Omega_\eta\|_{CB, p \rightarrow q} = \sup_d \sup_{\|X\|(t,p) = \|Y\|(t',q')} |\text{tr} ((|\chi\rangle\langle\chi|_{CC'} \otimes \eta_{AB})(X_{CA} \otimes Y_{C'B}))|. \quad (16) \]

where \(\mathcal{H}_C = \mathcal{H}_{C'}\) is a Hilbert space with dimension \(d\). Here we use the fact that the dual norm of \((t, p)\) is \((t', p')\) (see Appendix A for more details) as well as the fact that \(|\chi\rangle\langle\chi|\) is the Choi-Jamiołkowski representation of identity super-operator.

We now have all the tools required to state our results.

3. The Hypercontractivity Ribbon

For a given bipartite state \(\rho_{AB} \in \mathbf{L}(\mathcal{H}_A) \otimes \mathbf{L}(\mathcal{H}_B)\) and \(1 \leq p, q \leq \infty\) let

\[ \tilde{\rho}_{AB}^{(p,q)} = \left( \rho_A^{-\frac{1}{2p}} \otimes \rho_B^{-\frac{1}{2q}} \right) \rho_{AB} \left( \rho_A^{-\frac{1}{2p}} \otimes \rho_B^{-\frac{1}{2q}} \right)^* . \quad (17) \]

When it is clear from the context, we will drop the superscript \((p, q)\) and subscript \(AB\) and simply write \(\tilde{\rho}\). From the definition the 'tilde' operator corresponding to \(\rho_{AB}^{\otimes n}\) is equal to \(\tilde{\rho}_{AB}^{\otimes n}\).
Since $\tilde{\rho}_{AB}$ is in $\mathbf{L}(\mathcal{H}_A) \otimes \mathbf{L}(\mathcal{H}_B)$ we may consider the corresponding super-operator $\Omega_{\tilde{\rho}} : \mathbf{L}(\mathcal{H}_A) \rightarrow \mathbf{L}(\mathcal{H}_B)$ via the Choi–Jamiołkowski representation:

$$\tilde{\rho}_{AB} = \sum_{i,j=0}^{d_A-1} |i\rangle\langle j|_A \otimes \Omega_{\tilde{\rho}}(|i\rangle\langle j|)_B. \quad (18)$$

Observe that since $\tilde{\rho}_{AB}$ for every density matrix $\rho_{AB}$ is positive semi-definite, the corresponding super-operator $\Omega_{\tilde{\rho}}$ is CP.

**Definition 4.** For every bipartite quantum state $\rho_{AB}$ and integer $n \geq 1$ define

$$\mathcal{R}^{(n)}_{A\rightarrow B}(\rho_{AB}) := \left\{(p, q') \in \mathbb{R}^2 : q' \geq p \geq 1, \|\Omega^{\otimes n}_{\rho_{AB}}\|_{p\rightarrow q'} \leq 1 \right\}, \quad (19)$$

and let

$$\mathcal{R}_{A\rightarrow B}(\rho_{AB}) := \bigcap_{n \geq 1} \mathcal{R}^{(n)}_{A\rightarrow B}(\rho_{AB}). \quad (20)$$

Moreover define

$$\mathcal{R}_{CB,A\rightarrow B}(\rho_{AB}) := \left\{(p, q') \in \mathbb{R}^2 : q' \geq p \geq 1, \|\Omega^{\otimes (p,q)}_{\rho_{AB}}\|_{CB,p\rightarrow q} \leq 1 \right\}. \quad (21)$$

Here $q'$ is the Hölder conjugate of $q$ defined by (6). Following [6,9] we call $\mathcal{R}_{A\rightarrow B}(\rho_{AB})$ and $\mathcal{R}_{CB,A\rightarrow B}(\rho_{AB})$ the hypercontractivity ribbons (HR).

From the definition and (8) it is clear that

$$\mathcal{R}^{(nk)}_{A\rightarrow B}(\rho_{AB}) \subseteq \mathcal{R}^{(k)}_{A\rightarrow B},$$

and then

$$\mathcal{R}_{A\rightarrow B}(\rho^{\otimes k}_{AB}) = \mathcal{R}_{A\rightarrow B}(\rho_{AB}). \quad (22)$$

Moreover, by Theorem 1 the completely bounded norm is multiplicative on CP maps. Then we also have

$$\mathcal{R}_{CB,A\rightarrow B}(\rho^{\otimes k}_{AB}) = \mathcal{R}_{CB,A\rightarrow B}(\rho_{AB}). \quad (23)$$

Using (15), for a pair $(p, q')$ we have $(p, q') \in \mathcal{R}^{(1)}_{A\rightarrow B}(\rho_{AB})$ if and only if

$$\text{tr} \left( \tilde{\rho}_{AB}^{(p,q')} X \otimes Y \right) \leq 1, \quad (24)$$

for all $X, Y$ such that $\|X\|_p = \|Y\|_q = 1$. From this equation it is clear that if we similar to $\Omega_{\tilde{\rho}^{(p,q)}}$ define a super-operator $\Lambda_{\tilde{\rho}} : \mathbf{L}(\mathcal{H}_B) \rightarrow \mathbf{L}(\mathcal{H}_A)$, then $\|\Omega_{\tilde{\rho}^{(p,q)}}\|_{p\rightarrow q'} \leq 1$ if and only if $\|\Lambda_{\tilde{\rho}^{(p,q)}}\|_{q\rightarrow p'} \leq 1$. In fact we may define the hypercontractivity ribbon from $B$ to $A$ by

$$\mathcal{R}^{(n)}_{B\rightarrow A}(\rho_{AB}) := \left\{(q, p') \in \mathbb{R}^2 : p' \geq q \geq 1, ||\Lambda^{\otimes n}_{\tilde{\rho}_{AB}}||_{q\rightarrow p'} \leq 1 \right\}. \quad (24)$$

We then have

$$\text{If and only if } (p, q') \in \mathcal{R}^{(n)}_{A\rightarrow B}(\rho_{AB}) \iff (q, p') \in \mathcal{R}^{(n)}_{B\rightarrow A}(\rho_{AB}). \quad (25)$$

The same argument goes through for the completely bounded norm using (16), so we have

$$\text{If and only if } (p, q') \in \mathcal{R}_{CB,A\rightarrow B}(\rho_{AB}) \iff (q, p') \in \mathcal{R}_{CB,B\rightarrow A}(\rho_{AB}). \quad (26)$$
Remark 5. In definitions (19) and (21) we put the restriction $q' \geq p$ which seems unnecessary. In Theorem 13 we will justify this assumption by showing that for $q' \leq p$ the super-operator norms $\|\Omega_{\rho_{AB}}^{\otimes n}\|_{p\rightarrow q'}$ and $\|\Omega_{\tilde{\rho}_{CB}}^{\otimes n}\|_{CB,p\rightarrow q'}$ are always equal to 1, and give no information about $\rho_{AB}$.

Remark 6. By the definition of the completely bounded norm and Theorem 1 we have

$$\|\Omega_{\rho_{AB}}^{\otimes n}\|_{p\rightarrow q'} \leq \|\Omega_{\tilde{\rho}_{AB}}^{\otimes n}\|_{CB,p\rightarrow q'} = \|\Omega_{\tilde{\rho}_{AB}}^{\otimes n}\|_{p\rightarrow q'}.$$ 

Therefore if $(p, q') \in \mathcal{R}_{CB,A\rightarrow B}(\rho_{AB})$ then $(p, q') \in \mathcal{R}_{A\rightarrow B}(\rho_{AB})$. In fact we always have

$$\mathcal{R}_{CB,A\rightarrow B}(\rho_{AB}) \subseteq \mathcal{R}_{A\rightarrow B}(\rho_{AB}).$$  

(27)

Remark 7. Observe that

$$\text{tr} \left( \tilde{\rho}^{(p,q)} \rho_A^{1/p} \otimes \rho_B^{1/q} \right) = 1.$$ 

This means that in (24) if we let $X = \rho_A^{1/p}$ and $Y = \rho_B^{1/q}$ we get equality. As a result for all $p, q$ both $\|\Omega_{\tilde{\rho}}^{\otimes n}\|_{p\rightarrow q'}$ and $\|\Omega_{\tilde{\rho}}^{\otimes n}\|_{CB,p\rightarrow q'}$ are at least 1. Therefore $(p, q') \in \mathcal{R}_{A\rightarrow B}(\rho_{AB})$ and $(p, q') \in \mathcal{R}_{CB,A\rightarrow B}(\rho_{AB})$ indeed mean $\|\Omega_{\tilde{\rho}}^{\otimes n}\|_{p\rightarrow q'} = 1$ and $\|\Omega_{\tilde{\rho}}^{\otimes n}\|_{CB,p\rightarrow q'} = 1$ respectively.

Remark 8. By Theorem 19 the two ribbons $\mathcal{R}_{A\rightarrow B}(\rho_{AB})$ and $\mathcal{R}_{CB,A\rightarrow B}(\rho_{AB})$ coincide when $\rho_{AB}$ is classical.

3.1. Hypercontractivity ribbons under CPTP maps. In this section by studying the behavior of HRs under CPTP maps we show that they are indeed measures of correlation. But before that we need to derive an expression for $\tilde{\rho}_{AB}$.

By definition we have

$$\rho_{AB} = \sum_{i,j=0}^{d_A-1} |i\rangle \langle j| \otimes \Omega_{\rho}(|i\rangle \langle j|)$$  

(28)

Therefore,

$$\tilde{\rho}_{AB} = (\rho_A^{1/2p} \otimes \rho_B^{1/2q}) \rho_{AB} (\rho_A^{1/2p} \otimes \rho_B^{1/2q})$$

$$= \sum_{i,j} \rho_A^{1/2p} |i\rangle \langle j| \rho_A^{1/2p} \otimes \rho_B^{1/2q} \Omega_{\rho}(|i\rangle \langle j|) \rho_B^{1/2q}$$

$$= \sum_{i,j,k,l} |k\rangle \langle k| \rho_A^{1/2p} |i\rangle \langle j| \rho_A^{1/2p} |l\rangle \langle l| \otimes \rho_B^{1/2q} \Omega_{\rho}(|i\rangle \langle j|) \rho_B^{1/2q}$$

$$= \sum_{i,j,k,l} |k\rangle \langle i| \rho_A^{* -1/2p} |k\rangle \langle l| \rho_A^{* -1/2p} \Omega_{\rho}(|i\rangle \langle j|) \rho_B^{1/2q}$$

$$= \sum_{i,j,k,l} |k\rangle \langle l| \otimes \rho_B^{1/2q} \Omega_{\rho}(|i\rangle \langle j|) \rho_A^{* -1/2p} |k\rangle \langle l| \rho_A^{* -1/2p} \Omega_{\rho}(|i\rangle \langle j|) \rho_B^{-1/2q}$$

$$= \sum_{k,l} |k\rangle \langle l| \otimes \rho_B^{1/2q} \Omega_{\rho} (\rho_A^{* -1/2p} |k\rangle \langle l| \rho_A^{* -1/2p} \rho_B^{1/2q}).$$
Comparing to (18) we conclude that

$$\Omega_{\overline{\rho}}(X) = \rho_B^{-\frac{1}{p}} \Omega_{\rho}(\rho_A^{-\frac{1}{p}} X \rho_A^{-\frac{1}{p}}) \rho_B^{-\frac{1}{p}}.$$ 

For positive semi-definite $\tau \in \mathcal{L}(\mathcal{H}_A)$ define the super-operator $\Gamma_\tau : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$ by

$$\Gamma_\tau(X) = \tau^{\frac{1}{2}} X \tau^{\frac{1}{2}}.$$ 

Then observe that $\Gamma_\tau^\alpha(X) = \tau^{\alpha/2} X \tau^{\alpha/2}$ for all $\alpha$. Using this notation we have

$$\Omega_{\overline{\rho}} \approx \Gamma_1^{-\frac{1}{q}} \rho \circ \Gamma_1 \rho \circ \Gamma_1^{-\frac{1}{p}} \rho.$$ (29)

Before proving the main result of this section we need the following important lemma.

**Lemma 9.** For a CPTP map $\Phi$, density matrix $\tau$ and $1 \leq p, q \leq \infty$ define

$$\overline{\Phi} = \overline{\Phi}^{(p,q)} = \Gamma_\Phi(\tau) \circ \Phi \circ \Gamma_1^{\frac{1}{p}}.$$ (30)

Then if $q \geq p$,

$$\|\overline{\Phi}\|_{p' \to q'} = \|\overline{\Phi}\|_{CB, p' \to q'} \leq 1.$$ 

**Proof.** Observe that $\overline{\Phi}$ is CP. Then by part (c) of Theorem 1, $\|\overline{\Phi}\|_{p' \to q'} = \|\overline{\Phi}\|_{CB, p' \to q'}$ since $q \geq p$ implies $p' \geq q'$. Thus we need to prove $\|\overline{\Phi}\|_{p' \to q'} \leq 1$.

Let $1 \leq p_0 < p_1 \leq \infty$ and $1 \leq q_0 < q_1 \leq \infty$, and define $p_z$ and $q_z$ by

$$\frac{1}{p_z} = \frac{1}{p_0} \frac{z}{p_1} \quad \text{and} \quad \frac{1}{q_z} = \frac{1}{q_0} \frac{z}{q_1}.$$ 

Observe that

$$\frac{1}{p_z'} = \frac{1}{p_0'} \frac{z}{p_1'} \quad \text{and} \quad \frac{1}{q_z'} = \frac{1}{q_0'} \frac{z}{q_1'},$$

where $1/p_z + 1/p_z' = 1$ and $1/q_z + 1/q_z' = 1$. Now define

$$T_z = \frac{1}{q_z'} \circ \Phi \circ \Gamma_1^{\frac{1}{p_z}}.$$ 

$T_z$ satisfies the assumptions of Theorem 2. As a result, for every $0 < \theta < 1$ we have

$$\|T_\theta\|_{p_0' \to q_0'} \leq \left( \sup_{t \in \mathbb{R}} \|T_{1+t}\|_{p_0' \to q_0'} \right)^{(1-\theta)} \left( \sup_{t \in \mathbb{R}} \|T_{1+t}\|_{p_1' \to q_1'} \right)^{\theta}.$$ 

Observe that for every $t \in \mathbb{R}$ there are unitaries $U, V$ such that

$$T_{1+t}(X) = UT_0(VXV^\dagger)U^\dagger.$$
Theorem 10. sandwiched (quantum) Rényi relative entropy equality for the

Here we use the fact that \( \tau \) and \( \Phi(\tau) \) are hermitian and then \( \tau^{\pm i\tau} \) and \( \Phi(\tau)^{\pm i\tau} \) are unitary.

As a result we have \( \|T_{i\tau}\|_{p'_0\rightarrow q'_0} = \|T_0\|_{p'_0\rightarrow q'_0} \). We similarly have \( \|T_{1+i\tau}\|_{p'_1\rightarrow q'_1} = \|T_1\|_{p'_1\rightarrow q'_1} \). Then we arrive at

\[
\|T_\theta\|_{p'_0\rightarrow q'_0} \leq \|T_0\|_{p'_0\rightarrow q'_0}^{(1-\theta)} \|T_1\|_{p'_1\rightarrow q'_1}^\theta.
\]

(31)

This means that if \( \|T_0\|_{p'_0\rightarrow q'_0} \) and \( \|T_1\|_{p'_1\rightarrow q'_1} \) are at most 1, then \( \|T_\theta\|_{p'_0\rightarrow q'_0} \) is at most 1 too.

Based on this observation if we prove the lemma in the special cases of \((p, q) = (1, q)\) and \((p, q) = (q, q)\) for arbitrary \(1 \leq q \leq \infty\), then we have the result for all \(1 \leq p \leq q \leq \infty\). For these two cases we can again use (31). If we prove the result for the three cases \((p, q) \in \{(1, 1), (1, \infty), (\infty, \infty)\}\) then we obtain a proof for all \(1 \leq p \leq q \leq \infty\).

The case \((p, q) = (\infty, \infty)\) is verified noting that \( \Phi \) is completely positive, so using (7) we can restrict the supremum over positive inputs, and also that \( \Phi \) is trace preserving. For the case \(p = 1\) and \(q = \infty\), note that \( \Phi \) is CP, so the maximum of \( \|\Phi(X)\|_{q'} \) over all \(X\) with \(\|X\|_\infty = 1\) is obtained at \(X = I\) (see [22]). The same argument works for \(p = q = 1\) too. \(\square\)

The special case of this lemma for \(p = q\) is equivalent to the data processing inequality for the sandwiched (quantum) Rényi relative entropy [23].

**Theorem 10.** For a CPTP map \( \Phi : \mathcal{L}(\mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_B') \) let \( \sigma_{AB'} = I_A \otimes \Phi(\rho_{AB}) \). Then for all \(n \geq 1\) we have

\[
\mathcal{R}_{A\rightarrow B}^{(n)}(\rho_{AB}) \subseteq \mathcal{R}_{A\rightarrow B'}^{(n)}(\sigma_{AB'}),
\]

and

\[
\mathcal{R}_{CB,A\rightarrow B}^{(2)}(\rho_{AB}) \subseteq \mathcal{R}_{CB,A\rightarrow B'}^{(2)}(\sigma_{AB'}).
\]

**Proof.** We need to show that for a pair \((p, q)\), if \(\|\Omega_\rho\|_{CB,p\rightarrow q'} \leq 1\) then \(\|\Omega_\sigma\|_{CB,p\rightarrow q'} \leq 1\), and that if \(\|\Omega_{\rho}^{\otimes n}\|_{p\rightarrow q'} \leq 1\) then \(\|\Omega_{\sigma}^{\otimes n}\|_{p\rightarrow q'} \leq 1\).

By

\[
\sigma_{AB'} = I \otimes \Phi(\rho_{AB}) = \sum |i\rangle\langle j| \otimes \Phi(\Omega_\rho(|i\rangle\langle j|)),
\]

we have \(\Omega_\sigma = \Phi \circ \Omega_\rho\). Moreover, \(\sigma_A = tr_{B'}(\sigma_{AB'}) = tr_{B'}(I \otimes \Phi(\rho_{AB})) = \rho_A\), and \(\sigma_{B'} = \Phi(\rho_B)\). Then using (29) we compute

\[
\Omega_\sigma = \Gamma_{\sigma_{B'}}^{-\frac{1}{q}} \circ \Omega_\sigma \circ \Gamma_{\sigma_A}^{-\frac{1}{p}}
\]

\[
= \Gamma_{\sigma_{B'}}^{-\frac{1}{q}} \circ \Phi \circ \Gamma_{\sigma_A}^{-\frac{1}{p}}
\]

\[
= \left( \Gamma_{\Phi(\rho_B)}^{-\frac{1}{q}} \circ \Phi \circ \Gamma_{\rho_B}^{-\frac{1}{q}} \right) \circ \left( \Gamma_{\rho_B}^{-\frac{1}{q}} \circ \Omega_\rho \circ \Gamma_{\rho_A}^{-\frac{1}{p}} \right)
\]

\[
= \tilde{\Phi} \circ \Omega_{\tilde{\sigma}},
\]

where we set

\[
\tilde{\Phi} = \Gamma_{\Phi(\rho_B)}^{-\frac{1}{q}} \circ \Phi \circ \Gamma_{\rho_B}^{-\frac{1}{q}}.
\]

(32)
We now have
\[
\|\Omega_\sigma\|_{CB,p\rightarrow q'} = \|\Phi \circ \Omega_{\tilde{\rho}}\|_{CB,p\rightarrow q'} \\
\leq \|\Phi\|_{CB,q'\rightarrow q} \|\Omega_{\tilde{\rho}}\|_{CB,p\rightarrow q'} \\
\leq \|\Omega_{\tilde{\rho}}\|_{CB,p\rightarrow q'},
\]
where the last inequality is implied by Lemma 9. As a result, if \((p, q') \in R_{CB,A\rightarrow B}(\rho_{AB})\) then \((p, q') \in R_{A\rightarrow B}(\sigma_{A'B'})\).

The proof of \(\|\Omega_\sigma^{\otimes n}\|_{p\rightarrow q'} \leq \|\Omega_\rho^{\otimes n}\|_{p\rightarrow q'}\) is identical noting that \(\sigma_{A'B'}^{\otimes n} = \mathcal{I}^{\otimes n} \otimes \Phi^{\otimes n}(\rho_{AB}^{\otimes n})\). \(\square\)

We are now ready to prove the main result of this paper.

**Corollary 11.** Suppose that there are CPTP maps \(\Phi_{A^n \rightarrow A'}\) and \(\Psi_{B^n \rightarrow B'}\) such that \(\sigma_{A'B'} = \Phi \otimes \Psi(\rho_{AB}^{\otimes n})\). Then we have
\[
R_{CB,A\rightarrow B}(\rho_{AB}) \subseteq R_{CB,A'\rightarrow B'}(\sigma_{A'B'}),
\]
and
\[
R_{A\rightarrow B}^{(n)}(\rho_{AB}) \subseteq R_{A'\rightarrow B'}^{(n)}(\sigma_{A'B'}),
\]
which in particular gives \(R_{A\rightarrow B}(\rho_{AB}) \subseteq R_{A'\rightarrow B'}(\sigma_{A'B'})\).

**Proof.** Let \(\tau_{A^nB'} = \mathcal{I}^{\otimes n} \otimes \Psi(\rho_{AB}^{\otimes n})\). Then we just need to prove
\[
R_{A\rightarrow B}^{(n)}(\rho_{AB}) \subseteq R_{A'\rightarrow B'}^{(n)}(\tau_{A^nB'} \subseteq R_{A'\rightarrow B'}^{(n)}(\sigma_{A'B'}),
\]
and
\[
R_{CB,A\rightarrow B}(\rho_{AB}) \subseteq R_{CB,A^n\rightarrow B'}(\tau_{A^nB'} \subseteq R_{CB,A'\rightarrow B'}(\sigma_{A'B'}).
\]
The first inclusions are straightforward consequences of Theorem 10. The second inclusions are similarly proved by exchanging the roles of registers A and B and using (25) and (26). \(\square\)

We say that a resource state \(\rho_{AB}\) can be asymptotically transformed to \(\sigma_{A'B'}\) under local transformations, if for every \(\epsilon > 0\) there exists \(n\) and local operations \(\Phi_{A^n \rightarrow A'}\) and \(\Psi_{B^n \rightarrow B'}\) such that
\[
\|\sigma_{A'B'} - \Phi \otimes \Psi(\rho_{AB}^{\otimes n})\|_1 \leq \epsilon.
\]
Note that if \(\Phi, \Psi\) exist for some \(n\), then such local operations exist for all \(m > n\) (simply ignore the extra \(m - n\) copies of \(\rho_{AB}\)).

**Corollary 12.** Suppose that \(\rho_{AB}\) can be asymptotically transformed to \(\sigma_{A'B'}\) under local transformations. Then we have
\[
R_{CB,A\rightarrow B}(\rho_{AB}) \subseteq R_{CB,A'\rightarrow B'}(\sigma_{A'B'}),
\]
and
\[
R_{A\rightarrow B}(\rho_{AB}) \subseteq R_{A'\rightarrow B'}(\sigma_{A'B'}),
\]
Proof. Let \( \tau_\epsilon \) be the bipartite state for which there are \( n \) and \( \Phi, \Psi \) such that \( \Phi \otimes \Psi(\rho_{AB}^{\otimes n}) = \tau_\epsilon \), and

\[
\| \sigma - \tau_\epsilon \|_1 \leq \epsilon.
\]

Then \( \tau_\epsilon \) tends to \( \sigma \) as \( \epsilon \to 0 \) in 1-norm. This implies that for every \( p, q \) and \( m \), \( \tau_\epsilon \otimes m \) tends to \( \sigma \otimes m \) in 1-norm, and in fact in any other norm. Here we use the fact that in finite dimensions all norms are equivalent. This in particular gives that for every \( p, q \) and \( m \)

\[
\lim_{\epsilon \to 0} \| \Omega_{\tau_\epsilon}^{\otimes m} \|_{p \to q'} = \| \Omega_{\sigma}^{\otimes m} \|_{p \to q'},
\]

Therefore, if \( (p, q') \in R_{A \rightarrow B}(\rho_{AB}) \), using Corollary 11, \( (p, q') \in R_{A \rightarrow B}(\tau_\epsilon) \), hence \( \| \Omega_{\tau_\epsilon}^{\otimes m} \|_{p \to q'} \leq 1 \) for all \( m \). Fixing \( m \) and taking the limit \( \epsilon \to 0 \) and using the above equality, for all \( m \) we have \( \| \Omega_{\sigma}^{\otimes m} \|_{p \to q'} \leq 1 \). Thus \( (p, q') \in R_{A' \rightarrow B'}(\sigma_{A'B'}) \) and \( R_{A \rightarrow B}(\rho_{AB}) \subseteq R_{A' \rightarrow B'}(\sigma_{A'B'}) \). Similarly we have

\[
\lim_{\epsilon \to 0} \| \Omega_{\tau_\epsilon} \|_{CB, p \to q'} = \| \Omega_{\sigma} \|_{CB, p \to q'},
\]

and again using Corollary 11 and by sending \( \epsilon \) to zero, we obtain \( R_{CB, A \rightarrow B}(\rho_{AB}) \subseteq R_{CB, A' \rightarrow B'}(\sigma_{A'B'}) \).

3.2. Some properties of hypercontractivity ribbons. In this section we further investigate properties of HRs. These properties may be useful in computing the ribbons and also to compare the ribbons, as measures of correlation, to other such measures.

First as announced in Remark 5 we justify the assumption \( q' \geq p \) in the definition of HRs.

**Theorem 13.** For all \( 1 \leq q' \leq p \leq \infty \) we have

\[
\| \Omega_\eta \|_{p \to q'} = \| \Omega_\eta \|_{CB, p \to q'} \leq 1.
\]

Proof. Let

\[
\Phi = \Omega_\rho \circ \Gamma^{-1}_{\rho_A}.
\]

\( \Phi \) is obviously CP. Moreover according to (13) we have

\[
\tr (\Phi(X)) = \tr \left( \rho_{AB} \left( \Gamma^{-1}_{\rho_A}(X)^T \otimes I_B \right) \right)
\]

\[
= \tr \left( \rho_{AB} \left( \rho_A^{-1/2} X^T \rho_A^{-1/2} \otimes I_B \right) \right)
\]

\[
= \tr \left( \rho_A \left( \rho_A^{-1/2} X^T \rho_A^{-1/2} \right) \right)
\]

\[
= \tr (X).
\]

This means that \( \Phi \) is also trace preserving and then CPTP. Repeating similar calculations shows that \( \Phi(\rho_A^\eta) = \rho_B \). Then the proof is finished using Lemma 9 and noting that

\[
\Phi(p', q') = \Gamma^{-1}_{\rho_A} \circ \Phi \circ \Gamma_{\rho_A} = \Gamma^{-1}_{\rho_A} \circ \Omega_\rho \circ \Gamma_{\rho_A} = \Omega_\eta.
\]
Theorem 14. The regions

\[ \left\{ \frac{1}{p}, \frac{1}{q'} : (p, q') \in \mathcal{R}_{\text{CB,} A \rightarrow B}(\rho_{AB}) \right\}, \]

and

\[ \left\{ \frac{1}{p}, \frac{1}{q'} : (p, q') \in \mathcal{R}_{A \rightarrow B}^{(n)}(\rho_{AB}) \right\}, \]

for every \( n \) are convex.

Proof. We should show that if \((p_0, q_0'), (p_1, q_1')\) are in \( \mathcal{R}_{A \rightarrow B}^{(n)}(\rho_{AB}) \) (or \( \mathcal{R}_{\text{CB,} A \rightarrow B}(\rho_{AB}) \)) then \((p_0, q_0')\) is also in \( \mathcal{R}_{A \rightarrow B}^{(n)}(\rho_{AB}) \) (or \( \mathcal{R}_{\text{CB,} A \rightarrow B}(\rho_{AB}) \)) where \( 0 < \theta < 1 \) and

\[
\frac{1}{\rho_0} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \text{and} \quad \frac{1}{q_0'} = \frac{1 - \theta}{q_0'} + \frac{\theta}{q_1'}.
\]

The proof is based on Theorem 2. In fact following similar steps as in the proof of Lemma 9 we obtain

\[
\| \Omega_{\rho_{\theta}(p_0 \rho_{0}, q_0)} \|_{p_0 \rightarrow q_0'} \leq \| \Omega_{\rho_{\theta}(p_0, q_0)} \|_{p_0, q_0} \|_{1 - \theta} \|_{p_0 \rightarrow q_0} \|_{\rho_{\theta}(p_1, q_1)} \|_{\theta} \|_{p_1 \rightarrow q_1'};
\]

and

\[
\| \Omega_{\rho_{\theta}(p_0 \rho_{0}, q_0)} \|_{\text{CB}, p_0 \rightarrow q_0'} \leq \| \Omega_{\rho_{\theta}(p_0, q_0)} \|_{\text{CB}, p_0, q_0} \|_{\theta} \|_{\rho_{\theta}(p_1, q_1)} \|_{\theta} \|_{\text{CB}, p_1 \rightarrow q_1'}.
\]

We are done. □

The following lemma is sometimes useful in estimating HRs.

Lemma 15. Assume that \((p, q') \in \mathcal{R}_{A \rightarrow B}^{(1)}(\rho_{AB})\). Then for \( M, N \geq 0 \) we have

\[
\text{tr} \left( \rho_{AB} M \otimes N \right) \leq \text{tr} \left( \rho_{A} M^p \right)^{1/p} \text{tr} \left( \rho_{B} N^q \right)^{1/q}. \tag{33}
\]

As a conclusion, if \( 0 \leq M \leq I \) and \( 0 \leq N \leq I \) we have,

\[
\text{tr} \left( \rho_{AB} M \otimes N \right) \leq \text{tr} \left( \rho_{A} M \right)^{1/p} \text{tr} \left( \rho_{B} N \right)^{1/q}.
\]

Proof. We compute

\[
\text{tr} \left( \rho_{AB} M \otimes N \right) = \text{tr} \left( \tilde{\rho}_{AB} \left( \rho_{A}^{1/2} M \rho_{A}^{1/2} \right) \otimes \left( \rho_{B}^{1/2} N \rho_{B}^{1/2} \right) \right)
= \text{tr} \left( \Omega_{\tilde{\rho}} \left( \rho_{A}^{1/2} M^{T} \rho_{A}^{1/2} \right) \right) \left( \rho_{B}^{1/2} N \rho_{B}^{1/2} \right)
\leq \| \Omega_{\tilde{\rho}} \| \left( \rho_{A}^{1/2} M^{T} \rho_{A}^{1/2} \right) \|_{q'} \left( \rho_{B}^{1/2} N \rho_{B}^{1/2} \right) \|_{q}
\leq \| \rho_{A}^{1/2} M^{T} \rho_{A}^{1/2} \| \left( \rho_{B}^{1/2} N \rho_{B}^{1/2} \right) \|_{q}
\leq \| \rho_{A}^{1/2} M \rho_{A}^{1/2} \| \| \rho_{B}^{1/2} N \rho_{B}^{1/2} \| \|_{q}
\leq \left( \text{tr} \left( \rho_{A} M \right) \right)^{1/p} \left( \text{tr} \left( \rho_{B} N \right) \right)^{1/q}.
\]

Here in the third line we use Hölder’s inequality, and in the last line we use the Lieb–Thirring trace inequality [24]. □
The next theorem draws a connection between HR and the maximal correlation. This statement in the classical case was first proved in [8].

**Theorem 16.** Assume that \((p, q') \in \mathcal{R}_{\to B}^{(1)}(\rho_{AB})\). Then we have

\[
\frac{p - 1}{q' - 1} = \frac{pq}{p'q'} \geq \mu^2,
\]

where \(\mu = \mu(\rho_{AB})\) is the maximal correlation defined in (1).

Before giving a proof note that by this theorem, both the regions \(\mathcal{R}_{\to B}(\rho_{AB})\) and \(\mathcal{R}_{CB,\to B}(\rho_{AB})\) are in between lines \(x = y\) and \(x - 1 = \mu^2(y - 1)\) in the real plane.

**Proof.** Assume that \(X\) and \(Y\) are the optimal matrices that achieve the maximum in the definition of maximal correlation. Therefore,

\[
\mu = |\text{tr} \left( \rho_{AB} X \otimes Y^\dagger \right)|
\]

\[
\text{tr} (\rho_A X) = \text{tr} (\rho_B Y) = 0
\]

\[
\text{tr} \left( \rho_A XX^\dagger \right) = \text{tr} \left( \rho_B YY^\dagger \right) = 1.
\]

(34)

Note that as proved in [5] without loss of generality we may assume that \(X, Y\) are hermitian.

For \(\alpha, \beta, x \in \mathbb{R}\) let

\[
M_x = I + x\alpha X, \quad \text{and} \quad N_x = I + x\beta Y.
\]

Observe that \(M_x, N_x\) are positive semi-definite for small enough \(|x|\) (for fixed \(\alpha, \beta\)). Then by Lemma 15 we have

\[
\text{tr} \left( \rho_{AB} M_x^{1/p} \otimes N_x^{1/q} \right) \leq \text{tr} \left( \rho_A M_x \right)^{1/p} \text{tr} \left( \rho_B N_x \right)^{1/q}.
\]

Using \(\text{tr} (\rho_A X) = \text{tr} (\rho_B Y) = 0\) this inequality is simplified to

\[
f(x) := \text{tr} \left( \rho_{AB} M_x^{1/p} \otimes N_x^{1/q} \right) - 1 \leq 0,
\]

for small \(|x|\). Note that \(f(0) = 0\) and

\[
f'(x) = \frac{d}{dx} f(x) = \frac{\alpha}{p} \text{tr} \left( \rho_{AB} \left( X M_x^{1/p-1} \otimes N_x^{1/q} \right) \right) + \frac{\beta}{q} \text{tr} \left( \rho_{AB} \left( M_x^{1/p} \otimes Y N_x^{1/q-1} \right) \right).
\]

To compute this derivative we use the fact that the pairs \(X, M_x\) and \(Y, N_x\) commute. As a result, \(f'(0) = 0\). Then using the fact that \(f(x)\) is not positive in a neighborhood of 0, we should have \(f''(0) \leq 0\). The second derivative of \(f(x)\) is computed as

\[
f''(x) = \frac{\alpha^2}{p} \left( \frac{1}{p} - 1 \right) \text{tr} \left( \rho_{AB} \left( X^2 M_x^{1/p-2} \otimes N_x^{1/q} \right) \right)
\]

\[
+ \frac{2\alpha\beta}{pq} \text{tr} \left( \rho_{AB} \left( X M_x^{1/p-1} \otimes Y N_x^{1/q-1} \right) \right)
\]

\[
+ \frac{\beta^2}{q} \left( \frac{1}{q} - 1 \right) \text{tr} \left( \rho_{AB} \left( M_x^{1/p} \otimes Y^2 N_x^{1/q-2} \right) \right).
\]
Then using (34) and the fact that $X, Y$ are hermitian we have

$$f''(0) = \alpha^2 \frac{1}{p} \left( \frac{1}{p} - 1 \right) + \frac{2}{pq} \alpha \beta \mu + \frac{1}{q} \left( \frac{1}{q} - 1 \right) \beta^2$$

$$= -\alpha^2 \frac{1}{pp'} + \alpha \beta \frac{2}{pq} \mu - \beta^2 \frac{1}{qq'}$$

$$\leq 0.$$  

This inequality should hold for all $\alpha, \beta \in \mathbb{R}$. Therefore the determinant of the coefficient matrix

$$\left( \begin{array}{cc} \frac{1}{pp'} & -\frac{\mu}{pq} \\ -\frac{\mu}{pq} & \frac{1}{qq'} \end{array} \right)$$

should be non-negative. This gives the desired result. \( \square \)

Finally the following theorem in the classical case was proved in [8] (see also [9]).

**Theorem 17.** Assume that $(p, q') \in \mathcal{R}_{A \to B}^{(1)}(\rho_{AB})$ for $p, q' \geq 1$. Then for all density matrices $\rho_A \neq \sigma_A \in \mathcal{L} (\mathcal{H}_A)$ we have

$$\frac{D(\sigma_B \| \rho_B)}{D(\sigma_A \| \rho_A)} \leq \frac{q}{p'}, \quad (35)$$

where $\sigma_B := \Omega_{\rho_B} \circ \Gamma_{\rho_A}^{-1} (\sigma_A^*)$. Here $D(\cdot \| \cdot)$ denotes the KL divergence.

**Proof.** First note that as mentioned in the proof of Theorem 13, $\Omega_{\rho_B} \circ \Gamma_{\rho_A}^{-1}$ is CPTP and $\Omega_{\rho_B} \circ \Gamma_{\rho_A}^{-1} (\sigma_A^*) = \rho_B$. Thus $\sigma_B := \Omega_{\rho_B} \circ \Gamma_{\rho_A}^{-1} (\sigma_A^*)$ is also a density matrix.

Let $(p_1, q'_1) = (p, q')$ and $(p_0, q'_0) = (1, 1)$, and define

$$\frac{1}{p_0} = 1 - \theta \frac{p_0}{p_1} + \theta \frac{p_0}{p_1}, \quad \text{and} \quad \frac{1}{q'_0} = 1 - \theta \frac{q'_0}{q'_1} + \theta \frac{q'_0}{q'_1}.$$  

By Theorem 13 we have $\|\Omega_{\rho(p_0,q_0)}\|_{p_0 \to q_0'} \leq 1$. Then by Theorem 14 we obtain $\|\Omega_{\rho(p_0,q_0)}\|_{p_0 \to q_0'} \leq 1$, for all $0 \leq \theta \leq 1$. This means that for all $X$ we have

$$\|\Gamma_{\rho_B}^{-1/q_0} \circ \Omega_{\rho_B} \circ \Gamma_{\rho_A}^{-1/p_0} (X)\|_{q'_0} \leq \|X\|_{p_0},$$

or equivalently

$$\|\Gamma_{\rho_B}^{-1/q_0} \circ \Omega_{\rho_B} \circ \Gamma_{\rho_A}^{-1/p_0} (X)\|_{q'_0} \leq \|\Gamma_{\rho_A}^{-1/q_0} \circ \Omega_{\rho_B} (X)\|_{p_0}.$$  

In particular for $X = \sigma_A^*$ we find that

$$h(\theta) = \|\Gamma_{\rho_B}^{-1/p_0'} (\sigma_A^*)\|_{p_0} - \|\Gamma_{\rho_B}^{-1/q_0} (\sigma_B)\|_{q'_0},$$

is non-negative, i.e., $h(\theta) \geq 0$ for all $0 \leq \theta \leq 1$.  

Observe that,
\[
h(0) = \|\sigma_A^*\|_1 - \|\sigma_B\|_1 = 0.
\]
Therefore we should have \( h'(0) \geq 0 \).

The derivative \( h'(\theta) \) can be computed using formulas provided in [25] (see also [26]). Using these formulas we find that
\[
0 \leq h'(0) = \frac{1}{p'_1} D(\sigma_A \| \rho_A) - \frac{1}{q_1} D(\Gamma_{p_B}^1 \circ \Omega_{\bar{\rho}}(\sigma_A^*) \| \Gamma_{p_B}^1 \circ \Omega_{\bar{\rho}}(\sigma_A^*)) = \frac{1}{p'} D(\sigma_A \| \rho_A) - \frac{1}{q} D(\sigma_B \| \rho_B).
\]

We are done. \( \square \)

Note that by Theorem 13, \( \|\Omega_{\bar{\rho}}(\rho_{p,q})\|_{p \to q'} \leq 1 \) for \( p = q' \). Then the above theorem in particular gives the data processing inequality for KL divergence.

4. Some Examples

In this section we compute HRs for some bipartite states \( \rho_{AB} \). We first start with the extreme cases where \( \rho_{AB} \) is a product state and \( \rho_{AB} \) is a pure entangled state. In the former case \( \rho_{AB} \) contains no correlation and only product states can be generated from copies of \( \rho_{AB} \). Then Corollary 12 suggests that such a state should have the largest HRs. On the other hand, pure entangled states are the most correlated states so they should have the smallest HRs.

4.1. Product states. Assume that \( \rho_{AB} = \rho_A \otimes \rho_B \), thus
\[
\tilde{\rho}_{AB} = \left( \rho_A^{-1/2p} \otimes \rho_B^{-1/2q} \right) (\rho_A \otimes \rho_B) \left( \rho_A^{-1/2p} \otimes \rho_B^{-1/2q} \right) = \rho_A^{1/p'} \otimes \rho_B^{1/q'},
\]
and by (13) we have
\[
\Omega_{\bar{\rho}}(X) = \text{tr} \left( \rho_A^{1/p'} \rho_B^{1/q'} \right) X \rho_B^{1/q'}.
\]

Therefore,
\[
(\mathcal{I}_C \otimes \Omega_{\bar{\rho}})(Y_{CA}) = \text{tr}_A \left( (I \otimes \rho_A^{1/p'}) Y_{CA} \right) \otimes \rho_B^{1/q'}.
\]

Now we compute
\[
\| (\mathcal{I}_C \otimes \Omega_{\bar{\rho}})(Y) \|_{(t,q')} = \| \text{tr}_A \left( (I \otimes \rho_A^{1/p'}) Y_{CA} \right) \otimes \rho_B^{1/q'} \|_{(t,q')}
\]
\[
= \| \text{tr}_A \left( (I \otimes \rho_A^{1/p'}) Y_{CA} \right) \|_t \| \rho_B^{1/q'} \|_{(q')}
\]
\[
= \| \text{tr}_A \left( (I \otimes \rho_A^{1/p'}) Y_{CA} \right) \|_t
\]
to provide useful tools for computing the HRs. Indeed for every $p \leq q'$, by considering the Schmidt decomposition of $|\varphi\rangle_{AB}$ one can find (a rank-one) $X$ such that $\|\Omega_{\tilde{\rho}}(X)\|_{q'} > \|X\|_{p}$. This implies (37). Here we provide an indirect argument for this fact.

Suppose that $p \leq q'$ and $\|\Omega_{\tilde{\rho}}\|_{p \rightarrow q'} \leq 1$. As shown in [5] the maximal correlation of entangled pure states is equal to $\mu(|\varphi\rangle_{AB}) = 1$. Then by Theorem 16 we have

$$p - 1 \geq q' - 1.$$  

Given that $p \leq q'$ we find that $p = q'$. This gives (37).

4.3. Hypercontractivity via log-Sobolev inequalities. Computing HRs is a hard problem in general. To compute $\mathcal{R}_{A \rightarrow B}(\rho_{AB})$ we should compute the norm $\|\Omega_{\tilde{\rho}}^{\otimes n}\|_{p \rightarrow q'}$ for all integers $n$. Likewise, computing $\mathcal{R}_{CB,A \rightarrow B}(\rho_{AB})$ involves a supremum over the dimension of an auxiliary system which makes the computation of $\|\Omega_{\tilde{\rho}}\|_{CB,p \rightarrow q'}$ intractable. Here by giving an important example we show that quantum log-Sobolev inequalities [25,26] provide useful tools for computing the HRs.

Consider the bipartite state

$$\rho_{AB}^{(\alpha)} = \alpha |\psi\rangle \langle \psi| + (1 - \alpha) I_{AB}/4,$$

where $\dim \mathcal{H}_A = \dim \mathcal{H}_B = 2$ and $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is the maximally entangled state. Our goal in this section is to compute the HRs of $\rho_{AB}^{(\alpha)}$. 
The maximal correlation of $\rho_{AB}^{(\alpha)}$ can easily be computed \[5\]

$$\mu(\rho_{AB}^{(\alpha)}) = \alpha.$$ 

Then by Theorem 16 for all $(p, q') \in R_{A\to B}(\rho_{AB}^{(\alpha)})$ we have

$$\frac{p - 1}{q' - 1} \geq \alpha^2. \quad (38)$$

Note that independent of the value of $\alpha$, $\rho_A = \rho_B = I/2$. Therefore,

$$\tilde{\rho}_{AB}^{(\alpha)} = 2^{(1/p+1/q)} \rho_{AB}^{(\alpha)},$$

and we have

$$\Omega_{\tilde{\rho}}(X) = 2^{(1/p+1/q)} \Omega_{\rho}(X)$$

$$= 2^{(1/p+1/q)} (\alpha \Omega_{\rho|\psi}(X) + (1 - \alpha) \Omega_{I/4}(X))$$

$$= 2^{(1/p+1/q)} \left( \frac{\alpha}{2} X + \frac{1 - \alpha}{2} \text{tr} (X) \frac{I}{2} \right)$$

$$= 2^{1/p-1/q'} \Delta_\alpha(X),$$

where $\Delta_\alpha$ denotes the depolarizing channel

$$\Delta_\alpha(X) = \alpha X + (1 - \alpha) \text{tr} (X) \frac{I}{2}.$$

Let $\mathcal{L}$ be the super-operator defined by $\mathcal{L}(X) = X - \text{tr} (X) \frac{I}{2}$. Then we have

$$e^{-t \mathcal{L}} = \Delta_{e^{-t}}.$$

That is, depolarizing channels belong to a semigroup of super-operators, and then their hypercontractivity can be studied based on log-Sobolev inequalities, as done by King \[27\].

**Theorem 18** \[27, Corollary 2\]. For every $1 < p \leq q'$ and all $n \geq 1$ we have

$$2^{-n/q' + n/p} \left\| \Delta_{e^{i\cdot n}} \right\|_{p \to q'} = 1 \quad \text{if and only if} \quad e^{-t} \leq \frac{\sqrt{p - 1}}{\sqrt{q' - 1}}.$$

Rephrasing this theorem in our notation we conclude that the bound (38) is indeed tight, i.e.,

$$(p, q') \in R_{A\to B}(\rho_{AB}^{(\alpha)}) \quad \text{if and only if} \quad \alpha^2 \leq \frac{p - 1}{q' - 1}.$$ 

This fact can be considered as a quantum analogue of Bonami-Beckner inequality from which one can compute the HR of the classical analogue of $\rho_{AB}^{(\alpha)}$, i.e., a mixture of perfectly correlated coins and completely random coins \[6\].

We now can resolve the problem mentioned in the introduction. Let $\zeta_{UV}$ be the bipartite distribution defined by

$$\zeta_{00} = \zeta_{01} = \zeta_{10} = 1/3 \quad \text{and} \quad \zeta_{11} = 0.$$
The maximal correlation of $\zeta_{UV}$ is equal to $\mu(\zeta_{UV}) = 1/2$. This means that if $\mu(\rho_{AB}^{(\alpha)}) = \alpha < 1/2$, then $\zeta_{UV}$ cannot be generated from copies of $\rho_{AB}^{(\alpha)}$ under local measurement. Using hypercontractivity ribbons we now argue that this task is not doable for $\alpha \leq 0.6075$. Suppose that by local measurement on $\sigma_{A^n B^n} = \rho_{AB}^{(\alpha)} \otimes \cdots \otimes \rho_{AB}^{(\alpha)}$ we may generate $\zeta_{UV}$. That is, there are POVM measurements $\{M_0, M_1 = I - M_0\}$ and $\{N_0, N_1 = I - N_0\}$ such that for $i, j \in \{0, 1\}$ we have

$$\zeta_{ij} = \text{tr} \left( \sigma_{A^n B^n} M_i \otimes N_j \right).$$

This in particular implies that

$$\zeta(U=i) = \text{tr} \left( \sigma_{A^n M_i} \right) \quad \text{and} \quad \zeta(V=j) = \text{tr} \left( \sigma_{B^n N_j} \right).$$

Let $(p, q') \in \mathcal{R}_{A \to B}(\rho_{AB}^{(\alpha)}) \subseteq \mathcal{R}_{A \to B}^{(1)}(\rho_{AB})$. Then by Lemma 15 we obtain

$$\text{tr} \left( \sigma_{A^n B^n} M_1 \otimes N_0 \right) \leq \text{tr} \left( \sigma_{A^n M_1} \right)^{1/p} \text{tr} \left( \sigma_{B^n N_0} \right)^{1/q},$$

or equivalently

$$\frac{1}{3} \leq \left( \frac{2}{3} \right)^{1/p} \left( \frac{1}{3} \right)^{1-1/q'}. \quad (39)$$

Note that $(p, q') = (1 + \log 2 \alpha, 1 + k)$ is in $\mathcal{R}_{A \to B}(\rho_{AB}^{(\alpha)})$ for all $k \geq 0$. Putting in (39) we find that

$$3 \frac{k(1-\alpha^2)}{k+1} \leq 2, \quad \forall k \geq 0.$$

But this inequality does not hold if

$$\alpha < \sqrt{1 - \frac{\log 2}{\log 3}} \simeq 0.6075.$$

Observe that although, for instance, the maximal correlation of $\rho_{AB}^{(0.6)}$ is greater than the maximal correlation of $\zeta_{UV}$, local transformation of $n$ copies of $\rho_{AB}^{(0.6)}$ to $\zeta_{UV}$ is impossible even in the asymptotic limit.

5. Conclusion

In this paper we defined two hypercontractivity ribbons, one corresponding to the usual super-operator norm $\mathcal{R}_{A \to B}(\rho_{AB})$, and the other corresponding to the completely bounded norm $\mathcal{R}_{CB, A \to B}(\rho_{AB})$. By proving a data processing type property we concluded that these ribbons are indeed measures of bipartite correlation. These two ribbons coincide in the classical case, and we do not know of any quantum state $\rho_{AB}$ such that $\mathcal{R}_{A \to B}(\rho_{AB}) \neq \mathcal{R}_{CB, A \to B}(\rho_{AB})$. Note that the completely bounded norm and the usual super-operator norm are really different [16].

We also studied some properties of the ribbons. In particular we showed that maximal correlation gives a bound on HRs. Moreover we proved a relation between KL divergence and hypercontractivity ribbons. Here we should mention that in the classical case the
maximum of the left hand side of (35) over all states $\sigma_A$ is equal to the infimum of the right hand side over all $(p, q') \in \mathcal{R}_{A \rightarrow B}(\rho_{AB})$ (see [8] and also [9]). But we do not know whether such an equality holds in the quantum case or not.

The idea of reverse hypercontractivity [28] is applied in [6] to study the ribbons for values $p, q < 1$. We leave such an extension to the quantum case for future works.

It is argued in [10] that the hypercontractivity ribbon in the classical case can equivalently be characterized in terms of Rényi divergence. Given the recently proposed sandwiched quantum Rényi divergence [32,33] it is not hard to see that such an equivalency holds in the quantum case as well.

In this paper we employed non-commutative vector valued Schatten spaces, and used completely bounded norms because the usual super-operator norm is not multiplicative in the quantum case. Such spaces have already been shown to be useful in quantum information theory [16,29]. We hope that this work will serve as a motivation for employing such normed spaces in quantum information theory.

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### A. Non-Commutative Vector Valued $L_p$ Spaces

For $1 \leq q \leq p \leq \infty$ there exists $1 \leq r \leq \infty$ such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. Then for $X_{AB} \in \mathbf{L}(\mathcal{H}_A) \otimes \mathbf{L}(\mathcal{H}_B)$ define

$$\|X\|_{(p,q)} := \sup_{U,V \in \mathbf{L}(\mathcal{H}_A)} \frac{\|(U \otimes I_B)X(V \otimes I_B)\|_q}{\|U\|_{2r}\|V\|_{2r}},$$

(40)

and

$$\|X\|_{(q,p)} := \inf_{X = (U \otimes I_B)Y(V \otimes I_B)} \|U\|_{2r}\|V\|_{2r}\|Y\|_p,$$

(41)

where in (41) the infimum is taken over all $U, V \in \mathbf{L}(\mathcal{H}_A)$ and $Y \in \mathbf{L}(\mathcal{H}_A) \otimes \mathbf{L}(\mathcal{H}_B)$ such that $X = (U \otimes I_B)Y(V \otimes I_B)$. We can compute $\|X\|_{(p,p)}$ (when $p = q$) from both (40) and (41), but there is no ambiguity here since they coincide.

$\|\cdot\|_{(p,q)}$ defined by Eqs. (40) and (41) is indeed a norm on the tensor product space $\mathbf{L}(\mathcal{H}_A) \otimes \mathbf{L}(\mathcal{H}_B)$ for every $1 \leq p, q \leq \infty$. It is clear from the definitions that

$$\|X^T\|_{(p,q)} = \|X^*\|_{(p,q)} = \|X^\dagger\|_{(p,q)} = \|X\|_{(p,q)}.$$

Here we summarize some of the main properties of these norms. For proofs and details see [16–18,30].

(a) If $X_{AB} = M_A \otimes N_B$ then $\|X\|_{(p,q)} = \|M\|_p\|N\|_q$.

(b) $\|X\|_{(p,p)} = \|X\|_p$.

(c) If $X \in \mathbf{L}(\mathcal{H}_A) \otimes \mathbf{L}(\mathcal{H}_B)$ is block diagonal with diagonal blocks $M_i \in \mathbf{L}(\mathcal{H}_B)$, i.e.,

$$X = \sum_{i=0}^{d_A-1} |i\rangle \langle i| \otimes M_i,$$

then

$$\|X\|_{(p,q)} = \left(\sum_{i=0}^{d_A-1} \|M_i\|_p^p\right)^{\frac{1}{p}}.$$
(d) If $X$ is positive semi-definite then in optimizations (40) and (41) we may assume that $U = V$ and that they are positive semi-definite.

(e) $\| \cdot \|_{(p',q')}$ is the dual norm of $\| \cdot \|_{(p,q)}$, i.e., for every $X$ we have

$$\|X\|_{(p,q)} = \sup_{\|Y\|_{(p',q')} = 1} |\text{tr}(YX)|.$$  

Here $p', q'$ are the Hölder conjugates of $p, q$ respectively, i.e. $1/p + 1/p' = 1$.

(f) $\|X\|_{(p,q)} = \inf_{X = (U \otimes I_B)Y(V \otimes I_B)} \|U\|_2 \|V\|_2 \|Y\|_{(\infty,q)}$.

(g) $\|X\|_{(\infty,q)} = \sup_{U,V} \frac{\|U \otimes I_X (V \otimes I_X) \|_{(p,q)}}{\|U\|_2 \|V\|_2}$.

Now we can define the completely bounded norms as follows. For a super-operator $\Phi : L(\mathcal{H}_A) \rightarrow L(\mathcal{H}_B)$ and $t = \infty$ define

$$\|\Phi\|_{\text{CB},p \rightarrow q} := \sup_d \|I_d \otimes \Phi\|_{(t,p) \rightarrow (t,q)} = \sup_d \sup_X \frac{\|I_d \otimes \Phi(X)\|_{(t,q)}}{\|X\|_{(t,p)}}.$$  

(42)

Here $I_d$ is the identity super-operator corresponding to a $d$-dimensional Hilbert space.

In the definition of the completely bounded norm the value $t = \infty$ can be replaced with any $1 \leq t \leq \infty$; no matter what $t$ is chosen we get to the same number [17]. This fact can be easily proved using the last two properties mentioned above (see also [30]).

We now show that the completely bounded norm and the usual super-operator norm coincide for classical channels.

**Theorem 19.** Let $T : L(\mathcal{H}_A) \rightarrow L(\mathcal{H}_B)$ be a classical channel of the form

$$T(\rho) = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} c_{ij} \langle i|\rho|i\rangle \langle j|\langle j|,$$

where $c_{ij} \geq 0$. Then for all $1 \leq p, q \leq \infty$ we have $\|T\|_{\text{CB},p \rightarrow q} = \|T\|_{p \rightarrow q}$.

**Proof.** By Theorem 1 there is nothing to prove when $q \leq p$. So we assume that $q \geq p$ and define $r$ by

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}.$$  

It suffices to show that $\|I_C \otimes T\|_{(p,p) \rightarrow (p,q)} \leq \|T\|_{p \rightarrow q}$ for any finite dimensional space $\mathcal{H}_C$. Equivalently we show that

$$\|I_C \otimes T(\rho_{CA})\|_{(p,q)} \leq \|T\|_{p \rightarrow q} \|\rho_{CA}\|_p.$$  

(43)

Note that by the ‘pinching inequality’ [12] we have $\|\rho_{CA}'\|_p \leq \|\rho_{CA}\|_p$ where

$$\rho_{CA}' = \sum_{i=0}^{d_A-1} (I \otimes \langle i|) \rho_{CA}(I \otimes |i\rangle \otimes |i\rangle \langle i|,$$

and that $I_C \otimes T(\rho_{CA}) = I_C \otimes T(\rho_{CA}')$. As a result to prove (43) we assume that $\rho_{CA}$ has the form

$$\rho_{CA} = \sum_{i=0}^{d_A-1} \rho_i \otimes |i\rangle \langle i|_A.$$
Moreover, by Theorem 1 we may assume that $\rho_{CA}$ is positive semi-definite. Then observe that
\[
\|\rho_{CA}\|_p = \left( \sum_i \|\rho_i\|_p^p \right)^{1/p} = \left( \sum_i \left\| U_i \rho_i U_i^\dagger \right\|_p^p \right)^{1/p} = \left\| \sum_i U_i \rho_i U_i^\dagger \otimes |i\rangle\langle i| \right\|_p,
\]
where $U_i$'s are arbitrary unitary matrices.

Assume that $q \geq p \geq 1$, and let $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. The other case where $p \geq q$ is similar (or one may use Theorem 1 in this case since $T$ is completely-positive). We now compute
\[
\| I \otimes T(\rho_{CA}) \|_{(p,q)} = \left\| \sum_{i,j} c_{ij} \rho_i \otimes |j\rangle\langle j| \right\|_{(p,q)}
\]
\[
= \inf_{M \geq 0, \|M\|_2 = 1} \left\| \sum_{i,j} c_{ij} M^{-1} \rho_i M^{-1} \otimes |j\rangle\langle j| \right\|_q
\]
\[
\leq \inf_{M \geq 0, \|M\|_2 = 1} \sup_{U_i} \left\| \sum_{i,j} c_{ij} M^{-1} U_i \rho_i U_i^\dagger M^{-1} \otimes |j\rangle\langle j| \right\|_q
\]
\[
= \inf_{M \geq 0, \|M\|_2 = 1} \sup_{U_i} \left( \sum_j \left\| \sum_i c_{ij} M^{-1} U_i \rho_i U_i^\dagger M^{-1} \right\|_q^q \right)^{1/q}
\]
By Lidskii’s theorem [12] for every $M$, the term
\[
\left\| \sum_i c_{ij} M^{-1} U_i \rho_i U_i^\dagger M^{-1} \right\|_q
\]
is maximized when $U_i$’s are chosen in such a way that $U_i \rho_i U_i^\dagger$’s commute with $M$, and commute with each other. In other words we may assume from the beginning that $\rho_i$’s mutually commute.

So let us assume that $\rho_{CA} = \sum_{i,k} a_{ik} |k\rangle\langle k| \otimes |i\rangle\langle i|$. Then we have
\[
\| I \otimes T(\rho_{CA}) \|_{(p,q)} = \left( \sum_k \left\| \sum_{i,j} a_{ik} c_{ij} |j\rangle\langle j| \right\|_q^p \right)^{1/p}
\]
\[
= \left( \sum_k \left\| T \left( \sum_i a_{ik} |i\rangle\langle i| \right) \right\|_q^p \right)^{1/p}
\]
\[
\leq \| T \|_{p\rightarrow q} \left( \sum_k \left\| \sum_i a_{ik} |i\rangle\langle i| \right\|_p^p \right)^{1/p}
\]
\[
= \| T \|_{p\rightarrow q} \| \rho_{CA} \|_p.
\]
\[\square\]
B. Proof of Riesz–Thorin Theorem for Schatten Norms

To prove this theorem we use Hadamard’s three-line theorem \[31\].

**Theorem 20.** Let \( f: S \to \mathbb{C} \) be a bounded function that is holomorphic in the interior of \( S \) and continuous on the boundary. For \( k = 0, 1 \) let

\[
M_k = \sup_{t \in \mathbb{R}} |f(k + it)|.
\]

Then for every \( 0 \leq \theta \leq 1 \) we have \( |f(\theta)| \leq M_0^{1-\theta} M_1^\theta \).

**Proof of Theorem 3.** First note that by Hölder’s duality

\[
\|T_\theta\|_{p_0 \to q_0} = \left( \sup_{\|X\|_{p_0} = 1, \|Y\|_{q_0} = 1} |\text{tr}(YT_\theta(X))| \right)^{\frac{1}{\theta}}.
\]

So we need to show that for every \( X, Y \) with \( \|X\|_{p_0} = \|Y\|_{q_0} = 1 \) we have

\[
|\text{tr}(YT_\theta(X))| \leq \left( \sup_{t \in \mathbb{R}} \|T_{\text{ir}}\|_{p_0 \to q_0} \right)^{1-\theta} \left( \sup_{t \in \mathbb{R}} \|T_{1+\text{ir}}\|_{p_1 \to q_1} \right)^\theta.
\]

(44)

For an arbitrary matrix \( X \) and complex number \( z \) we may define \( X^z \) using the singular value decomposition of \( X \). That is, assume that \( X = UDV \) where \( U, V \) are unitary and \( D \) is diagonal with non-negative entries. Then define \( X^z := UD^zV \). Using this notation, to prove (44) we equivalently need to show that for every \( X, Y \) such that \( \|X\|_1 = \|Y\|_1 = 1 \) we have

\[
|\text{tr}\left( Y^\frac{1}{q_0} T_\theta(X^\frac{1}{p_0}) \right) | \leq \left( \sup_{t \in \mathbb{R}} \|T_{\text{ir}}\|_{p_0 \to q_0} \right)^{1-\theta} \left( \sup_{t \in \mathbb{R}} \|T_{1+\text{ir}}\|_{p_1 \to q_1} \right)^\theta.
\]

(45)

Fix \( X, Y \) with \( \|X\|_1 = \|Y\|_1 = 1 \), and for \( z \in S \) define

\[
f(z) = \text{tr}\left( Y^\frac{1}{q_0} T_z(X^\frac{1}{p_0}) \right).
\]

Note that

\[
f(\theta) = \text{tr}\left( Y^\frac{1}{q_0} T_\theta(X^\frac{1}{p_0}) \right).
\]

\( f(z) \) satisfies the assumptions of Hadamard’s three-line theorem. Therefore we have

\[
|\text{tr}\left( Y^\frac{1}{q_0} T_\theta(X^\frac{1}{p_0}) \right) | = |f(\theta)| \leq \left( \sup_{t \in \mathbb{R}} |f(it)| \right)^{1-\theta} \left( \sup_{t \in \mathbb{R}} |f(1+it)| \right)^\theta.
\]

(46)

Observe that for \( t \in \mathbb{R} \)

\[
|f(it)| = |\text{tr}\left( Y^{1-q_0} T_{it}(X^{1-p_0} + \frac{it}{p_1}) \right)|
\]

\[
\leq \|Y^{1-q_0} T_{it}\|_{q_0} \|T_{it}\|_{p_0 \to q_0} \|X^{1-p_0} + \frac{it}{p_1}\|_{p_0}
\]

\[
\leq \|Y^{1-q_0} T_{it}\|_{q_0} \|T_{it}\|_{p_0 \to q_0} \|X^{1-p_0} + \frac{it}{p_1}\|_{p_0}
\]

\[
= \|T_{it}\|_{p_0 \to q_0}.
\]
Here in (a) we use Hölder’s inequality, in (b) we use the definition of super-operator norm, and in (c) we use the fact that $p$-norms are invariant under multiplication by unitary matrices.

We similarly have

$$|f(1 + it)| \leq \|T_{1+it}\|_{p_1 \to q_1}.$$ 

Putting these two bounds in (46) gives the desired result (45). □

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