A Stochastic Description for Extremal Dynamics

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We show that extremal dynamics is very well modelled by the “Linear Fractional Stable Motion” (LFSM), a stochastic process entirely defined by two exponents that take into account spatio-temporal correlations in the distribution of active sites. We demonstrate this numerically and analytically using well-known properties of the LFSM. Further, we use this correspondence to write an exact expression for an n-point correlation function as well as an equation of fractional order for interface growth in extremal dynamics.

Since the eighties, several models have been proposed to describe in a common language diverse physical situations such as roughening of a crack front in fracture, wetting front motion on heterogeneous surfaces, dynamics of a ferromagnetic domain wall driven by an external magnetic field, motion of vortices in type-II superconductors, fluid invasion in porous media, solid friction or even biological evolution. All these models propose to explain the dissipative behaviour of the system by the competition between an elastic restoring force and a non linear, randomly distributed, time-independent, pinning force. In the case of the spreading of a partially wetting liquid for example, the pinning force is due to surface chemical heterogeneities or roughness, and the elastic restoring force is a result of the surface tension at the liquid/vapor interface. In the case of strong pinning it is well known that, when subjected to an external driving, the wetting front displays local instabilities that force it to advance quasistatically. This property makes it difficult to handle the problem in the continuum. In fact, in the stationary regime, the main contribution to the global displacement is from jumps of local parts of the chain resulting from these instabilities. Recently, Tanguy et al. have proposed to describe this sort of evolution by an extremal model: only the site closest to its instability threshold advances. After a jump, the instability thresholds of all the sites are modified by the (elastic) couplings between sites. More precisely, in their model (hereafter LREM for Long Range Elastic Model), the interface of size $L$ is defined on a discrete lattice $(x,h)$. Initially the front $h(x) = 0$, and the pinning forces $f_p(x, t = 0) = f_0(x, h = 0)$ are assigned independently from a flat distribution. The site $x_0$ subjected to the minimum pinning force (and hence closest to its instability threshold) advances first, thus $h(x_0) \to h(x_0) + \Delta h$. At this new position, a new random pinning force is encountered $f_p(x_0, t + \delta t) = f_0(x_0, h(x_0) + \Delta h)$. The external loading $F$ on the system, and interactions along the front, produce a local driving force on each site $x$ proportional to $f(x, t) = F \int G(x-y)h(y, t) \, dy$ where the kernel $G(x) \propto |x|^{-b-1}$ accounts for long range interactions mediated by the medium. The loading $F$ is then adjusted so that only one site depins $f(x, t) = f_p(x, t)$; the others remain trapped since $f(y, t) \leq f_p(y, t)$ for $y \neq x$. The dynamics of advancing the minimum site and readjusting the others is continued indefinitely.

A wide class of extremal models have already been studied extensively by Paczuski et al. These models include the Bak-Sneppen evolution model and the Sneppen Interface Model. All these models try to explain driven motion under strong pinning by means of a discrete, deterministic dynamics. Only one site is active at every instant of time. However the “time” is only a way to index the sequence of events. Further the extremal condition can be thought of as a way to retain the information of the spatially quenched heterogeneities that determine the evolution of the front. All the information in this sort of dynamics is clearly contained in the “activity” map: a space-time plot of where the front is active at every instant of time.

Previous studies regarding extremal models have shown that most of the relevant information is contained in the probability density function (hereafter, pdf) of the activity map. In the stationary regime, assuming that the activity was located at $x_0$ at time $t_0$, the probability that it is at $x$ at time $t$ is:

$$p(|x - x_0|, t - t_0) = (t - t_0)^{-1/\zeta} \phi \left( \frac{|x - x_0|}{(t - t_0)^{1/\zeta}} \right)$$ (1)

with
While the exponent $\alpha$ controls the asymptotic behavior of the time independent function $\phi$, $z$ controls the propagation of the activity along the system as a function of time and is therefore termed the “dynamical exponent”. The above distribution is self-affine and therefore its temporal evolution is completely defined through the exponents $z$ and $\alpha$ [13]. Moreover, as can be seen from Eq. (1), the form of the distribution of $|x(t) - x_0|$ is independent of $t$ for large enough $t$: it is “$\alpha$-stable” [13]. It has been shown in [14] that $\alpha = b$. It is hence also possible to consider the activity as performing a “Brownian-like” motion similar in spirit to studies of anomalous diffusion [14]. The relevant parameters here are: the exponent $\alpha$ appearing in the stationary distribution of the distance between successive active sites (distribution of increments):

$$p\left(\left|x(t + 1) - x(t)\right| = l\right) \propto l^{-\alpha - 1}, \forall t$$  \hspace{1cm} (3)

and the exponent $H$, characteristic of the moments:

$$\langle|x(t) - x(t')|^\alpha\rangle^{1/\alpha} \approx |t - t'|^H \text{ for } a < \alpha.$$  \hspace{1cm} (4)

$H$ accounts for possible temporal statistical dependence between jumps. When there is no temporal correlation between jumps, this distribution is Brownian for $\alpha > 2$ and a “Levy flight” for $0 < \alpha < 2$. In the former case $H = 1/2$ and in the latter case it is easy to show, for example from the asymptotic expression of $p(x,t)$ [13], that $H = 1/\alpha$. However when there are temporal correlations, as in some of the models studied in [13], or in the case when elastic interactions are long ranged [15], this is no longer true and $H \neq 1/\alpha$. The value of $H$ is hence indicative of the presence or absence of temporal correlations in the jumps of the activity. Attempts have been made in the past to understand the space-time plot of the activity in extremal models as an uncorrelated Levy flight [13,15], by keeping only the exponent $\alpha$ and thus assuming $H = 1/\alpha$. However, numerically, this hypothesis leads to erroneous quantitative predictions. Thus there are long range time correlations which have to be incorporated in the description. In this Letter, we introduce and study such a model.

We define here a Linear Fractional Stable Motion. Let $x$ denote a process generated in the following manner:

$$x(t) = \sum f(t,u)\eta(u),$$  \hspace{1cm} (4)

where $\eta(u)$ is an uncorrelated noise with a symmetric distribution $p(\eta = x) \sim |x|^{-\alpha - 1}$. Since we would like to consider stationary processes (if our eventual aim is to describe steady states) we assume that $f(t,u) = f(t-u)$. The definition above implies basically that $x$ consists of a sum of uncorrelated levy jumps weighted in time by $f(t-u)$. This weight function therefore controls the time dependence of the statistical properties of $x$. It is easy to show that the sum in Eq. (4) can be performed much as for independent Levy flights and the random variable $X = x(t)$ (given that $X = 0$ at $t = 0$) has a probability density function

$$p(X,t) = \int \exp(ikX) \exp(-|\sigma|^{|\alpha|}k^{|\alpha|})dk/2\pi.$$  \hspace{1cm} (5)

where the $\alpha^\alpha = \sum_{n=0}^\infty |f(t-u)|^\alpha$. For the so-called LFSM [13], the function $f(t,u)$ reads:

$$f(t,u) = (t-u)^d$$

where the parameter $d$ satisfies $-1/\alpha < d < 1 - 1/\alpha$. The LFSM is thus a self-similar process with stationary increments. Further the exponent $H$ defined earlier is here $d + 1/\alpha$ and $\delta^\alpha = \sigma^\alpha |t|^{\alpha H}$. Thus $x(t)$ can be seen as an integral or derivative of fractional order, of the noise $\eta$.

In order to have a prescription for constructing the LFSM given the activity plot of the extremal models that we studied, it is important to determine carefully the $H$ exponent in these models. We have done this by performing first a wavelet transform [16], which consists of computing wavelet coefficients $d_x(a,k) = \langle x, \psi_{a,k} \rangle$ where $\psi_{a,k}(t) = \sqrt{a}\psi(a(t-k))$ is a collection of dilated and translated templates of the mother wavelet $\psi(t)$. The wavelet transform is a relevant tool [17] to analyse self-similarity because the $d_x(a,k)$ of a self-similar process with stationary increments i) reproduce exactly the self-similarity, ii) form stationary sequences, iii) are quasi independent statistically one from another. For the LFSM, it has been shown [19] that $H$ can be estimated by performing a linear fit in a $(\log_2(a), Y_a)$ plot, where $Y_a = \langle \log_2 |d_x(a,k)| \rangle_k$. Confidence interval for the estimate of $H$ can be theoretically derived [18]. Figure 2 illustrates this estimation procedure on data produced by an LREM using $b = 1$ [13]. Only one such example is shown, but many other trials were performed using other values of $b$, or other models such as the Bak-Sneppen model. All the models produced power-laws of similar quality, thus proving that the activity map of these models can be modelled as an LFSM.

We now try to compare the exponents $\alpha$ and $z$ predicted by the scaling form (9) obtained for an extremal model (such as the LREM) and its LFSM description. From the definition of the LFSM given by Eq. (9) and using the fact that $\sigma^\alpha = \sigma^\alpha |t|^{\alpha H}$, it is easy to show that the probability density function for the LFSM satisfies the scaling form (9) with $z = 1/H$. This is also very well verified on numerical simulations of LFSMs (fig.2b) and gives an estimate of $z$ with a precision of $\pm 10^{-2}$ for a signal of $10^5$ time steps. To compare, this analysis has also been applied on the LREM (see fig.2a) allowing estimates on $z$ with a precision of $\pm 0.1$. The two estimates of $z$ give us the same result.
To further justify the fact that the LFSM models accurately the activity map of extremal models, we studied another property, namely the distribution of return times, which had been characterized on LREM as power-laws, but never on LFSM. The first return time is the time \( T \) elapsed between two subsequent activities at a given site. It has been shown for LREM, that \( T \) is distributed as a power-law \( p(T) \propto T^{-\gamma} \) up to a maximum time \( T^* \) such that the activity has spread over the entire system, \( T^* \propto L^{2} \). For the LFSM, we have already seen that the activity plot generated has a dynamic exponent \( z \equiv 1/H \). When \( H < 1 \), \( z \) is also the fractal dimension of the activity plot. From this it is easy to show that the first return time exponent (which is simply a one dimensional cut of a fractal set) should be

\[
\tau_f = 2 - H \equiv 2 - 1/z. \tag{6}
\]

This is a relation already known to numerically hold for extremal models \([3]\). Hence we see that the two are again identical in this regard. We have also checked this numerically for different extremal models and their corresponding LFSM’s. It is important to note that, when \( H > 1 \), \( \tau_{FIRST} = 1 \) for all \( H \). Thus our identification of the LFSM nature of the activity is indeed justified, and allows us to access a novel property of the LFSM which has never been reported so far in the literature.

We now turn to some of the interesting consequences of having an analytical expression (such as Eq. (3)) for the space-time plot of the activity. Consider the two-point function \( P(l_1,l_2) \): the probability of having a jump \( l_1 \) and \( l_2 \) at two consecutive instants. Using Eq. (2) we find that the expression of this function in Fourier space is just

\[
\sim \int \exp(i k (l_1 + l_2)) \exp(-2|\alpha|^H |k|^\alpha) dk.
\]

In real space this is just the function \( 1/(l_1 + l_2)^{\alpha+1} \) asymptotically. We can use this to compute the conditional probability \( P(l_2/l_1) \) — the probability of having a jump of length \( l_2 \) at time \( t = 2 \) given that there was a jump \( l_1 \) at time \( t = 1 \) — using the well known relation \( P(l_2/l_1) = P(l_1,l_2)/P(l_1) \). We see that this is just \( \sim 1/(1 + l_2/l_1)^{\alpha+1} \) thus theoretically confirming the numerical scaling obtained in \([3]\). We can now generalise to the full \( n \)-point function \( P(l_1,l_2,l_3,...,l_n) \) (discussed in \([3]\)) of having a jump of length \( l_1 \) at time \( t = 1 \), a jump of length \( l_2 \) at time \( t = 2 \) and so on till \( t = n \). Systematic expansions can be obtained for the conditional probabilities just as for the two-point function and can also be verified numerically on the models. The theoretical understanding of these correlation functions is hence one of the achievements of our mapping.

A further very interesting consequence of the definition of the LFSM is the following equation that we can write down for the front propagation under this dynamics. To do this, we define the height of an interface at time \( t \) at a spatial location \( X \) as simply the accumulated activity there upto time \( t \):

\[
h(X,t) = \Delta h. \int_0^t \delta(X - x(t')) dt'. \tag{7}
\]

Note that we can use with this definition \([3]\) to perform the scale transformation commonly used for self-affine surfaces: \( X \to bX, t \to b^\alpha t \) and \( h \to b^\chi h \), where \( \chi \) is the so called roughness exponent for the height. A power counting on both sides of this equation gives the relation \( \chi = z - 1 \), known to hold for extremal dynamics \([4]\).

Using the definition of the LFSM \([3]\) and the following definition of the fractional derivative \([19]\):

\[
\frac{\partial^\beta \psi}{\partial x^\beta}(X - x(t)) = \frac{1}{\Gamma(1 - \beta)} \lim_{\delta \to 0} \frac{d}{d\delta} \int_{x(t)}^{x(t)+\delta} \psi(u) - \psi(x(t)) \frac{1}{(x(t) + \delta - u)^\beta} du.
\]

we obtain the following equation for the height after tedious calculations (the details of which we leave for a longer paper \([23]\)):

\[
\frac{\partial^\alpha H h(X,t)}{\partial t^\alpha H} = \frac{\Gamma(\alpha H + 1)}{\Gamma(\alpha + 1)} \frac{\partial^\alpha}{\partial X^\alpha} h(X,t) + \frac{c}{\Gamma(2 - \alpha H)} \int_0^t du \eta(u) \int_0^t du' \frac{\partial^2 h(X,u')}{\partial X \partial u'} |u' - u|^{d-1} \tag{9}
\]

In deriving the above, we have basically used some of the techniques developed for derivatives of fractional order such as the fractional Taylor expansion \([19]\) and fractional integration by parts. \( c \) is a constant with \( c \sim \langle \Delta t \rangle^{1-\alpha H} \) taking into account the right dimensionality of the noise.

We have thus changed a problem with quenched disorder to one with a multiplicative annealed noise, \( \eta(t) \), with our approach. There are several points worth mentioning in this regard. Using the fact that \( p(X,t) \) is proportional to the number of times the site \( X \) has been visited at time \( t \) (for different realisations of the process), that is \( p(X,t) \sim \langle \delta h(X,t)/\partial \rangle \), we can also conclude from Eq. (1) that an effective Fokker Planck equation for the activity should be one of fractional order in space and time:

\[
\frac{\partial^\alpha H p(X,t)}{\partial t^\alpha H} \sim \frac{\partial^\alpha}{\partial X^\alpha} p(X,t). \tag{10}
\]

We note that the expression Eq. (3) is consistent with the above equation in the long-time limit \([23]\). For an uncorrelated process the above equation reduces to the one studied in \([23]\). An equation such as the above has also been derived for a process with long-ranged jumps and a power-law waiting time between jumps \([23]\). It would be interesting to see the connection between the above process and Eq. (1) more microscopically.

There are also several other interesting points to investigate. As we have proved above, any extremal model can be modelled by a tangent process, which we call the LFSM, for a suitable value of \( H \) and \( \alpha \). In Fig. \([3]\) we have indicated the above parameter values for all the extremal
models mentioned in this letter. It is interesting to note that for the models studied in [6] a further relation exists between $H$ and $\alpha$: $H = 3/2(\alpha + 1)$ as indicated in Fig. 3. The two other extremal models studied (The Bak Sneppen model and the Sneppen Interface model) however do not seem to obey this relation (Fig. 3). It would be interesting to understand why the above relation exists for the LREM.

In summary we have proposed the LFSM as a stochastic model for extremal processes. We argue that the self-affinity and $\alpha$-stability of this process, together with the stationarity of its increments makes it an accurate description of extremal dynamics. We demonstrate this by first determining the value $H$ and $\alpha$ for all the extremal models listed above by using a wavelet analysis of the activity. Subsequently we show that a LFSM process generated using these values shows the same scaling predicted by Eq. 1 because of a simple relation between the dynamical exponent $z$ and the exponent $H$. In order to test such a correspondence further we also investigate a property which has never been studied in the LFSM, namely the time interval distribution between recurrences of activity at a particular site. Further the LFSM is amenable to an analytical treatment much more easily than extremal dynamics. We show how the scaling form is a simple consequence of the definition of the process. We comment on how the process allows us to understand the full $n$-point probability distribution for the increments. Finally we conclude by proposing a fractional partial differential equation for a front subjected to extremal dynamics.

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FIG. 1. Log-log plot of the amplitude of the wavelet component versus scale factor using a Daubechies3 wavelet on time series produced by a LREM [6]. It exhibits a quasi perfect power-law revealing self-affinity. The slope is $H + 1/2$ where $H$ is the self-affinity parameter.

FIG. 2. Data collapse of eight different probability distributions $p(x, \Delta t)$ with time intervals $\Delta t$ ranging from 1 to 128 (a) for the extremal model of $\{L \rangle=16384$. $\alpha = 1.45$ and the best data collapse is obtained for $z = 1.55$. Idem for a LFSM (b) with $\alpha = 1.5$ and $H = 0.562$. The best collapse is obtained with $z = 1/H$.

FIG. 3. This diagram represents the (wavelet-based) estimated $H$s (and their confidence intervals) for various $\alpha$ and various extremal models. The dot-dash curve $1/\alpha$ is the estimate for an uncorrelated levy-flight. The function $3/2(\alpha + 1)$ is the numerically found best estimate for $z$ in the LREM. The two other extremal models studied however do not seem to obey this relation.