New Codes for OFDM with Low PMEPR

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Abstract—In this paper new codes for orthogonal frequency-division multiplexing (OFDM) with tightly controlled peak-to-mean envelope power ratio (PMEPR) are proposed. We identify a new family of sequences occurring in complementary sets and show that such sequences form subsets of a new generalization of the Reed–Muller codes. Contrarily to previous constructions we present a compact description of such codes, which makes them suitable even for larger block lengths. We also show that some previous constructions just occur as special cases in our construction.

I. INTRODUCTION

Let us consider an $n$-subcarrier orthogonal frequency-division multiplexing (OFDM) system. The signal

$$s_A(t) = \sum_{i=0}^{n-1} A_i e^{2\pi \sqrt{-1}(f_c+i f_s) t} \quad (0 \leq t < T)$$

is called the complex envelope of the transmitted signal. Here $T$ denotes the symbol duration, $f_s$ is the subcarrier spacing, and $f_c$ is the radio carrier frequency. In an ideal situation it is commonly assumed that $f_s = 1/T$. The vector $A = (A_0 A_1 \ldots A_{n-1})$ is called the modulating codeword of the OFDM symbol. Let us assume that each subcarrier is modulated with a $q$-ary phase-shift-keying (PSK) constellation. Our concern is the envelope power of the transmitted signal $P_A(t) = |s_A(t)|^2$. An important characteristic of an OFDM signal is the peak-to-mean power ratio (PMEPR), which is for PSK-modulated subcarriers defined as

$$\text{PMEPR}(A) = \frac{1}{n} \sup_{0 \leq t < T} P_A(t).$$

For uncoded transmission the PMEPR is typically much higher than 1 and can grow up to as much as $n$. The high PMEPR of uncoded OFDM signals can be considered as the major drawback of the OFDM technique. Due to the high signal dynamics, the power amplifier should have a large linear range causing inefficient operation. On the other hand, a nonlinear power amplifier may result in severe signal distortion, such as interferences between the subcarriers and out-of-band radiation, where the latter issue is subject to strong regulations.

There exists a number of approaches to alleviate the problem of high PMEPR. A promising one remains the use of coding across the subcarriers [1]. The employed code should comprise only those codewords having low PMEPR, and in addition, it should provide a certain level of error protection. Let $C$ denote such a code and define the PMEPR of the code $C$

$$\text{PMEPR}(C) := \max_{A \in C} \text{PMEPR}(A).$$

So, for a given $n$, we aim to find codes with low PMEPR, good error protection, and high rate. In [2] good codes with PMEPR at most 2 were constructed for small $n$ by establishing a link between Golay complementary pairs [3] and certain second-order cosets of a generalized first-order Reed–Muller code. This technique was extended and generalized in [4] by including sequences lying in complementary sets [5]. However the codes are still unions of quadratic cosets of a generalized first-order Reed–Muller code and are a bit unwieldy, which makes them only suitable for small $n$. Recently, in [6], progress has been made in constructing sequences lying in complementary sets, which are not necessarily of quadratic order. These sequences in connection with new generalizations of the classical Reed–Muller code will be used in this paper to build powerful codes with bounded PMEPR and good error protection properties.

The remainder of this paper is organized as follows. In the next section we merely adopt some useful notation. In Section III we present a new family of complementary sequences. A new generalization of the Reed–Muller codes is introduced in Section IV. In Section V we present our code constructions. Section VI concludes the paper.

II. NOTATION AND PRELIMINARIES

Let $A = (A_0 A_1 \ldots A_{n-1})$ and $B = (B_0 B_1 \ldots B_{n-1})$ be two complex-valued vectors. Then the aperiodic cross-correlation of $A$ and $B$ at a displacement $\ell$ is given by

$$C(A, B)(\ell) := \begin{cases} \sum_{i=0}^{n-\ell-1} A_{i+\ell} B_i^* & 0 \leq \ell < n \\ \sum_{i=0}^{n+\ell-1} A_i B_{i-\ell}^* & -n < \ell < 0 \\ 0 & \text{otherwise} \end{cases}$$

where $(\cdot)^*$ denotes complex conjugation. The aperiodic auto-correlation of $A$ at a displacement $\ell$ is then conveniently written as

$$A(A)(\ell) := C(A, A)(\ell).$$

A generalized Boolean function $f$ is defined as a mapping $f : \mathbb{Z}_2^n \to \mathbb{Z}_q$. Such a function can be uniquely written in
its algebraic normal form, i.e. \( f \) is a sum of the \( 2^m \) weighted monomials
\[
f = f(x_0, x_1, \ldots, x_{m-1}) = \sum_{i=0}^{2^m-1} c_i \prod_{\alpha=0}^{m-1} x_{j_{\alpha}^i},
\]
where \( c_0, \ldots, c_{2^m-1} \in \mathbb{Z}_q \) and \((i_0 i_1 \cdots i_{m-1})\) is the binary expansion of the integer \( i \), such that \( i = \sum_{j=0}^{m-1} i_j 2^j \).\( \) The order of the \( i \)th monomial is defined as \( x_j \) for \( j \neq 0 \), and the order of a generalized Boolean function is equal to the highest order of the monomials with a nonzero coefficient in the algebraic normal form of \( f \).

A generalized Boolean function may be equally represented by vectors of length \( 2^m \). We shall define the vector \( f = (f_0 f_1 \cdots f_{2^m-1}) \) and the vector \( F = \xi f = (\xi f_0 \xi f_1 \cdots \xi f_{2^m-1}) \) as the \( \mathbb{Z}_q \)-valued vector and the polyphase vector associated with \( f \), respectively. Here \( \xi = \exp(2\pi \sqrt{-1}/q) \) is a primitive \( q \)th root of unity, and \( f_i = f(i_0 i_1 \cdots i_{m-1}) \) is the binary expansion of the integer \( i \). Throughout this paper \( q \) is assumed to be even.

We shall now define the restriction of polyphase vectors of length \( 2^m \) and their corresponding generalized Boolean functions. This technique was introduced in [4] and it will be useful to prove the results in this paper. Let \( f : \mathbb{Z}_{2^m}^n \to \mathbb{Z}_q \) be a generalized Boolean function in the variables \( x_{01}, x_{11}, \ldots, x_{m-1} \), and let \( F \) be its associated polyphase vector. Suppose \( 0 \leq j_0 < j_1 < \cdots < j_k < m \) is a list of indices and write \( x = (x_{j_0} x_{j_1} \cdots x_{j_k}) \). Let \( d \) be an arbitrary binary vector of length \( k \), and let \((i_0 i_1 \cdots i_{m-1})\) be the binary expansion of the integer \( \leq i < 2^m \). Then the restricted vector \( F|_{x=d} \) is a vector of length \( 2^m \) with its elements \( (F|_{x=d})_i \) for \( i = 0, 1, \ldots, 2^m - 1 \) being defined as
\[
(F|_{x=d})_i := \begin{cases} F_i & \text{if } (i_0 i_1 \cdots i_{m-1}) = (d_0 d_1 \cdots d_k) \\ 0 & \text{if } (i_0 i_1 \cdots i_{m-1}) \neq (d_0 d_1 \cdots d_k) \end{cases}.
\]
For the case \( k = 0 \) we fix \( F|_{x=d} = F \).

A vector that is restricted in \( k \) variables comprises \( 2^m - 2^m-k \) zero entries and \( 2^m-k \) nonzero entries. Those nonzero entries are determined by a function, which we shall denote as \( f|_{x=d} \). This function is a Boolean function in \( m-k \) variables and is obtained by replacing the variables \( x_{j_\alpha} \) by \( d_{\alpha} \) for all \( 0 \leq \alpha < k \) in the original function \( f \). The restricted vector \( F|_{x=d} \) is then found by associating a polyphase vector of length \( 2^m-k \) with \( f|_{x=d} \) and inserting \( 2^m - 2^m-k \) zeros at the corresponding positions. Similarly to a disjunctive normal form of a Boolean function [7], the original function \( f \) can be reconstructed from the functions \( f|_{x=d} \) by
\[
f = \sum_{d} f|_{x=d} \prod_{\alpha=0}^{k-1} x_{j_{\alpha}^d}(1 - x_{j_{\alpha}^d})(1 - d_\alpha)
\]
Then the polyphase vectors associated with the functions
\[ f + \frac{q}{2} \left( \sum_{a=0}^{k-1} c_a x_a + c' e \right) \quad c_0, \ldots, c_{k-1}, c' \in \mathbb{Z}_2 \]
form a complementary set of size \(2^{k+1}\).

**Proof:** Write \(c = (c_0, c_1, \ldots, c_{k-1})\) and denote the \(2^{k+1}\) vectors in the complementary set as \(F_{cc'}\). We have to show that the sum of auto-correlations \(\sum_{c,c'} A(F_{cc'})(\ell)\) is zero for \(\ell \neq 0\). We employ Lemma [4] and write
\[
\sum_{c,c'} A(F_{cc'})(\ell) = \sum_{c,c'} \sum_d A(F_{cc'}|x=d)(\ell)
+ \sum_{c,c',d_1 \neq d_2} C(F_{cc'}|x=d_1,F_{cc'}|x=d_2)(\ell) = S_1 + S_2.
\]
We first focus on the term \(S_1\), which becomes
\[
S_1 = \sum_c \sum_d (A(F_{c0}|x=d)(\ell) + A(F_{c1}|x=d)(\ell)).
\]
Note that \(e|x=d = x_{d_0}\). Thus the functions corresponding to \(F_{c0}|x=d\) and \(F_{c1}|x=d\) are
\[
f|_{x=d} = \frac{q}{2} \sum_{a=0}^{k-1} c_a d_a \quad \text{and} \quad f|_{x=d} = \frac{q}{2} \sum_{a=0}^{k-1} c_a d_a + \frac{q}{2} x_{d_0},
\]
respectively. Notice that the sum over \(\alpha\) is just a constant occurring in both functions. Hence, by hypothesis and by Theorem [4] \(F_{c0}|x=d\) and \(F_{c1}|x=d\) form a complementary pair. It follows that the inner term of \(S_1\) is zero for \(\ell \neq 0\), and thus, also \(S_1\) itself is zero for \(\ell \neq 0\).

Next we focus on the term \(S_2\) and rearrange the sum as follows
\[
S_2 = \sum_{d_1 \neq d_2} \sum_{c,c'} \sum_c C(F_{cc'}|x=d_1,F_{cc'}|x=d_2)(\ell).
\]
For fixed \(d_1, d_2\), and \(c', c\) we consider the inner sum. The functions corresponding to \(F_{cc'}|x=d_1\) and \(F_{cc'}|x=d_2\) are
\[
\left( f + \frac{q}{2} c' e \right) |_{x=d_1} = \frac{q}{2} h_1 \quad \text{and} \quad \left( f + \frac{q}{2} c' e \right) |_{x=d_2} = \frac{q}{2} h_2,
\]
respectively, where
\[
h_1 = \sum_{a=0}^{k-1} c_a d_{1,a} \quad \text{and} \quad h_2 = \sum_{a=0}^{k-1} c_a d_{2,a}.
\]
Let us consider the terms \(h_1\) and \(h_2\) themselves as Boolean functions in the variables \(c_0, c_1, \ldots, c_{k-1}\). Since \(h_1\) and \(h_2\) are multiplied with \(q/2\) in [4], an inversion of \(h_1\) and \(h_2\) implies a sign change of \(F_{cc'}|x=d_1\) and \(F_{cc'}|x=d_2\), respectively. Now write \(g_1 = (f + q/2 c' e)|_{x=d_1}\) and \(g_2 = (f + q/2 c' e)|_{x=d_2}\), and let \(G_1\) and \(G_2\) be their associated vectors, respectively. Then the inner sum of \(S_2\) comprises terms of the form \(C(\pm G_1, \pm G_2)(\ell)\), where \(C(\pm G_1, \pm G_2)(\ell)\) and \(C(-G_1, -G_2)(\ell)\) occur if \(h_1 = h_2\), and \(C(\pm G_1, -G_2)(\ell)\) and \(C(-G_1, +G_2)(\ell)\) occur if \(h_1 \neq h_2\). It is easy to show that \(C(\pm G_1, -G_2)(\ell)\) = \(-C(-G_1, -G_2)(\ell)\) = \(-C(-G_1, +G_2)(\ell)\) = \(-C(\pm G_1, G_2)(\ell)\). In order to prove that the inner sum of \(S_2\) is zero, we have to show that \(h_1 = h_2\) and \(h_1 \neq h_2\) occur equally often as \(c\) runs through all possible values. Recall that \(d_1 \neq d_2\). Hence the difference
\[
h_2 - h_1 = \sum_{a=0}^{k-1} c_a (d_{2,a} - d_{1,a})
\]
is a nonzero linear Boolean function in the variables \(c_0, c_1, \ldots, c_{k-1}\). According to the randomization lemma [7, page 372], such a function produces the values ‘0’ and ‘1’ equally often as \(c\) takes all possible values. Thus \(h_1\) and \(h_2\) are distinct for half of all cases. It follows that the inner sum of \(S_2\), and hence, also \(S_2\) itself is zero for all \(\ell\).

The following corollary is a direct consequence of Theorem [4] and Result [5] and provides a general upper bound on the PMEPR of polyphase sequences of length \(2^m\).

**Corollary 6:** Let \(f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_d\) be a generalized Boolean function. If there exists a set of \(k\) variables \(x = (x_0, x_1, \ldots, x_{k-1})\), such that for each \(d \in \mathbb{Z}_2^k\) the function \(f|_{x=d}\) is of the form [1], then the PMEPR of the polyphase vector associated with \(f\) is at most \(2^{k+1}\).

### IV. Reed–Muller Codes and Generalizations

A code \(C\) of length \(n\) over the ring \(\mathbb{Z}_2\) is defined as a subset \(C \subseteq \mathbb{Z}_2^n\). Such a code \(C\) is said to be linear if each \(\mathbb{Z}_2\)-linear combination of the codewords of \(C\) yields again a codeword of \(C\). Let \(a\) be a codeword of \(C\). The Hamming weight \(w_T(a)\) is defined as the number of nonzero entries in \(a\). The Lee weight of \(a\) is defined as \(w_L(a) = \sum_{i=0}^{n} \min(a_i, 1 - a_i)\). For linear codes the minimum Hamming distance \(d_H(C)\) (minimum Lee distance \(d_L(C)\)) of a code \(C\) is defined as the minimum Hamming (Lee) weight of the nonzero codewords of \(C\). We next consider codes defined by generalized Boolean functions.

**Definition 7:** The Reed–Muller code \(RM(r, m)\) of order \(r\) and length \(2^m\) is the set of all binary vectors that can be associated with a Boolean function of order at most \(r\). The code \(RM(r, m)\) is linear, comprises \(2^{\sum_{i=0}^{m} \binom{m}{i}}\) codewords, and has minimum Hamming (and Lee) distance \(2^m - r\). For further details see [7]. Next we define a new generalization of the classical Reed–Muller codes. Notice that in the following we restrict \(q\) to be a power of 2, i.e. \(q = 2^h\).

**Definition 8:** For \(h > p\) and \(r \geq p\) we define the code \(ZRM^{p}(r, m)\) as the set of all vectors of length \(n = 2^m\) that can be associated with a generalized Boolean function \(ZRM^{p} \rightarrow \mathbb{Z}_{2^n}\) comprising the monomials of order at most \(r - p\) and \(2^p\) times the monomials of order \(r - p - i + i = 1, 2, \ldots, p\).

Apparently the code \(ZRM^{p}(r, m)\) is linear, and a simple counting argument shows that
\[
\log_2 |ZRM^{p}(r, m)| = \sum_{i=0}^{r-p} h \left( \binom{m}{i} \right) + \sum_{i=1}^{p} (h - i) \left( \binom{m}{i + r - p} \right).
\]
We remark that the code \(ZRM^{p}(r, m)\) generalizes the codes \(RM^{p}(r, m)\) and \(ZRM^{p}(r, m)\) from [2]. The code \(ZRM^{p}(r, m)\) simply reads \(ZRM^{p}(r, m)\) and \(ZRM^{p}(r, m)\) is in our notation \(ZRM^{p}(r, m)\). For \(p > 1\) the code
ZRM\textsuperscript{p}\textsubscript{2h}(r, m) yields a new generalization of the Reed–Muller code, that was, to our best knowledge, not mentioned before.

**Theorem 9:** The minimum Hamming distance of ZRM\textsuperscript{p}\textsubscript{2h}(r, m) is equal to $2^{m-r}$ and the minimum Lee distance of ZRM\textsuperscript{p}\textsubscript{2h}(r, m) is equal to $2^{m-r+p}$.

**Proof:** Since ZRM\textsuperscript{p}\textsubscript{2h}(r, m) is linear, we need to find the minimum weights of the nonzero codewords. We first prove a lower bound for the weights. Then we show that there exists at least one codeword that attains this bound.

The proof is by induction on $p$ and $h$, where we take the statement in the above theorem as a hypothesis. The base case for the induction is $p = 0$ and $h = 1$. Then ZRM\textsuperscript{p}\textsubscript{2h}(r, m) is equal to RM(r, m) and has minimum Hamming and Lee weight $2^{m-r}$. Suppose $a = (a_0 a_1 \ldots a_{n-1})$ is a codeword of ZRM\textsuperscript{p}\textsubscript{2h}(r, m) and let $b = (b_0 b_1 \ldots b_{n-1})$ with $b_i = a_i \mod 2^{h-1}$ be codeword over $\mathbb{Z}_{2^{h-1}}$. We will use the easily verified inequalities $w_t H(a) \geq w_t H(b)$ and $w_L(a) \geq w_L(b)$. The first relation is immediately clear and the latter one follows because $a_i \in \{b_i, b_i + 2^{h-1}\}$ and $\min(a_i, 2^h - a_i) \geq \min(b_i, 2^{h-1} - b_i)$.

**Case 1:** $b = 0$. In this case, $a$ comprises only values of either 0 or $2^{h-1}$. Then $2^{h-1} a$ is a codeword of $\text{RM}(r, m)$. Hence $w_t H(a) \geq 2^{m-r}$ and $w_L(a) \geq 2^{h-1} 2^{m-r} \geq 2^{m-r+p}$, since by Definition 3 $h > p$.

**Case 2:** $b \neq 0$ and $h = p + 1$. Now $b$ is a nonzero codeword of ZRM\textsuperscript{p}\textsubscript{2h}(r, m). Let us first consider the case $p = 1$. Then we have $h = 2$. Hence $b$ belongs to ZRM\textsuperscript{p}\textsubscript{2h}(r, m), which is equal to $\text{RM}(r, m)$. Thus we have $w_t H(a) \geq w_t H(b)$ and $w_L(a) \geq w_L(b) \geq 2^{h-1} 2^{m-r} \geq 2^{m-r+p}$. Now consider $p > 1$. Then, by induction, we obtain $w_t H(a) \geq w_t H(b)$ and $w_L(a) \geq w_L(b) \geq 2^{m-r+p}$.

**Case 3:** $b \neq 0$ and $h > p + 1$. In this case $b$ is a nonzero codeword of ZRM\textsuperscript{p}\textsubscript{2h}(r, m). By induction we eventually arrive at Case 1 or 2, and thus, we have $w_t H(b) \geq 2^{m-r}$ and $w_L(a) \geq w_L(b) \geq 2^{m-r+p}$. This implies that $w_t H(a) \geq 2^{m-r}$ and $w_L(a) \geq 2^{m-r+p}$.

Now consider the codeword corresponding to the Boolean function $2^p x_0 x_1 \cdots x_{r-1}$. This codeword has Hamming weight $2^{m-r}$ and Lee weight $2^{m-r+p}$. These weights attain the lower bounds derived above, which completes the proof.

V. **OFDM Codes with Low PMEPR**

We define two fixed lists of indices $I = \{i_0 i_1 \cdots i_{m-k}\}$ and $J = \{j_0 j_1 \cdots j_{k-1}\}$ such that $I \cap J = \emptyset$ and $I \cup J = \{0, 1, \ldots, m-1\}$. Suppose $g_0, g_1, \ldots, g_{m-k-1}, g' : \mathbb{Z}_2^k \rightarrow \mathbb{Z}_q$ are $m-k+1$ generalized Boolean functions in $k$ variables. Let $a : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_{2^h}$ a generalized Boolean function in $m$ variables, be given by

$$a = \sum_{\alpha = 0}^{m-k-1} x_{i_\alpha} g_\alpha(x_{j_0}, \ldots, x_{j_{k-1}}) + g'(x_{j_0}, \ldots, x_{j_{k-1}}).$$

Clearly the set of vectors that can be associated with a function of type (4) forms a linear subspace of ZRM\textsuperscript{p}\textsubscript{2h}(k + 1, m). Denote this subspace as $\mathcal{L}_{2^h}(k, m)$. Notice that $\mathcal{L}_{2^h}(0, m)$ is identical to ZRM\textsuperscript{p}\textsubscript{2h}(1, m), the first-order generalized Reed–Muller code.

Now suppose $d$ has binary expansion $(d_0 d_1 \cdots d_{k-1})_{2}$ and let $\pi_0, \pi_1, \cdots, \pi_{2^k-1}$ be $2^h$ permutations of $\{0, 1, \cdots, i_{m-k-1}\}$. Then consider the functions

$$b = 2^{h-1} \sum_{d=0}^{2^h-1} \sum_{\alpha=0}^{k-1} x_{\pi_d(\alpha)} x_{\pi_d(\alpha+1)} \prod_{\beta=0}^{k-1} x_{\pi_d(\beta)} (1 - x_{\beta})^{1-d_\beta},$$

and let the set of all vectors corresponding to a form of type (5) form a code $\mathcal{R}_q^h(k, m)$. Clearly the order of $b$ is at most $k+2$, and the elements of the associated $\mathcal{Z}_{2^h}$-valued codewords are either 0 or $2^{h-1}$. Since a codeword is completely determined by the $2^h$ permutations $\pi_0, \pi_1, \cdots, \pi_{2^k-1}$, there exist $[(m - k)!/2]^{2^h}$ codewords in $\mathcal{R}_q^h(k, m)$.

Now suppose $b$ is a codeword in $\mathcal{R}_q^h(k, m)$. Then the set of codewords

$$\{b + a | a \in \mathcal{L}_q^h(k, m)\}$$

is a coset of the linear code $\mathcal{L}_q^h(k, m)$ with $b$ being its coset representative.

**Theorem 10:** Let $b \in \mathcal{R}_q^h(k, m)$. Then each polyphase codeword of the coset (6) has PMEPR at most $2^{k+1}$.

**Proof:** Consider the notations above. Let $a$ and $b$ be the generalized Boolean functions corresponding to the codewords $a$ and $b$, respectively. Recall the definition of $I$ and $J$ and write $x = (x_{j_0} x_{j_1} \cdots x_{j_{k-1}})$. According to Corollary 3, we have to show that for each binary vector $d = (d_0 d_1 \cdots d_{k-1})$ the expression $\alpha = a_{|x = d} = a_{|x = d} + b_{|x = d}$ is of the form (1). Considering (4) we have

$$a_{|x = d} = \sum_{\alpha = 0}^{m-k-1} x_{i_\alpha} g_\alpha |x = d| + g'|x = d|,$$

where $g_0, g_1, \ldots, g_{k-1}$, and $g'$ are arbitrary generalized Boolean functions in the variables $x_{j_0}, x_{j_1}, \cdots, x_{j_{k-1}}$. Apparently, after restriction in $x$, each of the restricted functions $g_\alpha |x = d|$ and $g' |x = d|$ are constants, and $a_{|x = d}$ becomes an affine function. Now consider $b_{|x = d}$. It is easy to verify that

$$b_{|x = d} = 2^{h-1} \sum_{\alpha = 0}^{m-k-2} x_{\pi_d(\alpha)} x_{\pi_d(\alpha+1)},$$

which has the form of the quadratic part in (1). Thus, by Corollary 4, the polyphase vectors associated with $(a + b)$ have PMEPR at most $2^{k+1}$.

Before we state our new constructions let us define subcodes of $\mathcal{L}_q^h(k, m)$. Let $A_{2^h}^h(k, r, m) = \mathcal{L}_q^h(k, m) \cap ZRM_{2^h}(r, m)$. Of course $A_{2^h}^h(k, r, m)$ is linear and $A_{2^h}^0(k, m) = \mathcal{L}_q^h(k, m)$. By inspecting (3) and using (3), we have

$$\log_2 |A_{2^h}^h(k, r, m)| = \sum_{i=0}^{r-1} h(\begin{pmatrix} k \\ i \end{pmatrix}) + \sum_{h=0}^{r} (h) (\begin{pmatrix} k \\ i + r - p \end{pmatrix}),$$

and

$$+ \sum_{i=0}^{r-p-1} h(\begin{pmatrix} k \\ i \end{pmatrix}) + \sum_{i=0}^{r} (h) (\begin{pmatrix} k \\ i + r - p - 1 \end{pmatrix}).$$

(7)
Now we use cosets of the code $\mathcal{A}_{2h}^p(k, r, m)$ to construct three code classes with PMEPR at most $2^{k+1}$.

**Class I Codes:** A very simple code can be constructed by using just a single coset of $\mathcal{A}_{2h}^p(k, r, m)$, i.e.

$$\{ b + a \mid a \in \mathcal{A}_{2h}^p(k, r, m) \}, \quad b \in \mathcal{R}_{2h}(k, m).$$

Clearly the number of encodable bits is given by $\binom{m}{r}$. Since the constant offset leaves the distance properties unchanged and $\mathcal{A}_{2h}^p(k, r, m) \subseteq \mathcal{ZRM}_{2h}^p(r, m)$, this code has minimum Hamming distance $2^{m-r}$ and minimum Lee distance $2^{m-r+p}$.

**Class II Codes:** Consider the functions corresponding to $\mathcal{R}_{2h}(k, m)$, and set $\pi = \pi_0 = \cdots = \pi_{2k-1}$ in the definition of those functions in (3). Then we obtain quadratic forms of type

$$b' = 2^{h-1} \sum_{\alpha=0}^{m-k-2} x_{\pi(\alpha)} x_{\pi(\alpha+1)}.$$

There exist $(m-k)/2$ vectors associated with such a quadratic form. Let $\mathcal{R}_{2h}^c(k, m)$ denote this set. Then we define the code

$$\bigcup_{b \in \mathcal{R}_{2h}^c(k, m)} \{ b + a \mid a \in \mathcal{A}_{2h}^p(k, r, m) \}.$$

For $k > 0$ and $r > 1$ the above code is a union of $(m-k)/2$ cosets of $\mathcal{A}_{2h}^p(k, r, m)$ inside $\mathcal{ZRM}_{2h}^p(r, m)$, and hence, it has minimum Hamming distance $2^{m-r}$ and minimum Lee distance $2^{m-r+p}$. For $k = 0$ and $r = 1$ the code is a union of $ml/2$ cosets of $\mathcal{ZRM}_{2h}^p(1, m)$ inside $\mathcal{ZRM}_{2h}^p(2, m)$. In this case it has minimum Hamming distance $2^{m-2}$ and minimum Lee distance $2^{m-1+p}$. The maximal number of encodable bits amounts to $\log_2 |\mathcal{A}_{2h}^p(k, r, m)| + \lfloor \log_2 (m-k)!/2 \rfloor$.

**Class III Codes:** Recall that (3) identified $(m-k)/22^k$ coset representatives. Note that these coset representatives have order at most $k+2$ and its elements are either 0 or $2^{h-1}$. Then, for $p > 0$, we define the Class III codes as follows

$$\bigcup_{b \in \mathcal{R}_{2h}(k, m)} \{ b + a \mid a \in \mathcal{A}_{2h}^{p-1}(k, k+1, m) \}.$$

This code is a union of $(m-k)/22^k$ cosets of $\mathcal{A}_{2h}^{p-1}(k, k+1, m)$ inside $\mathcal{ZRM}_{2h}^p(k+2, m)$. Hence it has minimum Hamming distance $2^{m-k-2}$ and minimum Lee distance $2^{m-k-2+p}$.

With such a code one can encode $\log_2 |\mathcal{A}_{2h}^{p-1}(k, k+1, m)| + [2k \log_2 (m-k)!/2]$ bits.

Remarks: Some relations to previous constructions are given below.

1) Setting $k = 0$ and $p = 0$ in the Class II codes results in codes that coincide with those constructed in [2]. Then the codes comprise $ml/2$ cosets of $\mathcal{ZRM}_{2h}^p(1, m)$ inside $\mathcal{ZRM}_{2h}^p(2, m)$ and have PMEPR at most 2.

2) Setting $p = 1$ in the Class III codes, then each codeword can be obtained by interleaving $2^p$ codewords of length $m-k$ from the codes considered in [2] (see 1).

3) If $\mathcal{A}_{2h}^0(k, 2, m)$ is chosen as the underlying code for the Class II codes, the resulting codes are similar to those considered in [4]. Then we obtain a subcode of $\mathcal{ZRM}_{2h}^0(2, m)$.

The difference between our construction and that in [4] is that we apply the permutation of the variable indices only to those indices from the set $I$. This way we can guarantee that the codewords are generated exactly once, since all quadratic forms corresponding to the coset representatives are permutation invariant. In contrast to that, in [4] the permutation was applied to the indices $\{0, 1, \ldots, m-1\}$. Then, in order to avoid multiple generations of codewords, it was necessary to introduce some constraints on the quadratic parts of the associated functions, which made the handling of those codes a bit unwieldy.

Notice also that, contrarily to the codes considered in [2] and [4], our codes are not unions of cosets of a generalized first-order Reed–Muller code $\mathcal{ZRM}_{2h}^p(1, m)$, but instead are unions of cosets of the linear code $\mathcal{A}_{2h}^p(k, r, m)$, which has, for $k > 0$ and $p > 0$, more codewords than $\mathcal{ZRM}_{2h}^p(1, m)$. Hence, compared to the approaches in [2] and [4], we need less cosets to achieve about the same code rate. This way the encoding and decoding procedures become simpler, in particular for larger block lengths. For the details about encoding and decoding of the proposed code classes we refer the reader to [9].

VI. CONCLUSION

In this paper a large family of sequences lying in complementary sets have been presented. Moreover the classical Reed–Muller codes have been generalized in a novel manner. We have shown that the family of sequences lying in complementary sets form cosets of a linear code, which are contained in the generalized Reed–Muller code $\mathcal{ZRM}_{2h}^p(r, m)$. This way new codes for OFDM with low PMEPR have been proposed, which are not limited to be a subcode of the second-order generalized Reed–Muller code. A number of code families has been presented, where PMEPR, code rate, minimum distance, and encoding/decoding complexity can be traded against each other.

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