ANALYSES OF THREE-DIMENSIONAL MAGNETOHYDRODYNAMIC INSTABILITY OF ANTISOLAR LATITUDINAL DIFFERENTIAL ROTATION IN F, G, AND K STARS

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ABSTRACT

Motivated by observations that only a very few stars have been found to have antisolar differential rotation, much weaker in amplitude than that of the Sun, we analyze the stability of antisolar and solar-type latitudinal differential rotations in the tachoclines of typical F, G, and K stars. We employ two three-dimensional thin-shell models, one for a Boussinesq but nonhydrostatic system and the other for a hydrostatic but non-Boussinesq system. We find that, in general, the combination of toroidal field band and differential rotation is more unstable, and unstable for lower toroidal fields, for antisolar than for solar-type differential rotation. In the antisolar case, the instability is always found to weaken the differential rotation, even if the primary energy source for the instability is the magnetic field. This favors surface antisolar differential rotations in stars being weaker than solar types, if the instability in the tachocline is felt at the surface of the star. This is most likely to happen in F stars, whose convection zones are much thinner than they are in G and K stars. This effect could help explain why the antisolar differential rotations that have been found are very weak compared with the rotation of the Sun.

Key words: instabilities – magnetohydrodynamics (MHD) – stars: interiors – stars: magnetic field – stars: rotation

1. INTRODUCTION

The internal rotation pattern of a star has most extensively been studied observationally and theoretically for our nearest G star, the Sun. For the surface of the Sun, Carrington (1858) showed that the equator rotates faster than the pole by systematically monitoring the rotation of sunspots over the 11 year cycle. In the twentieth century, the Sun’s positive pole-to-equator differential rotation was established from Doppler shift measurements (DeLury 1939; Plaskett 1959). A review of early observations of solar surface differential rotation can be found in Gilman (1974). Quantitative analyses of long-term Mount Wilson Observatory data indicate that the equator rotates about 132 hHz faster than the pole (Ulrich et al. 1988) at the surface. Helioseismic measurements revealed that the Sun’s positive pole-to-equator differential rotation pattern persists from the surface down to the core-envelope interface, through the bulk of the convective envelope—the details can be found in an extensive review by Thompson et al. (2003).

Non-axisymmetric instability of solar-like, positive pole-to-equator differential rotation in the presence or absence of magnetic fields has been extensively studied in global two-dimensional (2D), quasi-three-dimensional (3D), and 3D thin-shell models (Watson 1981; Dziembowski & Kosovichev 1987; Gilman & Fox 1997; Dikpati & Gilman 1999, 2001; Charbonneau et al. 1999; Garaud 2000; Cally 2001, 2003; Gilman & Dikpati 2002; Cally et al. 2003; Dikpati et al. 2003, 2004; Arlt et al. 2005, 2007; Miesch et al. 2007; Gilman et al. 2007). Most of these calculations were applied to the solar tachocline, the interface between the solar convection zone and interior, where the stratification is thought to be either slightly subadiabatic, as in the overshoot layer where convection penetrates from above, or much more subadiabatic, as in the radiative layer immediately below the overshoot layer. But differential rotation anywhere in a star may be subject to the instabilities described in the above references.

Axisymmetric instability of latitudinal differential rotation cannot occur in 2D hydrodynamic (HD) or magnetohydrodynamic (MHD) models (see discussions in Gilman & Fox 1997; Dikpati & Gilman 1999; Cally et al. 2003) and is hard to excite in quasi-3D shallow-water models for which about a 2 million Gauss peak toroidal magnetic field is required. In a 3D thin-shell model, however, axisymmetric instabilities can be excited with a much lower toroidal field of about 5000 G. A few recent calculations have been done to investigate axisymmetric instabilities in solar tachocline (Cally et al. 2008; Dikpati et al. 2009; Hollerbach & Cally 2009). These studies show that latitudinal differential rotation of up to 18% pole-to-equator amplitude will be hydrodynamically stable in 2D, but is unstable in 3D or in the presence of magnetic fields.

Various observations, including light curves of rotating starspots (Messina & Guinan 2003), spectroscopic measurements (Reiners & Schmitt 2003), long-term changes in Ca II H and K fluxes (Donahue et al. 1996), and Doppler imaging (Barnes et al. 2000), indicate the existence of differential rotation in many stars. Among the stellar population of F, G, and K stars, solar-like positive pole-to-equator differential rotation, as well as an antisolar pattern in which the equator rotates slower than the pole, is observed. For example, multiwavelength studies of G0–G5V stars by Messina & Guinan (2003) and Zeeman–Doppler imaging by Jeffers & Donati (2008) have revealed the existence of solar-like (positive) and antisolar (negative) pole-to-equator differential rotation.

For the spot-dominated late-type stars, inversions of light curves for images of dark starspots on the surface reveal the differential rotation pattern of K stars; so far all of them were found to have solar-like profiles (Rottenboxer et al. 2011). Antisolar differential rotation has been found among F and G stars, but occurs much less frequently than solar-like differential rotation. Furthermore, the amplitude of antisolar differential rotation in those stars is also found to be smaller than solar-type pole-to-equator differential rotation.

Theoretical modeling has enabled us to understand how the solar-like positive pole-to-equator differential rotation is formed in the solar and in stellar convection zones (Gilman & Miller...
Stars with deeper convection zones are likely to have broader differential rotations with latitude (Gilman 1980). Except for anomalously weak rotators, such stars are likely to have equatorial acceleration like the Sun does. On the other hand, stars with shallow convection zones, like F stars, may not have broad differential rotation, but rather a much more structured pattern. Even if they rotate much faster than the Sun, if the turnover time in their shallow convection zone is short enough, they could have antisolar differential rotations.

HD and MHD instability of solar-like differential rotation in stars has also been explored (Knobloch & Spruit 1982; Urpinen et al. 1996; Spruit 1999, 2002; Braithwaite 2006), but to our knowledge, for the case of antisolar differential rotation, neither detailed theory of formation nor the stability analyses have been carried out yet.

The aim of this paper is to analyze the instability of antisolar differential rotation in order to answer the following questions. (1) Why have only about a dozen antisolar stars been found? (2) Is antisolar differential rotation more unstable than the solar-like pattern, and could that be the reason? (3) What is the limiting amplitude of a negative pole-to-equator differential rotation that can be stable? (4) Why has no antisolar K star been found yet?

The observations of stellar rotation are made at the surface of a star, and we do not know the differential rotation pattern in their tachoclines. Using the analogy that the Sun’s positive pole-to-equator surface latitudinal differential rotation persists down to the tachocline, we assume that the stellar tachoclines reflect the same sign of the pole-to-equator latitudinal differential rotation as observed at their surfaces. The differential rotation formed in the stellar convection zone imposes an upper boundary condition on the tachocline and provides one source for the energy needed to drive instabilities there.

But the precise relation between stellar rotation at the surface and in the stellar tachocline may be more complex and depend on such factors as the thickness of the stellar convection zone. In the case of the Sun, and probably K stars, the inertia of the convection zone is so large compared with the tachocline below it that it is unlikely that instability in the tachoclines of such stars could significantly change the surface differential rotation pattern. Rempel (2005) did show that thermal effects in the solar tachocline could significantly modify the rotation contours of the convection zone, moving them away from being constant on cylinders as global convection theory tends to create. But with the thin, low density convection zones of F stars, instability in the tachocline could have a much stronger feedback on the surface rotation, resulting in a lower overall differential rotation for F stars, including at their photospheres. This is a reason to study tachocline instabilities in stars as one determinant of their surface differential rotations.

We will use for our calculations a 3D thin-shell model of the solar/stellar tachocline to analyze axisymmetric and non-axisymmetric MHD instability of the pole-to-equator latitudinal differential rotation for a wide range of positive and negative amplitudes. We explore the MHD instability problem as opposed to the purely HD problem because it is likely that most stellar tachoclines contain substantial toroidal fields. Also, in the previous calculations on magneto-shear instabilities by us and many others, as referred earlier in this section, major findings in 2D, quasi-3D, and 3D models were that the magneto-shear instability exists for a wide range of toroidal field bands of latitudinal widths of above 2.7°, located at a wide range of latitudes above ~13°, and for a wide range of peak field strength from a few hundred Gauss and above.

Magneto-shear instabilities occur for differential rotation amplitudes that are characteristic of both radiative and overshoot tachoclines. In the case of the Sun, surface observations can give some guidance about the bandwidth of the tachocline toroidal field bands to be of 6°–10°, if active regions are being produced from them (see Zwaan 1978). Rising flux tube studies (see, for example, Fan et al. 1993) indicate that these bands also perhaps spend their maximum time in the latitude range of 0°–45°. So, in order to pick a typical case for the Sun and solar-like stars for investigating magneto-shear instabilities with solar and antisolar differential rotation, we consider in this study a 10° toroidal band located at 30° latitude and explore the details in the parameter space of solar/antisolar differential rotation amplitude and toroidal field strength.

2. PHYSICAL CONTEXT AND PARAMETER CHOICES

Full 3D and 3D thin-shell models of the solar tachocline have been developed by several authors (Zhang et al. 2003; Cally 2003; Arlt et al. 2007; Gilman et al. 2007; Miesch et al. 2007) to explore the global instability of tachocline latitudinal differential rotation in the presence of toroidal magnetic fields. For investigating global instability of antisolar differential rotations in F, G, and K stars, we perform the analyses in a 3D thin-shell model of the solar/stellar tachocline, but using two different approaches, namely, (1) a hydrostatic, non-Boussinesq system and (2) a nonhydrostatic, Boussinesq system. For (1), we start from the 3D thin-shell perturbation equations (16)–(25) of Gilman et al. (2007), linearized about a reference state containing differential rotation ($\theta_0$), toroidal field ($a_0$), pressure ($p_0$), and temperature ($\theta_0$). In the latitude, longitude, depth coordinate system ($\phi$, $\lambda$, $z$), the latitudinal and vertical force balance of the reference state can be described in the following equations:

$$\frac{a_0^2 - \alpha_0^2}{2} \cos \phi \sin \phi = \frac{\partial \pi_0}{\partial \phi},$$

$$\frac{\partial \pi_0}{\partial z} = -G^{1/2} \frac{\delta \phi}{\delta \theta} p_0 = -\frac{\delta \phi}{\gamma} p_0 + G^{1/2} \theta_0,$$

$$p_0 = \gamma G^{1/2} [\delta E_{-1}^{-1} \rho_0 + \delta S_{-1}^{-1} \theta_0],$$

$$\pi_0 = p_0 + \frac{a_0^2 \cos^2 \phi}{2}.$$

These are Equations (11)–(14) in Gilman et al. (2007). We specify $a_0$ and $\alpha_0$, which are, respectively, the toroidal field and differential rotation in angular measure. Assuming the perturbation variables are of the form, $u$, $v$, $w$, $a$, $b$, $c$, $\rho$, $p$, $\pi$, $\theta$, $\sim e^{i m(\lambda - \sigma)}$, in which $m$ is the longitudinal wavenumber, and representing each variable as $u = u_c(\mu) \cos(n \pi z) + u_s(\mu) \sin(n \pi z)$, for $0 \leq z \leq 1$, in order to separate $z$ from $\mu$, we obtain the following linearized first-order perturbation equations, which we will solve using parameters chosen to represent F, G, and K stars.

The five equations from the coefficients of $\cos(n \pi z)$ are

$$im(a_0 - \sigma) u_c - im \alpha_0 a_c - 2\mu(a_0 v_c - a_0 b_c)$$

$$+ (1 - \mu^2) \left( \frac{\partial a_0}{\partial \mu} v_c - \frac{\partial a_0}{\partial \mu} b_c \right) + \frac{im}{(1 - \mu^2)^{1/2}} \pi_c = 0,$$

(5)
The parameter $G = g|\nabla - \nabla_{\text{ad}}|H^2/2(r_0\omega_c)^2H_m$, in which $g$ is gravity at tachocline depth, $|\nabla - \nabla_{\text{ad}}|$ is the fractional departure of the temperature gradient from the adiabatic value, $r_0$ is the stellar radius of the tachocline, and $\omega_c$ is the angular velocity of the star beneath the tachocline. $G$ is a measure of the strength of the negative buoyancy force in the tachocline and is sometimes referred to as the “effective” gravity. For adiabatic stratification this effective gravity is zero. For each star we consider, we will use two values of $G$, one for the radiative tachocline and the other (much smaller value) for the convective overshoot layer just above the radiative zone.

In solving Equations (5)–(14), we use rigid top and bottom boundary conditions, which mean that the boundaries are not allowed to deform. There is no stress at the boundary, because our present calculation is nonviscous. In the future, when we attempt to do a viscous calculation, we will have to implement a stress-free boundary condition explicitly if we do not want to allow the model to form boundary layers. For the magnetic field, we have assumed perfectly conducting top and bottom boundaries in order to confine the magnetic field to the fluid shell. This is physically reasonable for the bottom boundary, but somewhat artificial for the top boundary, since we expect flux to escape upward. To account for this upward escape of flux is beyond the scope of the present model.

We use two independent numerical methods to analyze the instabilities for $m = 0$ as well as for $m > 0$, namely a shooting method and a spectral method. The details of the solution techniques have been described in Gilman et al. (2007) and in Cally et al. (2003) for $m > 0$ and in Cally et al. (2008) and Dikpati et al. (2009) for $m = 0$.

To find the values of $G$ and $\delta_s$, we use in our stability calculations, we use the parameters described in the following subsections. Certain parameters are chosen using the zero-age main sequence (ZAMS) stellar interior model of MacGregor et al. (2007).

### 2.1. A Typical F Star with 1.4 Solar Mass and 25 Times Solar Rotator

Most F stars have a small convective core and a convective outer envelope which is thinner than the Sun’s. Our instabilities will be applicable to the base of the outer convective envelope. For a typical F star, we take a radius of $1.38 \, R_\odot = 9.59 \times 10^{10}$ cm and a mass of $2.78 \times 10^{33}$ g. We assume that the outer 5% of the star is the convective envelope; therefore, the thickness of the convection zone is $4.79 \times 10^4$ km.

Assuming that the tachocline of a typical F star is 7.5% of the convection zone (CZ)-thickness (like the Sun), its tachocline thickness is $3.6 \times 10^{4}$ km. The density at the base of CZ of this typical F star would be $8.85 \times 10^{-5}$ g cm$^{-3}$. To estimate the pressure scale height, we use an $X = 0.71$, $Y = 0.27$ H/He mixture, which leads to an atomic weight $\mu$ for the mixture of $0.71 \times 0.5 + 0.27 \times 2 = 0.90$. We take the temperature at the base of the convection zone $T = 2.96 \times 10^5$ K. Then for the universal gas constant, $R = 8.31 \times 10^7$ erg deg$^{-1}$ mole$^{-1}$ and gravity ($g$) at the base of the convection zone $g = 2.245 \times 10^5$ cm s$^{-1}$, the pressure scale height $(RT/\mu g) = 1.22 \times 10^4$ km, and the ratio of tachocline thickness to pressure scale height $\delta_s = 0.294$. From the table of values for our typical F star, we take $|\nabla - \nabla_{\text{ad}}| = 10^{-1}$ and $10^{-4}$ for the radiative and overshoot tachoclines, respectively, leading to $G = 0.12$ and $1.2 \times 10^{-4}$.

\[
\begin{align*}
\text{im}(\omega_c - \sigma)v_c - \text{im}a_0b_c - 2\mu(a_0v_c - \omega_0b_c) + (1 - \mu^2)^{1/2} \frac{\partial \sigma}{\partial r} = 0, \\
\text{im}(\omega_c - \sigma)a_c - \text{im}a_0u_c + v_c(1 - \mu^2) \frac{\partial \sigma}{\partial r} - b_c(1 - \mu^2) \frac{\partial \sigma}{\partial r} = 0, \\
\text{im}(\omega_c - \sigma)b_c - \text{im}a_0v_c = 0,
\end{align*}
\]
Figure 1. Growth rate contours as a function of $s$ and $a$ for a typical G star. Frames (a) and (b) present growth rates for $m = 1$ symmetric and antisymmetric modes with radial wavenumber, $n = 1$ in the overshoot tachocline with effective gravity, $G = 0.01$, and frames (c) and (d) for the radiative tachocline ($G = 10$). A growth rate of 0.01 corresponds to an $e$-folding growth time of $\sim 1$ year in the Sun.

2.2. A Typical G Star Like the Sun

Tachocline instabilities for solar-type G stars have been studied for a wide range of $G$ values in Gilman et al. (2007). For comparison here we take $G = 10$ and 0.01 for radiative and overshoot effective gravities.

2.3. A Typical K Star with 0.75 Solar Mass and 1/2 Times Solar Rotator

Most K stars have convective outer envelopes that are thicker than the Sun’s. For this case we take a radius of $0.6801 R_\odot = 4.7301 \times 10^{10} \text{ cm}$. The outermost 32.8% is the convective envelope so the thickness of the CZ is $1.55 \times 10^3 \text{ km}$ and the tachocline thickness is $1.163 \times 10^4 \text{ km}$. The density at the base of the CZ is $1.61 \text{ g cm}^{-3}$. The mixture of H and He is the same as for the F star, for which $\mu = 0.8956$. In this star the temperature at the base of the convection zone $= 2.8867 \times 10^6 \text{ K}$ and gravity ($g$) at the base of the convection zone $= 9.848 \times 10^4 \text{ cm s}^{-2}$. Thus the pressure scale height $= 2.72 \times 10^4 \text{ km}$. Then $\delta_i = 0.428$. For this star $|V - V_{\text{ad}}| = 5 \times 10^{-3}$ and $5 \times 10^{-9}$ for radiative and overshoot tachoclines, respectively. Therefore $G = 2.5 \times 10^4$ for the radiative tachocline and $2.5 \times 10^{-3}$ for the overshoot tachocline.

2.4. Magnetic Field Scaling

Because of the different rotation rates and densities at tachocline depth in F, G, and K stars, the magnetic fields scale somewhat differently, but the effects of differing rotation and density tend to cancel each other out. In particular, in the Sun $a = 1$ corresponds to $B = 10^5 \text{ G}$, whereas in F stars $a = 1$ is for $B = 7.24 \times 10^4 \text{ G}$ and in K stars $B = 9.65 \times 10^4 \text{ G}$.

3. RESULTS

Considering a typical case for the dynamo-generated toroidal band of $10^\circ$ latitudinal width and band center located at $30^\circ$ latitude (Dikpati & Gilman 1999), we compute the unstable eigenmodes for this band in the parameter space of $-0.18 \leq s \leq 0.18$ and $0.0 \leq a \leq 4$. We look for axisymmetric ($m = 0$) as well as non-axisymmetric ($m > 0$) instabilities for various radial wavenumbers $n$ in the overshoot and radiative tachocline of a star.

We sequentially present our results for a typical G star, F star, and K star, for non-axisymmetric instabilities. Then we give some examples of axisymmetric instabilities that could occur.

3.1. Instabilities of Solar and Antisolar Differential Rotation in G Stars

Figure 1 displays the global MHD instability for a G star such as the Sun, as a function of the differential rotation parameter $s$ and the peak toroidal field $a$. Frames (a) and (b) show the growth rates for symmetric and antisymmetric modes, respectively, for an “overshoot” tachocline for which the effective gravity $G = 0.01$. 

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We see that for large $a$, the growth rates are high and nearly independent of the amount or even the sign of the differential rotation present. Symmetric and antisymmetric modes have virtually the same growth rates. This implies that the nonlinear interaction between these modes is very likely to occur, triggering the flip-flop mechanisms observed in many rapidly rotating stars (Fluri & Berdyugina 2004, 2005). We infer that for high $a$ the instability is totally governed by the toroidal field.

By contrast, for relatively weak toroidal fields, the amplitude and sign of the differential rotation determines the growth rate. Modes of both symmetries are unstable for much lower toroidal fields for antisolar differential rotation than for solar differential rotation. The difference in minimum field strength needed for instability is about a factor of 10. When applied to the tachocline of a G star, this result should mean that both antisolar differential rotation and tachocline toroidal fields are kept at much smaller values by the instability than they could be in a star with solar-like differential rotation. This may be one reason why so few cases of antisolar rotation have been reported.

Frames (c) and (d) display growth rates for a radiative tachocline of a G star. Here we again see that for high toroidal fields, the instability is magnetically dominated and not strongly dependent on differential rotation or mode symmetry. At low toroidal field, antisolar differential rotation is no longer unstable for much lower toroidal fields than is solar-type differential rotation. For both types of differential rotation, larger toroidal fields are needed to excite instability for lower differential rotation. For solar differential rotation, the antisymmetric mode becomes unstable for a lower toroidal field than does the symmetric mode, especially for larger differential rotation, whereas the antisymmetric differential rotation modes of both symmetries are about equally unstable.

The difference between low $G$ and high $G$ results comes from the difference in disturbance structure. In particular, for high $G$, the radial motion in disturbances is largely suppressed by the negative buoyancy. Its pattern is nearly 2D (longitude–latitude) and the radial motion is subject to relatively large longitudinal pressure gradients that produce longitudinal torques. By contrast, at low $G$, radial motions are much more prominent, and this allows the disturbances to more nearly conserve angular momentum locally. Such a disturbance is much easier to excite in a differential rotation profile that varies less with latitude. This favors antisolar over solar differential rotation. The larger the angular momentum gradient, the greater the perturbation centrifugal force pushing the displaced fluid elements back toward where they started. The perturbation $\mathbf{J} \times \mathbf{B}$ force has to overcome this effect to create the instability, and less $\mathbf{J} \times \mathbf{B}$ force is needed in the antisolar case.

Figure 2 shows a line plot of growth rate for the antisymmetric modes depicted in Figure 1 for three differential rotations ($s = -0.09, 0, 0.09$) for $a$ from zero to ten. These results show the dominance of the magnetic field in determining the growth rate as the field is made stronger. At what field strength the toroidal field dominates depends on the effective gravity—a stronger field is needed to overcome the stronger negative buoyancy effects associated with a larger effective gravity. We can see from Figure 2 that the magnetically dominated instability is very powerful: $e$-folding growth times are measured in days or even hours compared with years when the toroidal field is weak enough that differential rotation dominates.

Figure 2(b) displays the phase velocity in longitude of the unstable modes, measured in fractions of the inertial frame of core rotation rate, which is zero in the rotating reference frame. The $\phi$ parameter $a$ is the torque axis in the rotating frame, which is zero in the rotating reference frame.

We see that when the field is very weak, unstable modes in the overshoot tachocline propagate at approximately the rate of the core, which is very close to that of the latitude of the band. For extremely low toroidal field, modes in the radiative tachocline propagate at a rate somewhat retrograde relative to the core rate, typical of Rossby waves in a rotating spherical shell. This is to be expected because the perturbations in the radiative tachocline are nearly 2D (longitude–latitude). In both tachoclines the sign and amplitude of the differential rotation has relatively little effect on the propagation speed. For increased toroidal field, the phase speed in both tachoclines is still essentially independent of the differential rotation, while differing from each other. This difference comes from the very different mode structure in the two tachoclines. Similar retrograde propagation was found in previous analyses by Gilman et al. (2007).

### 3.2. Disturbance Planforms for Radiative Tachocline in G Stars

Examining the longitude–latitude structure of unstable disturbances can help us understand the physics of the instability seen in Figures 1 and 2. Figures 3–5 display for G star cases the longitude–latitude patterns of disturbance velocities, magnetic fields, and fluid pressure. Figure 3 is for antisymmetric disturbances in a radiative tachocline for relatively high toroidal field parameter $a$ of 1.0 and 2.5. The left column of panels (a), (c), and (e) is the antisolar case, the right column (panels b, d, f) the solar case.

We see in panels (a) and (b) of Figure 3 that the perturbation magnetic fields, represented by arrows, are tightly confined to the neighborhood of the toroidal band, as we should expect. These field lines are essentially closed in an oval, meaning that there is very little perturbation field that is vertical. We see further that the perturbation field vectors are oriented clockwise around the highs (red contours) in perturbation fluid pressure and counterclockwise around the lows (blue contours). This is
Figure 3. Left and right columns respectively show eigenfunctions for antisolar ($s = -0.18$) and solar-like ($s = 0.18$) disturbance patterns in latitude–longitude plane, for a typical radiative tachocline. Magnetic vectors superimposed on color contours of total pressure perturbations are presented in top two frames, (a) and (b), for $a = 2.5$. Horizontal velocities (black arrows) superimposed on fluid pressure perturbations, $\pi$, (color contours) have been plotted for $a = 2.5$ (frames c and d) and for $a = 1$ (frames e and f).

The opposite orientation to the so-called geostrophic balance between velocity vectors and fluid pressure. In that case there is a near balance of the Coriolis and pressure gradient forces. Here there is a tendency for the perturbation $J \times B$ and fluid pressure gradient forces to balance. For high field strengths in the neighborhood of the toroidal band, Coriolis forces are less important.

The direction of the arrows implies that the total horizontal field (undisturbed, positive, toroidal band plus perturbations) is displaced in latitude toward where the perturbation east–west field is positive (toward the right) and away from where it is negative (pointed toward the left). Given the position of high (red) and low fluid perturbation pressure (blue) centered at the latitude of the undisturbed band, we infer that the peaks and troughs of total perturbation pressure (fluid plus magnetic) are displaced in latitude in the same way as the total field.

The perturbation magnetic patterns are very similar for antisolar and solar differential rotation, but away from the band, the fluid pressure patterns are rather different. This is where HD processes are dominant. Panels (c) and (d) display the perturbation velocity vectors for antisolar and solar cases. We see that away from the latitude of the toroidal band, the flow is nearly geostrophic—clockwise around the highs of fluid pressure, counterclockwise around the lows. But within the band, the flow easily crosses the fluid pressure contours. Here it is being driven across by the perturbation $J \times B$ force, carrying field with it to generate the poleward and equatorward displacements of total field.

The “tilts” of the velocity vectors from the E–W direction, seen most strongly on the poleward side of the band, are of particular interest. In both antisolar and solar cases they imply angular momentum transport toward the equator. In the antisolar case, this implies angular momentum transport down the angular velocity gradient, which feeds kinetic energy to the disturbances away from the band, and would reduce the differential rotation. By contrast the tilts in the solar case imply angular momentum transport up the gradient, which would actually increase the differential rotation. The energy for this increase comes from the toroidal field. So there will be further tendency for antisolar differential rotation to be destroyed by this instability. In the solar
case, these high latitude HD perturbations must be driven by the low latitude $\mathbf{J} \times \mathbf{B}$ forces and associated fluid pressure gradients, because they are giving up energy to the differential rotation.

Panels (e) and (f) display the velocity and pressure perturbations for a weaker peak toroidal field parameter $a = 1$, which is much closer to the value for onset of this instability. Here we see the tilts have largely disappeared. The velocity patterns still show the near geostrophic balance, but are largely present only in high latitudes in the antisolar case, mostly in equatorial latitudes in the solar case.

### 3.3. Disturbance Planforms for Overshoot Tachocline in G Star

Figures 4 and 5 show planforms for unstable disturbances in the overshoot tachocline of a G star, with effective gravity $G = 0.01$, for modes that are symmetric about the equator. Figure 4 is for a strong toroidal field case, namely $a = 0.6$, for both antisolar and solar-type differential rotations. Figure 5 is for a weak field case, $a = 0.1$, for which, from Figure 1, only the antisolar case is unstable.

For $a = 0.6$ in Figure 4, we see the perturbation fluid pressure and velocity structures (frames c and d) are very different from the case of the radiative tachocline. Peaks and troughs of fluid pressure occur at nearly the same longitude, and there are two latitudinally narrow additional fluid pressure perturbations on the poleward side of the band, one near it and the other near the poles. Most of the magnetic perturbations (frames a and b) of both signs occur on the poleward side of the toroidal band. High fluid pressure is still on the left of the perturbation field arrows, low pressure on the right, but the perturbation fields are not closed ovals. Rather there are strong points of horizontal convergence and divergence of field. This means the vertical fields are quite significant. Similarly, the flow fields are not simply counterclockwise around lows and clockwise around highs. Instead there is much flow both up and down the fluid pressure gradients, and substantial areas of horizontal convergence and divergence. This implies the vertical motions are substantial.
Figure 6. Growth rate contours as a function of $s$ and $a$ for a typical F star. Frames (a) and (b) present growth rates for $m = 1$ symmetric and antisymmetric modes with radial wavenumber, $n = 1$ in the overshoot tachocline with effective gravity, $G = 0.00012$, and frames (c) and (d) for the radiative tachocline ($G = 0.12$). A growth rate of 0.01 here corresponds to an $e$-folding growth time of $\sim 15$ days in a typical F star rotating 25 times faster than the Sun.

Overall, the flow pattern is one in which, in each half-wavelength in longitude, there is a local “roll” with E–W axis, the sense of which reverses in the next half-wavelength. Within each roll, flow toward the equator acquires a negative component relative to the rotating frame, whereas flow toward the pole acquires a positive component. This means that locally the circulation in the roll is tending to conserve angular momentum, only weakly opposed by any longitudinal fluid pressure torques. In the radiative tachocline with high effective gravity, the strong negative buoyancy forces strong longitudinal fluid pressure torques that are balanced by the Coriolis force on the nearly horizontal motion. The limited latitudinal extent of the velocity and fluid pressure perturbations in the overshoot case is preferred because that minimizes the stabilizing effect of the perturbations conserving angular momentum.

Figure 5 for the weak field antisolar case shows an even more complex perturbation pattern. Here the flow again (frame b) has a tendency to be geostrophic, with counterclockwise flow around the low fluid pressure and clockwise around the high. This happens because the magnetic field is too weak to force the flow to do otherwise. Now the perturbation field peak vectors tend to coincide with the peaks in fluid pressure. At this low field, HD is more dominant everywhere. The energy for instability in this case is coming primarily from the differential rotation. The tilt in the velocity vectors is such that flow toward the pole has a strong component (in the rotating frame) opposite to the direction of rotation, whereas flow toward the equator has a component in the direction of rotation. These tilts imply a Reynolds stress that transports angular momentum from high latitudes with high angular velocity to low latitudes with low angular velocity, thereby extracting kinetic energy from the antisolar differential rotation. The same tilt in the case of solar-type differential rotation would do the opposite, and perturbations in this case would not grow.

3.4. Instabilities of Solar and Antisolar Differential Rotation in F Stars

Figure 6 displays growth rates for overshoot (frames a and b) and radiative (frames c and d) tachoclines in an F star whose convection zone is about 5% of its radius. The dimensionless vertical scale for the peak toroidal field $a$ is the same as for G stars shown in Figure 1, but the dimensional fields on the right-hand scale are smaller for the same $a$. In general, the effective gravity of the F star’s overshoot tachocline is lighter than that of G and K stars. Although the nonhydrostatic-Boussinesq and the hydrostatic-non-Boussinesq models produce very close results except for $G \approx 0$, it often becomes numerically difficult to compute the growth rates in a hydrostatic model with a very light effective gravity. The results presented in frames (a) and (b) of Figure 6 are obtained using Boussinesq system.

Comparing frames of Figure 6 to the corresponding ones in Figure 1, we see that the instability for F stars is qualitatively similar to those for G stars, but there are substantial quantitative differences. For high toroidal fields the instability is essentially
independent of differential rotation, whereas for low toroidal fields in the overshoot tachocline antisolar differential rotations are unstable for much lower toroidal fields than are solar-type differential rotations. In the radiative tachocline of an F star, antisolar and solar-type differential rotations are about equally unstable, similar to G stars.

In both types of tachocline, instability sets in at substantially lower toroidal field (both dimensional and dimensionless) than in G stars, and for the same differential rotation and dimensionless toroidal field, the growth rates are much larger than those for G stars. These differences arise from the fact that the effective gravities in F star overshoot and radiative tachoclines are substantially smaller than those of their G star counterparts.

These results imply that in F stars even relatively weak toroidal fields will cause instability that should largely wipe out the differential rotation (nonlinear calculations would be needed to test this conclusion). This inference implies that substantial differential rotation would be very hard to maintain in the tachocline of an F star, particularly if it is of antisolar type. Since F star convection zones are thin and their tachoclines are not far below their photospheres, we infer this instability could severely limit the amplitude of surface differential rotation in F stars.

### 3.5. Instabilities of Solar and Antisolar Differential Rotation in K Stars

Figure 7 displays growth rates for overshoot (frames a and b) tachoclines of K stars, again for disturbances of both symmetries about the equator. We see that in the overshoot case, the result is qualitatively similar to that for G stars. For weak field only antisolar differential rotations lead to instability for low peak field in the toroidal band. Instability occurs for somewhat lower toroidal field for all differential rotations, because the effective gravity G in K star overshoot tachoclines is about 25% that for G stars. Therefore in overshoot tachoclines for all three classes of stars, antisolar differential rotations with banded toroidal fields are unstable for much smaller peak toroidal fields than are solar-type differential rotations.

Instability in radiative tachoclines of K stars is mostly present only for \( s > 0.10 \) or so. These instabilities proved to be much more difficult to calculate than for F and G stars, using both shooting and spectral methods. The result obtained shows significant amounts of fine structure in growth rates that are almost all very weak, especially for antisolar rotation—we do not display them here. However, we note that for solar-type rotation with \( s > 0.10 \) more accurate solutions can be obtained, and the results correspond to growth rates of typically up to 0.006 in the antisymmetric case, and about double that for symmetric instabilities. Growth rates are apparently much smaller than this for antisolar rotations. There seems to be little dependence on \( \alpha \) in all cases, suggesting the instabilities are predominantly HD. The noisy eigenvalues in the parameter space could be due to the extremely high effective gravity of K star, which makes the equations solved very stiff by taking them to the computational limitation of the methods used.

### 3.6. Axisymmetric Instabilities of Solar and Antisolar Differential Rotation in F, G, and K Stars

Instability of axisymmetric (\( m = 0 \)) modes is almost impossible to excite in radiative tachoclines of F, G, or K stars, because vertical displacements are required and the negative buoyancy force from the strongly subadiabatic temperature gradient opposes such displacements. In the overshoot tachoclines of such stars, however, it is possible to excite axisymmetric modes. Figure 8 gives the result for these cases, for modes that are symmetric about the equator. Antisymmetric modes give very similar results.

We see that instability occurs essentially independent of the differential rotation for \( \alpha \) values significantly above 0.7 (F stars), 0.9 (G stars), and 0.8 (K stars). This is a purely magnetic instability in which the differential rotation plays very little role. So does the symmetry of the modes. In general, the instability for non-zero \( m \) in overshoot tachoclines described earlier in this paper occurs for much lower values of toroidal field parameter \( \alpha \). So we infer that these \( m = 0 \) modes of instability are less important for these classes of stars, though there is a weak tendency for instability to set in at lower field strengths for antisolar compared with solar differential rotation.

### 3.7. Instabilities of Solar and Antisolar Differential Rotation with Higher Radial Wavenumbers

Although we focused primarily on instability with radial wavenumbers 1 corresponding to half a wavelength across the layer, our instability equations allow modes of any radial
wavenumber $n$, and we have explored some cases with $n = 10$ and $n = 100$ (plots not presented). In general the higher the $n$ chosen, the shorter are the radial displacements. Shorter displacements imply reduced resistance by any negative buoyancy force. For overshoot tachoclines the low effective gravity implies there is little buoyant resistance to vertical motions. So the growth rate of unstable modes is nearly independent of $n$.

For radiative tachoclines, the strong negative buoyancy due to the high effective gravity $G$ suppresses the radial motions. So it becomes easier to excite unstable modes with higher $n$, because modes with higher $n$ contain smaller radial displacements, and hence should feel less resistance from the strong negative buoyancy. Therefore they could be more unstable, or be unstable for parameter values for which $n = 1$ modes are stable.

Our results for radiative tachoclines in F, G, and K stars represent the minimum amount of instability in the system which is for $n = 1$—there could be even more for $n > 1$. If diffusion were included in the model, which is certainly present in stellar tachoclines, modes with higher $n$ would become less unstable or even stable. Higher $n$ modes will also be less unstable if the toroidal bands are twisted (Fisher et al. 1999).

4. CONCLUDING COMMENTS

From analyses of global MHD instabilities in 3D thin-shell models of the solar/stellar tachoclines, we find that solar-like and antisolar-type latitudinal differential rotations with coexisting toroidal bands are unstable. Antisolar differential rotation is, in general, more unstable than solar-like differential rotation in all F, G, and K stars—the pole rotating 1% or 2% faster than the equator can make the latitudinal differential rotation unstable in weakly magnetized overshoot tachoclines, although it requires much stronger magnetic fields to be unstable in the radiative tachoclines in G and K stars.

High effective gravity in the radiative tachoclines largely suppresses the radial motions by the negative magnetic buoyancy and therefore, it is difficult to grow the perturbations. By contrast, low effective gravity allows the radial motions to grow, and more so when the variation in differential rotation in latitude is smaller. Thus antisolar-type differential rotation becomes more unstable than solar-type differential rotation, because less $J \times B$ force is needed to overcome the perturbation centrifugal force that tries to stabilize the displacement.

The eigenfunctions for the disturbances with solar-like and antisolar-type differential rotations have similarities in the high $G$ case, but have important differences in the low $G$ case. In the high $G$ case, the velocity and pressure perturbations are in nearly geostrophic balance and horizontal divergences in flow and field are very small. The east–west tilts of the velocity vectors indicate angular momentum transport toward the equator, but this means that the instability tries to destroy the antisolar-type gradient in latitudinal differential rotation, whereas it builds the solar-type gradient. However, in both cases, the primary energy source for the disturbances is the toroidal field, so for high $G$, solar and antisolar differential rotations require a similar amplitude of toroidal field to become unstable.

For low $G$, by contrast, the eigenfunctions have more radial motions and fields and associated horizontal divergences. For weak toroidal fields, the HD tend to dominate at all latitudes including at the locations of the toroidal bands. The energy for the growth of the disturbances is primarily coming from the differential rotation. The east–west tilts in horizontal velocity vectors for antisolar-type differential rotations imply angular momentum transport toward the equator, which takes kinetic energy out of the differential rotation. Disturbances with the same tilts would actually reinforce a solar-type differential rotation, damping the disturbances. This is why instability for solar-type differential rotations does not occur until a substantially higher toroidal field is included.

For F stars, antisolar latitudinal differential rotation of only a few percent pole-to-equator amplitude is unstable in both the overshoot and radiative tachoclines, because the effective gravity is much smaller in F stars compared with G and K.
stars. Therefore, antisolar differential rotation is unlikely to be large in F stars. By contrast, in K stars, antisolar differential rotation is stable in the radiative tachocline but unstable in the overshoot tachocline. This implies that we should find more K stars with antisolar differential rotation compared to F stars, but the observations have so far indicated the opposite, namely no K stars and about half a dozen of F stars are found to have antisolar differential rotations.

We might be able to detect more antisolar K stars in future. One important point we should restate again is that the stability analyses we are performing are in the tachoclines of these stars, and stellar differential rotations are generally observed at the surface. The deeper convection zone in the K stars might mean that the surface is not reflecting the tachocline differential rotation pattern, whereas a very thin convection zone in F stars may be reflecting a 1% or 2% stable antisolar tachocline differential rotation at the surface. In any case, antisolar stars are found to have a very small amount of positive pole-to-equator differential rotation.

In this paper, we did not attempt to understand what generates a solar-like or an antisolar-type differential rotation, but we investigated their stability to disturbances with various longitudinal and radial wavenumbers once they have been created. We focused on a 10° longitudinal and radial wavenumbers once they have been created.

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