Superheated integrability and multisoliton survival through scattering off barriers

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A true soliton is a solitary wave—an isolated wave that maintains its shape due to a balancing of dispersion and nonlinear attraction—with a key additional property [1]: when a soliton collides with another local disturbance in the same medium, and, in particular, with another soliton [1], then asymptotically far from the collision, the soliton will regain its exact initial shape and velocity. It is this maintaining of identity through collisions that is the observable signature of integrability of the underlying nonlinear system, of the presence of infinitely many “higher” conservation laws[1]. But then, if solitons meet on top of a strong integrability-breaking barrier, one would expect the solitons to undergo some process consistent with energy conservation but not with higher conservation laws, such as the larger soliton cannibalizing the smaller one. However, here we show that when a strongly-coupled “breather” of the integrable nonlinear Schrödinger equation is scattered off a strong barrier, the solitons constituting the breather separate but survive the collision: as we launch a breather with a fixed impact speed at barriers of lower and lower height, at first all constituent solitons are fully reflected; then, at a critical barrier height, the smallest soliton gets to be fully transmitted, while the other ones are still fully reflected. This persists as the barrier is lowered some more until, at another critical height, the second smallest soliton begins to be fully transmitted as well, etc., resulting in a staircase-like transmission plot, with quantized plateaus. We show how this effect makes tangible the inverse scattering transform: the powerful, but otherwise physically opaque mathematical formalism for solving completely integrable partial differential equations. Furthermore, if such collisions can be realized experimentally in ultracold Bose gases[2,3], they could become a basis of improved atom interferometers[4,5].

Completely integrable partial differential equations (PDEs) have played a central conceptual role in studies of nonlinear dynamical systems, such as those of integrability-to-chaos transition and thermalization [1]. At the same time, many physical systems are well-modeled by them, so that if a PDE supports solitons, they are also objects of interest in their own right. Here we focus on the 1D nonlinear Schrödinger equation (NLSE), Eq. (1) below. As one of universal equations describing envelope dynamics of quasi-monochromatic waves in weakly nonlinear dispersive media with negligible dissipation[6], the NLSE describes a variety of systems, including light propagating in nonlinear planar waveguides and optical fibers[6], small-amplitude gravity waves on the surface of deep inviscid water[6], the Langmuir waves in hot plasmas[6]; storage and transfer of vibrational energy in α-helix proteins via Davydov’s solitons[7]; and, of special relevance to us, ultracold quantum Bose gases in the mean-field regime in highly elongated traps that make them effectively one-dimensional[8] (1D). In particular, since their first experimental realizations[2,3,9], bright matter-wave solitons have been extensively studied in the context of matter-wave interferometry[9,10]. Their use may improve sensitivity by several orders of magnitude[4,5], especially in precise force sensing[5,11] and in the measurement of small magnetic field gradients[5,12]. Much-studied are the soliton collisions with potential barriers, resulting in splitting and subsequent recombination of solitons[13–21].

Here we will be interested in the regimes where a single soliton does not measurably split due to the collision: when it is, for all intents and purposes, either completely reflected or completely transmitted. Since we consider the classical field description, which is mean-field theory in the matter-wave context, this corresponds to the following regime: let the velocity of the soliton center of mass be \( v \); let \( N_a \) be the total number of particles, each of mass \( m \), in the soliton; and let \( E_{\text{rest}}/N_a \) be the soliton energy per particle in the frame in which its center of mass is at rest (so that this is also the interatomic interaction energy per particle). The regime of interest is defined by the center-of-mass kinetic energy per particle, \( E_{k,1} = \frac{1}{2}mv^2 \), being much less than \( E_{\text{rest}}/N_a \). The solitary wave is then energetically allowed to lose a certain number of particles, \( \Delta N \), but the fraction it is allowed to lose, \( \Delta N/N_a \), goes down as \( 1/N_a^2 \) if we increase the \( N_a \) while keeping \( v \) constant (see the Supplementary materials). The plot of transmission vs. incident kinetic energy will be almost a step function, with quantized plateaus at 0 and 1.

But now suppose we scatter a composite soliton, consisting of, e.g., two solitons propagating together, one on top of the other. The mental picture outlined above would suggest that during the collision, the larger constituent soliton (i.e. its solitary-wave descendant in the nonintegrable regime) should absorb at least some portion of the particles from the smaller one. Not only is this process energetically favorable, but since the interactions within the constituent solitons are of the same order as the interactions between the constituent solitons, one would expect such processes to occur as soon as there are no conservation laws to forbid them.

Nevertheless, we will now show that there are collision regimes when the constituent solitons are in fact preserved; and this despite the fact that the collision process is strongly nonperturbative, in the sense that the configuration of the nonlinear field at the end of the process is very different from what it was at the beginning. In our numerical experiments, we initialize a “breather” state [22,25] (see Fig. 1), which is a multi-soliton solution of the NLSE where the centers and the velocities of the constituent solitons coincide at the initial time (unlike in the more widely-known case of the breather in

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the sine-Gordon equation, where the breather is a distinct kind of solution from the two-soliton solutions \[1\]. We then send it to collide with a barrier—a repulsive potential bump, in our case either Gaussian or rectangular. As Fig. 2 shows, there exists a parameter regime in which the scenario outlined in the introductory paragraph takes place. In all cases, the sizes of the emerging solitons are precisely the sizes of the soliton constituents of the original breather, so that the transmission of each individual soliton is all or nothing. This results in a staircase-like transmission plot of Fig. 3. The cases where at least one soliton is fully reflected are particularly clear indicators that the process is nonperturbative. The decomposition of breathers has been studied previously, but, at least when it comes to decomposition into the constituent 1-solitons, only for weak integrability-breaking perturbations\[24,25\].

Details of our system are as follows. In the context of ultracold Bose gases, the 1D nonlinear Schrödinger equation (also called the Gross-Pitaevskii equation in that setting) is

\[
i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m} \partial_x^2 \Psi + g_{1D} N_a |\Psi|^2 \Psi + \tilde{V}(z) \Psi,
\]

where \( \int_{-\infty}^{\infty} |\Psi(z, t)|^2 \, dz = 1 \). It describes the dynamics of the order parameter \( \Psi(z, t) \) of the Bose-Einstein (quasi)condensate of \( N_a \) atoms of mass \( m \) confined to a cigar-shaped waveguide, in which the transverse confinement (a harmonic potential, \( \frac{1}{2} m \omega_r^2 r^2 \)) is so tight that the energy spacing \( \hbar \omega_r \) between the ground and the first excited transverse state is much greater than the typical energy of the confined particles, making the system effectively 1D. The atoms interact with an effective pairwise interaction \( g_{1D} \delta(z_j - z_k) \), where \( z_j \) and \( z_k \) are the positions of the atoms. For us, \( g_{1D} < 0 \). \( N_a \) is assumed large enough that we expect that quantum corrections to be negligible \[4,8\] (we thus ignore the fact that it really should enter as \( N_a - 1 \)). The potential \( \tilde{V}(z) \) will

FIG. 1. Figure 1 | The 1:3 odd-number ratio breather [see Eq. (3)], at three points of time. Here \( T = 32\pi \) (in natural units; see text) is the period of density oscillations. This is a particular two-soliton solution of the NLSE, where the constituent solitons have coinciding centers and zero velocities; the norms of the constituent solitons are 1/4 and 3/4.

FIG. 2. Figure 2 | Three kinds of outcome following a collision of a 1:3 ONRB with a Gaussian barrier. All numbers are in natural units (see text). The barrier is at \( x = 0 \), has peak height \( V_0 \), and \( 1/e^2 \) half-width \( w = 0.9 \). The breather has impact speed of 0.025, and chemical potential which is much higher than its kinetic energy per particle. The main plots show the details of the collision process, where the various density profiles correspond to different times \( t \); the insets are the final outcomes. \( a \), total reflection; \( b \), the constituent soliton of norm 1/4 is transmitted, while the soliton of norm 3/4 is reflected; \( c \), both constituent solitons are transmitted. The underlying experimental values are as in the current lithium experiments, except for the impact velocity and barrier width, which are ten times lower and larger, respectively. This boosts the ratio of the chemical potential to kinetic energy per particle, without which the conservation of the shapes of the individual solitons would not be as perfect (see Extended Data Figure 1 in the Supplementary Information).
be our barrier: \( \tilde{V}(z) = \tilde{V}_0 \exp(-2(z/\tilde{w})^2) \), where \( \tilde{w} \) is the half-width of the beam at the \( 1/e^2 \) intensity point. We have numerically verified that a rectangular barrier of comparable dimensions gives qualitatively the same results. If \( \tilde{V}(z) = 0 \), the equation is integrable, both on the whole real line and on a ring (i.e. with periodic boundary conditions) [1]. Best known are the exact solutions on the real line, which may be obtained through the inverse scattering transform [1][22], and which we will use in our work. It is convenient to work in the “natural units” (see the Supplementary materials) in which the equation takes the form

\[
    i \partial_t \psi = -\frac{1}{2} \partial_x^2 \psi - |\psi|^2 \psi + V(z) \psi, \tag{2}
\]

\[
    \int_{-\infty}^{\infty} |\psi(z, t)|^2 \, dz = N, \tag{2}
\]

where the new normalization \( N \) may be chosen at will. Our parameters are chosen so that the system is in a regime that is, on the one hand, as close as possible to that obtained in actual experiments with lithium[26], but in which, on the other hand, we do see the “staircase” transmission plot. Below we will discuss how to observe this experimentally.

Let us consider the integrable case, \( V(z) = 0 \). The basic 1-soliton solution, of norm 1, is \( \frac{1}{2} \exp(it/8) \text{sech}(z/2) \). Note that if \( \psi(z, t) \) (of norm \( \sqrt{N} \)) is a solution of Eq. (2), then so are the Galilei-transformed solution \( \exp[i\xi(z - z_0) - i\frac{1}{2} \xi^2 t] \psi(z - vt - z_0, t) \) and the renormalized solution \( \xi \psi(\xi z, \xi^2 t) \) of norm \( \sqrt{N} \); these two transformations may be used to produce 1-solitons of arbitrary norm, initial position, and velocity. \( N \)-soliton solutions (see the Supplementary Discussion for details on what follows) are parametrized by \( 4N \) real parameters, four per constituent soliton[27]: \( A_j, v_j, z_j, \phi_{j,0}, j = 1, \ldots, N \). The norm of \( \psi \) is \( \sum_{j=1}^{N} A_j \), and at \( t \to \pm \infty \), if the velocities \( v_j \) are all distinct, \( \psi \) becomes a sum of \( N \) 1-solitons, of norms \( A_j \), traveling at velocities \( v_j ; z_j,0, \) and \( \phi_{j,0} \) are almost the position and phase, at \( t = 0 \), of the \( j \)th soliton. We do not consider degenerate cases when two or more constituent solitons have both the same norm and the same velocity.

In the numerical experiments discussed so far, the initial state of \( \psi \) is a (Galilei-boosted) two-soliton breather belonging to a particular sequence multisoliton solutions which we will call the odd-norm-ratio breathers (ONRBs) [23]. The \( n \)th element of this sequence consists of \( n \) overlapping solitons, of norm ratios \( 1 : 3 : \cdots : (2n - 1) \), which periodically beat (“breathe”) in space due to interference and the fact that the angular velocities of the phases of individual solitons depend on their norm. All the ONRBs have the property that at least once per breathing period, their waveform is a sech, i.e. the same as a one-soliton solution for a NLSE with a different coupling constant. As we will discuss below, this provides a way to excite them experimentally. The two-soliton ONRB has the form

\[
    \psi_{\text{ONRB}}(x, t) = e^{i \frac{\pi}{4} \left( \cosh \frac{3x}{8} + 3 e^{i \frac{\pi}{4}} \cosh \frac{x}{8} \right)} \frac{6 \cos \frac{3x}{8} + 8 \cosh \frac{x}{4} + 2 \cos \frac{x}{4}}{4} ; \tag{3}
\]

it may be obtained from the general two-soliton solution discussed above if one sets \( A_1 = 3/4 \), \( A_2 = 1/4 \), \( \phi_{1,0} = 0 \), \( \phi_{2,0} = \pi \), and \( v_1 = v_2 = z_{1,0} = z_{2,0} = 0 \). The breather is sometimes referred to as a bound state of the two solitons [22], but note that the dissociation energy is zero [23]: if, instead of \( v_1 = v_2 = 0 \), we have \( |v_1 - v_2| = \epsilon > 0 \), no matter how small, then for \( t \gg 1/\epsilon \), the solitons will be well-separated and ballistically receding from each other with a speed that tends to \( \epsilon \) as \( t \to \infty \); constituent solitons of true multisoliton solutions experience no mutual force. (This is in contrast to the well-known [27] force between simple solitons, whose sign depends on relative phase; but in that case, the initial state is not an exact multisoliton solution but rather a linear superposition of simple solitons, relevant when simple solitons are injected into the system from the outside.) We have also made sure the breather always impacts the barrier at the same phase in its breathing cycle.

Several additional points are in order:

(i) We have checked that once the solitons move away from the barrier, the field \( \psi \) is numerically of the solitonic form; e.g., when the constituent solitons are spatially separated, their form is that of 1-soliton solutions, Galilei-boosted, of norms 1/4 and 3/4.

(ii) From Fig. 2 we see that the nonlinear interactions are not negligible at any point during the collision, since the field density is not close to zero anywhere over the barrier.

(iii) The stable plateaus in the transmission plots disappear if the underlying equation is not integrable; see Extended Data Figure 1 in the Supplementary Information.

(iv) The phase in the breathing cycle at which the collision occurs does affect where the plateau at 1/4 begins and ends, but not its existence; see Extended Data Figure 2.

(v) Similar results may be obtained with multisolitons of any number of constituent solitons; for example, superpositions of more than two solitons also transmit constituent soli-
tons “one by one,” resulting in multiple plateaus; see Extended Data Figure 3.

Here is why the effect happens. Given the NLSE of Eq. (2), the inverse scattering transform (IST) introduces, for each time \( t \), an auxiliary linear problem, in which the field \( \psi(z, t) \) of the NLSE plays the role of a potential:

\[
\begin{pmatrix}
-i\partial_z & \psi^*(z, t) \\
-\psi(z, t) & i\partial_z
\end{pmatrix}
\begin{pmatrix}
u(z, t) \\
v(z, t)
\end{pmatrix} = \lambda \begin{pmatrix}
u(z, t) \\
v(z, t)
\end{pmatrix}.
\]

(4)

The matrix on the left-hand side is called a Lax operator; the scattering data for this problem consist of: 1. the so-called reflection coefficient; 2. the normalization coefficients (which are really the prefactors of the long-distance asymptotics of the bound states); and 3. the discrete eigenvalues \( \lambda \), which are the only component of the scattering data that will enter our analysis below. Scattering problems of this kind can formally be introduced for any nonlinear PDE, but only for an integrable PDE can the reflection and normalization coefficients of the corresponding Lax operator be propagated in time easily (using a second Lax operator, which we will not discuss further), while the discrete eigenvalues \( \lambda \) remain time independent[28]. In fact, it turns out that each distinct \( \lambda \) describes a constituent soliton of the field \( \psi(z, t) \): up to numerical prefactors of order 1 in the natural units, the real part is the soliton velocity \( v \), and the imaginary part, the norm \( A \) (see above). The reflection coefficient, in turn, corresponds to the radiation component of the field \( \psi \), and is strictly zero if \( \psi \) consists purely of solitons. To solve the initial value problem for the NLSE, one proceeds in a way that has often been called a nonlinear generalization of solving a PDE via Fourier transforms, where the scattering data play the role of the Fourier coefficients: one first finds the scattering data corresponding to the “scattering potential” \( \psi(z, t = 0) \); this is a nonlinear analog of taking the direct Fourier transform. One then (easily) propagates the scattering data forward to time \( t \). Finally, one solves the inverse scattering problem for the propagated scattering data to obtain the “scattering potential” \( \psi(z, t) \) at any later time \( t \). This final part reduces to a linear integral equation, which is analytically tractable in many physically important cases, and which is the nonlinear analog of taking the inverse Fourier transform.

In the presence of an integrability-breaking potential \( V(z) = \epsilon v_{\text{ext}}(z) \), where \( v_{\text{ext}}(z) \sim 1 \) and \( \epsilon \) positive but not (yet) assumed small, the \( \lambda \)s develop time dependence. Using a generalization of the Hellmann-Feynman theorem, one can derive (see Supplementary Discussion) the following exact relationship:

\[
d\lambda = \frac{\epsilon}{2} \int_{-\infty}^{\infty} dz \frac{\nu^2 v^* + v^2 \nu^*}{\int_{-\infty}^{\infty} dz \nu v},
\]

(5)

where \( \psi(z, t) \) is the exact solution of the perturbed NLSE. This is consistent with certain well-known relations[28], which however are not as convenient for our purposes. The total change in \( \lambda \) is \( \Delta \lambda = \int_{-\infty}^{\infty} (d\lambda/dt) dt \).

In this language, the effect we are studying—the preservation of soliton identity through the collision—translates to the imaginary parts of \( \lambda \)s remaining unaffected by the collision, while their real parts (soliton velocities) change substantially. To see the mechanism for this, let us estimate how \( \Delta \lambda \) scales with \( \epsilon \). In the regime of interest, the norm of the breather (and so of its constituent solitons) is \( \sim 1 \), while the impact kinetic energy of the breather is of the same order as the potential, meaning that the impact velocity of the breather \( \sim \sqrt{\epsilon} \ll 1 \). Note that the barrier appreciably affects the solution only while the breather is on top of it. To proceed, we make the following assumption, which seems very physically reasonable even though we cannot, at present, justify it with any degree of rigor: as long as the sizes of the constituent solitons (the imaginary parts of the discrete eigenvalues of the Lax operator) are not much different from their initial values, the centers of mass of the constituent solitons will, at least qualitatively, move as if they were non-mutually-interacting particles undergoing a classical collision with an effective barrier (a convolution[29] of the actual barrier and \( |\psi|^2 \), even while the constituent solitons are not well separated. The typical collision time of a single soliton can be estimated as follows: since the barrier is much narrower than the breather, the width of the effective potential is the same as that of the breather, i.e. \( \sim 1 \). Thus the time that constituent solitons spend atop the barrier scales as width/speed \( \sim 1/\sqrt{\epsilon} \). But then \( \Delta \lambda \sim \sqrt{\epsilon} \). Now recall that \( \lambda \sim \text{velocity} + \text{norm}; \) thus, \( \Delta \lambda \) is comparable in magnitude to the velocity, but is much smaller than the norm, which thus remains unaffected, for all intents and purposes. And in the case when one of the solitons has just the critical velocity to spend a long time atop the barrier, the other solitons, having different norms, will not, and will leave the barrier in the time we just estimated. Thus, as the breather begins to climb the barrier, the changes in the norms of the constituent solitons (imaginary parts of their \( \lambda \)s) begin to accumulate according to Eq. (5), but the solitons will leave the barrier before the change in their norms can become significant—with a possible exception of one of the solitons having just the right size-velocity combination to linger atop the barrier. However, since the other solitons leave the barrier quickly, the remaining soliton is energetically protected from fragmenting.

In the usual sense of the word, the process is heavily non-perturbative, since it results in a dramatic change in the shape of the field \( \psi \). Almost all of the parameters that determine the properties of the constituent solitons substantially change as well, save for the norms. If we metaphorically picture the perturbation as a kind of heating up of the previously conserved (“frozen”) quantities, then we may say that the norms are “superheated”: though one might think they should start flowing, they do not. And the reason is that the IST, counterintuitively, treats the soliton velocity and norm as a unit, as the real and imaginary parts of a coordinate \( \lambda \) in the “IST space”: the magnitude of a change in \( \lambda \) may be much smaller than the imaginary part of \( \lambda \), while being substantial compared to the real part. The “superheating” of the norms is thus a direct manifestation of the underlying IST structure of the problem.

Let us discuss the possible experimental realization of the effect we have just described (we thank R. G. Hulet’s group for many useful discussions and for providing us with their
current experimental parameters). The protocol for the creation of the 1:3 breather is based on the observation that, at \( t = 0 \), its wavefunction coincides with the wavefunction of the stationary soliton but for the NLSE whose coupling constant is one quarter the coupling constant for the actual NLSE [23]. \( \psi_{\text{ONRB}}(x, 0) = \frac{1}{2} \text{sech} \left( \frac{x}{a} \right) \). Since a single soliton is usually first created very close to the stability threshold[30] and the breather is even more unstable than a single soliton, the magnitude of the coupling constant should first be adiabatically reduced by a factor of four (rough variational estimates suggest that the resulting breather should be well within stable regime). In the \(^{7}\text{Li} \) experiments [26], the parameters of Eq. (1) have the following values: \( \omega_r = 2\pi \times 254 \text{ Hz} \); \( \omega_z = 2\pi \times 31 \text{ Hz} \); \( g_{1\text{D}} = 2\hbar \omega_r a \), where \( a \) is the (3D) scattering length of the atoms, experimentally \( -1.0 \times a_0 \) (\( a_0 \) being the Bohr radius). It can be tuned down in magnitude by at least a factor of 10 by using the magnetic Feshbach resonance [26]. \( N_a = 3 \times 10^4 \), which is indeed at the collapse threshold. The barrier is produced by a 900 GHz blue-detuned Gaussian laser beam resulting in the half-width \( \tilde{\omega} \) of the beam at the \( 1/e^2 \) intensity point of \( \tilde{\omega} = 4.2 \mu \text{m} \), while \( \tilde{V}_0/\hbar \) is of the order of 200–3000 Hz (where \( h \) is the Planck’s constant). In experiments, \( \tilde{V}(z) \) will, in addition to the barrier, also include a longitudinal harmonic confinement \( \frac{1}{2} m \omega_z^2 z^2 \); we will omit it from the analysis, but we have checked that if its value what it in the experiments (see below), it does not qualitatively affect our results. The typical impact speed in the experiments is 514 \( \mu \text{m/s} \).

In our numerical experiments, we took \( a = -0.41 a_0 \); in order to produce very sharp transitions in transmission plots, we lowered the impact velocity tenfold from the typical experimental value of 514 \( \mu \text{m/s} \). For the smaller soliton, this resulted in \( (E_{\text{rest}}/N_a)/E_{K,1} \approx 8 \). However, stable plateaus can be obtained even with larger velocities, though the transitions will no longer be as sharp. Also, to prevent the computational grid from becoming impractically big, we increased the width of the barrier tenfold from the experimental value (results of Fig. 3 suggest this will not matter qualitatively).

When the superheating of soliton norms does become experimentally accessible, the presence of a transmission plateau could be used to stabilize atom interferometers against both the parasitic mean-field effects and the fluctuations of the profile of the beam splitter potential[4,5].

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**Supplementary Discussion**

1. **Energetically allowed fractional loss of particles for a single soliton**

Consider the stationary, single-soliton solution (normalized to 1) of Eq. (1) with \( \tilde{V} = 0 \) in the main text, \( \tilde{\Psi}_1(z) = \sqrt{e^2/2 \text{sech}(cz)} \), where \( c = |g_{1\text{D}}| N_a m / (2\hbar v) \).

The energy-per-particle of the soliton is given by

\[
E/N_a = \int_{-\infty}^{\infty} dz \left( \frac{h^2}{2m} |\partial_z \tilde{\Psi}_1(z)|^2 + \frac{g_{1\text{D}} N_a}{2} |\tilde{\Psi}_1(z)|^4 \right) = -d N_a^2,
\]

with

\[
d = \frac{m g_{1\text{D}}^2}{24 \hbar v^2},
\]

where \( g_{1\text{D}} < 0 \). A fracturing of a single soliton into two smaller ones, \( N_a \rightarrow N_1 + N_2 \), will raise the energy-per-particle by \( 2d N_1 N_2 \). If the fracturing is to occur due to the soliton (with center-of-mass speed \( v \)) colliding with a barrier, this energy can come only from the kinetic energy per particle of the center of mass, \( E_{K,1} = \frac{1}{2} m v^2 \). Let us write \( N_1 = N_a - \Delta N \) and \( N_2 = \Delta N \), where, without loss of generality, we impose the condition that \( 0 \leq \Delta N < N_a/2 \). We therefore have \( 0 \leq 2d(N_a - \Delta N) \Delta N \leq E_{K,1} \); dividing through by \( d N_a^2 \), we get

\[
0 \leq \frac{\Delta N}{N_a} - \left( \frac{\Delta N}{N_a} \right)^2 \leq \frac{E_{K,1}}{d N_a^2}.
\]

Now we invoke the assumption that \( E_{K,1} \ll |E/N_a| \). Thus the right-hand side of the above relation is much less than one.

The equation \( x - x^2 = \epsilon \) has the solutions \( x_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{2} - \epsilon} \), i.e. one close to zero, the other close to one. Because we imposed that \( \Delta N < N_a/2 \), it follows that \( \Delta N/N_a \) must be much less than one; so we may neglect the quadratic term in Eq. (1), obtaining

\[
0 \leq \frac{\Delta N}{N_a} \leq \frac{E_{K,1}}{d N_a^2}.
\]

2. **The natural units for the 1D nonlinear Schrödinger equation**

Consider the 1D nonlinear Schrödinger equation

\[
\text{i} h \partial_t \Psi = -\frac{\hbar^2}{2m} \partial_z^2 \Psi + g_{1\text{D}} N_a \Psi^2 \Psi + \tilde{V}(z) \Psi,
\]

where \( \int_{-\infty}^{\infty} |\tilde{\Psi}(z, t)|^2 dz = 1 \). By choosing to work in “natural units,” it may be written as

\[
\text{i} \partial_t \psi = -\frac{1}{2} \partial_z^2 \psi \pm |\psi|^2 \psi + V(z) \psi,
\]

where the sign of the \( |\psi|^2 \psi \) term is the sign of \( g_{1\text{D}} \). In the present work, this is negative (i.e. attractive, or “focusing”), so that it can support bright solitons. \( \int_{-\infty}^{\infty} |\psi(z, t)|^2 dz = \hat{N} \) is the new normalization, which may be chosen at will.

In the resulting system of units, the unit of length is \( u_L = \sqrt{\hbar^2 / (m g_{1\text{D}} N_a)} \), of time, \( u_T = \sqrt{\hbar / (m g_{1\text{D}} N_a^2)} \), and the field is transformed as \( \Psi(z, t) = \psi(z/u_L, t/u_T) / \sqrt{\hat{N} u_L} \).
3. The \( N \)-soliton solution of the nonlinear Schrödinger equation

The multisoliton solutions of the equation
\[
\imath \partial_t \psi(z, t) = -\frac{1}{2} \partial^2_z \psi(z, t) - |\psi(z, t)|^2 \psi(z, t)
\]
may be constructed as follows\(^{27} \text{:}\) given the \( 4N \) real parameters \( A_j > 0, v_j, z_{j,0}, \) and \( \phi_{j,0}, j = 1, \ldots, N \), we have that
\[
\psi(z, t) = \sum_{j=1}^{N} u_j(z, t),
\]
where the \( u_j \)'s are the solutions of the system of \( N \) equations
\[
\sum_{k=1}^{N} \frac{1}{\lambda_j + \lambda_k^*} u_k = 1, \quad j = 1, \ldots, N,
\]
where
\[
\lambda_j = A_j/2 + iv_j
\]
and
\[
\gamma_j = \exp \left[ \lambda_j (z - z_{j,0}) + \imath \lambda_j^2 t/2 + \imath \phi_{j,0} \right].
\]

Here the parameters \( z_{j,0}, \) and \( \phi_{j,0} \) are almost but not quite the position and phase, at \( t = 0 \), of the \( j \)th soliton when it is spatially separated from the others (see below), but the first two are precisely its norm and velocity, as we now explain. The norm of \( \psi \) is \( \sum_{j=1}^{N} A_j \); as \( t \to \pm \infty \), if the velocities \( v_j \) are all distinct, \( \psi \) becomes a sum of \( N \) 1-soliton solution, of norms \( A_j \), traveling at velocities \( v_j \). More precisely, in the limit as the \( j \)th soliton becomes more and more spatially separated from the others, its form converges to
\[
\frac{A_j}{2} \sech \left[ \frac{A_j}{2} (z - z_j) + q_j \right] \exp[i(\phi_j + \Psi_j)],
\]
where
\[
z_j = z_{j,0} + v_j t, \quad \phi_j = v_j (z - z_j) + \frac{1}{2} \left( A_j^2/4 + v_j^2 \right) t + \phi_{j,0},
\]
and the real numbers \( q_j \) and \( \Psi_j \) capture the interaction with the other solitons. They are given as
\[
q_j + i \Psi_j = \sum_{k=1}^{N} \text{sign}(z_k - z_j) \ln \frac{A_j + A_k + 2i(v_j - v_k)}{A_j - A_k + 2i(v_j - v_k)},
\]
where \( \text{sign} z \) is \(-1, 0, \) or \(+1\) if \( z < 0, \) \( z = 0, \) or \( z > 0 \), respectively.

Now we see that the parameters \( z_{j,0}, \) and \( \phi_{j,0} \) differ, through \( q_j \) and \( \Psi_j \), respectively, from the actual position and phase at \( t = 0 \) of the \( j \)th soliton when it is spatially separated from the others. But the actual position and phase at \( t = 0 \) is easily characterized: since the displacements \( q_j \) depend only on the ordering of the spatial positions of the solitons (and not on the magnitudes of the relative distances), it follows that if one wants isolated solitons to sit at \( z_{0,j} \) at \( t = 0 \), one may proceed as follows: first set each \( z_{0,j} \) to \( z_{0,j} \), and compute the \( q_j \)'s. Then set \( z_{j,0} = z_{0,j} + 2q_j/A_j \), and proceed to compute the \( N \)-soliton solution; a similar correction may be applied for the initial phases.

The case of degenerate \( \lambda_j \), when two or more constituent solitons have both the same norm and the same velocity, may be treated by taking the appropriate limit of the system of \( N \) equations above. In this case one obtains solutions that are qualitatively different from those discussed thus far. For example, in the two-soliton case, as \( t \to \pm \infty \), one finds\(^{22} \) that the distance between the solitons increases proportionally to \( \ln(A^2 t) \) (in natural units). Since here the solitons separate on their own time scale of \( 1/A^2 \), the collision experiment should last shorter than that. On the other hand, the breather needs to start sufficiently far from the barrier so that it begins in an approximately integrable regime, and it needs to be sufficiently slow so that the kinetic energy per particle is much less than the chemical potential. It turns out that these constraints are impossible to satisfy simultaneously, and thus the degenerate case is not of interest for us.

4. Derivation of the exact expression for \( d\lambda/dt \), Eq. (5) in the main text

We are dealing with the 1D nonlinear Schrödinger equation in Eq. (1) in the main text,
\[
i\hbar \frac{\partial}{\partial t} \Psi(z, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \Psi(z, t) + g_{1D} N \Psi(z, t)^2 \Psi(z, t) + \hat{V}(z, t) \Psi(z, t), \quad (S1)
\]
where \( g_{1D} < 0 \), in the presence of the external integrability-breaking potential \( \hat{V}(z, t) \) (note that here we will allow this external potential to explicitly depend on time). In order to facilitate comparison with Ref. 28, which treats the same kind of problem, we will work in the units in which \( \hbar = 1, m = 1/2, \) and \( |g_{1D}|N \) are \( 2 \). Thus the NLSE becomes
\[
i\hbar \frac{\partial}{\partial t} \psi(z, t) = \left[ -\frac{\partial^2}{\partial z^2} - 2|\psi(z, t)|^2 \right] \psi(z, t) + \epsilon \psi_{ext}(z, t) \psi(z, t), \quad (S2)
\]
where the external potential in \( (S1) \) is factorized as
\[
V_{ext}(z, t) \equiv V_0 \psi_{ext}(z, t) \max_z |\psi_{ext}(z, 0)| = 1,
\]
and the small parameter \( \epsilon \) is
\[
\epsilon \equiv \frac{2\hbar^2 V_0}{(g_{1D} N)^2 m}.
\]

The first Lax operator reads
\[
\hat{L} = \left( \begin{array}{cc} \hat{L}_1 & \hat{M} \\ -\hat{M}^T & -\hat{L} \end{array} \right) \quad (S3)
\]
with
\[ \hat{L} = -i \frac{\partial}{\partial z} \]  
\( M = \psi^\dagger (z, t) \).  
(\text{S4})

For each instance of time \( t \), one can set up an eigenstate-eigenvalue problem, which is the central object of interest of this derivation:
\[ \hat{L} |w\rangle = \lambda |w\rangle \, , \]
where
\[ |w\rangle = \begin{pmatrix} u(z, t) \\ v(z, t) \end{pmatrix} \, . \]

The property above justifies a standard notation \( (|w_1\rangle, \hat{L}|w_2\rangle) \equiv \langle w_1| \hat{L} |w_2\rangle - \text{ that we are going to employ below.} \]

The pseudo-Hermiticity property (\text{S7}) implies the following properties of the eigenstates of \( \hat{L} \): Let \( \hat{L}|w_1\rangle = \lambda_1 |w_1\rangle \), \( \hat{L}|w_2\rangle = \lambda_2 |w_2\rangle \), and \( \hat{L}|w\rangle = \lambda |w\rangle \). Then

1. eigenvectors whose eigenvalues are not complex conjugates of each other are mutually orthogonal,
\[ \lambda_1^* \neq \lambda_2 \Rightarrow \langle w_1|w_2\rangle = 0 \, ; \]

2. non-zero norm eigenstates of \( \hat{L} \) correspond to real eigenvalues (a corollary of the above):
\[ \lambda^* \neq \lambda \Rightarrow \langle w|w\rangle = 0 \, . \]

Note that the eigenspectrum of \( \hat{L} \) is not necessarily complete. In cases when the Lax operator (\text{S3}) represents a linear stability analysis equation of a nonlinear PDE, the missing states are associated with the continuous symmetries of the PDE that is broken by the solution in question (\text{S1}). (For a “flat” condensate, \( \psi(z) = \text{const} \), we found one missing state; there could be more. There seem to be none for a single soliton.)

The operator (\text{S3}) also possesses properties specific to a particular form of the matrix elements, Eqs. (\text{S4}) and (\text{S5}):
\[ \hat{L}^\dagger = \hat{L} \, . \]

This property implies that:

1. real eigenvalues \( \lambda \) are doubly degenerate. The corresponding eigenstates,
\[ |w\rangle = \begin{pmatrix} u(z) \\ v(z) \end{pmatrix} \]
\[ |\bar{w}\rangle = \begin{pmatrix} \bar{u}(z) \\ \bar{v}(z) \end{pmatrix} \, , \]
are related by
\[ \bar{u}(z) = -u^*(z) \]
\[ \bar{v}(z) = +u^*(z) \, . \]

Here,
\[ \hat{L}|w\rangle = \lambda |w\rangle \]
\[ \hat{L}|\bar{w}\rangle = \lambda |\bar{w}\rangle \, . \]

2. Complex eigenvalues \( \lambda \) come in complex conjugate pairs, \( \lambda_+ \), \( \lambda_- \) such that \( \lambda_+ = (\lambda_-)^* \). The corresponding eigenstates,
\[ |w_+\rangle = \begin{pmatrix} u_+ (z) \\ v_+ (z) \end{pmatrix} \]
\[ |w_-\rangle = \begin{pmatrix} u_- (z) \\ v_- (z) \end{pmatrix} \, , \]
\[ L = -i \frac{\partial}{\partial z} \]
\[ M = \psi^\dagger (z, t) \, . \]
are related by
\[
\begin{align*}
    u_-(z) &= -(v_+)^*(z) \\
v_-(z) &= +(u_+)^*(z) .
\end{align*}
\] (S8)

Here,
\[
\mathcal{L}|w_+\rangle = \lambda_+ |w_+\rangle \\
\mathcal{L}|w_-\rangle = \lambda_- |w_-\rangle \\
\lambda_+ = (\lambda_+)^*. 
\]

Within the context of the Inverse Scattering Transform, the wavefunction \(\psi(x, t)\) in the parent NLSE, Eq. (S1), is assumed to be localized in space, while the eigenstates of the Lax operator \(\mathcal{L}\) of Eq. (S3) are required to be finite at \(x = \pm\infty\). In this case, the real eigenvalues of \(\mathcal{L}\) form a continuum spectrum, while the complex eigenvalues are discrete. Finally, in the parent NLSE, the complex eigenvalues correspond to the solitonic part of the scattering data, while the real eigenvalues correspond to the thermal noise.

From now on, we will assume that the eigenstates of \(\mathcal{L}\) with substantially complex eigenvalues (i.e., the “discrete spectrum”, or “bound states”) are normalized as
\[
\langle w_- | w_+ \rangle = 1 . \quad \text{(S9)}
\]

For the case of a single soliton,
\[
\psi(z) = -i \, \text{sech}(x) ,
\]
the corresponding eigenvalues and eigenstates are
\[
\lambda_+ = -\frac{i}{2} \\
|w_+\rangle = \frac{-i}{2} \exp(x/2) \left( -1 + \tanh(x) \right) \text{sech}(x)
\]
and
\[
\lambda_- = \frac{i}{2} \\
|w_-\rangle = \frac{i}{2} \exp(x/2) \left( -\text{sech}(x) \right) \left( -1 + \tanh(x) \right) .
\]

4.2. The Hellmann-Feynman theorem

Let \(\mathcal{L}\) depend on a parameter \(\xi\): \(\mathcal{L} = \mathcal{L}(\xi)\). Its discrete eigenvalues \(\lambda_{\pm}\) and the corresponding eigenstates, \(|w_{\pm}\rangle\) then also depend on \(\xi\): \(\lambda_{\pm} = \lambda_{\pm}(\xi)\), and \(|w_{\pm}\rangle = |w_{\pm}(\xi)\rangle\).

Let us express the eigenvalue \(\lambda_+\) as
\[
\lambda_+(\xi) = \langle w_-(\xi) | \mathcal{L}(\xi) | w_+(\xi) \rangle .
\]

Then, the derivative of \(\lambda_+\) with respect to \(\xi\) becomes
\[
\frac{d}{d\xi} \lambda_+ = \left( \frac{d}{d\xi} |w_-\rangle, \mathcal{L} |w_+\rangle \right) + \left( |w_-\rangle, \frac{d}{d\xi} \mathcal{L} \right) |w_+\rangle + \left( |w_-\rangle, \mathcal{L} \frac{d}{d\xi} |w_+\rangle \right) .
\]

As in the proof of the usual Hellmann-Feynman theorem, the sum of the first and the last term will turn out to be proportional to the derivative of the norm; and since the norm is one, the derivative is zero. Indeed, the first term gives
\[
\left( \frac{d}{d\xi} |w_-\rangle, \mathcal{L} |w_+\rangle \right) = \left( \frac{d}{d\xi} |w_-\rangle, \lambda_+ |w_+\rangle \right) = \lambda_+ \left( \frac{d}{d\xi} |w_-\rangle, |w_+\rangle \right) .
\]

Similarly, the last term gives
\[
\left( |w_-\rangle, \mathcal{L} \frac{d}{d\xi} |w_+\rangle \right) = \left( \lambda_- |w_-\rangle, \frac{d}{d\xi} |w_+\rangle \right) = \left( \lambda_- \right)^* \left( |w_-\rangle, \frac{d}{d\xi} |w_+\rangle \right) .
\]

But \((\lambda_-)^* = \lambda_+\); thus, the sum of the two terms gives
\[
\lambda_+ \left[ \left( \frac{d}{d\xi} |w_-\rangle, |w_+\rangle \right) + \left( |w_-\rangle, \frac{d}{d\xi} |w_+\rangle \right) \right] = \lambda_+ \frac{d}{d\xi} |w_-\rangle, |w_+\rangle = \lambda_+ \frac{d}{d\xi} 1 = 0 .
\]

Thus, we get the following generalization of the Hellmann-Feynman theorem:
\[
\frac{d}{d\xi} \lambda_+ = \langle w_- | \left( \frac{d}{d\xi} \mathcal{L} \right) |w_+ \rangle . \quad \text{(S10)}
\]

4.3. The exact expression for \(d\lambda/dt\) from the Hellmann-Feynman theorem

Let us set
\[
\xi = t ; \quad \mathcal{L}(t) = \left( \begin{array}{cc} -i \frac{\partial}{\partial z} & \psi^*(z, t) \\ -\psi(z, t) & +i \frac{\partial}{\partial z} \end{array} \right) ; \\
\mathcal{L}(t)|w(t)\rangle = \lambda(t)|w(t)\rangle ; \\
\frac{d}{dt} \mathcal{L}(t) = \left( \begin{array}{cc} 0 & +(F[\psi] + \epsilon P[\psi]^*)^*(z, t) \\ -(F[\psi] + \epsilon P[\psi])(z, t) & 0 \end{array} \right) ; \\
|w(t)\rangle = \left( \begin{array}{c} u(z, t) \\ v(z, t) \end{array} \right) ; \\
2 \int_{-\infty}^{+\infty} u(z, t)v(z, t) \, dz = 1 ,
\]
where \(F[\psi](z, t) = -i \left[ -\frac{\partial^2}{\partial z^2} - 2\psi(z, t)^2 \right] \psi(z, t)\), and \(P[\psi](z, t) = -i \sigma_{ext}(z, t) \psi(z, t)\). According to the Hellmann-Feynman theorem, Eq. (S10), the time derivative
of the Lax eigenvalue is

\[
\frac{\partial}{\partial t} \lambda = - \int_{-\infty}^{+\infty} dz \left\{ u_+^2(z, t) eP[\psi](z, t) - v_+^2(z, t) e^* P^*[\psi](z, t) \right\}
\]

\[
= (+i) \int_{-\infty}^{+\infty} dz \left\{ e u^2(z, t) \psi(z, t) + e^* v^2(z, t) \psi^*(z, t) \right\} v_{\text{ext}}(z, t)
\]

Notice that the contribution to \( d\lambda/dt \) from \( F[\psi] \) disappears. Indeed this contribution describes the time derivative of the Lax eigenvalue in the time evolution according to the unper-
turbed NLS; this derivative indeed vanishes as a consequence of integrability of the NLSE.

A translation to the Kivshar-Malomed notation system of Ref. 28 gives

\[
\frac{\partial}{\partial t} \lambda_n = - \frac{1}{2} \int_{-\infty}^{+\infty} dz \psi^{(1)}(x, t) \psi^{(2)}(x, t)
\]

\[
\times \int_{-\infty}^{+\infty} dz \left\{ (\psi^{(1)}(x, t, \lambda_n) eP[\psi](x, t) - (\psi^{(2)}(x, t) e^* P^*[\psi](x, t) \right\}
\]

---

**Extended Data**

**Extended Data Figure 1.** Integrable vs. nonintegrable case. The initial state for the nonintegrable case was prepared by time-
propagating the breather at rest while the nonlinearity was slowly ramped from \(|\psi|^2\) to \(|\psi|^{2p}\) with \(p = 3/2\). The result was Galilei-
boosted and scattered off of a Gaussian barrier whose width (for numerical reasons) was twice the reference value. All other parame-
ters were at their reference values, including the number of particles. Also plotted is the integrable case, all of whose parameters are
at their reference values. In particular, the number of particles four times smaller than that for Fig. 3 in the main text, resulting in a
degraded, but still noticeable plateau at around 25% transmission. In the nonintegrable case, no plateau can be discerned.
Extended Data Figure 2. Dependence on the phase of the breathing cycle. The y-axis: the value of $E_{\text{kin}}/V_0$ for which transmission jumps from 0 to 1/4; the x-axis: the time offset in the breathing cycle for the initial state, relative to that in Fig. 2 in the main text; all other parameters are as in that Figure.

Extended Data Figure 3. Transmission plot for a three-soliton breather solution. The constituent solitons have norms 1/6, 1/3, and 1/2 (1:2:3 norm ratio, which does not belong to the sequence of odd number ratios). The dash-dotted horizontal lines are at transmissions values of 1/6, 1/2, and 1, corresponding, respectively, to only the smallest, norm-1/6 soliton being transmitted, to the norm-1/6 and norm-1/3 solitons being transmitted, and to all three constituent solitons being transmitted.
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