NONRATIONAL DEL PEZZO FIBRATIONS

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Abstract. Let \( X \) be a general divisor in \( |3M + nL| \) on the rational scroll \( \text{Proj}(\oplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}(d_i)) \), where \( d_i \) and \( n \) are integers, \( M \) is the tautological line bundle, \( L \) is a fibre of the natural projection to \( \mathbb{P}^1 \), and \( d_1 \geq \cdots \geq d_4 = 0 \). We prove that \( X \) is rational \( \iff \) \( d_1 = 0 \) and \( n = 1 \).

1. Introduction.

The rationality problem for threefolds\(^1\) splits in three cases: conic bundles, del Pezzo fibrations, and Fano threefolds. The cases of conic bundles and Fano threefolds are well studied.

Let \( \psi: X \rightarrow \mathbb{P}^1 \) be a fibration into del Pezzo surfaces of degree \( k \geq 1 \) such that \( X \) is smooth and \( \text{rk Pic}(X) = 2 \). Then \( X \) is rational if \( k \geq 5 \). The following result is due to \([11]\) and \([12]\).

**Theorem 1.1.** Suppose that fibres of \( \psi \) are normal and \( k = 4 \). Then \( X \) is rational if and only if \( \chi(X) \in \{0, -8, -4\} \), where \( \chi(X) \) is the topological Euler characteristic.

The following result is due to \([8]\).

**Theorem 1.2.** Suppose that \( K_X^2 \not\in \text{Int} \overline{\text{NE}}(X) \) and \( k \leq 2 \). Then \( X \) is nonrational.

In the case when \( k \leq 2 \) and \( K_X^2 \in \text{Int} \overline{\text{NE}}(X) \), the threefold \( X \) belongs to finitely many deformation families, whose general members are nonrational (see \([13]\), \([4]\), \([6]\), Proposition 1.5).

Suppose that \( k = 3 \). Then \( X \) is a divisor in the linear system \( |3M + nL| \) on the scroll

\[
\text{Proj}\left( \oplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}(d_i) \right),
\]

where \( n \) and \( d_i \) are integers, \( M \) is the tautological line bundle, and \( L \) is a fibre of the natural projection to \( \mathbb{P}^1 \). Suppose that \( d_1 \geq d_2 \geq d_3 \geq d_4 = 0 \).

Suppose that \( X \) is a general\(^2\) divisor in \( |3M + nL| \). The following result is due to \([8]\).

**Theorem 1.3.** Suppose that \( K_X^2 \not\in \text{Int} \overline{\text{NE}}(X) \). Then \( X \) is nonrational.

It follows from \([5]\), \([11]\), \([2]\), \([13]\), \([3]\), \([4]\) that \( X \) is nonrational when

\[
(d_1, d_2, d_3, n) \in \left\{ (0, 0, 0, 2), (1, 0, 0, 0), (2, 1, 1, -2), (1, 1, 1, -1) \right\}.
\]

We prove the following result in Section 3

**Theorem 1.4.** The threefold \( X \) is rational \( \iff \) \( d_1 = 0 \) and \( n = 1 \).

Therefore, the threefold \( X \) is nonrational in the case when \( \chi(X) \neq -14 \). Indeed, we have

\[
\chi(X) = -4K_X^2 - 54 = -4 \left( 18 - 6(d_1 + d_2 + d_3) - 8n \right) - 54 = 18 - 24(d_1 + d_2 + d_3) - 32n,
\]

and \( \chi(X) = -14 \) implies that \( (d_1, d_2, d_3, n) = (0, 0, 0, 1) \) or \( (d_1, d_2, d_3, n) = (2, 1, 1, -2) \).

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\(^1\)All varieties are assumed to be projective, normal, and defined over \( \mathbb{C} \).

\(^2\)A complement to a countable union of Zariski closed subsets.
The inequality $5n \geq 12 - 3(d_1 + d_2 + d_3)$ holds when $K_X^2 \notin \text{Int} \mathbb{NE}(X)$. For $n < 0$, the inequality $5n \geq 12 - 3(d_1 + d_2 + d_3)$ implies that $K_X^2 \notin \text{Int} \mathbb{NE}(X)$ (see Lemma 36 in [3]). Hence, the threefold $X$ does not belong to finitely many deformation families in the case when $K_X^2 \in \text{Int} \mathbb{NE}(X)$ (see Section 2).

Let us illustrate our methods by proving the following result.

**Proposition 1.5.** Let $X$ be double cover of the scroll

$$\text{Proj} \left( \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \right)$$

that is branched over a general divisor $D \in |4M - 2L|$, where $M$ is the tautological line bundle, and $L$ is a fibre of the natural projection to $\mathbb{P}^1$. Then $X$ is nonrational.

**Proof.** Put $V = \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$. The divisor $D$ is given by the equation

$$\alpha_6 x_1^4 + \alpha_6^1 x_1^3 x_2 + \alpha_4 x_1^3 x_2 + \alpha_6^2 x_1^2 x_2 + \alpha_4^2 x_1^2 x_2 + \alpha_2 x_1^2 x_2 + \alpha_6^3 x_1 x_2^2$$

$$+ \alpha_2^3 x_1 x_2^3 + \alpha_1 x_1 x_2^3 + \alpha_0 x_1 x_2^3 + \alpha_4 x_1 x_2^3 + \alpha_1^2 x_2^3 + \alpha_0^2 x_2^3 + \alpha_0^3 x_2^3 = 0$$

in bihomogeneous coordinates on $V$ (see §2.2 in [3]), where $\alpha_d^i = \alpha_d^i(\alpha_1, \alpha_2)$ is a sufficiently general homogeneous polynomial of degree $d \geq 0$. Let

$$\chi: Y \longrightarrow \text{Proj} \left( \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \right)$$

be a double cover branched over a divisor $\Delta \subset V$ that is given by the same bihomogeneous equation as of divisor $D$ with the only exception that $\alpha_0 = \alpha_0^1 = 0$. Then $Y$ is singular, because the divisor $\Delta$ is singular along the curve $Y_3 \subset V$ that is given by the equations $x_1 = x_2 = 0$.

The Bertini theorem implies the smoothness of $\Delta$ outside of the curve $Y_3$.

Let $C$ be a curve on the threefold $Y$ such that $\chi(C) = Y_3$. Then the threefold $X$ has singularities of type $\mathbb{A}_1 \times \mathbb{C}$ at general point of the curve $C$. We may assume that the system

$$\alpha_2(t_1, t_2) = \alpha_1^2(t_1, t_2) = \alpha_2^2(t_1, t_2) = 0$$

has no non-trivial solutions. Then $Y$ has singularities of type $\mathbb{A}_1 \times \mathbb{C}$ at every point of $C$.

Let $\alpha: \tilde{V} \to V$ be the blow up of $Y_3$, and $\beta: \tilde{Y} \to Y$ be the blow up of $C$. Then the diagram

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{\chi}} & \tilde{V} \\
\beta \downarrow & & \alpha \downarrow \\
Y & \xrightarrow{\chi} & V \\
\end{array}
$$

commutes, where $\tilde{\chi}: \tilde{Y} \to \tilde{V}$ is a double cover. The threefold $\tilde{Y}$ is smooth.

Let $E$ be the exceptional divisor of $\alpha$, and $\tilde{\Delta}$ be the proper transform of $\Delta$ via $\alpha$. Then

$$\tilde{\Delta} \sim \alpha^* (4M - 2L) - 2E,$$

which implies that $\tilde{\Delta}$ is nef and big, because the pencil $|\alpha^* (M - 2L) - E|$ does not have base points. The morphism $\tilde{\chi}$ is branched over $\tilde{\Delta}$. Then $\text{rk Pic}(\tilde{Y}) = 3$ by Theorem 2 in [10].

The linear system $|g^* (M - L) - E|$ does not have base points and gives a $\mathbb{P}^1$-bundle

$$\tau: \tilde{V} \longrightarrow \text{Proj} \left( \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \right) \cong \mathbb{F}_0,$$

which induces a conic bundle $\tilde{\tau} = \tau \circ \tilde{\chi}: \tilde{Y} \to \mathbb{F}_0$.

Let $Y_2 \subset V$ be the subscroll given by $x_1 = 0$, and $S$ be a proper transform of $Y_2$ via $\alpha$. Then

$$Y_2 \cong \text{Proj} \left( \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \right) \cong \mathbb{F}_2,$$

\footnote{A complement to a countable union of Zariski closed subsets.}
and \( S \cong Y_2 \). But \( \tau \) maps \( S \) to the section of \( F_0 \) that has trivial self-intersection.

Let \( \tilde{S} \) be a surface in \( \tilde{Y} \) such that \( \tilde{\chi}(\tilde{S}) = S \), and \( Z \subset \tilde{Y} \) be a general fibre of the natural projection to \( \mathbb{P}^1 \). Then \(-K_Z\) is nef and big and \( K_Z^2 = 2 \). But the morphism

\[
\alpha \circ \tilde{\chi}|_{\tilde{S}} : \tilde{S} \to Y_2
\]

is a double cover branched over a divisor that is cut out by the equation

\[
\alpha_0^2(t_0, t_1)x_3^2 + \alpha_1^2(t_0, t_1)x_2x_3 + \alpha_2^2(t_0, t_1)x_2^2 = 0.
\]

Let \( \Xi \subset F_0 \) be a degeneration divisor of the conic bundle \( \tau \). Then

\[
\Xi \sim \lambda\tau(\tilde{S}) + \mu\tau(Z),
\]

where \( \lambda \) and \( \mu \) are integers. But \( \lambda = 6 \), because \( K_Z^2 = 2 \). We have \( \tau(\tilde{S}) \not\subset \Xi \). Then

\[
\mu = \tau(\tilde{S}) \cdot \Xi = 8 - K_Z^2,
\]

because \( \mu \) is the number of reducible fibres of the conic bundle \( \tau|_{\tilde{S}} \). These fibers are given by

\[
\left( \alpha_4^3(t_0, t_1) \right)^2 = 4\alpha_2^2(t_0, t_1)\alpha_6^4(t_0, t_1),
\]

which implies that \( \mu = \tau(\tilde{S}) \cdot \Xi = 8 \). Then \( \tilde{Y} \) is nonruled by Theorem 10.2 in [11], which implies the nonrationality of the threefold \( X \) by Theorem 1.8.3 in §IV of the book [7]. \( \square \)

2. Preliminaries.

All results of this section follow from [3]. Take a scroll

\[
V = \text{Proj} \left( \bigoplus_{i=1}^4 O_{\mathbb{P}^1}(d_i) \right),
\]

where \( d_i \) is an integer, and \( d_1 \geq d_2 \geq d_3 \geq d_4 = 0 \). Let \( M \) and \( L \) be the tautological line bundle and a fibre of the natural projection to \( \mathbb{P}^1 \), respectively. Then \( \text{Pic}(V) = \mathbb{Z}M \oplus \mathbb{Z}L \).

Let \((t_1 : t_2 ; x_1 : x_2 : x_3 : x_k)\) be bihomogeneous coordinates on \( V \) such that \( x_i = 0 \) defines a divisor in \( |M - d_iL| \), and \( L \) is given by \( t_1 = 0 \). Then \(|aM + bL| \) is spanned by divisors

\[
c_{ij}i_1^i_2^i_3^i_4(t_1, t_2)x_1^i_1x_2^i_2x_3^i_3x_k^i_4 = 0,
\]

where \( \sum_{i=1}^4 i_j = a \) and \( c_{ij}i_1^i_2^i_3^i_4(t_1, t_2) \) is a homogeneous polynomial of degree \( b + \sum_{i=1}^4 i_jd_j \).

Let \( Y_1 \subset V \) be a subscroll \( x_1 = \cdots = x_{i-1} = 0 \). The following result holds (see §2.8 in [9]).

**Corollary 2.1.** Take \( D \in |aM + bL| \) and \( q \in \mathbb{N} \), where \( a \) and \( b \) are integers. Then

\[
\text{mult}_{Y_1}(D) \geq q \iff ad_j + b + (d_1 - d_j)(q - 1) < 0.
\]

Let \( X \) be a general\(^4\) divisor in \(|3M + nL|\), where \( n \) is an integer.

**Lemma 2.2.** Suppose \( X \) is smooth and \( \text{rk} \text{Pic}(X) = 2 \). Then \( d_1 \geq -n \) and \( 3d_3 \geq -n \).

**Proof.** We see that \( Y_2 \not\subset X \). Then \( Y_3 \not\subset X \), because \( \text{rk} \text{Pic}(X) = 2 \). But \( \text{mult}_{Y_3}(X) \leq 1 \), because the threefold \( X \) is smooth. The assertion of Corollary 2.1 concludes the proof. \( \square \)

**Lemma 2.3.** Suppose \( X \) is smooth and \( \text{rk} \text{Pic}(X) = 2 \). Then either \( d_1 = -n \) or \( d_2 \geq -n \).

**Proof.** Suppose that \( r = d_1 + n > 0 \) and \( d_2 < -n \). Then \( X \) can be given by the equation

\[
\sum_{i+j+k=2, i,j,k \geq 0} \gamma_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k x_4 = \alpha_r(t_1, t_2)x_1 x_4^2 + \sum_{i+j+k=3, i,j,k \geq 0} \beta_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k,
\]

\(^4\)A complement to a Zariski closed subset in moduli.
where \( \alpha_r(t_1, t_2) \) is a homogeneous polynomial of degree \( r \), \( \beta_{ijk} \) and \( \gamma_{ijk} \) are homogeneous polynomial of degree \( n + id_1 + jd_2 + kd_3 \). Then every point of the intersection

\[
x_1 = x_2 = x_3 = \alpha_r(t_1, t_2) = 0
\]

must be singular on the threefold \( X \), which is a contradiction. \( \square \)

**Lemma 2.4.** Suppose \( X \) is smooth, \( d_2 = d_3, n < 0 \) and \( \text{rk Pic}(X) = 2 \). Then \( 3d_3 \neq n \).

**Proof.** Suppose that \( 3d_3 = -n \). Then \( X \) can be given by the the bihomogeneous equation

\[
\sum_{j, k \geq 0 \atop i + j + k = 2} \gamma_{jkl}(t_0, t_2)x_1^j x_2^k x_3^l = f_3(x_2, x_3) + \alpha_r(t_0, t_2)x_1^3 + \sum_{j, k \geq 0 \atop j + k + l = 1} \beta_{jkl}(t_0, t_2)x_1^j x_2^k x_4^l,
\]

where \( f_3(x_2, x_3) \) is a homogeneous polynomial of degree 3, \( \beta_{jkl} \) and \( \gamma_{jkl} \) are homogeneous polynomial of degree \( n + 2d_1 + jd_2 + kd_3 \) and \( n + 1d_1 + jd_2 + kd_3 \), respectively, \( \alpha_r \) is a homogeneous polynomial of degree \( r = 3d_1 + n \). The threefold \( X \) contains 3 subscrolls given by the equations

\[
x_1 = f_3(x_2, x_3) = 0,
\]

which is impossible, because \( \text{rk Pic}(X) = 2 \). \( \square \)

The following result follows from Lemmas 2.2, 2.3 and 2.4.

**Lemma 2.5.** The threefold \( X \) is smooth and \( \text{rk Pic}(X) = 2 \) whenever

1. in the case when \( d_1 = 0 \), the inequality \( n > 0 \) holds,
2. either \( d_1 = -n \) and \( 3d_3 \geq -n \), or \( d_1 > 0 \), \( d_2 > -n \) and \( 3d_3 \geq -n \),
3. in the case when \( d_2 = d_3 \) and \( n < 0 \), the inequality \( 3d_3 > -n \) holds.

**Proof.** Suppose that all these conditions are satisfied. We must show that \( X \) is smooth, because the equality \( \text{rk Pic}(X) = 2 \) holds by Proposition 32 in [3].

The linear system \( |3M + nL| \) does not have base points if \( n \geq 0 \). So, the threefold \( X \) is smooth by the Bertini theorem in the case \( n \geq 0 \). Therefore, we may assume that \( n < 0 \).

The base locus of \( |3M + nL| \) consists of the curve \( Y_4 \), which implies that \( X \) is smooth outside of the curve \( Y_4 \) and in a general point of \( Y_4 \) by the Bertini theorem and Corollary 2.1, respectively.

In the case when \( d_1 = -n \) and \( d_2 < -n \), the bihomogeneous equation of the threefold \( X \) is

\[
\sum_{i, j, k \geq 0 \atop i + j + k = 2} \gamma_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k x_4 = \alpha_0 x_1 x_4^2 + \sum_{i, j, k \geq 0 \atop i + j + k = 3} \beta_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k x_4,
\]

where \( \beta_{ijk} \) and \( \gamma_{ijk} \) are homogeneous polynomials of degree \( n + id_1 + jd_2 + kd_3 \) and \( \alpha_0 \) is a nonzero constant. The curve \( Y_4 \) is given by \( x_1 = x_2 = x_3 = 0 \), which implies that \( X \) is smooth.

In the case when \( d_1 > -n \) and \( d_2 < 0 \), the bihomogeneous equation of \( X \) is

\[
\sum_{i, j, k \geq 0 \atop i + j + k = 2} \gamma_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k x_4 = 3 \sum_{i = 1}^{3} \alpha_i(t_0, t_2)x_1 x_4^2 + \sum_{i, j, k \geq 0 \atop i + j + k = 3} \beta_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k x_4,
\]

where \( \alpha_i \) is a homogeneous polynomial of degree \( d_i + n \), and \( \beta_{ijk} \) and \( \gamma_{ijk} \) are homogeneous polynomials of degree \( n + id_1 + jd_2 + kd_3 \). Therefore, either \( \alpha_1 x_1 x_4^2 \) or \( \alpha_2 x_2 x_4^2 \) does not vanish at any given point of the curve \( Y_4 \), which implies that \( X \) is smooth. \( \square \)

Thus, there is an infinite series of quadruples \( (d_1, d_2, d_3, n) \) such that the threefold \( X \) is smooth, the equality \( \text{rk Pic}(X) = 2 \) holds, the inequality \( 5n < 12 - 3(d_1 + d_2 + d_3) \) holds and \( n < 0 \).
3. Nonrationality.

We use the notation of Section 2. Let $X$ be a general divisor in $|3M + nL|$, and suppose that the threefold $X$ is smooth, $\text{rk} \text{Pic}(X) = 2$, and $X$ is rational. Let us show that $d_1 = 0$ and $n = 1$.

The threefold $X$ is given by a bihomogeneous equation

$$\sum_{i=0}^{3} \alpha_i(t_0, t_2)x_3^{i}x_4^{3-i} + x_1F(t_0, t_1, x_1, x_2, x_3, x_4) + x_2G(t_0, t_1, x_1, x_2, x_3, x_4) = 0,$$

where $\alpha_i$ is a general homogeneous polynomial of degree $n + id_3$, and $F$ and $G$ stand for

$$\sum_{i, j, k, l \geq 0 \atop i + j + k + l = 2} \beta_{ijkl}(t_0, t_2)x_1^{i}x_2^{j}x_3^{k}x_4^{l} \quad \text{and} \quad \sum_{i, j, k, l \geq 0 \atop i + j + k + l = 2} \gamma_{ijkl}(t_0, t_2)x_1^{i}x_2^{j}x_3^{k}x_4^{l}$$

respectively, where $\beta_{ijkl}$ is a general homogeneous polynomial of degree $n + (i + 1)d_1 + jd_2 + kd_3$, and $\gamma_{ijkl}$ is a general homogeneous polynomial of degree $n + id_1 + (j + 1)d_2 + kd_3$.

Let $Y$ be a threefold given by $x_1F + x_2G = 0$. Then $Y_3 \subset Y$, where $Y_3$ is given by $x_1 = x_2 = 0$.

**Lemma 3.1.** The threefold $Y$ has $2d_1 + 2d_2 + 4d_3 + 4n > 0$ isolated ordinary double points.

**Proof.** The threefold $Y$ is singular exactly at the points of $V$ where

$$x_1 = x_2 = F(t_0, t_1, x_1, x_2, x_3, x_4) = G(t_0, t_1, x_1, x_2, x_3, x_4) = 0$$

by the Bertini theorem. But $Y_3 \cong \text{Proj}(\mathcal{O}_P(d_3) \oplus \mathcal{O}_P) \cong \mathbb{P}_{d_3}$, where $(t_0 : t_1 : x_3 : x_4)$ can be considered as natural bihomogeneous coordinates on the surface $Y_3$.

Let $C$ and $Z$ be the curves on $Y_3$ that are cut out by the equations $F = 0$ and $G = 0$, respectively. Then $C$ and $Z$ are given by the equations

$$\sum_{k, l \geq 0 \atop k + l = 2} \beta_{kl}(t_0, t_2)x_3^{k}x_4^{l} = 0 \quad \text{and} \quad \sum_{k, l \geq 0 \atop k + l = 2} \gamma_{kl}(t_0, t_2)x_3^{k}x_4^{l} = 0$$

respectively, where $\beta_{kl} = \beta_{00kl}$ and $\gamma_{kl} = \gamma_{00kl}$.

The degrees of $\beta_{kl}$ and $\gamma_{kl}$ are $n + d_1 + kd_3$ and $n + d_2 + kd_3$, respectively.

Let $O$ be a point of the scroll $V$ such that the set

$$x_1 = x_2 = F(t_0, t_1, x_1, x_2, x_3, x_4) = G(t_0, t_1, x_1, x_2, x_3, x_4) = 0$$

contains the point $O$. Then $O \in C \cap Z$ and $O \in \text{Sing}(Y)$.

It is easy to see that $O$ is an isolated ordinary double point of the threefold $Y$ in the case when the curves $C$ and $Z$ are smooth and intersect each other transversally at the point $O$.

Put $M = M|_{Y_3}$ and $L = L|_{Y_3}$. Then $C \in |2M + (n + d_1)L|$ and $Z \in |2M + (n + d_2)L|$. But

$$|2M + (n + d_1)L|$$

does not have base points, because $d_1 + n \geq 0$ by Lemma 2.2. So, the curve $C$ is smooth.

The linear system $|2M + (n + d_2)L|$ may have base components, and $Z$ may not be reduced or irreducible. We have to show that $C$ intersects $Z$ transversally at smooth points of $Z$, because

$$|C \cap Z| = C \cdot Z = 2d_1 + 2d_2 + 4d_3 + 4n,$$

where $2d_1 + 2d_2 + 4d_3 + 4n > 0$ by Lemmas 2.2, 2.3 and 2.4.

Suppose that $d_1 > -n$. Then $d_2 \geq -n$ by Lemma 2.3. We see that $|2M + (n + d_2)L|$ does not have base points. Then $Z$ is smooth and $C$ intersects $Z$ transversally at every point of $C \cap Z$.

We may assume that $d_1 = -n$. Let $Y_4 \subset Y_3$ be a curve given by $x_3 = 0$. Then

$$C \cap Y_4 = \emptyset$$

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and either the linear system $|2M + (n + d_2)L|$ does not have base points, or the base locus of the linear system $|2M + (n + d_2)L|$ consist of the curve $Y_4$. However, we have
\[ C \cap Z \subset Y_3 \setminus Y_4, \]
which implies that $C$ intersects the curve $Z$ transversally at smooth points of $Z$. \qed

Let $\pi: \tilde{Y} \to V$ be the blow up of $Y_3$, and $\tilde{Y}$ be and the proper transforms of $Y$ via $\pi$. Then
\[ \tilde{Y} \sim \pi^*(3M + nL) - E, \]
where $E$ is and exceptional divisor of $\pi$. The threefold $\tilde{Y}$ is smooth.

**Lemma 3.2.** The equality $\text{rk} \, \text{Pic}(\tilde{Y}) = 3$ holds.

**Proof.** The linear system $|\pi^*(M - d_2L) - E|$ does not have base points. Thus, the divisor
\[ \tilde{Y} \sim \pi^*(3M + nL) - E \]
is nef and big when $n \geq 0$ by Lemmas 2.2, 2.3 and 2.4. Hence, the equality $\text{rk} \, \text{Pic}(\tilde{Y}) = 3$ holds in the case when $n \geq 0$ by Theorem 2 in [10]. So, we may assume that $n < 0$.

Let $\omega: \tilde{Y} \to \mathbb{P}^1$ be the natural projection and $S$ be the generic fibre of $\omega$, which is considered as a surface defined over the function field $\mathbb{C}(t)$. Then $S$ is a smooth cubic surface in $\mathbb{P}^3$, which contains a line in $\mathbb{P}^3$ defined over the field $\mathbb{C}(t)$, because $Y_3 \subset Y$. Then $\text{rk} \, \text{Pic}(S) \geq 2$.

To conclude the proof we must prove that $\text{rk} \, \text{Pic}(S) = 2$, because there is an exact sequence
\[ 0 \longrightarrow \mathbb{Z}[\pi^*(L)] \longrightarrow \text{Pic}(\tilde{Y}) \longrightarrow \text{Pic}(S) \longrightarrow 0, \]
because every fibre of $\tau$ is reduced and irreducible (see the proof of Proposition 32 in [3]).

Let $\tilde{S}$ be an example of the surface $S$ that is given by the equation
\[ x(q(t)x^2 + p(t)w^2) + y(r(t)y^2 + s(t)z^2) = 0 \subset \text{Proj} \left( \mathbb{C}[x, y, z, t] \right), \]
where $q(t), p(t), r(t), s(t)$ are polynomials such that the inequalities
\[ \deg(q(t)) > 0, \quad \deg(p(t)) \geq 0, \quad \deg(r(t)) > 0, \quad \deg(q(t)) \geq 0 \]
hold. The existence of the surface $\tilde{S}$ follows from the equation of the threefold $Y$.

Let $K$ be an algebraic closure of the field $\mathbb{C}(t)$, let $L$ be a line $x = y = 0$, and let
\[ \gamma: \tilde{S} \to \mathbb{P}^1 \]
be a projection from $L$. Then $\gamma$ is a conic bundle defined over $\mathbb{C}(t)$. But $\gamma$ has five geometrically reducible fibres $F_1, F_2, F_3, F_4, F_5$ defined over $\mathbb{F}$ such that
- $F_i = \tilde{F}_i \cup \bar{F}_i$, where $\tilde{F}_i$ and $\bar{F}_i$ are geometrically irreducible curves,
- the curve $L \cup F_i$ is cut out on the surface $\tilde{S}$ by the equation
\[ y = e^i \sqrt[3]{\frac{q(t)}{r(t)}} x, \]
where $\epsilon = -(1 + \sqrt{-3})/2$ and $i \in \{1, 2, 3\}$,
- the curve $F_4 \cup L$ is cut out on the surface $\tilde{S}$ by the equation $x = 0$,
- the curve $F_5 \cup L$ is cut out on the surface $\tilde{S}$ by the equation $y = 0$.

The group $\text{Gal}(K/\mathbb{C}(t))$ naturally acts on the set
\[ \Sigma = \{ \tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4, \tilde{F}_5, \bar{F}_1, \bar{F}_2, \bar{F}_3, \bar{F}_4, \bar{F}_5 \}, \]
because the conic bundle $\gamma$ is defined over $\mathbb{C}(t)$. The inequality $\text{rk} \, \text{Pic}(\tilde{S}) > 2$ implies the existence of a subset $\Gamma \subset \Sigma$ consisting of disjoint curves such that $\Gamma \subset \Sigma$ is $\text{Gal}(K/\mathbb{C}(t))$-invariant.
The action of $\text{Gal}(\mathbb{K}/\mathbb{C}(t))$ on the set $\Sigma$ is easy to calculate explicitly. Putting
\[ \Delta = \{ F_1, F_2, F_3, F_4, F_5 \}, \quad \Lambda = \{ F_4, F_5 \}, \quad \Xi = \{ F_2, F_3 \}, \]
we see that the group $\text{Gal}(\mathbb{K}/\mathbb{C}(t))$ acts transitively on each subset $\Lambda$, $\Xi$, $\Delta$, because we may assume that $q(t)$, $p(t)$, $r(t)$, $s(t)$ are sufficiently general. But each subset $\Lambda$, $\Xi$, $\Delta$ does not consist of disjoint curves. Hence, the equality $\text{rk Pic}(\bar{S}) = 2$ holds, which implies that $\text{rk Pic}(S) = 2$. □

The linear system $|\pi^* (M - d_2 L) - E|$ does not have base points and induces a $\mathbb{P}^2$-bundle
\[ \tau : \bar{\mathcal{V}} \rightarrow \text{Proj} \left( \mathcal{O}_{\mathbb{P}_1}(d_1) \oplus \mathcal{O}_{\mathbb{P}_1}(d_2) \right) \cong \mathbb{F}_r, \]
where $r = d_1 - d_2$. Let $l$ be a fibre of the natural projection $\mathbb{F}_r \rightarrow \mathbb{P}^1$, and $s_0$ be an irreducible curve on the surface $\mathbb{F}_r$ such that $s_0^2 = r$, and $s_0$ is a section of the projection $\mathbb{F}_r \rightarrow \mathbb{P}^1$. Then
\[ \pi^* (M - d_2 L) - E \sim \tau^*(s_0) \]
and $\pi^*(L) \sim \tau^*(l)$. The morphism $\tau$ induces a conic bundle $\bar{\tau} = \tau|_{\bar{\mathcal{V}}} : \bar{\mathcal{V}} \rightarrow \mathbb{F}_r$.

Let $\Delta$ be the degeneration divisor of the conic bundle $\bar{\tau}$. Then
\[ \Delta \sim 5s_\infty + \mu l, \]
where $\mu$ is a natural number, and $s_\infty$ is the exceptional section of the surface $\mathbb{F}_r$.

Let $S$ be a surface in $\bar{\mathcal{V}}$ and $B$ be a threefold in $\bar{\mathcal{V}}$ dominating the curve $s_0$. Then
\[ B \cong \text{Proj} \left( \mathcal{O}_{\mathbb{P}_1}(d_1) \oplus \mathcal{O}_{\mathbb{P}_1}(d_3) \oplus \mathcal{O}_{\mathbb{P}_1} \right) \]
and $\pi(B) \cong B$. But $\pi(B) \cap Y = \pi(S) \cup Y_3$.

The surface $Y_3$ is cut out on $\pi(B)$ by the equation $x_1 = 0$, where $\pi(B) \in |M - d_2 L|$. We have
\[ S \sim 2T + (d_1 + n)F, \]
where $T$ is a tautological line bundle on $B$, and $F$ is a fibre of the projection $B \rightarrow \mathbb{P}^1$. Then
\[ K^2 = -5d_1 + 2d_3 - 4d_2 - 3n + 8 \]
and $\mu = s_0 \cdot \Delta = 5d_1 - 2d_3 + 4d_2 + 3n$.

It follows from the equivalence $2K_{\mathbb{F}_r} + \Delta \sim s_\infty + (3d_1 - 2d_3 + 6d_2 + 3n - 4)l$ that
\[ \left| 2K_{\mathbb{F}_r} + \Delta \right| \neq \emptyset \iff 3d_1 - 2d_3 + 6d_2 + 3n \geq 4, \]
which implies that $Y$ is nonrational by Theorem 10.2 in [11] if $3d_1 - 2d_3 + 6d_2 + 3n < 4$.

The threefold $Y$ is nonruled if and only if it is nonrational, because the threefold $Y$ is rationally connected. So, the threefold $X$ is nonrational by Theorem 1.8.3 in §IV of the book [2] whenever
\[ 3d_1 - 2d_3 + 6d_2 + 3n \geq 4, \]
which implies that $3d_1 - 2d_3 + 6d_2 + 3n < 4$, because we assume that $X$ is rational.

We see that either $d_1 = 0$ and $n = 1$ or $d_1 = 1$ and $d_2 = n = 0$ by Lemmas 2.2, 2.3 and 2.4 but the threefold $X$ is birational to a smooth cubic threefold in the case when $d_1 = 1$ and $d_2 = n = 0$, which is nonrational by [5]. Then $d_1 = 0$ and $n = 1$. The assertion of Theorem [1,4] is proved.

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