GENERALIZED POWER SUM AND NEWTON-GIRARD IDENTITIES

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Abstract. In this article we prove an algebraic identity which significantly generalizes the formula for sum of powers of consecutive integers involving Stirling numbers of the second kind. Also we have obtained a generalization of Newton-Girard power sum identity.

1. Introduction

The sum of powers of consecutive integers has a long and fascinating history. Historically the first ever formula for the sum was obtained by Swiss mathematician Jacob Bernoulli (1654-1705), who proved the following:

\[ 1^m + 2^m + \cdots + (n-1)^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}, \quad m \geq 0, n \geq 1, \]

where \( B_k \)'s are the famous Bernoulli numbers. There is also a surprising relationship between the sum of powers and the Stirling numbers of second kind \([3]\). In fact,

\[ 1^m + 2^m + \cdots + n^m = \frac{1}{m+1} \sum_{k=0}^{n} \binom{n+1}{k+1} S(m, k) k!, \quad \text{where} \]

\[ S(m, k) = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^m. \]

In \([3]\), the author proved (2) along with its many generalizations using the so called binomial transform. In fact, the author proved the following general statement, and obtained various power sum identities as a corollary of the following:

Lemma 1.1 (Lemma 2.1, \([3]\)). Let \( c_1, c_2, \cdots, \) be a sequence of complex numbers. Then for every positive integer \( m, \) we have

\[ \sum_{k=1}^{m} k^\alpha c_k = \sum_{j=1}^{m} j! S(\alpha, j) \sum_{k=j}^{m} \binom{k}{j} c_k. \]

In this article, we prove a general identity, which proves Lemma 1.1 for positive integer \( \alpha \) as a corollary and consequently many other well known power sum identities. Before stating our result, let us fix some notations.

Let \( \{ x_i^{(j)} : 1 \leq j \leq r, 1 \leq i \leq m \} \) and \( \{ y_{\ell} : 2 \leq \ell \leq m + 1 \} \)

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be two sets of variables, and $P \subseteq [n]$ (where $[n] = \{1, 2, \ldots, n\}$). Define
\[
\Pi_r P = \prod_{j=1}^{r} \left(\sum_{i \in P} x_i^{(j)}\right),
\]
and for any finite set $Q$ of positive integers, the maximum element of $Q$ is denoted by $\text{Max}(Q)$. Then we have the following:

**Theorem 1.2.**

\[
\sum_{k=1}^{m} \Pi_r[k] y_{k+1} = \sum_{U \subset [m+1], |U| \geq 2} \left(\sum_{\emptyset \neq V \subset U \setminus \{\text{Max}(U)\}} (-1)^{|U|-|V|-1} \Pi_r V\right) y_{\text{Max}(U)}.
\]

We call this theorem “the generalized power sum theorem”. Note that, if we put $x_i^{(j)} = 1$ for all $1 \leq j \leq r$, $1 \leq i \leq m$, and $y_t = c_{t-1}$ and $r = \alpha$ in our Theorem 1.2, we obtain Lemma 1.1 for positive integer $\alpha$. In particular, if we put $x_i^{(j)} = 1 = y_t$ for all $1 \leq j \leq r$, $1 \leq i \leq m$, and $2 \leq t \leq m + 1$, we obtain the classical formula for the sum of powers (2).

Newton-Girard identity is a very important result occurring many places in algebra and combinatorics. A combinatorial proof of Newton-Girard identity was first given by Doron Zeilberger in [4]. In [2], the present authors gave a graph theoretic formulation of the Newton-Girard identity exhibiting a relation between weighted sum of closed walks and weighted sum of linear subdigraphs (defined later) of a weighted digraph $\Gamma$. In this paper, based on [2], we give a “colored” version of the graph theoretic formulation mentioned above and as a corollary we obtain a significant generalization of the classical Newton-Girard identity.

Before proceeding to the statement of our theorem, let us define some graph theoretic notions.

**Definition 1.1.** A weighted $k$-colored digraph, denoted by $\Gamma_{k,C}$ is a digraph equipped with a finite set $C = \{c_1, c_2, \ldots, c_k\}$ called the set of colors such that for any ordered pair of vertices $(i, j)$ in $\Gamma_{k,C}$, either there are no directed edges from $i$ to $j$ or there are precisely $k$ directed edges from $i$ to $j$ each receiving distinct colors from $C$ (i.e. no two of the directed edges from $i$ to $j$ receive the same color) and the edge from $i$ to $j$ with color $c_r$ is assigned a nonzero weight $a_{ij}^{(r)}$.

**Definition 1.2.** A colored linear subdigraph $\gamma$ of a weighted $k$-colored digraph $\Gamma_{k,C}$ is a collection of pairwise vertex-disjoint cycles such that any two different edges of $\gamma$ have different colors. Define $\text{Col}(\gamma)$ to be the set of colors of the edges in $\gamma$. $L(\gamma)$ is defined to be the length of $\gamma$ i.e. the number of edges in $\gamma$. The weight of a colored linear subdigraph $\gamma$, written as $W(\gamma)$, is the product of the weights of all its edges. The number of cycles contained in $\gamma$ is denoted by $c(\gamma)$. The set of all colored linear subdigraphs of $\Gamma_{k,C}$ is denoted by $\text{CLSD}(\Gamma_{k,C})$.

**Remark 1.** Colored digraphs have appeared in the literature before. See, for example [1]. Here we have defined this notion in a way, suitable to our purpose.

**Definition 1.3.** A colored closed walk $w$ of length $L(w) = m$ in a weighted $k$-colored digraph $\Gamma_{k,C}$ is a sequence of vertices $x_0, x_1, \ldots, x_{m-1}, x_m$ such that $x_0 = x_m$ and for each $0 \leq i \leq m - 1$, there is a directed edge from $x_i$ to $x_{i+1}$, and for $i \neq j$ the color of the directed edge from $x_i$ to $x_{i+1}$ is distinct from that of the directed edge from $x_j$ to $x_{j+1}$. Define $\text{Col}(w)$ to be the set of colors of the edges in $w$. The weight $W(w)$ of a colored closed walk $w$ is the
product of all weights of the edges present in that walk. The set of all colored closed walks of $\Gamma_{k,C}$ is denoted by $CCW(\Gamma_{k,C})$.

Let $\Gamma_{k,C}$ be a weighted $k$-colored digraph and $S$ and $T$ be subsets of $C$ such that $|S| = p$ and $|T| = q$. We define the following:

$$\ell_{p,S} \triangleq \begin{cases} \sum_{\gamma \in CLSD(\Gamma_{k,C}) \text{ such that } L(\gamma) = p, \text{ and } \text{Col}(\gamma) = S} (-c(\gamma))W(\gamma), & \text{if } S \neq \emptyset \\ 1, & \text{if } S = \emptyset \text{ or equivalently } p = 0 \\ 0, & \text{if there } \not\exists \text{ any } \gamma \in CLSD \text{ such that } \text{Col}(\gamma) = S. \end{cases}$$

$$c_{q,T} \triangleq \begin{cases} \sum_{w \in CCW(\Gamma_{k,C}) \text{ such that } L(w) = q, \text{ and } \text{Col}(w) = T} W(w), & \text{if } T \neq \emptyset \\ 1, & \text{if } T = \emptyset \text{ or equivalently } q = 0 \\ 0, & \text{if there } \not\exists \text{ any } w \in CCW \text{ such that } \text{Col}(w) = T. \end{cases}$$

Now we have the following:

**Theorem 1.3.** Let $\Gamma_{k,C}$ be a weighted $k$-colored digraph. Then

1. $\sum_{p+q=r, S \cap T = \emptyset} c_{q,T} \ell_{p,S} = 0$, if $r > n$
2. $\sum_{p+q=r, q > 0, S \cap T = \emptyset} c_{q,T} \ell_{p,S} + r \ell_{r,C} = 0$, if $r \leq n$.

As a corollary of this theorem we obtain the following very generalized form of classical Newton-Girard identity.

**Theorem 1.4.** Let $r$ and $n$ be two positive integers and $\{\alpha_j^{(i)} : 1 \leq j \leq n, 1 \leq i \leq r\}$ be a set of variables. Then our theorem states the following:

1. If $r > n$, $\sum_{k=0}^{r} (-1)^k \sum_{1\leq i_1 < i_2 < \cdots < i_k \leq r} (r - k)! \left( \sum_{j=1}^{n} \alpha_j^{(i_1)} \alpha_j^{(i_2)} \cdots \alpha_j^{(i_{r-k})} \right) X = 0$,
2. If $r \leq n$, $\sum_{k=0}^{r-1} (-1)^k \sum_{1\leq i_1 < i_2 < \cdots < i_k \leq r} (r - k)! \left( \sum_{j=1}^{n} \alpha_j^{(i_1)} \alpha_j^{(i_2)} \cdots \alpha_j^{(i_{r-k})} \right) X + rY = 0$,

where

$$X = \sum_{\gamma_1, \ldots, \gamma_k} \alpha_{\gamma_1}^{(i_1)} \alpha_{\gamma_2}^{(i_2)} \cdots \alpha_{\gamma_k}^{(i_k)}$$

and

$$Y = \sum_{\gamma_1, \ldots, \gamma_r} \alpha_{\gamma_1}^{(i')} \alpha_{\gamma_2}^{(i')} \cdots \alpha_{\gamma_r}^{(i')}$$

such that each $i_j' \in [r] \setminus \{i_1, \ldots, i_{r-k}\}$ and $i_p' \neq i_q'$ for $p \neq q$.
Note that if we put $a_j^{(i)} = a_j$, for all $1 \leq j \leq n$ and $1 \leq i \leq r$ in Theorem 1.3, we immediately obtain the following:

**Corollary 1** (Newton-Girard identity). Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be roots of the polynomial $f(x) = x^n + e_1 x^{n-1} + e_2 x^{n-2} + \cdots + e_t x^{n-t} + \cdots + e_n$. Suppose $p_r = \alpha_1^r + \alpha_2^r + \cdots + \alpha_n^r$ ($r = 0, 1, \cdots$). Then Newton-Girard identity says that

1. If $r > n$, $p_r + e_1 p_{r-1} + e_2 p_{r-2} + \cdots + p_1 e_{r-1} + e_n p_{r-n} = 0$
2. If $r \leq n$, $p_r + e_1 p_{r-1} + e_2 p_{r-2} + \cdots + p_1 e_{r-1} + re_r = 0$.

2. PROOF OF THE THEOREMS

In this section we prove the generalized power sum theorem. As a recipe to do so, let us describe some terminology. Let $A = \{a_1, a_2, \cdots, a_n\}$ be a finite set. Think $A$ to be the set of letters. The **free monoid** $A^*$ is the set of all finite sequences of elements of $A$, usually called words with the operation of concatenation. Construct an algebra from $A^*$ by taking formal sum of elements of $A$ with coefficient in $\mathbb{Z}$, extending the multiplication by usual distributivity. For example, in this algebra,

$$(a_1 + a_2) a_3 = a_1 a_3 + a_2 a_3$$
$$(a_1 + a_2) (a_1 + a_2) = a_1 a_1 + a_1 a_2 + a_2 a_1 + a_2 a_2.$$

**Proof of Theorem 1.2** Let

$$L = \{x_i^{(j)}, 1 \leq j \leq r, 1 \leq i \leq m\} \cup \{y_\ell, 2 \leq \ell \leq m + 1\}.$$ 

Take $L$ to be the set of letters. Then the left hand side of Theorem 1.2 can be interpreted as the sum of all words in $L^*$ of the form $x_i^{(1)} x_i^{(2)} \cdots x_i^{(r)} y_t$, where each $i_p < t$, for all $1 \leq p \leq r$. Let us call the word of this form, good word, and let $G$ be the set of all good words. Now let us evaluate the sum of all good words in another way. For any $U \subset [m+1]$ with $|U| \geq 2$, define $G_U = \{x_i^{(1)} x_i^{(2)} \cdots x_i^{(r)} y_t : \text{each } i_p \in U \setminus \text{Max}(U), t = \text{Max}(U)\}$, and for any $u \in U \setminus \text{Max}(U)$, there exists $q \in [r]: i_q = u$. It is clear that

$$G = \bigcup_{U \subset [m+1], |U| \geq 2} G_U.$$ 

Now by the principle of inclusion and exclusion, the sum of all words in $G_U$ is

$$\sum_{\emptyset \neq T \subset U \setminus \{\text{Max}(U)\}} (-1)^{|U|-|T|-1} \prod_{\ell \in T} y_{\text{Max}(U)}.$$ 

This completes the proof.

Now we proceed to the proof of Theorem 1.3. Before getting into the proof we need the following:

Let $\Gamma$ be a digraph (not necessarily colored). A walk $w$ in $\Gamma$ is defined to be a sequence of vertices $w = v_0, v_1, \cdots, v_t$ such that for each $i \in [0, t-1] \cap \mathbb{Z}$, there is a directed edge from $v_i$ to $v_{i+1}$. Now let $w_1 = v_0, v_1, \cdots, v_t$ and $w_2 = v_t, v_{t_1}, \cdots, v_{t_k}$ be two walks in $\Gamma$. Then the concatenation of $w_1$ and $w_2$, denoted by $w_1 \circ w_2$, is the walk $v_0, v_1, \cdots, v_t, v_{t_1}, \cdots, v_{t_k}$. □
Proof of Theorem 1.3. First we prove the case $r > n$. To prove this, consider all ordered pairs $(w, \gamma)$, where $w$ is a colored closed walk and $\gamma$ is a colored linear subdigraph (possibly empty), such that $L(w) + L(\gamma) = r$ and $\text{Col}(w) \cap \text{Col}(\gamma) = \emptyset$. Define the weight $W$ of $(w, \gamma)$ to be $W((w, \gamma)) = (-1)^{\epsilon(\gamma)} W(w) W(\gamma)$. Note that the left hand side of (1) in Theorem 1.3 is precisely equal to $\sum_{(w, \gamma)} W((w, \gamma))$, where the summation runs over all ordered pairs $(w, \gamma)$ as described above.

Now the crucial observation is that, since $r > n$, either $w$ and $\gamma$ share a common vertex or $w$ is not a “simple” closed walk (here simple means the graph structure of the closed walk is a directed cycle). Now take a particular pair $(w, \gamma)$ satisfying the above conditions. Suppose that $x$ is the initial and terminal vertex of $w$. Start moving from $x$ along $w$. There are two possibilities: either, first we meet a vertex $y$ which is a vertex of $\gamma$ or, we complete a closed directed cycle $\hat{w}$ which is a subwalk of $w$ and during this journey from $x$ up to the completion of $\hat{w}$ we have not met any vertex of $\gamma$. Now if the first case holds, we form a new ordered pair $(\hat{w}, \hat{\gamma})$, where $\hat{w} = xy|w_0 \gamma y \cup \hat{x}|w_0$ and $\hat{\gamma} = \gamma \setminus \{\gamma_y\}$, where $\hat{x}|w_0$ is the walk from $x$ to $y$ along $w$ and $\gamma_y$ is the directed cycle of $\gamma$ containing the vertex $y$. Note that $W((\hat{w}, \hat{\gamma})) = -W((w, \gamma))$. Now if the second case holds, then form a new ordered pair $(\tilde{w}, \tilde{\gamma})$, where $\tilde{w}$ is formed by removing the directed cycle $\hat{w}$ from $w$ and $\tilde{\gamma}$ is $\gamma \cup \hat{w}$. Note also that $W((\tilde{w}, \tilde{\gamma})) = -W((\tilde{w}, \tilde{\gamma}))$. It is easy to see that, this is in fact a sign reversing involution by additionally noting that $\text{Col}(\tilde{w}) \cap \text{Col}(\tilde{\gamma}) = \emptyset$ and $\text{Col}(\hat{w}) \cap \text{Col}(\hat{\gamma}) = \emptyset$. This completes the first part of the proof.

Now we prove the case $r \leq n$. Let $B = \{ (w, \gamma) : w$ is a colored closed walk of length $\geq 1, \gamma$ is a colored linear subdigraph (possibly empty), $L(w) + L(\gamma) = r$ and $\text{Col}(w) \cap \text{Col}(\gamma) = \emptyset \}$. Consider the following sum $D = \sum_{(w, \gamma) \in B} W((w, \gamma)) + r\ell_{r,C}$. Note that the left hand side of (2) in Theorem 1.3 is precisely equal to $D$.

Consider the subset of $B$ consisting of ordered pair $(w, \gamma)$ satisfying the conditions: either $w \cap \gamma \neq \emptyset$ or $w$ is not a simple closed walk. Call this subcollection BAD. So the GOOD members of $B$ are the ordered pairs $(w, \gamma)$ satisfying $w \cap \gamma = \emptyset$ and $w$ is a colored simple closed walk. Now observe that, the weights of the BAD members cancel among themselves just like the previous case (case, $r > n$). Now let us see, how a GOOD member looks like. As a directed graph it is just a disjoint collection of distinct cycles with vertex set, say, $\{v_1, v_2, \ldots, v_r\}$ i.e. it is a colored linear subdigraph $\gamma$ with vertex set $\{v_1, v_2, \ldots, v_r\}$. Now for this $\gamma$ with vertex set $\{v_1, v_2, \ldots, v_r\}$, we claim that there are precisely $r$ GOOD members $(w, \gamma)$. For the proof, take any vertex say $v_i$ from $\gamma$. Consider the cycle $w$ in $\gamma$ containing the vertex $v_i$. Let $\gamma_1 = \gamma \setminus \{w\}$. Now the cycle $w$ can be thought of as a closed walk $w_{v_i}$ starting and ending at the vertex $v_i$. So we get a GOOD member $(w_{v_i}, \gamma_1)$. Since $v_i$ is arbitrary the claim follows.

The main observation is that the sum of the weights of all the GOOD members, found in this way from $\gamma$ is $r(-1)^{\epsilon(\gamma) - 1} W(\hat{\gamma})$. This cancels with the term $r(-1)^{\epsilon(\gamma)} W(\hat{\gamma})$ in the equation $D = \sum_{(w, \gamma) \in B} W((w, \gamma)) + r\ell_{r,C}$.

Proof of Theorem 1.4. The proof immediately follows by applying Theorem 1.3 to the weighted colored digraph $\Gamma_{r,C}$ defined as follows:
The vertex set of the graph $\Gamma_{r,C}$ is $V(\Gamma_{r,C}) = \{v_1, v_2, \ldots, v_n\}$ and $C = \{c_1, c_2, \ldots, c_r\}$ is the set of colors. For each vertex $v_j$ there are precisely $r$ directed edges $e_j^{(1)}, e_j^{(2)}, \ldots, e_j^{(r)}$ from $v_j$ to itself such that $e_j^{(k)}$ is colored with the color $c_k$, for all $k = 1, 2, \ldots, r$. For $j \neq j'$ there is no directed edge from $v_j$ to $v_{j'}$. Also for any $j$, the edge from $v_j$ to itself with color $c_i$, is given a weight $\alpha_j^{(i)}$.

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