SOME APPLICATION OF GRUNSKY COEFFICIENTS IN THE THEORY OF UNIVALENT FUNCTIONS

MILUTIN OBRADOVIĆ AND NIKOLA TUNESKI

Abstract. Let function $f$ be normalized, analytic and univalent in the unit disk $D = \{ z : |z| < 1 \}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Using a method based on Grusky coefficients we study several problems over that class of univalent functions: upper bound of the third logarithmic coefficient, upper bound of the coefficient difference $|a_4| - |a_3|$, the special case of the generalised Zalcman conjecture $|a_2a_3 - a_4|$ and upper bounds of the second and the third Hankel determinant. Obtained results improve the previous ones.

1. Introduction and definitions

Let $A$ be the class of functions $f$ which are analytic in the open unit disc $\mathbb{D} = \{ z : |z| < 1 \}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$

and let $S$ be the subclass of $A$ consisting of functions that are univalent in $\mathbb{D}$.

Although the famous Bieberbach conjecture $|a_n| \leq n$ for $n \geq 2$, was proved by de Branges in 1985 [2], a great many other problems concerning the coefficients $a_n$ remain open.

One of them is finding sharp estimates of logarithmic coefficient, $\gamma_n$, of a univalent function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n.$$ 

Relatively little exact information is known about the coefficients. The natural conjecture $|\gamma_n| \leq 1/n$, inspired by the Koebe function (whose logarithmic coefficients are $1/n$) is false even in order of magnitude (see Duren [3, Section 8.1]). For the class $S$ the sharp estimates of single logarithmic coefficients $S$ are known only for $\gamma_1$ and $\gamma_2$, namely,

$$|\gamma_1| \leq 1 \quad \text{and} \quad |\gamma_2| \leq \frac{1}{2} + \frac{1}{e} = 0.635 \ldots,$$

and are unknown for $n \geq 3$. In this paper we give the estimate $|\gamma_3| \leq 0.5566178 \ldots$ for the general class of univalent functions. This is an improvement of $|\gamma_3| \leq 0.7688 \ldots$ obtained in [12]. For the subclasses of univalent functions the situation

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Hayman in [7] showed that if $f$ has coefficients $a_n$, then it is natural to conjecture that $|a_{n+1}| - |a_n| \leq 1$. But this is false even when $n = 2$, due to Fekete and Szegő (4) who obtained the sharp bounds

$$-1 \leq |a_3| - |a_2| \leq \frac{3}{4} + e^{-\lambda_0}(2e^{-\lambda_0} - 1) = 1.029\ldots,$$

where $\lambda_0$ is the unique solution of the equation $4\lambda = e^\lambda$ on the interval $(0, 1)$. Hayman in [7] showed that if $f \in \mathcal{S}$, then $|a_{n+1}| - |a_n| \leq C$, where $C$ is an absolute constant and the best estimate of $C$ is $3.61\ldots$ (5). In the case when $n = 3$ in [13], the authors improved this to $2.1033299\ldots$. In this paper we improve it even further, to the value $1.751853\ldots$.

Another problem is finding sharp upper and lower bounds of the coefficient difference $|a_{n+1}| - |a_n|$ over the class of univalent functions. Since the Keobe function has coefficients $a_n = n$, it is natural to conjecture that $|a_{n+1}| - |a_n| \leq 1$. But this is false even when $n = 2$, due to Fekete and Szegő (4) who obtained the sharp bounds

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Other, still open problem, is the generalized Zalcman conjecture

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1),$$

$n \geq 2$, $m \geq 2$, closed by Ma for the class of starlike functions and for the class of univalent functions with real coefficients and by Ravichandran and Verma in [16] for the classes of starlike and convex functions of given order and for the class of functions with bounded turning. In [14] the authors studied the generalized Zalcman conjecture for the class

$$\mathcal{U} = \left\{ f \in \mathcal{A} : \left| \frac{z}{f(z)} \right|^2 f'(z) - 1 < 1, z \in \mathbb{D} \right\}$$

and proved it for the cases $m = 2, n = 3$; and $m = 2, n = 4$. In this paper we prove the estimate $2.10064\ldots$ for the general class when $m = 2$ and $n = 3$ which is close to the conjectured value $2$.

The upper bound of the Hankel determinant is a problem rediscovered and extensively studied in recent years. Over the class $\mathcal{A}$ of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ analytic on the unit disk, this determinant is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

where $q \geq 1$ and $n \geq 1$. The second order Hankel determinants is

$$H_2(2) = \begin{vmatrix} a_2 & a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2,$$

and the third order one is

$$H_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_5 - a_2^2).$$

For the general class $\mathcal{S}$ of univalent functions in the class $\mathcal{A}$ these are very few results concerning the Hankel determinant. The best known for the second order case is due to Hayman (6), saying that $|H_2(n)| \leq An^{1/2}$, where $A$ is an absolute constant, and that this rate of growth is the best possible. Another one is [15], where it was...
proven that $|H_2(2)| \leq A$, where $1 \leq A \leq \frac{11}{3} = 3.66 \ldots$, and $|H_3(1)| \leq B$, where $\frac{4}{5} \leq B \leq \frac{32 + \sqrt{385}}{15} = 3.258796 \ldots$. There are much more results for the subclasses of $S$ and some references are [6, 9, 10, 17]. In this paper we improve the general estimates for the second and third order determinant with the values $1.3614356 \ldots$ and $2.321434 \ldots$, respectively.

For the study of the probes defined above we will use method based on Grunsky coefficients. In the proofs we will use mainly the notations and results given in the book of N. A. Lebedev ([11]).

Here are basic definitions and results.

Let $f \in S$ and let

$$\log \frac{f(t) - f(z)}{t - z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,$$

where $\omega_{p,q}$ are called Grunsky’s coefficients with property $\omega_{p,q} = \omega_{q,p}$. For those coefficients we have the next Grunsky’s inequality ([3, 11]):

$$\sum_{q=1}^{\infty} \left| \sum_{p=1}^{\infty} \omega_{p,q} t^p x^q \right|^2 \leq \sum_{p=1}^{\infty} \left| x_p \right|^2 \frac{p}{p + 1},$$

where $x_p$ are arbitrary complex numbers such that last series converges.

Further, it is well-known that if $f$ given by (1) belongs to $S$, then also

$$f_2(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \cdots$$

belongs to the class $S$. Then for the function $f_2$ we have the appropriate Grunsky’s coefficients of the form $\omega_{2p-1,2q-1}$ and the inequality (3) has the form:

$$\sum_{q=1}^{\infty} (2q - 1) \left| \sum_{p=1}^{\infty} \omega_{2p-1,2q-1} x_{2p-1} \right|^2 \leq \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p - 1}.$$  

Here, and futher in the paper we omit the upper index (2) in $\omega_{2p-1,2q-1}$ if compared with Lebedev’s notation.

From inequality (5), when $x_{2p-1} = 0$ and $p = 2, 3, \ldots$, we have

$$|\omega_{11} x_1 + \omega_{31} x_3|^2 + 3|\omega_{13} x_1 + \omega_{33} x_3|^2 + 5|\omega_{15} x_1 + \omega_{35} x_3|^2 \leq |x_1|^2 + \frac{|x_3|^2}{3}.$$  

As it has been shown in [11, p.57], if $f$ is given by (1) then the coefficients $a_2, a_3, a_4$ and $a_5$ are expressed by Grunsky’s coefficients $\omega_{2p-1,2q-1}$ of the function $f_2$ given by (1) in the following way:

$$a_2 = 2\omega_{11},$$

$$a_3 = 2\omega_{13} + 3\omega_{11}^2,$$

$$a_4 = 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3,$$

$$a_5 = 2\omega_{35} + 8\omega_{11}\omega_{33} + 5\omega_{13}^2 + 18\omega_{11}\omega_{13} + \frac{7}{3}\omega_{11}^4,$$

$$0 = 3\omega_{15} - 3\omega_{11}\omega_{13} + \omega_{11}^3 - 3\omega_{33}.$$
2. The third logarithmic coefficient

We now give upper bound of the third logarithmic coefficient over the class $S$.

**Theorem 1.** Let $f \in S$ and be given by (1). Then

$$|\gamma_3| \leq 0.5566178 \ldots$$

**Proof.** From (2), after differentiation and comparison of coefficients we receive

$$\gamma_3 = \frac{1}{2} \left( a_4 - a_2 a_3 + \frac{1}{3} a_2^3 \right).$$

The fifth relation in (7) gives

$$\omega_{33} = \omega_{15} - \omega_{11} \omega_{13} + \frac{1}{3} \omega_{11}^3,$$

which, together with the other expressions from (7) implies

$$\gamma_3 = \omega_{33} + 2 \omega_{11} \omega_{13} = \omega_{15} + \omega_{11} \omega_{13} + \frac{1}{3} \omega_{11}^3.$$

Therefore,

$$|\gamma_3| \leq \frac{1}{3} |\omega_{11}|^3 + |\omega_{11}| |\omega_{13}| + |\omega_{15}|.$$  

Now, choosing $x_1 = 1$ and $x_3 = 0$ in (6) we have

$$|\omega_{11}|^2 + 3 |\omega_{13}|^2 + 5 |\omega_{15}|^2 \leq 1,$$

and also from here

$$|\omega_{11}|^2 + 3 |\omega_{13}|^2 \leq 1.$$  

The last two relations imply

$$|\gamma_3| \leq \frac{1}{3} |\omega_{11}|^3 + |\omega_{11}| |\omega_{13}| + \frac{1}{3} \sqrt{1 - |\omega_{11}|^2 - 3 |\omega_{13}|^2} \equiv f_1(|\omega_{11}|, |\omega_{13}|),$$

where

$$f_1(x, y) = \frac{1}{3} x^3 + x y + \frac{1}{\sqrt{3}} \sqrt{1 - x^2 - 3 y^2}$$

and $0 \leq x \leq 1, 0 \leq y \leq \frac{1}{\sqrt{3}} \sqrt{1 - x^2}$ ($|a_2| = 2 |\omega_{11}| \leq 2$ implies $0 \leq |\omega_{11}| \leq 1$).

So, we need to find maximum of the function $f_1$ over the region $E = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{\sqrt{3}} \sqrt{1 - x^2}\}$.

The system

$$\begin{align*}
\frac{\partial f_1}{\partial x} &= x^2 + y - \frac{x}{\sqrt{7} \sqrt{1 - x^2 - 3 y^2}} = 0, \\
\frac{\partial f_1}{\partial y} &= x - \frac{3 y}{\sqrt{7} \sqrt{1 - x^2 - 3 y^2}} = 0,
\end{align*}$$

has only one solution in the interior of $E$, that is $(x_1, y_1) = (0.81267 \ldots, 0.243532 \ldots)$ such that $f_1(x_1, y_1) = 0.5566178 \ldots$.

Now, let consider the function $f_1$ on the boundary of $E$:
Theorem 2. Let $f \in S$ and be given by (1). Then
\[ |a_4| - |a_3| \leq 1.751853 \ldots \]

Proof. Since
\[ |a_4| - |a_3| \leq |a_4| - |\omega_{11}|a_3| \leq |a_4 - \omega_{11}a_3| = \left|2\omega_{33} + 6\omega_{11}\omega_{33} + \frac{1}{3}\omega_1^2\right|, \]
and using (3) and (10) we have
\[ |a_4| - |a_3| \leq \left|2\omega_{15} + 4\omega_{11}\omega_{13} + \omega_1^2\right| \]
\[ \leq |\omega_{11}|^3 + 4|\omega_{11}||\omega_{13}| + 2|\omega_{15}| \]
\[ \leq |\omega_{11}|^3 + 4|\omega_{11}||\omega_{13}| + \frac{2}{\sqrt{5}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2} \equiv f_2(|\omega_{11}|, |\omega_{13}|), \]
where
\[ f_2(x, y) = x^3 + 4xy + \frac{2}{\sqrt{5}}\sqrt{1 - x^2 - 3y^2} \]
with $0 \leq x \leq 1$, $0 \leq y \leq \frac{1}{\sqrt{3}}\sqrt{1 - x^2}$.

For finding the maximum of $f_2$ over the region $E = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{\sqrt{3}}\sqrt{1 - x^2}\}$ we solve the system $\frac{\partial f_2}{\partial x} = 0$, $\frac{\partial f_2}{\partial y} = 0$, and realize that inside $E$ it has only one solution $(x_2, y_2) = (0.8343643 \ldots, 0.2872063 \ldots)$ with $f_2(x_2, y_2) = 1.751853 \ldots$.

On the boundary of $E$ we have
- $f_2(x, 0) = x^3 + \frac{2}{\sqrt{5}}\sqrt{1 - x^2} \leq 1.13666 \ldots$ for $0 \leq x \leq 1$, obtained for $x = 0.94941 \ldots$;
- $f_2(0, y) = \frac{2}{\sqrt{5}}\sqrt{1 - 3y^2} \leq \frac{2}{\sqrt{5}} < 1$ for $0 \leq y \leq 1/\sqrt{3}$;
- $f_2(1, y) = f_2(1, 0) = 1$.

Summarizing the above analysis brings the conclusion that
\[ |\gamma_3| \leq f_1(x_1, y_1) = 0.5566178 \ldots. \]

\[ \square \]

3. Coefficient difference

We now give upper bound of $|a_4| - |a_3|$ over the class $S$ which improves the bound $2.103299 \ldots$ given in [13].

Theorem 2. Let $f \in S$ and be given by (1). Then
\[ |a_4| - |a_3| \leq 1.751853 \ldots \]
Using all the considerations, we conclusion that
\[ |a_4| - |a_3| \leq f_2(x_2, y_2) = 1.751853 \ldots \]

\[ \Box \]

4. Generalized Zalcman conjecture

In this section we consider the generalized Zalcman conjecture in the case \( n = 2 \) and \( m = 3 \).

**Theorem 3.** If \( f \in S \) is given by (1), then
\[ |a_{2a3} - a_4| \leq 2.10064 \ldots \]

**Proof.** Using (7) we have
\[ |a_{2a3} - a_4| = \left| 2\omega_{33} + 4\omega_{11}\omega_{13} - \frac{8}{3}\omega_{11}^3 \right|, \]
and further by (8)
\[ |a_{2a3} - a_4| = \left| 2\omega_{15} + (2\omega_{12} - \omega_{11}^2)\omega_{11} - \omega_{11}^3 \right| \leq 2|\omega_{15}| + |2\omega_{12} - \omega_{11}^2||\omega_{11}| + |\omega_{11}|^3. \]

Since \( |2\omega_{12} - \omega_{11}^2| = |a_3 - a_2^2| \leq 1 \) (see [17, p.5]) for the class \( S \) and using (10), from (11) we obtain
\[ |a_{2a3} - a_4| \leq x + x^3 + \frac{2}{\sqrt{3}}\sqrt{1 - x^2 - 3y^2} \equiv f_3(x, y), \]
where we put \( |\omega_{11}| = x, |\omega_{13}| = y \) and \((x, y) \in \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{\sqrt{3}}\sqrt{1 - x^2}\} \equiv E \). Since \( \frac{\partial f_3}{\partial x} = \frac{x + x^3 + \frac{2}{\sqrt{3}}\sqrt{1 - x^2 - 3y^2}}{\sqrt{3}\sqrt{1 - x^2 - 3y^2}} = 0 \) if, and only if, \( y = 0 \), we realize that \( f_3 \) has no singular points inside \( E \). On the boundary we have
- \( f_3(x, 0) = x + x^3 + \frac{2}{\sqrt{3}}\sqrt{1 - x^2} \leq 2.10064 \ldots \) for \( 0 \leq x \leq 1 \), obtained for \( x = 0.9740 \ldots \);
- \( f_3(0, y) = \frac{2}{\sqrt{3}}\sqrt{1 - 3y^2} \leq \frac{2}{\sqrt{3}} < 1 \) for \( 0 \leq y \leq 1/\sqrt{3} \);
- \( f_3(1, y) = f_3(1, 0) = 2 \);
- \( f_3 \left( x, \frac{1}{\sqrt{3}}\sqrt{1 - x^2} \right) = x + x^3 \leq 2 \) for \( 0 \leq x \leq 1 \).

Finally
\[ |a_{2a3} - a_4| \leq 2.10064 \ldots \]

\[ \Box \]

**Remark 1.** We believe that \( |a_{2a3} - a_4| \leq 2 \) is true for the class \( S \).
5. The second and the third Hankel determinant

**Theorem 4.** Let \( f \in S \) be given by (11). Then

(i) \( |H_2(2)| \leq 1.3614356 \ldots \);
(ii) \( |H_3(1)| \leq 1.83056 \ldots \).

**Proof.**

(i) Since by (17),

\[
H_2(2) = 4\omega_{11}\omega_{33} + 4\omega_{13}^2 - 4\omega_{13}^2 - \frac{7}{3}\omega_{11}^4, \\
\]

then using (18) we have

\[
H_2(2) = 4\omega_{11}\omega_{15} - 4\omega_{13}^2 - \omega_{11}^4,
\]

which implies

\[
|H_2(2)| \leq 4|\omega_{11}||\omega_{15}| + 4|\omega_{13}|^2 + |\omega_{11}|^4.
\]

This, together with relation (10), implies

\[
|H_2(2)| \leq \frac{4}{\sqrt{5}}|\omega_{11}|\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2 + 4|\omega_{13}|^2 + |\omega_{11}|^4} \equiv f_4(|\omega_{11}|, |\omega_{13}|),
\]

where

\[
f_4(x, y) = \frac{4}{\sqrt{5}}x\sqrt{1 - x^2 - 3y^2 + x^4 + 4y^2},
\]

\(|\omega_{11}| = x, |\omega_{13}| = y \) and \((x, y) \in E \) (set \( E \) is defined in the proofs of the previous theorems).

The system

\[
\begin{cases}
\frac{\partial f_4}{\partial x} = 0 \\
\frac{\partial f_4}{\partial y} = 0
\end{cases}
\]

has only one solution inside region \( E \), that is \((x_4, y_4) = (\sqrt{\frac{11}{30}}, \sqrt{\frac{281}{1500}}) = (0.60553 \ldots, 0.395108 \ldots) \) such that \( f(x_4, y_4) = 1079/900 = 1.19888 \ldots \).

On the boundary we have

- \( f_4(x, 0) = x^4 + \frac{4}{\sqrt{5}}x\sqrt{1 - x^2} \leq 1.3614356 \ldots \) for \( 0 \leq x \leq 1 \), obtained for \( x = 0.918107 \ldots \);
- \( f_4(0, y) = 4y^2 \leq \frac{4}{3}(1 - x^2) \leq \frac{4}{3} = 1.333 \ldots \) for \( 0 \leq y \leq 1/\sqrt{3} \);
- \( f_4(1, y) = f_4(1, 0) = 1 \);
- \( f_4 \left( x, \frac{1}{\sqrt{3}}\sqrt{1 - x^2} \right) = x^4 + \frac{4}{3}(1 - x^2) \leq \frac{4}{3} \) for \( 0 \leq x \leq 1 \).

By using all this facts, we conclude that

\[
|H_2(2)| \leq 1.3614356 \ldots
\]

(ii) By using (17), after some transformations, we have

\[
H_3(1) = (2\omega_{13} - \omega_{11}^2) \left( 2\omega_{35} + \omega_{13}^2 - 2\omega_{11}^2\omega_{13} \right) - \left( 2\omega_{33} - \frac{2}{3}\omega_{11}^3 \right)^2
\]
and from here

|H_3(1)| \leq \left|\frac{2\omega_{13} - \omega_{11}^2}{B_1}\right|\left|\frac{2\omega_{35} + \omega_{13}^2 - 2\omega_{11}\omega_{13}}{B_2}\right| + \left|\frac{2\omega_{33} - \frac{2}{3}\omega_{11}^2}{B_2}\right|.

(12)

Since for the class \(S\), \(|a_3 - a_2^2| \leq 1\) (see [17], p. 5), follows \(|2\omega_{13} - \omega_{11}^2| = |a_3 - a_2^2| \leq 1\), and further

\[B_1 \leq |2\omega_{35} + \omega_{13}^2 - 2\omega_{11}\omega_{13}|,
\]
i.e.,

(13)

\[B_1 \leq 2|\omega_{35}| + |\omega_{13}|^2 + 2|\omega_{11}|^2|\omega_{13}|.
\]

From (6), for \(x = 0\) and \(x = 1\) we obtain

\[|\omega_{13}|^2 + 3|\omega_{33}|^2 + 5|\omega_{35}|^2 \leq \frac{1}{3},
\]

and from here

\[|\omega_{35}| \leq \frac{1}{\sqrt{15}} \sqrt{1 - 3|\omega_{13}|^2 - 9|\omega_{33}|^2} \leq \frac{1}{\sqrt{15}} \sqrt{1 - 3|\omega_{13}|^2}.
\]

By using this, from (13) we obtain

(14)

\[B_1 \leq \frac{2}{\sqrt{15}} \sqrt{1 - 3|\omega_{13}|^2 + |\omega_{13}|^2 + 2|\omega_{11}|^2|\omega_{13}| =: \varphi_1(|\omega_{11}|, |\omega_{13}|),
\]

where

\[\varphi_1(x, y) = 2x^2y + y^2 + \frac{2}{\sqrt{15}} \sqrt{1 - 3y^2},
\]

with \(0 \leq x = |\omega_{11}| \leq 1\) and \(0 \leq y = |\omega_{13}| \leq \frac{1}{\sqrt{3}} \sqrt{1 - x^2}\). It is evident that the function \(\varphi_1\) has no interior critical points, while on the boundary we have

- \(\varphi_1(x, 0) = \frac{2}{\sqrt{15}} = 0.51639 \ldots\);
- \(\varphi_1(0, y) = y^2 + \frac{2}{\sqrt{15}} \sqrt{1 - 3y^2} \leq \frac{8}{15} = 0.533 \ldots\) (we can put here \(s = \sqrt{1 - 3y^2}\), etc.);
- \(\varphi_1 \left(\frac{1}{\sqrt{3}}, \sqrt{1 - x^2}\right) = \frac{2}{\sqrt{3}} x^2 \sqrt{1 - x^2} + \frac{1}{3}(1 - x^2) + \frac{2}{\sqrt{15}} x \leq 0.977238 \ldots\),

attained for \(x = 0.813 \ldots\).

For the analysis of the expression \(B_2\), we use the last relation from (7) and we get \(\omega_{33} = \omega_{15} - \omega_{11}\omega_{13} + \frac{1}{3}\omega_{11}^3\). This, together with (12), leads to \(B_2 = 4|\omega_{15} - \omega_{11}\omega_{13}|^2\). Therefore

\[B_2 \leq 4 \left(|\omega_{15}| + |\omega_{11}|\omega_{13}\right)^2
\]

\[= 4 \left(\frac{1}{\sqrt{3}} \sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2 + |\omega_{11}|\omega_{13}}\right)^2
\]

\[= 4\varphi_2^2(|\omega_{11}|, |\omega_{13}|),
\]

where \(\varphi_2(x, y) = \frac{1}{\sqrt{3}} \sqrt{1 - x^2 - 3y^2 + xy}\), with \(0 \leq x \leq 1\) and \(0 \leq y \leq \frac{1}{\sqrt{3}} \sqrt{1 - x^2}\).
From the system of equations \[ \varphi'_2(x, y) = 0 \text{ and } \varphi'_2(x, y) = 0 \] we receive the unique interior critical point \( \left( \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \) and \( \varphi_2 \left( \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) = \frac{4}{5\sqrt{3}} = 0.46188\ldots \) On the boundary we have

- \( \varphi_2(x, 0) = \frac{\sqrt{5}}{\sqrt{3}} \sqrt{1 - x^2} \leq \frac{1}{\sqrt{3}} = 0.44721\ldots ; \)
- \( \varphi_2(0, y) = \frac{\sqrt{5}}{\sqrt{3}} \sqrt{1 - 3y^2} \leq \frac{1}{\sqrt{3}} < \frac{1}{\sqrt{5}} \), for \( 0 \leq x \leq 1 \).

From all this considerations we conclude that

\begin{equation}
B_2 \leq 4 \cdot \left( \frac{4}{5\sqrt{3}} \right)^2 = \frac{64}{75} = 0.85(3) .
\end{equation}

Finally, from \( (12) \):

\[ |H_3(1)(f^{-1})| \leq B_1 + B_2 \leq 0.977238\ldots + 0.85(3) = 1.83056\ldots . \]

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