EQUIVALENCE OF LITTLEWOOD–PALEY SQUARE FUNCTION AND AREA FUNCTION CHARACTERIZATIONS OF WEIGHTED PRODUCT HARDY SPACES ASSOCIATED TO OPERATORS

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Abstract. Let $L_1$ and $L_2$ be non-negative self-adjoint operators acting on $L^2(X_1)$ and $L^2(X_2)$, respectively, where $X_1$ and $X_2$ are spaces of homogeneous type. Assume that $L_1$ and $L_2$ have Gaussian heat kernel bounds. This paper aims to study some equivalent characterizations of the weighted product Hardy spaces $H^p_{w,L_1,L_2}(X_1 \times X_2)$ associated to $L_1$ and $L_2$, for $p \in (0, \infty)$ and the weight $w$ belongs to the product Muckenhoupt class $A_\infty(X_1 \times X_2)$. Our main result is that the spaces $H^p_{w,L_1,L_2}(X_1 \times X_2)$ introduced via area functions can be equivalently characterized by Littlewood–Paley $g$-functions, Littlewood–Paley $g^\ast_{\lambda_1,\lambda_2}$-functions, and Peetre type maximal functions, without any further assumptions beyond the Gaussian upper bounds on the heat kernels of $L_1$ and $L_2$. Our results are new even in the unweighted product setting.

1. Introduction

The theory of Hardy spaces has been a successful story in modern harmonic analysis in the last fifty years. In the classical case of the Euclidean space $\mathbb{R}^n$, it is well known that among other equivalent characterizations the Hardy space $H^p(\mathbb{R}^n)$ are characterized by area functions, by Littlewood–Paley $g$-functions and by atomic decomposition \cite{14, 24}. Concerning Hardy spaces $H^p(X)$ on a space of homogeneous type $X$, a new approach to show the equivalence between characterizations of $H^p(X)$ by area functions and $g$-functions is to use the Plancherel–Polya type inequality, which requires the Hölder continuity and cancellation conditions \cite{8}. About the more recent Hardy spaces $H^p_L(X)$ associated to an operator $L$ on a space of homogeneous type $X$, one used to need extra assumptions to show that the characterizations by area functions and by Littlewood-Paley $g$-functions are equivalent, for example, Hölder continuity was assumed in \cite{10} and Moser type estimate in \cite{12}. Only recently, the equivalence of the characterizations of $H^p_L(X)$ by area functions and by Littlewood–Paley $g$-functions was obtained in \cite{19} under no further assumption beyond the Gaussian heat kernel bounds. Actually, the work in \cite{19} was done in the weighted setting.

The aim of the current paper is to prove the equivalence between the characterizations of the weighted product Hardy spaces $H^p_{w,L_1,L_2}(X_1 \times X_2)$ in terms of the area functions and Littlewood–Paley square functions, see Theorems 1.4 and 1.5 where we assume only that the operators $L_1$ and $L_2$ are non-negative self-adjoint and have Gaussian upper bounds on their heat kernels. This extends the main result in \cite{19} to the product setting. The strength of our results is that not only they are new for the setting of product spaces and covers larger classes of operators $L_1$ and $L_2$ but also recover a number of known results whose proofs rely on extra regularity of the semigroups. In particular, our Theorems 1.4 and 1.5

(i) give a direct proof for the equivalent characterizations via Littlewood–Paley square functions of the classical product Hardy space by Chang–Fefferman in \cite{6},

(ii) provide a new proof of equivalent characterizations via Littlewood–Paley square functions of the product Hardy spaces on spaces of homogeneous type in \cite{18} whose proofs required the Hölder continuity and cancellation condition,

(iii) provide the missing characterizations of product Hardy spaces via Littlewood–Paley square functions in the setting developed in \cite{9} and \cite{12}, and
(iv) recover the recent related known results in the setting of Bessel operators in [11] whose proofs relied on the Hölder regularity, and results for Bessel Schrödinger operators in [2] whose proofs used the Moser type inequality.

For more details and explanations of (iii) and (iv), we refer to Section 4.

We now recall some basic facts concerning spaces of homogeneous type. Let \((X, \rho)\) be a metric space, and \(\mu\) be a positive Radon measure on \(X\). Write \(V(x, r) := \mu(B(x, r))\), where \(B(x, r)\) denotes the open ball centered at \(x\) with radius \(r\). We say that \((X, \rho, \mu)\) is a space of homogeneous type if it satisfies the volume doubling property:

\[
V(x, 2r) \leq V(x, r)
\]

for all \(x \in X\) and \(r > 0\). An immediate consequence of (1.1) is that there exist constants \(C\) and \(n\) such that

\[
V(x, \lambda r) \leq C\lambda^n V(x, r)
\]

for all \(x \in X\), \(r > 0\) and \(\lambda \geq 1\). The constant \(n\) plays the role of an upper bound of the dimension, though it need not even be an integer, and we want to take \(n\) as small as possible. There also exist constants \(C\) and \(D\), \(0 \leq D \leq n\), so that

\[
V(y, r) \leq C \left(1 + \frac{\rho(x, y)}{r}\right)^D V(x, r)
\]

uniformly for all \(x, y \in X\) and \(r > 0\). Indeed, property (1.3) with \(D = n\) is a direct consequence of \([12]\).

In the case where \(X\) is the Euclidean space \(\mathbb{R}^n\) or a Lie group of polynomial growth, \(D\) can be chosen to be 0.

Throughout this paper, we assume that, for \(i = 1, 2\), \((X_i, \rho_i, \mu_i)\) is a space of homogeneous type with \(\mu(X_i) = \infty\). The constant \(n\) (resp. \(D\)) in (1.2) (resp. (1.3)) for \((X_i, \rho_i, \mu_i)\) is denoted by \(n_i\) (resp. \(D_i\)). Let \(L_i, i = 1, 2\), be a linear operator on \(L^2(X_i, d\mu_i)\) satisfying the following properties:

(H1) Each \(L_i\) is a non-negative self-adjoint operator on \(L^2(X_i, d\mu_i)\);

(H2) The kernel of the semigroup \(e^{-tL_i}\), denoted by \(p_t^{(i)}(x, y)\), is a measurable function on \(X_i \times X_i\) and obeys a Gaussian upper bound, that is,

\[
\left|p_t^{(i)}(x, y)\right| \leq \frac{C_i}{V(x, \sqrt{t})} \exp\left(-\frac{\rho_i(x, y)^2}{c_it}\right)
\]

for all \(t > 0\) and a.e. \((x, y) \in X_i \times X_i\), where \(C_i\) and \(c_i\) are positive constants, for \(i = 1, 2\).

Definition 1.1. Let \(\Phi_1, \Phi_2 \in \mathcal{S}(\mathbb{R})\).

a) Given a function \(f \in L^2(X_1 \times X_2)\), we define the product type Littlewood–Paley \(g\)-function 
\(g_{\Phi_1, \Phi_2, L_1, L_2}(f)\) associated to \(L_1\) and \(L_2\) by

\[
g_{\Phi_1, \Phi_2, L_1, L_2}(f)(x_1, x_2) := \left(\int_0^\infty \int_0^\infty \left|\Phi_1(t_1 \sqrt{L_1}) \otimes \Phi_2(t_2 \sqrt{L_2}) f(x_1, x_2)\right|^2 dt_1 dt_2\right)^{1/2}.
\]

b) The product type area function \(S_{\Phi_1, \Phi_2, L_1, L_2}(f)\) associated to \(L_1\) and \(L_2\) is defined by

\[
S_{\Phi_1, \Phi_2, L_1, L_2}(f)(x_1, x_2) := \left(\int_{\Gamma_i(x_1) \times \Gamma_i(x_2)} \left|\Phi_1(t_1 \sqrt{L_1}) \otimes \Phi_2(t_2 \sqrt{L_2}) f(y_1, y_2)\right|^2 \frac{dt_1 dt_2}{V(x_1, t_1) V(x_2, t_2)}\right)^{1/2},
\]

where \(\Gamma_i(x_i) := \{(y_1, t_1) \in X_i \times (0, \infty) : \rho_i(x_1, y_1) < t_1\}\) for \(i = 1, 2\).

c) For \(\lambda_1, \lambda_2, t_1, t_2 > 0\), the product Peetre type maximal functions associated to \(L_1\) and \(L_2\) is defined by

\[
\left[\Phi_1(t_1 \sqrt{L_1}) \otimes \Phi_2(t_2 \sqrt{L_2})\right]_{\lambda_1, \lambda_2} f(x_1, x_2) := \text{ess sup}_{(y_1, y_2) \in X_1 \times X_2} \left|\Phi_1(t_1 \sqrt{L_1}) \otimes \Phi_2(t_2 \sqrt{L_2}) f(y_1, y_2)\right|^{\lambda_1}, (x_1, x_2) \in X_1 \times X_2.
\]

If \((y_1, y_2) \in X_1 \times X_2\),

d) The product type Littlewood–Paley \(g_{\lambda_1, \lambda_2}\)-function associated to \(L_1\) and \(L_2\) is defined by

\[
g^*_{\Phi_1, \Phi_2, L_1, L_2, \lambda_1, \lambda_2}(f)(x_1, x_2)
\]
Definition 1.3. Let \( w \) be a given \( \mathcal{A}_p(\mathbb{R}) \) function, if there is a constant \( C \) such that for all balls \( B_1 \subset X_1 \) and \( B_2 \subset X_2 \),
\[
\left( \int_{B_1 \times B_2} \frac{\left| f(x_1, x_2) \right|^p}{w(x_1, x_2)} \, dx_1 \, dx_2 \right)^{1/p} \leq C \cdot \left( \int_{B_1 \times B_2} \frac{\left| f(x_1, x_2) \right|^p}{w(x_1, x_2)} \, dx_1 \, dx_2 \right)^{1/p}.
\]
Following [15, 16], we introduce product Muckenhoupt weights on spaces of homogeneous type.

Theorem 1.4. A non-negative locally integrable function \( w \) on \( X_1 \times X_2 \) is said to belong to the product Muckenhoupt class \( A_p(X_1 \times X_2) \) for a given \( p \in (1, \infty) \), if there is a constant \( C \) such that for all balls \( B_1 \subset X_1 \) and \( B_2 \subset X_2 \),
\[
\left( \frac{1}{\mu_1(B_1) \mu_2(B_2)} \int_{B_1 \times B_2} \frac{1}{w(x_1, x_2)} \, dx_1 \, dx_2 \right) \left[ \int_{B_1 \times B_2} w(x_1, x_2) \, dx_1 \, dx_2 \right]^{1/(p-1)} \leq C.
\]

The class \( \mathcal{A}_p(X_1 \times X_2) \) is defined to be the collection of all non-negative locally integrable functions \( w \) on \( X_1 \times X_2 \) such that
\[
\left( \frac{1}{\mu_1(B_1) \mu_2(B_2)} \int_{B_1 \times B_2} \frac{1}{w(x_1, x_2)} \, dx_1 \, dx_2 \right) \left[ \int_{B_1 \times B_2} w(x_1, x_2) \, dx_1 \, dx_2 \right]^{1/(p-1)} \leq C.
\]

We let \( A_{p,w}(X_1 \times X_2) := \bigcup_{1 \leq p < \infty} A_p(X_1 \times X_2) \) and, for any \( w \in A_{p,w}(X_1 \times X_2) \), define
\[
q_w := \inf \{ q \in [1, \infty) : w \in A_q(X_1 \times X_2) \},
\]
the critical index for \( w \) (see, for instance, [16]). For \( 1 < p < \infty \), the weighted Lebesgue space \( L^p_w(X_1 \times X_2) \) is defined to be the collection of all measurable functions \( f \) on \( X_1 \times X_2 \) for which
\[
\left\| \int_{X_1 \times X_2} |f(x_1, x_2)|^p w(x_1, x_2) \, dx_1 \, dx_2 \right\|^{1/p} < \infty.
\]

We next introduce a class of functions on \( \mathbb{R} \) which will play a significant role in our formulation.

Definition 1.5. A function \( \Phi \in \mathcal{S}(\mathbb{R}) \) is said to belong to the class \( \mathcal{A}(\mathbb{R}) \) if it satisfies the Tauberian condition, namely,
\[
|\Phi(\lambda)| > 0 \quad \text{on} \quad \{ \epsilon/2 < |\lambda| < 2\epsilon \}
\]
for some \( \epsilon > 0 \).

Now we are ready to state our main results.

Theorem 1.4. Let \( \Phi_1, \Phi_2, \tilde{\Phi}_1, \tilde{\Phi}_2 \in \mathcal{A}(\mathbb{R}) \) be even functions satisfying
\[
\Phi_1(0) = \Phi_2(0) = \tilde{\Phi}_1(0) = \tilde{\Phi}_2(0) = 0.
\]
Let \( p \in (0, \infty) \) and \( w \in A_{p,w}(X_1 \times X_2) \). Then there exists a constant \( C = C(p, w, \Phi_1, \Phi_2, \tilde{\Phi}_1, \tilde{\Phi}_2) \) such that for all \( f \in L^2(X_1 \times X_2) \),
\[
C^{-1} \| g_{\tilde{\Phi}_1, \tilde{\Phi}_2, L_2} \|_{L_X^p(X_1 \times X_2)} \leq \| g_{\Phi_1, \tilde{\Phi}_2, L_1, L_2} \|_{L_X^p(X_1 \times X_2)} \leq C \| g_{\Phi_1, \Phi_2, L_1, L_2} \|_{L_X^p(X_1 \times X_2)}.
\]

Theorem 1.5. Let \( \Phi_1, \Phi_2 \in \mathcal{A}(\mathbb{R}) \) be even functions. Let \( p \in (0, \infty) \), \( \lambda_i > \frac{2p}{\min(p, 2)} \), \( i = 1, 2 \). Then for \( f \in L^2(X_1 \times X_2) \) we have the following (quasi)-norm equivalence:
\[
\left\| \int_0^\infty \int_0^\infty \left[ \Phi_1(t_1 \sqrt{L_1}) \otimes \Phi_2(t_2 \sqrt{L_2}) \right] \, dt_1 \, dt_2 \right\|_{L^p_{\psi_1}(X_1 \times X_2)} \sim \left\| \Phi_1(t_1 \sqrt{L_1}) \otimes \Phi_2(t_2 \sqrt{L_2}) \right\|_{L^p_{\psi_1}(X_1 \times X_2)}.
\]
Definition 1.6. Let $p \in (0, \infty)$, $w \in A_{\infty}(X_1 \times X_2)$, and $\Phi_1, \Phi_2 \in A(\mathbb{R})$ be even functions satisfying
\[ \Phi_1(0) = \Phi_2(0) = 0. \]
The weighted product Hardy space $H^p_{w,L_1,L_2}(X_1 \times X_2)$ associated to $L_1$ and $L_2$ is defined to be the completion of the set
\[ \{ f \in L^2(X_1 \times X_2) : S_{\Phi_1,\Phi_2,L_1,L_2}(f) \in L^p_w(X_1 \times X_2) \} \]
with respect to the (quasi-)norm
\[ \| f \|_{H^p_{w,L_1,L_2}(X_1 \times X_2)} := \| S_{\Phi_1,\Phi_2,L_1,L_2}(f) \|_{L^p_w(X_1 \times X_2)}. \]

Remark 1.7. Combining Theorems 3.4 and 3.5 we see that the definition of $H^p_{w,L_1,L_2}(X_1 \times X_2)$ is independent of the choice of the even functions $\Phi_1, \Phi_2$, as long as $\Phi_1, \Phi_2 \in A(\mathbb{R})$ and satisfy $\Phi_1(0) = \Phi_2(0) = 0$. In particular, if we choose $\Phi_1(\lambda) = \Phi_2(\lambda) = \lambda^2 e^{-\lambda^2}$ for $\lambda \in \mathbb{R}$, then the (quasi-)norm of $H^p_{w,L_1,L_2}(X_1 \times X_2)$ can be written as
\[
\| f \|_{H^p_{w,L_1,L_2}(X_1 \times X_2)} := \left( \int_{X_1 \times X_2} \left| (t_1^2 L_1 e^{-t_1^2 L_1}) \otimes (t_2^2 L_2 e^{-t_2^2 L_2}) f_y(y_1,y_2) \right|^2 \frac{d\mu(y_1) d\mu(y_2) dt_2}{V(x_1,t_1) V(x_2,t_2)} \right)^{\frac{1}{2}}.
\]
Furthermore, from Theorem 1.5 we see that each quantity in (1.5) can be used as an equivalent (quasi-)norm of the space $H^p_{w,L_1,L_2}(X_1 \times X_2)$.

As mentioned above, we make no further assumption on the heat kernel of $L_1$ or $L_2$ beyond the Gaussian upper bounds. Thus, the approach in [10] which uses a Plancherel-Polya type inequality and the approach in [12] which uses a discrete characterization cannot be applied directly to our setting. To achieve our goal, we will follow the approach in [3, 4, 21], whose key ingredient is a sub-mean value property; see Lemma 3.4 below. This approach has recently been used in [19] to derive the equivalence of Littlewood–Paley $g$-function and area function characterisations of one-parameter Hardy spaces associated to operators. However, the Littlewood–Paley $g$-function and area function in [19] are only defined via the heat semigroup, which are less general than those defined in the current paper.

2. Preliminaries

In this section we collect some facts and technical results which will be needed in the subsequent section. We start by noting that, if $(X, \rho, \mu)$ is a space of homogeneous type, then for any $N > n$, there exists a constant $C = C(N)$ such that
\[
\int_X \left( 1 + \frac{\rho(x,y)}{t} \right)^{-N} d\mu(y) \leq CV(x,t)
\]
for all $x \in X$ and $t > 0$.

Lemma 2.1. Assume that $(X, \rho, \mu)$ is a space of homogeneous type and $L$ is a non-negative self-adjoint operator on $L^2(X, d\mu)$ whose heat kernel obeys the Gaussian upper bound. Let $\Phi \in S(\mathbb{R})$ be even functions. Then for every $N > 0$, there exists a constant $C = C(\Phi, N)$ such that the kernel $K_{\Phi(t\sqrt{L})}(x, y)$ of the operator $\Phi(t\sqrt{L})$ satisfies
\[
|K_{\Phi(t\sqrt{L})}(x, y)| \leq \frac{C}{V(x,t)} \left( 1 + \frac{\rho(x,y)}{t} \right)^{-N}.
\]
Proof. For the proof, we refer to [5, Lemma 2.3]. See also [23, Lemma 2.1].

Lemma 2.2. Assume that $(X, \rho, \mu)$ is a space of homogeneous type and $L$ is a non-negative self-adjoint operator on $L^2(X, d\mu)$ whose heat kernel obeys the Gaussian upper bound. Let $\Phi, \Psi \in S(\mathbb{R})$ be even functions and let $\Psi$ satisfy
\[
\Psi^{(\nu)}(0) = 0, \quad \nu = 0, 1, \cdots, m
\]
for some positive odd integer $m$. Then for every $N > 0$, there exists a constant $C = C(\Phi, \Psi, N, m)$ such that for all $s \geq t > 0$,

$$
(2.3) \quad \left| K_{\Phi(s\sqrt{t})}(x,y) \right| \leq C \left( \frac{t}{s} \right)^{m+1} \frac{1}{V(x,s)} \left( 1 + \frac{\rho(x,y)}{s} \right)^{-N}.
$$

**Proof.** First note that the property (2.2) implies that the function $\lambda \mapsto \lambda^{-(m+1)} \Psi(\lambda)$ is an even function, smooth at 0, and belongs to $\mathcal{S}(\mathbb{R})$. We set $\Phi_m(\lambda) := \lambda^{m+1} \Psi(\lambda)$ and $\Psi_m(\lambda) := \lambda^{-(m+1)} \Psi(\lambda)$ for $\lambda \in \mathbb{R}$. Then both $\Phi_m$ and $\Psi_m$ are even functions and belong to $\mathcal{S}(\mathbb{R})$. Since

$$
\Phi(s\sqrt{t}) \Psi(t\sqrt{t}) = \left( \frac{t}{s} \right)^{m+1} \Phi_m(s\sqrt{t}) \Psi_m(t\sqrt{t}),
$$

it follows from Lemma 2.1 that

$$
(2.4) \quad \left| K_{\Phi(s\sqrt{t})}(x,y) \right| = \left( \frac{t}{s} \right)^{m+1} \left| K_{\Phi_m(s\sqrt{t})}(x,y) \right| = \left( \frac{t}{s} \right)^{m+1} \int_X \frac{1}{V(x,s)} \left( 1 + \frac{\rho(x,z)}{s} \right)^{-N} \frac{1}{V(y,t)} \left( 1 + \frac{\rho(z,y)}{t} \right)^{-(N+1)} d\mu(z).
$$

For $s \geq t > 0$, we have

$$
\left( 1 + \frac{\rho(x,z)}{s} \right)^{-N} \left( 1 + \frac{\rho(z,y)}{t} \right)^{-(N+1)} \leq \left( 1 + \frac{\rho(x,y)}{s} \right)^{-N}.
$$

This along with (2.1) yields

$$
\int_X \left( 1 + \frac{\rho(x,z)}{s} \right)^{-N} \left( 1 + \frac{\rho(z,y)}{t} \right)^{-(N+1)} d\mu(z)
$$

$$
\leq C \left( 1 + \frac{\rho(x,y)}{s} \right)^{-N} V(y,t).
$$

Combining (2.4) and (2.5) we obtain (2.3). \qed

**Lemma 2.3.** Suppose $\Phi \in \mathcal{A}(\mathbb{R})$ is an even function. Then there exist even functions $\Psi, \Upsilon, \Theta \in \mathcal{S}(\mathbb{R})$ such that

$$
\text{supp } \Upsilon \subset \{ |\lambda| \leq 2\varepsilon \},
$$

$$
\text{supp } \Theta \subset \{ \varepsilon/2 \leq |\lambda| \leq 2\varepsilon \}
$$

and

$$
\Psi(\lambda) \Upsilon(\lambda) + \sum_{k=1}^{\infty} \Phi(2^{-2k}\lambda) \Theta(2^{-2k}\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R},
$$

where $\varepsilon$ is a constant from (1.8).

**Proof.** Define $\Psi(\lambda) := e^{-\lambda^2}$, $\lambda \in \mathbb{R}$. Obviously, $\Psi \in \mathcal{S}(\mathbb{R})$ and $\Psi$ is even. Choose nonnegative even functions $\Omega, \Gamma \in \mathcal{S}(\mathbb{R})$ such that

$$
\Omega(\lambda) \neq 0 \iff |\lambda| < 2\varepsilon,
$$

$$
\Gamma(\lambda) \neq 0 \iff \varepsilon/2 < |\lambda| < 2\varepsilon.
$$

Then we set

$$
(2.6) \quad \Xi(\lambda) := \Psi(\lambda) \Omega(\lambda) + \sum_{k=1}^{\infty} \Phi(2^{-k}\lambda) \Gamma(2^{-k}\lambda), \quad \lambda \in \mathbb{R}.
$$
From the properties of $\Phi$, $\Psi$, $\Omega$, and $\Gamma$ it follows that $\Xi$ is strictly positive on $\mathbb{R}$. In addition, from the properties of $\Omega$ and $\Gamma$ we see that for any fixed $\lambda_0 \in \mathbb{R}\setminus\{0\}$, the number of those $k$’s for which $\Phi(2^{-k}\lambda)\Gamma(2^{-k}\lambda)$ do not vanish identically in $(4\lambda_0,2^{-k}\lambda)$ is no more than 4, which implies that $\Xi$ is smooth in $(\lambda_0,2^{-k}\lambda)$ and hence $\Xi \in C^\infty(\mathbb{R}\setminus\{0\})$. It is obvious that $\Xi$ is also smooth at the origin 0. Therefore $\Xi \in C^\infty(\mathbb{R})$. Now define the functions $\Upsilon$ and $\Theta$ respectively by

$$
\Upsilon(\lambda) := \frac{\Omega(\lambda)}{\Xi(\lambda)} \quad \text{and} \quad \Theta(\lambda) := \frac{\Gamma(\lambda)}{\Xi(\lambda)}.
$$

Then it is straightforward to verify that $\Psi$, $\Upsilon$, and $\Theta$ satisfy the desired properties.

Lemma 2.4. Suppose $\Phi \in A(\mathbb{R})$ is an even function. Then there exists an even functions $\Theta \in S(\mathbb{R})$ such that

$$
\text{supp} \Theta \subset \{|\xi|/2 \leq |\lambda| \leq 2|\xi|\}
$$

and

$$
\sum_{k=-\infty}^{\infty} \Phi(2^{-k}\lambda)\Theta(2^{-k}\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R}\setminus\{0\},
$$

where $\xi$ is a constant from (1.8).

Proof. The proof is analogous to that of Lemma 2.3 and thus we omit the details.

Lemma 2.5. Assume that $(X, \rho, \mu)$ is a space of homogeneous type with $\mu(X) = \infty$ and $L$ is a non-negative self-adjoint operator on $L^2(X, \mu)$ whose heat kernel obeys the Gaussian upper bound. Let $\{E(\lambda) : \lambda \geq 0\}$ be spectral resolution of $L$. Then the spectral measure of the set $\{0\}$ is zero, i.e., the point $\lambda = 0$ may be neglected in the spectral resolution.

Proof. Assume by contradiction that $E(\{0\}) \neq 0$, then there exists $g \in L^2(X)$ such that $f := E(\{0\})g$ is not the zero element in $L^2(X, \mu)$. Since $E(\{0\})$ is an orthogonal projection, $E(\{0\})f = E(\{0\})E(\{0\})g$. Therefore we must have $E(\{0\}) = 0$.

It follows that for all $t > 0$,

$$
e^{-tL}f = \int_{0}^{\infty} e^{-t\lambda}dE(\lambda)f = \int_{0}^{\infty} e^{-t\lambda}dE(\lambda)E(\{0\})f = \int_{\{0\}} e^{-t\lambda}dE(\lambda)f = E(\{0\})f = f.
$$

Hence, for a.e. $x \in X$ and all $t > 0$, we have

$$
|f(x)| = |e^{-tL}f(x)| \leq \int_{X} |p_{t}(x,y)||f(y)|d\mu(y)
\leq ||f||_{L^{2}(X,\mu)} \left(\int_{X} |p_{t}(x,y)|^{2}d\mu(y)\right)^{1/2}
\leq C||f||_{L^{2}(X,\mu)} \left(\int_{X} V(x,\sqrt{t})V(x,\sqrt{t})^{-1/2}ight)^{-1/2}.
$$

Since $\mu(X) = \infty$, letting $t \to \infty$ in the above yields that $f(x) = 0$. Hence $f = 0$ in $L^2(X, \mu)$, which leads to a contradiction. Therefore we must have $E(\{0\}) = 0$.

The following two lemmas are two-parameter counterparts of Lemma 2 and Lemma 3 in [21], respectively. These can be proved by slightly modifying the proofs of the corresponding one-parameter results. We omit the details here.

Lemma 2.6. ([21] Lemma 2] Let $0 < p,q < \infty$ and $\sigma_1,\sigma_2 > 0$. Let $w$ be arbitrary weight (i.e., non-negative locally integrable function) on $X_1 \times X_2$. Let $\{g_{j_1,j_2}\}_{j_1,j_2=-\infty}^{\infty}$ be a sequence of non-negative measurable functions on $X_1 \times X_2$ and put

$$
h_{j_1,j_2}(x_1,x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} 2^{-|k_1-j_1|\sigma_1}2^{-|k_2-j_2|\sigma_2}g_{k_1,k_2}(x_1,x_2)
$$

(2.7)
for \((x_1, x_2) \in X_1 \times X_2\) and \(j_1, j_2 \in \mathbb{Z}\). Then, there exists a constant \(C = C(q, \sigma_1, \sigma_2)\) such that
\[
\left\| \left\{ h_{j_1, j_2} \right\}_{j_1, j_2 = -\infty}^\infty \right\|_{L^p_w(\ell^q)} \leq C \left\| \left\{ g_{j_1, j_2} \right\}_{j_1, j_2 = -\infty}^\infty \right\|_{L^p_w(\ell^q)},
\]
where
\[
(2.8) \quad \left\| \left\{ g_{j_1, j_2} \right\}_{j_1, j_2 = -\infty}^\infty \right\|_{L^p_w(\ell^q)} := \left\| \left\{ \sum_{j_1 = -\infty}^\infty \sum_{j_2 = -\infty}^\infty |g_{j_1, j_2}(x_1, x_2)|^q \right\}^{1/q} \right\|_{L^p_w(X_1 \times X_2)}.
\]

**Lemma 2.7.** ([21 Lemma 3]) Let \(0 < r \leq 1\), and let \(\{b_{j_1, j_2}\}_{j_1, j_2 = -\infty}^\infty\) and \(\{d_{j_1, j_2}\}_{j_1, j_2 = -\infty}^\infty\) be two sequences taking values in \((0, \infty)\) and \((0, \infty)\) respectively. Assume that there exists \(N_0 > 0\) such that
\[
(2.9) \quad d_{j_1, j_2} = O(2^{j_1 N_0} 2^{j_2 N_0}), \quad j_1, j_2 \to \infty,
\]
and that for every \(N > 0\) there exists a finite constant \(C = C_N\) such that
\[
(2.10) \quad d_{j_1, j_2} \leq C_N \sum_{k_1 = j_1}^\infty \sum_{k_2 = j_2}^\infty 2^{j_1 - k_1} N_0 2^{j_2 - k_2} N b_{k_1, k_2} d_{k_1, k_2}^{1 - r}, \quad j_1, j_2 \in \mathbb{Z}.
\]
Then for every \(N > 0\),
\[
(2.11) \quad d_{j_1, j_2}^r \leq C_N \sum_{k_1 = j_1}^\infty \sum_{k_2 = j_2}^\infty 2^{j_1 - k_1} N r 2^{j_2 - k_2} N r b_{k_1, k_2}, \quad j_1, j_2 \in \mathbb{Z},
\]
with the same constants \(C_N\).

For a locally integrable function \(f\) on \(X_1 \times X_2\), the strong maximal function is defined by
\[
\mathcal{M}_w(f)(x_1, x_2) := \sup_{(x_1, x_2) \in B_1 \times B_2} \frac{1}{\mu_1(B_1) \mu_2(B_2)} \int_{B_1 \times B_2} |f(y_1, y_2)| \, d\mu_1(y_1) \, d\mu_2(y_2),
\]
where \(B_i\) runs over all balls in \(X_i\), \(i = 1, 2\). Using \([13]\) and the volume doubling property, one can easily show that if \(N_i > n_i + D_i\) for \(i = 1, 2\), then
\[
(2.12) \quad \int_{X_1 \times X_2} \prod_{i=1}^2 V(y_i, t_i)(1 + t_i^{-1}) \rho_i(x_i, y_i) N_i d\mu_1(y_1) d\mu_2(y_2) \leq C \mathcal{M}_w(f)(x_1, x_2).
\]
We will also need the following weighted vector-valued inequality for strong maximal functions on spaces of homogeneous type. See, for instance, [16] and [22].

**Lemma 2.8.** Suppose \(1 < p < \infty\), \(1 < q \leq \infty\) and \(w \in A_p(X_1 \times X_2)\). Then there exists a constant \(C\) such that
\[
\left\| \left\{ \mathcal{M}_w(f_{j_1, j_2}) \right\}_{j_1, j_2 = -\infty}^\infty \right\|_{L^p_w(\ell^q)} \leq C \left\| \left\{ f_{j_1, j_2} \right\}_{j_1, j_2 = -\infty}^\infty \right\|_{L^p_w(\ell^q)}
\]
for all sequences \(\{f_{j_1, j_2}\}_{j_1, j_2 = -\infty}^\infty\) on \(X_1 \times X_2\), where the space \(L^p_w(\ell^q)\) is defined by \([23]\).

3. Proofs of Theorems [14] and [15]

We divide the proof of Theorems [14] and [15] into a sequence of lemmas.

**Lemma 3.1.** Let \(\Phi_1, \Phi_2 \in \mathcal{S}(\mathbb{R})\) be even functions. Let \(p \in (0, \infty)\), \(w \in A_\infty(X_1 \times X_2)\), and \(\lambda_1, \lambda_2 > \frac{2p}{2p - 1}\). Then there exists a constant \(C\) such that for all \(f \in L^2(X_1 \times X_2)\),
\[
\| S_{\Phi_1, \Phi_2, L_1, L_2, \lambda_1, \lambda_2}(f) \|_{L^p_w(X_1 \times X_2)} \leq C \| S_{\Phi_1, \Phi_2, L_1, L_2}(f) \|_{L^p_w(X_1 \times X_2)}.
\]

**Proof.** This can be proved by a standard argument; see, for instance, [25] Theorem 4 in Ch. 4. We omit the details here. □
Lemma 3.2. Let $\Phi_1, \Phi_2 \in S(\mathbb{R})$ be even functions. Let $p \in (0, \infty)$, $\lambda_1, \lambda_2 > 0$, and $w$ be arbitrary weight (i.e., non-negative locally integrable function) on $X_1 \times X_2$. Then there exists a constant $C$ such that for all $f \in L^2(X_1 \times X_2)$,
$$
\|S_{\Phi_1, \Phi_2, L_1, L_2}(f)\|_{L^p_w(X_1 \times X_2)} \leq C \left( \int_0^\infty \int_0^\infty \left[ \int_0^\infty \left[ f \left( t_1 \sqrt{L_1} \right) \right] \star \left( \int_0^\infty \left[ f \left( t_2 \sqrt{L_2} \right) \right] \right) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right]^{1/2} \right).
$$

Proof. Observe that for all $\lambda_1, \lambda_2, t_1, t_2 > 0$ and all $(x_1, x_2) \in X_1 \times X_2$,
$$
\frac{1}{V(x_1, t_1) V(x_2, t_2)} \int_{B(x_1, t_1) \times B(x_2, t_2)} \frac{\Phi_1(t_1 \sqrt{L_1}) \otimes \Phi_2(t_2 \sqrt{L_2}) f(y_1, y_2)}{|(y_1, y_2)|} \frac{dy_1}{dy_2} \leq 2^{2\lambda_1} 2^{2\lambda_2} \left( \int_0^\infty \int_0^\infty \left[ f \left( t_1 \sqrt{L_1} \right) \right] \star \left( \int_0^\infty \left[ f \left( t_2 \sqrt{L_2} \right) \right] \right) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^2.
$$
Taking the norm $\int_0^\infty \int_0^\infty \left| \frac{dy_1}{dy_2} \right|$ on both sides gives the pointwise estimate
$$
\left( \int_0^\infty \int_0^\infty \left[ f \left( t_1 \sqrt{L_1} \right) \right] \star \left( \int_0^\infty \left[ f \left( t_2 \sqrt{L_2} \right) \right] \right) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^2 \leq 2^{2\lambda_1} 2^{2\lambda_2} \int_0^\infty \int_0^\infty \left[ f \left( t_1 \sqrt{L_1} \right) \right] \star \left( \int_0^\infty \left[ f \left( t_2 \sqrt{L_2} \right) \right] \right) \frac{dt_1}{t_1} \frac{dt_2}{t_2},
$$
which readily yields the desired estimate. \qed

Lemma 3.3. Suppose $\Phi_1, \Phi_2, \tilde{\Phi}_1, \tilde{\Phi}_2 \in A(\mathbb{R})$ are even functions satisfying
$$
\Phi_1(0) = \Phi_2(0) = \tilde{\Phi}_1(0) = \tilde{\Phi}_2(0) = 0.
$$
Let $p \in (0, \infty)$, $\lambda_1, \lambda_2 > 0$, and $w$ be arbitrary weight (i.e., non-negative locally integrable function) on $X_1 \times X_2$. Then there exists a constant $C$ such that for all $f \in L^2(X_1 \times X_2)$,
$$
\left( \int_0^\infty \int_0^\infty \left[ f \left( t_1 \sqrt{L_1} \right) \right] \star \left( \int_0^\infty \left[ f \left( t_2 \sqrt{L_2} \right) \right] \right) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{1/2} \leq C \left( \int_0^\infty \int_0^\infty \left[ f \left( t_1 \sqrt{L_1} \right) \right] \star \left( \int_0^\infty \left[ f \left( t_2 \sqrt{L_2} \right) \right] \right) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{1/2}.
$$

Proof. For $i = 1, 2$, since $\Phi_i \in A(\mathbb{R})$ and $\Phi_i$ is even, by Lemma 2.4 there exists an even function $\Theta_i \in S(\mathbb{R})$ such that $\text{supp} \Theta_i \subset \{ \varepsilon_i / 2 \leq |\lambda| \leq 2 \varepsilon_i \}$
$$
\sum_{k=-\infty}^{\infty} \Phi_i(2^{-k} \lambda) \Theta_i(2^{-k} \lambda) = 1 \quad \text{for} \quad \lambda \in \mathbb{R} \setminus \{0\},
$$
where $\varepsilon_i$ is the constant in the Tauberian condition (1.8) corresponding to $\Phi_i$. Hence it follows from Lemma 2.5 and the spectral theorem that for all $f \in L^2(X_1 \times X_2)$ and $t_1, t_2 \in [1, 2]$,
$$
f = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \left( \Phi_1(2^{-k_1} t_1 \sqrt{L_1}) \Theta_1(2^{-k_1} t_1 \sqrt{L_1}) \right) \otimes \left( \Phi_2(2^{-k_2} t_2 \sqrt{L_2}) \Theta_2(2^{-k_2} t_2 \sqrt{L_2}) \right) f(y_1, y_2)
$$
with convergence in the sense of $L^2(X_1 \times X_2)$ norm. Consequently, for all $j_1, j_2 \in \mathbb{Z}$, all $t_1, t_2 \in [1, 2]$ and a.e. $(y_1, y_2) \in X_1 \times X_2$,
$$
\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \tilde{\Phi}_2(2^{-j_2} t_2 \sqrt{L_2}) f(y_1, y_2)
$$
$$
= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \left( \Phi_1(2^{-k_1} t_1 \sqrt{L_1}) \Phi_2(2^{-k_1} t_1 \sqrt{L_1}) \Theta_1(2^{-k_1} t_1 \sqrt{L_1}) \right)
$$
$$
\otimes \left( \tilde{\Phi}_2(2^{-k_2} t_2 \sqrt{L_2}) \Phi_2(2^{-k_2} t_2 \sqrt{L_2}) \Theta_2(2^{-k_2} t_2 \sqrt{L_2}) \right) f(y_1, y_2)
$$
$$
= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \int_{X_1 \times X_2} K_{\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \Theta_1(2^{-k_1} t_1 \sqrt{L_1})} \left( y_1, z_1 \right)
$$
$$
\times K_{\tilde{\Phi}_2(2^{-j_2} t_2 \sqrt{L_2}) \Theta_2(2^{-k_2} t_2 \sqrt{L_2})} \left( y_2, z_2 \right).
\[ \times (\Phi_1(2^{-k_2}t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2 \sqrt{L_2})) f(z_1, z_2) d\mu_1(z_1) d\mu_2(z_2) \]

Since \( \bar{\Phi}_i \) is an even function on \( \mathbb{R} \), we have \( \bar{\Phi}_i'(0) = 0 \), for \( i = 1, 2 \). Thus \( \bar{\Phi}_i'(0) = \bar{\Phi}_i''(0) = 0 \) for \( i = 1, 2 \).

On the other hand, since \( \Theta_i \) vanishes near the origin, we have \( \Theta_i^{(\nu)}(0) = 0 \) for every non-negative integer \( \nu \). Hence it follows from Lemma 2.2 that for any positive integer \( m \) and any \( N > 0 \),

\[
\left| K_{\bar{\Phi}_i(2^{-h_1}t_1 \sqrt{L_1}) \otimes \Theta_i(2^{-h_1}t_1 \sqrt{L_2})}(y_1, z_2) \right| \\
\leq \begin{cases} 
C(\bar{\Phi}_i, \Theta_i, N)2^{-j_1 - k_1} V(y_1, 2^{-k_1}t_1)^{-1}(1 + 2^{k_1}t_1^{-1} \rho_i(y_1, z_1))^{-N}, & j_1 \geq k_i, \\
C(\bar{\Phi}_i, \Theta_i, N, m)2^{-m-j_1 - k_1} V(y_1, 2^{-k_1}t_1)^{-1}(1 + 2^{k_1}t_1^{-1} \rho_i(y_1, z_1))^{-N}, & j_1 < k_i.
\end{cases}
\]

Choose \( N \geq \max\{\lambda_1 + n_1 + 1, \lambda_3 + n_2 + 1\} \), then from (3.4) and the inequality

\[
\left| \left( \Phi_1(2^{-k_2}t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2 \sqrt{L_2}) \right) f(z_1, z_2) \right| \\
\leq \left[ \Phi_1(2^{-k_2}t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2 \sqrt{L_2}) \right]^* \lambda_1, \lambda_2 f(x_1, x_2) \\
\times (1 + 2^{k_2}t_1^{-1} \rho_1(x_1, z_1))^{\lambda_1} (1 + 2^{k_2}t_2^{-1} \rho_2(x_2, z_2))^{\lambda_2},
\]

we infer that

\[
\left\lbrack \bar{\Phi}_1(2^{-j_1}t_1 \sqrt{L_1}) \otimes \bar{\Phi}_2(2^{-j_2}t_2 \sqrt{L_2}) \right\rbrack^* \lambda_1, \lambda_2 f(x_1, x_2)
\leq \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \gamma_{j_1, k_1, j_2, k_2} \left\lbrack \Phi_1(2^{-k_2}t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2 \sqrt{L_2}) \right\rbrack^* \lambda_1, \lambda_2 f(x_1, x_2) \\
\times \operatorname{ess} \sup_{(y_1, y_2) \in X_1 \times X_2} \left\{ \prod_{i=1}^{j_1} (1 + 2^{k_1}t_1^{-1} \rho_i(x_1, z_1))^{\lambda_1} (1 + 2^{j_1}t_1^{-1} \rho_i(y_1, z_1))^{\lambda_1 + n_1 + 1} V(y_1, 2^{-j_1}t_1) \right\}.
\]

where \( j_1 \wedge k_i := \min\{j_1, k_i\} \) and

\[
\gamma_{j_1, k_1, j_2, k_2} := \begin{cases} 
2^{-2j_1 - k_1} & \text{if } j_1 \geq k_1 \text{ and } j_2 \geq k_2, \\
2^{-2j_1 - k_1} & \text{if } j_1 \geq k_1 \text{ and } j_2 < k_2, \\
2^{-mj_1 - k_1} & \text{if } j_1 < k_1 \text{ and } j_2 \geq k_2, \\
2^{-mj_1 - k_1} & \text{if } j_1 < k_1 \text{ and } j_2 < k_2.
\end{cases}
\]

Using (2.1) and the fundamental inequality

\[
(1 + 2^{k_1}t_1^{-1} \rho_i(x_1, z_1))^{\lambda_1} \leq \begin{cases} 
(1 + 2^{k_1}t_1^{-1} \rho_i(x_1, y))^{\lambda_1} (1 + 2^{k_2}t_1^{-1} \rho_i(y_1, z_1))^{\lambda_1}, & j_1 \geq k_i, \\
2^{-(k_1 - j_1)} \lambda_1 (1 + 2^{j_1}t_1^{-1} \rho_i(x_1, y))^{\lambda_1} (1 + 2^{k_2}t_1^{-1} \rho_i(y_1, z_1))^{\lambda_1}, & j_1 < k_i.
\end{cases}
\]

it follows that

\[
\left\lbrack \bar{\Phi}_1(2^{-j_1}t_1 \sqrt{L_1}) \otimes \bar{\Phi}_2(2^{-j_2}t_2 \sqrt{L_2}) \right\rbrack^* \lambda_1, \lambda_2 f(x_1, x_2)
\leq \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \gamma'_{j_1, k_1, j_2, k_2} \left\lbrack \Phi_1(2^{-k_2}t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2 \sqrt{L_2}) \right\rbrack^* \lambda_1, \lambda_2 f(x_1, x_2),
\]

where

\[
\gamma'_{j_1, k_1, j_2, k_2} := \begin{cases} 
2^{-2j_1 - k_1} & \text{if } j_1 \geq k_1 \text{ and } j_2 \geq k_2, \\
2^{-2j_1 - k_1} & \text{if } j_1 \geq k_1 \text{ and } j_2 < k_2, \\
2^{-(m - \lambda_2)j_1} & \text{if } j_1 < k_1 \text{ and } j_2 \geq k_2, \\
2^{-(m - \lambda_2)j_1} & \text{if } j_1 < k_1 \text{ and } j_2 < k_2.
\end{cases}
\]

Now let us choose \( m > \max\{\lambda_1, \lambda_2\} \) and set \( \sigma := \min\{m - \lambda_1, m - \lambda_2, 2\} \). Then (3.5) implies that

\[
\left\lbrack \bar{\Phi}_1(2^{-j_1}t_1 \sqrt{L_1}) \otimes \bar{\Phi}_2(2^{-j_2}t_2 \sqrt{L_2}) \right\rbrack^* \lambda_1, \lambda_2 f(x_1, x_2)
\leq \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} 2^{-|j_1| - k_1} 2^{-|j_2| - k_2} \sigma \left\lbrack \Phi_1(2^{-k_2}t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2 \sqrt{L_2}) \right\rbrack^* \lambda_1, \lambda_2 f(x_1, x_2).
\]


Taking on both sides the norm \( \left( \int_1^2 \int_1^2 \left\{ \left[ \hat{\Phi}_1(2^{-j_1}t_1 \sqrt{L_1}) \otimes \hat{\Phi}_2(2^{-j_2}t_2 \sqrt{L_2}) \right]_{\lambda_1, \lambda_2}^* f(x_1, x_2) \right\}^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{1/2} \) and using Minkowski’s inequality, we get
\[
\leq C \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} 2^{-|j_1-k_1|\sigma} 2^{-|j_2-k_2|\sigma} \\
\times \left( \int_1^2 \int_1^2 \left\{ \left[ \hat{\Phi}_1(2^{-k_1}t_1 \sqrt{L_1}) \otimes \hat{\Phi}_2(2^{-k_2}t_2 \sqrt{L_2}) \right]_{\lambda_1, \lambda_2}^* f(x_1, x_2) \right\}^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{1/2}.
\]

Finally, applying Lemma 2.6 in \( L^p(\ell^2) \) yields
\[
\left\| \left( \int_0^\infty \int_0^\infty \left[ \hat{\Phi}_1(t_1 \sqrt{L_1}) \otimes \hat{\Phi}_2(t_2 \sqrt{L_2}) \right]_{\lambda_1, \lambda_2}^* f \right\|^{1/2} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\|_{L^p(\ell^2)}^{1/2} \\
\leq C \left\| \left( \int_0^\infty \int_0^\infty \left[ \hat{\Phi}_1(t_1 \sqrt{L_1}) \otimes \hat{\Phi}_2(t_2 \sqrt{L_2}) \right]_{\lambda_1, \lambda_2}^* f \right\|^{1/2} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\|_{L^p(\ell^2)}^{1/2}.
\]
By symmetry, the converse inequality of (3.6) also holds. The proof of the lemma is complete. \( \Box \)

**Lemma 3.4.** Let \( \Phi_1, \Phi_2 \in \mathcal{A}(\mathbb{R}) \) be even functions. Then for any \( r > 0, \sigma > 0, \lambda_1 > D_1/2 \) and \( \lambda_2 > D_2/2, \) there exists a constant \( C \) such that for all \( f \in L^2(X_1 \times X_2), \) all \( (x_1, x_2) \in X_1 \times X_2 \) and all \( t_1, t_2 \in [1, 2], \)
\[
\left\{ \left[ \Phi_1(2^{-j_1}t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2}t_2 \sqrt{L_2}) \right]_{\lambda_1, \lambda_2}^* f(x_1, x_2) \right\}^r \\
\leq C \sum_{k_1 = j_1}^{\infty} \sum_{k_2 = j_2}^{\infty} 2^{(j_1-k_1)\sigma} 2^{(j_2-k_2)\sigma} \\
\times \int_{X_1 \times X_2} \frac{[\Phi_1(2^{-k_1}t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2 \sqrt{L_2})]^* f(z_1, z_2)}{V(z_1, 2^{-k_1}t_1)(1 + 2^{k_1}t_1^2 \rho(z_1, z_1)\lambda_1 \rho(z_2, 2^{-k_2}t_2)(1 + 2^{k_2}t_2^2 \rho(z_2, z_2))^{2\lambda_2}} \ dv_1(z_1)dv_2(z_2).
\]

**Proof.** By Lemma 2.3 for \( i = 1, 2 \) there exist even functions \( \Psi_i, \Upsilon_i, \Theta_i \in \mathcal{S}(\mathbb{R}) \) such that \( \supp \Psi_i \subset \{|\lambda| \leq 2 \varepsilon_i\}, \supp \Upsilon_i \subset \{\varepsilon_i/2 \leq |\lambda| \leq 2 \varepsilon_i\}, \) and
\[
\Psi_i(\lambda)\Upsilon_i(\lambda) + \sum_{k=1}^{\infty} \Psi_i(2^{-k_i}\lambda)\Theta_i(2^{-k_i}\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R},
\]
where \( \varepsilon_i \) is the constant in the Tauberian condition (1.18) corresponding to \( \Phi_i. \) Replacing \( \lambda \) with \( 2^{-j_1}t_1 \) in (3.3), we see that for all \( j_i \in \mathbb{Z} \) and \( t_i \in [1, 2], \)
\[
\Psi_i(2^{-j_i}t_i)\Upsilon_i(2^{-j_i}t_i) + \sum_{k_i=1}^{\infty} \Psi_i(2^{-(k_i+j_i)}t_i)\Theta_i(2^{-(k_i+j_i)}t_i) = 1.
\]
It then follows from the spectral theorem that for all \( f \in L^2(X_1 \times X_2), \) all \( j_1, j_2 \in \mathbb{Z} \) and all \( t_1, t_2 \in [1, 2], \)
\[
f = (\Psi_1(2^{-j_1}t_1 \sqrt{L_1})\Upsilon_1(2^{-j_1}t_1 \sqrt{L_1})) \otimes (\Psi_2(2^{-j_2}t_2 \sqrt{L_2})\Upsilon_2(2^{-j_2}t_2 \sqrt{L_2})) f \\
+ \sum_{k_1=1}^{\infty} (\Psi_1(2^{-(k_1+j_1)}t_1 \sqrt{L_1})\Theta_1(2^{-(k_1+j_1)}t_1 \sqrt{L_1})) \otimes (\Psi_2(2^{-j_2}t_2 \sqrt{L_2})\Upsilon_2(2^{-j_2}t_2 \sqrt{L_2})) f \\
+ \sum_{k_2=1}^{\infty} (\Psi_1(2^{-j_1}t_1 \sqrt{L_1})\Upsilon_1(2^{-j_1}t_1 \sqrt{L_1})) \otimes (\Psi_2(2^{-(k_2+j_2)}t_2 \sqrt{L_2})\Theta_2(2^{-(k_2+j_2)}t_2 \sqrt{L_2})) f \\
+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} (\Psi_1(2^{-(k_1+j_1)}t_1 \sqrt{L_1})\Theta_1(2^{-(k_1+j_1)}t_1 \sqrt{L_1})) \otimes (\Psi_2(2^{-(k_2+j_2)}t_2 \sqrt{L_2})\Theta_2(2^{-(k_2+j_2)}t_2 \sqrt{L_2})) f.
\]
with convergence in the sense of $L^2(X_1 \times X_2)$ norm. Hence, for all $j_1, j_2 \in \mathbb{Z}$ and a.e. $(y_1, y_2) \in X_1 \times X_2$, we have

\begin{align*}
&\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2}) f(y_1, y_2) \\
&= (\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \Psi_1(2^{-j_1} t_1 \sqrt{L_1}) \Upsilon_1(2^{-j_1} t_1 \sqrt{L_1})) \\
&\quad \otimes (\Phi_2(2^{-j_2} t_2 \sqrt{L_2}) \Psi_2(2^{-j_2} t_2 \sqrt{L_2}) \Upsilon_2(2^{-j_2} t_2 \sqrt{L_2})) f(y_1, y_2) \\
&\quad + \sum_{k_1=1}^{\infty} (\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \Phi_1(2^{-(k_1+j_1)} t_1 \sqrt{L_1}) \Theta_1(2^{-(k_1+j_1)} t_1 \sqrt{L_1})) \\
&\quad \otimes (\Phi_2(2^{-j_2} t_2 \sqrt{L_2}) \Psi_2(2^{-j_2} t_2 \sqrt{L_2}) \Upsilon_2(2^{-j_2} t_2 \sqrt{L_2})) f(y_1, y_2) \\
&\quad + \sum_{k_2=1}^{\infty} (\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \Psi_1(2^{-j_1} t_1 \sqrt{L_1}) \Upsilon_1(2^{-j_1} t_1 \sqrt{L_1})) \\
&\quad \otimes (\Phi_2(2^{-j_2} t_2 \sqrt{L_2}) \Phi_2(2^{-(k_2+j_2)} t_2 \sqrt{L_2}) \Theta_2(2^{-(k_2+j_2)} t_2 \sqrt{L_2})) f(y_1, y_2) \\
&\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} (\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \Phi_1(2^{-(k_1+j_1)} t_1 \sqrt{L_1}) \Theta_1(2^{-(k_1+j_1)} t_1 \sqrt{L_1})) \\
&\quad \otimes (\Phi_2(2^{-j_2} t_2 \sqrt{L_2}) \Phi_2(2^{-(k_2+j_2)} t_2 \sqrt{L_2}) \Theta_2(2^{-(k_2+j_2)} t_2 \sqrt{L_2})) f(y_1, y_2) \\
&= \int_{X_1 \times X_2} K_{\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \Upsilon_1(2^{-j_1} t_1 \sqrt{L_1})}(y_1, z_1) K_{\Phi_2(2^{-j_2} t_2 \sqrt{L_2}) \Upsilon_2(2^{-j_2} t_2 \sqrt{L_2})}(y_2, z_2) \\
&\times (\Phi_1(2^{-(0+j_1)} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-(0+j_2)} t_2 \sqrt{L_2})) f(z_1, z_2) d\mu(z_1) d\mu(z_2) \\
&\quad + \sum_{k_1=1}^{\infty} \int_{X_1 \times X_2} K_{\Phi_1(2^{-(k_1+j_1)} t_1 \sqrt{L_1}) \Theta_1(2^{-(k_1+j_1)} t_1 \sqrt{L_1})}(y_1, z_1) K_{\Phi_2(2^{-j_2} t_2 \sqrt{L_2}) \Theta_2(2^{-(k_2+j_2)} t_2 \sqrt{L_2})}(y_2, z_2) \\
&\times (\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2})) f(z_1, z_2) d\mu(z_1) d\mu(z_2) \\
&\quad + \sum_{k_2=1}^{\infty} \int_{X_1 \times X_2} K_{\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \Theta_1(2^{-(k_1+j_1)} t_1 \sqrt{L_1})}(y_1, z_1) K_{\Phi_2(2^{-j_2} t_2 \sqrt{L_2}) \Theta_2(2^{-(k_2+j_2)} t_2 \sqrt{L_2})}(y_2, z_2) \\
&\times (\Phi_1(2^{-(0+j_1)} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-(k_2+j_2)} t_2 \sqrt{L_2})) f(z_1, z_2) d\mu(z_1) d\mu(z_2) \\
&\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \int_{X_1 \times X_2} K_{\Phi_1(2^{-(k_1+j_1)} t_1 \sqrt{L_1}) \Theta_1(2^{-(k_1+j_1)} t_1 \sqrt{L_1})}(y_1, z_1) \\
&\times K_{\Phi_2(2^{-j_2} t_2 \sqrt{L_2}) \Theta_2(2^{-(k_2+j_2)} t_2 \sqrt{L_2})}(y_2, z_2) \\
&\times (\Phi_1(2^{-(k_1+j_1)} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-(k_2+j_2)} t_2 \sqrt{L_2})) f(z_1, z_2) d\mu(z_1) d\mu(z_2).
\end{align*}

For $i = 1, 2$, let $N_i \geq \lambda_i$ and $m_i$ be any integer such that $m_i - \lambda_i - n_i/r > 0$. Since $\Theta_i$ vanishes near the origin, it follows from Lemma 22 that there exists a constant $C = C(\Phi_i, \Theta_i, m_i, N_i)$ such that for all $j_i \in \mathbb{Z}$, all $k_i \in \{1, 2, \cdots\}$, and all $t_i \in [1, 2]$, 

\begin{equation}
|K_{\Phi_i(2^{-j_i} t_i \sqrt{L_1}) \Theta_i(2^{-(k_i+j_i)} t_i \sqrt{L_1})}(y_i, z_i)| \leq C 2^{-k_i m_i} V(z_i, 2^{-j_i} t_i)^{-1}(1 + 2^{j_i} t_i^{-1} \rho_i(y_i, z_i))^{-N_i}.
\end{equation}

Analogously, for $i = 1, 2$, we have

\begin{equation}
|K_{\Psi_i(2^{-j_i} t_i \sqrt{L_1}) \Upsilon_i(2^{-(k_i+j_i)} t_i \sqrt{L_1})}(y_i, z_i)| \leq C V(z_i, 2^{-j_i} t_i)^{-1}(1 + 2^{j_i} t_i^{-1} \rho_i(y_i, z_i))^{-N_i}.
\end{equation}

Putting (3.10) and (3.11) into (3.9), we obtain

\begin{align*}
&\left| \Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2}) f(y_1, y_2) \right| \\
&\leq C \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} 2^{-k_1 m_1} 2^{-k_2 m_2} \\
&\times \int_{X_1 \times X_2} \frac{\left| \Phi_1(2^{-(k_1+j_1)} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-(k_2+j_2)} t_2 \sqrt{L_2}) f(z_1, z_2) \right|}{\prod_{i=1}^{2} V(z_i, 2^{-j_i} t_i)^{-1}(1 + 2^{j_i} t_i^{-1} \rho_i(y_i, z_i))^{N_i}} d\mu_1(z_1) d\mu_2(z_2).
\end{align*}
\[ C \sum_{k_1}^{\infty} \sum_{k_2=0}^{\infty} 2^{(j_1-k_1)m_1+2(j_2-k_2)m_2} \times \int_{X_1 \times X_2} \frac{|\Phi_1(2^{-k_1}t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2}t_2 \sqrt{L_2})f(z_1, z_2)|}{\prod_{i=1}^{2} V(z_i, 2^{-k_i} t_i) (1 + 2^{j_i} t_i^{-1} \rho_l(y_i, z_i))^{\lambda_i}} \, d\mu_1(z_1) \, d\mu_2(z_2). \]

To prove the desired inequality, we first consider the case \(0 < r \leq 1\). Dividing both sides of (3.14) by \((1 + 2^{k_2} t_2^{-1} \rho_l(x_1, z_1))(1 + 2^{j_2} t_2^{-1} \rho_l(y_1, z_1))\), taking the supremum over \((y_1, y_2) \in X_1 \times X_2\) in the left-hand side, and using the inequalities \(V(z, 2^{-k_i} t_i) \geq V(z_i, 2^{-k_i} t_i) (\forall k_i \geq j_i)\) and \((1 + 2^{k_2} t_2^{-1} \rho_l(x_1, z_1))(1 + 2^{j_2} t_2^{-1} \rho_l(x_1, z_1)) \geq (1 + 2^{j_2} t_2^{-1} \rho_l(x_1, z_1)) (\forall t_i \in [1, 2])\) in the right-hand side, we get that, for all \(t_i \in [1, 2]\) and \(x_i \in X_i\),

\[ \left(1 + 2^{j_2} t_2^{-1} \rho_l(x_1, z_1)\right)^{\lambda_i} \leq 2^{(k_2-j_2) \lambda_i}(1 + 2^{j_2} t_2^{-1} \rho_l(x_1, z_1))^{\lambda_i} \quad (\forall k_i \geq j_i, \forall t_i \in [1, 2]), \]

it follows that

\[ \left|\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2})f(z_1, z_2)\right| \leq C \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} 2^{(j_1-k_1)(m_1-\lambda_1) + 2(j_2-k_2)(m_2-\lambda_2)} \times \int_{X_1 \times X_2} \frac{|\Phi_1(2^{-k_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2} t_2 \sqrt{L_2})f(z_1, z_2)|^{1-r}}{\prod_{i=1}^{2} V(z_i, 2^{-k_i} t_i) (1 + 2^{j_i} t_i^{-1} \rho_l(x_i, z_i))^{\lambda_i}} \times \left(1 + 2^{j_2} t_2^{-1} \rho_l(x_1, z_1)\right)^{(1-r) \lambda_i} \, d\mu_1(z_1) \, d\mu_2(z_2). \]

From \((3.13), (3.14),\) and the inequality \((1 + 2^{j_2} t_2^{-1} \rho_l(x_1, z_1))^{\lambda_i} \leq 2^{(k_2-j_2) \lambda_i}(1 + 2^{j_2} t_2^{-1} \rho_l(x_1, z_1))^{\lambda_i}\) \((\forall k_i \geq j_i, \forall t_i \in [1, 2]),\)

it follows that

\[ \left|\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2})f(z_1, z_2)\right| \leq C \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} 2^{(j_1-k_1)(m_1-\lambda_1) + 2(j_2-k_2)(m_2-\lambda_2)} \times \int_{X_1 \times X_2} \frac{|\Phi_1(2^{-k_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2} t_2 \sqrt{L_2})f(z_1, z_2)|^{1-r}}{\prod_{i=1}^{2} V(z_i, 2^{-k_i} t_i) (1 + 2^{j_i} t_i^{-1} \rho_l(x_i, z_i))^{\lambda_i}} \times \left|\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2})\right|^{1-r} \, d\mu_1(z_1) \, d\mu_2(z_2). \]

We claim that for any \(f \in L^2(X_1 \times X_2)\), \(\lambda_i > D_i/2, x_i \in X_i, t_i \in [1, 2]\), and \(j_i \in \mathbb{Z}_n\),

\[ \left|\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2})f(x_1, x_2)\right| < \infty, \]

and there exists \(N_0 > 0\) such that

\[ \left|\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2})\right|^{1-r} = O(2^{-j_1 N_0} 2^{-j_2 N_0}) \]

as \(j_1, j_2 \to +\infty\). Indeed, for \(i = 1, 2\), by Lemma 2.1, we have

\[ \left|\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2})(y_1, z_1)\right| \leq CV(y_1, 2^{-j_1} t_1)^{-1}(1 + 2^{j_1} t_1^{-1} \rho_l(y_1, z_1))^{-(n_1+1)/2}. \]

Hence, by the Cauchy-Schwarz inequality and 2.1, we have

\[ \left|\Phi_1(2^{-j_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j_2} t_2 \sqrt{L_2})(y_1, y_2)\right| \leq C \int_{X_1 \times X_2} |K_{\Phi_1(2^{-j_1} t_1 \sqrt{L_1})}(y_1, z_1)||K_{\Phi_2(2^{-j_2} t_2 \sqrt{L_2})}(y_2, z_2)||f(z_1, z_2)|| \, d\mu(z_1) \, d\mu(z_2). \]

\[ \leq C \|f\|_{L^2(X_1 \times X_2)} V(y_1, 2^{-j_1} t_1)^{-1/2} V(y_2, 2^{-j_2} t_1)^{-1/2}. \]
This along with (3.13) yields that for \( \lambda_i \geq D_i/2, \)
\[
\left( \Phi_1(2^{-j} t_i \sqrt{L_1}) \otimes \Phi_2(2^{-j} t_2 \sqrt{L_2}) \right)^{j_1, j_2} f(x_1, x_2)
\leq C \sup_{(y_1, y_2) \in X_1 \times X_2} \left\| f \right\|_{L^2(X_1 \times X_2)}
\]
\[
= C \sup_{(y_1, y_2) \in X_1 \times X_2} \left\| f \right\|_{L^2(X_1 \times X_2)}
\]
\[
\leq C \left\| f \right\|_{L^2(X_1 \times X_2)} (x_1, 2^{-j} t_1)^{-1/2} (x_2, 2^{-j} t_2)^{-1/2} \cdot
\]
\[
Hence (3.16) is true. Moreover, if \( j_1, j_2 \geq 1, \) by (3.12) we have
\[
\left[ \Phi_1(2^{-j} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j} t_2 \sqrt{L_2}) \right]^{j_1, j_2} f(x_1, x_2)
\leq C \left\| f \right\|_{L^2(X_1 \times X_2)} (x_1, 2^{-j} t_1)^{-1/2} (x_2, 2^{-j} t_2)^{-1/2}
\]
\[
\leq C \left\| f \right\|_{L^2(X_1 \times X_2)} (x_1, 1)^{-1/2} (x_2, 1)^{-1/2},
\]
which verifies (3.17) with \( N_0 = \max \{n_1/2, n_2/2 \}. \)

Since \( m_1, m_2 \) in (3.15) can be chosen to be arbitrarily large, it follows from (3.15), (3.16), (3.17)
and Lemma 2.7 that for any \( r > 0, \)
\[
\left\{ \left[ \Phi_1(2^{-j} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-j} t_2 \sqrt{L_2}) \right]^{j_1, j_2} f(x_1, x_2) \right\}^r
\leq C \sum_{k_1 = j_1}^{\infty} \sum_{k_2 = j_2}^{\infty} 2^{(j_1 - k_1) \sigma} 2^{(j_2 - k_2) \sigma}
\]
\[
\times \int_{X_1 \times X_2} \frac{\left| \Phi_1(2^{-k_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2} t_2 \sqrt{L_2}) f(z_1, z_2) \right|^r}{\left\| f \right\|_{L^2(X_1 \times X_2)}}
\]
\[
\times \int_{X_1 \times X_2} \left( \left\| f \right\|_{L^2(X_1 \times X_2)} (z_1, 2^{-k_1} t_1)^{-1/2} \lambda_1 r \right)^{1/r} \frac{d \mu_1(z_1) d \mu_2(z_2)}{\left\| f \right\|_{L^2(X_1 \times X_2)}}
\]
\[
\leq C \sum_{k_1 = j_1}^{\infty} \sum_{k_2 = j_2}^{\infty} 2^{(j_1 - k_1) \sigma} 2^{(j_2 - k_2) \sigma}
\times \left( \int_{X_1 \times X_2} \left( \left\| f \right\|_{L^2(X_1 \times X_2)} \right)^{1/r} \frac{d \mu_1(z_1) d \mu_2(z_2)}{\left\| f \right\|_{L^2(X_1 \times X_2)}} \right)^{1/r},
\]
where we applied Hölder’s inequality for the integrals and the sums, and used (3.13) and (3.14). Raising
both sides to the power \( r, \) dividing both sides by \( (1 + 2^{j_1} t_i^{-1} \rho_1(x_1, y_i))^{\lambda_i r} (1 + 2^{j_2} t_i^{-1} \rho_2(x_2, y_i))^{\lambda_2 r} \),
in the left-hand side taking the supremum over \( (y_1, y_2) \in X_1 \times X_2, \) and in the right-hand side using
the inequalities
\[
(1 + 2^{j_i} t_i^{-1} \rho_i(x_i, y_i))^{\lambda_i r} \geq (1 + 2^{j_i} t_i^{-1} \rho_i(x_i, z_i))^{\lambda_i r},
\]
and
Proof. Since $\lambda_i > \frac{(n_i + D_i)q_i}{\min(p,2)}$, there exists a number $r$ such that $0 < r < \frac{\min(p,2)}{q_i}$ and $\lambda_i r > n_i + D_i$. From Lemma 3.4 we see that for any $j_1, j_2 \in [1,2]$, and $t_i \in [0,1]$, we obtain (3.7) for $r > 1$.

**Lemma 3.5.** Let $\Phi_1, \Phi_2 \in \mathcal{A}(\mathbb{R})$ be even functions. Let $p \in (0, \infty)$ and $\lambda_i > \frac{(n_i + D_i)q_i}{\min(p,2)}$, $i = 1, 2$. Then there exists a constant $C$ such that for all $f \in L^2(X_1 \times X_2)$,

$$
\left\| \left( \int_0^\infty \int_0^\infty \left[ \Phi_1(t_1 \sqrt{L_1}) \otimes \Phi_2(t_2 \sqrt{L_2}) \right] \right)_\lambda \cdot f \left( \frac{dt_1}{t_1}, \frac{dt_2}{t_2} \right)^{1/2} \right\|_{L^p_\infty(X_1 \times X_2)} 
\leq C \left\| \left( \int_0^\infty \int_0^\infty \left[ \Phi_1(t_1 \sqrt{L_1}) \otimes \Phi_2(t_2 \sqrt{L_2}) \right] \right)_\lambda \cdot f \left( \frac{dt_1}{t_1}, \frac{dt_2}{t_2} \right)^{1/2} \right\|_{L^p_\infty(X_1 \times X_2)}.
$$

Proof. From Lemma 3.4, we see that for any $j_1, j_2 \in [1,2]$, and $t_i \in [0,1]$, we obtain (3.7) for $r > 1$.
Lemma 3.6. Let $\Phi_1, \Phi_2 \in \mathcal{A}(\mathbb{R})$ be even functions. Let $p \in (0, \infty)$ and $\lambda_i > 0$, $i = 1, 2$. Let $w$ be arbitrary weight (i.e., non-negative locally integrable function) on $X_1 \times X_2$. Then there exists a constant $C$ such that for all $f \in L^2(X_1 \times X_2)$,

$$
\left\| \left( \int_0^\infty \int_0^\infty \left| \left[ \Phi_1(t_1^2 \sqrt{L_1}) \otimes \Phi_2(t_2^2 \sqrt{L_2}) \right]_{\lambda_1 + D_{i}/2, \lambda_2 + D_{i}/2} f(x_1, x_2) \right|^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\|^{1/2}_{L^p(X_1 \times X_2)} \leq C \left\| g_{\Phi_1, \Phi_2, L_1, L_2}(2/n_1 \lambda_1, 2/n_2 \lambda_2) \left( f \right) \right\|_{L^p(X_1 \times X_2)}.
$$

Proof. Let $\sigma > 0$. By Lemma 3.4 with $\tau = 2$, we see that there exists a constant $C$ such that for all $f \in L^2(X_1 \times X_2)$, $j_i \in \mathbb{Z}$ and $t_i \in [1, 2]$,

$$
\left( \int_0^\infty \int_0^\infty \left| \left[ \Phi_1(t_1^2 \sqrt{L_1}) \otimes \Phi_2(t_2^2 \sqrt{L_2}) \right]_{\lambda_1 + D_{i}/2, \lambda_2 + D_{i}/2} f(x_1, x_2) \right|^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{1/2} \leq C \sum_{k_1 = j_1}^{\infty} \sum_{k_2 = j_2}^{\infty} 2^{(j_1 - k_1) \sigma} 2^{(j_2 - k_2) \sigma} \times \int_{X_1 \times X_2} \frac{\Phi_1(2^{-k_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2} t_2 \sqrt{L_2}) f(z_1, z_2)}{\prod_{i=1}^{\infty} V(z_i, 2^{-k_i+1} t_i)(1 + 2^{k_i} t_i^{-1} p(x_i, z_i))^{2 \lambda_i + D_i}} \frac{d\mu_1(z_1)}{\lambda_1 + D_{i}/2, \lambda_2 + D_{i}/2} f(x_1, x_2)
$$

where for the last line we used \([3.33]\). Taking the norm $\int_1^2 \int_1^2 \left| \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right|$ on both sides of \([3.18]\) gives

$$
\int_1^2 \int_1^2 \left\| \left( \int_0^\infty \int_0^\infty \left| \left[ \Phi_1(t_1^2 \sqrt{L_1}) \otimes \Phi_2(t_2^2 \sqrt{L_2}) \right]_{\lambda_1 + D_{i}/2, \lambda_2 + D_{i}/2} f(x_1, x_2) \right|^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\|^{1/2}_{L^p(X_1 \times X_2)} \leq C \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} 2^{-|k_1 - j_1| \sigma} 2^{-|k_2 - j_2| \sigma} \times \int_1^2 \int_1^2 \int_{X_1 \times X_2} \frac{\Phi_1(2^{-k_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2} t_2 \sqrt{L_2}) f(z_1, z_2)}{\prod_{i=1}^{\infty} (1 + 2^{k_i} t_i^{-1} p(x_i, z_i))^{2 \lambda_i}} \frac{d\mu_1(z_1)}{\lambda_1 + D_{i}/2, \lambda_2 + D_{i}/2} f(x_1, x_2) \frac{d\mu_2(z_2)}{\lambda_1 + D_{i}/2, \lambda_2 + D_{i}/2} f(x_1, x_2) \frac{dt_1}{t_1} \frac{dt_2}{t_2}.
$$

Applying Lemma 2.6 in \(L^{p/2}(\ell^1)\) we obtain

$$
\left\| \left( \int_0^\infty \int_0^\infty \left| \left[ \Phi_1(t_1^2 \sqrt{L_1}) \otimes \Phi_2(t_2^2 \sqrt{L_2}) \right]_{\lambda_1 + D_{i}/2, \lambda_2 + D_{i}/2} f(x_1, x_2) \right|^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\|^{1/2}_{L^p(X_1 \times X_2)} \leq C \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} 2^{-|k_1 - j_1| \sigma} 2^{-|k_2 - j_2| \sigma} \times \int_1^2 \int_1^2 \int_{X_1 \times X_2} \frac{\Phi_1(2^{-k_1} t_1 \sqrt{L_1}) \otimes \Phi_2(2^{-k_2} t_2 \sqrt{L_2}) f(z_1, z_2)}{\prod_{i=1}^{\infty} (1 + 2^{k_i} t_i^{-1} p(x_i, z_i))^{2 \lambda_i}} \frac{d\mu_1(z_1)}{\lambda_1 + D_{i}/2, \lambda_2 + D_{i}/2} f(x_1, x_2).$$


Let $\Phi$.

Proof of Theorem 1.4. Let $\Phi_1, \Phi_2 \in A(\mathbb{R})$ be even functions satisfying $\Phi_1(0) = \Phi_2(0) = 0$.

Having the above lemmas, we are ready to give the proofs of Theorems 1.4 and 1.5.

Proof of Theorem 1.4. Let $\Phi_1, \Phi_2, \tilde{\Phi}_1, \tilde{\Phi}_2 \in A(\mathbb{R})$ be even functions satisfying $\Phi_1(0) = \Phi_2(0) = 0$.

Let $p \in (0, \infty)$ and $\lambda_i > \frac{(n_i + D_i)\lambda_{\min}}{\min(|p|, 2)}$, $i = 1, 2$. Note that for a.e. $(x_1, x_2) \in X_1 \times X_2$,

$$f_i \in (x_1, x_2)$$

Using (3.19), Lemma 3.6, Lemma 3.1, Lemma 3.2 and Lemma 3.3, we infer

$$\|\tilde{\Phi}_1(t_1 \sqrt{L_1}) \odot \tilde{\Phi}_2(t_2 \sqrt{L_2})f(x_1, x_2) \|_{L^p(X_1 \times X_2)}.$$ 

By symmetry, there also holds $\|\Phi_1(0, \Phi_2, \tilde{\Phi}_1, \tilde{\Phi}_2(0, L_2) (f) \|_{L^p(X_1 \times X_2)} \leq C\|\Phi_1(0, \Phi_2, L_2, f) \|_{L^p(X_1 \times X_2)}$. Hence the assertion of Theorem 1.4 is true.

Proof of Theorem 1.5. Let $\Phi_1, \Phi_2 \in A(\mathbb{R})$ be even functions. Let $p \in (0, \infty)$, $\lambda_i > \frac{2\lambda_{\min}}{\min(|p|, 2)}$ and $\lambda' \geq \frac{(n_i + D_i)\lambda_{\min}}{\min(|p|, 2)}$, $i = 1, 2$. Then, for all $f \in L^2(X_1 \times X_2)$, by (3.19), Lemma 3.6, Lemma 3.1, Lemma 3.2 and Lemma 3.3, we have

$$\|\Phi_1(0, \Phi_2, L_2, f) \|_{L^p(X_1 \times X_2)}.$$ 

which yields (1.19). The proof of Theorem 1.5 is complete.
4. Applications of Theorems 1.4 and 1.5

1. In [9] and [12], the theory of product Hardy space $H^1_{L_1, L_2}(\mathbb{R}^n \times \mathbb{R}^m)$ via the Littlewood–Paley area functions were established, where $L_1$ and $L_2$ are two non-negative self-adjoint operators that satisfy only the Gaussian heat kernel bound. To be more specific, $H^1_{L_1, L_2}(\mathbb{R}^n \times \mathbb{R}^m)$ is defined as the closure of

$$\{ f \in L^2(\mathbb{R}^n \times \mathbb{R}^m) : S_{L_1, L_2}(f) \in L^1(\mathbb{R}^n \times \mathbb{R}^m) \}$$

under the norm $\|f\|_{H^1_{L_1, L_2}(\mathbb{R}^n \times \mathbb{R}^m)} := \|S_{L_1, L_2}(f)\|_{H^1_{L_1, L_2}(\mathbb{R}^n \times \mathbb{R}^m)}$, where

$$S_{L_1, L_2}(f)(x_1, x_2) = \left( \int \int_{\Gamma_1(x_1) \times \Gamma_2(x_2)} (t_1^2 L_1 e^{-t_1^2 L_1}) \otimes (t_2^2 L_2 e^{-t_2^2 L_2}) f(y_1, y_2) \right)^{1/2} dt_1 dt_2,$$

Then, by applying our main result Theorem 1.5 (also Remark 1.7), we obtain a direct proof of the equivalence without using the H"{o}lder regularity and the cancellation property.

2. In 1965, Muckenhoupt and Stein in [20] introduced a notion of conjugacy associated with the Bessel operator $\Delta_\lambda$ on $\mathbb{R}^+ := (0, \infty)$ defined by

$$\Delta_\lambda f(x) := -\frac{d^2}{dx^2} f(x) - \frac{2\lambda}{x} \frac{d}{dx} f(x), \quad x > 0,$$

and the Bessel Schrödinger operator $S_\lambda$ on $\mathbb{R}^+$

$$S_\lambda f(x) := -\frac{d^2}{dx^2} f(x) + \frac{\lambda^2 - \lambda}{x^2} f(x), \quad x > 0.$$

In [11], Duong et al. established the product Hardy space $H^p_{\Delta_\lambda}(\mathbb{R}^+ \times \mathbb{R}^+)$ associated with $\Delta_\lambda$ via the Littlewood–Paley area function and square functions. Note that the measure on $\mathbb{R}^+$ related to $\Delta_\lambda$ is $dx(x) = e^{2\lambda x} dx$. We point out that the kernel of $t^2 \Delta_\lambda e^{-t^2 \Delta_\lambda}$ satisfies the Gaussian upper bounds with respect to the measure $dy_\lambda$, the H"{o}lder regularity and the cancellation property. Hence, by using the approach in [18] via the Plancherel–Polya type inequality, they obtained the equivalence of the characterizations of $H^p_{\Delta_\lambda}(\mathbb{R}^+ \times \mathbb{R}^+)$ via Littlewood–Paley area function and square functions. By applying our main result Theorem 1.5 (also Remark 1.7), we obtain a direct proof of the equivalence without using the H"{o}lder regularity and the cancellation property.

In [2], Betancor et al. established the product Hardy space $H^p_{S_\lambda}(\mathbb{R}^+ \times \mathbb{R}^+)$ associated with $\Delta_\lambda$ via the Littlewood–Paley area function and square functions. To prove the equivalence, they need to use the Poisson semigroup $\{e^{-t^2 \Delta_\lambda}\}$, the subordination formula and the Moser type inequality as a bridge. By applying our main result Theorem 1.5 (also Remark 1.7), we obtain a direct proof of this equivalence without using the Moser type inequality.

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