On Physical Properties of Cylindrically Symmetric Self-Similar Solutions

M. Sharif * and Sehar Aziz †
Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan.

Abstract

This paper is devoted to discuss some of the features of self-similar solutions of the first kind. We consider the cylindrically symmetric solutions with different homotheties. We are interested in evaluating the quantities acceleration, rotation, expansion, shear, shear invariant and expansion rate. These kinematical quantities are discussed both in co-moving as well as in non-co-moving coordinates (only in radial direction). Finally, we would discuss the singularity feature of these solutions. It is expected that these properties would help in exploring some interesting features of the self-similar solutions.

Keywords: Self-Similar Solutions.

1 Introduction

Einstein field equations (EFEs) given by

\[ R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab}, \quad (a, b = 0, 1, 2, 3) \]  

*hasharif@yahoo.com
†sehar_aziz@yahoo.com
where $R_{ab}$, $R$, $T_{ab}$, $\kappa$ are the Ricci tensor, Ricci scalar, matter tensor and the gravitational constant respectively, are highly non-linear partial differential equations. To simplify these equations, we frequently impose some symmetry on the concerned system. Self-similarity is very helpful in simplifying the field equations. In Newtonian gravity or General Relativity (GR), there does not exist any characteristic scale. A set of field equations remains invariant under a scale transformation for an appropriate matter field. This indicates that there exist scale invariant solutions to the EFEs. These solutions are known as self-similar solutions.

We study self-similar solutions because of its two important features. The first special feature of self-similar solutions is that, by a suitable coordinate transformations, the number of independent variables can be reduced by one and hence reduces the field equations. This variable is a dimensionless combination of the independent variables, namely the space coordinates and the time. In other words, self-similarity refers to an invariance which simply allows the reduction of a system of partial differential equations to ordinary differential equations. Secondly, self-similar solutions play an important role in describing the asymptotic properties of more general models. For example, the expansion of universe from big bang and the critical and gravitational collapse of a star might have self-similarity in some sense as we expect that the initial conditions have been lost.

In order to obtain realistic solutions of gravitational collapse leading to star formation, self-similar solutions have been investigated by many authors in Newtonian gravity [1]. However, in GR, these solutions were first studied by Cahill and Taub [2]. They studied these solutions in the cosmological context and under the assumption of spherically symmetric distribution of a self-gravitating perfect fluid. In GR, self-similarity is defined by the existence of a homothetic vector (HV) field. Such similarity is called the first kind (or homothety or continuous self-similarity). There exists a natural generalization of homothety called kinematic self-similarity, which is defined by the existence of a kinematic self-similar (KSS) vector field. A KSS vector $\xi$ satisfies the following conditions

$$L_\xi h_{ab} = 2\delta h_{ab},$$  \hspace{1cm} (2) $$L_\xi u_a = \alpha u_a,$$  \hspace{1cm} (3)  

where $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor, $\alpha$ and $\delta$ are constants. The similarity transformation is characterized by the scale independent ratio, $\alpha/\delta$, which is known as the similarity index and can be classified into three kinds.
The idea of self-similarity by Cahil and Taub corresponded to Newtonian self-similarity of the homothetic class. Carter and Henriksen [3,4] defined the other kinds of self-similarity namely second, zeroth and infinite kind. In the context of kinematic self-similarity, homothety is considered as the first kind. Several authors have explored KSS perfect fluid solutions. The only barotropic equation of state which is compatible with self-similarity of the first kind is \( p = k \rho \).

Carr [5] has classified the self-similar perfect fluid solutions of the first kind for the dust case \((k = 0)\). The case \(0 < k < 1\) has been studied by Carr and Coley [6]. Coley [7] has shown that the FRW solution is the only spherically symmetric homothetic perfect fluid solution in the parallel case. McIntosh [8] has discussed that a stiff fluid \((k = 1)\) is the only compatible perfect fluid with the homothety in the orthogonal case. Benoit and Coley [9] have studied analytic spherically symmetric solutions of the EFEs coupled with a perfect fluid and admitting a KSS vector of the first, second and zeroth kinds.

Carr et al. [10] have considered the KSS associated with the critical behavior observed in the gravitational collapse of spherically symmetric perfect fluid with equation of state \( p = k \rho \). Carr et al. [11], further, investigated solution space of self-similar spherically symmetric perfect fluid models and physical aspects of these solutions. They combine the state space description of the homothetic approach with the use of the physically interesting quantities arising in the co-moving approach. Coley and Goliath [12] have investigated self-similar spherically symmetric cosmological models with a perfect fluid and a scalar field with an exponential potential.

Recently, Maeda et al. [13,14] investigated the KSS vector of the second kind in the tilted case. In their recent paper [15], the same authors discussed the classification of the spherically symmetric KSS perfect fluid and dust solutions. The existence of self-similar solutions of the first kind is related to conservation laws and to the invariance of the problem with respect to the group of similarity transformations of quantities with independent dimensions. This can be characterized in GR by the existence of a homothetic vector.

Qadir et al. [16] have classified cylindrically symmetric static manifold according to their homotheties and metrics. In each case the homothety vector field and the corresponding metrics are obtained explicitly by solving the homothety equations. They found 10 solutions admitting 4, 5, 7 or 11 homotheties. There is only one metric admitting 7 homotheties and three
metrics correspond to 4, three correspond to 5 and three correspond to 11 homotheties.

In a recent paper, Sharif and Sehar [17] have explored some physical properties of the spherically symmetric self-similar solutions of the first kind. This has provided some interesting features of self-similar solutions. In this paper, we are extending the same analysis for the cylindrically symmetric self-similar solutions of the first kind. The paper can be outlined as follows. In section 2, we shall write down the self-similar solutions of cylindrically symmetric spacetimes. Section 3 is devoted for the discussion of the physical properties of these self-similar solutions of the first kind both in co-moving and non-co-moving coordinates. In section 4, we shall explore the singularity feature of these solutions. Finally, we shall summarize and discuss all the results.

2 Cylindrically Symmetric self-similar solution of first kind

The general cylindrically symmetric static spacetime is given by the line element [18]

\[ ds^2 = e^{\nu(r)} dt^2 - dr^2 - a^2 e^{\lambda(r)} d\theta^2 - e^{\mu(r)} dz^2, \quad (4) \]

where \( \nu, \lambda \) and \( \mu \) are arbitrary functions of \( r \) and \( a \) has units of length.

Qadir et al. [17] have solved the homothetic equations and found self-similar solutions of the first kind (i.e. homothetic solutions). There are four classes of such solutions admitting 4, 5, 7 or 11 homotheties respectively. The class \( A \) admitting 4 homotheties has three metrics.

The first metric is of Petrov type \( I \), Segré type \([1,111] \) [19] and is given by

\[ ds^2 = e^{2\alpha \ln(r/\tau_0)} dt^2 - dr^2 - a^2 e^{2\ln(r/\tau_0)} d\theta^2 - e^{2\ln(r/\tau_0)} dz^2, \quad (\alpha \neq 0, 1). \quad (5) \]

The second metric is of Petrov type \( D \), Segré type \([1,(11)1] \) and it follows

\[ ds^2 = e^{2\alpha \ln(r/\tau_0)} dt^2 - dr^2 - a^2 e^{2\beta \ln(r/\tau_0)} d\theta^2 - e^{2\beta \ln(r/\tau_0)} dz^2, \quad (\alpha \neq \beta, \alpha, \beta \neq 0, 1). \quad (6) \]

The third metric in this class is of Petrov type \( I \), Segré type \([1,(11)1] \) and takes the form

\[ ds^2 = e^{2\alpha \ln(r/\tau_0)} dt^2 - dr^2 - a^2 d\theta^2 - e^{2\ln(r/\tau_0)} dz^2. \quad (7) \]
The class $B$ of homothetic solutions corresponding to 5 homotheties also has three metrics. The first and the second solutions are of Petrov type $D$, Segré type $[1,111]$ and are given by
\[ ds^2 = e^{2\ln(\frac{r}{r_0})} dt^2 - dr^2 - a^2 e^{2\ln(\frac{r}{r_0})} d\theta^2 - dz^2, \] (8)
\[ ds^2 = e^{2\ln(\frac{r}{r_0})} dt^2 - dr^2 - a^2 d\theta^2 - e^{2\ln(\frac{r}{r_0})} dz^2. \] (9)

The third metric is of Petrov type $D$, Segré type $[(1,1)(11)]$ and has the form
\[ ds^2 = dt^2 - dr^2 - a^2 e^{2\ln(\frac{r}{r_0})} d\theta^2 - e^{2\ln(\frac{r}{r_0})} dz^2. \] (10)

The class $C$ admitting 7 homotheties corresponds to only one metric which is of Petrov type $D$, Segré type $[1,(11)1]$ and is given by
\[ ds^2 = e^{2\alpha \ln(\frac{r}{r_0})} dt^2 - dr^2 - a^2 e^{2\alpha \ln(\frac{r}{r_0})} d\theta^2 - e^{2\alpha \ln(\frac{r}{r_0})} dz^2. \] (11)

This metric represents a tachyonic fluid and could be re-interpreted as an anisotropic fluid with an appropriately chosen cosmological constant.

Finally, in the class, $D$ there are three metrics admitting 11 homotheties other than Minkowski space. These are given by the following three metrics
\[ ds^2 = dt^2 - dr^2 - a^2 d\theta^2 - e^{2\ln(\frac{r}{r_0})} dz^2, \] (12)
\[ ds^2 = dt^2 - dr^2 - a^2 d\theta^2 - dz^2, \] (13)
\[ ds^2 = e^{2\ln(\frac{r}{r_0})} dt^2 - dr^2 - a^2 d\theta^2 - dz^2. \] (14)

### 3 Kinematics of the velocity field

In this section, we shall discuss some of the kinematical properties of the self-similar solutions of the first kind, given by Eqs.(5)-(14), both in co-moving and non-co-moving coordinates. The kinematical properties [18] to be discussed can be listed as follows. The acceleration can be defined as
\[ \dot{u}_a = u_{a;b} u^b. \] (15)

The rotation is given by
\[ \omega_{ab} = u_{[a;b]} + \dot{u}_{[a} u_{b]}. \] (16)
The volume behavior of the fluid can be determined by the expansion scalar defined by
\[ \Theta = u^a_{;a}. \] (17)

The shear tensor, which provides the distortion arising in the fluid flow leaving the volume invariant, can be found by
\[ \sigma_{ab} = u_{(a;b)} + \dot{u}_{(a} u_{b)} - \frac{1}{3} \Theta h_{ab}. \] (18)

Shear scalar, which gives the measure of anisotropy, is defined as
\[ \sigma = \sigma_{ab} \sigma^{ab}. \] (19)

The rate of change of expansion with respect to proper time is given by Raychaudhuri’s equation [19]
\[ \frac{d\Theta}{d\tau} = -\frac{1}{3} \Theta^2 - \sigma_{ab} \sigma^{ab} + \omega_{ab} u^a u^b - R_{ab} u^a u^b. \] (20)

Now we discuss these properties for the above mentioned self-similar solutions.

### 3.1 Kinematic Properties in Co-moving Coordinates

First we evaluate the kinematical properties of the self-similar solutions in co-moving coordinates. We discuss these properties for all the classes of metrics mentioned above.

#### 3.1.1 Class A admitting four homotheties

There are three metrics in this class given by Eqs.(5)-(7). In co-moving coordinates, the five quantities, i.e., acceleration, rotation, expansion scalar, shear and shear invariant turn out to be same for the three metrics in this class. The expansion scalar is zero for this class. The non-zero component of acceleration is
\[ \dot{u}_1 = -\frac{\alpha}{r}. \] (21)

The non-vanishing rotation component takes the form
\[ \omega_{01} = 2\frac{\alpha}{r} e^{\alpha \ln\left(\frac{t}{r_0}\right)} \] (22)
and the shear component is
\[ \sigma_{01} = -\omega_{01}. \]

Finally, the shear invariant becomes
\[ \sigma = -4\alpha^2 r^2. \]

For this class, the rate of change of expansion is different for all the three solutions. Using Raychaudhuri’s equation, given by Eq.(20), the rate of expansion for these solutions turns out to be
\[ \frac{d\Theta}{d\tau} = 3\frac{\alpha^2}{r^2}, \]
\[ \frac{d\Theta}{d\tau} = \frac{\alpha}{r^2}(1 - 2\beta + 3\alpha), \]
\[ \frac{d\Theta}{d\tau} = \frac{\alpha}{r^2}(1 + 3\alpha) \]
respectively.

3.1.2 Class B admitting five homotheties

For this class, all the kinematic quantities turn out to be the same for the two metrics given by Eqs.(8)-(9). For these two solutions, the expansion scalar vanishes. The only component of acceleration is
\[ \dot{u}_1 = -\frac{1}{r} \]
while the rotation component is given by
\[ \omega_{01} = \frac{2}{r} e^{\ln(\frac{r}{r_0})}. \]

The non-zero component of shear takes the form
\[ \sigma_{01} = -\omega_{01} \]
and the shear invariant becomes
\[ \sigma = -4\frac{\alpha}{r^2}. \]

The rate of expansion turns out to be
\[ \frac{d\Theta}{d\tau} = \frac{3}{r^2}. \]
All the quantities vanish for the third solution given by Eq.(10).
3.1.3 Class C admitting seven homotheties

There is only one metric in this class given by Eq.(11) The expansion scalar is zero for this spacetime. The component of acceleration is

\[ \dot{u}_1 = -\frac{\alpha}{r}. \]  

(33)

The only component of rotation is given by

\[ \omega_{01} = 2\frac{\alpha}{r} e^{\alpha \ln \frac{r}{r_0}}. \]  

(34)

The shear component turns out to be

\[ \sigma_{01} = -\omega_{01} \]  

(35)

while the shear invariant takes the form

\[ \sigma = -4\frac{\alpha^2}{r^2} = 4\dot{u}_1^2. \]  

(36)

The expansion rate becomes

\[ \frac{d\Theta}{d\tau} = \frac{\alpha}{r^2}(\alpha + 1). \]  

(37)

3.1.4 Class D admitting eleven homotheties

All the kinematical properties turn out to be zero for the first two solutions given by Eqs.(12) and (13). For the third solution, all the quantities are exactly similar to the quantities given for the solutions (8) and (9) except the rate of expansion. The rate of expansion is given by

\[ \frac{d\Theta}{d\tau} = \frac{4}{r^2}. \]  

(38)

3.2 Kinematic Properties in Non-Co-moving Coordinates

In this section we shall discuss the kinematical properties of the self-similar solution in non-co-moving coordinates only in the radial direction.
3.2.1 Class $A$ admitting four homotheties

For this class, the acceleration and rotation for all the solutions turn out to be the same. The two components of acceleration are

$$\dot{u}_0 = \frac{\alpha}{r} e^{\alpha \ln(\frac{r}{r_0})}, \quad \dot{u}_1 = -\frac{\alpha}{r}. \quad (39)$$

The only rotation component is

$$\omega_{01} = \frac{\alpha}{r} e^{\alpha \ln(\frac{r}{r_0})}. \quad (40)$$

The expansion for the first solution turns out to be

$$\Theta = -\frac{(1 + \alpha)}{r}. \quad (41)$$

The components of shear become

$$\sigma_{00} = 4 \frac{\alpha}{r} e^{2\alpha \ln(\frac{r}{r_0})}, \quad \sigma_{11} = \frac{2}{3r} (2\alpha - 1), \quad \sigma_{22} = -\frac{\alpha^2}{3r} e^{2\ln(\frac{r}{r_0})},$$

$$\sigma_{33} = -\frac{(1 + \alpha)}{3r}, \quad \sigma_{01} = -\frac{\alpha}{3r} e^{2\alpha \ln(\frac{r}{r_0})}. \quad (42)$$

The measure of anisotropy is given by

$$\sigma = \frac{1}{9r^2} (113\alpha^2 - 14\alpha + 53). \quad (43)$$

The rate of expansion turns out to be

$$\frac{d\Theta}{d\tau} = -\frac{4}{9r^2} (29\alpha^2 - 2\alpha + 14) - 2\frac{\alpha}{r}. \quad (44)$$

For the second solution, given by Eq.(6), the non-zero expansion becomes

$$\Theta = -\frac{(2\beta + \alpha)}{r}. \quad (45)$$

The shear components are

$$\sigma_{00} = 4 \frac{\alpha}{r} e^{2\alpha \ln(\frac{r}{r_0})}, \quad \sigma_{11} = \frac{4}{3r} (\alpha - \beta), \quad \sigma_{22} = -\frac{(\alpha + 8\beta)}{3r} e^{2\beta \ln(\frac{r}{r_0})},$$

$$\sigma_{33} = -\frac{(\alpha + 8\beta)}{3r} e^{2\beta \ln(\frac{r}{r_0})}, \quad \sigma_{01} = -\frac{2}{3r} (4\alpha - \beta) e^{\alpha \ln(\frac{r}{r_0})}. \quad (46)$$
The shear invariant turns out to be

\[ \sigma = \frac{2}{9r^2}(49\alpha^2 + 16\alpha\beta + 70\beta^2). \]  

(47)

The rate of change of expansion becomes

\[ \frac{d\Theta}{d\tau} = -\frac{1}{9r^2}(101\alpha^2 + 9\alpha r + 62\alpha\beta + 64\beta^2 + 18\beta). \]  

(48)

For the third metric, given by Eq.(7), the expansion becomes the same as for the first solution. The non-vanishing components of shear are

\[ \sigma_{00} = \frac{4\alpha}{r}e^{2\alpha\ln(\frac{r}{r_0})}, \quad \sigma_{11} = \frac{2}{3r}(2\alpha - 1), \quad \sigma_{22} = -\frac{1}{3r}(\alpha + 1)a^2, \]
\[ \sigma_{33} = -\frac{1}{3r}(\alpha + 7)e^{2\alpha\ln(\frac{r}{r_0})}, \quad \sigma_{01} = -\frac{1}{3r}(8\alpha - 1)e^{\alpha\ln(\frac{r}{r_0})}. \]  

(49)

The measure of anisotropy is given by

\[ \sigma = \frac{1}{9r^2}(98\alpha^2 + 16\alpha + 53) \]  

and the expansion rate turns out to be

\[ \frac{d\Theta}{d\tau} = \frac{5}{9r^2}(19\alpha^2 - 2\alpha - 10) - \frac{\alpha}{r}. \]  

(51)

### 3.2.2 Class B admitting five homotheties

Now we discuss the kinematical properties for the solutions given by Eq.(8)-(10) of the class B. The expansion scalar remains the same for all the solutions in this class and is given by

\[ \Theta = -\frac{2}{r}. \]  

(52)

The acceleration and rotation components are same for the metrics given by Eqs.(8) and (9). The non-zero components of acceleration are

\[ \dot{u}_0 = \frac{1}{r}e^{\ln(\frac{r}{r_0})}, \quad \dot{u}_1 = -\frac{1}{r}. \]  

(53)

The rotation component turns out to be

\[ \omega_{01} = \frac{1}{r}e^{\ln(\frac{r}{r_0})}. \]  

(54)
The non-zero components of shear for the metric, given by Eq.(8), are
\[
\begin{align*}
\sigma_{00} &= \frac{4}{r} e^{2 \ln(r/r_0)}, \\
\sigma_{11} &= \frac{2}{3r}, \\
\sigma_{22} &= -\frac{8}{3r} a^2 e^{2 \ln(r/r_0)}, \\
\sigma_{33} &= -\frac{2}{3r}, \\
\sigma_{01} &= -\frac{7}{3r} e^{\ln(r/r_0)}. \\
\end{align*}
\]
(55)

The components of shear for the metric, given by Eq.(9), have the form
\[
\begin{align*}
\sigma_{00} &= \frac{4}{r} e^{2 \ln(r/r_0)}, \\
\sigma_{11} &= \frac{2}{3r}, \\
\sigma_{22} &= -\frac{8}{3r} a^2, \\
\sigma_{33} &= -\frac{8}{3r} e^{2 \ln(r/r_0)}, \\
\sigma_{01} &= -\frac{7}{3r} e^{\ln(r/r_0)}. \\
\end{align*}
\]
(56)

The shear invariant is same for the metrics, given by Eqs.(8) and (9), which turns out to be
\[
\sigma = \frac{167}{9 r^2}.
\]
(57)

The rate of expansion also remains the same for the two solutions given by Eqs.(8) and (9) and is the following
\[
\frac{d\Theta}{d\tau} = -\frac{188}{9 r^2} - \frac{1}{r}.
\]
(58)

For the third solution, given by Eq.(10), the acceleration and rotation components are zero. The non-zero components of shear are
\[
\begin{align*}
\sigma_{11} &= -\frac{4}{3r}, \\
\sigma_{22} &= -\frac{8}{3r} a^2 e^{2 \ln(r/r_0)}, \\
\sigma_{33} &= -\frac{8}{3r} e^{2 \ln(r/r_0)}, \\
\sigma_{01} &= \frac{2}{3r}.
\end{align*}
\]
(59)

The measure of anisotropy is
\[
\sigma = \frac{140}{9 r^2}
\]
(60)

and the expansion rate for this metric turns out to be
\[
\frac{d\Theta}{d\tau} = -\frac{152}{9 r^2}.
\]
(61)

### 3.2.3 Class C admitting seven homotheties

The solution for this class has two components of acceleration. These turn out to be
\[
\begin{align*}
\dot{u}_0 &= \frac{\alpha}{r} e^{\alpha \ln(r/r_0)}, \\
\dot{u}_1 &= -\frac{\alpha}{r}.
\end{align*}
\]
(62)
The expansion is given by
\[ \Theta = -3\frac{\alpha}{r} = 3\dot{u}_1 \] (63)
and the rotation component is
\[ \omega_{01} = \frac{\alpha}{r} e^{\alpha \ln\left(\frac{r}{r_0}\right)} . \] (64)

The components of shear are
\[ \sigma_{00} = 4\frac{\alpha}{r} e^{2\alpha \ln\left(\frac{r}{r_0}\right)} , \quad \sigma_{22} = -3\frac{\alpha}{r} a^2 e^{2\alpha \ln\left(\frac{r}{r_0}\right)} , \quad \sigma_{33} = \frac{1}{a^2} \sigma_{22}, \]
\[ \sigma_{01} = -2 \omega_{01} . \] (65)

The shear invariant becomes
\[ \sigma = 30\frac{\alpha^2}{r^2} = 30\dot{u}_1^2 \] (66)
and the rate of expansion turns out to be
\[ \frac{d\Theta}{d\tau} = -\frac{\alpha}{r^2} (33\alpha + 2) - \frac{\alpha}{r} . \] (67)

### 3.2.4 Class D admitting eleven homotheties

For the first metric in this class, given by Eq.(12), the components of acceleration and rotation turn out to be zero. The expansion becomes
\[ \Theta = -\frac{1}{r} . \] (68)

The components of shear take the form
\[ \sigma_{11} = -\frac{2}{3r} , \quad \sigma_{22} = -\frac{a^2}{3r} , \quad \sigma_{33} = -\frac{7}{3r} e^{2\ln\left(\frac{r}{r_0}\right)} , \quad \sigma_{01} = \frac{1}{3r} . \] (69)

The measure of anisotropy is given by
\[ \sigma = \frac{53}{9r^2} \] (70)
while the expansion rate turns out to be
\[ \frac{d\Theta}{d\tau} = -\frac{56}{9r^2} . \] (71)
For the second solution, given by Eq.(13), all the quantities vanish which is exactly similar to the co-moving case.

For the third metric, given by Eq.(14), two non-zero components of acceleration turn out to be

$$\dot{u}_0 = \frac{1}{r} e^{\ln(\frac{\tau}{\tau_0})}, \quad \dot{u}_1 = -\frac{1}{r}.$$  \hspace{1cm} (72)

The expansion becomes

$$\Theta = -\frac{1}{r} = \dot{u}_1$$  \hspace{1cm} (73)

and the rotation component is

$$\omega_{01} = \frac{1}{r} e^{\ln(\frac{\tau}{\tau_0})}.$$  \hspace{1cm} (74)

The shear components turn out to be

$$\sigma_{00} = \frac{4}{r} e^{2\ln(\frac{\tau}{\tau_0})}, \quad \sigma_{11} = \frac{4}{3r}, \quad \sigma_{22} = -\frac{1}{3r} a^2,$$

$$\sigma_{33} = -\frac{1}{3r}, \quad \sigma_{01} = -\frac{8}{3r} e^{\ln(\frac{\tau}{\tau_0})}.$$  \hspace{1cm} (75)

The measure of anisotropy is given by

$$\sigma = \frac{98}{9r^2}$$  \hspace{1cm} (76)

and the rate of expansion yields

$$\frac{d\Theta}{d\tau} = -\frac{101}{9r^2} - \frac{1}{r}.$$  \hspace{1cm} (77)

4 Singularities

In this section, we shall explore the singularities of the self-similar solutions given by Eqs.(5)-(14). We shall use the Kretschmann scalar to find the singularities of these solutions. The Kretschmann scalar is defined by

$$K = R_{abcd} R^{abcd},$$  \hspace{1cm} (78)

where $R_{abcd}$ is the Riemann tensor. For the solution, given by Eq.(5), it reduces to

$$K = 2 \frac{\alpha^2}{r^4} (\alpha^2 - 2\alpha + 2).$$  \hspace{1cm} (79)
Since $K$ diverges at $r = 0$ hence this solution is singular at $r = 0$.

For the second solution, given by Eq.(6), the Kretschmann scalar becomes

$$K = \frac{2}{r^4}(2\alpha^2\beta^2 + 3\beta^4 + \alpha^4 + \alpha^2(1 - 2\alpha) + 2\beta^2(1 - 2\beta)).$$

This also shows that the solution is singular at $r = 0$.

For the third solution, given by Eq.(7), the Kretschmann scalar turns out to be

$$K = \frac{2}{r^4}\alpha^2(\alpha^2 - 2\alpha + 2)$$

which again gives the singularity at $r = 0$.

For all the solutions of the class $B$, given by Eqs.(8)-(10), the Kretschmann scalar remains the same and is given by

$$K = \frac{2}{r^4}.$$ (82)

It is clear that $K$ diverges at the point $r = 0$. Thus the solutions are singular at $r = 0$.

For the solution in class $C$, given by Eq.(11), the Kretschmann scalar reduces to

$$K = 48\frac{\alpha^2}{r^4}(1 + 2\alpha^3 - 2\alpha).$$ (83)

It is obvious from here that $K$ diverges at $r = 0$ and consequently the solution is singular at $r = 0$.

For the last class, the Kretschmann scalar turns out to be zero for all the solutions.

## 5 Conclusion

Self-similar solutions in GR are very important and it is interesting to discuss their physical features. Keeping this point in mind, we have explored some kinematic properties and the singularity feature of such solutions representing cylindrically symmetric spacetime. We have discussed their properties both in co-moving as well as in non-co-moving coordinates (only in radial direction). We have explored acceleration, expansion, rotation, shear, rate of change of expansion and finally the singularity.

We obtain zero expansion in co-moving coordinates for all the solutions while in non-co-moving coordinates we have expansion negative/positive or
zero depending upon the value of $\alpha$ and $\beta$. There is an exceptional solution in the last class, given by Eq.(13), for which all the kinematical properties become zero in co-moving coordinates as well as in non-co-moving coordinates. In co-moving coordinates, it follows only one component of acceleration whereas there exist two components of acceleration in non-co-moving coordinates. The rotation component remains the same in both coordinates except the factor of half in non-co-moving coordinates. We see that only one component of shear exists in co-moving coordinates which is equal to negative of rotation in co-moving coordinates while four or five components of shear exist in non-co-moving coordinates. The measure of anisotropy is given by shear scalar and this scalar is much larger in non-co-moving coordinates than in co-moving coordinates. The rate of change of expansion is zero for the solutions given by Eqs.(10)-(13). Thus we can say that these solutions are in the state of equilibrium in co-moving coordinates. All the non-zero properties of each solution become infinite at $r = 0$.

We find that all the self-similar solutions of the first kind in classes $A$, $B$, $C$ are singular at $r = 0$ while in class $D$, the Kretschmann scalar becomes zero. We conclude that when we use non-co-moving coordinates, the quantities may become more complicated. However, we get simpler results in co-moving coordinates.

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