Structure of Supersymmetric Gauge Theories

Developments of theory of effective prepotential from extended Seiberg–Witten system and matrix models

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This is a semi-pedagogical review of a medium size on the exact determination of and the role played by the low energy effective prepotential $\mathcal{F}$ in quantum field theory with (broken) extended supersymmetry, which began with the work of Seiberg and Witten in 1994. While paying attention to an overall view of the over two decades of development of this subject, we probe several corners marked in the three major stages of the developments, emphasizing uses of the deformation theory on the attendant Riemann surface as well as its close relation to matrix models. Examples picked here in different contexts tell us that the effective prepotential is to be identified as the suitably defined free energy $F$ of a matrix model: $\mathcal{F} = F$.

Subject Index B06, B10, B16, B83

1. Introduction

The notion of effective action plays a vital role in the modern treatment of quantum field theory (see, for instance, [1,2]). In this review article, we deal with a special class of low energy effective actions that are controlled by (broken) extended rigid supersymmetry in four spacetime dimensions and permit exact determination exploiting integrals on the Riemann surface in question. A main object in such study is the low energy effective prepotential, to be denoted by $\mathcal{F}$ generically in this paper, which has proven to be central not only in the original case of unbroken $\mathcal{N} = 2$ supersymmetry initiated by the work of Seiberg–Witten [3,4] but also in the case where this symmetry is broken by the vacuum or by the superpotential. The review will be presented basically in a chronological order, following the three major stages of the developments that took place during the periods 1994~, 2002~, and 2009~. Each of the three subsequent sections will explain pieces of work done in its respective period.

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An emphasis will be put on the deformation theory of the effective prepotential on the Riemann surface as an extension of the Seiberg–Witten system consisting of the curve, the meromorphic differential, and the period, as well as its close relation to matrix models.

We conclude from the examples taken here in the different contexts that the effective prepotential is in fact identified as the suitably defined free energy $F$ of a matrix model: $\mathcal{F} = F$. While this is hardly a surprising conclusion from the point of view of mathematics of integrable systems and soliton hierarchies, the number of examples in quantum field theory (QFT) where this is explicitly materialized is not large enough. This note may serve to improve the situation.

In the next section, after presenting the curve for $\mathcal{N} = 2$, SU($N$) pure super Yang–Mills theory as a spectral curve of the periodic Toda chain, we discuss the deformation of the effective prepotential by placing higher order poles to the original meromorphic differential. We give a derivation of the formula which the meromorphic differential extended in this way obeys.

In Sect. 3, we discuss the degeneration phenomenon of the Riemann surface necessary to describe the $\mathcal{N} = 1$ vacua that lie in the confining phase and introduce the prepotential having gluino condensates as variables. We apply the formalism in Sect. 2 here, and describe the situation by the use of mixed second derivatives. After discussing the emergence of the matrix model curve and giving a sample calculation, we finish the section with the case of spontaneously broken $\mathcal{N} = 2$ supersymmetry in order to illustrate the role played by the two distinct singlet operators, one of which is the QFT counterpart of the matrix model resolvent.

In Sect. 4, we go back to the situation of $\mathcal{N} = 2$ and discuss the developments associated with the AGT relation and the upgraded treatment of the all-genus instanton partition function and therefore the deformation of the Seiberg–Witten curve to its noncommutative counterpart. A finite $N$ and $\beta$-deformed matrix model with filling fractions specified emerges as an integral representation of the conformal/W block and we discuss the direct evaluation of its $q$-expansion as the Selberg integral. We finish the section by mentioning some of the more recent developments.

Please note that the model or theory hops from one to the other as the sections proceed and that each section has an open ending, indicating calls for further developments of this long-lasting subject.

2. Effective prepotential from extended Seiberg–Witten system

We will not give here an account of the construction of the curve itself [3–9] (for a recent review, see, for instance, [10]), nor its connection to classical integrable systems [11–26]. Also omitted is the discussion associated with the WDVV equation, for which we direct the readers to [27–29] as well as references contained in [30,31].

2.1. Curves, periods, and meromorphic differentials

The list of papers which discuss subjects closely related to that of this subsection include [3–10,14–25,31–57].

Let us recall the most typical situation and consider the low energy effective action (LEEA) for $\mathcal{N} = 2$, SU($N$) pure super Yang–Mills theory. The symmetry of LEEA at a scale much smaller than that of the W boson mass is $U(1)^{N-1}$. The relevant curve is a hyperelliptic Riemann surface of genus $N - 1$ (see Fig. 1) described as

$$Y^2 = P_N^2(x) - 4A^{2N},$$  (2.1)
where
\[ P_N(x) = \langle \det(xI - \Phi) \rangle = \prod_{i=1}^{N} (x - p_i) = x^N - \sum_{k=2}^{N} u_k x^{N-k} = \sum_{k=0}^{N} s_k(h_\ell) x^{N-k}. \] (2.2)

Here,
\[ h_\ell = \frac{1}{\ell} (\text{tr} \Phi^\ell) = \frac{1}{\ell} \sum_{i=1}^{N} p_i^\ell, \quad \ell = 2, 3, \ldots, N, \] (2.3)

and \( s_k(h_\ell) \) are the appropriate Schur polynomials. Introducing the spectral parameter \( z \), we write the curve as that of the periodic Toda chain:
\[ P_N(x) = z + \frac{\Lambda^{2N}}{z}, \] (2.4)
\[ Y = z - \frac{\Lambda^{2N}}{z}. \] (2.5)

The distinguished meromorphic differential for the construction of the effective prepotential is given by
\[ \tilde{S}_{SW} \bigg|_{\frac{z}{\Lambda}} = x \log z = x t(x) dx, \quad t(x) = \frac{P_N'}{\sqrt{P_N^2 - 4 \Lambda^2}}. \] (2.6)

The characteristic feature of this is the existence of double poles at \( \infty \pm \). Later in this section, we interpret this to be the case where only \( T_1 \) has been turned on.

The defining property is that the moduli derivatives are holomorphic:
\[ \frac{\partial}{\partial u_k} \tilde{S}_{SW} \bigg|_{x, \Lambda} = \frac{x^{N-k}}{Y} dx, \] (2.6)
\[ \frac{\partial}{\partial u_k} \tilde{S}_{SW} \bigg|_{x, \Lambda} = \frac{x^{N-k}}{Y} dx - d \left( \frac{x^{N-k+1}}{Y} \right). \] (2.7)

The prepotential \( F_{SW} \) is introduced implicitly by the A cycle and B cycle integrations on the Riemann surface:
\[ a_i = \oint_{A_i} d\tilde{S}_{SW}, \quad \frac{\partial F_{SW}}{\partial a_i} = a_i^D = \oint_{B_i} d\tilde{S}_{SW}. \] (2.8)

While \( u_k \) possess invariant meaning both in the moduli space of the Riemann surface and in the integrable system, it is these constant background fields or Coulomb moduli \( a_i, a_i^D = \frac{\partial F_{SW}}{\partial a_i} \) which are directly related to the observables through the BPS formula. The moduli derivatives are coordinate dependent as we see in Eqs. (2.6) and (2.7). The final expression for \( F_{SW} \) is going to be
coordinate independent. This is supported by the pieces of evidence we present here that the effective prepotential is identified as the free energy of a matrix model.

2.2. Whitham deformation of the prepotential and the appearance of the “thermodynamic” relation

The list of papers which discuss subjects closely related to that of this subsection include [24,31,57–69].

We would now like to review the deformation of the effective prepotential above, which we have denoted by \( F_{SW} \). The basic idea of this extended theory of effective prepotential often referred to as Whitham deformation is to deform both moduli of the Riemann surface and the meromorphic differential above consistently without losing the defining properties:

\[
d\hat{S}_{SW} \to d\hat{S}; \quad \frac{\partial}{\partial h_k} d\hat{S}\bigg|_{*,\Lambda} = \text{holomorphic.} \quad (2.9)
\]

We have adopted the choice that \( z \) is fixed when the moduli derivatives are taken. We carry out the deformation by adding higher order poles to the original meromorphic differential containing the double poles. Let us denote the local coordinates in their neighborhood generically by \( \xi \) and

\[
\xi = z \pm \frac{1}{\pi} \quad \text{or} \quad x^{-1}. \quad (2.10)
\]

In order to describe the deformation, let us introduce a set of meromorphic differentials \( d\Omega_\ell \) that satisfy

\[
d\Omega_\ell = \xi^{-\ell-1}d\xi + \text{non-singular part} \quad \ell = 1, 2, 3, \ldots \quad (2.11)
\]

We are still left with the ambiguities that any linear combination of the canonical holomorphic differentials \( d\omega_i \) can be added to the right-hand side. In order to remove these, let us require a set of conditions

\[
\oint_{A_i} d\Omega_\ell = 0. \quad (2.12)
\]

The ones which are not subject to the conditions (2.12) are denoted by \( d\hat{\Omega}_\ell \).

Let us first state the formula,

\[
d\hat{S} = \sum_{i=1}^{g} a^i d\omega_i + \sum_{\ell \geq 1} T_\ell d\Omega_\ell, \quad (2.13)
\]

and outline its derivation below. As before, \( a^i \) are defined to be the local coordinates in the moduli space

\[
a^i \equiv \oint_{A_i} d\hat{S}, \quad (2.14)
\]

while \( T_\ell \), referred to as time variables or T moduli, are given by

\[
T_\ell = \text{res}_{\xi=0} \xi^{-\ell}d\hat{S}, \quad (2.15)
\]

once Eq. (2.13) is established. One then regards \( a^i \) and \( T_\ell \) as independent, taking \( h_k \) dependent: \( h_k = h_k(a^i, T_\ell) \). The (extended) effective prepotential \( F(a^i, T_\ell) \) is introduced via

\[
\frac{\partial F}{\partial a^i} = \oint_{B^i} d\hat{S}, \quad \frac{\partial F}{\partial T_\ell} = \frac{1}{2\pi i \ell} \text{res}_{\xi} \xi^{-\ell}d\hat{S} \equiv \mathcal{H}_{\ell+1}(h_k). \quad (2.16)
\]
The derivation of (2.13) begins with the introduction of the time variables $T_\ell$ via a solution $d\hat{S}(T_\ell|h)$ to Eq. (2.9),

\[ \frac{\partial d\hat{S}}{\partial T_\ell} = d\Omega_\ell, \quad \text{and hence} \quad \frac{\partial a_i}{\partial T_\ell} = 0. \]

(2.17)

In terms of our intermediate bases $d\hat{\Omega}_\ell$, Eq. (2.9) reads

\[ \frac{\partial}{\partial h_k} d\hat{\Omega}_\ell = \sum_{i=1}^{g} \sigma_{ki}^{(\ell)} d\omega_i, \]

while

\[ d\hat{\Omega}_\ell = d\Omega_\ell + \sum_{i=1}^{g} c_i^{(\ell)} d\omega_i, \]

(2.18)

(2.19)

as the difference between $d\hat{\Omega}_\ell$ and $d\Omega_\ell$ can be spanned by the holomorphic differentials. Expand the solutions as

\[ d\hat{S} = \sum_m \beta_m(T) d\hat{\Omega}_m(h), \]

hence

\[ \frac{\partial d\hat{S}}{\partial T_n} = \sum_m \left( \frac{\partial \beta_m}{\partial T_n} d\hat{\Omega}_m + \beta_m \sum_k \frac{\partial h_k}{\partial T_n} \sum_{i=1}^{g} \sigma_{ki}^{(m)} d\omega_i \right). \]

(2.21)

Exploiting Eqs. (2.17), (2.19), and (2.21), we obtain

\[ \frac{\partial \beta_m}{\partial T_n} = \delta_{m,n} \quad \text{i.e.} \quad \beta_m(T) = T_m \]

(2.22)

as well as

\[ \sum_k \frac{\partial h_k}{\partial T_n} \left( \sum_m T_m \sigma_{ki}^{(m)} \right) = -c_i^{(n)}. \]

(2.23)

Substituting Eqs. (2.22) and (2.19) into Eq. (2.20), we obtain

\[ d\hat{S} = \sum_m T_m d\Omega_m + \sum_m T_m \sum_i c_i^{(m)} d\omega_i, \]

(2.24)

whose integrations over the $A_i$ cycles yield

\[ a_i = \sum_m T_m c_i^{(m)}. \]

(2.25)

This shows Eq. (2.13).

2.3. Connection with the planar free energy of matrix models

Already at this stage of the developments, a connection between the extended Seiberg–Witten system and the construction of matrix models in general, or more specifically, the similarity of the effective prepotentials with the (planar) free energy of matrix models, was visible. In fact, starting from the homogeneity of the moduli and the prepotential, it is possible to derive an integral expression for $\mathcal{F}$ which resembles that of the matrix model planar free energy in terms of the density one-form on the eigenvalue coordinate (see Eq. (4.12) of [24]). See also [14,16,18].

One of the goals of the present review is to put together several subsequent developments that took place and have made this phenomenon more prominent. These are presented in the next two sections.
Table 1. Picture we want to materialize as prepotential theory.

| $\mathcal{N} = 2$ | $\mathcal{N} = 1$ |
|-------------------|-------------------|
| $U(N)$ pure SYM | \( \prod_{i=1}^{n} U(N_i) \) |
| deformed by superpotential | \( g_{n+1} \int d^2\theta \text{tr} W_{n+1}(\Phi) \) such that \( W'_{n+1}(x) = \prod_{i=1}^{n} (x - \alpha_i) \) |
| LEEA | \( U(1)^{N-1} \times U(1) \) \( \longrightarrow \) \( U(1)^{n-1} \times U(1) \times \prod_{i=1}^{n} SU(N_i) \) |
| Coulomb | Coulomb |
| RS | \( \text{degeneration} \) |
| | \( \text{n groups} \ldots \) |

3. Gluino condensate prepotential

One major use of the deformation theory of the effective prepotential presented above took place in the context of gluino condensate prepotential built on various $\mathcal{N} = 1$ vacua in contrast to $\mathcal{F}_{SW}$ and its extension in Sect. 2. We first consider the case in which the breaking to $\mathcal{N} = 1$ from $\mathcal{N} = 2$ supersymmetry is caused by the superpotential in the action. Later we will contrast this with the case in which $\mathcal{N} = 2$ is broken spontaneously to $\mathcal{N} = 1$ at the tree level [70–74].

3.1. Degeneration phenomenon and mixed second derivatives

The list of papers which discuss subjects closely related to that of this subsection include [30,75–106].

Let’s fix an action to work with: it is a $U(N)$ gauge theory consisting of adjoint vector superfields and chiral superfields with canonical kinematic factors, and the superpotential turned on in the $\mathcal{N} = 2$ action drives the system to its $\mathcal{N} = 1$ vacua.

As a phenomenon occurring on a Riemann surface, we consider the situation where a degeneration takes place and some of the cycles coalesce to form a new set of cycles. As for the description of the LEEA, some of the original Coulomb moduli disappear and the product of these $U(1)$s gets replaced by non-Abelian gauge symmetry $\prod_{i=1}^{n} SU(N_i)$. We tabulate these pictures in Table 1. The $\mathcal{N} = 1$ vacua are labelled by the set of order parameters representing gluino condensates:

\[
S_i \propto \text{Tr}_{SU(N_i)} W^\alpha W_\alpha, \quad i = 1, \ldots, n.
\] (3.1)

The proportionality constant will be fixed in subsequent subsections.

We now review, following the observation made in [105], that the condition for a curve to degenerate or factorize is given by the kernel of the matrix made of the mixed second derivatives of the deformed prepotential being nontrivial.

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1 Actually, supersymmetry is broken dynamically in the metastable vacua in both cases as was demonstrated in [75,76] in the Hartree–Fock approximation.
Continuing with the general discussion of Sect. 2.2, let us first note that we obtain two different expressions for the mixed second derivatives from Eq. (2.16):

\[ \frac{\partial^2 F}{\partial a^i \partial T_\ell} = \oint_{B_i} \frac{1}{2\pi i \ell} \text{res} \xi^{-\ell} d\omega_i, \quad i = 1, \ldots, N - 1, \quad \ell : \text{positive integers.} \]  

We impose the condition

\[ \ker \frac{\partial^2 F}{\partial a^i \partial T_\ell} \neq 0, \quad \text{or rank} \frac{\partial^2 F}{\partial a^i \partial T_\ell} \leq N - 2. \]  

Equation (3.3) has following straightforward implications:

(i) there exists a nonvanishing column vector \((e^1, e^2, \ldots, e^{N-1}, \ldots)\)' such that

\[ 0 = \sum_{\ell} \frac{\partial^2 F}{\partial a^i \partial T_\ell} c^\ell = \sum_{\ell} \oint_{B_i} d\Omega_\ell c^\ell = \frac{1}{2\pi i \ell} \text{res} \left( \sum_{\ell} c^\ell \xi^{-\ell} \right) d\omega_i. \]  

Here, we have exploited Eq. (2.17) in the second equality and Eq. (2.10) in the third equality. The former equality implies that \(d\tilde{\Omega} = \sum c^\ell d\Omega_\ell\) has vanishing periods over all \(A_i\) and \(B^i\) cycles. Then one can integrate this form along any path ending with a point \(z\) to define a function holomorphic except at punctures. As for the order of the poles at the punctures, it is generically arbitrary according to the construction. But this is contradictory to the Weierstrass gap theorem\(^2\) derived from the Riemann–Roch theorem. To avoid a contradiction, we must have a degeneration.

(ii) there exists a nonvanishing row vector \((\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_{N-1})\) such that

\[ 0 = \sum_{i=1}^{N-1} \tilde{c}_i \frac{\partial^2 F}{\partial a^i \partial T_\ell} = \sum_{i} \tilde{c}_i \frac{\partial H_{\ell+1}}{\partial a^i} \]  

in accordance with the second formula of Eq. (2.16). Equation (3.7) follows from

\[ \sum_{i} \tilde{c}_i \frac{\partial h_{\ell+1}}{\partial a^i} = 0, \]  

which is regarded as the statement of the vanishing discriminant. The moduli actually depend on less than \(N - 1\) arguments.

3.2. Emergence of the matrix model curve

The list of papers which discuss subjects closely related to that of this subsection include [30,100, 105,107–123].

Once we are convinced of the degeneration of the surface, we can proceed further by factorizing the original curve, which, in the current example, is the hyperelliptic one.

\[ ^2\text{The Weierstrass gap theorem states that} \]

for a given Riemann surface \(M\), with genus \(g\), and a point \(P \in M\),

\[ \begin{align*}
\text{(3.5)}
\text{and } g \text{ integers satisfying } 1 < n_1 < n_2 < \cdots < n_g < 2g, \quad \text{(3.6)}
\end{align*} \]

there does NOT exist a function \(f\) holomorphic on \(M \setminus \{P\}\) with a pole of order \(n_j\) at \(P\).
Let \( n - 1 \) be the number of genuses after the degeneration. Following \[88,89\], we state

\[
\begin{align*}
Y^2 &= H_{N-1}(x)^2 F_{2n}(x), \\
P'_N(x) &= H_{N-n}(x) R_{n-1}(x).
\end{align*}
\]

(3.9)

Finally, let us examine the last equality of Eq. (3.4). Let

\[
\sum_{\ell=1}^{N-1} c_{i,\ell} \left( \ell^{-\ell} \right) = W'_{k+1}(x) \equiv \prod_{j=1}^{k}(x - \alpha_j),
\]

(3.10)

and \( \sqrt{F_{2n}} \) serve as bases of the holomorphic differentials of the reduced Riemann surface. Actually, only the \( j = 1 \ldots n - 1 \) differentials are holomorphic and the \( j = n \) one has been added through the blow-up process, which physically implies that the overall \( U(1) \) fails to decouple. We obtain

\[
0 = \operatorname{res}_{x=\infty} \left( W'_{k+1}(x) \frac{x^{-j-1}}{\sqrt{F_{2n}}} \right),
\]

(3.11)

and therefore

\[
\frac{W'_{k+1}}{\sqrt{F_{2n}}} = Q_{k-n}(x) + \sum_{\ell>n} \beta_{\ell} x^{-\ell}.
\]

(3.12)

We obtain

\[
y^2 \equiv F_{2n} Q_{k-n}^2 = W'_{k+1} + f_{k-1}.
\]

(3.13)

Here, \( f_{k-1} \) is a polynomial of degree \( k - 1 \). This is the curve appearing in the \( k \)-cut solution of the matrix model.

We still need to see that \( W_{k+1}(x) \) introduced above is in fact a tree level superpotential. This is easily done by taking the classical limit \( \Lambda = 0 \):

\[
Y = z = \prod_{\ell=1}^{N}(x - p_{\ell}).
\]

(3.14)

The original Seiberg–Witten differential becomes

\[
d\hat{S}^{\text{(class)}}_{SW} = x \sum_{i=1}^{N} \frac{1}{x - p_i} dx,
\]

(3.15)

which is equal to \( \sum_{i=1}^{N} p_i d\omega_i + N dx \).

(3.16)

Here, we have used that the canonical holomorphic differential becomes

\[
d\omega_i^{\text{(class)}} = \frac{dx}{x - p_i}
\]

(3.17)

in this limit. The period integrals over the \( A_i \) cycles just pick up the residues at the poles \( p_i \):

\[
a_i^{\text{(class)}} = p_i.
\]

(3.18)

The degeneration in this limit is described as

\[
z = \prod_{j=1}^{n}(x - \beta_j)^{N_j}, \quad \sum_{j=1}^{n} N_j = N.
\]

(3.19)
Fig. 2. The reduced curve of $g = n - 1$.

In fact, the $N_j$ poles coalesce at $\beta_j$, $j = 1, \ldots, n$ and the canonical holomorphic differentials on the degenerate curve are

$$d\omega_{j}^{\text{class,red}} = \frac{dx}{x - \beta_j}.$$  \hspace{1cm} (3.20)

The condition Eq. (3.11) becomes

$$0 = \text{res}_{x=0} \left( W'_k(x)d\omega_{j}^{\text{class,red}} \right), \quad j = 1, \ldots, n,$$  \hspace{1cm} (3.21)

which tells us that $\beta_j$ must coincide with one of the roots $\alpha_j$ of $W'_k$. The vacuum expectation values (VEVs) of the adjoint scalar fields are thus constrained to the extrema of $W_k$.

Let us set $k = n$ for simplicity. We have the reduced curve of $g = n - 1$ (see Fig. 2):

$$y^2 = W'_{n+1}(x; \alpha_j)^2 + f_{n-1}(x),$$  \hspace{1cm} (3.22)

and let us denote the coefficients of the polynomial $f_{k-1}$ by $b_{\ell}(\alpha_j)$, temporarily forgetting the $\alpha_j$ dependence. We also mention here that the full set of parameters (moduli) of the model realized by the curve Eq. (3.22) is $2n$ dimensional and can be represented by the cut lengths and cut positions:

$$\text{dim(moduli)} = 2n \approx \text{cut lengths} + \text{cut positions}.$$  \hspace{1cm} (3.23)

3.3. Practical calculation

The list of papers which discuss subjects closely related to that of this subsection include [97,100,124–133].

Let us now proceed to discuss the use of this machinery in calculation. As the condensates $S_i$ are quantum mechanical in nature, one can develop loop expansion using these, including the Veneziano–Yankielowicz term which contains the logarithmic singularity [77]. The first question to be raised is what the distinguished meromorphic differential is that is to be used for such a calculation. It must be “almost” holomorphic after the $b_{\ell}$ derivatives are taken. Recall that the bases of the “holomorphic” differentials are taken as $\frac{1}{y^{j-1}}$, $j = 1, \ldots, n - 1, n$. Rather obviously, such differential is found as

$$d\hat{S}_{\text{mat}} = y(x)dx, \quad \text{with } T_2, \ldots, T_n, T_{n+1} \text{ turned on.}$$  \hspace{1cm} (3.24)

As before, the effective prepotential is introduced through the period integrals

$$S_i = \oint_{A_i} d\hat{S}_{\text{mat}} \quad i = 1, \ldots, n, \quad \text{and}$$

$$\frac{\partial \mathcal{F}}{\partial S_i} = 2 \int_{\text{cutoff}}^{\text{edge}} d\hat{S}_{\text{mat}}.$$  \hspace{1cm} (3.25)
We have, however, no reason to set
\[ S = \sum_{i=1}^{n} S_i = \int_{\prod_{i=1}^{n} \cup A_i} d\hat{S}_{\text{mat}} \]  
(3.26)
equal to zero. This tells us the presence of the cutoff at the infinities of the surface.

The expansion of \( F \) in \( S_i \) was done in [97], exploiting Eq. (3.25) and the small cut expansion as an intermediate step originally. This provided the answer given below for \( F \) to the cubic order in \( S_i [\text{Eqs. } (3.34)-(3.38)] \). Yet, there exists a simpler procedure, namely, a calculus from \( T \) moduli thanks to the machinery discussed in the present review. The \( T \) moduli are easily identified as
\[ T_{m+1} = \text{res}_x x^{-m-1} d\hat{S}_{\text{mat}} = g u_m, \quad u_m = (-)^{n-m} e_n^{(\alpha)}, \]  
(3.27)
where
\[ e_m(\alpha) = \sum_{i_1 < \cdots < i_m} \alpha_{i_1} \cdots \alpha_{i_m}. \]  
(3.28)
The dependence of the prepotential on the \( T \) moduli is determined by the equations
\[ \frac{1}{g} \frac{\partial F}{\partial u_\ell} = \frac{\partial F}{\partial T_{\ell+1}} = \frac{1}{\ell + 1} \text{res}_x (x^{\ell+1} - \Lambda^{\ell+1}) d\hat{S}_{\text{mat}}. \]  
(3.29)
Here, \( \Lambda^{\ell+1} \) is the term introduced in [100] in order to match with the computation done earlier.

In order to carry out this task, we introduce intermediate expansion variables \( \tilde{S}_i \) and parameterize the matrix model curve Eq. (3.22) by
\[ f_{n-1}(x) = \sum_{i=1}^{n} \tilde{S}_i \prod_{j \neq i} (x - \alpha_j) = W'_{n+1}(x) \sum_{i=1}^{n} \frac{\tilde{S}_i}{x - \alpha_i}. \]  
(3.30)
The differential \( d\hat{S}_{\text{mat}} \) of Eq. (3.24) has a straightforward expansion in \( \tilde{S}_i \). Therefore, \( A_i \) cycle integrations followed by the inversion provide an expansion of \( \tilde{S}_i \) in \( S_j \):
\[ \tilde{S}_i = S_i + \frac{1}{2g} \sum_{j,k} \frac{1}{\alpha_{i_j} \alpha_{i_k} \Delta_i} S_j S_k + \cdots. \]  
(3.31)
Here, we have introduced \( \alpha_{i_j} = \alpha_i - \alpha_j \), and \( \Delta_i = \prod_{j \neq i} \alpha_{ij} \). Other useful machinery is the \( T_m \) moduli derivatives of the roots \( \alpha_i \) of the superpotential, which read \( \frac{\partial \alpha_i}{\partial u_m} = -\frac{\alpha_i^m}{\Delta_i} \). Using these, the right-hand side of Eq. (3.29) is evaluated as
\[ \sum_{i} S_i \left( \frac{\partial W_{n+1} (\alpha_i)}{\partial u_\ell} - \frac{\partial W_{n+1} (\Lambda)}{\partial u_\ell} \right) - \frac{1}{4} \sum_{j<k} \left( S_j^2 + S_k^2 - 4S_j S_k \right) \frac{\partial}{\partial u_\ell} \log \alpha_{jk} + \cdots, \]  
(3.32)
which is trivially integrated in \( u_m \) to provide an answer. Let us mention that this procedure is straightforwardly generalizable to higher order contributions in \( S_i \) and that the terms independent of \( \alpha_i \) can be easily obtained by several other methods.
The expansion form of $\mathcal{F}(S|\alpha)$ which we managed to have proposed in [97] is

$$2\pi i \mathcal{F}(S|\alpha) = 4\pi i g_{n+1} \left( W_{n+1}(\Lambda) \sum_i S_i - \sum_i W_{n+1}(\alpha_i) S_i \right) - \left( \sum_i S_i \right)^2 \log \Lambda$$

$$+ \frac{1}{2} \sum_{i=1}^{n} S_i^2 \left( \log \frac{S_i}{4} - \frac{3}{2} \right) - \frac{1}{2} \sum_{i<j}^{n} \left( S_i^2 - 4S_iS_j + S_j^2 \right) \log \alpha_{ij} + \sum_{k=1}^{\infty} \frac{\mathcal{F}_{k+2}(S|\alpha)}{(i\pi g_{n+1})^k}.$$  

(3.33)

Here, we have denoted by $\mathcal{F}_{k+2}(S|\alpha)$ the contributions of the $k+2$ order polynomials in $S_i$. The explicit answer for $\mathcal{F}_3(S|\alpha)$ is

$$\mathcal{F}_3(S|\alpha) = \sum_{i=1}^{n} u_i(\alpha) S_i^3 + \sum_{i\neq j}^{n} u_{ij}(\alpha) S_i^2 S_j + \sum_{i<j<k}^{n} u_{ijk}(\alpha) S_i S_j S_k,$$  

(3.34)

$$u_i(\alpha) = \frac{1}{6} \left( -\sum_{j\neq i}^{\alpha_i/j} \frac{1}{\alpha_{ij}^2 \Delta_j} + \frac{1}{4\Delta_i} \sum_{j,k \neq i}^{\alpha_{ij} \alpha_{jk}} \alpha_{ij} \alpha_{jk} \right),$$  

(3.35)

$$u_{ij}(\alpha) = \frac{1}{4} \left( -\frac{3}{\alpha_{ij}^2 \Delta_i} + \frac{2}{\alpha_{ij}^2 \Delta_j} - \frac{2}{\alpha_{ij} \alpha_{jk}} \sum_{k \neq i,j}^{1} \alpha_{jk} \right),$$  

(3.36)

$$u_{ijk}(\alpha) = \frac{1}{\alpha_{ij} \alpha_{ik} \Delta_i} + \frac{1}{\alpha_{ij} \alpha_{jk} \Delta_j} + \frac{1}{\alpha_{ik} \alpha_{jk} \Delta_k},$$  

(3.37)

$$\Delta_i = W_{n+1}''(\alpha_i) = \prod_{j \neq i}^{n} \alpha_{ij}.$$  

(3.38)

For the computation of higher orders as well as the inclusion of matter, see, for instance, [128,131–133].

### 3.4. The case of spontaneously broken $N = 2$ supersymmetry and the Konishi anomaly equation

The list of papers which discuss subjects closely related to that of this subsection include [70–74,87,134–174].

The $N = 2$ effective action is completely characterized by the effective prepotential while, in the $N = 1$ case, a typical observable is (the matter induced part of) the effective superpotential. The interplay of these two upon the degeneration of the original Riemann surface is most clearly seen by dealing with the case of spontaneously broken $N = 2$ supersymmetry. This case accomplishes a continuous deformation from one to the other by tuning the electric and magnetic Fayet–Iliopoulos (FI) parameters. The action $S_{N=2}^{F}\in$ realizing this is given by

$$S_{N=2}^{F\in} = \int d^4x d^4\theta \left[ -\frac{i}{2} \text{Tr} \left( \Phi e^{a\Phi} \frac{\partial \mathcal{F}_{\text{in}}(\Phi)}{\partial \Phi} - h.c. \right) + \xi V^0 \right]$$

$$+ \left[ \int d^4x d^2\theta \left( -\frac{i}{4} \frac{\partial^2 \mathcal{F}_{\text{in}}(\Phi)}{\partial \Phi^a \partial \Phi^b} W^a_{\alpha} W^b_{\alpha} + e\Phi^0 + m \frac{\partial \mathcal{F}_{\text{in}}(\Phi)}{\partial \Phi^0} + h.c. \right) \right].$$  

(3.39)
Here, $\xi$, $e$, and $m$ are the electric and magnetic FI terms and we vary these to interpolate the two ends, keeping $\tilde{g}_\ell = mg_\ell \ (\ell \geq 2)$ fixed:

\[
\begin{align*}
\text{large } (\xi, e, m) & \quad \mathcal{S}_{N=1} \\
\text{small } (\xi, e, m) & \quad \mathcal{S}_{N=2}
\end{align*}
\]

In this subsection, we have denoted by the symbol $\mathcal{F}_{\text{in}}$ an input function in the effective action Eq. (3.39). For definiteness, we let the function $\mathcal{F}_{\text{in}}$ be a single trace function of a polynomial in $\Phi$,

\[
\mathcal{F}_{\text{in}}(\Phi) = \sum_{\ell=1}^{n+1} \frac{g_\ell}{(\ell + 1)!} \text{Tr}\Phi^{\ell+1}, \quad \deg \mathcal{F}_{\text{in}} = n + 2,
\]

and the matter induced part of the effective superpotential $W_{\text{eff}}$ be

\[
\exp \left\{ i \int d^4x \left( d^2 \theta W_{\text{eff}} + \text{h.c.} + (\text{D-term}) \right) \right\} = \int \mathcal{D}\Phi \mathcal{D}\Phi e^{i\mathcal{F}_{\text{in}}}.
\]

Let us now turn to the generalized Konishi anomaly equation. It is the anomalous Ward identity of the theory given by Eq. (3.39) and is derived by considering a response of the system under for the general local transformation $\delta \Phi = f(\Phi, \mathcal{W})$:

\[
-\left\langle \frac{1}{64\pi^2} \left[ \mathcal{W}^a, \left[ \mathcal{W}_a, \frac{\partial f}{\partial \Phi_{ij}} \right] \right] \right\rangle = \left\langle \text{Tr} f W'(\Phi) \right\rangle - \left\langle \frac{i}{4} \text{Tr} \left( f \mathcal{F}_{\text{in}}''(\Phi) \mathcal{W}^a \mathcal{W}_a \right) \right\rangle.
\]

The left-hand side is the contribution of the Konishi anomaly [80], which arises from the behavior of the functional integral measure under the transformation [175,176]. Introducing the two generating functions, we recast this into the following set of equations [161]:

\[
R(z) \equiv -\frac{1}{64\pi^2} \left( \text{Tr} \frac{\mathcal{W}^a \mathcal{W}_a}{z - \Phi} \right),
\]

\[
T(z) \equiv \left( \text{Tr} \frac{1}{z - \Phi} \right),
\]

\[
R(z)^2 = W'(z)R(z) + \frac{1}{4} f(z),
\]

\[
2R(z)T(z) = W'(z)T(z) + 16\pi^2 i \mathcal{F}_{\text{in}}'''(z)R(z) + \frac{1}{4} c(z),
\]

where $f(z)$ and $c(z)$ are polynomials of degree $n - 1$ and, with some abuse of notation,

\[
\mathcal{F}_{\text{in}}'''(z) = \sum_{\ell=2}^{n+1} \frac{g_\ell z^{\ell-2}}{(\ell - 2)!} = \frac{W'''(z)}{m}.
\]
Our final goal in this subsection is to derive a formula for the effective superpotential. Let us define the one point functions as
\[
v_\ell = -\frac{1}{64\pi^2} \left( \text{Tr} W'^{\alpha} W^{\alpha} \Phi^{\ell} \right), \quad u_\ell = \left\langle \text{Tr} \Phi^{\ell} \right\rangle, \quad \text{for } 1 \leq \ell \leq n + 1.
\] (3.47)

In terms of \( v_\ell \) we define \( F \) as
\[
\frac{\partial F}{\partial g_\ell} = \frac{m}{\ell!} v_\ell, \quad \text{for } 1 \leq \ell \leq n + 1.
\] (3.48)

Using \( F \), we can state the relation to be proven:
\[
W_{\text{eff}} = \sum_i N_i \frac{\partial F}{\partial S_i} + \frac{16\pi^2 i}{m} \sum_{\ell=2}^{n+1} g_\ell \frac{\partial F}{\partial g_{\ell-1}}.
\] (3.49)

Before proceeding to the proof of this relation, let us go back to Eqs. (3.44) and (3.45) to obtain the complete information. We consider the most general case that the gauge symmetry \( U(N) \) is broken to \( \prod_{i=1}^k U(N_i) \) with \( k < n \), \( \sum_{i=1}^k N_i = N \). The indices \( i, j, \ldots \) run from 1 to \( k \), while the indices \( I, J, \ldots \) run from 1 to \( N \). Of course, \( N_I = 0 \) \( (k + 1, \ldots, n) \). Solving Eq. (3.44), we obtain
\[
R(z) = \frac{1}{2} \left( W'(z) - \sqrt{W'(z)^2 + f(z)} \right),
\] (3.50)

where the Riemann surface \( \Sigma \) is genus \( n - 1 \) but its \( A_I \) cycles for \( I = k + 1, \ldots, n \) are vanishing. We conclude that the meromorphic function lives on a factorized curve
\[
y^2 = W'(z)^2 + f(z) = N_{n-k}(z)^2 F_{2k}(z),
\] (3.51)

\[
y^2_{\text{red}} = F_{2k}(z).
\] (3.52)

Here, \( N_{n-k}(z) \) and \( F_{2k}(z) \) are polynomials of degree \( n - k \) and \( 2k \) respectively. On the other hand, substituting Eq. (3.50) into Eq. (3.45), we obtain
\[
T(z) = -\frac{c(z)}{4\sqrt{W'(z)^2 + f(z)}} + 8\pi^2 i \left( F'''_{\text{in}}(z) - \frac{W'(z) F'''_{\text{in}}(z)}{\sqrt{W'(z)^2 + f(z)}} \right).
\] (3.53)

Let us list a few formulas that are obtained from Eq. (3.50) directly. The first set is [117]
\[
\frac{\partial R(z)}{\partial S_i} = \frac{g_i(z)}{4\sqrt{F_{2k}(z)}}, \quad \frac{\partial f(z)}{\partial S_i} = N_{n-k}(z) g_i(z), \quad i = 1, \ldots, k.
\] (3.54)

Here, \( \frac{g_i(z)}{4\sqrt{F_{2k}(z)}} \), \( i = 1, \ldots, n \) is a set of normalized holomorphic functions, as is easily seen by taking the derivatives of the \( A \) cycle integrations. Also, define \( h(z) = -\sum_i N_i g_i(z) \). The second one is
\[
\frac{16\pi^2 i}{m} \sum_{\ell=1}^{n} g_{\ell+1} \frac{\partial R(z)}{\partial g_\ell} = 8\pi^2 i \left( F'''_{\text{in}}(z) - \frac{W'(z) F'''_{\text{in}}(z)}{\sqrt{W'(z)^2 + f(z)}} \right) + \frac{16\pi^2 i}{m} \left( -\sum_{\ell=1}^{n} g_{\ell+1} \frac{\partial f(z)}{\partial g_\ell} \right).
\] (3.55)

where we have used Eq. (3.46).
The proof of Eq. (3.49) goes by observing that it is equivalent to the truncation of the following equation up to the first \( n + 1 \) terms in the \( 1/z \) expansion:

\[
T(z) = \sum_i N_i \frac{\partial R(z)}{\partial S_i} + \frac{16\pi^2 i}{m} \sum_{\ell=2}^{n+1} g_{\ell} \frac{\partial R(z)}{\partial g_{\ell-1}}. \tag{3.56}
\]

Substituting Eqs. (3.50), (3.53), (3.54), and (3.55) into Eq. (3.56), we see that the proof becomes complete as soon as we obtain

\[
D(z) = \frac{16\pi^2 i}{m} \sum_{\ell=1}^{n} g_{\ell+1} \frac{\partial f(z)}{\partial g_{\ell}}, \tag{3.57}
\]

where

\[
D(z) \equiv c(z) - N_{n-k} h(z). \tag{3.58}
\]

Observe that there are two expressions for \( N_i \):

\[
N_i = \oint_{A_i} T(z) dz = -\oint_{A_i} \frac{h(z)}{4\sqrt{F_{2k}(z)}} dz, \quad i = 1, \ldots, k, \tag{3.59}
\]

and therefore

\[
\oint_{A_i} \left( T(z) + \frac{h(z)}{4\sqrt{F_{2k}(z)}} \right) = 0, \quad I = 1, \ldots, n. \tag{3.60}
\]

Another consistency condition is

\[
0 = \frac{\partial S_I}{\partial g_{\ell}} = \frac{\partial}{\partial g_{\ell}} \oint_{A_I} R(z), \quad I = 1, \ldots, n. \tag{3.61}
\]

Eliminating \( F'''(z) - \frac{W'(z)F'''(z)}{\sqrt{W'(z)^2 + f(z)}} \) in the integrand of Eq. (3.60) and that of Eq. (3.61), we obtain

\[
0 = \oint_{A_I} D(z) - \frac{16\pi^2 i}{m} \sum_{\ell=1}^{n} g_{\ell+1} \frac{\partial f(z)}{\partial g_{\ell}} dz. \tag{3.62}
\]

Expanding the integrand of this equation by a set of holomorphic differentials \( \oint_{A_I} \frac{z^{\ell} dz}{\sqrt{W'(z)^2 + f(z)}} \), \( \ell = 0, \ldots, n - 1 \), of the original curve, we deduce Eq. (3.57).

### 4. AGT relation and 2d–4d connection via matrices

The contents of the two preceding sections later had the upgraded treatments mentioned in the introduction. In this section we outline these developments triggered by Ref. [177].

#### 4.1. Instanton partition function: What is \( Z_{\text{inst}}^{\epsilon_1, \epsilon_2} \)?

The list of papers which discuss subjects closely related to that of this subsection include [178–186].

Let us recall that the LEEA of \( \mathcal{N} = 2 \ SU(N_c) \) SUSY gauge theory is specified by the effective prepotential denoted in this section by \( F_{\text{SW}}(a_i) \), and that it has undetermined VEV called Coulomb
moduli \( a_i = \langle \phi_i \rangle \). The bare gauge coupling and the \( \theta \) parameter are grouped into

\[
q_{\text{bare}} = e^{\pi i \tau_{\text{bare}}}, \quad \tau_{\text{bare}} = \frac{\theta}{\pi} + \frac{8\pi i}{g_{\text{bare}}^2},
\]

and \( F_{SW}(a_i) \) consists of the one-loop contribution and the instanton sum

\[
F_{SW} = F_{1-\text{loop}} + F_{\text{inst}}^{(SW)}.
\]

It was shown in [181] that \( F_{\text{inst}}^{(SW)} \) is microscopically calculable in the presence of \( \Omega \) background equipped with the deformation parameters \( \epsilon_1 \) and \( \epsilon_2 \) as

\[
Z_{\text{inst}}(\epsilon_1, \epsilon_2, a_i; q) = \exp \left( \frac{1}{\epsilon_1 \epsilon_2} F_{\text{inst}}(\epsilon_1, \epsilon_2, a_i) \right), \quad F_{\text{inst}}(0, 0, a_i) = F_{\text{inst}}^{(SW)}.
\]

The corrections to the original \( F_{\text{inst}}^{(SW)} \) are regarded as higher orders in the genus expansion with \( g_s^2 = -\epsilon_1 \epsilon_2 \). Its expansion in \( q \) is computable by the localization technique with \( \epsilon_1, \epsilon_2 \) acting as Gaussian cutoffs.

\[
Z_k \equiv \int_{M_k} 1_{\epsilon_1, \epsilon_2, a_i},
\]

where

\[
Z_k \equiv \sum_{|\vec{Y}|=k} Z_{\vec{Y}},
\]

is the “volume” of the \( k \)-instanton moduli space.

Let \( T^{N_c-1} \) be the maximal torus of the gauge group \( SU(N_c) \). Since we also have the maximal torus \( T^2 \) of \( SO(4) \), namely, the global symmetry of \( \mathbb{R}^4 \), the \( T = T^2 \times T^{N_c-1} \) action can be defined on the instanton moduli space. Then the integral in Eq. (4.5) is computed \( T \)-equivariantly and consequently we obtain the regularized results. According to the localization formula, Eq. (4.5) is reduced to the summation of the contribution from the fixed points which are parametrized by \( N_c \) Young diagrams \( \vec{Y} = (Y^{(1)}, \ldots, Y^{(N_c)}) \),

\[
Z_{\vec{Y}} = \sum_{|\vec{Y}|=k} Z_{\vec{Y}},
\]

where \( |\vec{Y}| = \sum_{i=1}^{N_c} |Y^{(i)}| \) is the total number of boxes. Each \( Z_{\vec{Y}} \) is provided through a combinatorial method.

### 4.2. \( \beta \)-ensemble of quiver matrix model and noncommutative curve

The list of papers which discuss subjects closely related to that of this subsection include [187–233].

In this subsection, we give a general discussion of \( \beta \)-deformed matrix models at finite \( N \) (size of matrices) and with generic potentials and the attendant noncommutative curve. The curve at the planar level, which the original S–W curve for the \( SU(N_c) \) gauge group with \( 2N_c \) flavours is relevant to, turn out to come out in a relatively transparent way in the limit.

Let us begin with the \( \beta \)-deformed (\( \beta \)-ensemble of) the one-matrix model:

\[
Z = \int d^N \lambda \ (\Delta(\lambda))^{+2b_E^2} \exp \left( \frac{b_E}{g_s} \sum_{I=1}^{N} W(\lambda_I) \right),
\]

where

\[
\Delta(\lambda) = \prod_{1 \leq I < J \leq N} (\lambda_I - \lambda_J)
\]

is the van der Monde determinant.
The Virasoro constraints [192–194,197], namely the Schwinger–Dyson equations of this model for the resolvent, are obtained by inserting $\sum_{i=1}^{N_c} \frac{\partial}{\partial \lambda_i} \frac{1}{z - \lambda_i}$ into $Z$. Adopting the operator notation of conformal field theory,

$$J(z) = i \partial \phi(z) = \frac{1}{\sqrt{2} g_s} W'(z) + \sqrt{2} b_E \text{Tr} \frac{1}{z - M},$$

$$T(z) = -\frac{1}{2} : \partial \phi(z)^2 : + \frac{i Q_E}{\sqrt{2}} \partial^2 \phi(z), \quad Q_E = b_E - \frac{1}{b_E},$$

eye{4.10}

they can be written as the vanishing VEV of the non-negative part of $T(z)$,

namely, $\langle T(z) \rangle_+ \equiv \langle T(z) \rangle_{z > \lambda_i} = 0$. \hfill (4.11)

Equation (4.11) can, therefore, be written as

$$\langle \frac{g_s^2 T(z)}{g_s^2} \rangle = \frac{1}{4} W'(z)^2 - \frac{Q_E}{2} g_s W''(z) - f(z),$$ \hfill (4.12)

$$f(z) \equiv \left\langle b_E g_s \sum_{i=1}^{N_c} \frac{W'(z) - W'(\lambda_i)}{z - \lambda_i} \right\rangle.$$ \hfill (4.13)

Quite separately, let us introduce the “curve” $(x, z) = (y(z), z)$ by

$$\langle \frac{g_s^2 T(z)}{g_s^2} \rangle = x^2 - g_s^2 \langle T(z) \rangle = 0. \hfill (4.14)$$

Two remarks are in order. First of all, in order for the first equality to be true, $x$ and $z$ must satisfy the noncommutative algebra

$$[x, z] = Q_E g_s. \hfill (4.15)$$

Second, in order for Eq. (4.14) to be algebraic, the singularities in $\langle T(z) \rangle$ must be absent. This condition is ensured by the Schwinger–Dyson equation, Eq. (4.11).

Let us turn to the $A_{N_c - 1}$ quiver matrix model ($\beta$ deformed) which the effective prepotential for the $SU(N_c)$ gauge theory with $2N_c$ flavours is relevant to. This matrix model has been constructed [203] such that $W_{N_c}$ automatically obeys the $W_{N_c}$ constraints at finite $N_a, a = 1, \ldots, r, r = N_c - 1$;

$$Z = \int \prod_{a=1}^{r} \prod_{i=1}^{N_a} d\lambda_i \exp \left( \frac{b_E}{g_s} \sum_{a=1}^{r} \sum_{i=1}^{N_a} W_a(\lambda_i) \right), \hfill (4.16)$$

$$\Delta_{A_{N_c - 1}}(\lambda) = \prod_{a=1}^{r} \prod_{1 \leq i < J \leq N_a} \left( \lambda_i - \lambda_j \right)^2 \prod_{1 \leq a < b \leq r} \prod_{1 \leq n < N_b} \prod_{1 \leq l < J} (\lambda_l(a) - \lambda_l(b))^{(a, a_b)}. \hfill (4.17)$$

We follow the logic of the $\beta$-deformed one-matrix model at finite $N_a$. In this model, there exists $N_c$ spin 1 currents that satisfy $\sum_{i=1}^{N_c} J_i(z) = 0$:

$$J_i(z) = i \partial \phi_i(z) = \frac{1}{g_s} t_i(z) + b_E \sum_{a=1}^{N_c - 1} \left( \delta_{i,a} - \delta_{i,a+1} \right) \text{Tr} \frac{1}{z - M_a}, \hfill (4.18)$$

$$t_i(z) = \sum_{a=i}^{N_c - 1} W_a'(z) - \frac{1}{n} \sum_{a=1}^{N_c - 1} a W'_a(z). \hfill (4.19)$$

Note that

$$: \det(x - ig_s \partial \phi(z)) := \prod_{1 \leq i < N_c} (x - g_s J_i(z)) : \hfill (4.20)$$
contains \( W_{N_c} \) generators and the \( W_{N_c} \) constraints are expressible as

\[
\left\langle \det(x - ig_\Sigma \partial \phi(z)) \right\rangle = 0. \tag{4.21}
\]

The curve \( \Sigma (x = y_i(z), z) \) that we postulate in [233] is

\[
\left\langle \det(x - ig_\Sigma \partial \phi(z)) \right\rangle = 0. \tag{4.22}
\]

The isomorphism with the Witten–Gaiotto curve has been established by taking the planar limit of this construction, as we will see in the next subsection. In fact, the planar limit implies the singlet factorization which assigns the \( c \) number value to the operator \( \partial \phi(z) \), and the curve factorizes as

\[
0 = \prod_{i=1}^{N_c} (x - y_i(z)) \sim \lim_{g_\Sigma \to 0} J_i, \quad (x, z) = (y_i(z), z), \tag{4.23}
\]

where

\[
y_i(z) := \lim_{g_\Sigma \to 0} ig_\Sigma \left\langle \partial \phi_i(z) \right\rangle. \tag{4.24}
\]

### 4.3. The three Penner potential and the agreement with the Witten–Gaiotto curve

The list of papers which discuss subjects closely related to that of this subsection include [177,233–270]. Let us specialize our discussion to the three Penner model. Choose the potential as

\[
W_a(z) = \sum_{p=1}^3 (\mu_p, \alpha_a) \log(q_p - z), \quad q_0 = \infty, q_1 = 0, q_2 = 1, q_3 = q. \tag{4.25}
\]

The matrix integrals of this case realize the integral representation of the conformal block and the size of each matrix corresponds with the number of screening charges we have to insert to build the block. As is clear from the discussion above, the planar spectral curve of the \( A_{N_c-1} \) quiver matrix model takes the form

\[
x^{N_c} = \sum_{k=2}^{N_c} (-1)^{k-1} \frac{Q_k(z)}{(z(z-1)(z-q))^k} x^{N_c-k}, \tag{4.26}
\]

for some polynomials \( Q_k(z) \) in \( z \).

On the other hand, the Seiberg–Witten curve for the case of \( SU(N_c) \) gauge theory with \( 2N_c \) massive flavour multiplets, originally proposed in [236], can get converted into the Gaiotto form [238] by

\[
x^{N_c} = \sum_{k=2}^{N_c} \frac{P_{2k}^{(k)}(t)}{(t(t-1)(t-q_{\text{bare}}))^k} x^{N_c-k}, \tag{4.27}
\]

where \( P_{2k}^{(k)}(t) \) are degree \( 2k \) polynomials in \( t \). The two curves, Eqs. (4.26) and (4.27), are evidently similar. We can also see that the residues of \( y_i(z)dz \) \( (i = 1, \ldots, N_c) \) at \( z = 1, q, 0, \infty \) and those of \( xdt \) at \( t = 1, q_{\text{bare}}, 0, \infty \) on the \( i \)th sheet can be equated.

For general \( N_c \), these residues in fact match if the weights of the vertex operators are identified with the mass parameters of the gauge theory by the following relations [233]:

\[
\mu_0 = \sum_{a=1}^{N_c-1} (-m_a + m_{a+1}) \Lambda^a, \quad \mu_1 = \sum_{a=1}^{N_c-1} (\tilde{m}_a - \tilde{m}_{a+1}) \Lambda^a, \tag{4.28}
\]

\[
\Xi_0 = \sum_{a=1}^{N_c-1} (-m_a + m_{a+1}) \Lambda^a, \quad \Xi_1 = \sum_{a=1}^{N_c-1} (\tilde{m}_a - \tilde{m}_{a+1}) \Lambda^a.
\]
\[ \mu_2 = \left( \sum_{i=1}^{N_c} m_i \right) \Lambda^1, \quad \mu_3 = \left( \sum_{i=1}^{N_c} \bar{m}_i \right) \Lambda^{N_c-1}. \] (4.29)

The matrix model potentials \( W_a(z) \) \((a = 1, 2, \ldots, N_c - 1)\) are fixed as

\[ W_a(z) = (\tilde{m}_a - \tilde{m}_{a+1}) \log z + \delta_{a,1} \left( \sum_{i=1}^{N_c} m_i \right) \log (1 - z) + \delta_{a,N-1} \left( \sum_{i=1}^{N_c} \bar{m}_i \right) \log (q_{\text{UV}} - z). \] (4.30)

With this choice of the multi-log potentials, the \( A_{N_c-1} \) quiver matrix model curve in the planar limit coincides with the \( SU(N_c) \) Seiberg–Witten curve with \( 2N_c \) massive hypermultiplets.

### 4.4. Direct evaluation of the matrix integral as a Selberg integral

The list of papers which discuss subjects closely related to that of this subsection include [271–350].

In this subsection, we consider 2-d conformal field theory which has the Virasoro symmetry with the central charge \( c \). The correlation functions for primary operators \( \Phi_{\Delta}(z, \bar{z}) \) with the conformal weight \( \Delta \) are strongly constrained by this symmetry. We are interested in the four-point functions which can be expressed as

\[ \langle \Phi_{\Delta_1}(\infty, \infty) \Phi_{\Delta_2}(1, 1) \Phi_{\Delta_3}(q, \bar{q}) \Phi_{\Delta_4}(0, 0) \rangle = \sum_{I} C_{\Delta_2 \Delta_3 \Delta_4} C_{\Delta_1 \Delta_2} |C_{\Delta_3 \Delta_4}|^2 \mathcal{F}(q; c; \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_f | E) \] (4.31)

The sum on \( I \) is taken over all possible internal states. Here \( K_{\Delta} \) and \( C_{\Delta_1 \Delta_2} \) are the model-dependent factors. In contrast, the conformal block\(^4\) denoted by \( \mathcal{F}(q; c; \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_f) \) is a model-independent and purely representation theoretic quantity,

\[ \mathcal{F}(q; c; \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_f) = \sum_{|Y|=|Y'|} q^{|Y'|} Y_{\Delta_1, \Delta_2}(Y) Q_{\Delta_f}^{-1}(Y, Y') Y_{\Delta_3, \Delta_4}(Y'), \] (4.32)

where \( Q_{\Delta}(Y, Y') = \langle \Delta|L_Y L_{-Y'}\Delta \rangle \) is the Shapovalov form with \( L_Y = L_{k_1} L_{k_2} \cdots L_{k_\ell} \) for partition \( Y = (k_1, k_2, \ldots, k_\ell) \) and

\[ Y_{\Delta_1, \Delta_2}(Y) = \prod_{i=1}^{\ell} \left( \Delta + \sum_{j < i} k_j \right). \] (4.33)

Let us consider the four-point conformal block on sphere

\[ \mathcal{F}(q; c; \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_f) \] (4.34)

with

\[ c = 1 - 6Q_E^2, \quad \Delta_1 = \frac{1}{4} \alpha_i (\alpha_i - 2Q_E), \quad \Delta_f = \frac{1}{4} \alpha_f (\alpha_f - Q_E). \] (4.35)

The parameter \( \alpha_4 \) is determined by the following momentum conservation condition which comes from the zero-mode part:

\[ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2(N_L + N_R) b_E = 2Q_E. \] (4.36)

\(^4\) For a review, see [351,352].
The internal momentum $\alpha_I$ is given by

$$\alpha_I = \alpha_1 + \alpha_2 + 2N_L b_E = -\alpha_3 - \alpha_4 - 2N_R b_E + 2Q_E.$$  \hfill (4.37)

Equation (4.34) has an integral representation as a version of the $\beta$-deformed matrix model. Actually, the Dotsenko–Fateev multiple integrals,

$$Z_{\text{pert–(Selberg)}}^2 \left( q \mid b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3 \right) = q^{\Delta_I - \Delta_1 - \Delta_2} (1 - q)^{(1/2)\alpha_2 \alpha_3}$$

$$\times \left( \prod_{I=1}^{N_L} \int_0^1 dx_I \right) \prod_{I=1}^{N_L} x_I^{b_E \alpha_1} (1 - x_I)^{b_E \alpha_2} (1 - q x_I)^{b_E \alpha_3} \prod_{1 \leq I < J \leq N_L} |x_I - x_J|^{2b_E^2}$$

$$\times \left( \prod_{J=1}^{N_R} \int_0^1 dy_J \right) \prod_{J=1}^{N_R} y_J^{b_E \alpha_4} (1 - y_J)^{b_E \alpha_5} (1 - q y_J)^{b_E \alpha_3} \prod_{1 \leq I < J \leq N_R} |y_I - y_J|^{2b_E^2}$$

$$\times \prod_{I=1}^{N_L} \prod_{J=1}^{N_R} (1 - q x_I y_J)^{2b_E^2},$$  \hfill (4.38)

are regarded as a free field representation of Eq. (4.34).

From now on, we follow the discussion of [298]. Equation (4.38) is in fact a partition function of the “perturbed double-Selberg matrix model.” If we forget the Veneziano factor $q^{\Delta_I - \Delta_1 - \Delta_2} (1 - q)^{(1/2)\alpha_2 \alpha_3}$, we see that at $q = 0$ this expression decouples into two independent Selberg integrals. In order to develop its $q$-expansion, it is more convenient to interpret these multiple integrals as perturbations of the products of the two Selberg integrals.

We have the following expression of the perturbed double-Selberg model:

$$Z_{\text{pert–(Selberg)}}^2 \left( q \mid b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3 \right)$$

$$= q^{\Delta_I - \Delta_1 - \Delta_2} B_0 \left( b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3 \right) B \left( q \mid b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3 \right),$$  \hfill (4.39)

where

$$B_0 \left( b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3 \right)$$

$$= S_{N_L} \left( 1 + b_E \alpha_1, 1 + b_E \alpha_2, b_E^2 \right) S_{N_R} \left( 1 + b_E \alpha_4, 1 + b_E \alpha_3, b_E^2 \right),$$  \hfill (4.40)

$$B \left( q \mid b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3 \right)$$

$$= (1 - q)^{(1/2)\alpha_2 \alpha_3} \left[ \prod_{I=1}^{N_L} \left( 1 - q x_I \right)^{b_E \alpha_3} \prod_{J=1}^{N_R} \left( 1 - q y_J \right)^{b_E \alpha_2} \prod_{I=1}^{N_L} \prod_{J=1}^{N_R} \left( 1 - q x_I y_J \right)^{2b_E^2} \right]^{N_L, N_R},$$  \hfill (4.41)

$$\prod_{I=1}^{N_L} \prod_{J=1}^{N_R} \left( 1 - q x_I y_J \right)^{2b_E^2}.$$

Here, $S_{N_L}$ and $S_{N_R}$ are the celebrated Selberg integral

$$S_N (\beta_1, \beta_2, \gamma) = \left( \prod_{I=1}^{N} \int_0^1 dx_I \right) \prod_{I=1}^{N} x_I^{\beta_1 - 1} (1 - x_I)^{\beta_2 - 1} \prod_{1 \leq I < J \leq N} |x_I - x_J|^{2\gamma},$$  \hfill (4.43)

and the averaging $\langle \cdots \rangle_{N_L, N_R}$ is taken with respect to the unperturbed Selberg matrix model,

$$Z_{\text{(Selberg)}}^2 (b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3)$$

$$= Z_{\text{Selberg}} (b_E; N_L, \alpha_1, \alpha_2) Z_{\text{Selberg}} (b_E; N_R, \alpha_4, \alpha_3)$$

$$:= S_{N_L} \left( 1 + b_E \alpha_1, 1 + b_E \alpha_2, b_E^2 \right) S_{N_R} \left( 1 + b_E \alpha_4, 1 + b_E \alpha_3, b_E^2 \right).$$  \hfill (4.44)

Below we also use $\langle \cdots \rangle_{N_L}$ and $\langle \cdots \rangle_{N_R}$ which imply the averaging with respect to $Z_{\text{Selberg}} (N_L)$ and to $Z_{\text{Selberg}} (N_R)$, respectively.
The function \( B(q) = B(q \mid b_E; N_L, \alpha_1, \alpha_2; N_R, \alpha_4, \alpha_3) \) has the following \( q \)-expansion [298]:

\[
B(q) = 1 + \sum_{\ell=1}^{\infty} q^{\ell} B_{\ell} \\
= \left\langle \exp \left[ -2 \sum_{k=1}^{\infty} \frac{q^k}{k} \left( b_E \sum_{l=1}^{N_L} x_l^k + \frac{1}{2} \alpha_2 \right) \left( b_E \sum_{J=1}^{N_R} y_J^k + \frac{1}{2} \alpha_3 \right) \right] \right\rangle_{N_L,N_R} \\
= (1 - q^{(1/2)\alpha_2\alpha_3}) A(q),
\]

where we have defined \( A(q) \) by

\[
A(q) = 1 + \sum_{\ell=1}^{\infty} q^{\ell} A_{\ell} \\
= \left\langle \exp \left[ -\sum_{k=1}^{\infty} \frac{q^k}{k} \left( \alpha_2 + b_E \sum_{l=1}^{N_L} x_l^k \right) \left( b_E \sum_{J=1}^{N_R} y_J^k \right) \\
- \sum_{k=1}^{\infty} \frac{q^k}{k} \left( b_E \sum_{l=1}^{N_L} x_l^k \right) \left( \alpha_3 + b_E \sum_{J=1}^{N_R} y_J^k \right) \right] \right\rangle_{N_L,N_R}.
\]

It takes form

\[
A(q) = \sum_{k=0}^{\infty} q^{k} \sum_{|Y_1|+|Y_2|=k} A_{Y_1,Y_2}.
\]

Note that a pair of partitions \((Y_1, Y_2)\) naturally appears.

In general, the following correlation function is calculable:

\[
Z_{\text{pert–Selberg}}(\beta_1, \beta_2, \gamma; \{g_i\}) := S_N(\beta_1, \beta_2, \gamma) \left\langle \exp \left( \sum_{I=1}^{N} W(x_I; g) \right) \right\rangle_N,
\]

with

\[
W(x; g) = \sum_{i=0}^{\infty} g_i x^i.
\]

The averaging is with respect to the Selberg integral Eq. (4.43). The exponential of the potential is expanded by the Jack polynomial

\[
\exp \left( \sum_{I=1}^{N} W(x_I; \{g_i\}) \right) = \sum_{\lambda} C_{\lambda}^{(\gamma)}(g) P_{\lambda}^{(1/\gamma)}(x),
\]

where \( P_{\lambda}^{(1/\gamma)}(x) \) is a polynomial of \( x = (x_1, \ldots, x_N) \) and \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a partition: \( \lambda_1 \geq \lambda_2 \geq \cdots \geq 0 \). The Jack polynomial is characterized as the eigenstates of

\[
\sum_{I=1}^{N} \left( x_I \frac{\partial}{\partial x_I} \right)^2 + \gamma \sum_{1 \leq I < J \leq N} \left( \frac{x_I + x_J}{x_I - x_J} \right) \left( x_I \frac{\partial}{\partial x_I} - x_J \frac{\partial}{\partial x_J} \right),
\]

with homogeneous degree \( |\lambda| = \lambda_1 + \lambda_2 + \cdots \), and is normalized such that

\[
P_{\lambda}^{(1/\gamma)}(x) = m_\lambda(x) + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu(x).
\]
Here $\mu < \lambda$ stands for the dominance ordering defined by
\[ |\mu| = |\lambda| \quad \text{and} \quad \mu_1 + \mu_2 + \cdots + \mu_n < \lambda_1 + \lambda_2 + \cdots + \lambda_n \quad \text{for all} \ n \geq 1, \] (4.54)
and $m_\lambda(x)$ is the monomial symmetric function. Explicit forms of the Jack polynomials for $|\lambda| < 2$ are as follows:
\[
P^{(1/\gamma)}_{(1)}(x) = m_{(1)}(x) = \sum_{I=1}^{N} x_I,
\]
\[
P^{(1/\gamma)}_{(2)}(x) = m_{(2)}(x) + \frac{2\gamma}{1+\gamma} m_{(1^2)}(x) = \sum_{I=1}^{N} x_I^2 + \frac{2\gamma}{1+\gamma} \sum_{1 \leq I < J \leq N} x_I x_J,
\]
\[
P^{(1/\gamma)}_{(1^2)}(x) = m_{(1^2)}(x) = \sum_{1 \leq I < J \leq N} x_I x_J. \tag{4.55}
\]
The Selberg average for a single Jack polynomial is known as a Macdonald–Kadell integral [276,278,282], which implies that
\[
\left\langle P^{(1/\gamma)}_{\lambda}(x) \right\rangle_{N_L} = \prod_{i \geq 1} \frac{\left( \beta_1 + (N-i)\gamma \right)_{\lambda_i} \left( (N+1-i)\gamma \right)_{\lambda_i}}{\left( \beta_1 + \beta_2 + (2N-1-i)\gamma \right)_{\lambda_i}} \prod_{(i,j) \in \lambda} \frac{1}{(\lambda_i' - j + (\lambda_j' - i + 1)\gamma)}, \tag{4.56}
\]
where $(a)_n$ is the Pochhammer symbol:
\[(a)_n = a(a + 1) \cdots (a + n - 1), \quad (a)_0 = 1, \tag{4.57}\]
and $\lambda'$ stands for the conjugate partition of $\lambda$.

In order to apply this to Eq. (4.39), let us set $\gamma = b_E^2$ and $N \rightarrow N_L$, $\beta_1 \rightarrow 1 + b_E \alpha_1$, $\beta_2 \rightarrow 1 + b_E \alpha_2$, for the “left” part. Similar replacement yields the expression for the “right” part. We obtain
\[
\left\langle P^{(1/b_E^2)}_{\lambda}(x) \right\rangle_{N_L} = \prod_{i \geq 1} \frac{\left( 1 + b_E \alpha_1 + b_E^2 (N-i) \right)_{\lambda_i} \left( b_E^2 (N_L+1-i) \right)_{\lambda_i}}{\left( 2 + b_E (\alpha_1 + \alpha_2) + b_E^2 (2N_L-1-i) \right)_{\lambda_i}} \prod_{(i,j) \in \lambda} \frac{1}{(\lambda_i' - j + b_E^2 (\lambda_j' - i + 1))}. \tag{4.59}
\]
From the explicit form of Jack polynomials for $|\lambda| < 2$ listed in Eq. (4.55), we obtain [298]
\[
\left\langle b_E \sum_{I=1}^{N_L} x_I \right\rangle_{N_L} = \frac{b_E N_L (b_E N_L - Q_E + \alpha_1)}{(\alpha_I - 2Q_E)}, \tag{4.60}
\]
\[
2 \left\langle b_E^2 \sum_{1 \leq I < J \leq N_L} x_I x_J \right\rangle_{N_L} = \frac{b_E N_L (b_E N_L - b_E) (\alpha_1 + b_E N_L - Q_E) (\alpha_1 + b_E N_L - Q_E - b_E)}{(\alpha_I - 2Q_E) (\alpha_I - 2Q_E - b_E)}, \tag{4.61}
\]
\[
\left\langle b_E \sum_{I=1}^{N_L} x_I (1 - x_I) \right\rangle_{N_L} = \frac{b_E N_L (\alpha_1 + b_E N_L - Q_E) (\alpha_2 + b_E N_L - Q_E) (\alpha_1 + \alpha_2 + b_E N_L - 2Q_E)}{(\alpha_I - 2Q_E) (\alpha_I - 3Q_E + b_E) (\alpha_I - 2Q_E - b_E)}. \tag{4.62}
\]
Recall, at \( q = 0 \), the perturbed double-Selberg matrix model reduces to a pair of decoupled Selberg integrals. The original model \((q \neq 0)\) is built through the resolvents as in Eq. (4.42). For definiteness, let us consider the left part,

\[
Z_{\text{Selberg}}(b_E; N_L, \alpha_1, \alpha_2) = \left( \prod_{I=1}^{N_l} \int_{0}^{1} dx_I \right) \prod_{1 \leq I < J \leq N_L} |x_I - x_J|^2 b_E^{2} \exp \left( b_E \sum_{I=1}^{N_L} \widetilde{W}(x_I) \right). \tag{4.63}
\]

where

\[
\widetilde{W}(x) = \alpha_1 \log x + \alpha_2 \log (1 - x).
\]

By inserting

\[
\sum_{I=1}^{N_l} \frac{1}{\partial x_I z - x_I},
\]

into the integrand, we obtain the loop equation at finite \( N \),

\[
\left\langle \left( \omega_{NL}(z) \right)^2 \right\rangle_{NL} + \left( \tilde{W}(z) + Q_E \frac{d}{dz} \right) \left\langle \omega_{NL}(z) \right\rangle_{NL} - \tilde{f}_{NL}(z) = 0,
\]

where

\[
\omega_{NL}(z) := b_E \sum_{I=1}^{N_l} \frac{1}{z - x_I}, \quad \tilde{f}_{NL}(z) := \left\langle b_E \sum_{I=1}^{N_l} \frac{\tilde{W}'(z) - \tilde{W}'(x_I)}{z - x_I} \right\rangle_{NL}.
\]

The expectation value of \( \omega_{NL}(z) \) is the finite \( N \) resolvent

\[
\omega_{NL}(z) := \left\langle \omega_{NL}(z) \right\rangle_{NL} = \left\langle b_b \sum_{I=1}^{N_l} \frac{1}{z - x_I} \right\rangle_{NL}.
\]

By looking at \( O(1/z) \), \( O(1/z^2) \), and \( O(1/z^3) \), we obtain the exact results:

\[
\left\langle \left( b_E p_{(1)}(\mu) \right) \right\rangle_{NL} = \left\langle b_E \sum_{I=1}^{N_l} x_I \right\rangle_{NL} = \frac{b_E N_L (b_E N_L - Q_E + \alpha_1)}{(\alpha_1 + \alpha_2 + 2b_E N_L - 2Q_E)},
\]

\[
\tilde{f}_{NL}(z) = - \frac{b_E N_L (\alpha_1 + \alpha_2 + b_E N_L - Q_E)}{z(z-1)},
\]

\[
-\omega_{NL}(0) = \left\langle b_E \sum_{I=1}^{N_l} \frac{1}{x_I} \right\rangle_{NL} = \frac{b_E N_L (\alpha_1 + \alpha_2 + b_E N_L - Q_E)}{\alpha_1},
\]

\[
\omega_{NL}(1) = \left\langle b_E \sum_{I=1}^{N_l} \frac{1}{1 - x_I} \right\rangle_{NL} = \frac{b_E N_L (\alpha_1 + \alpha_2 + b_E N_L - Q_E)}{\alpha_2}.
\]

The first one agrees with Eq. (4.60).

Now, let us determine the 0d–4d dictionary. In the matrix model (0d side), we have seven parameters with one constraint Eq. (4.36):

\[
b_E, N_L, \alpha_1, \alpha_2, N_R, \alpha_4, \alpha_3,
\]

while in \( \mathcal{N} = 2, SU(2) \), \( N_f = 4 \) gauge theory (4d side), there exist six unconstrained parameters:

\[
\frac{\epsilon_1}{g_s}, \frac{a}{g_s}, \frac{m_1}{g_s}, \frac{m_2}{g_s}, \frac{m_3}{g_s}, \frac{m_4}{g_s}.
\]

Here, \( a \) is the vacuum expectation value of the adjoint scalar, \( m_i \) are mass parameters, and \( \epsilon_1 \) is one of Nekrasov’s deformation parameters. By looking at \( B_1 = A_1 + \frac{1}{2} \alpha_2 \alpha_3 \) and the explicit form of
we obtain
\[ b_{ENL} = \frac{a - m_2}{g_s}, \quad b_{ERN} = -\frac{a + m_3}{g_s}, \]
\[ \alpha_1 = \frac{1}{g_s} (m_2 - m_1 + \epsilon), \quad \alpha_2 = \frac{1}{g_s} (m_2 + m_1), \]
\[ \alpha_3 = \frac{1}{g_s} (m_3 + m_4), \quad \alpha_4 = \frac{1}{g_s} (m_3 - m_4 + \epsilon). \] (4.74)

The first two formulas tell us clearly the necessity that the filling fractions of the \( \beta \)-deformed matrix model must be explicitly specified at finite \( N \) in order to exhibit the Coulomb moduli.

In the next order, the expansion coefficients \( A_2 \) are rearranged as
\[ A_2 = \sum_{|Y_1| + |Y_2| = 2} A_{Y_1,Y_2} = A_{(2),(0)} + A_{(1^2),(0)} + A_{(1),(1)} + A_{(0),(2)} + A_{(0),(1^2)} + A_{(0),(2)}, \] (4.75)

where
\[ A_{Y_1,Y_2} = \left\langle \left\langle M_{Y_1,Y_2}(x) \right|_{N_L} \left\langle \widetilde{M}_{Y_1,Y_2}(y) \right|_{N_R} \right\rangle. \] (4.76)

Unfortunately, finding \( M \) and \( \widetilde{M} \) is not straightforward. But at least for \( |Y_1| + |Y_2| \leq 2 \), the explicit forms for them have been obtained. For example,
\[ \widetilde{M}_{(2),(0)}(y) = b_E^2 \chi^{(1/b^2)}_{(2)}(y), \quad \widetilde{M}_{(1^2),(0)}(y) = \frac{2b_E^2}{1 + b_E^2} \chi^{(1/b^2)}_{(1^2)}(y). \] (4.77)

We illustrate our discussion in this section in Fig. 3.
4.5. More recent developments

The list of papers which discuss subjects closely related to that of this subsection include [262,301,353–398].

We have reviewed the 2d–4d connection from the viewpoint of the matrix model. In this subsection, we comment on some of the more recent developments.

In the last subsection, we have presented the connection between the Virasoro conformal blocks and the four-dimensional \( SU(2) \) instanton partition functions via the matrix model and the Selberg integral. This discussion has been generalized in part to that between the \( W_N \) blocks and the \( SU(N) \) partition functions [377].

Both sides also have a natural generalization as a \( q \)-lift [365]. The Virasoro/\( W_N \) symmetry in the two-dimensional CFT side is deformed to the \( q \)-deformed Virasoro/\( W_N \) symmetry, while the four-dimensional \( SU(N) \) gauge theory is lifted to the five-dimensional theory. It is interesting to consider the root of unity limit \( q \to e^{2\pi i}r \) of the \( q \)-Virasoro/\( W_N \) algebras. The appropriate limiting procedure [387,392] to the root of unity exhibits the connection between the super Virasoro \( (r = 2) \) orthogonal \( Z_r \)-parafermionic CFT and the gauge theory on \( \mathbb{R}^4/Z_r \) [369,371].

There are several pieces of work [301,364,380,385] which prove the 2d–4d connection. The explicit identification can be established in the case of \( \beta = 1 \) [367,368]. In order to apply to the \( \beta \neq 1 \) case, the conformal blocks have to be expanded by the generalized Jack polynomial [386] that modifies the standard one. For some lower rank cases, this has been explicitly constructed [389].

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