Intermediate temperature superfluidity in an atomic Fermi gas with population imbalance

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We derive the underlying finite temperature theory which describes Fermi gas superfluidity with population imbalance in a homogeneous system. We compute the pair formation temperature and superfluid transition temperature \( T_c \), and superfluid density in a manner consistent with the standard ground state equations, and thereby present a complete phase diagram. Finite temperature stabilizes superfluidity, as manifested by two solutions for \( T_c \), or by low \( T \) instabilities. At unitarity the polarized state is an “intermediate temperature superfluid”.

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Excitement in the field of ultracold Fermi gases has to do with their remarkable tunability. As a magnetic field is varied from weak to strong this system undergoes a transition from a Bose Einstein condensate (BEC) to a BCS-based superfluid. Recently [1, 2] another tunability has emerged; one can vary the population imbalance between the two “spin” species. This capability has led to speculations about new phases of superfluidity, quantum critical points, and has repercussions as well in other sub-fields of physics [3, 4]. A nice body of theoretical work on this subject [5, 6, 7, 8] has focused on zero temperature (\( T \)) studies of the simplest mean field wavefunction [9], with population imbalance. Additional important work presents [10] a \( T = 0 \) two-channel, mean-field approach for very narrow Feshbach resonances, as well as a study of finite \( T \) effects [11], albeit without a determination of superfluid order.

Superfluidity is, generally, a finite \( T \) phenomenon and it is the purpose of the present paper to explore finite temperature effects [12, 13, 14] based on the BCS-Leggett ground state with population imbalance. We determine the behavior of the pair formation temperature \( T^* \), the superfluid transition temperature, \( T_c \), and superfluid density \( n_s(T) \), and, thereby arrive at a phase diagram which addresses general \( T \). Importantly, we find that in the fermionic regime, superfluidity exists at finite \( T \) (although not at \( T = 0 \)), leading to the new concept of an “intermediate temperature superfluid”. Because temperature acts in this rather unexpected fashion, we reduce the complexity and confine our attention to the homogeneous system.

The approach which we outline below, importantly, includes what we call “pseudogap effects”. For \( T \neq 0 \), the excitation gap \( \Delta \) is different from the order parameter, due to the contribution to \( \Delta \) from noncondensed pairs [12, 13]. In this way, the solution for \( T_c \) is necessarily different than that obtained in the literature. Generally, \( \Delta^2 \) contains two additive contributions [13, 14] from the condensate (\( \Delta^2_g \)) and noncondensed pairs (\( \Delta^2_{nc} \)), and they are proportional to the total, condensed, and noncondensed pair densities, respectively. This decomposition is analogous to the particle number constraint in ideal BEC. We emphasize that the central equations derived below are not compatible [13] with the \( T = 0 \) formalism of Ref. [14]. In addition, the “naive mean field theory” with the unphysical assumption that \( \Delta(T) \equiv \Delta_{nc}(T) \) is not a correct rendition of \( T \neq 0 \) effects associated with the BCS-Leggett ground state.

We define the noncondensed pair propagator, as \( t(Q) = U/[1 + U\chi(Q)] \), where, as in Ref. [14], the pair susceptibility, given by \( \chi(Q) = \frac{1}{\pi} \sum_K [G_0(Q - K)G_1(K) + G_0(Q - K)G_1(K)] \), can be derived from equations for the Green’s functions, consistent with the BCS-Leggett ground state equations. Here \( G_{\sigma}(K) = G_{0,\sigma}(K) = i\omega_n - \xi_{K,\sigma} \) are the full and bare Green’s functions (with \( \sigma = \uparrow, \downarrow \), \( \xi_{K,\sigma} = \epsilon_k - \mu_{\sigma} \)). We adopt a one-channel approach since the \( ^6 \text{Li} \) resonances studied thus far are broad and consider a Fermi gas of two spin species with kinetic energy \( \epsilon_k = \hbar^2 k^2/2m \) and chemical potential \( \mu_{\uparrow} \) and \( \mu_{\downarrow} \), subject to an attractive contact potential (\( U < 0 \)) between the different spin states. We take \( h = 1, k_B = 1, \) and \( K = (\omega_n, k) \equiv (i\Omega_n, q) \), \( \sum_{Q, K} = T \sum_n \sum_k \), etc, where \( \omega_n(\Omega_n) \) is the standard odd (even) Matsubara frequency.

In the superfluid state, the “gap equation” is given by \( U^{-1} + \chi(0) = 0 \), which is equivalent to \( \mu_{pair} = 0 \), the BEC condition of the pairs. Below \( T_c \), the self-energy can be well approximated [13] by the BCS form, \( \Sigma_{\sigma}(K) = -\Delta^2 G_{0,\sigma}(-K) \), where \( \sigma = -\sigma \). Therefore, \( G_{\uparrow,\uparrow}(K) = \frac{\omega_n + \hbar \xi_{K,\uparrow}}{\omega_n + \hbar \xi_{K,\uparrow} + \hbar \xi_{K,\downarrow}} \), where \( \xi_{K,\uparrow} = \sqrt{\epsilon_k^2 + \Delta^2_{sc}} \), \( \xi_{K,\downarrow} = \epsilon_k - \mu_{\uparrow} \), \( \mu = (\mu_\uparrow + \mu_\downarrow)/2 \), \( h = (\mu_\uparrow - \mu_\downarrow)/2 \), and \( \Delta^2 = (1 + \xi_{K,\uparrow}/E_K)/2 \). Since the polarization \( p > 0 \), we always have \( h > 0 \).

The “gap equation” can then be rewritten in terms of the two-body s-wave scattering length \( a \), leading to

\[
\frac{m}{4\pi a} = \sum_k \left[ \frac{1}{2\epsilon_k} - \frac{1 - 2\bar{f}(E_k)}{E_k} \right].
\]

where \( \bar{f}(x) \equiv [f(x+h) + f(x-h)]/2 \), and \( m/4\pi a = 1/U + \sum_k (2a)^{-1} \). Here \( f(x) \) is the Fermi distribution function. We define \( p_{\alpha} = \delta n = n_\uparrow - n_\downarrow > 0 \), where \( n = n_\uparrow + n_\downarrow \) is the total atomic density, and \( p = \delta n/n \). Similarly, using \( n_\sigma = \sum_K G_{\sigma}(K) \), one can write

\[
\begin{align}
\frac{n}{2} &= \sum_k \left[ v_k^2 + \frac{\xi_k}{E_k} \bar{f}(E_k) \right], \\
\frac{pn}{2} &= \sum_k \left[ f(E_k - h) - f(E_k + h) \right]
\end{align}
\]
Note that, except for the number difference [Eq. (2)], all equations including those below can be obtained from their unpolarized counterparts by replacing \( f(x) \) and its derivative \( f'(x) \) with \( \bar{f}(x) \) and \( \bar{f}'(x) \), respectively.

While Eqs. (1)-(2) have been written down in the literature \([5,11]\), the present derivation can be used to go further and to determine the dispersion relation and the number density for noncondensed pairs. We find

\[
\Delta_{pg}^2 \equiv - \sum_{Q \neq 0} t(Q),
\]

which vanishes at \( T = 0 \), where \( \Delta = \Delta_{sc}^2. \) In the superfluid phase, \( t^{-1}(Q) = \chi(Q) - \chi(0) \approx Z(\Omega - \Omega_q) \) to first order in \( \Omega \), and after analytical continuation \((i\Omega_q \rightarrow \Omega + i0^+)\). 

Here \( \chi(Q) = \sum_k \left[ \frac{1}{E_k + \xi_{q,k} + \mu} \right] u_k^2 - \frac{\bar{f}(E_k) - \bar{f}(\xi_{q,k} + \mu)}{2E_k} \right], \]

It follows that the inverse residue \( Z = \left[ n - 2 \sum_k \bar{f}(\xi_k) / 2\Delta^2 \right]. \)

Thus \( \Delta_{sc}^2 = Z^{-1} \sum_{Q \neq 0} b(\Omega_q), \) where \( b(x) \) is the Bose distribution function. To lowest order in \( q, \) the pair dispersion \( \Omega_q = q^2 / 2M^*, \) where the effective pair mass \( M^* \) can be computed from a low \( q \) expansion of \( \Omega_q. \) This \( q^2 \) dispersion is associated \([12]\) with BCS-type ground states, which have been the basis for essentially all population imbalance work.

Importantly, Eqs. (1)-(3) can be used to determine \( T_c \) as the extremal temperature(s) in the normal state at which noncondensed pairs exhaust the total weight of \( \Delta^2 \) so that \( \Delta_{pg}^2 = \Delta^2. \)

Solving for the “transition temperature” in the absence of pseudogap effects leads to the quantity \( T_{cMF}. \) More precisely, \( T_{cMF}^* \) is defined to be the temperature at which \( \Delta(T) \) vanishes within Eqs. (1) and (2). This provides a reasonable estimate for the pairing onset temperature \( T^* \), when a stable superfluid phase exists. It should be noted that \( T^* \) represents a smooth crossover rather than an abrupt phase transition, and that Eq. (1) must be altered \([17]\) above \( T_c \) to include finite \( \mu_{pair}. \) We will see that understanding the behavior of \( T_c^{MF} \) is a necessary first step en route to understanding the behavior of \( T_c \) itself.

The superfluid density \( n_s(T) \) is also required to vanish at the same value(s) for \( T_c \), as deduced above. Our calculation of \( n_s \) closely follows previous work \([14,18]\) for the case of the unpolarized superfluid. There is an important cancellation between the current vertex and self-energy contributions involving \( \Delta_{pg}^2 \) so that, as expected, \( n_s(T) \) varies with the order parameter \( \Delta_{sc}^2. \) It is given by

\[
n_s(T) = \frac{4}{3} \Delta_{sc}^2 \sum_k \frac{c_k}{E_k} \left[ \frac{1 - 2\bar{f}(E_k)}{2E_k} + \bar{f}'(E_k) \right],
\]

which at \( T = 0 \) agrees with Ref. \([5]\).

The stability requirements for the superfluid phase have been discussed in the literature \([5]\). In general, one requires that the superfluid density be positive and that the \( 2 \times 2 \) “number susceptibility” matrix for \( \partial n_s / \partial \mu_s \) have only positive eigenvalues when the gap equation is satisfied. The \( \Delta \) dependence of \( n_s \) introduces into the matrix the overall factor \( \left( \frac{\mu_s}{\Delta^2} \right)_{\mu,h}. \)

Thus, the second stability requirement is equivalently to the condition that

\[
\left( \frac{\partial^2 \Omega}{\partial \Delta^2} \right)_{\mu,h} = 2 \sum_k \frac{\Delta^2}{E_k^2} \left[ \frac{1 - 2\bar{f}(E_k)}{2E_k} + \bar{f}'(E_k) \right] > 0.
\]

Here \( \Omega \) is the thermodynamical potential. A third stability requirement, specific to the present calculations, is that the pair mass \( M^* > 0. \)

In Fig. 1 we present a plot of \( T_c^{MF} \) as a function of \( 1/k_{FA} \) for a range of \( p. \) In the inset we plot \( \Delta(T) \) at different \( 1/k_{FA} \) for \( p = 0.3. \) For \( p < 0.9 \) and sufficiently low \( T_c^{MF} \), the curves for \( T_c^{MF} \) develop an unexpected structure, as one sweeps toward the BCS regime. Once \( 1/k_{FA} \) is less than a critical value, \( (1/k_{FA})_{c}, \) where \( T_c^{MF} \) vanishes, there are two \( T_c^{MF} \) lines. The lower branch starts from \( (1/k_{FA})_{c} \) and increases as \( 1/k_{FA} \) decreases. This structure implies that \( \Delta \) is nonmonotonic \([19]\) in \( T, \) as indicated by the bottom curve in the inset of Fig. 1. The two zeroes of \( \Delta \) represent the two values of \( T_c^{MF}. \) In contrast to the more conventional behavior (shown in the top curve for stronger pairing interaction), \( \Delta \) increases with \( T \) at low temperature when \( 1/k_{FA} \) is sufficiently small. This indicates that temperature enables pairing. This was also inferred in Ref. \([11]\). In general superfluids, one would argue that these two effects compete.

Insight into this important phenomenon in the fermionic regime \((\mu > 0), \) is provided by studying the momentum distribution \( n_{s\sigma}(k) \) at \( T = 0 \) and finite \( T. \) At \( T = 0, \) pairing is present only for \( c_k \) below \( \epsilon_1 = \text{Max}(0, \mu - \sqrt{\hbar^2 - \Delta^2}) \) and above \( \epsilon_2 = \mu + \sqrt{\hbar^2 - \Delta^2}. \) This polarized \( T = 0 \) state requires that pairs persist to relatively high energies \( c_k > \epsilon_2, \) as a result of the Pauli principle which pushes these states out of the “normal” regime occupied by the majority species. This kinetic energy cost competes with the gain from condensation energy and for sufficiently weak attraction this “breached pair” structure \([20]\) becomes unstable at \( T = 0. \) By contrast, at finite \( T \) the regime originally occupied exclusively by the
majority species between $\epsilon_1$ and $\epsilon_2$ is no longer completely filled and pairs can “spill over” from both lower and higher energy states into this regime. This not only helps lower the kinetic energy but allows the “normal” regime to participate in pairing and thus lowers the potential energy. In this way temperature can enhance pairing. It should be noted that the majority species between $\epsilon_1$ and $\epsilon_2$ contains a pairing self-energy and is different from a free Fermi gas.

Figure 2 represents solutions for $T_c$ of our central equation set [Eq. (5)] as a function of $1/k_F a$ for the entire range of $p$. If the solution for $T_c$ falls into the shaded region, there is no stable superfluid (since $\partial^2 \Omega / \partial \Delta^2 < 0$, through Eq. (5)). For low polarizations $p \lesssim 0.185$, the behavior of $T_c$ is similar to that of $T_{c}^{MF}$ when one approaches the BCS regime. There may be one or two $T_c$’s which, when stable, will be associated with intermediate temperature superfluidity. When $p > 0.185$, however, no solution can be found for the regime $1/k_F a \lesssim 0.18$, because $M^* < 0$ there. We stress that the origin of the intermediate temperature superfluid we find here lies in a very early stage of the calculations; it can already be seen as a consequence of the constraints imposed on the pairing gap in the low $T$ regime when there is a delicate energetic balance between normal and paired states [see, Eqs. (1) and (2), and, Fig. 1].

In Fig. 3 we summarize our observations in the form of a general temperature phase diagram. In region I, the system is normal and superfluidity is absent. However, this normal phase need not be a Fermi gas. Close to the boundary, as shown in the inset to Fig. 1 (bottom curve) finite $T$ pairing may occur with or without phase coherence. In region III, stable superfluidity is present for all $T \leq T_c$. Finally in region II, within the shaded region (IIC and IID), we find a stable polarized superfluid phase for intermediate temperatures, not including $T = 0$, which we refer to as intermediate temperature superfluidity. In IIA and IIB no stable polarized superfluid is found. The nearly vertical blue line shown in the figure represents the line $T_c = 0$ which appears around $1/k_F a \approx 0.18$ (See Fig 2) and is roughly independent of polarization.

Finite momentum condensates [11, 21] may well occur in any of the regimes in II, particularly IIA and IIB for which our equations do not yield stable zero momentum condensation. Future work will explore the nature of the stable phases in these regimes. The boundaries of the region denoted II can be compared with other $T = 0$ phase diagrams in the literature [5, 10]. In contrast to Ref. [5], we find that the most stringent criterion for stability at $T = 0$ is the positivity of the second order partial derivative of $\partial^2 \Omega / \partial \Delta^2$ [as given in Eq. (5)]. This defines the boundary between II and III. This is substantially different from the line associated with $n_s(0) = 0$ (used in Ref. [5]) which is described by a nearly vertical line from $(p, 1/k_F a) = (1, 0.3)$ to $(0, 0.6)$. Similarly the locus of points in the two-dimensional parameter space $(p, 1/k_F a)$ where $T_{c}^{MF}$ vanishes defines the boundary between I and II, as is consistent with its counterpart in Ref. [5].

In Region IID, we define $T_{unstable}$ (which is below the single $T_c$) as the temperature where the system becomes unstable, via Eq. (5). At a given $1/k_F a$, $T_{unstable}$ decreases with decreasing $p$, and approaches 0 as $p \to 0$. This is shown in the inset to Fig. 3 for $1/k_F a = 0.5$. In region IIC, the lower $T_c$ approaches 0 as $p$ approaches 0. Thus, there is an important distinction between the $p = 0$ and $p \to 0^+$ limits, especially at $T = 0$. For $p$ small but finite, calculations readily encounter instabilities at strictly $T = 0$ and here superfluidity is very fragile to the introduction of small imbalance. By contrast, at
finite $T$ this fragility is not as pronounced. We conclude that only $T \equiv 0$ is a problematic temperature for weakly polarized superfluidity.

Figure 4 presents $n_s(T)$ for $p = 0.1$. Except for the case $1/k_Fa = 1.5$, Fig. 4 shows the typical behavior in region II (of Fig. 3), corresponding to intermediate temperature superfluidity. The observations here (and associated nonmonotonicities) for $n_s(T)$ are similar in many ways to what is seen for $\Delta(T)$ in the inset to Fig. 1. In region IIC, $n_s$ goes to zero at the upper and lower $T_c$, whereas in IID, $n_s(T)$ abruptly stops at $T_{\text{unstable}}$. The dotted lines indicate that they are in the unstable regime. Throughout region III, $n_s$ is found to be monotonically decreasing with increasing $T$, as in conventional superfluids. Finally in the inset of Fig. 4 we plot $n_{s0} \equiv n_s(0)$ as a function of $1 - p = 2n_s/n$. Only in the deep BEC regime is the dependence linear. This plot reflects that the excess unpaired fermions interact with the paired states, leading to a reduced superfluid density at $T = 0$ relative to $2n_s$.

The experimental situation regarding the stability of a unitary polarized superfluid (UPS) is currently being unraveled [1, 2]. If one includes the trap, within the local density approximation it appears [3] that the local polarization $p(r)$, in effect, increases continuously from a small value at the trap center to 100% at the trap edge. It follows from this paper that at very low $T$ the superfluid trap center will not support polarization, but for a range of $T$ closer to $T_c$, polarization can penetrate the core. We estimate from Fig. 2 (assuming the central $p \approx 0.05$), that there exists a UPS for $T \sim 0.05 - 0.25T_F$. Given the temperature range in experiment 1 this appears to be not inconsistent with current data ($T_F = 1.9\mu K$, $T = 300 \sim 505nK = 0.16 \sim 0.27T_F$ on resonance). In the near-BEC regime our predictions also appear consistent with new data in Ref. [22]. More generally, because the local $p(r = 0)$ is small the unstable region is suppressed to very low $T$ as $p \rightarrow 0$, this may explain why superfluidity in atomic traps can be observed experimentally. Future theory including the trap will be required to provide quantitative comparison with experiment.

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[1] M. W. Zwierlein, A. Schirotzek, C. H. Schunck, and W. Ketterle, Science 311, 492 (2006).
[2] G. B. Partridge, W. Li, R. I. Kamar, Y. A. Liao, and R. G. Hulet, Science 311, 503 (2006).
[3] W. V. Liu and F. Wilczek, Phys. Rev. Lett. 90, 047002 (2003).
[4] M. M. Forbes, E. Gubankova, W. V. Liu, and F. Wilczek, Phys. Rev. Lett. 94, 017001 (2005).
[5] C. H. Pao, S. T. Wu, and S. K. Yip, Phys. Rev. B 73, 132506 (2006).
[6] T. N. De Silva and E. J. Mueller, Phys. Rev. A 73, 051602(R) (2006).
[7] M. Haque and H. T. C. Stoof, cond-mat/0601321 (2006).
[8] P. Pieri and G. C. Strinati, Phys. Rev. Lett. 96, 150404 (2006).
[9] A. J. Leggett, in Modern Trends in the Theory of Condensed Matter, edited by A. Pekalski and J. Przystawa (Springer-Verlag, Berlin, 1980), pp. 13–27.
[10] D. Sheehy and L. Radzihovsky, Phys. Rev. Lett. 96, 060401 (2006).
[11] W. Yi and L. M. Duan, Phys. Rev. A 73, 031604(R) (2006).
[12] Q. J. Chen, J. Stajic, S. N. Tan, and K. Levin, Phys. Rep. 412, 1 (2005).
[13] I. Kosztin, Q. J. Chen, B. Jankó, and K. Levin, Phys. Rev. B 58, R5936 (1998).
[14] Q. J. Chen, I. Kosztin, B. Jankó, and K. Levin, Phys. Rev. Lett. 81, 4708 (1998).
[15] P. Pieri and G. C. Strinati, Phys. Rev. B. 71, 094520 (2005).
[16] P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. 59, 195 (1985).
[17] Q. J. Chen, J. Stajic, and K. Levin, Phys. Rev. Lett. 95, 260405 (2005).
[18] J. Stajic, A. Iyengar, Q. J. Chen, and K. Levin, Phys. Rev. B 68, 174517 (2003).
[19] A. Sedrakian and U. Lombardo, Phys. Rev. Lett. 84, 602 (2000).
[20] G. Sarma, J. Phys. Chem. Solids 24, 1029 (1963).
[21] T. Mizushima, K. Machida, and M. Ichioka, Phys. Rev. Lett. 94, 060404 (2005).
[22] M. W. Zwierlein, C. H. Schunck, A. Schirotzek, and W. Ketterle, Nature (London) 442, 54 (2006).