The BPS spectrum of the 4d $\mathcal{N} = 2$ SCFT’s
$H_1, H_2, D_4, E_6, E_7, E_8$

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Abstract

Extending results of 1112.3984, we show that all rank 1 $\mathcal{N} = 2$ SCFT’s in the sequence $H_1, H_2, D_4, E_6, E_7, E_8$ have canonical finite BPS chambers containing precisely $2h(F) = 12(\Delta - 1)$ hypermultiplets. The BPS spectrum of the canonical BPS chambers saturates the conformal central charge $c$, and satisfies some intriguing numerology.

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1 Introduction

Consider the seven rank 1 4d $\mathcal{N} = 2$ SCFT’s which may be engineered in $F$–theory using the Kodaira singular fibers $[1–9]$

\[ H_0, H_1, H_2, D_4, E_6, E_7, E_8. \] (1.1)

$H_0$ has trivial global symmetry and will be neglected in the following. The other six theories have flavor group $F$ equal, respectively, to $SU(2), SU(3), SO(8), E_6, E_7$ and $E_8$. (1.2)

We note that (1.2) is precisely the list of all simply–laced simple Lie groups $F$ with the property

\[ h(F) = 6 \frac{r(F) + 2}{10 - r(F)}, \] (1.3)

where $r(F)$ and $h(G)$ are, respectively, the rank and Coxeter number of $F$. Physically, the relation (1.3) is needed for consistency with the 2d/4d correspondence of [10], and is an example of the restrictions on the flavor group $F$ of a 4d $\mathcal{N} = 2$ SCFT following from that principle.

Neglecting $H_0$, let us list the numbers $2h(F)$ for the other six models

\[ 4, 6, 12, 24, 36, 60. \] (1.4)

The first four numbers in this list have appeared before in the non–perturbative analysis of the corresponding SCFT’s: it is known [11, 12] that the (mass–deformed) SCFT’s $H_1, H_2, D_4$ and $E_6$ have a finite BPS chamber in which the BPS spectrum consists precisely of (respectively) 4, 6, 12 and 24 hypermultiplets. The $H_1$ SCFT is the $D_3(SU(2))$ model of [12,13], while the $H_2, D_4$ and $E_6$ SCFT’s coincide, respectively, with the models $D_2(SU(3))$, $D_2(SU(4))$, and $D_2(SO(8))$ of those papers; then the above statement is a special instance of the general fact that, for all simply–laced Lie groups $G = ADE$, the $D_2(G)$ SCFT has a finite chamber with $r(G) h(G)$ hypermultiplets [12], while, for all $p \in \mathbb{N}$, the model $D_p(SU(2))$ has a special BPS chamber with $2(p-1)$ hypermultiplets\(^1\).

For the four SCFT’s $H_1, H_2, D_4, E_6$, the number of hypermultiplets in the above preferred chamber, $n_h$, may be written in a number of intriguing ways: we list just a few

\[ n_h = 2h(F) = \frac{12 r(F) + 24}{10 - r(F)} = 12(\Delta - 1) = n_7 \Delta, \] (1.5)

\(^1\) Note that $H_1$ is the Argyres–Douglas (AD) model of type $A_3$ [14] which has BPS chambers with any number $n_h$ of BPS hypers in the range $3 \leq n_h \leq 6$; likewise $H_2$ is the Argyres–Douglas model of type $D_4$, in both cases it is neither the AD minimal (3 resp. 4 hypers) nor the AD maximal (6 resp. 12 hypers) BPS chamber which is singled out by the property of being $c$–saturating, but rather their canonical chamber as a $D_p(SU(2))$ resp. a $D_2(G)$ theory [12] (for $D_4$ AD these two chambers are equivalent).
Table 1: Numerical invariants for the six SCFT’s $H_1$, $H_2$, $D_4$, $E_6$, $E_7$ and $E_8$. The rank of the flavor group, $r(F)$, is equal to the index in the SCFT symbol.

where $\Delta$ is the dimension of the field parametrizing the Coulomb branch of the rank 1 SCFT, and $n_7$ is the number of parallel 7–branes needed to engineer the SCFT in $F$–theory [1–9]; see Table 1.

The special finite BPS chambers with $n_h = 2 h(F)$ hypers have the particular property of saturating the conformal central charge $c$ of the strongly–coupled SCFT. By this we mean that, for these theories, the exact $c$ is equal to the value for $n_h$ free hypermultiplets plus the contribution from the massless photon vector multiplet

$$c = \frac{1}{12} n_h + \frac{1}{6},$$

that is, $c$ has the same value as the system of free fields with the same particle content as the BPS spectrum in the special chamber. In fact, the $c$–saturating property holds in general for the standard BPS chamber of all $D_2(G)$ SCFT’s [12], and also for all $D_p(SU(2))$. It was conjectured by Xie and Zhao [15] that a finite BPS chamber with this property exists for a large class of $\mathcal{N} = 2$ models (their examples are close relatives of the present ones). At the level of numerology, for the four SCFT’s $H_1$, $H_2$, $D_4$, $E_6$ we also have a simple relation between the number of hypers in our special chamber, $n_h$, and the $a$, $k_F$ conformal central charges: in facts, for all the above SCFT’s the central charge $a$ is given by the photon contribution, $5/24$, plus three–halves the contribution of $n_h$ free hypers

$$a = \frac{1}{24} \frac{3 n_h}{2} + \frac{5}{24},$$

$$k_F = \frac{n_h + 12}{6}. \quad (1.8)$$

In view of all this impressive numerology involving $n_h$, it is tempting to conjecture that the last two SCFT’s in the sequence (1.1), $E_7$ and $E_8$, also have canonical finite BPS chambers with, respectively, 36 and 60 hypermultiplets. This will extend our observations, eqn.(1.5)–(1.8), to the full SCFT sequence (1.1), suggesting that the numerology encodes deep physical properties of rank 1 SCFTs.
The purpose of the present short note is to prove the above conjecture, by constructing explicitly the canonical chambers with $2 \, h(F)$ hypers. To get the result we use the BPS quivers for the $E_7$ and $E_8$ Minahan–Nemeshanski theories identified in [12] together with the mutation algorithm of [11].

The rest of the note is organized as follows. In section 2 we recall the basics of the mutation algorithm. In section 3 we describe the relevant (class of) quivers and apply the mutation method to get the BPS spectra of the six SCFT’s, giving full details for the $E_6$, $E_7$ and $E_8$ theories.

## 2 Basics of the mutation algorithm

We recall the basics of the mutation algorithm referring the reader to [10, 11, 16, 17] for the details. Suppose we have a 4d $\mathcal{N} = 2$ model which has the BPS quiver property [18] [11, 17].

Let $\Gamma$ be the lattice of its quantized charges (electric, magnetic, and flavor). The susy central charge defines an additive function $Z(\cdot) : \Gamma \to \mathbb{C}$, so that a BPS particle of charge $v \in \Gamma$ has central charge $Z(v)$. Fix a half-plane $H_\theta = \{z \in \mathbb{C} : \text{Im}(e^{-i\theta}z) > 0\}$ such that no BPS particle has central charge laying on its boundary. We say (conventionally) that the BPS states with central charges in $H_\theta$ are particles, while those with central charges in $-H_\theta$ are their PCT–conjugate anti–particles. The charges of particles span a strict convex cone $\Gamma_\theta \subset \Gamma$ which, in a model with the BPS property [18], has the form $\Gamma_\theta \simeq \bigoplus_{i=1}^{r} \mathbb{Z}_{\geq 0} \epsilon_i^{(\theta)}$, where $r$ is the rank of $\Gamma$. The quiver $Q_\theta$ of the $\mathcal{N} = 2$ theory is obtained by picking one node per each generator $\epsilon_i^{(\theta)}$ of the positive cone $\Gamma_\theta$ and connecting nodes $\epsilon_i^{(\theta)}$, $\epsilon_j^{(\theta)}$ with $\langle \epsilon_i^{(\theta)}, \epsilon_j^{(\theta)} \rangle_{\text{Dirac}} \equiv B_{ij}^{(\theta)}$ oriented arrows, where $\langle \cdot, \cdot \rangle_{\text{Dirac}}$ is the Dirac electro–magnetic pairing of the charges$^{2}$. The integral skew–symmetric matrix $B_{ij}^{(\theta)}$ is called the exchange matrix of the quiver $Q_\theta$. $Q_\theta$ is supplemented with a superpotential $W_\theta$ (a sum, with complex coefficients, of cycles on $Q_\theta$) [11].

The BPS particles (as contrasted with antiparticles) correspond to stable representations $X$ of the quiver $Q_\theta$ subjected to the relations $\partial W_\theta = 0$. A representation $X$ is stable iff, for all non–zero proper subrepresentation $Y$, one has $\arg(e^{-i\theta}Z(Y)) < \arg(e^{-i\theta}Z(X))$, where we take $\arg(e^{-i\theta}H_\theta) = [0, \pi]$. The charge $v \in \Gamma_\theta$ of the BPS particle is given by the dimension vector $\sum_i \dim X_i \epsilon_i^{(\theta)}$ of the corresponding stable representation $X$.

In particular, the representations $S_i$ with dimension vector equal to a generator $\epsilon_i^{(\theta)}$ of $\Gamma_\theta$ are simple, and hence automatically stable for all choices of the function $Z(\cdot)$ (consistent with the given positive cone $\Gamma_\theta \subset \Gamma$). Therefore, $r$ BPS states are determined for free; they are necessarily hypermultiplets, since $Q_\theta$ has no loop [11, 19].

The above construction depends on an arbitrary choice, the angle $\theta$. Choosing a different angle $\theta'$, we get a different convex cone $\Gamma_{\theta'}$ with a different set of generators $\epsilon_i^{(\theta')}$. Since the

$^{2}$ Strictly speaking $\langle \epsilon_i^{(\theta)}, \epsilon_j^{(\theta)} \rangle_{\text{Dirac}}$ is only the number of net arrows (i.e. the number of $\epsilon_i^{(\theta)} \to \epsilon_j^{(\theta)}$ arrows minus the number of the $\epsilon_i^{(\theta)} \leftarrow \epsilon_j^{(\theta)}$ ones). For generic superpotentials pairs of opposite arrows $\epsilon_i^{(\theta)} \leftrightharpoons \epsilon_j^{(\theta)}$ get massive and may be integrated out, leaving the 2–acyclic quiver in the text [11].
physics does not depend on the conventional choice of the half plane $H_\theta$, the $e_{i_1}^{(\theta')}$'s should also be charge vectors of stable BPS hypermultiplets. The idea of the mutation algorithm is to get the full BPS spectrum by collecting all states with charges of the form $e_{i_1}^{(\theta')}$ for all $\theta$. It is easy to see that this gives the full BPS spectrum provided it consists only of hypermultiplets (i.e. particles of spin $\leq 1/2$) and their number $n_\mu$ is finite.

More concretely, we notice that the BPS particle of larger $\arg(e^{-i\theta}Z(v))$, has a charge $v$ which is a generator $e_{i_1}^{(\theta)}$ of $\Gamma_\theta$ (associated to some node $i_1$ of $Q_\theta$). We may tilt clockwise the boundary line of $H_\theta$ just past the point $Z(e_{i_1}^{(\theta)})$, producing a new half–plane $H_{\theta'}$. In the new frame the state with charge $e_{i_1}^{(\theta)}$ is an anti–particle, while its PCT–conjugate of charge $-e_{i_1}^{(\theta)}$ becomes a particle, and in facts a generator of the new positive cone $\Gamma_{\theta'}$. The generators $e_{i}^{(\theta')}$ of $\Gamma_{\theta'}$ are linear combinations with integral coefficients of the old ones $e_{i}^{(\theta)}$. The explicit expression of the $e_{i}^{(\theta')}$'s in terms of the $e_{i}^{(\theta)}$ is known as the Seiberg duality in physics [20,21], while in mathematics [22] it is called the basic quiver mutation of $Q_\theta$ at the $i_1$ node, written $\mu_{i_1}$,

$$e_{i}^{(\theta')} = \mu_{i_1}(e_{i}^{(\theta)}) = \begin{cases} -e_{i_1}^{(\theta)} & \text{if } i = i_1 \\ e_{i}^{(\theta)} + \max\{B_{i_1,i}^{(\theta)}, 0\} e_{i_1}^{(\theta)} & \text{otherwise.} \end{cases} \quad (2.1)$$

The mutated quiver $Q_{\theta'} = \mu_{i_1}(Q_\theta)$ is specified by the exchange matrix $B_{ij}^{(\theta')} \equiv \langle e_i^{(\theta')}, e_j^{(\theta')} \rangle_{\text{Dirac}}$. Under the quiver mutation $\mu_{i_1}$, the superpotential $W_{\theta}$ changes according to the rules of Seiberg duality [20,21] (which is equivalent to the the DWZ rule [22]).

The new generators $e_{i}^{(\theta')}$ are also charge vectors of stable hypers. We may reiterate the procedure by mutating $Q_{\theta'}$ at the node $i_2$ corresponding to the hypermultiplet with maximal $\arg(e^{-i\theta'}Z(v))$. Again we conclude that the BPS spectrum also contains stable hypers with charges $e_{i}^{(\theta')}$. Now suppose that after $m$ mutations we end up with the positive cone $\Gamma_{\theta^{(m)}} \equiv -\Gamma_{\theta}$; we conclude that $\theta^{(m)} = \theta + \pi$ and hence, with our sequence of $m$ half–plane tiltings, we have scanned the full complex half–plane $H_\theta$, picking up all the BPS particles, one at each step, according to their (decreasing) phase order in the central charge plane. Thus, whenever this happens, we conclude that we have a BPS chamber in which the BPS spectrum consists of precisely $m$ hypermultiplets.

This happens iff there is a sequence of $m$ basic quiver mutation such that [11]

$$\mu_{i_m} \circ \mu_{i_{m-1}} \circ \cdots \circ \mu_{i_2} \circ \mu_{i_1}(e_{i}^{(\theta)}) = -\pi(e_{i}^{(\theta)}) \quad \forall i, \quad (2.2)$$

where $\pi$ is a permutation of the $r$ generators $e_{i}^{(\theta)}$. If, for the given quiver $Q_\theta$, we are able to find a sequence of quiver mutations satisfying equation (2.2) (for some $\pi \in \mathfrak{S}_r$) we may claim to have found a finite BPS chamber consisting of $m$ hypermultiplets only, and list the quantum numbers $v_\ell \in \Gamma$ of all BPS particles

$$v_\ell = \mu_{i_{\ell-1}} \circ \mu_{i_{\ell-2}} \circ \cdots \circ \mu_{i_1}(e_{i}^{(\theta)}) \quad \ell = 1, 2, \ldots, m. \quad (2.3)$$

In the the rest of the paper we shall work at fixed $\theta$, and write the positive cone generators
simply as \( e_i \), omitting the angle.

There are a few strategies to find particular solutions to eqn. (2.2). An elegant one is the complete families of sink/source factorized subquivers of \( Q_\theta \) introduced in [16] and reviewed in [12]; as explained in these references, this is particular convenient when the factorized subquivers are Dynkin ones endowed with the standard Coxeter sink/source sequences.

For general quivers \( Q \), we may perform a systematic search for solutions on a computer; Keller’s quiver mutation applet [23] is quite helpful for both procedures. In doing this, it is convenient to rephrase eqn. (2.2) in terms of tropical \( y \)-seed mutations [24–26]. We recall that the tropical semifield \( \text{Trop}(u_1, u_2, \ldots, u_r) \) is the free multiplicative Abelian group generated by the indeterminates \( u_i \) endowed with the operation \( \oplus \) defined by

\[
(\prod u_i^{l_i}) \oplus (\prod u_i^{m_i}) = \prod u_i^{\min(l_i, m_i)}. \tag{2.4}
\]

To a BPS state of charge \( \sum_i n_i e_i \) we associate the tropical \( y \)-variable \( \prod u_i^{n_i} \in \text{Trop}(u_1, u_2, \ldots, u_r) \).

We start with the initial \( y \)-seed in which we assign to the \( i \)-th node of \( Q \) the variable associated to the generator \( e_i \) of the positive cone, namely \( y_i(0) \equiv u_i \), and we perform the sequence of mutations in eqn. (2.2) on the \( y \)-seed using the Fomin–Zelevinski rules [24–26]

\[
y_j(s) = \begin{cases} 
y_{i_0}(s-1)^{-1} & \text{if } j = i_s \\
y_j(s-1) y_i(s-1)^{B_{i,s}(s-1)} + (1 \oplus y_i(s-1))^{-B_{i,s}} & \text{otherwise}.
\end{cases} \tag{2.5}
\]

(here \( s = 1, 2, \ldots, m \), and \([x]_+ = \max(x, 0)\)). Since the tropical variables \( y_i(s-1) \) correspond to BPS particles with charges in the positive cone \( \Gamma_\theta \), one has \( 1 \oplus y_i(s-1) \equiv 1 \), and eqn. (2.5) reduces to the transformation rule (2.1). In terms of tropical \( y \)-variables, then eqn. (2.2) becomes

\[
y_j(m) = y_{\pi(j)}(0)^{-1}, \tag{2.6}
\]

supplemented by the condition that the tropical quantities \( y_i(s-1) \) are monomials in the \( u_i \)’s. This is the equation we actually solve in the next section.

For many purposes, it is convenient to rephrase the algorithm in the language of [10]. If the sequence of basic quiver mutations \( \mu_i \) satisfies eqn. (2.2), the associated composition of basic quantum cluster mutations satisfies

\[
Q_{i_m} \circ Q_{i_{m-1}} \circ \cdots Q_{i_2} \circ Q_{i_1} = I_{\pi} K(q) \tag{2.7}
\]

where \( K(q) \) is the quantum half–monodromy [10] and \( I_{\pi} \) is the unitary operator acting on the generators \( Y_i \) of the quantum torus algebra of \( Q_\theta \) as

\[
I_{\pi} Y_i I_{\pi}^{-1} = Y_{\pi(i)}^{-1}. \tag{2.8}
\]

The (finite) BPS spectrum may be read directly from the factorization of \( K(q) \) in quantum
dilogaritms [10], which is explicit in the LHS of eqn.(2.7).

In conclusion: given a solution to eqn.(2.2) we have determined a BPS chamber $C_{\text{fin}}$ (which may be formal in the sense of [18]) containing finitely many hypers, as well as the quantum numbers $v \in \Gamma$ of all these hypers.

In addition, the algorithm specifies the (cyclic) phase order of the central charges $Z(v)$ of the BPS states. From this last information we may read the domain $D_{\text{fin}} \subset \mathbb{C}^r \equiv (\Gamma \otimes \mathbb{C})^\vee$ of central charges $Z(\cdot) \in (\Gamma \otimes \mathbb{C})^\vee$ for which $C_{\text{fin}}$ is the actual BPS chamber, that is, we may determine the region in the space of the ‘physical’ parameters of the theory which corresponds to the finite chamber $C_{\text{fin}}$.

At a generic point in $D_{\text{fin}}$ the unbroken flavor symmetry is just $U(1)^{\text{rank} F}$. At particular points in parameter space the flavor symmetry may have a non–Abelian enhancement. Let $F_{\text{fin}}$ be the flavor symmetry group at a point of maximal enhancement in the domain $D_{\text{fin}}$. Clearly, the BPS hypers of $C_{\text{fin}}$ should form representations of $F_{\text{fin}}$. The fact that they do is a non–trivial check of the procedure.

## 3 Computing the BPS spectra

### 3.1 The quivers $Q(r, s)$

We begin by fixing uniform and convenient representatives of the quiver mutation–classes for the six $\mathcal{N} = 2$ models in eqn.(1.1) with $F \neq 1$. We define $Q(r, s)$ to be the quiver with $(r + s + 2)$ nodes

\[
\begin{array}{cccccccc}
a_1 & a_2 & \cdots & a_r & c_1 & b_1 & b_2 & \cdots & b_s \\
\uparrow & \uparrow & & \uparrow & \downarrow & \downarrow & & \downarrow & \downarrow \\
c_2
\end{array}
\]  

(3.1)

Then the (representative) quivers for our six SCFT’s are

\[
\begin{array}{cccccccc}
\text{SCFT} & H_1 & H_2 & D_4 & E_6 & E_7 & E_8 \\
quiver & Q(0, 1) & Q(1, 1) & Q(2, 2) & Q(3, 3) & Q(3, 4) & Q(3, 5)
\end{array}
\]  

(3.2)

(cfr. ref. [18] for $H_1$, $H_2$ and $D_4$, ref. [11] for $E_6$, and ref. [12] for $E_7$ and $E_8$).

The simplest way to get the table (3.2) is by implementing the flavor groups $F$ in eqn.(1.2) directly on the quiver. Indeed, given a $Q(r, s)$ quiver the flavor group $F$ of the corresponding $\mathcal{N} = 2$ QFT is canonically identified by the property that its Dynkin graph is the star with
three branches of lengths\textsuperscript{3} \([r, s, 2]\). Indeed, a simple computation shows that, if a quiver of the form \(Q(r, s)\) is consistent with the 2\(d/4d\) correspondence \([10]\) — that is, if \(Q(r, s)\) is the BPS quiver of a 2\(d\) (2,2) theory with \(\hat{c} < 2\) — then the star graph \([r, s, 2]\) is a Dynkin diagram (while 2\(d\) (2,2) models whose BPS quivers has the form \(Q(r, s)\), with \([r, s, 2]\) an affine Dynkin graph, necessarily have \(\hat{c} = 2\)).

### 3.2 The \(c\)-saturating chamber for \(H_1\), \(H_2\), \(D_4\) and \(E_6\)

The first four quivers in (3.2) may be decomposed into Dynkin subquivers in the sense of \([16]\)

\[
Q(1, 0) = A_2 \sqcup A_1, \quad Q(1, 1) = A_2 \sqcup A_2, \\
Q(2, 2) = A_3 \sqcup A_3, \quad Q(3, 3) = D_4 \sqcup D_4. \tag{3.3}
\]

For a quiver \(G \sqcup G'\) the charge lattice is \(\Gamma = \Gamma_G \oplus \Gamma_{G'}\), where \(\Gamma_G\) is the root lattice of the Lie algebra \(G\). Since the decomposition has the Coxeter property \([12, 16]\), there is a canonical chamber in which the BPS spectrum consists of one hypermultiplet per each of the following charge vectors \([16]\)

\[
\left\{ \alpha \oplus 0 \in \Gamma_G \oplus \Gamma_{G'}, \ \alpha \in \Delta^+(G) \right\} \bigcup \left\{ 0 \oplus \beta \in \Gamma_G \oplus \Gamma_{G'}, \ \beta \in \Delta^+(G') \right\}, \tag{3.4}
\]

where \(\Delta^+(G)\) is the set of the positive roots of \(G\). Then the number of hypermultiplets in this canonical finite chamber is

\[
n_h = \frac{1}{2} (r(G) \, h(G) + r(G') \, h(G')), \tag{3.5}
\]

which for the four cases in eqn. (3.3) gives (respectively)

\[
4, \ 6, \ 12, \ 24, \tag{3.6}
\]

i.e. \(n_h = 2 \, h(F)\) as expected for a \(c\)-saturating chamber.

For sake of comparison with the \(E_7\), \(E_8\) cases in the next subsection, we give more details on the computation of the above spectrum for the \(E_6\) Minahan–Nemeshanski theory \([7]\) using the mutation algorithm. The cases \(H_1\), \(H_2\) and \(D_4\) are similar and simpler.

The two \(D_4\) subquivers of \(Q(3, 3)\) are the full subquivers over the nodes \{\(a_1, a_2, a_3, c_1\)\} and, respectively, \{\(b_1, b_2, b_3, c_1\)\}. The quiver \(Q(3, 3)\) has an automorphism group \(\mathbb{Z}_2 \ltimes (\mathfrak{S}_3 \times \mathfrak{S}_3)\), where the two \(\mathfrak{S}_3\) are the triality groups of the \(D_4\) subgraphs, while \(\mathbb{Z}_2\) interchanges the two \(D_4\) subquivers (and hence the two \(\mathfrak{S}_3\)'s). The quiver embedding \(D_4 \oplus D_4 \to Q(3, 3)\) induces

\textsuperscript{3} As always, in the length of each branch we count the node at the origin of the star; in particular, a branch of length one is no branch at all, while a branch of length zero means that we delete the origin of the star itself. Note that, for all \(s\), the quiver \(Q(2, s)\) is mutation equivalent to the quiver of \(SU(2)\) SQCD with \(N_f = s + 2\) fundamental flavors which has flavor symmetry group \(SO(2s + 4)\), whose Dynkin graphs is the star with three branches of lengths \([2, s, 2]\).
an embedding of flavor groups

\[ SU(3) \times SU(3) \rightarrow F \equiv E_6, \quad (3.7) \]

where \( SU(3) \) is the flavor group of the Argyres–Douglas theory of type \( D_4 \) characterized by the fact that \( \text{Weyl}(SU(3)) \equiv \mathfrak{S}_3 \equiv \) the triality group of \( D_4 \).

The two \( D_4 \) subquivers have the ‘subspace’ orientation; in both bi–partite quivers \( Q(3, 3) \) and \( D_4 \) we call even the nodes \( a_i \) and \( b_i \) and odd the \( c_i \) ones. Then, by standard properties of the Weyl group, the quiver mutation ‘first all even then all odd’

\[ \mu_{c_2} \mu_{c_1} \prod_{i=1}^{3} \mu_{b_i} \prod_{i=1}^{3} \mu_{a_i} \quad (3.8) \]

transforms the quiver \( Q(3, 3) \) into itself while acting on \( \Gamma_{D_4} \oplus \Gamma_{D_4} \) as \( \text{Cox} \oplus \text{Cox} \), where \( \text{Cox} \in \text{Weyl}(D_4) \) is the Coxeter element of \( D_4 \). Since \( (\text{Cox})^3 = -1 \), the quiver mutation

\[ (\mu_{c_2} \mu_{c_1} \prod_{i=1}^{3} \mu_{b_i} \prod_{i=1}^{3} \mu_{a_i})^3 \quad (3.9) \]

is a solution to eqn.(2.2) with \( \pi = \text{Id} \). Since there are 24 \( \mu \)'s in eqn.(3.9), we have found a finite BPS chamber with 24 hypers. Eqn.(3.9) is invariant under the automorphism group \( \mathbb{Z}_2 \ltimes (\text{Weyl}(SU(3)) \times \text{Weyl}(SU(3))) \) so that there are points in the parameter domain \( D_{\text{fin}} \) corresponding to the above chamber which preserve a flavor group

\[ F_{\text{fin}} \supseteq \mathbb{Z}_2 \ltimes (SU(3) \times SU(3) \times U(1)^2), \quad (3.10) \]

where \( \mathbb{Z}_2 \) acts by interchanging the two \( SU(3) \)'s and inverting the sign of the first \( U(1) \) charge. The 24 BPS states may be classified in a collection of irrepresentations of the group in the large parenthesis of eqn.(3.10) which form \( \mathbb{Z}_2 \) orbits. From eqn.(3.9) we read the phase ordering of the particles in the 24 BPS hypers (in addition we have, of course, the PCT conjugate anti–particles). Ordered in decreasing phase order, we have

\[
\begin{align*}
&\overbrace{(3, 1)_{1,0}, (1, 3)_{-1,0}}^{(3, 1)_{1,0}}, \quad \overbrace{(1, 1)_{3,1}, (1, 1)_{-3,1}}^{(1, 1)_{3,1}}, \quad \overbrace{(3, 1)_{2,1}, (1, 3)_{-2,1}}^{(3, 1)_{2,1}}, \\
&\overbrace{(1, 1)_{3,2}, (1, 1)_{-3,2}}^{(1, 1)_{3,2}}, \quad \overbrace{(3, 1)_{1,1}, (1, 3)_{-1,1}}^{(3, 1)_{1,1}}, \quad \overbrace{(1, 1)_{0,1}, (1, 1)_{0,1}}^{(1, 1)_{0,1}},
\end{align*}
\quad (3.11)
\]

where overbraces collect representations forming a \( \mathbb{Z}_2 \)–orbit. In terms of dimension vectors of the corresponding quiver representations, the quantum numbers of the 24 BPS particles
(in decreasing phase order) is
\[
\begin{align*}
&\quad\quad a_1, a_2, a_3; b_1, b_2, b_3; a_1 + a_2 + a_3 + c_1; b_1 + b_2 + b_3 + c_2; \\
&\quad\quad a_2 + a_3 + c_1, a_1 + a_3 + c_1, a_1 + a_2 + c_1; b_2 + b_3 + c_2, b_1 + b_3 + c_2, b_1 + b_2 + c_2; \\
&\quad\quad a_1 + a_2 + a_3 + 2c_1; b_1 + b_2 + b_3 + 2c_2; \\
&\quad\quad a_1 + c_1, a_2 + c_1, a_3 + c_1; b_1 + c_2, b_2 + c_2, b_3 + c_2; c_1; c_2,
\end{align*}
\]
(3.12)

where, for notational convenience, the positive cone generators $e_{a_i}, e_{b_j}, e_{c_k}$ are written simply as $a_i, b_j, c_k$, respectively.

### 3.3 The 36–hyper BPS chamber of $E_7$ MN

The quiver $Q(3,4)$ has no obvious useful decomposition into Dynkin subquivers. However, with the help of Keller’s quiver mutation applet it is easy to check that the composition of the 36 basic quiver mutations at the sequence of nodes
\[
\begin{align*}
&\quad\quad a_1 a_2 a_3 b_1 b_2 b_3 c_1 c_2 \\
&\quad\quad a_1 a_2 b_1 b_2 b_3 c_1 c_2 a_1 a_2 a_3 b_2 c_1 c_2 b_1 b_2 b_3 c_1 c_2 \\
&\quad\quad a_1 a_2 a_3 b_1 b_2 b_3 c_1 c_2
\end{align*}
\]
(3.13)

is a solution to eqn.(2.2) for $Q(3,4)$ with\(^4\)
\[
\pi = (a_1 a_2)(a_3 b_1 b_4)(b_2 b_3)(c_1 c_2).
\]
(3.14)

Moreover no proper subsequence of mutations is a solution to eqn.(2.2). Note the similarity with the sequence for $E_6$ which is a three fold repetition of the first line of (3.13) (the Coxeter sequence of $D_4 \sqcup D_4$). Passing from $E_6$ to $E_7$ we simply replace the second repetition of the Coxeter sequence for $D_4 \sqcup D_4$ with the second line of (3.13) which may also be interpreted as a chain of Coxeter sequences (see remark after eqn.(3.17)).

The solution (3.13) corresponds to the finite BPS chamber for the $E_7$ Minahan Nemeshanski theory \([8]\) with 36 hypermultiplets we were looking for. The (manifest) automorphism of this finite chamber is given by the centralizer of $\pi$ in the $Q(3,4)$ automorphism group $G_3 \times G_4$, which is the subgroup $G_2 \times G_2$ generated by the involutions $(a_1 a_2)$ and $(b_2 b_3)$. Then the BPS hypers in this finite chamber form representations of $F_{\text{fin}} = SU(2) \times SU(2) \times U(1)^5$. From the list of charge vectors of the 36 hypers in Table 2 we see that this is indeed true.

We stress that the 36–hyper chamber above is far from being unique; a part for the other $m = 36$ solutions to eqn.(2.2) obtained from (3.13) by applying an automorphism of the

\[^4\] The fact that $\pi$ is not an involution implies that this mutation cannot arise from Coxeter–factorized subquivers as in the previous examples.
Weyl group is realized as permutations of the charge vector sets.

For the quiver $Q$ suitable $G$, $\vdash$ chambers look very similar, in particular they are expected to have isomorphic the sequence of nodes.

Table 2: The charge vectors of the 36 BPS particles in the chamber $C_\text{fin}$ of the $E_7$ MN theory. To simplify the notation, the positive cone generators $e_{a_i}, e_{b_j}, e_{c_k}$ are written simply as $a_i, b_j, c_k$, respectively. The particles are listed in decreasing BPS phase order. To get the full BPS spectrum, add the PCT conjugate anti–particles.

For the quiver $Q(3,4)$, there are other ones; for instance, the sequence of 36 mutations at the nodes

$$c_1 c_2 a_1 a_2 b_1 b_2 c_2 c_1 a_3 b_3 b_4 b_1 c_1 b_4 b_3 a_1 a_2$$

$$b_1 b_2 c_1 a_2 b_1 a_1 b_4 a_3 b_1 c_1 b_4 b_3 a_1 a_2 b_3.$$  \hspace{1cm} (3.15)

is a solution to (2.2) with $\pi = (a_1 a_2)(a_3 b_1 b_2)(b_3 b_4)(c_1 c_2)$. The properties of all these chambers look very similar, in particular they are expected to have isomorphic $F_\text{fin}$.

3.4 The 60–hyper BPS chamber of $E_8$ MN

For the quiver $Q(3,5)$ one checks that the composition of the 60 basic quiver mutations at the sequence of nodes

$$a_1 a_2 a_3 b_1 b_2 b_3 c_1 c_2 || a_1 a_2 b_4 b_1 b_2 b_3 c_1 c_2 ||$$

$$a_1 a_2 a_3 b_5 b_1 b_4 c_1 c_2 || b_1 b_2 b_3 a_3 c_1 c_2 ||$$

$$a_1 a_2 b_2 b_3 b_4 b_5 c_1 c_2 || a_1 a_2 a_3 b_2 b_3 b_4 c_1 c_2 ||$$

$$a_3 b_1 b_2 b_3 b_4 b_5 c_1 c_2 || a_1 a_2 b_4 b_5 c_1 c_2$$

is a solution to eqn.(2.2) with

$$\pi = (a_1 a_2)(a_3 b_1)(b_2 b_3)(b_4 b_5)(c_1)(c_2).$$  \hspace{1cm} (3.17)

while no proper subsequence solves it. In eqn.(3.16) the sequence of mutation is divided into pieces by the dividing symbol $||$; again each piece may be seen as a Coxeter sequence for a suitable $G \vdash G$ (sub)quiver with $G$ either $A_3$ or $D_4$.

Thus we have constructed a 60–hyper $c$–saturating chamber for the $E_8$ MN theory [8]. The manifest unbroken flavor symmetry in this finite chamber is $SU(2) \times SU(2) \times U(1)$ whose Weyl group is realized as permutations of the charge vector sets $\{e_{a_1}, e_{a_2}\}$ and $\{e_{b_2}, e_{b_3}\}$.

Again the 60–hyper chamber is not unique; for instance, another 60–mutation solution
\[ a_1, a_2, a_3, b_1, b_2, b_3, a_1 + a_2 + a_3 + c_1, b_1 + b_2 + b_3 + c_2, a_2 + a_3 + c_1, a_1 + a_3 + c_1, a_1 + a_2 + a_3 + b_4 + c_1, b_2 + b_3 + c_2, b_1 + b_3 + c_2, b_1 + b_2 + c_2, a_1 + a_2 + 2 a_3 + b_4 + 2 c_1, b_1 + b_2 + b_3 + 2 c_2, a_1 + a_3 + b_4 + c_1, a_2 + a_3 + b_4 + c_1, a_1 + a_2 + b_1 + b_2 + b_3 + c_1 + 2 c_2, a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + b_5 + c_1 + 2 c_2, b_1 + c_2, a_3 + c_1, a_3 + b_4 + c_1, 2 a_1 + 2 a_2 + a_3 + 2 b_1 + b_2 + b_3 + b_5 + 2 c_1 + 3 c_2, 2 a_1 + 2 a_2 + a_3 + b_2 + b_3 + b_5 + 2 c_1 + 2 c_2, a_3 + b_2 + b_4 + c_1 + c_2, a_3 + b_3 + b_4 + c_1 + c_2, a_1 + a_2 + a_3 + b_1 + b_5 + c_1 + c_2, a_3 + b_2 + b_3 + b_4 + c_1 + 2 c_2, a_1 + a_2 + a_3 + b_5 + c_1, a_2 + a_3 + b_5 + c_1, a_1 + a_3 + b_5 + c_1, b_3 + c_2, b_2 + c_2, a_1 + a_2 + a_3 + b_4 + b_5 + c_1, a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + b_4 + 2 c_1 + 3 c_2, a_1 + a_2 + b_1 + b_2 + b_3 + c_1 + 3 c_2, a_1 + a_2 + 2 a_3 + b_4 + 2 b_5 + 2 c_1, a_1 + a_3 + b_4 + b_5 + c_1, a_1 + a_3 + b_4 + b_5 + c_1, a_1 + a_2 + b_2 + b_3 + c_1 + 2 c_2, a_1 + a_2 + b_1 + b_2 + c_1 + 2 c_2, a_1 + a_2 + b_1 + b_3 + c_1 + 2 c_2, a_3 + b_5 + c_1, 2 a_1 + 2 a_2 + b_1 + b_2 + b_3 + 2 c_1 + 3 c_2, a_3 + b_4 + b_5 + c_1, a_1 + a_2 + b_1 + c_1 + c_2, a_3 + b_4 + b_5 + c_1, a_1 + a_2 + b_1 + c_1 + c_2, a_1 + a_2 + b_3 + c_1 + c_2, a_1 + a_2 + b_2 + c_1 + c_2, a_1 + a_2 + 2 c_1, a_4 + b_1 + b_5 + c_2, a_2 + c_1, a_1 + c_1, a_1 + b_5 + c_2, b_4 + 2 c_2. \]

Table 3: Charge vectors of the 60 BPS particles in the chamber \( \mathcal{C}_{\text{fin}} \) of the \( E_8 \) MN theory (in decreasing phase order). One has to add the PCT conjugate anti–particles.

is given by the node sequence

\[ c_2 c_1 b_1 b_4 b_2 a_1 a_3 a_2 c_1 c_2 b_5 b_3 b_2 c_4 c_1 a_1 a_3 a_2 b_1 b_5 c_1 c_2 a_1 a_3 a_2 b_2 b_5 b_3 \]
\[ c_2 c_1 b_2 b_5 b_3 b_4 b_1 a_1 c_2 a_3 a_2 a_1 b_1 c_2 a_3 a_2 b_2 b_5 b_3 c_1 c_2 a_3 a_2 b_2 b_5 b_3. \] (3.18)

### 3.5 Decoupling and other finite chambers

Sending the mass parameter dual to a charge \( e_{b_j} \) to infinity, \( |Z(e_{b_j})| \to +\infty \), all BPS states with non–zero \( e_{b_j} \)–charge get infinitely massive and decouple. The surviving states correspond to representations \( X \) of the \( Q(r,s) \) quiver with \( \text{dim } X_{b_j} = 0 \), which are stable representations of the quiver \( Q(r,s-1) \).

Assuming the decoupling limit may be taken while remaining in the domain \( \mathcal{D}_{\text{fin}} \), consistency requires that if we delete from the list of states in Table 3 all those which contain the given \( b_j \) \( (j = 1, 2, \ldots, 5) \) with non–zero coefficient, what remains should be the BPS spectrum of the \( E_7 \) Minahan–Nemeschanski model in some (not necessarily canonical) finite chamber. In the same vein, a similar truncation of the list in Table 2 should produce a finite BPS spectrum of \( E_6 \) MN. The fact the all the BPS spectra so obtained are related by the Wall Crossing Formula [27–31] to the canonical chamber determined above is a highly non–trivial check on the procedure.

This decoupling procedure applied to \( E_7 \) produces BPS chambers with 27 hypermulti-
plets, which are easily shown to be equivalent to the canonical 24–hypers one. In the $E_8$ case we get a chamber of $E_7$ with either 42 or 43 hypers, depending of which $e_{b_i}$ charge we make infinitely heavy.

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