EINSTEIN–CARTAN GRAVITY WITH TORSION FIELD SERVING AS AN ORIGIN FOR THE COSMOLOGICAL CONSTANT OR DARK ENERGY DENSITY

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Received 2016 July 2; revised 2016 July 10; accepted 2016 July 12; published 2016 September 21

ABSTRACT

We analyse the Einstein–Cartan gravity in its standard form \( R = R + \kappa^2 \), where \( R \) and \( R \) are the Ricci scalar curvatures in the Einstein–Cartan and Einstein gravity, respectively, and \( \kappa^2 \) is the quadratic contribution of torsion in terms of the contorsion tensor \( \kappa \). We treat torsion as an external (or background) field and show that its contribution to the Einstein equations can be interpreted in terms of the torsion energy–momentum tensor, local conservation of which in a curved spacetime with an arbitrary metric or an arbitrary gravitational field demands a proportionality of the torsion energy–momentum tensor to a metric tensor, a covariant derivative of which vanishes owing to the metricity condition. This allows us to claim that torsion can serve as an origin for the vacuum energy density, given by the cosmological constant or dark energy density in the universe. This is a model-independent result that may explain the small value of the cosmological constant, which is a long-standing problem in cosmology. We show that the obtained result is valid also in the Poincaré gauge gravitational theory of Kibble, where the Einstein–Hilbert action can be represented in the same form: \( R = R + \kappa^2 \).

Key words: cosmology: theory – dark energy – large-scale structure of universe

1. INTRODUCTION

Torsion is a natural geometrical quantity additional to a metric tensor. It is accepted (Hehl et al. 1976; Hammond 2002; Shapiro 2002; Kostelecky 2004; Hehl & Obukhov 2007; Ni 2010; Hehl 2012; Blagojević & Hehl 2013) that torsion characterizes spacetime geometry through spin–matter interactions, which allow us to probe the rotational degrees of freedom of spacetime in terrestrial laboratories (Rumpf 1979; Lämmerzahl 1997; Kostelecky et al. 2008; Lehert et al. 2014, 2015; Obukhov et al. 2014; Ivanov & Wellenzohn 2015a, 2015b, 2015c, 2016a). However, as has been shown recently (Ivanov & Wellenzohn 2016a), the requirement of linking torsion and fermion spin through torsion–fermion minimal couplings is violated in the low-energy approximation in curved spacetimes with rotation (see Equation (22) of Ivanov & Wellenzohn 2016a). The latter allows us to admit the existence of torsion even without spinning matter. In such an approach torsion can be treated as an external (or background) field, defined by a third-order tensor \( T_{\mu\nu\rho} \), antisymmetric with respect to indices \( \mu \) and \( \nu \), i.e., \( T_{\mu\nu\rho} = -T_{\nu\mu\rho} \) (Shapiro 2002; Kostelecky et al. 2008; Lehert et al. 2014, 2015; Ivanov & Wellenzohn 2015a, 2015b, 2015c, 2016a), which can be introduced into the Einstein–Cartan gravitational theory as an antisymmetric part of the affine connection through the metricity condition (Rebhan 2012). Such a torsion tensor field possesses 24 independent components, which can be decomposed into 4-vector \( \mathbf{E}_\mu = (\mathbf{E}_0, -\mathbf{E}) \), 4-axial-vector \( \mathbf{B}_\mu = (\kappa, -\mathbf{B}) \) and 16-tensor \( \mathcal{M}_{\mu\nu\rho} \) components (Shapiro 2002; Kostelecky et al. 2008; see also Ivanov & Wellenzohn 2015b). As has been shown in Ivanov & Wellenzohn (2015b), only torsion axial-vector \( \mathbf{B}_\mu \) components are present in the torsion–fermion minimal couplings in curved spacetimes with metric tensors, providing vanishing time–space (spacetime) components of the vierbein fields. The torsion vector \( \mathbf{E}_\mu \) and tensor \( \mathcal{M}_{\mu\nu\rho} \) components, coupled to Dirac fermions, appear through torsion–fermion non-minimal couplings with phenomenological coupling constants (Kostelecky et al. 2008; see also Ivanov & Wellenzohn 2015b). The presence of phenomenological coupling constants screens real values of the torsion vector \( \mathbf{E}_\mu \) and tensor \( \mathcal{M}_{\mu\nu\rho} \) components. Nevertheless, an observation of these non-minimal torsion–fermion interactions should testify to the existence of torsion and correctness of Einstein–Cartan gravitational theory. It should be emphasized that, as has been shown in Ivanov & Wellenzohn (2015b) some effective low-energy interactions of torsion 4-vector \( \mathbf{E}_\mu = (\mathbf{E}_0, -\mathbf{E}) \) and tensor \( \mathcal{M}_{\mu\nu\rho} \) components, caused by non-minimal torsion–fermion couplings, do not depend on a fermion spin. Then, as shown in Ivanov & Wellenzohn (2015c, 2016a), torsion vector and tensor components can be probed in terrestrial laboratories through torsion–fermion minimal couplings in spacetimes with rotation (Hehl & Ni 1990; Landau & Lifschitz 2008; Obukhov et al. 2009, 2011). Some steps toward the creation of such spacetimes in terrestrial laboratories have been made by Atwood et al. (1984) and Mashhoon (1988), who used rotating neutron interferometers. Estimates of constant torsion, coupled to Dirac fermions, have been carried out by Lämmerzahl (1997), Kostelecky et al. (2008) and Obukhov et al. (2014) and discussed by Ivanov & Wellenzohn (2015b). Recently, Lehert et al. (2014) have measured in liquid a rotation angle \( \phi_{PV} \) of the neutron spin about a neutron 3-momentum \( \mathbf{p} \) per unit length \( d\phi_{PV}/dL \). Using the results obtained by Kostelecky et al. (2008), Lehert et al. (2014) found that \( d\phi_{PV}/dL = 2\xi_\zeta \). The parameter \( \zeta \) is a superposition of the scalar \( T_0 \sim \xi_0 \) and pseudoscalar \( A_0 \sim \kappa \) torsion components equal to \( \zeta = (2m\xi_5^0 - \xi_6^0)T_0 + (2m\xi_5^0 - \xi_6^4)A_0 \), where \( m \) is the neutron mass and \( \xi_5^0, \xi_6^0, \xi_5^4, \xi_6^4 \) are phenomenological constants introduced by Kostelecky et al. (2008). The experiment by Lehert et al. (2014) is based on the phenomenon of neutron optical activity, related to a rotation of the plane of polarization of a transversely polarized slow-neutron beam moving through matter. As has been reported by Lehert et al. (2014), \( \zeta \) is restricted from above by \( |\zeta| < 9.1 \times 10^{-14} \text{ eV} \) at 68% of C.L. Such an estimate is by a factor $10^5$ larger than the upper bound \( |\zeta| < 10^{-18} \text{ eV} \).
calculated by Ivanov & Wellenzohn (2015b) using the estimates of Kostelecky et al. (2008).

In this paper we analyse Einstein–Cartan gravitational theory without fermions. The aim is to show that torsion as a geometrical characteristic of a curved spacetime additional to a metric tensor can exist independently of spinning matter and play an important role in the evolution of the universe. Torsion in such an approach is treated as an external (or background) field (Shapiro 2002; Kostelecky et al. 2008; Lehnert et al. 2014; Obukhov et al. 2014; Ivanov & Wellenzohn 2015a, 2015b, 2015c, 2016a). In Section 2 we show that the gravity-torsion part of the Einstein–Hilbert action of Einstein–Cartan gravitational theory can be given in the additive form \( \int d^4x \sqrt{-g} \mathcal{R} = \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{C} \), where \( \mathcal{R} = g^{\mu\nu} R_{\mu\nu} \) and \( R = g^{\mu\nu} R_{\mu\nu} \) are scalar curvatures in the Einstein–Cartan and Einstein gravity, respectively, with the Ricci tensor \( R_{\mu\nu} \) defined in terms of the metric tensor \( g_{\mu\nu} \) only (Rebhan 2012). Then, \( \mathcal{C} = g^{\mu\nu} C_{\mu\nu} = g^{\mu\nu}(\mathcal{K}^{\rho\sigma\alpha\beta} K_{\rho\sigma\alpha\beta} - \mathcal{K}^{\rho\sigma\alpha} K_{\rho\sigma\alpha}) \) is defined by torsion in terms of the contorsion tensor \( K_{\mu\nu\rho} = \frac{1}{2}(T_{\mu\rho\nu} + T_{\rho\nu\mu} + T_{\nu\mu\rho}) \) (Kostelecky 2004), and \( g = \det(g_{\mu\nu}) \). The raising and lowering of indices are performed with metric tensors \( g^{\mu\nu} \) and \( g_{\mu\nu} \), respectively. In Section 3, for a curved spacetime with an arbitrary metric tensor, we derive the Einstein equations in independent components. Thus, such a torsion-induced cosmological constant is able to explain the small value of the cosmological constant, which is a long-standing problem in cosmology (Weinberg 1989; see also Peebles & Ratra 2003). Of course, probing the torsion tensor field components can be possible only through interactions with spin particles, in particular with Dirac fermions (Rumpf 1979; Lämmerzahl 1997; Kostelecky 2004; Kostelecky et al. 2008; Lehnert et al. 2014, 2015; Obukhov et al. 2014; Ivanov & Wellenzohn 2015a, 2015b, 2015c, 2016a). Nevertheless, we emphasize that not all of torsion–fermion interactions are defined by a fermion spin. As has been shown in Ivanov & Wellenzohn (2016a) in curved spacetimes with rotation, torsion scalar and tensor components couple to massive Dirac fermions through low-energy non-spin interactions, caused by minimal torsion–fermion couplings. In Section 4 we discuss the obtained results and the equivalence between Einstein–Cartan gravitational theory, analyzed in this paper, and Poincaré gauge gravitational theory (Kibble 1961; see also Utiyama 1956; Sciama 1961; Sciama 1964; Blagojević 2001; Hehl et al. 1976; Hehl & Obukhov 2007; Hehl 2012; Blagojević & Hehl 2013; Obukhov et al. 2014) without spinning matter. In Appendix A we calculate the covariant divergence of the energy–momentum tensor of the chameleon (quintessence) field and show that it vanishes in a curved spacetime with an arbitrary metric tensor. In Appendix B we analyse the results obtained from Poincaré gauge gravitational theory. We show that the integrand of the Einstein–Hilbert action \( e R = e' \epsilon_{ab} R_{\mu\nu}^{ab} \) of Poincaré gauge gravitational theory, where \( e = \sqrt{-g} \) and \( R_{\mu\nu}^{ab} \) is its gravitational field strength tensor defined in terms of the vierbein fields \( e_a^\mu \) and \( e^b_\mu \) and torsion can be represented in the additive form \( e(R + \mathcal{C}) \), where \( R = e_a^\mu e'^b_\mu R_{\mu\nu}^{ab} \) and \( R_{\mu\nu}^{ab} \) is the gravitational field strength tensor defined only in terms of vierbein fields, and \( \mathcal{C} = \mathcal{K}^{\rho\sigma\alpha} \mathcal{K}^{\alpha\beta}_{\rho\sigma} - \mathcal{K}^{\rho\sigma\alpha} \mathcal{K}^{\rho\sigma}_{\alpha} \). This allows us to determine the contribution of torsion to the Einstein equations through the torsion energy–momentum tensor, local conservation of which demands its proportionality to a metric tensor.
2. EINSTEIN–HILBERT ACTION IN EINSTEIN–CARTAN GRAVITY WITH TORSION AND WITHOUT A CHAMELEON FIELD

We take the Einstein–Hilbert action $S_{EH}$ of Einstein–Cartan gravity with torsion in the standard model-independent form

$$S_{EH} = \frac{1}{2} M_{Pl}^2 \int d^4x \sqrt{-g} \; \mathcal{R},$$

(1)

where $M_{Pl} = 1/\sqrt{8\pi G_N}$ is the reduced Planck mass, $G_N$ is the Newtonian gravitational constant (Olive et al. 2014), and $g$ is the determinant of the metric tensor $g_{\mu\nu}$.

The scalar curvature $\mathcal{R}$ is defined by (Kostelecky 2004)

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu} = g^{\nu\rho}(\partial_\nu \Gamma^\alpha_{\rho\mu} - \partial_\rho \Gamma^\alpha_{\nu\mu} + \Gamma^\alpha_{\nu\xi} \Gamma^\xi_{\rho\mu})$$

$$- \Gamma^\alpha_{\nu\xi} \Gamma^\xi_{\rho\mu},$$

(2)

where $R_{\mu\nu}$ and $R_{\rho\sigma}$ are the Riemann and Ricci tensors in the Einstein–Cartan gravitational theory, respectively, and $\Gamma^\alpha_{\nu\xi}$ is the affine connection

$$\Gamma^\alpha_{\mu\nu} = \{^\alpha_{\nu\mu}\} + K^\alpha_{\mu\nu} = \{^\alpha_{\nu\mu}\} + g^{\alpha\beta} K_{\beta\mu\nu}.$$ (3)

Here $\{^\alpha_{\nu\mu}\}$ are the Christoffel symbols (Rebhan 2012)

$$\{^\alpha_{\nu\mu}\} = \frac{1}{2} g^{\alpha\lambda}(\partial_\nu b_{\lambda\mu} + \partial_\mu b_{\lambda\nu} - \partial_\lambda b_{\nu\mu}).$$ (4)

and $K_{\beta\mu\nu}$ is the contorsion tensor, related to torsion $T_{\mu\nu\beta}$ by

$$K_{\beta\mu\nu} = \frac{1}{2}(T_{\mu\nu\beta} - T_{\mu\beta\nu} + T_{\nu\beta\mu})$$

and

$$T_{\mu\nu\beta} = \Gamma^\alpha_{\nu\beta} - \Gamma^\alpha_{\nu\mu}$$ (Kostelecky 2004).

In the case of zero torsion the Riemann and Ricci tensors reduce to their standard form (Rebhan 2012). The integrand of the Einstein–Hilbert action, Equation (1), can be represented in the following form:

$$\sqrt{-g} \; R = \sqrt{-g} \; R + \sqrt{-g} \; \mathcal{C}$$

$$+ \partial_\mu \left( \sqrt{-g} \; K^\alpha_{\nu\mu} \right) - \sqrt{-g} \; g^{\mu\nu}$$

$$\times \left( \frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} \; K^\alpha_{\nu\mu} \right) - \{^\alpha_{\nu\mu}\} K^\alpha_{\nu\xi} - \{^\alpha_{\nu\xi}\} K^\xi_{\rho\mu} \right),$$

(5)

where we have denoted

$$\mathcal{C} = g^{\mu\nu} C_{\mu\nu} = g^{\mu\nu} (K^\alpha_{\nu\mu} K^\alpha_{\nu\xi} - K^\alpha_{\nu\xi} K^\xi_{\rho\mu}).$$ (6)

In Equation (5), removing the total derivatives and integrating by parts, we may delete the third term and transcribe the fourth term into the form $\sqrt{-g} \; g^{\mu\nu} a_{\alpha} K^\alpha_{\nu\mu}$, where $g^{\mu\alpha}$ is the covariant derivative of the metric tensor $g_{\mu\nu}$, vanishing because of the metricity condition $g^{\mu\nu} a_{\alpha} = 0$. Thus, Equation (1) with the scalar curvature Equation (2) can be represented in the following additive form:

$$S_{EH} = \frac{1}{2} M_{Pl}^2 \int d^4x \sqrt{-g} \; R + \frac{1}{2} M_{Pl}^2 \int d^4x \sqrt{-g} \; \mathcal{C}. $$ (7)

Below we use the Einstein–Hilbert–action, Equation (7) for the derivation of the Einstein equations in Einstein–Cartan gravitational theory with a chameleon (quintessence) field, spinless matter and torsion as an external (or background) field (Rumpf 1979; Lämmerzahl 1997; Shapiro 2002; Kostelecky 2004; Kostelecky et. al. 2008; Lehner et al. 2014, 2015; Obukhov et al. 2014; Ivanov & Wellenzohn 2015a, 2015b, 2015c, 2016a).

3. EINSTEIN’S EQUATIONS IN EINSTEIN–CARTAN GRAVITY WITH A CHAMELEON FIELD AND SPINLESS MATTER

3.1. Einstein’s Equations and the Torsion Energy–Momentum Tensor

Using Equation (7) we take the action of the Einstein–Cartan gravity with torsion and chameleon fields coupled to spinless matter in the form

$$S_{EH} = \frac{1}{2} M_{Pl}^2 \int d^4x \sqrt{-g} \; R + \frac{1}{2} M_{Pl}^2 \int d^4x \sqrt{-g} \; \mathcal{C}$$

$$+ \int d^4x \sqrt{-g} \; \mathcal{L}[\phi] + \int d^4x \sqrt{-g} \; \mathcal{L}[g],$$

(8)

where $\mathcal{L}[\phi]$ is the Lagrangian of the chameleon field

$$\mathcal{L}[\phi] = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi),$$ (9)

where $V(\phi)$ is the potential of the chameleon self-interaction. Spinless matter is described by the Lagrangian $\mathcal{L}_m[\tilde{g}_{\alpha\beta}]$. The interaction of spinless matter with the chameleon field runs through the metric tensor $\tilde{g}_{\alpha\beta}$ in the Jordan frame (Khouri & Weltman 2004a, 2004b; Mota & Shaw 2007a, 2007b; Dicke 1962), which is conformally related to the Einstein–frame metric tensor $g_{\alpha\beta}$ by $\tilde{g}_{\alpha\beta} = f^2 g_{\alpha\beta}$ (or $\tilde{g}^{\mu\nu} = f^{-2} g^{\mu\nu}$) and $\sqrt{-\tilde{g}} = f^4 \sqrt{-g}$ with $f = e^{\beta\phi/M_\phi}$, where $\beta$ is the chameleon–matter coupling constant (Khouri & Weltman 2004a, 2004b; Mota & Shaw 2007a, 2007b). The factor $f = e^{\beta\phi/M_\phi}$ can be interpreted also as a conformal coupling to matter (Dicke 1962; see also Khouri & Weltman 2004a, 2004b; Mota & Shaw 2007a, 2007b; Ivanov & Wellenzohn 2015a). Varying the action Equation (8) with respect to the metric tensor $\partial_\mu \tilde{g}^{\mu\nu}$ (see, for example, Rebhan 2012) we arrive at the Einstein equations, modified by the contribution of the chameleon field and torsion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - \frac{1}{M_{Pl}^2} T_{\mu\nu},$$ (10)

where the Ricci tensor $R_{\mu\nu}$ and the scalar curvature $R$ are expressed in terms of the Christoffel symbols only $\{^\alpha_{\nu\mu}\}$ and the metric tensor $g_{\mu\nu}$ in the Einstein frame (Rebhan 2012). Then, $T_{\mu\nu}$ is the tensor

$$T_{\mu\nu} = T_{\mu\nu}^{(sp)} + f T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(tors)}.$$ (11)

which can be identified as the energy–momentum tensor of the torsion–chameleon–matter system, where $T_{\mu\nu}^{(sp)}$ and $T_{\mu\nu}^{(m)}$ are the chameleon field and matter (dark and baryon matter) energy–momentum tensors. As has been shown by Ivanov & Wellenzohn (2016b), the matter energy–momentum tensor $T_{\mu\nu}^{(m)}$ appears in the right-hand side (rhs) of the Einstein equations multiplied by the conformal factor $f$. In the CDM model, accepted for the description of spinless matter in our analysis of Einstein–Cartan gravitational theory, the energy–momentum tensor $T_{\mu\nu}^{(m)}$ has only a time–time component $T_{00}^{(m)} = \rho$, where $\rho$ is the spinless matter density in the Einstein frame. In turn, the energy–momentum tensor $T_{\mu\nu}^{(sp)}$ of the scalar
field is defined by

$$T^{(\phi)}_{\mu \nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu \nu}} (\sqrt{-g} \mathcal{L}[\phi])$$

$$= \partial_{\mu} \phi \partial_{\nu} \phi - g_{\mu \nu} \left( \frac{1}{2} g^{\rho \lambda} \partial_{\rho} \phi \partial_{\lambda} \phi - V(\phi) \right).$$ (12)

Then, the tensor $T^{(\text{tors})}_{\mu \nu}$ arises from the contribution of the torsion field and is defined by

$$T^{(\text{tors})}_{\mu \nu} = \frac{M^2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu \nu}} (\sqrt{-g} \mathcal{C}).$$ (13)

We identify this tensor as the torsion energy–momentum tensor and investigate its properties below. Now we would like to rewrite the energy–momentum tensor of the scalar field in terms of the energy momentum tensor of the chameleon field. For this we have to take into account the equation of motion for the chameleon field (Ivanov & Wellenzohn 2015a)

$$\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} \partial^\mu \phi) + \frac{\partial V_{\text{eff}}(\phi)}{\partial \phi} = 0,$$ (14)

where $V_{\text{eff}}(\phi)$ is the effective potential for the chameleon field given by (Khoury & Weltman 2004a, 2004b; Mota & Shaw 2007a, 2007b; Ivanov & Wellenzohn 2016b)

$$V_{\text{eff}}(\phi) = V(\phi) + \rho(f(\phi) - 1),$$ (15)

and to replace in Equation (12) the potential $V(\phi)$ of self-interaction of the scalar field by the effective potential $V(\phi) = V_{\text{eff}}(\phi) - \rho(f(\phi) - 1)$. As a result, the first two terms in the total energy–momentum tensor (Equation (11)) are represented in the following form:

$$T^{(\phi)}_{\mu \nu} + f T^{(m)}_{\mu \nu} = T^{(\text{ch})}_{\mu \nu} + T^{(\text{matter})}_{\mu \nu},$$ (16)

where $T^{(\text{ch})}_{\mu \nu}$ is the energy–momentum tensor of the chameleon field. It is defined by Equation (12) with the replacement $V(\phi) \to V_{\text{eff}}(\phi)$. Then, $T^{(\text{matter})}_{\mu \nu}$ is the modified matter energy–momentum tensor, given by

$$T^{(\text{matter})}_{\mu \nu} = f T^{(m)}_{\mu \nu} - g_{\mu \nu} \rho(f - 1).$$ (17)

Now we may proceed to the analysis of local properties of the Einstein equations, i.e., the Einstein tensor $G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R$, and the total energy–momentum tensor $T_{\mu \nu} = T^{(\text{ch})}_{\mu \nu} + T^{(\text{matter})}_{\mu \nu} + T^{(\text{tors})}_{\mu \nu}$, respectively.

### 3.2. Bianchi Identity and Local Conservation of the Total Energy–Momentum Tensor

The important property of the left-hand side (lhs) of the Einstein equations is that the Einstein tensor $G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R$ obeys the Bianchi identity $G_{\mu \nu ; \rho} = 0$ in a curved spacetime with an arbitrary metric $g_{\mu \nu}$ (Rebhan 2012). This implies that the rhs of the Einstein equations, i.e., the total energy–momentum tensor $T_{\mu \nu}$, should also possess a vanishing covariant divergence, i.e., $T_{\mu \nu ; \rho} = 0$. As we show in Appendix A, the energy–momentum tensor of the chameleon field $T^{(\text{ch})}_{\mu \nu}$ possesses a vanishing covariant divergence $T^{(\text{ch})}_{\mu \nu ; \rho} = 0$ in a curved spacetime with an arbitrary metric $g_{\mu \nu}$. Since torsion is independent of the chameleon field and matter, the torsion energy–momentum tensor $T^{(\text{tors})}_{\mu \nu}$ and the matter energy–momentum tensor $T^{(\text{matter})}_{\mu \nu}$ should fulfill the constraints

$$T^{(\text{tors})}_{\mu \nu} = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} T^{(\text{tors})}_{\mu \nu}) + \{\nu_{\mu \lambda}\} T^{(\text{tors})}_{\lambda \nu} = 0,$$

$$T^{(\text{matter})}_{\mu \nu} = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} T^{(\text{matter})}_{\mu \nu}) + \{\nu_{\mu \lambda}\} T^{(\text{matter})}_{\lambda \nu} = 0$$ (18)

independently of each other. As has been shown by Ivanov & Wellenzohn (2016b), local conservation of the matter energy–momentum tensor leads to the evolution equation for the matter density. Since in the CDM model, which we accept here for the description of matter, the matter energy–momentum tensor $T^{(\text{matter})}_{\mu \nu}$ is equal to

$$T^{(\text{matter})}_{\mu \nu} = f \rho g_{\mu \nu} g^{00} - \rho(\ell - 1) g_{\mu \nu},$$ (19)

the evolution equation for the matter density $\rho$ in a curved spacetime with an arbitrary metric $g_{\mu \nu}$ is

$$\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} f \rho g_{\mu \nu} g^{00})$$

$$+ \{\nu_{\mu \lambda}\} f \rho g_{\mu \nu} g^{00} = g_{\mu \nu} \partial_{\nu}(\rho(f - 1)),$$ (20)

where we have used the metricity condition $g^{\mu \nu} : ; \nu = 0$. Then, Equation (20) can be rewritten in the more convenient form

$$\partial_{\nu}(\rho + (g_{\mu 0} \partial^{0}(f \rho) - \partial_{\nu}(f \rho))) + \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g_{\mu 0} g^{00})$$

$$+ \{\nu_{\mu \lambda}\} g_{\mu 0} g^{00}(f \rho) = 0.$$ (21)

In the Friedmann flat spacetime the evolution equation Equation (21) reduces to the form (Ivanov & Wellenzohn 2016b)

$$\dot{\rho} + 3H \rho = 0,$$ (22)

where $H = \dot{a}/a$ is the Hubble rate. Now we may proceed to the analysis of local conservation of the torsion energy–momentum tensor $T^{(\text{tors})}_{\mu \nu}$.

### 3.3. Local Conservation of the Torsion Energy–Momentum Tensor

Since torsion is an external field, which does not obey any equation of motion or boundary conditions, the requirement of local conservation of the torsion energy–momentum tensor in a curved spacetime with an arbitrary metric tensor can be fulfilled if and only if the torsion energy–momentum tensor is proportional to a metric tensor $T^{(\text{tors})}_{\mu \nu} \sim g_{\mu \nu}$. In this case local conservation of the torsion energy–momentum tensor $T^{(\text{tors})}_{\mu \nu} = 0$ arises from the metricity condition $g_{\mu \nu} : ; \nu = 0$ (Rebhan 2012), which is valid in the Einstein–Cartan gravitational theory under consideration (Hehl et al. 1976). Thus, we may set the torsion energy–momentum tensor equal to

$$T^{(\text{tors})}_{\mu \nu} = \Lambda_C M^2 \delta_{\mu \nu} = -p_{\text{tomp}} \delta_{\mu \nu},$$ (23)

where $\Lambda_C$ is the cosmological constant and $p_{\text{tomp}} = -\Lambda_C M^2$ can be interpreted as torsion pressure. According to the
standard definition of the “matter” energy–momentum tensor (Rebhun 2012), if the torsion energy–momentum tensor is defined by Equation (23), torsion obeys the equation of state \( p_{\text{tors}} = -\rho_{\text{tors}} \), where \( \rho_{\text{tors}} \) is torsion density, in agreement with the properties of dark energy (Peebles & Ratra 2003; Copeland et al. 2006). This gives the following equation for \( C \):

\[
\frac{M^2_{\text{Pl}}}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left( \sqrt{-g} C \right) = \Lambda C M^2_{\text{Pl}} g_{\mu\nu}. \tag{24}
\]

Solving this equation we obtain

\[
C = g^\mu\nu C_{\mu\nu} = g^\mu\nu \left( K^\alpha_\mu \partial_\mu K^\alpha_\nu - K^\alpha_\nu \partial_\mu K^\alpha_\mu \right) = -2 \Lambda C, \tag{25}
\]

where we have used Equation (6). The cosmological constant \( \Lambda C \) is related to the relative dark energy density at the present time as follows: \( \Lambda C = 3H_0^2 \Omega_\Lambda \), where \( H_0 = 1.437(26) \times 10^{-33} \text{eV} \) and \( \Omega_\Lambda \simeq 0.685 \) are the Hubble constant and the relative dark energy density at the present time (Olive et al. 2014).

Equation (25) can be treated as a surface in the 24-dimensional space of torsion tensor field \( T_{\mu\nu\rho} \) components, where the raising and lowering of indices are performed with the metric tensors \( g^\mu\nu \) and \( g_{\mu\nu} \), respectively.

4. DISCUSSION AND CONCLUSIONS

We have analyzed Einstein–Cartan gravitational theory in the standard model-independent form \( R = R + K^2 \), where \( R \) and \( K^2 \) are the contributions of Einstein gravity and torsion, respectively. We have also extended the Einstein–Cartan gravity by the contribution of a chameleon (quintessence) field and spinless matter (dark and baryon matter), described in the CDM model in terms of a matter density \( \rho \) in the Einstein frame. We have added the chameleon field and spinless matter because of their important role in the evolution of the universe (Brax et al. 2004; Ivanov & Wellenzohn 2016b). We have shown that (i) torsion does not couple to spinless matter and (ii) one may interpret the contribution of torsion to the Einstein equations in terms of the torsion energy–momentum tensor as a part of the total energy–momentum tensor \( T^{\mu\nu} = T^{(\text{ch})\mu\nu} + T^{(\text{tors})\mu\nu} \) of the system, including the chameleon field \( T^{(\text{ch})\mu\nu} \) spinless matter \( T^{(\text{br})\mu\nu} \) and torsion \( T^{(\text{tors})\mu\nu} \). The important property of the total energy–momentum tensor is its local conservation, which is equivalent to a vanishing covariant divergence \( T^{\mu\nu}_{\mu\nu} = 0 \) as a consequence of the Bianchi identity \( G^{\mu\nu} = 0 \) for the Einstein tensor \( G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \). Since the Bianchi identity \( G^{\mu\nu} = 0 \) is valid in a curved spacetime with an arbitrary metric tensor \( g_{\mu\nu} \) or an arbitrary gravitational field (Rebhun 2012), the total energy–momentum tensor \( T^{\mu\nu} \) should fulfills the constraint \( T^{\mu\nu}_{\mu\nu} = 0 \) also in a curved spacetime with an arbitrary metric tensor. We show (see Appendix A) that the energy–momentum tensor of the chameleon field fulfills the constraint \( T^{(\text{ch})\mu\nu}_{\mu\nu} = 0 \) identically for an arbitrary metric. Then, the constraint \( G^{(\text{br})\mu\nu}_{\mu\nu} = 0 \) is equivalent to the equation of evolution of matter. In the CDM model and in the Friedmann flat spacetime, such an evolution equation reduces to the evolution equation of a pressureless matter density \( \dot{\rho} + 3H \rho f = 0 \), which has been recently derived and analyzed by Ivanov & Wellenzohn (2016b), where \( H \) is the Hubble rate. As discussed in that paper, the presence of the conformal factor \( f \) in the evolution equation testifies to the important role of the chameleon field in matter evolution in the universe during its expansion. The traces of this influence may be found in the Cosmological Microwave Background (CMB) (Ivanov & Wellenzohn 2016b). The local properties of the energy–momentum tensors of the chameleon field and spinless matter imply that the torsion energy–momentum tensor \( T^{(\text{tors})\mu\nu} \) should also possess a vanishing covariant divergence \( T^{(\text{tors})\mu\nu}_{\mu\nu} = 0 \). Moreover, such a covariant divergence should vanish in a curved spacetime with an arbitrary metric tensor. Since torsion does not obey any equation of motion or boundary conditions, the only one possibility to fulfill the constraint \( T^{(\text{tors})\mu\nu}_{\mu\nu} = 0 \) is to set \( T^{(\text{tors})\mu\nu} \sim g^{\mu\nu} \). In this case the constraint \( T^{(\text{tors})\mu\nu}_{\mu\nu} = 0 \) is fulfilled identically because of the metricity condition \( g^\mu\nu;\lambda = 0 \) (Hehl et al. 1976; Kostelecky 2004; Rebhan 2012). Setting \( T^{\mu\nu}_{\text{tors}} = \Lambda C M^2_{\text{Pl}} g_{\mu\nu} \), leading to Equation (25), one may argue that torsion, serving as an origin of the cosmological constant \( \Lambda C \), may explain the latter’s small value, which is a long-standing problem in cosmology (Weinberg 1989; Peebles & Ratra 2003).

Equation (25) can be interpreted as a surface in the 24-dimensional space of torsion components. It is obvious that the constraint Equation (25) is not very stringent and allows variations of torsion components in sufficiently broad limits. Of course, any measurement of torsion components is possible only through their interactions with spin particles, for example, Dirac fermions (Lämmerzahl 1997; Kostelecky et al. 2008; Obukhov et al. 2014; Ivanov & Wellenzohn 2015b, 2015c, 2016a). As has been shown by Ivanov & Wellenzohn (2016a), in curved spacetimes with rotation one may, in principle, observe all torsion components through low-energy torsion–fermion effective potentials. However, some low-energy torsion–fermion interactions are not defined by torsion–spin–fermion couplings (see Equation (22) of Ivanov & Wellenzohn 2016a). As has been shown by Lehnert et al. (2014), cold neutrons can be a good tool for measurements of torsion–spin–fermion interactions. As also discussed by Ivanov & Wellenzohn (2015c, 2016a), qBounce experiments can provide a precision analysis of all torsion–neutron low-energy interactions at the level of sensitivity \( \Delta E \sim (10^{-12} - 10^{-21}) \text{eV} \) (Abele et al. 2010).

According to Kostelecky (2004), torsion, treated as an external (or background) field, should be responsible for violation of local Lorentz invariance or CPT invariance (Colladay & Kostelecky 1997, 1998; Kostelecky & Pottering 2009). A proportionality of the torsion energy–momentum tensor to a metric tensor, required by local conservation in a curved spacetime with an arbitrary metric tensor, should be of use to avoid a no-go issue with the Bianchi identities discovered by Kostelecky (2004). In effect, fixing torsion to a background value may mean that torsion tensor components should behave like Standard Model extension coefficients for Lorentz violation, so their couplings to any matter or forces are constrained by the various searches for Lorentz violation reported by Kostelecky & Mewes (2016).

An attempt to relate the cosmological constant to torsion has been undertaken by Poplawski (2011, 2013). In Einstein–Cartan gravitational theory with Dirac–quark fields Poplawski has varied the Einstein–Hilbert action with respect to the contorsion tensor and replaced the torsion–Dirac–quark interactions by the four-quark axial-vector–axial-vector interaction, which he has equated with the cosmological constant.
According to Poplawski (2011), the vacuum expectation value of such a four-quark interaction should correspond to the cosmological constant, whereas spacetime fluctuations of the quark fields should describe its spacetime dependence. However, as has been pointed out by Poplawski (2013), the value of the cosmological constant, defined by the quark condensate (Poplawski 2011), is a factor of 8 larger than the observable one (Olive et al. 2014). Thus, in comparison with our result the analysis of the torsion-induced cosmological constant, proposed by Poplawski (2011), seems to be model-dependent, which does not reproduce the observable value of the cosmological constant. One may find references to other dynamical approaches for the description of cosmological constant in the papers by Poplawski (2011, 2013). The discussion of these approaches goes beyond the scope of our paper.

Finally we would like to discuss the results given in Appendix B, where we have analyzed the Poincaré gauge gravitational theory (Kibble 1961; see also Utiyama 1956; Sciama 1961; Sciama 1964; Blagojević 2001; Hehl et al. 1976; Hehl & Obukhov 2007; Hehl 2012; Blagojević & Hehl 2013; Obukhov et al. 2014). We have shown that the integrand of the Einstein–Hilbert action $e \mathcal{R} = e e^{\alpha \beta} e^{\gamma \delta} R^{\alpha \beta \gamma \delta}$ of Poincaré gauge gravitational theory, where $e = \sqrt{-g}$ and $R^{\alpha \beta \gamma \delta}$ is its gravitational field strength tensor defined in terms of the vierbein fields $e^{\alpha a}$ and $e^{\beta b}$ and torsion can be represented in the additive form $e (R + \mathcal{C})$, where $R = e^{\alpha a} e^{\beta b} R^{ab \alpha \beta}$ and $R^{ab \alpha \beta}$ is its gravitational field strength tensor defined only in terms of vierbein fields, and $\mathcal{C} = K^{\alpha \beta \gamma \delta} - K^{\alpha \beta} K^{\gamma \delta}$. This allows us to get the contribution of torsion to the Einstein equations in the form of the torsion energy–momentum. A requirement of local conservation of the torsion energy–momentum imposes its proportionality to a metric tensor in complete agreement with the result obtained in Einstein–Cartan gravitational theory discussed in this paper.

We thank Hartmut Abele for interest in our work. We are grateful to Friedrich Hehl for interesting discussions and critical comments and to Alan Kostecky for fruitful and encouraging discussions. This work was supported by the Austrian “Fonds zur Förderung der Wissenschaftlichen Forschung” (FWF) under the contracts I689-N16, I862-N20 and P26781-N20.

**APPENDIX A**

**ANALYSIS OF LOCAL CONSERVATION OF THE ENERGY–MOMENTUM TENSOR OF THE SCALAR FIELD**

In this appendix we calculate the covariant divergence of the energy–momentum tensor of the chameleon field $T^{(ch)\mu \nu}$, defined by

\[ T^{(ch)\mu \nu} = \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} - g^{\mu \nu} \mathcal{L}_{\text{eff}}[\phi]. \]  

(26)

where we have denoted

\[ \mathcal{L}_{\text{eff}}[\phi] = \frac{1}{2} g^{\alpha \beta} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} + V_{\text{eff}}(\phi). \]  

(27)

The requirement of local conservation of the energy–momentum tensor of the chameleon field demands a vanishing covariant divergence

\[
T^{(ch)\mu \nu}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} T^{(ch)\mu \nu} \right) 
+ \{ \nu \mu \} T^{(ch)\rho \phi} = 0. \tag{28}
\]

Using the equation of motion for the chameleon field

\[
1 \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu \nu} \frac{\partial \phi}{\partial x^\nu} \right) + \frac{\partial V_{\text{eff}}(\phi)}{\partial \phi} = 0 \tag{29}
\]

the calculation of the covariant divergence runs as follows:

\[
T^{(ch)\mu \nu}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} \left( \frac{\partial \phi}{\partial x_\nu} \frac{\partial \phi}{\partial x_\rho} - g^{\mu \rho} \mathcal{L}_{\text{eff}}^{(ch)}[\phi] \right) \right) 
+ \{ \nu \mu \} \frac{\partial \phi}{\partial x_\nu} \frac{\partial \phi}{\partial x_\rho} - g^{\mu \rho} \mathcal{L}_{\text{eff}}^{(ch)}[\phi] \tag{30}
\]

where we have used the relation \( \mathcal{L}_{\text{eff}}^{(ch)}[\phi] \) (Rebhan 2012)

\[
g^{\mu \nu} \{ \rho \mu \} = - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\lambda} (\sqrt{-g} g^{\lambda \lambda}). \tag{31}
\]

Cancelling like terms and using Equation (29) we arrive at the expression

\[
T^{(ch)\mu \nu}_{;\mu} = \frac{\partial \phi}{\partial x_\mu} \frac{\partial^2 \phi}{\partial x_\mu \partial x_\nu} - \frac{\partial}{\partial x_\nu} \left( \frac{1}{2} \frac{\partial \phi}{\partial x_\rho} \frac{\partial \phi}{\partial x_\lambda} \right) 
+ \{ \nu \mu \} \frac{\partial \phi}{\partial x_\nu} \frac{\partial \phi}{\partial x_\rho}. \tag{32}
\]

Because of the relation \( \mathcal{L}_{\text{eff}}^{(ch)}[\phi] \) (Rebhan 2012)

\[
\{ \nu \mu \} \frac{\partial \phi}{\partial x_\nu} \frac{\partial \phi}{\partial x_\rho} = \left( \frac{\partial \phi}{\partial x_\nu} \right)_\rho \phi \frac{\partial \phi}{\partial x_\nu} - \frac{\partial^2 \phi}{\partial x_\nu \partial x_\rho} \frac{\partial \phi}{\partial x_\nu}. \tag{33}
\]
we may transcribe the rhs of Equation (32) into the form

\[ T^{(\text{ch})\mu\nu} = \left( \frac{\partial \phi}{\partial x_\mu} \right)_\nu \left( \frac{\partial \phi}{\partial x_\nu} \right)_\mu - \frac{1}{2} \left( \frac{\partial \phi}{\partial x_\rho} \right)_\nu \left( \frac{\partial \phi}{\partial x_\rho} \right)_\mu \]

\[ = \left( \frac{\partial \phi}{\partial x_\mu} \right)_\nu \left( \frac{\partial \phi}{\partial x_\nu} \right)_\mu - \left( \frac{\partial \phi}{\partial x_\rho} \right)_\nu \left( \frac{\partial \phi}{\partial x_\rho} \right)_\mu \]

\[ = g^{\nu\lambda} \left\{ \left( \frac{\partial \phi}{\partial x_\lambda} \right)_\rho \left( \frac{\partial \phi}{\partial x_\rho} \right)_\mu - \left( \frac{\partial \phi}{\partial x_\mu} \right)_\rho \left( \frac{\partial \phi}{\partial x_\rho} \right)_\nu \right\}, \quad (34) \]

where we have used the relation (Rebhan 2012)

\[ \frac{\partial}{\partial x_\mu} \left( \frac{1}{2} \frac{\partial \phi}{\partial x_\rho} \frac{\partial \phi}{\partial x_\rho} \right) = \left( \frac{\partial \phi}{\partial x_\mu} \right)_\nu \left( \frac{\partial \phi}{\partial x_\nu} \right)_\nu. \quad (35) \]

Since the covariant derivatives \((\partial_\lambda \phi)_\rho\) and \((\partial_\rho \phi)_\lambda\) are equal, i.e., \((\partial_\lambda \phi)_\rho = (\partial_\rho \phi)_\lambda\), we get \(T^{(\text{ch})\mu\nu} = 0\). This confirms local conservation of the energy–momentum tensor of the chameleon field in a curved spacetime with an arbitrary metric tensor.

**APPENDIX B**

**EQUIVALENCE BETWEEN EINSTEIN–CARTAN GRAVITATIONAL THEORY, CONSIDERED IN THIS PAPER, AND POINCARÉ GAUGE GRAVITATIONAL THEORY**

In this appendix we show that the Poincaré gauge gravitational theory field strength tensor \(R_{\mu\nu}^{ab}\), expressed in terms of the spin connection \(\omega_{\mu}^{ab}\) (or local Lorentz connection) (Kibble 1961; see also Kostelecký 2004)

\[ R_{\mu\nu}^{ab} = \partial_\mu \omega_{\nu}^{ab} - \partial_\nu \omega_{\mu}^{ab} + \omega_{\nu}^{a} \epsilon_{\mu}^{a} \epsilon_{\nu}^{b} - \omega_{\mu}^{a} \epsilon_{\nu}^{a} \epsilon_{\nu}^{b} \quad (36) \]

is related to the Riemannian curvature tensor \(R_{\nu\rho\lambda\sigma}^{ab}\) of Einstein–Cartan gravitational theory as

\[ R_{\nu\rho\lambda\sigma}^{ab} = \partial_\nu \Gamma_{\rho\lambda\sigma}^{a} - \partial_\rho \Gamma_{\nu\lambda\sigma}^{a} + \Gamma_{\nu\rho}^{a} \Gamma_{\rho\lambda\sigma}^{c} - \Gamma_{\nu\lambda}^{a} \Gamma_{\rho\nu\sigma}^{c} \quad (37) \]

by the relation \(R_{\mu\nu}^{ab} = e_{a}^{\mu} e_{b}^{\nu} R_{\nu\rho\lambda\sigma}^{ab}\), where \(e_{a}^{\mu}\) and \(e_{b}^{\nu}\) are the vierbein fields. The indices \(a = 0, 1, 2, 3\) are in Minkowski spacetime. The lowering and raising of the indices \(a\) is performed with the Minkowski metric tensors \(\eta_{ab}\) and \(\eta^{ab}\), respectively. In turn, the indices \(\mu = 0, 1, 2, 3\) are in a curved spacetime and the lowering and raising of the indices \(\mu\) are performed with the metric tensors \(g_{\mu\nu}\) and \(g^{\mu\nu}\), respectively.

For the derivation of the relation \(R_{\mu\nu}^{ab} = e_{a}^{\mu} e_{b}^{\nu} R_{\nu\rho\lambda\sigma}^{ab}\) we define the spin affine connection as (Kostelecký 2004; see also Ivanov & Wellenzohn 2015b)

\[ \omega_{\mu}^{ab} = -\partial_\mu \epsilon_{a}^{b} + \Gamma_{\mu\lambda}^{a} \epsilon_{a}^{b}, \quad (38) \]

Plugging Equation (38) into Equation (36) we arrive at the expression

\[ R_{\mu\nu}^{ab} = -\partial_\mu \epsilon_{a}^{b} + \Gamma_{\nu\lambda}^{a} \epsilon_{a}^{b}, \quad \epsilon_{a}^{b} \epsilon_{a}^{b} = \eta^{ab} \quad (39) \]

Using the properties of the vierbein fields (Ivanov & Wellenzohn 2015b) we get \(R_{\mu\nu}^{ab} = e_{a}^{\mu} e_{b}^{\nu} R_{\nu\rho\lambda\sigma}^{ab} + O_{\mu\nu}^{ab}\), where \(O_{\mu\nu}^{ab}\) is defined by

\[ O_{\mu\nu}^{ab} = -\partial_\mu \epsilon_{a}^{c} \epsilon_{\nu}^{b} + \partial_\nu \epsilon_{a}^{c} \epsilon_{\mu}^{b} \]

Thus, we have confirmed the relations between the Riemannian curvature tensor \(R_{\nu\rho\lambda\sigma}^{ab}\) and the Poincaré gauge gravitational field strength tensor \(R_{\mu\nu}^{ab}\), proposed for the first time by Kibble (1961; see also Kostelecký 2004). Equation (10) testifies to the equivalence between Einstein–Cartan gravitational theory with the Riemannian curvature tensor, Equation (37), defined in terms of the affine connection, Equation (3), and Poincaré gauge gravitational theory (Kibble 1961; see also Hehl et al. 1976; Blagojević & Hehl 2013; Obukhov et al. 2014) with the Poincaré gauge gravitational field strength tensor, Equation (36), defined in terms of the spin (or local Lorentz) connection \(\omega_{\mu}^{ab}\) and the vierbein field \(e_{a}^{\mu}\) and \(e_{a}^{\nu}\). Indeed, the Einstein–Hilbert action, Equation (1), can be written as follows (Kostelecký 2004)

\[ S_{\text{EH}} = \frac{1}{2} M_{5}^{-1} \int d^{4}x \sqrt{-g} \ R \]

\[ = \frac{1}{2} M_{5}^{-1} \int d^{4}x \ e^{\nu} \epsilon_{\nu}^{a} \epsilon_{\nu}^{b} R_{\mu\nu}^{ab}, \quad (41) \]

where the Poincaré gauge gravitational field strength tensor \(R_{\mu\nu}^{ab}\) is given by Equation (36) as a functional of the spin connection \(\omega_{\mu}^{ab}\) and the vierbein fields \(e_{a}^{\mu}\) and \(e_{a}^{\nu}\), respectively. Then, \(e\) is the determinant \(e = \det \{e_{a}^{\mu}\}\), i.e., \(\sqrt{-g} = \sqrt{-\det \{g_{\mu\nu}\}} = \sqrt{-\det \{\eta_{ab} e_{\nu}^{a} e_{\mu}^{b}\}} = e\). Now we
may show that the Einstein–Hilbert action, Equation (42), can be represented in an additive form analogous to Equation (7).

To this end we define the spin affine connection $\omega_{\mu}^{ab}$ as follows:

$$\omega_{\mu}^{ab} = E_{\mu}^{ab} + \mathcal{K}_{\mu}^{ab},$$

where $E_{\mu}^{ab}$ and $\mathcal{K}_{\mu}^{ab}$ are given by (Kostelecky 2004)

$$E_{\mu}^{ab} = \frac{1}{2} e^{a\alpha}(\partial_{\mu} e_{\beta}^{\alpha} - \partial_{\beta} e_{\alpha}^{\mu}) - \frac{1}{2} e^{b\beta}(\partial_{\mu} e_{\alpha}^{\beta} - \partial_{\alpha} e_{\beta}^{\mu}) - \frac{1}{2} e^{\alpha\beta} e^{a\beta} e_{\mu}^{\alpha} e_{\kappa}^{\mu} (\partial_{\kappa} e_{\mu}^{\nu} - \partial_{\nu} e_{\mu}^{\kappa}),$$

$$\mathcal{K}_{\mu}^{ab} = \mathcal{K}_{\mu
abla}^{a} e^{\alpha} e^{b\beta}.$$  

(44)

Plugging Equation (44) into Equation (42) we arrive at the Einstein–Hilbert action

$$S_{\text{EH}} = \frac{1}{2} M_{P}^{2} \int d^{4}x \, e \, e^{a\alpha} e_{a\alpha} e_{b\beta} R_{\mu
u}^{ab}$$

$$= \frac{1}{2} M_{P}^{2} \int d^{4}x \, e \, R + \frac{1}{2} M_{P}^{2} \int d^{4}x \, e \, C + \tilde{S}_{\text{EH}}.$$  

(45)

where $R = e^{a\alpha} e_{a\alpha} e_{b\beta} R_{\mu
u}^{ab}$ is the functional of $E_{\mu}^{ab}$. It is defined only in terms of the vierbein fields and corresponds to the contribution of the scalar curvature in the Einstein gravity, whereas $C$ is given by $C = K_{\mu
abla}^{a} K_{\nu
abla}^{a} - K_{\mu
abla}^{a} K_{\nu
abla}^{a}$ and corresponds to the contribution of torsion (see Equation (6)).

Then, the term $\tilde{S}_{\text{EH}}$ is equal to

$$\tilde{S}_{\text{EH}} = \frac{1}{2} M_{P}^{2} \int d^{4}x \, e \, e^{a\alpha} e_{a\alpha} e_{b\beta} (\partial_{\mu} K_{\nu}^{ab} - \partial_{\nu} K_{\mu}^{ab} + \partial_{\mu} e_{a\alpha} e_{b\beta} K_{\nu}^{\alpha\beta} + e_{a\alpha} e_{b\beta} K_{\mu}^{\alpha\beta} - e_{b\beta} e_{a\alpha} e_{a\alpha} e_{b\beta} K_{\nu}^{\alpha\beta}),$$

(46)

Below we show that $\tilde{S}_{\text{EH}} = 0$. The first step is to define the Christoffel symbols in terms of the vierbein fields. We get

$$\{^{\alpha}_{\mu
abla}\} = \frac{1}{2} e^{a\alpha}(\partial_{\mu} e_{\beta}^{a} + \partial_{\beta} e_{\alpha}^{\mu}) + \frac{1}{2} e^{b\beta} e_{a\beta} e_{b\beta}$$

$$\times (e_{a\beta} \partial_{\mu} e_{b\beta} + e_{b\beta} \partial_{\mu} e_{a\beta}) - \frac{1}{2} e^{a\alpha} e^{\alpha\beta} e_{\mu}^{\alpha} e_{\kappa}^{\beta} (\partial_{\kappa} e_{\mu}^{\nu} - \partial_{\nu} e_{\mu}^{\kappa}),$$

(47)

Then, using the definitions for $E_{\mu}^{ab}$ and $\{^{\alpha}_{\mu
abla}\}$, given by Equations (44) and (47), respectively, one may show that the covariant derivative of the vierbein field $e_{a\alpha}^{\nu} : \mu$ and $e_{a\alpha}^{\nu} : \nu$ is defined by (Sciama 1961; Sciama 1964; Kostelecky 2004)

$$e_{a\alpha}^{\nu} : \mu = \partial_{\mu} e_{a\alpha}^{\nu} - \{^{\alpha}_{\mu
abla}\} e_{a\alpha}^{\nu} + E_{a\beta}^{b} e_{b\beta}^{\nu}$$

$$e_{a\alpha}^{\nu} : \nu = \partial_{\nu} e_{a\alpha}^{\nu} + \{^{\nu}_{\mu
abla}\} e_{a\alpha}^{\nu} + E_{a\beta}^{b} e_{b\beta}^{\nu},$$

(48)

equal to zero, i.e., $e_{a\alpha}^{\nu} : \mu = 0$ and $e_{a\alpha}^{\nu} : \nu = 0$. Integrating by parts in Equation (46) we arrive at the expression

$$S_{\text{EH}} = \frac{1}{2} M_{P}^{2} \int d^{4}x \, e \, e^{a\alpha} e_{a\alpha} e_{b\beta} (\partial_{\nu} K_{\mu}^{ab} - \partial_{\mu} K_{\nu}^{ab} + e_{a\alpha} e_{b\beta} K_{\nu}^{\alpha\beta} + e_{a\alpha} e_{b\beta} K_{\mu}^{\alpha\beta} - e_{b\beta} e_{a\alpha} e_{a\alpha} e_{b\beta} K_{\nu}^{\alpha\beta}),$$

(49)

Calculating the first-order derivatives we get

$$\tilde{S}_{\text{EH}} = \frac{1}{2} M_{P}^{2} \int d^{4}x \, (e^{a\alpha} e_{a\alpha} e_{b\beta} + \partial_{\nu} e_{a\alpha} e_{b\beta} + e_{a\alpha} e_{b\beta} K_{\nu}^{\alpha\beta} + e_{a\alpha} e_{b\beta} K_{\mu}^{\alpha\beta} - e_{b\beta} e_{a\alpha} e_{a\alpha} e_{b\beta} K_{\nu}^{\alpha\beta}).$$

(50)

where we may combine some terms into the covariant divergences of the vierbein fields

$$\tilde{S}_{\text{EH}} = \frac{1}{2} M_{P}^{2} \int d^{4}x \, (e^{a\alpha} e_{a\alpha} e_{b\beta} + \partial_{\nu} e_{a\alpha} e_{b\beta} + e_{a\alpha} e_{b\beta} K_{\nu}^{\alpha\beta} + e_{a\alpha} e_{b\beta} K_{\mu}^{\alpha\beta}).$$

(51)

Since $e^{\nu} : b_{\nu} = e^{\nu} : a_{\mu}$, we get

$$\tilde{S}_{\text{EH}} = \frac{1}{2} M_{P}^{2} \int d^{4}x \, (e^{a\alpha} e_{a\alpha} e_{b\beta} + \partial_{\nu} e_{a\alpha} e_{b\beta} + e_{a\alpha} e_{b\beta} K_{\nu}^{\alpha\beta} + e_{a\alpha} e_{b\beta} K_{\mu}^{\alpha\beta}).$$

(52)

We rewrite the integrand of Equation (52) as follows:

$$S_{\text{EH}} = \frac{1}{2} M_{P}^{2} \int d^{4}x \, (e^{a\alpha} e_{a\alpha} e_{b\beta} + \partial_{\nu} e_{a\alpha} e_{b\beta} + e_{a\alpha} e_{b\beta} K_{\nu}^{\alpha\beta} + e_{a\alpha} e_{b\beta} K_{\mu}^{\alpha\beta} - e_{b\beta} e_{a\alpha} e_{a\alpha} e_{b\beta} K_{\nu}^{\alpha\beta})$$

(53)

Thus, we have shown that $\tilde{S}_{\text{EH}} \equiv 0$. This means that the Einstein–Hilbert action, Equation (42), can be written in the additive form

$$S_{\text{EH}} = \frac{1}{2} M_{P}^{2} \int d^{4}x \, e \, e^{a\alpha} e_{a\alpha} e_{b\beta} (\partial_{\nu} R_{\mu\nu}^{ab} + \partial_{\mu} R_{\nu\mu}^{ab} + e_{a\alpha} e_{b\beta} R_{\nu\mu}^{ab} + e_{a\alpha} e_{b\beta} R_{\mu\nu}^{ab} - e_{b\beta} e_{a\alpha} e_{a\alpha} e_{b\beta} R_{\nu\mu}^{ab}),$$

(54)

where $R = e^{a\alpha} e_{a\alpha} e_{b\beta} R_{\mu\nu}^{ab}$ is defined only in terms of the vierbein fields and corresponds to the contribution of the scalar curvature in Einstein gravity, whereas $C$ is given by $C = K_{\mu
abla}^{a} K_{\nu
abla}^{a} - K_{\mu
abla}^{a} K_{\nu
abla}^{a}$ and corresponds to the contribution of torsion (see Equation (6)). For the derivation of Equation (54) we have used the definition of the covariant derivatives of the vierbein fields, Equation (48), and the properties of the contorsion tensor $K_{\mu
abla}^{a} = -K_{\mu\nu}^{b\alpha} K_{\alpha
abla}^{c}$ (Kostelecky 2004).

The obtained result, Equation (54), confirms a complete equivalence between Einstein–Cartan gravitational theory, analyzed in this paper, and Poincaré gauge gravitational theory by Kibble (1961) (see also Utiyama 1956; Sciama 1961; Sciama 1964; Blagojević 2001; Hehl et al. 1976; Hehl & Obukhov 2007; Hehl 2012; Blagojević & Hehl 2013; Obukhov et al. 2014). This also confirms the identification of the torsion contribution to the Einstein equations with the torsion energy–
momentum tensor, Equation (23), local conservation of which can be attained only through Equation (24), allowing us to set \( C = -2 \Lambda C \) (see Equation (25)).

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