Introduction. Superconducting states with spontaneously broken time-reversal symmetry (BTRS) have been recently in the focus of interest. First such states have been studied in connection with the chiral $p$-wave order parameter in the superfluid $^3$He A phase\cite{1} and Sr$_2$RuO$_4$ superconducting compound\cite{2}. More recently, $s+id$ and $s+is$ states have been suggested as the candidate order parameters in multiband iron pnictide compounds\cite{3–9}. Recent experiment\cite{10} supports this hypothesis demonstrating the presence of spontaneous currents in the ion irradiated samples of Ba$_{1-x}$K$_x$Fe$_2$As$_2$ in the certain doping level interval.

Spontaneous currents were predicted to exist near impurities in $s+id$ superconducting states which spontaneously break the $C_4$ crystalline symmetry of the parent compound\cite{3}. As for the $s+is$ states, initially it has been claimed that magnetic field can appear only in samples subjected to strain\cite{11}. However, this conclusion was made based on the specific circularly-symmetric model of the impurity.

More general consideration has shown\cite{12,13} that magnetic fields in the $s+is$ state can be generated without strain in the presence of the general-form inhomogeneities of the order parameter. They can be induced e.g. by the domain wall between $s+is$ and $s-is$ states\cite{14}, attached to the sample edge or by any external controllable perturbation such as the local heating. Later the particular case of two-dimensional defects elongated along the crystal $c$-axis and forming square shapes in the ab plane have been studied\cite{15}. In such system the spontaneous magnetic field generated in $s+is$ state is several order of magnitude smaller than in the $s+id$ one. The purpose of the present paper is threefold. First, we show that multiband superconductors with broken time-reversal symmetry can produce spontaneous currents and magnetic fields in response to the local variations of pairing constants. Considering the iron pnictide superconductor Ba$_{1-x}$K$_x$Fe$_2$As$_2$ as an example we demonstrate that both the point-group symmetric $s+is$ state and the $C_4$-symmetry breaking $s+id$ states produce in general the same magnitudes of spontaneous magnetic fields. In the $s+is$ state these fields are polarized mainly in ab crystal plane, while in the $s+id$ state their ab-plane and c-axis components are of the same order. The same is true for the random magnetic fields which are produced by the and the order parameter fluctuations near the critical point of the time-reversal symmetry breaking phase transition. Our findings can be used as a direct test of the $s+is/s+id$ dichotomy and the additional discrete symmetry breaking phase transitions deep in the superconducting state.

Figure 1. (Color online) The schematic picture of spontaneous magnetic fields generated in the $s+is$/$s+id$ superconductors by the cylindrical inhomogeneities with the axis directed either perpendicular (ca-defect) or along the crystal anisotropy axis z(c) (ab-defect). In the general case when both types of defects are present the $s+is$ state has dominant component of $B$ in ab-plane, while in the $s+id$ all components are of the same amplitude.
s + is states. This qualitative prediction can be used for resolving the s + id/s + is dichotomy in real materials. Third, we demonstrate that the order parameter fluctuations near the BTRS phase transition generate random magnetic fields with the critical correlation radius. Thus, the discrete symmetry breaking phase transition can be revealed through the magnetic field fluctuations.

**Three-band model.** Here we develop general treatment of spontaneous magnetic fields in BTRS states further considering inhomogeneities created by the spatial variation of pairing constants in the minimal three-band microscopic model [6, 16, 17] with three distinct superconducting gaps $\Delta_{1,2,3}$ residing in different bands. The pairing which leads to the BTRS state is dominated near the BTRS phase transition generate random states. This qualitative prediction can be used for both the s + id states when all three spatial components of these tensors are different.

The magnetic field can be generated even in the particular case of s + is state with 2D spatial inhomogeneities in the ab-plane. However, this mechanism relies on the non-linear coupling between the current and the gradients if pairing coefficients. Indeed in the linear response regime we can assume that $\gamma_{ki} = \gamma_k i$, where $\gamma_k$ is the unit matrix and $\gamma_{ki}$ = const when the second term in Eq.(2) vanishes. The non-linear coupling is realized when the relative density gradients $\nabla \gamma_{ki}(r)$ are non-collinear with that of relative-phase gradients $\nabla \theta_{ki}$. Below we show that such a coupling generically appear in s + is states described by the model (1) with the spatially-dependent pairing coefficient $\eta_2 = \eta_2(r)$, although the magnitude of the spontaneous magnetic field is significantly smaller than in the s + id state.

**Ginzburg-Landau calculation.** To go beyond local approximation we can calculate spontaneous magnetic fields using Ginzburg-Landau theory derived for s + id states [13, 18] corresponding to the model (1). The general free energy density, normalized to $B_0^2$ is given by $F = F_0 + B^2/8\pi$, where the Ginzburg-Landau (GL) free energy describing both the s + is and s + id states is given by

$$F_s = \frac{1}{2} \sum_{j=1}^{2} \left( \mathbf{\Phi} \psi_j^* \mathbf{\Phi} \psi_j + \alpha_j |\psi_j|^2 + \beta_j^2 |\psi_j|^4 \right) + 2(\mathbf{\Phi} \psi_1)^* \mathbf{\Phi} \psi_2 + |\psi_1|^2 |\psi_2|^2 + \delta |\psi_1|^4 |\psi_2|^4 + c.c.,$$

where $\mathbf{\Phi} = \nabla - i\hbar \mathbf{A}$. This model is formulated in terms of critical field at zero temperature[19]. Length is normalized by the Cooper pair size $\bar{\xi}_0 = \hbar v_F/\epsilon_c$, and we introduce the dimensionless Cooper pair charge is $e_0 = 2e\bar{\xi}_0^2 B_0$. The London penetration depth is given by $\lambda_L^2 = \sum \lambda_i^2$ and $\gamma_{ki} = \lambda_i^2 (\Delta^2 - \lambda_i^2)$ is the tensor coefficient characterising magnetic field generated by the variations of interband phase differences.
of the two order parameters $\psi_1$ and $\psi_2$ which are related to the individual gap functions within separate bands as $(\Delta_1, \Delta_2, \Delta_3) = (a\psi_1 - \psi_1, a\psi_2 + \psi_1, \psi_2)$, where $a = (\eta_1 - \sqrt{\eta_1^2 + 8\eta_2^2})/4\eta_2$.

The coefficients that determine gradient terms in Eq.(4) are combined from the anisotropy tensors characterising each superconducting band as follows

$$k_{11} = \rho(\hat{K}_1 + \hat{K}_2)$$

$$k_{22} = \rho[a^2(\hat{K}_1 + \hat{K}_2) + \hat{K}_3]$$

$$k_{12} = a\rho(\hat{K}_2 - \hat{K}_1)$$

The difference between $s+is$ and $s+id$ symmetries is determined by the structure of mixed-gradient coefficients (7) in ab plane. That is for $s+is$ state $k_{12,x} = k_{12,y} \equiv k_{12}^{eb}$ and for $s+id$ state $k_{12,x} = -k_{12,y} \equiv k_{12}^{ab}$. Despite having quite different properties in the ab plane both states are characterized by the same anisotropy along the c axis determined by the coefficients $K^c_\alpha \equiv K^c_\alpha \neq K^c_\beta$. The 122 iron pnictide compounds has been shown to feature moderate anisotropy [20] therefore we will use for estimations that $K^c_\alpha/K^c_\beta \approx 1 - 3$. In case of the $s+is$ superconductor it is of the crucial importance for the linear coupling between the magnetic field and pairing constant inhomogeneities that the at least two bands should have different anisotropies, e.g. $K^c_1/K^c_2 \neq K^c_3/K^c_2$. Otherwise, by using the scale transformation the problem is reduced to the case of the fully isotropic $s+is$ superconductor when the non-linear coupling is possible yielding much smaller spontaneous currents.

$$\eta_2(x, y) = 1 + 0.5 \sin(x/2) \sin(y/2).$$

This model allows for demonstrating differences between the linear and non-linear mechanisms of the spontaneous current generation, where the former takes place for $s+id$ and the latter is $s+is$ pairings. The magnetic field produced by xy-inhomogeneities has only x-component (Fig.1). The calculated distribution of $B_x = B_x(x, y)$ is shown in Fig.2. The vector potential in the numerical calculations is normalized by the value of $\Phi_0/(2\pi \xi_0)$ where $\Phi_0$ is the superconducting flux quantum, so the magnetic field is given in the units of $\Phi_0 B_0$ which has an order of the second critical field $H_{c2}$. One can see that for one and the same set of parameters $s+is$ state yields the spontaneous magnetic field response about $10^3$ times smaller than $s+id$, which is consistent with the results obtained before [15].

 Except of the special case of ab-plane inhomogeneities, in general the $s+is$ and $s+id$ states produce the magnetic fields of comparable amplitudes. To demonstrate this we compare responses produced by the Gaussian pairing constant variation given by

$$\eta_2 = 1 + 0.5e^{-(x^2+y^2)/2} \quad s+id \, \text{state}$$

$$\eta_2 = 1 + 0.5e^{-(x^2+z^2)/2} \quad s+is \, \text{state}$$

The former inhomogeneity (13) corresponds to the ab-plane defect, while the latter (14) is the ca-plane defect. To obtain spontaneous fields produced by ca-plane Gaussian defects in $s+is$ case we assume that there is c-axis anisotropy set by the choice of coefficient ratio in different bands $K^c_1 = 1, K^c_2 = 1.5, K^c_3 = 0.5, K^c_4 = 1.5, K^c_5 = 2, K^c_6 = 0.75$. Such system yields the magnetic field component $B_y(x, z)$ shown in Fig.3c. One can compare it with the qualitatively similar distribution of $B_z$ component produced by the Gaussian ab-plane inhomogeneity (13) in the $s+id$ state shown in Fig.3f. These two characteristic cases demonstrate that spontaneous magnetic responses of $s+is$ and $s+id$ in general are the
same. However, one can conclude that the largest spontaneous magnetic field in the $s + is$ case appears in the direction perpendicular to the anisotropy axis. Therefore one can distinguish between these states by analysing the polarization of spontaneous magnetic fields.

Based on the above analysis one can suggest the polarization-sensitive test of the superconducting state symmetry based. That is, under general conditions, the spontaneous magnetic field in $s + is$ state is directed mostly in the ab-plane, with the typical ratio of components $B_z/B_{ab} \sim 10^{-3}$. On the other hand, $s + id$ state produces spontaneous fields which have in general all components with the same order $B_z/B_{ab} \sim 1$.

**Critical magnetic fluctuations.** The spontaneous magnetic field produced by the order parameter inhomogeneities allows for the direct observation of the critical phenomena and fluctuations near the phase transition to the broken time-reversal symmetry state. To demonstrate that we introduce the order parameter $\eta = -i\langle \psi^2 \rangle /|\psi|$. Further we assume that $\psi_1 = |\psi_1| e^{i\theta_1}$ and $|\psi_1| = \text{const}$. Introducing the gauge-invariant momentum $Q = A - \nabla \psi_1/\tilde{e}$, the real and imaginary parts of the complex order parameter $\eta = \eta_r + i \eta_im$ allows for representing the GL free-energy as follows

$$F(\eta, Q) = \tilde{\alpha}_r \eta_r^2 + \tilde{\alpha}_{im} \eta_{im}^2 + \frac{\beta_2}{2} |\eta|^4 + \frac{\nabla \times Q^2}{8\pi} + |\psi_1|^2 \tilde{e}^2 Q k_{11} Q + 2 |\psi_1| \tilde{e} Q k_{12} (\nabla \eta_r + \tilde{e} Q \eta_{im}) + (\nabla + i \tilde{e} Q) \eta_2 \tilde{e}^2 Q \eta_{22} (\nabla - i \tilde{e} Q) \eta_r.$$  (15)

Here $\tilde{\alpha}_r = \alpha_r + |\psi|^2(\delta - \gamma)$ and $\tilde{\alpha}_{im} = \alpha_2 + |\psi_1|^2(\delta + \gamma)$. Equation $\tilde{\alpha}_r(T) = 0$ gives the critical temperature of $Z_2$ transition $T = T_{Z2}$. In the vicinity of this transition only the fluctuations of $\eta_r$ are important as the $\tilde{\alpha}_{im}$ is positive and non-vanishing. Therefore we can describe the time-reversal symmetry breaking phase transition in terms of the real-valued order parameter $\eta_r$:

$$F(\eta_r, Q) = \tilde{\alpha}_r \eta_r^2 + \frac{\beta_2}{2} \eta_r^4 + \nabla \eta_r k_{22} \nabla \eta_r + 2 \tilde{e} |\psi_1| Q k_{12} \nabla \eta_r.$$  (16)

Note that the real order parameter $\eta_r$ is still coupled to the magnetic field, which can be demonstrated as follows. The superconducting current obtained from the functional (16) is given by $j = -2 |\psi_1|^2 \tilde{e}^2 k_{11} Q - 2 \tilde{e} |\psi_1| k_{12} \nabla \eta_r$, up to the lowest non-vanishing order of $\eta_r$. For simplicity let us assume that the coefficients $k_{ii}$ for $i = 1, 2$ are isotropic and the anisotropy is determined by

![Figure 3. The order parameter modulation (a,b,d,e) and spontaneous fields (c,f) produced by the Gaussian inhomogeneities of the pairing constant $\eta_2(r)$ (13,14) and $\eta_1 = 1$.

The upper row corresponds to the ca-plane defect in $s + is$ superconductor with anisotropy parameters for $s + is$ are $K_{11}^a = 1$, $K_{22}^a = 1.5$, $K_{33}^a = 0.5$, $K_1^z = 1.5$, $K_2^z = 2$, $K_3^z = 0.75$. The lower row corresponds to ab-plane defects in $s + id$ state characterized by $K_{11}^{ab} = 1$, $K_{22}^{ab} = 1.5$, $K_{33}^{ab} = 0.5$. GL parameter $\tilde{e} = 4$ for both cases.](image)
\[ \dot{k}_{12} \]. Then, going to the Fourier transform \( \eta_\alpha(r) = V \int e^{iqr} \eta_\alpha(q) d^3q / (2\pi)^3 \), where \( V \) is the system volume we obtain the magnetic field \( B(q) = \eta_\alpha(q) \sqrt{8\pi}/|k_{12}|(q \times \dot{k}_{12})/[\lambda(q^2 + \lambda^2)] \), where \( \lambda = 1/(\sqrt{8\pi}k_{11}|\psi_1|\dot{\epsilon}) \) is the London penetration length.

The above derivation demonstrates that the spatial inhomogeneities of \( \eta_\alpha \), with necessity produce the spontaneous magnetic field. That means fluctuations near the \( \mathbb{Z}_2 \) critical point become magnetic. The variance of magnetic field components, e.g. the one lying in ab-plane \( B_y \) is given by

\[
\frac{k_{11}}{8\pi} \langle B_y^2(q) \rangle = (\eta_\alpha^2(q))(k_{122} - k_{122})^2 \frac{\lambda^2 q^2 q^2}{(\lambda^2 q^2 + 1)^2} \quad (17)
\]

For simplicity we consider the limiting case when the cross-coupling gradient terms in the functional (16) are rather small \( k_{12} \ll k_{11}k_{22} \) when the feedback of magnetic field fluctuations can be neglected. Then, fluctuations of the order parameter \( \eta_\alpha \) near the critical temperature \( T_{Z2} \) can be calculated using the conventional expression \([21]\) \( \langle \eta_\alpha^2(q) \rangle = T / [2V(k_{222} + |\alpha_r|)] \) .

Now, we can calculate the correlation function of the spontaneous magnetic field component \( \langle B_y(0)B_y(r) \rangle = V \int d^3q \langle B_y^2(q) \rangle e^{-iqr} \).

Considering the long-range part of the fluctuations we obtain \( \langle B_y(0)B_y(r) \rangle = \frac{2\pi k_{12}^2}{4k_{11}k_{22}}(k_{122} - k_{122})^2 \partial^2_{xxzz}G(r) \), where \( G(r) = (T / 8\pi k_{22}) e^{-r/\eta_c} / r \) is the order parameter correlation function with the correlation radius \( r_c = \sqrt{k_{22}|\alpha_r|} \). Thus one can see that fluctuation of magnetic field has the same critical radius as the time-reversal symmetry breaking order parameter. These spontaneous fields provide therefore the direct access to the previously hidden critical behaviour near the discrete symmetry-breaking phase transitions.

**Conclusion.** To summarize, we have shown that in the \( s+id \) and \( s+is \) phases in multiband superconductors can produce spontaneous currents and magnetic fields in response to the spatial inhomogeneities caused by either the fluctuations of the pairing constants or the critical fluctuations of the order parameter components. This is in contrast to the previous predictions that \( s+is \) state has much weaker magnetic signatures. However, the spontaneous field polarization is found to be drastically different in \( s+is \) and \( s+id \) states making it possible to distinguish between them experimentally. The random magnetic fields produced by the scalar order parameter fluctuations can reveal the critical behaviour near the BTRS transition and in general any additional discrete-symmetry phase transition deep in the superconducting state.

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