Compositional Sparsity, Approximation Classes, and Parametric Transport Equations*  

Dedicated to Ronald DeVore on the occasion of his 80th birthday

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Abstract

Approximating functions of a large number of variables poses particular challenges often subsumed under the term “Curse of Dimensionality” (CoD). Unless the approximated function exhibits a very high level of smoothness the CoD can be avoided only by exploiting some typically hidden structural sparsity. In this paper we propose a general framework for new model classes of functions in high dimensions. They are based on suitable notions of compositional dimension-sparsity quantifying, on a continuous level, approximability by compositions with certain structural properties. In particular, this describes scenarios where deep neural networks can avoid the CoD. The relevance of these concepts is demonstrated for solution manifolds of parametric transport equations. For such PDEs parameter-to-solution maps do not enjoy the type of high order regularity that helps to avoid the CoD by more conventional methods in other model scenarios. Compositional sparsity is shown to serve as the key mechanism for proving that sparsity of problem data is inherited in a quantifiable way by the solution manifold. In particular, one obtains convergence rates for deep neural network realizations showing that the CoD is indeed avoided.

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1 What this is about

Background: Like hardly any other topic have Deep Neural Networks (DNNs) been recently influencing numerous vibrant research activities constantly broadening the scope of

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applications of machine learning. In a nutshell, DNNs are compositions of simple finitely parametrized mappings, each one being in turn a composition of an affine map and a componentwise acting nonlinear “activation” function. Thus, DNNs are highly nonlinear mappings differing in many respects from classical approximation systems, see \[9, 11\] for DNNs in the general context of nonlinear approximation.

The empirically observed stunning success of deep networks in unsupervised and supervised learning, in particular, in “Big Data” application scenarios, has raised high expectations of a similar impact in other, more science related areas which are, however, typically less error-tolerant.

A challenge common to either regime is the need to recover or approximate functions of a large number of variables. In the latter science oriented sector examples are partial differential equations (PDEs) in high-dimensional phase space like Schrödinger equations or Fokker-Planck equations, describing the evolution of probability distributions. Similarly, the formulation of “background models” in terms of parameter dependent families of PDEs is ubiquitous in Uncertainty Quantification, especially, in connection with inverse tasks like state- or parameter-estimation. Here, the states of interest are functions of time, space, and parametric variables. They form what is often called the “solution manifold” comprised of all parameter dependent solutions obtained when traversing the parameter domain. We will interpret this notion in a broader sense including the dependence of solutions on “problem data” such as coefficients, initial or boundary conditions, and right hand sides, ranging over suitable compact sets. Thus, such data either have already a parametric form or can be approximated within any given tolerance by expressions depending on a finite number of parameters. In either case, the states in solution manifolds can be viewed as functions of spatio-temporal as well as (possibly infinitely) many parametric variables. Being able to efficiently explore such solution manifolds, i.e., to approximate the parameter-to-solution map is at the heart of numerous applications.

Generally, the so-called \textit{Curse of Dimensionality} (CoD) describes an obstruction, encountered in such high-dimensional approximation tasks (meaning approximation of functions of many variables). It roughly expresses an exponential dependence of computational complexity on the number of variables. A startling early result by Novak and Wojniakowsky \[28\] states, for instance, that the “information complexity” suffers from the CoD even when the model class consists of all infinitely differentiable functions on \([0, 1]^d\) with all derivatives bounded by one and approximation cost is measured by the number of functional evaluations used to construct a reconstruction algorithm. This means the classical paradigm of characterizing approximability of functions of a few variables by smoothness is no longer relevant in high-dimensional approximation. It also hints at the fact that avoiding the CoD does not depend on a particular method alone but refers to the interplay or combination of a particular \textit{model class} of approximands and a particular \textit{approximation/information system} applied to this class.

A general perception is that an increasing level of nonlinerarity of an approximation method increases the chance to avoid or mitigate the CoD hopefully for a larger scope
of model classes. A by now famous result concerns the greedy construction of shallow neural networks with a single hidden layer [1]. It is known to realize dimension independent approximation rates when the model class is a Barron class, see e.g. [3, 34]. Note though that this model class becomes “smaller” with increasing dimension as smoothness needs to be proportional to the spatial dimension. In view of their celebrated universality and even higher level of nonlinearity, deep neural networks (DNNs) are therefore often perceived as providing the “silver bullet” for high-dimensional approximation. This is one motivation behind intense research over the past years on the expressive power of DNNs which can roughly be grouped as follows: (i) “emulation” strategies, (ii) super-convergence results, (iii) approximating parameter/data-to-solution maps.

(i) is based on proving that classical approximation tools have efficient representations in terms of DNNs. As a consequence the approximation power of nearly all known “classical” methods are (perhaps up to some log-factors) matched by deep neural networks (see e.g. [9, 23, 56, 33, 11, 32, 18, 15, 20, 29] and the general exposition [11] in the context of nonlinear approximation). This applies to classical generic model classes, defined through smoothness properties as well as to known ways of capturing solution manifolds. Regarding the CoD these results show success only when the emulated schemes do too.

What goes beyond? It is remarkable that DNNs can well approximate very rough (fractal-like) functions as well as very smooth ones like holomorphic mappings, [36, 29]. Even more stunning are the results under (ii) establishing strictly better DNN approximation rates for certain smoothness classes than those obtained by classical methods, see e.g. [36, 26]. But even these results come with a grain of salt. The finding in [6] on so-called stable widths show that algebraic convergence rates of any (nonlinear) method that is stable in a certain sense, applied to a given compact class, is lower bounded by the rates of entropy numbers for that class. The entropy numbers of classical smoothness classes, in turn, are known to suffer from the CoD. This questions whether the super-convergence rates (which wouldn’t break the CoD for smoothness classes either) can be stably realized.

Nevertheless, the prospect of having a single approximation system that can address essentially any approximation problem with near-optimal expressive power, otherwise realized only by specialized systems, may strike one as a significant advantage. However, classical methods come typically with well-understood stable algorithmic realizations. By contrast, training DNNs is haunted by a significant remaining uncertainty of optimization success let alone a satisfactory stability assessment. In fact, it is shown in [19] that (for ReLU networks) theoretically possible high approximation orders cannot be realized based only on point samples as training input. So, one may wonder:

for which model classes of interest can DNNs avoid the Curse of Dimensionality while state of the art methods don’t?

in which case one may see good reasons beyond “black box comfort” to put up with the disadvantages of DNN approximation. This turns the spotlight to less generic model classes like those under (iii). In fact, interesting rigorous results have been obtained for DNN approximations of solutions of certain spatially high-dimensional partial differential equations.
(PDEs) like Kolmogorov’s of Hamilton-Jacobi-Bellman equations where the use of DNNs is shown to break the Curse of Dimensionality, see e.g. in [4, 17]. Especially [10] quantifies DNN approximability of solutions when the problem data are given as DNNs which is close to the point of view taken here as well.

Comparable results about DNN approximation of solution manifolds in the context of parameter dependent PDEs seem to be so confined to emulation, see [23]. That is, there exist alternate methods performing at least as well. These include low-rank approximation, reduced basis methods, or sparse polynomial approximations. [2, 10, 5]. These methods exploit the fact that the Kolmogorov $n$-widths of the solution manifolds decay robustly with respect to the parametric dimension. This, in turn, hinges on the holomorphy of the underlying parameter-to-solution maps warranting an efficient approximability from linear spaces. This is a typical asset of elliptic models and their close relatives, see [10] and references therein.

Examples of PDEs, so to speak, at the extreme other end of the spectrum are transport equations where no dissipative effects come to aid. Unless imposing rather restrictive assumptions on domain smoothness or initial conditions subject to compatibility constraints, for affine parameter representations (see [21]), parameter-to-solution maps are “generically” no longer holomorphic and have low regularity. In fact, the approximation of solutions to parameter dependent transport equations by (DNNs) has been analyzed already in the interesting work [30]. It is important to note though that the convergence rates established there still reflect the full Curse of Dimensionality. Roughly, achieving accuracy $\varepsilon$ requires a network complexity of the order $\varepsilon^{-\frac{m+1+dy}{\alpha}}$ where $m+1, d_y$ denote the number of space-time and parametric variables and $\alpha$ represents the smoothness of the solutions resulting from suitable smoothness assumptions on the problem data (initial conditions, right hand sides, convection coefficients). In the light of the preceding discussion, this is, Under such generic assumptions, best possible.

**Objectives:** The angle taken in this paper, regarding solution manifolds of parametric transport equations, is therefore quite different. It is more in the spirit of classical regularity theory for PDEs where smoothness of solutions is deduced from smoothness of problem data. Only, as indicated earlier, in high-dimensions, smoothness needs to be replaced by a new type of “regularity” that manifests itself by some suitable notion of “structural sparsity”. The objective of this paper is therefore twofold: (1) establish such “heredity results” - sparsity of problem data implies sparsity of solutions - for parametric transport equations forming an extreme adversary to existing model reduction concepts; (2) develop corresponding structural sparsity notions, whose relevance is argued in the end to go beyond the particular case study of transport equations.

**Layout:** The relevance of *compositions* for neural approximation is by no means new and rather apparent, see e.g. [13, 26, 27, 33]. Although somewhat inspired by the work in [33] in the context of nonlinear regression and statistical estimation, the present approach is quite different. It is more in line with [7] where it is shown that a certain notion of tensor-sparsity of data is inherited the solutions to high-dimensional diffusion equations. It is important
to note that this sparsity notion does not require the data to have a fixed finite rank or can be described by a finite number of parameters. This is also the case here where the central objects are compositional approximation classes. These are based on what we call tamed dimension-sparse compositional representations, see §2 especially §2.2 §2.3. The notion of compositional approximability is then formulated first on a continuous level in terms of intrinsic structural properties of functions. The principal mechanism of how to obtain from this in the end finitely parametrized DNN approximations is shown by Theorem 2.15 (primarily for the purpose of conceptual orientation).

With these preparatory sections in place we formulate in §4 our results concerning the DNN approximability of solution manifolds for transport equations. In particular, we show that dimension-sparse compositional approximability of the data (convection field, initial conditions, right hand side) is inherited by the parameter dependent solutions. Under such circumstances the parameter-to-solution map admits DNN approximations that do not suffer from the Curse of Dimensionality. Since solutions result from compositions with initial data and right hand side a key is to establish first efficient approximability of characteristics. For instance, when the parameter dependence of the convection field is affine, the characteristics, as functions of \(m\) spatial and \(d_y\) parametric variables, can be approximated by DNNs as follows

\[
\|z - \mathcal{N}_\varepsilon\|_{L_\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^m)} \leq \varepsilon, \quad \#\mathcal{N}_\varepsilon \lesssim d_y \left(\frac{e^{LT}}{\varepsilon}\right)^{m+1} \left|\log_2 \frac{e^{LT}}{\varepsilon}\right|^2.
\]

Our results yield, in particular, upper bounds for nonlinear manifold widths of corresponding solution manifolds. All proofs are given in §6. They are at times technical and tedious which is the price for keeping track of how problem parameters effect complexity.

In §5 we close with indicating several directions of future research suggested by the present findings and their bearing on a wider problem scope.

Notational Conventions: In what follows we often write \(a \lesssim b\) to indicate that \(a\) is bounded by a constant multiple of \(b\) where the constant is independent of any parameters \(a\) and \(b\) may depend on, unless specified otherwise. Accordingly \(a \asymp b\) means \(a \lesssim b\) and \(b \lesssim a\).

For notational brevity and convenience we will use, for any pair of finite dimensional metric spaces \(X, Y\) and any continuous function \(g : X \subset \mathbb{R}^{d_0} \to Y \subset \mathbb{R}^{d_1}\), the shorthand notation

\[
\|g\|_X = \|g\|_{L_\infty(X; Y)} = \sup_{x \in X} \sup_{i=1,\ldots,d_1} |g_i(x)| = \sup_{x \in X} |g(x)|_\infty,
\]

when the particular domains and ranges don’t matter. Likewise we use the domain- and dimension independent notation \(\|g\|_{\text{Lip}_p}, |\cdot|_{\text{Lip}_p}\) to denote the full Lipschitz norm, respectively semi-norm

\[
|g|_{\text{Lip}_p} := \sup_{x,z \in X} \frac{|g(x) - g(z)|}{|x - z|_\infty}, \quad \|g\|_{\text{Lip}_1} := \max\{\|g\|_\infty, |g|_{\text{Lip}_p}\}.
\]

Our default meaning of \(|\cdot|\) for vector-valued arguments is the max-norm.
2 Compositions

The role of compositions for DNN approximation has been addressed from different perspectives in a number of earlier studies, see e.g., [13, 27, 26, 33]. From a rather different angle the aim here is to put forward a general framework that allows one to quantify compositional approximability, first on a continuous infinite-dimensional functional level. Here approximants are not yet determined by a finite number of degrees of freedom but are subjected to certain structural conditions revolving on the notion of dimension-sparsity, see in §2.1.

In §2.2 we introduce metric properties of (finite) compositions, that allow us to introduce in §2.3 compositional approximation classes. Later those are to serve as model classes for characterizing approximability by DNNs.

2.1 Basic and Notation and Structural Properties

2.1.1 Compositional Representations

We will interpret “compositions” - denoted in what follows (at times in slight abuse of notation) by the symbol “◦” in \((g \circ h)(z) := g(h(z))\) in a broad sense, covering iterated applications of global operators. A prominent albeit not exclusive role is, however, played by compositions in a pointwise sense, defined for continuous functions. Specifically, we consider compositions of mappings

\[ g^1 : D_0 \subset \mathbb{R}^{d_0} \to \mathbb{R}^{d_1}, \quad g^\ell : \mathbb{R}^{d_{\ell-1}} \to \mathbb{R}^{d_\ell}, \quad \ell = 2, \ldots, n, \]

where we always require that the last factor is linear, i.e., for some \(\alpha_i \in \mathbb{R}, i = 1, \ldots, d_{n-1}\)

\[ g^n = \sum_{i=1}^{d_{n-1}} \alpha_i g_i^{n-1}. \]

When writing \(h \circ g\) it will always be implicitly assumed that the two factors \(g, h\) are “dimensionally compatible” in the above sense. It will be convenient to abbreviate the ordered array of mappings by \(g = (g^j)_{j=1}^n\) to provide a particular realization

\[ G(z) = (g^n \circ \cdots \circ g^1)(z) =: G_g(z), \]

(2.1)
of a mapping from \(D_0 \subset \mathbb{R}^{d_0} \to \mathbb{R}^{d_n}\). We sometimes simply refer to \(g\) instead of \(G_g\).

A prominent example of compositional representations are DNNs whose formal definition can be found in numerous texts, see e.g. [11, 20, 16, 29, 36]. Here we are content with mentioning that DNN realizations (denoted by \(\mathcal{N}\)) are (in their simplest feed-forward version) of the form (2.1) with factors \(g^j(\cdot) = \sigma(A^j \cdot + b^j)\) where \(A^j \in \mathbb{R}^{d_j \times d_{j-1}}, b^j \in \mathbb{R}^{d_j}\), and the activation function or rectifier \(\sigma\) acts componentwise. An important examples is the ReLU rectifier \(\sigma(t) = \max\{0, t\} =: t_+\). The entries in \(A^j, b^j\) or in the linear output layer are called weights and their number \#\(\mathcal{N}\) is the size or complexity of the DNN \(\mathcal{N}\). There are numerous important “architectural variants” like ResNet structures. We will address those as well as slight generalizations later below when the need arises.
2.1.2 Lipschitz-Stability

When dealing with pointwise compositions, the first important constraint on (2.1) in what follows is that we require all factors \( g^j \) (and hence \( G \) itself) in \( g \) to be Lipschitz continuous with Lipschitz constants \( L_j \). One reason lies in the following simple folklore perturbation bound. Consider dimensionally compatible Lipschitz functions \( g, h \) with constants \( L_g, L_h \) and \( \varepsilon_g, \varepsilon_h \) accurate approximations \( \tilde{g}, \tilde{h} \). Then

\[
\| g \circ h - \tilde{g} \circ \tilde{h} \|_X \leq \| g \circ h - g \circ \tilde{h} \|_X + \| g \circ \tilde{h} - \tilde{g} \circ \tilde{h} \|_X \leq L_g \varepsilon_h + \varepsilon_g. \tag{2.2}
\]

Given a compositional representation \( G_g \) of the form (2.1), we denote for \( k \leq n \) by \( L[n,k] = L[n,k](g) \) the Lipschitz constant of partial compositions \( g^n \circ g^{n-1} \circ \cdots \circ g^1 \). It will be convenient to set \( L[n,n+1] = 1 \). Then, using the above argument inductively yields the following familiar facts, see e.g. [33].

**Remark 2.1** Assume that we have mappings \( \tilde{g}^j : \mathbb{R}^{d_j-1} \to \mathbb{R}^{d_j}, j = 1, \ldots, n, \) such that

\[
\| g^j - \tilde{g}^j \|_X \leq \varepsilon_j, \quad j = 1, \ldots, n.
\]

Then

\[
\| g^n \circ \cdots \circ g^1 - \tilde{g}^n \circ \cdots \circ \tilde{g}^1 \|_X \leq \varepsilon_n + \sum_{j=1}^{n-1} \varepsilon_j L[n,j+1] = \sum_{j=1}^{n} \varepsilon_j L[n,j+1].
\]

By a symmetric argument we can replace \( L[n,j+1](g) \) by \( L[n,j+1](\tilde{g}) \).

In terms of the individual Lipschitz constants, one has, of course, \( L[n,k] \leq \prod_{j=k}^{n} L_j \). Finally, the estimates remain valid for more general iterated applications of operators as long as an estimate like (2.2) holds.

An important context where this will be used later is the following result.

**Proposition 2.2** For any \( \delta > 0 \), and any \( g \in \text{Lip}_1((0,1)^s) \), there exist a ReLU network \( \mathcal{N}_\delta \) such that

\[
\| g - \mathcal{N}_\delta \|_\infty \leq \delta, \quad \| \mathcal{N}_\delta \|_{\text{Lip}_1} \leq c_3(1 + \| g \|_\infty) \| g \|_{\text{Lip}_1}, \tag{2.3}
\]

and

\[
\#\mathcal{N}_\delta \leq c_1 \| g \|_{\text{Lip}_1} \delta^{-s} \log_2 \frac{1}{\delta}, \quad \text{depth of } \mathcal{N}_\delta \leq c_2 \log_2 \frac{1}{\delta}, \tag{2.4}
\]

where the constants \( c_1, c_2, c_3 \) depend only on \( s \). Using if necessary Lipschitz-stable continuation from bounded domains to hypercubes, analogous results hold for more general domains under mild geometric constraints (see e.g. [20]).

[20] establishes the existence of ReLU networks that approximate functions of higher Sobolev regularity in weaker Sobolev norms without compromising the standard complexity bounds. If we imposed more regularity than just \( g \in \text{Lip}_1 \), These results would imply (2.3) with a constant tending to one. In the present context we prefer to avoid assuming such “excess regularity” and sketch corresponding arguments in Appendix A for completeness, building on some of the concepts in [20].
2.1.3 Compositional Complexity

Since
\[ f \circ g = (f \circ h) \circ (h^{-1} \circ g) =: \tilde{f} \circ \tilde{g}, \] (2.5)
a mapping \( G \) may have infinitely many compositional representations. In slight abuse of terminology we sometimes write \( g_G \) to express that \( g \) is a representation of the mapping \( G \). Hence, one can always “reshape” compositional factors by writing where the factors in the new representation could have unfavorable regularity or stability properties, or in fact vice versa. In particular, we are interested in large input-dimension \( d_0 \) and ask whether specific compositional structures \( g_G \) may help avoiding the Curse of Dimensionality. In this regard, the role of sparsely connected neural networks has been already observed in [33] in the context of nonlinear regression and has been a source of motivation for what follows.

To that end, for a scalar valued function \( v : \mathbb{R}^d \to \mathbb{R} \), let \( s(v) \leq d \) denote the actual number of variables, \( v \) depends on explicitly. Then, for \( g^j : \mathbb{R}^{d_j-1} \to \mathbb{R}^{d_j} \) we define
\[
s_\infty(g^j) =
\begin{cases} 
0, & \text{if } g^j \text{ is the identity,} \\
1, & \text{if } g^j \text{ is multi-linear,} \\
\max_{i \leq d_j} s(g^j_i), & \text{else.}
\end{cases}
\] (2.6)

Low weighting of linear, bilinear factors or the identity “rewards” the fact that such factors are already finitely parametrized and further approximations are not needed.

**Definition 2.3** For a representation \( g \) of a mapping \( G \) let
\[
s_\infty(g) := \max_{j=1, \ldots, n(g)-1} s_\infty(g^j).
\]

A representation \( g_G \) of the form (2.1) is called \( s \)-dimension sparse if \( s_\infty(g) \leq s \).

This gives rise to the following measure for the compositional complexity of a representation \( g \)
\[
\mathcal{N}(g) := \sum_{j=1}^{n} d_j \sum_{i=1}^{s_\infty(g^j)} s(g^j_i) \leq \sum_{j=1}^{n} d_{j-1} d_j.
\]

**Remark 2.4** Thus, when \( g \) represents a DNN \( \mathcal{N} \) dimension-sparsity corresponds to sparse connectivity and \( \mathcal{N}(g) \approx \# \mathcal{N} \) which will be frequently used in what follows.

We say that two representations \( g, g' \) are dimensionally compatible (in this order) if \( G_g, G_{g'} \) are, i.e., if the output dimension \( d_{\text{out}(g)} \) agrees with the input-dimension \( d_0(g') \).

**Remark 2.5** For any two dimensionally compatible representations \( g, g' \), the composition \( \hat{G} := G_{g'} \circ G_g \) has a representation \( \hat{g} = (g'|g) \) satisfying
\[
\mathcal{N}(\hat{g}) = \mathcal{N}(g) + \mathcal{N}(g').
\] (2.7)
Likewise, when $g, g'$ have equal in- and output dimension, i.e., $d_0(g) = d_0(g')$ and $d_n(g) = d_n(g')$, the sum $\tilde{G} := G_g + G_{g'}$ has a representation $\tilde{g} = (\tilde{g}_g)$ satisfying

$$\mathcal{N}(\tilde{g}) = \mathcal{N}(g) + \mathcal{N}(g').$$

(2.8)

This follows easily by (what in the DNN context is called) parallelization (as hinted at by the notation $(\tilde{g}_g)$), upon possibly inserting identity factors in the representation of smaller depth.

Given any integer $s$,

$$\mathcal{C}_{N,s}(D, d') := \left\{ G = G_g : D \rightarrow \mathbb{R}^{d' \times d} : \mathcal{N}(g) \leq N, s_\infty(g) \leq s, g^j \in \text{Lip}_{1} \right\}$$

denotes then the collection of mappings with $s$-dimension-sparse representations of complexity at most $N$. When $D, d'$ is clear from the context we write $\mathcal{C}_{N,s}$. $\mathcal{C}_{N,s}$ is of course not a linear set but, by Remark 2.5, one has (for compatible in- and output dimensions)

$$\mathcal{C}_{N,s} + \mathcal{C}_{{N}',s} \subset \mathcal{C}_{N+N',s}.$$  

(2.9)

Likewise for mappings $G \in \mathcal{C}_{N,s}(D, d')$, $\tilde{G} \in \mathcal{C}_{{N}',s}(\mathbb{R}^{d'}, d'')$ we infer from (2.7) that

$$\tilde{G} \circ G \in \mathcal{C}_{N+N',s}(D, d''),$$

(2.10)

The following remarks motivate the discussion in the next section.

**Remark 2.6** When $s \geq d_0$, $D \subset \mathbb{R}^{d_0}$, the constraint of $s$-dimension sparsity is, of course void. In this case one simply has that $\mathcal{C}_{N,s}(D, d') = \text{Lip}_1(D; \mathbb{R}^{d'})$. In fact, any $G \in \text{Lip}_1(D, d')$ has a trivial representation

$$G(x) = (g^2 \circ g^1)(x), \quad g^1_i = G_i, \quad i = 1, \ldots, d' \quad g^2 = \text{id}_{d'}.$$  

(2.11)

So, $s$-dimension-sparsity with $s < d = d_0$ is essential for such a framework to offer interesting information.

On the other hand, dimension-sparsity per se is also not yet a sufficient instrument which can be seen from the famous Arnold-Kolmogorov Superposition Theorem. In a variant established by G.G. Lorentz (25), it states that every continuous function $G$ on $D_0 = [0, 1]^d$ has a representation

$$G(x_1, \ldots, x_d) = \sum_{q=0}^{2d} \Phi \left( \sum_{p=1}^{d} \phi_{q,p}(x_p) \right),$$

(2.12)

where $\Phi, \phi_{q,p}$ are continuous functions. Thus, $G = g^4 \circ g^3 \circ g^2 \circ g^1$, where $g^1(x) = (\phi_{q,p}(x_p))_{q,p=0,1}^{2d}, \quad i.e., \quad d_1 = (2d + 1)d$, $g^2(z) = (\sum_{p=1}^{d} z_{q,p})_{q=0}^{2d}$, i.e., $d_2 = 2d + 1$, $g^3(z) = (\Phi(z_q))_{q=0}^{2d}$, $d_3 = 2d + 1$, $g^4(z) = \sum_{q=0}^{2d} z_q$. Hence, every $G \in \text{Lip}_1([0, 1]^d)$ belongs to $\mathcal{C}_{N_{d},1}([0, 1]^d, 1)$ where $N_d = (2d + 1) + (2d + 1) + (2d + 1) + 1$. In the above terminology the representation is 1-dimension sparse. Unfortunately, $\Phi$ could have arbitrarily low
regularity beyond continuity (even for smooth $G$), an approximation of these functions by
finitely parametrized objects could still be arbitrarily expensive. This limits the direct use
of the Superposition Theorem for practical purposes, see [14, 22] for somewhat controversial
views in this regard. In summary, $s$-dimension sparsity just by itself is still not sufficient
either. In addition, one needs to “tame” $s$-dimension-sparse representations to arrive at a
meaningful sparsity concept.

2.2 Tamed Compositions and Compactness

In view of the preceding comments, we need to regularize representations of elements in $\mathcal{C}_{N,s}$. Assume that $\mathcal{R} : g \mapsto \mathcal{R}(g) \in \mathbb{R}_+$ complies with addition and composition in the sense that

$$\mathcal{R}(g) \leq \max\{\mathcal{R}(g), \mathcal{R}(g')\}, \quad \mathcal{R}(G_g \circ G_{g'}) \leq \max\{\mathcal{R}(g), \mathcal{R}(g'), \mathcal{R}(g) \cdot \mathcal{R}(g')\},$$

(2.13)

where we assume dimensional compatibility in the second relation. Then for any $G \in \mathcal{C}_{N,s}$ let

$$\|G\|_{N,s,\mathcal{R}} = \|G\|_{N,s} := \inf \left\{ \mathcal{R}(g) : G_g = G, s(g) \leq s, \mathcal{R}(g) \leq N \right\}.$$  

(2.14)

We supress reference to $\mathcal{R}$ when this is clear from the context. We refer to $\| \cdot \|_{N,s,\mathcal{R}}$ as
“compositional norm” although it is not a norm but close to one. In fact, the following
relations follow from Remark 2.5, (2.9) and (2.10), combined with (2.13).

Remark 2.7 For any $G \in \mathcal{C}_{N,s}$ and $\tilde{G} \in \tilde{\mathcal{C}}_{N,s}$ with the same in- and output dimensions one has

$$\|G + \tilde{G}\|_{N+N,s} \leq \max\{\|G\|_{N,s}, \|\tilde{G}\|_{N,s}\}.$$  

(2.15)

Similarly, for dimensionally compatible mappings $G_i \in \mathcal{C}_{N_i,s}, i = 1, 2$, one has

$$\|G_2 \circ G_1\|_{N_1 + N_2,s} \leq \max\left\{\|G_1\|_{N_1,s}, \|G_2\|_{N_2,s}, \|G_1\|_{N_1,s} \cdot \|G_2\|_{N_2,s}\right\}.$$  

(2.16)

Although other variants of $\mathcal{R}$ are conceivable we focus here on

$$\mathcal{R}(g) := \max\left\{\|g^\ell\|_{\text{Lip}}, \operatorname{L}_{[n(g),\ell+1]}(g) : \ell = 1, \ldots, n(g)\right\},$$  

(2.17)

where $\operatorname{L}_{[n(g),\ell+1]}(g)$ denotes again the Lipschitz constant of the partial compositions $g^{n(g)} \circ \cdots \circ g^{\ell+1}$. One easily verifies that $\mathcal{R}$ satisfies (2.13). Controlling $\| \cdot \|_{N,s}$, obviously constrains representations of elements further. In particular, for $\mathcal{R}$, given by (2.17),

$$\|G\|_{\text{Lip}} \leq \|G\|_{N,s}, \quad G \in \mathcal{C}_{N,s}.$$  

(2.18)

Remark 2.8 It will be at times useful to consider alternate weaker regularizers. A natural
alternative would be

$$\mathcal{R}^o(g) := \max\left\{\|g^\ell\|_{\text{Lip}} : \ell = 1, \ldots, n(g)\right\},$$  

(2.19)

so that trivially

$$\|G\|_{N,s,\mathcal{R}^o} \leq \|G\|_{N,s,\mathcal{R}} \leq \max\{1, \|G\|_{N,s,\mathcal{R}^o}\}.$$  

(2.20)
Another way of weakening $\| \cdot \|_{N,s}$ is to replace the compositional Lipschitz constants $L_{[n(g),t+1]}(g)$ in (2.17) by $\zeta(N)L_{[n(g),t+1]}(g)$, where $\zeta(N) \to 0$. This would permit some growth of the $L_{[n(g),t+1]}(g)$ and hence no longer confines compositions to stay in $\text{Lip}_1$. In view of the applications to come we focus in what follows on the stronger version, which implies (2.18), see also Remark 2.12.

Proposition 2.9 For any fixed constant $B < \infty$ and any $s \in \mathbb{N}$ the set
\[ \mathcal{C}_{N,s}(B) := \{ G \in \mathcal{C}_{N,s} : \| G \|_{N,s} \leq B \} \] (2.21)
is compact in $X_0 = L_\infty(D)$ for either regularizer $\mathcal{R}$ or $\mathcal{R}^o$. Hence, a minimizing representation in (2.14) exist.

The proof of Proposition 2.9 is given in Appendix A.

Again, in view of Remark 2.6 and (2.18), compactness of $\mathcal{C}_{N,s}(B)$ is trivial when $s \geq d_0$ or $\mathcal{R}$ is given by (2.17).

We are now prepared to formulate compositional approximability.

### 2.3 Approximation Classes

Unless stated otherwise (2.17) is used in the definition of (2.14) in what follows. In the spirit of [7], consider the “K-functional"
\[ K_s(v, N, \delta) := \inf_{G \in \mathcal{C}_{N,s}} \| v - G \|_X + \delta \| G \|_{N,s}. \] (2.22)

Obviously, $K_s(v, N, \delta) \leq C\delta$ means that for some $\bar{G} \in \mathcal{C}_{N,s}$ one has $\| v - \bar{G} \|_X \leq C\delta$ and $\| \bar{G} \|_{N,s} \leq C$, i.e., accuracy $\delta$ is achieved with a controlled “composition-norm”.

Interrelating $N$ and $\delta$ is then a way to define collections of functions with a certain quantifiable compositional approximability. A (smooth) strictly increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{s \to \infty} \gamma(s) = \infty$, is called a growth function. Its inverse $\gamma^{-1}$ exists and is also a growth function. This is to be distinguished from $\gamma(\cdot)^{-1} = 1/\gamma(\cdot)$. Given such a growth function $\gamma$, we consider the class
\[ \mathcal{A}_{\gamma,s} := \{ v \in X : \| v \|_{\mathcal{A}_{\gamma,s}} := \| v \|_\infty + |v|_{\mathcal{A}_{\gamma,s}} < \infty \}, \] (2.23)
where $|v|_{\mathcal{A}_{\gamma,s}} := \sup_{N \in \mathbb{N}} \gamma(N)K_s(v, N, \gamma(N)^{-1})$.

**Remark 2.10** The definition of $\mathcal{A}_{\gamma,s}$ makes sense for any regularizer satisfying (2.13). If we want to specify any other regularizer than the default one (2.17) we indicate this by a corresponding subscript such as e.g. $\mathcal{A}_{\mathcal{R}^o,s}$.

Since trivially
\[ \| v - G_N \|_X \leq \gamma(N)^{-1} \gamma(N) \{ \| v - G_N \|_X + \gamma(N)^{-1} \| G_N \|_{N,s} \} \leq \gamma(N)^{-1} |v|_{\mathcal{A}_{\gamma,s}}, \]
we will use the information $v \in \mathcal{A}_{\gamma,s}$ often in the form that for each $N \in \mathbb{N}$ there exists $G_N \in \mathfrak{C}_{N,s}$ such that
\[ \|v - G_N\|_X \leq \gamma(N)^{-1}\|v\|_{\mathcal{A}_{\gamma,s}}, \quad \|G_N\|_{N,s} \leq \|v\|_{\mathcal{A}_{\gamma,s}}, \quad N \in \mathbb{N}. \] (2.24)

To put it differently, realizing target accuracy $\varepsilon$ by a composition $G_{\varepsilon}$ is achievable within a complexity $N = N_\varepsilon$ of the order
\[ N_\varepsilon = \left\lceil \gamma^{-1}\left(\frac{\|v\|_{\mathcal{A}_{\gamma,s}}}{\varepsilon}\right) \right\rceil. \] (2.25)

A natural candidate for a model class can now be defined as follows. Let for any fixed $B < \infty$
\[ \mathcal{K}_{\gamma,s}(B) := \{ v \in \mathcal{A}_{\gamma,s} : \|v\|_{\mathcal{A}_{\gamma,s}} \leq B \}. \] (2.26)

Again, on account of Remark 2.6, the classes $\mathcal{A}_{\gamma,s}^{\gamma,s}, \mathcal{A}_{\gamma,s}^{\gamma,s}$ agree with $\text{Lip}_1$ when $s \geq d_0$ and hence do not provide any useful information. In particular, by (2.18) and (2.24), balls in these classes are compact when $s \geq d_0$. Since $\|v\|_{\text{Lip}_1} \leq \|v\|_{\mathcal{A}_{\gamma,s}}$ for all $s \in \mathbb{N}$, precompactness of $\mathcal{K}_{\gamma,s}(B)$ is also immediate. Since in general $\mathcal{A}_{\gamma,s}^{\gamma,s}$ need not be contained in $\text{Lip}_1$ for $s < d$ the following claim requires an argument provided in Appendix A.

**Remark 2.11** The sets $\mathcal{K}_{\gamma,s,\mathcal{R}_s}(B)$ (see (2.26)) are for $B < \infty$ and $s \in \mathbb{N}$ compact.

**Remark 2.12** Finer scales $\mathcal{A}_{\gamma,s,\beta}$, $\beta \in (0, 1]$, of approximation classes can be obtained by defining $\|v\|_{\mathcal{A}_{\gamma,s,\beta}} := \sup_{n \in \mathbb{N}} \gamma(N)^{\beta}K_{\gamma,s}(v, N, \gamma(N)^{-1})$. This implies the existence of $G_N \in \mathfrak{C}_{N,s}$ such that $\|v - G_N\|_X \leq \gamma(N)^{-\beta}\|v\|_{\mathcal{A}_{\gamma,s,\beta}}$, while $\gamma(N)^{-\beta}\|G_N\|_{N,s} \leq \|v\|_{\mathcal{A}_{\gamma,s}}$. So, for $\beta < 1$ the composition norms are allowed to grow like at most $\gamma(N)^{1-\beta}$, $N \in \mathbb{N}$. Since in subsequent applications a uniform control on composition norms matter, we confine the discussion henceforth to the special case $\beta = 1$.

We record a few elementary properties of compositional approximation classes.

**Remark 2.13** (a) Obviously one has
\[ \gamma(\cdot) \preceq \bar{\gamma}(\cdot) \Rightarrow \mathcal{A}_{\bar{\gamma},s} \subseteq \mathcal{A}_{\gamma,s}. \] (2.27)

(b) Whenever $\gamma$ satisfies $\gamma(N) \geq c_\gamma(2N)$ for $N \in \mathbb{N}$, one has $\mathcal{A}_{\gamma,s} + \mathcal{A}_{\gamma,s} \subseteq \mathcal{A}_{\gamma,s}$.

**Remark 2.14** (a) For each $N \in \mathbb{N}$ (fixed) $\mathfrak{C}_{N,s} \subseteq \mathcal{A}_{\gamma,s}$ for every growth function $\gamma$.

(b) Assume that $\gamma \preceq \bar{\gamma}$ and $\gamma(N) \geq c_\gamma(2N)$ for a fixed $c_\gamma > 0$. For respective equal input- and output-dimension and $\mathcal{R}$ according to (2.17), $\mathcal{A}_{\gamma,s}^{\gamma,s}$ is closed under composition with elements from $\mathcal{A}_{\gamma,s}^{\gamma,s}$. An analogous statement holds when $v \in \mathcal{A}_{\gamma,s}(D, d')$, $w \in \mathcal{A}_{\gamma,s,\mathcal{R}_s}^{\gamma,s}(D', d''')$, $\dim D' = d'$. The growth range covers any polynomial growth.
Proof: (a) is obvious.

Regarding (b), for $v \in \mathcal{A}^{γ,s}$ and $w \in \mathcal{A}^{γ,s}$, $N \in \mathbb{N}$, there exist $G_w, G_v \in \mathcal{C}_{N,s}$ (with respective in- and output-dimensions), such that $\|v - G_v\|_X \leq \|v\|_{\mathcal{A}^{γ,s}} \gamma(N)^{-1}$, and $\|w - G_w\|_X \leq \|w\|_{\mathcal{A}^{γ,s}} \tilde{\gamma}(N)^{-1}$. Thus, since by (2.10), $G_w \circ G_v \in \mathcal{C}_{2N,s}$, we use (2.18) to conclude

$$\|w \circ v - G_w \circ G_v\|_X \leq \|(w - G_w) \circ v\|_X + \|G_w(v) - G_w(G_v)\|_X \leq \|w\|_{\mathcal{A}^{γ,s}} \tilde{\gamma}(N)^{-1} + |G_w|_{\text{Lip}_1}\|v\|_{\mathcal{A}^{γ,s}} \gamma(N)^{-1} \leq C^{-1}\|w\|_{\mathcal{A}^{γ,s}} \left\{1 + \|v\|_{\mathcal{A}^{γ,s}}\right\} \gamma(N)^{-1}. \quad (2.28)$$

Since by (2.16) and (2.24), $\|G_w \circ G_v\|_{2N,s} \leq \max\{\|w\|_{\mathcal{A}^{γ,s}}, \|v\|_{\mathcal{A}^{γ,s}}, \|w\|_{\mathcal{A}^{γ,s}} \cdot \|v\|_{\mathcal{A}^{γ,s}}\}$, (b) follows.

2.3.1 Implanting finitely-parametrized components:

The approximation classes introduced above characterize approximability by “tamed” or “regularized” compositions which themselves are not yet described by finitely many parameters. However, the compositional structure leads, in a second step, to a finitely parametrized approximation, see also [33] for an approach in the same spirit. We refer to this as “implanting Lipschitz-stable neural networks” which means that every component $g_j^0$ with $s(g_j^0) > 1$ is replaced by a neural network (with in- and output-dimension one), so as to produce an expanded composition whose factors are either at most bilinear or neural networks. So in total they form a finitely parametrized function which we still refer to as a neural network. Note that this does preserve s-dimension-sparsity.

The next result, although not directly applied in this form later in applications, reflects the underlying guiding principle.

Theorem 2.15 Assume that $v \in \mathcal{A}^{γ,s}$, then for every $\varepsilon > 0$ there exists a DNN $\mathcal{N}_\varepsilon$, such that

$$\|v - \mathcal{N}_\varepsilon\|_X \leq \varepsilon, \quad (2.29)$$

where

$$\#\mathcal{N}_\varepsilon \lesssim \left(\frac{\|a\|_{\mathcal{A}^{γ,s}}}{\varepsilon}\right)^s \gamma^{-1}\left(\frac{2\|v\|_{\mathcal{A}^{γ,s}}}{\varepsilon}\right)^{s+1} \left(\log_2 \gamma^{-1}\left(\frac{2\|v\|_{\mathcal{A}^{γ,s}}}{\varepsilon}\right)\right). \quad (2.30)$$

Moreover, one has

$$\|\mathcal{N}_\varepsilon\|_{\mathcal{N}_\varepsilon,s} \lesssim \|v\|_{\mathcal{A}^{γ,s}}, \quad \mathcal{N}_\varepsilon \approx \gamma^{-1}(2\|v\|_{\mathcal{A}^{γ,s}}/\varepsilon), \quad (2.31)$$

while for the weaker regularization $\mathcal{R}^0$ one has $\|\mathcal{N}_\varepsilon\|_{\mathcal{N}_\varepsilon,\mathcal{R}^0} \lesssim \|v\|_{\mathcal{A}_{\mathcal{R}^0}^{γ,s}}$.

Thus, unless $\|v\|_{\mathcal{A}^{γ,s}}$ hides an exponential dependence on the large input dimension $d_0$, unit balls $U\mathcal{A}^{γ,s}$ are model classes for which DNN approximation avoids the Curse of Dimensionality when $s \ll d_0$.

It is instructive to specialize these estimates for two types of growth functions

(alg): $\gamma(r) \approx C_0 r^a$, or (exp): $\gamma(r) \approx C_0 e^{ar}$, \quad (2.32)
for some $\alpha > 0$. (With a bit more technical effort the arguments extend to more refined scales like $\gamma(r) \sim e^{\alpha r^\beta}$, for some $0 < \beta \leq 1$.) For convenience we record for frequent future use

$$
\gamma^{-1}(s) \approx \begin{cases} 
C a^{-1/\alpha} s^{1/\alpha}, & \gamma \sim \text{(alg)}, \quad \alpha \geq \alpha_0 > 0; \\
\frac{1}{\alpha} \ln \frac{s}{\gamma}, & \gamma \sim \text{(exp)}, \quad \alpha > 0. 
\end{cases}
$$

(2.33)

Hence (2.30) takes the form

$$
N_\varepsilon \preceq \begin{cases} 
\|v\|_{A^{\gamma,s}}^2 \left( \frac{|v|_{A^{\gamma,s}}}{\varepsilon} \right)^{\frac{(\alpha+1)s}{\alpha}} \log_2 \frac{|v|_{A^{\gamma,s}}}{\varepsilon}, & \gamma \sim \text{(alg)}, \\
\left( \frac{|v|_{A^{\gamma,s}}}{\varepsilon} \right)^{\frac{s}{\alpha}} \ln \frac{|v|_{A^{\gamma,s}}}{\varepsilon}, & \gamma \sim \text{(exp)}. 
\end{cases}
$$

(2.34)

When $\gamma \sim \text{(exp)}$ strong stability in (2.31) deteriorates only slowly according to $\|v\|_{A^{\gamma,s}}^\ln \varepsilon$.

In general, the stronger the algebraic growth order the closer the dominating complexity factor comes to the rate $\varepsilon^{-s}$ which is attained for exponential growth (up to logarithmic factors). This rate is what one can expect for Lipschitz functions of $s$ variables.

The assertion of Theorem 2.15 hinges on the following Lemma which we state here for later reference in several applications of similar type.

**Lemma 2.1** Assume that for some $s \leq d_0$, the mapping $G$ belongs to $C_{N,s}$, i.e., is $s$-dimension-sparse (see Definition 2.3). Let $G = G_{\mathbf{g}} \in C_{N,s}$. Let the DNN $\mathcal{N}$ be obtained by replacing each component $g_i^j$ of each factor $g^j$ with $s(g^j) > 1$, by a $\delta_j$ accurate Lipschitz stable network $\mathcal{N}_j^i$, i.e.,

$$
\|g_i^j - \mathcal{N}_j^i\|_{X_i} \leq \delta_j, \quad \|\mathcal{N}_j^i\|_{\text{Lip}_1} \leq c_3(1 + \|g_i^j\|_{L_\infty})\|g_i^j\|_{\text{Lip}_1}, \quad i = 1, \ldots, d_j, \quad j = 1, \ldots, n(g) - 1.
$$

Then one has

$$
\|G - \mathcal{N}\|_{L_\infty} \leq \delta_n + \sum_{j=1}^{n(D) - 1} \delta_j L_{[n(D), j + 1]}, \quad \#\mathcal{N} \leq \sum_{j=1}^{n(D) - 1} \|g^j\|_{\text{Lip}_C} d_j \delta_j^s \log_2 \delta_j. 
$$

(2.35)

and

$$
\#\mathcal{N} \leq N \|G\|_{C_{N,s}} \max_{j=1, \ldots, n(D)} \delta_j^s \log_2 \delta_j.
$$

(2.36)

**Proof:** The first relation in (2.35) follows from Remark 2.3 while the second one is a consequence of Proposition 2.2 together with Remark 2.4. Since $\sum_{j=1}^{n(g)} d_j \leq \mathfrak{N}(g)$, (2.36) follows from the definition of the compositional norms. \[\square\]

We return to the proof of Theorem 2.15. By (2.24) we can find for each $N \in \mathbb{N}$ a $G_N \in C_{N,s}$ satisfying (2.24). Given $\varepsilon > 0$, (2.25) says that $N_\varepsilon = \gamma^{-1}(2|v|_{A^{\gamma,s}}/\varepsilon)$ (we ignore the ceil-operation) suffices to ensure that

$$
\|v - G_{N_\varepsilon}\|_X \leq \frac{\varepsilon}{2}.
$$
Let $\mathcal{N}_{N,\delta}$ denote the DNN, obtained by Lemma 2.1, with implant-tolerances $\delta = \delta_j$ all equal. to conclude that for any minimizing representation $g_{N,\varepsilon}$ of $G_{N,\varepsilon}$

$$\|v - \mathcal{N}_{N,\delta}\|_X \leq \frac{\varepsilon}{2} + \delta n(D(g_{N,\varepsilon})) \|G_{N,\varepsilon}\|_{N,\varepsilon,s} \leq \frac{\varepsilon}{2} + N_\varepsilon \delta \|v\|_{A^\gamma,s}.$$ 

Choosing $\delta = \delta(\varepsilon) := \varepsilon(2\|a\|_{A^\gamma,s}^{-1}(2\|v\|_{A^\gamma,s}/\varepsilon))^{-1}$, produces a network $\mathcal{N}_\varepsilon := \mathcal{N}_{N,\delta(\varepsilon)}$ satisfying (2.29), on account of (2.35). Regarding (2.30), we infer now from $\|G_{N,\varepsilon}\|_{N,\varepsilon,s} \leq \|v\|_{A^\gamma,s}$ and (2.36) that

$$\#\mathcal{N}_\varepsilon \leq N_\varepsilon \|v\|_{A^\gamma,s}^s \delta(\varepsilon)^{-s} |\log_2 \delta(\varepsilon)| \lesssim \gamma^{-1} \left( \frac{2\|v\|_{A^\gamma,s}}{\varepsilon} \right)^s \left( \|a\|_{A^\gamma,s}^{-1}(2\|v\|_{A^\gamma,s}/\varepsilon) \right)^s |\log_2 \gamma^{-1}(2\|v\|_{A^\gamma,s}/\varepsilon)|,$$

which is (2.30).

Finally, regarding the stability of the networks $\mathcal{N}_\varepsilon$, we employ the (possibly over-pessimistic) estimate (2.20) to obtain (2.31). \hfill \Box

**Remark 2.16** Time-stepping in discretized dynamical systems is not the only context where one can expect to encounter compositional sparsity. More generally, solutions to operator equations can often be approximated by iterative processes such as fixed-point iterations that may help to assert membership to a compositional approximation class. This is exemplified next for a specific scenario where standard reduced modeling concepts suffer from slowly decaying Kolmogorov widths.

### 3 Linear Parametric Transport Equations

#### 3.1 A Model Problem

We consider the Cauchy problem for a linear (scalar) transport equation in $m$ spatial dimensions ($m \in \{1, 2, 3\}$, say) with parameter dependent data

\[
\begin{align*}
\partial_t u(t, x, y) &+ a(t, x, y) \cdot \nabla_x u(t, x) - f(t, x, y) = 0, \quad x \in \mathbb{R}^m, \ t \in [0, T_\infty), \ y \in \mathcal{Y}, \\
u(0, x, y) &= u_0(x, y), \quad x \in \mathbb{R}^m, \ y \in \mathcal{Y},
\end{align*}
\]

which is a standard format for models with uncertain data. We assume for convenience that supp $u_0 = D \times \mathcal{Y}$ where $D$ is a bounded domain. Hence, for a fixed time horizon $T_\infty$ the solution, as a function of $t, x$ can take values different from zero only in a bounded subset of $[0, T_\infty) \times \mathbb{R}^m$. In what follows $T_\infty$ should be viewed as a fixed finite but possibly large time horizon whose order of magnitude is expected to affect the complexity of the envisaged parameter-to-solution maps $y \mapsto u(y)$.

We shall sometimes view $u(y)$ for each $y \in \mathcal{Y}$ as a function of $(t, x) \in [0, \hat{T}) \times \mathbb{R}^m$, i.e., as a “point” in $L_\infty([0, \hat{T}) \times \mathbb{R}^m)$. To generate in the end efficient surrogates for the parameter-to-solution map, it will nevertheless be useful to view $u$ as a function of all variables $(t, x, y)$

\[
u : \Omega := [0, \hat{T}) \times \mathbb{R}^m \times \mathcal{Y} \to \mathbb{R},
\]
so that for high parameter-dimensionality $\mathcal{Y} \subset \mathbb{R}^{d_y}$, $d_y \gg 1$, one faces approximation problems in high dimensions. Since we are interested in conditions other than smoothness that may help avoiding the Curse of Dimensionality we impose only low or moderate smoothness conditions on the problem data. Specifically, we assume throughout:

$$a \in C(0, \hat{T}; \text{Lip}_1(\mathbb{R}^m \times \mathcal{Y})), \quad \|a\|_{L_\infty([0,T]; \mathbb{R}^m)} \leq A,$$

(3.3)

$$|a(t, z; y) - a(t, z', y')| \leq L \max\{|z - z'|, |y - y'|\}, \quad (t, z, y), (t, z', y') \in \Omega.$$  

In addition we require at times in addition Lipschitz-in-time continuity of $a$

$$a \in \text{Lip}_1(0, \hat{T}; C(\mathbb{R}^m \times \mathcal{Y})), \quad |a(\cdot, w)|_{\text{Lip}_1([0, \hat{T}])} \leq L_t.$$  

(3.4)

We separate the Lipschitz-conditions (3.3) and (3.4) because (3.4) is, under certain circumstances not necessary, see Remark 3.1 in the next section.

To see what one can expect regarding sparsity of solutions, the very special case, where $a$ is independent of $t, x$, is instructive. E.g. when $f = 0$ the solution $u(t, x, y) = u_0(x - ta(y), y)$ is a simple composition of $u_0$ with a linear function in $(t, x)$ involving, however, a $y$-dependent coefficient. Even when $u_0$ does not depend on $y$ but $a(y)$ can be any element in $\text{Lip}_1(\mathcal{Y})$ the solution can be in essence an arbitrary Lipschitz function and stable approximations will suffer from the CoD. The same holds, if $a(y) = a$ is constant but $u_0(\cdot, y)$ is an arbitrary element in a $\text{Lip}_1(\mathcal{Y})$-ball. Analogous considerations apply to the right hand side $f$ when only smoothness conditions are imposed. This is in agreement with the findings in [30] where the only conditions on the convection field are given in terms of classical smoothness properties.

In conclusion, more specific structural constraints on the data are needed to ensure that $u$ can be approximated without suffering from the Curse. In brief, all one can expect is a "heredity" effect where some structural sparsity of the data leads to a structural solution sparsity that allows one to avoid the CoD.

### 3.2 Dimension-Sparse Compositional Convection Fields

In the light of the preceding comments we consider convection fields $a$ that belong to the Bochner-type space of functions that are continuous in time with uniformly controlled values in $\mathcal{A}^{\gamma,m} = \mathcal{A}^{\gamma,s}_{\mathbb{R}^m; \mathbb{R}^m}$ given by (2.17)

$$a \in L_\infty([0, T_\infty); \mathcal{A}^{\gamma,s}_{\mathbb{R}^m \times \mathcal{Y}; \mathbb{R}^m}).$$

(3.5)

The regularization (2.17) is used in the definition of $\mathcal{A}^{\gamma,s}$ because, under the above assumptions, solutions and characteristics belong to $\text{Lip}_1(\Omega)$.

Here and below $m \leq s \leq m + 1 + d_y$ marks some dimension-sparsity with respect to the total number of variables. The fact that we do not assume $a \in \mathcal{A}^{\gamma,s}(\Omega; \mathbb{R}^m)$ indicates that the time variable receives a special treatment.
Remark 3.1 Time-Lipschitz continuity (3.4) is not always necessary. For our purposes it would suffice to know that compositional approximability is inherited by time-averages 
\[ a_I(z; y) := \frac{1}{|I|} \int_I a(s, z; y) ds \in \mathcal{A}_r^\beta, \quad i.e., \]
\[ \|a_I\|_{\mathcal{A}_r^\beta} \lesssim \|a\|_{L_\infty(0, \hat{T}; \mathcal{A}_r^\beta)} \tag{3.6} \]
This condition holds e.g. in the case of affine parameter dependence, introduced next.

A particular case of interest concerns affine parametric expansions for the convection field
\[ a(t, x; y) = \sum_{j=1}^{d_y} y_j a_j(t, x), \quad \mathcal{Y} = [-1, 1]^{d_y}, \tag{3.7} \]
i.e., \( a : \mathbb{R}^{m+1+d_y} \to \mathbb{R}^m \). Such representations arise, for instance, from Karhunen-Loéve expansions of random convection fields in which case the \( a_j \) have some decay properties.

Notice that the second relation in (3.3) now reads
\[ \|a\|_{L_\infty(\Omega)} = \sup_{(t, x) \in \Omega} \sum_{j=1}^{d_y} |a_j(t, x)| \leq A. \tag{3.8} \]
More specifically, we choose the following representation format that allows us later to explore several possible regimes
\[ a_j(t, w) = \omega_j a_j^\circ(t, w), \quad \|a_j^\circ\|_{L_\infty([0, \hat{T}] \times \mathbb{R}^m)} \leq A^\circ, \quad \Lambda := \max_{t \in [0, \hat{T}], j=1, \ldots, d_y} |a_j^\circ(t, \cdot)|_{\text{Lip}_1(\mathbb{R}^m)}, \tag{3.9} \]
where \( \omega = (\omega_1, \ldots, \omega_{d_y}) \in \mathbb{R}_{d_y}^+, \quad |\omega|_1 := \sum_{j=1}^{d_y} \omega_j, \quad \text{so that} \quad A = |\omega|_1 A^\circ. \)

Then, one has for all \((x, y), (x', y') \in \mathbb{R}^m \times \mathcal{Y}\)
\[ |a(t, x; y) - a(t, x'; y')| \leq \sum_{j=1}^{d_y} |y_j - y'_j||a_j(t, x)| + |y'_j||a_j(t, x) - a_j(t, x')| \]
\[ \leq A|y - y'| + \Lambda|\omega|_1|x - x'|, \]
and therefore
\[ \sup_{t \in [0, \hat{T}]} |a(t, \cdot; \cdot)|_{\text{Lip}_1(\mathbb{R}^m \times \mathcal{Y})} \leq A + \Lambda|\omega|_1 =: L \tag{3.10} \]
is a valid Lipschitz constant permitted in (3.3). Note that we do not require here the validity of (3.4).

Note also that (3.8) is possible even when \( \omega_j = 1 \), i.e., \( |\omega|_1 = d_y \) in which case \( A \) and \( L \) are proportional to \( d_y \). The case \( |\omega|_1 = 1 \) ensures a dimension-independent boundedness and regularity of the convection field.

However, in either of the two “extreme” regimes (R1) \( |\omega|_1 = 1 \), (R2) \( |\omega|_1 = d_y \), the convection field \( a \) is \( m \)-dimension sparse. More specifically, one has:
Remark 3.2 Assume that the convection field $a$ is of the form (3.7), satisfying (3.8) and (3.9). Then $a$ has an $m$-dimension sparse compositional representation of depth two

$$a(t, \cdot; \cdot) \in \mathcal{C}_{N_a,m}, \quad \text{depth}(a) = 2, \quad N_a := \mathcal{N}(a) = d_y(1 + m^2) + 1,$$

and

$$\|a(t, \cdot; \cdot)\|_{N,m} \leq A + \Lambda |\omega|_1, \quad N \geq N_a.$$  

Hence $a$ belongs to $L_\infty(0, \hat{T}; A^{\gamma,m})$ for every growth function $\gamma$.

To see this, note that

$$a(t, x^1, y) = (g^2 \circ g^1)(t, x^1; y), \quad (3.13)$$

where, in view of (2.6), for $S_A := \{(r^1, \ldots, r^{d_y}) \in \mathbb{R}^{md_y} : \sum_{j=1}^{d_y} |r^j| \leq A\}$,

$$g^1 : (t, x, y) \mapsto (y, a_1(t, x), \ldots, a_d(t; x)) \in \mathbb{R}^{d_y(1+m)},$$

$$g^2 : (y, r^1, \ldots, r^{d_y}) \in \mathcal{Y} \times S_A \mapsto \sum_{j=1}^{d_y} y_j r^j, \quad (3.14)$$

This shows (3.11). Moreover, we infer from (3.10) and (3.8) that

$$|g^1|_{\text{Lip}} \leq \max\{1, \Lambda\}, \quad |g^2|_{\text{Lip}(\mathcal{Y} \times S_A)} \leq (A + \Lambda |\omega|_1) \max\{|r - r'|, |y - y'|\}. \quad (3.15)$$

Since by assumption $|g^2 \circ g^1|_{\text{Lip}} = |a|_{\text{Lip}} \leq (A + \Lambda |\omega|_1) = L$ we see that uniformly in $t$, as a function of $x, y$, one has $a \in \mathcal{C}_{N_a,m}$ where $N_a := \mathcal{N}(a) = d_y(1 + m^2) + 1$. This confirms (3.12).

Remark 3.3 To reduce technicalities when tracking the dependence of constants on problem parameters we assume from now on that

$$1 \leq L_t, A \leq L, \quad (3.16)$$

because a large $L$ will be seen to have the most adverse effect. Finally, recall that, by definition

$$A, L \leq \|a\|_{L_\infty(0, \hat{T}; A^{\gamma,s})} =: \|a\|, \quad (3.17)$$

where this latter notational abbreviation will be used whenever reference to $\gamma, s$ is clear from the context.

### 3.3 Characteristics

The field of characteristics, given by the family of ODEs

$$\dot{z}(t) = a(t, z(t); y), \quad z(0) = x,$$

plays a pivotal role in what follows. Note that the characteristics have a natural semi-group property, namely that they can be obtained by composing individual characteristic segments.
More precisely, suppressing the dependence on \(y\) for a moment, we consider the solution of the more general initial value problem
\[
\dot{z}(t, \tau; \bar{z}) = a(t, z(t)), \quad z(\tau) = \bar{z}.
\] (3.19)

Later concatenations of characteristic segments necessitates including a specific initial time \(\tau\) in the notation. If \(\tau = 0\) and there is no risk of confusion we often abbreviate \(z(t, 0; \bar{z}) = z(t; \bar{z})\). Thus, one has for any \(\tau\)
\[
z(t, x) = z(t, \tau; z(\tau; x)) = (z(\cdot; \tau; \cdot) \circ z(\cdot; 0; x))(t).
\] (3.20)

In slight abuse of terminology we refer to this as *composing* characteristic segments.

There is a second angle regarding compositional approximations to characteristics, namely that (3.18) is equivalent to the fixed-point relation
\[
z(t, \tau; x; y) = x + \int_\tau^t a(s, \tau; z(s; \tau; x; y); y)ds.
\] (3.21)

Both, the semi-group property and the fixed-point relation will be combined to construct compositional approximations to the characteristics.

Under the above assumptions characteristics don’t cross, i.e., the value of the solution to (3.7) can be determined by tracing back along characteristics. In fact, in view of (3.18), one has for the solution \(u\) of (3.1) (suppressing again the dependence on \(y\) for a moment)
\[
\frac{\partial}{\partial t} u(t, z(t)) = f(t, z(t)).
\]
Hence, recalling that \(u(0, z(0, x; y); y) = u_0(x; y),\)
\[
u(t, z(t, x; y), y) = u_0(x; y) + \int_0^t f(s, z(s, x; y))ds,
\] (3.22)
or equivalently, using (3.20) and noting that when \(x = z(t, 0; \bar{x})\) one has \(z(s, 0; \bar{x}) = z(-(t-s), t; x) = z(s-t, t; x),\) (3.22) takes the form
\[
u(t, x, y) = u_0(z(-t, t; x, y), y) - \int_0^t f(s, z(s-t, t; x; y))ds.
\] (3.23)

In summary, if the characteristics have “good (pointwise) compositional approximability”, for \(f = 0,\) the solution results from one additional composition.

The central objective in what follows is to construct finitely parametrized surrogates \(\mathcal{N}(t, z, y)\) for the map
\[
(t, z, y) \in \Omega \mapsto u(t, z, y),
\] (3.24)
that are determined by possibly few degrees of freedom. The general flavor of the following results is: membership of the problem data (convection field, initial conditions, right hand side) to an approximation class (see §2.3) imply membership of characteristics and solution to a certain approximation class.
4 Main Results

In favor of an easier interpretability we focus, in view of (3.23), on the exemplary types of growth functions \( \gamma \sim (\text{alg}) \) and \( \gamma \sim (\text{exp}) \), defined in (2.32).

Since the spatial dimension \( m \) is fixed and at most three we do not always mark the dependence of estimates on \( m \).

The proofs of the following results can all be found in § 6. The point of the first result is that dimension-sparsity of the convection field is inherited by the characteristic field.

**Theorem 4.1** Let \( \tilde{T} := [0, \tilde{T}] \) and assume that the convection field \( a \) satisfies (3.5) and in addition (3.6) or (3.4) for some growth function \( \gamma \) of either type in (2.32). Abbreviating as before \( \|a\| := \|a\|_{L^\infty}(\tilde{T}, A_{\gamma,s}) \), one has

\[
z \in \text{Lip}_1(\Omega) \cap L^\infty(\tilde{T}, A_{\gamma,s}), \quad \|z\|_{L^\infty(\tilde{T}, A_{\gamma,s})} \lesssim e^{\|a\|_{\tilde{T}}}. \tag{4.1}
\]

where

\[
\tilde{\gamma}(r) := \begin{cases} 
\left( \frac{rC_1^{1/\alpha}}{AT} \right)^{\alpha r} \log_2 \left( \frac{rC_1^{1/\alpha}}{AT} \right)^{1+\alpha}, & \gamma \sim (\text{alg}), \\
\frac{\alpha}{AT} \log_2 \left( \frac{\alpha}{AT} \right), & \gamma \sim (\text{exp}).
\end{cases} \tag{4.2}
\]

In particular, the parameter dependent characteristic field satisfies

\[
\inf_{C \in C_{N,s}} \|z - C\|_{L^\infty(\Omega)} \lesssim e^{\tilde{T}\|a\|} \tilde{\gamma}^{-1} \tag{4.3}
\]

where \( \tilde{T}^{\|a\|} \). The tamed compositional approximations in (4.3) are not yet characterized by a finite number of degrees of freedom which is done in a next step similar to Theorem 2.15.

In what follows we adopt a generous understanding of deep neural networks regarding the dependence on the time variable \( t \). We allow in essence layers that are piecewise affine in \( t \) and hence still enjoy the basic properties of DNNs regarding evaluation and back-propagation.

**Theorem 4.2** Under the same assumptions on the convection field \( a \) and growth functions \( \gamma \) according to (2.32) there exists for each \( \varepsilon > 0 \) a deep neural network (DNN) \( N_\varepsilon \) such that

\[
\|z - N_\varepsilon\|_{L^\infty(\Omega; \mathbb{R}^m)} \leq \varepsilon, \tag{4.4}
\]

and

\[
\# N_\varepsilon \lesssim AT^{2^s\|a\|} \begin{cases} 
C_{a}^{-\frac{2^s\|a\|}{\varepsilon}} \left( \frac{\varepsilon^{1+\alpha}}{\varepsilon} \right)^{1+\alpha} \log_2 \left( \frac{\varepsilon^{1+\alpha}}{\varepsilon} \right)^{2}, & \gamma \sim (\text{alg}), \\
C_{a}^{-\frac{2^s\|a\|}{\varepsilon}} \left( \frac{\varepsilon^{1+\alpha}}{\varepsilon} \right)^{1+\alpha} \log_2 \left( \frac{\varepsilon^{1+\alpha}}{\varepsilon} \right)^{3+\alpha}, & \gamma \sim (\text{exp}).
\end{cases} \tag{4.5}
\]

It is instructive to reformulate these findings in terms of convergence rates.
Corollary 4.3 For each $N \in \mathbb{N}$, there exists a DNN $\mathcal{N}_N$ of complexity $\#\mathcal{N}_N \leq N$, such that
\[ \|z - \mathcal{N}_N\|_{L_\infty(\Omega; \mathbb{R}^m)} \lesssim e^{T\|a\|}\tilde{\gamma}(N)^{-1}, \] where
\[ \tilde{\gamma}(r) \approx \begin{cases} \frac{B r^{\frac{1}{1+\alpha}}}{(1+\alpha)(1+s)} \log_2 r \left| \frac{2^\alpha}{(1+\alpha)(1+s)} \right|, & \text{when } \gamma(r) \sim r^{\alpha}, \\ C \alpha r^{\frac{1}{1+s}} \log_2 r \left| \frac{3+\alpha}{4+\alpha} \right|, & \text{when } \gamma(r) \sim e^{ar}, \end{cases} \] with
\[ B = C_a r^{\frac{1}{1+\alpha}} \left( A \tilde{T} 2^s \|a\|^2 \right)^{-\frac{\alpha}{1+\alpha}}, \quad C = \left( A \tilde{T} 2^s \|a\|^2 \right)^{-\frac{1}{1+\alpha}}. \]

Remark 4.4 It seems that one cannot expect in general a uniform bound on the composition norms $\|\mathcal{N}_\varepsilon\|_{\#\mathcal{N}_\varepsilon, s}$, see Theorem 2.13 and the comments preceding Lemma 6.7 in §6.3, unless the compositional approximations of $a(t, \cdot, \cdot)$ have uniformly bounded depth. For the growth-types $(\text{alg})$ and $(\text{exp})$ in (2.32), the following holds
\[ \|\mathcal{N}_\varepsilon\|_{\text{Lip}\left([0,\tilde{T}]_\varepsilon; \mathbb{R}^m \times y\right)} \lesssim \max\{1, \|a\|\}, \quad \|\mathcal{N}_\varepsilon\|_{\#\mathcal{N}_\varepsilon, s, \text{Lip}} \lesssim e^{L_\varepsilon \tilde{T}}, \quad \varepsilon > 0, \] where
\[ L_\varepsilon \lesssim \begin{cases} (c_3(1 + A)\|a\|)\left(\frac{1}{\alpha} \log_2 \frac{1}{\varepsilon}\|a\|\frac{1}{\tilde{T}}\right), & \text{in case } \gamma \sim (\text{alg}), \\ (c_3(1 + A)\|a\|)\left(\frac{1}{\alpha} \log_2 \frac{1}{\varepsilon}\|a\|\frac{1}{\tilde{T}}\right), & \text{in case } \gamma \sim (\text{exp}). \end{cases} \]
Here $c_3$ is the constant from (2.3). Thus, Lipschitz continuity with respect to $x, y$ degrades when $\varepsilon$ decreases, the less though, the stronger the growth order of $\gamma$.

In both theorems the exponential case can be seen as a formal “limit $\alpha \to \infty$” of algebraic rates. For $s = m + d_y$ the obtained rate would reflect the full CoD. So, for a non-trivial dimension sparsity $s \ll m + 1 + d_y$ the CoD does not show in the convergence rates.

Nevertheless, some adverse dependence on (the potentially large) parametric dimension $d_y$ may still be hidden in $\|a\| = \|a\|_{L_\infty(I; \mathcal{A}^{\gamma, s})}$. This will be illustrated by specializing $a$ to affine representations of the form (3.7).

Recalling from Remark 3.2 that $a$ belongs to $C(\tilde{I}; \mathcal{A}^{\gamma, m})$ for any growth function $\gamma$ a first result follows from Theorem 4.2 and Corollary 4.3 by a judicious choice of $\gamma$.

Corollary 4.5 Assume that $a$ is of the form (3.7) and satisfies (3.8) and (3.9). Then,
\[ \|a\| = \|a\|_{L_\infty(\tilde{I}, \mathcal{A}^{\gamma, m})} \leq 2A + \Lambda |\omega|_1, \] and the characteristic field belongs to $\text{Lip}_1(\Omega) \cap C(\tilde{I}; \mathcal{A}^{\gamma, m})$ where
\[ \tilde{\gamma}(r) \approx \frac{r}{d_y A \tilde{T}} \log_2 \frac{r}{d_y A \tilde{T}} \left| \frac{3}{1+\alpha} \right|, \quad \|z\|_{\infty([0,\tilde{T}], \mathcal{A}^{\gamma, m})} \leq e^{(2A + \Lambda |\omega|_1)\tilde{T}}. \] Moreover, for each $N \in \mathbb{N}$ there exists a network $\mathcal{N}_N$ such that
\[ \|z - \mathcal{N}_N\|_{L_\infty(\Omega \times \gamma)} \lesssim d_y F e^{(2A + \Lambda |\omega|_1)\tilde{T}} N^{-\frac{1}{m+\tau}}, \] where $F = (A \tilde{T} 2^m(2A + \Lambda |\omega|_1)^2)^{\frac{1}{1+\tau}}$. 

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While the rates do not suffer from the CoD, to gain traction, \( N \) has to exceed \( d_y \). Although this delay effect is only algebraic in \( d_y \), this dependence is not optimal since the choice of any growth function for \( a \) does not fully exploit the special structure \((3.7)\), see the proof in §6.4. A more direct reasoning yields the following better results with regard to the stability of the networks, the scaling in \( \hat{T} \), and the dependence on \( d_y \).

**Theorem 4.6** Assume that \( a \) is of the form \((3.7)\) and satisfies \((3.3)\) and \((3.9)\). Recall from \((3.10)\) that

\[
L := A + \Lambda|\omega|_1. 
\]

Then, for any \( \varepsilon > 0 \) there exists a DNN \( \mathcal{N}_\varepsilon \) such that

\[
\|z - \mathcal{N}_\varepsilon\|_{L^\infty(\Omega; \mathbb{R}^m)} \leq \varepsilon, \quad \#\mathcal{N}_\varepsilon \lesssim d_y m^2 A \hat{T} \left( \frac{e^{L\hat{T}}}{\varepsilon} \right)^{m+1} \left| \log_2 \frac{e^{L\hat{T}}}{\varepsilon} \right|^2.
\]

Moreover, there exists a DNN \( \mathcal{N}_\varepsilon \) with complexity \( \#\mathcal{N}_\varepsilon \leq N \) such that

\[
\|z - \mathcal{N}_\varepsilon\|_{L^\infty(\Omega; \mathbb{R}^m)} \lesssim e^{L\hat{T}} \gamma(N)^{-1}, \quad N \in \mathbb{N}, \quad \gamma(r) = C \left( \frac{r}{d_y} \right)^{\frac{1}{m+1}} \log_2 \left( \frac{r}{d_y} \right)^{-\frac{1}{m+1}},
\]

where \( C = (A \hat{T} m^2)^{-\frac{1}{m+1}} \). The networks belong to \( \text{Lip}_1([0, \hat{T}]; C(\Omega; \mathbb{R}^m)) \) and are stable with \( \|\mathcal{N}_\varepsilon\|_{\mathcal{N},m} \lesssim e^{L\hat{T}} \) where \( \varepsilon \leq A + \hat{T}^{-1} + C_3 (1 + A^\gamma) |\omega|_1 \) whenever \( \varepsilon \leq 1 \).

If, on the other hand, we consider regime (R2) \( |\omega|_1 = d_y \) the Lipschitz constant \( \|a\| \geq L \) scales like \( d_y \) so that the constant \( e^{L\|a\|} \) depends exponentially on \( d_y \) (see \((3.10)\)). Hence, the Curse of dimensionality still strikes through an exponential delay in gaining accuracy.

We discuss next approximability of the parameter dependent solutions themselves.

**Theorem 4.7** Under the same hypotheses on the conception field \( a \) as in Theorem 4.6 assume that the data \( u_0, f \) satisfy

\[
u_0 \in \mathcal{A}^{\gamma,m}, \quad f \in L^\infty(\hat{T}; \mathcal{A}^{\gamma,m}) \cap \text{Lip}_1(\hat{T}; C(\mathbb{R}^m \times \mathcal{Y})), \quad \gamma(r) \sim r^\alpha.
\]

and let

\[
\beta := \max \left\{ 1, \frac{m+1}{\alpha} \right\}.
\]

Then, for any \( \varepsilon > 0 \) there exits a DNN \( \mathcal{N}_{u,\varepsilon} \) such that for the exact solution \( u \) of the transport equation \((3.7)\)

\[
\|u - \mathcal{N}_{u,\varepsilon}\|_{L^\infty(\Omega)} \leq \varepsilon, \quad \#\mathcal{N}_{u,\varepsilon} \lesssim B d_y \left( \frac{e^{L\hat{T}}}{\varepsilon} \right)^{(m+1+\beta)} \left| \log_2 \frac{e^{L\hat{T}}}{\varepsilon} \right|^2,
\]

where \( B \) depends on \( m, L, \alpha, \max\{1, \|u_0\|, \|f\| \} \) with \( \|u_0\| := \|u_0\|_{\mathcal{A}^{\gamma,m}}, \|f\| := \|f\|_{L^\infty(\hat{T}; \mathcal{A}^{\gamma,m})} \).

Moreover, for \( N \in \mathbb{N} \) there exists a stable DNN \( \mathcal{N}_\varepsilon \) with \( \#\mathcal{N}_\varepsilon \leq N \) such that

\[
\|u - \mathcal{N}_\varepsilon\|_{L^\infty(\Omega)} \lesssim e^{L\hat{T}} \left( d_y B \right)^{\frac{1}{m+1+\beta}} N^{-\frac{1}{m+1+\beta}} \left| \log_2 \frac{N}{B} \right|^{\frac{2}{m+1+\beta}},
\]

with \( \|\mathcal{N}_\varepsilon\|_{\mathcal{N},m} \lesssim \max\{1, \|u_0\|, \|f\|\} e^{L\hat{T}}. \) Thus, for \( \hat{\gamma} \) defined in \((4.1\theta)\), we have \( u \in C(\hat{T}; \mathcal{A}^{\hat{\gamma},m}) \cap \text{Lip}_1(\hat{T}; C(\mathbb{R}^m \times \mathcal{Y})). \)
Remark 4.8 Although irrespective of the CoD the rate \( \{4.20\} \) becomes arbitrarily bad when the algebraic order \( \alpha \) gets small below the space-time dimension \( m + 1 \). The best value \( \beta = 1 \) for \( \alpha \geq m + 1 \), as opposed to a value tending to zero when \( \alpha \) grows as in \( \{3.7\} \), is due to the additional time-integration on the source field \( f \). If one replaces the algebraic order \( \gamma(r) \sim r^\alpha \) in \( \{4.14\} \) by an exponential growth order \( \gamma(r) \sim e^{\alpha r} \) one can show that

\[
\|u - N_{u,\varepsilon}\|_{L^\infty(\Omega)} \leq \varepsilon, \quad \#N_{u,\varepsilon} \lesssim B_d \left( \frac{e^{\hat{T}L}}{\varepsilon} \right)^{(m+2)} \left| \log_2 \frac{e^{\hat{T}L}}{\varepsilon} \right|^2,
\]

with the same dependencies of \( B \) on problem parameters. Finally, we could have replaced \( f \in \text{Lip}_1(\hat{I}; C(\mathbb{R}^m \times \mathcal{Y})) \) by an assumption like \( \{3.6\} \).

5 Comments and Outlook

The common trait of the above results is a uniform approximation rate for the characteristic field close to \( \varepsilon^{-\frac{1}{m+1}} \) (the closer the stronger the approximability order of the convection field) which is the rate one can expect for a ball in \( \text{Lip}_1(\hat{I} \times D; \mathbb{R}^m) \). For the solutions themselves it seems that one cannot quite benefit from increasing algebraic growth orders beyond \( \alpha = m + 1 \). Since again the range of possible solutions \( u \) is dense in a Lipschitz ball of \( \text{Lip}_1(\hat{I} \times D) \) the obtained rate in Theorem \( \{4.7\} \) seems to be close to optimal. Moreover, whenever the problem data have some compositional dimension-sparsity, in all scenarios the constructed approximations avoid the CoD. In general, emphasis has been on weak dependence on \( d_y \) not on high order rates.

We conclude with indicating some ramifications of the preceding findings whose detailed treatment is postponed to forthcoming work. Let \( \mathcal{M}(a, \mathcal{Y}) \) denote the set of characteristic fields \( z(\cdot, \cdot; y) \) obtained when \( y \) traverses \( \mathcal{Y} \) for a fixed given convection field \( a \), while the solution manifold \( \mathcal{M}(a, u_0, f_0, \mathcal{Y}) \) is comprised of all solutions to \( \{3.1\} \) for fixed data \( a, u_0, f \). To capture stability with respect to those data as well, let \( \mathfrak{A} \) denote the set of all convection fields with fixed bounds for \( L, L_t, A, \|a\|_{L^\infty(\hat{I}; A^{-\gamma})} \). Likewise let \( \mathfrak{Y} \) denote the set of all \((u_0, f)\) with \( \|u_0\|_{A^{\gamma,m}}, \|f\|_{L^\infty(\hat{I}; A^{\gamma,m})} \leq M \). Obviously, \( \mathfrak{A}, \mathfrak{Y} \) are compact in \( C(\Omega) \). For the Lipschitz-regularizer \( \mathcal{R} \) from \( \{2.17\} \) the preceding results say that all sets

\[
\mathcal{M}(a, \mathcal{Y}), \quad \mathcal{M}(a, u_0, f, \mathcal{Y}), \quad \mathcal{M}(\mathfrak{A}, \mathcal{Y}) := \bigcup_{a \in \mathfrak{A}} \mathcal{M}(a, \mathcal{Y}),
\]

\[
\mathcal{M}(\mathfrak{A} \times \mathfrak{Y}, \mathcal{Y}) := \bigcup_{(a,u_0,f) \in \mathfrak{A} \times \mathfrak{Y}} \mathcal{M}(a, u_0, f, \mathcal{Y}), \tag{5.1}
\]

are contained in bounded balls of spaces of the type \( L^\infty(\hat{I}; A^{\gamma,a}) \cap \text{Lip}_1(\Omega) \) for some growth function \( \gamma \).

A common way to characterize the complexity of these collections is to determine their metric entropy or suitable versions of widths, among those so-called (nonlinear) manifold widths, introduced in \( \{12\} \). More precisely, for a compact set \( \mathcal{K} \) in a Banach space \( X \) they are defined as

\[
\delta_n(\mathcal{K})_X := \inf_{D, E \in \mathcal{K}} \sup_{v \in X} \|v - D(E(v))\|_X, \tag{5.2}
\]
where the infinimum is taken over all encoder-decoder pairs \( E: \mathcal{K} \to \mathbb{R}^n \), \( D: \mathbb{R}^n \to X \), that are at least continuous. Denoting by \( \theta_N. \in \mathbb{R}^N \) the collection of weights defining the respective DNN approximations \( N_{N,z}, N_{N,u} \) in Theorems 4.2, 4.6, 4.7 respectively, the functions
\[
D_N(t, x; \theta_N(a; y)) := N_{N,z}(t, x; y; \theta_{N,z}), \quad D_N(t, x; \theta_N(a, y, u_0, f)) := N_{N,u}(t, x; y; \theta_{N,u}(u_0; f)),
\]
are valid candidates for encoder-decoder pairs \( D_N \circ E_N \), where \( E_N(a, y) = \theta_N(a; y) \in \mathbb{R}^N \), \( E_N(a, y, u_0, f) = \theta_N(a, y, u_0, f) \in \mathbb{R}^N \), are the mappings that take \( z(\cdot, \cdot; y) \), respectively \( u(\cdot, \cdot; y) \) into \( \theta_N(a; y), \theta_N(a, y, u_0, f) \). Confining the discussion to \( a \) according to (3.7), for \( K \in \{ M(a, Y), M(A, Y) \} \) the continuity of \( E_N, D_N \) can be established based on the presented results. In fact, continuity in \( y \) follows from the constructive proofs which is all that is needed for fixed \( a \in A \). As a next step, continuity in \( a \in A \) follows from the continuity of the construction of the implanted Lipschitz stable networks from Proposition 2.2 as can be seen by inspecting the proof in Appendix A. To extend these arguments to the remaining sets in (5.1), one yet has to establish the existence of continuous metric (or near metric) selections on the level of dimension sparse compositional approximations prior to implanting Lipschitz stable DNNs. In particular, this would yield bounds for the manifold widths of compact sets of the type (2.26).

Knowing the manifold-widths does not allow one to infer directly on the entropy numbers of the sets in (5.1) (and hence on the number of bits needed to encode the centers of respective \( \varepsilon \)-covers). For a strengthened version of manifold widths, so called stable widths, introduced in [6], a version of Carl’s inequality is known which asserts that an algebraic order of stable widths implies the corresponding algebraic order of the entropy numbers. These stable widths require both factors \( E_N, D_N \) to be Lipschitz continuous. For fixed \( a, u_0, f, \) the above findings assert (uniform) Lipschitz continuity of the compositions \( D_N \circ E_N \) (for \( a \) of the type (3.7)). It is known that DNNs are Lipschitz continuous with respect to the weights under size constraints on the weights, see e.g. [31]. In general corresponding Lipschitz constants are expected to be very large which impedes an inference from approximation rates to entropy numbers. This gives rise to the notion of Lipschitz widths studied in [31]. There, among other things, bounds on entropy numbers are derived from DNN approximation rates which are (necessarily) somewhat weaker than those in Carl’s inequality, see [31, § 6.2]. Since they are derived under specific architecture constraints (either widths or depths stay bounded) they do not apply directly to the scenarios discussed here. Specifying (and perhaps refining) such results to the current situation would be interesting as they may shed light on how the entropy numbers of the solution manifolds in (5.1) relate to those of the accommodating balls of type (2.26). Finally, it would be interesting to quantify the difference between DNN approximation spaces and compositional approximation spaces.

In a different direction, in principle, the framework allows us to treat even less regular data leading to solutions that are no longer Lipschitz continuous. This may require weaker regularizations than (2.17) or refined notions of approximation classes that allow gradually increasing Lipschitz constants in compositional approximations, as indicated in Remarks 2.8, 2.12. Remark 2.16 already indicates a wider scope of applications. For instance, it would
be interesting to apply the above concepts to nonlinear conservation laws by exploiting their equivalent kinetic formulations as linear parametric transport equations, see [24]. An obvious obstacle here is that the right hand sides are measure-valued. However, solutions do satisfy linear transport equations with zero right hand side on regions separated by shocks. Alternatively, one may consider constructing compositional approximations generated through the fixed-point iterations considered in [35]. Splitting methods for more involved kinetic models may serve as another starting point for generating compositional approximations. Finally, the above concepts apply as well to high-dimensional transport equations and solution manifolds induced by source terms and initial conditions. Aside from their role in Fokker-Planck equations, the correspondence between nonlinear high-dimensional dynamical systems and linear transport PDEs opens another interesting perspective.

Finally, one may consider the case of smooth data for which one could expect better rates. However, in the end one may have to resort to training concepts, typically based on point samples to determine DNN approximations, perhaps in combination with pre-structured architectures suggested by the constructive proofs. It has been shown, however, in [19] that there is no hope then to realize higher convergence orders.

6 Proofs for § 4

This section consists on two major parts. In § 6.1 we collect several technical preliminaries that will find repeated use in subsequent sections § 6.2 – § 6.5 containing the actual the proofs the above theorems. Moreover, some of the elementary technical arguments, needed in these sections, are exported to an Appendix for completeness. The key idea pursued in § 6.2 – § 6.5 is to generate first dimension-sparse compositional approximations to the characteristic fields by combining the semi-group property (3.20) with their fixed point property (3.21). Note that the lengths of the underlying characteristic segments - macro time steps, so to speak - depends only on problem parameters but not on the target accuracy which is controlled by the number of fixed point iterations. It also shows that a straightforward discretization based on the ODEs (3.19) would provide less favorable complexity estimates.

6.1 Some Technical Prerequisites

6.1.1 “Inverting” growth functions

Remark 6.1 Given $g \in X$, suppose we have found for each $\varepsilon > 0$ an approximation $g_\varepsilon$, depending on at most $N_\varepsilon$ degrees of freedom, that satisfies $\|g - g_{N_\varepsilon}\|_X \leq \varepsilon$. If $N_\varepsilon \approx \phi(Q/\varepsilon)$ for some strictly increasing function $\phi$ of at most algebraic growth, then one has

$$\|g - g_N\|_X \lesssim Q\gamma(N)^{-1}, \quad N \in \mathbb{N},$$

(6.1)

where $\gamma(r)$ is any growth function satisfying

$$\gamma(\phi(s)) \approx s.$$  (6.2)

We often briefly write then $\gamma \approx \phi^{-1}$. This will be repeatedly used as follows: Suppose the $g_N$ in (6.1) belong to $C_{N,s}$ and $\|g_N\|_{N,s} \leq Q$ for all $N \in \mathbb{N}$. Then $g \in A^{\gamma,s}$ with $\|g\|_{A^{\gamma,s}} \lesssim Q$. 

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To see the last conclusion, just note that $\gamma(N)\left\{\|g - g_N\|_X + \gamma(N)^{-1}\|g_N\|_{N,s,Lip}\right\} \lesssim 2Q$. □

Appropriate “near-inverses” $\gamma$ will be needed for growth functions $\phi$ of the following form.

**Lemma 6.1** Assume that for positive $b_1, b_2, \zeta$ and real $\beta$

$$\phi(s) = b_1 s^c \left| \log_2 b_2 s \right|^\beta, \quad s \geq s_0 > 0. \quad (6.3)$$

Then

$$\phi^{-1}(r) \approx b_1^{-1/c} \zeta^{\beta/c} r^{1/\zeta} \left| \log_2 (b_2^c r/b_1) \right|^{-\frac{\beta}{\zeta}}, \quad r \geq r_0 > 0. \quad (6.4)$$

**Proof:** Making the ansatz $\phi^{-1}(r) \approx Fr^{\frac{1}{\zeta}} \left| \log_2 (Qr) \right|^\theta$, we have

$$s = \phi^{-1}(\gamma(s)) \approx F \left( b_1 s^c \left| \log_2 (b_2 s) \right|^\beta \right) \left| \log_2 (b_2 s) \right|^\beta \left| \log_2 (Qb_2 s^c) \right|^{\beta/\zeta} \left( \log_2 (Qb_2 s^c) \right)^{-\beta/\zeta} \left| \log_2 (Qb_2 s^c) \right| \left( \log_2 (Qb_2 s^c) \right)^{-\beta/\zeta}. \quad (6.5)$$

Equating coefficients yields $F = b_1^{-1/c} \zeta^{\beta/c}$, $Q = b_2^c b_1$, $\theta = -\frac{\beta}{\zeta}$, which confirms the claim. □

Note that in the above situation the proportionality constants in (6.2) tend to one when the argument increases. For our purposes uniformly bounded proportionality constants suffice so that in later applications we can drop the constant $\zeta^{\beta/c}$ in (6.4).

### 6.1.2 Fixed-point iterations

In what follows we denote by $I := [t, \bar{t}]$ a fixed time interval whose length $|I| := \bar{t} - t$ depends on $L$. In what follows we fix the “macro-time-step” $|I|$ so that

$$|I| \|a\| = \frac{1}{2}. \quad (6.6)$$

For a given time horizon $\hat{T}$ one then needs $K := \left\lceil \hat{T}/|I| \right\rceil$ such steps and we assume for convenience that $K = \hat{T}/|I|$ is already an integer. In addition we denote by

$$\Omega(I) := I \times \mathbb{R}^m \times \mathcal{Y} \subset \Omega$$

the spatio-parametric time-slab determined by $I$.

To find approximate compositions we recall the fixed-point relation (3.21) and consider the corresponding mapping $\Phi_{x,I} : L_\infty(\Omega(I); \mathbb{R}^m) \to L_\infty(\Omega(I); \mathbb{R}^m)$, defined by,

$$\Phi_{x,I}(t, \bar{z}; y) := x + \int_{\bar{t}}^{t} a(s, \bar{z}(s), y)ds, \quad t \in I = [t, \bar{t}]. \quad (6.7)$$
A natural strategy is to approximate the fixed point of (3.21) by iterates of the mapping \( \Phi_{x,I}(\cdot; t, y) \). In this case the arguments \( x, \bar{z} \) sometimes depend on each other. In fact, a natural initialization would be the constant-in-time function

\[
\bar{z}_x(s) = x, \quad s \in I,
\]

i.e., the initial value \( x \) is frozen in time throughout \( I \). Then, we always use the notational convention

\[
\Phi_{x,I}^k(t, \bar{z}; y) := \Phi_{x,I}(t, \Phi_{x,I}^{k-1}(\cdot, \bar{z}; y); y), \quad \bar{z} \in L_\infty(I; \mathbb{R}^m).
\]

Condition (6.6) and \( L \leq \|a\| \) say that \( \Phi_{x,I} \) is a contraction in \( \bar{z} \) since

\[
|\Phi_{x,I}(t; \bar{z}; y) - \Phi_{x,I}(t; \bar{z}'; y)| \leq \int_I |a(s; \bar{z}(s); y) - a(s; \bar{z}'(s); y)| ds \leq (t - t) L \|\bar{z} - \bar{z}'\|_{L_\infty(I; \mathbb{R}^m)} \\
\leq \frac{1}{2} \|\bar{z} - \bar{z}'\|_{L_\infty(I; \mathbb{R}^m)}.
\]

Since by (3.3), \( |z(t; x; y) - x| = \left| \int_I a(s, z(s; x); y) ds \right| \leq (t - t) A \leq A|I| \leq \frac{1}{2} \), this implies

\[
|z(t; x; y) - \Phi_{x,I}^k(t; \bar{z}; y)| = |\Phi_{x,I}(t; \bar{z}; y) - \Phi_{x,I}(t; \Phi_{x,I}^{k-1}(\cdot; \bar{z}; y))| \\
\leq 2^{-k} \|\bar{z}; \bar{z}_x(s)\|_{L_\infty(I; \mathbb{R}^m)} \leq 2^{-k} A|I| \leq 2^{-k-1},
\]

where we have used \( A \leq L \) (see (3.16) and (6.6)). Hence, by (6.6), it takes roughly \( |\log_2 \eta| \) steps to achieve accuracy \( \eta \)

\[
|z(t; x; y) - \Phi_{x,I}^\mu(t; \bar{z}; y)| \leq \eta, \quad (t, x, y) \in \Omega(I), \quad \mu = \mu(\eta) = \left\lfloor \log_2(2\eta)^{-1} \right\rfloor.
\]

In view of (3.20), it is natural to concatenate next iterates \( \Phi_{x,I}^\mu \) in time for successive time intervals \( I \). To that end, consider an equally spaced (for simplicity) partition

\[
[0, \hat{T}] = \bigcup_{k=1}^K [t_{k-1}, t_k], \quad t_k := \frac{k\hat{T}}{K},
\]

where \( K \) is chosen in compliance with (6.6). Along with the sequence of intervals consider the vector of tolerances with corresponding sufficient iteration numbers

\[
\eta^k = (\eta_1, \ldots, \eta_k) \in \mathbb{R}_+^k, \quad \mu_k := \mu(\eta_k), \quad k = 1, \ldots, K.
\]

Then define, for \( j < k, w \in \mathbb{R}^m \)

\[
\Psi_{[k,j]}(t, w; y) := \Phi_{w_{k-1,j}}^\mu(t; \bar{z}_{w_{k-1,j}}; y), \quad w_{k-1,j} := \Psi_{[k-1,j]}(T_{k-1}; w; y), \quad t \in I_k' \\
\Psi_{[j+1,j]}(t; w; y) := \Phi_{w_{j+1,j}}^\mu(t; \bar{z}_w; y),
\]

(6.14)
i.e., $\mu_k$ iterates of $\Phi_{x,I}$ are applied to the result of a $\mu_{k-1}$-fold application of $\Phi_{x,I}$ evaluated at the last time-junction $t_{k-1}$.

Specifically,

$$\Psi_{\alpha_k}(t, x; y) := \Psi_{[k,0]}(t, x; y) \quad (6.15)$$

is a natural candidate for approximating $z(\cdot, x; y)$ on $I_k$.

To estimate $|z - \Psi_{\alpha_k}|$ on $I_k$ we invoke Remark 2.1. Viewing $\Psi_{\alpha_k}$ as a perturbation of the characteristic field, we need bounds for the Lipschitz constants of the exact characteristics $z(t; w; y)$. Recall that under the above assumptions on the convection field $a$, it follows from a classical Gronwall inequality that one has

$$\|z(\cdot, x; y) - z(\cdot, \bar{x}; y)\|_{L^\infty(I; \mathbb{R}^m)} \leq e^{L|I|}|x - \bar{x}| \leq e^{1/2}|x - \bar{x}|, \quad (6.16)$$

so that in terms of Remark 2.1 we have $L_{[k,j+1]} \leq e^{L(t_{j+1} - j)} = e^{L(t_{k} - j)} = e^{(k-j)/2}$. Thus,

$$|z(t; x; y) - \Psi_{\alpha_k}(t; x; y)| \leq \eta_k + \sum_{j=1}^{k-1} \eta_j e^{(k-j)/2}. \quad (6.17)$$

It remains to choose the intermediate tolerances $\eta_j$. The simplest option is to take them all equal

$$\eta_j = \eta(\varepsilon) := (e^{1/2} - 1)\varepsilon e^{-K/2}, \quad j = 1, \ldots, K, \quad (6.18)$$

which yields

$$\|z(\cdot; x; y) - \Psi_{\alpha_k}(t; x; y)\|_{L^\infty(I_k; \mathbb{R}^m)} \leq \varepsilon e^{(k-K)/2} \leq \varepsilon, \quad k = 1, \ldots, K. \quad (6.19)$$

In summary, we have

$$\|z - Z_\varepsilon\|_{L^\infty(\Omega; \mathbb{R}^m)} \leq \varepsilon, \quad \text{where} \quad Z_\varepsilon(t; x; y) := \sum_{k=1}^{K} \chi_k(t)\Psi_{\alpha_k}(t; x; y). \quad (6.20)$$

### 6.1.3 Lipschitz bounds

The mappings $Z_\varepsilon$ from (6.20) are still global operators. To analyze their approximation by pointwise compositions via Remark 2.1, we need to bound the Lipschitz constants of partial compositions.

**Lemma 6.2** Under the above assumptions one has for $k \in \mathbb{N}$, $\bar{z}, \bar{z}' \in L^\infty(I; \mathbb{R}^m)$, $y, y' \in Y$

$$|\Phi_{x,I}^k(t, \bar{z}; y) - \Phi_{x,I}^k(t, \bar{z}'; y')| \leq e^{1/2}|x - x'| + \frac{2^{-k}}{k!} \max\{\|\bar{z} - \bar{z}'\|_{L^\infty(I; \mathbb{R}^m)}, |y - y'|\}. \quad (6.21)$$

In particular, one has for all $(t, x, y), (t', x', y') \in \Omega(I)$

$$|\Phi_{x,I}^k(t, \tilde{z}; y) - \Phi_{x,I}^k(t', \tilde{z}'; y')| \leq A|t - t'| + \max\{|x - x'|, |y - y'|\} e^{1/2}, \quad k \in \mathbb{N}. \quad (6.22)$$
\textbf{Proof:} By our assumptions (3.3) on the convection field we conclude that for \( t \in I \)
\[
|\Phi_{x,I}(t, \tilde{z}; y) - \Phi_{x,I}(t, \tilde{z}'; y')| \leq (t - \frac{t}{2}) L \max\{\|\tilde{z} - \tilde{z}'\|_{L^\infty(I; \mathbb{R}^m)}, |y - y'|\},
\]
so that \( |\Phi_{x,I}(t, \tilde{z}; y) - \Phi_{x',I}(t, \tilde{z}'; y')| \leq |x - x'| + (t - \frac{t}{2}) L \max\{\|\tilde{z} - \tilde{z}'\|_{L^\infty(I; \mathbb{R}^m)}, |y - y'|\} \). Hence
\[
\Phi_{x,I}(t; \Phi_{x,I}(\cdot; \tilde{z}; y); y) - \Phi_{x',I}(t; \Phi_{x',I}(\cdot; \tilde{z}'; y'); y')
= x + \int_{\frac{t}{2}}^{t} a\left(s, \left(x + \int_{\frac{t}{2}}^{s} a(s', \tilde{z}(s'); y) ds'\right)\right) ds - \left\{x' + \int_{\frac{t}{2}}^{t} a\left(s, \left(x' + \int_{\frac{t}{2}}^{s} a(s', \tilde{z}'(s'); y') ds'\right)\right) ds\right\}.
\]
This yields
\[
|\Phi_{x,I}(t; \Phi_{x,I}(\cdot; \tilde{z}; y); y) - \Phi_{x',I}(t; \Phi_{x',I}(\cdot; \tilde{z}'; y'); y')|
\leq |x - x'| + \int_{\frac{t}{2}}^{t} |a\left(s, \left(x + \int_{\frac{t}{2}}^{s} a(s', \tilde{z}(s'); y) ds'\right)\right) - a\left(s, \left(x' + \int_{\frac{t}{2}}^{s} a(s', \tilde{z}'(s'); y) ds'\right)\right)| ds
\leq |x - x'| + \int_{\frac{t}{2}}^{t} L |x - x'| + L \int_{\frac{t}{2}}^{t} |a(s', \tilde{z}(s'); y) - a(s', \tilde{z}'(s'); y')| ds' ds
\leq (1 + (t - \frac{t}{2}) L) |x - x'| + L^2 \int_{\frac{t}{2}}^{t} \int_{\frac{t}{2}}^{t} \max\{|\tilde{z}(s') - \tilde{z}'(s')|, |y - y'|\} ds' ds
\leq (1 + (t - \frac{t}{2}) L) |x - x'| + \frac{(t - \frac{t}{2}) L^2}{2} \max\{\|\tilde{z} - \tilde{z}'\|_{L^\infty(I; \mathbb{R}^m)}, |y - y'|\}.
\]
One then easily verifies inductively that
\[
|\Phi_{x,I}^k(t, \tilde{z}; y) - \Phi_{x',I}^k(t, \tilde{z}'; y')| \leq \sum_{\nu=0}^{k-1} \frac{(t - \frac{t}{2}) L^\nu}{\nu!} |x - x'| + \frac{(L(t - \frac{t}{2}))^k}{k!} \max\{\|\tilde{z} - \tilde{z}'\|_{L^\infty(I; \mathbb{R}^m)}, |y - y'|\},
\]
which implies (6.21). Specifically, when \( \tilde{z}(s) = \tilde{x}_x(s) = x \) for \( s \in I \), (6.21) gives
\[
|\Phi_{x,I}^k(t, \tilde{x}; y) - \Phi_{x',I}^k(t, \tilde{x}'(s); y')| \leq \sum_{\nu=0}^{k} \frac{(t - \frac{t}{2}) L^\nu}{\nu!} \max\{|x - x'|, |y - y'|\}
\leq \max\{|x - x'|, |y - y'|\} e^{L|I|}.
\]
from which (6.22) follows for \( t = t' \) since \( L|I| \leq 1/2 \).
Since (for \( t' < t \)), keeping (333) in mind \( |\Phi_{x,t}(t, \bar{z}; y) - \Phi_{x,t}(t', \bar{z}; y)| \leq \int_{t'}^t |a(s, \bar{z}(s); y)| ds \leq A|t - t'| \), we have

\[
|\Phi_{x,t}^k(t, \bar{z}; y) - \Phi_{x,t}^k(t', \bar{z}; y)| \leq \int_{t'}^t |a(s, \Phi_{x,t}^{k-1}(s, \bar{z}; y); y)| ds \leq A|t - t'|, \quad t, t' \in I, \quad (6.26)
\]

proving (6.22) and hence the assertion.

To approximate the \( \Psi_{[k,j]} \) by pointwise compositions we need the following bounds

**Corollary 6.2** For \((t, x, y), (t', x', y') \in \Omega(I_k)\), one has

\[
|\Psi_{[k,j]}(t, w; y) - \Psi_{[k,j]}(t', w'; y')| \leq |t - t'| + e^{(k-j)/2} \max\{|w - w'|, |y - y'|\}. \quad (6.27)
\]

Moreover, for \( Z_\varepsilon \) defined by (6.63), one has

\[
|Z_\varepsilon(t, x, y) - Z_\varepsilon(t', x', y')| \leq A|t - t'| + \max\{|x - x'|, |y - y'|\} e^{\|a\|/T}. \quad (6.28)
\]

**Proof:** Since

\[
\Psi_{[k,j]}(t, w; y) - \Psi_{[k,j]}(t', w'; y') = \Phi_{w_{k-1,j}, t_k}(t, \bar{z}_{w_{k-1,j}}; y) - \Phi_{w_{k-1,j}, t_k}(t', \bar{z}_{w_{k-1,j}}; y'),
\]

where \( w_{k-1,j} = \Psi_{[k-1,j]}(t_{k-1}; w; y) \), we infer from (6.26) that

\[
|\Psi_{[k,j]}(t, w; y) - \Psi_{[k,j]}(t', w'; y')| \leq A|t - t'|
\]

\[
+ \max \{ |\Psi_{[k-1,j]}(t_{k-1}; w; y) - \Psi_{[k-1,j]}(t_{k-1}; w'; y')|, |y - y'| \} e^{1/2}.
\]

Again one concludes inductively that

\[
e^{1/2} \max\{|\Psi_{[k-1,j]}(t_{k-1}; w; y) - \Psi_{[k-1,j]}(t_{k-1}; w'; y')|, |y - y'| \}
\]

\[
\leq e^{1/2} \max \left\{ e^{1/2} \max \left\{ |\Psi_{[k-2,j]}(t_{k-2}; w; y) - \Psi_{[k-2,j]}(t_{k-2}; w'; y')|, |y - y'| \right\}, |y - y'| \right\}
\]

\[
\leq e^{(k-j-1)/2} \left\{ |\Psi_{[j+1,j]}(t_{j+1}; w; y) - \Psi_{[j+1,j]}(t_{j+1}; w'; y')|, |y - y'| \right\},
\]

and since

\[
|\Psi_{[j+1,j]}(t_{j+1}; w; y) - \Psi_{[j+1,j]}(t_{j+1}; w'; y')|
\]

\[
= |\Phi_{w_{l+1,j}, t_{j+1}}(t_{j+1}; t_j, \bar{z}_w; y) - \Phi_{w_{l+1,j}, t_{j+1}}(t_{j+1}; t_j, \bar{z}_w; y')| \leq e^{1/2} \max\{|w - w'|, |y - y'| \},
\]

(6.24) follows.

Concerning (6.28), recall from (6.6) that \( k \leq K \leq 2\|a\|/T \). Then, (6.28) follows for any \( t, t' \in I_k, \ k \leq K \), from (6.27). The general case is again obtained by using the triangle inequality, inserting intermediate time-segments. This completes the proof.

\[\square\]
6.1.4 Pointwise compositions

We wish to pass from compositions of global operators (integral operators) to compositions of pointwise mappings. Consider an equidistant partition of $I = [\overline{t}, \overline{t}]$ with breakpoints $\tau_i = \tau_i(\overline{I}, q) := \frac{\overline{t} + i|\overline{I}|}{q}$, for some $q \in \mathbb{N}$. Let $\xi_i$ denote the respective midpoints of the intervals $[\tau_{i-1}, \tau_i] := J_i = J_i(\overline{I}, q) \subset \overline{I}$, $i = 1, \ldots, q$ and define

$$\rho_{i,I}(t) = \rho_i(t) := \int_{\overline{\xi_i}}^t \chi_{J_i}(s) \, ds. \quad (6.29)$$

The following simple facts will be used frequently.

**Lemma 6.3** Adhering to the above notation, the following holds:

(a) For $t \in \overline{I}_k$

$$\sum_{i=1}^{q} \rho_i(t) = \sum_{i=1}^{k-1} |J_i| + t - \tau_{k-1} \leq \frac{k|I|}{q}. \quad (6.30)$$

and for $t, t' \in \overline{I}$

$$\sum_{i=1}^{q} |\rho_i(t) - \rho_i(t')| \leq |t - t'|. \quad (6.31)$$

(b) Assume that $g \in L_\infty(\overline{I})$ and let $g_{J_i} := |J_i|^{-1} \int_{J_i} g(s) \, ds$, $i = 1, \ldots, q$. Then,

$$\left| \int_{\overline{\xi_i}}^t g(s) \, ds - \sum_{i=1}^{q} \rho_i(t) g_{J_i} \right| \leq \frac{|I||g|_{L_\infty(\overline{I})}}{2q}, \quad t \in \overline{I}. \quad (6.32)$$

(c) Assume that $g \in \text{Lip}_1(\overline{I})$ with Lipschitz constant $L'$. Then

$$\left| \int_{\overline{\xi_i}}^t g(s) \, ds - \sum_{i=1}^{q} \rho_i(t) g(\xi_i) \right| \leq \frac{|I|^2 L'}{2q}. \quad (6.33)$$

The proof is elementary and given for completeness in Appendix B.

We approximate now $\Phi_{\overline{x},I}$ in a first step by the piecewise affine-in-time function

$$P_{\overline{x},I,q}(t; \overline{z}; y) := x + \sum_{i=1}^{q} \rho_i(t) \overline{a}_i(\overline{z}(\xi_i); y), \quad (6.34)$$

where, depending on our hypothesis on $a$ we set

$$\overline{a}_i(\overline{z}; y) := \begin{cases} a(\xi_i, \overline{z}; y), & \text{(A1) in case } (3.4) \text{ holds,} \\ a_{J_i}(\overline{z}; y) := |J_i|^{-1} \int_{J_i} a(s; \overline{z}; y) \, ds, & \text{(A2) in case } (3.6) \text{ holds.} \end{cases} \quad (6.35)$$
We record for later use that, by (6.30), the following analog to (6.10) holds

\[ |P_{x,I,q}(t; w; y) - P_{x,I,q}(t; \bar{w}; y)| \leq \sum_{i=1}^{q} \rho_i(t) |\bar{a}_i(w(\xi_i); y) - \bar{a}_i(\bar{w}(\xi_i); y)| \]

\[ \leq L \|w - \bar{w}\|_{L_\infty(I)} \frac{i(t)}{q} \leq \frac{1}{2} \|w - \bar{w}\|_{L_\infty(I)}, \quad (6.36) \]

where we have used (6.3), \( L \leq \|a\| \), and the fact that for either version of \( \bar{a}_i \) Lipschitz constants with respect to \( \mathbb{R}^m \times Y \) are preserved.

Next we estimate the deviation between \( \Phi_{x,I} \) and \( P_{x,I,q} \).

**Lemma 6.4** Assume that (3.3) holds and that \( \bar{z} \in L_\infty(I; \mathbb{R}^m) \) satisfies

\[ \| \bar{z} - \bar{z}(\xi) \|_{L_\infty(J; \mathbb{R}^m)} \leq \frac{A|I|}{2q}. \quad (6.37) \]

Then one has

\[ \left| \Phi_{x,I}(t; \bar{z}; y) - P_{x,I,q}(t; \bar{z}; y) \right| \leq \left\{ \begin{array}{ll} \frac{(1+|A|L)|I|^2}{2q}, & \text{when } \bar{a}_i = a(\xi_i), \\ \frac{(L|I|+1)A|I|}{2q}, & \text{when } \bar{a}_i = a_{J_i}. \end{array} \right\} \leq \frac{A|I|}{q} \leq \frac{1}{2q}. \quad (6.38) \]

The second but last inequality is relevant when \( L \gg A \) so that \( |I| \) is correspondingly small.

**Proof:** Let \( \bar{z}(\xi) \) denote the piecewise constant \( \bar{z}(\xi)|_{J_i} = \bar{z}(\xi_i) \) to obtain from (3.3) and (3.16)

\[ \left| \Phi_{x,I}(t; \bar{z}; y) - P_{x,I,q}(t; \bar{z}; y) \right| \leq \left| \Phi_{x,I}(t; \bar{z}; y) - \Phi_{x,I}(t, \bar{z}(\xi); y) \right| + \left| \Phi_{x,I}(t, \bar{z}(\xi); y) - P_{x,I,q}(t, \bar{z}; y) \right| \]

\[ \leq |I| L \|\bar{z}(\xi) - \bar{z}\|_{L_\infty(I; \mathbb{R}^m)} + \left| \Phi_{x,I}(t, \bar{z}(\xi); y) - P_{x,I,q}(t, \bar{z}; y) \right| \]

\[ \leq \frac{LA|I|^2}{2q} + \left| \Phi_{x,I}(t, \bar{z}(\xi); y) - P_{x,I,q}(t, \bar{z}; y) \right|, \quad (6.39) \]

where we have used (6.37). In case (A2), i.e., \( \bar{a}_i(\cdot; \cdot) = a_{J_i}(\cdot; \cdot) \), (6.32) yields, in view of (3.16),

\[ \left| \Phi_{x,I}(t, \bar{z}(\xi); y) - P_{x,I,q}(t, \bar{z}; y) \right| \leq \frac{A|I|}{2q}. \quad (6.40) \]

Thus, in this case

\[ \left| \Phi_{x,I}(t, \bar{z}; y) - P_{x,I,q}(t, \bar{z}; y) \right| \leq \frac{(L|I|+1)A|I|}{2q}. \quad (6.41) \]

Now suppose (A1), i.e., \( \bar{a}_i(\cdot; \cdot) = a(\xi_i; \cdot; \cdot) \) under assumption (3.4). Then, we apply Lemma 6.3 (c), to \( g(s) = a(s, \bar{z}; y) \) and, by (3.4) and (3.16), \( (L' \leq L) \), obtain

\[ \left| \Phi_{x,I}(t, \bar{z}(\xi); y) - P_{x,I,q}(t, \bar{z}; y) \right| \leq \frac{L|I|^2}{2q}, \]

which confirms the first inequality. On account of the assumption \( 1 \leq A \leq L \) (see (3.16), (3.17), (6.6)) ensures that the first case is bounded by \( A|I|/(2q) \) while the second case is bounded by \( A|I|/2q \). Again (6.6) concludes the proof. \( \Box \)

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Remark 6.3 The hypothesis (6.37) in Lemma 6.4 is valid in the following cases:

(i) \( \bar{z} = \bar{z}_w \) for some \( w \in \mathbb{R}^m \) on \( I \);

(ii) \( \bar{z} \) results from applying \( \Phi_{x,t} \), i.e., \( \bar{z}(t) = \Phi_{x,t}(t; w; y) \) for some \( w \in L_\infty(I; \mathbb{R}^m), y \in \mathcal{Y} \).

(iii) \( \bar{z}(t) = P_{x,t,q}(t; w; y) \) results from applying \( P_{x,t,q} \) to some \( w, y \) as above.

In fact, in case (i) one has \( \bar{z}_w(s) - \bar{z}_w(\xi_i) = 0 \). In case (ii) one has for \( s \in J_i \)
\[
|\bar{z}(s) - \bar{z}(\xi_i)| = |\Phi_{x,t}(s; w; y) - \Phi_{x,t}(\xi_i; w; y)| \leq \int_{\xi_i}^s |a(s', w(s'); y)|ds' \leq \frac{A|I|}{2q},
\]
where we have again used (3.16). Finally for (iii) we have for \( s \in J_i \), by (6.31) and (3.16),
\[
|\bar{z}(s) - \bar{z}(\xi_i)| = |P_{x,t,q}(s; w; y) - P_{x,t,q}(\xi_i; w; y)| \leq \sum_{k=1}^q |\rho_k(s) - \rho_k(\xi_i)||\bar{a}_k(w(\xi_k); y)| \leq \frac{A|I|}{2q},
\]
which confirms the claim.

Compositional representation of \( P_{x,q} \): Note that \( P_{x,q} \) can be written as a composition
\[
P_{x,q}(t; \bar{z}; y) = (g_{1,q} \circ g_{1,q})(t, x, \bar{z}; y).
\]
In slight abuse of notation we identify a piecewise constant \( \bar{z} \) with the vector \( \bar{z}(\xi) := (\bar{z}(\xi_1), \ldots, \bar{z}(\xi_q)) \in \mathbb{R}^{qm} \) when writing
\[
g_{1,q} : (t, x, \bar{z}; y) \mapsto (\rho_1(t), \ldots, \rho_q(t), x, \bar{a}_1(\bar{z}(\xi_1); y), \ldots, \bar{a}_q(\bar{z}(\xi_q); y)) \in \mathbb{R}^{(q+1)m+q},
\]
and the bi-linear map
\[
g_{2,q} : (r_1, \ldots, r_q, x, w_1, \ldots, w_q) \mapsto x + \sum_{i=1}^q r_iw_i \in \mathbb{R}^m.
\]

6.1.5 Dimension-sparse approximation

The approximation \( P_{x,t,q} \) to \( \Phi_{x,t} \) still involves the functions \( \bar{a}_i(\bar{z}; y) \) which eventually need to be approximated by finitely parametrized expressions. Here we use the structural assumptions on the convection field. In case (A1) from (6.35) \( a \in L_\infty([0, \bar{T}]; A^{\gamma,s}) \) immediately implies that \( \bar{a}_i(\cdot; \cdot) = a(\xi_i; \cdot; \cdot) \) belong to \( A^{\gamma,s} \), uniformly in \( i = 1, \ldots, q, q \in \mathbb{N} \). Hence, for each \( i = 1, \ldots, q, N \in \mathbb{N} \), there is a composition \( \bar{A}_{N,i} \in \mathcal{C}_{N,s} \) such that
\[
\max_{i=1,\ldots,q} |\bar{a}_i(\bar{z}; y) - \bar{A}_{N,i}(\bar{z}; y)| \leq \gamma(N)^{-1}\|a\|_{L_\infty(I; A^{\gamma,s})}, \|\bar{A}_{N,i}\|_{N,s} \leq \|a\|, \tag{6.45}
\]
recall \( \|a\| := \|a\|_{L_\infty(I; A^{\gamma,s})} \). In case (A2), the same conclusion holds, due to (3.6).
Lemma 6.5  We adhere to the definitions \( \tilde{a}_i(\cdot; \cdot) = a(\xi_i; \cdot; \cdot) \) or \( a_i = a_{1,i} \) when (3.4), respectively (3.6), hold and let

\[
A_{x,I,q,N}(t, \bar{z}; y) := x + \sum_{i=1}^{q} \rho_i(t) \tilde{A}_{N,i}(\bar{z}; y). \tag{6.46}
\]

Then, for either version of \( \tilde{a}_i \) one has

\[
|\Phi_{x,I}(t; \bar{z}; y) - A_{x,I,q,N}(t, \bar{z}; y)| \leq |I| \left\{ \frac{A}{q} + \frac{\|a\|}{\gamma(N)} \right\}. \tag{6.47}
\]

In particular, choosing

\[
q = q(\tau) := \left\lceil \frac{2A|I|}{\tau} \right\rceil, \quad N = N(\tau) = \left\lceil \gamma^{-1}(2|I|\|a\|/\tau) \right\rceil, \tag{6.48}
\]

we have

\[
|\Phi_{x,I}(t; \bar{z}; y) - A_{x,I,q,N}(\tau, N)(t, \bar{z}; y)| \leq \tau, \tag{6.49}
\]

In what follows we write briefly \( A_{x,I,\tau} := A_{x,I,q(\tau),N(\tau)} \) with dimensionality vector \( D_{x,I,\tau} \).

Proof: For (A1) it follows from (6.38) and Lemma 6.3, (a), (see also (3.3))

\[
|\Phi_{x,I}(t; \bar{z}; y) - A_{x,I,q,N}(t, \bar{z}; y)| \leq \frac{A|I|}{q} + \sum_{i=1}^{q} \rho_i(t)|\tilde{a}_i(\bar{z}, y) - \tilde{A}_{N,i}(\bar{z}, y)| \leq \frac{A|I|}{q} + \frac{|I|\|a\|}{\gamma(N)} = |I| \left\{ \frac{A}{q} + \frac{\|a\|}{\gamma(N)} \right\}. \tag{6.50}
\]

The remainder of the assertion is an obvious consequence. \( \square \)

Remark 6.4 (a) Suppose that \( D_{i,q,N} \) is the dimensionality vector of \( \tilde{A}_{N,i} \). Then, a corresponding realization of \( A_{x,I,q,N} \) results from parallelization of the \( \tilde{A}_{N,i} \). One easily concludes from Remark 2.7 that the resulting dimensionality vector \( D_{x,I,q,N} \) of \( A_{x,I,q,N} \) is bounded by

\[
\mathcal{H}(D_{x,I,q,N}) \leq q \max_{i=1,\ldots,q} \mathcal{H}(D_{i,q,N}) \leq qN. \tag{6.51}
\]

Thus, by (6.48), one has for an absolute constant (depending only on \( m \))

\[
\mathcal{H}(D_{x,I,\tau}) \leq q(\tau)\gamma^{-1}(\|a\|2|I|/\tau) \leq \frac{2A|I|\gamma^{-1}(2|I|\|a\|/\tau)}{\tau} \leq \frac{A\gamma^{-1}(1/\tau)}{\|a\|/\tau}, \tag{6.52}
\]

where we have used (6.6).

(b) \( A_{x,I,q,N} \) has a compositional representation analogous to (6.42), obtained by replacing \( a(\xi_i; \cdot; \cdot) \) by \( \tilde{A}_{N,i} \). Since by assumption \( s \geq m \), one can see from (6.42), that \( s \)-dimension sparsity of the \( \tilde{A}_{N,i} \) is inherited by the mappings \( A_{x,I,q,N} \) and hence by their compositions.
6.1.6 Lipschitz continuity of pointwise compositions

As a final prerequisite, to eventually control the stability of compositions of $A_{x,I,q,N}$, we need bounds for the Lipschitz constants of such compositions. To that end, suppose that

$$\tilde{A}_{N,i} = (\tilde{A}_{N,i})^{n_i} \circ \cdots \circ (\tilde{A}_{N,i})^1, \quad i = 1, \ldots, q,$$

(6.53)

where, by definition of $\mathcal{C}_{N,s}$, each component $(\tilde{A}_{N,i})^j$, $1 \leq \nu \leq d_j$ depends for $j < n_i$ only on at most $s$ variables or is at most bilinear.

To proceed, recall also that the Lipschitz constants of the factors in $\tilde{A}_{N,i}$ as well as the Lipschitz constants $L_{[n_i,j]}(\tilde{A}_{N,i})$ of the partial compositions $(\tilde{A}_{N,i})^{n_i} \circ \cdots \circ (\tilde{A}_{N,i})^j$ are controlled by

$$\max_{1 \leq i \leq q} \|\tilde{A}_{N,i}\|_{N,Lip} \leq \|a\| := \|a\|_{Lip(I;A^\alpha_{N,i})}.$$

(6.54)

**Lemma 6.6** For any $q \in \mathbb{N}$, $k \in \mathbb{N}$, and $t \in I$, one has

$$|A_{x,I,q,N}^k(t, \tilde{z}; y) - A_{x,I,q,N}^k(t, \tilde{z}'; y')| \leq \frac{(\|a\| |I|)^k}{k!} \max \{|y - y'|, \|\tilde{z} - \tilde{z}'\|_{Lip(I;\mathbb{R}^m)}\},$$

(6.55)

for $x \in D, \tilde{z}(\xi), \tilde{z}'(\xi) \in \mathbb{R}^{mq}$. Similarly, when $\tilde{z} = \tilde{z}_x, \tilde{z}' = \tilde{z}_{x'}$, one has for all $x, x' \in D$,

$$|A_{x,I,q,N}^k(t, \tilde{z}_x; y) - A_{x,I,q,N}^k(t, \tilde{z}_{x'}; y')| \leq \|a\| |t - t'| + \max \{|y - y'|, |x - x'|\} c\|a\| |I|.$$

(6.56)

Finally, the $A_{x,I,q,N}^k$ belong to $\mathcal{C}_{CkqN,s,Lip}$, where $C$ is a fixed constant.

The reasoning is analogous to the proof of Lemma 6.2 based on the smoothing effect of multiple integration, here in terms of multiple summation. The proof is therefore deferred to Appendix B.

6.2 Proof of Theorem 4.1

**Step 1 - construction of an $\varepsilon$-accurate pointwise composition:** Given the $\varepsilon$-accurate approximation of the characteristic field by compositions of global operators $Z_\varepsilon$ from (6.63), we construct now a pointwise compositional counterpart. Specifically, we define approximations $\tilde{Z}_{[k,j]}, \tilde{Z}^k_{[k,j]} = \tilde{Z}^k_{[k,0]}$ to the (global) counterparts $\tilde{Z}^k_{[k,j]}, \tilde{Z}^k_{[k,j]}$ from (6.57), (6.58). We adhere to the meaning of $\mu_k, \eta_k$ from (6.13), and replace $\Phi_{x,I}$ by $A_{x,I,\tau} = A_{x,I,N(\tau),q(\tau)}$. Precisely, let for $j < k, w \in \mathbb{R}^m$, and for a new vector of tolerances

$$\tau = (\tau_1, \ldots, \tau_K), \quad \text{with sections } \tau^k = (\tau_1, \ldots, \tau_k), \quad 1 \leq k \leq K,$$

yet to be chosen. We define for $t \in I_k$

$$\tilde{Z}^k_{[k,j]}(t, w; y) := A_{w_k-I_{k-1};j}^{m_k}(t, k_{k-1}; \tilde{z}_w-k_{k-1}; y), \quad w_{k-1,j} := \tilde{Z}^k_{[k-1;j]}(t_{k-1}; w; y),$$

(6.57)

$$\tilde{Z}^k_{[j+1,j]}(t; w; y) := A_{w_{j+1};j}^{m_{j+1}}(t, t_{j}; \tilde{z}_w; y).$$

We denote as before

$$\tilde{Z}^k_{[k,0]}(t, x; y) := \tilde{Z}^k_{[k,0]}(t, x; y).$$

(6.58)
We choose \( \tau_k = \tau \) all equal so that for \( \eta_k \) given by (6.18)

\[
|\Phi^{\mu_k}_{w,I_k}(t, t_{k-1}; \bar{z}_w; y) - A^{\mu_k}_{w,I_k,\tau_k}(t, t_{k-1}; \bar{z}_w; y)| \leq \eta_k, \quad k \leq K. \tag{6.59}
\]

Since, by Lemma 6.2 (6.21), the Lipschitz constants \( L_{[k,j]} \) of \( k - j \) partial compositions of \( \Phi_{w,I} \) in Remark 2.1 are bounded by \( 2^{-\frac{(k-j)}{(j-k)}} \), we conclude

\[
|\Phi^{\mu_k}_{w,I_k}(t, T_{k-1}; \bar{z}_w; y) - A^{\mu_k}_{w,I_k,\tau_k}(t, T_{k-1}; \bar{z}_w; y)| \leq \tau + \tau \sum_{j=1}^{\frac{\mu_k-1}{\mu_k-j}} 2^{-\frac{j}{\mu_k-j}} \leq \tau e^{1/2}. \tag{6.60}
\]

On account of (6.18), choosing \( \tau = \tau(\varepsilon) \) such that \( \tau e^{1/2} \leq \eta(\varepsilon) \) from (6.18), i.e.,

\[
\tau_k(\varepsilon) = \tau(\varepsilon) = e^{-1/2}\eta(\varepsilon) = \varepsilon e^{-K/2}(1 - e^{-1/2}), \quad k = 1, \ldots, K, \tag{6.61}
\]

yields via the same reasoning as in (6.19)

\[
\|\Psi_{\mu_k}(\varepsilon) - \tilde{\Psi}_{\mu_k}(\varepsilon)\|_{L_{\infty}(\Omega(I_k))} \leq \varepsilon. \tag{6.62}
\]

In summary, we obtain as before

\[
\|z - \tilde{z}_\varepsilon\|_{L_{\infty}(\Omega; \mathbb{R}^m)} \leq 2\varepsilon, \quad \text{where} \quad \tilde{z}_\varepsilon(t; x; y) := \sum_{k=1}^{K} \chi(I_k(t))\tilde{\Psi}_{\mu_k}(\varepsilon)(t; x; y). \tag{6.63}
\]

Step 2 - Complexity of \( \tilde{z}_\varepsilon \): It follows from Remark 2.5 that

\[
\mathcal{N}(\tilde{z}_\varepsilon) \leq \sum_{k=1}^{K} \mu_k(\varepsilon)\mathcal{N}(A_{\mu_k,\tau_k}(\varepsilon)). \tag{6.64}
\]

On account of (6.18) and (6.13) we have (recall \( |I_k| = \frac{\hat{T}}{K} \) and \( K/2 = \frac{\hat{T}}{\|a\|} \) by (6.6))

\[
\mu_k(\varepsilon) = \left| \log_2 \left( \frac{e^{K/2}}{2(1 - e^{K/2})\varepsilon} \right) \right| \approx \log_2 \left( \frac{e^{\|a\|\hat{T}}}{\varepsilon} \right). \tag{6.65}\]

Furthermore, (6.61) in conjunction with Remark 6.4 yields (for the range of \( \gamma \) under consideration, see (2.33))

\[
\mathcal{N}(A_{\mu_k,\tau_k}(\varepsilon)) \approx \frac{A e^{K/2}}{\|a\|} \gamma^{-1} \left( \frac{e^{K/2}}{e^{1/2}\varepsilon} \right) \approx \frac{A e^{\|a\|\hat{T}}}{\|a\|} \gamma^{-1} \left( \frac{e^{\|a\|\hat{T}}}{\varepsilon} \right).
\]

Substituting this into (6.64), yields

\[
\mathcal{N}(\tilde{z}_\varepsilon) \approx K \left| \log_2 \left( \frac{e^{\|a\|\hat{T}}}{\varepsilon} \right) \right| A e^{\|a\|\hat{T}} \gamma^{-1} \left( \frac{e^{\|a\|\hat{T}}}{\varepsilon} \right), \tag{6.66}
\]

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By (2.33), we obtain
\[
\gamma^{-1}(e^{\|a\|\hat{T}_\varepsilon^{-1}}) \sim \begin{cases} 
C_a^{-1/\alpha} e^{\|a\|\hat{T}/\alpha \varepsilon^{-\frac{1}{\alpha}}}, & \text{for } \gamma \sim \text{(alg)}, \\
\frac{1}{\alpha} \ln \left(e^{\|a\|\hat{T}} / C_a \varepsilon\right), & \text{for } \gamma \sim \text{(exp)}. 
\end{cases} \tag{6.67}
\]
We conclude that for \(\gamma \sim \text{(alg)}\), (see (2.32))
\[
\mathfrak{N}(\tilde{Z}_\varepsilon) \approx AC_a^{-1/\alpha} \hat{T} \log_2 \left(\frac{e^{\|a\|\hat{T}}}{\varepsilon}\right) \left(\frac{e^{\|a\|\hat{T}}}{\varepsilon}\right)^{\frac{\alpha+1}{\alpha}} := \phi_{\text{alg}}(e^{\|a\|\hat{T}} / \varepsilon). \tag{6.68}
\]
For exponential growth \(\gamma \sim \text{(exp)}\) (since \(\|a\| \geq L \geq 1\), (3.16)) we obtain
\[
\mathfrak{N}(\tilde{Z}_\varepsilon) \approx \frac{1}{\alpha} A \hat{T} \left(\log_2 \left(\frac{e^{\|a\|\hat{T}}}{\varepsilon}\right)\right) \left(\frac{e^{\|a\|\hat{T}}}{\varepsilon}\right)^{\frac{\alpha+1}{\alpha}} := \phi_{\text{exp}}(e^{\|a\|\hat{T}} / \varepsilon) \tag{6.69}
\]
provided that \(C_a \geq 1\).

**Remark 6.5** On account of Remark 6.4 (b) and Lemma 6.6, we conclude that \(\tilde{Z}_\varepsilon \in \mathcal{C}_{CN_\varepsilon,s}\) for some uniform constant \(C\) and \(\|\tilde{Z}_\varepsilon\|_{N_\varepsilon,s} \leq e^{\|a\|\hat{T}}\) for \(N_\varepsilon\), defined by the respective right hand sides in (6.68), (6.69).

**Step 3 - Convergence rates:** To determine the convergence rates, corresponding to (6.63), we apply Remark 6.1 and Lemma 6.1 to \(N_\varepsilon = \phi_{\text{alg}} / \exp \left(e^{\|a\|\hat{T}} / \varepsilon\right)\) from (6.68) and (6.69). (4.7) follows then by straightforward calculations.

The following statement follows now from Remark 6.1 and the above observations.

**Remark 6.6** In summary we shown that for each \(N \in \mathbb{N}\) there exists an \(s\)-dimension-sparse compositional representation \(\tilde{Z}_N\) satisfying
\[
\|z - \tilde{Z}_N\|_{L_\infty(\Omega; \mathbb{R}^m)} \lesssim e^{\|a\|\hat{T}} \tilde{\gamma}(N)^{-1}, \quad N \in \mathbb{N}. \tag{6.70}
\]
with \(\tilde{\gamma}\) is from (4.7). This confirms (4.3).

**Step 4 - Stability of \(\tilde{Z}_\varepsilon\):** It follows from Lemma 6.6 (6.56), that the Lipschitz constants of partial compositions of the approximations \(\tilde{\Psi}_{e_k}\) from (6.58) remain uniformly bounded by \(\|a\| + e^{\|a\|/\hat{T}_k}\). Specifically,
\[
|\tilde{Z}_\varepsilon(t, x, y) - \tilde{Z}_\varepsilon(t, x', y')| \leq \max\{|x - x'|, |y - y'|\} e^{\|a\|\hat{T}}, \quad t \in [0, \hat{T}].
\]
i.e.,
\[
\|\tilde{Z}_\varepsilon(t, \cdot, \cdot)\|_{N_\varepsilon,s} \leq e^{\|a\|\hat{T}}, \quad t \in [0, \hat{T}]. \tag{6.71}
\]
Moreover, the growth functions \(\hat{\gamma}\) satisfy \(\hat{\gamma}(N_\varepsilon) \approx \varepsilon\). By Remark 6.1 this finishes the proof of Theorem 4.1. \(\square\)
6.3 Proof of Theorem 4.2

The main step is to invoke Lemma 2.1 and Proposition 2.2 to approximate each $\tilde{A}_{N,i}(z, y) \in \mathcal{C}_{N,s}$ in (6.46) by a DNN. In essence we follow the same steps as in the preceding section.

Approximation of data - accuracy: Recall from (6.53) that $\tilde{A}_{N,i}$ has a compositional representation $\tilde{A}_{N,i} = (\tilde{A}_{N,i})^{n_1} \circ \cdots \circ (\tilde{A}_{N,i})^1$ of some depth $n_i \leq \mathfrak{M}(A_{N,i})$.

For a given tolerance $\delta > 0$, we construct next a network $\mathcal{N}_{i,\delta,N,q}(z, y)$ approximating $\tilde{A}_{N,i}$ with accuracy $\|a\|\delta$. Specifically, we approximate each component $(\tilde{A}_{N,i})^j$, $\nu = 1, \ldots, d_j$, in the $j$th composition factor by a $\delta/n_i$-accurate neural network ($\mathcal{N}_j$). One infers from Lemma 2.1 (2.35), (6.54), that (since $\|a\| \geq 1$)

$$\|\tilde{A}_{N,i} - \mathcal{N}_{i,\delta,N,q}\|_{L_\infty(\mathbb{R}^m \times \mathbb{Y}; \mathbb{R}^m)} \leq n_i^{-1} \left\{ \delta + \sum_{r=1}^{n_i-1} \delta L_{[n_i,r+1]}(\tilde{A}_{N,i}) \right\}$$

$$\leq n_i^{-1} \left\{ \delta + \|a\| \sum_{r=1}^{n_i-1} \delta \right\} \leq \|a\|\delta, \quad i = 1, \ldots, q. \quad (6.72)$$

On the other hand, we invoke (2.36) in Lemma 2.1 to conclude

$$\# \mathcal{N}_{i,\delta,N,q} \leq C_s \|a\|^s N n_i^s \delta^{-s} \|\log_2 \delta\| + \log_2 n_i, \quad i = 1, \ldots, q, \quad (6.73)$$

where the constant $C_s$ depends only on $s$. Now we define in analogy to (6.46)

$$\mathcal{N}_{x,I,q,N,\delta}(z, t; y) := x + \sum_{i=1}^q \rho_i(t) \mathcal{N}_{i,\delta,N,q}(z, y), \quad t \in I, \quad (6.74)$$

where we recall that for $I = [t, T]$, $J_i = J_i(I, q) := t + \left[ (i-1)/q, i/q \right]$ and $\rho_i(t) := \int_t^T \chi_{J_i}(s) ds$. Taking

$$\mathcal{N}_{x,I,\tau}(z; t; y) := \mathcal{N}_{x,I,q(t/2),N(\tau/2),\delta(\tau)}(z; t; y) \quad (6.75)$$

with $q(\tau/2), N(\tau/2)$, defined according to (6.48), and

$$\delta = \delta(\tau) := \frac{\tau}{2\|a\|}, \quad (6.76)$$

we conclude that $\|A_{x,I,\tau} - \mathcal{N}_{x,I,\tau}\|_{L_\infty(\Omega(t); \mathbb{R}^m)} \leq \tau/2$. Hence, by (6.49),

$$\|\Phi_{x,I}(z; t; y) - \mathcal{N}_{x,I,\tau}(z; t; y)\|_{L_\infty(\Omega(t); \mathbb{R}^m)} \leq \tau. \quad (6.77)$$

To estimate the complexity of $\mathcal{N}_{x,I,\tau}$ we use (6.73) and bound the depths $n_i$ by $N(\tau/2) \sim \gamma^{-1}(4I\|a\|/\tau) = \gamma^{-1}(2/\tau)$. Arguing as in (6.51) and recalling (6.48), we then have (since $q(\tau) = 2A|I|/\tau = \frac{A}{\|a\|\tau}$)

$$\# \mathcal{N}_{x,I,\tau} \approx q(\tau/2) \max_{i=1, \ldots, q(\tau/2)} \# \mathcal{N}_{i,\delta(t)(\tau/2),N(\tau/2)} \approx 2A \|a\|\max_{i=1, \ldots, q(\tau/2)} \# \mathcal{N}_{i,\delta(t)(\tau/2),N(\tau/2)}$$

$$\approx 2A \|a\|^{-s} \gamma^{-1}(2/\tau)^{1+s} \delta^{-s} \|\log_2 \delta\| + \log_2 N(\tau/2) \approx A 2^s \|a\|^{-s} \tau^{-1+s} \gamma^{-1}(2/\tau)^{1+s} \|\log_2 \frac{2\|a\|\gamma^{-1}(2/\tau)}{\tau}\|. \quad (6.78)$$
For algebraic growth \( \gamma(r) = C_a r^\alpha \) we obtain \( \gamma^{-1}(2/\tau) = (2/C_a)^{1/\alpha} \tau^{-1/\alpha} \), and \( \log_2 \gamma^{-1}(2/\tau) \sim \alpha^{-1} \log_2(2/C_a \tau) \), so that \( \log_2 \frac{2\|a\|^\alpha \gamma^{-1}(2/\tau)}{\tau} = \frac{\alpha+1}{\alpha} \log_2 \frac{2\tau}{c_a \alpha^{\alpha+1}} \). For \( \gamma \sim (\exp) \), we have
\[
\log_2 \frac{2\|a\|^\alpha \gamma^{-1}(2/\tau)}{\tau} = \log_2 \frac{2\|a\| \ln(2/C_a \tau)}{\alpha \tau} .
\]
Thus, applying Remark 6.1 and Lemma 6.1 to the bounds on \( \gamma \), using\( N \),
\[
\#N_{x,I,\tau} = \frac{\#N_{x,I,\tau}}{A^2 \|a\|^{2s-1}} \left\{ \begin{array}{ll}
C_a^{-1/\alpha} \tau^{-\frac{(1+\alpha)(1+s)}{\alpha}} \log_2 \tau(| \gamma \sim (\text{alg}) |), & \\
\alpha^{-s+1} \tau^{-s+1} \log_2 | \gamma \sim (\text{exp}) |^{2+s} & \end{array} \right.
\]
where we recall that \( C_a \gtrsim 1 \), accepting a logarithmic dependence of the proportionality constant on \( \|a\| \) (or assume that \( \tau \leq \tau_0(\|a\|) \)).

We can now define \( \hat{\Psi}_{[k,j]}(t, w; y) \) in analogy to (6.57) with \( A_{w,I,\tau_k}(t, w; y) \) replaced by \( N_{w,I,\tau_k}(t, w; y) \) and likewise \( \hat{\Psi}_{z,k} = \hat{\Psi}_{[k,0]} \) in analogy to (6.56). With the same tolerances \( \tau_k(\varepsilon) = \tau(\varepsilon) \), given by (6.61). The same reasoning as in § 6.2 yields (see (6.62)
\[
\|\hat{\Psi}_{z,k}(\varepsilon) - \hat{\Psi}_{z,k}(\varepsilon)\|_{\infty,(\Omega(I_k))} \leq \varepsilon, \quad k = 1, \ldots, K,
\]
and hence
\[
\|z - N_\varepsilon\|_{\infty,(\Omega; \mathbb{R}^m)} \leq 2\varepsilon, \quad N_\varepsilon(t, x; y) := \sum_{k=1}^{K} \chi_{I_k}(t) \hat{\Psi}_{z,k}(\varepsilon)(t, x; y).
\]

\textbf{Complexity:} It remains to bound \# \( N_\varepsilon \). Invoking Remark 2.5 as before, one obtains from (6.79) with \( \mu_k(\varepsilon) \) from (6.65) (see also (6.13), (6.18)
\[
\#N_\varepsilon \approx \sum_{k=1}^{K} \mu_k(\varepsilon) \#N_{x,I,\tau_k} \approx K \log_2 \left( \frac{\#N_{x,I,\tau(\varepsilon)}}{A^2 \|a\|^{2s-1}} \right) \left\{ \begin{array}{ll}
C_a^{-1/\alpha} \tau^{-\frac{(1+\alpha)(1+s)}{\alpha}} \log_2 \tau(\varepsilon) | \gamma \sim (\text{alg}) |, & \\
\alpha^{-s+1} \tau^{-s+1} \log_2 \tau(\varepsilon)^{2+s} \quad | \gamma \sim (\text{exp}) | & \end{array} \right.
\]
By (6.61),
\[
C_a^{-1/\alpha} \tau(\varepsilon)^{-\frac{(1+\alpha)(1+s)}{\alpha}} \log_2 \tau(\varepsilon) \sim C_a^{-1/\alpha} \left( \frac{\#N_{x,I,\tau(\varepsilon)}}{\varepsilon} \right)^{(1+s)(1+\alpha)} \log_2 \left( \frac{\#N_{x,I,\tau(\varepsilon)}}{\varepsilon} \right),
\]
while
\[
\alpha^{-s+1} \tau^{-s+1} \log_2 \tau(\varepsilon)^{2+s} \sim \alpha^{-s+1} \left( \frac{\#N_{x,I,\tau(\varepsilon)}}{\varepsilon} \right)^{-(1+s)} \log_2 \left( \frac{\#N_{x,I,\tau(\varepsilon)}}{\varepsilon} \right)^{2+s} .
\]
Inserting these estimates into (6.82), confirms (4.5).

The convergence rates stated in Corollary 4.3 follow now in the same way as before by applying Remark 6.1 and Lemma 6.1 to the bounds on \#(N), given in (4.5).

Regarding the stability of the networks \( N_\varepsilon \), there is a principal obstacle related to the fact that the precise dimension-vectors of the compositions \( \tilde{A}_{N,i} = \tilde{A}_{N,i} \circ \cdots \circ \tilde{A}_{N,i} \), especially their depths \( n_i \) are not known. Although, the Lipschitz constants of the implanted networks
in each factor $\tilde{A}^\nu_{N,i}$ are controlled by $\|a\|$ it is not clear whether the Lipschitz constants of their compositions also remain controlled by $\|a\|$. Such network approximations exist by Proposition 2.2 but need no longer be s-dimension sparse. So, the only guaranteed general bound for the Lipschitz constants of the partial compositions in view of Proposition 2.2

\[
L_{N,\delta} := \max_{j < n_i} L_{[n_i, j+1]}(N_{i,\delta,N,q}) \leq (c_3(1 + A)\|a\|)^{n-i},
\]

(6.83)

where we have applied the (perhaps too pessimistic) bound $n_i \leq N$. If on the other hand, the depths $n_i$ remain uniformly bounded by $\bar{n}$, say, one obtains a uniform Lipschitz-bound $L_{N,\delta} \leq (c_3(1 + A)\|a\|)^{\bar{n}}$.

**Lemma 6.7** The network approximations $N_{i,q,N,\delta}$ from (6.74) have the following Lipschitz continuity properties: for $(t, x, y)$, $(t', x', y') \in \Omega(I)$, $\bar{z}, \bar{z}' \in L_\infty(I; \mathbb{R}^m)$:

\[
|N_{x,i,q,N,\delta}(t, \bar{z}; y) - N_{x,i,q,N,\delta}(t, \bar{z}'; y')| \leq \frac{(L_{N,\delta}|I|)^{\ell}}{\ell!} \{|y - y'|, \|\bar{z} - \bar{z}'\|_{L_\infty(I; \mathbb{R}^m)}\},
\]

(6.84)

where $L_{N,\delta}$ is given by (6.83). Similarly, when $\bar{z} = \bar{z}_x$, $\bar{z}' = \bar{z}'_x$, one has

\[
|N_{x,i,q,N,\delta}(t, \bar{z}_x; y) - N_{x,i,q,N,\delta}(t, \bar{z}'_x; y')| \leq (1 + \delta)\|a\||t - t'| + \max \{|y - y'|, |x - x'|\}e^{L_{N,\delta}|I|}.
\]

(6.85)

**Proof:** Recall from (6.74) that $N_{x,i,q,N,\delta}(t, \bar{z}; y) := x + \sum_{i=1}^q \rho_i(t)N_{i,q,N,\delta}(\bar{z}, y)$. Then, we have for $i = 1, \ldots, q$,

\[
|N_{i,q,N,\delta}(\bar{z}, y) - N_{i,q,N,\delta}(\bar{z}', y')| \leq |N_{i,q,N,\delta}|_{Lip_1(\mathbb{R}^m \times \mathcal{Y})} \max\{|\bar{z} - \bar{z}'|_{L_\infty(I; \mathbb{R}^m)}, |y - y'|\}.
\]

(6.86)

By the comments preceding the lemma, we obtain

\[
|N_{x,i,q,N,\delta}(t, \bar{z}; y) - N_{x,i,q,N,\delta}(t, \bar{z}'; y')| \leq \sum_{i=1}^q \rho_i(t)L_{N,\delta} \max\{|\bar{z} - \bar{z}'|_{L_\infty(I; \mathbb{R}^m)}, |y - y'|\}.
\]

(6.87)

Hence, we are in the same situation as in (6.133). Finally,

\[
|N_{x,i,q,N,\delta}(t, \bar{z}; y) - N_{x,i,q,N,\delta}(t, \bar{z}'; y)| \leq \|a\||t - t'| + \sum_{i=1}^q |\rho_i(t) - \rho_i(t')| |\bar{z}_i(y) - N_{i,q,N,\delta}(\bar{z}_i; y)| \leq (1 + \delta)\|a\||t - t'|,
\]

where we have used (6.72). Therefore, the claim follows by the same arguments as used in the proof of Lemma 6.6.

Regarding Remark 4.4, recall that $N(\varepsilon) \sim \gamma^{-1}(2/\tau(\varepsilon))$, where $\tau(\varepsilon) \approx \varepsilon e^{-\|a\|^2/\tilde{T}}$, $\delta(\varepsilon) = \tau(\varepsilon)/2\|a\|$ so that by (6.83),

\[
L_{N(\varepsilon),\delta(\varepsilon)} \approx \begin{cases} (c_3(1 + A)\|a\|)^{\frac{1}{2} + \frac{\gamma}{2}} e^{-\|a\|^2/\tilde{T}} & \text{if } \gamma \sim (\text{alg}), \\ (c_3(1 + A)\|a\|)^{\frac{1}{2} + \frac{\gamma}{2}} (\|\varepsilon_c + \|a\|^2\tilde{T}) & \text{if } \gamma \sim (\text{exp}). \end{cases}
\]

(6.88)

This confirms Remark 4.4. 

\[\square\]
6.4 Proof of Corollary 4.5 and Theorem 4.6

We first prove Corollary 4.5. To apply Theorem 4.2 and Corollary 4.3, note first that condition (3.6) is applicable, i.e., version (A2) can be used. In fact, \( a_j(t; \cdot) = \omega_j a_j^\circ(t; \cdot) \in \operatorname{Lip}_1(\mathbb{R}^m) \), uniformly in \( t \in [0, \tilde{T}] \), immediately implies that \( \bar{a}_{j,i} = a_{j,i} = \omega_j a_j^\circ_{j,i} \) belongs to \( \operatorname{Lip}_1(\mathbb{R}^m) \), for \( i = 1, \ldots, q \), with the same Lipschitz constants \( \Lambda_{j,i} \) from (3.17).

Next we recall from Remark 3.2 that \( a \in \mathcal{C}_{N_a,m} \) with \( N_a = 1 + d_y(1 + m^2) \). Hence, the simplest compositional approximations \( A_N(t; \cdot) \) to \( a \) is

\[
A_N(t, x, y) = \begin{cases} 
0, & N < N_a, \\
 a(t; x; y), & N \geq N_a.
\end{cases}
\]

In view of Remark 3.2 (3.12), one obtains for \( \gamma(r) = Ce^{\alpha r} \) and all \( t \in [0, \tilde{T}] \),

\[
\|a\|_{L_\infty([0, \tilde{T}], A^\circ_{j,i})} \leq \max_{N \in \mathbb{N}} \gamma(N) \left\{ \|a(t) - A_N(t)\|_{L_\infty(\mathbb{R}^m \times \mathbb{R}^m)} + \gamma(N)^{-1} \|A_N(t)\|_{N,m} \right\} \\
\leq \begin{cases} 
ACe^{\alpha N} + (A + \Lambda|\omega|_1), & N < N_a, \\
 A + \Lambda|\omega|_1, & N \geq N_a.
\end{cases}
\]

Taking \( \alpha := N_a^{-1}, \ C_e := 1 \), yields \( \|a\| \leq \tilde{L} := 2(A + \Lambda|\omega|_1) \). By (3.11), one has \( \alpha \approx d_y^{-1} \) (with \( m \)-dependent proportionality). Theorem 4.1 yields then \( z \in L_\infty([0, \tilde{T}], A^\circ_{j,i}) \) with

\[
\tilde{\gamma}(r) \approx \frac{r}{d_y A}_{\tilde{T}} \left| \frac{d_y A_{\tilde{T}}}{d_y A_T} \right|^{-2}, \quad \|z\|_{L_\infty([0, \tilde{T}], A^\circ_{j,i})} \leq e^{2(A + \Lambda|\omega|_1)\tilde{T}},
\]

and hence (4.11). Now (4.13) and the expression for \( F \) follow from Corollary 4.3.

We now turn to the proof of Theorem 4.6 approximating \( \Phi_{x,t} \) in a first step by

\[
P_{x,t,q}(t, \bar{z}; y) := x + \sum_{j=1}^{d_y} \sum_{i=1}^{q} y_j \, \rho_i(t) a_{j,i}(\bar{z}(\xi_i)) = a_{j,i}(\cdot) := \frac{\omega_j}{|J_i|} \int_{J_i} a_j^\circ(s; \cdot) ds, \quad (6.89)
\]

recalling that \( J_i = J_i(I, q) = t + \left[ (i - 1)/q, i/q \right] \) and \( \rho_i(t) = \int_{J_i} \chi_{s \leq t}(s) ds \). For \( L \) from (3.10) and \( \bar{z} \) as in Lemma 6.4, we infer from (6.38) that \( |\Phi_{x,t}(t, \bar{z}; y) - P_{x,t,q}(t, \bar{z}; y)| \leq \frac{A|I|}{q} \). Invoking Proposition 2.2, we approximate the low-dimensional functions \( a_j^\circ_{j,i}(\cdot) \) by finitely parametrized functions such as neural networks. Specifically, there exist networks (suppressing the reference to \( I \)) \( N_{j,i,\delta} \) of depth \( \lesssim \log_2 \delta^{-1} \) such that for \( j = 1, \ldots, d_y, i = 1, \ldots, q, \)

\[
\|a_j^\circ_{j,i} - N_{j,i,\delta}\|_{L_\infty(\mathbb{R}^m \times \mathbb{R}^m)} \leq \delta, \quad \#N_{j,i,\delta} \lesssim \Lambda^m \delta^{-m} \log_2 \delta. \quad (6.90)
\]

Then

\[
N_{x,t,q,\delta}(t, \bar{z}; y) := x + \sum_{j=1}^{d_y} \sum_{i=1}^{q} y_j \, \rho_i(t) \omega_j N_{j,i,\delta}(\bar{z}(\xi_i)), \quad (6.91)
\]
is indeed an \( m \)-dimension-sparse neural network. To that end, we keep viewing \( t \) as a parameter and the input-variables \( x, y \) are passed across layers, formally in a “skip-connection” format. Thus, formally we have

\[
\mathcal{N}_{x,I,q,\delta}(t, \bar{z}, y) = (G_2 \circ G_1)(t, x, \bar{z}, y). \tag{6.92}
\]

For better readability the following representation groups variables in a formally incorrect way and should be viewed as a \( t \)-dependent mapping into \( \mathbb{R}^{1+m+d_y+qd_y} \)

\[
G_1 : (t, x, \bar{z}, y) \mapsto \begin{pmatrix}
    x, y, \rho_1(t), \ldots, \rho_q(t) \\
    \omega_1 \mathcal{N}_{1,1,\delta}(\bar{z}), \ldots, \omega_{d_y} \mathcal{N}_{d_y,1,\delta}(\bar{z}), \\
    \vdots \\
    \omega_1 \mathcal{N}_{1,q,\delta}(\bar{z}), \ldots, \omega_{d_y} \mathcal{N}_{d_y,q,\delta}(\bar{z})
\end{pmatrix} \in \mathbb{R}^{q+m+d_y+qd_y}, \tag{6.93}
\]

which is obviously \( m \)-dimension sparse. Hence \( G_1 \) itself is a neural network whose depth is bounded by \( \log_2 \delta^{-1} \). The tri-linear factor \( G_2 \) reads then

\[
G_2 : (x, y, r_1, \ldots, r_q, \zeta^{1,1}, \ldots, \zeta^{d_y,q}) \mapsto \left( x + \sum_{i=1}^{q} \sum_{j=1}^{d_y} r_j y \zeta^{j,i} \right) \in \mathbb{R}^m. \tag{6.94}
\]

Assessing the accuracy of \( \mathcal{N}_{x,I,q,\delta} \) follows in essence the same lines as before. In view of (6.38) and (6.90),

\[
|\Phi_{x,I}(t, \bar{z}; y) - \mathcal{N}_{x,I,q,\delta}(t, \bar{z}; y)| \leq |\Phi_{x,I}(t, \bar{z}; y) - P_{x,I,q}(t, \bar{z}; y)|
\]

\[
+ \sum_{j=1}^{d_y} |y_j| \sum_{i=1}^{q} \rho_i(t) \omega_j |\mathcal{N}_{j,i,\delta}(\bar{z}(\xi_i)) - a_{j,i}^{\infty}(\bar{z}(\xi_i))| \leq \frac{A|I|}{q} + |\omega| |I| \delta. \tag{6.95}
\]

Thus, given any target tolerance \( \tau > 0 \), choosing

\[
q(\tau) = \left\lceil \frac{2A|I|}{\tau} \right\rceil, \quad \delta(\tau) = \frac{\tau}{2|I||\omega|},
\]

and abbreviating \( \mathcal{N}_{x,I,\tau} := \mathcal{N}_{x,I,q(\tau),\delta(\tau)} \), we obtain

\[
|\Phi_{x,I}(t, \bar{z}, y) - \mathcal{N}_{x,I,\tau}(t, \bar{z}; y)| \leq \tau, \quad x \in \mathbb{R}^m, \bar{z} \in L_\infty(I; \mathbb{R}^m), t \in I. \tag{6.97}
\]

Regarding the complexity of \( \mathcal{N}_{x,I,\tau} \) we see from (6.93) that \( \mathcal{N}(G_1) = m + d_y + q + qd_y m^2 \), \( \mathcal{N}(G_2) = 1 \) because of bilinearity. Thus, \( \mathcal{N}(G_2 \circ G_1) \approx m^2 qd_y \) so that by (6.90) and (6.96),

\[
\# \mathcal{N}_{x,I,\tau}(t, \cdot, \cdot) \lesssim m^2 d_y q(\tau) \Lambda^m \delta(\tau)^{-m} \log_2 \delta(\tau)|
\]

\[
\lesssim m^2 d_y \frac{2A|I|}{\tau} \left( \frac{2|I|A|\omega|}{|\tau|^m} \right)^{m} \log_2 \frac{2|I||\omega|}{|\tau|}, \quad t \in I. \tag{6.98}
\]

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In view of (3.12), the earlier role of \( \|a\| \) will now be played by \( \|a\|_{N_\varepsilon,m} = A + \Lambda \|\omega\|_1 =: L \), and since by (6.6) \( |I|L = \frac{1}{2} \), we obtain from (6.98)

\[
\#N_{x,I,I}(t, \cdot, \cdot) \lesssim m^2 d_y A|I|\tau^{-(m+1)} \log_2 \frac{2|I\|\omega_1}{\tau}.
\]  

(6.99)

Thus, with the same number \( \mu = \mu(\eta) \geq \log_2 (2\eta)^{-1} \) from (6.13) we get \( |z(t; x, y) - \Phi_\mu^\varepsilon(t; x, y)| \leq \eta \). Hence, the same \( \tilde{Z}_\varepsilon \), defined by (6.63), based on tolerances \( \eta(\varepsilon) \) from (6.18), provide \( \varepsilon \)-accuracy of time-catenated iterates of \( \Phi_{x,J_k} \) where the number \( K \) of macro-time-steps still equals \( 2\hat{T}L \). We can therefore choose the same vectors of tolerances \( \tau^k(\varepsilon) \) from (6.61) (i.e., \( \tau(\varepsilon) \approx \eta(\varepsilon) \approx \varepsilon^{-K/2} \)) as well as tolerances \( \delta(\eta(\varepsilon)) \approx \varepsilon e^{-K/2}/(2|I|\omega_1) \).

We then define \( N_\varepsilon \) in complete analogy to \( \tilde{Z}_\varepsilon \) from (6.63), with \( A_{w,I,\tau(\varepsilon)} \) replaced by \( N_{w,I,\tau(\varepsilon)} \) to obtain \( \|z - N_\varepsilon\|_{L_\infty(\Omega; \mathbb{R}^m)} \leq 2\varepsilon \).

Hence, on account of (6.99), since \( |I|K = \hat{T}, K/2 = \hat{L} \)

\[
\#N_\varepsilon \lesssim m^2 d_y K \mu(\eta(\varepsilon)) A|I|\tau^{-(m+1)} \log_2 \frac{2|I\|\omega_1}{\tau} \lesssim A\hat{T} m^2 d_y \tau^{-(m+1)} \log_2 \frac{2|I\|\omega_1}{\tau} \\
\lesssim A\hat{T} m^2 d_y \left( \frac{e\hat{T}}{\varepsilon} \right)^{m+1} \log_2 \frac{2|I|e\hat{T}|\omega_1|}{\varepsilon} \lesssim A\hat{T} m^2 d_y \left( \frac{e\hat{T}}{\varepsilon} \right)^{m+1} \log_2 \frac{e\hat{T}}{\varepsilon} \\
=: \phi \left( \frac{e\hat{T}}{\varepsilon} \right),
\]

where we have used that, by (6.6), \( 1 \lesssim |I|\Lambda \|\omega\|_1 \leq 1/2 \). This confirms (6.13).

We apply Remark 6.1 and Lemma 6.1 to conclude that (surpressing a logarithmic dependence on \( A\hat{T} m^2 \))

\[
\tilde{\gamma}(r) := (A\hat{T} m^2)^{-\frac{1}{m+1}} \left( \frac{r}{d_y} \right)^{\frac{1}{m+1}} \log_2 \frac{r}{d_y} \left( \frac{2|I|}{\varepsilon} \right)^{\frac{1}{m+1}},
\]

(6.100)

satisfies \( \tilde{\gamma}(\phi(s)) \approx s \). Hence, there exists a network \( N_N \) with \( \#N_N \leq N \) such that

\[
\|z - N_N\|_{L_\infty(\Omega; \mathbb{R}^m)} \lesssim e^{\hat{T}\hat{T}} \tilde{\gamma}(N)^{-1}, \quad N \in \mathbb{N},
\]

where the constant depends only on \( m \). This proves (4.16).

It remains to estimate \( \|N_\varepsilon\|_{N_\varepsilon,m} \) where \( N_\varepsilon := \#N_\varepsilon \).

**Lemma 6.8** For \( N_{x,I,q,\delta}(t, \cdot, \cdot) \) from (6.99) the following statements hold: Let

\[
L_\delta := (A + (\Lambda + \delta) \|\omega\|_1), \quad \Lambda := c_3(1 + A^2)\Lambda,
\]

with \( c_3 \) from (2.3) (see also (3.4)). Then, for (\( t, x, y \), (\( t', x', y' \)) \( \in \Omega(I) \),

\[
|N_{x,I,q,\delta}(t, z; y) - N_{x,I,q,\delta}(t, z'; y')| \leq \frac{(L_\delta |I|)^{\varepsilon}}{\varepsilon !} \left\{ |y - y'|, \|z - z'\|_{L_\infty(I; \mathbb{R}^m)} \right\}.
\]

(6.102)

Similarly, when \( \bar{z} = \bar{z}_x, \bar{z}' = \bar{z}_x' \), one has for all \( x, x' \in \mathbb{R}^m, t, t' \in I \),

\[
|N_{x,I,q,\delta}(t, \bar{z}; y) - N_{x',I,q,\delta}(t, \bar{z}; y')| \leq (A + |\omega|_1 \delta)|t - t'| \\
+ \max \left\{ |y - y'|, |x - x'| \right\} e^{L_\delta |I|}.
\]

(6.103)
while, by Proposition 2.2, (2.3),

\[ \sum_{j=1}^{d_y} \omega_j |N_j,i,\delta(\bar{z}(\xi_i))| \leq \sum_{j=1}^{d_y} |a_{j,i}(\bar{z}(\xi_i))| + \omega_j |a_{j,i}(\bar{z}(\xi_i)) - N_j,i,\delta(\bar{z}(\xi_i))| \leq A + |\omega|\delta, \quad (6.104) \]

uniformly in \( i = 1, \ldots, d_y \), with \( \bar{\lambda} \) from (6.101) where. Then

\[ \frac{d_y}{d_y} \sum_{j=1}^{d_y} \omega_j |N_j,i,\delta(\bar{z}(\xi_i))| \leq \sum_{j=1}^{d_y} \omega_j |y_j N_j,i,\delta(\bar{z}(\xi_i)) - y_j N_j,i,\delta(\bar{z}'(\xi_i))| \]

\[ \leq \sum_{i=1}^{q} \rho_i(t) \left\{ \sum_{j=1}^{d_y} \omega_j |y_j - y_j'| N_j,i,\delta(\bar{z}(\xi_i)) + \sum_{j=1}^{d_y} \omega_j |y_j'| N_j,i,\delta(\bar{z}'(\xi_i)) - N_j,i,\delta(\bar{z}'(\xi_i)) \right\} \]

\[ \leq \sum_{i=1}^{q} \rho_i(t) \left\{ (A + |\omega|\delta) |y - y'| + \bar{\lambda} |\omega| |\bar{z} - \bar{z}'| L_\infty(I;\mathbb{R}^m) \right\} \]

\[ \leq \sum_{i=1}^{q} \rho_i(t) L_\delta \max \{|y - y'|, |\bar{z} - \bar{z}'| L_\infty(I;\mathbb{R}^m)\}. \quad (6.106) \]

Similarly, for \( t, t' \in I \)

\[ |N_x,I,q,\delta(t, \bar{z}; y) - N_x,I,q,\delta(t', \bar{z}; y)| \leq \sum_{i=1}^{q} |\rho_i(t) - \rho_i(t')| \sum_{j=1}^{d_y} |y_j \omega_j N_j,i,\delta(\bar{z}(\xi_i))| \]

\[ \leq (A + |\omega|\delta)|t - t'|, \]

where we have used (6.31) and (6.104). Applying this to \( \bar{z} = N_x,I,q,\delta \) extends this to iterates of \( N_x,I,q,\delta \).

Hence, we are in the same situation as in (6.133). Therefore, (6.102) and (6.103) follow by the same arguments as used in the proof of Lemma 6.6. \( \square \)

Now recall that \( \delta(\varepsilon) \approx \frac{e e^{K/2}}{2|\lambda| \lambda} \). Hence, \( \delta(\varepsilon) |\omega| \lesssim \frac{e e^{K/2}}{2|\lambda| |\lambda|} \lesssim \bar{T}^{-1} \varepsilon Ke^{K/2} \lesssim \varepsilon / \bar{T} \). Then

\[ L_{\delta(\varepsilon)} \leq A + \bar{\lambda} |\omega| + \bar{T}^{-1} \varepsilon \leq \bar{L} := \bar{A} + \bar{\lambda} |\omega|, \quad \bar{A} := \max_{\varepsilon \leq 1} A + \bar{T}^{-1} \varepsilon, \]

and the same arguments as used earlier provide

\[ |\hat{\Psi}_{\delta(\varepsilon)}(t, x, y) - \hat{\Psi}_{\delta(\varepsilon)}(t', x', y')| \leq \bar{L} |t - t'| + \bar{L} \max \{|x - x'|, |y - y'|\}. \quad (6.107) \]

From these observations it follows that \( \|N_x\|_{N_x,m} \leq e^{\bar{T} L} \) and for \( \gamma(r) := \left( \frac{r}{\log_2 \frac{1}{r}} \right)^{m+1} \log_2 \frac{1}{r} \)

\[ \gamma(N_x) K_m(z, N, \gamma(N)^{-1}) \leq \gamma(N_x) |z - N_x| L_\infty(\Omega) + \|N_x\|_{N_x,m} \lesssim e^{LT} + e^{\bar{T} L}, \]

which confirms the remainder of the assertion. \( \square \)
6.5 Proof of Theorem 4.7

By assumption, given \( \varepsilon > 0 \), there exists an \( f_\varepsilon \in L_\infty(\tilde{I}; C_{N_{\varepsilon}}(f_{\varepsilon}(\cdot), m), \cap \text{Lip}_1(\tilde{I}; C(\mathbb{R}^m \times \mathcal{Y})), \) piecewise affine in time, and a composition \( u_{0,\varepsilon} \in C_{N_{\varepsilon}(u_{0}), m}, \) so that (identifying for notational convenience in what follows mappings and representations)

\[
\|u_0 - u_{0,\varepsilon}\|_{L_\infty(\mathbb{R}^m \times \mathcal{Y})} \leq \varepsilon,
\mathcal{M}(u_{0,\varepsilon}) \leq \gamma^{-1}(\|u_0\|_{A^{\gamma, m}/\varepsilon}) \approx \|u_0\|_{A^{\gamma, m}}^{1/\alpha} \varepsilon^{-1/\alpha},
\]

and likewise, since \( \mathcal{M}(f_\varepsilon(t, \cdot)) \leq \gamma^{-1}(\|f(t, \cdot)\|_{A^{\gamma, m}/\varepsilon}) \) for \( t \in \tilde{I} \)

\[
\|f - f_\varepsilon\|_{L_\infty(\bar{I}; C(\mathbb{R}^m \times \mathcal{Y})))} \leq \varepsilon,
\mathcal{M}(f_\varepsilon(t, \cdot)) \leq \|f(t, \cdot)\|_{A^{\gamma, m}}^{1/\alpha} \varepsilon^{-1/\alpha}.
\]

Next, we use that the compositional factors in \( u_{0,\varepsilon}, f_\varepsilon \) are Lipschitz continuous with constants controlled by \( \|u_0\| := \|u_0\|_{A^{\gamma, m}}, \|f\| := \|f\|_{L_\infty(\bar{I}; A^{\gamma, m})} \), respectively. We employ Lemma 2.1 to implant \( \eta \)-accurate Lipschitz controlled DNNs into \( u_{0,\varepsilon}, f_\varepsilon \), respectively. Invoking Remark 2.1 we obtain

\[
\|u_{0,\varepsilon} - \mathcal{N}_{u_{0,\varepsilon}, \eta}\|_{L_\infty(D)} \leq \eta \left\{ 1 + \sum_{j=1}^{n(D(u_{0,\varepsilon}))} \|u_0\| \right\} \leq \eta \mathcal{M}(u_{0,\varepsilon})\|u_0\|,
\]

as well as

\[
\|f_\varepsilon(t, \cdot) - \mathcal{N}_{f_\varepsilon, \eta}(t, \cdot)\|_{L_\infty(\mathbb{R}^m \times \mathcal{Y})} \leq \eta \left\{ 1 + \sum_{j=1}^{n(D(f_\varepsilon)))} \|f_0\| \right\} \leq \eta \mathcal{M}(f_\varepsilon(t))\|f_0\|. \quad t \in \bar{I},
\]

By Proposition 2.2 (6.108), and (6.109), it follows that

\[
\#\mathcal{N}_{u_{0,\varepsilon}, \eta} \leq \|u_{0,\varepsilon}\|_{\mathcal{M}(u_{0,\varepsilon})}^m \mathcal{M}(u_{0,\varepsilon}) \eta^{-m} |\log_2 \eta| \leq \|u_0\|^{m+1} \varepsilon^{-1} \eta^{-m} |\log_2 \eta|,
\]

and, uniformly in \( t \in \bar{I} \),

\[
\#\mathcal{N}_{f_\varepsilon, \eta}(t, \cdot) \leq \|f_\varepsilon(t, \cdot)\|_{\mathcal{M}(f_\varepsilon(t))}^m \mathcal{M}(f_\varepsilon(t)) \eta^{-m} |\log_2 \eta| \leq \|f\|^{m+1} \varepsilon^{-1} \eta^{-m} |\log_2 \eta|. \quad (6.113)
\]

Employing again a time-discretization of size \( q \), \( (3.23) \) suggests the following DNN approximation to \( u \) which yields, on account of Lemma 6.3

\[
\left| \int_0^t f_\varepsilon(s, w, y) ds - \sum_{i=1}^q \rho_i(t) f_\varepsilon(\xi_i, w, y) \right| \leq \frac{\|f\|_{\tilde{T}^2}}{2q}.
\]

Finally, we know from Theorem 4.6 that there exists a DNN \( \mathcal{N}_{z, \tilde{\varepsilon}} \) that approximates the characteristic field \( z \) within accuracy \( \tilde{\varepsilon} \), i.e., in view of (4.15),

\[
\|z - \mathcal{N}_{z, \tilde{\varepsilon}}\|_{L_\infty(\Omega)} \leq \tilde{\varepsilon}, \quad \#\mathcal{N}_{z, \tilde{\varepsilon}} \leq \#d_\gamma \hat{T}(e^{L\hat{T}})^{m+1} \frac{\log_2 \left( \frac{e^{L\hat{T}}}{\tilde{\varepsilon}} \right)^2}{2},
\]

where we suppress in what follows the dependence on \( A, m \) and where \( L \) is given by (4.14).
In summary, the network $\mathcal{N}_{u, \bar{e}, \eta, q}$ formed by composing the DNNs $\mathcal{N}_{u_0, \bar{e}, \eta}, \mathcal{N}_{f_\varepsilon, \eta}(\xi, \cdot)$, $i = 1, \ldots, q$, with the approximate characteristics $\mathcal{N}_{z, \bar{e}, \varepsilon}$ satisfies, on account of (6.118), (2.77), (6.112), and (6.113),

$$
\#\mathcal{N}_{u, \bar{e}, \eta, q} \leq \#(\mathcal{N}_{u_0, \bar{e}, \eta} \circ \mathcal{N}_{z, \bar{e}}) + q \max_{i=1, \ldots, q} \#(\mathcal{N}_{f_\varepsilon, \eta}(\xi, \cdot) \circ \mathcal{N}_{z, \bar{e}})
$$

$$
\leq \#\mathcal{N}_{u_0, \bar{e}, \eta} + \#\mathcal{N}_{z, \bar{e}} + q \left( \max_{i=1, \ldots, q} \#\mathcal{N}_{f_\varepsilon, \eta}(\xi, \cdot) + \#\mathcal{N}_{z, \bar{e}} \right)
$$

$$
\lesssim \left( \|u_0\|^{m^* \frac{1}{\alpha} \bar{e}^{-\frac{1}{\alpha}}} + q \|f\|^{m^* \frac{1}{\alpha} \bar{e}^{-\frac{1}{\alpha}}} \right) \eta^{-m} \log_2 \eta + (1 + q)d_y \hat{T} \left( \frac{e^{L_T \bar{T}}}{\bar{e}} \right)^{m+1} \left| \log_2 \left( \frac{e^{L_T \bar{T}}}{\bar{e}} \right) \right|^2 \right),
$$

where we set $M := \max\{1, \|u_0\|, \|f\|\}$. To determine $\eta$, we have by (3.23),

$$
\|u(t, \cdot) - \mathcal{N}_{u, \bar{e}, \eta, q}(t, \cdot)\|_{L_\infty(\mathbb{R}^m \times \mathcal{Y})} \leq \|u_0(z(-t, t, \cdot)) - \mathcal{N}_{u_0, \bar{e}, \eta} \circ \mathcal{N}_{z, \bar{e}}(-t, t, \cdot)\|_{L_\infty(\mathbb{R}^m \times \mathcal{Y})}
$$

$$
+ \sup_{s,y} \int_0^t f(s, z(t-s, t, x, y))ds - \sum_{i=1}^q \rho_i(t)\mathcal{N}_{\xi_i, \eta}(\xi(t - \xi_i, \xi, x, y))
$$

$$
=: Q_1 + \sup_{x,y} Q_2(x, y).
$$

Regarding $Q_1$, let $L_0 := \|u_0\|_{\text{Lip}_1(\mathbb{R}^m \times \mathcal{Y})} \leq \|u_0\|$. Because of (6.110) and (6.108),

$$
Q_1 \leq \|u_0(z(-t, t, \cdot)) - u_0(\mathcal{N}_{z, \bar{e}}(-t, t, \cdot))\|_{L_\infty(\mathbb{R}^m \times \mathcal{Y})}
$$

$$
+ \|u_0(\mathcal{N}_{z, \bar{e}}(-t, t, \cdot)) - u_\bar{e}, 0(\mathcal{N}_{z, \bar{e}}(-t, t, \cdot))\|_{L_\infty(\mathbb{R}^m \times \mathcal{Y})}
$$

$$
+ \|u_\bar{e}, 0(\mathcal{N}_{z, \bar{e}}(-t, t, \cdot)) - \mathcal{N}_{u_0, \bar{e}, \eta}(\mathcal{N}_{z, \bar{e}}(-t, t, \cdot))\|_{L_\infty(\mathbb{R}^m \times \mathcal{Y})}
$$

$$
\leq (1 + L_0)\bar{e} + \|u_0\| \eta \mathcal{R}(u_0, \bar{e}) \lesssim (1 + L_0)\bar{e} + \|u_0\|^{1 + \frac{1}{\alpha}} \eta \bar{e}^{-\frac{1}{\alpha}}
$$

$$
\leq \|u_0\| \left\{ 2\bar{e} + \eta \bar{e}^{-\frac{1}{\alpha}} \|u_0\|^{\frac{1}{\alpha}} \right\},
$$

where we have used $\|u_0\| \geq 1$ and (4.17) in the last step. Similarly, by (6.109) (6.111), and (6.114), abbreviating $L_f := |f|_{\text{Lip}_1(\hat{F}; C(\mathbb{R}^m \times \mathcal{Y}))}$,

$$
Q_2(x, y) \leq \hat{T} L_f \bar{e} + \int_0^t \|f(s, \mathcal{N}_{z, \bar{e}}(t - s, t, x, y) - f_\bar{e}(s, \mathcal{N}_{z, \bar{e}}(t - s, t, x, y)) |ds
$$

$$
+ \int_0^t \left| f_\bar{e}(s, \mathcal{N}_{z, \bar{e}}(t - s, t, x, y)) - \sum_{i=1}^q \rho_i(t)f_\bar{e}(\xi_i, \mathcal{N}_{z, \bar{e}}(t - \xi_i, \xi, x, y)) \right|
$$

$$
+ \sum_{i=1}^q \rho_i(t)|f_\bar{e}(\xi_i, \mathcal{N}_{z, \bar{e}}(t - \xi_i, x, y)) - \mathcal{N}_{f_\bar{e}, \eta}(\xi_i, \mathcal{N}_{z, \bar{e}}(t - \xi_i, x, y))|.
$$

(6.118)
On account of (6.33) and the assumption $L_f \leq \|f\|$, this gives

$$Q_2(x, y) \leq (1 + \|f\|)\hat{T}\tilde{e} + \frac{\|f\|\hat{T}^2}{2q} + \hat{T}\eta\mathfrak{M}(f\tilde{\epsilon})\|f\| \leq 2\|f\|\hat{T}\tilde{e} + \frac{\|f\|\hat{T}^2}{2q} + \hat{T}\frac{\eta}{\tilde{e}^{1/\alpha}}\|f\|^{1+\frac{1}{\alpha}}$$

$$\leq \hat{T}\|f\|\left\{2\tilde{e} + \frac{\hat{T}}{2q} + \|f\|\frac{\hat{T}}{\tilde{e}}\tilde{e}^{-\frac{1}{\alpha}}\right\}.$$  (6.119)

Now recall that $M = \max\{1, \|f\|, \|u_0\|\}$ and let

$$q(\tilde{e}) = \frac{\hat{T}}{2\tilde{e}}, \quad \eta(\tilde{e}) = M^{-\frac{1}{\alpha}}\tilde{e}^{1+\frac{1}{\alpha}},$$  (6.120)

to conclude that $\max_{x, y} Q_2(x, y) \leq 4\hat{T}\|f\|\tilde{e}$. Hence, we derive from (6.117) and (6.119) that the network $\mathcal{N}_{u, \tilde{e}} := \mathcal{N}_{u, \tilde{e}, \eta(\tilde{e}), q(\tilde{e})}$ satisfies (recall that by assumptions $\|u_0\|, \|f\| \geq 1$)

$$\|u - \mathcal{N}_{u, \tilde{e}}\|_{\infty} \leq \{4\hat{T}\|f\| + 3\|u_0\|\}\tilde{e} \leq 7\hat{T}M\tilde{e}. $$  (6.121)

This confirms the first part of (4.19) with $\tilde{e} := 7\hat{T}M\tilde{e}$.

Now we infer from (6.116), (6.108), (6.109) that

$$\#\mathcal{N}_{u, \tilde{e}} \lesssim \hat{T}\tilde{e}^{-1}\left\{M^{\frac{m+1}{\alpha}}\tilde{e}^{-\frac{1}{\alpha}}M^{\tilde{e}^{-\frac{m(1+\alpha)}{\alpha}}} \left|\log_2 \frac{M}{\tilde{e}^{1+\alpha}}\right| + d_y\hat{T}e\left(e^{\hat{T}\tilde{e}}\right)^{m+1} \left|\log_2 \left(e^{\hat{T}\tilde{e}}\right)\right|^2 \right\}$$

$$\lesssim \hat{T}\tilde{e}^{-1}\left\{M^{\frac{m+1}{\alpha}}\tilde{e}^{-\frac{1}{\alpha}}M^{\tilde{e}^{-\frac{m(1+\alpha)}{\alpha}}} + d_y\hat{T}e\hat{T}(m+1)\tilde{e}^{-(m+1)} \left|\log_2 \left(e^{\hat{T}\tilde{e}}\right)\right|^2 \right\}$$

$$= \hat{T}\left\{M^{\frac{(m+1)m+1}{\alpha}}\tilde{e}^{-(1+\alpha)(m+1)} + d_y\hat{T}e\hat{T}(m+1)\tilde{e}^{-(m+2)} \left|\log_2 \left(e^{\hat{T}\tilde{e}}\right)\right|^2 \right\}$$

Introducing $\beta := \max\{1, (m + 1)/\alpha\}$, (see (4.18)) and substituting $\tilde{e} = \tilde{e}/(7\hat{T}M)$, yields upon elementary calculations

$$\#\mathcal{N}_{u, \tilde{e}} \lesssim \left\{M^{\frac{(m+1)m+1}{\alpha}}\hat{T}^{m+2+\beta}e^{-L\hat{T}(m+1+\beta)} + d_y\hat{T}^{m+4}e^{-L\hat{T}\beta} \left|\log_2 \left(e^{\hat{T}\tilde{e}}\right)\right|^2 \right\}$$

The terms $\hat{T}^{m+2+\beta}e^{-L\hat{T}(m+1+\beta)}$, $\hat{T}^{m+4}e^{-L\hat{T}\beta}$ remain uniformly bounded for all $\hat{T} > 0$ with a constant that actually decreases when $L$ gets large. Thus, fixing $M$, a large parametric dimension in the second summand dominates, giving

$$\#\mathcal{N}_{u, \tilde{e}} \lesssim \max\{M^{\frac{(m+1)m+1}{\alpha}}, d_y\left(Me^{\hat{T}\tilde{e}}\right)^{m+1+\beta} \left|\log_2 \left(e^{\hat{T}\tilde{e}}\right)\right|^2 \} =: \phi(Me^{\hat{T}/\tilde{e}}),$$  (6.122)

which proves (4.19).

Regarding the remainder of the claim, recall from Theorem 4.6 that the approximations $\mathcal{N}_{z, \tilde{e}}$ have uniformly bounded composition norms $\|\mathcal{N}_{z, \tilde{e}}\|_{\#\mathcal{N}_{z, \tilde{e}, m}} \lesssim e^{\hat{T}\tilde{e}}$, see also Lemma 6.8

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To bound the composition norms of $\mathcal{N}_{u, \tilde{\varepsilon}}$, we recall that the composition norms of the network approximations to $u_0$ and $f$ are bounded by $M = \max\{1, \|u_0\|, \|f\|\}$. We then infer from Remark 2.14 (see also (2.16) and (2.15)), applied to the first line of (6.116), that

$$\|\mathcal{N}_{u, \tilde{\varepsilon}}\|_{\# \mathcal{N}_{u, \varepsilon, m}} \lesssim M e^{LT},$$

which is the asserted stability estimate. The coonvergence rate (4.20) follows from Remark 6.1 and Lemma 6.1 applied to the growth function $\phi$ in (6.122).

Regarding Remark 4.8, The same reasoning applies (with slightly simpler technicalities), replacing (6.120) by $\eta(\tilde{\varepsilon}) := \tilde{\varepsilon}/(\ln((\|u_0\| + \|f\|)/\tilde{\varepsilon}))$ while keeping $q(\tilde{\varepsilon})$ the same, $\square$

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Proof of Proposition 2.2  In this section we build mainly on findings from \([20, 36]\). Consider the “hat-function” \(\phi(x) := (1 - |x|)_+ = \max\{0, 1 - |x|\}\), \(x \in \mathbb{R}\), as well as the scaled and shifted versions \(\phi_{i,h}(x) := \phi(h^{-1}x - i), i \in \mathbb{Z}\), with support \(S_i = [(i-1)h, (i+1)h]\). We let \(h = 1/q\) for some integer \(q \in \mathbb{N}\), so that the restrictions of the \(\phi_{i,h}\) to \([0, 1]\) form a stable basis for all piecewise linears on \((0, 1)\) subordinate to the partition induced by the nodes \(\{ih = i/q : i = 0, \ldots, q\}\). Since each \(\phi_{i,h}\) is a second order divided difference of the ReLU rectifier \(\sigma(x) := x^+\) with respect to the nodes \(ih\) it has an exact representation as a univariate neural network of fixed finite depth and a fixed finite number of weights. The Lipschitz constant of \(\phi_{i,h}\) and hence of this network is clearly \(h^{-1}\). Abbreviating \(i := (i_1, \ldots, i_s) \in \{0, \ldots, q\}^s\) we consider next for \(x = (x_1, \ldots, x_s) \in \mathbb{R}^s\) the tensor products \(\phi_{i,h} := \phi_{i_1,h}(x_1) \cdots \phi_{i_s,h}(x_s)\), which obviously satisfy \(\phi_{i,h}(i') = \delta_{i,i'}\) for any \(i, i' \in \mathcal{I}_h := \{0, h, \ldots, hq\}^s\) while we still have \(\|\partial_j \phi_{i,h}\|_{\infty} \leq h^{-1}\). The next step consists in approximating each \(\phi_{i,h}\), viz. a product of univariate ReLU networks of fixed depth and number of weights by a ReLU network of input dimension \(s\). This is where one uses that the function \(M : \nu = (\nu_1, \ldots, \nu_s) \mapsto \prod_{j=1}^s \nu_s\) can be approximated by a ReLU network \(\mathcal{N}_{M,\delta}\) according to

\[
\|M - \mathcal{N}_{M,\delta}\|_{W^k((0,1)^s)} \leq \delta, \quad k \in \{0,1\},
\]

where the depth of \(\mathcal{N}_{M,\delta}\) as well as \#\(\mathcal{N}_{M,\delta}\) is bounded by a constant multiple of \(\log_2 \delta^{-1}\), with constants depending only on \(s\). Moreover, \(\mathcal{N}_{M,\delta}(0) = 0\). The case \(k = 0\) in (6.123) appears already in [36]. A key observation in [20, § C] is that \(k = 1\) still holds under the same complexity bounds. This is then used to show that for each \(i \in \mathcal{I}_h\) there exists a ReLU network \(\mathcal{N}_{i,\delta}\) such that

\[
\|\phi_{i,h} - \mathcal{N}_{i,\delta}\|_{W^k((0,1)^s)} \leq c^k \delta^{-1}\chi, \quad k \in \{0,1\},
\]

\[
\|\mathcal{N}_{i,\delta}\|_{\text{Lip}} \leq c\delta^{-1}, \quad \text{supp} \mathcal{N}_{i,\delta} \subseteq \text{supp} \phi_{i,h},
\]

(with a constant \(c\), depending on \(s\)) and

\[
\#\mathcal{N}_{i,\delta}, \text{ depth of } \mathcal{N}_{i,\delta} \preceq \log_2 \frac{1}{\delta},
\]

with constants depending only on \(s\). Now, given \(g\), consider the interpolant

\[
g_h := \sum_{i \in \mathcal{I}_h} g(ih)\phi_{i,h}.
\]

Obviously \(\|g_h\|_{\infty} \leq \|g\|_{\infty}\). We claim that \(g_h\) is also Lipschitz continuous. To see this, let \(\mathcal{I}_h(x) := \{i \in \mathcal{I}_h : \phi_{i,h}(x) \neq 0\}\) denote the collection of those nodes whose basis functions contain \(x \in (0, 1)^s\) in the interior of their support. Then for \(x \in (0, 1)^s\) let \(i(x) \in \mathcal{I}_h(x)\) denote the node closest to the Chebyshev center of the convex hull \([\mathcal{I}_h(x)]\) of \(\mathcal{I}_h(x)\). Then, since \(g(i(x)) \sum_{i \in \mathcal{I}_h(x)} \phi_{i,h}(x)\) is constant in a neighborhood of \(x\), one has

\[
|\partial_j g_h(x)| = \left| \sum_{i \in \mathcal{I}_h(x)} (g(ih) - g(i(x)h))\partial_j \phi_{i,h}(x) \right| \leq \max_{i \in \mathcal{I}_h(x)} |g(ih) - g(i(x)h)| \frac{1}{h} \#(\mathcal{I}_h(x)).
\]
Since $\text{diam } \mathcal{I}_h(x) \leq c h$ for a constant depending on $s$, this yields
\begin{equation}
|\partial_j g_h(x)| \leq c^{-1} \|g\|_{\text{Lip}_1}, \quad x \in (0,1)^s,
\end{equation}
from which it follows that (weakly)
\begin{equation}
|g_h(x) - g_h(x')| = \left| \int_0^1 \nabla g_h(x + t(x' - x)) \cdot (x' - x) dt \right| \leq |x - x'| \sqrt{5} c^{-1} \|g\|_{\text{Lip}_1}.
\end{equation}
Moreover,
\begin{equation}
|g(x) - g_h(x)| = \left| \sum_{i \in \mathcal{I}_h(x)} (g(x) - g(ih)) \phi_{i,h} \right| \leq C h \|g\|_{\text{Lip}_1},
\end{equation}
since $\max \{|x - ih| : i \in \mathcal{I}_h(x)\} \leq Ch$ with $C$ depending only on $s$. Given $\delta > 0$, the choice $h = h(\delta) \leq \frac{\delta}{2c \|g\|_{\text{Lip}_1}}$ ensures $|g(x) - g_h(x)| \leq \frac{\delta}{2}$. Now approximate each $\phi_{i,h(\delta)}$ by a ReLU network $\mathcal{N}_{i,\delta}$ with accuracy $\|\phi_{i,h(\delta)} - \mathcal{N}_{i,\delta}\|_\infty \leq c^* \delta$, with $c^*$ to be determined in a moment. We obtain, by (6.124)
\begin{equation}
\left\| g - \sum_{i \in \mathcal{I}_h(\delta)} g(ih) \mathcal{N}_{i,\delta} \right\|_\infty \leq \|g - g_h\|_\infty + \sup_{x \in (0,1)^s} \sum_{i \in \mathcal{I}_h(\delta)} \|g(ih)\| \|\phi_{i,h(\delta)}(x) - \mathcal{N}_{i,\delta}(x)\|
\end{equation}
\begin{equation}
\leq \frac{\delta}{2} + \sup_{x \in (0,1)^s} \# \mathcal{I}_h(\delta)(x) c^* \delta.
\end{equation}
Thus, choosing $c^* = (2 \sup_{x \in (0,1)^s} \# \mathcal{I}_h(\delta)(x))^{-1}$, we have confirmed
\begin{equation}
\left\| g - \sum_{i \in \mathcal{I}_h(\delta)} g(ih) \mathcal{N}_{i,\delta} \right\|_\infty \leq \delta.
\end{equation}
Moreover, defining $\mathcal{N}_\delta := \sum_{i \in \mathcal{I}_h} g(ih) \mathcal{N}_{i,\delta}$, we obtain, again by (6.124) and (6.127),
\begin{equation}
|\partial_j \mathcal{N}_\delta(x)| \leq |\partial_j (\mathcal{N}_\delta(x) - g_h(x))| + c^{-1} \|g\|_{\text{Lip}_1} = \left| \sum_{i \in \mathcal{I}_h(\delta)} g(ih) \left(\partial_j \mathcal{N}_{i,\delta}(x) - \partial_j \phi_{i,h}\right) \right|
\end{equation}
\begin{equation}
\leq c \|g\|_\infty \#(\mathcal{I}_h(\delta)) h^{-1} \delta + c^{-1} \|g\|_{\text{Lip}_1} \leq c'' \|g\|_\infty + 1 \|g\|_{\text{Lip}_1},
\end{equation}
where $c''$ depends only on $s$ and where we have used that $\frac{\delta}{2} \leq 4C \|g\|_{\text{Lip}_1}$. This confirms (2.3). Regarding the complexity (2.4) of $\mathcal{N}_\delta$, we have $\#(\mathcal{I}_h(\delta)) = h(\delta)^{-s} \leq c \delta^{-s} \|g\|_{\text{Lip}_1}^s$, which completes the proof because $\#\mathcal{N}_\delta \lesssim \#(\mathcal{I}_h(\delta)) \log_2 \frac{1}{\delta}$. \hfill $\square$

**Proof of Proposition 2.9** For each finite $N$ there is only a finite number of feasible dimensionality vectors $D$ with $\mathfrak{m}(D) \leq N$ which representations of $G \in \mathcal{C}_N$ may have. For each such $D$ consider first the following auxiliary classes. Let $\mathbb{F}_j \subset C(\mathbb{R}^{d_j-1}; \mathbb{R}^{d_j})$ be compact and let $\mathcal{C}(D, \mathbb{F}) := \{G \in \mathcal{X}_0, G = G_g, D(g) = D : g^j \in \mathbb{F}_j, j = 1, \ldots, n(D)\}$.

**Lemma 6.9** The collection
\begin{equation}
\mathcal{C}_N(\mathbb{F}) := \bigcup_{\mathfrak{m}(D) \leq N} \mathcal{C}(D, \mathbb{F}),
\end{equation}
is compact in $C(D_0; \mathbb{R}^{d_n(D)})$ (and so are the subsets $\mathcal{C}(D, \mathbb{F})$).
To establish first this Lemma, it is enough to confirm compactness of \( \mathcal{C}(D, F) \) for each of the eligible \( D \). To this end, let \( F, F' \), be dimensionally compatible compact subclasses so that for \( g \in F, h \in F' \) compositions \( h \circ g \) are defined. Then \( \{ h \circ g : h \in F', g \in F \} \) is compact in \( \Xi \). In fact, let \( (g_n)_{n \in \mathbb{N}}, (h_n)_{n \in \mathbb{N}} \) be uniformly bounded sequences in \( F, F' \), respectively. By Arzela-Ascoli’s Theorem they are equicontinuous and have a convergent subsequence with continuous limits \( g, h \), say. Denote these subsequences again by \( (g_n)_{n \in \mathbb{N}}, (h_n)_{n \in \mathbb{N}} \). Then,
\[
\| h_n \circ g_n - h \circ g \|_{\Xi(D_\alpha)} \leq \| h_n \circ g_n - h_n \circ g \|_{\Xi(D_\alpha)} + \| h_n \circ g - h \circ g \|_{\Xi(D_\alpha)}.
\]
By uniform convergence of the \( g_n \) and equicontinuity of the \( h_n \) the first summand becomes arbitrarily small for \( n \) large enough. By uniform convergence of the \( h_n \) the second summand gets small as well. Iterating this argument, shows that \( \mathcal{C}(D, F) \) is compact. Since \( \mathcal{C}(F) \) is a finite union of such sets the assertion of Lemma 6.9 follows.

The proof of Proposition 2.9 follows now from noticing that membership to \( \mathcal{C}(B) \) requires all composition factors to have uniformly bounded Lipschitz norm and hence belong to compact classes.

**Proof of Remark 2.11:** Fix \( B < \infty \) and let \( (f_j)_{j \in \mathbb{N}} \) be a sequence in \( K_{\gamma, s, \mathcal{R}^0}(B) \). Take a sequence \( (\varepsilon_k)_{k \in \mathbb{N}} \) of numbers decreasing monotonically to zero. For each \( f_j \) let \( g_{j,k} \) denote a compositional representation of some function in \( \mathcal{C}_{N_{\xi_k}, s} \) such that \((\text{see } (2.27)) \| G_{g_{j,k}} \|_{N_{\xi_k}, s, \mathcal{R}^0} \leq B \| f_j - G_{g_{j,k}} \|_{\Xi} \leq \varepsilon_k \). The complexity function \( \mathcal{N}(g_{j,k}) \) is controlled uniformly in \( j \) by \( N_{\xi_k} \), defined by (2.27). For fixed \( k \) the class \( \mathcal{C}_{N_{\xi_k}, s} \) is compact (Proposition 2.9). Therefore, for fixed \( k \), \( (G_{g_{j,k}})_{j \in \mathbb{N}} \) contains a subsequence (again denoted by \( (G_{g_{j,k}})_{j \in \mathbb{N}} \), converging uniformly to some \( G_k \in \mathcal{C}_{N_{\xi_k}, s} \). Now one can take a diagonalization argument, letting \( k \) tend to infinity, extracting a convergent subsequence from \( (f_j)_{j \in \mathbb{N}} \).

**Appendix B**

**Proof of Lemma 6.3** As for (a), let \( i(t) := \text{argmin}_{i=1,\ldots,q} |t - \xi_i| \) and since
\[
\rho_i(t) = \begin{cases} 
0, & t < \tau_{i-1}, \\
-t - \tau_{i-1}, & t \in J_i, \\
|I|/q, & t > \tau_i.
\end{cases}
\] (6.132)
(6.30) follows. Regarding (6.31), without loss of generality assume that \( t \leq t' \) so that
\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\rho_i(t) - \rho_i(t') & t, t' \leq \xi_{i-1} & t, t' \geq \xi_i & t < \xi_{i-1}, t' \in J_i & t \leq \xi_{i-1}, t' > \xi_i & t, t' \in J_i & t \in J_i, t' > \xi_i \\
\hline 
0 & \xi_i - t' & |J_i| = |I|/q & t - t' & t - \xi_{i-1} - |I|/q \\
\end{array}
\]
Hence \( \rho_i(t) \geq \rho_i(t'), i = 1, \ldots, q \). Specifically, assume that \( t' \in J_\nu, t \in J_\ell \). Then
\[
\sum_{i=1}^{q} |\rho_i(t) - \rho_i(t')| = \sum_{i=t}^{\nu} \rho_i(t) - \rho_i(t') = |J_\ell| - (t - \tau_{\ell-1}) + |J_{\ell+1}| + \cdots + |J_{\nu-1}| + t' - \tau_{\nu-1} \\
= t' - t + \tau_\nu - \tau_\ell - (\tau_{\nu-1} - \tau_{\ell-1}) = t' - t,
\]

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confirming claim (a).

As for (b), we obtain
\[
\int_0^t g(s)ds - \sum_{i=1}^q \rho_i(t)g_{j_i} = \sum_{i=1}^q \int_{J_i} \left( \chi_{s \leq t}(s) - \frac{\rho_i(t)}{|J_i|} \right)g(s)ds.
\]

Now suppose that \( t \in J_k \). By (6.132), we have \( \left( \chi_{s \leq t}(s) - \frac{\rho_i(t)}{|J_i|} \right)|_{J_i} = 0 \) for \( i \leq k - 1 \) while elementary calculations yield
\[
\left| \int_{\tau_{k-1}}^t \left( g(s) - \frac{\rho_k(t)}{|J_k|} \right)ds \right| = \left| \frac{\tau_k - t}{\tau_k - \tau_{k-1}} \int_{\tau_{k-1}}^t g(s)ds - \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}} \int_{t}^{\tau_k} g(s)ds \right|
\leq \frac{\tau_k - t}{\tau_k - \tau_{k-1}} \left\{ (t - \tau_{k-1})\|g\|_{L_\infty(J_k)} + (t - \tau_{k-1})\|g\|_{L_\infty(J_k)} \right\}
\leq 2\|g\|_{L_\infty(J_k)} \frac{(t - \tau_{k-1})(\tau_k - t)}{\tau_k - \tau_{k-1}} \leq \frac{|J_k|\|g\|_{L_\infty(J_k)}}{2},
\]
which is (6.32).

Concerning (c), By assumption \( \int_{J_i} |g(s) - g(\xi_i)| ds \leq |J_i|^2 L'/2 \) so that
\[
\left| \int_0^t g(s)ds - \sum_{i=1}^q \rho_i(t)g(\xi_i) \right| \leq \sum_{i=1}^{i(t)-1} \int_{J_i} |g(s) - g(\xi_i)| ds + \int_{\tau_{i(t)-1}}^t |g(s) - g(\xi_{i(t)})| ds
\leq i(t) \frac{|J_i|^2 L'}{2} = \frac{i(t) |J|^2 L'}{2q}.
\]

**Proof of Lemma 6.6** We consider first the case of fixed \( x \). Using (6.43) and (6.54), one finds for \( t \in J_k \)
\[|A_{x,I,q,N}(t, z; y) - A_{x,I,q,N}(t, z'; y')| \leq \sum_{i=1}^\ell \rho_i(t)\|a\| \max\{|y - y'|, |\bar{z}(\xi_i) - \bar{z}'(\xi_i)|\}, \quad (6.133)\]
where \( \|a\| \) plays the role of \( L \). To see the pattern,
\[|A_{x,I,q,N}^2(t, z; y) - A_{x,I,q,N}^2(t, z'; y')| \leq \sum_{i_1=1}^\ell \rho_{i_1}(t)\|a\| \max\{ \sum_{i_2=1}^{i_1} \rho_{i_2}(t)\|a\| \max\{|y - y'|, |\bar{z}(\xi_{i_1}) - \bar{z}'(\xi_{i_1})|\}\}.
\]
Inductively it follows that for \( t \in J_k \) and \( k \in \mathbb{N} \)
\[|A_{x,I,q,N}^k(t, z; y) - A_{x,I,q,N}^k(t, z'; y')| \leq \|a\|^k \sum_{\ell_i \geq 1, i_1 \geq i_2 \geq \cdots \geq i_k \geq 1} \rho_{i_1}(t) \cdots \rho_{i_k}(t) \max\{|y - y'|, |\bar{z}(\xi_{i_k}) - \bar{z}'(\xi_{i_k})|\}.
\]
Invoking (6.33), yields
\[
\sum_{\ell \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 1} \rho_{i_1}(t) \cdots \rho_{i_k}(t) \leq |I|^k \sum_{\ell \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 1} \frac{i_1 \cdots i_k}{q^k} \leq \frac{|I|^k}{k!},
\]
providing
\[
|A_{x,I,q,N}^k(t, \tilde{z}; y) - A_{x,I,q,N}^k(t, \tilde{z}'; y')| \leq \frac{(||a|||I||)^k}{k!} \max\{\|y - y'\|, \|\tilde{z} - \tilde{z}'\|_{L_{\infty}(I;\mathbb{R}^m)}\}. \tag{6.134}
\]
Similarly, for \(\tilde{z} = \tilde{z}_x, \tilde{z}' = \tilde{z}'_x\) (see (6.8)), we obtain
\[
|A_{x,I,q,N}(t, \tilde{z}; y) - A_{x',I,q,N}(t, \tilde{z}'; y')| \leq |x - x'| + \sum_{i=1}^\ell \rho_i(t)\|a\| \max\{\|y - y'\|, \|\tilde{z} - \tilde{z}'\|_{L_{\infty}(I;\mathbb{R}^m)}\},
\]
and hence inductively
\[
|A_{x,I,q,N}(t, \tilde{z}; y) - A_{x',I,q,N}(t, \tilde{z}'; y')| \leq \max \{\|y - y'\|, |x - x'|\} \sum_{\nu=0}^k \frac{(||a|||I||)^\nu}{\nu!}. \tag{6.135}
\]
Finally, by (6.46) and (6.31), one has for any \(t, t' \in I\)
\[
|A_{x,I,q,N}(t, \tilde{z}; y) - A_{x,I,q,N}(t', \tilde{z}; y)| \leq \sum_{i=1}^q |\rho_i(t) - \rho_i(t')| |\tilde{A}_{N,i}(\tilde{z}; y)| \leq \|a\| |t - t'|,
\]
where we have used (6.31) and the definition of \(\|a\|\). This confirms (6.56). The remaining claim follows from Remark 6.3, (6.51). \(\square\)