In the arena of the discrete phase space and continuous time, the theory of S-matrix is formulated. In the special case of Quantum-Electrodynamics (QED), the Feynman rules are precisely developed. These rules in the four-momentum turn out to be identical to the usual QED, except for the vertex function. The new vertex function is given by an infinite series which can only be treated in an asymptotic approximation at the present time. Preliminary approximations prove that the second order self-energies of a fermion and a photon in the discrete model have convergent improper integrals. In the final section, a sharper asymptotic analysis is employed. It is proved that in case the number of external photon or fermion lines is at least one, then the S-matrix elements converge in all orders. Moreover, there are no infra-red divergences in this formulation.

1. Introduction

It is well known that the Quantum-Electrodynamic (QED) has been verified to an extraordinary degree of precision by many experiments. However, it can never explain in its renormalized version the theoretical values of charge, mass etc. Moreover, it cannot produce in the classical limit the exact, non-perturbative solution of the coupled Maxwell-Dirac equations. Therefore, it is worthwhile to investigate the S-matrix, QED etc. in the discrete phase space and continuous time which involve non-singular Green’s functions, in order to obtain convergent results.

In the section-II, we compare and contrast the continuous and discrete methods involving perturbation series and Green’s functions for extremely simple mathematical problems. We conclude that in linear problems, both methods incur similar divergence difficulties. However, in non-linear problems, the continuous methods encounter divergences whereas the discrete methods avoid divergences in the lower order terms. Furthermore, the classical potential energy for two particles, as well as the Green’s functions are non-singular in the discrete model. Discussions of this section offer
suggestive insights into the complicated problems of QED in subsequent sections.

We assume that the free electromagnetic fields and Dirac fields obey the difference-differential equations of the paper-II. In the section-III, the interaction picture is introduced and the relativistic S-matrix is derived in the setting of the discrete phase space and continuous time.

In the next section, we introduce the trilinear interaction energy density of QED in the discrete model. We carefully derive Feynman rules in the four-momentum space following the difference-differential equations. The four-momentum variables do not have cut off limits in spite the presence of a characteristic length in the theory. The only difference with the usual continuous model is the appearance of the vertex function \( \delta^# \) instead of the usual delta functions \( \delta \). The distribution function \( \delta^# \) is both the joy and the pain of this research project. It is a joy since \( \delta^# \) is different from \( \delta \) (see the Appendix) and there is a slim chance of obtaining the divergence-free S-matrix. On the other hand, it is a pain because of the exceedingly difficult mathematics involved. We have obtained a simple asymptotic approximation \( d^# \) for \( \delta^# \) and apply it to the computations of the next section.

In the section-V, we apply the Feynman rules for the discrete phase-space to evaluate the second order fermionic as well as the photon self energy. In both cases, the improper integrals converge. Since the second order terms are the most dominant terms in the corresponding S-matrix series, these convergences are welcome news. In section-VI, the asymptotic analysis of the distribution function \( \delta^# \) and the infinite series representing an S-matrix element is sharpened considerably. We investigated an S-matrix element corresponding to the physical process involving \( E_B \) external boson lines and \( E_F \) external fermion lines. We prove that in the framework of the discrete phase space, the S-matrix elements converge in all orders provided \( E_B + (3/2)E_F > 1 \).

In the present formalism, the basic physical framework remains unchanged. Only the representations of the quantum mechanics and the quantum field theory are altered. Moreover, there are no infra-red divergences.

2. Comparison of continuous and discrete methods in simple problems

Let us consider a continuous wave function \( \exp(ikx) \) in a one dimensional space. It behaves as the following:

\[
\lim_{x \to \infty} \exp(ikx) \text{ is not defined;}
\]

\[
\lim_{k \to \infty} \exp(ikx) \text{ is not defined;}
\]

\[
\lim_{L \to \infty} \int_{-L}^{L} |\exp(ikx)|^2 dk \to \infty.
\] (1)
Now consider a momentum wave function in the discrete model (see equation II-A.1.6):

\[ \xi_n(k) := \frac{(i)^n \exp(-k^2/2) H_n(k)}{\pi^{1/2} 2^n \sqrt{n!}}, \]

\[ n \in \{0, 1, 2, 3, \ldots \} =: N. \tag{2} \]

Here, \( H_n(k) \) is a Hermite polynomial. In contrast to (2), the wave function \( \xi_n(k) \) satisfy:

\[ \lim_{n \to \infty} \xi_n(k) = \lim_{k \to \infty} \xi_n(k) = 0, \]

\[ \lim_{L \to \infty} \int_{-L}^{L} |\xi_n(k)|^2 dk = 1. \tag{3} \]

Now let us try to solve an extremely simple, first order, linear, non-homogeneous ordinary differential equation

\[ \psi'(x) = \phi(x), \quad x \in \mathbb{R}, \tag{4} \]

by the method of Green’s function. Here, \( \phi(x) \) is a prescribed differentiable complex-valued function over \( \mathbb{R} \).

The Green’s function for (4) is given by

\[ G(x - \hat{x}) := (2\pi i)^{-1} (\text{C.P.V.}) \int_{-\infty}^{\infty} k^{-1} \exp \{ ik(x - \hat{x}) \} dk \tag{5} \]

\[ = \frac{1}{2} \frac{(x - \hat{x})}{|x - \hat{x}|} \quad \text{for } x \neq \hat{x}, \]

\[ 0 \quad \text{for } x = \hat{x}; \]

\[ \frac{\partial}{\partial x} G(x - \hat{x}) = \delta(x - \hat{x}) \]

\[ \lim_{\hat{x} \to x} |\frac{\partial}{\partial x} G(x - \hat{x})| \to \infty. \tag{6} \]

The general solution of (4), with help of (6), is furnished by

\[ \psi(x) = \alpha + (\text{C.P.V.}) \int_{-\infty}^{\infty} G(x - \hat{x}) \phi(\hat{x}) d\hat{x}, \]

\[ = \alpha + \frac{1}{2} \lim_{L \to \infty} \left[ \int_{-L}^{x} \phi(\hat{x}) d\hat{x} - \int_{L}^{x} \phi(\hat{x}) d\hat{x} \right], \tag{7} \]

where \( \alpha \) is an arbitrary complex constant of integration.
Now we shall solve the corresponding complex, linear, nonhomogeneous difference equation

$$\Delta^# \psi(n) := \frac{1}{\sqrt{2}} \left[ \sqrt{n+1} \psi(n+1) - \sqrt{n} \psi(n-1) \right] = \phi(n),$$

$$n \in \mathbb{N}. \quad (8)$$

The Green’s function for this problem is furnished by

$$G(n, \hat{n}) := (i)^{-1} (C.P.V.) \int_{-\infty}^{\infty} (k)^{-1} \xi_n(k) \xi_{\hat{n}}(k) \, dk,$$

$$\Delta^# G(n, \hat{n}) = \delta_{n\hat{n}},$$

$$\Delta^# G(n, \hat{n})|_{\hat{n}=n} \equiv 1. \quad (9)$$

The general solution of (8) is given by

$$\psi(n) = \alpha \xi_n(0) - i \sum_{\hat{n}=0}^{\infty} \{ \phi(\hat{n}) \left[ (C.P.V.) \int_{-\infty}^{\infty} (k)^{-1} \xi_n(k) \xi_{\hat{n}}(k) \, dk \right] \}, \quad (10)$$

where \( \alpha \) is an arbitrary constant.

Let us apply the solutions (7) as well as (10) to the momentum eigenfunction problem in quantum mechanics, namely

$$P\psi = e\psi. \quad (11)$$

Here, we assume that the parameter \( e \neq 0 \) is sufficiently small. In the continuous (Schroedinger) representation of quantum mechanics (11) yields the differential equation

$$\psi'(x) = ie\psi(x), \quad x \in \mathbb{R}, \quad \psi(x) \neq 0. \quad (12)$$

The general solution of the above equation is obviously given by

$$\psi(x) = \alpha \exp(ie x) = \alpha \left[ 1 + iex + \frac{(iex)^2}{2} \right] + O(e^3). \quad (13)$$

Here, \( \alpha \) is an arbitrary, non-zero, complex constant.

Now we try to solve (12) by the perturbative expansion and the method of Green’s function in (7). Substituting an expansion

$$\psi(x) = \sum_{j=0}^{\infty} (ie)^j \psi_j(x) \quad (14)$$

into (12), we derive an infinite string of differential equations:

$$\psi'_0(x) = 0, \psi'_1(x) = \psi_0(x), \psi'_2(x) = \psi_1(x), ..., \psi'_j(x) = \psi_{j-1}(x), ... \quad (15)$$
Solving the first equation we obtain

$$\psi_0(x) = \alpha,$$  \hspace{1cm} (16)

where $\alpha \neq 0$ is otherwise arbitrary. In solutions of other equations, we shall ignore arbitrary constants. Solutions of the next two equations in (15) by the method of (17) are listed below.

$$\psi_1(x) = \left( \frac{\alpha}{2} \right) \left[ \lim_{L \to \infty} \left( \int_{-L}^{x} \hat{x} \, dx - \int_{x}^{L} \hat{x} \, dx \right) \right] = \alpha x,$$  \hspace{1cm} (17)

In the second order solution $\psi_2(x)$ we encounter a divergent term! Ignoring this term ("renormalizing"), by (16) and (17) we can recover the first three terms of the expansion in the RHS of (13).

Now, we shall try to solve the same problem (11) with the discrete representation of quantum mechanics. The corresponding difference equation and the exact general solution are provided by

$$\Delta^\# \psi(n) = i e \psi(n),$$

$$\psi(n) = \alpha \xi_n(e) = \alpha \left[ \xi_n(0) + e \xi'_n(0) + \frac{e^2}{2} \xi''_n(0) \right] + O(e^3),$$

$$\alpha \neq 0.$$  \hspace{1cm} (18)

Using a perturbative expansion

$$\psi(n) = \sum_{j=0}^{\infty} (i e)^j \psi_j(n),$$  \hspace{1cm} (19)

the difference equation in (18) yields the following infinite string of difference equations:

$$\Delta^\# \psi_0(n) = 0, \Delta^\# \psi_1(n) = \psi_0(n), ..., \Delta^\# \psi_j(n) = \psi_{j-1}(n), ...$$  \hspace{1cm} (20)

The first of the equations (20) is solved by (see Appendix-I of paper-II)

$$\psi_0(n) = \alpha \xi_n(0), \quad \psi_0(2n) = \alpha \xi_{2n}(0), \quad \psi_0(2n+1) = \alpha \xi_{2n+1}(0) \equiv 0.$$  \hspace{1cm} (21)
The second and third equations of (20) can be solved using (10) and (21). Ignoring arbitrary constants, these solutions are:

\[
\psi_1(n) = -i\alpha \left[ (C.P.V.) \int_{-\infty}^{\infty} k^{-1} \xi_n(k) \delta(k) \, dk \right],
\]

\[
\psi_1(2n) \equiv 0,
\]

\[
\psi_1(2n + 1) = -i\alpha \left\{ \lim_{k \to \infty} k^{-1} [\xi_{2n+1}(k) - \xi_{2n+1}(0)] \right\}
\]

\[
= -i\alpha \xi'_{2n+1}(0),
\]

\[
\psi_2(2n) = -\alpha \left\{ (C.P.V) \int_{-\infty}^{\infty} k^{-1} \xi_{2n}(k) \left[ \sum_{n=0}^{\infty} \xi'_{2n+1}(0) \xi_{2n+1}(k) \right] \, dk \right\}
\]

\[
= -\alpha \xi''(0) - \lim_{k \to 0} [k^{-2} \xi_{2n}(k)],
\]

\[
\psi_2(2n + 1) \equiv 0.
\]

(22)

(In deriving equations (22), we have used

\[
\sum_{-\infty}^{\infty} \xi'_{2n+1}(0) \xi_{2n+1}(k) = -\delta'(k), \int_{-\infty}^{\infty} f(k) \delta'(k) \, dk = -f'(k)
\]

etc.) The RHS of the equation for \( \psi_2(2n) \) has a divergent constant. Ignoring it and using equations (19), (21) and (22) we can recover the three terms of the RHS of the series in equation (18). So, we discover that the solution of the linear quantum mechanical problem in (11) with perturbative expansion and Green’s functions produces exactly similar divergence difficulties in the continuous or the discrete representation.

Now let us investigate a simple non-linear toy model by the two methods. Consider a first order non-linear ordinary differential equation:

\[
\psi'(x) = e [\psi(x)]^2.
\]

(24)

The exact general solution is given by

\[
\psi(x) = \alpha \left[ 1 - e\alpha x \right]^{-1},
\]

(25)

where \( \alpha \) is an arbitrary complex constant. In the case where \( \alpha \) is a non-zero real constant, the solution in (25) has a singularity at \( x = (e\alpha)^{-1} \). In case \( |\alpha x| < 1 \), we can elicit from (25) a series expansion

\[
\psi(x) = \alpha \sum_{j=0}^{\infty} (e\alpha x)^j.
\]

(26)

A perturbative expansion

\[
\psi(x) = \sum_{j=0}^{\infty} (e)^j \psi_j(x)
\]

(27)
leads to the following string of differential equations:

\[ \psi_0'(x) = 0, \psi_1'(x) = [\psi_0(x)]^2, \psi_2'(x) = 2\psi_0(x)\psi_1(x), \ldots. \] (28)

Using the solution (7) involving the Green’s function and ignoring arbitrary constants after the first solution, we obtain

\[
\begin{align*}
\psi_0(x) &= \alpha, \\
\psi_1(x) &= \alpha^2 x, \\
\psi_2(x) &= \alpha^3 \lim_{L \to \infty} \left[ \int_{-L}^{x} \hat{x} \, d\hat{x} - \int_{x}^{L} \hat{x} \, d\hat{x} \right] \\
&= \alpha^3 x^2 - \alpha^3 \lim_{L \to \infty} L^2.
\end{align*}
\] (29)

Ignoring the divergent term in the last equation, using equations (27) and (29), we recover the first three terms in the expansion of (26).

The corresponding non-linear difference equation is furnished by

\[ \Delta\# \psi(n) = e[\psi(n)]^2. \] (30)

With a perturbative expansion

\[ \psi(n) = \sum_{j=0}^{\infty} e^j \psi_j(n), \] (31)

the equation (30) implies that

\[ \Delta\# \psi_0(n) = 0, \Delta\# \psi_1(n) = [\psi_0(n)]^2, \Delta\# \psi_0(n) = 2\psi_0(n)\psi_1(n), \ldots. \] (32)

The solutions of these equations by the method of Green’s function in (10) lead to the following expressions:

\[
\begin{align*}
\psi_0(n) &= \alpha \xi_n(0), \\
\psi_1(2n) &= -i\alpha^2 \sum_{n=0}^{\infty} \{[\xi_{2n}(0)]^2 \left[ (C.P.V.) \int_{-\infty}^{\infty} k^{-1} \xi_{2n}(k) \xi_{2n}(k) \, dk \right] \} \\
&\equiv 0,
\end{align*}
\] (33)

\[
\begin{align*}
\psi_1(2n+1) &= -i\alpha^2 (C.P.V.) \int_{-\infty}^{\infty} \left\{ k^{-1} \xi_{2n+1}(k) \left[ \sum_{n=0}^{\infty} [\xi_{2n}(0)]^2 \xi_{2n}(k) \right] \right\} dk \approx -i\alpha^2 (C.P.V.) \int_{-\infty}^{\infty} (2\pi k)^{-1} \xi_{2n}(k)[4\sqrt{|k|}\delta(k) + |k|^{-1/2} dk].
\end{align*}
\] (34)

(Here we have used A.18.) In the last integral there are neither “ultraviolet” nor “infrared” divergences.
The third equation in \((32)\) yields the solution

\[
\psi_2(n) = -2\alpha^3 \sum_{n=0}^{\infty} \left\{ \left[ \xi_{2n}(0) \right] \left[ \xi_n(0) \right] \right\} \left( \text{C.P.V.} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\hat{k}k)^{-1} \xi_{2n}(\hat{k}) \xi_{2n}(k) \xi_n(k) \xi_{2n}(k) \, d\hat{k} \, dk \right\}
\]

\[
\equiv 0. \quad (35)
\]

In the RHS of equation \((35)\), the (C.P.V.) integral converges to zero since the integrand is odd with respect to the variable \(\hat{k}\). Therefore we conclude that for a non-linear problem, convergence is more likely in the discrete method compared to the continuous case.

No we shall discuss the vertex function in the one dimensional momentum space in the case of a trilinear interaction term. In the usual continuous case, it is given by

\[
(2\pi)^{-1} \text{ (C.P.V.) } \int_{-\infty}^{\infty} \exp \left[ i (p - q - k) x \right] \, dx = \delta(p - q - k). \quad (36)
\]

The above equation indicates a sharp conservation of momenta or wave numbers for three particles. In case functions are defined in the half line \(x > 0\), the corresponding vertex function is furnished by

\[
(2\pi)^{-1} \int_{0}^{\infty} \exp \left[ i (p - q - k) x \right] \, dx = \frac{1}{2} \left\{ \delta(p - q - k) + i \text{ sgn}(p - q - k) \frac{2}{|p - k - k|} \right\}. \quad (37)
\]

In the RHS, there is one term indicating the sharp conservation and another is representing one soft conservation. In the field theory in an one dimensional lattice space, the corresponding vertex function is

\[
(2\pi)^{-1} \sum_{n=-\infty}^{\infty} \exp[i(p - q - k)n] = \sum_{j=-\infty}^{\infty} \delta(p - q - k + 2\pi j). \quad (38)
\]

The above indicates a denumerably infinite sharp conservation of wave numbers. In a single discrete phase plane, the corresponding vertex function is specified by (see equations A.1):

\[
\delta^\#(p, -q, -k) := \sum_{n=0}^{\infty} \xi_n(p) \xi_n(q) \xi_n(k) \simeq d^\#(p, -q, -k).
\]

In the special case of \(p = 0\), it is explicitly furnished by (see A.18):

\[
d^\#(0, -q, -k) = \frac{1}{\pi} \left\{ \sqrt{|q + k|} \delta(q + k) + \sqrt{|q - k|} \delta(q - k) + \frac{1}{4} \left[ |q + k|^{-1/2} + |q - k|^{-1/2} \right] \right\}. \quad (39)
\]
The RHS of the above equation demonstrates two sharp as well as two soft conservations. Moreover, spontaneous reflections of momenta (on lattice points) are allowed. (These reflections are analogous to Bragg reflections on the lattice planes of a crystal.)

Now we investigate the potential equation in our discrete model. It is given by:

$$\delta^{ab} \Delta^a \Delta^b V(\mathbf{n}) = 0, \ (\mathbf{n}) := (n^1, n^2, n^3) \in \mathbb{N}^3. \quad (40)$$

The corresponding Green’s function (which is the potential energy between two unit charges) satisfy:

$$G(\mathbf{n}, \hat{\mathbf{n}}) = \int_{\mathbb{R}^3} (\mathbf{k} \cdot \mathbf{k})^{-1} \left[ \prod_{a=1}^{3} \xi_{n^a}(k_a) \xi_{\hat{n}^a}(k_a) \right] d^3 k,$$

$$\lim_{\hat{\mathbf{n}} \to \mathbf{n}} |G(\mathbf{n}, \hat{\mathbf{n}})| < \infty. \quad (41)$$

The convergences of the integrals above are due to the facts that $H_{n^a}(k_a)$ is a polynomial and there is a decaying weight factor $\exp[-(k \cdot k)]$ in products of $\xi_{n^a}(k_a)$.

Now we consider the Green’s functions (equation (II-A.II.1.B)) for the difference-differential Klein-Gordon equation. The coincidence limits are provided by:

$$\Delta_{(a)}(\mathbf{n}, t; \mathbf{n}, t; \mu) = (2\pi)^{-1} \pi^{-3/2} \int_{\mathbb{R}^3} \left\{ \exp[-k_1^2 - k_2^2 - k_3^2] \left[ \prod_{j=1}^{3} \frac{[H_{n^j}(k_j)]^2}{2^{n^j}(n_j)!} \right] \right\} d^3 k \times \left[ \int_{C_{(a)}} (\eta^\alpha k_\alpha k_\beta + \mu^2)^{-1} \right] d^4 k. \quad (42)$$

Because of the weight factor $\exp[-k_1^2 - k_2^2 - k_3^2]$, the integral on the RHS of (42) converges. All of the Green’s functions for free fields in our discrete model are non-singular.

The discussions of this present section will provide valuable insights into the complicated topics of the following sections.

3. The interaction picture and the S-matrix

We shall follow the same notations as in the previous papers I and II. The equations of the relativistic quantum fields are expressed exclusively by the partial difference-differential equations in this paper. We shall now derive the S-matrix for interacting fields in the sequel. In the interaction picture, the time-evolution of a Hilbert space (state) vector $|\Psi_I(t)\rangle$ (representing a many particle system) is governed by the differential equation

$$i\partial_t |\Psi_I(t)\rangle = H_I(t) |\Psi_I(t)\rangle. \quad (43)$$
Here, \( H_I(t) \) represents the hermitian operator corresponding to the \textit{total interaction energy} at the instant \( t \). We can express \( H_I(t) \) in terms of the \textit{interaction energy density operator} \( \mathcal{H}_I(n, t) \) by the triple sum:

\[
H_I(t) := \sum_{n=0}^{\infty} \mathcal{H}_I(n, t), \tag{44}
\]

\[
\mathcal{H}_I(t) = -\mathcal{L}_I(n, t). \tag{45}
\]

Here, \( \mathcal{L}_I(n, t) \) stands for the interaction Lagrangian density and it is a relativistic invariant operator. (The equation (45) holds for most of the useful cases.) A necessary micro-causality requirement is

\[
[\mathcal{H}_I(n, t), \mathcal{H}_I(\hat{n}, t)] \equiv 0 \quad \text{for} \quad n \neq \hat{n}. \tag{46}
\]

Usually, the interaction energy density operator \( \mathcal{H}_I(n, t) \) has an overall multiplier which is small. It is customary to solve (43) by perturbative series involving this small parameter. Moreover, the S-matrix is the operator which takes a prescribed initial state \( |i\rangle \) into another prescribed final state \( |f\rangle \) consistent with the evolution equation (43). The perturbative series expansion for the S-matrix is furnished by

\[
S = I + \sum_{j=1}^{\infty} S_j = I + \sum_{j=1}^{\infty} [(-i)^j/j!] \sum_{n_1=0}^{\infty} \int_{\mathbb{R}} dt_1 \sum_{n_2=0}^{\infty} \int_{\mathbb{R}} dt_2 \ldots \sum_{n_j=0}^{\infty} \int_{\mathbb{R}} dt_j \{ T[H_I(n_1, t_1)H_I(n_2, t_2)\ldots H_I(n_j, t_j)] \}. \tag{47}
\]

Here, \( T \) stands for Wick’s \textit{time ordering operator}. There is another operator \( N \) called \textit{normal ordering}. It arranges creation operators to the left of annihilation operators. Furthermore, there is still another operation called \textit{contraction} between two operators and it is defined by:

\[
A(n, t)B(\hat{n}, \hat{t}) := T[A(n, t)B(\hat{n}, \hat{t})] - N[A(n, t)B(\hat{n}, \hat{t})]. \tag{48}
\]

We can extract from the commutation and anti-commutation relations (II-3.10B), (II-4.9B), (II-5.11Bi-vii) and the linear relationships (II-A.II.4B), (II-A.II.8c), the following examples of contractions:

\[
A_\mu(n, t)A_\nu(\hat{n}, \hat{t}) = -i\eta_{\mu\nu}D_F(n, t; \hat{n}, \hat{t}) I, \tag{49}
\]

\[
\Psi(n, t)\bar{\Psi}(\hat{n}, \hat{t}) = \bar{\Psi}(n, t)\Psi(\hat{n}, \hat{t}) \equiv 0, \]

\[
\Psi(n, t)\bar{\Psi}(\hat{n}, \hat{t}) = -\bar{\Psi}(\hat{n}, \hat{t})\Psi(n, t) = iS_F(n, t; \hat{n}, \hat{t}; m) I, \]

\[
\Psi(n, t)A_\mu(\hat{n}, \hat{t}) = \bar{\Psi}(n, t)A_\mu(\hat{n}, \hat{t}) \equiv 0.
\]

The Green’s functions \( \Delta_F(\cdots) \), \( D_F(\cdots) \), and \( S_F(\cdots) \) are all defined in Appendix-II of paper-II. (We should mention that these Green’s functions are analogues
of the causal Green’s functions of Stuckelberg in the +2 signature and differ from the corresponding Feynman-Dyson propagators by a factor $(−i2)$.

Now, we shall state the Wick’s theorem on the decomposition of a chronological product in the arena of the discrete phase space and continuous time. It can be succinctly stated as:

\[
T[A(n_1, t_1)B(n_2, t_2)C(n_3, t_3)..... J(n_j, t_j)]
= N[A(n_1, t_1)B(n_2, t_2)C(n_3, t_3)..... J(n_j, t_j)]
+ N\{[A(n_1, t_1)B(n_2, t_2)C(n_3, t_3)..... J(n_j, t_j)] + all other single contractions\}
+ ....
+ N\{[A(n_1, t_1)B(n_2, t_2)C(n_3, t_3)..... J(n_j, t_j)] + all other double contractions\}
+ ....
+ N\{terms with maximal number of contractions\}
\] (50)

(This theorem is provable by induction.)

4. Feynman rules of Q.E.D. in discrete phase-space and continuous time

We choose the interaction density as

\[
H_I(n, t) := −ieN[\bar{\Psi}(n, t)\gamma^\mu\Psi(n, t)A_\mu(n, t)].
\] (51)

Here, the parameter $|e| = \sqrt{4\pi/137}$ is a small positive number. The interaction term in (51) is derived from the principle of minimal electromagnetic interaction. According to this principle, the difference and differential operators $\Delta_a^\#$, $\partial_t$ in the free Lagrangian density (II-5.3) are replaced by $\Delta_a^\# − ieA_a(n, t)$ and $\partial_t − ieA_4(n, t)$ respectively. Moreover, from the discussions in section-IV of paper-I, it is amply clear that the interaction energy density in (51) is a relativistic invariant. Furthermore, the micro-causality condition (3.3) is satisfied by interaction (51) due to equations (II-3.10B), (II-4.9B), and (II-5.11B i-vii).

The equations (3.4) and (51) yield for the S-matrix:

\[
S = I + \sum_{j=1}^{\infty} S_j = I + \sum_{j=1}^{\infty} [(-e)^j/j!] \sum_{n_1=0}^{\infty} \int_R dt_1... \sum_{n_j=0}^{\infty} \int_R dt_j T\{N[\bar{\Psi}(n_1, t_1)\gamma^\mu_1\Psi(n_1, t_1)A_{\mu_1}(n_1, t_1)]...
N[\bar{\Psi}(n_j, t_j)\gamma^\mu_j\Psi(n_j, t_j)A_{\mu_j}(n_j, t_j)]\}.\] (52)

The R.H.S. of (52) is a relativistic invariant operator. Let us consider the second order term $S_2$ in (52) for the sake of simplicity. By the Wick’s decomposition (50) and equations (49i-v), $S_2$ can be reduced to the following
Two of the operators $\Gamma_{(A)}(\ldots)$ are explicitly furnished below as:

$$
\Gamma_{(5)}(n_1, t_1; n_2, t_2) = -N[\gamma^\mu \Psi(n_1, t_1)S_F(n_2, t_2; n_1, t_1; m)\eta_{\mu\nu}D_F(n_2, t_2; n_1, t_1)\bar{\Psi}(n_2, t_2)\gamma^\nu],
\Gamma_{(6)}(n_1, t_1; n_2, t_2) = \Gamma_{(5)}(n_2, t_2; n_1, t_1).
$$

The second order terms $S_{2(5)} = S_{2(6)}$ contribute towards the self-energy of an electron.

Let us now work out the matrix entry $\langle f | \Gamma_{(5)}(\ldots) | i \rangle$ for the initial state of one electron $|i\rangle := \alpha^\dagger_\nu(p)|\Psi_0\rangle$ and the final state of one electron $|f\rangle := \alpha^\dagger_\nu(\tilde{p})|\Psi_0\rangle$.

Using equations (II-5.7Bii), (II-5.8), (II-5.10), (II-A.II.5B), and (II-A.II.6B), the operator $\Gamma_{(5)}(\ldots)$ in (54) yields

$$
\langle f | \Gamma_{(5)}(\ldots) | i \rangle = m[\hat{E}(\tilde{p})E(p)]^{-1/2}\hat{u}_\nu(\tilde{p})\gamma^\mu\{\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} d^4k d^4q
\left[\prod_{b=1}^{3} \xi_{n_2}^j(\hat{p}_b)\xi_{n_2}^j(k_b)\xi_{n_2}^j(q_b)\right](2\pi)^{-1}\exp[i(\hat{E} + k_4 + q_4)t_2]\eta_{\mu\nu}
(k^\alpha k_\alpha - \epsilon^\gamma \epsilon_{\gamma\rho}\eta_{\mu\nu} - mI)(q^\rho q_\rho + m^2 - \epsilon^\gamma \epsilon_{\gamma\rho})^{-1}\gamma^\nu\}
\int_{\mathbb{R}^4} d^4q : = d^3q dq^4 = -d^3q dq_4.
$$

We shall now define a new distribution function (see the Appendix) by the triple sum

$$
\delta_3^+(p, q, k) := \sum_{n=0}^\infty \prod_{j=1}^{3} \xi_{n_j}(p_j)\xi_{n_j}(q_j)\xi_{n_j}(k_j)]
= \sum_{n^{(1)}=0}^\infty \sum_{n^{(2)}=0}^\infty \sum_{n^{(3)}=0}^\infty \xi_{n^{(1)}}(p_1)\xi_{n^{(2)}}(q_1)\xi_{n^{(3)}}(k_1)]
[\xi_{n^{(2)}}(p_2)\xi_{n^{(2)}}(q_2)\xi_{n^{(2)}}(k_2)]\xi_{n^{(3)}}(p_3)\xi_{n^{(3)}}(q_3)\xi_{n^{(3)}}(k_3).
$$

Therefore, by the equations (53), (56), and (56), we obtain the second order contributions for a fermionic self-energy as

$$
\langle f | S_2 | i \rangle = \langle f | S_{2(5)} + S_{2(6)} | i \rangle = (2!)(f | S_{2(5)} | i)$$

12
\[ = e^2m[E(\hat{p})E(p)]^{-1/2}\hat{u}_s(\hat{p})\{ \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} d^4q d^4k \delta^3_3(\hat{p}, -q, -k) \delta(\hat{E} - q^4 - k^4)\gamma^\mu(q_\alpha q_\alpha + m^2 - i\epsilon)^{-1}(i\gamma^\rho q_\rho - mI)\eta_{\mu\nu}(k^\beta k^\beta - i\epsilon)^{-1}\gamma^\nu \delta^3_3(p, -q, -k)\delta(q^4 + k^4 - E)\}u_r(p). \]

A graphic way to represent the right hand side of the equation (57) is by Fig.1.

![Figure 1: The second order fermion self-energy graph in the momentum space](image)

This expression and other matrix elements give rise to table -1 of Feynman Rules.

The Feynman rules in the four-momentum space (instead of the rules in the discrete phase space and continuous time) will be used in the sequel for the sake of simplicity. Integration of all the internal (or virtual) bosons and fermions (or anti-fermions) must be performed over \( \mathbb{R}^4 \). (The fourth component of the momentum is integrated over the real axis because of the addition of \(-i\epsilon\) in the propagators.) The limit \( \epsilon \to 0^+ \) is taken after all the internal integrations are performed.

However, the actual construction of a matrix element \( \langle f|S_j|i \rangle \) in (52) from Table-I is somewhat incomplete until we determine the correct numerical factor which multiplies the ordered product of operators with the same \( j \) and the same physical process. It can be deduced that the multiplicative factor must be

\[ \kappa_j := (-1)^l \text{sgn}(\sigma)(-q|e|^j). \]  

Here, \( l \) is the number of closed loops, \( \sigma \) is the permutation of the final fermions, and \( q = -1 \) for electrons, \( \pm1/3 \) for quarks etc. The integer \( j \) is the number of vertices. In case, \( \delta^3_3(p, -k, -q) \) in Table-I is replaced by \((2\pi)^3\delta^3(p - k - q)\), we obtain exactly the same Feynman rules which emerge out of the usual theory. However, we shall prove in the Appendix
Table-1: The second order photon self-energy graph in the momentum space.

That $\delta_3^\#(p, -k, -q) \neq (2\pi)^3\delta^3(p - k - q)$. Feynman rules in Table-I are all manifestly relativistic except possibly the vertex term.

According to the Feynman rules in Table-I, energy is precisely conserved at each vertex. Therefore, we can introduce a physically meaningful matrix $M^\#$ by:

$$\langle f|M^\#|i \rangle := \langle f|\sum_{j=1}^{\infty} M_j^\#|i \rangle,$$

$$\langle f|S_j|i \rangle = i(2\pi)\delta(E_{(f)} - E_{(i)})\langle f|M_j^\#|i \rangle,$$

$$\langle f|S - I|i \rangle = i(2\pi)\delta(E_{(f)} - E_{(i)})\langle f|M^\#|i \rangle,$$

(59)

where $E_{(i)}$ and $E_{(f)}$ are the initial and final energies respectively.

The transition probability from the initial state $|i \rangle$ into the final state $|f \rangle$ per unit time is provided by:

$$\omega_{(f)(i)} := (2\pi)^2\delta(E_{(f)} - E_{(i)})|\langle f|M^\#|i \rangle|^2.$$

(60)
5. The second order self-energies of electron and photon

For the analysis of the second order self-energy of an electron we start with the equation (57). Moreover, utilizing (59) we can write

\[ \langle f | S_2^2 | i \rangle = i (2\pi) \delta (\hat{E} - E) \langle f | M^2_2 | i \rangle \]

(61)

\[ = -2e^2 m [E(\hat{p})E(p)]^{-1/2} \tilde{u}_s(\hat{p}) \sum_{(2)} (p, \hat{p}) u_r(p), \]

\[ -2 \sum_{(2)} (p, \hat{p}) : = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} d^4q d^4k \delta_3(\hat{p}, -q, -k) \delta(\hat{E} - q^4 - k^4) \]

\[ \times \delta^2(\hat{p}, -q, -k) \delta(-E + q^4 + k^4) \gamma^\mu (i\gamma^\rho q^\rho - m I) \]

\[ \times (q^\alpha q_\alpha + m^2 - i\epsilon)^{-1} (k^\beta k_\beta - i\epsilon)^{-1} \gamma_\mu. \]

(62)

Using the properties of the Dirac matrices and the asymptotic approximation \(d_3^\# (p, q, k)\) in equation (A.18), we obtain from (62) that

\[ \sum_{(2)} (p, \hat{p}) \simeq \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} d^4q d^4k \delta_3(\hat{p}, -q, -k) \delta(\hat{E} - q^4 - k^4) \]

\[ \delta(-E + q^4 + k^4) (i\gamma^\rho q^\rho + 2m I) (q^\alpha q_\alpha + m^2 - i\epsilon)^{-1} \]

\[ \times (k^\beta k_\beta - i\epsilon)^{-1}. \]

(63)

The last two factors in the above integrand can be combined by the parameter \(x\) into

\[ (q^\alpha q_\alpha + m^2 - i\epsilon)^{-1} (k^\beta k_\beta - i\epsilon)^{-1} = \int_0^1 dx \left[ x(q^\alpha q_\alpha + m^2) + (1 - x)k^\beta k_\beta - i\epsilon \right]^{-2}. \]

(64)

Moreover, we make the simplifying assumptions

\[ \hat{p} = p = 0. \]

(65)

By (63 - 65) and (A.15) we obtain
\[
\sum_{(2)}^{\#} (0, m; 0, m) = \lim_{\epsilon \to 0^+} \int_{R^4} \int_{R^4} \int_{0^1} d^4 q d^4 k d x |d^\#_{(2)} (0, -q, -k)|^2
\]

\[
[\delta (m - q^4 - k^4)]^2 (i \gamma^\rho q_\rho + 2mI) \left[ x(q^\alpha q_\alpha + m^2) + (1 - x) k^\beta k_\beta - i \epsilon \right]^{-1}
\]

\[
= (4)^{-1} (4\pi)^{-6} \lim_{\epsilon \to 0^+} \int_{R^4} \int_{R^4} \int_{0^1} d^4 k d^4 q d x \left[ \delta (m - q^4 - k^4) \right]^2 \prod_{a=1}^3 (4^3 \{ |k_a + q_a| [\delta(k_a + q_a)^2 + 2|k_a^2 - q_a^2|^{1/2} \delta(k_a + q_a) \delta(k_a - q_a) \})
\]

\[
+ 2(4)^2 \{ \delta(k_a + q_a) + \delta(k_a - q_a) + |(k_a + q_a)/(k_a - q_a)|^{1/2} \delta(k_a + q_a) + |(k_a - q_a)/(k_a + q_a)|^{1/2} \delta(k_a - q_a) \}
\]

\[
+ 4 \left[ |k_a + q_a|^{-1} + |k_a - q_a|^{-1} + 2|k_a^2 - q_a^2|^{-1/2} \right] (i \gamma^\rho q_\rho + 2mI) \left[ x(q^\alpha q_\alpha + m^2) + (1 - x) k^\beta k_\beta - i \epsilon \right]^{-2}.
\] (66)

To analyze the above integral we employ some mathematical devices. (i) We put \( \sqrt{|k|} \delta(k) = |k| \delta(k)|^2 \equiv 0 \), \( \sqrt{|(k + q)/(k - q)|} \delta(k + q) = 0 \) for \( k \neq -q \). (ii) Whenever there is a factor in the integrand which is odd with respect to some integration variable, we drop that term. With such tricks, the RHS of (66) can be reduced to the form

\[
4 \sum_{(2)}^{\#} (0, m; 0, m) = (4)^3 (4\pi)^{-6} \delta(0) \left[ 4^6 A + 4^4 (B_1 + B_2 + B_3) + 4^2 (C_1 + C_2 + C_3) + D \right],
\] (67)

\[
A := \lim_{\epsilon \to 0^+} \int_{R^4} d^4 k \int_{0^1} dx \left[ -i \gamma^4 (m - k^4) + 2mI \right] [k^b k_b - (k^4 - mx)^2 + m^2 x^2 - i \epsilon]^{-2},
\] (68)

\[
B_1 := \lim_{\epsilon \to 0^+} \int_{R^2} d q_1 \int_{R^4} d^4 k \int_{0^1} dx \left[ -i \gamma^4 (m - k^4) + 2mI \right] [k^b k_b - (k^4 - mx)^2 + m^2 x^2 + x(q_1^2 - k_1^2 - i \epsilon)^{-2} [k_1 + q_1] + |k_1 - q_1| + 2|k_1^2 - q_1^2|^{-1/2}] + 2mI,
\]

\[
C_1 := \lim_{\epsilon \to 0^+} \int_{R^2} d q_1 d q_2 \int_{R^2} d^4 k \int_{0^1} dx \left[ -i \gamma^4 (m - k^4) + 2mI \right] [k^b k_b - (k^4 - mx)^2 + m^2 x^2 + x(q_1^2 + q_2^2 - k_1^2 - k_2^2 - i \epsilon)^{-2} [k_2 + q_2]^{-1} + |k_2 - q_2|^{-1} + 2|k_2^2 - q_2^2|^{-1/2}] + 2mI,
\]

\[
D := \lim_{\epsilon \to 0^+} \int_{R^2} d^3 q \int_{R^4} d^4 k \int_{0^1} dx \left[ -i \gamma^4 (m - k^4) + 2mI \right] [k^b k_b - (k^4 - mx)^2 + m^2 x^2 + x(q^c q_c - k^c k_c - i \epsilon)^{-2} \prod_{a=1}^3 (|k_a + q_a|^{-1} + |k_a - q_a|^{-1} + 2|k_a^2 - q_a^2|^{-1/2})].
\] (69)
In the above expression, a mathematically unacceptable symbol, \( \delta(0) \) occurs. We can explain it by physical arguments of energy conservation \( \delta(\hat{E} - E) = \delta(0) \) in (63) and (59).

We note that \( A, B_1, C_1, D \) are all “logarithmically divergent” integrals. Therefore, symbolically we can write:

\[
\sum_{(2)}^\# (0, m; 0, m) = m \delta(0) \text{[log. div. terms].} \quad (70)
\]

From the \( S_2 \)-matrix element of the usual quantum field theory in the space-time continuum, we obtain

\[
\sum_{(2)} (0, m; 0, m) = m \left[ \delta(0) \right]^4 \int_{\mathbb{R}^4} d^4k \int_0^1 dx (k^\beta k_\beta + m^2 x^2)^{-2} \left[ 2I - i\gamma^4 (1 - x) \right] \\
= m \left[ \delta(0) \right]^4 \text{[log. div. terms].} \quad (71)
\]

Comparing (70) and (71) we conclude that the \( S_2 \) matrix element for the fermionic self-energy case in the discrete phase space and continuous time does converge!

We shall now work out the second order photon self-energy (see fig.2. Below.)

![Figure 2: The second order photon self-energy graph in the momentum space.](image)

This self energy is given by the Feynman rules in Table-I as proportional to

\[
\Pi^{\# \mu \nu} (k) : = \lim_{\epsilon \to 0_+} \{ \text{Tr} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} d^4p d^4\hat{p} \gamma^\mu \left[ \frac{i\gamma^\rho \hat{p}_\rho - mI}{\hat{p}^\rho \hat{p}_\rho + m^2 - i\epsilon} \right] \\
\times \left[ \frac{i\gamma^\rho \hat{p}_\rho - mI}{\hat{p}^\rho \hat{p}_\rho + m^2 - i\epsilon} \right] \gamma^\nu \left[ \delta(-\hat{p}^4 + \hat{p}^4 + \nu)^2 \right] \\
\times \prod_{c=1}^3 \left| \delta^\# (-p_c, \hat{p}_c, k_c) \right|^2 . \quad (72)
\]
We use the asymptotic approximation (A-I.21) to evaluate $\Pi^{#\mu\nu}(k)$ above. It is given by (dropping $\epsilon$)

\[
\Pi^{#\mu\nu}(k) \approx -(4\pi)^{-6} \delta(0) \text{Tr}\{\gamma^\mu \left[ 8\hat{A} + 2 \left( \hat{B}_1 + \hat{B}_2 + \hat{B}_3 \right) \right] \gamma^\nu \},
\]

\[
\hat{A} := \int_{\mathbb{R}^4} d^4p \left[ i\gamma^\alpha p_\alpha - mI \right] \left[ i\gamma^4(p^4 - \nu) + mI \right] \left[ p^\beta p_\beta + m^2 \right]^{-1}
\times \left\{ \left[ (p_1 + k_1)^2 + (p_2 + k_2)^2 + (p_3 + k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
+ \left[ (p_1 + k_1)^2 + (p_2 + k_2)^2 + (p_3 - k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
+ \left[ (p_1 - k_1)^2 + (p_2 + k_2)^2 + (p_3 + k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
+ \left[ (p_1 - k_1)^2 + (p_2 - k_2)^2 + (p_3 + k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
+ \left[ (p_1 - k_1)^2 + (p_2 - k_2)^2 + (p_3 - k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
+ \left[ (p_1 - k_1)^2 + (p_2 - k_2)^2 + (p_3 - k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
\right\},
\]

\[
\hat{B}_1 := \int_{\mathbb{R}^4} d^4p \int_{\mathbb{R}^2} dp_1 \left[ i\gamma^\alpha p_\alpha - mI \right] \left[ i\gamma^4(p^4 - \nu) + mI \right] \left[ p^\beta p_\beta + m^2 \right]^{-1}
\times \left\{ \left[ p_1^2 + (p_2 + k_2)^2 + (p_3 + k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
+ \left[ p_1^2 + (p_2 + k_2)^2 + (p_3 - k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
+ \left[ p_1^2 + (p_2 - k_2)^2 + (p_3 + k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
+ \left[ p_1^2 + (p_2 - k_2)^2 + (p_3 - k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
\right\}
\times \left\{ \left[ |p_1 - \hat{p}_1 - k_1|^{-1} + |p_1 + \hat{p}_1 + k_1|^{-1} + |p_1 - \hat{p}_1 + k_1|^{-1} + |p_1 + \hat{p}_1 - k_1|^{-1}
+ \frac{(p_1 - \hat{p}_1 - k_1)(p_1 + \hat{p}_1 + k_1)}{1 + \text{sgn}(p_1 - \hat{p}_1 - k_1) \text{sgn}(p_1 + \hat{p}_1 + k_1)}
+ \frac{(p_1 - \hat{p}_1 - k_1)(p_1 - \hat{p}_1 + k_1)}{1 + \text{sgn}(p_1 - \hat{p}_1 - k_1) \text{sgn}(p_1 - \hat{p}_1 + k_1)}
+ \frac{(p_1 - \hat{p}_1 - k_1)(p_1 + \hat{p}_1 - k_1)}{1 + \text{sgn}(p_1 - \hat{p}_1 - k_1) \text{sgn}(p_1 + \hat{p}_1 - k_1)}
+ \frac{(p_1 + \hat{p}_1 + k_1)(p_1 - \hat{p}_1 + k_1)}{1 + \text{sgn}(p_1 + \hat{p}_1 + k_1) \text{sgn}(p_1 - \hat{p}_1 + k_1)}
+ \frac{(p_1 + \hat{p}_1 + k_1)(p_1 + \hat{p}_1 - k_1)}{1 + \text{sgn}(p_1 + \hat{p}_1 + k_1) \text{sgn}(p_1 + \hat{p}_1 - k_1)}
+ \frac{(p_1 + \hat{p}_1 + k_1)(p_1 - \hat{p}_1 - k_1)}{1 + \text{sgn}(p_1 + \hat{p}_1 + k_1) \text{sgn}(p_1 - \hat{p}_1 - k_1)} \right\}
\]

\[
\hat{C} := \int_{\mathbb{R}^4} d^4p \int_{\mathbb{R}^2} dp_1 dp_2 \left[ i\gamma^\alpha p_\alpha - mI \right] \left[ i\gamma^4(p^4 - \nu) + mI \right] \left[ p^\beta p_\beta + m^2 \right]^{-1}
\times \left\{ \left[ p_1^2 + p_2^2 + (p_3 + k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
+ \left[ p_1^2 + p_2^2 + (p_3 - k_3)^2 - (p_4 - \nu)^2 + m^2 \right]^{-1}
\right\}
\]
\[
\times \left\{ \prod_{A=1}^{2} \left[ |p_A + \hat{p}_A + k_A|^{-1} + |p_A - \hat{p}_A - k_A|^{-1} + |p_A - \hat{p}_A - k_A|^{-1} + |p_A + \hat{p}_A - k_A|^{-1} \right] + |(p_A + \hat{p}_A + k_A)(p_A - \hat{p}_A - k_A)|^{-1/2} \left( 1 + \text{sgn}(p_A + \hat{p}_A + k_A) \text{sgn}(p_A - \hat{p}_A - k_A) \right) \\
+ |(p_A + \hat{p}_A + k_A)(p_A + \hat{p}_A - k_A)|^{-1/2} \left( 1 + \text{sgn}(p_A + \hat{p}_A + k_A) \text{sgn}(p_A + \hat{p}_A - k_A) \right) \right. \\
+ |(p_A + \hat{p}_A + k_A)(p_A - \hat{p}_A + k_A)|^{-1/2} \left( 1 + \text{sgn}(p_A + \hat{p}_A + k_A) \text{sgn}(p_A - \hat{p}_A + k_A) \right) \\
+ |(p_A - \hat{p}_A + k_A)(p_A + \hat{p}_A - k_A)|^{-1/2} \left( 1 - \text{sgn}(p_A - \hat{p}_A + k_A) \text{sgn}(p_A + \hat{p}_A - k_A) \right) \\
+ |(p_A - \hat{p}_A - k_A)(p_A - \hat{p}_A + k_A)|^{-1/2} \left( 1 - \text{sgn}(p_A - \hat{p}_A - k_A) \text{sgn}(p_A - \hat{p}_A + k_A) \right) \\
+ |(p_A + \hat{p}_A - k_A)(p_A - \hat{p}_A + k_A)|^{-1/2} \left( 1 + \text{sgn}(p_A + \hat{p}_A - k_A) \text{sgn}(p_A - \hat{p}_A + k_A) \right) \right] \\
\]

\[
\hat{D} := \int_{\mathbb{R}^4} d^4p \int_{\mathbb{R}^3} d^3\mathbf{p} \left[ i\gamma^\alpha p_\alpha - mI \right] \left[ i\gamma^4 (p^4 - \nu) + mI \right] \\
\times \left[ p^2 p_\beta + m^2 \right]^{-1} \left[ \hat{p}_b \hat{p}_b - (p^4 - \nu)^2 + m^2 \right]^{-1} \\
\times \left\{ \prod_{c=1}^{3} \left[ |p_c - \hat{p}_c - k_c|^{-1} + |p_c + \hat{p}_c + k_c|^{-1} + |p_c - \hat{p}_c + k_c|^{-1} + |p_c + \hat{p}_c - k_c|^{-1} \right] + |(p_c - \hat{p}_c - k_c)(p_c - \hat{p}_c - k_c)|^{-1/2} \left( 1 + \text{sgn}(p_c - \hat{p}_c - k_c) \text{sgn}(p_c + \hat{p}_c + k_c) \right) \\
+ |(p_c - \hat{p}_c - k_c)(p_c - \hat{p}_c + k_c)|^{-1/2} \left( 1 - \text{sgn}(p_c - \hat{p}_c - k_c) \text{sgn}(p_c - \hat{p}_c + k_c) \right) \\
+ |(p_c - \hat{p}_c + k_c)(p_c + \hat{p}_c - k_c)|^{-1/2} \left( 1 - \text{sgn}(p_c - \hat{p}_c + k_c) \text{sgn}(p_c + \hat{p}_c - k_c) \right) \\
+ |(p_c + \hat{p}_c + k_c)(p_c - \hat{p}_c + k_c)|^{-1/2} \left( 1 - \text{sgn}(p_c + \hat{p}_c + k_c) \text{sgn}(p_c - \hat{p}_c + k_c) \right) \\
+ |(p_c + \hat{p}_c + k_c)(p_c + \hat{p}_c + k_c)|^{-1/2} \left( 1 - \text{sgn}(p_c + \hat{p}_c + k_c) \text{sgn}(p_c + \hat{p}_c - k_c) \right) \right] \right) \\
\]

(74)

The analysis of the integrands in \( \hat{A}, \hat{B}_a, \hat{C}_a, \hat{D} \) imply that

\[
\Pi^{\#\mu\nu} = \delta(0) \left[ \text{quadratically divergent terms} \right]. \\
\]

(75)

However, the expression from the usual theory in continuous space-time indicates that

\[
\Pi^{\mu\nu} = [\delta(0)]^4 \left[ \text{quadratically divergent terms} \right]. \\
\]

(76)

Comparing (75) and (76), we conclude that the second order photon self-energy integral converges in the discrete phase space and continuous time formulation.

6. **Convergence of the S-matrix element** \( \langle f | S_j | i \rangle \)

We start this section with some simple examples. Let us consider the improper Riemann integral \( \int_{-\infty}^{\infty} k \, dk \). This integral does not converge. However, the Cauchy principal value C.P.V. \( \int_{-\infty}^{\infty} k \, dk \) “converges” to zero. Now
consider another improper integral \( \int_{-\infty}^{\infty} k^2 \, dk \). It diverges (strongly) and its Cauchy principal value does not converge either. Now let us investigate the infinite series \( 1 - x + 2x^2 - (3!)x^3 + (4!)x^4 - \ldots \). It is a divergent series for \( x \neq 0 \). However, it is Borel-summable to the value \( \int_0^{\infty} e^{-t}[1 + xt]^{-1} \, dt \) (which converges). Consider another (complex) infinite series \( \sum_{n=0}^{\infty} \exp[i2^n x] \).

It is strongly divergent for all \( x \in \mathbb{R} \) and cannot be “summed” in any sense.

In this section, our primary goal is to investigate the convergence of the \( j \)-th order S-matrix element \( \langle f \mid S_j \mid i \rangle \), where \( |i> \) and \( |f> \) are initial and final states respectively. This matrix element is furnished by the equation (72). We apply Wick’s theorem inherent in (50) and consider only terms with a specific number of contractions. There exist many terms even in that category. We pick a typical term \( \langle f \mid \overline{S}_j \mid i \rangle \) in a particular category. Using the Table-I, we can express the matrix element as the following infinite series of integrals:

\[
\langle f \mid \overline{S}_j \mid i \rangle = (\text{const}) \sum_{n_1=0}^{\infty(3)} \int_{\mathbb{R}} dt_1 \ldots \sum_{n_j=0}^{\infty(3)} \int_{\mathbb{R}} dt_j \left\{ \langle f \mid N [\gamma^{\mu_1} \psi(n_1,t_1)A_{\mu_1}(n_1,t_1) \right.
\left. \gamma^{\mu_2} S_F(n_1,t_1;n_2,t_2;m)\eta_{\mu_2 \mu_3} D_F(n_2,t_2;n_3,t_3)S_F(n_2,t_2;n_3,t_3;m)\gamma^{\mu_3} \ldots \right.
\left. S_F(n_{j-1},t_{j-1};n_j,t_j;m)\tilde{\psi}\gamma^{\mu_j} A_{\mu_j}(n_j,t_j) \rangle \right\}. \tag{77}
\]

The right hand side of the above equation contains multiple series and multiple integrals. There exist three possibilities. It can be convergent, or summable in some sense, or strongly divergent. Before embarking upon such investigations, let us consider a similar problem for a much simpler series of integrals:

\[
\sum_{n=0}^{\infty} \left[ c_n \int_{-\infty}^{\infty} f_n(k) \, dk \right].
\]

A necessary criterion for convergence is that \( \int_{-\infty}^{\infty} f_n(k) \, dk \) converges (or converges as Cauchy principal value) for each \( n \in \{0,1,2,\ldots\} \) to \( s_n \). If furthermore, \( \sum_0^{\infty} (c_n s_n) \) converges, then the original series of integrals is said to converge. In case the series \( \sum_0^{\infty} (c_n s_n) \) does not converge, we should determine whether it is summable or strongly divergent. (In the case of the right hand side of (77), such analysis is extremely cumbersome.) In case the analysis of \( \sum_0^{\infty} (c_n s_n) \) is very difficult, we can try the asymptotic analysis. In this strategy, criteria for “convergence” of \( \sum_{n=0}^{\infty} \left[ c_n \int_{-\infty}^{\infty} f_n(k) \, dk \right] \) is summarized by the following steps:

i) Prove that C.P.V. \( \int_{-\infty}^{\infty} f_n(k) \, dk \) converges to \( s_n \) for every \( n \in \{0,1,2,\ldots\} \).

ii) Express the series as \( \sum_{n=0}^{N} \left[ c_n \int_{-\infty}^{\infty} f_n(k) \, dk \right] + \sum_{n=N+1}^{\infty} \left[ c_n \int_{-\infty}^{\infty} f_n(k) \, dk \right] \) for a sufficiently large positive integer \( N \).

iii) Prove that the series \( \sum_{n=1}^{\infty} c_{N+n} f_{N+n}(k) \) converges to \( \sigma_N(k) \).
iv) Check that \( \lim_{N \to \infty} \sigma_N(k) = 0 \).

v) Finally, prove that C.P.V. \( \int_{-\infty}^{\infty} \sigma_N(k) \, dk = \int_{-\infty}^{\infty} \left[ \sum_{n=N+1}^{\infty} c_n f_n(k) \right] \, dk \) converges.

We shall follow a similar five step strategy to investigate the convergence of the matrix element \( \langle f \mid \mathbf{S}_j \mid i \rangle \) in (77). For the step \( i \), we have to investigate improper integrals representing propagators (See the Appendix-II of the paper-II.) These are furnished by:

\[
D_{(a)}(\mathbf{n}, t; \hat{n}, \hat{t}) = (2\pi)^{-1} \int_{\mathbb{R}^3} \left[ \prod_{b=1}^{3} \xi_{n_b}(k_b) \xi_{n_b}(k_b) \right] \left[ \int_{C_{(a)}} (\eta^{\alpha\beta} k_{\alpha} k_{\beta})^{-1} \exp \left[ ik_1(t - \hat{t}) \right] \, dk^4 \right] d^3k, \quad (78)
\]

\[
S_{(a)}(\mathbf{n}, t; \hat{n}, \hat{t}) = (2\pi)^{-1} \int_{\mathbb{R}^3} \left[ \prod_{b=1}^{3} \xi_{n_b}(k_b) \xi_{n_b}(k_b) \right] \left[ \int_{C_{(a)}} (i\gamma^\mu p_\mu + mI)^{-1} \exp \left[ ip_1(t - \hat{t}) \right] \, dp^4 \right] d^3p. \quad (79)
\]

With the help of reference\(^{10}\), we have integrated explicitly the right hand side of (78) in case of \( D_+(\mathbf{n}, t; \hat{n}, \hat{t}) \). The result is the following:

\[
D_+(\mathbf{n}, t; \hat{n}, \hat{t}) = -i \left[ 2(2\sqrt{\pi})^3 \right]^{-1} \exp \left[ -(t - \hat{t})^2/8 \right] \prod_{b=1}^{3} (i)^{n_b - \bar{n}_b} \sqrt{n_b! \bar{n}_b!} \\
\times \sum_{j_1=0}^{[n_1/2]} \sum_{j_3=0}^{[\bar{n}_3/2]} (-2)^{-j_1+j_3} \left[ 1 + (-1)^{n_1+\bar{n}_1-2(j_1+j_3)} \right] \left[ 1 + (-1)^{n_2+\bar{n}_2-2(j_2+j_3)} \right] \\
\times \left[ 1 + (-1)^{n_3+\bar{n}_3-2(j_3+j_3)} \right] \left[ j_1! \ldots j_3!(n_1 - 2j_1)! \ldots (\bar{n}_3 - 2\bar{j}_3)! \right]^{-1} \\
\times \Gamma \left[ 2^{-1} \left( n_1 + \bar{n}_1 - 2(j_1 + \hat{j}_1) \right) + 1 \right] \Gamma \left[ 2^{-1} \left( n_2 + \bar{n}_2 - 2(j_2 + \hat{j}_2) \right) + 1 \right] \\
\times \Gamma \left[ 2^{-1} \left( n_3 + \bar{n}_3 - 2(j_3 + \hat{j}_3) \right) + 1 \right] \left[ \Gamma \left[ 2^{-1} (n_1 + \ldots + \bar{n}_3 - 2(j_1 + \ldots + \hat{j}_3)) + 3 \right] \right]^{-1} \\
\times D_-(n_1 + \ldots + \bar{n}_3 - 2(j_1 + \ldots + \hat{j}_3) + 2) \left( i(t - \hat{t})/\sqrt{2} \right); \quad (80)
\]

\[
\left[ \frac{n}{2} \right] : = n/2 \quad \text{for } n \text{ even}, \quad (n-1)/2 \quad \text{for } n \text{ odd}. \quad (81)
\]

Here, \( D_{-p}(iz) \) is the parabolic cylinder function\(^{10}\).
The right hand side of (80) is defined (or non-singular) everywhere. In fact, in the coincident point,

\[
D_+(0,0;0,0) = -i\sqrt{\pi} = -D_-(0,0;0,0).
\]

(82)

Thus, by equations (A.II.4B) and (A.II.5B) of the paper-II, \(D_F(n,t;\hat{n},\hat{t})\) in (78) is non-singular everywhere. Similarly, the integral representing \(S_F(n,t;\hat{n},\hat{t};m)\) in (79) converges by the equations

\[
S_F(n,t;\hat{n},\hat{t};m) = (\gamma^\mu\Delta^\nu - mI) \Delta_F(n,t;\hat{n},\hat{t};m);
\]

\[
D_F(n,t;\hat{n},\hat{t}) := \Delta_F(n,t;\hat{n},\hat{t};0).
\]

(83)

In fact, in the coincident points,

\[
S_+(0,0;0,0;m) = 2^{-1} \left[ -\gamma^4 + im^3\Psi(3/2,2;m^2)I \right],
\]

\[
S_-(0,0;0,0;m) = 2^{-1} \left[ -\gamma^4 - im^3\Psi(3/2,2;m^2)I \right].
\]

(84)

Here, \(\Psi(\alpha,\beta;z)\) is a degenerate hypergeometric function\(^{10}\).

In the integrations for \(D_+(n,t;\hat{n},\hat{t})\) in (80) and \(S_\pm(0,0;0,0;m)\) in (6.8) neither ultraviolet nor infrared divergences are encountered.

Since wave functions \(A_\mu(n,t)\) and \(\psi(n,t)\) are defined everywhere, we can conclude that (initial) finite part of the infinite series in (77), namely

\[
(const) \sum_{n_1=0}^{N_1(3)} \int_{R} dt_1 \ldots \sum_{n_j=0}^{N_j(3)} \int_{R} dt_j \{ <f|..|i> \}
\]

(85)

is always non-singular. We have to investigate the convergence of the remaining tail-end:

\[
(const) \sum_{n_1=N_1+1}^{\infty} \int_{R} dt_1 \ldots \sum_{n_j=N_j+1}^{\infty} \int_{R} dt_j \{ <f|..|i> \};
\]

(86)

where \(1 := (1,1,1)\). For that purpose we take recourse to the analogue of step iii) in the strategy. We have to prove that

\[
\sum_{n_1=N_1+1}^{\infty} \int_{R} dt_1 \ldots \sum_{n_j=N_j+1}^{\infty} \int_{R^4} dt_j \int_{R^4} d^4k_1 \ldots \int_{R^4} d^4p_1 \ldots \{ \text{Fourier Transform of } <f|..|i> \};
\]

(87)

converges. In this endeavour, we use Fourier transforms of all fields (as in (4.5)). Thus we encounter at each vertex the modified distribution function:

\[
\delta^\#_{\Delta N}(p,q,k) := \sum_{n_1=N_1+1}^{\infty} \sum_{n_2=N_2+1}^{\infty} \sum_{n_3=N_3+1}^{\infty} \left[ \prod_{b=1}^{3} \xi_{n_b}(p_b)\xi_{n_b}(q_b)\xi_{n_b}(k_b) \right].
\]

(88)
By the theorem A.1 of the Appendix, the infinite oscillatory series converges

\[ \sum_{n=N+1}^{\infty} \xi_n(p)\xi_n(q)\xi_n(k), \quad (p, q, k) \in \mathbb{R}^3. \]  

The exact sum in (89) is intractable presently. Therefore, we try the asymptotic approximation for a sufficiently large positive \( N \), as in the Appendix, by the expression

\[ \delta_N^N(p, q, k) \approx d_N^N(p, q, k) := \sum_{n=N+1}^{\infty} \xi_n(p)\xi_n(q)\xi_n(k) \]  

\[ \approx (2\sqrt{\pi})^{-3} \sum_{n=N+1}^{\infty} n^{-3/4} \left[ e^{i2\sqrt{n}(p+q+k)} + e^{-i2\sqrt{n}(p+q+k)} ight. 
\left. + e^{-i2\sqrt{n}(p+q-k)} + e^{i2\sqrt{n}(p+q-k)} \right]. \]

By the theorem A.1 of the Appendix, the infinite oscillatory series converges as

\[ \sum_{n=N+1}^{\infty} n^{-3/4} e^{i2k\sqrt{n}} \approx \int_{N}^{\infty} y^{-3/4} e^{i2k\sqrt{y}} dy. \]

The improper integral in (91) converges\(^{10} \) (except for \( k \neq 0 \)) to

\[ \int_{N}^{\infty} y^{-3/4} e^{i2k\sqrt{y}} dy \approx \left\{ \sqrt{|k|} \delta(k) + (1/4)|k|^{-1/2} \times \left[ (1 - i)\Gamma \left( -1/2, i2\sqrt{N}|k| \right) + (1 + i)\Gamma \left( -1/2, -i2\sqrt{N}|k| \right) \right] 
\right. 
\left. - N^{-1/4}|k|^{-1} \sin(2\sqrt{N}|k|) 
\right. 
\left. + i2^{-1}|k|^{-1/2}\text{sgn}(k)[(1 + i)\Gamma(1/2, i2\sqrt{N}|k|) 
\right) 
\left. + (1 - i)\Gamma(1/2, -i2\sqrt{N}|k|)] \right\}. \]

Here, \( \Gamma(\alpha, iz) \) is the incomplete gamma function. Using the asymptotic representation of the incomplete gamma function\(^{10} \) for very large \( \sqrt{N}|k| \), we derive from (90) (91) (92) that

\[ \int_{N}^{\infty} y^{-3/4} e^{i2k\sqrt{y}} dy = N^{-1/4} \cos(2\sqrt{N}|k|)\delta(k) + isgn(k) \exp[i2\sqrt{N}k] + O(1/N^{3/4}|k|^2) \]

\[ \approx \left( N^{-1/4}|k|^{-1} \right) \left[ N^{-1/2}\delta(k) + isgn(k) \right]. \]

Therefore, by the equations (92) we conclude that

\[ (2\sqrt{\pi})^3 d_N^N(p, q, k) \approx N^{-1/4} \left\{ |p + q + k|^{-1} \left[ N^{-1/2}\delta(p + q + k) + isgn(p + q + k) \right] 
\right. 
\left. + |q + k - p|^{-1} \left[ N^{-1/2}\delta(q + k - p) + isgn(q + k - p) \right] 
\right. 
\left. + |q - k + p|^{-1} \left[ N^{-1/2}\delta(q - k + p) + isgn(q - k + p) \right] 
\right. 
\left. + |q - k - p|^{-1} \left[ N^{-1/2}\delta(q - k - p) + isgn(q - k - p) \right] \right\} 
\left. \right\} \approx : \sigma_N(p, q, k). \]
Thus, by equations (90) and (93) we obtain the asymptotic approximation

\((8\pi^{3/2})^3 \delta_{3N}^\# (p, q, k) \simeq (8\pi^{3/2})^3 d_{3N}^\# (p, q, k)\)

\[

\simeq \prod_{b=1}^{3} \sigma_{N_b} (p_b, q_b, k_b)
\]

\[

= \prod_{b=1}^{3} \{N_b^{-1/4} |p_b + q_b + k_b|^{-1} [N_b^{-1/2} \delta (p_b, q_b, k_b) + i \text{sgn}(p_b + q_b + k_b)] + .. + ..\}. \tag{95}
\]

For completion of the step iii) in the strategy, we need to consider \(j\) products of the triple series as in (6.12). These multiple series obviously converge asymptotically to the products of terms as in [95].

As for the step iv) of the strategy, we notice that the analogous necessary condition

\[

\lim_{(N_1, N_2, N_3) \to (\infty, \infty, \infty)} \left[ \prod_{b=1}^{3} \sigma_{N_b} (p_b, q_b, k_b) \right] = 0 \tag{96}
\]

is satisfied by [95].

Now we have to investigate the most important step v) of the strategy. Suppose that for the S-matrix element \(\langle f | S_j | i \rangle\) in (77) yielding the equation (6.10), the corresponding Feynman graph involves the following specifications:

- \(E_F\) = The number of external electron-positron or fermion lines,
- \(E_B\) = The number of external photon or boson lines,
- \(I_F\) = The number of internal fermion lines,
- \(I_B\) = The number of internal boson lines,
- \(j\) = The number of corner or vertices.

For a trilinear interaction as in the equation (51), the following equations among various numbers hold:

\[

I_F = j - (E_F/2) \geq 0,

I_B = (j - E_B)/2 \geq 0,

j = I_F + (E_F/2) = 2I_B + E_B \geq 0. \tag{97}
\]

The typical S-matrix element in (6.10), with the help of the modified Feynman rules in momentum space (Table-I), yields the asymptotic approximation:

\[

< f | S_j | i > \simeq (\text{const}) \int_{\mathbb{R}^{4I_F}} d^4 k_{(1)}..d^4 k_{(I_B)} \int_{\mathbb{R}^{4I_F}} d^4 p_{(1)}..d^4 p_{(I_F)}
\]
Integrating over all total energy. The number of integration variables in (99) is 3(\[\hat{\kappa}\] alone, is

From (100) and (101) we derive that

There exist other degrees of divergences arising from the mixed products of

of divergence with product of all

\(|p|^{-1}\) factors

The Dirac delta function \(\delta\left(\sum (E_L + \nu(L))\right)\) indicates the conservation of total energy. The number of integration variables in (99) is \(3(I_B + I_F) + (I_B + I_F - j + 1)\). At high energy-momentum, the factor with products of all \(p^{-1}, k^{-2}, |p|^{-1}\text{sgn}(p)\) and \(|k|^{-1}\text{sgn}(k)\) in the integrand, behaves asymptotically as the power \(-I_F - 2I_B - 3j\). Therefore, counting the degree, \(\kappa\), of ultra-violet divergence with \(p^{-1}, k^{-2}, |k|^{-1}\text{sgn}(k)\) and \(|p|^{-1}\text{sgn}(p)\) factors alone, is

\[\kappa := 3(I_B + I_F) + (I_B + I_F - j + 1) - I_F - 2I_B - 3j = 1 - E_B - (3/2)E_F.\] (100)

The degree, \(\hat{\kappa}\) of divergence with product of all \(|p|^{-1}\delta(p)\) and \(|k|^{-1}\delta(k)\) functions alone (after integrating out all \(\delta\text{-functions}\)) is

\[\hat{\kappa} := 4(I_B + I_F - j + 1) - I_F - 2I_B - 3j = 4 - 3j - E_B - (3/2)E_F.\] (101)

From (100) and (101) we derive that

\[\kappa - \hat{\kappa} = 3(j - 1) \geq 0.\] (102)

There exist other degrees of divergences arising from the mixed products of sgn(\(..\)) and \(\delta(\..\)). These degrees are all less or equal to \(\kappa\). Therefore, the final criterion for “convergence” of the S-matrix element \(<f|\Sigma_{jN}|i>\) or \(<f|\Sigma_{j}|i>\) is that

\[\kappa < 0,\] (3/2)E_F + E_B > 1. (103)

25
The above inequality proves that in case the number of external boson lines is at least two, the S-matrix elements “converge” in every order of $j$. However, in case the number of external bosonic lines is one, the S-matrix element vanishes by Furry’s theorem. Therefore if there is at least one bosonic external line, the S-matrix elements converge in every order. Now, the number of incoming external fermionic lines always equals to the number of outgoing fermionic lines. Therefore, we conclude from the inequality (102) that in case the number of incoming external fermion lines is at least one, the S-matrix elements “converge” in every order $j$. Thus, our final conclusion is that the S-matrix elements in the discrete phase space formulation converges in every order provided there is at least one external line in the process.

Let us consider some specific examples. In case of the self-energy of an electron, as discussed in section-5, the relevant numbers are as follows:

$$E_F = 2, \quad E_B = 0, \quad \kappa = -2 < 0,$$

$$(3/2)E_F + E_B = 3 > 1. \quad (104)$$

Therefore, by the equations (102), the S-matrix elements representing the self-energy of the electron “converges” in every order.

As a second example, consider the self-energy of the photon. By the treatment in section-5, we deal with the numbers:

$$E_F = 0, \quad E_B = 2, \quad \kappa = -1 < 0,$$

$$(3/2)E_F + E_B = 2 > 1. \quad (105)$$

Again, the corresponding S-matrix elements for the self-energy of the photon “converges” in every order.

It can be shown that each of the so called primitive diagrams, “converges” in every order within the arena of the discrete phase space formulation of the S-matrix.

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A Appendix: The distribution function $\delta^#(p, q, k)$

We define the distribution function $\delta^#$ by the following equation;

$$\delta^#(p, q, k) := \sum_{n=0}^{\infty} \xi_n(p) \xi_n(q) \xi_n(k),$$

$(p, q, k) \in \mathbb{R}^3$.

(106)

We assume that the right-hand side is at least summable in some sense. The distribution function $\delta^#$ so defined, is totally symmetric with respect to the real variables $p, q, k$ and thus satisfies

$$\delta^#(p, q, k) = \delta^#(q, p, k) = \delta^#(p, k, q) = \delta^#(k, q, p).$$

(107)

It has also the following properties by the consequences (A.7) of paper-II:

$$\delta^#(p, q, k) = \delta^#(p, -q, -k) = \delta^#(-p, q, -k),$$

(108)

$$\delta^#(-p, -q, -k) = \delta^#(p, q, -k) = \delta^#(-p, q, k) = \delta^#(k, q, p),$$

(109)

Now we shall state and prove in a simple manner the following theorem.

**Theorem A.1:** The distribution function $\delta^#$ is distinct from the Dirac $(2\pi)\delta$-distribution.

**Proof:** Assuming the interchangeability of the summation and integration, and using the equations (II-A.I.1) and (II-A.I.19) we obtain

$$\int_{\mathbb{R}} \xi_m(k) \delta^#(p, q, k) \, dk = \sum_{n=0}^{\infty} \left[ \int_{\mathbb{R}} \xi_m(k) \xi_n(k) \, dk \right] \xi_n(p) \xi_n(q),$$

$$= \xi_m(p) \xi_m(q).$$

(110)

(Here the subscript $m$ is not summed!) However, by the equation (II-A.I.16), we have

$$\int_{\mathbb{R}} \xi_m(k) \delta(p + q + k) \, dk = \xi_m(-p - q) = \xi_m(p + q)$$

$$= (\pi)^{4-2m/2} \exp[(p - q)^2/2]$$

$$\times \sum_{j=0}^{m} \sqrt{\binom{m}{j}} \xi_{m-j}(\sqrt{2}p) \xi_j(\sqrt{2}q).$$

(111)

Comparing the equations (110) and (111) we conclude that $\delta^#(p, q, k) \neq (2\pi)\delta(p + q + k)$.

Now we shall introduce another sequence of functions

$$\zeta_n(k) := \xi_n(k) \text{ for } n \in \{0, 1\},$$

$$\pi^{-1/2} n^{-1/4} \cos(2k\sqrt{n}) \text{ for } n \in \{2, 4, 6, \ldots\},$$

$$\pi^{-1/2} n^{-1/4} \sin(2k\sqrt{n}) \text{ for } n \in \{3, 5, 7, \ldots\}.$$  (112)
If we introduce a new inner-product and norm for a non-separable space \( l^2 \) by
\[
\langle \xi | \eta \rangle^\# : = \lim_{N \to \infty} \left( \frac{\pi}{2\sqrt{N}} \sum_{n=0}^{2N+1} \xi_n \eta_n \right),
\]
\[
||\xi||^\# : = \sqrt{\langle \xi | \xi \rangle},
\]
then it can be shown by (112) that
\[
||\xi(k) - \zeta(k)|| = 0.
\]
Therefore, it is reasonable to introduce an asymptotic approximation of \( \delta^\#(p,q,k) \) in (106) by the expression
\[
d^\#(p,q,k) : = \sum_{n=0}^{\infty} \zeta_n(p) \zeta_n(q) \zeta_n(k)
\approx \pi^{-3/4} e^{-(p^2+q^2+k^2)/2} \left[ 1 - i \frac{3}{2} \frac{p q k}{2} \right]
+(1/2) \sum_{n=2}^{\infty} \pi^{-3/4} \left[ e^{i2\sqrt{n}(p+q)} + e^{-i2\sqrt{n}(q-p)}
+ e^{-i2\sqrt{n}(q-k+p)} + e^{i2\sqrt{n}(q-k-p)} \right].
\]

The convergence or summability of the above series is not obvious. However, we shall discuss some related results. We can cite\(^9\) the following theorem.

**Theorem A.2:** Let \( f \) and \( p \) be two real-valued and continuous functions defined over \([a, \infty)\). Let \( \{b_n\}_0^\infty \) be a monotone increasing sequence such that \( b_0 = a \) and \( \lim_{m \to \infty} b_m \to \infty \). Moreover, let \( f(x) > 0 \) and \((-1)^m p(x) \geq 0 \) for \( x \in [b_m, b_{m+1}] \subset [a, \infty) \) Then, the improper (Riemann or Lebesgue) integral \( \int_a^\infty f(x)p(x) \, dx \) converges to \( \alpha \) iff the alternating series
\[
\sum_{n=0}^{\infty} c_n := \sum_{n=0}^{\infty} \left[ \int_{b_n}^{b_{n+1}} f(x)p(x) \, dx \right]
\]
converges to \( \alpha \). (For proof see the reference 9.) Now we shall prove the following corollary.

**Corollary A.1:** There exists a sequence \( \{\Theta_n(k)\}_0^\infty \) with the property \( \Theta_n(k) \in (0,1) \) for every \( n \in \mathbb{N} \) such that the series
\[
\sum_{n=0}^{\infty} \frac{\sin[(n+\Theta_n(k))\pi]}{[n+\Theta_n(k)]^\beta}
\]
converges to \( \frac{\Gamma(1-\beta) \sin[(1-\beta)\pi/2]}{[k^{1-\beta}]} \) for all \( k > 0 \) and all \( \beta \in (0,1) \).

**Proof:** It is known\(^{10}\) that for \( k > 0 \) and \( 0 < \beta < 1 \), the improper integral
\[
\int_0^\infty \frac{\sin(kx)}{x^\beta} \, dx = \frac{\Gamma(1-\beta) \sin[(1-\beta)\pi/2]}{[k^{1-\beta}]^{1-\beta}}
\]
\[
= \frac{2(k^{1-\beta} \cos[(1-\beta)\pi/2] \Gamma(\beta)}{.}
\]

\[116\]
If we choose $a = 0$, $f(x) = x^{-\beta}$, $p(x) = \sin(kx)$ ($k > 0$), $b_n = (n\pi/k)$, and apply the preceding theorem-A.2, then we obtain that
\[
\Gamma(1 - \beta) \sin[(1 - \beta)\pi/2] = \frac{\Gamma(1 - \beta) \sin[(1 - \beta)\pi/2]}{|k|^{1-\beta}}.
\] (117)

Applying the mean-value theorem for an integral, we conclude that
\[
\int_{n\pi/k}^{(n+1)\pi/k} \frac{\sin(kx)}{x^{\beta}} dx = \frac{\sin[(n + \Theta_n(k))\pi]}{[n + \Theta_n(k)]^{\beta}} \frac{\Gamma(1 - \beta) \sin[(1 - \beta)\pi/2]}{|k|^{1-\beta}} \text{ for some } \Theta_n(k) \in (0,1). \quad (118)
\]

We can generalize the preceding corollary for any $k \neq 0$ and $\beta \in (0,1)$ to the following equations:
\[
\sum_{n=0}^{\infty} \frac{\sin[(n + \Theta_n(k))\pi]}{[n + \Theta_n(k)]^{\beta}} = \int_0^{\infty} \frac{\sin(kx)}{x^{\beta}} dx = \frac{\text{sgn}(k) \Gamma(1 - \beta) \sin[(1 - \beta)\pi/2]}{|k|^{1-\beta}};
\] (119)
\[
\sum_{n=0}^{\infty} \frac{\cos[(n + \Theta_n(k))\pi]}{[n + \Theta_n(k)]^{\beta}} = \int_0^{\infty} \frac{\cos(kx)}{x^{\beta}} dx = \cos[(1 - \beta)\pi/2] \{ \frac{\Gamma(1 - \beta)}{|k|^{1-\beta}} - 2\text{sgn}(k) k^{\beta} \delta(k) \};
\]
\[
\sum_{n=0}^{\infty} \frac{\exp\{i[(n + \Theta_n^h(k))\pi)]}{[n + \Theta_n^h(k)]^{\beta}} = \int_0^{\infty} \frac{\exp\{i(kx)}{x^{\beta}} dx
\]
\[
= \frac{\Gamma(1 - \beta)}{|k|^{1-\beta}} \exp\{i\text{sgn}(k)(1 - \beta)\pi/2\} - 2\cos[(1 - \beta)\pi/2] \{ 2\text{sgn}(k) k^{\beta} \delta(k) \}.
\] (120)

Here, $\delta(k)$ is the Dirac delta distribution.

It is very plausible from (119) that the series $d^\#(p, q, k)$ in (115) converge. In fact, from (119) ii, iii) we shall choose an approximation for $d^\#(p, q, k)$ by the following relation:
\[
8d^\#(p, q, k) \approx \pi^{-3/2} \int_0^{\infty} y^{-3/4} \left( \exp\{i[2(p + q + k)\sqrt{y}]\} + \exp\{-i[2(q + k - p)\sqrt{y}]\} + \exp\{-i[2(q - k + p)\sqrt{y}]\} + \exp\{i[2(q - k - p)\sqrt{y}]\} \right) dy
\]
\[
= \sqrt{2} \pi^{-3/2} \int_0^{\infty} x^{-1/2}(\exp\{i(p + q + k)x]\)
\]
\[
+ \exp[-i(q + k - p)x] + \exp[-i(q - k + p)x] + \exp[i(q - k - p)x]) dx.
\] (121)
Now, choosing the special case $\beta = 1/2$ and recalling $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(-1/2) = -2\sqrt{\pi}$, we obtain from (A.12iii)

\[
\int_{0^+}^{\infty} x^{-1/2} e^{ikx} \, dx = \sqrt{\pi} \{ 2\sqrt{2|k|}\delta(k) + \frac{[1 + i \text{sgn}(k)]}{\sqrt{2|k|}} \}.
\] (122)

Substituting (122) into (121), we finally obtain an asymptotically "closed form":

\[
2\pi d^\#(p, q, k) = \{ \sqrt{|p + q + k|}\delta(p + q + k) + \sqrt{|q + k - p|}\delta(q + k - p) + \sqrt{|q - k + p|}\delta(q - k + p) + \sqrt{|q - k - p|}\delta(q - k - p) \}
+ (1/4)\{ \frac{[1 + i \text{sgn}(p + q + k)]}{\sqrt{|p + q + k|}} + \frac{[1 - i \text{sgn}(q + k - p)]}{\sqrt{|q + k - p|}} + \frac{[1 - i \text{sgn}(q - k + p)]}{\sqrt{|q - k + p|}} + \frac{[1 + i \text{sgn}(q - k - p)]}{\sqrt{|q - k - p|}} \}.
\] (123)

We shall now state and sketch briefly the proof of the remarkable properties of the distribution function $d^\#(p, q, k)$ in (123).

**Theorem A.3**: The distribution function $d^\#(p, q, k)$ satisfies exactly the symmetry properties (A.107, 108, 109) of $\delta^\#(p, q, k)$.

**Proof**: The proof is straightforward (using (123)) and is skipped.

We define the three dimensional distribution function

\[
d^\#_3(p, q, k) := \prod_{j=1}^{3} d^\#(p_j, q_j, k_j).
\] (124)

In section-V, we shall use the asymptotic approximation of the $\delta^\#_3$-distribution function by putting

\[
\delta^\#_3(p, q, k) \approx d^\#_3(p, q, k).
\] (125)
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