Multiplicative rule in the Grothendieck cohomology of a flag variety

Dedicated to Professor Yifeng Sun on his 80th birthday

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Abstract

In the Grothendieck cohomology of a flag variety $G/H$ there are two canonical additive bases, namely, the Demazure basis [D] and the Grothendieck basis [KK]. We present explicit formulae that reduce the multiplication of these basis elements to the Cartan numbers of $G$.

1 Introduction

Let $G$ be a compact connected Lie group and let $H \subset G$ be the centralizer of a one–parameter subgroup in $G$. The homogeneous space $G/H$ is known as a flag variety. We fix a maximal torus $T \subseteq H$ and write $W$ (resp. $W'$) for the Weyl group of $G$ (resp. $H$).

In the founding article [AH] of topological $K$–theory as a “generalized cohomology theory”, Atiyah and Hirzebruch raised also the problem to determine the ring $K(G/H)$, with the expectation that “the new cohomology theory can be applied to various topological questions and may give better results than the ordinary cohomology theory”. As an initial step they showed that the $K(G/H)$ is a free $\mathbb{Z}$-module with rank equal to the quotient of the order of $W$ by the order of $W'$ [AH, Theorem 3.6].

In the subsequent years two distinguished additive bases of $K(G/H)$ have emerged from algebraic geometry. The first of these, valid for the case $H = T$ and indexed by elements from the Weyl group: $\{a_w \in K(G/T) \mid w \in W\}$,
is due to Demazure [D, Proposition 7]. The basis element \( a_w \) will be called the Demazure class relative to \( w \in W \).

The second basis goes back to the classical works of Grothendieck and Chevalley. Let \( l : W \to \mathbb{Z} \) is the length function relative to a fixed maximal torus \( T \subset G \) and identify the set \( W/W' \) of left cosets of \( W' \) in \( W \) with the subset \( \mathcal{W} = \{ w \in W \mid l(w) \leq l(w_1) \text{ for all } w_1 \in wW' \} \) of \( W \). According to Chevalley [Ch] the flag variety \( G/H \) admits a canonical partition into Schubert varieties, indexed by elements of \( \overline{W} \),

\[
G/H = \bigcup_{w \in \mathcal{W}} X_w(H), \quad \dim X_w(H) = 2l(w).
\]

The coherent sheaves \( \Omega_w(H) \in K(G/H) \) of the Schubert variety Poincare dual to the \( X_w(H) \), \( w \in \mathcal{W} \), form a basis for the \( \mathbb{Z} \)-module \( K(G/H) \) by Grothendieck (cf. [C, Lecture 4]). The \( \Omega_w(H) \) is called the Grothendieck class relative to \( w \in \mathcal{W} \).

A complete description of the ring \( K(G/T) \) (resp. \( K(G/H) \)) now amounts to specify the structure constants \( C^w_{u,v} \in \mathbb{Z} \) (resp. \( K^w_{u,v}(H) \in \mathbb{Z} \)) required to express the products of basis elements

\[
(1.1) \quad a_u \cdot a_v = \sum C^w_{u,v} a_w \quad \text{(resp. } \Omega_u(H) \cdot \Omega_v(H) = \sum K^w_{u,v}(H)\Omega_w(H)\text{)},
\]

where \( u, v, w \in W \) (resp. \( \in \overline{W} \)).

Based on combinatorial methods, partial information concerning the ring \( K(G/H) \) has been achieved during the past decade. Generalizing the classical Pieri-Chevalley formula in the ordinary cohomology [Ch], various combinatorial formulae expressing the product \( L \cdot \Omega_u \) in terms of the \( \Omega_w \) were obtained, where \( L \) is a line bundle on \( G/T \). This was originated by Fulton and Lascoux [FL] for the unitary group \( U(n) \) of rank \( n \) (see also Lenart [L]), extended to general \( G \) by Pittie and Ram [PR1,2], Mathieu [M], Littelmann and Seshadri [LS] by using very different methods. Another progress is when \( G = U(n) \) and \( H = U(k) \times U(n-k) \), the flag variety \( G/H \) is the Grassmannian \( G_{n,k} \) of \( k \)-planes through the origin in \( \mathbb{C}^n \) and a combinatorial description for the \( K^w_{u,v}(H) \) was obtained by Buch [Bu].

In general, using purely geometric approach, Brion [Br1,Br2] proved that the \( K^w_{u,v} \) have alternating signs, confirming a conjecture of Buch in [Bu].

In this paper we present a formula that expresses the constants \( C^w_{u,v} \) (resp. the \( K^w_{u,v}(H) \)) by the Cartan numbers of \( G \), see Theorem 1 in §2 (resp. Theorem 2 in §5). These results are natural generalization of the formula in [Du2] for multiplying Schubert classes in the cohomology of \( G/H \) for, in the
special cases \( l(w) = l(u) + l(v) \), the number \( C_{u,v}^w \) (resp. the \( K_{u,v}^w(H) \)) agrees with the coefficient of the Schubert class \( P_w \) in the product \( P_u \cdot P_v \) (see in the notes after Theorem 1 in §2).

An important problem in algebraic combinatorics is to find a combinatorial description of the \( C_{u,v}^w \) (resp. the \( K_{u,v}^w(H) \)) that has the advantage to reveal their signs [L,Bu]. Apart from the combinatorial concern, effective computation in the \( K \)-theory of such classical manifolds as the \( G/H \) is decidedly required by many problems from geometry and topology. We emphasize that the formulae as given in this paper are computable, although it is not readily to reveal the signs of the constants. With some additional works the existing program [DZ2] for multiplying Schubert classes can be extended to implement the \( C_{u,v}^w \) (resp. \( K_{u,v}^w(H) \)). This program uses the Cartan matrix of \( G \) as the only input and, therefore, is functional uniformly for all \( G/H \).

The author feels very grateful to S. Kumar for valuable communication. Indeed, our exposition benefits a lot from certain results (cf. Lemma 4.2; Lemma 5.1) developed in the classical treatise [KK] on this subject.

Thanks are also due to my referee for many improvements on the earlier version of the paper, and for his kindness in informing me the work [GR] by S. Griffeth and A. Ram, where a combinatorial method to multiply two elements of the Grothendieck basis of the equivariant K–ring of a flag variety is given, and the article [W2] by M. Willems, where similar methods are used in the setting of equivariant K–theory.

2 The formula for \( C_{u,v}^w \)

We introduce notations (from Definition 1 to 4) in terms of which the formula for \( C_{u,v}^w \) will be presented in Theorem 1.

Fix once and for all a maximal torus \( T \) in \( G \). Set \( n = \dim T \). Equip the Lie algebra \( L(G) \) with an inner product \( (, \) ) so that the adjoint representation acts as isometries of \( L(G) \).

The restriction of the exponential map \( \exp : L(G) \to G \) to \( L(T) \) defines a set \( D(G) \) of \( \frac{1}{2}(\dim G - n) \) hyperplanes in \( L(T) \), i.e. the set of singular hyperplanes through the origin in \( L(T) \). The reflections of \( L(T) \) in these planes generate the Weyl group \( W \) of \( G \) ([Hu, p.49]).

Take a regular point \( \alpha \in L(T) \) and let \( \Delta \) be the set of simple roots relative to \( \alpha \) [Hu,p.47]. For a \( \beta \in \Delta \) the reflection \( r_\beta \) in the hyperplane \( L_\beta \in D(G) \) relative to \( \beta \) is called a simple reflection. If \( \beta, \beta' \in \Delta \), the Cartan number

\[
\beta \circ \beta' = 2(\beta, \beta')/\langle \beta', \beta' \rangle
\]

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is always an integer (only 0, ±1, ±2, ±3 can occur). It is also customary to use $(\beta, \beta')$ instead of $\beta \circ \beta'$.

It is well known that the set of simple reflections $\{r_\beta \mid \beta \in \Delta\}$ generates $W$. That is, any $w \in W$ admits a factorization of the form

\begin{equation}
2.1 \quad w = r_{\beta_1} \cdots r_{\beta_m}, \quad \beta_i \in \Delta.
\end{equation}

**Definition 1.** The length $l(w)$ of a $w \in W$ is the least number of factors in all decompositions of $w$ in the form (2.1). The decomposition (2.1) is said reduced if $m = l(w)$.

If (2.1) is a reduced decomposition, the $m \times m$ (strictly upper triangular) matrix $A_w = (a_{i,j})$ with

$$a_{i,j} = \begin{cases} 0 & \text{if } i \geq j; \\ \beta_j \circ \beta_i & \text{if } i < j \end{cases}$$

will be called the Cartan matrix of $w$ relative to the decomposition (2.1).

**Definition 2.** Given a sequence $(\beta_1, \cdots, \beta_m)$ of simple roots and a $w \in W$, let $[i_1, \cdots, i_k] \subseteq [1, \cdots, m]$ be the subsequence maximal in the inverse-lexicographic order so that

$$l(w) > l(wr_{\beta_{i_k}}) > l(wr_{\beta_{i_k}}r_{\beta_{i_{k-1}}}) > \cdots > l(wr_{\beta_{i_k}}r_{\beta_{i_{k-1}}} \cdots r_{\beta_{i_1}}).$$

We call $(\beta_1, \cdots, \beta_m)$ a derived (simple root) sequence of $w$, written $(\beta_1, \cdots, \beta_m) \sim w$, if $k = l(w)$ (i.e. $r_{\beta_{i_k}} \cdots r_{\beta_{i_1}}$ is a reduced decomposition of $w$).

**Remark 1.** It is clear that $(\beta_1, \cdots, \beta_m) \sim w$ implies $m \geq l(w)$, while the equality holds if and only if $w = r_{\beta_1} \cdots r_{\beta_m}$.

The Definition 2 implies also that if $e \in W$ is the group unit, then $(\beta_1, \cdots, \beta_m) \sim e$ for any sequence $(\beta_1, \cdots, \beta_m)$ of simple roots.

Let $\mathbb{Z}[y_1, \cdots, y_m]$ be the ring of integral polynomials in $y_1, \cdots, y_m$, graded by $|y_i| = 1$, and let $\mathbb{Z}[y_1, \cdots, y_m]_{(n)}$ be the submodule of all polynomials of degree $\leq n$. We introduce the additive maps

$$(n) : \mathbb{Z}[y_1, \cdots, y_m] \to \mathbb{Z}[y_1, \cdots, y_m]_{(n)}, \quad n \geq 0,$$

by the following rule. If $f \in \mathbb{Z}[y_1, \cdots, y_m]$, then
\[ f = f^{(n)} + \text{a sum of monomials of degree} > n. \]

**Definition 3.** Let \( A = (a_{i,j})_{m \times m} \) be a strictly upper triangular matrix (with integer entries). In terms of the entries of \( A \) define two sequences \( \{q_1, \ldots, q_m\}, \{\overline{q}_1, \ldots, \overline{q}_m\} \subset \mathbb{Z}[y_1, \ldots, y_m] \) of polynomials inductively as follows. Put \( q_1 = \overline{q}_1 = 1 \), and for \( k > 1 \) let

\[
q_k = \prod_{a_{i,k} > 0} (y_i + 1)^{a_{i,k}} \prod_{a_{i,k} < 0} (-q_i y_i + 1)^{-a_{i,k}},
\]

\[
\overline{q}_k = \prod_{a_{i,k} > 0} (-q_i y_i + 1)^{a_{i,k}} \prod_{a_{i,k} < 0} (y_i + 1)^{-a_{i,k}}.
\]

**Remark 2.** As an example if \( A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \), then

\[
\begin{align*}
q_2 &= y_1 + 1, \\
\overline{q}_2 &= 1 - y_1; \quad \text{and} \quad q_3 &= (y_1 + 1)^2 [(y_1 + 1)y_2 + 1], \\
\overline{q}_3 &= [(y_1 + 1)y_1 + 1]^2 (y_2 + 1).
\end{align*}
\]

Note also that, since \( A \) is strictly upper triangular, we always have

\[
q_k, \overline{q}_k \in \mathbb{Z}[y_1, \ldots, y_{k-1}, q_1, \ldots, q_{k-1}] = \mathbb{Z}[y_1, \ldots, y_{k-1}].
\]

**Definition 4.** Given a strictly upper triangular matrix \( A = (a_{i,j})_{m \times m} \) of rank \( m \) define the operator \( \Delta_A : \mathbb{Z}[y_1, \ldots, y_m]_{(m)} \to \mathbb{Z} \) as the composition

\[
\mathbb{Z}[y_1, \ldots, y_m]_{(m)} \xrightarrow{D_{m-1}} \mathbb{Z}[y_1, \ldots, y_{m-1}]_{(m-1)} \xrightarrow{D_{m-2}} \cdots \xrightarrow{D_1} \mathbb{Z}[y_1]_{(1)} \xrightarrow{D_0} \mathbb{Z},
\]

where the operator \( D_{k-1} : \mathbb{Z}[y_1, \ldots, y_k]_{(k)} \to \mathbb{Z}[y_1, \ldots, y_{k-1}]_{(k-1)} \) is given by the following elimination rule.

Expand each \( f \in \mathbb{Z}[y_1, \ldots, y_k]_{(k)} \) in terms of the powers of \( y_k \)

\[
f = h_0 + h_1 y_k + h_2 y_k^2 + \cdots + h_k y_k^k, \quad h_i \in \mathbb{Z}[y_1, \ldots, y_{k-1}],
\]

then put

\[
D_{k-1}(f) = [h_1 + h_2 (\overline{q}_k - 1) + \cdots + h_k (\overline{q}_k - 1)^{k-1}]_{(k-1)},
\]

where the \( \overline{q}_k \) is given by \( A \) as in Definition 3 (see also Remark 2).

**Remark 3.** The \( \Delta_A \) can be easily evaluated, as the formula shows.
\[ D_{k-1}(f) = \left[ \sum_{n \geq 1} \left( \frac{1}{n!} \frac{\partial^n f}{\partial y_k^n} \right) |_{y_k=0} (q_k - 1)^{n-1} \right] (k-1). \]

**Remark 4.** The operator \( \Delta_A \) extends the idea of **triangular operator** \( T_A \) in \([Du_1,Du_2,DZ_1,DZ_2] \) in the following sense. If \( f \in \mathbb{Z}[y_1, \ldots, y_m] \) is of homogeneous degree \( m \), then \( \Delta_A(f) = T_{-A}(f) \).

The operator \( T_A \) appears to be a useful tool in computing with the cohomology of \( G/H \). It was applied to evaluate the degrees of Schubert varieties in \([Du_1] \), to present a formula for multiplying Schubert classes in \([Du_2,DZ_2] \), and to compute the Steenrod operations on the \( \mathbb{Z}_p \)-cohomologies of \( G/H \) in \([DZ_1]\). Apart from the \( \Delta_A \), another generalization of the \( T_A \) was given by Willems \([W, \text{Definition 5.2.1}] \), which was useful for multiplying Schubert classes in the \( T \)-equivariant cohomology of \( G/T \) \([W_1, \text{Theorem 5.3.1}] \).

Assume that \( w = r_{\beta_1} \cdots r_{\beta_m} \), \( \beta_i \in \Delta \), is a reduced decomposition of \( w \in W \), and let \( A_w = (a_{i,j})_{m \times m} \) be the associated Cartan matrix. For a subsequence \( L = [i_1, \ldots, i_k] \subseteq [1, \ldots, m] \) we set
\[
\beta(L) = (\beta_{i_1}, \ldots, \beta_{i_k}), \quad y_L = y_{i_1} \cdots y_{i_k} \in \mathbb{Z}[y_1, \ldots, y_m].
\]

**Theorem 1.** For \( u, v \in W \) we have
\[
(2.2) \quad (-1)^{l(w)} C_{u,v} = (-1)^{l(u)+l(v)} \Delta_{A_u} \left[ \left( \sum_{\beta(L) \sim u} y_L \right) \left( \sum_{\beta(K) \sim v} y_K \right) \right]_{(m)}
\]
\[
- \sum_{l(u)+l(v)=l(x) \leq l(w)-1} \sum_{\beta(1, \ldots, m) \sim x} (-1)^{l(x)} C_{x,u,v},
\]
where \( L, K \subseteq [1, \ldots, m] \).

As suggested by Theorem 1, the job to compute all \( C_{u,v} \) for given \( u, v \in W \) may be organized as follows.

1. If \( l(w) = m < l(u) + l(v) \), then
\[
\left[ \left( \sum_{\beta(L) \sim u} y_L \right) \left( \sum_{\beta(K) \sim v} y_K \right) \right]_{(m)} = 0
\]
implies that \( C_{u,v} = 0 \).

2. If \( l(w) = m = l(u) + l(v) \) the formula becomes
\[
C_{u,v} = \Delta_{A_u} \left[ \left( \sum_{\beta(L) \sim u} y_L \right) \left( \sum_{\beta(K) \sim v} y_K \right) \right]_{(m)}
\]
\[
= \Delta_{A_u} \left[ \left( \sum_{r_L = u, |L| = l(u)} y_L \right) \left( \sum_{r_K = v, |K| = l(v)} y_K \right) \right] \quad (\text{cf. Remark 1})
\]
\[
= T_{-A_u} \left[ \left( \sum_{r_L = u, |L| = l(u)} y_L \right) \left( \sum_{r_K = v, |K| = l(v)} y_K \right) \right] \quad (\text{cf. Remark 4}),
\]
where \( r_L = r_{\beta_1} \cdots r_{\beta_k}, \mid L \mid = k \) if \( L = [i_1, \ldots, i_k] \). This recovers the formula for multiplying Schubert classes in the ordinary cohomology of \( G/T \) [Du_2, DZ_1].

(3) In general, assuming that all the constants \( C_{u,v}^x \) with \( l(x) < m \) have been obtained, the theorem gives \( C_{u,v}^w \) with \( l(w) = m \) in terms of the operator \( \Delta_{A_w} \) as well as those \( C_{u,v}^x \ (l(x) < m, \left( \beta_1, \cdots, \beta_m \right) \sim x) \) calculated before.

It is clear from the discussion that Theorem 1 reduces the \( C_{u,v}^w \) to the operators \( \Delta_{A_w} \), hence to the matrices \( A_x \) formed by Cartan numbers.

Theorem 1 is originated from the celebrated Bott-Samelson cycles on flag manifolds [BS]. This may be seen from the geometric consideration that underlies the algebraic formation from Definition 1 to 4. Indeed, the Cartan matrix of a \( w \) (Definition 1) with respect to the decomposition (2.1) gives the structural data characterizing the Bott-Samelson cycle \( S(\alpha; \beta_1, \cdots, \beta_m) \) as a twisted products of 2–spheres (Lemma 4.3); the polynomials \( \overline{q}_k \)'s (Definition 3) provide the relations in describing the \( K \)–ring of \( S(\alpha; \beta_1, \cdots, \beta_m) \) as the quotient of a polynomial ring (Lemma 4.4); the operator \( \Delta_A \) (Definition 4) handles the integration along the top cell of \( S(\alpha; \beta_1, \cdots, \beta_m) \) in the \( K \)–theory (Lemma 4.4); and the idea of derived sequence of a Weyl group element (Definition 2) is required to specify the induced map of a Bott-Samelson cycle in \( K \)–theory (Lemma 4.5).

The remaining sections are so arranged. Before involving the specialities of flag manifolds, Section 3 studies the \( K \)–theory of \( \text{twisted products of 2-spheres} \), a family of manifolds that generalizes the classical Bott-Samelson cycles on \( G/T \) [BS] (Lemma 3.4). In addition, \( \text{divided difference} \) in \( K \)–theory is introduced for \( \text{spherical represented involutions} \) (cf. 3.2). In Section 4, by resorting to the geometry of the adjoint representation, we interpret the Bott-Samelson cycles on \( G/T \) as certain twisted products of 2-spheres, and describe their \( K \)–rings in terms of the Cartan numbers of \( G \) (Lemma 4.4). After determining the image of Demazure classes in the \( K \)–ring of a Bott-Samelson cycle (Lemma 4.5), the theorems are established in Section 5.

3 Preliminaries in topological \( K \)-theory

All homologies (resp. cohomologies) will have integer coefficients unless otherwise stated. If \( f : M \rightarrow N \) is a continuous map between two topological spaces \( M \) and \( N \), \( f_* \) (resp. \( f^* \)) is the homology (resp. cohomology) map induced by \( f \), and \( f^! : K(N) \rightarrow K(M) \) is the induced map on the Grothendieck
groups of topological complex bundles. The involution on \( K(M) \) by the complex conjugation is denoted by \( \xi \mapsto \overline{\xi}, \xi \in K(M) \).

Write \( S^2 \) for the 2–dimensional sphere. If \( M \) is an oriented closed manifold \([M] \in H_{\dim M}(M)\) stands for the orientation class. The Kronecker pairing, between cohomology and homology of a space \( M \), will be denoted by \( <, >: H^*(M) \times H_*(M) \to \mathbb{Z} \).

Let \( L_C(M) \) be the set of isomorphism classes of complex line bundles (i.e. of oriented real 2–plane bundles) over \( M \). The trivial complex line bundle on \( M \) is denoted by \( 1 \). It is well known that

\[ \text{(3.1) sending a complex line bundle } \xi \text{ to its first Chern class } c_1(\xi) \text{ yields a one-to-one correspondence } c_1 : L_C(M) \to H^2(M) \text{ which is natural with respect to maps } M \to N. \]

Recall also from [AH] that

\[ \text{(3.2) if } M \text{ is a cell complex with even dimensional cells only, the Chern character } Ch : K(M) \to H^*(M; \mathbb{Q}) \text{ is injective.} \]

3.1. *\( S^2 \)–bundle with a section.* Let \( p : E \to M \) be a smooth, oriented \( S^2 \)–bundle over an oriented manifold \( M \) with a section \( s : M \to E \). As the normal bundle \( \eta \) of the embedding \( s \) is oriented by \( p \) and has real dimension 2, we may regard \( \eta \in L_C(M) \). We put \( \xi = p^!(\eta) \in L_C(E) \), \( c = c_1(\xi) \in H^2(E) \).

The integral cohomology \( H^*(E) \) can be described as follows. Denote by \( i : S^2 \to E \) the fiber inclusion over a point \( z \in M \), and write \( J : E \to E \) for the involution given by the antipodal map in each fiber sphere.

**Lemma 3.1** (cf. [Du2, Lemma 3.1]). *There exists a unique class \( x \in H^2(E) \) such that \( s^*(x) = 0 \in H^*(M) \) and \( < i^*(x), [S^2] > = 1 \).*

Furthermore

1. \( H^*(E) \), as a module over \( H^*(M) \), has the basis \( \{1, x\} \) subject to the relation \( x^2 + cx = 0 \);
2. the \( J^* \) acts identically on \( H^*(M) \subset H^*(E) \) and \( J^*(x) = -x - c \).

Let \( X \in L_C(E) \) be with \( c_1(X) = x \in H^2(E) \), where \( x \) as that in Lemma 3.1. Put \( y = X - 1 \in K(E) \). The next result is seen as the \( K \)–theoretic analogous of Lemma 3.1, in which (1) is classical (cf. [A, Proposition 2.5.3]).

**Lemma 3.2.** Assume that \( M \) has a cell decomposition with even dimensional cells only. Then
(1) $K(E)$, as a module over $K(M)$, has the basis $\{1, y\}$ subject to the relation $y^2 = (\xi - 1)y$. Furthermore

(2) $\overline{y} = -\xi y$;

(3) the $J^t : K(E) \to K(E)$ acts identically on $K(M)$ and satisfies $J^t(y) = -y + (\xi - 1)$.

**Proof.** Since $X$ restricts to the Hopf-line bundle on the fiber sphere, $K(E) = K(M)[1, y]$. To verify the relation in (1) we compute

$$Ch(y) = e^x - 1 = x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$= -\frac{\xi}{c}(e^{-c} - 1) \quad \text{(for } x^n = (-c)^{n-1}x \text{ by (1) of Lemma 3.1).}$$

It follows that

$$Ch(y^2) = [\frac{\xi}{c}(e^{-c} - 1)]^2 = -\frac{\xi}{c}(e^{-c} - 1)^2 \quad (x^2 = -cx \text{ by Lemma 3.1})$$

$$= Ch[(\xi - 1)y].$$

This implies that $y^2 = (\xi - 1)y$ by (3.2). This completes (1).

In view of (1) we may assume that $\overline{y}(= \overline{X} - 1) = a + by$, $a, b \in K(M)$. Multiplying both sides by $X = y + 1$ yields

$$-y = (a + by)(y + 1) = a + (a + b\overline{\xi})y \quad \text{(by (1)).}$$

Coefficients comparison gives $a = 0$; $a + b\overline{\xi} = -1$. This shows (2).

Finally we show (3). From $J^t(x) = -x - c$ (by Lemma 3.1) and $c_1J^t = J^t c_1$ (by the naturality of (3.1)) we get $J^t(X) = \overline{X} \cdot \overline{\xi}$. From this one obtains

$$J^t(y) = \overline{X} \cdot \overline{\xi} - 1 = (\overline{y} + 1)(\overline{\xi} - 1) = (-\xi y + 1)(\overline{\xi} - 1) \quad \text{(by (2))}$$

$$= -y + (\xi - 1). \Box$$

**3.2. Divided difference in $K$–theory.** A self-map $r$ of a manifold $M$ is called an involution if $r^2 = id : M \to M$. A 2–spherical representation of the involution $(M; r)$ is a system $f : (E; J) \to (M; r)$ in which $p : E \to M$ is an oriented $S^2$–bundle with a section $s$; and $f$ is a continuous map $E \to M$ that satisfies the following two constrains

$$f \circ s = id : M \to M; \quad f \circ J = r \circ f : E \to M,$$
where $J$ is the involution on $E$ given by the antipodal map on the fibers.

In view of the $K(M)$–module structure on $K(E)$ (cf. (1) of Lemma 3.2), a 2–spherical representation (3.3) of the involution $(M, r)$ gives rise to an additive operator $\Lambda_f : K(M) \to K(M)$ characterized uniquely by (3.4) below, where the constraint $f \circ s = id$ implies that the coefficient of 1 is $p^1(z)$.

The induced homomorphism $f^! : K(M) \to K(E)$ satisfies

\begin{equation}
(\ref{3.4}) \quad f^!(z) = p^1(z) \cdot 1 + p^1(\Lambda_f(z)) \cdot y
\end{equation}

for all $z \in K(M)$.

The operator $\Lambda_f$ will be called the divided difference associated to the 2–spherical representation $f$ of the involution $(M; r)$.

**Lemma 3.3.** Let $\eta \in L_C(M)$ the normal bundle of the section $s$. Then

1. $r^! = Id + (\overline{\eta} - 1)\Lambda_f : K(M) \to K(M)$;
2. $\Lambda_f \circ r^! = -\Lambda_f$.

**Proof.** Applying $J^!$ to (3.4) and using (3) of Lemma 3.2 to write the resulting equality yield (note that $\xi = p^!(\eta)$ in Lemma 3.2)

\[ J^! f^!(z) = p^![z + (\overline{\eta} - 1)\Lambda_f(z)] - p^![-\Lambda_f(z)y]. \]

On the other hand one gets from (3.4) that

\[ f^!(r^!(z)) = p^!(r^!(z)) + p^![\Lambda_f(r^!(z))]y. \]

Since $J^! f^! = f^! r^!$ by (3.3), coefficients comparison shows (1) and (2). □

**3.3. The $K$-theory of a twisted product of $S^2$.** We determine the $K$-rings for a class of manifolds specified below.

**Definition 5.** A smooth manifold $M$ is called an oriented twisted product of 2–spheres of rank $m$, written $M = \bigotimes_{1 \leq i \leq m} S^2$, if there is a tower of smooth maps

\[ M = M_m \xrightarrow{p_m} M_{m-1} \xrightarrow{p_{m-1}} \cdots \xrightarrow{p_3} M_2 \xrightarrow{p_2} M_1 \xrightarrow{p_1} M_0 = \{z_0\} \]

in which

1) $M_0$ consists of a single point (as indicated);
2) each $p_k$ is an oriented $S^2$–bundle with a fixed section $s_k : M_{k-1} \to M_k$. 

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Let $M = \bigoplus_{1 \leq i \leq m} S^2$ be a twisted product of 2-spheres. Assign each $M_k$ with the base point $z_k = s_k \circ \cdots \circ s_1(z_0) \in M_k$ and denote by $h_k : S^2 \to M_k$ the inclusion of the fiber sphere of $p_k$ over the point $z_k$, $1 \leq k \leq m$. Consider the embedding $\iota_k : S^2 \to M$ given by the composition

$$s_m \circ \cdots \circ s_{k+1} \circ h_k : S^2 \to M_k \to M.$$  

Then the set $\{\iota_1[S^2], \ldots, \iota_m[S^2] \in H_2(M)\}$ of 2-cycles forms a basis of $H_2(M)$ [Du, Lemma 3.3]. Consequently, if we let $x_i \in H^2(M)$, $1 \leq i \leq m$, be the classes Kronecker dual to $\iota_k[S^2]$ as $< x_i, \iota_k[S^2] > = \delta_{ik}$, then

(3.5) the set $\{x_1, \ldots, x_m\}$ is a basis of $H^2(M)$ that satisfies

$$(s_m \circ \cdots \circ s_k)^*(x_k) = 0, 1 \leq k \leq m.$$  

A set of numerical invariants for $M$ can now be extracted as follows. Let $\eta_k \in L_C(M_{k-1})$ be the normal bundle of the embedding $s_k : M_{k-1} \to M_k$ with the induced orientation. Put $\xi_k = (p_k \circ \cdots \circ p_m)^1 \eta_k \in L_C(M)$. In view of (3.5) we must have the expression in $H^2(M)$

$$c_1(\xi_1) = 0;$$
$$c_1(\xi_2) = a_{1,2}x_1;$$
$$c_1(\xi_3) = a_{1,3}x_1 + a_{2,3}x_2;$$
$$\vdots$$
$$c_1(\xi_m) = a_{1,m}x_1 + \cdots + a_{m-1,m}x_{m-1}$$

with $a_{i,j} \in \mathbb{Z}$.

**Definition 6.** With $a_{i,j} = 0$ for all $i \geq j$ being understood, the strictly upper triangular matrix $A = (a_{i,j})_{m \times m}$ is called the structure matrix of $M = \bigoplus_{1 \leq i \leq m} S^2$ relative to the basis $\{x_1, \ldots, x_m\}$ of $H^2(M)$.

The ring $K(M)$ is determined by $A$ as follows. Let $X_i \in L_C(M)$ be defined by $c_1(X_i) = x_i$ (cf. (3.1), (3.5)) and set $y_i = X_i - 1 \in K(M)$. For a subset $I = [i_1, \ldots, i_k] \subseteq [1, \ldots, m]$ we put

$$y_I = \begin{cases} 1 \text{ if } k = 0 \text{ (i.e. } I = \{\emptyset\}), \\ y_I = y_{i_1} \cdots y_{i_k} \text{ if } k \geq 1. \end{cases}$$

Let $q_k, \overline{q}_k \in \mathbb{Z}[y_1, \ldots, y_m]$ be defined in terms of $A$ as in Definition 3.

**Lemma 3.4.** If $M$ has the structure matrix $A = (a_{i,j})_{m \times m}$, then

(1) the set $\{y_I : I \subseteq [1, \ldots, m]\}$ is a basis for $K(M)$;

(2) $K(M) = \mathbb{Z}[y_1, \ldots, y_m]/ < y_k^2 = (\overline{q}_k - 1)q_k, 1 \leq k \leq m >$. 

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Proof. If \( m = 1 \) then \( M = S^2 \) and it is clearly done. Assume next that Lemma 3.4 holds for \( m = n - 1 \), and consider now the case \( m = n \).

Applying Lemma 3.2 to the \( S^2 \)-bundle \( M = M_m \to M_{m-1} \) we get

\[
K(M) = K(M_{m-1})[1, y_m] (y_m^2 = (\xi_m - 1)y_m).
\]

This already shows (1).

For (2) it remains only to show \( \xi_m = q_m \in \mathbb{Z}[y_1, \ldots, y_{m-1}] \). In view of (1) of Lemma 3.2 and by the inductive hypothesis, we can assume that

\[
(3.6) \quad \xi_k = q_k, \quad \bar{\xi}_k = \bar{q}_k \text{ for all } k \leq m - 1.
\]

From \( c_1(\xi_m) = a_{1,m}x_1 + \cdots + a_{m-1,m}x_{m-1} \) we find that

\[
\bar{\xi}_m = \prod_{a_{i,m} > 0} \xi_i^{a_{i,m}} \prod_{a_{i,m} < 0} X_i^{-a_{i,m}}
\]

\[
= \prod_{a_{i,m} > 0} (y_i + 1)^{a_{i,m}} \prod_{a_{i,m} < 0} (y_i + 1)^{-a_{i,m}} \quad \text{(since } y_i = X_i - 1) \]

\[
= \prod_{a_{i,m} > 0} (-\xi_i y_i + 1)^{a_{i,m}} \prod_{a_{i,m} < 0} (y_i + 1)^{-a_{i,m}} \quad \text{(by Lemma 3.2)} \]

\[
= \prod_{a_{i,m} > 0} (-q_i y_i + 1)^{a_{i,m}} \prod_{a_{i,m} < 0} (y_i + 1)^{-a_{i,m}} \quad \text{(by (3.6))} \]

\[
= \bar{q}_m \quad \text{(cf. Definition 3)}.
\]

Similarly, we have \( \xi_m = \prod_{a_{i,m} > 0} X_i^{a_{i,m}} \prod_{a_{i,m} < 0} \Xi_i^{-a_{i,m}} = q_m. \square \)

According to (1) of Lemma 3.4, any polynomial \( f \in \mathbb{Z}[y_1, \ldots, y_m] \) can be expanded (uniquely) as a linear combination of the \( y_I \)

\[
f = \sum a_I(f) y_I, \quad I \subseteq [1, \ldots, m],
\]

where the correspondences \( a_I : \mathbb{Z}[y_1, \ldots, y_m] \to \mathbb{Z} \) by \( f \to a_I(f) \) are clearly additive. Indeed, problems concerning computing in the \( K(M) \) ask an effective algorithm to evaluate the \( a_I \). The case \( I = [1, \ldots, m] \) will be relevant to us and whose solution brings us the operator \( \Delta_A \) given by Definition 4.

**Lemma 3.5.** If \( M \) has the structure matrix \( A = (a_{i,j})_{m \times m} \), then

\[
a_{[1,\ldots,m]} = \Delta_A \circ (m) : \mathbb{Z}[y_1, \ldots, y_m] \to \mathbb{Z}[y_1, \ldots, y_m] \to \mathbb{Z}.
\]

In particular, \( a_{[1,\ldots,m]}(f) = 0 \) if \( f_{(m)} = 0 \).

**Proof.** This is parallel to the proof of Proposition 2 in [Du1]. \( \square \)
4 Bott-Samelson cycles and Demazure classes

With respect to the fixed regular point $\alpha \in L(T)$ the adjoint representation $Ad : G \to L(G)$ gives rise to a smooth embedding

$$\varphi : G/T \to L(G) \quad \text{by} \quad \varphi(gT) = Ad_g(\alpha).$$

In this way $G/T$ becomes a submanifold of the Euclidean space $L(G)$. By resorting to the geometry of this embedding we recover the Demazure operators on $K(G/T)$ in 4.3, and the classical Bott-Samelson cycle $\varphi_{0, \beta_1, \cdots, \beta_k} : S(\alpha; \beta_1, \cdots, \beta_k) \to G/T$ (associated to a sequence of simple roots $\beta_1, \cdots, \beta_k$) on $G/T$ in 4.4. As application of Lemma 3.4, the ring $K(S(\alpha; \beta_1, \cdots, \beta_k))$ is described in terms of the Cartan numbers of $G$ (cf. Lemma 4.4). The main result in this section is Lemma 4.5, which specifies the $\varphi_0^!_{0, \beta_1, \cdots, \beta_k}$–image of a Demazure class in $K(S(\alpha; \beta_1, \cdots, \beta_k))$.

4.1 Geometry from the adjoint representation. Let $\Phi^+ \subset L(T)$ (resp. $\Delta \subset L(T)$) be the set of positive (resp. simple) roots relative to $\alpha$ ([Hu, p.35]). Assume that the Cartan decomposition of the Lie algebra $L(G)$ relative to $T \subset G$ is

$$L(G) = L(T) \oplus_{\beta \in \Phi^+} F_\beta,$$

where $F_\beta$ is the root space, viewed as a real 2-plane, belonging to the root $\beta \in \Phi^+ ([Hu, p.35])$. It is known (cf. [HPT, p.426-427; or Du_2, Sect.4]) that

(4.1) The subspaces $\oplus_{\beta \in \Phi^+} F_\beta$ and $L(T)$ of $L(G)$ are tangent and normal to $G/T$ at $\alpha$ respectively;

(4.2) The tangent bundle to $G/T$ has a canonical orthogonal decomposition into the sum of integrable 2-plane bundles $\oplus_{\beta \in \Phi^+} E_\beta$ with $E_\beta(\alpha) = F_\beta$.

(4.3) The leaf of the integrable subbundle $E_\beta$ through a point $z \in G/T$, denoted by $S(z; \beta)$, is a 2-sphere that carries a preferred orientation.

(4.4) Via the embedding $\varphi$, the canonical action of $W$ on $G/T$ can be given in terms of the $W$ action on $L(T)$ as $w(z) = Ad_g(w(\alpha))$ if $z = Ad_g(\alpha) \in G/T$, $w \in W$ (cf. [BS]).

4.2. Demazure basis of $K(G/T)$. For a complete subvariety $Y \subset G/T$ the Euler–Poincaré characteristic relative to $Y$ is the homomorphism $\chi(Y, -) : K(G/T) \to \mathbb{Z}$ defined by

$$[F] \to \chi(Y, F) = \sum_j (-1)^j h^j(F \mid Y),$$

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where $F | Y$ means the restriction of $F$ on $Y$, and where $h^j(F | Y)$ denotes the dimension of the $j$th cohomology group of $h^j(F | Y)$. The following characterization of Demazure basis is due to B. Kostant and S. Kumar (compare [KK, (3.39) Proposition] with [D, Proposition 7]).

**Definition 7.** The Demazure basis \( \{ a_w \in K(G/T) \mid w \in W \} \) of the ring \( K(G/T) \) is defined by

\[
\chi(X_w, a_u) = \delta_{w,u}, \quad w, u \in W,
\]

where \( X_w \) is the Schubert class on \( G/T \) associated to \( w \).

### 4.3. Divided difference on \( K(G/T) \) associated to a root.

Each root \( \beta \in \Phi^+ \) gives rise to an involution \( r_{\beta} : G/T \to G/T \) in the fashion of (4.4), and defines also the subspace

\[
S(\beta) = \{(z, z_1) \in G/T \times G/T \mid z_1 \in S(z; \beta)\}
\]

in view of (4.3). Projection \( p_{\beta} : S(\beta) \to G/T \) onto the first factor is easily seen to be a \( S^2 \)-bundle (with \( S(z; \beta) \) as the fiber over \( z \in G/T \)). The map \( s_{\beta} : G/T \to S(\beta) \) by \( s_{\beta}(z) = (z, z) \) furnishes \( p_{\beta} \) with a ready-made section.

Let \( x \in H^2(S(\beta)) \) be such that \( s_{\beta}^*(x) = 0 \) and \( < i^*(x), [S(z; \beta)] > = 1 \) (cf. Lemma 3.1) and set \( y = X - 1 \in K(S(\beta)) \), where \( X \in L_\mathbb{C}(S(\beta)) \) is defined by \( c_1(X) = x \) (cf. (3.1)). Since the normal bundle of the embedding \( s_{\beta} \) is easily seen to be \( E_{\beta} \in L_\mathbb{C}(G/T) \), one infers from Lemma 3.2 that

**Lemma 4.1.** \( K(S(\beta)) = K(G/T)[1, y]/ < y^2 = (E_{\beta} - 1)y >. \)

Let \( J_{\beta} \) be the involution on \( S(\beta) \) given by the antipodal maps in the fiber spheres, and let \( f_{\beta} : S(\beta) \to G/T \) be the projection onto the second factor. Then, as is clear,

\[
f_{\beta} \circ s_{\beta} = id : G/T \to G/T; \quad f_{\beta} \circ J_{\beta} = r_{\beta} \circ f_{\beta} : S(\beta) \to G/T.
\]

That is, the map \( f_{\beta} : (S(\beta), J_{\beta}) \to (G/T, r_{\beta}) \) is a 2-spherical representation of the involution \( (G/T, r_{\beta}) \) (cf. 3.2).

Abbreviate the divided difference \( \Lambda f_{\beta} : K(G/T) \to K(G/T) \) associated to \( f_{\beta} \) by \( \Lambda_{\beta} \). The next result, essentially due to Kostant and Kumar [KK], is the key in the proof of Lemma 4.5.

**Lemma 4.2.** Let \( \{ a_w \in K(G/T) \mid w \in W \} \) be the Demazure basis of \( K(G/T) \), and let \( \beta \in \Delta \) be a simple root. Then
\[ \Lambda_{\beta}(aw) = \begin{cases} E_{\beta} \cdot awr_{\beta} & \text{if } l(w) > l(wr_{\beta}); \\ -E_{\beta} \cdot aw & \text{if } l(w) < l(wr_{\beta}). \end{cases} \]

**Proof.** Recall from [KK, PR1, D] that the classical Demazure operator \( T_{\beta} : K(G/T) \to K(G/T) \) is given by

\[ T_{\beta}(u) = \frac{E_{\beta}u - r_{\beta}(u)}{E_{\beta}-1}. \]

Substituting in \( r_{\beta} = Id + (E_{\beta} - 1)\Lambda_{\beta} \) (lemma 3.3) yields

\[ T_{\beta}(u) = u + E_{\beta}\Lambda_{\beta}(u). \]

That is \( \Lambda_{\beta} = E_{\beta}(T_{\beta} - Id) : K(G/T) \to K(G/T) \).

On the other hand combining [KK, Proposition (2.22),(d)] with [KK, Proposition (3.39)] one gets

\[ T_{\beta}(aw) = \begin{cases} aw + awr_{\beta} & \text{if } l(w) > l(wr_{\beta}); \\ 0, & \text{otherwise}. \end{cases} \]

This completes the proof. \( \square \)

### 4.4. Bott-Samelson cycles and their \( K \)-rings.

Given an ordered sequence \( (\beta_1, \cdots, \beta_k) \) of simple roots (repetitions like \( \beta_i = \beta_j \) for some \( 1 \leq i < j \leq k \) may occur), we set

\[ S(\alpha; \beta_1, \cdots, \beta_k) = \{(z_0, z_1, \cdots, z_k) \in G/T \times \cdots \times G/T \mid z_0 = \alpha; \\
\quad z_i \in S(z_{i-1}; \beta_i)\}. \]

It is furnished with the structure of oriented twisted product of 2-spheres of rank \( k \) by the maps

\[ S(\alpha; \beta_1, \cdots, \beta_i) \xrightarrow{p_i} S(\alpha; \beta_1, \cdots, \beta_{i-1}); \\
p_i(z_0, \cdots, z_i) = (z_0, \cdots, z_{i-1}); \quad s_i(z_0, \cdots, z_{i-1}) = (z_0, \cdots, z_{i-1}, z_{i-1}). \]

One has also the ready-made maps

\[ \varphi_{0,\beta_1,\cdots,\beta_k} : S(\alpha; \beta_1, \cdots, \beta_k) \to G/T \text{ by } (z_0, \cdots, z_k) \to z_k \\
\bar{\varphi}_{0,\beta_1,\cdots,\beta_k} : S(\alpha; \beta_1, \cdots, \beta_k) \to S(\beta_k) \text{ by } (z_0, \cdots, z_k) \to (z_{k-1}, z_k) \]

that clearly satisfy

\[ (4.5) \quad \varphi_{0,\beta_1,\cdots,\beta_k} = f_{\beta_k} \circ \bar{\varphi}_{0,\beta_1,\cdots,\beta_{k-1}} : S(\alpha; \beta_1, \cdots, \beta_k) \to G/T; \]
(4.6) the commutative diagrams

\[
\begin{array}{ccc}
S(\alpha; \beta_1, \ldots, \beta_k) & \xrightarrow{\varphi_{0, \beta_1, \ldots, \beta_{k-1}}} & S(\beta_k) \\
p_{k-1} \downarrow s_{k-1} & & f_{\beta_k} \\
S(\alpha; \beta_1, \ldots, \beta_{k-1}) & \xrightarrow{\varphi_{0, \beta_1, \ldots, \beta_{k-1}}} & G/T \quad f_{\beta_k} \circ s_{\beta_k} = \text{id} \\
G/T & \xrightarrow{f_{\beta_k}} & G/T
\end{array}
\]

in which \(\varphi_{0, \beta_1, \ldots, \beta_{k-1}}\) is a bundle map over \(\varphi_{0, \beta_1, \ldots, \beta_{k-1}}\).

**Definition 8 ([Du2, 7.2])**. The map (4.5) is called the Bott-Samelson cycle associated to the sequence \(\beta_1, \ldots, \beta_k\) of simple roots.

Let \(\iota_i : S(\alpha, \beta_i) \rightarrow S(\alpha; \beta_1, \ldots, \beta_k)\) be the embedding specified by

\[\iota_i(\alpha, z') = (z_0, \ldots, z_k),\]

where \(z_0 = \cdots = z_{i-1} = \alpha, z_i = \cdots = z_k = z'\). Then the cycles \(\iota_{i*}[S(\alpha, \beta_i)] \in H_2(S(\alpha; \beta_1, \ldots, \beta_k)), 1 \leq i \leq k\), form a basis of \(H_2(S(\alpha; \beta_1, \ldots, \beta_k))\) (cf. 3.3). Let \(x_i \in H^2(S(\alpha; \beta_1, \ldots, \beta_k))\) be the basis Kronecker dual to the \(\iota_{j*}[S(\alpha, \beta_j)]\) as \(< x_i, \iota_{j*}[S(\alpha, \beta_j)] >= \delta_{ij}\). The next result was shown in [Du2, Lemma 4.5].

**Lemma 4.3.** The structure matrix \(A = (a_{i,j})_{k \times k}\) of \(S(\alpha; \beta_1, \ldots, \beta_k)\) relative to \(\{x_1, \ldots, x_k\}\) is given by the Cartan numbers of \(G\) as

\[a_{i,j} = \left\{ \begin{array}{ll} 
\beta_j \circ \beta_i & \text{if } i < j; \\
0 & \text{if } i \geq j.
\end{array} \right.\]

Let \(X_i \in L_C(S(\alpha; \beta_1, \ldots, \beta_k))\) be defined by \(c_1(X_i) = x_i\). Set \(y_i = X_i - 1 \in K(S(\alpha; \beta_1, \ldots, \beta_k))\). Let \(\overline{y}_k \in \mathbb{Z}[y_1, \ldots, y_m]\) be defined in terms of \(A\) as in Definition 3. Combining Lemma 3.4, Lemma 3.5 with Lemma 4.3 yields the next result.

**Lemma 4.4.** Let \(M = S(\alpha; \beta_1, \ldots, \beta_k)\). Then

1. the set \(\{y_I \mid I \subseteq [1, \ldots, m]\}\) is a basis of \(K(M)\);
2. \(K(M) = \mathbb{Z}[y_1, \ldots, y_k]/ < y_i^2 = (\overline{y}_i - 1)y_i, 1 \leq i \leq k >\);
3. if \(f \in \mathbb{Z}[y_1, \ldots, y_k]\) with \(f = \sum a_I(f)y_I\), then

\[a_{[1, \ldots, k]}(f) = \Delta_A(f_{(k)}).\]
In particular, \( a_{[1, \ldots, k]}(f) = 0 \) if \( f(k) = 0 \).

### 4.5. The induced map of a Bott-Samelson cycle.

Given a sequence \( \beta_1, \ldots, \beta_k \) of simple roots consider the induced ring map
\[
\varphi_{\alpha, \beta_1, \ldots, \beta_k} : K(G/T) \to K(S(\alpha; \beta_1, \ldots, \beta_k)).
\]
The ring \( K(S(\alpha; \beta_1, \ldots, \beta_k)) \) has the additive basis \( \{ y_I \mid I \subseteq [1, \ldots, k] \} \) by Lemma 4.4. Let \( \{ a_w \mid w \in W \} \) be the Demazure basis of \( K(G/T) \).

**Lemma 4.5.** The induced map \( \varphi_{\alpha, \beta_1, \ldots, \beta_k}^1 \) is given by
\[
[\varphi_{\alpha, \beta_1, \ldots, \beta_k}]^1(a_w) = (-1)^{l(w)} \sum_{I \subseteq [1, \ldots, k], \beta(I) \sim w} y_I.
\]

**Proof.** It suffices to show that
\[
[\varphi_{\alpha, \beta_1, \ldots, \beta_k}]^1(a_w) = (-1)^{l(w)} \sum_{I \subseteq [1, \ldots, k], \beta(I) \sim w} \overline{y}_I,
\]
for, the complex conjugation of (4.7) yields the Lemma. To this end we compute
\[
\varphi_{\alpha, \beta_1, \ldots, \beta_k}^1(a_w) = \varphi_{\alpha, \beta_1, \ldots, \beta_{k-1}}^1(f_{\beta_k}^1((a_w)))) \text{ (by (4.5))}
= \varphi_{\alpha, \beta_1, \ldots, \beta_{k-1}}^1(\varphi^1_{\alpha, \beta_1, \ldots, \beta_{k-1}}(a_w) + \varphi^1_{\alpha, \beta_1, \ldots, \beta_{k-1}}(a_w) E_{\beta_k} \cdot y_k) \text{ (by (3.4))}
= \varphi_{\alpha, \beta_1, \ldots, \beta_{k-1}}^1(a_w) + \varphi_{\alpha, \beta_1, \ldots, \beta_{k-1}}^1(\lambda_{\beta_k}(a_w)) y_k \text{ (by (4.6))}
= \{ \begin{array}{ll}
\varphi_{\alpha, \beta_1, \ldots, \beta_{k-1}}^1(a_w) & \text{if } l(w) > l(wr_{\beta}); \\
\varphi_{\alpha, \beta_1, \ldots, \beta_{k-1}}^1(a_w) + \varphi_{\alpha, \beta_1, \ldots, \beta_{k-1}}^1(a_w) E_{\beta_k} \cdot y_k, & \text{otherwise},
\end{array}
\]
where the last equality is by Lemma 4.2. From \( \overline{y}_k = -E_{\beta_k} \cdot y_k \) (Lemma 3.2) we obtain
\[
\varphi_{\alpha, \beta_1, \ldots, \beta_k}^1(a_w) = \{ \begin{array}{ll}
\varphi_{\alpha, \beta_1, \ldots, \beta_{k-1}}^1(a_w) - \varphi_{\alpha, \beta_1, \ldots, \beta_{k-1}}^1(a_w) \overline{y}_k & \text{if } l(w) > l(wr_{\beta}); \\
\varphi_{\alpha, \beta_1, \ldots, \beta_{k-1}}^1(a_w) + \varphi_{\alpha, \beta_1, \ldots, \beta_{k-1}}^1(a_w) \overline{y}_k, & \text{otherwise}.
\end{array}
\]
This, together with an easy induction on \( k \), reduces the proof of (4.7) to the general properties of the Demazure classes. Let \( f : X \to G/T \) be a continuous map from a connected 2m-dimensional CW-complex \( X \). Then
\[
(1) \ f^1(a_w) = 0 \text{ whenever } l(u) > m; \\
(2) \ f^1(\overline{a}) = 1 \text{ if } X \text{ is a single point, where } e \in W \text{ is the group unit}.
\]
Indeed, (1) follows from Assertion III in [KK, (3.26)] and [KK, (3.39)]; (2) can be deduced from (1) and
\[
(1) \ \Omega_e = 1 \text{ (cf. [PR], Corollary 2.5) and the footnote in 5.2};
(2) \ \Omega_e = \sum_{w \in W} a_w \text{ (cf. Lemma 5.1 below)}. \]
Corollary 1. Let \( e \in W \) be the group unit. Then
\[
(1) \quad [\varphi_{0,\beta_1,\ldots,\beta_k}]^l(a_e) = \prod_{i=1}^k (1 + y_i)
\]
and for \( w \neq e \)
\[
(2) \quad [\varphi_{0,\beta_1,\ldots,\beta_k}]^l(a_w) = \begin{cases} 
0 & \text{if } l(w) > k; \\
(-1)^k \delta_{w,r_{\beta_1},\ldots,r_{\beta_k},y[1,\ldots,m]} & \text{if } l(w) = k.
\end{cases}
\]

Example 1. Let \( \Delta = \{\alpha_1, \alpha_2\} \) be a set of simple roots of \( G_2 \) in which \( \alpha_1 \) is the short root [Hu, p.57]. The Weyl group \( W \) of \( G_2 \) is generated by \( \sigma_i \), \( i = 1, 2 \), the reflection in the hyperplane \( L_i \subset L(T) \) perpendicular to \( \alpha_i \). If we take \( u = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \), then Lemma 4.5 yields that
\[
\begin{align*}
(a) & \quad [\varphi_{0,\alpha_1,\alpha_2,\alpha_1,\alpha_2}]^l(a_u) = x_1 x_2 x_3 x_4 x_5; \\
(b) & \quad [\varphi_{0,\alpha_2,\alpha_1,\alpha_2,\alpha_1}]^l(a_u) = 0; \\
(c) & \quad [\varphi_{0,\alpha_1,\alpha_2,\alpha_1,\alpha_2}]^l(a_u) = x_1 x_2 x_3 x_4 x_5 (1 + x_6).
\end{align*}
\]

5 Multiplication in the ring \( K(G/H) \)

Lemma 4.5 enables us to establish Theorem 1 (resp. Theorem 2) by computation in the simpler ring \( K(S(\alpha; \beta_1, \ldots, \beta_k)) \) (cf. Lemma 4.4).

5.1. Proof of Theorem 1 (cf. §2). Let \( w = r_{\beta_1} \cdots r_{\beta_m}, \beta_i \in \Delta \) be a reduced decomposition of \( w \in W \), and let \( A_w = (a_{i,j})_{m \times m} \) be the associated Cartan matrix (cf. Definition 1). Consider the Bott-Samelson cycle \( \varphi_{0,\beta_1,\ldots,\beta_m} : S(\alpha; \beta_1, \ldots, \beta_m) \to G/T \) associated to the sequence \( \beta_1, \ldots, \beta_m \) of simple roots. Applying the induced ring map \( \varphi_{0,\beta_1,\ldots,\beta_m}^l \) to (1.1) yields in \( K(S(\alpha; \beta_1, \ldots, \beta_m)) \)
\[
\varphi_{0,\beta_1,\ldots,\beta_m}^l[a_u \cdot a_v] = \sum_{x \in W} C_{u,v}^x \varphi_{0,\beta_1,\ldots,\beta_m}^l[a_x] = (-1)^l w C_{u,v}^w y[1,\ldots,m] + \sum_{l(x) \leq l(w)-1} (-1)^{l(x)} C_{u,v}^{x} \varphi_{0,\beta_1,\ldots,\beta_m}^l[a_x],
\]
where the second equality follows from (2) of Corollary 1. Using Lemma 4.5 to rewrite this equation yields
\[
(-1)^{l(u)+l(v)} \sum_{\beta(L) \sim u} y_L \sum_{\beta(K) \sim v} y_K]
\]
\[
= (-1)^{l(w)} C_{u,v}^w y[1,\ldots,m] + \sum_{l(x) \leq l(w)-1} (-1)^{l(x)} C_{u,v}^{x} \sum_{\beta(J) \sim x} y_J; \]

\[
18
\]
where \( L, K, J \subseteq [1, \cdots, m] \). Finally, comparing the coefficients of the monomial \( y_{[1, \cdots, m]} \) on both sides by using (3) of Lemma 4.4, we obtain
\[
(-1)^{(l(u)+l(v))} \Delta_A \left[ \sum_{\beta(L) \sim u} y_L \left( \sum_{\beta(K) \sim v} y_K \right) \right]_{(m)} = (-1)^{l(u)} C^w_{u,v} + \sum_{l(u)+l(v) \leq l(w)-1} (-1)^{l(x)} C^x_{u,v}.
\]
This finishes the proof. \( \square \)

**Example 2.** Continuing from Example 1 we take
\[
u = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1; \quad v = \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2; \quad w = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2.
\]
Then the \( u, v \in W \) are the only elements of length 5, and \( w \) is the element of highest length. Applying Theorem 1 we compute the structure constants appearing in the expansion
\[
a_e a_u = C^u_{e,u} a_u + C^v_{e,u} a_v + C^w_{e,u} a_w.
\]
In views of (a) in Example 1 and (1) in Corollary 1 we have
\[
C^u_{e,u} = \Delta_A u [x_1 x_2 x_3 x_4 x_5 \cdot \prod_{1 \leq i \leq 5} (1 + x_i)]_{(5)} = \Delta_A u (x_1 x_2 x_3 x_4 x_5) = 1.
\]
Similarly, we get
\[
C^v_{e,u} = \Delta_A v [0 \cdot \prod_{1 \leq i \leq 5} (1 + x_i)]_{(5)} = 0;
\]
\[
C^w_{e,u} = (-1)^5 \Delta_A w [x_1 x_2 x_3 x_4 x_5 (1 + x_6) \cdot \prod_{1 \leq i \leq 5} (1 + x_i)]_{(6)} - (-1)^5 (C^u_{e,u} + C^v_{e,u})
\]
\[
= -\Delta_A w [x_1 x_2 x_3 x_4 x_5 (x_1 + \cdots + x_5 + 2x_6)] + 1
\]
\[
= -2 + 1 = -1,
\]
where the three \( \Delta_A(f) \)'s concerned in the above computation are directly evaluated from Definition 4 without resorting to the specialities of \( A \), thanks to the simplicities of the polynomials \( f \) involved. Summarizing we get, in the ring \( K(G_2/T) \), that
\[
a_e a_u = a_u - a_w.
\]

**5.2.** The method establishing Theorem 1 is directly applicable to find a formula for the structure constants \( K_{u,v}^w(H) \) for multiplying Grothendieck classes in the \( K(G/H) \).

We begin with the simpler case \( H = T \) (a maximal torus in \( G \)). Abbreviate \( X_w(H) \) by \( X_w \), \( \Omega_w(H) \) by \( \Omega_w \) and \( K_{u,v}^w(H) \) by \( K_{u,v}^w \). The transition
\[\text{2} \text{The } \Omega_w \text{ corresponds to } O_{w_0w} \text{ in } [Br_2,KK,PR_2].\]
between the two bases \( \{ a_w \mid w \in W \} \) and \( \{ \Omega_w \mid w \in W \} \) of \( K(G/T) \) has been determined by Kostant and Kumar in [KK, Proposition 4.13; 3.39].

**Lemma 5.1.** In the ring \( K(G/T) \) one has

\[
\Omega_w = \sum_{w \leq u} a_u, \quad a_w = \sum_{w \leq u} (-1)^{l(u)-l(w)}\Omega_u
\]

where \( w \leq u \) means \( X_w \subseteq X_u \).

Combining Lemma 5.1 with Lemma 4.5 gives the next result.

**Lemma 5.2.** Let \( \beta_1, \ldots, \beta_k \) be a sequence of simple roots. With respect to the Grothendieck basis the induced map \( \varphi^!_{0, \beta_1, \ldots, \beta_k} \) is given by

\[
[\varphi^!_{0, \beta_1, \ldots, \beta_k}](\Omega_w) = \sum_{I \subseteq [1, \ldots, k]} b_I(w) y_I,
\]

where \( b_I(w) = \sum_{\beta(I) \sim u, u \geq w} (-1)^{l(u)} \).

Based on Lemma 5.2, an argument parallel to the proof of Theorem 1 yields

**Theorem 2.** Assume that \( w = r_{\beta_1} \cdots r_{\beta_m} \), \( \beta_i \in \Delta \), is a reduced decomposition of a \( w \in W \), and let \( A_w = (a_{i,j})_{m \times m} \) be the associated Cartan matrix. For \( u, v \in W \) we have

\[
(-1)^{l(w)}K^w_{u,v} = \Delta_{A_w}[(\sum b_I(u)y_I)(\sum b_L(v)y_L)](m)
- \sum_{I \subseteq [1, \ldots, m]} b_{[1, \ldots, m]}(x)K^x_{u,v},
\]

where \( I, L \subseteq [1, \ldots, m] \), and where the numbers \( b_K(x) \), \( K \subseteq [1, \ldots, m] \), \( x \in W \), are given as that in Lemma 5.2.

Proceeding to the general case let \( H \subset G \) be the centralizer of a one–parameter subgroup in \( G \). Take a maximal torus \( T \subset H \) and consider the standard fibration \( p : G/T \to G/H \). It is well known that (cf. [PR2, Proposition 1.6])

The induced ring map \( p^! : K(G/H) \to K(G/T) \) is injective and satisfies \( p^! [\Omega_w(H)] = \Omega_w, w \in \overline{W} \).
Consequently one gets

**Corollary 2.** $K^w_{u,v}(H) = K^w_{u,v}$ for $u, v, w \in \mathbb{W}$.

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