Infinite order decoupling of random chaoses in Banach space

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We prove a number of decoupling inequalities for nonhomogeneous random polynomials with coefficients in Banach space. Degrees of homogeneous components enter into comparison as exponents of multipliers of terms of certain Poincaré-type polynomials. It turns out that the fulfillment of most of types of decoupling inequalities may depend on the geometry of Banach space.

KEY WORDS: decoupling principle, symmetric tensor products, random polynomials, multiple random series, multiple stochastic integrals, random multilinear forms, random chaos, Gaussian chaos, Rademacher chaos, stable chaos, multiple Wiener integral, multiple stable integral, Mazur-Orlicz polarization formula, symmetrization, Banach space, Banach lattice, Krivine’s type, rearrengement invariant space, convexity, Orlicz space, Rademacher sequence, Gaussian law, Walsh polynomials, empirical measure.

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1 INTRODUCTION

The concept of decoupling stems from the martingale theory (cf. the survey [Bur86]). First decoupling inequalities for multiple random forms were proved in [MT86, MT87, Kwa87], and many variants have been published since the time when the above papers were published (to name but a sample, cf. [DA87, Hit88, Zin86, NP87, dlPn90, KS89, RST91, Szu91a, RW86, Szu92], and further references in there). So far, all known results have involved a two-sided estimate of $L^p$-(or Orlicz) norms of suitable $k$-homogeneous multilinear forms (or multiple integrals), where $k$ is arbitrary but fixed. Decoupling constants are degree-dependent and escape to infinity. If the degree increases, the strength of the decoupling principle seems to decline.

In this paper we will show how to overcome this deficiency (one cannot expect that decoupling constants remain bounded). Our approach is based on a suitable normalization of polynomials

$$Q(X; t) = Q_0 + tQ_1(X) + \ldots + t^nQ_n(X),$$

where $X = [X_{ij}]$ is a matrix of random variables, with independent rows, and $Q_k$ is a Banach space-valued homogeneous polynomial of degree $k$ (a $k$-linear form). Under suitable integrability and symmetry assumptions the presented decoupling principle compares norms (e.g., Orlicz norms) of $Q(X, t)$ and of the polynomial $Q(X)$,

$$\|Q(X)\| \sim \|Q(X)\|$$

where the matrix $X$ is a “decoupled” version of $X$, i.e., the columns of $X$ are replaced by their independent copies. For example, on the real line, for Rademacher or standard Gaussian random variables, we check directly that

$$E|\sum_{k \geq 0} Q_k|^2 = E|\sum_{k \geq 0} \overline{Q}_k|^2$$

where

$$\overline{Q}_k = \begin{cases} \frac{1}{\sqrt{k!}} \sum_{i_1, \ldots, i_k, j \neq j'} f_k(i_1, \ldots, i_k)X_{i_1i_1} \cdots X_{ki_k}, & \text{if } f_k \text{ is symmetric, or} \\ \sum_{i_1 < \ldots < i_k} f_k(i_1, \ldots, i_k)X_{i_1i_1} \cdots X_{ki_k}, & \text{if } f_k \text{ is tetrahedral} \end{cases}$$

and $Q_k$ follows the same pattern, respectively, but without the multiplier $1/\sqrt{k!}$ in the symmetric case. Also, $L^2$-norm can be replaced by $L^p$-norm, $p > 1$, at cost of multiplying each $k$-homogeneous polynomial by a constant $\epsilon_p^k$.

Degrees of specific components will enter into formulas as exponents of certain multipliers. We replace, so to speak, external constants by internal constants or, more precisely, by sequences of constants. Asymptotic behavior of such sequences is of interest, and the exponential growth is most desirable. If columns of $X$ are identical, we call $Q(X)$ a “random chaos”, and when they are independent (desirably – identically distributed), a “decoupled random chaos”. A tetrahedral decoupled chaos can be written as a sort of lacunary chaos, by a monotone (non-unique, in general) change of ordering on all tetrahedra, from the coordinatewise ordering to a linear ordering. An analogous procedure for symmetric chaoses is possible “locally”, i.e., for a fixed and finite order, and when only a finite number of random variables is involved.
Thus, decoupling inequalities can be viewed as embedding-projection theorems. In the infinite order decoupling we require that projections are contractions. We will observe new phenomena, absent in the homogeneous (or finite order) case. First of all, in the infinite order decoupling, two types of inequalities (“the lower decoupling” – domination by the chaos, and “the upper decoupling” – domination of the chaos) determine two distinct problems. Already homogeneous tetrahedral and symmetric chaoses behave differently (cf. Bourgain’s example in [MT87]).

A robust lower decoupling inequality is satisfied, i.e., the inequality is fulfilled in any Banach space and for an arbitrary symmetric integrable polynomial chaos. At the same time, the fulfillment of a robust upper decoupling inequality is still uncertain. In order to study the upper decoupling principle, we introduce a class of Banach spaces that are characterized by a certain inequality involving linear forms in independent random variables with vector coefficients (in some aspects, the property is similar to classical properties of Banach spaces, like Rademacher type and cotype, smoothness or convexity of norm, etc.

The main feature of the new class of spaces is that random polynomials admit a “horizontal slicing”, reducing the study to that of sums of independent random variables. In the introduced class of Banach spaces a sign-randomized upper decoupling inequality holds. The class of spaces allowing slicing of random polynomials is, unfortunately, geometry-dependent, and very fragile. It is sensitive to an equivalent renorming (just an addition of a two-dimensional normed space with \( \ell^\infty \)- or \( \ell^1 \)-norm terminates the property), in contrast to the homogeneous case. Therefore, a positive result will always require the existence of a suitable equivalent norm (cf. “smooth” vs. “smoothable”, or “convex” vs. “convexifiable”). The family of Banach spaces, satisfying the slicing requirements, contains Banach lattices of finite cotype and sufficiently convex norm. This class turns out to be suitable even for the tetrahedral lower decoupling.

In Section 2 we introduce the nonhomogeneous tensor product notation, define the domination, and derive some basic relations. We refer to [PA91] and the literature included there for a treatment of Gaussian symmetric tensors. The comparison will be given in terms of the aforementioned Poincaré-type polynomials or, equivalently, in terms of a semigroup of contractions, associated with a random polynomial.

The new results are gathered in Sections 3 and 4. The employed techniques in the nonhomogeneous case are different from techniques related to the homogeneous case. First of all, the applicability of conditional expectations is limited. Secondly, the type of domination forbids use of external constants. We will use a “slicing technique”, which reduces the study to a case of certain sums of random variables.

In Section 5 we will indicate some directions in a further study of decoupling inequalities, and show that many assertions can be directly obtained from results of this paper. For example, one can formulate decoupling inequalities in the language of infinite order stochastic multiple integrals. We will also collect some observations that do not fit into the main line of the paper, although may be of some interest.

Let us point out that a widely understood convexity is the principal feature implicit in most of applied techniques. This includes the setting of Banach spaces and existence of moments (integrability) of involved random variables. One can find a number of decoupling principles in the
literature, where the convexity is of no concern, and the focus is on positivity, not on symmetry. However, most of the known results have been obtained so far at the cost of the limitation to the real line (see [KS89] for a discussion on the latter topic). In some special cases, e.g., for Gaussian homogeneous polynomials, a decoupling principle applies to probabilities $P(\cdot \notin K)$, where $K$ is a convex symmetric set in a Banach space $\mathbb{K}$; the aforementioned paper [DA87] deals with $p$-stable random variables and spaces $L^r$, $0 < r < p$, etc.

2 RANDOM TENSOR PRODUCTS

2.1 Notation

Throughout the paper $\varepsilon = (\varepsilon_i)$ denotes a Rademacher sequence, that is, $\varepsilon_i$ are independent random variables taking values $\pm 1$ with probability $1/2$. Walsh functions are products of Rademacher random variables. We will denote by $X$ a sequence, and by $X = [X_1, \ldots, X_n]$ a matrix, of real random variables. $(E, \|\cdot\|_E)$ denotes a real Banach space. By $(L, \|\cdot\|_L)$ we denote a rearrangement invariant Banach space of integrable random variables, $L \subset L^1(P)$, defined on a probability space $(\Omega, \mathcal{F}, P)$, rich enough to carry independent sequences, and with a separable $\sigma$-field. In fact, we will use only specific properties of $L$, ensured by the above restrictions.

(L) Conditional expectations are contractions acting in $L$

$L(E)$ denotes the Banach space of $E$-valued random variables (i.e., strongly measurable mappings from $\Omega$ into $E$) whose norms belong to $L$ a.s., and let $\|\theta\|_{L(E)} = \|\|\theta\|_E\|_L$. Whenever it causes no ambiguity, we omit the subscript.

Let $N = \{1, 2, \ldots\}$ be the set of natural numbers, and $\overline{N} = N \cup \{0\}$. For $m, n \in \overline{N}$, put $[m, n] = \{m, m+1, \ldots, n\}$. Throughout the paper, the bold-face Greek characters $\alpha, \beta, \ldots$, etc., will denote subsets of $N$, identified with $\{0,1\}$-valued sequences:

$$N \supset \alpha \leftrightarrow \alpha = (\alpha_1, \alpha_2, \ldots) \in \{0,1\}^N.$$

Denote $|\alpha| = \# \alpha = \sum_i \alpha_i$, and $\alpha' = (1 - \alpha_1, 1 - \alpha_2, \ldots)$. The following convention will be very helpful. Let $E$ be a nonvoid set. Suppose that $0, 1 \in E$. Define two operations $\{0,1\} \times E \rightarrow E$:

$$0x = 0, \quad 1x = x \quad \text{and} \quad x^0 = 1, \quad x^1 = x \quad \text{(by convention, } 0^0 = 1).$$

If $*$ is any operation in an abstract set $Z$, then we will use the same for functions taking values in $Z$. In particular, if $*: E \times Y \rightarrow Z$, then, by writing $*: E^N \times Y^N \rightarrow Z^N$, we understand the action of $*$ coordinatewise. For example, $(X * y)_i = x_i * y_i$, $i \in N$, $X = (x_i)$, $y = (y_i)$. Also, for $X \in E^N$ and $\alpha \in \{0,1\}^N$, $\alpha X = (\alpha_i x_i)$ and $X^\alpha = (x_i^{\alpha_i})$. In section 3.4, the term $SX$, where $S$ and $X$ are $(N \times n)$ matrices, according to our convention, will denote a new $(N \times n)$ matrix, whose entries are products of entries of $S$ and $X$.

Identifying $\alpha$ and a constant sequence $(\alpha, \alpha, \ldots)$, we have then $\alpha X = (\alpha x_1, \alpha x_2, \ldots)$. For $\alpha \subset N$, we identify $N^\alpha$ and the subset $\{\alpha i : i \in N\}$ of $\overline{N}^N$ (the empty set is identified with $\{(0,0,0,\ldots)\}$). We will consider functions $f = (f_\alpha : \alpha \in \{0,1\}^N)$, where $f_\alpha : N^\alpha \rightarrow E$ ($f_0$ is
an element of $E$, and, if necessary, $f$ may be identified with suitable functions $f : 2^N \times \mathbb{N}^N \to E$, $f(\alpha, i) = f_{\alpha}(\alpha i)$. For definiteness, we request that all functions $f = (f_{\alpha})$ under consideration have a finite support (i.e. $f(\alpha, i) = 0$ for all but finitely many $\alpha$ and $i$).

In this paper we will see an abundance of summation, averaging, and integration on several levels, in order to diminish the notational burden, we introduce a variety of summing brackets (of course, we might replace all following brackets by just one but, by doing so, we would cause a serious visual dissonance). Define

$$\langle f_{\alpha} \rangle = \langle f_{\alpha} \rangle_\alpha = \sum_{\alpha i} f_{\alpha}(\alpha i) \quad \text{and} \quad \langle f \rangle = \sum_{\alpha} \langle f_{\alpha} \rangle_\alpha.$$ 

and, for $x = (x_i) \in E^N$,

$$\langle x \rangle = \sum_{i} x_i$$

All functions $f = (f_{\alpha})$, appearing in the sequel are assumed to vanish on diagonals, i.e., $f_{\alpha}(\alpha i) = 0$ if at least two nonzero-arguments $(\alpha i)_k$ are equal. Define the symmetrizator $\hat{f}$, which unifies values of functions $f_{\alpha}$ on distinct tetrahedra, by the formula

$$\hat{f}_{\alpha} = \frac{1}{|\alpha|!} \sum_{\sigma} \sigma f_{\alpha},$$

where the sum is taken over all permutations $\sigma$ of $\alpha$, and $\sigma f_{\alpha} = f_{\alpha} \circ \sigma$. If $\hat{f} = f$, then the functions is called symmetric. Call a function $f$ tetrahedral, if it may take nonzero values only on the main tetrahedron: $\alpha = \{1, \ldots, |\alpha|\}$, $i_1 < \ldots < i_{|\alpha|}$.

The random matrix $X = (X_{ij} : i \in [1,n], j \in N) \in R^{[1,n] \times N}$, considered before, can be written as $X = [X_1, X_2, \ldots] \in (R^N)^{[1,n]}$, where $X_i = (X_{ij} : j \in N)$. Define a tensor product $X^{\otimes} = (X^{\otimes})$ on $R^{[1,n] \times N}$ by the formula

$$X^{\otimes}(\alpha i) = X_{i_1}^{\alpha_{i_1}} \cdots X_{i_n}^{\alpha_{i_n}},$$

and the symmetric tensor product, by $X^{\otimes} = \hat{X}^{\otimes}$. By convention, a single sequence $X$ can be viewed as a matrix (the sequence) $[X_1, \ldots, X]$. Whence the tensor products $X^{\otimes}$ and its symmetrizations are well defined. One can consider other type of symmetrizers $U$ and the induced symmetric tensor products $U^{\otimes}$ (see Section 5).

The Mazur-Orlicz polarization formula can be written as follows (it is fulfilled in any commutative algebra).

$$X^{\otimes}(\alpha i) = \frac{1}{k!} \sum_{\beta \subset [1,n]} (-1)^{|\beta|} \langle \beta X \rangle^{\otimes}(\alpha i),$$

$$= 2^{-n} \frac{1}{k!} \sum_{\beta \subset [1,n]} (-1)^{|\beta\alpha|} \langle 2\beta\alpha X \rangle^{\otimes}(\alpha i),$$

(2.1)

where $|\alpha| = k$. Endowing $2^{[1,n]}$ with the uniform probability, the functions

$$r_i(\beta) = (-1)^{\beta_i}, \quad \beta \in 2^{[1,n]}, \quad i = 1, \ldots, n$$
are representations of the first $n$ Rademacher random variables. We define \textit{Walsh functions} as products of Rademacher functions:

$$w_{\alpha}(\beta) \overset{df}{=} \prod_{i \in \alpha} r_i(\beta) = (-1)^{\alpha \cdot \beta}, \quad \beta \in 2^{[1,n]}.$$  \hfill (2.2)

Notice the presence of Walsh functions in the Mazur-Orlicz polarization formula.

By $S^0_k = S^0_k(X)$ denote the $\sigma$-field generated by the family

$$\{ g_\alpha(X^\alpha) : \alpha \subset [1,n], |\alpha| = k, g_\alpha = \hat{g}_\alpha, g : R^k \to R \},$$

and let $S_k = S_k(X)$ be the $\sigma$-field generated by $S^0_1 \cup \ldots \cup S^0_k$, and $S$ be spanned by $\bigcup_k S_k$.

\textbf{Proposition 2.1} Let $[X, X]$ have independent rows and i.i.d columns. Let $g_\alpha$ be a function vanishing on diagonals and equal 1 off diagonals. Then the following equalities hold.

$$g_\alpha(\beta X) \hat{\otimes} \alpha = E[g_\alpha(\alpha X) \otimes \alpha | X^\beta]; \quad \hfill (2.3)$$

$$g_\alpha(X + X') \hat{\otimes} \alpha = E[g_\alpha(2X) \otimes \alpha | X + X'], \quad \hfill (2.4)$$

where $X'$ is an independent copy of $X$;

$$g_\alpha X \hat{\otimes} \alpha = E[g_\alpha (X + X') \otimes \alpha | X] \quad \hfill (2.5)$$

provided $EX' = 0$, and $X$ and $X'$ are independent;

$$\left\langle f X^\otimes \right\rangle = \left\langle f E[X^\otimes | S(X)] \right\rangle = E[\left\langle f X^\otimes \right\rangle | S(X)] \quad \hfill (2.6)$$

\textbf{Proof.} Conditions (2.3), (2.4), and (2.5) follow immediately. In order to see the fulfillment of the remaining condition, it suffices to implement the following simple rule. For two $\sigma$-fields $F_1$, $F_2$, and random variables $Z_m$, $m = 1, \ldots, M$, if $E[Z_m | F_1] = E[Z_1 | F_1]$, and $\sum_m Z_m$ is $F_1$-measurable, then $E[Z_i | \sigma(F_1 \cup F_2) = \sum_m Z_i / M$. This rule allows us to reduce each situation to the homogeneous case, and then the proof is direct. \hfill \blacksquare

\subsection*{2.2 Domination of random polynomials}

The decoupling principle for nonhomogeneous polynomials will be defined in terms of a more general concept of domination, applied to certain Poincaré-type polynomials (e.g., cf. various variants of hypercontraction or Malliavin’s calculus, cf. \cite{Sug88}).

First, we decide a setting of domination. Let $(E, \| \cdot \|)$ be a Banach space, $L \subset L^0(P)$ be an algebra of real random variables, endowed with a positive functional $\varphi$, and $L(E) = \{ \xi \in L^0(E) : \| \xi \| \in L \}$ a.s. Define the functional $\Phi(\xi) = \varphi(\| \xi \|)$.

Usual examples consist of $L^p$-norms or quasi-norms, $0 \leq p \leq \infty$, Orlicz (or more general rearrangement invariant) norms, distribution tails $\phi(\theta; t) = P(\| \theta \| > t)$, etc. (cf. \cite{KW92} Chapters 3, 5) for more examples).

Consider $E$-valued random polynomials $\left\langle f X^\otimes \right\rangle$, where $f = (f_\alpha : \alpha \subset N, \; f_\alpha : N^\alpha \to E, \; f_\alpha \equiv 0$ for all but finitely many finite sets $\alpha$, and $f$ belong to a certain category $F$ of functions,
realized on the class of Banach spaces. The role of a constant is to be played by a real valued function \( c = (c_\alpha) : 2^N \times \mathbb{N} \to R \in \mathcal{U} \), where \( \mathcal{U} \) is realized on \( R \). The system \( S = (E, L, \varphi, \mathcal{F}) \) becomes the setting of domination.

**Definition.** Say that \( X \) is dominated by \( X' \) in the setting \( S \) with a constant \( c \) \((X \preceq_S c X', \text{in short})\), if, for every \( n \geq 0 \),

\[
\Phi(\left\langle fX^\otimes \right\rangle) \leq \Phi(\left\langle fcX'^\otimes \right\rangle), \quad f \in \mathcal{F}.
\]

If the constant is of the form \( c_\alpha = c^{|\alpha|} \), where \( c \) is a positive number we will say that the \( X \) is exponentially dominated by \( X' \).

It is easy to see that linear forms in zero mean integrable random variables are comparable with their symmetrized counterparts. A similar property is enjoyed by random chaoses. Let \( \Phi \) be a positive convex functional defined on \( L^1(E) \) turning conditional expectations into contractions, i.e.,

\[
\Phi(\mathbb{E}[X|\mathcal{F}]) \leq \Phi(X) \quad (2.7)
\]

(for example, \( \Phi(X) = \|X\|_{L(E)} \), where \( L \) is a rearrangement invariant space of real random variables, or \( \Phi(X) = \mathbb{E}\varphi(\|X\|_E) \), where \( \varphi \) is an increasing convex function).

**Proposition 2.2** Let \( X \) and \( X' \) be independent identically distributed matrices with independent rows and interchangeable columns (this includes the case of matrices with identical columns). let \( f = (f_\alpha) \) be an \( E \)-valued function vanishing on diagonals.

1. Then

\[
\Phi(\left\langle f(X - EX)^\otimes \right\rangle) \leq \Phi(\left\langle f(X' - X')^\otimes \right\rangle). \quad (2.8)
\]

2. If \( f \) is symmetric, and \( X \) has independent columns, then there exists a Walsh system \( w_\alpha \), independent of \( X \) and \( X' \), such that

\[
\Phi(\left\langle f(X - X')^\otimes \right\rangle) \leq \Phi(\left\langle fw(2X)^\otimes \right\rangle). \quad (2.9)
\]

where \((fw)_\alpha(i) = f_\alpha(i)w_\alpha\).

3. If \( f \) is symmetric, and \( \mathbb{E}X = 0 \), then

\[
\Phi(\left\langle f(X - X')^\otimes \right\rangle) \leq \Phi(\left\langle fw(4X)^\otimes \right\rangle). \quad (2.10)
\]

**Proof.** Inequality \((2.8)\) follows by convexity and contractivity of conditional expectations.
The proof of (2.10) is similar. However, the assumption $EX = 0$ is essential in the following argument. We have

$$
\Phi\left( \left\langle f(X - X')^\otimes \right\rangle \right) = \Phi\left( \sum_{\alpha \subset [1,n]} \left\langle f\alpha(X - X')^{\otimes \alpha} \right\rangle \right)
$$

$$
= \Phi\left( \sum_{\alpha \subset [1,n]} \left\langle f\alpha \sum_{\beta \leq \alpha} X^{\otimes \beta} \otimes (-X')^{\otimes (\alpha - \beta)} \right\rangle \right)
$$

$$
= \Phi\left( \sum_{\alpha \subset [1,n]} \left\langle f\alpha^2|\alpha|^{-n} \sum_{\beta \subset [1,n]} X^{\otimes \beta \alpha} \otimes (-X')^{\otimes (\alpha - \beta)} \right\rangle \right)
$$

$$
\leq \frac{1}{2^n} \sum_{\beta \subset [1,n]} \Phi\left( \sum_{\alpha \subset [1,n]} \left\langle f\alpha^2|\alpha|^{-n} \sum_{\beta \subset [1,n]} X^{\otimes \beta \alpha} \otimes (-X')^{\otimes (\alpha - \beta)} \right\rangle \right)
$$

$$
\leq \frac{1}{2^n} \sum_{\beta \subset [1,n]} \Phi\left( \sum_{\alpha \subset [1,n]} \left\langle f\alpha^2|\alpha|^{-n} \sum_{\beta \subset [1,n]} (X + X')^{\otimes (\alpha - \beta)} \otimes (-1)^{\otimes (\alpha - \beta)} \right\rangle \right)
$$

$$
= \frac{1}{2^n} \sum_{\beta \subset [1,n]} \Phi\left( \sum_{\alpha \subset [1,n]} w_{\alpha}(\beta)^2|\alpha| \langle f\alpha E[(2X)^{\otimes \alpha}|X + X'] \rangle \right)
$$

$$
\leq \Phi\left( \sum_{\alpha \subset [1,n]} \left\langle f\alpha w_{\alpha}(4X)^{\otimes \alpha} \right\rangle \right).
$$

The proof has been completed.

The following contraction principle is well known in the one dimensional case (cf. [Kah68] for the real case, and [HJ74], for the vector case).

**Theorem 2.3** Let $\varphi : R_+ \rightarrow R_+$ be a convex increasing function. Let $X$ be a matrix of real symmetric random variables with independent rows and either independent or identical columns. Then for every $E$-valued function $f \in F_S$ (or $F_T$), and bounded real function $g = (g\alpha)$ with $c = \|g\|_\infty$, of the form $g_k(i) = g_{k1}(i_1) \cdots g_{kk}(i_k)$, we have

$$
E\varphi \left( \| \left\langle f\alpha g X^{\otimes} \right\rangle \| \right) \leq E\varphi \left( \| \left\langle f\alpha g X \right\rangle \| \right)
$$
Proof. In the case of a homogeneous chaos, i.e., when columns of the matrix $X$ are identical, the result appeared in [Kwa87], while for nonhomogeneous chaoses, in [KS88]. The case with independent columns follows by a spreading argument. That is, there exists an enumeration of functions $f$ such that the polynomial $\langle f X \rangle$ can be written as a polynomial $\langle f' X \rangle$. Hence, we arrive in the previous situation. 

Say that tails of two random variable $X$ and $Y$ are comparable, if, for some constants $K > 0$ and $t_0 \geq 0$

$$P(|X| > t) \leq K P(|Y| > Kt) \quad \text{and} \quad P(|Y| > t) \leq K P(|X| > Kt), \quad t \geq t_0.$$ 

Notice that, at cost of increasing the constant $K$, one may assume that the above estimates hold for every $t > 0$. Thus, there exist probability spaces and copies $X'\prime, X''\prime, X'\prime', X''\prime'$ and $Y'\prime, Y''\prime, Y'\prime', Y''\prime'$ of $X$ and $Y$, respectively, such that $|X'| \leq K'|Y'|$ and $|Y''| \leq K'|X''|$ a.s. In particular, the upper decoupling inequality is satisfied simultaneously for chaoses spanned by $X$ and $Y$, provided components of both sequences are independent and have comparable tails.

**Corollary 2.4** Let $X_1$ and $X_2$ be matrices of real symmetric random variables with independent rows and independent or identical columns. Suppose that corresponding entries of both matrices have comparable tails. Then, for any symmetric or tetrahedral function $f$, polynomials in $X_1$ and $X_2$ are comparable, i.e., for any increasing function $\phi : R_+ \rightarrow R_+$,

$$E\phi\left(\|\langle f X_i \rangle\|\right) \leq E\phi\left(\|\langle f(c X_j) \rangle\|\right),$$

$i, j \in \{1, 2\}$, for some constant $c$, depending on the tail domination constant $K$.

### 2.3 Lower and upper decoupling inequalities

**Definition.** Let $X$ be a sequence of real independent random variables and $X$ be a matrix whose columns are independent copies of $X$. Let $F$ be a class of functions $f_\alpha = (f_\alpha) : N^\alpha \rightarrow E$. Denote by $UD = UD(E; \Phi; F)$ (respectively, $LD = LD(E; \Phi; F)$) the class of sequences $X = (X_i)$ of independent random variables (more exactly, the class of product probability measures) such that that the upper decoupling inequality (respectively, the lower decoupling inequality) holds on $F$, i.e., there exists a constant $c$ such that, for every $n \in N$ and $f \in F$, one has

$$E\Phi\left(\langle f X \rangle\right) \leq E\Phi\left(\langle f(c X) \rangle\right) \quad (\text{respectively,} \quad E\Phi\left(\langle f(c X) \rangle\right) \leq E\Phi\left(\langle f X \rangle\right)).$$

If the considered sequences have components with the same probability distribution $\mu$, we will say that $\mu$ (or a random variable with the distribution $\mu$) satisfies the upper (respectively, lower) decoupling inequality.

The most important classes are $FS$, the class of symmetric functions, and $FT$, the class of tetrahedral functions (recall that we always assume that functions vanish on diagonals). One can consider also other classes (cf. Section 5). More precisely, the decoupling introduced above is understood in the sense of the exponential domination. Note that in most cases of interest that is a desired property. By the same token one can discuss the decoupling in a weaker sense (with a
“constant” \(c_\alpha\) being not necessarily of the exponential type), but then both sides of decoupling, the lower and upper inequality, should be treated separately.

Proposition 2.2 indicated that in case of insufficient symmetry it is necessary to randomize signs of consecutive homogeneous components of a random chaos. In the proposition, such a randomization does not affect internal components of homogeneous polynomials. However, as will be shown, frequently one needs random signs within each and every homogeneous term, and the intricacy of such a randomization may vary. For example, one may use Walsh multipliers induced either by one sequence \(\varepsilon\), or by a matrix \([\varepsilon_1, \varepsilon_2, \ldots]\) with independent Rademacher columns, or, instead of Walsh functions, one may require plain Rademacher family, indexed by the multi-index \(\alpha_i\). The latter sign-randomization is the only known way, so far, of extending Theorem 2.3 to functional multipliers \(g\) whose arguments are not separated (cf. [KS86]).

3 SLICING AND DECOUPLING

3.1 Slicing

Let \(\mathcal{C}\) be a class of finite random real sequences and \(N\) be an integer. Say that an \(n \times N\) random real matrix \(X\) is \(\mathcal{C}\)-sliceable, if its rows are independent and belong to \(\mathcal{C}\). For \(\alpha \subset [1, N]\), denote \(\alpha^* = \max \{i \in [1, N]: \alpha_i = 1\}\) (\(\max \emptyset \text{ def } 0\)).

Lemma 3.1 Let \(E\) be a measurable vector space and \(\Phi : E \to R_+\) be a measurable function. Let \(\mathcal{C}\) and \(\tilde{\mathcal{C}}\) be classes of finite random sequences such that

\[
E\Phi \left( x + \sum_{i=1}^{m} \xi_i x_i \right) \leq E\Phi \left( x + \sum_{i=1}^{m} \tilde{\xi}_i x_i \right) \tag{3.1}
\]

for every integer \(m\), \(\{x_i\} \subset E\), \((\xi_i) \in \mathcal{C}\), and \((\tilde{\xi}_i) \in \tilde{\mathcal{C}}\). Let \(X\) and \(\tilde{X}\) be \(\mathcal{C}\)- and \(\tilde{\mathcal{C}}\)-sliceable \(n \times N\) random matrices, respectively, and \(f = (f_\alpha) : \alpha \subset [1, N]\) be an \(E\)-valued symmetric function.

Then

\[
E\Phi \left( \langle fX_\alpha \rangle \right) \leq E\Phi \left( \langle f\tilde{X}_\alpha \rangle \right) \tag{3.2}
\]

Proof. The statement will be proved by induction with respect to \(n\). Without loss of generality, we may assume that \(X\) and \(\tilde{X}\) are independent and defined on a product space, and functions \(f_\alpha(\alpha i)\) vanish unless \(i_1 < i_2 < i_3\ldots\).

For \(n = 1\), (3.2) coincides with (3.1). Suppose that (3.2) holds for every \(\mathcal{C}\)-sliceable matrix \(X\) and every \(\tilde{\mathcal{C}}\)-sliceable matrix \(\tilde{X}\). We note the decomposition:

\[
f_\alpha(\alpha i) = f_\alpha(\alpha i) \mathbb{I}_{\{i_\alpha^* \leq n-1\}} + f_\alpha(\alpha i) \mathbb{I}_{\{i_\alpha^* = n\}}
\]

Hence, denoting

\[
f^n_\alpha(\alpha i) = f_\alpha(\alpha i) \mathbb{I}_{\{i_\alpha^* \leq n\}},
\]

and

\[
\tilde{f}_{\alpha \setminus \{\alpha^*\}}(\alpha i) = \tilde{f}^{(n-1)}_{\alpha \setminus \{\alpha^*\}} \mathbb{I}_{\{i_\alpha^* = n\}}
\]
we have
\[ \left\langle X^{\alpha} \right\rangle = \sum_{\alpha \subset [1,N]} \left\langle f_{\alpha} X^{\alpha} \right\rangle = \sum_{\alpha \subset [1,N]} \left\langle f_{\alpha}^{(n-1)} X^{\alpha} \right\rangle + \sum_{\alpha \subset [1,N]} \left\langle \tilde{f}_{\alpha(\alpha^*)} X^{\alpha(\alpha^*)} \right\rangle X^{\alpha(\alpha^*)}. \]

Now, we use the Fubini’s theorem, combined, first, with the inductive assumption, and then, with condition (3.1). This completes the proof.

**Remarks 1** Several special cases and variations of the above lemma will be of particular interest.

1. Let \( X = [X_1, \ldots, X_n] \), where \( X_k \) is an \( n \times k \) random matrix, and \( \bar{X} \) have the same structure. Assume that both matrices are \( \mathcal{C} \)- and \( \tilde{\mathcal{C}} \)-sliceable, respectively, and let \( N = 1 + \ldots + n \). Let \( E \) and \( \Phi \) be as in Lemma 3.1.

   (a) If condition (3.1) is fulfilled then, for every symmetric function \( f = (f_k : 0 \leq k \leq n) \), we have
   \[ E \Phi \left( \sum_{k=0}^n \left\langle f_k X_{k}^{\otimes k} \right\rangle \right) \leq E \Phi \left( \sum_{k=0}^n \left\langle f_k \bar{X}_{k}^{\otimes k} \right\rangle \right) \]  
   (3.3)

   (b) Assume, additionally, that columns of \( X \) and \( \bar{X} \) are independent, and the classes \( \mathcal{C} \) and \( \tilde{\mathcal{C}} \) are closed under independent extensions (i.e., if \( \xi, \xi' \in \mathcal{C} \), and \( \xi \) is independent of \( \xi' \), then \( (\xi, \xi') \in \mathcal{C} \)). Then the following inequality is sufficient for (3.3).
   \[ E \Phi(x + \xi y) \leq E \Phi(x + \tilde{\xi} y) \]  
   (3.4)

2. Let \( 0 < q < p < \infty \) and \( \mathcal{C}, \tilde{\mathcal{C}}, X, \bar{X} \) be as in the lemma or as in the special case described above (in Remark 1.1). Assume that

   \[ \|x + \sum_{i=1}^m \xi_i x_i\|_p \leq \|x + \sum_{i=1}^m \tilde{\xi}_i x_i\|_q. \]  
   (3.5)

Then

   \[ \| \left\langle f X^{\otimes} \right\rangle \|_p \leq \| \left\langle f \bar{X}^{\otimes} \right\rangle \|_q, \]  
   (3.6)

and, in the special case (Remark 1.1),

   \[ \| \sum_{k=0}^n \left\langle f_k X_{k}^{\otimes k} \right\rangle \|_p \leq \| \sum_{k=0}^n \left\langle f_k \bar{X}_{k}^{\otimes k} \right\rangle \|_q. \]  
   (3.7)

Consider the assumption in Remark 1.2. Then (3.7) is fulfilled provided

   \[ \|x + \xi y\|_p \leq \|x + \tilde{\xi} y\|_q \]  
   (3.8)

holds. If there exists a constant \( c \) such that \( \tilde{\xi} = c\xi \), relations (3.3)–(3.8) are called hypercontraction inequalities, and \( \xi \) is called a hypercontractive random variable. Gaussian and Rademacher random variables are hypercontractive with constants \( c = c_{p,q} = ((p-1)/(q-1))^{1/2}, 1 < q < p < \infty \) (cf. [Bor84, Gro73, KS88, KS91]). A symmetric \( \alpha \)-stable random variable is hypercontractive in any normed space with exponents \( q, p \in (h_\alpha, \alpha) \), where \( h_\alpha = 0 \), for \( \alpha \leq 1 \), and \( h_\alpha < 1 \), for every \( \alpha < 2 \) [Szu90].
3.2 Tail estimates

In [KW92] the following relation between two $E$-valued random vectors $X$ and $Y$ is called the $\Phi$ domination of $X$ by $Y$:

$$E\Phi(x + X) \leq E\Phi(x + Y), \quad x \in E.$$  

In case when $\Phi(\cdot) = \|\cdot\|^p$, for some $p > 0$, we will use the phrase “$E + L^p(E)$-domination (to distinguish the notion from the comparison of moments).

The fulfillment of $\Phi$-domination yields immediately the same relation for sums of finite copies of $X$ and $Y$. In [Szu92] we used that fact to prove that the $\Phi$-domination of two type of random chaoses generated by hypercontractive random variables implies the tail domination. We will rephrase that result in a more general manner, pointing out the assumptions needed for the fulfillment of the tail decoupling.

**Theorem 3.2** Let a class $\mathcal{X}$ of random vectors $X$ be $(E + L^p(E))$-dominated by a class $\mathcal{Y}$ of random vectors $Y$ in $c_0$ (or, equivalently, in every separable Banach space). Let $\mathcal{Y}$ satisfy the, so called, Marcinkiewicz-Paley-Zygmund (MPZ) condition, i.e.

$$m = \sup_{Y \in \mathcal{Y}} \frac{\|Y\|^p}{\|Y\|^q} < \infty$$

for some (equivalently, all) $q < p$. Then, for some constants $c, C > 0$, for every $Y \in \mathcal{Y}$, there exists $t_0 = t_0(\mathcal{L}(\|Y\|))$, such that

$$P(\|X\| > ct) \leq CP(\|Y\| > t), \quad t \geq t_0.$$  

If, additionally, $\mathcal{Y}$ is bounded in $L^0(E)$, then the class $\mathcal{X}$ is tail-dominated by the class $\mathcal{Y}$ (i.e., the number $t_0$ above does not depend on a particular choice of $Y \in \mathcal{Y}$).

We omit the proof, since its steps are exactly the same as steps in the proof of Theorem 5.3 in [Szu92]. Also, as in [Szu92], we obtain immediately the following corollaries.

**Corollary 3.3** Let assumptions of Theorem 3.2 be fulfilled, including the boundedness of $\mathcal{Y}$ in $L^0(E)$.

1. Let $\varphi : R_+ \to R_+$ be an increasing function of moderate growth, and $\phi(0) = 0$. Then, for some $C' > 0$,

$$E\varphi(\|X\|) \leq C'E\varphi(\|Y\|), \quad X \in \mathcal{X}, \quad Y \in \mathcal{Y},$$

If the growth is not moderate, then we still preserve the implication

$$E\varphi(\|Y\|) < \infty \Rightarrow E\varphi(c\|Y\|) < \infty, \quad X \in \mathcal{X}, \quad Y \in \mathcal{Y},$$

for some universal constant $c > 0$.

2. The $L^0$-boundedness of $\mathcal{Y}$ implies the $L^0$-boundedness of $\mathcal{X}$. If $\mathcal{Y}$ is tight, so is $\mathcal{X}$.

3. The domination in the sense of tightness also holds in any separable Fréchet (i.e., metrizable complete locally convex) space, with the topology generated by a countable family of seminorms (cf. [Rud73]), provided the $(E + L^p(E))$-domination is fulfilled and the uniform Marcinkiewicz-Paley-Zygmund condition is fulfilled for all seminorms.
It is clear how this pattern applies to decoupling inequalities. If a (lower or upper) decoupling inequality holds in every Banach space, and a random chaos is induced by hypercontractive random variables, then the same type of decoupling holds by means described in the above corollary.

3.3 Lower decoupling

We assume in this subsection that $L$ is an Orlicz space $L^\varphi$ such that $\varphi$ satisfies a strong convexity condition

$$\text{for some } a < 1, \varphi^a \text{ is convex}$$  \hfill (3.9)

Note that (3.9) means that, for some $p > 1$, $\lim_{t \to \infty} \varphi(t)/t^p = \infty$. In particular, for a moderately increasing $\varphi$ (i.e., for separable $L^\varphi$), $L$ is uniformly convex. We begin with an auxiliary result.

**Lemma 3.4** Let $L$ and $\varphi$ satisfy (3.9). Let $\theta = (\theta_i)$ be a sequence of integrable independent identically distributed random variables. Put $S_n = \theta_1 + \ldots + \theta_n$ and $\xi = \sup_n |S_n|/n$, and let $(\varepsilon_1, \varepsilon_2, \ldots)$ be a Rademacher sequence independent of $(\theta_i)$. Then, there exists a constant $c_\varphi$, such that

(i) For every $x, y \in E$

$$E\varphi(\|x + \varepsilon y\|) \leq E\varphi(\|x + c_\varphi \varepsilon \theta y\|).$$

(ii) For every $n \in N, x, x_1, \ldots, x_n \in E$,

$$E\varphi(\|x + \sum_{i=0}^{n} \varepsilon_i S_i x_i\|) \leq E\varphi(\|x + c_\varphi \theta \sum_{i=0}^{n} \varepsilon_i x_i\|).$$

**Proof.** Assertion (ii) follows immediately from (i), the Fubini’s theorem, and the contraction principle for a Rademacher sequence.

We will prove (i). The function $[0, \infty) \ni t \mapsto \psi(t) = E\varphi(\|x + \varepsilon ty\|) - \varphi(\|x\|)$ is convex and increasing. Hence

$$E\psi(\xi) \leq CE\psi(\theta),$$

since the sequence $M_1 = S_n/n, M_2 = S_{n-1}/(n-1), \ldots, M_{n-1} = S_2/2, M_n = S_1 = \theta_1$ forms a martingale with respect to the natural filtration.

It is an elementary exercise to prove that the following transformations inherit property (3.9): the shift $\phi - a$, the composition $\phi \circ \psi$ with another convex function, averages $\int \phi_{\omega} (\cdot) \mu(d\omega)$ with respect to probability measures $\mu$ and a (measurable) family $\{\phi_{\omega}\}$ of functions with property (3.9). Hence the function $[0, \infty) \ni t \mapsto E\phi(\|x + \varepsilon ty\|) - \phi(\|x\|)$ has property (3.9), where $x, y \in E$ and $\varepsilon$ is a Rademacher random variable. Therefore, by Doob’s inequality,

$$P(\xi > t) = P(\psi(\xi) > \psi(t)) \leq \frac{E[\psi^a(\theta); \xi > t]}{\psi^a(t)}.$$  \hfill (3.10)

Then, we infer from (3.10) and Hölder’s inequality that

$$E\phi(\|x + \varepsilon y\|) - \phi(1) = E\psi(\xi) = \int_0^\infty P(\xi > t) d\psi(t)$$

$$\leq \int_0^\infty \frac{E[\psi^a(\theta); \xi > t]}{\psi^a(t)} d\psi(t)$$

$$= (1 - a)^{-1} E[\psi^a(\theta)]^a [E[\psi(\theta)]]^{1-a}$$

$$\leq (1 - a)^{-1} E[\psi^a(\theta)]^a [E[\psi(\theta)]]^{1-a}.$$
Define $a_0 = \inf \{ a \in (0,1) : \phi^a \text{ is convex} \}$. Then, letting $a \to a_0$, and using the convexity, we obtain

$$E \psi(\xi) \leq (1-a_0)^{-1/a_0} E \psi(\theta) \leq E \psi((1-a_0)^{-1/a_0} \theta).$$

The lemma has been proved. \hfill \blacksquare

**Remark 2** The constant $c = c_\varphi$ depends on the exponent $a$, appearing in (3.9), or more precisely, on $a_0$. Also, $c = \infty$, if $a_0 = 1$, in general. The convexity assumption concerning $\varphi$ is necessary for (i), if we do not restrict the class of distributions of $\theta$. Consider, for example, $L = L^1$. Then (i) implies that $E \sup_1 |\theta|/i < \infty$, if $E|\theta| < \infty$. A symmetric random variable $\theta$ with the tail $P(|\theta| > t) = (t \log^2 t)^{-1}, t \geq e$, produces a quick counterexample.

We will let the generality of the proof of the following theorem slightly exceed our current needs. The reason will be explained in Section 5. Recall (see (1.1)) that in the real case the lower decoupling for Gaussian or Rademacher chaoses holds with a constant $c_k = 1/\sqrt{k!}$.

**Theorem 3.5** Let the matrix $[X, X]$ have i.i.d. columns. Let $\varphi$ satisfy (3.9). Then the sign-randomized weak lower decoupling inequality holds, i.e., for Walsh function $w = (w_k)$, independent of $[X, X]$, we have

$$\| \sum_{k=0}^n w_k \langle f_k X^{\otimes k} \rangle \| \leq \| \sum_{k=0}^n w_k \frac{(2ck)^k}{k!} \langle f_k X^{\otimes k} \rangle \|,$$

for every symmetric function $f = (f_k)$ vanishing on diagonals, where $c = c_\varphi$ depends only on the function $\varphi$. If the underlying random variables are symmetric, then the Walsh functions can be omitted.

**Proof.** We begin with the Mazur-Orlicz polarization formula.

$$\| \sum_{\alpha \subset [1,n]} \langle f_\alpha X^{\otimes \alpha} \rangle \| = \| \sum_{\alpha \subset [1,n]} \langle f_\alpha \frac{1}{|\alpha|!} \sum_{\beta \leq \alpha} (-1)^{|\alpha-\beta|} \langle \beta X \rangle^{\otimes \alpha} \rangle \|$$

$$= \| 2^{-n} \sum_{\alpha \subset [1,n]} \sum_{|\alpha|!} \langle \sum_{\beta \leq \alpha} (-1)^{|\alpha-\beta|} f_\alpha \langle 2\beta X \rangle^{\otimes \alpha} \rangle \|$$

$$\leq 2^{-n} \sum_{\alpha \subset [1,n]} \| \sum_{\beta \leq \alpha} (-1)^{|\alpha-\beta|} \langle f_\alpha \langle 2\beta X \rangle^{\otimes \alpha} \rangle \|$$

For a fixed $\beta$, we have (cf. (2.3))

$$\langle \beta X \rangle^{\otimes \alpha} = E[\langle \alpha X \rangle^{\otimes \alpha} | X^I \beta].$$

Recall that $w_\alpha(\beta) = (-1)^{|\alpha-\beta|}$ are Walsh functions. Since the mapping $\beta \mapsto \beta' = 1 - \beta$ is measure preserving, hence, by the contractivity of conditional expectations and Fubini’s theorem, we have

$$\| \sum_{\alpha \subset [1,n]} \langle f_\alpha X^{(\otimes \alpha)} \rangle \| \leq \| \sum_{\alpha \subset [1,n]} w_\alpha^{|\alpha|} \langle f_\alpha \langle \alpha X \rangle^{\otimes \alpha} \rangle \|.$$ 

At this moment we give up the generality and notice that $f_\alpha (|\alpha| = k)$ vanish unless $\alpha = [1,k]$. 


Now it suffices to apply the Slicing Lemma 3.1. Let \( g = (g_k) \) be a symmetric function, \( g_k : N^k \to E \), (3.9) be fulfilled and \( w = (w_k) \) be a Walsh sequence independent of \( X \). Then, for a constant \( c = c_\varphi \),
\[
\| \sum_{k=0}^n w_k \langle g_k \left( \frac{X_1 + \ldots + X_k}{k} \right)^\otimes k \rangle \| \leq \left\| \sum_{k=0}^n w_k \langle g_k (cX)^\otimes k \rangle \right\|.
\] (3.12)

The proof is completed.

**Corollary 3.6** Let assumptions of Theorem 3.5 be fulfilled, where \( \varphi(t) = t^p \), \( p > 1 \). Denote by \( Q(f) \) the coupled, and by \( \overline{D}(f) \), the decoupled chaos, as appear, respectively, in the right and left hand side of inequality (3.11). Assume that components \( X_i \) of \( X \) are hypercontractive, with hypercontractivity constants uniformly bounded away from 0. Then the following conditions are fulfilled.

1. There is a constant \( C \), depending only on the hypercontractivity constants, and a sequential constant \( c = (c_k) \), depending only on \( p \), such that, for any non-decreasing moderately growing function \( \varphi : R_+ \to R_+ \), every \( f \in F_S \),
\[
E\varphi(\| Q(f) \|) \leq C E\varphi(\| Q(cf) \|) \]
(if \( \varphi \) does not grow moderately, the finitness of the Orlicz modular is preserved).

2. The stochastic boundedness of a family of coupled polynomial chaos \( \{ Q(f_a) : a \in A \} \) implies the same for \( \{ \overline{D}(df_a) : a \in A \} \), where \( d = c^{-1} \), and \( c \) appears in the preceding statement. By the same token, the tightness of the first family yields the tightness of the second family.

**Proof.** It suffices to interpret appropriately Corollary 2.4.

### 3.4 Reduction to Rademacher chaoses

We will focus on a search of reasonably wide classes of Banach spaces, which support the exponential upper decoupling. Recall that \( X \in \mathcal{UD} = \mathcal{UD}(E; \Phi; \mathcal{F}) \) (\( X \) satisfies the upper decoupling inequality), if
\[
E\Phi(\left\langle fX^\otimes \right\rangle) \leq E\Phi(\left\langle f(cX)^\otimes \right\rangle)
\]
for every function from class \( \mathcal{F} \). The most important are classes \( F_S \), of symmetric functions, and \( F_T \), of tetrahedral functions. Denote by \( \mu = \mathcal{L}(X) \) the distribution of a sequence \( X \). Let \( \varphi : R_+ \to R_+ \) be a measurable function. Denote by \( \mathcal{R}_U = \mathcal{R}_U(\mu, \varphi) \) (respectively, \( \mathcal{R}_L = \mathcal{R}_L(\mu, \varphi) \) the class of Banach spaces such that, for some constant \( c > 0 \), the inequality
\[
E\varphi(\| x + X \sum_i \varepsilon_i x_i \|) \leq E\varphi(\| x + c \sum X_i x_i \|),
\] (3.13)
(respectively,
\[
E\varphi(\| x + \sum X_i x_i \|) \leq E\varphi(\| x + cX \sum \varepsilon_i x_i \|)
\] (3.14)
is fulfilled, for every $x \in E$, and for all finite families $\{ x_i \} \subset E$.

The following result shows the importance of the introduced classes. Its proof is a direct consequence of the Slicing Lemma 3.1, and, for tetrahedral functions, of equality (2.4) from Proposition 2.1.

**Proposition 3.7** Let $X$ be a sequence of independent symmetric random variables, and $X$ be a matrix whose columns are independent copies of $X$, $f = (f_\alpha)$ be a symmetric function with values in $E$, $\varphi : E \to \mathbb{R}_+$ be a measurable function. Denote by $\varepsilon = (\varepsilon_i)$ be a Rademacher sequence independent of $X$, and by $S$, a Rademacher matrix, independent of $[X, X]$. If $E \in \mathcal{R}_U(\mu, \varphi)$ (respectively, $E \in \mathcal{R}_L(\mu, \varphi)$), then

$$E\varphi \left( \left\| \mathbf{f}(X S) \right\| \right) \leq E\varphi \left( \left\| \mathbf{f}(cX) \right\| \right)$$

(respectively, the converse implication is valid, with $c$ replaced by $c^{-1}$). If, additionally, $\varphi$ is convex, the latter inequality is fulfilled also for tetrahedral functions (respectively, the fulfillment of the latter inequality for tetrahedral functions implies the same, for symmetric functions).

### 3.5 Limitations of the reduction

Inequalities (3.14) and (3.13) may fail in some Banach spaces, and for some random sequences. First, we note the following immediate consequence of Proposition 3.7.

**Lemma 3.8** Let $X, X$, and $\varphi$ be as in Proposition 3.7. Let $E \in \mathcal{R}_U(\mu, \varphi)$ (respectively, $E \in \mathcal{R}_L(\mu, \varphi)$). Then

$$E\varphi \left( \left\| x + \sum_{j=1}^m \sum_{i=1}^n X_j \varepsilon_{ij} x_{ij} \right\| \right) \leq E\varphi \left( \left\| x + c \sum_{j=1}^m \sum_{i=1}^n X_{ij} x_{ij} \right\| \right)$$

(3.15)

(respectively,

$$E\varphi \left( \left\| x + \sum_{j=1}^m \sum_{i=1}^n X_{ij} x_{ij} \right\| \right) \leq E\varphi \left( \left\| x + c \sum_{j=1}^m \sum_{i=1}^n X_j \varepsilon_{ij} x_{ij} \right\| \right)$$

(3.16)

for every $m, n \in N$, and every matrix $[x_{ij}]$ of vectors of $E$.

**Proposition 3.9** Let $X, X$ be as in Proposition 3.7, $\varphi(t) = t^2$, and $G = (G, G_1, G_2, \ldots)$, and $Y = (Y, Y_1, Y_2, \ldots)$, respectively, be a sequence of i.i.d. standard normal, and a sequence of i.i.d. exponential random variables with parameter 2, respectively, such that $G$ and $Y$ are independent of $X$.

1. Let $X$ be square integrable. Assume that $E \in \mathcal{R}_U(\mu, \varphi)$. Then the following conditions are fulfilled.

   (i)

   $$E\| x + XGy \|^2 \leq E\| x + cGy \|^2, \quad x, y \in E;$$

   (3.17)

   (ii)

   $$E\| x + \varepsilon Y y \|^2 \leq E\| x + c' G y \|^2, \quad x, y \in E,$$

   (3.18)

   where $c'$ may be a new constant;
(iii) \[ E\|x + \sum_i \epsilon_i Y_i x_i\|^2 \leq E\|x + c' \sum_i G_i x_i\|^2, \quad x, y \in E \] (3.19)

(iv) \( E \) does not contain isomorphic copies of \( \ell_\infty^n \), uniform in \( n \), neither it contains two-dimensional subspaces isometric to \( \ell_\infty^2 \) or \( \ell_1^2 \).

2. Assume that \( E \in \mathcal{R}L(\mu, \varphi) \), where \( \varphi \) be a nondegenerate nondecreasing function such that \( \limsup_{t \to \infty} \varphi(t)e^{-at^2} = 0 \) for some \( a > 0 \). Then \( X \) is square integrable.

Proof. 1. Inequality (3.17) follows by Lemma 3.8 and the (real) Central Limit Theorem. The passage to the limit can be justified by a routine uniform integrability argument (cf., e.g., [Bil68, Theorems 5.3 and 5.4]). By the same token, one may assume that \( X \) in (3.16) has the normal distribution. An exponential random variable \( Y \) with intensity \( \lambda = 2 \) has the tail comparable to the tail of the product of two independent Gaussian random variable (cf. [Yos80, pp. 243-244]), which proves (3.18), in view of Corollary 2.4. Estimate (3.19) follows by the Fubini’s theorem and iteration.

Suppose \( E = \ell_\infty^2 \) (i.e., \( E \) is just \( \mathbb{R}^2 \) with sup-norm). Take orthogonal \( y, x \) in (3.18) with \( \|x\| = 1, \|y\| = 1/u < 1 \), and subtract 1 from both sides of (3.17). Then the right hand side is of order \( \exp\{-u^2/2\} \), while the left hand side is of order \( \exp\{-2u\} \), for \( u \to \infty \), which produces a contradiction.

Inequality (3.19) yields the domination of sums of symmetrized independent exponential random variables by sums of independent Gaussian random variables. Clearly, this is impossible in \( \ell_\infty^n \) (it suffices to take orthogonal \( x_i \)’s, and apply the classical estimates for suprema of independent random variable, cf., e.g., [VCT87, Lemma V.3.2]).

Finally, inequality (3.18) does not hold in \( \ell_1^2 \), since by the Ferguson-Hertz embedding theorem every two-dimensional normed space can be isometrically embedded into \( L^1 \) (cf. [Per62, Her63], see also [KS91]). One can construct a direct counterexample, too.

2. By choosing \( x_i = t_n x_i/\sqrt{n}, i = 1, \ldots, n \), where \( \|x\| = 1 \), in the defining inequality of the class \( \mathcal{R}L \), we infer that, for some constant \( c' \)

\[ E\varphi(t_n \frac{\sum_{i=1}^n X_i}{\sqrt{n}}) \leq c'E\varphi(t_n \frac{\sum_{i=1}^n \epsilon_i}{\sqrt{n}}), \] (3.20)

for every real sequence \( t_n \to 0 \). Because of the regular variability of \( \varphi \) at \( \infty \), the right hand side converges to 0, hence the sequence \( (\sum_{i=1}^n X_i/\sqrt{n}) \) is bounded in \( L^0 \) (i.e., tight), by Chebyshev’s inequality. This is possible only if \( X \in L^2 \).

3.6 Reduction in some spaces

3.6.1 Rademacher versus Gaussian chaos

So far, we have established a class of Banach spaces, where the exponential upper decoupling inequality is fulfilled. Now, we will show that class is reasonably wide. In general, the upper decoupling may depend also on distributions of involved random variables (whether it does, is an
open question at this time). Before we proceed further, in order to avoid unnecessary redundancy, we will determine some dependence (far from being complete) between decoupling inequalities for random chaoses spanned by random variables with different distributions.

**Proposition 3.10**

(i) The class $\mathcal{UD}_S = \mathcal{UD}(E; \| \cdot \|_p, \mathcal{F}_S)$ is closed under products and sums of i.i.d. sequences, i.e., if $X$ and $X'$ are equidistributed independent sequences, and $X, X' \in \mathcal{UD}_S$ with a constant $c$, then $XX' = (X_iX'_i) \in \mathcal{UD}$ and $X + X' \in \mathcal{UD}_S$ with the constant $c$.

(ii) In addition to the above properties, the class $\mathcal{UD}_T$ is also closed under linear combinations of independent sequences, i.e., if $X$ and $X'$ are independent sequences, and $X, X' \in \mathcal{UD}_T$ with constants $c, c'$, then, for every numerical sequences $a$ and $b$, $aX + bX' \in \mathcal{UD}_T$ with the constant $c$.

(iii) Denote $\psi(t) = \psi_{x,y}(t) = E\Phi(x + \varepsilon ty)$. If $X^{(m)} \in \mathcal{UD}$ with the same constant $c$, the distributions of $X^{(m)}$ converge weakly to the distribution of $X$, and the family $\{\psi(X^{(m)})\}$ is uniformly integrable, then $X \in \mathcal{UD}$ with a constant which is less or equal $c$.

(iv) If $\Phi$ is convex, then $\mathcal{UD}_S \subset \mathcal{UD}_T$

**Proof.** The closeness under the product is easy to see and follows immediately by Fubini’s theorem.

In order to prove the additivity in (i), let us consider a $(2n \times 2n)$-matrix

$$\begin{bmatrix} X & Y \\ X' & Y' \end{bmatrix}$$

where $Y$ and $Y'$ are independent copies of $X$, and the sequence $(X, Y)$, where $Y$ is an independent copy of $X$. Then it suffices to change the enumeration of arguments of functions $f_k(\cdot)$, putting, in particular $f_k = 0$ for $k \in [n+1, 2n]$.

For additivity in (ii), we rather use the following $(2n^2 \times n)$-matrix

$$\begin{bmatrix} X_1 & * & * & \cdots & * \\ X'_1 & * & * & \cdots & * \\ * & X_2 & \cdots & * \\ * & X'_2 & \cdots & * \\ * & \cdots & \cdots & \cdots & * \\ * & \cdots & \cdots & \cdots & * \\ * & \cdots & \cdots & \cdots & * \\ * & * & * & \cdots & X_n \\ * & * & * & \cdots & X'_n \end{bmatrix}$$

where the symbols $*$ indicate the presence of mutually independent copies of corresponding portions of columns.

Note that the lack of symmetry assumption in (ii) enables us to use arbitrary sequential multipliers $a$ and $b$, while both sequences must be constant under the symmetry assumption.

Assertion (iii) follows from basic properties of weak convergence (cf. [Bil98, Theorems 5.3 and 5.4]).
Assertion (iv) follows from (2.6).

**Corollary 3.11**  If the upper decoupling inequality for symmetric (or triangular functions) is satisfied for some zero-mean probability law with finite variance, e.g., by a Rademacher random variable, then it is satisfied by the Gaussian law.

**Proof.** The assertion follows from Proposition 3.10 (i) or (ii), the Central Limit Theorem, and Proposition 3.10 (iii).

**Corollary 3.12**  Let the assumptions of Theorem 3.11 be fulfilled. Then the upper decoupling inequality of the same type (i.e., either for symmetric or triangular functions) is satisfied by all symmetrized Gamma(m)-distributions, \( m = 1 \) (exponential law), 2, 3, ...

**Proof.**  Indeed, the product of two independent Gaussian random variables is comparable to a random variable with exponential distribution (cf. e.g. [Yos80, pp.243-244]). Hence the assertion follows by Proposition 3.10.

### 3.6.2 Banach lattices

Let \( E \) be a Banach lattice. Then, for every continuous positive homogeneous function \( \psi : R^n \to R \), the expression \( \psi(x_1, \ldots, x_n) \in E, x_1, \ldots, x_n \in E \), is well defined, in particular,

\[
\left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad \text{and} \quad \left( \mathbb{E} \left( \sum_{i=1}^{n} |x_i\theta|^p \right)^{1/p} \right),
\]

where \( \theta_i \in L^p \) are real random variables, and \( 0 < p \leq \infty \) ([Kri74], also see [LT79]). Any inequality or equality that is valid in the real case, carries over to Banach lattices (when \( E \) is a space of functions, these constructions, in general, can be viewed pointwise, both intuitively and rigorously).

Recall the Krivine’s notion of type \( \geq p \) and \( \leq p \) (\( p \)-convex and \( p \)-concave in [LT79]). A Banach lattice is said to be of type \( \geq p \) (respectively, of type \( \leq p \)), \( 1 \leq p \leq \infty \) if

\[
\| \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \| \leq C \left( \sum_{i=1}^{n} \| x_i \|^p \right)^{1/p}
\]

(respectively, the inverse inequality holds). These properties refer to a degree of convexity of the unit sphere, compared to the unit sphere in \( L^p \). For example, \( L^r \) is of type \( \leq p \), when \( r \leq p \), and of type \( \geq p \), when \( r \geq p \), \( 0 < r \leq \infty \).

**Theorem 3.13**  Every Banach lattice \( E \) of type \( \leq q < \infty \) and type \( p > 1 \) admits an equivalent norm for which both upper and lower (and both symmetric and tetrahedral) decoupling inequalities hold for the Rademacher (hence Gaussian) law, by means of comparison in \( L^r \), \( 1 < r < \infty \). More precisely, for such a norm, there exists a constant \( c \) such that

\[
\| \sum_{k \geq 0} \langle Q_k(f/c) \rangle \| \leq \| \sum_{k \geq 0} \langle Q_k(f) \rangle \| \leq \| \sum_{k \geq 0} \langle Q_k(cf) \rangle \|,
\]
where
\[
\overline{Q}_k(f_k) = \begin{cases} 
\frac{1}{\sqrt{k!}} \langle f_k X^\otimes k \rangle & \text{if } f_k \text{ is symmetric} \\
\langle f_k X^\otimes k \rangle & \text{if } f_k \text{ is tetrahedral},
\end{cases}
\]
and \(Q_k(f_k) = \langle f_k X^\otimes k \rangle\) in both cases.

**Proof.** Essentially, we reduce the problem to the situation on the real line (cf. [1.3]). By Figiel and Johnson theorem ([FJ74], see also [LT73, Theorem 1.d.8]), a Banach lattice \(E\), which is of type \(\geq p\) and \(\leq q\), \(1 < p \leq q < \infty\), admits an equivalent norm, making both constants \(C\), appearing in the definition, equal to 1. So, assume that is the case. Then, we have
\[
\| (E^n \sum_{i=1}^n x_i \theta_i^p)^{1/p} \| \leq (E^n \sum_{i=1}^n x_i \theta_i^p)^{1/p}
\]
and
\[
(\| E^n \sum_{i=1}^n x_i \theta_i^q \|^{1/q} \leq (E^n \sum_{i=1}^n x_i \theta_i^q)^{1/q}\]
for any collection of suitably integrable random variables \((\theta_i)\). We will apply both inequalities to Rademacher chaos and use the hypercontractivity of Rademacher chaos. Denote \(c_{r,q} = \max(1,((r-1)/(q-1))^{1/2})\). The proof is similar in the symmetric and tetrahedral case, and also for the upper and lower decoupling. We will give details only in one case, say, for tetrahedral \(f\) and the upper inequality. We have
\[
\left( E \| f \|_{\theta^\otimes}^r \right)^{1/r} \leq \left( E \| f(c_{r,q}\theta) \|_{\theta^\otimes}^q \right)^{1/q}
\]
\[
\leq \left( E \| f(c_{r,q}c_{q,2}\theta) \|_{\theta^\otimes}^2 \right)^{1/2} \leq \left( E \| f(c_{r,q}c_{q,2},p) \|_{\theta^\otimes}^p \right)^{1/p}
\]
\[
\leq \left( E \| f(c_{r,q}c_{q,2},p) \|_{\theta^\otimes}^n \right)^{1/n}.
\]
Other cases follow by an almost verbatim argument. 

The class of Banach lattices, appearing in the theorem, can be enlarged to uniformly convex spaces with a local unconditional structure (LUST) (i.e., such that \(E\) can be embedded into the dual of a Banach lattice, cf., e.g., [GL74]). In fact, the context of Banach lattices, as appear in the theorem, makes the problem of decoupling rather trivial. Any tetrahedral (respectively, symmetric and of finite order) Rademacher or Gaussian chaos is exponentially equivalent, in the sense of the introduced domination in any \(L^r\), \(1 < r < \infty\), to an infinite (respectively, finite) Rademacher or Gaussian sum
\[
\sum_{k=0}^{\infty} \sum_{i \in N^k} f_k(i) X_i,
\]
where \(\{X_i : i \in N^N, i \text{ finite}\}\) is a family of independent Rademacher or Gaussian variables. In particular, for the aforementioned class of Banach spaces, after a renorming, in a trivial manner an infinite order contraction principle holds for Rademacher or Gaussian chaos
\[
E\| f g X^\otimes \| \|^r \leq E\| f(c X)^\otimes \| \|^r,
\]
where $c$ is a suitable constant, and $g = (g_k)$, $g_k : \mathbb{N}^k \rightarrow [-1,1]$ is an arbitrary measurable function. Such the contraction principle fails in general, e.g., if $E = c_0$, and even for a single $k$-homogeneous component, $k \geq 2$ [KSS86].

Remark 3 A related procedure can be applied for random variables with sufficiently high moments. That is, if $E$ is as in Proposition 3.13, and $\theta \in L^r$, $r \geq 2$, is a symmetric random variable such that $r > q_0 = \inf \{ q : E \text{ is of type } \leq q \}$, then $\theta$ is hypercontractive with constants $c_{r,q}(\theta) = ||\theta||_r/||\theta||_q$ [KSS88], which would replace constants $c_{r,q}$ in the proof. By a similar argument to the one used in the proof of Proposition 3.3, one can show that $L^q \notin R_{e}(\mathcal{L}(X), L^r)$, if $q \geq q_0 > r_0$ $(r_0 = \sup \{ r : \theta \in L^r \})$ ($L^q$ in the latter formula can be replaced by any Banach spaces containing isomorphic copies of finite dimensional spaces $\ell_n^q$, uniform in $n$). This may suggest that the upper decoupling inequality fails in such spaces yet the problem remains open.

We do not know whether the upper decoupling inequalities for Gaussian and Rademacher chaoses are equivalent. For tetrahedral functions, even in the homogeneous case, the lower decoupling inequality may fail (cf. Bourgain’s example included in [MT87], or [KW92, Section 6.9]).

4 STABLE CHAUSES

4.1 Auxiliary definitions and inequalities

In this section $X$ denotes a symmetric standard $\alpha$-stable ($S\alpha S$, in short) random variable, i.e., $\mathbb{E} \exp \{ i t X \} = \exp \{-|t|^\alpha\}$, and $Y$ denotes a symmetric $\alpha$-Pareto random variable, i.e., $P(|Y| > t) = t^{-\alpha}$, $t \geq 1$ ($S\alpha P$, in short). It is known that tails of $S\alpha S$ and $S\alpha P$ random variables are comparable, i.e. $P(|X| > t) \leq K P(|Y| > Kt)$ and $P(|Y| > t) \leq K P(|X| > Kt)$, $t \geq t_0$ (we may assume that the above estimate are valid for all $t \geq 0$). Hence, for tetrahedral or symmetric infinite order polynomial $Q(f, \cdot)$ we have

$$\|Q(f, Y/c)\|_p \leq \|Q(f, X)\|_p \leq \|Q(f, cY)\|_p.$$ (4.1)

This remark will enable us to switch freely (in the sense of the exponential domination) between stable and Pareto chaoses, and benefit from algebraic properties of stable random variables, or analytic properties of Pareto random variables. We will need also the following estimate (cf. KSS88, Szu90, Szu91b) for similar inequalities.

Lemma 4.1 Let $0 < s < \alpha < 2$. There exists a constant $a = a(\alpha, s)$ such that, for every sequence of i.i.d. $S\alpha S$ (or $S\alpha P$) random variables, the inequality

$$(\|x\|^\alpha + a \sum \|x_i\|^\alpha)^{1/\alpha} \leq \|x + \sum x_i x_i\|_s$$

is fulfilled, for all $x, x_1, x_2, \ldots \in E$.

Proof. We will apply a hypercontractive iteration for Pareto random variables, and then use the fact that $S\alpha P$ law belongs to the normal domain of attraction of the $S\alpha S$ law (cf. the aforementioned papers for details). It suffices to verify the inequality

$$(1 + at^\alpha)^{s/\alpha} \leq \mathbb{E}\|x + Y t y\|^s,$$ (4.2)
where $\|x\| = \|y\| = 1$, $0 < t \leq 1$. The inequality follows by combining the estimate

$$E\|x + Yty\|^{s} - 1 \geq t^{\alpha}\inf_{\|x\| = \|y\| = 1} E[\|x + Yy\|^{s} - 1; |Y| \geq 2] \geq t^{\alpha}E(|Y| - 1|^{s} - 1)_{+}$$

with the inequality $(1 + t^{\alpha})^{s/\alpha} - 1 \leq s/\alpha t^{\alpha}$, which holds for all $t \geq 0$. Put

$$a = \alpha E(|Y| - 1|^{s} - 1)_{+}/s.$$ 

This completes the proof.

We will see that the fulfillment of decoupling inequalities may depend on the convexity and smoothness of the norm. A norm of a Banach space $E$ is called $p$-smooth (cf. [Ass75, LT79]), $1 < p \leq 2$, if

$$(E\|x + \varepsilon ty\|^{2})^{1/2} \leq (1 + Ct^{p})^{1/p}$$

where $\|x\| = \|y\| = 1$, $t > 0$ (it suffices to consider only $t \leq 1$), and $\varepsilon$ is a Rademacher random variable. By hypercontractivity, the $L^{2}$-norm on the left hand side can be replaced by any $L^{s}$-norm, $1 < s < \infty$. A Banach $E$ is called $p$-smoothable, if it admits an equivalent $p$-smooth norm. It will be convenient to extend trivially the notion of smoothness to the case $p = 1$ (every norm is 1-smooth).

For a Banach lattice $E$, let

$$k_{0} = \inf \{ q : E \text{ is of Krivine’s type } \leq q \} \leq \infty, k_{0} = \sup \{ p : E \text{ is of Krivine’s type } \geq p \} \geq 1.$$ 

Clearly, $k_{0} \leq k_{0}$. Say that a Banach space is of infinite cotype, if it contains subspaces isomorphic to $\ell_{n}^{\infty}$ uniform in $n$. Otherwise, $E$ is said to be a space of finite cotype. A Banach lattice is of finite cotype if and only if it is of Krivine’s type $\leq q$, for some $q < \infty$ [LT79].

### 4.2 Symmetric decoupling

Let us extract a suitable fragment from Theorem 3.5.

**Theorem 4.2 (Lower Symmetric Decoupling)** Let $1 < p < \alpha < 2$. The lower decoupling inequality in $L^{p}$ for symmetric $S_{\alpha}S$ and $S_{\alpha}P$ chaoses is fulfilled with constants $c_{k} = d^{k}$, for some $d > 0$.

The obtained constant is the best we know, even in the real case. However, for a single homogeneous chaos, the estimate can be significantly improved (cf., e.g., [DA80]). Like before, in the Rademacher or Gaussian case, we can prove the upper decoupling inequality only in some Banach spaces. Surprisingly, the upper decoupling inequality for nonintegrable stable chaoses is a trivial consequence of the slicing techniques, and holds in an arbitrary Banach space. Recall that any $S_{\alpha}S$ (or $S_{\alpha}P$) random variable is hypercontractive in any normed space with exponents $q, p \in (h_{\alpha}, \alpha)$, and $h_{\alpha} = 0$, for $\alpha \leq 1$.

**Theorem 4.3 (Upper Symmetric Decoupling)** Let $E$ be a Banach space. Consider $S_{\alpha}S$ (or $S_{\alpha}P$) chaoses in symmetric functions and the norm $L^{s}$, $h_{\alpha} < s < \alpha$. 

(i) For $\alpha \leq 1$, the exponential upper decoupling inequality for symmetric $S\alpha S$ (or $S\alpha P$) chaoses holds in every Banach space.

(ii) Let $E$ be $p$-smoothable, $1 \leq p \leq 2$, and $0 < \alpha \leq p$. Then $E \in R_U(\mu, s)$. In particular, if an upper symmetric decoupling inequality holds for Rademacher chaoses, with constants $(c_k^{(R)})$, then, for an equivalent norm, an upper decoupling inequality for symmetric $S\alpha S$ (or $S\alpha P$) chaoses holds, with constants $(ac_k^{(R)})$, where $a = a(\alpha, s, \|\cdot\|_E, p)$.

(iii) Let $E$ be a Banach lattice, $0 < s < \alpha < 2$. If $k^0 < \infty$, and $\alpha > k^0$ or $k_0 > \alpha$, then $E \in R_U(\mu, s)$, and the upper decoupling inequality with exponential constants $c_k = a^k$ holds.

Proof. (i): Let $0 < s < \alpha \leq 1$, and $\|x\| = 1$, $x_1, \ldots, x_m \in E$. Put $t = \sum_i \|x_i\|$. Then, by the triangle inequality and [Szu90, Cor. 3.2],

$$\|x + Y \sum_{i=1}^m x_i\|_s \leq \|1 + |Yt|\|_s \leq (1 + c_1t^{\alpha})^{1/\alpha}$$

for some constant $c_1 = c_1(\alpha, s)$. On the other hand, since the $\ell^1$-norm dominates the $\ell^\alpha$-norm, and by Lemma [1.3], we have

$$\|x + \sum_{i=1}^m Y_ix_i\|_s \geq (1 + c_2^p \sum_{i=1}^m \|x_i\|^{\alpha})^{1/\alpha} \geq (1 + C_2t^{\alpha})^{1/\alpha},$$

(4.3)

which yields the assertion of the theorem, by virtue of Lemma 3.7, where all entries of $[S, S]$ are equal 1.

(ii): Almost the same argument works for integrable stable chaoses. Denote now $t = (\sum_i \|x_i\|^q)^{1/q}$, where $q \geq \alpha > 1$. By Fubini’s theorem, and the smoothness property (assuming that the norm is already $q$-smooth), and by the hypercontractivity of Rademacher random variables, we have

$$\|x + Y \sum_{i=1}^m x_i\|_s \leq \|(1 + |c_3Yt|^q)^{1/q}\|_s \leq (1 + c_4^{\alpha})^{1/\alpha}$$

for some constants $c_3$ and $c_4$. Since the right inequality in (1.3) holds also for $\alpha > 1$, we complete the proof of the second assertion, in view of Lemma [2.7], with Rademacher multipliers.

(iii): Assume that $k^0 < \infty$, i.e., $E$ is of Krivine’s type $\leq q < \infty$. First, let $k^0 < \alpha$, and choose $q$ such that $1 < q < \alpha$. By the Figiel-Johnson renorming theorem [FJ74], there exists an equivalent norm such that the type $\leq q$-constant is equal 1.

By the hypercontractivity of the $S\alpha S$ (or $S\alpha P$) law, we may use any $s$-norm, for $h_\alpha < s < p$, with a constant $c = c_{\alpha, q, s}(Y)$, cf. [Szu90]. It is important that $h_\alpha < 1$. Choose $s = q$.

We will check the following inequality in the real case

$$(E|x + X \sum_i x_i\xi_i|^q)^{1/q} \leq (E|x + b \sum_i x_iX_i|^q)^{1/q},$$

(4.4)

where $b = b_{\alpha, q}$. Indeed, assuming that $x = 1$, we obtain the following upper bounds of the left hand side, by virtue of the Fubini’s theorem and the hypercontractivity of Rademacher random variables ($h = h_{q, \alpha} \cdot h_{\alpha, 2} = \sqrt{(q - 1)/(\alpha - 1)}$),

$$(E(1 + |\sum_i x_i\xi_i|^q)^{\alpha/q})^{1/q} \leq (E(1 + |\sum_i x_i\xi_i|^\alpha)^{q/\alpha})^{1/q} \leq (1 + h(\sum_i |x_i|^2)^{\alpha/2})^{1/\alpha}.$$
The right hand side is estimated from below as follows:

\[(E|1 + b \sum_i x_iX_i|^q)^{1/q} \geq (1 + ba_{\alpha,q}(\sum_i |x_i|^\alpha)^{1/\alpha})\]

in view of Lemma [1.3]. These estimates prove (4.4), with \(b = h/a\).

Whence, and also by the Fubini’s theorem and hypercontractivity of \(S\alpha S\) (or \(S\alpha P\)) law, we have

\[
\|x + Y \sum x_i \varepsilon_i\|_q \leq \|(E|x + X \sum x_i \varepsilon_i|^q)^{1/q}\|
\]

\[
\leq \|(E|x + b \sum_i x_iX_i|^q)^{1/q}\| \leq \|E|x + bc_{q,1} \sum_i x_iX_i|| \leq \|x + bc_{q,1} \sum_i x_iX_i||
\]

\[
\leq \|x + bc_{q,1} \sum_i x_iX_i\|_q
\]

By applying Lemma 3.7, we complete the proof of assertion (ii) in the case \(k^0 < \alpha\).

Let now \(\alpha < k_0\) (\(k_0 \leq k^0\)). Choose \(q \in (\alpha, k_0)\). This case follows immediately from assertion (ii), since in presence of finite cotype, there is an equivalent \(q\)-smooth norm (cf. [FJ74] or [LT79, Theorem 1.f.1]).

**Remark 4** Consider \(S\alpha S\) (or \(S\alpha P\)) symmetric chaoses. That the Krivine’s classification does not fully describe the fulfillment of the upper inequality follows from the following observation. Consider the case \(k_0 \leq \alpha \leq k^0\).

Note that an upper decoupling inequality for \(S\alpha S\) (or \(S\alpha P\)) chaoses, held in Banach spaces \((E_1, \| \cdot \|_1)\) and \((E_2, \| \cdot \|_2)\), with constants \((c_{1,k})\) and \((c_{2,k})\), respectively, holds also in \(E_1 \oplus_s E_2\), \(1 \leq s\alpha\), endowed with the norm \(\| \cdot \| = (\| \cdot \|^s_1 + \| \cdot \|^s_2)^{1/s}\), with constants \(c_k = \max(c_{1,k}, c_{2,k})\). Thus, since by assertion (ii) an upper decoupling inequality holds in every \(L^p\), \(p \neq \alpha\), it will be fulfilled in every \(L^q \oplus_s L^r\), \(q < \alpha < r\).

**Proposition 4.4** Let \(E\) be a Banach lattice of finite cotype such that \(k_0 > \alpha\). Then there exists an equivalent renorming such that all lower and upper, symmetric and tetrahedral, decoupling constants for stable chaoses are equivalent to the corresponding constants in the real line, i.e.,

\(c_k(E) = a_{\alpha,s,c_k}(R)\).

**Proof.** By a result from [FJ74], one can choose an equivalent norm of type \(\geq q\) with the constant equal to 1, \(q < \alpha\). We will use the hypercontractivity of stable (or Pareto) (one may use any \(s\)-norm, for \(h_\alpha < s < \alpha\) (where \(h_\alpha < 1\), with a constant \(a_{\alpha,s}\) [Szu90]). Now, denoting by \(Q\) and \(Q'\) two type of chaoses under interest, and combining the estimates

\[
\|\sum_k Q_k\|_s \leq \|(E|\sum_k Q_k|^s)^{1/s}\| \leq \|(E|\sum_k (c_k(R))^k Q'_k|^s)^{1/s}\|
\]

and

\[
\|(E|\sum_k Q'_k|^s)^{1/s}\| \leq \|(E|\sum_k (a_{\alpha,s})^k Q'_k|^s)^{1/s}\|
\]

\[
\leq E\|\sum_k (a_{\alpha,s})^k Q'_k\| \leq \|\sum_k (a_{\alpha,s}a_{\alpha,s})^k Q'_k\|,
\]

we complete the proof. ■
5 CONCLUDING REMARKS

In this section we display some further features of infinite order decoupling and domination. Some properties or generalizations can be obtained by well known routines, while other properties, enjoyed by homogeneous chaoses, yield to the dead end. Yet a number of open problems arise that have no counterparts for homogeneous chaoses. At this time, the infinite order approach to random chaoses is still in a preliminary stage.

5.1 Multiple stochastic integrals

Decoupling inequalities for infinite order Gaussian or stable polynomials can be carried over to infinite order multiple stochastic integrals, preserving all constants, the dependence on geometry, and subjection to the presence or lack of symmetry of underlying functions. These results follow by a routine approximation (integrals of simple functions are random chaoses).

The real case does not require any comments, since the theory is classical. In the vector case, one needs an appropriate construction of $k$-tuple stochastic integrals of deterministic functions with respect to a Gaussian (or more generally, a second order symmetric) process. One may apply the Dunford-Bartle approach, which reduces the integration in Banach space to that with respect to an $L^2$-valued vector measure (cf., e.g. [DU77]).

5.2 Non-multiplicative functions

In [Szu92, Theorem 4.1] (and before, in [MT87, lIPn90]), a nonmultiplicative version of the decoupling principle for homogeneous chaoses was proved. In such a version, a term $f(i_1, \ldots, i_k) \cdot X_{i_1} \cdots X_{i_k}$ was replaced by a term $F(i, X_{i_1} \cdots X_{i_k})$. Let us consider a nonhomogeneous analog of such a decoupling principle (as in [Szu92, 4.1]). Let $L$ be an Orlicz space induced by a strongly convex function $\varphi$ (3.9). For the sake of simplicity of formulations, assume that $\varphi$ grows moderately. Let $F = (F_{\alpha})$ be a function whose components are functions $F_{\alpha} : N^\alpha \times R^\alpha \to E$ satisfying conditions [Szu92]

\begin{align*}
\text{(F1)} & \quad F(i, \cdot) = 0 \quad \mu^k\text{-a.s. for all but finitely many } i; \\
\text{(F2)} & \quad F(i; X_{i_1}, \ldots, X_{i_k}) \in L^\varphi(E) \text{ for every } i \in N^k. 
\end{align*}

(5.1)

Put

$$F(X^\otimes) = \sum_\alpha F_{\alpha}(X^\otimes\alpha).$$

If $w = (w_{\alpha})$ is a Walsh sequence, write $Fw = (F_{\alpha}w_{\alpha})$ (i.e. $[F_{\alpha}w_{\alpha}](\alpha i) = F_{\alpha}(\alpha i)w_{\alpha}$). Then the analog of Theorem 3.3 holds, where $F_{\alpha}$ vanish, unless $\alpha = [1, k]$.

**Theorem 5.1** Let $L$ be an Orlicz space induced by a strongly convex function $\varphi$ (3.9), $F = (F_k)$ satisfy (F1)-(F2), $\|F_k(X^\otimes k)\| \in L^\varphi$, $k \geq 0$, and $[X, X]$ be as in Theorem 3.3. Then

$$E\varphi(\| \sum_{k \geq 0} F_k(X^\otimes k) \|) \leq E\varphi(\| \sum_{k \geq 0} \frac{(2ck)^k}{k!} F_k(X^\otimes k) \|),$$

where $c$ depends on the convexity of $\varphi$. 
The upper decoupling inequality for functions $F$ shares all deficiencies of the corresponding decoupling inequality for homogeneous chaoses. But there arise significant difficulties that cannot be removed by using techniques based on hypercontractivity, since the latter method works efficiently only for symmetric random variables. In the proof of [Szu92, Theorem 4.1], nonsymmetric random variables were used, which does not allow one to proceed as in the proof of Theorem 3.13. A very limited, almost trivial, real line- version of the upper decoupling inequality can be seen as follows.

\[ E|\sum_k F_k((\varepsilon X)\otimes k)|^2 = E|\sum_k c_k F_k((S X)\otimes k)|^2, \]

where $F_k(X\otimes k) \in L^2$, and $c_k = 1$ for tetrahedral functions, and $c_k = k!$ for symmetric functions. Any non-trivial extension (beyond Hilbert space and $L^2$-norm) would require some intrinsic symmetry of functions $F_k$. Therefore, at this stage it is meaningless to look at these kinds of decoupling inequalities from the view point of integration with respect to empirical measures (as in [Szu92]), even though other types of domination might be still of interest.

### 5.3 Cesàro averages

There exists a variety of operators acting on the entire matrix $X$. For example, one may use the operator $D$, which nullifies diagonal values of functions $f_\alpha$. For the sake of consistency, denote the basic symmetrizator by $S$, $S(f) = \hat{f}$. Many an operator do not have meaning for a single homogeneous polynomial. We will consider a certain multilinear analog (one of many) of Cesàro averages. Let us confine ourselves to subsets $\alpha \subset [1, n]$, and functions $f = (f_\alpha : \alpha \subset [1, n])$. We introduce the “index average” operator $A = (A_\alpha)$, which unifies values of functions $f_\alpha$ along sets $\alpha$ with the same cardinality.

First, we define the symmetrizator $A' = (A'_k)$, which transforms $f$ into a function $g = (g_k : k = 0, 1, \ldots, n)$, where $g_k : N^k = N^{[0,k]} \to E$.

Let $|\alpha| = k$. Denote by $s_\alpha$ the “stretching map” which embeds $N^k = N^{[0,k]}$ into $N^\alpha$ by moving the elements of a sequence $i_k = (i_1, \ldots, i_k) = (i_1, \ldots, i_k, 0, \ldots)$ into the places marked by the consecutive ones of the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$, and filling up the remaining places by zeros. Put, for $i_k = (i_1, \ldots, i_k)$,

\[ A'_k(f)(i_k) = \frac{1}{(n)} \sum_{|\alpha| = k} f_\alpha(s_\alpha i_k). \]

Clearly,

\[ \langle f \rangle = \sum_{k=0}^n \binom{n}{k} \langle A'_k(f) \rangle. \]

Denote by $c_\alpha$ the “contracting” mapping from $N^\alpha$ onto $N^k$, which just cancels all elements marked by zeros of the sequence $\alpha$. Now, we define the “inverse” mapping $A'' = (A''_\alpha)$ transforming functions $g = (g_k)$ into functions $f = (f_\alpha)$, according to the formula

\[ A''_\alpha(g)(\alpha i) = g_k(c_\alpha(\alpha i)), \quad |\alpha| = k. \]

Define $A = A'' A'$. The operators $D$, $S$, and $A$ are idempotent and commute with each other.
Let $E_1, E_2$ be additive abelian groups. Denote by $x_1x_2$ a bi-additive mapping from $E_1 \times E_2$ into $E$. Use the same notation $f_1 f_2$ for functions taking values in $E_1$ and $E_2$, respectively. If $\, U$ and $V$ are compositions of selected symmetrizators $D, A, S$, then the following symmetrization formulas hold:

$$
\left\langle U(f_1) V(f_2) \right\rangle = \left\langle f_1 U V (f_2) \right\rangle = \left\langle U V(f_1) f_2 \right\rangle = \left\langle V f_1 U f_2 \right\rangle. 
$$

(5.2)

Note that $X^\otimes$ is $A$-symmetric. By $A_k$ denote the $\sigma$-field generated by the family of random variables

$$
\left\{ h(X^\alpha : |\alpha| = k) : h = \hat{h}, h : (R^k)^n \rightarrow R \right\}.
$$

Notice that the symmetry assumption is applied to $h$ as to a function of $\binom{n}{k}$ vector variables, and $A_k$ is ascending $\sigma$-fields. The symmetrizer $A$ can be expressed as a conditional expectation. Note the following equalities:

$$
\left\langle f X A^{(\otimes)} \right\rangle = \left\langle f E[X^\otimes | A(X)] \right\rangle = E[\left\langle f X^\otimes \right\rangle | A(X)],
$$

(5.3)

$$
E \left[ D(X_1 + \ldots + X_k)^{\otimes k} | A(X) \right] = D A_{\alpha} \left( (\alpha X)^{\otimes \alpha} \right),
$$

(5.4)

or equivalently,

$$
E \left[ D(\frac{X_1 + \ldots + X_k}{k})^{\otimes k} | A(X) \right] = D A_{\alpha} \left( \alpha X \right)^{\otimes \alpha}.
$$

(5.5)

Now, Theorem 3.3 holds for $A$-convex functions. That is,

$$
E \varphi(\| \sum_{\alpha \subseteq [1, n]} w_{\alpha} \langle f_{\alpha} X^{\otimes \alpha} \rangle \|) \leq E \varphi(\| \sum_{\alpha \subseteq [1, n]} w_{\alpha} h_{\alpha} \langle f_{\alpha} X^{\otimes \alpha} \rangle \|),
$$

(5.6)

for every $D-, A, & S$-symmetric function $f$, where, for $|\alpha| = k$, $h_{\alpha} = h_k = (2ck)^k/k!$, and $c = c_{\varphi}$.

However, the $A$-symmetry is too strong for the upper inequality of arbitrary order to be fulfilled. For integrable symmetric random variables, by examining just polynomials of the first degree, we would obtain the inequality

$$
E \varphi(\| x + \sum_{i \leq K} X_i x_i \|) \leq E \varphi(\| x + c_1 \sum_{i \leq K} \frac{\sum_{j=1}^n X_{ji} x_i}{n} \|)
$$

which is impossible, as can be seen by applying the strong law of large numbers and Fatou’s lemma. Yet, the above observations open a new, even in the real case, direction in a study of $A$-symmetric chaoses. Clearly, any domination “constant” is expected to depend on $n$, which makes the concept of infinite order much more difficult.

**Problem.** Describe the closure in $L^2$ and limit distributions of real Gaussian (or Rademacher) $A$-symmetric decoupled chaoses

$$
\left\{ \sum_{\alpha \subseteq [1, n]} \left\langle f_{\alpha} X^{A(\alpha)} \right\rangle : n \in N \right\}.
$$

Note that the metric ($L^2$-) problem is easy for $S$-symmetric or tetrahedral functions (cf. the first subsection of this section). For $S$-symmetric functions, a related limit theorem for coupled Gaussian random chaoses, obtained in [DM83], brought up infinite order Wiener integrals.
INFINITE ORDER DECOUPLING

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