Schrödinger Operators on Regular Metric Trees with Long Range Potentials: Weak Coupling Behavior

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Abstract

Consider a regular $d$-dimensional metric tree $\Gamma$ with root $o$. Define the Schrödinger operator $-\Delta - V$, where $V$ is a non-negative, symmetric potential, on $\Gamma$, with Neumann boundary conditions at $o$. Provided that $V$ decays like $x^{-\gamma}$ at infinity, where $1 < \gamma \leq d \leq 2$, $\gamma \neq 2$, we will determine the weak coupling behavior of the bottom of the spectrum of $-\Delta - V$. In other words, we will describe the asymptotical behavior of $\inf \sigma(-\Delta - \alpha V)$ as $\alpha \to 0^+$. 

Keywords: Schrödinger operators, metric trees, Fourier-Bessel transformation, weak coupling.

1 Introduction

It is a well known fact that the weak coupling asymptotics for a Schrödinger operator $-\Delta - \alpha V$ in $\mathbb{R}^n$ depends on the dimension $n$ of the underlying space, \cite{14, 3}. In case $n = 1$ it has been shown in \cite{14, 3} that if an integrable potential $V$ decays at infinity faster than $x^{-2}$, then for $\alpha$ small enough the operator $-\Delta - \alpha V$ has a unique eigenvalue $E_1(\alpha)$ which satisfies the asymptotic equation

$$E_1(\alpha) \sim \left( \alpha \int_{\mathbb{R}} V \right)^2, \quad \alpha \to 0.$$ (1)
Note that as long as $V$ satisfies the above criteria then the behavior of $E_1(\alpha)$ is uniform in order of $\alpha$, i.e. proportional to $\alpha^2$, and the potential $V$ enters only in the coefficient.

If $V$ decays more slowly than $x^{-2}$ at infinity, the picture is different, since the operator $-\Delta - \alpha V$ now has infinitely many eigenvalues for any $\alpha > 0$. However, Klaus has shown in [7] that the ground state $E_1(\alpha)$ has different asymptotics than the rest of the discrete spectrum but is still proportional to $\alpha^2$, provided $V$ decays faster than $x^{-1}$. Finally, when $V(x) \sim x^{-\gamma}$ with $\gamma \leq 1$, then the quadratic dependence on $\alpha$ fails to hold and the corresponding power of $\alpha$ in the asymptotics of $E_1(\alpha)$ is fully determined by the parameter $\gamma$, see [7] for details.

In the present paper the goal is to study these questions in the case of Schrödinger operators defined on metric trees. Such trees represent a particular case of the so-called quantum graphs, that provide mathematical models for nanotechnological devices consisting of connected thin strips. Spectral analysis of Laplace and Schrödinger operators on these structures has therefore attracted a lot of attention, see e.g. [4, 6, 10, 8, 11, 15, 16].

A metric tree $\Gamma$ consists of a set of vertices and a set of edges (branches), the edges being one dimensional intervals connecting the vertices; see Section 3.1 for details. In [9] it was proved that the behavior of $E_1(\alpha)$ then depends on the global structure of the tree. More precisely, if $V$ decays fast enough, then the asymptotic behavior of $E_1(\alpha)$ is again uniform and given by

$$E_1(\alpha) \sim \left( \alpha \int_{\Gamma} V \right)^{-\frac{2}{d-2}}, \quad \alpha \to 0, \quad 1 \leq d < 2. \quad (2)$$

where $d$ is the so-called dimension of the tree. Roughly speaking, the value of $d$ tells us how fast the number of branches of the tree increases as a function of the distance from its root, see Section 3.1 for a precise definition.

In this work we will study the interplay between the global structure of the tree, i.e. its dimension $d$, and the decay of $V$. In particular, it will be shown that if $V$ decays as $x^{-\gamma}$ with $1 < \gamma \leq d \leq 2$, $\gamma \neq 2$, then the corresponding asymptotics of $E_1(\alpha)$ is again fully determined by the behavior of $V$ at infinity, that is, by the value of $\gamma$, see Theorem 2.1. It is easily seen that such potentials are not integrable on $\Gamma$, which means that the method of [14, 3, 9] cannot be applied and a different approach is needed. In order to prove the main result we proceed in several steps, which we now briefly outline.

In Section 4 we show that $E_1(\alpha)$ is bounded from above and from below by the infimum of the spectra of certain auxiliary operators $H^{-}_\alpha$ and $H^{+}_\alpha$ which act in a weighted $L^2$ space on the half-line $(0, \infty)$; see Lemma 4.1. This enables us to reduce the problem of the asymptotical behavior of the ground state of $-\Delta - \alpha V$ on $\Gamma$ to the problem of finding the asymptotical behavior of the ground state of certain Schrödinger operators in weighted $L^2$ spaces on the half-line with a weight which depends parametrically on $d$. These operators are analysed in Section 5; see Theorem 5.2.

The main technical tools used in the paper are the Birman-Schwinger principle and the Fourier-Bessel transformation. In fact, the derivation of the latter transform may be considered interesting on its own merits. It provides an
explicit formula for the unitary operator which transforms the one-dimensional Laplace operator on $L^2((0, \infty), (1 + t)^{\alpha}dt)$, $\alpha \geq 0$ to a multiplication operator on $L^2(0, \infty)$. We therefore believe that it might be of a general interest also for other applications. For the convenience of the reader, this material together with the Weyl-Titchmarsh-Kodaira Theorem is described in the Appendices.

2 Main Result

For functions $f, g : (0, \infty) \to \mathbb{R}$, we will use the notation $f(x) \approx g(x)$ to mean that there are constants $C_1, C_2 > 0$ such that $C_1g(x) \leq f(x) \leq C_2g(x)$ for any $x > 0$, and the notation $f(x) \asymp g(x), x \to 0+$ to mean that there are constants $D_1, D_2 > 0$ such that $D_1g(x) < f(x) < D_2g(x)$ for every sufficiently small $x > 0$.

Having introduced the necessary notation we are in a position to state the main result of the paper:

**Theorem 2.1.** Let $1 < \gamma \leq d \leq 2$, $\gamma \neq 2$. Suppose that $\Gamma$ is a regular metric tree of dimension $d$. Define the Neumann Laplacian $-\Delta$ in $L^2(\Gamma)$. Suppose that $V$ is non-negative, measurable, and such that $V(t) \approx (1 + t)^{\gamma}$.

Let $\tilde{V} : \Gamma \to \mathbb{C}$ be defined by $\tilde{V}(x) = V(|x|)$ for $x \in \Gamma$. Then we have:

(i) If $1 < \gamma < d \leq 2$, then

$$\inf \sigma \left( -\Delta - \alpha \tilde{V} \right) \asymp -\alpha^{\frac{1}{d-\gamma}}, \quad \alpha \to 0^+.$$  

(ii) If $1 < \gamma = d < 2$, then

$$\inf \sigma \left( -\Delta - \alpha \tilde{V} \right) \asymp -|\alpha \log \alpha|^{\frac{1}{d-\gamma}}, \quad \alpha \to 0^+.$$  

**Remark 2.2.** Note that in contrast to the case treated in [9], the asymptotics of $\inf \sigma \left( -\Delta - \alpha \tilde{V} \right)$, for $\gamma < d$, are independent of $d$ and fully determined by the parameter $\gamma$. In other words, they are determined by the behavior of $V$ at infinity. This is analogous to the regime $\gamma < 1$ for one-dimensional Schrödinger operators, [7].

Moreover, in the border-line case $\gamma = d$ there is a logarithmic correction to the power-like law. This is again reminiscent of the behavior of one-dimensional Schrödinger operators in the case where $V(x) \sim 1/(1 + |x|)$, corresponding to $\gamma = 1$; see [7].

As mentioned in the introduction, the proof of Theorem 2.1 will proceed in several steps. The main result below is Theorem 5.2 that together with Lemma 4.1 proves the main theorem. We start with the preliminary section introducing Schrödinger operators on regular metric trees.
3 Spectral Theory on Regular Metric Trees

The geometry of regular metric trees and the definition of the Laplacian on those trees are discussed thoroughly in [16, 11]. We give here a brief summary that serves our purposes.

3.1 Regular Metric Trees

Let $\Gamma \subset \mathbb{R}^2$ be a rooted metric tree with root $o$. Let $\preceq$ be the natural partial ordering on $\Gamma$ defined by letting $x \preceq y$ mean that either $x = y$ or that $x$ is on the path from $o$ to $y$. We use the notation $x < y$ to mean that $x \preceq y$ and $x \neq y$.

For a point $x \in \Gamma$, let us denote by $|x|$ the length of the path in $\Gamma$ from $o$ to $x$.

Let $V = V(\Gamma)$ be the set of vertices in $\Gamma$ and $E = E(\Gamma)$ the set of edges in $\Gamma$. Clearly $V \subset \Gamma$ and $e \subset \Gamma$ for each $e \in E$.

For a point $x \in \Gamma$, let $b(x)$ be its branching number defined by

$$b(x) = \lim_{\epsilon \to 0^+} \# \{ y \in \Gamma ; y \succeq x, |y| = |x| + \epsilon \}.$$

To clarify, if $x$ is a vertex of $\Gamma$, then $b(x)$ is the number of edges emanating from $x$, and if $x$ is not a vertex, then $b(x) = 1$. Naturally, we will assume throughout that $\Gamma$ is such that $b(x) > 1$ for any vertex $x \neq o$, and that $b(o) = 1$. In other words, there are no vertices, except $o$, that have only one emanating edge.

**Definition 3.1.** The tree $\Gamma$ is said to be **regular** if $b(x) = b(y)$ for all points $x, y \in \Gamma$ satisfying $|x| = |y|$.

Now we define the branching function $g_\Gamma$ by

$$g_\Gamma(t) = \# \{ x \in \Gamma ; |x| = t \}, \quad t \geq 0.$$

**Definition 3.2.** The tree $\Gamma$ is said to have **dimension** $d$ if there are constants $C_1, C_2 > 0$ such that

$$C_1 (1 + t)^{d-1} \leq g_\Gamma(t) \leq C_2 (1 + t)^{d-1}, \quad t \geq 0.$$

Often one studies the **height** $h(\Gamma)$ and the **reduced height** $L(\Gamma)$ of $\Gamma$ defined respectively by

$$h(\Gamma) = \sup_{x \in \Gamma} |x| \quad \text{and} \quad L(\Gamma) = \int_0^{h(\Gamma)} \frac{dt}{g_\Gamma(t)}.$$

We will be interested in regular $d$-dimensional trees of infinite reduced height, for which $g_\Gamma$ is growing. Given the above definitions, this means that $\Gamma$ is of infinite height and that $1 < d \leq 2$. 


3.2 The Laplace and Schrödinger operators on $\Gamma$

Here, $\Gamma$ will be a fixed, regular, $d$-dimensional metric tree, where $1 < d \leq 2$. Define the Hilbert space $L^2(\Gamma)$ as the space of functions $f : \Gamma \to \mathbb{C}$ satisfying that $f \upharpoonright e \in L^2(e)$ for any $e \in \mathcal{E}(\Gamma)$ and that

$$\|f\|^2 := \sum_{e \in \mathcal{E}(\Gamma)} \int_e |f(x)|^2 \, dx < \infty.$$  

Similarly, the Sobolev space $H^1(\Gamma)$ is defined as containing the continuous functions $f : \Gamma \to \mathbb{C}$ such that $f \upharpoonright e \in H^1(e)$ for any $e \in \mathcal{E}(\Gamma)$ and such that

$$\|f\|_{H^1}^2 := \sum_{e \in \mathcal{E}(\Gamma)} \int_e \left( |f'(x)|^2 + |f(x)|^2 \right) \, dx < \infty.$$  

The Neumann Laplacian $-\Delta_{\Gamma}$ is the (unique) self-adjoint operator on $L^2(\Gamma)$ associated with the closed quadratic form

$$h_{\Gamma}[f] = \sum_{e \in \mathcal{E}(\Gamma)} \int_e |f'(x)|^2 \, dx$$  

with domain $D[h_{\Gamma}] = H^1(\Gamma)$.

4 Reduction to an Operator on the Half-Line

As before, $\Gamma$ is a fixed, regular, $d$-dimensional metric tree, where $1 < d \leq 2$. We will be interested in the Schrödinger operator $-\Delta_{\Gamma} - \alpha \tilde{V}$ for $\alpha > 0$. It turns out that bottom of the spectrum of $-\Delta_{\Gamma} - \alpha \tilde{V}$ is described by studying a certain Schrödinger operator on the half-line. Recall that by definition, there are constants $C_1, C_2 > 0$ such that

$$C_1(1 + t)^{d-1} \leq g_{\Gamma}(t) \leq C_2(1 + t)^{d-1}.$$  

Let

$$E^+ = \frac{C_1}{C_2} \quad \text{and} \quad E^- = \frac{C_2}{C_1}.$$  

Lemma 4.1. Let $V$ be a bounded, non-negative and measurable function on $(0, \infty)$. Define a function $\tilde{V}$ on $\Gamma$ by

$$\tilde{V}(x) = V(|x|), \quad x \in \Gamma,$$

and the Hilbert space $\mathcal{H} = L^2((0, \infty), (1 + t)^{d-1} \, dt)$. For $\alpha > 0$, let $H^\pm_{\alpha}$ be the self-adjoint operator in $\mathcal{H}$ associated with the closed quadratic form

$$h^\pm_{\alpha}[u] = \int_0^\infty \left( |u'(t)|^2 - \alpha E^\pm V(t) |u(t)|^2 \right) (1 + t)^{d-1} \, dt,$$

with domain $D[h^\pm_{\alpha}] = H^1((0, \infty), (1 + t)^{d-1} \, dt)$. Then, for any $\alpha > 0$ for which $\inf \sigma (H^-_{\alpha}) < 0$,

$$\inf \sigma (H^-_{\alpha}) \leq \inf \sigma \left( -\Delta_{\Gamma} - \alpha \tilde{V} \right) \leq \inf \sigma (H^+_{\alpha}).$$
Remark 4.2. If, as in Theorem 2.1, \( V(t) \approx 1/(1 + t)^\gamma \), where \( 1 < \gamma \leq d \leq 2 \), \( \gamma \neq 2 \), then Lemma 5.4 below shows that \( \inf \sigma (H_\alpha^+) < 0 \) as \( \alpha \to 0^+ \). Therefore the lemma is applicable.

Proof. For \( \alpha > 0 \), define the quadratic form \( a_\alpha \) in \( L^2((0, \infty), g_T(t)\, dt) \) by

\[
a_\alpha[u] = \int_0^\infty \left( |u'(t)|^2 - \alpha V(t)|u(t)|^2 \right) g_T(t) \, dt,
\]

with domain \( D[a_\alpha] = H^1((0, \infty), g_T) \). Let \( A_\alpha \) be the self-adjoint operator associated with \( a_\alpha \). By using the standard orthogonal decomposition of the operator \( -\Delta_T - \alpha \tilde{V} \), as described in [11, 12] and the arguments of [9, Sec. 5.3], it is seen that

\[
\inf \sigma \left( -\Delta_T - \alpha \tilde{V} \right) = \inf \sigma (A_\alpha) .
\]

(3)
The following is a variant of the techniques employed in [9]. Assume that \( \alpha > 0 \) is such that \( \inf \sigma (H_\alpha^+) < 0 \). Choose an arbitrary \( \epsilon \) with \( 0 < \epsilon < |\inf \sigma (H_\alpha^+)| \).

Let \( u \in D[\hat{h}_\alpha^+] \) be such that

\[
\hat{h}_\alpha^+[u] = \int_0^\infty \frac{|u(t)|^2(1 + t)^{\alpha - 1} dt}{|u(t)|^2(1 + t)^{\alpha - 1} dt} \leq \inf \sigma (H_\alpha^+) + \epsilon.
\]

Now, \( u \in D[a_\alpha] \) and

\[
a_\alpha[u] \leq C_2 \hat{h}_\alpha^+[u].
\]

In particular,

\[
\hat{h}_\alpha^+[u] < 0 \quad \text{and} \quad a_\alpha[u] < 0.
\]

We also have that

\[
\int_0^\infty |u(t)|^2 g_T(t) \, dt \leq C_2 \int_0^\infty |u(t)|^2(1 + t)^{d-1} dt,
\]

and therefore,

\[
\inf \sigma (A_\alpha) \leq \frac{a_\alpha[u]}{\int_0^\infty |u(t)|^2 g_T(t) \, dt} \leq \frac{\hat{h}_\alpha[u]}{\int_0^\infty |u(t)|^2(1 + t)^{d-1} dt} \leq \inf \sigma (H_\alpha^+) + \epsilon.
\]

Since \( \epsilon \) was arbitrary, it follows that

\[
\inf \sigma (A_\alpha) \leq \inf \sigma (H_\alpha^+).
\]

Similarly, it is shown that

\[
\inf \sigma (H_\alpha^-) \leq \inf \sigma (A_\alpha).
\]

5 Estimates of the Bottom of the Spectrum

Consider the space \( \mathcal{H} = L^2((0, \infty); (1 + x)^{d-1} \, dx) \) and a measurable, bounded potential \( V \). For \( \alpha > 0 \), let \( H_\alpha \) be the operator in \( \mathcal{H} \) determined by the closed quadratic form

\[
h_\alpha[u] = \int_0^\infty \left( |u'(x)|^2 - \alpha V(x)|u(x)|^2 \right) (1 + x)^{d-1} \, dx
\]
with domain 
\[ D[h_\alpha] = \{ u \in \mathcal{H} : u' \in \mathcal{H} \}. \]

Here, \( d \in (1, 2] \) is the dimension of the underlying tree.

**Remark 5.1.** It can be seen by the standard arguments, see e.g. [9], that under the above conditions on the potential \( V \), the essential spectrum of \( H_\alpha \) is \([0, \infty)\). Hence the negative spectrum of \( H_\alpha \) can only contain eigenvalues of finite multiplicity. Furthermore,

\[
\inf \sigma(H_\alpha) \to 0, \quad \alpha \to 0^+. 
\]

**Theorem 5.2.** Suppose that \( V \) is measurable and that there are positive constants \( C_1 \) and \( C_2 \) such that for any \( x > 0 \),

\[
\frac{C_1}{(1 + x)^\gamma} \leq V(x) \leq \frac{C_2}{(1 + x)^\gamma}. 
\]

(i) If \( 1 < \gamma < d \leq 2 \), then there are constants \( D_1, D_2 > 0 \) such that

\[ -D_1 \alpha^{2 - \gamma} < \inf \sigma(H_\alpha) < -D_2 \alpha^{2 - \gamma}, \quad \alpha \to 0^+. \]

(ii) If \( 1 < \gamma = d < 2 \), then there are constants \( D_1, D_2 > 0 \) such that

\[ -D_1 |\alpha \log \alpha|^{2 - \gamma} < \inf \sigma(H_\alpha) < -D_2 |\alpha \log \alpha|^{2 - \gamma}, \quad \alpha \to 0^+. \]

The proof of Theorem 5.2 relies on the following two lemmas, of which the first is proved in Appendix B.

**Lemma 5.3.** Suppose that \( 1 < \gamma \leq d \leq 2 \), \( \gamma \neq 2 \), and that there is a \( C > 0 \) such that \( V \) satisfies

\[ 0 \leq V(x) \leq \frac{C}{(1 + x)^\gamma}. \]

Then there is a constant \( D > 0 \) such that for any \( E > 0 \), there is a non-negative trace-class operator \( Q_E \) on \( L^2(0, \infty) \) whose trace satisfies

\[
\text{Tr} Q_E \leq \begin{cases} 
DE^{\frac{2 - \gamma}{2}}, & \gamma < d, \\
DE^{\frac{2 - \gamma}{2}} (1 + |\log E|), & \gamma = d, 
\end{cases}
\]

and such that \( \alpha^{-1} \) is an eigenvalue of \( Q_E \) if and only if \(-E\) is an eigenvalue of the operator \( H_\alpha \).

**Lemma 5.4.** Suppose that \( V \) is bounded and that there is a \( C > 0 \) such that

\[ V(x) \geq \frac{C}{(1 + x)^\gamma}. \]

(i) If \( 1 < \gamma < d \leq 2 \), then there is a \( D > 0 \) such that

\[ \inf \sigma(H_\alpha) < -D \alpha^{2 - \gamma}, \quad \alpha \to 0^+. \]
(ii) If \( 1 < \gamma = d < 2 \), then there is a \( D > 0 \) such that
\[
\inf \sigma (H_\alpha) < -D |\alpha \log \alpha|^{\frac{2}{d-\gamma}}, \quad \alpha \to 0 + .
\]

Proof.

(i) For the first case, assume that \( 1 < \gamma < d \leq 2 \). Choose \( \alpha > 0 \) and \( \delta \) with \( 0 < \delta < 1 \). Consider the function
\[
u_\delta(x) = e^{-\delta x}.
\]
Clearly \( u_\delta \in D[h_\alpha] \) and
\[
h_\alpha[u_\delta] \leq \int_0^\infty \left( |u_\delta'(x)|^2 (1 + x)^{d-1} - C\alpha |u_\delta(x)|^2 (1 + x)^{d-\gamma-1} \right) dx
\]
\[
= \delta^{-d} e^{2\delta} \int_0^\infty e^{-2x} x^{d-1} dx - \alpha \delta^{\gamma-d} e^{2\delta} C \int_0^\infty e^{-2x} x^{d-\gamma-1} dx \tag{4}
\]
\[
\leq \delta^{-d} e^{2\delta} \left( \delta^2 \int_0^\infty e^{-2x} (1 + x) dx - \alpha \delta^{\gamma} C \int_1^{\infty} \frac{e^{-x}}{x} dx \right) .
\]
Furthermore,
\[
\|u_\delta\|_\infty^2 = \delta^{-d} e^{2\delta} \int_0^\infty e^{-2x} x^{d-1} dx
\]
\[
\leq \delta^{-d} e^{2\delta} \int_0^\infty e^{-2x} x^{d-1} dx \tag{5}
\]
\[
= \delta^{-d} e^{2\delta} 2^{-d} \Gamma(d).
\]

Now choose a constant \( K > 0 \) such that
\[
\tilde{K}_\gamma := K\gamma C \int_1^{\infty} \frac{e^{-2x}}{x} dx - K^2 \int_0^{\infty} e^{-2x} (1 + x) dx > 0 .
\]
Assume that \( \alpha > 0 \) is small enough to ensure that
\[
\delta := K\alpha^{\frac{2}{\gamma}} < 1 .
\]
Then we have that
\[
\alpha \delta^{\gamma} = K\gamma \alpha^{\frac{2}{\gamma}}
\]
and that
\[
\delta^2 = K^2 \alpha^{\frac{2}{\gamma}} .
\]
Therefore, by (4),
\[
h_\alpha[u_\delta] \leq -\delta^{-d} e^{2\delta} \tilde{K}_\gamma \alpha^{\frac{2}{\gamma}} .
\]
In particular \( h_\alpha[u_\delta] < 0 \) and by combining (5) with (6) we obtain
\[
\inf \sigma (H_\alpha) \leq \frac{h_\alpha[u_\delta]}{\|u_\delta\|_\infty^2} \leq -\frac{2^d \tilde{K}_\gamma}{\Gamma(d)} \alpha^{\frac{2}{\gamma}}.
\]
To prove the second statement, let \( 1 < \gamma = d < 2 \). Let \( \beta > 2 \) be large enough to guarantee that

\[
M := \frac{C}{8(2 - d)} \log \frac{\beta}{2} - \frac{2^d}{d} \beta^{d-2} > 0
\]

Choose any \( \alpha \) with \( 0 < \alpha < 1 \) small enough to satisfy

\[
\nu := \frac{1}{|\alpha \log \alpha|^{\frac{1}{1 - d}}} \geq e.
\]

Note that \( x \log x \geq -1/2 \) for any \( x > 0 \). Hence

\[
y - \log y = (1 + y^{-1} \log y^{-1}) y \geq \frac{1}{2} y
\]

for any \( y > 0 \) and therefore, since \( 0 < \alpha < 1 \),

\[
\alpha \log \nu = \frac{1}{2 - d} \alpha (|\log \alpha| - |\log \log \alpha|) \geq \frac{1}{2(2 - d)} |\alpha \log \alpha|.
\]

Let

\[
\mu = \beta \nu.
\]

Note that \( \mu > 2e \) and define

\[
w(x) = \begin{cases} 
1 - \frac{x}{\mu}, & 0 < x < \mu, \\
0, & x \geq \mu.
\end{cases}
\]

Clearly \( w \in D[h_{\alpha}] \) and since \( \mu \geq 1 \),

\[
\int_0^\infty |w'(x)|^2 (1 + x)^{d-1} \, dx = \frac{1}{d \mu^2} \left( (1 + \mu)^d - 1 \right)
\]

\[
\leq \frac{1}{d \mu^2} (1 + \mu)^d
\]

\[
\leq \frac{2^d}{d} \mu^{d-2}.
\]

On the other hand, using the assumption on \( V \) we get a lower bound on the potential energy by

\[
\alpha \int_0^\infty \left| w(x) \right|^2 V(x)(1 + x)^{d-1} \, dx \geq C \alpha \int_0^\mu \frac{\left| 1 - \frac{x}{\mu} \right|^2}{1 + x} \, dx
\]

\[
\geq C \alpha \int_0^{\mu/2} \frac{\left| 1 - \frac{x}{\mu} \right|^2}{1 + x} \, dx
\]

\[
\geq C \alpha \int_0^{\mu/2} \frac{1}{1 + x} \, dx
\]

\[
= \frac{C \alpha}{4} \log \left( 1 + \frac{\mu}{2} \right)
\]

\[
\geq \frac{C \alpha}{4} \log \frac{\mu}{2}.
\]
Using (8), (9) and (7) we get that
\[ h_\alpha[w] \leq \frac{2^d}{d} \mu^{d-2} - \frac{C\alpha}{4} \log \frac{\mu}{2} \]
\[ = \frac{2^d}{d} \beta^{d-2} \nu^{d-2} - \frac{C}{4} \log \frac{\beta}{2} \cdot \alpha \log \nu \]
\[ \leq |\alpha \log \alpha| \left( \frac{2^d}{d} \beta^{d-2} - \frac{C}{4} \log \frac{\beta}{2} \cdot \frac{1}{2(2-d)} \right) \]
\[ = -M |\alpha \log \alpha|. \]

Finally, since \(|w| \leq 1\),
\[ \|w\|_H^2 \leq \int_0^\mu (1 + x)^{d-1} \, dx \leq \frac{2^d}{2} \mu^d \]
\[ = \frac{2^d}{d} \beta^d |\alpha \log \alpha|^{-\frac{d}{2-d}}. \]

It thus follows that
\[ \inf \sigma(H_\alpha) \leq \frac{h_\alpha[w]}{\|w\|_H^2} \leq - \frac{dM}{(2\beta)^d} |\alpha \log \alpha|^{\frac{2}{2-d}}. \]

**Proof of Theorem 5.2.** Start by assuming that \(1 < \gamma \leq d \leq 2\), \(\gamma \neq 2\). Let
\[ E(\alpha) = -\inf \sigma(H_\alpha) \]
for every \(\alpha > 0\). By Lemma 5.4, \(E(\alpha) > 0\). Since the negative spectrum of \(H_\alpha\) is discrete, \(-E(\alpha)\) is an eigenvalue of \(H_\alpha\). By Lemma 5.3 there is a \(D = D(\gamma, d) > 0\) such that for any \(\alpha > 0\), there is a non-negative trace-class operator \(Q_{E(\alpha)}\) whose trace is estimated by
\[ \text{Tr} Q_{E(\alpha)} \leq \begin{cases} D \cdot (E(\alpha))^{\frac{2-d}{2}}, & \gamma < d, \\ D \cdot (E(\alpha))^{\frac{2-d}{2}} (1 + |\log E(\alpha)|), & \gamma = d, \end{cases} \]
and such that \(\alpha^{-1}\) is an eigenvalue of \(Q_{E(\alpha)}\). Since \(Q_{E(\alpha)}\) is non-negative, we get that \(\alpha^{-1} \leq \text{Tr} Q_{E(\alpha)}\) and therefore
\[ \frac{1}{\alpha} \leq \begin{cases} D \cdot (E(\alpha))^{\frac{2-d}{2}}, & \gamma < d, \\ D \cdot (E(\alpha))^{\frac{2-d}{2}} (1 + |\log E(\alpha)|), & \gamma = d. \end{cases} \]

(i) Now consider the first case, where \(1 < \gamma < d \leq 2\). Choose \(\alpha > 0\). By (10), \(D\alpha \geq (E(\alpha))^{(2-\gamma)/2}\). From the fact that \(2-\gamma > 0\) it follows that
\[ E(\alpha) \leq (D\alpha)^{\frac{2}{2-d}} \]
for any \(\alpha > 0\). To complete the proof of the first statement it now remains to apply Lemma 5.4.
(ii) Let us turn to the second case, where $1 < \gamma = d < 2$. Introduce the function $W \in C^\infty(0, \infty)$, defined as the inverse of the function $y \mapsto ye^y, y > 0$. The function $W$ is sometimes called the Lambert W-function. Since $W$ is increasing, we get that if $y, z > 0$ are such that $z \leq ye^y$, then

$$y = W(ye^y) \geq W(z). \quad (11)$$

For simplicity, let $\alpha_0 > 0$ be such that $E(\alpha) < 1/2$ for any $\alpha$ with $0 < \alpha < \alpha_0$. Then, by (10) there is a constant $\tilde{D} = \tilde{D}(\gamma)$ such that

$$\frac{1}{\tilde{D} \alpha} \leq - (E(\alpha))^{\gamma^2/2} \log E(\alpha) \quad (12)$$

for any $\alpha$ with $0 < \alpha < \alpha_0$. Let $y(\alpha) = \log \left( (E(\alpha))^{\gamma^2/2} \right)$. Then by (12),

$$\frac{1}{\tilde{D} \alpha} \leq \frac{2}{2 - \gamma} y(\alpha)e^{y(\alpha)}$$

whenever $0 < \alpha < \alpha_0$. By (11),

$$y(\alpha) \geq W \left( \frac{2 - \gamma}{2 \tilde{D} \alpha} \right)$$

and therefore, since $W(x)e^{W(x)} = x$ for any $x > 0$,

$$E(\alpha) \leq \left( \exp \left( -W \left( \frac{2 - \gamma}{2 \tilde{D} \alpha} \right) \right) \right)^{\frac{2}{2 - \gamma}}$$

$$= \left( \frac{W \left( \frac{2 - \gamma}{2 \tilde{D} \alpha} \right)}{W \left( \frac{2 - \gamma}{2 \tilde{D} \alpha \exp \left( W \left( \frac{2 - \gamma}{2 \tilde{D} \alpha} \right) \right) \right)} \right)^{\frac{2}{2 - \gamma}}$$

$$= \left( \frac{2 \tilde{D} \alpha W \left( \frac{2 - \gamma}{2 \tilde{D} \alpha} \right)}{2 - \gamma} \right)^{\frac{2}{2 - \gamma}}$$

(13)

for such $\alpha$. Now, note that for any $x > 0$, $W(1/x) \exp(W(1/x)) = 1/x$. Therefore $\log(1/x) = \log W(1/x) + W(1/x)$ and since $W(1/x) \to \infty$ as $x \to 0+$ we get that

$$\log \frac{1}{x} \sim W \left( \frac{1}{x} \right), \quad x \to 0 + .$$

In particular, letting $x = 2\tilde{D}\alpha/(2 - \gamma)$, we get that

$$W \left( \frac{2 - \gamma}{2 \tilde{D} \alpha} \right) \leq K \log \frac{2 - \gamma}{2 \tilde{D} \alpha}, \quad \alpha \to 0 + ,$$

for any fixed $K > 1$. Together with (13) this gives

$$E(\alpha) \leq \left( \frac{2K \tilde{D} \alpha}{2 - \gamma} \log \left| \log \alpha \right| + \log \left( \frac{2 - \gamma}{2 \tilde{D}} \right) \right)^{\frac{2}{2 - \gamma}}, \quad \alpha \to 0 + .$$

Applying Lemma 5.4 completes the proof of the second case. \qed

11
A Appendix: The Fourier-Bessel Transform

A.1 Properties of the Bessel functions

Let $\nu$ be a real number. Denote by $J_\nu$ and $Y_\nu$ the Bessel functions of the first and second type, respectively. Also, let

$H^{(1)}_\nu = J_\nu + iY_\nu$ and $H^{(2)}_\nu = J_\nu - iY_\nu$

denote the Hankel functions. We will use the following properties of these functions, as listed in Chapter 9 of [1]:

**Proposition A.1.** Let $\nu \in \mathbb{R}$. The functions $J_\nu$, $Y_\nu$, $H^{(1)}_\nu$ and $H^{(2)}_\nu$ are analytic in $\mathbb{C} \setminus (-\infty, 0]$ and satisfy the following:

(i) The functions $J_\nu$ and $Y_\nu$ are linearly independent.

(ii) Let $C_\nu$ denote $J_\nu$, $Y_\nu$, $H^{(1)}_\nu$, $H^{(2)}_\nu$ or any linear combination of these functions. Then we have that

$$C'_\nu(z) = C_{\nu-1}(z) - \frac{\nu}{z} C_\nu(z) \quad \text{and} \quad C'_\nu(z) = -C_{\nu+1}(z) + \frac{\nu}{z} C_\nu(z).$$

(iii) If $\nu > 0$ and $z \to 0$, then

$$J_\nu(z) \text{ is bounded and } Y_\nu(z) \sim -\frac{1}{\pi} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu}.$$

(iv) If $\nu = 0$ and $z \to 0$, then

$$J_0(z) \text{ is bounded and } Y_0(z) \sim -\frac{2}{\pi} \log z.$$

(v) If $x$ is real and $x \to \infty$, then

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left(x - \frac{\nu \pi}{2} - \frac{\pi}{4}\right) \quad \text{and} \quad Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left(x - \frac{\nu \pi}{2} - \frac{\pi}{4}\right).$$

(vi) If $z \in \mathbb{C} \setminus (-\infty, 0]$ and $z \to \infty$, then

$$H^{(1)}_\nu(z) \sim \sqrt{\frac{2}{\pi z}} e^{iz\nu/2 - \pi/4} \quad \text{and} \quad H^{(2)}_\nu(z) \sim \sqrt{\frac{2}{\pi z}} e^{-iz(\nu/2 - \pi/4)},$$

(vii) We have that $\overline{J_\nu(z)} = J_\nu(\bar{z})$ and $\overline{Y_\nu(z)} = Y_\nu(\bar{z})$.

(viii) The following Wronskian formula holds:

$$J_{\nu+1}(z)Y_{\nu}(z) - J_{\nu}(z)Y_{\nu+1}(z) = 2/(\pi z).$$
A.2 The Weyl-Titchmarsh-Kodaira Theorem

Let \( \tau \) be a formally self-adjoint formal differential operator of order \( n \) on the interval \((0, \infty)\), and let \( T \) be a self-adjoint realization of \( \tau \) in \( L^2(0, \infty) \). We will assume that \( T \geq 0 \). For such operators it is sometimes possible to give an explicit description of the spectral measure, using the following technique from Chapter XIII of \([5]\):

**Proposition A.2.** Let \( U \) be a fixed open neighborhood of \((0, \infty)\) and let the functions \( \sigma_1(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda) \) be continuous on \((0, \infty) \times U\), analytically dependent on \( \lambda \) for \( \lambda \in U \), and form a basis for the solutions of the equation

\[
\tau \sigma = \lambda \sigma, \quad \lambda \in U.
\]

Suppose that for \( \lambda \in U \setminus [0, \infty) \), the resolvent \((T - \lambda)^{-1}\) is an integral operator with kernel \( K_\lambda \) that satisfies

\[
K_\lambda(x, y) = \theta(\lambda)\sigma_1(x, \lambda)\overline{\sigma_1(y, \lambda)} + \sum_{i,j=1}^{n} a_{ij}\sigma_i(x, \lambda)\overline{\sigma_j(y, \lambda)} \quad \text{if } y < x,
\]

for some function \( \theta \) and complex numbers \( a_{ij} \). Then \( \theta \) is analytic in \( U \setminus [0, \infty) \) and

\[
\rho(a, b) = \lim_{\delta \to 0} \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} (\theta(x + i\epsilon) - \theta(x - i\epsilon)) \, dx, \quad 0 < a < b,
\]

defines a positive Borel measure \( \rho \) on \((0, \infty)\) that satisfies:

(i) There is an isometric isomorphism \( \mathcal{V} \) from \( L^2(0, \infty) \) onto \( L^2((0, \infty), \rho) \), given by

\[
(\mathcal{V} \varphi)(y) = \int_0^\infty \varphi(x)\overline{\sigma_1(x, y)} \, dx, \quad \text{supp } \varphi \subseteq (0, \infty).
\]

(ii) The inverse of \( \mathcal{V} \) is given by

\[
(\mathcal{V}^{-1} f)(x) = \int_0^\infty f(y)\sigma_1(x, y) \, d\rho(y), \quad \text{supp } f \subseteq (0, \infty).
\]

(iii) We have that

\[
\mathcal{V} D(T) = \{ f(y) : yf(y) \in L^2((0, \infty), \rho) \}
\]

and

\[
(\mathcal{V} T \mathcal{V}^{-1} f)(y) = yf(y), \quad f \in \mathcal{V} D(T).
\]

A.3 An Auxiliary Schrödinger Operator on the Half-Line

Let \( 1 < d \leq 2 \) and consider the self-adjoint operator

\[
H_0 = -\frac{d^2}{dx^2} + \frac{(d-1)(d-3)}{4(1+x)^2}
\]

in the space \( L^2(0, \infty) \) defined by the introduction of the boundary condition

\[
\varphi'(0) = \frac{d-1}{2} \varphi(0).
\]
Lemma A.3. Let the space $\mathcal{H}$, the operator $H_\alpha$ and the potential $V$ be as in Section 5. Then $H_\alpha$ is unitarily equivalent to $H_0 - \alpha V$ for any $\alpha > 0$.

**Proof.** Introduce the isometric isomorphism $\mathcal{U}$ from $L^2(0, \infty)$ onto $\mathcal{H}$ defined by

$$(\mathcal{U}\varphi)(x) = \varphi(x)(1 + x)^{(1 - d)/2}, \quad \varphi \in L^2(0, \infty).$$

Recall that $h_\alpha$ is the closed quadratic form corresponding to the operator $H_\alpha$. Choose $\varphi, \psi \in D(H_0)$ and let $u = \mathcal{U}\varphi$ and $v = \mathcal{U}\psi$. Clearly $u, v \in D[h_\alpha]$ and partial integration gives us

$$h_\alpha[u, v] = - \int_0^\infty \varphi''(x)\overline{\psi(x)}\,dx + \int_0^\infty \left(\frac{(d - 1)(d - 3)}{4(1 + x)^2} - \alpha V(x)\right)\varphi(x)\overline{\psi(x)}\,dx.$$

Therefore the operator $\mathcal{U}(H_0 - \alpha V)\mathcal{U}^{-1}$ is associated to the quadratic form $h_\alpha$, which proves the statement.

Lemma A.4. The transformation $\mathcal{U} : L^2(0, \infty) \to L^2(0, \infty)$ given by

$$(\mathcal{U}\varphi)(p) = \int_0^\infty \varphi(x)\sqrt{p(1 + x)}f_d(p, x)\,dx, \quad \text{supp } \varphi \subseteq (0, \infty),$$

where

$$f_d(p, x) = \frac{J_\frac{d}{2}(p)Y_{\frac{3-d}{2}}(p(1 + x)) - Y_{\frac{3-d}{2}}(p)J_{\frac{d}{2}}(p(1 + x))}{\left(\left(J_\frac{d}{2}(p)\right)^2 + \left(Y_{\frac{3-d}{2}}(p)\right)^2\right)^{1/2}}$$

is a unitary isomorphism under which $H_0$ is equivalent to multiplication by the function $p \mapsto p^2$. The inverse of $\mathcal{U}$ is given by

$$(\mathcal{U}^{-1}\psi)(x) = \int_0^\infty \psi(p)\sqrt{p(1 + x)}f_d(p, x)\,dp, \quad \text{supp } \psi \subseteq (0, \infty).$$

**Proof.** Throughout, the complex number $\lambda$ will be chosen from a sufficiently small, fixed, open neighborhood in $\mathbb{C}$ of $(0, \infty)$. Let

$$\sigma_1(x, \lambda) = \sqrt{1 + x} \left(J_{\frac{d}{2}}(\lambda^{1/2})(1 + x)) - Y_{\frac{3-d}{2}}(\lambda^{1/2})(1 + x)\right)$$

and

$$\sigma_2(x, \lambda) = \sqrt{1 + x} J_{\frac{3-d}{2}}(\lambda^{1/2})(1 + x)$$

and

$$\chi(x, \lambda) = \begin{cases} \sqrt{1 + x} H^{(1)}_{\frac{3-d}{2}}(\lambda^{1/2}(1 + x)) & \text{if } \text{Im } \lambda > 0 \\ \sqrt{1 + x} H^{(2)}_{\frac{3-d}{2}}(\lambda^{1/2}(1 + x)) & \text{if } \text{Im } \lambda < 0. \end{cases}$$

Using Proposition A.3 it is seen that the functions $\sigma_1(\cdot, \lambda)$, $\sigma_2(\cdot, \lambda)$ and $\chi(\cdot, \lambda)$ all satisfy the equation

$$-\varphi''(x) + \frac{(d - 1)(d - 3)}{4(1 + x)^2}\varphi(x) = \lambda\varphi(x).$$
The same Proposition also shows that

\[
\sigma_1'(0, \lambda) = \frac{d - 1}{2} \sigma_1(0, \lambda) \quad \text{and} \quad \lim_{R \to \infty} \int_R^{\infty} |\chi(x, \lambda)|^2 \, dx = 0
\]

whenever \(\text{Im} \lambda \neq 0\). Hence, if we let

\[
K_{\lambda}(x, y) = \frac{1}{\sigma_1'(0, \lambda) \chi(0, \lambda) - \sigma_1(0, \lambda) \chi'(0, \lambda)} \begin{cases}
\chi(x, \lambda) \sigma_1(y, \lambda) & \text{if } y < x \\
\sigma_1(x, \lambda) \chi(y, \lambda) & \text{if } y > x,
\end{cases}
\]

it follows that the resolvent \((H_0 - \lambda)^{-1}\) is an integral operator with kernel \(K_{\lambda}\) for \(\lambda\) with \(\text{Im} \lambda \neq 0\). Some calculations give us that if \(y < x\) and \(\text{Im} \lambda > 0\) then

\[
K_{\lambda}(x, y) = \frac{\pi}{2J_{-\frac{d}{2}}(\lambda^{1/2})} \left( -i \sigma_1(x, \lambda) \sigma_1(y, \lambda) \right. \left. \frac{J_{-\frac{d}{2}}(\lambda^{1/2}) - iY_{-\frac{d}{2}}(\lambda^{1/2})}{J_{-\frac{d}{2}}(\lambda^{1/2}) + iY_{-\frac{d}{2}}(\lambda^{1/2})} + \sigma_2(x, \lambda) \sigma_1(y, \lambda) \right)
\]

and if \(y > x\) and \(\text{Im} \lambda > 0\) then

\[
K_{\lambda}(x, y) = \frac{\pi}{2J_{-\frac{d}{2}}(\lambda^{1/2})} \left( -i \sigma_1(x, \lambda) \sigma_1(y, \lambda) \right. \left. \frac{J_{-\frac{d}{2}}(\lambda^{1/2}) - iY_{-\frac{d}{2}}(\lambda^{1/2})}{J_{-\frac{d}{2}}(\lambda^{1/2}) + iY_{-\frac{d}{2}}(\lambda^{1/2})} + \sigma_2(x, \lambda) \sigma_1(y, \lambda) \right)
\]

Proposition \(\text{A.1}\) provides that \(\sigma_1(\cdot, \lambda) = \sigma_1(\cdot, \lambda^*)\) and that \(\sigma_1\) and \(\sigma_2\) are linearly independent. Hence, applying Proposition \(\text{A.2}\) we get that the operator \(V\) defined by

\[
(V \varphi)(y) = \int_0^\infty \varphi(x) \sigma_1(x, y) \, dx, \quad \text{supp } \varphi \subseteq (0, \infty)
\]

is an isometric isomorphism from \(L^2(0, \infty)\) onto \(L^2((0, \infty), \rho)\) such that the operator \(H_0\) is equivalent to multiplication by the identity function under this isomorphism. Here, the measure \(\rho\) is given by

\[
d\rho(y) = \frac{dy}{2 \left( \sqrt{J_{-\frac{d}{2}}(\sqrt{y})^2 + Y_{-\frac{d}{2}}(\sqrt{y})^2} \right)^2}.
\]

Furthermore, the inverse of \(\mathcal{Y}\) is given by

\[
(\mathcal{Y}^{-1} f)(x) = \int_0^\infty f(y) \sigma_1(x, y) \, d\rho(y), \quad \text{supp } f \subseteq (0, \infty).
\]

It remains to set \(\mathcal{W} = \mathcal{W} \mathcal{Y}\), where the isometry \(\mathcal{W}\) from \(L^2((0, \infty), \rho)\) onto \(L^2(0, \infty)\) is given by

\[
(\mathcal{W} f)(p) = f(p^2) \cdot \frac{\sqrt{p}}{\left( \left( J_{-\frac{d}{2}}(p) \right)^2 + \left( Y_{-\frac{d}{2}}(p) \right)^2 \right)^{1/2}}, \quad f \in L^2((0, \infty), \rho).
\]

**B Appendix: Integral Kernel of the Birman-Schwinger Operator**

In this appendix, we will use the operator \(H_0\) as defined in Section \(\text{A.3}\) and apply the Birman-Schwinger principle, [2, 13], to prove Lemma \(\text{A.3.5}\). Moreover,
$d$ and $\gamma$ will be fixed numbers satisfying $1 < \gamma \leq d \leq 2$ and $\gamma \neq 2$, and the measurable potential function $V$ will satisfy

$$0 \leq V(x) \leq \frac{C}{(1 + x)^{\gamma}},$$

for some $C > 0$. Let $\Omega = (0, \infty) \times (0, \infty)$ and consider the function $l_E$ on $\Omega$ given by

$$l_E(p, x) = f_d(p, x) \left( \frac{p(1 + x)V(x)}{p^2 + E} \right)^{1/2},$$

for $E > 0$. Here, as in Lemma A.4, we have that

$$f_d(p, x) = \frac{J_{-\frac{d}{2}}(p)Y_{-\frac{d}{2}}(p(1 + x)) - Y_{-\frac{d}{2}}(p)J_{-\frac{d}{2}}(p(1 + x))}{\left( \left( J_{-\frac{d}{2}}(p) \right)^2 + \left( Y_{-\frac{d}{2}}(p) \right)^2 \right)^{1/2}}.$$

Also define the bounded integral operator $L_E$ by

$$(L_E\psi)(p) = \int_{0}^{\infty} l_E(p, x)\psi(x) \, dx, \quad \psi \in L^2(0, \infty).$$

That $L_E$ is well-defined and even a Hilbert-Schmidt operator is provided by the following lemma:

**Lemma B.1.** Suppose that $1 < \gamma \leq d \leq 2$, $\gamma \neq 2$ and that $V$ satisfies (14). Then the following holds:

(i) If $1 < \gamma < d \leq 2$ then there is a constant $B_1 > 0$ such that for any $E > 0$,

$$\iint_{\Omega} |l_E(p, x)|^2 \, dx \, dp < B_1 E^{\frac{2-\gamma}{2}}.$$

(ii) If $1 < \gamma = d < 2$ then there is a constant $B_2 > 0$ such that for any $E > 0$,

$$\iint_{\Omega} |l_E(p, x)|^2 \, dx \, dp < B_2 E^{\frac{2-\gamma}{2}} \left( 1 + |\log E| \right).$$

**Proof.** Making a change of variables twice, we get that

$$\iint_{\Omega} |l_E(p, x)|^2 \, dx \, dp \leq C \int_{\int_{\Omega}} \frac{p}{(p^2 + E)(1 + x)^{\gamma - 1}} (f_d(p, x))^2 \, dx \, dp$$

$$= CE^{\frac{-\gamma}{2}} \int_{0}^{\infty} \int_{pE^{1/2}}^{\infty} \frac{p^{\gamma - 1}}{(p^2 + 1)x^{\gamma - 1}} F_d(p, x) \, dx \, dp,$$

where

$$F_d(p, x) = \left( f_d \left( \frac{x}{pE^{1/2}} - 1 \right) \right)^2 = \left( \frac{J_{-\frac{d}{2}}(pE^{1/2})Y_{-\frac{d}{2}}(x) - Y_{-\frac{d}{2}}(pE^{1/2})J_{-\frac{d}{2}}(x)}{\left( J_{-\frac{d}{2}}(pE^{1/2}) \right)^2 + \left( Y_{-\frac{d}{2}}(pE^{1/2}) \right)^2} \right)^2.$$
Recall that
\[ J_{\nu}(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \quad \text{and} \quad Y_{\nu}(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right), \]

as \( x \to \infty \) for any \( \nu \) and therefore there is a \( c_1 > 0 \), such that for any \( x \geq 1/2 \) and any \( d \),
\[ 2 \left( J_{\frac{1}{2}}(x) \right)^2 \leq \frac{c_1}{x} \quad \text{and} \quad 2 \left( J_{\frac{1}{2}}(x) \right)^2 \leq \frac{c_1}{x}. \quad (16) \]

Now assume that \( 1 < d \leq 2 \). We have that \( Y_{\nu}(x) \sim -(1/\pi)\Gamma(\nu)(x/2)^{-\nu} \) and that \( J_{\nu}(x) \) is bounded as \( x \to 0^+ \), for any \( \nu > 0 \). Hence there is a constant \( c_2 > 0 \) such that for any \( x \) with \( 0 < x < 1 \) and any \( d \) with \( 1 < d < 2 \),
\[ 2 \left( J_{\frac{1}{2}}(x) \right)^2 \leq \frac{c_1}{x^{2-d}} \quad \text{and} \quad 2 \left( J_{\frac{1}{2}}(x) \right)^2 \leq \frac{c_1}{x^{2-d}}. \]

Combine this with (16) and obtain that as soon as \( 1 < d < 2 \),
\[ F_d(p, x) \leq \frac{2 \left( J_{\frac{1}{2}}(pE^{1/2}) \right)^2 \left( J_{\frac{1}{2}}(pE^{1/2}) \right)^2 \left( J_{\frac{1}{2}}(pE^{1/2}) \right)^2 \left( J_{\frac{1}{2}}(pE^{1/2}) \right)^2}{\left( J_{\frac{1}{2}}(pE^{1/2}) \right)^2 + \left( J_{\frac{1}{2}}(pE^{1/2}) \right)^2} \]
\[ \leq \frac{c_2}{x^{2-d}} \chi(0,1)(x) \quad \text{and} \quad \frac{c_1}{x^{2-d}} \chi(1,\infty)(x), \quad (17) \]

for any \( x, p > 0 \). If \( 1 < \gamma < d < 2 \), we get from (15) and (17) that
\[
\int_{\Omega} |l_E(p, x)|^2 \, dx \, dp \leq C E^{2-2\gamma} \int_0^\infty \int_0^\infty \frac{p^{\gamma-1}}{(p^2 + 1)x^{\gamma-1}} F_d(p, x) \, dx \, dp \leq D_1 E^{2-2\gamma},
\]

where
\[ D_1 = D_1(\gamma, d) = C \left( c_2 \int_0^1 \frac{p^{\gamma-1}}{(p^2 + 1)x^{\gamma-1}} \, dp + c_1 \int_1^\infty \frac{p^{\gamma-1}}{(p^2 + 1)x^{\gamma}} \, dp \right) < \infty. \]

Moreover, if \( 1 < \gamma = d < 2 \), then (15) and (17) give that
\[
\int_{\Omega} |l_E(p, x)|^2 \, dx \, dp \leq \int_0^\infty \frac{p^{\gamma-1}}{p^2 + 1} \left( c_2 \int_0^1 \frac{dx}{p^{\gamma/2}} + c_1 \int_1^\infty \frac{dx}{x^{\gamma}} \right) \, dp
\]
\[
= \frac{c_1}{\gamma-1} \int_0^{p^{\gamma-1}} \frac{dp}{p^2 + 1} \frac{p^{\gamma-1}}{p^2 + 1} \left( \log p + \frac{1}{2} \log E \right) \, dp,
\]

which means that
\[
\int_{\Omega} |l_E(p, x)|^2 \, dx \, dp \leq D_2 E^{2-2\gamma} + D_3 E^{2-2\gamma} |\log E|,
\]

where
\[ D_2 = D_2(\gamma) = C \left( \frac{c_1}{\gamma-1} \int_0^\infty \frac{p^{\gamma-1}}{p^2 + 1} \, dp + c_2 \int_0^\infty \frac{p^{\gamma-1}|\log p|}{p^2 + 1} \, dp \right) < \infty \]
and
\[ D_3 = D_3(\gamma) = C \frac{c_2}{2} \int_0^\infty \frac{p^{\gamma-1}}{p^2 + 1} \, dp < \infty. \]
Finally, assume that $d = 2$ and that $1 < \gamma < d$. Since $Y_0(x) \sim -2/\pi \cdot \log x$ and $J_0(x)$ is bounded as $x \to 0+$, there is a constant $c_3$ such that

$$2 (Y_0(x))^2 \leq c_3 |\log x| \quad \text{and} \quad 2 (J_0(x))^2 \leq c_3 |\log x|$$

for any $x$ with $0 < x < 1/2$. Together with (16) this gives

$$F_2(p, x) \leq c_3 |\log x| \chi_{(0,1/2)}(x) + \frac{c_1}{x} \chi_{[1/2, \infty)}(x),$$

(18)

for any $p, x > 0$. If $1 < \gamma < d = 2$, equations (15) and (18) shows that

$$\int \int_\Omega |l_E(p, x)|^2 \, dx \, dp \leq CE^{\gamma - 2} \int_0^\infty \left( \frac{p^{\gamma - 1}}{(p^2 + 1)^{\gamma - 1}} \right) F_2(p, x) \, dx \, dp \leq D_4 E^{\gamma - 2},$$

where

$$D_4 = D_4(\gamma) = C \int_0^\infty \frac{p^{\gamma - 1}}{p^2 + 1} \, dp \left( c_3 \int_0^{1/2} \frac{|\log x|}{x^{\gamma - 1}} \, dx + c_1 \int_{1/2}^\infty \frac{1}{x^{\gamma}} \, dx \right) < \infty. \quad \Box$$

**Lemma B.2.** Let the unitary transformation $\mathcal{U} : L^2(0, \infty) \to L^2(0, \infty)$ be as in Lemma A.4 and $E > 0$. Assume that $1 < \gamma \leq d \leq 2$, $\gamma \neq 2$ and that $V$ satisfies (14). Then

$$\mathcal{U} \left( (H_0 + E)^{-1/2} V (H_0 + E)^{-1/2} \right) \mathcal{U}^{-1} = L_E L_E^*.$$

**Proof.** Choose $\psi \in L^2(0, \infty)$ with supp $\psi \Subset (0, \infty)$. Note that

$$\mathcal{U} V^{1/2} (H_0 + E)^{-1/2} \mathcal{U}^{-1} \psi = \mathcal{U} V^{1/2} \mathcal{U}^{-1} \mathcal{U} (H_0 + E)^{-1/2} \mathcal{U}^{-1} \psi$$

$$= \mathcal{U} V^{1/2} \mathcal{U}^{-1} \left( \frac{\psi(\cdot)}{(\cdot^2 + E)^{1/2}} \right)$$

$$= \mathcal{U} V^{1/2} \left( \int_0^\infty \frac{\psi(p)}{(p^2 + E)^{1/2}} \sqrt{p(1 + \cdot)} f_d(p, \cdot) \, dp \right)$$

$$= \mathcal{U} L_E^* \psi$$

Since the operators in question are bounded, this is enough to prove that

$$\mathcal{U} V^{1/2} (H_0 + E)^{-1/2} \mathcal{U}^{-1} = \mathcal{U} L_E^*.$$

It follows that

$$\mathcal{U} (H_0 + E)^{-1/2} V (H_0 + E)^{-1/2} \mathcal{U}^{-1}$$

$$= \left( \mathcal{U} V^{1/2} (H_0 + E)^{-1/2} \mathcal{U}^{-1} \right)^* \left( \mathcal{U} V^{1/2} (H_0 + E)^{-1/2} \mathcal{U}^{-1} \right)$$

$$= (\mathcal{U} L_E^*)^* \mathcal{U} L_E^*$$

$$= L_E L_E^*. \quad \Box$$

We are now finally in a position to prove Lemma 5.3.
Proof of Lemma 5.3. It is well-known that if \( l \in L^2(\Omega) \) and \( L \) is the Hilbert-Schmidt operator defined by
\[
(L\psi)(p) = \int_0^\infty l(p,x)\psi(x) \, dx,
\]
then the operator \( Q = LL^* \) is non-negative, trace class and satisfies
\[
\text{Tr} \, Q \leq \int_\Omega |l(p,x)|^2 \, dx \, dp.
\]
It remains to choose \( E > 0 \), set
\[
Q_E = (H_0 + E)^{-1/2} V (H_0 + E)^{-1/2},
\]
and use Lemmas A.3, B.1 and B.2.

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