SPECTRAL ACMS: A ROBUST LOCALIZED APPROXIMATED COMPONENT MODE SYNTHESIS METHOD

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Abstract. We consider finite element methods of multiscale type to approximate solutions for two-dimensional symmetric elliptic partial differential equations with heterogeneous $L^\infty$ coefficients. The methods are of Galerkin type and follow the Variational Multiscale and Localized Orthogonal Decomposition–LOD approaches in the sense that it decouples spaces into multiscale and fine subspaces. In a first method, the multiscale basis functions are obtained by mapping coarse basis functions, based on corners used on primal iterative substructuring methods, to functions of global minimal energy. This approach delivers quasi-optimal a priori error energy approximation with respect to the mesh size, but it is not robust with respect to high-contrast coefficients. In a second method, edge modes based on local generalized eigenvalue problems are added to the corner modes. As a result, optimal a priori error energy estimate is achieved which is mesh and contrast independent. The methods converge at optimal rate even if the solution has minimum regularity, belonging only to the Sobolev space $H^1$.

1. Introduction

Let $u : \Omega \to \mathbb{R}$ be the weak solution of

$$
\begin{align*}
-\text{div} \mathcal{A} \nabla u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^2$, and is an open bounded domain with polygonal boundary $\partial\Omega$, the symmetric tensor $\mathcal{A} \in [L^\infty(\Omega)]^{2\times2}_{\text{sym}}$ is uniformly positive definite almost everywhere, and $f \in L^2(\Omega)$ is given. For almost all $\mathbf{x} \in \Omega$ let the positive constants $a_{\min}$ and $a_{\max}$ be such that

$$
a_{\min} |\mathbf{v}|^2 \leq a_{\mathcal{A}}(\mathbf{x}) |\mathbf{v}|^2 \leq \mathcal{A}(\mathbf{x}) \mathbf{v} \cdot \mathbf{v} \leq a_{\max} |\mathbf{v}|^2 \leq a_{\mathcal{A}}(\mathbf{x}) |\mathbf{v}|^2 \quad \text{for all } \mathbf{v} \in \mathbb{R}^2,
$$

where $a_{\mathcal{A}}(\mathbf{x})$ and $a_{\max}(\mathbf{x})$ are the smallest and largest eigenvalues of $\mathcal{A}(\mathbf{x})$. Let $\rho \in L^\infty(\Omega)$ be chosen by the user such that $\rho(\mathbf{x}) \in [\rho_{\min}, \rho_{\max}]$ almost everywhere for some positive constants $\rho_{\min}$ and $\rho_{\max}$. Consider $g$ such that

$$f = \rho g,$$
and then the $\rho$-weighted $L^2(\Omega)$ norm $\|g\|_{L^2(\Omega)} := \|\rho^{1/2}g\|_{L^2(\Omega)} = \|f\|_{L^2_{\rho}(\Omega)}$ is finite. The introduction of the weight $\rho$ is to balance $u$ and $f$ with respect to the tensor $A$, adding flexibility to the method and making error estimates more meaningful. It allows for a \textit{fairer measure} of the error when the right hand side of error estimates depends on $f$ only. Also, for high-contrast problems, it might compensate for local low coercivity of $A$; see the end of Section 4 for more details. We note that a related numerical work, without any proofs, was presented in the conference paper [44]. Here the main goal is the corresponding analysis as sharp as possible without any hidden constants.

For $v, w \in H^1(\Omega)$ let

$$a(v, w) = \int_\Omega A \nabla v \cdot \nabla w \, dx,$$

and denote by $(\cdot, \cdot)$ the $L^2(\Omega)$ inner product.

Although the solution $u$ of (11) in general only belongs to the Sobolev space $H^1(\Omega)$, a priori error analyses of multiscale methods established on the literature often rely on solution regularity; see [3, 10, 19–21, 27, 28, 32, 34, 63, 65] and references therein.

Considering the low contrast case, some methods require minimum regularity, as the generalized finite element methods [2], the rough polyharmonic splines [53], the variational multiscale method [35, 36], and the Localized Orthogonal Decomposition (LOD) [49–51]. The general idea is to decompose the solution spaces as a direct sum of fine (local) and multiscale (low dimensional, nonlocal) spaces. The final approximate solution belongs to the multiscale space. The LOD approximation [31,60] also works for the high contrast cases when the local Poincaré inequality is not large; see Remark 1.

However, there are several domain decomposition solvers that are optimal with respect to mesh and contrast, relying on coarse basis functions from local generalized eigenvalue problems. The \textit{adaptive choice of primal constraints} method was introduced to ensure robustness with respect to contrast for non-overlapping domain decomposition methods based on FETI-DP [24,41] and BDDC [14]. References [4, 9, 13, 8, 40, 46, 52, 55, 58] elaborate on this approach. For overlapping domain decomposition see [15,22,25,8,67]. Some of this ideas were incorporated in [11,23] to obtain discretizations that depend only logarithmically on the contrast.

In [45] we introduced the Localized Spectral Decomposition–LSD method for mixed and hybrid-primal methods [62], that is, we re-frame the LOD version in [50] into the non-overlapping domain decomposition framework, and consider the Multiscale Hybrid Method–MHM [1, 28, 29], which falls in the BDDC and FETI-DP classes, and then explore adaptive
choice of primal constraints to generate the multiscale basis functions. We obtain a discretization that is robust with respect to contrast.

In this paper we propose an *Approximated Component Mode Synthesis*–ACMS type method [6, 7, 12, 30, 32, 33, 37]. In general, these methods require extra solution regularity and do not work for high contrast, and the goal here is to develop a discretization that has optimal and robust a priori error approximation, assuming minimum regularity on the solution, $A$ and $\rho$.

To consider the LOD approach with Galerkin-Ritz projection, we use conforming primal iterative substructuring techniques [5, 8, 17, 18, 42, 43, 61, 64, 69] rather than BDDC and FETI-DP methods. Two versions are under consideration here, both of Galerkin type and based on edges and local harmonic extensions. The first method is simpler and converges at quasi-optimal rates, even under minimal regularity of the solution. We note, however, this method has a weak singularity at the coarse nodes and its properties deteriorate if the contrast of the coefficients increases. To circumvent these two issues, we modify the method by incorporating solutions of specially designed local eigenfunction problems, yielding optimal convergence rate uniformly with respect to contrast.

The remainder of the this paper is organized as follows. Section 2 describes the substructuring decomposition into interior and interface unknowns, while our methods for low and high contrast coefficients are considered in Sections 3 and 4 respectively. In Section 5 we consider how to deal with local, elementwise problems. Numerical tests and some of the results of this paper, for the case of high-contrast only, were presented without proofs in [44].

## 2. Substructuring Formulation

Let $\mathcal{T}_h$ be a finite element regular partition of $\Omega$ based on triangles, with elements of characteristic length $H > 0$. We denote the mesh skeleton by $\partial \mathcal{T}_h$, and denote by $\mathcal{N}_h$ the set of nodes on $\partial \mathcal{T}_h \setminus \partial \Omega$. For $h < H$, let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$, in the sense that every (coarse) edge in $\partial \mathcal{T}_H$ is a union of edges of elements in $\mathcal{T}_h$. Let $\mathcal{N}_h$ be the set of nodes of $\mathcal{T}_h$ on the skeleton $\partial \mathcal{T}_h \setminus \partial \Omega$. Therefore, all nodes in $\mathcal{N}_h$ belong to edges of elements in $\mathcal{T}_H$.

For $v \in H^1(\Omega)$ and a given set of elements $\mathcal{T} \subset \mathcal{T}_H$, let

$$
|v|_{H^1_A(\Omega)}^2 = \|A^{1/2} \nabla v\|_{L^2(\Omega)}^2, \\
|v|_{H^1_A(\mathcal{T})}^2 = \sum_{\tau \in \mathcal{T}} \|A^{1/2} \nabla v\|_{L^2(\tau)}^2.
$$

Let $V_h \subset H^1_0(\Omega)$ be the space of continuous piecewise linear functions associated with the fine mesh $\mathcal{T}_h$. For the sake of reference, let $u_h \in V_h$ such that

$$
a(u_h; v_h) = (\rho g, v_h) \quad \text{for all } v_h \in V_h.
$$
We assume that \( u_h \) approximates \( u \) well. Our numerical schemes yield good approximations for \( u_h \) without ever computing it. Assume the decomposition \( u_h = u_h^B + u_h^\text{H} \) in its bubble and harmonic components, where \( u_h^B \in V_h^B \), \( u_h^\text{H} \in V_h^\text{H} \), and

\[
V_h^B = \{ v_h \in V_h : v_h = 0 \text{ on } \partial \tau, \tau \in \mathcal{T}_H \},
\]

\[
V_h^\text{H} = \{ u_h^\text{H} \in V_h : a(u_h^\text{H}, v_h^B) = 0 \text{ for all } v_h^B \in V_h^B \},
\]
i.e., \( V_h^\text{H} = (V_h^B)^{\perp_a} \). It follows immediately from the definitions that

\[
a(u_h^\text{H}, v_h^\text{H}) = (\rho g, v_h^\text{H}) \quad \text{for all } v_h^\text{H} \in V_h^\text{H}, \quad a(u_h^B, v_h^B) = (\rho g, v_h^B) \quad \text{for all } v_h^B \in V_h^B.
\]

The problems for the bubble solution \( u_h^B \) are local and uncoupled and are considered in Section 5.

We now proceed to approximate \( u_h^\text{H} \), and start by noting that the functions in \( V_h^\text{H} \) are uniquely determined by their traces on the boundary of elements in \( \mathcal{T}_H \). Let

\[
\Lambda_h = \{ v_h|_{\partial \mathcal{T}_H} : v_h \in V_h^\text{H} \} \subset H^{1/2}(\partial \mathcal{T}_H),
\]

and the local discrete-harmonic extension operator \( T : \Lambda_h \to V_h^\text{H} \) such that, for \( \mu_h \in \Lambda_h \),

\[
(T \mu_h)|_{\partial \mathcal{T}_H} = \mu_h, \quad \text{and} \quad a(T \mu_h, v_h^B) = 0 \quad \text{for all } v_h^B \in V_h^B.
\]

Define the bilinear forms \( s_\tau, s : \Lambda_h \times \Lambda_h \to \mathbb{R} \) such that, for \( \mu_h, \nu_h \in \Lambda_h \),

\[
s_\tau(\mu_h, \nu_h) = \int_\tau \mathcal{A} \nabla T \mu_h \cdot \nabla T \nu_h \, d\mathbf{x} \quad \text{for } \tau \in \mathcal{T}_H, \quad s(\mu_h, \nu_h) = \sum_{\tau \in \mathcal{T}_H} s_\tau(\mu_h, \nu_h).
\]

Let \( \lambda_h = u_h|_{\partial \mathcal{T}_H} \). Then \( u_h^\text{H} = T \lambda_h \) and

\[
s(\lambda_h, \mu_h) = (\rho g, T \mu_h) \quad \text{for all } \mu_h \in \Lambda_h.
\]

3. The Low-Contrast Multiscale Case

We now propose a scheme to approximate [5] based on LOD techniques. Define the fine-scale subspace \( \widetilde{\Lambda}_h \subset \Lambda_h \) by

\[
\widetilde{\Lambda}_h = \{ \tilde{\lambda}_h \in \Lambda_h : \tilde{\lambda}_h(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in \mathcal{N}_H \}.
\]

Let the multiscale space \( \Lambda_h^{\text{ms}} \subset \Lambda_h \) be such that \( \widetilde{\Lambda}_h \perp_s \Lambda_h^{\text{ms}} \) and \( \Lambda_h = \widetilde{\Lambda}_h \oplus \Lambda_h^{\text{ms}} \). Our numerical method is defined by \( \lambda_h^{\text{ms}} \in \Lambda_h^{\text{ms}} \) such that

\[
s(\lambda_h^{\text{ms}}, \mu_h^{\text{ms}}) = (\rho g, T \mu_h^{\text{ms}}) \quad \text{for all } \mu_h^{\text{ms}} \in \Lambda_h^{\text{ms}},
\]

and we set \( u_h^{\text{ms}} = T \lambda_h^{\text{ms}} \) as an approximation for \( u_h^\text{H} \).

To make the definition of \( \Lambda_h^{\text{ms}} \) explicit, let the coarse-scale space \( \Lambda_H \subset \Lambda_h \) be the trace of piecewise continuous linear functions on the \( \partial \mathcal{T}_H \) triangulation. Thus, a function \( \lambda_H \in \Lambda_H \).
is uniquely determined by its nodal values and is linear on each edge. A basis \( \{ \theta^i_H \}_{i=1}^{N_H} \) for \( \Lambda_H \) can be obtained by imposing that \( \theta^i_H \) be continuous and piecewise linear on \( \partial \mathcal{T}_H \) and \( \theta^i_H(x_j) = \delta_{ij} \) for all \( x_j \in \mathcal{N}_H \). The support of \( \theta^i_H \) is on all edges of elements \( \tau \in \mathcal{T}_H \) for which \( x_i \in \tau \). If \( \mu_H = \sum_{i=1}^{N_H} \mu_H(x_i) \theta^i_H \) is such that \( \mu_H(x_i) = 0 \) for all \( x_i \in \mathcal{N}_H \), then \( \mu_H(x) = 0 \) for all \( x \in \mathcal{N}_H \). Hence, \( \Lambda_h = \Lambda_H \oplus \tilde{\Lambda}_h \), and then \( \dim \Lambda_h = \dim \Lambda_h^{ms} \).

Now, for each \( K \in \mathcal{T}_H \) and \( \nu_h \in \Lambda_h \), let \( P^K : \Lambda_h \rightarrow \tilde{\Lambda}_h \) be such that
\[
(7) \quad s(P^K \nu_h, \tilde{\mu}_h) = s_K(\nu_h, \tilde{\mu}_h) \quad \text{for all } \tilde{\mu}_h \in \tilde{\Lambda}_h,
\]
and \( P : \Lambda_h \rightarrow \tilde{\Lambda}_h \) be such that
\[
(8) \quad P \nu_h = \sum_{K \in \mathcal{T}_H} P^K \nu_h.
\]
Note that
\[
s(P \nu_h, \tilde{\mu}_h) = \sum_{K \in \mathcal{T}_H} s(P^K \nu_h, \tilde{\mu}_h) = \sum_{K \in \mathcal{T}_H} s_K(\nu_h, \tilde{\mu}_h) = s(\nu_h, \tilde{\mu}_h).
\]
It follows from the above that \( \Lambda_h^{ms} = \{ (I-P)\theta_H : \theta_H \in \Lambda_H \} \). A basis for \( \Lambda_h^{ms} \) is defined by \( \lambda_i^{ms} = (I-P)\theta^i_H \in \Lambda_h^{ms} \), and by construction, \( \lambda_i^{ms}(x_j) = \delta_{ij} \) for all \( x_j \in \mathcal{N}_H \).

An alternative to (6) is to find \( \lambda_H \in \Lambda_H \) such that
\[
(9) \quad s((I-P)\lambda_H, (I-P)\mu_H) = (\rho g, T(I-P)\mu_H) \quad \text{for all } \mu_H \in \Lambda_H,
\]
and then \( \lambda_H^{ms} = (I-P)\lambda_H \). We name it as ACMS–NLOD (Approximated Component Mode Synthesis Non-Localized Orthogonal Decomposition) method.

Albeit being well-defined, the method (9) is not “practical”, in the sense that the operators and \( P^K \) and \( P \) are nonlocal, and computing (9) is as hard as solving (1). To circumvent that, we use the fact that the solutions of (7) actually decay exponentially to zero away from \( K \). That allows the definition of a local approximation \( P^{K,j} \) for \( P^K \), having support at a patch of width \( j \) around \( K \). Next, before proving the exponential decay, we investigate the convergence rates for the ideal nonlocal solution \( u_h^{ms} \).

In what follows, \( \gamma_1, \gamma_2 \), etc denote positive constants that do not depend on \( A \), \( f \), \( \rho \), \( h \) and \( H \), depending only on the shape regularity of elements on \( \mathcal{T}_h \) and \( \mathcal{T}_H \). Let
\[
(10) \quad \| T \mu_h \|_{L^2_\rho(\Omega)} \leq C_{P,G} |T\mu_h|_{H^1_\rho(\Omega)}.
\]
Lemma 1. Let \( c_{P,L} \) and \( c_{P,L}^\tau \) be the smallest constants such that

\[
\|T\tilde{\mu}_h\|_{L^2(\tau)} \leq c_{P,L}^\tau H|T\tilde{\mu}_h|_{H^1(\tau)} \quad \text{for all } \tilde{\mu}_h \in \tilde{\Lambda}_h.
\]

\( \text{Lemma 1.} \) Let \( \tau \in \mathcal{T}_H \) and \( c_{P,L}^\tau \) as in (11). Then, an upper bound for \( c_{P,L}^\tau \) is given by

\[
(c_{P,L}^\tau)^2 \leq \gamma_1 (1 + \log(H/h)) \frac{\rho_{\text{max}}^\tau}{\alpha_{\text{min}}^\tau}.
\]

\( \text{Proof.} \) Using that \( \tilde{\mu}_h \) vanishes at the \( \mathcal{N}_H \) nodes, we have [69]

\[
\|T\tilde{\mu}_h\|_{L^2(\tau)}^2 \leq \gamma_1 H^2 \rho_{\text{max}}^\tau \|T\tilde{\mu}_h\|_{L^\infty(\tau)}^2 \\
\leq \gamma_1 H^2 (1 + \log(H/h)) \rho_{\text{max}}^\tau |T\tilde{\mu}_h|_{H^1(\tau)}^2 \leq \gamma_1 H^2 (1 + \log(H/h)) \frac{\rho_{\text{max}}^\tau}{\alpha_{\text{min}}^\tau} |T\tilde{\mu}_h|_{H^1(\tau)}^2.
\]

\( \Box \)

\( \text{Lemma 2.} \) Given \( \mu_h \in \Lambda_h \) let \( I_H \mu_h \in \Lambda_H \) be its Lagrange \( \mathcal{N}_H \)-nodal linear interpolation on \( \partial\mathcal{T}_H \). Then

\[
|TI_H \mu_h|_{H^1(\Omega)}^2 \leq \gamma_2 \kappa (1 + \log(H/h)) |T\mu_h|_{H^1(\Omega)}^2.
\]

\( \text{Proof.} \) Let \( T \chi \) be defined by (4) with \( \mathcal{A} = \mathcal{I} \), the identity matrix. It follows [69] for each \( \tau \in \mathcal{T}_H \) that

\[
|TI_H \mu_h|_{H^1(\tau)}^2 \leq |T\chi e I_H \mu_h|_{H^1(\tau)}^2 \leq \alpha_{\text{max}}^\tau |T\chi e I_H \mu_h|_{H^1(\tau)}^2 \leq \gamma_2 \alpha_{\text{max}}^\tau (1 + \log(H/h)) |T\tilde{\mu}_h|_{H^1(\tau)}^2 \\
\leq \gamma_2 \alpha_{\text{max}}^\tau (1 + \log(H/h)) |T\tilde{\mu}_h|_{H^1(\tau)}^2 \leq \gamma_2 \kappa (1 + \log(H/h)) |T\tilde{\mu}_h|_{H^1(\tau)}^2.
\]

\( \Box \)

We know extend the \( \text{Face Lemma} \) [69] Subsection 4.6.3] to variable coefficients.

\( \text{Lemma 3.} \) Let \( \tau \in \mathcal{T}_H \), \( e \) an edge of \( \partial\tau \) and \( \chi_e \) be the characteristic function of \( e \) being identically equal to one on \( e \) and zero on \( \partial\tau \setminus e \). Then given \( \tilde{\mu}_h \in \tilde{\Lambda}_h \) we have

\[
|T\chi_e \tilde{\mu}_h|_{H^1(\tau)}^2 \leq \gamma_3 \kappa (1 + \log(H/h))^2 |T\tilde{\mu}_h|_{H^1(\tau)}^2.
\]

\( \text{Proof.} \) We have

\[
|T\chi_e \tilde{\mu}_h|_{H^1(\tau)}^2 \leq |T\chi_e I_H \mu_h|_{H^1(\tau)}^2 \leq \alpha_{\text{max}}^\tau |T\chi_e I_H \mu_h|_{H^1(\tau)}^2 \leq \gamma_3 \alpha_{\text{max}}^\tau (1 + \log(H/h))^2 |T\tilde{\mu}_h|_{H^1(\tau)}^2 \\
\leq \gamma_3 \alpha_{\text{max}}^\tau (1 + \log(H/h))^2 |T\tilde{\mu}_h|_{H^1(\tau)}^2 \leq \gamma_3 \kappa (1 + \log(H/h))^2 |T\tilde{\mu}_h|_{H^1(\tau)}^2.
\]

\( \Box \)
Let $\lambda_h = u_h|_{\partial T_H}$, and $\lambda_h^{ms}$ solution of \((6)\). Then $\lambda_h - \lambda_h^{ms} \in \tilde{\Lambda}_h$ and

$$|u_h^m - u_h^{ms}|_{H^2(\Omega)} \leq c_{PL} H \|g\|_{L^2(\Omega)},$$

where we recall that $u_h = T\lambda_h$ and $u_h^{ms} = T\lambda_h^{ms}$.

**Proof.** First note that $\lambda_h - \lambda_h^{ms} \in \tilde{\Lambda}_h$ since it follows from the Galerkin orthogonality that $s(\lambda_h - \lambda_h^{ms}, \mu^{ms}_h) = 0$ for all $\mu^{ms}_h \in \Lambda_h^{ms}$. Using the local Poincaré’s inequality \((\Pi)\) we obtain

$$|u_h^m - u_h^{ms}|^2_{H^2(\Omega)} = s(\lambda_h - \lambda_h^{ms}, \lambda_h - \lambda_h^{ms}) = s(\lambda_h - \lambda_h^{ms}, \lambda_h) = \langle \rho g, T(\lambda_h - \lambda_h^{ms}) \rangle \leq \|g\|_{L^2(\Omega)} \|T(\lambda_h - \lambda_h^{ms})\|_{H^2(\Omega)},$$

and the result follows. \hfill \Box

### 3.1. Decaying Low-Contrast

We next prove exponential decay of $P^K \nu_h$ for $K \in T_H$. Denote

$$T_1(K) = \{K\}, \quad T_{j+1}(K) = \{\tau \in T_H : \tau \cap T_j \neq \emptyset \text{ for some } \tau_j \in T_j(K)\}.$$

The following estimate is fundamental to prove exponential decay.

**Lemma 5.** Assume that $K \in T_H$ and $\nu_h \in \Lambda_h$, and let $\tilde{\phi}_h = P^K \nu_h \in \tilde{\Lambda}_h$. Then, for any integer $j \geq 1$,

$$|T\tilde{\phi}_h|_{H^2_A(T_H \setminus T_{j+1}(K))}^2 \leq \alpha |T\tilde{\phi}_h|_{H^2_A(T_{j+1}(K) \setminus T_j(K))}^2,$$

where $\alpha = \gamma_3 h (1 + \log H/h)^2$.

**Proof.** Choose $\tilde{\nu}_h \in \tilde{\Lambda}_h$ such that $\tilde{\nu}_h|_{\partial \tau} = \tilde{\phi}_h$ if $\tau \in T_H \setminus T_{j+1}(K)$, and $\tilde{\nu}_h = 0$ on the remaining edges. We obtain

$$|T\tilde{\phi}_h|_{H^2_A(T_H \setminus T_{j+1}(K))}^2 \leq \sum_{\tau \in T_{j+1}(K) \setminus T_j(K)} s_{\tau}(\tilde{\nu}_h, \tilde{\phi}_h) + \sum_{\tau \in T_{j+1}(K) \setminus T_j(K)} s_{\tau}(\tilde{\nu}_h, \tilde{\phi}_h)$$

$$\leq \sum_{\tau \in T_{j+1}(K) \setminus T_j(K)} |T\tilde{\nu}_h|_{H_A^2(\tau)} |T\tilde{\phi}_h|_{H^2_A(\tau)},$$

where we used that $s(\tilde{\nu}_h, \tilde{\phi}_h) = s_K(\tilde{\nu}_h, \nu_h) = 0$ since the support of $\tilde{\nu}_h$ does not intersect with $K$. For each edge $e$ of $\partial \tau$, let $\chi_e$ be the characteristic function of $e$ being identically equal to one on $e$ and zero on $\partial \tau \setminus e$. For $\tau \in T_{j+1}(K) \setminus T_j(K)$,

$$|T\tilde{\nu}_h|_{H^2_A(\tau)} \leq 3 \sum_{e \subset \partial \tau} |T(\chi_e \tilde{\nu}_h)|_{H^2_A(\tau)}^2 \leq 9 \gamma_3 h (1 + \log H/h)^2 |T\tilde{\phi}_h|_{H^2_A(\tau)}^2,$$

where we have used the Face Lemma [69, Subsection 4.6.3]. \hfill \Box
Corollary 6. Assume that $K \in \mathcal{T}_H$ and $\nu_h \in \Lambda_h$ and let $\tilde{\varphi}_h = P^K \nu_h \in \tilde{\Lambda}_h$. Then, for any integer $j \geq 1$,

$$|T \tilde{\varphi}_h|_{H^1(\mathcal{T}_h \setminus \mathcal{T}_{j+1}(K))}^2 \leq e^{-\frac{j}{1+9\alpha}} |T \tilde{\varphi}_h|_{H^1(\mathcal{T}_h)}^2,$$

where $\alpha$ is as in Lemma 5.

Proof. Using Lemma 5 we have

$$|T \tilde{\varphi}_h|_{H^1(\mathcal{T}_h \setminus \mathcal{T}_{j+1}(K))}^2 \leq 9\alpha |T \tilde{\varphi}_h|_{H^1(\mathcal{T}_h \setminus \mathcal{T}_j(K))}^2 - 9\alpha |T \tilde{\varphi}_h|_{H^1(\mathcal{T}_h \setminus \mathcal{T}_{j+1}(K))}^2,$$

and then

$$|T \tilde{\varphi}_h|_{H^1(\mathcal{T}_h \setminus \mathcal{T}_{j+1}(K))}^2 \leq \frac{9\alpha}{1 + 9\alpha} |T \tilde{\varphi}_h|_{H^1(\mathcal{T}_h \setminus \mathcal{T}_j(K))}^2 \leq e^{-\frac{j}{1+9\alpha}} |T \tilde{\varphi}_h|_{H^1(\mathcal{T}_h \setminus \mathcal{T}_{j+1}(K))}^2,$$

and the theorem follows. □

Remark 1. The $\alpha$ in this paper, defined in Lemma 5, is estimated as the worst case scenario. For particular cases of coefficients $A$ and $\rho$, sharper estimates for $\alpha$ can be derived using weighted Poincaré inequalities techniques and partitions of unity that conform with $A$ in order to avoid large energies on the interior extensions [16, 17, 32, 54, 59, 66]; see [31, 60] for examples.

Inspired by the exponential decay stated in Corollary 6 we define the operator $P^j$ as follows. First, for a fixed $K \in \mathcal{T}_H$, let

$$\tilde{\Lambda}^{K,j}_h = \{ \tilde{\mu}_h \in \tilde{\Lambda}_h : T \tilde{\mu}_h = 0 \text{ on } \mathcal{T}_h \setminus \mathcal{T}_j(K) \}.$$

Given $\mu_h \in \Lambda_h$, define then $P^{K,j} \mu_h \in \tilde{\Lambda}^{K,j}_h$ such that

$$s(P^{K,j} \mu_h, \tilde{\mu}_h) = s_K(\mu_h, \tilde{\mu}_h) \quad \text{for all } \tilde{\mu}_h \in \tilde{\Lambda}^{K,j}_h,$$

and let

$$P^j \mu_h = \sum_{K \in \mathcal{T}_H} P^{K,j} \mu_h. \quad (14)$$

We define the approximation $\lambda_h^j \in \Lambda_H$ of $\lambda_H$ by

$$s((I - P^j) \lambda_H^j, (I - P^j) \mu_H) = (\rho g, T(I - P^j) \mu_H) \quad \text{for all } \mu_H \in \Lambda_H, \quad (15)$$

and then let $\lambda_h^{ms,j} = (I - P^j) \lambda_H^j$ and $u_h^{ms,j} = T \lambda_h^{ms,j}$. We name the scheme as ACMS–LOD (Approximated Component Mode Synthesis Localized Orthogonal Decomposition) method.
We now analyze the approximation error of the method, starting by a technical result essential to obtain the final estimate. Let \( c_\gamma \) be a constant depending only on the shape regularity of \( \mathcal{T}_H \) such that

\[
\sum_{\tau \in \mathcal{T}_H} |v|^2_{H^1(\tau)} \leq (c_\gamma j)^2 |v|^2_{H^1(\mathcal{T}_H)},
\]

for all \( v \in H^1(\mathcal{T}_H) \).

Lemma 7. Consider \( \nu_h \in \Lambda_h \) and the operators \( P \) defined by (8) and \( P^j \) by (14) for \( j > 1 \). Then

\[
|T(P - P^j)\nu_h|^2_{H^1_A(\mathcal{T}_H)} \leq (9c_\gamma j\alpha)^2 e^{-\frac{j^2}{1+9\alpha}} |T\nu_h|^2_{H^1_A(\mathcal{T}_H)}.
\]

Proof. Let \( \tilde{\psi}_h = (P - P^j)\nu_h = \sum_{K \in \mathcal{T}_H} (P^K - P^K_j)\nu_h \). For each \( K \in \mathcal{T}_H \), let \( \tilde{\psi}_h^K \in \tilde{\Lambda}_h \) be such that \( \tilde{\psi}_h^K \mid_e = 0 \) if \( e \) is a face of an element of \( \mathcal{T}_j(K) \) and \( \tilde{\psi}_h^K \mid_e = \tilde{\psi}_h \mid_e \), otherwise. We obtain

\[
|T\tilde{\psi}_h|^2_{H^1_A(\mathcal{T}_H)} = \sum_{K \in \mathcal{T}_H} \sum_{\tau \in \mathcal{T}_H} s_\tau(\tilde{\psi}_h - \tilde{\psi}_h^K, (P^K - P^K_j)\nu_h) + s_\tau(\tilde{\psi}_h^K, (P^K - P^K_j)\nu_h).
\]

See that the second term of (17) vanishes since

\[
\sum_{\tau \in \mathcal{T}_H} s_\tau(\tilde{\psi}_h^K, (P^K - P^K_j)\nu_h)_{\partial \tau} = \sum_{\tau \in \mathcal{T}_H} s_\tau(\tilde{\psi}_h^K, P^K\nu_h)_{\partial \tau} = 0.
\]

For the first term of (17), as in Lemma 3

\[
\sum_{\tau \in \mathcal{T}_H} s_\tau(\tilde{\psi}_h - \tilde{\psi}_h^K, (P^K - P^K_j)\nu_h)_{\partial \tau} \leq \sum_{\tau \in \mathcal{T}_{j+1}(K)} |T(\tilde{\psi}_h - \tilde{\psi}_h^K)|_{H^1_A(\tau)} |T(P^K - P^K_j)\nu_h|_{H^1_A(\tau)}
\]

\[
\leq 3\alpha^{1/2}|T\tilde{\psi}_h|^2_{H^1_A(\mathcal{T}_{j+1}(K))} |T(P^K - P^K_j)\nu_h|^2_{H^1_A(\mathcal{T}_{j+1}(K))}.
\]

Let \( \nu_h^K \in \tilde{\Lambda}_h^K \) be equal to zero on all faces of elements of \( \mathcal{T}_H \setminus \mathcal{T}_j(K) \) and equal to \( P^K \nu_h \) otherwise. Using Galerkin best approximation property and Corollary 6 we obtain

\[
|T(P^K - P^K_j)\nu_h|^2_{H^1_A(\mathcal{T}_{j+1}(K))} \leq |T(P^K - P^K_j)\nu_h|^2_{H^1_A(\mathcal{T}_H)} \leq |T(P^K \nu_h - \nu_h^K)|^2_{H^1_A(\mathcal{T}_H)}
\]

\[
\leq 9\alpha |TP^K \nu_h|^2_{H^1_A(\mathcal{T}_H \setminus \mathcal{T}_{j+1}(K))} \leq 9\alpha e^{-\frac{j^2}{1+9\alpha}} |TP^K \nu_h|^2_{H^1_A(\mathcal{T}_H)}.
\]

We gather the above results to obtain

\[
|T\tilde{\psi}_h|^2_{H^1_A(\mathcal{T}_H)} \leq 9\alpha e^{-\frac{j^2}{2(1+9\alpha)}} \sum_{K \in \mathcal{T}_H} |T\tilde{\psi}_h|_{H^1_A(\mathcal{T}_{j+1}(K))} |TP^K \nu_h|_{H^1_A(\mathcal{T}_H)}
\]

\[
\leq 9\alpha e^{-\frac{j^2}{2(1+9\alpha)}} c_\gamma j |T\tilde{\psi}_h|_{H^1_A(\mathcal{T}_H)} \left( \sum_{K \in \mathcal{T}_H} |TP^K \nu_h|^2_{H^1_A(\mathcal{T}_H)} \right)^{1/2}.
\]
We finally gather that
\[ |TP^K \nu_h|_{H^1_{\Lambda}(T_H)}^2 = s(P^K \nu_h, P^K \nu_h)_{\partial T_H} = s_K(P^K \nu_h, \nu_h) = \int_K A \nabla (TP^K \nu_h) \cdot \nabla \nu_h \, dx \]
and from Cauchy–Schwarz, \( |TP^K \nu_h|_{H^1_{\Lambda}(T_H)} \leq |\nu_h|_{H^1_{\Lambda}(T_H)} \), we have
\[
\sum_{K \in T_H} |TP^K \nu_h|_{H^1_{\Lambda}(T_H)}^2 \leq |\nu_h|_{H^1_{\Lambda}(T_H)}^2.
\]

\[ \square \]

**Theorem 8.** Define \( u_h^H \) by (3) and let \( u_h^{ms,j} = (I - P^j) \lambda^j_H \), where \( \lambda^j_H \) is as in (15). Then
\[ |u_h^H - u_h^{ms,j}|_{H^1_{\Lambda}(T_H)} \leq H \left( c_{P,L} + [\gamma_2 \kappa (1 + \log(H/h))]^{1/2} \right) 2 \frac{9 \alpha}{\pi^2} e^{-\frac{(\log(c_{P,G}/H))^2}{10}} \|g\|_{L^2(\Omega)}.
\]

**Proof.** First, from the triangle inequality,
\[
|u_h^H - u_h^{ms,j}|_{H^1_{\Lambda}(T_H)} \leq |u_h^H - u_h^{ms}|_{H^1_{\Lambda}(T_H)} + |u_h^{ms} - u_h^{ms,j}|_{H^1_{\Lambda}(T_H)},
\]
and for the first term we use Theorem 4. For the second term, we first define \( \tilde{u}_h^{ms,j} = \sum_i \lambda^{ms}_h(x_i) T(I - P^j) \theta^j_H \), and then
\[
|u_h^{ms} - \tilde{u}_h^{ms,j}| = (P - P^j) \sum_i \lambda^{ms}_h(x_i) T \theta^j_H = T(P - P^j) T I H \lambda^{ms}_h,
\]
where \( I_H \) is as in Lemma 2. Relying on the Galerkin best approximation we gather from Lemma 7 that
\[
|u_h^{ms} - \tilde{u}_h^{ms,j}|^2_{H^1_{\Lambda}(T_H)} \leq (c_{\gamma j})^2 (9 \alpha)^2 e^{-\frac{(\log(c_{P,G}/H))^2}{10}} |TI_H \lambda^{ms}_h|^2_{H^1_{\Lambda}(T_H)}.
\]
Since \( u_h^{ms} = T \lambda^{ms}_h \) the result follow from Lemma 2 and the global Poincaré’s inequality (10).
\[ \square \]

4. The High-Contrast Multiscale Case

The main bottle-neck in dealing with high-contrast coefficients is that \( \alpha \) becomes too large, therefore \( j \) has to be large as well, cf. Theorems 1 and 8. Furthermore, the large local Poincaré inequality constant \( c_{P,L} \) deteriorates the a priori error estimate in Theorem 4.

Also, we would like to remove the \((1 + \log(H/h)) \) term that appears in these estimates due to the mismatch between \( H^{1/2}(e) \) and \( H^{0/2}(e) \). To deal with these issues, we replace \( \tilde{\Lambda}_h \) by a subspace \( \Lambda_{\Lambda} \subset \tilde{\Lambda}_h \) by removing a subspace spanned by some eigenfunctions associated to an appropriated generalized eigenvalue problem, on each edge of the mesh \( T_H \). We first introduce some notation.

Given an edge \( e \) of an elements \( \tau \in T_H \), let \( \tilde{\Lambda}_h = \tilde{\Lambda}_h|_e \) and \( \tilde{\Lambda}_h = \tilde{\Lambda}_h|_{\partial \tau} \) be the restrictions of functions on \( \tilde{\Lambda}_h \) to \( e \) and on \( \tilde{\Lambda}_h \) to \( \partial \tau \). Since \( \tilde{\mu}_e^e \in \tilde{\Lambda}_h \) vanishes at the end-points of \( e \), it...
is possible to continuously extend it by zero for all nodes \( x_i \in \mathcal{N}_{\partial \mathcal{R}} \) := \((\mathcal{N}_h \setminus \mathcal{N}_H) \cap (\partial \mathcal{R} \setminus e)\). Let \( R_{e,\tau}^T : \tilde{\Lambda}_h^r \to \tilde{\Lambda}_h^r \) be such extension. Conversely, we define the restriction operator \( R_{e,\tau} : \tilde{\Lambda}_h^r \to \tilde{\Lambda}_h^r \), such that \( R_{e,\tau} \nu_h(x_i) = \nu_h(x_i) \) for all nodes \( x_i \in \mathcal{N}_e := (\mathcal{N}_h \setminus \mathcal{N}_H) \cap e \).

Denote by \((\cdot, \cdot)_e\) the \(L^2(e)\) inner product and define \( S^\tau : \tilde{\Lambda}_h^r \to \left( t\Lambda^\tau \right)', \) where \( \left( \tilde{\Lambda}_h^r \right)' \) is the dual space of \( \tilde{\Lambda}_h^r \), such that

\[
(\mu_h^r, S^\tau \nu_h^r)_{\partial \mathcal{R}} = \int_{\mathcal{R}} A \nabla T \mu_h^r \cdot \nabla T \nu_h^r \, dx \quad \text{for all } \mu_h^r, \nu_h^r \in \tilde{\Lambda}_h^r.
\]

Also let \( S_{ee}^\tau : \tilde{\Lambda}_h^r \to \left( \tilde{\Lambda}_h^r \right)' \) be such that

\[
(\bar{\mu}_h^e, S_{ee}^\tau \bar{\nu}_h^e)_{e} = (R_{e,\tau}^T \bar{\mu}_h^e, S^\tau R_{e,\tau}^T \bar{\nu}_h^e)_{\partial \mathcal{R}} \quad \text{for all } \bar{\mu}_h^e, \bar{\nu}_h^e \in \tilde{\Lambda}_h^e.
\]

Similarly we define \( S_{e'\tau e}^\tau, S_{ee}^\tau \) and \( S_{e'\tau e}^\tau \), related to the degrees of freedom on \( e^c = \mathcal{N}_{\partial \mathcal{R} \setminus e} \).

Let us introduce \( M_{ee}^\tau \) by

\[
(\bar{\mu}_h^e, M_{ee}^\tau \bar{\nu}_h^e)_e = \int_{\mathcal{R}} \rho (T R_{e,\tau}^T \bar{\mu}_h^e) (T R_{e,\tau}^T \bar{\nu}_h^e) \, dx
\]

and define \( \bar{S}_{ee}^\tau = \mathcal{K}^{-2} M_{ee}^\tau + S_{ee}^\tau \), where \( \mathcal{K} \) is the target precision of the method, that can be set by the user.

We finally consider the Schur complement

\[
\bar{S}_{ee}^\tau = S_{ee}^\tau - S_{ee}^\tau (S_{e'\tau e}^\tau)^{-1} S_{e'\tau e}^\tau,
\]

and then

\[(18) \quad (\bar{\nu}_h^e, \bar{S}_{ee}^\tau \bar{\nu}_h^e) \leq (\nu_h, S^\tau \nu_h) \quad \text{for all } \nu_h \in \tilde{\Lambda}_h^r \text{ such that } R_{e,\tau} \nu_h = \bar{\nu}_h^e.\]

See [45] for a similar computation.

We are ready then to define a generalized eigenvalue problem that takes into account high contrast coefficients. For a given edge \( e \) shared by elements \( \tau \) and \( \tau' \), find eigenpairs \((\alpha_i^e, \tilde{\psi}_{h,i}^e) \in (\mathbb{R}, \tilde{\Lambda}_h^e)\), where \( \alpha_1^e \geq \alpha_2^e \geq \alpha_3^e \geq \cdots \geq \alpha_{\mathcal{N}_e}^e > 1 \), such that

\[(19) \quad (\bar{S}_{ee}^\tau + \bar{S}_{ee}^e) \tilde{\psi}_{h,i}^e = \alpha_i^e (\bar{S}_{ee}^\tau + \bar{S}_{ee}^e) \tilde{\psi}_{h,i}^e.
\]

We impose that the eigenfunctions \( \tilde{\mu}_{h,i}^e \) are orthonormal with respect to \((\cdot, (\bar{S}_{ee}^\tau + \bar{S}_{ee}^e)' \cdot)\).

Now we decompose \( \tilde{\Lambda}_h^e := \tilde{\Lambda}_h^{e,\Delta} \oplus \tilde{\Lambda}_h^{e,\Pi} \) where for a given \( \alpha_{\text{stab}} > 1 \),

\[
(20) \quad \tilde{\Lambda}_h^{e,\Delta} := \text{span}\{ \tilde{\mu}_{h,i}^e : \alpha_i^e < \alpha_{\text{stab}} \}, \quad \tilde{\Lambda}_h^{e,\Pi} := \text{span}\{ \tilde{\mu}_{h,i}^e : \alpha_i^e \geq \alpha_{\text{stab}} \}.
\]

We remark that \( \alpha_{\text{stab}} \) is chosen by the user and replaces \( \alpha \) in the proof of Lemma [5] the counterpart of Lemma [5].
To define our ACMS–NLSD (Approximated Component Mode Synthesis Non-Localized Spectral Decomposition) method for high-contrast coefficients, let

\[
\tilde{\Lambda}_h^\Pi = \{ \tilde{\mu}_h \in \tilde{\Lambda}_h : \tilde{\mu}_h|_e \in \tilde{\Lambda}_h^e | e \in \partial T_H \}, \\
\tilde{\Lambda}_h^\Delta = \{ \tilde{\mu}_h \in \tilde{\Lambda}_h : \tilde{\mu}_h|_e \in \tilde{\Lambda}_h^e | e \in \partial T_H \}.
\]

Note that \( \Lambda_h = \Lambda_h^\Pi \oplus \tilde{\Lambda}_h^\Delta \), where

\[
\Lambda_h^\Pi = \Lambda_h^0 \oplus \tilde{\Lambda}_h^\Pi
\]

and \( \Lambda_h^0 \) is the set of functions on \( \Lambda_h \) which vanish on all nodes of \( \mathcal{N}_h \setminus \mathcal{N}_H \). Denote

\[
(v_h, S \mu_h)_{\partial T_H} = \sum_{\tau \in T_H} (v_h^\tau, S^\tau \mu_h^\tau)_{\partial \tau}.
\]

We now introduce the ACMS–NLSD multiscale functions. For \( \tau \in T_H \), consider the operators \( P^{\tau,\Delta}, P^\Delta : \Lambda_h \to \tilde{\Lambda}_h^\Delta \) as follows: Given \( \mu_h \in \Lambda_h \), find \( P^{\tau,\Delta} \mu_h \in \tilde{\Lambda}_h^\Delta \) and define \( P^\Delta \) such that

\[
(\tilde{v}_h^\Delta, S P^{\tau,\Delta} \mu_h)_{\partial T_H} = (\tilde{v}_h^\Delta, S^\tau \mu_h^\tau)_{\partial \tau} \quad \text{for all} \quad \tilde{v}_h^\Delta \in \tilde{\Lambda}_h^\Delta, \quad P^\Delta = \sum_{\tau \in T_H} P^{\tau,\Delta}.
\]

Consider \( \Lambda_h^{ms,\Pi} = (I - P^\Delta)\Lambda_h^\Pi \). The ACMS–NLSD method is defined by: Find \( \lambda_h^{ms,\Pi} \in \Lambda_h^{ms,\Pi} \) such that

\[
(v_h^{ms,\Pi}, S \lambda_h^{ms,\Pi})_{\partial T_H} = (\rho g, T v_h^{ms,\Pi}) \quad \text{for all} \quad v_h^{ms,\Pi} \in \Lambda_h^{ms,\Pi}.
\]

Note that

\[
(v_h^{ms,\Pi}, S \lambda_h^{ms,\Pi})_{\partial T_H} = \int_\Omega A \nabla T v_h^{ms,\Pi} \cdot \nabla T \lambda_h^{ms,\Pi} \, dx = \int_\Omega \rho g T v_h^{ms,\Pi} \, dx.
\]

**Remark 2.** A similar approach was followed by [32, 33], where different local eigenvalue problems are introduced to construct the approximation spaces. The analysis of the method however requires extra regularity of the coefficients, and the error estimate is not robust with respect to contrast.

The counterpart of Lemma 1 follows.

**Lemma 1’.** Let \( \tilde{\mu}_h^\Delta \in \tilde{\Lambda}_h^\Delta \). Then

\[
\| T \tilde{\mu}_h^\Delta \|_{L^2(\Omega)} \leq (9 \alpha_{\text{stab}})^{1/2} \mathcal{K} | T \tilde{\mu}_h^\Delta |_{H_0^1(\Omega)}.
\]
Proof. We have for $\tau \in T_H$, 
\[ \mathcal{H}^{-2} \| T \tilde{\mu}_h^\triangle \|^2_{L^2_\tau} \leq 3 \mathcal{H}^{-2} \sum_{e \subset \partial \tau} \| T R^T_{e,\tau} \tilde{\mu}_h^\triangle \|^2_{L^2_\tau}. \]

Fixing the edge $e$ of both $\tau$ and $\tau'$, we have 
\[ \mathcal{H}^{-2} \| T R^T_{e,\tau} \tilde{\mu}_h^\triangle \|^2_{L^2_\tau} + \mathcal{H}^{-2} \| T R^T_{e,\tau'} \tilde{\mu}_h^\triangle \|^2_{L^2_\tau} \leq (\tilde{\mu}_h^\triangle, \tilde{\tilde{\tau}}_e e \tilde{\mu}_h^\triangle) + (\tilde{\mu}_h^\triangle, \tilde{\tilde{\tau}}_{ee} e \tilde{\mu}_h^\triangle) \]
\[ \leq \alpha_{\text{stab}}(\tilde{\mu}_h^\triangle, (\tilde{\tilde{\tau}}_e + \tilde{\tilde{\tau}}_{ee}) e \tilde{\mu}_h^\triangle) \leq \alpha_{\text{stab}}(\|T \tilde{\mu}_h^\triangle\|^2_{H^1_\Lambda(\tau)} + \|T \tilde{\mu}_h^\triangle\|^2_{H^1_\Lambda(\tau')}) \]
from (19), (20) and (18). By adding all $\tau \in T_H$, the results follows. \hfill \square

Note that we added $\mathcal{H}^{-2} M_{ee}$ to define $\tilde{\tilde{\tau}}_{ee}$. This is necessary otherwise we might have a few modes that would make the local Poincaré’s inequality constant in (24) too large.

Now we concentrate on the counterpart of Lemma 2.

Lemma 2’. Let $\mu_h \in \Lambda_h$ and let $\mu_h = \mu_h^\Pi + \tilde{\mu}_h^\triangle$. Then 
\[ |T \mu_h^\Pi|_{H^1_\Lambda(\Omega)} \leq (2 + 18 \alpha_{\text{stab}})^{1/2} |T \mu_h|_{H^1_\Lambda(\Omega)}. \]

Proof. We have 
\[ |T \mu_h^\Pi|_{H^1_\Lambda(\Omega)}^2 \leq 2 |T \mu_h|_{H^1_\Lambda(\Omega)}^2 + |T \tilde{\mu}_h^\triangle|_{H^1_\Lambda(\Omega)}^2. \]

Consider the decomposition 
\[ \mu_h|_\tau = \mu_h^{0,\triangle} + \sum_{e \subset \partial \tau} \tilde{\mu}_h^{e,\triangle} \]
where $\mu_h^{0,\triangle} \in \Lambda^0_h$ and $\tilde{\mu}_h^{e,\triangle} = \tilde{\mu}_h^{e,\Pi} + \tilde{\mu}_h^{e,\triangle}$. Then 
\[ |T \mu_h^{e,\triangle}|_{H^1_\Lambda(\tau)}^2 \leq 3 \sum_{e \subset \partial \tau} |T R^T_{e,\tau} \mu_h^{e,\triangle}|_{H^1_\Lambda(\tau)}^2 = 3 \sum_{e \subset \partial \tau} (\mu_h^{e,\triangle}, (\tilde{\tilde{\tau}}_{ee} + \tilde{\tilde{\tau}}_{ee}) \mu_h^{e,\triangle})_e. \]

Now we use that, if $e$ is an edge of both $\tau$ and $\tau'$, 
\[ (\mu_h^{e,\triangle}, (\tilde{\tilde{\tau}}_{ee} + \tilde{\tilde{\tau}}_{ee}) \mu_h^{e,\triangle})_e \leq \alpha_{\text{stab}} (\mu_h^{e,\triangle}, (\tilde{\tilde{\tau}}_{ee} + \tilde{\tilde{\tau}}_{ee}) \mu_h^{e,\triangle})_e. \]

In addition, due to the orthogonality condition of the spaces $\Lambda^e_h$ and $\Lambda^{e,\Pi}_h$ with respect to the inner product $(\cdot, (\tilde{\tilde{\tau}}_{ee} + \tilde{\tilde{\tau}}_{ee}) \cdot)_e$, and (18), we have 
\[ (\mu_h^{e,\triangle}, (\tilde{\tilde{\tau}}_{ee} + \tilde{\tilde{\tau}}_{ee}) \mu_h^{e,\triangle})_e \leq (\mu_h^{e,\triangle}, (\tilde{\tilde{\tau}}_{ee} + \tilde{\tilde{\tau}}_{ee}) \mu_h^{e,\triangle})_e \leq |T \mu_h^{e,\Pi}|_{H^1_\Lambda(\tau)}^2 + |T \mu_h^{e,\Pi}|_{H^1_\Lambda(\tau')}^2. \]

Adding all terms together we obtain the result. \hfill \square

We now state the counterpart of the Face Lemma [69, Subsection 4.6.3]. The lemma follows directly from the definition of the generalized eigenvalue problem and properties of $\Lambda^{e,\triangle}_h$ and (18).
Lemma 3'. Let $e$ be a common edge of $\tau$, $\tau' \in \mathcal{T}_H$, and $\widetilde{\mu}^\triangle_h \in \widetilde{\Lambda}^\triangle_h$. Then, defining $\bar{\mu}^\triangle_h = R_{e,\tau}\widetilde{\mu}^\triangle_h|_\partial_T$ and $\bar{\mu}^\triangle_h = \bar{\mu}^\triangle_h|_\partial_T$ it follows that

$$|TR_{e,\tau}^T\bar{\mu}^\triangle_h|^2_{H^1(\tau)} + |TR_{e,\tau'}\bar{\mu}^\triangle_h|^2_{H^1(\tau')} \leq \alpha_{\text{stab}}(|T\bar{\mu}^\triangle_h|^2_{H^1(\tau)} + |T\bar{\mu}^\triangle_h|^2_{H^1(\tau')}).$$

Proof. We have

$$|TR_{e,\tau}^T\bar{\mu}^\triangle_h|^2_{H^1(\tau)} + |TR_{e,\tau'}\bar{\mu}^\triangle_h|^2_{H^1(\tau')} = (\bar{\mu}^\triangle_h, S_{ee}\bar{\mu}^\triangle_h)_{\partial_T} + (\bar{\mu}^\triangle_h, S_{ee}\bar{\mu}^\triangle_h)_{\partial_T'} \leq \alpha_{\text{stab}}(\bar{\mu}^\triangle_h, S\bar{\mu}^\triangle_h)_{\partial_T} + (\bar{\mu}^\triangle_h, S\bar{\mu}^\triangle_h)_{\partial_T'}.$$ 

$\square$

Theorem 4'. Let $\lambda_h = u_h|_{\partial\mathcal{T}_H}$, and $\lambda_{h,\text{ms}}$ solution of (23). Then $\lambda_h - \lambda_{h,\text{ms}} \in \widetilde{\Lambda}^\triangle_h$ and

$$|u_h - u_{h,\text{ms}}|^2_{H^1(\Omega)} \leq 9\alpha_{\text{stab}}\mathcal{H}^2\|g\|^2_{L^2(\Omega)}.$$

Proof. First note that $\lambda_h - \lambda_{h,\text{ms}} \in \widetilde{\Lambda}^\triangle_h$ since it follows from the Galerkin orthogonality that $s(\lambda_h - \lambda_{h,\text{ms}}, p_{h,\text{ms}}) = 0$ for all $p_{h,\text{ms}} \in \Lambda_{h,\text{ms}}$. Using Lemma 3 we obtain

$$|u_h - u_{h,\text{ms}}|^2_{H^1(\Omega)} = (\alpha_{\text{stab}})^{1/2}\mathcal{H}^2\|g\|_{L^2(\Omega)}|T(\lambda_h - \lambda_{h,\text{ms}})|_{H^1(\Omega)},$$

and the result follows. $\square$

4.1. Decaying High-Contrast. We next prove exponential decay of $P^K,\triangle v_h$ for $K \in \mathcal{T}_H$.

Lemma 5'. Let $\mu_h \in \Lambda_h$ and let $\bar{\phi}^\triangle_h = P^K,\triangle \mu_h$ for some fixed element $K \in \mathcal{T}_H$. Then, for any integer $j \geq 1$,

$$|T\bar{\phi}^\triangle_h|^2_{H^1(\Omega \setminus \mathcal{T}_{j+1}(K))} \leq 9\alpha_{\text{stab}}|T\bar{\phi}^\triangle_h|^2_{H^1(\Omega \setminus \mathcal{T}_j(K))}.$$ 

Proof. Following the steps of the proof of Lemma 3 we gather that

$$|T\bar{\phi}^\triangle_h|^2_{H^1(\Omega \setminus \mathcal{T}_j(K))} \leq \sum_{\tau \in \mathcal{T}_{j+1}(K) \setminus \mathcal{N}_j(K)} |T\bar{v}_h|^2_{H^1(\tau)} + \sum_{\tau' \in \mathcal{T}_{j+1}(K) \setminus \mathcal{N}_j(K)} |T\bar{\phi}^\triangle_h|^2_{H^1(\tau)},$$

where $\bar{v}_h \in \widetilde{\Lambda}^\triangle_h$ is such that $\bar{v}_h|_{\partial_T} = \bar{\phi}^\triangle_h$ if $\tau \in \mathcal{T}_H \setminus \mathcal{T}_{j+1}(K)$, and $\bar{v}_h = 0$ on the remaining edges. If $e$ is an edge of $\partial\tau$ and $\partial\tau'$, and $\chi$ the characteristic function of $e$, for $\tau \in \mathcal{T}_{j+1}(K) \setminus \mathcal{N}_j(K)$ and $\tau' \in \mathcal{T}_j(K) \setminus \mathcal{T}_{j-1}(K)$, then, for $\bar{\mu}^\triangle_h = \bar{\mu}^\triangle_h|_e$, 

$$|T(\bar{v}_h)|^2_{H^1(\tau)} \leq 3 \sum_{e \in \partial\tau} |T(\chi_e \bar{\mu}_h)|^2_{H^1(\tau)}$$

and

$$|T(\chi_e \bar{\mu}_h)|^2_{H^1(\tau)} = \alpha_{\text{stab}}(\bar{\mu}^\triangle_h, S_{ee}\bar{\mu}^\triangle_h)|e| \leq \alpha_{\text{stab}}((\bar{\mu}^\triangle_h, S_{ee}\bar{\mu}^\triangle_h)|e| \leq \alpha_{\text{stab}}(\bar{\mu}^\triangle_h, S_{ee}\bar{\mu}^\triangle_h)_{\partial_T} + (\bar{\mu}^\triangle_h, S_{ee}\bar{\mu}^\triangle_h)_{\partial_T'} = \alpha_{\text{stab}}(|T\bar{\mu}^\triangle_h|^2_{H^1(\tau)} + |T\bar{\mu}^\triangle_h|^2_{H^1(\tau')}),$$

$\square$
where we have used \([18]\).

Note that now the bound is in terms of \(\mathcal{T}_{j+2}(K) \setminus \mathcal{T}_j(K)\) rather than \(\mathcal{T}_{j+1}(K) \setminus \mathcal{T}_j(K)\). This means that the \(j\) in Corollary \([3]\) is replaced below by the integer part of \((j + 1)/2\).

**Corollary 6’.** Assume that \(K \in \mathcal{T}_H\) and \(\nu_h \in \Lambda_h\) and let \(\tilde{\phi}_h^\Delta = P^K_{\Delta} \nu_h \in \tilde{\Lambda}_h^\Delta\). Then, for any integer \(j \geq 1\),

\[
|T \tilde{\phi}_h^\Delta|_{H^1_A(\mathcal{T}_H \setminus \mathcal{T}_j+1(K))}^2 \leq e^{-\frac{(j+1)/2}{1+9\alpha_{\text{stab}}}} |T \tilde{\phi}_h^\Delta|_{H^1_A(\mathcal{T}_H \setminus \mathcal{T}_j(K))}^2.
\]

where \([s]\) is the integer part of \(s\).

**Proof.** Using Lemma \([5]\) we have

\[
|T \tilde{\phi}_h^\Delta|_{H^1_A(\mathcal{T}_H \setminus \mathcal{T}_j+1(K))}^2 \leq |T \tilde{\phi}_h^\Delta|_{H^1_A(\mathcal{T}_H \setminus \mathcal{T}_j(K))}^2 \\
\leq 9\alpha_{\text{stab}} |T \tilde{\phi}_h^\Delta|_{H^1_A(\mathcal{T}_H \setminus \mathcal{T}_j-1(K))}^2 - 9\alpha_{\text{stab}} |T \tilde{\phi}_h^\Delta|_{H^1_A(\mathcal{T}_H \setminus \mathcal{T}_j+1(K))}^2,
\]

and then

\[
|T \tilde{\phi}_h^\Delta|_{H^1_A(\mathcal{T}_H \setminus \mathcal{T}_j+1(K))}^2 \leq \frac{9\alpha_{\text{stab}}}{1+9\alpha_{\text{stab}}} |T \tilde{\phi}_h^\Delta|_{H^1_A(\mathcal{T}_H \setminus \mathcal{T}_j-1(K))}^2 \leq e^{-\frac{1}{1+9\alpha_{\text{stab}}}} |T \tilde{\phi}_h^\Delta|_{H^1_A(\mathcal{T}_H \setminus \mathcal{T}_j-1(K))}^2.
\]

Inspired by the exponential decay stated in Corollary \([6]\) we define the operator \(P_{\Delta,j}\) as follows. First, for a fixed \(K \in \mathcal{T}_H\), let

\[
\tilde{\Lambda}_{h,j}^\Delta = \{ \tilde{\mu}_h \in \tilde{\Lambda}_h^\Delta : T \tilde{\mu}_h = 0 \text{ on } \mathcal{T}_H \setminus \mathcal{T}_j(K) \}.
\]

For \(\mu_h \in \Lambda_h\), define \(P_{\Delta,j} \mu_h \in \tilde{\Lambda}_{h,j}^\Delta\) such that

\[
s(P_{\Delta,j} \mu_h, \tilde{\mu}_h) = s_K(\mu_h, \tilde{\mu}_h) \text{ for all } \tilde{\mu}_h \in \tilde{\Lambda}_{h,j}^\Delta,
\]

and let

(25)

\[
P_{\Delta,j} \mu_h = \sum_{K \in \mathcal{T}_H} P_{\Delta,j} \mu_h.
\]

Finally, define the approximation \(\lambda_{H}^{\Pi,j} \in \Lambda_{H}^{\Pi}\) such that

(26) \(s((I - P_{\Delta,j}) \lambda_{H}^{\Pi,j}, (I - P_{\Delta,j}) \mu_{H}^{\Pi}) = (\rho g, T(I - P_{\Delta,j}) \mu_{H}^{\Pi}) \quad \text{for all } \mu_{H}^{\Pi} \in \Lambda_{H}^{\Pi},\)

and then let \(\lambda_{h}^{m,s,\Pi,j} = (I - P_{\Delta,j}) \lambda_{H}^{\Pi,j}\) and \(u_{h}^{m,s,\Pi,j} = T \lambda_{h}^{m,s,\Pi,j}\). We name as ACMS–LSD (Approximated Component Mode Synthesis Localized Spectral Decomposition) method.

We now analyze the approximation error of the method, starting by a technical result essential to obtain the final estimate.
Lemma 7'. Consider \( \nu_h \in \Lambda_h \) and the operators \( P^\Delta \) defined by (22) and \( P^\Delta,j \) by (23) for \( j > 1 \). Then
\[
|T(P^\Delta - P^\Delta,j)\nu_h|_{H_\Delta^1(\mathcal{T}_H)}^2 \leq (c_\gamma,j)^2(9\alpha_{\text{stab}})^2 e^{-(j-1)/2} |T\nu_h|_{H_\Delta^1(\mathcal{T}_H)},
\]
where \( c_\gamma \) is as in (16).

Proof. Let \( \tilde{\psi}_h^\Delta = (P^\Delta - P^\Delta,j)\nu_h = \sum_{K \in \mathcal{T}_H} (P^K\Delta - P^K\Delta,j)\nu_h \). For each \( K \in \mathcal{T}_H \), let \( \tilde{\psi}_h^{K,\Delta} \in \tilde{\Lambda}_h \) be such that \( \tilde{\psi}_h^{K,\Delta}|_e = 0 \) if \( e \) is an edge of an element of \( \mathcal{T}_j(K) \) and \( \tilde{\psi}_h^{K,\Delta}|_e = \tilde{\psi}_h^\Delta|_e \), otherwise. We obtain
\[
(27)
|T\tilde{\psi}_h^\Delta|_{H_\Delta^1(\mathcal{T}_H)}^2 = \sum_{K \in \mathcal{T}_H} \sum_{T \in \mathcal{T}_H} s_\tau(\tilde{\psi}_h^\Delta - \tilde{\psi}_h^{K,\Delta}, (P^K\Delta - P^K\Delta,j)\nu_h) + s_\tau(\tilde{\psi}_h^{K,\Delta}, (P^K\Delta - P^K\Delta,j)\nu_h).
\]
See that the second term of (27) vanishes since
\[
\sum_{\tau \in \mathcal{T}_H} s_\tau(\tilde{\psi}_h^{K,\Delta}, (P^K\Delta - P^K\Delta,j)\nu_h)_{\partial \tau} = 0.
\]
For the first term of (17), as in Lemma 3
\[
\sum_{\tau \in \mathcal{T}_H} s_\tau(\tilde{\psi}_h^\Delta - \tilde{\psi}_h^{K,\Delta}, (P^K\Delta - P^K\Delta,j)\nu_h)_{\partial \tau}
\leq \sum_{\tau \in \mathcal{T}_{j+1}(K)} |T(\tilde{\psi}_h^\Delta - \tilde{\psi}_h^{K,\Delta})|_{H_\Delta^1(\tau)} |T(P^K\Delta - P^K\Delta,j)\nu_h|_{H_\Delta^1(\tau)}
\leq 3\alpha_{\text{stab}}^{1/2} |T\tilde{\psi}_h^\Delta|_{H_\Delta^1(\mathcal{T}_{j+1}(K))} |T(P^K\Delta - P^K\Delta,j)\nu_h|_{H_\Delta^1(\mathcal{T}_{j+1}(K))}.
\]
Let \( \nu_h^{K,\Delta,j} \in \tilde{\Lambda}_h^{K,\Delta,j} \) be equal to zero on all faces of elements of \( \mathcal{T}_H \setminus \mathcal{T}_j(K) \) and equal to \( P^K\Delta\nu_h \) otherwise. Using Galerkin best approximation property, Lemma 3 and Corollary 6, we obtain
\[
|T(P^K\Delta - P^K\Delta,j)\nu_h|_{H_\Delta^1(\mathcal{T}_{j+1}(K))} \leq |T(P^K\Delta - P^K\Delta,j)\nu_h|_{H_\Delta^1(\mathcal{T}_H)}
\leq |T(P^K\Delta\nu_h - P^K\Delta,j)\nu_h|_{H_\Delta^1(\mathcal{T}_H)} \leq 9\alpha_{\text{stab}} |TP^K\Delta\nu_h|_{H_\Delta^1(\mathcal{T}_H)}
\leq 9\alpha_{\text{stab}} e^{-(j-1)/2} |TP^K\Delta\nu_h|_{H_\Delta^1(\mathcal{T}_H)}.
\]
We gather the above results to obtain
\[
|T\tilde{\psi}_h^\Delta|_{H_\Delta^1(\mathcal{T}_H)}^2 \leq 9\alpha_{\text{stab}} e^{-(j-1)/2} |TP^K\Delta\nu_h|_{H_\Delta^1(\mathcal{T}_H)} \sum_{K \in \mathcal{T}_H} |T\tilde{\psi}_h^\Delta|_{H_\Delta^1(\mathcal{T}_{j+1}(K))} |TP^K\Delta\nu_h|_{H_\Delta^1(\mathcal{T}_H)}
\leq 9\alpha_{\text{stab}} e^{-(j-1)/2} c_{\gamma,j} |T\tilde{\psi}_h^\Delta|_{H_\Delta^1(\mathcal{T}_H)} \left( \sum_{K \in \mathcal{T}_H} |TP^K\Delta\nu_h|_{H_\Delta^1(\mathcal{T}_H)} \right)^{1/2}.
\]
We finally gather that

\[ |TP^{K,\Delta}v_h|^2_{H^1(\mathcal{T}_H)} = s(P^{K,\Delta}v_h, P^{K,\Delta}v_h)_{\partial\mathcal{T}_H} = s_K(P^{K,\Delta}v_h, v_h) \]

\[ = \int_K \mathcal{A} \nabla (TP^{K,\Delta}v_h) \cdot \nabla T v_h \, dx, \]

and from Cauchy–Schwarz, \( |TP^{K,\Delta}v_h|_{H^1(\mathcal{T}_H)} \leq |Tv_h|_{H^1(\mathcal{K})} \), we have

\[ \sum_{K \in \mathcal{T}_H} |TP^{K,\Delta}v_h|^2_{H^1(\mathcal{T}_H)} \leq |Tv_h|^2_{H^1(\mathcal{T}_H)}. \]

\[ \square \]

**Theorem 8’.** Define \( u_h^{32} \) by (3) and let \( u_h^{ms,\Pi,j} = T(I - P^{\Delta,j})\lambda_H^{\Pi,j} \), where \( \lambda_H^{\Pi,j} \) is as in (26). Then

\[ |u_h^\Pi - u_h^{ms,\Pi,j}|_{H^1(\mathcal{T}_H)} \leq \mathcal{H} \left( 3(\alpha_{\text{stab}})^{1/2} + c_j \varrho \alpha_{\text{stab}} e^{-\left(\frac{(j-1/2)}{2(1+\varrho\alpha_{\text{stab}})} - \log(c_{P,G}/\mathcal{H})\right)} \right) \|g\|_{L^2(\Omega)}. \]

**Proof.** First, from the triangle inequality,

\[ |u_h^\Pi - u_h^{ms,\Pi,j}|_{H^1(\mathcal{T}_H)} \leq |u_h^\Pi - u_h^{ms,\Pi}|_{H^1(\mathcal{T}_H)} + |u_h^{ms,\Pi} - u_h^{ms,\Pi,j}|_{H^1(\mathcal{T}_H)}, \]

and for the first term we use Theorem 4. For the second term, we define \( u_h^{ms,\Pi,j} = T(I - P^{\Delta,j})\lambda_h^{ms,\Pi} \), and then

\[ u_h^{ms} - u_h^{ms,\Pi,j} = T(P - P^j)\lambda_h^{ms,\Pi}. \]

Relying on the Galerkin best approximation we gather from Lemma 7 that

\[ |u_h^{ms} - u_h^{ms,\Pi,j}|^2_{H^1(\mathcal{T}_H)} \leq |u_h^{ms} - u_h^{ms,\Pi,j}|^2_{H^1(\mathcal{T}_H)} \leq (c_j)^2\varrho \alpha_{\text{stab}} e^{-\left(\frac{(j-1/2)}{2(1+\varrho\alpha_{\text{stab}})} - \log(c_{P,G}/\mathcal{H})\right)}|T\lambda_h^{ms,\Pi}|^2_{H^1(\mathcal{T}_H)}. \]

Since \( u_h^{ms} = T\lambda_h^{ms} \) the result follow from Lemma 1 and the global Poincaré’s inequality (10). \( \square \)

**Remark 3.** The localization required for the a priori error estimate of the previous theorem depends just on the logarithmic of the global Poincaré’s inequality \( c_{P,G} \) defined in (10). Furthermore, it is common on the literature to assume the global condition \( a_{\text{min}} \geq 1 \) and \( \rho = 1 \), which implies \( c_{P,G} = O(1) \). We can weaken this condition by choosing \( \rho(x) = a_-(x) \) for almost all \( x \in \Omega \) so that when the weighted Poincaré inequality holds, then \( c_{P,G} = O(1) \); a simple example of such a case is when \( \Omega \) is composed of several inclusions with small permeabilities and surrounded by a material with a large permeability.
Remark 4. The complexity of the proposed method depends on the number of eigenvalues of (19) below a threshold $\alpha_{stab}$ associated to each edge $e$. There are several works on the literature discussing this issue for similar eigenvalues problems \cite{26,70,71} and the number is related to the amount of channels of high permeability crossing the edge $e$. For the numerical experiments tested in \cite{44}, just one eigenvalue per edge was enough for $H = H$, $H = H/2$ and $H = H/4$ with $\alpha_{stab} = 1.5$

5. Spectral Multiscale Problems inside Substructures

Recall the decomposition $u_h = u_h^B + u_h^H$, and so far we derived an scheme that approximates $u_h^H$ only. From (3), we gather that $u_h^B$ is defined locally. Fixing an element $\tau \in T_H$, we introduce a multiscale method by first building the approximation space $V_{ms}^\tau := \text{Span}\{\psi_1^\tau, \psi_2^\tau, \ldots, \psi_N^\tau\}$ generated by the following generalized eigenvalue problem: Find the eigenpairs $(\alpha_i, \psi_i^\tau) \in (\mathbb{R}, V_h^B(\tau))$ such that

$$a_\tau(v_h, \psi_i^\tau) = \lambda_i(\rho v_h, \psi_i^\tau)_\tau \quad \text{for all } v_h \in V_h^B(\tau)$$

where

$$a_\tau(v_h, \psi_i^\tau) = \int_\tau A \nabla v_h \cdot \nabla \psi_i^\tau \, dx \quad \text{and} \quad (\rho v_h, \psi_i^\tau)_\tau = \int_\tau \rho v_h \psi_i^\tau \, dx$$

and $0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_{N_r} < 1/\mathcal{H}^2$ and $\lambda_{N_r+1} \geq 1/\mathcal{H}^2$. Again, $\mathcal{H}$ is the user defined target accuracy target. For instance $\mathcal{H} = H$ or $\mathcal{H} = h^r$, $0 < r \leq 1$. The local multiscale problem is defined by: Find $u_h^{B,ms} \in V_{ms}^\tau$ such that

$$a_\tau(u_h^{B,ms}, v_h) = (\rho g, v_h)_\tau \quad \text{for all } v_h \in V_h^{B,ms}.$$ 

It then follows that

$$|u_h^B - u_h^{B,ms}|_{H_\lambda^1(\tau)}^2 = (\rho g, u_h^B - u_h^{B,ms})_\tau \leq \mathcal{H}|u_h^B - u_h^{B,ms}|_{H_\lambda^1(\tau)} \|g\|_{L_\rho^2(\tau)}$$

and $|u_h^B - u_h^{B,ms}|_{H_\lambda^1(\tau)} \leq \mathcal{H}\|g\|_{L_\rho^2(\tau)}$. See \cite{43} Section 4 for a similar computation.

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