Abstract. We prove a necessary optimality condition of Euler–Lagrange type for quantum variational problems involving Hahn’s derivatives of higher-order.

1. Introduction

Many physical phenomena are described by equations involving nondifferentiable functions, e.g., generic trajectories of quantum mechanics [15]. Several different approaches to deal with nondifferentiable functions are followed in the literature of variational calculus, including the time scale approach, which typically deal with delta or nabla differentiable functions [14] [20] [23], the fractional approach, allowing to consider functions that have no first order derivative but have fractional derivatives of all orders less than one [3] [12] [16], and the quantum approach, which is particularly useful to model physical and economical systems [8] [11] [22].

Roughly speaking, a quantum calculus substitute the classical derivative by a difference operator, which allows to deal with sets of nondifferentiable functions. Several dialects of quantum calculus are available [13] [18]. For motivation to study a nondifferentiable quantum variational calculus we refer the reader to [4] [8] [10].

In 1949 Hahn introduced the difference operator $D_{q, \omega}$ defined by

$$D_{q, \omega} [f] (t) := \frac{f(qt + \omega) - f(t)}{(q - 1)t + \omega},$$

where $f$ is a real function, and $q \in (0, 1)$ and $\omega > 0$ are real fixed numbers [17]. The Hahn difference operator has been applied successfully in the construction of families of orthogonal polynomials as well as in approximation problems [6] [11] [25]. However, during 60 years, the construction of the proper inverse of Hahn’s difference operator remained an open question. Eventually, the problem was solved in 2009 by Aldwoah [1] (see also [2] [7]). Here we introduce the higher-order Hahn’s quantum variational calculus, proving the Hahn quantum analog of the higher-order Euler–Lagrange equation. As particular cases we obtain the $q$-calculus Euler–Lagrange equation [8] and the $h$-calculus Euler–Lagrange equation [9] [19].

Variational functionals that depend on higher derivatives arise in a natural way in applications of engineering, physics, and economics. Let us consider, for example, the equilibrium of an elastic bending beam. Let us denote by $y(x)$ the deflection...
of the point $x$ of the beam, $E(x)$ the elastic stiffness of the material, that can vary with $x$, and $\xi(x)$ the load that bends the beam. One may assume that, due to some constraints of physical nature, the dynamics does not depend on the usual derivative $y'(x)$ but on some quantum derivative $D_{q, \omega}[y](x)$. In this condition, the equilibrium of the beam correspond to the solution of the following higher-order Hahn's quantum variational problem:

$$\int_{0}^{L} \left[ \frac{1}{2} \left( E(x) D_{q, \omega}^{2}[y](x) \right)^{2} - \xi(x)y \left( q^{2} x + q \omega + \omega \right) \right] dx \rightarrow \min.$$  

Note that we recover the classical problem of the equilibrium of the elastic bending beam when $(\omega, q) \to (0, 1)$. Problem (1.1) is a particular case of the problem (P) investigated in Section 3. Our higher-order Hahn’s quantum Euler–Lagrange equation (Theorem 3.10) gives the main tool to solve such problems.

The paper is organized as follows. In Section 2 we summarize all the necessary definitions and properties of the Hahn difference operator and the associated $q, \omega$-integral. In Section 3 we formulate and prove our main results: in §3.1 we prove a higher-order fundamental Lemma of the calculus of variations with the Hahn operator (Lemma 3.8); in §3.2 we deduce a higher-order Euler–Lagrange equation for Hahn’s variational calculus (Theorem 3.10); finally we provide in §3.3 a simple example of a quantum optimization problem where our Theorem 3.10 leads to the global minimizer, which is not a continuous function.

2. Preliminaries

Let $q \in (0, 1)$ and $\omega > 0$. We introduce the real number

$$\omega_{0} := \frac{\omega}{1 - q}.$$  

Let $I$ be a real interval containing $\omega_{0}$. For a function $f$ defined on $I$, the Hahn difference operator of $f$ is given by

$$D_{q, \omega}[f](t) := \frac{f(qt + \omega) - f(t)}{(q - 1)t + \omega}, \quad \text{if } t \neq \omega_{0},$$  

and $D_{q, \omega}[f](\omega_{0}) := f'(\omega_{0})$, provided $f$ is differentiable at $\omega_{0}$. We sometimes call $D_{q, \omega}[f]$ the $q, \omega$-derivative of $f$, and $f$ is said to be $q, \omega$-differentiable on $I$ if $D_{q, \omega}[f](\omega_{0})$ exists.

Remark 2.1. The $D_{q, \omega}$ operator generalizes (in the limit) the forward $h$-difference and the Jackson $q$-difference operators [13, 18]. Indeed, when $q \to 1$ we obtain the forward $h$-difference

$$\Delta_{h}[f](t) := \frac{f(t + h) - f(t)}{h},$$  

when $\omega \to 0$ we obtain the Jackson $q$-difference operator

$$D_{q}[f](t) := \frac{f(qt) - f(t)}{t(q - 1)}, \quad \text{if } t \neq 0,$$

and $D_{q}[f](0) = f'(0)$ provided $f'(0)$ exists. Notice also that, under appropriate conditions,

$$\lim_{\omega \to 0, q \to 1} D_{q, \omega}[f](t) = f'(t).$$

The Hahn difference operator has the following properties:
Theorem 2.2 ([1] [2] [7]). Let \( f \) and \( g \) be \( q, \omega \)-differentiable on \( I \) and \( t \in I \). One has:

1. \( D_{q, \omega} f(t) \equiv 0 \) on \( I \) if and only if \( f \) is constant;
2. \( D_{q, \omega} (f + g)(t) = D_{q, \omega} f(t) + D_{q, \omega} g(t) \);
3. \( D_{q, \omega} f[t] = D_{q, \omega} f(t) g(t) + f(qt + \omega) D_{q, \omega} g(t) \);
4. \( D_{q, \omega} \left[ \frac{f}{g} \right](t) = \frac{D_{q, \omega} f(t) g(t) - f(t) D_{q, \omega} g(t)}{g(t) g(qt + \omega)} \) if \( g(t) g(qt + \omega) \neq 0 \);
5. \( f(qt + \omega) = f(t) + (t(q - 1) + \omega) D_{q, \omega} f(t) \).

For \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) define \( [k]_q := \frac{1 - q^k}{1 - q} \) and let \( \sigma(t) = qt + \omega, t \in I \). Note that \( \sigma \) is a contraction, \( \sigma(I) \subseteq I, \sigma(t) < t \) for \( t > \omega_0 \), \( \sigma(t) > t \) for \( t < \omega_0 \), and \( \sigma(\omega_0) = \omega_0 \). The following technical result is used several times in our paper:

Lemma 2.3 ([1] [7]). Let \( k \in \mathbb{N} \) and \( t \in I \). Then,

1. \( \sigma^k(t) = \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{k \text{-times}}(t) = q^k t + \omega \left[ k \right]_q \);
2. \( \left( \sigma^k(t) \right)^{-1} = \sigma^{-k}(t) = \frac{t - \omega \left[ k \right]_q}{q^k} \).

\( \sigma \) is defined on \( I \) and \( \omega \). Following [1] [2] [7] we define the notion of \( q, \omega \)-integral (also known as the Jackson–Nörlund integral) as follows:

Definition 2.4. Let \( a, b \in I \) and \( a < b \). For \( f : I \rightarrow \mathbb{R} \) the \( q, \omega \)-integral of \( f \) from \( a \) to \( b \) is given by

\[
\int_a^b f(t) \, d_{q, \omega} t := \int_{\omega_0}^b f(t) \, d_{q, \omega} t - \int_{\omega_0}^a f(t) \, d_{q, \omega} t,
\]

where

\[
\int_{\omega_0}^x f(t) \, d_{q, \omega} t := (x(1-q)-\omega) \sum_{k=0}^{+\infty} q^k f \left( xq^k + \omega \left[ k \right]_q \right), \quad x \in I,
\]

provided that the series converges at \( x = a \) and \( x = b \). In that case, \( f \) is called \( q, \omega \)-integrable on \([a, b] \). We say that \( f \) is \( q, \omega \)-integrable over \( I \) if it is \( q, \omega \)-integrable over \([a, b] \) for all \( a, b \in I \).

Remark 2.5. The \( q, \omega \)-integral generalizes (in the limit) the Jackson \( q \)-integral and the Nörlund’s sum [18]. When \( \omega \to 0 \), we obtain the Jackson \( q \)-integral

\[
\int_a^b f(t) \, dq t := \int_0^b f(t) \, dq t - \int_0^a f(t) \, dq t,
\]

where

\[
\int_0^x f(t) \, dq t := x(1-q) \sum_{k=0}^{+\infty} q^k f \left( xq^k \right).
\]

When \( q \to 1 \), we obtain the Nörlund’s sum

\[
\int_a^b f(t) \, \Delta_\omega t := \int_{+\infty}^b f(t) \, \Delta_\omega t - \int_{+\infty}^a f(t) \, \Delta_\omega t,
\]
where
\[ \int_{-\infty}^{x} f(t) \Delta_\omega t := -\omega \sum_{k=0}^{+\infty} f(x + k\omega). \]

It can be shown that if \( f : I \to \mathbb{R} \) is continuous at \( \omega_0 \), then \( f \) is \( q, \omega \)-integrable over \( I \).

**Theorem 2.6** (Fundamental Theorem of Hahn’s Calculus). Assume that \( f : I \to \mathbb{R} \) is continuous at \( \omega_0 \) and, for each \( x \in I \), define
\[ F(x) := \int_{\omega_0}^{x} f(t) d_{q, \omega} t. \]
Then \( F \) is continuous at \( \omega_0 \). Furthermore, \( D_{q, \omega}[F](x) \) exists for every \( x \in I \) with \( D_{q, \omega}[F](x) = f(x) \). Conversely, \( \int_{a}^{b} D_{q, \omega}[f](t) d_{q, \omega} t = f(b) - f(a) \) for all \( a, b \in I \).

The \( q, \omega \)-integral has the following properties:

**Theorem 2.7** (\([\text{II, II}]\)). Let \( f, g : I \to \mathbb{R} \) be \( q, \omega \)-integrable on \( I \), \( a, b, c \in I \) and \( k \in \mathbb{R} \). Then,
\begin{enumerate}
\item \( \int_{a}^{b} f(t) d_{q, \omega} t = 0; \)
\item \( \int_{a}^{b} k f(t) d_{q, \omega} t = k \int_{a}^{b} f(t) d_{q, \omega} t; \)
\item \( \int_{a}^{b} f(t) d_{q, \omega} t = -\int_{a}^{b} f(t) d_{q, \omega} t; \)
\item \( \int_{a}^{b} f(t) d_{q, \omega} t = \int_{c}^{b} f(t) d_{q, \omega} t + \int_{a}^{c} f(t) d_{q, \omega} t; \)
\item \( \int_{a}^{b} (f(t) + g(t)) d_{q, \omega} t = \int_{a}^{b} f(t) d_{q, \omega} t + \int_{a}^{b} g(t) d_{q, \omega} t; \)
\item Every Riemann integrable function \( f \) on \( I \) is \( q, \omega \)-integrable on \( I \);
\item If \( f, g : I \to \mathbb{R} \) are \( q, \omega \)-differentiable and \( a, b \in I \), then
\[ \int_{a}^{b} f(t) D_{q, \omega}[g](t) d_{q, \omega} t = f(b)g(b) - f(a)g(a) - \int_{a}^{b} D_{q, \omega}[f](t) g(qt + \omega) d_{q, \omega} t. \]
\end{enumerate}

Property 7 of Theorem 2.7 is known as \( q, \omega \)-integration by parts. Note that
\[ \int_{\sigma(t)}^{t} f(\tau) d_{q, \omega} \tau = (t(1-q) - \omega) f(t). \]

**Lemma 2.8** (cf. \([\text{I, II}]\)). Let \( b \in I \) and \( f \) be \( q, \omega \)-integrable over \( I \). Suppose that
\[ f(t) \geq 0 \quad \forall t \in \left\{ q^n b + n_0 : n \in \mathbb{N}_0 \right\}. \]
\begin{enumerate}
\item If \( \omega_0 \leq b \), then
\[ \int_{\omega_0}^{b} f(t) d_{q, \omega} t \geq 0. \]
\item If \( \omega_0 > b \), then
\[ \int_{b}^{\omega_0} f(t) d_{q, \omega} t \geq 0. \]
\end{enumerate}

**Remark 2.9.** There is an inconsistency in \([\text{I, II}]\). Indeed, Lemma 6.2.7 of \([\text{I}]\) is only valid if \( b \geq \omega_0 \) and \( a \leq b \). Similarly with respect to Lemma 3.7 of \([\text{II}]\).

**Remark 2.10.** In general it is not true that
\[ \left| \int_{a}^{b} f(t) d_{q, \omega} t \right| \leq \int_{a}^{b} \left| f(t) \right| d_{q, \omega} t, \quad a, b \in I. \]
For a counterexample see [11, 7]. This illustrates well the difference with other non-quantum integrals, e.g., the time scale integrals [21, 24].

For $s \in I$ we define

$$\{s\}_{q,\omega} := \left\{ q^n s + \omega [n]_q : n \in \mathbb{N}_0 \right\} \cup \{\omega_0\}. \tag{2.1}$$

The following definition and lemma are important for our purposes.

**Definition 2.11.** Let $s \in I$ and $g : I \times (-\theta, \theta) \rightarrow \mathbb{R}$. We say that $g(t, \cdot)$ is differentiable at $\theta_0$ uniformly in $\{s\}_{q,\omega}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |\theta - \theta_0| < \delta \Rightarrow \left| \frac{g(t, \theta) - g(t, \theta_0)}{\theta - \theta_0} - \partial_2 g(t, \theta_0) \right| < \varepsilon$$

for all $t \in \{s\}_{q,\omega}$, where $\partial_2 g = \frac{\partial g}{\partial \theta}$.

**Lemma 2.12** (cf. [22]). Let $s \in I$. Assume that $g : I \times (-\theta, \theta) \rightarrow \mathbb{R}$ is differentiable at $\theta_0$ uniformly in $\{s\}_{q,\omega}$, and $\frac{\partial^2 g}{\partial t^2}(t, \theta_0) d_{q,\omega}t$ exist. Then,

$$G(\theta) := \int_{\omega_0}^s \frac{\partial^2 g}{\partial t^2}(t, \theta_0) d_{q,\omega}t,$$

for $\theta$ near $\theta_0$, is differentiable at $\theta_0$ with $G'(\theta_0) = \int_{\omega_0}^s \frac{\partial^2 g}{\partial t^2}(t, \theta_0) d_{q,\omega}t$.

### 3. Main Results

We define the $q,\omega$-derivatives of higher-order in the usual way: the $r$th $q,\omega$-derivative ($r \in \mathbb{N}$) of $f : I \rightarrow \mathbb{R}$ is the function $D^r_{q,\omega}[f] : I \rightarrow \mathbb{R}$ given by $D^r_{q,\omega}[f] := D^r_{q,\omega}[D_{q,\omega}^{r-1}[f]]$, provided $D_{q,\omega}^{r-1}[f]$ is $q,\omega$-differentiable on $I$ and where $D^0_{q,\omega}[f] := f$.

Let $a, b \in I$ and $a < b$. We introduce the linear space $\mathcal{Y}^r = \mathcal{Y}^r([a, b], \mathbb{R})$ by

$$\mathcal{Y}^r := \{ y : I \rightarrow \mathbb{R} \mid D^r_{q,\omega}[y], i = 0, \ldots, r, \text{ are bounded on } [a, b] \text{ and continuous at } \omega_0 \}$$

endowed with the norm $||y||_{r, \infty} := \sum_{i=0}^r ||D^i_{q,\omega}[y]||_{\infty}$, where $||y||_{\infty} := \sup_{t \in [a, b]} |y(t)|$.

The following notations are in order: $\sigma(t) = qt + \omega$, $\sigma^r(t) = \sigma^{r-1}(t) = (y \circ \sigma)(t) = y(qt + \omega)$, and $y^r = y \circ y^{r-1}$, $k = 2, 3, \ldots$. Our main goal is to establish necessary optimality conditions for the higher-order $q,\omega$-variational problem

$$L[y] = \int_a^b L \left( t, y^r(t), D^{r-1}_{q,\omega}[y](t), \ldots, D^r_{q,\omega}[y](t) \right) d_{q,\omega}t \rightarrow \text{extr}$$

$$y \in \mathcal{Y}^r([a, b], \mathbb{R})$$

(\text{P})

$$y(a) = \alpha_0, \quad y(b) = \beta_0,$$

$$\vdots$$

$$D^{r-1}_{q,\omega}[y](a) = \alpha_{r-1}, \quad D^{r-1}_{q,\omega}[y](b) = \beta_{r-1},$$

where $r \in \mathbb{N}$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 0, \ldots, r-1$, are given.

**Definition 3.1.** We say that $y$ is an admissible function for (\text{P}) if $y \in \mathcal{Y}^r([a, b], \mathbb{R})$ and $y$ satisfies the boundary conditions $D^i_{q,\omega}[y](a) = \alpha_i$ and $D^i_{q,\omega}[y](b) = \beta_i$ of problem (\text{P}), $i = 0, \ldots, r-1$.\footnote{In problem (\text{P}) “extr” denotes “extremize” (i.e., minimize or maximize).}
The Lagrangian $L$ is assumed to satisfy the following hypotheses:

\begin{itemize}
  \item [(H1)] $(u_0, \ldots, u_r) \to L(t, u_0, \ldots, u_r)$ is a $C^1(\mathbb{R}^{r+1}, \mathbb{R})$ function for any $t \in [a, b]$;
  \item [(H2)] $t \to L(t, y(t), D_{q,\omega}[y](t), \ldots, D_{q,\omega}^r[y](t))$ is continuous at $\omega_0$ for any admissible $y$;
  \item [(H3)] functions $t \to \partial_t L(t, y(t), D_{q,\omega}[y](t), \ldots, D_{q,\omega}^r[y](t))$, $i = 0, 1, \ldots, r$, belong to $Y^1([a, b], \mathbb{R})$ for all admissible $y$.
\end{itemize}

**Definition 3.2.** We say that $y_*$ is a local minimizer (resp. local maximizer) for problem (P) if $y_*$ is an admissible function and there exists $\delta > 0$ such that
\[ \mathcal{L}[y_*] \leq \mathcal{L}[y] \quad \text{(resp. } \mathcal{L}[y_*] \geq \mathcal{L}[y]) \]
for all admissible $y$ with $\|y_* - y\|_{C,\infty} < \delta$.

**Definition 3.3.** We say that $\eta \in Y^\nu([a, b], \mathbb{R})$ is a variation if $\eta(a) = \eta(b) = 0, \ldots, D_{q,\omega}^{\nu-1}[\eta](a) = D_{q,\omega}^{\nu-1}[\eta](b) = 0$.

We define the $q, \omega$-interval from $a$ to $b$ by
\[ [a, b]_{q,\omega} := \left\{ q^n a + \omega [n]_q : n \in \mathbb{N}_0 \right\} \cup \left\{ q^n b + \omega [n]_q : n \in \mathbb{N}_0 \right\} \cup \{ \omega_0 \}, \]
i.e., $[a, b]_{q,\omega} = [a]_{q,\omega} \cup [b]_{q,\omega}$, where $[a]_{q,\omega}$ and $[b]_{q,\omega}$ are given by (2.1).

### 3.1. Higher-order fundamental lemma of Hahn’s variational calculus

The chain rule, as known from classical calculus, does not hold in Hahn’s quantum context (see a counterexample in [11, 17]). However, we can prove the following.

**Lemma 3.4.** If $f$ is $q, \omega$-differentiable on $I$, then the following equality holds:
\[ D_{q,\omega} \left[ f^\sigma \right](t) = q \left( D_{q,\omega} \left[ f \right] \right)^\sigma(t), \quad t \in I. \]

**Proof.** For $t \neq \omega_0$ we have
\[ (D_{q,\omega} \left[ f \right])^\sigma(t) = \frac{f(q(t + \omega) + \omega) - f(qt + \omega)}{(q - 1)(qt + \omega) + \omega} = \frac{f(q(t + \omega) + \omega) - f(qt + \omega)}{q((q - 1)t + \omega)} \]
and
\[ D_{q,\omega} \left[ f^\sigma \right](t) = \frac{f^\sigma(q(t + \omega) + \omega) - f^\sigma(qt + \omega)}{(q - 1)t + \omega} = \frac{f(q(t + \omega) + \omega) - f(qt + \omega)}{(q - 1)t + \omega}. \]
Therefore, $D_{q,\omega} \left[ f^\sigma \right](t) = q \left( D_{q,\omega} \left[ f \right] \right)^\sigma(t)$. If $t = \omega_0$, then $\sigma(\omega_0) = \omega_0$. Thus,
\[ (D_{q,\omega} \left[ f \right])^\sigma(\omega_0) = (D_{q,\omega} \left[ f \right])(\omega_0) = f'(\omega_0) \]
and $D_{q,\omega} \left[ f^\sigma \right](\omega_0) = [f^\sigma]'(\omega_0) = f'(\sigma(\omega_0)) \sigma'(\omega_0) = q f'(\omega_0)$.

**Lemma 3.5.** If $\eta \in Y^\nu([a, b], \mathbb{R})$ is such that $D_{q,\omega}^i[\eta](a) = 0$ (resp. $D_{q,\omega}^i[\eta](b) = 0$) for all $i \in \{0, 1, \ldots, r\}$, then $D_{q,\omega}^{i-1}[\eta^\sigma](a) = 0$ (resp. $D_{q,\omega}^{i-1}[\eta^\sigma](b) = 0$) for all $i \in \{1, \ldots, r\}$.

**Proof.** If $a = \omega_0$ the result is trivial (because $\sigma(\omega_0) = \omega_0$). Suppose now that $a \neq \omega_0$ and fix $i \in \{1, \ldots, r\}$. Note that
\[ D_{q,\omega}^i[\eta](a) = \frac{(D_{q,\omega}^{i-1}[\eta])^\sigma(a) - D_{q,\omega}^{i-1}[\eta](a)}{(q - 1)a + \omega}. \]
Since, by hypothesis, \( D_{q,\omega}^i [\eta](a) = 0 \) and \( D_{q,\omega}^{i-1} [\eta](a) = 0 \), then \( (D_{q,\omega}^{i-1} [\eta])^\sigma(a) = 0 \). Lemma 3.4 shows that

\[
(D_{q,\omega}^{i-1} [\eta])^\sigma(a) = \left( \frac{1}{q} \right)^{i-1} D_{q,\omega}^{i-1} [\eta^\sigma](a).
\]

We conclude that \( D_{q,\omega}^{i-1} [\eta^\sigma](a) = 0 \). The case \( t = b \) is proved in the same way. □

**Lemma 3.6.** Suppose that \( f \in \mathcal{Y}^1([a, b], \mathbb{R}) \). One has

\[
\int_a^b f(t) D_{q,\omega} [\eta](t) \, d_{q,\omega} t = 0
\]

for all functions \( \eta \in \mathcal{Y}^1([a, b], \mathbb{R}) \) such that \( \eta(a) = \eta(b) = 0 \) if and only if \( f(t) = c, c \in \mathbb{R}, \) for all \( t \in [a, b]_{q,\omega} \).

**Proof.** The implication “\( \Rightarrow \)” is obvious. We prove “\( \Leftarrow \)”. We begin noting that

\[
\int_a^b f(t) D_{q,\omega} [\eta](t) \, d_{q,\omega} t = f(t) \eta(t) \bigg|_a^b - \int_a^b D_{q,\omega} [f](t) \eta^\sigma(t) \, d_{q,\omega} t.
\]

Hence,

\[
\int_a^b D_{q,\omega} [f](t) \eta(qt + \omega) \, d_{q,\omega} t = 0
\]

for any \( \eta \in \mathcal{Y}^1([a, b], \mathbb{R}) \) such that \( \eta(a) = \eta(b) = 0 \). We need to prove that, for some \( c \in \mathbb{R}, f(t) = c \) for all \( t \in [a, b]_{q,\omega} \), that is, \( D_{q,\omega}[f](t) = 0 \) for all \( t \in [a, b]_{q,\omega} \).

Suppose, by contradiction, that there exists \( p \in [a, b]_{q,\omega} \) such that \( D_{q,\omega}[f](p) \neq 0 \).

(1) If \( p \neq \omega_0 \), then \( p = q^k a + \omega[k]_q \) or \( p = q^k b + \omega[k]_q \) for some \( k \in \mathbb{N}_0 \). Observe that \( a(1-q) - \omega \) and \( b(1-q) - \omega \) cannot vanish simultaneously.

(a) Suppose that \( a(1-q) - \omega \neq 0 \) and \( b(1-q) - \omega \neq 0 \). In this case we can assume, without loss of generality, that \( p = q^k a + \omega[k]_q \) and we can define

\[
\eta(t) = \begin{cases} 
D_{q,\omega}[f] \left( q^k a + \omega[k]_q \right) & \text{if } t = q^{k+1} a + \omega[k+1]_q \\
0 & \text{otherwise.}
\end{cases}
\]

Then,

\[
\int_a^b D_{q,\omega}[f](t) \cdot \eta(qt + \omega) \, d_{q,\omega} t = - a(1-q) - \omega \, q^k D_{q,\omega}[f] \left( q^k a + \omega[k]_q \right) \cdot D_{q,\omega}[f] \left( q^k a + \omega[k]_q \right) \neq 0,
\]

which is a contradiction.

(b) If \( a(1-q) - \omega \neq 0 \) and \( b(1-q) - \omega = 0 \), then \( b = \omega_0 \). Since \( q^k \omega_0 + \omega[k]_q = \omega_0 \) for all \( k \in \mathbb{N}_0 \), then \( p \neq q^k b + \omega[k]_q \) \( \forall k \in \mathbb{N}_0 \) and, therefore,

\[
p = q^k a + \omega[k]_{q,\omega} \text{ for some } k \in \mathbb{N}_0.
\]

Repeating the proof of (a) we obtain again a contradiction.

(c) If \( a(1-q) - \omega = 0 \) and \( b(1-q) - \omega \neq 0 \) then the proof is similar to (b).
(2) If \( p = \omega_0 \) then, without loss of generality, we can assume \( D_{q,\omega} [f] (\omega_0) > 0 \). Since

\[
\lim_{n \to +\infty} \left( q^n a + \omega [k]_q \right) = \lim_{n \to +\infty} \left( q^n b + \omega [k]_q \right) = \omega_0
\]

(see [1]) and \( D_{q,\omega} [f] \) is continuous at \( \omega_0 \), then

\[
\lim_{n \to +\infty} D_{q,\omega} [f] \left( q^n a + \omega [k]_q \right) = \lim_{n \to +\infty} D_{q,\omega} [f] \left( q^n b + \omega [k]_q \right) = D_{q,\omega} [f] (\omega_0) > 0.
\]

Thus, there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) one has \( D_{q,\omega} [f] \left( q^n a + \omega [k]_q \right) > 0 \) and \( D_{q,\omega} [f] \left( q^n b + \omega [k]_q \right) > 0 \).

(a) If \( \omega_0 \neq a \) and \( \omega_0 \neq b \), then we can define

\[
\eta(t) = \begin{cases} 
D_{q,\omega} [f] \left( q^{N+1} b + \omega [N+1]_q \right) & \text{if } t = q^{N+1} a + \omega [N+1]_q \\
D_{q,\omega} [f] \left( q^{N+1} a + \omega [N+1]_q \right) & \text{if } t = q^{N+1} b + \omega [N+1]_q \\
0 & \text{otherwise.}
\end{cases}
\]

Hence,

\[
\int_a^b D_{q,\omega} [f] (t) \eta (qt + \omega) \, dq,\omega t
\]

\[
= (b - a) \left( 1 - q \right) q^N D_{q,\omega} [f] \left( q^N b + \omega [N]_q \right) \cdot D_{q,\omega} [f] \left( q^N a + \omega [N]_q \right) \neq 0,
\]

which is a contradiction.

(b) If \( \omega_0 = b \), then we define

\[
\eta(t) = \begin{cases} 
D_{q,\omega} [f] (\omega_0) & \text{if } t = q^{N+1} a + \omega [N+1]_q \\
0 & \text{otherwise.}
\end{cases}
\]

Therefore,

\[
\int_a^b D_{q,\omega} [f] (t) \eta (qt + \omega) \, dq,\omega t
\]

\[
= - \int_{\omega_0}^a D_{q,\omega} [f] (t) \eta (qt + \omega) \, dq,\omega t
\]

\[
= - (a (1 - q) - \omega) q^N D_{q,\omega} [f] \left( q^N a + \omega [k]_q \right) \cdot D_{q,\omega} [f] (\omega_0) \neq 0,
\]

which is a contradiction.

(c) When \( \omega_0 = a \), the proof is similar to (b). \( \square \)

**Lemma 3.7** (Fundamental lemma of Hahn’s variational calculus). Let \( f, g \in \mathcal{Y}^1 ([a, b], \mathbb{R}) \). If

\[
\int_a^b \left( f(t) \eta^2 (t) + g(t) D_{q,\omega} [\eta] (t) \right) \, dq,\omega t = 0
\]

for all \( \eta \in \mathcal{Y}^1 ([a, b], \mathbb{R}) \) such that \( \eta (a) = \eta (b) = 0 \), then

\[
D_{q,\omega} [g] (t) = f(t) \quad \forall t \in [a, b]_{q,\omega}.
\]
Lemma 3.7. Assume that $\int_a^b f(t) \, dq, \omega \, t$. Then $D_{q, \omega} [A] (t) = f(t)$ for all $t \in [a, b]$ and

$$
\int_a^b A(t) \, D_{q, \omega} [\eta] (t) \, dq, \omega \, t = A(t) \, \eta(t) \bigg|_a^b - \int_a^b D_{q, \omega} [A] (t) \, \eta^\sigma(t) \, dq, \omega \, t = -\int_a^b D_{q, \omega} [A] (t) \, \eta^\sigma(t) \, dq, \omega \, t = -\int_a^b f(t) \, \eta^\sigma(t) \, dq, \omega \, t.
$$

Hence,

$$
\int_a^b (f(t) \, \eta^\sigma(t) + g(t) \, D_{q, \omega} [\eta](t)) \, dq, \omega \, t = 0
$$

$$
\iff \int_a^b (-A(t) + g(t)) \, D_{q, \omega} [\eta] (t) \, dq, \omega \, t = 0.
$$

By Lemma 3.6 there is a $c \in \mathbb{R}$ such that $-A(t) + g(t) = c$ for all $t \in [a, b]_{q, \omega}$. Hence $D_{q, \omega} [A](t) = D_{q, \omega} [g](t)$ for $t \in [a, b]_{q, \omega}$, which provides the desired result: $D_{q, \omega} [g](t) = f(t) \forall t \in [a, b]_{q, \omega}$.

We are now in conditions to deduce the higher-order fundamental Lemma of Hahn’s quantum variational calculus.

Lemma 3.8 (Higher-order fundamental lemma of Hahn’s variational calculus). Let $f_0, f_1, \ldots, f_r \in \mathcal{Y}^1 ([a, b], \mathbb{R})$. If

$$
\int_a^b \left( \sum_{i=0}^r f_i(t) \, D_{q, \omega}^i [\eta^{r+1-i}] (t) \right) \, dq, \omega \, t = 0
$$

for any variation $\eta$, then

$$
\sum_{i=0}^r (-1)^i \left( \frac{1}{q} \right)^{i+1} \, D_{q, \omega}^i [f_i] (t) = 0
$$

for all $t \in [a, b]_{q, \omega}$.

Proof. We proceed by mathematical induction. If $r = 1$ the result is true by Lemma 3.7. Assume that

$$
\int_a^b \left( \sum_{i=0}^{r+1} f_i(t) \, D_{q, \omega}^i [\eta^{r+1-i}] (t) \right) \, dq, \omega \, t = 0
$$

for all functions $\eta$ such that $\eta(a) = \eta(b) = 0, \ldots, D_{q, \omega}^r [\eta](a) = D_{q, \omega}^r [\eta](b) = 0$. Note that

$$
\int_a^b f_{r+1}(t) \, D_{q, \omega}^{r+1} [\eta](t) \, dq, \omega \, t
$$

$$
= f_{r+1}(t) \, D_{q, \omega}^r [\eta](t) \bigg|_a^b - \int_a^b D_{q, \omega} [f_{r+1}] (t) \, (D_{q, \omega}^r [\eta])^\sigma(t) \, dq, \omega \, t
$$

$$
= -\int_a^b D_{q, \omega} [f_{r+1}] (t) \, (D_{q, \omega}^r [\eta])^\sigma(t) \, dq, \omega \, t
$$
and, by Lemma 3.4
\[
\int_a^b f_{r+1}(t) D_{q,\omega}^{r+1} [\eta](t) \, dq_\omega t = - \int_a^b D_{q,\omega} [f_{r+1}](t) \left( \frac{1}{q} \right)^r D_{q,\omega}^r [\eta^r](t) \, dq_\omega t.
\]
Therefore,
\[
\int_a^b \left( \sum_{i=0}^{r+1} f_i(t) D_{q,\omega}^i [\eta^{r+1-i}](t) \right) \, dq_\omega t
= \int_a^b \left( \sum_{i=0}^r f_i(t) D_{q,\omega}^i [\eta^{r+1-i}](t) \right) \, dq_\omega t
- \int_a^b D_{q,\omega} [f_{r+1}](t) \left( \frac{1}{q} \right)^r D_{q,\omega}^r [\eta^r](t) \, dq_\omega t
= \int_a^b \left( \sum_{i=0}^{r-1} f_i(t) D_{q,\omega}^i [(\eta^r)^{r-i}](t) \right) \, dq_\omega t
+ \left( f_r - \left( \frac{1}{q} \right)^r D_{q,\omega} [f_{r+1}] \right) (t) D_{q,\omega}^r [\eta^r](t) \, dq_\omega t.
\]
By Lemma 3.5, \( \eta^r \) is a variation. Hence, using the induction hypothesis,
\[
\sum_{i=0}^{r-1} (-1)^i \left( \frac{1}{q} \right)^{i(r-1)} D_{q,\omega}^i [f_i](t)
+ (-1)^r \left( \frac{1}{q} \right)^{(r-1)r} D_{q,\omega}^r \left( \left( f_r - \frac{1}{q} \right)^r D_{q,\omega} [f_{r+1}] \right)(t)
= \sum_{i=0}^{r-1} (-1)^i \left( \frac{1}{q} \right)^{i(r-1)} D_{q,\omega}^i [f_i](t) + (-1)^r \left( \frac{1}{q} \right)^{(r-1)r} D_{q,\omega}^r [f_r](t)
+ (-1)\left( \frac{1}{q} \right)^r D_{q,\omega}^r [D_{q,\omega} [f_{r+1}]](t)
= 0
\]
for all \( t \in [a, b]_{q,\omega} \), which leads to
\[
\sum_{i=0}^{r+1} (-1)^i \left( \frac{1}{q} \right)^{i(r+1)} D_{q,\omega}^i [f_i](t) = 0, \quad t \in [a, b]_{q,\omega}.
\]
\[\square\]

3.2. **Higher-order Hahn’s quantum Euler–Lagrange equation.** For a variation \( \eta \) and an admissible function \( y \), we define the function \( \phi : (\epsilon, \bar{\epsilon}) \to \mathbb{R} \) by
\[
\phi(\epsilon) = \phi(\epsilon, y, \eta) := \mathcal{L} [y + \epsilon \eta].
\]
The first variation of the variational problem (P) is defined by
\[
\delta \mathcal{L} [y, \eta] := \phi'(0).
\]
Observe that
\[
\mathcal{L} [y + \epsilon \eta] = \int_a^b L \left( t, y^{\sigma r} (t) + \epsilon \eta^{\sigma r} (t), D_{q, \omega} \left[ y^{\sigma r - 1} \right] (t) \right) \left( \ldots, D_{q, \omega}^r [y] (t) + \epsilon D_{q, \omega}^r [\eta] (t) \right) d_{q, \omega} t
\]

\[= \mathcal{L}_b [y + \epsilon \eta] - \mathcal{L}_a [y + \epsilon \eta] \]

with
\[
\mathcal{L}_\xi [y + \epsilon \eta] = \int_{\omega_0}^\xi L \left( t, y^{\sigma r} (t) + \epsilon \eta^{\sigma r} (t), D_{q, \omega} \left[ y^{\sigma r - 1} \right] (t) \right) \left( \ldots, D_{q, \omega}^r [y] (t) + \epsilon D_{q, \omega}^r [\eta] (t) \right) d_{q, \omega} t,
\]

\[\xi \in \{a, b\}. \]

Therefore,
\[\delta \mathcal{L} [y, \eta] = \mathcal{L}_b [y, \eta] - \mathcal{L}_a [y, \eta].\]

Considering (3.1), the following lemma is a direct consequence of Lemma 2.12:

**Lemma 3.9.** For a variation \(\eta\) and an admissible function \(y\), let
\[g (t, \epsilon) := L \left( t, y^{\sigma r} (t) + \epsilon \eta^{\sigma r} (t), D_{q, \omega} \left[ y^{\sigma r - 1} \right] (t) \right) \left( \ldots, D_{q, \omega}^r [y] (t) + \epsilon D_{q, \omega}^r [\eta] (t) \right),\]

\[\epsilon \in (-\bar{\epsilon}, \bar{\epsilon}). \]

Assume that:
(1) \(g (t, \cdot)\) is differentiable at 0 uniformly in \(t \in [a, b]_{q, \omega};\)
(2) \(\mathcal{L}_a [y + \epsilon \eta] = \int_{\omega_0}^a g (t, \epsilon) d_{q, \omega} t\) and \(\mathcal{L}_b [y + \epsilon \eta] = \int_{\omega_0}^b g (t, \epsilon) d_{q, \omega} t\) exist for \(\epsilon \approx 0;\)
(3) \(\int_{\omega_0}^a \partial_2 g (t, 0) d_{q, \omega} t\) and \(\int_{\omega_0}^b \partial_2 g (t, 0) d_{q, \omega} t\) exist.

Then
\[\phi' (0) = \delta \mathcal{L} [y, \eta] = \int_a^b \left( \sum_{i=0}^r \partial_i + 2 L \left( t, y^{\sigma r} (t), D_{q, \omega} \left[ y^{\sigma r - 1} \right] (t) \right) \ldots, D_{q, \omega}^r [y] (t) \right) \left( \ldots, D_{q, \omega}^i [\eta^{\sigma r - 1}] (t) \right) d_{q, \omega} t,
\]

where \(\partial_i L\) denotes the partial derivative of \(L\) with respect to its \(i\)th argument.

The following result gives a necessary condition of Euler–Lagrange type for an admissible function to be a local extremizer for (P).

**Theorem 3.10** (Higher-order Hahn’s quantum Euler–Lagrange equation). Under hypotheses (H1)–(H3) and conditions (1)–(3) of Lemma 3.9 on the Lagrangian \(\mathcal{L}\),
if \( y_* \in \mathcal{V}^r \) is a local extremizer for problem \( \mathcal{P} \), then \( y_* \) satisfies the \( q, \omega \)-Euler–Lagrange equation

\[
(3.2) \quad \sum_{i=0}^{r} (-1)^i \left( \frac{1}{q} \right) \frac{(i-1)!}{2^i} D_q^i [\partial_{i+2} L] \left( t, y^{(i)}(t), D_q \left[ y^{(i-1)}(t) \right] (t), \ldots, D_q^r [y] (t) \right) = 0
\]

for all \( t \in [a, b] \).

**Proof.** Let \( y_* \) be a local extremizer for problem \( \mathcal{P} \) and \( \eta \) a variation. Define \( \phi : (-\epsilon, \epsilon) \to \mathbb{R} \) by \( \phi(\epsilon) := \mathcal{L}[y_* + \epsilon \eta] \). A necessary condition for \( y_* \) to be an extremizer is given by \( \phi'(0) = 0 \). By Lemma 3.9 we conclude that

\[
\int_a^b \left( \sum_{i=0}^{r} \partial_{i+2} L \left( t, y^{(i)}(t), D_q \left[ y^{(i-1)}(t) \right] (t), \ldots, D_q^r [y] (t) \right) \right) \cdot D_q^i [\eta^{(i-1)}] (t) \, dq, \omega \, dt = 0
\]

and (3.2) follows from Lemma 3.8.

**Remark 3.11.** In practical terms the hypotheses of Theorem 3.10 are not so easy to verify *a priori*. One can, however, assume that all hypotheses are satisfied and apply the \( q, \omega \)-Euler–Lagrange equation (3.2) heuristically to obtain a *candidate*. If such a candidate is, or not, a solution to problem \( \mathcal{P} \) is a different question that always requires further analysis (see an example in 3.3).

When \( \omega \to 0 \) one obtains from (3.2) the higher-order \( q \)-Euler–Lagrange equation:

\[
(3.3) \quad \sum_{i=0}^{r} (-1)^i \left( \frac{1}{q} \right) \frac{(i-1)!}{2^i} \Delta_q^i [\partial_{i+2} L] \left( t, y^{(i)}(t), D_q \left[ y^{(i-1)}(t) \right] (t), \ldots, D_q^r [y] (t) \right) = 0
\]

for all \( t \in \{ a q^n : n \in \mathbb{N}_0 \} \cup \{ b q^n : n \in \mathbb{N}_0 \} \cup \{ 0 \} \). The higher-order \( h \)-Euler–Lagrange equation is obtained from (3.2) taking the limit \( q \to 1 \):

\[
(3.4) \quad \sum_{i=0}^{r} (-1)^i \Delta_h^i [\partial_{i+2} L] \left( t, y^{(i)}(t), \Delta_h \left[ y^{(i-1)}(t) \right] (t), \ldots, \Delta_h^r [y] (t) \right) = 0
\]

for all \( t \in \{ a + n h : n \in \mathbb{N}_0 \} \cup \{ b + n h : n \in \mathbb{N}_0 \} \). The classical Euler–Lagrange equation (26) is recovered when \( (\omega, q) \to (0, 1) \):

\[
(3.5) \quad \sum_{i=0}^{r} (-1)^i \frac{d^i}{dt^i} \partial_{i+2} L \left( t, y(t), y'(t), \ldots, y^{(r)}(t) \right) = 0
\]

for all \( t \in [a, b] \).

We now illustrate the usefulness of our Theorem 3.10 by means of an example that is not covered by previous available results in the literature.

**3.3. An Example.** Let \( q = \frac{1}{2} \) and \( \omega = \frac{1}{2} \). Consider the following problem:

\[
(3.3) \quad \mathcal{L}[y] = \int_{-1}^1 \left( y^2(t) + \frac{1}{2} \right)^2 \left( (D_{q, \omega} [y] (t))^2 - 1 \right)^2 \, dq, \omega t \to \min
\]

over all \( y \in \mathcal{V}^1 \) satisfying the boundary conditions

\[
(3.4) \quad y(-1) = 0 \quad \text{and} \quad y(1) = -1.
\]
This is an example of problem \([\text{P}]\) with \(r = 1\). Our \(q, \omega\)-Euler–Lagrange equation \((3.2)\) takes the form
\[
D_{q, \omega} \left[ \partial_t L \right] (t, y^\sigma (t), D_{q, \omega} [y] (t)) = \partial_y L (t, y^\sigma (t), D_{q, \omega} [y] (t)) .
\]
Therefore, we look for an admissible function \(y^\ast\) of \((3.3)-(3.4)\) satisfying
\[
(3.5) \quad D_{q, \omega} \left[ 4 \left( y^\sigma + \frac{1}{2} \right)^2 \left( (D_{q, \omega} [y])^2 - 1 \right) D_{q, \omega} [y] \right] (t) = 2 \left( y^\sigma (t) + \frac{1}{2} \right) \left( (D_{q, \omega} [y] (t))^2 - 1 \right)
\]
for all \(t \in [-1, 1]_{q, \omega}\). It is easy to see that
\[
y^\ast (t) = \begin{cases} 
  -t & \text{if } t \in (-1, 0) \cup (0, 1] \\
  0 & \text{if } t = -1 \\
  1 & \text{if } t = 0
\end{cases}
\]
is an admissible function for \((3.3)-(3.4)\) with
\[
D_{q, \omega} [y^\ast] (t) = \begin{cases} 
  -1 & \text{if } t \in (-1, 0) \cup (0, 1] \\
  1 & \text{if } t = -1 \\
  -3 & \text{if } t = 0, 
\end{cases}
\]
satisfying the \(q, \omega\)-Euler–Lagrange equation \((3.5)\). We now prove that the candidate \(y^\ast\) is indeed a minimizer for \((3.3)-(3.4)\). Note that here \(\omega_0 = 1\) and, by Lemma 2.8 and item \(3\) of Theorem 2.7
\[
(3.6) \quad \mathcal{L} [y] = \int_{-1}^{1} \left( y^\sigma (t) + \frac{1}{2} \right)^2 \left( (D_{q, \omega} [y] (t))^2 - 1 \right)^2 \, dt \geq 0
\]
for all admissible functions \(y \in Y^1 ([\mathbb{R} \setminus \mathbb{Q}]_{q, \omega} \setminus \mathbb{Q}). \) Since \(\mathcal{L} [y^\ast] = 0\), we conclude that \(y^\ast\) is a minimizer for problem \((3.3)-(3.4)\).

It is worth to mention that the minimizer \(y^\ast\) of \((3.3)-(3.4)\) is not continuous while the classical calculus of variations \([26]\), the calculus of variations on time scales \([14, 20, 23]\), or the nondifferentiable scale variational calculus \([4, 5, 10]\), deal with functions which are necessarily continuous. As an open question, we pose the problem of determining conditions on the data of problem \((3.3)-(3.4)\) assuring, \textit{a priori}, the minimizer to be regular.

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