The Cohomology Class of the Mod 4 Braid Group

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Abstract

The mod 4 braid group, \( \mathbb{Z}_n \), is defined to be the quotient of the braid group by the subgroup of the pure braid group generated by squares of all elements, \( PB_n^2 \). Kordek and Margalit proved \( \mathbb{Z}_n \) is an extension of the symmetric group by \( \mathbb{Z}/(n^2\mathbb{Z}) \). For \( n \geq 1 \), we construct a 2-cocycle in the group cohomology of the symmetric group with twisted coefficients classifying \( \mathbb{Z}_n \). We show this cocycle is the mod 2 reduction of the 2-cocycle corresponding to the extension of the symmetric group by the abelianization of the pure braid group. We also construct the 2-cocycle corresponding to this second extension and show it represents an order two element in the cohomology of the symmetric group. Furthermore, we give presentations for both extensions and a normal generating set for \( PB_n^2 \).

1 Introduction

In 2014 Brendle and Margalit proved the level 4 congruence subgroup, \( B_n[4] \), of the braid group, \( B_n \), is the subgroup of the pure braid group generated by squares of all elements, \( PB_n^2 \). More recently, Kordek and Margalit showed the quotient of the braid group by \( PB_n^2 \), which we define as the mod 4 braid group and denote \( \mathbb{Z}_n \), is an extension of the symmetric group, \( S_n \), by \( \mathbb{Z}/(n^2\mathbb{Z}) \). Eilenberg and MacLane proved we can classify group extensions by low dimensional cohomology classes in the group cohomology with twisted coefficients \[7\]. In particular, \( \mathbb{Z}_n \) is classified by a class in the second cohomology of the symmetric group with coefficients in \( \mathbb{Z}/(n^2\mathbb{Z}) \) twisted by the action of the symmetric group permuting unordered pairs of integers. We begin by constructing a CW-complex which is the two-skeleton of an Eilenberg-MacLane space for the symmetric group. This Eilenberg-MacLane space gives us a low dimensional approximation for a resolution of \( \mathbb{Z} \) over \( \mathbb{Z}S_n \). Then we construct a presentation for the extension of the symmetric group by the abelianization of the pure braid group. We obtain our main results from classifying the extension of the symmetric group by the abelianization of the pure braid group and composing this 2-cocycle with the mod 2 reduction of integers.

Let \( G \) be a group and \( A \) be a \( G \)-module. We represent the, possibly nontrivial, action of \( G \) on \( A \) by a map \( \theta : G \to \text{Aut}(A) \). For an extension \( E \) of \( G \) by \( A \), with \( \iota : A \to E \) and \( \pi : E \to G \), we say \( E \) gives rise to \( \theta \) if conjugating an element of \( \iota(A) \) by any \( e \in E \) is determined by \( \theta(\pi(e)) \). The cohomology of \( G \) with coefficients in \( A \) is equivalent to the cohomology of an Eilenberg-MacLane space for \( G \), referred to as a \( K(G, 1) \)-space, with a local coefficients system of \( A \) determined by \( \theta \). We use \( H^2(G; A) \) to denote the second cohomology group of \( G \) with coefficients in \( A \) twisted by \( \theta \). Fixing the action of \( G \) on \( A \), Eilenberg and MacLane proved there exists a bijection between \( H^2(G; A) \) and the set of group extensions \( E \), up to equivalence, of \( G \) by \( A \) which give rise to \( \theta \). In this paper we will only consider the standard action of the symmetric group on unordered
pairs of integers between 1 and n; therefore we often omit θ. We give a more explicit explanation of the background for group cohomology in section 2.2.

Extensions of the symmetric group arise naturally while studying quotients of the braid group, \( B_n \). The level \( m \) congruence subgroup of the braid group, \( B_n[m] \), is defined more generally as the kernel of a map: \( B_n \to GL_n(\mathbb{Z}_m) \) (more information is given in section 2.3). In 2018 Kordek and Margalit showed \( \mathbb{Z}_n = B_n/B_n[4] \) is an extension of \( \mathbb{Z}_n \) by \( \mathbb{Z}_2^{(2)} \) where \( \mathbb{Z}_n \) permutes the generators of \( \mathbb{Z}_2^{(2)} \) by the standard action on unordered pairs of \( \{1, \ldots, n\} \). Further work by Appel, Bloomquist, Gravel, and Holden prove that for \( m \) odd, \( B_n[m]/B_n[4m] = \mathbb{Z}_n \). In this paper we will give descriptions of \( \mathbb{Z}_n \) by both the corresponding 2-cocycle in \( H^2(\mathbb{Z}_n; \mathbb{Z}_2^{(2)}) \) and by a group presentation. Our first theorem shows the cocycle classifying \( \mathbb{Z}_n \) is determined by a cocycle which classifies an extension of \( \mathbb{Z}_n \) by \( \mathbb{Z}_2^{(2)} \).

**Theorem 1.1.** If \( n \geq 1 \), then the cohomology class \([κ] ∈ H^2(\mathbb{Z}_n; \mathbb{Z}_2^{(2)})\) classifying \( \mathbb{Z}_n \) as an extension of \( \mathbb{Z}_n \) by \( \mathbb{Z}_2^{(2)} \) is the mod 2 reduction of an element \([φ] ∈ H^2(\mathbb{Z}_n; \mathbb{Z}_2^{(2)})\) of order 2.

We define representatives of our cohomology classes by constructing a low dimensional approximation for a cellular chain complex of the universal cover of a \( K(\mathbb{Z}_n, 1) \)-space. Furthermore, we show that the 2-chains of this chain complex are generated by the \( \mathbb{Z}_n \) orbits of three classes of elements: \( \tilde{c}_{i,j} \), \( \tilde{d}_{i,j,k,ℓ} \), and \( \tilde{e}_{i,k,j} \) where \( 1 ≤ i < j ≤ n \) and \( 1 ≤ k < ℓ ≤ n \). Respectively, each of these three types of generators correspond to the squaring, commuting, and braid relations in the presentation for the symmetric group. Note that the generators \( \tilde{e}_{i,k,j} \) are only required for \( n ≥ 3 \) while \( \tilde{d}_{i,j,k,ℓ} \) are required only for \( n ≥ 4 \). Therefore, the cocycle representing \([κ]\) is determined as a function from the \( \mathbb{Z}_n \) module generated by these three classes of 2-chains to \( \mathbb{Z}_2^{(2)} \). Using the presentation:

\[
\mathbb{Z}_2^{(2)} = \langle \{g_{i,j}\}_{1 ≤ i < j ≤ n} \mid g_{i,j}^2 = 1, [g_{i,j}, g_{k,ℓ}] = 1 \rangle
\]  

our second theorem defines the 2-cocycle classifying \( \mathbb{Z}_n \).

**Theorem 1.2.** Let \( i < j \) and \( k < ℓ \). If \( n ≥ 1 \), then a representative for the cocycle classifying \( \mathbb{Z}_n \) as an extension of \( \mathbb{Z}_n \) by \( \mathbb{Z}_2^{(2)} \) is given by:

\[
κ(\tilde{c}_{i,j}) = \tilde{g}_{i,j}
\]

\[
κ(\tilde{d}_{i,j,k,ℓ}) = \begin{cases} \tilde{g}_{k,j} + \tilde{g}_{k,ℓ} + \tilde{g}_{i,ℓ} + \tilde{g}_{i,j} & i < k < j < ℓ \text{ or } k < i < ℓ < j \\ 0 & \text{otherwise} \end{cases}
\]

\[
κ(\tilde{e}_{i,k,j}) = \begin{cases} \tilde{g}_{i,j} + \tilde{g}_{j,k} & i < k < j, j < k < i \text{ or } k < j < i \\ 0 & \text{otherwise} \end{cases}
\]

In the construction of Theorem 1.2 we give a presentation for \( \mathbb{Z}_n \) with relations in Table 3. Furthermore, considering Artin’s original presentation for the braid group given by:

\[
B_n = \left\{ b_1, \ldots, b_n \mid \begin{array}{c} b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1} \text{ for all } i \\ b_ib_j = b_jb_i \text{ if } |i - j| > 1 \end{array} \right\}
\]  

our presentation for \( \mathbb{Z}_n \) yields a normal generating set for \( PB_2^2 \) as a subgroup of \( B_n \). Since Brendle and Margalit proved \( PB_2^2 = B_n[4] \), we get a normal generating set for \( B_n[4] \) in Theorem 1.3. To simplify notation in the statement of Theorem 1.3 let \( b_{i,j} \) be the conjugation of \( b_{j-1} \) by \( b_ib_{i+1}\cdots b_{j-2} \).
Theorem 1.3. For all \( n \geq 1 \), \( B_n[4] \) is normally generated as a subgroup of the braid group by elements of the form:

1. \([b_i^2, b_{i+1}^2] \) for all \( 1 \leq i \leq n - 1 \).
2. \([b_i^2, b_{i+1}^2, b_{i+2}^2] \) for all \( 1 \leq i \leq n - 3 \)
3. \( b_i^4 \) for all \( 1 \leq i \leq n - 1 \)

Outline This paper will begin with preliminaries which will help with constructing group presentations, group cohomology, and braid/symmetric groups. In section 3 we build a truncated resolution for the chain complex corresponding to the universal cover of a \( K(S_n, 1) \) space and define the maps needed to construct a 2-cocycle using this resolution. Then in section 4 we construct a representative for the cohomology class \([\phi]\) ∈ \( H^2(S_n; \mathbb{Z}^{(i)}) \) and compute the order of this element. In section 5 we prove \( \kappa \) is determined by \( \phi \), finishing the proof of Theorems 1.1 and 1.2. The proof of Theorem 1.3 is finished at the end of the paper with an explanation of the difficulties which arise in finding a finite generating set for \( B_n[4] \).

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2 Preliminaries

2.1 Building a Group Presentation

Given a group extension

\[
1 \longrightarrow K \overset{\iota}{\longrightarrow} G \overset{\pi}{\longrightarrow} Q \longrightarrow 1
\]

and group presentations for \( K \) and \( Q \), we describe the process to build a group presentation for \( G \). This construction is previously known and included for completeness. Let \( \langle S_K \mid R_K \rangle \) and \( \langle S_Q \mid R_Q \rangle \) be group presentations for \( K \) and \( Q \) respectively. Note that, without loss of generality, we may assume \( 1_Q \in S_Q \). For any set \( S \), let \( F(S) \) denote the free group generated by \( S \). Since we consider \( K \) as a subgroup of \( G \) we will not distinguish between \( K \) and \( \iota(K) \).

Generators: Since \( Q = G/K \), we may lift any element \( \bar{g} \in Q \) to an element \( g \in G \) by choosing a \( g \) such that \( \pi(g) = \bar{g} \). Fix \( S_Q \subset G \) as choice of lifts from elements in \( S_Q \) such that \( \pi \) restricted to \( S_Q \) is a bijection. Note that if \( 1_Q \in S_Q \), we may choose the lift of \( 1_Q \) to be \( 1_G \). Therefore we may assume \( 1_G \notin S_Q \). Define \( S = S_K \cup S_Q \) as the generators of \( G \).

Relations: Since \( K = \langle S_K \mid R_K \rangle \), any element \( k \in K \) can be represented by a word \( k_1 \cdots k_q \in F(S_K) \) for some \( q \geq 1 \) and \( k_i \in S_K \) for all \( i \). Furthermore, since \( K \) is normal in \( G \), both \( sks^{-1} \) and \( s^{-1}ks \) are in \( K \) for any \( s \in S_Q \) and \( k \in K \); the choice of \( s \in S_Q \) determines element \( k', k'' \in K \) such that \( sks^{-1} = k' \) and \( s^{-1}ks = k'' \). For each \( s \in S_Q \) and \( k \in S_K \), fix a choice of words \( x_k \) and \( y_k \) in \( F(S_K) \) representing \( k' \) and \( k'' \) respectively. For all \( s \in S_Q \) and \( k \in S_K \), let \( R' \) be the relations in \( G \) defined by \( sks^{-1} = x_k \) and \( s^{-1}ks = y_k \).
Now, any relation in $\tilde{R}_Q$ is given by $\bar{s}_1 \cdots \bar{s}_q = 1_Q$ where $q \geq 1$ and $\bar{s}_i \in \bar{S}_Q$ for all $1 \leq i \leq q$. Since, for each $i$, $s_i \in S_Q$ is defined to be a lift of $\bar{s}_i$ and $G$ is an extension of $Q$ by $K$, there exists some $k \in K$ such that $s_1 \cdots s_q = k$. Furthermore, the choices of $s_i$ as a lift of $\bar{s}_i$ uniquely determines $k \in K$. Therefore, $k$ is expressed as a word $k_1 \cdots k_t$ on the generators of $S_K$, for some $t \geq 1$. Hence $s_1 \cdots s_q = k_1 \cdots k_t$ determines the relation $s_1 \cdots s_q k_t^{-1} \cdots k_1^{-1} = 1_G$ in $G$. Let $R_Q$ be the set of all relations in $G$ determined by lifting relations of $\tilde{R}_Q$ using this method. Define $R = R_K \cup R_Q \cup R'$ and $G' = \langle S \mid R \rangle$.

**Isomorphism**: Given the definition for $G'$ above, it remains to show $G' \approx G$. Consider the natural map $f : G' \to G$ defined by $f(s) = s$ for all $s \in S$, it suffices to show there exists homomorphisms $\iota'$ and $\pi'$ such that following diagram has exact rows and commutes:

\[
\begin{array}{ccc}
1 & \longrightarrow & K \\
\downarrow & \searrow & \downarrow \pi_Q \\
1 & \longrightarrow & K \\
\end{array}
\]

\[
\begin{array}{ccc}
& & 1 \\
& & \longrightarrow \\
& & \longrightarrow \\
& & \downarrow \iota_Q \\
& & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
\iota' \quad & \quad \pi' & \quad \iota \\
\pi' & = & \pi \\
\iota' & = & \iota \\
\end{array}
\]

**Lemma 2.1.** Let $G$ be the extension of $Q$ by $K$ above and $G' = \langle S \mid R \rangle$ be the group described in the preceding paragraphs. There exists $\iota' : K \to G'$ such that $\iota'$ is injective and commutes with $\pi$.

**Proof.** Recall $(S_K \mid R_K)$, $(\bar{S}_Q \mid \bar{R}_Q)$ and $(S \mid R)$ are group presentations for $K, Q$ and $G'$ respectively. Since $K \cong (S_K \mid R_K)$, any element $k \in K$ can be written as $k_1 \cdots k_t$ where $k_i \in S_K$ for all $i$. Define $\iota' : K \to G'$ by $\iota'(k) = k_1 \cdots k_t$. Since $R_K \subset R$, $\iota'$ is a well defined homomorphism. Furthermore, since $f : G' \to G$ defined above is natural, $f \circ \iota'(k) = \iota \circ id_K(k)$ for all $k \in K$.

**Lemma 2.2.** Let $K, G', Q$ be groups with presentations be defined as above. Any element $g \in G'$ can be written as $g = w_1 w_2$ where $w_1$ and $w_2$ are words in the generators of $Q$ and $K$ respectively.

**Proof.** Let $g \in G'$, then $g = s_1 \cdots s_t$ where $s_i \in S$. By induction on $t$, suppose $g = s_1 s_2$. In this case the statement is trivial unless $s_1 \in S_K$ and $s_2 \in S_Q$. By the relations of $R'$ (conjugating $S_K$ by elements of $S_Q$ and the inverses) there exists $k \in K$ such that $s_2^{-1} s_1 s_2 = k$. Consider

\[
s_1 s_2 = s_2 s_2^{-1} s_1 s_2 = s_2 k
\]

Since $K$ is generated by $S_K$, $k$ is represented by a word $w_2 \in F(S_K)$ and hence the Lemma holds for $t = 2$.

Now, suppose the Lemma is true for any element of $G'$ that can be represented by a reduced word of length $t - 1$ in the generators of $S$. Suppose $g = s_1 \cdots s_t$, then by induction $s_1 \cdots s_{t-1} = w_1 w_2$ where $w_1$ and $w_2$ are words in the generators of $S_Q$ and $S_K$ respectively. If $s_t \in S_K$, then we are done. Therefore assume $s_t \in S_Q$, then we have $g = w_1 w_2 s_t$. In particular, $w_2 = k_1 \cdots k_t$ for $k_1, \ldots, k_t \in S_K$. By the relations of $R'$ there exists $h_i \in K$ such that $s^{-1}_i k_i s_i = h_i$ for every $i$ such that $1 \leq i \leq \ell$. Thus we have:

\[
s_1 \cdots s_t = w_1 w_2 s_t = w_1 k_1 \cdots k_t s_t \cdots = w_1 k_1 \cdots k_{t-1} s_t^{-1} k_t s_t = w_1 k_1 \cdots k_{t-2} s_t^{-1} k_{t-1} s_t h_t \cdots = w_1 s_t h_1 \cdots h_t
\]
Noting that each $h_i$ can be written as a word in $F(SK)$ for all $i$ proves the Lemma. \hfill $\Box$

**Lemma 2.3.** Let $K, G', Q$ with group presentations defined previously. There exists $\pi' : G' \to Q$ such that \( [R] \) commutes.

**Proof.** Let $K, G, Q,$ and $G'$ be as above. Recall $G' = \langle S \mid R \rangle$ where $S = SQ \cup SK$ and $R = R' \cup RQ \cup RK$. Without loss of generality, assume $1_Q \notin S_Q$. Define $\pi' : G \to Q$ by the following:

$$\pi'(s) = \begin{cases} \bar{s} & s \in SQ \\ 1 & s \in SK \end{cases}$$

We first show $\pi'$ is well defined by proving $\pi'$ respects the relations in $R$. If $r$ is a relation in $R_K$, then $r$ can be expressed as a word in $F(SK)$ and $\pi'(r) = 1$. By construction any relation $r \in R_Q$ is $s_1s_2\cdots s_qk_1^{-1}k_2^{-1}\cdots k_1^{-1} = 1$ where $s_i \in SQ$ for all $1 \leq i \leq q$ and $k_j \in SK$ for all $1 \leq j \leq t$.

Furthermore, for each $i$,

$$\pi'(s) = \begin{cases} \bar{s} & s \in SQ \\ 1 & s \in SK \end{cases}$$

By the choice of $S_Q$ as a lift of $\bar{S}_Q$, $\bar{s}_1\bar{s}_2\cdots \bar{s}_q = 1$ is a relation in $Q$.

Now, suppose $r$ is a relation in $R'$. Then $r$ is of the form $sks^{-1} = k'$ for some $s \in SQ$, $k \in SK$, and $k' \in K$. Since $k' \in K$, $k'$ can be represented by a word $k_1k_2\cdots k_t$ in $F(SK)$, $\pi'(k') = 1$. It remains to show $\pi'(sks^{-1}) = 1$.

Thus $\pi'$ is a well defined homomorphism. Furthermore, by the definition of $\mathcal{S}$, $\pi'$ is a surjection.

It remains to show $\pi'$ commutes with \([R]\). Let $g \notin G'$, then $g$ can be represented by $s_1s_2\cdots s_q$ for some word in $F(S)$. Since we assume $1_Q \notin S_Q$, notice that $\pi \circ f(g) = \bar{s}_1\bar{s}_2\cdots \bar{s}_q$ where $\bar{s}_1 = 1_Q$ if and only if $s_i \in SK$. This is the definition of $\pi'$, so $\pi'$ commutes with \([R]\). \hfill $\Box$

**Theorem 2.4.** Let $K, G', Q$ be as above with $\iota'$ and $\pi'$ as in the previous lemmas. Then the top row of \([R]\) is exact.

**Proof.** By Lemma 2.1 and Lemma 2.3 it suffices to show $\im \iota' = \ker \pi'$. By the definition of $\pi'$, $\im \iota' \subseteq \ker \pi'$. It remains to show $\ker \pi' \subseteq \im \iota'$. Suppose $g \in \ker \pi'$, by Lemma 2.2 $g = w_1w_2$ for some words $w_1$ and $w_2$ on the generators of $S_Q$ and $SK$ respectively. Furthermore, $w_1 = \bar{s}_1\cdots \bar{s}_q$ where $s_i \in S_Q$ for all $i$. Since $w_2$ can be expressed as a word in $F(SK)$, $\pi'(g) = \bar{s}_1\bar{s}_2\cdots \bar{s}_q$ where $\bar{s}_i \in S_Q$ for all $i$. But $g$ is in $\ker \pi'$, so $\bar{s}_1\cdots \bar{s}_q = 1$ is a relation of $Q$. Since $R_Q$ normally generates all relations of $Q$ in $F(S_Q)$:

$$\bar{s}_1\cdots \bar{s}_q = \bar{r}_1\bar{r}_2\bar{r}_3\cdots \bar{r}_t$$

where $\bar{r}_i \in F(S_Q)$ and $\bar{r}_i \in R_Q$ for all $i$. For each $i$, $\bar{r}_i$ lifts to a word $r_i \in F(S_K)$ and $\bar{r}_i$ lifts to the relation $r_i = k_i$ where $k_i \in K$. Therefore $w_1$ is equivalent to the following in $G'$:

$$x_1k_1x_1^{-1}x_2k_2x_2^{-1}\cdots x_tx_t^{-1}$$

Furthermore, for each $i$, $k_i$ can be represented by $k_{i,1}k_{i,2}\cdots k_{i,\ell_i}$ where each $k_{i,j} \in SK$. Therefore, for each $i$, $x_ik_ix_i^{-1}$ can be represented by:

$$x_1k_{i,1}x_1^{-1}x_2k_{i,2}x_2^{-1}\cdots x_\ell_i$$
Since each \( x_i \in F(S_Q) \), applying relations of \( R' \) to \( x_i k_{i,j} x_i^{-1} \) yields \( x_i k_{i,j} x_i^{-1} = k_{i,j}' \) for some \( k_{i,j}' \in K \). For each \( i \) we get \( k_i = k_{i,1} k_{i,2} \cdots k_{i,n} = k_i' \) for some \( k_i' \in K \). Therefore \( g = k_i' k_i'' \cdots k_i' w_2 \) where \( k_i' \) and \( w_2 \) are words in \( F(S_K) \) for all \( i \). Therefore \( g \) can be expressed as a word in \( F(S_K) \) and \( g \in K \). Thus \( \ker \pi' \subseteq \im \iota' \) and the top row of (3) is exact.

**Theorem 2.5.** Let \( K, G, \) and \( Q \) be defined above with \( S \) and \( R \) as in the paragraphs on generators and relations. Then \( (S | R) \) is a presentation for \( G \).

**Proof.** Consider the diagram (3):

\[
\begin{array}{ccc}
1 & \rightarrow & K \\
\downarrow & & \downarrow \\
1 & \rightarrow & K
\end{array}
\]

\[
\begin{array}{ccc}
K & \rightarrow & G' \\
\downarrow & & \downarrow \\
G & \rightarrow & Q
\end{array}
\]

\[
\begin{array}{ccc}
Q & \rightarrow & 1 \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1
\end{array}
\]

with \( \iota' \) and \( \pi' \) defined as in Lemmas 2.1 and 2.3. It suffices to show \( f : G' \rightarrow G \) is an isomorphism. Since \( G \) is an extension of \( Q \) by \( K \), the bottom row of (3) is exact. By Theorem 2.4, the top row of (3) is exact. Therefore both rows of (3) are exact. Furthermore, \( f \) is a natural map so \( f \) makes the diagram commute. Thus \( f \) is an isomorphism by the 5-Lemma.

### 2.2 Group cohomology

In this section we will review the general group cohomology required to classify group extensions by 2-cocycles. We begin with constructing the normalized bar resolution, then give the definition by 2-cocycles. We begin with constructing the normalized bar resolution, then give the definition of equivalent group extensions, and describe the construction of a cocycle from an extension. The results of this section are known and can be found more thoroughly in chapters I and IV of Brown’s text [5].

**Normalized standard resolution** Let \( G \) be a group and let \( P_t \) be the free \( \mathbb{Z} \) module generated by \( t + 1 \) tuples \((p_0, \ldots, p_t)\) where \( p_i \in G \) for all \( i \). \( G \) acts on \((p_0, \ldots, p_n)\) by \( p \cdot (p_0, \ldots, p_n) = (p_0 \cdot p, \ldots, p_n \cdot p) \) for any \( p \in G \). To construct a chain complex, we use the boundary operator \( \partial_t^P : P_t \rightarrow P_{t-1} \) determined by \( \partial_t^P = \sum_{i=0}^{t} (-1)^i d_i \) where:

\[
d_i(p_0, \ldots, p_t) = (p_0, \ldots, p_{i-1}, p_{i+1}, \ldots, p_t)
\]

As \( \mathbb{Z}G \) modules, \( P_t \) is freely generated by elements \((1, p_1, \ldots, p_t)\) which represent the \( G \) orbits of the \( t + 1 \) tuples where \( p_0 = 1 \). We use the following bar notation to represent elements of \( P_t \) as \( G \)-orbits: \([p_1 | p_2 | \ldots | p_t] = (1, p_1 p_2, \ldots, p_1 p_2 \cdot p_t)\). The change of basis results in the following change to \( d_i \) in the boundary operator:

\[
d_i[p_1 | \cdots | p_t] = \begin{cases} 
  p_1 [p_2 | \cdots | p_t] & i = 0 \\
  [p_1 | \cdots | p_{i-1} | p_i p_{i+1} | p_{i+2} | \cdots | p_t] & 0 < i < t \\
  [p_1 | \cdots | p_{i-1}] & i = t
\end{cases}
\]

In particular \( \partial_t^P([p_1 | p_2]) = p_1 [p_2] - [p_1 p_2] + [p_1] \) and \( \partial_t^P([p_1]) = (p_1 - 1)[1] \). The augmentation map \( \varepsilon_P : P_0 \rightarrow \mathbb{Z} \) defined by \( \varepsilon_P([1]) = 1 \) implies

\[
\mathcal{P} : \cdots \xrightarrow{\partial_t^P} P_2 \xrightarrow{\partial_{t-1}^P} P_1 \xrightarrow{\partial_{t-2}^P} P_0 \xrightarrow{\varepsilon_P} \mathbb{Z} \rightarrow 0
\]
is a free resolution of \( \mathbb{Z} \) over \( \mathbb{Z} G \) modules. Now, let \( D_t \) be the subcomplex of \( P_t \) generated over \( \mathbb{Z} G \) by elements \( [p_1 | \ldots | p_t] \) such that \( p_i = 1 \) for some \( i \). Then \( P_t = P_t/D_t \) with the maps of the chain complex \( \mathcal{P} \) induced from \( \mathcal{P} \) defines a free resolution of \( \mathbb{Z} \) over \( \mathbb{Z} G \) modules, called the normalized standard resolution.

Furthermore, two projective resolutions of \( \mathbb{Z} \) over \( \mathbb{Z} G \) are chain homotopy equivalent. For a proof of this fact the interested reader is referred to chapter one of Brown’s text [5].

**Equivalent extensions** Let \( K \) be an abelian group and let \( Q \) be a group which acts on \( K \) by the map \( \theta : Q \to Aut(K) \). An extension of \( Q \) by \( K \) giving rise to \( \theta \) is a short exact sequence:

\[
1 \longrightarrow K \overset{i}{\longrightarrow} E \overset{\pi}{\longrightarrow} Q \longrightarrow 1
\]

with the following condition: for any \( k \in K \) and \( \tilde{g} \in E \) such that \( \pi(\tilde{g}) = g \in G \), then \( \tilde{g}i(k)\tilde{g}^{-1} = i(\theta(g)(k)) \). Two group extensions \( E_1 \) and \( E_2 \) are equivalent if there exists an isomorphism \( \varphi : E_1 \to E_2 \) such that the diagram:

\[
\begin{array}{ccc}
1 & \longrightarrow & K \\
\downarrow & & \downarrow \varphi \\
E_1 & \longrightarrow & Q & \longrightarrow & 1 \\
\uparrow & & & & \uparrow \\
E_2 & \longrightarrow & & &
\end{array}
\]

commutes. Let \( \mathcal{E}(Q;K) \) denote the set of equivalence classes of these extensions.

**Constructing 2-cocycles** Suppose \( K \) is an abelian group and \( Q \) acts on \( K \) by the action of \( \theta \) as in the preceding paragraph. Consider the extension:

\[
0 \longrightarrow K \overset{i}{\longrightarrow} E \overset{\pi}{\longrightarrow} Q \longrightarrow 1
\]

A section, \( s \), is a function \( s : Q \to E \) such that \( \pi \circ s = id_Q \); furthermore \( s \) is normalized if \( s(1_Q) = 1_E \). Since \( s \) is a set theoretical function, \( s(p_1)s(p_2) \) is not necessarily equal to \( s(p_1p_2) \) in \( E \). Therefore we can measure the failure of \( s \) to be a homomorphism by a function \( \kappa \in Hom_Q(P_2,K) \) such that \( s(p_1)s(p_2) = i(\kappa([p_1 | p_2]))s(p_1p_2) \). Hence we have a formula:

\[
\kappa([p_1 | p_2]) = s(p_1)s(p_2)s(p_1p_2)^{-1}
\]

which determines a function corresponding to \( E \) as an extension of \( Q \) by \( K \). A thorough explanation that \( \kappa \) satisfies the cocycle condition can be found in chapter IV, section 3 of Brown [5].

**Corresponding Extensions** Let \( \kappa \) be a representative for a cohomology class in \( H^2(Q;K) \) determined by the normalized standard resolution. Suppose \( Q \) acts on \( K \) by the action \( \theta \), define \( E_\kappa \) to be the twisted semi-direct product \( K \rtimes_\kappa Q \) with multiplication defined by:

\[
(a, g) \cdot (b, h) = (a + g \cdot b + \kappa(g, h), gh)
\]

Note that multiplication in \( E_\kappa \) satisfies associativity since \( \kappa \) is a cocycle (page 92 of [5]). Thus \( E_\kappa \) is a representative for the equivalence class of group extensions of \( Q \) by \( K \) corresponding to \([\kappa] \).

The following theorem by Eilenberg and MacLane provides the classification of group extensions by 2-cocycles constructed above [7].
Theorem 2.6 (Eilenberg MacLane 1947). Suppose $Q$ and $K$ are groups with $K$ abelian such that $Q$ acts on $K$ by $\theta$. There exists a bijection between $E(Q, K)$ and $H^2(Q; K)$.

The choice of section determines the representative of the cohomology class in $H^2(Q; K)$. Furthermore, if the extension, $E$, is split, then $E \cong K \rtimes Q$ corresponds to the cohomology class represented by the trivial cocycle. Changing the choice of projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ yields a corresponding representative in an isomorphic cohomology group.

2.3 Braid groups and symmetric groups

Symmetric Group The standard presentation for the symmetric group generated by adjacent transpositions is:

$$S_n = \left\{ \sigma_1, \ldots, \sigma_{n-1} \middle| \begin{array}{l}
\sigma_i^2 = 1, \quad [\sigma_i, \sigma_j] = 1 \text{ if } |i - j| > 1 \\
\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}
\end{array} \right\}$$

(4)

Let $\sigma_{i,j}$ represent the transposition which permutes $i$ and $j$. As an element of (4) we take the convention:

$$\sigma_{i,j} = \sigma_{i} \cdots \sigma_{j-2} \sigma_{j-1} \sigma_{j-2}^{-1} \cdots \sigma_i^{-1}$$

where $i < j$. Throughout this paper, we will use the following presentation for the symmetric group:

$$S_n = \left\{ \{\sigma_{i,j}\}_{1 \leq i < j \leq n} \middle| \begin{array}{l}
\sigma_{i,j}^2 = 1, \quad [\sigma_{i,j}, \sigma_{k,\ell}] = 1 \text{ if } \{i, j\} \cap \{k, \ell\} = \emptyset \\
\sigma_{i,j}\sigma_{j,k}\sigma_{i,j}^{-1} = \sigma_{i,k} \text{ for all } i, j, k
\end{array} \right\}$$

(5)

Braid groups Let $x_1, \ldots, x_n$ be marked points in $\mathbb{C}$. Elements of $B_n$ can be represented by a collection of $n$ non-colliding paths $f_i : [0,1] \to \mathbb{C} \times [0,1]$ such that $f_i(0), f_i(1) \in \{x_1, \ldots, x_n\}$ and $f_i(t) \in \mathbb{C} \times \{t\}$. Then the collection $\{f_i\}$ determines a permutation on $\{1, \ldots, n\}$. Throughout this paper we will represent elements of $B_n$ by strand diagrams representing the paths $f_1(t), \ldots, f_n(t)$. A positively oriented twist between strands $i$ and $i+1$, denoted by $b_i$, corresponds to a clockwise twist in the strand diagram, where the $(i+1)st$ strand passes over the $i^{th}$ strand.

For $1 \leq i < j \leq n$, we define the half twist between the $i$ and $j$ strands by:

$$b_{i,j} = b_i \cdots b_{j-2} b_{j-1}^{-1} b_{j-2} \cdots b_i^{-1}$$

Under the strand diagrams, this is equivalent to pulling all strands between the $i^{th}$ and $j^{th}$ strands over the $i^{th}$ strand, half twist the $i^{th}$ and $j^{th}$ strands by pulling the $j^{th}$ strand over the $i^{th}$ strand, then pull all strands between $i$ and $j$ over the $j^{th}$ strand. This choice of $b_{i,j}$ yields the following presentation for $B_n$ which is equivalent to the Birman-Ko-Lee presentation [3]:

$$B_n = \left\{ \{b_{i,j}\}_{1 \leq i < j \leq n} \middle| \begin{array}{l}
[b_{i,j}, b_{k,\ell}] = 1 \text{ if } (j-k)(j-\ell)(i-\ell)(i-\ell) > 0 \\
b_{i,j}b_{j,k}b_{i,j}^{-1} = b_{i,k} \text{ if } i < j < k, \quad k < i < j, \quad j < k < i \\
b_{i,j}^{-1}b_{j,k}b_{i,j} = b_{i,k} \text{ if } j < i < k, \quad i < k < j, \quad j < k < i
\end{array} \right\}$$

(6)

Note that the third relation can be determined by the second relation. Conjugating the second relation by $b_{i,j}^{-1}$ and renaming the indices provides the third relation. However the third relation is included for clarity in later computations.
**Pure braids** As a subgroup of the braid group, elements of the pure braid group, \( PB_n \), are braids in which the induced permutation on \( \{1, \ldots, n\} \) is trivial. Considering strand diagrams, every strand begins and ends at the same point of \( \mathbb{C} \). In terms of \([4]\), \( PB_n \) is generated by all \( b_i^2 \). For any pure braid, we can define the winding number of the \( i^{th} \) and \( j^{th} \) strand to be the number of positively oriented full twists between those two strands.

**Level \( m \) braid group** Let \( \mathbb{Z}_m \) denote \( \mathbb{Z}/m\mathbb{Z} \) and \( B_n \) be the braid group. Evaluating the unreduced Burau representation (See Birman \([2]\)), at \( t = -1 \), and reducing mod \( m \) (for any \( m \geq 0 \)) yields the following map from \( B_n \) to \( GL_n(\mathbb{Z}_m) \):

\[
B_n \xrightarrow{\rho} GL_n(\mathbb{Z}[t,t^{-1}]) \xrightarrow{t \mapsto t^{-1}} GL_n(\mathbb{Z}) \xrightarrow{} GL_n(\mathbb{Z}_m)
\]

The map \( \rho : B_n \to GL_n(\mathbb{Z}[t,t^{-1}]) \) is the unreduced Burau representation of the braid group defined on the generators of \([2]\) by:

\[
\rho(b_i) \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 0 & 1 \end{pmatrix} \oplus I_{n-i-1}
\]

For \( m \geq 0 \), the level \( m \) braid group \( B_n[m] \) is defined as the kernel of this composition, which is a finite index subgroup of \( B_n \). A thorough topological description of the Burau representation and alternative descriptions of \( B_n[m] \) can be found in Breckle and Margalit’s paper \([4]\).

**Definition 2.1.** The mod 4 braid group, denoted \( \mathbb{Z}_n \), is the quotient of the braid group by the level 4 congruence subgroup of the braid group, \( B_n/B_n[4] \).

For arbitrary \( m \), little is known about the algebraic structure of \( B_n[m] \). In the case \( m = 4 \), Breckle and Margalit proved \( B_n[4] = PB_n^2 \), the subgroup of \( PB_n \) generated by squares of all elements \([4]\). The algebraic structure of quotients of level \( m \) braid groups is better understood. Stylianakis proved that for each odd prime \( p \), \( B_n[p]/B_n[2p] \cong \mathbb{Z}_n \) \([11]\). Appel, Bloomquist, Gravel, and Holden generalized Stylianakis’ result to \( B_n[\ell]/B_n[2\ell] \cong \mathbb{Z}_n \) for every odd, positive integer \( \ell \) \([1]\). In the same work they also proved the following Theorem, which yields greater context for the group presentation of \( \mathbb{Z}_n \) we give \([4]\):

**Theorem 2.7** (Appel, Bloomquist, Gravel, Holden). For each \( n \) and each \( \ell \) odd:

\[
B_n[\ell]/B_n[4\ell] = \mathbb{Z}_n
\]

**Formalization of Theorem** \([13]\) Define \( \mathbb{Z}_n = B_n/PB_n^2 \) and \( \mathcal{P}Z_n \) be the image of \( PB_n \) in \( \mathbb{Z}_n \).

The standard surjection of \( B_n \) onto the symmetric group \( S_n \) yields the following non split group extension \([4]\):

\[
1 \to \mathcal{P}Z_n \to \mathbb{Z}_n \to S_n \to 1
\]

Kordek and Margalit also proved \( \mathcal{P}Z_n \cong \mathbb{Z}_n^2 \) where the action of \( S_n \) on \( \mathcal{P}Z_n \), represented by \( \theta \), is induced by the action of \( B_n \) on \( PB_n \) \([4]\). Let \([\kappa] \in H^2(S_n; \mathcal{P}Z_n) \) be the nontrivial cohomology class corresponding to this extension. Since \( \mathbb{Z}_n^2 \cong H_1(PB_n; \mathbb{Z}) \) is the abelianization of \( PB_n \) (page 252 of \([8]\)), consider the group extension:

\[
0 \to H_1(PB_n; \mathbb{Z}) \to G_n \to S_n \to 1
\]

where \( G_n \) is the quotient of \( B_n \) by the commutator subgroup of \( PB_n \). Let \([\phi] \in H^2(S_n; H_1(PB_n; \mathbb{Z})) \) be the cohomology class corresponding to the extension of \( S_n \) by \( H_1(PB_n; \mathbb{Z}) \). In this paper we prove a representative of \([\phi] \) composed with the mod 2 reduction of integers determines a representative for \([\kappa] \). Furthermore, we show \([\phi] \) is order 2 in \( H^2(PB_n; H_1(PB_n; \mathbb{Z})) \).
3 The universal cover of a $K(S_n, 1)$-space

In this section we construct a truncated resolution of $\mathbb{Z}$ over $\mathbb{Z}S_n$ corresponding to the universal cover of a $K(S_n, 1)$-space. We approximate the cellular chain complex in dimensions 0, 1, and 2 for the universal cover, $\tilde{X}$, of the 2-skeleton for the Cayley complex of $S_n$. Let $R$ be the chain complex of $\tilde{X}$ and recall $\mathcal{P}$ is the normalized bar resolution. Since each dimension of $R$ is a free $S_n$ module and $\mathcal{P}$ is acyclic, for each $i$ there exists $\gamma_i : R_i \to P_i$ which fits into the following commutative diagram:

\[ \cdots \xrightarrow{\partial^n_0} R_2 \xrightarrow{\partial^n_1} R_1 \xrightarrow{\partial^n_0} R_0 \xrightarrow{\varepsilon_R} \mathbb{Z} \xrightarrow{id} 0 \]

\[ \cdots \xrightarrow{\partial^n} P_2 \xrightarrow{\partial^n_1} P_1 \xrightarrow{\partial^n_0} P_0 \xrightarrow{\varepsilon_P} \mathbb{Z} \xrightarrow{id} 0 \]

The existence of $\gamma_i$ yields the following theorem:

**Theorem 3.1.** Let $0 \to K \to E \to S_n \to 1$ be a group extension which corresponds to $[\kappa] \in H^2(S_n; K)$ under the normalized standard resolution. There exists $\kappa' \in Hom(R_2, K)$ such that $[\kappa'] = [\kappa]$ in $H^2(S_n; K)$ defined by:

\[ \kappa' = \kappa \circ \gamma_2 \]

**Cayley Complex** To describe the universal cover of a $K(S_n, 1)$-space, we use the group presentation of the symmetric group generated by all possible transpositions given by $[5]$. Let $X$ be a $K(S_n, 1)$-space which has the presentation complex as its 2-skeleton. Then $X$ has one 0-cell, a 1-cell for each generator of $S_n$, and a 2-cell for each relation in $S_n$. Let $x_0$ denote the 0-cell of $X$ and $x_{i,j}$ denote the 1-cell of $X$ corresponding to the generator $\sigma_{i,j}$ of $S_n$. Note that we only use positive generators to label 1-cells. Let $c_{ij}$ be the two cell glued by the relation $\sigma_{ij} = 1$ and $d_{i,j,k,\ell}$ be the two cell glued by the relation $[\sigma_{ij}]$ if $\{i, j\} \cap \{k, \ell\} = \emptyset$. Consider the relation $\sigma_{ij}\sigma_{jk}\sigma_{ij}^{-1} = \sigma_{i,k}$ from $[5]$ as $\sigma_{ij}\sigma_{jk}\sigma_{ik}^{-1}\sigma_{i,k}^{-1} = 1$; let $c_{i,j,k}$ be the 2-cell that glues along this relation in $X$. Note we have one 2-cell $c_{i,j,k}$ for each distinct, ordered triple in $\{1, 2, \ldots, n\}$ while $c_{i,j}$ and $d_{i,j,k,\ell}$ assume $i < j$ and $k < \ell$.

Let $\tilde{X}$ be the universal cover of $X$ with the basepoint, $\tilde{x}_0$, chosen to be the lift of $x_0$. For each $i$-cell of $X$, there are $n!$ choices of lifts in $\tilde{X}$. Then the 0-cells of $\tilde{X}$ are of the form $g \cdot \tilde{x}_0$ for $g \in S_n$ (with $g = 1_{S_n}$ representing $\tilde{x}_0$) and $R_0$ is generated as a $\mathbb{Z}S_n$ module by $\tilde{x}_0$.

Now, lift $x_{i,j}$ to the 1-cell, $\tilde{x}_{i,j}$, of $\tilde{X}$ which begins at $\tilde{x}_0$ and ends at $\sigma_{ij} \cdot \tilde{x}_0$. Notice that for each 0-cell, $g \cdot \tilde{x}_0$, there is a 1-cell, denoted $g \cdot \tilde{x}_{i,j}$, beginning at $g \cdot \tilde{x}_0$ and ending at $g \sigma_{i,j} \cdot \tilde{x}_0$. Therefore $R_1$ is generated as a $\mathbb{Z}S_n$ module by the set of $\tilde{x}_{i,j}$ where $1 \leq i < j \leq n$. Furthermore, the gluing map determines the differential $\partial^2_i : R_1 \to R_0$ by $\partial^2_i (\tilde{x}_{i,j}) = (\sigma_{ij} - 1) \tilde{x}_0$. 
Let \( \tilde{c}_{i,j} \) denote the lift of \( c_{i,j} \) glued into \( \tilde{X} \) along the loop which begins at \( \tilde{x}_0 \), follows \( \tilde{x}_{i,j} \) to \( \sigma_{i,j} \cdot \tilde{x}_0 \), then follows \( \sigma_{i,j} \cdot \tilde{x}_{i,j} \) back to \( \tilde{x}_0 \). Define \( \tilde{d}_{i,j,k,l} \) to be the lift of \( d_{i,j,k,l} \) glued into \( \tilde{X} \) by the loop which starts at \( \tilde{x}_0 \), follows \( \tilde{x}_{i,j} \) to \( \sigma_{i,j} \cdot \tilde{x}_0 \), then follows \( \sigma_{i,j} \cdot \tilde{x}_{i,j} \) in reverse to \( \sigma_{k,l} \cdot \tilde{x}_0 \), and follows \( \tilde{x}_{k,l} \) in reverse back to \( \tilde{x}_0 \). Furthermore, let \( \tilde{e}_{i,j,k} \) denote the lift of \( e_{i,j,k} \) glued into \( \tilde{X} \) by the loop which begins at \( \tilde{x}_0 \), follows \( \tilde{x}_{i,j} \) to \( \sigma_{i,j} \cdot \tilde{x}_0 \), then follows \( \sigma_{i,j} \cdot \tilde{x}_{i,j} \) to \( \sigma_{i,j} \sigma_{j,k} \cdot \tilde{x}_0 \), then follows \( \sigma_{j,k} \cdot \tilde{x}_{i,j} \) in reverse to \( \sigma_{i,k} \cdot \tilde{x}_0 \), and finally follows \( \tilde{x}_{i,k} \) in reverse back to \( \tilde{x}_0 \). Notice the paths of \( \tilde{d}_{i,j,k,l} \) and \( \tilde{e}_{i,j,k} \) are given in Figure 1.

Furthermore, for each \( g \cdot \tilde{x}_0 \), there is a lift of \( \tilde{c}_{i,j} \), \( \tilde{d}_{i,j,k,l} \), and \( \tilde{e}_{i,j,k} \) glued into \( \tilde{X} \) by a corresponding loop which begins and ends at \( g \cdot \tilde{x}_0 \), denoted \( g \cdot \tilde{c}_{i,j} \), \( g \cdot \tilde{d}_{i,j,k,l} \), and \( g \cdot \tilde{e}_{i,j,k} \) respectively. Therefore, as a free \( \mathbb{Z}S_n \) module, \( R_2 \) is generated by the set of all \( \tilde{c}_{i,j} \), \( \tilde{d}_{i,j,k,l} \), and \( \tilde{e}_{i,j,k} \). To determine \( \partial^R_2 : R_2 \to R_1 \), we use the gluing of 2-cells in \( \tilde{X} \). Therefore \( \partial^R_2 (\tilde{c}_{i,j}) = (\sigma_{i,j} \cdot 1) \tilde{x}_{i,j} \) and by Figure 1 we have:

\[
\partial^R_2 (\tilde{d}_{i,j,k,l}) = \tilde{x}_{i,j} + \sigma_{i,j} \cdot \tilde{x}_{k,l} - \sigma_{k,l} \cdot \tilde{x}_{i,j} - \tilde{x}_{k,l} \\
\partial^R_2 (\tilde{e}_{i,j,k}) = \tilde{x}_{i,j} + \sigma_{i,j} \cdot \tilde{x}_{j,k} - \sigma_{i,k} \cdot \tilde{x}_{i,j} - \tilde{x}_{i,k}
\]

Therefore we have proven the following lemma.

**Lemma 3.2.** As a \( \mathbb{Z}S_n \) module, \( R_1 \) is generated by \( \{ \tilde{x}_{i,j} \} \) and \( R_0 \) by \( \tilde{x}_0 \). Furthermore \( R_2 \) is generated by \( \{ \tilde{c}_{i,j} \} \cup \{ \tilde{d}_{i,j,k,l} \} \cup \{ \tilde{e}_{i,j,k} \} \) where \( i < j \) and \( k < l \).

By Lemma 3.2, defining \( \gamma_0 \) and \( \gamma_1 \) by \( \gamma_0 (\tilde{x}_0) = [ \ ] \) and \( \gamma_1 (\tilde{x}_{i,j}) = [ \sigma_{i,j} ] \) implies \( \gamma_0 \) and \( \gamma_1 \) commute with the differential.

**Theorem 3.3.** Define \( \gamma_2 : R_2 \to \tilde{P}_2 \) by:

\[
\gamma_2 (\tilde{c}_{i,j}) = [ \sigma_{i,j} \mid \sigma_{i,j} ] \\
\gamma_2 (\tilde{d}_{i,j,k,l}) = [\sigma_{i,j} \mid \sigma_{k,l}] - [\sigma_{k,l} \mid \sigma_{i,j}] \\
\gamma_2 (\tilde{e}_{i,j,k}) = [\sigma_{i,j} \mid \sigma_{j,k}] - [\sigma_{i,k} \mid \sigma_{i,j}]
\]

Then \( \gamma_2 \) commutes with the differentials:

\[
\begin{array}{cccc}
R_2 & \xrightarrow{\partial^R_2} & R_1 & \xrightarrow{\partial^R_1} & R_0 & \xrightarrow{\epsilon_R} & \mathbb{Z} & \xrightarrow{id} & 0 \\
\downarrow{\gamma_2} & & \downarrow{\gamma_1} & & \downarrow{\gamma_0} & & \downarrow{\epsilon_0} & & 0
\end{array}
\]

**Proof.** \( \gamma_2 \) is a \( \mathbb{Z}S_n \) module homomorphism defined on generators. Therefore it suffices to show \( \gamma_0 \circ \partial^R_2 = \partial^P_2 \circ \gamma_2 \). Recall the generators of \( R_1 \) are \( \tilde{x}_{i,j} \) and \( \gamma_1 (\tilde{x}_{i,j}) = [\sigma_{i,j}] \). By following the gluing maps in Figure 1 we have:

\[
\gamma_0 \circ \partial^R_2 (\tilde{c}_{i,j}) = [\sigma_{i,j}] + \sigma_{i,j} [\sigma_{i,j}] \\
\gamma_0 \circ \partial^R_2 (\tilde{d}_{i,j,k,l}) = [\sigma_{i,j}] + \sigma_{i,j} [\sigma_{k,l}] - [\sigma_{k,l}][\sigma_{i,j}] - [\sigma_{k,l}] \\
\gamma_0 \circ \partial^R_2 (\tilde{e}_{i,j,k}) = [\sigma_{i,j}] + \sigma_{i,j} [\sigma_{j,k}] - [\sigma_{i,k}] [\sigma_{j,k}] - [\sigma_{i,k}]
\]

Recall we are using the normalized bar resolution, therefore \([p_1 \cdots p_l] = 0\) if \( p_i \) is trivial for any \( i \). Now, since \( \partial^P_2 ([p_1 \mid p_2]) = p_1 [p_2] - [p_1 \cdot p_2] + [p_1] \) for any \( p_1, p_2 \in S_n \) and \( \partial^P_2 \) is a \( \mathbb{Z}S_n \) module homomorphism, we have:

\[
\partial^P_2 ([\sigma_{i,j} \mid \sigma_{i,j}]) = \sigma_{i,j} [\sigma_{i,j}] - [\sigma_{i,j}^2] + [\sigma_{i,j}] = \sigma_{i,j} [\sigma_{i,j}] - [1] + [\sigma_{i,j}] = [\sigma_{i,j}] + \sigma_{i,j} [\sigma_{i,j}]
\]
Thus \( \partial_R^2 \circ \gamma_2 = (\gamma_1 \circ \partial_R^2)(\tilde{c}_{i,j}) \). To compute \( \partial_R^2(\gamma_2(\tilde{d}_{i,j,k,l})) \), recall that \( \{i,j\} \cap \{k,l\} = \emptyset \), therefore \( \sigma_{i,j} \sigma_{k,l} = \sigma_{k,l} \sigma_{i,j} \):

\[
\partial_R^2([\sigma_{i,j} | \sigma_{k,l}]) - [\sigma_{k,l} | \sigma_{i,j}] = \sigma_{i,j}[\sigma_{k,l}] - [\sigma_{i,j} \sigma_{k,l}] - [\sigma_{i,j}][\sigma_{k,l}]
\]

Therefore, by the truncated resolution for the universal cover of a \( H \)

Thus \( \partial_R^2(\tilde{c}_{i,j,k,l}) \) recall that in \( S_n \), \( \sigma_{i,k} = \sigma_{i,j} \sigma_{j,k}^{-1} \) and therefore \( \sigma_{i,k} \sigma_{i,j} = \sigma_{i,j} \sigma_{j,k} \).

Finally, to compute \( \partial_R^2(\tilde{c}_{i,j,k,l}) \) recall that in \( S_n \), \( \sigma_{i,k} = \sigma_{i,j} \sigma_{j,k}^{-1} \) and therefore \( \sigma_{i,k} \sigma_{i,j} = \sigma_{i,j} \sigma_{j,k} \).

Hence we have:

\[
\partial_R^2([\sigma_{i,j} | \sigma_{j,k}]) - [\sigma_{j,k} | \sigma_{i,j}] = \sigma_{i,j}[\sigma_{j,k}] - [\sigma_{i,j} \sigma_{j,k}] - [\sigma_{i,j}][\sigma_{j,k}]
\]

**Theorem 3.4.** Let \( K \) be any \( S_n \) module and let \( E \) be an extension of \( S_n \) by \( K \). Suppose \( \kappa \in Hom(P_2, K) \) is a representative for the cohomology class in \( H^2(S_n; K) \) corresponding to \( E \) determined by the normalized bar resolution. Define \( \kappa' \in Hom(R_2, K) \) by:

\[
\kappa'(\tilde{c}_{i,j}) = s(\sigma_{i,j}) (\sigma_{i,j}) \\
\kappa'(\tilde{d}_{i,j,k,l}) = s(\sigma_{i,j}) s(\sigma_{k,l}) s(\sigma_{i,j} \sigma_{k,l}^{-1}) - s(\sigma_{k,l}) s(\sigma_{i,j} \sigma_{k,l}^{-1}) \\
\kappa'(\tilde{e}_{i,j,k,l}) = s(\sigma_{i,j}) s(\sigma_{j,k}) s(\sigma_{i,j} \sigma_{j,k}^{-1}) - s(\sigma_{i,j}) s(\sigma_{j,k} \sigma_{i,j}^{-1})
\]

Then \( \kappa' \) is the 2-cocycle determined by the resolution corresponding to \( X \) such that \([\kappa'] \) and \([\kappa] \) represent the same group extension.

**Proof.** Recall that \( \mathcal{P} \) is the normalized bar resolution and \( \mathcal{R} \) is the resolution corresponding to \( X \). Evaluating \( \kappa \circ \gamma \) on the generators of \( R_2 \) we get:

\[
\kappa([\sigma_{i,j} | \sigma_{i,j}]) = s(\sigma_{i,j}) s(\sigma_{i,j}) \\
\kappa([\sigma_{i,j} | \sigma_{k,l}]) - [\sigma_{k,l} | \sigma_{i,j}] = s(\sigma_{i,j}) s(\sigma_{k,l}) s(\sigma_{i,j} \sigma_{k,l}^{-1}) - s(\sigma_{k,l}) s(\sigma_{i,j} \sigma_{k,l}^{-1}) \\
\kappa([\sigma_{i,j} | \sigma_{k,l}]) = s(\sigma_{i,j}) s(\sigma_{j,k}) s(\sigma_{i,j} \sigma_{j,k}^{-1}) - s(\sigma_{i,j}) s(\sigma_{j,k} \sigma_{i,j}^{-1})
\]

Therefore \( \kappa' = \kappa \circ \gamma \) and by Section 2.2 both \([\kappa'] \) and \([\kappa] \) correspond to the same extension. \( \square \)

### 4 Cohomology class of \( S_n \) with coefficients in \( \mathbb{Z}^{(2)} \)

**Extension of \( S_n \) by \( \mathbb{Z}^{(2)} \)** Let \( K_n \) be the commutator subgroup of \( PB_n \). Since \( H_1(PB_n; \mathbb{Z}) \approx \mathbb{Z}^{(2)} \) is the abelianization of \( PB_n \), we have \( \mathbb{Z}^{(2)} \approx H_1(PB_n; \mathbb{Z}) \approx PB_n/K_n \). Furthermore, \( K_n \) is a characteristic subgroup of \( PB_n \) and \( PB_n \) is normal in \( B_n \), therefore \( K_n \) is normal in \( B_n \).

By the third isomorphism theorem,

\[
(B_n/K_n)/(PB_n/K_n) \approx B_n/PB_n \approx S_n
\]
Table 1: Relations of $G_n$

|   |   |
|---|---|
| R1: | $[g_{i,j}, g_{k,\ell}] = 1$ for all $i, j, k, \ell$ |
| R2: | $\sigma_{i,j}^2 = g_{i,j}$ for all $i$ |
| R3: | $\sigma_{i,j} \sigma_{k,j} \sigma_{i,j}^{-1} = \sigma_{i,k}$ if $k < i < j$; $i < j < k$; or $j < k < i$ |
| R4: | $\sigma_{i,j}^{-1} \sigma_{j,k} \sigma_{i,j} = \sigma_{i,k}$ if $i < j < k$; $i < j < k$; or $k < j < i$ |
| R5: | $[\sigma_{i,j}, \sigma_{k,\ell}] = \begin{cases} g_{k,\ell} \sigma_{i,j} g_{j,\ell}^{-1} \sigma_{i,j}^{-1} & i < j < \ell \\ 1 & \text{otherwise} \end{cases}$ |
| R6: | $\sigma_{i,j} g_{k,\ell} \sigma_{i,j}^{-1} = g_{\sigma_{i,j}(k),\sigma_{i,j}(\ell)}$ for all $i, j, k, \ell \in \{1, \ldots, n\}$ |

Let $G_n = B_n/K_n$, then $G_n$ is an extension of $S_n$ by $H_1(PB_n; \mathbb{Z})$. Therefore we have the group extension:

$$0 \longrightarrow \mathbb{Z}^{(n)} \overset{i_1}{\longrightarrow} G_n \overset{\pi_1}{\longrightarrow} S_n \longrightarrow 1$$

where the action of $S_n$ on $\mathbb{Z}^{(n)}$ is induced by the conjugation of $B_n$ on $PB_n$. Notice the induced action of $S_n$ on the abelianization of the pure braid group permutes the strands of pure braids. So $S_n$ permutes the generators of $\mathbb{Z}^{(n)}$ by acting on the indices with the standard action of $S_n$ on unordered pairs of integers.

**Normalized section** Since $G_n$ is a quotient of $B_n$, let $\tilde{\sigma}_{i,j}$ represent the projection of the positively oriented half twist between the $i$th and $j$th strands, $b_{i,j}$ from (6), in $G_n$. In particular, we need to fix a choice of normal form for each element in $S_n$ and choose an algorithm which takes any element of $S_n$, and produces the chosen normal form in terms of the generators $\sigma_{i,j}$.

Note, since we consider multiplication in $G_n$ from left to right, we will also consider composition of permutations in $S_n$ from left to right for consistency. Let $p \in S_n$ such that $p(n) = k_n$, then $p \cdot \sigma_{k_n,n} \in S_{n-1}$. Suppose $p \cdot \sigma_{k_n,n}(n-1) = k_{n-1}$, then $p \cdot \sigma_{k_n,n} \cdot \sigma_{k_{n-1},n-1} \in S_{n-2}$. Inductively:

$$p \cdot \sigma_{k_n,n} \cdot \sigma_{k_{n-1},n-1} \cdot \cdots \sigma_{1,k_1} = 1$$

Therefore $p = \sigma_{1,k_1} \cdot \sigma_{2,k_2} \cdots \sigma_{k_n,n}$. Define the normal section $s : S_n \rightarrow G_n$ by:

$$s(p) = \tilde{\sigma}_{1,k_1} \tilde{\sigma}_{2,k_2} \cdots \tilde{\sigma}_{k_n,n}$$

**4.1 Presentation of $G_n$**

**Generators of $G_n$** Let $g_{i,j}$ ($1 \leq i < j \leq n$) be the commuting generators of $\mathbb{Z}^{(n)}$. Since $\mathbb{Z}^{(n)} = PB_n/K_n$, each $g_{i,j}$ represents the projections of a pure braid, $b_{i,j}^2$, in $G_n$. In particular $g_{i,j}$ is the positively oriented full twist of the $i$th and $j$th strands while $g_{i,j}^{-1}$ represents the negatively oriented full twist between the $i$th and $j$th strands. By the choice of section, each $\sigma_{i,j}$ ($\sigma_{i,j}^{-1}$) in the generating set of $S_n$ lifts to the positively (negatively) oriented half twist between the $i$th and $j$th strands, denoted $\tilde{\sigma}_{i,j}$ ($\tilde{\sigma}_{i,j}^{-1}$). Thus by section 2.1 a generating set for $G_n$ is:

$$\{g_{i,j}\} \cup \{\tilde{\sigma}_{i,j}\}$$

where $1 \leq i < j \leq n$. 

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Relations of \( G_n \)

Note that we will remove the assumption \( i < j \) for many of the statements and proofs in this section for more efficient notation. Recall from section 2.1 to construct a presentation for \( G_n \), we need to include relations of \( \mathbb{Z}^{(2)} \), lift relations of \( S_n \), and conjugate generators of \( \mathbb{Z}^{(2)} \). Therefore a full list of all the relations in \( G_n \) are given in Table 1. Notice R1 is the included relation of \( \mathbb{Z}^{(2)} \), R2-R5 are the lifted relations of \( S_n \), and R6 is the conjugation of relations of generators.

The rest of this section will prove the relations given in Table 1. Since \( \mathbb{Z}^{(2)} \) embeds into \( G_n \), the relation \([g_{i,j}, g_{k,\ell}]\) is preserved in \( G_n \). Therefore R1 holds in \( G_n \) as the inclusion of the relation in \( \mathbb{Z}^{(2)} \). We begin with a theorem describing how to determine elements of \( \mathbb{Z}^{(2)} \) in \( G_n \).

**Theorem 4.1.** Let \( k \in \iota_1(\mathbb{Z}^{(2)}) \). Then \( k \) is determined by the winding numbers of a strand diagram.

**Proof.** Let \( k \in \iota_1(\mathbb{Z}^{(2)}) \), then \( k \) is represented by a pure braid since \( \mathbb{Z}^{(2)} \approx PB_n/K_n \). Since \([g_{i,j}, g_{k,\ell}] = 1\), \( k \) can be written uniquely as:

\[
k = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} g_{i,j}^{\epsilon_{i,j}}
\]

where \( \epsilon_{i,j} \in \mathbb{Z} \) is the number of positively oriented full twists between the \( i \) and \( j \) strands. By definition \( \epsilon_{i,j} \) is the winding number between the \( i \)th and \( j \)th strands. Therefore the winding numbers of all combinations of strands determine \( k \). \( \Box \)

Since \( \tilde{\sigma}_{i,j} \) represents the positively oriented half twist between the \( i \)th and \( j \)th strands, R2 is a relation of \( G_n \). We begin by proving relations R3 and R4.

**Lemma 4.2.** If \( k < i < j, i < j < k, \) or \( j < k < i \), then

\[
\tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} = \tilde{\sigma}_{i,k}
\]

**Proof.** Note that \( b_{i,j} b_{j,k} b_{i,j}^{-1} = b_{i,k} \) if \( i < j < k \), \( k < i < j \), or \( j < k < i \) is a relation of \( B_n \) in (6). Since \( G_n \) is a quotient of \( B_n \) with \( \tilde{\sigma}_{i,j} \) representing the projection of \( b_{i,j} \) in \( G_n \), \( \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} = \tilde{\sigma}_{i,k} \) is a relation of \( G_n \). \( \Box \)

**Lemma 4.3.** If \( i < k < j, j < i < k, \) or \( k < j < i \), then

\[
\tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{i,k}^{-1} = g_{j,k} g_{k,j}^{-1}
\]

**Proof.** By rewriting the relations as \( \sigma_{i,j} \sigma_{j,k} \sigma_{i,j}^{-1} \sigma_{i,k}^{-1} = 1 \) in \( S_n \), we get \( \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{i,k}^{-1} \) is in ker \( \pi \). Therefore \( \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{i,k}^{-1} \epsilon \in \mathbb{Z}^{(2)} \) and by Theorem 4.1 is determined by Figure 2. Suppose \( i < k < j \), notice that by Figure 2 strands \( i \) and \( k \) have both a clockwise and counterclockwise twist, therefore the winding number of \( g_{i,k} \) is zero. Furthermore, strand \( j \) passes over strand \( i \) then under, so \( g_{i,j} \) has a winding number of 1. Also, strand \( j \) passes under strand \( k \) then over it, thus \( g_{j,k} \) has a winding number of \(-1\).

If \( j < i < k \), \( g_{j,i} \) has a winding number of 1 in Figure 2. Now \( k \) passes over \( i \) twice and \( g_{i,k} \) has a winding number of zero. Furthermore \( k \) passes under then over strand \( j \), so \( g_{j,k} \) has a winding number of \(-1\).

Suppose \( k < j < i \), by Figure 2 strand \( k \) passes under \( i \), over \( j \), then under both \( i \) and \( j \). Therefore \( g_{k,i} \) has a winding number of zero while \( g_{k,j} \) has a winding number of \(-1\). Now, \( i \) passes over then under \( j \), so \( g_{j,i} \) has winding number 1 since \( i > j \). \( \Box \)
Figure 2: $\tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{i,k}^{-1}$
Relation 3 Notice that relation $R_3$ is proven by Lemma 4.2. To prove relation $R_4$, we need to show Lemma 4.3 implies $\tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{k,\ell} \tilde{\sigma}_{i,j} = \tilde{\sigma}_{i,k}$. To prove this we need to prove the case of $R_6$ when $\{|i,j\} \cap \{k,\ell\} = 1$.

Theorem 4.4. Suppose $\{|i,j\} \cap \{k,\ell\} = 1$, then:

$$\tilde{\sigma}_{i,j} g_{j,k} \tilde{\sigma}_{i,j}^{-1} = g_{\sigma_{i,j}(k),\sigma_{i,j}(\ell)}$$

Proof. Since $\mathbb{Z}(n)$ is normal in $G_n$, $\tilde{\sigma}_{i,j} g_{j,k} \tilde{\sigma}_{i,j}^{-1} \in \mathbb{Z}(n)$. Suppose $\max\{|i,j\} \in \{k,\ell\}$, without loss of generality assume $j = k$. Then there are three cases: $i < j < k$, $i < k < j$ and $k < i < j$.

By Figure 3 for both cases $i < j < k$ and $k < i < j$, the winding number of strand $j$ associated to both $i$ and $k$ is zero. Furthermore, in both cases there is a full clockwise twist between the $i$th and $j$th strands, so the resulting twist for both cases are $g_{i,k} = g_{\sigma_{i,j}(k),\sigma_{i,j}(\ell)}$.

If $i < k < j$, notice that the $j$th strand passes under the $k$th strand twice and over the $i$th strand twice. Therefore there is no twist between the $j$th strand and either of the other two. Furthermore, the $k$th strand passes over the $i$th strand, then under it, and over strand $i$ twice more. Therefore there is one full clockwise twist between strands $i$ and $k$. Thus $\tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} = g_{i,k}$.

Now, suppose $\min\{|i,j\} \in \{k,\ell\}$ and without loss of generality assume $i = \ell$. There are the three cases: $i < j < k$, $i < k < j$, and $k < i < j$ as shown in Figure 4. In both cases $i < k < j$ and $k < i < j$ the $i$th strand only passes under all other strands, therefore the winding number of any twist involving the $i$th strand is zero. If $k < i < j$ then Figure 4 shows a clockwise full twist between strands $k$ and $j$. If $i < k < j$, notice the $j$th strand passes under the $k$th strand twice,
Figure 4: $\tilde{\sigma}_{i,j} g_{i,k} \tilde{\sigma}_{i,j}^{-1}$
then over it, then under again. Therefore strands \( k \) and \( j \) have the equivalent of a full clockwise twist and the winding number between the \( k^{th} \) and \( j^{th} \) strands is 1.

If \( i < j < k \), Figure 4 shows the \( i^{th} \) and \( j^{th} \) strands have both a full clockwise and counterclockwise twist; therefore the winding number for \( g_{i,j} \) is zero. Furthermore, Figure 4 shows strands \( k \) and \( j \) have a positive full twist while strands \( i \) and \( k \) have a winding number of zero. Thus:

\[
\sigma_{i,j} g_{k,l} \sigma_{i,j} = g_{\sigma_{i,j}(k), \sigma_{i,j}(l)}
\]

if \( |\{i, j\} \cap \{k, l\}| = 1 \).

**Theorem 4.5.** Relation R4 holds in \( G_n \).

**Proof.** By Theorem 4.4, we have \( \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,k}^{-1} = g_{i,j} g_{k,j}^{-1} \). Multiplying on the right by \( \tilde{\sigma}_{i,k} \) and on the left by \( g_{i,j}^{-1} \), by relation R1 we have

\[
\tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,k}^{-1} \tilde{\sigma}_{i,k}^{-1} = g_{i,j} g_{k,j}^{-1} g_{i,j}^{-1} \\
g_{i,j}^{-1} \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} = g_{k,j} \sigma_{i,k}
\]

By applying R2 and Theorem 4.3 we have:

\[
g_{i,j}^{-1} \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} = g_{k,j} \sigma_{i,k} \\
\tilde{\sigma}_{i,j}^{-2} \tilde{\sigma}_{i,j} \tilde{\sigma}_{k,j} \tilde{\sigma}_{i,j}^{-1} = \tilde{\sigma}_{i,k} g_{k,j}^{-1} \\
\tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{k,j} \tilde{\sigma}_{i,j}^{-1} \tilde{g}_{i,j} = \tilde{\sigma}_{i,k}
\]

By R2: \( g_{i,j} = \tilde{\sigma}_{i,j}^2 \). Therefore the theorem and R4 are proven.

**Theorem 4.6.** Suppose \( |\{i, j\} \cap \{k, l\}| \neq 1 \). Then

\[
[\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,l}] = \begin{cases} 
g_{i,k} g_{k,i}^{-1} g_{j,l} g_{j,i} & i < k \land j \land l \land i < k \land j \land l \\
g_{i,k} g_{k,i}^{-1} g_{j,l} g_{j,i} & k < i \land j \land l \land k < i \land j \land l \\
1 & \text{otherwise}
\end{cases}
\]

**Proof.** Suppose \( \{i, j\} \cap \{k, l\} = \varnothing \). Without loss of generality, assume \( i < j \) and \( k < l \). If \( j < k \) or \( \ell < i \) then \( [\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,l}] = 1 \) since \( G_n \) is a quotient of \( B_n \). Since \( [\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,l}] = [\tilde{\sigma}_{k,l}, \tilde{\sigma}_{i,j}]^{-1} \), it suffices to show the cases \( i < k < j \) and \( i < k < \ell < j \).

If \( i < k < \ell < j \), then \( (i-k)(j-\ell)(j-k) > 0 \). Therefore \( [b_{i,j}, b_{k,l}] = 1 \) in \( \mathbb{Q} \). Since \( G_n \) is a quotient of \( B_n \), \( [\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,l}] = 1 \). Therefore it remains to show the case \( i < k < j < \ell \).

Now, suppose \( i < k < j < \ell \). By Figure 5 the \( i^{th} \) strand passes under the \( k^{th} \) strand then over, therefore \( g_{i,k} \) has a winding number of 1. Furthermore strand \( i \) passes under the \( j^{th} \) strand twice; also the \( j^{th} \) strand passes over the \( \ell^{th} \) strand once then under it and therefore has a counterclockwise full twist. Therefore \( g_{i,j} \) has a winding number of zero and \( g_{i,\ell} \) has a winding number \(-1\). Now, the \( j^{th} \) strand passes under the \( k^{th} \) strand then over it, therefore \( g_{k,j} \) has a winding number of \(-1\). Strand \( j \) passes under strand \( \ell \) then under it and thus \( g_{j,\ell} \) has a winding number of 1. Finally strand \( \ell \) passes over strand \( k \) twice, therefore \( g_{k,\ell} \) has a winding number of zero. Thus the theorem is proven.

To finish the proof this presentation for \( G_n \) is correct, it remains to show the second part of R5.
Figure 5: $[\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,\ell}]$
Figure 6: $\tilde{\sigma}_{i,j} g_{k,\ell} \tilde{\sigma}_{i,j}^{-1}$
Theorem 4.7. If $|\{i,j\} \cap \{k,\ell\}| \neq 1$, then
\[
\tilde{\sigma}_{i,j}g_{k,\ell}\tilde{\sigma}_{i,j}^{-1} = g_{k,\ell}
\]

Proof. First suppose $\{i,j\} \cap \{k,\ell\} = \{i,j\}$. Then by relation R3 we have
\[
\tilde{\sigma}_{i,j}g_{k,\ell}\tilde{\sigma}_{i,j}^{-1} = \tilde{\sigma}_{i,j}\tilde{\sigma}_{i,j}^{-1}\tilde{\sigma}_{i,j}^{-1} = \tilde{\sigma}_{i,j}^{-1} = \tilde{g}_{i,j}
\]

Now, suppose $\{i,j\} \cap \{k,\ell\} = \emptyset$. If $i < k < j < \ell$, then by Figure 6 strands $i$ and $k$ have one full clockwise twist followed by a full counterclockwise twist, therefore the winding number corresponding to $g_{i,k}$ is zero. Also, the $\ell$th strand has one full clockwise twist with the $k$th strand and does not interact with either of the other strands. Therefore $g_{i,\ell}$ and $g_{j,\ell}$ have winding numbers zero while $g_{k,\ell}$ has a winding number of 1. Furthermore, strand $j$ passes under strand $k$ twice, so $g_{j,k}$ has a corresponding winding number of zero as well. Thus, if $i < k < j < \ell$ then the theorem holds.

Now, suppose $k < i < \ell < j$. By Figure 6 the $i$th strand always passes under the other strands, therefore $g_{k,i}, g_{i,\ell},$ and $g_{j,\ell}$ all have winding number zero. Furthermore, the $j$th strand passes under the strand $\ell$ twice and over strand $k$ twice, therefore $g_{k,j}$ and $g_{\ell,j}$ also have winding numbers zero. Finally, Figure 6 shows one full clockwise twist between the $k$th and $\ell$th strands. Therefore $\tilde{\sigma}_{i,j}g_{k,\ell}\tilde{\sigma}_{i,j}^{-1} = g_{k,\ell}$ if $k < i < \ell < j$.

Otherwise $(i-k)(i-\ell)(j-k)(j-\ell) > 0$, therefore by R3 and R5 we have:
\[
\tilde{\sigma}_{i,j}g_{k,\ell}\tilde{\sigma}_{i,j}^{-1} = \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j}^{-1}
\]

4.2 Cohomology class in $H^2(S_n; \mathbb{Z}(2))$

Given the group presentation for $G_n$ with relations in Table 1, we can determine the cohomology class representing the group extension of $S_n$ by $\mathbb{Z}(2)$ by Theorem k. Recall the algorithm described at the beginning of Section 4 to determine the chosen normal form of a permutation $p \in S_n$ does the following:

- Rewrites $p$ as a word in the generators of $S_n$ given in 5.
- For all $1 \leq i < j \leq n$, replaces $\sigma_{i,j}^2$ with $1_{S_n}$.
- If $\{i,j\} \cap \{k,\ell\} = \emptyset$ and $\max\{k,\ell\} < \max\{i,j\}$, replaces $\sigma_{i,j}\sigma_{k,\ell}$ with $\sigma_{k,\ell}\sigma_{i,j}$.
- If $j = \max\{i,j,k\}$, replaces $\sigma_{i,j}\sigma_{j,k}$ with $\sigma_{i,k}\sigma_{i,j}$.
- If $i = \max\{i,j,k\}$, replaces $\sigma_{i,j}\sigma_{j,k}$ with $\sigma_{j,k}\sigma_{i,k}$.
The normal section \( s : S_n \to G_n \) defined on generators of \( S_n \) by \( s(\sigma_{i,j}) = \tilde{\sigma}_{i,j} \) determines the lift of permutations in \( S_n \) by the chosen normal form. Therefore Theorem 4.10 implies we can compute the image for a cocycle representing the corresponding cohomology class by the relations of \( G_n \). In particular, we can use the relations in Table 1. Since we lift any element of \( S_n \) by the choice of normal form, we first determine the normal forms for products of two transpositions.

**Lemma 4.8.** Suppose \( \{i, j\} \cap \{k, \ell\} = \emptyset \), then

\[
s(\sigma_{k,\ell} \sigma_{i,j}) = s(\sigma_{i,j} \sigma_{k,\ell}) = \begin{cases} 
\tilde{\sigma}_{i,j} \tilde{\sigma}_{k,\ell} & j < \ell \\
\tilde{\sigma}_{k,\ell} \tilde{\sigma}_{i,j} & \ell < j
\end{cases}
\]

**Proof.** Both \( \sigma_{k,\ell} \sigma_{i,j} \) and \( \sigma_{i,j} \sigma_{k,\ell} \) have the same normal form since they both describe the same permutation. If \( \ell < j \) then we take \( \sigma_{i,j} \sigma_{k,\ell} \sigma_{i,j} \sigma_{k,\ell} = 1 \) and \( \sigma_{i,j} \sigma_{k,\ell} \) becomes \( \sigma_{k,\ell} \sigma_{i,j} \). Therefore the normal form of \( \sigma_{i,j} \sigma_{k,\ell} \) is the same as the normal form for \( \sigma_{k,\ell} \sigma_{i,j} \) and \( s(\sigma_{i,j} \sigma_{k,\ell}) = \tilde{\sigma}_{k,\ell} \tilde{\sigma}_{i,j} \).

Now, suppose \( j < \ell \). Then we take the transposition fixing \( k \) first, yielding \( \sigma_{i,j} \sigma_{k,\ell} \sigma_{k,\ell} \sigma_{i,j} \) and thus \( s(\sigma_{i,j} \sigma_{k,\ell}) = \tilde{\sigma}_{k,\ell} \tilde{\sigma}_{i,j} \). \( \Box \)

**Lemma 4.9.** The normal form for products of transpositions with intersection are:

\[
s(\sigma_{i,k} \sigma_{i,j}) = s(\sigma_{i,j} \sigma_{j,k}) = \begin{cases} 
\tilde{\sigma}_{i,k} \tilde{\sigma}_{i,j} & \max\{i, k, j\} = j \\
\tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} & \max\{i, k, j\} = k \\
\tilde{\sigma}_{k,j} \tilde{\sigma}_{k,i} & \max\{i, k, j\} = i
\end{cases}
\]

The proof of Lemma 4.9 is a computation similar to the proof of Lemma 4.8 and omitted. Now that we have the normal forms for products of generators, it is possible to compute a representative for the cohomology class describing \( G_n \) as an extension of \( S_n \) by \( \mathbb{Z}^2(2) \).

**Theorem 4.10.** Let:

\[
0 \longrightarrow \mathbb{Z}^2(2) \overset{i_1}{\longrightarrow} G_n \overset{\pi_1}{\longrightarrow} S_n \longrightarrow 1
\]

be the group extension where the action of \( S_n \) on \( \mathbb{Z}^2(2) \) is determined by the conjugation of pure braids by half twists in \( B_n \). The cocycle, \( \phi \), defined by:

\[
\phi(\tilde{e}_{i,j}) = g_{i,j} \\
\phi(\tilde{d}_{i,j,k,\ell}) = \begin{cases} 
g_{i,k} - g_{i,\ell} - g_{k,j} + g_{j,\ell} & i < j < \ell \\
g_{i,k} + g_{k,j} + g_{j,\ell} - g_{i,\ell} & k < i < j \\
0 & \text{otherwise}
\end{cases}
\]

\[
\phi(\tilde{d}_{i,k,j}) = \begin{cases} 
g_{i,j} - g_{k,j} & i < k < j, \quad j < i < k, \quad k < j < i \\
0 & \text{otherwise}
\end{cases}
\]

is a representative for the cohomology class of \( H^2(S_n; \mathbb{Z}^2) \) corresponding to this extension.

Before we prove Theorem 4.10 note that the image of our cocycle is the abelian group \( \mathbb{Z}^2(2) \) with additive notation. However the computations to determine elements of \( \mathbb{Z}^2(2) \) is done within the extension \( G_n \) using multiplicative notation.
Proof. First, by Theorem 3.4 we have

\[ \phi(\tilde{c}_{i,j}) = s(\sigma_{i,j})s(\sigma_{i,j}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{i,j} = g_{i,j} \]

Thus for \( e \)

\[ s(\sigma_{i,j})s(\sigma_{i,j}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{i,j} \]

By Lemma 4.8, Theorem 3.4, and R5

\[ \phi(\tilde{d}_{i,j,k,\ell}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell}(\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell})^{-1} - \tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j}(\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell})^{-1} \]

Now, consider \( \tilde{d}_{i,j,k,\ell} \), without loss of generality assume \( i < j \) and \( k < \ell \). By Lemma 4.8

\[ s(\sigma_{k,\ell}\sigma_{i,j}) = s(\sigma_{i,j}\sigma_{k,\ell}) \]

depends on \( \max\{j, \ell\} \). Suppose \( j < \ell \), then by Lemma 4.8

\[ s(\sigma_{k,\ell}\sigma_{i,j}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell} \]

Therefore by Theorem 3.4 and relation R5

\[ \phi(\tilde{d}_{i,j,k,\ell}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell}(\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell})^{-1} - \tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j}(\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell})^{-1} \]

Thus for \( \phi(\tilde{d}_{i,j,k,\ell}) \) it remains to show the case \( \ell < j \). By Lemma 4.8, Theorem 3.4 and R5

\[ \phi(\tilde{d}_{i,j,k,\ell}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell}(\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell})^{-1} - \tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j}(\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell})^{-1} \]

Thus it remains to compute the image of \( \tilde{c}_{i,k,j} \). By Lemma 4.9 there are three cases for the normal form of the permutation \( \sigma_{i,k}\sigma_{i,j} = \sigma_{i,j}\sigma_{i,k} \) depending on \( \max\{i, k, j\} \). Suppose \( \max\{i, k, j\} = j \), then by Theorem 3.4 and Lemma 4.9 we have:

\[ \phi(\tilde{c}_{i,k,j}) = s(\sigma_{i,j})s(\sigma_{j,k})s(\sigma_{i,j}\sigma_{j,k})^{-1} - s(\sigma_{i,k})s(\sigma_{j,k})s(\sigma_{i,j}\sigma_{j,k})^{-1} \]

\[ = \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,j}(\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,j})^{-1} - \tilde{\sigma}_{i,k}\tilde{\sigma}_{i,j}(\tilde{\sigma}_{i,k}\tilde{\sigma}_{i,j})^{-1} \]

\[ = \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,j}\tilde{\sigma}_{i,j}^{-1}\tilde{\sigma}_{i,k} \]
By relations R3 and R4 together with Theorem 4.4 we get:

\[ \phi(\tilde{e}_{i,k,j}) = \tilde{\sigma}_{i,j} \tilde{\sigma}_{k,j} \tilde{\sigma}_{i,k}^{-1} \tilde{\sigma}_{i,k} \]

\[
= \begin{cases} 
\tilde{\sigma}_{i,k} \tilde{\sigma}_{k,k}^{-1} & k < i < j \\
\tilde{g}_{i,j} \tilde{\sigma}_{i,j} \tilde{\sigma}_{k,j} \tilde{g}_{i,j} \tilde{\sigma}_{i,k}^{-1} & i < k < j 
\end{cases}
\]

\[
= \begin{cases} 
\tilde{g}_{i,j} \tilde{\sigma}_{i,k} \tilde{g}_{i,j}^{-1} \tilde{\sigma}_{k,k}^{-1} & i < k < j 
\end{cases}
\]

\[
= \begin{cases} 
\tilde{g}_{i,j} \tilde{\sigma}_{i,k} \tilde{g}_{i,j}^{-1} & i < k < j 
\end{cases}
\]

Hence we are done if \( \max\{i, k, j\} = j \).

Now, let \( \max\{i, k, j\} = k \), then by Lemma 4.9, \( s(\sigma_{i,k}, \sigma_{i,j}) = \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \). Therefore, as above we have

\[
\phi(\tilde{e}_{i,k,j}) = \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} (\tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k})^{-1} - \tilde{\sigma}_{i,k} \tilde{\sigma}_{j,k} (\tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k})^{-1}
\]

\[= -\tilde{\sigma}_{i,k} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k}^{-1} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} \]

Now, if \( i < j < k \) then R3 applies to \( \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k}^{-1} \) and if \( j < i < k \) then R4 applies:

\[
\phi(\tilde{e}_{i,k,j}) = \begin{cases} 
\tilde{\sigma}_{i,k} \tilde{\sigma}_{k,k}^{-1} & i < j < k \\
\tilde{\sigma}_{i,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & i < j < k 
\end{cases}
\]

\[
= \begin{cases} 
\tilde{g}_{i,j} \tilde{\sigma}_{i,k} \tilde{g}_{i,j}^{-1} & i < j < k 
\end{cases}
\]

\[
= \begin{cases} 
\tilde{g}_{i,j} \tilde{\sigma}_{i,k} \tilde{g}_{i,j}^{-1} & i < j < k 
\end{cases}
\]

\[
= \begin{cases} 
\tilde{g}_{i,j} \tilde{\sigma}_{i,k} \tilde{g}_{i,j}^{-1} & j < i < k 
\end{cases}
\]

\[
= \begin{cases} 
\tilde{g}_{i,j} \tilde{\sigma}_{i,k} \tilde{g}_{i,j}^{-1} & j < i < k 
\end{cases}
\]

It remains to prove the result if \( \max\{i, k, j\} = i \). By Lemma 4.9, \( \max\{i, k, j\} = i \) implies \( s(\sigma_{i,j}, \sigma_{i,j}) = \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \). Thus

\[
\phi(\tilde{e}_{i,k,j}) = \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} (\tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k})^{-1} - \tilde{\sigma}_{i,k} \tilde{\sigma}_{j,k} (\tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k})^{-1}
\]

\[= \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k}^{-1} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} \]

If \( j < i < k \), then relation R3 applies to \( \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k} \) while R4 applies if \( k < j < i \). Therefore,

\[
\tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} = \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{j,k}^{-1} & j < i \\
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{j,k}^{-1} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[
= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[
= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[
= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[
= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]

\[= \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k}^{-1} & j < i 
\end{cases}
\]
Furthermore, if \( j < k < i \) then R3 applies to \( \tilde{\sigma}_{k,i}, \tilde{\sigma}_{j,i}, \tilde{\sigma}_{k,j}^{-1} \) while R4 applies if \( k < j < i \). Hence

\[
\tilde{\sigma}_{k,i}, \tilde{\sigma}_{j,i}, \tilde{\sigma}_{k,j}^{-1} \tilde{\sigma}_{j,k}^{-1} = \begin{cases} 
\tilde{\sigma}_{j,k} \tilde{\sigma}_{j,k}^{-1} & j < k < i \\
g_{k,i} \tilde{\sigma}_{k,i}^{-1} \tilde{\sigma}_{j,i} g_{k,i}^{-1} \tilde{\sigma}_{k,j}^{-1} & k < j < i 
\end{cases}
\]

Therefore:

\[
\phi(\tilde{e}_{i,k,j}) = \begin{cases} 
0 - 0 & j < k < i \\
g_{k,i} \tilde{\sigma}_{k,i}^{-1} - g_{k,i} \tilde{\sigma}_{j,i}^{-1} & k < j < i 
\end{cases}
\]

\[
= \begin{cases} 
0 & j < k < i \\
g_{k,i} - g_{k,j} - (g_{k,i} - g_{j,i}) & k < j < i 
\end{cases}
\]

\[
= \begin{cases} 
0 & j < k < i \\
g_{j,i} - g_{k,j} & k < j < i 
\end{cases}
\]

\[ \square \]

### 4.3 Order of \([\phi]\)

To determine the order of \([\phi]\) we build the extensions \( G_n^t \) of \( S_n \) by \( \mathbb{Z}(3) \) which correspond to the class of \( t \cdot [\phi] \) and show that if \( t \in 2\mathbb{Z} \) then \( G_n^t \) is a split extension. To construct extensions corresponding to multiples of \([\phi]\) consider Theorem [4.4]. Notice the image of \( \phi \) is determined by lifting the relations of \( S_n \) to relations in \( G_n \) since we are using the resolution from the universal cover of a \( K(S_n, 1) \)-space.

**Remark** Recall from section 2.2, the extension corresponding to a cohomology class \([\theta]\) is determined by \( E_\theta = \mathbb{Z}(3) \ltimes \theta S_n \) where multiplication of group elements is:

\[
(a, g) \cdot (b, h) = (a + g \cdot b + \theta(g, h), gh)
\]

To prove \( G_n^t \) is an extension of \( S_n \) by \( \mathbb{Z}_2^t \) corresponding to \( t \cdot \phi \), it suffices to show \( G_n^t \cong E_t \cdot \phi \). The following Lemma proves this isomorphism.

**Lemma 4.11.** Let \( G_n^t \) be the group with generating set \( \{ \tilde{\sigma}_{i,j}, \{ g_{k,\ell} \} \) for all \( i, j, k, \ell \) such that \( 1 \leq i < j \leq n \) and \( 1 \leq k < \ell \leq n \) with relations given in Table 2. Then \( G_n^t \) is the extension of \( S_n \) by \( \mathbb{Z}(3) \) corresponding to the cohomology class of \( t \cdot [\phi] \in H^2(S_n; \mathbb{Z}(3)) \).

**Proof.** We first prove:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}(3) & \stackrel{\iota}{\longrightarrow} & G_n^t & \stackrel{\pi}{\longrightarrow} & S_n & \longrightarrow & 1 \\
& \downarrow{\text{id}_{\mathbb{Z}(3)}} & & \downarrow{f_t} & & \downarrow{\text{id}_{S_n}} & & \\
0 & \longrightarrow & \mathbb{Z}(3) & \stackrel{\iota'}{\longrightarrow} & E_{t, \phi} & \stackrel{\pi'}{\longrightarrow} & S_n & \longrightarrow & 1
\end{array}
\]

(7)
is a commuting diagram of short exact sequences. Since $\mathbb{Z}^\prime = \{\{g_{i,j}\}_{1 \leq i < j \leq n} \mid [g_{i,j}, g_{k,\ell}] = 1\}$, define a map $\iota_t : \mathbb{Z}^\prime \to G_n^t$ by $\iota_t(g_{i,j}) = g_{i,j}$ for all $1 \leq i < j \leq n$. Since $g_{i,j}$ is a generator of $G_n^t$ and relation $R^t 1$, $\iota_t$ is well defined. Recall from section 2.2, $\pi : E_{t, \varphi} \to S_n$ defined by $\pi_t((a, g)) = g$ is a well defined, surjective homomorphism. Define $f_t : G_n^t \to E_{t, \varphi}$ on the generators of our presentation by:

$$f_t(s) = \begin{cases} (s, 1) & \text{if } s \in \{g_{i,j}\}_{1 \leq i < j \leq n} \\ (0, s) & \text{if } s \in \{\tilde{\sigma}_{i,j}\}_{1 \leq i < j \leq n} \end{cases}$$

By the definitions of $\iota_t$, $\iota'_t$, and $f_t$:

$$f_t \circ \iota_t(g_{i,j}) = f_t(\iota_t(g_{i,j}))
= f_t(g_{i,j})
= (g_{i,j}, 1)$$

Therefore $f_t \circ \iota_t = \iota'_t \circ \phi_t$. Suppose $a \in \ker \iota_t$, then $f_t \circ \iota_t(a) = \iota'_t(a)$ and $f_t \circ \iota_t(a) = (0, 1)$. Since $\iota'_t$ is injective, $a = 0$ and $\iota_t$ is injective. Let $s$ be a generator of $G_n^t$. By the definitions of $f_t$, $\phi_t$, and $\pi_t$, if $s \in \{g_{i,j}\}_{1 \leq i < j \leq n}$:

$$\pi'_t \circ f_t(g_{i,j}) = \pi'_t(f_t(g_{i,j}))
= \pi'_t((g_{i,j}, 1))
= 1$$

Table 2: Relations of $G_n^t$

| Relation | Expression | Condition |
|----------|------------|-----------|
| R^t 1 | $[g_{i,j}, g_{k,\ell}] = 1$ | for all $i, j, k, \ell$ |
| R^t 2 | $\tilde{\sigma}_{i,j}^2 = g_{i,j}$ | for all $i$ |
| R^t 3 | $\tilde{\sigma}_{i,j} \tilde{\sigma}_{k,\ell} \tilde{\sigma}_{i,j}^{-1} = \tilde{\sigma}_{i,k}$ | if $k < i < j$; $i < j < k$; or $j < k < i$ |
| R^t 4 | $\tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{j,k} \tilde{\sigma}_{i,j} = \tilde{\sigma}_{i,k}$ | if $i < k < j$; $j < i < k$; or $k < j < i$ |
| R^t 5 | $[\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,\ell}] = \begin{cases} g_{i,k} g_{j,\ell} g_{i,j} & \text{if } i < k < j \text{ and } j < \ell \text{ or } \ell < j < i \\ 1 & \text{otherwise} \end{cases}$ | $k < i < j$ |
| R^t 6 | $\tilde{\sigma}_{i,j} g_{k,\ell} \tilde{\sigma}_{i,j}^{-1} = g_{\sigma_{i,j,k}(k), \sigma_{i,j}(\ell)}$ for all $i, j, k, \ell \in \{1, \ldots, n\}$ | |
Similarly, if $s \in \{\tilde{\sigma}_{i,j}\}_{1 \leq i < j \leq n}$:

$$
\pi_i \circ f_i(\tilde{\sigma}_{i,j}) = \pi_i(\tilde{f}_i(\tilde{\sigma}_{i,j})) = \pi_i((0,\sigma_{i,j})) = \sigma_{i,j} = \pi_t(\tilde{\sigma}_{i,j})
$$

Therefore $f_t$ makes the diagram (4) commute. It remains to show the top row of (7) is exact. By the definitions of $t_t$ and $\pi_t$, $\text{im} \ t_t \subseteq \ker \pi_t$. Furthermore, by adding the relation $g_{i,j} = 1$ to the relations in Table 2 we obtain a presentation for $S_n$. Therefore $\ker \pi_t$ is the normal closure of $\text{im} \ t_t$. Note the image of $t_t$ is generated by the $g_{i,j}$’s in $G_n^t$. By relation $R^t6$, the $g_{i,j}$’s generate a normal subgroup of $G_n^t$. Thus $\text{im} \ t_t$ is normal in $G_n^t$. In particular, the normal closure of $\text{im} \ t_t$ is equal to $\text{im} \ t_t$. Therefore $\text{im} \ t_t = \ker \pi_t$ and the top row of (7) is exact. Thus $G_n^t$ is an extension of $S_n$ by $Z^2(t)$.

Since $E_{t,\phi}$ is an extension of $S_n$ by $Z^2(t)$, the bottom row of (7) is exact as well. By the 5-Lemma, $G_n^t \cong E_{t,\phi}$. Furthermore, since $f_t$ makes the diagram commute, $G_n^t$ and $E_{t,\phi}$ are equivalent group extensions. Therefore $G_n^t$ corresponds to the cohomology class of $t \cdot [\phi] \in H^2(S_n;Z^2(t))$. \qed

**Remark** An alternative, and more formal, process of constructing the cohomology class corresponding to $t \cdot [\phi]$ in $H^2(S_n;Z^2(t))$ would be to construct new extensions from $G_n$ using a technique analogous to Baer sums. Note that Baer sums do not generalize when the cokernel of an extension is non-abelian, however no obstructions arise when taking the pullback construction of equivalent extensions with the same group presentation.

**Theorem 4.12.** If $t \in 2\mathbb{Z}$, then the group extension:

$$
0 \to \mathbb{Z}^2(t) \to G_n^t \to S_n \to 1
$$

is split.

**Proof.** First, note that the existence of a splitting is independent of group presentation for $S_n$. Therefore it suffices to define a homomorphism $\omega : S_n \to G_n^t$ using the presentation (4) of $S_n$. Set:

$$
\omega(\sigma_i) = \tilde{\sigma}_{i,i+1}^{g_{i,i+1}^{-t/2}}
$$

First note that:

$$
\omega(\sigma_i)^2 = \tilde{\sigma}_{i,i+1}^{-t/2} \tilde{\sigma}_{i,i+1}^{g_{i,i+1}^{-t/2}} = \tilde{\sigma}_{i,i+1}^{-t/2} \tilde{\sigma}_{i,i+1} = \tilde{\sigma}_{i,i+1}^{g_{i,i+1}^{-2}}
$$

Now, if $|i-j| > 1$, then by the relations in Table 2

$$
\omega(\sigma_i)\omega(\sigma_j)\omega(\sigma_i)^{-1}\omega(\sigma_j)^{-1} = 1
$$

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Thus it only remains to show the braid relation is preserved under $\omega$. Applying relations $R^1$, $R^2$, and $R^6$ together with $R^3$ on the image we get:

\[
\begin{align*}
\tilde{\sigma}_{i,j} &= g_{i,j} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \\
&= g_{i,j} \tilde{\sigma}_{i,j} \tilde{\sigma}_{i,j}^{-1} \\
&= 1
\end{align*}
\]

Thus $\omega$ is well defined and the short exact sequence is split. □

5 $\mathbb{Z}_n$ and $B_n[4]$

5.1 Cohomology class corresponding to $\mathbb{Z}_n$

Recall the group extension

\[
0 \longrightarrow \mathbb{Z}_2^{(2)} \longrightarrow \mathbb{Z}_n \longrightarrow S_n \longrightarrow 1
\]

Since $\mathbb{Z}_2^{(2)} \approx \mathcal{P} \mathbb{Z}_n$ and $\mathbb{Z}_n \approx B_n/B_n[4]$, the action of $S_n$ on $\mathbb{Z}_2^{(2)}$ is induced by the conjugation action of $B_n$ on $PB_n$. Furthermore, $\mathbb{Z}_2^{(2)} \approx \mathcal{P} \mathbb{Z}_n$, so each $g_{i,j}$ corresponds to the pure braid between strands $i$ and $j$. 

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\textbf{Theorem 5.1.} Let $A_n$ be the subgroup of $G_n$ normally generated by $\{g^2_{i,j}\}_{1 \leq i < j \leq n}$. Then $G_n/A_n \cong \mathbb{Z}_n$.

\textit{Proof.} Since $A_n$ is generated by $g_{i,j}^2$ for $1 \leq i < j \leq n$ and $g_{i,j}$ corresponds to the pure braid between the $i^{th}$ and $j^{th}$ strands, $A_n$ is the subgroup of $G_n$ generated by squares of standard generators of pure braids. Recall $B_n[4]$ is the subgroup of $PB_n$ generated by squares of all elements \[1\]. Since the $g_{i,j}$’s commute in $G_n$, the image of $PB_n$ is an abelian subgroup of $G_n$. Therefore the image of $B_n[4]$ in $G_n$ is generated by squares of generators of the pure braid group.

Therefore $A_n$ is the image of $B_n[4]$ in $G_n$. Thus $G_n/A_n \cong \mathbb{Z}_n$. \qed

\textbf{Presentation of $\mathbb{Z}_n$} Let $s$ be the normalized section lifting $\sigma_{i,j}$ to $\tilde{\sigma}_{i,j}$ where $\tilde{\sigma}$ represents the image of $b_{i,j}$ in $\mathbb{Z}_n$. Since $A_n$ is the image of $B_n[4]$ in $G_n$, we obtain a presentation for $\mathbb{Z}_n$ by adding the relation $g_{i,j}^2 = 1$ to the relations in Table 1. We simplify the notation by taking $g_{i,j}$ and $\sigma_{i,j}$ with $\tilde{g}_{i,j}$ and $\tilde{\sigma}_{i,j}$ for clarity.

Therefore we get the following generating set for $\mathbb{Z}_n$:

\[
\{\tilde{g}_{i,j}\} \cup \{\tilde{\sigma}_{i,j}\}
\]

where $1 \leq i < j \leq n$. A full list of relations is given in Table 3. The following theorem completes the proofs of both Theorem 1.1 and Theorem 1.2.

\textbf{Remark} Notice that $\mathbb{R}4$ can be determined from $\mathbb{R}3$.

\textbf{Theorem 5.2.} Let $\kappa \in \text{Hom}(R_2, \mathbb{Z}_2^{(2)})$ be the representative for the cohomology class in $H^2(S_n; \mathbb{Z}_2^{(2)})$ corresponding to $\mathbb{Z}_n$ as an extension of $S_n$ by $\mathbb{Z}_2^{(2)}$ given by the usual construction. Let $\eta : \mathbb{Z}_2^{(2)} \to \mathbb{Z}_2^{(2)}$ be the mod 2 reduction map. Then $\kappa = \eta \circ \phi$ where $\phi$ is the representative for $G_n$ defined in Theorem 4.10.

\textit{Proof.} Define $f : G_n \to \mathbb{Z}_n$ by $f(\tilde{\sigma}_{i,j}) = \tilde{\sigma}_{i,j}$ and $f(\tilde{g}_{i,j}) = \tilde{g}_{i,j}$ for all $i,j$ such that $1 \leq i < j \leq n$. Thus $f$ is a well defined group homomorphism which commutes with the following diagram:

\[
\begin{array}{c}
\begin{array}{c}
0 \longrightarrow \mathbb{Z}_2^{(2)} \longrightarrow G_n \longrightarrow S_n \longrightarrow 1
\end{array}
\end{array}
\]

Furthermore, by Theorem 3.4

\[
\kappa(\tilde{e}_{i,j}) = s'(\sigma_{i,j})s'(\sigma_{i,j})
\]

\[
\kappa(\tilde{d}_{i,j,k}) = s'(\sigma_{i,j})s'(\sigma_{j,k})s'(\sigma_{i,j})^{-1} - s'(\sigma_{j,k})s'(\sigma_{i,j})s'(\sigma_{j,k})^{-1} - s'(\sigma_{k,i})s'(\sigma_{i,j})^{-1}
\]

\[
k'(\tilde{e}_{i,j,k}) = s'(\sigma_{i,j})s'(\sigma_{j,k})s'(\sigma_{i,j})^{-1} - s'(\sigma_{j,k})s'(\sigma_{i,j})s'(\sigma_{i,j})^{-1}
\]

Since $f$ commutes with the diagram and by the choices of $s$ and $s'$, we have $s'(\sigma_{i,j}) = f \circ s(\sigma_{i,j})$. Thus $\kappa = \phi \circ f$. Furthermore, the induced map on cohomology is $\eta$ by the definition of $f$. \qed

5.2 Generating sets of $B_n[4]$ \hspace{1cm}

In this section we begin with a well known consequence of Schreier’s formula which guarantees the existence of a finite generating set for $B_n[4]$. Then we prove Theorem 1.1 and give a summary describing the difficulties in refining our normal generating set to a finite generating set.
Proposition 5.3. If $G$ is a group with a generating set of size $j$ and $H$ is a subgroup of finite index $k$, then there exists a finite generating set for $H$ of size at most $k(j - 1) + 1$.

This proposition is a well known fact and a proof can be found in Lyndon and Schupp (page 164 of [10]). Since $\mathbb{Z}_n$ is an extension of $S_n$ by $2\mathbb{Z}_2^{(3)}$, $\mathbb{Z}_n$ is finite of order $n! \cdot 2^{(3)}$. Furthermore, $B_n$ is finitely generated with a generating set of size $n - 1$, therefore there exists a generating set for $B_n[4]$ of size at most:

$$n! \cdot 2^{(3)} \cdot (n - 2) + 1$$

Now we give a proof of Theorem 5.3.

Proof of Theorem 5.3. Consider the quotient of the Artin presentation of $B_n$ given in (2) by the normal subgroup generated by $b_i^4$, $[b_i^2, b_i^{i+1}]$, and $[b_{i+2}, b_{i+1}^2, b_{i+3}]$. Then this quotient has a group presentation:

$$\left\{ b_1, \ldots, b_{n-1} \mid b_i^4 = 1, [b_i^2, b_i^{i+1}] = 1, [b_{i+2}, b_{i+1}^2, b_{i+3}] = 1 \right\}$$

(8)

Now, consider the presentation for $\mathbb{Z}_n$ with generators $\{\tilde{g}_{i,j}\}$ for all $1 \leq i < j \leq n$ with relations given by Table 5. By relation $\mathcal{R}2$, we can replace $\tilde{g}_{i,j}$ with $\tilde{a}_{i,j}^2$ for all $1 \leq i < j \leq n$. Therefore $\mathbb{Z}_n$ is generated by $\{\tilde{g}_{i,j}\}$ where $1 \leq i < j \leq n$. Furthermore, by the case of relation $\mathcal{R}3$ with $i < j < k$, $\{\tilde{a}_{i,j}\}$ is generated by the set $\{\tilde{a}_{i+1,j}\}$. Thus we can take a generating set of $\{\tilde{a}_{i+1,j}\}_{1 \leq i \leq n-1}$ for $\mathbb{Z}_n$.

Now, since $\tilde{a}_{i,j}^2 = \tilde{g}_{i,j}$ and $\tilde{g}_{i,j}^2 = 1$, therefore we can replace $\mathcal{R}0$ and $\mathcal{R}2$ with $\tilde{a}_{i+1,j}^4 = 1$ in $\mathbb{Z}_n$. By relation $\mathcal{R}3$ and $\mathcal{R}4$, we have $\tilde{a}_{i+1,j} \tilde{a}_{i+1,j+2} \tilde{a}_{i,j+1}^{-1} = \tilde{a}_{i,j+2} \tilde{a}_{i+1,j} \tilde{a}_{i+1,j+2} = \tilde{a}_{i,j+2}$ respectively. Therefore:

$$\tilde{a}_{i+1,j} \tilde{a}_{i+1,j+2} \tilde{a}_{i,j+1}^{-1} = \tilde{a}_{i+1,j+2} \tilde{a}_{i,j+1} \tilde{a}_{i+1,j+2}$$

and we can replace relations $\mathcal{R}3$ and $\mathcal{R}4$ with the relation braid relation. Furthermore, $\mathcal{R}5$ implies $[\tilde{a}_{i+1,j}, \tilde{a}_{j+1,j}] = 1$ for $|i - j| > 1$. Replacing $\tilde{g}_{i,j}$ with $\tilde{a}_{i,j}^2$ in relation $\mathcal{R}1$ and restricting to $\{\tilde{a}_{i+1,j}\}_{1 \leq i \leq n-1}$ we get $[\tilde{a}_{i+1,j}, \tilde{a}_{i,j+1}] = 1$ and $[\tilde{a}_{i+1,j+2}, \tilde{a}_{i,j+2}] = 1$. Thus we get the same presentation as (5) with $\tilde{a}_{i+1,j}$ replacing $b_i$. Recall, $\mathbb{Z}_n = B_n/B_n[4]$; therefore the relations of (8) which are not relations of (2) normally generate $B_n[4]$. 

Finite generating set for $B_n[4]$ Proposition 5.3 guarantees the existence of a finite generating set for $B_n[4]$, however the size of this generating set grows faster than exponentially in $n$. We would hope for a generating set which has polynomial growth in $n$ and the natural method of finding this would be to add necessary generators to our normal generating set until we obtain a finite generating set. Notice that our normal generating set contains only squares and commutators of pure braids. Therefore we would hope to take the union of a finite generating sets for squares of pure braids and the commutator subgroup of the pure braid group.

Cohen, Falk, and Randell proved there exists an epimorphism from the pure braid group to the free group of rank 2 [9]. Since the commutator subgroup of the free group of rank 2 is not finitely generated, the existence of an epimorphism implies the commutator subgroup of the pure braid group is not finitely generated. Therefore we must first determine the intersection of the commutator subgroup of the pure braid group with the subgroup generated by squares of pure braids.

For odd integers $m \geq 1$ and $k \in \{2, 4\}$, the work of Apple, Bloomquist, Gravel, and Holden determine certain quotient structures of $B_n[m]/B_n[km]$ [4]. An alternative approach to determining a finite generating set for $B_n[4]$ would be to determine $B_n[4]$ as a group extensions and use the methods similar to section 2.4 to construct a group presentation. The first step in this method is to understand $B_n[8]$ or $B_n[16]$. 

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