APPLICATION OF THE SUBHARMONIC MELNIKOV METHOD TO PIECEWISE-SMOOTH SYSTEMS

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Abstract. We extend a refined version of the subharmonic Melnikov method to piecewise-smooth systems and demonstrate the theory for bi- and trilinear oscillators. Fundamental results for approximating solutions of piecewise-smooth systems by those of smooth systems are given and used to obtain the main result. Special attention is paid to degenerate resonance behavior, and analytical results are illustrated by numerical ones.

1. Introduction. Greenspan and Holmes [11] developed a perturbation method for analyzing (homoclinic and) subharmonic orbits in time-periodic perturbations of planar Hamiltonian systems. It is now called “Melnikov’s method” since its original idea was found in [13]. Existence of periodic orbits and their saddle-node bifurcations were analyzed but some difficulty arose in determination of their stability. Subsequently, some extensions have been made for the stability of subharmonics and their Hopf (Neimark-Sacker) and Bogdanov-Takens bifurcations in [17, 20, 21]. Especially, simple formulas for determining their stability and these bifurcations were given, and degenerate resonance behavior, in which the derivative of the unperturbed frequency with respect to the Hamiltonian energy disappears, was appropriately treated. Piecewise-smooth systems can be studied by using the extended approaches, as demonstrated below.

Piecewise-smooth systems also naturally arise in many engineering and physical applications due to impact, friction, collision and switching, and have attracted much attention for more than several decades. Modern theories in dynamical systems have been applied and developed to uncover and understand nonlinear phenomena such as bifurcations and chaos in these systems. The nonsmoothness of systems yields many interesting behaviors which never occur in smooth system. See, e.g., [6] and references therein for more details. Traditional techniques such as the averaging and multiple-scale methods [14, 15], which were developed for smooth systems, have also been used to study these piecewise-smooth systems (see,
e.g., [2, 4]). These analytical results are helpful to understand some behaviors in piecewise-smooth systems even though they are not mathematically rigorous. In fact, in some cases, such analyses were carried out rigorously [3, 8].

In this paper, we extend the refined version of the subharmonic Melnikov method [17, 21] to piecewise-smooth systems and demonstrate the theory for bi- and trilinear oscillators. Here we mainly treat two-dimensional systems, for which the subharmonic Melnikov method has been well developed although it was also extended to a special class of four-dimensional systems in [5, 19]. Fundamental results for approximating solutions of piecewise-smooth systems by those of smooth systems are given and used to obtain the main result. Special attention is paid to degenerate resonance behavior and analytical results are also illustrated by numerical ones via the computer software AUTO97 [7]. Similar analyses were given earlier for a mathematical model of vibrating microcantilevers in tapping mode atomic force microscopy in [22, 23]. The analytical results succeeded in explaining many nonlinear behaviors which were experimentally and numerically observed in the microcantilevers. Our result gives a mathematical basis for the theoretical analyses.

The outline of this paper is as follows: In Section 2 we review the extended version of the subharmonic Melnikov method. The existence, stability and saddle-node bifurcation results for periodic orbits are stated and degenerate resonances are briefly discussed. In Section 3 we give the fundamental results in general settings and our main result for piecewise-smooth systems. We apply the theory to bi- and trilinear oscillators in Sections 4 and 5, respectively. Discontinuous and continuous cases are, respectively, considered for bi- and trilinear oscillators. Moreover, the analytical results are compared with numerical ones via AUTO97 for both oscillators. Finally, we give a summary and some comments in Section 6.

2. Subharmonic Melnikov method.

2.1. Setup. We consider systems of the form

\[ \dot{x} = JDH(x) + \epsilon g(x, \omega t; \mu), \quad (x, \mu) \in \mathbb{R}^2 \times \mathbb{R}, \]  

(1)

where \( 0 < \epsilon \ll 1 \), \( H : \mathbb{R}^2 \to \mathbb{R} \) is \( C^{r+1} \) (\( r \geq 2 \)), \( g : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \) is \( C^r \) and \( 2\pi \)-periodic in \( \theta = \omega t \), and \( J \) is the \( 2 \times 2 \) symplectic matrix

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

We often drop out the dependence of (1) on the parameter \( \mu \) below. When \( \epsilon = 0 \), Eq. (1) becomes

\[ \dot{x} = JDH(x), \]  

(2)
which is a planar Hamiltonian system with a Hamiltonian $H(x)$. Thus, Eq. (1) is a time-periodic perturbation of the Hamiltonian system (2). We make the following assumption on the unperturbed system (2):

(A) There is a one-parameter family of periodic orbits, $x^\alpha(t)$, $\alpha \in [\alpha_\ell, \alpha_u] \subset \mathbb{R}$, with period $T^\alpha$. See Fig. 1. The family $x^\alpha(t)$ is also $C^r$ with respect to $\alpha$.

Note that $x^\alpha(t)$ is automatically $C^r$ with respect to $t$ since the vector field of (2) is $C^r$.

2.2. Outline of the approaches. Here we give an outline of the approaches used to obtain the results below in this section. See Section 2 of [17] for the proofs and technical details.

We first introduce action-angle coordinates $(I, \phi) \in \mathbb{R}_+ \times \mathbb{S}^1$ for the unperturbed system (2), where $\mathbb{R}_+$ is the set of non-negative real numbers and $\mathbb{S}^1 = \mathbb{R}/2\pi$ is the circle of length $2\pi$. The action variable $I$ is defined as

$$ I = \frac{1}{2\pi} \int_{x^\alpha} x_2 \, dx_1 = \int_0^{T^\alpha} x_2^\alpha(t) \dot{x}_1^\alpha(t) \, dt $$

for the periodic orbit $x^\alpha(t)$ (e.g., [1]). Note that the value of $I$ depends only on $\alpha$ in (3). The Hamiltonian is constant on $x^\alpha(t)$, so that it can be regarded as a function of only $I$:

$$ H = H(I). $$

By a well-known result on integrable Hamiltonian systems (see, e.g., Section 50 of [1]), we define the angle variable $\phi$ such that Eq. (2) is rewritten as

$$ \dot{I} = 0, \quad \dot{\phi} = \Omega(I), $$

where

$$ \Omega(I) = \frac{dH}{dI}(I). $$

Hence, we see that the symplectic transformation $(I, \phi) \mapsto x$ is given by

$$ x = x^{\alpha(I)} \left( \frac{\phi}{\Omega(I)} \right) $$

if the angle variable $\phi$ is chosen such that $\phi(0) = 0$, where $\alpha(I)$ represents the inverse of the relation (3). See p. 1723 of [17] for the proof that the transformation (6) is actually symplectic.

We next transform (1) into the action-angle coordinates $(I, \phi)$ to obtain

$$ \dot{I} = \epsilon F(I, \phi, \omega t; \mu), \quad \dot{\phi} = \Omega(I) + \epsilon G(I, \phi, \omega t; \mu), $$

where

$$ F(I, \phi, \theta; \mu) = -J \frac{\partial}{\partial \phi} x^{\alpha(I)} \left( \frac{\phi}{\Omega(I)} \right) \cdot g \left( x^{\alpha(I)} \left( \frac{\phi}{\Omega(I)} \right), \theta; \mu \right) $$

$$ = \frac{1}{\Omega(I)} \frac{dH}{dI} \left( x^{\alpha(I)} \left( \frac{\phi}{\Omega(I)} \right) \right) \cdot g \left( x^{\alpha(I)} \left( \frac{\phi}{\Omega(I)} \right), \theta; \mu \right), $$

$$ G(I, \phi, \theta; \mu) = J \frac{\partial}{\partial I} x^{\alpha(I)} \left( \frac{\phi}{\Omega(I)} \right) \cdot g \left( x^{\alpha(I)} \left( \frac{\phi}{\Omega(I)} \right), \theta; \mu \right), $$

where “$\cdot$” represents the inner product. Define the Poincaré map $P_\epsilon$ for (7) as

$$ P_\epsilon : (I(0), \phi(0)) \mapsto (I(T), \phi(T)) $$
where \((I, \phi) = (I(t), \phi(t))\) is a solution of (7) and \(T = 2\pi/\omega\). We also write \(P_0\) for \(P_\epsilon\) when \(\epsilon = 0\), so that \(P_0\) represents the Poincaré map for (4) and yields a simple rotation. The \(m\)th iterate of \(P_\epsilon\) is estimated up to \(O(\epsilon)\) as

\[
P_\epsilon^m : (I_0, \phi_0) \mapsto (I_0, \phi_0 + mT\Omega(I_0)) + \epsilon(Q^m(I_0, \phi_0; \omega, \mu), R^m(I_0, \phi_0; \omega, \mu)),
\]

where

\[
Q^m(I_0, \phi_0; \omega, \mu) = \int_0^{mT} F(I_0, \Omega(I_0)t + \phi_0, \omega t, \mu) \, dt,
\]

\[
R^m(I_0, \phi_0; \omega, \mu) = \frac{\partial \Omega}{\partial t}(I_0) \int_0^{mT} F(I_0, \Omega(I_0)v + \phi_0, \omega v, \mu) \, dv \, dt
\]

\[+ \int_0^{mT} G(I_0, \Omega(I_0)t + \phi_0, \omega t, \mu) \, dt.
\]

We reduce the study of \(m\)th-order subharmonic orbits in (1) to that of fixed points of \(P_\epsilon^m\).

Let \(I^\alpha\) denote the value of \(I\) for \(x^\alpha(t)\) and let \(\Omega^\alpha = \Omega(I^\alpha)\). We define the subharmonic Melnikov functions as

\[
M^{m/n}(\alpha, \theta; \mu) = \int_0^{mT} DH(x^\alpha(t)) \cdot g \left( x^\alpha(t), \frac{m}{n} \Omega^\alpha t + \theta; \mu \right) \, dt,
\]

\[
L^{m/n}(\alpha, \theta; \mu) = \int_0^{mT} \det D_x g \left( x^\alpha(t), \frac{m}{n} \Omega^\alpha t + \theta; \mu \right) \, dt,
\]

\[
N^{m/n}(\alpha, \theta; \mu) = \frac{n^2}{\Omega^\alpha} \frac{\partial \Omega}{\partial I}(I^\alpha) \int_0^{2\pi} F(I^\alpha, nu, mu + \theta; \mu) \, ds
\]

\[+ n \int_0^{2\pi} G(I^\alpha, ns, ms + \theta; \mu) \, ds.
\]

In general, the function \(N^{m/n}\) is difficult to estimate directly, compared with \(M^{m/n}\) and \(L^{m/n}\), but we have the following result.

**Proposition 2.1.** \(N^{m/n}(\alpha, \theta; \mu)\) satisfies

\[
\frac{\partial N^{m/n}}{\partial \theta}(\alpha, \theta; \mu) = \frac{\pi n}{\Omega^\alpha} \frac{d\Omega}{dI}(I^\alpha) \frac{\partial M^{m/n}}{\partial \theta}(\alpha, \theta; \mu)
\]

\[+ \frac{n}{m} \frac{d\alpha}{dI}(I^\alpha) \frac{\partial M^{m/n}}{\partial \alpha}(\alpha, \theta; \mu) - \frac{n}{m} \Omega^\alpha L^{m/n}(\alpha, \theta).
\]

See Appendix A of [17] for the proof but a small typographical error exists in the formula (A.2) of that paper.

For \(m, n \in \mathbb{N}\) relatively prime, let \(\alpha^{m/n}\) and \(I^{m/n}\) be, respectively, the value of \(\alpha\) and \(I^\alpha\) such that

\[nT^\alpha = mT \quad (m\Omega^\alpha = n\omega).
\]

The \(m\)th iterate \(P_0^m\) has fixed points on \(I_0 = I^{m/n}\). We can show that the first-order terms \(Q^m, R^m\) in (8) are expressed by the Melnikov functions \(M^{m/n}, N^{m/n}\) on \(I_0 = I^{m/n}\) (see Proposition 2.2 of [17]). Using this fact, we can analyze fixed points of \(P_\epsilon^m\) near \(I_0 = I^{m/n}\) in (8) to obtain existence, stability and bifurcation.
results for mth-order subharmonic orbits in (1), as given below. We also define
\[ \hat{M}^{m/n}(\theta; \mu) = M^{m/n}(\alpha^{m/n}, \theta; \mu) \]
\[ = \int_0^{mT} D H(x^{\alpha^{m/n}}(t)) \cdot g(x^{\alpha^{m/n}}(t), \omega t + \theta; \mu) \, dt \]
and
\[ \hat{L}^{m/n}(\theta; \mu) = L^{m/n}(\alpha^{m/n}, \theta; \mu). \]
In particular, \( \hat{M}^{m/n}(\theta; \mu) \) is the standard, subharmonic Melnikov function, which
was originally used in [11] (see also [12, 16]).

2.3. Existence, stability and saddle-node bifurcations. We begin with existence and stability theorems for subharmonics in (1) (see Section 3.1 of [17] for the proof). In this subsection we assume the nondegenerate condition
\[ \frac{d\Omega}{dI}(I^{m/n}) \neq 0. \] (10)

**Theorem 2.2.** Suppose that \( \hat{M}(\theta) \) has a simple zero at \( \theta = \theta_0 \), i.e.,
\[ \hat{M}^{m/n}(\theta_0) = 0, \quad \frac{dM^{m/n}}{d\theta}(\theta_0) \neq 0. \]
Then for \( \epsilon > 0 \) sufficiently small there exists a subharmonic orbit of period \( mT \) near \( x^{\alpha^{m/n}}(t - \theta_0/\omega) \). Moreover, it is of a saddle type if
\[ \frac{d\Omega}{dI}(I^{m/n}) \frac{dM^{m/n}}{d\theta}(\theta_0) < 0 \]
and of a sink type (resp. of a source type) if
\[ \frac{d\Omega}{dI}(I^{m/n}) \frac{dM^{m/n}}{d\theta}(\theta_0) > 0 \text{ and } \frac{d\hat{L}^{m/n}}{d\theta}(\theta_0) < 0 \text{ (resp. > 0)}. \]

The first part of Theorem 2.2 is standard in the subharmonic Melnikov method [11, 12, 16] but the second one is not and was originally obtained in [17].

We next state a saddle-node bifurcation theorem for subharmonics in (1) (see Section 4.1 of [17] for the proof).

**Theorem 2.3.** Suppose that at some point \( (\theta_0, \mu_0) \) the following conditions hold:
(i) \( \hat{M}^{m/n}(\theta_0; \mu_0) = 0; \)
(ii) \( \hat{M}_\mu^{m/n}(\theta_0; \mu_0) = 0; \)
(iii) \( \hat{M}_{\theta_0}^{m/n}(\theta_0; \mu_0) \neq 0; \)
(iv) \( \hat{M}_{\theta_0}^{m/n}(\theta_0; \mu_0) \neq 0. \)
Then at \( \mu = \mu_0 + O(\epsilon) \) a saddle-node bifurcation of subharmonics of period \( mT \) occurs. Moreover, it is supercritical (resp. subcritical) if
\[ \hat{M}_{\theta_0}^{m/n}(\theta_0; \mu_0) \hat{M}_\mu^{m/n}(\theta_0; \mu_0) < 0 \text{ (resp. > 0)}. \] (11)

This result is basically standard but the formula (11) was not given in [11, 12, 16].
2.4. Degenerate resonances. Assume that for some $I = \bar{I}$

\[
\overline{\Omega} = \Omega(\bar{I}) \neq 0, \quad \bar{\Omega}' = \frac{d\Omega}{d\bar{I}}(\bar{I}) = 0, \quad \bar{\Omega}'' = \frac{d^2\Omega}{d\bar{I}^2}(\bar{I}) \neq 0.
\]

The standard Melnikov method \cite{11, 12, 16} generally detects such saddle-node bifurcation curves near $(\omega, \mu) = (\bar{\omega}, \mu_0)$ in the $(\omega, \mu)$-parameter space as shown in Fig. 2, depending whether $\bar{I}$ exists or not. So, when $\bar{I}$ exists, it is unclear what behavior occurs near the degenerate point $(\bar{\omega}, \mu_0)$. Based on Theorem 2.3, for example, if Eq. (11) is negative, then in Fig. 2 there are four subharmonics above the upper curve; two subharmonics between the upper and lower curves; and no subharmonics under the lower curve and in the left (resp. right) region of $\omega < \bar{\omega}$ (resp. of $\omega > \bar{\omega}$) for $\bar{\Omega}'' > 0$ (resp. for $\bar{\Omega}'' < 0$).

Let $I^\alpha = \bar{I}$, $\omega = m\Omega(\bar{I})/m$, $\epsilon \nu = \omega - \bar{\omega}$ and

\[
\{M, \bar{N}, \bar{L}\}^{m/n} (\theta; \mu) = \{M, N, L\}^{m/n} (\bar{\alpha}, \theta; \mu).
\]

We have the following result in the situation of (12).

**Theorem 2.4.** Suppose that at some point $(\theta_0, \mu_0)$ the following conditions hold:

(i) $\bar{M}^{m/n}(\theta_0; \mu_0) = 0$;

(ii) $\bar{M}'^{m/n}(\theta_0; \mu_0) = 0$;

(iii) $\bar{M}_0^{m/n}(\theta_0; \mu_0) \neq 0$;

(iv) $\bar{M}_0''^{m/n}(\theta_0; \mu_0) \neq 0$.

Then in the $(\nu, \mu)$-parameter plane there exist three saddle-node bifurcation curves for subharmonics of period $mT$ in a neighborhood of $(\nu_0, \mu_0)$, where

\[
\nu_0 = \frac{m}{2\pi n^2} \bar{N}^{m/n}(\theta_0; \mu_0),
\]
See Section 4.1 of [17] or Section 2.2 of [21] for the proof. We can also compute the cusp bifurcation point \((\nu_*, \mu_*)\) and three saddle-node bifurcation curves in the \((\nu, \mu)\)-parameter space up to \(O(\epsilon^{1/3})\) and \(O(\epsilon^{2/3})\) on \(\nu\) and \(\mu\), respectively (see Theorem 2.6 of [21] for the details).

2.5. Example. Consider general single-degree-of-freedom nonlinear oscillators of the form
\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -\varphi(x_1) + \epsilon(-\delta x_2 + \gamma \cos \omega t),
\] (14)
as an example to illustrate the above theories, where \(\gamma, \delta\) are positive constants, \(\varphi : \mathbb{R} \rightarrow \mathbb{R}\) is \(C^r\), \(\varphi(0) = 0\) and \(\varphi'(0) > 0\). When \(\epsilon = 0\), Eq. (14) is Hamiltonian with a Hamiltonian function
\[
H(x) = \int \varphi(x_1) \, dx_1 + \frac{1}{2} x_2^2
\]
and has a one-parameter family of periodic orbits around a center-type equilibrium at the origin. Moreover, the periodic orbits can be chosen such that \(x_1^\alpha(t)\) and \(x_2^\alpha(t)\), respectively, are even and odd functions of \(t\).

We can express the Melnikov functions (9) as
\[
M^{m/1}(\alpha, \theta) = -\gamma A^m(\alpha) \sin \theta - \delta B(\alpha), \quad L^{m/1}(\alpha, \theta) = -\delta T^\alpha < 0,
\] (15)
where
\[
A^m(\alpha) = \int_0^T x_2^\alpha(t) \sin(m \Omega^\alpha t) \, dt, \quad B(\alpha) = \int_0^T [x_2^\alpha(t)]^2 \, dt.
\]
Note that
\[
M^{m/n}(\alpha, \theta) = -n \delta B(\alpha) \neq 0
\]
for \(\forall \alpha, \theta\) when \(n \neq 1\), since \(x^\alpha(t)\) is periodic in \(t\) with period \(2\pi / \Omega^\alpha\).

Choose \(\mu = \gamma\) and \(\alpha = H(x^\alpha(t))\) and let \(\delta\) be fixed. Applying Theorems 2.2 and 2.3, we see that a supercritical saddle-node bifurcation occurs at
\[
\gamma = \frac{B(\alpha^{m/1})}{A^m(\alpha^{m/1})} \delta + O(\epsilon)
\] (16)
at which stable and unstable subharmonics are born, if \(d\Omega^\alpha d\alpha / d\alpha \neq 0\). Here the parameter value \(\alpha^{m/1}\) of \(\alpha\) was defined in (5) with \(n = 1\). Moreover, if \(d\Omega^\alpha / d\alpha = 0\) and \(d^2\Omega^\alpha / d\alpha^2 \neq 0\) at \(\alpha = \tilde{\alpha} = \alpha^{m/1}\), then by Theorem 2.4 the degenerate resonance

\[\text{Figure 3. Degenerate resonance: (a) } \bar{\Omega}'' > 0; \text{ (b) } \bar{\Omega}'' < 0.\]
behavior shown in Fig. 3 occurs at \( \omega = \tilde{\omega} = m\Omega^{m/1} \) and (16). Using Proposition 2.1 and noting that \( d\alpha/dI = \Omega(I) \), we also have
\[
\bar{N}^{m/1}(\theta) = \frac{\gamma \tilde{\omega}}{m} d^m(\tilde{\alpha}) \cos \theta
\]
in (13) (cf. Section 3.1 of [21]).

3. Main result. The subharmonic Melnikov method described above is originally applicable only to smooth systems. In this section we extend the method to piecewise-smooth systems. To this end, we give fundamental results to approximate solutions of piecewise-smooth systems by those of smooth systems in general settings. As stated in Section 1, the dynamics of piecewise-smooth systems are often different from those of smooth systems. These results guarantee that such a difference does not occur under some conditions.

3.1. Smooth approximations of piecewise-smooth systems. Let \( \Pi = \{ x \in \mathbb{R}^n \mid h(x) = 0 \} \) be an \((n-1)\)-dimensional smooth surface in \( \mathbb{R}^n \), where \( h : \mathbb{R}^n \to \mathbb{R} \) is \( C^r \) and \( Dh(x) \neq 0 \) on \( \Pi \). For simplicity we assume that \( h(x) \) is independent of the parameter \( \mu \). Consider piecewise-smooth systems of the form
\[
\dot{x} = f(x, t; \mu), \quad x \in \mathbb{R}^n,
\]
where \( f(x, t; \mu) \) is \( C^r \) in \((t, \mu)\) while it is \( C^r \) about \( x \) in \( \mathbb{R}^n \setminus \Pi \) and nonsmooth on \( \Pi \). We assume that an initial value problem for (18) has a unique solution which is \( C^r \) outside \( \Pi \) and continuous on \( \Pi \). Again, we drop out the dependence of (18) on the parameter \( \mu \) below.

**Theorem 3.1.** Let \( T > 0 \) and let \( x(t) \) be a solution of (18). Let \( D(\subset \mathbb{R}^n) \) be a bounded region containing \( \{ x(t) \mid t \in [0, T] \} \) and let \( U_\rho \) be a \( \rho \)-neighborhood of \( \Pi \cap D \). Suppose that a function \( f_\rho(x, t) \) is \( C^r \) in \((x, t)\), and satisfies
\[
f_\rho(x, t) = f(x, t) \quad \text{for } x \in D \setminus U_\rho
\]
and
\[
f_\rho(x, t) \cdot Dh(x) \neq 0 \quad \text{on } x \in \Pi \cap D.
\]
Let \( x_\rho(t) \) be a solution of
\[
\dot{x} = f_\rho(x, t)
\]
with \( x_\rho(0) = x(0) + O(\rho) \). If \( x(t) \) passes through \( \Pi \) only finitely many times on \([0, T]\) and the left- and right-hand limits
\[
\lim_{t \to t_0 \pm 0} f(x(t), t) \cdot Dh(x(t))
\]
are nonzero and have the same sign as (20) at any \( t = t_0 \in [0, T] \) such that \( x(t_0) \in \Pi \), then for \( \rho > 0 \) sufficiently small \( x(t) = x_\rho(t) + O(\rho) \) uniformly in \([0, T]\).

This theorem is very fundamental but has been previously unpublished at least, to the author’s knowledge.

**Proof.** For simplicity we assume that \( x(t) \) passes through \( \Pi \) once. It is obvious that the general case follows from this case.

Let \((t_1, t_2)\) be an interval on which \( x(t) \) stays in \( U_\rho \) and let \( u_+ \) be the left- and right-hand limits of (22). Specifically, we assume that \( u_+ > 0 \), so that \( h(x(t)) < 0 \) for \( t < t_1 \) and \( h(x(t)) > 0 \) for \( t > t_2 \) since \((\dd/dt)h(x(t)) = f(x(t), t) \cdot Dh(x(t))\). Let \( D_1 = D \setminus \{ h(x) < 0 \} \) and \( D_2 = D \setminus \{ h(x) > 0 \} \). Thus, \( x(t) \in D_1 \) for \( t \leq t_1 \) and \( x(t) \in D_2 \) for \( t \geq t_2 \).
Let $0 < t'_1 < t''_1 < t_1$ and suppose that $x_\rho(t)$ stays in $D_1 \setminus U_\rho$ for $0 \leq t \leq t'_1$, in $U_\rho$ for $t'_1 < t < t''_1$ and in $D_2 \setminus U_\rho$ for $t''_1 \leq t \leq t_1$. For $0 \leq t \leq t'_1$ we have
\[
|x(t) - x_\rho(t)| \leq |x(0) - x_\rho(0)| + \int_0^t |f(x(s), s) - f(x_\rho(s), s)|ds
\]
\[
\leq |x(0) - x_\rho(0)| + L_1 \int_0^t |x(s) - x_\rho(s)|ds,
\]
so that by Gronwall’s inequality (see, e.g., Lemma 4.1.2 of [12])
\[
|x(t) - x_\rho(t)| \leq |x(0) - x_\rho(0)| e^{L_1 t} = O(\rho),
\]
where $L_1$ is the Lipschitz constant of $f$ in $D_1$. On the other hand, let
\[
u_\rho = \inf_{x \in U_\rho} \frac{f_\rho(x) \cdot Dh(x)}{|Dh(x)|},
\]
which is positive by assumption. We estimate the length of the time interval $(t'_1, t''_1)$ as
\[
t''_1 - t'_1 < \frac{2\rho}{\nu_\rho} + O(\rho^2).
\]
For $t'_1 < t \leq t''_1$ we have
\[
|x(t) - x_\rho(t)| \leq |x(0) - x_\rho(0)| + \int_0^{t'_1} |f(x(s), s) - f(x_\rho(s), s)|ds
\]
\[
+ \int_{t'_1}^t |f(x(s), s)|ds + \int_{t'_1}^t |f_\rho(x_\rho(s), s)|ds
\]
\[
\leq L_1 \int_0^t |x(s) - x_\rho(s)|ds + O(\rho),
\]
so that by Gronwall’s inequality
\[
|x(t) - x_\rho(t)| = O(\rho).
\]
Since $x(t''_1)$ is $O(\rho)$-close to the boundary of $U_\rho$, we estimate $t_1 - t''_1 = O(\rho)$ to obtain (24) for $t''_1 < t \leq t_1$, although $x_\rho(t) \not\in U_\rho$. Replacing $x(t)$ and $x_\rho(t)$ in the above arguments, we prove (24) generally when $x(t)$ or $x_\rho(t) \in D_1 \setminus U_\rho$.

Let $0 < t'_1 < t_1 < t'_2 < t_2$ and suppose that $x_\rho(t)$ stays in $U_\rho$ for $t'_1 \leq t \leq t'_2$ and in $D_2 \setminus U_\rho$ for $t'_2 < t \leq t_2$. We easily see that $t_2 - t'_1 = O(\rho)$. Hence, for $t_1 < t \leq t_2$ we still have (23), which yields (24). For $t > t_2$ we have
\[
|x(t) - x_\rho(t)| \leq |x(0) - x_\rho(0)| + \int_0^{t'_2} |f(x(s), s) - f(x_\rho(s), s)|ds
\]
\[
+ \int_{t'_2}^{t_1} |f(x(s), s)|ds + \int_{t'_1}^t |f_\rho(x_\rho(s), s)|ds
\]
\[
+ \int_{t_2}^t |f(x(s), s) - f(x_\rho(s), s)|ds
\]
\[
\leq L \int_0^t |x(s) - x_\rho(s)|ds + O(\rho),
\]
from which Eq. (24) follows by Gronwall’s inequality, where $L = \max(L_1, L_2)$ with $L_2$ the Lipschitz constant of $f$ in $D_2$. Finally, we replace $x(t)$ and $x_\rho(t)$ in these arguments to complete the proof. \qed
Remark 3.2. (i) From the above proof, we can replace \( x(t) \) with \( x_\rho(t) \) in the hypotheses of Theorem 3.1: If \( x_\rho(t) \) passes through \( \Pi \) only finitely many times on \([0,T]\) and the corresponding left- and right-hand limits (22) with \( x(t) = x_\rho(t) \) are nonzero and have the same sign as (20) at any \( t = t_0 \in [0,T] \) such that \( x_\rho(t_0) \in \Pi \), then the conclusion of Theorem 3.1 holds.

(ii) If the system (18) is continuous, then for the statement of Theorem 3.1 to hold, we only have to assume

\[
\begin{cases}
  f_\rho(x,t) = f(x,t) + O(\rho) & \text{for } x \in U_\rho; \\
  f_\rho(x,t) = f(x,t) & \text{for } x \not\in U_\rho
\end{cases}
\]

without the assumptions that \( x(t) \) passes \( \Pi \) finitely many times. Actually, we have

\[
\begin{align*}
|x(t) - x_\rho(t)| & \leq |x(0) - x_\rho(0)| + \int_0^t |f_\rho(x(s),s) - f_\rho(x_\rho(s),s)|\,ds \\
& \quad + \int_0^t |f(x(s),s) - f(x(s),s)|\,ds \\
& \leq L_\rho \int_0^t |x(s) - x_\rho(s)|\,ds + O(\rho)
\end{align*}
\]

for \( t \in [0,T] \), where \( L_\rho \) is the Lipschitz constant of \( f_\rho \).

(iii) The statement of Theorem 3.1 also holds even when \( f \) is only continuous in \( t \).

We now consider the case in which the piecewise-smooth system (18) is periodic in \( t \) and depends on the parameter \( \mu \).

Theorem 3.3. Let \( f(x,t;\mu) \) be \( T \)-periodic in \( t \) and suppose that \( f_\rho(x,t;\mu) \) is \( T \)-periodic in \( t \) and \( C^r \) in \( (x,t,\mu) \), and satisfies (19). If for \( \rho > 0 \) sufficiently small the smooth system (21) has a hyperbolic periodic orbit \( \bar{x}_\rho(t) \) satisfying the hypotheses of Theorem 3.1 with \( x(t) = \bar{x}_\rho(t) \) (see Remark 3.2(ii)), then the piecewise-smooth system (18) has a hyperbolic periodic orbit of the same stability type as \( \bar{x}_\rho(t) \) in its \( O(\rho) \)-neighborhood. Moreover, if the periodic orbit \( \bar{x}_\rho(t) \) undergoes a local bifurcation at \( \mu = \mu_0 \), then the same local bifurcation of the corresponding periodic orbit occurs in (18) at \( \mu = \mu_0 + O(\rho) \).

Proof. For simplicity we assume that \( \bar{x}_\rho(t) \) passes through \( \Pi \) once as in the proof of Theorem 3.1. We first note that the period of \( x_\rho(t) \) must be \( mT \) for some positive integer \( m \) (see, e.g., Exercise 1.5.3 of [12]). We apply Theorem 3.1 to \( \bar{x}_\rho(t) \) to obtain an orbit \( \bar{x}(t) \) of (18), which may not be periodic, such that \( \bar{x}(t) = \bar{x}_\rho(t) + O(\rho) \) uniformly in \([0,mT]\).

Let \( x(t) \) and \( x_\rho(t) \) be solutions of (18) and (21) with \( x(0) \) and \( x_\rho(0) \) near \( \bar{x}(0) \) and \( \bar{x}_\rho(0) \), respectively. Define \( mT \)-time Poincaré maps \( \bar{P} \) and \( \bar{P}_\rho \) near \( \bar{x}(0) \) and \( \bar{x}_\rho(0) \), respectively, in a standard manner as

\[
\bar{P}: x(0) \mapsto x(mT), \quad \bar{P}_\rho: x_\rho(0) \mapsto x_\rho(mT).
\]

By assumption, \( x(t) \) also passes through \( \Pi \) once, say, at \( t = t_0 \), and it is \( C^r \) on \([0,t_0]\) and \([t_0,mT]\) if only right- or left-hand higher-order continuous differentiability is required at the end of the intervals. Since it is a root of \( h(x(t_0)) = 0 \), \( t_0 \) depends on \( x(0) \) in a \( C^r \) manner. Hence, two maps

\[
\bar{P}_1: x(0) \mapsto x(t_0), \quad \bar{P}_2: x(t_0) \mapsto x(mT)
\]

are \( C^r \), so that \( \bar{P} = \bar{P}_2 \bar{P}_1 \) is also \( C^r \), although the system (18) is not smooth.
Since $\bar{x}_\rho(0)$ is a hyperbolic fixed point of $\bar{P}_\rho$ and $\bar{P} = \bar{P}_\rho + O(\rho)$ by Theorem 3.1, it follows from the persistence of hyperbolic fixed points [12] that $\bar{P}$ has a hyperbolic fixed point of the same stability type as $\bar{x}_\rho(0)$ in its $O(\rho)$-neighborhood. This means the first part. We use a standard result for smooth diffeomorphisms to prove the second part.

Remark 3.4. Obviously, we have the same result as Theorem 3.3 even when the system (18) is autonomous.

3.2. Applications for the subharmonic Melnikov analyses. Now we reconsider the system (1) but only assume that $H(x)$ is $C^{r+1}$, $g(x, t; \mu)$ is $C^r$ in $x \in \mathbb{R}^2 \setminus \Pi$ and they are nonsmooth for $x \in \Pi$ while $g(x, t; \mu)$ is $C^r$ in $t$ and $\mu$, where $\Pi = \{x \in \mathbb{R}^2 \mid h(x) = 0\}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ is $C^r$. We also modify Assumption (A) as follows:

(A1) There is a one-parameter family of periodic orbits, $x^\alpha(t)$, $\alpha \in [\alpha_l, \alpha_u]$, with period $T^\alpha$. They are $C^r$ about $t$ and $\alpha$ in $\mathbb{R}^2 \setminus \Pi$ but only continuous on $\Pi$.

(A2) The left- and right-hand limits

$$\lim_{t \to t_0 \pm 0} JDH(x^\alpha(t), t) \cdot Dg(x^\alpha(t))$$

are nonzero and have the same sign at any $t = t_0 \in [0, T^\alpha)$ such that $x^\alpha(t_0) \in \Pi$.

Note that $T^\alpha$ and $I^\alpha$ are $C^r$ if Assumption (A1) holds since the sum of $C^r$ functions is also $C^r$ (see Eq. (3)). Letting $\Omega^\alpha = 2\pi/T^\alpha$, we can also define the Melnikov functions using (9) for the piecewise-smooth case.

Using Theorem 3.3, we can prove the following result.

Theorem 3.5. Under Assumptions (A1) and (A2), the statements of Theorems 2.2-2.4 also hold for the piecewise-smooth case of (1).

Proof. We first approximate solutions of (1) up to $O(\rho)$ by those of a smooth system

$$\dot{x} = JDH_\rho(x) + \epsilon g_\rho(x, \omega t; \mu),$$

where $H_\rho$ is $C^{r+1}$ and $g_\rho$ is $C^r$. Here $f_\rho = JDH_\rho$ and $g_\rho$ satisfy (19) for $f = JDH$ and $g$, respectively. Note that $H_\rho$ and $g_\rho$ can be chosen independently of $\epsilon$. Hence, we see that the small constant $\rho$ can be taken uniformly in $[0, \epsilon_0]$ for $\epsilon_0 > 0$ sufficiently small. Moreover, the continuous family $x^\alpha(t)$ is approximated by a smooth family of periodic orbits $\bar{x}_\rho^\alpha(t)$, via Theorem 3.3 and Remark 3.4. The periodic orbits $x_\rho^\alpha(t)$ also satisfy Assumption (A2) with $H = H_\rho$ for $\rho > 0$ sufficiently small.

We apply Theorems 2.2-2.4 to the smooth system (25). Note that the Melnikov functions for (25) can also be approximated by those for the piecewise-smooth system (1). For instance,

$$M^{m/n}_\rho(\alpha, \theta; \mu) = \int_0^{mT} DH_\rho(x_\rho^\alpha(t)) \cdot g\left(x_\rho^\alpha(t), \frac{m}{n} \Omega_\rho t + \theta; \mu \right) dt$$

$$= M^{m/n}(\alpha, \theta; \mu) + O(\rho),$$

where $M^{m/n}$ denotes the Melnikov function estimated for the piecewise-smooth system (1). The following result gives a key to the proof of Theorem 3.5 and guarantees that the size of $\epsilon$ for which the statements of Theorems 2.2-2.4 hold does not tend to zero as $\rho \to 0$. 
Lemma 3.6. Suppose that the hypotheses of Theorems 2.2-2.4 are satisfied in (25) for \( \rho \in (0, \rho_0) \) with \( \rho_0 > 0 \) sufficiently small. Then for \( \epsilon \in (0, \epsilon_0) \) with \( \epsilon_0 > 0 \) sufficiently small the conclusions of Theorems 2.2-2.4 hold in (25) for \( \rho \in (0, \rho_0) \).

Proof. As in Section 2.2, consider the Poincaré map \( P_{\epsilon,\rho} \) for the action-angle coordinates \((I, \phi)\) in (25). We also define \( P_{\epsilon,0} = \lim_{\rho \to 0} P_{\epsilon,\rho} \) for \( \epsilon \in [0, \epsilon_1] \) with some small \( \epsilon_1 > 0 \).

Recall that \( x^\alpha(t) \) is \( C^r \) in \((t, \alpha)\) except on \( \Pi \) by Assumption (A1) and \( I^\alpha \) is \( C^r \) if \( x \notin \Pi \). Noting this fact and using the argument in the proof of Theorem 3.3, we take \( P_{\epsilon,\rho} \) to be \( C^r \) not only in \((I, \phi)\) but also in \( \epsilon \in [0, \epsilon_1] \) for \( \rho \in (0, \rho_0) \). For \( \rho \in (0, \rho_0) \) we also estimate its \( n \)th iterate as (8) up to \( O(\epsilon) \) and easily see that the statements of Theorems 2.2-2.4 hold in the \( O(\epsilon) \)-approximation. Applying standard results for smooth diffeomorphisms as in the proof of Theorem 3.3, we obtain the result.

For \( \epsilon > 0 \) sufficiently small, periodic orbits detected by the Melnikov method in (25) for \( \rho \in (0, \rho_0) \) satisfy the hypotheses of Theorem 3.1 (they pass through \( \Pi \) only finitely many times and the left- and right-hand limits (19) are nonzero and have the same sign when they do) like the unperturbed orbits \( x^\alpha_0(t) \). From the proof of Lemma 3.6 we see that \( P_{\epsilon,0} \) is the Poincaré map for the action-angle coordinates \((I, \phi)\) in (1). Using Lemma 3.6 and taking the limit \( \rho \to 0 \), we complete the proof by Theorem 3.3.

In the next two sections, we apply Theorem 3.5 to bi- and trilinear oscillators.

4. Bilinear oscillator. We first consider a bilinear oscillator

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -\varphi(x_1) + \epsilon(-\delta x_2 + \gamma \cos \omega t),
\]

where

\[
\varphi(x) = \begin{cases} 
\xi & \text{for } x < \xi_1; \\
\xi_1 + \varphi_1 & \text{for } x \geq \xi_1.
\end{cases}
\]

When \( k_1 \xi_1 + \varphi_1 \neq \xi_1 \), Eq. (26) represents a discontinuous piecewise-linear system. See Fig. 4(a) for the restoring characteristic \( \varphi(x) \).
For $\epsilon = 0$ there exists a one-parameter family of periodic orbits which are expressed as

$$x^\alpha(t) = \begin{cases} 
(a_1 \cos \omega_1 t - \varphi_1 k_1, -\omega_1 a_1 \sin \omega_1 t) & \text{for } t \in [0, \tau_1); \\
(a \cos(t + t_1), -a \sin(t + t_1)) & \text{for } t \in [\tau_1, \frac{1}{2} T^\alpha]; \\
(x^\alpha_1(-t), -x^\alpha_2(-t)) & \text{for } t \in [-\frac{1}{2} T^\alpha, 0),
\end{cases}$$

(27)

where $\omega^2 = k_1$, $T^\alpha = 2(\tau_0 + \tau_1)$ and

$$\alpha_1^2 = \left(\xi_1 + \frac{\varphi_1}{k_1}\right)^2 + \frac{\alpha^2 - \xi_1^2}{k_1}, \quad \tau_0 = \pi - \arccos\left(\xi_1 \alpha\right),$$

$$\tau_1 = \frac{1}{\omega_1} \arccos\left(\frac{1}{\alpha_1} \left(\xi_1 + \frac{\varphi_1}{k_1}\right)\right), \quad t_1 = \arccos\left(\frac{\xi_1}{\alpha}\right) - \tau_1.$$
Figure 6. Numerical bifurcation analysis by AUTO97 for the bilinear oscillator (26) with $k_1 = 0.5$, $\xi_1 = 1$, $\phi_1 = 1.5$, $m = 1$, $\epsilon \delta = 0.1$: (a) Saddle-node bifurcation sets; (b) bifurcation diagram for $\epsilon \gamma = 0.3$. In plate (a) saddle-node bifurcation and border collision curves are drawn in black and in blue, respectively, while the theoretical prediction by the standard Melnikov method is drawn in red and the degenerate resonance point is plotted as ‘●’. In plate (b) stable and unstable periodic orbits are drawn in red and in blue, respectively.

in the form of (15) and (17), as in Section 2.5, where for $\alpha > 1$

$$A^m(\alpha) = \frac{2 m \Omega^\alpha \sqrt{\alpha^2 - \xi_1^2} \cos m \Omega^\alpha \tau_1 - 2 (k_1 \xi_1 + \phi_1) \sin m \Omega^\alpha \tau_1}{(m \Omega^\alpha)^2 - k_1}$$

$$+ \frac{2 \xi_1 \sin m \Omega^\alpha \tau_1 - 2 m \Omega^\alpha \sqrt{\alpha^2 - \xi_1^2} \cos m \Omega^\alpha \tau_1}{(m \Omega^\alpha)^2 - 1}.$$  

$$B(\alpha) = \alpha^2 \tau_0 + k_1 \alpha_1^2 \tau_1 - \frac{\phi_1}{k_1} \sqrt{\alpha^2 - \xi_1^2}.$$  

So we see that a supercritical saddle-node bifurcation occurs at (16) if $d\Omega(\alpha^{m/1})/d\alpha \neq 0$, and a degenerate resonance behavior shown in Fig. 3 occurs at $\omega = \bar{\omega} = m \Omega^{m/1}$ and (16) if $d\Omega(\alpha)/d\alpha = 0$ and $d^2\Omega(\alpha)/d\alpha^2 \neq 0$ at $\alpha = \bar{\alpha} = \alpha^{m/1}$.

We give a numerical example for $k_1 = 0.5$, $\xi_1 = 1$, $\phi_1 = 1.5$, $m = 1$. Figure 5 shows numerical computations of $\Omega(\alpha) = 2\pi/T^\alpha$, $A^1(\alpha)$, $B(\alpha)$ and $\gamma/\delta = A^1(\alpha)/B(\alpha)$. It follows by the piecewise-smooth versions of Theorems 2.2 and 2.3 that a saddle-node bifurcation occurs near the computed value of $\gamma/\delta$, and stable and unstable periodic orbits exist above the value when $d\Omega(\alpha)/d\alpha \neq 0$. Moreover, since $d\Omega(\alpha)/d\alpha = 0$ at $\alpha \approx 1.52753$, by the piecewise-smooth version of Theorem 2.4, a degenerate resonance as shown in Fig. 3(b) occurs near $(\omega, \gamma/\delta) = (1.10906, 1.57846)$.

Figure 6 shows numerical bifurcation results by AUTO97 [7] for $\epsilon \delta = 0.1$. See Appendix A for some details on the approach of the numerical bifurcation analysis.

The starting solution was chosen as a periodic orbit in the linear case of $k_1 = 1$ and $\phi_1 = 0$ and continued to $k_1 = 0.5$ and $\phi_1 = 1.5$. In Fig. 6(a) saddle-node bifurcation curves are drawn in black as well as a border collision curve on which the periodic orbit touches the border $x_1 = \xi_1$ is drawn in blue. The theoretical prediction by the
standard Melnikov method, $\gamma/\delta = A^4(\alpha)/B(\alpha)$, is drawn in red (cf. Fig. 2(c)) and the degenerate resonance point is plotted as ‘•’. In Fig. 6(b) a bifurcation diagram is given for $\epsilon\gamma = 0.3$ and stable and unstable periodic orbits are plotted in red and in blue, respectively.

We see that there are three saddle-node bifurcation curves and a cusp bifurcation point in the $(\omega, \epsilon\gamma)$-parameter space of Fig. 6(a), as predicted by the piecewise-smooth version of Theorem 2.4 (see also Fig. 3(b)). A good agreement between the theoretical prediction and numerical computation is found far from the degenerate resonance point. Three saddle-node bifurcations corresponding to the bifurcation curves of Fig. 6(a) are observed in the bifurcation diagram of Fig. 6(b). Moreover, one of the saddle-node bifurcation curves suddenly disappears when it collides with the border collision curve in Fig. 6(a).

5. **Trilinear oscillator.** We next consider a trilinear oscillator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\varphi(x_1) + \epsilon(-\delta x_2 + \gamma \cos \omega t),$$

(28)

where

$$\varphi(\xi) = \begin{cases} 
  k_2 \xi + \varphi_2 & \text{for } x < -\xi_2; \\
  \xi & \text{for } -\xi_2 \leq x < \xi_1; \\
  k_1 \xi + \varphi_1 & \text{for } x \geq \xi_1.
\end{cases}$$

We also assume that $-k_2\xi_2 + \varphi_2 = -\xi_2$ and $k_1\xi_1 + \varphi_1 = \xi_1$, so that Eq. (28) represents a continuous piecewise-linear system. See Fig. 7(a) for the restoring characteristic $\varphi(\xi)$.

When $\epsilon = 0$, we have a one-parameter family of periodic orbits

$$x^\alpha(t) = \begin{cases} 
  \left(\alpha_1 \cos \omega_1 t - \frac{\varphi_1}{k_1}, -\omega_1 \alpha_1 \sin \omega_1 t\right) & \text{for } t \in [0, \tau_1); \\
  \left(\alpha \cos(t + t_1), -\alpha \sin(t + t_1)\right) & \text{for } t \in [\tau_1, \tau_0 + \tau_1]; \\
  \left(\alpha_2 \cos \omega_2(t + t_2) - \frac{\varphi_2}{k_2}, -\omega_2 \alpha_2 \sin \omega_2(t + t_2)\right) & \text{for } t \in [\tau_0 + \tau_1, \frac{\tau}{2} T^\alpha]; \\
  (x_1^\alpha(-t), -x_2^\alpha(-t)) & \text{for } t \in [-\frac{\tau}{2} T^\alpha, 0),
\end{cases}$$

\[ \text{Figure 7. Continuous trilinear oscillator: (a) Restoring force characteristic; (b) unperturbed periodic orbit} \]
where $\omega^2 = k_i^2$ (i = 1, 2), $T_\alpha = 2(\tau_0 + \tau_1 + \tau_2)$, and
\[
\alpha_1 = \left(\xi_1 + \frac{\varphi_1}{k_1}\right)^2 + \frac{\alpha^2 - \xi_1^2}{k_1}, \quad \alpha_2 = \left(\xi_2 - \frac{\varphi_2}{k_2}\right)^2 + \frac{\alpha^2 - \xi_2^2}{k_2},
\]
\[
\tau_0 = \pi - \arccos\left(\frac{\xi_2}{\alpha}\right) - \arccos\left(\frac{\xi_1}{\alpha}\right), \quad \tau_1 = \frac{1}{\omega_1} \arccos\left(\frac{1}{\alpha_1}\left(\xi_1 + \varphi_1\right)\right),
\]
\[
\tau_2 = \frac{1}{\omega_2} \arccos\left(\frac{1}{\alpha_2}\left(\xi_2 - \varphi_2\right)\right), \quad t_1 = \arccos\left(\frac{\xi_1}{\alpha}\right) - \tau_1,
\]
\[
t_2 = \tau_2 - \tau_0 - \tau_1.
\]

See Fig. 7(b). Here the parameter $\alpha$ represents the amplitude of the periodic orbit in the center range.

As in Section 4, we apply Theorem 3.5. Using the expression (27), we obtain the Melnikov functions (9) in the form of (15) and (17), as in Section 2.5, where for $\alpha > 1$
\[
A^m(\alpha) = \frac{2}{(m\Omega^2)^2 - k_1} \left[ m\Omega^2 \sqrt{\alpha^2 - \xi_1^2} \cos m\Omega^\alpha \tau_1 - (k_1 \xi_1 + \varphi_1) \sin m\Omega^\alpha \tau_1 \right]
\]
\[
+ \frac{2}{(m\Omega^2)^2 - k_2} \left[ \xi_2 \sin m\Omega^\alpha (\tau_0 + \tau_1) + m \sqrt{\alpha^2 - \xi_2^2} \cos m\Omega^\alpha (\tau_0 + \tau_1) \right]
\]
\[
+ \xi_1 \sin m\Omega^\alpha \tau_1 - m\Omega^\alpha \sqrt{\alpha^2 - \xi_1^2} \cos m\Omega^\alpha \tau_1 \right]
\]
\[
+ \frac{2}{(m\Omega^2)^2 - k_2} \left[ (-k_2 \xi_2 + \varphi_2) \sin m\Omega^\alpha (\tau_0 + \tau_1) \right]
\]
\[
- m\Omega^\alpha \sqrt{\alpha^2 - \xi_2^2} \cos m\Omega^\alpha (\tau_0 + \tau_1) \right],
\]

\[
B(\alpha) = \alpha^2 \tau_0 + k_1 \alpha_1^2 \tau_1 + k_2 \alpha_2^2 \tau_2 - \frac{\varphi_1}{k_1} \sqrt{\alpha^2 - \xi_1^2} + \frac{\varphi_2}{k_2} \sqrt{\alpha^2 - \xi_2^2}.
\]

Again, a supercritical saddle-node bifurcation occurs at (16) if $d\Omega(\alpha^{m/1})/d\alpha \neq 0$ and a degenerate resonance behavior occurs at $\omega = \tilde{\omega} = m\Omega^{m/1}$ and (16) if $d\Omega(\alpha)/d\alpha = 0$ and $d^2\Omega(\alpha)/d\alpha^2 \neq 0$ at $\alpha = \tilde{\alpha} = \alpha^{m/1}$.

We give a numerical example for $k_1 = 0.8$, $k_2 = 1.6$, $x_1 = 0.5$, $x_2 = 1$, $\varphi_1 = 0.1$, $\varphi_2 = 0.6$, $m = 1$. Figure 8 shows numerical computations of $\Omega(\alpha) = 2\pi/T$, $A^1(\alpha)$, $B(\alpha)$ and $\gamma/\delta = A^1(\alpha)/B(\alpha)$. Again, by the piecewise-smooth versions of Theorems 2.2 and 2.3, a saddle-node bifurcation occurs near the computed value of $\gamma/\delta$, and stable and unstable periodic orbits exist above the value when $d\Omega(\alpha)/d\alpha \neq 0$. Moreover, since $d\Omega(\alpha)/d\alpha = 0$ at $\alpha \approx 1.0155$, by the piecewise-smooth version of Theorem 2.4, a degenerate resonance as shown in Fig. 3(a) occurs near $(\omega, \gamma/\delta) = (0.978614, 1.00973)$.

Figure 9 shows numerical bifurcation results by AUTO97 [7] for $\epsilon \delta = 0.01$. The starting solution was chosen as a periodic orbit in the linear case of $k_1 = k_2 = 1$ and $\varphi_1 = \varphi_2 = 0$ and continued to $k_1 = 0.8$, $k_2 = 1.6$, $\varphi_1 = 0.1$ and $\varphi_2 = 0.6$. In Fig. 9(a) saddle-node bifurcation curves are drawn in black as well as border collision curves on which the periodic orbit touches the border $x_1 = \xi_1$ and $x_1 = -\xi_2$ are drawn in blue and in green, respectively. The theoretical prediction by the standard Melnikov method, $\gamma/\delta = A^1(\alpha)/B(\alpha)$, is drawn in red (cf. Fig. 2(b)) and the degenerate resonance point is plotted as ‘•’. In Fig. 9(b) a bifurcation diagram is given for $\epsilon \gamma = 0.05$ and $\epsilon \delta = 0.01$ and stable and unstable periodic orbits are plotted in red and in blue, respectively.
Figure 8. Numerical computations of $\Omega(\alpha), A^1(\alpha), B(\alpha)$ and $\gamma/\delta = A^1(\alpha)/B(\alpha)$ for the trilinear oscillator (28) with $k_1 = 0.8$, $k_2 = 1.6$, $x_1 = 0.5$, $x_2 = 1$, $\varphi_1 = 0.1$, $\varphi_2 = 0.6$.

Figure 9. Numerical bifurcation analysis by AUTO97 for the trilinear oscillator (28) with $k_1 = 0.8$, $k_2 = 1.6$, $\xi_1 = 0.5$, $\xi_2 = 1$, $\varphi_1 = 0.1$, $\varphi_2 = 0.6$, $m = 1$, $\epsilon \delta = 0.01$ (a) Saddle-node bifurcation sets; (b) bifurcation diagram for $\epsilon \gamma = 0.05$. In plate (a) saddle-node bifurcation curves are drawn in black as well as border collision curves for $x_1 = \xi_1$ and $x_1 = -\xi_2$ are plotted in blue and in green, respectively. The theoretical prediction by the standard Melnikov method is drawn in red and the degenerate resonance point is plotted as ‘•’. In plate (b) stable and unstable periodic orbits are drawn in red and in blue, respectively.

We see that there are four saddle-node bifurcation curves and two cusp bifurcation points in the $(\omega, \epsilon \gamma)$-parameter space of Fig. 9(a). A fairly good agreement
between the theoretical prediction and numerical computation is found not near the degenerate resonance point. In comparison with the prediction of Fig. 3(a), additional bifurcation curve and cusp exist. Such behavior typically occur in weakly forced nonlinear oscillators and overlooked by the version of the subharmonic Melnikov method described here: We have to rely on the subharmonic Melnikov method for weakly nonlinear systems (see Section 5 of [17]) or the averaging method (see, e.g., [18]). Note that \( \epsilon \) is taken to be very small here, compared with the numerical example of Section 4. This is the reason why similar behavior was not observed in the numerical example of Section 4. Moreover, in the bifurcation diagram of Fig. 9(b) four saddle-node bifurcations are observed. Three of them correspond to the bifurcation curves passing through the line \( \epsilon \gamma = 0.05 \) in Fig. 9(a), and the other one near \((\omega, \epsilon \gamma) = (0.95, 0.01)\) is due to the nonsmoothness of the system. Such a nonsmooth saddle-node or fold bifurcation is typical in nonsmooth systems (see, e.g., [6]). Again, one of the saddle-node bifurcation curves suddenly disappears when it collides with the border collision curve in Fig. 9(a).

6. Conclusions. In this paper, we have extended the refined version of the subharmonic Melnikov method developed in [17, 21] and demonstrated its usefulness and validity for bi- and trilinear oscillators. Fundamental results for approximating solutions of piecewise-smooth systems by those of smooth systems were given and used to obtain the main result. Special attention was paid to degenerate resonance behavior, and analytical results were illustrated by numerical ones via the computer software AUTO97 [7]. In particular, we showed that the degenerate resonance behavior can occur in continuous trilinear oscillators as well as in discontinuous bilinear oscillators.

Although demonstrated only for piecewise-linear systems, our theory is also valid for more general piecewise-smooth nonlinear systems, as shown for a mathematical model of vibrating microcantilevers in tapping-mode atomic force microscopy in [22, 23]. Especially, a degenerate bifurcation behavior can also occur in such systems (cf. [23]). Theorems 3.1 and 3.3 also give a mathematical basis for applying other techniques such as the averaging and multiple-scale methods [14, 15], which were developed for smooth systems, to piecewise-smooth systems. Moreover, by referring to the results of [9, 10], our result may be extended to the case in which the unperturbed periodic orbits have jumps, i.e., are discontinuous.

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Appendix A. Approach for numerical bifurcation analysis. We describe our approach for numerical bifurcation analysis by AUTO97 [7]. For simplicity, we only consider the bilinear oscillator (26) for \( m = 1 \) here. The trilinear oscillator (28) and the case of \( m > 1 \) can be similarly treated.

We reduce the problem of computing periodic orbits with period \( 2\pi/\omega \) in (26) to a boundary value problem given by

\[
\begin{align*}
\dot{x}_1 &= x_2, & \dot{x}_2 &= -x_1 - \delta x_2 + \gamma \cos x_3, & \dot{x}_3 &= \omega, \\
\dot{x}_4 &= x_5, & \dot{x}_5 &= -k_1 x_4 - \varphi_1 - \delta x_5 + \gamma \cos x_6, & \dot{x}_6 &= \omega,
\end{align*}
\]
with boundary conditions
\[
\begin{align*}
    x_1(0) &= x_1(T_1) = x_4(0) = x_4(T_2) = \xi_1, \\
    x_2(T_1) &= x_5(0) > 0, \quad x_2(0) = x_5(T_2) < 0, \\
    x_3(T_1) &= x_6(0), \quad x_3(0) = x_6(T_2) - 2\pi,
\end{align*}
\]
where \( \bar{\delta} = \epsilon \delta, \bar{\gamma} = \epsilon \gamma, T_j \geq 0, j = 1, 2, \) and \( T_1 + T_2 = 2\pi/\omega. \) The stability of the computed periodic orbits can be determined as follows.

Let \( \phi(t, x_{10}, x_{20}, x_{30}) \) and \( \psi(t, x_{40}, x_{50}, x_{60}) \) be, respectively, the flows generated by \( (x_1, x_2, x_3)- \) and \( (x_4, x_5, x_6)- \) components of \((A.1)\). Take the Poincaré section as \( \Sigma = \{(x_1, x_2, x_3) | x_1 = \xi_1\} \) for \((26)\), where the variable \( x_3 \) denotes the phase of the cosine function as in \((A.1)\). Then we can represent the Poincaré map \( P : \Sigma \rightarrow \Sigma \) as
\[
P(\xi_1, x_{20}, x_{30}) = \psi(\tau_j, \phi(\xi_1, x_{20}, x_{30}))
\]
ear the periodic orbit, where \( \tau_j > 0, j = 1, 2, \) depend on \( x_{3j-1,0} \) and \( x_{3j,0} \) and satisfy
\[
\phi_1(\tau_1, x_{10}, x_{20}, x_{30}) = \psi_1(\tau_2, x_{10}, x_{20}, x_{30}) = \xi_1
\]
with \( x_{j+3,0} = \phi_j(\tau_1, x_{10}, x_{20}, x_{30}), j = 2, 3. \) Here \( \phi_j \) and \( \psi_j \) represent \( j \)th components of \( \phi \) and \( \psi \), respectively. We have
\[
\frac{\partial P_j}{\partial x_{k0}} = \left( \frac{\partial \psi_j}{\partial \tau_2} \frac{\partial \tau_2}{\partial x_{50}} + \frac{\partial \psi_j}{\partial x_{50}} \right) \left( \frac{\partial \phi_2}{\partial \tau_1} \frac{\partial \tau_1}{\partial x_{k0}} + \frac{\partial \phi_2}{\partial x_{k0}} \right)
\]
\[
+ \left( \frac{\partial \psi_j}{\partial \tau_2} \frac{\partial \tau_2}{\partial x_{60}} + \frac{\partial \psi_j}{\partial x_{60}} \right) \left( \frac{\partial \phi_3}{\partial \tau_1} \frac{\partial \tau_1}{\partial x_{k0}} + \frac{\partial \phi_3}{\partial x_{k0}} \right)
\]
for \( j, k = 2, 3. \) It also follows from \((A.4)\) that
\[
\begin{align*}
    \frac{\partial \tau_1}{\partial x_{k0}} &= - \frac{\partial \phi_1}{\partial x_{k0}} / \frac{\partial \phi_1}{\partial \tau_1}, \quad k = 2, 3, \\
    \frac{\partial \tau_2}{\partial x_{k0}} &= - \frac{\partial \psi_1}{\partial x_{k0}} / \frac{\partial \psi_1}{\partial \tau_2}, \quad k = 5, 6.
\end{align*}
\]
Moreover, we can easily obtain general solutions of \((A.1)\) and estimate
\[
\begin{align*}
    \frac{\partial \phi_1}{\partial \tau_1} &= x_{50}, \quad \frac{\partial \phi_1}{\partial x_{20}} = \frac{1}{\omega_1} e^{-\bar{\delta} \tau_1/2} S_1, \\
    \frac{\partial \phi_1}{\partial x_{30}} &= -a_1 \sin(x_{60} - \theta_1) \\
    &\quad + a_1 e^{-\bar{\delta} \tau_1/2} \left[ \sin(x_{30} - \theta_1) C_1 + \frac{\omega}{\omega_1} \cos(x_{30} - \theta_1) S_1 \right], \\
    \frac{\partial \phi_2}{\partial \tau_1} &= -\bar{\xi}_1 - \bar{\delta} x_{50} + \bar{\gamma} \cos(x_{60}), \quad \frac{\partial \phi_2}{\partial x_{20}} = e^{-\bar{\delta} \tau_1/2} \left( C_1 - \frac{\bar{\delta}}{2 \omega_1} S_1 \right), \\
    \frac{\partial \phi_2}{\partial x_{30}} &= -\omega a_1 \cos(x_{60} - \theta_1) \\
    &\quad + a_1 e^{-\bar{\delta} \tau_1/2} \left( \omega \cos(x_{30} - \theta_1) - \frac{1}{2} \bar{\delta} \sin(x_{30} - \theta_1) \right) C_1 \\
    &\quad - \left( \omega_1 \sin(x_{30} - \theta_1) + \frac{\bar{\delta} \omega}{2 \omega_1} \cos(x_{30} - \theta_1) \right) S_1, \\
    \frac{\partial \phi_3}{\partial \tau_1} &= \omega, \quad \frac{\partial \phi_3}{\partial x_{20}} = 0, \quad \frac{\partial \phi_3}{\partial x_{30}} = 1,
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial \psi_1}{\partial \tau_2} &= x_{20}, \quad \frac{\partial \psi_1}{\partial x_{50}} = \frac{1}{\omega_2} e^{-\delta \tau_2/2} S_2, \\
\frac{\partial \psi_1}{\partial x_{60}} &= -a_2 \sin(x_{30} - \theta_2) \\
&\quad + a_2 e^{-\delta \tau_2/2} \left( \sin(x_{60} - \theta_2) C_2 + \frac{\omega}{\omega_2} \cos(x_{60} - \theta_2) S_2 \right), \\
\frac{\partial \psi_2}{\partial \tau_2} &= -k_1 \xi_1 - \varphi_1 - \tilde{\delta} x_{20} + \tilde{\gamma} \cos x_{30}, \quad \frac{\partial \psi_2}{\partial x_{50}} = e^{-\delta \tau_2/2} \left( C_2 - \frac{\tilde{\delta}}{2\omega_2} S_2 \right), \\
\frac{\partial \psi_2}{\partial x_{60}} &= -\omega a_2 \cos(x_{30} - \theta_2) \\
&\quad + a_2 e^{-\delta \tau_2/2} \left( \left( \omega \cos(x_{60} - \theta_2) - \frac{1}{2} \tilde{\delta} \sin(x_{60} - \theta_2) \right) C_2 \\
&\quad - \left( \omega \sin(x_{60} - \theta_2) + \frac{\tilde{\delta} \omega}{2\omega_2} \cos(x_{60} - \theta_2) \right) S_2 \right), \\
\frac{\partial \psi_3}{\partial \tau_2} &= \omega, \quad \frac{\partial \psi_3}{\partial x_{50}} = 0, \quad \frac{\partial \psi_3}{\partial x_{60}} = 1,
\end{align*}
\] (A.8)

for the periodic orbit, where \( C_j = \cos \omega_j T_j, \) \( S_j = \sin \omega_j T_j, \) \( j = 1, 2, \) and
\[
\begin{align*}
a_1 &= \sqrt{1 - \omega^2} \sqrt{\frac{\tilde{\gamma}}{(\delta \omega)^2}} + (\delta \omega)^2, \quad a_2 = \sqrt{\frac{\tilde{\gamma}}{(k_1 - \omega^2)^2} + (\delta \omega)^2}, \\
\theta_1 &= \arctan \left( \frac{\tilde{\delta} \omega}{1 - \omega^2} \right), \quad \theta_2 = \arctan \left( \frac{\tilde{\delta} \omega}{k_1 - \omega^2} \right), \\
\omega_1 &= \sqrt{1 - \frac{1}{4} \tilde{\delta}^2}, \quad \omega_2 = \sqrt{k_1 - \frac{1}{4} \tilde{\delta}^2}.
\end{align*}
\]

Here we have assumed that \( \tilde{\delta} < \min(2, 2\sqrt{k_1}) \) and can similarly estimate these partial derivatives for the other cases. Substituting (A.6)-(A.8) into (A.5), we compute the Jacobian matrix \( DP \) of the Poincaré map \( P = (P_2, P_3) \) to determine the stability of the periodic orbit: It is stable if \( DP \) has only eigenvalues less than one in modulus, and unstable if it has an eigenvalue greater than one in modulus. Here \( P_j \) represents \( j \)th component of \( P \) in (A.3).

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