The second Yamabé invariant with singularities

Mohammed Benalili and Hichem Boughazi

Abstract. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\). We suppose that \(g\) is a metric in the Sobolev space \(H^p_2(M, T^*M \otimes T^*M)\) with \(p > \frac{n}{2}\) and there exist a point \(P \in M\) and \(\delta > 0\) such that \(g\) is smooth in the ball \(B_\delta(\delta)\). We define the second Yamabe invariant with singularities as the infimum of the second eigenvalue of the singular Yamabe operator over a generalized class of conformal metric to \(g\) and of volume 1. We show that this operator is attained by a generalized metric, we deduce nodal solutions to a Yamabe type equation with singularities.

1. Introduction

Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\). The problem of finding a metric conformal to the original one with constant scalar curvature was first formulated by Yamabe [9] and a variational method was initiated by this latter in an attempt to solve the problem. Unfortunately or fortunately a serious gap in the Yamabe was pointed out by Trudinger who addressed the question in the case of non positive scalar curvature ([8]). Aubin ([2]) solved the problem for arbitrary non locally conformally flat manifolds of dimension \(n \geq 6\). Finally Shoen ([7]) solved completely the problem using the positive-mass theorem founded previously by Shoen himself and Yau. The method to solve the Yamabe problem could be described as follows: let \(u\) be a smooth positive function and let \(\overline{g} = u^{N-2}g\) be a conformal metric where \(N = 2n/(n-2)\). Up to a multiplying constant, the following equation is satisfied

\[ L_g(u) = S_{\overline{g}}|u|^{N-2}u \]

where

\[ L_g = \frac{4(n-1)}{n-2} \Delta + S_g \]
and $S_g$ denotes the scalar curvature of $g$. $L_g$ is conformally invariant called the conformal operator. Consequently, solving the Yamabe problem is equivalent to find a smooth positive solution to the equation

$$L_g(u) = ku^{N-1}$$

where $k$ is a constant.

In order to obtain solutions to this equation, Yamabe defined the quantity

$$\mu(M, g) = \inf_{u \in C^\infty(M), u > 0} Y(u)$$

where

$$Y(u) = \frac{\int_M \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + S_g u^2 \right) dv_g}{\left( \int_M |u|^N dv_g \right)^{2/N}}.$$

$\mu(M, g)$ is called the Yamabe invariant, and $Y$ the Yamabe functional. In the sequel we write $\mu$ instead of $\mu(M, g)$. Writing the Euler-Lagrange equation associated to $Y$, we see that there exists a one to one correspondence between critical points of $Y$ and solutions of equation (1). In particular, if $u$ is a positive smooth function such that $Y(u) = \mu$, then $u$ is a solution of equation (1) and $g = u^{(N-2)/N}g$ is metric of constant scalar curvature. The key point to solve the Yamabe problem is the following fundamental results due to Aubin (2). Let $S_n$ be the unit euclidean sphere.

**Theorem 1.** Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 3$. If $\mu(M, g) < \mu(S_n)$, then there exists a positive smooth solution $u$ such that $Y(u) = \mu(M, g)$.

This strict inequality $\mu(M, g) < \mu(S_n)$ avoids concentration phenomena. Explicitly $\mu(S_n) = n(n - 1)\omega_n^{2/n}$ where $\omega_n$ stands for the volume of $S_n$.

**Theorem 2.** Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 3$. Then $\mu(M, g) \leq \mu(S_n)$. Moreover, the equality holds if and only if $(M, g)$ is conformally diffeomorphic to the sphere $S_n$.

Amman and Humbert (1) defined the second Yamabe invariant as the infimum of the second eigenvalue of the Yamabe operator over the conformal class of the metric $g$ with volume 1. Their method consists in considering the spectrum of the operator $L_g$

$$\text{spec}(L_g) = \{\lambda_{1,g}, \lambda_{2,g}, \ldots\}$$

where the eigenvalues $\lambda_{1,g} < \lambda_{2,g} \ldots$ appear with their multiplicities. The variational characterization of $\lambda_{1,g}$ is given by

$$\lambda_{1,g} = \inf_{u \in C^\infty(M), u > 0} \frac{\int_M \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + S_g u^2 \right) dv_g}{\int_M u^2 dv_g}.$$

Let

$$[g] = \{u^{N-2}g, u \in C^\infty(M), u > 0\}$$
Then they defined the \( k \th \) Yamabe invariant with \( k \in \mathbb{N}^* \), by

\[
\mu_k = \inf_{\bar{g} \in [g]} \lambda_k \text{Vol} (M, \bar{g})^{2/n}.
\]

With these notations \( \mu_1 \) is the Yamabe invariant. They studied the second Yamabe invariant \( \mu_2 \), they found that contrary to the Yamabe invariant, \( \mu_2 \) cannot be attained by a regular metric. In other words, there does not exist \( \bar{g} \in [g] \), such that

\[
\mu_2 = \lambda_2 \text{Vol} (M, \tilde{g})^{2/n}.
\]

In order to find minimizers, they enlarged the conformal class to a larger one. A generalized metric is the one of the form \( \bar{g} = u^{N-2} g \), which is not necessarily positive and smooth, but only \( u \in L^N(M) \), \( u \geq 0 \), \( u \neq 0 \) and where \( N = 2n/(n - 2) \).

The definitions of \( \lambda_2 \) and of \( \text{Vol} (M, \bar{g})^{2/n} \) can be extended to generalized metrics. The key points to solve this problem is the following theorems (11).

**Theorem 3.** Let \((M, g)\) be a compact Riemannian manifold of dimension \( n \geq 3 \), then \( \mu_2 \) is attained by a generalized metrics in the following cases.

\[
\mu > 0 \ , \ \mu_2 < \left[ (\mu^{n/2} + (\mu(S_n))^{n/2}) \right]^{2/n}
\]

and

\[
\mu = 0 \ , \ \mu_2 < \mu(S_n)
\]

**Theorem 4.** The assumptions of the last theorem are satisfied in the following cases

- If \((M, g)\) in not locally conformally flat and, \( n \geq 11 \) and \( \mu > 0 \)
- If \((M, g)\) in not locally conformally flat and, \( \mu = 0 \) and \( n \geq 9 \).

**Theorem 5.** Let \((M, g)\) be a compact Riemannian manifold of dimension \( n \geq 3 \), assume that \( \mu_2 \) is attained by a generalized metric \( \tilde{g} = u^{N-2} g \) then there exist a nodal solution \( w \in C^{2,\alpha}(M) \) of equation

\[
L_g(w) = \mu_2 |u|^{N-2} w
\]

such that

\[
|w| = u
\]

where \( \alpha \leq N - 2 \).

In (5), recently F. Madani studied the Yamabe problem with singularities when the metric \( g \) admits a finite number of points with singularities and smooth outside these points. Let \((M, g)\) be a compact Riemannian manifold of dimension \( n \geq 3 \), assume that \( g \) is a metric in the Sobolev space \( H^p_2(M, T^* M \otimes T^* M) \) with \( p > \frac{n}{2} \) and there exist a point \( P \in M \) and \( \delta > 0 \) such that \( g \) is smooth in the ball \( B_p(\delta) \), and let \((H)\) be these assumptions. By Sobolev’s embedding, we have for \( p > \frac{n}{2} \), \( H^p_2(M, T^* M \otimes T^* M) \subset C^{1-[n/p],\beta}(M, T^* M \otimes T^* M) \), where \( [n/p] \) denotes the entire part of \( n/p \). Hence the metric satisfying assumption \((H)\) is of class \( C^{1-[n/p],\beta} \) with \( \beta \in (0, 1) \) provided that \( p > n \). The Christoffels symbols belong to \( H^p_2(M) \) ( to \( C^{\alpha}(M) \) in case \( p > n \)), the Riemannian curvature tensor, the Ricci tensor and scalar curvature are in \( L^p(M) \). F. Madani proved under the assumption \((H)\) the existence of a metric \( \bar{g} = u^{N-2} g \) conformal to \( g \) such that \( u \in H^p_2(M), u > 0 \) and the scalar curvature \( S_{\bar{g}} \) of \( \bar{g} \) is constant and \((M, g)\) is not conformal.
to the round sphere. Madani proceeded as follows: let \( u \in H^2_2(M) \), \( u > 0 \) be a function and \( \gamma = u^{N-2}g \) a particular conformal metric where \( N = 2n/(n-2) \). Then, multiplying \( u \) by a constant, the following equation is satisfied

\[
L_g u = \frac{n-2}{4(n-1)} S_g |u|^{N-2}u
\]

where

\[
L_g = \Delta_g + \frac{n-2}{4(n-1)} S_g
\]

and the scalar curvature \( S_g \) is in \( L^p(M) \). Moreover \( L_g \) is weakly conformally invariant hence solving the singular Yamabe problem is equivalent to finding a positive solution \( u \in H^2_2(M) \) of

\[
L_g u = k|u|^{N-2}u
\]

where \( k \) is a constant. In order to obtain solutions of equation (2) we define the quantity

\[
\mu = \inf_{u \in H^2_2(M), \ u > 0} Y(u)
\]

where

\[
Y(u) = \frac{\int_M \left( |\nabla u|^2 + \frac{(n-2)}{4(n-1)} S_g u^2 \right) dv_g}{(\int_M |u|^N dv_g)^{2/N}}
\]

\( \mu \) is called Yamabe invariant with singularities. Writing the Euler-Lagrange equation associated to \( Y \), we see that there exists a one to one correspondence between critical points of \( Y \) and solutions of equation (2). In particular, if \( u \in H^2_2 \) is a positive function which minimizes \( Y \), then \( u \) is a solution of equation (2) and \( \gamma = u^{N-2}g \) is a metric of constant scalar curvature and \( \mu \) is attained by a particular conformal metric. The key points to solve the above problem are the following theorems (5).

**Theorem 6.** If \( p > n/2 \) and \( \mu < K^{-2} \) then equation (2) admits a positive solution \( u \in H^2_2(M) \subset C^{1-\lfloor n/p \rfloor, \beta}(M) ; \lfloor n/p \rfloor \) is the integer part of \( n/p \), \( \beta \in (0, 1) \) which minimizes \( Y \), where \( K^2 = \frac{4}{n(n-1)} \omega_n^{-2/n} \) with \( \omega_n \) denotes the volume of \( S_n \). If \( p > n \) , then \( u \in H^p_2(M) \subset C^1(M) \).

**Theorem 7.** Let \( (M, g) \) be a compact Riemannian manifold of dimension \( n \geq 3 \). \( g \) is a metric which satisfies the assumption (H). If \( (M, g) \) is not conformal to the sphere \( S_n \) with the standard Riemannian structure then

\[
\mu < K^{-2}
\]

**Theorem 8.** (5) On an \( n \)-dimensional compact Riemannian manifold \( (M, g) \), if \( u \geq 0 \) is a non trivial weak solution in \( H^2_2(M) \) of equation \( \Delta u + hu = 0 \), with \( h \in L^p(M) \) and \( p > n/2 \), then \( u \in C^{1-\lfloor n/p \rfloor, \beta} \) and \( u > 0 \); \( \lfloor n/p \rfloor \) is the integer part of \( n/p \) and \( \beta \in (0, 1) \).

For regularity argument we need the following results
Lemma 1. Let \( u \in L^N(M) \) and \( v \in H^2_p(M) \) a weak solution to \( L_g(v) = u^{N-2} v, \) then

\[ v \in L^{N+\varepsilon}(M) \]

for some \( \varepsilon > 0. \)

The proof is the same as in (5) with some modifications. As a consequence of Lemma 7, \( v \in L^s(M), \forall s \geq 1. \)

Proposition 1. If \( g \in H^2_p(M, T^*M \otimes T^*M) \) is a Riemannian metric on \( M \) with \( p > n/2. \) If \( \varphi = u^{N-2} g \) is a conformal metric to \( g \) such that \( u \in H^2_p(M), u > 0 \) then \( L_g \) is weakly conformally invariant, which means that \( \forall v \in H^2_p(M), \) \( |u|^{N-1}L\varphi(v) = L_g(uv) \) weakly. Moreover if \( \mu > 0, \) then \( L_g \) is coercive and invertible.

In this paper, let \((M, g)\) be a compact Riemannian manifold of dimension \( n \geq 3. \) We suppose that \( g \) is a metric in the Sobolev space \( H^p(M, T^*M \otimes T^*M) \) with \( p > n/2 \) and there exist a point \( P \in M \) and \( \delta > 0 \) such that \( g \) is smooth in the ball \( B_P(\delta) \) and we call these assumptions the condition \((H)\).

In the smooth case the operator \( L_g \) is an elliptic operator on \( M \) self-adjoint, and has a discrete spectrum \( Spec(L_g) = \{ \lambda_{1,g}, \lambda_{2,g}, \ldots \} \), where the eigenvalues \( \lambda_{1,g} < \lambda_{2,g}, \ldots \) appear with their multiplicities. These properties remain valid also in the case where \( S_g \in L^p(M). \) The variational characterization of \( \lambda_{1,g} \) is given by

\[
\lambda_{1,g} = \inf_{u \in H^2_p, u > 0} \frac{\int_M (|\nabla u|^2 + \frac{(n-2)}{4(n-1)} S_g u^2) \, dv_g}{\int_M u^2 \, dv_g}
\]

Let \( [g] = \{ u^{N-2} g : u \in H^2_p \text{ and } u > 0 \}, \) Let \( k \in N^* \), we define the \( k^{\text{th}} \) Yamabe invariant with singularities \( \mu_k \) as

\[
\mu_k = \inf_{\varphi \in [g]} \lambda_k \varphi \text{Vol}(M, \varphi)^{2/n}
\]

with these notations, \( \mu_1 \) is the first Yamabe invariant with singularities.

In this work we are concerned with \( \mu_2. \) In order to find minimizers to \( \mu_2 \) we extend the conformal class to a larger one consisting of metrics of the form \( \varphi = u^{N-2} g \) where \( u \) is no longer necessarily in \( H^2_p(M) \) and positive but \( u \in L^N(M) = \{ L^N(M), u \geq 0, u \neq 0 \} \) such metrics will be called for brevity generalized metrics. First we are going to show that if the singular Yamabe invariant \( \mu \geq 0 \) then \( \mu_1 \) it is exactly \( \mu \) next we consider \( \mu_2. \) \( \mu_2 \) is attained by a conformal generalized metric.

Our main results state as follows:

Theorem 9. Let \((M, g)\) be a compact Riemannian manifold of dimension \( n \geq 3. \) We suppose that \( g \) is a metric in the Sobolev space \( H^2_p(M, T^*M \otimes T^*M) \) with \( p > n/2. \) There exist a point \( P \in M \) and \( \delta > 0 \) such that \( g \) is smooth in the ball \( B_P(\delta), \) then

\[
\mu_1 = \mu.
\]
THEOREM 10. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\), we suppose that \(g\) is a metric in the Sobolev space \(H^p_2(M, T^*M \otimes T^*M)\) with \(p > n/2\). There exist a point \(P \in M\) and \(\delta > 0\) such that \(g\) is smooth in the ball \(B_P(\delta)\). Assume that \(\mu_2\) is attained by a metric \(\overline{g} = u^{N-2}g\) where \(u \in L^N_+ (M)\), then there exist a nodal solution \(w \in C^{1-[n/p], \beta}\), \(\beta \in (0,1)\), of equation
\[ L_g w = \mu_2 u^{N-2}w. \]
Moreover there exist real numbers \(a, b > 0\) such that
\[ u = aw_+ + bw_- \]
with \(w_+ = \sup(w,0)\) and \(w_- = \sup(-w,0)\).

THEOREM 11. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\), suppose that \(g\) is a metric in the Sobolev space \(H^p_2(M, T^*M \otimes T^*M)\) with \(p > n/2\). There exist a point \(P \in M\) and \(\delta > 0\) such that \(g\) is smooth in the ball \(B_P(\delta)\) then \(\mu_2\) is attained by a generalized metric in the following cases:
If \((M, g)\) in not locally conformally flat and, \(n \geq 11\) and \(\mu > 0\)
If \((M, g)\) in not locally conformally flat and, \(\mu = 0\) and \(n \geq 9\).

2. Generalized metrics and the Euler-Lagrange equation

Let
\[ L^N_+(M) = \{u \in L^N_+(M) : u \geq 0, u \neq 0\} \]
where \(N = \frac{2n}{n-2}\).

As in (1)

DEFINITION 1. For all \(u \in L^N_+(M)\), we define \(Gr_k^u(H^2_2(M))\) to be the set of all \(k\)-dimensional subspaces of \(H^2_2(M)\) with span\((v_1, v_2, ..., v_k)\) \(\in Gr_k^u(H^2_2(M))\) if and only if \(v_1, v_2, ..., v_k\) are linearly independent on \(M - u^{-1}(0)\).

Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\). For a generalized metric \(\overline{g}\) conformal to \(g\), we define
\[ \lambda_{k, \overline{g}} = \inf_{V \in Gr_k^u(H^2_2(M))} \sup_{v \in V} \frac{\int_M v L_g (v) dv_g}{\int_M |u|^{N-2} v^2 dv_g}. \]
We quote the following regularity theorem

THEOREM 12. \([7]\) On a \(n\)-dimensional compact Riemannian manifold \((M, g)\), if \(u \geq 0\) is a non trivial weak solution in \(H^2_2(M)\) of the equation
\[ \Delta u + hu = cu^{N-1} \]
with \(h \in L^p(M)\) and \(p > n/2\), then
\[ u \in H^p_2(M) \subseteq C^{1-[n/p], \beta}(M) \]
and \(u > 0\), where \([n/p]\) denotes the integer part of \(n/p\) and \(\beta \in (0,1)\).

PROPOSITION 2. Let \((v_m)\) be a sequence in \(H^2_2(M)\) such that \(v_m \to v\) strongly in \(L^2(M)\), then for all any \(u \in L^N_+(M)\)
\[ \int_M u^{N-2} (v^2 - v_m^2) dv_g \to 0. \]

PROOF. The proof is the same as in (3). \(\square\)
Proposition 3. If $\mu > 0$, then for all $u \in L^N_+ (M)$, there exist two functions $v, w$ in $H^2 (M)$ with $v \geq 0$ satisfying in the weak sense the equations

\[ L_g v = \lambda_1 g u^{N-2} v \]

and

\[ L_g w = \lambda_2 g u^{N-2} w \]

Moreover we can choose $v$ and $w$ such that

\[ \int_M u^{N-2} v^2 dv_g = \int_M u^{N-2} w^2 dv_g = 1 \quad \text{and} \quad \int_M u^{N-2} vdv_g = 0. \]

Proof. Let $(v_m)_m$ be a minimizing sequence for $\lambda_1, \tilde{g}$ i.e. a sequence $v_m \in H^2$ such that

\[ \lim_{m \to \infty} \int_M v_m L_g (v_m) dv_g = \lambda_1, \tilde{g} \]

It is well known that $(|v_m|)_m$ is also minimizing sequence. Hence we can assume that $v_m \geq 0$. If we normalize $(v_m)_m$ by

\[ \int_M |u|^{N-2} v_m^2 dv_g = 1. \]

Now by the fact that $L_g$ is coercive

\[ c \|v_m\|_{H^2} \leq \int_M v_m L_g (v_m) dv_g \leq \lambda_1, \tilde{g} + 1. \]

$(v_m)_m$ is bounded in $H^2 (M)$ and after restriction to a subsequence we may assume that there exist $v \in H^2 (M), v \geq 0$ such that $v_m \to v$ weakly in $H^2 (M)$, strongly in $L^2 (M)$ and almost everywhere in $M$, then $v$ satisfies in the sense of distributions

\[ L_g v = \lambda_1 g u^{N-2} v. \]

If $u \in H^2_+ (M) \subset C^{1-\frac{1}{p}, \beta} (M)$ then

\[ \int_M u^{N-2} (v^2 - v_m^2) dv_g \to 0 \]

and

\[ \int_M u^{N-2} v^2 dv_g = 1. \]

Then $v$ is non trivial and nonnegative minimizer of $\lambda_1, \tilde{g}$, by Lemma 7

\[ h = S_g - \lambda_1 g u^{N-2} \in L^p (M) \]

and by Theorem 8

\[ v \in C^{1-\frac{1}{p}, \beta} (M) \]

and

\[ v > 0. \]

If $u \in L^N_+ (M)$, by Proposition 2, we get

\[ \int_M u^{N-2} (v^2 - v_m^2) dv_g \to 0 \]
so
\[
\int_M u^{N-2} v^2 dv_g = 1.
\]

\(v\) is a non negative minimizer in \(H^2_1\) of \(\lambda_{1,\pi}\) such that \(\int_M u^{N-2} v^2 dv_g = 1\).

Now consider the set
\[
E = \{ w \in H^2_1 : \text{such that} \ u^{N-2} w \neq 0 \ \text{and} \ \int_M u^{N-2} w dv_g = 0 \}
\]
and define
\[
\lambda'_{2,g} = \inf_{w \in E} \frac{\int_M w L_g(w) dv_g}{\int_M |u|^{N-2} w^2 dv_g}.
\]

Let \((w_m)\) be a minimizing sequence for \(\lambda'_{2,g}\) i.e. a sequence \(w_m \in E\) such that
\[
\int_M u^{N-2} w_m dv_g = 0
\]
and taking account of \(\int_M u^{N-2} w_m dv_g = 0\) and the fact that \(w_m \to w\) weakly in \(L^N(M)\) and since \(u^{N-2} v \in L^{\frac{N}{N-1}}(M)\), we infer that
\[
\int_M u^{N-2} w dv_g = 0.
\]

Hence \((8)\) and \((9)\) are that satisfied with \(\lambda'_{2,g}\) instead of \(\lambda_{2,\pi}\).

**Proposition 4.** We have \(\lambda'_{2,g} = \lambda_{2,\pi}\).

**Proof.** The Proof is the same as in \((3)\) so we omit it.

**Remark 1.** If \(p > n\) then \(u \in H^p_2(M) \subset C^1(M)\), by Theorem 2, \(v\) and \(w \in C^1(M)\) with \(v > 0\).

**Remark 2.** If \(p > n\) then \(u \in H^p_2(M) \subset C^1(M)\) and \(\lambda_{2,\pi} = \lambda_{1,\pi}\), we see that \(|w|\) is a minimizer for the functional associated to \(\lambda_{1,\pi}\), then \(|w|\) satisfies the same equation as \(v\) and by Theorem 2 we get \(|w| > 0\), this contradicts relation \((9)\) necessarily
\[
\lambda_{2,\pi} > \lambda_{1,\pi}.
\]
3. Variational characterization and existence of $\mu_1$

In this section we need the following results

**Theorem 13.** Let $(M, g)$ be a compact $n$-dimensional Riemannian manifold. For any $\varepsilon > 0$, there exists $A(\varepsilon) > 0$ such that $\forall u \in H^2_1(M)$,

$$\|u\|_N^2 \leq (K^2 + \varepsilon)\|\nabla u\|_2^2 + A(\varepsilon)\|u\|_2^2$$

where $N = 2n/(n-4)$ and $K^2 = 4/(n(n-2)) \omega_n^{2/n}$. $\omega_n$ is the volume of the round sphere $S_n$.

Let $[g] = \{u^{N-2}g : u \in H^p_2(M) \text{ and } u > 0\}$, we define the first singular Yamabe invariant $\mu_1$ as

$$\mu_1 = \inf_{[g] \in \mathcal{P}} \lambda_1 \text{Vol}(M, \tilde{g})^{2/n}$$

then we get

$$\mu_1 = \inf_{u \in H^p_2, V \in \text{Gr}_{u^1}(H^p_2)} \sup_{v \in V} \frac{\int_M vL_g(v)dv_g}{\int_M |u|^{N-2}v^2dv_g} \left( \int_M u^N dv_g \right)^{\frac{2}{n}}.$$

**Lemma 2.** We have

$$\mu_1 \leq \mu < K^{2}.$$

**Proof.** If $p \geq 2n/(n+2)$, the embedding $H^p_2(M) \subset H^1_2(M)$ is true, so

$$\mu_1 = \inf_{u \in H^p_2, V \in \text{Gr}_{u^1}(H^p_2(M))} \sup_{v \in V} \frac{\int_M vL_g(v)dv_g}{\int_M |u|^{N-2}v^2dv_g} \left( \int_M u^N dv_g \right)^{\frac{2}{n}} \leq \inf_{u \in H^p_2, V \in \text{Gr}_{u^1}(H^p_2(M))} \sup_{v \in V} \frac{\int_M vL_g(v)dv_g}{\int_M |v|^{N-2}v^2dv_g} \left( \int_M v^N dv_g \right)^{\frac{2}{n}}.$$

in particular for $p > \frac{n}{2}$ and $u = v$ we get

$$\mu_1 \leq \inf_{v \in H^p_2, V \in \text{Gr}_{u^1}(H^p_2(M))} \sup_{v \in V} \frac{\int_M vL_g(v)dv_g}{\int_M |v|^{N-2}v^2dv_g} \left( \int_M u^N dv_g \right)^{\frac{2}{n}} = \mu$$

i.e

$$\mu_1 \leq \mu < K^{2}.$$

**Theorem 14.** If $\mu > 0$, there exits conform metric $\tilde{g} = u^{N-2}g$ which minimizes $\mu_1$.

**Proof.** The proof will take several steps.

**Step 1.**

We study a sequence of metrics $g_m = u_m^{N-2}g$ with $u_m \in H^p_2(M)$, $u_m > 0$ which minimize $\mu_1$ i.e. a sequence of metrics such that

$$\mu_1 = \lim_{m} \lambda_{1,m} \text{Vol}(M, g_m)^{2/n}.$$
Without loss of generality, we may assume that $\text{Vol}(M, g_m) = 1$ i.e.

$$\int_M u_m^N dv_g = 1.$$ 

In particular, the sequence of functions $u_m$ is bounded in $L^N(M)$ and there exists $u \in L^N(M)$, $u \geq 0$ such that $u_m \to u$ weakly in $L^N(M)$. We are going to prove that the generalized metric $u^{N-2}g$ minimizes $\mu_1$. Proposition 3 implies the existence of a sequence $(v_m)$ of class $H^2_1(M)$, $v_m > 0$ such that

$$L_g(v_m) = \lambda_{1,m} u_m^{N-2}v_m$$

and

$$\int_M u_m^{N-2} v_m^2 dv_g = 1.$$ 

Now since $\mu > 0$, by Proposition 1, $L_g$ is coercive and we infer that

$$c \|v_m\|_{H^2_1} \leq \int_M v_m L_g(v_m) dv_g = \lambda_{1,m} \leq \mu_1 + 1.$$ 

The sequence $(v_m)_m$ is bounded in $H^2_1(M)$, we can find $v \in H^2_1(M)$, $v \geq 0$ such that $v_m \to v$ weakly in $H^2_1(M)$. Together with the weak convergence of $(u_m)_m$, we obtain in the sense of distributions

$$L_g(v) = \mu_1 u^{N-2}v.$$ 

**Step 2.**

Now we are going to show that $v_m \to v$ strongly in $H^2_1(M)$.

We put

$$z_m = v_m - v$$

then $z_m \to 0$ weakly in $H^2_1(M)$ and strongly in $L^q(M)$ with $q < N$, and writing

$$\int_M |\nabla v_m|^2 dv_g = \int_M |\nabla z_m|^2 dv_g + \int_M |\nabla v|^2 dv_g + 2 \int_M \nabla z_m \nabla v dv_g$$

we see that

$$\int_M |\nabla v_m|^2 dv_g = \int_M |\nabla z_m|^2 dv_g + \int_M |\nabla v|^2 dv_g + o(1).$$

Now because of $2p/(p-1) < N$, we have

$$\int_M \frac{n-2}{4(n-1)} S_g (v_m - v)^2 dv_g \leq \frac{n-2}{4(n-1)} \|S_g\|_p \|v_m - v\|_{p}^{2} \to 0$$

so

$$\int_M \frac{n-2}{4(n-1)} S_g v_m^2 dv_g = \int_M \frac{n-2}{4(n-1)} S_g v^2 dv_g + o(1)$$

and

$$\int_M |\nabla v_m|^2 dv_g + \int_M \frac{n-2}{4(n-1)} S_g (v_m)^2 dv_g$$

$$= \int_M |\nabla z_m|^2 dv_g + \int_M |\nabla v|^2 dv_g + \int_M \frac{n-2}{4(n-1)} S_g (v)^2 dv_g + o(1).$$
\[
\int_M v_m L_g v_m dv_g = \int_M |\nabla z_m|^2 dv_g + \int_M |\nabla v|^2 dv_g + \int_M \frac{n-2}{4(n-1)} S_g v^2 dv_g + 0(1)
\]

Then

\[
\int_M |\nabla v|^2 dv_g + \int_M \frac{n-2}{4(n-1)} S_g (v)^2 dv_g \geq \mu (\int_M v^N dv_g)^\frac{2}{N} \geq \mu_1 (\int_M v^N dv_g)^\frac{2}{N}
\]

and by the definition of $\mu$ and Lemma 2 we get

\[
\int_M |\nabla v|^2 dv_g + \int_M \frac{n-2}{4(n-1)} S_g (v)^2 dv_g \geq \mu_1 (\int_M v^N dv_g)^\frac{2}{N} + 0(1).
\]

And since

\[
\int_M v_m L_g (v_m) dv_g = \lambda_{1,m} \leq \mu_1 + 0(1)
\]

and

\[
\int_M |\nabla z_m|^2 dv_g + \mu_1 (\int_M v^N dv_g)^\frac{2}{N} \leq \mu_1 + 0(1)
\]

i.e.

\[
\mu_1 \|v\|^2_N + \|\nabla z_m\|^2_N \leq \mu_1 + 0(1)
\]

Now by Brezis-Lieb lemma, we get

\[
\lim_m \int_M v_m^N + z_m^N dv_g = \int_M v^N dv_g
\]

i.e.

\[
\lim_m \|v_m\|^N_N - \|z_m\|^N_N = \|v\|^N_N.
\]

Hence

\[
\|v_m\|^N_N + 0(1) = \|z_m\|^N_N + \|v\|^N_N.
\]

By Hölder’s inequality and $\int_M u_m^{N-2} v_m^2 dv_g = 1$, we get

\[
\|v_m\|^N_N \geq 1
\]

i.e.

\[
\int_M v^N + z_m^N dv_g = \int_M v_m^N dv_g + 0(1) \geq 1 + 0(1).
\]

Then

\[
\left(\int_M v^N dv_g\right)^\frac{1}{N} + \left(\int_M z_m^N dv_g\right)^\frac{1}{N} \geq 1 + 0(1)
\]

i.e.

\[
\|z_m\|^2_N + \|v\|^2_N \geq 1 + 0(1).
\]

Now by Theorem 13 and the fact $z_m \to 0$ strongly in $L^2$, we get

\[
\|z_m\|^2_N \leq (K^2 + \varepsilon)\|\nabla z_m\|^2_N + o(1)
\]

\[
1 + o(1) \leq \|z_m\|^2_N + \|v\|^2_N \leq \|v\|^2_N + (K^2 + \varepsilon)\|\nabla z_m\|^2_N + o(1).
\]

So we deduce
1 + o(1) \leq \|v\|^2_N + (K^2 + \varepsilon)\|\nabla z_m\|^2_2 + o(1)

and from inequality (10), we get

\|\nabla z_m\|^2_2 + \mu_1 v^2_N \leq \mu_1((K^2 + \varepsilon)\|\nabla z_m\|^2_2 + v^2_N) + o(1).

So if \( \mu_1 K^2 < 1 \), we get

\( (1 - \mu_1(K^2 + \varepsilon))\|\nabla z_m\|^2_2 \leq 0(1) \)

i.e. \( v_m \to v \) strongly in \( H^2_1(M) \).

**Step 3.** We have

\[
\lim_m \int_M (u_m^{N-2}v_m^2 - u^{N-2}v^2 + u_m^{N-2}v_2 - u_m^{N-2}v_2) \, dv_g = \lim_m \int (u_m^{N-2}(v_m^2 - v^2) + (u_m^{N-2} - u^{N-2})v^2) \, dv_g.
\]

Now since \( u_m \to u \) a.e. so does \( u_m^{N-2} \to u^{N-2} \) and \( \int_M u_m^{N-2}dv_g \leq c \), hence \( u_m^{N-2} \) is bounded in \( L^{N/(N-2)} \) and up to a subsequence \( u_m^{N-2} \to u^{N-2} \) weakly in \( L^{N/(N-2)} \). Because of \( v^2 \in L^2(M) \), we have

\[
\lim_m \int (u_m^{N-2} - u^{N-2})v^2dv_g = 0
\]

and by Hölder’s inequality

\[
\lim_m \int u_m^{N-2}(v_m - v)^2dv_g \leq (\int u_m^{N}dv_g)^{(N-2)/N}(\int |v_m - v|^N dv_g)^{1/N} \leq 0.
\]

By the strong convergence of \( v_m \) in \( L^N(M) \), we get \( \int_M u^{N-2}v^2dv_g = 1 \), then \( v \) and \( u \) are non trivial functions.

**Step 4.**

Let \( \tau = av \in L^N_+(M) \) with \( a > 0 \) a constant such that \( \int_M \tau^Ndv_g = 1 \) with \( v \) a solution of

\[ L_g(v) = \mu_1 u^{N-2}v \]

with the constraint

\[ \int_M u^{N-2}v^2dv_g = 1. \]

We claim that \( u = v \); indeed,

\[
\mu_1 \leq \frac{\int_M vL_g(v)dv_g}{\int_M u^{N-2}v^2dv_g} \leq \frac{\int_M vL_g(v)dv_g}{\int_M (av)^{N-2}v^2dv_g} = \frac{a^2 \mu_1 \int_M u^{N-2}v^2dv_g}{\int_M \tau^{N-2}(av)^2dv_g}
\]

and Hölder’s inequality lead

\[
\leq \mu_1 \int_M (u)^{N-2}(av)^2dv_g
\]
\[ \mu_1 \left( \int_M (u)^{N-2} \frac{u}{N-2} \right)^\frac{2}{N-2} \left( \int_M (av)^2 \frac{N}{2} \right)^\frac{2}{N} \leq \mu_1. \]

And since the equality in Hölder’s inequality holds if \( \overline{u} = u = av \)
then \( a = 1 \) and \( u = v \).

Then \( v \) satisfies \( L_g v = \mu_1 v^{N-1} \), by Theorem \( \text{[12]} \) we get \( v = u \in H^p_2(M) \subset C^{1,\frac{n}{2}}(M) \) with \( \beta \in (0,1) \) and \( v = u > 0 \),
Resuming, we have
\[ L_g(v) = \mu_1 v^{N-1}, \quad \int_M v^N dv_g = 1 \quad \text{and} \quad v = u \in H^p_2(M) \subset C^{1,\frac{n}{2}}(M) \]
so the metric \( \tilde{g} = u^{N-2}g \) minimizes \( \mu_1 \).

4. Yamabe conformal invariant with singularities

**Theorem 15.** If \( \mu \geq 0 \), then \( \mu_1 = \mu \)

**Proof.** Step 1
If \( \mu > 0 \). Let \( v \) such that \( L_g v = \mu_1 v^{N-1} \) and \( \int_M v^N dv_g = 1 \) then
\[ \mu_1 = \int_M vL_g(v)dv_g \geq c\|v\|_{H^1_2} \]
and \( v \) in non trivial function then \( \mu_1 > 0 \). On the other hand
\[ \mu = \inf \frac{\int_M vL_g(v)dv_g}{(\int_M v^N dv_g)^\frac{2}{N}} \]
\[ \leq \int_M vL_g(v)dv_g = \mu_1 \]
and by Lemma \( \text{[2]} \), we get
\[ \mu_1 = \mu \]
Step 2
If \( \mu = 0 \), Lemma \( \text{[2]} \) implies that \( \mu_1 \leq 0 \), hence
\[ \mu_1 = 0. \]

5. Variational characterization of \( \mu_2 \)

Let \( \{ g \} = \{ u^{N-2}g, \quad u \in H^p_2(M) \text{ and } u > 0 \} \), we define the second Yamabe invariant \( \mu_2 \) as
\[ \mu_2 = \inf_{\mathcal{F} \in [g]} \lambda_2 Vgol(M, \mathcal{F})^{2/n} \]
or more explicitly
\[ \mu_2 = \inf_{u \in H^p_2, V \in Gr_2(H^p_2(M))} \sup_{v \in V} \frac{\int_M vL_g(v)dv_g}{\int_M |u|^{N-2}v^2 dv_g} \left( \int_M u^N dv_g \right)^\frac{2}{N} \]
Theorem 16. On compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\), we have for all \(v \in \mathcal{H}_1^2(M)\) and for all \(u \in L_+^N(M)\),

\[
2^{\frac{n}{2}} \int_M |u|^{N-2} v^2 dv_g \leq (K^2 \int_M |\nabla v|^2 dv_g + \int_M B_0 v^2 dv_g)(\int_M u^N dv_g)^{\frac{2}{n}}
\]

Or

\[
2^{\frac{n}{2}} \int_M |u|^{N-2} v^2 dv_g \leq \mu_1(S_n)(\int_M C_n |\nabla v|^2 + B_0 u^2 dv_g)(\int_M u^N dv_g)^{\frac{2}{n}}
\]

Theorem 17. For any compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\), there exists \(B_0 > 0\) such that

\[
\mu_1(S_n) = n(n - 1) \omega_n^{2/n} = \inf_{H^2_1} \frac{\int_M 4(n-1)(n-2) |\nabla u|^2 + B_0 u^2 dv_g}{(\int_M |u|^N dv_g)^{2/N}}
\]

where \(\omega_n\) is the volume of the unit round sphere.

Or

\[
(\int_M |u|^N dv_g)^{2/N} \leq K^2 \int_M |\nabla u|^2 dv_g + \int_M B_0 u^2 dv_g
\]

\[K^2 = \mu_1(S_n)^{-1} C_n \text{ and } C_n = (4(n-1))/(n-2)\]

6. Properties of \(\mu_2\)

We know that \(g\) is smooth in the ball \(B_p(\delta)\) by assumption (H), this assumption is sufficient to prove that Aubin’s conjecture is valid. The case \((M, g)\) is not conformally flat in a neighborhood of the point \(P\) and \(n \geq 6\), let \(\eta\) is a cut-off function with support in the ball \(B_p(2\varepsilon)\) and \(\eta = 1\) in \(B_p(\varepsilon)\), where \(2\varepsilon \leq \delta\) and

\[
v_\varepsilon(q) = (\frac{\varepsilon}{r^2 + \varepsilon^2})^{\frac{n-2}{2}}
\]

with \(r = d(p, q)\). We let \(u_\varepsilon = \eta v_\varepsilon\) and define

\[
Y(u) = \frac{\int_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)} S g u^2\right) dv_g}{(\int_M |u|^N dv_g)^{2/N}}.
\]

We obtain the following lemma

Lemma 3. \([1]\)

\[
\mu = Y(v_\varepsilon) \leq \begin{cases} \{(K^{-2} - c|w(P)|^2 \varepsilon^4 + 0(\varepsilon^4) \text{ if } n > 6 \} \\ K^{-2} - c|w(P)|^2 \varepsilon^4 \log \frac{1}{\varepsilon} + 0(\varepsilon^4) \text{ if } n = 6 \end{cases}
\]

where \(|w(P)|\) is the norm of the Weyl tensor at the point \(P\) and \(c > 0\).

Theorem 18. If \((M, g)\) in not locally conformally flat and \(n \geq 11\) and \(\mu > 0\), we find

\[
\mu_2 < (\frac{\mu}{2} + (K^{-2})^{\frac{2}{n}})
\]

and if \(\mu = 0\), \(n \geq 9\) then

\[
\mu_2 < K^{-2}
\]

Proof. With the same method as in \([1]\), this lemma follows from theorem \([8]\)
7. Existence of a minimizer to \( \mu_2 \)

**Lemma 4.** Assume that \( v_m \to v \) weakly in \( H_1^2 (M) \), \( u_m \to u \) weakly in \( L^N (M) \) and \( \int_M u_m^{-2} v_m^2 dv_g = 1 \) then

\[
\int_M u_m^{-2} (v_m - v)^2 dv_g = 1 - \int_M u^{-2} v^2 dv_g + o(1)
\]

**Proof.** We have

\[
\int_M u_m^{-2} (v_m - v)^2 dv_g = \int_M u_m^{-2} v_m^2 dv_g + \int_M u_m^{-2} v^2 dv_g - \int_M 2 u_m^{-2} v_m v dv_g
\]

(15)

\[
= 1 + \int_M u_m^{-2} v^2 dv_g - \int_M 2 u_m^{-2} v_m v dv_g .
\]

Now \( (u_m^{-2}) \) is bounded in \( L_{\infty}^N (M) \) and \( u_m^{-2} \to u^{-2} \) a.e., then \( u_m^{-2} \to u^{-2} \) weakly in \( L_{\infty}^N (M) \) and \( \forall \phi \in L_0^N \)

\[
\int_M \phi u_m^{-2} dv_g \to \int_M \phi u^{-2} dv_g
\]

in particular for \( \phi = v^2 \)

\[
\int_M v^2 u_m^{-2} dv_g \to \int_M v^2 u^{-2} dv_g.
\]

\[
\int_M u_m^{-2} v_m dv_g \text{ is bounded in } L_{\infty}^N (M), \text{ because of}
\]

\[
\int_M u_m^{-2} v_m^N dv_g \leq (\int_M u_m^{-2} dv_g)^{N-2} \left( \int_M v_m^N dv_g \right)^{1/N}
\]

and \( u_m^{-2} v_m \to u^{-2} v \) a.e., then \( u_m^{-2} v_m \to u^{-2} v \) weakly in \( L_{\infty}^N (M) \). Hence

\[
\int_M u_m^{-2} v_m v dv_g \to \int_M u^{-2} v^2 dv_g
\]

and

\[
\int_M u_m^{-2} (v_m - v)^2 dv_g = 1 - \int_M u^{-2} v^2 dv_g + o(1).
\]

\[\square\]

**Theorem 19.** If \( 1 - 2^{-\frac{2}{N}} K^2 \mu_2 > 0 \), then the generalized metric \( u^{-2} g \) minimizes \( \mu_2 \)

**Proof.** Step 1.

We study a sequence of metrics \( g_m = u_m^{-2} g \) with \( u_m \in H_2^p (M), u_m > 0 \) which minimizes the infimum in the definition of \( \mu_2 \) i.e. a sequence of metrics such that

\[
\mu_2 = \lim \lambda_{2,m}(Vol(M, g_m)^{2/N}).
\]

Without loss generality, we may assume that \( Vol(M, g_m) = 1 \) i.e. that \( \int_M v_m^N dv_g = 1 \). In particular, the sequence of functions \( (u_m)_m \) is bounded in \( L^N (M) \) and there
exists \( u \in L^N(M) \), \( u \geq 0 \) such that \( u_m \to u \) weakly in \( L^N \). We are going to prove that the generalized metric \( u^{N-2}g \) minimizes \( \mu_2 \). Proposition \( \text{[3]} \), implies the existence of \( v_m, w_m \in H^2_1(M) \), \( v_m > 0 \) such that

\[
L_g(v_m) = \lambda_{1,m} u_m^{N-2} v_m
\]

and such that

\[
\int_M u_m^{N-2} v_m^2 dg = \int_M u_m^{N-2} w_m^2 dg = 1, \int_M u_m^{N-2} v_m w_m dg = 0.
\]

The sequence \( v_m, w_m \) is bounded in \( H^2_1(M) \), we can find \( v, w \in H^2_1(M) \), \( v \geq 0 \) such that \( v_m \rightharpoonup v \), \( w_m \rightharpoonup w \) weakly in \( H^2_1(M) \). Together with the weak convergence of \( (u_m) \), we get in weak sense

\[
L_g(v) = \widehat{\mu}_1 u^{N-2} v
\]

and

\[
L_g(w) = \mu_2 u^{N-2} w
\]

where

\[
\widehat{\mu}_1 = \lim \lambda_{1,m} \leq \mu_2.
\]

**Step 2.**

Now we show \( v_m \rightharpoonup v \), \( w_m \rightharpoonup w \) strongly in \( H^2_1(M) \). Applying Theorem \( \text{[10]} \) to the sequence \( v_m - v \), we get

\[
\int_M |u_m|^{N-2}(v_m - v)^2 dv_g \leq (2 - \hat{\mu}_1) K^2 \int_M |\nabla (v_m - v)|^2 dv_g + \int_M B_0(v_m - v)^2 dv_g (\int_M u^N dv_g) \hat{\mu}_1
\]

and since \( v_m \rightharpoonup v \) strongly in \( L^2 \),

\[
\int_M |u_m|^{N-2}(v_m - v)^2 dv_g \leq (2 - \hat{\mu}_1) K^2 \int_M |\nabla (v_m - v)|^2 dv_g + o(1)
\]

\[ \leq (2 - \hat{\mu}_1) K^2 \int_M |\nabla (v_m)|^2 + |\nabla v|^2 - 2\nabla v_m \nabla vdvg + o(1). \]

By the weak convergence of \( (v_m) \), \( \int_M \nabla v_m \nabla v dg = \int_M |\nabla v|^2 dg + o(1) \)

\[
\int_M |u_m|^{N-2}(v_m - v)^2 dv_g \leq (2 - \hat{\mu}_1) K^2 \int_M |\nabla (v_m)|^2 - |\nabla v|^2 dv_g + o(1)
\]

and since

\[
\int_M \frac{n - 2}{4(n - 1)} S_g v_m^2 dv_g = \int_M \frac{n - 2}{4(n - 1)} S_g v^2 dv_g + 0(1)
\]

we get

\[
\int_M |u_m|^{N-2}(v_m - v)^2 dv_g \leq 2 - \hat{\mu}_1 K^2 (\int_M |\nabla (v_m)|^2 - |\nabla v|^2 dv_g)
\]

\[ + \int_M \frac{n - 2}{4(n - 1)} S_g (v_m^2 - v^2) dv_g + o(1) \leq 2 - \hat{\mu}_1 K^2 (\int_M v_m L_g(v_m) - v L_g(v) dv_g + o(1)
\]

\[ \leq 2 - \hat{\mu}_1 K^2 (\lambda_{1,m} - \mu_1) \int_M u^{N-2} v^2 dv_g + o(1) \]
By the fact $\hat{\mu}_1 = \lim \lambda_{1,m} \leq \mu_2$

$$\leq 2^{-\frac{2}{N}} K^2 \mu_2(1 - \int_M u^{N-2} v^2 dv_g) + o(1)$$

Then

$$\int_M |u_m|^{N-2}(v_m - v)^2 dv_g \leq 2^{-\frac{2}{N}} K^2 \mu_2(1 - \int_M u^{N-2} v^2 dv_g) + o(1)$$

Now using the weak convergence of $(v_m)$ in $H^2_1(M)$ and the weak convergence of $(u_m)$ in $L^N(M)$, we infer by Lemma 4 that

$$\int_M |u_m|^{N-2}(v_m - v)^2 dv_g = 1 - \int_M u^{N-2} v^2 dv_g + o(1)$$

then

$$1 - \int_M u^{N-2} v^2 dv_g \leq 2^{-\frac{2}{N}} K^2 \mu_2(1 - \int_M u^{N-2} v^2 dv_g) + o(1)$$

and

$$1 - 2^{-\frac{2}{N}} K^2 \mu_2 \leq (1 - 2^{-\frac{2}{N}} K^2 \mu_2) \int_M u^{N-2} v^2 dv_g + o(1).$$

So if $1 - 2^{-\frac{2}{N}} K^2 \mu_2 > 0$ then

$$\int_M u^{N-2} v^2 dv_g \geq 1.$$

and by Fatou’s lemma, we obtain

$$\int_M u^{N-2} v^2 dv_g \leq \lim \int_M u_m^{N-2} v_m^2 dv_g = 1.$$

We find that

(16) $$\int_M u^{N-2} v^2 dv_g = 1.$$

So $u$ and $v$ are not trivial.

Moreover

$$\int_M |\nabla (v_m - v)|^2 dv_g = \int_M \left( |\nabla (v_m)|^2 + |\nabla v|^2 - 2 \nabla v_m \nabla v \right) dv_g$$

$$= \int_M |\nabla (v_m)|^2 - |\nabla v|^2 dv_g + o(1)$$

and since $\int_M \nabla g(v_m^2 - v^2) dv_g = o(1)$, we get

$$\int_M |\nabla (v_m - v)|^2 dv_g = \int_M v_m L_g(v_m) - v L_g(v) dv_g + o(1)$$

$$\leq \mu_2 (1 - \int_M u^{N-2} v^2 dv_g) + o(1)$$

Then, by relation (16)

$$\int_M |\nabla (v_m - v)|^2 dv_g = o(1)$$
and \( v_m \to v \) strongly in \( H^2(M) \). The same argument holds with \( (w_m) \), hence \( w_m \to w \) strongly in \( H^2(M) \) and \( \int_M u^{N-2}w^2dv_g = 1 \).

To show that \( \int_M u^{N-2}vwdv_g = 0 \), first writing and using Hölder’s inequality, we get

\[
\int_M (u_m^{N-2}v_m w_m - u^{N-2}v) dv_g = \int_M (u_m^{N-2}v_m w_m - u_m^{N-2}v w_m + u_m^{N-2}v w_m - u^{N-2}v) dv_g
\]

\[= \int_M u_m^{N-2}(v_m - v) w_m dv_g + \int_M (u_m^{N-2}v w_m - u^{N-2}v) dv_g\]

\[= \int_M \frac{N-2}{u_m^2} w_m (u_m^{N-2}v_m - v) dv_g + \int_M (u_m^{N-2}v w_m - u^{N-2}v) dv_g\]

\[\leq \left( \int_M u_m^{N-2}w^2 dv_g \right)^{\frac{N-2}{N}} \left( \int_M |v_m - v|^N dv_g \right)^{\frac{N}{N}} + \int_M (u_m^{N-2}v w_m - u^{N-2}v) dv_g\]

\[\leq \left( \int_M |v_m - v|^N dv_g \right)^{\frac{N}{N}} + \int_M (u_m^{N-2}v w_m - u^{N-2}v) dv_g\]

\[\leq \left( \int_M |v_m - v|^N dv_g \right)^{\frac{N}{N}} + \int_M (u_m^{N-2}v (m - w) + (u_m^{N-2} - u^{N-2}) v w) dv_g\]

\[\leq \left( \int_M |v_m - v|^N dv_g \right)^{\frac{N}{N}} + \int_M \left( \frac{N-2}{u_m^2} (u_m^{N-2}v m - w) + (u_m^{N-2} - u^{N-2}) v w \right) dv_g\]

\[\leq \left( \int_M |v_m - v|^N dv_g \right)^{\frac{N}{N}} + \left( \int_M u_m^{N-2}v^2 dv_g \right)^{\frac{1}{2}} \left( \int_M u_m^{N-2}(w_m - w)^2 dv_g \right)^{\frac{1}{2}} + \int_M (u_m^{N-2} - u^{N-2}) v w dv_g\]

Now noting that

\[
\int_M u_m^{N-2}v^2 dv_g \leq \left( \int_M u_m^N dv_g \right)^{\frac{N-2}{N}} \left( \int_M v^N dv_g \right)^{\frac{1}{N}} < +\infty
\]
and taking account of \( u_m^{N-2} \to u^{N-2} \) weakly in \( L^{\frac{2N}{N-2}}(M) \) and the fact that \( vw \in L^{\frac{2}{N}}(M) \), we deduce
\[
\int_M (u_m^{N-2} - u^{N-2}) v w dv_g \to 0
\]
hence
\[
\int_M u^{N-2} v w dv_g = 0.
\]
Consequently the generalized metric \( u^{N-2} g \) minimizes \( \mu_2 \).

**THEOREM 20.** If \( \mu_2 < K^{-2} \), then generalized metric \( u^{N-2} g \) minimizes \( \mu_2 \)

**Proof.** Step 1.

We study a sequence of metrics \( g_m = u_m^{N-2} g \) with \( u_m \in H^2_2(M), u_m > 0 \) which attained \( \mu_2 \) i.e. a sequence of metrics such that
\[
\mu_2 = \lim_{m} \lambda_{2,m} (Vol(M,g_m))^{2/n}.
\]
Without loss of generality, we may assume that \( Vol(M,g_m) = 1 \) i.e. \( \int_M u_m^N dv_g = 1 \).

In particular, the sequence \( (\mu_2) \) is bounded in \( H^2_2(M) \) and there exists \( \mu_2 \) such that \( \mu_2 \leq \mu_2 \).

The sequences \( (v_m) \) and \( (w_m) \) are bounded in \( H^2_1(M) \), we can find \( v, w \in H^2_1(M), v > 0 \) such that
\[
L_g(v_m) = \lambda_{1,m} u_m^{N-2} v_m
\]
and
\[
L_g(w_m) = \lambda_{2,m} u_m^{N-2} w_m
\]
and
\[
\int_M u_m^{N-2} v_m^2 dv_g = \int_M u_m^{N-2} w_m^2 dv_g = 1, \int_M u_m^{N-2} v_m w_m dv_g = 0.
\]

The sequences \( (v_m) \) and \( (w_m) \) are bounded in \( H^2_1 \), we can find \( v, w \in H^2_1 \) with \( v > 0 \) such that \( v_m \to v, w_m \to w \) weakly in \( H^2_1 \).

Together with the weak convergence of \( (u_m) \), we get in the weak sense
\[
L_g(v) = \mu_1 u^{N-2} v
\]
and
\[
L_g(w) = \mu_2 u^{N-2} w
\]
where
\[
\mu_1 = \lim_{m} \lambda_{1,m} \leq \mu_2.
\]

Step 2.

Now we are going to show that \( v_m \to v, w_m \to w \) strongly in \( H^2_1 \).

By Hölder’s inequality, Theorem 13, strong convergence of \( v_m \) in \( L^2 \), we get
\[
\int_M |v_m|^{N-2}(v_m - v)^2 dv_g \leq \|v_m - v\|_N^2 \leq K^2 \|\nabla(v_m - v)\|_2^2 + o(1)
\]

\[
\leq K^2 \int_M |\nabla(v_m)|^2 + |\nabla v|^2 - 2\nabla v_m \nabla v dv_g + o(1)
\]
\[ \leq K^2 \int_M |\nabla (v_m)|^2 - |\nabla v|^2 \, dv_g + o(1) \]
\[ \leq K^2 \int_M v_m L_g(v_m) - v L_g(v) \, dv_g + o(1) \]
\[ \leq K^2 \mu_2 (1 - \int_M u^{N-2} v^2 \, dv_g) + o(1) \]

and with Lemma 4
\[ \int_M |u_m|^{N-2} (v_m - v)^2 \, dv_g = 1 - \int_M u^{N-2} v^2 \, dv_g + o(1) \]
then
\[ 1 - \int_M u^{N-2} v^2 \, dv_g \leq K^2 \mu_2 (1 - \int_M u^{N-2} v^2 \, dv_g) + o(1) \]
i.e.
\[ 1 - K^2 \mu_2 \leq (1 - K^2 \mu_2) \int_M u^{N-2} v^2 \, dv_g \]
so if \( 1 - K^2 \mu_2 > 0 \),
\[ \int_M u^{N-2} v^2 \, dv_g \geq 1. \]

On the other hand since by Fatou’s lemma
\[ \int_M u^{N-2} v^2 \, dv_g \leq \lim \int_M v_m^{N-2} v_m^2 \, dv_g = 1. \]
Then
\[ \int_M u^{N-2} v^2 \, dv_g = 1. \]
and
\[ \int_M |\nabla (v_m - v)|^2 \, dv_g = o(1) \]
Hence \( v_m \to v \) strongly in \( H^2_1 \subset L^N \).
The same conclusion also holds for \( (w_m)_m \).

**Lemma 5.** Let \( u \in L^N \) with \( \int_M u^N \, dv_g = 1 \) and \( z, w \) nonnegative functions in
\( H^2_1 \) satisfying
\[ \int_M w L_g(w) \, dv_g \leq \mu_2 \int_M u^{N-2} w^2 \, dv_g \]
and
\[ \int_M z L_g(z) \, dv_g \leq \mu_2 \int_M u^{N-2} z^2 \, dv_g \]
And suppose that \( (M - z^{-1}(0)) \cap (M - w^{-1}(0)) \) has measure zero. Then \( u \) is a linear combination of \( z \) and \( w \) and we have equality in (20) and (21).

**Proof.** The proof is the same as that of Aumann and Humbert in [1].
THE SECOND YAMABÉ INVARIANT WITH SINGULARITIES

THEOREM 21. If a generalized metric \( u^{N-2}g \) minimizes \( \mu_2 \), then there exist a nodal solution \( w \in H^p_2 \subset C^{1-[n/p],\beta} \) of equation

\[
L_g(w) = \mu_2 u^{N-2}w
\]

More over there exist \( a, b > 0 \) such that

\[
u = aw_+ + bw_-
\]

With \( w_+ = \sup(w, 0) \) and \( w_- = \sup(-w, 0) \).

PROOF. Step 1. Applying Lemma 5 to \( w_+ = \sup(w, 0) \) and \( w_- = \sup(-w, 0) \), we get the existence of \( a, b > 0 \) such that

\[
u = aw_+ + bw_-
\]

Now by Lemma 1, \( w_+, w_- \in L^\infty \) i.e. \( u \in L^\infty \), \( u^{N-2} \in L^\infty \), then

\[
h = S_g - \mu_2 u^{N-2} \in L^p
\]

and from Theorem 12 we obtain

\[
w \in H^p_2 \subset C^{1-[n/p],\beta}.
\]

Step 2. If \( \mu_2 = \mu_1 \), we see that \( |w| \) is a minimizer to the functional associated to \( \mu_1 \), then \( |w| \) satisfies the same equation as \( \nu \) and Theorem 12 shows that \( |w| = w \in H^p_2 \subset C^{1-[n/p],\beta} \) that is \( |w| > 0 \) everywhere, which contradicts the condition (9) in Proposition 3, then

\[
\mu_2 > \mu_1.
\]

Step 3. The solution \( w \) of the equation (22) changes sign. Since if it does not, we may assume that \( w \geq 0 \), by step 2 the inequality in (20) is strict and by Lemma 5 we have the equality: a contradiction. \( \square \)

REMARK 3. Step 1 shows that \( u \) is not necessarily in \( H^p_2(M) \) and by the way the minimizing metric is not in \( H^p_2(M, T^*M \otimes T^*M) \) contrary to the Yamabe invariant with singularities.

References

[1] B. Ammann, E.Humbert. The second Yamabe invariant, J. Funct. Anal., 235 (2006), 2, 377-412.
[2] T. Aubin. Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl., 55 (1976), 269-296.
[3] M.Benalili, H.Boughazi. On the second Paneitz-Branson invariant, Houston, J. Math.,36, (2010), 2, 393-420.
[4] E.Hebey. Introduction à l’analyse non linéaire sur les variétés, Diderot Editeur Arts Sciences (1997).
[5] F. Madani. Le problème de Yamabe avec singularités. Bull. Sci. Math. 132 (2008), 7, 575-591.
[6] R. Shoen. Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differential Geom. 20 (1984), 2, 479-495.
[7] R. Shoen, S.T.Yau. Conformally flat manifolds, Kleinian groups and scalar curvature. Invent. Math, 9247-71, 1988.
[8] N.S.Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa 32,3, (1968)
[9] H.Yamabe. On a deformation of Riemannian structures on compact manifolds. Osaka Math. J. 12 (1960) 21-37.
E-mail address: m_benalili@mail.univ-lemcen.dz

Current address: Université Aboubekr Belkaïd, Faculty of Sciences, Dept. of Math. B.P. 119, Tlemcen, Algeria.