ON FUTURE DRAWDOWNS OF LÉVY PROCESSES

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Abstract. For a given stochastic process \( X = (X_t)_{t \in \mathbb{R}_+} \) the future drawdown process \( D^* = (D^*_{t,s})_{t,s \in \mathbb{R}_+} \) is defined by
\[
D^*_{t,s} = \inf_{t \leq u < t+s} (X_u - X_t).
\]
For fixed non-negative \( s \), two path-functionals that describe the fluctuations of \( D^*_{t,s} \) are the running supremum \( (\overline{D}^*_{t,s})_{t,s \in \mathbb{R}_+} \) and running infimum \( (\overline{D}^*_{u,s})_{t,s \in \mathbb{R}_+} \) of \( D^*_{t,s} \),
\[
\overline{D}^*_{t,s} = \sup_{0 \leq u \leq t} D^*_{u,s}, \quad \overline{D}^*_{t,s} = \inf_{0 \leq u \leq t} D^*_{u,s}.
\]
The path-functionals \( \overline{D}^*_{t,s} \) and \( \overline{D}^*_{t,s} \) are of interest in various areas of application, including financial mathematics and queueing theory. In the case that \( X \) is a Lévy process with strictly positive mean, we find the exact asymptotic decay as \( x \to \infty \) of the tail probabilities \( P(\overline{D}^*_t < x) \) and \( P(\overline{D}^*_t < x) \) of \( \overline{D}^*_t \) and \( \overline{D}^* \) both when the jumps satisfy the Cramér assumption and in the heavy-tailed case. Furthermore, when the jumps of the Lévy process \( X \) are of single sign, we identify the one-dimensional distributions in terms of the scale function of \( X \). By way of example, we derive explicit results for the Black–Scholes–Samuelson model.

1. Introduction

In recent times, various pricing models with jumps have been put forward which address the shortcomings of diffusion models in representing the risk related to large market movements (see e.g. [7]). Such models allow for a more realistic representation of price dynamics and a greater flexibility in calibration to market prices and in reproducing a wide variety of implied volatility skews and smiles. An important indicator for the riskiness and effectiveness of an investment strategy is the drawdown, which is the distance of the current value away from the maximum value it has attained to date. The drawdown may also be seen as a measure of regret as it represents the hypothetical gain the investor would have made had he sold the asset at the moment the current maximum was attained. A number of commonly used trading rules are based on the drawdown (see e.g. [24]), while (maximum) drawdowns have also been applied as risk-measures (see [12, 28, 27]). Drawdown processes (also called reflected processes) are also encountered in various other areas, such as applied probability, queueing theory and mathematical genetics. See [5, 19, 20] and references therein for further applications and results concerning drawup and drawdown processes.

In this paper we consider a number of path-functionals of the increments of a given stochastic process \( X = (X_t)_{t \in \mathbb{R}_+} \) that are closely related to the drawdown and drawup processes. In particular, we consider a future drawdown process \( D^* = (D^*_{t,s})_{t,s \in \mathbb{R}_+} \) which is defined as a two-parameter process with value at \( (t,s) \) equal to the smallest increment \( X_u - X_t \) for \( u \) in the interval \( [t,t+s] \), i.e.
\[
D^*_{t,s} = \inf_{t \leq u < t+s} (X_u - X_t), \quad t, s \in \mathbb{R}_+,
\]

together with its running supremum and infimum described by
\[
\overline{D}^*_{t,s} = \sup_{0 \leq u \leq t} D^*_{u,s}, \quad \overline{D}^*_{t,s} = \inf_{0 \leq u \leq t} D^*_{u,s}.
\]

Similarly, we define a future drawup process of \( X \) by
\[
U^*_{t,s} = \sup_{t \leq u < t+s} (X_u - X_t)
\]
We denote the corresponding quantities for an infinite horizon by
\[ \mathcal{U}_t^{\ast} = \sup_{0 \leq u \leq t} U^*_u, \quad \mathcal{U}_t^{\ast} = \inf_{0 \leq u \leq t} U^*_u. \]

We denote the corresponding quantities for an infinite horizon by
\[ \mathcal{U}^*_t = \lim_{s \to \infty} \mathcal{U}^*_{t,s}, \quad \mathcal{U}^*_t = \lim_{s \to \infty} U^*_{t,s}, \quad \mathcal{D}^*_t = \lim_{s \to \infty} \mathcal{D}^*_{t,s}, \quad \mathcal{D}^*_t = \lim_{s \to \infty} D^*_{t,s}. \]

Note that for any nonnegative \( s \) we have \( \mathcal{D}^*_{t,s} = -\mathcal{U}^*_{t,s} \) and \( \mathcal{D}^*_{t,s} = -\hat{\mathcal{U}}^*_{t,s} \), where \( \hat{\cdot} \) denotes the quantity calculated for the dual process \( \hat{X} = -X \), so that in the analysis we may restrict to either the future drawdown or drawup process. In this paper we focus on the case that \( X \) is a Lévy process.

The future drawdown and drawup processes are of interest in various applications, including in financial risk analysis and queueing models. We note that, under an exponential Lévy model \( P_t = P_0 \exp(X_t) \) for the stock price, the random variables \( \mathcal{D}^*_{t,s} \) and \( \mathcal{D}^*_{t,s} \) are path-dependent risk indicators: \( \mathcal{D}^*_{t,s} \) is the lowest future log-return \( \log(P_u/P_s) \) achieved at some \( u \) in the time-window \([t, t + s]\), while \( \mathcal{D}^*_{t,s} \) and \( \mathcal{D}^*_{t,s} \) are the maximal and minimal such future returns for \( t \) ranging over \([0, T]\).

Another application comes from telecommunications and queueing models, where \( \mathcal{U}^*_t = \lim_{s \to \infty} \mathcal{U}^*_{t,s} \) and \( \mathcal{U}^*_t = \lim_{s \to \infty} \mathcal{U}^*_{t,s} \) describe the supremum and the infimum of the workload process over a finite time horizon \( t \) in a fluid model with netput \( X \), respectively (see for details [9]).

In the mentioned applications it is of interest to characterise the laws, and in particular the tail-probabilities, of the random variables \( \mathcal{D}^*_{t,s} \) and \( \mathcal{D}^*_{t,s} \), respectively (see [9]).

We also describe the law of \( \mathcal{S}^* \) conditional on \( \mathcal{U}^*_t \) or \( \mathcal{U}^*_t \) being large, which gives rise to “asymptotic drawdown measures” \( \mathcal{P}^*(\gamma) \) and \( \mathcal{P}^*(\gamma, v) \). Similar results are obtained for \( \mathcal{U}^*_t \) and \( \mathcal{U}^*_t \) leading to “asymptotic drawup measures” \( \mathcal{P}^*(\gamma) \) and \( \mathcal{P}^*(\gamma, v) \).

In a complementary case that the Lévy measure \( V \) of \( X \) is in the class \( S(\alpha) \) of tail-equivalent measures then, under some additional regularity conditions, we show
\[
\lim_{x \to \infty} \frac{\mathbb{P}(\mathcal{D}^*_t < x)}{\mathbb{P}(\mathcal{U}^*_t < x)} = C(3)
\]
for some explicit constant \( C(3) \), and derive a similar result for \( \mathcal{D}^*_t \).

When the jumps of \( X \) are all of single sign, we explicitly identify the Laplace transform in time of the one-dimensional distributions in terms of the scale function of \( X \). As example, we analyze in detail (future) drawdowns and drawups under the Black-Scholes model, identifying in particular the mean of the value \( P_T = P_0 \exp(X_T) \) under the measure \( \mathbb{P}(\gamma) \) and the laws of \( \mathcal{D}^*_T \) and \( \mathcal{D}^*_T \).

The remainder of the paper is organized as follows. In Section 2, we present the main representation of drawup and drawdown processes. In Section 3, the Cramér asymptotics is identified which produce the drawup and drawdown measures described in Section 3.1. The heavy-tailed case is analyzed in Section 4. In Section 5, we find further results in the spectrally one-sided case and we conclude the paper with more explicit expressions of future drawups and drawdowns in the Black-Scholes–Samuelson model.
2. Extremal increments

Let \((X_t)_{t \in \mathbb{R}_+}\) be a general Lévy process (i.e. a process with stationary and independent increments with càdlàg paths such that \(X_0 = 0\)) defined on some filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_+}, \mathbb{P})\) with \(\mathcal{F}_t = \sigma\{X_s, s \leq t\}\) denoting the completed filtration generated by \(X\). The law of \(X\) is determined by its characteristic exponent \(\Psi\) which is the map \(\Psi : \mathbb{R} \to \mathbb{C}\) of \(X\) that satisfies \(\mathbb{E}[e^{i\theta X_1}] = \exp(\Psi(\theta))\).

The drawdown and drawup processes of \(X\), \((D_t)_{t \in \mathbb{R}_+}\) and \((U_t)_{t \in \mathbb{R}_+}\), are path-functional of the increments of \(X\) given by

\[ D_t = \overline{X}_t - X_t, \quad U_t = X_t - \underline{X}_t, \]

with \(\overline{X}_t = \sup_{0 \leq s \leq t} X_s\) and \(\underline{X}_t = \inf_{0 \leq s \leq t} X_s\). We note that the drawdown \(D_t\) and drawup \(U_t\) at time \(t\) are equal to the largest of all increments \(X_{u+t} - X_t\) and the negative of the smallest increment of such increments. By replacing the interval \([0, t]\) by intervals of the form \([t, t+s]\), we arrive at the following generalizations which we will call future drawdown and drawup processes:

**Definition 1.** The future drawdown and future drawup process of \(X\), \(D^* = (D^*_{t,s})_{t,s \in \mathbb{R}_+}\) and \(U^* = (U^*_{t,s})_{t,s \in \mathbb{R}_+}\), are given by

\[ D^*_{t,s} = \inf_{t \leq u < t+s} (X_u - X_t), \quad U^*_{t,s} = \sup_{t \leq u < t+s} (X_u - X_t). \]

Further, we will use the following notation: \(U^*_t = \lim_{s \to \infty} U^*_{t,s}\) and \(D^*_t = \lim_{s \to \infty} D^*_{t,s}\).

Before turning to the analysis of the future drawdown and drawup processes, we first note a number of facts concerning drawup and drawdown processes that follow from the fluctuation theory of Lévy processes. The marginal distributions of the drawup \(U_t\) and drawdown \(D_t\), \(t \in \mathbb{R}_+\), can be expressed in terms of the marginal distributions of \(X\) by deploying the Wiener–Hopf factorisation of \(X\), according to which the characteristic exponent \(\Psi\) of \(X\) is related to the marginal distributions of the running supremum and running infimum of \(X\) at an exponential random time \(e_q\) (independent of \(X\)) as follows:

\[ \frac{q}{\Psi(\theta)} = \mathbb{E}[e^{i\theta X^q}] \mathbb{E}[e^{-i\theta \Delta_q}], \quad q > 0, \theta \in \mathbb{R}. \]

Using the duality lemma (see e.g. [2, Prop. VI.3]), the Wiener–Hopf factorization may be expressed in terms of the drawdown and drawup processes by

\[ \frac{q}{\Psi(\theta)} = \mathbb{E}[e^{i\theta U_q}] \mathbb{E}[e^{-i\theta D_q}], \quad q > 0, \theta \in \mathbb{R}. \]

If \(\mathbb{E}[X_1]\) is strictly negative, it follows from the duality lemma that \(U_t\) converges in distribution to a random variable \(U_\infty\) as \(t\) tends to infinity, and that \(U_\infty\) is in distribution equal to \(U_0^\ast\). The Laplace transform of \(U_\infty^\ast\) is given explicitly in terms of the Laplace exponent \(\kappa\) of the ascending ladder process \((L^{-1}, H)\). The ladder time process \(L^{-1} = \{L_t^{-1}\}_{t \in \mathbb{R}_+}\) is the right-continuous inverse of the local time \(L\) of \((D_t)_{t \in \mathbb{R}_+}\) at zero, while the corresponding ladder-height process \(H = (H_t)_{t \geq 0}\) is given by \(H_t = X_{L_t^{-1}}\) for all \(t \geq 0\) for which \(L_t^{-1}\) is finite, and defined to be \(H_t = +\infty\) otherwise. The Laplace transform of \(U_\infty^\ast\) (and thus \(U_\infty\)) is given as follows:

\[ \mathbb{E}[e^{-uU_\infty}] = \mathbb{E}[e^{-uU_0^\ast}] = \frac{\kappa(0,0)}{\kappa(0,u)}, \quad u > 0, \theta \in \mathbb{R}. \]

where \(1_A\) denotes the indicator of the set \(A\). The Laplace exponent of the ascending ladder process \((\tilde{L}^{-1}, \tilde{H})\) corresponding to the dual process \(\tilde{X} = -X\) of \(X\) we denote by \(\tilde{\kappa}(\theta, \theta) = -\log \mathbb{E}[\exp\{-\tilde{\beta}\tilde{L}_1^{-1} - \theta \tilde{H}_1\} 1_{\{\tilde{H}_1 < \infty\}}]\). The fluctuations of the processes \(U^*_{t,s} = (U^*_{t,s})_{t,s \in \mathbb{R}_+}\) and \(D^*_{t,s} = (D^*_{t,s})_{t,s \in \mathbb{R}_+}\) for fixed nonnegative \(s\) are described by their running supremum processes \(\mathcal{U}^*_{t,s}, \mathcal{D}^*_{t,s}\) and running infimum processes \(\mathcal{U}^*_{t,s}, \mathcal{D}^*_{t,s}\) which are defined for \(t \geq 0\) by

\[ \mathcal{U}^*_{t,s} = \sup_{0 \leq u \leq t} U^*_{u,s}, \quad \mathcal{U}^*_{t,s} = \inf_{0 \leq u \leq t} U^*_{u,s}, \quad \mathcal{D}^*_{t,s} = \sup_{0 \leq u \leq t} D^*_{u,s}, \quad \mathcal{D}^*_{t,s} = \inf_{0 \leq u \leq t} D^*_{u,s}. \]

We denote the infinite-horizon versions by

\[ \mathcal{U}^*_{t} = \lim_{s \to \infty} \mathcal{U}^*_{t,s}, \quad \mathcal{U}^*_{t} = \lim_{s \to \infty} \mathcal{U}^*_{t,s}, \quad \mathcal{D}^*_{t} = \lim_{s \to \infty} \mathcal{D}^*_{t,s}, \quad \mathcal{D}^*_{t} = \lim_{s \to \infty} \mathcal{D}^*_{t,s}. \]
Figure 1. Drawn are two schematic pictures of a part of the path of $X$ in the cases that (i) the largest value of $X$ up to time $T + S$ has already been attained before time $T$ so that the path-functional $U^*_{T,S}$ is zero (top picture) or (ii) $X$ attains a new maximum between $T$ and $T + S$ and the path-functional $U^*_{T,S}$ is strictly positive (bottom picture). These pictures help clarify the expression for $U^*_{T,S}$ in Proposition 1. For further details, see its proof below.

A first step in the study of the random variables $U^*_{t,s}$, $U^*_{t,s}$, $D^*_{t,s}$ and $D^*_{t,s}$ is the following result.

**Proposition 1.** Let $t,s \in \mathbb{R}^+$. We have the following representations:

\begin{align}
U^*_{t,s} &\overset{(d)}{=} \max \{ \tilde{U}_s + U_t, \tilde{U}_t \}, & U^*_{t,s} &\overset{(d)}{=} \max \{ \tilde{U}_s - D_t, 0 \}, \\
D^*_{t,s} &\overset{(d)}{=} \min \{ U_t - \tilde{D}_s, 0 \}, & D^*_{t,s} &\overset{(d)}{=} - \max \{ \tilde{D}_s + D_t, 0 \},
\end{align}

where $\overset{(d)}{=}$ denotes equality in distribution, $\hat{U}_t = \sup_{0 \leq s \leq t} U_s$, $\hat{D}_t = \sup_{0 \leq s \leq t} D_s$ and where $\tilde{U}_s \overset{(d)}{=} U_s$ and $\tilde{D}_s \overset{(d)}{=} D_s$ are independent of $\mathcal{F}_t$. As a consequence, when $\mathbb{E}[X_1] \in (-\infty, 0)$ (or $\mathbb{E}[X_1] \in (0, \infty)$), then $\hat{U}_t^*$ and $\hat{U}_t^*$ ($\hat{D}_t^*$ and $\hat{D}_t^*$) are finite $\mathbb{P}$-a.s., respectively.

**Remark 1.**

(i) As noted in the Introduction, it suffices to analyze $\hat{U}_t^*$ and $\hat{U}_t^*$, since we have $\hat{D}_t^* = -\tilde{U}_t^*$ and $\hat{D}_t^* = -\tilde{U}_t^*$, where $\tilde{U}_t^*$ and $\tilde{U}_t^*$ denote the extrema $U_t^*$ and $U_t^*$ corresponding to $\hat{X}$.

(ii) We observe that, as $X$ has independent increments, $U_t$ and $D_t$ are independent of $U^*_{t,s}$ and $D^*_{t,s}$ for any $t,s \in (0, \infty]$. Furthermore, we note that, for any $t \geq 0$, $D_t^*$ is finite a.s. if and only if $\mathbb{E}[X_1]$ is strictly positive, as this condition implies that $X_t$ converges to infinity a.s. as $t \to \infty$. Similarly, $U_t^*$ is finite a.s. precisely if $\mathbb{E}[X_1]$ is strictly negative.
(iii) Extending $X$ from $\mathbb{R}_+$ to a two-sided version on $\mathbb{R}$ and using a time-reversal argument we find that

$$
\mathcal{U}_T^+ \overset{(d)}{=} \sup_{0 \leq t \leq T} \sup_{-\infty < s \leq t} (X_t - X_s), \quad \mathcal{U}_T^- \overset{(d)}{=} \inf_{0 \leq t \leq T} \sup_{-\infty < s \leq t} (X_t - X_s).
$$

Indeed, using the change of variables $t' = T - t$ and $s' = T - s$ we see that

$$
\sup_{0 \leq t \leq T} \sup_{-\infty < s \leq t} (X_t - X_s) = \sup_{0 \leq t' \leq T, t' \geq s'} (X_T - X_{T - t'}) = \sup_{0 \leq t' \leq T, t' \geq s'} (X_{s'} - X_{t'}). 
$$

The result for $\mathcal{U}_T^+$ follows similarly.

The random variables $\mathcal{U}_T^+$ and $\mathcal{U}_T^-$ arise in a queueing application. Indeed, the workload process $Q_t$ of a queue with net input process $X$ (i.e. input less output) evolves according to the process $X$ reflected at its infimum, i.e., $Q_t = X_t - \inf_{s \leq t} X_s$. If we assume that the workload process is stationary (i.e., $Q_0$ follows the stationary distribution, which is equal to $\bar{X}_\infty = -\inf_{-\infty < s \leq 0} X_s$), then the workload $Q_t$ is given by:

$$
Q_t = \sup_{-\infty < s \leq t} (X_t - X_s)
$$

and $\mathcal{U}_T^+$ and $\mathcal{U}_T^-$ describe the supremum and infimum of the workload process $Q$ over a finite time horizon $T$, respectively. For details on queues driven by a Lévy process; see the review paper [9].

(iv) We note $\mathbb{P}(\mathcal{U}_T^- = 0) = \mathbb{P}(\int_0^\infty 1(X_s < s) \, ds < t)$ (see for example [2] Lem. 15, p. 170] and [2] Th. 13, p. 169].

**Proof of Proposition 7** In view of Remark (iii) it suffices to establish the statements concerning $\mathcal{U}_T^+$ and $\mathcal{U}_T^-$. Writing $[t, T + S] = [t, T] \cup [T, T + S]$ for given $t, T, S \in \mathbb{R}_+$ leads to

$$
\mathcal{U}_{T,S}^+ = \sup_{0 \leq t \leq T} \max \left\{ \sup_{s \in [T, T + S]} (X_s - X_T) + X_T - X_t, \sup_{0 \leq s \leq T} (X_s - X_t) \right\}.
$$

Since $\bar{U}_s := \sup_{s \in [T, T + S]} (X_s - X_T)$ is independent of $\mathcal{F}_T$ and is equal in distribution to $U_S$, we find that $\mathcal{U}_{T,S}^+$ is equal in distribution to

$$
\sup_{0 \leq t \leq T} \max \left\{ \bar{U}_S + X_T - X_t, \sup_{0 \leq s \leq T} (X_s - X_t) \right\} = \max \left\{ \bar{U}_S + U_T, \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq T} (X_s - X_t) \right\} = \max \left\{ \bar{U}_S + U_T, \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq S} (X_s - X_t) \right\},
$$

which proves the first identity in (2.4).

For the second identity in (2.4) note that, for fixed nonnegative $S$, the function $t \mapsto U_{t,S}^+$ attains its infimum over $[0, T]$ at $G_{T-}$ or $G_T$ where $G_T = \sup \{ u \leq T : X_u = \bar{X}_T \}$. In the case that $G_{T+S} \leq T$ (i.e., when $\bar{X}_{T+S} = \bar{X}_T$) we have $G_T = G_{T+S}$ (see Figure 1 top picture) and

$$
\mathcal{U}_{T,S}^+ = \min \{ U_{G_{T+S} - S, T}, U_{G_{T+S}, S} \} = 0,
$$

while in the case that $G_{T+S} > T$ (see Figure 1 bottom picture) we find

$$
\mathcal{U}_{T,S}^+ = \bar{X}_{T+S} - \bar{X}_T > 0.
$$

Combining these two observations we deduce that

$$
\mathcal{U}_{T,S}^+ \overset{(d)}{=} \max \left\{ X_T - \sup_{0 \leq s \leq T} X_s + \sup_{0 \leq s \leq S} \bar{X}_s, 0 \right\},
$$

where $\bar{X}$ denotes an independent copy of $X$. Since $\sup_{0 \leq s \leq T} X_t$ is equal in distribution to $U_S$, the expression for $\mathcal{U}_{T,S}^+$ follows.

Taking $s \to \infty$ in (2.4) and noting that $\sup_{s \geq 0} X_s$ is finite $\mathbb{P}$-a.s. if $\mathbb{E}[X_1] \in [-\infty, 0)$ we conclude that also $\mathcal{U}_T^+$ and $\mathcal{U}_T^-$ are $\mathbb{P}$-a.s. finite.

$\square$
3. Asymptotic future drawdown — the light-tailed case

In this section we study the asymptotics of the tail probabilities \( \mathbb{P}(U^*_t > x) \) and \( \mathbb{P}(U^*_t > x) \) in the case that the Lévy measure is light-tailed. More precisely, in this section we will make the following assumptions.

**Assumption 1.** The Cramér condition holds, i.e.,

\[
\text{(3.1)} \quad \exists \gamma \in (0, \infty) \text{ satisfying } \mathbb{E}[e^{\gamma X_1}] = 1,
\]

The mean of \( X_1 \) is negative and finite, \( \mathbb{E}[X_1] \in (-\infty, 0) \), and \( \mathbb{E}[e^{\gamma X_1} | X_1] < \infty \).

**Assumption 2.** \( X \) has non-monotone paths and either 0 is regular for \((0, \infty)\) or the Lévy measure of \( X \) is non-lattice.

**Assumption 3.** The \((-\gamma)\)-exponential moment of \( X \) is finite, i.e.,

\[
\text{(3.2)} \quad \mathbb{E}[e^{-\gamma X_1}] < \infty.
\]

Under condition \(3.1\) the characteristic exponent \( \Psi(\theta) \) can be extended to the strip \( S_{\gamma} = \{ \theta \in \mathbb{C} : \Im(\theta) \in [-\gamma, 0] \} \) in the complex plane, by analytical continuation and continuous extension. If in addition \(3.2\) is satisfied, \( \Psi \) can be extended to \( S_{\gamma}' = \{ \theta \in \mathbb{C} : \Im(\theta) \in [-\gamma, \gamma] \} \). The Laplace exponent \( \psi(\theta) = \log \mathbb{E}[e^{\theta X_1}] \) of \( X \) is finite on the maximal domain \( \Theta = \{ \theta \in \mathbb{R} : \psi(\theta) < \infty \} \), which contains the intervals \([0, \gamma]\) and \([-\gamma, 0]\) under conditions \(3.1\) and \(3.2\) respectively. Restricted to the interior \( \Theta^o \), the map \( \theta \mapsto \psi(\theta) \) is convex and differentiable, with derivative \( \psi'(\theta) \). By the strict convexity of \( \psi \), it follows that \( \psi' \) is strictly increasing on \( \Theta^o \) and we denote by \( \xi : \psi'(\Theta^o) \to \Theta^o \) its right-inverse function.

Under \(3.1\) [\(3.1\), \(3.2\)] the Wiener–Hopf factorisation \(2.1\) remains valid for \( \theta \) in the strip \( S_{\gamma} [S_{\gamma}', \text{respectively}] \).

**Lemma 1.** (i) If Assumption \(1\) is satisfied, it holds \( \mathbb{E}[e^{\gamma U_{t\gamma}}] = \mathbb{E}[e^{-\gamma D_{\gamma t}}]^{-1} \) and

\[
\text{(3.3)} \quad \mathbb{E}[e^{\gamma U_{t\gamma}}] < \infty.
\]

(ii) If Assumption \(2\) is satisfied, it holds

\[
\text{(3.4)} \quad \mathbb{E}[e^{\gamma D_{\gamma t}}] < \infty.
\]

(iii) Furthermore, \(3.3\) and \(3.4\) remain valid when \( \gamma \) is replaced by any \( \theta \in \Theta \).

**Proof.** It follows from the Wiener–Hopf factorisation that

\[
\mathbb{E}[e^{\theta U_{t\gamma}}] = q(q - \Psi(\theta))^{-1} \mathbb{E}[e^{-\theta D_{\gamma t}}]^{-1}
\]

for all \( \theta \) in the strip \( S_{\gamma} \). The continuity of \( \mathbb{E}[e^{-\theta D_{\gamma t}}] \) in the half-plane \( \Im(\theta) \leq 0 \), the convexity of \( \theta \mapsto \Psi(\theta) \) restricted to \( \{i\theta : \theta \in [-\gamma, 0]\} \) and the fact that \( \Psi(-i\gamma) = 0 \) then yield \(3.3\). The proof of the remaining assertions is similar and omitted. \( \Box \)

In \([3]\) it was shown that under Assumptions \(1\) and \(2\) Cramér’s estimate holds for the Lévy process \( X \), i.e.,

\[
\text{(3.5)} \quad \mathbb{P}(U^*_0 > y) \simeq C_{\gamma} e^{-\gamma y}, \quad C_{\gamma} = \frac{\kappa(0, 0)}{\gamma \left[ \frac{\partial}{\partial \theta} \Psi(0, -\theta) \right]_{\theta = -\gamma}} > 0, \quad \text{as } y \to \infty,
\]

where we write \( f(x) \simeq g(x) \) as \( x \to \infty \) if \( \lim_{x \to \infty} f(x)/g(x) = 1 \). Cramér’s estimate can be extended to the decay of the finite time probability \( \mathbb{P}(U_t > x) \) when \( x, t \) jointly tend to infinity in fixed proportion. For a given proportion \( v \) the rate of decay is either equal to \( \gamma \) or to \( \gamma(v) = \psi^*(v)/v \) which is given in terms of the convex conjugate \( \psi^* \) of \( \psi \) given by

\[
\psi^*(u) = \sup_{\alpha \in \mathbb{R}} (\alpha u - \psi(\alpha)).
\]

Here, the proportions \( v \) are to be positive and lie in the range of \( \psi' \). This leads to the following definition.

**Definition 2.** A proportion \( v \in (0, \infty) \) is feasible if there exists a \( \xi_v \in \Theta^o \) such that \( \psi'(\xi_v) = v \).

\(1\)For \( \theta \in \Theta \setminus \Theta^o \), \( \psi'(\theta) \) is understood to be \( \lim_{\eta \to \theta, \eta \in \Theta^o} \psi'(\eta) \).
More specifically, in [22] it was shown that the Höghlund’s estimates hold for $X$, i.e., if Assumptions 1 and 2 are satisfied and the proportion $v$ is feasible, then for $x$ and $s$ tending to infinity such that $x = vs + o(s^{1/2})$ we have

\[(3.6) \quad \mathbb{P}(U_s > x) = \mathbb{P}(U_{s,t}^* > x) \sim \left\{ \begin{array}{ll}
C_\gamma e^{-\gamma x} & \text{if } 0 < v < \psi'(\gamma), \\
\tilde{C}_v e^{1/2}e^{-\gamma(v)x} & \text{if } v > \psi'(\gamma),
\end{array} \right. \]

with

\[(3.7) \quad \tilde{C}_v = -\log \frac{\mathbb{E}[e^{-\gamma L_v^1} \mathbf{1}_{(H_1 < \infty)}]}{\mathbb{E}[e^{\gamma L_v^1} \mathbf{1}_{(H_1 < \infty)}]} \times \frac{1}{\xi_v \sqrt{2\pi \psi''(\xi_v)}}, \]

and with $\eta_v = \psi'(\xi_v)$, where we write $f \sim g$ if $\lim_{x \to \infty} x = \lim_{x \to \infty} x = 1$ and we used that $U_s \overset{(d)}{=} U_{s,t}^*$ for $s \in \mathbb{R}_+$. In particular, note that, if $v$ is feasible with $v > \psi'(\gamma)$, then $\gamma(v) = \xi_v - \psi'(\xi_v)/\xi_v$ so that $\gamma(v) \in \Theta^v$.

The exact asymptotic decay of the tail probabilities of $U_{s,t}^*$ and $U_{t,s}^*$ is given as follows:

**Theorem 1.** Suppose that Assumptions 1 and 2 hold, and let $t > 0$.

(i) Then the following limit hold true:

\[(3.8) \quad \mathbb{P}(U_{t,s}^* > x) \to C_\gamma \mathbb{E}[e^{\gamma D_t}] e^{-\gamma x}, \quad x \to \infty.\]

If furthermore Assumption 3 holds, then we have

\[(3.9) \quad \mathbb{P}(U_{t,s}^* > x) \sim \left\{ \begin{array}{ll}
C_\gamma \mathbb{E}[e^{\gamma D_t}] e^{-\gamma x} & \text{if } 0 < v < \psi'(\gamma), \\
\tilde{C}_v \mathbb{E}[e^{\gamma(v) D_t}] e^{1/2}e^{-\gamma(v)x} & \text{if } v > \psi'(\gamma),
\end{array} \right. \]

(ii) If $x$ and $s$ tend to infinity such that $x = vs + o(s^{1/2})$ for some feasible proportion $v$ then we have the following limits:

\[(3.10) \quad \mathbb{P}(U_{t,s}^* > x) \sim \left\{ \begin{array}{ll}
C_\gamma \mathbb{E}[e^{\gamma U_t}] e^{-\gamma x} & \text{if } 0 < v < \psi'(\gamma), \\
\tilde{C}_v \mathbb{E}[e^{\gamma(v) U_t}] e^{1/2}e^{-\gamma(v)x} & \text{if } v > \psi'(\gamma),
\end{array} \right. \]

**Remark 2.** In specific cases the Wiener–Hopf factors are known, so that the constants in (3.7) can be identified.

(i) If $X$ is spectrally negative, then $C_\gamma = 1$ and

\[(3.11) \quad \mathbb{E}[e^{\gamma U_t}] = \frac{\Phi(q)}{\Phi(q) - \gamma}, \quad q > 0,\]

where $\gamma = \Phi(0)$, with $\Phi(q)$, $q \geq 0$, the largest root of the equation $\psi(\theta) = q$ where $\psi(\theta) = \log \mathbb{E}[e^{\theta X_1}]$ is the Laplace exponent. These expressions hold since $U_{0,t}^*$ follows an exponential distribution with parameter $\Phi(q)$. By inverting the Laplace transforms in $q$ we find the following explicit expression in terms of the one-dimensional distributions of $X$:

\[(3.12) \quad \mathbb{E}[e^{\gamma U_t}] = 1 + \gamma \int_0^t \mathbb{E}[e^{\gamma X_s^+}] s^{-1} ds.\]

Here we used that, on account of Kendall’s identity ($\mathbb{P}(\tau_x^+ \in dt) = \frac{1}{\gamma} \mathbb{P}(X_t \in dx)$) for the first passage time $\tau_x^+ = \inf\{t \geq 0 : X_t > x\}$, it follows that

\[(3.13) \quad \int_0^\infty e^{-qt} \mathbb{E}[e^{\gamma X_t^+}] t^{-1} dt = \frac{1}{\Phi(q) - \gamma}.\]

Moreover, if $\psi(-\gamma) < \infty$,

\[(3.14) \quad \mathbb{E}[e^{\gamma D_t}] = \frac{q}{q - \psi(-\gamma)} \left[ 1 + \frac{\gamma}{\Phi(q)} \right].\]

Hence, we have

\[(3.15) \quad \mathbb{E}[e^{\gamma D_t}] = e^{\psi(-\gamma)} + \int_0^t e^{(s-t)\psi(-\gamma)} \mathbb{E}[e^{\gamma X_s^+}] s^{-1} ds.\]
(ii) If $X$ is spectrally positive, then $C_\gamma = \frac{\psi'(0)}{\psi(0)}$ and
\begin{equation}
\mathbb{E}[e^{\gamma U_{s,t}}] = \mathbb{E}[e^{-\gamma D_{s,t}}]^{-1} = \frac{\Phi(q) + \gamma}{\Phi(q)},
\end{equation}
where $\gamma$ and $\Phi(q)$, $q \geq 0$, are the largest roots of $\psi(\theta) = 0$ and $\psi(-\theta) = q$ for $\psi(\theta) = \log \mathbb{E}[e^{\theta X_1}]$, and we used that $D_{s,t} \overset{(d)}{=} \hat{U}_{s,t}$. Hence
\begin{equation}
\mathbb{E}[e^{\gamma U_{s,t}}] = 1 + \gamma \int_0^t \mathbb{E}[X_{s,t}^{-1}] \mathbb{E}[e^{\gamma D_{t,s}}] = 1 - \gamma \int_0^t \mathbb{E}[e^{-\gamma X_s X_t^{-1}}] s^{-1} ds.
\end{equation}

(iii) The Wiener–Hopf factors may also be identified for the meromorphic Lévy processes \cite{Def. 1}:
\begin{equation}
\mathbb{E}[e^{\gamma U_{s,t}}] = \prod_{n \geq 1} \frac{1 - \frac{1}{\gamma \rho_n} \Gamma(\frac{1}{\gamma \rho_n})}{\Gamma(\frac{1}{\gamma \rho_n})}, \quad \mathbb{E}[e^{\gamma D_{s,t}}] = \prod_{n \geq 1} \frac{1 - \frac{1}{\gamma \rho_n} \Gamma(\frac{1}{\gamma \rho_n})}{\Gamma(\frac{1}{\gamma \rho_n})},
\end{equation}
where $\{-i\rho_n, i\rho_n\}_{n \geq 1}$ are the poles of $\Psi$ (which is meromorphic) and $\{-i\zeta_n(q), i\zeta_n(q)\}_{n \geq 1}$ are the roots of $q + \Psi(\theta) = 0$. The above Laplace transforms in $q$ can be numerically inverted giving $\mathbb{E}[e^{\gamma U_{s,t}}]$ and $\mathbb{E}[e^{\gamma D_{s,t}}]$ (see for details \cite{Sec. 8}).

**Proof of Theorem 7** (i) From Proposition 1 it follows that for $s, t \in \mathbb{R}_+$
\begin{equation}
P(U_{s,t}^* \leq x) = \int_{[0,x]} P(U_s \leq x-z) P(U_t \in dz, U_t \leq x) \Leftrightarrow \quad P(U_{s,t}^* > x) = P(U_t > x) + \int_{[0,x]} P(U_s > x-z) P(U_t \in dz, U_t \leq x)
\end{equation}

By letting $s \to \infty$ in (3.16) and using $U_\infty \overset{(d)}{=} U_\infty^*$ (see (2.2)) we arrive at the identity
\begin{equation}
P(U_t^* > x) = P(U_t > x) + \int_{[0,x]} P(U_t^* > x-z) P(U_t \in dz, U_t \leq x).
\end{equation}

Denote by $P(\gamma)$ the Cramér measure that is defined on $(\Omega, \mathcal{F})$ by $P(\gamma)(A) = \mathbb{E}[e^{\gamma X_t} 1_A], A \in \mathcal{F}$. The Cramér asymptotic decay (3.5) implies that
\begin{equation}
e^{\gamma x} P(U_0^* > x) = \mathbb{E}(\gamma)[e^{-\gamma(\tau_x^+ - x)}] \simeq C_\gamma, \quad \text{as } x \to \infty,
\end{equation}
where $\tau_x^+ = \inf \{ t \geq 0 : X_t > x \}$ is the first-passage time of $X$ over the level $x$. In view of the facts that $t \mapsto X_t := \inf_{0 \leq s \leq t} X_s$ is non-increasing and $U_{t^+}^* - x \geq 0$ for $T_{t^+}^\gamma = \inf \{ t \geq 0 : U_t > x \}$ and any $x \in (0, \infty)$ we find
\begin{equation}
P(U_t^* \leq x) = P(T^U_X < t) = e^{-\gamma x} \mathbb{E}(\gamma)[e^{-\gamma(\tau_x^+ - x)}] \chi_{[T^\gamma_{t^+} < t]} = e^{-\gamma x} \mathbb{E}(\gamma)[e^{-\gamma(U_{t^+}^* + X_{t^+}^* - x)}] \chi_{[T^\gamma_{t^+} < t]} \leq e^{-\gamma x} \mathbb{E}(\gamma)[e^{-\gamma \Delta_{X_t}^*} \chi_{[T^\gamma_{t^+} < t]}] = o(e^{-\gamma x}), \quad \text{as } x \to \infty,
\end{equation}

where the expectation in (3.19) converges to zero by virtue of the dominated convergence theorem and the facts that $\mathbb{E}(\gamma)[e^{-\gamma \Delta_{X_t}^*}] = \mathbb{E}[e^{\gamma U_{t^+}^*}] < \infty$ (by Lemma 1) and $T^\gamma_{t^+} \to \infty$ $P(\gamma)$-a.s. as $x \to \infty$ (as $X$ drifts to infinity $P(\gamma)$-a.s.). Combining (3.17) with (3.19), the Cramér asymptotics (3.18) and the dominated convergence theorem yield
\begin{equation}
limit_{x \to \infty} e^{\gamma x} P(U_t^* > x) = C_\gamma \int_{[0, \infty)} e^{\gamma z} P(U_t \in dz) = C_\gamma e^{\gamma U_t}, \quad t \in \mathbb{R}_+.
\end{equation}

As far as $U_{t^+}^*$ is concerned, we deduce from Proposition 1 the Cramér asymptotics (3.5), Lemma 1(ii) and the dominated convergence theorem that
\begin{equation}
P(U_{t^+}^* > x) = \int_{[0, \infty)} P(U_{t^+}^* > x-z) P(D_t \in dz) \simeq C_\gamma e^{-\gamma x} \int_{[0, \infty)} e^{\gamma z} P(D_t \in dz) = C_\gamma e^{-\gamma x} \mathbb{E}[e^{\gamma D_t^*}].
\end{equation}
Suppose Assumptions 1 and 2 hold, and let Corollary 1. 

By following a similar line of reasoning as the proof of Theorem 1 it is straightforward to show that when \( 0 < v < \psi'(\gamma) \)

\[ e^{\gamma z} \mathbb{P}(U_{t,s}^* > x) \sim C_\gamma \int_{[0,\infty)} e^{\gamma z} \mathbb{P}(U_t \in dz) = C_\gamma e^{\gamma U_t}. \]

Turning to the case \( v > \psi' \), we note that a similar reasoning as was used to prove (3.19) and the fact that \( v \) is feasible shows that, for some \( \delta > 0 \) sufficiently small, \( \mathbb{P}(U_t > x) \) and \( \mathbb{P}(U_t > x) \) are \( o(e^{-\gamma(x+\delta)x}) \) as \( x \to \infty \). Hence, for any \( \epsilon > 0 \) and \( x > \epsilon \) we have

\[ x^{1/2} e^{\gamma(v)x} \mathbb{P}(U_{t,s}^* > x) \sim \int_{[0,\infty)} (x - z)^{1/2} e^{\gamma(v)(x - z)} \mathbb{P}(U_s > x - z) e^{\gamma(v)z} \mathbb{P}(U_t \in dz) + I_s(x) \]

with \( I_s(x) = x^{1/2} e^{\gamma(v)x} \int_{[x-\epsilon,x]} \mathbb{P}(U_x > x-z) \mathbb{P}(U_t \in dz). \)

We have that \( I_s(x) \sim 0 \) in view of the bounds

\[ 0 \leq I_s(x) \leq x^{1/2} e^{\gamma(v)x} \mathbb{P}(U_t > x - \epsilon), \]

the upper bound of which converges to zero when \( x \) tends to infinity. Hence, using (3.6) and the dominated convergence theorem yields

\[ x^{1/2} e^{\gamma(v)x} \mathbb{P}(U_{t,s}^* > x) \sim \tilde{C}_v \int_{[0,\infty)} e^{\gamma(v)z} \mathbb{P}(U_t \in dz) = \tilde{C}_v e^{\gamma U_t} \]

for any \( t \in \mathbb{R}_+ \), where the expectation is finite by Lemma [1(iii)] and the fact \( \gamma(v) \in \Theta^o \). The proof of the asymptotics of the tail-probability \( \mathbb{P}(U_{t,s}^* > x) \) is similar and omitted. \( \square \)

3.1. Asymptotic drawdown and drawup measures. Conditional on \( U_{t,s}^* \) being large, or on \( U_{t,s}^* \) being large \( X_t \) admits a limit in distribution, as is shown next. These limits are given by the “drawup-measures” \( \mathbb{P}^{(s)} \) and the “drawdown measures” \( \mathbb{P}^{(v)} \), \( s \in \Theta \), that are defined as follows on the measurable space \( (\Omega, \mathcal{F}_t) \):

\[ \mathbb{P}^{(s)}(A) = \mathbb{E} \left[ e^{sU_t} \mathbbm{1}_A \right], \quad \mathbb{P}^{(v)}(A) = \mathbb{E} \left[ e^{vD_t} \mathbbm{1}_A \right], \quad A \in \mathcal{F}_t, s \in \Theta. \]

**Corollary 1.** Suppose Assumptions 2 and 3 hold, and let \( t > 0 \).

(i) Then, conditional on \( \{U_t \to x\} \) and on \( \{U_t^* \to x\} \), \( X_t \) converges in distribution as \( x \to \infty \):

\[ \mathbb{P}[X_t \leq x|U_t^* > x] \sim \mathbb{P}^{(\gamma)}[X_t \leq x], \quad 0 < v < \psi'(\gamma), \]

\[ \mathbb{P}[X_T \leq x|U_T^* > x] \sim \mathbb{P}^{(\gamma)}[X_t \leq x]. \]

(ii) If \( x \) and \( s \) tend to infinity such that \( x = vs + o(s^{1/2}) \) where \( v \) is feasible then the following limits hold true:

\[ \mathbb{P}[X_t \leq x|U_{t,s}^* > x] \sim \begin{cases} \mathbb{P}^{(\gamma)}[X_s \leq x] & \text{if } 0 < v < \psi'(\gamma), \\ \mathbb{P}^{(\gamma)}[X_t \leq x] & \text{if } \psi'(\gamma) > v, \end{cases} \]

\[ \mathbb{P}[X_T \leq x|U_{T,s}^* > x] \sim \begin{cases} \mathbb{P}^{(\gamma)}[X_s \leq x] & \text{if } 0 < v < \psi'(\gamma), \\ \mathbb{P}^{(\gamma)}[X_t \leq x] & \text{if } \psi'(\gamma) > v. \end{cases} \]

**Proof of Corollary 2 (i)** By following a similar line of reasoning as the proof of Theorem 1 it is straightforward to show that for \( u \in [0,\gamma] \), as \( x \to \infty \),

\[ \mathbb{E}[e^{uX_t} \mathbbm{1}_{U_t^* > x}] \sim C_\gamma e^{-\gamma x} \mathbb{E}[e^{uX_t+\gamma U_t}], \]

\[ \mathbb{E}[e^{uX_t} \mathbbm{1}_{U_T^* > x}] \sim C_\gamma e^{-\gamma x} \mathbb{E}[e^{uX_t+\gamma D_t}]. \]

Bayes’ lemma then yields the stated identities. The proof of (ii) is similar and is therefore omitted. \( \square \)
4. Asymptotic future drawdown — the heavy-tailed case

We continue the study of the asymptotic behaviour of the tail probabilities of $U_i^+$ and $U_i^-$ in the case that the Lévy measure $\mathcal{V}$ of $X$ belongs to the class $S^{(\alpha)}$ of convolution-equivalent measures which, we recall, is a subset of the following class $L^{(\alpha)}$.

**Definition 3.** (Class $L^{(\alpha)}$) For a parameter $\alpha \geq 0$ we say that measure $G$ with tail $\mathcal{G}(u) := G(u, \infty)$ belongs to class $L^{(\alpha)}$ if

(i) $\mathcal{G}(u) > 0$ for each $u \geq 0$,
(ii) $\lim_{u \to \infty} \frac{\mathcal{G}(u-y)}{\mathcal{G}(u)} = e^{\alpha y}$ for each $y \in \mathbb{R}$, and $G$ is nonlattice,
(iii) $\lim_{u \to \infty} \frac{\mathcal{G}(\alpha/u)}{\mathcal{G}(u)} = e^{\alpha}$ if $G$ is lattice (then assumed of span 1).

**Definition 4.** (Class $S^{(\alpha)}$) We say that $G$ belongs to class $S^{(\alpha)}$ if

(i) $G \in L^{(\alpha)}$;
(ii) for some $M_0 < \infty$, we have

$$\lim_{u \to \infty} \frac{G^{*2}(u)}{G(u)} = 2M_0,$$

where $G^{*2}(u) = G^{*2}(u, \infty)$ and $*$ denotes convolution.

The asymptotics are derived under conditions on the Lévy measure $\Pi_H$ of the ladder height process $H$, which according to the Vigon [26] identity is related to the Lévy measures $\mathcal{V}$ of $X$ by

$$\Pi_H(u) = \Pi_H(u, \infty) = -\int_{(-\infty,0)} \mathcal{V}(u-y)d\tilde{V}(y), \quad u \geq 0,$$

for the renewal measure $d\tilde{V}(y) = \int_{-\infty}^{\infty} \mathbb{P}(\tilde{H}(t) \in dy)dt$. Throughout this section we assume that for some fixed $\alpha > 0$ the following three conditions hold true:

\begin{align*}
(4.2) & \quad \Pi_H \in S^{(\alpha)}; \\
(4.3) & \quad \psi(\alpha) < 0; \\
(4.4) & \quad \kappa(0,0) + \kappa(0,-\alpha) > 0.
\end{align*}

**Theorem 2.** Assume that $\mathbb{E}[X_1] < 0$ and let $t > 0$. Under conditions (4.2)-(4.4) we have:

$$\mathbb{P}(U_i^+ > x) \simeq \text{const}^+ \Pi_H(x), \quad \mathbb{P}(U_i^- > x) \simeq \text{const}^- \Pi_H(x),$$

where $\text{const}^+$ and $\text{const}^-$ are given by

$$\text{const}^+ = \mathbb{E}[e^{\alpha X_1}] + \int_0^t \mathbb{E}[e^{\alpha \tilde{S}_r}]^{-1} \mu(ds), \quad \text{const}^- = \mathbb{E}[e^{-\alpha X_1}],$$

with the Borel measure $\mu$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ given by

$$\mu(ds) = \int_0^\infty \mathbb{P}(L^{-1}(u) \in ds) e^{-\kappa(0,-\alpha)u} [1 - u\kappa(0,-\alpha)] du.$$

**Remark 3.** (i) By straightforward calculations it can be verified that

$$\ell_\mu \| = \frac{1}{\ell^2},$$

where $\ell_\mu$ denotes the Laplace–Stieltjes transform of the measure $\mu$.

(ii) If $\mathcal{V} \in S^{(\alpha)}$ for $\alpha > 0$ then (4.2) holds and

$$\Pi_H(x) \simeq \frac{1}{\kappa(0,-\alpha)} \tilde{V}(x);$$

see [14], Prop. 5.3.
(iii) If \( X \) is spectrally positive, then from (3.15):

\[
\mathbb{E}[e^{\alpha X_t}] = 1 + \alpha \int_0^t \mathbb{E}[X_s^-] \, ds,
\]

Moreover, since \( \kappa(q, 0) = q/\Phi(q) \) and \( \kappa(0, -\alpha) = -\psi(\alpha)/(\Phi(0) + \alpha) \), we have

\[
q \left( \mathcal{L}_\mu(q) \right) = \frac{\kappa(q, 0)}{(\kappa(q, 0) + \kappa(0, -\alpha))^2} = \frac{q\Phi(q)}{(q - \phi(q)^2 \psi(\alpha)/\phi(0))^2}.
\]

**Proof of Theorem 3** We first prove the statement concerning \( \overline{U}^* \). The starting point of the proof is to take the identity noted earlier in (3.17) and replace the fixed time \( T \) by an independent exponential random variable \( e_q \) with parameter \( q \), which yields

\[
\mathbb{P}(\overline{U}_{e_q} > x) = \mathbb{P}(\overline{U}_{e_q} > x) + \int_{[0,x]} \mathbb{P}(U_e^+_q > x - z)\mathbb{P}(U_{e_q} \in dz, \overline{U}_{e_q} \leq x).
\]

We show that both terms on the right-hand side of (4.8) are asymptotically equivalent to the tail-measure \( \Pi_H(x) \) of the ladder process \( H \) as \( x \to \infty \) and identify the constant. Throughout the proof we denote the first upward and downward passage times of \( X \) across the level \( x \) by \( \tau_x^+ = \inf\{t \geq 0 : X_t > x\} \) and \( \tau_x^- = \inf\{t \geq 0 : X_t < x\} \).

To establish this result suffices to show asymptotic equivalence of the two terms on the right-hand side of (4.8) to the probability \( \mathbb{P}(\tau_x^+ < e_q) \), since it is known from \([13\] Th. 4.1 and \([21\] Lem. 5.4, eq. (5.6)] that under the conditions stated in the theorem

\[
\mathbb{P}(\tau_x^+ < e_q) \approx \frac{\kappa(q, 0)}{(\kappa(q, 0) + \kappa(0, -\alpha))^2} \cdot \Pi_H(x), \quad q \geq 0,
\]

with the interpretation \( \mathbb{P}(\tau_x^+ < \infty) = \mathbb{P}(\tau_x^+ < e_0) \) for \( q = 0 \). Note that the constant in (4.9) is strictly positive for all \( q \geq 0 \) by the condition \([14\] and \( \kappa(0, 0) > 0 \) (as \( \mathbb{E}[X_1] \) is strictly negative).

We treat both terms separately, starting with the first term. We first derive upper and lower bounds for the ratio \( \mathbb{P}(\overline{U}_{e_q} > x)/\mathbb{P}(\tau_x^+ < e_q) \). By an application of the strong Markov property and the definition of \( \overline{U} \) we have

\[
\mathbb{P}(\overline{U}_{e_q} > x) \geq \mathbb{P}(\tau_x^+ < \tau_x^- \wedge e_q, \overline{U}_{e_q} > x) + \mathbb{P}(\tau_x^- < \tau_x^+ \wedge e_q, \overline{U}_{e_q} > x)
\]

(4.10) 

\[
= \mathbb{P}(\tau_x^+ < \tau_x^- \wedge e_q) + \mathbb{P}(\tau_x^- < \tau_x^+ \wedge e_q) \mathbb{P}(\overline{U}_{e_q} > x) \quad \text{and}
\]

\[
\mathbb{P}(\overline{U}_{e_q} > x) \leq \mathbb{P}(\tau_x^+ < \tau_x^- \wedge e_q) + \mathbb{P}(\tau_x^- < \tau_x^+ \wedge e_q) \mathbb{P}(\overline{U}_{e_q} > x) + A_q
\]

(4.11) 

\[
A_q = \mathbb{P}(X_{e_q} > -\epsilon, x + \epsilon \geq X_{e_q} - X_{e_q} \geq x) = \mathbb{P}(X_{e_q} > -\epsilon) \mathbb{P}(x + \epsilon \geq X_{e_q} \geq x),
\]

where in the last line we used that \( X_{e_q} \) and \( X_{e_q} - X_{e_q} \) are independent (by the Wiener–Hopf factorisation) and \( X_{e_q} - X_{e_q} \) have the same distribution. Hence we find from (4.10) and (4.11) that

\[
\mathbb{P}(\overline{U}_{e_q} > x) \geq \frac{\mathbb{P}(\tau_x^+ < \tau_x^- \wedge e_q)}{\mathbb{P}(\tau_x^+ < \tau_x^- \wedge e_q) \mathbb{P}(\overline{U}_{e_q} > x)} \quad \text{and}
\]

\[
\mathbb{P}(\overline{U}_{e_q} > x) \leq \frac{\mathbb{P}(\tau_x^+ < \tau_x^- \wedge e_q)}{\mathbb{P}(\tau_x^+ < \tau_x^- \wedge e_q) \mathbb{P}(\overline{U}_{e_q} > x)} + \frac{\mathbb{P}(\tau_x^+ < e_q) - \mathbb{P}(\tau_x^+ < e_q)}{\mathbb{P}(\tau_x^+ < e_q)}
\]

(4.12) 

(4.13) 

The first terms on the right-hand sides of (4.12) and (4.13) may be simplified by using that, by the Markov property, we have

\[
\mathbb{P}(\tau_x^+ < \tau_x^- \wedge e_q) = \mathbb{P}(\tau_x^+ < e_q) - \mathbb{P}(\tau_x^- < \tau_x^+ < e_q)
\]

(4.14) 

\[
\mathbb{P}(\tau_x^+ < e_q) = \mathbb{E}\left[1_{\tau_x^- < \tau_x^+ \wedge e_q} \mathbb{P}(X_{\tau_x^+} < e_q)\right].
\]

Furthermore, since \( \Pi_H \in \mathcal{S}(\alpha) \) we note that

\[
\lim_{x \to \infty} \mathbb{P}(\tau_x^+ < e_q) = e^{-\alpha x}, \quad \epsilon > 0.
\]

(4.15)

From the dominated convergence theorem and Definition 3(ii)–(iii) it then follows that

\[
\lim_{x \to \infty} \frac{\mathbb{P}(\tau_x^+ < e_q)}{\mathbb{P}(\tau_x^+ < e_q)} = \mathbb{E}\left[e^{-\alpha x} \mathbb{1}_{\tau_x^- < e_q}\right],
\]

(4.16)
and an application of the Markov property yields

\[(4.17) \quad \mathbb{E} \left[ e^{\alpha X_{\tau^-}} 1_{\{\tau^- < e_q\}} \right] = \mathbb{E} \left[ e^{\alpha X_{\tau^-}} 1_{\{\tau^- < e_q\}} \right].\]

Taking first \( x \to \infty \) in \([4.12]\) and \([4.13]\) and using \([4.14]\), \([4.15]\), \([4.16]\) and \([4.17]\) and that \( \mathbb{P}[\tau^- = e_q] = 0 \) we find

\[
\mathbb{E} \left[ e^{\alpha X_{\tau^-}} \mid \tau^- > e_q \right] \leq \liminf_{x \to \infty} \frac{\mathbb{P}(U_{e_q} > x)}{\mathbb{P}(\tau^+_x < e_q)} \leq \limsup_{x \to \infty} \frac{\mathbb{P}(U_{e_q} > x)}{\mathbb{P}(\tau^+_x < e_q)} \leq \mathbb{E} \left[ e^{\alpha X_{\tau^-}} \mid \tau^- > e_q \right] + 1 - e^{-\alpha}. \]

Letting subsequently \( \epsilon \downarrow 0 \) and using

\[
\lim_{\epsilon \downarrow 0} \mathbb{E} \left[ e^{\alpha X_{\tau^-}} \mid \tau^- > e_q \right] = 1, \]

which in turn holds as the conditional expectation is bounded above by 1 and bounded below by \( e^{-\alpha \epsilon} \), we get the following asymptotics:

\[(4.18) \quad \mathbb{P}(U_{e_q} > x) \simeq B_q \mathbb{P}(\tau^+_x), \quad \text{with} \quad B_q = \frac{\kappa(q,0)}{(\kappa(q,0) + \kappa(0,-\alpha))^2} \mathbb{E} \left[ e^{\alpha X_{e_q}} \right].\]

Next, we turn to the proof of the asymptotic decay of the second term on the right-hand side of \((4.18)\). Note that

\[
\int_{[0,x]} \mathbb{P}(X_{\infty} > x - z) \mathbb{P}(U_{e_q} \in dz, \tau^+_x \leq x) = \left( \int_{[0,y']} + \int_{(y',x-y')} + \int_{(x-y',x)} \right) \mathbb{P}(X_{\infty} > x - z) \mathbb{P}(U_{e_q} \in dz, \tau^+_x \leq x). \quad \tag{4.19}
\]

We next show that the second and third integral of the right-hand side of \((4.19)\) tend to zero as we let first \( x \) and then \( y \) tend to infinity. Indeed, concerning the second integral we use \((4.2)\), Definition \([3] \)(ii)–(iii) and \((4.9)\) to show that

\[
\lim_{x \to \infty} \int_{(y',x-y')} \mathbb{P}(X_{\infty} > x - z) \mathbb{P}(U_{e_q} \in dz, \tau^+_x \leq x) = \int_{(y',\infty)} e^{\alpha z} \mathbb{P}(X_{e_q} \in dz),
\]

which tends to 0 as \( y' \to \infty \).

For the third integral, we obtain the bound

\[
\int_{(x-y',x]} \mathbb{P}(X_{\infty} > x - z) \mathbb{P}(U_{e_q} \in dz, \tau^+_x \leq x) \leq \mathbb{P}(X_{\infty} > y') \mathbb{P}(X_{e_q} > x - y') \leq \mathbb{P}(\tau^+_y < \infty) \mathbb{P}(\tau^+_{y-y'} < \infty).
\]

After dividing the integral in the display by \( \mathbb{P}(\tau^+_y < \infty) \) and letting first \( x \to \infty \) and then \( y' \to \infty \), it tends to zero.

Finally, the first integral on the right-hand side of \((4.19)\) is asymptotically of the same order as the left-hand side. Indeed, using \((4.2)\) and Definition \([3] \)(ii)–(iii) and \((4.9)\) and the dominated convergence theorem we find

\[
\lim_{x \to \infty} \int_{[0,y']} \mathbb{P}(X_{\infty} > x - z) \mathbb{P}(U_{e_q} \in dz, \tau^+_x \leq x) = \int_{[0,y']} e^{\alpha z} \mathbb{P}(U_{e_q} \in dz),
\]

which converges to \( \int_0^\infty e^{\alpha z} \mathbb{P}(U_{e_q} \in dz) = \mathbb{E}[e^{\alpha X_{e_q}}] := \tilde{B}_q \) as \( y' \to \infty \).

By combining the previous estimates we have the following asymptotics of the tail probability \( \mathbb{P}(U_{e_q} > x) \):

\[
\lim_{x \to \infty} \frac{\mathbb{P}(U_{e_q} > x)}{q \mathbb{P}(\tau^+_x)} = q^{-1}(B_q + \tilde{B}_q). \tag{4.21}
\]

Noting that the right-hand side of \((4.21)\) is a pointwise limit of Laplace transforms of measures and is itself such a Laplace transform, it follows from (an extension of) the continuity theorem (see \([13] \) Th. 15.5.2) that the corresponding measures also converge to the limiting measure with Laplace transform given by \( q^{-1}(B_q + \tilde{B}_q) \). Hence the first assertion of the theorem follows by inverting the Laplace transform \( q^{-1}(B_q + \tilde{B}_q) \) (see Remark \([3] \)).
Concerning $U^*_+$, note that by (3.20) we have

$$
P(U^*_+ > x) = \int_{(-\infty, 0]} \mathbb{P}((\tau^+_x) < \infty) \mathbb{P}(X_t \in dz).
$$

Asymptotics (4.9), the dominated convergence theorem and part (ii) and (iii) of Definition 3 establish that the asymptotic decay of $\mathbb{P}(U^*_+ > x)$ is as stated. \hfill \Box

5. Exact distributions

From Proposition 1 it follows that the distributions of $U_{T,S}^+$ and $U_{T,S}^*$ can be identified if one is able to identify the laws of the finite time supremum and the resolvent of the Lévy process reflected at its infimum. In the case of a spectrally one-sided Lévy process $X$ such explicit expressions are provided by existing fluctuation theory.

In this section we suppose that $X$ is spectrally negative (the case of spectrally positive Lévy process follows from considering the dual of $X$; see Remark 1(i)). Many fluctuation results for $X$ can be conveniently formulated in terms of its scale function $W(q)$ that is defined as the unique continuous increasing function on $[0, \infty)$ with Laplace transform

$$
\int_0^\infty e^{-\lambda x} W(q)(x) \, dx = \frac{1}{\psi(\lambda) - q} \quad \text{for any } \lambda > \Phi(q).
$$

Let $e_\beta$ be an exponentially distributed random variable with parameter $\beta$ (independent of $e_q$ and $X$).

**Proposition 2.** (i) If $E[X_1] \in [-\infty, 0)$ then

$$
P(U^*_{e_\beta, e_\beta} > x) = \frac{1}{\Phi(q)} \left[ 1 + q \int_0^x e^{-\Phi(\beta) z} W(q)(z) \, dz \right] \quad \text{and}
$$

$$
P(U^*_+ > x) = \frac{q}{\beta} e^{-\Phi(\beta) x} \frac{\Phi(\beta) - \Phi(q)}{\Phi(q)}.
$$

(ii) If $E[X_1] \in (0, \infty)$ then

$$
P(-D^*_{e_\beta, e_\beta} > x) = \Phi(q) \int_0^\infty e^{-\Phi(q) z} Z(q)(x + z) \, dz - \frac{\beta}{\Phi(\beta)} \Phi(q) \int_0^\infty e^{-\Phi(q) z} W(q)(x + z) \, dz \quad \text{and}
$$

$$
P(-D^*_+ > x) = q \frac{\beta}{\Phi(\beta)} \int_{[0, x]} (W(q)(x - z) - \beta Z(q)(x - z)) W(q)(z) \, dz
$$

$$
- \frac{\beta}{\Phi(\beta)} \frac{W(q)(x)}{W^+_q(x)} \int_{[0, x]} (W(q)(x - z) - \beta Z(q)(x - z)) W(q)(z) \, dz
$$

$$
+ Z(q)(x) - q \frac{W(q)(x)^2}{W^+_q(x)},
$$

where $W^+_q(x)$ denotes the right-derivative of $W(q)$ at $x$.

The proof of Proposition 2 is based on the representations derived in Proposition 1 and the form of the $q$-resolvent measures $R^U_q$ and $R^D_q$ of $U$ and $-D$ killed upon crossing the level $x > 0$, which are defined by

$$
R^U_q(dy) = \int_0^\infty e^{-y t} \mathbb{P}(U_t \in dy, T^U_x > t) \, dt \quad \text{and} \quad R^D_q(dy) = \int_0^\infty e^{-y t} \mathbb{P}(D_x \in dy, T^D_x > t) \, dt,
$$

where $T^U_x$ and $T^D_x$ are the first-passage times of $U$ and $D$ over $x$, $T^U_x = \inf\{t \geq 0 : U_t > x\}$, $T^D_x = \inf\{t \geq 0 : D_t > x\}$. In [23] Thm. 1 it was shown that these resolvent measures have a density a version of which is given by

$$
R^U_q(dy) = \frac{W(q)(x - y)}{Z(q)(x)} \, dy, \quad y \in [0, x],
$$

$$
R^D_q(dy) = \frac{W(q)(y) W(q)(dy)}{W^q(x)} - W(q)(y) \, dy, \quad y \in [0, x].
$$

**Proof.** Recall that by Proposition 1 we have

$$
P(U^*_{e_\beta, e_\beta} > x) = E \left[ e^{-\Phi(\beta) x} + q \int_{[0, x]} P(X_{e_\beta} > x - z) R^U_q(dy) \right].
$$
where by [23] Prop. 2,

$$
\mathbb{E} \left[ e^{-\eta T_{x}} \right] = \frac{1}{Z(q)(x)}
$$

and

$$
P(U_{a_{0}} > x - z) = \mathbb{P}(X_{a_{0}} > x - z) = e^{-\Phi(\beta)(x-z)}.
$$

Similarly,

$$
P(U_{a_{0}^*} > x) = \int_{0}^{\infty} P(U_{a_{0}} > x + z) P(D_{a_{0}} \in dz),
$$

where by [18]:

$$
P(D_{a_{0}} \in dz) = P(-X_{a_{0}} \in dz) = \frac{q}{\Phi(q)} W^{(q)}(dz) - q W^{(q)}(z) dz.
$$

Straightforward calculations complete the proof of (i).

The proof of (ii) follows by a similar reasoning using the identity

$$
\mathbb{E} \left[ e^{-\eta T_{x}} \right] = Z^{(q)}(x) - q W^{(q)}(x)^2;
$$

see [23] Prop. 2.

\[\square\]

**Corollary 2.** (i) If \( \mathbb{E}[X_{1}] \in [-\infty, 0) \)

$$
P(U_{a_{0}} > x) = \frac{1}{Z^{(q)}(x)} \left[ 1 + q \int_{0}^{x} e^{-\Phi(0)z} W^{(q)}(z) dz \right]
$$

and

$$
P(U_{a_{0}} > x) = e^{-\Phi(0)x} \frac{\Phi(q) - \Phi(0)}{\Phi(q)}.
$$

where \( Z^{(q)}(x) = 1 + q \int_{0}^{x} W^{(q)}(y) dy \).

(ii) If \( \mathbb{E}[X_{1}] \in (0, \infty) \) then

$$
P(-D_{a_{0}} > x) = 1 - \psi'(0) \Phi(q) \int_{0}^{x} e^{-\Phi(0)z} W(x+z) dz
$$

and

$$
P(-D_{a_{0}} > x) = 1 + q \psi'(0) \int_{[0,x]} W(x-z) W^{(q)}(z) dz
$$

$$
- \psi'(0) \frac{W^{(q)}(x)}{W_{+}^{(q)}(x)} \int_{[0,x]} W(x-z) W^{(q)}(dz).
$$

**Proof.** Since \( U_{0}^{*} \) follows an exponential distribution with parameter \( \Phi(0) \), we have for any \( x \geq 0 \)

$$
P(U_{a_{0}}^{*} \leq x) = \int_{[0,x]} \int_{z}^{y} \Phi(0) e^{-\Phi(0)z} dz \mathbb{P}(z + U_{a_{0}} \in dy, e_{q} < T_{x})
$$

$$
= \frac{q \Phi(0)}{Z^{(q)}(x)} \int_{0}^{x} \int_{0}^{y} e^{-\Phi(0)z} W^{(q)}(x - y + z) dz \ dy
$$

$$
= \frac{1}{Z^{(q)}(x)} \left[ q \int_{0}^{x} (1 - e^{-\Phi(0)y}) W^{(q)}(y) \ dy \right].
$$

Furthermore, from (3.11),

$$
P(U_{a_{0}^*}^{*} \leq x) = P(\bar{U}_{0}^{*} - D_{a_{0}} \leq x)
$$

$$
= \Phi(0) \int_{(0,\infty)} \int_{y}^{x} e^{-\Phi(0)(z+y)} dz \mathbb{P}(D_{a_{0}} \in dy) = 1 - \Phi(0) \Phi(q) \Phi(q).
$$

The proof of (ii) follows by a similar reasoning, using the fact \( \mathbb{P}|D_{0}^{*} < x| = \psi'(0)^{-1} W(x) \) and the form of the resolvent.

\[\square\]

**Remark 4.** (i) By inverting the Laplace transform we find that

$$
P(U_{a_{0}}^{*} > x) = e^{-\Phi(0)x} (1 - \Phi(0) \mathbb{E}[U_{1}]).
$$
(ii) Straightforward calculations show that the double Laplace transforms $\mathcal{L}_U(r, s)$ and $\mathcal{L}_D(r, s)$ of $\mathbb{P}(U_T^+ \leq u)$ and $\mathbb{P}(\bar{U}_T^* \leq u)$ in $T$ and $u$ are given by:

\[
\mathcal{L}_U(r, s) = \frac{\Phi(0)(\Phi(s) + r)}{(\Phi(0) + r)s\Phi(s)} , \quad \mathcal{L}_D(r, s) = r\Phi'(0)\Phi(s) \frac{\psi(r) - s}{s^2\psi(r)(r - \Phi(s))}.
\]

This agrees with the form of $\mathcal{L}_D(r, s)$ obtained in \cite{[8]}.

(iii) From the proofs of the propositions above it is clear that we can identify the bivariate Laplace transform of $U_{t,s}, U_{t,s}^+, D_{t,s}$ and $\bar{D}_{t,s}^*$ with respect of $t$ and $s$ as long as the laws of $X_{\lambda^+}, X_{\lambda^-}$ and resolvents of reflected process $R_{\lambda}, R_{\lambda}^+$ are known. This could be done not only for one-sided Lévy processes. For example one can consider the Kou model, where the log-price $X = (X_t)_{t \in \mathbb{R}^+}$ is modelled by a jump-diffusion with constant drift $\mu$ and volatility $\sigma > 0$, with the upward and downward jumps arriving at rate $\lambda_+$ and $\lambda_-$ with sizes following exponential distributions with means $1/\alpha_+$ and $1/\alpha_-$.

$$X_t = \mu t + \sigma W_t + \sum_{j=1}^{N^+_t} U_j^+ - \sum_{j=1}^{N^-_t} U_j^-,$$

where $N^\pm$ are independent standard Poisson processes with rates $\lambda^\pm$, independent of a Brownian motion $W$, and $U_t^\pm \sim \text{Exp}(\alpha^\pm)$ are independent. Then the important ingredients are identified in \cite[Prop. 3]{[1]} (also applied for the dual process).

6. **(Future) drawdowns and drawups under Black–Scholes–Samuelson model**

Consider a risky asset whose price process $P = (P_t)_{t \in \mathbb{R}^+}$ is given as follows:

\[
P_t = P_0 \exp(X_t), \quad t \in \mathbb{R}^+,
\]

where $X_t$ is a Lévy process. In the case of the Black–Scholes–Samuelson model, $P$ is a geometric Brownian motion, with rate of appreciation $\mu \in \mathbb{R}$ and the volatility $\sigma$, and $X = (X_t)_{t \in \mathbb{R}^+}$ is given by the linear Brownian motion

\[X_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t.
\]

Let $\mu > \sigma^2/2$. This model is widely used in practice as a benchmark for other models.

For this model we have $\psi(\theta) = \sigma^2 \theta^2/2 + (\mu - \sigma^2/2)\theta$, $\Phi(q) = -\omega + \delta(q)$ with

$$\delta(q) = \sigma^{-2} \sqrt{(\mu - \sigma^2/2)^2 + 2\sigma^2q}$$

and $\omega = \frac{\mu}{\sigma^2} - \frac{1}{2}$ and

$$W(q)(x) = \frac{1}{\delta(q)\sigma^2} \left[ e^{-\omega(q)x} - e^{-(\omega(q)+\delta(q))x} \right],$$

$$Z(q)(x) = \frac{q}{\delta(q)\sigma^2} \left[ -\frac{1}{\omega - \delta(q)} e^{-\omega(q)x} + \frac{1}{\omega + \delta(q)} e^{-(\omega(q)+\delta(q))x} \right].$$

Hence from Corollary \cite{[3]} we have

$$\mathbb{P}(\bar{D}_{\alpha^+}^* > x) = 1 + \frac{1}{\sigma^4 \delta(q)\omega} (\mu - \sigma^2/2)(Z(q)(x) - 1) + \frac{q}{\delta(q)\sigma^2} (\mu - \sigma^2/2) \left( -\frac{1}{\delta(q) - \omega} e^{-\omega(q)x} - \frac{1}{\delta(q) + \omega} e^{\delta(q)x} - \frac{2\omega}{\delta^2(q) - \omega^2} e^{-2\omega x} \right)$$

and

$$\mathbb{P}(\bar{D}_{\alpha^+}^* > x) = 1 - \frac{\mu - \sigma^2/2}{\sigma^2 \omega} \left( -\frac{1}{\omega - \delta(q)} e^{-\omega(q)x} + \frac{1}{\omega + \delta(q)} e^{-(\omega(q)+\delta(q))x} - \frac{2\delta(q)}{\delta^2(q) - \omega^2} e^{-2\omega x} \right), \quad q > 0.$$

Hence we find for $t \in \mathbb{R}^+$

$$\mathbb{P}(\bar{D}_t^* > x) = 1 - \frac{\mu - \sigma^2/2}{\sigma^2 \omega} \left( 1 - \mathbb{E}[e^{-2\omega U_t}] e^{-2\omega x} \right),$$

\footnote{Note that our expression for $\mathcal{L}_U(r, s)$ slightly differs from the one stated in \cite{[8]} Thm. 2], which is incorrect due to a small error in the calculation in the proof of \cite[Thm. 2]{[8]}.}
where by \([3.12]\) and \([4] (1.1.3), \text{p. 250}\) we have
\[
\mathbb{E}[e^{-2\omega X_t}] = \mathbb{E}[e^{-2\omega X^+_t}] = 1 - 2\omega \int_0^t \mathbb{E}[e^{-2\omega X_s} X^+_s] s^{-1} ds
\]
\[
(6.2)\]
\[
\frac{2 - \sigma^2}{2 - 2\sigma^2} \frac{\omega(2 - \sigma^2)}{\sqrt{2\sigma}} \text{Erfc} \left( \frac{\omega(2 - \sigma^2)}{\sqrt{2\sigma}} \sqrt{t} \right) - \frac{\sigma^2}{2 - 2\sigma^2} \text{Erfc} \left( \frac{\omega\sigma}{\sqrt{2}} \sqrt{t} \right).
\]
Moreover,
\[
\mathbb{E}[P_t] = P_0 \mathbb{E} \left[ e^{\gamma U_t + X_t} \right] = P_0 e^{\psi(1)t} \mathbb{E} \left[ e^{\gamma U_t} \right],
\]
\[
\mathbb{E}[P_t] = P_0 e^{\psi(1)t} \mathbb{E} \left[ e^{\gamma D_t} \right],
\]
where the measure \(\mathbb{P}(1)\) is defined via \(\mathbb{P}(1)(A) = \mathbb{E}[e^{X_t - \psi(1) t} 1_A]\) for \(A \in \mathcal{F}_t\) and \(\gamma = 2\omega\). Under \(\mathbb{P}(1)\) we have
\[
X_t = \left( \mu - \frac{3}{2} \sigma^2 \right) + \sigma W_t.
\]

Straightforward calculations based on \([4] (1.1.3), (1.2.3) \text{p. 250-251}\) give:
\[
\mathbb{E}[e^{\gamma U_t}] = \frac{2 + \sigma^2}{2 + 2\sigma^2} \frac{e^{\omega(2 + \sigma^2)/\sqrt{2\sigma}}} {\text{Erfc} \left( \frac{\omega(2 + \sigma^2)}{\sqrt{2\sigma}} \sqrt{t} \right)} + \frac{\sigma^2}{2 + 2\sigma^2} \text{Erfc} \left( \frac{\omega\sigma}{\sqrt{2}} \sqrt{t} \right),
\]
\[
\mathbb{E}[e^{\gamma D_t}] = \frac{2 + \sigma^2}{2 + 2\sigma^2} \frac{e^{\omega(2 + \sigma^2)/\sqrt{2\sigma}}} {\text{Erfc} \left( \frac{\omega(2 + \sigma^2)}{\sqrt{2\sigma}} \sqrt{t} \right)} + \frac{\sigma^2}{2 + 2\sigma^2} \text{Erfc} \left( -\frac{\omega\sigma}{\sqrt{2}} \sqrt{t} \right),
\]
\[
\mathbb{E}^{(1)}[e^{\gamma U_t}] = \frac{1 + \sigma^2}{1 + 2\sigma^2} \frac{e^{\omega(2\mu - \sigma^2)/(\sigma^2 + 1) - \sigma^4/4}} {\text{Erfc} \left( -\frac{(2\mu - \sigma^2)/(\sigma^2 + 2) - 2\sigma^4/2\sigma^3}{\sqrt{2\sigma^3}} \sqrt{t} \right)} + \frac{\sigma^2(2\mu - 3\sigma^2)}{4(\mu - 2\sigma^2)(1 + \sigma^2) - 4\sigma^4} \text{Erfc} \left( \frac{2\mu - 3\sigma^2}{2\sqrt{2\sigma}} \sqrt{t} \right),
\]
\[
\mathbb{E}^{(1)}[e^{\gamma D_t}] = \frac{1 + \sigma^2}{1 + 2\sigma^2} \frac{e^{\omega(2\mu - \sigma^2)/(\sigma^2 + 1) - \sigma^4/4}} {\text{Erfc} \left( -\frac{(2\mu - \sigma^2)/(\sigma^2 + 2) - 2\sigma^4/2\sigma^3}{\sqrt{2\sigma^3}} \sqrt{t} \right)} + \frac{\sigma^2(2\mu - 3\sigma^2)}{4(\mu - 2\sigma^2)(1 + \sigma^2) - 4\sigma^4} \text{Erfc} \left( -\frac{2\mu - 3\sigma^2}{2\sqrt{2\sigma}} \sqrt{t} \right).
\]

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