Single-particle relaxation time versus transport scattering time in a two-dimensional graphene layer

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We theoretically calculate and compare the single particle relaxation time ($\tau_s$) defining the quantum level broadening and the transport scattering time ($\tau_i$) defining the Drude conductivity in two-dimensional (2D) graphene layers in the presence of screened charged impurity scattering and short-range defect scattering. We find that the ratio $\tau_i/\tau_s$ strongly increases with increasing $k_F z_i$ and $\kappa$, where $k_F$, $z_i$, and $\kappa$ are, respectively, the Fermi wave vector, the separation of the substrate charged impurities from the graphene layer, and the background lattice dielectric constant. A critical quantitative comparison of the $\tau_i/\tau_s$ results for graphene with those for the corresponding modulation-doped semiconductor structures is provided, showing significant differences between these two 2D carrier systems.

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I. INTRODUCTION AND BACKGROUND

In an impure disordered conductor (either a metal or a doped semiconductor), scattering by static impurities and defects, in general, leads to two distinct momentum relaxation times: the scattering lifetime or the transport relaxation time (denoted by $\tau_i$ in this paper, but often called just $\tau$) and the quantum lifetime or the single particle relaxation time (denoted by $\tau_s$ in this paper, but often also called $\tau_0$). Although $\tau_i$ and $\tau_s$ both arise from impurity scattering in the metallic regime, they are, in general, distinct and unique with no direct analytical relationship connecting them, except for the simple (and often unrealistic) model of completely isotropic s-wave zero-range impurity scattering when they are equal. In particular, $\tau_i$ determines the conductivity, $\sigma = n e \mu \times \tau_i$, where $n$ is the carrier density and $\mu$ is the mobility, whereas $\tau_s$ determines the quantum level broadening, $\Gamma = \hbar/2\tau_s$, of the momentum eigenstates (i.e., the band states defined by the momentum $\hbar k$, with $k$ as the conserved wave vector). The difference between $\tau_i$ and $\tau_s$ arises from the subtle effect of the wave vector dependent impurity potential $u_i(k)$, which distinguishes between momentum scattering in all directions contributing to $\tau_s^{-1}$ and transport relaxation $\tau_i^{-1}$ which is unaffected by forward scattering (or more generally, small-angle scattering).

In this paper, we theoretically study $\tau_i$ and $\tau_s$ in a two-dimensional (2D) graphene layer due to long-range and short-range impurity scattering, finding interesting behavior in the ratio $\tau_i/\tau_s$ as a function of impurity location, carrier density, and system environment (e.g., the background lattice dielectric constant of the substrate). We compare graphene $\tau_s/\tau_i$ with the corresponding situation in extensively studied 2D semiconductor systems, finding significant difference.

In three-dimensional (3D) metals or semiconductors, the effective impurity-electron interaction potential is likely short ranged since carrier-induced electronic screening of the charged impurities is highly effective. This strongly screened short-range impurity potential in 3D systems (as well as in 2D systems when electrons and impurities are not spatially separated) has led to the almost universal theoretical adoption of the uncorrelated white-noise zero-range impurity potential model for studying the impurity scattering effect on “metallic” transport properties. Since such a white-noise disorder, by definition, leads to an isotropic impurity scattering dominated entirely by large-angle scattering, $\tau_i = \tau_s$ in ordinary metals and semiconductors for such a zero-range disorder model as forward scattering simply plays no special role here. Graphene is, however, qualitatively different even for this short-range white noise disorder model due to its chiral sublattice symmetry, which suppresses backward (i.e., a scattering induced wave vector change by $2k_F$ from $+k_F$ to $-k_F$) scattering,2 thus introducing an intrinsic chiral preference for forward scattering over backward scattering. As we will show later in this paper, due to this suppression of backward scattering, the zero-range white-noise disorder potential model leads to $\tau_i = 2\tau_s$ in graphene, in contrast to the $\tau_i = \tau_s$ in ordinary metals and semiconductors. This is due to the importance of $k_F$ (rather than $2k_F$) scattering in dominating graphene transport properties, whereas the small-angle scattering always dominates $\tau_s$.

For long-range impurity scattering, however, it was shown1 that $\tau_i$ can exceed $\tau_s$ by a large factor (i.e., $\tau_i \gg \tau_s$) in high-mobility 2D modulation-doped semiconductor structures, where the predominant scattering is the small-angle scattering due to charged impurities located far from the 2D electron layer. In this paper, we theoretically calculate $\tau_i/\tau_s$ (as well as $\tau_s$ and $\tau_i$ individually) for 2D graphene electrons (and holes) due to both long-range and short-range impurity scatterings, taking into account the dependence on the impurity location. We find that there are significant differences between the 2D graphene (with its linear chiral dispersion) and 2D semiconductors, showing that $\tau_i/\tau_s$, in general, could also be very large for graphene although the behavior is quite different from that of the corresponding 2D semiconductors.

Our work is entirely restricted to extrinsic or doped (or gated) graphene, wherein the Fermi level is away from the charge neutral Dirac point (taken to be the energy zero). There has been extensive recent theoretical work3−7 on the charged impurity scattering limited transport properties in gated extrinsic graphene, but the single particle quantum relaxation time $\tau_s$ (and its relationship to the transport scattering time $\tau_i$) has not been discussed. We note that while $\tau_i$ has
been recently studied in extrinsic graphene through measurements of density-dependent conductivity, the corresponding \( \tau_s \) has not yet been directly experimentally measured. The most direct way of experimentally measuring \( \tau_s \) is by measuring the quantum level broadening \( \Gamma = h/2\tau_s \) through the Dingle temperature \( T_D = \Gamma/\pi k_B \). The Dingle temperature can be measured in low-field shubnikov–de Haas magnetoresistance oscillations as has been extensively done in 2D semiconductor structures. Any measurement of the single particle level broadening, e.g., magnetization, also gives an estimate for \( \tau_s \). Experimental comparisons of \( \tau_s/\tau \) for various system parameters (e.g., carrier density, impurity location, material parameters such as effective mass and lattice dielectric constant) have been carried out on semiconductor based 2D systems but not in graphene since measurements of \( \tau_s \) are unavailable in graphene. Extensive measurements of \( \tau_s \) as a function of carrier density, i.e., gate voltage, however, have been recently carried out in graphene.

In our theoretical calculations (and in the numerical results), we have used two complementary models of bare disorder: Long-range random charged impurity centers at or near the graphene-substrate interface and short-range model white-noise disorder in the graphene layer. We also include screening (by the graphene carriers themselves) within the random phase approximation (RPA) in the theory. In Sec. II, we describe theoretical details and present our results. In Sec. III, we discuss the relevance and significance of our theory in the context of transport experiments in gated or doped graphene and provide a conclusion.

II. THEORY AND RESULTS

In the relaxation time approximation, the graphene transport scattering time \( \tau_s \) by randomly distributed impurity centers is given by

\[
\frac{1}{\tau_s} = \frac{2\pi n_i}{\hbar} \sum_{\lambda,\lambda'} \int d^2k' \left( \frac{\langle V_{el}(q) \rangle^2}{\epsilon(q)^2} \right) \Gamma_{\lambda\lambda'}(k,k')(1 - \cos \theta_{k,k'})
\]

where \( n_i \) is the concentration of the impurity center, \( q = |k - k'| \), \( \theta_{k,k'} \) is the scattering angle between the scattering in and out wave vectors \( k \) and \( k' \), \( \lambda, \lambda' = \pm 1 \) denote the band indices, \( \epsilon_{k,k'} = \hbar v_F |k| \) is a single particle energy (\( \gamma = \hbar v_F \) being the graphene band velocity), and \( F_{\lambda\lambda'}(k,k') \) is the overlap of states and given by \( F_{\lambda\lambda'}(k,k') = (1 + \lambda' \cos \theta_{k,k'})/2 \). In Eq. (1), \( V_{el}(q) \) is the matrix elements of the scattering potential between an electron and an impurity. For charged impurities, we use the Coulomb interaction \( V_{el}(q) = 2\pi e^2/\kappa q \), where \( \kappa \) is the dielectric constant of the surrounding materials, and for short-range point defect scatterers, \( V_{el}(q) = v_0 \), which is a constant. In Eq. (1), \( \epsilon(q) \) is the static RPA dielectric (screening) function appropriate for graphene, which is calculated to be

\[
\epsilon(q) = 1 + v_0(q) \Pi(q),
\]

where \( v_0(q) = 2\pi e^2/\kappa q \) is the electron-electron Coulomb potential, and \( \Pi(q) \) is the polarizability function of graphene, which is calculated to be

\[
\Pi(q) = \begin{cases} 
1 & \text{if } q \leq 2k_F \\
\frac{\pi q}{8k_F} - \frac{\sqrt{q^2 - 4k_F^2}}{2q} - \frac{q}{4k_F} & \text{if } q > 2k_F,
\end{cases}
\]

which \( N_F = 2E_F/\pi k_F^2 \) is the density of states of graphene. Since we consider elastic scattering, we can neglect interband scattering processes (\( \lambda' \neq \lambda \)). Then, we have the leading-order (in impurity disorder \( n_i \)) transport scattering time,

\[
\frac{1}{\tau_s} = \frac{n_i E_F}{2\pi \hbar} \int_0^{\pi} d\theta \frac{\langle V_{el}(q) \rangle^2}{\epsilon(q)^2} (1 - \cos^2 \theta),
\]

where \( q = 2k_F \sin(\theta/2) \).

From a many-body-theory viewpoint, the single particle relaxation time \( \tau_s \) can be calculated from the electron self-energy of the coupled electron-impurity system. The electron self-energy due to the impurity scattering is given by

\[
\Sigma(k,\omega) = \sum_{\lambda'k'} \frac{\langle V_{el}(q) \rangle^2}{\epsilon(q)^2} F_{\lambda\lambda'}(k,k') G_{\lambda}(k',\omega),
\]

with the single particle quantum (impurity induced) level broadening \( \Gamma_s = \text{Im} \Sigma(k_F,\omega) \), i.e., \( \tau_s = \hbar/2\Gamma_s \). In the leading-order disorder approximation, we have the single particle relaxation time \( \tau_s \),

\[
\frac{1}{\tau_s} = \frac{2}{\hbar} \text{Im} \Sigma(k_F,\omega),
\]

Thus, the only difference between the scattering time \( \tau_s \) [Eq. (4)] and the single particle relaxation time \( \tau_p \) [Eq. (7)] is the weighting factor \( 1 - \cos \theta \) in the transport scattering time. The factor \( 1 - \cos \theta \) weights the amount of backward scattering of the electron by the impurity. Small-angle (or forward) scattering, where \( \cos \theta = 1 \), is relatively unimportant in contributing to \( \tau_s^{-1} \) and contributes little to the resistivity. In normal 2D systems, the factor \( 1 - \cos \theta \) obviously favors large angle scattering events, which are more important for the electrical resistivity. However, in graphene, the large angle scattering is also suppressed due to the wave function overlap factor \( 1 + \cos \theta \). The transport scattering time thus gets weighted by an angular contribution factor of \( (1 - \cos \theta)(1 + \cos \theta) \), which suppresses both small-angle scattering and large-angle scattering contributions in the transport scattering rate. However, the single particle relaxation time is weighted only by \( 1 + \cos \theta \) term. Therefore, \( \tau_s \) is insensitive to both small and large angle scatterings, while \( \tau_s \) is only sensitive to small angle scattering events. In fact, the dominant contribution to \( \tau_s \) comes from \( \cos^2 \theta = 0 \), i.e., \( \theta \)}
SINGLE-PARTICLE RELAXATION TIME VERSUS

where

By using the RPA screening function [Eq. (2)] at \( T = 0 \), we calculate the scattering times due to charged impurities completely randomly distributed at the interface between graphene and the substrate with density \( n_{ic} \),

\[
\frac{1}{\tau_{sc}} = \frac{r_s^2}{\tau_{sc}} I_n(2r_s),
\]

where \( r_s = e^2/\gamma \kappa \) and

\[
\frac{1}{\tau_{sc}} = \frac{r_s^2}{\tau_{sc}} I_n(2r_s),
\]

(8a)

(8b)

In Eqs. (8a) and (8b), \( I_n(x) \) and \( I_{sc}(x) \) are calculated to be

\[
I_n(x) = \frac{\pi}{2} - 2\frac{d}{dx}[x^2 g(x)],
\]

\[
I_{sc}(x) = -\frac{d}{dx} g(x),
\]

(10a)

(10b)

where \( g(x) \) is given by

\[
g(x) = -1 + \frac{\pi}{2} x + (1-x^2)f(x),
\]

(11)

with

\[
f(x) = \begin{cases} 
\frac{1}{\sqrt{1-x^2}} \ln \left[ 1 + \sqrt{1-x^2} \right] & \text{for } x < 1 \\
1 & \text{for } x = 1 \\
\frac{1}{\sqrt{x^2-1}} \cos^{-1} \frac{1}{x} & \text{for } x > 1.
\end{cases}
\]

(12)

From Eqs. (11) and (12), we have

\[
\frac{dg(x)}{dx} = \frac{\pi}{2} - \frac{1}{x} - xf(x).
\]

(13)

The limiting forms of the scattering times in the small and large \( r_s \) regimes are given by

\[
\frac{\tau_{0c}}{\tau_{sc}} = \begin{cases} 
\frac{r_s^2}{\pi/2 + 12r_s + 8r_s \ln(r_s) + \cdots} & \text{for } r_s \ll 1 \\
\pi/32 - 1/(15r_s) + O(1/r_s^2) & \text{for } r_s \gg 1
\end{cases}
\]

(14)

and

\[
\frac{\tau_{0c}}{\tau_{sc}} = \begin{cases} 
r/2 - \pi r_s^2/2 - 2r_s^3 \ln(r_s) + \cdots & \text{for } r_s \ll 1 \\
\pi/16 - 1/(12r_s) + O(1/r_s^2) & \text{for } r_s \gg 1.
\end{cases}
\]

(15)

For large \( r_s \), the ratio approaches 2, which indicates that scattering is not isotropic even though the screening is strong. On the other hand, \( \tau_{0c}/\tau_{sc} \) diverges as \( 1/r_s \) for small \( r_s \). In \( r_s \to 0 \) limits, wherein the screening is very weak, the transport scattering rate is much weaker than the single particle scattering rate, which indicates that the mobility in these limits is very high.

For short-ranged (\( \delta \)-function) scatterers with impurity density \( n_{i0} \) and potential strength \( v_{0\delta} \), we have the scattering times (considering screening effects)

\[
\frac{1}{\tau_{i\delta}} = \frac{1}{\tau_{0\delta}} I_\delta(2r_s),
\]

(17a)

FIG. 1. (Color online) Calculated ratio of the transport scattering time to the single particle scattering time \( \tau_{i\delta}/\tau_{sc} \) (a) as a function of the parameter \( r_s = e^2/\gamma \kappa \) and (b) as a function of \( k_F/q_{TF} \). The top (bottom) solid line represents the ratio for charged impurity (short-ranged neutral impurity) scattering. The top (bottom) dashed line represents the ratio of the normal 2D system with parabolic dispersion for charged impurity (short-ranged neutral impurity) scattering. We note that for graphene, \( k_F/q_{TF} = 1/4r_s \).
where $1/\tau_{\delta}=2n_i\sqrt{\frac{\hbar^2}{2m}\gamma}$, and $I_{\delta}$ and $I_{s}$ are given by

$$I_{\delta}(x) = \frac{\pi}{8} - \frac{4}{3}x + \frac{3\pi}{2}x^2 - 2\frac{d}{dx}[x^2g(x)],$$

(18a)

$$I_{s}(x) = \frac{\pi}{4} - \frac{d}{dx}[x^2g(x)].$$

(18b)

In Fig. 1, we show the ratio of the scattering times for the short-ranged scatterers, $\tau_{\delta}/\tau_{s}$, as a function of $r_s$, and the limiting forms are given by

$$\tau_{\delta}/\tau_{s} = \begin{cases} 2 + 32r_s\ln(r_s)/\pi & \text{for } r_s \ll 1 \\ 1 + 128r_s/105\pi & \text{for } r_s \gg 1. \end{cases}$$

(19)

We find that $1 \leq \tau_{\delta}/\tau_{s} \leq 2$ (see Fig. 1). The ratio does not exceed 2 for short-ranged scatterers, but for charged impurity scattering, the ratio is always greater than 2. Thus, the ratio of the scattering times may offer the possibility of determining the relevant scattering mechanisms in disordered graphene layers.

Now, we compare the calculated scattering time ratio of graphene with the scattering time ratio of a normal 2D electron system with parabolic energy dispersion. We have the scattering time ratio for charged impurity (see the Appendix) in a normal 2D layer as

$$\tau_{\delta}/\tau_{s} = \begin{cases} 1 - \frac{4}{\pi}\ln(r_s)/\pi & \text{for } r_s \ll 1 \\ 1 + \frac{4}{3\pi}r_s/\pi & \text{for } r_s \gg 1, \end{cases}$$

(20)

where $r_s=1/4\sqrt{\pi n}$ ($a_B$ is an effective Bohr radius of the system and $n$ is a carrier density). Unlike graphene, in normal 2D systems, $r_s$ depends on the carrier density. In small $r_s$ (or high density) limit, the ratio shows the same leading-order behavior as graphene (i.e., $\tau_{\delta}/\tau_{s} \approx 1/r_s$), but in large $r_s$ (low density) limit, $\tau_{\delta}/\tau_{s} \rightarrow 1$. The difference between the normal 2D system and graphene in the strong screening limit can be traced back to the suppression of the backscattering in graphene. We also find the difference in the scattering time ratio for short-ranged impurity. The ratio becomes $2/3 \leq \tau_{\delta}/\tau_{s} \leq 1$ (see Fig. 1 and the Appendix). Thus, the ratio is always less than 1. In experiment, the ratio of the transport time to the single particle time has been found to be smaller than 1 in Si metal oxide semiconductor field effect transistor (MOSFET) systems. Thus, in a Si MOSFET system, the short-ranged scattering (such as interface roughness scattering) dominates.

It is, in principle, possible for the charged impurities to be at a distance $z_i$ away from the 2D graphene layer. In fact, in 2D modulation-doped GaAs-AlGaAs semiconductor heterostructures, the charged dopants are put a distance $d(=z_i)$ away (inside the GaAlAs insulting barrier region) from the 2D electron gas in order to minimize the degradation of the electron mobility due to remote dopant scattering. As discussed in Sec. I, paper, a large separation $(k_Fz_i>1)$ leads to a very strong enhancement in the $\tau_{\delta}/\tau_{s}$ in the GaAs-GaAlAs heterostructure since large-angle scattering by the remote impurities is strongly suppressed by the separation. To see the effect of separating the impurities from the graphene layer, all we need to do is to modify the form of the $q$-space Fourier transform of the electron-impurity Coulomb interaction by the factor $e^{-qz_i}$ arising from the separation between the 2D carriers in the graphene layer and the charged impurity centers, leading to $V_{ei}(q,z_i)=V_{ei}(q)e^{-qz_i}$. With this simple modification for $V_{ei}$, we can calculate the results for $\tau_{\delta}$ and $\tau_{s}$ for remote scatterers by using the same formalism as above.

In Figs. 1–4, we individually present our theoretical results for $\tau_{\delta}/\tau_{s}$ as well as $\tau_{\delta}$ and $\tau_{s}$ in graphene, comparing it with the corresponding regular 2D results (i.e., nonchiral 2D electron system with parabolic band dispersion). We exactly use the same materials parameters for both graphene and the regular parabolic 2D system. The results are plotted as a function of the dimensionless parameter $r_s$ (or $k_F/q_{TF}$), the impurity location from the interface $z_i$, or the density $n$, where $k_F$ ($q_{TF}$) are, respectively, the Fermi wave vector (Thomas–Fermi wave vector) for the system. We give the expressions for $k_F$ and $q_{TF}$ (with $g$ as the spin and valley degeneracy factor): $k_F=(4\pi n/m)^{1/2}$ for both graphene and parabolic 2D systems, $q_{TF}=\gamma e^{2}k_F/(\lambda s e^2/(kF^2)$ for graphene and $q_{TF}=\gamma e^{2}k_F/(\lambda s e^2/(kF^2)$ for 2D parabolic systems. We note that $q_{TF}/k_F=r_s$ for $g=1$ and $q_{TF}/k_F=x_{TF}$ for any value of $g$ for both the graphene and parabolic 2D systems (in general, $g=4$ for graphene).

The most important qualitative feature of Figs. 1 and 2 is that, in general, $\tau_{\delta}/\tau_{s}$ is larger in graphene than in the corre-
SINGLE-PARTICLE RELAXATION TIME VERSUS...

In Fig. 3, we individually show our calculated graphene transport and single particle times individually as a function of carrier density for various values of the impurity location $z_i$. In Fig. 3(a), wherein charged impurity Coulomb scattering is involved, both $\tau_r$ and $\tau_s$ increase with carrier density, but $\tau_s$ increases much faster, making $\tau_s/\tau_r$ increase with increasing density. However, for short-range scattering [Fig. 3(b)], the reverse is true, with both $\tau_r$ and $\tau_s$ decreasing with increasing density but $\tau_s$ decreasing faster, again leading to an increasing $\tau_s/\tau_r$ with increasing carrier density.

Finally, in Fig. 4, we show our calculated level broadening $\Gamma = \hbar/2\tau_r$ as a function of the carrier density in graphene. We note that for a charged impurity scattering, $\Gamma/\hbar$ monotonically decreases with increasing the carrier density. However, the scaled damping rate due to the short-ranged impurity scattering is independent of the carrier density.

**III. DISCUSSION AND CONCLUSION**

We calculate the transport scattering time ($\tau_i$) and the single particle relaxation time ($\tau_s$) for disordered graphene in the lowest order of the electron-impurity interaction (Born approximation). The scattering mechanisms that we consider are screened charged impurity scattering and short-range potential (e.g., caused by lattice defects). For the screening function, we use the RPA. We find that for short-ranged scatterers, the ratio of the scattering time to the single particle relaxation time is always smaller than 2, but for a charged impurity scattering, the ratio is always greater than 2. These theoretical results provide a technique to experimentally discriminate between short-range and Coulomb scatterings as to the relevant scattering mechanism in disordered graphene layers. We also find a strong dependence of the ratio $\tau_s/\tau_r$ in graphene on $z_i$, which is the separation of the impurities from the 2D graphene layer. Somewhat surprisingly, the dependence of $\tau_s/\tau_r$ on $z_i$ is stronger in graphene than in the corresponding parabolic 2D system, leading to the possibility that an accurate measurement of $\tau_s/\tau_r$ in graphene could lead to better understanding of the impurity location underlying graphene disorder. In particular, $\tau_s/\tau_r$ rapidly increases with increasing $z_i$ in graphene (similar to nonchiral 2D GaAs-GaAlAs modulation-doped heterostructure) and this could be directly experimentally tested.

Before concluding, we will discuss the scattering time due to electron-electron interaction. In this paper, we consider the elastic scattering times due to impurities at zero temperature, so the scatterings take place at the Fermi surface. In general, the quasiparticle scattering time due to electron-electron interaction at the Fermi surface is infinite due to the limitation of the phase space and the quasiparticle is always undamped at the Fermi surface. Thus, the electron-electron relaxation time is much larger than other relaxation times and irrelevant at $T=0$. However, the electron-electron scattering dominates over impurity scatterings at high temperatures and for high energy electrons. The inelastic scattering time due to electron-electron interaction is of the order of 0.1 ps for an energetic electron with an energy of 1 eV above Fermi energy for an electron density $n = 5 \times 10^{12} \text{ cm}^{-2}$, and it decreases as the electron energy increases.\textsuperscript{15}
We conclude by emphasizing that independent measurements of $\tau_i$ and $\tau_s$ in graphene samples could lead to a detailed useful insight into the nature of disorder scattering of graphene carriers.

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**APPENDIX**

Here, we provide the scattering times of a normal 2D system with a parabolic band. By using the 2D RPA screening function\(^{13}\) at $T=0$, we have the scattering times for charged impurity centers,

\[
\frac{1}{\tau_{ic}} = \frac{1}{\tau_{0c}} I_{ic}(q_0),
\]

(A1a)

\[
\frac{1}{\tau_{sc}} = \frac{1}{\tau_{0c}} I_{sc}(q_0),
\]

(A1b)

where

\[
\frac{1}{\tau_{0c}} = 2\pi \hbar n_{ic} \left( \frac{2}{g} \right)^2 q_0^2,
\]

(A2)

$n_{ic}$ is the density of a charged impurity, $q_0 = q_{TF}/2k_F$ ($q_{TF} = g/a_B$ is a 2D Thomas–Fermi wave vector with an effective Bohr radius $a_B = \hbar^2/mv_0^2$), and $I_{ic}(q_0)$ and $I_{sc}(q_0)$ are given by

\[
I_{ic}(q_0) = \pi - 2 \frac{d}{dq_0} \left[ q_0^2 f(q_0) \right], \quad (A3)
\]

\[
I_{sc}(q_0) = - \frac{d}{dq_0} f(q_0), \quad (A4)
\]

where $f(x)$ is given in Eq. (12) and its derivative is given by

\[
\frac{df}{dx} = \frac{1}{x^2 - 1} [1 - x^2 f(x)]. \quad (A5)
\]

For short-ranged impurity centers, we have

\[
\frac{1}{\tau_{i\delta}} = \frac{1}{\tau_{0\delta}} I_{i\delta}(q_0), \quad (A6a)
\]

\[
\frac{1}{\tau_{s\delta}} = \frac{1}{\tau_{0\delta}} I_{s\delta}(q_0), \quad (A6b)
\]

where

\[
\frac{1}{\tau_{0\delta}} = \frac{2n_{i\delta}}{\hbar \hbar} \frac{mv_0^2}{2}. \quad (A7)
\]

$n_{i\delta}$ is the density of a short-ranged impurity, $n_{i\delta}$, $v_0$ is the potential strength, and $I_{i\delta}(q_0)$ and $I_{s\delta}(q_0)$ are given by

\[
I_{i\delta}(q_0) = \frac{\pi}{2} - 4q_0 + 3\pi q_0^2 - 2 \frac{d}{dq_0} \left[ q_0^2 \right] f(q_0), \quad (A8)
\]

\[
I_{s\delta}(q_0) = \frac{\pi}{2} - \frac{d}{dq_0} \left[ q_0^2 \right] f(q_0). \quad (A9)
\]