Toward formalizing ologs: Linguistic structures, instantiations, and mappings

Marco A. Pérez  
Department of Mathematics  
Massachusetts Institute of Technology  
maperez@mit.edu

David I. Spivak  
Department of Mathematics  
Massachusetts Institute of Technology  
dspivak@math.mit.edu

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Abstract

We define the notion of linguistic structure on a small category, in order to provide a more formal description of ontology logs, also known as ologs, introduced in [18] by R. E. Kent and the second author. Out of our formalism emerges a new notion of linguistic functor, which can be understood with almost no category-theoretic background, thus adhering to the aesthetic of [18], and also extending the concept of meaningful functor defined in [14] by the second author. We also present the notion of an olog complex, a network of overlapping ologs arranged as a simplicial complex.

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1 Introduction

The theory of ontology logs (ologs for short) was introduced by Robert Kent and the second author in their paper [18], as a framework for knowledge representation. Ologs are basically mathematical categories that have been wrapped in natural-language English. They have been applied in several branches of science and engineering, [3, 5, 8, 9, 12, 17], as a tool for various kinds of formal modeling.

Typically, a person who wishes to record and document some of her knowledge or ideas will do so in prose, e.g., a scientist publishes ideas in the form of research papers. Ologs offer the ability to express complex ideas using a special type of diagrams. Namely, the objects of study and the relationships between them can be represented as the objects and arrows in a category. The difference between an olog and a category is that an olog has additional structure: each object is labeled with a noun phrase (thus forming a type) and each arrow is labeled with a verb phrase (thus forming an aspect), so that reading source-arrow-target yields an English sentence. There is a formula for composing sentences end-to-end into a new sentence, when following a path of arrows through an olog, and a pair of equivalent paths (also known as a commutative diagram) in the category is understood as a declared fact equating the two English sentences.

The primary goal of this paper is to present a linguistic description of mappings between ologs. As ologs are defined from categories, mappings between ologs should be defined in terms of mappings between categories, namely functors. However, these functors need to respect the linguistic descriptions on each node and arrow. For example, there is an obvious functor (namely the identity) between the following ologs:

\[
\begin{array}{c}
\text{a man} \\
\downarrow \text{is} \\
\text{an object}
\end{array} 
\quad \Rightarrow 
\begin{array}{c}
\text{a woman} \\
\downarrow \text{has as weight (in kilograms)} \\
\text{a number between 20 and 500}
\end{array}
\]

But would this functor mean anything? We will rephrase this question in § 3.1 like this: “is there an author willing to endorse that this functor is linguistic?” In this paper we explain the difference between a (linguistic) map of ologs and a mere functor between their underlying categories. We will address this particular case (1) in Example 3.1.11.

To some extent, this issue was considered in [14], where the authors introduced the concept of meaningful functor. One limitation of their approach is that it depends on a strong assumption, namely that every olog \( \mathcal{C} \) is equipped with a functor \( I: \mathcal{C} \rightarrow \text{Set} \), and that this
functor somehow controls the meaning of the olog. Such a set-valued functor is called an \textit{instantiation}, a term coming from database theory [15]. The idea is that $I$ represents a kind of database of examples, or instances, for the various types, aspects, and facts in the olog. For example, if an author is writing an olog $C$ describing a familiar real-world situation, then for a type $c$ (e.g., $c = \text{a cat}$) the set $I(c)$ represents all the examples of $c$ (all cats) known by the author. Somewhat strangely, however, there is no requirement in [14] that the database functor $I$ should in any way correspond to the linguistic structure on the olog $C$. Similar issues existed for morphisms between ologs: they were not required to respect the linguistic structures, which are what distinguish ologs from categories.

In this paper, we remedy these issues. First, we allow ologs to exist without being instantiated; that is we disentangle ologs and their instantiations. This way, the set of documented examples can evolve over time, without changing the olog to which they refer. On the other hand, we add a constraint to instantiations: for a set-valued functor to count as an instantiation of an olog $C$, it must conform to the linguistic structure, the labelings, on $C$. The same goes for functors between ologs: in order for a functor to count as a mapping between ologs, it must conform to the linguistic structures involved.

We also take more care to explain the relationship between an olog and its authors. We introduce the concept of \textit{endorsement}: an author can endorse that a certain concept or relationship between concepts makes sense, that a certain fact is true, etc. The author set endorsing an olog is part of its structure. This allows us to explicitly define the notion of \textit{olog complex}, which is a network of connected ologs, endorsed by overlapping author sets.

Here is an expert-level view of this paper. To every category $C$, we define a category $\text{Ling}(C)$ of linguistic structures $\mathcal{L}$ on $C$. An olog is defined as a pair $(C, \mathcal{L})$, consisting of a category and a linguistic structure on it. This construction is contravariant in the base category $C$, and allowing the base category to vary, we get a fibration $\text{Olog} \rightarrow \text{Cat}$, where $\text{Olog}$ denotes the category of ologs, and $\text{Cat}$ the category of small categories. Instantiated ologs are defined similarly: to each category $C$ we define a category $\text{Inst}(C)$ of instantiated linguistic structures $(\mathcal{L}, I)$ on $C$, and again we obtain a fibration $\text{InstOlog} \rightarrow \text{Cat}$, where $\text{InstOlog}$ denotes the category of instantiated ologs. An (instantiated) olog complex is roughly a simplicial complex whose $n$-simplices are equipped with ologs that are agreed upon by $n + 1$ authors. All of this will be explained in the main sections of the paper.

This paper is organized as follows: § 2 introduces the category of linguistic structures $\text{Ling}(C)$, as well as the category $\text{Inst}(C)$ of instantiated linguistic structures, on a category $C$. In § 3 we define mappings between ologs and also mappings between instantiated ologs. To do so, we introduce the notion of \textit{linguistic functors} and \textit{instantiated functors}, as mentioned above. The latter of these is an adaptation of the “meaningful functor” notion defined in [14].
Most of the results have a linguistic element and are not purely mathematical. However, with the help of several linguistic postulates scattered throughout the paper, fairly formal proofs are possible. All of these proofs are given in full; however, being written mainly for a non-mathematical audience, they are saved for the end; see the Appendix § A.

1.1 Background and Notation

We will assume the reader is familiar with some basic concepts from category theory, such as opposite categories, isomorphisms, functors, and natural transformations. Readers without category-theoretic background may still benefit from reading the less categorical definitions and results, skimming the category theory, and trying to digest the examples. The books [1, 16] are good sources for category theory, with many illustrations. Some knowledge on fibrations, Grothendieck constructions, and cartesian arrows will be helpful, although certainly not vital, to have a deeper understanding of § 3. We refer the reader to the books [2, 11, 10], or online to [13, 20], to recall these concepts.

The word “category” will always mean a small category unless otherwise stated, i.e., the collections $\text{Ob}(C)$ and $\text{Mor}(C)$ of objects and morphisms (also called arrows), respectively, are sets. By a path of length $n$ in a category we mean a composition of $n$ morphisms $f = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$. Suppose that $p, q \in \text{Ob}(C)$ are objects and that $f$ and $g = g_m \circ g_{m-1} \circ \cdots \circ g_2 \circ g_1$ are paths between them. We will represent that there is an equality $f = g$ of these composites in $C$ by drawing a checkmark symbol $\checkmark$, as in the following commutative diagram:

$$
\begin{array}{ccccccc}
& c_2 & f_2 & c_3 & \cdots & c_{n-1} & f_{n-1} & c_n \\
\lower{1.5em}\hline
f & & & & & & & f \\
\lower{1.5em}\hline
p & c_1 & f_1 & c_2 & \cdots & c_{n-1} & f_{n-1} & c_n \\
p & & & & & & & \checkmark \\
p & & & & & & & g \\
g & & & & & & & q \\
g & d_2 & g_2 & d_3 & \cdots & d_{m-1} & g_{m-1} & d_m \\
g & & & & & & & \checkmark \\
g & & & & & & & p \\
g & & & & & & & f \\
g
\end{array}
$$

Figure 1: Commutative diagram: equality between two paths.

2 Ologs and instantiated ologs

This section is devoted to formalizing the definition of ologs found in [18], using a new construction that we will call a linguistic structure. We suggest that readers who are unfamiliar with ologs consult [14, §1, 2, 3, and 4] or [18, §1, 2, and 3]. Note that the later sections of
these papers also discuss pullbacks and pushouts within an olog; we do not address these
notions in this paper, for the sake of “brevity”.

2.1 Linguistic structures

Recall that an olog, as defined in [18], is a category whose objects and morphisms are
labeled with noun phrases (i.e., types) and verb phrases (i.e., aspects), respectively, in
order to model a conceptual situation. Commutative diagrams in an olog are called facts;
they are equivalences between two sentences in the English language. Importantly, these
types, aspects, and facts must follow certain guidelines, to ensure that the category-theoretic
meaning is aligned with the conceptual and linguistic meaning intended by the authors.
The Rules of Good Practice stated in [18, 2.1.2, 2.2.3, and 2.3.4] are such a set of guidelines.
The most important of these rules is that the label of each arrow $a \rightarrow b$ corresponds to
a mathematical function [21]. All of these notions are author-dependent, as we will make
explicit using the notion of endorsement, as in Definition 2.1.1.

Functional sentences

Suppose that $N$ represents the noun phrase “a person”. Then we denote this by $\langle N \rangle$=“a
person”; we call $\langle N \rangle$ the reading of $N$ (and say that $N$ is read $\langle N \rangle$). Similarly for verb
phrases. If $\langle V \rangle$=“is” and $\langle N' \rangle$=“a mammal”, then we denote the concatenated string, “a
person is a mammal” by $\langle N \rangle \langle V \rangle \langle N' \rangle$ or simply by $\langle NVN' \rangle$.

Definition 2.1.1 Let $N$ be a noun phrase. We will say that an author $s$ endorses $N$, denoted $s \models N$, if $\langle N \rangle$ refers to a distinction made and recognizable by $s$, for which examples
can be documented by $s$.

The idea that $s$ can provide examples of any type he or she endorses will be better explained
in Definition 2.2.2.

Definition 2.1.2 Given two noun phrases $N_1$ and $N_2$ and a verb phrase $V$, one says that
$V$ functionally connects $N_1$ to $N_2$ if the concatenation $\langle N_1VN_2 \rangle$ is an English sentence
that refers to a mathematical function. In this case one says that $\Sigma = (N_1,V,N_2)$ is a
functional sentence.

If an author $s$ is willing to declare that $\Sigma$ is a functional sentence and understands how
examples of $N_1$ correspond via $V$ to examples of $N_2$, we write $s \models \Sigma$, and say that $s$
endorses $\Sigma$. 

For example, given two noun phrases [a person] and [a woman],\(^1\) the arrow
\[
\begin{array}{c}
\text{a person} \\
\text{has as mother} \\
\text{a woman}
\end{array}
\]
can be read as a sentence, “a person has as mother a woman”. This sentence expresses that for anything that could be called “a person” there is something it “has as mother” that can itself be called “a woman”. This seems true, but more important for us, it seems to represent a function: each person has only one mother. Each person who understands these concepts probably has an example of a person (namely himself) and the woman that corresponds to the mother he has. Thus the arrow text is a verb phrase that functionally connects [a person] and [a mother] in the sense of Definition 2.1.2.

We can contrast the above situation with the following:
\[
\begin{array}{c}
\text{a woman} \\
\text{has as dog?} \\
\text{a canine}
\end{array}
\]

Because not every woman has a dog, and some women have two dogs, the arrow text is a verb phrase that does not functionally connect its source and target noun phrases.

**Definition 2.1.3** Let \(\Sigma_1 = (N_1, V_1, N_2)\) and \(\Sigma_2 = (N'_2, V_2, N_3)\) be two sentences, with \(N_2 = N'_2\). In this case we say that \(\Sigma_1\) and \(\Sigma_2\) are **concatenable** and that their **concatenation**, denoted \(\Sigma_1; \Sigma_2\), is the sentence read
\[
\langle \Sigma_1; \Sigma_2 \rangle := \langle N_1 V_1 N_2 \rangle \text{“which”} \langle V_2 N_3 \rangle.
\]

In the concatenation \(\Sigma_1; \Sigma_2\), the verb phrase \(\langle V_1 N_2 \rangle \text{“which”} \langle V_2 \rangle\) will be called the **concatenated verb phrase** and may be denoted by the symbol \(V_1; V_2\).

It follows that for any two concatenatable sentences \(\Sigma_1\) and \(\Sigma_2\) as in Definition 2.1.3, we have
\[
\langle \langle \Sigma_1; \Sigma_2 \rangle \rangle = \langle \langle N_1 V_1; V_2 N_3 \rangle \rangle.
\]

**Example 2.1.4** Consider the diagram below:
\[
\begin{array}{c}
\text{an amino acid} \\
\text{has} \\
\text{an amine group} \\
\text{includes} \\
\text{a nitrogen atom}
\end{array}
\]
The sentences \(\Sigma_1 := \text{“an amino acid has an amine group”}\) and \(\Sigma_2 := \text{“an amine group includes a nitrogen atom”}\) are concatenatable, and their concatenation is \(\Sigma_1; \Sigma_2 := \text{“an amino acid has an amine group, which includes a nitrogen atom”}\).

\(^1\)As in [18, §2.1], we write [a person] instead of \(\text{a person}\) because some typographical problems emerge when writing a text-box in a line of text. Note that the text-box seems out of place in this paragraph, and so many in-line text-boxes are troublesome for the aesthetic of this paper.
A reader may not be able to endorse either of these as functional (does every amine group really contain exactly one amino acid, or can their be none or many?). But in fact, they are broadly endorsed. Postulate (2.1.7) says that an author who endorses concatenatable sentences Σ₁ and Σ₂ must also endorse their concatenation Σ₁; Σ₂.

Declared equivalences

Definition 2.1.5 Let \{Σᵢ = (Nᵢ, Vᵢ, Nᵢ₊₁)\}_{1≤ᵢ≤n} and \{Γᵢ = (Mᵢ, Wᵢ, Mᵢ₊₁)\}_{1≤ᵢ≤m} be two families of concatenatable sentences satisfying N₁ = M₁ and Nₙ₊₁ = Mₘ₊₁. Let Σ := Σ₁; ...; Σₙ and Γ := Γ₁; ...; Γₘ be their concatenations, obtained according to Definition 2.1.3. One says that Σ and Γ are equivalent, denoted Σ ≡ Γ, if the two concatenated sentences Σ and Γ refer to the same function relationship. We refer to the equivalence Σ ≡ Γ as a fact. If Σ ≡ Γ, then their concatenated verb phrases are equivalent as well, V₁; ...; Vₙ ≡ W₁; ...; Wₘ.

Let s be an author such that \(s \equiv Σᵢ\) and \(s \equiv Γⱼ\), for every 0 ≤ i ≤ n and 0 ≤ j ≤ m (and so \(s \equiv Σ\) and \(s \equiv Γ\), by Postulate 2.1.7). If s is willing to declare that Σ ≡ Γ is an equivalence, i.e., a fact, we write s ≡ [Σ ≡ Γ], and say that s endorses Σ ≡ Γ.

As in [16, §2.3.3.4], an equivalence Σ ≡ Γ can be given the following English-language interpretation (where \(x, y₁,\) and \(y₂\) are just symbols, to be copied verbatim):

“For any” Ω(\(N₁\)) \(x\), “we know that” Ω(\(V₁\)\(N₂\)), “which” Ω(\(V₂\)\(N₃\)), “which” Ω(... Ω(\(Vₙ₋₁\)\(Nₙ\)), “which” Ω(\(Vₙ\)\(Nₙ₊₁\)) “that we call” \(y₁\), “and we know that” Ω(\(W₁\)\(M₂\)), “which” Ω(\(W₂\)\(M₃\)), “which” Ω(... Ω(\(Wₘ₋₁\)\(Mₘ\)), “which” Ω(\(Wₘ\)\(Mₘ₊₁\)) “that we call” \(y₂\); and the fact is, \(y₁\) and \(y₂\) are the same for any \(x\).

Figure 2: How facts are interpreted as English.

Example 2.1.6 The following diagram represents a fact, as indicated by the checkmark symbol ✓:

```
\begin{tikzpicture}
    \node (a) {a person};
    \node (b) [right of=a] {an address};
    \node (c) [below of=b] {a city};
    \draw [->] (a) -- (b) node [midway, above] {lives at};
    \draw [->] (b) -- (c) node [midway, below] {includes};
    \draw [->, double] (a) to (c) node [midway, left] {lives in};
\end{tikzpicture}
```

First, note that the concatenation of the top and right arrows yields the sentence “a person lives at an address, which includes a city”, and the diagonal arrow is read “a person lives in a city”. According to Figure 2, the equivalence between these two sentences is read as the assertion
For any person \( x \), we know that \( x \) lives at an address which includes a city \( y_1 \), and we know that \( x \) lives in a city \( y_2 \); and the fact is, \( y_1 \) and \( y_2 \) are the same for any \( x \).

Contrast that fact to the following diagram

![Diagram]

which *does not* represent a fact (note the absence of the checkmark symbol \( \checkmark \)). In this case, the assertion indicating the equivalence between the two involved sentences would be read

For any person \( x \), we know that \( x \) lives in a city, which has its most affluent residence at a house number \( y_1 \), and we know that \( x \) lives at an address, which includes a house number \( y_2 \); and the fact is, \( y_1 \) and \( y_2 \) are the same for any \( x \).

This assertion is quite dubious, because there seem to be many city-dwellers whose house numbers are different from those of the most affluent resident in their city. Hence, we as authors cannot endorse it as a fact.

**Some postulates about endorsement**

Recall that a postulate is an idea suggested or assumed as true as the basis for reasoning, discussion, or belief. We make a few linguistic postulates throughout this document, starting with the next one.

**Linguistic postulate 2.1.7** Let \( \Sigma_1 \) and \( \Sigma_2 \) be two concatenatable sentences as in Definition 2.1.3. If \( s \) is an endorsing author for both sentences \( s \models \Sigma_1 \) and \( s \models \Sigma_2 \), then we will assume that \( s \) endorses their concatenation, \( s \models \Sigma_1 ; \Sigma_2 \).

**Definition 2.1.8** We say that a symbol \( L \) is a **linguistic expression** if it represents a noun phrase, a verb phrase, a sentence or a fact.

**Definition 2.1.9** We say that a linguistic expression \( L \) is **universally endorsed** if, for every author \( s \) with any conceptual scheme, we have \( s \models L \).
For example, the following postulate says that every olog author must endorse the sentence “a bottle is of course a bottle” as functional. We will later postulate (in 2.2.6) that this sentence encodes the identity function.

**Linguistic postulate 2.1.10** We will assume that there is a unique verb phrase $e$, read $\langle e \rangle = \text{“is of course”}$ such that for every noun phrase $N$, the sentence $(N, e, N)$ is universally endorsed. We call $e$ the **unit verb phrase**.

The following postulate says that the unit verb phrase introduced above is unital with respect to the string concatenation defined in Definition 2.1.3.

**Linguistic postulate 2.1.11** Let $e$ be the unit verb phrase as in Postulate 2.1.10. For every sentence $\Sigma = (N_1, V, N_2)$, the equivalences

$$\Sigma; (N_2, e, N_2) \simeq \Sigma$$

and

$$(N_1, e, N_1); \Sigma \simeq \Sigma$$

are universally endorsed.

**Remark 2.1.12** Associativity of concatenation is universally endorsed as well, but this does not require a postulate. It follows from the associativity of string concatenation.

**Definition 2.1.13** Let $L$ be a linguistic expression. We say that an author set $S$ **endorses** $L$, denoted $S \models L$, if

- for each $s \in S$ there is an endorsement $s \models L$, and
- the members of $S$ have agreed to a common understanding of $L$.

**Remark 2.1.14**

1. Note that if $S \models L$ and $S' \subseteq S$, then $S' \models L$. In other words, endorsement by an author set requires a sort of honesty: if an author $s$ endorses a concept when he or she is part of a larger group $T$, he or she must endorse it when she is in any subgroup $S \subseteq T$.

2. Postulates 2.1.7, 2.1.10, and 2.1.11 will be also assumed for every author set $S$. Specifically, the following statements are valid for every author set $S$:

   a. For every pair $\Sigma_1$ and $\Sigma_2$ of concatenatable sentences, if $S \models \Sigma_1$ and $S \models \Sigma_2$ then $S \models \Sigma_1; \Sigma_2$.

   b. $S \models (N, e, N)$, for every noun phrase $N$.

   c. $S \models [\Sigma; (N_2, e, N_2) \simeq \Sigma]$ and $S \models [(N_1; e, N_1); \Sigma \simeq \Sigma]$, for every $\Sigma = (N_1, V, N_2)$. 


**Definition of linguistic structure on \( C \)**

We are now ready to define the notion of linguistic structure on a category. In Definition 2.1.20 we will discuss how two linguistic structures on the same category can be related.

**Definition 2.1.15** A **linguistic structure** \( \mathcal{L} \) on a category \( C \) consists of a finite set of authors \( S \), together with the following data:

1. Every object \( c \in \text{Ob}(C) \) is assigned a noun phrase, denoted \( \mathcal{L}(c) \), such that \( S \upharpoonright \mathcal{L}(c) \).
2. Every arrow \( f : c \rightarrow c' \) in \( \text{Mor}(C) \) is assigned a verb phrase, denoted \( \mathcal{L}(f) \), such that \( S \upharpoonright \mathcal{L}(f) \).

For each object \( c \in \text{Ob}(C) \), the identity arrow \( \text{id}_c : c \rightarrow c \) is assigned the verb phrase \( \mathcal{L}(\text{id}_c) = e \), with \( e \) the unit verb phrase as in Definition 2.1.5.

3. For every commutative diagram in \( C \) of the form specified in Figure 1, the corresponding concatenated sentences, obtained according to Definition 2.1.3, are endorsed as equivalent by \( S \), i.e.,

\[
S = [(\mathcal{L}(p), (\mathcal{L}(f_1), \ldots, \mathcal{L}(f_n)), \mathcal{L}(q)) = (\mathcal{L}(p), (\mathcal{L}(g_1), \ldots, \mathcal{L}(g_m)), \mathcal{L}(q))] \]

We call \( S \) the **author set** of \( \mathcal{L} \) and denote it by \( \text{Auth}(\mathcal{L}) = S \).

An **olog** is a pair \((C, \mathcal{L})\), where \( C \) is a category and \( \mathcal{L} \) is a linguistic structure on \( C \).

**Example 2.1.16** We run through Definition 2.1.15 in the case of linguistic structure shown here:

Each object has been assigned a noun phrase, and each arrow has been assigned a verb phrase. The two paths from 1 to 3, namely 1 \( \rightarrow \) 3 and the composition of 1 \( \rightarrow \) 2 followed by 2 \( \rightarrow \) 3, have been declared equivalent. This fact is read.
For any person $x$, we know that $x$ has as parents a pair $(w, m)$ where $w$ is a woman and $m$ is a man, which yields as $w$ a woman $y_1$, and we know that $x$ has as mother a woman $y_2$; and the fact is, $y_1$ and $y_2$ are the same for any $x$.

Identity morphisms are not drawn in our pictures, but for example the identity on $[\text{a person}]$ must be assigned the sentence “a person is of course a person” by $\mathcal{L}$. There is a universally-endorsed fact that the

For any person $x$, we know that $x$ is of course a person which has as mother a woman $y_1$, and we know that $x$ has as mother a woman $y_2$; and the fact is, $y_1$ and $y_2$ are the same for any $x$.

**Morphisms of linguistic structures on $\mathcal{C}$**

**Definition 2.1.17** Let $S$ and $T$ be two author sets and let $d: S \to T$ be a function. For any $S' \subseteq S$, let $d(S') \subseteq T$ denote its image. We say that a function $d$ is a **delegation** if for every $S' \subseteq S$ and every linguistic expression $L$, the following implication is true:

$$\text{if } d(S') \models L, \text{ then } S' \models L.$$

**Remark 2.1.18** Note that Definition 2.1.17 implies that if $T$ is an author set containing $S$, then the inclusion $S \to T$ is a delegation, and thus every author in $S$ is his or her own delegate, as a member of $T$ (See Remark 2.1.14).

Like all results in this paper, the proof of the following proposition can be found in § A.

**Proposition 2.1.19** The collection of author sets and delegations, denoted $\text{Del}$, is a category. There is a faithful functor $U: \text{Del} \to \text{Set}$, sending each author set to its underlying set.

Thus Remark 2.1.18 says that if $U(S) \subseteq U(T)$, then there is an associated delegation $d: S \subseteq T$.

**Definition 2.1.20** Suppose given two linguistic structures $\mathcal{L}$ and $\mathcal{L}'$ on a category $\mathcal{C}$, with $S = \text{Auth}(\mathcal{L})$ and $S' = \text{Auth}(\mathcal{L}')$. A morphism of linguistic structures, denoted by $\alpha: \mathcal{L} \to \mathcal{L}'$, consists of the following data:

1. A delegation $\alpha_0: S' \to S$.

2. Every object $c \in \text{Ob}(\mathcal{C})$ is assigned a verb phrase $\alpha(c)$, called the $c$-component of $\alpha$, and denoted $\alpha(c): \mathcal{L}(c) \to \mathcal{L}'(c)$, such that $S' \equiv (\mathcal{L}(c), \alpha(c), \mathcal{L}'(c))$.

---

Footnote 2: One may wonder why the delegation $\alpha_0$ maps $S'$ to $S$ rather than the other direction. In fact, both directions are mathematically reasonable. We prefer the one shown here because it seems to fit better with practice, as will be discussed in Remark 4.2.4.
(3) For every arrow $f: c \to c'$ in $\mathcal{C}$, there is an endorsement $S' = [\alpha(c); \mathcal{L}'(f) \simeq \mathcal{L}(f); \alpha(c')]$ as shown in Figure 3.

![Figure 3: A morphism between linguistic structures on the same category](image)

The first condition in Definition 2.1.20 says that the authors $S' = \text{Auth}(\mathcal{L}')$ agree that both $\mathcal{L}$ and $\mathcal{L}'$ are valid linguistic structures whenever their corresponding delegates $\alpha_0(S')$ also agree. They can thus evaluate whether they endorse each component $\alpha(c)$ as a functional verb phrase, and each equivalence $\alpha(c); \mathcal{L}'(f) \simeq \mathcal{L}(f); \alpha(c')$ as a fact. If they do, then they endorse $\alpha: \mathcal{L} \to \mathcal{L}'$ as a morphism of linguistic structures.

**Remark 2.1.21** Both the arrows between objects in an olog, denoted $\rightarrow$, and the component arrows for maps between ologs, denoted $\rightrightarrows$, are assigned functional verb phrases. Although we use differently-shaped arrows to denote them, the equivalence in Figure 3 is of the usual kind, as in Definition 2.1.5. It can be read in English, as in Figure 2:

"For any" $\langle \mathcal{L}(c) \rangle x$, "we know that" $x \mathcal{L}(f) \mathcal{L}(c')$, "which" $\langle \alpha(c') \mathcal{L}'(c') \rangle$ "that we call" $y_1$, "and we know that" $x \langle \alpha(c) \mathcal{L}(c) \rangle$, "which" $\langle \mathcal{L}'(f) \mathcal{L}'(c') \rangle$ "that we call" $y_2$; and the fact is, $y_1$ and $y_2$ are the same for any $x$.

**Example 2.1.22** Consider the following linguistic structures $\mathcal{L}$ and $\mathcal{L}'$ on the category $\bullet_2 \leftarrow \bullet_1 \rightarrow \bullet_3$:

![Linguistic structures](image)
The phrase “legitimate child” is an old-fashioned term for a child who was born in a marriage, which itself also required to be between a man and a woman. Let \( \text{Auth}(\mathcal{L}) = \text{Auth}(\mathcal{M}) \) be an author set who endorse these definitions, and let \( \alpha_0 : \text{Auth}(\mathcal{M}) \to \text{Auth}(\mathcal{L}) \) be the identity function. Suppose these authors also endorse the following sentences as functional:

"a legitimate child was born in a marriage"

"a father is a man"

"a mother is a woman".

In other words, they endorse \( \langle \alpha(1) \rangle := \text{"was born in"} \); \( \langle \alpha(2) \rangle := \text{"is"} \); and \( \langle \alpha(3) \rangle := \text{"is"} \) as component verb phrases \( \alpha(c) \) for \( c \in \{1, 2, 3\} = \text{Ob}(\mathcal{C}) \).

At this point, they have endorsed every object and arrow (both \( \to \) and \( \leadsto \) in the diagram below:

```
2
|-----------------|-----------------|
| a father        | a man           |
|                 |                 |
| is              |                 |
|                 |                 |
| has             | includes        |
|                 |                 |
| 1               |                 |
|                 |                 |
| a legitimate child | was born in   |
|                 |                 |
| has             | includes        |
|                 |                 |
| 3               |                 |
|                 |                 |
| a mother        | a woman         |
|                 |                 |
| is              |                 |
```

Figure 4: Example of a morphism between two linguistic structures on \( \mathcal{C} \).

The checkmarks are drawn if the authors also endorse the corresponding facts. For example, a legitimate child has a father who is a man, and a legitimate child was born in a marriage, which includes a man. The point is, they had better be the same man!

If the authors \( \text{Auth}(\mathcal{L}) \) endorse the three components of \( \alpha \) and the two facts, as shown above, then they have endorsed a morphism of linguistic structures \( \alpha : \mathcal{L} \to \mathcal{M} \).
The category of linguistic structures on $\mathcal{C}$

Now that we have a solid notion of linguistic structures on a category $\mathcal{C}$ and the morphisms between them, we are ready to define a category.

**Proposition 2.1.23** With objects as in Definition 2.1.15 and morphisms as in Definition 2.1.20, $\text{Ling}(\mathcal{C})$ forms a category, and authorship defines a functor $\text{Auth}: \text{Ling}(\mathcal{C}) \rightarrow \text{Def}^{op}$.

Different ologs on the same underlying category have been considered before in other fields, such as biology and materials science [7]. In [7, Figure 2] for example, such ologs are compared via the notion of functor, and two ologs are said to be analogous if there exists a functorial isomorphism between them. Other interpretations of analogies are given in [4, 17]. Motivated by this, we present next in Definition 2.1.24 a notion of analogy which fits all these and which is based on Definition 2.1.20. Later on we will use this notion of analogy to introduce a special type of linguistic functors (See Definition 3.1.8).

**Definition 2.1.24** A morphism $\alpha: \mathcal{L} \rightarrow \mathcal{L}'$ of linguistic structures on $\mathcal{C}$ is said to be a formal analogy if it is an isomorphism in $\text{Ling}(\mathcal{C})$, i.e., if there is a bijection between $\text{Auth}(\mathcal{L})$ and $\text{Auth}(\mathcal{L}')$, and if there exists a morphism $\alpha': \mathcal{L}' \rightarrow \mathcal{L}$ of linguistic structures on $\mathcal{C}$, endorsed by all authors, such that for every $c \in \text{Ob}(\mathcal{C})$ one has that $\alpha'(c): \alpha(c) \simeq e$ and $\alpha(c): \alpha'(c) \simeq e$. In this case we say that $\mathcal{L}$ and $\mathcal{L}'$ are (formally) analogous.

**Example 2.1.25** We present below two analogous ologs on the same category. Those who live in a presidential republic identify the current president as the head of government. The US, Mexico, and Brazil are examples of such countries with a presidential system. The situation is different in countries for which the government system is parliamentary, such as the UK, Australia, or Spain, where the head of government is represented by the prime minister. Many people would agree that the title of president is analogous to that of prime minister. We describe this perspective by two ologs on $\mathcal{C} = \bullet_1 \rightarrow \bullet_2$ and a formal analogy between them:
A morphism $\alpha: \mathcal{L} \to \mathcal{L}'$ between these linguistic structures is showed below:

In other words, the head of a parliamentary government $X$ is a prime minister, which in a presidential system corresponds to a president, namely the head of the presidential government corresponding to $X$. Because this morphism has an inverse $\beta: \mathcal{L}' \to \mathcal{L}$, constructed analogously, for which the composites $\alpha \circ \beta$ and $\beta \circ \alpha$ are “of course” the identity, the linguistic structures $\mathcal{L}$ and $\mathcal{L}'$ are formally analogous.

### 2.2 Instantiations

In this section we study instantiated ologs. These are ologs for which each type has been assigned a set of examples. The authors of an olog should, and generally do, know more than just some types and relationships; they should also have in mind some examples of these types and relationships. For example, someone who writes an olog about dogs, say including the arrow $\text{a dog} \rightarrow \text{a name}$, probably knows some examples of dogs and their names. This information can be stored in an instantiation of the olog, which we will define in Definition 2.2.7.

Instantiating an olog—filling it with conforming data—serves three purposes:

- **i.** It gives users a place to store data about—examples of—their subject of interest.
- **ii.** It validates the olog as a mathematical structure.
- **iii.** It differentiates between different author sets who endorse the same conceptual scheme (olog).

The first of these purposes is probably the most important, but it is also straightforward, so we briefly explain the other two. Issues of functional connectivity and endorsed facts (see Definition 2.1.2 and 2.1.5) rely on the authors’ understanding of mathematical functions and their compositions. By instantiating an olog, the users validate that understanding.
Another reason to instantiate an olog is to differentiate one group of authors from another, even if they use the same conceptual scheme. For example, consider the following linguistic structure:

\[ L = \begin{array}{c}
  \text{a person} \\
  \xrightarrow{\text{has}} \\
  \text{a father}
\end{array} \]  

One author set may be interested in the fathers of US politicians (e.g., George W. Bush’s father is George H. W. Bush), whereas another set may be interested in the fathers of famous mathematicians (e.g., Emmy Noether’s father is Max Noether). The same olog can house multiple instantiations.

The mathematical motivation behind instantiations comes from the concept of set-valued functors \( C \to \text{Set} \) as database instances, introduced by [6] and rediscovered by the second author in [15]. The same notion was defined for ologs in [18], where it was assumed that every olog, say \( (C, L) \), comes equipped with such a functor \( C \to \text{Set} \). We find three problems with this:

i. An olog can exist before one has recorded the corresponding examples.

ii. The examples should have something to do with the linguistic structure.

iii. Two authors may have the same olog but different examples.

We have commented on iii above, and i is straightforward. We explain ii., which is probably the most important, in Remark 2.2.1.

**Remark 2.2.1** In [18], there was no assurance that the functor \( I: C \to \text{Set} \) had anything to do with the linguistic structure on \( C \). So a type \( c = [\text{a dog}] \) would be mapped to a set, but there was nothing ensuring that it was a set of dogs. Of course, such a thing cannot be ensured mathematically, but in Definition 2.2.2 we do the next best thing, and provide a sentence for authors to endorse.

**Definition 2.2.2** Recall the notion of endorsement for noun phrases and functional verb phrases, from Definitions 2.1.1 and 2.1.2.

1. Let \( N \) be a noun phrase. One says that \( x \) is a **token** of \( N \) if \( x \) is an example to which the noun phrase \( \{N\} \) applies. If \( s \) is an endorsing author \( s \models N \), and \( x \) is a token of \( N \) according to \( s \), we say that \( s \) **endorses** \( x \) **as a token of** \( N \); this will be denoted by \( s \models (x : N) \).

2. Let \( s \) be an author who endorses a sentence \( s \models (N_1, V, N_2) \) and tokens \( s \models (x : N_1) \)

---

\(^3\)The motivation behind the term “token” comes from the usual usage in philosophy, e.g., [19]. Tokens are real-world examples and instances of abstract types. In this paper the word “example” has a wider connotation, while the term “instance” will be reserved for Definition 2.2.7.
and \( s = (y : N_2) \). Suppose that \( s \) agrees that the following sentence, denoted \( V(x, y) \), is true:

\[
V(x, y) \equiv x \text{ “is” } \langle N_1 \rangle \text{ “which” } \langle V N_2 \rangle, \text{ “namely” } y.
\]

(4)

In other words, \( s \) agrees that \( x \) corresponds to \( y \) via the function corresponding to \( V \). In this case, we write \( s \equiv V(x, y) \) and say that \( s \) **endorses the correspondence** \( V(x, y) \) **between** \( x \) and \( y \).

**Example 2.2.3** Consider the person-father olog \((C, \mathcal{L})\) from Figure (3), reproduced here:

```
| a person | has | a father |
```

An author \( s \) might endorse that George W. Bush is (a token of the type) a person, *i.e.*, \( s \equiv (\text{George W. Bush} : \text{a person}) \), and similarly that George H. W. Bush is (a token of the type) a father. Suppose, following (4) that \( s \) also agrees with the sentence:

“George W. Bush is a person, which has a father, namely George H. W. Bush.”

Then \( s \) endorses that “has” is a correspondence between George W. Bush as a person and George H. W. Bush as a father.

**Definition 2.2.4** Let \( S \) be an author set.

(1) Let \( N \) be a noun phrase and \( x \) be a token of \( N \). We say that \( S \) **endorses \( x \) as a token of \( N \)**, denoted \( S \equiv (x : N) \), if \( S \equiv N \) as in Definition 2.1.13, and all \( s \in S \) have agreed to a common sense in which \( x \) is an example of the concept \( \langle N \rangle \) (and thus \( s \equiv (x : N) \) for every \( s \in S \)).

(2) Let \((N_1, V, N_2)\) be a sentence, and \( x \) and \( y \) be tokens of \( N_1 \) and \( N_2 \), respectively. We say that \( S \) **endorses the correspondence** \( V(x, y) \) **between** \( x \) and \( y \), denoted \( S \equiv V(x, y) \), if \( S \equiv V \) as in Definition 2.1.13, and all \( s \in S \) have agreed to a common sense in which \( V(x, y) \) is a valid relationship of examples (and thus \( s \equiv V(x, y) \) for every \( s \in S \)), and that they can defend it as members of any \( S' \subseteq S \).

**Remark 2.2.5** Note that we have \( S' \equiv (x : N) \) and \( S' \equiv V(x, y) \), for every \( S' \subseteq S \).

We now postulate that the unit verb phrase and composition of verb phrases act like the identity function and function composition for instances.

**Linguistic postulate 2.2.6** Let \( S \) be an author set.

(1) Suppose given endorsements for concatenatable sentences, \( S \equiv (N_1, V_1, N_2) \) and \( S \equiv (N_2, V_2, N_3) \). Note that \( S \equiv (N_1, (V_1; V_2), N_3) \) by Postulate 2.1.7 and Remark 2.1.14
(2) Suppose also that $S$ endorses tokens $S \models (x : N_1), (y : N_2), (z : N_3)$ and correspondences $S \models V_1(x, y)$ and $S \models V_2(y, z)$. Then we will assume that $S \models (V_1; V_2)(x, z)$.

(2) Suppose given an endorsement $S \models (x : N)$ for some noun phrase $N$. Then we will assume that $S \models e(x, x)$, with $e$ as the unit verb phrase defined in Postulate 2.1.10.

Definition 2.2.7 Let $\mathfrak{L}$ be a linguistic structure on a category $\mathcal{C}$, with $S = \text{Auth}(\mathfrak{L})$. We say that a functor $I: \mathcal{C} \to \text{Set}$ conforms to $\mathfrak{L}$ if the following two conditions are satisfied:

1. For every object $c \in \text{Ob}(\mathcal{C})$, each element $x \in I(c)$ is endorsed by $S$ as a token of $\mathfrak{L}(c)$, i.e., $S \models (x : \mathfrak{L}(c))$.

2. For every arrow $f: c \to c'$ in $\mathcal{C}$ and every element $x \in I(c)$, the correspondence $\mathfrak{L}(f)(x, I(f)(x))$ between $x$ and $I(f)(x)$ is endorsed by $S$, i.e., $S \models \mathfrak{L}(f)(x, I(f)(x))$.

Given an olog $(\mathcal{C}, \mathfrak{L})$ and a functor $I: \mathcal{C} \to \text{Set}$ conforming to $\mathfrak{L}$, we refer to $I$ as an instantiation of $(\mathcal{C}, \mathfrak{L})$, to the pair $(\mathfrak{L}, I)$ as an instantiated linguistic structure on $\mathcal{C}$, and to the whole triple $(\mathcal{C}, \mathfrak{L}, I)$ as an instantiated olog.

It is this notion, that an instance should conform to the linguistic structure, which we find missing in [18]; see Remark 2.2.1.

Example 2.2.8 Recall again the person-father olog $(\mathcal{C}, \mathfrak{L})$ from Figure 3, reproduced here:

We adopt a tabular description similar to that used in [15], except with column headings taken from (4). Using it, we can record the data of a functor $I: \mathcal{C} \to \text{Set}$ as follows:

| a person | has | a father |
|----------|-----|----------|
| George W. Bush | a father, namely | George W. Bush |
| Jeb Bush | George H. W. Bush |
| Emmy Noether | Max Noether |

This table then shows two correspondences, associated to the arrow labeled “has”:

- George W. Bush is a person, which has a father, namely George H. W. Bush; and
- Emmy Noether is a person, which has a father, namely Max Noether.

An author who endorses the four tokens and two correspondences shown here then also endorses $I$ as an instantiation of the olog $(\mathcal{C}, \mathfrak{L})$.

\footnote{For a sentence $(N_1, V, N_2)$, the instance will fit into a table with $\langle N_1 \rangle$ as the head of the first column and $\langle VN_2 \rangle$ as the head of the second column.}
Definition 2.2.9 Let $S$ and $T$ be two author sets. A delegation $d : S \to T$, as in Definition 2.1.17, is said to be **instantiated** if the following two implications hold for every $S' \subseteq S$:

1. For every noun phrase $N$, if $d(S') \models (x : N)$ then $S' \models (x : N)$.
2. For every sentence $(N_1, V, N_2)$, if $d(S') \models V(x, y)$ then $S' \models V(x, y)$.

Proposition 2.2.10 The collection of author set and instantiated delegations forms a category, denoted $\text{InstDel}$.

Definition 2.2.11 Suppose $\alpha : \mathcal{L} \to \mathcal{L}'$ is a morphism of linguistic structures on $\mathcal{C}$, where $S = \text{Auth}(\mathcal{L})$ and $S' = \text{Auth}(\mathcal{L}')$, and suppose that the delegation $\alpha_0$ is instantiated as in Definition 2.2.9. Suppose that $I, I' : \mathcal{C} \to \text{Set}$ are instantiations conforming to $\mathcal{L}$ and $\mathcal{L}'$, respectively. We say that a natural transformation $p : I \Rightarrow I'$ **conforms** to $\alpha$ if for every every $c \in \text{Ob}(\mathcal{C})$ and every element $x \in I(c)$, the correspondence is endorsed $S' \models \alpha(c)(x, p_c(x))$. In this case we say that

$$(\alpha, p) : (\mathcal{L}, I) \Rightarrow (\mathcal{L}', I')$$

is a **morphism of instantiated linguistic structures**.

We denote by $\text{Inst}(\mathcal{C})$ the collection instantiated linguistic structures on $\mathcal{C}$ and morphisms between them.

Proposition 2.2.12 With objects as in Definition 2.2.7 and morphisms as in Definition 2.2.11, $\text{Inst}(\mathcal{C})$ forms a category.

### 3 Mappings between ologs

In the previous section we studied how two different linguistic structures $\mathcal{L}$ and $\mathcal{M}$, on the same category $\mathcal{C}$, can be connected by a morphism $\mathcal{L} \to \mathcal{M}$ in $\text{Ling}(\mathcal{C})$. In this section we move to a more general setting in which we show how to relate linguistic structures that exist on possibly different categories. In the first part of § 3.1 we describe how to pull a linguistic structure back along a functor. We use this notion of pullback in the second part of § 3.1, where we introduce the notion of **linguistic functor** as a mapping between ologs. We later extend this concept in § 3.2 to obtain a notion of mapping between instantiated ologs, which we call **instantiated functors**.

Remark 3.0 It should be remarked that (linguistic) functors, as defined in Definition 3.1.8 do not provide the most general way we know to connect two ologs. In general, it is rare that one olog, $\mathcal{O}_1$, would map entirely into another olog, $\mathcal{O}_2$. The reason is that this would
not only that require every linguistic expression endorsed by authors of $O_1$ to be endorsed by the authors of $O_2$, but also that the entire concern of olog $O_1$ be of concern in olog $O_2$. This will rarely be the case in practice for two ologs trying to communicate.

As discussed in [18, §4], the proper approach, in general, is to find common ground, a third olog $O$ with linguistic functors to both others,

$$O_1 \leftarrow O \rightarrow O_2.$$

We will discuss this idea more in § 4.

3.1 The category of ologs

Given a functor $F: C \rightarrow D$, category theory provides many examples in which structures on $D$ can be pulled back along $F$ to structures on $C$. In this section, we discuss how this is done for linguistic structures. In § 3.2 we study the analogous process for instantiated linguistic structures.

Suppose we are given two ologs, $(C, \mathcal{L})$ and $(D, \mathcal{M})$. A mapping between them begins with a functor $F: C \rightarrow D$. Using a pullback construction associated to $F$, defined in Definition 3.1.2, we obtain a linguistic structure $F^*(\mathcal{M})$ on $C$. At this point, we have two linguistic structures on $C$, so we can relate them using a morphism $\mathcal{L} \rightarrow F^*\mathcal{M}$ in $\text{Ling}(C)$ (see Definition 2.1.20). Putting these pieces together, we arrive at the notion of olog morphism, or linguistic functor; see Definition 3.1.8.

The approach will be to construct a functor $\text{Ling}: \text{Cat} \rightarrow \text{Cat}^{\text{op}}$, where $\text{Cat}$ denotes the category of small categories. This functor was defined on objects $C \in \text{Ob}(\text{Cat})$ above, in Proposition 2.1.23. The goal of the present section is to define it on morphisms in $\text{Cat}$, i.e., functors $F: C \rightarrow D$. Once $\text{Ling}$ is defined as a functor, the category of ologs will be its Grothendieck construction, $\text{Olog} = \int \text{Ling}$. It may be useful to consult [16] or [10].

Pullback of linguistic structures

Proposition 3.1.1 Suppose that $F: C \rightarrow D$ is a functor and that $\mathcal{M}$ is a linguistic structure on $D$, as in Definition 2.1.15. Define a new collection $\mathcal{L}$ of linguistic expressions as follows:

1. Every $c \in \text{Ob}(C)$ is assigned the noun phrase $\mathcal{L}(c) := \mathcal{M}(Fc)$.
2. Every $f \in \text{Mor}(C)$ is assigned the verb phrase $\mathcal{L}(f) := \mathcal{M}(Ff)$. 

20
(3) Every commutative diagram in \( \mathcal{C} \), as in Figure 1, is assigned the fact
\[
(\mathcal{L}(p), (\mathcal{L}(f_1); \ldots; \mathcal{L}(f_n)), \mathcal{L}(q)) \simeq (\mathcal{L}(p), (\mathcal{L}(g_1); \ldots; \mathcal{L}(g_m)), \mathcal{L}(q)).
\]
If we set \( \text{Auth}(\mathcal{L}) := \text{Auth}(\mathcal{M}) \), then \( \mathcal{L} \) satisfies the conditions of Definition 2.1.15, making it a linguistic structure on \( \mathcal{C} \).

**Definition 3.1.2** Let \( F: \mathcal{C} \rightarrow \mathcal{D} \) be a functor, and let \( \mathcal{M} \) be a linguistic structure on \( \mathcal{D} \). We define the **pullback of \( \mathcal{M} \) along \( F \)**, denoted \( F^*(\mathcal{M}) \), to be the linguistic structure denoted \( \mathcal{L} \) in Proposition 3.1.1.

**Example 3.1.3** In this example we show how pulled back linguistic structures look in general for a functor \( \mathcal{C} \rightarrow \mathcal{D} \). We will see that each morphism in \( \mathcal{C} \) is labeled with either a unital verb phrase or a concatenation of verb phrases in \( \mathcal{D} \).

Consider the following functor \( F: \mathcal{C} \rightarrow \mathcal{D} \) where \( \mathcal{C} = \mathcal{D} = \bullet^1 \rightarrow \bullet^2 \rightarrow \bullet^3 \):

\[
\begin{array}{c}
\bullet^1 \\
\downarrow \\
\bullet^2 \\
\downarrow \\
\bullet^3
\end{array}
\xrightarrow{F}
\begin{array}{c}
\bullet^1 \\
\downarrow \\
\bullet^2 \\
\downarrow \\
\bullet^3
\end{array}
\]

Now suppose \( \mathcal{D} \) is equipped with the following linguistic structure:

\[
\mathcal{M} = \begin{array}{c}
\text{a woman} \\
\text{lives at} \\
\text{an address} \\
\text{includes} \\
\text{a city}
\end{array}
\]

(5)

We have the following computations, where \( \mathcal{L} := F^*(\mathcal{M}) \):

\[
\begin{align*}
\langle \mathcal{L}(1) \rangle &= \langle \mathcal{M}(F(1)) \rangle = \langle \mathcal{M}(1) \rangle = "\text{a woman}" , \\
\langle \mathcal{L}(2) \rangle &= \langle \mathcal{L}(F(2)) \rangle = \langle \mathcal{M}(3) \rangle = "\text{a city}" , \\
\langle \mathcal{L}(3) \rangle &= \langle \mathcal{L}(F(3)) \rangle = \langle \mathcal{M}(3) \rangle = "\text{a city}" , \\
\langle \mathcal{L}(1 \rightarrow 2) \rangle &= \langle \mathcal{M}(F(1) \rightarrow F(2)) \rangle = \langle \mathcal{M}(1 \rightarrow 3) \rangle = "\text{lives at an address, which includes}" , \\
\langle \mathcal{L}(2 \rightarrow 3) \rangle &= \langle \mathcal{M}(F(2) \rightarrow F(3)) \rangle = \langle \mathcal{M}(3 \rightarrow 3) \rangle = "\text{is of course}" .
\end{align*}
\]

Then the pullback \( F^*(\mathcal{M}) \) of \( \mathcal{M} \) along \( F \) is the following linguistic structure on \( \mathcal{C} \):

\[
\mathcal{L} = \begin{array}{c}
\text{a woman} \\
\text{which includes} \\
\text{a city} \\
\text{is of course} \\
\text{a city}
\end{array}
\]
**Proposition 3.1.4** Suppose given a functor \( F: C \to D \) and a morphism \( \beta: \mathcal{M} \to \mathcal{M}' \) in \( \text{Ling}(D) \), as in Definition 2.1.20, and let \( \mathcal{L} := F^*(\mathcal{M}) \) and \( \mathcal{L}' := F^*(\mathcal{M}') \) be their pullbacks as in Definition 3.1.2. Define a function \( \alpha_0: \text{Auth}(\mathcal{L}') \to \text{Auth}(\mathcal{L}) \) by

\[
\alpha_0(s) := \beta_0(s)
\]

for every \( s \in \text{Auth}(\mathcal{L}') \). Finally, suppose that to each object \( c \in \text{Ob}(C) \) we assign the component verb phrase \( \alpha(c): \mathcal{L}(c) \to \mathcal{L}'(c) \) given by

\[
\alpha(c) := \beta(Fc).
\]

Then the family \( \{\alpha(c) : c \in \text{Ob}(C)\} \) of verb phrases and the function \( \alpha_0 \) together constitute a morphism \( \alpha: \mathcal{L} \to \mathcal{L}' \) of linguistic structures on \( C \).

**Definition 3.1.5** Let \( F: C \to D \) be a functor and let \( \beta: \mathcal{M} \to \mathcal{M}' \) be a morphism of linguistic structures on \( D \). We define the **pullback of \( \beta \) along \( F \)**, denoted \( F^*(\beta) \), to be the morphism \( \alpha: F^*(\mathcal{M}) \to F^*(\mathcal{M}') \) of linguistic structures on \( C \), as defined in Proposition 3.1.4.

**Proposition 3.1.6** Let \( F: C \to D \) be a functor. The pullback construction \( F^* \), defined on objects and morphisms of \( \text{Ling}(D) \) as in Definitions 3.1.2 and 3.1.5 is functorial:

\[
F^*: \text{Ling}(D) \to \text{Ling}(C).
\]

We sometimes denote this functor by \( \text{Ling}(F) := F^* \).

**Theorem 3.1.7** There is a functor

\[
\text{Ling}: \text{Cat} \to \text{Cat}^{\text{pp}}
\]

acting on objects as in Proposition 2.1.23 and on morphisms as in Proposition 3.1.6.

**Linguistic functors**

We begin this section by defining the category of ologs in Definition 3.1.8. Abstractly, this definition arises as a kind of reformulation (called the Grothendieck construction) of Theorem 3.1.7; see Remark 3.1.10. In Example 3.1.11 we give two ologs and two functors between them, one of which makes linguistic sense and the other does not, which recapitulates the issue we presented in our introductory example (1).

**Definition 3.1.8** Let \( (C, \mathcal{L}) \) and \( (D, \mathcal{M}) \) be ologs, as in Definition 2.1.15. A **linguistic functor** between them, denoted \( (F, F^\sharp):(C, \mathcal{L}) \to (D, \mathcal{M}) \) consists of a functor \( F: C \to D \) together with a morphism \( F^\sharp: \mathcal{L} \to F^*(\mathcal{M}) \) of linguistic structures on \( C \).
We define the category of ologs, denoted \( \text{Olog} \), to be the category whose objects are ologs and whose morphisms are linguistic functors.

**Remark 3.1.9** Let \((F,F^\dagger):(C,L) \rightarrow (D,M)\) be a linguistic functor. Then we have a function \(F^\dagger_0: \text{Auth}(M) \rightarrow \text{Auth}(L)\) between author sets, since \(F^\dagger\) is a morphism in \(\text{Ling}(C)\).

In other words, we can consider \(\text{Auth}\) as a functor as shown below:

\[
\text{Olog} \xrightarrow{\text{Auth}} \text{Del}^{\text{op}} \xrightarrow{L^{\text{op}}} \text{Set}^{\text{op}}.
\]

Suppose that \((F,F^\dagger):(C,L) \rightarrow (D,M)\) is a linguistic functor. We briefly discuss the morphism \(F^\dagger : \text{L}(C) \rightarrow \text{M}(Fc)\) of linguistic structures on \(C\). It includes a delegation function \(F^\dagger_0: \text{Auth}(M) \rightarrow \text{Auth}(L)\), so that every author in \(\text{Auth}(M)\) endorses both linguistic structures, and can thus evaluate possible morphisms between them. The map \(F^\dagger\) also includes, for each \(c \in \text{Ob}(C)\), a \(c\)-component verb phrase

\[
F^\dagger(c): \text{L}(c) \rightarrow \text{M}(Fc).
\]

And for every morphism \(f:c \rightarrow c'\) in \(C\), it includes an equivalence

\[
\begin{array}{ccc}
\text{L}(c) & \xrightarrow{F^\dagger(c)} & \text{M}(Fc) \\
\text{L}(f) \downarrow & & \downarrow \text{M}(Ff) \\
\text{L}(c') & \xrightarrow{F^\dagger(c')} & \text{M}(Fc')
\end{array}
\]

This equivalence has an English-language interpretation, as in Remark 2.1.21.

**Remark 3.1.10** An expert will recognize the category \(\text{Olog}\) as the Grothendieck construction applied to the functor \(\text{Ling}: \text{Cat} \rightarrow \text{Cat}^{\text{op}}\). The functor \(\text{Olog} \rightarrow \text{Cat}\) sending an olog \((C,L)\) to its underlying category \(C\) is a fibration of categories. The linguistic functors \((F,F^\dagger)\) for which \(F^\dagger\) is an isomorphism \(i.e.,\) what we call a formal analogy in Definition 2.1.24\) are the cartesian morphisms of this fibration. For more on this, the reader may consult \([2, 11, 10, 13, 20]\).

In the Introduction, we displayed two ologs (1) and a functor between them, and we said that it would be difficult to find an author to endorse that this functor carried linguistic meaning. By now we have enough definitions in place to be more precise about this.

**Example 3.1.11** In this example we will show two ologs \((C,L)\) and \((D,M)\), and two functors \(F,G:C \rightarrow D\) between their underlying categories. We will propose that \(G\) has very
little chance of being extended to a linguistic functor, and explain our introductory com-
ments about (1) in the process. On the other hand, we will find that it is straightforward
to attach to $F$ a morphism $\mathcal{L} \rightarrow F^*(\mathcal{M})$ of linguistic structures.

Consider the following ologs:

$$(\mathcal{C}, \mathcal{L}) := \begin{array}{c}
1 \\
a \text{ man} \\
\text{is} \\
2 \\
a \text{ object}
\end{array}$$

$$(\mathcal{D}, \mathcal{M}) := \begin{array}{c}
a \\
a \text{ woman} \\
\text{is} \\
c \\
a \text{ animal}
\end{array}$$

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be the unique functor such that $F(1) = c$ and $F(2) = d$, and let $G: \mathcal{C} \rightarrow \mathcal{D}$ be the unique functor such that $G(1) = a$ and $G(2) = b$. Then the pulled back linguistic structures $F^*(\mathcal{M})$ and $G^*(\mathcal{M})$ on $\mathcal{C}$ are shown below.

$$(\mathcal{C}, F^*\mathcal{M}) := \begin{array}{c}
1 \\
a \text{ animal} \\
\text{has as weight (in kilograms)} \\
2 \\
a \text{ number}
\end{array}$$

$$(\mathcal{C}, \mathcal{L}) := \begin{array}{c}
1 \\
a \text{ man} \\
\text{is} \\
2 \\
a \text{ object}
\end{array}$$

$$(\mathcal{C}, G^*\mathcal{M}) := \begin{array}{c}
1 \\
a \text{ woman} \\
\text{has as weight (in kilograms)} \\
2 \\
a \text{ number between 20 and 120}
\end{array}$$

In order to extend $F$ and $G$ to linguistic functors, we need maps $F^\sharp: \mathcal{L} \rightarrow F^*(\mathcal{M})$ and $G^\sharp: \mathcal{L} \rightarrow G^*(\mathcal{M})$, as indicated in (6). In the Introduction, we said that it will be difficult to
find $G^\sharp$. We begin by endorsing a certain $F^\sharp$, and finally return to $G^\sharp$. 

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To provide $F^\sharp$, we first need two functional component verb phrases; we use those labeling the top and bottom maps here:

\[
\begin{array}{cccc}
2 & \text{a man} & \overset{\text{is}}{\sim} & 2 \\
1 & \text{an object} & \overset{\text{has as weight}}{\sim} & 1 \\
\end{array}
\]

It just suffices to ask whether, if we measure the weight (in kilograms) of a man, regarding him either as an object or as an animal, we get the same number. We endorse that fact, thus providing a linguistic functor $(F, F^\sharp):(C, L) \rightarrow (D, M)$.

It is difficult to do the same for $G^\sharp$. One can find functional verb phrases connecting [a man] to [a woman], for example “has as mother”. But it is not straightforward to find a functional verb phrase connecting [an object] to [a number between 20 and 120] making the necessary diagram (analogous to (7)) commute. The reason, roughly, is that once one has regarded a man as an object, there is no aspect that applies to an arbitrary object which will correspond (i.e., commute with) with the mother-having aspect of an arbitrary man.

The upshot of this example is that linguistic structures give the necessary semantics to constrain mappings between ologs. This has important applications to databases, as we will show in Example 3.2.9.

### 3.2 The category of instantiated ologs

In this section, we provide a notion of mapping between two instantiated ologs $(C, L, I)$ and $(D, M, J)$ by constraining the set of linguistic functors $(C, L) \rightarrow (D, M)$ to those that respect the tokens and their correspondences (See Definition 2.2.2).

**Pullback of instantiations**

We first explain how instantiations are pulled back along a functor. The idea is to complement the pullback construction introduced in the previous section (see Definition 3.1.2) with the notion of data-pullback explained in [15, §2].
**Proposition 3.2.1** Suppose that $F: C \to D$ is a functor and that $(\mathcal{M}, J)$ is an instantiated linguistic structure on $D$, as in Definition 2.2.7. Define a functor $I: C \to \text{Set}$ as follows:

1. For every object $c \in \text{Ob}(C)$, we put $I(c) := J(Fc)$.
2. For every morphism $f \in \text{Mor}(C)$, we put $I(f) = J(Ff)$.

Then $I$ conforms to the linguistic structure $F^*(\mathcal{M})$ given in Definition 3.1.2. In other words, $(F^*(\mathcal{M}), I)$ is an instantiated linguistic structure on $C$.

**Definition 3.2.2** Let $F: C \to D$ be a functor, and let $(\mathcal{M}, J)$ be an instantiated linguistic structure on $D$. We define the **pullback of $(\mathcal{M}, J)$ along $F$**, denoted $F^*(\mathcal{M}, J)$, to be the instantiated linguistic structure denoted $(F^*(\mathcal{M}), I)$ in Proposition 3.2.1. We sometimes denote the instantiation $I$ by $F^*(J)$.

**Proposition 3.2.3** Suppose given a functor $F: C \to D$ and a morphism $(\beta, r): (\mathcal{M}, J) \to (\mathcal{M}', J')$ in $\text{Inst}(D)$, as in Definition 2.2.11. Let $(\mathcal{L}, I) := F^*(\mathcal{M}, J)$ and $(\mathcal{L}', I') := F^*(\mathcal{M}', J')$ be their pullbacks as in Definition 3.2.2, and let $\alpha = F^*(\beta)$, as in Definition 3.1.5. Finally, let $q: I \Rightarrow I'$ be the natural transformation given by $q_c := r_{Fc}$, for every $c \in \text{Ob}(C)$.

Then $q$ conforms to $\alpha$ in the sense of Definition 2.2.11; in other words, $(\alpha, q): F^*(\mathcal{M}, J) \to F^*(\mathcal{M}', J')$ is a morphism of instantiated linguistic structures on $C$.

**Definition 3.2.4** Let $F: C \to D$ be a functor and let $(\beta, r): (\mathcal{M}, J) \to (\mathcal{M}', J')$ be a morphism of instantiated linguistic structures on $D$. We define the **pullback of $(\beta, r)$ along $F$**, denoted $F^*(\beta, r)$ to be the morphism $(\alpha, q)$, as defined in Proposition 3.2.3. We sometimes denote the natural transformation $q$ by $F^*(r)$.

**Proposition 3.2.5** Let $F: C \to D$ be a functor. The pullback construction $F^*$, defined on objects and morphisms of $\text{Inst}(D)$, as in Definitions 3.2.2 and 3.2.4, is functorial:

$$F^*: \text{Inst}(D) \to \text{Inst}(C).$$

We sometimes denote this functor by $\text{Inst}(F) := F^*$.

**Theorem 3.2.6** There is a functor

$$\text{Inst}: \text{Cat} \to \text{Cat}^{\text{op}}$$

acting on objects as in Proposition 2.2.12 and on morphisms as in Proposition 3.2.5.
Instantiated functors

We begin this section by defining the category of instantiated ologs in Definition 3.1.8. This definition arises as the Grothendieck construction applied to Theorem 3.2.6, as in Remark 3.1.10.

Definition 3.2.7 Let \((\mathcal{C}, \mathcal{L}, I)\) and \((\mathcal{D}, \mathcal{M}, J)\) be instantiated ologs, as in Definition 2.2.7. A instantiated linguistic functor between them, denoted 
\[(F, F^\dagger, F^\flat): (\mathcal{C}, \mathcal{L}, I) \rightarrow (\mathcal{D}, \mathcal{M}, J)\]
consists of a functor \(F: \mathcal{C} \rightarrow \mathcal{D}\), a morphism \(F^\dagger: \mathcal{L} \rightarrow F^*(\mathcal{M})\) of linguistic structures on \(\mathcal{C}\), and a morphism \(F^\flat: I \Rightarrow F^*(J)\) of instantiations conforming to \(F^\dagger\), as in Definition 2.2.11.

We define the category of instantiated ologs, denoted \(\text{InstOlog}\), to be the category whose objects are instantiated ologs and whose morphisms are instantiated linguistic functors.

Remark 3.2.8 As we noted for \(\text{Olog}\) and \(\text{Ling}\) in Remark 3.1.10, the category \(\text{InstOlog}\) can be recognized as the Grothendieck construction applied to the functor \(\text{Inst} \circ \text{Cat} \rightarrow \text{Cat}^{\text{op}}\).

The functor \(\text{InstOlog} \rightarrow \text{Cat}\) sending an instantiated olog \((\mathcal{C}, \mathcal{L}, I)\) to its underlying category \(\mathcal{C}\) is a fibration of categories, whose cartesian morphisms are very similar to “strongly meaningful” functors found in [14, §4]. See Remark 2.2.1.

Example 3.2.9 Suppose we are given two categories \(\mathcal{C}, \mathcal{D}\), which we think of as database schemas (as in [15]), a functor \(F: \mathcal{C} \rightarrow \mathcal{D}\), and two instantiations \(I: \mathcal{C} \rightarrow \text{Set}\) and \(J: \mathcal{D} \rightarrow \text{Set}\). Suppose that these two databases are to be merged. We are asked to find a morphism \(F^\flat: I \Rightarrow F^*(J)\). In this example, we show that our job will be easier if \(F\) has been equipped with a morphism \(F^\dagger\) of linguistic structures.

In order to emphasize the issue, we suppose that \(\mathcal{C}\) and \(\mathcal{D}\) are both single-object categories
\[
\mathcal{C} = \bullet \quad \text{and} \quad \mathcal{D} = \bullet
\]
equipped with the following linguistic structures:
\[
(\mathcal{C}, \mathcal{L}) = \begin{array}{c}
1 \\
a \text{human}
\end{array}
\quad \text{and} \quad
(\mathcal{D}, \mathcal{M}) = \begin{array}{c}
a \\
a \text{person}
\end{array}
\]
Suppose we want to compare two instantiations, \(I: \mathcal{C} \rightarrow \text{Set}\) and \(J: \mathcal{D} \rightarrow \text{Set}\), which are represented by the following tables:
To compare instances on different schemas, we first need a functor between them. In our case there is a unique functor $F: C \rightarrow D$ (it sends $1 \mapsto a$), so this is not an issue. With this functor in hand, we can pull back $J$ to an instantiation $F^*(J)$ on $C$, and attempt to compare it to $I$.

The purpose of this example is to show that the choice of linguistic structure $F^\sharp$ on $F$ is an important aid to making this comparison, i.e., to choosing a database homomorphism $I \Rightarrow F^*(J)$ out of the $5^2 = 25$ possible choices. Consider the following two linguistic structures for $F$:

\[
\begin{array}{ccc}
\text{a human} & \sim & \text{a person} \\
\text{Emmy Noether} & \sim & \text{Max Noether} \\
\text{George W. Bush} & \sim & \text{George H. W. Bush}
\end{array}
\]

We denote the first by $\alpha$ and the second by $\beta$, i.e., $\alpha, \beta: L \rightarrow F^*M$. Clearly, these linguistic structures give a useful hint at the intended semantics for the mapping. The only natural transformation $I \Rightarrow F^*J$ we endorse as conforming to $\alpha$ (see Definition 2.2.11) is $p$, and the only one that conforms to $\beta$ is $q$, as shown below:

\[
(\alpha,p):= \begin{array}{cc}
\text{a human} & \text{is a person, namely} \\
\text{Emmy Noether} & \text{Emmy Noether} \\
\text{George W. Bush} & \text{George W. Bush}
\end{array}
\]

\[
(\beta,q):= \begin{array}{cc}
\text{a human} & \text{has as father a person, namely} \\
\text{Emmy Noether} & \text{Max Noether} \\
\text{George W. Bush} & \text{George W. H. Bush}
\end{array}
\]

This example complements, but also extends the scope of “meaningful functors” found in [14, §4].
4 Networks of interconnected ologs

In this section we discuss simplicial complexes of ologs, as in [18, §4]. The idea here is to have multiple authors, various subsets of which endorse different ologs. The network of ologs and their authors takes the shape of a simplicial complex, in which the vertices are single-author ologs, the edges are 2-author ologs, the triangles are 3-author ologs, etc.

4.1 Background on simplicial complexes

We begin by recalling the definition of simplicial complex. For a set $V$, we denote by $\mathbb{P}_+(V)$ the positive power set, i.e.,

$$\mathbb{P}_+(V) := \{S \subseteq V \mid S \neq \emptyset\}.$$

If $S' \subseteq S \subseteq V$, we write $S' \leq S$ as elements of $\mathbb{P}_+(V)$, giving it the structure of a partial order. There is a function $V \rightarrow \mathbb{P}_+(V)$, sending each element $v \in V$ to the singleton subset $\{v\} \in \mathbb{P}_+(V)$. Recall that for any function $f: V \rightarrow V'$ there is an induced monotonic function

$$\mathbb{P}_+(f): \mathbb{P}_+(V) \rightarrow \mathbb{P}_+(V'),$$

sending a subset $S \subseteq V$ to its image $f(S) \subseteq V'$.

Definition 4.1.1 Let $V$ be a finite set, and $X \subseteq \mathbb{P}_+(V)$ a set of nonempty subsets of $V$. We say that $(V, X)$ is a simplicial complex if

- $X$ contains each singleton, i.e., $\{v\} \in X$ for every $v \in V$, and
- if $S \in X$ and $S' \subseteq S$, then $S' \in X$.

We call an element of $V$ a vertex and an element of $X$ a simplex.

A morphism of simplicial complexes $f: (V, X) \rightarrow (W, Y)$, denoted $(f, \tilde{f}): (V, X) \rightarrow (W, Y)$ consists of a function $f: V \rightarrow W$, and a function $\tilde{f}: X \rightarrow Y$, making the diagram commute:

$$\begin{array}{cc}
X & \xrightarrow{\tilde{f}} & Y \\
\downarrow & & \downarrow \\
\mathbb{P}_+(V) & \xrightarrow{\mathbb{P}_+(f)} & \mathbb{P}_+(W)
\end{array}$$

Note that if $\tilde{f}$ exists then it is unique, and it is monotonic. The uniqueness implies that we can specify a morphism $(f, \tilde{f})$ in terms only of the function $f: V \rightarrow W$ on vertices.
The idea will be to put an olog at every simplex in a simplicial complex; the resulting network of ologs will be called an olog complex.

**Example 4.1.2** Here is a picture of a simplicial complex with \( V = \{A, B, \ldots, M\} \).

Note, for example, that the triangle with vertices \( A, B, C \) is filled in; this indicates that \( \{A, B, C\} \subseteq X \) is a simplex, which of course implies that \( \{A, B\}, \{A, C\}, \) and \( \{B, C\} \) are also simplices, as indicated by lines. Because there is no line connecting \( A \) and \( F \), we know that the subset \( \{A, F\} \in \mathbb{P}_+(V) \) is not a simplex. Neither is \( \{C, F, I\} \).

### 4.2 Olog complexes

Recall that a partially ordered set \((X, \leq)\) can be considered as a category, whose objects are the elements of \( X \), and which has a unique morphism \( S' \to S \) if and only if \( S' \leq S \) in \( X \); see [16, §5.2.1]. In this sense, a morphism of simplicial complexes induces a functor between the associated categories. If \((V, X)\) is a simplicial complex, we will use the following evident functors (or their opposites) below:

\[
X \to \mathbb{P}_+(V) \to \text{Set}
\]

We will also use the following proposition to restrict author sets along the face maps of a simplex.

**Proposition 4.2.1** The authorship functor

\[
\text{Auth: Olog} \to \text{Def}^{op}
\]

(resp. \(\text{Auth: InstOlog} \to \text{Def}^{op}\)) is an op-fibration of categories. That is, for any olog \( \mathcal{O} \) and delegation \( d: S \to \text{Auth}(\mathcal{O}) \), there is an olog \( d^* \mathcal{O} \), and an olog morphism \( \mathcal{O} \to d^*(\mathcal{O}) \) which is cocartesian (see [13]).

---

5In this section, we denote ologs by \( \mathcal{O} \), denoted by pairs \((C, \mathcal{L})\) in the previous sections. The purpose of this is to shorten the statements and proofs of the current section. By \(\text{Auth}(\mathcal{O})\) we mean \(\text{Auth}(\mathcal{L})\) if \(\mathcal{O} = (C, \mathcal{L})\).
Recall from Remark 2.1.18 that if $T$ is any author set then any subset $S \subseteq T$ is a delegation $d: S \to T$. If $O$ is an olog with $\text{Auth}(O) = T$, we denote its pullback $d^*O$, as in Proposition 4.2.1, by $O|_S$. We do the same for instantiated linguistic structures and instantiated ologs.

**Definition 4.2.2** An olog complex (resp., an instantiated olog complex) consists of a simplicial complex $(V, X)$ together with a functor $\omega: X^{\text{op}} \to \text{Olog}$ (resp., $\omega: X^{\text{op}} \to \text{InstOlog}$), such that the following diagram commutes:

\[
\begin{array}{ccc}
X^{\text{op}} & \xleftarrow{\pi}(V)^{\text{op}} & \\
\omega & \downarrow & \\
\text{Olog} & \xrightarrow{\text{Auth}} & \text{Set}^{\text{op}}
\end{array}
\]

( resp.,
\[
\begin{array}{ccc}
X^{\text{op}} & \xleftarrow{\pi}(V)^{\text{op}} & \\
\omega & \downarrow & \\
\text{InstOlog} & \xrightarrow{\text{Auth}} & \text{Set}^{\text{op}}
\end{array}
\]

In other words, to every simplex $S \in X$ we assign an (instantiated) olog $\omega(S)$, whose author set is $S$.

A morphism of (instantiated) olog complexes from $(V_1, X_1, \omega_1)$ to $(V_2, X_2, \omega_2)$ consists of a morphism $(f, \bar{f}): (V_2, X_2) \to (V_1, X_1)$ of simplicial complexes, together with a natural transformation $f^! (\omega_1 \circ \bar{f}) \Rightarrow \omega_2$,

\[
\begin{array}{ccc}
X_1^{\text{op}} & \xleftarrow{\bar{f}} & X_2^{\text{op}} \\
\omega_1 & \xrightarrow{f} & \omega_2
\end{array}
\]

(resp.,
\[
\begin{array}{ccc}
X_1^{\text{op}} & \xleftarrow{\bar{f}} & X_2^{\text{op}} \\
\omega_1 & \xrightarrow{f} & \omega_2
\end{array}
\]

**Proposition 4.2.3** The collection of (instantiated) olog complexes and morphisms of (instantiated) olog complexes, denoted $\text{Ologx}$ (resp. $\text{InstOlogx}$), forms a category.

**Remark 4.2.4** It is reasonable to wonder why, for a morphism $\alpha: \mathcal{L} \to \mathcal{L}'$ of linguistic structures on a category $\mathcal{C}$, the authorship delegation function points “backwards”, i.e., $\alpha_0: \text{Auth}(\mathcal{L}') \to \text{Auth}(\mathcal{L})$; see Definition 2.1.20. Indeed, everything we have done in this paper could have been done with that map pointing the opposite way, and some definitions would have been cleaner, including fewer “op’s” (for example Definition 4.2.2).

We make the case for our “contravariant” convention as follows. Suppose we have authors 1 and 2, and that each of them store some of their knowledge as an olog, say $\mathcal{O}_1$ and $\mathcal{O}_2$. Suppose also that they share some common ground knowledge, also represented as an olog $\mathcal{O}_{12}$. Since both authors agree on it, it is likely a subset of their own individual ologs, i.e.,
there should be morphisms (linguistic functors)

$$\mathcal{O}_1 \leftrightarrow \mathcal{O}_{12} \rightarrow \mathcal{O}_2.$$  

But since $\text{Auth}(\mathcal{O}_{12}) = \{1, 2\}$, whereas $\text{Auth}(\mathcal{O}_1) = \{1\}$ and $\text{Auth}(\mathcal{O}_2) = \{2\}$, the delegation functions indeed need to point “the opposite way”.

It is possible to argue that instead of common ground ologs, akin to “intersections”, we should be using “unions of ologs”; this way, delegation functions would be covariant. For example, we can think of $\mathcal{O}_{12}$ as an olog containing the knowledge of 1 and 2. In this case, there are inclusions

$$\mathcal{O}_1 \rightarrow \mathcal{O}_{12} \leftarrow \mathcal{O}_2,$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ are related by aspects and facts endorsed by 1 and 2.

Before explaining our reasons not to prefer this convention, we run through the story anyway, in case it is of use to some readers. For any category $\mathcal{C}$, we denote the category of covariantly delegated linguistic structures on $\mathcal{C}$ by $\text{Ling}^\dagger(\mathcal{C})^6$ This, as well as $\text{Inst}(\mathcal{C})$, is still contravariant in the base category $\mathcal{C}$, i.e., we have $\text{Ling}^\dagger : \text{Cat}^{\text{op}} \rightarrow \text{Cat}$ and $\text{Inst}^\dagger : \text{Cat}^{\text{op}} \rightarrow \text{Cat}$, and the associated Grothendieck construction of covariantly delegated (instantiated) ologs will be denoted $\text{Olog}^\dagger$ (respectively, $\text{InstOlog}^\dagger$).

In terms of sharing ideas, the covariant conception is unwieldy. For two authors to share ideas, they must see and endorse the entirety of the union olog. This reduces privacy, as well as the chances that such sharing will occur. It adds a great deal of unnecessary clutter. Covariant olog complexes would contain very large amounts of data, most of which is redundant, because the individual ologs, plus their $n$-fold unions, are all necessarily stored. For these reasons, we felt compelled to adopt the contravariant delegation convention.

We conclude this section with a sanity check; namely, we construct a faithful functor $\text{Olog} \rightarrow \text{Ologx}$. The idea is that each olog $\mathcal{O}$ has a finite set $V = \text{Auth}(\mathcal{O})$ of authors, and thus we can assign that same olog $\mathcal{O}$ (with various subsets of its authors) to each sub-simplex of $\mathbb{P}_+(V)$, forming an olog complex. This construction is functorial.

**Proposition 4.2.5** There exists a functor $\Phi : \text{Olog} \rightarrow \text{Ologx}$ mapping every olog $(\mathcal{C}, \mathcal{L})$ to an olog complex $(V, \mathbb{P}_+(V), \omega)$, where $V = \text{Auth}(\mathcal{L})$, and $\omega : \mathbb{P}_+(V)^{\text{op}} \Rightarrow \text{Olog}$ is the constant functor, mapping every $S \subseteq V$ to the olog $\mathcal{O}|_S$, as in Proposition 4.2.1.

---

$^6$Note that our symbol $\dagger$ has nothing to do with dagger structures on categories.
A Proofs

Proof of Proposition 2.1.19: First, note that for every author set $S$, the identity function $\text{id}_S$ is clearly a delegation. The composition in $\text{Del}$ is the composition of functions, which we know is associative. So it suffices to verify that the composition of delegations is a delegation, but that follows straightforward by definition.

Proof of Proposition 2.1.23: The functoriality $\text{Auth} : \text{Ling}(C) \to \text{Del}^{\text{op}}$ is obvious by definition, once we establish that $\text{Ling}(C)$ is a category. We first define composition in $\text{Ling}(C)$.

Let $\mathcal{L}$, $\mathcal{L}'$ and $\mathcal{L}''$ be three linguistic structures on $C$, with $S = \text{Auth}(\mathcal{L})$, $S' = \text{Auth}(\mathcal{L}')$, and $S'' = \text{Auth}(\mathcal{L}'')$, and let $\alpha : \mathcal{L} \to \mathcal{L}'$ and $\alpha' : \mathcal{L}' \to \mathcal{L}''$ be morphisms between linguistic structures on $C$. Define a morphism $\beta := (\alpha; \alpha') : \mathcal{L} \to \mathcal{L}''$ by the following data (parallel to Definition 2.1.20):

1. On authors, we are given two delegations $\alpha_0 : S' \to S$ and $\alpha'_0 : S'' \to S'$. Then the composite function $\beta_0 := \alpha_0 \circ \alpha'_0 : S'' \to S$ is a delegation by Proposition 2.1.19, and thus satisfying Definition 2.1.20 (1).

2. Every object $c \in \text{Ob}(C)$ is assigned the concatenated component verb phrase $\beta(c) := \alpha(c); \alpha'(c)$. Then $S'' = (\mathcal{L}'(c), \alpha'(c), \mathcal{L}''(c))$ since $\alpha' : \mathcal{L}' \to \mathcal{L}''$ is a morphism between linguistic structures. On the other hand, $\alpha'_0(S'') = (\mathcal{L}(c), \alpha(c), \mathcal{L}(c))$ since $\alpha_0(S'') \subseteq S'$ and $S' = (\mathcal{L}(c), \alpha(c), \mathcal{L}(c))$ (see Remark 2.1.14 (1)). By Definition 2.1.20 (1), we have $S'' = (\mathcal{L}(c), \alpha(c), \mathcal{L}(c))$, and so we obtain $S'' \equiv (\mathcal{L}(c), (\alpha_0(c); \alpha'(c)), \mathcal{L}''(c))$ by Postulate 2.1.7, i.e., $S'' \equiv (\mathcal{L}(c), \beta(c), \mathcal{L}''(c))$. Therefore, $\beta(c)$ satisfies Definition 2.1.20 (2).

3. Now suppose we are given an arrow $f : c \to c'$ in $C$. Then $S'' = [\alpha'(c); \mathcal{L}''(f) \simeq \mathcal{L}'(f); \alpha'(c')]$ by Definition 2.1.20 (3). On the other hand, $\alpha'_0(S'') = [\alpha(c), \mathcal{L}'(f) \simeq \mathcal{L}(f); \alpha(c')]$ since $\alpha'_0(S'') \subseteq S'$ and $S' \equiv [\alpha_0(c), \mathcal{L}'(f) \simeq \mathcal{L}(f); \alpha(c')]$ (again, see Remark 2.1.14 (1)). Then by Definition 2.1.20 (1), it follows $S'' \equiv [\alpha(c), \mathcal{L}'(f) \simeq \mathcal{L}(f); \alpha(c')]$. Then it is easy to check that $(\alpha; \alpha')(c) : \mathcal{L}'(f) = \mathcal{L}(f); (\alpha; \alpha')(c')$. It follows $S'' \equiv [(\alpha; \alpha')(c); \mathcal{L}''(f) \simeq \mathcal{L}(f); (\alpha; \alpha')(c')]$, i.e., $S'' \equiv [\beta(c); \mathcal{L}''(f) \simeq \mathcal{L}(f); \beta(c')]$.

It follows that $\beta$ defines a morphism between linguistic structures $\mathcal{L} \to \mathcal{L}''$ on $C$. The composition of morphisms in $\text{Ling}(C)$ defined by (1), (2) and (3) above is associative, since the concatenation of verb phrases and the composition of functions are associative.

Now consider a linguistic structure $\mathcal{L}$ on $C$. We define a morphism $\text{id}_\mathcal{L} : \mathcal{L} \to \mathcal{L}$ in $\text{Ling}(C)$ by setting $(\text{id}_\mathcal{L})_0 := \text{id}_S$, which is clearly a delegation, and $\text{id}_\mathcal{L}$ as the collection of verb phrases $\text{id}_\mathcal{L}(c) = c$, with $c$ the unit verb phrase as in Postulate 2.1.10. It is easy to check that this morphism is unital with respect to the composition just defined.

\hfill $\square$
Proof of Proposition 2.2.10: Analogous to the proof of Proposition 2.1.19. □

Proof of Proposition 2.2.12: We first define composition in \( \text{Inst}(C) \). Suppose we are given two morphisms \((\alpha, p):(\mathcal{L}, I) \rightarrow (\mathcal{L}', I')\) and \((\alpha', p'):(\mathcal{L}', I') \rightarrow (\mathcal{L}'', I'')\) between instantiated linguistic structures on \( C \), where \( S = \text{Auth}(\mathcal{L}) \), \( S' = \text{Auth}(\mathcal{L}') \) and \( S'' = \text{Auth}(\mathcal{L}'') \). Define the composite morphism \((\beta, q): (\mathcal{L}, I) \rightarrow (\mathcal{L}'', I'')\) by the pair \((\beta, q)\) with \( \beta = \alpha; \alpha' \) as in the proof of Proposition 2.1.23, and \( q = p' \circ p \) if \( I \Rightarrow I'' \) as the natural transformation defined by the family \( \{q_c = p'_c \circ p_c: I(c) \rightarrow I''(c) \mid c \in \text{Ob}(C)\} \). We check that \( q \) conforms to \( \beta \).

- First, note that \( \beta_0 \) is an instantiated delegation by Proposition 2.2.10.

- For every \( c \in \text{Ob}(C) \) and \( x \in I(c) \), we have that \( S'' \models \alpha'(c)(p_c(x), p'_c(p_c(x))) \) and \( \alpha'_0(S'') \models \alpha(c)(x, p_c(x)) \), since \( \alpha'_0(S'') \models S' \) and \( S' \models \alpha(c)(x, p_c(x)) \) (see Remark 2.2.5). It follows that \( S'' \models \alpha'(c)(x, p_c(x)) \) since \( \alpha'_0 \) is an instantiated delegation (see Definition 2.2.9 (2)). It follows by Postulate 2.2.6 (1) that \( S'' \models (\alpha(c); \alpha'(c))(x, p'_c(p_c(x))) \), i.e., \( S'' \models \beta(c)(x, q_c(x)) \).

As the composition of morphisms in \( \text{Ling}(C) \), and the composition of functions are associative, we have that the composition defined above in \( \text{Inst}(C) \) is associative.

For each instantiated linguistic structure \((\mathcal{L}, I)\), the identity morphism on \((\mathcal{L}, I)\) is given by the pair \((\text{id}_{\mathcal{L}}, \text{id}_I)\), with \( \text{id}_{\mathcal{L}} \) as defined in the proof of Proposition 2.1.23, and where \( \text{id}_I \) is the identity natural transformation defined by \( \text{id}_I(c) = \text{id}_{I(c)} \), for every \( c \in \text{Ob}(C) \). Note that \((\text{id}_{\mathcal{L}})_0 \) is an instantiated delegation, and that \( \text{id}_I \) conforms to \( \text{id}_{\mathcal{L}} \), since \( s \models \text{id}_{\mathcal{L}}(c)(x, \text{id}_I(x)) \) for every \( c \in \text{Ob}(C) \) and \( s \in \text{Auth}(\mathcal{L}) \), by Postulate 2.2.6 (2). It is easy to check that the morphisms \((\text{id}_{\mathcal{L}}, \text{id}_I)\) are unital with respect to the composition defined above. Therefore, the collection \( \text{Inst}(C) \) defines a category. □

Proof of Proposition 3.1.1: Conditions (1), (2), and (3) in the statement of this proposition are endorsed by \( \text{Auth}(\mathcal{L}) = \text{Auth}(\mathcal{M}) \). On the other hand, it is clear that \( \mathcal{L}(\text{id}_c) = c \) for every \( c \in \text{Ob}(C) \). Then the result follows. □

Proof of Proposition 3.1.4: Conditions (1), (2) and (3) of Definition 2.1.20 follow directly from the definition of \( \alpha \) and the fact that \( \text{Auth}(\mathcal{L}) = \text{Auth}(\mathcal{M}) \). □

Proof of Proposition 3.1.6: Follows easily by Definition 3.1.5. □

Proof of Theorem 3.1.7: Suppose we are given two composable functors \( F:C \rightarrow D \) and \( G:D \rightarrow \mathcal{E} \). We verify that following equality:
\( \text{Ling}(GF) = \text{Ling}(F) \circ \text{Ling}(G). \)  

(1) First, let \( \mathfrak{M} \in \text{Ling}(\mathcal{E}) \) with \( S = \text{Auth}(\mathfrak{M}) \). Then for every \( c \in \text{Ob}(\mathcal{C}) \) and every \( f: c \to c' \in \text{Mor}(\mathcal{C}) \), it is easy to see that

\[
(\text{Ling}(F)(\text{Ling}(G)(\mathfrak{M}))(c) = \text{Ling}(GF)(\mathfrak{M})(c) \quad \text{and} \quad (\text{Ling}(F)(\text{Ling}(G)(\mathfrak{M}))(f) = \text{Ling}(GF)(\mathfrak{M})(f)).
\]

Note that these equalities are endorsed by \( S \), since \( S = \text{Auth}((GF)^*(\mathfrak{M})) \). On the other hand, we have:

\[
S = [(\text{Ling}(GF)(\mathfrak{M})(c), \text{Ling}(F)(\text{Ling}(G)(\mathfrak{M}))(f), \text{Ling}(GF)(\mathfrak{M})(c')) = (\text{Ling}(GF)(\mathfrak{M})(c), \text{Ling}(GF)(\mathfrak{M})(f), \text{Ling}(GF)(\mathfrak{M})(c'))].
\]

Then we have the equality \( \text{Ling}(F) \circ \text{Ling}(G)(\mathfrak{M}) = \text{Ling}(GF)(\mathfrak{M}) \) between linguistic structures.

(2) Now if \( \gamma: \mathfrak{M} \to \mathfrak{M}' \) is a morphism in \( \text{Ling}(\mathcal{E}) \), with \( S' = \text{Auth}(\mathfrak{M}') \), it is easy to see that the equality \( \text{Ling}(F)(\text{Ling}(G)(\gamma))(c) = \text{Ling}(GF)(\gamma)(c) \) holds for every \( c \in \text{Ob}(\mathcal{C}) \), which is endorsed by \( S' \). Then we have the equality \( \text{Ling}(GF)(\gamma) = \text{Ling}(F) \circ \text{Ling}(G)(\gamma) \) between morphisms in \( \text{Ling}(\mathcal{C}) \).

By (1) and (2), the equality \( \text{8} \) follows.

The equality \( \text{Ling}(\text{id}_\mathcal{C}) = \text{id}_{\text{Ling}(\mathcal{C})} \) for every identity functor \( \text{id}_\mathcal{C} \) follows similarly.

\[\square\]

Proof of Proposition 3.2.1: We show that the functor \( I: \mathcal{C} \to \text{Set} \) conforms to \( F^*(\mathfrak{M}) \). Let \( S = \text{Auth}(F^*(\mathfrak{M})) \). Then by definition of \( F^*(\mathfrak{M}) \), we have \( S = \text{Auth}(\mathfrak{M}) \). We check conditions (1) and (2) in Definition 2.2.7. Condition (1) follows easily. To check (2), let \( f: c \to c' \) be a morphism in \( \mathcal{C} \), and \( x \in I(c) \). Since \( J \) conforms to \( \mathfrak{M} \), \( Ff \in \text{Mor}(\mathcal{D}) \) and \( x \in J(Fc) \), we have \( S = M(Ff)(x, J(Ff)(x)) \), or in other words \( S = F^*(\mathfrak{M})(f)(x, I(f)(x)) \), and hence condition (2) follows.

\[\square\]

Proof of Proposition 3.2.3: We only need to check that \( q \) conforms to \( \alpha \). Let \( S = \text{Auth}(\mathfrak{L}) \), \( c \in \text{Ob}(\mathcal{C}) \) and \( x \in I(c) \). Then \( Fc \in \text{Ob}(\mathcal{D}) \), and \( x \) is a token of \( Fc \) in the set \( J(Fc) \). Since \( r \) conforms to \( \beta \), we have that \( S = \alpha(c)(x, q_r(x)) \). In other words, \( S = \alpha(c)(x, q_r(x)) \).

\[\square\]

Proof of Proposition 3.2.5: Consider composable morphisms \((\beta, r): (\mathfrak{M}, J) \to (\mathfrak{M}', J')\) and \((\beta', r'): (\mathfrak{M}', J') \to (\mathfrak{M}'', J'')\) in \( \text{Inst}(\mathcal{D}) \). In order to show that the equality

\[\square\]
holds, it suffices to check that \( F^* (r' \circ r) = F^* (r') \circ F^* (r) \) for every object \( c \in \text{Ob}(\mathcal{C}) \), by Proposition 3.1.6. But this equality follows easily, and so does \( \text{Inst}(\mathcal{F})(\text{id}_\mathfrak{M}, \text{id}_J) = (\text{id}_{F^* (\mathfrak{M})}, \text{id}_{F^* (J)}) \), for every instantiated linguistic structure \((\mathfrak{M}, J)\) on \( \mathcal{D} \).

\[
\text{Proof of Theorem 3.2.6:}\quad \text{To prove } \text{Inst}(GF) = \text{Inst}(F) \circ \text{Inst}(G) \text{ for every pair of functors } F: \mathcal{C} \rightarrow \mathcal{D} \text{ and } G: \mathcal{D} \rightarrow \mathcal{E}, \text{ it suffices to check for every } (\mathfrak{M}, K) \in \text{Ob}(\text{Inst}(\mathcal{E})) \text{ and } (\gamma, v) \in \text{Mor}(\text{Inst}(\mathcal{E})) \text{ that } F^* (G^* (K)) = (GF)^* (K) \text{ and } F^* (G^* (v)) = (GF)^* (v). \text{ On the other hand, the equality } \text{Inst}(\text{id}_C) = \text{id}_{\text{Inst}(C)} \text{ for every } C \in \text{Ob} (\mathbf{Cat}) \text{ is also easy to verify.}\]

\[
\text{Proof of Proposition 4.2.1:}\quad \text{Let } \mathcal{O} \text{ be an olog formed by a category } \mathcal{C} \text{ and a linguistic structure on it, also denoted } \mathcal{O}. \text{ Let } d: S \rightarrow \text{Auth}(\mathcal{O}) \text{ be a delegation. Define a new collection } d^*(\mathcal{O}) \text{ of linguistic expressions as follows:}
\]

i. Every \( c \in \text{Ob}(\mathcal{C}) \) is assigned the noun phrase \( d^*(\mathcal{O})(c) := \mathcal{O}(c) \).

ii. Every \( f \in \text{Mor}(\mathcal{C}) \) is assigned the verb phrase \( d^*(\mathcal{O})(f) := \mathcal{O}(f) \).

iii. Every commutative diagram in \( \mathcal{C} \), as in Figure 1, is assigned the fact

\[
(d^*(\mathcal{O})(p), (d^*(\mathcal{O})(f_1); \ldots; d^*(\mathcal{O})(f_n)), d^*(\mathcal{O})(q)) \cong (d^*(\mathcal{O})(p), (d^*(\mathcal{O})(g_1); \ldots; d^*(\mathcal{O})(g_m)), d^*(\mathcal{O})(q)).
\]

If we set \( S := \text{Auth}(d^*(\mathcal{O})) \), then it is not hard to check that \( d^*(\mathcal{O}) \) satisfies the conditions of Definition 2.1.15 (since \( d \) is a delegation), making \( d^*(\mathcal{O}) \) a linguistic structure on \( \mathcal{C} \) endorsed by \( S \). We allow some abuse of notation, and denote the olog \((\mathcal{C}, d^*(\mathcal{O}))\) also by \( d^*(\mathcal{O}) \).

Associated to this olog \( d^*(\mathcal{O}) \), we define a linguistic functor \( \phi: \mathcal{O} \rightarrow d^*(\mathcal{O}) \) by the pair \( \phi = (\text{id}_\mathcal{C}, \gamma) \), where \( \text{id}_\mathcal{C}: \mathcal{C} \rightarrow \mathcal{C} \) is the identity functor on \( \mathcal{C} \), and \( \gamma: \mathcal{O} \rightarrow d^*(\mathcal{O}) \) is the morphism of linguistic structures on \( \mathcal{C} \) whose \( c \)-component verb phrases (with \( c \in \text{Ob}(\mathcal{C}) \)) are read \( \langle \gamma(c) \rangle \) := “is of course”.

We show that \( \phi \) is a cocartesian arrow. Suppose we are given a linguistic functor \( \phi' = (F, F^\sharp): \mathcal{O}' \rightarrow d^*(\mathcal{O}) \) such that there exists a delegation \( h^{\text{op}}: \text{Auth}(\mathcal{O}') \rightarrow \text{Auth}(\mathcal{O}) \) satisfying the equality \( \text{Auth}(\phi')^{\text{op}} \circ h^{\text{op}} = \text{Auth}(\phi') \) (in \( \text{Def}^{\text{op}} \)). Define a linguistic functor \( \psi: \mathcal{O}' \rightarrow \mathcal{O} \) by the pair \( (F, \delta) \), where \( \delta: \mathcal{O}' \rightarrow F^*(\mathcal{O}) \) is the morphism of linguistic structures on the underlying category of \( \mathcal{O}' \), say \( \mathcal{D} \), given by \( \delta(d) = F^1(d) \) for every \( d \in \text{Ob}(\mathcal{D}) \), such that \( \delta_0 := h: \text{Auth}(\mathcal{O}) \rightarrow \text{Auth}(\mathcal{O}') \). It is not hard to check that \( \psi \) is the only linguistic functor satisfying the equality \( \phi \circ \psi = \phi' \). Therefore, \( \phi \) is cocartesian.
Proof of Proposition 4.2.3: Let \((V,X,\omega)\) be an olog complex. Then it is clear that the identity \(\text{id}_{(V,X,\omega)}\), given by the triple \((\text{id}_V, \text{id}_X, \text{id}_\omega)\), defines a morphism of complexes of ologs.

Now suppose we are given two morphisms

\[
(f, \tilde{f}, f^\dagger): (V_1, X_1, \omega_1) \to (V_2, X_2, \omega_2), \quad \text{and} \quad (g, \tilde{g}, g^\dagger): (V_2, X_2, \omega_2) \to (V_3, X_3, \omega_3).
\]

Then we have two morphisms of simplicial complexes \((f^{\text{op}}, \tilde{f}^{\text{op}}): (V_2, X_2^{\text{op}}) \to (V_1, X_1^{\text{op}})\) and \((g^{\text{op}}, \tilde{g}^{\text{op}}): (V_3, X_3^{\text{op}}) \to (V_2, X_2^{\text{op}})\), along with natural transformations \(f^\dagger: \omega_1 \circ \tilde{f}^{\text{op}} \Rightarrow \omega_2\) and \(g^\dagger: \omega_2 \circ \tilde{g}^{\text{op}} \Rightarrow \omega_3\). Since \(\mathbb{P}_+(g \circ f)^{\text{op}} = \mathbb{P}_+(f^{\text{op}}) \circ \mathbb{P}_+(g^{\text{op}})\), we have that

\[
((g \circ f)^{\text{op}}, \tilde{g} \circ \tilde{f}^{\text{op}}): (V_3, X_3^{\text{op}}) \to (V_1, X_1^{\text{op}})
\]

defines a morphism of simplicial complexes, where \(\tilde{g} \circ \tilde{f}^{\text{op}}\) denotes the composite function \(\tilde{f}^{\text{op}} \circ \tilde{g}^{\text{op}}\).

Now, define \((g \circ f)^\dagger: \omega \circ \tilde{g} \circ \tilde{f}^{\text{op}} \Rightarrow \omega_3\) as the natural transformation given by

\[
(g \circ f)^\dagger_S := g^\dagger_3 \circ f^\dagger_3(S), \text{ for every } S \in X_3
\]

where the previous composition is defined in \(\text{Olog}\).

Define the composition of \((f, \tilde{f}, f^\dagger)\) and \((g, \tilde{g}, g^\dagger)\) in \(\text{OlogX}\) by the triple

\[
(g, \tilde{g}, g^\dagger) \circ (f, \tilde{f}, f^\dagger) := (g \circ f, \tilde{g} \circ \tilde{f}, (g \circ f)^\dagger).
\]

This composition is clearly associative, for which the identity morphisms \((\text{id}_V, \text{id}_X, \text{id}_\omega)\) defined above are unital.

\[\square\]

Proof of Proposition 4.2.5: We divide this proof into three parts (1), (2) and (3). In (1), we define how \(\Phi\) acts on objects. We do the same for morphisms in (2). Finally, in (3) we verify that \(\Phi\) defined by (1) and (2) is indeed a functor.

(1) Let \(\mathcal{O}\) be an olog, with \(V = \text{Auth}(\mathcal{O})\). We construct a functor \(\omega: \mathbb{P}_+(V)^{\text{op}} \to \text{Olog}\) in order to get a olog complex \((V, \mathbb{P}_+(V), \omega)\).

- a. For every simplex \(S \in \mathbb{P}_+(V)\), set \(\omega(S) := \mathcal{O}|_S\) (see the paragraph after Proposition 4.2.1).

- b. For every inclusion \(j: S \to T\) with \(S\) and \(T\) nonempty subsets of \(V\), i.e., \(j\) is a morphism in \(\mathbb{P}_+(V)\), define \(\omega(j): \omega(T) \to \omega(S)\) as the linguistic functor formed by the identity functor \(\text{id}_C: \mathcal{C} \to \mathcal{C}\), and the morphism \(\mathcal{O}|_T \to \mathcal{O}|_S\) of linguistic
structures on $\mathcal{C}$ whose $c$-components (with $c \in \text{Ob}(\mathcal{C})$) are all given by the unit verb phrase $e$.

Then $\omega : \mathbb{P}_{+}(V)^{\text{op}} \rightarrow \text{Olog}$ acting on objects as in $a.$, and on morphisms as in $b.$, defines a functor.

It is not hard to see that the triple $(V, \mathbb{P}_{+}(V), \omega)$ defines an olog complex, that will be denoted $\Phi(\mathcal{O})$.

(2) Now suppose we are given a linguistic functor $(F, F^{I}) : \mathcal{O}_{1} \rightarrow \mathcal{O}_{2}$, where $\Phi(\mathcal{O}_{1}) = (V_{1}, \mathbb{P}_{+}(V_{1}), \omega_{1})$ and $\Phi(\mathcal{O}_{2}) = (V_{2}, \mathbb{P}_{+}(V_{2}), \omega_{2})$. Then we have a morphism of simplicial complexes $(F_{1}^{b}, \mathbb{P}_{+}(F_{1}^{b})): (V_{2}, \mathbb{P}_{+}(V_{2})) \rightarrow (V_{1}, \mathbb{P}_{+}(V_{1}))$. So in order to obtain a morphism $\Phi(\mathcal{O}_{1}) \rightarrow \Phi(\mathcal{O}_{2})$ of complexes of ologs, it suffices to construct a natural transformation $\varphi^{F}_{*} : \omega_{1} \circ \mathbb{P}_{+}(F_{1}^{b}) \Rightarrow \omega_{2}$.

Let $\mathcal{C}$ be the underlying category of $\mathcal{O}_{1}$. For every $T \subseteq V_{2}$, define the linguistic functor $\varphi^{F}_{T} : \omega_{1}(F_{1}^{b}(T)) \rightarrow \omega_{2}(T)$, where $\omega_{1}(F_{1}^{b}(T)) = \mathcal{O}_{1}|_{F_{1}^{b}(T)}$ and $\omega_{2}(T) = \mathcal{O}_{2}|_{T}$, by the pair $\varphi^{F}_{T} = (F_{1}^{b}, \varphi^{F}_{T})$, where $(\varphi^{F}_{T}) : \mathcal{O}_{1}|_{F_{1}^{b}(T)} \rightarrow F^{*}(\mathcal{O}_{2}|_{T})$ is the morphism of linguistic structures whose $c$-components are given by $(\varphi^{F}_{T})_{c}(c) = F_{1}^{b}(c)$, for every $c \in \text{Ob}(\mathcal{C})$, and whose author set delegation $(\varphi^{F}_{T})_{A}^{b} = F_{1}^{b}|_{T}$ is defined by the restriction $(\varphi^{F}_{T})_{A}^{b} := F_{1}^{b}|_{T}$. It is easy to check that $\varphi^{F}_{T}$ defines a natural transformation $\omega_{1} \circ \mathbb{P}_{+}(F_{1}^{b}) \Rightarrow \omega_{2}$. Thus, we set $\Phi(F, F^{I}) : (F_{1}^{b}, \mathbb{P}_{+}(F_{1}^{b}), \varphi^{F}_{T})$.

(3) First, suppose we are given two linguistic functors $(F, F^{I}) : \mathcal{O}_{1} \rightarrow \mathcal{O}_{2}$ and $(G, G^{I}) : \mathcal{O}_{2} \rightarrow \mathcal{O}_{3}$. We show the equality

\[
\Phi((G, G^{I}) \circ (F, F^{I})) = \Phi(G, G^{I}) \circ \Phi(F, F^{I}),
\]

where $\Phi(F, F^{I}) = (F_{1}^{b}, \mathbb{P}_{+}(F_{1}^{b}), \varphi^{F})$ and $\Phi(G, G^{I}) = (G_{1}^{b}, \mathbb{P}_{+}(G_{1}^{b}), \varphi^{G})$, with natural transformations $\varphi^{I} : \omega_{1} \circ \mathbb{P}_{+}(F_{1}^{b}) \Rightarrow \omega_{2}$ and $\varphi^{G} : \omega_{2} \circ \mathbb{P}_{+}(G_{1}^{b}) \Rightarrow \omega_{3}$. On the one hand, recall that $(GF)^{b} = F_{1}^{b} \circ G_{1}^{b}$, and so we have $\mathbb{P}_{+}((GF)^{b}) = \mathbb{P}_{+}(F_{1}^{b}) \circ \mathbb{P}_{+}(G_{1}^{b})$. On the other hand, the natural transformation $\varphi^{GF} : \omega_{1} \circ \mathbb{P}_{+}((GF)^{b}) \Rightarrow \omega_{3}$ corresponding to $(G, G^{I}) \circ (F, F^{I})$ is defined as follows: for every $W \subseteq V_{3}$, $\varphi^{GF}_{W}$ is the linguistic functor formed by $GF$ and the morphism $(\varphi^{GF}_{W})_{c} : \omega_{1}((GF)^{b}(W)) \rightarrow \omega_{3}(W)$ of linguistic structures on $\mathcal{C}$ given by

\[
(\varphi^{GF}_{W})_{c} = (GF)^{b}(c) = F^{I}(c) \circ G^{I}(Fc) = (\varphi^{F}_{G_{1}^{b}(W)})_{c} \circ (\varphi^{G}_{W})_{c},
\]

for every $c \in \text{Ob}(\mathcal{C})$. Then the previous equality implies that $\varphi^{GF}_{W} = (\varphi^{F}_{G_{1}^{b}(W)}) \circ \varphi^{G}_{W}$. Hence, the equality (9) follows.

The equality $\Phi(\text{id}_{\mathcal{C}}, \text{id}_{\mathcal{C}}) = \text{id}_{(V, \mathbb{P}_{+}(V), \omega)}$ is easy to verify.

Therefore, $\Phi$ defines a functor $\text{Olog} \rightarrow \text{Ologx}$. 

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