Nonclassicality and entanglement: observable conditions

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Abstract. A unified characterization of continuous-variable quantum states is presented that gives general insight in their nonclassical features and, in particular, in their entanglement properties. Nonclassicality is characterized in terms of observable characteristic functions. Equivalently, the nonclassicality conditions are reformulated in terms of moments. The latter approach turns out to be a powerful tool for deriving a complete set of entanglement conditions for bipartite quantum systems, which are fully equivalent to the nonpositivity of the partially transposed density operator. The possibilities of observing the needed moments by balanced homodyne correlation measurements are analyzed.

1. Introduction

During the last decades the progress in the experimental techniques led to an manifold of realizations of basic phenomena of quantum theory. Initiated by the field of quantum optics, the demonstrations of photon antibunching [1], sub-Poissonian radiation [2] and squeezed light [3] opened new perspectives for the generation and application of quantum states of radiation which have no counterpart in classical physics. Moreover, nonclassical states were realized under various conditions, e.g. in cavity QED, the quantized motion of trapped ions and in a variety of other systems.

More recently, the research efforts on quantum information processing and quantum computation lead to new and rapidly developing topics, for example see [4]. In this context entanglement is considered as the main resource. In many applications of this type the entangled states to be used are prepared in finite dimensional Hilbert spaces, for a review see [5]. In the following we will provide an overview on some recent attempts of characterizing nonclassical and entangled quantum states of continuous-variable (CV) systems, such as harmonic oscillators, in infinite dimensional Hilbert spaces. Particular emphasis will be paid on the possibility to reformulate the conditions for nonclassicality and entanglement in a rather general form. The derived conditions are throughout related to measurement principles.

The paper is organized as follows. In Sec. 2 we provide necessary and sufficient conditions for the nonclassicality of quantum states, based on the failure of the $P$-function to be a probability density, in terms of both observable characteristic functions and moments. Entanglement conditions are formulated in Sec. 3, which represent a complete CV reformulation of the nonpositivity of partial transposition (NPT) condition of the density operator. In Sec. 4 we deal with the measurement of the moments to be used in the nonclassicality and entanglement conditions. A summary and some conclusions are given in Sec. 5.
2. Nonclassicality conditions
Let us start with a density operator \( \hat{\rho} \), represented in the form of the Glauber-Sudarshan \( P \)-representation [6, 7],
\[
\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2 \alpha,
\]
where \( P(\alpha) \) is a generalized function. The expectation value of a normally ordered operator function, \( \langle \hat{F}(\hat{a}^\dagger, \hat{a}) \rangle \), on the state (1) is given by the integral
\[
\langle \hat{F}(\hat{a}^\dagger, \hat{a}) \rangle = \int P(\alpha)F(\alpha^*, \alpha) d^2 \alpha.
\]
In the field of quantum optics, the quantum state \( \hat{\rho} \) is usually called nonclassical if its \( P \)-function \( P(\alpha) \) fails to be a classical probability on the complex plane [8].

Typical example of such states are the squeezed states, which are characterized by the condition
\[
\exists \varphi : \langle (\Delta \hat{x}_\varphi)^2 \rangle < 0.
\]
The quadrature operator, \( \hat{x}_\varphi = \hat{a} e^{-i\varphi} + \hat{a}^\dagger e^{i\varphi} \), is given in terms of the annihilation (creation) operator \( \hat{a} (\hat{a}^\dagger) \) of the considered harmonic oscillator, \( \varphi \) is a phase parameter. States with a sub-Poissonian number statistics,
\[
\langle (\Delta \hat{n})^2 \rangle < 0,
\]
are another example of nonclassical states that clearly fulfill the above definition. In fact, for any operator \( \hat{F}(\hat{a}^\dagger, \hat{a}) \) its normally ordered variance cannot obey the condition
\[
\langle (|\Delta \hat{F}(\hat{a}^\dagger, \hat{a})|^2 \rangle \rangle = \int P(\alpha)[\Delta F(\alpha^*, \alpha)]^2 d^2 \alpha < 0,
\]
as long as the \( P \)-function has the properties of a classical probability density.

The condition (5) can be considered to be an observable condition for nonclassicality, provided that the normally ordered variance of the operator \( \hat{F} \) represents an observable quantity. However, in general the consideration of normally ordered variances of special observables cannot characterize the nonclassical properties of the \( P \)-function completely. Since \( P(\alpha) \) can be highly singular and thus cannot be observed in experiments or reconstructed from measured data, one needs other approaches for the experimental verification of the nonclassical nature of a given quantum state.

2.1. Nonclassical characteristic functions
The approach presented in the following is based on the normally-ordered characteristic function, defined by
\[
\Phi(\beta) = \int P(\alpha)e^{\alpha \hat{\beta}^* - \hat{\alpha} \beta^*} d^2 \alpha.
\]
The conditions for the \( P \)-function to have all the properties of a classical probability \( P(\alpha) \) can be reformulated in terms of the characteristic function \( \Phi(\beta) \) by using the Bochner theorem [9]: \( P(\alpha) \) is a probability distribution on the complex plane if and only if the inequality
\[
\sum_{i,j=1}^{n} \Phi(\beta_i - \beta_j)\xi_i^* \xi_j \geq 0 \quad (7)
\]
is valid for any integer \( n \) and all complex numbers \( \beta_i, \xi_i, i = 1, \ldots, n \). According to the Sylvester criterion for the nonnegativity of a quadratic form, the condition (7) is equivalent to the following one:
\[
D_k(\beta_1, \ldots, \beta_k) = \begin{vmatrix} 1 & \Phi_{12} & \ldots & \Phi_{1k} \\ \Phi_{12}^* & 1 & \ldots & \Phi_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{1k}^* & \Phi_{2k}^* & \ldots & 1 \end{vmatrix} \geq 0,
\]
Figure 1. The quadrature distributions $p(x, \varphi)$ and the absolute values of the related characteristic functions, $|G(k, \varphi)|$, are shown for the vacuum state (dotted line), the Fock state $|n = 4\rangle$ (full line), and the even coherent state, $|\alpha\rangle_+ \sim |\alpha\rangle + |-\alpha\rangle$, for $\alpha = 2$ and $\varphi = \pi/2$ (dashed line).

for any integer $k$ and for all complex numbers $\beta_i$, $i = 1, \ldots, k$, where $\Phi_{ij} = \Phi(\beta_i - \beta_j)$. Note that the Sylvester criterion states that the condition (7) is equivalent to the nonnegativity of the determinants (8), as well as any of its principal subdeterminants. As one can easily see, any principal subdeterminant of the determinants (8) has the same structure, and due to this the conditions (7) and (8) are equivalent.

This allows us to formulate a condition that completely characterizes the nonclassicality of the $P$-function. A given quantum state is nonclassical if there exists an integer $k$ and together with $k$ complex numbers $\beta_1, \ldots, \beta_k$, such that the conditions

$$D_k(\beta_1, \ldots \beta_k) < 0$$

are fulfilled [10]. In this form, however, the conditions for the nonclassicality are still not formulated in terms of observable functions.

The normally-ordered characteristic function $\Phi$ can be easily related to the characteristic function $G(k, \varphi)$ of the observable quadrature distribution via

$$G(k, \varphi) = G_{gr}(k) \Phi(ike^{i\varphi}),$$

cf. e.g. [11]. In the ground state (or vacuum state) of the harmonic oscillator the corresponding functions read as

$$\Phi_{gr} = 1, \quad G_{gr}(k) = e^{-k^2/2}.$$  

By combining Eqs (8) – (11) we can readily express the nonclassicality conditions in terms of the observable quadrature characteristic functions $G(k, \varphi)$.

Let us consider the simplest form of the corresponding condition for nonclassicality, which is based on the second order determinant, $D_2 < 0$. Since this is the lowest order that yields insight in nonclassical effects, we may speak about first-order nonclassicality. Using the quadrature characteristic function we obtain the nonclassicality condition in the form [12]

$$|G(k, \varphi)| > G_{gr}(k).$$

It is interesting that this simple condition is fulfilled by a manifold of nonclassical states, such as squeezed states, Fock states, superpositions of coherent states, and so forth.

Examples for the quadrature distributions $p(x, \varphi)$ and the corresponding characteristic functions $G(k, \varphi)$ are shown in Fig. 1 for some typical nonclassical states, such as Fock states and superpositions of coherent states. They clearly fulfill the condition (12) for first-order nonclassicality. For quadrature squeezed states it is trivial that this condition is fulfilled as well. Experimental demonstrations of the condition (12) have been given in [13] and [14]. In the latter case even the second-order condition, $D_3 < 0$, has been studied.
2.2. Nonclassical moments

According to Eq. (2) the condition that $P$-function $P(\alpha)$ is a probability distribution can be formulated as follows: any operator $\hat{f}$ whose normally ordered form exists satisfies the inequality

$$\langle \hat{f}^\dagger \hat{f} \rangle \geq 0.$$  \hspace{1cm} (13)

By choosing a normally-ordered Fourier representation of $\hat{f}$,

$$\hat{f} = \int d^2 \alpha \ f(\alpha) : \hat{D}(\alpha) :,$$  \hspace{1cm} (14)

with $\hat{D}(\alpha)$ being the coherent displacement operator, Eq. (13) represents a continuous version of the Bochner condition, with $f(\alpha)$ being a continuous function with compact support [15].

Under these conditions the normally ordered expansion of $\hat{f}$ exists and converges,

$$\hat{f} = \sum_{n,m=0}^{+\infty} c_{nm} \hat{a}^n \hat{a}^m.$$  \hspace{1cm} (15)

The nonnegativity condition (13) now reads as the nonnegativity of the following quadratic form [16]:

$$\langle \hat{f}^\dagger \hat{f} \rangle = \sum_{n,m,k,l=0}^{+\infty} c_{nm} c_{kl} \langle \hat{a}^m \hat{a}^{n+l} \rangle \geq 0.$$  \hspace{1cm} (16)

This condition is equivalent to the nonnegativity of the infinite matrix

$$M = \begin{pmatrix} 1 & \langle \hat{a} \rangle & \langle \hat{a}^2 \rangle & \langle \hat{a}^3 \rangle & \vdots \\ \langle \hat{a}^1 \rangle & \langle \hat{a}^2 \rangle & \langle \hat{a}^3 \rangle & \langle \hat{a}^4 \rangle & \vdots \\ \langle \hat{a}^2 \rangle & \langle \hat{a}^3 \rangle & \langle \hat{a}^4 \rangle & \langle \hat{a}^5 \rangle & \vdots \\ \langle \hat{a}^3 \rangle & \langle \hat{a}^4 \rangle & \langle \hat{a}^5 \rangle & \langle \hat{a}^6 \rangle & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. $$  \hspace{1cm} (17)

Leading principal minors of this matrix we denote $d_k$, where $k$ is the order of the minor. If any principal minor of the matrix (17) is negative, then the state under study is nonclassical.

For example, the third order leading principal minor

$$d_3 = \begin{vmatrix} 1 & \langle \hat{a} \rangle & \langle \hat{a}^2 \rangle \\ \langle \hat{a} \rangle & \langle \hat{a}^2 \rangle & \langle \hat{a}^3 \rangle \\ \langle \hat{a}^2 \rangle & \langle \hat{a}^3 \rangle & \langle \hat{a}^4 \rangle \end{vmatrix}.$$  \hspace{1cm} (18)

can be rewritten in terms of normally ordered variances of quadratures as

$$d_3 = \frac{1}{4} \min_{x} \left\{ \langle (\Delta x)^2 \rangle \right\} \max_{\varphi} \left\{ \langle (\Delta x_{\varphi})^2 \rangle \right\}. $$  \hspace{1cm} (19)

One can easily see that the last factor in the product (19) is always nonnegative, so the condition $d_3 < 0$ is the squeezing condition. This condition can be generalized to describe $k$-th power amplitude squeezing, which is characterized by the condition [17]

$$\Delta_k = \begin{vmatrix} 1 & \langle \hat{a}^k \rangle \\ \langle \hat{a}^k \rangle & \langle \hat{a}^{2k} \rangle \end{vmatrix} < 0.$$  \hspace{1cm} (20)
Figure 2. Minimum uncertainty states of the third (left) and the fourth (right) order.

As before, the determinant $\Delta_k$ can be factorized,

$$\Delta_k = \frac{1}{4} \min_{\varphi} \langle (\Delta \hat{f}_\varphi^{(k)})^2 : \rangle \max_{\varphi} \langle (\Delta \hat{f}_\varphi^{(k)})^2 : \rangle,$$

where $\hat{f}_\varphi^{(k)} = \hat{a}^k e^{-i\varphi} + \hat{a}^{\dagger k} e^{i\varphi}$. The particular case of $k = 2$ represents amplitude-squared squeezing [18].

Examples of $Q$-functions of quantum states showing $k$-th power amplitude squeezing are shown in Fig. 2. They represent minimum uncertainty $k$-th power squeezed states for $k = 3$ and $k = 4$, for more details see [17]. Note that some special cases of our nonclassicality conditions in terms of moments were introduced in [19, 20].

3. Entanglement conditions

A bipartite state $\hat{\varrho}$ is called separable if it is a convex combination of factorizable states [21],

$$\hat{\varrho} = \sum_{n=0}^{+\infty} p_n \hat{\varrho}_1^{(n)} \otimes \hat{\varrho}_2^{(n)},$$

where $p_n \geq 0$ and $\sum_{n=0}^{+\infty} p_n = 1$. A bipartite state $\hat{\varrho}$ is called entangled if it cannot be represented in such a form. Given a quantum state $\hat{\varrho}$ it is a very difficult computational problem to find out whether it is separable or not since all the states $\hat{\varrho}_i^{(n)}$, $i = 1, 2$, and the coefficients $p_n$, $n \geq 0$ in the decomposition (22) are unknown. From the decomposition in the $P$-representation,

$$\hat{\varrho} = \iint P(\alpha, \beta) |\alpha\rangle \langle \alpha| \otimes |\beta\rangle \langle \beta| d^2\alpha d^2\beta,$$

it follows that any bipartite classical state is separable. On the other hand, there exist also nonclassical separable states so that the test of nonclassicality of a quantum state usually gives no insight in entanglement.

3.1. Complete characterization of NPT

There is a simple but in many cases powerful condition for entanglement, the so-called Peres-Horodecki condition [22, 23], which is based on the NPT of the density operator. The partial transposition with
respect to one subsystem (it is equivalent to transpose the first or the second one) of any separable state is positive semidefinite, what can be easily seen from the decomposition (22),

$$\hat{\varrho}^{\text{PT}} = \sum_{n=0}^{+\infty} p_n \hat{\varrho}_1^{(n)} \otimes \hat{\varrho}_2^{(n)\text{PT}} \geq 0.$$  \hfill (24)

Let us analyze this condition in more detail. An operator $\hat{A}$ is positive semidefinite if and only if for all operators $\hat{f}$ the following mean value is nonnegative:

$$\langle \hat{f}^\dagger \hat{f} \rangle_{\hat{A}} = \text{tr}(\hat{A} \hat{f}^\dagger \hat{f}) \geq 0.$$  \hfill (25)

It was proven in [24] that it is enough to test the condition (25) only for operators $\hat{f}$ whose normally-ordered form exists. In such a case one can write the following expansion:

$$\hat{f} = \sum_{n,m,k,l=0}^{+\infty} c_{nmkl} \hat{a}^\dagger_n \hat{a}^\dagger_m \hat{b}^\dagger_k \hat{b}^\dagger_l.$$  \hfill (26)

For the mean value (25), with $\hat{A} = \hat{\varrho}^{\text{PT}}$, we have

$$\langle \hat{f}^\dagger \hat{f} \rangle = \sum_{n,m,k,l,p,q,r,s=0}^{+\infty} c_{pqrs}^* c_{nmkl} \langle \hat{a}^\dagger_q \hat{a}^\dagger_p \hat{a}^\dagger_n \hat{a}^\dagger_m \hat{b}^\dagger_s \hat{b}^\dagger_r \hat{b}^\dagger_k \hat{b}^\dagger_l \rangle.$$  \hfill (27)

On the other hand, when the mean value (25) is calculated for $\hat{A} = \hat{\varrho}^{\text{PT}}$ and denoted by $\langle \hat{f}^\dagger \hat{f} \rangle^{\text{PT}}$, one obtains the quadratic form

$$\langle \hat{f}^\dagger \hat{f} \rangle^{\text{PT}} = \sum_{n,m,k,l,p,q,r,s=0}^{+\infty} c_{pqrs}^* c_{nmkl} M_{pqrs,nmkl},$$  \hfill (28)

where $M_{pqrs,nmkl} = \langle \hat{a}^\dagger_q \hat{a}^\dagger_p \hat{a}^\dagger_n \hat{a}^\dagger_m \hat{b}^\dagger_s \hat{b}^\dagger_r \hat{b}^\dagger_k \hat{b}^\dagger_l \rangle^{\text{PT}}$. It is easy to check that the following relation is valid:

$$\langle \hat{a}^\dagger_q \hat{a}^\dagger_p \hat{a}^\dagger_n \hat{a}^\dagger_m \hat{b}^\dagger_s \hat{b}^\dagger_r \hat{b}^\dagger_k \hat{b}^\dagger_l \rangle^{\text{PT}} = \langle \hat{a}^\dagger_q \hat{a}^\dagger_p \hat{a}^\dagger_n \hat{a}^\dagger_m \hat{b}^\dagger_k \hat{b}^\dagger_l \hat{b}^\dagger_k \hat{b}^\dagger_l \rangle.$$  \hfill (29)

According to the Sylvester criterion, the nonnegativity of the quadratic form (28) is equivalent to the nonnegativity of the matrix

$$\begin{pmatrix}
M_{11} & \ldots & M_{1N} & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
M_{N1} & \ldots & M_{NN} & \ldots \\
\ldots & \ldots & \ldots & \ldots 
\end{pmatrix},$$  \hfill (30)

where $(n, m, k, l)$ is the $i$th index and $(p, q, r, s)$ is the $j$th. Explicitly this matrix reads as

$$\begin{pmatrix}
1 & \langle \hat{a} \rangle & \langle \hat{a}^\dagger \rangle & \langle \hat{b} \rangle & \langle \hat{b} \rangle & \ldots \\
\langle \hat{a}^\dagger \rangle & \langle \hat{a} \rangle \langle \hat{a} \rangle & \langle \hat{a}^\dagger \rangle \langle \hat{a}^\dagger \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \ldots \\
\langle \hat{a} \rangle & \langle \hat{a} \rangle \langle \hat{a} \rangle & \langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \ldots \\
\langle \hat{b} \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \ldots \\
\langle \hat{b} \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \langle \hat{b} \rangle \langle \hat{b} \rangle & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix}.$$  \hfill (31)

The Sylvester criterion states that all principal minors of this matrix are nonnegative. Consequently, entanglement is identified by demonstrating that there exists at least one principal minor which becomes
negative. Testing all the principal minors is fully equivalent to the Peres-Horodecki NPT criterion [24, 25]. For attempts to generalize this approach beyond NPT see also [26].

Let us consider the example of a noisy CV Bell-type state,

\[
\hat{\rho} = \eta |\psi\rangle \langle \psi| + (1 - \eta) |0\rangle \langle 0|,
\]

(32)

where

\[
|\psi\rangle = \mathcal{N}(\alpha, \beta) (|\alpha, -\beta\rangle - | -\alpha, \beta\rangle)
\]

and \(\mathcal{N}(\alpha, \beta) = [2(1 - e^{-2(|\alpha|^2 + |\beta|^2)})]^{-1/2}\). We may formulate special entanglement conditions of the form

\[
s, \tilde{s} < 0,
\]

(33)

where

\[
s = \begin{vmatrix}
1 & \langle \hat{b}^{\dagger}\rangle & \langle \hat{a}\hat{b}^{\dagger}\rangle \\
\langle \hat{b}\rangle & \langle \hat{b}\hat{b}^{\dagger}\rangle & \langle \hat{a}\hat{b}\hat{b}^{\dagger}\rangle \\
\langle \hat{a}\hat{b}\rangle & \langle \hat{a}\hat{b}\hat{b}^{\dagger}\rangle & \langle \hat{a}\hat{a}\hat{b}\hat{b}^{\dagger}\rangle
\end{vmatrix},
\]

\[
\tilde{s} = \begin{vmatrix}
1 & \langle \hat{a}^{2}\rangle & \langle \hat{a}\hat{b}^{\dagger}\rangle \\
\langle \hat{a}^{\dagger}\hat{a}\rangle & \langle \hat{a}\hat{a}^{\dagger}\rangle & \langle \hat{a}\hat{a}\hat{b}\hat{b}^{\dagger}\rangle \\
\langle \hat{a}\hat{b}\rangle & \langle \hat{a}\hat{b}\hat{a}\hat{b}^{\dagger}\rangle & \langle \hat{a}\hat{a}\hat{b}\hat{a}\hat{b}^{\dagger}\rangle
\end{vmatrix}.
\]

(34)

It is seen in Fig. 3 that \(\tilde{s}\) identifies the entanglement for all \(\eta\)-values, whereas \(s\) proofs entanglement only for large \(\eta\)-values.

3.2. Special entanglement conditions

In the following we will demonstrate that some previously known entanglement conditions can be obtained from the above complete set of conditions based on NPT. This also includes such conditions that have not directly been derived by using partial transposition.

The nonnegativity of the leading principal minor of the fifth order is exactly the Simon condition [27]. The minor

\[
d_5 = \begin{vmatrix}
1 & \langle \hat{a}\rangle & \langle \hat{a}\hat{b}\rangle \\
\langle \hat{a}^{\dagger}\rangle & \langle \hat{a}\hat{a}\rangle & \langle \hat{a}\hat{a}\hat{b}\rangle \\
\langle \hat{a}\hat{b}\rangle & \langle \hat{a}\hat{b}\hat{a}\rangle & \langle \hat{a}\hat{a}\hat{b}\hat{b}\rangle
\end{vmatrix},
\]

(35)

can be identified with

\[
d_5 = S,
\]

(36)
where $S$ is the quantity used to formulate the original form of the Simon condition as a necessary condition for separability as

$$S = \det A_1 \det A_2 + \left( \frac{1}{4} + \det C \right)^2 - \text{Tr}\left( A_1 J C J A_2 J C^T J \right)$$

$$-\frac{1}{4} \left( \det A_1 + \det A_2 \right) \geq 0,$$  \hfill (37)

with

$$A_i = \begin{pmatrix} \langle \Delta \hat{x}_i \Delta \hat{x}_i \rangle & \langle \Delta \hat{x}_i \Delta \hat{p}_i \rangle \\ \langle \Delta \hat{p}_i \Delta \hat{x}_i \rangle & \langle \Delta \hat{p}_i \Delta \hat{p}_i \rangle \end{pmatrix}$$

and

$$C = \begin{pmatrix} \langle \Delta \hat{x}_1 \Delta \hat{x}_2 \rangle & \langle \Delta \hat{x}_1 \Delta \hat{p}_2 \rangle \\ \langle \Delta \hat{p}_1 \Delta \hat{x}_2 \rangle & \langle \Delta \hat{p}_1 \Delta \hat{p}_2 \rangle \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  \hfill (38)

Any violation of the relation (37) is a sufficient condition for entanglement.

Another condition was obtained in [28]. For any non-zero real $r$ define

$$\hat{u} = |r| \hat{x}_1 + \frac{1}{r} \hat{x}_2, \quad \hat{v} = |r| \hat{p}_1 - \frac{1}{r} \hat{p}_2,$$  \hfill (40)

then for any separable state the following inequality holds true:

$$\langle \Delta \hat{u} \Delta \hat{u} \rangle + \langle \Delta \hat{v} \Delta \hat{v} \rangle - \left( r^2 + \frac{1}{r^2} \right) \geq 0.$$  \hfill (41)

Substituting the operators (40) into this inequality and optimizing it over $r$, one can write the condition (41) in an equivalent form as

$$\langle \Delta \hat{a}^\dagger \Delta \hat{a} \rangle \langle \Delta \hat{b}^\dagger \Delta \hat{b} \rangle - \text{Re}^2 \langle \Delta \hat{a} \Delta \hat{b} \rangle \geq 0.$$  \hfill (42)

This is a weaker version of the condition corresponding to a $3 \times 3$ principal minor,

$$\begin{vmatrix} 1 & \langle \hat{a} \rangle & \langle \hat{b} \rangle \\ \langle \hat{a}^\dagger \rangle & \langle \hat{a}^\dagger \hat{a} \rangle & \langle \hat{a}^\dagger \hat{b} \rangle \\ \langle \hat{b} \rangle & \langle \hat{a} \hat{b} \rangle & \langle \hat{b} \hat{b} \rangle \end{vmatrix} = \langle \Delta \hat{a}^\dagger \Delta \hat{a} \rangle \langle \Delta \hat{b}^\dagger \Delta \hat{b} \rangle - |\langle \Delta \hat{a} \Delta \hat{b} \rangle|^2 \geq 0.$$  \hfill (43)

Some other entanglement conditions were obtained in [29–32]. All of them are recovered as special cases of our hierarchy of conditions, for some details see [24].

We note that some limited insight in multipartite entanglement can be gained by considering all possibilities of partial transpositions [33]. However, genuine multipartite entanglement cannot be verified in this way. Only in special cases solutions of this complex problem are known [34].

4. Balanced homodyne correlation measurements

The moments $\langle \hat{a}^{m} \hat{a}^{n} \rangle$ of the signal field can be measured with the device shown in Fig. 4. It works as follows. The input signal is superimposed with a local oscillator which is in a coherent state $|\alpha\rangle$, where $\alpha = |\alpha| e^{i\varphi_{\text{LO}}}$. By changing the phase of the local oscillator and measuring the correlations of the photodetectors for different phases of the local oscillator one may extract the values of the moments $\langle \hat{a}^{m} \hat{a}^{n} \rangle$. This idea can be further extended to the multimode case, see [35].

By measuring different correlations of the photodetectors and combining them in a proper way, one may derive the moments in the following manner. The first (50:50) beam splitter combines the signal and the local oscillator as

$$\hat{a}_\pm = \frac{1}{\sqrt{2}} (\hat{a} \pm i \hat{a}_{\text{LO}}).$$  \hfill (44)
Similarly, the fields at the detectors are given by
\[ \hat{a}_{1,2} = \frac{1}{2} (\hat{a} + i\hat{a}_{\text{LO}}) \pm \text{vac}, \]
and
\[ \hat{a}_{3,4} = \frac{1}{2} (\hat{a} - i\hat{a}_{\text{LO}}) \pm \text{vac}, \]
where "vac" denotes the vacuum noise effects. Measuring the difference of the mean photon numbers \( \langle \hat{a}_{1}^{\dagger} \hat{a}_{1} \rangle \) and \( \langle \hat{a}_{3}^{\dagger} \hat{a}_{3} \rangle \),
\[ \langle \hat{a}_{1}^{\dagger} \hat{a}_{1} \rangle - \langle \hat{a}_{3}^{\dagger} \hat{a}_{3} \rangle = \frac{1}{2} |\alpha| |\bar{x}_{\phi}| = \frac{1}{2} |\alpha| (\langle \hat{a}e^{-i\phi} \rangle + \langle \hat{a}^{\dagger}e^{i\phi} \rangle), \]
we get the mean value of the quadrature operator, where \( \phi = \phi_{\text{LO}} + \pi/2 \).

Higher order correlations can also be detected and properly combined, for example, in the form
\[ \langle \hat{a}_{1}^{\dagger} \hat{a}_{2}^{\dagger} \hat{a}_{2} \hat{a}_{3} \rangle - 2\langle \hat{a}_{1}^{\dagger} \hat{a}_{1} \hat{a}_{3}^{\dagger} \hat{a}_{3} \rangle + \langle \hat{a}_{3}^{\dagger} \hat{a}_{3} \hat{a}_{4}^{\dagger} \hat{a}_{4} \rangle = \frac{1}{4} |\alpha|^2 \langle \hat{x}_{\phi}^2 \rangle \]
\[ = \frac{1}{4} |\alpha|^2 (\langle \hat{a}^2 \rangle e^{-2i\phi} + 2\langle \hat{a}^{\dagger} \hat{a} \rangle + \langle \hat{a}^{12} \rangle e^{2i\phi}). \]

It is seen that by Fourier transforming the \( \phi \)-dependence one obtains moments such as \( \langle \hat{a}^2 \rangle \), that are not measured directly by photon-correlation techniques. The method is easily extended by replacing each photodetector in Fig. 4 with a beam splitter (with vacuum noise in the unused input port) and a pair of photodetectors. Now we can extract the moments \( \langle \hat{a}^{1n} \hat{a}^{m} \rangle \) with \( n + m \leq 4 \) via Fourier transform. The method can be further extended to higher orders. In general, the measured correlation functions are combined in a binomial form. Extensions to moments of two and more modes are straightforward [35], this requires phase control for each of the modes.

5. Summary and Conclusion
We have formulated conditions for the nonclassicality of quantum states of CV systems. Our starting point is an old theorem by Bochner, which provides the necessary and sufficient conditions for a certain function to represent a classical characteristic function, that is the Fourier transform of a probability
density. Any violation of this condition, considered for the characteristic function of the Glauber-Sudarshan $P$-function, provides a sufficient condition for nonclassicality in the spirit of Titulaer and Glauber. The conditions are reformulated in terms of a hierarchy of conditions for the observable characteristic functions of quadrature distributions.

It is also shown that a reformulation of the Bochner condition opens the way to formulate nonclassicality conditions in terms of normally ordered moments of photon annihilation and creation operators. The formulation with moments also allows one to derive new conditions for the entanglement of bipartite quantum states in the infinite dimensional Hilbert space. One may derive a complete set of conditions that is equivalent to the NPT condition for the density operator. Some insight in multipartite entanglement can be gained as well. However, the problems of characterizing bound entanglement and genuine multipartite entanglement in CV systems still remain open.

The conditions for both nonclassicality and entanglement are based on moments containing unequal powers of photon annihilation and creation operators. In direct detection, including photon correlation measurements, only correlations of equal powers of annihilation and creation operators are accessible. We propose to perform balanced homodyne correlation measurements by connecting the advantages of correlation techniques with those of balanced homodyning. This new method yields insight in the moments needed for the characterization of nonclassicality and entanglement.

**Acknowledgments**

The authors gratefully acknowledge support by the Deutsche Forschungsgemeinschaft.

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