On Cartan Spaces with $m$-th Root Metrics

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Abstract

In this paper, we define some non-Riemannian curvature properties for Cartan spaces. We consider Cartan space with the $m$-th root metric. We prove that every $m$-th root Cartan space of isotropic Landsberg curvature, or isotropic mean Landsberg curvature, or isotropic mean Berwald curvature reduces to a Landsberg, weakly Landsberg and weakly Berwald space, respectively. Then we show that $m$-th root Cartan space of almost vanishing $H$-curvature satisfies $H = 0$.

Keywords: Landsberg curvature, mean Landsberg curvature, mean Berwald curvature, $H$-curvature.

1 Introduction

É. Cartan has originally introduced a Cartan space, which is considered as dual of Finsler space [4]. Then Rund and Brickell studied the relation between these two spaces [3][16]. The theory of Hamilton spaces was introduced by Miron [11]. He proved that Cartan space is a particular case of Hamilton space.

Let us denote the Hamiltonian structure on a manifold $M$ by $(M, H(x, p))$. If the fundamental function $H(x, p)$ is 2-homogeneous on the fibres of the cotangent bundle $T^*M$, then the notion of Cartan space is obtained [10][14][15]. Indeed, the modern formulation of the notion of Cartan spaces is due of the Miron [11][12]. Based on the studies of Kawaguchi [6], Miron [10], Hrimiuc-Shimada [5], Anastasiei-Antonelli [2], Mazetis [7][8][9], Urbonas [22] etc., the geometry of Cartan spaces is today an important chapter of differential geometry.

Under Legendre transformation, the Cartan spaces appear as the dual of the Finsler spaces [11]. Finsler geometry was developed since 1918 by Finsler, Cartan, Berwald, Akbar-Zadeh, Matsumoto, Shen and many others, see [1][17]. Using this duality several important results in the Cartan spaces can be obtained: the canonical nonlinear connection, the canonical metrical connection, the notion of $(\alpha, \beta)$-metrics, the theory of $m$-root metrics, etc [7][8][13]. Therefore, the theory of Cartan spaces has the same symmetry and beauty like Finsler geometry. Moreover, it gives a geometrical framework for the Hamiltonian theory of Mechanics or Physical fields.

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The theory of $m$-th root metric has been developed by H. Shimada [18], and applied to Biology as an ecological metric. It is regarded as a direct generalization of Riemannian metric in a sense, i.e., the second root metric is a Riemannian metric. Recently studies, shows that the theory of $m$-th root Finsler metrics play a very important role in physics, theory of space-time structure, general relativity and seismic ray theory [19][20][21].

An $n$-dimensional Cartan space $C^n$ with $m$-th root metric is a Cartan structure $C^n = (M^n, K(x, p))$ on differentiable $n$-manifold $M^n$ equipped with the fundamental function $K(x, p) = \sqrt{a^{i_1i_2...i_m}(x)p_{i_1}p_{i_2}...p_{i_m}}$ where $a^{i_1i_2...i_m}(x)$, depending on the position alone, is symmetric in all the indices $i_1, i_2, ..., i_m$ and $m \geq 3$. The Hessian of $K^2$ give us the fundamental tensor $g$ of Cartan space. Taking a vertical derivation of $g$ give us the Cartan torsion $C$. The rate of change of the Cartan torsion along geodesics, $L$ is said to be Landsberg curvature. A Cartan metric with $L = dFC$ is called isotropic Landsberg metric, where $c = c(x)$ is a scalar function on $M$. In this paper, we prove that every $m$-root Cartan spaces with isotropic Landsberg curvature is a Landsberg space.

**Theorem 1.1.** Let $(M, K)$ be an $m$-th root Cartan space. Suppose that $K$ is isotropic Landsberg metric, $L + cKC = 0$ for some scalar function $c = c(x)$ on $M$. Then $K$ reduces to a Landsberg metric.

Taking a trace of Cartan torsion $C_y$ and Landsberg curvature $L_y$ give us the mean Cartan torsion $I_y$ and mean Landsberg curvature $J_y$, respectively. A Cartan metric with $J = 0$ and $J = cFI$ is called weakly Landsberg and isotropic mean Landsberg metric, respectively, where $c = c(x)$ is a scalar function on $M$. We show that every $m$-root Cartan spaces of isotropic mean Landsberg curvature reduces to weakly Landsberg space.

**Theorem 1.2.** Let $(M, K)$ be an $m$-th root Cartan space. Suppose that $K$ has isotropic mean Landsberg curvature, $J + cKI = 0$ for some scalar function $c = c(x)$ on $M$. Then $K$ reduces to a weakly Landsberg metric.

Taking a trace of Berwald curvature of Cartan metric $K$ gives rise the $E$-curvature. The Cartan metric $K$ with $E = 0$ and $E = \frac{n+1}{2}cK\ h$ is called weakly Berwald and isotropic mean Berwald metric, respectively, where $c = c(x)$ is a scalar function on $M$ and $h = h^{ij}dx_i dx_j$ is the angular metric.

**Theorem 1.3.** Let $(M, K)$ be an $m$-th root Cartan space. Suppose that $K$ has isotropic mean Berwald curvature $E = \frac{n+1}{2}cK\ h$, for some scalar function $c = c(x)$ on $M$. Then $K$ reduces to a weakly Berwald metric.

Akbar-Zadeh introduces the non-Riemannian quantity $H$ which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. More precisely, the non-Riemannian quantity $H = H^{ij}dx_i \otimes dx_j$ is defined by $H^{ij} := E^{ij}p_x$. The Cartan metric $K$ is called of almost vanishing $H$-curvature if $H^{ij} = \frac{n+1}{2k}\theta h^{ij}$, where $\theta$ is a 1-form on $M$.

**Theorem 1.4.** Let $(M, K)$ be an $n$-dimensional $m$-th root Cartan space. Suppose that $K$ has almost vanishing $H$-curvature, $H = \frac{n+1}{2k}K^{-1}\theta h$ for some 1-form $\theta$ on $M$. Then $H = 0$. 

2
2 Preliminaries

A Cartan space is a pair \( C^n = (M^n, K(x, p)) \) such that the following axioms hold good:
1. \( K \) is a real positive function on the cotangent bundle \( T^*M \), differentiable on \( T^*M_0 := T^*M \setminus \{0\} \) and continuous on the null section of the canonical projection
   \[ \pi^*: T^*M \to M; \]
2. \( K \) is positively 1-homogenous with respect to the momenta \( p_i \);
3. The Hessian of \( K^2 \), with the elements
   \[ g^{ij}(x, p) = \frac{1}{2} \frac{\partial^2 K^2}{\partial p_i \partial p_j} \]
is positive-defined on \( T^*M_0 \).

An \( n \)-dimensional Cartan space \( C^n \) with \( m \)-th root metric is by definition a Cartan structure \( C^n = (M^n, K(x, p)) \) on differentiable \( n \)-manifold \( M^n \) equipped with the fundamental function \( K(x, p) \) such that
\[
K(x, p) = \sqrt[1]{a_{i_1i_2...i_m}(x)p_{i_1}p_{i_2}...p_{i_m}}
\]
where \( a_{i_1i_2...i_m}(x) \), depending on the position alone, is symmetric in all the indices \( i_1, i_2, ..., i_m \) and \( m \geq 3 \).

From \( K(x, y) \) we define Cartan symmetric tensors of order \( r \) (\( 1 \leq r \leq m-1 \)) with the components
\[
a_{i_1i_2...i_r}(x, p) = \frac{1}{K^{m-r}}a_{i_1i_2...i_rj_1j_2...j_{m-r}}p_{j_1}p_{j_2}...p_{j_{m-r}}
\]
Thus we have
\[
a^i = \frac{a_{i_1i_2...i_m}(x)p_{i_1}p_{i_2}...p_{i_m}}{K^{m-1}},
\]
\[
a^{ij} = \frac{a_{i_1i_2...i_mj_1j_2...j_{m-r}}p_{j_1}p_{j_2}...p_{j_{m-r}}}{K^{m-2}},
\]
\[
a^{ijk} = \frac{a_{i_1i_2...i_mj_1j_2...j_{m-r}}p_{j_1}p_{j_2}...p_{j_{m-r}}}{K^{m-1}}.
\]
The normalized supporting element is given by
\[
l^i = \dot{\nabla}^i K,
\]
where \( \dot{\nabla}^i = \frac{\partial}{\partial p_i} \). The fundamental metrical d-tensor is
\[
g^{ij} = \frac{1}{2} \dot{\nabla}^i \dot{\nabla}^j K^2
\]
and the angular metrical d-tensor is given by
\[
h^{ij} = K \dot{\nabla}^i \dot{\nabla}^j K.
\]
The following hold

\[ l^i = a^i, \]
\[ g^{ij} = (m - 1) a^{ij} - (m - 2) a^i a^j, \]
\[ h^{ij} = (m - 1) (a^{ij} - a^i a^j). \]

From the positively 1-homogeneity of the \( m \)-th root Cartan metrical function, it follows that

\[ K^2(x, p) = g^{ij}(x, p)p_i p_j = a^{ij}(x, p)p_i p_j. \]

Since \( \det(g^{ij}) = (m - 1)^n - 1 \det(a^{ij}) \), the regularity of the \( m \)-th metric is equivalent to \( \det(a^{ij}) \neq 0 \). Let us suppose now that the d-tensor \( a^{ij} \) is regular, that is there exists the inverse matrix \( (a^{ij})^{-1} = (a^i) \). Obviously, we have

\[ a_i a^i = 1, \]

where

\[ a_i = a_{is} a^s = p_i K. \]

Under these assumptions, we obtain the inverse components \( g_{ij}(x, p) \) of the fundamental metrical d-tensor \( g^{ij}(x, p) \), which are given by

\[ g_{ij} = \frac{1}{m - 1} a^{ij} + \frac{m - 2}{m - 1} a_i a_j. \]

We have

\[ \dot{\partial}^k (a^{ij}) = \frac{(m - 2)}{K} [a^{ijk} - a^{ij} a^k], \]
\[ \dot{\partial}^k (a^{i}) = \frac{(m - 1)}{K} [a^{ik} - a^i a^k], \]
\[ \dot{\partial}^k (a^i a^j) = \frac{(m - 1)}{K} [a^{ijk} a^j + a^{ijk} a^i - 2a^i a^j a^k]. \]

The Cartan tensor \( C^{ijk} = -\frac{1}{2} (\dot{\partial}^k g^{ij}) \) are given in the form

\[ C^{ijk} = -\frac{(m - 1)(m - 2)}{2K} (a^{ijk} - a^{ij} a^k - a^{kJ} a^i - a^i a^j + 2a^i a^j a^k). \]

The \( m \)-th Christoffel symbols is defined by

\[ \{i_1...i_m, j\} = \frac{1}{2(m - 1)} (\dot{\partial}^{i_1} a^{i_2...i_m j} + \dot{\partial}^{i_2} a^{i_3...i_m j} + \cdots + \dot{\partial}^{i_m} a^{i_1...i_{m-1} j} - \dot{\partial}^{j} a^{i_1...i_m}), \]

where the cyclic permutation is applied to \( (i_1...i_m) \) in the first \( m \) terms of the right-hand side.

Now, if we write the equations of geodesics in the usual form

\[ \frac{d^2 x_i}{ds^2} + 2G_i(x, \frac{dx}{ds}) = 0, \]
then the quantities \( G_i(x,y) \) are given by
\[
a^{hv}_r G_r = \frac{1}{mK^{m-2}} \{00...0, h\},
\]
where we denote by the index 0 the multiplying by \( p_i \) as usual, that is
\[
\{00...0, h\} = \{i_1i_2...i_m, h\}p_{i_1}p_{i_2}...p_{i_m}, a^{hv}_r = a^{hrio0...0}/K^{m-2}.
\]
Using the definition of \( a^{hv}_r \), we can write (5) in the form
\[
a^{hrio0...0}_r G_r = \frac{1}{m} \{00...0, h\}. \tag{6}
\]
Differentiating of (6) with respect to \( p_i \) yields
\[
a^{hrio0...0}_r G_r + (m-2)a^{hrio0...0}r G^r = \{i00...0, h\}, \tag{7}
\]
where \( G^r_i = \partial^r G_r \). By differentiating of (7) with respect to \( p_j \), we have
\[
a^{hrio0...0}ij G^r + (m-2)a^{hrio0...0}i G^r + (m-2)(m-3)a^{hrio0...0}r G^r = (m-1)\{ij00...0, h\},
\]
where \( G^i_j = \partial^i G^r_i \) constitute the coefficients of the Berwald connection \( B\Gamma = (G^i_j, G^r_i) \). The above equations can be written in the following forms
\[
K^{m-3}[K a^{hv}_r G^i_r + (m-2)a^{hr} G_r] = \{i00...0, h\}, \tag{8}
\]
and
\[
K^{m-3}[K a^{hv}_r G^{ij}_r + (m-2)(a^{hr} G^i_r + a^{hr} G^j_r)] + K^{m-4}(m-2)(m-3)a^{hrio3} G^r = (m-1)\{ij00...0, h\}. \tag{9}
\]
By differentiation of (9) with respect to \( p_k \), we get the Berwald curvature of Berwald connection as follows
\[
K^{m-3}[K a^{hv}_r G^{ijk}_r + (m-2)a^{hijr} G^x_r + (i,j,k)] + (m-2)(m-3)K^{m-5}[K a^{hijr} G^x_r + (i,j,k)] + (m-4)a^{hijkr} G_r = (m-1)(m-2)\{ijk00...0, h\}, \tag{10}
\]
where \( \{..., (ijk)\} \) shows the cyclic permutation of the indices \( i, j, k \) and summation. Multiplying (10) with \( p_k \) yields
\[
K^{m-2}[p^i G^{ij}_r + (m-2)a^{iv} G^j_r + (i,j,k)] + (m-2)(m-3)K^{m-4}[K a^{ijr} G^x_e + (i,j,k)] + (m-4)a^{ijkr} G_r = (m-1)(m-2)\{ijk00...0, 0\}. \tag{11}
\]
Remark 2.1. In the equations (10) and (11), we have some terms with coefficients \((m-3)\) and \((m-4)\). We shall be concerned mainly with cubic metric \((m=3)\) and quartic metric \((m=4)\)

\[ K^3 = a^{ijk}(x)p_ip_jp_k, \quad K^4 = a^{hijk}(x)p_hp_ip_jp_k. \]

For these metrics, it is supposed that the terms with \((m-3)\) and \((m-4)\) vanish, respectively. For instance, (10) of a cubic metric is reduced to following

\[ Ka^{hr}G^{ijk}_r + \{a^{hir}G^{jk}_r + (i,j,k)\} = \{ijk, h\}. \]

3 Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. First, we remark the following.

Lemma 3.1. (\cite{?}) Let \((M, K)\) be an \(m\)-th root Cartan space. Then the spray coefficients of \(K\) are given by following

\[ G_r = \frac{1}{m}(00...0, h) / a^{hr00...0}. \]

Now, we can prove the Theorem 1.1.

Proof of Theorem 1.1. By assumption, the Cartan metric \(K\) has isotropic Landsberg curvature \(L = cKC\) where \(c = c(x)\) is a scalar function on \(M\). By definition, we have

\[ L^{ijk} = -\frac{1}{2}p^sG^{sijk}, \]

where

\[ p^s = g^{sj}p_j = [(m-1)a^sj - (m-2)a^s]p_j = (m-1)Ka^s - (m-2)Ka^s = Ka^s = a^{s00...0} / K^{m-2}. \]

Then we get

\[ L^{ijk} = -\frac{1}{2}a^{s00...0} / K^{m-2}G^{sijk}. \]

By assumption, we have

\[ -\frac{1}{2K^{m-2}}a^{s00...0}G^{sijk} = cKC^{ijk} \]

or equivalently

\[ a^{s00...0}G^{sijk} = \frac{c}{K^{2-m}}(m-1)(m-2)(a^{ijk} - a^iak - a^jai - a^kaij + 2ai^ja^k). \]
By Lemma 3.1, the left-hand side of (13) is rational function, while its right-hand side is an irrational function. Thus, either \( c = 0 \) or \( a \) satisfies the following

\[
a^{ijk} - a^{ij} a^k - a^{ik} a^j - a^{ki} a^j + 2a^i a^j a^k = 0. \tag{14}
\]

Plugging (17) into (2) implies that \( C^{ijk} = 0 \). Hence, \( K \) is Riemannian metric, which contradicts with our assumption. Therefore, \( c = 0 \). This completes the proof.

4 Proof of Theorem 1.2

The quotient \( J^i/I^j \) is regarded as the relative rate of change of mean Cartan torsion \( I^j \) along Cartan geodesics. Then \( K \) is said to be isotropic mean Landsberg metric if \( J^i = cK^i \), where \( c = c(x) \) is a scalar function on \( M \). In this section, we are going to prove the Theorem 1.2. More precisely, we show that every \( m \)-th root isotropic mean Landsberg metric reduces to a weakly Landsbe rg metric.

**Proof of Theorem 1.2** The mean Cartan tensor of \( K \) is given by following

\[
I^i = g_{jk}C^{ijk} = -\frac{(m-2)}{2K}\left\{a^{ij} - \delta_k^i a^k - na^j - \delta_j^i a^j + 2a^j\right\} = -\frac{(m-2)}{2K}\left\{a^{ik} - na^i\right\}.
\]

The mean Landsberg curvature of \( K \) is given by

\[
J^i = g_{jk}L^{ijk} = \left[ \frac{1}{m-1}a_{jk} + \frac{m-2}{m-1}a_j a_k \right] \left[ -\frac{1}{2}a^{000\ldots 0}/K^{m-2}G_s^{ijk} \right].
\]

Since \( J = cFI \), then we have

\[
c(m-1)(m-2)\left\{a^{ijk} - a^{ij} a^k - a^{ik} a^j - a^{ki} a^j + 2a^i a^j a^k\right\} = a^{000\ldots 0}/K^{m-2}G_s^{ijk}.
\]

Thus we get

\[
a^{000\ldots 0}G_s^{ijk} = cK^{m-2}(m-1)(m-2)\left\{a^{ijk} - a^{ij} a^k - a^{ik} a^j - a^{ki} a^j + 2a^i a^j a^k\right\}. \tag{15}
\]

By Lemma 3.1, the left hand side of (15) is a rational function with respect to \( y \), while its right-hand side is an irrational function with respect to \( y \). Thus, either \( c = 0 \) or \( a \) satisfies the following

\[
a^{ijk} - a^{ij} a^k - a^{ik} a^j - a^{ki} a^j + 2a^i a^j a^k = 0.
\]

That implies that \( C^{ijk} = 0 \). Hence, \( K \) is Riemannian metric, which contradicts with our assumption. Therefore, \( c = 0 \). This completes the proof.
5 Proof of Theorem 1.3

Let \((M, K)\) be a Cartan space of dimension \(n\). Denote by \(\tau(x, y)\) the distortion of the Minkowski norm \(K_x\) on \(T_x^*M_0\), and \(\sigma(t)\) be the geodesic with \(\sigma(0) = x\) and \(\dot{\sigma}(0) = y\). The rate of change of \(\tau(x, y)\) along Cartan geodesics \(\sigma(t)\) called \(S\)-curvature. \(K\) is said to have isotropic \(S\)-curvature if

\[ S = (n + 1)cK, \]

where \(c = c(x)\) is a scalar function on \(M\). \(K\) is called of almost isotropic \(S\)-curvature if

\[ S = (n + 1)cK + \eta, \]

where \(c = c(x)\) is a scalar function and \(\eta = \eta^i(x)p_i\) is a 1-form on \(M\).

Remark 5.1. By taking twice vertical covariant derivatives of the \(S\)-curvature, we get the \(E\)-curvature

\[ E^{ij}(x, p) := \frac{1}{2} \frac{\partial^2 S}{\partial p_i \partial p_j}. \]

It is remarkable that, we can get the \(E\)-curvature by taking a trace of Berwald curvature of Cartan metric \(K\), also. The Cartan metric \(K\) is called weakly Berwald metric if \(E = 0\) and is said to have isotropic mean Berwald curvature if \(E = \frac{n+1}{2}cK\h\), where \(c = c(x)\) is a scalar function on \(M\) and \(\h = h^{ij}dx_i dx_j\) is the angular metric.

In this section, we are going to prove an extension of Theorem 1.3. More precisely, we prove the following.

Theorem 5.2. Let \((M, K)\) be an \(m\)-th root Cartan space. Then the following are equivalent:

a) \(K\) has isotropic mean Berwald curvature, i.e., \(E = \frac{n+1}{2}cK\h\);

b) \(K\) has vanishing \(E\)-curvature, i.e., \(E = 0\);

c) \(K\) has almost isotropic \(S\)-curvature, i.e., \(S = (n + 1)cK + \eta\);

where \(c = c(x)\) is a scalar function and \(\eta = \eta^i(x)p_i\) is a 1-form on \(M\).

To prove the Theorem 5.2, first we show the following.

Lemma 5.3. Let \((M, K)\) be an \(n\)-dimensional \(m\)-th root Cartan space. Then the following are equivalent:

a) \(S = (n + 1)cK + \eta\);

b) \(S = \eta\);

where \(c = c(x)\) is a scalar function and \(\eta = \eta^i(x)p_i\) is a 1-form on \(M\).
Proof. By lemma 5.1, the $E$-curvature of an $m$-th root metric is a rational function. On the other hand, by taking twice vertical covariant derivatives of the $S$-curvature, we get the $E$-curvature. Thus $S$-curvature is a rational function.

Suppose that $K$ has almost isotropic $S$-curvature, $S = (n + 1)cK + \eta$, where $c = c(x)$ is a scalar function and $\eta = \eta^i(x)p_i$ is a 1-form on $M$. Then the left hand side of $S - \eta = (n + 1)c(x)K$ is rational function while the right hand is irrational function. Thus $c = 0$ and $S = \eta$.

**Lemma 5.4.** Let $(M, K)$ be an $n$-dimensional $m$-th root Cartan space. Then the following are equivalent:

a) $E = \frac{n+1}{2}cK\theta$;

b) $E = 0$;

where $c = c(x)$ is a scalar function.

*Proof.* Suppose that $K$ has isotropic mean Berwald curvature

$$E = \frac{n+1}{2}cK\theta,$$  \hspace{1cm} (16)

where $c = c(x)$ is a scalar function. The left hand side of (16), is a rational function while the right hand is irrational function. Thus $c = 0$ and $E = 0$. \qed

**Proof of Theorem 5.2.** By Lemmas 5.3 and 5.4 we get the proof.

**Corollary 5.5.** Let $(M, K)$ be an $n$-dimensional $m$-th root Cartan space. Suppose that $K$ has isotropic $S$-curvature, $S = (n + 1)c(x)K$, for some scalar function $c = c(x)$ on $M$. Then $S = 0$.

6 Proof of Theorem 1.4

**Proof of Theorem 1.4.** Let $(M, K)$ be an $n$-dimensional $m$-th root Cartan space. Suppose that $K$ be of almost vanishing $H$-curvature, i.e.,

$$H^{ij} = \frac{n+1}{2K}\theta h^{ij},$$  \hspace{1cm} (17)

where $\theta$ is a 1-form on $M$. The angular metric $h^{ij} = g^{ij} - K^2 p^i p^j$ is given by the following

$$h^{ij} = (m-1)(a^{ij} - a^i a^j),$$  \hspace{1cm} (18)

Plugging (18) into (17) yields

$$H^{ij} = \frac{n+1}{2K}\theta[(m-1)(a^{ij} - a^i a^j)].$$  \hspace{1cm} (19)
By Lemma 3.1 and $H_{ij} = E_{ij}^p p_s$, it is easy to see that $H_{ij}$ is rational with respect to $y$. Thus, (19) implies that $\theta = 0$ or

$$ (m - 1)(a^{ij} - a^i a^j) = 0. \tag{20} $$

By (18) and (20), we conclude that $h_{ij} = 0$, which is impossible. Hence $\theta = 0$ and then $H_{ij} = 0$.

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