Symplectic sigma models in superspace

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Abstract

We discuss a special “symplectic” class of $\mathcal{N} = 4$ supersymmetric sigma models in $(0 + 1)$ dimension with $5r$ bosonic and $4r$ complex fermionic degrees of freedom. These models can be described off shell by $\mathcal{N} = 2$ superfields (so that only half of supersymmetries are manifest) and also by $\mathcal{N} = 4$ superfields in the framework of the harmonic superspace approach. Using the latter, we derive the general form of the relevant bosonic target metric.

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1 Introduction

Three classes of supersymmetric sigma models (in dimensions $d \leq 4$) are widely known: generic, Kähler, and hyper–Kähler. In all these models, the fermions are vectors in tangent space so that the number of dynamical (on-shell) fermionic degrees of freedom coincides with the dimension of the bosonic target manifold. The component action of all such models has the same generic form and the presence of extra supersymmetries is due to some special properties of Kähler resp. hyper–Kähler manifolds. A theorem proven in [1] claims that no other supersymmetric sigma model can be constructed. This is true under two assumptions: (i) the kinetic part of the Lagrangian depends only on the metric in the standard way \( \sim g_{ij}(\phi)\partial_{\mu}\phi^i\partial_{\mu}\phi^j \); (ii) the fields \( \phi^i \) depend on time and at least one spatial coordinate.

In two dimensions, the first condition can be relaxed by adding the torsion term \( \sim \epsilon_{\mu
u}h_{ij}(\phi)\partial_{\mu}\phi^i\partial_{\nu}\phi^j \). If such a term is allowed, one can construct a twisted supersymmetric sigma model, which enjoys \( N = 4 \) supersymmetry \(^1\) even though the target space is not hyper–Kähler (neither it is Kähler) \(^2\). If the second condition is not satisfied and we are dealing not with field theory, but with quantum mechanics, even a larger variety of supersymmetric sigma models can be constructed. A special interesting subclass of such \((0 + 1)\) sigma models is formed by those which cannot be directly reproduced by dimensional reduction from the higher \( d \) ones.

The simplest model of this type is defined on a conformally flat 3–dimensional manifold with fermions and bosons belonging, respectively, to the fundamental and adjoint representations of \( SO(3) \sim USp(2) \). The Lagrangian in components was constructed in [4] and the superfield description was given in [5] (see also [6]).

Let us dwell on the latter in some more details. We introduce the real superfield \( V_k(t, \theta_\alpha, \bar{\theta}^\alpha) \), \( (k = 1, 2, 3) \); \( \bar{\theta}^\alpha = (\theta_\alpha)^\dagger \), or, in spinor notation,

\[
V_{\alpha\beta} = i(\sigma_k)_\gamma^{\alpha\beta}V_k, \quad V_{\alpha\beta} = V_{\beta\alpha},
\]

(1.1)

where \( \epsilon_{\beta\gamma} = -\epsilon_{\gamma\beta} \), \( \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\gamma^\alpha \) and \( \epsilon_{12} = \epsilon^{21} = 1 \). Let us impose the constraint

\[
D_{(\alpha}V_{\beta\gamma)} = 0, \quad \bar{D}_{(\alpha}V_{\beta\gamma)} = 0
\]

(1.2)

where

\[
D_{\alpha} = \frac{\partial}{\partial \theta^\alpha} + i\bar{\theta}_\alpha \frac{\partial}{\partial t}, \quad \bar{D}_{\alpha} = \frac{\partial}{\partial \bar{\theta}^\alpha} - i\theta_\alpha \frac{\partial}{\partial t}
\]

(1.3)

are the covariant derivatives, \( \bar{D}_{\alpha} = (D^\alpha)^\dagger \), and the doublet \( SU(2) \) indices are raised and lowered with the help of the invariant tensors \( \epsilon^{\alpha\beta} \), \( \epsilon_{\alpha\beta} \). The constrained superfield \( V_k \) is expressed in terms of components as follows:

\[
V_k = A_k + \bar{\psi}\sigma_k\theta + \bar{\theta}\sigma_k\psi + \epsilon_{kj\mu}\dot{A}_j\theta\sigma_k\theta + D\bar{\theta}\sigma_k\theta
\]

\(^1\)In our convention here \( N \) always counts the number of complex supercharges; then \( N = 4 \) in \( d = 2 \) means two complex supercharges both in the left and right light-cone sectors.

\(^2\)The relevant geometry is sometimes called “hyper-Kähler with torsion” (HKT)\(^3\).
\[
\frac{\partial^2 \bar{\theta}}{\partial^2 \theta} + i(\tilde{\theta} \sigma_k \tilde{\psi} - \tilde{\psi} \sigma_k \theta) \theta + \frac{\hat{A}_k}{4} \theta^2 \bar{\theta}^2
\]

(\theta^2 = \theta^\alpha \theta_\alpha, \quad \bar{\theta}^2 = \bar{\theta}_\alpha \bar{\theta}^\alpha, \quad \theta \theta = \theta^\alpha \theta_\alpha). \] We have, as promised, 3 bosonic (\(A_k\)) and 2 complex fermionic (\(\psi_\alpha\)) variables. \(D\) is the auxiliary field. By construction, the action

\[
S = \int dt \int d^2 \theta d^2 \bar{\theta} \, F(V_k)
\]

enjoys off-shell \(\mathcal{N} = 2\) supersymmetry.

The action (1.5) is genetically related to the action of 4D \(\mathcal{N} = 1\) supersymmetric gauge models. The superfield \(V_{\alpha\beta}\) can be represented as

\[
V_{\alpha\beta} = (D_\alpha \bar{D}_\beta + D_\beta \bar{D}_\alpha) C
\]

where \(C\) is the standard vector superfield prepotential reduced to \((0+1)\) dimensions. The standard action of the dimensionally reduced supersymmetric photodynamical \(\propto \int dt \int d^2 \theta \, W^2_\alpha\) with \(W_\alpha \propto (\bar{D})^2 D_\alpha C\) can be cast in the form (1.5) with \(F_0(V) = V_k^2\). The point is that in the quantum mechanical limit \(V_k\) is \(gauge\) \(invariant\), and now \(any\) function \(F(V_k)\) is allowed as the invariant Lagrangian. Actually, \(V_k\) is just the spatial part of the initial 4D vector \(U(1)\) superconnection. In the case of general Lagrangian function \(F(V_k)\), it is impossible to rewrite the invariant action only in terms of \(W_\alpha\), so this action cannot be recovered from a 4D supersymmetric gauge-invariant action by the direct dimensional reduction \(3 + 1 \rightarrow 0 + 1\). The kinship to gauge models displays itself in the fact that the model (1.5) was first derived in \([4]\) (at the component level) as the effective action for supersymmetric QED in small spatial volume.

An obvious generalization of the model (1.5) is

\[
S = \int dt \int d^2 \theta d^2 \bar{\theta} \, F(V^A_k), \quad A = 1, \ldots, r.
\]

It involves \(r\) different fermion variables \(\psi_\alpha^A\) belonging to the fundamental representation of \(USp(2)\) and \(r\) bosonic variables belonging to the adjoint representation. Such a model (with \(F\) of some special form) represents the effective action of a non-Abelian gauge theory in small volume, with \(A\) running over Cartan generators and \(r\) being the rank of the gauge group \([7]\). A remarkable fact is that the model (1.7) describes also the dynamics of slowly moving maximally charged Reissner–Nordström black holes \([8]\). \(^3\) The correct expression of the target bosonic metric in the generic “multiflavor” case can be found in \([8, 10]\).

Another class of models was first introduced in ref. \([11]\) (for the accurate expression of the Lagrangian in components, see ref.\([12]\)). In its simplest variant, the

\(^3\)This duality is presumably a reflection of the general “Gauge theories/Strings” correspondence \([9]\).
model is defined on a 5–dimensional manifold $v_I \equiv (v_k, \phi, \bar{\phi})$ ($k = 1, 2, 3$) and involves 4 fermion degrees of freedom (belonging to the fundamental representation of $SO(5) \sim USp(4)$). The metric of the model is conformally-flat like in the model discussed earlier and it is derived from a prepotential $K(v_k, \phi, \bar{\phi})$ as

$$g_{IJ} = 2 \delta_{IJ} \frac{\partial^2 K}{\partial v_k \partial v_k} \equiv \delta_{IJ} h.$$ (1.8)

The function $K$ is subject to the constraint (5-dimensional Laplace equation)

$$\Delta^{(5)} K = \frac{\partial^2 K}{\partial v_k \partial v_k} + 2 \frac{\partial^2 K}{\partial \phi \partial \bar{\phi}} = 0.$$ (1.9)

In ref. [11], the Lagrangian was written as a function of the $\mathcal{N} = 2$ superfields $V_k$ discussed above and standard chiral superfields $\Phi, \bar{\Phi}$. The off-shell action has the form

$$S = \int dt \int d^2 \theta d^2 \bar{\theta} \ K(V_k, \bar{\Phi}, \Phi).$$ (1.10)

It is invariant under additional $\mathcal{N} = 2$ supersymmetry transformations

$$\delta \bar{\Phi} = \frac{2i}{3} \epsilon^\alpha (\sigma_k)_\alpha^\beta D_\beta V_k,$$

$$\delta \Phi = \frac{2i}{3} \bar{\epsilon}_\alpha (\sigma_k)^\alpha_\beta \bar{D}^\beta V_k,$$

$$\delta V_k = -i \epsilon^\alpha (\sigma_k)_\alpha^\beta \bar{D}_\beta \Phi - i \bar{\epsilon}_\alpha (\sigma_k)^\alpha_\beta \bar{D}^\beta \bar{\Phi},$$ (1.11)

provided that

$$\frac{\partial^2 K}{\partial V_k^2} + 2 \frac{\partial^2 K}{\partial \Phi \partial \bar{\Phi}} = 0.$$ (1.12)

When restricted to the bosonic target manifold $v_k = V_k |, \phi = \Phi |, \bar{\phi} = \bar{\Phi} |$ (hereafter, “X” stands for the lowest component of the superfield $X$), eq. (1.12) yields just the 5-dimensional Laplace equation (1.9). In the next section we explain in some more details how the constraint (1.12) can be derived. Then we consider the generalization of (1.10) to the “multiflavor” case, derive a set of constraints for the prepotential $K$ and present their solution. In Sect. 3 we present the description of these models in terms of $\mathcal{N} = 4$ harmonic superfields and discuss the relationship between the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ approaches.

In what follows, the $d = 1$ sigma models with $3r$- and $5r$-dimensional bosonic target manifolds we have defined above will be referred to as symplectic sigma models, respectively of the first and second types. This nomenclature emphasizes the property that their bosonic and fermionic fields belong to the adjoint and fundamental representations of the symplectic groups $USp(2) \sim SO(3)$ and $USp(4) \sim SO(5)$.

As explained in [12], these models are genetically related to 4D $\mathcal{N} = 2$ and 6D $\mathcal{N} = 1$ gauge theories: they describe low–energy dynamics of such theories in small box. The $SO(5)$ covariance of the $(0 + 1)$ dimensional models is related to spatial rotational symmetry of 6D theory.

To avoid a misunderstanding, let us point out that these symmetries can be explicitly broken in the sigma model actions and so are not isometries of the general bosonic target metric.
2 \( \mathcal{N} = 2 \) description

Consider the variation of (1.10) under the transformation (1.11). The term involving \( \epsilon^\alpha \) is proportional to

\[
\int dt \int d^4 \theta \left( \sigma_k \right)_\alpha^\beta \left[ \frac{2}{3} \frac{\partial \mathcal{K}}{\partial \Phi} D_\beta V_k - \frac{\partial \mathcal{K}}{\partial V_k} D_\beta \Phi \right].
\]  

(2.1)

For the integral to vanish, the integrand has to be a total (covariant) derivative, i.e.

\[
\left( \sigma_k \right)_\alpha^\beta \left[ \frac{2}{3} \frac{\partial \mathcal{K}}{\partial \Phi} D_\beta V_k - \frac{\partial \mathcal{K}}{\partial V_k} D_\beta \Phi \right] = D_\alpha R + \left( \sigma_k \right)_\alpha^\beta D_\beta L_k
\]  

(2.2)

where \( R \) and \( L_k \) are some complex functions of the involved superfields. Let us write

\[
D_\alpha R = \frac{\partial R}{\partial \Phi} D_\alpha \Phi + \frac{\partial R}{\partial V_k} D_\alpha V_k,
\]

\[
D_\beta L_k = \frac{\partial L_k}{\partial \Phi} D_\beta \Phi + \frac{\partial L_k}{\partial V_p} D_\beta V_p
\]

(2.3)

Then, comparing the coefficients of the independent structures in both sides of (2.3), we end up with the set of linear differential equations

\[
2 \frac{\partial \mathcal{K}}{\partial \Phi} - \frac{\partial L_k}{\partial V_k} = 0, \quad (a)
\]

\[
\epsilon_{kpq} \frac{\partial L_k}{\partial V_p} + \frac{\partial R}{\partial V_q} = 0, \quad (b)
\]

\[
\frac{\partial \mathcal{K}}{\partial V_k} + \frac{\partial L_k}{\partial \Phi} = 0, \quad (c)
\]

\[
\frac{\partial R}{\partial \Phi} = 0. \quad (d)
\]

(2.5)

Differentiating eq. (2.5a) with respect to \( \Phi \) and eq. (2.5c) with respect to \( V_k \) and summing up the obtained relations, we arrive at eq. (1.12) as an integrability condition for the set (2.5). The rest of equations imposes some constraints on the unknowns \( R \) and \( L_k \) the precise generic form of which is of no interest for our purposes. Nevertheless, we should show that eq. (1.12) is not only necessary, but also sufficient condition for the system (2.5) to have a nontrivial solution. It is not difficult to check that

\[
R = 0, \quad L_k = \frac{\partial Q}{\partial V_k}
\]

(2.6)
supplies a particular solution provided \( Q \) is a 5-dimensional harmonic function such that \( K = -\partial Q / \partial \Phi \). For the Green’s function of the Laplace equation

\[
K = \frac{1}{(2\Phi \Phi + V_k^2)^{3/2}} ,
\]

we have

\[
Q = \frac{1}{\Phi(2\Phi \Phi + V_k^2)^{1/2}} \quad \text{and} \quad L_k = -\frac{V_k}{\Phi(2\Phi \Phi + V_k^2)^{3/2}}
\]

(2.8)

(it is assumed that the range of the allowed values of the involved superfields does not include the singularity point \( \Phi = \bar{\Phi} = V_k = 0 \)). An arbitrary harmonic function \( Q \) can be represented as a linear superposition of the Green’s functions (2.7).

By the same token as for the symplectic \( \sigma \) model of the first kind discussed above, the Lagrangian (1.10) can be generalized to the “multiflavor” case:

\[
S = \int dt \int d^2\theta d^2\bar{\theta} \ K(V_k^A, \bar{\Phi}_A, \Phi_A) , \quad A = 1, \ldots, r .
\]

(2.9)

Let us find the constraints to be imposed on \( K \) for the action (2.9) to be \( \mathcal{N} = 4 \) supersymmetric. Proceeding along the same lines as before, we arrive at the system

\[
\begin{align*}
2 \frac{\partial K}{\partial V^A_k} - \frac{\partial L_k}{\partial V^A_k} &= 0 , \quad (a) \\
i\epsilon_{kpq} \frac{\partial L_k}{\partial V^p_k} + \frac{\partial R}{\partial V^q_k} &= 0 , \quad (b) \\
\frac{\partial K}{\partial V^A_k} + \frac{\partial L_k}{\partial \Phi^A} &= 0 , \quad (c) \\
\frac{\partial R}{\partial \Phi^A} &= 0 . \quad (d)
\end{align*}
\]

(2.10)

Comparing eqs. (2.10a) and (2.10c), we obtain a kind of the generalized harmonicity condition

\[
\left(2 \frac{\partial^2}{\partial \Phi^A \partial \Phi^B} + \frac{\partial^2}{\partial V^A_k \partial V^B_k} \right) K = 0 ,
\]

(2.11)

as the integrability condition for these equations. Eq. (2.11) actually amounts to two independent conditions which are obtained by symmetrizing and antisymmetrizing indices \( A, B \). The antisymmetric condition reads

\[
\frac{\partial^2 K}{\partial \Phi^A \partial \Phi^B} = 0 .
\]

(2.12)

This is not the end of the story. Differentiating eq. (2.10c) with respect to the argument \( \Phi^B \) and antisymmetrizing, we obtain the condition

\[
\frac{\partial^2 K}{\partial V^A_k \partial \Phi^B} = 0 .
\]

(2.13)
Its complex conjugate is
\[ \frac{\partial^2 K}{\partial V^k_k[A \partial \Phi^B]} = 0. \] (2.14)

Finally, differentiating eq. (2.10b) with respect to \( \Phi^B \) and eq. (2.10c) with respect to \( V^B_p \), and then making use of eq. (2.10d), we obtain one more integrability condition
\[ \frac{\partial^2 K}{\partial V^k_k[A \partial V^B_p]} = 0. \] (2.15)

The concise 5–dimensional form of all the conditions derived is
\[ \frac{\partial^2 K}{\partial V^A_I \partial V^B_J} = 0, \quad \frac{\partial^2 K}{\partial V^A_I \partial V^B_J} = 0, \] (2.16)
where \( V^A_I \equiv (V^k_A, \text{Re} \Phi^A, \text{Im} \Phi^A) \). The first condition is just the symmetric part of (2.11), while the second one encompasses (2.12) - (2.15).

An example of the function \( K \) satisfying the constraints (2.16) is a harmonic function \( h(V^I_1) \) (no dependence on \( V^2_I, \ldots, V^r_I \)). A more general solution is \( h(p^A_A V^A_I + X_I) \) with arbitrary \( X_I \) and \( p^A_A \) (at first glance, the constant shift \( X_I \) seems to be redundant; the reason why it was included is explained a few lines lower). Still more generally, one can write \( K \) as a sum of several terms
\[ K = \sum_n h(n) \left( p^{(n)}_A V^A_I + X^{(n)}_I \right). \] (2.17)

Eq. (2.17) can be further generalized by upgrading the sum \( \sum_n \) to an integral. Our conjecture is that in this way one can obtain a general solution for the set (2.16).

An important remark is in order. Mathematicians like to solve the Laplace equation and other elliptic equations in a finite region, imposing Dirichlet or Neumann, or mixed conditions at the boundary. For physical applications, we rather need to solve eqs.(2.16) in the whole \( \mathbb{R}^5 \), allowing certain singularities inside. The Green’s function (2.7) is such a solution for the Laplace equation with singularity at origin. The harmonic functions entering the sum in eq.(2.17) should be also thought of as such Green’s function which are singular at \( 5(r-1) \) – dimensional hyperplanes \( p^A_A V^A_I + X^{(n)}_I = 0 \). As soon as some region in \( \mathbb{R}^5 \) is free from singularities, the solution makes sense.

In fact, the model (2.9) was derived as the effective Lagrangian of 4D \( \mathcal{N} = 2 \) supersymmetric non–Abelian gauge theory in a small box [12]. In that case, the function \( K \) was shown to have the form (2.17) with \( X^{(n)}_I = 0 \) and \( p^A_A \) having the meaning of the roots in the Lie algebra of the corresponding gauge group.

When \( K \) is expressed as a sum over Green’s functions (2.7) of the argument \( p^{(n)}_A V^A_I + X^{(n)}_I \), the solution to the system of linear equations (2.10) for \( R, L_k \) can be easily found as a sum of the corresponding expressions for \( L_k \) (the function \( R \) can be set equal to zero). The metric that follows from eq.(2.9) has the form
\[ g_{ij}^{AB} = \delta_{ij} h^{AB} = 2 \delta_{ij} \frac{\partial^2 K}{\partial v^A_k \partial v^B_k}. \] (2.18)
3 $\mathcal{N} = 4$ description in harmonic superspace

Like the 3$\text{r}$-dimensional symplectic model (1.7) is related to Abelian $\mathcal{N} = 1$, 4$\text{D}$ supersymmetric photodynamics, the model (2.9) is related to $\mathcal{N} = 2$, 4$\text{D}$ Abelian supersymmetric photodynamics. Using the technique of harmonic superspace (HSS) [13, 14], the action for this model can be written in such a way that all 4 complex supersymmetries are manifest.

3.1 Overview of $\mathcal{N} = 2$ photodynamics in HSS

We start by recalling some salient features of the HSS description of $\mathcal{N} = 2$ photodynamics.

The basic building block is the harmonic analytic superfield

$$V^{++}(x^m_A, \theta^+_\alpha, \bar{\theta}^+_{\dot{\alpha}}, u^+_i) \equiv V^{++}(\zeta^M, u)$$

where $u^+_i$ are spherical spinorial harmonics parametrizing the $R$-symmetry coset $S^2 \sim SU(2)_R/U(1)_R$, $u^+_i = (u^{++i})^\dagger \implies u^{++i}u^-_i = 1$ $(i = 1, 2)$ and $\theta^+_\alpha = \theta^+_\alpha u^+_i$, $\bar{\theta}^+_{\dot{\alpha}} = \bar{\theta}^+_{\dot{\alpha}} u^+_i$, with $\bar{\theta}^+_{\dot{\alpha}} = (\theta_{\dot{\alpha}})^\dagger$. The superfield $V^{++}$ does not depend on another half of Grassmann coordinates, i.e. on $\theta^-_{\alpha} = \theta^-_{\alpha} u^+_i$, $\bar{\theta}^-_{\dot{\alpha}} = \bar{\theta}^-_{\dot{\alpha}} u^+_i$, and in this sense it is Grassmann-analytic [15] like a chiral superfield. The “analytic basis” coordinate $x^m_A$ is related to $x^m$ (the “central basis” coordinate) by a $\theta$-dependent shift,

$$x^m_A = x^m - i\theta^+ \sigma^m \bar{\theta}^- - i\theta^- \sigma^m \bar{\theta}^+,$$

similar to left and right coordinates in $\mathcal{N} = 1$ chiral superspace. The harmonic analyticity is covariantly expressed as the Grassmann Cauchi-Riemann conditions

$$D^+_\alpha V^{++} = 0, \quad \bar{D}^+_\dot{\alpha} V^{++} = 0,$$

(3.1)

where

$$D^+_\alpha = u^+_i D^i_\alpha, \quad \bar{D}^+_\dot{\alpha} = u^+_i \bar{D}^i_{\dot{\alpha}},$$

(3.2)

and $D^i_\alpha$ and $\bar{D}^i_{\dot{\alpha}}$ are standard spinor covariant derivatives with respect to Grassmann coordinates in the central basis $(x^m, \theta, \bar{\theta}^+_{\dot{\alpha}}, u^+_i) \equiv (X^M, u)$.

The harmonic projections (3.2) satisfy the following anticommutation relations

$$\{D^+_{\alpha}, \bar{D}^+_{\dot{\alpha}}\} = -\{D^+_{\dot{\alpha}}, \bar{D}^+_{\alpha}\} = 2i(\sigma^m)_{\alpha\dot{\alpha}} P_m,$$

(3.3)

all other (anti)commutators being zero. An important difference of the harmonic Grassmann analyticity from the chirality is that $V^{++}$ (and any other analytic superfield with the even $U(1)$ charge) can be chosen real with respect to a special

\[\text{We basically follow conventions and notation of ref. [14].}\]
involution which preserves the set of analytic coordinates and acts as the standard complex conjugation on any component field.

The fields in the \( \theta^+, \bar{\theta}^+ \)-expansion of \( V^{++} \) are expanded in a harmonic series over \( u_\pm^\pm \). Now, \( ++ \) upstairs means that only the terms of \( U(1) \) charge +2 (i.e. the terms \( \propto (u^+)^2, (u^+)^3 u^-, \) etc) are kept in the harmonic expansion of the lowest component \( v^{++} = V^{++} \). The rest of component fields carry \( U(1) \) charge \( q \) ranging from +1 to –2, in accord with the obvious rule that the sum of \( q \) and the charge of the relevant \( \theta \)-monomial is always +2. The corresponding harmonic expansions go over the \( u_\pm^\pm \)-monomials having the charge \( q \).

Because of harmonic expansions, \( V^{++} \) encompasses an infinite number of field components. Fortunately, most of them are pure gauge degrees of freedom thanks to the invariance under the gauge transformation

\[
V^{++} \Rightarrow V^{++'} = V^{++} + D^{++} \Lambda.
\]

Here, the gauge superparameter \( \Lambda \) is an unconstrained harmonic analytic superfield involving an infinite number of ordinary gauge parameters, and \( D^{++} \) is the harmonic derivative which preserves harmonic Grassmann analyticity

\[
D^{++} = \partial^{++} - 2i \theta^+ \sigma^m \bar{\theta}^+ \partial_m + \theta^+_\alpha \frac{\partial}{\partial \theta^-_\alpha} + \bar{\theta}^+_\dot{\alpha} \frac{\partial}{\partial \bar{\theta}^-_{\dot{\alpha}}}
\]

\[
\partial^{++} = u^{+i} \frac{\partial}{\partial u^{-i}}.
\]

This was written in the analytical basis \( (x^m_A, \theta^\pm, \bar{\theta}^\pm, u) \). In the central basis, \( D^{++} \) coincides with \( \partial^{++} \). One can equally well introduce the derivative \( D^{--} \) coinciding with

\[
\partial^{--} = u^{-i} \frac{\partial}{\partial u^{+i}}
\]

in the central basis. Irrespective of the choice of the basis, the following commutation relations hold:

\[
[D^{\pm\pm}, D^{\pm\pm}_\alpha] = 0, \quad [D^{\pm\pm}, D^{\mp\mp}_\alpha] = D^{\pm\pm}_\alpha, \quad \text{and c.c.}
\]

\[
[D^{++}, D^{--}] = D^0
\]

where \( D^0 \) is the operator counting external \( U(1) \) charges of functions on \( S^2 \), \( D^0 \Psi^q(u) = q \Psi^q(u) \). In the central basis it reads

\[
D^0 = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}}.
\]

An important property is

\[
\int du \, d^4 x D^{++} F(u, x) = \int du d^4 x \, D^{--} F(u, x) = 0,
\]

(3.8)
which allows one to integrate by parts with respect to $D^{++}$ and $D^{--}$.

The gauge freedom (3.4) can be used to gauge away all fields in $V^{++}$ except for $(8+8)$ fields of the off-shell vector $\mathcal{N} = 2$ multiplet [13, 14]. In such a gauge ($\mathcal{N} = 2$ version of the Wess-Zumino gauge), $V^{++}$ looks as

\[ V_{WZ}^{++}(\zeta^M, u) = \sqrt{2}(\theta^+)^2\bar{\phi}(x) + \sqrt{2}(\bar{\theta}^+)^2\phi(x) - 2i\theta^+\sigma^m\bar{\theta}^+ A_m(x) + 4(\bar{\theta}^+)^2\theta^+\psi^i(x)u_i + 4(3(\theta^+)^2(\bar{\theta}^+)^2G^{ik}(x)u_i - u_k = 3(\theta^+)^2(\bar{\theta}^+)^2G^{ik}(x)u_i - u_k - 3(\theta^+)^2(\bar{\theta}^+)^2G^{ik}(x)u_i - u_k \] (3.9)

where, for brevity, we suppressed the index “$A$” of $x^m_A$. The physical fields are $\phi, A_m, \psi^i$. The coefficients in (3.9) are chosen so that the kinetic terms for all these fields in the action (3.12) to be defined below are normalized in a standard way. Now, $G^{ik}$ is the $SU(2)_R$ triplet of auxiliary fields.

For constructing covariant superfield strength and invariant actions one needs also the general (non-analytic) harmonic superfield $V^{--}$ which is related to $V^{++}$ by the harmonic equation

\[ D^{++}V^{--} = D^{--}V^{++}. \] (3.10)

The requirement for eq. (3.10) to be gauge covariant dictates that $V^{--}$ is transformed as

\[ V^{--} \Rightarrow V^{--'} = V^{--} + D^{--}\Lambda \] (3.11)

(use the last relation in (3.7) and the fact that $\Lambda$ has zero $q$ charge). The action has the form

\[ S = \frac{1}{4} \int du d^{12}X V^{++}V^{--}. \] (3.12)

Its gauge invariance can be checked using (3.10) and the analyticity conditions (3.1) for $V^{++}$ and $\Lambda$.

The action (3.12) involves, besides $\int d^{12}X \equiv \int d^4x d^4\theta d^4\bar{\theta}$, also the integral $\int du$ over $S^2 \sim SU(2)_R/U(1)_R$. The integral $\int du$ can actually be explicitly done (see below), after which (3.12) takes the standard form of an integral over chiral $\mathcal{N} = 2$ superspace. It admits three equivalent representations

\[ S = \frac{1}{4} \int d^4xd^4\theta W^2 = \frac{1}{4} \int d^4xd^4\bar{\theta} \bar{W}^2 \]
\[ = \frac{1}{8} \int d^4xd^4\theta W^2 + \frac{1}{8} \int d^4xd^4\bar{\theta} \bar{W}^2. \] (3.13)

Here \footnote{\( (D^+)^2 \equiv D^{+\alpha}D^{+\alpha}, (\bar{D}^+)^2 \equiv \bar{D}^{+\dot{\alpha}}\bar{D}^{+\dot{\alpha}} \).}

\[ W = -\frac{1}{4}(\bar{D}^+)^2V^{--}, \quad \bar{W} = -\frac{1}{4}(D^+)^2V^{--} \] (3.14)
are mutually conjugated superfield strengths of Abelian $\mathcal{N} = 2$ vector multiplet. They satisfy the chirality conditions
\begin{equation}
\bar{D}_\alpha^\pm W = 0, \quad D_\alpha^\pm \bar{W} = 0
\end{equation}
and the additional irreducibility constraint
\begin{equation}
(D^\pm)^2 W = (\bar{D}^\pm)^2 \bar{W}, \quad (D^+ D^-) W = (\bar{D}^+ \bar{D}^-) \bar{W}.
\end{equation}

The chiral superfield strength (3.14) is an $\mathcal{N} = 2$ analog of the conventional $\mathcal{N} = 1$ field strength superfield $W_\alpha$. It is gauge invariant and does not depend on harmonics in the central basis. The latter follows from the property
\begin{equation}
D^{++} W = D^{++} \bar{W} = 0,
\end{equation}
which can be easily checked using (3.10), the (anti)commutation relations and the analyticity conditions (3.1). Eqs. (3.17) also imply, via the complex conjugation, that
\begin{equation}
D^{--} W = D^{--} \bar{W} = 0.
\end{equation}

The constraints (3.15) and (3.16) actually follow from the definition (3.14) and the property of harmonic independence of $W, \bar{W}$. From (3.14) one immediately deduces the $+$ and $++$ projections of (3.15) and (3.16), respectively, while the rest of these relations is obtained by applying $D^{--}$ to the $+$ and $++$ projections and using (3.18) together with the commutation relations. The basic difference from $\mathcal{N} = 1$ superfield strength $W_\alpha$ is that (3.14) involves 2 rather than 3 covariant derivatives on the gauge prepotential. As a result, the lowest component of $W$ is a complex scalar field rather than a spinor one. The explicit form of the bosonic part of $W$ in the $(0 + 1)$ case will be given later in the context of our main topic, the HSS description of symplectic sigma models of the second kind.

Let us outline in brief how to reveal the equivalence of (3.12) and (3.13). The proof essentially relies upon the definition (3.14) and the harmonic independence conditions (3.17), (3.18). As a first step, one represents the integration measure in (3.12) as
\begin{align*}
dud^{12}X &= dud^4x_L d^4\theta (\bar{D})^4 = dud^4x_R d^4\bar{\theta}(D)^4, \\
(D)^4 &\equiv \frac{1}{16} (D^+)^2 (D^-)^2, \quad (\bar{D})^4 \equiv \frac{1}{16} (\bar{D}^+)^2 (\bar{D}^-)^2,
\end{align*}
and then act by $(\bar{D})^4$ or $(D)^4$ on the integrand, in order to end up with integrals over chiral subspaces. The representation (3.13) is obtained after a multiple use of analyticity of $V^{++}$, the conditions (3.17), (3.18) and the commutation relations (3.3), (3.7) together with their corollary $[D^{++}, (D^\pm)^2] = 2 D^{\mp\alpha} D_{\pm\alpha}$. In the process of this calculation, one should integrate by parts with respect to harmonic derivatives using (3.8).
The point important for the following is that one can generalize the free action (3.12) and write [16]

$$S = \frac{1}{4} \int d^{12}x \left[ V^{++} V^{--} F(\bar{W}, W) \right], \quad F|_{W=\bar{W}=0} = 1 \quad (3.19)$$

where $F$ is some function of superfield strengths the form of which is not fixed by $\mathcal{N} = 2$ supersymmetry alone. However, the requirement of invariance under the gauge transformations (3.4), (3.11) forces $F$ to satisfy the stringent constraint [16]

$$\frac{\partial^2 F}{\partial W \partial \bar{W}} = 0 \quad \Rightarrow \quad F(W, \bar{W}) = f(W) + \bar{f}(\bar{W}) \quad (3.20)$$

where $f(W)$ is an arbitrary holomorphic function. After substituting this into (3.19) and performing the same manipulations which lead from (3.12) to (3.13), the action (3.19) takes the form

$$S = \frac{1}{4} \int d^{12}x d^8 \theta V^{++}V^{--} \left[ f(W) + \bar{f}(\bar{W}) \right] = \frac{1}{4} \int d^4x d^4\theta L d^4\bar{\theta} W^2 f(W) + \frac{1}{4} \int d^4x d^4\bar{\theta} \bar{W}^2 \bar{f}(\bar{W}). \quad (3.21)$$

The famous Seiberg–Witten effective action [17] can be presented in this form.  

Gauge invariance of (3.21) is manifest. The emergence of eq. (3.20) as the necessary condition for gauge invariance of (3.19) can be explained as follows. The variation of $S$ under the gauge transformations (3.4), (3.11) can be reduced to

$$\delta S = -\frac{1}{2} \int d^{12}x \left[ \Lambda D^{--} V^{++} F(W, \bar{W}) \right]. \quad (3.22)$$

Then one represents the integration measure as

$$d^{12}x = d^4x_A (D^-)^4(D^+)^4, \quad (D^\pm)^4 = \frac{1}{16} (D^\pm)^2 (\bar{D}^\pm)^2,$$

and pulls $(D^+)^4$ through $V^{++} V^{--}$ to $F(W, \bar{W})$ using the analyticity of $V^{++}$ and (anti)commutation relations (3.3), (3.7). In this process, one obtains several terms, the most crucial of which is

$$\propto \int d^4x_A (D^-)^4 \left\{ \Lambda \partial_m V^{++} [D^+ \sigma^m \bar{D}^+ F(W, \bar{W})] \right\}.$$  

It cannot be cancelled by any other term in $\delta S$, so the necessary condition of vanishing of $\delta S$ is

$$D_\alpha^+ D_\bar{\alpha}^+ F(W, \bar{W}) = 0, \quad (3.23)$$

which, in virtue of the chirality conditions (3.15), amounts just to (3.20). The remaining terms in $\delta S$ can be shown to vanish in consequence of (3.20).

---

For the first time the action of the type (3.21) (in its second, harmonic-independent form) appeared in [18].
3.2 Manifestly $\mathcal{N} = 4$ supersymmetric $(0+1)$ sigma models

The $(0 + 1)$ version of the actions (3.12), (3.21) can be readily recovered by neglecting the spatial derivatives in $V^{\pm \pm}, W$ and reducing $d^4x_A \to dt_A$. However, in the case when there is no dependence on spatial coordinates, the action (3.21) can be generalized yet further, giving a manifestly $\mathcal{N} = 4$ supersymmetric form of the sigma models discussed in Sect.2. Before explaining this point, we need to give some necessary formulas related to $\mathcal{N} = 4$ HSS in $(0 + 1)$ dimension. Actually, the HSS formalism for this case can be constructed in a self-contained way in its own right, without reference to its $4D$ parent.

We shall work in the analytic basis of $\mathcal{N} = 4, (0 + 1)$ HSS, parametrized by coordinates

$$(X^M, u^\pm_i) \equiv (t, \theta^+_\alpha, \bar{\theta}^+_{\alpha}, \theta^-_\alpha, \bar{\theta}^-_{\alpha}, u^\pm_i)$$

(3.24)

where $\theta^\pm_\alpha = \theta^\dagger_\alpha u^\pm_i, \bar{\theta}^\pm_{\alpha} = -\bar{\theta}^\dagger_{\alpha} u^\pm_i$ and $(\theta^\dagger_{\alpha})_i = \bar{\theta}^\dagger_{\alpha}$. The analytic subspace closed under $\mathcal{N} = 4$ supersymmetry is defined as

$$(\zeta^M, u^\pm_i) \equiv (t, \theta^+_\alpha, \bar{\theta}^+_{\alpha}, u^\pm_i).$$

(3.25)

Both (3.24) and (3.25) are “real” under the pseudoconjugation $\tilde{}$ which is the product of the standard complex conjugation $\dagger$ and the Weyl reflection of $S^2 \sim SU(2)_R/U(1)_R$ [13, 14],

$$\tilde{t}_A = t_A, \quad \tilde{\theta}^\pm_{\alpha} = -\bar{\theta}^\pm_{\alpha}, \quad \tilde{\bar{\theta}}^\pm_{\alpha} = \theta^\pm_{\alpha}, \quad u^\pm_i = u^\pm_i.$$ (3.26)

In what follows we omit the index $A$ on $t$, hoping that this will not be a source of confusion.

The spinor and harmonic covariant derivatives in the analytic basis are given by

$$D^+_\alpha = \frac{\partial}{\partial \theta^-_{\alpha}}, \quad \tilde{D}^+_\alpha = \frac{\partial}{\partial \bar{\theta}^+_{\alpha}}, \quad D^-_\alpha = -\frac{\partial}{\partial \theta^+_{\alpha}} + 2i\bar{\theta}^-_{\alpha} \partial_t,$$

$$D^-_\alpha = -\frac{\partial}{\partial \theta^+_{\alpha}} + 2i\theta^-_{\alpha} \partial_t,$$

(3.27)

$$\{D^+_\alpha, \tilde{D}^-_\beta\} = -2i\epsilon_{\alpha\beta} \partial_t, \quad \{D^-_\alpha, D^+_\beta\} = 2i\epsilon_{\alpha\beta} \partial_t,$$

(3.28)

$$D^{++} = \partial^{++} - 2i(\theta^+_{\alpha} \tilde{\theta}^+_{\alpha}) \partial_t + \theta^+_{\alpha} \frac{\partial}{\partial \theta^-_{\alpha}} + \tilde{\theta}^+_{\alpha} \frac{\partial}{\partial \bar{\theta}^+_{\alpha}},$$

$$D^{--} = \partial^{--} - 2i(\theta^-_{\alpha} \tilde{\theta}^-_{\alpha}) \partial_t + \theta^-_{\alpha} \frac{\partial}{\partial \theta^+_{\alpha}} + \tilde{\theta}^-_{\alpha} \frac{\partial}{\partial \bar{\theta}^-_{\alpha}}.$$ (3.29)

The non-vanishing commutation relations between spinor and harmonic derivatives are as follows

$$[D^{\pm\pm}, D^{\mp}_\alpha] = D^{\pm}_\alpha$$

(3.30)
and the same for $D_\alpha^\pm$.

The basic object which encompasses the irreducible off-shell set of fields of the sigma model we are dealing with is the $(0 + 1)$ reduction of the gauge potential $V^{++}$. It is subject to the gauge transformations (3.4), so we can choose the same WZ gauge as in the $d = 4$ case, i.e. (3.9). Since we will be interested in the bosonic target space metrics only, we can neglect all fermions. The bosonic target space coordinates are also singlets of the R-symmetry group $SU(2)_R$, so we can omit the auxiliary $SU(2)_R$ triplet $C^{(ik)}$ as well for it cannot couple to the $SU(2)_R$ singlet sector. Thus we shall use as input the following $V^{++}\ WZ$

$$V^{++}_{WZ} = \sqrt{2}(\theta^+)^2 \phi(t) + \sqrt{2}(\bar{\theta}^+)^2 \bar{\phi}(t) + 2\theta^{+\alpha} \bar{\theta}^{+\beta} v_{\alpha\beta}(t) - 2i(\theta^+ \bar{\theta}^+) v_0(t) \tag{3.31}$$

with symmetric $v_{\alpha\beta}$. One can introduce $v_k$ related to $v_{\alpha\beta}$ as in eq.(1.1) and observe that $v_k$ and $v_0$ are genetically related to the spatial and temporal components of the 4D gauge vector potential $A_m$, correspondingly. It follows from (3.4) and the explicit form of $D^{++}$ in (3.29), that after dimensional reduction to $(0+1)$ space, the only residual gauge freedom is

$$v_0 \to v'_0 = v_0 + \dot{\lambda}(t) \tag{3.32}$$

where $\lambda(t) = \Lambda(\zeta, u)$. The remaining five gauge invariant fields in (3.31) are just the bosonic target space coordinates. It is clear that any supergauge invariant constructed out of $V^{++}$ and $V^{--}$ cannot involve $v_0$. So we can from the very beginning set $v_0 = 0$ in (3.31)

To construct the superfield invariants, we need to know the precise form of the non-analytic potential corresponding to the choice (3.31). It can be found by solving eq. (3.10). This is straightforward, though a little bit boring. One substitutes the WZ expression (3.31) for $V^{++}$ and the most general $\theta$ and $u$ expansion of the non-analytic harmonic superfield $V^{--}$ into (3.10). Then one uses the explicit expressions (3.29) for $D^{++}$ and solves (3.10) in components by equating to zero the coefficients in front of independent $\theta$ and $u$ monomials. The answer for $V^{--}$ in the considered bosonic approximation is as follows

$$V^{--}(t, \theta^\pm, \bar{\theta}^\pm) = v^{--} + \theta^{+\alpha} v_{-\alpha}^{-3} + \bar{\theta}^{+\alpha} \bar{v}^{-3\alpha} + (\theta^+)^2 v^{-4} + (\bar{\theta}^*)^2 \bar{v}^{-4} + \theta^{+\alpha} \bar{\theta}^{+\beta} C^{-4}_{(\alpha\beta)} \tag{3.33}$$

where

$$v^{--} = \sqrt{2}(\theta^-)^{2} \phi + \sqrt{2}(\bar{\theta}^-)^{2} \bar{\phi} + 2\theta^{-\alpha} \bar{\theta}^{-\beta} v_{(\alpha\beta)} ,$$

$$v_{-\alpha}^{-3} = -2i(\bar{\theta}^-)^2 \theta^{-\beta} \bar{v}_{(\alpha\beta)} + 2\sqrt{2}i(\theta^-)^2 \bar{\theta}^{-\alpha} \bar{\phi} ,$$

$$\bar{v}^{-3}_{\alpha} = 2i(\theta^-)^2 \bar{\theta}^{-\beta} \bar{v}_{(\alpha\beta)} - 2\sqrt{2}i(\bar{\theta}^-)^2 \theta^{-\alpha} \bar{\phi} ,$$

$$v^{-4} = -\sqrt{2}(\theta^-)^2 \bar{\theta}^{-2} \bar{\phi} , \quad \bar{v}^{-4} = -\sqrt{2}(\bar{\theta}^-)^2 \theta^{-2} \phi ,$$

$$C^{-4}_{(\alpha\beta)} = -2(\theta^-)^2 (\bar{\theta}^-)^2 \bar{v}_{(\alpha\beta)} . \tag{3.34}$$
Now it is straightforward to find $\bar{W}$ and $W$ defined by eqs. (3.14)

$$\bar{W} = \sqrt{2} \dot{\phi} + 2\sqrt{2i} (\theta^+ \bar{\theta}^-) \dot{\phi} - 2i \bar{\theta}^+ \theta^- \bar{v}_{(\alpha \beta)}$$

$$- \sqrt{2} (\bar{\theta}^-)^2 \left[(\theta^+)^2 \dot{\phi} + (\theta^+)^2 \phi + \sqrt{2}(\theta^+ \bar{\theta}^+ \bar{\theta}^-) \bar{v}_{(\alpha \beta)}\right],$$

\hspace{1cm} (3.35)

and

$$W = \sqrt{2} \phi + 2\sqrt{2i} (\theta^- \bar{\theta}^+) \dot{\phi} - 2i \theta^+ \theta^- \bar{v}_{(\alpha \beta)}$$

$$- \sqrt{2} (\theta^-)^2 \left[(\theta^+)^2 \dot{\phi} + (\theta^+)^2 \phi + \sqrt{2}(\theta^+ \bar{\theta}^+ \bar{\theta}^-) \bar{v}_{(\alpha \beta)}\right].$$

\hspace{1cm} (3.36)

The specificity of the considered $(0 + 1)$ dimensional case is the appearance of one additional gauge invariant object

$$W_{\alpha \beta} = \frac{1}{2} \bar{D}^{+}_{(\alpha} \bar{D}^{+}_{\beta)} V^{-},$$

\hspace{1cm} (3.37)

Indeed, using (3.28), (3.30) and the analyticity of the gauge parameter $\Lambda(\zeta, u)$ it is easy to check invariance of (3.37) under the gauge transformation (3.11). Like in the case of $3\sigma$ sigma models, this invariant is none other than the spatial part of the original vector superconnection. From the definition (3.37) it also follows that $W_{\alpha \beta}$ does not depend on harmonics

$$D^{\pm \pm} W_{\alpha \beta} = 0,$$

\hspace{1cm} (3.38)

and obeys the constraints

$$D^{\pm (\alpha} W_{\beta \gamma)} = \bar{D}^{\pm (\alpha} W_{\beta \gamma)} = 0.$$

\hspace{1cm} (3.39)

In terms of the bosonic components in WZ gauge it is written as

$$W_{(\alpha \beta)} = v_{(\alpha \beta)} + 2i \theta^+ \delta_{(\alpha} \bar{\theta}^-_{(\beta)} \dot{\phi} + 2i \bar{\theta}^+ \delta_{(\alpha} \theta^-_{(\beta)} \dot{\phi} + 2\sqrt{2}i \theta^+ \bar{\theta}^- \bar{v}_{(\alpha \beta)}$$

$$+ 2\sqrt{2}i \bar{\theta}^+ (\theta^-)^2 \dot{\phi} + 2\sqrt{2}(\theta^+)^2 \bar{\theta}^- \bar{v}_{(\alpha \beta)} + 2\sqrt{2}(\theta^+ \bar{\theta}^+ \bar{\theta}^-) \bar{v}_{(\alpha \beta)}$$

$$+ 4\bar{\theta}^- \theta^+ \theta^+ \bar{\theta}^- \bar{v}_{(\alpha \beta)}. $$

\hspace{1cm} (3.40)

Before proceeding further, let us point out that the above $\mathcal{N} = 4$ harmonic formalism in $d = (0 + 1)$ actually reveals a hidden $USp(4) \sim SO(5)$ covariance, besides the manifest covariance under $SU(2)_R$ acting on the indices $i, k$ of harmonics and fields in the harmonic expansions and $SU(2)$ realized on the indices $\alpha, \beta$ (the latter is a remnant of four-dimensional $SL(2, C)$). This $USp(4) \sim SO(5)$ contains as subgroups both the latter $SU(2)$ and the $U(1)$ R-symmetry group which acts as a phase transformation on the $\theta$ s and their conjugate (also on $W$ and $\bar{W}$ which have the same $U(1)$ charges as $(\bar{D})^2$ and $(D)^2$). The remaining coset $SO(5)/U(2)$ transformations act e.g. on spinor derivatives as

$$\delta D^\pm_{\alpha} = \lambda_{\alpha \beta} \bar{D}^{\pm \beta}, \quad \delta \bar{D}^\pm_{\alpha} = -\bar{\lambda}_{\alpha \beta} D^{\pm \beta}. $$

\hspace{1cm} (3.41)
Here $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}$ is a complex triplet comprising 6 real group parameters. The harmonic potentials $V^{\pm\pm}$ are $SO(5)$ singlets, while the superfield strengths $W, \bar{W}, W_{\alpha\beta}$ constitute an $SO(5)$ vector, as it follows from their definition (3.14) and (3.37):

$$\delta\bar{W} = \lambda^{\alpha\beta} W_{\alpha\beta}, \delta W = \bar{\lambda}^{\alpha\beta} W_{\alpha\beta}; \delta W_{\alpha\beta} = -\lambda_{\alpha\beta} W - \bar{\lambda}_{\alpha\beta} \bar{W}. \quad (3.42)$$

One can make the $SO(5)$ symmetry manifest introducing real superfields $W_I$ with the components $W_{1,2,3}$ related to $W_{\alpha\beta}$ as in eq.(1.1) and $W = W_4 + iW_5$. The $SO(5)$ covariance will be used later to derive in the HSS approach the metric of symplectic sigma models of the second kind defined on 5$r$–dimensional target spaces.

Let us discuss first the case $r = 1$. Our basic statement is that a generic $(0 + 1)$ action is written in the same way as the general $4D$ action (3.19), but the function $F$ may depend, besides $W$ and $\bar{W}$, on additional gauge invariants $W_{\alpha\beta}$:

$$S = \frac{1}{4} \int dtd^8\theta du V^{++} V^{--} F(W, \bar{W}, W_{\alpha\beta}). \quad (3.43)$$

Like in the $4D$ case, the requirement of gauge invariance of (3.43) imposes stringent constraints on the function $F(W, \bar{W}, W_{\alpha\beta})$. To derive them, one can proceed in the same way as in deriving eq. (3.20). Using the relation between the integration measures in the full harmonic superspace and its analytic subspace

$$dt du d^8\theta = dt du (D^-)^4 (D^+)^4 \equiv du d\zeta^{-4} (D^+)^4,$$

as well as the analyticity of the gauge parameter $\Lambda$ and the relations (3.28), (3.30), one can cast the gauge variation of (3.43) into the form

$$\delta S = -\frac{1}{2} \int dt du d\zeta^{-4} \Lambda (D^+)^4 \left[ D^{-} V^{++} F(W, \bar{W}, W_{\alpha\beta}) \right],$$

whence the general condition of gauge invariance of (3.43) is obtained as

$$(D^+)^4 \left[ D^{-} V^{++} F(W, \bar{W}, W_{\alpha\beta}) \right] = 0. \quad (3.44)$$

When acting by the spinor derivatives from $(D^+)^4$ on the expression within square brackets, one obtains a few terms which include derivatives of $F$ with respect to its superfield arguments, starting with the second-order ones, and some independent superfield projections of $V^{++}$ resulting from the action of spinor derivatives on $D^{-} V^{++}$. These independent contributions should disappear on their own right. Fortunately, a careful analysis shows that the common condition of vanishing of all these terms is the following constraint which is the $(0 + 1)$ analog of (3.20) [16]

$$\left( \frac{\partial^2}{\partial W^{\alpha\beta} \partial W_{\alpha\beta}} + 2 \frac{\partial^2}{\partial W \partial \bar{W}} \right) F = 0. \quad (3.45)$$

The simplest way to deduce (3.45) is to consider in (3.44) one of the terms which are produced when $(D^+)^4$ as a whole is pulled out through $D^{-} V^{++}$ and hits the
function $F$. This specific term contains only bosonic superfields and is a collection of the terms which are obtained either when $(D^+)^2$, $(\bar{D}^+)^2$ hit, respectively, $W$, $\bar{W}$ or $D^+_\alpha \bar{D}^+_\beta$ hits $W_{\gamma\lambda}$. Using the relations

$$(D^+)^2 W = (D^+)^2 W \equiv G^{+2}, \quad D^+_\alpha \bar{D}^+_\beta W_{\gamma\lambda} = \frac{1}{4} (\epsilon_{\alpha\gamma} \epsilon_{\beta\lambda} + \epsilon_{\alpha\lambda} \epsilon_{\beta\gamma}) G^{+2}$$

(3.46)

which follow from the explicit expressions (3.14) and (3.37), one can show that this term in (3.44) is represented as

$$\propto D^{--} V^{++} (G^{+2})^2 \left( \frac{\partial^2}{\partial W^{\alpha\beta} \partial W_{\alpha\beta}} + 2 \frac{\partial^2}{\partial W \partial \bar{W}} \right) F, \quad (3.47)$$

whence (3.45) is obtained as the condition of its vanishing. All other terms involve the same structure or its derivatives and vanish together with (3.47).

Eq. (3.45) is of course nothing but a familiar 5-dimensional Laplace equation like in (1.9). Note that in the $\mathcal{N} = 2$ approach of sect. 2, the harmonicity constraint is imposed on the prepotential $K$, while in the $\mathcal{N} = 4$ approach of the present section it is imposed on the function $F(W_I)$. It remains to learn how these two functions are related.

The most explicit way to do this is to express the action (3.43) in components. The computations in the general case become a little bit sophisticated since (3.43) involves the integral $\int d^8\theta$, which cannot be reduced to an integral over chiral superspace like it was the case in four dimensions (see (3.21)): in contrast to $W$ and $\bar{W}$, $W_{\alpha\beta}$ is not chiral. Nevertheless, the calculations are feasible. After substituting the explicit expressions (3.31), (3.33), (3.34), (3.35), (3.36) and (3.40) into (3.43), doing there the $u$- and $\theta$-integrals, and going over to $SO(5)$ vector notation, one obtains

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} \dot{v}_I^2 \left( 1 + \frac{3}{2} \hat{D} + \frac{1}{2} \hat{D}^2 \right) F(v_I) \equiv \frac{1}{2} \dot{v}_I^2 h(v_I) \quad (3.48)$$

where

$$v_I \equiv (v_k, v_4, v_5), \quad v_{\alpha\beta} = i \epsilon_{\alpha\gamma}(\sigma_k)_\gamma^\beta v_k, \quad \phi = \frac{v_4 + iv_5}{\sqrt{2}}, \quad (3.49)$$

and

$$\hat{D} \equiv v_I \frac{\partial}{\partial v_I} \quad (3.50)$$

is the operator of the target space scale transformation. Now, the commutator of $\hat{D}$ and the 5$D$ Laplacian $\Delta^{(5)}$ gives again $\Delta^{(5)}$, and this means that the function

$$h(v_I) = \left( 1 + \frac{3}{2} \hat{D} + \frac{1}{2} \hat{D}^2 \right) F \quad (3.51)$$
is harmonic provided \( F \) is harmonic. It can be identified, up to a numerical coefficient, with the harmonic function \( h \) introduced in (1.8). The simplest example is \( SO(5) \) invariant metric \([11]\) for which

\[
h(v_I) = F(v_I) = c_0 + c_1 (v_I v_I)^{-3/2}
\]  

(3.52)

where \( c_0, c_1 \) are some constants.

The prepotential \( K \) can be restored by solving the 3-dim Laplace equation

\[
2 \frac{\partial^2 K}{\partial v_k^2} = h
\]  

(3.53)

(cf. eq. (2.10) in ref.\([12]\)).

There is another way to arrive at the same result (3.51), which exploits \( SO(5) \) covariance of the \( \mathcal{N} = 4, (0 + 1) \) formalism. Let us start from the \( (0 + 1) \) reduction of the Seiberg–Witten action (3.21) having no dependence on the superfield \( W_k \), i.e. with \( F(W, \bar{W}) = f(W) + \bar{f}(\bar{W}) \). The metric in the space of scalar fields \( \phi, \bar{\phi} \) is just

\[
h(\bar{\phi}, \phi) = \frac{1}{2} \partial^2 \big[ \phi^2 f(\phi) \big] + \text{c.c.}
\]

\[
= \left[ 1 - 2 \left( \phi \frac{\partial}{\partial \phi} + \bar{\phi} \frac{\partial}{\partial \bar{\phi}} \right) + \frac{1}{2} \left( \phi^2 \frac{\partial^2}{\partial \phi^2} + \bar{\phi}^2 \frac{\partial^2}{\partial \bar{\phi}^2} \right) \right] F.
\]  

(3.54)

Bearing in mind that \( \phi = (v_4 + iv_5)/\sqrt{2} \), one can rewrite it as

\[
h = \left[ 1 + \frac{3}{2} \left( v_4 \frac{\partial}{\partial v_4} + v_5 \frac{\partial}{\partial v_5} \right) + \frac{1}{2} \left( v_4 \frac{\partial}{\partial v_4} + v_5 \frac{\partial}{\partial v_5} \right)^2 \right] F.
\]  

(3.55)

Allowing now for the dependence on \( v_k \), \( F(\phi, \bar{\phi}) \to F(\phi, \bar{\phi}, v_k) \), and requiring the \( SO(5) \) covariance, we arrive at eq.(3.51). To avoid a misunderstanding, we should point out that the general metric is not obliged to have \( SO(5) \) as an isometry. But the whole breaking of \( SO(5) \) is encoded in the structure of the function \( F \), while the relation between this function and metric is \( SO(5) \) covariant by construction. Indeed, all steps leading from (3.43) to the kinetic sigma model term (3.48) preserve \( SO(5) \) covariance.

The action (3.43) can be easily generalized to the case of several multiplets. We write

\[
S = \int dt d^8\theta du \ V_A^+ V_B^- F^{AB}(\bar{W}^C, W^C, W_k^C)
\]  

(3.56)

where \( F^{AB} \) can be assumed to be symmetric under \( A \leftrightarrow B \). To show this, one should express \( V_B^- \) via \( V_B^+ \) to derive (cf. eq.(7.26) in ref.\([14]\))

\[
S = \int dt d^8\theta du_1 du_2 \frac{V_A^+(t, \theta, u_1) V_B^+(t, \theta, u_2)}{(u_1^+ u_2^+)^2} F^{AB}(W_I^C).
\]  

(3.57)
Using the fact that $W_C^I$ do not depend on harmonics, we see that the antisymmetric part of $F^{AB}$ does not contribute.

The action (3.56) is gauge invariant provided

$$\frac{\partial^2 F^{AB}}{\partial W^C_I \partial W^{CD}_J} = \frac{\partial^2 F^{AB}}{\partial W^I_C \partial W^{B}^J} = 0.$$ (3.58)

These conditions can be proved like in the earlier considered case of one multiplet.

To derive the metric, we proceed in the same way as in the one-flavor case. Supressing the dependence on $W^C_k$, we arrive at

$$S = \int dt d^4 \theta W^A W^B \tilde{f}^{AB}(W^C) + \text{c.c.}$$ (3.59)

(The Seiberg–Witten effective action for a group of rank $r$ has such a form). Deriving the metric $h^{AB}(\tilde{\phi}^C, \phi^C)$ and restoring the $SO(5)$ covariant dependence on $v^C_k$, we arrive at the result

$$h^{AB} = F^{AB} + \frac{3}{4}(R^{CB} F^{CA} + R^{CA} F^{CB})$$

$$+ \frac{1}{4}(R^{CA} R^{DB} + R^{DB} R^{CA}) F^{CD}$$ (3.60)

where $R^{CA} = v^C_I \partial^A_I$. It is straightforward to check that this $h^{AB}$ satisfies the same conditions (3.58) as $F^{AB}$. This is in the full agreement with the conditions (2.16) derived in the $\mathcal{N} = 2$ superfield approach.

To summarize, the multilavor analog of the sigma model Lagrangian (3.48) is

$$L_{\text{kin}}^{\text{mf}} = \frac{1}{2} v^A_I v^B_I h^{AB}(v).$$ (3.61)

The symmetric matrix function $h^{AB}(v)$ satisfies eqs. (3.58) (with $W_C^I \rightarrow v^C_I$) and is arbitrary otherwise. Its precise relation to the function $\mathcal{K}$ of the $\mathcal{N} = 2$ superfield formalism remains to be clarified.

4 Conclusions

In this paper we studied the general off-shell actions of $\mathcal{N} = 4$, (0+1) supersymmetric sigma models associated with the $\mathcal{N} = 4$ multiplets $\{5, 8, 3\}$ (symplectic sigma models of the second type), both for a single supermultiplet and for several such multiplets. We performed our analysis in the $\mathcal{N} = 2$ superspace where only half of the underlying $\mathcal{N} = 4$ supersymmetry is manifest and in the harmonic $\mathcal{N} = 4$ superspace which makes manifest the entire supersymmetry. In the first formalism the multiplet in question is represented by the $\mathcal{N} = 2$ superfields $V_I \propto (V_k, \Phi, \bar{\Phi})$ with the off-shell contents $(3, 4, 1)$ and $(2, 4, 2)$, respectively. The most general $\mathcal{N} = 2$
action is given by (2.9) where the Lagrangian $\mathcal{K}(V^A_I)$ obeys the constraints (2.16). In the harmonic $\mathcal{N} = 4$ formalism the same multiplet is presented by the harmonic analytic gauge superfield $V^{++}$, from which in $(0+1)$ dimension one can construct the covariant superfield strengths $(W_{\alpha\beta}, W, \bar{W}) \equiv W_I$ with 5 physical bosonic fields as their first components. However, the most general action cannot be written as a superspace integral of some gauge invariant Lagrangian composed out of these covariant strengths; it is inevitably of the Chern-Simons type and is gauge invariant only modulo a shift of the integrand by a total harmonic derivative. It is given by the expression (3.56), with the function $F^{AB}(W_C^I)$ obeying the constraints (3.58). We explicitly computed the target bosonic metric in terms of this function in the one-flavor case (eq. (3.48)) and restored the metric in the multiflavor case by using the $SO(5)$ covariance of the harmonic formalism (eqs. (3.60), (3.61)).

It would be interesting to understand the target bosonic geometry associated with the actions (2.9) and (3.56) along the line of e.g. [19], [20] and to reveal the geometric meaning of the constraints (2.16), (3.58). It is also desirable to realize how these models are inscribed into the general “Gauge Theories/Strings” correspondence. As mentioned in Introduction, for the symplectic sigma models of the first type such an interpretation is quite likely (they emerge both in the context of the low-energy description of $\mathcal{N} = 1, d = 4$ gauge theories and in the black holes stuff). Meanwhile, so far only the first, gauge theory “face” of the symplectic models of the second type has been clarified [12]. It is still unclear what could be the string theory “face” of these sigma models. Some hints are given in [11] and [21]. In particular, it was argued in [21] that the superconformal versions of such models can describe the near-horizon AdS$_2 \times S^4$ geometry of a D5-brane in an orthogonal D3-brane background. Note that the superconformal action for one-flavor symplectic model of the second type was constructed in the framework of the $\mathcal{N} = 2$ superfield formalism in a recent preprint [22] (its bosonic sector corresponds to the choice $c_0 = 0, c_1 \neq 0$ in (3.52)).

As a final remark we note that there exists an alternative $\mathcal{N} = 2$ description of the same $\mathcal{N} = 4$ multiplet $(5, 8, 3)$ in terms of the multiplets with the off-shell contents $(1, 4, 3)$ and $(4, 4, 0)$ [22] (the latter multiplet involves no auxiliary fields at all). It is of obvious interest to see what the generic action of the considered class of symplectic sigma models looks like in this alternative formulation.

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