I. Introduction

An interesting question in algebraic geometry is: *In what ways can a smooth projective variety X degenerate?* Here one imagines a situation

\[(I.1) \quad X \xrightarrow{\pi} S\]

where \(X\) and \(S\) are complex manifolds with \(X \subset \mathbb{P}^N\) and where \(\pi\) is a proper holomorphic mapping with \(X\) a smooth fibre. Then \(\pi\) is a holomorphic submersion over a Zariski open set \(S^* \subset S\), and one is interested in which varieties \(X_s = \pi^{-1}(s)\) can arise when \(s \in S \setminus S^*\). The question of course needs refinement; e.g., by assuming some sort of semi–stable reduction for (I.1) (cf. [1]).
Hodge theory provides an invariant associated to (I.1). Namely there is a period mapping

\[(I.2) \quad \Phi : S^* \to \Gamma \backslash D\]

where \(D = G_{\mathbb{R}}/R\) is a Mumford–Tate domain and for \(s \in S^*\),

\[\Phi(s)\] is the polarized Hodge structure on \(H^n(X_s, \mathbb{Q})_{\text{prim}}\).

The ambiguity in the identification of \(H^n(X_s, \mathbb{Q})_{\text{prim}}\) with a fixed vector space \(V\) is given by the image of the monodromy representation

\[\rho : \pi_1(S^*) \to \Gamma \subset G.\]

We note that the invariant (I.2) of (I.1) only depends on the family over \(S^*\); it does not depend on the generally non-unique semi-stable reduction, although as we shall see it strongly limits what the singular fibres can be.

There are two ways of attaching Hodge–theoretic data to the limits

\[(I.3) \quad \lim_{s \to s_0} \Phi(s); \]

these data will then reflect the specialization \(X_s \to X_{s_0}\). The first, and traditional, way is to think of (I.3) as giving a \textit{limiting mixed Hodge structure}. Specifically, following [7] and [14] and taking \(S \backslash S^*\) to be a local normal crossing divisor, one attaches to \(D\) a set \(B(\Gamma)\) of equivalence classes of limiting mixed Hodge structures and extends (I.2) to an \textit{extended period mapping}

\[\Phi_e : S \to \Gamma \backslash (D \cup B(\Gamma)).\]

One may roughly think of \(\Phi_e(s_0)\) as containing a \textit{maximal} amount of Hodge–theoretic information in the limit.

To explain the second, more recent method we assume that \(S = \Delta\) is the unit disc and \(S^* = \Delta^*\) the punctured disc so that (I.1) becomes

\[(I.4) \quad \Phi : \Delta^* \to \Gamma_T \backslash D\]

where \(T\) is the unipotent monodromy transformation with logarithm \(N\) and \(\Gamma_T = \{T^k\}_{k \in \mathbb{N}}\). Then (I.4) may be lifted to a mapping of the upper–half plane

\[\tilde{\Phi} : \mathbb{H} \to D,\]
and following [13, Appendix to Lecture 10] and [16] we define\(^{(1)}\) the reduced limit period mapping associated to (I.4) by
\[
(I.5) \quad \lim_{z \to \infty} \tilde{\Phi}(z) \in \partial D
\]
where \(D\) is embedded in its compact dual \(\tilde{D}\) and \(\partial D \subset \tilde{D}\), cf. Section II.G.

To explain this a bit more, in the situation (I.4) the boundary component \(B(\Gamma)\) referred to above becomes
\[
B(N) = \left\{ \text{equivalence classes of limiting mixed Hodge structures } (V, W^\bullet(N), F^\bullet) \text{ with monodromy weight filtration } W^\bullet(N) \right\}.
\]
There is then a mapping
\[
\Phi_\infty : B(N) \to \partial D,
\]
and the reduced limit period mapping (I.5) is the composition of this mapping with \(\Phi_\epsilon(s_o)\), where \(s_o = 0 \in \Delta\). We will abbreviate it by \(\Phi_\infty(s_o)\). It is well-defined since (I.5) is a fixed point of \(T\). We may roughly think of \(\Phi_\infty(s_o)\) as containing the minimal amount of Hodge-theoretic information in the limit (I.3). For the classical case of weight \(n = 1\) polarized Hodge structures, \(\Phi_\epsilon\) corresponds to a toroidal compactification [2] and \(\Phi_\infty\) to the Satake–Bailey–Borel compactification [3, 20].

One advantage of the reduced limit period mapping is that it maps to a space on which the group \(G_\mathbb{R}\) acts. One may then use the rich and well understood structure of the partially ordered lattice of \(G_\mathbb{R}\)-orbits in \(\partial D\) to define what is meant by extremal degenerations of a polarized Hodge structure, cf. Section III.A. Specifically, a \(G_\mathbb{R}\)-orbit \(O \subset \partial D\) is said to be polarized relative to the Mumford–Tate domain \(D\) in case there is a period mapping (I.4) whose reduced limit period lies in \(O\), cf. Definition II.31. When the infinitesimal period relation is bracket-generating, all orbits in \(\partial D\) of real codimension one in \(\tilde{D}\) are polarizable relative to some Mumford–Tate domain structure on the open \(G_\mathbb{R}\)-orbit \(D\), cf. [13, 16] or Section III.C. The unique closed orbit in \(\partial D\) is sometimes, but not always, polarizable. The general question of polarizability is discussed in [16, 17].

A degeneration (I.4) of a polarized Hodge structure \(\Phi(s)\) is said to be minimal if the reduced limit period lies in a codimension-one \(G_\mathbb{R}\)-orbit; it is said to be maximal if its reduced limit lies in an orbit whose closure does not lie in a proper sub-orbit

\(^{(1)}\)Precise definitions of all notions discussed in this introduction are given in later sections.
that is polarizable relative to $D$, cf. Definition III.4. One way to think of this is the following: Points of $D$ are given by polarized Hodge structures $(V, Q, F^\bullet)$. Then arbitrary period mappings may be well–approximated by nilpotent orbits with the same limit period mapping. Thus a minimal degeneration of $F^\bullet \in D$ is given by a nilpotent orbit such that

$$\lim_{z \to \infty} e^{zN} \cdot F^\bullet$$

lies in a codimension–one $G_\mathbb{R}$–orbit in $\partial D$.

Intuitively these are the least degenerate limiting mixed Hodge structures that $F^\bullet \in D$ can specialize to. Similarly, maximal degenerations are the most degenerate that $F^\bullet$ can specialize to.

The main results of this paper will describe the extremal — the minimal and maximal — degenerations of polarized Hodge structures in a number of cases. These will be described in terms of the types of limiting mixed Hodge structure that maps to the reduced limit period point in $\partial D$. We shall deal with two types of polarized Hodge structures.

**Type I.** These are polarized Hodge structures $(V, Q, F^\bullet)$ of weight $n > 0$ that we think of as $H^n(X, \mathbb{C})_{\text{prim}}$ for a smooth algebraic variety $X$ of dimension $n$.

**Type II.** These are polarized Hodge structures $(\mathfrak{g}, Q_\mathfrak{g}, F^\bullet_\mathfrak{g})$ of weight $n = 0$ and where, unless otherwise mentioned, $-Q_\mathfrak{g}$ is the Cartan–Killing form.

We think of polarized Hodge structures of Type I as directly related to algebraic geometry. Limiting mixed Hodge structures $(V, W_\bullet(N), F^\bullet)$ arising from polarized Hodge structures of Type I may be pictured in the first quadrant of the $(p, q)$–plane in terms of the Deligne splitting

$$V_{\mathbb{C}} = \bigoplus_{0 \leq p+q \leq 2n} I^{p,q}$$

where dots indicate a possibly nonzero $I^{p,q}$. For example, a pure Hodge structure of weight $n = 5$ is depicted as

![Diagram](image-url)
A polarized Hodge structure of Type I gives one of Type II with \( \mathfrak{g} \subset \text{End}(V, Q) \).

In this case the corresponding Mumford–Tate domains are the same. The polarized Hodge structures of Type II are especially convenient when studying the geometry of the \( G_\mathbb{R} \)-orbits \( \mathcal{O} \) in \( \tilde{D} \). For example, suppose that the limiting mixed Hodge structures \((V, W_\bullet(N), F^\bullet)\) is \( \mathbb{R} \)-split. The induced adjoint limiting mixed Hodge structures \((\mathfrak{g}, W_\bullet(N)_g, F^\bullet_g)\) is also \( \mathbb{R} \)-split. Let \( \mathfrak{g}_\mathbb{C} = \oplus I^{p,q}_g \) be the Deligne splitting. Let

\[
F^\bullet = \lim_{z \to \infty} e^{z N} F^\bullet \in \mathcal{O}.
\]

Then the tangent and normal spaces are naturally identified with

(I.6a) \[
T_{F^\bullet} \mathcal{O} = \bigoplus_{p>0 \text{ or } q>0} \left(I^{p,q}_g \oplus I^{q,p}_g\right)_{\mathbb{R}},
\]

(I.6b) \[
N_{F^\bullet} \mathcal{O} = i \bigoplus_{p,q>0} \left(I^{p,q}_g \oplus I^{q,p}_g\right)_{\mathbb{R}},
\]

cf. Section III.B. In particular, much of the geometry (such as dimension and codimension, CR–tangent space, and intrinsic Levi form) associated with the \( G_\mathbb{R} \)-orbit \( \mathcal{O} \subset \tilde{D} \) can be “read off” from the Deligne splitting. Moreover, each \( I^{p,q}_g \) may be realized as a direct sum of root spaces (and a Cartan subalgebra if \( p = q = 0 \)), and this Lie theoretic structure plays an essential rôle in the analysis.

**I.A. Minimal degenerations.** We begin with a result for period domains.

**Theorem I.7.** Given a period domain \( D \) parameterizing polarized Hodge structures of weight \( n \), the minimal degenerations have either

\[
N^2 = 0 \quad \text{and} \quad \text{rank } N \in \{1, 2\}, \quad \text{or}
\]

\[
N^2 \neq 0, \quad N^3 = 0 \quad \text{and} \quad \text{rank } N = 2.
\]

We shall describe the nonzero \( I^{p,q} \). Figure I.1 illustrates the possibilities for weights one through four; from these the reader will easily guess (correctly) what the general case will be.
Figure I.1. Minimal degenerations for weight \( n \) period domains

\[ n = 1 \]
\[ \begin{array}{ccc}
N = 0 & \text{---} & N \neq 0 \\
\end{array} \]

\[ n = 2 \]
\[ \begin{array}{ccc}
N = 0 & \text{---} & N \neq 0 \\
N^2 \neq 0 & \text{---} & \\
\end{array} \]

\[ n = 3 \]
\[ \begin{array}{ccc}
N = 0 & \text{---} & N \neq 0 \\
N \neq 0 & \text{---} & \\
\end{array} \]

\[ n = 4 \]
\[ \begin{array}{ccc}
N = 0 & \text{---} & N \neq 0 \\
N^2 \neq 0 & \text{---} & \\
\end{array} \]
The general rule is this: Let $h_{p,q}$ be the Hodge numbers of the polarized Hodge structure $(V, Q, F^\bullet)$ and 

$$i^{p,q} = \dim I^{p,q}.$$ 

If $N \neq 0$ and $N^2 = 0$, then for one pair $p_o < q_o$ with $p_o + q_o = n$ we have:

- $i^{p_o+1,q_o}, i^{p_o,q_o-1} = 1$;
- $i^{p_o,q_o} = h^{p_o,q_o} - 1$ and $i^{p_o+1,q_o-1} = h^{p_o+1,q_o-1} - 1$;
- for all other $p < q$, $i^{p,q} = h^{p,q}$.

Put another way, for some $p_o < q_o$ one class in $V^{p_o,q_o}$ and one class in $V^{p_o+1,q_o-1}$ disappear in $\text{Gr}^W_n(N)$ and reappear as classes

$$\alpha \in I^{p_o+1,q_o},$$
$$N\alpha \in I^{p_o,q_o-1}.$$ 

If $N^2 \neq 0$ and $N^3 = 0$, then $n = 2m$ is even and we have:

- $i^{m-1,m-1}, i^{m+1,m+1} = 1$;
- $i^{m-1,m+1} = h^{m-1,m+1} - 1$ and $i^{m+1,m-1} = h^{m+1,m-1} - 1$;
- for all other $p < q$, $i^{p,q} = h^{p,q}$.

In this case, one class in $V^{m-1,m+1}$ and one class in $V^{m+1,m-1}$ disappear in $\text{Gr}^W_n(N)$ and reappear as classes

$$\alpha \in I^{m+1,m+1},$$
$$N^2\alpha \in I^{m-1,m-1}.$$ 

The pictures above are particularly revealing when

(I.8) 

$$h^{p,q} = \begin{cases} 
1 & \text{for all } p \neq q \\
2 & \text{if } p = q.
\end{cases}$$

In this case they are as pictured in Figure I.2; in these figures a uncircled node indicates $i^{p,q} = 1$, a circled node indicates $i^{p,q} = 2$.

From the algebro–geometric perspective, Theorem I.7 (we will prove the more precise Theorem IV.1) may at first glance seem surprising. For example, for a smooth threefold $X$ a “generic” specialization might be thought to be $X \to X_o$ where $X_o$ has a node. In the case that (I.8) holds, this is the right–most picture of Figure I.2 for $n = 3$. The middle picture would arise from $X_o$ having a smooth double surface $D,$
Figure I.2. Minimal degenerations for period domains with (I.8)
where there is an $\omega \in H^0(\Omega^3_X)$ that specializes to $\omega_o \in H^0(\Omega^3_{X_o}(\log D))$ whose residue goes to the class in $I^{2,0}$.

In the case that $n = 5$, the second and fourth picture of Figure I.2 may be interpreted as in the $n = 3$ case. For the third we may think of a five-fold $X$ specializing to $X_o$ with the local equation $\{x_1x_2 + x_3x_4 = 0\}$ in $\mathbb{C}^6$; that is, $X_o$ has a double point along a surface. Then the $h^{4,1}$ drops by one. In summary, the three degenerations (in the $n = 5$ case) correspond to the local equations

\[
x_1x_2 = 0, \quad \text{(double four-fold)}
\]
\[
x_1x_2 + x_3x_4 = 0, \quad \text{(double surface)}
\]
\[
x_1x_2 + x_3x_4 + x_5x_6 = 0, \quad \text{(double point)}.
\]

In the more general setting of Mumford–Tate domains, the $I^{p,q}$ may be more complicated than those of Figure I.1: there may be more $N$–strings, and they may have length greater than two. What we can say, in terms of generalizing Theorem I.7, is Proposition IV.5. Nonetheless, from the Lie theoretic perspective, the codimension one orbits all possess a uniform structure in the following sense: for an appropriate choice of Cartan subalgebra $h_R$ (essentially one may think of this as reflecting a “good choice” of basis of $V$), the nilpotent $N$ will be a root vector and the normal space may be identified with a real root space $[13, 16]$. That is,

\[ N \in g_R^\alpha, \]

where $\alpha \in h_C^*$ is a root and the root space $g^\alpha \subset g_C$ is defined over $\mathbb{R}$, and (I.6b) becomes

\[ N_{F^*}O = i I^{1,1}_{g^*}(\mathbb{R}) = i g_R^\alpha. \]

I.B. Maximal degenerations. We recall that mixed Hodge structure $(V, W_\bullet, F^\bullet)$ is of Hodge–Tate type if the $I^{p,q} = 0$ for all $p \neq q$. (The Hodge structures in row 4 of Figure II.1 are of Hodge–Tate type.) For limiting mixed Hodge structures there is the general

**Proposition I.9.** The limiting mixed Hodge structure $(V, W_\bullet(N), F^\bullet)$ is of Hodge–Tate type if and only if the associated adjoint limiting mixed Hodge structure $(g, W_\bullet(N)_g, F^\bullet_0)$ is of Hodge–Tate type.
The proposition is proved in Section V.B.

For the description here of the maximal degenerations we shall make the assumption

the closed $G_R$–orbit $O_{cl}$ is polarizable relative to $D$.

Thus the image (I.5) of (I.4) under the reduced limit period map is a point $F_{\infty} \in O_{cl}$.

**Theorem I.10.** The following are equivalent:

(a) The orbit $O_{cl}$ is totally real; i.e., the Cauchy–Riemann tangent space $T^{\text{CR}}O_{cl} = 0$.

(b) The real dimension of the closed orbit is the complex dimension of the compact dual: $\dim_R O_{cl} = \dim_C \mathcal{D}$.

(c) The stabilizer $P = \text{Stab}_{G_C}(F_{\infty}^\bullet)$ is $\mathbb{R}$–split and $O_{cl} = G_R/P_R$.

Any of these imply that the limiting mixed Hodge structures $(V, W_\bullet(N), F^\bullet)$ and $(g, W_\bullet(N)_g, F^\bullet_g)$ are of Hodge–Tate type.

The theorem is proved as Theorem III.22 and Corollary V.4.

From an algebro–geometric perspective it is not surprising that the most degenerate limiting mixed Hodge structure is one of Hodge–Tate type. More interesting is that there are both Hodge theoretic (Lemma V.7) and Lie theoretic (Lemma V.10 and Remark V.12) obstructions to a given type of polarized Hodge structure being able to degenerate to one of Hodge–Tate type.

In general there is the following

**Theorem I.11.** Suppose that the limiting mixed Hodge structure $(V, W_\bullet(N), F^\bullet)$ is sent to the closed orbit under the reduced limit period mapping (I.5) in the closed orbit. Then Deligne splitting $g_C = \bigoplus I^p_{\mathfrak{g}}$ associated with the induced limiting mixed Hodge structure $(g, W_\bullet(N)_g, F^\bullet_g)$ satisfies:

$I^p_{\mathfrak{g}} = 0$ for all $p \neq q > 0$,

$I^p_{\mathfrak{g}} = 0$ for all odd $p \geq 3$,

$I^p_{\mathfrak{g}} = 0$ for all $p + q \neq 0$ with $|p - q| > 2$.

The constraints of Theorem I.11 are illustrated in Figure III.3.b. We will prove the slightly stronger Theorem V.1.

For period domains one may reconstruct the limiting mixed Hodge structure $(V, W_\bullet(N), F^\bullet)$ from $(g, W_\bullet(N)_g, F^\bullet_g)$; doing so, Theorem I.11 yields
Theorem I.12. Let $D$ be a period domain parameterizing weight $n$ Hodge structures. If there exists a limiting mixed Hodge structure $(V, W_\bullet(N), F^\bullet)$ that maps to the closed $G_R$–orbit in $\tilde{D}$, but is not of Hodge–Tate type, then $n = 2m$ is even and:

(a) For $k \neq 0$, $\text{Gr}^{W_\bullet(N)}_{n+k, \text{prim}}$ is of Hodge–Tate type. (Thus $k$ is even.)

(b) For $k \neq 0$, $\text{Gr}^{W_\bullet(N)}_{n+k, \text{prim}} \neq 0$ implies $k \equiv 2 \pmod{4}$.

(c) $\text{Gr}^{W_\bullet(N)}_{n, \text{prim}} \neq 0$, and the only nonzero $I^{p,q}_{\text{prim}}$, with $p + q = n$, are

\[ I^{m+1,m-1}_{\text{prim}} \text{ and } I^{m-1,m+1}_{\text{prim}}. \]

The theorem is proved in Section V.D. A convenient schematic to picture a limiting mixed Hodge structures is its decomposition into $N$–strings

\[
\begin{array}{c}
H^0(n) \xrightarrow{N} H^0(n-1) \xrightarrow{N} \cdots \xrightarrow{N} H^0(1) \xrightarrow{N} H^0 \\
H^1(n-1) \xrightarrow{N} \cdots \xrightarrow{N} H^1 \\
\vdots \\
H^{n-1}(1) \xrightarrow{N} H^{n-1} \\
H^n
\end{array}
\]

(I.13)

where

\[ H^k = N^{n-k} \text{Gr}^{W_\bullet(N)}_{2n-k, \text{prim}} \]

is a polarized Hodge structure of weight $k$. (It may happen that $H^k$ is a Tate twist of a lower weight Hodge structure.) Under the schematic (I.13), the possibilities in Theorem I.12 are:

\[ n = 2 \]

We have $H^1 = 0$, $H^0 \neq 0$ and $H^2$ has type $(\ast, 0, \ast)$. In terms of the $i^{p,q}_{\text{prim}}$ the Hodge numbers are

\[ h^{2,0} = i^{2,0}_{\text{prim}} + i^{2,2}_{\text{prim}}, \quad h^{1,1} = i^{2,2}_{\text{prim}}. \]

\[ n = 4 \]

We have $H^0 = H^1 = H^3 = 0$, $H^2 \neq 0$ is of Hodge–Tate type and $H^4$ has type $(0, \ast, 0, \ast, 0, 0)$.

\[ n = 6 \]

We have $H^1 = H^2 = H^3 = H^5 = 0$, at least one of $H^0$ and $H^4$ is nonzero and of Hodge–Tate type, and $H^6$ has type $(0, 0, \ast, 0, \ast, 0, 0)$. 
Corollary I.14. If the limiting mixed Hodge structure corresponds to a point in the closed orbit, then $\text{Gr}_W^{W^\bullet}(N)$ is rigid in the sense that it does not admit a non–trivial variation of Hodge structure.

The reason is that the infinitesimal period relation is trivial for the $\text{Gr}_W^{W^\bullet}(N)$.

The above results were under the assumption that the limit period map sends (I.4) to a point in the closed orbit. As we have discussed, in some, but not all, cases the limiting mixed Hodge structure is of Hodge–Tate type. In the other direction we have

Proposition I.15. If the limiting mixed Hodge structure is of Hodge–Tate type, then the limit period is a point in the closed orbit.

The proposition is proved in Section V.B.

Given a degeneration of polarized Hodge structures (I.4) whose reduced limit mapping (I.5) goes to a point in a $G_R$–orbit $\mathcal{O} \subset \partial D$, for the limiting mixed Hodge structure $(V, W_\bullet(N), F^\bullet)$ the above may be summarized as follows:

(a) If $\mathcal{O}$ is the closed orbit, and is totally real, then $(V, W_\bullet(N), F^\bullet)$ is Hodge–Tate.

(b) If $(V, W_\bullet(N), F^\bullet)$ is Hodge–Tate, then $\mathcal{O}$ is the closed orbit.

(c) If $\mathcal{O}$ is the closed orbit and $(V, W_\bullet(N), F^\bullet)$ is not Hodge–Tate, then the nonzero $\text{Gr}_{n+k,\text{prim}}^{W_\bullet(N)}$ have $k \equiv 2 \pmod{4}$, and are Hodge–Tate, and $\text{Gr}_{n,\text{prim}}^{W_\bullet(N)}$ is as close to being Hodge–Tate as the Hodge numbers of $(V, Q, F^\bullet)$ will allow.

I.C. Notation.

- $\Delta = \{ t \in \mathbb{C} : |t| < 1 \}$ is the unit disc, $\Delta^* = \{ t \in \Delta : t \neq 0 \}$ is the punctured unit disc, and $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ is the upper–half plane.
- $X \rightarrow \Delta$ is a semi–stable reduction.
- $\tilde{D} = G_C/P$ is a generalized flag variety containing $D = G_R/R$ as an open $G_R$–orbit with a compact isotropy group $R = G_R \cap P$.
- $\Phi : \Delta^* \rightarrow \Gamma_T \setminus D$ is a period mapping with lift $\tilde{\Phi} : \mathbb{H} \rightarrow D$ and $\tilde{\Phi}_* : T\mathbb{H} \rightarrow I \subset T\tilde{D}$, where $I$ denotes the infinitesimal period relation.
- $\tilde{B}(N) \subset \tilde{D}$ is the set of $N$–polarized limiting mixed Hodge structures, and $B(N)$ the boundary component consisting of equivalence classes of limiting mixed Hodge structures.
\( \Phi_{\infty} : \partial D \to \partial D \) is the reduced limit period map.

\( F_{\text{lim}}^\bullet = \lim_{t \to 0} (e^{-\ell(t)N} F_t^\bullet) = \lim_{\text{Im}(z) \to \infty} e^{-zN}\tilde{\Phi}(z) \) where \( \ell(t) = \frac{1}{2\pi i} \log t \) and \( t = e^{2\pi i z} \in \Delta^* \) for \( z \in \mathbb{H} \).

\( F_{\infty}^\bullet = \lim_{\text{Im}(z) \to \infty} \tilde{\Phi}(z) \).

\( \mathcal{T} \subset R \) is a compact maximal torus of \( G_{\mathbb{R}} \).

\( g_{\mathbb{C}} \) and \( g_{\mathbb{R}} \) are the Lie algebras of \( G_{\mathbb{C}} \) and \( G_{\mathbb{R}} \).

\( (V, W_\bullet(N), F^\bullet) \) a limiting mixed Hodge structure with \( F^\bullet = F_{\text{lim}}^\bullet \), \( (V, W_\bullet, \tilde{F}^\bullet) \) is the associated \( \mathbb{R} \)-split limiting mixed Hodge structure associated, and \( (g, W_\bullet(N)_{g}, F_g^\bullet) \) the induced adjoint limiting mixed Hodge structures on \( g \).

\( V_C = \bigoplus I^{p,q} \) and \( g_C = \bigoplus I^{p,q}_g \) are the Deligne splittings of a limiting mixed Hodge structure \( (V, W_\bullet(N), F^\bullet) \) and the induced adjoint limiting mixed Hodge structure \( (g, W_\bullet(N)_{g}, F_g^\bullet) \).

\( -Q_\mathfrak{g} \) is the Killing form on \( \mathfrak{g} \).

\( \mathfrak{h} \subset g_{\mathbb{C}} \) is a Cartan subalgebra with roots \( \Delta \subset \mathfrak{h}^* \).

Given \( x \in \check{D} \), \( \mathcal{O}_x = G_{\mathbb{R}} \cdot x \) is the \( G_{\mathbb{R}} \)-orbit.

II. Reduced limit period mappings

II.A. Generalized flag varieties. A generalized flag variety is a homogeneous complex manifold

\[ \check{D} = G_{\mathbb{C}} / P \]

where \( G_{\mathbb{C}} \) is a complex, semi-simple Lie group and \( P \) is a parabolic subgroup. When \( P = B \) is a Borel subgroup we shall use the term flag variety. At the reference point \( x_o = P1 \) there is a natural identification of the holomorphic tangent space

\[ T_{x_o} \check{D} = g_{\mathbb{C}} / p. \]

\footnote{We shall use the notation \( \Delta \) for both the set of roots of \((g_{\mathbb{C}}, \mathfrak{h})\) and for the unit disc in \( \mathbb{C} \); the context should make it clear which use of \( \Delta \) is being made.}

\footnote{We use the notation \( \check{D} \) because we shall mainly think of it as the compact dual of a generalized flag domain.}
We shall describe \( p \) in terms of the roots associated to a Cartan subalgebra \( h \subset p \), which always exists. The root space decomposition of \((g, h)\) is

\[
g = h \oplus \bigoplus_{\alpha \in \Delta} g^\alpha
\]

where \( \Delta \subset h^* \) is the set of roots and \( g^\alpha \) is the one–dimensional root space. We let \( \Delta^+ \subset \Delta \) denote a choice of positive roots. This determines a Borel subalgebra \( b \supset h \) by

\[
b = h \oplus \bigoplus_{\alpha \in \Delta^+} g^{-\alpha}.
\]

Conversely, a choice of Borel \( b \supset h \) determines the positive roots by

\[
\Delta^+ = \{ \alpha \in \Delta : g^{-\alpha} \subset b \}.
\]

Let \( \Delta^+_s \subset \Delta^+ \) denote the set of simple roots relative to that choice.

The first description of parabolic subalgebras \( p \subset g \) is in terms of subsets \( \Sigma \subset \Delta^+ \). Denoting by \( \langle \Sigma \rangle \subset \Delta \) the roots spanned by \( \Sigma \), we set

\[
p_{\Sigma} = h \oplus \left( \bigoplus_{\alpha \in \langle \Sigma \rangle} g^\alpha \right) \oplus \left( \bigoplus_{\beta \in \Delta^- \setminus \langle \Sigma \rangle^+} g^\beta \right)
\]

where \( \Delta^- = -\Delta^+ \) and \( \langle \Sigma \rangle^- = \langle \Sigma \rangle \cap \Delta^- \). Then \( p_{\Sigma} \) is a parabolic subalgebra with reductive Levi factor \( p_r \) and nilpotent radical \( p_n \). Note that

\[
h \subset b \subset p_{\Sigma}.
\]

When a choice of Cartan and Borel has been made, any parabolic of the form \( p_{\Sigma} \) is a standard parabolic. Every parabolic subalgebra \( p \subset g \) is \( \text{Ad}(G) \)–conjugate to a standard parabolic. Using the identification (II.1) of the holomorphic tangent space we have

\[
(T_{x_0} \hat{D} = \bigoplus_{\beta \in \Delta^+ \setminus \langle \Sigma \rangle^+} g^\beta =: p_n^+).
\]

The second description of parabolic subalgebras \( p \subset g \) is in terms of the set \( \text{Hom}(\Lambda_{rt}, \mathbb{Z}) \) of grading elements where \( \Lambda_{rt} = \langle \Delta \rangle \subset h^* \) is the root lattice. If
\[ \Delta_+^+ = \{ \alpha_1, \ldots, \alpha_r \}, \text{ then there is a dual basis } \{ L_1, \ldots, L_r \} \text{ for the set of grading elements given by} \]

\[(\text{II.3}) \quad \alpha_j(L_i) = \delta_{ij}. \]

Given a grading element \( L \), under the action of \( h \) on \( \mathfrak{g}_C \) we have an eigenspace decomposition

\[(\text{II.4}) \quad \mathfrak{g}_C = g_{-k} \oplus \cdots \oplus g_0 \oplus \cdots \oplus g_k, \]

with \( g_\ell = \{ \xi \in \mathfrak{g} : [L, \xi] = \ell \xi \}. \)

Because the roots are integral linear combination of the simple roots, we see from (II.3) that the eigenvalues of \( L \) are integers \( \ell \in \mathbb{Z} \). We note that

- Each \( g_\ell \) is a direct sum of root spaces (and \( h \) when \( \ell = 0 \)). Explicitly,

\[(\text{II.5}) \quad g_\ell = \bigoplus_{\alpha(L)=\ell} g^\alpha \quad \text{if } \ell \neq 0, \]

\[ g_0 = h \oplus \bigoplus_{\alpha(L)=0} g^\alpha. \]

- The eigenspaces \( g_\ell \) and \( g_{-\ell} \) pair non–degenerately under the Cartan–Killing form.
- The Jacobi identity yields

\[(\text{II.6}) \quad [g_\ell, g_m] \subset g_{\ell+m}. \]

- The Cartan subalgebra \( h \subset g_0 \) and \( g_0 \) is a reductive subalgebra. (Indeed, \( g_0 \) is a Levi factor the parabolic subalgebra (II.8).) By (II.6), each \( g_\ell \) is a \( g_0 \)–module.
- More generally, any representation \( U \) of \( \mathfrak{g}_C \) admits an \( L \)–eigenspace decomposition. Since the weights of \( \mathfrak{g}_C \) are rational linear combinations of the roots, the eigenvalues are rational \( U = \oplus_{m \in \mathbb{Q}} U_m. \)

**Remark II.7.** Grading elements may be defined without reference to a choice of positive roots (equivalently, a choice of Cartan and Borel \( h \subset \mathfrak{b} \)). In general, a grading
element is any semisimple endomorphism of $g_C$ with integer eigenvalues and the property that the eigenspace decomposition (II.4) satisfies (II.6). If $g_C$ is semisimple, any such endomorphism is necessarily a grading element [6, Proposition 3.1.2(1)].$^{(4)}$

Given $L$ we may define the parabolic subalgebra

$$p_L = g_0 \oplus g_-$$

where $g_- = g_{-1} \oplus \cdots \oplus g_{-k}$. Then (II.8) is the Levi decomposition of $p_L$. This associates to every grading element a standard parabolic subalgebra. Conversely, given a standard parabolic $p_\Sigma$, there is a canonically associated grading element

$$L_\Sigma = \sum_{\alpha \notin \Sigma} L_\alpha.$$

In particular, the relationship between the root and grading element descriptions is:

Given $L$ define

$$\Sigma_L = \{ \alpha \in \Delta^+_s : \alpha(L) = 0 \} \subset \Delta^+_s.$$

With the grading element notation we note that (II.2) is

$$T_{x_o \check{D}} = g_+ = g_1 \oplus \cdots \oplus g_k.$$

II.B. Generalized flag domains. A real form $g_\mathbb{R}$ of $g_\mathbb{C}$ is the set of fixed points of a conjugation

$$\sigma : g_\mathbb{C} \rightarrow g_\mathbb{C} \; \text{satisfying} \; \sigma(\lambda X) = \bar{\lambda}(X) \; \text{for all} \; \lambda \in \mathbb{C} \; \text{and} \; X \in g_\mathbb{C}. $$

We denote by $G_\mathbb{R} \subset G_\mathbb{C}$ the corresponding connected real Lie group. A generalized flag domain is defined to be an open $G_\mathbb{R}$–orbit $D \subset \check{D}$ whose isotropy group is compact. A period domain is an example of a generalized flag domain. If $D = G_\mathbb{R} \cdot x_o \subset \check{D} = G_\mathbb{C}/P$, then the isotropy group is

$$R = G_\mathbb{R} \cap P.$$

Conversely, given a homogeneous complex manifold $D = G_\mathbb{R}/R$ with $R \subset G_\mathbb{R}$ the compact centralizer of a torus, its compact dual is a generalized flag variety $\check{D} = G_\mathbb{C}/P$ as above in which $D$ is an open $G_\mathbb{R}$–orbit. It is known that $R$ contains a compact $\chi$-stable conjugacy class.

$^{(4)}$The definition of grading element in [6] is more restrictive than ours: it imposes the condition that $g_1$ generate the Lie subalgebra $g_+$. Nonetheless the proof of [6, Proposition 3.1.2(1)] applies to our looser notion.
maximal torus $\mathcal{T}$ whose complexified Lie algebra $\mathfrak{t}_\mathbb{C}$ is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The roots $\Delta$ of $\mathfrak{h}$ take purely imaginary values on $\mathfrak{t}$, which gives that

$$\overline{\mathfrak{g}}^{\alpha} = \mathfrak{g}^{-\alpha}.$$ 

We have the identification of the complexified (real) tangent space

$$T_{x_0,\mathbb{C}}D = \bigoplus_{\alpha \in \Delta \setminus \langle \Sigma \rangle} \mathfrak{g}^{\alpha} = \mathfrak{g}_+ \oplus \mathfrak{g}_-,$$

and with a suitable choice of positive roots we have

$$T_{x_0}^{1,0}D = \bigoplus_{\alpha \in \Delta^+ \setminus \langle \Sigma \rangle^+} \mathfrak{g}^{\alpha} = T_{x_0}D.$$ 

If we specify $\check{D}$ (equivalently, $P \subset G_\mathbb{C}$) by a grading element $L$ as above, then we have

$$(II.10) \quad \overline{L} = -L.$$ 

II.C. **Polarized Hodge structures.**

II.C.1. **Definition.** Let $(V, Q, F^\bullet)$ denote a polarized Hodge structure of weight $n$ on a real vector space $V$. That is, the polarization $Q: V \times V \to \mathbb{R}$ is a nondegenerate bilinear form such that

$$Q(u, v) = (-1)^n Q(v, u) \quad \text{for all } u, v \in V,$$

and the Hodge filtration $F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 = V_\mathbb{C}$ is a decreasing filtration on $V_\mathbb{C}$ with the properties that

$$(HR1) \quad Q(F^p, F^{n-p+1}) = 0,$$

$$(HR2) \quad Q(Cv, \bar{v}) > 0 \quad \text{for all } 0 \neq v \in V_\mathbb{C}$$

where $C$ denotes the Weil operator. The equation (HR1) is the first Hodge–Riemann bilinear relation (HR1); the inequality (HR2) is the second Hodge–Riemann bilinear relation.

A $Q$–isotropic flag is any filtration $F^\bullet$ of $V_\mathbb{C}$ satisfying (HR1). Let

$$f = (f^p = \dim F^p)$$
denote the Hodge numbers. We may regard the Hodge structure as a point in the $\text{Aut}(V_C, Q)$–homogeneous generalized flag variety $\text{Flag}^Q_f(V_C)$ of $Q$–isotropic filtrations $F^\bullet$ of $V_C$ with dimension $\dim F^p = f^p$. The $\text{Aut}(V_R, Q)$–orbit of any one of these Hodge structures is the period domain of $Q$–polarized Hodge structures with Hodge numbers $f$. It is an open subset of $\text{Flag}^Q_f(V_C)$; the latter is the compact dual of the period domain.

The Hodge decomposition is

$$V_C = \bigoplus_{p+q=n} V^{p,q} \quad \text{with} \quad V^{p,q} = F^p \cap F^q$$

and satisfies

$$Q(V^{p,q}, V^{r,s}) \neq 0 \quad \text{only if} \quad (p, q) = (s, r),$$

$$C^1_{V^{p,q}} = i^{p-q} 1.$$  

We will also refer to

$$h = (h^{p,q} = \dim V^{p,q})$$

as the Hodge numbers of the Hodge structure. Note that $f^p = \sum_{q \geq p} h^{q,n-q}$.

A polarized Hodge structure $(V, Q, F^\bullet)$ of weight $n$ is equivalent to a homomorphism

$$\varphi : S^1 \rightarrow \text{Aut}(V_R, Q)$$

of real algebraic groups with the properties that $\varphi(-1) = (-1)^n 1$ and $Q(\varphi(i)v, \bar{v}) > 0$ for all $0 \neq v \in V_C$. The Hodge decomposition is the $\varphi$–eigenspace decomposition

$$V^{p,q} = \{ v \in V_C : \varphi(z)v = z^{p-q}v, \ \forall \ z \in S^1 \}.$$  

The Mumford–Tate group $G$ of the Hodge structure is the $Q$–algebraic closure of $\varphi(S^1)$ in $\text{Aut}(V_R, Q)$.(5) The stabilizer of $F^\bullet$ in $G_R$ is the compact $R = Z_\varphi = \{ g \in G_R : g\varphi(z) = \varphi(z)g \ \forall \ z \in S^1 \}$.  

(5) The Mumford–Tate group is the subgroup of $\text{Aut}(V, Q)$ stabilizing the Hodge tensors [12, (I.B.1)].
II.C.2. **Induced PHS on** $\text{End}(V, Q)$. There is an induced Hodge structure on the Lie algebra $\text{End}(V, Q)$ of $\text{Aut}(V, Q)$ defined by

$$F^p \text{End}(V, Q) = \{ \xi \in \text{End}(V, Q) : \xi(F^q) \subset F^{p+q} \forall q \}.$$  

Equivalently,

$$\text{End}(V, Q)^{p,q} = \{ \xi \in \text{End}(V, Q) : \xi(V^{r,s}) \subset V^{p+r,q+s} \forall r, s \}.$$  

Note that $\text{End}(V, Q)^{p,q} = 0$ if $p + q \neq 0$, so that

$$\text{End}(V, Q) = \bigoplus_{p \in \mathbb{Z}} \text{End}(V, Q)^{p,-p}$$

is a weight zero Hodge structure.

II.C.3. **PHS in terms of grading elements.** We may view grading elements as “infinitesimal Hodge structures” as follows.\(^{(6)}\) Note that the induced Hodge structure on $\text{End}(V, Q)$ satisfies

$$[\text{End}(V, Q)^{p,-p}, \text{End}(V, Q)^{q,-q}] \subset \text{End}(V, Q)^{p+q,-p-q}.$$  

So, if we define $L$ to be the semisimple endomorphism of $\text{End}(V, Q)$ that acts on $\text{End}(V, Q)^{p,-p}$ by the eigenvalue $-p$, then the discussion of Remark II.7 implies that $L \in \text{End}(V, Q)$ is a grading element.

The grading element $L$ also induces the original Hodge structure on $V$. In particular, the standard representation $V_C$ of $\text{Aut}(V, Q)$ decomposes into a direct sum $\bigoplus_m V_m$ of $L$–eigenvectors with rational eigenvalues (Remark II.7). This eigenspace decomposition is the Hodge decomposition

$$V^{p,q} = V_{(q-p)/2} = \{ v \in V_C : L(v) = \frac{1}{2}(q-p)v \}.$$  

We say that the grading element defines a polarized Hodge structure on $V$.

II.D. **Mumford–Tate domains.**

\(^{(6)}\)“Infinitesimal” because, appropriately rescaled, $\varphi'(1)$ is a grading element, and conversely every grading element may be realized as $\varphi'(1)$, cf. [19].
II.D.1. **Definition.** A Mumford–Tate domain is a generalized flag domain \( D = G_{\mathbb{R}}/R \) with additional data arising from a *Hodge representation*

\[ \rho : G_{\mathbb{R}} \to \text{Aut}(V, Q). \]

Specifically, in addition to \( \rho \) we are given a grading element \( L \in \text{Hom}(\Lambda_{rt}, \mathbb{Z}) \) such that

- \( \rho_*(L) \) defines a polarized Hodge structure \((V, Q, F^\bullet)\);
- The Mumford–Tate group of \((V, Q, F^\bullet)\) is equal to \( \rho(G_{\mathbb{R}}) \), and
- the isotropy group in \( G_{\mathbb{R}} \) of \( F^\bullet \) is equal to \( R \).

By definition the *Mumford–Tate domain* \( D \) is the set of polarized Hodge structures \( \{(V, Q, \rho(g)F^\bullet) : g \in G_{\mathbb{R}}\} \).\(^{(7)}\) Note that \( D \) is the \( G_{\mathbb{R}}^- \)-orbit of \( F^\bullet \) in the period domain containing the polarized Hodge structure \((V, Q, F^\bullet)\). We may think of \( D \) as the set of polarized Hodge structures \((V, Q, F^\bullet_x)\), with \( x \in D \), such that the Mumford–Tate group of each \((V, Q, F^\bullet_x)\) is contained in \( \rho(G) \), and equality holds for general \( x \).

A Mumford–Tate domain structure gives an embedding of the compact dual

\[ \tilde{D} \hookrightarrow \text{Flag}^Q_f(V_{\mathbb{C}}) \]

as the \( G_{\mathbb{C}}^- \)-orbit of any \( F^\bullet \in D \). Specifically, \( x \in \tilde{D} \) gives a flag \( F^\bullet_x \) that satisfies the first Hodge–Riemann bilinear relation \( Q(F^p_x, F^{n-p+1}_x) = 0 \). The second Hodge–Riemann bilinear relation defines the open \( G_{\mathbb{R}}^- \)-orbit \( D \subset \tilde{D} \).

**Example II.12.** A special case of a Mumford–Tate domain is the period domain of Section II.C.1. In this case \( G_{\mathbb{R}} = \text{Aut}(V_{\mathbb{R}}, Q) \) is \( \text{SO}(2a, b) \) for even weight, and \( \text{Sp}(2g, \mathbb{R}) \) for odd weight.

In the case when \( V \) is an irreducible \( G \)-module another way of think of a Mumford–Tate domain is that it is given by a pair \((\varphi, \chi)\) consisting of a co-character \( \varphi \) and a character \( \chi \) of \( \mathcal{Z} \subset G_{\mathbb{R}} \). Specifically, for \( S^1 = \mathbb{R}/2\pi i\mathbb{Z} \) and

\[ \varphi : S^1 \to \mathcal{Z} \]

\(^{(7)}\)A more precise term would be a *Mumford–Tate domain structure on the generalized flag domain* \( D \).
the isotropy subgroup \( R = Z_{G_R}(\varphi(S^1)) \) is the centralizer in \( G_R \) of the circle \( \varphi(S^1) \). The \( G_R \)-invariant complex structure on \( D = G_R/R \) is given by

\[
\text{Ad} \circ \varphi : S^1 \to g_R/r \simeq T_{x_o,R}G_R/R.
\]

The character \( \chi \) is the highest weight of the representation \( \rho : G \to \text{Aut}(V, Q) \). There are conditions, not spelled out here, on the pair \((\varphi, \chi)\). See [12] for details.

II.D.2. Induced PHS on \( g \). From the description of Section II.D.1 it is clear that a given homogeneous complex manifold \( D \), corresponding to \( \varphi \) above, may be realized as a Mumford–Tate domain in multiple ways, corresponding to the \( \chi \)'s above (see Section II.D.4). For this work a particularly important pair of such realizations is the following: Given a generalized flag domain \( D \) realized as a Mumford–Tate domain for polarized Hodge structure \((V, Q, F^\bullet)\), another realization is as induced polarized Hodge structures on \( g \subset \text{End}(V, Q) \). Generalizing Section II.C.2 from \( \text{Aut}(V, Q) \) to the more general Mumford–Tate groups \( G \), these are defined by

\[
F^p_g = \{ \xi \in g_C : \xi(F^q) \subset F^{p+q} \forall q \}.
\]

Equivalently,

\[
\mathfrak{g}^{p,q} = \{ \xi \in g_C : \xi(V^r,s) \subset V^{p+r,q+s} \forall r,s \}.
\]

Note that \( \mathfrak{g}^{p,q} = 0 \) if \( p + q \neq 0 \), so that

\[
(II.13) \quad g_C = g^{-k,k} \oplus \cdots \oplus g^{0,0} \oplus \cdots \oplus g^{k,-k}
\]

is a weight zero Hodge structure.

The polarization \( Q \) on \( V \) induces a polarization \( Q_g \) on \( g \). The latter is invariant under \( G \). Therefore, if \( g \) is simple, then \(-Q_g\) is necessarily a positive multiple of the Killing form. Unless otherwise stated,

\( -Q_g \) will denote the Killing form throughout.

Notice that \( F^0_g \) is the Lie algebra of the stabilizer \( P \subset G_C \) of both the Hodge structure \( F^\bullet \) on \( V \) and the induced Hodge structure \( F^\bullet_g \) on \( g \). In particular,

the \( G_C \)-orbits of \( F^\bullet \in \text{Flag}^Q(V_C) \) and \( F^\bullet_g \in \text{Flag}^Q_g(g_C) \) both realize the generalized flag variety \( \tilde{D} = G_C/P \) as a projective variety.
Moreover, the two infinitesimal period relations agree under this identification, cf. [12]. Likewise, if
\[ R = G_R \cap P, \]
then
\[ \text{the } G_R\text{-orbits of } F^\bullet \in \text{Flag}_Q^Q(V_C) \text{ and } F^\bullet_0 \in \text{Flag}_Q^Q(g_C) \text{ both realize the homogeneous manifold } D = G_R/R \text{ as a Mumford–Tate domain.} \]

By slight abuse of terminology we refer to the \( G_R \)-orbit of \( F^\bullet \) as the \emph{adjoint Mumford–Tate domain} \( D_\varnothing \) associated to \( D \), where the latter is viewed as the Mumford–Tate domain for the Hodge structure on \( V \). The reason for doing this is that coming from algebraic geometry one thinks of \((V, Q, F^\bullet)\) as arising from \( H^n(X, Q)_{\text{prim}} \) where \( X \) is a smooth projective variety. However, in order to study (i) the geometry of the \( G_R \)-orbits in \( \hat{D} \) and (ii) Lie–theoretic aspects of the Hodge structure, it is necessary to work with the polarized Hodge structures \((g, Q_\varnothing, F^\bullet_0)\).

II.D.3. \emph{PHS in terms of grading elements}. In analogy with Section II.C.3, given a Hodge structure on \( V \) with Mumford–Tate group \( G \), there is a canonical choice of grading element. To be precise, given the induced Hodge structure (II.13) on \( g \), define \( L \) by
\[
L|_{g^{-p,p}} = p \mathbb{1}.
\]

Since \( L \) is a derivation and \( g_C \) is semisimple, \( L \) is necessarily an element of \( g_C \). Moreover, since \( L \) is semisimple, it is necessarily contained in a Cartan subalgebra. And since the eigenvalues of \( L \) on \( g_C \) are integers, \( L \) is necessarily a grading element (Remark II.7).

Conversely, given a complex semisimple Lie algebra \( g_C \) and a grading element \( L \in \text{Hom}(\Lambda_\text{rt}, \mathbb{Z}) \), there is a canonical choice of real form \( g_\mathbb{R} \) (which we may take to be defined over \( \mathbb{Z} \)) with the property that the \( L \)-eigenspace decomposition (II.4) of \( g_C \) defines a weight zero Hodge structure (II.13) by the assignment
\[
g^{-p,p} = g_p,
\]
cf. [19, Proposition 2.36]. This Hodge structure on \( g \) is related to the initial Hodge structure on \( V \) by the grading element: the subspaces \( V^{p,q} \) are also \( L \)-eigenspaces;
that is, (II.11) holds. Observe, that while the $L$-eigenvalues on $\mathfrak{g}$ are integers, on $V$ they lie in $\frac{1}{2}\mathbb{Z}$.

The Hodge structure (II.15) on $\mathfrak{g}$ is polarized by $Q_{\mathfrak{g}}$; equivalently, if

$$\mathfrak{t}_{\mathbb{C}} = \bigoplus \mathfrak{g}_{2\ell} \quad \text{and} \quad \mathfrak{t}_{\mathbb{C}}^\perp = \bigoplus \mathfrak{g}_{2\ell+1},$$

then $\mathfrak{t}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{t}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}^\perp = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{t}_{\mathbb{C}}^\perp$ define a Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} \oplus \mathfrak{t}_{\mathbb{R}}^\perp$.

We see from (II.5) and (II.15) that each $\mathfrak{g}^{p,-p}$ is a direct sum of root spaces (and $\mathfrak{h}$ if $p = 0$). Define compact and noncompact roots by

$$\Delta_c = \{ \alpha \in \Delta : \mathfrak{g}^\alpha \subset \mathfrak{t}_{\mathbb{C}} \} = \{ \alpha \in \Delta : \alpha(L) \text{ is even} \},$$
$$\Delta_{nc} = \{ \alpha \in \Delta : \mathfrak{g}^\alpha \subset \mathfrak{t}_{\mathbb{C}}^\perp \} = \{ \alpha \in \Delta : \alpha(L) \text{ is odd} \}.$$  

Note that

$$\Delta = \Delta_c \cup \Delta_{nc}.$$

II.D.4. Realizations of $D$. In Section II.D.3 we observed that both the Hodge structure on $V$ and the induced Hodge structure on $\mathfrak{g}$ are given by a common grading element $L$. This fact may be used to deduce that the two Mumford–Tate domains $D$ and $D_\mathfrak{g}$ parameterizing Hodge structures of these types are isomorphic as $G_{\mathbb{R}}$-homogeneous complex submanifolds $G_{\mathbb{R}}/R \subset G_{\mathbb{C}}/P$, cf. [12]. This is one example of a general method to realize $G_{\mathbb{R}}/R$ as a Mumford–Tate domain, which we now outline.

Given a generalized flag variety $G_{\mathbb{C}}/P$ there are canonically defined Hodge structures that realize the variety as a compact dual. Let $L = L_p$ be the grading element (II.9) associated with $\mathfrak{p}$. Let $\Lambda_{\text{wt}} \subset \mathfrak{h}^*$ be the weight lattice of $\mathfrak{g}_{\mathbb{C}}$, and let $\{\omega_1, \ldots, \omega_r\}$ be the fundamental weights with respect to the simple roots $\{\alpha_1, \ldots, \alpha_r\}$. Given any dominant integral weight $\lambda \in \Lambda_{\text{wt}}$, let $U^\lambda$ denote the corresponding irreducible representation of $\mathfrak{g}_{\mathbb{C}}$ with highest weight $\lambda$. If $\lambda$ is a weight of $G_{\mathbb{C}},(8)$ then the parabolic $P$ is the stabilizer of the highest weight line in $U^\lambda$ if and only if

$$\lambda = \sum_{\alpha_i \in \Sigma} \lambda^i \omega_i \quad \text{with} \quad 0 < \lambda^i \in \mathbb{Z}. \quad (II.16)$$

(8) This will always be the case if $G_{\mathbb{C}}$ is simply connected.
In this case, the $G_C$–orbit of the highest weight line is a homogeneous embedding $G_C/P \hookrightarrow \mathbb{P}U^\lambda$ that realizes the homogeneous complex manifold $G_C/P$ as a homogeneous projective variety.

Let $g_\mathbb{R}$ be the real form determined by $L$ (Section II.D.3). Given an irreducible representation $V_\mathbb{R}$ of $g_\mathbb{R}$ there exists an irreducible representation $U$ of $g_C$ such that one of the following holds:

- $V_C = U$, in which case $U$ is real;
- $V_C = U \oplus U^*$ and $U \simeq U^*$, in which case $U$ is quaternionic;
- $V_C = U \oplus U^*$ and $U \not\simeq U^*$, in which case $U$ is complex.

In the case that $\lambda$ is the highest weight of $U$, the representation $V_\mathbb{R}$ admits a polarized Hodge structure $(F^\bullet, Q)$ with Mumford–Tate group $G$ if and only if

\[(\text{II.17}) \quad L(\lambda) \in \frac{1}{2}\mathbb{Z},\]

cf. [12]. (A priori, we have only $L(\lambda) \in \mathbb{Q}$.) In this case, the Hodge structure $(V, Q, F^\bullet)$ is given by the $L$–eigenspace decomposition (II.11). In particular,

$G_C/P$ is realized as the compact dual for any of these polarized Hodge structures.

From this perspective, a very natural realization is given by any $\lambda$ that minimizes the coefficients $\lambda^i$ of (II.16) subject to the constraint (II.17). In many cases it is possible to take $\lambda^i = 1$;\(^{(9)}\) this corresponds to the minimal homogeneous embedding $G/P \hookrightarrow \mathbb{P}U^\lambda$ of $G/P$ as a rational homogeneous variety.

Likewise,

the $G_\mathbb{R}$–orbit of any of these Hodge filtrations $F^\bullet$ realizes $G_\mathbb{R}/R$ as a Mumford–Tate domain.

\(^{(9)}\)The issue is the following: In the case that $g_C$ is simple, the weights of $g_C$ lie in the $\frac{1}{2}\mathbb{Z}$–span of the simple roots, where $1 \leq d \in \mathbb{Z}$ is the determinant of the Cartan matrix. So, when $d \in \{1, 2\}$ (II.17) will hold with $\lambda^i = 1, \alpha_i \not\in \Sigma$. We have $d \in \{1, 2\}$ when $g_C$ is one of $\mathfrak{so}_{2r+1}\mathbb{C}, \mathfrak{sp}_{2r}\mathbb{C}, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4$ or $\mathfrak{g}_2$. In the case that $g_C = \mathfrak{so}_d\mathbb{C}$, the determinant is $n + 1$ and we will be able to satisfy (II.17) with values $\lambda^i \in \{1, \ldots, n + 1\}$; in the case that $g_C = \mathfrak{so}_{2r}\mathbb{C}$, we have $d = 4$ and will be able to satisfy (II.17) with $\lambda^i \in \{1, 2\}$; in the case that $g_C = \mathfrak{e}_6$, we have $d = 3$ and will be able to satisfy (II.17) with $\lambda^i \in \{1, 2, 3\}$. 
Remark II.18. As the highest weight of the adjoint representation, the highest root \( \tilde{\alpha} \) is a dominant integral weight. By definition the grading element is integer–valued on \( \tilde{\alpha} \). So (II.17) holds with \( \lambda = \tilde{\alpha} \). Whence, the construction above yields a polarized Hodge structure \( (\mathfrak{g}, \tilde{Q}_g, \tilde{F}^*_g) \). This agrees with the polarized Hodge structure \( (\mathfrak{g}, Q_g, F^*_g) \) induced (as in Section II.D.2) from any of the \( (V, Q, F^*) \) constructed in this section.

II.E. Period mappings and nilpotent orbits. We shall only consider period mappings corresponding to a one–parameter family of degenerating polarized Hodge structures. Such is given by a Mumford–Tate domain \( D \), a unipotent monodromy transformation \( T \in G \) and a locally liftable holomorphic mapping

\[
(II.19) \quad \Phi : \Delta^* \to \Gamma_T \setminus D.
\]

which satisfies the infinitesimal period relation. Here \( \Gamma_T = \{ T^k : k \in \mathbb{Z} \} \).

Denoting by \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) the upper–half plane with covering map \( \mathbb{H} \to \Delta^* \) given by

\[
t = e^{2\pi i z},
\]

we may lift (II.19) to give

\[
\tilde{\Phi} : \mathbb{H} \to D,
\]

\[
\tilde{\Phi}(z + 1) = T \cdot \tilde{\Phi}(z).
\]

Setting \( N = \log(T) \in \mathfrak{g}^{\text{nilp}} \), we may then “unwind” \( \Phi \) to

\[
\tilde{\Psi} : \mathbb{H} \to \tilde{D}
\]

by defining

\[
\tilde{\Psi}(z) = e^{-zN} \cdot \tilde{\Phi}(z).
\]

Then \( \tilde{\Psi}(z + 1) = \tilde{\Psi}(z) \) so that there is an induced map

\[
\Psi : \Delta^* \to \tilde{D}.
\]

A basic result is that \( \Psi \) extends across the origin \( t = 0 \). Then setting \( \ell(t) = \log(t)/2\pi i \), the original period mapping is well approximated by the nilpotent orbit

\[
(II.20) \quad t \mapsto e^{\ell(t)N} \cdot \Psi(0),
\]
see [21]. Implicit here is the statement that $e^{\ell(t)N} \cdot \Psi(0) \in D$ for $0 < |t| < \varepsilon$. We shall further explain this below.

We shall sometimes write $\Phi(t) = F_t^\bullet$ for the multi-valued filtration on $V_\mathbb{C}$. The lift $\tilde{\Phi}$ to $\mathbb{H} \to D$ will be denoted by

$$z \mapsto F_z^\bullet$$

where $F_{z+1}^\bullet = T \cdot F_z^\bullet$.

Because of the strong approximation of (II.19) by a nilpotent orbit, we shall replace $\Phi$ by the nilpotent orbit (II.20). For this we set

$$\Psi(0) = F_{\lim}^\bullet.$$ 

Note that $F_{\lim}^\bullet$ is defined only up to the action of $\Gamma_T$ and a choice of coordinate $t$. Rescaling $t$ by $t \mapsto e^{2\pi i \lambda t}$ induces the change

$$F_{\lim}^\bullet \to e^{\lambda N} \cdot F_{\lim}^\bullet.$$ 

Thus what is well-defined is a map

$$\{\text{period mappings (II.19)}\} \times T_0^* \Delta \to \text{nilpotent orbits}.$$ 

The conditions that $\Phi$ define a period mapping translate into

(II.21)

$$e^zN \cdot F_{\lim}^\bullet \in D \quad \text{for } \Im(z) \gg 0,$$

$$N \cdot F_{\lim}^p \subset F_{\lim}^{p-1} \quad \text{(infinitesimal period relation).}$$

**Definition** II.22. A nilpotent orbit is given by $(F^\bullet, N)$ where $F^\bullet \in \mathcal{D}$, $N \in \mathfrak{g}_{\mathbb{R}}^{\text{nilp}}$ and where the conditions (II.21) are satisfied with $F^\bullet$ in place of $F_{\lim}^\bullet$.

Two nilpotent orbits $(F^\bullet, N)$ and $(F'^\bullet, N)$ are equivalent if

$$F'^\bullet = e^{\lambda N} F^\bullet$$

for some $\lambda \in \mathbb{C}$. We set

- $\hat{B}(N) = \{\text{nilpotent orbits } (F^\bullet, N)\}$,
- and let $B(N) = e^{CN} \backslash \hat{B}(N)$ denote the set of equivalence classes of nilpotent orbits.
We next set
\[ D_N = D \cup B(N) \]
and observe that the action of \( \Gamma_T \) extends naturally to \( D_N \). In [14] the structure of a log-analytic variety with slits is defined on \( D_N \), and a basic result is that the period mapping (II.19) extends to
\begin{equation}
\Phi_e : \Delta \to \Gamma_T \setminus D_N
\end{equation}
where the origin is mapped to the equivalence class of \( (F_{\text{lim}}^\bullet, N) \). We shall refer to (II.23) as the extended period mapping.

**Definition II.24.** The mapping
\[ \{ \text{period mappings (II.19)} \} \to \Phi_e(0) \in \Gamma_T \setminus D_N \]
will be called the limit period mapping.

**II.F. Nilpotent orbits and limiting mixed Hodge structures.** Let \( D \) be a Mumford–Tate domain parameterizing weight \( n \), \( Q \)-polarized Hodge structures on \( V \) whose generic Mumford–Tate group is \( G \). Associated to a nilpotent orbit \( (F^\bullet, N) \) there is a special type of mixed Hodge structure \( (V, W^\bullet(N), F^\bullet) \) called a limiting mixed Hodge structure (limiting mixed Hodge structures). There will then be a bijection of sets
\[ \{ \text{nilpotent orbits} \} \overset{(\text{II.27)}}{\longleftrightarrow} \{ \text{limiting mixed Hodge structures} \}, \]
which will pass to the quotient by taking equivalence classes.

Since \( N \) is nilpotent, there exists \( 0 \leq m \leq n \) such that \( N^m \neq 0 \) and \( N^{m+1} = 0 \). Then the weight filtration
\[ W_{-m}(N) \subset \cdots \subset W_0(N) \subset \cdots \subset W_m(N) = V \]
is the unique filtration that satisfies
\begin{align*}
N : W_k(N) &\longrightarrow W_{k-2}(N), \\
N^k : \text{Gr}_{k}^{W^\bullet(N)} &\longrightarrow \text{Gr}_{k-2}^{W^\bullet(N)} \quad k \geq 0.
\end{align*}
It is always possible to complete \( N \) to a \( \mathfrak{sl}_2 \)-triple
\begin{equation}
\{ N, Y, N^+ \} \subset \mathfrak{g}
\end{equation}
where
\[
[Y, N] = -2N, \\
[Y, N^+] = 2N^+, \\
[N^+, N] = Y.
\]
The span of (II.25) is a three-dimensional semisimple subalgebra (TDS) of \(\mathfrak{g}\) that is isomorphic to \(\mathfrak{sl}_2\). Denoting by \(V_\ell = \{ v \in V : Yv = \ell v \}\) the \(\ell\)-weight space for the semisimple action of \(Y\), we have
\[
(W_k(N) = \bigoplus_{\ell \leq k} V_\ell \quad \text{and} \quad Gr^{W_\bullet(N)}_k \simeq V_k.
\]
The primitive spaces are defined, for \(k \geq 0\), by
\[
Gr^{W_\bullet(N)}_{k, \text{prim}} = \ker \left\{ N^{k+1} : Gr^{W_\bullet(N)}_k \to Gr^{W_\bullet(N)}_{k-2} \right\}.
\]
Decomposing \(V\) under the action of the TDS, the primitive spaces are the highest weight spaces. There are nondegenerate bilinear forms, of parity \(k\),
\[
Q_k : Gr^{W_\bullet(N)}_k \times Gr^{W_\bullet(N)}_k \to \mathbb{R}
\]
defined by
\[
Q_k(v, w) = \epsilon_k Q(v, N^k w)
\]
where \(\epsilon_k = \pm 1\).

The basic result \([7]\) is
\[
(F^\bullet, N) \text{ is a nilpotent orbit if and only if } (V, W_\bullet(N), F^\bullet) \text{ is a polarized mixed Hodge structure.}
\]
The filtration \(F^\bullet\) induces a Hodge structure of weight \(k\) on \(Gr^{W_\bullet(N)}_k\) and \(N\) is of Hodge type \((-1, -1)\). The polarization condition means that the summand \(Gr^{W_\bullet(N)}_{k, \text{prim}}\) of \(Gr^{W_\bullet(N)}_k\) is polarized by the form \(Q_k\). We shall generally suppress mention of the polarization conditions, which we always take to be understood.

The equivalence relation on limiting mixed Hodge structures is induced by rescaling \(F^\bullet \mapsto e^{\lambda N}F^\bullet\). The Hodge structures on \(Gr^{W_\bullet(N)}_k\) are unchanged, but some of the extension data in the mixed Hodge structure will be altered.
II.G. **Reduced limit period mapping.** This section summarizes material developed in [13, Appendix to Lecture 10] and [16]; the interested reader should consult those references for additional detail and proofs.

Given a period mapping (II.19) with extension (II.23) the limiting mixed Hodge structure given by the point

\[ \Phi_e(0) \in \Gamma_T \setminus B(N) \]

represents, in a precise sense, the maximal amount of information in the limit of a degenerating family of polarized Hodge structures. For some purposes the other extreme of describing a minimal amount of information in the limit is useful. In that direction we will consider two notions of a reduced limit period mapping. For the first we lift (II.19) to

\[ \tilde{\Phi} : \mathbb{H} \to D. \]

**Definition** II.28 (First notion). The reduced limit period mapping is defined by

\[ \Phi \mapsto \lim_{\text{Im}(z) \to \infty} \tilde{\Phi}(z) \in \partial D. \]

The map was introduced in [16] under the term naïve limit. It has been further discussed in [13, Appendix to Lecture 10] and [17]. If we think of \( \tilde{\Phi}(z) = F_z \cdot F^\bullet \) as a filtration on \( V_\mathbb{C} \), then we will set

\[ \lim_{\text{Im}(z) \to \infty} F_z \cdot F^\bullet = F^\bullet. \]

The reduced limit period mapping has the properties:

(a) It is the same for \( \Phi \) and for the approximating nilpotent orbit (II.20). So, without loss of generality, we shall assume that \( \tilde{\Phi}(z) = e^{zN} \cdot F^\bullet \).

(b) It is independent of the lifting \( \tilde{\Phi} \). In fact, \( F^\bullet_\infty \in \partial D \) is a fixed point of \( T \) and the differential

\[ T_z : T_{F^\bullet_\infty} \hat{D} \to T_{F^\bullet_\infty} \hat{D} \]

is the identity. Equivalently, the vector field on \( \hat{D} = G_\mathbb{C}/P \) defined by \( N \in g \) vanishes to second order at \( F^\bullet_\infty \).

(c) The mapping

(II.29a) \[ \Phi_\infty : B(N) \to \partial D \]
defined on $\tilde{B}(N)$ by

\[(II.29b) \quad \Phi_\infty(F^\bullet, N) = \lim_{\text{Im}(z) \to \infty} \Phi(z) = F^\bullet\]

is well–defined on $B(N)$, and it is $Z(N)_R$–equivariant. The image of $B(N)$ lies in a $G_\mathbb{R}$–orbit in $\partial D$

$$\Phi_\infty(B(N)) \subset O_{F^\bullet} = G_\mathbb{R} \cdot F^\bullet.$$  

**Definition II.30 (Second notion).** Given a Mumford–Tate domain $D \subset \tilde{D}$ with boundary component $B(N)$, the reduced limit period mapping is (II.29).

Note that the first notion (Definition II.28) is the composition $\Phi_\infty \circ \Phi_e(0)$.

**Definition II.31 ([15]).** A $G_\mathbb{R}$–orbit $O \subset \partial D$ for a generalized flag domain $D$ is polarizable relative to $D$ if there is a Mumford–Tate domain structure on $D$ and a nilpotent orbit $(F^\bullet, N)$ such that $F^\bullet_\infty \in O$. The orbit $O$ is polarizable if there is a Mumford–Tate domain $D$ with $O \subset \partial D$ relative to which $O$ is polarizable.

There are examples of generalized flag domains $D$ and $D'$ in a generalized flag variety $\tilde{D}$ and a $G_\mathbb{R}$–orbit $O \subset \partial D \cap \partial D'$ such that $O$ is polarizable relative to $D$, but not relative to $D'$, cf. [13, Appendix to Lecture 10].

Let $N \subset g_\mathbb{R}$ denote the $\text{Ad}(G_\mathbb{R})$–orbit of $N$. Then

$$O_{F^\bullet} = \bigcup_{N' \in N} \Phi_\infty(B(N')).$$

In this case we say that $O_{F^\bullet}$ is polarized by $N$ relative to $D$. In this sense, the $G_\mathbb{R}$–orbits in $\tilde{D}$ separate into those that have Hodge–theoretic significance, meaning that over $\mathbb{R}$ every point is realized as a reduced limit for some Mumford–Tate domain structure, and those that don’t have Hodge–theoretic significance in this sense.

To explain this a bit more, given a Mumford–Tate domain $D$, for every $F^\bullet_x \in \tilde{D}$ we may consider the intersection

$$V_x^{p,q} = F_x^p \cap \overline{F_x^q}.$$  

The $x$ for which

$$F_x^p \oplus \overline{F_x^{n-p+1}} \xrightarrow{\sim} V_c$$
for $0 \leq p \leq n$ give Hodge structures, perhaps with indefinite polarizations meaning that the Hermitian forms in the second Hodge–Riemann bilinear relation are nonsingular but may not be positive definite. These $O_x$ are exactly the open $G_\mathbb{R}$–orbits in $\hat{D}$. For the lower–dimensional orbits, the $V_x^{p,q}$ lead to mixed Hodge structures, but without the presence of an $N$ whose weight filtration together with the $F^p_x$ give a polarized limiting mixed Hodge structure these seem relatively uninteresting.

In light of the equivalence (II.27) between nilpotent orbits and limiting mixed Hodge structures, one may ask: What point of $\partial D$ does a limiting mixed Hodge structure map to? To answer this we first need two general facts about mixed Hodge structures. Given a mixed Hodge structure $(V, W^\bullet, F^\bullet)$ there is the canonical Deligne splitting

$$V_C = \bigoplus I^{p,q},$$

$$I^{p,q}_+ \equiv I^{q,p} \mod W_{p+q-2}$$

with

$$W_k = \bigoplus_{p+q \leq k} I^{p,q} \quad \text{and} \quad F^p = \bigoplus_{q \geq p} I^{q,*}.$$  

The mixed Hodge structure is split over $\mathbb{R}$, or $\mathbb{R}$–split, if

$$I^{p,q}_+ = I^{q,p}.$$  

In this case $(V, W_*, F^\bullet)$ is a direct sum over $\mathbb{R}$ of pure Hodge structures $\oplus_{p+q=k} I^{p,q}$ of weight $k$.

For the second property, canonically associated to a mixed Hodge structure $(V, W_*, F^\bullet)$ there is an $\mathbb{R}$–split mixed Hodge structure $(V, W_*, \tilde{F}^\bullet)$ given by

$$\tilde{F}^\bullet = e^{-2i\delta} \cdot F^\bullet$$

where $\delta \in \oplus_{p+q<0} I^{p,q}_g$. Here, $I^{p,q}_g$ is the Deligne splitting of the induced limiting mixed Hodge structure $(g, W_{\ast,g}, F^\bullet_g)$.

If $(V, W_\ast(N), F^\bullet)$ is a limiting mixed Hodge structure, then so is $(V, W_\ast(N), \tilde{F}^\bullet)$, and conversely. Moreover, we have

$$F^\bullet_\infty = \tilde{F}^\bullet_\infty.$$  

(II.33)
that is, denoting by $B(N)_{\mathbb{R}}$ the equivalence classes containing an $\mathbb{R}$–split limiting mixed Hodge structure, the reduced limit period mapping factors

$$
\begin{align*}
B(N) \xrightarrow{\Phi_\infty} & \partial D \\
\downarrow & \\
B(N)_{\mathbb{R}} & \xrightarrow{} 
\end{align*}
$$

(II.34)

Next, in terms of the Deligne splitting

$$
V_C = \bigoplus \tilde{I}^{p,q}
$$

associated with $(V, W_\bullet(N), \tilde{F}^\bullet)$ we have

$$
F_\infty^p = \bigoplus_{q \leq n-p} \tilde{I}^{\bullet,q}.
$$

(II.35)

The picture is this

Starting with $F^\bullet \in D$ such that $(V, W_\bullet(N), F^\bullet)$ is a limiting mixed Hodge structure, we have for the reduced limit period mapping (the solid arrow emanating from $F^\bullet \in D$)

$$
\lim_{y \to \infty} e^{byN} F^\bullet = F_\infty^\bullet.
$$

We also have the map $F^\bullet \to \tilde{F}^\bullet$ (the dashed arrow), and then

$$
\lim_{y \to \infty} e^{yN} \tilde{F}^\bullet = F_\infty^\bullet.
$$

Thus, $F_\infty^\bullet$ is reached from $\tilde{F}^\bullet$ by traveling along the real one–parameter subgroup $\exp(\mathbb{R}N)$ in a $G_{\mathbb{R}}$–orbit in $\partial D$.

Because of (II.33) and the subsequent factorization (II.34), henceforth, unless mentioned otherwise, we shall adopt the
Convention. We shall assume that a limiting mixed Hodge structure is \( \mathbb{R} \)-split.

Because of this we may drop the tilde in (II.35) to have

\[
F^p_\infty = \bigoplus_{q \leq n-p} I^{\bullet,q}.
\]

II.H. Reduced limit period mapping for \((\mathfrak{g}, Q_\mathfrak{g}, F^\bullet)\). The limiting mixed Hodge structure \((V, W_\bullet(N), F^\bullet)\) determines a limiting mixed Hodge structure \((\mathfrak{g}, W_\bullet(N)_\mathfrak{g}, F^\bullet_\mathfrak{g})\) by

\[
F^p_\mathfrak{g} = \{ \xi \in \mathfrak{g}_\mathbb{C} : \xi(F^q) \subset F^{p+q} \forall q \},
\]

\[
W_\ell(N)_\mathfrak{g} = \{ \xi \in \mathfrak{g}_\mathbb{R} : \xi(W_m(N)) \subset W_{m+\ell}(N) \forall m \}.
\]

As above \( V_\mathbb{C} = \bigoplus I^{p,q} \) will denote the Deligne splitting of \( V_\mathbb{C} \), and

\[
(II.37) \quad \mathfrak{g}_\mathbb{C} = \bigoplus I^{p,q}_\mathfrak{g}
\]

will denote the Deligne splitting on \( \mathfrak{g}_\mathbb{C} \). If the initial limiting mixed Hodge structure \((V, W_\bullet(N), F^\bullet)\) is \( \mathbb{R} \)-split, so is the induced limiting mixed Hodge structure \((\mathfrak{g}, W_\bullet(N)_\mathfrak{g}, F^\bullet_\mathfrak{g})\) on \( \mathfrak{g} \); indeed,

\[
(II.38) \quad I^{p,q}_\mathfrak{g} = \{ \xi \in \mathfrak{g}_\mathbb{C} : \xi(I^{r,s}) \subset I^{p+r,q+s} \forall r, s \}.
\]

In particular, (II.36) implies

\[
F^{p,\infty}_\mathfrak{g} = \bigoplus_{q \leq -p} I^{\bullet,q}_\mathfrak{g}.
\]

Example II.39 (A \( G_2 \) Mumford–Tate domain). The exceptional simple Lie group \( G_2 \) of rank two may be realized as the Mumford–Tate group of a weight 6 Hodge structure with Hodge numbers \( h = (1, 1, 1, 1, 1, 1, 1, 1) = (1^7) \), [16, Section 6.1.3]. The associated Mumford–Tate domain \( D \) is an open \( G_2(\mathbb{R}) \)-orbit in the flag variety \( G_2(\mathbb{C})/B \).

Including \( D \) (which is polarized by \( N = 0 \)) there are four \( G_2(\mathbb{R}) \)-orbits that are polarized relative to \( D \). The Deligne splittings \( V_\mathbb{C} = \bigoplus I^{p,q} \) (left column) and \( \mathfrak{g}_\mathbb{C} = \bigoplus I^{p,q}_\mathfrak{g} \) (right column) for each of these orbits are pictured in Figure II.1. A circled node indicates \( i^{p,q} = 2 \), and an uncircled node indicates \( i^{p,q} = 1 \).

The boundary \( \partial D \) contains seven \( G_2(\mathbb{R}) \)-orbits, four of them are not polarized relative to \( D \) [16, Section 6.1.3].
Figure II.1. Deligne splittings for polarized orbits in $G_2(\mathbb{C})/B$

Remark II.40 (Jacobson–Morosov parabolics). Let $Y$ be the neutral element of an $\mathfrak{sl}_2$–triple (II.25) containing the given nilpotent $N$. In analogy with (II.26) we have

$$W_k(N)_\mathfrak{g} = \bigoplus_{\ell \leq k} \mathfrak{g}_\ell, \quad \text{where} \quad \mathfrak{g}_\ell = \{\xi \in \mathfrak{g} : [Y, \xi] = \ell \xi\}$$

is the $\ell$–eigenspace of $Y$. In particular,

$$W_0(N)_\mathfrak{g} = \bigoplus_{\ell \geq 0} \mathfrak{g}_{-\ell}$$
is a parabolic subalgebra of $\mathfrak{g}$. Not every parabolic $\mathfrak{p} \subset \mathfrak{g}$ may be realized as $\mathfrak{p} = W_0(N)_{\mathfrak{g}}$ for some $N$; \(^{(10)}\) those that can are Jacobson–Morosov parabolics. When the eigenvalues of $Y$ are even, that is $\mathfrak{g}_{2k+1} = 0$ for all $k \in \mathbb{Z}$, the we say that $W_0(N)$ is an even Jacobson–Morosov parabolic. A Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ may always be realized as an even Jacobson–Morosov parabolic by taking $N$ to be a principal nilpotent \([8]\).

Recall that, up to the action of $\text{Ad}(G_{\mathbb{C}})$, the parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$ are indexed by (possibly empty) subsets $\Sigma \subset \Delta^+_s$ of the simple roots (Section II.A). The subset $\Sigma$ indexing a Jacobson–Morosov parabolic $W_0(N)$ may be determined as follows. The fact that the eigenvalues of $Y$ are integers implies that $Y$ is a grading element. (The Jacobson–Morosov parabolic $W_0(N)$ is even if and only if $\alpha_i(Y)$ is even for all simple roots.) We may choose $\mathfrak{h} \subset \mathfrak{b} \subset W_0(N)$ so that $\alpha_i(Y) \geq 0$ for all simple roots $\alpha_i \in \Delta^+_s$. (Alternatively, ‘conjugate’ $Y$ so that it lies in a chosen positive Weyl chamber.) Then

$$W_0(N)_{\mathfrak{g}} = \mathfrak{p}_{\Sigma}, \quad \text{where} \quad \Sigma = \{\alpha_i : \alpha_i(Y) = 0\}.$$

In fact, given the normalization $\alpha_i(Y) \geq 0$, we have

$$\alpha_i(Y) \in \{0, 1, 2\},$$

and the $\text{Ad}(G_{\mathbb{C}})$–orbits of nilpotent $N \in \mathfrak{g}_{\mathbb{C}}$ are indexed by the characteristic vectors $(\alpha_1(Y), \ldots, \alpha_r(Y))$, cf. \([8, 10, 18]\).

III. Extremal degenerations of polarized Hodge structures

III.A. Precise definition of extremal degenerations.

Definition III.1. Let $(V, Q, F^\bullet)$ be a polarized Hodge structure. The following data constitute a degeneration of the polarized Hodge structure $(V, Q, F^\bullet)$:

(i) A Mumford–Tate domain $D$ such that $F^\bullet = F^\bullet_x$ for some $x \in D$.

(ii) A period mapping

$$\Phi : \Delta^* \to \Gamma_T \setminus D$$

\(^{(10)}\)For example, if $n > 2$, then the parabolic $P \subset \text{SL}_n \mathbb{C}$ stabilizing a line in $\mathbb{C}^n$ is not a Jacobson–Morosov parabolic, cf. \([8]\).
such that for the lift $\tilde{\Phi} : \mathbb{H} \to D$ we have

$$\tilde{\Phi}(z) = x$$

for some $z \in \mathbb{H}$.

For $t = e^{2\pi i z} \in \Delta^*$, we think of $\Phi(t) = F_t^\bullet \in \Gamma_T \setminus D$ as defining a one–parameter family of $\Gamma_T$–equivalence classes of polarized Hodge structures whose Mumford–Tate groups are contained in $G \subset \text{Aut}(V,Q)$ and which tend to a “singular point” as $t \to 0$. Because of the constraints on the Mumford–Tate groups, this is a more refined notion than just a family of equivalence classes of polarized Hodge structures over the punctured disc.

In this paper two types of limits associated to (III.2) have been discussed. One is the equivalence class of limiting mixed Hodge structures $(V, W_s(N), F_{\text{lim}}^\bullet)$ viewed as the image of the origin $\Phi_t(0)$ under the extended period map

$$\Phi_t : \Delta \to \Gamma_T \setminus (D \cup B(N))$$

(Section II.E). The other is the image

$$F_\infty^\bullet \in \partial D$$

of the reduced limit period mapping (Section II.G)

$$\Phi_\infty : B(N) \to \partial D$$

applied to $\Phi_t(0) \in B(N)$. As noted above, $\Phi_\infty$ is invariant under $T$ so that $F_\infty^\bullet \in \partial D$ is well–defined.

The orbits $O_x = G_\mathbb{R} \cdot x$ of the action of $G_\mathbb{R}$ on the generalized flag variety form a partially ordered set by the relation “contained in the closure of”

$$O' \prec O \text{ if } O' \subset \overline{O}.$$  

Aside from the open orbits, at one extreme are the real codimension–one orbits; at the other extreme is the unique closed orbit $O_{\text{cl}}$. In first approximation, an extremal degeneration of a polarized Hodge structure is a degeneration whose reduced limit period $F_\infty^\bullet$ lies in one of these two extremes. However, since there may be no degeneration with $F_\infty^\bullet$ in the closed orbit, a refinement of this notion is required.
Definition III.3. We shall say that a $G_{\mathbb{R}}$-orbit $\mathcal{O} \subset \partial D$ is maximal relative to the Mumford–Tate structure on the generalized flag domain $D$ if it is polarizable relative to $D$ and if there exists no other orbit $\mathcal{O}' \nsubseteq \mathcal{O}$ that is polarizable relative to $D$.

Definition III.4. A degeneration of a polarized Hodge structure is extremal if the reduced limit period lies in either a codimension-one orbit (in which case the degeneration is minimal), or in an orbit that is maximal relative to the Mumford–Tate domain structure on $D$ (in which case the degeneration is maximal).

Thus extremal degenerations of a polarized Hodge structure represented by a point $x \in D$ are the least and most degenerate the reduced limit period map can realize for a period map (III.2) with $\Phi(z) = x$ for some $z \in \mathbb{H}$. There are a few subtleties in the concept.

(a) We cannot talk of the degenerations of a generic $x \in D$. As a trivial example, the infinitesimal period relation $I \subset TD$ may be zero. Even if we make the reasonable assumption that $I$ is bracket–generating, we know of no result that ensures the existence of a period mapping (III.2).

(b) Given (III.2), we may replace it by the corresponding nilpotent orbit without changing the limit $F_{\infty}^*$. If (III.2) arises from a family of algebraic varieties, the corresponding nilpotent orbit usually does not; however, this doesn’t matter in the sense that the limiting mixed Hodge structure, constructed algebro–geometrically in [22], will be the same as that constructed analytically from the nilpotent orbit. Thus we may speak unambiguously of the degeneration of the Hodge structure on $H^n(X_t, \mathbb{Q})_{\text{prim}}$ for $X_t = \pi^{-1}(t)$, $t \neq 0$, in a family $X \xrightarrow{\pi} \Delta$.

The study of extremal degenerations requires that we understand the $G_{\mathbb{R}}$-orbit structure of $\tilde{D} = G_{\mathbb{C}}/P$. The necessary material is presented in Sections III.B–III.E.

III.B. $G_{\mathbb{R}}$–orbit structure. The representation theory of Lie groups and Lie algebras plays an essential rôle in the analysis of the geometry of the compact dual $G_{\mathbb{C}}/P$ and its $G_{\mathbb{R}}$-orbits. In this section we outline that structure.

Fix a generalized flag variety $\tilde{D} = G_{\mathbb{C}}/P$ and a $G_{\mathbb{R}}$-orbit $\mathcal{O}$. Given a point $x \in \mathcal{O}$ the Lie algebra $p_x$ of the stabilizer $P_x = \text{Stab}_{G_{\mathbb{C}}}(x)$ contains a Cartan subalgebra $h_x$ that is stable under conjugation [11, Corollary 2.1.3]. In particular, $h_x = h_x(\mathbb{R}) \otimes \mathbb{C}$ is defined over $\mathbb{R}$.
Set $p = p_x$ and $h = h_x$. Choose a Borel subalgebra $h \subset b \subset p$, and let $L \in h$ be the grading element (II.9) associated with this triple. The fact that $h$ is closed under conjugation implies that $\mathfrak{h}(\mathbb{R}) = h \cap \mathfrak{g}_\mathbb{R}$ is Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$. It also implies that $\bar{\alpha}$ is a root whenever $\alpha$ is a root. It follows that $\bar{L}$ is a grading element.\(^{(11)}\)

Additionally, the $h$–roots of $\mathfrak{g}$ decompose into three types: we say $\alpha$ is real if $\bar{\alpha} = \alpha$; we say $\alpha$ is imaginary if $\bar{\alpha} = -\alpha$; and we say $\alpha$ is complex otherwise.

As elements of the Cartan subalgebra $L$ and $\overline{L}$ define a bigraded eigenspace decomposition

$$\mathfrak{g} = \bigoplus \mathfrak{g}^{p,q}$$

given by

$$\mathfrak{g}^{p,q} = \{ \xi \in \mathfrak{g} : [L, \xi] = p\xi, [\overline{L}, \xi] = q\xi \} .$$

This is the bigrading of [16, Lemma 3.2]. If the infinitesimal period relation is bracket–generating, then the filtration corresponding to $x$ is

$$F_{\mathfrak{g},x}^p = \bigoplus_{q \leq -p} \mathfrak{g}^{q,*}. $$

From this point on:

(III.7) \textit{We assume that the infinitesimal period relation is bracket–generating.}\(^{(12)}\)

In the event that $F_{\mathfrak{g},x}^{*,*} = F_{\mathfrak{g},\infty}^{*,*}$ lies in the image of a reduced limit period mapping, the bigrading (III.5) is related to the Deligne splitting (II.37) by

$$I_{\mathfrak{g}}^{p,q} = \mathfrak{g}^{q,p},$$

cf. [13, 16]. If $\mathcal{O}$ is polarized by $N$ relative to $D$ (Definition II.31), then (III.8) implies

(III.9) \hspace{1cm} N \in I_{\mathfrak{g}}^{-1,-1} = \mathfrak{g}^{-1,-1} .

\(^{(11)}\)We may assume that $h \subset \mathfrak{k}$ if and only if $x$ lies in an open $G_\mathbb{R}$–orbit. When the orbit is not open, (II.10) will fail.

\(^{(12)}\)In the event that the IPR is not bracket–generating one may either modify the definition of $L$ so that (III.6) holds, as in [16], or reduce to the case that the IPR is bracket–generating as in [19, Section 3.3].
Note that $\mathfrak{g}^{p,q}$ is a direct sum of root spaces (and $\mathfrak{h}$ if $p, q = 0$). In particular,

$$
\mathfrak{g}^{p,q} = \bigoplus_{\alpha(L) = p, \bar{\alpha}(L) = q} \mathfrak{g}^{\alpha}, \quad \text{if } (p, q) \neq (0, 0),
$$

(III.10)

$$
\mathfrak{g}^{0,0} = \mathfrak{h} \oplus \bigoplus_{\alpha(L) = 0, \bar{\alpha}(L) = 0} \mathfrak{g}^{\alpha}.
$$

Observe that

$$
\mathfrak{g}^{p,q} = \mathfrak{g}^{q,p};
$$

(III.11)

that is, \textit{complex conjugation corresponds to reflection about the line } p = q.

From (III.11) we see that $\mathfrak{g}^{p,q} \oplus \mathfrak{g}^{q,p}$ is defined over $\mathbb{R}$. Let $(\mathfrak{g}^{p,q} \oplus \mathfrak{g}^{q,p})_{\mathbb{R}} = (\mathfrak{g}^{p,q} \oplus \mathfrak{g}^{q,p}) \cap \mathfrak{g}_{\mathbb{R}}$ denote the real form. Then the tangent, CR–tangent and normal spaces are given by

$$
T_x \mathcal{O} = \bigoplus_{p > 0, q > 0} (\mathfrak{g}^{p,q} \oplus \mathfrak{g}^{q,p})_{\mathbb{R}}
$$

(III.12)

$$
T_x^{\text{CR}} \mathcal{O} = \bigoplus_{p > 0, q \geq 0} (\mathfrak{g}^{p,-q} \oplus \mathfrak{g}^{-q,p})_{\mathbb{R}}
$$

$$
N_x \mathcal{O} = i \bigoplus_{p, q > 0} (\mathfrak{g}^{p,q} \oplus \mathfrak{g}^{q,p})_{\mathbb{R}}.
$$

In the case that $F_{\mathbb{R}}^{\bullet,x} = F_{\mathbb{R}}^{\bullet,\infty}$, equations (III.8) and (III.12) yield (I.6). Note also that Lie algebra of the stabilizer $R \subset G_{\mathbb{R}}$ is

$$
r = (p \oplus \bar{p})_{\mathbb{R}} = \bigoplus_{p, q \geq 0} (\mathfrak{g}^{-p,-q} \oplus \mathfrak{g}^{-q,-p})_{\mathbb{R}}.
$$

It is convenient to visualize this structure in the $(p, q)$–plane: see Figure III.1.

**III.C. Codimension–one orbits.** Note that, if $\mathcal{O}$ is of codimension–one, then (III.10) and (III.12) imply that there exists a real root $\alpha$ such that

$$
N_x \mathcal{O} = i \mathfrak{f}^p = i \mathfrak{g}^{\alpha}_{\mathbb{R}}, \quad \text{with } p = \alpha(L).
$$

The assumption (III.7) forces $p = 1$ [16, Corollary 4.4].
**Figure III.1.** Visualization of tangent and normal spaces

Suppose that $\mathcal{O} \subset \partial D$. Then for a suitably scaled root vector $N^+ \in \mathfrak{g}_R^0$ (which is necessarily nilpotent) we have

$$e^{iN^+} \cdot F_x^\bullet \in D.$$  

We may complete $N^+$ to an $\mathfrak{sl}_2$–triple (II.25) with $N \in \mathfrak{g}_R^{-\alpha}$. Note that the $\mathfrak{sl}_2 = \text{span}\{N, Y, N^+\}$ gives an SL$_2(\mathbb{C})$–homogeneous embedding of $\mathbb{P}^1 = \mathbb{CP}^1$ into $\hat{D}$ with $H = \mathbb{P}^1 \cap D$ and $\mathbb{RP}^1 = \mathbb{P}^1 \cap \mathcal{O}$. In particular,

$$\lim_{z \to \infty} e^{zN} \cdot \left( e^{iN^+} \cdot F_x^\bullet \right) \in \mathcal{O}.$$  

It now follows from Definition II.31 that we have recovered [16, Proposition 5.16]:

*Every codimension–one orbit $\mathcal{O} \subset \partial D$ is polarizable relative to $D$ by a root vector $N \in \mathfrak{g}_R^{-\alpha}$.)*

Moreover, in this case

$$L + \overline{L} = Y,$$

and the Deligne splitting $\mathfrak{g}_C = \bigoplus I_p^q$ is *explicitly* given by (III.8) as an eigenspace decomposition

$$I_p^q = \{ \xi \in \mathfrak{g}_C : [L, \xi] = q\xi, \ [Y, \xi] = (p + q)\xi \}.$$  

Moreover, in the event that the Deligne splitting on $\mathfrak{g}_C$ is induced from one on $V_C$ as in (II.38), we have

$$I_p^q = \{ v \in V_C : L(v) = qv, \ Y(v) = (p + q)v \}.$$
These decompositions are in practice straightforward to compute, cf. [17] or Example IV.6.

Remark III.14. Each $G_R$–orbit in $\mathring{D}$ contains a distinguished set of “Matsuki points” (Section III.D.1). The open orbit $D$ is related to the codimension–one orbits $O \subset \partial D$ by an application of a Cayley transform to a Matsuki point: Given a Matsuki point $x_0 \in D$ one may apply a Cayley transform $c_\alpha \in G_C$ to obtain a Matsuki point $x = c_\alpha(x_0)$ in the codimension one orbit $O$, [13, 16, 17].

III.D. Orbit dimensions. Fix a maximal, connected compact Lie subgroup $K_R \subset G_R$. Let $\mathfrak{t}_R \subset \mathfrak{g}_R$ denote the Lie algebra of $K_R$, and let

$$\mathfrak{g}_R = \mathfrak{t}_R \oplus \mathfrak{t}_R^\perp.$$ 

denote the Cartan decomposition. The complexification will be written as

$$\mathfrak{g}_C = \mathfrak{t}_C \oplus \mathfrak{t}_C^\perp.$$ 

Let $\theta$ denote the Cartan involution

(III.15) \hspace{1cm} \theta|_{\mathfrak{t}_C} = 1 \ \text{and} \ \theta|_{\mathfrak{t}_C^\perp} = -1.

The dimensions of the $G_R$ and $K_R$–orbits through $x$ are given by certain $\mathfrak{h}$–root “counts,” cf. (III.18) and (III.19) below. A comparison of these root counts yields a characterization (Lemma III.20) of the unique closed $G_R$–orbit in $\mathring{D}$.

III.D.1. Matsuki points. We say that a point $x \in \mathring{D} = G/P$ is a Matsuki point if the Lie algebra $\mathfrak{p}_x$ of the stabilizer $P_x = \text{Stab}_{G_C}(x)$ contains a Cartan subalgebra $\mathfrak{h}_x$ that is stable under conjugation and the Cartan involution (III.15). This implies that the $K_R$–orbit is equal to the intersection of the $K_C$–orbit with the $G_R$–orbit: $K_R \cdot x = (G_R \cdot x) \cap (K_C \cdot x)$, cf. [11, Chapter 8]. That is, the $G_R$–orbit and the $K_C$–orbit are Matsuki dual. Every $G_R$–orbit (resp. $K_C$–orbit) contains a Matsuki point.

The fact that $\mathfrak{h}$ is closed under the Cartan involution implies that $\theta \alpha$ is a root whenever $\alpha$ is a root. Moreover,

(III.16) \hspace{1cm} -\alpha = \theta \bar{\alpha} = \overline{\theta \alpha},
and the complex roots appear in quartets
\[ \{ \alpha, \bar{\alpha}, \theta \alpha, \theta \bar{\alpha} = \overline{\theta \alpha} \} = \{ \pm \alpha, \pm \bar{\alpha} \} . \]

Note that (III.16) implies that the bigrading (III.5) satisfies

(III.17) \[ \theta(g^{p,q}) = g^{-q,-p} ; \]

that is,

*the Cartan involution corresponds to reflection about the line \( p = -q \).*

III.D.2. *The \( G_R \)-orbit.* Let \( \mathcal{O} = G_R \cdot x \) be the \( G_R \)-orbit of \( x \). Then

\[ \dim_R \mathcal{O} = \dim_R g_R - \dim_R g_R \cap p . \]

Note that

\[ g_R \cap p = (p \cap \bar{p})_R = h(R) \oplus \sum_{\alpha(L), L(\bar{\alpha}) \leq 0} (g^\alpha + g^{\bar{\alpha}})_R . \]

The sum above is over roots in (closed) lower–left quadrant

\[ \Delta(\leq 0, \leq 0) = \{ \alpha \in \Delta : \alpha(L), L(\bar{\alpha}) \leq 0 \} , \]

cf. Figure III.2. Set

\[ \Delta(\mathcal{O}) = \Delta \setminus \Delta(\leq 0, \leq 0) . \]

Then the (real) tangent space

\[ T_x \mathcal{O} \simeq g_R / p \simeq \sum_{\alpha \in \Delta(\mathcal{O})} (g^\alpha + g^{\bar{\alpha}})_R \]

as a vector space, and we conclude that

(III.18) \[ \dim_R \mathcal{O} = \frac{1}{2} \left| \{ \alpha \in \Delta(\mathcal{O}) \mid \alpha \neq \bar{\alpha} \} \right| + \left| \{ \alpha \in \Delta(\mathcal{O}) \mid \alpha = \bar{\alpha} \} \right| . \]
III.D.3. The $K_\mathbb{R}$-orbit. Likewise, we may compute the dimension of the $K_\mathbb{R}$-orbit

$$\mathcal{O}^{K_\mathbb{R}} = K_\mathbb{R} \cdot x$$

as follows. Decompose

$$\Delta(\mathcal{O}) = \Delta(\geq 0, \geq 0) \times \bigcup \Delta(-, +) \cup \Delta(+, -)$$

into three disjoint sets by

$$
\Delta(\geq 0, \geq 0) = \{ \alpha \in \Delta(\mathcal{O}) : -\alpha \in \Delta(\leq 0, \leq 0) \} = -\Delta(\geq 0, \geq 0) \setminus \{(0, 0)\},
\Delta(+, -) = \{ \alpha \in \Delta : \alpha(L) > 0, L(\bar{\alpha}) < 0 \},
\Delta(-, +) = \{ \alpha \in \Delta : \alpha(L) < 0, L(\bar{\alpha}) > 0 \}.
$$

We may visualize $\Delta(+, -)$ and $\Delta(-, +)$ as open quadrants in the $(p, q)$-plane, and $\Delta(\geq 0, \geq 0)$ as the closed upper–right quadrant minus the origin, cf. Figure III.2.

We have an identification $T_o\mathcal{O}^{K_\mathbb{R}} \simeq \mathfrak{t}_\mathbb{R}/\mathfrak{p}$, as vector spaces. So the dimension of $\mathcal{O}^{K_\mathbb{R}}$ is equal to the dimension of $\mathfrak{t}_\mathbb{R}/\mathfrak{p}$. Recall that $\mathfrak{t}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$ is the fix point locus of the Cartan involution $\theta : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$. In particular, $\mathfrak{t}_\mathbb{C} = \{ x + \theta x : x \in \mathfrak{g}_\mathbb{C} \}$. Then, making use of (III.11) and (III.17), we have

$$
\dim \mathcal{O}^{K_\mathbb{R}} = |\{ \alpha \in \Delta(\mathcal{O}) : \alpha = \bar{\alpha} \}| + |\{ \alpha \in \Delta(-, +) : \alpha = \theta \alpha \}|
+ \frac{1}{2} |\{ \alpha \in \Delta(-, +) : \alpha \neq \theta \alpha \}|
+ \frac{1}{2} |\{ \alpha \in \Delta(\leq 0, \leq 0) : \alpha \neq \bar{\alpha} \}|
$$

(III.19)
III.E. **Characterization of closed orbits.** The flag manifold \( \tilde{D} = G/P \) contains a unique closed \( G_{\mathbb{R}} \)-orbit [23, Theorem 3.3], which is contained in the closure of every other \( G_{\mathbb{R}} \)-orbit [23, Corollary 3.4]. It follows that the closed \( G_{\mathbb{R}} \)-orbit is the unique orbit for which \( O = O^{K_{\mathbb{R}}} \). Therefore \( O \) is closed if and only if the two dimensions (III.18) and (III.19) are equal. Whence we obtain the following characterization of the closed \( G_{\mathbb{R}} \)-orbit.

**Lemma III.20.** Let \( x \in \tilde{D} \) be a Matsuki point, and let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \) contained in the Lie algebra \( \mathfrak{p} \) of the stabilizer \( P = \text{Stab}_G(x) \) that is stable under complex conjugation and the Cartan involution. Then the \( G_{\mathbb{R}} \)-orbit \( O \) through \( x \) is closed (equivalently, \( O = O^{K_{\mathbb{R}}} \)) if and only if the \( \mathfrak{h} \)-roots satisfy

\[
(\text{III.21}) \quad \theta \alpha = \alpha \text{ for all } \alpha \in \Delta(-,+). 
\]

That is, by (III.16), all the roots of \( \Delta(-,+) \) are imaginary and compact.

Keeping in mind that \( \alpha \mapsto \bar{\alpha} \) is a bijection between \( \Delta(+,-) \) and \( \Delta(-,+), \) we see that (III.21) holds if and only if

\[
\theta \alpha = \alpha \text{ for all } \alpha \in \Delta(+,-). 
\]

Whence the lemma may be visualized as in Figure III.3.a, where the potentially nonzero \( \mathfrak{g}^{p,q} \) are indicated by a node at the point \((p,q)\). Equivalently, we will say that

**Figure III.3.** Visualization of (polarized) closed \( G_{\mathbb{R}} \)-orbit

![Figure III.3](image-url)

the bigraded \((L,\overline{L})\)-eigenspace decomposition (III.5) must be of the form depicted in Figure III.3.a.
Remark. We remark that Figure III.3.a is necessary, but not sufficient, for an orbit to be closed. That is, there are examples in which the bigrading takes this form, but (III.21) fails.

III.F. Totally real orbits. A $G_{\mathbb{R}}$–orbit $O \subset \tilde{D}$ is totally real if the Cauchy–Riemannian tangent bundle $T^{CR}O$ is trivial. Visually, the orbit is totally real if there are no $\ast$’s in Figure III.1. By (III.12) and Lemma III.20 any totally real orbit is necessarily the unique closed orbit. Moreover, recalling that

$$p = \bigoplus_{\rho \geq 0} g^{-\rho \cdot} = g^{<0, \cdot},$$

the first half of Theorem I.10 is now evident:

**Theorem III.22.** The following are equivalent:

(i) The orbit $O_{\text{cl}}$ is totally real.

(ii) The real dimension of the closed orbit is the complex dimension of the compact dual: $\dim_{\mathbb{R}} O_{\text{cl}} = \dim_{\mathbb{C}} \tilde{D}$.

(iii) The stabilizer $P$ is $\mathbb{R}$–split and $O_{\text{cl}} = G_{\mathbb{R}}/P_{\mathbb{R}}$.

IV. Minimal degenerations of polarized Hodge structures

Let $D$ be a Mumford–Tate domain. In Section III.C we saw that every codimension—one $G_{\mathbb{R}}$–orbit $O \subset \partial D$ is polarizable relative to $D$, and gave explicit descriptions of Deligne splittings as eigenspace decompositions. In Section IV.A we specialize to the case that $D$ is a period domain and give a refined description. In Example IV.6 we give an example illustrating the (in general) more complicated structure in the Mumford–Tate domain case.

IV.A. Minimal degenerations in period domains.

Definition. A type I basic boundary component for weight $n$ consists of limiting mixed Hodge structures that are of the form

$$H^{n-1-2k}(k+1) \xrightarrow{N} H^{n-1-2k}(k) \xrightarrow{H^n}$$
Figure IV.1. Limiting mixed Hodge structures in basic boundary component

for some \( k \) with \( 0 \leq 2k \leq n - 1 \). A type II basic boundary component for weight \( n = 2m \) consists of limiting mixed Hodge structures that are of the form

\[
H^0(m + 1) \overset{N}{\rightarrow} H^0(m) \overset{N}{\rightarrow} H^0(m - 1) \rightarrow H^n.
\]

Any \( G_\mathbb{R} \)-orbit \( \mathcal{O} \subset \partial D \) containing the image of a basic boundary component \( B(N) \) under the reduced limit period mapping is a basic boundary orbit.

The Deligne splittings of the limiting mixed Hodge structures in a basic boundary component are illustrated in Figure IV.1.

**Theorem IV.1.** Given a period domain \( D \subset \hat{D} \), the codimension–one boundary orbits are basic. Additionally, the nonzero off–diagonal \( I^{p,q} \) all have dimension one. Moreover, the type II basic boundary components occur only for even weight Hodge structures with \( h^{m,m} \) odd.

**Proof.** Any codimension–one orbit \( \mathcal{O} \subset \text{bd}(D) \) contains the image under the reduced limit period mapping of a boundary component \( B(N) \); moreover we may take \( N \) to be a root vector (Section III.C). Therefore, for an appropriate choice of basis \( \{e_i\}_{i=1}^d \) of \( V \) we may take \( N \) to be of one of the following normal forms. Let \( \{e^i\}_{i=1}^d \) be the dual basis of \( V^* \), and let \( \{e^i_j = e_j \otimes e^i\}_{i,j=1}^d \) be the corresponding basis of \( \text{End}(V) \).

Suppose the weight \( n \) is odd, so that \( d = 2c \). The nilpotent \( N \) is of one of the following forms

\[
(IV.2) \quad e^i_j - e^{c+j}_c, \quad e^{c+i}_c + e^{c+j}_c, \quad e^i_{c+j} + e^{j}_{c+i}, \quad e^i_{c+i}, \quad e^{c+i}_c, \quad \text{with } i, j \leq c.
\]
If the weight $n$ is even, and the dimension $d = 2c$ of $V$ is even, then the nilpotent $N$ is of one of the following forms

\[(IV.3)\quad e^j_i - e^{c+j}_{c+i}, \quad e^{c+i}_i - e^{c+j}_{c+i}, \quad e^i_{c+j} - e^j_{c+i}, \quad \text{with} \quad i, j \leq c.\]

In the event that the dimension $d = 2c + 1$ is odd, then the nilpotent $N$ is either of one of the three forms in (IV.3), or of one of the following forms

\[(IV.4)\quad e^d_i - e^{c+i}_d, \quad e^d_{c+i} - e^i_d, \quad \text{with} \quad i \leq c.\]

It is clear from these explicit expressions that:

- If $N$ is of one of the forms in (IV.2) or (IV.3), then $N^2 = 0$ and $N$ has rank at most two, so that the boundary component is type I basic.
- If $N$ is of one of the forms in (IV.4), then $N^3 = 0$ while $N^2 \neq 0$, and $N$ has rank two, so that the boundary component is type II basic. Moreover, in this case the weight $n = 2m$ is even, and $d = 2c + 1 = \sum h^{p,q}$ forces $h^{m,m}$ to be odd.

\[\square\]

**IV.B. Minimal degenerations in Mumford–Tate domains.** In the case of Mumford–Tate domains the $I^{p,q}$ corresponding to a minimal degeneration may be more complicated than those arising in the case of period domains (Figure IV.1). Illustrations of this are given in Examples II.39 (see, in particular, the rows 2 and 3 of Figure II.1) and IV.6. What we can say in general is

**Proposition IV.5.** Fix a flag domain $G_R/R$ admitting the structure of a Mumford–Tate domain $D$. Then there exists a Hodge representation $(V, Q)$ of $G$ that realizes $G_R$ as Mumford–Tate domain $D$ and with the property that each codimension–one $G_R$–orbit $O \subset \partial D \subset \breve{D}$ is polarized by a limiting mixed Hodge structure $(V, W_\bullet(N), F^\bullet)$ such that

\[N^3 = 0 \quad \text{as an element of} \quad g_R \subset \text{End}(V_R, Q).\]

In fact, if $G_C \neq G_2$ is simple, then we may take $V$ to be the adjoint representation $g$; if $G_C = G_2(C)$, then we may take $V$ to be the standard representation $V_R = \mathbb{R}^7$.

The proof follows Example IV.6.
Example IV.6 (An $F_4$ Mumford–Tate domain). Consider the case that the Mumford–Tate group is the exceptional simple Lie group of rank four. The smallest representation $V_\mathbb{C}$ of $G_\mathbb{C} = F_4(\mathbb{C})$ is of dimension 26. There exist real forms $V_\mathbb{R}$ and $G_{\mathbb{R}} = F_4^\mathbb{R}$, the latter of real rank four and with maximal compact subalgebra $\mathfrak{t}_\mathbb{R} = \mathfrak{sp}(3) \oplus \mathfrak{su}(2) \subset \mathfrak{g}_\mathbb{R}$, with the property that $V_\mathbb{R}$ admits the structure of a $G_{\mathbb{R}}$–Hodge representation with Hodge numbers

$$h = (1^4, 2^9, 1^4).$$

In this case the compact dual $G_\mathbb{C}/B$ is the flag variety and has dimension 24. The Deligne splittings associated with the four codimension one boundary orbits are depicted in Figure IV.2; these are computed by (III.13). Each circled node has dimension two, the remainder have dimension one. Note that in the first row of Figure IV.2 we have $N^2 = 0$; in the second row $N^2 \neq 0$ and $N^3 = 0$.

![Figure IV.2. Deligne splittings associated with codimension one boundary orbits for $D \subset F_4(\mathbb{C})/B$.](image)

Proof of Proposition IV.5. Given a polarized Hodge structure $(V, Q, F^\bullet) \in D$, let $(\mathfrak{g}, Q_\mathfrak{g}, F^\bullet_\mathfrak{g})$ denote the induced polarized Hodge structure on $\mathfrak{g}_\mathbb{R}$ (Section II.D.2). Any codimension–one boundary $G_{\mathbb{R}}$–orbit $\mathcal{O} \subset \partial D$ is polarized by a root vector $N \in \mathfrak{g}_{\mathbb{R}}^{-\alpha}$. 
It is well-known that root vectors satisfy $N^4 = 0$ as elements of $\text{End}(g)$. (Equivalently, every $\alpha$–string has length at most four.)

- If $\alpha$ is not a root of the exceptional Lie algebra $g_2$, then $N^3 = 0$ on $g_C$ (every $\alpha$–string has length at most three).
- If $\alpha$ is a root of $g_2$, then it is possible that $N^3 \neq 0$ on $g_C$; this is illustrated in Figure II.1 (row 3, column 2). However, as is also illustrated in Figure II.1 (rows 2 and 3, column 1), any root vector will satisfy $N^3 = 0$ as an element of $g_C \subset \text{End}(\mathbb{C}^7)$ acting on the standard representation.

\[\square\]

V. Maximal degenerations of polarized Hodge structures

V.A. Polarizable closed orbits. In the case that the closed orbit is polarizable, Lemma III.20 and Figure III.3.a may be refined. We now recall Theorem I.11 and prove the slightly stronger

**Theorem V.1.** Suppose that the limiting mixed Hodge structure $(g, W_\bullet(N), F^\bullet)$ is sent to the closed orbit under the reduced limit period mapping. Then Deligne splitting $g_C = \bigoplus I_p^q g$ satisfies:

\[(V.2a)\quad I_{g,p}^{p-q} = 0 \quad \text{for all} \quad p \neq q > 0,\]
\[(V.2b)\quad I_{g,p}^{p-p} = 0 \quad \text{for all odd} \quad p \geq 3,\]
\[(V.2c)\quad I_{g,p}^{p,q} = 0 \quad \text{for all} \quad p + q \neq 0 \quad \text{with} \quad |p - q| > 2.\]

Moreover, any $N$–string in $\bigoplus_{q-p=2} g^{p,q}$ has length $\equiv 3 \mod 4$. In particular, if $G^{W_\bullet(N)}_{k,\text{prim}} \neq 0$, then $k$ is even.

The Deligne splitting is pictured in Figure III.3.b. The circled nodes indicate those $I_{g,p}^{p,q}$ with $p - q = \pm 2$ that may (but need not) have a non–trivial primitive component. (There are no constraints on the primitive components with $p - p \neq \pm 2$, aside from the obvious $I_{g,p}^{p-p} = I_{g,p}^{p,-p}_{\text{prim}}$ for all $|p| \geq 2$.)

**Proof.** Lemma III.20 implies (V.2a). Assertion (V.2c) follows from the the property $N : g^{p,q} \to g^{p-1,q-1}$ and the fact that the $N$–strings are uninterrupted. Given this, it
follows that

\[ \Gamma = \bigoplus_{p \geq 2} (\mathfrak{g}^{-p,p} \oplus \mathfrak{g}^{p,-p}) \subset \text{Gr}^{W\bullet}_{0,\text{prim}}(\mathfrak{g}) . \]

Recall (Section II.F) that \( \text{Gr}^{W\bullet}_{0,\text{prim}}(\mathfrak{g}) \) carries a weight zero Hodge structure polarized by \( Q_\mathfrak{g} \). So given a nonzero \( v \in \mathfrak{g}^{-p,p} \subset \Gamma \), we have \( (-1)^p Q_\mathfrak{g}(v, \bar{v}) > 0 \). By (III.10), \( \Gamma \) is a direct sum of root spaces. Moreover, Lemma III.20 asserts that the roots are imaginary and compact. Recall that \( -Q_\mathfrak{g} \) is the Killing form. Therefore, given a nonzero root vector \( v \in \mathfrak{g}^\alpha \subset \mathfrak{g}^{-p,p} \), the inequality \( (-1)^p Q_\mathfrak{g}(v, \bar{v}) > 0 \) implies that \( \alpha \) is compact if \( p \) is even and noncompact if \( p \) is odd. Therefore, \( p \) is necessarily even; this establishes (V.2b).

Any \( N \)–string in \( \oplus_{q-p=2} \mathfrak{g}^{p,q} \) is necessarily of the form

\[ u, Nu, N^2u, \ldots, N^{2k}u, \]

with \( 0 \neq u \in \mathfrak{g}^{k-1,k+1}, N^{k}u \in \mathfrak{g}^{-1,1} \) and \( N^{2k+1}u = 0 \). By the polarization hypothesis,

\[ 0 < -Q_\mathfrak{g}(u, N^{2k}\bar{u}) = (-1)^{k+1}Q_\mathfrak{g}(N^{k}u, N^{k}\bar{u}) . \]

Lemma III.20 implies \( 0 \neq v = N^{k}u \in \mathfrak{g}^{-1,1} \) satisfies \( Q_\mathfrak{g}(v, \bar{v}) > 0 \). Therefore \( k \) is odd, and the length of the \( N \)–string is \( 2k + 1 = 4\ell + 3 \equiv 3 \pmod{4} \).

**V.B. The Hodge–Tate case.** The polarized limiting mixed Hodge structure \((V, W_\bullet(N), F^\bullet)\) is **Hodge–Tate** if

\[ I^{p,q} = 0 \quad \text{for all } p \neq q . \]  

Given that the \( N \)–strings are uninterrupted and centered on the line \( p = -q \), we obtain the following corollary to Theorem I.11:

**Corollary V.4.** If a limiting mixed Hodge structure \((V, W_\bullet(N), F^\bullet)\) polarizes a totally real (and necessarily closed) \( G_\mathbb{R} \)–orbit, then the limiting mixed Hodge structure is of **Hodge–Tate** type.

We will now recall and prove

**Proposition I.9.** The polarized limiting mixed Hodge structure \((V, W_\bullet(N), F^\bullet)\) is **Hodge–Tate** if and only if the induced limiting mixed Hodge structure \((\mathfrak{g}, W_\bullet(N)_\mathfrak{g}, F^\bullet_\mathfrak{g})\) is **Hodge–Tate**.
Together Corollary V.4 and Proposition I.9 yield the second half of Theorem I.10. Proposition I.9 is a consequence of the following lemma.

**Lemma V.5.** Fix a real form $\mathfrak{g}_\mathbb{R}$ and let $\mathfrak{h} \subset \mathfrak{g}_\mathbb{C}$ be a Cartan subalgebra that is stable under complex conjugation. Let $L \in \mathfrak{h}$ be any grading element. Let $V$ be a representation of $\mathfrak{g}_\mathbb{C}$, and let $\Lambda(V) \subset \Lambda_{wt}$ denote the weights of $V$. Suppose that $\lambda(L) = \lambda(\bar{L})$ for all $\lambda \in \Lambda(V)$. (Equivalently, $\lambda(L) = \bar{\lambda}(L)$.) Then $\mu(L) = \mu(\bar{L})$ for all weights $\mu \in \Lambda_{wt}$ of $\mathfrak{g}_\mathbb{C}$.

**Proof of Lemma V.5.** The weight lattice $\Lambda_{wt} \subset \mathfrak{h}^*$ is spanned over $\mathbb{Q}$ by the weights of $V$; that is, $\Lambda_{wt} = \text{span}_\mathbb{Q} \Lambda(V)$. □

**Proof of Proposition I.9.** The nontrivial $I^{p,q}$ are precisely those with $(p, q) = (\lambda(L), \lambda(\bar{L}))$ for some $\lambda \in \Lambda(V)$. □

**Proof of Proposition I.15.** Follows directly from Proposition I.9 and Lemma III.20. □

There are a number of conditions that a Mumford–Tate domain $D$ must satisfy in order to admit a Hodge–Tate degeneration. Some of the conditions are Hodge–theoretic in nature (Lemma V.7), while others are representation theoretic (Lemma V.10 and Remark V.12).\(^{(13)}\)

Given weight $n$, set

\[(V.6a) \quad m = \lfloor n/2 \rfloor ;\]

that is, $m$ is defined by

\[(V.6b) \quad n \in \{2m, 2m + 1\} .\]

**Lemma V.7.** Let $D$ be a Mumford–Tate domain for polarized Hodge structures of weight $n$ and with $\mathfrak{h} = (h^{n,0}, h^{n-1,1}, \ldots, h^{1,n-1}, h^{0,n})$. If a point of $D$ admits a Hodge–Tate degeneration, then the Hodge numbers satisfy

\[(V.8) \quad h^{n,0} \leq h^{n-1,1} \leq \cdots \leq h^{n-m,m} .\]

\(^{(13)}\)With respect to the latter we present only an illustrative sketch. The general story will be discussed in a later work.
The converse to Lemma V.7 holds when $D$ is a period domain (Theorem V.15), but fails for Mumford–Tate domains in general (Remark V.16).

**Proof.** If there is a Hodge–Tate degeneration of a point in $D$, then the limiting mixed Hodge structure looks like

\[
\begin{align*}
H^0(n) &\to H^0(n-1) \to H^0(n-2) \to \cdots \to H^0(2) \to H^0(1) \to H^0 \\
H^0_2(n-1) &\to H^0_2(n-2) \to \cdots \to H^0_2(2) \to H^0_2(1) \\
H^0_4(n-2) &\to \cdots \to H^0_4(2) \\
&\vdots
\end{align*}
\]  

(V.9)

where $H^0_{2k}$ is a Hodge–Tate structure of weight zero. The inequalities (V.8) are then simple consequences of this picture: the Deligne splitting must satisfy

\[
i^{n,n} \leq i^{n-1,n-1} \leq \cdots \leq i^{n-m,n-m}.
\]

Since $h^{p,q} = i^{p-n,q}$, this yields (V.8).

For example, when $n = 4$, the $I^{p,q}$ picture of the Hodge–Tate limiting mixed Hodge structure is

\[
\begin{array}{c}
\circ \\
\circ
\end{array}
\]

and $h^{4,0} = i^{4,4} \leq h^{3,1} = i^{3,3} \leq h^{2,2} = i^{2,2}$.

\[\square\]

A second condition that $D$ must satisfy to admit a Hodge–Tate degeneration is

**Lemma V.10.** If a Mumford–Tate domain $D \subset \tilde{D} = G_{\mathbb{C}}/P$ admits a Hodge–Tate degeneration, then it is necessarily the case that $p$ is an even Jacobson–Morosov parabolic (Remark II.40).

**Example V.11** (Even Jacobson–Morosov parabolics in $G_2$). The exceptional simple Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_2$ has three conjugacy classes of (proper) parabolic subalgebras. Each may be realized as Jacobson–Morosov parabolics, but only two are even; they are indexed by $\Sigma = \{\alpha_2\}$ (the simple root $\alpha_2$ is long) and $\Sigma = \{\alpha_1, \alpha_2\}$ (indexing the Borel), cf. [8, Section 8.4]. In these two cases there exist Mumford–Tate domains
$D \subset G_C/P$, homogeneous with respect to the split real form of $G_2$, admitting Hodge–Tate degenerations, cf. [16, Section 6.1.3].

**Proof.** Suppose that $(V, W_\bullet(N), F^\bullet)$ is Hodge–Tate limiting mixed Hodge structure for the period domain $D$. Then the induced limiting mixed Hodge structure $(\mathfrak{g}, W_\bullet(N)_g, F_g^\bullet)$ is also Hodge–Tate (Proposition I.9). Therefore the Lie algebra of the parabolic stabilizing $F^\bullet$ is

$$\mathfrak{p} = F_g^0 \overset{(II.32)}{=} \bigoplus_{p \geq 0} I_{g}^{p,\bullet} \overset{(V.3)}{=} \bigoplus_{p \geq 0} I_{g}^{p,p} \overset{(II.32)}{=} W_0(N^+)_g$$

where $N^+$ is the nilpositive element of an $\mathfrak{sl}_2$–triple (II.25) containing $N$. It follows that $\mathfrak{p}$ is a Jacobson–Morosov parabolic. Moreover, since the neutral element $Y$ acts on $I_{g}^{p,p}$ by the even scalar $2p$, it follows that $\mathfrak{p}$ is an even Jacobson–Morosov parabolic. □

**Remark V.12** (Hodge–Tate degenerations in flag varieties $\tilde{D} = G_C/B$). Lemma V.10 imposes constraints on the compact duals without reference to the real form $G_R$. Once we pick a Jacobson–Morosov parabolic $P \subset G_C$, there are constraints on the real forms $G_R$ that may arise as Mumford–Tate groups with Hodge representation admitting a Hodge–Tate degeneration. To illustrate this we consider the case that $G_C$ is one of the classical simple Lie groups, the principal nilpotents are nicely characterized by the decomposition of the standard representation under an $\mathfrak{sl}_2 \subset \mathfrak{g}_C$. To be precise, given any nilpotent $N \in \mathcal{N}$, let $\mathfrak{sl}_2 \subset \mathfrak{g}_C$ be the Lie subalgebra spanned by an $\mathfrak{sl}_2$–triple (II.25) containing $N$. Then:

(i) Let $V_C = \mathbb{C}^n$ be the standard representation of any one of $\mathfrak{sl}_n \subset \mathfrak{sl}_n \mathbb{C}$ with $n = 2m$, or $\mathfrak{so}_{2m+1} \mathbb{C}$ with $n = 2m + 1$. Then the nilpotent $N$ is principal if and only $V_C$ is irreducible as an $\mathfrak{sl}_2$–module.
(ii) If $G_C = \text{SO}_{2m} \mathbb{C}$, then $N$ is principal if and only if the standard representation $V_C = \mathbb{C}^{2m}$ decomposes as $\mathbb{C} \oplus \mathbb{C}^{2m-1}$ (a trivial subrepresentation plus an irreducible subrepresentation).

The challenge to the real form $G_{\mathbb{R}}$ is that:

An open $G_{\mathbb{R}}$–orbit $D \subset G_C/B$ can admit the structure of a Mumford–Tate domain with a Hodge–Tate degeneration only if $\mathfrak{g}_{\mathbb{R}} \cap \mathcal{N}_{\text{prin}}$ is nonempty.

From the classification [8, Section 9.3] of Ad($G_{\mathbb{R}}$)–orbits of nilpotent elements in $\mathfrak{g}_{\mathbb{R}}$ we deduce that $\mathfrak{g}_{\mathbb{R}} \cap \mathcal{N}_{\text{prin}}$ is nonempty only for the following real forms (in the classical case):

$$\mathfrak{su}(m,m), \mathfrak{su}(m,m+1); \mathfrak{sp}(n,\mathbb{R}); \mathfrak{so}(m,m), \mathfrak{so}(m+1,m), \mathfrak{so}(m+2,m).$$

The classification also determines the open $G_{\mathbb{R}}$–orbits $D \subset G_C/B$:

(a) Suppose that $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n,\mathbb{R}) = \text{End}(\mathbb{R}^{2n},Q)$. From the classification [8, Theorem 9.3.5] we may draw the following conclusions. The intersection $\mathfrak{g}_{\mathbb{R}} \cap \mathcal{N}_{\text{prin}}$ consists of two Ad($G_{\mathbb{R}}$)–orbits $\mathcal{N}_{\text{prin},\mathbb{R}}^{+}$ and $\mathcal{N}_{\text{prin},\mathbb{R}}^{-}$. Given $N \in \mathcal{N}_{\text{prin},\mathbb{R}}^{\pm}$, there exists $v \in \mathbb{R}^{2n}$ such that $\{v, Nv, \ldots, N^{2n-1}v\}$ is a basis of $\mathbb{R}^{2n}$ and

$$\pm Q(N^a v, N^b v) = (-1)^a \delta_{2m-1}^{a+b}.$$

So we see that, if $N \in \mathcal{N}_{\text{prin},\mathbb{R}}^{+}$ and

(V.13) $$F^p = \text{span}_{\mathbb{C}} \{N^a v : a \geq p\}$$

then $(\mathbb{C}^{2n},W_\bullet(N), F^\bullet)$ is a limiting mixed Hodge structure for the period domain parameterizing weight $2n-1$, $Q$–polarized Hodge structures with Hodge numbers $h = (1,1,\ldots,1)$. Note that

$I^{p-p} = \text{span} \{N^{2n-1-p}v\}.$

(b) In the case that $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(m+1,m) = \text{End}(\mathbb{R}^{2m+1},Q)$, the classification [8, Theorem 9.3.4] asserts that the intersection $\mathfrak{g}_{\mathbb{R}} \cap \mathcal{N}_{\text{prin}}$ consists of one Ad($G_{\mathbb{R}}$)–orbit $\mathcal{N}_{\text{prin},\mathbb{R}}$, and given an element $N$ in that orbit, $\mathbb{R}^{2m+1}$ admits a basis $\{v, Nv, \ldots, N^{2m}v\}$ such that

$$(-1)^m Q(N^a v, N^b v) = (-1)^a \delta_{2m}^{a+b}.$$
Taking again the definition (V.13), we see that \((C^2_n, W_\bullet(N), F^\bullet)\) is a limiting mixed Hodge structure for the period domain parametrizing weight 2m, \((-1)^m Q\)-polarized Hodge structures with Hodge numbers \(h = (1, \ldots, 1)\).

(c) In the case that \(g_\mathbb{R} = \text{so}(m, m) = \text{End}(\mathbb{R}^{2m}, Q)\), the classification [8, Theorem 9.3.4] asserts that the intersection \(g_\mathbb{R} \cap N_{\text{prin,} \mathbb{R}}^+\) consists of two Ad\((G_\mathbb{R})\)-orbits \(N_{\text{prin,} \mathbb{R}}^+\) and \(N_{\text{prin,} \mathbb{R}}^-\). Given \(N \in N_{\text{prin,} \mathbb{R}}^\pm\) there exists a \(Q\)-orthogonal decomposition \(\mathbb{R}^{2m} = \mathbb{R} \oplus \mathbb{R}^{2m-1}\) into \(\mathfrak{sl}_2\)-submodules admitting bases \(\{w\}\) and \(\{v, Nv, \ldots, N^{2m-2}v\}\) such that

\[\pm Q(w, w) = 1 \quad \text{and} \quad \pm Q(N^a v, N^b v) = (-1)^{m+a} \delta^{a+b}_{2m-2}.\]

Let

\[F^p = \text{span}\{N^a v : a \geq p\}, \quad p \geq m,\]

\[F^p = \text{span}\{w, N^a v : a \geq p\}, \quad p \leq m - 1.\]

(V.14)

If \(m\) is even and \(N \in N_{\text{prin,} \mathbb{R}}^+\), then \((C^{2m}, W_\bullet(N), F^\bullet)\) is a limiting mixed Hodge structure for the period domain parametrizing weight 2m - 2, \(Q\)-polarized Hodge structures with Hodge numbers \(h = (1, \ldots, 1, 2, 1 \ldots, 1)\). Note that

\[I_{p,p} = \left\{ \begin{array}{ll} \text{span}\{N^{2m-2-p}v\}, & p \neq m - 1, \\
\text{span}\{w, N^{m-1}v\}, & p = m - 1. \end{array} \right.\]

(d) In the case that \(g_\mathbb{R} = \text{so}(m+2, m) = \text{End}(\mathbb{R}^{2m}, Q)\), the classification [8, Theorem 9.3.4] asserts that the intersection \(g_\mathbb{R} \cap N_{\text{prin,} \mathbb{R}}\) consists of a single Ad\((G_\mathbb{R})\)-orbit \(N_{\text{prin,} \mathbb{R}}\). Given \(N \in N_{\text{prin,} \mathbb{R}}\) there exists a \(Q\)-orthogonal decomposition \(\mathbb{R}^{2m} = \mathbb{R} \oplus \mathbb{R}^{2m-1}\) into \(\mathfrak{sl}_2\)-submodules admitting bases \(\{w\}\) and \(\{v, Nv, \ldots, N^{2m-2}\}\) such that

\[Q(w, w) = 1 \quad \text{and} \quad Q(N^a v, N^b v) = (-1)^{m+a} \delta^{a+b}_{2m-2}.\]

Taking (V.14) as above, we see that if \(m\) is even, then \((C^{2m}, W_\bullet(N), F^\bullet)\) is a limiting mixed Hodge structure for the period domain parametrizing weight 2m - 2, \(Q\)-polarized Hodge structures with Hodge numbers \(h = (1, \ldots, 1, 2, 1 \ldots, 1)\).
V.C. **Hodge–Tate degenerations in period domains.** In the event that $D$ is a period domain, the converse to Lemma V.7 holds.

**Theorem V.15.** Let $D$ be the period domain for polarized Hodge structures of weight $n$ and with Hodge numbers $\mathbf{h} = (h^{n,0}, h^{n-1,1}, \ldots, h^{1,n-1}, h^{0,n})$. If (V.8) holds, then a point of $D$ admits a Hodge–Tate degeneration.

*Proof.* This is an existence question, and we will construct a Hodge–Tate limiting mixed Hodge structure $(V, W_\bullet(N), F^\bullet)$. The construction will be given as direct sum of “atomic” Hodge–Tate limiting mixed Hodge structures with dimensions

$$i_{k,d} = (i_{k,d}^{n,n}, \ldots, i_{k,d}^{0,0}) = \left(0, \ldots, 0, d, \ldots, d, 0, \ldots, 0\right);$$

the latter are illustrated in Figure V.1. To make this precise, given $0 \leq d \in \mathbb{Z}$ and $0 \leq k \leq m$, with $m$ as in (V.6), define

$$H^0_{k,d}(s) = \text{span}_\mathbb{R}\{e_1^s, \ldots, e_d^s\} \cong \mathbb{R}^d, \quad k \leq s \leq n-k.$$ 

Set

$$V_{k,d}(\mathbb{R}) = \bigoplus_{k \leq s \leq n-k} H^0_{k,d}(s),$$

and define a nilpotent $N_{k,d} \in \text{End}(V_{k,d}(\mathbb{R}))$ by

$$N_{k,d}(e_a^s) = \begin{cases} e_a^{s-1}, & s > k, \\ 0, & s = k. \end{cases}$$
Note that
\[ N_{k,d}(H_{k,d}^0(s)) = \begin{cases} 
H_{k,d}^0(s-1), & s > k, \\
0, & s = k.
\end{cases} \]

Define a \((-1)^n\)-symmetric \(Q_{k,d}\) on \(V_{k,d}\) by
\[ (-1)^{n-k-s} Q(e_a^s, e_b^t) = \delta_{ab} \delta_n^{s+t}. \]
Then \(N_{k,d} \in \text{End}(V_{k,d}, Q_{k,d})\) is nilpotent with weight filtration
\[ W_\ell(N_{k,d}) = \text{span}\{e_a^s : 1 \leq a \leq d, s \leq \ell\}. \]
Setting
\[ F^p_{k,d} = \text{span}\{e_a^s : 1 \leq a \leq d, s \geq p\} \]
defines an \(\mathbb{R}\)-split, polarized, Hodge–Tate limiting mixed Hodge structure \((V_{k,d}, W_\bullet(N_{k,d}), F_{k,d})\), with dimensions \(i_{k,d}\).

We now define \((V, W_\bullet(N), F^\bullet)\) as follows. Set
\[ d_k = \begin{cases} 
h_{n,0}, & k = 0, \\
h_{n-k,k} - h_{n-k+1,k-1}, & 1 \leq k \leq m,
\end{cases} \]
and
\[ V = \bigoplus_{0 \leq k \leq m} V_{k,d_k}, \]
\[ N = \bigoplus_{0 \leq k \leq m} N_{k,d_k}, \]
\[ F^p = \bigoplus_{0 \leq k \leq m} F^p_{k,d_k}. \]
Then \((V, W_\bullet(N), F^\bullet)\) is an \(\mathbb{R}\)-split, polarized, Hodge–Tate limiting mixed Hodge structure with dimensions
\[ i = \sum i_{k,d_k} = h. \]
Therefore \(e^z N \cdot F^\bullet \in D\) for all \(\text{Im}(z) > 0\).

**Remark V.16.** Theorem V.15 is false for for Mumford–Tate domains in general. That is, the inequalities (V.8) do not imply the existence of a Hodge–Tate degeneration. To see this suppose that \((V, Q)\) and \((V', Q')\) are two Hodge representations of \(G\) realizing \(G_{\mathbb{R}}/R\) as a Mumford–Tate domain (Section II.D.4); denote the two (isomorphic) realizations by \(D\) and \(\tilde{D}\).
(a) Any limiting mixed Hodge structure \((V, W_\bullet(N), F^\bullet)\) on \((V, Q)\) determines a limiting mixed Hodge structure \((\tilde{V}, \tilde{W}_\bullet(N), \tilde{F}^\bullet)\) on \((\tilde{V}, \tilde{Q})\). Moreover, these two limiting mixed Hodge structures induce the same limiting mixed Hodge structure \((g, W_\bullet(N)_g, F^\bullet_g)\).

(b) The reduced limit period mapping \(\Phi_\infty\) sends \((F^\bullet, N)\) to the closed orbit if and only if it sends \((\tilde{F}^\bullet, N)\) to the closed orbit.

(c) By Proposition I.9 the limiting mixed Hodge structure \((V, W_\bullet(N), F^\bullet)\) is Hodge–Tate if and only if \((\tilde{V}, \tilde{W}_\bullet(N), \tilde{F}^\bullet)\) is Hodge–Tate.

(d) Suppose that (V.8) fails for \((\tilde{V}, \tilde{Q})\). Then \(\tilde{D}\), as the Mumford–Tate domain for \((\tilde{V}, \tilde{Q})\), contains no point admitting a Hodge–Tate degeneration. Therefore, even if (V.8) holds for \((V, Q)\), the Mumford–Tate domain \(D\) can not have a point admitting a Hodge–Tate degeneration. So to disprove the theorem if suffices to exhibit a pair \((V, Q)\) and \((\tilde{V}, \tilde{Q})\) such that (V.8) holds for one but not the other. Here is a counter–example.

**Example V.17.** Fix a symmetric nondegenerate bilinear form \(Q\) on \(\mathbb{R}^5\) of signature \((1, 4)\). Set \(G_\mathbb{R} = \text{SO}(1, 4) = \text{Aut}(\mathbb{R}^5, Q)\). Then \(G_\mathbb{C} = \text{SO}(5, \mathbb{C})\). Let the compact dual \(\tilde{D} = G_\mathbb{C}/P = \text{Gr}^Q(2, \mathbb{C}^5)\) be the variety of \(Q\)-isotropic 2–planes in \(\mathbb{C}^5\), and let \(D\) be the period domain parameterizing \(Q\)-polarized Hodge structures on \(\mathbb{C}^5\) with Hodge numbers \(h = (2, 1, 2)\). Note that (V.8) fails. On the other hand the induced \(Q_0\)-polarized Hodge structure on \(g_\mathbb{R}\) has Hodge numbers \(h_0 = (1, 2, 4, 2, 1)\); in this case (V.8) holds.

**V.D. Non–Hodge–Tate degenerations in period domains.** We recall and prove Theorem I.12. Let \(D\) be a period domain parameterizing weight \(n\) Hodge structures. If there exists a limiting mixed Hodge structure \((V, W_\bullet(N), F^\bullet)\) that maps to the closed \(G_\mathbb{R}\)-orbit in \(\tilde{D}\), but is not of Hodge–Tate type, then \(n = 2m\) is even and:

(a) For \(k \neq 0\), \(\text{Gr}^{W_\bullet(N)}_{n+k, \text{prim}}\) is of Hodge–Tate type. (Thus \(k\) is even.)

(b) For \(k \neq 0\), \(\text{Gr}^{W_\bullet(N)}_{n+k, \text{prim}} \neq 0\) implies \(k \equiv 2 \pmod{4}\).

(c) \(\text{Gr}^{W_\bullet(N)}_{n, \text{prim}} \neq 0\), and the only nonzero \(I^p,q_{\text{prim}}\), with \(p + q = n\), are

\[I^{m+1, m-1}_{\text{prim}} \quad \text{and} \quad I^{m-1, m+1}_{\text{prim}}.\]
The proof of Theorem I.12 proceeds in four steps.

**Step 1: odd weight.** For odd weight \( n = 2m + 1 \), we have \( G_{\mathbb{R}} = \text{Sp}(2g, \mathbb{R}) \). This real form is \( \mathbb{R} \)-split; whence the closed orbit \( O_{\text{cl}} \) is totally real. It follows from Theorem I.10 that, if a degeneration to \( O_{\text{cl}} \) exists, then the limiting mixed Hodge structure must be Hodge–Tate. So from this point on we assume that

\[ n = 2m. \]

**Step 2: preliminaries for even weight.** By Theorem I.11 we are given the form of the induced limiting mixed Hodge structure \((g, W_\bullet(N)_g, F_\bullet)\). The issue to is extract from the inclusion

\[ g \subset \text{End}(V, Q) \]

the form of original limiting mixed Hodge structure \((V, W_\bullet(N), F^\bullet)\). So, the question is: *given the Deligne splitting for \((g, W_\bullet(N)_g, F_\bullet)\), what can we infer about the Deligne splitting for \((V, W_\bullet(N), F^\bullet)\)*? An especially convenient way to do this is to use the relationship between the weights of \( V_C \) and the roots of \( g_C = \text{End}(V_C, Q) \); this will amount to comparing eigenvalues between the two spaces.

Define \( L \in g_C \) by

\[ L|_{I_p^g,q} = q \mathbb{1}. \]

Then (III.8) implies the Deligne splitting is the \((L, L)\)-eigenspace decomposition of \( g_C \),

\[ (L, L)|_{I_p^g,q} = (q, p) \mathbb{1}. \]

Likewise, in analogy with the discussion of Sections II.C.2 and II.D.2, the Deligne splitting of \( V_C \) is an \((L, L)\)-eigenspace decomposition. To be precise, \((L, L)\) acts on \( I_p^{m,q+m} \) by the scalars \((q, p)\). It will be convenient to shift the bigrading by

\[ I_p^{g,q} = I_p^{m,q+m}, \quad \text{so that} \quad (L, L)|_{I_p^{g,q}} = (q, p) \mathbb{1}. \]

Note that, like \( \bigoplus I_p^{g,q} \), the splitting \( \bigoplus I_p^{g,q} \) is symmetric about the line \( p + q = 0 \).

Let \( h \) be a Cartan subalgebra containing the grading elements \( L \) and \( \overline{L} \). We may assume without loss of generality that \( h \) is closed under conjugation. Let \( \Lambda(V) \subset h^* \) denote the weights of the standard representation \( V_C \), and given \( \lambda \in \Lambda(V) \), let \( V^\lambda \subset \)
\( V_C \) denote the weight space. Then the \((L, \overline{L})\)-eigenvalues of \( V_C \) are \( \{ (L(\lambda), \overline{L}(\lambda)) : \lambda \in \Lambda(V) \} \); equivalently,

\[
(V.18a) \quad \tilde{I}^{p,q} = \bigoplus_{\overline{L}(\lambda) = p} V^\lambda.
\]

Likewise, the \((L, \overline{L})\)-eigenvalues of \( g_C \) are \( \{ (\alpha(L), \alpha(\overline{L})) : \alpha \in \Delta \cup \{0\} \} \); that is,

\[
(V.18b) \quad I^{p,q}_g = \bigoplus_{\alpha(L) = q} g^\alpha \left( \bigoplus \mathfrak{h} \text{ if } (p, q) = (0, 0) \right).
\]

The relationship between the roots of \( g_C \) and the weights \( \Lambda(V) \) of \( V_C \) is

\[
(V.18c) \quad \Delta \cup \{0\} = \{ \alpha = \lambda + \mu : \lambda \neq \mu \in \Lambda(V) \}.
\]

This yields relationships between the \((L, \overline{L})\)-eigenvalues on \( V_C \) and \( g_C \). In particular, \( (V.18) \) implies the following:

\[
(V.19) \quad \text{Suppose } \tilde{I}^{p,q}, \tilde{I}^{r,s} \neq 0 \text{ and } (p, q) \neq (r, s).
\]

Then \( I^{p+q,r+s}_g \neq 0 \).

**Remark V.20** (Properties of \( I^{p,q}_g \)). In the arguments that follow it will be helpful to keep in mind that both \( \tilde{I}^{p,q} \) and \( I^{p,q}_g \) are symmetric about the \( p + q = 0 \) and \( p − q = 0 \) lines: if \( \tilde{I}^{p,q} \) is nonzero, then so are \( \tilde{I}^{q,p} \), \( \tilde{I}^{-q,-p} \) and \( \tilde{I}^{-p,-q} \). (Similarly for \( I^{p,q}_g \).) In fact they all have the same dimension. The symmetry about the line \( p − q = 0 \) is due to the fact that both the roots \( \Delta \) and the weights \( \Lambda(V) \) are closed under conjugation (because \( \mathfrak{h}, \mathfrak{g} \) and \( V \) are defined over \( \mathbb{R} \)). The symmetry about the line \( p + q = 0 \) is due to the fact that the \( N \)-strings are uninterrupted and centered on the line \( p + q = 0 \). This also implies the following: suppose that \( \tilde{I}^{p,q} \) is nonzero. Then \( \tilde{I}^{k,-k} \) is nonzero if \( p − q = 2k \), and \( \tilde{I}^{k+1,-k} \) and \( \tilde{I}^{k,-k−1} \) are nonzero if \( p − q = 2k + 1 \).

**Step 3:** Suppose there exists \( \tilde{I}^{p,q} \neq 0 \) with \( p − q = 2k + 1 \). Then \( \tilde{I}^{k+1,-k}, \tilde{I}^{k,-k−1} \neq 0 \) (Remark V.20). So \( (V.19) \) implies \( I^{2b+1,-2b−1}_g = 0 \), and \( (V.2b) \) forces \( c = \pm 1 \). So, suppose that \( \tilde{I}^{\pm 1,0} \) and \( \tilde{I}^{0,\pm 1} \) are all nonzero. (As discussed in Remark V.20, if any one of the four is nonzero, then all four are nonzero.) If \( \tilde{I}^{r,s} \neq 0 \), then \( (V.19) \) implies

\[
I^{r,s\pm 1}_g \text{ and } I^{r,s\pm 1}\text{ are nonzero.}
\]
Given (V.2c), this forces $|r - s| \leq 1$. Whence the Deligne splitting must be of the form depicted in Figure V.2. But this implies that (V.8) holds. Whence Theorem V.15 implies the limiting mixed Hodge structure is Hodge–Tate, contradicting our hypothesis. To conclude:

If $\tilde{I}^{p,q} \neq 0$, then $p - q$ is even.

**Figure V.2.** The Deligne splitting $V_C = \oplus I^{p,q}$.

**Step 4:** Suppose there exists $\tilde{I}^{p,q} \neq 0$ with $p - q = 2k > 0$. Then $\tilde{I}^{k,-k}$ and $\tilde{I}^{-k,k}$ are nonzero (Remark V.20). Since $N \neq 0$, there exists some nonzero $I^{r,s}$ with $r + s \neq 0$. Then (V.19) implies $I_{\tilde{g}}^{r-k,s+k}$ and $I_{\tilde{g}}^{r+k,s-k}$ are nonzero, and (V.2c) forces $|r - s \pm 2k| \leq 2$. By hypothesis $2k \geq 2$, so it must be the case that $k = 1$ and $r - s = 0$. Thus, the Deligne splitting $V_C = \oplus I^{p,q}$ is as depicted in Figure V.3.a. (The induced Deligne splitting $g_C = \oplus I^{p,q}_{\tilde{g}}$ is as in Figure V.3.b.) This establishes Theorem I.12(a).

**Figure V.3.** The Deligne splittings $V_C = \oplus I^{p,q}$ and $g_C = \oplus I^{p,q}_{\tilde{g}}$.

The hypothesis that $(V, W_*(N), F^*)$ is not Hodge–Tate implies

(V.21) $\tilde{I}^{-1,1} = \tilde{I}^{-1,1}_{\text{prim}} \neq 0$. 
Moreover, if $\tilde{I}_{p,p}^{p,p} \neq 0$, then (V.19) and (V.21) imply that $I_{p,p}^{p-1,p+1} \neq 0$. By Theorem V.1, this forces $p \geq 1$ to be odd. This establishes Theorem I.12(b). Finally, since $\tilde{I}_{p,p}^{p,p} = I_{m-1,m+1}^{m-1,m+1}$, we see that (V.21) establishes Theorem I.12(c) and completes the proof.

V.E. All degenerations are induced from maximal Hodge–Tate degenerations. In a suitably interpreted sense all degenerations are induced from a (maximal) degeneration of Hodge–Tate type. Some care must be taken with this statement, as it is not necessarily the case that the underlying degeneration arises algebro-geometrically: this is a statement about the orbit structure and representation theory associated with the $\text{SL}_2$-orbit approximating an arbitrary degeneration, which may or may not arise algebro-geometrically.

Fix a Mumford–Tate domain $D \subset \tilde{D}$. Let $N \in \mathfrak{g}_R$ be a nilpotent element and consider the corresponding boundary component $B(N) = \tilde{B}(N)/\exp(\mathbb{C}N)$. Given $F^* \in \tilde{B}(N)$, let $(F^*, W(N)_*)$ denote the corresponding limiting mixed Hodge structure on $\mathfrak{g}$. Let $\mathfrak{g}_C = \bigoplus I_{p,q}^{p,q}$ be the Deligne splitting. Without loss of generality, the limiting mixed Hodge structure is $\mathbb{R}$-split. The diagonal subalgebra

$$\mathfrak{s}_C = \bigoplus_p I_{p,p}^{p,p}$$

is a conjugation stable subalgebra of $\mathfrak{g}_C$ containing $N$. Let $\mathfrak{s}_R = \mathfrak{s}_C \cap \mathfrak{g}_R$ denote the real form. Moreover, as the zero eigenspace for the grading element $L - \mathcal{T}$, the subalgebra $\mathfrak{s}_R$ is necessarily a Levi subalgebra, and therefore reductive.

**Lemma V.22.** The limiting mixed Hodge structure $(F^*, W(N)_*)$ on $\mathfrak{g}_R$ induces a sub-limiting mixed Hodge structure $(F^*_s, W_s(N)_s)$ on $\mathfrak{s}_R$ by

$$F^*_s = F^* \cap \mathfrak{s}_C = \bigoplus_{q \geq p} I_{q,q}^{q,q} \quad \text{and} \quad W_s(N)_s = W_s(N)_s \cap \mathfrak{s}_R = \bigoplus_{q \leq p} I_{q,q}^{q,q}.$$

**Proof.** This follows directly from the definition of limiting mixed Hodge structures (Section II.F).

Let $S_C \subset G_C$ be the connected Lie subgroup with Lie algebra $\mathfrak{s}_C$, and set

$$\tilde{D} = S_C \cdot F^* \quad \text{and} \quad D = \tilde{D} \cap D.$$
By [CKS], $F^\bullet_g(z) = e^{zN}F^\bullet_g \in D$ for all $\text{Im}(z) > 0$; equivalently, the Hodge filtration $F^\bullet_g(z)$ defines a Hodge structure $\varphi_z$ on $\mathfrak{g}_R$ (Section II.C.1). Likewise, $F^\bullet_s(z) = e^{zN}F^\bullet_s \in D$ defines a Hodge structure $\varphi_z|_s$ on $\mathfrak{s}_R$ for all $\text{Im}(z) > 0$. This implies $D$ carries the structure of a Mumford–Tate subdomain of $D$ with compact dual $\hat{D}$.

In particular, $D$ is an open $S_R$–orbit in $\hat{D}$. From Proposition I.15, and the fact that $(F^\bullet_s, W^\bullet(N)_s)$ is Hodge–Tate, we see that $S_R \cdot F^\bullet$ is the closed $S_R$–orbit in $\hat{D}$.

In this sense,

the nilpotent orbit $(F^\bullet_s, N)$ is a maximal degeneration of the Hodge structure on $\mathfrak{s}_R$.

So far we have viewed $\mathfrak{s}_R$ as having sub–Hodge structures $\varphi_z|_s$ that are restrictions of Hodge structures $\varphi_z$ on $\mathfrak{g}_R$. In fact a stronger statement holds: the circle $\varphi_z$ is contained in $S_R$.

**Lemma V.23.** The Hodge structure $(\mathfrak{g}, \varphi_z)$ is given by a Hodge representation of $S_R$.

In this sense,

the degeneration of Hodge structure on $\mathfrak{g}$ given by $(F^\bullet_g, N)$ is induced from a maximal degeneration of Hodge structure on $\mathfrak{s}_R$.

**Proof.** We need to show that the circle $\varphi_z$ is contained in $S_C$, cf. [12]. The corresponding grading element $L_z$ (Sections II.C.3 and II.D.3) is equal to $\varphi'(1)/4\pi i$, cf. [19, Section 2.3]. So $\varphi_z \subset S_R$ if and only if $L_z$, a priori an element of $\mathfrak{g}_C$, is an element of $\mathfrak{s}_C$.

Decompose $\mathfrak{s}_C = \mathfrak{j}_C \oplus \mathfrak{s}_C^{ss}$ into its center $\mathfrak{j}_C$ and semisimple factor $\mathfrak{s}_C^{ss} = [\mathfrak{s}_C, \mathfrak{s}_C]$. Since $\mathfrak{s}_R$ is a sub–Hodge structure of $\mathfrak{g}_R$, with respect to $\varphi_z$, it follows that $\mathfrak{s}_R^{ss}$ is also a sub–Hodge structure. Therefore, $L_z$ determines a graded decomposition of $\mathfrak{s}_C^{ss}$. As discussed in Remark II.7, this graded decomposition is also induced by a grading element $L'_z \in \mathfrak{s}_C$. It follows that $L_z - L'_z \in \mathfrak{g}_C$ is contained in the centralizer of $\mathfrak{s}_C$. It is here that the fact that $\mathfrak{s}_C$ is a Levi subalgebra is key, for it is a well–known property of Levi subalgebras that the centralizer in $\mathfrak{g}_C$ is equal to the center $\mathfrak{j}_C$. Therefore, $L_z - L'_z \in \mathfrak{j}_C \subset \mathfrak{s}_C$. Whence $L_z \in \mathfrak{s}_C$. □
Remark V.24. Since $\mathfrak{s}$ is reductive, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}^\perp$ as a $\mathfrak{s}$-module. The idea here is that the essential structure/relationship is between $N$ and the Levi subalgebra $\mathfrak{s}$; the remaining structure on $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}^\perp$, that is the structure on $\mathfrak{s}^\perp$, is induced from the $\mathfrak{s}$-module structure on $\mathfrak{s}^\perp$. This sort of idea does back to Bala and Carter’s classification [4, 5] of nilpotent orbits $\mathcal{N} \subset \mathfrak{g}_C$, where the idea is to look at minimal Levi subalgebras $\mathfrak{l}$ containing a fixed $N \in \mathcal{N}$, and to classify the pairs $(N, \mathfrak{l})$. (In fact, the idea goes back farther to Dynkin [10], who looked at minimal reductive subalgebras containing $N$, but this approach does not seem to work as well.)

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