Multiplayer bandits without observing collision information

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Abstract

We study multiplayer stochastic multi-armed bandit problems in which the players cannot communicate, and if two or more players pull the same arm, a collision occurs and the involved players receive zero reward. We consider two feedback models: a model in which the players can observe whether a collision has occurred, and a more difficult setup when no collision information is available. We give the first theoretical guarantees for the second model: an algorithm with a logarithmic regret, and an algorithm with a square-root regret type that does not depend on the gaps between the means. For the first model, we give the first square-root regret bounds that do not depend on the gaps. Building on these ideas, we also give an algorithm for reaching approximate Nash equilibria quickly in stochastic anti-coordination games.

1 Introduction

The stochastic multi-armed bandit problem is a well-studied problem of machine learning: consider an agent that has to choose among several actions in each round of a game. To each action $i$ is associated a real-valued parameter $\mu_i$. Whenever the player performs the $i$-th action, she receives a random reward with mean $\mu_i$. If the player knew the means associated to the actions before starting the game, she would play an action with the highest mean during all rounds. The problem is to design a strategy for the player to maximize her reward in the setting where she

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does not know the means. The *regret* of the strategy is the difference between the accumulated rewards in the two scenarios.

This problem encapsulates the well-known exploration/exploitation trade-off: the player never learns the means exactly, but can estimate them. As the game proceeds, she learns that some of the actions *probably* have better means, so she can ‘exploit’ these actions to obtain a better reward, but at the same time she has to ‘explore’ other actions as well, since they *might* have higher means. We refer the reader to Bubeck and Cesa-Bianchi (2012) for a survey on this problem. Traditionally, actions are called ‘arms’ and ‘pulling an arm’ refers to performing an action.

We study a multiplayer version of this game, in which each player pulls an arm in each round, and if two or more players pull the same arm, a *collision* occurs and all players pulling that arm receive zero reward. The players’ goal is to maximize the collective received reward.

One application for this model is opportunistic spectrum access with multiple users in a cognitive radio network: we have a radio network with several channels (corresponding to the arms) that have been purchased by primary users. There are also secondary users (the players) that can try to use these channels during the rounds when the primary users are not transmitting. Successfully using a channel to transmit a message means a unit reward, and not transmitting means zero reward. If more than one secondary user tries to use the same channel in the same round, a collision occurs and none of them can transmit. If a unique secondary user tries to use a channel, she will succeed if the primary user owning that channel happens to be idle in that round, which happens with a certain probability. Thus, the reward of the secondary user is a Bernoulli random variable whose mean depends on the activity of the corresponding primary user, and whether other secondary users have tried to use the same channel. See Liu and Zhao (2010, Section I.D) for other applications.

One may consider (at least) two possible feedback models: in the first model, whenever a player pulls an arm, she observes whether a collision has occurred on that arm, and receives a reward. In the second model the player just receives a reward, without observing whether a collision has occurred (of course, if the reward is positive, she can infer that no collision has occurred, but if the reward is zero, it is not clear whether a collision has occurred or not). The first feedback model has been studied in a series of work where theoretical guarantees have been proved. The second feedback model was introduced by Bonnefoi, Besson, Moy, Kaufmann, and Palicot (2017), motivated by large scale IoT applications, and further studied by Besson and Kaufmann (2018), but for this model no theoretical guarantees have been proved.

Our main contributions are summarized as follows.

1. We offer the first theoretical guarantees for the second model, where the play-
ers do not observe collision information. We propose an algorithm with a logarithmic regret (in terms of the number of rounds), and we also give an algorithm with a sublinear regret that does not depend on the gaps between the means.

2. For the first model, in which the players observe collision information, we prove the first sublinear regret bounds that do not depend on the gaps between the means.

3. One may also view this setup as a stochastic anti-coordination game. Using the algorithmic ideas introduced here, we give an algorithm for reaching an approximate Nash equilibrium quickly in such games.

1.1 The model, results, and organization

Let \( K > 1 \) be a positive integer and let \( \mu_1, \ldots, \mu_K \) be nonnegative numbers corresponding to the arm means. Let \( Y_{i,t} \) be the reward of arm \( i \) in round \( t \), so the \( \{Y_{i,t}\} \) are independent, identically distributed, and \( EY_{i,t} = \mu_i \). We may assume, by relabelling the arms if necessary, that \( \mu_1 \geq \cdots \geq \mu_K \). The players are of course unaware of this labelling.

A set of \( m > 1 \) players play the following game for \( T > 0 \) rounds: in each round \( t = 1, \ldots, T \), player \( j \) chooses an arm \( A_j(t) \in \{1, \ldots, K\} = [K] \). Let \( C_i(t) \in \{0,1\} \) be the collision indicator for arm \( i \) in round \( t \), that is, \( C_i(t) = 1 \) if and only if there exist distinct \( j, j' \) with \( A_j(t) = A_{j'}(t) = i \). Player \( j \) receives the reward \( r_j(t) = Y_{A_j(t),t}(1 - C_{A_j(t)}(t)) \) in round \( t \).

We will also consider a stronger feedback model, in which each player \( j \) also observes \( C_{A_j(t)}(t) \) in each round \( t \); this is called ‘the model with collision information.’

The regret of a strategy is defined as

\[
\text{Regret} = T \sum_{i \in [m]} \mu_i - \sum_{t \in [T]} \sum_{j \in [m]} \mu_{A_j(t)}(1 - C_{A_j(t)}(t))
\]

Note that regret is a random variable (since the strategy can randomize hence \( A_j(t) \) can be random) and we will bound its expected value, although ‘with high probability’ bounds can also be derived from our proofs.

To simplify the statements and proofs of our main theorems, we make three additional assumptions, which can be relaxed at the expense of getting worse bounds, as discussed in Section 4.

(Assumption 1) \( K \geq m \): there are at least as many arms as players.
(Assumption 2) \( Y_{A(t)}, t \) is supported on \([0, 1]\) so the means \( \mu_i \) and the rewards \( r_j(t) \) are also in \([0, 1]\).

(Assumption 3) All players know the values of both \( T \) and \( m \).

Note that we assume no communication between the players, and our algorithms are totally distributed. All of our algorithms are explicit, simple, and efficient.

We can now state our main theorems. Let \( \Delta := \mu_m - \mu_{m+1} \). All the following results correspond to the weak feedback model (i.e., no collision information), except if stated otherwise. Certainly any regret upper bound for this model automatically carries over to the stronger feedback model as well.

**Theorem 1.1.** There is an algorithm with expected regret \( O(mK \log(T)/\Delta^2) \).

In this theorem and throughout, the notation \( f = O(g) \) means there exists an absolute constant \( C \) such that for all admissible parameters, \( f \leq Cg \).

A shortcoming of Theorem 1.1 is that it gives a vacuous bound if \( \Delta = 0 \), and gives a very bad bound if \( \Delta \) is very small. Moreover, one may wonder if a regret of the form \( \sqrt{T} \) is possible that is independent of the gaps, as in the single player case. The following theorem shows this is possible, under some weak assumptions. Let \( \Delta' := \max[\Delta, \min(|\mu_m - \mu_i| : \mu_m - \mu_i > 0)] \). Observe that \( \Delta' \geq \Delta \), and that \( \Delta' \) is positive unless all arms have the same mean.

**Theorem 1.2.**

(a) Suppose all players know a lower bound \( \mu \) for \( \mu_m \). Then there is an algorithm with expected regret \( O(K^2 m \log^2(T)/\mu + Km \min\{\sqrt{T \log T}, \log(T)/\Delta'\}) \).

(b) For the stronger feedback model, in which the players observe the collision information, there is an algorithm with expected regret

\[
O(K^2 m \log^2(T) + Km \min\{\sqrt{T \log T}, \log(T)/\Delta'\}) = O(K^2 m \sqrt{T \log T}).
\]

(c) Suppose each player has the option of leaving the game at any point; that is, she can choose not to pull from some round onwards. Then, for any \( \mu > 0 \), there exists an algorithm with expected regret \( O(\mu mT + K^2 m \log^2(T)/\mu + Km \min\{\sqrt{T \log T}, \log(T)/\Delta'\}) \). In particular, setting \( \mu = K \log(T)/\sqrt{T} \) gives an algorithm whose expected regret is \( O(Km \sqrt{T \log T}) \).

We do not know whether our regret upper bounds are tight; proving lower bounds is left for further work. Some asymptotic lower bounds for the stronger feedback model have been proved by Besson and Kaufmann (2018, Section 3).

Another interesting avenue for future research is the setting in which the rewards are not i.i.d., but are chosen by an adversary.
The three algorithms proving Theorem 1.2 are quite similar. All of our algorithms have the property that, eventually each player fixates on one arm. This can be viewed as ‘reaching an equilibrium’ in a game-theoretic framework, where the actions correspond to arms, and the outcome of each action is the mean of the arm if no two players choose that action, and zero otherwise. Games with the property that ‘if two or more players choose the same action then their outcome is zero’ are called ‘anti-coordination games.’ Using our techniques for multiplayer bandits, we also provide an algorithm for converging to an approximate Nash equilibrium quickly in such a game.

More precisely, we define a stochastic anti-coordination game as follows: for each player \( j \in [m] \) and action \( i \in [K] \), there is an outcome \( \mu^i_j \in [0,1] \), such that if player \( j \) performs action \( i \) while no other player performs it, she will get a random reward in \([0,1]\] with mean \( \mu^i_j \), while if two or more players perform the same action \( i \), all get reward 0. An assignment of players to actions is an \( \epsilon \)-Nash equilibrium if, no player can improve her expected reward by more than \( \epsilon \) by switching to another action, while other players’ actions are unchanged. Then, we would like to design an algorithm for each player that reaches an \( \epsilon \)-Nash equilibrium quickly. We prove the following theorem in this direction.

**Theorem 1.3.** There is a distributed algorithm that with probability at least \( 1 - \delta \) converges to an \( \epsilon \)-Nash equilibrium in any stochastic anti-coordination game within \( O(\log(K/\delta)(K/\epsilon^2 + K^2/\epsilon)) \) many rounds.

In proving this theorem we assume each player also has the option of choosing a ‘dummy’ action with zero reward, which is given index 0. This is a realistic assumption in most applications.

Next we review some related work. Theorems 1.1 and 1.2 are proved in Sections 2 and 3, respectively. In Section 4 we discuss how to relax Assumptions 1–3 above. Finally, the proof of Theorem 1.3 appears in Section 5.

### 1.2 Related work

There is little previous work on the model without observing collision information: the model was introduced by Bonnefoi et al. (2017) and further studied by Besson and Kaufmann (2018). These papers introduce an algorithm and study it empirically, but no theoretical guarantee is given. In particular, it is argued in Besson and Kaufmann (2018, Appendix E) that the expected regret of that algorithm is linear.

We now review previous work on the stronger feedback model with collision information available to the players. The multiplayer multi-armed bandit games were introduced by Anantharam, Varaiya, and Walrand (1987) and further studied by Komiyama, Honda, and Nakagawa (2015). They studied a cen-
entralized algorithm, that is, when there is a single centre that controls the players, and observes the rewards of all players. The distributed setting was introduced by [Liu and Zhao (2010)], where an algorithm was given with expected regret bounded by $\kappa \log T$, with $\kappa$ depending on the game parameters, that is, $m$, $K$, and the arm means. They also showed that any algorithm must have regret $\Omega(\log T)$.

The dependence of $\kappa$ on the parameters was further improved by [Anandkumar, Michael, Tang, and Swami (2011); Rosenski, Shamir, and Szlak (2016); Besson and Kaufmann (2018)].

In Rosenski et al. (2016) a ‘musical chairs’ subroutine was introduced to reduce the number of collisions; we have further developed and used this subroutine in our algorithms. Their final algorithm requires the knowledge of $\Delta$ and its regret is bounded by $O(\Delta^2 m^3 K^2)$, which is at least as large as the bound of Theorem 1.1.

Besson and Kaufmann (2018) tightened the previous lower bounds, and also developed an algorithm whose regret is bounded by

$$O(\log(T) \left( \sum_{i=m+1}^{K} \frac{m}{\text{kl}(\mu_i, \mu_m)} + \sum_{1 \leq i < j \leq K} \frac{m^3}{\text{kl}(\mu_j, \mu_i)} \right)),$$

where $\text{kl}(x, y) := x \log(x/y) + (1-x) \log((1-x)/(1-y))$. This bound is not comparable with the bound of Theorem 1.1 in general; however if $\mu_1 = \cdots = \mu_m = 1/2$ and $\mu_{m+1} = \cdots = \mu_{K} = 1/2 - \Delta$, then their bound becomes $O(m^3 K^2 \log(T)/\Delta^2)$, which is worse than our bound by a multiplicative factor of $m^2 K$.

We emphasize that all the previously known upper bounds become vacuous if $\Delta = 0$, whereas our Theorem 1.2 gives sublinear bounds in this case.

Finally, [Avner and Mannor (2014); Rosenski, Shamir, and Szlak (2016)] also study a dynamic version of the problem, in which the players can leave the game and new players can arrive, and prove sublinear regret bounds. We do not study such scenarios here.

**Preliminaries.** We denote $[n] := \{1, \ldots, n\}$. All logarithms are in the natural base. We will use the following versions of Chernoff-Hoeffding concentration inequalities; see, e.g., [McDiarmid (1998), Theorem 2.3]:

**Proposition 1.4.** Let the random variables $X_1, \ldots, X_n$ be independent, with $0 \leq X_k \leq 1$ for each $k$. Let $\tilde{\mu} = \sum X_k/n$ and $\mu = \mathbb{E}\tilde{\mu}$. Then we have,

(a) for any $t \geq 0$,

$$P(\|\tilde{\mu} - \mu\| > t) < 2 \exp(-2nt^2),$$

(b) and, for any $\varepsilon > 0$,

$$P(\tilde{\mu} < (1 - \varepsilon)\mu) < \exp(-\varepsilon^2 n\mu/2).$$
2 Proof of Theorem 1.1

Each of the players follow the same algorithm, which has four phases, described next. Note that the phases are not synchronized, that is, each phase may have different starting and stopping times for each player. Let \( g = 128K \log(3Km^2T^2) \).

Phase 1: The player pulls arms uniformly at random, and maintains an estimate for the mean of each arm: the estimate for arm \( i \) is the average reward received from arm \( i \), divided by \((1 - 1/K)^{m-1}\). Note that \((1 - 1/K)^{m-1}\) is precisely the probability of not getting a conflict for each pull provided that the other players are also pulling arms uniformly at random, hence this is indeed an unbiased estimate for \( \mu_i \). In other words, for any round \( t \) that arm \( i \) is pulled and reward \( r(t) \) is received, since conflicts and rewards are independent we have

\[
\mu_i = \mathbb{E}Y_{i,t} = \frac{\mathbb{E}r(t)}{\mathbb{E}(1 - C_i(t))} = \frac{\mathbb{E}r(t)}{(1 - 1/K)^{m-1}}.
\]

For each round \( t \), the player maintains a sorted list \( \hat{\mu}_{i_1,t} \geq \cdots \geq \hat{\mu}_{i_K,t} \) of estimated means, and let \( \tau \) be the first round when \( \hat{\mu}_{i_m,\tau} - \hat{\mu}_{i_{m+1},\tau} \geq 3\sqrt{g/\tau} \). The first phase finishes at the end of round \( \tau \). By this time, the player has learned the best \( m \) arms with high probability (as we prove later), and so has a list \( G \subseteq [K] \) of \( m \) arms with the highest means.

Phase 2: For \( 24\tau \) rounds, the player just pulls arms uniformly at random, without updating the estimates.

Phase 3: The player plays a so-called Musical Chairs algorithm until it occupies an arm: in each round, she pulls a uniformly random arm \( i \in G \). If she gets a positive reward (which means no other player has pulled arm \( i \)), we say the player has ‘occupied’ arm \( i \), and this phase is finished for the player. Note that, by construction, at most one player will occupy any given arm.

Phase 4: The player pulls the occupied arm forever.

The pseudocode is shown as Algorithm 1. We next analyze the regret of this algorithm, starting with some preliminary lemmas.

**Lemma 2.1.** Suppose \( T \geq 17 \). Consider any fixed player and let \( \hat{\mu}_{i,t} \) denote her estimated mean for arm \( i \) after \( t \) rounds of Phase 1. Then we have

\[
P\left( \exists i \in [K], t \in [\tau] : |\hat{\mu}_{i,t} - \mu_i| > \sqrt{g/t} \right) < 3KT \exp(-g/128K).
\]
**Algorithm 1: the algorithm for Theorem 1.1**

**Input:** number of players $m$, number of arms $K$, number of rounds $T$

1. $g \leftarrow 128K \log(3Km^2T^2)$
2. $\hat{\mu}_i \leftarrow 0$ for all $i \in [K]$

// Phase 1 starts

3. $\tau \leftarrow 0$

4. repeat
5. pull a uniformly random arm $i$
6. $\hat{\mu}_i \leftarrow$ average reward from arm $i$, divided by $(1 - 1/K)^{m-1}$
7. Sort the $\hat{\mu}$ vector as $\hat{\mu}_{i_1} \geq \cdots \geq \hat{\mu}_{i_K}$
8. $\tau \leftarrow \tau + 1$

9. until $\hat{\mu}_{i_m} - \hat{\mu}_{i_{m+1}} \geq 3\sqrt{g/\tau}$
10. Best-$m$-arms $\leftarrow \{i_1, i_2, \ldots, i_m\}$

// Phase 2 starts

11. for $24\tau$ rounds do pull arms uniformly at random

// Phase 3 starts

12. $i \leftarrow$ MusicalChairs1 ($\text{Best-}m$-arms)

// Phase 4 starts

13. Pull arm $i$ until end of game

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**Subroutine MusicalChairs1($A$)**

**Input:** set $A \subseteq [K]$ of target arms

**Output:** index of an occupied arm

1. while true do
2. pull an arm $i \in A$ uniformly at random
3. if positive reward is received then output $i$ // arm $i$ is occupied
4. end
Proof. Fix an arm $i \in [K]$. Observe that $\hat{\mu}_{i,t} \leq 1/(1 - 1/K)^{m-1} < 1/(1 - 1/K)^K \leq 4$, so we have $|\hat{\mu}_{i,t} - \mu_i| \leq 4$ deterministically, so for $t \leq g/16$ we have $|\hat{\mu}_{i,t} - \mu_i| \leq \sqrt{g/t}$. Now fix a $t > g/16$. Let $T_i(t)$ denote the number of times this player has pulled arm $i$ by round $t$, which is a binomial random variable with mean $t/K$, hence Proposition 1.4(b) implies $\Pr\{T_i(t) < t/(2K)\} < \exp(-t/8K)$. Thus, the union bound gives

$$\Pr\{|\hat{\mu}_{i,t} - \mu_i| > \sqrt{g/t}\} < \exp(-t/8K) + \max_{s \geq t} \Pr\{|\hat{\mu}_{i,t} - \mu_i| > \sqrt{g/t} \mid T_i(t) = s\}.$$ 

Also, conditioned on any $s \in [t]$, $|\hat{\mu}_{i,t} - \mu_i|$ is the difference between an empirical average of $s$ i.i.d. random variables bounded in $[0, 4]$ and their expected value, thus Proposition 1.4(a) gives

$$\Pr\{|\hat{\mu}_{i,t} - \mu_i| > \sqrt{g/t} \mid T_i(t) = s\} < 2\exp(-sg/16t),$$

giving

$$\Pr\{|\hat{\mu}_{i,t} - \mu_i| > \sqrt{g/t}\} < \exp(-t/8K) + 2\exp(-g/16K).$$

We now apply a union bound over $t \in [\lceil g/16 \rceil, \ldots, T]$ to get

$$\Pr\{\exists t \in [\tau] : |\hat{\mu}_{i,t} - \mu_i| > \sqrt{g/t}\} < \left( \sum_{t = \lceil g/16 \rceil}^{T} \exp(-t/8K) \right) + 2T \exp(-g/16K)$$

$$< 17\exp(-g/128K) + 2T \exp(-g/16K)$$

$$< 3T \exp(-g/128K),$$

since $K \geq 2$ and $T \geq 17$. Applying a union bound over the $K$ arms concludes the proof of the lemma. 

\[ \square \]

Corollary 2.2. With probability at least $1 - 1/mT$, the following are true:

(i) all players have learned the best $m$ arms by end of their Phase 1,

(ii) we have

$$g/\Delta^2 \leq \tau \leq 25g/\Delta^2$$

for all players, and

(iii) the first two phases are finished for all players after at most $625g/\Delta^2$ many rounds.

Proof. By the choice of $g = 128K \log(3Km^2T^2)$, Lemma 2.1 and a union bound over the $m$ players, we have that with probability at least $1 - 1/mT$, all players have mean estimates that are $\sqrt{g/t}$-close to the actual means, uniformly for all arms and all $t \in [\tau]$. If this event holds then the three parts follow.

Part (i) follows by noting that a player would stop Phase 1 when she has found a gap of size $3\sqrt{g/t}$ between the $m$th the and the $(m + 1)$th arm. However, by
this time she has learned the means of all arms within an error of $\sqrt{g/t}$, therefore by the triangle inequality, she has correctly determined that the $m$th mean is larger than the $(m + 1)$th mean, whence has learned the best $m$ arms.

For part (ii), using the triangle inequality and the definition of $\tau$ we have

$$\sqrt{\frac{3g}{\tau}} \leq \hat{\mu}_{m,\tau} - \mu_{m+1,\tau} \leq (\hat{\mu}_{m,\tau} - \mu_m) + (\mu_m - \mu_{m+1}) + (\mu_{m+1} - \mu_{m+1,\tau}) \leq \sqrt{\frac{g}{\tau}} + \Delta + \sqrt{\frac{g}{\tau}},$$

whence $\tau \geq g/\Delta^2$. On the other hand, for $t = 25g/\Delta^2$,

$$\hat{\mu}_{m,t} - \mu_{m+1,t} \geq (\mu_m - \mu_{m+1}) - |\hat{\mu}_{m,t} - \mu_m| - |\mu_{m+1} - \mu_{m+1,t}| \geq \Delta - 2\sqrt{g/t} \geq 3g/\sqrt{t},$$

whence $\tau \leq t$.

Part (iii) follows from part (ii) and noting that the duration of Phase 2 is $24\tau$. \hfill \square

The curious reader may wonder about the role of Phase 2 and ask why cannot a player proceed to Phase 3 right after she has learned the best $m$ arms? The reason is to help other players to find the best $m$ arms. Indeed, it is possible that a player finishes Phase 1 by round $g/\Delta^2$, but the algorithm asks her to continue pulling arms at random, so that the other players continue to have unbiased estimators for the means, for at least $24g/\Delta^2$ many more rounds, at which point we are guaranteed that all players have finished their Phase 1. Otherwise, if a player switches to Phase 3 too quickly, then this would skew the collision probabilities, and the other players will not have unbiased estimators of the means.

We now proceed to analyze Phase 3, the musical chairs subroutine. By this point all players have learned the best $m$ arms, hence they just want to share these $m$ arms between themselves as quickly as possible. Note that by definition of the subroutine, once this phase is finished, each player has occupied a distinct arm.

**Lemma 2.3.** With probability at least $1 - 1/mT$, Phase 3 takes at most $4m \log(m^2T)/\Delta$ many rounds for all players.

**Proof.** We use the fact that, each reward $Y_{i,t}$ is bounded in $[0, 1]$, hence $P\{Y_{i,t} > 0\} \geq EY_{i,t}$. Fix any player in her Phase 3 who has not occupied an arm, and suppose there are still $a$ unoccupied arms available for her. (There are $m$ players, and each occupies at most 1 arm, hence $a \geq 1$.) Whenever she tries to occupy an unoccupied arm, her success probability is at least

$$\frac{a}{m} \Delta (1 - 1/m)^{m-a} \geq \Delta/4m.$$

Here, $\frac{a}{m} \geq 1/m$ is the probability that she pulls an unoccupied arm, $\Delta \leq \mu_m$ is a lower bound for the probability that that arm produces a positive reward, and
$(1 - 1/m)^{m-a} \geq 1/4$ is the probability that, none of the other players pull that arm. Hence, the probability that the player has not occupied an arm after $t$ attempts can be bounded by $(1 - \Delta/4m)^t \leq \exp(-t\Delta/4m)$. Letting $t = 4m \log(m^2 T)/\Delta$ makes this probability $\leq 1/Tm^2$. Applying the union bound over all players completes the proof.

Proof of Theorem 1.1. By Corollary 2.2 and Lemma 2.3 with probability at least $1 - 2/mT$, the first three phases finish for all players after at most $625g/\Delta^2 + 4m \log(m^2 T)/\Delta = O(K \log(KT)/\Delta^2)$ many rounds, and after this time, each player has occupied one of the best $m$ arms, and different players have occupied distinct arms. During each round, the regret is at most $m$, hence the total regret incurred during the first three phases is bounded by $O(mK \log(KT)/\Delta^2)$, and the regret afterwards would be 0. On the other hand, with the remaining $2/mT$ probability, the regret is at most $mT$. Therefore, the expected regret is at most $O(mK \log(KT)/\Delta^2) + 2$, as required. The $O(\log(KT))$ can be replaced with $O(\log(T))$, noting that

$$\min\{mT, O(mK \log(KT)/\Delta^2)\} = O(mK \log(T)/\Delta^2).$$

(Similar replacements of $O(\log(KT))$ with $O(\log(T))$ will also be done in a few other places in the following.)

3 Proof of Theorem 1.2

3.1 The modified musical chairs subroutines

We need a modified version of the musical chairs algorithm, which we call MusicalChairs2. For any ‘target’ set $A$ of arms and any number $\alpha$ of rounds, this subroutine consists of precisely $\alpha$ rounds as follows: in each round the player pulls a uniformly random arm $j \in [K]$. If she gets a positive reward, and $j \in A$, then she occupies arm $j$, pulls arm $j$ for the remaining rounds, and outputs $j$. Otherwise she tries again. If after $\alpha$ rounds she has not occupied any arm, she outputs 0. See the pseudocode below. The following lemma bounds the failure probability of this subroutine.

Lemma 3.1. Let $\mu \in [0, 1]$ be arbitrary. Suppose a player executes MusicalChairs2$(A, \alpha)$, while any other player either has occupied an arm, or is executing MusicalChairs2, or is pulling arms uniformly at random during these $\alpha$ rounds. We say the player is ‘successful’ if after the execution of the subroutine she has occupied an arm, or each arm in $A$ with mean $\geq \mu$ is occupied by other players. The probability of ‘failure’ is upper bounded by $\exp(-\alpha \mu/4K)$ if $m \leq K$, and by $\exp(-\alpha \mu \exp(-2m/K)/K)$ in general.
Subroutine MusicalChairs2(A, α)

Input: set $A \subseteq [K]$ of target arms, number $α \in \{0, 1, \ldots\}$ of rounds
Output: index of an occupied arm, or 0 if no arm was occupied

1. for $i \leftarrow 1$ to $α$
   2. pull an arm $j \in [K]$ uniformly at random
   3. if $j \in A$ and positive reward is received then // arm $j$ is occupied
      4. pull arm $j$ for the remaining $α - i$ rounds
      5. output $j$
   6. end
7. end
// no arm is occupied
8. output 0

Proof. At any round during the subroutine, suppose the player has not occupied an arm and that there are still $a \geq 1$ unoccupied arms of mean $\geq \mu$ in $A$. Whenever she tries to occupy one of her target arms, her success probability is at least

$$\frac{a}{K}(1 - 1/K)^{m-1} \geq \mu \exp(-2m/K)/K.$$  

Here, $\frac{a}{K} \geq 1/K$ is the probability that she pulls an unoccupied arm in her target set with mean $\geq \mu$, $\mu$ is a lower bound for the probability that that arm produces a positive reward, and $(1 - 1/K)^{m-1} \geq \exp(-2m/K)$ is the probability that none of the other players pull the same arm. (Note that her success probability may indeed be larger than this, because she may also occupy arms in her target set with mean $< \mu$.) Hence, the probability that she has not occupied an arm after $α$ attempts can be bounded by $(1 - \mu \exp(-2m/K)/K)^α \leq \exp(-α\mu \exp(-2m/K)/K)$. The argument when $m \leq K$ is identical, except we would use the tighter bound $(1 - 1/K)^{m-1} > (1 - 1/K)^K \geq 1/4$. 

The following corollary provides a guarantee when many players execute MusicalChairs2 in parallel. The proof is via applying a union bound over the participating players.

Corollary 3.2. Suppose a subset of players execute MusicalChairs2 for the same number $α$ of rounds, but with potentially different target sets, while the other players are either pulling random arms or have occupied arms during these $α$ rounds. We say the subroutine is successful if all players are successful. The probability that the subroutine fails can be bounded by $m \exp(-α\mu/4K)$ if $m \leq K$, and $m \exp(-α\mu \exp(-2m/K)/K)$ in general.

In the stronger feedback model in which the players observe the collision information, we modify the MusicalChairs2 algorithm such that for a player to
occupy an arm, instead of receiving a positive reward, she would look at the collision information, and would occupy the arm if there was no collision. Call this subroutine MusicalChairs3. We get the following corollary for the failure probability, whose statement and proof are identical to that for Corollary 3.2 except there is no parameter \( \mu \).

**Subroutine MusicalChairs3** \((A, \alpha)\)

**Input:** set \( A \subseteq [K] \) of target arms, number \( \alpha \in \{0, 1, \ldots \} \) of rounds

**Output:** index of an occupied arm, or 0 if no arm was occupied

1. for \( i \leftarrow 1 \) to \( \alpha \) do
2. pull an arm \( j \in [K] \) uniformly at random
3. if \( j \in A \) and there was no collision then // arm \( j \) is occupied
   4. pull arm \( j \) for the remaining \( \alpha - i \) rounds
   5. output \( j \)
4. end
5. // no arm is occupied
6. output 0

**Corollary 3.3.** Consider the stronger feedback model with collision information available. Suppose a subset of players execute MusicalChairs3 for the same number \( \alpha \) of rounds, but with potentially different target sets, while the other players are either pulling random arms or have occupied arms during these \( \alpha \) rounds. We say the subroutine is successful if all players are successful. The probability that the subroutine fails is at most \( m \exp(-\alpha \mu/4K) \) if \( m \leq K \), and \( m \exp(-\alpha \mu \exp(-2m/K)/K) \) in general.

### 3.2 The whole algorithm

We focus on proving part (a), and then explain how the algorithm should be modified to prove parts (b) and (c). Recall that \( \mu \) is a lower bound for \( \mu_m \) that all players know in advance.

We describe the algorithm each player executes, first informally and then formally. The player maintains estimates \( \hat{\mu} \) for the means, which get closer to actual means as the algorithm proceeds. She also keeps a `confidence interval` for each arm \( j \), which is centred at \( \hat{\mu}_j \) and has the property that \( \mu_j \) lies in this interval with sufficiently high probability. If arm \( j \) has been pulled \( s \) times, this interval has length \( O(\sqrt{\log(T)/s}) \). Once the player makes sure that some arm is not among the best \( m \) arms, she marks it as bad and puts it in a set \( B \). This can happen either if it is determined that the arm has mean \( < \mu \), or if it is determined that there are at least \( m \) arms whose confidence intervals lie strictly above this arm’s interval.
(we say interval \([c,d]\) lies strictly above \([a,b]\) if \(b < c\)). On the other hand, once the player makes sure that some arm is within the best \(m\) arms, she marks it as a ‘golden’ arm, and puts it in a set \(G\). More precisely, this would happen as soon as there are at least \(K - m\) arms that are either determined to be bad, or whose confidence intervals lie strictly below this arm. Other arms (whose status is yet unknown) are called ‘silver’ arms and kept in a set \(S\).

Initially all arms are silver. The algorithm proceeds in epochs with increasing lengths. In each epoch, the player explores all the silver arms and hopes to characterize each silver arm as golden or bad at the end of the epoch. As time proceeds, arms whose means are far away from the \(m\)th arm will be characterized as either golden or bad. Golden arms will be occupied quickly, and bad arms will not be pulled again, and this will control the algorithm’s regret.

Special care is needed to ensure all players explore all the silver arms without conflicts, and this is done via careful sequences of MusicalChairs2 subroutines. In each epoch, each player maintains a set \(E\) of explored arms, which is empty when the epoch starts. The epoch has \(K + m - 1\) iterations. In each iteration, if there exists some arm in \(S \setminus E\) (i.e., an unexplored silver arm), the player tries to occupy such an arm; otherwise, the player has finished exploring the arms in \(S\), and she will try to occupy and pull an arbitrary arm from \(S\), while other players are exploring their silver arms. Say an arm is \(\mu\)-good if its mean is at least \(\mu\), and is \(\mu\)-bad otherwise. Note that by assumption, any \(\mu\)-bad arm is bad. The length of the MusicalChairs2 subroutines are chosen such that any \(\mu\)-good arm in \(S\), which is not marked as golden by any other player, will be explored in each epoch by each player. Thus, if an arm is not explored by the end of epoch (that is, if it lies in \(S \setminus E\)), that would mean the either the arm is \(\mu\)-bad or the arm is golden and is occupied by another player in the beginning of the epoch. The two cases can be distinguished by checking the empirical reward received from that arm.

The complete pseudocode appears as Algorithm 2 below. Note that this algorithm is synchronized: for all players the epochs and the iterations within the epochs begin and end at the same round.

To analyze the regret of the algorithm, we first define two bad events. The first bad event is that some of the MusicalChairs2 subroutines fail, and the second bad event is that some player’s confidence interval is incorrect, that is, the actual mean does not lie in the confidence interval. We start by bounding the probability of the bad events. Let \(\alpha = 4K \log(6Km^2T)/\mu\) and \(g = \log(4m^3T^2K)/2\).

**Lemma 3.4.** The probability that some bad event happens is at most \(1/mT\).

**Proof.** The probability that some MusicalChairs2 subroutine fails is bounded by \(m \exp(-\alpha M/4K)\) by Corollary 3.2. Applying the union bound over the (at most \(T\)
Algorithm 2: the algorithm for Theorem 1.2(a)

**Input:** number of players $m$, number of arms $K$, number of rounds $T$, lower bound $\mu$ for $\mu_i$

1. $g \leftarrow \log(4m^3T^2K)/2$, $\alpha \leftarrow 4K\log(6Km^2T)/\mu$
2. $\mu_i \leftarrow 0$ for all $i \in [K]$
3. $G \leftarrow \emptyset$, $B \leftarrow \emptyset$, $S \leftarrow [K]$

for epoch $i = 1, 2, \ldots,$ do

4. $j \leftarrow \text{MusicalChairs2}(G, \alpha)$ \slash occupy a golden arm if possible
5. if $j > 0$ then disregard the rest of the algorithm and pull arm $j$ forever
6. $E \leftarrow \emptyset$ \slash the set of arms that have been explored in this epoch
7. for $K + m - 1$ iterations do \slash goal: explore all silver arms by end of this loop

8. $j \leftarrow 0$
9. if $E \neq S$ then
10. \hspace{1em} $j \leftarrow \text{MusicalChairs2}(S \setminus E, \alpha)$ \slash occupy an unexplored arm
11. else \hspace{1em} randomly pull arms in the next $\alpha$ rounds \slash all silver arms have been explored in this epoch, so waste time for $\alpha$ rounds
12. end
13. if $j = 0$ then
14. \hspace{1em} $j \leftarrow \text{MusicalChairs2}(S, \alpha)$ \slash occupy any silver arm
15. else
16. \hspace{1em} pull arm $j$ for $\alpha$ rounds \slash keep on occupying arm $j$
17. end
18. pull arm $j$ for $2^i$ rounds and let $\hat{\mu}_j \leftarrow$ the average reward received
19. update confidence interval of arm $j$ to $[\hat{\mu}_j - \sqrt{g/2^i}, \hat{\mu}_j + \sqrt{g/2^i}]$
20. insert $j$ into $E$ \slash add $j$ to the set of explored arms

21. foreach $j \in S \setminus E$ do \slash put the unexplored arms in either $G$ or $B$
22. \hspace{1em} if $\hat{\mu}_j - \sqrt{g/2^{i-1}} > \mu$ then
23. \hspace{2em} move $j$ from $S$ to $G$ \slash arm $j$ is occupied by another player
24. \hspace{1em} else
25. \hspace{2em} move $j$ from $S$ to $B$ \slash arm $j$ has mean $< \mu$
26. \hspace{1em} end
27. \hspace{1em} end
28. foreach $j \in S$ do \slash update the golden and bad arms
29. \hspace{1em} if there exist at least $m - |G|$ arms $\ell \in S$ with $\mu_\ell - \sqrt{g/2^i} > \mu_j + \sqrt{g/2^i}$ then
30. \hspace{2em} move $j$ from $S$ to $B$
31. \hspace{1em} else if $\mu_j > \mu + 3\sqrt{g/2^i}$ and there exist at least $K - m - |B|$ arms $\ell \in S$ with $\mu_\ell + \sqrt{g/2^i} < \mu_j - \sqrt{g/2^i}$ then
32. \hspace{2em} move $j$ from $S$ to $G$
33. \hspace{1em} end
34. \hspace{1em} end
35. \hspace{1em} end
36. \hspace{1em} end
37. \hspace{1em} end
38. end
epochs and the $1 + 2(K + m - 1) \leq 3Km$ times MusicalChairs2 is executed in each epoch, gives the probability that some MusicalChairs2 subroutine fails is at most

$$3Km \times T \times m \exp(-\alpha M / 4K) \leq 1/2mT,$$

by the choice of $\alpha = 4K \log(6Km^2T)/\mu$.

Whenever a confidence interval for some arm $j$ is updated in some epoch $i$ (Line 21), the arm has been pulled precisely $2^i$ times right before that (Line 20). The probability that some confidence interval is incorrect for some player, say in epoch $i$, is hence bounded via Proposition 1.4(a) by

$$P\left\{ |\hat{\mu}_j - \mu_j| > \sqrt{g / 2^i} \right\} < 2 \exp(-2 \times 2^i g / 2^i) = 2 \exp(-2g).$$

Now applying the union bound over the $m$ players, the (at most $T$) epochs, and the $K + m - 1 \leq Km$ number of updating of the confidence intervals within each epoch, gives the probability of some incorrect confidence interval is at most

$$m \times T \times Km \times 2 \exp(-2g) \leq 1/2mT,$$

by the choice of $g = \log(4m^3T^2K)/2$, as required.

We are now ready to prove Theorem 1.2(a).

**Proof of Theorem 1.2(a).** We bound the regret assuming no bad events happen, and the bound for the expected regret follows as in the proof of Theorem 1.1.

Note that each epoch has two types of rounds: estimation rounds (Line 20) in which each arm is pulled by at most one player, during which she updates her estimate of its mean, and other rounds during which some of the players are executing MusicalChairs2, hence we call them MusicalChairs2 rounds.

Observe that, since there are at least $m \mu$-good arms, we always have $|G \cup S| \geq m$ and since the MusicalChairs2 subroutines are always successful, there can be no conflict during the estimation rounds.

The first claim is the following: consider a player that has just executed her Line 7 in epoch $i$, and has not occupied a golden arm by end of epoch $i$. Consider also a $\mu$-good arm $j$ that is silver, suppose this arm is not occupied by another player as a golden arm in their Line 5. Then the claim is that the player will pull arm $j$ at least $2^i$ times during epoch $i$ Line 20 and it will be put in $E$ at the end of the $K + m - 1$ iterations. To see this, note that the epoch has $K + m - 1$ iterations. In each iteration, if the player has any unexplored silver arm, in the first $\alpha$ rounds attempts to occupy one of those (Line 11), while the other players pull random arms. By Lemma 3.5 below and since the MusicalChairs2 subroutines are
successful, after the $K + m - 1$ iterations, each player has explored any such arm $j$. Therefore, the confidence interval of each such arm will have length $2\sqrt{g/2^i}$.

The second claim is that if no bad event occurs, then the algorithm never makes a mistake in characterizing the arms as golden/bad. First, the characterizations based on confidence intervals (Lines 31–35) are correct because all confidence intervals are correct. Next note that if $j \in S \setminus E$ on Line 24 then $j$ is not explored, and that can be because of one of two reasons: first, its mean may be smaller than $\mu$, hence it is not occupied during the MusicalChairs2 subroutines. Or, it may be a golden arm occupied by another player on Line 5. In the latter case, let $\hat{\mu}_j$ be the average reward received from this arm by that other player. Suppose the arm was marked as golden by the other player in epoch $i' \leq i - 1$. Then we must have $\hat{\mu}_j' > \mu + 3\sqrt{g/2^{i'}}$ (see Line 34). This implies

$$\mu_j \geq \hat{\mu}_j' - \sqrt{g/2^{i'}} > \mu + 2\sqrt{g/2^{i'}} \geq \mu + 2\sqrt{g/2^{i'-1}}.$$  

On the other hand, since $j$ was silver at the end of epoch $i - 1$, we have $\hat{\mu}_j \geq \mu_j - \sqrt{g/2^{i'-1}} > \mu + \sqrt{g/2^{i-1}}$. Hence Line 26 is executed and the player marks $j$ as golden. If this latter case has not happened, we are in the first case, so because the confidence intervals are correct, $\mu$ lies in the confidence interval for arm $j$, which has length $\sqrt{g/2^{i-1}}$. This means $\hat{\mu}_j - \sqrt{g/2^{i-1}} \leq \mu$, so Line 28 is executed and the player correctly marks $j$ as bad.

The third claim is that, any arm with mean $< \mu_m - 4\sqrt{g/2^i}$ has been marked as bad by all players by the end of epoch $i$. Let $j$ be such an arm and suppose we are at the end of epoch $i$. By definition of confidence intervals, it suffices to show there exists at least $m$ arms $\ell$ such that either $\ell \in G$ or $\hat{\mu}_\ell - \hat{\mu}_j > 2\sqrt{g/2^i}$. In fact this holds for all $\ell \in \{m\}$, since for any such $m$, either $\ell \in G$, or $\ell \in S$, which implies $\hat{\mu}_\ell - \hat{\mu}_j \geq \mu_\ell - \hat{\mu}_j - |\mu_\ell - \hat{\mu}_\ell| - |\hat{\mu}_j - \mu_j| > 4\sqrt{g/2^{i}} - \sqrt{g/2^{i}} - \sqrt{g/2^{i}} = 2\sqrt{g/2^{i}}$.

The fourth claim, whose proof is similar to the third claim, is that any arm with mean $\mu_m + 4\sqrt{g/2^i}$ has been marked as golden by all players by the end of epoch $i$. The only difference is the extra condition $\hat{\mu}_j > \mu + 3\sqrt{g/2^{i'}}$, which is satisfied by any such arm, since $\hat{\mu}_j \geq \mu_j - \sqrt{g/2^{i'}}$ by correctness of confidence intervals.

Now we bound the algorithm’s regret. First observe that the number of epochs is fewer than $\log_2(T) < 2\log(T)$. The number of iterations per epoch is $K + m - 1 < 2K$, whence the total number of MusicalChairs2 rounds can be bounded by $2\log(T)(\alpha + 4K\alpha) \leq 10K\alpha \log(T)$. Let us now bound the regret of the estimation rounds. The regret of the first epoch can be bounded by $m(K + m - 1) \leq 2Km$. Next note that any arm with mean $\mu_m + 4\sqrt{g/2^{i-1}}$ has been put in $G$ by the end of epoch $i - 1$ by all players (by fourth claim above), and so some player occupies it in the beginning of epoch $i$. During epoch $i$, each of the remaining players pull either a
silver or a golden arm, which are at most $8\sqrt{g/2^i - 1}$ away from the best available arms (by the third claim above). Since the probability that some bad event happens is $1/mT$ (Lemma 3.4), and in this case the total regret can be bounded by $mT$, the total expected regret can be bounded by

$$mT \times (1/mT) + 10mK\alpha \log(T) + 2Km + m \times \sum_{i=2}^{[\log_2(T)]} (2K \times 2^i \times 18\sqrt{g/2^{i-1}})$$

$$= O(K^2m\log^2(KT)/\mu + Km\sqrt{T\log(KT)}).$$

Recall that $\Delta' = \max\{\Delta, \min\{||\mu_m - \mu_i|: \mu_m - \mu_i > 0\}\}$. Let $j$ be the smallest integer that $4\sqrt{g/2^j} < \Delta'$. So after the first $j$ epochs, any remaining silver arm would have mean precisely $\mu_m$, and the regret will be zero after epoch $j$. The algorithm’s regret can be alternatively bounded by

$$mT \times (1/mT) + 10Km\alpha \log(T) + 2Km + \sum_{i=2}^{j} 8\sqrt{g/2^{i-1}(2Km)2^i}$$

$$= O(K^2m\log^2(KT)/\mu + Km\log(KT)/\Delta').$$

The following lemma is the last piece in completing the proof of Theorem 1.2(a).

**Lemma 3.5.** Fix an epoch and suppose all MusicalChairs2 subroutines of Line 11 are successful. Then, during the $K + m - 1$ iterations of the epoch, each player will occupy each $\mu$-good silver arm at least once.

**Proof.** Build a bipartite graph with one part being the players and the other part the arms, with an edge between a player and an arm if the arm is silver for that player. Say an edge is **good** if the corresponding arm is $\mu$-good. Say two edges are **neighbours** if they share a vertex, and the degree of an edge is its number of neighbours. Initially, the degree of each edge is at most $K + m - 2$. Observe that, whenever the MusicalChairs2 subroutine in Line 11 is executed, the set of edges corresponding to players and their occupied arms forms a matching in this graph, that is, a set of edges such that no two of them are neighbours. Moreover, since the MusicalChairs2 subroutine is successful by assumption, this matching $M$ has the property that, for any good edge $e$, either $e \in M$ or some neighbour of $e$ lies in $M$. After the execution of this subroutine, we delete this matching from the graph, and hence the degree of each good edge decreases by 1. In particular, the maximum degree of good edges decrease by 1, and once this maximum degree is 0, in the next iteration all good edges will be deleted. Therefore, after at most $K + m - 1$ iterations, all good edges will be deleted, as required.
The proof of Theorem 1.2(b) is identical, except we would choose \( \alpha := 4K \log(6Km^2T) \) in the algorithm, replace MusicalChairs2 with MusicalChairs3, and use Corollary 3.3 instead of Corollary 3.2.

**Proof of Theorem 1.2(c).** The algorithm would be similar, except that if a player has not occupied an arm when she wants to start an estimation period, she would simply leave the game and never pull any arm again. To be more precise, add the following line before line 20: `if \( j = 0 \) then leave the game.` This could happen if there are fewer than \( m \) many \( \mu \)-good arms, and so players may fail to find and occupy an arm. Suppose \( m' \) of the best \( m \) arms are \( \mu \)-bad. Once \( m' \) players have left the game, we will have \( m - m' \) players and \( m - m' \) many \( \mu \)-good arms, so the algorithm will work as in part (a), and the same analysis works. We would only lose a reward of at most \( m' \mu T \), due to players who have left the game.

\[ \square \]

### 4 Relaxing the assumptions

#### 4.1 Unknown time horizon

If \( T \) is not known, we can apply a simple doubling trick: we execute the algorithm assuming \( T = 1 \), then we execute it assuming \( T = 2 \times 1 \), and so on, until the actual time horizon is reached. If the expected regret of the algorithm for a known time horizon \( T \) is \( R(T) \), then the expected regret of the modified algorithm for unknown time horizon would be \( R'(T) \leq \sum_{i=0}^{[\log_2(T)]} R(2^i) \leq \log_2(T) \times R(T) \). For example, if the players have the option of leaving the game, we can apply Theorem 1.2(c) with \( \mu = K \log(T)/\sqrt{T} \) to get the regret upper bound

\[
R'(T) \leq \sum_{i=0}^{[\log_2(T)]} O(Km \log(2^i) \sqrt{2^i}) \leq O(Km \log(T)) \times \sum_{i=0}^{[\log_2(T)]} O(2^{i/2}) \leq O(Km \sqrt{T} \log(T)),
\]

which is within a constant multiplicative factor of the upper bound for \( R(T) \).

#### 4.2 Other reward distributions

In Theorems 1.1 and 1.2 we assumed the rewards are supported on \([0, 1]\). We used this assumption in three ways: first, we used that the expected regret incurred any round can be bounded by \( m \), second, that the rewards satisfy the Chernoff-Hoeffding concentration inequality (Proposition 1.4(a)), and third, for correctness proofs of the MusicalChairs1,2 subroutines we used the fact that \( \mathbb{P}[X > 0] \geq EX \) for any random variable \( X \in [0, 1] \).
A random variable $X$ is $\sigma$-sub-Gaussian if $\max\{P\{X - \mathbb{E}X < -t\}, P\{X - \mathbb{E}X > t\}\} < \exp(-t^2/2\sigma^2)$, for example a standard normal random variable is 1-sub-Gaussian. The first two facts hold for $\sigma$-sub-Gaussian random variables whose means lie in a bounded interval $[0, c]$ (with appropriate adjustments). For the proofs, see Vershynin (2018, Section 2.5). The third fact also holds up to a logarithmic factor, see Lemma 4.1 below. Hence, our main theorems can be readily extended to such distributions, with appropriate adjustments.

**Lemma 4.1.** Let $X \geq 0$ be a random variable with mean $\mu$ that satisfies $P\{X > \mu + t\} < \exp(-t^2/2\sigma^2)$. Then we have $P\{X > 0\} \geq \min\{|\mu/(\sigma \log(\mu/\sigma))|, 1\}/99$.

*Proof.* By dividing $X$ by $\sigma$ we may assume $\sigma = 1$. Let $t \geq 0$ be a parameter to be chosen later, and define $Y = X \cdot 1[X > t + \mu]$ and $Z = X \cdot 1[X \leq t + \mu]$. Note that $\mu = \mathbb{E}X = \mathbb{E}Y + \mathbb{E}Z$ and $\mathbb{E}Z \leq P\{X > 0\}(t + \mu)$. We next write $\mathbb{E}Y$ as

$$\mathbb{E}Y = \int_0^{t+\mu} P\{Y > s\} ds + \int_{t+\mu}^{\infty} P\{Y > s\} ds$$

For $0 \leq s \leq t + \mu$, we have $Y > s$ if and only if $Y > 0$ if and only if $X > t + \mu$, whence $\int_0^{t+\mu} P\{Y > s\} ds = (t + \mu)P\{X > t + \mu\} < (t + \mu)\exp(-t^2/2)$. For the second term we have

$$\int_{t+\mu}^{\infty} P\{Y > s\} ds < \int_{t}^{\infty} \exp(-s^2/2)ds < \exp(-t^2/2)/2t.$$  

Consequently,

$$\mu = \mathbb{E}Y + \mathbb{E}Z < (t + \mu + 1/2t)\exp(-t^2/2) + P\{X > 0\}(t + \mu),$$

which implies

$$P\{X > 0\} > \frac{\mu - (t + \mu + 1/2t)\exp(-t^2/2)}{t + \mu}.$$  

Now, if $\mu \leq 0.05$ then setting $t = \log(1/\mu)$ gives that the right-hand side is greater than $\mu/(5\log(1/\mu)) = |\mu/(5\log(1/\mu))|$. (Here, we have used the inequality $5\mu\log(\mu) + 5\log(\mu)\exp(-\log^2\mu/2)(\log(\mu) - \mu + 1/(2\log(\mu)) - \mu\log(\mu) + \mu^2 < 0$ which holds for all $0 < \mu \leq 0.05$.)

On the other hand, if $\mu > 0.05$, setting $t = 4$ gives that the right-hand side is greater than $1/99$, as required. (Here, we have used the inequality $(98 - e^{-8})\mu > 4 + 33 \times e^{-8}/8$, which holds for any $\mu > 0.05$.)

### 4.3 More players than arms

If $m > K$, the term $\sum_{i \in [m]} \mu_i$ in the definition of regret is not well defined, hence we must redefine the regret. There are two natural ways to do this.
4.3.1 Original model

In the original model, the best strategy of the players had they known the means would be for \( K - 1 \) of them to occupy the best \( K - 1 \) arms, and for the rest to occupy the worst arm; so the regret in this case can be defined as

\[
\text{Regret} = T \sum_{i \in [K-1]} \mu_i - \sum_{t \in [T]} \sum_{j \in [m]} \mu_{A_j(t)}(1 - C_{A_j(t)}(t)).
\]

Let \( \Delta := \mu_{K-1} - \mu_K \). We give an algorithm with regret \( O(mK \log(T) \exp(4m/K)/\Delta^2) \).

The algorithm is similar to Algorithm 1. Let \( p := (1 - 1/K)^{m-1} \geq \exp(-2m/K) \) be the probability of no-conflict, when the players pull arms uniformly at random, and let \( g = CK \log(KT)/p^2 \), for a sufficiently large constant \( C \). Each player pulls arms randomly until at some round \( \tau \) she finds a gap of \( 3g/\tau \) between the \((K - 1)\)th and \( K \)th arm, and continues for \( 24\tau \) more rounds to make sure that all others have also found this gap. An argument similar to that of Corollary 2.2 gives that these two phases will take \( O(K \log(KT)/p^2 \Delta^2) \) many rounds. Moreover, each player has learned that \( \mu_{K-1} \geq \Delta \geq \sqrt{g/\tau} \) and that \( \sqrt{\tau/g} \leq 5/\Delta \) (see Corollary 2.2(ii)). Then the player executes MusicalChair2 on the set of \( K - 1 \) best arms, for \( \alpha = CK \log(KT)\sqrt{\tau/g}/p \) many rounds, for a large enough constant \( C \). Since \( m \exp(-\alpha \mu_{K-1}p/K) \leq m \exp(-\alpha \sqrt{g/\tau p/K}) < 1/mT \), Lemma 3.1 implies that with probability at least \( 1 - 1/mT \) all players will be successful, meaning that the best \( K - 1 \) arms are occupied. After MusicalChair2 is finished, if the player has occupied an arm she will pull it until the end of game, otherwise she pulls the worst arm for the rest of game. Thus the regret will be zero after at most \( O(K \log(KT)/p^2 \Delta^2) + O(K \log(KT)\sqrt{\tau/g}/p) \leq O(K \log(KT)/p^2 \Delta^2) \) many rounds, giving a total regret of \( O(mK \log(KT)/p^2 \Delta^2) \leq O(mK \log(KT) \exp(4m/K)/\Delta^2) \).

4.3.2 Model allowing players to leave

Alternatively, if we allow the players to leave the game, the best strategy had we known the means would be for \( m - K \) players to leave the game, and for the rest of the players to occupy distinct arms. The regret in this model can be defined as

\[
\text{Regret} = T \sum_{i \in [K]} \mu_i - \sum_{t \in [T]} \sum_{j \in [m]} \mu_{A_j(t)}(1 - C_{A_j(t)}(t)).
\]

We propose a simple algorithm in this case: each player executes the MusicalChair2 algorithm for a certain number of rounds, and if she has not occupied an arm by that time, she leaves the game.

If the players know a lower bound \( \mu \) for all the arm means, they play MusicalChair2 for \( O(\log(TK)K \exp(2m/K)/\mu) \) many rounds, and by Lemma 3.1 with
probability at least $1 - 1/mT$ all the $K$ arms are occupied, whence the expected regret is bounded by $O(mK \log(T) \exp(2m/K)/\mu)$.

Otherwise, if the players observe the collision information, they play MusicalChairs3 for $O(\log(TK)K \exp(2m/K))$ many rounds, and the expected regret is upper bounded by $O(mK \log(T) \exp(2m/K))$.

Finally, if the players have none of the above information, they play MusicalChairs2 for $O(\log(TK)K \exp(2m/K)/\mu)$ many rounds. With high probability arms with means $\geq \mu$ are occupied, and any other arm contributes a regret of at most $\mu T$. So the total expected regret can be bounded by $O(mK \log(T) \exp(2m/K)/\mu) + K\mu T$, which after optimizing $\mu$ gives the bound $O(K \exp(2m/K) \sqrt{mT \log(TK)})$ for the regret.

4.4 Unknown number of players

In this section we assume $m \leq K$. We do not know how to handle the case $m > K$, although if $m \leq CK$ for some absolute constant $C$, then the analysis in this section works after appropriate adjustments, and all the derived asymptotic bounds hold.

If the players observe the collision information, then it is shown by Rosenski et al. (2016, Lemma 2) that there is a simple algorithm with $O(K^2 \log(1/\delta))$ many rounds using which each player learns $m$ with probability $\geq 1 - \delta$. Setting $\delta = 1/K^2 T$ makes sure this simultaneously holds for all players with probability $\geq 1 - 1/KT$, and after this estimation is done, the players can run the algorithm of Theorem 1.2(b). The additional regret due to these estimation rounds is $O(K^2 m \log(KT))$, which is dominated by the final regret upper bound of Theorem 1.2(b).

For the model without the collision information, we shall assume the players know that at least one arm has mean at least $\mu$, and we give an algorithm with $O(K^3 \log^2(K/\mu\delta)/\mu^2)$ many rounds such that if all players execute it, they all learn $m$ with probability $1 - \delta$. Setting $\delta = 1/K^2 T$ makes sure this simultaneously holds for all players with probability $\geq 1 - 1/KT$, and after this estimation is done, the players can execute Algorithm 1 or Algorithm 2. The additional regret due to estimation rounds is bounded by $O(K^3 m \log^2(KT/\mu)/\mu^2)$.

Here is the algorithm each player executes (with pseudocode in Algorithm 3):
We claim that uniformly with probability 1 Lemma 4.2.

Let

\[ \varepsilon = \frac{1}{4} \times \frac{1}{3} \times ((1 - 1/K)^{-2/5} - 1) \]

\[ < \frac{1}{4} \times (1 - 1/K)^{m-1} \times ((1 - 1/K)^{-2/5} + 1) \times ((1 - 1/K)^{-2/5} - 1) \]

\[ = \frac{1}{4} \times (1 - 1/K)^{m-1} \times \frac{(1 - 1/K)^{-2/5} - 1}{(1 - 1/K)^{-2/5} + 1} \]

\[ < \frac{1}{4} \times (1 - 1/K)^{m-1}. \]

First, the player pulls random arms for \(8K\log(K^2/9\delta)/\varepsilon^2\) rounds, and estimates the quantities \(\mu_j(1 - 1/K)^{m-1}\). By an argument similar to that of Lemma 2.1, she obtains estimates \(\sigma_j\) such that \(|\mu_j(1 - 1/K)^{m-1} - \sigma_j| \leq \varepsilon\) for all arms and for all players, uniformly with probability \(1 - \delta/3\). Then let \(\ell\) be the arm with maximum \(\sigma\) value. We claim that \(\mu_\ell \geq \mu/2\). To justify this, note that

\[(1 - 1/K)^{m-1} \mu_\ell \geq \sigma_\ell - \varepsilon \geq \sigma_1 - \varepsilon \geq (1 - 1/K)^{m-1} \mu_1 - 2\varepsilon \geq (1 - 1/K)^{m-1} \mu - 2\varepsilon,\]

whence \(\mu_\ell \geq \mu - 2\varepsilon/(1 - 1/K)^{m-1} \geq \mu/2,\) since \(\varepsilon \leq \mu/4 \times (1 - 1/K)^{m-1}\).

Then the player tries to estimate \(\mu_\ell\) itself, and uses the ratio \(\mu_\ell/\sigma_\ell\) for estimating \(m\). For this, she tries \(4K\log(6K/\mu\delta)\) times to explore the arm \(\ell\) alone, using a musical chairs type algorithm: she chooses an arm uniformly at random and pulls it \(\log(6K/\mu\delta)/\varepsilon^2\) times. If this arm was arm \(\ell\) and she receives a positive reward, then she has the unbiased estimate for \(\mu_\ell\) that she wants, otherwise she tries again. Using an analysis similar to that of MusicalChairs2, after \(4K\log(6K/\mu\delta)\) iterations, with probability at least \(1 - \delta/3\) all players have explored their arm \(\ell\).

The exploration has \(\log(6K/\mu\delta)/\varepsilon^2\) rounds, and with probability \(1 - \delta/3\), each player obtains an estimate \(\hat{\mu}\) for \(\mu_\ell\) such that \(|\hat{\mu} - \mu_\ell| \leq \varepsilon\).

Therefore, we have \(\mu_\ell \in [\hat{\mu} - \varepsilon, \hat{\mu} + \varepsilon]\) and that \(\mu_\ell(1 - 1/K)^{m-1} \in [\sigma_\ell - \varepsilon, \sigma_\ell + \varepsilon]\). Given the two intervals, we want to recover \(m\). Since \(\varepsilon < \mu/4 \times (1 - 1/K)^{m-1} \times (1 - 1/K)^{-2/5} \), we have

\[ \frac{\hat{\mu} + \varepsilon}{\hat{\mu} - \varepsilon} \leq \frac{\sigma_\ell + \varepsilon}{\sigma_\ell - \varepsilon} \leq \frac{\mu_\ell(1 - 1/K)^{m-1} + 2\varepsilon}{\mu_\ell(1 - 1/K)^{m-1} - 2\varepsilon} \leq \frac{\mu(1 - 1/K)^{m-1} + 2\varepsilon}{\mu(1 - 1/K)^{m-1} - 2\varepsilon} < (1 - 1/K)^{-2/5}, \]

hence Lemma 4.2 below shows that \(m\) can be recovered uniquely.

**Lemma 4.2.** Let \(a,b,c,d,p > 0\). Consider intervals \([a,b]\) and \([c,d]\) with \(\max\{b/a,d/c\} \leq p^{2/5}\), and suppose there exists \(x \in [a,b], y \in [c,d]\) such that \(xp^2 = y\) for some integer \(z\). Then there exists a unique integer \(n\) such that \([ap^n,bp^n]\) ∩ \([c,d]\) ≠ \(\emptyset\).

**Proof.** The existence of such an \(n\) follows from existence of \(x\) and \(y\) and that \(xp^2 = y\) for some integer \(z\). For the uniqueness, note that we have \([ap^n,bp^n]\) ∩ \([c,d]\) ≠ \(\emptyset\).
if and only if \([\log a/\log p + n, \log b/\log p + n] \cap [\log c/\log p, \log d/\log p]\) ≠ \(\emptyset\). Now note that the interval \([\log c/\log p, \log d/\log p]\) has length \(2/5\). Each interval \(I_n = [\log a/\log p + n, \log b/\log p + n]\) also has length \(2/5\), hence \(I_n\) and \(I_{n+1}\) are at least \(3/5\) apart from each other, hence \([\log c/\log p, \log d/\log p]\) cannot intersect both of them!

To bound the number of rounds of the algorithm, note that

\[
(1 - 1/K)^{-2/5} - 1 = \left(1 + \frac{1}{K - 1}\right)^{2/5} - 1 \geq 1 + \frac{2/5}{K - 1} - 1 = \frac{2}{5(K - 1)} > 2/5,
\]

thus \(\varepsilon \geq \mu/120K\). So the number of rounds of the algorithm is \(8K \log(K^2/9\delta)/\varepsilon^2 + 4K \log(6K/\mu\delta) \log(6/\delta)/\varepsilon^2 = O(K^3 \log^2(K/\mu\delta)/\mu^2)\), as required.

Algorithm 3: algorithm for estimating the number of players \(m\)

| Input: number of arms \(K\), lower bound \(\mu\) on \(\mu_1\), failure probability \(\delta\) |
| Output: number of players \(m\) |
| \(\varepsilon \leftarrow \mu((1 - 1/K)^{-2/5} - 1)/48\) |
| \(\text{for } 8K \log(K^2/9\delta)/\varepsilon^2 \text{ rounds do} \) |
| \(\text{pull a uniformly random arm } j\) |
| \(\sigma_j \leftarrow \text{average reward received from arm } j\) |
| \(\text{end}\) |
| \(\ell \leftarrow \text{arg max}_j \sigma_j\) |
| \(\text{for } 4K \log(6K/\mu\delta) \text{ iterations do} \) |
| \(\text{Pick arm } j \text{ uniformly at random}\) |
| \(\text{Pull } j \text{ for } \log(6/\delta)/\varepsilon^2 \text{ times and let } \hat{\mu} \leftarrow \text{average reward received}\) |
| \(\text{if } j = \ell \text{ and } \hat{\mu} > 0 \text{ then output the unique } m \text{ satisfying}\) |
| \([\hat{\mu} - \varepsilon](1 - 1/K)^{m-1}, (\hat{\mu} + \varepsilon)(1 - 1/K)^{m-1}] \cap [\sigma_{\ell} - \varepsilon, \sigma_{\ell} + \varepsilon] \neq \emptyset\) |
| \(\text{end}\) |

5 Proof of Theorem 1.3

First, the player pulls arms uniformly at random, and maintains an estimate for the mean of each arm. An argument similar to that of Lemma 2.1 gives that, after \(512K \log(6mK/\delta)/\varepsilon^2\) rounds, with probability at least \(1 - \delta/2m\) all estimated means are within \(\varepsilon/2\) of the actual means. By a union bound over all players, this is true uniformly over all players with probability at least \(1 - \delta/2\).

The player then sorts the \(\hat{\mu}_i\) as \(\hat{\mu}_i_1 \geq \cdots \geq \hat{\mu}_{i_K}\). Then for \(j = 1, \ldots, K\), she plays MusicalChairs2 on \(\{i_j\}\) (in this order) for \(4K \log(2mK/\delta)/\varepsilon\) many rounds. If during
any of these subroutines she occupies an arm, she chooses that action. Otherwise, she chooses the dummy action 0. By Corollary 3.2, all the MusicalChairs2 subroutines for all players are successful with probability $1 - \delta/2$. The pseudocode is given as Algorithm 4.

Algorithm 4: algorithm for reaching an $\varepsilon$-approximate Nash Equilibrium in an anti-coordination game

**Input:** number of players $m$, number of arms $K$, accuracy $\varepsilon$, failure probability $\delta$

**Output:** action $\ell$

1. $\hat{\mu}_i \leftarrow 0$ for all $i \in [K]$
2. for $512K \log(6mK/\delta)/\varepsilon^2$ rounds do
   3. pull a uniformly random arm $j$
   4. $\hat{\mu}_j \leftarrow$ average reward received from arm $j$, divided by $(1 - 1/K)^{m-1}$
   5. end
3. Sort the $\hat{\mu}$ vector as $\hat{\mu}_{i_1} \geq \cdots \geq \hat{\mu}_{i_K}$
4. $\ell \leftarrow 0$
5. for $j = 1$ to $K$ do
   6. $\ell \leftarrow$ MusicalChairs2 $([i_j], 4K \log(2mK/\delta)/\varepsilon)$
   7. if $\ell \neq 0$ then pull arm $\ell$ for the remaining $K - j$ rounds
   8. end
9. Output $\ell$

We now show that the resulting assignment is an $\varepsilon$-Nash Equilibrium. Fix any player $p$, and first suppose that she has output a non-dummy action $i_j$. This means all actions $i_1, i_2, \ldots, i_{j-1}$ were either occupied by other players, or had mean < $\varepsilon$, or both. On the other hand, since the estimated means are within $\varepsilon/2$ of the actual means, for any $s \in \{i_1, i_2, \ldots, i_{j-1}\}$ we have $\mu^p_s \leq \hat{\mu}_{i_j} + (\mu^p_s - \hat{\mu}_s) + (\hat{\mu}_{i_j} - \mu^p_{i_j}) \leq \mu^p_{i_j} + \varepsilon$, hence the player cannot increase her outcome by more than $\varepsilon$ by switching to action $s$. Finally, if the player has chosen the dummy action 0, that would mean for each $j \in [K]$, either action $i_j$ is occupied or $\mu^p_{i_j} \leq \varepsilon$, or both. Thus, there is no unoccupied action $s$ with $\mu^p_s > \varepsilon$, so again the player cannot increase her outcome by more than $\varepsilon$ by switching.

The total number of rounds is $512K \log(6mK/\delta)/\varepsilon^2 + K \times 4K \log(2mK/\delta)/\varepsilon = O(K \log(K/\delta)/\varepsilon^2 + K^2 \log(K/\delta)/\varepsilon)$, as required.
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References

Animashree Anandkumar, Nithin Michael, Ao Kevin Tang, and Ananthram Swami. Distributed algorithms for learning and cognitive medium access with logarithmic regret. *IEEE Journal on Selected Areas in Communications*, 29(4):731–745, 2011.

Venkatachalam Anantharam, Pravin Varaiya, and Jean Walrand. Asymptotically efficient allocation rules for the multiarmed bandit problem with multiple plays—part I: IID rewards. *IEEE Transactions on Automatic Control*, 32(11):968–976, 1987.

Orly Avner and Shie Mannor. Concurrent bandits and cognitive radio networks. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, pages 66–81. Springer, 2014.

Lilian Besson and Emilie Kaufmann. Multi-player bandits revisited. In Firdaus Janoos, Mehryar Mohri, and Karthik Sridharan, editors, *Proceedings of Algorithmic Learning Theory*, volume 83 of *Proceedings of Machine Learning Research*, pages 56–92. PMLR, 07–09 Apr 2018. URL [http://proceedings.mlr.press/v83/besson18a.html](http://proceedings.mlr.press/v83/besson18a.html).

Rémi Bonnefoi, Lilian Besson, Christophe Moy, Emilie Kaufmann, and Jacques Palicot. Multi-armed bandit learning in IoT networks: Learning helps even in non-stationary settings. In Paulo Marques 0002, Ayman Radwan, Shahid Mumtaz, Dominique Noguet, Jonathan Rodriguez, and Michael Gundlach, editors, *CrownCom*, volume 228 of *Lecture Notes of the Institute for Computer Sciences, Social Informatics and Telecommunications Engineering*, pages 173–185. Springer, 2017. ISBN 978-3-319-76206-7; 978-3-319-76207-4. doi: 10.1007/978-3-319-76207-4. URL [https://doi.org/10.1007/978-3-319-76207-4](https://doi.org/10.1007/978-3-319-76207-4).
Sébastien Bubeck and Nicolo Cesa-Bianchi. Regret analysis of stochastic and non-stochastic multi-armed bandit problems. *Foundations and Trends® in Machine Learning*, 5(1):1–122, 2012.

Junpei Komiyama, Junya Honda, and Hiroshi Nakagawa. Optimal regret analysis of Thompson sampling in stochastic multi-armed bandit problem with multiple plays. In Francis Bach and David Blei, editors, *Proceedings of the 32nd International Conference on Machine Learning*, volume 37 of *Proceedings of Machine Learning Research*, pages 1152–1161, Lille, France, 07–09 Jul 2015. PMLR. URL http://proceedings.mlr.press/v37/komiyama15.html.

Keqin Liu and Qing Zhao. Distributed learning in multi-armed bandit with multiple players. *IEEE Trans. Signal Process.*, 58(11):5667–5681, 2010. ISSN 1053-587X. doi: 10.1109/TSP.2010.2062509. URL https://doi.org/10.1109/TSP.2010.2062509.

Colin McDiarmid. Concentration. In *Probabilistic methods for algorithmic discrete mathematics*, volume 16 of *Algorithms Combin.*, pages 195–248. Springer, Berlin, 1998. doi: 10.1007/978-3-662-12788-9_6. URL https://doi.org/10.1007/978-3-662-12788-9_6.

Jonathan Rosenski, Ohad Shamir, and Liran Szlak. Multi-player bandits–a musical chairs approach. In *International Conference on Machine Learning*, pages 155–163, 2016.

Roman Vershynin. *High-Dimensional Probability*. Cambridge University Press, 2018. To appear, available at https://www.math.uci.edu/~rvershyn/papers/HDP-book/HDP-book.pdf.