LETTER

Logarithmic observables in critical percolation

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Abstract. Although it has long been known that the proper quantum field theory description of critical percolation involves a logarithmic conformal field theory (LCFT), no direct consequence of this has been observed so far. Representing critical bond percolation as the $Q \to 1$ limit of the $Q$-state Potts model, and analyzing the underlying $S_Q$ symmetry of the Potts spins, we identify a class of simple observables whose two-point functions scale logarithmically for $Q \to 1$. The logarithm originates from the mixing of the energy operator with a logarithmic partner that we identify as the field that creates two propagating clusters. In $d = 2$ dimensions this agrees with general LCFT results, and in particular the universal prefactor of the logarithm can be computed exactly. We confirm its numerical value by carrying out extensive Monte Carlo simulations.

Keywords: conformal field theory, correlation functions, loop models and polymers, classical Monte Carlo simulations

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1. Introduction

The analysis of two-dimensional critical geometrical problems such as percolation or self-avoiding walks (SAW) in terms of conformal field theory (CFT) involves features more complicated than those appearing in minimal (e.g., Ising or three-state Potts) models. One might think that these complications are due to the non-local nature of the interesting geometric observables, such as connectivities of clusters and loops. This viewpoint is however somewhat misleading, since genuine non-locality would break the invariance under conformal transformations. In fact the above problems definitely are conformally invariant and their non-locality is only apparent: it can be traded for non-unitarity, by reformulating these models in terms of vertex models or supersymmetric spin chains. The origin of the difficulty in building a CFT description of percolation or SAWs is precisely this non-unitarity, which turns out to have deeper consequences than that of certain minimal models, such as the Lee–Yang edge singularity.

One of these consequences is that the operator algebra underlying the lattice description—or the Virasoro algebra that emerges in the continuum limit—of percolation or SAW problems involves representations that are not fully reducible. This leads to indecomposability and, ultimately, to the appearance of logarithms in correlation functions. This feature follows from the fact that the models cited above have central charge $c = 0$, and hence it is necessary to go ‘outside’ their minimal Kac table (made up of just the identity operator) to describe non-trivial physical observables. This extension leads to logarithmic CFTs (LCFTs) whose study has attracted considerable interest over the last few years, as their potential role in condensed matter as well as string theory applications has become more evident.

The best known property of LCFTs at $c = 0$ is the existence of a logarithmic partner to the stress–energy tensor, whose presence is necessary (under some circumstances at least) for avoiding the ‘$c = 0$ catastrophe’ [1]. Associated with this partner is a universal number, the so-called ‘$b$ number’ or logarithmic coupling. It was suggested that $b$ is a sort
of effective central charge \( c = 0 \) theories \([3]\). While there has been a lot of work on the determination of the \( b \) number and its relation with abstract module properties of the Virasoro algebra \([4]\), it is only very recently that methods have been devised to measure it in a numerical experiment. These methods \([5, 6]\) are however rather indirect—in particular, they require a proper quantization scheme—and thus to this day no simple physical procedure for determining \( b \) in a real experiment has been proposed.

The indecomposability that leads to the existence of a logarithmic partner to the stress–energy tensor can be traced back to the peculiarities of the \( Q \to 1 \) (resp. \( n \to 0 \)) limit that relate percolation (resp. SAWs) to the \( Q \)-state Potts (resp. \( O(n) \) spin) model. It is then possible to predict other striking consequences \([7]–[9]\) that should be observed at \( c = 0 \), such as logarithmic terms in certain correlation functions. These logarithms arise from degeneracies, where several operators that are distinguishable at \( Q \) (resp. \( n \)) generically ‘mix’ in the limit \( Q \to 1 \) (resp. \( n \to 0 \)). This mixing is the physical phenomenon resulting from the indecomposability in the formal algebraic description. Nevertheless, a clear physical interpretation of such logarithmic correlators is often difficult to find, and also hard to access numerically since the logarithm multiplies a much stronger power law singularity. Direct consequences of the logarithmic nature of the underlying CFT have thus, to this day, remained unobserved.

The purpose of this letter is to present a new example of a physical observable in the \( Q \)-state Potts model which, on the one hand, exhibits a pure logarithmic dependence (with no multiplying power law) in the \( Q \to 1 \) percolation limit, whose universal prefactor we compute analytically in \( d = 2 \) dimensions, and, on the other hand, has a sufficiently simple geometric formulation in terms of percolation clusters to make possible a numerical study. And indeed our thorough Monte Carlo simulations nicely confirm both the logarithmic scaling and our result for the universal prefactor.

2. Percolation, the Potts model, and the continuum limit

We first recall the well-known reformulation of bond percolation as the \( Q \to 1 \) limit of the Potts model. The partition function of the Potts model reads

\[
Z = \sum_{\sigma} \prod_{(ij) \in E} \exp(K \delta_{\sigma_i, \sigma_j}),
\]

where \( K \) is the coupling between spins \( \sigma_i = 1, 2, \ldots, Q \) along the edges \( E \) of some lattice \( G \). The Kronecker symbol \( \delta_{\sigma_i, \sigma_j} \) equals 1 if \( \sigma_i = \sigma_j \), and 0 otherwise. Universal properties depend only on the dimension \( d \), and not on the precise choice of \( G \). Although the main conclusions of this letter should be valid in any dimension, for the sake of simplicity, we shall restrict ourselves to \( d = 2 \) in the following. The generalization of our results in higher dimensions will be discussed in section 6. We therefore suppose that \( G \) is the square lattice. The transition line—which gives rise to a critical theory for \( 0 \leq Q \leq 4 \)—is then given by the self-duality criterion \( e^K = 1 + \sqrt{Q} \).

We can expand \( Z \) by rewriting the local Boltzmann weight as \( \exp(K \delta_{\sigma_i, \sigma_j}) = 1 + (p/(1-p))\delta_{\sigma_i, \sigma_j} \), with \( p = 1 - e^{-K} \). The set of edges \((ij) \in E\) for which we have the term \((p/1-p)\delta_{\sigma_i, \sigma_j}\) are called ‘occupied bonds’. They define a graph \( H \), whose connected components are known as Fortuin–Kasteleyn (FK) clusters \([10]\). Since spins belonging to
the same FK cluster are aligned, we can perform the sum over \( \{\sigma_i\} \) in (1) to obtain

\[
Z \propto \sum_H Q^{|\text{clusters}(H)|} p^{|\text{edges}(H)|} (1 - p)^{|E| - |\text{edges}(H)|}.
\]

(2)

In this formulation, the number of colors \( Q \) can be thought of as a real parameter, and the limit \( Q \to 1 \) yields a sum over bond percolation configurations, with a probability \( p \) per occupied bond. This model is critical for \( p = p_c = \frac{1}{2} \), and although the partition function \( Z = 1 \) is trivial, the correlation functions of the model capture the salient geometrical properties of critical percolation clusters.

The continuum limit of the critical Potts model is described by a CFT. To establish this standard result, one first notices that the (outer and inner) hulls of the FK clusters constitutes a gas of loops. This in turn defines a height model (the loops being contour lines of the height) which can be argued to renormalize towards a Coulomb gas (CG), that is, a compactified free boson with Lagrangian density \( \mathcal{L} = (g/2\pi)(\nabla \phi)^2 \), along with additional ‘electric charges’ at infinity. The stiffness \( g \in [2, 4] \) is given by \( Q = 2 + 2 \cos(\pi g/2) \), so percolation has \( g = \frac{3}{2} \). Over the past thirty years, this mapping has allowed physicists to compute many interesting geometrical properties for percolation (and for the Potts model in general), including crossing probabilities and critical exponents. More recently, the continuum limit has been studied by mathematicians under the name SLE\(_{\kappa}\), where \( \kappa = 16/g \).

3. The symmetric group \( S_Q \) and operators in the Potts model

To understand how logarithms appear in percolation, the key idea is the study of the symmetric group \( S_Q \) in the (formal) limit \( Q \to 1 \) (see also [7]). We expect scaling fields of the underlying CFT to transform as irreducible representations (irreps) under the \( S_Q \) symmetry of the Potts model.

Consider first observables acting on a single spin \( \mathcal{O}(\sigma_i) \), for \( Q \) integer. Obviously, any such operator can be decomposed onto a basis of \( Q \) generators as \( \mathcal{O}(\sigma_i) = \sum_{a=1}^Q c_a \delta_{\sigma_i,a} \). The action of \( S_Q \) defines a representation of dimension \( Q \), which however is reducible. Indeed, \( \delta_{\sigma_i,a} \) can be decomposed onto an invariant \( 1 = \sum_{a=1}^Q \delta_{\sigma_i,a} \), and an irrep \( \varphi_a = \delta_{\sigma_i,a} - 1/Q \) with \( \sum_{a=1}^Q \varphi_a = 0 \) of dimension \( Q - 1 \). Obviously 1 corresponds to the identity operator, while \( \varphi_a \) is the magnetization, or order parameter operator, of the Potts model. The two-point function \( \langle \varphi_a(\sigma_i) \varphi_b(\sigma_j) \rangle \) vanishes if \( i \) and \( j \) belong to different FK clusters (the sums over \( \sigma_i \) and \( \sigma_j \) being independent). At the critical point, \( \langle \varphi_a(\sigma_i) \varphi_b(\sigma_j) \rangle \) decays algebraically as \( r^{-2\Delta_{\varphi}} \), where the (bulk) critical exponent \( \Delta_{\varphi} = (6 - g)(g - 2)/8g \) can be computed within the CG setup [11, 12].

Nothing particular happens when one takes \( Q \to 1 \) in these expressions. However, the limit \( Q \to 0 \) in \( \langle \varphi_a(\sigma_i) \varphi_b(\sigma_j) \rangle \) is ill-defined, and although we will not discuss it in detail here, this is actually responsible for the occurrence of logarithms at the level of the identity operator. The \( Q \to 0 \) field theory of free (symplectic) fermions can be interpreted geometrically in terms of dense polymers, or spanning trees on \( G \), and in this latter context several similar results were obtained using exact combinatorial methods [13]–[15]. This theory is also related to the logarithmic form of the Gaussian propagator for \( d = 2 \) and to the asymptotic behavior of the equivalent resistance in an infinite network of resistors via the Kirchhoff theorem [16].
In the remainder of the letter we focus on the $Q \to 1$ percolation case for which one needs to consider observables acting on two nearest-neighbor spins $O(r_i) \equiv O(\sigma_i, \sigma_{i+1})$ in order to recover logarithms. We impose the constraint $\sigma_i \neq \sigma_{i+1}$, whence these observables are $Q \times Q$ (symmetric) matrices with zero elements on the diagonal. As before, the starting point is the basis elements $\delta_{\sigma_i,a} \delta_{\sigma_{i+1}, b}$ which should be symmetrized in $\{\sigma_i, \sigma_{i+1}\}$ in order to obtain symmetric matrices. The decomposition of this representation of $S_Q$ is straightforward and we find

$$\frac{Q(Q-1)}{2} = (1) \oplus (Q-1) \oplus \frac{Q(Q-3)}{2},$$

where we denoted the representations by their dimensions. The explicit expression for the generators of these representations reads

$$E(\sigma_i, \sigma_{i+1}) = \delta_{\sigma_i, \neq \sigma_{i+1}},$$

$$\phi_a(\sigma_i, \sigma_{i+1}) = \delta_{\sigma_i, \neq \sigma_{i+1}}(\varphi_a(\sigma_i) + \varphi_a(\sigma_{i+1})), \quad (4a)$$

$$\hat{\psi}_{ab}(\sigma_i, \sigma_{i+1}) = \delta_{a \neq b} \left( \delta_{\sigma_i,a} \delta_{\sigma_{i+1}, b} + \delta_{\sigma_i,b} \delta_{\sigma_{i+1},a} - \frac{1}{Q-2} (\phi_a + \phi_b) - \frac{2}{Q(Q-1)} E \right), \quad (4c)$$

where we recall that $\varphi_a(\sigma_i) = \delta_{\sigma_i,a} - 1/Q$. The ‘scalar’ $E$ is obviously a one-dimensional irrep. The operator $\phi_a$ satisfies $\sum_a \phi_a = 0$, and transforms like the ‘vector’ $\varphi_a$. The ‘tensor’ $\hat{\psi}_{ab} = \hat{\psi}_{ba}$ satisfies $\sum_{a(\neq b)} \hat{\psi}_{ab} = 0$ for any $b$. This last irrep thus has $Q(Q-1)/2 - Q = Q(Q-3)/2$ independent elements, as expected from (3).

The fact that we obtained three independent operators acting on two spins can be understood physically in terms of the fusion of one-spin operators (see also [17]). As $\sigma_i \neq \sigma_j$, all three operators act on two distinct FK clusters. The operator with highest symmetry, $\hat{\psi}$, will then impose that the two clusters propagate until they encounter another $\psi$ operator. In section 5 we further develop this geometric interpretation.

4. The percolation limit and Jordan cell for the energy operator

We can already see from (4) that $\hat{\psi}_{ab}$ becomes ill-defined in the (formal) $Q \to 1$ limit. We shall now see that a well-defined limit is obtained by mixing $\hat{\psi}_{ab}$ with $E$. Let us first study these operators from a quantum field theory point of view. The energy operator is given by $\varrho(r_i) \equiv E(\sigma_i, \sigma_{i+1}) - \langle E \rangle$, where we subtracted the bulk expectation value of $E$ so as to obtain a well-defined scaling field. This subtraction does not change the representation theoretic considerations, so $\varrho(r_i)$ remains an irrep. This field corresponds to the thermal perturbation of the Potts model, and its bulk scaling dimensions is $\Delta_\varrho = 6/g - 1$ [18, 12]. In the CFT with generic real $Q$, we expect the following form for the two-point function:

$$\langle \varrho(r) \varrho(0) \rangle = \tilde{A}(Q)(Q-1)r^{-2\Delta_\varrho(Q)},$$

where $\tilde{A}(Q)$ is a regular function of $Q$, with a finite non-zero limit $\tilde{A}(1)$ for $Q \to 1$. The reason why $\langle \varrho(r) \varrho(0) \rangle$ should vanish at $Q = 1$ is very natural from a lattice point of view, and was recently argued in the context of bulk logarithmic CFT [19].

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The above fusion considerations imply that the ‘tensor’ operator \( \hat{\psi}_{ab} \) can be identified, with the so-called four-leg watermelon operator [12] (the two propagating clusters correspond to four propagating hulls). Its bulk scaling dimension \( \Delta_\hat{\psi} = (4+g)(3g-4)/8g \) follows from a CG computation [20]. We thus deduce the form of the two-point function

\[
\langle \hat{\psi}_{ab}(r) \hat{\psi}_{cd}(0) \rangle = \frac{2A(Q)}{Q^2} \left( \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \frac{1}{Q-2} (\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd}) \right) + \frac{2}{(Q-1)(Q-2)} r^{-2\Delta_\hat{\psi}(Q)},
\]

where \( A(Q) \) is again a regular function of \( Q \) when \( Q \to 1 \), and the factor \( 2/Q^2 \) is purely conventional. However, the Kronecker symbol combination is completely fixed by (4c).

In the formal limit \( Q \to 1 \) the two-point function (6) diverges. To cure this, we introduce a new field

\[
\tilde{\psi}_{ab}(r) = \hat{\psi}_{ab}(r) + \frac{2}{Q(Q-1)} \varepsilon(r), \quad a \neq b.
\]

Its two-point function is easily computed and in order to have a finite \( Q \to 1 \) limit, we must require \( A(1) = \tilde{A}(1) \), and that \( \Delta_\varepsilon = \Delta_\hat{\psi} \) at \( Q = 1 \). The latter condition is indeed satisfied, since for \( g = \frac{8}{3} \) the CG results read \( \Delta_\varepsilon = \Delta_\hat{\psi} = \frac{5}{4} \). Assuming also the former, the two-point function of \( \tilde{\psi}_{ab} \) has a well-defined \( Q \to 1 \) limit:

\[
\langle \tilde{\psi}_{ab}(r) \tilde{\psi}_{cd}(0) \rangle = 2A(1)r^{-5/2} \left[ (\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \frac{4\sqrt{3}}{\pi} \log r \right],
\]

where we have used

\[
\lim_{Q \to 1} \frac{\Delta_\hat{\psi} - \Delta_\varepsilon}{Q-1} = \frac{\sqrt{3}}{\pi}.
\]

We have thus identified a logarithmic two-point function (8) as a result of the mixing (7) of the energy operator \( \varepsilon \) and the four-leg operator \( \hat{\psi} \) of the Potts model. This mixing is consistent with recent algebraic results in bulk LCFTs [19, 21], but was obtained here from very simple physical arguments based on the ‘\( S_{Q=1} \) symmetry’ of the theory.

Logarithms in LCFTs can be associated with the non-diagonalizability of the scale transformation generator—i.e., the Hamiltonian in the usual radial quantization—of the theory. To show that this logarithm corresponds to a Jordan cell, we now study the change of the field \( \tilde{\psi}_{ab} \) under a scale transformation \( r \to \Lambda r \). Recall [12] that \( \varepsilon \) and \( \hat{\psi}_{ab} \) are both primary operators for generic \( Q \) and thus transform as

\[
\varepsilon(\Lambda r) = \Lambda^{-\Delta_\varepsilon} \varepsilon(r), \quad (10a)
\]

\[
\hat{\psi}_{ab}(\Lambda r) = \Lambda^{-\Delta_\hat{\psi}} \hat{\psi}_{ab}(r). \quad (10b)
\]

Using this, one can readily show that at \( Q = 1 \),

\[
\tilde{\psi}_{ab}(\Lambda r) = \Lambda^{-5/4} \left( \hat{\psi}_{ab}(r) + \frac{2\sqrt{3}}{\pi} \log \Lambda \varepsilon(r) \right).
\]

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Figure 1. Percolation configurations contributing to (a) \( P_0(r) \), (b) \( P_1(r) \) (one cluster propagating a distance \( r = |r_1 - r_2| \)), (c) \( P_2(r) \) (two propagating clusters).

The field \( \tilde{\psi}_{ab} \) is therefore mixed with the energy operator \( \varepsilon(r) \) after a scale transformation. We stress that the appearance of logarithms in the two-point function (8) of the field \( \tilde{\psi} \) is fully compatible with scale and conformal invariance of the critical theory. The point is that some operators are not pure scaling operators, and get mixed with others under a scale transformation. In other words, the scale transformation generator (or Hamiltonian) is non-diagonalizable, with a rank-2 Jordan cell mixing the two fields \( \tilde{\psi}_{ab} \) and \( \varepsilon \).

5. Geometrical interpretation

We now show that (8) has a very nice physical meaning in terms of geometrical observables in the percolation problem. Let \( i_1 \) and \( i_1 + 1 \) be two nearest-neighbor points, separated from two other nearest-neighbor points \( i_2 \) and \( i_2 + 1 \) by a distance \( r = |r_1 - r_2| \). To be consistent with the bulk calculation of section 4, we consider percolation defined on a torus (i.e., with doubly periodic boundary conditions) and we assume that \( r \) is much smaller than either period. We define \( \mathbb{P}_x \) as the (position independent) probability that two neighboring points belong to different FK clusters. Let \( \mathbb{P}_0(r) \) be the probability that \( \{i_1, i_1 + 1, i_2, i_2 + 1\} \) belong to four different FK clusters. Note that \( \mathbb{P}_0(r) \) is an increasing function of \( r \), with limit \( (\mathbb{P}_x)^2 \) for \( r \rightarrow \infty \). Further let \( \mathbb{P}_1(r) \) be the probability that \( \{i_1, i_1 + 1, i_2, i_2 + 1\} \) belong to three different FK clusters, of which one ‘propagating’ cluster contains one point from \( \{i_1, i_1 + 1\} \) and one point from \( \{i_2, i_2 + 1\} \)—there are four ways to make this choice. Finally, let \( \mathbb{P}_2(r) \) be the probability that \( \{i_1, i_1 + 1, i_2, i_2 + 1\} \) belong to two different FK clusters, i.e., with two distinct clusters each containing one point from \( \{i_1, i_1 + 1\} \) and one point from \( \{i_2, i_2 + 1\} \)—this can be done in two ways. Configurations contributing to these probabilities are shown schematically in figure 1.

Using these probabilities, one can interpret geometrically the Potts operators previously introduced. For example, it is straightforward to show that \( \langle E \rangle = ((Q - 1)/Q) \mathbb{P}_x \). Recall that the energy operator \( \varepsilon \) was constructed by subtracting this expectation value from the operator \( E \). A careful analysis shows that the two-point
correlator of $\hat{\psi}_{ab}$ is proportional to $\mathbb{P}_2(r)$:
\[
(\hat{\psi}_{ab}(\sigma_i, \sigma_i+1) \hat{\psi}_{cd}(\sigma_i, \sigma_i+1)) = \frac{2}{Q^2} \left( \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc} - \frac{1}{Q-2} (\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd}) \right) + \frac{2}{(Q-1)(Q-2)} \mathbb{P}_2(r).
\]  
\[
(12)
\]
We infer from (6) that $\mathbb{P}_2(r) \sim A(Q) r^{-2\Delta_\psi(Q)}$. Other correlation functions follow from the lattice description in the same way. A direct computation shows that $\langle \hat{\psi}_{ab} \rangle = \langle \phi_a \rangle = 0$. This also follows from the representation theory of $S_Q$, which moreover implies the vanishing of the ‘crossed’ correlation functions: $\langle \varepsilon \hat{\psi}_{ab} \rangle = \langle \varepsilon \phi_a \rangle = \langle \hat{\psi}_{ab} \phi_c \rangle = 0$. The vanishing of one-point functions and crossed two-point functions is consistent with fundamental CFT results.

To analyze the $Q \to 1$ limit from a CFT perspective, we studied the correlation functions of the field $\tilde{\psi}_{ab}$ defined in (7). One can repeat the very same steps from the lattice perspective. It is convenient to write $\tilde{\psi}_{ab}(r_i) \equiv \tilde{\psi}_{ab}(\sigma_i, \sigma_{i+1}) + (2/Q(Q-1)) \varepsilon(\sigma_i, \sigma_{i+1}) = \psi_{ab}(\sigma_i, \sigma_{i+1}) - \langle \psi \rangle$, where we have introduced
\[
\psi_{ab}(\sigma_i, \sigma_{i+1}) = \delta_{\sigma_i,a} \delta_{a} \delta_{\sigma_{i+1},b} + \delta_{\sigma_i,\delta} \delta_{\sigma_{i+1},a} - \frac{1}{Q-2} (\phi_a(\sigma_i, \sigma_{i+1}) + \phi_b(\sigma_i, \sigma_{i+1})).
\]  
\[
(13)
\]
Note that $\psi_{ab}$ is not a scaling field (since $\langle \psi \rangle = (2/Q^2) \mathbb{P}_\neq \neq 0$), whereas $\tilde{\psi}_{ab}$ is. One can show that the two-point function of $\tilde{\psi}_{ab}$ reads
\[
\langle \psi_{ab}(r_1) \psi_{cd}(r_2) \rangle = \frac{4}{Q^4} \left( \mathbb{P}_0(r) + \mathbb{P}_1(r) \right) + \mathbb{P}_2(r) \times \frac{1}{Q^2} \left[ \frac{8}{Q(Q-2)} + \frac{2}{2-Q} (\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd}) + 2(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \right],
\]  
\[
(14)
\]
whereas that of the corresponding scaling field $\tilde{\psi}_{ab}$ is, in the limit $Q = 1$,
\[
\langle \tilde{\psi}_{ab}(r_1) \tilde{\psi}_{cd}(r_2) \rangle = 2 \left( \delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} \right) \mathbb{P}_2(r)
\]  
\[+ 4 \left[ \mathbb{P}_0(r) + \mathbb{P}_1(r) - 2 \mathbb{P}_2(r) - \mathbb{P}_\neq^2 \right].
\]  
\[
(15)
\]
Comparing with (8) we deduce that $\mathbb{P}_2(r) \sim A(1) r^{-5/2}$, as was of course expected from its relation to the four-leg operator. Meanwhile, the logarithmic term in (8) can be identified with
\[
\mathbb{P}_0(r) + \mathbb{P}_1(r) - \mathbb{P}_\neq^2 \sim A(1) \left( \theta + \frac{2\sqrt{3}}{\pi} \log r \right) r^{-5/2},
\]  
\[
(16)
\]
where we have added a subdominant non-universal (i.e., lattice dependent) term $\theta$. Finally, the following combination:
\[
F(r) \equiv \frac{\mathbb{P}_0(r) + \mathbb{P}_1(r) - \mathbb{P}_\neq^2}{\mathbb{P}_2(r)} \sim \theta + \frac{2\sqrt{3}}{\pi} \log r,
\]  
\[
(17)
\]
cancels out the dominant power law ($r^{-5/2}$), leaving a pure logarithmic scaling which should be observable in numerical simulations (see below). The number $2\sqrt{3}/\pi \approx 1.1026$ is a universal constant that can be traced back to (9). Although the combination \(17\)
may look slightly complicated, it is important to keep in mind that the logarithmic term that we are after resides in the disconnected part $P_0(r)$—a similar observation holds true for LCFTs with other values of $Q$, such as $Q \to 0$.

We have checked the validity of (17) by performing extensive Monte Carlo simulations on square lattices of various sizes ranging from $150 \times 150$ to $300 \times 300$, with doubly periodic boundary conditions. We checked that different pseudo-random number generators—including a Mersenne twister algorithm [22]—led to consistent results. Statistics were obtained on $\sim 10^3$ independent runs of $10^7$ percolation configurations each. Results are shown in figure 2, and are in good agreement with (17). Careful extrapolations removing successively the first few short-distance points yields a slope $1.15 \pm 0.05$ in good agreement with our prediction $2\sqrt{3}/\pi \simeq 1.1026$.

Note that although all the calculations of this letter were made for the bulk, the derivation in the boundary case presents only minor differences. In this case the scaling dimension of the energy operator should be replaced by $\Delta_\epsilon = 2$, since $\epsilon$ becomes degenerate with the stress–energy tensor $T$ [23]. Similarly, the four-leg watermelon exponent should be changed [24] to $\Delta_{\tilde{\psi}} = 3g/2 - 2$, and the operator $\tilde{\psi}_{ab}$ would be proportional to the well-known logarithmic partner $t(z)$ of the stress–energy tensor introduced by Gurarie [2] and Gurarie and Ludwig [3, 25] in their work on CFTs with central charge $c = 0$. However, it is easy to see that the limit (9) remains unchanged, so we expect (17) to hold true also if the points lie at a boundary. It would be interesting to check this numerically as well.

6. Conclusion

We have found a simple geometrical observable in percolation that provides a lattice version of a logarithmic two-point function in a $c = 0$ (L)CFT. The method that we used is very similar to what was done by Cardy for disordered systems and for the $\mathcal{O}(n)$ model [7], and is far more general in that respect than the specific logarithmic solutions [13]–[15] found in the case of the free-fermion dense polymers ($Q = 0$). Meanwhile, our result (17) for percolation allows for a direct numerical verification, and the prefactor in front of the
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logarithmic term turns out to be universal. It should be noticed that this coefficient is actually closely related to what is known as the logarithmic coupling or indecomposability parameter in the context of LCFT [3, 25, 4, 6]. We also note that, remarkably, in all the other examples that we have studied, logarithmic terms tend to appear in disconnected observables such as \( P_0(r) \). This is similar to results for disordered systems [7], and we will get back to this issue elsewhere.

The main parts of our derivation—notably the representation theory of section 3—are not restricted to the \( d = 2 \)-dimensional case. Indeed, we expect \( \Delta \hat{\psi} = \Delta \varepsilon \) at \( Q = 1 \) also for \( d > 2 \),\(^5\) and only the derivative (9) will change. Accordingly we expect (17) to remain correct for \( d > 2 \), albeit with a different universal prefactor that could be, in principle, computed in a \( \epsilon = 6 - d \) expansion.

Our results are not restricted to percolation. In particular, studying the \( Q \to 2 \) limit should also yield logarithmic observables, involving geometrical properties of Ising spin clusters. Higher-rank correlation functions and observables acting on more spins—including the generalization of \( 2n \)-leg watermelon operators to \( d > 2 \)—can be worked out along the same lines. There are indications that matching these more physical observations with formal algebraic developments should lead to further progress in our understanding of LCFTs.

In conclusion, it is important to stress that logarithmic terms such as those that we have identified would not be present for generic \( Q \), and occur solely because of the special degeneracies present at \( Q = 1 \). This is of course quite different from the case of logarithmic dependences in other non-local quantities—see e.g. [27, 28]—which are obtained as derivatives of correlation functions with respect to the Boltzmann weights (such as \( Q \)).

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\(^5\) This implies that the fractal dimension \( d_{RB} \) of the so-called ‘red bonds’ (also called ‘cutting bonds’) is related to the thermal exponent \( \nu \) via \( d_{RB} = \nu^{-1} \). This is indeed a well-known percolation result [26].

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