RIGIDITY OF BACH-FLAT GRADIENT SCHOUTEN SOLITONS

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Abstract. In this paper we show that a complete Schouten soliton whose Ricci tensor has at most two eigenvalues at each point is rigid. This allows the classification of both shrinking and expanding Bach-flat Schouten solitons for \( n \geq 4 \). When \( n = 3 \) we are able to conclude rigidity under a more general condition, namely when the Bach tensor is divergence free. These results imply rigidity of locally conformally flat Schouten solitons for \( n \geq 3 \).

1. Introduction and main results

A Riemannian manifold \((M^n, g)\) endowed with a smooth function \( f \in C^\infty(M)\) and for which there is a constant \( \lambda \) satisfying the tensorial equation
\[
Ric + \nabla \nabla f = \left( \frac{R}{2(n-1)} + \lambda \right) g
\]
is called a gradient Schouten soliton, where \( f \) is called its potential function. In the equation above \( \nabla \nabla f \) is the Hessian of \( f \), \( R \) is the scalar curvature and \( Ric \) is the Ricci tensor of \( M \). The soliton is called shrinking, steady or expanding, provided \( \lambda \) is positive, zero or negative, respectively. In this case we use the notation \((M^n, g, f, \lambda)\).

Schouten solitons were introduced in [4] by Catino and Mazzieri. In the same paper they introduced the \( \rho \)-Einstein solitons, which are depicted as the Riemannian manifolds for which given \( \rho \in \mathbb{R} \), there are \( f \in C^\infty(M) \), also called potential function, and \( \lambda \in \mathbb{R} \), satisfying
\[
Ric + \nabla \nabla f = (\rho R + \lambda) g.
\]
For \( \rho = 1/2(n-1) \) one recovers (1.1), while for \( \rho = 0 \) one obtains the famous Ricci solitons [9], which have been intensively studied in the recent years. It is important to mention that different values of \( \rho \) may give rise to quite different objects. For instance, if \( \rho \neq 0 \) then the corresponding \( \rho \)-Einstein manifold is rectifiable (see [4] for the definition) and when \( \rho \in \{1/2, 1/2(n-1), 1/n\} \) the corresponding ones which are compact must have constant potential function. Another difference is that, while it is known that for \( \rho \notin \{1/2(n-1), 1/n\} \) the corresponding solitons are analytic, nothing is known in the remaining cases. For the proofs see [4, 5].

\( \rho \)-Einstein solitons are closely related to certain geometric evolution equations, for they are the static formulations of self-similar solutions of such equations [5], whose definition has dynamic nature. It is important to mention that among these

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evolution equations, the one for which (1.1) represents a self-similar solution is still an enigma concerning basic properties such as short time existence [6].

Besides rectifiability [4], other general results on Schouten solitons were obtained by the author in the manuscript [1]. Among other results, the author has shown in [1] that the scalar curvature of such metrics are bounded and have a defined sign. It has also been proved that the potential function controls the growth of the squared norm of its gradient. See Theorem 2.2 in Section 2 below.

Concerning examples of Schouten solitons, the simplest ones are Einstein manifolds. Other examples are obtained as follows.

Example 1.1. Given \( n \geq 3 \), \( k \leq n \) and \( \lambda \in \mathbb{R} \), consider an Einstein manifold \( (N^k, g) \) of dimension \( k \leq n \) and scalar curvature

\[
R_N = \frac{2(n-1)k \lambda}{2(n-1) - k}.
\]

Now, if \( (x, p) \in \mathbb{R}^{n-k} \times N^k \), \( \|x\|^2 \) denotes the Euclidean norm, and

\[
f(x, p) = \frac{1}{2} \left( \frac{R_N}{2(n-1) + \lambda} \right) \|x\|^2,
\]

it follows that \( (\mathbb{R}^{n-k} \times_{\Gamma} N^k, g, f, \lambda) \) is an \( n \) dimensional Schouten soliton, where \( g = \langle , \rangle + g_N \), and \( \Gamma \) acts freely on \( N \) and by orthogonal transformations on \( \mathbb{R}^{n-k} \).

If \( k = 0 \), \( (\mathbb{R}^n, g, f, \lambda) \) will be addressed as the Gaussian soliton.

A Riemannian manifold is called rigid if it is isometric to one of those described in Example 1.1. The following classes of Schouten solitons were proven to be rigid in [4]: compact; complete noncompact with \( \lambda = 0 \) and \( n \geq 3 \); complete noncompact with \( \lambda > 0 \) and \( n = 3 \); complete warped products \( B^1 \times_h N^{n-1} \) (including those with rotational symmetry), where \( B^1 \) is a one dimensional manifold, \( h : B^1 \to \mathbb{R} \) is a positive smooth function and \( N^{n-1} \) a space form.

In this paper we will add to the list above Schouten solitons whose Ricci tensors have at most two principal curvatures at each point. Namely,

Theorem 1.2. Let \( (M^n, g, f, \lambda) \), \( \lambda \neq 0 \), be a complete Schouten soliton with \( n \geq 3 \) and assume that its Ricci tensor has at most two eigenvalues at each point of \( M \). Then one of the following is true

1. \( M \) is Einstein with scalar curvature given by (1.3) for \( k = n \), and \( f \) is constant;
2. \( M \) is isometric to a finite quotient of the Gaussian Soliton, with potential function given by (1.4);
3. \( M \) is isometric to a finite quotient of \( \mathbb{R} \times N^{n-1} \), where \( N^{n-1} \) is Einstein if \( n \geq 4 \), and a space form if \( n = 3 \). Furthermore, the scalar curvature and the potential function are given by (1.3) and (1.4) for \( k = n - 1 \).

In particular, \( (M^n, g, f, \lambda) \) is rigid.

In order to prove the theorem above, we will show that the scalar curvature of such manifolds are constant, which implies that their Ricci tensors have constant rank equals to 0, \( n - 1 \) or \( n \). Previous results on rigidity of Ricci solitons are used to conclude the proof [8].

In what follows we will rewrite the condition of Theorem 1.2 on the eigenvalues of the Ricci tensor. As it was proved by Catino and Mazzieri [4], wherever \( \nabla f \) does not
vanish, it is an eigenvector of $\text{Ric}$, whose corresponding eigenvalue is 0. Therefore, the hypothesis of having at most two eigenvalues turns out to be equivalent to
\begin{equation}
R^2 - (n - 1)|\text{Ric}|^2 = 0
\end{equation}
on the set of regular points of $M$. In [1] the author has shown that the set of regular points of $f$ is dense in $M$ (see also Proposition 3.1 below), which implies that (1.5) is equivalent to the Ricci curvature of $M$ having at most two eigenvalues at each point of $M$.

Now we turn to the classification of Bach-flat Schouten solitons. Recall that steady Schouten solitons were already proven to be Ricci flat. Therefore, we concentrate our study in classifying shrinking and expanding Schouten solitons. There is a wildly known classification of Bach-flat Ricci solitons. In [2], Cao and Chen showed rigidity of Bach-flat shrinking Ricci solitons. In [3], Cao, Catino and coworkers classified steady Ricci solitons, showing that they must be either rigid or homothetic to the Bryant soliton. It was also proved in [3] that Bach-flat expanding Ricci solitons with nonnegative Ricci curvature are rotationally symmetric. Although, there are non rigid examples in the latter case (see page 12 of [3]).

The rigidity of Bach-flat Schouten solitons is a consequence of Theorem 1.2. More precisely, we will show that $\nabla f$ is an eigenvalue of the Bach tensor of a Schouten soliton at regular points of $f$, whose corresponding eigenvalue is, up to a constant, given by the expression in left hand side of (1.5). This connects Theorem 1.2 and Theorem 1.3 and shows that the same result is true if the Bach tensor vanishes in the direction of $\nabla f$.

\textbf{Theorem 1.3.} A complete noncompact Bach-flat Schouten soliton $(M^n, g, f, \lambda)$ with $n \geq 4$ and $\lambda \neq 0$ is isometric to one of those listed in Theorem 1.2.

Notice that unlike expanding Bach-flat Ricci solitons, an expanding Schouten soliton is rigid under the vanishment of its Bach tensor.

For $n = 3$ we follow the same definition of Bach tensor introduced in [3] (see (4.5)). With this definition, Theorem 1.3 is also true for a 3 dimensional Schouten soliton, and the proof does not change. But a stronger result is true, as we state below.

\textbf{Theorem 1.4.} A complete noncompact Schouten soliton $(M^3, g, f, \lambda)$ with $\lambda \neq 0$, whose Bach tensor is divergence-free, is isometric to one of those listed in Theorem 1.2. Besides, $N^2$ in item (3) is a space form.

\textbf{Remark 1.5.} Shrinking Schouten solitons have already been classified in dimension 3 by Catino et all (see Theorem 5.4 in [4]). Therefore, when $n = 3$ the novelty of the theorems above concerns only expanding Schouten solitons.

It is well known that locally conformally flat manifolds are necessarily Bach-flat. Therefore, a direct consequence of the previous results is the classification of locally conformally flat shrinking and expanding Schouten solitons.

\textbf{Corollary 1.6.} A locally conformally flat complete noncompact Schouten soliton $(M^n, g, f, \lambda)$ with $n \geq 3$ and $\lambda \neq 0$ is isometric to one of those listed in Theorem 1.2, where $N^n$ in item (3) is a space form.

This paper is organized in the following way. In Section 2 we introduce notation, definition and provide the basic tools to prove the main results of this paper. In
Section 3 we deal with Schouten solitons whose Ricci tensors have at most two eigenvalues at each point. We close this section with the proof of Theorem 1.2. In Section 4 we recall the definition of classical tensors as the Weyl, Cotton and Bach tensor (including its extension to $n = 3$) present in Theorem 1.3 Theorem 1.4 and Corollary 1.6 and present their proofs. It is in this section that we show that $\nabla f$ is an eigenvalue of the Bach tensor of $M$ at regular points of $f$.

2. Preliminary Results

The proposition below collects some important identities on Schouten solitons.

**Proposition 2.1 ([4]).** If $(M^n, g, f, \lambda)$ is a gradient Schouten soliton, then

\[ \Delta f = n\lambda - \frac{n-2}{2(n-1)} R, \]

\[ \text{Ric}(\nabla f, X) = 0, \forall X \in \mathfrak{X}(M), \]

\[ \langle \nabla f, \nabla R \rangle + \left( \frac{R}{n-1} + 2\lambda \right) R = 2|Ric|^2. \]

In [1] the author proved the following result, which can be seen as the analog of Hamilton’s identity for Ricci solitons. The latter plays a fundamental role to Ricci soliton’s theory, as one can see for example in [7, 10] and references therein.

**Theorem 2.2 ([1]).** Let $(M^n, g, f, \lambda), \lambda \neq 0$, be a complete noncompact Schouten soliton with $f$ nonconstant. If $\lambda > 0$ ($\lambda < 0$, respectively), then the potential function $f$ attains a global minimum (maximum, respectively) and is unbounded above (below, respectively). Furthermore,

\[ 0 \leq \lambda R \leq 2(n-1)\lambda^2, \]

\[ 2\lambda(f - f_0) \leq |\nabla f|^2 \leq 4\lambda(f - f_0), \]

with $f_0 = \min_{p \in M} f(p)$, if $\lambda > 0$, ($f_0 = \max_{p \in M} f(p)$, if $\lambda < 0$, respectively).

For applications of the result above see [1], where volume growth of geodesic balls are investigated for shrinking Schouten solitons.

In order to prove Theorem 2.2 an ordinary differential inequality satisfied by $|\nabla f|^2$ along suitable curves was important. Let us recall such inequality, since it will be used in next sections.

Let $p \in M$ be a regular point of $f$ and $\alpha : (\omega_1, \omega_2) \to M$ the maximal integral curve of $\frac{\nabla f}{|\nabla f|^2}$ through $p$. It is not hard to see that

\[ (f \circ \alpha)'(s) = 1, \forall s \in (\omega_1, \omega_2), \]

that is, $f \circ \alpha$ is a linear function of $s$.

**Proposition 2.3 ([1]).** Let $(M^n, g, f, \lambda), \lambda \neq 0$, be a Schouten soliton with $f$ nonconstant and $\alpha(s), s \in (\omega_1, \omega_2)$, a maximal integral curve of $\frac{\nabla f}{|\nabla f|^2}$. The function $b : (\omega_1, \omega_2) \to \mathbb{R}$, defined by

\[ b(s) = |\nabla f(\alpha(s))|^2. \]

satisfies the differential inequality

\[ bb'' - (b')^2 + 6\lambda b' - 8\lambda^2 \geq 0, \]
where $b'$ and $b''$ are the first and the second derivative of $b$ with respect to $s$. Furthermore, equality holds in (2.8) at $s_1$ if and only if $(n - 1)|\text{Ric}|^2 = R^2$ holds on $f^{-1}(s_1)$.

Proof. Consider the smooth function $a : (\omega_1, \omega_2) \to \mathbb{R}$ given by $a(s) = R(\alpha(s))$. From $d(|\nabla f|^2)(X) = 2\nabla \nabla f(X, \nabla f)$ and equation (1.1) one has

\begin{equation}
(2.9) \quad b'(s) = d(|\nabla f|^2)(\alpha'(s)) = \frac{a(s)}{n-1} + 2\lambda,
\end{equation}

which after differentiating gives $a'(s) = (n - 1)b''(s)$. Consequently,

\begin{equation}
(2.10) \quad (\nabla f(\alpha(s)), \nabla R(\alpha(s))) = b(s)a'(s) = (n - 1)b(s)b''(s).
\end{equation}

Putting (2.8), (2.9) and (2.10) together we have

\begin{equation}
(2.11) \quad (n - 1)(b(s)b''(s) + b'(s)(b'(s) - 2\lambda)) = 2|\text{Ric}|^2(\alpha(s))
\end{equation}

\begin{equation}
\geq 2(n - 1)(b'(s) - 2\lambda)^2,
\end{equation}

where in the second line we have used the inequality $(n - 1)|\text{Ric}|^2 \geq R^2$, which is a consequence of (2.2) and (2.9) once again. Consequently,

\begin{equation}
(2.12) \quad b'' \geq (b' - 2\lambda)(b' - 4\lambda).
\end{equation}

Suppose there is an $s_1$ where equality is reached in (2.8). Then equality is also obtained in (2.11), what is only possible if $(n - 1)|\text{Ric}|^2 = R^2$ on $f^{-1}(s_1)$. \hfill \Box

Example 2.4. Let $(\mathbb{R}^{n-k} \times \Gamma, \mathcal{N}^k, g, f, \lambda)$ be the Schouten soliton of Example 1.1. Notice that it has constant scalar curvature $R = \frac{2(n-1)k\lambda}{2(n-1) - k}$ and, if $k \leq n - 1$, its potential function $f$ is not constant and satisfies $|\nabla f|^2 = \left(\frac{R}{n-1} + 2\lambda\right)f$. After a linear change of coordinates using (2.0) with the condition $f(\alpha(0)) = 0$ and replacing $f$ by $s$, we obtain the function $b(s) = \left(\frac{R}{n-1} + 2\lambda\right)s$. Now, a simple computation shows that $b'(s) = \frac{4(n-1)\lambda}{2(n-1) - k}$, and then

\begin{equation}
(2.13) \quad (b' - 2\lambda)(b' - 4\lambda) = -\frac{8k(n-1-k)\lambda^2}{(2(n-1) - k)^2} \leq 0.
\end{equation}

Once $bb'' = 0$, for $b'$ is constant, we see that $b(s)$ is a solution of (2.8). Observe that equality holds in (2.13) for $k = 0$ and $k = n - 1$.

We take the opportunity to mention that the inequality in (2.13), that is,

\begin{equation}
(b' - 2\lambda)(b' - 4\lambda) \leq 0,
\end{equation}

was proven to be true for all complete Schouten solitons whose potential function is not constant \cite{1}. This is an important tool used in the proof of Theorem 2.2.

3. Equality for (2.8) and proof of Theorem 1.2

In this section we investigate the geometry of $M$ when equality holds in (2.8) for any $s \in (\omega_1, \omega_2)$. More precisely, we will assume that $b(s) = |\nabla f(\alpha(s))|^2$ is a solution of

\begin{equation}
(3.1) \quad bb'' - (b')^2 + 6\lambda b' - 8\lambda^2 = 0,
\end{equation}

for each $s \in (\omega_1, \omega_2)$. As we know from Proposition 2.3 this is equivalent to $(n - 1)|\text{Ric}|^2 = R^2$ on the set of regular points of $f$. If we could approximate any
critical point by a sequence of regular points, this equivalence would be true all over \( M \). In the next proposition we show that this is actually the case.

**Proposition 3.1.** Let \((M^n, g, f, \lambda), \lambda \neq 0\), be a complete Schouten soliton with \( f \) nonconstant. The set of regular points of \( f \) is dense in \( M \).

**Proof.** Denote by \( R \) the set of regular points of \( f \). Suppose by contradiction that the open set \( U = M \setminus R \) is nonempty. Then \( \Delta f \) vanishes in \( U \), what from (2.1) and (2.4) gives, at \( p \in U \), the following inequality

\[
\frac{2(n - 1)n\lambda^2}{n - 2} = R(q)\lambda \leq 2(n - 1)\lambda^2,
\]

which cannot be true. Then \( \overline{R} = M \), as the proposition claims. \(\square\)

The lemma below shows an algebraic relation between a solution of (3.1) and its first derivative, which can be seen as a first integral of (3.1).

**Lemma 3.2.** Let \( b(s) \) be a smooth solution of (3.1) so that \( b'(s) \neq 2\lambda \) on an interval \( I \subset (\omega_1, \omega_2) \). Then there is a constant \( \sigma_0 \) so that

\[
(b'(s) - 4\lambda)^2 = \sigma_0 b(s)(b'(s) - 2\lambda), \forall s \in I.
\]

**Proof.** A straightforward computation gives

\[
\frac{\left(\frac{(b'(s) - 4\lambda)^2}{b(s)(b'(s) - 2\lambda)}\right)'}{b^2(b' - 2\lambda)^2} = \frac{b'(b' - 4\lambda)(2(b' - 2\lambda)bb'' - (b' - 4\lambda)(b'(b' - 2\lambda) + bb''))}{b^2(b' - 2\lambda)^2} = \frac{b'(b' - 4\lambda)}{b^2(b' - 2\lambda)^2}(bb'' - (b' - 4\lambda)(b' - 2\lambda)) = 0,
\]

for all \( s \in I \), where we have used (3.1). This proves the lemma. \(\square\)

Using the relations \(|\nabla f|^2 = b\) and \( R = (n - 1)(b' - 2\lambda) \) we have the following corollary, which rewrites (3.2).

**Corollary 3.3.** Let \((M^n, g, f, \lambda), \lambda \neq 0\), be a Schouten soliton with \( f \) nonconstant for which (1.5) happens, and let \( A \subset M \) be a set of regular points of \( f \) where \( R \) does not vanish. Then there is a constant \( \sigma_0 \) so that

\[
(R - 2(n - 1)\lambda)^2 = \sigma_0(n - 1)R|\nabla f|^2
\]

on \( A \).

We are finally ready to set the main ingredient to prove Theorem 1.2.

**Proposition 3.4.** Let \((M^n, g, f, \lambda), \lambda \neq 0\), be a Schouten soliton with \( f \) nonconstant and \( a(s), s \in (\omega_1, \omega_2) \), a maximal integral curve of \( \frac{\nabla f}{|\nabla f|^2} \). Assume that the function \( b : (\omega_1, \omega_2) \to \mathbb{R} \), defined by \( b(s) = |\nabla f(a(s))|^2 \), satisfies the ODE (3.1) on \((\omega_1, \omega_2) \). Then \( b' \equiv 2\lambda \) or \( b' \equiv 4\lambda \).

**Proof.** First notice that (3.1) can be rewritten in the following two ways

\[
(b(b' - 2\lambda))^2 = \frac{2b}{b^2}(b(b' - 2\lambda))^2 \quad \text{and} \quad (b(b' - 4\lambda))^2 = \frac{2(b' - \lambda)}{b}b(b' - 4\lambda).
\]
Since \( b \) never vanishes on \((\omega_1, \omega_2)\), the ODE's above, satisfied by \( b(b' - 2\lambda) \) and \( b(b' - 4\lambda) \), respectively, imply that if \( b'(s_0) \in \{2\lambda, 4\lambda\} \) for some \( s_0 \in (\omega_1, \omega_2) \), then \( b' = 2\lambda \) or \( b' = 4\lambda \) on \((\omega_1, \omega_2)\). The same conclusion holds if \( b''(s_0) \) vanishes.

Suppose that \( b' \) is not constant. In this case, \( b'(s) \notin \{2\lambda, 4\lambda\}, \forall s \in (\omega_1, \omega_2) \), which by Proposition 3.1 and Corollary 3.3 implies that \( 3.3 \) is true on \( M \) and \( \sigma_0 \neq 0 \). In particular, at a critical point \( p_0 \) of \( f \) this implies that \( R(p_0) = 2(n-1)\lambda \). Computing the Laplacian of both sides of \((3.3)\) gives

\[
\sigma_0(n - 1)(R(\Delta) |\nabla f|^2 + |\nabla f|^2 \Delta R + 2 (\nabla |\nabla f|^2, \nabla R)) =
2(R - 2(n - 1)\lambda) \Delta R + 2|\nabla R|^2,
\]

which at \( p_0 \) reveals that \( 2\sigma_0(n - 1)\lambda \Delta |\nabla f|^2(p_0) = 2|\nabla R(p_0)|^2 = 0 \), where we have used that \( p_0 \) is a critical point of \( R \), a consequence of Theorem 2.2. Using \((2.2)\) and Bochner’s formula we conclude that

\[
|\nabla \nabla f|^2(p_0) = 12 \lambda |\nabla f|^2(p_0) - (\Delta f(p_0), \nabla R(p_0)) = 0,
\]

from where it follows that

\[
0 = n|\nabla \nabla f|^2(p_0) \geq (\Delta f(p_0))^2 = 4\lambda^2 > 0,
\]

which is not possible. This proves that \( b' \) must be constant. \( \square \)

**Proof of Theorem 1.2.** If \( f \) is constant, then \( M \) is Einstein. Now assume that \( f \) is not constant and consider \( b(s) \) defined as in \((2.7)\). In view of \((2.7)\), \( 0 \) is always an eigenvalue of \( Ric \) at the regular points of \( f \). Since the set of regular points of \( f \) is dense in \( M \) (Proposition 3.1), \( Ric \) has at most two eigenvalues at each point if and only if \((1.5)\) holds on \( M \). This, according to Proposition 2.3, implies that \( b \) is a solution of \((3.1)\). Applying Proposition 3.4 and using the relation

\[ R = (n - 1)(b' - 2\lambda) \] (see \((3.9)\)), we conclude that \( R \) is constant, either equals to 0 or \( 2(n - 1)\lambda \). Then \((M^n, g, f, \mu), \mu \in \{\lambda, 2\lambda\} \), is a Ricci soliton whose Ricci tensor, according to \((1.5)\), has constant rank equals either to 0 or \( n - 1 \). Now we apply Theorem 2 of \([8]\) to conclude that \((M, g)\) is rigid, and the theorem is proved. \( \square \)

4. The Bach Tensor of a Schouten Soliton

In this section we present the proof of Theorem 1.3, Theorem 1.4 and Corollary 1.6.

First let us recall some definitions. Let \((M^n, g)\) be a Riemannian manifold. For any \( n \geq 3 \) its \textit{Weyl} and \textit{Cotton} tensors are defined, respectively, by

\[
W_{ijkl} = R_{ijkl} - \frac{1}{n - 2} (g_{ik} R_{jlt} - g_{jl} R_{ikt} + g_{jl} R_{ikt}) + \frac{R}{(n - 1)(n - 2)} (g_{ik} g_{jl} - g_{ij} g_{kl}),
\]

\[
C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n - 1)} (g_{jk} \nabla_i R - g_{ik} \nabla_j R),
\]

and for \( n \geq 4 \) the \textit{Bach} tensor is defined by

\[
B_{ij} = \frac{1}{n - 3} \nabla^k \nabla^l W_{ikjl} + \frac{1}{n - 2} R_{kl} W_{ijk}^l.
\]

In terms of the Cotton tensor, \( B \) can be rewritten by

\[
(n - 2) B_{ij} = \nabla^k C_{kij} + R_{kl} W_{ijk}^l.
\]
As in [2], we use the fact that (4.4) is well defined for \( n = 3 \) to define the Bach tensor in this dimension. More precisely, the Bach tensor of \((M^3, g)\) is defined by
\[
B_{ij} = \nabla^k C_{kij}.
\]

One says that a Riemannian manifold \(M^n\) is \textit{locally conformally flat} if \( n \geq 4 \) and its Weyl tensor vanishes or, if \( n = 3 \) and its cotton tensor vanishes. \(M^n\) is said to be \textit{Bach-flat} when its Bach tensor vanishes, \( n \geq 3 \).

The proposition below shows that whenever \( p \in M \) is a regular point of \( f \), its gradient is an eigenvector of the Bach tensor at this point and gives the corresponding eigenvalue.

**Proposition 4.1.** Let \((M^n, g, f, \lambda)\) be a gradient Schouten soliton with \( n \geq 3 \). At a regular point of \( f \) the Bach tensor of \( M \) satisfies
\[
B(\nabla f, X) = \frac{R^2 - (n-1)|Ric|^2}{(n-1)(n-2)g} g(\nabla f, X),
\]
for all \( X \in \mathfrak{X}(M) \). If \( n = 3 \) we have, in addition,
\[
(div B)(X) = \frac{R^2 - 2|Ric|^2}{2} g(\nabla f, X).
\]

**Proof.** In order to prove the proposition we need to find an expression for the Bach tensor of a Schouten soliton. We will first compute the Cotton tensor of such a manifold, and then use definitions (4.4) and (4.5) to achieve our goal. According to (1.1) and (4.2) we have
\[
C_{ijk} = \frac{\nabla_i R}{2(n-1)} g_{jk} - \nabla_i \nabla_j \nabla_k f - \frac{\nabla_j R}{2(n-1)} g_{ik} + \nabla_j \nabla_i \nabla_k f
- \frac{1}{2(n-1)} (\nabla_i R g_{jk} - \nabla_j R g_{ik})
= \nabla_j \nabla_i \nabla_k f - \nabla_i \nabla_j \nabla_k f
= R_{jikl} \nabla_l f.
\]

(4.8)

and then, as required in (4.4) and (4.5), we have the expression
\[
\nabla_i C_{ijk} = \nabla_i R_{klj} \nabla_l f + \left( \frac{R}{2(n-1)} + \lambda \right) R_{jik} - R_{jikl} R_{li}.
\]

On the other hand, since
\[
\nabla_i R_{klj} = \nabla_i R_{klij} - \nabla_k R_{lij}
= -\nabla_i \nabla_k \nabla_j f + \nabla_k \nabla_i \nabla_j f + \frac{1}{2(n-1)} (\nabla_i R g_{kj} - \nabla_k R g_{ij})
= R_{klj} \nabla_i f + \frac{1}{2(n-1)} (\nabla_i R g_{kj} - \nabla_k R g_{ij})
\]
we conclude that
\[
\nabla_i R_{klj} \nabla_l f = R_{klj} \nabla_i f \nabla_l f + \frac{1}{2(n-1)} ((\nabla R, \nabla f) g_{kj} - \nabla_k R \nabla_j f).
\]
Consequently,

\[ \nabla_i C_{ijk} = R_{klji} \nabla_l f \nabla_i f + \frac{1}{2(n-1)} (\nabla R, \nabla f) g_{kj} - \nabla_k R \nabla_j f \]

+ \left( \frac{R}{2(n-1)} + \lambda \right) R_{jk} - R_{ijkl} R_{ld} \]

To prove (4.6) when \( n \geq 4 \), notice that (4.1), (4.4) and (4.9) together imply

\[ (n-2)B_{kj} = R_{klji} \nabla_l f \nabla_i f + \frac{1}{2(n-1)} (\nabla R, \nabla f) g_{kj} - \nabla_k R \nabla_j f \]

+ \left( \frac{R}{2(n-1)} + \lambda \right) R_{jk} + \frac{1}{(n-1)(n-2)} (R_{kj} - R_{gkj}) \]

+ \frac{1}{n-2} (2R_{kl} R_{lj} - R R_{kj} - |Ric|^2 g_{kj}). \]

Using (2.2) we finally have

\[ (n-2)B_{kj} \nabla_k f = \frac{2R^2 - (n-1)|Ric|^2}{(n-1)(n-2)} \nabla_j f, \]

as it was claimed. Similar computations show the result for \( n = 3 \), where the Bach tensor is defined by (4.5).

Now we turn to the divergence of \( B \) when \( n = 3 \). We use the following formula

\[ (\text{div} B)_j = -R_{ji} C_{jil}, \]

true in this dimension, proved in [3]. Using (4.1) and (4.8) we then get

\[ (\text{div} B)_j = -R_{il} R_{lijr} \nabla_i f = -R_{il} R_{lijr} \nabla_i f \]

\[ = (|Ric|^2 g_{jr} - 2R_{ij} R_{ir} + RR_{jr} - \frac{R}{2} (R g_{jr} - R_{jr})) \nabla_i f \]

\[ = \left( |Ric|^2 - \frac{R^2}{2} \right) \nabla_j f, \]

proving (4.7).

\[ \square \]

Proofs of Theorem 1.3 and Theorem 1.4 Let \( (M^n, g, f, \lambda) \) be a Schouten Soliton with \( n \geq 3 \), whose Bach tensor vanishes. Since by hypothesis we have \( B_{ij} \nabla_j f = 0 \) for \( n \geq 3 \), or \( (\text{div} B)_j = 0 \) when \( n = 3 \), one has on \( M \), according to Proposition 4.1 and Proposition 5.1, that

\[ R^2 = (n-1)|Ric|^2. \]

Now we apply item (3) Theorem 1.2 to finish the proof.

\[ \square \]
Proof of Corollary 1.6. Let \((M^n, g, f, \lambda)\) be a locally conformally flat Schouten Soliton with \(n \geq 3\). It follows from (4.3) and (4.5) that these manifolds are Bach-flat, from where Theorem 1.3 and Theorem 1.4 imply that these manifolds are isometric to those of Theorem 1.2. In addition, if item (3) happens, locally conformally flatness implies that \(N^{n-1}\) must be a space form, and the corollary is proved.

\[\square\]

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