Relative (functionally) Type I spaces and narrow subspaces

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August 22, 2022

Abstract

An open chain cover \( \mathcal{U} = \{ U_\alpha : \alpha \in \kappa \} \) (\( \kappa \) a cardinal) of a space \( X \) is a systematic cover if \( U_\alpha \subset U_\beta \) when \( \alpha < \beta \), and \( X \) is Type I if \( \kappa = \omega_1 \) and each \( U_\alpha \) is Lindelöf. A closed subspace \( D \subset X \) is narrow in \( X \) if for each systematic cover \( \{ V_\alpha : \alpha \in \omega_1 \} \) of \( X \), either there is \( \alpha \) such that \( D \subset V_\alpha \), or \( V_\alpha \cap D \) is Lindelöf for each \( \alpha \). Taking systematic covers given by \( s^{-1}([0, \alpha)) \) for a continuous \( s : X \to \mathbb{L}_{\geq 0} \) (where \( \mathbb{L}_{\geq 0} \) is the longray) defines functionally Type I spaces and functionally narrow subspaces. For instance, \( \mathbb{L}_{\geq 0} \) and \( \omega_1 \) are narrow in themselves and any other space.

We investigate these properties and relative versions, as well as their relationship, and show in particular the following. There are functionally Hausdorff Type I spaces which are not functionally Type I while regular Type I spaces are functionally Type I. We exhibit examples of spaces which are narrow in some but not in other spaces. There are subspaces of a Tychonoff space \( Y \) that are functionally narrow but not narrow in \( Y \), while both notions agree if \( Y \) is normal. Under \textbf{PFA} and using classical results, any \( \omega_1 \)-compact locally compact countably tight Type I space contains a non-Lindelöf subspace narrow in it (a copy of \( \omega_1 \), actually), while a Suslin tree does not. There are spaces with subspaces narrow in them that are essentially discrete. Finally, we investigate natural partial orders on (functionally) narrow subspaces and when these orders are \( \omega \)- or \( \omega_1 \)-closed.

1 Introduction & definitions

This paper is about the notions of Type I, functionally Type I, narrow and functionally narrow spaces (and relative versions), and their relations. Type I spaces were introduced by P. Nyikos \cite{Nyikos1980} in his study of non-metrizable manifolds. Functionally narrow spaces were introduced by the author in the preprint \cite{Baillif2013} under another name (they were called \textit{directions}, we believe that the new terminology fits better). A good portion of the contents of the present paper comes from another preprint \cite{Baillif2013} which is almost ten years old and has gathered a grand total of zero citations (well, now: one), thus exhibiting the intensity of the community’s interest on the subject. (Containing some unspotted inaccuracies might have been a hindrance.) But since we keep going back to these questions again and again despite the passage of the years, we could not help but gather our results (old and new) in a more polished and publishable form. Our ambition being limitless, we are not afraid to state publicly that we aim for a strictly positive number of readers.\footnote{The boldness of this statement is alas diminished by the fact that a failure would be without witness. Referees do not count.}

1.1 Informalities

Let us try to present informally the main ideas of this paper, precise definitions can be found in the next subsection, where the reader is encouraged to jump directly should they have an
aversion to vagueness and shaky analogies.

A space is Type I (in itself) iff it is an increasing union of length $\omega_1$ of closed Lindelöf subspaces whose interiors contain the preceding members of the union. For functionally Type I spaces, the increasing union is given by preimages of initial segments of the longray $\mathbb{L}_{\geq 0}$ for a function called a slicer. A functionally Type I space is Type I. If one sees spaces as human beings, a (functionally) Type I space grows slowly out of childhood (i.e. Lindelöfness) and thus tends to behave more reasonably than those who see one aspect of their life jump at once into adulthood (going from being smooth-chinned to growing a beard overnight, for instance), having a Lindelöf subspace with non-Lindelöf closure.

Given some ambient space $X$, a closed subspace $D$ is functionally narrow in $X$ if, given $f : X \to \mathbb{L}_{\geq 0}$, whenever a closed non-Lindelöf part of $D$ is sent by $f$ into a strict initial (Lindelöf) segment then all of $D$ is sent to a strict initial segment. Narrow subspaces are defined by a similar property, but with increasing unions of closed sets (whose interior contain the preceding members) of length $\omega_1$ of $X$ instead of a function. A subspace that is narrow is functionally narrow. Going back to our analogy, a narrow subspace cannot be at the same time scolded (having a grown-up aspect treated as that of a child) and educated gradually into adulthood.

Being (functionally) narrow is very sensitive to the ambient space, as a copy of a space can be narrow in one and non-narrow in another space. Prototypical examples of spaces that are narrow (in themselves and any other space) are $\omega_1$ and the longray.

These concepts inspired us questions that felt natural, we try to answer some of them in the present paper. Here is a short list.

- When does being Type I imply being functionally Type I (both in a fixed ambient space) ? We will see that it is the case if the ambient space is normal or regular and Type I (in itself).
- When does being functionally narrow imply being narrow (both in a fixed ambient space) ? This is again the case if the ambient space is normal, but there are Tychonoff counterexamples.
- Given a Type I space, one may ask if it contains a subspace (functionally) narrow in it. Counter-examples are very easy to find (for instance, the discrete space of cardinality $\omega_1$), but it gets trickier if one imposes global and local conditions such as countable compactness and first countability, in which case some results depend on the model of set theory. This is the subject of Section 4.
- Since a prototypical example of a space that is non-narrow (in itself) is the discrete space of cardinality $\omega_1$, it sounds fun (out of contrariness, obviously) to try to find spaces whose narrow subspaces are essentially discrete. This is done in Section 5.
- The definitions of (functional) narrowness yield two natural partial orders on closed non-Lindelöf subspaces of a given space $X$. One is defined simply as $C \preceq_f D$ iff given any $f$ from $X$ to the longray, then $f$ sends $C$ into an initial segment whenever $f$ sends $D$ into an initial segment. For instance, in the square of the longray, the diagonal is strictly $\preceq_f$-bigger than any horizontal or vertical line. These orders generalize the inclusion relation. Section 6 is dedicated to a short study of these orders on the set of (functionally) narrow subspaces.

1.2 Definitions

By space we mean a topological Hausdorff space, hence normal and regular spaces are assumed to be Hausdorff. Throughout the paper, every function is assumed to be continuous if not stated otherwise. We follow the set-theoretic tradition of denoting the set of positive integers by $\omega$ and the first uncountable ordinal by $\omega_1$, and $\overline{U}$ denotes the closure of the subspace $U$ in some topological space clear from the context. We use the term club as a shorthand for closed and

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2We are aware that various aspects of this analogy are open to debate.
3Well, we should maybe forget completely about this analogy.
unbounded. The restriction of a function \( f \) to a subset \( A \) of its domain is denoted by \( f \upharpoonright A \). We use brackets \( (, ) \) for ordered pairs and parenthesis \( (, ) \) for open intervals in (partially) ordered sets. Letters \( \alpha, \beta, \gamma, \delta \) are reserved exclusively for ordinals, and an ordinal is always assumed to be the set of its predecessors. When seen as topological spaces, ordinals are endowed with the order topology (unless specified).

**Definition 1.1.** A cover of a space is a cover by open sets. A chain cover is linearly ordered by the inclusion relation. A good cover of \( X \) is a chain cover \( \{U_\alpha : \alpha \in \kappa \} \) where \( U_\alpha \subseteq U_\beta \) when \( \alpha < \beta, \kappa \) is a regular cardinal and \( \bigcup_{\gamma < \alpha} U_\gamma = U_\alpha \) when \( \alpha \) is a limit ordinal. A systematic cover is a chain cover \( \{U_\alpha : \alpha \in \kappa \} \) such that \( \overline{U_\alpha} \subseteq U_\beta \) when \( \alpha < \beta \).

We write down the next lemma for the record since we use it throughout the paper (most of the time implicitly). Its proof is immediate.

**Lemma 1.2.** Any chain cover has a subcover which either contains only one element (the whole space) or is indexed by a regular cardinal, in the latter case it can become a good chain cover by adding relevant unions at limit ordinal (when needed).

**Definition 1.3.** The longray \( \mathbb{L}_{\geq 0} \) is \( \omega_1 \times [0,1) \) with lexicographic order topology. A subset of \( \mathbb{L}_{\geq 0} \) or \( \omega_1 \) is bounded iff it is contained in a strict initial segment (or equivalently iff it is Lindelöf), and unbounded otherwise. A function into \( \mathbb{L}_{\geq 0} \) or \( \omega_1 \) is bounded iff its range is bounded, and unbounded otherwise.

When convenient, we see \( \omega_1 \) as a subset of \( \mathbb{L}_{\geq 0} \), identifying \( \alpha \in \omega_1 \) with \( (\alpha,0) \in \mathbb{L}_{\geq 0} \). Hence, “\( \alpha \in \mathbb{L}_{\geq 0} \)” should be understood as “\( (\alpha,0) \in \mathbb{L}_{\geq 0} \)”. The notions of stationary and club subsets of \( \omega_1 \) extend in an obvious way to subsets of \( \mathbb{L}_{\geq 0} \). Recall the following classical lemma, whose proof can be found in any textbook about set theory.

**Lemma 1.4 (Fodor’s Lemma).** Let \( S \subseteq \omega_1 \) be stationary and \( f : S \to \omega_1 \) be (non necessarily continuous and) regressive, that is \( f(\alpha) < \alpha \ \forall \alpha \in S \). Then there is \( \alpha \in \omega_1 \) such that \( f^{-1}(\{\alpha\}) \) is stationary.

**Definition 1.5.** Let \( Y \) be a space and \( X \subseteq Y \) be a closed subspace.

(a) \( X \) is Type I in \( Y \) iff there is a systematic cover \( \{U_\alpha : \alpha \in \omega_1 \} \) of \( Y \) such that \( \overline{U_\alpha} \cap X \) is Lindelöf for each \( \alpha \). When \( X = Y \), we say that \( X \) is Type I in itself, or just Type I.

(b) A cover of \( Y \) witnessing that \( X \) is Type I in \( Y \) is a canonical cover of \( X \) in \( Y \) if it is good and systematic.

(c) \( X \) is functionally Type I in \( Y \) (in short: \( \text{f-Type I in } Y \)) iff there is \( s : Y \to \mathbb{L}_{\geq 0} \) such that \( X \cap s^{-1}([0,\alpha]) \) is Lindelöf for each \( \alpha \in \omega_1 \). When \( X = Y \), we say that \( X \) is \( \text{f-Type I in itself} \), or just \( \text{f-Type I} \). In that case \( s \) is called a slicer of \( X \).

The term canonical is motivated by the fact that two canonical covers agree (on \( X \)) on a club set of indices, see Corollary 2.14 below. The choice of \( \mathbb{L}_{\geq 0} \) for the range of \( f \) in (c) (instead of another Type I space) is because it plays a role similar to that of the interval \( [0,1] \) in the classical Urysohn Lemma, or in the definition of Tychonoff spaces. Indeed, if a space \( X \) is Type I in itself and regular, it is Tychonoff and f-Type I, with a slicer “following” a given canonical cover of \( X \). See Lemma 2.3 below for details. Let us first clear up some trivialities.

**Lemma 1.6.** Let \( X \) be closed in a space \( Y \).

(a) If \( \{U_\alpha : \alpha \in \omega_1 \} \) is a systematic cover of \( Y \) such that \( \overline{U_\alpha} \cap X \) is Lindelöf for each \( \alpha < \beta \), then there is a canonical cover of \( X \) in \( Y \).

(b) If \( X \) is \( \text{f-Type I in } Y \) then it is Type I in \( Y \).

(c) If \( X \) is Type I in itself, then any closed subset of \( X \) is Type I in \( X \).

(d) If \( X \) is \( \text{f-Type I in itself} \), then any closed subset of \( X \) is \( \text{f-Type I in } X \).

(e) The Lindelöf number of a space that is Type I in another is at most \( \aleph_1 \).
Proof.
(a) The only missing property is that $\bigcup_{\gamma<\alpha} U_\gamma = U_\alpha$ for limit $\alpha$. But in any chain cover we have $\bigcup_{\gamma<\alpha} U_\gamma \subset \overline{U_\alpha}$, the latter being Lindelöf, we may thus replace $U_\alpha$ by $\bigcup_{\gamma<\alpha} U_\gamma$, yielding a canonical cover of $X$.
(b) If $s : Y \to \mathbb{L}_{\geq 0}$ witnesses that $X$ is f-Type I in $Y$, then $\{s^{-1}([0, \alpha)) : \alpha \in \omega_1\}$ is canonical for $X$ in $Y$. Indeed, $s^{-1}([0, \alpha))$ is contained in $s^{-1}([0, \alpha))$ whose intersection with $X$ is Lindelöf.
(c), (d) & (e) Immediate. \qed

Definition 1.7. Let $Y$ be a space and $D \subset Y$ be a closed subspace.
(a) We say that $D$ is narrow in $Y$ iff given any systematic cover $\{U_\alpha : \alpha \in \omega_1\}$ of $Y$ then either $D \subset U_\alpha$ for some $\alpha$ or $\overline{U_\alpha} \cap D$ is Lindelöf for each $\alpha$.
(b) We say that $D$ is functionally narrow in $Y$ (in short: f-narrow in $Y$) iff given any $f : Y \to \mathbb{L}_{\geq 0}$, either $f \upharpoonright D$ is bounded or $f^{-1}([0, \alpha)) \cap D$ is Lindelöf for each $\alpha$.

Remark. Slightly more general definitions arise by considering non-necessarily closed subsets and asking for Lindelöfness of $\overline{U_\alpha} \cap D$ and $f^{-1}([0, \alpha)) \cap D$, but we did not see any gain in choosing these versions.

As above, we often abbreviate “(f-)narrow in itself” by just “(f-)narrow”. Let us immediately get rid of more trivialities. Point (c) is the main reason to restrict (f-)narrow subspaces to closed subsets.

Lemma 1.8. Let $D$ be closed in a space $Y$.
(a) If $D$ is narrow in $Y$, then $D$ is f-narrow in $Y$.
(b) If $D$ is narrow (resp. f-narrow) in itself, then $D$ is narrow (resp. f-narrow) in $Y$.
(c) If $C \subset D$ is closed and $D$ is narrow (resp. f-narrow) in $Y$, then $C$ is narrow (resp. f-narrow) in $Y$.

Proof.
(a) Given $s : Y \to \mathbb{L}_{\geq 0}$ which is unbounded on $D$, $\{s^{-1}([0, \alpha)) : \alpha \in \omega_1\}$ is a systematic chain cover of $Y$. Then $s^{-1}([0, \alpha + 1)) \cap D$ is Lindelöf and contains $s^{-1}([0, \alpha]) \cap D$, thus so is the latter.
(b) & (c) Immediate. \qed

Notice that the direction of the implication is the opposite between (f-)Type I and (f-)narrow spaces, as we have, in short (with the obvious warning about not writing explicitly the assumptions):

$$f\text{-Type I} \implies \text{Type I}$$
$$\text{narrow} \implies \text{f-narrow}.$$ 

Also, notice that our definitions allow for trivial examples (take $U_\alpha$ to be the whole space for each $\alpha \in \omega_1$):

Example 1.9. Every closed Lindelöf subspace of some space $Y$ is f-Type I and narrow, both in itself and in $Y$.

While we did not want to rule them out in order to avoid cumbersome exceptions in many proofs, these are examples which we are not very interested in.

2 Type I and functionally Type I subspaces

Let us start by stating an almost obvious useful lemma.
Lemma 2.1. Let \( \{U_\alpha : \alpha \in \kappa \} \) be a systematic chain cover of \( Y \). Then a subset is closed [resp. open] iff its intersection with \( \bigcup_\alpha [\text{resp. } U_\alpha] \) is closed [resp. open] for each \( \alpha \), and \( f: Y \to \mathbb{L}_{\geq 0} \) is continuous iff its restriction to \( U_\alpha \) is continuous for each \( \alpha \) iff its restriction to \( \overline{U_\alpha} \) is continuous for each \( \alpha \).

Proof. Let \( A \subset Y \). If \( A \) is not closed, then there is \( x \in \overline{A} - A \), choose \( \alpha \) such that \( x \in U_\alpha \), then \( A \cap \overline{U_\alpha} \) is not closed. If \( A \) is not open then there is \( x \in A - \text{int}(A) \), choose \( \alpha \) such that \( x \in U_\alpha \), then \( A \cup U_\alpha \) is not open. The rest follows immediately. \( \square \)

Lemma 2.2. Let \( \mathcal{U} = \{U_\alpha : \alpha \in \omega_1 \} \) be a systematic good cover of the space \( Y \) such that \( U_\alpha \neq U_\beta \) whenever \( \alpha \neq \beta \). If each \( \overline{U_\alpha} \) is normal, then there is \( s : Y \to \mathbb{L}_{\geq 0} \) such that \( s^{-1}([\alpha, \alpha + 1]) = \overline{U_{\alpha+1}} - U_\alpha \) for each \( \alpha \in \omega_1 \).

Notice that \( s \) sends \( \overline{U_\alpha} - U_\alpha \) (if nonempty) to \( \{\alpha\} \) for each \( \alpha \in \omega_1 \).

Proof. Since \( A = \overline{U_{\alpha+1}} - U_{\alpha+1} \) and \( B = U_\alpha \) are disjoint closed subsets of \( \overline{U_{\alpha+1}} \), there is a Urysohn function \( s_\alpha : \overline{U_{\alpha+1}} \to [\alpha, \alpha + 1] \subset \mathbb{L}_{\geq 0} \) sending \( A \) to \( \alpha + 1 \) and \( B \) to \( \alpha \). (If \( B \) is empty, just take a constant function on \( \alpha + 1 \), if \( A \) is empty but \( B \) is not, a constant one on \( \alpha \).) We may then define \( s : Y \to \mathbb{L}_{\geq 0} \) as equal to \( s_\alpha \) on \( \overline{U_{\alpha+1}} - U_\alpha \). Then \( s \) is well defined, continuous, and satisfies the claimed properties. \( \square \)

As in Definition [1.5] (c), such an \( s \) is called a slicer of \( \mathcal{U} \), because it cuts the systematic cover into slices \( s^{-1}(\{x\}) \), respecting the inclusion relation and sending the boundary of \( U_\alpha \) to \( \{\alpha\} \). If \( \mathcal{U} \) is a canonical cover of a Type I space \( X \) and \( s \) is a slicer of \( \mathcal{U} \), we say that \( s \) is a slicer of \( X \) as well.

Lemma 2.3. Let \( X \) be a closed subset of \( Y \).

(a) If \( Y \) is regular and Type I (in itself) with canonical cover \( \{V_\alpha : \alpha \in \omega_1\} \), then \( Y \) is Tychonoff, each \( \overline{V_\alpha} \) is normal, and \( Y \) is f-Type I (in itself).

(b) If \( Y \) is normal and \( X \) is Type I in \( Y \), then \( X \) is f-Type I in \( Y \).

Proof. If \( X \) is Lindelöf, there is nothing to do (see Example [1.9]). We thus assume that \( X \) in non-Lindelöf.

(a) Almost immediate: a regular Lindelöf space is normal, hence so is each \( \overline{V_\alpha} \). Then \( s \) given by Lemma 2.2 makes \( Y \) f-Type I. Given \( x \) in some open \( U \), take \( \alpha \) so that \( x \in V_\alpha \), by normality of \( \overline{V_\alpha} \) there is a real valued function that is 0 on \( x \) and 1 on \( \overline{V_\alpha} - (U \cap V_\alpha) \). Extend it to all of \( Y \) by 1 outside of \( \overline{V_\alpha} \). This shows that \( Y \) is Tychonoff.

(b) By Lemma 2.2. \( \square \)

It is trivial (and recorded in Lemma [1.6] (c)-(d)) that (f-)Type I-ness (in a given space) is hereditary with respect to closed subspaces. Let us now exhibit some examples that illustrate how this relative (f-)Type I-ness may turn out in more general situations. First, we note that a non-Type I space might contain a closed subspace which is f-Type I in it.

Example 2.4. \( Y = \omega_2 - \{\omega_1\} \) is not Type I (in itself) but \( \omega_1 \subset Y \) is f-Type I in \( Y \). (More generally, the disjoint union of a f-Type I space and a space of Lindelöf number \( > \aleph_1 \) has the same property.)

Details. It is obvious that \( Y \) is not Type I (its Lindelöf number is \( \aleph_2 > \aleph_1 \)). The function \( f: Y \to \omega_1 \) which is the identity on \( \omega_1 \) and constant with any value on the rest shows that \( \omega_1 \) is f-Type I in \( Y \).

A space can become f-Type I with the removal of a single point.

Example 2.5. The cone over \( \mathbb{L}_{\geq 0} \) defined as \( Y = \mathbb{L}_{\geq 0} \times [0,1]/\sim \), where \( \langle x,1 \rangle \sim \langle y,1 \rangle \) for each \( x,y \in \omega_1 \) is first countable and not Type I (in itself), but removing its “apex” (the point with second coordinate 1) yields an f-Type I space.
Details. First countability follows from properties of the product $\mathbb{L}_{\geq 0} \times [0,1]$: \{ $L_{\geq 0} \times (1-1/n,1]$ : $n \in \omega \} $ is a neighborhood base of $\mathbb{L}_{\geq 0} \times \{1\} $, see Lemma 2.3.1(a) below. $Y$ is not Type I because no neighborhood of the apex point is Lindelöf. The projection on the first factor shows that $\mathbb{L}_{\geq 0} \times \{0,1\} $ is $f$-Type I.

A space can be Type I in itself but not in another space.

**Example 2.6.** The closed subspace $\omega_1 \times \{\omega\} $ is not Type I in the Tychonoff plank $T = (\omega_1 + 1) \times (\omega + 1) - \{(\omega_1,\omega)\} $ but is $f$-Type I in itself.

Details. $(\omega_1 + 1) \times \omega$ is Lindelöf (even $\sigma$-compact) and dense in $T$, hence any systematic chain cover of $T$ must stagnate (and contain the whole space) after at most countably many steps. 

One may believe that having a normal ambient space does prevent this situation, but it is not the case.

**Example 2.7.** The copy of $\omega_1$ in any $\gamma \mathbb{N}$ is not Type I in $\gamma \mathbb{N}$, but is obviously $f$-Type I in itself. Moreover, $\gamma \mathbb{N}$ is a normal space.

Details. $\gamma \mathbb{N}$ is a symbol used in particular by P. Nyikos to denote a normal space that is the union of a countable dense subset of isolated points (identified with the integers) and a closed copy of $\omega_1$, see e.g. [15]. Such spaces exist in ZFC. Any systematic chain cover of $\gamma \mathbb{N}$ stagnates after at most countably many steps.

We now turn to the problem of when a (relative) Type I space is ( relatively) $f$-Type I. The simplest instance is for the ambient space itself: when does Type I in itself imply $f$-Type I in itself. Lemma 2.3 shows that this is the case for regular spaces. We now present two examples of Hausdorff spaces for which the implication fails. The first one is given by “piling up” copies of Roy’s lattice space (see e.g. [15] Ex. 126)), whose real-valued maps are constant, preventing $f$-Type I-ness. The second one is even functionally Hausdorff (also called completely Hausdorff), that is, given $x, y \in X$, there is a real valued function on $X$ with $f(x) = 0$, $f(y) = 1$, but its construction is a bit more cumbersome.

**Example 2.8.** A first countable connected Hausdorff Type I space $R$ which is not $f$-Type I.

(Note: This is a corrected version of the space constructed in [2] Lemma 10.6 whose description is flawed.)

Details. As a set, $R$ is a subset of $\mathbb{L}_{\geq 0} \times (\omega + 1)$. Let $\{C_n : n \in \omega\}$ be a disjoint collection of dense subsets of $\mathbb{Q} \cap [0, 1)$, We assume that $0 \in C_0$. We then let $L_n$ to be the subspace of $\mathbb{L}_{\geq 0}$ defined as $\omega_1 \times C_n$ (with lexicographic order topology). Define $R$ as $\cup_{n \in \omega} L_n \times \{n\} \cup \omega_1 \times \{\omega\}$, with the following topology. Denote by $q_a$ the point $(\alpha, \omega) \in R$. Given $a < b \in \mathbb{L}_{\geq 0}$ and $n < \omega$, denote by $(a,b)_n$ the set $L_n \cap (a, b) \subset \mathbb{L}_{\geq 0}$ and set $U_{a,b,n} = (a,b)_n \times \{n\}$. Let $z = (x, n) \in R$, with $n < \omega$. A neighborhood of $z$ is given by $U_{a,b,n}$ if $n$ is even and by a stack of 3 intervals $U_{a,b,n-1} \cup U_{a,b,n} \cup U_{a,b,n+1}$ if $n$ is odd, with $a < x < b$ in each case. Finally, a neighborhood of $q_a$ is $\{q_a\} \cup \cup_{n \geq m}(\alpha, \alpha + 1)_n \times \{n\}$, for $m$ even. Set $R_\alpha = R \cap [0, \alpha) \times (\omega + 1)$.

**Claim 2.8.1.** $R$ is Hausdorff, Type I and each $R_\alpha$ is connected (and thus $R$ too).

**Proof.** Hausdorffness is immediate since the $C_n$ are disjoint. By construction, since $0 \in C_0$, $R_\alpha = R_\alpha \cup \{\alpha, 0\} \ (\alpha \text{ seen as a member of } \mathbb{L}_{\geq 0} \text{ here})$ is countable and thus Lindelöf, so $R$ is Type I. Now, $R_{\alpha+1} - R_\alpha$ is exactly Roy’s lattice space and is thus connected (see details in [15] Ex. 126)). Since $R_{\alpha+1} = (R_{\alpha+1} - R_\alpha) \cup R_\alpha$, $(R_{\alpha+1} - R_\alpha) \cap R_\alpha = \{\{\alpha, 0\}\} \neq \emptyset$, and $R_\alpha = \cup_{\beta < \alpha} R_\alpha$ when $\alpha$ is limit, it follows by induction that $R_\alpha$ is connected (and thus so is $R$).

**Claim 2.8.2.** Any $g : R \to \mathbb{L}_{\geq 0}$ is constant.
Proof. Since $R_\alpha$ is connected and countable, $g$ must send $R_\alpha$ to a connected and at most countable subspace of $\mathbb{L}_{\geq 0}$. The only such subspaces of $\mathbb{L}_{\geq 0}$ are singletons, hence $g$ must be constant on $R_\alpha$ and thus on all of $R$.

It follows at once from Claim 2.8.2 that $R$ cannot be f-Type I.

Our next example is a subset of a (set-theoretic) tree. Recall that a tree is a partially ordered set such that the set of predecessors of each member is well ordered. Our terminology is standard, see e.g. [13] if in need of definitions of the terms height, level, chain and antichains.

Example 2.9. A first countable functionally Hausdorff Type I space $T$ which is not f-Type I.

Details. $T$ is a subset of a tree homeomorphic to $<\omega_1 \omega$, but its topology is weaker than the order topology. We first recall an example of a functionally Hausdorff non-regular countable space [15, Ex. 79] which we will use as a building brick (see Figure 1, left). (Beware that functionally Hausdorff spaces are called Urysohn, and completely Hausdorff means another closely related property, in [15].)

Let $B$ be the subset of the lattice of integers $\omega \times \omega$ defined as $\{\langle 0, 0 \rangle\} \cup A \cup S$, where $A = \{(n, m) : n, m \in \omega, n, m \geq 1\}$ and $S = \{(n, 0) : n \in \omega, n \geq 1\}$. Points in $A$ are isolated, while a neighborhood base for $\langle n, 0 \rangle \in S$ is given by $\{\langle n, m \rangle : m \in \omega, m > k\}$ for $k \in \omega$. That is, points in the vertical line above $\langle n, 0 \rangle$ converge to it. A neighborhood base of $\langle 0, 0 \rangle$ is given by $\{\langle 0, 0 \rangle\} \cup \{\langle n, m \rangle : n, m \in \omega, n, m > k\}$ for $k \in \omega$. It is easy to see that $S$ is functionally Hausdorff but non-regular since $S$ is closed in it and cannot be separated from $\langle 0, 0 \rangle$ (see [15, Ex. 79] for details). In fact, any neighborhood of $\{\langle 0, 0 \rangle\}$ has all the points in $S$ with first coordinate $> k$ (for some $k$) in its closure. Set $B_0$ to be $B - \{\langle 0, 0 \rangle\} = A \cup S$.

We now define $T$, which is a subset of $<\omega_1 B_0$. We see members of $T$ as sequences of length $< \omega_1$ with entries in $B_0$. Recall that the order $\leq$ on $<\omega_1 B_0$ is given by sequence extension, and that $\sigma^* b$ denotes the extension of the sequence $\sigma \in <\omega_1 B_0$ by adding $b \in B_0$ as the last member. We define $T$ (and its topology) by induction on the levels. As one may guess, $T_\alpha$, $T_{<\alpha}$ and $T_{\leq \alpha}$ respectively mean the elements at level $\alpha$, $< \alpha$ and $\leq \alpha$. A canonical cover for $T$ will be given by $\{T_{<\alpha} : \alpha \in \omega_1, \alpha \text{ is limit}\}$. Each $\sigma \in T_{\leq \alpha}$ will have a countable local base in $T_{\leq \alpha}$ denoted by $B(\sigma, \alpha)$.

The idea of the construction is the following. We start with a copy of $B$, and at successor

\textsuperscript{4} Separation axioms are a bit like the moitié-moitié, which may be understood as a dish of melted cheese or an alcoholic beverage depending on whether you are in Geneva or in Fribourg, actually even depending on the establishment on which you are sitting in Geneva. But we digress.
levels glue copies of $B$ to each member $s$ of $S$, identifying $s$ with $(0, 0)$ in the glued copy. The four first stages of this construction are illustrated in Figure 4 right. At limit levels, since we are in a subtree of a tree homeomorphic to $<\omega_1\omega$, we take the topology of cylinders (actually, equivalent to it) to obtain (at this level) a space homeomorphic to $\mathbb{R} - \mathbb{Q}$. $S$ being a closed discrete subspace of $B$ and the copies of $B_0$ being discrete as well in $T$ at successor levels, we cannot have uncountably many of them if we want the space to be of Type I. Hence, after a limit level, we extend only countably many sequences, choosing a countable subset which is dense in this limit level.

We now describe the construction in more details. The set $C_\alpha \subset T_\alpha$ will contain the sequences that continue in the next levels. $T_0 = C_0$ contains the empty sequence, which is open in $T_0$. The $\alpha + 1$th level $T_{\alpha + 1}$ is given by the set of $\sigma \upharpoonright b$ for each $b \in B_0$ and each $\sigma \in C_\alpha$, and $C_{\alpha + 1}$ is the subset of those $\sigma \upharpoonright b$ with $b \in S$. Then $\{\sigma \} \cup \{\sigma \upharpoonright b : b \in B_0\}$ will be a copy of $B$, with $\sigma$ playing the role of $(0, 0)$. The local bases (in $T_{\leq \alpha + 1}$) of points at levels $\alpha$ do not change. Given $\sigma \in C_\alpha$ and $O \in B(\sigma, \alpha)$, set

$$V_k(O) = O \cup \{\mu \upharpoonright \langle n, m \rangle : \mu \in O \cap C_\alpha, n, m > k\}.$$  

Then $B(\sigma, \alpha + 1) = \{V_k(O) : O \in B(\sigma, \alpha), k \in \omega\}$. For $\langle n, 0 \rangle \in S$, $B(\sigma \upharpoonright \langle n, 0 \rangle, \alpha + 1)$ is the set containing $\{\sigma \upharpoonright \langle n, 0 \rangle\} \cup \{\sigma \upharpoonright \langle n, m \rangle : m > k\}$ for each $k \in \omega$. Finally, if $a \in A$, then $\sigma \upharpoonright a$ is isolated. If $\alpha$ is limit, we set $T_\alpha$ to be $\{\sigma \in ^\omega B_0 : \sigma \upharpoonright \beta \in C_\beta \forall \beta < \alpha\}$. The neighborhood bases in $T_{\leq \alpha}$ do not change for points strictly below level $\alpha$. We add to the topology the bases for points in $T_\alpha \subset T_{\geq \alpha}$ given by $\{U_\sigma : \sigma \in \cup_{\beta < \alpha} C_\beta\}$, where $U_\sigma = \{\mu \in T_\alpha : \mu > \sigma\}$. Notice that $U_\sigma$ and $U_\nu$ are either disjoint (when $\sigma$ and $\nu$ are incomparable) or included one in the other (when $\sigma$ and $\nu$ are comparable), and that $U_\sigma$ is open in $T_{<\alpha}$. This topology is stronger than the usual cylinder topology on $T_\alpha$, where one fixes only finitely many values, but still second countable. Actually, we have:

**Claim 2.9.1.** $T_\alpha$ is homeomorphic to $\mathbb{R} - \mathbb{Q}$ when $\alpha$ is limit.

**Proof.** Let $\alpha_n$ be an increasing sequence of ordinals with supremum $\alpha$. Then $T_\alpha$ is homeomorphic to the space $M$ of $\omega$-sequences whose $n$th members are is $C_\alpha$, with the initial segment topology – that is, open sets are defined by fixing the first entries of the sequence. (Just forget the entries of the $\alpha$-sequence $\sigma \in T_\alpha$ that are in between levels $\alpha_n$.) By construction, each $C_\alpha$ is countable and given $\mu \in C_\alpha$, there are $\omega$-many members of $C_{\alpha + 1}$ above $\mu$. Hence, $M$ is homeomorphic to $^\omega \omega$ with the initial segment topology, which in turn is homeomorphic to $\mathbb{R} - \mathbb{Q}$.

We then let $C_\alpha$ be a countable dense subset of $T_\alpha$. This defines $T$ and its topology. By construction, each $T_{<\alpha}$ is second countable.

**Claim 2.9.2.** $U = \{T_{<\alpha} : \alpha$ is limit $\}$ is a canonical cover making $T$ a Type I space.

**Proof.** By construction, $\overline{T_{<\alpha}} = T_{\leq \alpha}$ when $\alpha$ is limit.

Notice that $T_{<\alpha + 1} = T_{\leq \alpha}$ is not open in $T$, but $T_{\leq \alpha} \cup \{\mu \upharpoonright \langle n, m \rangle : \mu \in C_\alpha, n, m > 0\}$ is.

**Claim 2.9.3.** $T$ is functionally Hausdorff.

**Proof.** Notice that any real valued function $f$ on $T_{\leq \alpha}$ can be continuously extended to all of $T$ by letting $f(\sigma) = f(\sigma \upharpoonright \alpha)$ for $\sigma$ at a level higher than $\alpha$. Let $\sigma, \mu$ be distinct members of $T$. If one of them is isolated, we are over. Otherwise, none of their entries (as sequences) is in $A$. If $\sigma$ and $\mu$ are incomparable in $<\omega_1 B_0$, let $\alpha$ be minimal such that $\sigma \upharpoonright \alpha \neq \mu \upharpoonright \alpha$. Then $\alpha = \beta + 1$ for some $\beta$, hence $\sigma \upharpoonright \alpha = \nu \upharpoonright \langle n, 0 \rangle$ and $\mu \upharpoonright \alpha = \nu \upharpoonright \langle m, 0 \rangle$ for some $\nu$ at level $\beta$, and $n \neq m$. Define $f : T_{<\alpha} \to \mathbb{R}$ to take value 1 on $\{\nu \upharpoonright \langle n, k \rangle : k \in \omega\}$ and 0 elsewhere. Then $f$ is continuous, extending it above $T_{\leq \alpha}$ yields a map $f : T \to \mathbb{R}$ with $f(\sigma) = 1$, $f(\mu) = 0$. If $\sigma < \mu$, $f(\sigma) - f(\mu) = 1$. This is functionally Hausdorff.


let \( n \) be such that \( \sigma^n(n,0) \leq \mu \). Let \( f \) take value 1 on \( \{\sigma^n(n,k) : k \in \omega\} \) and all the points above \( \sigma^n(n,0) \), and 0 elsewhere. Then \( f \) is continuous, and takes value 0 on \( \sigma \) and 1 on \( \mu \). \( \square \)

**Claim 2.9.4.** If \( \sigma \in C_\alpha \subset T \) and \( f : T \to \mathbb{R} \), then for any interval \((a,b) \ni f(\sigma)\) and any \( \beta \geq \alpha \), there is \( \mu \in C_\beta, \mu \geq \sigma \), such that \( f(\mu) \in [a,b] \).

**Proof.** By induction on \( \beta \). If \( \beta = \alpha \), it is obvious. If \( \beta = \gamma + 1 \), fix \( c,d \in \mathbb{R} \) such that \( f(\sigma) \in (c,d) \subset [c,d] \subset (a,b) \) and \( \mu \in f^{-1}([c,d]) \cap C_\gamma, \mu \geq \sigma \). By definition of the neighborhood base, \( f^{-1}((a,b]) \) contains \( \mu^n(n,0) \) if \( n \) is big enough, hence so does \( f^{-1}([a,b]) \). Assume now that \( \beta \) is limit and let \( \beta_n \) be an increasing sequence converging to \( \beta \). Let \( d \) be the minimum between \( f(\sigma) - a, b - f(\sigma) \) and let \( d_n \) be a sequence of strictly positive numbers whose sum is less than \( d/2 \). By induction, for each \( n \) there is \( \nu_n \in C_{\beta_n} \) such that \( \nu_{n+1} \geq \nu_n \) (hence \( \nu_{n+1} \) extends \( \nu_n \)) and \( f(\nu_n) \in [f(\nu_n) - d_n, f(\nu_n) + d_n] \). By construction, \( \nu \) defined as the supremum (in \( T \)) of the \( \nu_n \) (which thus converge to \( \nu \)) is a member of \( T_\beta \), and \( |f(\sigma) - f(\nu)| \leq d/2 \). By density (and the fact that the members of \( T_\beta \) above \( \sigma \) form an open set), there is a member \( \mu \) of \( C_\beta \) above \( \sigma \) in \( f^{-1}((f(\nu) - d/2, f(\nu) + d/2)) \). Since \( |f(\mu) - f(\sigma)| \leq d, f(\mu) \in [a,b] \). \( \square \)

If follows that \( T \) is not \( f \)-Type I. Indeed, given \( f : T \to \mathbb{R}_{>0} \), take any \( \sigma \in C_\alpha \subset T \) and \( \gamma \) such that \( f(\sigma) \in [0,\gamma] \). Define \( g : T \to [0,\gamma+2] \) as the minimum between \( \gamma + 2 \) and \( f \), then \( f^{-1}([0,\gamma+1]) = g^{-1}([0,\gamma+1]) \) is unbounded in the tree by the previous claim and hence non-Lindelöf. \( \square \)

**Remark.** The space \( B \) has a weaker regular topology, but there is no weaker regular topology on \( T \) such that \( \{T_{\alpha} : \alpha \in \omega_1\} \) is still a canonical cover. Indeed, if \( X \) with topology \( \tau \) is Type I with canonical cover \( \{U_\alpha : \alpha \in \omega_1\} \) and there is a weaker topology \( \rho \subset \tau \) such that \( (X,\rho) \) is regular, \( U_\alpha \) is \( \rho \)-open for each \( \alpha \), and the \( \rho \)-closure of \( U_\alpha \) is included in \( U_\gamma \) for some \( \gamma \geq \alpha \), then by Lemma 2.3 \( (X,\tau) \) is \( f \)-Type I.

By Lemma 2.3 (b), being relatively Type I is equivalent to being relatively \( f \)-Type I when the ambient space is normal. We were not able to decide the following question, although we anticipate a positive answer.

**Question 2.10.** Is there a regular (or even Tychonoff) non-Type I space \( Y \) containing a subspace \( X \) which is Type I in \( Y \) but not \( f \)-Type I in \( Y \) ?

Let us now briefly mention a partial order on systematic covers of spaces, whose study we have not pursued (except for the simple lemmas below), but which gives another light on the concept of a canonical cover. If \( U \) is a cover of \( X \) and \( B \subset X \), we let \( U \cap B \subset \{U \cap B : U \in U\} \).

**Definition 2.11.** Let \( U, V \) be covers of some space \( X \), and \( B \subset X \) be closed. Then \( U \leq_B V \) iff \( U \cap B \) is a refinement of \( V \cap B \), that is, for each \( U \in U \) there is \( V \in V \) with \( U \cap B \subset V \cap B \). If \( U \leq_B V \leq_B U \) we write \( U \asymp_B V \). We abbreviate \( \leq_{X,\asymp_X} \) by \( \leq, \asymp \).

Our first lemma shows that canonical covers are in a sense \( \leq \)-minimal.

**Lemma 2.12.** Let \( B \subset X \) be Type I in \( X \), \( U \) be a canonical cover of \( B \) in \( X \) and \( V = \{V_\alpha : \alpha \in \omega_1\} \) be a chain cover of \( X \). Then \( U \leq_B V \).

**Proof.** The closure of each member of \( U \) intersected with \( B \) is Lindelöf, and hence included in some \( V_\alpha \). \( \square \)

**Lemma 2.13.** Let \( U = \{U_\alpha : \alpha \in \lambda\}, V = \{V_\alpha : \alpha \in \kappa\} \) be good systematic covers of some space and \( B \) be a closed subset. Assume that \( U_\alpha \neq U_\beta \) and \( V_\alpha \neq V_\beta \) when \( \alpha < \beta \). Then \( U \asymp_B V \) iff \( \lambda = \kappa \) and \( \{\alpha \in \kappa : U_\alpha \cap B = V_\alpha \cap B\} \) is club in \( \kappa \).
Proof. The reverse implication is immediate. For the direct implication, recall that \( \lambda, \kappa \) are regular cardinals since the covers are good. For each \( \alpha \in \lambda \) there are \( \beta \in \kappa \) and \( \gamma \in \kappa \) such that
\[
U_\alpha \cap B \subset V_\beta \cap B \subset V_\gamma \cap B \subset U_\gamma \cap B,
\]
hence \( \kappa \) and \( \lambda \) have the same cofinality and are thus equal. A leapfrog argument then shows that \( \{ \alpha \in \kappa : U_\alpha \cap B = V_\alpha \cap B \} \) is club.

The following well known corollary explains the choice of terminology for canonical covers. (This terminology comes from P. Nyikos, who uses the slightly different term canonical sequence in [12].)

**Corollary 2.14.** Let \( B \) be closed and Type I in the space \( X \). If \( U = \{ U_\alpha : \alpha \in \omega_1 \}, V = \{ V_\alpha : \alpha \in \omega_1 \} \) are canonical covers of \( B \) in \( X \), then \( \{ \alpha : U_\alpha \cap B = V_\alpha \cap B \} \) is club in \( \omega_1 \).

**Proof.** Lemma 2.12 implies \( U \nsubseteq V \), and we conclude with Lemma 2.13.

We finish this section with some definitions and results that will be useful later. If \( U = \{ U_\alpha : \alpha \in \omega_1 \} \) is a canonical cover of a Type I subspace \( D \) of \( X \), its set of bones is \( \text{Bo}(U, D) = \{ (U_\alpha - U_\gamma) \cap D : \alpha \in \omega_1 \} \), and its skeleton \( \text{Sk}(U, D) \) is the union of the bones. Notice that \( \text{Sk}(U, D) \) is closed in \( X \). We denote \( \text{Bo}(U, X), \text{Sk}(U, X) \) by \( \text{Bo}(U), \text{Sk}(U) \). (This terminology is also due to P. Nyikos.) Recall that a space is \( \omega_1 \)-compact (or has countable extent) iff its closed discrete subspaces are at most countable.

**Lemma 2.15.** Let \( D \subset X \subset Y \), where \( D \) is closed non-Lindelöf, \( X \) closed and Type I in the space \( Y \). Let \( U \) be a canonical cover of \( X \) in \( Y \). Then the following hold.

(a) If \( X \) is countably compact, then \( D \) intersects the members of \( \text{Bo}(U) \) on a club set of indices.
(b) If \( X \) is \( \omega_1 \)-compact, then \( D \) intersects the members of \( \text{Bo}(U) \) on a stationary set of indices.

**Proof.** This is well known, and (a) is straightforward. For (b), assume that \( X \) is \( \omega_1 \)-compact. If \( D \) misses the bones of \( U \) on a club set of indices, taking one point of \( D \) (when available) between each member of this club set yields a closed discrete uncountable subset of \( X \).

Recall that a space is \( \omega \)-bounded iff each countable subset has a compact closure. A product of \( \omega \)-bounded spaces is \( \omega \)-bounded. The following lemma is immediate.

**Lemma 2.16.** Let \( X \) be closed and Type I in the space \( Y \). Then \( X \) is countably compact iff \( X \) is \( \omega \)-bounded iff each member of a canonical cover of \( X \) has compact closure when intersected with \( X \).

## 3 Narrow and functionally narrow subspaces

Notice that the Tychonoff plank (Example 2.6) and any \( \gamma \mathbb{N} \) (Example 2.7) are narrow in themselves but not Type I in themselves. We will be more interested in cases where narrow (sub)spaces are also Type I. We start with the most basic examples. Recall that the longline \( L \) consists of two copies of \( \mathbb{L}_{\geq 0} \) glued at their 0 point, with the reverse order on one of the copies.

**Example 3.1.** \( L_{\geq 0}, \omega_1 \) are Type I and narrow in themselves. \( L, (L_{\geq 0})^2 \) are Type I in themselves but not f-narrow.

**Details.** Anyone who read until this point should be convinced, or see the next result.

The example of \( L \) shows in passing that the union of two closed subsets which are (f-)narrow in some space is not always (f-)narrow. It might be interesting to note the following (easy) theorem.
Lemma 3.4. The notions “stationary” and “club” extend naturally to copies of \( \omega \).

Theorem 3.2. Let \( S \) be an uncountable subset of \( \omega_1 \) endowed with the subspace topology. Then \( S \) is f-Type I and the following are equivalent.

(a) \( S \) is stationary,
(b) \( S \) is \( \omega_1 \)-compact,
(c) \( S \) is narrow in itself.

Proof. It is immediate that \( S \) is f-Type I, and (a) \( \leftrightarrow \) (b) is classical.

(a) \( \rightarrow \) (c). If \( A \) is closed non-Lindelöf in \( S \), it is unbounded. Hence \( A \) contains \( C \cap S \) for a club \( C \subset \omega_1 \), hence a stationary subset of \( \omega_1 \). By Fodor’s lemma, any open \( V \supset A \) contains a terminal part of \( S \). Since the rest of the space is countable, any systematic (actually, chain) cover of \( S \) with a non-Lindelöf member has a member containing all of \( S \).

(c) \( \rightarrow \) (a). If \( S \) is non-stationary, it misses a club \( C \subset \omega_1 \). Take one isolated point of \( S \) between each member of \( C \) (when available), this yields a clopen uncountable discrete subset \( E \) of \( S \).

By throwing away some points, we may assume that \( E \) has uncountable complement in \( S \). Then \( \{(0, \alpha) \cap S \} \cup E : \alpha \in \omega_1 \} \) is a systematic cover of \( S \) whose members have non-Lindelöf closure.

We now define two spaces which we use as building blocks in many of our examples. When considering the space \( \mathbb{L}_{\geq 0} \cup \{\omega_1\} \), the neighborhoods of \( \{\omega_1\} \) are of course final segments of \( \mathbb{L}_{\geq 0} \) union \( \{\omega_1\} \).

Definition 3.3. The octant is defined as \( \mathbb{O} = \{\langle x, y \rangle \in (\mathbb{L}_{\geq 0})^2 : y \leq x \} \), and its diagonal and horizontal “boundaries” are denoted \( \Delta = \{\langle x, x \rangle : x \in \mathbb{L}_{\geq 0}\} \), \( H = \{\langle x, 0 \rangle : x \in \mathbb{L}_{\geq 0}\} \).

The “horizontally compactified octant” is \( \widehat{\mathbb{O}} = \{\langle x, y \rangle \in \mathbb{L}_{\geq 0} \cup \{\omega_1\} \times \mathbb{L}_{\geq 0} : y \leq x \} \), with its diagonal, horizontal and vertical “boundaries” \( \widehat{\Delta}, \widehat{H} = \{\langle x, 0 \rangle : x \in \mathbb{L}_{\geq 0} \cup \{\omega_1\}\} \), \( V = \{\langle \omega_1, x \rangle : x \in \mathbb{L}_{\geq 0}\} \).

We also define \( \mathbb{O}_{\geq \alpha} = \{\langle x, y \rangle \in \mathbb{O} : x, y \geq \alpha \} \) (and \( \Delta_{\alpha}, H_{\alpha} \)) and \( \widehat{\mathbb{O}}_{\geq \alpha} = \{\langle x, y \rangle \in \widehat{\mathbb{O}} : x, y \geq \alpha \} \) (and \( \Delta_{\alpha}, H_{\alpha}, V_{\alpha} \) defined accordingly).

\( \mathbb{O}_{\geq \alpha}, \widehat{\mathbb{O}}_{\geq \alpha} \) are represented as in Figure 2. \( \mathbb{O}_{\geq \alpha} \) is homeomorphic to \( \mathbb{O}_{\geq \beta} \) for each \( \alpha, \beta \in \omega_1 \), and so are \( \widehat{\mathbb{O}}_{\geq \alpha} \) and \( \widehat{\mathbb{O}}_{\geq \beta} \). Notice that \( \mathbb{O}_{\geq \alpha} \) is first countable while \( \widehat{\mathbb{O}}_{\geq \alpha} \) is not at the points of \( V_{\alpha} \), hence the bolder line used for representing \( V_{\alpha} \) in Figure 2. The arrow from \( H_{\alpha} \) to \( \Delta_{\alpha} \) is inspired by Point (c) in the next well known lemma, which contains most of the basic topological properties of these octants. The notions “stationary” and “club” extend naturally to copies of \( \omega_1 \) or \( \mathbb{L}_{\geq 0} \) in \( (\mathbb{L}_{\geq 0})^2 \) such as \( \Delta, H, V \).

Lemma 3.4 (Properties of \( (\mathbb{L}_{\geq 0})^2, \mathbb{O}, \widehat{\mathbb{O}} \)).

(a) Let \( U \subset (\mathbb{L}_{\geq 0})^2 \) be open. If \( U \) intersects the diagonal in a stationary subset, then \( U \supset [\alpha, \omega_1]^2 \) for some \( \alpha \). If \( U \) intersects a horizontal (resp. vertical) line in a stationary subset, then \( U \)
contains a horizontal strip \((\alpha, \omega_1) \times (a, b)\) (resp. vertical strip \((a, b) \times (\alpha, \omega_1)\)) for some \(\alpha \in \omega_1\) and \(a < b\) in \(L_{\geq 0}\).

(b) A non-Lindelöf closed subset of \((L_{\geq 0})^2\) intersects either the diagonal, an horizontal line or a vertical line on a club subset.

(c) If \(f : \hat{\Omega} \to L_{\geq 0}\) is unbounded on any horizontal line, then it is unbounded on each horizontal line and on \(\Delta\).

(d) If \(U \subset \hat{\Omega}\) is open and intersects \(\Delta\) or \(V\) in a stationary subset, then \(\overline{U} \supset [\alpha, \omega_1] \times [\alpha, \omega_1] \cap \hat{\Omega}\) for some \(\alpha\).

(e) Given \(g : \hat{\Omega} \to \mathbb{R}\), there is \(c \in \mathbb{R}\) such that \(g^{-1}(\{c\}) \supset [\alpha, \omega_1] \times [\alpha, \omega_1] \cap \hat{\Omega}\).

(f) If \(g : \hat{\Omega} \to \mathbb{R}\) is such that \(g^{-1}((a, b)) \cap V\) is non-Lindelöf, then \(g^{-1}((a, b))\) contains a terminal part of \(\Delta\).

(g) A non-Lindelöf closed subset of \(\hat{\Omega}\) has a club intersection with \(\Delta\) or \(V\).

**Proof.** Each property is well known and easy, for instance (a) and (b) are essentially proved in [12] Lemma 3.4 or [10] Lemma B.29 p. 194], (d), (e), (f) and (g) in [11] 8L p.158]. For (c), notice that (a) (for horizontal lines) implies that the set \(\{y \in L_{\geq 0} : f\) is bounded on \([y, \omega_1] \times \{y\}\}\) is clopen in \(L_{\geq 0}\), and hence either empty or the whole space. Applying (a) again (for the diagonal) yields the result.

**Example 3.5.** \(L_{\geq 0} \times [0, 1]\) and \(L_{\geq 0} \times \mathbb{R}\) are f-Type I and narrow (in themselves).

**Details.** Both are obviously f-Type I (use the horizontal projection). Assume \(\{U_\alpha : \alpha \in \omega_1\}\) is a systematic cover of \(L_{\geq 0} \times [0, 1]\). Suppose that \(\overline{U_\alpha}\) is non-Lindelöf for some \(\alpha\). We show that this implies that \(L_{\geq 0} \times [0, 1]\) is contained in some \(U_\delta\), which entails its narrowness. By Lemma 3.4 (a) and (b), \(U_{\alpha+1}\) contains a strip \((\beta_\alpha, \omega_1) \times (a_\alpha, b_\alpha)\) for \(\beta_\alpha \in \omega_1\), \(a_\alpha, b_\alpha \in [0, 1]\). Since \(\overline{U_{\alpha+1}} \supset [\beta_\alpha, \omega_1] \times (a_\alpha, b_\alpha)\), \(U_{\alpha+2}\) contains a strictly wider (but maybe shorter) strip \((\beta_{\alpha+1}, \omega_1) \times (a_{\alpha+1}, b_{\alpha+1})\), that is, \(\beta_{\alpha+1} \geq \beta_\alpha\) and \(a_{\alpha+1} < a_\alpha < b_\alpha < b_{\alpha+1}\). Proceeding by induction we obtain an \(\omega_1\)-sequence of wider and wider strips. But since \([0, 1]\) is hereditarily Lindelöf, the vertical projection of these strips must cover all of it after at most countably many steps, hence some \(U_\gamma\) contains \((\beta, \omega_1) \times [0, 1]\) for some \(\gamma, \beta\). Now, the space not covered by \(U_\gamma\) is compact, hence \(L_{\geq 0} \times [0, 1]\) is contained in \(U_\delta\) for some \(\delta > \gamma\). The proof \(L_{\geq 0} \times \mathbb{R}\) is the same.

The fact that \([0, 1]\) and \(\mathbb{R}\) are connected is crucial, as \(L_{\geq 0} \times X\) is never narrow in itself when \(X\) is the disjoint union of two (non-empty) clopen subsets.

**Example 3.6.** \(\emptyset\) is f-Type I but not f-narrow, \(\hat{\Omega}\) is f-Type I and narrow.

**Details.** For \(\emptyset\), consider the horizontal and vertical projections: the former yields f-Type I-ness, the latter the non-f-narrowness. For \(\hat{\Omega}\), the vertical projection shows that it is a f-Type I space. Let \(\{U_\alpha : \alpha \in \omega_1\}\) be a systematic cover of \(\emptyset\). If some \(U_\alpha\) is non-Lindelöf, by Lemma 3.4 (g) and (d), it intersects either \(\Delta\) or \(V\) on a club set and thus \(U_{\alpha+1}\) contains \([\alpha, \omega_1] \times [\alpha, \omega_1] \cap \hat{\Omega}\) for some \(\alpha\). Since the rest of \(\hat{\Omega}\) is compact, \(\hat{\Omega}\) is contained in \(U_\gamma\) for some \(\alpha < \gamma < \omega_1\).

Being (f-)narrow is quite sensitive to the ambient space: a space might be (f-)narrow in some spaces but not in others. Obviously, given any space, a closed copy of it is narrow in any narrow-in-itself space. But a non-narrow-in-itself space can become narrow in another non-narrow-in-itself space.

**Example 3.7.** Any copy of \(L \times [0, 1]\) in \(\emptyset\) is narrow in it, although neither \(L \times [0, 1]\) nor \(\emptyset\) are f-narrow in themselves.

**Details.** By Lemma 3.4 (a), any such copy must be contained in \((L_{\geq 0} \times [0, \alpha]) \cap \emptyset\) for some \(\alpha\). But the latter is Type I and narrow in itself (Example 3.5).
Notice however that it is not true for \( \mathbb{L} \) itself, as \( \Delta \cup H \) is a copy of \( \mathbb{L} \) not f-narrow in \( \mathbb{O} \), as the vertical projection shows. Our next result gives many properties equivalent to f-narrowness.

**Theorem 3.8.** \( D \subset X \) be closed and f-Type I in \( X \). Then the following properties are all equivalent.

(a) \( D \) is f-narrow in \( X \).

(b) Given a map \( f : X \to \mathbb{L}_{\geq 0} \) and \( E_1, E_2 \subset D \) two closed non-Lindelöf subsets, then \( f \upharpoonright E_1 \) is bounded iff \( f \upharpoonright E_2 \) is bounded.

(c) There is no map \( f : X \to \mathbb{L}_{\geq 0} \) with \( f \upharpoonright D \) unbounded and \( f^{-1}(\{0\}) \cap D \) non-Lindelöf.

(d) Let \( f : X \to \mathbb{L}_{\geq 0} \) be such that \( f \upharpoonright D \) is unbounded and \( \{U_\alpha : \alpha \in \omega_1\} \) be a systematic chain cover of \( X \) with \( \bigcup_{\alpha \in D} \) Lindelöf for each \( \alpha \). Then for each \( \alpha \in \omega_1 \) there is \( \gamma(\alpha) \in \omega_1 \) such that \( f(D - U_{\gamma(\alpha)}) \subset [\alpha, \omega_1) \).

(e) Let \( \{U_\alpha : \alpha \in \omega_1\} \) be a systematic chain cover of \( X \) with \( \bigcup_{\alpha \in D} \) Lindelöf for each \( \alpha \). Then for every \( f : X \to \mathbb{L}_{\geq 0} \) with \( f \upharpoonright D \) unbounded there is aclub \( C \subset \omega_1 \) such that \( f(D - U_\alpha) \subset [\alpha, \omega_1) \) for each \( \alpha \in C \).

(f) Given \( f : X \to \mathbb{L}_{\geq 0} \), then there is \( \alpha \in \omega_1 \) such that for any \( \beta \geq \alpha \) either \( f^{-1}([0, \beta]) \cap D \) or \( f^{-1}((\beta, \omega_1]) \cap D \) is (empty or) Lindelöf.

**Proof.** If \( D \) is Lindelöf, each point holds trivially. We thus assume throughout the proof that \( D \) is non-Lindelöf.

(a) \( \to \) (b). Assume that \( D \) is f-narrow, and \( f, E_1, E_2 \) be as in (b). We show that \( f \upharpoonright E_i \) is bounded iff \( f \upharpoonright D \) is bounded, for \( i = 1, 2 \). If \( f \upharpoonright E_i \) is bounded, then \( f^{-1}((0, \alpha]) \cap D \supset E_i \) for some \( \alpha \). By f-narrowness, \( f \upharpoonright D \) must then be bounded. The conversely is immediate.

(b) \( \to \) (a). Let \( f : X \to \mathbb{L}_{\geq 0} \) and \( \alpha \in \omega_1 \) be given, and set \( E_1 = D \) and \( E_2 = f^{-1}((0, \alpha]) \cap D \). If \( E_2 \) is non-Lindelöf (b) implies that \( f \upharpoonright D \) is bounded. This shows that \( D \) is f-narrow in \( X \).

(a) \( \to \) (c). Immediate.

(c) \( \to \) (a). If \( D \) is not f-narrow, then there is a \( g : X \to \mathbb{L}_{\geq 0} \) unbounded on \( D \) and an \( \alpha \) such that \( g^{-1}((0, \alpha]) \cap D \) is non-Lindelöf. Let \( p : \mathbb{L}_{\geq 0} \to \mathbb{L}_{\geq 0} \) be continuous such that \( p((0, \alpha]) = 0 \) and \( p \upharpoonright [\alpha + 1, \omega_1) = id \), then \( f = p \circ g \) yields a contradiction.

(b) \( \to \) (d). If there is some \( \alpha \) such that \( E_1 = f^{-1}([0, \alpha]) \) contains points in \( X - U_\gamma \) for arbitrarily high \( \gamma \), then \( E_1 \) is non-Lindelöf. Since \( f \) is unbounded on \( D = E_2 \), this contradicts (b).

(d) \( \to \) (a). Immediate.

(d) \( \to \) (e). By a leapfrog argument and continuity of \( f \).

(e) \( \to \) (d). Immediate.

(b) \( \to \) (f). If \( f : X \to \mathbb{L}_{\geq 0} \) is bounded on \( D \), then, for some \( \alpha \), \( f^{-1}((\alpha, \omega_1]) \cap D \) is empty. If \( f \upharpoonright D \) is unbounded, then \( f^{-1}((\alpha, \omega_1]) \cap D \) is non-Lindelöf for each \( \alpha \). By (b) \( f^{-1}((0, \alpha]) \cap D \) must be Lindelöf.

(f) \( \to \) (c). Let \( f : X \to \mathbb{L}_{\geq 0} \) be unbounded on \( D \) and let \( \alpha \) be as in (f). Since \( f^{-1}([\alpha, \omega_1]) \cap D \) is non-Lindelöf, \( f^{-1}((0, \alpha]) \) is Lindelöf, and hence so is \( f^{-1}({0}) \).

Type I and f-Type I spaces coincide in the realm of regular spaces, this is not the case for f-narrow and narrow spaces for which a stronger property (e.g. normality) is needed.

**Lemma 3.9.** Let \( X \) be a normal space and \( D \) be closed in \( X \). Then \( D \) is narrow in \( X \) iff \( D \) is f-narrow in \( X \).

**Proof.** Only the reverse implication needs a proof, and it follows directly from Lemma 2.2.

**Example 3.10.** There is a Type I Tychonoff countably compact locally compact space \( Y \) which is f-narrow but not narrow (in itself).

**Details.** The ideas are similar to that of Example 2.9 (but the construction is simpler). As a set, \( Y \) is the union of (non-disjoint) copies \( \hat{O}_\alpha \) of \( \hat{O}_{\geq \alpha} \) for each \( \alpha \in \omega_1 \). Given \( (x, y) \in \hat{O}_{\geq \alpha} \), we write \( (x, y)_{\alpha} \) for its copy in \( \hat{O}_\alpha \), but (somewhat abusively) still write \( V_{\alpha}, \Delta_{\alpha} \) for their respective
Figure 3: Piling up copies of $O_{\geq \alpha}$

copies in $\hat{O}_\alpha$. The topology of each $\hat{O}_\alpha$ outside of $\Delta_\alpha$ and $V_\alpha$ is that of $\hat{O}_{\geq \alpha}$. We glue $\hat{O}_{\alpha+1}$ “on top” of $\hat{O}_\alpha$ by identifying $(x, x)_\alpha \in \Delta_\alpha \subset \hat{O}_\alpha$ with $x \geq \alpha + 1$ with $(\omega_1, x)_{\alpha+1} \in V_{\alpha+1} \subset \hat{O}_{\alpha+1}$, as seen on Figure 3 (left). For $\alpha$ limit, a neighborhood of $(\omega_1, x)_\alpha \in V_\alpha \subset \hat{O}_\alpha$ is given by fixing $z < x < y$ in $L_{\geq 0}$ and $\gamma < \alpha$ in $\omega_1$ and taking the union for $\gamma < \beta < \alpha$ of the horizontal bands between height $z, y$ in $\hat{O}_\beta$, together with an open set in $\hat{O}_\alpha$ whose intersection with $V_\alpha$ is the open segment between $(\omega_1, z)_\alpha$ and $(\omega_1, y)_\alpha$ (or the segment $\{\omega_1\} \times [\alpha, z]$ if $x = (\alpha, \alpha) \in \hat{O}_\alpha$). It should be clear $(\alpha, \alpha)_\alpha$ is the limit point of the sequence $(\alpha_n, \alpha_n)_{\alpha_n}$ for any sequence of ordinals $\alpha_n \nearrow \alpha$. Actually, it is not hard to see that $Y$ is countably compact and that the following claim holds.

Claim 3.10.1. If $z_n \in Y$ ($n \in \omega$) is a sequence such that $z_n = (x_n, y_n)_{\alpha_n}$ and $\alpha_{n+1} > \max(y_n, \alpha_n)$, then this sequence converges to $(\alpha, \alpha)_\alpha$ for $\alpha = \sup_{n \in \omega} \alpha_n$.

Define $W$ as $\{(\alpha, \alpha)_\alpha : \alpha \in \omega_1\}$. By the previous claim, $W$ is a copy of $\omega_1$ in $Y$. Let $E_{\alpha, \beta}$ be $\{(x, y)_\alpha \in \hat{O}_\alpha : y \geq \beta\}$, that is, the points in $\hat{O}_\alpha$ with vertical coordinate $\geq \beta$.

Claim 3.10.2. Let $C \subset Y$. (i) If there is some $\beta$ such that $C \cap E_{\gamma, \beta} \neq \emptyset$ for a subset of $\gamma$ which is unbounded below some limit $\alpha < \omega_1$, then $C \cap V_\alpha \cap E_{\alpha, \beta} \neq \emptyset$. (ii) If the set of $\gamma < \omega_1$ such that $C \cap \hat{O}_\gamma \neq \emptyset$ is unbounded in $\omega_1$, then $C \cap W$ is club.

Proof. (i) Fix $\gamma_n \nearrow \alpha$ and $(x_n, y_n)_{\gamma_n} \in E_{\gamma_n, \beta} \cap C$. Then take a converging subsequence to obtain a point in $V_\alpha$. (ii) For $W$, fix $\gamma_0 \in \omega_1$, and by induction choose $(x_n, y_n)_{\alpha_n} \cap C$ such that $\alpha_{n+1} > \max(y_n, \alpha_n)$. Then apply Claim 3.10.1.

It should be clear that $Y$ is locally compact (and thus Tychonoff). Moreover, $Y$ is Type I (functionally, thanks to Lemma 2.3) with canonical cover

$$U_\alpha = \cup_{\beta < \alpha} \{(x, y)_\beta \in \hat{O}_\beta : y < \alpha\}$$

(see Figure 3 right). Actually, $\overline{U_\alpha}$ is compact for each $\alpha$. The systematic cover

$$\{(\cup_{\beta < \alpha} \hat{O}_\beta) - V_\alpha : \alpha \in \omega_1\}$$

shows that $Y$ is not narrow it itself. However, $Y$ is functionally narrow in itself, as we shall now see.
Claim 3.10.3. (i) For any open set $U$ whose intersection with $V_\alpha$ is stationary for some $\alpha < \omega_1$ there are $\gamma < \alpha$ and $\delta < \omega_1$ such that $\overline{U}$ contains $\bigcup_{\gamma \leq \delta < \omega_1} E_{\gamma, \delta}$. (ii) Any open set whose intersection with $W$ is stationary (that is, it contains $\langle \alpha, \alpha \rangle_\alpha$ for a stationary set of $\alpha$) contains $\bigcup_{\gamma > 0} \hat{O}_\gamma$ for some $\gamma$.

Proof. We show (ii) first. Let $U$ be open and intersect $W$ on a stationary set and assume that the claim does not hold. Then the complement of $U$ has a club intersection with $W$ by Claim 3.10.2 and thus intersects $U$, a contradiction. (i) If $\alpha = \gamma + 1$, then $\overline{U}$ contains $E_{\gamma, \beta}$ for some $\beta$ by Lemma 3.3 (d). If $\alpha$ is limit, it follows from the previous claim. \qed

Claim 3.10.4. If $g: Y \to \mathbb{R}_{\geq 0}$ is such that $g^{-1}(\{0, a]\})$ is non-Lindelöf for some $a \geq 0$, then $g^{-1}(\{0, b]\}) \cap W$ is club for any $b > a$ and contains each $\hat{O}_\alpha$ above some $\gamma$.

Proof. Let $b > a$ be given. If $g^{-1}(\{0, a]\}) \cap W$ is stationary, we are over by Claim 3.10.3. If not, by Claim 3.10.2 there is $\gamma < \omega_1$ such that $g^{-1}(\{0, b]\}) \cap \hat{O}_\alpha = \emptyset$ for each $\alpha \geq \gamma$. Hence $g^{-1}(\{0, a]\}) \cap \hat{O}_\alpha$ is non-Lindelöf for some $\alpha < \gamma$. By Lemma 3.3 (d) & (f), $g^{-1}(\{0, c]\})$ contains a terminal part of $V_\alpha$ for any $c > a$. Let $e: \gamma \to \mathbb{R}$ be a monotone embedding of the ordinal $\gamma + 1$ such that $e(\alpha) = a$ and $a < e(\gamma) < b$. Applying inductively Lemma 3.3 (d) & (f), we see that $g^{-1}(\{0, e(\beta)\})$ contains a terminal part of $V_\beta$ for each $\beta$ between $\alpha$ and $\gamma$ (included). Hence, $g^{-1}(\{0, b\})$ must contain a terminal part of $V_\gamma$, contradicting $g^{-1}(\{0, b\}) \cap \hat{O}_\gamma = \emptyset$. \qed

Now let $f: Y \to L_{\geq 0}$ be given with $f^{-1}(\{0, a]\})$ non-Lindelöf. Set $g(x) = \min\{f(x), \alpha + \omega\}$, then $g$ has range $\subset [0, \alpha + \omega + 1] \simeq \mathbb{R}_{\geq 0}$, and by the previous claim $f^{-1}(\{0, \alpha + 1\}) = g^{-1}(\{0, \alpha + 1\})$ contains each $\hat{O}_\alpha$ above some $\gamma$. By Claim 3.10.3 applied recursively, there is $\in \omega$ and $\beta < \omega_1$ such that $f^{-1}(\{0, \alpha + n\})$ contains $\bigcup_{\alpha < \gamma} E_{\alpha, \beta} \cup \bigcup_{\gamma \leq 0 < \omega_1} \hat{O}_\alpha$. But the complement of this set is compact, hence $f^{-1}(\{0, b\})$ contains all of $Y$ for some $b \in \omega_1$ bigger than $\alpha + n$, that is $f$ is bounded. This shows that $Y$ is $\text{f-narrow}$. \qed

Question 3.11. Is there a first countable space as in Example 3.10? \qed

4 Type I spaces without (f-)narrow non-Lindelöf subspaces

There are trivial examples of f-Type I spaces without non-Lindelöf narrow subspaces.

Example 4.1. A discrete space of cardinality $\aleph_1$ is a Type I and does not contain any non-Lindelöf subspace f-narrow in it.

It is more difficult to find examples if one imposes further properties on the space such as countable compactness or $\omega_1$-compactness. If one adds to the mix some local demands (first countability or weakenings of it), the existence of such spaces ends up depending on the axioms of set theory. Let us first show that there is a countably compact f-Type I space without non-Lindelöf subspaces f-narrow in it. This example is not countably tight, as none can be exhibited in $\text{ZFC}$ alone since the proper forcing axiom $\text{PFA}$ prevents their existence (see Theorem 4.8 below). Recall that $\beta \omega$ is the Čech-Stone compactification of the integers.

Example 4.2. There is an $\omega$-bounded f-Type I subspace $Q$ of $\omega^* = \beta \omega - \omega$ containing no closed non-compact subspace f-narrow in it.

Recall that $\omega$-boundedness and countable compactness are equivalent for Type I spaces.
Details. Let $A = \{A_\alpha : \alpha \in \omega_1\}$ be a $\subset^*$ strictly increasing $\omega_1$-sequence of infinite subsets of $\omega$, that is: $A_\alpha - A_\beta$ is finite and $A_\beta - A_\alpha$ infinite when $\alpha < \beta$. (It is a classical fact that $A$ can be defined by induction.) Set $U_\alpha = \overline{A_\alpha - \omega}$, where the closure is taken in $\beta \omega$. Then each $U_\alpha$ is clopen in $\omega^*$ and $U_\beta$ contains each $U_\alpha$ whenever $\alpha < \beta$ (see e.g. [11, 6Q & 6S]). Hence, $Q = \cup_{\alpha<\omega_1} U_\alpha$ is a Tychonoff Type I-in-itself (and hence f-Type-I-in-itself) subspace of $\omega^*$. Notice that $Q$ is open in $\omega^*$. A canonical cover of $Q$ is given by replacing $U_\alpha$ by $\cup_{\beta<\alpha} U_\beta$ at limit levels. Since each $U_{\alpha+1}$ is compact, $Q$ is $\omega$-bounded. Let $C \subset Q$ be closed and non-Lindelöf. We show that $C$ is not f-narrow in $Q$. Up to taking a subcover of the canonical cover, we may assume that $C \cap (U_{\alpha+1} - \overline{U_\alpha})$ contains at least two points. We shall define closed subsets $D_\alpha, E_\alpha \subset C \cap \overline{U_\alpha}$ and $f_\alpha : \beta \omega \rightarrow [0, \alpha] \subset \mathbb{I}_{\geq 0}$ such that the following holds:

1. $f_\alpha(D_\alpha) = \{0\}$, $f_\alpha(E_\alpha) = \{0, 1, 2, \ldots, \alpha\} \subset \mathbb{I}_{\geq 0}$,
2. $f_\alpha \upharpoonright \overline{U_\gamma} = f_\gamma \upharpoonright \overline{U_\gamma}$ whenever $\gamma \leq \alpha$,
3. $D_\alpha \cap \overline{U_\gamma} = D_\gamma$ and $E_\alpha \cap \overline{U_\gamma} = E_\gamma$ whenever $\gamma \leq \alpha$.

Once these are defined, set $D = \cup_{\alpha<\omega_1} D_\alpha$, $E = \cup_{\alpha<\omega_1} E_\alpha$. Then the map $f : Q \rightarrow \mathbb{I}_{\geq 0}$ defined by $f(x) = f_\alpha(x)$ for $\alpha$ such that $x \in U_\alpha$ is continuous, unbounded on $E$ and constant on 0 on $D$. This shows that $C$ is not f-narrow in $X$ by Theorem 3.8 (c).

So, start with $E_0 = F_0 = \{e_0\} \subset U_0$, $f_0 : U_0 \rightarrow \mathbb{I}_{\geq 0}$ be constant on 0. Let $\alpha = \gamma + 1$. Take $d, e \in C \cap (U_{\gamma+1} - \overline{U_\gamma})$. The map $g : U_\gamma \cup \{d, e\} \rightarrow [0, \gamma + 1] \subset \mathbb{I}_{\geq 0}$ defined as

$$g(x) = \begin{cases} f_\gamma(x) & \text{if } x \in \overline{U_\gamma} \\ 0 & \text{if } x = d \\ \gamma + 1 & \text{if } x = e \end{cases}$$

is continuous. Since $\overline{U_\gamma} \cup \{d, e\}$ is closed and $\beta \omega$ normal, there is some $f_{\gamma+1} : \beta \omega \rightarrow [0, \gamma + 1]$ extending $g$. Set $D_{\gamma+1} = D_\gamma \cup \{d\}$, $E_{\gamma+1} = E_\gamma \cup \{e\}$, by construction the above conditions are fulfilled. Let now $\alpha$ be limit. The map $g : U_\alpha \rightarrow [0, \alpha]$ defined by $g \upharpoonright U_\gamma = f_\gamma \upharpoonright U_\gamma$ for $\gamma < \alpha$ is continuous. As well known (see e.g. [11, Theorem 1.5.2]), if $M \subset \beta \omega$ is Lindelöf, then any continuous function $M \rightarrow [0, 1]$ can be continuously extended to a function $\beta \omega \rightarrow [0, 1]$. Hence, there is $f_\alpha : \beta \omega \rightarrow [0, \alpha]$ extending $g$. Set

$$D_\alpha = (\cup_{\gamma<\alpha} D_\gamma) \cup (f^{-1}_\alpha(\{0\}) \cap C \cap (\overline{U_\alpha} - U_\alpha)), \quad E_\alpha = (\cup_{\gamma<\alpha} E_\gamma) \cup (f^{-1}_\alpha(\{\alpha\}) \cap C \cap (\overline{U_\alpha} - U_\alpha)).$$

By construction $D_\alpha$ and $E_\alpha$ are closed in $X$ and satisfy the above conditions (1)–(3). This finishes the proof.

We now show that there are (very classical) consistent examples of $\omega_1$-compact first countable Type I spaces without any closed non-Lindelöf subspace f-narrow in it, namely: Suslin trees. Recall that a Suslin tree is an tree of height $\omega_1$ whose chains and antichains are at most countable. They cannot be shown to exist by ZFC alone but pop up in many models of set theory. In what follows, a (set theoretic) tree $T$ is always given the order (also called interval) topology, that is, a basis for the open sets is given by the “intervals” $I_{x,y} = \{z \in T : x < z < y\}$. Any tree is locally compact and a tree of height $\leq \omega_1$ is first countable in this topology. We shall often use the fact that antichains are closed discrete in trees, and that a closed discrete subset of a Hausdorff tree is a countable union of antichains (see e.g. Theorem 4.11 in [13]). A Suslin tree is $\omega_1$-compact. Recall also that a tree is Hausdorff iff two members at a limit level are equal whenever they have the same predecessors. The next lemma shows a little bit more than what we really need.

**Lemma 4.3.** A Hausdorff tree $T$ of height $\omega_1$ without an uncountable branch does not contain any closed non-Lindelöf subspace which is both Type I in $T$ and f-narrow in $T$.

**Proof.** Let $D \subset T$ be closed. If $D$ has points as high as one wants in the tree, since $T$ has no uncountable branch, there are $x_0, x_1 \in T$ at same level $\alpha$ such that $\{y \in D : y > x_1\}$ is
Another proof of the high level of involvement of experts in the present subject.

Proof. If \( D \) is not \( f \)-narrow, consider its smallest uncountable level. Pick one point in \( D \) above each member in this level. This yields an uncountable antichain.

\[ \exists \alpha \in D \quad \text{whose levels are countable.} \]

Recall that an \( \omega_1 \)-tree is a tree of height \( \omega_1 \) whose levels are countable.

\begin{align*}
\text{Lemma 4.4.} & \quad \text{Let } D \text{ be an uncountable subset of a tree } T \text{ of height } \omega_1. \text{ Then either } D^\downarrow = \{ t \in T : \exists d \in D \text{ with } t < d \} \text{ is an } \omega_1 \text{-tree, or } D \text{ contains an uncountable antichain.} \\

\text{Proof.} & \quad \text{If } D^\downarrow \text{ is not an } \omega_1 \text{-tree, consider its smallest uncountable level. Pick one point in } D \text{ above each member in this level. This yields an uncountable antichain.} \\

\text{Lemma 4.5.} & \quad \text{Any closed subspace of an } \omega_1 \text{-tree } T \text{ is } f \text{-Type I in } T. \\

\text{Proof.} & \quad \text{The level function works.} \\

\end{align*}

The above lemmas yield immediately our claimed example.

\begin{align*}
\text{Example 4.6.} & \quad \text{A Suslin tree is an } \omega_1 \text{-compact locally compact first countable } f \text{-Type I space containing no } f \text{-narrow non-Lindelöf subspace.} \\

\text{Question 4.7.} & \quad \text{Consistently, is there an } \omega \text{-bounded first countable } (f-) \text{Type I space containing no closed non-Lindelöf subspace } (f-) \text{narrow in it?} \\

\text{This question dates back to 2006, see [1], Problem 6.3]. Peter Nyikos, in 2006 also, claimed to have found an } \omega \text{-bounded surface (i.e. 2-manifold) containing no closed non-Lindelöf subspace } f \text{-narrow in it under } \diamond^+, \text{ but he seems to have unfortunately been sidetracked by many other problems before completing a first draft of his construction}, \text{ which we were not able to emulate. Question 4.7 asks for a consistent example because of the following theorem.} \\

\text{Theorem 4.8} \quad [3, 6], \text{in effect}. (\text{PFA}) \quad \text{If } X \text{ is countably compact, countably tight and Type I in some space } Y, \text{ then } X \text{ contains a closed copy of } \omega_1 \text{ and thus a subspace narrow in it.} \\

\text{Proof.} & \quad \text{Given a canonical cover } \mathcal{U} \text{ of } X \text{ in } Y, \text{ its skeleton } \text{Sk}(\mathcal{U}, X) \text{ is a perfect preimage of } \omega_1. \text{ (Send the } \alpha \text{-th bone to } \alpha, \text{ this gives a continuous closed map with compact fibers.) Under } \text{PFA}, \text{ it contains a closed copy of } \omega_1. \text{ This was shown by Balogh in [3] for spaces of character } \leq R_1 \text{ and Eisworth proved it for countably tight spaces in [6].} \\

\text{Notice that for first countable spaces, the conclusion holds in a model of } \text{ZFC + CH without inaccessible cardinals by a result of Eisworth andNyikos}. \text{ The paternity (through an oral account) of the next result should be attributed to P. Nyikos alone.} \\

\text{Theorem 4.9.} \quad (\text{PFA}) \quad \text{If } X \text{ is } \omega_1 \text{-compact, countably tight, locally compact and Type I in some space } Y, \text{ then } X \text{ contains a closed copy of } \omega_1 \text{ and thus a subspace narrow in } X \text{ (and in } Y). \\

\text{Example [4,6] shows that this result does not hold in } \text{ZFC.}

\text{Another proof of the high level of involvement of experts in the present subject.}
\textbf{Proof.} If }X\text{ is Lindelöf, there is nothing to prove, hence we assume that it is not. The first
Trichotomy Theorem of Eisworth and Nyikos \cite{EisworthNyikos} shows that under \textbf{PFA} a locally compact space
is either a countable union of }\omega\text{-bounded subspaces, or has a closed uncountable discrete space,
or has a Lindelöf subset with a non-Lindelöf closure. The latter two are impossible if }X\text{ is Type
I in }Y\text{ and }\omega_1\text{-compact. Thus }X\text{ is a countable union of }\omega\text{-bounded subspaces, one of which
must be non-Lindelöf, and we evoke Theorem }\ref{thm:mainresult}\text{ to conclude. (Note that a copy of }\omega_1\text{ is closed
in a countably tight Type I subspace of }Y.\)

We note in passing that there are models of set-theory in which there are }\omega\text{-bounded first
countable spaces (2-manifolds, even) containing no copies of }\omega_1\text{ but which are however narrow
in themselves, for instance \cite{Hodel} Example 6.9] (built with the help of }\diamondsuit\text{). This brings the next
question.

\textbf{Question 4.10.} Is there a model of set theory where first countable }\omega\text{-bounded Type I spaces
necessarily contain a non-Lindelöf closed subspace }f\text{-narrow in it, but not necessarily a copy
of }\omega_1\text{ ?}

Another question, inspired by Example \ref{ex:spwH}

\textbf{Question 4.11.} Is there a space which contains a non-Lindelöf closed subspace }f\text{-narrow in it
but no closed non-Lindelöf subspace narrow in it ? Is there an }f\text{-Type I example ?}

Notice that each subspace }V_\alpha\text{ (and }W)\text{ is narrow in the space of Example }\ref{ex:spwH}\text{ and that
any subset is }f\text{-narrow in Example }\ref{ex:spwH}\text{ (since functions with range }\subset \mathbb{L}_{\geq 0}\text{ are constant), but this
space also contains a copy of }\omega_1\text{.}

\section{Discrete narrow and }f\text{-narrow non-Lindelöf subspaces}

Recall that a space }X\text{ is (strongly) \textit{collectionwise Hausdorff }((s)\text{cwH} \text{ for short) iff any closed
discrete subspace }\{d_\alpha : \alpha \in \kappa\}\text{ can be expanded to a disjoint (resp. discrete) collection of open
sets. That is, there are open }O_\alpha \ni d_\alpha\text{ such that }\{O_\alpha : \alpha \in \kappa\}\text{ is a disjoint (resp. discrete)
collection in }X.

There are simple examples of spaces containing discrete narrow non-Lindelöf subspaces, for
instance }\mathbb{L}_{\geq 0} \times [0,1] - \omega_1 \times \{0\}\text{ where }\omega_1\text{ is seen as a subspace of }\mathbb{L}_{\geq 0},\text{ see Example }\ref{ex:spwH}\text{ for
details. The main goal of this section is to exhibit more elaborate examples whose narrow-in
the-space non-Lindelöf subspaces are all discrete outside of a Lindelöf subset. We first show how
scwH-ness does not allow such behaviours and how }f\text{-narrow subspaces interact with skeletons
of canonical covers. At the same time, we investigate how much information about (}f\text{-)narrow
subspaces does the skeleton possess.}

\textbf{Lemma 5.1.} Let }D\text{ be a closed subspace of }Y\text{ and }\{V_\alpha : \alpha \in \omega_1\}\text{ be a good systematic cover of
}Y\text{ such that }V_\alpha\text{ is Tychonoff and }D \notin V_\alpha\text{ for each }\alpha.\text{ Let }U \supset D\text{ be open. If }D\text{ is }f\text{-narrow in
}Y,\text{ then }\bigcup \cap (\overline{V_\alpha} - V_\alpha) \neq \emptyset\text{ for a stationary set of }\alpha.

\textbf{Proof.} By contradiction, suppose that }\bigcup \cap (\overline{V_\alpha} - V_\alpha) = \emptyset\text{ for a club set of }\alpha.\text{ Since }D \notin V_\alpha,\text{ up to taking a subcover, we may assume that that }\bigcup \cap (\overline{V_\alpha} - V_\alpha) = \emptyset\text{ and }D \cap U_\alpha \neq \emptyset\text{ for each }\alpha,\text{ where }U_\alpha = U \cap (V_{\alpha+1} - \overline{V_\alpha}).\text{ If }y \in \overline{U},\text{ take }\alpha\text{ minimal such that }y \in V_\alpha,\text{ then since
the cover is good }\alpha\text{ must be a successor ordinal }\beta + 1. \text{ Then }y \in V_{\beta+1} - \overline{V_\beta},\text{ which intersects
at most one member of the family }U = \{U_\alpha : \alpha \in \omega_1\}\text{, which is thus discrete. Picking a point }x_\alpha\text{ in each
}D \cap U_\alpha\text{ yields a club discrete subset of }D\text{ which we may partition in two disjoints
uncountable club discrete subsets }D_i = \{x_\alpha : \alpha \in E_i\}, i = 0, 1.\text{ Since }V_\alpha\text{ is Tychonoff and
contains }U_\alpha,\text{ we may define }f_\alpha : \overline{V_\alpha} \rightarrow [0, \alpha] \subset \mathbb{L}_{\geq 0}\text{ which takes value }\alpha\text{ on }x_\alpha\text{ and }0\text{ outside of}
Given a closed discrete subset that is narrow in $X$, expand it to a discrete collection of open sets $O$. Then as in Lemma 5.1 define a function $f: X \rightarrow \mathbb{L}_{\geq 0}$ which is 0 on the complement of $\bigcup O_1$ and takes higher and higher values in each member of $O_1$. Discreteness ensures that the function is continuous.

“Strongly” cannot be omitted in Lemma 5.4, as $cwH$ is not enough to rule out closed discrete subspaces narrow in the whole space.

**Example 5.5.** $Y = \mathbb{L}_{\geq 0} \times \mathbb{R} - \omega_1 \times \mathbb{Q}$ is $cwH$, $f$-Type I and narrow in itself. Any open non-Lindelöf subset contains an uncountable closed discrete subset that is narrow in $Y$.

Of course, $\omega_1$ is seen here as a subspace of $\mathbb{L}_{\geq 0}$.

**Details.** The projection on the first factor ensures that $Y$ is functionally Type I. Let us show that $Y$ is narrow in itself. The proof is very similar to that of Example 3.5.

**Claim 5.5.1.** Let $U, V \subset Y$ be open such that $U$ intersects some horizontal $\mathbb{L}_{\geq 0} \times \{y\} \cap Y$ unboundedly and $V \supset \overline{U}$. Then $V \supset \{x, \omega_1\} \times ([a, b] - \mathbb{Q})$ for some $x \in \mathbb{L}_{\geq 0}$ and $a < y < b \in \mathbb{R}$.

**Proof.** If $y \in \mathbb{R} - \mathbb{Q}$, this is a consequence of Lemma 5.4 (a) (and subspace topology) since $\mathbb{L}_{\geq 0} \times \{y\} \cap Y$ is a copy of $\mathbb{L}_{\geq 0}$. If $y \in \mathbb{Q}$, take a family $\{x_\alpha \in \mathbb{L}_{\geq 0} : \alpha \in \omega_1\}$ such that $x_\alpha > \alpha$ and $(x_\alpha, y) \in U$. Then $U \supset \{x_\alpha\} \times (a_\alpha, b_\alpha)$ for some $a_\alpha, b_\alpha \in \mathbb{Q}$, $a_\alpha < y < b_\alpha$, and some interval $(a, b)$ must appear uncountably many times. It follows that $\overline{U}$ intersects $C \times [a, b] \cap Y$ for some club $C \subset \mathbb{L}_{\geq 0}$ and $a < y < b$ in $\mathbb{R}$. Hence for each $y \in (a, b) - \mathbb{Q}$, $V$ contains $[x_y, \omega_1] \times (a_y, b_y) \cap Y$. By Lindelöfness, there is a countable family of $y$ such that the union of $(a_y, b_y)$ cover $(a, b) - \mathbb{Q}$, taking the supremum of the $x_y$ yields the claim.
Suppose that \( \{ U_\alpha : \alpha \in \omega_1 \} \) is a systematic cover of \( Y \) with \( \overline{U_0} \) non-Lindelöf, hence \( U_0 \) is horizontally unbounded. Since \( \mathbb{R} \) is second countable there is some interval \((a,b)\) such that \( U_0 \) contains \( \{x\} \times (a,b) \) for unboundedly many \( x \). We see by induction and the above claim that \( \overline{U_\beta} \) contains \( [x_\beta, \omega_1) \times [a_\beta, b_\beta) \cap Y \) with \( x_\beta \geq x_\alpha \) and \( a_\beta < a_\alpha < b_\alpha < b_\beta \) whenever \( \alpha < \beta \).

By second countability again, there is some \( \beta < \omega_1 \) such that \( U_\beta \cap (\omega_2 - \omega_1) \times \mathbb{R} \cap Y \). Since the rest of the space is Lindelöf, it is contained in some \( U_\gamma \) for \( \gamma < \delta < \omega_1 \).

It follows that \( Y \) is narrow in itself. Also, if \( U \) is open and non-Lindelöf, there is \( q \in \mathbb{Q} \) such that \( U \cap (\mathbb{L}_{\geq 0} - \omega_1) \times \{ q \} \) is unbounded. It is then easy to find an uncountable closed discrete subset of \( U \cap (\mathbb{L}_{\geq 0} - \omega_1) \times \{ q \} \). To finish, we are left only with the following claim to prove.

Claim 5.5.2. \( Y \) is cwH.

Proof. By [10], Lemma B.29 p. 194] (or arguing as above) a closed subset \( C \) of \( \mathbb{L}_{\geq 0} \times (\mathbb{R} - \mathbb{Q}) \) which has unbounded projection on the first factor must intersect \( \mathbb{L}_{\geq 0} \times \{ c \} \) on a club set for some \( c \in \mathbb{R} - \mathbb{Q} \). If \( C \) is discrete, its intersection with \( \mathbb{L}_{\geq 0} \times (\mathbb{R} - \mathbb{Q}) \) must then be horizontally bounded, say by \( \gamma \). Since \( Y \cap [0, \gamma + 2] \times \mathbb{R} \) is metrizable, it is scwH and we may separate the points of \( C \cap [0, \gamma] \times \mathbb{R} \) by open sets inside of it. The rest of \( C \) lies in \( \bigcup_{\gamma < \alpha < \omega_1} (\alpha, \alpha + 1) \times \mathbb{R} \), which is a disjoint (non-discrete) union of open metrizable subspaces of \( Y \). We may thus separate the rest of \( C \) in each piece by open sets.

This finishes the proof.

To obtain more, that is, spaces \( X \) where all narrow-in-\( X \) non-Lindelöf subspaces are discrete after removal of a Lindelöf subset (and some other variations), we introduce a general construction procedure to be applied later in concrete cases.

Definition 5.6. Let \( X \) be f-Type I with \( s : X \to \mathbb{L}_{\geq 0} \) a slicer of \( X \). We set:

\[
\Down(X, s) = \{ (x, y) \in X \times \mathbb{L}_{\geq 0} : y \leq s(x) \}
\]

\[
\Delta(X, s) = \{ (x, y) \in X \times \mathbb{L}_{\geq 0} : y = s(x) \}
\]

\( \Delta(X, s) \) is homeomorphic to \( X \) (see below), and \( \Down(X, s) \) is a closed subspace of \( X \times \mathbb{L}_{\geq 0} \).

If \( X = \mathbb{L}_{\geq 0} \) and \( s \) is the identity, then \( \Down(X, s) \) is the octant \( \mathbb{O} \) and \( \Delta(X, s) \) its diagonal \( \Delta \).

For our later examples, we only need the case where \( X \) is an \( \omega_1 \)-tree and \( s \) is the height function, but it is not difficult to obtain general properties of \( \Down(X, s) \).

Lemma 5.7. Let \( X \) be f-Type I with slicer \( s : X \to \mathbb{L}_{\geq 0} \), \( D \subset \Down(X, s) \) be closed and \( \pi : \Down(X, s) \to X \) be the projection on the first factor. Then the following hold.

(a) \( \pi \) is a closed map and \( \pi \) restricted to \( \Delta(X, s) \) is a homeomorphism.

(b) \( s \circ \pi \) is a slicer of \( \Down(X, s) \).

(c) If there is no closed non-Lindelöf subset f-narrow in \( X \), then there is no closed non-Lindelöf subset f-narrow in \( \Down(X, s) \).

(d) If \( D \) is closed discrete, so is \( \pi(D) \).

(e) If \( \pi(D) \) contains a closed discrete subset, then so does \( D \).

(f) \( X \) is countably compact iff \( \Down(X, s) \) is countably compact.

(g) \( X \) is \( \omega_1 \)-compact iff \( \Down(X, s) \) is \( \omega_1 \)-compact.

(h) \( X \) is locally compact iff \( \Down(X, s) \) is locally compact.

Proof.

(a) The homeomorphism claim is immediate since \( \Delta(X, s) \) is the graph of \( s \) in \( X \times \mathbb{L}_{\geq 0} \). Set \( U_\alpha = s^{-1}((0, \alpha]) \) for each \( \alpha \). Let \( C \subset \Down(X, s) \) be closed and \( x \in \pi(C) \subset X \). Let \( \alpha \) be such that \( U_\alpha \ni x \). Then \( x \) is in the closure of \( \pi(C \cap (U_\alpha \times [0, \alpha])) \). By compactness of \( [0, \alpha] \) the projection \( U_\alpha \times [0, \alpha] \to U_\alpha \) is closed (see e.g. [9], Thm 3.1.16), hence \( x \in \pi(C) \).

(b) Immediate since \( (s \circ \pi)^{-1}([0, \alpha]) \) is contained in \( s^{-1}([0, \alpha]) \times [0, \alpha] \) which is Lindelöf.
(c) Let \( C \subset \text{Down}(X, s) \) be closed and non-Lindelöf. Then \( \pi(C) \) is closed non-Lindelöf, let \( f : X \to \mathbb{L}_{\geq 0} \) witness the fact that \( \pi(C) \) is not f-narrow in \( X \). Then \( f \circ \pi : \text{Down}(X, s) \to \mathbb{L}_{\geq 0} \) witnesses that \( C \) is not f-narrow in \( \text{Down}(X, s) \).

(d) Let \( x = \pi(y) \) for \( y \in D \). By definition, \( \pi^{-1}(\{x\}) = \{x\} \times [0, s(x)) \). For each \( y \in \pi^{-1}(\{x\}) \) there is an open \( V_y \subset X \) and an interval \( (a_y, b_y) \subset [0, s(x)) \) such that \( V_y \times (a_y, b_y) \) intersects at most one point of \( D \) and contains \( y \). Then \([0, s(x))\] is covered by finitely many intervals \((a_y, b_y)\), taking the intersection of the corresponding \( V_y \) yields a neighborhood of \( x \) intersecting \( \pi(D) \) in at most finitely many points, hence \( \pi(D) \) is discrete.

(e) If \( C \subset \pi(D) \) is closed discrete, taking one member in \( \pi^{-1}(\{c\}) \cap D \) for each \( c \in C \) yields a closed discrete subset of \( D \).

(f) One direction is immediate. A Type I countably compact space is \( \omega \)-bounded, and \( \omega \)-boundedness is productive (Lemma 2.10). It follows that \( X \times \mathbb{L}_{\geq 0} \) is \( \omega \)-bounded, and this holds as well for any closed subspace.

(g) Immediate by (d) and (e).

(h) Local compactness is productive and \( \text{Down}(X, s) \) is a closed subspace of \( X \times \mathbb{L}_{\geq 0} \). \( \square \)

We now pile up copies of \( \text{Down}(X, s) \) as follows. We define \( D^{(\omega)}(X, s) \) (sometimes shortened as \( D^{(\omega)} \) if there is no ambiguity) as the disjoint union of copies \( D_n \) \((n \in \omega)\) of \( \text{Down}(X, s) \) where we identify \((x, s(x)) \in D_n \) with \((x, 0) \in D_{n+1} \). (Notice that \((x, s(x)) \in \Delta(X, s) \). If \( X = \mathbb{L}_{\geq 0} \) and \( s \) is the identity, this consists of piling up \( \omega \) copies of \( \emptyset \), identifying \( \Delta \) in one copy with \( H \) in the next.) By a slight abuse of notation we still call \( D_n \) the images of these subspaces in \( D^{(\omega)} \) (they are thus not disjoint). The map \( s^{(\omega)} : D^{(\omega)} \to \mathbb{L}_{\geq 0} \) defined as \( s \circ \pi \) in each \( D_n \) is a slicer. We write \( \Delta_n \) for the copy of \( \Delta(X, s) \) inside \( D_n \).

We now define \( \Omega(X, s) \) to be the disjoint union of \( D^{(\omega)} \) and a copy \( L \) of \( \mathbb{L}_{\geq 0} \) with the following topology. First, \( D^{(\omega)} \) is open in \( \Omega(X, s) \). Given \( a < b \in \mathbb{L}_{\geq 0} \) and \( n \in \omega \), set

\[ V_{a, b, n} = (s^{(\omega)})^{-1} \left( \{(a, b)\} \right) - \bigcup_{k < n} D_k. \]

Then the neighborhoods of \( x \in L \) are \((a, b) \subset L \) union \( V_{a, b, n} \), for \( a < x < b \) and \( n \in \omega \). In words: the points in \( D_n \) whose value under \( s \circ \pi \) is \( x \in \mathbb{L}_{\geq 0} \) converge to \( (\text{the copy of}) \ x \in L \) when \( n \) grows. These points may be thought as being under \( x \). If \( A \subset L \), we write \( \Omega(X, A, s) \) for the subspace \( D^{(\omega)} \cup A \) of \( \Omega(X, s) \). Notice that the function \( s^{(\omega)} \) defined by the identity on \( L \) and \( s^{(\omega)} \) on \( D^{(\omega)} \) is a slicer for \( \Omega(X, s) \) (the preimage of \([0, \alpha)\] is a countable union of Lindelöf spaces and is thus Lindelöf), which is thus f-Type I.

**Lemma 5.8.** Let \( X \) be f-Type I with slicer \( s : X \to \mathbb{L}_{\geq 0} \). Then the following hold.

(a) \( \Omega(X, s) \) and \( D^{(\omega)}(X, s) \) are Tychonoff iff \( X \) is regular.

(b) \( \Omega(X, s) \) is countably compact iff \( X \) is countably compact.

(c) \( D^{(\omega)}(X, s) \) is \( \omega_1 \)-compact iff \( X \) is \( \omega_1 \)-compact.

(d) \( D^{(\omega)}(X, s) \) is locally compact iff \( X \) is locally compact.

**Proof.**

(a) If \( X \) is regular, it is straightforward to check that \( s^{-1}_{\Omega}([0, \alpha]) \) is regular for each \( \alpha \). Then apply Lemma 2.3.

(b) By Lemma 5.7 (f) and construction: a countable set either has infinite intersection with some \( D_k \) or has an accumulation point in \( L \).

(c) An uncountable subset of \( D^{(\omega)}(X, s) \) has an uncountable intersection with some \( D_k \), which is a closed subset of \( D^{(\omega)}(X, s) \). The result follows by Lemma 5.7 (g).

(d) Immediate by Lemma 5.7 (h) and construction. \( \square \)

Note however that \( \Omega(X, A, s) \) is not locally compact at points of \( A \subset L \) if the bones of \( X \) do not have compact neighborhoods. The following lemma is reminiscent of Theorem 3.2 and Lemma 5.1.
Lemma 5.9. Let $X$ be f-Type I with slicer $s$, and let $B_\alpha = s^{-1}([0, \alpha]) - s^{-1}([0, \alpha))$ for $\alpha \in \omega_1$ be the bones of the canonical cover given by $s$. If $B_\alpha \neq \emptyset$ for a stationary set of $\alpha \in \omega_1$ and $A \subset L$. Then $A$ is narrow in $\Omega(X, A, s)$.

Proof. We assume $A$ to be unbounded in $L$, otherwise the result is immediate. For $x \in L$ and $n \in \omega$, let $N(x, n) = \{s^{(\omega)}(x) \} - \cup_{k<n} D_k$, in words: the points of $D^{(\omega)}$ which are under $x$ but not in the $n$ first $D_k$. Let $U = \{U_\alpha : \alpha \in \omega_1\}$ be a systematic cover of $\Omega(X, A, s)$. Suppose that there is some $\alpha$ such that $A_0 = U_\alpha \cap A$ is non-Lindelöf. If $A - U_{\alpha+2}$ is bounded in $L$ (hence Lindelöf), then $A$ is contained in $U_\beta$ for some $\beta > \alpha$ and we are over. Otherwise, $A_1 = A - U_{\alpha+2}$ is unbounded in $L$. Notice that $A_0 \cap A_1 = \emptyset$ and both are closed. For $i = 0, 1$, $A_i$ contains an uncountable subset $B_i = \{b_\alpha : \alpha \in \omega_1\}$ such that $b_\alpha > \alpha$ (recall that $A \subset L$ which is a copy of $\mathbb{L}_{\geq 0}$). Up to taking subsets, we may assume that there is $n \in \omega$ such that $U_\alpha$ contains $N(b_{0, n})$ and $\Omega(X, A, s) - U_{\alpha+1}$ contains $N(b_{\alpha, n})$ for each $\alpha$. The closure of $B_i$ in $L$ contains a club $C_i \subset \omega_1 \subset L$. Let now $C_i = C_0 \cap C_1$. Since $C$ is club and $B_\alpha \neq \emptyset$ for a stationary set $S$ of $\alpha$, by construction and closedness both $\overline{U_\alpha}$ and $\Omega(X, A, s) - U_{\alpha+1}$ contain all of $N(\alpha, n)$ for those $\alpha \in S \cap C$, which is clearly impossible. It follows that $\overline{U_\alpha} \cap A$ is Lindelöf for each $\alpha$, hence $A$ is narrow in $\Omega(X, A, s)$.

Lemma 5.10. Let $X$ be f-Type I with slicer $s$ and $f: \text{Down}(X,s) \to \mathbb{L}_{\geq 0}$ be such that $f(x) \leq s \circ \pi(x)$ for all $x \in X$. For each $n \in \omega$ there is $g: \Omega(X,s) \to \mathbb{L}_{\geq 0}$ which is equal to $f$ on $D_n$ and equal to $s_0$ on $L$ and $D_L$ for each $\ell > n + 1$.

Proof. Set $h: \text{Down}(X,s) \to \mathbb{L}_{\geq 0}$ to be the function $h((x,y)) = \max\{f((x,y)), y\}$. Then $h$ is equal to $f$ on $X \times \{0\}$ and to $s \circ \pi$ on $\Delta(X,s)$. Define $j: \text{Down}(X,s) \to \mathbb{L}_{\geq 0}$ as $j((x,y)) = f((x,0))$. Finally, define $g$ as being equal to $h$ on $D_{n+1}$, to $f$ in $D_n$, to $j$ in $D_k$ with $k < n$ and to $s_0$ on $L$ and $D_L$ for each $\ell > n + 1$.

Corollary 5.11. Let $X$ be f-Type I with slicer $s$ and $A \subset \mathbb{L}_{\geq 0}$ be unbounded. If $X$ does not contain any non-Lindelöf closed subspace $f$-narrow in $X$, then any non-Lindelöf closed subset $f$-narrow in $\Omega(X, A, s)$ is contained in $A$ outside of $s^{-1}_\omega([0, \alpha])$ (for some $\alpha$).

Proof. By Lemma 5.7 (c), $\text{Down}(X,s)$ has no closed non-Lindelöf subset $f$-narrow in it. Let $B \subset \Omega(X, A, s)$ be closed. If $B \cap D_k$ is non-Lindelöf for some $k$, by Theorem 5.8 there is some $f: D_k \to \mathbb{L}_{\geq 0}$ which is unbounded on $B \cap D_k$ with $f^{-1}([0, \alpha]) \cap B$ non-Lindelöf. A classical leapfrog argument shows that the set of $\alpha$ such that $f((s \circ \pi)^{-1}([0, \alpha])) \subset [0, \alpha]$ is club in $\omega_1$. It follows that $h(x) = \min\{s \circ \pi(x), f(x)\}$ is (continuous and) unbounded on $B \cap D_k$. Replacing $f$ by $h$ if needed, we may assume that $f(x) \leq s \circ \pi(x)$. The function $g$ given by Lemma 5.10 is unbounded on $B$ and $g^{-1}([0, \alpha])$ is non-Lindelöf. Hence $B$ is not $f$-narrow in $\Omega(X, A, s)$. It follows that any closed subspace $f$-narrow in $\Omega(X, A, s)$ has a Lindelöf intersection with $D^{(\omega)}$.

Example 5.12. There are f-Type I locally metrizable spaces $Y$ which contain a non-Lindelöf closed subspace $f$-narrow in $Y$ such that any closed subspace $f$-narrow in $Y$ is discrete outside of a Lindelöf subset, and with the additional following properties.

(a) Each non-Lindelöf closed subset of $Y$ contains an uncountable closed discrete subset.

(b) Under $\Theta^*$, $Y$ can be made as in (a) and cuH.

(c) If there is a Suslin tree, $Y$ can be made to have an open dense $\omega_1$-compact and locally compact subspace.

Recall that a tree is Aronszajn if it is an $\omega_1$-tree whose chains are at most countable.

Details. In each case, $Y = \Omega(T, A, s)$ where $T$ is an Aronszajn tree, $s$ the height function and $A$ the subset of successor ordinals in $\omega_1 \subset L_{\geq 0}$. Then $\Omega(T, A, s)$ is f-Type I and locally metrizable since $s^{-1}_\Omega([0, \alpha])$ is second countable for each $\alpha$. By construction, $A$ is closed discrete and Lemma
implies its narrowness in $\Omega(T, A, s)$. By Corollary 5.11 and the fact that an Aronszajn tree does not contain a non-Lindelôf closed f-narrow space (Lemma 4.3), any subspace f-narrow in $\Omega(T, A, s)$ is contained in A outside of a Lindelôf subset. This proves the general claim.

(a) Take T to be $\mathbb{R}$-special (that is, there is a non-necessarily continuous order preserving map $T \to \mathbb{R}$). $\mathbb{R}$-special Aronszajn trees exist in ZFC. Any non-Lindelôf closed subset of an Aronszajn $\mathbb{R}$-special tree contains a uncountable closed discrete subset (see e.g. [4, Thm 2]). Let $C \subset \Omega(T, A, s)$ be closed and non-Lindelôf. If $C \cap D_k$ is non-Lindelôf for some $k$, by Lemma 5.7 (e), $C$ contains an uncountable closed discrete subset. If not, its intersection with A is non-Lindelôf, and we conclude the same.

(b) In [5], Devlin and Shelah use $\diamondsuit^*$ to construct an $\mathbb{R}$-special cwH tree.

Claim 5.12.1. If T is a cwH $\omega_1$-tree and A is the successor ordinals, then $\Omega(T, A, s)$ is cwH.

Proof. Let B be closed discrete in $\Omega(T, A, s)$. Then $B \cap D_n$ is closed discrete for each $n$, and by Lemma 5.7 (d) the projection of $B \cap D_n$ on the first factor T of $D_n$ is closed discrete. Since T is cwH (and $s^{-1}([0, \alpha])$ is countable hence ccc), by Lemma 5.3,

$$B \cap D_n \cap \left( (s \circ \pi)^{-1}([0, \alpha]) - (s \circ \pi)^{-1}([0, \alpha]) \right) = \emptyset$$

for a club set $E_n \subset \omega_1$ of limit ordinals $\alpha$. Hence, B avoids $s^{-1}_\Omega([0, \alpha]) - s^{-1}_\Omega([0, \alpha])$ for $\alpha$ in $E = \cap_{n \in \omega} E_n$. Enumerating $E$ as $\{\alpha_\beta : \beta \in \omega_1\}$, B is contained in the disjoint union of the open second countable hence metrizable subspaces $s^{-1}_\Omega((\alpha_\beta, \alpha_{\beta+1}))$. We may then separate B by open sets inside each piece, yielding the result.

(c) Take T to be a Suslin tree. Then $D^{(\omega)}(T, s)$ is open dense in $\Omega(T, A, s)$, $\omega_1$-compact and locally compact by Lemma 5.8 (c)–(d).

A way to describe (b) is to say that given a subspace narrow in Y, there is a skeleton which is totally blind to it as their intersection is empty, as shown in the proof.

6 Two partial orders on closed (non-Lindelôf) subsets of a space

Definition 6.1. Let Y be a space and C, D be closed subsets of Y.

(a) Set $C \preceq D$ iff given any systematic cover $\{U_\alpha : \alpha \in \omega_1\}$ of Y, if $D \subset U_\alpha$ for some $\alpha$, then $C \subset U_\beta$ for some $\beta$. If $C \preceq D \preceq C$, we write $D \equiv C$ and say that D and C are sc-equivalent (sc is for systematic cover).

(b) Set $C \preceq f D$ iff for any $f : Y \to \mathbb{L}_{\geq 0}$, $f \mid D$ is bounded implies $f \mid C$ is bounded. If $C \preceq f D \preceq f C$, we write $D \equiv f C$ and say that D and C are uf-equivalent (uf is for unbounded function).

If $C \preceq D$ and $C \not\preceq D$, we write $C \prec D$, and idem for $\preceq f$. The order $\preceq f$ is written $\preceq$ in [2] and called the UFO, but being some years older than when [2] was written enables us to slither away from names that sound too cool. Notice that any closed subset D is sc- and uf-equivalent to the closure of $D - E$, where E is Lindelôf. The following lemmas are immediate.

Lemma 6.2. Let Y be a space and $D \subseteq Y$ be closed and non-Lindelôf. If $D$ is Type I [resp. $f$-Type I] in Y, then $D$ is narrow [resp. $f$-narrow] in Y iff $D \equiv C$ [resp. $D \equiv f C$] for any closed non-Lindelôf C $\subseteq D$.

Let $C \subseteq_L D$ iff $\overline{C} - \overline{D}$ is Lindelôf.

We remark that skeletons being blind is in spectacular accordance with cutting edge research in the biology of vertebrates.
Lemma 6.3. Let $C, D \subset Y$ be closed. Then

$$C \subset D \iff C \subset_L D \iff C \prec D \iff C \prec_f D.$$ 

Moreover, if $C$ and $D$ are Lindelöf, then $C \equiv_f D \equiv C$.

Example 6.4.
(a) In the octant $\mathbb{O}$, all horizontal lines are se-equivalent and $\prec$ the diagonal, and any non-Lindelöf closed set narrow in it is se-equivalent to the horizontals or to the diagonal.

(b) By construction.

Details. (a) follows essentially by Lemma 3.4 and Example 3.5, and (b) by construction.

Since the whole space is always sc- and uf-maximal, one looks only at (f-)narrow subsets.

Definition 6.5.
$\mathfrak{N}(D, X)$ is the poset ordered by $\leq$ of se-equivalence classes of subsets of $D \subset X$ which are narrow in $X$.

$\mathfrak{N}_f(D, X)$ is the poset ordered by $\leq_f$ of uf-equivalence classes of subsets of $D \subset X$ which are $f$-narrow in $X$.

We denote $\mathfrak{N}(X, X)$ by $\mathfrak{N}(X)$, and the same for $\mathfrak{N}_f(X, X)$.

Lemma 6.6. If $X$ is a normal space and $D$ is closed in $X$, then $\mathfrak{N}(D, X) = \mathfrak{N}_f(D, X)$ as posets.

Proof. By Lemma 3.9, a closed subset of $D \subset X$ is narrow iff it is f-narrow. The slicers provided by Lemma 2.2 enable to pass from systematic covers to functions $X \to L_{\geq 0}$ with the same properties, hence $C_0 \leq C_1 \iff C_0 \leq_f C_1$ for closed subsets.

We show in Lemma 6.9 that $\mathfrak{N}(X)$ and $\mathfrak{N}_f(X)$ are countably closed when $X$ is Type I and countably compact, this is not the case if $X$ is only $\omega_1$-compact.

Example 6.7. An $\omega_1$-compact surface (i.e. 2-manifold) $X$ such that $\mathfrak{N}(X) = \mathfrak{N}_f(X) \simeq \mathbb{Z}$, hence $\mathfrak{N}(X)$ has neither a maximal nor a minimal element.

Details. We take copies $O_n$ of $\mathbb{O}$ for $n \in \mathbb{Z}$, gluing the copy of $H$ in $O_{n+1}$ to that of $\Delta$ in $O_n$. More precisely: we identify $\langle x, x \rangle \in O_n$ with $\langle x, 0 \rangle \in O_{n+1}$ for each $n \in \mathbb{Z}$. (This is very similar to $D(\omega)/(\mathbb{O}, \pi)$ where $\pi$ is the projection on the first factor, but we take copies ordered by $\mathbb{Z}$ instead of $\omega$.) The resulting space is $\omega_1$-compact since it is a countable union of countably compact subspaces. It is not difficult to show that it is also normal, hence $\mathfrak{N}(X)$ and $\mathfrak{N}_f(X)$ agree by Lemma 6.6. Any closed non-Lindelöf subset must intersect one of the $O_n$ on a non-Lindelöf subset, and proceeding as in Example 6.4 and Lemma 5.10 it is easy to show that the copy of $\Delta$ in $O_n$ is $\prec_f$ the one in $O_{n+1}$, and that any non-Lindelöf narrow subspace is se-equivalent to one of those diagonals.

By adding a copy of $L_{\geq 0}$ at the top and/or the bottom, we may continue piling up copies of the octant and turn $\mathfrak{N}(X)$ to be (for instance) any countable ordinal. By piling up uncountably many copies of $\mathbb{O}_{\geq \alpha}$ for $\alpha \in \omega_1$, as in the construction of $S^{(\omega_1, \omega)}$ in Example 6.10 below, we may obtain $\omega_1$, or actually any ordinal $< \omega_2$. (The resulting space is countably compact exactly when the ordinal is successor, $\omega_1$-compact when it has countable cofinality, not $\omega_1$-compact when the ordinal has uncountable cofinality.) Hence, it is not completely trivial to come up with a countably compact space such that $\mathfrak{N}(X)$ has no maximum or minimum, and our next results are geared towards this precise problem. We first state the following easy lemma whose proof is left to the reader (or see [7, Prop. 2.2]).
Lemma 6.8. If $C_0 \supset C_1 \supset C_2 \supset \ldots$ is an $\omega$-sequence of closed non-Lindelöf sets of a countably compact Type I space, then $\bigcap_{n \in \omega} C_n$ is closed and non-Lindelöf.

Lemma 6.9. If $X$ is Type I and countably compact, then $\mathcal{M}(X)$ and $\mathcal{M}_f(X)$ are up- and downwards countably closed.

Proof. Let $D_n$ be narrow in $X$ for $n \in \omega$. We may assume that $D_n$ is non-Lindelöf for each $n$. Let

$$D = \bigcap_{n \in \omega} \bigcup_{k \geq n} D_k.$$  

Then $D$ is closed and non-Lindelöf by Lemma 6.8.

Claim 6.9.1. If $\{U_\alpha : \alpha \in \omega_1\}$ is a systematic cover of $X$ such that $\overline{U_0} \cap D$ is non-Lindelöf, then there is strictly increasing sequence $k_n (n \in \omega)$ and $\alpha \in \omega_1$ such that $U_\alpha \supset D_{k_n}$ for each $n$.

Proof. By induction on $n$. Let $\kappa_{\kappa-1} = \alpha_{\kappa-1} = 0$. Since the closure of Lindelöf subspaces are Lindelöf in Type I spaces, if $U_{\alpha_{m+1}} \cap D_m$ is Lindelöf for each $m > k_n$, then $D \cap U_0 \subset U_{\alpha_{m+1}} \cap \bigcup_{n \geq k_n} D_n$ would be Lindelöf. Hence there is $k_{n+1} \geq k_n$ such that $U_{\alpha_{m+1}} \cap D_{k_{n+1}}$ is non-Lindelöf, and then $D_{k_{n+1}} \subset U_{\alpha_{n+1}}$ for some $\alpha_{n+1}$ since $D_{k_{n+1}}$ is narrow in $X$. Set $\alpha$ to be the suprema of the $\alpha_n$ to conclude.

Assume now that $D_n \prec D_{n+1}$ for each $n$. Then $D \supset D_n$ for each $n \geq 0$ by the previous claim and transitivity of $\preceq$. Actually, the claim also shows that $D$ is narrow in $X$, because if the intersection of $D$ with the closure of a member of a systematic cover $\mathcal{U}$ is non-Lindelöf, then each $D_n$ lies in some member of $\mathcal{U}$, and $D$ is then inside the closure of their union. To finish, it is enough to show that $D \npreceq D_n$ for each $n$. For this, let $\{U_\alpha : \alpha \in \omega_1\}$ be a systematic cover of $X$ witnessing that $D_n \prec D_{n+1}$, that is, $D_n \subset U_0$ but $D_{n+1} \cap U_0$ is Lindelöf for each $\alpha$. It follows that $D_k \cap U_\alpha$ is Lindelöf for each $k \geq n+1$ by transitivity of $\preceq$. Again, the closure of a Lindelöf set is Lindelöf, hence $D \cap U_\alpha \subset \bigcup_{k \geq n+1} D_k \cap U_\alpha$ is Lindelöf as well. This shows that $D$ is an $\preceq$-upper bound of the $D_n$.

Now, if $D_n \succ D_{n+1}$ for each $n$, fix $n$ and let $\mathcal{U} = \{U_\alpha : \alpha \in \omega_1\}$ be a systematic cover such that $U_0 \supset D_{n+1}$. Then, there is $\alpha$ such that $D_k \subset U_\alpha$ for each $k > n$. Since $D \subset \bigcup_{k \geq n+1} D_k \cap U_\alpha$, $D \preceq D_n$. If $\mathcal{U}$ is such that moreover $U_\alpha \cap D_n$ is Lindelöf for each $\alpha$, then we see that $D \npreceq D_n$, and it follows that $D \prec D_n$ for each $n$. The previous claim easily shows that $D$ is narrow in $X$, which concludes the proof for $\mathcal{M}(X)$. The proof for $\mathcal{M}_f(X)$ is the same, replacing systematic covers with appropriate functions $X \rightarrow \mathbb{N}_{\geq 0}$.

It is however not very difficult to find an example without a $\preceq$-minimal element.

Example 6.10. A Type I countably compact surface $Y$ such that $\mathcal{M}(Y) = \mathcal{M}_f(Y)$ has no minimal element.

Details. The construction is very similar to that of Examples 8.10 and 6.7 and is shown in Figure 4.1 (center). We start with piling up (non-disjoint) copies $O_\alpha$ of $O_{\geq \alpha}$ for $\alpha \in \omega_1$, gluing $\Delta_{\alpha+1}$ to $H_\alpha$, that is, we identify $\langle x, x \rangle \in \Delta_{\alpha+1}$ with $\langle x, 0 \rangle \in H_\alpha$ (in a sense, we put $O_{\geq \alpha+1}$ below $O_{\geq \alpha}$). At limit levels we define neighborhoods similarly as in Example 6.10, points with first coordinate $x \in \mathbb{L}_{\geq 0}$ in $O_\alpha$ for $\alpha < \beta$ with limit $\beta$ accumulate to $\langle x, x \rangle \in O_\beta$. When $\alpha$ is limit or 0, we let $\Lambda_\alpha$ be the copy of $\Delta_\alpha$ in $O_\alpha$. For successors, we define $\Lambda_{\alpha+1}$ to be the copy of $H_\alpha$ in $O_\alpha$ (which is identified with $\Delta_{\alpha+1} \subset O_{\alpha+1}$ above $\alpha + 1$). Then, as in Example 6.7, $\Lambda_\alpha \prec \Lambda_\beta$ if $\alpha < \beta$. Call $S(\omega_{\alpha+1})$ the space thus obtained, whose first stages of construction are depicted in Figure 4.1 (left). Set $\Lambda_{\omega_1}$ to be the union of the segments $\{\langle y, y \rangle \in \Lambda_\alpha : \alpha \leq y \leq \alpha + 1\}$. Then $\Lambda_\omega_1$ is a copy of $\mathbb{L}_{\geq 0}$ uf-uncomparable (and thus, sc-uncomparable) with each $\Lambda_\alpha$. To see this, let $f_\alpha : O_{\geq \alpha} \rightarrow [0, \alpha + 1] \subset \mathbb{L}_{\geq 0}$ be a continuous map which takes value $\alpha$ on the segment $\Gamma$ and above the dashed line depicted in Figure 4.1 (right), is equal to the projection on the first factor on the segment $Z$ and constant with value $\alpha + 1$ on the bottom boundary $H_\alpha$ above $\langle \alpha + 1, 0 \rangle$. 

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Then \( f : S^{(\omega_1, \downarrow)} \to L_{\geq 0} \) defined as \( f \upharpoonright O_\alpha = f_\alpha \) is well defined, continuous, bounded on each \( \Lambda_\alpha \) when \( \alpha < \omega_1 \) and unbounded on \( \Lambda_{\omega_1} \), showing that \( \Lambda_\alpha \not\leq_f \Lambda_{\omega_1} \). Conversely, the map \( g \) defined as constant on \( \alpha \) on \( O_\beta \) when \( \alpha < \beta \), to the projection on the second coordinate on \( O_\alpha \) and to the projection on the first coordinate on \( O_\beta \) with \( \alpha > \beta \) is continuous, unbounded on \( \Lambda_\alpha \) and bounded on \( \Lambda_{\omega_1} \). It follows that \( \Lambda_\alpha \not\leq \Lambda_{\omega_1} \). Notice that any closed non-Lindelöf subset of \( S^{(\omega_1, \downarrow)} \) intersects either one of the \( O_\alpha \) or \( \Lambda_{\omega_1} \) in a non-Lindelöf set (this is similar to Claims 3.10.1–3.10.2).

Now, define \( Y \) by gluing two copies of \( S^{(\omega_1, \downarrow)} \) by identifying pointwise \( \Lambda_0 \) in the first copy with \( \Lambda_{\omega_1} \) in the second, and vice-versa. To simplify, shift the indexing in the second copy by \( \omega_1 \), as in Figure 4 (middle). Then any non-Lindelöf space narrow in \( Y \) must be sc-equivalent to \( \Lambda_\alpha \) for some \( \alpha < \omega_1 \cdot 2 \). It is easy to adapt the maps \( S^{(\omega_1, \downarrow)} \to L_{\geq 0} \) witnessing that \( \Lambda_{\omega_1} \) and \( \Lambda_\alpha \) are uncomparable to have domain all of \( Y \). It follows that \( N_f(Y) \) is the partially ordered set consisting of two disjoint copies of \( \omega_1 \) with reverse order, elements in distinct copies being uncomparable. One may check that \( Y \) is a normal space, hence \( N(Y) = N_f(Y) \), and we are over.

On the other hand, we have:

**Theorem 6.11 (PFA).** In a Type I countably tight countably compact space \( X \), \( \mathfrak{N}(X) \) and \( \mathfrak{N}_f(X) \) are \( \omega_1 \)-upwards closed.

**Proof.** As Lemma 6.9, we prove it for \( \mathfrak{N}(X) \), the proof for \( \mathfrak{N}_f(X) \) being basically the same. \( \text{PFA} \) is needed only through Theorem 4.8 at the very end of the proof. Let \( D = \{ D_\alpha \subseteq X : \alpha \in \omega_1 \} \) be a family of closed subspaces narrow in \( X \), with \( D_\alpha \prec D_\beta \) whenever \( \alpha < \beta \). Let \( \mathcal{U} = \{ U_\alpha : \alpha \in \omega_1 \} \) be a canonical cover for \( X \). By Lemma 2.4, each \( D_\beta \) intersects the members of \( \text{Bo}(\mathcal{U}) \) on a club set \( C_\beta \) of indices. Set \( C \) to be the diagonal intersection of the \( C_\beta \). This means that if \( \alpha \in C \), then \( D_\beta \cap (\overline{U_\alpha} - U_\alpha) \neq \emptyset \) for each \( \beta < \alpha \). We may assume that \( C \) contains only limit ordinals. We then set:

\[
\bigtriangleup D = \bigcup_{\alpha \in C} H_\alpha, \text{ where } H_\alpha = (\overline{U_\alpha} - U_\alpha) \cap \bigcap_{\beta < \gamma < \alpha} D_\gamma.
\]

(1)
Then \( \Delta D \) is a closed unbounded subset of the skeleton and is thus non-Lindelöf, and each \( H_\alpha \) is compact. Notice that setting \( D_\delta = \{ D_\beta - U_\delta : \delta < \beta < \omega_1 \} \) for \( \delta \in \omega_1 \), have \( \Delta D_\delta = \Delta D - U_\delta \).

**Claim 6.11.1.** Let \( \{ V_\alpha : \alpha \in \omega_1 \} \) be a systematic cover of \( X \). If \( \bigcap \Delta D \) is non-Lindelöf for some \( \beta \), then for each \( \alpha \in \omega_1 \), there is a \( \beta(\alpha) \) such that \( \bigcap \{ V_\beta(\alpha) \} \subset \Delta D \).

**Proof.** By construction, \( \bigcap \Delta D_\beta \) intersects \( H_\alpha \) in a club \( B \subset C \). It follows that for each \( \alpha \in B \), \( \bigcap \Delta D_\beta \) intersects \( \{ U_\alpha - U_\beta \} \cap D_\gamma(\alpha) \) for some \( \gamma(\alpha) < \alpha \). Fodor’s Lemma then implies that there is \( \gamma \in \omega_1 \) such that \( \bigcap \Delta D_\beta \) intersects \( \{ U_\alpha - U_\beta \} \cap D_\gamma \) for (stationary many, hence) club many \( \alpha \). Since \( D_\gamma \) is narrow in \( X \), there is some \( \gamma \) such that \( V_\delta \supset D_\gamma \), and the same holds for each ordinal \( \gamma' \) smaller than \( \gamma \) since \( D_\gamma \supset D_{\gamma'} \). By replacing \( D \) with \( D_{\gamma'} \), we may ensure that \( \gamma > \gamma' \) for any \( \gamma' < \omega_1 \), proving the claim. \( \square \)

By definition of \( \preceq \), it follows that if \( E \subset \Delta D \) is closed and non-Lindelöf, then \( E \supset D_\alpha \) for each \( \alpha \in \omega_1 \). Moreover, \( E \not\supset D_\alpha \) for each \( \alpha \in \omega_1 \). Indeed, since \( D_\alpha \preceq D_{\alpha+1} \), there is a systematic cover \( \{ V_\alpha : \alpha \in \omega_1 \} \) of \( X \) with \( D_\alpha \subset V_0 \) and \( D_{\alpha+1} = \bigcap V_\alpha \). The previous claim implies that \( \bigcap \Delta D_\alpha \) is Lindelöf for each \( \alpha \), and thus so is \( \bigcap \Delta D \). Since \( \Delta D \) is closed non-Lindelöf in \( X \), by Theorem 4.8 there is a copy of \( \omega_1 \) in it. This copy is an \( \preceq \)-upper bound of the \( D_\alpha \) in \( \mathfrak{M}(X) \).

**Remark.** We use PFA only to ensure that \( \Delta D \) contains a closed non-Lindelöf subset narrow in itself, and thus in \( X \) since \( X \) is of Type I.

We were frustratingly unable to find the answer to the following question (but have a nagging sensation of overlooking something simple):

**Question 6.12.** Is there a “reasonable” (i.e., countably tight, first countable, etc) Type I space \( X \) such that \( \mathfrak{M}(X) \) (and/or \( \mathfrak{R}(X) \)) contains a chain of cofinality \( \omega_2 \) (that is, a collection \( \{ D_\alpha : \alpha \in \omega_2 \} \subset \mathfrak{D}(\beta) \) when \( \alpha < \beta \) ?

(Actually, we do not know the answer even for unreasonable spaces.) A negative answer to this question implies that PFA ensures (through Zorn’s Lemma, Lemma 6.9 and Theorem 6.11) that \( \mathfrak{M}(X) \) has a maximal element when \( X \) is Type I, countably compact and countably tight.

Let us end this paper with an example showing that \( \mathfrak{M}(M) \) can be of cardinality \( \aleph_1 \).

**Example 6.13.** A Type I countably compact surface \( M \) such that \( \mathfrak{M}(M) \) is the Cantor set.

**Details.** Start with \( \mathbb{L}_{\geq 0} \times I \) where \( I \subset \mathbb{R} \) is a compact interval with irrational boundary points, delete \( \mathbb{L}_{\geq 0} \times \{ q \} \) for rational \( q \) and replace it by a copy \( \mathcal{O}_q = \bigcirc \times \{ q \} \) of the octant. We thus write points of \( M \) as pairs \( (x, r) \) where \( x \) is in \( \mathbb{L}_{\geq 0} \) or \( \bigcirc \) according to whether \( r \) is irrational or rational. Write \( \mathcal{D}_q, \mathcal{H}_q \) for the subsets \( \Delta \times \{ q \} \), \( \mathbb{H} \times \{ q \} \), and \( \mathcal{L}_r \) for \( \mathbb{L}_{\geq 0} \times \{ r \} \) (with \( q \) rational and
r irrational, obviously). Given \( a, b \in \mathbb{L}_0 \) and \( r \in [0,1] \), set \( W_{a,b} \) to be \( \{ (x,y) \in \mathbb{O} : a < x < b \} \) (see Figure 3.3 left) and \( N_{a,b,r} \) be \( (a,b) \times \{ r \} \subset \mathcal{L}_r \) if \( r \) is irrational and and \( W_{a,b} \times \{ r \} \subset \mathcal{O}_r \) if \( r \) is rational. A neighborhood of \( (x,r) \) for irrational \( r \) is given by \( \cup_{r' \in (s,t)} N_{a,b,r'} \), for some \( a < x < b \in \mathbb{L}_0 \) and \( s < r < t \in [0,1] \) (with the obvious subtleties when \( r \) is in the boundary of \( I \)). Fix a rational \( q \in I \). Neighborhoods of points in the interior of each \( \mathcal{O}_q \) (that is: away from \( \mathcal{D}_q, \mathcal{H}_q \)) are those of \( \mathbb{O} \). Those of points in \( \mathcal{H}_q \) are given by \( \cup_{r' \in (s,q)} N_{a,b,r'} \) for some \( s < q \) union \( U \times \{ q \} \), for \( U \) open in \( \mathbb{O} \) whose intersection with \( \mathcal{H} \) is an interval \( (a,b) \times \{ 0 \} \) containing \( x \). See Figure 3.3 right. Neighborhoods of points in \( \mathcal{D}_q \) are defined similarly, taking \( \cup_{r' \in (q,t)} N_{a,b,r'} \) for some \( t > q \).

Let \( \tau : M \to \mathbb{L}_0 \times I \) be the map obtained by sending \( \langle (x,y), q \rangle \) to \( \langle x, q \rangle \) when \( q \) is rational and the identity on \( \mathcal{L}_r \) when \( r \) is irrational. Let \( \pi : \mathbb{L}_0 \times I \to \mathbb{L}_0 \) be the projection on the first factor. Then \( \pi \circ \tau \) is a slicer for \( M \), which is easily seen to be a countably compact f-Type I surface (with boundary).

Claim 6.13.1. \( \tau \) is a perfect closed quotient map.

Proof. The fact that it is perfect (i.e. preimages of points are compact) and a quotient map should be clear by definition. Any perfect map with range a locally compact space is closed, hence so is \( \tau \).

Therefore, if \( C \subset M \) is closed and non-Lindelöf, so is \( \tau(C) \), and by Lemma 3.3 (b) its intersection with some horizontal line in \( \mathbb{L}_0 \times I \) is non-Lindelöf. Hence \( C \) has a non-Lindelöf intersection with either \( \mathcal{L}_r \) for some irrational \( r \) or \( \mathcal{O}_q \) for some rational \( q \), and it follows from the proof of Claim 6.13.3 below that a closed non-Lindelöf subspace narrow in \( M \) is sc- and uf-equivalent to \( \mathcal{L}_r, \mathcal{H}_r \) or \( \mathcal{D}_r \) for some \( r \in I \).

Claim 6.13.2. Let \( U \subset M \) be open, \( r \) be irrational and \( q \) rational. If \( U \supset \mathcal{L}_r \), then \( \tau(U) \supset \mathbb{L}_0 \times (t,s) \) for some \( t < r < s \). If \( U \supset \mathcal{H}_q \), then \( \tau(U) \supset \mathbb{L}_0 \times (t,q) \) for some \( t < q \). If \( U \supset \mathcal{D}_q \), then \( \tau(U) \supset \mathbb{L}_0 \times (q,t) \) for some \( t > q \).

Proof. Suppose that \( U \supset \mathcal{L}_r \) and \( \tau(U) \not\supset \mathbb{L}_0 \times (t,r) \) for any \( t < r \). Then there is an \( \omega \)-sequence \( (x_n, r_n) \notin \tau(U) \) with \( r_n \not\succ r \). Let \( x \in \mathbb{L}_0 \) be an accumulation point of the \( x_n \). By the previous claim \( (x,r) \in \tau(M-U) \), which is impossible since \( \tau(M-U) \cap \mathbb{L}_0 \times \{ r \} = \emptyset \). The same holds if \( \tau(U) \not\supset \mathbb{L}_0 \times (r,s) \) for any \( r < s \), with obvious changes if \( r \) is a boundary point of \( I \).

Suppose now that \( U \supset \mathcal{H}_q \) and \( \tau(U) \not\supset \mathbb{L}_0 \times (t,q) \) for any \( t < q \). Forget all of \( \mathcal{O}_q \) except \( \mathcal{H}_q \), and every point with vertical coordinate \( > q \) and argue as above to obtain the desired contradiction. The proof for \( \mathcal{D}_q \) is the same.

Claim 6.13.3. Let \( t < p < r \), with \( t, r \) irrational and \( p \) rational. Then

\[
\mathcal{L}_t \prec \mathcal{H}_p \prec \mathcal{D}_p \prec \mathcal{L}_r.
\]

Proof. Example 6.4 shows that \( \mathcal{H}_p \preceq \mathcal{D}_p \). Let \( R_s \) denote \( \mathcal{L}_s \) if \( s \) is irrational and \( \mathcal{H}_s \) if it is rational. We show that \( R_s \preceq \mathcal{D}_p, \mathcal{H}_q, \mathcal{L}_t \) whenever \( t, p < s \), which implies \( \preceq \) instead of \( < \). Let \( \{ U_\alpha : \alpha \in \mathcal{W}_1 \} \) be a systematic cover of \( M \) such that \( U_0 \supset R_s \). By Claim 6.4, \( \tau(U_0) \) contains \( \mathbb{L}_0 \times (u,s) \) for some \( u < s \). If \( u \) is rational, it implies by construction that \( U_0 \supset \mathcal{D}_u \). Hence, by Example 6.4, \( \mathcal{H}_u \subset U_\gamma \) for some \( \gamma_0 \). If \( u \) is irrational, then \( U_1 \supset U_0 \supset \mathcal{L}_u \). Proceeding by induction, we obtain a strictly decreasing sequence \( u_\alpha \in I \) and ordinals \( \gamma_\alpha \) such that \( U_{\gamma_\alpha} \) contains all the points of \( M \) with vertical coordinate between \( t_0 \) and \( s \). The sequence \( u_\alpha \) must reach the lower boundary point of \( I \) in at most countably many steps, hence all \( \mathcal{D}_q, \mathcal{H}_p, \mathcal{L}_t \) are contained in some \( U_\gamma \) whenever \( t, p < s \).

To pass from \( \preceq \) to \( < \), notice that it is easy to define a map \( f_p : M \to \mathbb{L}_0 \) which is identically 0 on each \( \mathcal{L}_t, \mathcal{O}_r \) for \( t < p \) and on \( \mathcal{H}_p \) and equal to \( \pi \circ \tau \) on \( \mathcal{D}_p \) and \( \mathcal{L}_r, \mathcal{O}_r \) for \( r > p \) (take the vertical projection in \( \mathcal{O}_p \)). This shows that \( \mathcal{L}_t, \mathcal{H}_p \not\preceq f \mathcal{D}_p \not\preceq f \mathcal{L}_r \). The remaining \( \mathcal{L}_t \not\preceq f \mathcal{H}_p \) is provided by \( f_q \) for some \( t < q < p \).
Hence, $\mathcal{R}(M)$ is the ordered set obtained by doubling each rational point in $I$, with no point in between new pairs. This is a way of describing the Cantor space. 

If we insert copies of $(\mathbb{L}_{\geq 0})^2$ instead of copies of $\mathbb{O}$ in the previous example, then $\mathcal{R}(M)$ is the poset obtained by taking $\mathbb{R}$ and replacing each rational $q$ by a triplet $q_0, q_1, q_2$, where $q_0, q_2 < q_1$, $q_0 \perp q_2$, and each other pair of points are incomparable. In particular, $\mathcal{R}(M)$ contains an antichain of cardinality $\mathfrak{c}$ (the irrational points). Notice also that in Example 6.13 countably many octants were inserted at once. We may instead proceed inductively by removing (final parts of) copies of $\mathbb{L}_{\geq 0}$ above level $\alpha$ and insert copies of $\mathbb{O}_{\geq \alpha}$ in the scar to obtain more complicated posets. It is probably possible to use Example 6.13 as a blueprint to obtain some kind of general theorem about inverse limits of spaces obtained this way, thus producing many other examples. We did not pursue this road (except for a small number of specimens described with variable rigor in [2]) and let the inquisitive reader forge their way through this swamp of possibilities if it suits their curiosity.

Acknowledgements. Some of the work presented here dates back to 2006, when David Gauld invited the author to Auckland’s university. We express our gratitude to him and this institution for their hospitality. We also thank P. Nyikos for sharing some of his insights and ideas about the subject, and Jean-Luc le Ténia for his tenacity.

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