The circular Dyson Brownian motion model refers to the stochastic dynamics of the log-gas on a circle. It also specifies the eigenvalues of certain parameter-dependent ensembles of unitary random matrices. This model is considered with the initial condition that the particles are non-interacting (Poisson statistics). Jack polynomial theory is used to derive a simple exact expression for the density-density correlation with the position of one particle specified in the initial state, and the position of one particle specified at time $\tau$, valid for all $\beta > 0$. The same correlation with two particles specified in the initial state is also derived exactly, and some special cases of the theoretical correlations are illustrated by comparison with the empirical correlations calculated from the eigenvalues of certain parameter-dependent Gaussian random matrices. Application to fluctuation formulas for time displaced linear statistics is made.

1 Introduction

The Dyson Brownian motion model [1] refers to the overdamped Brownian dynamics of the one-dimensional log-gas. This dynamics is specified by the Fokker-Planck equation

$$\frac{\partial p}{\partial \tau} = \mathcal{L} p \quad \text{where} \quad \mathcal{L} = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left( \frac{\partial W}{\partial x_j} + \beta^{-1} \frac{\partial}{\partial x_j} \right)$$

(1.1)

with the particular potential

$$W = \frac{1}{2} \sum_{j=1}^{N} x_j^2 - \sum_{1 \leq j < k \leq N} \log |x_k - x_j|,$$

(1.2)

or its periodic version

$$W = - \sum_{1 \leq j < k \leq N} \log |e^{2\pi i x_k/L} - e^{2\pi i x_j/L}|.$$

(1.3)
The formulation of this model has its origin in the theory of parameter dependent Gaussian random matrices. We recall that a random matrix $H$ is of this type if its joint distribution of elements is proportional to

$$\exp\left(-\beta \text{Tr}(H - e^{-\tau}H(0))^2/(2(1 - e^{-2\tau}))\right), \quad (1.4)$$

where $\beta = 1, 2$ or 4 according to $H$ being symmetric, Hermitian or self-dual real quaternion respectively. This means that each part (real or imaginary) of each independent element of $H$ is chosen with a particular Gaussian distribution. The matrix $H(0)$ is some prescribed (random) matrix. Note that as $\tau \to 0$, $H = H(0)$, while as $\tau \to \infty$ (1.4) reduces to $\exp(-\beta \text{Tr}H^2/2)$. With $\beta = 1, 2$ and 4 this latter distribution specifies the Gaussian Orthogonal, Unitary and Symplectic Ensembles (GOE, GUE, GSE) respectively.

Let $p = p(x_1, \ldots, x_N; \tau)$ denote the eigenvalue probability density function (p.d.f.) corresponding to (1.4). By using second order perturbation theory, Dyson [1] showed that $p$ satisfies the Fokker-Planck equation (1.1) with $W$ given by (1.2). This equation must be solved subject to the initial condition that $p$ corresponds to the eigenvalue p.d.f. of $H$ at $\tau = 0$. Dyson also formulated a parameter dependent theory of random unitary matrices from the so-called circular ensembles. Here the situation is more abstract in the sense that no explicit construction of such unitary random matrices is known. Nonetheless, in the framework of the abstract formulation it was shown that the corresponding eigenvalue p.d.f. (eigenvalues $e^{2\pi i x_j/L}$, $0 \leq x_j < L$) satisfies the Fokker-Planck equation (1.3) with $W$ given by (1.3). It turns out that upon introducing the scaling [2]

$$x \mapsto \pi \rho x/\sqrt{2N}, \quad \tau \mapsto (\pi \rho)^2 \tau/(2N) \quad (1.5)$$

into the Fokker-Planck equation with $W$ given by (1.2), and taking the $N \to \infty$ limit, the results obtained for the correlation functions and other observable quantities are identical to those obtained with the choice of $W$ (1.3). In fact for correlations over one or two eigenvalue spacings in the bulk of the spectrum, the agreement already becomes apparent for matrix dimensions $N = 11$ (see e.g. [3]).

Our specific interest in this paper is the exact calculation of certain correlation functions in the Fokker-Planck equation (1.1) with $W$ given by (1.3) and subject to the initial condition $p(x_1, \ldots, x_N; 0) = 1/LN$. This initial condition corresponds to non-interacting particles; in the applied random matrix literature it is referred to as Poisson initial conditions, which is consistent since a gas of non-interacting particles exhibits Poisson statistics. Thus for the random matrix couplings $\beta = 1, 2$ and 4 and $N$ large, the Fokker-Planck equation with this initial condition describes the transition between an eigenvalue p.d.f. with Poisson statistics to the eigenvalue p.d.f. of the appropriate Gaussian ensemble. Such a transition is relevant to the description of the statistical properties of the eigenvalue spectrum in a quantum system which is initially integrable, but becomes chaotic as a parameter (usually identified with $\tau^{1/2}$; see e.g. [4]) is varied. Because of this application, this problem has received a lot of recent attention [5, 6, 7, 8, 9].

In the next section we will specify the particular correlation functions to be calculated (density-density correlation with the position of $n$ particles specified in the initial state, and the
position of one particle specified at ‘time’ $\tau$), and then proceed to calculate this correlation in the case $n = 1$ for general $\beta$. Comparison with the empirical evaluation of this correlation for $\beta = 1$ obtained from numerically generated parameter dependent random matrices is also made. In Section 3 we consider the same correlation with $n = 2$, and provide its exact value for $\beta = 2$ and 4, as well as all $0 < \beta < 2$. A discussion of these results, including asymptotic properties and their relationship to fluctuation formulas for linear statistics, is given in Section 4.

2 Density-density correlation

2.1 Formalism

In general the correlation functions for the Brownian motion described by (1.1) can be specified in terms of the Green function $G(x_1^{(0)}, \ldots, x_N^{(0)}; x_1, \ldots, x_N; \tau)$, which is by definition the solution of (1.1) subject to the initial condition

$$p(x_1, \ldots, x_n; 0) = \prod_{j=1}^N \delta(x_j - x_j^{(0)}).$$

For a general initial condition with p.d.f. $f(x_1^{(0)}, \ldots, x_N^{(0)})$ (assumed symmetric) the particular density-density correlation $\rho_{(n,1)}^T(x_1^{(0)}, \ldots, x_n^{(0)}; x; \tau)$ is given in terms of $G$ by

$$\rho_{(n,1)}^T(x_1^{(0)}, \ldots, x_n^{(0)}; x; \tau) = N(N-1) \cdots (N-n+1) \int_I dx_{n+1} \cdots \int_I dx_N f(x_1^{(0)}, \ldots, x_N^{(0)})$$

$$\times \int_I dx_1 \cdots \int_I dx_N \left( \sum_{j=1}^N \delta(x - x_j^{(1)}) \right) G(x_1^{(0)}, \ldots, x_N^{(0)}; x_1, \ldots, x_N; \tau)$$

$$- \rho_{(n)}(x_1^{(0)}, \ldots, x_n^{(0)}; 0) \rho_{(1)}(x; \tau).$$

Note that $\rho_{(n,1)}^T(x_1^{(0)}, \ldots, x_n^{(0)}; x; \tau)/\rho_{(n)}(x_1^{(0)}, \ldots, x_n^{(0)}; 0) + \rho_{(1)}(x; \tau)$ represents the density at position $x$ after time $\tau$, given that there are particles at $x_1^{(0)}, \ldots, x_n^{(0)}$ initially.

To obtain a useful expression for the Green function, one uses the general fact [10] that after conjugation with $e^{-\beta W}$ the Fokker-Planck operator $\mathcal{L}$ transforms into a Hermitian operator. In fact (see e.g. [11]) for the choices of (1.2) and (1.3) one has

$$e^{\beta W/2} \mathcal{L} e^{-\beta W/2} = -\frac{1}{\beta} (H - E_0),$$

where $H$ is the Schrödinger operator for a quantum mechanical system with one and two body interactions only, and $E_0$ is the corresponding ground state energy. Explicitly, for $W$ given by (1.2)

$$H = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \beta (\beta/2 - 1) \left( \frac{\pi}{T} \right)^2 \sum_{1 \leq j < k \leq N} \frac{1}{\sin^2 \pi(x_k - x_j)/L},$$

which is an example of the Calogero-Sutherland quantum many body system (quantum particles with $1/r^2$ pair interaction). Now in general for the imaginary time Schrödinger equation

$$\frac{\partial \psi}{\partial \tau} = -\frac{1}{\beta} (H - E_0) \psi$$

(2.5)
in which $H$ has a complete set of orthogonal eigenfunctions $\{\psi_\kappa(x)\}_\kappa$ (here $x := (x_1, \ldots, x_N)$ and $\kappa$ represents an $n$-tuple of ordered integers $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_N$) with corresponding eigenvalues $\{E_\kappa\}_\kappa$, the method of separation of variables gives that the Green function solution is

$$G^{(H)}(x_1^{(0)}, \ldots, x_N^{(0)}; x_1, \ldots, x_N; \tau) = \sum_\kappa \frac{\psi_\kappa(x^{(0)})}{\langle \psi_\kappa | \psi_\kappa \rangle} e^{-(E_\kappa-E_0)\tau/\beta}. \quad (2.6)$$

In (2.4)

$$\langle \psi_\kappa | \psi_\kappa \rangle := \int_I dx_1 \cdots \int_I dx_N |\psi_\kappa(x)|^2, \quad (2.7)$$

and * denotes complex conjugation.

Since, according to (2.3), the Green function solution $G(x^{(0)}; x; \tau)$ of the Fokker-Planck equation (1.1) is related to the Green function solution $G^{(H)}(x^{(0)}; x; \tau)$ of (2.5) by

$$G(x^{(0)}; x; \tau) = \frac{e^{-\beta W(x_1, \ldots, x_N)/2}}{e^{-\beta W(x_1^{(0)}, \ldots, x_N^{(0)}/2)}} G^{(H)}(x^{(0)}; x; \tau), \quad (2.8)$$

$G(x^{(0)}; x; \tau)$ is determined to the extent that the quantities in (2.7) are known. Independent of such explicit knowledge, substituting (2.6) in (2.8) and substituting the result in (2.2) gives

$$\rho_{(n,1)}(x_1^{(0)}, \ldots, x_n^{(0)}; x; \tau) = N(N-1) \cdots (N-n+1) \times \sum_{\kappa, \kappa \neq 0} A_\kappa(x_1^{(0)}, \ldots, x_n^{(0)}) \langle \psi_\kappa | \psi_\kappa \rangle \sum_{j=1}^N \delta(x-x_j) \psi_\kappa(x) e^{-(E_\kappa-E_0)\tau/\beta} \quad (2.9)$$

where

$$A_\kappa(x_1^{(0)}, \ldots, x_n^{(0)}) = \int_I dx_{n+1} \cdots \int_I dx_N f(x_1^{(0)}, \ldots, x_n^{(0)}) \psi_\kappa(x_1^{(0)}) e^{\beta W/2} \quad (2.10)$$

$$\langle e^{-\beta W/2} \sum_{j=1}^N \delta(x-x_j) \psi_\kappa \rangle = N \int_I dx_2 \cdots \int_I dx_N e^{-\beta W(x)/2} \psi_\kappa(x). \quad (2.11)$$

### 2.2 Explicit formulas

For the Schrödinger operator (2.4), it is known [12, 13] that the (unnormalized) eigenfunctions can be written in the form

$$\psi_\kappa(x) = e^{-\beta W/2} P^{(2/\beta)}_\kappa(z), \quad z := e^{2\pi i x/L}, \quad (2.13)$$

where $\kappa = (\kappa_1, \ldots, \kappa_N)$ forms a partition and thus has all parts non-negative and ordered as specified below (2.5), and $P^{(2/\beta)}_\kappa(z)$ is a particular polynomial known as the Jack polynomial. Each Jack polynomial has the special structure

$$P^{(2/\beta)}_\kappa(z) = m_\kappa + \sum_{\mu < \kappa} b_{\kappa \mu} m_\mu \quad (2.14)$$

where $m_\kappa$ refers to the monomial symmetric function (in the variables $z_1, \ldots, z_N$) corresponding to the partition $\kappa$, the $b_{\kappa \mu}$ are coefficients, while $\mu < \kappa$ refers to the dominance ordering of
partitions. For $\kappa$ possessing negative parts, (2.14) has no immediate meaning. In such cases the Jack polynomials are defined by the relation

$$P^{(2/\beta)}_\kappa(z) = \left(\prod_{l=1}^{N} z_l^{-c}\right) P^{(2/\beta)}_{\kappa+c}(z), \quad (2.15)$$

where $\kappa$ is a partition while $c := (c, \ldots, c), \ c \in \mathbb{Z}^+$. Also, for general $\kappa$, the energy eigenvalues are given by

$$E_\kappa - E_0 = \left(\frac{2\pi}{L}\right)^2 \left(\sum_{j=1}^{N} \kappa_j^2 + \frac{\beta}{2} \sum_{j=1}^{N} \kappa_j(N - 2j + 1)\right). \quad (2.16)$$

Although (2.14) is a very special property associated with (2.4), the true utility of Jack polynomial theory in regard to computing correlation functions lies with our knowledge of the integrals (2.7) and (2.12) (see e.g. [11] and references therein). First consider the normalization integral (2.7). In the case of the ground state ($\kappa = 0$) we have

$$\langle \psi_0 | \psi_0 \rangle := \int_0^L dx_1 \cdots \int_0^L dx_N \prod_{1 \leq j < k \leq N} |e^{2\pi i x_j/L} - e^{2\pi i x_k/L}|^\beta \quad \Rightarrow \quad (2.17)$$

This can be conveniently factored out of the normalization integral for general $\kappa$. In addition, the general $\kappa$ case involves the generalized factorial

$$[u]^{(2/\beta)}_\kappa := \prod_{j=1}^{N} \frac{\Gamma(u - \frac{\beta}{2}(j - 1) + \kappa_j)}{\Gamma(u - \frac{\beta}{2}(j - 1))}, \quad (2.18)$$

the magnitude of the partition,

$$|\kappa| := \sum_{j=1}^{N} \kappa_j, \quad (2.19)$$

the Jack polynomial which each variable set equal to unity $P^{(2/\beta)}_\kappa(1^N)$, and the quantity

$$d'_\kappa := \prod_{(i,j) \in \kappa} \left(\kappa'_j - i + \frac{2}{\beta}(\kappa'_i - j + 1)\right). \quad (2.20)$$

In (2.20) it is necessary that $\kappa$ be a partition The product is over all squares in the diagram of $\kappa$, and $\kappa'$ refers to the partition conjugate to $\kappa$. In terms of this notation, for $\kappa$ a partition we have

$$\langle \psi_\kappa | \psi_\kappa \rangle = \langle \psi_0 | \psi_0 \rangle \left(\frac{\beta}{2}\right)^{|\kappa|} d'_\kappa P^{(2/\beta)}_\kappa(1^N) \frac{|\kappa|^{(2/\beta)}}{[\frac{\beta}{2}(N - 1) + 1]^{(2/\beta)}} \quad (2.21)$$

The normalization of the states in which $\kappa$ possesses negative parts and so does not form a partition are reducible to the latter case by the simple formula

$$\left\langle \left(\prod_{l=1}^{N} z_l^{-c}\right) \psi_\kappa \right| \left(\prod_{l=1}^{N} z_l^{-c}\right) \psi_\kappa = \langle \psi_\kappa | \psi_\kappa \rangle. \quad (2.22)$$
We turn now to the integral (2.12). This quantity has the evaluation
\[
\langle e^{-\beta W/2} \sum_{j=1}^{N} \delta(x - x_j) | \psi_\kappa \rangle = e^{2\pi i|\kappa|/L} \langle \psi_0 | \psi_0 \rangle \frac{[|\kappa|][|\kappa_1| - 1]!}{L} \frac{\prod_{j=2}^{\ell(\kappa)} \left(-\frac{\beta}{2}(j - 1)\right)^{\kappa_j}}{\left[\frac{\beta}{2}(N - 1) + 1\right]^{(2/\beta)}},
\]
where \(\ell(\kappa)\) denotes the length of the partition \(\kappa\) (i.e. the number of non-zero parts) and
\[(a)_n := a(a + 1) \cdots (a + n - 1).
\]
For the \(n\)-tuples of non-positive integers
\[-\bar{\kappa} := (-\kappa_N, -\kappa_{N-1}, \ldots, -\kappa_1)
\]we have the simple formula
\[
\langle \psi_0 \sum_{j=1}^{N} \delta(x - x_j) | \psi_{-\bar{\kappa}} \rangle = \langle \psi_0 \sum_{j=1}^{N} \delta(x - x_j) | \psi_\kappa \rangle^*,
\]
and thus (2.23) gives the evaluation. Furthermore, for \(\beta\) rational, we have the result [14] that (2.12) is non-zero only if all parts of \(\kappa\) are non-negative or all parts are non-positive. Hence, if we restrict attention to \(\beta\) rational (which we will do henceforth), the results (2.23) and (2.26) suffice. In this regard we note also that it is a simple result that
\[
\langle \psi_{-\bar{\kappa}} | \psi_{-\bar{\kappa}} \rangle = \langle \psi_\kappa | \psi_\kappa \rangle,
\]
which when combined with (2.26) implies that we can restrict the summation in (2.3) to partitions provided twice the real part is taken.

Finally we come to consider the integral \(A_\kappa\) defined by (2.11). For the case under investigation, this reads
\[
A_\kappa(x_1^{(0)}, \ldots, x_n^{(0)}) = \frac{1}{L^N} \int_0^L dx_{n+1} \cdots \int_0^L dx_N \left( P_{\kappa}^{(2/\beta)}(z_1^{(0)}, \ldots, z_N^{(0)}) \right)^*,
\]
which is proportional to the term independent of \(z_{n+1}^{(0)}, \ldots, z_N^{(0)}\) in the power series expansion of \(P_{\kappa}^{(2/\beta)}(z^{(0)})\). Since we have now reduced the cases to be considered down to those for which \(\kappa\) is a partition, this term can be determined by setting \(z_{n+1}^{(0)} = \cdots = z_N^{(0)} = 0\) and we therefore have
\[
A_\kappa(x_1^{(0)}, \ldots, x_n^{(0)}) = L^{N-n} \left( P_{\kappa}^{(2/\beta)}(z_1^{(0)}, \ldots, z_n^{(0)}, 0, \ldots, 0) \right)^* \begin{cases} L^{N-n} \left( P_{\kappa}^{(2/\beta)}(z_1^{(0)}, \ldots, z_n^{(0)}) \right)^* \quad \ell(\kappa) \leq n, \\ 0 \quad \text{otherwise}, \end{cases}
\]
where the first expression in the second equality follows from the fact that the coefficients \(b_{\kappa\mu}\) in (2.14) are independent of \(N\).

Substituting (2.24), (2.26) and (2.28) into (2.9) and taking twice the real part we obtain the formula
\[
\rho^T_{(n,1)}(x_1^{(0)}, \ldots, x_n^{(0)}; x; \tau) = N(N - 1) \cdots (N - n + 1) 2 \text{Re} \sum_{\substack{\text{partitions } \kappa \ni 0, \ell(\kappa) \leq n}} u_\kappa \left( P_{\kappa}^{(2/\beta)}(z_1^{(0)}, \ldots, z_n^{(0)}) \right)^* e^{2\pi i x/L} e^{-\tau(E_\kappa - E_0)/\beta}
\]
(2.29)
where
\[
u_\kappa := \frac{1}{L} \left| \kappa \right| (\kappa_1 - 1)! \prod_{j=2}^{\ell(\kappa)} \left( -\frac{\beta}{2} (j - 1) \right)_{\kappa_j},
\] (2.30)
valid for \(\beta\) rational at least. In fact this result must be valid for all \(\beta > 0\), by continuity of \(\rho^T_{(n,1)}\) in \(\beta\).

### 2.3 The case \(n = 1\)

In the case \(n = 1\) the only partitions contributing to (2.29) are \((\kappa_1, 0, \ldots, 0)\) and we have
\[
P^{(2/\beta)}_\kappa(z^{(0)}_1) = \left(z^{(0)}_1\right)^{\kappa_1}, \quad E_\kappa - E_0 = \left(\frac{2\pi}{L}\right)^2 \left(\kappa_1^2 + \frac{\beta}{2} (N - 1) \kappa_1\right), \quad \nu_\kappa = 1.
\] (2.31)
Thus (2.29) reduces to the remarkably simple formula
\[
\rho^T_{(1,1)}(x^{(0)}_1; x; \tau) = \frac{2(N-1)\rho}{NL} \sum_{\kappa_1=0}^{\infty} e^{-\frac{2\pi}{L} (\kappa_1^2 + (\beta/2)(N-1)\kappa_1)\tau/\beta} \cos 2\pi \kappa_1 (x - x^{(0)}_1)/L
\] (2.32)
where \(\rho := N/L\). In the thermodynamic limit \(N, L \to \infty, \rho\) fixed, (2.32) being a Riemann sum tends to the definite integral
\[
\rho^T_{(1,1)}(x^{(0)}_1; x; \tau) = 2\rho^2 \int_0^\infty e^{-2\pi \rho^2 (s^2 + (\beta/2)\kappa_1^2)\tau/\beta} \cos 2\pi s \rho (x - x^{(0)}_1) ds.
\] (2.33)

As mentioned in the Introduction, the correlation function given analytically by (2.32) with \(\beta = 1, 2\) or 4 can be calculated empirically using parameter-dependent random matrices. For definiteness consider the case \(\beta = 1\). We construct initially a diagonal random matrix \(H^{(0)} = \text{diag}(h^{(0)}_{11}, \ldots, h^{(0)}_{NN})\) with each diagonal entry chosen from a Gaussian distribution with mean zero and standard deviation \(N(2\pi)^{-1/2}\). This standard deviation is chosen so that the eigenvalue density, which equals \(N\) times the Gaussian distribution for a diagonal element, is unity at the origin. Then, following the prescription (1.4) with \(\beta = 1\) and the scaling (1.5), we construct a real symmetric random matrix in which the diagonal entries have mean \(e^{-t h^{(0)}_{jj}}\) and standard deviation \((\sqrt{2N}/\pi)(1 - e^{-2t})^{1/2}\), \(t := \pi^2 \tau/(2N)\), while the independent off diagonal elements (upper triangular elements say) have mean zero and standard deviation equal to \(1/\sqrt{2}\) of that of the diagonal elements. Note in particular that the factor of \(\sqrt{2N}/\pi\) in the standard deviation is chosen so that at the centre of the spectrum the theoretical density remains equal to unity.

We choose a specific value of \(N\) and \(t\), and numerically generate many such random matrices with a fixed initial diagonal entry \(h^{(0)}_{11} = 0\), together with their eigenvalues. The corresponding empirical eigenvalue density in the neighbourhood of the origin can be computed and compared directly against the theoretical result (2.32) with \(\rho = 1, \beta = 1\) and the equilibrium density \(\rho = 1\) added. The results of such a calculation are given in Figure 1. We see that the empirical and theoretical results agree to statistical accuracy.
Figure 1: Comparison between the empirical density for 2,500 Gaussian real symmetric parameter-dependent random matrices of dimension $13 \times 13$ with $t = .025$ and a zero eigenvalue initially, and the theoretical value computed from (2.32) with $\beta = 1$.

3 The correlation $\rho_{(2,1)}^T(x_1^{(0)}, x_2^{(0)}; x; \tau)$

In this section we will consider the correlation (2.29) with $n = 2$. Explicit formulas are possible in this case because of the formula 13

$$P^{(2/\beta)}_{(\kappa_1, \kappa_2)}(z_1^{(0)}, z_2^{(0)}) = \frac{2^{\kappa_1 - \kappa_2}}{a_{\kappa_1 - \kappa_2}} (z_1^{(0)} z_2^{(0)})^{(\kappa_1 + \kappa_2)/2} P^{(\gamma, \gamma)}_{\kappa_1 - \kappa_2} \left( \frac{1}{2} (z_1^{(0)} + z_2^{(0)}) (z_1^{(0)} z_2^{(0)})^{-1/2} \right), \quad (3.1)$$

where on the r.h.s. $P^{(\alpha, \beta)}_n(x)$ denotes the Jacobi polynomial, $\gamma : = (\beta - 1)/2$ and

$$a_n : = \left( \frac{n + \gamma}{n} \right) \left( \frac{n + 2\gamma + 1}{\gamma + 1} \right)^n 2^{-n}. \quad (3.2)$$

Of particular relevance is the asymptotic form of (3.1) in the limit $\kappa_1, \kappa_2, L \to \infty$, $\rho$ fixed. The leading behaviour of the term (3.2) is determined using Stirling’s formula, while the Jack polynomial is estimated by noting that

$$\frac{1}{2} (z_1^{(0)} + z_2^{(0)}) (z_1^{(0)} z_2^{(0)})^{-1/2} = \cos \pi (x_1^{(0)} - x_2^{(0)}) / L, \quad (3.3)$$

and making use of the asymptotic formula

$$P^{(\gamma, \gamma)}_n(\cos \theta) \sim \left( \frac{2}{\theta} \right)^\gamma J_\gamma(n\theta), \quad (3.4)$$

where $J_\gamma$ denotes the Bessel function. Thus we have

$$P^{(2/\beta)}_{(\kappa_1, \kappa_2)}(z_1^{(0)}, z_2^{(0)}) \sim N^{\beta/2} \left( \pi (s_1 - s_2) \right)^{1/2} \times e^{\pi i (s_1 + s_2) \rho(x_1^{(0)} + x_2^{(0)})} \frac{1}{\left( 2 \pi \rho(x_1^{(0)} - x_2^{(0)}) \right)^{\beta - 1/2}} J_{(\beta - 1)/2} \left( (\kappa_1 - \kappa_2) \pi \rho(x_1^{(0)} - x_2^{(0)}) \right) \quad (3.5)$$

where $s_1 : = \kappa_1 / N$, $s_2 : = \kappa_2 / N$. 
Due to the factor \((-\beta/2)\kappa_2\) in (2.30), the cases for which \(\beta\) is even require special consideration since then \(\kappa_2\) is restricted to the range 0 to \(\beta/2 - 1\) for a non-zero contribution. In fact, due to cancellation effects within this range, each even value of \(\beta\) must be considered separately. We will consider in detail the cases \(\beta = 2\) and 4.

### 3.1 \(\beta = 2\)

In this case we see from (2.30) and (2.20) that with \(\kappa_2 = 0\), \(u_\kappa = 1/L\) while for \(\kappa_2 = 1\), \(u_\kappa = -1/L\). This fact, together with (3.5) and (2.16) shows

\[
\sum_{\kappa_2=0}^1 u_\kappa (P_{(\kappa_1,\kappa_2)}(\tilde{z}_1^{(0)}, \tilde{z}_2^{(0)})) e^{2\pi i \kappa x/L} e^{-\tau(E_\kappa - E_0)/\beta} \sim \frac{\rho \pi^{1/2}}{N} e^{-\pi i \kappa \rho(\tilde{x}_1^{(0)} + \tilde{x}_2^{(0)})} \\
\times e^{-2\pi \rho^2 (s_1^2 + s_1)} e^{-2\pi \rho^2 (s_2^2 + s_2)} \times e^{-2\pi \rho^2 (s_1^2 + s_1)} e^{-2\pi \rho^2 (s_2^2 + s_2)} \times \left( \frac{s_1}{2\rho \rho(x_1^{(0)} - x_2^{(0)})} \right)^{1/2} J_{1/2}(\pi \kappa \rho(\tilde{x}_1^{(0)} - \tilde{x}_2^{(0)}))
\]

(3.6)

where \(\tilde{x}_1^{(0)} := x_1^{(0)} - x, \tilde{x}_2^{(0)} := x_1^{(0)} - x\). Making use of the formula

\[
J_{1/2}(z) = \left( \frac{2}{\pi z} \right)^{1/2} \sin z,
\]

(3.7)

summing over \(\kappa_1\) and substituting the result in (2.29) gives that in the thermodynamic limit

\[
\rho_{(2,1)}^{T}(x_1^{(0)}, x_2^{(0)}; x; \tau) = 2 \rho^3 \int_0^\infty ds \cos \pi \kappa \rho(\tilde{x}_1^{(0)} + \tilde{x}_2^{(0)}) e^{-2\pi \rho^2 (s^2 + s)^\tau/2} \\
+ 2 \rho^3 \int_0^\infty ds \left( \frac{\sin \pi \kappa \rho(\tilde{x}_1^{(0)} - \tilde{x}_2^{(0)})}{\pi \rho(\tilde{x}_1^{(0)} - \tilde{x}_2^{(0)})} \right) \left( \frac{\pi \rho(\tilde{x}_1^{(0)} + \tilde{x}_2^{(0)})}{\pi \rho(\tilde{x}_1^{(0)} + \tilde{x}_2^{(0)})} \right) \sin \pi \kappa \rho(\tilde{x}_1^{(0)} + \tilde{x}_2^{(0)})) e^{-2\pi \rho^2 (s^2 + s)^\tau/2}.
\]

(3.8)

As an analytic check on (3.8), consider the limit \(x_2^{(0)} \to \infty\). Rewriting the trigonometric products as sums shows that

\[
\lim_{x_2^{(0)} \to \infty} \rho_{(2,1)}^{T}(x_1^{(0)}, x_2^{(0)}; x; \tau) = \rho_{(1,1)}^{T}(x_1^{(0)}; x; \tau)
\]

(3.9)

as expected.

A case of special interest is when \(x_1^{(0)} = x_2^{(0)}\), so that two particles coincide in the initial state. Making use of the formula

\[
P_{n;\gamma}(1) = \binom{n + \gamma}{n}
\]

(3.10)
in (3.1) and then proceeding as in the derivation of (3.8) we have that in the finite system

\[
\rho_{(2,1)}^{T}(x_1^{(0)}, x_1^{(0)}; x; \tau) = \frac{2(N-1)(N-2)}{L^3} \text{Re} \sum_{\kappa_1=1}^\infty e^{-2\pi i \kappa_1 \rho(x_1^{(0)} - x)/N} e^{-(2\pi \rho/N)^2(N-1)(N-2)\tau/2} \\
\times \left( (\kappa_1 + 1) - \kappa_1 e^{-2\pi i \rho(x_1^{(0)} - x)/N} e^{-(2\pi \rho/N)^2(N-2)\tau/2} \right). \quad (3.11)
\]
Figure 2: Comparison between the empirical density for 2,000 Gaussian Hermitian parameter-dependent random matrices of dimension $13 \times 13$ with $t = .05$ and a doubly degenerate zero eigenvalue initially, and the theoretical value computed from (3.11).

It is easy to see that in the thermodynamic limit (3.11) agrees with (3.8) with $x_1^{(0)} = x_2^{(0)}$, thus providing another check on the latter result. We can also compare the correlations obtained from the analytic formula (3.11) with those obtained empirically from parameter-dependent Hermitian random matrices with the initial condition of a doubly degenerate zero eigenvalue. The parameter-dependent matrices are constructed in an analogous way to that described at the end of the previous section. Thus the initial matrix $H^{(0)}$ is again diagonal and real, with the elements $h_{jj}^{(0)}$, $j > 2$, chosen with the same Gaussian distribution as before, while the elements $h_{11}$ and $h_{22}$ are set equal to zero. The matrix $H$ is chosen so that the diagonal entries have mean $e^{-t}h_{jj}^{(0)}$ and standard deviation $(\sqrt{N}/\pi)(1 - e^{-2t})^{1/2}$, $t := \pi^2\tau/(2N)$, while the independent off diagonal elements (real and imaginary parts of the upper triangular elements) have mean zero and standard deviation equal to $1/\sqrt{2}$ of that of the diagonal elements. Again one feature of this prescription is that the theoretical density at the origin is unity.

In Figure 2 we give a comparative plot of the empirical density obtained from generating the eigenvalues of such matrices for a certain choice of $N$ and $t$ with the corresponding theoretical formula (3.11).

3.2 $\beta = 4$

Here the formulas (2.30) and (2.20) give

$$u_\kappa|_{\kappa_2=0} = \frac{1}{L}, \quad u_\kappa|_{\kappa_2=1} = -\frac{2}{L}(1 - \frac{1}{\kappa_1 + 2}), \quad u_\kappa|_{\kappa_2=2} = \frac{1}{L}\left(1 - \frac{2}{\kappa_1 + 1}\right). \quad (3.12)$$

These values suggest we write the corresponding integrand for the sum over $\kappa_1$ in the form of a difference approximation to the second derivative. Doing this, and making use of (3.5) we find
that in the thermodynamic limit
\[
\rho_{(2,1)}^T(x_1^{(0)}, x_2^{(0)}; x; \tau) = 2\rho^3 \pi^{1/2} \text{Re} \int_0^\infty \! ds \, e^{-2\pi i \rho(x_1^{(0)} + x_2^{(0)}) s} e^{-(2\pi \rho)^2(s^2 + 2s)\tau/4} e^{-(2\pi \rho)^2 s\tau/2}
\times \left( \frac{\partial^2}{\partial s^2} f(s) - \left(\tau/2\right)(2\pi \rho)^2 f(s) + 2 \frac{\partial}{\partial s} \left( \frac{f(s)}{s} \right) \right)
\] (3.13)

where
\[
f(s) := s^{1/2} \left( \frac{1}{2\pi \rho(x_1^{(0)} - x_2^{(0)})} \right)^{3/2} J_{3/2} \left( \pi \rho(x_1^{(0)} - x_2^{(0)}) s \right) e^{\pi i \rho(x_1^{(0)} + x_2^{(0)}) s} e^{(2\pi \rho)^2 s\tau/2}.
\] (3.14)

Regarding checks on this formula, from the large-\(x\) expansion \(J_{3/2}(x) \sim -\left(2/(\pi x)\right)^{1/2} \cos x\) it is straightforward to show that the limiting behaviour (3.3) is exhibited. Furthermore, use of (3.10) as well as (3.12) in (2.29) gives that in the finite system with \(x_1^{(0)} = x_2^{(0)},\)
\[
\rho_{(2,1)}^T(x_1^{(0)}, x_2^{(0)}; x; \tau) = \frac{\rho^3}{3N} \text{Re} \sum_{\kappa_1=1}^\infty e^{-(2\pi \rho/N)^2(\kappa_1^2 + 2(N-1)\kappa_1)\tau/4} e^{-2\pi i \kappa_1 \rho x_1^{(0)}/N} \left( (\kappa_1 + 3)(\kappa_1 + 1) - 2(\kappa_1 + 1)^2 e^{-2\pi \rho/N)^2 (1+2(N-3))\tau/4} e^{2\pi i \rho x_1^{(0)}/N} + (\kappa_1 - 1)\kappa_1 e^{-(2\pi \rho/N)^2 (4+4(N-3))\tau/4} e^{4\pi i \rho x_1^{(0)}/N} \right).
\] (3.15)

In the thermodynamic limit this agrees with (3.13) in the case \(x_1^{(0)} = x_2^{(0)}\) (the formula \(J_\nu(x) \sim x^\nu/(2^{\nu}\Gamma(1+\nu))\) as \(x \to 0\) is required in the checking).

### 3.3 General \(0 < \beta < 2\)

In the cases \(\beta = 2\) and \(\beta = 4\) we have seen that cancellation takes place within the summand of (2.29) due to the factor \((-\beta/2)\kappa_2\) having varying sign. This is again true for general \(0 < \beta < 2\).

Thus for \(\kappa_2 = 0, (-\beta/2)\kappa_2 = 1\), while for \(\kappa_2 \geq 1, (-\beta/2)\kappa_2 = -(\beta/2)\Gamma(\kappa_2 - \beta/2)/\Gamma(1 - \beta/2) < 0\).

However in the case of general \(0 < \beta < 2\) the factor \((-\beta/2)\kappa_2\) is non-zero for all \(\kappa_2 \geq 0\), so the method of grouping together terms for cancellation used in the cases \(\beta = 2\) and \(\beta = 4\) is no longer applicable.

To gain some insight into the necessary grouping, let us consider the behaviour of \(u_\kappa\), as specified by (2.30) with \(\ell(\kappa) = 2\), for large \(\kappa_1, \kappa_2\). Use of (2.20) and Stirling’s formula shows
\[
u_\kappa = \frac{(\beta/2)(\kappa_1 + \kappa_2)(\kappa_1 - 1)! \Gamma(\kappa_2 - \beta/2)}{L \kappa_1! (\kappa_1 - \kappa_2)! \Gamma(\kappa_2 - \beta/2)} \frac{\Gamma(\beta/2 + \kappa_1 - \kappa_2 + 1)}{\Gamma(\beta/2 + \kappa_1 + 1)}
\sim \frac{(\beta/2)}{L \Gamma(1 - \beta/2)^{\beta/2}} \frac{(\kappa_1 - \kappa_2)^{\beta/2}}{\kappa_1^{1+\beta/2} \kappa_2^{1+\beta/2}}.
\] (3.16)

Writing this in terms of continuous variables \(s_1 := \kappa_1/N, s_2 := \kappa_2/N\) we see that (3.16) is non-integrable at \(s_1 = 0\) and \(s_2 = 0\). It is this singularity that we must cancel out using the \(\kappa_2 = 0\) term.

We proceed as follows. Define
\[
u_{(\kappa_1, \kappa_2)} = \frac{(\kappa_1 + \kappa_2)(\kappa_1 - 1)! \Gamma(\beta/2 + \kappa_1 - \kappa_2 + 1)}{L (\kappa_1 - \kappa_2)! \Gamma(\beta/2 + \kappa_1 + 1)}
\] (3.17)
so that
\[ u_\kappa = g_{\kappa_2} v_{(\kappa_1, \kappa_2)}, \quad g_{\kappa_2} := -\frac{(\beta/2)\Gamma(\kappa_2 - \beta/2)}{\Gamma(1 - \beta/2)\kappa_2!} \] (3.18)
and
\[ \rho^T_{(2,1)}(x_1(0), x_2(0); x; \tau) = \frac{N(N-1)}{L^2} \times 2\text{Re} \left( \sum_{\kappa_1=1}^{\infty} v_{(\kappa_1,0)} B_{(\kappa_1,0)}(z_1(0), z_2(0)) + \sum_{\kappa_1 \geq \kappa_2 \geq 1} g_{\kappa_2} v_{(\kappa_1, \kappa_2)} B_{(\kappa_1, \kappa_2)}(z_1(0), z_2(0)) \right), \] (3.19)
where
\[ B_{(\kappa_1, \kappa_2)}(z_1(0), z_2(0)) := \left( P^{(2/\beta)}(z_1(0), z_2(0)) \right)^* e^{2\pi i(\kappa_1+\kappa_2)x/L} e^{-\tau(E_\kappa - E_0)/\beta}. \] (3.20)
Now write
\[ g_{\kappa_2} = G_{\kappa_2+1} - G_{\kappa_2}, \quad G_0 := 0, \ G_1 := 1. \] (3.21)

From the general formula
\[ \sum_{k=0}^{N} \left( a_k(b_{k+1} - b_k) + b_{k+1}(a_{k+1} - a_k) \right) = a_{N+1}b_{N+1} - a_0b_0, \] (3.22)
we see that (3.19) can be rewritten
\[ \rho^T_{(2,1)}(x_1(0), x_2(0); x; \tau) = \frac{N(N-1)}{L^2} \times 2\text{Re} \left( \sum_{\kappa_1 \geq \kappa_2 \geq 1} G_{\kappa_2} \left( v_{(\kappa_1, \kappa_2)} B_{(\kappa_1, \kappa_2)}(z_1(0), z_2(0)) - v_{(\kappa_1, \kappa_2-1)} B_{(\kappa_1, \kappa_2-1)}(z_1(0), z_2(0)) \right) \\
+ \sum_{\kappa_1=1}^{\infty} G_{\kappa_1+1} v_{(\kappa_1, \kappa_1)} B_{(\kappa_1, \kappa_1)}(z_1(0), z_2(0)) \right). \] (3.23)

The utility of (3.23) is that the summand is well behaved in the large \( \kappa_1, \kappa_2 \) limit. To see this, note from the definition (3.18) that for large \( \kappa_2, \)
\[ g_{\kappa_2} \sim -\frac{(\beta/2)}{\Gamma(1 - \beta/2)\kappa_2^{1+\beta/2}}, \] (3.24)
and so (3.21) is asymptotically satisfied with
\[ G_{\kappa_2} \sim \frac{1}{\Gamma(1 - \beta/2)} \frac{1}{\kappa_2^{\beta/2}}. \] (3.25)

This has an integrable singularity at the origin, as distinct from the non-integrable singularity in (3.16). Making use of the asymptotic formulas (3.25) and (3.5) and noting from (3.17) that
\[ v_\kappa \sim \frac{\rho}{N} (\kappa_1 + \kappa_2) |\kappa_1 - \kappa_2|^{\beta/2} \frac{1}{\kappa_1^{1+\beta/2}}, \] (3.26)
we see from (3.23) that in the thermodynamic limit
\[ \rho^T_{(2,1)}(x_1(0), x_2(0); x; \tau) = \frac{\rho^3 \pi^{1/2}}{\Gamma(-\beta/2)^2} \frac{1}{|2\rho(x_1(0) - x_2(0))|^{\beta-1/2}} \int_0^\infty ds_1 \int_0^\infty ds_2 \left( \frac{1}{s_1s_2} \right)^{\beta/2} \times \frac{\partial^2}{\partial s_1 \partial s_2} \left( (s_1 + s_2)|s_1 - s_2|^{\beta+1/2} J_{(\beta-1)/2}(|s_1 - s_2|\pi |\rho(x_1(0) - x_2(0))|) \right) \times \cos \left( \pi |\rho(x_1(0) + x_2(0))(s_1 + s_2)\right) e^{-\left(2\pi \rho^2(s_1^2 + s_2^2 + (\beta/2)(s_1 + s_2))\right)^{\beta}} \right), \] (3.27)
where an integration by parts in \( s_1 \) has been carried out.
4 Discussion

4.1 Asymptotics

In the guise of the Calogero-Sutherland model, the density-density correlation $\rho_{(1,1)}^{T}(x_1^{(0)}; x; \tau)$ for the Dyson Brownian motion model in the case that the initial state equals the equilibrium state (in. eq.) is known for all rational $\beta$ (see [16] or [1] for the explicit connection between the correlations of the two models). For $\beta/2 = p/q$ ($p$ and $q$ relatively prime) its value in the thermodynamic limit is given in terms of a $(p+q)$-dimensional Dotsenko-Fateev-type integral. This contrasts with the simple one-dimensional integral (2.33) for the same correlation with Poisson initial conditions.

Although the expression for $\rho_{(1,1)}^{T}(x_1^{(0)}; x; \tau)$ in the case in. eq. is complicated, its non-oscillatory leading order large $|x_1^{(0)} - x|$ and/or $\tau$ expansion is very simple, being given by [16, 13]

$$\rho_{(1,1)}^{T}(x_1^{(0)}; x; \tau)\big|_{\text{in. eq.}} \sim \frac{4\rho^2}{\beta} \text{Re} \left( \frac{1}{\{\frac{1}{2}(2\pi \tau)^2 + 2\pi i \rho \bar{x}_1^{(0)}\}^2} \right).$$

(4.1)

Fixing $\tau$, (4.1) shows that the leading non-oscillatory portion of $\rho_{(1,1)}^{T}(x_1^{(0)}; x; \tau)$ falls off as $-1/(\beta(\pi \bar{x}_1^{(0)})^2)$, while with $\bar{x}_1^{(0)}$ fixed the decay $1/(\beta(\pi \rho)^2 \tau)^2$ is exhibited. Note that the decay in $\bar{x}_1^{(0)}$ is independent of the density.

Let us compare (4.1) with the corresponding asymptotic expansion in the case of Poisson initial conditions (in. Po.i.). By rewriting the cosine term as a complex exponential, linearizing the exponent about $s = 0$, and changing variables we see from (2.33) that

$$\rho_{(1,1)}^{T}(x_1^{(0)}; x; \tau)\big|_{\text{in. Po.i.}} \sim 2\rho^2 \text{Re} \left( \frac{1}{\frac{1}{2}(2\pi \rho)^2 - 2\pi i \rho \bar{x}_1^{(0)}} \right)$$

(4.2)

Notice that there is no $\beta$ dependence in this expression. For $\bar{x}_1^{(0)}$ fixed the decay here is given by $1/(\pi^2 \tau)$, independent of $\rho$, while for $\tau$ fixed the decay is $\rho^2 \tau/(\bar{x}_1^{(0)})^2$. These behaviours have distinct features from those noted above for (4.1).

The leading order large $\bar{x}_1^{(0)}$ and/or $\tau$ asymptotic expansion of (3.8) ($\beta = 2$ result) with $\bar{x}_1^{(0)} = \bar{x}_2^{(0)}$ can readily be computed. We find the same behaviour as in (4.2), except that the prefactor $2\rho^2$ is replaced by $4\rho^3$. For general $0 < \beta < 2$, setting $\bar{x}_1^{(0)} = \bar{x}_2^{(0)}$ in (3.27), then repeating the analysis which led to (4.1), we find

$$\rho_{(2,1)}^{T}(x_1^{(0)}; x_1^{(0)}; x; \tau) \sim 4\rho^3 A(\beta) \text{Re} \left( \frac{1}{\frac{1}{2}(2\pi \rho)^2 - 2\pi i \rho \bar{x}_1^{(0)}} \right)$$

(4.3)

where

$$A(\beta) := \frac{\pi^{1/2}}{2^{\beta+1} \beta/2 \Gamma(-\beta/2) \Gamma(\beta/2 + 1/2)}$$

$$\times \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\rho^{\beta/2}}{(s_1 + s_2)^{(\beta/2) + 1/2}} \frac{\partial^2}{\partial s_1 \partial s_2} \left( (s_1 + s_2)|s_1 - s_2|^{\beta} e^{-(s_1+s_2)} \right).$$

(4.4)
The integral in (4.4) is evaluated in the Appendix. Substituting its value gives $A(\beta) = 1$, so the asymptotic behaviour found at $\beta = 2$ persists independent of $\beta$ (for $\beta < 2$ at least) analogous to (4.3).

### 4.2 Fluctuation formulas

Let us now turn attention to the application of the density-density correlation $\rho_{(1,1)}^T(x_1^{(0)}; \tau)$ in the study of fluctuation formulas. In general [19] the time displaced covariance of two linear statistics

$$A_\tau = \sum_{j=1}^N a(x_j(\tau)), \quad B_\tau = \sum_{j=1}^N b(x_j(\tau))$$  \tag{4.5}

is given in terms of the Fourier transform

$$\tilde{S}(k; \tau) := \int_{-\infty}^{\infty} \rho_{(1,1)}^T(x_1^{(0)}; x; \tau)e^{i\bar{x}_1^{(0)}k} d\bar{x}_1^{(0)}$$  \tag{4.6}

(assuming a fluid state so $\rho_{(1,1)}^T(x_1^{(0)}; x; \tau)$ is a function of $\bar{x}_1^{(0)} := x_1^{(0)} - x$) according to the formula

$$\text{Cov}(A_0, B_\tau) = \int_{-\infty}^{\infty} \tilde{a}(k)\tilde{b}(-k)\tilde{S}(k; \tau) dk.$$  \tag{4.7}

Now, from (2.33), for the Dyson Brownian motion model with Poisson initial conditions

$$\tilde{S}(k; \tau) = \rho e^{-k^2\tau/\beta - \pi \rho |k| \tau},$$  \tag{4.8}

so (4.3) is known explicitly.

Rigorous studies [19] of the infinite Dyson Brownian motion model for $\beta = 2$ and with ineq., and of the same model on a circle [20] for general $\beta > 0$ have shown that after appropriate scaling the joint distribution of $(A_0, B_\tau)$ is a Gaussian. In the infinite system the covariance is given by (4.7) with $\tilde{S}(k; \tau)$ replaced by its scaled form, while for the circle system, a discrete version of (4.7) applies. Explicitly, in the infinite system a small parameter $\epsilon$ is introduced in (4.5) so that the linear statistics become

$$A_\tau = \sum_{j=1}^N a(\epsilon x_j(\tau)), \quad B_\tau = \sum_{j=1}^N b(\epsilon x_j(\tau)).$$  \tag{4.9}

Physically this means that the variation of $a$ and $b$ is macroscopic. Also, $\tau$ is scaled by writing

$$t = \tau/\epsilon.$$  \tag{4.10}

Then it is proved in [19] that in the limit $\epsilon \to 0$ and with $\beta = 2$ the joint distribution of $(A_0, B_t)$ is a Gaussian with covariance

$$\frac{1}{\pi \beta} \int_{-\infty}^{\infty} \tilde{a}(k)\tilde{b}(-k)|k|e^{-|k|\tau} dk.$$
Also relevant to the present discussion is the corresponding result for the Brownian dynamics specified by (1.1) in which the equilibrium state is compressible (gas with a short range potential). Here one introduces the scaled linear statistics by

\[ A_\tau = \epsilon^{1/2} \sum_{j=1}^{N} a(\epsilon x_j(\tau)), \quad B_\tau = \epsilon^{1/2} \sum_{j=1}^{N} b(\epsilon x_j(\tau)) \]

(4.11)

and scales \( \tau \) by writing

\[ t = \tau / \epsilon^2. \]

(4.12)

In this setting it is proved in [21] that the joint distribution of \( (A_0, B_t) \) is a Gaussian with covariance

\[ \chi \int_{-\infty}^{\infty} \tilde{a}(k) \tilde{b}(-k) e^{-k^2 \rho t/(2\chi)} dk, \]

where \( \chi \) denotes the compressibility. Note that as well as the different scaling for \( \tau \), the linear statistics are suppressed by a factor \( \epsilon^{1/2} \) which is not required in (4.9) for the Dyson model. This means that in the Dyson model the fluctuations are naturally suppressed by the long-range nature of the pair potential.

In the present work we are considering the Dyson model with Poisson initial conditions. Thus initially the particles are non-interacting so the system is compressible. A simple calculation using (4.7) and (2.33) shows that if we scale the linear statistics according to (4.11) as in a compressible gas, but scale \( \tau \) according to (4.10) as in the Dyson model with \( \text{in.}= \text{eq.} \), then the covariance in the limit \( \epsilon \to 0 \) becomes equal to

\[ \rho \int_{-\infty}^{\infty} \tilde{a}(k) \tilde{b}(-k) e^{-\pi \rho |\kappa_1| t} dk. \]

(4.13)

We conjecture that in this limit the joint distribution of \( (A_0, B_t) \) is a Gaussian.

Following [21] we can also consider the covariance for the system on a circle. For this purpose one first scales \( x_1^{(0)} \) and \( x \) by multiplying each by \( N \), and also scales \( \tau \) by making the replacement (4.10). Then, according to (2.32), in the thermodynamic limit

\[ \rho_T^{T(1,1)}(x_1^{(0)}, x) \sim \rho^2 N \sum_{\kappa_1 = -\infty}^{\infty} e^{-2(\pi \rho)^2 |\kappa_1| t} e^{2\pi i \rho (x_1^{(0)} - x) \kappa_1}, \]

(4.14)

where now \( x_1^{(0)}, x \in [0, \rho] \). In this setting, analogous to (4.11) we choose for the scaled linear statistics

\[ A_\tau = \frac{1}{N^{1/2}} \sum_{j=1}^{N} a(x_j), \quad B_\tau = \frac{1}{N^{1/2}} \sum_{j=1}^{N} b(x_j). \]

(4.15)

The formula for the covariance is then

\[ \text{Cov}(A_0, B_\tau) = \frac{1}{N^2} \int_{0}^{1/\rho} dx_1^{(0)} a(x_1^{(0)}) \int_{0}^{1/\rho} dx b(x) \rho_T^{T(1,1)}(x_1^{(0)}, x), \]

(4.16)

where the factor \( 1/N \) results from the factors of \( 1/N^{1/2} \) in (4.15), while the factor \( N^2 \) results from the change of scale \( x_1^{(0)} \to N x_1^{(0)} \), \( x \to N x \). Substituting (4.14) in (4.16) gives that as \( N \to \infty \)

\[ \text{Cov}(A_0, B_\tau) = \sum_{\kappa_1 = -\infty}^{\infty} a_{\kappa_1} b_{-\kappa_1} e^{-2(\pi \rho)^2 |\kappa_1| t}. \]

(4.17)
where
\[ a_{\kappa_1} := \rho \int_0^{1/\rho} a(x) e^{2\pi i \rho x \kappa_1} \, dx \]
and similarly the meaning of \( b_{-\kappa_1} \). Again we would expect the joint distribution of \((A_0, B_\tau)\) to be Gaussian.

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**Appendix**

Here we will evaluate the integral
\[ I(\mu, z, \beta) := \int_0^\infty ds_1 \int_0^\infty ds_2 (s_1 s_2)^{\mu+1} \frac{\partial^2}{\partial s_1 \partial s_2} (s_1 + s_2)s_1 - s_2|^\beta e^{-z(s_1 + s_2)}, \tag{A.1} \]
which includes the integral in (4.4) as a special case. Assuming temporarily that Re(\( \mu \)) > -1, integration by parts gives
\[ I(\mu, z, \beta) = (\mu + 1)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 (s_1 s_2)^{\mu+1}(s_1 + s_2)s_1 - s_2|^\beta e^{-z(s_1 + s_2)}. \tag{A.2} \]
Furthermore, the term \((s_1 + s_2)\) can be obtained from \( e^{-z(s_1 + s_2)} \) by partial differentiation with respect to \( z \). Doing this, then scaling out the \( z \) dependence by changing variables and computing the derivative gives
\[ I(\mu, z, \beta) = (\mu + 1)^2(2 + 2\mu + \beta)z^{-(\beta+2\mu+\beta)}L(\mu, \beta) \tag{A.3} \]
where
\[ L(\mu, \beta) := \int_0^\infty ds_1 \int_0^\infty ds_2 (s_1 s_2)^{\mu+1}(s_1 + s_2)s_1 - s_2|^\beta e^{-(s_1 + s_2)} \]
\[ = \frac{\Gamma(1 + \mu)\Gamma(1 + \beta)\Gamma(1 + \mu + \beta/2)}{\Gamma(1 + \beta/2)} \tag{A.4} \]
where the second equality in (A.4) follows because \( L(\mu, \beta) \) is a two-dimensional example of a well known limiting case of the \( n \)-dimensional Selberg integral \[22\], which in general can be evaluated as a product of gamma functions.

To obtain the integral in (A.4) we must put \( z = 1 \) and \( \mu = -1 - \beta/2 \). However we see from (A.4) that \( L(\mu, \beta) \) diverges with this choice of \( \mu \). This is compensated for by a vanishing factor in (A.3), so the correct procedure is to take the limit \( \mu \to -1 - \beta/2 \) (an analogous procedure has been necessary in the evaluation of similar integrals ocuring in random matrix problems \[18\]). Doing this, and using the gamma function identity
\[ 2^{2x-1}\Gamma(x)\Gamma(x + 1/2) = \sqrt{\pi}\Gamma(2x) \]
shows that
\[ I(-1 - \beta/2, 1, \beta) = \frac{1}{\sqrt{\pi}}2^{\beta+1}(\beta/2)^2\Gamma(-\beta/2)\Gamma(\beta/2 + 1/2). \tag{A.5} \]
References

[1] F.J. Dyson. *J. Math. Phys.*, 3:1191 (1962).

[2] P.J. Forrester *Physica A*, 223: 365 (1996).

[3] P.J. Forrester and T. Nagao. *J. Stat. Phys.*, 89: 69 (1997).

[4] C.W.J. Beenakker and B. Rejaei. *Physica A*, 203: 61 (1994).

[5] F. Haake. *Quantum Signatures of Chaos*, (Springer, Berlin, 1992).

[6] E. Brézin and S. Hikami. *Nucl. Phys. B*, 479: 697 (1996).

[7] A. Pandey. *Chaos, Solitons & Fractals*, 5: 1275 (1995).

[8] T. Guhr. *Phys. Rev. Lett.*, 76: 2258 (1996); T. Guhr and A. Müller-Groeling. *J. Math. Phys.*, 38: 1870 (1997).

[9] H. Kunz and B. Shapiro. cond-mat/9802263.

[10] H. Risken. *The Fokker-Planck Equation*, (Springer, Berlin, 1992).

[11] P.J. Forrester. in *Quantum Many Problems and Representation Theory*, MSJ Memoirs Vol. 1 (Math. Soc. Japan, 1998).

[12] B. Sutherland. *Phys. Rev. A* 5: 1372 (1972).

[13] P.J. Forrester. *Nucl. Phys. B* 416: 377 (1994).

[14] P.J. Forrester. *Mod. Phys. Lett B* 9: 359 (1995).

[15] Z. Yan. *Contem. Math.* 138:239 (1992).

[16] Z.N.C. Ha. *Nucl. Phys. B* 435: 604 (1995).

[17] P.J. Forrester and B. Jancovici. *Physica A* 238: 405 (1997).

[18] P.J. Forrester and J.A. Zuk. *Nucl. Phys. B* 473: 616 (1996).

[19] H. Spohn. in: *Hydrodynamic Behaviour and Interacting Particle Systems* (ed. G. Papanicolau) (Springer Verlag, Berlin,1987)

[20] H. Spohn. mp_arc/98-198.

[21] H. Spohn. *Commun. Math. Phys.* 103:1 (1986).

[22] A. Selberg. *Norsk. Mat. Tidsskr.* 26: 71 (1944).