Coloring the squares of graphs whose maximum average degrees are less than 4

Seog-Jin Kim
Department of Mathematics Education
Konkuk University
Seoul, Korea
skim12@konkuk.ac.kr

Boram Park*
Department of Mathematics
Ajou University
Suwon, Korea
borampark@ajou.ac.kr

June 16, 2015

Abstract

The square $G^2$ of a graph $G$ is the graph defined on $V(G)$ such that two vertices $u$ and $v$ are adjacent in $G^2$ if the distance between $u$ and $v$ in $G$ is at most 2. The maximum average degree of $G$, $\text{mad}(G)$, is the maximum among the average degrees of the subgraphs of $G$.

It is known in [2] that there is no constant $C$ such that every graph $G$ with $\text{mad}(G) < 4$ has $\chi(G^2) \leq \Delta(G) + C$. Charpentier [5] conjectured that there exists an integer $D$ such that every graph $G$ with $\Delta(G) \geq D$ and $\text{mad}(G) < 4$ has $\chi(G^2) \leq 2\Delta(G)$. Recent result in [1] implies that $\chi(G^2) \leq 2\Delta(G)$ if $\text{mad}(G) < 4 - \frac{1}{c}$ with $\Delta(G) \geq 40c - 16$.

In this paper, we show for $c \geq 2$, if $\text{mad}(G) < 4 - \frac{1}{c}$ and $\Delta(G) \geq 14c - 7$, then $\chi(G^2) \leq 2\Delta(G)$, which improves the result in [1]. We also show that for every integer $D$, there is a graph $G$ with $\Delta(G) \geq D$ such that $\text{mad}(G) < 4$ and $\chi(G^2) \geq 2\Delta(G) + 2$, which disproves Charpentier’s conjecture. In addition, we give counterexamples to Charpentier’s another conjecture in [5], stating that for every integer $k \geq 3$, there is an integer $D_k$ such that every graph $G$ with $\text{mad}(G) < 2k$ and $\Delta(G) \geq D_k$ has $\chi(G^2) \leq k\Delta(G) - k$.

1 Introduction

A proper $k$-coloring $\phi : V(G) \rightarrow \{1, 2, \ldots, k\}$ of a graph $G$ is an assignment of colors to the vertices of $G$ so that any two adjacent vertices receive distinct colors. The chromatic number $\chi(G)$ of a graph $G$ is the least $k$ such that there exists a proper $k$-coloring of $G$. A list assignment on $G$ is a function $L$ that assigns each vertex $v$ a set $L(v)$ which is a list of available colors at $v$. A graph $G$ is said to be $k$-choosable if for any list assignment $L$ such that $|L(v)| \geq k$ for every vertex $v$, there exists a proper coloring $\phi$ such that $\phi(v) \in L(v)$ for every $v \in V(G)$. The list chromatic number $\chi_L(G)$ of a graph $G$ is the least $k$ such that $G$ is $k$-choosable.

The square $G^2$ of a graph $G$ is the graph defined on $V(G)$ such that two vertices $u$ and $v$ are adjacent in $G^2$ if the distance between $u$ and $v$ in $G$ is at most 2. The maximum average

*Corresponding author: borampark@ajou.ac.kr
degree of $G$, $\text{mad}(G)$, is the maximum among the average degrees of the subgraphs of $G$. That is, $\text{mad}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|}$.

The study of $\chi(G^2)$ was initiated in [3], and has been actively studied. From the fact that $\chi(G^2) \geq \Delta(G) + 1$ for every graph $G$, a naturally arising problem is to find graphs $G$ which satisfy $\chi(G^2) = \Delta(G) + 1$. A lot of research has been done to find sufficient conditions in terms of girth or $\text{mad}(G)$ to be $\chi(G^2) = \Delta(G) + 1$. Also, given a constant $C$, determining graphs $G$ which satisfy $\chi(G^2) \leq \Delta(G) + C$ is also an interesting research topic. See [1] [3] [8] for more information.

Bonamy, Lévêque, Pinlou [1] showed that $\chi_\ell(G^2) \leq \Delta(G) + 2$ if $\text{mad}(G) < 3$ and $\Delta(G) \geq 17$. However, it was reported in [2] that there is no constant $C$ such that every graph $G$ with $\text{mad}(G) < 4$ has $\chi(G^2) \leq \Delta(G) + C$. On the other hand, Bonamy, Lévêque, Pinlou [2] showed the following result.

**Theorem 1.1** ([2]). There exists a function $h(\epsilon)$ such that every graph $G$ with $\text{mad}(G) < 4 - \epsilon$ satisfies $\chi_\ell(G^2) \leq \Delta(G) + h(\epsilon)$, where $h(\epsilon) \sim \frac{40}{\epsilon}$ as $\epsilon \to 0$.

It is known in [2] that for arbitrarily large maximum degree, there exists a graph $G$ such that $\text{mad}(G) < 4$ and $\chi(G^2) \geq \frac{3\Delta(G)}{2}$. On the other hand, Charpentier [5] proposed the following conjectures.

**Conjecture 1.2** ([5]). There exists an integer $D$ such that every graph $G$ with $\Delta(G) \geq D$ and $\text{mad}(G) < 4$ has $\chi(G^2) \leq 2\Delta(G)$.

**Conjecture 1.3** ([5]). For each integer $k \geq 3$, there exists an integer $D_k$ such that every graph $G$ with $\Delta(G) \geq D_k$ and $\text{mad}(G) < 2k$ has $\chi(G^2) \leq k\Delta(G) - k$.

It was mentioned in [5] that Conjecture 1.2 and Conjecture 1.3 are best possible, if they are true. In this paper, we disprove Conjecture 1.2 by showing that for any positive integer $D$, there is a graph $G$ with $\Delta(G) \geq D$ and $\text{mad}(G) < 4$ such that $\chi(G^2) \geq 2\Delta(G) + 2$. Precisely, for arbitrarily positive integer $d \geq 2$, there exists a graph $G_d$ with $\Delta(G) = d + 1$ such that $\text{mad}(G) = 4 - \frac{10}{d+1}$ and the maximum clique size of $G_d^2$ is $2\Delta(G_d) + 2$. It means that there is no constant $D_0$ such that every graph $G$ with $\text{mad}(G) < 4$ and $\Delta(G) \geq D_0$ satisfies that $\chi(G^2) \leq 2\Delta(G)$. In addition, we give counterexamples to Conjecture 1.3 by using similar idea.

As a modification of Conjecture 1.2, we are interested in finding the optimal value $h(c)$ such that $\chi(G^2) \leq 2\Delta(G)$ (or $\chi_\ell(G^2) \leq 2\Delta(G)$) for every graph $G$ with $\text{mad}(G) < 4 - \frac{1}{c}$ and $\Delta(G) \geq h(c)$. Our main theorem of this paper is the following, which shows that $h(c) \leq 14c - 7$.

**Theorem 1.4.** Let $c$ be an integer such that $c \geq 2$. If a graph $G$ satisfies $\text{mad}(G) < 4 - \frac{1}{c}$ and $\Delta(G) \geq 14c - 7$, then $\chi(G^2) \leq 2\Delta(G)$.

Note that Theorem 1.1 implies that if $G$ is a graph with $\text{mad}(G) < 4 - \frac{1}{c}$ and $\Delta(G) \geq 40c - 16$, then $\chi_\ell(G^2) \leq 2\Delta(G)$. Thus Theorem 1.3 gives a better bound on $h(c)$ than Theorem 1.1 when $14c - 7 \leq \Delta(G) \leq 40c - 17$.

Next, we will show that $h(c) \geq 2c + 2$. Thus the current bound on $h(c)$ is $2c + 2 \leq h(c) \leq 14c - 7$. Hence it would be interesting to solve the following problem.
Problem 1.5. Given a positive integer \( c \geq 1 \), there is a function \( h(c) \) such that \( \chi(G^2) \leq 2\Delta(G) \) (or \( \chi(G^2) \leq 2\Delta(G) \)) whenever a graph \( G \) satisfies \( mad(G) < 4 - \frac{1}{c} \) and \( \Delta(G) \geq h(c) \). What is the optimal value of \( h(c) \)? Or, reduce the gap in \( 2c + 2 \leq h(c) \leq 14c - 7 \).

Remark 1.6. Yancey [10] showed that for \( t \geq 3 \), if \( G \) is a graph with \( mad(G) < 4 - \frac{4}{t+1} - \epsilon \) for some \( \frac{4}{t+1} > \epsilon > 0 \), then \( \chi(G^2) \leq \max\{\Delta(G) + t, 16t^2c^{-2}\} \). We can convert \( mad(G) < 4 - \frac{4}{t+1} - \epsilon \) into \( mad(G) < 4 - \frac{1}{c} \) form by setting \( \frac{4}{t+1} + \epsilon = \frac{1}{c} \). Then from \( 0 < \epsilon < \frac{4}{t(t+1)} < 1 \), we have that \( \epsilon < \frac{1-c}{c} \times \frac{4-c}{4+c+\epsilon-1} \), and consequently \( 0 < \epsilon < \frac{1}{c(4c+1)} < 1 \). Thus \( 0 < 1 - \epsilon \epsilon < 1 \), and consequently, we have \( t = \frac{4c}{1-\epsilon} - 1 > 4c - 1 \). Hence, when \( 16t^2\epsilon^{-2} \leq 2\Delta(G) \), we have

\[
\Delta(G) \geq 8t^2\epsilon^{-2} > 8(4c - 1)^2c^2(4c + 1)^2
\]

since \( \epsilon < \frac{1}{c(4c+1)} \). Thus Yancey’s result implies that \( \chi(G^2) \leq 2\Delta(G) \) only when \( mad(G) < 4 - \frac{1}{c} \) and \( \Delta(G) \geq t_0c^6 \) for some constant \( t_0 \). But, note that in our result, the lower bound on \( \Delta(G) \) is linear as \( \Delta(G) \geq 14c - 7 \).

This paper is organized as follows. In Section 2 we will give a construction which is a counterexample to Conjecture 1.2 and in Section 3 we will prove Theorem 1.4 using discharging method. In Section 4, we modify the construction in Section 2 slightly, and show that for any positive integer \( c \), there exists a graph \( G \) such that \( mad(G) < 4 - \frac{1}{c} \), \( \Delta(G) = 2c + 1 \), and \( \chi(G^2) = 2\Delta(G) + 1 \), which implies that \( h(c) \geq 2c + 2 \). And next, in Appendix, we will give counterexamples to Conjecture 1.3.

2 Construction

We will show that for any positive integer \( n \geq 2 \), there is a graph \( G \) with \( \Delta(G) = n + 1 \) such that \( mad(G) < 4 \), and \( \chi(G^2) > 2\Delta(G) \). Let \( [n] = \{1, \ldots, n\} \).

Construction 2.1. Let \( n \geq 2 \) be a positive integer. Let \( S = \{u_1, u_2, \ldots, u_n\} \), \( T = \{v_1, \ldots, v_n\} \), and \( X = \{x_{ij} \mid (i, j) \in [n] \times [n]\} \). We define a graph \( G_n \) by

\[
\begin{align*}
V(G_n) &= \{u, v\} \cup S \cup T \cup X \\
E(G_n) &= \{uv\} \cup \{uu_i \mid u_i \in S\} \cup \{vv_i \mid v_i \in T\} \\
&\quad \cup \left( \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} \{u_ix_{ij}, v_jx_{ij}\} \right) \cup \left( \bigcup_{i=2}^{n} \{x_{11}x_{ii}\} \right) \cup \left( \bigcup_{i=2}^{n} \{x_{12}x_{i(i+1)}\} \right),
\end{align*}
\]

where \( x_{n(n+1)} = x_{n1} \). See Figure 1 for an illustration.

We have the following simple observations.

- For \( y \in \{u, v\} \cup S \cup T \), \( d(y) = n + 1 \).
Figure 1: Construction of $G_n$

- For $x_{ij} \in X$,

$$d(x_{ij}) = \begin{cases} 
  n + 1 & \text{if } (i, j) = (1, 1), \text{ or } (1, 2) \\
  3 & \text{if } j = i + r \text{ for } r \in \{0, 1\} \text{ and } i \geq 2, \\
  2 & \text{otherwise},
\end{cases}$$

where $x_{n(n+1)} = x_{n1}$.

Therefore $\Delta(G_n) = n + 1$ and $\{u, v, x_{11}, x_{12}\} \cup S \cup T$ is a clique in $G_n^2$ with $2n + 4$ vertices. Thus $\chi(G_n^2) \geq 2\Delta(G_n) + 2$.

From now on, we denote $G_n$ by $G$ for simplicity. Next, we will show that $mad(G) < 4$. Denote the number of edges of the subgraph of $G$ induced by $A$ by $||A||$, that is, $|E(G[A])| = ||A||$. Define a potential function $\rho_G: 2^{V(G)} \rightarrow \mathbb{Z}$ by for $A \subset V(G)$,

$$\rho_G(A) = 2|A| - ||A||.$$ 

Note that $\rho_G(A) \geq 1$ for every subset $A \subset V(G)$ is equivalent to $mad(G) < 4$.

We will show that $\rho_G(A) \geq 1$ for all $A \subset V(G)$. A vertex of degree $k$ is called a $k$-vertex, and a vertex of degree at least $k$ (at most $k$) is called a $k^+$-vertex ($k^-$-vertex).

**Claim 2.2.** For all $A \subset V(G)$, $\rho_G(A) \geq 1$.

**Proof.** Suppose that there is $A \subset V(G)$ such that $\rho_G(A) \leq 0$. Let $A$ be a smallest subset of $V(G)$ among all subsets of $V(G)$ with minimum value $\rho_G(A)$. That is, $A$ is a minimal counterexample to Claim 2.2.

If $G[A]$ contains a $2^-$-vertex $v$ then $\rho_G(A \setminus \{v\}) \leq \rho_G(A)$, which is a contradiction to the minimality of $\rho_G(A)$ or the minimality of $|A|$. Thus $G[A]$ does not have a $2^-$-vertex.

Let $X_3$ be the set of $3^+$-vertices in $X$. Then every vertex in $X \setminus X_3$ does not belong to $A$, since each vertex in $X \setminus X_3$ is a 2-vertex.
If \( a \notin A \) and \( a \) has at least three neighbors in \( A \), then \( \rho_G(A \cup \{a\}) < \rho_G(A) \), a contradiction to the minimality of \( \rho_G(A) \). Thus every vertex not in \( A \) has at most two neighbors in \( A \).

Next, we will show that \( \{u, v\} \subset A \). If \( |A \cap S| \leq 1 \), then any vertex in \( A \cap T \) is a 2\(^{-}\) vertex of \( G[A] \), a contradiction. Thus \( |A \cap S| \geq 2 \). Suppose that \( |A \cap S| = 2 \). If \( v \notin A \), then \( A \cap T \) has a 2-vertex of \( G[A] \), which is forbidden. Thus \( v \in A \), and then \( u \) is adjacent to three vertices of \( A \), and so \( u \in A \). Therefore \( \{u, v\} \subset A \). Similarly, if \( |A \cap T| = 2 \), then \( \{v, u\} \subset A \). On the other hand, if \( |A \cap S| \geq 3 \) and \( |A \cap T| \geq 3 \), then \( \{v, u\} \in A \), since every vertex not in \( A \) has at most two neighbors in \( A \). Therefore, we can conclude that \( \{u, v\} \subset A \).

Let \( X'_3 \) be the set of 3-vertices of \( G \) in \( X \cap A \). That is, \( X'_3 = (X_3 \cap A) \setminus \{x_{11}, x_{12}\} \). As we noted that every vertex in \( X \setminus X_3 \) does not belong to \( A \), in fact, \( X'_3 = (X \cap A) \setminus \{x_{11}, x_{12}\} \). Note that every vertex in \( X'_3 \) is also a 3-vertex of \( G[A] \). Since any two vertices in \( X'_3 \) are not adjacent in \( G \), we have

\[
\rho_G(A \setminus X'_3) = 2|A \setminus X'_3| - |A \setminus X'_3| = 2|A| - 2|X'_3| - |A| + 3|X'_3| = \rho_G(A) + |X'_3|.
\]

Since \( \rho_G(A) \leq 0 \),

\[
\rho_G(A \setminus X'_3) \leq |X'_3|.
\]

Note that each vertex in \( (A \setminus X'_3) \cap X = A \cap \{x_{11}, x_{12}\} \) has degree at most 2 in \( G[A \setminus X'_3] \), and therefore, we have \( \rho_G(A \setminus X'_3) \geq \rho_G(A \setminus X) \).

Let \( \alpha = |A \cap (S \cup T)| \) for simplicity. Note that \( \alpha \geq |X'_3| \), since for vertex \( x \) in \( X'_3 \), \( x \) is a 3-vertex in both \( G \) and \( G[A] \), and so \( N_G(x) \subset A \). Now note that \( G[A \setminus X] \) has \( \alpha + 2 \) vertices and has \( \alpha + 1 \) edges. Thus

\[
\rho_G(A \setminus X'_3) \geq \rho_G(A \setminus X) \geq 2\alpha + 4 - (\alpha + 1) \geq \alpha + 3 \geq |X'_3| + 3,
\]

a contradiction to (1). Therefore \( \rho_G(A) \geq 1 \) for every subset \( A \subset V(G) \). This completes the proof of Claim 2.2.

**Remark 2.3.** In Appendix, we will also show that Conjecture 1.3 is not true. That is, for any integers \( k \) and \( n \) such that \( k \geq 2 \) and \( n \geq k^2 - k \), there exists a graph \( G \) such that \( mad(G) < 2k \), \( \Delta(G) \geq n \), and \( \chi(G^2) \geq k\Delta(G) + k \). The construction for \( k \geq 3 \) is similar to Construction 2.1.

### 3 Proof of Theorem 1.4

We use double induction on the number of 3\(^{+}\)-vertices first, and then on the number of edges.

**Definition 3.1.** Let \( n_3(G) \) be the number of 3\(^{+}\)-vertices of \( G \). We order graphs as follows. Give two graphs \( G \) and \( G' \), say that \( G' \) is smaller than \( G \) if (1) \( n_3(G') < n_3(G) \), or (2) \( n_3(G') = n_3(G) \) and \( |E(G')| < |E(G)| \).

Throughout this section, we let \( G \) be a minimal counterexample to Theorem 1.4.

**Lemma 3.2.** If a vertex \( u \) has a neighbor of degree 2, then

\[
\sum_{x \in N(u)} d(x) \geq 2\Delta(G).
\]


Proof. Suppose that $\sum_{x \in N(u)} d(x) < 2\Delta(G)$. Let $v$ be a neighbor of $u$ whose degree is 2. Let $H = G - uv$. The number of $3^+$-vertices of $H$ is not greater than that of $G$, and the number of edges of $H$ is less than that of $G$. Thus, $\chi_l(H^2) \leq 2\Delta(H)$. Note that $2\Delta(H) \leq 2\Delta(G)$. Now uncolor $u$ and $v$. Then the number of forbidden colors at $v$ is at most $2\Delta(G) - 1$ and so we can assign a color of $v$. And the number of forbidden colors at $u$ is at most $\sum_{x \in N(u)} d(x) < 2\Delta(G)$, and so we can give a color to $u$. Thus $G^2$ is $2\Delta(G)$-choosable. This is a contradiction. \hfill $\Box$

Corollary 3.3. Let $u$ be a vertex having a neighbor of degree 2. Then

(i) if $d(u) \leq \frac{2\Delta(G)}{3}$, then $u$ has at least one neighbor of degree at least 4;

(ii) if $d(u) \leq \frac{\Delta(G)}{3}$, then $u$ has at least two neighbors of degree at least 4;

(i) a 2-vertex is not adjacent to a 2-vertex.

Lemma 3.4. Every 3-vertex has a neighbor of degree at least $4c$.

Proof. For any graph $G'$, we define a potential function $\rho_{G'}: 2^{V(G')} \to \mathbb{Z}$ by

$$\rho_{G'}(A) = (4c - 1)|A| - 2c||A||.$$

Note that $\rho_{G'}(A) \geq 1$ for every subset $A \subset V(G')$ is equivalent to $\text{mad}(G') \leq 4 - \frac{1}{c}$.

Let $u$ be a 3-vertex of $G$, and let $N(u) = \{x_1, x_2, x_3\}$. Suppose that $\max\{d(x_1), d(x_2), d(x_3)\} \leq 4c - 1$. Let $H$ be the graph obtained by deleting the vertex $u$ and adding three vertices $y_1, y_2, y_3$ such that $N(y_1) = \{x_1, x_2\}, N(y_2) = \{x_2, x_3\}$, and $N(y_3) = \{x_1, x_3\}$.

Now we will show that $\rho_H(S) \geq 1$ for every $S \subset V(H)$. Suppose that there exists $S \subset V(H)$ such that $\rho_H(S) \leq 0$. We take a smallest such $S$. If $S \cap \{y_1, y_2, y_3\} = \emptyset$, then $\rho_H(S) = \rho_G(S) \geq 1$. Thus $S \cap \{y_1, y_2, y_3\} \neq \emptyset$. If a vertex $y$ in $S \cap \{y_1, y_2, y_3\}$ is a 1-vertex, then $\rho_H(S) > H(S \setminus \{y\})$, a contradiction to the minimality of $|S|$. Thus, any vertex in $S \cap \{y_1, y_2, y_3\}$ is a 2-vertex.

Let $S' = S \setminus \{y_1, y_2, y_3\}$ and $|S \cap \{y_1, y_2, y_3\}| = \alpha$. If $\alpha = 1$ then $\rho_H(S) = \rho_G(S' \cup \{u\}) \geq 1$. If $\alpha \geq 2$, then $\{x_1, x_2, x_3\} \subset S$ and so

$$\rho_H(S) = \rho_H(S') + (4c - 1)\alpha - 4c\alpha$$

$$= \rho_G(S') - \alpha$$

$$= \rho_G(S' \cup \{u\}) - (4c - 1) + 6c - \alpha$$

$$= \rho_G(S' \cup \{u\}) + 2c - \alpha + 1 \geq 1,$$

where the last inequality is from the fact that $\rho_G(S' \cup \{u\}) \geq 1$ and $\alpha \leq 3$. Hence $\rho_H(S) \geq 1$ for every $S \subset V(H)$.

Note that each $x_i$ has degree in $H$ at least 3 by Lemma 3.2. Hence the number $3^+$-vertices of $H$ is smaller than the number of $3^+$-vertices of $G$. Thus by the minimality of $G$, we have $\chi_l(H^2) \leq 2\Delta(H)$. Since the degrees of $x_1, x_2, x_3$ in $G$ are at most $4c - 1$ and $\Delta(G) \geq 14c - 7$, $\Delta(H) = \Delta(G)$. Thus $\chi_l(H^2) \leq 2\Delta(G)$. Now, since the number of 2-distance neighbors of $u$ is at most $12c < 2\Delta(G)$, the number of forbidden colors at $u$ is less than $2\Delta(G)$. Thus $G^2$ is $2\Delta(G)$-choosable. This is a contradiction. This completes the proof of Lemma 3.4. \hfill $\Box$
Discharging Rules

R1: If \( d(u) \geq 8c - 2 \), then \( u \) sends \( 1 - \frac{1}{2c} \) to each of its neighbors.

R2: If \( 4c \leq d(u) < 8c - 2 \), \( u \) sends \( 1 - \frac{1}{2c} \) to each of neighbors of degree 2, and sends \( 1 - \frac{1}{c} \) to each of neighbors of degree 3.

R3: If \( 4 \leq d(u) < 4c \) and \( u \) has exactly one neighbor of degree at least 4, then \( u \) does not send any charge to its neighbors.

R4: If \( 4 \leq d(u) < 4c \) and \( u \) has at least two neighbors of degree at least 4, then \( u \) sends \( 1 - \frac{1}{2c} \) to each of its neighbors.

R5: If a 3-vertex \( u \) has two neighbors of degree at least \( 8c - 2 \) and one neighbor of degree 2, then \( u \) sends \( 1 - \frac{1}{2c} \) to its neighbor whose degree is 2.

Let \( d^*(u) \) be the new charge after discharging. We will show that \( d^*(u) \geq 4 - \frac{1}{c} \) for all \( u \). Note that \( \Delta(G) \geq 14c - 7 \).

1. When \( d(u) \geq 8c - 2 \),

\[
d^*(u) \geq d(u) - d(u) \left( 1 - \frac{1}{2c} \right) = \frac{d(u)}{2c} \geq 4 - \frac{1}{c}.
\]

2. If \( 4c \leq d(u) \leq 8c - 3 \) and \( u \) has no neighbor of degree 2, then

\[
d^*(u) \geq d(u) - d(u) \left( 1 - \frac{1}{c} \right) = \frac{d(u)}{c} \geq 4 - \frac{1}{c}.
\]

3. Suppose that \( 4c \leq d(u) \leq 8c - 3 \) and \( u \) is adjacent to a 2-vertex. Note that by (i) of Corollary 3.3 \( u \) is adjacent to at least one \( 4^+ \)-vertex.
   - If \( 6c - 1 \leq d(u) \leq 8c - 3 \), then
     \[
d^*(u) \geq d(u) - (d(u) - 1) \left( 1 - \frac{1}{2c} \right) = 1 + \frac{d(u)}{2c} - \frac{1}{2c} \geq 4 - \frac{1}{c},
     \]
     since \( u \) is adjacent to at least one \( 4^+ \)-vertex.
   - If \( 4c \leq d(u) \leq 6c - 2 \) and \( u \) has exactly one neighbor \( z \) of degree at least 4, then
     \[
     \sum_{x \in N(u)} d(x) \leq d(z) + 3 \cdot (d(u) - 2) + 2,
     \]
and by Lemma 3.2

\[ 2\Delta(G) \leq \sum_{x \in N(u)} d(x) \leq d(z) + 3 \cdot (d(u) - 2) + 2. \]

Thus

\[ d(z) \geq 2\Delta(G) - 3 \cdot (d(u) - 2) - 2 \geq 2 \cdot (14c - 7) - 3 \cdot (6c - 4) - 2 \geq 8c - 2, \]

which implies that \( u \) receives charge \( 1 - \frac{1}{2c} \) from \( z \). Thus

\[ d^*(u) \geq d(u) - (d(u) - 1) \left( 1 - \frac{1}{2c} \right) + 1 - \frac{1}{2c} = 2 + \frac{d(u)}{2c} - \frac{1}{c} \geq 4 - \frac{1}{c}. \]

- If \( 4c \leq d(u) \leq 6c - 2 \) and \( u \) is adjacent to at least two \( 4^+ \)-vertices, then

\[ d^*(u) \geq d(u) - (d(u) - 2) \left( 1 - \frac{1}{2c} \right) = 2 + \frac{d(u)}{2c} - \frac{1}{c} \geq 4 - \frac{1}{c}. \]

Thus \( d^*(u) \geq 4 - \frac{1}{c}. \)

(4) Suppose that \( 2c + 1 \leq d(u) < 4c \). If \( u \) has no neighbor of degree 2, then \( u \) does not send any charge to others. Next, consider the case when \( u \) has a neighbor of degree 2. By (ii) of Corollary 3.3, \( u \) is adjacent to at least two \( 4^+ \)-vertices.

- Suppose that \( u \) has exactly two neighbors of degree at least 4, say \( z_1 \) and \( z_2 \). Then by Lemma 3.2

\[ d(z_1) + d(z_2) + 3 \cdot (d(u) - 2) \geq 2\Delta(G). \]

Note that

\[ d(z_1) + d(z_2) \geq 2\Delta(G) - 3d(u) + 6 \geq 2 \cdot (14c - 7) - 3 \cdot (4c - 1) + 6 = 16c - 5 = 2 \cdot (8c - 2) - 1. \]

Thus at least one of \( d(z_1) \) and \( d(z_2) \) is at least \( 8c - 2 \), implies that \( u \) receives charge at least \( 1 - \frac{1}{2c} \) from \( z_1 \) and \( z_2 \). Thus

\[ d^*(u) \geq d(u) - (d(u) - 2) \left( 1 - \frac{1}{2c} \right) + 1 - \frac{1}{2c} = \frac{d(u)}{2c} + 3 - \frac{3}{2c} \geq 4 - \frac{1}{c}. \]

- If \( u \) is adjacent to at least three \( 4^+ \)-vertices, then

\[ d^*(u) \geq d(u) - (d(u) - 3) \left( 1 - \frac{1}{2c} \right) = \frac{d(u)}{2c} + 3 - \frac{3}{2c} \geq 4 - \frac{1}{c}. \]
Suppose that \( u \) has at least \((d(u) - 3)\) neighbors of degree 2. Let \( z_1, z_2 \) and \( z_3 \) be the other neighbors. By Lemma 3.2, since \( u \) has a neighbor of degree 2, 
\[
d(z_1) + d(z_2) + d(z_3) + 2 \cdot (d(u) - 3) \geq 2\Delta(G).
\]
Note that 
\[
d(z_1) + d(z_2) + d(z_3) \geq 2\Delta(G) - 2d(u) + 6 \geq 2 \cdot (14c - 7) - 2 \cdot 2c + 6 \geq 3 \cdot (8c - 2) - 2.
\]
Thus we can conclude that at least one of \(d(z_1), d(z_2), d(z_3)\) is at least \(8c - 2\), and so \( u \) receives charge at least \(1 - \frac{1}{2c}\) from \( z_1, z_2, z_3 \). Thus 
\[
d^*(u) \geq d(u) - (d(u) - 3) \left(1 - \frac{1}{2c}\right) + 1 - \frac{1}{2c} = \frac{d(u)}{2c} + 4 - \frac{4}{2c} \geq 4.
\]

(5) Suppose that \( 4 \leq d(u) < 2c + 1 \). If \( u \) has no neighbor of degree 2, then \( u \) does not send any charge to others. Consider the case when \( u \) has a neighbor of degree 2. If \( u \) has at most \((d(u) - 4)\) neighbors of degree 2, then
\[
d^*(u) \geq d(u) - (d(u) - 4) \left(1 - \frac{1}{2c}\right) = \frac{d(u)}{2c} + 4 - \frac{2}{c} \geq 4 - \frac{1}{c}.
\]
Suppose that \( u \) has at least \((d(u) - 3)\) neighbors of degree 2. Let \( x \) be a neighbor of degree at least 4, the it violates (ii) of Corollary 3.3. Thus the case of \( 4 \) neighbors of degree 2, then \( c \) receives charge \( 1 - \frac{1}{2c} \) from its neighbors. Even though we consider \( R5 \), we have \( d^*(u) \geq 4 - \frac{1}{c} \).

(6) When \( d(u) = 3 \), by Lemma 3.4, \( u \) has at least one neighbor of degree at least \( 4c \). Thus \( u \) receives charge at least \( 1 - \frac{1}{2c} \) from its neighbors. Even though we consider \( R5 \), we have \( d^*(u) \geq 4 - \frac{1}{c} \).

(7) Suppose that \( d(u) = 2 \). We will show that \( u \) receives \( 1 - \frac{1}{2c} \) from both neighbors. Let \( x \) be a neighbor of \( u \). Suppose that \( d(x) \leq 3 \). Then \( d(x) = 3 \) by (iii) of Corollary 3.3. By Lemma 3.2, each neighbor of \( x \) other than \( u \) has degree at least \( \Delta(G) - 2 \), which implies that \( u \) receives charge \( 1 - \frac{1}{2c} \) from \( x \) by \( R5 \). Suppose that \( d(x) \geq 4 \). If \( 4 \leq d(x) < 4c \) and \( x \) has exactly one neighbor of degree at least 4, the it violates (ii) of Corollary 3.3. Thus the case of \( R3 \) does not happen to \( x \). That is, \( x \) must send \( 1 - \frac{1}{2c} \) to \( u \). Then \( d^*(u) \geq 4 - \frac{1}{c} \).

4 Remark on a condition for \( \Delta(G) \) to be \( \chi(G^2) \leq 2\Delta(G) \)

Given a positive integer \( c \geq 2 \), let \( h(c) \) be the smallest value such that \( \chi(G^2) \leq 2\Delta(G) \) whenever \( G \) is a graph with mad\((G) < 4 - \frac{1}{c} \) and \( \Delta(G) \geq h(c) \). Theorem 1.4 shows that \( h(c) \leq 14c - 7 \).

In the following, we will see that \( h(c) \geq 2c + 2 \) by showing that for any integer \( c \geq 2 \), there is a graph \( G \) such that \( mad(G) < 4 - \frac{1}{c} \), \( \Delta(G) = 2c + 1 \), and \( \chi(G^2) \geq 2\Delta(G) + 1 \). Hence \( 2c + 2 \leq h(c) \leq 14c - 7 \). Thus, it would be interesting to find the optimal value of \( h(c) \) or to reduce the gap in \( 2c + 2 \leq h(c) \leq 14c - 7 \).

Now, given a positive integer \( c \geq 2 \), we give a graph \( G \) such that \( mad(G) < 4 - \frac{1}{c} \), \( \Delta(G) = 2c + 1 \), and \( \chi(G^2) \geq 2\Delta(G) + 1 \). Let consider \( G_n \) in Section 2 when \( n = 2c \), and then let
\[
G = G_n - \{x_{12}x_{i(i+1)} : 2 \leq i \leq n\} \cup \{x_{12}x_{n1}\}.
\]
Then \( |V(G)| = 4c^2 + 4c + 2 \) and \( |E(G)| = 8c^2 + 6c \). Therefore, \( \Delta(G) = n + 1 \) and \( \{u, v, x_{11}\} \cup S \cup T \) is a clique in \( G^2 \) with \( 2n + 3 \) vertices. Thus \( \chi(G^2) \geq 2\Delta(G) + 1 \).
Claim 4.1. $\text{mad}(G) < 4 - \frac{1}{c}$.

Proof of Claim. Define a potential function $\rho^*_G(A) = (4c - 1)|A| - 2c||A||$ for $A \subseteq V(G)$. Note that $\text{mad}(G) < 4 - \frac{1}{c}$ is equivalent to $\rho^*_G(A) \geq 1$ for all $A \subseteq V(G)$.

Now, we will show that $\rho^*_G(A) \geq 1$ for all $A \subseteq V(G)$. Suppose that there is $A \subseteq V(G)$ such that $\rho^*_G(A) \leq 0$, and take such $A$ with minimum value $\rho^*(A)$. If $G[A]$ contains a $1^-$-vertex $v$ then $\rho^*_G(A \setminus \{v\}) < \rho^*_G(A)$, which is a contradiction to the minimality of $\rho^*(A)$. Thus $G[A]$ does not have a $1^-$-vertex. If $a \notin A$ and $a$ has at least two neighbors in $A$, then $\rho^*(A \cup \{a\}) < \rho^*(A)$, a contradiction to the minimality of $\rho^*(A)$. Therefore, if $a \notin A$, then $a$ has at most one neighbor in $A$. Thus for $i, j$ such that $i \neq j$,

$$x_{ij} \in A \iff \{u_i, v_j\} \subset A.$$ \hfill (2)

Without loss of generality, we may assume that $|S \cap A| \leq |T \cap A|$. From (2), it is easy to check that if $|A \cap S| \leq 1$ or $|A \cap T| \leq 1$, then $\rho^*(A) \geq 1$. Thus we can assume that $|T \cap A| \geq |S \cap A| \geq 2$, and so $\{u, v\} \in A$.

For simplicity, let $s = |S \cap A|$ and $t = |T \cap A|$. Then $s \leq t$. By (2), $|A \cap X| = |S \cap A| \cdot |T \cap A| = st$. Thus we have that $|A| \geq st + s + t + 2$. On the other hand,

$$||A|| \leq 2st + s + t + 1 + (s - 1) = 2 \cdot (st + s + t + 2) - (t + 4).$$

Thus,

$$\rho^*(A) = (4c - 1)|A| - 2c||A||$$

$$\geq (4c - 1)(st + s + t + 2) - 2c \cdot (2 \cdot (st + s + t + 2) - (t + 4))$$

$$= (4c - 1)(st + s + t + 2) - 4c \cdot (st + s + t + 2) + 2c \cdot (t + 4)$$

$$= -(st + s + t + 2) + 2c(t + 4)$$

$$= -(st - s - t - 2 + 2tc + 8c)$$

$$= (2ct - st) + (2c - s) + (2c - t) + (4c - 2) \geq 1,$$

where the last equality is from the fact that $2c \geq \max\{s, t\}$. This is a contradiction to the assumption that $\rho^*(A) \leq 0$. Thus $\rho^*(A) \geq 1$ for every subset $A \subseteq V(G)$.

References

[1] M. Bonamy, B. Lévêque, and A. Pinlou, Graphs with maximum degree $\Delta \geq 17$ and maximum average degree less than 3 are list 2-distance $(\Delta + 2)$-colorable, *Discrete Math.*, **317** (2014), 19–32.

[2] M. Bonamy, B. Lévêque, and A. Pinlou, 2-distance coloring of sparse graphs, *J. Graph Theory*, **77** (2014), 190–218.

[3] M. Bonamy, B. Lévêque, and A. Pinlou, List coloring the square of sparse graphs with large degree, *European J. Combin.*, **41** (2014), 128–137.
Appendix: Counterexamples to Conjecture 1.3

Fix an integer \( k \geq 3 \) and let \( n \) be a prime with \( n \geq k^2 - k \). We will define a graph \( G = G_{k,n} \) such that \( \Delta(G) = k + n - 1 \), \( mad(G^2) < 2k \), and \( G^2 \) contains a clique of size \( k\Delta(G) + k \). The idea is exactly same as Construction 2.1 in Section 2.

For \( \ell \in [k - 2] \), we define a Latin square \( L_\ell \) of order \( n \) by

\[
L_\ell(i,j) = j + \ell(i - 1) \pmod{n}, \quad \text{for } (i,j) \in [n] \times [n].
\]

That is, the \((i,j)\)-entry of the Latin square of \( L_\ell \) is \( L_\ell(i,j) \). (See page 252 in [7] for detail.)

**Construction 4.2.** For \( i \in [k] \), let \( U_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,n}\} \). Let \( U = \{u_1, u_2, \ldots, u_k\} \) and \( X = \{x_{i,j} \mid (i,j) \in [n] \times [n]\} \). Define

\[
V(G) = U \cup \left( \bigcup_{i=1}^{n} U_i \right) \cup X
\]

\[
E(G) = \{u_iu_j \mid 1 \leq i < j \leq k\} \cup \left( \bigcup_{i=1}^{k} \{u_i x \mid x \in U_i\} \right)
\]

\[
\cup \left( \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} \{x_{i,j} y \mid y \in \{u_{1,i}, u_{2,j}, u_{3,L_1(i,j)}, u_{4,L_2(i,j)}, \ldots, u_{k,L_{k-2}(i,j)}\}\} \right)
\]

\[
\cup \left( \bigcup_{r=0}^{k^2-k-1} \bigcup_{i=2}^{n} \{x_{1,1+r} x_{i,i+r}\} \right)
\]

where the subscripts of \( x_{i,j} \) are computed by modulo \( n \).
Then we have the following observations.

- For \( u_i \in U \), \( d(u_i) = n + k - 1 \) and for \( u_{i,j} \in U_i \), \( d(u_{i,j}) = n + 1 \).
- For \( x_{i,j} \in X \),

\[
d(x_{i,j}) = \begin{cases} 
2k - 1 & \text{if } i = 1 \\
k + 1 & \text{if } j = i + r \text{ for some } 0 \leq r \leq k^2 - k - 1 \text{ and } i \geq 2 \\
k & \text{otherwise}
\end{cases}
\]

where the subscript of \( x_{i,j} \) are computed by modulo \( n \).

Therefore \( \Delta(G) = n + k - 1 \).

**Claim 4.3.** \( \chi(G^2) \geq k\Delta(G) + k \).

**Proof.** We will show that \{\( x_{1,1}, x_{1,2}, x_{1,3}, \ldots, x_{1,k^2-k} \)\} \( \cup U \cup U_1 \cup \cdots \cup U_k \) is a clique in \( G^2 \). From the orthogonality of Latin squares, we know that \( u_{i,j} \) and \( u_{i',j'} \) are adjacent in \( G^2 \) if \( i \neq i' \). For each \( i \in [k] \), since \( u_{i,j} \) and \( u_{i,j} \) share a neighbor \( u_i \), they are adjacent in \( G^2 \). In addition, \( u_i \) and \( u_{i,j} \) are adjacent in \( G^2 \), since they share a neighbor \( u_i \). Therefore, \( U \cup U_1 \cup \cdots \cup U_k \) is a clique in \( G^2 \).

Note that the vertices in \{\( x_{1,1}, x_{1,2}, \ldots, x_{1,k^2-k} \)\} share a neighbor \( u_{1,1} \), and so they form a clique in \( G^2 \). Furthermore, each vertex in \( U \) is adjacent to each vertex in \( X \) in \( G^2 \) since they share a neighbor in \( U_1 \cup \cdots \cup U_k \). Thus, it remains to show that for each integer \( r \) such that \( 0 \leq r \leq k^2 - k - 1 \), \( x_{1,1+r} \) is adjacent to each vertex in \( U_1 \cup \cdots \cup U_k \).

Let \( r \) be an integer with \( 0 \leq r \leq k^2 - k - 1 \). Since for \( i \in [n] \),

\[
N_G(x_{1,i+r}) \supseteq \{u_{1,i}, u_{2,i+r}, u_{3,L_1(i,i+r)}, u_{4,L_2(i,i+r)}, \ldots, u_{k,L_{k-2}(i,i+r)}\}.
\]

Thus \( N_G(x_{1,1+r}) \cup N_G(x_{2,2+r}) \cup \cdots \cup N_G(x_{n,n+r}) \) contains

\[
\begin{align*}
\{u_{1,1}, u_{2,1+r}, u_{3,L_1(1,1+r)}, u_{4,L_2(1,1+r)}, \ldots, u_{k,L_{k-2}(1,1+r)}\} \\
\cup \{u_{1,2}, u_{2,2+r}, u_{3,L_1(2,2+r)}, u_{4,L_2(2,2+r)}, \ldots, u_{k,L_{k-2}(2,2+r)}\} \\
\vdots \\
\cup \{u_{1,n}, u_{2,n+r}, u_{3,L_1(n,n+r)}, u_{4,L_2(n,n+r)}, \ldots, u_{k,L_{k-2}(n,n+r)}\}.
\end{align*}
\]

Since for each \( \ell \in [k-2] \),

\[
\{L_\ell(1,1+r), L_\ell(2,2+r), \ldots, L_\ell(n,n+r)\} = [n],
\]

we can conclude that

\[
N_G(x_{1,1+r}) \cup N_G(x_{2,2+r}) \cup \cdots \cup N_G(x_{n,n+r}) \supseteq U_1 \cup \cdots \cup U_k.
\]

Since \( x_{1,1+r} \) is adjacent to every vertex in \( \{x_{2,2+r}, x_{3,3+r}, \ldots, x_{n,n+r}\} \), \( x_{1,1+r} \) is adjacent to each vertex in \( U_1 \cup \cdots \cup U_k \) in \( G^2 \).

Consequently, \( \{x_{1,1}, x_{1,2}, x_{1,3}, \ldots, x_{1,k^2-k}\} \cup U \cup U_1 \cup \cdots \cup U_k \) is a clique in \( G^2 \) with \( kn + k + (k^2 - k) = k\Delta(G) + k \) vertices. Thus \( \chi(G^2) \geq k\Delta(G) + k \). \qed
Next, we will show that $\text{mad}(G) < 2k$. Define a potential function $\rho_G : 2^V(G) \rightarrow \mathbb{Z}$ by for $A \subset V(G)$,

$$\rho_G(A) = k|A| - ||A||.$$ 

Note that $\rho_G(A) \geq 1$ for every $A \subset V(G)$ is equivalent to $\text{mad}(G) < 2k$.

Now, we will show that $\rho_G(A) \geq 1$ for all $A \subset V(G)$.

**Claim 4.4.** For all $A \subset V(G)$, $\rho_G(A) \geq 1$.

**Proof.** Suppose that there is $A \subset V(G)$ such that $\rho_G(A) \leq 0$. Let $A$ be a smallest subset of $V(G)$ among all subsets of $V(G)$ with minimum value $\rho_G(A)$. That is, $A$ is a minimal counterexample to Claim 4.4.

If there is a $k^{-}$-vertex $v$ of $G[A]$, then $\rho_G(A \setminus \{v\}) \leq \rho_G(A)$, which is a contradiction to the minimality of $\rho_G(A)$ or the minimality of $|A|$. Thus there is no $k^{-}$-vertex in $G[A]$. Thus if a vertex $x$ in $X \cap A$ has degree $k+1$ in $G$, then $N_G(x) \subset A$.

Therefore if $x \in X$ has degree $k$ in $G$, then $x \not\in A$, and if a vertex $x$ in $X \cap A$ has degree $k+1$ in $G$, then $N_G(x) \subset A$. Let $X'$ be the set of $(k+1)$-vertices of $G$ in $X \cap A$. Thus every vertex in $X'$ is also a $(k+1)$-vertex in $G[A]$. Since any two vertices in $X'$ are not adjacent in $G$, we have

$$\rho_G(A \setminus X') = k|A \setminus X'| - ||A \setminus X'|| = k|A| - k|X'| - ||A|| + (k+1)||X'|| = \rho_G(A) + |X'|.$$ 

Since $\rho_G(A) \leq 0$,

$$\rho_G(A \setminus X') \leq |X'|. \tag{3}$$

On the other hand, all the vertices in $(A \setminus X') \cap X$ have degree at most $k$ in $G[A \setminus X']$. Then $\rho_G(A \setminus X') \geq \rho_G(A \setminus X)$. Then all those vertices in $A \setminus (X \cup U)$ are pendent vertices in $G[A \setminus X]$. Let $\alpha = |A \setminus (X \cup U)| = |A \cap (U_1 \cup \cdots U_k)|$ for simplicity. Note that $\alpha \geq |X'|$, since for vertex $x$ in $X'$, $N_G(v) \subset A$. Let $u = |A \cap U|$. Therefore, $G[A \setminus X]$ has $u + \alpha$ vertices and has $\frac{u^2 - u}{2} + \alpha$ edges. Thus

$$\rho_G(A \setminus X') \geq \rho_G(A \setminus X) \geq ku + k\alpha - \frac{u^2 - u}{2} - \alpha \geq u^2 + k\alpha - \frac{u^2 - u}{2} - \alpha = \frac{u^2 + u}{2} + (k-1)\alpha \geq 1 + (k-1)|X'| \geq 1 + |X'|,$$

a contradiction to (3). Thus $\rho_G(A) \geq 1$ for every subset $A \subset V(H)$. Hence, $\text{mad}(G) < 2k$. \qed