The solutions and the effects of Brans-Dicke field in the GBD theory

Jianbo Lu,1 Yabo Wu,1 Weiqiang Yang,1 Molin Liu,2 and Xin Zhao1

1Department of Physics, Liaoning Normal University, Dalian 116029, P. R. China
2College of Physics and Electronic Engineering, Xinyang Normal University, Xinyang, 464000, P. R. China

A generalized Brans-Dicke (GBD) theory is studied in this paper. The GBD theory is obtained with replacing the Ricci scalar $R$ in the original Brans-Dicke (BD) action by an arbitrary function $f(R)$. In this theory, the gravitational field equation and the motion equation of BD scalar field are given. And then, we investigate the cosmological equations and their solutions in the GBD. The solutions show that they are consistent with the observational Hubble data. Also, the effects of the Brans-Dicke scalar field in the GBD are probed, it is found that the modification to the cosmological quantities from the dynamical BD scalar field is remarkable when they are compared with the $f(R)$ theory. In addition, the forms of the effective potential of Brans-Dicke field are investigated in the GBD model.

PACS numbers: 98.80.-k
Keywords: Modified gravity; Brans-Dicke theory; Cosmological solution; Effective potential of field.

I. Introduction

Observations indicate that the Newton gravitational constant $G$ maybe depends on time, such as the observations from white dwarf star [1, 2], pulsar [3], supernovae [4], neutron star [5], and so on. Brans-Dicke (BD) theory is a popular one to describe the time-variable $G(t)$ gravity. And as a simple theory among the scalar-tensor theories [6], BD theory is apparently compatible with Mach’s principle [7]. But in the original BD theory [8], it is hard to interpret the cosmic acceleration indicated by the observations [9–11]. In order to obtain an accelerating universe, one usually modified this theory at three aspects: (1) introducing the invisible component—dark energy in universe [12], (2) assuming the coupling constant $\omega$ to be variable with respect to time [13, 14], (3) adding a potential term to the original BD theory [15]. In this paper, we investigate other way to explain the cosmic acceleration in the framework of the BD theory—generalize the Ricci scalar $R$ to be an arbitrary function $f(R)$ in the BD-type time-variable gravitational constant theory (abbreviate as GBD), which is different from the studies on equivalence between the BD theory and the modified $f(R)$ theory [16]. We show the field equations and the cosmological equations in the GBD theory. And then the cosmological solutions of GBD model are given, which are consistent with the observational Hubble data. Also, we investigate the effects on the cosmological quantities induced by the BD field, the properties of the geometrical dark energy, the effective potential of the BD scalar field in the GBD model.

*Electronic address: lvjianbo819@163.com
II. Field equations in GBD theory

In the framework of time-variable gravitational constant, we study a generalized Brans-Dicke theory by using a function \(f(R)\) to replace the Ricci scalar \(R\) in the original BD action. The action of system is written

\[
S = S_g(g_{\mu\nu}, \phi) + S_m(g_{\mu\nu}, \psi) = \frac{1}{2} \int d^4x \mathcal{L}_T,
\]

where the total Lagrange quantity \(\mathcal{L}_T\) has a form

\[
\mathcal{L}_T = \mathcal{L}_g + \mathcal{L}_\phi + \mathcal{L}_m = \sqrt{-g}[\phi f(R) - \frac{\omega}{2\phi} \partial^\mu \phi \partial_\mu \phi + \frac{16\pi}{c^4} L_m].
\]

Obviously, the system contains three dynamical variable: the gravitational field \(g_{\mu\nu}\), the matter field \(\psi\) and the scalar field \(\phi\). \(\omega\) is the couple constant. From the action equation \([22]\), it is easy to see that GBD theory can be considered as a special case of the more general \(f(R, \phi)\) theory \([17, 19]\). As shown in several references, the so-called \(f(\phi)R\) theory \([20, 21]\) can also be seen as the special cases of the \(f(R, \phi)\) theory, and the \(f(\phi)R\) theory have been widely studied \([22, 24]\). Given that \(f(R, \phi)\) is complex, and usually the more simple theory is more favored by the researcher in physics, here we investigate the GBD model motivated by the directly observational motivation of the accelerating universe, and discuss some interesting cosmological contents in this model, including the cosmological solutions of the GBD model, the effective potential and the effects of the BD scalar field in the GBD theory, etc..

Taking \(c = 1\) and varying the action with respect to metric \(\frac{\delta S}{\delta g_{\mu\nu}} = \frac{\delta S_g(g_{\mu\nu}, \phi)}{\delta g_{\mu\nu}} + \frac{\delta S_m(g_{\mu\nu}, \psi)}{\delta g_{\mu\nu}} = 0\), one can get the gravitational field equation

\[
\phi \left[ f_R R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} \right] - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box)(\phi f_R) + \frac{1}{2} \omega \partial_\mu \phi \partial^\mu \phi - \frac{\omega^2}{\phi^2} \partial_\mu \phi \partial^\mu \phi = 8\pi T_{\mu\nu},
\]

where \(f_R \equiv \partial f/\partial R\), \(\nabla_\mu\) is the covariant derivative associated with the Levi-Civita connection of the metric, \(\Box \equiv \nabla^\mu \nabla_\mu\), and \(T_{\mu\nu} = \frac{1}{2\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}\) is the energy momentum tensor of the matter. Varying the action \([11]\) with respect to the scalar field \(\phi\) gives

\[
f(R) + 2\omega \frac{\Box \phi}{\phi} - \frac{\omega^2}{\phi^2} \partial_\mu \phi \partial^\mu \phi = 0.
\]

Varying the action \([11]\) with respect to the matter field \(\psi\) gives

\[
\frac{\delta S}{\delta \psi} = \frac{\delta S_\phi(g_{\mu\nu}, \phi)}{\delta \psi} + \frac{\delta S_m(g_{\mu\nu}, \psi)}{\delta \psi} = 0.
\]

The trace of Eq. \([8]\) is

\[
f_R R - 2f(R) + \frac{3\Box (\phi f_R)}{\phi} + \frac{\omega^2}{\phi^2} \partial_\mu \phi \partial^\mu \phi = \frac{8\pi T}{\phi}.
\]

Obviously, if one takes BD scalar field \(\phi = \)constant and \(f(R) = R\), above equations reduce to the Einstein’s general relativity (GR). In GR, one has \(R = 0\) for \(T = 0\). If one takes \(\phi = \)constant, the standard \(f(R)\) modified gravity is recovered. Furthermore, if no hypothesis are put, according to Eqs. \([14]\) and \([11]\) we have the result that the curvature of the spacetime could be caused by the motion of \(\phi\) (besides matter). Also, it is shown from Eq. \([3]\) that the BD scalar field does not exert any direct influence on matter, while it couples with another scalar field \(f_R\).

Combining Eqs. \([14]\) and \([11]\), we get

\[
\Box \phi - \frac{\partial_\mu \phi \partial^\mu \phi}{4\phi} = \frac{1}{4\omega} [8\pi T - \phi f_R - 3\Box (\phi f_R)].
\]

One can read from Eq. \([17]\), for \(\omega \rightarrow \infty\) the constant-G theory can be recovered, which is same to the result in the standard BD theory.
III. The cosmological equations and their solutions in GBD theory

In the flat FLRW background

\[ ds^2 = -dt^2 + a^2(t) \, d\vec{x}^2, \]

one can obtain the components of connection \( \Gamma_{11}^0 = a \dot{a}, \Gamma_{10}^1 = \frac{a}{c} = H \), and \( R = 6 \left( 2H^2 + \dot{H} \right) \). Here \( a \) is the cosmic scale factor, \( H \) is the Hubble parameter, and ", dot" denotes the derivative with respect to cosmic time \( t \). Then using Eqs. (3) and (4), one can derive the cosmological evolutional equations in the GBD theory,

\[ 3f_R H^2 = \frac{8\pi \rho_m}{\phi} + \frac{f_R R - f(R)}{2} - 3H \dot{f}_R + \frac{1}{2} \omega \left( \frac{\dot{\phi}}{\phi} \right)^2 - 3H f_R \frac{\dot{\phi}}{\phi}, \]

\[ -2f_R \ddot{H} = \frac{8\pi}{\phi} (\rho_m + p_m) + \dot{f}_R - H \dot{f}_R + \omega \left( \frac{\dot{\phi}}{\phi} \right)^2 - H f_R \frac{\dot{\phi}}{\phi} + f_R \frac{\ddot{\phi}}{\phi} + 2 \frac{\dot{\phi}}{\phi} \ddot{f}_R, \]

\[ f(R) - \omega \left( \frac{\dot{\phi}}{\phi} \right)^2 + 2 \frac{\dot{\phi}}{\phi} + 6 \omega H \frac{\dot{\phi}}{\phi} = 0. \]

For \( \phi = \text{constant} (\dot{\phi} = 0 \text{ and } \ddot{\phi} = 0) \) in Eqs. (9), (11), they are reduced to the \( f(R) \) theory, while for \( f(R) = R \) they are reduced to the original Brans-Dicke theory. For solving the cosmological equations, we define the dimensionless variables

\[ y_H = H^2/m^2 - a^{-3}, \]

\[ y_R = R/m^2 - 3a^{-3}, \]

\[ y_\phi = \phi/\phi_0, \]

\[ y'_\phi = \phi'/\phi_0, \]

and using Eqs. (9) and (11), we get the differential equations for \( \{ y_H, y_R, y_\phi, y'_\phi \} \)

\[ y'_H = \frac{1}{3} y_R - 4y_H, \]

\[ y'_R = \frac{\left[ (y_H + a^{-3}) f_R - \frac{f_R}{6}(y_R + 3a^{-3}) + \frac{f}{6m^2} - \frac{\omega}{2} (y'_\phi)^2 (y_H + a^{-3}) + f_R y'_\phi (y_H + a^{-3}) \right] - a^{-3}}{(y_H + a^{-3}) m^2 f_{RR}} + 9a^{-3}, \]

\[ y'_\phi = \frac{\phi'}{\phi_0}, \]

\[ y''_\phi = \frac{\phi}{2\omega(y_H + a^{-3})} \left[ \frac{\omega}{m^2} (y'_\phi)^2 (y_H + a^{-3}) - \omega \frac{y'_\phi}{y_\phi} (\frac{1}{3} y_R - 4y_H - 3a^{-3}) - 6 \omega \frac{y'_\phi}{y_\phi} (y_H + a^{-3}) \right]. \]
Here the subscript ”0” denotes the current value of parameters, the superscript ”prime” denotes the derivative with respect to $ln a$, $m^2 = (8315 \text{Mpc})^{-2} (\Omega_{m0} a^2)$ and $\Omega_{m0}$ is the current dimensionless energy density of the matter. The solutions of these equations describe the background evolution for a GBD theory. For solving above differential equations, the initial conditions ($a_0 = 1$) are expressed respectively

$$y_H|_{a=1} = H_0^2/m^2 - 1,$$

$$y_R|_{a=1} = 6H_0^2(1 - q_0)/m^2 - 3,$$

$$y_{\phi}|_{a=1} = 1,$$

$$y_{\phi}'|_{a=1} = 0.01.$$

Here $q = -\frac{a}{aH}$ is the deceleration parameter, and its current value $q_0$ can be given by the cosmic observations. The value of the initial condition $y_{\phi}'|_{a=1}$ can be investigated by the following observations. For example, the limits on the variation of $G$: $|\frac{\Delta G}{G}| = |\frac{\dot{G}}{G}| \leq 4.1 \times 10^{-10}y^{-1}$ from Pulsating white dwarf G117-B15A [1], $-4 \times 10^{-11}y^{-1} \leq \dot{G}/G \leq 2.5 \times 10^{-10}y^{-1}$ from Nonradial pulsations of white dwarfs [2], $|\frac{\dot{G}}{G}| \leq 2.3 \times 10^{-11}y^{-1}$ from Millisecond pulsar PSR J0437-4715 [3], $|\frac{\dot{G}}{G}| \leq 10^{-11}y^{-1}$ from Type-Ia supernovae [4], $\frac{\dot{G}}{G}(0.6 \pm 4.2) \times 10^{-12}y^{-1}$ from Neutron star masses [5], $|\frac{\dot{G}}{G}| \leq 1.6 \times 10^{-12}y^{-1}$ from Helioseismology [25], and $\frac{\dot{G}}{G} = (4 \pm 9) \times 10^{-13}y^{-1}$ from Lunar laser ranging experiment [26], etc. Taking a stringent bound $|\frac{\dot{G}}{G}| \leq 10^{-12}y^{-1}$ and considering the current value of the dimensionless Hubble constant $h = 0.673 \pm 0.010$ from the Planck 2015 results [27], we can calculate the limit: $|y_{\phi}'(a = 1)| \leq 0.015$ with using the center value $H_0 = 67.3 \text{km}^{-1}\text{Mpc}^{-1} = 6.87 \times 10^{-11}y^{-1}$. Here we take $y_{\phi}'(a = 1) = 0.01$ as an initial condition shown in Eq. (23). Also for comparison, other initial values of $y_{\phi}'(a = 1)$ (less than 0.01) are discussed.

To find a cosmological solution of GBD theory, we have to take a concrete form of $f(R)$ function at prior. As an example, we consider an interesting model called exponential gravity

$$f(R) = R - \beta R_s(1 - e^{-R/R_s}),$$

which is proposed by Refs. [28,30]. Here $\beta$ and $R_s$ are two constants with $\beta R_s \simeq 12H_0^2\Omega_{m0}$ [30]. This model has an important feature that it has only one more parameter than the $\Lambda$CDM model. The first two derivatives of Eq. (24) with respect to $R$ are

$$f_R = 1 - \beta e^{-R/R_s},$$

$$f_{RR} = -\frac{\beta}{R_s}e^{-R/R_s}.$$

Thus using the system of ordinary differential equations (12)-(19) and the initial conditions (20)-(23), we can numerically obtain the solutions: $H(a)$ and $\phi(a)$ in the GBD theory, which are illustrated in Fig.1 and Fig.2.

In Fig.1 (left), we show the dependence of $H(a)$ on the parameter $\beta$. Fig.2 (right) illustrates the modification on $H(a)$ by the dynamical BD scalar field $\phi$. In the following, we use $a_0$ to denotes $a = 1$. We can see that the evolutions of $H(a)$ almost have the same trajectory for two cases: $y_{\phi}'(a_0) = 0.01$ and $y_{\phi}'(a_0) = -0.01$, while the evolutions of
FIG. 1: The numerical solutions of $H(a)/H_0$ for the GBD model with the different model parameter $\beta$ or the different initial condition $y_{\phi}'(a_0)$.

$H(a)$ are obviously different for two cases: $y_{\phi}'(a_0) = 0.01$ and $y_{\phi}'(a_0) = 0$. The effect on $H(a)$ from the dynamical BD scalar field is notable. Using the observational Hubble data listed in table I, we display these observational $H(z)$ value to Fig. 1 (right). Here $z = (1 - a)/a$ is the cosmic redshift. It is shown from Fig. 1 (right) that the most observational $H(z)$ data are located in the region between $y_{\phi}'(a_0) = 0$ and $y_{\phi}'(a_0) = \pm 0.01$. It seems that the case of $y_{\phi}'(a_0) = 0.003$ for the evolution of $H(a)$ in the GBD model is well consistent with observations. From Fig. 2 (right), one can see that evolutional tendency of BD scalar field depends on the initial value of $y_{\phi}'(a_0)$.

IV. Effective equation of state of geometrical dark energy in the GBD

Probing the properties of the dark energy is important, and it has been studied in the standard cosmology or the several modified gravity theories [42–58]. Next we investigate the properties of geometrical dark energy in this GBD
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi T_{\mu\nu}^{\text{eff}}}{\phi_0}, \]  

(27)

with

\[ T_{\mu\nu}^{\text{eff}} = \frac{\phi_0 T_{\mu\nu}}{\phi f_R} + \frac{\phi_0}{8\pi f_R} (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) f_R + \frac{1}{2} f(R) g_{\mu\nu} - \frac{1}{2} f_R R g_{\mu\nu} + \frac{f_R}{\phi} (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) \phi \\
- \frac{1}{2} \omega \frac{\phi_0}{\phi^2} g_{\mu\nu} \phi_0 \partial_\mu \partial_\nu \phi + \frac{\omega}{\phi^2} \partial_\mu \phi_0 \partial_\nu \phi, \]

(28)

then the effective energy density and the effective pressure are derived

\[ \rho^{\text{eff}} = \frac{\phi_0 \rho_m}{\phi f_R} + \frac{\phi_0}{8\pi f_R} \left[ -3H f_R - \frac{1}{2} f(R) + \frac{1}{2} f_R R - 3H f_R f_\phi + \frac{1}{2} \omega (\frac{\phi_0}{\phi})^2 \right], \]

(29)

\[ p^{\text{eff}} = \frac{\phi_0 p_m}{\phi f_R} + \frac{\phi_0}{8\pi f_R} \left[ f_R + 2H f_R + \frac{1}{2} f(R) - \frac{1}{2} f_R R + f_R f_\phi + 2H f_R f_\phi + \frac{1}{2} \omega (\frac{\phi_0}{\phi})^2 + 2 \frac{\phi_0}{\phi} f_R \right]. \]

(30)

Here \( \rho_m \) and \( p_m \) are the energy density and the pressure of matter, respectively. The effective equation of state for geometrical dark energy has a form

\[ w_g^{\text{eff}} = \frac{p_g^{\text{eff}} - p_m}{\rho_g^{\text{eff}} - \rho_m} = \frac{8\pi \phi_0 p_m - 8\pi \phi p_m f_R + \phi_0 \phi \left[ f_R + 2H f_R + \frac{1}{2} f(R) - \frac{1}{2} f_R R + f_R f_\phi + 2H f_R f_\phi + \frac{1}{2} \omega (\frac{\phi_0}{\phi})^2 + 2 \frac{\phi_0}{\phi} f_R \right]}{8\pi \phi_0 \rho_m - 8\pi \phi \rho_m f_R + \phi_0 \phi \left[ -3H f_R - \frac{1}{2} f(R) + \frac{1}{2} f_R R - 3H f_R f_\phi + \frac{1}{2} \omega (\frac{\phi_0}{\phi})^2 \right]} \]

(31)

Taking the function \( f(R) = R + \beta R_d (1 - e^{-R/R_d}) \) as an example, we plot the evolution of \( w_g^{\text{eff}}(a) \) in Fig.3 by using the different values of model parameter \( \beta \) and the initial values \( y_\phi'(a_0) \). In the GBD model, the dependence of \( w_g^{\text{eff}}(a) \) on the model parameter \( \beta \) are illustrated in Fig.3(left). In the Fig.3(right), it is shown that \( w_g^{\text{eff}}(a) \) almost have the same evolutions for the two cases: \( y_\phi'(a_0) = \pm 0.01 \), i.e. the trajectories of \( w_g^{\text{eff}}(a) \) is not sensitive to the symbol of initial condition \( y_\phi'(a_0) \), while the effect on \( w_g^{\text{eff}}(a) \) by the BD scalar field is notable since the evolution of
One knows that the potential of a scalar field usually play an important role in the early inflation universe and the the late accelerating universe. Determining the forms of the potential function is significative for a scalar field, since the potential display some properties for a scalar field. Obviously, this GBD theory is similar to the two scalar-fields theory. The equations of motion for the Brans-Dicke scalar field \( \phi \) and the other scalar field \( \varphi = f_R \) are expressed as follows

\[
\Box \phi = \frac{\partial \mu \partial \mu \phi}{2 \phi} - \frac{\phi \dot{f}}{2 \omega} = -V_\phi(\phi). \tag{32}
\]

\[
\Box f_R = \frac{1}{3} \left[ \frac{8\pi T}{\phi} - f_R R + 2f - 3f_R \frac{\Box \phi}{\phi} - 6f_R \frac{\dot{\phi}}{\phi} - \omega \frac{\partial \mu \partial \mu \phi}{\phi^2} \right] = -V_\varphi(\varphi). \tag{33}
\]

Here \( V(\phi) \) and \( V(\varphi) \) are the effective potential of field, and the subscript \( \phi \) (or \( \varphi \)) denotes the derivative with respect to scalar field. Next we investigate the effective potential \( V(\phi) \) of the BD scalar field. Integrating Eq. (32), we can give the form of \( V(\phi) \)

\[
V(\phi) = -\frac{H^2 \phi^2}{2} \ln \phi + \frac{f}{4\omega} \phi^2 + C(\phi). \tag{34}
\]

Using Eq. (34) and taking \( C(\phi) = 0 \), we can plot the shapes of BD effective potential in Fig. 4. We can see from Fig. 4 that the trajectories of BD effective potential \( V(\phi) \) are not sensitive to the variation of \( \beta \) values, while the shapes
of \( V(\phi) \) much depend on the initial condition \( y'_\phi(a_0) \) for the smaller \( \ln a \) \((\ln a < -0.4)\), and for \( \ln a > -0.4 \) one has \( V(\phi) \sim 0 \). Obviously, \( C(\phi) \) in Eq. (34) is a undetermined freedom, whose uncertainty can be used to modify the trajectories of the BD effective potential.

\[ \begin{align*}
V(\phi) & \sim 0.1, \quad \beta = 1.0,
V(\phi) & \sim -0.01, \quad \beta = 1.5,
V(\phi) & \sim 0.0, \quad \beta = 1.0.
\end{align*} \]

**FIG. 4:** The evolutions of effective potential for Brans-Dicke scalar field with the different model parameter \( \beta \) and the different initial condition \( y'_\phi(a_0) \).

### VI. Conclusion

The GBD theory is investigated in this paper, which is obtained with replacing the Ricci scalar \( R \) in the original Brans-Dicke action by an arbitrary function \( f(R) \). This theory can be reduced to the original BD theory and the \( f(R) \) modified gravity under certain conditions. We give the gravitational field equation and the BD scalar-field equation in the GBD theory. Using the FLRW metric and the field equations, we can get the cosmological equations in this theory, and then these equations are numerically solved by taking a concrete form of \( f(R) \) function as an example. It is shown that the modification to \( H(a) \) from the dynamical BD scalar field is notable, and the evolution of \( H(a) \) in the GBD model is well consistent with the observations. The effective state-parameter equation and its trajectory for the geometrical dark energy in the GBD universe is studied, which indicates that the evolutions of \( w^\text{eff}_g(a) \) with \( y'_\phi(a_0) \neq 0 \) can vary from radiation \((w^\text{eff}_g \sim 1/3)\) to dark energy \((w^\text{eff}_g < 0)\). In addition, the effective potential of Brans-Dicke field is investigated in the GBD model. One can see that the evolutional tendency of the BD effective potential depends on the initial value of \( y'_\phi(1) \), especially it is sensitive to the given symbol of \( y'_\phi(1) \). The same result can be found for the evolution of BD scalar field \( \phi \). Also, we investigate the effects of the Brans-Dicke scalar field on the evolutions of the cosmological quantities, and it is found that the modifications of the Hubble parameter (or the state parameter of geometrical dark energy) from the BD scalar field is remarkable, while the evolutions of these two parameter are almost same for the case of taking the opposite number of initial value \( y'_\phi(1) \).

**Acknowledgments** The research work is supported by the National Natural Science Foundation of China
(11645003,11705079,11575075,11475143).

[1] M. Biesiada and B. Malec, Mon. Not. Roy. Astron. Soc. 350, 644 (2004) [astro-ph/0303489].
[2] O. G. Benvenuto et al., Phys. Rev. D, 69, 082002, (2004).
[3] J.P.W. Verbiest et al., Astrophys. J. 679, 675 (2008) [arXiv:0801.2589].
[4] E. Gaztanaga, et al, Phys. Rev. D 65, 023506, 2002 [arXiv:astro-ph/0109299].
[5] S. E. Thorsett, Phys. Rev. Lett. 77, 1432 (1996) [astro-ph/9607003].
[6] K. Bamba, D. Momeni, R. Myrzakulov, Int.J.Geom.Meth.Mod.Phys. 12(10), 1550106 (2015) [arXiv:1404.4255]
[7] L. Qiang, Y. Ma, M. Han, D. Yu, Phys. Rev. D 71, 061501 (2005).
[8] C. Brans, R.H. Dicke, Phys. Rev. 124, 925 (1961).
[9] A.G. Riess et al., Astron. J. 116, 1009 (1998).
[10] S. Perlmutter et al., Astrophys. J. 517, 565 (1999).
[11] D.N. Spergel et al., Astrophys. J. Suppl. 148, 175 (2003).
[12] L.X. Xu, W.B Li, J.B. Lu, Eur. Phys. J. C 60, 135 (2009).
[13] N. Banerjee, D. Pavon, Phys. Rev. D 63, 043504 (2001).
[14] A.D. Felice, S. Tsujikawa, JCAP 07, 024 (2010).
[15] J. Lu, S. Gao, Y. Zhao and Y. Wu, Eur. Phys. J. Plus 127, 154 (2012).
[16] T. P. Sotiriou, Class. Quant. Grav. 23, 5117 (2006).
[17] J.C. Hwang, Class. Quantum Grav. 7, 1613-1631 (1990).
[18] J.C. Hwang, H. Noh, Phys. Rev. D 54, 1460 (1996).
[19] J.C. Hwang, H. Noh, Phys. Rev. D 71, 063536 (2005).
[20] S.D. Odintsov, V.K. Oikonomou, Nuclear Physics B 929, 79-112 (2018) [arXiv:1801.10529].
[21] Y. Huang, Y. Gong, D. Liang, Z. Yi, Eur. Phys. J. C 75, 351 (2015) [arXiv:1504.01271].
[22] B. Boisseau, Phys.Rev.D 83:043521 (2011) [arXiv:1011.2915).
[23] B. Boisseau, H. Giacomini, D. Polarski, A. A. Starobinsky, JCAP 07, 002 (2015) [arXiv:1504.07927].
[24] T. Chiba, M. Yamaguchi, JCAP 10, 040 (2013) [arXiv:1308.1142].
[25] D.B. Guenther, L.M. Krauss, P. Demarque, Astrophys. J. 498, 871 (1998).
[26] J.G. Williams, S.G. Turyshhev, D.H. Boggs, Phys. Rev. Lett. 93, 261101 (2004), [arXiv:gr-qc/0411113].
[27] A.G. Riess et al., Astrophys. J. 699, 539 (2009) [arXiv:0905.0695].
[28] P. Zhang, M. Liguori, R. Bean, S. Dodelson, Phys.Rev.Lett. 99, 141302 (2007) [arXiv:0704.1932].
[29] E.V. Linder, Phys.Rev.D 80, 123528 (2009) [arXiv:0906.2962].
[30] K. Bamba, C.G. Geng, C.C. Lee, JCAP 08, 021 (2010) [arXiv:1005.4574].
[31] C. Zhang, H. Zhang, S. Yuan, S. Liu, T.J. Zhang and Y.C. Sun, Research in Astronomy and Astrophysics 14 (2014) 1221-1233, [arXiv:1207.4541].
[32] R. Jimenez, L. Verde, T. Treu and D. Stern, Astrophys. J. 593 (2003) 622-629, [arXiv:0302560].
[33] J. Simon, L. Verde and R. Jimenez, Gravit. Cosmol. 71 (2005) 123001, [arXiv:0412269].
[34] M. Moresco, L. Verde, L. Pozzetti, R. Jimenez and A. Cimatti, JCAP (2012) 53, [arXiv:1201.0665].
[35] E. Gaztanaga, A. Cabrera and L. Hui, [arXiv:0807.3551].
[36] X. Xu, A. J. Cuesta, N. Padmanabhan, D. J. Eisenstein and C. K. McBride, Mon. Not. R. Astron. Soc. 431 (2013) 2834-2860, [arXiv:1206.6732].
[37] M. Moresco, L. Pozzetti and A. Cimatti et al., JCAP 5 (2016) 014, [arXiv:1601.01701].
[38] C. Blake, S. Brough and M. Colless, et al., Mon. Not. R. Astron. Soc. 425 (2012) 405-414, arXiv:1204.3674.
[39] D. Stern, R. Jimenez and L. Verde, et al., J. Cosmol. Astropart. Phys. 2 (2010) 8, arXiv:0907.3149.
[40] L. Samushia, B. A. Reid and M. White, et al., Mon. Not. R. Astron. Soc. 429 (2013) 1514-1528, arXiv:1206.5309.
[41] M. Moresco, Mon. Not. R. Astron. Soc. Lett. 450 (2015) L16-L20, arXiv:1503.01116.
[42] R.G. Cai, S.J. Wang, Phys. Rev. D 93, 023515 (2016) arXiv:1511.00627.
[43] J.J. Guo, J.F. Zhang, Y.H. Li, D.Z. He, X. Zhang, Sci. China-Phys. Mech. Astron. 61, 030011 (2018) arXiv:1710.03068.
[44] T. Yang, Z.K. Guo, R.G. Cai, Phys. Rev. D 91, 123533 (2015) arXiv:1505.04443.
[45] H. Wei, X.B. Zou, H.Y. Li, D.Z. Xue, Eur. Phys. J. C 77 (2017) 14 arXiv:1605.04571.
[46] Q.G. Huang, Eur. Phys. J. C (2014) 74, 2964 arXiv:1403.0655.
[47] Y. Fan, P.X. Wu, H.W. Yu, Physical Review D 92, 083529 (2015) arXiv:1510.04010.
[48] X. Zhang, Sci. China-Phys. Mech. Astron. 60, 060431 (2017) arXiv:1703.00651.
[49] L.X. Xu, Phys. Rev. D.87, 043503(2013) arXiv:1210.7413.
[50] S. Li, Y.G. Ma, Eur.Phys.J.C 68, 227-239 (2010) arXiv:1004.4350.
[51] L. Feng, J.F. Zhang, X. Zhang, Sci. China-Phys. Mech. Astron. 61, 050411 (2018) arXiv:1706.06913.
[52] L.X. Xu, Phys. Rev. D.87, 043525(2013) arXiv:1302.2291.
[53] J.B. Lu, G.Y. Chee, JHEP 05, 024 (2016).
[54] M. Hohmann, L. Jarv, P. Kuusk, E. Randla, O. Vilson, Phys. Rev. D 94, 124015 (2016) arXiv:1607.02356.
[55] S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Phys.Rept. 692 (2017) 1-104 arXiv:1705.11098.
[56] A. de la Cruz-Dombriz, E. Elizalde, S. D. Odintsov, D. Saez-Gomez, JCAP 05, 060 (2016) arXiv:1603.05537.
[57] J. Lu, D. Geng, L. Xu, Y. Wu, M. Liu, JHEP 02, 071 (2015) arXiv:1312.0770.
[58] J. Lu, M. Liu, Y. Wu, Y. Wang, W. Yang, Eur. Phys. J. C 76, 679 (2016) arXiv:1606.02987.