Cosmological Evolution of Homogeneous Universal Extra Dimensions

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The lightest Kaluza-Klein particle appearing in models with universal extra dimensions has recently been proposed as a viable dark matter candidate when the extra dimensions are compactified on a scale of the order of 1 TeV. Underlying assumptions of this proposal are that the size of the extra dimensions stays constant and that the evolution of the universe is given by standard cosmology. Here we investigate, both analytically and numerically, whether this is possible without introducing an explicit stabilization mechanism. By analyzing Einstein’s field equations for a \((3+n+1)\)-dimensional homogeneous, but in general anisotropic universe, we find that approximately static extra dimensions arise naturally during radiation domination. For matter domination, however, there are no solutions to the field equations that allow static extra dimensions or the usual behavior of the scale factor for ordinary three-dimensional space. We conclude that an explicit mechanism is needed in order to stabilize the extra dimensions and reproduce standard cosmology as we know it.

I. INTRODUCTION

Everyday experience seems to suggest that our world consists of four space-time dimensions. However, at the beginning of the 20th century Nordström, Kaluza, and Klein (KK) already realized that this may in fact not be the case. In the last few years there has again been a great deal of interest in models with extra dimensions, most notably due to the influence of string (or M) theory which in its usual formulation requires more than four dimensions (see e.g. [4]). In particular, this has led to a number of brane-world scenarios where all, or only some, of the gauge bosons are allowed to propagate in the extra dimensions while matter fields are restricted to \((3+1)\)-dimensional branes. For a review of different models with extra dimensions, see, for example, [5] and references therein.

In this context, a specific model of so-called universal extra dimensions (UED) has recently been proposed by Appelquist et al. [6], in which all standard model fields are allowed to propagate in the extra dimensions. As usual, quantization of the extra-dimensional momentum leads to a tower of KK states that appear as new massive particles in the effective four-dimensional theory. The existing constraints on electroweak observables translate into bounds on the compactification scale \(R\), which is related to the mass of the lowest excitations by \(M \sim 1/R\). For one or two UED these bounds are of the order of a few hundred GeV and thus within reach of the Large Hadron Collider (LHC) or the Fermilab Tevatron run II [7, 8, 9].

The UED model is not only of great interest from the point of view of particle physics [10, 11, 12, 13], but might also provide a solution to one of the most outstanding puzzles in modern cosmology – the nature of dark matter [14, 15, 16]. In the UED scenario the lightest KK particle (LKP) is stable because of KK parity conservation and could therefore still be present today as a thermal relic. Furthermore, if it is also neutral and nonbaryonic it has all the properties of a weakly interacting massive particle (WIMP), one of the most promising dark matter candidates (see [17] for a nice introduction to WIMP dark matter). According to [10], both the KK photon (\(B(1)\)) and the KK neutrino could account for dark matter with \(\Omega_M \sim 0.3\), as suggested by the current cosmological concordance model [18], if one assumes a compactification scale of about \(R \sim 1\) TeV\(^{-1}\) size. Indirect and direct detection properties of such KK dark matter candidates are promising for the next generation detectors [19, 20, 21, 22]. In fact, the KK neutrino seems to be ruled out already by present data [21].

However, when considering the freeze-out process and the further cosmological evolution of thermally produced LKPs, the size of the extra dimensions has so far been assumed to stay constant – although no explicit stabilization mechanism has been given. As already noted in [14, 23], the resulting relic density today depends crucially on this assumption, and it is therefore important to investigate whether it can be justified within the UED framework. Since the UED may be relatively large (in fact, they must be if the LKPs are not to overclose the universe), their evolution should be governed by Einstein’s field equations. The aim of this work is therefore to carefully study the dynamics of the extra dimensions.
of the universe as described by an appropriate extension of the usual Friedmann equations that results from the field equations in higher dimensions. Specifically, we focus on solutions with constant or only slowly varying extra dimensions in the absence of any explicit stabilization mechanism. However, our analysis is more general in that we investigate whether there is any time evolution of the extra dimensions that corresponds to standard cosmology in four dimensions; in this context, we also include a numerical study of the transition regime between the radiation- and matter-dominated eras. For earlier work on the stability properties of higher-dimensional cosmologies, often referred to as Kaluza-Klein or multidimensional cosmology, see, for example, [24, 25, 26, 27, 28, 29, 30, 31, 32] and references therein.

This paper is organized as follows: In Sec. II we introduce the cosmological solutions to Einstein’s field equations for a (3+n+1)-dimensional homogeneous, but in general anisotropic, universe. Here, we also comment on the interpretation of pressure in higher dimensions and derive a general relation between pressure and energy density in UED cosmology. Necessary conditions that every solution with static extra dimensions in such a model must satisfy are then derived in Sec. III. In the next two sections we study the existence of solutions with (nearly) constant extra dimensions during radiation and matter domination, respectively. A possible transition between these two regimes is then outlined in Sec. VI. Finally, Sec. VII discusses the implications for the UED scenario and concludes.

II. SETUP

A. Basic equations

We introduce $n$ universal extra dimensions and adopt coordinates $X^A$, $A = 0, 1, \ldots, 3 + n$, with

$$x^\mu \equiv X^\mu \quad (\mu = 0, 1, 2, 3)$$

and

$$x^i \equiv X^i \quad (i = 1, 2, 3)$$

being the coordinates for ordinary four-dimensional spacetime and three-dimensional (3D) space respectively, and

$$y^p \equiv X^{3+p} \quad (p = 1, \ldots, n)$$

the coordinates for the UED. In the absence of a cosmological constant, Einstein’s field equations are given by

$$G^A_B = R^A_B - \frac{1}{2} R \delta^A_B = \kappa^2 T^A_B .$$

Here, $\kappa^2$ is defined as

$$\kappa^2 = \frac{8\pi}{M^{2+n}} ,$$

where $M$ is the higher-dimensional Planck mass. In the case of compactified extra dimensions with volume $V_{(n)}$ it is related to the usual Planck mass by

$$M_{Pl}^2 = V_{(n)} M^{2+n} .$$

In the UED scenario, there is no localization mechanism that confines particles to a brane, so we assume that the energy density is distributed homogeneously throughout all dimensions. Thus, we are looking for homogeneous solutions to the field equations that are isotropic in ordinary three-dimensional space and also – but separately – in the space of extra dimensions. This can be described by the standard Friedmann-Robertson-Walker (FRW) metric if we allow for different scale factors in 3D and the UED:

$$ds^2 = -dt^2 + a^2(t) \gamma_{ij} dx^i dx^j + b^2(t) \tilde{\gamma}_{pq} dy^p dy^q ,$$

where $\gamma_{ij}$ and $\tilde{\gamma}_{pq}$ are maximally symmetric metrics in three and $n$ dimensions, respectively. Spatial curvature is thus parametrized in the usual way by $k_a = -1, 0, 1$ in ordinary space and $k_b = -1, 0, 1$ in the UED. Of course, one could imagine a model that is not described by the metric $\Box$. For instance, there is no theoretical or observational argument against having separate scale factors for each extra dimension. We choose this metric because it is the simplest realistic alternative for studying dynamical extra dimensions.
With our choice of metric, the energy-momentum tensor must take the following form:

\[
T^A_{\ B} = \begin{pmatrix}
-\rho & 0 & 0 \\
0 & -\gamma_{ij} p_a & 0 \\
0 & 0 & -\gamma_{ij} p_b
\end{pmatrix},
\]  
(8)

which describes a homogeneous but in general anisotropic perfect fluid in its rest frame. The pressure in ordinary space (UED) is related to the energy density by an equation of state \( p_a = w_a \rho \) and \( p_b = w_b \rho \).

The nonzero components of the field equations (4) are then given by

\[
3 \left( \frac{\dot{a}}{a} \right)^2 + 3 \frac{k_a}{a^2} + 3 n \frac{\dot{b}}{b} + \frac{n(n-1)}{2} \left[ \left( \frac{\dot{b}}{b} \right)^2 + \frac{k_b}{b^2} \right] = \kappa^2 \rho,
\]
(9a)

\[
2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k_a}{a^2} + n \frac{\ddot{\gamma}}{\gamma} + 2 n \frac{\dot{a}}{a} \frac{\dot{b}}{b} + \frac{n(n-1)}{2} \left[ \left( \frac{\dot{b}}{b} \right)^2 + \frac{k_b}{b^2} \right] = -\kappa^2 w_a \rho,
\]
(9b)

\[
3 \frac{\ddot{a}}{a} + 3 \left( \frac{\dot{a}}{a} \right)^2 + 3 \frac{k_a}{a^2} + (n-1) \frac{\ddot{b}}{b} + 3(n-1) \frac{\dot{a}}{a} \frac{\dot{b}}{b} + \frac{(n-1)(n-2)}{2} \left[ \left( \frac{\dot{b}}{b} \right)^2 + \frac{k_b}{b^2} \right] = -\kappa^2 w_b \rho,
\]
(9c)

where an overdot denotes differentiation with respect to cosmic time \( t \). From conservation of energy \( T^A_{\ 0;A} = 0 \) we find, furthermore,

\[
\frac{\dot{\rho}}{\rho} = -3(1 + w_a) \frac{\dot{a}}{a} - n(1 + w_b) \frac{\dot{b}}{b}.
\]
(10)

For constant equations of state this can be integrated to give

\[
\rho = \rho_i \left( \frac{a}{a_i} \right)^{-3(1+w_a)} \left( \frac{b}{b_i} \right)^{-n(1+w_b)}.
\]
(11)

We will use a subscript \( i \) to indicate arbitrary initial values throughout.

### B. On energy density and pressure

The energy density and pressure appearing in the above equations are not the usual three-dimensional quantities but their higher-dimensional analogues. The pressure in some direction \( X^A \) is conventionally defined as the momentum flux through hypersurfaces of constant \( X^A \). This can be expressed as

\[
p_a = \left\langle \frac{k_A^2}{E} \right\rangle = g \int \frac{k_A^2}{E} f(k, \mathbf{x}, t) d^{3+n}k,
\]
(12)

where \( k_A \) is the momentum in direction \( X^A \), \( g \) is the statistical weight and \( f(k, \mathbf{x}, t) \) gives the phase space probability distribution. Isotropy in our model means that \( p_a = \left\langle k_t^2/E \right\rangle = \left\langle k_3^2/E \right\rangle = \left\langle k_3^2/E \right\rangle \) and \( p_b = \left\langle k_2^2/E \right\rangle = \ldots = \left\langle k_{n+3}^2/E \right\rangle \). Therefore we find

\[
3 p_a + n p_b = \rho - \left\langle \frac{m^2}{E} \right\rangle
\]
(13)

where \( \rho = \left\langle E \right\rangle \) and \( m \) is the mass of the particles producing the pressure. In the case of different particle species one has to sum over all of them in Eq. (12), and the mass appearing in Eq. (13) can then be interpreted as the effective mass of all particles. For highly relativistic particles, Eq. (13) reduces to

\[
3 w_a + n w_b = 1.
\]
(14)

Setting \( w_a = w_b \) (corresponding to a completely isotropic \( 3+n \)-dimensional universe) would then result in the equation of state

\[
p = \frac{\rho}{3+n}.
\]
(15)
As expected, for \( n = 0 \) we find the well-known relation for a relativistic gas in \((3+1)\) dimensions.

How can we recover standard cosmology with this setup? Let us first consider the case of highly relativistic particles and static, compact extra dimensions. Equations (9a) and (9b) are then equivalent to the ordinary Friedmann equations with three-dimensional energy density \( \rho^{(3)} \), pressure \( p^{(3)} \), and an effective cosmological constant \( \Lambda_{\text{eff}} \) given by

\[
\begin{align*}
\rho^{(3)} &= V(n) \rho, \\
p^{(3)} &= w_a \rho^{(3)}, \\
\Lambda_{\text{eff}} &= -\frac{n(n - 1) k_b}{2 b^2}.
\end{align*}
\]

Moreover, from Eq. (14) we then find the standard cosmological evolution of \( \rho^{(3)} \propto a^{-3(1+w_a)} \). For vanishing extra-dimensional curvature, all we need in order to recover the familiar case of \((3+1)\)-dimensional radiation domination is to set \( w_a = 1/3 \). However, this forces us to allow for different pressures in ordinary space and the UED, since according to Eq. (14) \( w_b \) must then be close to zero. That is, the extra dimensional pressure (and momentum) must be negligible, and thus the KK tower must be largely unpopulated compared to the 3D radiation in order to reproduce standard cosmology for a radiation-dominated stage with \( w_a \approx \frac{1}{3} \).

We also note that if LKPs do form a substantial part of the dark matter, then from a \((3+n+1)\)-dimensional point of view (ignoring a possible epoch of vacuum energy domination) the universe is always dominated by relativistic particles. This is because any standard model particle with extra-dimensional momentum is automatically relativistic since \( m \ll 1/R \sim 1 \text{ TeV} \). During ordinary radiation domination, on the other hand, the contribution from dark matter is negligible -- but then the dominant part of the energy density is relativistic anyway. Therefore, Eq. (14) is always valid in our model. Of course, we still want \( w_a \approx 0 \) in order to describe what looks like a 3D matter dominated universe, so we have to set

\[ w_b \approx \frac{1}{n} \]

in that case.

On the other hand, for a time-dependent scale factor \( b \), Eqs. (9a) and (9b) can still be cast in the standard cosmological form by absorbing all terms containing factors of \( \dot{b}/b \) and its derivatives into an effective three-dimensional energy density and pressure:

\[
\begin{align*}
\rho^{(3)}_{\text{eff}} &= \frac{M_{Pl}^2}{8\pi} \left\{ \kappa^2 \rho - 3 \frac{\dot{a}}{a} \frac{\dot{b}}{b} - \frac{n(n-1)}{2} \left[ \frac{\dot{b}}{b} \right]^2 + \frac{k_b}{b^2} \right\}, \\
p^{(3)}_{\text{eff}} &= \frac{M_{Pl}^2}{8\pi} \left\{ \kappa^2 w_a \rho + n \frac{\dot{b}}{b} + 2n \frac{\dot{a}}{a} \frac{\dot{b}}{b} + \frac{n(n-1)}{2} \left[ \frac{\dot{b}}{b} \right]^2 + \frac{k_b}{b^2} \right\}.
\end{align*}
\]

Note, however, that the actual three-dimensional energy density does not evolve in the standard manner:

\[
\rho^{(3)} = V(n) \rho \propto a^{-3(1+w_a)} b^{-n w_b},
\]

and there is no reason to expect that \( \rho^{(3)}_{\text{eff}} \) would either, if at the same time we want to keep the standard behavior of \( a \). Finally, an era of effective radiation (matter) domination corresponding to \( a \sim t^{1/2} \) \((a \sim t^{2/3})\) and \( w_{\text{eff}} \sim 1/3 \) \((w_{\text{eff}} \sim 0)\) need not correspond to actual radiation (matter) domination, i.e. \( w_a \gg w_b \) \((w_b \gg w_a)\).

Finally, we would like to mention that according to Eq. (16) one expects the gravitational coupling constant \( M_{Pl}^2 \) to vary with a time-dependent \( V(n) \approx b^n \). Since in the UED model all particles are allowed to propagate in all dimensions, a similar case can be made for other interactions. Therefore, any nonstatic solution for \( b \) must obey the tight observational bounds on the allowed cosmological variation of the gravitational and electromagnetic coupling constants (see, for example, [35, 36, 37] and references therein).

1 For an alternative definition of the effective pressure, see [32].
III. SOLUTIONS WITH STATIC EXTRA DIMENSIONS

Static extra dimensions is the only case considered so far in the UED context and we saw above that there may be severe problems reproducing standard cosmology otherwise. Let us therefore study whether the field equations (9) admit static solutions for \( b \). Taking the difference of twice Eq. (9c) and the sum of Eqs. (9a) and three times (9b) gives

\[
\ddot{b} \frac{b}{b} + 3 \frac{\dot{a}}{a} \frac{b}{b} + (n - 1) \left( \frac{\dot{b}}{b} \right)^2 + (n - 1) \frac{k_b}{b^2} + \frac{3w_a - 2w_b - 1}{n + 2} \kappa^2 \rho = 0. \tag{20}
\]

From this we can immediately read off a necessary condition for exactly static extra dimensions:

\[
(n + 2)(n - 1) \frac{k_b}{b^2} = (1 - 3w_a + 2w_b) \kappa^2 \rho. \tag{21}
\]

If the extra dimensions are flat (for \( n = 1 \), the curvature is automatically zero), this requires the universe to be empty (\( \rho = 0 \)) or the equations of state to satisfy the following constraint:

\[
1 - 3w_a + 2w_b = 0, \tag{22}
\]

which agrees with the five-dimensional models considered in [38, 39]. In both cases, setting \( \dot{b} = \ddot{b} = 0 \) reduces Eqs. (9a) and (9b) to the ordinary Friedmann equations for a (3+1)-dimensional universe and Eq. (9c) to a linear combination of these. The static solutions are therefore consistent with the full set of field equations. The particular combination \( w_a = 1/3, w_b = 0 \) has been found and verified before [23, 26, 40]. For flat extra dimensions, there are thus two ways of getting static solutions – although the case of an empty universe is, of course, not particularly interesting.

If, on the other hand, the extra dimensions are curved, Eq. (21) requires \( \rho \) to be constant for static \( b \). Unless \( a \) is also static, Eq. (10) then implies \( w_a = -1 \). The origin of such an energy density could, for example, be a (3+n+1)-dimensional cosmological constant \( \Lambda \), for which \( \rho = \Lambda / \kappa^2 \) and \( w_a = w_b = -1 \). Setting \( \dot{b} = \ddot{b} = 0 \) then reduces Eqs. (9) to the ordinary Friedmann equations for a (3+1)-dimensional de Sitter universe, with an effective energy density

\[
\rho_{\text{eff}}^{(3)} = \frac{M_p^2}{8 \pi} \left( \frac{\Lambda - n(n - 1) k_b}{2} \right) = \frac{M_p^2 \Lambda}{4 \pi (n + 2)}, \tag{23}
\]

where the second equality follows from Eq. (21). Curved, static extra dimensions are thus possible in principle, but only for constant \( \rho \).

IV. RADIATION DOMINATION

Recent measurements of the cosmic microwave background indicate that the universe is flat to a high degree of accuracy [18]. We therefore set the 3D curvature to zero. Moreover, we have shown that static extra dimensions are incompatible with extra-dimension curvature in the case of non-negative pressure in 3D. Although the latest results from type Ia supernovae observations strongly suggest that the presently dominating energy component indeed does have negative pressure [41], such a component, be it a cosmological constant or a quintessence field, is believed to be negligible until relatively recently and it therefore cannot provide static, curved extra dimensions at earlier times. Thus we take \( k_a = k_b = 0 \) from here on.

Shortly after freeze-out the LKP contribution should be negligible and the total energy density of the universe is therefore dominated by ordinary radiation, i.e., the KK tower is largely unpopulated compared to relativistic particles with no extra-dimensional momentum. Equation (14) then implies that the extra-dimensional pressure should be negligible. Now, combining Eqs. (14) and (22) we find that \( w_a = 1/3, w_b = 0 \) is the only choice for the equations of state in a universe that is dominated by relativistic particles and has exactly static UED:

\[
b(t) = b_i. \tag{24}
\]

\[\text{2 Strictly speaking, this is true only for constant } w_a, \text{ which implies a constant } w_b \text{ according to Eq. (14). From standard cosmology, however, we expect long periods of approximately constant } w_a \text{ (for example, during matter domination).}\]
Furthermore, we have already seen that this choice reproduces standard cosmological radiation domination in that the scale factor and energy density evolve as

\[
a(t) = a_i \left( \frac{t}{t_i} \right)^{\frac{1}{2}}, \tag{25}
\]

\[
\rho(t) = \rho_i \left( \frac{a(t)}{a_i} \right)^{-4}. \tag{26}
\]

Thus in the UED scenario, static extra dimensions arise naturally during radiation domination.

However, from our discussion on pressure, we do not expect \( w_a \) and \( w_b \) to take exactly these values and we must therefore consider perturbations of the static solution. So let us assume that \( 0 \neq w_b \ll 1 \) and look for solutions of the form

\[
a(t) = a_s(t) + \delta_a(t), \tag{27a}
\]

\[
b(t) = b_s + \delta_b(t), \tag{27b}
\]

\[
\rho(t) = \rho_s(t) + \delta_\rho(t), \tag{27c}
\]

where a star denotes the unperturbed solutions \(24\)–\(26\) and \( |\delta_a|/a_s, |\delta_b|/b_s, |\delta_\rho|/\rho_s \ll 1 \), with corresponding relations for the time derivatives of these quantities. The linearized versions of the field equations \(\text{[1]}\) are then given by

\[
\left( \frac{\dot{a_s}}{a_s} \right)^2 + 2 \left( \frac{\dot{a_s}}{a_s} \right) \left( \frac{\ddot{a_s}}{a_s} - \frac{2}{3} \frac{\ddot{a}}{a} \right) + n \frac{\dot{a_s} \dot{b_s}}{a_s b_s} = \frac{\kappa^2}{3} \rho,
\]

\[
2 \frac{\ddot{a_s}}{a_s} \left( 1 + \frac{\ddot{b_s}}{b_s} - \frac{\ddot{a_s}}{a_s} \right) + \left( \frac{\dot{a_s}}{a_s} \right)^2 \left( 1 + 2 \frac{\dot{b_s}}{b_s} - \frac{2}{3} \frac{\dot{a_s}}{a_s} \right) + \frac{\dot{b_s}}{b_s} + 2n \frac{\dot{a_s} \dot{b_s}}{a_s b_s} = -\frac{\kappa^2}{3} (1 - nw) \rho,
\]

\[
\frac{\ddot{a_s}}{a_s} \left( 1 + \frac{\ddot{b_s}}{b_s} - \frac{\ddot{a_s}}{a_s} \right) + \left( \frac{\dot{a_s}}{a_s} \right)^2 \left( 1 + 2 \frac{\dot{b_s}}{b_s} - \frac{2}{3} \frac{\dot{a_s}}{a_s} \right) + \frac{n - 1}{3} \frac{\dot{b_s}}{b_s} + n \left( \frac{\dot{a_s}}{a_s} \right)^2 + \frac{\dot{b_s}^2}{b_s} = -\frac{\kappa^2}{3} w \rho.
\]

Subtracting the unperturbed equations this can be rewritten as

\[
2 \left( \frac{\dot{a_s}}{a_s} \right)^2 \left( \frac{\ddot{a_s}}{a_s} - \frac{2}{3} \frac{\ddot{a}}{a} \right) + \frac{\dot{a_s} \dot{b_s}}{a_s b_s} = \frac{\kappa^2}{3} \delta_\rho,
\]

\[
\frac{\ddot{a_s}}{a_s} \left( \frac{\dot{b_s}}{b_s} - \frac{\dot{a_s}}{a_s} \right) + 2 \left( \frac{\dot{a_s}}{a_s} \right)^2 \left( \frac{\ddot{a_s}}{a_s} - \frac{2}{3} \frac{\ddot{a}}{a} \right) = -\frac{\kappa^2}{3} nw \rho,
\]

\[
\frac{\ddot{b_s}}{b_s} + \frac{\dot{a_s} \dot{b_s}}{a_s b_s} = \kappa^2 w \rho.
\]

Equations \(29b\) and \(29c\) are uncoupled differential equations for \(\delta_a\) and \(\delta_b\) which can be solved after inserting the unperturbed solutions \(24\), \(26\). Equation \(29a\) can be interpreted as defining \(\delta_\rho\) and is thus automatically satisfied.

The general solutions are found to be:

\[
\delta_a(t) = A_1 \left( \frac{t}{t_i} \right)^{1/2} + A_2 \left( \frac{t}{t_i} \right)^{-1/2} - \frac{3}{2} n \kappa^2 \rho_i t_i a_i \left( \frac{t}{t_i} \right)^{1/2} \int_{t_i}^{t} w_b(x)(x^{-1} - t^{-1}) dx,
\]

\[
\delta_b(t) = B_1 + B_2 \left( \frac{t}{t_i} \right)^{-1/2} + 2 \kappa^2 \rho_i t_i b_i \int_{t_i}^{t} w_b(x)(x^{-1} - t^{-1/2} x^{-1/2}) dx,
\]

where \(A_1, A_2, B_1, B_2\) are integration constants fixed by the initial conditions at time \(t_i\). Note that \(A_1\) and \(B_1\) can just as well be regarded as part of the initial conditions for the unperturbed solutions \(a_s\) and \(b_s\) respectively. Thus for \(w_b = 0\) we find only decaying solutions for the perturbations in both scale factors – i.e. the static solution given by Eqs. \(24\), \(26\) is stable under small perturbations, as claimed previously (without proof) in \(\text{[4]}\).
In the case of \( w_b \neq 0 \), however, there might also exist a growing solution. For example, a constant \( w_b \) gives

\[
\delta_a(t) \sim -\frac{1}{3} w_b n \kappa^2 \rho_i t_i^2 a_i \left( \frac{t}{t_i} \right)^{1/2} \ln \left( \frac{t}{t_i} \right),
\]

\[
\delta_b(t) \sim 2 w_b \kappa^2 \rho_i t_i^2 b_i \ln \left( \frac{t}{t_i} \right).
\]

Since the relative perturbations of the scale factors grow only logarithmically in both cases,

\[
\frac{a_i}{|\delta_a|}, \frac{b_i}{|\delta_b|} \propto \ln t,
\]

these perturbations can still be expected to remain relatively small during radiation domination after LKP freeze-out.

\[3\]

V. MATTER DOMINATION

We have shown that there are approximately static solutions for the UED during radiation domination. In fact, we found that \( w_a = 1/3 \) and \( w_b = 0 \) is the only possible choice for the equations of state that can give exactly static, flat extra dimensions in a universe dominated by relativistic particles. However, as we remarked before, the universe is always dominated by relativistic particles if a significant amount of the dark matter is made up of LKPs. It is therefore clear that there are no exactly static solutions for the UED during matter domination, i.e. \( w_a \approx 0 \).

Having excluded exactly static extra dimensions, the next case of interest would be slowly evolving solutions:

\[
\left| \frac{\dot{b}}{b} \right| \ll \left| \frac{\dot{a}}{a} \right|.
\]

But what ansatz should we make for \( a \)? A significant, long-term deviation from the usual time evolution during matter domination would most likely alter the predictions of standard cosmology concerning, e.g., large-scale structure formation. Let us therefore first examine whether there are any solutions to the field equations that give

\[
a(t) = a_\star(t) \equiv a_i \left( \frac{t}{t_i} \right)^{\hat{\gamma}}.
\]

We noted before that \( w_a = 0 \) implies \( w_b = 1/n \). The field equations \( 31 \) can then be rewritten as two homogeneous equations for \( \dot{a} \) and \( \dot{b} \) and one defining equation for \( \rho \). With the above expression for \( a \) we get

\[
\left( \frac{\dot{a}_\star}{a_\star} \right)^2 + n \frac{\dot{a}_\star \dot{b}}{a_\star b} + \frac{n(n-1)}{6} \left( \frac{\dot{b}}{b} \right)^2 = \frac{n^2}{3} \rho,
\]

\[
\frac{\ddot{b}}{b} + 2 \frac{\dot{a}_\star \dot{b}}{a_\star b} + \frac{n-1}{2} \left( \frac{\dot{b}}{b} \right)^2 = 0,
\]

\[
\frac{3 \dot{a}_\star}{2 n a_\star} + \frac{\dot{b}}{b} = 0.
\]

Differentiating Eq. \( 35 \), one finds that the last two equations are inconsistent with each other for \( n \neq -3 \), so there are no solutions with \( a(t) = a_\star(t) \) and \( w_a = 0 \). Neither can there be any solutions with \( w_a \approx 0 \) and \( a \) of the form

\[
a(t) = a_\star(t) + \delta_a(t),
\]

where \( |\delta_a|/a_\star, |\dot{\delta}_a|/\dot{a}_\star, |\ddot{\delta}_a/\ddot{a}_\star| \ll 1 \). This is because, to zeroth order, inserting such an ansatz would leave the unperturbed equations unchanged. Of course, if we allow for rapidly oscillating \( a(t) \) then \( \delta_a \) and \( \dot{\delta}_a \) need not be small. We do not consider such behavior here.

\[3\] A more realistic time dependence of \( w_b \) is considered in Sec. \( 34 \) where we find a more rapid growth of \( \delta_b \).
So are there any solutions to the field equations at all during matter domination? With the quite general ansatz

\[ a(t) = a_i \left( \frac{t}{t_i} \right)^x, \]

we find that all solutions are of the form

\[ b(t) = b_i \left( \frac{t + B}{t_i + B} \right)^y. \]

Some of these, namely

\[
\begin{cases}
  x = \frac{3 \pm \sqrt{3n(n+2)}}{3(n+3)}, \\
  y = \frac{n \pm \sqrt{5n(n+2)}}{n(n+3)}, \\
  B \equiv 0,
\end{cases}
\]

are vacuum solutions. In fact, they are known as Kasner-type solutions and have been found before under the assumption of an empty universe and a power-law behaviour of both scale factors \cite{42}. The only additional solutions appear when \( n \neq 1 \) and are given by

\[
\begin{cases}
  x = 0, \\
  y = \frac{2}{n+1},
\end{cases}
\]

with \( B \) being an arbitrary integration constant. Although they describe a nonempty universe they have a static scale factor \( a \). Of course, for \( n = 3 \) we knew of this solution beforehand, since \( w_a = 0 \) and \( w_b = \frac{1}{3} \) is the static, radiation dominated solution with \( a \) and \( b \) interchanged.

With no suitable solutions describing actual matter domination (\( w_a \ll w_b \)), we turn to the possibility of an era of effective matter domination, i.e., \( p^{(3)}_{\text{eff}} = 0 \). Inserting the defining Eqs. (18) into the field equations (9) then yields the familiar result \( a \propto t^{2/3} \) by construction. Using Eq. (14) we can write the remaining equations as

\[
\begin{align}
  \frac{\dot{b}}{b} &= -\frac{2}{3nt^2} - \frac{2}{7} \left( \frac{\dot{b}}{b} \right) - \frac{n-1}{2} \left( \frac{\dot{b}}{b} \right)^2, \quad (41a) \\
  w_a &= \frac{4(1 + nt \dot{b} / b)}{8 + 12nt \dot{b} / b + 3n(n-1)t^2 \left( \frac{\dot{b}}{b} \right)^2}, \quad (41b) \\
  \kappa^2 \rho &= \frac{4}{3t^2} + \frac{2n \dot{b}}{t \dot{b}} + \frac{n(n-1)}{2} \left( \frac{\dot{b}}{b} \right)^2. \quad (41c)
\end{align}
\]

Equation (41a) has the general solution

\[ b = B_1 \left( \frac{t}{t_i} \right)^{-1} \cos^2 \left( \sqrt{\frac{n+4}{12n}} \ln \frac{t}{t_i} + B_2 \right)^{1/(n+1)}, \]

where \( B_1 \) and \( B_2 \) are integration constants to be fixed by the initial conditions. This corresponds to decaying, bouncing extra dimensions. Although such a behavior of \( b \) might be possible, the corresponding evolution of \( w_a \) is not – in fact it is singular. Indeed, in our model there is no physical motivation for why \( p^{(3)}_{\text{eff}} \) should vanish for nonstatic \( b \), and it is therefore no surprise that we get unphysical solutions from imposing it.

VI. TRANSITION PERIOD

So far we have focused on the two extreme cases of having zero pressure in either the UED (radiation domination) or 3D (matter domination). In a final attempt to find solutions which resemble the standard cosmological evolution of \( a \) (in particular during matter domination), we will now make a more general numerical study of the transition from an era of radiation domination with approximately static UED to one with a sizable energy density contribution.
from LKPs. In order to do this we make the approximation that the LKPs have only extra-dimensional momentum, which should be valid for temperatures below $\sim 1$ TeV. This allows us to split the energy density and pressure into two parts:

\begin{align}
\rho &= \rho_r + \rho_m, \\
p_a &= w^r_r \rho_r + w^m_m \rho_m = \frac{\rho_r}{3}, \\
p_b &= w^r_r \rho_r + w^m_m \rho_m = \frac{\rho_m}{n},
\end{align}

where $r$ and $m$ denote ordinary particles (radiation) and LKPs (matter), respectively. Neglecting interactions, the energy-momentum is separately conserved and Eq. (10) gives:

\begin{align}
\rho_r &= \rho_{r,i} \left( \frac{a}{a_i} \right)^{-4} \left( \frac{b}{b_i} \right)^{-n}, \\
\rho_m &= \rho_{m,i} \left( \frac{a}{a_i} \right)^{-3} \left( \frac{b}{b_i} \right)^{-(n+1)}.
\end{align}

Now introduce the dimensionless variable $t' \equiv t/t_i$ and rescale $a \rightarrow \frac{a}{a_i}, \ b \rightarrow \frac{b}{b_i}$.

Using a prime to denote differentiation with respect to $t'$, Eqs. (48) become

\begin{align}
2a'' + \left( \frac{a'}{a} \right)^2 + \frac{b''}{b} + 2n \frac{a' b'}{a b} + \frac{n(n-1)}{2} \left( \frac{b'}{b} \right)^2 &= -\frac{\kappa^2 t_i^2 \rho_{r,i}}{3a^3 b^n}, \\
3a'' + 3 \left( \frac{a'}{a} \right)^2 + (n-1) \frac{b''}{b} + 3(n-1) \frac{a' b'}{a b} + \frac{(n-1)(n-2)}{2} \left( \frac{b'}{b} \right)^2 &= -\frac{\kappa^2 t_i^2 \rho_{r,i}}{n a^3 b^{n+1}},
\end{align}

where $\epsilon \equiv \rho_{m,i}/\rho_{r,i}$, and from Eq. (50a) we get

\begin{equation}
\kappa^2 t_i^2 \rho_{r,i} = \frac{3}{1 + \epsilon} \left[ \left( \frac{a'}{a} \right)^2 + n \left( \frac{a' b'}{a b} \right) + \frac{n(n-1)}{6} \left( \frac{b'}{b} \right)^2 \right] \bigg|_{t' = 1}.
\end{equation}

Starting from the solutions (27), (30) for radiation domination and approximately static extra dimensions, the appropriate initial conditions are given by

\begin{align}
a(1) &= 1, \quad a'(1) = \frac{1}{2} - \frac{n}{3} \epsilon, \\
b(1) &= 1, \quad b'(1) = \frac{3}{4} \epsilon, \\
\epsilon &\ll 1.
\end{align}

Here, we keep track of terms linear in $\epsilon$ in order to be consistent with the expected behavior of slowly growing extra dimensions (as opposed to the case of exactly static extra dimensions that would result from $\epsilon \equiv 0$). Of course, one could in principle imagine different initial conditions, but the ones chosen above correspond naturally to the setup presented here and in Sec. [IV].

The numerical solutions to Eqs. (48) are plotted in Figs. 1 and 2 for different numbers of extra dimensions and values of $\epsilon$. In the beginning, we find the behavior expected from our discussion in Sec. [IV] - very slowly growing extra dimensions and an expansion of 3D that corresponds to the usual radiation domination $a \propto t_i^{3/2}$. However, as soon as the LKPs make up roughly 10% of the total energy density of the universe, the extra dimensions start to expand at a rate comparable to $a$. Such a rapid expansion of the UED is already ruled out by present bounds on the

\begin{footnote}
With the explicit time dependence of $w_b$ given by Eq. (10), we find that $\delta_b$ grows as $t_i^{1/2}$, which is much faster than logarithmically. The radiation dominated era is therefore shorter than expected from Eqs. (9), although the qualitative behavior remains the same.
\end{footnote}
FIG. 1: The evolution of the scale factors $a$ (dashed) and $b$ (dash-dotted), as well as the LKP energy density $\rho_m/\rho$ (solid) for $n=1$ (thin), 2 (medium), and 7 (thick) with $\epsilon \equiv \rho_m/\rho_r = 10^{-7}$. For $\rho_m/\rho \lesssim 0.1$ the extra dimensions are nearly static and the evolution of $a$ reproduces the radiation dominated regime of standard cosmology to a very good approximation. For $\rho_m/\rho > 0.1$ however, neither $a$ nor $b$ show the desired behaviour.

The behavior of the scale factors as described above is stable against perturbations in the initial conditions of $a'$ and $b'$ of the order of $\epsilon$. Allowing for even larger perturbations, the only qualitatively different behavior we find is collapsing $b$ and collapsing or inflating $a$ - these solutions are obviously not viable alternatives either.

VII. DISCUSSION AND CONCLUSIONS

The identity of dark matter is one of the most challenging puzzles in modern particle physics and cosmology. Recently, it has been noted that models with universal extra dimensions provide a natural WIMP candidate that could make up a significant amount of the dark matter today. This subsequently led to a great deal of interest in studying the detection properties of these particles. However, the estimates for today’s LKP abundance depend crucially on the underlying assumption of static extra dimensions and a standard cosmological evolution history of the universe since the time of freeze-out of these particles.

In this article we have studied in detail whether one can expect such behavior without adding an explicit stabilization mechanism. To this end we have analyzed cosmological solutions to Einstein’s field equations in $(3+n+1)$ dimensions that are appropriate to describe a universe with UED. More specifically, our setup is given by a homogeneous FRW metric with two different scale factors and the assumption that LPKs make up the dominant part of the dark matter, i.e., the universe is always dominated by relativistic particles.

We find that a natural – and in fact the only – way to get exactly static extra dimensions in this scenario is to set $w_a = \frac{1}{3}$ and $w_b = 0$. This also reproduces the usual radiation-dominated behavior of the scale factor in 3D. Allowing for $0 \neq w_b \ll 1$, which is much more realistic in the UED scenario, we still find approximately constant extra dimensions and $a \sim t^{1/2}$.

However, during matter domination ($w_a = 0$) there are no static solutions for the extra dimensions. Even worse, there are no solutions at all that are consistent with the standard matter dominated behavior of the scale factor in
3D, \( a(t) \propto t^{2/3} \). With a more general ansatz \( a(t) \propto t^x \) we do find solutions, but for \( x \neq 0 \) they all describe an empty universe. Demanding \( w_{\text{eff}} = 0 \) instead of \( w_a = 0 \) we get the usual behavior of \( a(t) \) by construction, but the corresponding solutions for \( b(t) \) and \( \rho(t) \) are unphysical.

In the reasonable approximation that the LKPs have only extra-dimensional momentum, we have also performed a numerical analysis of the transition from a radiation dominated universe with approximately static UED to one with a sizable energy density contribution from LKPs. The evolution of \( b \) is generically found to be much too rapid given the present bounds on the time variation of the electromagnetic and gravitational coupling constants, and \( a \) does not show the standard behavior either.

To summarize, we have shown that, within our framework, an explicit mechanism is needed not only to stabilize the UED but also to reproduce standard cosmology in 3D during matter domination. Although one could consider more complex models, e.g., with different scale factors for each extra dimension, we believe that finding static solutions – or indeed any solutions which give standard cosmology for both radiation and matter domination without obviously violating experimental bounds on the evolution of the extra dimensions – is a generic difficulty of this scenario.

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