ON THE DIFFRACTION MEASURE OF \(k\)-FREE INTEGERS

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Abstract. We show that the scaling of the total diffraction intensity for the set of \(k\)-free integers near the origin is asymptotically approximated by a power law. This considerably improves on recent work by Baake and Coons whose main result provides upper and lower bounds for the same quantity. Our method is different and relies on introducing an auxiliary variable that enables the use of results concerning the zero-free region of the Riemann zeta function.

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References

1. Introduction

The theory of aperiodic order has been a rapidly expanding area of mathematics in recent years \[3\]. In this paper we are concerned with the intersection of this theory with number theory. The \(k\)-free integers have received much attention in this area as they are weak model sets with maximal density which display pure point diffraction, see \[5\] (c.f. the following subsection for an introduction to some of these ideas). In \[1\], Baake and Coons studied the fluctuation of the density of this set by considering the scaling behaviour of the diffraction measure \(\nu_k\), given by

\[
Z_k(\varepsilon) = \frac{\nu_k((0, \varepsilon])}{\nu_k(\{0\})}\frac{\zeta(2)}{(k-1)\zeta(2)}(1 + o(1)), \quad (0 < \varepsilon \leq 1),
\]

where \(\zeta\) denotes the Riemann zeta function and every \(o(1)\)-term is taken as \(\varepsilon \to 0^+\). Baake and Coons’ main result \[1\], Theorem 1.1 reads that for each fixed \(k \geq 2\) we have

\[
\frac{\varepsilon^{2 - \frac{1}{k} + o(1)}}{(k-1)\zeta(2)}(1 + o(1)) \leq Z_k(\varepsilon) \leq \frac{\varepsilon^{2 - \frac{1}{k} - o(1)}}{(k-1)\zeta(2)}(1 + o(1)), \quad (0 < \varepsilon \leq 1),
\]

(1.1)

where \(\zeta\) denotes the Riemann zeta function and every \(o(1)\)-term is taken as \(\varepsilon \to 0^+\). This is equivalent to finding asymptotics for \(\log Z_k(\varepsilon)\) rather than \(Z_k(\varepsilon)\) itself. For example, (1.1) does not rule out situations like

\[
Z_k(\varepsilon) = \beta_k\varepsilon^{2 - \frac{1}{k}} + O_k\left(\varepsilon^{2 - \frac{1}{k}} \exp\left(-\gamma' k^{-2} \left(\log \frac{1}{\varepsilon}\right)^{3/5} \left(\log \log \frac{1}{\varepsilon}\right)^{-1/5}\right)\right), \quad (0 < \varepsilon \leq 1),
\]

for instance. Our main result determines the precise asymptotic behavior of \(Z_k(\varepsilon)\).

Theorem 1.1. Fix \(k \in \mathbb{Z}\) with \(k \geq 2\). There exists a positive absolute constant \(\gamma'\) such that

\[
Z_k(\varepsilon) = \beta_k\varepsilon^{2 - \frac{1}{k}} + O_k\left(\varepsilon^{2 - \frac{1}{k}} \exp\left(-\gamma' k^{-2} \left(\log \frac{1}{\varepsilon}\right)^{3/5} \left(\log \log \frac{1}{\varepsilon}\right)^{-1/5}\right)\right), \quad (0 < \varepsilon \leq 1),
\]

where the implied constant depends at most on \(k\). The constant \(\beta_k\) is defined by

\[
\frac{\zeta(2 - 1/k)}{\zeta(2)(2k-1)} \prod_p \left(1 - \frac{1}{(p^k - 1)^2}\right) \left(1 + \frac{2}{(p+1)(p^k-2)}\right) \left(1 + \frac{I_{[3,\infty)}(k)p^{-3+\frac{1}{k}}}{(1 - 2p^{-k})(1 + \frac{1}{p} + \frac{2}{p(p^k-2)})}\right),
\]

where \(I_{[3,\infty)}\) is the indicator function of the interval \([3, \infty)\) and the product is over all primes \(p\).
This result replaces the $o(1)$ in the exponent of $\varepsilon$ in (1.1) by 0 and the remaining $o(1)$ terms by an error term that exhibits polynomial decay. In particular, the diffraction measure of the $k$-free integers is provably approximated by a power law. Furthermore, our result shows that an asymptotic equality with leading constant $\frac{1}{(k-1)\zeta(2)}$ does not hold, but one with $\beta_k$ does. For example, the approximate value of $\beta_2$ is 0.6756\ldots, while the approximate value of $\frac{1}{\zeta(2)}$ is 0.6079\ldots.

1.1. Method of proofs and contents of the paper. Our method is different from the one used by Baake and Coons. We approximate $Z_k(\varepsilon)$ by $Z_k(1/N)$ for certain appropriate integers $N$ in Lemma 2.1. We then observe that the auxiliary variable $N$ divides certain integers, which helps to transform $Z_k(1/N)$ into an infinite sum of the number-theoretic quantities $z_k(c)$, see Lemma 2.2. These quantities are more amenable to methods in analytic number theory than $Z_k(\varepsilon)$ itself, see Lemma 2.4 and Proposition 2.5, where these quantities are estimated. By taking the validity of Proposition 2.5 for granted we prove Theorem 1.1 at the end of §2.1. The content in §2.1 is the least standard part of our paper.

The proof of Proposition 2.5 is given in §2.2 and it uses a result of Walfisz [8] regarding the distribution of square-free numbers. The main input in the proof of Walfisz is the zero-free region of the Riemann zeta function.

The constant $\beta_k$ asymptotically behaves as $\beta\infty/k$ when $k \to +\infty$, where $\beta\infty$ is an absolute constant that is given by

$$\beta\infty := \frac{1}{2} \prod_p \left( 1 + \frac{1}{(p + 1)(p^2 - 1)} \right) = 0.579\ldots,$$

see Proposition 3.1. It would be interesting to explain the value of $\beta\infty$ directly from the theory of aperiodic order. The proof of Proposition 3.1 is contained in §3.

Our approach of using an auxiliary variable firmly establishes that the approximation of the diffraction intensity by a power law is connected to the Riemann hypothesis. Indeed, in §4 we show that the Riemann hypothesis implies that these approximate each other faster than is shown in Theorem 1.1. We are not aware of a previous connection between the Riemann hypothesis and aperiodic structures.

1.2. Mathematical diffraction for number theorists. Diffraction refers to the physical phenomenon of a wave passing through an aperture or around an obstacle. The interference caused by this process creates a pattern, one might hope that properties of this pattern could be used to deduce something about the properties of the aperture through which the wave passed. An important example of this is X-ray diffraction. An X-ray is passed through a molecule creating a diffraction pattern and one hopes to deduce something about the atomic structure of the molecule. Typically one expects that highly ordered patterns, i.e. very clearly defined and distinct peaks and troughs, result from highly ordered atomic structures and vice versa. It has long been known that crystals give rise to such so-called pure point diffraction spectra, however it has recently been shown that such ordered patterns can be created by quasicrystals, which in particular exhibit an aperiodic molecular structure.

We can model this situation mathematically by viewing the atomic structure as a point set and using the wave equation to try to predict the diffraction pattern. However, in practice, it might be difficult to compute an analytic solution. To rectify this, one often appeals to Fraunhofer’s far field limit which is the physically observed principle that the diffraction image is often well predicted by the Fourier transform of the indicator function for the aperture. The intensity of a peak then corresponds to the modulus squared of the Fourier transform. When mathematically modelling obstructions given by continuous distributions it is natural to replace the indicator function of the aperture with some measure. In order to investigate the prediction of the far field approximation, it then becomes necessary to develop a theory of Fourier transforms of measures associated to point
sets. We provide a quick intro to this theory here, leaning heavily on [2] and [3] to which the curious reader is referred for details.

If $S$ is a uniformly discrete subset of $\mathbb{R}^n$ then one can associate to it a Dirac comb,

$$\omega_S = \sum_{x \in S} \delta_x,$$

where $\delta_x$ denotes the Dirac delta function at $x$. Associated to this Dirac comb is a tempered distribution, called the auto-corellation measure given by

$$\gamma_\omega = \lim_{R \to \infty} \frac{1}{\text{vol}(B_R(0))} \sum_{z \in S - S} \sharp\{(y, y') \in (S \cap B_R(0))^2 : y - y' = z\} \delta_z,$$

where $B_R(0)$ is the ball of radius $R$ with centre $0$. Since this distribution is tempered it makes sense to take its Fourier transform and this quantity, $\nu := \hat{\gamma_\omega}$, is the diffraction measure, which gives us the diffraction pattern in which we are interested.

A pure point of a measure $\nu$ is a point $x \in \mathbb{R}^n$ such that $\nu(\{x\}) \neq 0$. Since $\mu$ is both a tempered distribution and a positive measure it follows that it can be decomposed into its pure point and continuous parts. This means we may write

$$\nu = \nu_{pp} + \nu_c,$$

where $\nu_{pp}$ is supported on pure points. A set $S$ displays pure point diffraction if $\nu_c = 0$.

Most currently known examples of aperiodic point sets which give rise to pure point diffraction spectra have come from Meyer sets, point sets $S$ which are uniformly discrete and relatively dense and whose difference sets $S - S$ also satisfy these properties, or sets closely related to these. However two examples of point sets with pure point diffraction which vary significantly from this setting were recently given by Baake–Moody–Pleasants [2]. These sets are natural objects in number theory: the set of primitive (or visible) points in a lattice and the set of $k$-free integers. Both sets contain arbitrarily large holes and so are uniformly discrete but not relatively dense. In this note we investigate the latter set which following [1] we denote $V_k$.

It was proven in [2] that the diffraction measure associated to $V_k$ is given by

$$\nu_k = \sum_{z \in L_k} I_k(z) \delta_z,$$

where $L_k = \{\frac{m}{q} \in \mathbb{Q} : \gcd(m, q) = 1 \text{ and } q \text{ is } k\text{-free}\}$ and

$$I_k\left(\frac{m}{q}\right) = \left(\frac{1}{\zeta(k)} \prod_{p|q} \frac{1}{p^k - 1}\right)^2.$$

Now that the diffraction pattern is known, Baake and Coons studied in [1] the scaling behaviour given by

$$Z_k(\varepsilon) := \frac{\nu_k([0, \varepsilon])}{\nu_k([0])} = \sum_{q \gg 1/\varepsilon} \sum_{1 \leq m \leq q\varepsilon \text{ \ and } \gcd(m, q) = 1} \prod_{p|q} \frac{1}{(p^k - 1)^2}.$$

The motivation behind this is that studying the diffraction measure as $\varepsilon \to 0^+$ yields insight into fluctuations of the structure of $S$. In particular, for periodic sets the scaling function becomes 0 for sufficiently small $\varepsilon$, but for aperiodic sets this often decays according to some power law, as seen in Theorem [11]. Lastly, it must be mentioned that scaling questions as the one in the present paper are currently heavily investigated by several researchers in physics and applied mathematics, see the recent work of Baake and Grimm [4] and the references they provide.
Notation. All implied constants in the Landau/Vinogradov $O()$, $\ll$ are absolute. Any further dependence on a further quantity $h$ will be recorded by the use of a subscript $O_h()$, $\ll_h$. The number of positive integer divisors of an integer $n$ is denoted by $\tau(n)$, the Möbius function by $\mu(n)$ and the indicator function of the $k$-free integers $n$ by $\mu_k(n)$.

Acknowledgements. We thank Michael Baake and Michael Coons for helpful comments that helped improve the exposition of this work.

2. The proof of Theorem 1.1

2.1. The auxiliary variable. Throughout this paper $k$ denotes any integer with $k \geq 2$. For any $N \in \mathbb{N}$ we let

$$\tilde{Z}_k(N) := \sum_{q \in \mathbb{N}} \mu_{k+1}(q) \left( \prod_{p \text{ prime}, p|q} \frac{1}{p^{k-1} - 1} \right) \left\lfloor \frac{1}{m} \right\rceil_{m \in \mathbb{N}, \frac{q}{N}} : \gcd(m, q) = 1 \right\}.$$  \hspace{1cm} (2.1)

where for any $t \in \mathbb{N}$ the function $\mu_t : \mathbb{N} \to \{0, 1\}$ is the indicator function of the $t$-free integers. The function $\tilde{Z}_k(N)$ is well-defined because its modulus is at most

$$\sum_{q \in \mathbb{N}} \mu_{k+1}(q) \left( \prod_{p \text{ prime}, p|q} \frac{1}{p^{k-1} - 1} \right) q \leq \prod_{p \text{ prime}} \left( 1 + \sum_{n=1}^{k} \frac{p^n}{(p^k - 1)^2} \right) \leq \prod_{p \text{ prime}} \left( 1 + \frac{k}{p^k - 2} \right) < \infty. \hspace{1cm} (2.2)$$

Lemma 2.1. For any $\varepsilon \in (0, 1)$ let $N$ be the integer part of $1/\varepsilon$. Then $\tilde{Z}_k(N+1) \leq Z_k(\varepsilon) \leq \tilde{Z}_k(N)$.

Proof. As explained in [1, §2] the main result in the work of Baake, Moody and Pleasants [2] can be used to show that

$$Z_k(\varepsilon) = \sum_{q \geq 1/\varepsilon} \sum_{1 \leq m \leq q \varepsilon} \prod_{p|q} \frac{1}{p^{k-1} - 1}.$$  \hspace{1cm} (2.3)

The condition $q \geq 1/\varepsilon$ is implied by the presence of the sum over $m$ and it can therefore be omitted. The inequality $N \leq \frac{1}{\varepsilon} < N + 1$ shows that

$$\tilde{Z}_k(N+1) = \sum_{q=1}^{\infty} \sum_{1 \leq m \leq q/(N+1)} \prod_{p|q} \frac{1}{p^{k-1} - 1} \leq Z_k(\varepsilon) \leq \sum_{q=1}^{\infty} \sum_{1 \leq m \leq q/N} \prod_{p|q} \frac{1}{p^{k-1} - 1} = \tilde{Z}_k(N). \hspace{1cm} \square$$

Lemma 2.2. For any positive integer $N$ we have

$$\tilde{Z}_k(N) = \sum_{\text{N divides } c} z_k(c), \hspace{1cm} (2.3)$$

where

$$z_k(c) := \sum_{d \in \mathbb{N}} \sum_{c \mid d, c \geq c} \mu(d) \mu_{k+1}(dr) \prod_{p|dr} \frac{1}{p^{k-1} - 1}.$$  \hspace{1cm} (2.3)

Proof. The changes in the order of summation in the following arguments are justified by the absolute convergence of the sum in (2.1), which is proved in (2.2). It is a standard fact that the expression

$$\sum_{d \in \mathbb{N}} \mu(d)$$

is a standard fact that the expression
This transforms $z_k$ of a positive integer. It would be interesting to check whether such a plan can work when the set $\delta, r$ of unique integers $r/r$ is replaced by a set whose associated function $Z$ is the indicator function of the event $\gcd(m,q) = 1$. Injecting it into (2.1) yields

$$\tilde{Z}_k(N) = \sum_{d \in \mathbb{N}} \mu(d) \sum_{q \in \mathbb{N}} \mu_{k+1}(q) \left[ \frac{q}{dN} \prod_{p|q} \frac{1}{(p^k - 1)^2} \right],$$

where $[x]$ denotes the integer part of a real number $x$. The integers $q$ appearing above are of the form $dq$ for some $r \in \mathbb{N}$, hence,

$$\tilde{Z}_k(N) = \sum_{d \in \mathbb{N}} \mu(d) \sum_{r \in \mathbb{N}} \mu_{k+1}(dr) \left[ \frac{r}{N} \prod_{p|dr} \frac{1}{(p^k - 1)^2} \right].$$

We now replace the term $[r/N]$ by $\mathbb{Z}\{ c \in \mathbb{N} \cap [1,r] : c \equiv 0 \pmod{N} \}$, thus obtaining

$$\tilde{Z}_k(N) = \sum_{c \in \mathbb{N}} \sum_{r \in \mathbb{N}} \sum_{d \in \mathbb{N}} \mu(d) \mu_{k+1}(dr) \prod_{p|dr} \frac{1}{(p^k - 1)^2}. \quad \square$$

**Remark 2.3.** Our goal is to find asymptotic expressions for $z_k(c)$ and then inject them into (2.3). It is clear that this rests crucially on the presence of the integer variable $N$ in (2.3). It will succeed because we need to estimate $Z_k(\varepsilon)$ only for special values of $\varepsilon$, namely when $\varepsilon$ is the inverse of a positive integer. It would be interesting to check whether such a plan can work when the set of $k$-free integers is replaced by a set whose associated function $Z(\varepsilon)$ oscillates more wildly than $Z_k(\varepsilon)$.

We next express $z_k(c)$ via an average of an arithmetic function over the integers exceeding $c$.

**Lemma 2.4.** For any positive integer $c$ we have

$$z_k(c) = \xi_k \sum_{m \in \mathbb{N} \cap (c,\infty)} \lambda_k(m),$$

where $\xi_k := \prod_p (1 - (p^k - 1)^{-2})$ and

$$\lambda_k(m) := \sum_{(\delta, r_0, r_2, \ldots, r_k) \in \mathbb{N}^{1+k}} \mu(\delta r_0 r_2 \ldots r_k)^2 \mu(\delta) \prod_{p|m} \frac{1}{(p^k - 1)^2 - 1}.$$

**Proof.** The integers $r$ in the definition of $z_k(c)$ are $(k+1)$-free, hence can be written uniquely as

$$r = \prod_{i=1}^{k} r_i^i,$$

where $r_i \in \mathbb{N}$ are square-free and coprime in pairs. The integer $d$ in the definition of $z_k(c)$ is square-free and therefore coprime to $r/r_1$. Therefore, letting $\delta := \gcd(r,d)$ we infer that there are unique integers $\delta, r_0, d_0$ with the properties

$$r_1 = \delta r_0, d = \delta d_0, \mu(\delta d_0 r_0)^2 = 1.$$

This transforms $z_k(c)$ into

$$\sum_{(\delta, r_0, r_2, \ldots, r_k) \in \mathbb{N}^{1+k}} \mu(\delta r_0 r_2 \ldots r_k)^2 \sum_{d_0 \in \mathbb{N}} \frac{\mu(d_0 \delta) \mu_{k+1}}{\gcd(d_0, \delta r_2 \ldots r_k) = 1} \frac{1}{(d_0^2 r_0 \prod_{i=2}^{k} r_i^i)} \prod_{p|d_0 \delta^2 r_0} \frac{1}{(p^k - 1)^2}.$$
Using the fact that all new variables are coprime in pairs allows to rewrite this as

\[ \sum_{(\delta, \gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathbb{N}^{1+k}} \mu(\delta \gamma_1 \gamma_2 \cdots \gamma_k) \mu(\delta) \left( \prod_{p \mid \delta} \frac{1}{(p^k - 1)^2} \right) \sum_{d_0 \in \mathbb{N}} \mu(d_0) \prod_{p \mid d_0} \frac{1}{(p^k - 1)^2}, \]

because \( d_0 \delta \gamma_1 \gamma_2 \cdots \gamma_k \) is \((k+1)\)-free. Finally, the sum over \( d_0 \) is absolutely convergent and equals

\[ \prod_{p \mid \delta \gamma_1 \gamma_2 \cdots \gamma_k} \left( 1 - \frac{1}{(p^k - 1)^2} \right) = \xi_k \prod_{p \mid \delta \gamma_1 \gamma_2 \cdots \gamma_k} \left( 1 - \frac{1}{(p^k - 1)^2} \right)^{-1}. \]

**Proposition 2.5.** There exists a positive absolute constant \( \gamma' \) such that for fixed \( k \in \mathbb{N}, k \geq 2 \) and all \( z \geq 1 \) we have

\[ \sum_{1 \leq m \leq z} m^2 \lambda_k(m) = \gamma_k z^{1/k} + O_k \left( \frac{z^{1/(2k)}}{\exp \left( \gamma' \kappa^{-2} \log \left( \log \log z \right)^{-1/5} \right)} \right), \]

where the implied constant depends at most on \( k \) and

\[ \gamma_k = \frac{1}{\zeta(2)} \prod_p \left( 1 + \frac{2}{(p + 1)(p - 2)} \right) \prod_p \left( 1 + \frac{1}{p^{2k} (1 - 2p^{-k}) (1 + \frac{1}{p} + \frac{2}{p(1 - 2p^{-k})}) \sum_{j=2}^{k-1} p^{2(2-\frac{1}{k})} \right). \]

The proof of Proposition 2.5 will be our main aim in §2.2. We conclude this section by deducing Theorem 1.1 from Proposition 2.5.

**Proof of Theorem 1.1.** Define for all \( z > 0 \),

\[ \Lambda_k(z) := \sum_{1 \leq m \leq z} m^2 \lambda_k(m), \quad (2.4) \]

so that Lemma 2.4 and Abel’s summation formula gives

\[ \frac{z_k(c)}{\xi_k} = -\Lambda_k(c - 1) \frac{1}{(c - 1)^2} + 2 \int_{c-1}^{\infty} \frac{\Lambda_k(z)}{z^3} \, dz. \quad (2.5) \]

By Proposition 2.5 this turns into

\[ \frac{\gamma_k}{(2k - 1)} \frac{1}{(c - 1)^{2 - \frac{1}{k}}} + O_k \left( \frac{(c - 1)^{-2+1/(2k)} + \int_{c-1}^{\infty} z^{-3+1/(2k)} \, dz}{\exp \left( \gamma' \kappa^{-2} \log(c - 1) \left( \log \log(c - 1) \right)^{-1/5} \right)} \right), \]

therefore,

\[ z_k(c) = \frac{\gamma_k \xi_k}{(2k - 1)} \frac{1}{c^{2 - \frac{1}{k}}} + O_k \left( \frac{c^{-2+1/(2k)}}{\exp \left( \gamma' \kappa^{-2} \log(c) \left( \log \log(c) \right)^{-1/5} \right)} \right). \]

Feeding this into (2.3) produces

\[ \bar{Z}_k(N) = \sum_{b \in \mathbb{N}} z_k(Nb) = \frac{\gamma_k \xi_k}{(2k - 1)} \frac{\zeta(2 - \frac{1}{k})}{N^{2 - \frac{1}{k}}} + O_k \left( \frac{N^{-2+1/(2k)}}{\exp \left( \gamma' \kappa^{-2} \left( \log N \right)^{3/5} \left( \log \log N \right)^{-1/5} \right)} \right). \]

Finally, invoking Lemma 2.1 concludes the proof because the inequality \( N \leq \frac{1}{c} < 1 + N \) implies that both \((N + 1)^{-2+1/k}\) and \(N^{-2+1/k}\) are \(c^{2 - \frac{1}{k}} + O_k(c^{3 - \frac{1}{k}})\). \(\square\)
2.2. Using the zero-free region of the Riemann zeta function. In this section we use results regarding the distribution of square-free integers to prove Proposition 2.5. Namely, we use the following result, the proof of which is based on the best known zero-free region for the Riemann zeta function.

Lemma 2.6 (Walfisz, [8]). There exists an absolute constant $\gamma_0 > 0$ such that

$$
\sum_{n \in \mathbb{N} \cap [1, x]} \mu(n)^2 = \frac{x}{\zeta(2)} + O \left( x \frac{1}{2} \exp \left( -\gamma_0 (\log x)^{3/5} (\log \log x)^{-1/5} \right) \right).
$$

In the proof we shall need a version of this result with equally strong error term regarding $x$ and with an explicit dependence of the error term in the parameter $a$.

Corollary 2.7. There exists an absolute constant $\gamma > 0$ such that for every $a \in \mathbb{N}, x \geq 1$ we have

$$
\sum_{n \in \mathbb{N} \cap [1, x]} \mu(n)^2 = \left( \prod_{p \mid a} \left( 1 + \frac{1}{p} \right) \right) \frac{x}{\zeta(2)} + O \left( \tau(a)^3 x \frac{1}{2} \exp \left( -\gamma (\log x)^{3/5} (\log \log x)^{-1/5} \right) \right),
$$

where the implied constant is absolute.

Proof. The Dirichlet series of $\mathbb{1}_{\gcd(a,n)=1}(n)\mu(n)^2$ is

$$
\sum_{\substack{n=1 \\ \gcd(n,a)=1}}^\infty \frac{\mu(n)^2}{n^s} = \prod_p \left( 1 + \frac{1}{p^s} \right) \prod_{p \mid a} \left( 1 + \frac{1}{p^s} \right)^{-1} = \prod_p \left( 1 + \frac{1}{p^s} \right) \prod_{p \mid a} \left( 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{p^{ks}} \right).
$$

This is the product of the Dirichlet series of $\mu(n)^2$ by the Dirichlet series of the multiplicative function $g_a(n)$, where

$$
g_a(n) := \mathbb{1}_{p \mid n \Rightarrow p \mid a}(n)(-1)^{\Omega(n)}
$$

and $\Omega(n)$ is the number of prime divisors of $n$ counted with multiplicity. We get

$$
\mathbb{1}_{\gcd(a,n)=1}(n)\mu(n)^2 = (\mu^2 * g_a)(n) = \sum_{c,d \in \mathbb{N} \atop \gcd(c,d)=n} g_a(c)\mu(d)^2,
$$

where $*$ is the Dirichlet convolution. Hence, we can write

$$
\sum_{n \in \mathbb{N} \cap [1, x]} \mu(n)^2 = \sum_{1 \leq c \leq x} g_a(c) \sum_{1 \leq d \leq x/c} \mu(d)^2.
$$

Let $Y := x^{7/10}$. The terms with $Y < c \leq x$ contribute at most

$$
x \sum_{c > Y} \frac{|g_a(c)|}{c} \leq \frac{x}{Y^{3/4}} \sum_{c > Y} \frac{|g_a(c)|}{c^{1/4}} \leq \frac{x}{Y^{3/4}} \prod_{p \mid a} \left( 1 + \sum_{k=1}^{\infty} \frac{1}{p^{k/4}} \right) \leq \frac{x}{Y^{3/4}} \left( \prod_{p \mid a} s_p \right) \leq \frac{x \tau(a)^3}{Y^{3/4}},
$$

which equals $\tau(a)^3 x^{10/58}$. By Lemma 2.6 the terms with $c \leq Y$ contribute

$$
\frac{x}{\zeta(2)} \sum_{1 \leq c \leq Y} \frac{g_a(c)}{c} + O \left( x \frac{1}{2} \sum_{1 \leq c \leq Y} \frac{|g_a(c)|}{c^2} \exp \left( -\gamma_0 (\log x/c)^{3/5} (\log \log x/c)^{-1/5} \right) \right).
$$

Note that $x/c \geq x^{3/10}$, therefore

$$
(\log x/c)^{3/5} (\log \log x/c)^{-1/5} \geq 5^{-3/5} (\log x)^{3/5} (\log \log x/c)^{-1/5}.
$$
Letting $\gamma := 5^{-3/5} \gamma_0$ we infer that the error term contribution is
\[
\ll x^{\frac{1}{4}} \exp \left( -\gamma (\log x)^{3/5} (\log \log x)^{-1/5} \right) \sum_{1 \leq c \leq Y} \frac{|g_a(c)|}{c^2} \ll \tau(a)^3 x^{\frac{1}{10}} \exp \left( -\gamma (\log x)^{3/5} (\log \log x)^{-1/5} \right).
\]
To complete the summation over $c > Y$ note that
\[
x \sum_{c > Y} \frac{g_a(c)}{c} \ll \tau(a)^3 x^{\frac{1}{10}}.
\]
can be proved as earlier in this proof. Finally, the proof is concluded by noting that
\[
\sum_{c \in \mathbb{N}} \frac{g_a(c)}{c} = \prod_{p \mid a} \left( 1 + \frac{1}{p} \right)^{-1}.
\]
\[
\Box
\]

The following result is a generalisation of Lemma 2.6 and its proof uses Corollary 2.7.

**Lemma 2.8.** There exists a positive absolute constant $\gamma'$ such that the following holds: for each multiplicative function $\delta : \mathbb{N} \to \mathbb{R}$ with $|\delta(p)| \leq \frac{1}{p}$ for every prime $p$, all $u \geq 1$ and $\ell \in \mathbb{N}$ we have
\[
\sum_{1 \leq m \leq u} \mu(m)^2 \sum_{d \mid m} \delta(d) = \frac{u}{\zeta(2)} \left( \prod_p \left( 1 + \frac{\delta(p)}{p + 1} \right) \right) \left( \prod_{p \mid \ell} \frac{p}{p + 1 + \delta(p)} \right) + O \left( \frac{\tau(\ell)^3 u^{1/2}}{\exp \left( \gamma' (\log u)^{3/5} (\log \log u)^{-1/5} \right)} \right),
\]
where the implied constant is absolute.

**Proof.** Switching the order of summation, the sum in the lemma is
\[
\sum_{d \leq u} \delta(d) \sum_{\substack{1 \leq m \leq u \\text{gcd}(m, \ell) = 1 \\text{gcd}(d, \ell) = 1}} \mu(m)^2 = \sum_{\substack{1 \leq m \leq u \\text{gcd}(m, \ell) = 1 \\text{gcd}(d, \ell) = 1}} \delta(d) \mu(d)^2 \sum_{\substack{1 \leq m' \leq u/d \\text{gcd}(m', \ell) = 1}} \mu(m')^2.
\]
The contribution of $d > u^{3/4}$ is
\[
\ll \sum_{d > u^{3/4}} \delta(d) \mu(d)^2 \frac{u}{d} \leq u \sum_{d > u^{3/4}} \frac{\mu(d)^2}{d^2} \ll u^{1/4},
\]
which is admissible. To the remaining range, $1 \leq d \leq u^{3/4}$, we apply Corollary 2.7 with $x = u/d$ and $a = d\ell$. It gives
\[
\sum_{\substack{d \leq u^{3/4} \\text{gcd}(d, \ell) = 1}} \delta(d) \mu(d)^2 \left( \frac{u}{d\zeta(2)} \left( \prod_{p \mid \ell} \left( 1 + \frac{1}{p} \right)^{-1} \right) + O \left( \frac{\tau(d)^3 \tau(\ell)^3 u^{\frac{1}{2}}}{d^{\frac{1}{2}} \exp \left( \gamma' (\log u)^{3/5} (\log \log u)^{-1/5} \right)} \right) \right)
\]
Using $d \leq u^{3/4}$ we see that $\log \frac{u}{d} \geq \frac{1}{4} \log u$, hence the error term is
\[
\frac{\tau(\ell)^3 u^{\frac{1}{2}}}{\exp \left( \gamma' (\log u)^{3/5} (\log \log u)^{-1/5} \right)} \sum_{d \leq u^{3/4}} \frac{\tau(d)^3}{d^{\frac{1}{2}}} \ll \frac{\tau(\ell)^3 u^{\frac{1}{2}}}{\exp \left( \gamma' (\log u)^{3/5} (\log \log u)^{-1/5} \right)}
\]
from some positive absolute constant \(\gamma'\). To complete the summation in the main term we use the bound \(\tau(d) \ll d^{1/4}\) to obtain
\[
\sum_{d > u^{3/4} \text{gcd}(d,\ell) = 1} \frac{\delta(d) \mu(d)^2}{d} \prod_{p | d} \left(1 + \frac{1}{p}\right)^{-1} \leq \tau(\ell) \sum_{d > u^{3/4}} \frac{\tau(d)}{d^2} \ll \tau(\ell) \sum_{d > u^{3/4}} \frac{1}{d^{1/4}} \ll \tau(\ell) u^{-1/16}. \tag{2.8}
\]

We turn to analysing \(\Lambda_k(z)\). First, observe that \(\lambda_k\) is a multiplicative function, which follows from the definition. Moreover on primes it is in fact 0. Indeed, we have
\[
\lambda_k(p) = \sum_{(\delta_0 \cdot r_2 \cdot \ldots \cdot r_k) \in \mathbb{N}^{1+k}} \mu(\delta_0) r_2 \cdot \ldots \cdot r_k \mu(\delta) \frac{1}{(p^k - 1)^2 - 1} = \sum_{\delta_0 = p} \mu(\delta) \mu^2(\delta_0) \frac{1}{(p^k - 1)^2 - 1} = 0.
\]

Similarly, we find that for all primes \(p\) and positive integers \(e\), we have
\[
\lambda_k(p^e) = \frac{1_{[2,k]}(e)}{(p^k - 1)^2 - 1}, \tag{2.6}
\]
where \(1_{[2,k]}(e)\) is the indicator function of the interval \([2, k]\). Therefore, every \(m\) making a contribution to \(2.4\) is square-full and \((k + 1)\)-free, hence it can be factored in a unique way as
\[
m = m_2^2 m_3^3 \cdots m_k^k
\]
for some positive integers \(m_2, \ldots, m_k\) that are square-free and coprime in pairs. In particular,
\[
\Lambda_k(z) = \sum_{m = (m_2, m_3, \ldots, m_{k-1}) \in \mathbb{N}^{k-2}} \mu(m_2 m_3 \cdots m_{k-1})^2 \Upsilon(z, m) \prod_{j=2}^{k-1} m_j^{\frac{2j - 2k}{p \mid m_j} \prod_{p \mid m_j} \frac{1}{1 - 2p^{-k}}}, \tag{2.7}
\]
where
\[
\Upsilon(z, m) := \sum_{1 \leq m_k \leq (z m_2^2 m_3^3 \cdots m_{k-1}^k)^{1/k}} \frac{\mu(m_k^2 \prod_{p \mid m_k} \frac{1}{1 - 2p^{-k}}}}{\text{gcd}(m_k, m_2 m_3 \cdots m_{k-1}) = 1}.
\]

**Lemma 2.9.** There exists a positive absolute constant \(\gamma'\) such that for all \(m \in \mathbb{N}^{k-2}\) and \(z \geq 1\) as in \(2.7\) we have
\[
\Upsilon(z, m) = \frac{z^{1/k}}{\zeta(2)} \left( \prod_p \left(1 + \frac{2}{(p+1)(p^k-2)}\right) \right) \prod_{j=2}^{k-1} m_j^{\frac{2j - 2k}{p \mid m_j} \prod_{p \mid m_j} \frac{1}{1 - 2p^{-k}}} \right) + O \left( \frac{\tau(m_2^3 \cdots \tau(m_{k-1})^3}{\exp(\gamma' k^{-2}(\log u)^3/5(\log \log u)^{-1/5})} \right),
\]
where the implied constant depends at most on \(k\).

**Proof.** We use Lemma 2.8 with
\[
\delta(p) := -1 + \frac{1}{1 - 2p^{-k}} = \frac{2}{p^k - 2}, \quad \ell := m_2 \cdots m_{k-1} \quad \text{and} \quad u := \left(\frac{z}{m_2^2 m_3^3 \cdots m_{k-1}^k}\right)^{1/k}
\]
to see that \(\Upsilon(z, m)\) equals
\[
\frac{z^{1/k}}{\zeta(2)} \left( \prod_p \left(1 + \frac{2}{(p+1)(p^k-2)}\right) \right) \prod_{j=2}^{k-1} m_j^{\frac{2j - 2k}{p \mid m_j} \prod_{p \mid m_j} \frac{1}{1 - 2p^{-k}}} \right) + O \left( \frac{\tau(m_2^3 \cdots \tau(m_{k-1})^3}{\exp(\gamma' k^{-2}(\log u)^3/5(\log \log u)^{-1/5})} \right).
\]
up to a term whose modulus is
\[ \ll \tau(m_2)^3 \cdots \tau(m_{k-1})^3 z^{3/5} \exp \left( -\gamma' k^{-2} (\log u)^{3/5} (\log \log u)^{-1/5} \right). \]

Here we used that the error term supplied by Lemma 2.8 is a non-decreasing function of \( u \), hence one can use it with \( u = z^{1/k} \) instead of \( u = z^{1/k} m_2^{-2/k} \cdots m_{k-1}^{-(k-1)/k} \).

**Proof of Proposition 2.5.** We inject the asymptotic supplied by Lemma 2.9 into (2.7). The error term contribution is
\[ \ll \frac{z^{1/(2k)}}{\exp \left( \gamma' k^{-2} (\log z)^{3/5} (\log \log z)^{-1/5} \right)} \sum_{m \in \mathbb{N}^{k-2}} \prod_{m_j \leq z} \tau(m_j)^3 m_j^{2j - 2k} \prod_{p | m_j} \frac{1}{1 - 2p^{-k}}. \]

The inequality \( 2j - 2k \leq -2 \) and the bound
\[ \prod_{p | m_j} \frac{1}{1 - 2p^{-k}} \ll \prod_{p} \frac{1}{1 - 2p^{-k}} = O_k(1) \]
show that the sum over \( m_j \) is absolutely convergent, hence the sum over \( m \) is \( O_k(1) \). The main term contribution is \( \eta_k z^{1/k} \mathcal{S}_k(z) \), where
\[ \eta_k := \frac{1}{\zeta(2)} \prod_p \left( 1 + \frac{2}{(p + 1)(p^k - 2)} \right) \]
and \( \mathcal{S}_k(z) \) is given by
\[ \sum_{m \in \mathbb{N}^{k-2}} \mu(m_2 \cdots m_{k-1})^2 \prod_{j=2}^{k-1} m_j^{2j - 2k - j/k} \prod_{p | m_j} \frac{1}{(1 - 2p^{-k})(1 + \frac{1}{p} + \frac{2}{p(p^k - 2)})}. \]

Each exponent \( 2j - 2k - j/k \) in \( \mathcal{S}_k(z) \) is at most \(-3 + \frac{1}{k} \leq -2 \), hence the sum \( \mathcal{S}_k(z) \) is absolutely convergent. To bound its tail we note that there exists an absolute positive constant \( \kappa \) such that for all \( k \in \mathbb{N} \) and primes \( p \) we have
\[ \frac{1}{(1 - 2p^{-k})(1 + \frac{1}{p} + \frac{2}{p(p^k - 2)})} \leq 2^k. \]

Therefore, writing \( m = \prod_{j=2}^{k} m_j^j \) we deduce that the tail is at most
\[ \sum_{m \geq z} \mu_k(m) m^{2 - 1/k} \frac{\tau(m)^k}{\text{rad}(m)^{2k}} \leq \frac{1}{z^{1/k}} \sum_{m \in \mathbb{N}} \mu_k(m) m^{2} \frac{\tau(m)^k}{\text{rad}(m)^{2k}}, \]
where \( \text{rad}(m) \) is the radical of \( m \). We have
\[ \sum_{m \in \mathbb{N}} \frac{\mu_k(m) m^{2} \tau(m)^k}{\text{rad}(m)^{2k}} \leq \prod_p \left( 1 + O_{\kappa,k} \left( \frac{p^{2(k-1)}}{p^{2k}} \right) \right) = \prod_p \left( 1 + O_{\kappa,k} \left( p^{-2} \right) \right) = O_{\kappa,k}(1), \]
where the implied constants depend at most on \( \kappa \) and \( k \). Hence, completing the sum \( \mathcal{S}_k(z) \) introduces an error term that is at most \( O_{\kappa,k}(z^{1/k} z^{-1/k}) = O_{\kappa,k}(1) \), which is admissible. The proof concludes by noticing that
\[ \sum_{m \in \mathbb{N}^{k}} \mu(m_2 \cdots m_{k-1})^2 \prod_{j=2}^{k-1} m_j^{2j - 2k - j/k} \prod_{p | m_j} \frac{1}{(1 - 2p^{-k})(1 + \frac{1}{p} + \frac{2}{p(p^k - 2)})} \]
can be factored as
\[ \prod_p \left( 1 + \frac{p^{-2k} \sum_{j=2}^{k-1} p^{j(2-\frac{j}{k})}}{(1 - 2p^{-k})(1 + \frac{1}{p} + \frac{2}{p^{p-2}})} \right) = \prod_p \left( 1 + \frac{p^{-3+\frac{1}{k}} \sum_{i=0}^{k-3} p^{-i(2-\frac{i}{k})}}{(1 - 2p^{-k})(1 + \frac{1}{p} + \frac{2}{p^{p-2}})} \right). \]

3. Analysis of the leading constant

We analyse the behavior of the constant \( \beta_k \) defined in Theorem 1.1 for large \( k \).

**Proposition 3.1.** The following holds for all \( k \geq 2 \) and with an absolute implied constant,
\[ \beta_k = \frac{\beta_{\infty}}{k} + O \left( \frac{1}{k^2} \right). \]

**Proof.** We begin by showing that
\[ \prod_p \left( 1 - \frac{1}{(p^k - 1)^2} \right) = 1 + O(2^{-2k}) \] 
with an absolute implied constant. To see this, note that \( p^k \geq 2^2 \) allows us to use the approximation \( \log(1-x)^{-1} \ll x \) for \( x = (p^k - 1)^{-2} \) to obtain
\[ 0 \leq \log \prod_p \left( 1 - \frac{1}{(p^k - 1)^2} \right)^{-1} \ll \sum_p p^{-2k} \ll 2^{-2k}, \]
with an absolute implied constant. We infer that
\[ \prod_p \left( 1 - \frac{1}{(p^k - 1)^2} \right)^{-1} = \exp \left( O(2^{-2k}) \right) = 1 + O(2^{-2k}), \]
thereby proving (3.1).

A similar argument, based on the bound
\[ 0 \leq \log \left( 1 - \frac{2}{(p+1)(p^k - 2)} \right) \ll p^{-k} \]
shows that the following holds with an absolute implied constant,
\[ \prod_p \left( 1 + \frac{2}{(p+1)(p^k - 2)} \right) = 1 + O(2^{-k}). \] 

The bound
\[ \left| \frac{p^{-3+\frac{1}{k}} \sum_{i=0}^{k-3} p^{-i(2-\frac{i}{k})}}{(1 - 2p^{-k})(1 + \frac{1}{p} + \frac{2}{p^{p-2}})} \right| \ll \frac{p^{-3+\frac{1}{k}} \sum_{i=0}^{\infty} 2^{-2i}}{(1 - 2^{-1-k})(1 + \frac{1}{2} + \frac{2}{2^2})} \ll p^{-2} \]
holds with an absolute implied constant, hence we have
\[ \prod_{p>k} \left( 1 + \frac{p^{-3+\frac{1}{k}} \sum_{i=0}^{k-3} p^{-i(2-\frac{i}{k})}}{(1 - 2p^{-k})(1 + \frac{1}{p} + \frac{2}{p^{p-2}})} \right) = 1 + O \left( \sum_{p>k} p^{-2} \right) = 1 + O \left( \frac{1}{k} \right). \]

To continue the proof, note that the following holds with an absolute implied constant,
\[ \sum_{i=0}^{k-3} p^{-i(2-\frac{i}{k})} = \frac{1 - p^{-(k-2)(2-\frac{1}{k})}}{1 - p^{-(2-\frac{1}{k})}} = \left( 1 - p^{-2+\frac{1}{k}} \right)^{-1} + O(p^{-2k+5}), \]

hence
\[ \frac{p^{-3+\frac{1}{k}} \sum_{i=0}^{k-3} p^{-i(2-\frac{i}{k})}}{(1 - 2p^{-k})(1 + \frac{1}{p} + \frac{2}{p^{p-2}})} = p^{-3+\frac{1}{k}} \left( \left( 1 - p^{-2+\frac{1}{k}} \right)^{-1} + O(p^{-2k+5}) \right) \left( 1 + O(p^{-k}) \right). \]
We now use the bound \( \frac{\log p}{k} \leq \frac{\log k}{k} \leq \frac{\log 2}{k} \) to obtain \( p^{1/k} = \exp\left(\frac{1}{k} \log p\right) = 1 + O\left(\frac{1}{k} \log p\right) \). Hence

\[
p^{-3+\frac{1}{k}} \left( \left(1 - p^{-2+\frac{1}{k}}\right)^{-1} + O\left(p^{-2k+5-\frac{2}{k}}\right) \right) = \frac{1}{p^3(1 - p^{-2})\left(1 + \frac{1}{p}\right)} + O\left(\frac{\log p}{kp^3} + p^{-2k+2}\right).
\]

Note that \( p^{-2k+2} \leq p^{-k} \), hence this is

\[
\frac{1}{(p + 1)(p^2 - 1)} \left(1 + O\left(\frac{\log p}{kp^3}\right)\right) \left(1 + O\left(p^{-k}\right)\right).
\]

We obtain

\[
\prod_{p \leq k} \left(1 + \frac{p^{-3+\frac{1}{k}} \sum_{i=0}^{k-3} p^{-i(2-\frac{i}{k})}}{(1 - 2p^{-k})(1 + \frac{1}{p} + \frac{2}{p(p^2-2)})}\right) = \prod_{p \leq k} \left(1 + \frac{1}{(p + 1)(p^2 - 1)}\right) \prod_{p \leq k} \left(1 + O\left(\frac{\log p}{kp^3} + \frac{1}{p^k}\right)\right),
\]

which we combine with the estimates

\[
\prod_{p \leq k} \left(1 + O\left(\frac{\log p}{kp^3} + \frac{1}{p^k}\right)\right) = \exp \left( O\left(\sum_{p \leq k} \frac{\log p}{kp^3} + \frac{1}{p^k}\right)\right) = \exp \left( O\left(\frac{1}{k^2} + \frac{k}{2^k}\right)\right) = 1 + O\left(\frac{1}{k}\right).
\]

We conclude the proof by using the bound \( \{\zeta'(\sigma) : \frac{3}{2} < \sigma < \frac{5}{2}\} = O(1) \) to infer that

\[
\zeta \left(2 - \frac{1}{k}\right) = \zeta(2) + O\left(\frac{1}{k}\right).
\]

\[\square\]

4. Approximations via the Riemann Hypothesis

The main aim in this section is to use the Riemann hypothesis to prove that the approximation of the diffraction intensity of the \( k \)-free integers by a power law is more precise than what is implied by Theorem 1.1.

**Theorem 4.1.** Assume the Riemann Hypothesis and let \( \delta > 0 \) be arbitrary and fixed. Then for every fixed integer \( k \geq 2 \) we have

\[
Z_k(\varepsilon) = \beta_k \varepsilon^{2 - \frac{1}{k}} + O_{\delta,k}\left(\varepsilon^{2 - \frac{1}{2k} - \delta}\right), \quad (0 < \varepsilon \leq 1),
\]

where \( \beta_k \) is as in Theorem 1.1 and the implied constant depends at most on \( \delta \) and \( k \).

The differences between Theorem 1.1 and Theorem 4.1 are that the latter has a stronger error term, but the former is unconditional. The main new input for Theorem 4.1 compared to the proof of Theorem 1.1 is the following result.

**Lemma 4.2** (Liu, [6]). Assume the Riemann Hypothesis. Then for every fixed \( \delta > 0 \) we have

\[
\sum_{n \in \mathbb{N} \cap [1, x]} \mu(n)^2 = \frac{x}{\zeta(2)} + O_{\delta}\left(x^{\frac{11}{18} + \delta}\right).
\]

The proof of Lemma 4.2 builds on the work of several authors; it combines ideas of Montgomery–Vaughan [7] with van der Corput’s method for estimating exponential sums.

**Corollary 4.3.** Assume the Riemann Hypothesis and let \( \delta > 0 \) be arbitrary and fixed. Then for every \( a \in \mathbb{N} \) and \( x \geq 1 \) we have

\[
\sum_{\substack{n \in \mathbb{N} \cap [1, x] \atop \gcd(n,a)=1}} \mu(n)^2 = \frac{x}{\zeta(2)} \left(\prod_{p \mid a} \left(1 + \frac{1}{p}\right)^{-1}\right) + O_{\delta}\left(\tau(a)^3 x^{\frac{11}{18} + \delta}\right),
\]

where the implied constant depends at most on \( \delta \).
Proof. We make use of the function $g_a(n)$ that is defined in the proof of Corollary 2.7. Thus the sum in our corollary equals

$$\sum_{1 \leq c \leq x} g_a(c) \sum_{1 \leq d \leq x/c} \mu(d)^2 = \frac{x}{\zeta(2)} \sum_{1 \leq c \leq x} \frac{g_a(c)}{c} + O\varepsilon \left(x^{\frac{11}{35} + \varepsilon} \sum_{1 \leq c \leq x} \frac{|g_a(c)|}{c^{\frac{11}{35} + \varepsilon}}\right),$$

where a use of Lemma 4.2 has been made. The bound $|g_a(c)| \leq 1_{p|c\Rightarrow p|a}(c)$ shows that the error term is

$$\ll \prod_{p|a} \left(1 + \sum_{k=1}^{c} \frac{1}{p^{\frac{k}{35} + \varepsilon}}\right) \leq \prod_{p|a} \left(1 + \sum_{k=1}^{\infty} \frac{1}{2^{\frac{k}{35}}}ight) \leq 8^{\omega(a)} \leq \tau(a)^3.$$

The same bound yields

$$\sum_{c>x} \frac{|g_a(c)|}{c} \leq \sum_{c \in \mathbb{N}} \left(\frac{c}{x}\right)^{\frac{24}{35}} \prod_{p|a} \left(1 + \sum_{k=1}^{\infty} \frac{1}{p^{\frac{k}{35}}}ight) \leq x^{-\frac{24}{35}} \prod_{p|a} \left(1 + \sum_{k=1}^{\infty} \frac{1}{2^{\frac{k}{35}}}ight) \leq \frac{\tau(a)^3}{x^{\frac{24}{35}}}.$$

Lemma 4.4. Assume the Riemann Hypothesis and let $\delta > 0$ be arbitrary and fixed. There exists a positive absolute constant $\gamma$ such that the following holds. Then for each multiplicative function $\psi : \mathbb{N} \rightarrow \mathbb{R}$ with $|\psi(p)| \leq \frac{1}{p}$ for every prime $p$, all $u \geq 1$ and $\ell \in \mathbb{N}$ we have

$$\sum_{1 \leq \mu \leq u \atop \gcd(m,\ell)=1} \mu(m)^2 \sum_{d|m} \psi(d) = \frac{u}{\zeta(2)} \left(\prod_{p} \left(1 + \frac{\psi(p)}{p + 1}\right)\right) \left(\prod_{p \mid \ell} \frac{p}{p + 1 + \psi(p)}\right) + O\delta \left(\tau(\ell)^3 u^{\frac{11}{35} + \delta}\right),$$

where the implied constant depends at most on $\delta$.

Proof. As in the proof of Lemma 2.8 we see that the sum in our lemma is

$$\sum_{d\leq u \atop \gcd(d,\ell)=1} \psi(d) \mu(d)^2 \sum_{1 \leq \mu' \leq u/d \atop \gcd(m',d)=1} \mu(m')^2,$$

which, by Corollary 4.3, is

$$\sum_{d\leq u \atop \gcd(d,\ell)=1} \psi(d) \mu(d)^2 \left(\frac{u}{d\zeta(2)} \left(\prod_{p\mid \ell} \left(1 + \frac{1}{p}\right)^{-1}\right)\right) + O\delta \left(\tau(d)^3 \tau(\ell)^{3} u^{\frac{11}{35} + \delta}\right).$$

The main term above matches the main term in our lemma up to a quantity that has modulus

$$\ll u \sum_{d>u} \psi(d) \mu(d)^2 \ll u \sum_{d>u} d^{-2} \ll 1.$$

The error term contribution is

$$\ll \delta \sum_{d\leq u} \psi(d) \mu(d)^2 \tau(d)^3 \ll \delta \sum_{d\leq u} \frac{1}{d^{\frac{11}{35} + \delta}} \ll \delta \sum_{d\leq u} d^{1 + \frac{1}{35}} \ll \delta \ll \delta.$$

Recall (2.8). The proof of the next lemma follows directly from Lemma 4.4.
Lemma 4.5. Assume the Riemann Hypothesis and let $\delta > 0$ be arbitrary and fixed. Then for all $m \in \mathbb{N}^{k-2}$ and $z \geq 1$ as in (2.7) we have

$$\Upsilon(z, m) = \frac{z^{1/k}}{\zeta(2)} \left( \prod_{p} \left( 1 + \frac{2}{(p+1)(p^k - 2)} \right) \right)^{k-1} \prod_{j=2}^{k-1} m_{j}^{-j/k} \left( \prod_{p|n_{j}} \frac{1}{1 + \frac{1}{p} + \frac{2}{p(p^k-2)}} \right)$$

$$+ O_{\delta,k} \left( \tau(m_{2})^{3} \cdots \tau(m_{k-1})^{3} z^{\frac{1}{4k}+\delta} \right),$$

where the implied constant depends at most on $\delta$ and $k$.

The next proposition can be proved as Proposition 2.5 by replacing the use of Lemma 2.9 by the use of Lemma 4.5.

Proposition 4.6. Assume the Riemann Hypothesis and let $\delta > 0$ be arbitrary and fixed. Then for all $z \geq 1$ and fixed $k \in \mathbb{N}, k \geq 2$ we have

$$\sum_{1 \leq m \leq z} m^{2} \lambda_{k}(m) = \gamma_{k} z^{\frac{1}{k}} + O_{\delta,k} \left( z^{\frac{1}{4k}+\delta} \right),$$

where the implied constant depends at most on $k$ and the constant $\gamma_{k}$ is as in Proposition 2.5.

Finally, the proof of Theorem 4.1 can be proved as Theorem 1.1 by replacing the use of Proposition 2.5 by Proposition 4.6.

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