\section{n-SCHUR FUNCTIONS AND DETERMINANTS ON AN INFINITE GRASSMANNIAN}

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Abstract. A set of functions is defined which is indexed by a positive integer \( n \) and partitions of integers. The case \( n = 1 \) reproduces the standard Schur polynomials. These functions are seen to arise naturally as a determinant of an action on the frame bundle of an infinite grassmannian. This fact is well known in the case of the Schur polynomials \((n = 1)\) and has been used to decompose the \( \tau \)-functions of the KP hierarchy as a sum. In the same way, the new functions introduced here \((n > 1)\) are used to expand quotients of \( \tau \)-functions as a sum with Plücker coordinates as coefficients.

Among their many important properties, the Schur polynomials arise naturally as the determinant of an exponential function acting on the frame bundle of the grassmannian of the Hilbert space \( H = L^2(S^1, \mathbb{C}) \). It is for this reason that the \( \tau \)-functions of the KP hierarchy can be expanded as a sum of Schur polynomials with the Plücker coordinates as coefficients (cf. \cite{14, 15}).

Quotients of \( \tau \)-functions have recently played a prominent role in several papers on bispectrality \cite{3, 7}, Darboux transformations \cite{2, 8} and random matrices \cite{1}. In \cite{8} these quotients themselves are computed as a determinant of the action of a matrix valued function on the frame bundle of the grassmannian \( Gr^n = Gr(H^n) \) (cf. \cite{12}).

This note will define the \( n \)-Schur functions which play an analogous role in this more general situation. In particular, as in the case \( n = 1 \), it is shown that the determinant of the action of a matrix function on the frame bundle of \( Gr^n \) can be expanded as a sum of \( n \)-Schur functions with the Plücker coordinates as coefficients. As an application one may expand quotients of \( \tau \)-functions in this manner.

Note that although the \( n \)-Schur functions are defined in the next section and only later shown to be related to determinants on the frame bundle, this relationship should not be seen as a surprise. It is actually this property which led to the definition of these functions. The definition is given separately in the hope that someone reading this paper, who might not have interest in infinite grassmannians, may recognize other ways in which these generalized Schur functions can be used.

\section{1. \( n \)-SCHUR FUNCTIONS}

Let \( n \in \mathbb{N} \) and consider the variables \( h_k^{i,j} \) \((1 \leq i, j \leq n, k \in \mathbb{N})\) which can be conveniently grouped into \( n \times n \) matrices

\[
H_k = \begin{pmatrix}
h_k^{1,1} & \cdots & h_k^{1,n} \\
\vdots & \ddots & \vdots \\
h_k^{n,1} & \cdots & h_k^{n,n}
\end{pmatrix}
\]

\( k = 0, 1, 2, 3, 4, \ldots \)
Moreover, these matrices will be grouped into the infinite matrix $M_\infty$

$$M_\infty = \begin{pmatrix} H_0 & H_1 & H_2 & H_3 & \cdots & \cdots \\ H_0 & H_1 & H_2 & H_3 & \cdots & \cdots \\ 0 & H_0 & H_1 & H_2 & H_3 & \cdots \\ 0 & 0 & H_0 & H_1 & H_2 & \cdots \\ 0 & 0 & 0 & H_0 & H_1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

(← rows 0 through $n - 1$

\[ \downarrow \text{positively indexed rows} \])

It is convenient to label the rows of this matrix by the integers with 0 being the first row with $H_0$ at the left and increasing downwards.

We wish to define a set of functions in these variables indexed by partitions of integers. Specifically, the index set $\mathcal{S}$ will be the set of increasing sequences of integers whose values are eventually equal to their indices

$$\mathcal{S} = \{(s_0, s_1, s_2, \ldots) \mid s_{j+1} > s_j \in \mathbb{Z} \text{ and } \exists N \text{ such that } j = s_j \forall j > N\}.$$

Of particular interest here will be the special case $0 = (0, 1, 2, 3, \ldots) \in \mathcal{S}$.

**Definition 1:** For any $S \in \mathcal{S}$ let $f_S^n := \det(M_S \cdot M_0^{-1})$ where the infinite matrix $M_S$ is the matrix whose $j^{th}$ row is the $s_j^{th}$ row of $M_\infty$. Then we have, for instance, that

$$M_0 = \begin{pmatrix} H_0 & H_1 & H_2 & H_3 & \cdots & \cdots \\ 0 & H_0 & H_1 & H_2 & H_3 & \cdots \\ 0 & 0 & H_0 & H_1 & H_2 & \cdots \\ 0 & 0 & 0 & H_0 & H_1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Equivalently, one could say that the element in position $(l, m)$ ($0 \leq l, m \leq \infty$) of the matrix $M_S$ is given by

$$(M_S)_{l,m} = h_k^{i,j} = 1 + (l \mod n), \quad j = 1 + (s_m \mod n), \quad c = \left\lfloor \frac{l}{n} \right\rfloor - \left\lfloor \frac{s_m}{n} \right\rfloor$$

where $h_k^{i,j} = 0$ if $k < 0$.

**Remark 2:** Note that given any $N \in \mathbb{N}$ such that $s_i = i$ for all $i > Nn$, the matrix $M_S \cdot M_0^{-1}$ looks like the identity matrix below the $Nn^{th}$ row. Consequently, the easiest way to actually compute these functions is as two finite determinants

$$f_S^n = \frac{\det(M_S|_{nN \times nN})}{(\det H_0)^N}$$

where $M_S|_{nN \times nN}$ denotes the top left block of size $nN \times nN$ of the matrix $M_S$. For example, regardless of $n$, $f_0^n \equiv 1$. However, for $S = (-2, 1, 2, 3, \ldots)$ one finds instead

$$f_1^1 = \frac{h_{1,1}^{1,1}}{h_{0,1}^{1,1}} \quad f_2^2 = \frac{h_{1,1}^{1,2} h_{0,1}^{2,1} - h_{1,2}^{1,2} h_{0,1}^{2,1}}{h_{1,1}^{1,2} h_{0,1}^{2,1} - h_{1,2}^{1,2} h_{0,1}^{2,1}}.$$

**Remark 3:** In fact, in the case $n = 1$ and $h_0^{1,1} = 1$, the functions $\{f_1^1\}$ are the famous Schur polynomials (cf. [1], [3]). Similarly, if we assume in general that $\det H_0 = 1$, then all $f_S^n$ are polynomials.
Remark 4: If we consider $h_{k}^{i,j}$ to have weight $kn + i - j$ then the function $f_{k}^{n}$ is homogenous of weight $\sum_{j=0}^{\infty} s_{j} - j$.

2. The Grassmannian $Gr^{n}$

In this section we will recall notation and some basic facts about an infinite dimensional Grassmannian. Please refer to \[8, 11, 12, 15\] for additional details.

Let $H^{n} = L^{2}(S^{1}, \mathbb{C})$ be the Hilbert space of square-integrable vector valued functions $S^{1} \to \mathbb{C}^{n}$, where $S^{1} \subset \mathbb{C}$ is the unit circle. Denote by $e_{i}$ ($0 \leq i \leq n-1$) the $n$-vector which has the value 1 in the $i+1$st component and zero in the others.

We fix as a basis for $H^{n}$ the set $\{e_{i} | i \in \mathbb{Z}\}$ with $e_{i} := z^{\frac{i}{n}}e_{(i \mod n)}$.

The Hilbert space has the decomposition
\[ H^{n} = H_{n}^{+} \oplus H_{n}^{-} \]
where these subspaces are spanned by the basis elements with non-negative and negative indices respectively. Then $Gr^{n}$ denotes the grassmannian of all closed subspaces $W \subset H^{n}$ such that the orthogonal projection $W \to H_{+}^{n}$ is a compact operator and such that the orthogonal projection $W \to H_{-}^{n}$ is Fredholm of index zero \[11, 12\].

Associate to any basis $\{w_{0}, w_{1}, \ldots\}$ for a point $W \in Gr^{n}$ the linear map $w$
\[
w : H_{+}^{n} \to W
\]
\[e_{i} \mapsto w_{i}.
\]

The basis is said to be admissible if $w$ differs from the identity by an element of trace class \[10\]. The frame bundle of $Gr^{n}$ is the set of pairs $(W, w)$ where $W \in Gr^{n}$ and $w : H_{+}^{n} \to W$ is an admissible basis.

There is a convenient way to embed $Gr^{n}$ in a projective space. Let $\Lambda$ denote the infinite alternating exterior algebra generated by the alternating tensors
\[
\{e_{s_{0}} \wedge e_{s_{1}} \wedge e_{s_{2}} \wedge \cdots | (s_{0}, s_{1}, s_{2}, \ldots) \in S\}.
\]

To any point $(W, w)$ in the frame bundle we associate the alternating tensor
\[
|w| := w_{0} \wedge w_{1} \wedge w_{2} \wedge \cdots \in \Lambda.
\]

Note in particular that $|\cdot|$ is projectively well defined on the entire fiber of $W$ (i.e. for two admissible bases of $W$ we have $|w| = \lambda|w'|$ for some non-zero constant $\lambda$). Consequently, $|W|$ is a well defined element of the projective space $\mathbb{P} \Lambda$.

The Plücker coordinates (cf. \[6\]) of $W$ are defined as the coefficients $\langle S | W \rangle$ in the unique expansion
\[
|W| = \sum_{S \in S} \langle S | W \rangle e_{s_{0}} \wedge e_{s_{1}} \wedge e_{s_{2}} \wedge \cdots
\]
and are therefore well defined as a set up to a common multiple. Alternatively, given an admissible basis $w$ for $W$, $\langle S | W \rangle$ is the determinant of the infinite matrix made of the rows of $w$ indexed by the elements of $S$. 


3. Determinants of Action on Frame Bundle

Let \( g \) be an \( n \times n \) matrix valued function of \( z \) with expansion

\[
g = \sum_{k=0}^{\infty} H_k z^k \quad H_k = \begin{pmatrix} h_{k,1} & \cdots & h_{k,n} \\ \vdots & \ddots & \vdots \\ h_{k,1} & \cdots & h_{k,n} \end{pmatrix}
\]

such that an inverse matrix \( g^{-1} \) exists for all \( z \). We will view \( g \in GL(H^n) \) as an operator on \( Gr^n \) and demonstrate that the \( n \)-Schur functions arise naturally in this context.

In general, an operator on the frame bundle \([11]\) is a pair \( A = (g,q) \) where \( g \in GL(H^n) \) with the form

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (7)
\]

relative to the splitting \([6]\) and \( q : H^n_+ \rightarrow H^n_+ \) such that \( a \cdot q^{-1} \) differs from the identity by an operator of trace class. The action is given by

\[
A : (W,w) \mapsto (gW,gwq^{-1}).
\]

In the particular case \([6]\) of interest here \( c = 0 \) and we simply let \( q = a \) so that \( aq^{-1} \) is the identity matrix. We will write \( |g| W \rangle = |gwa^{-1} \rangle \) for the action of \( g \) on the frame bundle. Moreover, since this action is well defined on projective equivalence classes we will write \( |g| W \rangle \) for the class containing \( |g| w \rangle \) with any admissible basis \( w \) of \( W \).

The main result of this paper is then the observation that for any point \( W \in Gr^n \) the determinant \( \langle 0|g|W \rangle \) can be written in terms of the Plücker coordinates of \( W \) and the \( n \)-Schur functions:

**Theorem 1.** For \( W \in Gr^n \) and \( g \) as in \([6]\)

\[
\langle 0|g|W \rangle = \sum_{S \in \mathcal{S}} \langle S|W \rangle f^n_S.
\]

The proof is elementary in the case \( W = W_S \) with basis \( \{e_{s_0}, e_{s_1}, e_{s_2}, \ldots \} \). In fact, this is essentially the definition of \( f^n_S \) since the matrix representation of the operator \( g \) is precisely \( M_\infty \). The general case follows from the observation that multilinearity of determinants is equivalent to the linearity of the map \( \langle 0|g \rangle : \Lambda \rightarrow \mathbb{C} \) and expanding \( |W \rangle \) as a sum.

4. Application: Expanding Quotients of \( \tau \)-functions

4.1. Basic Facts about the KP Hierarchy. A pseudo-differential operator of the form

\[
\mathcal{L} = \partial + u_1(x) \partial^{-1} + u_2(x) \partial^{-2} + \cdots \quad \partial = \partial/\partial x
\]

is said to be a solution of the KP hierarchy if its coefficients \( u_i \) depend on “time variables” \( t_1, t_2, \ldots \) so as to satisfy the equations

\[
\frac{\partial}{\partial t_i} \mathcal{L} = [\mathcal{L}, (\mathcal{L}^i)_+] \quad (9)
\]

where the “+” subscript indicates projection onto the differential operators by simply eliminating all negative powers of \( \partial \) and \([A,B] = A \circ B - B \circ A\) \([13]\).

Remarkably, there exists a convenient way to encode all information about the KP
solution \( \mathcal{L} \) in a single function of the time variables \( t_1, t_2, \ldots. \) Specifically, each of the coefficients \( u_i \) of \( \mathcal{L} \) can be written as a rational function of this function \( \tau(t_1, t_2, \ldots) \) and its derivatives. Alternatively, one can construct \( \mathcal{L} \) from \( \tau \) by letting \( W \) be the pseudo-differential operator

\[
W = \frac{1}{\tau} \tau t_1 - \partial^{-1}, t_2 - \frac{1}{2} \partial^{-2}, \ldots.
\]

and then \( \mathcal{L} := W \circ \partial \circ W^{-1} \) is a solution to the KP hierarchy.

4.2. **Expanding \( \tau \)-quotients with \( n \)-Schur functions.** Of particular interest for many situations are the solutions \( \mathcal{L} \) of the KP hierarchy that have the property that \( \mathcal{L} = \mathcal{L}^n = (\mathcal{L}^n)_+ \) is an ordinary differential operator. We say that solutions of the KP hierarchy with this property are solutions of the \( n \)-KdV hierarchy. Associated to any chosen solution \( \mathcal{L} = \mathcal{L}^n \) of the \( n \)-KdV hierarchy, we naturally associate an \( n \)-vector valued function of the variables \( \{t_i\} \) and the new spectral parameter \( z \). In particular, following \[12\] we define the vector Baker-Akhiezer function to be the unique function \( \vec{\psi}(z, t_1, t_2, \ldots) \) satisfying

\[
\mathcal{L}^n \vec{\psi} = L \vec{\psi} = z \vec{\psi}, \quad \frac{\partial}{\partial t_i} \vec{\psi} = (\mathcal{L}'^i)_+ \vec{\psi}
\]

and such that the \( n \times n \) Wronskian matrix

\[
\Psi(z, t_1, t_2, \ldots) := \begin{pmatrix}
\vec{\psi} \\
\frac{\partial}{\partial x} \vec{\psi} \\
\vdots \\
\frac{\partial^{n-1}}{\partial x^{n-1}} \vec{\psi}
\end{pmatrix}
\]

is the identity matrix when evaluated at \( 0 = t_1 = t_2 = \ldots \).

Defining \( g = \Psi^{-1} \in GL(H^n) \) where \( \Psi \) is the matrix above, the determinant \( \langle \emptyset | g | W \rangle \) is a (projective) function of the variables \( t_i \). It is shown in \[8\] (Definition 7.4 and Claim 7.12) that these functions are quotients of KP \( \tau \)-functions with a \( \tau \)-function corresponding to \( \mathcal{L} \) in the denominator. Consequently, we may write these quotients in terms of the \( n \)-Schur functions:

**Corollary 1.** Let \( \mathcal{L} \) be a solution of the \( n \)-KdV hierarchy with corresponding matrix \( \Psi \) given in \[10\] and \[11\]. Defining time dependent variables \( h_{k,j} \) by

\[
\Psi^{-1} = \sum_{k=0}^{\infty} H_k z^k.
\]

gives the \( n \)-Schur functions dependence on the time variables of the KP hierarchy. Then one has that every \( f_S^n \) is then a quotient of KP \( \tau \)-functions in the sense of \[8\] for any \( S \in \mathbb{S} \). Moreover, it follows that

\[
\tau_0 \cdot \left( \sum_{S \in \mathbb{S}} \pi_S f_S^n \right)
\]

is a \( \tau \) function of the KP hierarchy whenever \( \tau_0 \) is a \( \tau \)-function for \( \mathcal{L} \) and \( \pi_S \) are the Plücker coordinates of some point in \( Gr^\mathbb{S}_{rat} \).

The special case that \( \tau_0 = 1, L = \partial \) and \( \Psi = \exp(\sum t_i z^i) \) reproduces the well-known expansion of KP \( \tau \)-functions in terms of the Schur polynomials \[15\] (see also \[4\]). Explicit new examples based on the results presented here, including “higher
cases, can be found in [6] and [9] where the $n$-Schur functions are used to demonstrate the connection between the KP hierarchy and Plücker relations [5].

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