Magnetic energy spectrum produced by turbulent dynamo: effect of time irreversibility.

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We consider the kinematic stage of evolution of magnetic field advected by turbulent hydrodynamic flow. We use a generalization of the Kazantsev-Kraichnan model to investigate time irreversible flows. In the viscous range of scales, the infinite-time limit of the spectrum is a power law but its slope is more flat than that predicted by Kazantsev model. This result agrees with numerical simulations. The rate of magnetic energy growth is slower than that in the time-symmetric case. We show that for high magnetic Prandtl turbulent plasma, the formation of the power-law spectrum shape takes very long time and may never happen because of the nonlinearity. We propose another ansatz to describe the spectrum shape at finite time.

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I. INTRODUCTION

Magnetic fields are observed in a great variety of astrophysical objects of different scales, including stars and interstellar medium, galaxies and galaxy clusters. The origin of most of these fields is related to turbulent dynamo mechanism; the conventional point of view is that the amplification of seed small-scale magnetic field fluctuations is caused by their advection in a turbulent flow of the conducting fluid or plasma

\[ \nu \lesssim k \ll k_d \sim \nu^{-1/2}m \]

The statistical stationarity can be achieved at late stages of evolution as a result of non-linear interaction between the magnetic field and the flow; to the contrary, the most rapid increase of magnetic field takes place at the kinematic stage when the Lorentz force and the feedback of magnetic field are negligible. During this process, the flow is purely hydrodynamic. It obeys the Kolmogorov theory, which implies the existence of energy flux from larger to smaller scales

Mathematically, this results in nonzero third-order longitudinal velocity correlators, physically, this means statistical irreversibility of the turbulent flow at all scales.

Flows with high magnetic Prandtl numbers are typical for many astrophysical problems: e.g., in interstellar and intercluster media \( P_{m} \) varies from \( 10^{10} \) to \( 10^{22} \), Kazantsev-Kraichnan model is the most appropriate and conventional tool to investigate the magnetic field evolution in such flows. It allowed to calculate magnetic field correlators and to analyze the spectrum evolution. But this model assumes that velocity field is Gaussian (and, hence, time-symmetric) and delta-correlated in time. Several modifications of the model were proposed to take account of finite correlation time. In the account of the third order correlator was performed for the equation with additive noise (e.g., the driving force was considered non-Gaussian).

In this paper we develop the generalization of the Kazantsev-Kraichnan model for time asymmetric velocity statistics. The \( V^3 \) model was first proposed in, it describes small fast anisotropy by taking account of the non-zero third order correlator. Here we combine it with the Kraichnan approach and calculate the magnetic energy spectrum produced by a slightly irreversible flow (which corresponds to real hydrodynamic turbulence).

Our consideration is restricted to high magnetic Prandtl numbers, and we consider a wide range of wave numbers

\[ k_v \ll k \ll k_d \sim \nu^{-1/2}m \]

where \( k_v \) and \( k_d \) are viscous dissipation and magnetic diffusion characteristic wave numbers, respectively. We show that the existence of energy cascade results in more gradual slope of the spectrum: \( \propto k^1 \), as compared to \( k^{3/2} \) predicted by the ‘classical’ Kazantsev model. This agrees with the results of numerical simulations. However, estimates show that for high magnetic Prandtl numbers demanded in astrophysical problems, the characteristic time needed to saturate the power spectrum up to the largest (diffusive) wave number \( k_d \) is very long and practically unattainable. So, the power spectrum is only valid at either infinite time or rather small (but still \( \gg k_v \)) wavenumbers. We propose a more complicated ansatz to fit the spectrum profile.

II. EQUATION FOR MAGNETIC SPECTRUM

We start from the classical problem statement. Kinematic transport of magnetic flux density \( B(t, \mathbf{r}) \) advected by random statistically homogenous and isotropic non-divergent velocity field \( \mathbf{v}(t, \mathbf{r}) \), \( \mathbf{V} \cdot \mathbf{v} = 0 \), is described by the evolution equation

\[ \frac{\partial}{\partial t} B(t, \mathbf{r}) + (\mathbf{v} \nabla) B - (\mathbf{B} \nabla) \mathbf{v} = \kappa \nabla B. \tag{1} \]

where \( \kappa \) is the diffusivity. The random process \( \mathbf{v}(t, \mathbf{r}) \) is assumed to be stationary, and to have given statistical properties. The initial conditions for magnetic field are also stochastically isotropic and homogenous. The aim is to find statistical characteristics of the process \( \mathbf{B} \), in particular, its pair correlation function.

To get the equation for the pair correlator, one has to take the tensor product of \( \nabla \) and \( \mathbf{B}(\mathbf{r}) \), and take the average over
different realizations of the velocity field. The result contains cross-correlations of magnetic field and velocity. In the Kazantsev model the velocity field is assumed to be Gaussian and $\delta$-correlated; so, these cross-correlations are split by means of the Furutsu-Novikov theorem\textsuperscript{25,26}. To account higher-order correlations of velocity, one can use the generalization of this theorem for arbitrary statistics of $\mathbf{v}$:

\[ \langle v_p(\mathbf{r},t)g[\mathbf{v}] \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int K_{p_1 \ldots p_n}(\mathbf{r},t,\mathbf{r}_1,t_1,\ldots,\mathbf{r}_n,t_n) \times \left( \frac{\delta^n g[\mathbf{v}]}{\delta v_{i_1}(\mathbf{r}_1,t_1) \ldots \delta v_{i_n}(\mathbf{r}_n,t_n)} \right) d\mathbf{r}_1 d\mathbf{r}_2 \ldots d\mathbf{r}_n d\mathbf{t}_n, \quad (2) \]

where $g[\mathbf{v}]$ is some functional of $\mathbf{v}(\mathbf{r},t)$ (in our case it contains second order combinations of $\mathbf{B}$), $\frac{\delta^n g}{\delta v_{i_1}(\mathbf{r}_1,t_1) \ldots \delta v_{i_n}(\mathbf{r}_n,t_n)}$ is the functional derivative, and $K_{p_1 \ldots p_n}(\mathbf{r},t,\mathbf{r}_1,t_1,\ldots,\mathbf{r}_n,t_n)$ are $n+1$-th order connected correlators (cumulants) of velocity:

\[ K_{p_1 \ldots p_n}(\mathbf{r},t,\mathbf{r}_1,t_1,\ldots,\mathbf{r}_n,t_n) = \frac{\delta^{n+1} \ln \left\langle v_p(\mathbf{r},t) g[\mathbf{v}] \right\rangle}{\delta \psi_{p_1}(\mathbf{r}_1,t_1) \ldots \delta \psi_{p_n}(\mathbf{r}_n,t_n)} \bigg|_{\mathbf{v}=0} \]

We are interested in the influence of the third order correlator. So, in the frame of the $V^3$ model\textsuperscript{22,23,24} we consider the non-zero second and third order correlators,

\begin{align*}
K_p(\mathbf{r},t) & = 0, \\
K_{p_1}(\mathbf{r},t,\mathbf{r}_1,t_1) & = \langle v_p(\mathbf{r},t) v_{i_1}(\mathbf{r}_1,t_1) \rangle, \\
K_{p_1 p_2}(\mathbf{r},t,\mathbf{r}_1,t_1,\mathbf{r}_2,t_2) & = \langle v_p(\mathbf{r},t) v_{i_1}(\mathbf{r}_1,t_1) v_{i_2}(\mathbf{r}_2,t_2) \rangle
\end{align*}

and we neglect the contribution of the higher-order correlators. A vice of this simplification is that the probability density is negative in some range of its argument as only a finite number of connected correlators are unequal to zero\textsuperscript{25}. This artefact can be fixed in the case of small $F$ by addition negligibly small but non-zero higher-order correlators. These higher-order corrections would not affect the magnetic field increment and the slope of the spectrum.

Furthermore, we replace an arbitrary finite-correlation time process by the corresponding $\delta$-process. The reason for such substitution is that in the equation with multiplicative noise, the higher order connected correlators of the noise contribute to the long-time statistical properties of the solutions only via their integrals (see, e.g., Appendix A in\textsuperscript{26}). So, the $V^3$ model considers both second and third order correlators as $\delta$-correlated in time:

\begin{align*}
\langle v_{i_j}(\mathbf{r},t) v_{j_k}(\mathbf{r}-\mathbf{r}_i, t-\tau) \rangle & = D_{i_j}(r) \delta_{\tau}(\tau), \\
\langle v_{i_j}(\mathbf{r},t) v_{j_k}(\mathbf{r}_1,t_1) v_{j_k}(\mathbf{r}-\mathbf{r}_2,t-\tau) \rangle & = F_{i_j k}(\mathbf{r}_1,\mathbf{r}_2) \phi(\tau,\tau_1,\tau_2) & (4) \\
\phi & = \frac{1}{3} \left( \delta_{\tau}(\tau_1) \delta_{\tau}(\tau_2) + \delta_{\tau}(\tau_1) \delta_{\tau}(\tau_2 - \tau_1) + \delta_{\tau}(\tau_2) \delta_{\tau}(\tau_2 - \tau_1) \right)
\end{align*}

Here we also take account of statistical homogeneity of $\mathbf{v}$. The function $\delta_{\tau}(\tau)$ is the regularized $\delta$-functions, i.e., time-symmetric function with narrow support $\varepsilon$ that satisfy $\int \delta_{\tau}(\tau) d\tau = 1$. This regularization is needed for correct multiplication of these functions by the $\delta$-functions appearing from variational derivatives; after calculation of the convolution, we set $\varepsilon = 0$. The details of the procedure are described in\textsuperscript{25}. The complicated form of $\phi(\tau)$ in (4) preserves the permutation symmetry of the multipliers within the average brackets.

Taking the time integrals in (2), we arrive at

\[ \langle v_p(\mathbf{r},t)g[\mathbf{v}] \rangle = \frac{1}{2} \int D_{i_j}(\mathbf{r}-\mathbf{r}') \left( \frac{\delta g[\mathbf{v}]}{\delta \psi_{i_j}(\mathbf{r},t)} \right) d\mathbf{r}' + \frac{1}{6} \int F_{i j k}(\mathbf{r}-\mathbf{r}', \mathbf{r}-\mathbf{r}'') \left( \frac{\delta^2 g[\mathbf{v}]}{\delta \psi_{i_j}(\mathbf{r},t) \delta \psi_{i_k}(\mathbf{r}',t')} \right) d\mathbf{r}' d\mathbf{r}'' \quad (5) \]

The tensors $D_{i j}$ and $F_{i j k}$ are not arbitrary. They are restricted by the requirements of isotropy and non-divergency of the flow; also, since we consider the viscous range of scales ($k \gg k_v$), the velocity can be treated as a linear function of distance. These conditions reduce the freedom of each of the two tensors to one constant multiplier. In concordance with\textsuperscript{25}, we choose the normalization by fixation of two constants:

\begin{align*}
D & = - \frac{3}{4} d^2 \left( \frac{1}{2} r r_j D_{i j} \right) \bigg|_{r=0}, \\
F & = - \frac{3}{4} d^3 \left( \frac{1}{2} r_1 r_2 r_{i j} F_{i j k} \right) \bigg|_{r=0}
\end{align*}

The time scale $D^{-1}$ is of the order of the eddy turnover time at the viscous scale; $f = F/D$ reflects the time asymmetry of the flow.

The applicability of the $V^3$ model formally requires $F \ll D$ (in order to provide positive probability density by means of negligibly small higher-order contributions). On the other hand, in\textsuperscript{26} it was shown that the constants $D$ and $F$ are related to the Lyapunov indices\textsuperscript{30} of the flow, namely,

\[ \frac{\lambda_2}{\lambda_1} = \frac{2F}{2D - F} \]

(A relation between the second Lyapunov index and the third-order velocity correlator was also considered in\textsuperscript{32}). The numerical simulation of isotropic turbulence\textsuperscript{32} and the experiment\textsuperscript{33} give the ratio of Lyapunov exponents in hydrodynamic flow $\lambda_2/\lambda_1 \simeq 0.14$. This leads to

\[ f = F/D \approx 0.13 \quad (8) \]

So, we see that in real turbulence the ratio $F/D$ is small enough to use the $V^3$ model but may be essential for magnetic correlators evolution.

Returning to the equation (1), we note that the magnetic field is also statistically homogenous, isotropic and non-divergent. This reduces its second order correlator to one scalar function:

\[ \langle B_i(\mathbf{r},t) B_j(\mathbf{r}+\mathbf{r},t) \rangle = G(r,t) \delta_{ij} + \frac{1}{2} r G'(r,t) (\delta_{ij} - n_i n_j) \]

\[ (9) \]
and the magnetic correlation function

\[
\frac{\partial G(r,t)}{\partial t} = \frac{2}{3} D \left( r^2 G''_r + 6r G'_r + 10G \right) + \frac{4}{9} F \left( 2r^3 G''_r + 21r^2 G'' + 14r G'_r - 70G \right) + 2\kappa (G''_r + \frac{2}{3} G'_r)
\]  

The detailed derivation of this equation and the analysis of its grown modes is performed in \[19\]. To consider the magnetic energy spectrum, we proceed to the Fourier transform of $\mathbf{B}(r,t)$.

\[
\mathbf{B}(R,t) = \int \mathbf{B}(k,t)e^{i\mathbf{k}\cdot\mathbf{R}}dk
\]

and the magnetic correlation function

\[
\langle \tilde{B}_i(k,t)\tilde{B}_j(k',t) \rangle = \frac{M(k,t)}{4\pi k^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \delta(|k-k'|),
\]

where the function $M$ is related to $G(r,t)$ by

\[
G(r,t) = 2 \int_0^\infty \left( \frac{\cos k r}{(kr)^2} + \frac{\sin k r}{(kr)^3} \right) M(k,t)dk.
\]

Then the equation (10) transforms to

\[
\frac{\partial M(k,t)}{\partial t} = \frac{2}{3} D \left( k^2 M''_{kk} - 2k M'_k + 6M \right) + \frac{4}{9} F \left( -2k M''_{kk} + 3k^2 M'_{kk} + 34k M'_k - 54M \right) - 2\kappa k^2 M
\]

The case $F = 0$ corresponds to the Kazantsev-Kraichnan model and was derived and solved in \[5,14\]. Here we investigate the corrections produced by the term responsible for time asymmetry. We consider the long-time evolution of the Green function:

\[
M(k,0) = \delta(k - k_0).
\]

By the change of variables

\[
\xi = \ln k
\]

the Eq. (13) can be reduced to

\[
\frac{\partial M(\xi,t)}{\partial t} = \frac{2}{3} D \left( M''_{\xi\xi} - 3M'_\xi + 6M \right) + \frac{4}{9} F \left( -2M''_{\xi\xi\xi} + 9M'_\xi\xi + 27M''_{\xi\xi} - 54M \right) - 2\kappa e^{2\xi} M
\]

\[
M(\xi,0) = e^{-\xi_0} \delta(\xi - \xi_0)
\]

III. ZERO DIFFUSIVITY

First, consider the limit of zero magnetic diffusivity: $\kappa = 0$. By means of the Fourier transform, we find the formal solution:

\[
M(\xi,t) = \frac{e^{-\xi_0}}{2\pi} \int d\eta e^{-i\eta(\xi - \xi_0)} \times e^{\left[ \frac{1}{3} D(-\eta^2 + 3i \eta + 6) + \frac{4}{9} F(-2i \eta^3 - 9\eta^2 - 27i \eta - 54) \right]}
\]

\[16\]

The integral diverges formally at large $\eta$; this is an artefact of the $V^3$ model, this divergence is the result of the 'parasite' solution that is produced by the third-order term and tends to infinity as $F \to 0$. The account of higher order terms would evidently eliminate this divergence.

To calculate the integral, we use the saddle-point method. The equation for the saddle point is

\[
-\frac{i\delta\xi}{\delta \xi} + D \left( -\frac{4}{3} \eta^* + 2i \right) + F \left( -\frac{4}{3} i \eta^* - 2 \eta^* - 3i \right) = 0
\]

where

\[
\delta\xi = \xi - \xi_0
\]

The 'physical' solution is the one that is close to the Kraichnan solution corresponding to $F = 0$:

\[
\eta^* = i \left( \frac{3}{2} + \frac{1}{f} \left( 1 - \sqrt{1 + \frac{27}{4} f^2 + f \frac{3\xi}{2D_t}} \right) \right)
\]

Substituting this into (16) we obtain:

\[17\]

FIG. 1. The evolution of magnetic energy spectrum in the non-diffusivity approximation: the Kazantsev-Kraichnan model (blue dashed lines) and the $V^3$ model for $f = 0.13$ (red solid lines). The thick points correspond to the maxima of the spectra. The time moments (normalized by $D_t^{-1}$) are indicated along the vertical axis.
Returning from $\xi$ to $k$ and leaving only the second order in $f = F/D$ (which restricts us to $\xi/(Dt) \leq f^{-2}$) we finally get

$$M(k, t) \approx \exp \left[ \frac{3\delta \xi}{2} + \frac{18f^2}{(4\pi Dt)^{3/2}} (2(4 + 63f^2)Dt + 18f\delta \xi - \frac{(4 + 27f^2)Dt + 6f\delta \xi}{\sqrt{Dt}})^{3/2} \right]$$

(18)

This solution describes the evolution of spectrum in the absence of magnetic diffusivity. Its behavior and its comparison with the result of the Kazantsev-Kraichnan model is illustrated in Fig. 1 where we choose $k_0 = k_\nu$. We see that the spectrum grows exponentially (the first multiplier), and it exponentially spreads into the region of large wave numbers (the multiplier $\mu_{id}$). This situation coincides qualitatively with the one obtained in the Gaussian case. However, the slope of the spectrum is more flat than that in the Kraichnan model, the exponential growth is slower, and the speed of the maximum of the spectrum is smaller (for $F > 0$). Thus, the magnetic energy increases significantly slower than in the Gaussian case. Indeed, taking the integral of (18) we have:

$$E(t) = \int M(k, t)dk \propto e^{\left(\frac{27}{8}D - \frac{29j^2}{8}e^{\frac{1}{2}}\right)}$$

IV. SPECTRUM FORMATION

However, for much higher Prandtl numbers it appears that the stabilization time needed to reach the self-similar growth is extremely long; the non-linear stage would start much earlier than the stationary shape of the spectrum is formed. To prove this, we now restrict ourselves (for brevity) by the Kazantsev-Kraichnan approximation.

The diffusive term in the equation (15) suppresses the magnetic energy at $k \geq k_d$ and does not affect much the smaller wave numbers. In the variables $\xi$ this boundary is sharp, and, to take diffusivity into account, it is natural to simplify the problem by full suppression of the magnetic field at the wave numbers $\geq \xi_d$ and by neglecting diffusivity at smaller $\xi$. So, we consider the ‘absorption’ model with the boundary condition $M = 0$ for $\xi \geq \xi_d$ and with $\kappa = 0$ for $\xi < \xi_d$.

Calculating the Green function of the ‘non-diffusive’ equation (15) for this ‘absorption’ boundary condition and returning to the $k$ variables, we obtain (for $f = 0$):

$$M(k, t) = \frac{1}{2\sqrt{\pi t}} \frac{e^{\frac{2j^2}{k_0}}}{k_0} \left( \frac{k}{k_0} \right)^\frac{3}{2} \exp \left[ -\frac{3}{8Dt} \ln^2 \frac{k}{k_0} \right]$$

$$- \exp \left[ -\frac{3}{8Dt} \ln \left( \frac{k_0}{k_d} \right)^2 \right], \quad k < k_d$$

(20)

The phase of stationary self-similar growth of the spectrum means that the term in the brackets becomes roughly independent of $k$, i.e., the exponents are smaller than unity. Then the term in the brackets can be expanded into a Taylor series to the first order for all $k_\nu < k < k_d$:

$$M(k, t) \approx \frac{3}{4\sqrt{\pi Dt}^{3/2}} \frac{e^{\frac{2j^2}{k_0}}}{k_0} \left( \frac{k}{k_0} \right)^\frac{3}{2} \ln \frac{k_d}{k_0} \ln \frac{k}{k_d} \left\{ \begin{array}{c} k < k_d, \quad t > t_{st} \end{array} \right\}$$

(21)

But this time appears to be very long for large magnetic Prandtl numbers: for instance, for $Pr_m \sim 10^1$ (which is moderate value for cosmic plasma) the characteristic time of the spectrum shape stabilization is $Dt_{st} \approx 200$. The energy would increase by 200 orders of magnitude during this time! It is evident that the nonlinear feedback of magnetic field would start earlier than the spectrum saturates.

In Fig. 2a,b we show the evolution of the spectrum (20) for two different Prandtl numbers: $Pr_m = 10^3$ (a) and $Pr_m = 10^{10}$ (b). For definiteness, we take $k_0 = k_\nu$. We compare the solutions with the exact stationary growing solution of the Kazantsev equation. One can see that the self-similarly growing solution found from the ‘absorption’ model practically coincides in the region $k < k_d$ with the exact solution, which proves the validity of the ‘absorption’ approximation. Furthermore, in accordance with our expectations, the ‘moderate-Prandtl’ spectrum corresponding to $Pr_m = 10^1$ is practically saturated at $t = 18D^{-1}$, while the high-Prandtl spectrum with $Pr_m = 10^{10}$ is far from the saturation after the same time. To concentrate upon the slope of the spectrum, we plot dependence of the logarithmic derivative on the wave number:

$$\xi(k) = \frac{d\ln M(k)}{d\ln k}$$

(22)

for both values of the Prandtl number (Fig. 2c,d). One can see that the power law is rather rough approximation. Indeed, in the case of $Pr_m = 10^3$ the spectrum saturates quickly but its shape is far from a power law. In particular, only in the longest-wave region $k \sim k_\nu$ one can find the power predicted from the stationary-growing solution. For $Pr_m = 10^{10}$, the
\[
\ln M(k) = \ln k + \ln k_d \\
\ln k
\]

\[Dt = 2, 10, 18\]

\[
\zeta(k) = \frac{3}{2} - \frac{3}{4Dt} \ln \frac{k}{k_0}, \quad f = 0
\]

\[\frac{3}{2} - \frac{1}{f} \left( \sqrt{1 + \frac{27}{4} f^2} + \frac{3}{2} \frac{\ln k/k_0}{Dt} - 1 \right), \quad f \neq 0
\]

FIG. 2. Evolution of the spectrum \(M\) and the logarithmic derivative \(\zeta\) in the Kazantsev-Kraichnan model with diffusion taken into account by means of zero boundary condition at \(k = k_d\) (‘absorption’ model), (20): (a), (c) \(Pr_m = 10^3\); (b), (d) \(Pr_m = 10^{10}\). The scale is the same for all figures. The solid purple line corresponds to the stationary shape of the spectrum in the ‘absorption’ model (21), the dashed-dotted black line represents the exact solution for the stationary shape of the spectrum, both of them are normalized by \(Dt = 18\). The thin dashed horizontal lines in the figures (c) and (d) correspond to the theoretical predictions for the power of the self-similarly growing spectrum for the Kazantsev model with \(\kappa = 0\) and for the ‘absorption’ model (21) at \(k = k_\nu\).

Scaling region is wide but the power of the spectrum does not coincide with that predicted from the Kazantsev equation for any reasonable time.

The approximation of zero diffusivity is only valid until the maximum of the spectrum reaches the diffusion scale \(k_d\): \(t \leq t_d \sim D^{-1} \ln(k_d/k_0)\).

At longer time, the expansion of the spectrum into the short-wave region stops; however, the spectrum continues to grow exponentially inside the region \(k < k_d\). The conventional understanding is that, after some time, the power law shape would propagate from the long-wave to the short-wave part of the spectrum, and the spectrum would become self-similar and stationary-growing. This self-similarity would last until the dynamo becomes non-linear and starts acting back on the velocity field. In this picture this is proved numerically for Prandtl numbers \(\lesssim 10^3\).

While time is smaller than the saturation time \(t_d\), the second term in the brackets in (20) remains much smaller than the first term; hence, this during this time, the effect of the magnetic diffusivity at relatively large values of \(k (k \sim k_0)\) remains negligible. So, the solution (18) found in the approximation of zero diffusivity for \(k \nu \gtrsim k_0 \gg k_d\) appears to be valid much longer than \(t_d\). The logarithmic derivative of (18) is

\[
\zeta = \begin{cases} 
\frac{3}{2} - \frac{3}{4Dt} \ln \frac{k}{k_0}, & f = 0 \\
\frac{3}{2} - \frac{1}{f} \left( \sqrt{1 + \frac{27}{4} f^2} + \frac{3}{2} \frac{\ln k/k_0}{Dt} - 1 \right), & f \neq 0
\end{cases}
\]

One can see that \(\zeta\) depends on time and wave number only in the combination \(t^{-1} \ln(k/k_0)\). The dependence has a universal form (Fig. 3) and is completely determined by the asymmetry parameter \(f\). This property can be used to fit the experimental and numerical spectra in a more accurate way than the power fit.

V. DISCUSSION

So, in this paper we consider the evolution of magnetic energy spectrum produced by turbulent dynamo in the viscous range of scales, in the case of high Prandtl numbers, \(Pr_m \gg 1\).
Regrettably, many high-magnetic Prandtl dynamo simulations are performed with Reynolds numbers of the order of unity\(^{19,24}\); thus, the relation of the Lyapunov exponents in these investigations may differ essentially from the value\(^8\) calculated for Kolmogorov turbulence. This makes it difficult to compare them to our result. On the other hand, the Shell model simulations\(^{12,24}\) allow to get high magnetic Prandtl numbers together with high Reynolds numbers, which allows to check the predictions for the dynamo produced in a turbulent flow with energy cascade. Numerical simulations in a ‘synthetic’ velocity field with prescribed ratio of the Lyapunov indices could be another way to check the predictions of the \(V^3\) model. One could investigate both the deviation of the final power spectrum from the ‘3/2’ law of the Kazantsev’s model, and the intermediate behavior of the spectrum before it gets saturated.

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