Matrix Construction Using Cyclic Shifts of a Column

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Abstract— This paper describes the synthesis of matrices with good correlation, from cyclic shifts of pseudonoise columns. Optimum matrices result whenever the shift sequence satisfies the constant difference property. Known shift sequences with the constant (or almost constant) difference property are: Quadratic (Polynomial) and Reciprocal Shift modulo prime, Exponential Shift, Legendre Shift, Zech Logarithm Shift, and the shift sequences of some m-arrays. We use these shift sequences to produce arrays for watermarking of digital images. Matrices can also be unfolded into long sequences by diagonal unfolding (with no deterioration in correlation) or row-by-row unfolding, with some degradation in correlation.

I. INTRODUCTION

Sequences with good auto and cross-correlation have many applications in modern communications, whilst radar, sonar and digital watermarking of images also requires arrays (matrices) with similar properties. Some sequences can be naturally folded into arrays, whilst arrays can always be unfolded into sequences. Here, we consider the synthesis of arrays with good correlation properties. We focus on arrays where each column is a cyclic shift of one pseudonoise (or impulse) column, or a constant column. The shift sequence of an array lists the cyclic shift of each pseudonoise or impulse column, and lists - for each constant column. A sequence is considered pseudonoise if its autocorrelation takes on values restricted to: polynomials, exponentials, logarithms and the naturally occurring shift sequences of m/GMW arrays.

II. POLYNOMIAL SHIFT SEQUENCES

Consider a shift sequence calculated as a polynomial of order \( n > 1 \), \( \varphi_1(x) \pmod{p} \), with coefficients from \( \mathbb{Z}_p \):

\[
\varphi_1(x) = - a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0
\]

(1)

When column labeled \( x=0,1, \ldots, p-1 \), is cyclically shifted through \( \varphi_1(x) \) places downwards, this describes a \( p \times p \) matrix \( A_1 \). There are \( p^2 - 1 \) other matrices cyclically equivalent to \( A_1 \). These are obtained by all two-dimensional non-zero shifts to matrix \( A_1 \). Horizontal shifts are equivalent to the transformation \( x' = x + k \), whilst vertical shifts are achieved by the transformation \( \varphi'(x) = \varphi(x) + l \) where \( k \) and \( l \) are chosen from \( \mathbb{Z}_p \). Shift polynomials which differ only in \( a_0 \) and/or \( a_{n-1} \) describe matrices, which are cyclic shifts of each other. There are \( p^{n-1} - 1 \) inequivalent matrices.

A. Autocorrelation

The autocorrelation for shift \((k, l)\) is obtained from the number of matching columns: i.e. solutions of

\[
\varphi_1(x) - \varphi_1(x') = 0
\]

(2)

The above is a polynomial of degree at most \( n - 1 \) and this is an upper bound on the number of matching columns. For pseudonoise columns with peak autocorrelation of \( p \) and off-peak autocorrelation of \(-1\), the off-peak matrix autocorrelation takes on values restricted to: \(-p, +1, p+2, 2p+3, \ldots, (n-2)p + (n-1)\). The frequency of occurrence of each autocorrelation value depends on the number of distinct roots of the difference polynomial for the shift \((k, l)\). The lowest meaningful value of \( n \) is 2 \([1]\) results in three-valued matrix autocorrelation: \( p^2, -p+1 \). In case of a ternary Legendre column, with autocorrelation values of \(-1\) and \(-1\), the matrix autocorrelation value becomes: \( p(p-1), 0, -p \). The quadratic shift sequence is one which possesses the distinct difference property (DDP), in that all differences in the shift sequence taken a fixed distance apart appear exactly once. This is a result of the (linear) difference polynomial having exactly one solution for each \((k, l)\) shift. Subsequent sections describe other known...
shift sequences with DDP, where each difference is allowed to appear at most once.

B. Cross-correlation

Consider another matrix, $A_2$ built from identical column sequences, using the shift sequence:

$$\varphi_2(x) = b_m x^m + b_{m-1} x^{m-1} + ... + b_1 x^1 + b_0$$

(3)

The number of matching columns between the matrices is obtained by solving

$$\varphi_2(x) - \varphi_1(x') = 0$$

(4)

where $x'$ and $\varphi'$ are defined as above. The difference polynomial is of degree at most $\max(m,n)$, so this is an upper bound on the number of matching columns.

Example 1

A quadratic shift sequence is: 0,6,4,1,6,0 mod 7. A matrix built from this shift sequence using a ternary Legendre sequence column is shown below. The autocorrelation off-peak values are 0,−p. Another matrix can be constructed from the quadratic shift sequence: 0,5,1,2,1,5,0. It is shown below, as well as its cross-correlation with the first matrix, which is constrained to 0,−(p−1),+(p−1). There are $p − 1$

| 0 | 1 | -1 | -1 | -1 | 1 | 0 |
|---|---|----|----|----|---|---|
| 1 | 1 | 0 | 1 | 1 | -1 | -1 |
| -1 | -1 | 1 | -1 | -1 | 1 | -1 |
| 1 | 1 | 0 | -1 | 0 | -1 | -1 |
| -1 | 0 | -1 | 1 | 1 | 0 | -1 |
| -1 | 0 | 1 | -1 | -1 | 0 | -1 |

C. Column Sequences

The required sequence length of $p$ can be satisfied by a Legendre sequence, Hall sequence or binary m-sequence.

III. Exponential Shift Sequence

An exponential shift sequence, with $p−1$ entries of the form $g^j$, where $g$ is any primitive root in $Z_p$, has the distinct difference property [2]. There are $\phi(p−1)$ sequences, where $\phi$ is the Euler Totient Function. The treatment in §§II, III, of exponential and logarithmic functions, as shift sequences, is an adaptation of that for Costas Arrays [3]. Consider two arrays constructed using common column sequences and the shift sequences: $\varphi_1(j) = g^j$ and $\varphi_2(j) = h^j$, where $g$ and $h$ are primitive roots of $Z_p$. Columns match whenever

$$h^j = g^{j+k} + l$$

(5)

But $h = g^r$ where $r \in Z_p$ and $gcd(r,p−1)=1$. The columns which match are solutions of

$$g^{rj} - g^{j+k} - l = 0$$

(6)

Let $X = g^j$. Therefore

$$X^r - g^k X - l = 0$$

(7)

Equation (7) has at most $r$ solutions. Hence, an upper bound on the number of matching columns is $r$.

A. Autocorrelation

Here $r = 1$ so the number of matching columns is: $p − 1$, 1, 0. The matrix autocorrelation values are: $p^2−p, +2, 1−p$.

B. Cross-correlation

The best cross-correlation is achieved for $r = −1$. Then, equation (7) becomes:

$$X^{-1} - g^k X - l = 0 \quad \text{or} \quad g^k X^2 + X - 1 = 0$$

(8)

Hence, an upper bound on the number of matching columns is 2. The matrix cross-correlation values are: $p + 3$, $+2, 1 − p$.

C. Column Sequences

The required sequence of length $p$ can be any Legendre sequence, Hall sequence or some binary m-sequences.

IV. Logarithmic Shift

Two types of discrete logarithms arise:

1) \text{index function giving rise to Legendre Shift Arrays}
2) \text{Zech logarithm giving rise to Zech Shift Arrays}

The former applies to $Z_p$, whilst the latter to $GF(p^m)$

A. Legendre Arrays

Let a pseudonoise sequence of length $(p−1)$ be shifted cyclically as a function of column number $j$:

$$\varphi_r(j) = r \times ind_g(j), \text{where} \quad r \in 1,2,...,p−1$$

(9)

with $g$ being a primitive root of $Z_p$ and $ind_g$ being the $k0 \leq k \leq p−2$, such that $g^k = j$. Note that $\varphi(j)$ is expressed mod $(p−1)$. The column indices $(j)$ of the array range from 0 to $p−1$, so the matrix has $p$ columns with $(p−1)$ rows, with the first column being blank, since $ind(0)$ is not defined. There are $p−2$ sequences. For example, for $p = 7, g = 3$, the shift sequence is: -0,2,1,4,5,3.
B. Correlation

An appropriate choice of $g$ can make $r = 1$ for one of the shift sequences. For two shift sequences $\varphi_r(j)$ and $\varphi_1(j)$ the number of matching columns is the number of solutions of:

$$r \times \text{ind}_g(j) - \text{ind}_g(j + k) - l = 0 \quad (10)$$

or

$$j^r - g^j + g^l k = 0 \quad (11)$$

1) Autocorrelation: For autocorrelation, $r = 1$ and so, the number of matching columns is $p - 1$ for $k = l = 0$, 1 for $k \neq 0$ or 0 for $k = 0, l \neq 0$ and $l = 0, k \neq 0$. The matrix autocorrelation values are: $(p - 1)^2$ for full match, $+1$ for 1 column match, $-(p - 1)$ for 0 column match. These values can be altered, if the blank column is substituted by a constant column.

2) Cross-correlation: The best result of an upper bound of 2 columns matching is obtained for $r = 2$ or $-1$. This occurs for pairs of sequences only. The matrix cross-correlation values are: $p + 1, +1, -(p - 1)$.

3) Column Sequence: A suitable column sequence for such a construction is any m-sequence over $Z_p$ and its mapping onto roots of unity $S(i) = \omega^j$, Schroeder [2,26,19], where $g$ is a primitive root of $Z_p$, $\omega = e^{2\pi i/p}$ is a primitive root of unity. Here $p = 7$. Multiples of this angle are shown.

$$\begin{bmatrix}
-5 & 3 & 1 & 6 & 4 & 2 \\
-4 & 1 & 5 & 2 & 6 & 3 \\
-6 & 5 & 4 & 3 & 2 & 1 \\
-2 & 4 & 6 & 1 & 3 & 5 \\
-3 & 6 & 2 & 5 & 1 & 4 \\
-1 & 2 & 3 & 4 & 5 & 6 \\
\end{bmatrix}$$

C. Zech Arrays

Logarithmic arrays of size $(p^m - 1) \times (p^n - 1)$ can be constructed using column shifts determined by the Zech logarithm of elements of a Galois Field $GF(p^m)$. For the general (Golomb) construction, the shift sequence is

$$\varphi(j) = \log_\beta(1 - \alpha^j) \quad (12)$$

$\alpha$ and $\beta$ are primitive elements of $GF(p^m)$. The total number of distinct arrays is: $\frac{\log(p^m - 1)}{m}$. The special case of $\beta = \alpha$ is the Lempel construction.

1) Correlation: Consider two shift sequences:

$$\varphi_1(j) = \log_\beta(1 - \alpha^j) \quad \text{and} \quad \varphi_2(j) = \log_\gamma(1 - \delta^j) \quad (13)$$

where $\gamma = \beta^r$ and $\delta = \alpha^s$ and $\gcd(r, p^m - 1) = \gcd(s, p^n - 1) = 1$. We can interpret $r$ as a multiplier and $s$ as a sampler or decimation. An upper bound on the number of matching columns is determined by the number of solutions of:

$$\log_\beta(1 - \alpha^{j+k}) + l = \log_\gamma(1 - \delta^l) = r^{-1} \log_\beta(1 - \alpha^s) \quad (14)$$

so

$$\log_\beta \left( \frac{1 - \alpha^{j+k}}{1 - \alpha^s} \right) = -l$$

Finally $(1 - \alpha^{j+k})^r = \beta^{-l}(1 - \alpha^s)$

Once again, let $\alpha^j = X$. Therefore

$$\(1 - \alpha^k X)^r = \beta^{-l}(1 - X^s) \quad (15)$$

(15) is a polynomial equation of degree $\max(r,s)$.

2) Autocorrelation: $r = s = 1$, so the numbers of matching columns are: $p - 1, 1, 0$.

3) Cross-Correlation: $r = -1, s = 1$ or $r = 1, s = -1$ or $r = 2, s = 1$ or $r = 1, s = 2$ once again result in quadratic forms and hence limit the number of matching columns to 0, 1, 2.

Example 2 A shift sequence of length 15 over $GF(2^4)$ is:

$$-4,8,14,1,10,13,9,2,7,5,12,11,6,3.$$  

Another shift sequence is:

$$-8,1,3,2,5,11,3,4,14,10,9,7,12,6.$$  

The lower sequence could be obtained by decimation of the above sequence by 2 or multiplication of the entries by 2. The cross-correlation shows that 2 columns can match at most.

4) Column Sequences: Any m-sequence/GMW sequence of length $p^m - 1$ is suitable.

V. OTHER SHIFT SEQUENCES

A. Shift Sequences from M-arrays

An m-sequence of length $p^{km} - 1$ can be written row-by-row as a matrix, such that all columns are constants or cyclic shifts of one short m-sequence of length $p^m - 1$. This matrix can be described by a shift sequence and the column sequence. Formally, this shift sequence $f_j$ can be obtained from finite field theory [10]:

$$f_j = \text{ind}_s(T_r^m(\alpha^j)) \quad (16)$$

where $T_r^m$ is the trace function, which takes the sum of conjugates (as defined in [11]). This shift sequence can be converted into a perfect shift sequence for two-dimensional matrices by the addition of an appropriate linear function [4]. Then, all differences appear equally frequently for all cyclic shifts of the shift sequence. Such shift sequences can be used to construct new matrices, by substituting the columns by other pseudonoise sequences. For example, an m-sequence of length 120 ($p^2 - 1$, with $p = 11$) can be written as an array of 10 rows, each of length 12. Each column of the array is an m-sequence of length 10, except for the single null sequence. The twelve columns are described by a shift sequence modulo 10. This shift sequence can be converted into a perfect shift sequence modulo 5, [4]: $3,2,2,4,0,3,-3,0,4,4,2,2$. This produces a perfect matrix, by using it to shift (cyclically) a ternary Legendre sequence of length 5: 0,1,-1,-1,1. The choice of $a = \sqrt{11}$ for the entry $a$ in the constant column adjusts the autocorrelation numbers to : 55 for zero shift, and 0 for all other shifts, an example of perfect autocorrelation for the array and its diagonal sequences.
B. Hyperbolic Shift Sequences

Hyperbolic sequences have been used for designing Frequency Hop Patterns [8]. They can also be applied to construct matrices of the type described in this paper.

VI. Window Property

A column sequence is said to have the $n \times 1$ strong window property if each possible $n$-tuple occurs and occurs once only, as $n$ consecutive symbols in the column. The weak window property is that no window of $n$ symbols appears more than once. Each m-sequence as a column sequence in this paper has the weak window property. The shift sequences discussed in this paper also have the window property: the quadratic shift sequence mod $p$ has the strong window property, whilst all the others have the weak window property. For each shift sequence in this paper, each doubleton $(\varphi(j), \varphi(j+k))$ appears at most once. Hence, an array constructed using such a shift sequence, with a column sequence having the weak $n \times 1$ window property, has the weak $n \times 2$ window property, for any fixed separation of the two window columns [7]. Arrays with window properties are useful as registration patterns in structured light for medical imaging.

VII. Applications

A. Watermarking

Matrices have been used as watermarks in various images in spatial and transform domains [5]. Entries with real values are preferred, although matrices over complex numbers have also been embedded as watermarks [6]. To keep watermarks unobtrusive, and immune to correlation type attack, the matrices need to be as large, and as efficient as possible (least zero-value entries), with low off-peak autocorrelation. This makes use of the maximum processing gain. All matrices described in the previous sections satisfy these criteria. Some applications require additional immunity to cryptographic attack. Since most attacks involve linear processing, linear complexity is a good measure. Matrices constructed using exponential, logarithmic and hyperbolic shift sequences, applied to high linear complexity column sequences e.g. GMW, Bent sequences etc provide this feature. Other applications require large information storage in the watermark. Typically, information is stored in the value of the cyclic shift of the matrix. Information capacity can only be increased by combining more matrices with low mutual cross-correlation. The polynomial shift sequence appears to be the only one capable of this. The example of this feature is illustrated in the figures below. Figure 1 shows the result of adding 4 $127 \times 127$ matrices constructed using 4 different quadratic shift sequences and the same m-sequence column. The matrices all have different cyclic shifts, each one contributing a bit word to the information content. Figure 2 shows the original image. Figure 3 shows the watermarked image, where the watermark is just visible. Figure 4 shows the result of correlating the watermarked image with the four matrices, showing 4 distinct peaks, one for each array. Such a composite watermark can carry almost 4 ASCII characters. The performance of such watermarks is much better for full scale images, using RGB and can be extended to video.

B. Communications

Modern wireless communications require large sets of sequences with good auto and cross-correlation. Preferably, such sequences should be binary $(+/1)$ or ternary $(+/1,0)$ and have a high linear complexity. Our matrices can be unfolded to yield such sets, and many satisfy the above criteria. The unfolded sequences can be utilized in CDMA. We consider two methods of matrix unfolding.

1) Diagonal Unfolding: Matrix $A = \{a_{i,j}\}$ of $T$ columns each of length $v$ with $gcd(v,T) = 1$ can be unfolded along a diagonal $s_i = \{a_{qi,ri}\}$ where $gcd(v,q) = 1$ and $gcd(r,T) = 1$. This is because diagonal passes through every entry in the matrix exactly once, before repeating. Each one-dimensional cyclic shift of the diagonal is equivalent to a two-dimensional $(k,l)$ cyclic shift of the matrix. Therefore, correlation values for a matrix are equal to those of the diagonal sequence. The above applies to exponential and logarithmic shift sequences, because $gcd(p, (p-1)) = 1$. Such matrices can be unfolded to yield new sequences of length $p^2-p$ directly along diagonals of such matrices [7], without changing the correlation numbers.

2) Row-by-Row Unfolding: Any matrix $A = \{a_{i,j}\}$ of $T$ columns each of length $v$ can be unfolded row-by-row into a sequence $s_{i\times T} = \{a_{i,j}\}$ Here, there is no correspondence between cyclic shifts of the long sequence and two-dimensional matrix shifts. This kind of unfolding results in a doubling of the upper bound on the number of matching columns [9]. The worst case correlation are also doubled (approximately).

VIII. Conclusion

This paper shows how matrices with good two dimensional autocorrelation and cross-correlation can be synthesized by using cyclic shifts of pseudonoise columns. These sequences of shifts (shift sequences) are derived from finite fields using mappings from number theory. The properties of matrices constructed by shifts of a pseudonoise sequence based on polynomial type shift sequences are summarized in Table 1 below. Autocorrelation and cross-correlation entries refer to numbers of matching columns. L refers to Legendre sequence, H to Hall sequence and $M^t$ to M/GMW sequence of length $2^n-1$ which is a prime number (Mersenne Prime). $M^t$ refers to non-binary M/GMW sequence.
### Table 1

Matrices based on exponential and inverse exponential (logarithmic) shift sequences are summarized in Table 2.

| Construction $\varphi$ | Quadratic | Degree $n$ |
|------------------------|-----------|------------|
| Matrix Size $Z$         | $p \times p$ | $p \times p$ |
| Total Matrices $N$      | $p - 1$   | $p^{n-1} - 1$ |
| Optimum Set $Q$         | $p - 1$   | $p^{n-1} - 1$ |
| Autocorrelation $\Theta_{AA}$ | $p, 1, 0$ | $p, n - 1, n - 2, ..., 0$ |
| Cross $(A, B \in Q) = \Theta_{AB}$ | $2, 1, 0$ | $n, n - 1, ..., 0$ |
| Column Sequence $S$     | $L, H, M^*$ | $L, H, M^*$ |

### Table 2

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