Stability and neimark - sacker bifurcation for a discrete system of one - scroll chaotic attractor with fractional order

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Abstract. This present work investigates the dynamical behaviors of a new form of fractional order three dimensional system with a chaotic attractor of the one-scroll structure and its discretized counterpart. Firstly, existence and parametric conditions for local stability analysis of steady states of the model are addressed. Then the bifurcation theory is applied to investigate the presence of Neimark - Sacker (NS) bifurcations at the coexistence steady state taking the time delay as a bifurcation parameter in the discrete fractional order system. Also the trajectories, phase diagrams, limit cycles, bifurcation diagrams, attractors for period - 1, 2, 3, 4, 5 and a chaotic attractor with one scroll are exhibited for biologically meaningful sets of parameter values in the discretized system. Finally, several numerical examples are presented to assure the validity of the theoretical results and further rich dynamics of the model is explored as well.

Key words: Fractional order system, Steady states, Discretization, Stability analysis, Limit cycles, Bifurcations, Chaotic attractor.

1. Introduction

Over the past few centuries the explosion in scientific knowledge and technology has been a contributing factor and also to some extent a direct consequence in advancement of mathematical modeling. The development of more contemporary form of mathematical modeling has further enhanced our understanding of the world around us. A periodic motivation for those who develop and study mathematical models is the desire to form prediction \[1\]. The system of equations, that describes physical problems or phenomenon has been applied in physical and chemical science, economics, engineering and mathematical biology. Also the subject of mathematical modeling involves physical intuition, formulation of equations, solution methods, and analysis leading to accurate prediction. During the past centuries, several types of mathematical models have been developed to investigate the natural and social processes that enlarged over time. These models are referred as dynamical systems. Dynamical systems are divided into two general categories, i.e. deterministic models and stochastic models respectively. Deterministic models are ordinarily employed when the number of quantities involved in the process being modeled is relatively small and all the underlying scientific principles are fairly well understood \[4, 5\].

The rest of this work is organized as follows: In section 2, we obtain the discretization of the model with piecewise constant arguments from the continuous fractional - order dynamical system \((5)\). The existence of steady states of \((5)\) is discussed in section 3. In section 4, we discuss the various
types of dynamical nature at each steady state and some numerical examples are illustrated to show our theoretical results in section 5. In section 6, we investigate the Neimark - Sacker bifurcation around coexistence steady state by using bifurcation theory.

2. New form of fractional order system and its discretization

The fractional order systems have risen to prominence due to its distinct features namely they are realistic and practical in their approach, it has non-local property, as it takes into consideration the past and distributed effects of the model which the integer differential equations lack, and these systems are convenient to model biological systems with memory effect. The non-local property implies that the subsequent status of the model depends not only upon its present stage but also upon all its previous stages. Few mathematicians showed that a time delay could have considerable influence over the local stability of coexistence steady state and the cause of periodic doubling bifurcation (PDB) and Neimark - Sacker bifurcation (NSB) [7]. Lorenz and Rosler systems discussed the nonlinear dynamics, all kinds of bifurcations, synchronization and existence of chaotic attractors in a fractional order continuous dynamical system [8]. The motivation for this research work comes from the paper [9].

The model under consideration is a new form of fractional order continuous three dimensional system:

\[
\begin{align*}
\frac{d^\alpha}{dt^\alpha} x(t) &= \alpha x - y, x(0) = x_0 \\
\frac{d^\beta}{dt^\beta} y(t) &= x - \beta y - z, y(0) = y_0 \\
\frac{d^\eta}{dt^\eta} z(t) &= ax + (y - \mu)\eta z, z(0) = z_0
\end{align*}
\]

where \(0 < q \leq 1\), especially when \(q = 1\), the system (1) is a classical integer order system and \(\frac{d^\alpha}{dt^\alpha} \) is the Caputo fractional order derivative. In this model, \(x(t)\), \(y(t)\) and \(z(t)\) are state variables as functions of time \(t\), respectively. Here \(\alpha\), \(\beta\) and \(\eta\) are the system parameters, \(q\) is the fractional order and \(h > 0\) is the step size. Now, the system (1) is discretized with piecewise constant arguments process [2], [3] are given as

\[
\begin{align*}
\frac{d^\alpha}{dt^\alpha} x(t) &= \alpha x(t/h) - y(t/h) \\
\frac{d^\beta}{dt^\beta} y(t) &= x(t/h) - \beta y(t/h) - z(t/h) \\
\frac{d^\eta}{dt^\eta} z(t) &= ax(t/h) + (y(t/h) - \mu)\eta z(t/h).
\end{align*}
\]

First, we take \(0 < t < h\), so \(0 < (t/h) < 1\). Thus, we have

\[
\begin{align*}
\frac{d^\alpha}{dt^\alpha} x(t) &= \alpha x_0 - y_0 \\
\frac{d^\beta}{dt^\beta} y(t) &= x_0 - \beta y_0 - z_0 \\
\frac{d^\eta}{dt^\eta} z(t) &= ax_0 + (y_0 - \mu)\eta z_0.
\end{align*}
\]

The solution of (3) is

\[
\begin{align*}
x_1(t) &= x_0 + J^\alpha (\alpha x_0 - y_0) = x_0 + \frac{t^\alpha}{\Gamma(\alpha)} (\alpha x_0 - y_0) \\
y_1(t) &= y_0 + J^\beta (x_0 - \beta y_0 - z_0) = y_0 + \frac{t^\beta}{\Gamma(\beta)} (x_0 - \beta y_0 - z_0) \\
z_1(t) &= z_0 + J^\eta (ax_0 + (y_0 - \mu)\eta z_0) = z_0 + \frac{t^\eta}{\Gamma(\eta)} (ax_0 + (y_0 - \mu)\eta z_0)
\end{align*}
\]

Second, we take \(h \leq t < 2h\), so \(1 < (t/h) < 2\). Then we obtain

\[
\begin{align*}
\frac{d^\alpha}{dt^\alpha} x(t) &= \alpha x_1 - y_1 \\
\frac{d^\beta}{dt^\beta} y(t) &= x_1 - \beta y_1 - z_1 \\
\frac{d^\eta}{dt^\eta} z(t) &= ax_1 + (y_1 - \mu)\eta z_1
\end{align*}
\]

which have the following solution
\[
\begin{align*}
x_2(t) &= x_1(h) + \frac{(t-h)^\alpha}{q \Gamma(q)} (\alpha x_1(h) - y_1(h)) \\
y_2(t) &= y_1(h) + \frac{(t-h)^\beta}{q \Gamma(q)} (x_1(h) - \beta y_1(h) - z_1(h)) \\
z_2(t) &= z_1(h) + \frac{(t-h)^\eta}{q \Gamma(q)} (\alpha x_1(h) + (y_1(h) - \mu) \eta z_1(h))
\end{align*}
\]

Proceeding like this up to \( n \) times, we have
\[
\begin{align*}
x_{n+1}(t) &= x_n(nh) + \frac{(t-nh)^\alpha}{q \Gamma(q)} (\alpha x_n(nh) - y_n(nh)) \\
y_{n+1}(t) &= y_n(nh) + \frac{(t-nh)^\beta}{q \Gamma(q)} (x_n(nh) - \beta y_n(nh) - z_n(nh)) \quad (4) \\
z_{n+1}(t) &= z_n(nh) + \frac{(t-nh)^\eta}{q \Gamma(q)} (\alpha x_n(nh) + (y_n(nh) - \mu) \eta z_n(nh))
\end{align*}
\]

where \( nh \leq t < (n+1)h \). As \( t \to (n+1)h \), then the system (4) becomes
\[
\begin{align*}
x_{n+1} &= x_n + \frac{h^\alpha}{q \Gamma(q)} (\alpha x_n - y_n) \\
y_{n+1} &= y_n + \frac{h^\beta}{q \Gamma(q)} (x_n - \beta y_n - z_n) \quad (5) \\
z_{n+1} &= z_n + \frac{h^\eta}{q \Gamma(q)} (\alpha x_n + (y_n - \mu) \eta z_n)
\end{align*}
\]

3. Steady states of model (5)

First, we find the steady states of system (5) from
\[
\begin{align*}
x &= x + \frac{h^\alpha}{q \Gamma(q)} (\alpha x - y) \\
y &= y + \frac{h^\beta}{q \Gamma(q)} (x - \beta y - z) \\
z &= z + \frac{h^\eta}{q \Gamma(q)} (\alpha x + (y - \mu) \eta z).
\end{align*}
\]

Obviously, the model (5) has always two non negative steady states, (i) \( S_0 = (0,0,0) \), a trivial steady state. (ii) If \( \mu > \frac{\alpha}{\eta(1-\alpha \beta)} \) and \( \alpha < \frac{1}{\beta} \) then the model (5) has coexistence steady state \( S_1 = (x^*, y^*, z^*) \), where \( x^* = \left( \frac{\mu}{\alpha} - \frac{1}{\eta(1-\alpha \beta)} \right) \), \( y^* = \alpha x^* \) and \( z^* = (1-\alpha \beta)x^* \).

4. Dynamical nature of system (5)

In this section, we investigate the nonlinear dynamical behavior of the discretized new form of fractional-order system (5). Now, we study the criteria for stability analysis in the neighborhood of
each steady state. At any steady state \( S(x, y, z) \), the Variation matrix \( V(x, y, z) \) of the system (5) has the form:

\[
V(x, y, z) = \begin{bmatrix}
1 + \frac{h^q}{q\Gamma(q)} \alpha & -\frac{h^q}{q\Gamma(q)} \beta & 0 \\
\frac{h^q}{q\Gamma(q)} \alpha & 1 - \frac{h^q}{q\Gamma(q)} \beta & -\frac{h^q}{q\Gamma(q)} \\
\frac{h^q}{q\Gamma(q)} \alpha & \frac{h^q}{q\Gamma(q)} \alpha \xi & 1 + \frac{h^q}{q\Gamma(q)} \eta (y - \mu)
\end{bmatrix}.
\] (7)

In view of the local stability analysis for discrete fractional order system (5), the following theorem is presented.

**Theorem 4.1.** [6] We consider the cubic polynomial equation of the form:

\[
\lambda^3 + \sigma_1 \lambda^2 + \sigma_2 \lambda + \sigma_3 = 0, \quad (8)
\]

where \( \sigma_1 \), \( \sigma_2 \) and \( \sigma_3 \) are constants. The roots of the cubic equation (8) lie within the open unit disk iff the following conditions are satisfied:

\[
|\sigma_1 + \sigma_3| < 1 + \sigma_2, \quad |\sigma_1 - 3\sigma_3| < 3 - \sigma_2, \quad \text{and} \quad \sigma_2^2 + \sigma_2 - \sigma_3 \sigma_1 < 1.
\]

### 4.1. Trivial steady state

At the trivial steady state \( S_0 = (0,0,0) \), the Variation matrix \( V_{S_0} \) is

\[
V(S_0) = \begin{bmatrix}
1 + \frac{h^q}{q\Gamma(q)} \alpha & -\frac{h^q}{q\Gamma(q)} \beta & 0 \\
\frac{h^q}{q\Gamma(q)} \alpha & 1 - \frac{h^q}{q\Gamma(q)} \beta & -\frac{h^q}{q\Gamma(q)} \\
\frac{h^q}{q\Gamma(q)} \alpha & 0 & 1 - \frac{h^q}{q\Gamma(q)} \mu \eta
\end{bmatrix}
\]

The characteristic equation at \( S_0 \) becomes

\[
F(\lambda) = \lambda^3 + \delta_1 \lambda^2 + \delta_2 \lambda + \delta_3 = 0.
\] (9)

where \( \delta_1 = \left( \frac{h^q}{q\Gamma(q)} \right)(\beta - \alpha + \mu \eta) - 3, \)

\[
\delta_2 = 3 - 2 \left( \frac{h^q}{q\Gamma(q)} \right)(\beta - \alpha + \mu \eta) + \left( \frac{h^q}{q\Gamma(q)} \right)^2 \left[(1 - \alpha \beta) + (\beta - \alpha) \mu \eta\right]
\]

and

\[
\delta_3 = \left( \frac{h^q}{q\Gamma(q)} \right)(\beta - \alpha + \mu \eta) - \left( \frac{h^q}{q\Gamma(q)} \right)^2 \left[(1 - \alpha \beta) + (\beta - \alpha) \mu \eta\right] - \left( \frac{h^q}{q\Gamma(q)} \right)^3 \left[(1 - \alpha \beta) \mu \eta\right] - 1.
\] (10)

**Theorem 4.1.1.** The trivial state \( S_0 \) of system (5) is a sink and locally asymptotically (LAS) stable iff (i) \( |\delta_1 + \delta_3| < 1 + \delta_2 \), (ii) \( |\delta_1 - 3\delta_3| < 3 - \delta_2 \) and (iii) \( \delta_2^2 - \delta_2 \delta_3 < 1 \), where \( \delta_1, \delta_2 \) and \( \delta_3 \) are as in (10).

### 4.2. Coexistence steady state

At the coexisting steady state \( S_1 = (x^*, y^*, z^*) \), the Variation matrix \( V_{S_1} \) is

\[
V(S_1) = \begin{bmatrix}
1 + \frac{h^q}{q\Gamma(q)} \alpha & -\frac{h^q}{q\Gamma(q)} \beta & 0 \\
\frac{h^q}{q\Gamma(q)} \alpha & 1 - \frac{h^q}{q\Gamma(q)} \beta & -\frac{h^q}{q\Gamma(q)} \\
\frac{h^q}{q\Gamma(q)} \alpha & \frac{h^q}{q\Gamma(q)} \alpha \xi & 1 + \frac{h^q}{q\Gamma(q)} \eta \xi
\end{bmatrix}
\]
The characteristic equation at $S_1$ becomes $F(\lambda) = \lambda^3 + \tau_1 \lambda^2 + \tau_2 \lambda + \tau_3 = 0$, (11)
where $	au_i = \left( \frac{h^q}{q!} \right) (\beta - \alpha - \eta \xi_i) - 3$, 
\[
\tau_2 = 3 - 2 \left( \frac{h^q}{q!} \right) (\beta - \alpha - \eta \xi_2) + \left( \frac{h^q}{q!} \right)^2 [(1 - \alpha \beta) - (\beta - \alpha) \eta \xi_2 + \alpha \xi_1] \text{ and } \\
\tau_3 = \left( \frac{h^q}{q!} \right) (\beta - \alpha - \eta \xi_2) - \left( \frac{h^q}{q!} \right)^2 [(1 - \alpha \beta) - (\beta - \alpha) \eta \xi_2 + \alpha \xi_1] \\
- \left( \frac{h^q}{q!} \right)^3 [(\alpha + \alpha \xi_1) - (1 - \alpha \beta) \eta \xi_1] - 1 \tag{12}
\]
such that $\xi_1 = z^*$ and $\xi_2 = y^* - \mu$.

**Theorem 4.2.1.** If the conditions $\mu > \frac{\alpha}{\eta (1 - \alpha \beta)}$ and $\alpha < \frac{1}{\beta}$ are satisfied, then the coexistence state $S_1$ of system (5) is sink and it is locally asymptotically stable (LAS) if (i) $|\tau_1 + \tau_3| < 1 + \tau_2$, (ii) $|\tau_1 - 3\tau_3| < 3 - \tau_2$ and (iii) $\tau_2^2 + \tau_2 - \tau_1 \tau_3 < 1$, where $\tau_1$, $\tau_2$ and $\tau_3$ are as in (12).

### 5. Numerical results

In this section, we provide some numerical examples for the qualitative dynamical nature of a new form of discrete fractional order system (5) to verify the analytical results in Section 4. From the numerical study, it is clearly shown that the approximate solutions $x_i$, $y_i$ and $z_i$ depend on the fractional parameter $h$ and $q$, see figure 1 - figure 6. We choose the values $q = 0.95$, $h = 0.081$, $\alpha = 0.215$, $\beta = 0.387$, $\mu = 1.82$ and $\eta = 3.19$ with $x_0 = -0.5$, $y_0 = -0.2$ and $z_0 = -0.4$. Under such conditions, the coexisting steady state is $S_1 = (0.2831, 0.0609, 0.2596)$ and the eigenvalues are $\lambda_{1,2} = 0.9911 \pm i 0.0874$ and $\lambda_3 = 0.4756$ so that $|\lambda_{1,2}| = 0.9950 < 1$ and $|\lambda_3| < 1$. Also $|\tau_1 + \tau_3| = 2.9288 < 2.9328 = 1 + \tau_2$, $|\tau_1 - 3\tau_3| = 1.0452 < 1.0672 = 3 - \tau_2$ and $\tau_2^2 + \tau_2 - \tau_1 \tau_3 = 0.9971 < 1$. We observe that time plot is oscillatory but convergent. The corresponding trajectory spirals moving inwards and it approaches to steady state $S_1$. In this case, we get $S_1$ is sink and the system (5) is LAS according to Theorem 4.2.1, see figure 1. While with $q = 0.948$, $\beta = 0.273$ and keeping all other parameter values are fixed, we have $S_1 = (0.2755, 0.0592, 0.2593)$ and the eigenvalues are $\lambda_{1,2} = 0.9965 \pm i 0.0895$ and $\lambda_3 = 0.4720$ so that $|\lambda_{1,2}| = 1.0005 > 1$ and $|\lambda_3| < 1$. Also $|\tau_1 + \tau_3| = 2.9375 < 2.9417 = 1 + \tau_2$, $|\tau_1 - 3\tau_3| = 1.0475 < 1.0583 = 3 - \tau_2$ and $\tau_2^2 + \tau_2 - \tau_1 \tau_3 = 1.0002 > 1$. In this case, we observe that $S_1$ is a saddle state of index 2 and the model (5) is unstable, see figure 2. Also the approximate solutions of the state variables $x_i$ and $y_i$ depend on the system parameters $q$ and $\beta$ are displayed in figure 3. We can easily verify that whenever the values of $q$ and $\beta$ decreases, then $S_1$ moves from stability to unstable position and the trajectories spiral inwards but does not converge to steady state $S_1$. It settles down as a limit cycle, see figure 3.
Figure 1. (a) Time series and (b) phase portrait are stable at $S_1$.

Figure 2. (a) Time series and (b) phase portrait are unstable at $S_1$.

Figure 3. The attractor projected onto $x - y$ plane at the steady state $S_1$ (a) for $q = 0.95$, $\beta = 0.387$ & (b) for $q = 0.948$, $\beta = 0.273$.

We consider system (5) with $q = 0.98$, $h = 0.081$, $\alpha = 0.215$, $\beta = 0.03$, $\mu = 1.82$ and $\eta = 3.29$ with $x_0 = -0.5$, $y_0 = -0.2$ and $z_0 = -0.1$. In figure 4, we have the coexisting steady state $S_1 = (0.2512, 0.0540, 0.2496)$ and the eigenvalues are $\lambda_{1,2} = 1.0073 \pm 0.0837i$ and $\lambda_3 = 0.5023$ so that
Figure 4. (a) Time series is unstable at $S_1$ (b) 3D plot of the attractor when $q = 0.98 \& \eta = 3.29$.

Figure 5. (a) Time series is unstable at $S_1$ (b) 3D plot of the attractor when $q = 0.99 \& \eta = 2.29$.

Figure 6. The attractor projected onto $x - y$ plane at the steady state $S_1$ (a) for $q = 0.98, \eta = 3.29$ & (b) for $q = 0.99, \eta = 2.29$. 
$|\lambda_2|=1.0108>1$ and $|\lambda_1|<1$. Also $|\tau_1+\tau_3|=3.0301<3.0335=1+\tau_2$, $|\tau_1-3\tau_3|=0.9773>0.9665 =3-\tau_2$ and $\tau_2^2+\tau_2-\tau_1\tau_3=1.0051>1$. In this case, we get $S_1$ is a saddle state of index 2 (unstable) and the system (5) is one scroll chaotic attractor, see figure 4. While with $q=0.99$, $\eta=2.29$ and keeping all other parameter values are fixed, we have $S_1=(0.3609,0.0776,0.3586)$ and the eigenvalues are $\hat{\lambda}_1=1.0064\pm i0.0807$ and $\hat{\lambda}_3=0.6698$ so that $|\hat{\lambda}_1|=1.0097>1$ and $|\hat{\lambda}_3|<1$. Also $|\tau_1+\tau_3|=3.3654<3.3675=1+\tau_2$, $|\tau_1-3\tau_3|=0.6342<0.6325=3-\tau_2$ and $\tau_2^2+\tau_2-\tau_1\tau_3=1.0020>1$. The coexisting steady state $S_1$ is unstable and the system (5) is chaotic attractor, see figure 5. In dynamical systems, existence of chaos implies that the scrolls of a chaotic attractor are developed only around the saddle states of index 2. Furthermore, the saddle states of index 1 are responsible for connecting the scrolls [9]. Also the approximate solutions $x_t$ and $y_t$ depend on the system parameters $q$ and $\eta$ are displayed in figure 6. From the above analysis we can see that the coexisting steady state $S_1$ is a saddle states of index 2. Therefore, the system (5) is chaotic with one - scroll and it is displayed on $x-y$ plane, see figure 6.

6. Neimark – Sacker bifurcation analysis

In this section, we discuss the parametric conditions for the existence of Neimark-Sacker bifurcation (NSB) at the coexistence steady state and attractors for different values of $\alpha$ to support the analytical analysis and the complex dynamics of a new form of discrete fractional order system (5) with the help of numerical simulations. In order to discuss the NSB for the system (5) at the coexisting steady state $S_1$, we choose $\alpha$ as bifurcation parameter. From (11) it is easy to see that $F(\lambda)=0$ must have a complex conjugate root with modulus one. Clearly equation (11) will have two pure imaginary roots and one real root. Let $\tau_1=\tau_2$, for some values of $\alpha$, say $\alpha=\alpha^*,$ then equation (11) becomes $(\lambda^2+\tau_2)(\lambda+\tau_1)=0$ which has three roots $\hat{\lambda}_1=\pm i \sqrt{\tau_2}$ and $\hat{\lambda}_3=-\tau_1$. Hence the system (5) undergoes a NSB at the coexistence steady state $S_1$ if $\alpha$ varies in the small neighborhood of $NS_{S_1}$, where $NS_{S_1}=$\{(q,h,\alpha,\beta,\mu,\eta)\}.$\{\tau_2=1, \tau_1=\pm 1, \alpha, q, h, \beta, \mu, \eta>0\}$.

Figure 7. (a-c) Neimark-sacker bifurcation diagrams of (5) in $(\alpha-x)$, $(\alpha-y)$ and $(\alpha-z)$ planes. Figure 7 shows that Neimark - Sacker bifurcation of system (5) with the parameter values $q=0.95$, $h=0.181$, $\beta=0.327$, $\mu=1.2$, $\eta=4$ and $\alpha\in[0,0.4]$ and the initial values are $x_0=-0.5$, $y_0=-1$ and $z_0=0.3$. In this case the system (5) undergoes a NSB emerges at the coexisting steady state $S_1=(0.0622,0.0093,0.5092)$ in a small neighborhood of the bifurcation parameter $\alpha=0.1497108$. 
The corresponding bifurcation diagram is shown in figure 7. The characteristic equation evaluated at $S_1$ is
$$\lambda^3 - (2.0061)\lambda^2 + (1.0853)\lambda - 0.0431 = 0.$$ (13)

Furthermore, the roots of (13) are $\lambda_{1,2} = 0.9813 \pm i 0.1922$ and $\lambda_3 = 0.0435$ with $|\lambda_{1,2}| = |\tau_2| = 1$ and $\lambda_3 = \tau_3 = 0.0435 \neq \pm 1$. From figure 7, we observe that positive steady state $S_1$ of map (5) is stable for $\alpha < 0.1497108$ and its loses stability through a Neimark-Sacker bifurcation for $\alpha = 0.1497108$ and attracting invariant circle appears for $\alpha$ in the range of $[0.1497108, 0.187]$, see figures (8(c) & 8(d)). The phase diagrams with different values of $\alpha$ are plotted in figure 8 corresponding the value of $\alpha \in [0, 0.4]$ in figure 7, to illustrate these observations. When $\alpha = 0.22$, the quasi-periodic orbits appear and increasing the value of $\alpha$, it is also seen that the chaotic attractor sets are presented such as
period - 1 for $\alpha = 0.3$, period - 2 for $\alpha = 0.32$, period - 3 for $\alpha = 0.343$, period - 4 for $\alpha = 0.35$, period - 5 for $\alpha = 0.367$ and a chaotic attractor with one scroll for $\alpha = 0.395$ are plotted in figures 8(f) - 8(l) to illustrate these observations.

Finally the Neimark - Sacker bifurcation diagram for $\mu$ as bifurcation parameter of the system (5) at coexisting steady state $S_1$ with $x_0 = -0.3$, $y_0 = -0.2$ and $z_0 = -0.4$ as above and the selected parameter values $q = 0.99$, $h = 0.81$, $\alpha = 0.315$, $\beta = 1.2$, $\eta = 0.3$, $\mu \in [0.16, 0.7]$ with step size $\Delta \mu = 0.001$ in $(\mu-x)$, $(\mu-y)$ and $(\mu-z)$ planes are given in Figure 9. We observe that the bifurcation diagrams of steady state $S_1$ for larger value of system parameter $\mu$ of state variables $x$, $y$ and $z$, there is no possibility of chaotic dynamics of the above system (5). But for smaller value of $\mu$ the systems becomes chaotic. Finally a series of Neimark - Sacker bifurcation explores the system from non periodic (chaos) behavior tents to a stabilility.

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