Schwinger equation as singularly perturbed equation

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Abstract

A new approximation scheme for non-perturbative calculations in a quantum field theory is proposed. The scheme is based on investigation of solutions of the Schwinger equation with its singular character taken into account. As a necessary supplementary boundary condition the Green functions’ connected structures correspondence principle is used. Besides the usual perturbation theory expansion which is always available as a particular solution of our scheme some non-perturbative solutions of an equation for the propagator are found in the model of a self-interacting scalar field.

Introduction

As is well known there exists a wide class of physical phenomena (called non-perturbative effects) which cannot be described by a finite number of terms of the perturbation theory series. It is usually believed that the full or partial summation of the perturbative series solves this problem. Such an approach supposes implicitly that the sum of the perturbative series contains exhaustive information about a given quantum field model.

Meanwhile even simple mechanical models provide an example for the perturbative series to be unable to describe a physical situation properly.

Thus, it is well known from fluid mechanics that one cannot use an ideal liquid as the leading approximation to a viscous one (even if the viscosity is very small) when considering a process of flowing near the boundary of an immersed body. It is a consequence of the fact that the viscous liquid is a so called singularly perturbed system [1, 2] compared to the ideal liquid. A solution of equations of motion for the ideal liquid as well as the perturbation theory based on this solution cannot obey the boundary conditions for the viscous liquid in principle.

A characteristic feature of a singularly perturbed system is existence of an essential singularity with respect to an expanding parameter in solutions. If such a system is described by a differential equation then the highest derivative coefficient depends on an expanding parameter as follows: when this parameter is equal to zero the order of the equation decreases and the number of necessary boundary conditions reduces. This means that the perturbative series is a solution of the initial problem only for some special boundary conditions. If a physical situation goes beyond their
framework the result of the perturbation theory summation fails to describe such a situation in principle.

Let us consider an elementary example. Take the equation for a system in the form:

$$\lambda \dot{x} = t - x$$

with the boundary condition $x(0) = X$. At $\lambda = 0$ the order of the equation lowers so (1) is a singularly perturbed equation. In this case the perturbative series on $\lambda$ breaks at the second term and the result of its summation is

$$x_{\text{pert}}(t) \equiv \sum \lambda^n x_n = t - \lambda. \quad (2)$$

Though $x_{\text{pert}}$ is an exact solution of (1) with the special (perturbative) boundary condition $x_{\text{pert}}(0) = -\lambda$, it cannot serve as a solution of the problem (1) when $X \neq -\lambda$. The exact solution of (1) has the form

$$x(t) = t - \lambda + (X + \lambda) \exp\left(-\frac{t}{\lambda}\right) \quad (3)$$

and shows that in solving a problem with non-perturbative boundary conditions by iterations one should introduce a so called boundary series besides the usual perturbative one. In our simple example (1) this boundary series consists of the two terms containing an essential singularity on $\lambda$.

The boundary series contribution cannot be taken into account by perturbation theory and in general this contribution dominates near the point $t = 0$ that is outside the region of the Tikhonov’s theorem applicability [2]. This domination can be easily seen from our example if we calculate the derivatives of $x(t)$ at $t = 0$. We get $\dot{x}(0) = -\frac{1}{\lambda}X$, $\ddot{x}(0) = \frac{1}{\lambda^2}X + \frac{1}{\lambda}$ and so on. These derivatives are of obviously non-perturbative character (except for the case of the perturbative boundary condition $X = -\lambda$).

Now let us turn to the case of quantum field theory. An essential singularity with respect to a coupling constant is the wide known attribute of any field theory model with interaction. Moreover, the Schwinger equation for the generating functional contains a coupling constant as a coefficient at the highest functional derivative so from the point of view of the differential equations theory the Schwinger equation is a typical example of a singularly perturbed equation.

It is also worth mentioning that it is Green functions (vacuum expectation values) which are of physical interest. As these functions are the derivatives of the generating functional at the source switched off (hence, in the obviously non-Tikhonov region) so the boundary series contribution to Green functions will dominate except for the perturbative boundary conditions. In this work we make an attempt to treat the Schwinger equation as singularly perturbed one and to construct a new approximation scheme of solving of this equation.

The proposed scheme approximates the non-perturbative Green functions already at first step without summing of expansions and at the same time contains the standard perturbation theory as a particular case. The scope of admissible boundary conditions becomes wider and they do not limited by the framework of the perturbation theory any more. It opens new possibilities for describing of essentially...
non-perturbative effects (such as mass spectrum reconstruction) already at the leading orders of approximation. We hope that the proposed approximation scheme (or those similar to it) can prove to be useful when studying the problems connected with the spontaneous symmetry breaking and also in problems where one should take into account from the very beginning the nontrivial properties of the physical vacuum as media.

Our work is organized as follows. In part 1 necessary definitions are introduced and general principles of the approximation scheme construction are stated. The primary version of the scheme based on a general solution of the Schwinger equation is considerably simplified and in the form proposed here this scheme can be generalized to any quantum field model (in present work we deal with the simplest case of a self-interacting scalar field only). In part 2 the equation for the propagator at the leading approximation is solved. Brief discussion of the results contains in Conclusion.

1. Schwinger Equation and Approximation Scheme

We will consider the theory of complex scalar field \( \phi(x) \) in \( d \)-dimensional Euclidean space \( x \in E_d \) with nonlocal self-interaction:

\[
S_{\text{int}} = \int dx \, dy \, (\phi^* \phi)(x) \frac{\lambda(xy)}{2} (\phi^* \phi)(y).
\]  

(4)

The limit \( \lambda(xy) \to \lambda \delta(x - y) \) corresponds to the local theory with the quadric interaction.

To define the Green functions we introduce a bilocal source \( \eta(xy) \). Then the \( n \)-particle (2n-point) functions (vacuum expectation values) are the derivatives of the generating functional

\[
G = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} G_n \eta^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int \prod_{k=1}^{n} dx_k \, dy_k \, G_n \left( x_1 \ldots x_n \right) \eta(y_1 x_1) \ldots \eta(y_n x_n). 
\]

(5)

The first derivative is the propagator of the particle:

\[
\Delta(xy) = G_1 \left( \begin{array}{c} x \\ y \end{array} \right) = - \frac{\delta G}{\delta \eta(y x)} \bigg|_{\eta = 0},
\]

(6)

the second one is the two-particle function:

\[
G_2 \left( \begin{array}{c} x_1 \, x_2 \\ y_1 \, y_2 \end{array} \right) = \frac{\delta^2 G}{\delta \eta(y_2 x_2) \, \delta \eta(y_1 x_1)} \bigg|_{\eta = 0}.
\]

(7)

\footnote{The use of the bilocal source is made only for the sake of simplicity and compactness of calculations and is not a principal point in proposed formulation of the approximation scheme.}
The Schwinger equation for the generating functional is a consequence of the field equations and the quantization conditions. For the model involved it has the form:

$$\int dx' \lambda(xx') \frac{\delta^2 G}{\delta \eta(yx') \delta \eta(x'x)} =$$

$$\left( m^2 - \partial^2 \right) \frac{\delta G}{\delta \eta(yx')} + \int dx' \eta(xx') \frac{\delta G}{\delta \eta(yx')} + \delta(x - y) G. \quad (8)$$

Here \( m^2 \) stands for the mass of the free field \( \phi(x) \) and \( \partial^2 \equiv \sum_{n=1}^{d} \frac{\partial^2}{\partial x^2} \) is the Laplace operator.

The following notations will be subsequently used:

$$AB(x) \equiv \int dx' A(xx') B(x') \quad \text{and} \quad \frac{\delta}{\delta \eta(x)} \equiv \frac{\delta}{\delta \eta(yx)},$$

$$A * B(x) \equiv \int dx' A(xx') B(x'y). \quad (9)$$

In these notations the Schwinger equation (8) reads as follows:

$$\left( \lambda \frac{\delta G}{\delta \eta} \right) * \frac{\delta}{\delta \eta} = \left( m^2 - \partial^2 + \eta \right) * \frac{\delta G}{\delta \eta} + G. \quad (10)$$

Successive differentiation of (10) gives (on switching off the source) the infinite system of the Dyson equations for the Green functions \( G_n \).

The Schwinger equation (10) belongs to the class of singularly perturbed equations. Indeed, when \( \lambda = 0 \) the order of the equation reduces: the generating functional \( G(0) \) at the leading order of the perturbation theory on \( \lambda \) (that is the free field generating functional) is a solution of the first order equation:

$$\left( m^2 - \partial^2 + \eta \right) * \frac{\delta G^{(0)}}{\delta \eta} + G^{(0)} = 0. \quad (11)$$

The solution of (11) is uniquely fixed by the only boundary condition: the normalization of the generating functional

$$G[\eta = 0] = 1. \quad (12)$$

This solution is:

$$G^{(0)} = \det \left[ \left( m^2 - \partial^2 + \eta \right)^{-1} \right]. \quad (13)$$

The perturbative solution for the Schwinger equation

$$G_{\text{pert}} = \sum_{n=0}^{\infty} G^{(n)}, \quad (14)$$

where \( G^{(n)} = O(\lambda^n) \) is constructed by the iterations. The equation for \( G^{(n)} \) is again of the first order with respect to the functional derivatives

$$\left( m^2 - \partial^2 + \eta \right) * \frac{\delta G^{(n)}}{\delta \eta} + G^{(n)} = \left( \lambda \frac{\delta G^{(n-1)}}{\delta \eta} \right) * \frac{\delta}{\delta \eta} \quad (15)$$
and therefore to fix $G^{(n)}$ at any order of the perturbation theory on $\lambda$ one needs the only condition $G^{(n)}[0] = 0$ which is a simple consequence of the normalization condition (12).

Meanwhile at $\lambda \neq 0$ (even if small) the Schwinger equation (10) is that of the second order with respect to the functional derivatives and for its solution to be fixed uniquely one needs a supplementary boundary condition. The problem of boundary conditions is understood to be the key one for constructing of non-perturbative solutions of the Schwinger-Dyson equations [4, 5, 6].

Being the solution of the Schwinger equation (10), the perturbative series (14) has the doubtless advantage that there is no need for fixing a supplementary boundary condition. But this advantage can turn to be a serious lack: physical phenomena described in principle by a given field model can escape out of the consideration. The perturbation theory even being summed up can fail to obey the physical boundary conditions just as the ideal liquid is unable to provide the physical boundary conditions for the viscous liquid in the boundary layer. At the same time enlarging of the class of admissible boundary conditions can provide the description of non-perturbative phenomena after making a finite number of iterations and without summing of the corresponding expansions.

Let us turn to the construction of the approximation scheme. Since the perturbation theory is the only universal tool for calculating of the Green functions in quantum field theory we will use it as a base that is will construct our scheme in such a way that the perturbation theory expansion would be contained in it as a particular case.

Introduce a perturbative approximant:

$$G_{\text{pert}}^N \equiv \sum_{n=0}^{N} G^{(n)}$$

(16)

and define a non-perturbative approximant $G^N$ to be found as a solution of the equation:

$$\left(\lambda \frac{\delta G^N}{\delta \eta}\right) * \frac{\delta}{\delta \eta} - (m^2 - \partial^2 + \eta) * \frac{\delta G^N}{\delta \eta} - G^N =$$

$$\left(\lambda \frac{\delta G_{\text{pert}}^N}{\delta \eta}\right) * \frac{\delta}{\delta \eta} - (m^2 - \partial^2 + \eta) * \frac{\delta G_{\text{pert}}^N}{\delta \eta} - G_{\text{pert}}^N. \tag{17}$$

When $N \to \infty$ the perturbative approximant $G_{\text{pert}}^N$ tends to the perturbative solution of the Schwinger equation $G_{\text{pert}}$ and therefore $G^N$ tends to some exact solution of the same equation with, in general, arbitrary boundary conditions. That is why it would be more correct to speak about a set of approximants \{G^N\} every element of which obeying definite boundary conditions. Under $N \to \infty$ this set of approximants turns into the full set of the all solutions of the Schwinger equation. $G_{\text{pert}}^N \in \{G^N\}$ as the trivial solution of equation (17).

\[\text{We do not discuss the problem of convergency of expansions.}\]
To construct a solution of equation (17) one needs to formulate some criteria restricting the framework of the admissible supplementary boundary conditions so as in general case they are arbitrary. After such a restriction we obtain a certain subset \( \{G^N\} \) of the all possible solutions of equation (17): \( \{G^N\} \subset \{G^N\} \). We will choose the boundary conditions in such a way (see below) that the set \( \{G^N\} \) would include \( G^N_{pert} \). Our scheme will be nontrivial if it will contain other solutions too. In such a case one comes again to the question of the choice of the proper solution for the given model among the elements of \( \{G^N\} \). This choice can be made on taking into account some additional physical requirements. One of them is the principle of minimal energy value of the ground state.

So, summing all mentioned above, our goal is to reduce the full set of the all possible solutions \( \{G^N\} \) to some subset \( \{G^N\} \) by imposing additional boundary conditions on the solutions of equation (17).

In formulating such conditions we will base ourselves on those properties of the perturbative approximant which can be generalized to the non-perturbative case. In so doing we extremely restrict the admissible boundary conditions but the perturbative solution is still contained in \( \{G^N\} \).

One of the properties mentioned above is the connected structure of the perturbative approximant. General connected structure of Green functions can be found beyond the framework of the perturbation theory with the help of the theorem on the connectivity of the logarithm\footnote{It is worth mentioning that to find the connected structure one should introduce the simple sources. The derivatives of \( Z = \ln G \) with respect to the bilocal source are not in general the connected functions.}. For example, the connected structure of the two-particle function is:

\[
G_2(x_1, x_2, y_1, y_2) = \Delta(x_1, y_1) \Delta(x_2, y_2) + \Delta(x_1, y_2) \Delta(x_2, y_1) + G_{2con}^2(x_1, x_2, y_1, y_2),
\]

(18)

\( G_{2con}^2 \) being the connected part of the two-particle function and \( \Delta \) is the full propagator. For the three-particle function this structure is of the form

\[
G_3 = \text{Sym} \{6 \Delta \Delta \Delta + 9 \Delta G_{2con}^2 \} + G_{3con}^2
\]

(19)

where the notation Sym stands for the Bose-symmetrization. Formulae (18 - 19) (and those similar to them for higher functions) are valid not only at any order of the perturbation theory but also beyond its frames.

At \( N \)-th order of the perturbation theory all the connected parts of \( n \)-particle functions are equal to zero when \( n > N + 1 \). For example, at the leading order only the propagator is not equal to zero, at the order \( O(\lambda) \) (\( N = 1 \)) the propagator and the connected part of the two-particle function are non-vanishing, and so on.

To say it another, dynamics at \( N \)-th approximation is fully defined by \( N + 1 \) lowest Green functions. In this connection at \( N \)-th order one can confine oneself to considering the subsystem of \( N + 1 \) equations from the whole infinite Dyson system. So as at \( N \to \infty \) the \( N + 1 \) equations turn into the full system then the truncated approximant (a solution of \( N + 1 \) equations) still tends to the solution \( G_{pert} \).
We lay these properties into the base of our scheme construction. Physically this means that we consider the non-perturbative solutions of the Schwinger equation of such a type that the highest Green functions weakly affect dynamics defined by the lowest functions that is, for example, one can neglect the three-particle forces when considering the leading approximation to the two-particle processes.

Thus, to construct the \( N \)-th step of the approximation scheme one should proceed in the following way:

1. At the \( N \)-th step we consider \( N + 1 \) equations of the Dyson type following from (17) that is equation (17) itself and its \( N \) derivatives at \( \eta = 0 \).
2. Due to \( N + 1 \) equations contain \((N + 2)\) \( n \)-particle functions \( G_1^N, ..., G_{N+2}^N \) this system needs a supplementary boundary condition\[. \]To introduce such a condition we will demand the correspondence between the connected structures of the sought approximant and the perturbative one. This perturbative approximant consists of the set of \((N + 2)\) \( n \)-particle functions calculated with the perturbation theory which will be denoted as

\[ g_n^N \equiv (-1)^n \left. \frac{\delta^n G_{\text{pert}}^N}{\delta \eta^n} \right|_{\eta=0}. \]

For the perturbative approximant \([g_{N+2}^N]_{\text{con}} = 0\) therefore we put for our system

\[ [G_{N+2}^N]_{\text{con}} = 0. \] (20)

The connected structures correspondence condition (20) allows one to express \((N + 2)\)-particle function in terms of \( N + 1 \) lowest ones

\[ G_{N+2}^N = G_{N+2}^N \left[ G_1^N, \ldots, G_{N+1}^N \right] \] (21)

and by this to close the system of equations.

At every step of the approximation scheme we get, generally speaking, some collection of the \( n \)-particle functions sets \( \{G_1^N, ..., G_{N+2}^N\} \). The set of perturbative \( n \)-particle functions \( \{g_1^N, ..., g_{N+2}^N\} \) is also contained in the collection. \textit{A priori} the region of the scheme applicability is that of the small \( \lambda \). When \( N \to \infty \) each of these sets tends to the full (infinite) number of Green functions corresponding to some exact solution of the Schwinger equation (10). Each of these solutions has the true connected structure.

Let us elaborate the two primary steps in more detail. For \( N = 0 \) the perturbative approximant \( G_{\text{pert}}^0 = G^{(0)} \) is the generating functional of the free Green functions (13). The system consists of one equation — this is equation (17) at \( \eta = 0 \):

\[ \int dx' \lambda(x,x') G_2^0 \left( \frac{x x'}{y x'} \right) + (m^2 - \partial^2) \Delta^0 (x y) - \delta(x - y) = \int dx' \lambda(x,x') g_2^0 \left( \frac{x x'}{y x'} \right). \] (22)

\[ \text{Normalization condition (12) is supposed to be fulfilled at any step of consideration.} \]
Here
\[ \Delta^N = - \frac{\delta G^N}{\delta \eta} \bigg|_{\eta=0}, \]  
(23)
and
\[ g_2^0 \left( \frac{x_1 x_2}{y_1 x_2} \right) = \triangle_c (x_1 y_1) \triangle_c (x_2 y_2) + \triangle_c (x_1 y_2) \triangle_c (x_2 y_1) \]  
(24)
— the two-particle function of the free field where \( \triangle_c = (m^2 - \partial^2)^{-1} \) is the free propagator. In accordance with the connected structures correspondence condition \[ [G^0_2]_{\text{con}} = 0 \] that is we have for the two-particle function \( G^0_2 \):
\[ G^0_2 \left( \frac{x_1 x_2}{y_1 y_2} \right) = \triangle^0 (x_1 y_1) \triangle^0 (x_2 y_2) + \triangle^0 (x_1 y_2) \triangle^0 (x_2 y_1). \]  
(25)
So we have the equation for the propagator \( \triangle^0 \) at the leading approximation.

At the next step \( (N = 1) \) the system contains two equations. As it follows from \[ \lambda \delta G^0_{\text{pert}} + \triangle^0 \] so the system can be written as:
\[ \int dx' \lambda (x x') \left( \frac{x' x'}{y x'} \right) = \int dx' \lambda (x x') \left[ g_2^0 \left( \frac{x' x'}{y x'} \right) - g_2^0 \left( \frac{x' x'}{y x'} \right) \right], \]  
(26)
\[ \int dx' \lambda (x x') \left( \frac{x_1 x_2}{y_1 y_2} \right) = \int dx' \lambda (x x') \left[ g_2^0 \left( \frac{x_1 x_2}{y_1 y_2} \right) - g_2^0 \left( \frac{x_1 x_2}{y_1 y_2} \right) \right]. \]  
(27)

At \( N = 1 \) that is at the order \( O(\lambda) \) of the perturbation theory the connected part of the three-particle function \( g_3^1 \) is equal to zero. Hence the connected structures correspondence condition at this step reads:
\[ [g_3^1]_{\text{con}} = 0 \]  
(28)
and in accordance with \[ (18-19) \] equations \[ (26) \] and \[ (27) \] form the system for \( \triangle^1 \) and \[ [G^1_2]_{\text{con}}. \]

To the end of this section we point out another way of the approximation scheme construction based on reducing the order of the Schwinger equation.

Let us recall that if any particular solution of a linear \( n \)-th order differential equation is known the equation can be reduced to that of \( (n-1) \)-th order. This procedure can be applied to the Schwinger equation \[ (10) \] treated as a linear second order differential equation with respect to the functional derivatives \[ \frac{\delta}{\delta \eta}. \]
Define a functional $W$ in accordance with the formula

$$W(xy|\eta) = R(xy|\eta) G[\eta] - \frac{\delta G}{\delta \eta(yx)}$$ \hspace{1cm} (29)

where $R$ is a solution of the following equation:

$$(\lambda R) \frac{\delta}{\delta \eta} + (\lambda R) \ast R = 1 + (m^2 - \partial^2 + \eta) \ast R.$$ \hspace{1cm} (30)

The equation for $W$ follows from (10) and (30) and turns out to be of the first order:

$$(\lambda W) \frac{\delta}{\delta \eta} + (\lambda R) \ast W = (m^2 - \partial^2 + \eta) \ast W.$$ \hspace{1cm} (31)

To use this method one should find at least one exact solution of equation (30) or equivalently the particular solution $G_{part}$ of (10) (then $R_{part} = \frac{\delta}{\delta \eta} \ln G_{part}$). Such a solution was found in [6] (see also [3, 7]) and is of the form:

$$R_{part} = \int dx'dy' \lambda^{-1}(yy') \eta(y'x') + m^2 \int dy' \lambda^{-1}(yy').$$ \hspace{1cm} (32)

On the other hand one can perform “approximate” lowering of the order by using the perturbative approximant $R^N$ which is defined as being an approximate perturbative solution of (30) up to $O(\lambda^N)$ accuracy.

On solving the equation for $W^N$

$$(\lambda W^N) \frac{\delta}{\delta \eta} + (\lambda R^N) \ast W^N = (m^2 - \partial^2 + \eta) \ast W^N$$ \hspace{1cm} (33)

we obtain the sequence of approximants $\{R^N, W^N\}$ and from (29) can recover all the derivatives at $\eta = 0$ of the corresponding functional $\tilde{G}^N$. For the trivial solution $W^N = 0$ we get $\tilde{G}^N = G^N_{pert}$ — the perturbative approximant (13) and for any nontrivial $W^N \neq 0$ we find some non-perturbative approximation. The problem of supplementary boundary conditions can be solved with the help of the connected structures correspondence condition. The final equation for the propagator at the leading approximation coincides with (22).

But it should be mentioned that this way is more technically complicated because of the necessity to take into account nontrivial integrability and Bose-symmetry conditions for $W$. An advantage of this scheme is the possibility to use as a starting point the exact solution (32). But in so doing the question about the boundary condition should be solved anew because it is clear that $R_{pert} = \lim_{N \to \infty} R^N \neq R_{part}$.

2. The solution of the equation for the propagator

At the leading approximation ($N = 0$) equation (22) with (24) (25) gives the equation for the propagator $\triangle^0$ which will be denoted as $\triangle$ in this section.

Note, that at this step of the approximation we cannot use the principle of minimal energy value of the ground state in order to fix the physical solution because for
so doing one should investigate (as minimum) equations of the next step \((N = 1)\) in which the connected two-particle function is contained.

Nevertheless already at the leading approximation it is possible to solve the question of nontriviality of our scheme that is to verify if the equation for the propagator admits solutions different from \(\Delta_c\). If exist, these nontrivial solutions will be of the nonperturbative origin by the construction.

The equation for \(\Delta\) can be written in the form:

\[
(m^2 - \partial^2 + \Sigma) * \Delta = (m^2 - \partial^2 + \Sigma_c) * \Delta_c \tag{34}
\]

The operators \(\Sigma\) and \(\Sigma_c\) are formally defined as:

\[
\Sigma(xy) \equiv \lambda(xy) \Delta(xy) + \delta(x - y) \int dx_1 \lambda(xx_1) \Delta(x_1x_1), \tag{35}
\]

\[
\Sigma_c(xy) \equiv \lambda(xy) \Delta_c(xy) + \delta(x - y) \int dx_1 \lambda(xx_1) \Delta_c(x_1x_1), \tag{36}
\]

The structure of these quantities coincide with that of the mass operator at the order \(O(\lambda)\) of the perturbation theory. In the local limit \(\lambda(xy) \rightarrow \lambda \delta(x - y)\) they are the formal divergent quantities and the problem arises of equation (34) renormalizing.

Due to \(\Sigma\) is expressed in terms of the propagator to be sought it is sufficient to define \(\Sigma_c\). The usual way of handling such a quantity is to introduce a regularization \(\Sigma_c \rightarrow \text{reg} \Sigma_c\) and then to define a renormalized quantity by subtractions. However this technique is inconvenient and complicated in our case.

More simple recipe is to apply “the renormalization without subtraction” method [8]. This method as well as that of “differential renormalization” [3, 10] is one of the realizations of Bogolubov’s idea to define products of distributions without using senseless divergent quantities. The general idea of the method is to define such a product as a distribution from the Schwartz space being a solution of some equation well defined in this space.

In the perturbation theory this method is equivalent to the usual ones. The results thus obtained can differ only by finite renormalization as they do for different subtraction procedures.

The quantity \(\Sigma_c\) to be defined consist of the two terms. At the local limit these terms are equal to each other and \(\Sigma_c\) can be formally written as \(2\lambda \Delta_c(0) \delta(x - y)\). Due to \(\Delta_c(x)\) has a singularity in the origin when \(d \geq 2\) this formal expression needs to be correctly defined.

Define this as a solution of an equation in the Schwartz space. The equation can be found as follows. Let us consider the quantity:

\[
(x - y)^2 \Delta_c(xy).
\]

In \(d \leq 4\) this is regular at \(x = y\) because the singularity of the propagator \(\Delta_c\) is “killed” by the factor \((x - y)^2\). Therefore the expression \((x - y)^2 \Delta_c(xy)\lambda(xy)\) is a well defined distribution at the local limit \(\lambda(xy) \rightarrow \lambda \delta(x - y)\). We lay this quite a definite in sense of distributions limit into the base of \(\Sigma_c\) definition. Namely we will take \(\Sigma_c\) as a solution of the equation

\[
(x - y)^2 \Sigma_c(xy) = 2\lambda ((x - y)^2 \Delta_c(xy)) \bigg|_{x = y} \delta(x - y).
\]
i) $d = 2$. In the two-dimensional space we have under $x \to y$

$$\Delta_c (xy) = -\frac{1}{4\pi} \ln \left[ \frac{(x - y)^2 m^2}{4} \right] + \psi(1) + O \left( (x - y)^2 \ln \left[ (x - y)^2 m^2 \right] \right). \quad (37)$$

If follows from (37) that at local limit $\lambda(xy) \to \lambda \delta(x - y)$ the mass operator $\Sigma_c$ satisfy the equation:

$$(x - y)^2 \Sigma_c(xy) = 0. \quad (38)$$

We will define $\Sigma_c$ as a $O(2)$-invariant solution of (38) belonging to the Schwartz space $S'(E_2)$. Such a solution is

$$\Sigma_c(xy) = C \delta(x - y) \quad (39)$$

where $C$ is a constant. It is a finite mass renormalization as can be expected. From formula (37) one can also find that the difference of the two free propagators with masses $m_1$ and $m_2$

$$\Delta_c \left( xy|m_1 \right) - \Delta_c \left( xy|m_2 \right) = \frac{1}{4\pi} \ln \frac{m_2^2}{m_1^2} + O((x - y)^2 \ln[(x - y)^2 m^2]) \quad (40)$$

is non-singular at coinciding arguments. Therefore at $d = 2$

$$\Sigma_c \left( xy|m_1 \right) - \Sigma_c \left( xy|m_2 \right) = \frac{\lambda}{2\pi} \delta(x - y) \ln \frac{m_2^2}{m_1^2}. \quad (41)$$

That is a finite renormalization cancels in the difference of the two mass operators. It is a reflection of the fact that the divergencies of the corresponding integrals in the momentum space cancel.

ii) $d = 3$. This case is similar to the previous one. In the three dimensional space:

$$\Delta_c (xy) = \frac{1}{4\pi \sqrt{(x - y)^2}} \exp \{-m \sqrt{(x - y)^2}\} \quad (42)$$

and $\Sigma_c$ is $O(3)$-invariant solution of (38) belonging to the Schwartz space $S'(E_3)$ and is given by the same formula (39). The two mass operators difference also does not contain the renormalization arbitrariness and reads as follows

$$\Sigma_c (xy|m_1) - \Sigma_c (xy|m_2) = \frac{\lambda}{2\pi} \delta(x - y) (m_2 - m_1). \quad (43)$$

Here $m \equiv \sqrt{m^2}$.

iii) $d = 4$. In this case the theory $\phi^4$ contains quadratic divergencies and result changes. The free propagator is more singular at coinciding arguments

$$\Delta_c (xy) = \frac{1}{4\pi^2 (x - y)^2} + O(\ln[(x - y)^2 m^2]). \quad (44)$$
As a consequence we get an inhomogeneous equation for $\Sigma_c$ at the local limit

$$(x - y)^2 \Sigma_c(xy) = \frac{\lambda}{2\pi^2} \delta(x - y).$$

(45)

Its $O(4)$-invariant solution from the Schwartz space $S'(E_4)$ is

$$\Sigma_c(xy) = \frac{\lambda}{16\pi^2} \partial^2 \delta(x - y) + C \delta(x - y)$$

(46)

that is $\Sigma_c$ contains the finite term of the wave function renormalization besides the mass renormalization. This term reflects the presence of the quadratic divergencies. In contrast to the previous cases the difference of two $\Sigma_c$

$$\Sigma_c(xy|m_1) - \Sigma_c(xy|m_2) = C' \delta(x - y)$$

(47)

contains a finite arbitrariness. To say another, in the corresponding momentum integrals the quadratic divergency cancels but the logarithmic one still remains.

Having defined $\Sigma_c$ in (36) as described above let us turn to the solutions of equation (34). In what follows we will everywhere assume $m^2 > 0$. When $m^2 < 0$ (Goldstone case) a vacuum reconstruction is needed which leads to changing both the initial perturbation theory and all the equations of the approximation scheme.

i) $d = 2$. We will seek solutions for which the representation $\Sigma(xy) = \Sigma \delta(x - y)$ is valid, $\Sigma$ being a constant. Then equation (34) has the following solution in the momentum space:

$$\Delta = \frac{A}{\mu^2 + p^2} + \frac{B}{m^2 + p^2},$$

(48)

where

$$A = \frac{m^2 - \mu^2}{m^2 - \mu_2^2}, \quad B = \frac{m^2 - m^2}{m^2 - \mu^2} = 1 - A$$

(49)

and the notations are introduced:

$$\mu^2 = m^2 + \Sigma, \quad m^2 = m^2 + C = m^2 + \Sigma_c[m].$$

(50)

The operator $\Sigma$ built from $\Delta$ in accordance with (38), (37) and (18) reads

$$\Sigma = A\Sigma_c[\mu] + B\Sigma_c[m] = A(\Sigma_c[\mu] - \Sigma_c[m]) + \Sigma_c[m].$$

(51)

Taking into account formula (44) and definition (30) leads to an equation for $\mu^2$

$$\frac{\lambda}{2\pi} \frac{m^2 - \mu^2}{\mu^2 - m^2} \ln \frac{m^2}{\mu^2} = m^2 - \mu^2.$$  

(52)

Equation (52) has two solutions. One of them is $\mu^2 = m^2$ which implies $A = 0$, $B = 1$ hence $\Delta = \Delta_c$. This is the trivial solution. Another one is a solution of the equation:

$$\frac{\lambda}{2\pi} \frac{1}{\mu^2 - m^2} \ln \frac{m^2}{\mu^2} = 1.$$  

(53)
A solution of (53) exists if $\lambda$ is negative and at small $\lambda$ can be written as

$$\mu^2 \simeq m^2 \exp \left( -\frac{2\pi m^2}{|\lambda|} \right). \quad (54)$$

The corresponding values of $A$ and $B$ are

$$A \simeq \frac{m^2}{m^2} = 1 + O(\lambda), \quad B \simeq \frac{m^2 - m'^2}{m^2} = O(\lambda) \quad (55)$$

and propagator is:

$$\triangle \simeq \frac{1}{\mu^2 + p^2} + O(\lambda). \quad (56)$$

The following circumstances connected with solution (54 - 56) should be mentioned. First, the non-perturbative character of the solution is obvious. Second, the negative value of the $\lambda$ parameter does not obligatory mean that the value of the physical coupling constant is negative. To answer the question about the vacuum stability and the sign of physical coupling corresponding to this solution one should at least consider the next step of the approximation scheme when $[G_2]^{\text{con}} \neq 0$. Such a situation is not unusual one in the quantum field theory (see [11], for example).

ii) $d = 3$. Let us try to find a solution $\Sigma(xy) = \Sigma \delta(x - y)$ again. All formulae (48)-(51) remain to be valid in this case too. The only change consists in using formula (13) for the two mass operators difference. The equation for $\mu^2$ has the form:

$$m'^2 - \mu^2 = -\frac{\lambda}{2\pi} \frac{m'^2 - \mu^2}{m + \mu}. \quad (57)$$

Besides the trivial solution $\mu^2 = m'^2$ ($\triangle = \triangle_c$) the non-perturbative one exists:

$$\mu = -m - \frac{\lambda}{2\pi}. \quad (58)$$

Since $\mu \equiv \sqrt{\mu^2}$ this solution exists at $\lambda \leq -2\pi m$.

iii) $d = 4$. Now $\Sigma_c = -\bar{\lambda} p^2 + C$ where we have denoted

$$\bar{\lambda} = \frac{\lambda}{16\pi^2}.$$

A solution should be sought in the form $\Sigma = ap^2 + b$. It still looks as (18) but now

$$A = \frac{m'^2 - (1 - \bar{\lambda}) \mu^2}{(1 + a) (m'^2 - \mu^2)}, \quad B = \frac{(1 - \bar{\lambda}) m^2 - m'^2}{(1 + a) (m'^2 - \mu^2)}, \quad (59)$$

where

$$\mu^2 = \frac{m^2 + b}{1 + a}, \quad m'^2 = m^2 + C. \quad (60)$$
It follows from definitions (35-36) and equations (48) and (59) that
\[ \Sigma = a p^2 + b = A \Sigma_c[\mu] + B \Sigma_c[m] = A(\Sigma_c[\mu] - \Sigma_c[m]) + \frac{1 - \bar{\lambda}}{1 + a} \Sigma_c[m]. \] (61)

It leads to the system of equations for \( a \) and \( b \):
\[
\begin{aligned}
& a(1 + a) = \bar{\lambda}(\bar{\lambda} - 1) \\
& b = AC' + \frac{1 - \bar{\lambda}}{1 + a} C.
\end{aligned}
\] (62)

We find the two solutions for \( a \): \( a_1 = -\bar{\lambda} \) and \( a_2 = \bar{\lambda} - 1 \). For the first of them \( A + B = 1 \) and \( \mu^2 = m^2 + O(\lambda) \), \( a = 1 + O(\lambda) \). This solution is perturbatively close to the free one:
\[ \triangle = \triangle_c + O(\lambda). \] (63)

The second solution gives
\[ A + B = \frac{(1 - \bar{\lambda})}{\bar{\lambda}}, \quad A = \frac{1}{\bar{\lambda}} + O(1). \]
This solution is of the non-perturbative character:
\[ \triangle = \frac{1}{\bar{\lambda}(\mu^2 + p^2)} + O(1). \] (64)

Besides above mentioned solutions there can exist the ones with a dipole term:
\[ \triangle = \frac{A'}{m^2 + p^2} + \frac{B'}{(m^2 + p^2)^2}. \] (65)

We will not dwell upon this case which as a matter of fact is the particular case of the previous consideration when \( b \to m^2 a \). We only point out that the dipole term is always of the higher order of the smallness in comparison with the simple pole contribution: \( B' = o_\lambda(A') \).

To the end of this section we would like to discuss the regularization dependence of our results. First, such a dependence is obviously absent when \( d < 4 \). Indeed, introducing quantities
\[ \triangle' = \triangle - \triangle_c, \quad \Sigma' = \Sigma - \Sigma_c, \] (66)
in place of \( \triangle \) and \( \Sigma \), we obtain from (36) the following equation:
\[ \triangle' = -(m^2 - \partial^2 + \Sigma' + \Sigma_c)^{-1} \Sigma' * \triangle_c. \] (67)

One can see from (67) that at \( d < 4 \) the quantity \( \triangle'(x = 0) \) is a finite one and therefore so is \( \Sigma' \). Having solved equation (67) we get, of course, all the same results as we do above, irrespective of the regularization.
In case of $d = 4$ the situation is more complicated. When using the standard regularization schemes like cutoff one we find qualitatively the same results but the problem of cutoff removing arises. In our approach this problem do not appear at all.

**Conclusion**

The main idea of the proposed approach is to enlarge the class of solutions of the Schwinger equation by widening the set of admissible boundary conditions. But the problem of boundary conditions itself remains unsolved, of course. In essence, when constructing the approximation scheme in section 1 we constrained ourselves to the solutions which “copy” the connected structure of the perturbation theory. But even under such hard restriction it is possible to find the solutions of the non-perturbative character (section 2). An appropriate choice of the solution can be made in comparing it with the physical phenomena are to be described and on the base of general physical principles such as the minimal energy value of the ground state.

In this connection the investigation of the next step of the scheme containing the two-particle amplitude (equations (26) - (27)) is of undoubted interest. Also it would be interesting to apply this method to other more sophisticated quantum field models.

At last it is worth mentioning that one can construct an approximation scheme taking as an input the loop expansion or the kinetic expansion instead of the perturbation theory.

The summation of the perturbative series, independently of how far ahead this problem could be elaborated cannot give us all the variety of solutions of the quantum field equations. These equations contain much more information than the sum of the perturbation theory does. We hope that the development of approximation schemes taking into account the singular character of the interacting fields equations allows one to enlarge the number of the phenomena described by the quantum field theory.

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