OPERATOR POPOVICIU’S INEQUALITY FOR SUPERQUADRATIC AND CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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ABSTRACT. In this work, operator version of Popoviciu’s inequality for positive selfadjoint operators in Hilbert spaces under positive linear maps for superquadratic functions is proved. Analogously, using the same technique operator version of Popoviciu’s inequality for convex functions is obtained. Some other related inequalities are also deduced.

1. INTRODUCTION

Let \( B(\mathcal{H}) \) be the Banach algebra of all bounded linear operators defined on a complex Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) with the identity operator \( 1_{\mathcal{H}} \) in \( B(\mathcal{H}) \). Denotes \( B^+(\mathcal{H}) \) the convex cone of all positive operators on \( \mathcal{H} \). A bounded linear operator \( A \) defined on \( \mathcal{H} \) is selfadjoint if and only if \( \langle Ax, x \rangle \in \mathbb{R} \) for all \( x \in \mathcal{H} \). For two selfadjoint operators \( A, B \in \mathcal{H} \), we write \( A \leq B \) if \( \langle Ax, x \rangle \leq \langle Bx, x \rangle \) for all \( x \in \mathcal{H} \). Also, we define

\[
\|A\| = \sup_{\|x\|=1} |\langle Ax, x \rangle| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y \rangle|.
\]

If \( \varphi \) is any function defined on \( \mathbb{R} \), we define

\[
\|\varphi\|_A = \sup \{|\varphi(\lambda)| : \lambda \in \text{sp}(A)\}.
\]

If \( \varphi \) is continuous then we write \( \|\varphi\|_A = \|\varphi\| \).

Let \( A \in B(\mathcal{H}) \) be a selfadjoint linear operator on \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \). Let \( C(\text{sp}(A)) \) be the set of all continuous functions defined on the spectrum of \( A \) (\( \text{sp}(A) \)) and let \( C^*(A) \) be the \( C^* \)-algebra generated by \( A \) and the identity operator \( 1_{\mathcal{H}} \).

Let us define the map \( \mathcal{G} : C(\text{sp}(A)) \to C^*(A) \) with the following properties ([12], p.3):

1. \( \mathcal{G}(\alpha f + \beta g) = \alpha \mathcal{G}(f) + \beta \mathcal{G}(g) \), for all scalars \( \alpha, \beta \).
2. \( \mathcal{G}(fg) = \mathcal{G}(f)\mathcal{G}(g) \) and \( \mathcal{G}(f^*) = \mathcal{G}(f)^* \); where \( f^* \) denotes to the conjugate of \( f \) and \( \mathcal{G}(f)^* \) denotes to the Hermitian of \( \mathcal{G}(f) \).
3. \( \|\mathcal{G}(f)\| = \|f\| = \sup_{t \in \text{sp}(A)} |f(t)|. \)
4. \( \mathcal{G}(f_0) = 1_{\mathcal{H}} \) and \( \mathcal{G}(f_1) = A \), where \( f_0(t) = 1 \) and \( f_1(t) = t \) for all \( t \in \text{sp}(A) \).

Accordingly, we define the continuous functional calculus for a selfadjoint operator \( A \) by

\[
f(A) = \mathcal{G}(f) \text{ for all } f \in C(\text{sp}(A)).
\]

If both \( f \) and \( g \) are real valued functions on \( \text{sp}(A) \) then the following important property holds:

\[
(1.1) \quad f(t) \geq g(t) \text{ for all } t \in \text{sp}(A) \implies f(A) \geq g(A),
\]

in the operator order of \( B(K) \).

A linear map is defined to be \( \Phi : B(\mathcal{H}) \to B(K) \) which preserves additivity and homogeneity, i.e., \( \Phi(\lambda_1 A + \lambda_2 B) = \lambda_1 \Phi(A) + \lambda_2 \Phi(B) \) for any \( \lambda_1, \lambda_2 \in \mathbb{C} \) and \( A, B \in B(\mathcal{H}) \). The linear map is positive \( \Phi : B(\mathcal{H}) \to B(K) \) if it preserves the operator order, i.e., if \( A \in B^+(\mathcal{H}) \) then \( \Phi(A) \in B^+(K) \), and in this case we write \( \mathfrak{B}(B(\mathcal{H}), B(K)) \). Obviously, a positive linear map \( \Phi \) preserves the order relation, namely \( A \leq B \implies \Phi(A) \leq \Phi(B) \) and preserves the adjoint operation \( \Phi(A^*) = \Phi(A)^* \). Moreover, \( \Phi \) is said to be normalized (unital) if it preserves the identity operator, i.e. \( \Phi(1_{\mathcal{H}}) = 1_K \), in this case we write \( \mathfrak{B}_u(B(\mathcal{H}), B(K)) \).

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\end{itemize}
1.1. **Superquadratic functions.** A function \( f : J \to \mathbb{R} \) is called convex iff

\[
(1.2) \quad f (t \alpha + (1 - t) \beta) \leq tf (\alpha) + (1 - t) f (\beta),
\]

for all points \( \alpha, \beta \in J \) and all \( t \in [0, 1] \). If \(-f\) is convex then we say that \( f \) is concave. Moreover, if \( f \) is both convex and concave, then \( f \) is said to be affine.

Geometrically, for two points \((x, f(x))\) and \((y, f(y))\) on the graph of \( f \) are on or below the chord joining the endpoints for all \( x, y \in I, x < y \). In symbols, we write

\[
f (t) \leq \frac{f (y) - f (x)}{y - x} (t - x) + f (x)
\]

for any \( x \leq t \leq y \) and \( x, y \in J \).

Equivalently, given a function \( f : J \to \mathbb{R} \), we say that \( f \) admits a support line at \( x \in J \) if there exists a \( \lambda \in \mathbb{R} \) such that

\[
(1.3) \quad f (t) \geq f (x) + \lambda (t - x)
\]

for all \( t \in J \).

The set of all such \( \lambda \) is called the subdifferential of \( f \) at \( x \), and it’s denoted by \( \partial f \). Indeed, the subdifferential gives us the slopes of the supporting lines for the graph of \( f \). So that if \( f \) is convex then \( \partial f (x) \neq \emptyset \) at all interior points of its domain.

From this point of view Abramovich et al. [3] extend the above idea for what they called superquadratic functions. Namely, a function \( f : [0, \infty) \to \mathbb{R} \) is called superquadratic provided that for all \( x \geq 0 \) there exists a constant \( C_x \in \mathbb{R} \) such that

\[
(1.4) \quad f (t) \geq f (x) + C_x (t - x) + f (|t - x|)
\]

for all \( t \geq 0 \). We say that \( f \) is subquadratic if \(-f\) is superquadratic. Thus, for a superquadratic function we require that \( f \) lie above its tangent line plus a translation of \( f \) itself.

Prima facie, superquadratic function looks to be stronger than convex function itself but if \( f \) takes negative values then it may be considered as a weaker function. Therefore, if \( f \) is superquadratic and non-negative. Then \( f \) is convex and increasing [3] (see also [1]).

Moreover, the following result holds for superquadratic function.

**Lemma 1.** [3] Let \( f \) be superquadratic function. Then

1. \( f (0) \leq 0 \)
2. If \( f \) is differentiable and \( f (0) = f' (0) = 0 \), then \( C_x = f' (x) \) for all \( x \geq 0 \).
3. If \( f (x) \geq 0 \) for all \( x \geq 0 \), then \( f \) is convex and \( f (0) = f' (0) = 0 \).

The next result gives a sufficient condition when convexity (concavity) implies super(sub)quadracity.

**Lemma 2.** [3] If \( f' \) is convex (concave) and \( f (0) = f' (0) = 0 \), then is super(sub)quadratic. The converse of is not true.

**Remark 1.** Subquadraticity does always not imply concavity; i.e., there exists a subquadratic function which is convex. For example, \( f (x) = x^p, x \geq 0 \) and \( 1 \leq p \leq 2 \) is subquadratic and convex.

1.2. **Popoviciu’s inequality.** In 1906, Jensen in [15] proved his famous characterization of convex functions. Simply, for a continuous functions \( f \) defined on a real interval \( I \), \( f \) is convex if and only if

\[
f \left( \frac{x + y}{2} \right) \leq \frac{f (x) + f (y)}{2},
\]

for all \( x, y \in I \).

In 1965, a parallel characterization of Jensen convexity was presented by Popoviciu [27] (for more details see [26], p.6), where he proved his celebrated inequality, as follows:

**Theorem 1.** Let \( f : I \to \mathbb{R} \) be continuous. Then, \( f \) is convex if and only if

\[
(1.5) \quad \frac{2}{3} \left[ f \left( \frac{x + z}{2} \right) + f \left( \frac{y + z}{2} \right) + f \left( \frac{x + y}{2} \right) \right] \leq f \left( \frac{x + y + z}{3} \right) + \frac{f (x) + f (y) + f (z)}{3}
\]

for all \( x, y, z \in I \), and the equality occurred by \( f (x) = x, x \in I \).
In fact, Popoviciu characterization of convex function is sound and several mathematicians greatly received his work since that time and much of them considered his characterization as alternative approach to describe convex functions. For instance, the Popoviciu’s inequality can be considered as an elegant generalization of Hlawka’s inequality using convexity as a simple tool of geometry. Indeed, if \( f(x) = |x|, \ x \in \mathbb{R} \), then the Popoviciu inequality reduces to the famous Hlawka inequality, which reads:

\[
|x| + |y| + |z| + |x + y + z| \geq |x + z| + |z + y| + |x + y|. 
\]

Geometrically, Hlawka inequality means the total length over all sums of pairs from three vectors is not greater than the perimeter of the quadrilateral defined by the three vectors. This geometric meaning was given by D. Smiley & M. Smiley [32] (see also [28], p. 756). For other related results see [20] and [25].

Also, The extended version of Hlawka’s inequality to several variables was not possible without the help of Popoviciu’s inequality, as it inspired the authors of [7] to develop a higher dimensional analogue of Popoviciu’s inequality based on his characterization. Interesting generalizations and counterparts of Popoviciu inequality with some ramified consequences can be found in [13], [29], [30] and [31].

Recently, The corresponding version of Popoviciu inequality for GG-convex (Recall that: a positive real valued function \( f \) is GG-convex if and only if \( f \left( x^t y^{1-t} \right) \leq \left[ f(\langle x \rangle) \right]^t \left[ f(\langle y \rangle) \right]^{1-t} \) for all \( t \in [0,1] \) and all \( x, y \geq 0 \)) was discussed eighteen years ago by Niculescu in [24], where he proved that for all \( x, y, z \in I \subset [0,\infty) \), the inequality

\[
f^2(\sqrt{xyz}) \leq f^3(\sqrt[3]{xyz}) \quad \text{for all} \quad x, y, z \in I.
\]

Seeking the operator version of Popoviciu’s inequality (1.5), the expected version of (1.5) for selfadjoint operators is

\[
\frac{2}{3} \left[ f\left( \left\langle \frac{A+B}{2} u, u \right\rangle \right) + f\left( \left\langle \frac{B+D}{2} u, u \right\rangle \right) \right] + f\left( \left\langle \frac{A+D}{3} u, u \right\rangle \right) \leq f\left( \left\langle \frac{f(A) + f(B) + f(D)}{3} u, u \right\rangle \right)
\]

for every selfadjoint operators \( A, B, D \in \mathcal{B}(\mathcal{H}) \) whose spectra contained in \( I \) and every convex function \( f \) defined on \( I \) and this is valid for each \( u \in \mathcal{K} \) with \( \|u\| = 1 \). The proof of the above inequality is obvious by taking \( x = \langle Au, u \rangle \), \( y = \langle Bu, u \rangle \) and \( z = \langle Du, u \rangle \) in (1.5).

In this work, we offer two operator versions of Popoviciu’s inequality for positive selfadjoint operators in Hilbert spaces under positive linear maps for both superquadratic and convex functions with some other related results.

2. Main Result

Throughout this work and in all needed situations, \( f \) is real valued continuous function defined on \([0,\infty)\). In order to prove our main result, we need the following result concerning Jensen’s inequality for superquadratic functions. Let us don’t miss the chance here to mention that the next result was proved in [18] and originally in [17] for positive selfadjoint \((n \times n)\)-matrices with complex entries under unital completely positive linear maps. However, let us state down this result in more general Hilbert spaces for normalized positive linear maps.

**Theorem 2.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be a positive selfadjoint operator, \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a normalized positive linear map. If \( f : [0,\infty) \to \mathbb{R} \) is super(sub)quadratic, then we have

\[
\langle \Phi (f(A)) x, x \rangle \geq (\leq) f \left( \langle \Phi (A) x, x \rangle \right) + \langle \Phi (f(\|A - \langle \Phi (A) x, x \rangle 1_{\mathcal{H}}\|)) x, x \rangle
\]

for every \( x \in \mathcal{K} \) with \( \|x\| = 1 \).

For more recent results concerning inequalities for selfadjoint operators and other related result, we suggest [2], [4]-[12], [16], [19] and [22].

The operator version of Popoviciu’s inequality for superquadratic functions under positive linear maps is proved in the next result.
Theorem 3. Let $A, B, C \in \mathcal{B}(\mathcal{H})$ be three positive selfadjoint operators, $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a normalized positive linear map. If $f : [0, \infty) \to \mathbb{R}$ is superquadratic, then we have

\begin{equation}
\langle \Phi \left( \frac{f(A) + f(B) + f(D)}{3} \right) x, x \rangle + f \left( \langle \Phi \left( \frac{A + B + D}{3} \right) x, x \rangle \right)
\geq \frac{2}{3} \left[ f \left( \langle \Phi \left( \frac{A + B}{2} \right) x, x \rangle \right) + f \left( \langle \Phi \left( \frac{B + D}{2} \right) x, x \rangle \right) + f \left( \langle \Phi \left( \frac{A + D}{2} \right) x, x \rangle \right) \right]
+ \frac{1}{3} \left[ \langle \Phi \left( f \left( A - \langle \Phi \left( \frac{B + D}{2} \right) x, x \rangle 1_\mathcal{H} \right) \right) x, x \rangle + f \left( \langle \Phi \left( \frac{2A - B - D}{6} \right) x, x \rangle \right) \right]
+ \langle \Phi \left( f \left( D - \langle \Phi \left( \frac{A + B}{2} \right) x, x \rangle 1_\mathcal{H} \right) \right) x, x \rangle + f \left( \langle \Phi \left( \frac{2D - A - B}{6} \right) x, x \rangle \right)
+ \langle \Phi \left( f \left( B - \langle \Phi \left( \frac{A + D}{2} \right) x, x \rangle 1_\mathcal{H} \right) \right) x, x \rangle + f \left( \langle \Phi \left( \frac{2B - A - D}{6} \right) x, x \rangle \right) \right]
\end{equation}

for each $x \in \mathcal{K}$ with $\|x\| = 1$.

Proof. Since $f$ is superquadratic on $I$, then by utilizing the continuous functional calculus for the operator $E \geq 0$ we have by the property (1.1) for the inequality (1.4) we have

\[ f(E) \geq f(s) \cdot 1_\mathcal{H} + C_s (E - s \cdot 1_\mathcal{H}) + f(|E - s \cdot 1_\mathcal{H}|) \]

and since $\Phi$ is normalized positive linear map we get

\[ \Phi(f(E)) \geq f(s) \cdot 1_\mathcal{K} + C_s \Phi(E - s \cdot 1_\mathcal{H}) + \Phi(f(|E - s \cdot 1_\mathcal{H}|)) \]

and this implies that

\begin{equation}
\langle \Phi \left( f(E) \right) x, x \rangle \geq f(s) \langle x, x \rangle + C_s \langle \Phi \left( E - s \cdot 1_\mathcal{H} \right) x, x \rangle + \langle \Phi \left( f \left( |E - s \cdot 1_\mathcal{H}| \right) \right) x, x \rangle
\end{equation}

for each vector $x \in \mathcal{K}$ with $\|x\| = 1$.

Let $A, B, D$ be three positive selfadjoint operators in $\mathcal{B}(\mathcal{H})$. Since $f$ is superquadratic then by applying (2.3) for the operator $A \geq 0$ with $s_1 = \langle \Phi \left( \frac{B+D}{2} \right) x, x \rangle$, we get

\begin{equation}
\langle \Phi \left( f(A) \right) x, x \rangle \geq f \left( \langle \Phi \left( \frac{B+D}{2} \right) x, x \rangle \right) + C_s \langle \Phi \left( \frac{2A - B - D}{2} \right) x, x \rangle
+ \langle \Phi \left( f \left( A - \langle \Phi \left( \frac{B+D}{2} \right) x, x \rangle 1_\mathcal{H} \right) \right) x, x \rangle
\end{equation}

for each $x \in \mathcal{K}$ with $\|x\| = 1$.

Again applying (2.3) for the operator $D \geq 0$ with $s_2 = \langle \Phi \left( \frac{A+B}{2} \right) x, x \rangle$

\begin{equation}
\langle \Phi \left( f(D) \right) x, x \rangle \geq f \left( \langle \Phi \left( \frac{A+B}{2} \right) x, x \rangle \right) + C_s \langle \Phi \left( \frac{2D - A - B}{2} \right) x, x \rangle
+ \langle \Phi \left( f \left( D - \langle \Phi \left( \frac{A+B}{2} \right) x, x \rangle 1_\mathcal{H} \right) \right) x, x \rangle
\end{equation}

for each $x \in \mathcal{K}$ with $\|x\| = 1$.

Also, for the operator $B \geq 0$ with $s_3 = \langle \Phi \left( \frac{A+D}{2} \right) x, x \rangle$

\begin{equation}
\langle \Phi \left( f(B) \right) x, x \rangle \geq f \left( \langle \Phi \left( \frac{A+D}{2} \right) x, x \rangle \right) + C_s \langle \Phi \left( \frac{2B - A - D}{2} \right) x, x \rangle
+ \langle \Phi \left( f \left( B - \langle \Phi \left( \frac{A+D}{2} \right) x, x \rangle 1_\mathcal{H} \right) \right) x, x \rangle.
\end{equation}

for each $x \in \mathcal{K}$ with $\|x\| = 1$.
Adding the inequalities (2.4)--(2.6) and then multiplying by \( \frac{1}{3} \) we get

\[
\left\langle \Phi \left( \frac{f(A) + f(B) + f(D)}{3} \right), x, x \right\rangle
\]

\[
\geq \frac{1}{3} \left[ f \left( \left\langle \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle \right) + f \left( \left\langle \Phi \left( \frac{B + D}{2} \right), x, x \right\rangle \right) + f \left( \left\langle \Phi \left( \frac{A + D}{2} \right), x, x \right\rangle \right) \right]
\]

\[
+ \frac{1}{3} \left[ C_{s_1} \left\langle \Phi \left( \frac{2A - B - D}{2} \right), x, x \right\rangle + C_{s_2} \left\langle \Phi \left( \frac{2D - A - B}{2} \right), x, x \right\rangle \right.
\]

\[
+ C_{s_3} \left\langle \Phi \left( \frac{2B - A - D}{2} \right), x, x \right\rangle \right]
\]

\[
+ \left\langle \Phi \left( f \left[ A - \left\langle \Phi \left( \frac{B + D}{2} \right), x, x \right\rangle \right] \right), x, x \right\rangle + \left\langle \Phi \left( f \left[ D - \left\langle \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle \right] \right), x, x \right\rangle \right] .
\]

Setting \( C := \min \{C_{s_1}, C_{s_2}, C_{s_3}\} \), then (2.7) reduces to

\[
\left\langle \Phi \left( \frac{f(A) + f(B) + f(D)}{3} \right), x, x \right\rangle
\]

\[
\geq \frac{1}{3} \left[ f \left( \left\langle \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle \right) + f \left( \left\langle \Phi \left( \frac{B + D}{2} \right), x, x \right\rangle \right) + f \left( \left\langle \Phi \left( \frac{A + D}{2} \right), x, x \right\rangle \right) \right]
\]

\[
+ \frac{1}{3} \left[ \left\langle \Phi \left( \frac{2A - B - D}{2} \right), x, x \right\rangle + \left\langle \Phi \left( \frac{2D - A - B}{2} \right), x, x \right\rangle + \left\langle \Phi \left( \frac{2B - A - D}{2} \right), x, x \right\rangle \right.
\]

\[
+ \left\langle \Phi \left( f \left[ A - \left\langle \Phi \left( \frac{B + D}{2} \right), x, x \right\rangle \right] \right), x, x \right\rangle + \left\langle \Phi \left( f \left[ D - \left\langle \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle \right] \right), x, x \right\rangle \right] .
\]

Now, applying (1.4) three times for \( t = \left\langle \Phi \left( \frac{A + B + C}{3} \right), x, x \right\rangle \) with \( s_1, s_2, s_3 \), then for each unit vector \( x \in \mathcal{K} \), we get respectively,

\[
f \left( \left\langle \Phi \left( \frac{A + B + D}{3} \right), x, x \right\rangle \right)
\]

\[
\geq f \left( \left\langle \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle \right) + C_{s_1} \left\langle \Phi \left( \frac{A + B + D}{3} \right), x, x \right\rangle - C_{s_1} \left\langle \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle
\]

\[
+ f \left( \left\langle \Phi \left( \frac{A + B + D}{3} \right), x, x \right\rangle - \left\langle \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle \right)
\]

\[
= f \left( \left\langle \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle \right) + C_{s_1} \left\langle \Phi \left( \frac{2D - A - B}{6} \right), x, x \right\rangle + f \left( \left\langle \Phi \left( \frac{2D - A - B}{6} \right), x, x \right\rangle \right) .
\]
Multiplying each inequality by $\frac{1}{3}$ and summing up the inequalities (2.9)-(2.11), we get

\begin{equation}
\begin{aligned}
f \left( \langle \Phi \left( \frac{A+B+D}{2} \right), x, x \rangle \right) \\
\geq f \left( \langle \Phi \left( \frac{A+B}{2} \right), x, x \rangle \right) + C_{s_2} \left( \langle \Phi \left( \frac{A+B+D}{3} \right), x, x \rangle \right) - C_{s_2} \left( \langle \Phi \left( \frac{B+D}{2} \right), x, x \rangle \right) \\
+ f \left( \langle \Phi \left( \frac{A+B+D}{3} \right), x, x \rangle - \langle \Phi \left( \frac{B+D}{2} \right), x, x \rangle \right) \\
= f \left( \langle \Phi \left( \frac{A+B+D}{2} \right), x, x \rangle \right) + C_{s_2} \left( \langle \Phi \left( \frac{2A-B-D}{6} \right), x, x \rangle \right) + f \left( \langle \Phi \left( \frac{2A-B-D}{6} \right), x, x \rangle \right)
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
f \left( \langle \Phi \left( \frac{A+B+D}{3} \right), x, x \rangle \right) \\
\geq f \left( \langle \Phi \left( \frac{A+B}{2} \right), x, x \rangle \right) + C_{s_3} \left( \langle \Phi \left( \frac{A+B+D}{3} \right), x, x \rangle \right) - C_{s_3} \left( \langle \Phi \left( \frac{A+D}{2} \right), x, x \rangle \right) \\
+ f \left( \langle \Phi \left( \frac{A+B+D}{3} \right), x, x \rangle - \langle \Phi \left( \frac{A+D}{2} \right), x, x \rangle \right) \\
= f \left( \langle \Phi \left( \frac{A+B+D}{2} \right), x, x \rangle \right) + C_{s_3} \left( \langle \Phi \left( \frac{2B-A-D}{6} \right), x, x \rangle \right) + f \left( \langle \Phi \left( \frac{2B-A-D}{6} \right), x, x \rangle \right).
\end{aligned}
\end{equation}

But since $C := \min \{C_{s_1}, C_{s_2}, C_{s_3}\}$, then (2.12) becomes

\begin{equation}
\begin{aligned}
f \left( \langle \Phi \left( \frac{A+B+D}{3} \right), x, x \rangle \right) \\
\geq \frac{1}{3} f \left( \langle \Phi \left( \frac{A+B}{2} \right), x, x \rangle \right) + \frac{1}{3} f \left( \langle \Phi \left( \frac{A+D}{2} \right), x, x \rangle \right) + \frac{1}{3} f \left( \langle \Phi \left( \frac{A+B}{2} \right), x, x \rangle \right) \\
+ \frac{1}{3} C \left[ \langle \Phi \left( \frac{2D-A-B}{6} \right), x, x \rangle \right] + \frac{1}{3} \left[ \langle \Phi \left( \frac{2A-B-D}{6} \right), x, x \rangle \right] + \frac{1}{3} \left[ \langle \Phi \left( \frac{2B-A-D}{6} \right), x, x \rangle \right] \\
+ \frac{1}{3} \left[ \langle \Phi \left( \frac{2D-A-B}{6} \right), x, x \rangle \right] + \frac{1}{3} \left[ \langle \Phi \left( \frac{2A-B-D}{6} \right), x, x \rangle \right] \\
= \frac{1}{3} f \left( \langle \Phi \left( \frac{A+B}{2} \right), x, x \rangle \right) + \frac{1}{3} f \left( \langle \Phi \left( \frac{A+D}{2} \right), x, x \rangle \right) \\
+ \frac{1}{3} f \left( \langle \Phi \left( \frac{2D-A-B}{6} \right), x, x \rangle \right) + \frac{1}{3} f \left( \langle \Phi \left( \frac{2A-B-D}{6} \right), x, x \rangle \right) \\
+ \frac{1}{3} f \left( \langle \Phi \left( \frac{2B-A-D}{6} \right), x, x \rangle \right).
\end{aligned}
\end{equation}
Adding the inequalities (2.8) and (2.13) we get that
\[
\left\langle \Phi \left( \frac{f(A) + f(B) + f(D)}{3} \right), x, x \right\rangle + f \left( \left\langle \Phi \left( \frac{A + B + D}{3} \right), x, x \right\rangle \right) \\
\geq \frac{2}{3} \left[ f \left( \left\langle \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle \right) + f \left( \left\langle \Phi \left( \frac{B + D}{2} \right), x, x \right\rangle \right) + f \left( \left\langle \Phi \left( \frac{A + D}{2} \right), x, x \right\rangle \right) \right] \\
+ \frac{1}{3} \left[ f \left( \left\langle A - \Phi \left( \frac{B + D}{2} \right), x, x \right\rangle 1_\mathcal{H} \right) \left\rangle, x, x \right\rangle + f \left( \left\langle D - \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle 1_\mathcal{H} \right) \left\rangle, x, x \right\rangle \right] \\
+ f \left( \left\langle \Phi \left( \frac{2A - B - D}{6} \right), x, x \right\rangle \right) + f \left( \left\langle \Phi \left( \frac{2B - A - D}{6} \right), x, x \right\rangle \right)
\]
for each \( x \in \mathcal{K} \) with \( \|x\| = 1 \), which gives the required inequality in (2.2).  

\[\square\]

**Corollary 1.** Let \( A, B, C \in \mathcal{B}(\mathcal{H}) \) be three positive selfadjoint operators, \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a normalized positive linear map. If \( f : [0, \infty) \to \mathbb{R} \) is subquadratic, then we have
\[
\left\langle \Phi \left( \frac{f(A) + f(B) + f(D)}{3} \right), x, x \right\rangle + f \left( \left\langle \Phi \left( \frac{A + B + D}{3} \right), x, x \right\rangle \right) \\
\leq \frac{2}{3} \left[ f \left( \left\langle \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle \right) + f \left( \left\langle \Phi \left( \frac{B + D}{2} \right), x, x \right\rangle \right) + f \left( \left\langle \Phi \left( \frac{A + D}{2} \right), x, x \right\rangle \right) \right] \\
+ \frac{1}{3} \left[ f \left( \left\langle A - \Phi \left( \frac{B + D}{2} \right), x, x \right\rangle 1_\mathcal{H} \right) \left\rangle, x, x \right\rangle + f \left( \left\langle D - \Phi \left( \frac{A + B}{2} \right), x, x \right\rangle 1_\mathcal{H} \right) \left\rangle, x, x \right\rangle \right] \\
+ f \left( \left\langle \Phi \left( \frac{2A - B - D}{6} \right), x, x \right\rangle \right) + f \left( \left\langle \Phi \left( \frac{2B - A - D}{6} \right), x, x \right\rangle \right)
\]
for each \( x \in \mathcal{K} \) with \( \|x\| = 1 \).

*Proof.* Repeating the same steps in the proof of Theorem 3, by writing \( \leq \) instead of \( \geq \) and in this case we consider \( C := \max \{C_3, C_{3x}, C_{3z}\} \).

\[\square\]

A generalization of the result in Theorem 2 is deduced as follows:

**Corollary 2.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be a positive selfadjoint operator, \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a normalized positive linear map. If \( f : [0, \infty) \to \mathbb{R} \) is super(sub)quadratic, then we have
\[
f \left( \left\langle \Phi (A), x, x \right\rangle \right) \leq \left( \left\langle \Phi (f(A)), x, x \right\rangle - \left\langle \Phi (A - \Phi (A) x, x \rangle 1_\mathcal{H}) \right) x, x \right\rangle \right) - f(0)
\]
for each \( x \in \mathcal{K} \) with \( \|x\| = 1 \).

*Proof.* Setting \( D = B = A \) in (2.2) we get the required result.

\[\square\]

**Remark 2.** According to Corollary 2, if \( f \geq 0 \) (\( f \leq 0 \)) then the above inequality refines and improves Theorem 2.

The classical Bohr inequality for scalars reads that if \( a, b \) are complex numbers and \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then
\[
|a - b|^2 \leq p |a|^2 + q |b|^2.
\]

The first result regarding operator version of Bohr inequality was established in [14]. For refinements, generalizations and other related results see [8], [9], [11] and [33].

The following Bohr's type inequalities for positive selfadjoint operators under positive linear maps are hold:

**Corollary 3.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be a positive selfadjoint operator, \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a normalized positive linear map.
(1) If \( f : [0, \infty) \to \mathbb{R} \) is superquadratic, then we have
\[
\| \Phi (f (|A - \| \Phi (A) \| 1_H|)) \| \leq \| \Phi (f (A)) \| - f (\| \Phi (A) \|) - f (0) .
\]
In particular, let \( f(t) = t^r, r \geq 2, t \geq 0. \)
\[
\| \Phi (|A - \| \Phi (A) \| 1_H|) \| \leq \| \Phi (A^r) \| - \| \Phi (A) \| .
\]

(2) If \( f : [0, \infty) \to \mathbb{R} \) is subquadratic, then we have
\[
\| \Phi (f (|A - \| \Phi (A) \| 1_H|)) \| \geq \| \Phi (f (A)) \| - f (\| \Phi (A) \|) - f (0) .
\]
In particular, let \( f(t) = t^r, 0 < r \leq 2, t \geq 0. \)
\[
\| \Phi (|A - \| \Phi (A) \| 1_H|) \| \geq \| \Phi (A^r) \| - \| \Phi (A) \| .
\]

Proof. Taking the supremum in (2.15) over \( x \in K \) with \( \| x \| = 1 \) we obtain the required result(s). \( \square \)

Corollary 4. Let \( A_j, B_j, D_j \in \mathcal{B}(\mathcal{H}) \) be three positive selfadjoint operators for every \( j = 1, \ldots, n \). Let \( \Phi_j : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be positive linear maps such that \( \sum_{j=1}^{n} \Phi_j (1_H) = 1_K \). If \( f : [0, \infty) \to \mathbb{R} \) is superquadratic, then we have
\[
(2.16) \quad f \left( \sum_{j=1}^{n} \Phi_j \left( A_j + \frac{B_j + D_j}{3} \right) u, u \right) + f \left( \sum_{j=1}^{n} \Phi_j \left( \frac{B_j + D_j}{3} \right) u, u \right) \\
\geq \frac{2}{3} \left[ f \left( \sum_{j=1}^{n} \Phi_j \left( A_j + \frac{B_j + D_j}{2} \right) u, u \right) + f \left( \sum_{j=1}^{n} \Phi_j \left( \frac{B_j + D_j}{2} \right) u, u \right) \\
+ \frac{1}{3} \left( \sum_{j=1}^{n} \Phi_j \left( f \left( A_j - \sum_{j=1}^{n} \Phi_j \left( \frac{B_j + D_j}{2} \right) u, u \right) 1_H \right) \right) u, u \right) \\
+ f \left( \sum_{j=1}^{n} \Phi_j \left( \frac{2A_j - B_j - D_j}{6} \right) u, u \right) \\
+ \frac{1}{3} \left( \sum_{j=1}^{n} \Phi_j \left( f \left( B_j - \sum_{j=1}^{n} \Phi_j \left( \frac{A_j + B_j}{2} \right) u, u \right) 1_H \right) \right) u, u \right) \\
+ f \left( \sum_{j=1}^{n} \Phi_j \left( \frac{2D_j - A_j - B_j}{6} \right) u, u \right) .
\]

Proof. Let \( E \) stands for \( A, B, D \). Since \( E \in \mathcal{B}^+(\mathcal{H}) \), then there exists \( E_1, \ldots, E_n \in \mathcal{B}^+(\mathcal{H}) \) (where \( E_j \) stands for \( A_j, B_j, D_j \) for all \( j = 1, \ldots, n \)) such that \( E = E_1 \oplus \cdots \oplus E_n \in \mathcal{B}^+(\mathcal{H} \oplus \cdots \oplus \mathcal{H}) \) for every unit vector \( u = (u_1, \ldots, u_n) \in \mathcal{H} \oplus \cdots \oplus \mathcal{H} \). Let \( \Phi : \mathcal{B}^+(\mathcal{H} \oplus \cdots \oplus \mathcal{H}) \to \mathcal{B}^+(\mathcal{K}) \) be a positive normalized linear map defined by \( \Phi (E) = \sum_{j=1}^{n} \Phi_j (E_j) \). By utilizing Theorem 3 we get the desired result. \( \square \)

Corollary 5. Let \( A_j \in \mathcal{B}(\mathcal{H}) \) be positive selfadjoint operators for each \( j = 1, \ldots, n \). Let \( \Phi_j : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be positive linear maps such that \( \sum_{j=1}^{n} \Phi_j (1_H) = 1_K \). If \( f : [0, \infty) \to \mathbb{R} \) is superquadratic, then we have
\[
(2.17) \quad \left\langle \sum_{j=1}^{n} \Phi_j (f (A_j)) u, u \right\rangle \\
\geq (\leq) \left[ \sum_{j=1}^{n} \Phi_j (A_j) u, u \right\rangle + \sum_{j=1}^{n} \Phi_j \left( f \left( A_j - \sum_{j=1}^{n} \Phi_j (A_j) u, u \right) 1_H \right) \right) u, u \rangle + f (0) 
\]
for each \( x \in K \) with \( \| x \| = 1 \).

Proof. Setting \( D_j = B_j = A_j \) for each \( j = 1, \ldots, n \), in (2.16) we get the required result. \( \square \)

As a direct consequence of Theorem 3, the expected operator version Popoviciu’s inequality for convex functions would be as follows:
Proposition 1. Let \( A, B, C \in \mathcal{B}(\mathcal{H}) \) be three positive selfadjoint operators, \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a normalized positive linear map. If \( f : [0, \infty) \to \mathbb{R} \) is non-negative and superquadratic, then \( f \) is convex and

\[
\left\langle \frac{f(A) + f(B) + f(D)}{3}, x, x \right\rangle + f \left( \left\langle \frac{A + B + D}{3}, x, x \right\rangle \right) \geq \frac{2}{3} \left[ f \left( \left\langle \frac{A + B}{2}, x, x \right\rangle \right) + f \left( \left\langle \frac{B + D}{2}, x, x \right\rangle \right) + f \left( \left\langle \frac{A + D}{2}, x, x \right\rangle \right) \right]
\]

for each \( x \in \mathcal{K} \) with \( \|x\| = 1 \).

Proof. Since \( f \) is non-negative superquadratic then by Lemma 1 \( f \) is convex and so that from (2.2), we get

\[
\left\langle \frac{f(A) + f(B) + f(D)}{3}, x, x \right\rangle + f \left( \left\langle \frac{A + B + D}{3}, x, x \right\rangle \right) \geq \frac{2}{3} \left[ f \left( \left\langle \frac{A + B}{2}, x, x \right\rangle \right) + f \left( \left\langle \frac{B + D}{2}, x, x \right\rangle \right) + f \left( \left\langle \frac{A + D}{2}, x, x \right\rangle \right) \right]
\]

\[
+ \frac{1}{3} \left[ f \left( \left\langle A - \left( \frac{B + D}{2} \right)^2, x, x \right\rangle \right) + f \left( \left\langle \frac{B + D}{2}, x, x \right\rangle \right) + f \left( \left\langle \frac{A + D}{2}, x, x \right\rangle \right) \right]
\]

\[
+ \left\langle \frac{2A - B - D}{6}, x, x \right\rangle + f \left( \left\langle \frac{2D - A - B}{6}, x, x \right\rangle \right) + f \left( \left\langle \frac{2B - A - D}{6}, x, x \right\rangle \right)
\]

which gives (2.18).

\( \Box \)

Proposition 2. Let \( A, B, D \in \mathcal{B}(\mathcal{H}) \) be three selfadjoint operators, \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a normalized positive linear map and \( f : [0, \infty) \to \mathbb{R} \) be a differentiable function with \( f(0) = f'(0) = 0 \). If \( f' \) is convex (concave), then \( f \) is super(sub)quadratic and

\[
\left\langle f' \left( \frac{f'(A) + f'(B) + f'(D)}{3}, x, x \right) \right\rangle + f' \left( \left\langle \frac{A + B + D}{3}, x, x \right\rangle \right) \geq \frac{2}{3} \left[ f' \left( \left\langle \frac{A + B}{2}, x, x \right\rangle \right) + f' \left( \left\langle \frac{B + D}{2}, x, x \right\rangle \right) + f' \left( \left\langle \frac{A + D}{2}, x, x \right\rangle \right) \right]
\]

for each \( x \in \mathcal{K} \) with \( \|x\| = 1 \).

Proof. The superquadratic of \( f \) follows from Lemma 2. To obtain the inequality (2.19) we apply the same technique considered in the proof of Theorem 3, by applying (1.3) for \( f' \) instead of (1.4) for \( f \), so that we get the required result. \( \Box \)

Proposition 3. Let \( A, B, D \in \mathcal{B}(\mathcal{H}) \) be three selfadjoint operators, \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a normalized positive linear map and \( g : [0, \infty) \to \mathbb{R} \) be a continuous function. If \( g \) is convex (concave) and \( g(0) = 0 \), then

\[
\left\langle g \left( \frac{g(A) + g(B) + g(D)}{3}, x, x \right) \right\rangle + g \left( \left\langle \frac{A + B + D}{3}, x, x \right\rangle \right) \geq \frac{2}{3} \left[ g \left( \left\langle \frac{A + B}{2}, x, x \right\rangle \right) + g \left( \left\langle \frac{B + D}{2}, x, x \right\rangle \right) + g \left( \left\langle \frac{A + D}{2}, x, x \right\rangle \right) \right]
\]

for each \( x \in \mathcal{K} \) with \( \|x\| = 1 \).

Proof. Applying Corollary 5 for \( \Gamma = \int_0^t g(s) \, ds, t \in [0, \infty) \), then it’s easy to observe that \( \Gamma(0) = G'(0) = 0 \) and \( G'(t) = g(t) \) is convex (concave) for all \( t \in [0, \infty) \). \( \Box \)

Inequality (2.20) holds with more weaker conditions, indeed neither continuity assumption nor the image of 0 is needed, it is hold just with convexity assumption, as follows:

Theorem 4. Let \( A, B, D \in \mathcal{B}(\mathcal{H}) \) be three selfadjoint operators with \( \text{sp}(A), \text{sp}(B), \text{sp}(D) \subset [\gamma, \Gamma] \) for some real numbers \( \gamma, \Gamma \) with \( \gamma < \Gamma \). Let \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a normalized positive linear map. If \( f : [\gamma, \Gamma] \to \mathbb{R} \) is convex (concave) function, then (2.20) holds for each \( x \in \mathcal{K} \) with \( \|x\| = 1 \). The inequality is satisfied with \( f(t) = t \).
Proof. Applying the same technique considered in the proof of Theorem 3, by applying (1.3) for \( f \) instead of (1.4) for \( f \).

\[ \square \]

Remark 3. Employing (2.20) for \( g(x) = |x|, x \in \mathbb{R} \) then we observe that

\[
\begin{align*}
&\|\Phi(A + C)x, x\| + \|\Phi(B + C)x, x\| + \|\Phi(A + B)x, x\| \\
&\leq \|\Phi(A + B + C)x, x\| + \Phi(|A| + |B| + |C|)x, x),
\end{align*}
\]

(2.21)

which gives the operator version of Hlawka’s inequality for positive linear maps of selfadjoint operators in Hilbert space. Furthermore, by taking the supremum in (2.21) over \( x \in \mathcal{K} \) with \( \|x\| = 1 \), we obtain the following Hlawka’s norm inequality

\[
\|\Phi(A + C)\| + \|\Phi(B + C)\| + \|\Phi(A + B)\| \\
\leq \|\Phi(A + B + C)\| + \|\Phi(|A| + |B| + |C|)\|.
\]

Generally, the Popoviciu’s extension of Hlawka’s norm inequality can be presented in the form:

\[
\frac{2}{3} g \left( \frac{\|\Phi(A + C)\|}{3} \right) + g \left( \frac{\|\Phi(B + C)\|}{3} \right) + g \left( \frac{\|\Phi(A + B)\|}{3} \right)
\]

\[
\leq g \left( \frac{\|\Phi(A + B + C)\|}{3} \right) + \Phi \left( \frac{\|A\| + \|B\| + \|C\|}{3} \right).
\]

for every positive linear map \( \Phi \) and convex increasing function \( g \).

Corollary 6. Let \( A_j, B_j, D_j \in \mathcal{B}(\mathcal{H}) \) be three selfadjoint operators with \( \text{sp}(A_j), \text{sp}(B_j), \text{sp}(D_j) \subset [\gamma, \Gamma] \) for some real numbers \( \gamma, \Gamma \) with \( \gamma < \Gamma \) and for every \( j = 1, \ldots, n \). Let \( \Phi_j : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a positive linear map. such that \( \sum_{j=1}^{n} \Phi_j(1_H) = 1_K \). If \( f : [\gamma, \Gamma] \to \mathbb{R} \) is convex (concave) function, then

\[
f \left( \sum_{j=1}^{n} \Phi_j \left( \frac{A_j + B_j + D_j}{3} \right) u, u \right) + \sum_{j=1}^{n} \Phi_j \left( \frac{f(A_j) + f(B_j) + f(D_j)}{3} \right) u, u \right)
\] \[
\geq (\leq) \frac{2}{3} \left[ f \left( \sum_{j=1}^{n} \Phi_j \left( \frac{A_j + D_j}{2} \right) u, u \right) + f \left( \sum_{j=1}^{n} \Phi_j \left( \frac{B_j + D_j}{2} \right) u, u \right) \right]
\]

for each \( u \in \mathcal{K} \) with \( \|u\| = 1 \).

References

[1] S. Abramovich, On superquadraticity, J. Math. Inequal., 3 (3) (2009), 329–339.
[2] S. Abramovich, S. Ivelić and J. Pečarić, Improvement of Jensen-Steffensen’s inequality for superquadratic functions, Banach J. Math. Anal., 4 (1) (2010), 146-158.
[3] S. Abramovich, G. Jameson and G. Sinnamon, Refining Jensen’s inequality, Bull. Math. Soc. Sci. Math. Roumanie, 47 (2004), 3–14.
[4] R.P. Agarwal and S.S. Dragomir, A survey of Jensen type inequalities for functions of selfadjoint operators in Hilbert spaces, Comput. Math. Appl., 59 (2010), 3785–3812.
[5] J. Barić, A. Matković and J. Pečarić, A variant of the Jensen–Mercer operator inequality for superquadratic functions, Math. Comput. Modelling, 51 (2010) 1230–1239.
[6] S. Banić, J. Pečarić and S. Varošanec, Superquadratic functions and refinements of some classical inequalities, J. Korean Math. Soc., 45, (2) (2008), 513–525.
[7] M. Benche, C.P. Niculescu and F. Popovici, Popoviciu’s inequality for functions of several variables, J. Math. Anal. Appl., 365 (2010), 399–409.
[8] P. Chansangiam P. Hemchote and P. Pantaragphong, Generalizations of Bohr inequality for Hilbert space operators, J. Math. Anal. Appl., 356 (2009), 525–536.
[9] W.-S. Cheung and J. Pečarić, Bohr’s inequalities for Hilbert space operators, J. Math. Anal. Appl., 323 (2006), 403–412.
[10] S.S. Dragomir, Operator inequalities of the Jensen, Čebyšev and Grüss type, Springer, New York, 2012.
[11] M.Fuji and H. ZUO, Matrix order in Bohr inequality for operators, Banach J. Math. Anal, 4 (2010), 21–27.
[12] T. Furuta, J. Mićić, J. Pečarić and Y. Seo, Mond-Pečarić method in operator inequalities. Inequalities for bounded selfadjoint operators on a Hilbert space, Element, Zagreb, 2005.
[13] D. Grinberg, Generalizations of Popoviciu’s inequality, (2008), arXiv:0803.2958v1.
[14] O. Hirzallah, Non-commutative operator Bohr inequality, J. Math. Anal. Appl., 282 (2003), 578–583.
Journal of the Mathematical Society of Japan, 68(1) (2016), 205–216.

[11] S. Miyajima, H. Takagi, Some inequalities for convex functions, *Aequ. Math.*, 91(1) (2016), 143–166.

[12] M. Moslehian, R. Rajić, Generalizations of Bohr’s inequality in Hilbert $C^*$-modules, *Linear and Multilinear Algebra*, 58 (2010), 323–331.

[13] M. Stanković, Some inequalities for convex functions, *Math. Balkanica*, 6 (1976), 281–288.

[14] F. Zhang, On the Bohr inequality of operators, *J. Math. Anal. Appl.*, 333 (2007), 1264–1271.

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