A NOTE ON HALÁSZ’S THEOREM IN $\mathbb{F}_q[t]$

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Abstract. In the setting of the integers, Granville, Harper and Soundararajan showed that the upper bound in Halász’s Theorem can be improved for smoothly supported functions. We derive the analogous result for Halász’s Theorem in $\mathbb{F}_q[t]$, and then consider the converse question of when the general upper bound in this version of Halász’s Theorem is actually attained.

1. Introduction

1.1. Halász’s Theorem for the integers. Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function such that $f(1) = 1$, and such that its associated Dirichlet Series and Euler Product (respectively)
$$
\mathcal{F}(s) := \sum_{n \geq 1} \frac{f(n)}{n^s} = \prod_{\text{prime}} \sum_{k \geq 0} \frac{f(p^k)}{p^{ks}}
$$
are defined and absolutely convergent for $s \in \mathbb{C}$ with $\Re(s) > 1$. Then define $\Lambda_f(n)$, the von Mangoldt function associated to $f$, by
$$
- \frac{\mathcal{F}’(s)}{\mathcal{F}(s)} =: \sum_{n \geq 1} \frac{\Lambda_f(n)}{n^s}
$$
and consider the set of such functions $f$ such that, for some $\kappa > 0$, we have $|\Lambda_f(n)| \leq \kappa \Lambda(n)$ for all $n \geq 1$ (where $\Lambda$ is the usual von Mangoldt function), which we denote $\mathcal{C}(\kappa)$. In [2], Granville, Harper and Soundararajan generalise Halász’s Theorem to this class of functions:

**Theorem 1.1 (Halász’s Theorem).** Let $\kappa > 0$, $x$ large and $f \in \mathcal{C}(\kappa)$, and define $M = M(x)$ by
$$
e^{-M(\log x)\kappa} := \max_{|t| \leq (\log x)\kappa} \left| \frac{\mathcal{F}(1 + 1/\log x + it)}{1 + 1/\log x + it} \right|.
$$
Then we have that
$$
S(x) := \frac{1}{x} \sum_{n \leq x} f(n) \ll _\kappa (1 + M) e^{-M(\log x)\kappa - 1} + \frac{(\log \log x)^\kappa}{\log x}.
$$

**Remark 1.2.** Halász’s Theorem gives us a very general tool for understanding multiplicative functions, and provides another way to recover results associated to particular cases. Note, for example, that the non-vanishing of $\zeta(s)$ on $\Re(s) = 1$ implies, for $f = \mu$, the Möbius function, that $e^{-M} \asymp \frac{1}{\log x}$ and so by Halász’s Theorem we have
$$
\frac{1}{x} \sum_{n \leq x} \mu(n) \ll \frac{\log \log x}{\log x}
$$
which is equivalent to the Prime Number Theorem (albeit with a weak error term).

**Remark 1.3.** In the case of $f \in \mathcal{C}(1)$ the inequality in Theorem 1.1 becomes
$$
S(x) \ll (1 + M) e^{-M} + \frac{(\log \log x)}{\log x}.
$$
Now, for simplicity, consider the multiplicative functions $f$ with $f(1) = 1$ and $|f(n)| \leq 1$ for all $n$, which form a superset of $\mathcal{C}(1)$. For this set, the same authors show that if $f$ is supported only on primes of size $p \leq x^{1-\delta}$ for some $\delta > 0$, then we can improve the upper bound in equation (1) to get
$$
S(x) \ll _\delta e^{-M} + \frac{(\log \log x)}{\log x}.
$$
This observation is presented in Remark 3.2 of [4], albeit with a different set of notation associated to this setting.

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Remark 1.4. The upper bound in (2) does not hold for general multiplicative functions. It has been shown by a variety of authors (see [6], [7] and [5]), following the idea of Montgomery in [6], that there exists a multiplicative function \( f \) with \( f(1) = 1 \) and \( |f(n)| \leq 1 \) for all \( n \), such that

\[
S(x) \gg (1 + M)e^{-M} + \frac{(\log \log x)}{\log x}.
\]

1.2. Halász’s Theorem in \( \mathbb{F}_q[t] \). We work in the setting of polynomials over a finite field, and set up the quantities analogous to those in the setting of the integers by following the notation in [3]. Let \( \mathbb{F}_q \) be a finite field of order \( q \), \( \mathcal{M} = \{ F \in \mathbb{F}_q[t] \text{ monic} \} \) and \( \mathcal{I} = \{ P \in \mathcal{M} : P \text{ is irreducible} \} \). We define \( f : \mathcal{M} \to \mathbb{C} \) to be a multiplicative function such that \( f(1) = 1 \), and such that its associated Power Series and Euler Product (respectively)

\[
\mathcal{P}(z) := \sum_{P \in \mathcal{M}} f(P)z^{\deg P} = \prod_{P \in \mathcal{I}} \sum_{k \geq 0} f(P^k)z^k
\]

are defined and absolutely convergent for \( z \in \mathbb{C} \) with \( |z| < \frac{1}{q} \). By taking the logarithmic derivative of the latter, and multiplying by \( z \), we acquire a new power series through which we can define \( \Lambda_f(F) \) (the von Mangoldt function associated to \( f \)):

\[
\frac{z \mathcal{P}'(z)}{\mathcal{P}(z)}(z) := \sum_{F \in \mathcal{M}} \Lambda_f(F)z^{\deg F}.
\]

Then, we let \( \mathcal{M}_n = \{ F \in \mathcal{M} : \deg F = n \} \), and define

\[
\sigma(n) = \sigma(n; f) := \frac{1}{q^n} \sum_{F \in \mathcal{M}_n} f(F)
\]

to be the mean value of \( f \) over polynomials of degree \( n \) and

\[
\chi(n) = \chi(n; f) := \frac{1}{q^n} \sum_{F \in \mathcal{M}_n} \Lambda_f(F)
\]

to be the corresponding weighted average over prime powers.

As in the setting of the integers, we consider the set \( \mathcal{C}(\kappa) \) of such \( f \) such that, for some \( \kappa > 0 \), we have that \( |\Lambda_f(F)| \leq \kappa \Lambda(F) \) for all \( F \in \mathcal{M} \), where

\[
\Lambda(f) = \begin{cases} \deg P & \text{if } F = P^k \\ 0 & \text{else} \end{cases}.
\]

In particular, given the prime polynomial theorem in the form \( \sum_{F \in \mathcal{M}_n} \Lambda_f(F) = q^n \), for \( f \in \mathcal{C}(\kappa) \) we have that

\[
|\chi(n)| \leq \frac{1}{q^n} \sum_{F \in \mathcal{M}_n} |\Lambda_f(F)| \leq \kappa
\]

and so we consider the more general set \( \tilde{\mathcal{C}}(\kappa) \) of \( f \) with \( |\chi(j)| = |\chi(j; f)| \leq \kappa \) for all \( j \geq 1 \).

Finally, we define \( f^\perp = f^\perp_n \) by setting

\[
\Lambda_{f^\perp}(F) = \begin{cases} \Lambda_f(F) & \text{if } \deg F < n \\ 0 & \text{else} \end{cases}
\]

and then we set \( \mathcal{P}^\perp(z) := \sum_{F \in \mathcal{M}} f^\perp(F)z^{\deg F}, \sigma^\perp(j) := \frac{1}{q^n} \sum_{F \in \mathcal{M}_j} f^\perp(F) \) and \( \chi^\perp(j) := \frac{1}{q^n} \sum_{F \in \mathcal{M}_j} \Lambda_{f^\perp}(F) \). We observe that \( \chi^\perp(j) = \chi(j) \) if \( j < n \) and \( \chi^\perp(j) = 0 \) otherwise, and from equation (1.8) of [3] we have

\[
\sum_{k=1}^{j} \sigma(k) \chi(j-k) = \sum_{j \geq 1} \frac{\chi(j)}{j} \left( \frac{z}{q} \right)^j
\]

from which we conclude that \( \sigma^\perp(j) = \sigma(j) \) if \( j < n \) and \( \sigma^\perp(n) = \sigma(n) - \frac{\chi(n)}{n} \).

We also note that, from their definitions and our observations above, we have

\[
\mathcal{P} \left( \frac{z}{q} \right) = \sum_{j \geq 0} \sigma(j)z^j = \exp \left( \sum_{j \geq 1} \frac{\chi(j)}{j} z^j \right)
\]
In Theorem 1.1, we define
\[ n > m \]
We define a new quantity for
\[ (7) \]
Then we have that
\[ F \]
and
\[ \sigma \]
\[ \tilde{\chi} \]
Finally, inspired by the idea in the setting of the integers (see \[6\]), in section 3 we compute an example for which
\[ \text{Theorem 1 is actually asymptotically smaller than the upper bound in Theorem 1.5.} \]
Let
\[ \kappa > 0, \ n \geq 1 \text{ and } f \in \tilde{\mathcal{C}}(\kappa), \ 	ext{and define } M = M(n) \text{ by} \]
\[ e^{-M(2n)^\kappa} := \max_{|z|=\frac{1}{2}} |F^\perp(z)|. \]
Then we have that
\[ (7) \]
\[ |\sigma(n)| \leq 2\kappa(\kappa + 1 + M)e^{-M(2n)^{\kappa-1}}. \]

**Remark 1.6.** In Theorem 1.1 we define \( M \) in terms of the maximum value of the Dirichlet Series \( F(s) \) on the line segment \( \{ \Re(s) = 1 + 1/\log x + it : |t| \leq (\log x)^\kappa \} \). The restriction of \( t \) up to height \( (\log x)^\kappa \) comes from using a truncated version of Pellet’s formula in the proof of Theorem 1.1 and taking the real part of \( s \) to be \( 1 + 1/\log x \) is the ensure the convergence of \( F(s) \). The analogous proof of Theorem 1.5 uses Cauchy’s theorem, in which we integrate over the whole circle, and to ensure convergence, instead truncates the Power Series \( F(z) \) at height \( n \) (which is equivalent to its analogue, up to a multiplicative constant). This is why, in Theorem 1.5 we define \( M \) in terms of the maximum value of the Power Series \( F^\perp(z) \) on the circle \( |z| = 1/q \).

We consider the case analogous to that discussed in Remark 1.3 and show that the upper bound in Halász’s Theorem can be improved when \( M \) is smoothly supported.

**Theorem 1.** Let \( \kappa > 0, \ n \geq 1 \text{ and } f \in \tilde{\mathcal{C}}(\kappa), \ 	ext{and define } M = M(n) \text{ as in Theorem 1.5.} \text{ Suppose in addition that, for some small } \delta > 0, \ f \text{ is supported only on irreducibles } P \text{ of degree at most } (1 - \delta)(n - 1). \text{ Then we get that} \]
\[ |\sigma(n)| \leq 2\kappa^2 e^{-M(2n)^{\kappa-1}} \left( 1 - \log \left( 1 - e^{-\frac{1}{\log x}} \right) + \frac{1}{\kappa} \left( 1 - e^{M/\kappa} \right)^{\delta(n-1)} \right) + \frac{q}{q-1}^2 \frac{\kappa n^{\kappa-2}}{q(1-\delta)(n-1)^2} \]
\[ \ll_{\delta} \kappa(\kappa + 1)e^{-M(2n)^{\kappa-1}}. \]
Conversely, we derive a criterion for when the upper bound in equation (7) is asymptotically attained:

**Theorem 2.** Let \( \kappa > 0, \ n \geq 1 \text{ and } f \in \tilde{\mathcal{C}}(\kappa), \ 	ext{and define } M = M(n) \text{ as in Theorem 1.5.} \text{ Suppose that } \kappa + 1 = o(M), \text{ then } |\sigma(n)| \gg \kappa(\kappa + 1 + M)e^{-M(2n)^{\kappa-1}} \text{ if, and only if, for all } \delta > 1 \text{ we have} \]
\[ \sum_{|\chi(n)\sigma(n-j)| \leq n} \chi(j) \sigma(n-j) \gg \kappa M e^{-M(2n)^{\kappa-1}}. \]

**Remark 1.7.** The assumption that \( \kappa + 1 = o(M) \) in Theorem 2 is precisely the case in which the upper bound in Theorem 1 is actually asymptotically smaller than the upper bound in Theorem 1.5.

Finally, inspired by the idea in the setting of the integers (see \[9\]), in section 8 we compute an example for which the criterion in Theorem 2 holds, in the case of \( \kappa = 1 \).

## 2. Proofs of Theorems 1 and 2

Let \( \kappa > 0, \) and let \( f \in \tilde{\mathcal{C}}(\kappa). \) From equation (3.3) of \[8\] we have that
\[ (8) \]
\[ \sigma(n) - \frac{\chi(n)}{n} = \frac{1}{nq^s} \int_0^1 \frac{1}{2\pi i} \int_{|z| = \frac{1}{q^n}} \left( \sum_{j=1}^{n-1} \chi(j)(qz)^j \right)^{\left( \sum_{j=1}^{n-1} \chi(j)(qtz)^j \right)^{F^\perp(tz) \frac{dz}{z^{n+1}} \frac{dt}{t}.} \]
We define a new quantity for \( n > m \geq 1 \)
\[ (9) \]
\[ \sigma_m(n) := \frac{1}{nq^s} \int_0^1 \frac{1}{2\pi i} \int_{|z| = \frac{1}{q^m}} \left( \sum_{j=1}^{m} \chi(j)(qz)^j \right)^{\left( \sum_{j=1}^{n-1} \chi(j)(qtz)^j \right)^{F^\perp(tz) \frac{dz}{z^{n+1}} \frac{dt}{t}} \]
so that \( \sigma_{n-1}(n) = \sigma(n) - \frac{\chi(n)}{n}, \) and bound it following the strategy in \[8\].
Proposition 2.1. Let $\kappa > 0$, $n \geq 1$, $f \in \mathcal{C}(\kappa)$ and $M = M(n)$ as in Theorem 1.5. Then for $m < n - 1$ we have

$$|\sigma_m(n)| \leq 2\kappa^2 e^{-M(2n)^{\kappa}} \left(1 - \log \left(1 - e^{-\frac{(n-1)-m}{2\sqrt{m(n-1)}}}\right) + \frac{1}{\kappa} \left(1 - e^{M/\kappa} \right)^{(n-1)-m}\right).$$

Proof. First we use Cauchy-Schwarz on the inner integral in equation (9)

$$\frac{1}{2\pi i} \int_{|z| = \sqrt{\tau}} \left| \sum_{j=1}^{m} \chi(j)(qz)^j \right| \left( \sum_{j=1}^{n-1} \chi(j)(qtz)^j \right) F^\perp(tz) \frac{dz}{z^{n+1}} \right|$$

$$\leq (q\sqrt{t})^n \left( \max_{|z| = \sqrt{\tau}} |F^\perp(tz)| \right) \left( \sum_{j=1}^{m} \frac{1}{j^{\frac{1}{2}}} \left( \sum_{j=1}^{n-1} \frac{1}{j^{\frac{1}{2}}} \right) \right) \kappa^2 q^n \left( \max_{|z| = \frac{q}{\sqrt{\tau}}} |F^\perp(z)| \right) \frac{\sqrt{(1-t^m)(1-t^{n-1})}}{1-t}$$

where, for $s \geq 0$, have

$$I_n(s, R) := \frac{1}{2\pi} \int_{|z| = R} \left| \sum_{j=1}^{a} \chi(j)(qs)j \right|^2 \frac{dz}{|z|} = \sum_{j=1}^{a} \chi(j) |qsR|^{2j} \leq \kappa^2 \sum_{j=1}^{a} \chi(j)^2 (qsR)^{2j}$$

by Parseval's identity. Using this, we bound the inner integral by the quantity

$$\kappa^2 (q\sqrt{t})^n \left( \max_{|z| = \sqrt{\tau}} |F^\perp(tz)| \right) \left( \sum_{j=1}^{m} \frac{1}{j^{\frac{1}{2}}} \left( \sum_{j=1}^{n-1} \frac{1}{j^{\frac{1}{2}}} \right) \right) \kappa^2 q^n \left( \max_{|z| = \frac{q}{\sqrt{\tau}}} |F^\perp(z)| \right) \frac{(1-t^m)(1-t^{n-1})}{1-t}$$

and then recall the bound from equation (3.6) of [9], which for $t \in (0, 1)$, states that

$$\max_{|z| = \frac{q}{\sqrt{\tau}}} |F^\perp(z)| \leq \min(e^{-M(2n)^{\kappa}}, (1 - \sqrt{t})^{-\kappa})$$

where $e^{-M(2n)^{\kappa}} := \max_{|z| = \frac{q}{\sqrt{\tau}}} |F^\perp(z)|$.

Putting this back into the full integral we get

$$|\sigma_m(n)| \leq \frac{\kappa^2}{n} \int_0^1 \min(e^{-M(2n)^{\kappa}}, (1 - \sqrt{t})^{-\kappa}) \frac{(1-t^m)(1-t^{n-1})}{1-t} dt$$

and after the substitution $t = (1-u)^2$ we have

$$|\sigma_m(n)| \leq \frac{\kappa^2}{n} \int_0^1 \min(e^{-M(2n)^{\kappa}}, u^{-\kappa}) (1-u)^{(n-1)-m} \min \left( \frac{1}{u(1-u)} \right) 2(1-u) du$$

$$\leq \frac{\kappa^2}{n} \int_0^1 \min(e^{-M(2n)^{\kappa}}, u^{-\kappa}) (1-u)^{(n-1)-m} \min \left( \frac{2\sqrt{m(n-1)}(1)}{u} \right) du.$$
We can combine these two cases as follows
\[
|\sigma_m(n)| \leq K^2 n e^{-M(2n)^{\kappa}} + e^{-M(2n)^{\kappa}} \sum_{j=1}^{\sqrt{\log(n-1)}} \int_0^1 \frac{du}{u^{\frac{1}{2}}} 
\]
\[
\left(1 - \frac{e^{M/\kappa}}{2n}\right)^{(n-1)-m} \int_0^1 \frac{du}{u^{\kappa+1}}
\]
When \(m < n-1\) we get
\[
|\sigma_m(n)| \leq 2K^2 e^{-M(2n)^{\kappa}} \left(1 + \sum_{j=1}^{\sqrt{\log(n-1)}} \left(1 - \frac{\sqrt{\log(n-1)}}{2\sqrt{\log(n-1)}}\right)^{(n-1)-m} \log \left(1 + \frac{1}{j}\right) + \frac{1}{\kappa} \left(1 - \frac{e^{M/\kappa}}{2n}\right)^{(n-1)-m}\right)
\]
\[
\leq 2K^2 e^{-M(2n)^{\kappa}} \left(1 + \sum_{j=1}^{\sqrt{\log(n-1)}} \frac{\sqrt{\log(n-1)}}{j} + \frac{1}{\kappa} \left(1 - \frac{e^{M/\kappa}}{2n}\right)^{(n-1)-m}\right)
\]
\[
\leq 2K^2 e^{-M(2n)^{\kappa}} \left(1 - \log \left(1 - e^{-\frac{\sqrt{\log(n-1)}}{2\sqrt{\log(n-1)}}}\right) + \frac{1}{\kappa} \left(1 - \frac{e^{M/\kappa}}{2n}\right)^{(n-1)-m}\right).
\]

\[\square\]

**Corollary 2.2.** Let \(\kappa > 0, n \geq 1, f \in \tilde{C}(\kappa)\) and \(M = M(n)\) as in Theorem 1.5. Then for \(\delta > 0\) and \(m \leq (1-\delta)(n-1)\) we have that
\[
|\sigma_m(n)| \leq 2K^2 e^{-M(2n)^{\kappa}} \left(1 - \log \left(1 - e^{-\frac{\sqrt{\log(n-1)}}{2\sqrt{\log(n-1)}}}\right) + \frac{1}{\kappa} \left(1 - \frac{e^{M/\kappa}}{2n}\right)^{(n-1)-m}\right).
\]

Then we relate our quantity \(\sigma_m(n)\) to \(\sigma(n)\) with the following observation

**Lemma 2.3.** Let \(n > m \geq 1\). Then we have that
\[
\sigma(n) = \sigma_m(n) + \frac{1}{n} \sum_{j=m+1}^{n} \chi(j)\sigma(n-j).
\]

**Proof.** From the definition of \(\sigma_m(n)\) in equation (11), and our observation in equation (10) we have
\[
\sigma_m(n) = \frac{1}{nq^n} \int_0^1 \frac{1}{2\pi i} \int_{|z|=1} \left(\sum_{j=1}^{m} \chi(j)(qz)^j\right) \left(\sum_{k=1}^{n-1} \chi(k)(qtz)^k\right) \left(\sum_{l=0}^{n-1} \sigma(l)(qtz)^l\right) \frac{dz}{z^{n+1}} \frac{dt}{t}
\]
\[
= \frac{1}{nq^n} \int_0^1 \frac{1}{2\pi i} \int_{|z|=1} \left(\sum_{j=1}^{m} \chi(j)(qz)^j\right) \left(\sum_{k=1}^{n-1} \chi(k)(qtz)^k\right) \left(\sum_{l=0}^{n-1} \sigma(l)(qtz)^l\right) \frac{dz}{z^{n+1}} \frac{dt}{t}
\]
\[
= \frac{1}{nq^n} \int_0^1 \sum_{j \leq m} \chi(j)\chi(k)\sigma(l)q^{j+k+l+k+l} \frac{dt}{t}
\]
\[
= \frac{1}{n} \sum_{j+k+l=n} \chi(j)\chi(k)\sigma(l)q^{j+k+l} = \frac{1}{n} \sum_{j=1}^{m} \chi(j) \frac{1}{n-j} \sum_{k=1}^{n-j} \chi(k)\sigma(n-j-k) = \frac{1}{n} \sum_{j=1}^{m} \chi(j)\sigma(n-j)
\]
where in the second equality we use the fact that \(\sigma^+(j) = \sigma(j)\) for \(j < n\), and in the final equality we use equation (11). Finally, using equation (11) once more we get that
\[
\sigma(n) = \frac{1}{n} \sum_{j=1}^{n} \chi(j)\sigma(n-j) = \sigma_m(n) + \frac{1}{n} \sum_{j=m+1}^{n} \chi(j)\sigma(n-j).
\]

\[\square\]

This bring us to our proof of Theorem 11.
Remark 3.1. Let $m = [(1 - \delta)(n - 1)]$. Since $f$ is supported only on irreducibles $P$ of degree at most $m$, we have for $j > m$, that
\[
|\chi(j)| = \frac{1}{q^j} \left| \sum_{P \in M_j} \lambda_j(P) \right| \leq \frac{1}{q^j} \sum_{dj \leq m} \left| \sum_{P \in M_j} \lambda_j(P^{j/d}) \right| \leq \frac{1}{q^j} \sum_{d \leq \min(j/2, m)} q^d |\chi(d)| \leq \kappa \min(q^{-j/2}, q^{m-j}) \frac{q}{q-1}.
\]
Moreover, using equation (4), and our assumption that $|\chi(j)| \leq \kappa$ for all $j$, we can deduce inductively (given the base case $\sigma(0) = 1$) the trivial bound $|\sigma(j)| \leq (j + 1)\kappa^{-1}$ for all $j$. Now we can bound the following sum thus
\[
\left| \sum_{j=m+1}^{n} \chi(j)\sigma(n-j) \right| \leq \frac{\kappa \kappa^{-1}}{q-1} \sum_{j=m+1}^{n} q^{-j/2} \leq \left( \frac{q}{q-1} \right)^2 \frac{\kappa \kappa^{-1}}{q^{(m+1)/2}} \leq \left( \frac{q}{q-1} \right)^2 \frac{\kappa \kappa^{-1}}{q^{(1-\delta)(n-1)/2}}.
\]
We combine this with Lemma 2.3 and Corollary 2.2 to get that
\[
|\sigma(n)| \leq 2\kappa^2 e^{-M} (2n)^{\kappa^{-1}} \left( 1 - \log \left( 1 - e^{-\frac{1}{2}} \right) + \frac{1}{\kappa} \left( 1 - e^{-M/\kappa} \right)^{\delta(n-1)} \right) + \left( \frac{q}{q-1} \right)^2 \frac{\kappa \kappa^{-2}}{q^{(1-\delta)(n-1)/2}}.
\]
Finally, by the maximum modulus principle, $e^{-M} (2n)^{\kappa} = \max_{|z| = 1} |F^{\perp}(z)| \geq |F^{\perp}(0)| = 1$, which means that the second term in (10) is much smaller (asymptotically in $n$) than the first. So, we can conclude that $|\sigma(n)| \ll \kappa (1 + 1)e^{-M} (2n)^{\kappa^{-1}}$.

and our proof of Theorem 2.

Proof of Theorem 2. Let $\delta \gg 1$ and $m = [(1 - \delta)(n - 1)]$, so that, by Lemma 2.3 we have that
\[
\sigma(n) = \sigma_m(n) + \frac{1}{n} \sum_{j=m+1}^{n} \chi(j)\sigma(n-j) = \sigma_m(n) + \frac{1}{n} \sum_{(1-\delta)(n-1) < j \leq n} \chi(j)\sigma(n-j)
\]
and from Corollary 2.2 we know that $\sigma_m(n) \ll \kappa (1 + 1 + M)e^{-M} (2n)^{\kappa^{-1}}$. Therefore, if $\kappa + 1 = o(M)$, we have that $|\sigma(n)| \gg \kappa (1 + 1 + M)e^{-M} (2n)^{\kappa^{-1}}$ if, and only if,
\[
\left| \sum_{(1-\delta)(n-1) < j \leq n} \chi(j)\sigma(n-j) \right| \gg \kappa Me^{-M} (2n)^{\kappa}.
\]

3. A Sharp Example

We conclude with an example for which the criterion in Theorem 2 holds, and thus which attains the upper bound in Halász’s Theorem. For simplicity, we take $\kappa = 1$ throughout this example.

Remark 3.1. First we observe that, if $0 < \delta < \frac{1}{2} - \frac{1}{2n}$, then the values taken by $\sigma(n-j)$ in equation (11) are independent of those taken by $\chi(j)$. So, we may for example take $\chi(j) = e^{i(\theta - \phi_{n-j})}$ where $\sigma(j) = |\sigma(j)|e^{i\phi_j}$ (for some $\theta \in [0, 2\pi)$) for $j > (1-\delta)(n-1)$. In this case Theorem 2 says that if $1 = o(M)$ then
\[
|\sigma(n)| \gg (1 + M)e^{-M} \iff \sum_{j < 1 + \delta(n-1)} |\sigma(j)| \gg Me^{-M} n.
\]

We use this observation to construct the following example

Example 1. Let $n \geq 2$, $0 < \delta < \frac{1}{2} - \frac{1}{2n}$ and $M = M(n)$ as in Theorem 1.5. Let
\[
\chi(j) = \begin{cases} 
1 & \text{if } 1 \leq j < 1 + \delta(n-1) \\
0 & \text{if } 1 + \delta(n-1) \leq j < (1-\delta)(n-1) \\
e^{-i\phi_{n-j}} & \text{if } j > (1-\delta)(n-1)
\end{cases}
\]
where $\sigma(j) = |\sigma(j)|e^{i\phi_j}$. Then $1 = o(M)$ and
\[
\sum_{j < 1 + \delta(n-1)} |\sigma(j)| \gg Me^{-M} n.
\]
which means that, by Theorem \[2\] we have
\[|\sigma(n)| \gg (1 + M)e^{-M}.\]

**Proof.** In this case, we have from equation \[6\] that
\[
\max_{|z|=\frac{1}{q}} \log |\mathcal{F}^\perp(z)| = \max_{|z|=\frac{1}{q}} \Re \left( \sum_{j=1}^{n-1} \frac{\chi(j)}{j} (qz)^j \right)
\]
\[
= \max_{\theta \in [0,2\pi)} \left( \sum_{1 \leq j < 1+\delta(n-1)} -\sin(j\theta) + \sum_{(1-\delta)(n-1) < j \leq n-1} \cos(j(\theta - \phi - n-1)) \right).
\]

Now, we know that uniformly for \(\theta\) and \(x\), we have \(\left| \sum_{j \leq x} \sin(j\theta) \right| \ll 1\) and moreover
\[
\left| \sum_{(1-\delta)(n-1) < j \leq n-1} \frac{\cos(j(\theta - \phi - n-1))}{j} \right| \ll \sum_{(1-\delta)(n-1) < j \leq n-1} \frac{1}{j} \ll -\log(1 - \delta) \ll 1.
\]

Therefore, we have that \(\max_{|z|=\frac{1}{q}} \log |\mathcal{F}^\perp(z)| \ll 1\), and conversely, by the maximum modulus principle
\[
\max_{|z|=\frac{1}{q}} |\mathcal{F}^\perp(z)| \geq |\mathcal{F}^\perp(0)| = 1.
\]

This means that \(e^{-M}(2n) := \max_{|z|=\frac{1}{q}} |\mathcal{F}^\perp(z)| \asymp 1\) and \(M = \log 2n - \max_{|z|=\frac{1}{q}} \log |\mathcal{F}^\perp(z)| \sim \log n\) so that overall we get have that \(1 = o(M)\) and \(Me^{-M}n \asymp \log n\).

On the other hand, by Cauchy’s Theorem, we have for \(j < 1 + \delta(n-1)\) and \(R < 1\) that
\[
\sigma(j) = \frac{1}{q!} \frac{1}{2\pi i} \int_{|z|=\frac{1}{q}} \mathcal{F}(z) \frac{dz}{z^{j+1}} = \frac{1}{2\pi i} \int_{|w|=R} \mathcal{F} \left( \frac{w}{q} \right) \frac{dw}{w^{j+1}}
\]
\[
= \frac{1}{2\pi i} \int_{|w|=R} \exp \left( \sum_{k \geq 1} \frac{\chi(k)}{k} \frac{w^k}{w^{j+1}} \right) \frac{dw}{w^{j+1}}
\]
\[
= \frac{1}{2\pi i} \int_{|w|=R} \exp \left( \sum_{k=1}^{j} \frac{w^k}{k} \right) \frac{dw}{w^{j+1}}
\]
\[
= \frac{1}{2\pi i} \int_{|w|=R} \exp \left( \sum_{k \geq 1} \frac{w^k}{k} \right) \frac{dw}{w^{j+1}}
\]
\[
= \frac{1}{2\pi i} \int_{|w|=R} \frac{1}{(1-w)^j} \frac{dw}{w^{j+1}} \sim \frac{1}{\Gamma(j)} 
\]
where we use equation \[6\] in the third line. From this we conclude that
\[
\sum_{j < 1+\delta(n-1)} |\sigma(j)| \asymp \sum_{j < 1+\delta(n-1)} \frac{1}{j} \gg_{\delta} \log n \gg M e^{-M} n.
\]

\[\square\]

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References

[1] Ardavan Afshar. *Topics in the arithmetic of polynomials over finite fields*. PhD thesis, 2020.

[2] Andrew Granville, Adam J. Harper, and K. Soundararajan. A new proof of Halász’s theorem, and its consequences. *Compos. Math.*, 155(1):126–163, 2019.

[3] Andrew Granville, Adam J. Harper, and Kannan Soundararajan. Mean values of multiplicative functions over function fields. *Res. Number Theory*, 1:Paper No. 25, 18, 2015.

[4] Andrew Granville, Adam J. Harper, and Kannan Soundararajan. A more intuitive proof of a sharp version of Halász’s theorem. *Proc. Amer. Math. Soc.*, 146(10):4099–4104, 2018.

[5] Andrew Granville and K. Soundararajan. Decay of mean values of multiplicative functions. *Canad. J. Math.*, 55(6):1191–1230, 2003.

[6] H. L. Montgomery. A note on mean values of multiplicative functions. *Report No. 17, Institut Mittag-Leffler, Djursholm*, 1978.

[7] H. L. Montgomery and R. C. Vaughan. Mean values of multiplicative functions. *Period. Math. Hungar.*, 43(1-2):199–214, 2001.

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