AN ANALOGUE OF THE KOSTANT-RALLIS
MULTICIPICITY THEOREM FOR θ–GROUP
HARMONICS

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To Roger Howe with admiration.

Abstract. The main result in this paper is the generalization of
the Kostant-Rallis multiplicity formula to general θ–groups (in the
sense of Vinberg). The special cases of the two most interesting
examples one for E_6 (three qubits) and one for E_8 are given explicit
formulas.

1. Introduction

The purpose of this paper is to give proofs of analogues for a Vin-
berg θ–group of two results of Kostant-Rallis [KR] for the case when
θ is an involution of a semi–simple Lie algebra, \( g \), over \( \mathbb{C} \). To describe
the results we need some notation. Let \( V \) denote the \(-1\) eigenspace
of \( \theta \). Set \( H \) equal to the identity component of the centralizer of \( \theta \) in
the automorphism group of \( g \). We use the notation \( \mathcal{O}(V) \) for the alge-
bra of polynomials on \( V \) and \( \mathcal{O}(V)^H \) for the algebra of \( H \)-invariants.
Finally we set \( \mathcal{H} \) equal to the \( H \)-module of harmonics. The first re-
sult is that \( \mathcal{O}(V) \) is isomorphic with \( \mathcal{O}(V)^H \otimes \mathcal{H} \) as a \( H \) and
\( \mathcal{O}(V)^H \) module. The second result the generalization of their formula for the
\( H \)-multiplicities in \( \mathcal{H} \). If \( \theta \) an automorphism of \( g \) of order \( 0 < m < \infty \)
then a “\( \theta \)–group” is a pair \( (H, V) \) where \( V \) is the eigenspace for a prin-
cipal \( m \)-th root of unity in \( g \) and \( H \) is the identity component of the centralizer
of \( \theta \) in the automorphism group of \( g \) restricted to \( V \). We prefer to use the term Vinberg pair for \( (H, V) \). We will also give a proof
of Vinberg’s main theorem (that says that \( \mathcal{O}(V)^H \) is a polynomial alge-
bra over \( \mathbb{C} \)) for these pairs that does not rely on classification based on
a brilliant theorem of Panyushev [P]. The reasons for this inclusion are
that we have not seen this proof in full detail in the literature. Also
the other results of Vinberg (which do not depend on classification)
that are used in our argument are also needed in the proof of the mul-
tiplicity formula. Vinberg’s original argument is complicated whereas
both Panyushev’s argument and the reduction of Vinberg’s theorem to
it are not complicated. Also in light of the recent deep applications of Vinberg theory in characteristic $p$ we note that the arguments should extend to arbitrary fields where the Shephard-Todd theorem applies using deeper étale theoretic arguments. Vladimir Popov has communicated to us that he and Vinberg published a similar argument in their (much) earlier paper [PV]

Vinberg’s paper [V] was a major addition to the literature of geometric invariant theory. Even if the reader is not interested in the results that go beyond [V], the listing of the main results of that paper (with explicit references to the original) might be reason enough to read this one. The interested reader should also look at the tables in [V]. An expanded exposition of the results in of Vinberg quoted and the newer results in this paper will appear as part of my forthcoming book [W].

The paper is organized as follows: Section 2 describes the part of Vinberg’s work that is necessary for our proof of his main theorem. Section 3. is a description of the Shephard-Todd theorem, Panyushev’s result and a proof of Vinberg’s main theorem. Section 4 studies maximal compact subgroups of $H$ and a description of the Kempf-Ness set for a Vinberg pair. Section 5 contains several more results of Vinberg and our proof of the freeness theorem above and the multiplicity formula. I give two important examples of the multiplicity formula in an effectively computable form. The first is what is probably the most interesting non-symmetric example for $E_6$ (what the physicists call “three qutrits”) and the second is the example that was studied extensively by Vinberg and Elashvili [EV] for $E_8$.

We thank Hanspeter Kraft for his patient explanation of Panyushev’s result; Vladimir Popov for pointing out his work with Vinberg [PV] and the referee for this article for pointing out the paper of Dodak and Kac on what they call polar representations [DK]. It appears that many of the preliminary results in this paper are also true in this larger context. Thus it is likely that variant of the multiplicity formula is true in this context. Jeb Willenbring’s student, Alexander Heaton, will be doing part of his thesis work on this problem.

This paper is dedicated to my long time friend Roger Howe on the occasion of his seventieth birthday. We met in Berkeley in 1966 when I was a first year postdoc and he was a graduate student. I was asked by Cal Moore to teach the third quarter of his Lie groups course. I was awed by the quality of the students. In that exceptional group Roger stood out. Although we have written only one joint paper (joint also with Tom Enright) we have had many deep mathematical conversations. It was my good fortune that Roger’s parents spent their
retirement in San Diego. This led Roger to visit UCSD often. I miss Roger’s parents and I miss his visits.

2. Definitions and some of Vinberg’s results

A \( \theta \)-group is a pair \((H, V)\) of a finite dimensional vector space over \( \mathbb{C} \), \( V \), and a Zariski closed, connected, reductive subgroup of \( \text{GL}(V) \) constructed as follows: \( g \) is a semi-simple Lie algebra over \( \mathbb{C} \), \( \theta \) is an automorphism of finite order, \( m, \) of \( g \), \( \zeta \) is a primitive \( m \)-th root of unity, \( V \) is the \( \zeta \) eigenspace for \( \theta \), \( L \) is the connected subgroup of \( \text{Aut}(g) \) with \( \text{Lie}(L) = g^\theta \) (eigenspace for 1) and \( H = L|_V \). In this paper we will call \((H, V)\) a Vinberg pair.

Vinberg’s theory reduces the study of the orbit structure and invariant theory of a Vinberg pair to the case when \( g \) is simple. In this paper we will concentrate on this case. The purpose of this section is to give a listing of the results of Vinberg that we will need in our proof of his main theorem (all of which are proved without case by case checks).

First some general notation.

If \( U \) is a finite dimensional vector space over \( \mathbb{C} \) let \( \mathcal{O}(U) \) denote the polynomial functions on \( U \) and if \( X \) is Zariski closed in \( U \) then \( \mathcal{O}(X) = \mathcal{O}(U)|_X \) (the regular functions on \( U \)). If \( G \) is a algebraic group acting on \( X \) regularly then we have a representation of \( G \) on \( \mathcal{O}(X) \) by

\[
(gf)(x) = f(g^{-1}x), g \in G, x \in X.
\]

We note that if \( f \in \mathcal{O}(X) \) then the span of \( Gf \), \( Z \), is finite dimensional and the corresponding action of \( G \) on \( Z \) is regular. We set \( \mathcal{O}(X)^G = \{f \in \mathcal{O}(X) | gf = f\} \). This algebra is finitely generated over \( \mathbb{C} \) and so we can form the maximal spectrum, \( X//G \), of \( \mathcal{O}(X)^G \) which is an affine variety with \( \mathcal{O}(X//G) = \mathcal{O}(X)^G \). Null cone of \( X \) is the set \( \{x \in X | f(x) = f(0), f \in \mathcal{O}(X)^G\} \).

Let \((H, V)\) be a Vinberg pair that corresponds to a simple Lie algebra \( g \) and automorphism \( \theta \). Here are the results

1. Let \( v \in V \). Then \( Hv \) is closed in \( V \) if and only if \( v \) is semi-simple in \( g \) [3], Proposition 3.

2. \( v \) is an element of the null cone of \( V \) if and only if \( v \) is nilpotent in \( g \) [4], Proposition 1.

A Cartan subspace of \( V \) is a subspace, \( a \), such that

a) every element of \( a \) is semi-simple in \( g \),

b) \([a, a] = 0\) and

c) \( a \) is maximal with respect to a) and b).
3. All Cartan subspaces are conjugate under $H$. Define the common dimension of the Cartan subspaces to be the rank of the Vinberg pair. Theorem 1.

4. If $a$ is a Cartan subspace then $Ha$ is the union of the closed orbits of $H$ [V] Corollary to Theorem 1.

5. Set for $v \in V$ the variety $X_v = \{x \in V | f(x) = f(v), f \in \mathcal{O}(V)^H\}$ ($X_0$ is the null cone). If $l$ is the rank of $(H, V)$ then

$$\dim X_v = \dim V - l.$$ 

Furthermore every irreducible component of $X_v$ contains an open $H$-orbit, (indeed, $X_v$ is a finite union of $H$-orbits) [V] Theorems 4,5.

6. Let $a$ be a Cartan subspace of $V$ and let $N_H(a) = \{h \in H | ha = a\}$ and $W(a) = N_H(a)|a$. Then $W_H(a)$ is a finite group and $V//H$ is isomorphic with $a/W_H(a)$ as an affine variety. [V] Theorem 7.

3. **Complex reflections, Panyushev’s result and Vinberg’s main theorem**

Let $U$ be a finite dimensional vector space over $\mathbb{C}$ then a complex reflection on $U$ is a linear isomorphism of finite order such that $\dim \ker(U - I) = \dim U - 1$. Shephard-Todd in [ST] proved

**Theorem 1.** Let $U$ be a finite dimensional vector space and $G$ a finite group acting on $U$. Then $G$ is generated by complex reflections if and only if $U/G$ is isomorphic with $U$ as an affine variety.

The “only if” part of this theorem is usually stated

**Theorem 2.** If $U$ is a finite dimensional vector space and $G \subset GL(U)$ is a finite subgroup generated by complex reflections then $\mathcal{O}(U)^G$ is generated as an algebra over $\mathbb{C}$ by $\dim U$ algebraically independent homogenous polynomials.

The proof in [ST] of this part of the theorem was by a case by case check. The “if” part is proved without case by case checking by reducing to the “only if” part. Chevalley [C] gave a proof of this part without classification under the hypothesis that $G$ is generated by reflections of order 2. The literature seems unanimous that Serre pointed out to him that his proof of the special case proved the full result without any real change.

The theorem of Panyushev [P] rests on the full theorem above. In [W] a complete exposition of the proof of this theorem and the Shephard-Todd theorem is given.
Theorem 3. Let $V$ and $U$ be finite dimensional complex vector spaces and let $H$ be a reductive group acting on $V$ regularly and let $W \subset \text{GL}(U)$ be a finite subgroup. Let $p : V \to V//H$ be the natural surjection. Assume that if $X \subset V//H$ is Zariski closed and of codimension at least 2 then $p^{-1}(X)$ is of codimension at least 2 in $V$. If $V//H$ is isomorphic with $U/W$ as an affine variety then $W$ is generated by complex reflections.

Panyushev’s proof of this theorem is an ingenious application of the Shephard-Todd theorem. We note that it is an exercise to prove that if $H$ is semi-simple then the codimension assumption is satisfied. Thus 6. in the previous section implies that if $H$ is semi-simple that $W_H(a)$ is generated by reflections. This contains all of the cases Vinberg looks at in his tables for the exceptional groups.

The main theorem in [V] says

Theorem 4. Let $(H, V)$ be a Vinberg pair and let $a$ be a Cartan subspace of $V$. Then $W_H(a)$ is generated by complex reflections of $a$.

Proof. If $\dim a = 1$ then any map of of finite order of $a$ is a complex reflection.

We may now assume that $\dim a \geq 2$. Let $X \subset V//H$ be Zariski closed and of codimension at least 2 we show that $p^{-1}(X)$ has codimension at least 2. Let $l = \dim a$. Then $\dim X = l - k$ with $k \geq 2$. Let $Y \subset p^{-1}(X)$ be an irreducible component. Then since $H$ is connected $Y$ is $H$-invariant. We may assume that $p(Y) = X$ (this might increase $k$). If $x \in p(Y)$ then $\dim \left( p|_Y \right)^{-1}(x) \leq \dim p^{-1}(x) = \dim V - l$. The theorem of the fiber (c.f. [GW]) implies that there exists $x \in p(Y)$ such that

$$\dim \left( p|_Y \right)^{-1}(x) = \dim Y - \dim X.$$

Thus

$$\dim V - l \geq \dim Y - \dim X = \dim Y - (l - k).$$

So

$$\dim Y \leq \dim V - l + l - k = \dim V - k.$$

Thus the hypothesis of Panyushev’s theorem is satisfied in this case. This completes our proof of Vinberg’s main theorem. \qed

4. Maximal compact subgroups

If $\mathfrak{g}$ is a semi-simple Lie algebra over $\mathbb{C}$ then we may realize $\mathfrak{g}$ as a Lie sub-algebra of $M_n(\mathbb{C})$ with the property that if $x \in \mathfrak{g}$ then $x^* = \bar{x}^T \in \mathfrak{g}$
(i.e. the conjugate transpose is in \( g \)). On \( g \) we put the Hilbert space structure
\[
\langle x, y \rangle = \text{tr}(\text{ad}x \text{ad}y^*) = B(x, y^*)
\]
with \( B \) the Killing form. \( \text{Aut}(g) \) is closed under adjoint with respect to \( \langle ..., ... \rangle \).

4.1. \( \theta \) can be assumed unitary.

**Lemma 1.** \( \text{Aut}(g) \) is closed under adjoint with respect to \( \langle ..., ... \rangle \).

**Proof.** Let \( g \in \text{Aut}(g) \). Then
\[
\langle gx, y \rangle = B(gx, y^*) = B(x, g^{-1}y^*)
\]
\[
= B(x, ((g^{-1}y^*)^*)^*) = \langle x, (g^{-1}y^*)^* \rangle.
\]
Thus setting \( \sigma(x) = x^* \), the adjoint of \( g \) is \( \sigma g^{-1} \sigma \). We assert that this element is in \( \text{Aut}(g) \). To see this we calculate
\[
[\sigma g^{-1} \sigma x, \sigma g^{-1} \sigma y] = -[g^{-1} \sigma x, g^{-1} \sigma y]^* =
\]
\[
= -(g^{-1}[x^*, y^*])^* = (g^{-1}[x, y])^* = \sigma g^{-1} \sigma [x, y].
\]
\qed

Now let \( \theta \) be an automorphism of order \( m < \infty \) of \( g \) and let \( (H, V) \) be the corresponding Vinberg pair \( (V = g_\zeta, \zeta \) a primitive \( m \)-th root of 1. Let \( G = \text{Aut}(g) \) and let \( G^o \) denote the identity component of \( G \). Let \( U \) be the unitary group of \( g \) relative to \( \langle ..., ... \rangle \). Then \( G \cap U \) is a maximal compact subgroup of \( G \) and \( G^o \cap U \) is maximal compact in \( G^o \).

The conjugacy theorem of maximal compact subgroups implies that there exists \( g \in G \) such that \( g\theta g^{-1} \) is contained in \( G \cap U \). Thus we have proved

**Theorem 5.** There exists \( g \in G \) such that \( g\theta g^{-1} \) normalizes \( G^o \cap U \). Furthermore replacing \( \theta \) with \( g\theta g^{-1} \) then \( g_\zeta^* = g_{\zeta^{-1}} \).

Replace \( \theta \) with \( g\theta g^{-1} \). We note that if \( L \) is the connected subgroup of \( G^o \) corresponding to \( g^\theta \) then \( L^* = L \). This implies

**Lemma 2.** \( H \) is invariant under the adjoint corresponding to the restriction of \( \langle ..., ... \rangle \) to \( V \). Thus in particular \( U(V) \cap H \) is a maximal compact subgroup of \( H \).
4.2. Kempf-Ness theory. We recall a bit of the Kempf-Ness theory. Let $V$ be a finite dimensional complex Hilbert space and let $H \subset GL(V)$ be a Zariski closed subspace invariant under adjoint. Let $K = H \cap U(V)$ then $K$ is a maximal compact subgroup of $H$ and $H$ is the Zariski closure of $K$. We say that $v \in V$ is critical if $\langle Xv, v \rangle = 0$ for all $X \in \text{Lie}(H)$. We will use the notation $\text{Crit}(V)$ for the space of critical elements of $V$ (this set is usually called the Kempf-Ness set).

The Kempf-Ness theorem [KN] says

**Theorem 6.** Notation as above

1. $x \in V$ is critical if and only if $\|hx\| \geq \|x\|$ for all $h \in H$.
2. If $x \in V$ is critical then $\{y \in Hx | \|y\| = \|x\|\} = Kx$.
3. If $x \in V$ and $Hx$ is closed then $Hx$ contains a critical element.
4. If $x \in V$ is critical then $Hx$ is closed.

The hard part of this theorem is part 4. We will now apply this to $(H, V)$ which we can assume satisfies the hypotheses of the theorem in light of the material in the last sub-section. We now carry over $g, \theta, \langle \ldots, \ldots \rangle, H, V = g_\xi$. Set $K_H = H \cap V$.

**Lemma 3.** $\text{Crit}(V) = \{x \in V | [x, x^*] = 0\}$.

**Proof.** $x \in \text{Crit}(V)$ if and only if $\langle Xx, x \rangle = 0$ for all $X \in \text{Lie}(H)$. This condition is if and only if $B(Xx, x^*) = 0$ for all $X \in \text{Lie}(H)$. Hence $x \in \text{Crit}(V)$ if and only if $B([X, x], x^*) = B([x, x^*], X) = 0$ for all $X \in g^\theta$. But our assumptions imply that $[x, x^*] \in g^\theta$. □

5. Analogues of the Kostant-Rallis theorems

Let $(H, V)$ be a Vinberg pair corresponding (as above) to a simple Lie algebra $g$ with an automorphism of order $m, \theta$. Let $a$ be a Cartan subspace of $V$. We assume, as we may, that $g \subset M_n(\mathbb{C})$ is invariant under adjoint and thus we have the inner product $\langle x, y \rangle = B(x, y^*)$.

The restriction of this form yielding an inner product on $V$.

5.1. The freeness. Let $K_H = H \cap U(V)$. Then $K_H$ is a maximal compact subgroup of $H$. We set $V_1 = \{v \in V | \langle v, a \rangle = 0\}$. If $p_0$ and $p_1$ are the natural projections of respectively $V$ to $a$ and $V$ to $V_1$ we will identify $O(a)$ and $O(V_1)$ respectively with $p_0^*(O(a))$ and $p_1^*(O(V_1))$ thus we have the graded algebra isomorphism $O(a) \otimes O(V_1) \to O(V)$. 
under multiplication. Set $W = W_H(a)$ then using Théorème 4 ii p.115 of [Bour] there is a subspace $\mathcal{A}$ of $\mathcal{O}(a)$ such that the map
\[
\mathcal{O}(a)^W \otimes \mathcal{A} \rightarrow \mathcal{O}(a)
\]
given by multiplication is a graded isomorphism and the Shephard Todd theorem implies that we can take $\mathcal{A}$ to be a graded subspace of $\mathcal{O}(a)$ of dimension equal to $|W|$ and is a $W$ module equivalent to the regular representation. The next result is analogous to Lemma 12.4.11 in [GW].

**Proposition 1.** The map $\mathcal{O}(V)^H \otimes \mathcal{A} \otimes \mathcal{O}(V_1) \rightarrow \mathcal{O}(V)$ given by multiplication is a graded vector space isomorphism.

**Proof.** We have seen that the restriction map $p_0^* : \mathcal{O}(V)^H \rightarrow \mathcal{O}(a)^W$ is a graded algebra isomorphism. Thus if we grade the tensor products above by the tensor product grade then the graded components of $\mathcal{O}(V)^L \otimes \mathcal{A} \otimes \mathcal{O}(V_1)$ and $\mathcal{O}(a)^W \otimes \mathcal{A} \otimes \mathcal{O}(V_1)$ have the same dimension. Now the rest of the argument is identical to that of Lemma 12.4.11 [GW].

The following result is proved in exactly the same way as in the last paragraph of p.602 in [GW] by induction on the degree.

**Corollary 1.** We extend $\langle \ldots, \ldots \rangle$ to an inner product on $\mathcal{O}(V)$ and define $\mathcal{H}^j = (\mathcal{O}(V)^H)^j \perp \mathcal{O}(V)^j$ relative to this inner product. Then $\mathcal{H} = \bigoplus_{j=0}^\infty \mathcal{H}^j$ is an $H$–module isomorphic with $\mathcal{O}(V)/ (\mathcal{O}(V)^H)^j \mathcal{O}(V)^j$ and furthermore the map
\[
\mathcal{O}(V)^H \otimes \mathcal{H} \rightarrow \mathcal{O}(V)
\]
given by multiplication is a linear bijection.

We note that the ideal $\mathcal{O}(V)^H$ defines the null cone of $V$. Thus if we could show that the ideal $\mathcal{O}(V)^H$ is a radical ideal then $\mathcal{H}$ could be identified with $\mathcal{O}(N)$ with $N$ the null cone of $V$. This is one of the main results of Kostant and Rallis in the case when $\theta$ is an involution. A result of Panyushev (c.f. [KS]) proves that this ideal is reduced if $H$ is semi–simple. The technique of Kostant-Rallis [KR] does not work in the context of Vinberg pairs. However their multiplicity theorem does generalize as does the technique used in [GW] to prove the theorem.

5.2. **A few more results of Vinberg.** If $\lambda \in a^*$ set
\[
g^\lambda = \{x \in g | [h, x] = \lambda(h)x, h \in a\}.
\]
and $\Sigma(a) = \{ \lambda \neq 0|g^\lambda \neq 0 \}$. We set
\[
a' = \{ h \in a|\lambda(h) \neq 0, \lambda \in \Sigma(a) \}.
\]
Set $C_g(a) = g^0$, $C_g(h) = \ker \text{ad}h$ and $C_H(a) = \{ g \in H|gh = h, h \in a \}$. If $h \in a'$ then $C_g(h) = C_g(a)$. The following results are contained in subsection 2 of section 3 of \[V\].

**Theorem 7.** $C_g(a) \cap V = a \oplus n$ with $n$ a subspace of $V$ consisting of nilpotent elements. $(C_H(a)|_n, n)$ is a Vinberg pair of rank 0.

The second part of the theorem combined with 5. in Section 2 implies

**Corollary 2.** The space $n$ is a finite union of $C_H(a)$ orbits.

We also note that it implies

**Corollary 3.** If $h \in a'$ then $X_h$ contains a unique open $H$-orbit, $H(h + x)$ where $C_H(a)x$ is the unique open $C_H(a)$ orbit in $n$.

**Corollary 4.** If $h \in a'$ then $X_h = H(h + n)$, in particular, $X_h$ is irreducible.

Another result, of a different nature, that will be used in the next subsection is the content of subsection 1 of section 3 in \[V\].

**Theorem 8.** Let $T_a$ be the intersection of all Zariski closed subgroups of $G$ whose Lie algebra contains $a$. Then $T_a$ is a torus that is the center of the group $C_G(a)$. If $t_a = \text{Lie}(T_a)$ then
\[
t_a = \bigoplus_{1 \leq j < m} t_a \cap g_\zeta^j \quad \text{gcd}(j, m) = 1
\]
and each space $t_a \cap g_\zeta^j$ is a Cartan subspace of the Vinberg pair $(L|_{g_\zeta^j}, g_\zeta^j)$.

### 5.3. The critical set revisited

Let $a$ be a Cartan subspace of $V$ (= $g_\zeta$). Let $x \in a'$. Since $Hx$ is closed there exists $y \in \text{Crit}(V) \cap Hx$. Write $y = gx$. We replace $x$ with $y$ and $a$ with $qa$. Thus we may assume that $x \in a'$ is critical. Since $x$ is critical we have $[x, x^*] = 0$. Noting that $\text{Lie}(T_a)_{\zeta^{-1}}$ is a Cartan subspace for the Vinberg pair $(L_{|g_\zeta^{-1}}, g_\zeta^{-1})$ we see that $C_g(a)_{\zeta^{-1}} = \text{Lie}(T_a)_{\zeta^{-1}} \oplus u$ with $u$ consisting of nilpotent elements. This implies that $x^* \in \text{Lie}(T_a) \cap g_\zeta^{-1}$. Hence $\text{Lie}(T_a) \cap g_\zeta^{-1}$ is contained in the set of semi-simple elements in the centralizer of $x^*$ in $g_\zeta^{-1}$ which is $a^*$. Recalling that $\dim \text{Lie}(T_a) \cap g_\zeta^{-1} = \dim a = \dim a^*$.

We have proved

**Proposition 2.** We may choose a Cartan sub-algebra, $a \subset V$ such that $[a, a^*] = 0$.

**Proposition 3.** $\text{Crit}(V) = K_Ha$. 

Proof. The above lemma implies that $a \subset \text{Crit}(V)$. Suppose that $x \in \text{Crit}(V)$ then $Hx$ is closed. Thus there exists $g \in H$ such that $gx = y \in a$. Thus $\|x\| = \|y\|$ (since both are critical). Hence there exists $k \in K_H$ such that $ky = x$ by the Kempf-Ness theorem.

Proposition 4. If $w \in W_H(a)$ then there exists $k \in K_H$ such that $k|_a = w$.

Proof. Let $x \in a$ be such that if $\lambda, \mu \in \Sigma(a) \cup \{0\}$ then $\lambda(x) = \mu(x)$ implies $\lambda = \mu$. Such an $x \in a$ exists. Indeed, define

$$S = \{\lambda - \mu|\lambda, \mu \in \Sigma(a) \cup \{0\}, \lambda \neq \mu\}$$

then $S$ is a finite set in $a^*$ (here the super script means dual space) and $x$ is an element in $a$ such that $\xi(x) \neq 0$ for $\xi \in S$. Let $h \in H$ be such that $h|_a = w$. Then $hx \in Hx \cap \text{Crit}(V) = K_Hx$. So $hx = kx$ for some $k \in K_H$. Now $w^*\Sigma(a) = \Sigma(a)$ thus

$$C_g(x) \cap \text{Crit}(V) = C_g(a) \cap \text{Crit}(V) = a.$$ 

This implies that $ka = a$. Also $k^{-1}hx = x$. Thus the choice of $x$ implies that $k^{-1}h|_a$ is the identity. □

5.4. The structure of $X_h$ for $h$ generic. We maintain the notation of the previous subsection and we assume as we may that $[a, a^*] = 0$. The next result uses an argument in [GW] 12.4.12 in its proof.

We note that since $W_H(a)$ is a subgroup of $GL(a)$ the set of $x \in a$ such that $|W_H(a)x| = |W_H(a)|$ is a Zariski open dense subset $a'' \subset a$. We note that if $m = 2$ then $a'' = a'$.

Theorem 9. If $h \in a' \cap a''$ then $I_h$ is a radical ideal hence prime.

Proof. The part of the proof of Proposition 12.4.12 in [GW] that shows that the ideal $I_h$ (in the context of that book) is a radical ideal that starts on line 10 on p. 603 and continues through line -11 on p. 604 left unchanged in this more general context (except that we must replace $a'$ by $a' \cap a''$) proves the result. □

Lemma 4. The set of $h \in a' \cap a''$ such that $X_h$ is a smooth affine variety contains a Zariski open dense subset, $a''$.

Proof. We have seen that $X_h = H(h+n)$. Let $f_1, \ldots, f_r$ be algebraically independent homogeneous generators for $O(V)^H$. We note that if $g \in H, h \in a', v \in a, f \in O(V)^H$ and $x \in n$ then

$$df_{g(h+x)}(gv) = \frac{d}{dt=0} f(g(h + x + tv)) =$$

$$\frac{d}{dt=0} f(h + tv + x) = \frac{d}{dt=0} f(h + tv)$$
since $h + tv$ is semi-simple and $x$ is nilpotent and $[x, h + tv] = 0$. Thus if $z = g(h + x)$, if $u_j = f_{j|a}$ and if $v_1, ..., v_s$ is a basis of $a$ with corresponding linear coordinates $x_1, ..., x_n$ then we have

$$(df_{1_x} \wedge \cdots \wedge df_{r_x})(gv_1, ..., gv_s) = \det(\frac{\partial u_i}{\partial x_j}(h)).$$

Now $u_1, ..., u_r$ are algebraically independent on $a$ so the Jacobian criterion implies that the polynomial $\det(\frac{\partial u_i}{\partial x_j}(h))$ is not identically 0 on $a$. Take $a'' = \{h \in a' \cap a'' | \det(\frac{\partial u_i}{\partial x_j}(h)) \neq 0\}$. If $h \in a''$ then

$$\dim(T_z(X_h)) = \dim V - r$$

for all $z \in X_h$. \qed

We note that if $m = 2$ and if $h \in a'$ then $X_h = Hh$ so the lemma above is obvious in this case.

We set $M = C_H(a)$ and define for $m \in M$, $g \in H$, $x \in n$, $(g, x)m = (gm, m^{-1}x)$. Then $(H \times n)/M$ is the vector bundle $H \times_M n$ over $H/M$.

**Theorem 10.** Fix $h \in a''$. If we define $\Psi_h : H \times n \rightarrow X_h$ by $\Psi_h(g, x) = g(h + x)$ then $\Psi_h(g, x)$ depends only on $(g, x)M$ and the induced map of $H \times_M n$ to $X_h$ is an isomorphism of algebraic varieties.

**Proof.** Since $h \in a''$, $X_h$ is smooth hence it is a complex manifold of dimension $n = \dim V - \dim a$. We also note that $H \times_M n$ is also a smooth variety of the same dimension. Suppose $\Psi_h(g, x) = \Psi_h(g', x')$ with $g, g' \in H$ and $x, x' \in n$ then $g(h + x) = g'(h + x')$. This implies (using The Jordan decomposition) that $gh = g'h$. Thus $g^{-1}g'h = h$. So $g' = gm$ with $m \in M$. Also $gx = g'x'$. Thus $g'(h + x') = gm(h + m^{-1}x)$. This implies that

$$\Psi_h : H \times_M n \rightarrow X_h$$

is regular and bijective.

We calculate the differential of $\Psi_h$ at $g, x$ for $g \in H$ and $x \in n$. Let $X \in \text{Lie}(H)$ and $v \in n$. Then

$$(d\Psi_h)_{g,x} (X, v) = g(X(h + x) + v).$$

We assert that the dimension of the image of $(d\Psi_h)_{g,x}$ is $\dim V - \dim a$ for all $g \in H$ and $x \in n$. It is clearly enough to prove this for $g = I$. Let $y \in g_{\xi-1}$ be such that $B(y, z) = 0$ for all $z = X(h + x) + v$ as above. Then

$$0 = B(y, [X, h + x]) = B([h + x, y], X)$$

for all $X \in \text{Lie}(H)$. But $[h + x, y] \in \text{Lie}(H)$ so this implies that $[h + x, y] = 0$. This implies $y \in C_h(h)_{\xi-1} = C_a(\xi)_{\xi-1}$ since $(h + x)_a = h$ and $h \in a'$. Also $B(y, n) = 0$ implies that $y \in a^*$. This implies the
dimension estimate. We therefore see that \((d\Psi_h)_{g,x}\) is of maximal rank for all \(g \in H, x \in \mathfrak{n}\) so the inverse function theorem implies that

\[
\Psi_h : H \times_M \mathfrak{n} \to X_h
\]

is biholomorphic. We assert that \(\Psi_h\) is also birational. Indeed, if \(x \in \mathfrak{n}\) is such that \(H(h + x)\) is open in \(X_h\) then if \(g \in H\) is such that \(g(h + x) = h + x\) then the uniqueness of the Jordan decomposition implies that \(gh = h\) and \(gx = x\). Thus \(g \in C_H(a)_x\). Thus the open orbit is biregularly isomorphic with \(H/C_H(a)_x\). Thus \(g \in C_H(a)_x\). Thus the open orbit is biregularly isomorphic with \(H/C_H(a)_x\). We now consider the same \(x\) and the orbit under \(H\) of \((e, x)\) in \(H \times C_H(a)\). The stabilizer is the set of \(g \in H\) such that \(g \in C_H(a)\) and \(gx = x\). Thus it is exactly the same. Also \(\dim X_h = \dim H \times H\) so the orbit of \((e, x)M\) is Zariski open and is regularly isomorphic to the open orbit in \(V_h\) under the map \(\Psi_h\). Thus \(\Psi_h\) is a birational isomorphism. The result now follows from the following lemma. □

**Lemma 5.** Let \(X\) and \(Y\) be smooth irreducible affine varieties of the same dimension

\[ F : X \to Y \]

be regular, biholomorphic and birational then \(F\) is a regular isomorphism of varieties.

**Proof.** Let \(F^{-1} : Y \to X\) then \(F^{-1}\) is a rational map that is also holomorphic. We may assume that \(X \subset \mathbb{C}^n\) as a Zariski closed subset. Then \(F^{-1} = (\phi_1, ..., \phi_n)\) with \(\phi_j, j = 1, ..., n\) both rational and holomorphic on \(Y\). This implies that if \(p \in Y\) then the germ at \(p\) of each \(\phi_j\) is in \(O_{X,p}\) (see the Lemma on p.177 in [Sh] which follows from the fact that since \(X\) is smooth \(O_{X,p}\) is a unique factorization domain). Thus each \(\phi_j\) is regular and so \(F^{-1}\) is regular. □

5.5. The multiplicity formula. We consider the representation of \(H\) on the harmonics \(H\) (see Corollary [1]). Our generalization of the Kostant-Rallis decomposition of the harmonics is

**Theorem 11.** If \(U\) is an irreducible regular \(H\)-module then

\[
\dim \text{Hom}_H(U, \mathcal{H}) = \dim \text{Hom}_M(U, \mathcal{O}(\mathfrak{n})).
\]

We will call the Vinberg pair tame if \(\mathfrak{n} = \{0\}\). In particular, if \(\theta^2 = 1\) or the pair is regular \((T_a\) is a maximal torus in \(G\)) then the pair is tame. Thus the theorem in this context is an exact generalization to the multiplicity theorem of Kostant-Rallis.

**Corollary 5.** If the pair is tame and if \(U\) is an irreducible regular \(H\)-module then

\[
\dim \text{Hom}_H(U, \mathcal{H}) = \dim \text{Hom}_M(U, \mathbb{C}).
\]
Note that the corollary follows directly from the above theorem and Theorem 12.4.13 in [GW].

We will devote the rest of this subsection to the prove of this theorem. First we need

**Lemma 6.** The $H$–module $\mathcal{H}$ is equivalent to $\mathcal{O}(V)/I_h$ for any $h \in \mathfrak{a}$.

**Proof.** We put the natural filtration by degree on $\mathcal{O}(V)/I_h$. Then Lemma 12.4.9 in [GW] immediately implies that $\text{Gr}(\mathcal{O}(V)/I_h)$ is isomorphic with $\mathcal{O}(V)/I_0 = \mathcal{O}(V)/\mathcal{O}(V)\mathcal{O}_+(V)^H$. which we have seen is isomorphic with $\mathcal{H}$ as an $H$ module. \hfill \Box

We recall that if $h \in \mathfrak{a}''$ then $I_h$ is prime. So the lemma above implies that $\mathcal{H}$ is isomorphic with $\text{Gr}(\mathcal{O}(X_h))$ as a representation of $H$ for any $h \in \mathfrak{a}''$ (see the previous section). The theorem now follows from

**Proposition 5.** Let $h \in \mathfrak{a}''$. If $U$ is an irreducible regular $H$–module then

$$\dim \text{Hom}_H(U, \mathcal{O}(X_h)) = \dim \text{Hom}_M(U, \mathcal{O}(n)).$$

**Proof.** In light of Theorem [10] may replace $\mathcal{O}(X_h)$ for $h \in \mathfrak{a}''$ with $\mathcal{O}(H \times_M n)$. By our definition $H \times_M n = (H \times n)/M$ using the right action above, $\mathcal{O}((H \times n)/M) = \mathcal{O}(H \times n)^M$. Here $M$ acts on $\mathcal{O}(H \times n)$ by $mf(g, x) = f(gm, m^{-1}x)$ for $g \in H, x \in n$ and $m \in M$. Now

$$\mathcal{O}(H \times n) \cong \mathcal{O}(H) \otimes \mathcal{O}(n)$$

(under the map $(u \otimes v)(g, x) = u(g)v(x), u \in \mathcal{O}(H), v \in \mathcal{O}(n)$) and the action of $M$ is just the tensor product action relative to the right action on $H$ and the left action on $n$. Thus $M$ leaves invariant the grade on $\mathcal{O}(n)$. So

$$\mathcal{O}(H \times n)^M = \bigoplus_{j \geq 0} \left(\mathcal{O}(H) \otimes \mathcal{O}^j(n)\right)^M.$$

If $f \in (\mathcal{O}(H) \otimes \mathcal{O}^j(n))^M$ (as a subspace of $(\mathcal{O}(H) \otimes \mathcal{O}(n))^M$) then we define $f(g)(x) = f(g, x)$ for $g \in H$ and $x \in n$.Then

$$f : H \to \mathcal{O}^j(n)$$

is regular and $f(gm) = m^{-1}f(g)$. That is, as an $H$–module,

$$(\mathcal{O}(H) \otimes \mathcal{O}^j(n))^M \cong \text{Ind}_M^H(\mathcal{O}^j(n)).$$

The theorem now follows from Frobenius reciprocity. (See e.g. [GW] section 12.1.2 for the undefined terms and the reciprocity.) \hfill \Box
6. EXAMPLES FOR $E_6$ AND $E_8$

The full details of this discussion can be found in [W] also most of the preliminaries to the actual multiplicity formula can be found in [EV]. However the interested reader can take the unproved assertions in this paper to be exercises.

6.1. An $E_6$ example. We take $\mathfrak{g}$ to be simple of type $E_6$. Fix a Cartan subalgebra $\mathfrak{h}$ and a system of positive roots. The simple roots are $\alpha_1, \ldots, \alpha_6$ and the extended Dynkin diagram in the Bourbaki ordering is

```
 o  -\beta
 |    |
 o  \alpha_2
```

Let $H_1, \ldots, H_6$ be the dual basis of $\mathfrak{h}$ to the simple roots (i.e. $\alpha_i(H_j) = \delta_{ij}$). Then the automorphism $\mathfrak{g}$ given by $\theta = \exp(\frac{2\pi i}{3}\text{ad}H_4)$ is of order 3 since the coefficient of $\alpha_4$ in the expansion of the highest root, $\beta$, is 3. In this case we see that we have $H$ is locally isomorphic with $SL(3, \mathbb{C}) \times SL(3, \mathbb{C}) \times SL(3, \mathbb{C})$ (since its Dynkin diagram is gotten by deleting the node labeled $\alpha_4$) and $\mathfrak{g}$ is the direct sum of $\text{Lie}(H)$ and a direct sum of $H$–modules

$$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)^*$$

The corresponding Vinberg pair is

$$(SL(3, \mathbb{C}) \otimes SL(3, \mathbb{C}) \otimes SL(3, \mathbb{C}), \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3).$$

(Here the indicated group is the set of elements $g_1 \otimes g_2 \otimes g_3$ with $g_i \in SL(3, \mathbb{C})$.) Let $e_1, e_2, e_3$ denote the standard basis of $\mathbb{C}^3$. One can show that

$$v_1 = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 + e_3 \otimes e_3 \otimes e_3,$$

$$v_2 = e_1 \otimes e_2 \otimes e_3 + e_3 \otimes e_1 \otimes e_2 + e_2 \otimes e_3 \otimes e_1,$$

$$v_3 = e_3 \otimes e_2 \otimes e_1 + e_1 \otimes e_3 \otimes e_2 + e_2 \otimes e_1 \otimes e_3$$

then $v_1, v_2, v_3$ is a basis of a Cartan subspace, $\mathfrak{a}$, of $V$. To see this we note that if $T$ is the product of the diagonal Cartan subgroups then the weights of $V$ are all of multiplicity one. This implies that $\wedge^2 V$ is multiplicity free and since $V^* \text{occurs in } \wedge^2 V$ we see that the bracket of $\mathfrak{g}$ restricted to $V$ is up to scalar multiple given by

$$[x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3] = x_1 \wedge y_1 \otimes x_2 \wedge y_2 \otimes x_3 \wedge y_3$$
with $\wedge^2 C^3$ identified with $(C^3)^*$. Observe that this implies that $[v_i, v_j] = 0$ all $i, j$. Also a direct calculation shows that
\[
\langle E_{rs} v_i, v_j \rangle = 0
\]
and
\[
\langle (E_{rr} - E_{ss}) v_i, v_j \rangle = 0
\]
if $r \neq s$ for all $i, j$. Thus the span of the $v_i$ is abelian and consists of critical. hence semi–simple, elements. Finally since $\varphi(3) = 2$ and rank $E_6$ is 6 we see that since a Cartan subspace is at most of dimension $\frac{\text{rank}(g)}{\varphi(m)} = 3$ this span, indeed a Cartan subspace. We also note that

1. $C_g(a) \cap V = a$.
2. The group $M$ is the set of triples of matrices $M_1 \cup M_2 \cup M_3 \cup M_4$ with $\alpha, \beta, \delta, \mu$ third roots of 1.

\[
M_1 = \left\{ \left( \begin{array}{ccc}
\alpha & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right), \beta \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right), \frac{1}{\alpha \beta} \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \right\},
\]
\[
M_2 = \left\{ \left( \begin{array}{ccc}
\alpha & 0 & 0 \\
0 & \delta & 0 \\
0 & \frac{1}{\delta \mu} & 0
\end{array} \right), \beta \left( \begin{array}{ccc}
\delta & 0 & 0 \\
0 & \mu & 0 \\
0 & \frac{1}{\delta \mu} & 0
\end{array} \right), \frac{1}{\alpha \beta} \left( \begin{array}{ccc}
\delta & 0 & 0 \\
0 & \mu & 0 \\
0 & \frac{1}{\delta \mu} & 0
\end{array} \right) \right| \delta \neq \mu \right\},
\]
\[
M_3 = \left\{ \left( \begin{array}{ccc}
\alpha & 0 & 0 \\
0 & \delta & 0 \\
\frac{1}{\delta \mu} & 0 & 0
\end{array} \right), \beta \left( \begin{array}{ccc}
0 & \delta & 0 \\
0 & \mu & 0 \\
\frac{1}{\delta \mu} & 0 & 0
\end{array} \right), \frac{1}{\alpha \beta} \left( \begin{array}{ccc}
0 & \delta & 0 \\
0 & \mu & 0 \\
\frac{1}{\delta \mu} & 0 & 0
\end{array} \right) \right\},
\]
\[
M_4 = \left\{ \left( \begin{array}{ccc}
\alpha & 0 & 0 \\
0 & \delta & 0 \\
0 & \frac{1}{\delta \mu} & 0
\end{array} \right), \beta \left( \begin{array}{ccc}
0 & \delta & 0 \\
0 & \mu & 0 \\
0 & \frac{1}{\delta \mu} & 0
\end{array} \right), \frac{1}{\alpha \beta} \left( \begin{array}{ccc}
0 & \delta & 0 \\
0 & \mu & 0 \\
0 & \frac{1}{\delta \mu} & 0
\end{array} \right) \right\}.\]

3. The order of $M$ is 81
4. Every element of $M_i$ for $i > 1$ is conjugate to
\[
\left( \begin{array}{ccc}
\zeta^2 & 0 & 0 \\
0 & \zeta & 0 \\
0 & 0 & 1
\end{array} \right), \left( \begin{array}{ccc}
\zeta^2 & 0 & 0 \\
0 & \zeta & 0 \\
0 & 0 & 1
\end{array} \right), \left( \begin{array}{ccc}
\zeta^2 & 0 & 0 \\
0 & \zeta & 0 \\
0 & 0 & 1
\end{array} \right)
\]
in $H$ with $\zeta = e^{2\pi i}$.  

We parametrize the irreducible regular representations of $SL(3, \mathbb{C})$ by pairs of integers $m \geq n \geq 0$ as the restrictions of the irreducible representation of $GL(3, \mathbb{C})$ corresponding to $m \geq n \geq 0$ (c.f. [GW] Theorem 5.5.22). This parametrization is by the highest weight $m \varepsilon_1 + n \varepsilon_2$ restricted to the diagonal matrices of trace 0. We write the representation as $F^{m,n}$. Thus the irreducible regular representations of $H$ are of the form $F^{m_1,n_1} \otimes F^{m_2,n_2} \otimes F^{m_3,n_3}$.
We have
6. $F^{m_1,n_1} \otimes F^{m_2,n_2} \otimes F^{m_3,n_3}$ has a fixed vector for the group $M_1$ above if and only if $m_1 + n_1 \equiv m_2 + n_2 \equiv n_3 + m_3 \mod 3$.

**Proposition 6.** If the condition of Exercise 4 is not satisfied then $\text{Hom}_H(F^{m_1,n_1} \otimes F^{m_2,n_2} \otimes F^{m_3,n_3}, H) = \{0\}$. If it is satisfied then $\dim F^{m_1,n_1} \otimes F^{m_2,n_2} \otimes F^{m_3,n_3} \mod 9 \in \{0, 1, 8\}$.

Set $\varepsilon(F^{m_1,n_1} \otimes F^{m_2,n_2} \otimes F^{m_3,n_3}) = 0, 8, -8$ respectively if the congruence modulo 9 is 0, 1 or 8. Then

$$\text{Hom}_H(F^{m_1,n_1} \otimes F^{m_2,n_2} \otimes F^{m_3,n_3}, H) = \frac{\dim F^{m_1,n_1} \otimes F^{m_2,n_2} \otimes F^{m_3,n_3} + \varepsilon(F^{m_1,n_1} \otimes F^{m_2,n_2} \otimes F^{m_3,n_3})}{9}$$

We will prove the proposition using
7. Let $G$ be a group and $X$ a finite dimensional $G$–module with character $\chi_X$. If $M$ is a finite subgroup of $G$ then

$$\dim V^M = \frac{1}{|M|} \sum_{m \in M} \chi_X(m).$$

We can now prove the result. The order of $M$ is 81. If $X = F^{m_1,n_1} \otimes F^{m_2,n_2} \otimes F^{m_3,n_3}$ and it satisfies the congruence condition in 6. then the value of $\chi_X$ on each element of $M_1$ is $\dim X$. There are 9 such elements. Set $\chi_{m,n} = \chi_{F^{m,n}}$. 4. implies that the other 72 elements of $M$ all have the value

$$\chi_{m_1,n_1}(u)\chi_{m_2,n_2}(u)\chi_{m_3,n_3}(u)$$

with

$$u = \begin{bmatrix} \zeta^2 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the Weyl character formula, the Weyl denominator formula and 7. above the proposition follows as an exercise here is a hint

A similar result in the more complicated context of $E_8$ is proved in the next subsection with some of the same ideas. Let $T$ be the diagonal torus in $SL(3, \mathbb{C})$ and let

$$\varepsilon_i \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} = x_i, i = 1, 2, 3.$$
Then \( \rho \) the half sum of the positive roots is \( \varepsilon_1 - \varepsilon_3 \). Then

\[
H_\rho = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

so \( u = e^{2\pi i H_\rho} \). The Weyl denominator formula implies that

\[
\sum_{s \in S_3} sgn(s)e^{sp(H)} = e^{\rho(H)}(1-e^{-(\varepsilon_1-\varepsilon_2)(H)})(1-e^{-(\varepsilon_2-\varepsilon_3)(H)})(1-e^{-(\varepsilon_1-\varepsilon_3)(H)})
\]

The Weyl character formula says that if \( \Lambda = m\varepsilon_1 + n\varepsilon_2 \) then

\[
\chi_{m,n}(e^H) = \frac{\sum_{s \in S_3} sgn(s)e^{s(\Lambda + \rho)(H)}}{\sum_{s \in S_3} sgn(s)e^{sp(H)}}.
\]

So

\[
\chi_{m,n}(u) = \chi_{m,n}(e^{H_\rho}) = \frac{\sum_{s \in S_3} sgn(s)e^{s(\Lambda + \rho)(H_\rho)}}{\sum_{s \in S_3} sgn(s)e^{sp(H_\rho)}} = \frac{\sum_{s \in S_3} sgn(s)e^{sp(H_{\Lambda + \rho})}}{\sum_{s \in S_3} sgn(s)e^{sp(H_\rho)}}
\]

with

\[
H_{\Lambda + \rho} = \begin{bmatrix}
m + 1 & 0 & 0 \\
0 & n & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

now apply the denominator formula and calculate.

6.2. An \( E_8 \) example. We take \( g \) to be simple of type \( E_8 \). Fix a Cartan subalgebra \( h \) and a system of positive roots. The simple roots are \( \alpha_1, ..., \alpha_8 \) and the extended Dynkin diagram in the Bourbaki ordering is

\[
\circ \quad \alpha_2 \\
\circ \quad \alpha_3 \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \\
\alpha_1 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \quad \alpha_8 \quad -\beta
\]

As before we take the dual basis to the simple roots \( \alpha_i(H_j) = \delta_{ij} \). In this case the coefficient of \( \alpha_2 \) in \( \beta \) is 3 so \( \theta = \exp(2\pi i \text{ad} H_2) \) is a homomorphism of \( g \) of order 3. This yields the Vinberg pair \((H, V) = (SL(9, \mathbb{C}), \wedge^3 \mathbb{C}^9)\). For simplicity we will use the simply connected covering group \( SL(9, \mathbb{C}) \) then we note that the covering map \( H = SL(9, \mathbb{C}) \to \wedge^3 SL(9, \mathbb{C}) \) has kernel \( S = \{ zI | z^3 = 1 \} \). We also note that a Cartan subspace in \( V \) is the space \( a \) with basis

\[
\omega_1 = e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6 + e_7 \wedge e_8 \wedge e_9,
\]
\[ \omega_2 = e_1 \wedge e_4 \wedge e_7 + e_2 \wedge e_5 \wedge e_8 + e_3 \wedge e_6 \wedge e_9, \]
\[ \omega_3 = e_1 \wedge e_5 \wedge e_9 + e_2 \wedge e_6 \wedge e_7 + e_3 \wedge e_4 \wedge e_8, \]
\[ \omega_4 = e_1 \wedge e_6 \wedge e_8 + e_2 \wedge e_4 \wedge e_9 + e_3 \wedge e_5 \wedge e_7. \]

Indeed one checks that \( \langle X \omega_i, \omega_j \rangle = 0 \) for \( X = E_{i,j}, i < j \) and \( X = E_{ii} - E_{i+1,i+1}, i = 1, \ldots, 8 \). Thus every element of the span of \( \omega_1, \omega_2, \omega_3, \omega_4 \) is critical and so the Kempf-Ness theorem implies that \( \tilde{H}v \) is closed for every element in \( a \). Up to scalar multiple the bracket in \( E_8 \) of \( v, w \in V \) is given by \( v \wedge w \). Since the weights in \( \wedge^3 \mathfrak{C}_9 \) are multiplicity at most 1 (in general the multiplicity of an extreme weight \( \xi \) in a tensor product of \( F^{\Lambda} \otimes F^{\mu} \) is at most the multiplicity of the weight \( \xi - \Lambda \) in \( F^{\mu} \). This also implies that \( [\omega_i, \omega_j] = 0 \) for all \( i, j \).

The centralizer of \( a \) in \( H, C = C_H(a) \), is the intersection of \( H \) with \( T_a \). Thus \( C \) is abelian. We thus have the exact sequence
\[ 1 \to S \to \tilde{C}_H(a) \to C \to 1. \]

The following elements are obviously in \( \tilde{C}_H(a) \) (Here \( w \) and \( z \) are third roots of 1):

\[ A_{z,w} = w \begin{bmatrix} I & 0 & 0 \\ 0 & zI & 0 \\ 0 & 0 & z^2I \end{bmatrix}, \]

\[ B_{z,w} = w \begin{bmatrix} 1 & z^2 & 0 \\ z & 0 & 0 \\ 0 & z & 0 \end{bmatrix}. \]

Thus if \( g \in \tilde{C} \) and
\[ g = \begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{bmatrix} \]

If \( z \) is a primitive third root of 1 then
\[ A_{z,1}gA_{z^2,1} = wg \]
with \( w = 1, z, \) or \( z^2 \). We have the following three cases.

a) \( w = 1 \): \( g \) is block diagonal
\[ \begin{bmatrix} X_1 & 0 & 0 \\ 0 & Y_2 & 0 \\ 0 & 0 & Z_3 \end{bmatrix}. \]
We have a linear isomorphism

$$T : \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \to \wedge^3 \mathbb{C}^6$$
given by

$$e_i \otimes e_j \otimes e_k \mapsto e_i \wedge e_j \wedge e_k + e_i \wedge e_{j+3} \wedge e_{k+6}, 1 \leq i, j, k \leq 3.$$ 

Under this map we have the intertwining

$$T \circ g_1 \otimes g_2 \otimes g_3 = \bigwedge^3 \begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix}, g_i \in SL(3, \mathbb{C}), i = 1, 2, 3.$$ 

We also note that

$$T^{-1}(\omega_2) = v_1 = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 + e_3 \otimes e_3 \otimes e_3,$$

$$T^{-1}(\omega_3) = v_2 = e_1 \otimes e_2 \otimes e_3 + e_3 \otimes e_1 \otimes e_2 + e_2 \otimes e_3 \otimes e_1,$$

$$T^{-1}(\omega_4) = v_3 = e_3 \otimes e_2 \otimes e_1 + e_1 \otimes e_3 \otimes e_2 + e_2 \otimes e_1 \otimes e_3.$$ 

We will think of $(X, Y, Z)$ as the corresponding block diagonal matrix. The results of the previous subsection imply we will find all elements of the form in case a) if we find the elements in the sets $M_i$ that fix $\omega_1$. Here is the list as they come from the $M_i$:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} | w^3 = 1,$$

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix}, \begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix}, \begin{pmatrix} z & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} | z^3, w^3 = 1, z \neq 1,$$

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & z \\ z^2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z^2 & 0 \\ 0 & 0 & 1 \\ z & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & z^2 \\ 1 & 0 & 0 \end{pmatrix} \right\} | z^3, w^3 = 1.$$
Thus in case a) there are 27 elements.

We now observe that the elements

\[
U = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}
\]

are in \( C_{\tilde{H}}(a) \) and the product of \( V \) with the elements in case b) are in the case a) as are the products of \( U \) with the elements in case c) are in the case a). Thus we have a group of order 81.

Noting that \( H_\rho \) is the diagonal matrix \( \text{diag}(4, 3, 2, 1, 0, -1, -2, -3, -4) \) we have

**Lemma 7.** Any element in \( C_{\tilde{H}}(a) \) that is not a multiple of the identity is conjugate to

\[
\mu = e^{2\pi i H_\rho} = \text{diag}(\zeta, 1, \zeta^2, \zeta, 1, \zeta^2, \zeta, 1, \zeta^2)
\]

with \( \zeta = e^{2\pi i / 3} \).

We will label the irreducible representations of \( \tilde{H} = SL(9, \mathbb{C}) \) by their highest weight \( \Lambda = (\lambda_1, ..., \lambda_8, 0) \) restricted to the diagonal matrices of trace 0. Thus a necessary condition for \( F^\Lambda \) to occur in \( O(\wedge^3 \mathbb{C}^9) \) is that \( \sum_{i=1}^8 \lambda_i \equiv 0 \mod 3 \). Let \( \chi_\Lambda \) denote the character of \( F^\Lambda \).

**Lemma 8.** If \( \sum_{i=1}^8 \lambda_i \equiv 0 \mod 3 \) then denoting by \( \mathcal{H} \) the \( \tilde{H} \) harmonics in \( O(\wedge^3 \mathbb{C}^9) \) we have

\[
\dim \text{Hom}_{SL(9, \mathbb{C})}(F^\Lambda, \mathcal{H}) = \frac{\dim F^\Lambda + 26\chi_\Lambda(\mu)}{27}.
\]

**Proof.** Frobenius reciprocity and 7. in the previous subsection imply

\[
\dim \text{Hom}_{SL(9, \mathbb{C})}(F^\Lambda, \mathcal{H}) = \frac{1}{|C_{\tilde{H}}(a)|} \sum_{c \in C_{\tilde{H}}(a)} \chi_\Lambda(c).
\]

The above results imply that this expression is equal to

\[
\frac{3\chi_\Lambda(\mathbb{I}) + 78\chi_\Lambda(\mu)}{81}.
\]

\( \square \)
We will now use a variant of Weyl’s method of deriving his dimension formula to calculate $\chi_\Lambda(\mu)$. We first consider

$$\chi_\Lambda(e^{(2\pi i/3)H_\rho}) = \frac{\sum_{s \in S_9} sgn(s)e^{\Lambda+\rho}((2\pi i/3)H_\rho)}{\sum_{s \in S_9} sgn(s)e^{\rho}((2\pi i/3)H_\rho)}.$$ 

We want to apply Weyl’s denominator formula (using the usual positive roots of the diagonal Cartan subgroup that is $\varepsilon_i - \varepsilon_j$ with $i < j$) to both the numerator and the denominator. Since there are exactly 9, let $S_9$ denotes the set of roots. Note that

$$\chi_j = \frac{1}{\chi_{j-1}}.$$ 

Thus the value we want is gotten by taking the limit as $t \to 0$. In the denominator the factors that go to 0 are exactly the ones such that $\alpha(H_\rho) \equiv 0 \mod 3$.

There are 9 of these roots which correspond to $\varepsilon_i - \varepsilon_j$ with $j - i = 3$ or 6, with 6 roots for the value 3 and 3 for the value 6. Thus to take the limit we must have at least 9 positive roots with

$$\langle \alpha, \Lambda + \rho \rangle \equiv 0 \mod 3.$$ 

If there are more than 9 then the limit is 0 and thus in this case

$$\dim \text{Hom}_{SL(9,\mathbb{C})}(F^\Lambda, H) = \frac{\dim F^\Lambda}{27}.$$ 

So suppose that there are exactly 9. Let $S_j(\Lambda) = \{\alpha | \alpha > 0, \langle \alpha, \Lambda + \rho \rangle \equiv j \mod 3\}, j = 0, 1, 2$. Then $S_0(\Lambda) = 9 = S_0(0)$ and thus $S_1(\Lambda) + S_2(\Lambda) = S_1(0) + S_2(0) = 27$. With this notation $\chi_\Lambda(e^{(2\pi i/3)H_\rho})$ is given by

$$e^{(2\pi i/3)(\Lambda, \rho)} \frac{\prod_{\alpha \in S_0(\Lambda)}(1 - e^{-t(\alpha, \Lambda + \rho)})}{\prod_{\alpha \in S_0(0)}(1 - e^{-t(\alpha, \rho)})} \frac{\prod_{\alpha \in S_1(\Lambda)}(1 - \zeta^2 e^{-t(\alpha, \Lambda + \rho)})}{\prod_{\alpha \in S_1(0)}(1 - \zeta^2 e^{-t(\alpha, \rho)})} \times \frac{\prod_{\alpha \in S_2(\Lambda)}(1 - \zeta e^{-t(\alpha, \Lambda + \rho)})}{\prod_{\alpha \in S_2(0)}(1 - \zeta e^{-t(\alpha, \rho)})}.$$ 

Note that $|S_1(0)| = 15$ and $|S_2(0)| = 12$. Thus the limit as $t \to 0$ is

$$e^{2\pi i/3(\Lambda, \rho)} \frac{\prod_{\alpha \in S_0(\Lambda)}\langle \alpha, \Lambda + \rho \rangle}{\prod_{\alpha \in S_0(0)}\langle \alpha, \rho \rangle} \frac{(1 - \zeta^2)^{|S_1(\Lambda)|}}{(1 - \zeta)^{15}(1 - \zeta)^{12}} = e^{2\pi i/3(\Lambda, \rho)} \frac{\prod_{\alpha \in S_0(\Lambda)}\langle \alpha, \Lambda + \rho \rangle}{\prod_{\alpha \in S_0(0)}\langle \alpha, \rho \rangle} \frac{(1 + \zeta)^{|S_1(\Lambda)|}}{(1 + \zeta)^3}.$$
We consider the first factor
\[ e^{\frac{2\pi i}{3} \langle \Lambda, \rho \rangle} = e^{\frac{2\pi i}{3} \sum_{\alpha > 0} \langle \alpha, \Lambda \rangle} = e^{\frac{2\pi i}{3} \sum_{\alpha > 0} \langle \alpha, \Lambda + \rho \rangle} \]
since \( \sum_{\alpha > 0} \langle \rho, \alpha \rangle = 2 \langle \rho, \rho \rangle = 120 \). Now, if \( \alpha \in S_j(\Lambda) \) then \( \langle \alpha, \Lambda + \rho \rangle = 3k_\alpha + j \) for \( j = 0, 1, 2 \) and \( k_\alpha = \left\lfloor \frac{\langle \alpha, \Lambda + \rho \rangle}{3} \right\rfloor \). So, if we set \( \gamma = e^{\frac{2\pi i}{3}} = (1 + \zeta) \), we have
\[ e^{\frac{2\pi i}{3} \langle \Lambda, \rho \rangle} = (-1)^{\sum_{\alpha > 0} \left\lfloor \frac{\langle \alpha, \Lambda + \rho \rangle}{3} \right\rfloor} \gamma^{|S_1(\Lambda)|} \gamma^{|S_2(\Lambda)|} = -(-1)^{\sum_{\alpha > 0} \left\lfloor \frac{\langle \alpha, \Lambda + \rho \rangle}{3} \right\rfloor} \gamma^{|S_2(\Lambda)|}. \]

We are now ready to multiply out the formula and have
\[ -(-1)^{\sum_{\alpha > 0} \left\lfloor \frac{\langle \alpha, \Lambda + \rho \rangle}{3} \right\rfloor} \gamma^{|S_2(\Lambda)|} \gamma^{|S_1(\Lambda)|} - 3 \prod_{\alpha \in S_0(\Lambda)} \langle \alpha, \Lambda + \rho \rangle = \]
\[ -(-1)^{\sum_{\alpha > 0} \left\lfloor \frac{\langle \alpha, \Lambda + \rho \rangle}{3} \right\rfloor} \frac{\gamma^{|S_2(\Lambda)|} \gamma^{|S_1(\Lambda)|}}{27^{39}} \]
since \( |S_1(\Lambda)| + |S_2(\Lambda)| - 3 = 24 \). We therefore have

**Proposition 7.** Assume that \( \Lambda = (\lambda_1, ..., \lambda_8, 0) \) is dominant integral. If \( \sum \lambda_i \) is not divisible by 3 then \( \dim \text{Hom}_{SL(9, \mathbb{C})}(F^\Lambda, \mathcal{H}) = 0 \). Let \( S_j(\Lambda) = \{ \alpha > 0 | \langle \alpha, \Lambda + \rho \rangle \equiv j \mod 3 \} \) for \( j = 0, 1, 2 \). Assume \( \sum \lambda_i \equiv 0 \mod 3 \) then \( |S_0(\Lambda)| \geq 9 \) and

\[ \dim \text{Hom}_{SL(9, \mathbb{C})}(F^\Lambda, \mathcal{H}) = \begin{cases} \frac{\dim F^\Lambda}{27} \text{ if } |S_0(\Lambda)| > 9, \\ \frac{\dim F^\Lambda - 26(-1)^{\sum_{\alpha > 0} \left\lfloor \frac{\langle \alpha, \Lambda + \rho \rangle}{3} \right\rfloor} \prod_{\alpha \in S_0(\Lambda)} \langle \alpha, \Lambda + \rho \rangle}{2^{39}} \text{ if } |S_0(\Lambda)| = 9. \end{cases} \]

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