Classification of simple Harish-Chandra modules over the
$N = 1$ Ramond algebra

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Abstract

In this paper, we give a new approach to classify all Harish-Chandra modules for the $N = 1$ Ramond algebra $\mathfrak{s}$ based on the so called $A$-cover theory developed in [1].

Keywords: Virasoro algebra, $N = 1$ Ramond algebra, cuspidal module, $A$-cover

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1. Introduction

Superconformal algebras have a long history in mathematical physics. The simplest examples, after the Virasoro algebra itself (corresponding to $N = 0$) are the $N = 1$ superconformal algebras: the Neveu-Schwarz algebra and the Ramond algebra. These infinite dimensional Lie superalgebras are also called the super-Virasoro algebras as they can be regarded as natural super generalizations of the Virasoro algebra. Weight modules for the super-Virasoro algebras have been extensively investigated (cf. [4, 6, 7]), for more related results we refer the reader to [5, 8–11, 13–15, 17, 18, 20] and references therein. It is an important and challenging problem to give complete classifications of Harish-Chandra modules (simple weight modules with finite dimensional weight spaces) for superconformal algebras. In [3], all simple unitary weight modules with finite dimensional weight spaces over the $N = 1$ superconformal algebra were classified, which includes highest and lowest weight modules. Recently simple weight modules with finite dimensional weight spaces over the $N = 2$ superconformal algebra were classified in [12]. With the theory of the $A$-cover in [1] for the Virasoro algebra, [21] completed such classification for the Lie superalgebra $W_{m,n}$ (also see [2]). A complete classification for the $N = 1$ superconformal algebra was given by Su in [19]. However, the complicated computations in the proofs make it extremely difficult to follow. In this paper, we give a new approach to classify all Harish-Chandra modules for the $N = 1$ Ramond algebra $\mathfrak{s}$ based on the $A$-cover theory.

This paper is arranged as follows. In Section 2, we recall some notations and collect known facts about the $N = 1$ Ramond algebra $\mathfrak{s}$. In Section 3, we classify all simple cuspidal modules for $\mathfrak{s}$. With this classification we get the main result about the classification of Harish-Chandra modules over $\mathfrak{s}$ in Section 4.

Throughout this paper, we denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \mathbb{C}$ and $\mathbb{C}^*$ the sets of all integers, non-negative integers, positive integers, complex numbers, and nonzero complex numbers, respectively. All vector spaces and algebras in this paper are over $\mathbb{C}$. We denote by $U(\mathfrak{a})$ the universal enveloping algebra of the Lie superalgebra $\mathfrak{a}$ over $\mathbb{C}$. Also, we denote by $\delta_{i,j}$ the Kronecker delta.
2. Preliminaries

In this section, we collect some basic definitions and results for our study.

A vector superspace $V$ is a vector space endowed with a $\mathbb{Z}_2$-gradation $V = V_0 \oplus V_1$. The parity of a homogeneous element $v \in V_i$ is denoted by $|v| = i \in \mathbb{Z}_2$. Throughout this paper, when we write $|v|$ for an element $v \in V$, we will always assume that $v$ is a homogeneous element.

The $N = 1$ Ramond algebra $\mathfrak{s}$ is the Lie superalgebra with basis $\{L_n, G_n, C | n \in \mathbb{Z}\}$ and brackets

\[
\begin{align*}
[L_m, L_n] &= (n-m)L_{m+n} + \delta_{m+n,0}\frac{1}{12}(n^3-n)C, \\
[L_m, G_p] &= (p - \frac{m}{2})G_{p+m}, \\
[G_p, G_q] &= -2L_{p+q} + \delta_{p+q,0}\frac{4p^2-1}{12}C.
\end{align*}
\]

The even part of $\mathfrak{s}$ is spanned by $\{L_n, C | n \in \mathbb{Z}\}$, and is isomorphic to the Virasoro algebra, the universal central extension of the Witt algebra $\mathfrak{w}$. The odd part of $\mathfrak{s}$ is spanned by $\{G_n | n \in \mathbb{Z}\}$. Let $\tilde{\mathfrak{s}}$ be the quotient algebra $\mathfrak{s}/\mathfrak{c}\mathfrak{c}$.

Let $A = \mathbb{C}[t^{\pm 1}] \otimes \Lambda(1)$, where $\Lambda(1)$ is the Grassmann algebra in one variable $\xi$. $A$ is $\mathbb{Z}_2$-graded with $|t| = 0$ and $|\xi| = 1$. $A$ is an $\tilde{\mathfrak{s}}$-module with

\[
\begin{align*}
L_n \circ x &= t^{n+1} \partial_t(x) + \frac{n}{2} t^n \xi \partial_t(x), \\
G_n \circ x &= t^{n+1} \xi \partial_t(x) - t^n \partial_t(x),
\end{align*}
\]

where $n \in \mathbb{Z}$, $x \in A$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_\xi = \frac{\partial}{\partial \xi}$. So, we have Lie superalgebra $\tilde{\mathfrak{s}} = \tilde{\mathfrak{s}} \rtimes A$ with $A$ an abelian Lie superalgebra and $[x, y] = x \circ y, x \in \mathfrak{s}, y \in A$.

On the other hand, $\tilde{\mathfrak{s}}$ has a natural $A$-module structure

\[
t^i L_n := L_{n+i}, t^i G_n := G_{n+i}, \xi L_n = \frac{1}{2} G_n, \xi G_n = 0, \forall n, i \in \mathbb{Z}.
\]  \hspace{1cm} (2.1) \hspace{1cm} 

And $\tilde{\mathfrak{s}}$ is an $\tilde{\mathfrak{s}}$-module with adjoint $A$-actions and $A$ acting as (2.1):

\[
\begin{align*}
[L_m, t^i L_n] - t^i [L_m, L_n] &= [L_m, L_{n+i}] - (n-m)t^i L_{m+n} = i L_{m+n+i} = it^{m+i} L_n, \\
[L_m, t^i G_n] - t^i [L_m, G_n] &= [L_m, \frac{1}{2} G_n] - (n-m)t^i G_{m+n} = \frac{1}{4} m G_{m+n} = \frac{m}{2} t^m \xi L_n, \\
[L_m, t^i G_n] - t^i [L_m, G_n] &= [L_m, L_{n+i}] - (n-m)t^i L_{m+n} = i G_{m+n+i} = it^{m+i} G_n, \\
[L_m, \xi G_n] - \xi [L_m, G_n] &= 0, \\
[G_m, t^i L_n] - t^i [G_m, L_n] &= [G_m, L_{n+i}] + (m - \frac{n}{2})t^i L_{m+n} = \frac{i}{2} G_{m+n+i} = it^{m+i} \xi L_n, \hspace{1cm} \\
[G_m, t^i G_n] - t^i [G_m, G_n] &= [G_m, G_{n+i}] + 2t^i L_{m+n} = 0, \\
[G_m, \xi L_n] + \xi [G_m, L_n] &= \frac{1}{2} [G_m, G_n] = -L_{m+n} = -t^m L_n, \hspace{1cm} \\
[G_m, \xi G_n] + \xi [G_m, G_n] &= -2 \xi L_{m+n} = -G_{m+n} = -t^m G_n.
\end{align*}
\]
An $A\mathfrak{s}$-module is an $\mathfrak{s}$-module with $A$ acting associatively. Let $U = U(\mathfrak{s})$ and $I$ be the left ideal of $U$ generated by $t^i \cdot t^j - t^{i+j}, t^0 - 1, t^i \cdot \xi - t^i \xi$ and $\xi^i \cdot \xi$ for all $i, j \in \mathbb{Z}$. Then it is clear $I$ is an ideal of $U$. Let $\mathbb{U} = U/I$. Then the category of $A\mathfrak{s}$-modules is naturally equivalent to the category of $\mathbb{U}$-modules.

Let $\mathfrak{g}$ be any of $\mathfrak{s}, \mathfrak{s}, \tilde{\mathfrak{s}}$. A $\mathfrak{g}$-module $M$ is called a weight module if the action of $L_0$ on $M$ is diagonalizable. Let $M$ be a weight $\mathfrak{g}$-module. Then $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$, where $M_\lambda = \{v \in M \mid L_0 v = \lambda v\}$, called the weight space of weight $\lambda$. The support of $M$ is $\text{Supp}(M) := \{\lambda \in \mathbb{C} \mid M_\lambda \neq 0\}$. A weight $\mathfrak{g}$-module is called cuspidal or uniformly bounded if the dimension of weight spaces of $M$ is uniformly bounded, that is there is $N \in \mathbb{N}$ such that $\dim M_\lambda < N$ for all $\lambda \in \text{Supp}(M)$. Clearly, if $M$ is simple, then $\text{Supp}(M) \subseteq \lambda + \mathbb{Z}$ for some $\lambda \in \mathbb{C}$.

Let $\sigma : L \rightarrow L'$ be any homomorphism of Lie superalgebras or associative superalgebras, and $M$ be any $L'$-module. Then $M$ become an $L$-module, denoted by $M^\sigma$, under $x \cdot v := \sigma(x)v, \forall x \in L, v \in M$. Denote by $T$ the automorphism of $L$ defined by $T(x) := (-1)^{\|x\|}x, \forall x \in L$. For any $L$-module $M$, $\Pi(M)$ is the module defined by a parity-change of $M$.

A module $M$ over an associative superalgebra $B$ is called strictly simple if it is a simple module over the associative algebra $B$ (forgetting the $\mathbb{Z}_2$-gradation).

We need the following result on tensor modules over tensor superalgebras.

**Lemma 2.1** ([21, Lemma 2.1, 2.2]). Let $B, B'$ be associative superalgebras, and $M, M'$ be $B, B'$ modules, respectively.

1. $M \otimes M' \cong \Pi(M) \otimes \Pi(M')$ as $B \otimes B'$-modules.

2. If in addition that $B'$ has a countable basis and $M'$ is strictly simple, then

   (a) Any $B \otimes B'$-submodule of $M \otimes M'$ is of the form $N \otimes M'$ for some $B$-submodule $N$ of $M$;

   (b) Any simple quotient of the $B \otimes B'$-module $M \otimes M'$ is isomorphic to some $\overline{M} \otimes M'$ for some simple quotient $\overline{M}$ of $M$.

   (c) $M \otimes M'$ is a simple $B \otimes B'$-module if and only if $M$ is a simple $B$-module.

   (d) If $V$ is a simple $B \otimes B'$-module containing a strictly simple $B' = \mathbb{C} \otimes B'$ module $M'$, then $V \cong M \otimes M'$ for some simple $B$-module $M$.

Let $K$ be the Weyl superalgebra $A[\partial, \partial_k]$. All simple weight $K$-modules are classified in [21].

**Lemma 2.2** ([21, Lemma 3.5]). Any simple weight $K$-module is isomorphic to some $A(\lambda)$ for some $\lambda \in \mathbb{C}$ up to a parity-change, here $A(\lambda) \cong K/I_\lambda$ with $I_\lambda$ the left ideal of $K$ generated by $t\partial - \lambda, \partial_k$.

Also, the following results about $(t - 1)\tilde{\mathfrak{s}} \subset \tilde{\mathfrak{s}}$ follow from (2.1) directly.

**Lemma 2.3.** Let $k, \ell \in \mathbb{Z}_+$. Then for all $i, j \in \mathbb{Z}$,

\[
[(t - 1)^k L_i, (t - 1)^j L_j] = (\ell - k + j - i)(t - 1)^{k+j} L_{i+j} + (\ell - k)(t - 1)^{k+j-1} L_{i+j},
\]

\[
[(t - 1)^k L_i, (t - 1)^j G_j] = (j - \frac{i}{2})(t - 1)^{k+j} G_{i+j} + (\ell - \frac{k}{2})(t - 1)^{k+j-1} G_{i+j+1},
\]

\[
[(t - 1)^k G_i, (t - 1)^j G_j] = -2(t - 1)^{k+j} L_{i+j}.
\]
From Lemma 2.3, we get

**Lemma 2.4.** For \( k \in \mathbb{N} \), let \( a_k = (t - 1)^k \mathfrak{s} \). Then

1. \( a_1 \) is a Lie subsuperalgebra of \( \mathfrak{s} \);
2. \( a_k \) is an ideal of \( a_1 \) and \( a_1/a_2 \) is a two dimensional Lie superalgebra with bosonic basis \( X \) and fermionic basis \( Y \) and nontrivial brackets \([X,Y] = \frac{1}{2}Y\).
3. The ideal generated by \( \{(t - 1)^k L_m \mid m \in Z\} \) is \( a_k \).

**Lemma 2.5.** Let \( L = \mathbb{C}X + \mathbb{C}Y \) be the Lie superalgebra with \([X] = 0, [Y] = 1\) and \([X,Y] = \frac{1}{2}Y, [Y,Y] = 0\). Then any simple finite dimensional \( L \)-module is one dimensional with \( X.v = bv, Y.v = 0 \) for some \( b \in \mathbb{C} \).

**Lemma 2.6** ([16, Theorem 2.1], Engel’s Theorem for Lie superalgebras). Let \( V \) be a finite dimensional module for the Lie superalgebra \( L = L_0 \oplus L_1 \) such that the elements of \( L_0 \) and \( L_1 \) respectively are nilpotent endomorphisms of \( V \). Then there exists a nonzero element \( v \in V \) such that \( xv = 0 \) for all \( x \in L \).

3. Cuspidal modules

For \( m \in Z \setminus \{0\} \), let

\[
X_m := t^{-m} \cdot L_m + \frac{m}{2} t^{-m} \xi \cdot G_m - L_0, \\
Y_m := t^{-m} \cdot G_m - 2t^{-m} \xi \cdot L_m - G_0 + 2 \xi \cdot L_0 \in \mathcal{U}.
\]

And let \( \mathcal{T} \) be the subspace of \( \mathcal{U} \) spanned by \( \{X_m, Y_m \mid m \in Z \setminus \{0\}\} \). Then we have

**Proposition 3.1.**

1. \([\mathcal{T}, G_0] = [\mathcal{T}, A] = 0\).
2. \( \mathcal{T} \) is a Lie subsuperalgebra of \( \mathcal{U} \). Moreover, \( \mathcal{T} \) is isomorphic to the Lie superalgebra \((t - 1)\mathfrak{s}\).

**Proof.** The first statement follows from

\[
[G_0, X_m] = [G_0, t^{-m}] \cdot L_m + t^{-m} \cdot [G_0, L_m] + \frac{m}{2}([G_0, t^{-m} \xi] \cdot G_m - t^{-m} \xi \cdot [G_0, G_m]) \\
= -mt^{-m} \xi \cdot L_m + \frac{m}{2} t^{-m} \cdot G_m + \frac{m}{2}(-t^{-m} \cdot G_m + 2t^{-m} \xi \cdot L_m) \\
= 0,
\]

\[
[G_0, Y_m] = [G_0, t^{-m}] \cdot G_m + t^{-m}[G_0, G_m] - 2([G_0, t^{-m} \xi] \cdot L_m - t^{-m} \xi \cdot [G_0, L_m]) - [G_0, G_0] + 2[G_0, \xi] \cdot L_0 \\
= -mt^{-m} \xi \cdot G_m - 2t^{-m} \cdot L_m - 2(-t^{-m} \cdot L_m - \frac{m}{2} t^{-m} \xi \cdot G_m) + 2L_0 - 2L_0 \\
= 0,
\]

\[
[t^n, X_m] = t^{-m}[t^n, L_m] + \frac{m}{2} t^{-m} \xi [t^n, G_m] + [L_0, t^n] = -nt^n + nt^n = 0,
\]

4
\[ [t^n, Y_m] = t^{-m}[t^n, G_m] - 2t^{-m}\xi[t^n, L_m] - [t^n, G_0] + 2\xi[t^n, L_0] = -nt^n\xi + 2nt^n\xi + nt^n\xi - 2nt^n\xi = 0, \]

\[ [X_m, \xi] = t^{-m}[L_m, \xi] + \frac{m}{2}t^{-m}\xi[G_m, \xi] - [L_0, \xi] = \frac{m}{2}t^{-m}\xi - \frac{m}{2}t^{-m}\xi = 0,\]

\[ [Y_m, \xi] = t^{-m}[G_m, \xi] - 2t^{-m}\xi[L_m, \xi] - [G_0, \xi] + 2\xi[L_0, \xi] = -1 + 1 = 0. \]

And the second statement follows from

\[ [X_m, X_n] = [t^{-m}, L_m + \frac{m}{2}t^{-m}\xi \cdot G_m - L_0, t^{-n} \cdot L_n + \frac{n}{2}t^{-n}\xi \cdot G_n - L_0] \]

\[ = t^{-m}[L_m, t^{-n}\cdot L_n] - t^{-n}[L_n, t^{-m}\cdot L_m] + t^{-m-n}\cdot [L_m, L_n] \]

\[ + t^{-n} \cdot [L_m, G_m] - L_0, t^{-n} \cdot [G_m, t^{-m}\cdot L_m] + t^{-m-n}\cdot [G_m, L_n] \]

\[ + \frac{m}{2}(t^{-m}\cdot [G_m, t^{-n}\cdot L_n] - t^{-n}[G_m, t^{-m}\cdot L_n] + t^{-m-n}\cdot [G_m, L_n]) \]

\[ + \frac{n}{2}(t^{-m}\cdot [L_m, t^{-n}\cdot G_n] - L_0, t^{-n} \cdot [G_n, t^{-m}\cdot L_m] + t^{-m-n}\cdot [G_n, L_0]) \]

\[ + \frac{m}{2}t^{-m\cdot [L_m, t^{-n}\cdot G_n] - L_0, t^{-n} \cdot [G_n, t^{-m}\cdot L_m] + t^{-m-n}\cdot [G_n, L_0]} \]

\[ + \frac{n}{2}t^{-n}\cdot [L_m, t^{-m}\cdot G_n] - L_0, t^{-n} \cdot [G_n, t^{-m}\cdot L_m] + t^{-m-n}\cdot [G_n, L_0] \]

\[ = -nX_n + mX_m + (n - m)X_{m+n}, \]

\[ [X_m, Y_n] = [t^{-m}, L_m + \frac{m}{2}t^{-m}\xi \cdot G_m - L_0, t^{-n} \cdot L_n + \frac{n}{2}t^{-n}\xi \cdot G_n - L_0 + 2\xi \cdot L_0] \]

\[ = t^{-m}[L_m, t^{-n}\cdot L_n] - t^{-n}[L_n, t^{-m}\cdot L_m] + t^{-m-n}\cdot [L_m, L_n] \]

\[ - 2t^{-m}[L_m, t^{-n}\xi \cdot L_n] - t^{-n}\xi[L_n, t^{-m}\cdot L_m] + t^{-m-n}\xi \cdot [L_m, L_n] - [t^{-m}, G_0] \cdot L_m \]

\[ - t^{-m} \cdot [L_m, G_0] + 2[t^{-m}[L_m, \xi \cdot L_0 - \xi[L_0, t^{-m}\cdot L_m] + t^{-m}\xi \cdot [L_m, L_0] \]

\[ + \frac{m}{2}(t^{-m}\cdot [G_m, t^{-m}\xi \cdot L_n - t^{-n}\xi[L_n, t^{-m}\cdot G_m] - \frac{m}{2}(t^{-m}\cdot [G_m, G_0] - [G_0, t^{-m}\xi \cdot G_m]) \]

\[ + \frac{m}{2}(t^{-m}\cdot [G_m, \xi \cdot L_0 - \xi[L_0, t^{-m}\cdot G_n] - [L_0, t^{-m}\cdot G_n] \]

\[ - t^{-n} \cdot [L_0, G_n] + 2[L_0, t^{-n}\xi \cdot L_n + t^{-n}\xi \cdot [L_0, L_n] \]

\[ = -nt^{-n}\cdot G_n + mt^{-m}\xi \cdot L_m + (n - \frac{m}{2})t^{-m-n}\cdot G_{m+n} \]

\[ - 2((m - \frac{n}{2})t^{-m}\xi \cdot L_n + mt^{-m}\xi \cdot L_m + (n - m)mt^{-n}\xi \cdot L_{m+n} - mt^{-m}\xi \cdot L_n \]

\[ + \frac{m}{2}t^{-m} \cdot G_m + 2(m - \frac{n}{2} \cdot L_0 + mt^{-m}\xi \cdot L_n - mt^{-m}\xi \cdot L_m) \]
We have the associative superalgebra isomorphism $\mathcal{U} \cong \mathcal{K} \otimes U(\mathcal{T})$.

**Proof.** Note that $U(\mathcal{T})$ is an associative subalgebra of $\mathcal{U}$ and the map $\tau : A[G_0] \to K$ with $\tau|_A = \text{Id}_A, \tau(G_0) = \xi \partial_l - \partial_l \xi$ is a homomorphism of associative superalgebras. Define the map $\iota : A[G_0] \otimes U(\mathcal{T}) \to \mathcal{U}$ by $(t^m \xi^i)G_0 \otimes y = t^m \xi^i \cdot G_0^j \cdot y + 1, \forall i, j = 0, 1, k \in \mathbb{Z}_+, y \in U(\mathcal{T})$. Then the restrictions of $\iota$ on $A[G_0]$ and $U(\mathcal{T})$ are well-defined homomorphisms of associative superalgebras. Also, note that $[\mathcal{T}, A] = [\mathcal{T}, G_0] = 0, \iota$ is a well defined homomorphism of associative superalgebras. From

$$\iota(t^m \otimes X_m \cdot \frac{m}{2} t^m \xi \otimes Y_m + t^m L_0 \otimes 1 - \frac{m}{2} t^m \xi G_0 \otimes 1) = L_m,$$

$$\iota(t^m \otimes Y_m + 2 t^m \xi \otimes X_m + t^m G_0 \otimes 1) = G_m,$$

we can see that $\iota$ is surjective.

By PBW theorem we know that $\mathcal{U}$ has a basis consisting monomials in variables $\{L_m, G_m \mid m \in \mathbb{Z} \setminus \{0\} \}$ over $A[G_0]$. Therefore $\mathcal{U}$ has an $A[G_0]$-basis consisting monomials in the variables $\{t^{-m} \cdot L_m - L_0, t^{-m} \cdot G_m - G_0 \mid m \in \mathbb{Z} \setminus \{0\} \}$. So $\iota$ is injective and hence an isomorphism. ∎

For any $(t-1)\bar{s}$-module $V$, we have the $A\bar{s}$-module $\Gamma(\lambda, V) = (A(\lambda) \otimes V)^{\varphi_1}$, where $\varphi_1 : \mathcal{U} \xrightarrow{\iota^{-1}} \mathcal{K} \otimes U(\mathcal{T}) \xrightarrow{\iota \otimes k} \mathcal{K} \otimes U((t-1)\bar{s})$. More precisely, $\Gamma(\lambda, V) = A \otimes V$ with actions

$$t^m \xi^i \cdot (y \otimes u) := t^m \xi^i y \otimes u,$$

$$L_m \cdot (y \otimes u) := t^m y \otimes (L_m - L_0) u - (-1)^{|y|} \frac{m}{2} t^m \xi y \otimes (G_m - G_0) u,$$

$$G_m \cdot (y \otimes u) := (-1)^{|y|} \frac{m}{2} t^m \xi y \otimes (G_m - G_0) u.$$
Lemma 3.3. 1. For any \( \lambda \in \mathbb{C} \) and any simple \((t-1)\bar{s}\)-module \( V \), \( \Gamma(\lambda, V) \) is a simple weight \( A\bar{s} \)-module.

2. Any simple weight \( A\bar{s} \)-module \( M \) is isomorphic to some \( \Gamma(\lambda, V) \) for some \( \lambda \in \text{Supp}(M) \) and some simple \((t-1)\bar{s}\)-module \( V \).

Proof. The first statement follows from Lemma 2.1 and Lemma 2.2. For the second statement, let \( M \) be any simple weight \( A\bar{s} \)-module with \( \lambda \in \text{Supp}(M) \). Then \( M^{\bar{s}^{-1}} \) is a simple \( \mathcal{K} \otimes U((t-1)\bar{s}) \)-module. Fix a nonzero homogeneous element \( v \in (M^{\bar{s}^{-1}})_{\lambda} \), then \( \mathcal{C}[\partial_\xi]v \) is a finite dimensional supersubspace with \( \partial_\xi \) acting nilpotently. So there exists a nonzero element \( v'^{'} \in \mathcal{C}[\partial_\xi]v \) with \( I_\lambda v'^{'} = 0 \). Clearly, \( \mathcal{K}v'^{'} \) is isomorphic to \( A(\lambda) \) or \( \Pi(\lambda) \). Hence by Lemma 2.1 and Lemma 2.2, there exists a simple \( U((t-1)\bar{s}) \)-module \( N \) such that \( M^{\bar{s}^{-1}} \cong A(\lambda) \otimes N \) or \( M^{\bar{s}^{-1}} \cong \Pi(\lambda) \otimes N \cong A(\lambda) \otimes \Pi(N^T) \).

Thus, to classify all simple weight \( A\bar{s} \)-modules, it suffices to classify all simple \((t-1)\bar{s}\)-modules. In particular, to classify all simple cuspidal \( A\bar{s} \)-modules, it suffices to classify all finite dimensional \((t-1)\bar{s}\)-modules.

Lemma 3.4. 1. Let \( V \) be any finite dimensional \((t-1)\bar{s}\)-module. Then there exists \( k \in \mathbb{N} \) such that \( a_kV = 0 \).

2. Let \( V \) be any simple finite dimensional simple \((t-1)\bar{s}\)-module. Then \( a_2V = 0 \). In particular, \( \dim V = 1 \).

Proof. 1. Since \( V \) is a finite dimensional \((t-1)\bar{w}\)-module, there exists \( k \in \mathbb{N} \) such that \((t-1)^k\bar{w}V = 0 \). So the first statement follows from Lemma 2.4.

2. Consider the finite dimensional Lie superalgebra \( \mathfrak{g} = a_1/\text{ann}(V) \), then \( V \) is a finite dimensional \( \mathfrak{g}_0 \)-module and \( a_{2,0} + \text{ann}(V) \) acts nilpotently on \( V \). Since \([x,x] \in a_{2,1}\) for all \( x \in a_{2,1} \), every element in \( a_{2,1} + \text{ann}(V) \) acts nilpotently on \( V \). Hence, by Lemma 2.6, there is nonzero \( v \in V \) annihilated by \( a_{2} + \text{ann}(V) \). And therefore \( a_2V = 0 \), which means \( V \) is a simple finite dimensional module for \( a_1/a_2 \).

Corollary 3.5. Any simple cuspidal \( A\bar{s} \)-module is isomorphic to some \( \Gamma(\lambda, b) = A \otimes \mathbb{C}u \) with \( \lambda, b \in \mathbb{C} \) defined as follows:

\[
\begin{align*}
 t^i \xi^r (y \otimes u) &= t^i \xi^r y \otimes u, \\
 L_m(t^i \xi^r \otimes u) &= (\lambda + i + m(b + \frac{1}{2} \delta_{1,r})) t^{m+i} \xi^r \otimes u, \\
 G_m(t^i \otimes u) &= (\lambda + i + 2mb) t^{m+i} \xi \otimes u, \\
 G_m(t^i \xi \otimes u) &= -t^{m+i} \otimes u,
\end{align*}
\]

where \( i, m \in \mathbb{Z}, r = 0,1, y \in A \).
Next we are going to define the $A$-cover $\widehat{M}$ of a cuspidal $\mathfrak{g}$-module $M$. Consider $\mathfrak{g}$ as the adjoint $\mathfrak{g}$-module. Then the tensor product $\mathfrak{g} \otimes M$ is an $A\mathfrak{g}$-module by

$$x \cdot (y \otimes b) := (xy) \otimes v, \forall x \in A, y \in \mathfrak{g}, v \in M.$$  

Let $K(M) = \{ \sum_i x_i \otimes v_i \in \mathfrak{g} \otimes M \mid \sum_i (\omega x_i)v_i = 0, \forall \omega \in A \}$. Then $K(M)$ is an $A\mathfrak{g}$-submodule of $\mathfrak{g} \otimes M$. And hence we have the $A\mathfrak{g}$-module $\widehat{M} = (\mathfrak{g} \otimes M)/K(M)$, called the cover of $M$ when $\mathfrak{g}M = M$, as in [1]. Clearly, the linear map $\pi : \widehat{M} \to \mathfrak{g}M: x \otimes v + K(M) \mapsto xv$ is an $\mathfrak{g}$-module epimorphism.

Recall that in [1], the authors show that every cuspidal $W$-module is annihilated by the operators $\Omega_{k,s}^{(m)}$ for $m$ large enough.

**Lemma 3.6** ([1, Corollary 3.7]). For every $\ell \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for all $k, s \in \mathbb{Z}$ the differentiators $\Omega_{k,s}^{(m)} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} L_{k-i} L_{s+i}$ annihilate every cuspidal $W$-module with a composition series of length $\ell$.

Let $M$ be a cuspidal $\mathfrak{g}$-module. Then $M$ is a cuspidal $W$-module and hence there exists $m \in \mathbb{N}$ such that $\Omega_{k,p}^{(m)} M = 0, \forall k, p \in \mathbb{Z}$. Therefore, $[\Omega_{k,p}^{(m)}, G_j]M = 0, \forall j, k, p \in \mathbb{Z}, s \in S$. Thus, on $M$ we have

$$0 = [\Omega_{k-1}^{(m)}, G_{j+1}] - 2[\Omega_{k-1}^{(m)}, G_j] + [\Omega_{k+1}^{(m)}, G_{j-1}] - [\Omega_{k+1}^{(m)}, G_j]$$

$$+ 2[\Omega_{k+1}^{(m)}, G_{j-1}] - [\Omega_{k+1}^{(m)}, G_{j-2}]$$

$$= \sum_{i=0}^{m} (-1)^i \binom{m}{i} L_{k-i} L_{p-1+i} G_{j+1} - 2 \sum_{i=0}^{m} (-1)^i \binom{m}{i} L_{k-i} L_{p+i} G_j$$

$$+ \sum_{i=0}^{m} (-1)^i \binom{m}{i} L_{k-i} L_{p+1+i} G_{j-1} - \sum_{i=0}^{m} (-1)^i \binom{m}{i} L_{k-i} L_{p+1+i} G_j$$

$$+ 2 \sum_{i=0}^{m} (-1)^i \binom{m}{i} L_{k-i} L_{p+1+i} G_{j-1} - \sum_{i=0}^{m} (-1)^i \binom{m}{i} L_{k-i} L_{p+1+i} G_{j-2}$$

$$= \sum_{i=0}^{m} (-1)^i \binom{m}{i} (j + 1 - \frac{k-i}{2}) G_{k-i+j+1} L_{p-i+1} + (j + 1 - \frac{p-1+i}{2}) L_{k-i} G_{p+i+j}$$

$$- 2(j - \frac{k-i}{2}) G_{k-i+j} L_{p+i} - 2(j - \frac{p+i}{2}) L_{k-i} G_{p+i+j} + (j - \frac{k-i+1}{2}) G_{k-i+j+1} L_{p+i+1}$$

$$+ (j - 1 - \frac{p+i-1}{2}) L_{k-i} G_{p+i+j} - (j - \frac{k-i+1}{2}) G_{k-i+j+1} L_{p+i+1}$$

$$- (j - \frac{p+i-1}{2}) L_{k-i+1} G_{p+i+j-1} + 2(j - 1 - \frac{k-i+1}{2}) G_{k-i+j+1} L_{p+i}$$

$$+ 2(j - 1 - \frac{p+i}{2}) L_{k-i+1} G_{p+i+j-1} - (j - 2 - \frac{k-i+1}{2}) G_{k-i+j+1} L_{p+i+1}$$

$$- (j - 2 - \frac{p+i+1}{2}) L_{k+i+1} G_{p+i+j-1}$$

$$= \frac{3}{2} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (G_{k-i+j+1} L_{p+i-1} - 2 G_{k-i+j} L_{p+i} + G_{k-i+j-1} L_{p+i+1})$$
\[
\frac{3}{2} \sum_{i=0}^{m} (-1)^i \binom{m+2}{i} G_{k-i+j+1} L_{p+i-1}.
\]

That is, we have

**Lemma 3.7.** Let \( M \) be a cuspidal \( \hat{\mathfrak{g}} \)-module. Then there exists \( m \in \mathbb{N} \) such that for all \( j, p \in \mathbb{Z} \) the operators \( \Omega_{p, p}^{(m)} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} G_{j-i} L_{p+i} \) annihilate \( M \).

**Lemma 3.8.** For any cuspidal \( \hat{\mathfrak{g}} \)-module \( M, \hat{M} \) is also cuspidal.

**Proof.** Since \( \hat{M} \) is an \( A \)-module, it suffices to show that one of its weight spaces is finite dimensional. Fix a weight \( \alpha + p, p \in \mathbb{Z} \) and let us prove that \( \hat{M}_{\alpha+p} = \text{span} \{ L_{p-k} \otimes M_{\alpha+k}, G_{p-k} \otimes M_{\alpha+k} \mid k \in \mathbb{Z} \} \) is finite dimensional. Assume that \( \alpha = 0 \) when \( \alpha + Z = Z \).

From Lemma 3.6 and Lemma 3.7, there exists \( m \in \mathbb{N} \), such that \( \sum_{i=0}^{m} (-1)^i \binom{m}{i} L_{j-i} L_{p+i} v = \sum_{i=0}^{m} (-1)^i \binom{m}{i} G_{j-i} L_{p+i} v = 0, \forall j, p \in \mathbb{Z}, v \in M \). Hence,

\[
\sum_{i=0}^{m} (-1)^i \binom{m}{i} L_{j-i} \otimes L_{p+i}, \sum_{i=0}^{m} (-1)^i \binom{m}{i} G_{j-i} \otimes L_{p+i} v \in K(M). \tag{3.1}
\]

We are going to prove by induction on \( |q| \) for \( q \in \mathbb{Z} \) that for all \( u \in M_{\alpha+q} \),

\[
L_{p-q} \otimes u, G_{p-q} \otimes u \in \sum_{|k| \leq \frac{q}{2}} \left( L_{p-k} \otimes M_{\alpha+k} + G_{p-k} \otimes M_{\alpha+k} \right) + K(M).
\]

We only need to prove this claim for \( |q| > \frac{m}{2} \), and we may assume that \( q < -\frac{m}{2} \), the proof for \( q > \frac{m}{2} \) is similar. Since \( L_0 \) acts on \( M_{\alpha+q} \) with a nonzero scalar, we can write \( u = L_0 v \) for some \( v \in M_{\alpha+q} \). Then by (3.1) and induction hypothesis, we have

\[
L_{p-q} \otimes L_0 v = \sum_{i=0}^{m} (-1)^i \binom{m}{i} L_{p-q-i} \otimes L_i v - \sum_{i=1}^{m} (-1)^i \binom{m}{i} L_{p-q-i} \otimes L_i v \\
\in \sum_{|k| \leq \frac{q}{2}} \left( L_{p-k} \otimes M_{\alpha+k} + G_{p-k} \otimes M_{\alpha+k} \right) + K(M),
\]

\[
G_{p-q} \otimes L_0 v = \sum_{i=0}^{m} (-1)^i \binom{m}{i} G_{p-q-i} \otimes L_i v - \sum_{i=1}^{m} (-1)^i \binom{m}{i} G_{p-q-i} \otimes L_i v \\
\in \sum_{|k| \leq \frac{q}{2}} \left( L_{p-k} \otimes M_{\alpha+k} + G_{p-k} \otimes M_{\alpha+k} \right) + K(M).
\]

Now we can classify all simple cuspidal \( \hat{\mathfrak{g}} \)-modules.

**Theorem 3.9.** Any nontrivial simple cuspidal \( \hat{\mathfrak{g}} \)-module is isomorphic to a simple quotient of \( \Gamma(\lambda, b) \) for some \( \lambda, b \in \mathbb{C} \).
Proof. Let $M$ be any nontrivial simple cuspidal $\mathfrak{g}$-module. Then $\mathfrak{g}M = M$ and there is an epimorphism $\pi : \hat{M} \to M$. From Lemma 3.8, $\hat{M}$ is cuspidal. Hence $\hat{M}$ has a composition series of $A\mathfrak{g}$-submodules:

$$0 = \hat{M}^{(0)} \subset \hat{M}^{(1)} \subset \cdots \subset \hat{M}^{(s)} = \hat{M}$$

with $\hat{M}^{(i)}/\hat{M}^{(i-1)}$ being simple $A\mathfrak{g}$-modules. Let $k$ be the minimal integer such that $\pi(\hat{M}^{(k)}) \neq 0$. Then we have $\pi(\hat{M}^{(k)}) = M, \hat{M}^{(k-1)} = 0$ since $M$ is simple. So we have an $\mathfrak{g}$-epimorphism from the simple $A\mathfrak{g}$-module $\hat{M}^{(k)}/\hat{M}^{(k-1)}$ to $M$. Now theorem follows from Corollary 3.5.

4. Main results

In this section, we will classify all simple weight $\mathfrak{g}$-modules with finite dimensional weight spaces. First of all, from the representation theory of Virasoro algebra, we know that $C$ acts trivially on any simple cuspidal $\mathfrak{g}$-module, and hence the category of simple cuspidal $\mathfrak{g}$-modules is naturally equivalent to the category of simple cuspidal $A\mathfrak{g}$-modules. Thus, it remains to classify all all simple weight $\mathfrak{g}$-modules with finite dimensional weight spaces which is not cuspidal. From now on, we will assume $M$ is such an $\mathfrak{g}$-module. Let $\lambda \in \text{supp}(M)$.

The following result is well-known

Lemma 4.1. Let $M$ be a weight module with finite dimensional weight spaces for the Virasoro algebra with $\text{supp}(M) \subseteq \lambda + \mathbb{Z}$. If for any $v \in M$, there exists $N(v) \in \mathbb{N}$ such that $L_i v = 0, \forall i \geq N(v)$, then $\text{supp}(M)$ is upper bounded.

Lemma 4.2. Suppose $M$ is a simple weight $\mathfrak{g}$-module with finite dimensional weight spaces which is not cuspidal, then $M$ is a highest (or lowest) weight module.

Proof. Since $M$ is not cuspidal, then there is a $k \in \mathbb{Z}$ such that $\dim M_{\lambda + k} > 2(\dim M_{\lambda} + \dim M_{\lambda + 1})$. Without lost of generality, we may assume that $k \in \mathbb{N}$. Then there exists a nonzero element $w \in M_{\lambda + k}$ such that $L_kw = L_{k+1}w = G_kw = G_{k+1}w = 0$. Therefore, $L_i w = G_i w = 0$ for all $i \geq k^2$, since $[s_i, s_j] = s_{i+j}$.

It is easy to see that $M' = \{ v \in M \mid \dim s^+ v < \infty \}$ is a nonzero submodule of $M$, here $s^+ = \sum_{n \in \mathbb{N}} (CL_n + CG_n)$. Hence $M = M'$. So, Lemma 4.1 tells us that $\text{supp}(M)$ is upper bounded, that is $M$ is a highest weight module.

Combining with Lemma 4.2 and Theorem 3.9, we can get the following result, which was given in [19] by much complicated calculations.

Theorem 4.3. Let $V$ be a simple $\mathfrak{g}$-module with finite dimensional weight spaces. Then $V$ is a highest weight module, a lowest weight module, or a simple quotient of $\Gamma(\lambda, b)$ for some $\lambda, b \in \mathbb{C}$ (which is called a module of the intermediate series).

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