LAGRANGIAN DYNAMICS IN NON–FLAT UNIVERSES AND NON–LINEAR GRAVITATIONAL EVOLUTION

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Abstract

We present a new formalism which allows to derive the general Lagrangian dynamical equations for the motion of gravitating particles in a non-flat Friedmann universe with arbitrary density parameter \( \Omega \) and no cosmological constant. We treat the set of particles as a Newtonian collisionless fluid. The non-linear dynamical evolution of the fluid element trajectories is then described up to the third-order expansion in Lagrangian coordinates. This work generalizes recent investigations carried out by Bouchet et al. (1992) and Buchert (1994).

Subject headings: Galaxies: clustering – large-scale structure of the Universe

1 Introduction

The most widely accepted hypothesis about the formation of large-scale structures is that galaxies and clusters of galaxies formed by gravitational collapse around primordial slightly overdense mass fluctuations in the Universe. One way to link these initial conditions to the final mass distribution is to attempt a solution of the equations for the matter fluctuation field \( \delta \) and the peculiar velocity field \( \mathbf{v} \), namely the Euler equation, the continuity equation and the Poisson equation.

Linear theory easily provides solutions even for cosmological models with arbitrary density parameter \( \Omega \) (e.g. Peebles 1980), but when the perturbation amplitudes approach unity, non-linear gravitational effects become very important. It is undoubtedly impossible to follow the non-linear stage of the gravitational collapse in an exact analytical way, and one is forced to use approximation techniques such as perturbation theory where, for instance, the systematic expansion of the density solution is obtained by writing \( \delta = \sum \delta^{(n)} \), where \( \delta^{(n)} = O(\delta^{(1)n}) \), \( \delta^{(1)} \) corresponding to the linear solution (Fry 1984; Goroff et al. 1986; Juszkiewicz, Bouchet & Colombi 1993; Catelan & Moscardini 1994a,b; Bernardeau 1992,
1994; Catelan et al. 1995).

Usually, these analyses are intrinsically Eulerian, the fundamental quantities being the density and velocity fields evaluated at the (comoving) Eulerian coordinate $x$; also, with the exception of Bernardeau (1992, 1994) and Catelan et al. (1995), they have been confined within the limits of the Einstein–de Sitter cosmology, mainly because the condition $\Omega = 1$ enormously eases the investigations of solutions of the dynamical equations for $\delta$ and $v$: apart from theoretical expectations, there is however no definitive observational evidence that our Universe is really flat (see e.g. Peebles 1991; Coles & Ellis 1994).

It has been recognized that the problem of giving an analytical description of the non–linear process of gravitational clustering simplifies when formulated in terms of Lagrangian coordinates rather than the standard Eulerian ones: Zel’dovich (1970a, b) first proposed to approximately describe the weakly non–linear regime of density evolution in terms of the departure from to the Lagrangian (initial) positions of the fluid elements. The Zel’dovich approximation is now widely used in cosmology, showing also to be extremely useful in reconstruction methods of initial conditions from velocity data (e.g. Nusser & Dekel 1992).

However, only recently it has been fully understood that the whole dynamics of gravitational clustering may be suitably described in terms of the displacement field $S$, which turns out, in the Lagrangian approach, to be the only underlying fundamental field. Buchert (1989, 1992) indeed derives the exact equations governing the evolution of the displacement $S$ (therefore of the density and velocity fields): since, as in the Eulerian case, it is impossible to work out the general solution $S$, a perturbative approach is again introduced. The key novelty with respect to the Eulerian approach is that one searches for solutions of perturbed trajectories about the linear (initial) displacement $S^{(1)}$: $S = \sum S^{(n)}$ where $S^{(n)} = o(S^{(1)})$ (see Moutarde et al. 1991). The important point is that a slight perturbation of the Lagrangian particle paths carries a large amount of non–linear information about the corresponding Eulerian evolved observables, since the Lagrangian picture is intrinsically non–linear in the density field.
Solutions up to the third–order Lagrangian approximation have been obtained (the first–order solution corresponding to the Zel’dovich approximation), although limited to the case of an Einstein–de Sitter model (Buchert & Ehlers 1993; Buchert 1994). The higher accuracy of Lagrangian perturbative methods as compared to other currently studied approximation ansatzs \cite{1992MNRAS.254..403M, 1993MNRAS.262..108B, 1994ApJ...422..405B, 1994ApJ...421..499B, 1994PhRvD..50.1195B, 1994PhRvD..50.1245B} is discussed in Munshi & Starobinsky (1994), Bernardeau et al. (1994) and Munshi, Sahni & Starobinsky (1994), again in the framework of an Einstein–de Sitter cosmology. Comparisons with $N$–body simulations in the fully developed non–linear clustering are displayed in Moutarde et al. (1991), Coles, Melott & Shandarin (1993) and Melott et al. (1994).

The second–order Lagrangian solution for generic non–flat Friedmann models has been derived by Bouchet et al. (1992; hereafter BJCP), where particular emphasis on the connection with the Eulerian formulation has been given. Further attempts to extend the Lagrangian formalism to models with arbitrary density parameter may be found in Gramann (1993) and Lachièze–Rey (1993b), which, however, lead to not completely correct conclusions. Matarrese, Pantano & Saez (1994a,b), developed a relativistic Lagrangian treatment of the non–linear dynamics of an irrotational collisionless fluid, which reduced to the standard Newtonian approach on sub–horizon scales, but is also suitable for the description of perturbations on super–horizon scales.

In this work, we present a new Lagrangian formalism which enables one to easily derive the exact dynamical equations governing a pressureless Newtonian gravitating fluid in an expanding universe with arbitrary density parameter $\Omega$ and no cosmological constant. We then describe the non–linear evolution of perturbations up to the third–order Lagrangian approximation. The first–order solution corresponds, of course, to the Zel’dovich approximation, and the second–order one to the BJCP approximation. The third–order solution is then derived in detail for arbitrary initial conditions. It consists of three independent
(growing) modes, two being purely longitudinal and one purely transversal, in such a way that a fluid which is irrotational in Eulerian space is not so in the corresponding Lagrangian space. This problem was first addressed in Buchert (1994), where the consequences of requiring irrotationality of the fluid motion in Lagrangian space on the initial conditions are explored in detail.

The layout of this paper is as follows. In section 2 the Lagrangian approach is reviewed. In section 3, the general Lagrangian equations describing the dynamical evolution of a collisionless Newtonian self–gravitating fluid in an expanding universe with arbitrary density parameter $\Omega$ are obtained. In section 4, after rederiving in our formalism the Zel’dovich and the BJCP approximations, we work out the third–order Lagrangian approximation. Our conclusions are presented in section 5. To avoid an excess of mathematical contortions in the text, four technical appendices are given.

## 2 Lagrangian Formulation

Let us consider a Newtonian pressureless self–gravitating fluid embedded in an expanding universe with arbitrary density parameter $\Omega$. We assume that the cosmological constant is exactly zero. We have in mind that such a fluid mimics the behavior of matter on scales smaller than the horizon scale and that, around its primordial density perturbations $\delta$, the present luminous objects like galaxies or clusters of galaxies started to grow according to a gravitational instability process. We indicate by $\mathbf{x}$ the comoving Eulerian coordinates, from which physical distances may be obtained according to the law $\mathbf{r} = a(t)\mathbf{x}$, $a(t)$ being the expansion scale factor and $t$ the standard cosmic time.

According to the Lagrangian point of view, the path of each fluid element is followed during its evolution. A proper observer may label each neighbouring fluid particle by e.g. its initial (comoving) coordinate, say $\mathbf{q} \equiv \mathbf{x}_0$. At time $t$, the (Eulerian) position of the $\mathbf{q}$–particle will be

$$\mathbf{x} = \mathbf{x}(\mathbf{q}, t) \equiv \mathbf{x}_L. \quad (1)$$
Here the only independent variables are the labels $\mathbf{q}$ (apart from the time $t$) which therefore play the role of spatial coordinates in the Lagrangian $\{\mathbf{q}\}$-space. The vector $\mathbf{x}$, which in the Eulerian picture is an independent variable, is now introduced as a new real dynamical field: in the Lagrangian $\{\mathbf{q}\}$-space, the trajectories of the mass elements are fully described by the dynamical maps $\mathbf{x}(\mathbf{q},t)$, starting from the initial positions $\mathbf{q}$. The definition (1) implicitly assumes that there is a one-to-one correspondence between the Eulerian coordinate $\mathbf{x}$ and the Lagrangian coordinate $\mathbf{q}$: this is certainly the case for a cold non collisional fluid, at least until the stage of caustic formation (see e.g. Shandarin & Zel’dovich 1989). Mathematically, this is equivalent to the statement that the determinant $J$ of the Jacobian of the map $\mathbf{q} \rightarrow \mathbf{x}(\mathbf{q},t)$ is non-singular,

$$J(\mathbf{q},t) \equiv \det \left( \frac{\partial \mathbf{x}}{\partial \mathbf{q}} \right) \neq 0,$$

the map $\mathbf{x}(\mathbf{q},t)$ being thus reversible to $\mathbf{q}(\mathbf{x},t)$. Obviously, during the highly non-linear evolution, many particles coming from very different original positions will tend to arrive at the same Eulerian place: infinite-density regions (caustics) will form in Eulerian space and the map from Lagrangian to Eulerian space becomes singular (Shandarin & Zel’dovich 1989; see also the discussion in Kofman et al. 1994). This caustic formation process enormously limits the predictive power of any (perturbative) Eulerian method, in that the main requirement for it to work is the very restrictive condition $|\delta| \ll 1$. This is possibly the strongest reason for preferring the alternative Lagrangian picture: in this description indeed the density is not a dynamical variable and is fully integrated.

Following Zel’dovich (1970a, b), all Eulerian fields can be represented in terms of the only dynamical field $\mathbf{x}_L$ – or its derivatives. Thus, the peculiar velocity $\mathbf{u}$, the peculiar acceleration $\mathbf{f}$ and the density $\rho$ are respectively given by

$$\frac{d\mathbf{x}(\mathbf{q},t)}{dt} = \mathbf{u}(\mathbf{q},t) a(t)^{-1},$$

$$\frac{d^2\mathbf{x}(\mathbf{q},t)}{dt^2} = [\mathbf{f}(\mathbf{q},t) - 2H\mathbf{u}(\mathbf{q},t)] a(t)^{-1},$$

$$1 + \delta[\mathbf{x}(\mathbf{q},t),t] = J(\mathbf{q},t)^{-1},$$

1 + $\delta[\mathbf{x}(\mathbf{q},t),t] = J(\mathbf{q},t)^{-1},$
the quantity $H$ being the Hubble constant and $\delta = (\rho - \rho_b)/\rho_b$ the density fluctuation about the density background value $\rho_b(t)$. The operator $d/dt$ is the usual convective Lagrangian time derivative which follows the mass element, $\frac{d}{dt} \equiv \frac{\partial}{\partial t}|_{q} = \frac{\partial}{\partial t}|_{x} + \frac{dx}{dt} \cdot \nabla_x$. Recall that the operator $\frac{d}{dt}$ does not commute with the Eulerian nabla operator $\nabla_x$.

Equation (4) corresponds to the (comoving) Euler equation and eq.(5) to the continuity equation for the system. Furthermore, the equations (3) and (4) – governing the evolution of the field $x_L = x(q, t)$ – may be thought as definitions of the velocity $u$ and the acceleration $f$. The particular form of the mass conservation (4) derives from the fact that the observer follows the mass elements during their motion and that the initial matter distribution is assumed homogeneous.

The gravitational interaction among the mass elements is then introduced, requiring that the peculiar acceleration field $f$ is induced by the density fluctuations $\delta$ through the Poisson equation, $\nabla \cdot f = -4\pi G a \rho_b \delta$. For our purposes, however, the best form of the Poisson equation is the one obtained introducing the time variable (Doroshkevich, Ryabenki & Shandarin 1973; Shandarin 1980)

$$\tau \equiv \sqrt{-k (1 - \Omega)^{-1/2}},$$

where $k = -1$ for open universes and $k = 1$ for closed universes. The case $\Omega = 1 (k = 0)$ is a singular point for the transformation (6), and, in that case we take $\tau \equiv t^{-1/3}$. As we shall see below, the main advantage in using $\tau$ instead of the standard cosmic time $t$ is that the dynamical equations for the higher–order Lagrangian modes are considerably simplified, since the rescaled peculiar acceleration $g$ is now simply given by

$$\frac{d^2x}{d\tau^2} = g.$$  

Finally, gravity is introduced through the equation

$$\nabla_x \cdot g = -\alpha(\tau) \delta,$$

where the quantity $\alpha(\tau) \equiv 6/(\tau^2 + k)$ (Shandarin 1980). The time variable $\tau$ has been recently used also in BJCP and Lachièze–Rey (1993a,b).
Alternatively, if the original $f$ is expressed in terms of the gravitational potential $\Phi$, i.e. $f \equiv -a^{-1}\nabla\Phi$, then, because of (8),

$$\nabla^2_x \phi = \alpha(\tau) \delta ,$$

once the rescaled potential $\phi \equiv (2a/\Omega_0 a_0 \dot{a}^3)_2 \Phi$ such that $g = -\nabla\phi$ has been introduced.

Finally, if one assumes irrotational motions in Eulerian space, which is a plausible hypothesis for a collisionless (cold) dust, then the “irrotationality” condition may be expressed according to the relation

$$\nabla_x \wedge u = 0 .$$

(10)

Summarizing: the Lagrangian system of equations governing a gravitating collisionless fluid is given by eqs. (3) and (4) – the Euler equation – for the field $x(q, t)$, the mass conservation relation (5) and the Poisson equation.

### 3 Lagrangian Equations for the Displacement

To describe the departure of the mass elements from the initial positions $q$ one usually introduces the displacement vector $S$ such that

$$x(q, \tau) \equiv q + S(q, \tau) .$$

(11)

It is clear from this definition that the motion of the fluid elements may be completely described in terms of the displacement $S$, since the latter fully characterizes the map (1) between the Eulerian and the Lagrangian coordinates. In terms of $S$, the Euler equation and the continuity equation may be written, respectively, as

$$\frac{d^2 S(q, \tau)}{d\tau^2} = g[x(q, \tau), \tau] ,$$

(12)

$$1 + \delta[x(q, \tau), \tau] = \det(I + S)^{-1} .$$

(13)

Here $I = \text{diag}(1, 1, 1)$ and $S$ is a $3 \times 3$ matrix whose elements are $S_{\alpha\beta} \equiv \partial S_\alpha / \partial q_\beta$, also called the deformation tensor. In general the deformation tensor is not symmetric, i.e.
$S_{\alpha \beta} \neq S_{\beta \alpha}$. $S_{\alpha \beta}$ is symmetric iff the displacement $S$ is potential in the Lagrangian space.

The Eulerian “irrotationality” condition (10) – if taken into account – and the Poisson equation (8) may be written as

\[
\nabla \times \dot{S} = 0 , \\
\nabla \cdot \ddot{S} = -\alpha(\tau) \delta[x(q, \tau), \tau] ,
\]

the velocity field being defined by the relation $dx/d\tau = a(\tau) u(x, \tau)$.

The second equation clearly shows how the trajectories $S$ are deformed during the time evolution by the density perturbation $\delta[x(q, \tau), \tau]$. The dot indicates the operator $d/d\tau$. However, we cannot say that the previous equations are completely written in the Lagrangian $\{q\}$–space: indeed the operator $\nabla_x$ does not act on the Lagrangian coordinates $q$. The exact way in which the differentiation with respect to the Eulerian position $x$ is translated – through the map (11) – into differentiation with respect to the Lagrangian position $q$ is displayed in the following relation:

\[
J \frac{\partial}{\partial x_\beta} = \left[(1 + \nabla \cdot S) \delta_{\alpha \beta} - S_{\alpha \beta} + S^C_{\alpha \beta}\right] \frac{\partial}{\partial q_\alpha} ,
\]

where now $\nabla \equiv \nabla_q$. The symbol $\delta_{\alpha \beta}$ indicates the Kronecker tensor and the quantity $S^C_{\alpha \beta}$ is an element of the cofactor matrix $S^C$. Summation over repeated Greek indices (where $\alpha = 1, 2, 3$) is understood. We derive the above relation in detail in Appendix A.

The Newtonian Lagrangian equations for the collisionless fluid finally become

\[
\epsilon_{\alpha \beta \gamma} \left[(1 + \nabla \cdot S) \delta_{\beta \sigma} - S_{\beta \sigma} + S^C_{\beta \sigma}\right] \dot{S}_{\gamma \sigma} = 0 ,
\]

and

\[
\left[(1 + \nabla \cdot S) \delta_{\alpha \beta} - S_{\alpha \beta} + S^C_{\alpha \beta}\right] \ddot{S}_{\beta \alpha} = \alpha(\tau)[J(q, \tau) - 1] ,
\]

where $\epsilon_{\alpha \beta \gamma}$ is the totally antisymmetric Levi–Civita tensor of rank three, $\epsilon_{123} \equiv 1$.

We call the latter equation the Lagrangian Poisson equation. The irrotationality condition in Lagrangian space (17) and the Lagrangian Poisson equation (18) dynamically constrain (up to a constant vector) the field $S$. They are the (closed set of) general dynamical equations for the displacement vector $S$ describing the motion of a collisionless fluid.
fluid in the Lagrangian \( \{ q \} \)-space, embedded in an arbitrary non-flat universe and subject to the Newtonian gravitational influence of the mass fluctuations \( J^{-1} - 1 \). We stress that eqs.(17) and (18) are non-linear and non-local in the displacement \( S \) (see the discussion in Kofman & Pogosyan 1994). Analogous equations, although using a very different tensorial notation, are analyzed in Buchert (1989).

Some remarks are appropriate. Eq.(17) does not mean that the motion is potential in the Lagrangian space, since this would correspond to the condition \( \epsilon_{\alpha\beta\gamma} \dot{S}_{\beta\gamma} = 0 \). On the contrary, we can surely state from (17) that the motion is vortical in Lagrangian space and that, however, the departure from the irrotationality of the Lagrangian peculiar velocity \( u \) is gravitationally induced only at higher order in the displacement \( S \), as may be clearly seen if one writes

\[
\epsilon_{\alpha\beta\gamma} \dot{S}_{\beta\gamma} = -\epsilon_{\alpha\beta\gamma} \dot{S}_{\beta\sigma} [ (\nabla \cdot S) \delta_{\gamma\sigma} - S_{\gamma\sigma} + C_{\gamma\sigma}] .
\]

This implies that, at least to the first order in \( S \), the condition for irrotationality in Eulerian space means that the Lagrangian motion is also potential, as first noted by Zel’dovich (1970a, b). The irrotationality problem in Lagrangian coordinates has been recently addressed by Buchert (1992, 1994). We will discuss again it below for a general nonflat model. Considering now the Lagrangian Poisson equation, we stress the fact that only the left hand side of eq.(18) is the ‘dynamical’ part, containing the time derivatives of the field \( S \): note that a term proportional to \( \dot{S} \) is absent; furthermore, the Lagrangian Poisson equation is *intrinsically* non-linear in the density field, unlike the Eulerian Poisson equation, as it may be seen from the relation \( J - 1 = \frac{1}{\delta+1} - 1 \), where \( \delta \) is fully integrated.

It is remarkable the fact that the equations (17) and (18) hold for a generic non-flat model, in that \( \Omega \) – although fundamental – is a very poorly known cosmological parameter and, apart from theoretical expectations (essentially due to the implications of inflation), there is no definitive observational evidence that our Universe is flat (Peebles 1991; Coles & Ellis 1994). Furthermore, eqs.(17) and (18) are manifestly comoving, which is also convenient, because the overall expansion is usually subtracted in analytical or numerical studies of departures from the mean homogeneous and isotropic universe. In Appendix A,
we give more compact expressions for eqs. (17) and (18).

The irrotationality condition and the Lagrangian Poisson equation are exact equations in the Lagrangian description. It is undoubtedly very difficult to solve them in a rigorous way. A possible alternative is to seek for approximate solutions: the standard technique is to expand the trajectory $S$ in a perturbative series, the leading term being the linear displacement which corresponds indeed to the Zel’dovich approximation (see Moutarde et al. 1991). To approximate the Lagrangian Poisson equation implies that the gravitational interaction among the particles of the fluid is described only approximately.

4 Lagrangian Perturbative Approximation: Higher–Order Solutions

We now solve the dynamical equations for the displacements $S$ according to the following Lagrangian perturbative prescription:

$$S(q, \tau) = D(\tau)S^{(1)}(q) + E(\tau)S^{(2)}(q) + F(\tau)S^{(3)}(q) + \cdots.$$  \hspace{1cm} (19)

Here $S^{(1)}(q)$ corresponds to the first–order approximation, $S^{(2)}(q)$ to the second–order approximation, and so on: the dynamics of the evolution constrains in general both the temporal dependence as described by the functions $D, E, F, \ldots$, and the spatial displacements $S^{(n)}(q)$.

Recently Gramann (1993), to calculate an analytical expression relating the density to the velocity in a self–gravitating system, used a (second–order) perturbative expansion similar to the previous one, but with the assumption that $E \equiv D^2$: we will discuss below why this is a very restrictive hypothesis. Furthermore, we anticipate here – and we will demonstrate in Section 4.3 – that the third–order solution $S^{(3)}$ actually corresponds to three modes: we maintain here the expression as given in eq. (19) for the sake of simplicity.

We emphasize that the first– and the second–order perturbative solutions (and, as it will be shown, each mode of the third–order solution) are explicitly written in (19) as separable
with respect to the temporal and spatial coordinates. This is not an assumption, being just a property of the perturbative Lagrangian description. Indeed, this can be demonstrated to be just a direct consequence of the dynamical Lagrangian equations: while this turns out to be trivial for the first-order solution, it is not so e.g. for the second-order one. In Appendix B, it is shown that the separable solution \( E(\tau)S^{(2)}(q) \) is the most general second-order Lagrangian perturbative solution.

Conversely, in Catelan et al. (1995) the non-separability of the corresponding higher-order Eulerian perturbative solutions is thoroughly analyzed: the Eulerian perturbative solutions factorize in space and time only in the very special case of the flat universe, and not in a generic Friedmann model; intriguingly, it is also shown that the non-separability of the Eulerian solutions is fully consistent with the separability of the Lagrangian solutions, at least explicitly up to the second-order.

Because of their convenience in deriving the subsequent equations, we define the following scalars:

\[
\mu_1(A) \equiv \nabla \cdot A = A_{\alpha\alpha},
\]

\[
\mu_2(A, B) \equiv \frac{1}{2} [A_{\alpha\alpha}B_{\beta\beta} - A_{\alpha\beta}B_{\beta\alpha}],
\]

\[
\mu_2(A) \equiv \mu_2(A, A),
\]

\[
\mu_3(A) \equiv \det(A_{\alpha\beta}),
\]

where \( A \) and \( B \) are generic functions of the Lagrangian coordinate \( q \) and spatial derivatives are with respect to the variable \( q \). Note that the functions \( \mu_1, \mu_2 \) and \( \mu_3 \) are linear, quadratic and cubic in their arguments, respectively. Furthermore, the following expression for the Jacobian determinant \( J \) holds:

\[
J(q, \tau) = 1 + \mu_1(S) + \mu_2(S) + \mu_3(S).
\]

This is an exact relation for the displacement and it is cubic in \( S \), justifying investigation of the third-order solution. According to eqs.(12) and (19), the Lagrangian acceleration
may be perturbatively expressed as
\[
\mathbf{g}[\mathbf{x}(\mathbf{q}, \tau), \tau] = \ddot{D}(\tau) \mathbf{S}^{(1)}(\mathbf{q}) + \ddot{E}(\tau) \mathbf{S}^{(2)}(\mathbf{q}) + \ddot{F}(\tau) \mathbf{S}^{(3)}(\mathbf{q}) + \cdots ,
\] (25)
explicitly up to the third–order term: this expression corresponds to the Euler equation. We now solve the Lagrangian Poisson equation order–by–order.

### 4.1 First–Order Solution: Zel’dovich Approximation

We can easily find the first–order approximation truncating eq.(18) accordingly to the linear terms, \( \ddot{S}_{\alpha\alpha} = \alpha(\tau) \mu_1(\mathbf{S}) \), to obtain, in terms of the displacement \( \mathbf{S}^{(1)} \),
\[
\ddot{D}(\tau) \mathbf{S}^{(1)} = \alpha(\tau) D(\tau) \mu_1(\mathbf{S}^{(1)}) .
\] (26)
Given the definition of \( \mu_1 \), we immediately get
\[
\ddot{D}(\tau) - \alpha(\tau) D(\tau) = 0 .
\] (27)
The two linearly independent solutions coincide with the growing and decreasing modes of the linear density field: hereafter we will consider only the growing mode \( D_+ \equiv D \), since any perturbative approach is consistently applicable to the (mildly) non–linear regime, when the decreasing modes have already become negligible. We report here the linear growing solution
\[
D(\tau) = 1 + 3 \left( \tau^2 - 1 \right) \left[ 1 + \tau \ln \sqrt{\frac{\tau - 1}{\tau + 1}} \right] ,
\] (28)
where we remind that, in the case \( k = 1, \ln \left( \frac{1-i\tau}{1+i\tau} \right) = 2i \arctan(i\tau) \). Eq.(27) thus corresponds to the Zel’dovich approximation. Note that the linear regime does not constrain the vector \( \mathbf{S}^{(1)} \) at all, whose particular form has to be ascribed to the chosen initial conditions. Furthermore, the displacement \( \mathbf{S}^{(1)} \) is potential in Lagrangian space, since the irrotationality condition (17) in the linear regime reduces to
\[
\epsilon_{\alpha\beta\gamma} S^{(1)}_{\gamma\beta} = 0 ,
\] (29)
as already known (Zel’dovich 1970a, b). One can thus define a potential \( \psi^{(1)}(q) \) such that \( S^{(1)}(q) \equiv \nabla \psi^{(1)}(q) \), with \( \psi^{(1)} \) the velocity potential in the Zel’dovich approximation (see the discussion in Kofman 1991). As a consequence, the linear deformation tensor is symmetric. The particular growing solution (28) reduces to \( D = \tau^{-2} \) in the case of an Einstein–de Sitter model, since \( \alpha = 6\tau^{-2} \).

### 4.2 Second–Order Solution: BJCP approximation

Retaining only the quadratic terms in the Lagrangian Poisson equation and introducing the ansatz \( S = DS^{(1)} + ES^{(2)} \), one gets to second–order

\[
\ddot{E} \mu_1(S^{(2)}) + D\ddot{D} \mu_1(S^{(1)})^2 - D\dot{D} S^{(1)}_{\alpha\beta} S^{(1)}_{\beta\alpha} = \alpha E \mu_1(S^{(2)}) + \alpha D^2 \mu_2(S^{(1)}) \, ,
\]

from which, using the first–order results,

\[
\left[ \ddot{E}(\tau) - \alpha(\tau)E(\tau) \right] \mu_1(S^{(2)}) = -\alpha(\tau) D(\tau)^2 \mu_2(S^{(1)}) \, .
\] (30)

This is a separable differential equation leading to the system of equations (cf. BJCP 1992)

\[
\begin{aligned}
\ddot{E}(\tau) - \alpha(\tau)E(\tau) &= -\alpha(\tau) D(\tau)^2, \\
\mu_1(S^{(2)}) &= \mu_2(S^{(1)}) \, .
\end{aligned}
\] (31)

We stress that, unlike the first–order case, the second–order approximation constrains both the time and the spatial dependence of the solution. The solution \( E(\tau) \) of the temporal equation has been found by BJCP (1992). Its generality is discussed in Appendix B. We report here its explicit expression:

\[
E(\tau) = -\frac{1}{2} - \frac{9}{2} \left( \tau^2 - 1 \right) \left\{ 1 + \tau \ln\sqrt{\frac{\tau - 1}{\tau + 1}} + \frac{1}{2} \left[ \tau + (\tau^2 - 1) \ln\sqrt{\frac{\tau - 1}{\tau + 1}} \right]^2 \right\} .
\] (32)

In BJCP, this solution has been applied to describe the dependence of the skewness of the unfiltered density field on the density parameter \( \Omega \), in that, near \( \Omega = 1 \), \( E \approx -\frac{3}{7} \Omega^{-2/63} D^2 \).
For us it will be useful, because we will be able to express the third–order solutions in terms of the lower–order results.

The solution of the second–order spatial equation in (31) may be written as

\[ S^{(2)} = \frac{1}{2} \left[ S^{(1)} \left( \nabla \cdot S^{(1)} \right) - \left( S^{(1)} \cdot \nabla \right) S^{(1)} \right] + R^{(2)} , \]  

(33)

where \( R^{(2)}(q) \) is any divergence–free vector such that \( \nabla \wedge S^{(2)} = 0 \): if the first–order Lagrangian motion is assumed potential, this is indeed the result one obtains from the irrotationality equation (17) once only the second–order terms are retained

\[ \dot{\bar{E}}(\tau) \left( \nabla \wedge S^{(2)} \right)_\alpha = -\dot{D}(\tau)D(\tau) \epsilon_{\alpha\beta\gamma} S^{(1)}_{\beta\sigma} S^{(1)}_{\gamma\sigma} = 0 , \]  

(34)

where the last equality follows from the fact that the tensor \( S^{(1)}_{\beta\sigma} S^{(1)}_{\gamma\sigma} \) is symmetric. Thus, we have to conclude that the gravitational evolution does not induce vorticity in the second–order Lagrangian motion at all, whatever the initial conditions are: note that \( \nabla \wedge S^{(2)} = 0 \) is a purely spatial relation. It appears clear from (34) that the displacement \( S^{(2)} \) is potential once \( S^{(1)} \) is assumed to be potential too, and thus the second–order deformation tensor is symmetric: a potential \( \psi^{(2)} \) may be introduced, from which one obtains \( S^{(2)}(q) = \nabla \psi^{(2)}(q) \). An useful expression of the Fourier transform of \( \psi^{(2)}, \bar{\psi}^{(2)} \), in terms of the first–order potential \( \bar{\psi}^{(1)} \) may be found in Appendix C: it will clearly result that an explicit expression of the gauge–dependent vector \( R^{(2)} \) is, in practice, unnecessary: we nevertheless emphasize that the expression in (33) is the only compatible with the irrotationality condition (34).

A restricted class of second–order irrotational solutions, for which additional constraints on the initial conditions have to be fulfilled, is discussed in Buchert & Ehlers (1993) and Buchert (1994); the same constraints also allow the construction of local forms, as debated in Buchert (1994), again in the context of the Einstein–de Sitter cosmology. The solution in (32) reduces to the simple form \( E = -\frac{3}{7} \tau^{-4} \propto D^2 \) in the case of the flat model.
4.3 Third–Order Solution

Inserting the ansatz (19) in the Lagrangian Poisson equation, after some algebra one obtains the third–order expression

\[
\ddot{F}_1(S^{(3)}) + 2(D\ddot{E} + \dot{D}E) \mu_2(S^{(1)}, S^{(2)}) + 3\ddot{D}D^2 \mu_3(S^{(1)}) = \alpha(\tau) \left[ F_1(S^{(3)}) + 2DE \mu_2(S^{(1)}, S^{(2)}) + D^3 \mu_3(S^{(1)}) \right],
\]

where we used the relation

\[
S_{\alpha\beta}S_{\beta\alpha}^{(1)} = D^2 S_{\alpha\beta}^{(1)} S_{\beta\alpha}^{(1)} = 3D^2 \mu_3(S^{(1)}). \tag{35}
\]

On the light of the lower–order results, the former expression becomes

\[
\left[ \ddot{F}(\tau) - \alpha(\tau)F(\tau) \right] \mu_1(S^{(3)}) = -2\alpha(\tau)D(\tau) \left[ E(\tau) - D(\tau)^2 \right] \mu_2(S^{(1)}, S^{(2)}) - 2\alpha(\tau)D(\tau)^3 \mu_3(S^{(1)}). \tag{36}
\]

Note that the mixed invariant \( \mu_2 \) couples the first– and second–order solutions \( S^{(1)} \) and \( S^{(2)} \), while the last term \( \mu_3 \) is cubic in the argument \( S^{(1)} \). This fact forces us, as suggested by Buchert (1994), to split the third–order displacement \( S^{(3)} \) into two parts, one resulting from the interaction among the linear perturbations, the second from the interaction between the first– and second–order perturbations:

\[
S^{(3)}(q) = S_a^{(3)}(q) + S_b^{(3)}(q). \tag{37}
\]

According to this ansatz, the dynamical equation (36) splits into the systems

\[
\begin{cases}
\ddot{F}_a(\tau) - \alpha(\tau)F_a(\tau) = -2\alpha(\tau)D(\tau)^3, \\
\mu_1(S_b^{(3)}) = \mu_3(S^{(1)}),
\end{cases}
\]

and
\[
\begin{align*}
\dot{F}_b(\tau) - \alpha(\tau) F_b(\tau) &= -2\alpha(\tau) D(\tau) [E(\tau) - D(\tau)^2] , \\
\mu_1(S_b^{(3)}) &= \mu_2(S^{(1)}, S^{(2)}). 
\end{align*}
\] (39)

Note that, since now \(S^{(2)}\) may be considered a potential field, \(\mu_2(S^{(1)}, S^{(2)}) = \mu_2(S^{(2)}, S^{(1)})\).

It clearly appears that the assumption \(E \equiv D^2\) [see Gramann (1993)] implies a neglect of the interaction mode between the first– and second–order perturbations, in that \(E = D^2 \implies F_b \equiv 0\). This is in general a very restrictive hypothesis and possibly only (high–resolution) \(N\)–body simulations might quantify the real loss of accuracy at the third–order level.

Furthermore, Lachièze–Rey (1993b) argues that one formal third–order solution exists corresponding to the assumption that the deformation tensor \(S(q, \tau)\) remains proportional, at each Lagrangian position, to its initial value, being only multiplied by a scalar Lagrangian growth factor. This is surely true, but the problem is now that such a hypothesis notably restricts the form of the spatial part of the third–order solution: it is not difficult indeed to show that the Lachièze–Rey solution corresponds to assuming that, in our notation, \(\mu_3(S^{(1)}) = \mu_2(S^{(1)}, S^{(2)})\) and, after that, \(\mu_1(S^{(3)}) = \mu_3(S^{(1)})\). Of course these conditions depend also on the particular chosen initial configurations, but it is not clear to which type of realistic physical situation they may be applied.

After these comments, we give now the third–order solutions by quadrature:

\[
F_a(\tau) = -2 D(\tau) \int^\tau d\tau_1 D(\tau_1)^{-2} \int^{\tau_1} d\tau_2 \alpha(\tau_2) D(\tau_2)^4 , 
\] (40)

and

\[
F_b(\tau) = -2 D(\tau) \int^\tau d\tau_1 D(\tau_1)^{-2} \int^{\tau_1} d\tau_2 \alpha(\tau_2) D(\tau_2)^2 [E(\tau_2) - D(\tau_2)^2] . 
\] (41)

In this way the two solutions \(F_a(\tau)\) and \(F_b(\tau)\) are given in terms of the lower–order solutions \(D(\tau)\) and \(E(\tau)\): explicit versions of (40) and (41) are cumbersome, while approximate expressions may be recovered in the limit e.g. \(\Omega \rightarrow 1\) (Catelan 1995). In the case of an
Einstein–de Sitter model, we find $F_a = -\frac{1}{3} \tau^{-6}$ and $F_b = \frac{10}{27} \tau^{-6} \propto D^3$: the results in Buchert (1994) are particular cases of the general solutions (40) and (41).

A solution $S_a^{(3)}$ of (38) may be obtained recalling that, from (35), $S_{\alpha\beta}^{(1)C} S_{\beta\alpha}^{(1)} = 3\mu_3(S^{(1)})$; thus, since $\partial_\alpha S_{\alpha\beta}^{(1)C} = 0$, one gets $\partial_\alpha \left( S_{\alpha\beta}^{(3)} - \frac{1}{3} S_{\alpha\beta}^{(1)C} S_{\beta\alpha}^{(1)} \right) = 0$ and finally, for each component,

$$S_{a\alpha}^{(3)} = \frac{1}{3} S_{\alpha\beta}^{(1)C} S_{\beta\alpha}^{(1)} + R_{a\alpha}^{(3)} .$$

(42)

Here $R_{a}^{(3)}(q)$ is again a divergence–free vector such that $\nabla \wedge S_{a}^{(3)} = 0$. To understand why $S_{a}^{(3)}$ is a potential vector it is indeed enough to isolate, in the Lagrangian irrotationality condition (17), all the possible vortical terms induced by the interactions among the first–order displacements: it is not difficult to see that the only term

$$\epsilon_{\alpha\beta\gamma} S_{\beta\gamma}^{(1)} S_{\alpha\gamma}^{(1)C} = 0 ,$$

(43)

where the equality is justified by the assumption that the linear displacement is potential.

Another way to describe the irrotationality of $S_{a}^{(3)}$ is to note the that third–order interactions like $S_{\alpha}^{(1)C} S_{\beta}^{(1)C} S_{\gamma}^{(1)}$ can be only symmetric in $S_{\gamma}^{(1)}$ and not antisymmetric (here * indicate a generic Greek index), as any possible vortical–like interaction would be.

In a similar fashion, one can derive a solution $S_{b}^{(3)}$ from (39), assuming in particular that $S_{b}^{(3)}$ is induced by symmetric couplings between the first– and second–order solutions. It can be written as

$$S_{b}^{(3)} = \frac{1}{4} \left[ S^{(1)}(\nabla \cdot S^{(2)}) - (S^{(1)} \cdot \nabla)S^{(2)} + S^{(2)}(\nabla \cdot S^{(1)}) - (S^{(2)} \cdot \nabla)S^{(1)} \right] + R_{b}^{(3)} ,$$

(44)

where again $R_{b}^{(3)}(q)$ is a divergence–free vector such that $\nabla \wedge S_{b}^{(3)} = 0$: in fact, symmetric interactions between the first– and second–order modes cannot be vortical.

The solutions $S_{a}^{(3)}$ and $S_{b}^{(3)}$, being longitudinal, may be written in terms of the respective potentials $\psi_{a}^{(3)}$ and $\psi_{b}^{(3)}$, namely $S_{a}^{(3)}(q) \equiv \nabla \psi_{a}^{(3)}(q)$ and $S_{b}^{(3)}(q) \equiv \nabla \psi_{b}^{(3)}(q)$. In Appendix C, the expressions of the Fourier components $\tilde{\psi}_{a}^{(3)}$ and $\tilde{\psi}_{b}^{(3)}$ as functions of the Zel’ dovich potential $\tilde{\psi}^{(1)}$ are given. The significance of the vectors $R_{a}^{(3)}$ and $R_{b}^{(3)}$ is perfectly equivalent to that of the vector $R^{(2)}$ for the second–order solution $S^{(2)}$. 

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One now has to make the following inquiry: does any *antisymmetric* coupling between the Zel’dovich and the BJCP solutions exist at the third–order level? If it exists, then a third–order transverse component $T$ arises in the Lagrangian motion of the fluid mass elements, and this is the *only* vortical component which can be induced in the framework of the third–order approximation. Note that such a type of component cannot modify the amplitude of the density fluctuation field.

Let us suppose therefore that such a transverse mode actually exists: if the original ansatz (19) is now improved to

$$S(q, \tau) = S_\parallel + S_\perp = S_\parallel + F_c(\tau) T(q) ,$$  

where we recall that $S_\parallel \equiv DS^{(1)} + ES^{(2)} + F_a S^{(3)}_a + F_b S^{(3)}_b$, it follows that, in terms of $T$, the Lagrangian irrotationality condition becomes

$$\dot{F}_c(\tau) (\nabla \wedge T)_\alpha = \left[D(\tau)\dot{E}(\tau) - \dot{D}(\tau)E(\tau)\right] \epsilon_{\alpha\beta\gamma} S^{(1)}_\beta S^{(2)}_\gamma .$$  

(46)

We derive eq.(46) explicitly in Appendix D. Again, this equation may be split into a temporal and a spatial part:

$$\begin{align*}
\dot{F}_c(\tau) &= -\alpha(\tau) D(\tau^3) , \\
(\nabla \wedge T)_\alpha &= \epsilon_{\alpha\beta\gamma} S^{(1)}_\beta S^{(2)}_\gamma .
\end{align*}$$  

(47)

The growth factor $F_c(\tau)$ may be explicitly written by quadrature

$$F_c(\tau) = - \int^\tau d\tau_1 \int^{\tau_1} d\tau_2 \alpha(\tau_2) D(\tau_2)^3 .$$  

(48)

This solution reduces to the simple $F_c = -\frac{1}{4} \tau^{-6}$ in the case of an Einstein–de Sitter universe.

From the second equation in (47), we note – consistently – that the antisymmetric part of the third–order interaction between the modes $S^{(1)}$ and $S^{(2)}$ generates the transverse component

$$T(q) = \frac{1}{2} \left[(S^{(1)} \cdot \nabla)S^{(2)} - (S^{(2)} \cdot \nabla)S^{(1)}\right] + \nabla \varphi(q) .$$  

(49)
In Appendix C the Fourier components $\tilde{T}_\alpha$ are explicitly given. The term $\nabla \varphi$ is such that $\nabla \cdot \mathbf{T} = 0$ or, equivalently, the transverse component $\mathbf{T}$ can now be described as the curl of a vector potential, $\mathbf{T} \equiv \nabla \times \mathbf{A}$, which cannot in general be removed by a suitable gauge-fixing of the initial conditions: one can say that the Lagrangian motion is no longer purely potential from third-order onward.

This fact was first discovered by Buchert (1994) within the Einstein–de Sitter model, and therefore seems a universal feature of the Lagrangian motion, independently of the value of the density parameter $\Omega$ (i.e. of the underlying model of universe). However, it is important to stress again that the displacements $S^{(3)}_a$ and $S^{(3)}_b$ are in general purely potential just because they originate through symmetric couplings among the lower-order longitudinal perturbations; also, the transverse displacement $\mathbf{T}$ is purely vortical because it originates through the antisymmetric coupling between the first- and second-order solutions, independently of the peculiar initial conditions one is picking out. We summarize the results of this work in the next section.

5 Summary and Conclusions

In this work we studied, in the Lagrangian description, the behaviour of a self-gravitating collisionless fluid embedded in a generic non-flat expanding universe. We assumed the Newtonian limit and zero cosmological constant. We have mainly focussed on the formal aspects of this kind of analysis, also proposing a new Lagrangian formalism. In particular, we derived the general equations, in the forms given in (17) and (18), governing the Lagrangian motion of the mass elements in a universe with arbitrary density parameter $\Omega$ and directly in comoving coordinates. We consider this kind of derivation one of the main results of this work. An alternative derivation and notation may be found in Buchert (1989). Since it is in practice very difficult to find exact solutions of the foregoing general equations, we solved them according to a Lagrangian perturbative approach, namely we sought approximate solutions $S = \sum S^{(n)}$ about the linear displacement $S^{(1)}$ as pioneered
by Moutarde et al. (1991). Our formalism enables one to easily recover the already known lower–order solutions, in particular the linear Zel’dovich approximation (Zel’dovich 1970a, b) and the second–order BJCP approximation (Bouchet et al. 1992). Then, we explicitly worked out the third–order solution, generalizing to an arbitrary Friedmann universe the recent results obtained by Buchert (1994) in the context of the Einstein–de Sitter cosmology. The question of the irrotationality in the Lagrangian space has been analyzed too. In particular, we found that the spatial solutions $S^{(2)}, S_a^{(3)}, S_b^{(3)}$ are purely potential for any acceptable initial conditions – the only underlying hypothesis being that the linear displacement is longitudinal – as one finds if one carefully applies, order–by–order, the irrotationality condition (17). Similarly, the third–order transverse component $T$ is purely vortical for any realistic initial conditions. Obviously, the reason of the existence of the transverse component $T$ is that the transformation from Eulerian to Lagrangian coordinates $x \rightarrow q$ is in general a non–Galilean transformation, and fictitious forces are induced in the Lagrangian description: no new physics can indeed appear.

To avoid an exceeding proliferation of formulae in the main text of the article, we performed in Appendix C the Fourier analysis of the higher–order Lagrangian solutions: such a results can be important for numerical and practical applications. Specifically, we found that the third–order dynamics is fully described in terms of the tetra–potential $\Psi \equiv (\psi_a^{(3)}, \psi_b^{(3)}, A_1, A_2)$, since only two of the three components of the vector potential $\mathbf{A}$ are physically significative.

Our results offer a tool with which to follow the dynamics of the formation of the structures in the universe, as resulting from the non–linear gravitational instability.

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To Elena, she had the time of knowing it.

Note added in proof After the submission of this article, Thomas Buchert addressed our attention on the Appendix A in Buchert (1989) where, employing the same temporal parameter in (6), an alternative expression of the equation (18) is given. The temporal coordinate (6) has been recently used also in Bharadwaj, S. 1994, ApJ, 428, 419 and Bouchet, F.R., et al. 1994, A&A, submitted.

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Appendix A

In the first appendix we explicitly show how to obtain the general Lagrangian equations (17) and (18). Let us start by deriving the Lagrangian Poisson equation: from eq.(15), this can be written in the form

\[ J(q, \tau) \frac{\partial^2 S_\alpha (q, \tau)}{\partial x_\alpha} = \alpha(\tau)[J(q, \tau) - 1], \]  

(50)

where \( J \) is the determinant of the Jacobian \( \left( \frac{\partial x}{\partial q} \right) \), and \( x = q + S(q, \tau) \). To derive (18), all we need is to show that the following relation holds

\[ J \frac{\partial}{\partial x_\alpha} = \left[ (1 + S_{\gamma}) \delta_{\beta\alpha} - S_{\beta\alpha} + S_{\beta\alpha}^C \right] \frac{\partial}{\partial q_\beta}. \]  

(51)

This indeed corresponds to the implicit expression

\[ \frac{\partial}{\partial x_\alpha} = \left( \frac{\partial x_\alpha}{\partial q_\beta} \right)^{-1} \frac{\partial}{\partial q_\beta}. \]  

(52)

The quantities \( \left( \frac{\partial x_\alpha}{\partial q_\beta} \right)^{-1} \) are elements of the matrix \( \left( \frac{\partial x}{\partial q} \right)^{-1} = \left( \frac{\partial q}{\partial x} \right) \). Since, explicitly,

\[ \left( \frac{\partial x}{\partial q} \right) = \begin{pmatrix} 1 + S_{11} & S_{12} & S_{13} \\ S_{21} & 1 + S_{22} & S_{23} \\ S_{31} & S_{32} & 1 + S_{33} \end{pmatrix}, \]  

(53)

we obtain, in matrix notation,

\[ \left( \frac{\partial x}{\partial q} \right)^{-1} = \frac{1}{J} \left\{ (1 + \nabla \cdot S)I - S + S^C \right\}, \]  

(54)

where \( I = \text{diag}(1, 1, 1) \) is the identity, \( S \equiv (S_{\alpha\beta}) \) is the deformation matrix and \( S^C \) is the cofactor matrix we usefully write below:

\[ S^C = \begin{pmatrix} S_{22}S_{33} - S_{32}S_{23} & S_{32}S_{13} - S_{12}S_{33} & S_{12}S_{23} - S_{22}S_{13} \\ S_{31}S_{23} - S_{21}S_{33} & S_{11}S_{33} - S_{31}S_{13} & S_{21}S_{13} - S_{11}S_{23} \\ S_{21}S_{22} - S_{31}S_{22} & S_{31}S_{12} - S_{11}S_{32} & S_{11}S_{22} - S_{21}S_{12} \end{pmatrix}. \]  

(55)
With these results one now immediately recovers the Lagrangian Poisson equation from the original eq.(15).

In exactly the same way one can obtain the irrotationality condition in the Lagrangian space from the Eulerian eq.(10):

$$\epsilon_{\alpha\beta\gamma} \frac{\partial u_\gamma}{\partial x_\beta} = 0 . \quad (56)$$

Recalling eqs.(11), (12) and (52), this becomes

$$\epsilon_{\alpha\beta\gamma} (\frac{\partial x_\beta}{\partial q_\sigma})^{-1} \frac{\partial S_\gamma}{\partial q_\sigma} = 0 . \quad (57)$$

If now the relation (51) is inserted in the last equation, we get the general equation (17).

To conclude we propose here even more compact forms for the irrotationality condition (17) and the Lagrangian Poisson equation (18): in fact, defining the cofactor element of $x_{\alpha\beta}$

$$x^C_{\alpha\beta} \equiv J x^{-1}_{\alpha\beta} = \left[ (1 + \nabla \cdot S) \delta_{\alpha\beta} - S_{\alpha\beta} + S^C_{\alpha\beta} \right] , \quad (58)$$

where $x^{-1}_{\alpha\beta} \equiv (\frac{\partial x_\alpha}{\partial q_\beta})^{-1}$, we may finally write

$$\epsilon_{\alpha\beta\gamma} x^C_{\beta\sigma} \ddot{x}_{\gamma\sigma} = 0 , \quad (59)$$

$$x^C_{\alpha\beta} \dddot{x}_{\beta\alpha} = \alpha(\tau)[J - 1] , \quad (60)$$

respectively. We have nevertheless preferred to retain less compact and elegant versions in the main text, because the dependence on the displacement $S$ and its derivatives is there explicitly shown.

**Appendix B**

In this appendix it is demonstrated that the *separable* second–order solution $E(\tau)S^{(2)}(q)$ is the most general solution of the Lagrangian fluid equations, once only the second–order
terms are retained. Let us suppose that, ab absurdo, the second-order solution is non-separable (and longitudinal for simplicity), namely \( D(\tau)^2 \nabla \Phi_2(q, \tau) \): the factorization of the term \( D^2 \) does not alter the demonstration. The first order solution is given in Section 4.1. From the Lagrangian Poisson equation (18), one gets to second-order

\[
D^2 \partial_\tau^2 \Phi_2 + 4D \dot{D} \partial_\tau \Phi_2 + \left( 2\dot{D}^2 + 2D \ddot{D} - \alpha(\tau) D^2 \right) \Phi_2 = -\alpha(\tau) D^2 \tilde{P}_2, \tag{61}
\]

where the tilde “~” indicates the Fourier transformed quantities (see next appendix). The function \( \tilde{P}_2 \), whose explicit expression is superfluous to give here (the interested reader is addressed to the next appendix), is defined by the relation \( \tilde{P}_2 \equiv -p^{-2} \mu_2 (\mathcal{S}^{(1)}) \): the important point is to note that \( \mathcal{P}_2 \) depends only on the spatial variable: \( \mathcal{P}_2 = \mathcal{P}_2(p) \), where \( p \) indicates the comoving Lagrangian wavevector, and \( p \equiv |p| \). Now it is easy to verify that, if the function \( B(\tau) \) satisfying the differential equation

\[
K^2 \ddot{B} + 4K \dot{B} + (2 + \alpha K^2) B = 1,
\]

where \( K \equiv D/\dot{D} \), is introduced, the function \( \tilde{\Phi}_2 \) may be recast in the separable form

\[
\tilde{\Phi}_2(p, \tau) = [2B(\tau) - 1] \tilde{P}_2(p). \tag{62}
\]

Then the function \( E \) so defined, \( E(\tau) \equiv D(\tau)^2 [2B(\tau) - 1] \), satisfies the differential equation

\[
\ddot{E} - \alpha(\tau) E = -\alpha(\tau) D(\tau)^2, \]

which is the first equation in (31). This concludes the demonstration. Similar considerations may be extended to higher-order modes (Buchert, private communication; Ehlers & Buchert, 1995, in preparation).

Appendix C

In this third appendix, we perform the complete Fourier analysis of the Lagrangian motion described in the main text. The final results are useful for practical and numerical applications.

Let us indicate by \( p \) the comoving Lagrangian wavevector. The \( n \)th-order displacement potential \( \psi^{(n)}(q) \) may be written as a Fourier integral,

\[
\psi^{(n)}(q) = \frac{1}{(2\pi)^3} \int dp \tilde{\psi}^{(n)}(p) e^{i p \cdot q}, \tag{63}
\]
where it is understood that we restrict, in this appendix, to the cases \( n = 1, 2, 3 \). Observing that \( i p_{\alpha} \tilde{\psi}^{(n)}(p) = \tilde{S}^{(n)}_{\alpha}(p) \), it is immediate to obtain from the solution (33) that

\[
\tilde{\psi}^{(2)}(p) = -\frac{1}{p^2} \int \frac{dp_1 dp_2}{(2\pi)^6} \left[ (2\pi)^3 \delta_D(p_1 + p_2 - p) \right] \kappa^{(2)}(p_1, p_2) \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_2),
\]

where we have defined the kernel

\[
\kappa^{(2)}(p_1, p_2) \equiv \frac{1}{2} \left[ p_1^2 p_2^2 - (p_1 \cdot p_2)^2 \right] = \frac{1}{2} \left( p_1 p_2 \sin \theta_{12} \right)^2,
\]

being e.g. \( p \equiv |p| \) and \( \theta_{12} \equiv \arccos(p_1 \cdot p_2/p_1 p_2) \) the angle between the vectors \( p_1 \) and \( p_2 \); the presence of the Dirac–function \( \delta_D \) comes from momentum conservation in Fourier space. An alternative simpler expression of \( \tilde{\psi}^{(2)}(p) \) may be obtained performing one integration: the result may be written as follows,

\[
\tilde{\psi}^{(2)}(p) = -\frac{1}{p^2} \int \frac{dp'}{(2\pi)^3} \kappa^{(2)}(p, p') \tilde{\psi}^{(1)}(p') \tilde{\psi}^{(1)}(p - p') .
\]

We stress that to obtain the expression of \( \tilde{\psi}^{(2)} \) in terms of the first–order Zel’dovich potential \( \tilde{\psi}^{(1)} \) it is completely unnecessary, for practical uses, to specify the form of the divergence–free vector \( R^{(2)} \), for any realistic initial conditions. The kernel \( \kappa^{(2)}(p_1, p_2) \) describes in the Lagrangian Fourier space the effects of the non–linear dynamics.

Similar considerations and calculations may be easily extended to the third–order solutions. We give here the explicit expressions of the Fourier components of the Lagrangian potentials \( \psi_a^{(3)} \) and \( \psi_b^{(3)} \), originating the longitudinal motion, and of the vector potential \( A \), describing the transversal motion.

To show how the calculations progress, it is more simple to start with the derivation of the Fourier component \( \tilde{\psi}_b^{(3)} \). From the expression (44),

\[
\tilde{\psi}_b^{(3)}(p) = -\frac{\alpha}{2 p^2} \int \frac{dp_1 dp_2}{(2\pi)^6} \left[ (2\pi)^3 \delta_D(p_1 + p_2 - p) \right] \left( p_2^2 p_{1\alpha} - p_1 \cdot p_2 p_{2\alpha} \right) \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(2)}(p_2).
\]

Inserting the solution (66), we eventually obtain, in terms of the Zel’dovich potential, the expression

\[
\tilde{\psi}_b^{(3)}(p) = \int \frac{dp_1 dp_2}{(2\pi)^6} \kappa^{(3)}_{b}(p_1, p_2; p) \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_2 - p_1) \tilde{\psi}^{(1)}(p - p_2),
\]
where the kernel $\kappa_b^{(3)}$ has been introduced:

$$
\kappa_b^{(3)}(p_1, p_2; p) \equiv \frac{1}{2} \kappa^{(2)}(p_1, p_2) \left[ 1 - \left( \frac{p \cdot p_2}{p_2} \right)^2 \right].
$$

In a similar fashion, the Fourier component $\tilde{\psi}_a^{(3)}$ may be derived:

$$
\tilde{\psi}_a^{(3)}(p) = -\frac{1}{p^2} \int \frac{dp_1 dp_2}{(2\pi)^6} \kappa_a^{(3)}(p_1, p_2; p) \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_2) \tilde{\psi}^{(1)}(p - p_1 - p_2),
$$

where, again, we have defined a kernel $\kappa_a^{(3)}$ according to the definition

$$
\kappa_a^{(3)}(p_1, p_2; p) \equiv \frac{1}{6} \left[ \epsilon_{\alpha\gamma\delta} p_\alpha p_1 \epsilon_{\beta\eta\sigma} (p_\beta - p_1 \beta - p_2 \beta) p_1 \eta p_2 \sigma \right].
$$

The algebraic relation $S_{\alpha\beta}^{(1)}C = \frac{1}{2} \epsilon_{\alpha\gamma\delta} \epsilon_{\beta\eta\sigma} S_{\gamma\eta}^{(1)} S_{\delta\sigma}^{(1)}$ has to be used. Finally, the vortical component $\tilde{T}_\alpha$ along the $\hat{\alpha}$–direction is given by the integral

$$
\tilde{T}_\alpha(p) = i \int \frac{dp_1 dp_2}{(2\pi)^6} \iota_\alpha^{(3)}(p_1, p_2; p) \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_2 - p_1) \tilde{\psi}^{(1)}(p - p_2),
$$

where the kernel $\iota_\alpha^{(3)}$, which indeed depends upon the direction, is explicitly given by

$$
\iota_\alpha^{(3)}(p_1, p_2; p) \equiv \frac{1}{2} \kappa^{(2)}(p_1, p_2) \left( 1 - \frac{p \cdot p_2}{p_2^2} \right) (2p_2 - p)\alpha.
$$

Specifically, the vortical vector $\tilde{T}(p)$ has only two independent components, in that it satisfies the condition of transversality, $p \cdot \tilde{T} = 0$. Therefore it follows that the vector potential $A$ is fully specified by only two of its three components: without loss of generality we can decide that $A_z \equiv 0$. The significant Fourier components $\tilde{A}_x$ and $\tilde{A}_y$ are thus given by the relations:

$$
p_z \tilde{A}_y(p) = i \tilde{T}_x(p),
$$

and

$$
p_z \tilde{A}_x(p) = i \tilde{T}_y(p).
$$

These last two equations complete the analysis of the Lagrangian motion up to the third–order perturbative approximation: specifically, the third–order dynamics is fully described in terms of the tetra–potential $\Psi \equiv (\psi_a^{(3)}, \psi_b^{(3)}, A_1, A_2)$. The complications due to the non–linear evolution are summarized in the five kernels $\kappa^{(2)}, \kappa_a^{(3)}, \kappa_b^{(3)}$ and $\iota_\alpha^{(3)}$, the two last ones depending on the chosen direction.
Appendix D

In this last appendix, and as an example of how our formalism works, we want to show that the general irrotationality condition in Lagrangian space

\[ \epsilon_{\alpha\beta\gamma} \dot{S}_{\beta\sigma} \left[ (1 + \nabla \cdot S) \delta_{\gamma\sigma} - S_{\gamma\sigma} + S_{\gamma\sigma}^C \right] = 0, \quad (76) \]

leads to the eq.(46) for the third–order transverse component \( T \) of the displacement \( S \).

From the ansatz (45) one immediately obtains

\[ \dot{S}_{\beta\sigma} = \dot{D}(\tau) S_{\beta\sigma}^{(1)} + \dot{E}(\tau) S_{\beta\sigma}^{(2)} + \dot{F_c}(\tau) T_{\beta\sigma}, \quad (77) \]

(the \( F_a \)-mode and \( F_b \)-mode do not enter in the vortical couplings) and, from eq.(61), we get

\[ 0 = -\dot{F_c} \epsilon_{\alpha\beta\gamma} T_{\gamma\beta} + \]

\[ + \epsilon_{\alpha\beta\gamma} \left[ \dot{D} S_{\beta\sigma}^{(1)} + \dot{E} S_{\beta\sigma}^{(2)} \right] \left[ D\mu_1(1) \delta_{\gamma\sigma} + E\mu_1(2) \delta_{\gamma\sigma} - D S_{\gamma\sigma}^{(1)} - E S_{\gamma\sigma}^{(2)} + D^2 S_{\gamma\sigma}^{(1)C} \right], \quad (78) \]

i.e.

\[ \epsilon_{\alpha\beta\gamma} \left\{ \dot{D} D S_{\beta\sigma}^{(1)} S_{\gamma\sigma}^{(1)C} + \dot{D} E S_{\beta\sigma}^{(1)} \left[ \mu_1(2) \delta_{\gamma\sigma} - S_{\gamma\sigma}^{(2)} \right] + \dot{D} E S_{\beta\sigma}^{(2)} \left[ \mu_1(1) \delta_{\gamma\sigma} - S_{\gamma\sigma}^{(1)} \right] - \dot{F_c} T_{\gamma\beta} \right\} = 0, \quad (79) \]

where for brevity we wrote \( \mu_1(n) \equiv \mu_1(S^{(n)}) \). Recalling that \( \epsilon_{\alpha\beta\gamma} S_{\beta\gamma}^{(1)C} = 0 \), \( \epsilon_{\alpha\beta\gamma} S_{\beta\gamma}^{(2)} = 0 \) and \( \epsilon_{\alpha\beta\gamma} S_{\beta\gamma}^{(1)} = 0 \), we finally obtain

\[ \epsilon_{\alpha\beta\gamma} \left[ (D\dot{E} - \dot{D} E) S_{\beta\sigma}^{(1)} S_{\gamma\sigma}^{(2)} - \dot{F_c} T_{\gamma\beta} \right] = 0, \quad (80) \]

which coincides with the relation (46) in the main text.