SMOOTH ANALYTIC FUNCTIONS
AND MODEL SUBSPACES

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Dedicated to the memory of Cora Sadosky

Abstract. The main themes of this survey are as follows: (a) the canonical (Riesz–Nevanlinna) factorization in various classes of analytic functions on the disk that are smooth up to its boundary, and (b) model subspaces (i.e., invariant subspaces of the backward shift) in the Hardy spaces $H^p$ and in BMOA. It is the interrelationship and a peculiar cross-fertilization between the two topics that we wish to highlight.

1. Introduction

Our first topic in this survey is the multiplicative structure in spaces of smooth analytic functions. This phrase may sound somewhat redundant, if not downright confusing, since every analytic function is automatically smooth (in any reasonable sense) on its domain. The term becomes perfectly meaningful, though, if “smooth” is interpreted as “smooth up to the boundary”. It is indeed the boundary smoothness of analytic functions that interests us here.

Our functions will live on the disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$. Putting the smoothness issue aside (but only for a short while), let us now recall a bit of function theory on the disk. Suppose that $f$ is analytic on $D$ and not too large near the unit circle $T := \partial D$. Specifically, assume that $f$ lies in some Hardy space $H^p$ with $0 < p \leq \infty$.

By definition, this means – in addition to analyticity – that

$$
||f||_{H^p} := \sup_{0 < r < 1} \left( \int_T |f(r\zeta)|^p dm(\zeta) \right)^{1/p} < \infty
$$

if $0 < p < \infty$, or $||f||_{H^\infty} := \sup_D |f| < \infty$ if $p = \infty$. Here and below, $m$ denotes the normalized arclength measure on $T$. It is well known that $H^p$ functions have boundary values (nontangential limits) $m$-almost everywhere on $T$. We may then identify $H^p$ with a subspace of $L^p = L^p(T, m)$ bearing in mind that the above norm, $\| \cdot \|_{H^p}$, agrees on $H^p$ with the standard $L^p$-norm $\| \cdot \|_p$ over $T$ (see [16, Chapter II]). When $0 < p < 1$, the two quantities should actually be called quasinorms rather than norms.

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For $f$ as above, the function $\varphi := |f|_\mathbb{T}$ will satisfy $\varphi \in L^p$ and $\log \varphi \in L^1$. Moreover, these last two conditions characterize the moduli of $H^p$ functions on $\mathbb{T}$. Now, letting $u := \log \varphi$ and writing $P u$ for the harmonic extension (via the Poisson integral) of $u$ from $\mathbb{T}$ into $\mathbb{D}$, we define the outer function $O_\varphi$ as the (essentially unique) analytic function on $\mathbb{D}$ satisfying $\log |O_\varphi(z)| = P u(z)$. This done, we have $O_\varphi \in H^p$ and $|O_\varphi| = \varphi$ a.e. on $\mathbb{T}$. The ratio $f/O_\varphi =: \theta$ will then be an inner function; that is, $\theta \in H^\infty$ and $|\theta| = 1$ a.e. on $\mathbb{T}$. Thus we arrive at the Canonical Factorization Theorem: the general form of an $f \in H^p$ is given by $f = \theta F$, where $\theta$ is inner and $F$ outer (so that $F = O_\varphi$ for some $\varphi$ as above). A further factorization formula for inner functions allows us to express $\theta$ canonically in terms of its zeros $\{a_n\}$ (these are only required to satisfy $\sum_n (1 - |a_n|) < \infty$) and a certain singular measure $\mu$ on $\mathbb{T}$; see [16, Chapter II]. In summary, the original function $f \in H^p$ is fully described by the parameters $\varphi$, $\{a_n\}$ and $\mu$ that emerge; and any choice of parameters gives rise to an $f \in H^p$ via factorization.

The terms “inner function” and “outer function” were coined by Beurling. Why did he call them that? An amusing, but rather controversial, explanation I have heard is that the identity $f = \theta F$, when written in this specific form, has $\theta$ (the “inner factor”) inside and $F$ (the “outer factor”) outside. Observe that in some noncommutative generalizations, which we do not touch upon, the order may become crucial; and yes, it should be $\theta F$ rather than $F\theta$.

While quite a bit of modern 1-D complex analysis has evolved in an attempt to extend the $H^p$ theory to larger analytic spaces, one also feels tempted to look at smaller (nicer) classes, in particular, at those populated by smooth analytic functions. Here, the good news is that the canonical factorization theorem applies. The bad news is, however, that the parameters cannot be chosen freely. Indeed, most inner functions – actually, all the “interesting” (i.e., nonrational) ones – are highly oscillatory, hence discontinuous, at some points of $\mathbb{T}$. Consequently, the product $\theta F$ may only be smooth on $\mathbb{T}$ if the outer factor, $F$, is good enough and kills the singularities of the (bad) inner factor, $\theta$. To find an explicit quantitative expression of this interplay, for a given “smooth analytic space”, is therefore one problem to be dealt with.

Our second topic is the model subspaces, alias star-invariant subspaces, in $H^p$ and in $\text{BMOA} := \text{BMO} \cap H^1$, where $\text{BMO} = \text{BMO}(\mathbb{T})$ is the space of functions of bounded mean oscillation on $\mathbb{T}$ (see [16, Chapter VI]). In $H^2$, the model subspace $K_\theta$ generated by an inner function $\theta$ is, by definition, the orthogonal complement of the shift-invariant subspace $\theta H^2$. Thus,

\begin{equation}
K_\theta \ (= K_\theta^2) := H^2 \ominus \theta H^2.
\end{equation}

It is a reproducing kernel Hilbert space, whose kernel function $k_z$ associated with a point $z \in \mathbb{D}$ is given by

\[
k_z(\zeta) = \frac{1 - \overline{\theta(z)}\theta(\zeta)}{1 - z\overline{\zeta}}.
\]
This last function is therefore in $K_\theta$ for every $z$, and every $f \in K_\theta$ satisfies

$$f(z) = \int_T f(\zeta) \overline{k_z(\zeta)} \, dm(\zeta), \quad z \in \mathbb{D}.$$ 

It is straightforward to verify that $K_\theta = H^2 \cap \overline{\theta H^2}$, and we further define $K_\theta^p$ (the $H^p$-analogue of $K_\theta$) by putting

$$K_\theta^p := H^p \cap \overline{\theta H^p}, \quad 1 \leq p \leq \infty,$$

where $H^p_0 := \{ f \in H^p : f(0) = 0 \}$ and the bar denotes complex conjugation. For smaller $p$’s, a more reasonable definition appears to be

$$K_\theta^p := \text{clos}_{H^p} K_\theta, \quad 0 < p < 1.$$ 

These subspaces play a crucial role in the Sz.-Nagy–Foiaș operator model (see [20]), which accounts for the terminology. Now, the term “star-invariant” means invariant under the backward shift operator $f \mapsto (f - f(0))/z$, and it follows from Beurling’s theorem (see [16, Chapter II]) that the general form of a closed and nontrivial star-invariant subspace in $H^2$ is indeed given by (1.1), with $\theta$ inner. A similar fact is true for $H^p$ when $1 \leq p < \infty$.

Finally, we put

$$K_{\ast \theta} := K_\theta \cap \text{BMOA}.$$ 

When equipped with the BMO-norm $\| \cdot \|_s$, $K_{\ast \theta}$ becomes a star-invariant subspace of BMOA; in fact, it is the annihilator in BMOA of the shift-invariant subspace $\theta H^1$ in $H^1$. Of course, $K_{\ast \theta}$ contains $K_\infty^p$ and is contained in every $K_\theta^p$ with $0 < p < \infty$.

While each of the two topics just mentioned has received quite a bit of attention in its own right, the intimate interconnection between them does not seem to have been noticed (until recently) or explored in any detail. It is precisely the systematic exploitation of this interrelationship, perhaps a kind of duality, between the two subjects that is characteristic of our approach. In fact, the three stories told in the next three sections are intended to show that results and methods pertaining to one of our themes cast new light on the other, and vice versa.

Before moving any further, we need to recall the notions of Toeplitz and Hankel operators, since these will be crucial in what follows. We let $P_+$ and $P_-$ denote the orthogonal projections from $L^2$ onto $H^2$ and onto $\overline{H^2_0}$, respectively. Thus,

$$(P_+ F)(z) := \sum_{n \geq 0} \hat{F}(n) z^n \quad \text{and} \quad (P_- F)(z) := \sum_{n < 0} \hat{F}(n) z^n,$$

where $\hat{F}(n) := \int_T F(\zeta) \overline{\zeta^n} \, dm(\zeta)$ is the $n$th Fourier coefficient of $F$. These operators are then extended to $L^p$ with $1 < p < \infty$ (in which case they become bounded projections onto $H^p$ and $\overline{H^p_0}$, the classical M. Riesz theorem tells us) and furthermore to $L^1$ (even though $P_\pm(L^1) \not\subset L^1$). Next, given a measurable function $\psi$ on $T$, we write

$$T_\psi f := P_+(\psi f) \quad \text{and} \quad H_\psi f := P_-(\psi f),$$

whenever $f \in H^1$ and $\psi f \in L^1$. The mapping $T_\psi$ (resp., $H_\psi$) is called the Toeplitz (resp., Hankel) operator with symbol $\psi$. 


In the special case where $\psi$ is analytic (i.e., $\psi \in H^1$), $T_\psi$ reduces to the multiplication map $f \mapsto f\psi$, defined at least on $H^\infty$. The Toeplitz operators with symbols in $H^1$ are said to be \textit{coanalytic}. It is also worth mentioning that the model subspace $K_\theta^p$ (where $p \geq 1$) or $K_{\ast \theta}$, with $\theta$ an inner function, is precisely the kernel of the coanalytic Toeplitz operator $T_\theta$ acting on $H^p$ or BMOA.

Because Toeplitz and Hankel operators were among Cora Sadosky’s best beloved mathematical creatures, their appearance in this survey seems to be appropriate (and is, anyway, far from incidental to the subject matter).

We conclude this introduction with a brief outline of the rest of the paper. In Sections 2 and 3, we look at certain smooth analytic spaces $X$ and seek to characterize the pairs $(f, \theta)$, with $f \in X$ and $\theta$ inner, which satisfy

$$f\theta \in X. \quad (1.2)$$

Sometimes it is more natural to replace $(1.2)$ by

$$f\theta^k \in X \text{ for all } k \in \mathbb{N}, \quad (1.3)$$

and we are led to consider some other related conditions as well. In Section 2, the role of $X$ is played by the analytic Lipschitz–Zygmund spaces $A^\alpha$ (see the beginning of that section for definitions), and the pairs $(f, \theta)$ with property $(1.3)$ are then explicitly described by a certain \textit{smallness condition}, to be imposed on $|f|$ near the singularities of $\theta$. Furthermore, the same smallness condition ensures that the multiplication operator $g \mapsto fg$ acts nicely on the model space $K_\theta^p$, or perhaps on $K_{\theta n}^p$ with $n$ suitably large, by improving integrability properties of the functions therein. For instance, given $1 < p < q < \infty$ and $\alpha = p^{-1} - q^{-1}$, we prove that multiplication by a function $f \in A^\alpha$ maps $K_\theta^p$ into $H^q$ if and only if it maps $\theta$ into $A^\alpha$ (so that $(1.2)$ holds with $X = A^\alpha$). The case of smaller $p$’s and larger $\alpha$’s leads to a minor complication involving $(1.3)$ in place of $(1.2)$, and $K_{\theta n}^p$ in place of $K_\theta^p$.

In Section 3, our space $X$ is chosen from among the so-called Dirichlet-type spaces. Each of these is formed by the functions $f \in H^2$ whose coefficient sequence, $\{\hat{f}(n)\}$, lies in a certain weighted $\ell^2$. An important special case is the classical \textit{Dirichlet space} $\mathcal{D}$, the set of analytic functions $f$ on $\mathbb{D}$ whose derivative, $f'$, is square integrable over $\mathbb{D}$ with respect to the normalized area measure $A$; the (semi)norm $\|f\|_\mathcal{D}$ is then defined to be $(\int_\mathbb{D} |f'|^2 dA)^{1/2}$. Among other things we recover, for $f \in \mathcal{D}$ and $\theta$ inner, the identity

$$\|f\theta\|_\mathcal{D}^2 = \|f\|_\mathcal{D}^2 + \int_\mathbb{T} |f|^2|\theta'|dm, \quad (1.4)$$

which forms part of Carleson’s celebrated formula from [4]. Moreover, we obtain similar – but more sophisticated – formulas for general Dirichlet-type spaces; these yield the smallness conditions on $f$ (in relation to $\theta$) that are responsible for the interplay between the two factors in $(1.2)$, for the current choices of $X$. When $X = \mathcal{D}$, the corresponding smallness condition reads $\int_\mathbb{T} |f|^2|\theta'|dm < \infty$, as readily seen from $(1.4)$. Our approach to $(1.4)$ is based on the fact that the quantity $\|f\theta\|_\mathcal{D}$ coincides with the Hilbert–Schmidt norm of the Hankel operator $H\overline{\theta}^*\overline{g}$ acting from $H^2$ to $H_\theta^2$ (and similarly for $f$ in place of $f\theta$). Now let $\{g_n\}$ be an orthonormal basis.
in the model subspace $K_\theta$. Since $H^2 = \theta H^2 \oplus K_\theta$, the family $\{\theta z^k\}_{k \geq 0} \cup \{g_n\}$ is an orthonormal basis in $H^2$, and we may use it to compute the Hilbert–Schmidt norm of $H_{\theta}$. This gives
\[ \|f_\theta\|_2^2 = \sum_{k \geq 0} \|H_{\theta}(\theta z^k)\|_2^2 + \sum_n \|H_{\theta}g_n\|_2^2, \]
and a further calculation shows that the two sums above reduce to the two terms on the right-hand side of \((1.4)\). A modification of the same technique allows us to handle the case of a generic Dirichlet-type space.

In Section 4, we consider coanalytic Toeplitz operators on the model subspace $K_{\ast \theta}$, and we obtain a criterion for such an operator to act boundedly from $K_{\ast \theta}$ to a given analytic space $X$, under certain assumptions on the latter. Precisely speaking, the spaces $X$ that arise here naturally are those which enjoy the $K$-property of Havin. In other words, it will be assumed that every Toeplitz operator $T_h$ with $h \in H^\infty$ maps $X$ boundedly into itself and satisfies $\|T_h\|_{X \to X} \leq \text{const} \cdot \|h\|_\infty$. This property was introduced by Havin in [17], where he also verified it for a number of smooth analytic spaces. (It was further observed in [17] that every space $X$ with the $K$-property admits division by inner factors: whenever $f \in X$ and $I$ is an inner function such that $f/I \in H^1$, it follows that $f/I \in X$.) Now, the appearance of the $K$-property in connection with model subspaces of BMOA seems to reveal yet another link between the two topics of concern.

The content of Section 2 is essentially borrowed from the author’s papers [7, 8], while Sections 3 and 4 are based on [10] and [12], respectively. It seems that a bit of self-plagiarism is unavoidable – and hopefully pardonable – under the circumstances.

2. Factorization in Lipschitz–Zygmund spaces

This section deals with the Lipschitz–Zygmund spaces $\Lambda^\alpha = \Lambda^\alpha(\mathbb{T})$ and their analytic subspaces $A^\alpha$. For $0 < \alpha < \infty$, the space $\Lambda^\alpha$ is defined as the set of all (complex-valued) functions $f \in C(\mathbb{T})$ that satisfy
\[ \|\Delta^nf\|_\infty = O(|h|^\alpha), \quad h \in \mathbb{R}, \]
where $\| \cdot \|_\infty$ is the sup-norm on $\mathbb{T}$, $n$ is an integer with $n > \alpha$, and $\Delta^nf$ denotes the $n$th order difference operator with step $h$. (As usual, the difference operators $\Delta^nh$ are defined by induction: one puts $(\Delta^1f)(\zeta) := f(e^{ih}\zeta) - f(\zeta)$ and $\Delta^nh := \Delta^1\Delta^{n-1}f$.) It is well known that property \((2.1)\) does not depend on the choice of $n$, as long as $n > \alpha$, except possibly for the constant in the $O$-condition.

The corresponding analytic subspaces are
\[ A^\alpha := \Lambda^\alpha \cap H^\infty, \quad 0 < \alpha < \infty. \]
Equivalently, by a theorem essentially due to Hardy and Littlewood, $A^\alpha$ is formed by those holomorphic functions $f$ on $\mathbb{D}$ which obey the condition
\[ |f^{(n)}(z)| = O\left((1 - |z|)^{\alpha-n}\right), \quad z \in \mathbb{D}, \]
for some (and then every) integer $n$ with $n > \alpha$; here $f^{(n)}$ is the $n$th order derivative of $f$. The spaces $\Lambda^\alpha$ and $A^\alpha$ are then normed in a natural way.
The main result of this section is Theorem 2.1 below, which characterizes the pairs \((f, \theta)\), with \(f \in A^\alpha\) and \(\theta\) inner, such that \(f\) admits multiplication and/or division by every power of \(\theta\) in \(\Lambda^\alpha\). The characterization involves an explicit quantitative condition saying that \(|f(z)|\) must decay at a certain rate as \(z\) approaches the boundary along the sublevel set

\[
\Omega(\theta, \varepsilon) := \{z \in \mathbb{D} : |\theta(z)| < \varepsilon\}
\]

with \(0 < \varepsilon < 1\). Moreover, it turns out that the same decay condition provides a criterion for the multiplication operator \(T_f : g \mapsto fg\) to map the model subspace \(K^p_{\theta^n}\) continuously into \(H^q\), once the exponents are related appropriately.

**Theorem 2.1.** Suppose that \(0 < p < \infty\), \(\max(1, p) < q < \infty\), \(\alpha = p^{-1} - q^{-1}\), and \(n\) is an integer with \(np > 1\). Assume also that \(f \in A^\alpha\) and \(\theta\) is an inner function.

The following conditions are equivalent:

(i) \(f\theta^k \in \Lambda^\alpha\) for all \(k \in \mathbb{N}\).

(ii) \(f\theta^n \in \Lambda^\alpha\).

(iii) The multiplication operator \(T_f\) maps \(K^p_{\theta^n}\) boundedly into \(H^q\).

(iv) For some (or every) \(\varepsilon \in (0, 1)\), one has

\[
|f(z)| = O((1 - |z|)^\alpha) \quad \text{for} \quad z \in \Omega(\theta, \varepsilon).
\]

(v) \(f\theta^k \in A^\alpha\) for all \(k \in \mathbb{N}\).

(vi) \(f\theta^n \in A^\alpha\).

It should be noted that the set \(\Omega(\theta, \varepsilon)\) hits \(\mathbb{T}\) precisely at those points which are singular for \(\theta\). Thus, (2.3) tells us how strongly the good factor \(f\) must vanish on the bad set of the problematic (nonsmooth) factor \(\theta\) in order that the products in question be appropriately smooth.

Postponing the proof for a while, we first establish a few preliminary facts to lean upon. To begin with, we recall the Duren–Romberg–Shields theorem (see [6]) which allows us to identify \(A^\alpha\) with the dual of the Hardy space \(H^r\), where \(r = (1 + \alpha)^{-1}\), under the pairing

\[
\langle \varphi, \psi \rangle = \int_{\mathbb{T}} \varphi \overline{\psi} \, dm.
\]

For a given \(\psi \in A^\alpha\), the integral above is well defined at least when \(\varphi \in H^\infty\), and we have

\[
|\langle \varphi, \psi \rangle| \leq c_\alpha \|
\varphi
\|_r \|\psi\|_{\Lambda^\alpha}
\]

with some constant \(c_\alpha > 0\). Moreover, the norm of the functional induced by \(\psi\) on \(H^r\) is actually comparable to \(\|\psi\|_{\Lambda^\alpha}\).

The next three lemmas exploit this duality relation. The first of these was established by Havin in [17]; we also cite Shamoyan [24] in connection with part (b) below.

**Lemma 2.2.** Let \(0 < \alpha < \infty\).

(a) If \(h \in H^\infty\), then the Toeplitz operator \(T^*_h\) maps the space \(A^\alpha\) boundedly into itself, with norm at most \(\text{const} \cdot \|h\|_\infty\).
(b) If \( f \in H^1 \) and \( \theta \) is an inner function such that \( f \theta \in A^\alpha \), then \( f \in A^\alpha \) and 
\[ \| f \|_{\Lambda^\alpha} \leq \text{const} \cdot \| f \theta \|_{\Lambda^\alpha}. \]

The constants are allowed to depend only on \( \alpha \).

In Havin’s terminology, statements (a) and (b) can be rephrased by saying that \( A^\alpha \) has the \( K \)-property and the (weaker) \( f \)-property, respectively. To prove (a), one notes that \( T_\bar{\theta} \) is the adjoint of the multiplication operator \( T_h : g \mapsto gh \), which is obviously bounded on \( H^r \) with norm at most \( \| h \|_\infty \). To deduce (b) from (a), observe that \( f = T_\bar{\theta}(f \theta) \).

**Lemma 2.3.** Suppose that \( 0 < p < \infty, \max(1, p) < q < \infty, \) and \( \alpha = p^{-1} - q^{-1} \). If \( f \in A^\alpha \), then the Hankel operator \( H_{\bar{\theta}} \), defined by
\[ H_{\bar{\theta}}g = P_-(\bar{f} g), \quad g \in H^\infty, \]
can be extended to a bounded linear operator mapping \( H^p \) into \( \overline{H^q} \).

**Proof.** Put \( r = (1 + \alpha)^{-1} \) and \( q' = q/(q - 1) \). Given \( g \in H^\infty \) and \( h \in H^q \), we have
\[ \left| \int \tau(H_{\bar{\theta}}g) h \, dm \right| = \left| \int \tau P_-(\bar{f} g) \cdot h \, dm \right| = \left| \int \bar{f} g h \, dm \right| \leq c_\alpha \| f \|_{\Lambda^\alpha} \| gh \|_r \leq c_\alpha \| f \|_{\Lambda^\alpha} \| g \|_p \| h \|_{q'}. \]

Here, the last two inequalities rely on the Duren–Romberg–Shields duality theorem and on Hölder’s inequality. Taking the supremum over the unit-norm functions \( h \) in \( H^q \), we obtain
\[ \| H_{\bar{\theta}}g \|_q \leq c_\alpha \| f \|_{\Lambda^\alpha} \| g \|_p, \]
which proves the required result. \( \square \)

**Lemma 2.4.** Suppose that \( 0 < p < \infty, \max(1, p) < q < \infty, \) and \( \alpha = p^{-1} - q^{-1} \). Further, let \( f \in H^2 \) and let \( \theta \) be an inner function. If \( P_-(f \bar{\theta}) \in \Lambda^\alpha \), then the operator \( T_{\bar{\theta}}|_{K^p_\theta} \) can be extended to a bounded linear operator acting from \( K^p_\theta \) to \( H^q \).

**Proof.** Given \( g \in K^\infty_{\bar{\theta}} \), put \( h := \bar{\theta} \bar{g} \theta \) (so that \( h \in H^\infty \)) and \( \psi := P_-(\bar{f} \bar{\theta}) \). The elementary identity
\[ P_\tau F = zP_-(z\bar{F}), \quad F \in L^2, \]
shows that \( T_{\bar{\theta}}g = zH\psi h \). Using Lemma 2.3, we get
\[ \| T_{\bar{\theta}}g \|_q = \| H_{\bar{\psi}} h \|_q \leq \text{const} \cdot \| \psi \|_{\Lambda^\alpha} \| h \|_p = \text{const} \cdot \| \psi \|_{\Lambda^\alpha} \| g \|_p, \]
which completes the proof. \( \square \)

As a final preliminary result, we list some facts about the so-called Carleson curves associated with an inner function; see [16, Chapter VIII] for a proof.

**Lemma 2.5.** Given an inner function \( \theta \) and a number \( \varepsilon \in (0, 1) \), there exists a countable (possibly finite) system \( \Gamma_\varepsilon = \Gamma_\varepsilon(\theta) \) of simple closed rectifiable curves in \( \mathbb{D} \cup \mathbb{T} \) with the following properties.

(a) The interiors of the curves in \( \Gamma_\varepsilon \) are pairwise disjoint; the intersection of each of these curves with the circle \( \mathbb{T} \) has zero length.
(b) One has $\eta < |\theta| < \varepsilon$ on $\Gamma_\varepsilon \cap \mathbb{D}$ for some positive $\eta = \eta(\varepsilon)$.

(c) The arclength $|dz|$ on $\Gamma_\varepsilon \cap \mathbb{D}$ is a Carleson measure, i.e., $H^1 \subset L^1(\Gamma_\varepsilon, |dz|)$; moreover, the norm of the corresponding embedding operator is bounded by a constant $N(\varepsilon)$ depending only on $\varepsilon$.

(d) For every $F \in H^1$, the equality

$$\int_\Gamma F/\theta \, dz = \int_{\Gamma_\varepsilon} F/\theta \, dz$$

holds true, provided that the curves in the family $\Gamma_\varepsilon$ are oriented appropriately.

Now we are in a position to prove our main result in this section.

**Proof of Theorem 2.1** The implications (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (vi) being obvious, our plan is to show that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) $\Rightarrow$ (v) and also that (vi) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (iii). Write $u = \theta^n$ and let $g \in K^\infty_\theta$. Note that (2.4)

$$Tf g = T^\theta g + H^\theta g.$$

Since $f \in A^\alpha$, Lemma 2.3 tells us that

$$\|H^\theta g\|_q \leq c_\alpha \|f\|_{\Lambda^\alpha} \|g\|_p.$$  

Now, since $f^\theta \in \Lambda^\alpha$ by (ii), it follows that $P_-(f^\theta) \in \Lambda^\alpha$ (indeed, the operators $P_+$ and $P_-$ are known to map $\Lambda^\alpha$ into itself), and Lemma 2.4 gives

$$\|T^\theta g\|_q \leq c_\alpha \|P_-(f^\theta)\|_{\Lambda^\alpha} \|g\|_p.$$  

The last two inequalities, together with (2.4), imply

$$\|Tg\|_q \leq \text{const} \cdot \|g\|_p,$$

where the constant does not depend on $g$. Obviously,

$$\|Tg\|_q = \|fg\|_q = \|T^\theta g\|_q,$$

and since $K^\infty_u$ is dense in $K^p_u$, we conclude that $T^\theta$ is a bounded operator from $K^p_u$ to $H^q$.

(iii) $\Rightarrow$ (iv). Fix $z \in \mathbb{D}$ and consider the reproducing kernel $k_z$ (for $K^\varnothing_\theta$), given by

$$k_z(\zeta) = \frac{1 - \theta(z)\theta(\zeta)}{1 - \frac{z}{\zeta}}.$$  

Since $k_z \in K^\infty_\theta$, it follows that $k^n_z \in K^\infty_\theta(\subset K^p_{\theta^n})$; indeed,

$$k^n_z \theta^n = (k_z \theta)^n \in K^\infty_\theta.$$  

Therefore, by (iii),

$$\|fk^n_z\|_q \leq \text{const} \cdot \|k^n_z\|_p.$$  

In order to derive further information from this inequality, we now estimate its right-hand side from above, and the left-hand side from below. The elementary estimate

$$\int_\varnothing \frac{dm(\zeta)}{|\zeta - z|^\gamma} \leq \frac{C_\gamma}{(1 - |z|)^{\gamma - 1}} \quad (\gamma > 1)$$  

In view of (2.6) and (2.7), inequality (2.5) now yields

\begin{equation}
\|k^n_{\zeta}\|_p = \left( \int_T \left| \frac{1 - \bar{\theta}(\zeta)\theta(\zeta)}{1 - \bar{\zeta}} \right|^{np} dm(\zeta) \right)^{1/p} \\
\leq 2^n \left( \int_T \frac{dm(\zeta)}{|\zeta - z|^{np}} \right)^{1/p} \leq \frac{\text{const}}{(1 - |z|)^{n-1/p}},
\end{equation}

since \( np > 1 \).

Now let \( F \) stand for the outer factor of \( f \). Using the Cauchy integral formula, we get

\begin{equation}
\|fk^n_{\zeta}\|_q = \left( \int_T |F(\zeta)|^q \left| \frac{1 - \bar{\theta}(\zeta)\theta(\zeta)}{1 - \bar{\zeta}} \right|^{nq} dm(\zeta) \right)^{1/q} \\
\geq \left| \int_T F^q(\zeta) \frac{(1 - \bar{\theta}(\zeta)\theta(\zeta))^{nq} dm(\zeta)}{(1 - \bar{\zeta})^{nq-1} - z\zeta} \right|^{1/q} \\
= \left| F(z) \right|^q \frac{(1 - |\theta(z)|^{2})^{nq} \left(1 - |\bar{\theta}(\zeta)|^{2} \right)^{nq-1}}{(1 - |z|^2)^{n-1/q}} \\
\geq \text{const} \cdot |f(z)| \frac{(1 - |\theta(z)|)^n}{(1 - |z|)^{n-1/q}}.
\end{equation}

In view of (2.6) and (2.7), inequality (2.5) now yields

\[ |f(z)| \cdot (1 - |\theta(z)|)^n \leq \text{const} \cdot (1 - |z|)^{1/p - 1/q} = \text{const} \cdot (1 - |z|)^\alpha, \]

the constant being independent of \( z \). Hence, for \( 0 < \varepsilon < 1 \), we have

\[ |f(z)| \leq \text{const} \cdot (1 - \varepsilon)^{-n}(1 - |z|)^\alpha \]

whenever \( z \in \Omega(\theta, \varepsilon) \), so that (iv) holds true.

(iv) \( \implies \) (i). We begin by showing that if (iv) is fulfilled with some \( \varepsilon \in (0, 1) \), then \( f\bar{\theta} \in \Lambda^\alpha \). Since

\[ f\bar{\theta} = T_{\bar{\theta}}f + H_{\bar{\theta}}f \]

and \( T_{\bar{\theta}}f \in \Lambda^\alpha \) (recall Lemma 2.2), it suffices to check that \( H_{\bar{\theta}}f \in \Lambda^\alpha \). To this end, we take an arbitrary function \( g \in H_0^\infty \) with \( \|g\|_r = 1 \), where \( r = (1 + \alpha)^{-1} \), and verify that the integrals \( \int_T (H_{\bar{\theta}}f)g d\mu \) are bounded in modulus by a constant independent of \( g \). This will mean that the function \( zH_{\bar{\theta}}f \) generates a continuous linear functional on \( H^r \), and hence lies in \( \Lambda^\varepsilon \). Writing \( g_1 := g/z \) and using the Carleson curves \( \Gamma_{\varepsilon} = \Gamma_{\varepsilon}(\theta) \) as described in Lemma 2.5, we obtain

\[ \int_T (H_{\bar{\theta}}f)g d\mu = \int_T f\bar{\theta}g d\mu = \left| \frac{1}{2\pi i} \int_T \frac{f g_1}{\theta} d\zeta \right| \\
= \left| \frac{1}{2\pi i} \int_{\Gamma_{\varepsilon}} \frac{f g_1}{\theta} d\zeta \right| \leq \frac{1}{2\pi} \int_{\Gamma_{\varepsilon}} \left| \frac{f}{\theta} \right|^r |g_1| r d\zeta. \]
Because $g_1$ is a unit-norm function in $H^r$, it follows easily that $|g_1(z)|^r \leq (1 - |z|)^{-1}$, whence

$$|g_1(z)|^{1-r} \leq (1 - |z|)^{-(1-r)/r} = (1 - |z|)^{-\alpha}, \quad z \in \mathbb{D}.$$ 

Plugging this into the preceding estimate and recalling that $|\theta| \geq \eta(\varepsilon)$ on $\Gamma_\varepsilon \cap \mathbb{D}$, we find that

$$(2.8) \quad \left| \int_T (H_{\bar{\theta}} f) g \, dm \right| \leq \frac{1}{2\pi \eta(\varepsilon)} \cdot \left( \sup_{z \in \Gamma_\varepsilon \cap \mathbb{D}} \frac{|f(z)|}{(1 - |z|)^{\alpha}} \right) \cdot \int_{\Gamma_\varepsilon} |g_1|^r \, |dz|.$$ 

Since $\Gamma_\varepsilon \cap \mathbb{D}$ is contained in $\Omega(\theta, \varepsilon)$, the supremum in (2.8) is finite by virtue of (iv). Also,

$$\int_{\Gamma_\varepsilon} |g_1|^r \, |dz| \leq N(\varepsilon) \cdot \int_T |g_1|^r \, dm = N(\varepsilon).$$

Taking this into account, we deduce from (2.8) that

$$\sup \left\{ \left| \int_T (H_{\bar{\theta}} f) g \, dm \right| : g \in H _0^\infty, \|g\|_r = 1 \right\} \leq C N(\varepsilon) / 2\pi \eta(\varepsilon),$$

where $C$ is the constant coming from the $O$-condition in (iv). This means that $H_{\bar{\theta}} f \in \Lambda_\alpha$, and hence $f \bar{\theta} \in \Lambda_\alpha$.

Replacing $\theta$ by $\theta^k$ and $\varepsilon$ by $\varepsilon^k$ in the above argument, we similarly verify that $f \bar{\theta}^k \in \Lambda_\alpha$ for every $k \in \mathbb{N}$.

(i) $\implies$ (v). Assuming (i), we prove first that $f \theta \in A^\alpha$, or equivalently, that

$$(f \theta)^{(n)}(z) = O((1 - |z|)^{\alpha - n}) \quad \text{as} \quad |z| \to 1^-.$$ 

For $z \in \mathbb{D}$ and almost all $\zeta \in \mathbb{T}$, we have the elementary identity

$$\theta^{n+1}(\zeta) = (\theta(\zeta) - \theta(z))^{n+1} + \sum_{k=0}^n \varphi_k(z) \theta^k(\zeta),$$

where

$$\varphi_k(z) := (-1)^{n-k} \binom{n+1}{k} \theta^{n+1-k}(z).$$

Therefore,

$$(f \theta)^{(n)}(z) = \frac{n!}{2\pi i} \int_\mathbb{T} \frac{f(\zeta) \theta(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta = \frac{n!}{2\pi i} \int_\mathbb{T} \frac{f(\zeta) \bar{\theta}^n(\zeta) \theta^{n+1}(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

$$= \frac{n!}{2\pi i} \int_\mathbb{T} (f \bar{\theta}^n)(\zeta) \left( \frac{\theta(\zeta) - \theta(z)}{\zeta - z} \right)^{n+1} \, d\zeta + \frac{n!}{2\pi i} \sum_{k=0}^n \varphi_k(z) \int_\mathbb{T} \frac{f(\zeta) \bar{\theta}^{n-k}(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

$$= \frac{n!}{2\pi i} \int_\mathbb{T} (f \bar{\theta}^n)(\zeta) \cdot \Phi_z(\zeta) \, d\zeta + \sum_{k=0}^n \varphi_k(z) \cdot (T_{\bar{\theta}^{n-k}} f)^{(n)}(z),$$

where

$$\Phi_z(\zeta) := \left( \frac{\theta(\zeta) - \theta(z)}{\zeta - z} \right)^{n+1}.$$
In view of (i), \( f^{\alpha-k} \in \Lambda^\alpha \) for \( k = 0, \ldots, n \), so that \( T_{\theta^{\alpha-k}} f \in A^\alpha \), which implies that
\[
(T_{\theta^{\alpha-k}} f)^{(n)} (z) = O((1 - |z|)^{\alpha-n}) \quad \text{as} \quad |z| \to 1^{-}.
\]
The functions \( \varphi_k(z) \) are bounded in \( \mathbb{D} \), and to prove (2.9) it remains to verify that
\[
\int T_{\theta^{n-k}} f (\zeta) \cdot \Phi_z (\zeta) \frac{d\zeta}{2\pi i} = O((1 - |z|)^{\alpha-n}) \quad \text{as} \quad |z| \to 1^{-}.
\]
Denote the integral on the left-hand side by \( I_n(z) \). Since \( \Phi_z \in H^\infty \), it follows that
\[
|I_n(z)| = \left| \int T f^\alpha (\zeta) \cdot \zeta \Phi_z (\zeta) \frac{dm(\zeta)}{\zeta - z} \right| \leq c_\alpha \|P_-(f^\alpha)\|_{\Lambda^\alpha} \|\Phi_z\|_r;
\]
here, as before, \( r = (1 + \alpha)^{-1} \). Because \( n > \alpha \), we have \( (n+1)r > 1 \) and
\[
\|\Phi_z\|_r \leq 2^{n+1} \left( \int T \left| \frac{dm(\zeta)}{\zeta - z} \right| \right)^{1/r} \leq \frac{\text{const}}{(1 - |z|)^{n+1-1/r}} = \frac{\text{const}}{(1 - |z|)^{n-\alpha}},
\]
where the constant does not depend on \( z \). Consequently,
\[
|I_n(z)| \leq \text{const} \cdot \|P_-(f^\alpha)\|_{\Lambda^\alpha} (1 - |z|)^{\alpha-n}.
\]
Since
\[
\|P_-(f^\alpha)\|_{\Lambda^\alpha} \leq C_\alpha \|f^\alpha\|_{\Lambda^\alpha} < \infty
\]
by virtue of (i), the estimate (2.10) is thereby established.
Thus, we have proved the implication
\[
(f \in A^\alpha) \& (i) \implies f\theta \in A^\alpha.
\]
Applying this inductively to \( f\theta, f\theta^2, \text{etc.} \), in place of \( f \), we eventually deduce from (i) that \( f\theta^k \in A^\alpha \) for each \( k \in \mathbb{N} \).
(vi) \implies (iii). Write \( u := \theta^n \) and suppose that \( g \in K_{\alpha}^\infty \). Then \( g\theta \in H_{0}^\infty \), and hence
\[
T f\theta g = P_-(f\theta g) = H f\theta g.
\]
Therefore,
\[
\|fg\|_q = \|Tfg\|_q = \|H_{0} f\theta g\|_q \leq c_\alpha \|fu\|_{\Lambda^\alpha} \|g\|_p,
\]
where the last inequality is due to Lemma 2.3. The quantity \( \|fu\|_{\Lambda^\alpha} \) is finite in view of (vi), and (2.11) tells us that
\[
\|fg\|_q \leq \text{const} \cdot \|g\|_p
\]
with a constant independent of \( g \). Thus, the multiplication operator \( T_f : g \mapsto fg \) maps \( K_{\alpha}^p \) boundedly into \( H^q \), as required.

If we wish to restrict ourselves to the issue of multiplying or dividing a function \( f \in A^\alpha \) by an inner function \( \theta \) (and its powers), leaving out the model subspace part, we may state the result in a more concise form as follows.
Proposition 2.6. Suppose that $0 < \alpha < \infty$, $n \in \mathbb{N}$, and $n > \alpha$. Given $f \in A^\alpha$ and an inner function $\theta$, the four statements below are equivalent.

(i) $f \theta^n \in A^\alpha$.

(ii) $f \bar{\theta} \in \Lambda^\alpha$.

(iii) $f \theta^k \in \Lambda^\alpha$ for all $k \in \mathbb{Z}$.

(iv) Condition (2.3) holds for some (or every) $\varepsilon \in (0, 1)$.

To prove this, it suffices to choose exponents $p$ and $q$ (once $\alpha$ and $n$ are given) so as to make the hypotheses of Theorem 2.1 true, and then invoke the theorem.

Remarks. (1) An alternative route to Proposition 2.6 (but not to Theorem 2.1 in its entirety) via the pseudoanalytic extension method was found by Dyn’kin [15]. A similar technique was later used by the author in [11] to completely characterize the functions in $A^\alpha$, $0 < \alpha < 1$, and in more general Lipschitz-type spaces, in terms of their moduli. (In particular, some equivalent forms of the crucial condition (2.3) came out as a corollary.) Subsequently, Pavlović [21] gave a more elementary proof of that result from [11].

(2) Some of the conditions in Theorem 2.1 and Proposition 2.6 would become simpler if we could take $n = 1$. This can be done if $1 < p < \infty$ in Theorem 2.1 or if $0 < \alpha < 1$ in Proposition 2.6, but not in the general case. Indeed, it follows from Shirokov’s work (see [28, 29]) that for each $\alpha > 1$, one can find $f \in A^\alpha$ and a Blaschke product $\theta$ such that $f/\theta \in A^\alpha$, but $f \theta \not\in A^\alpha$. This means, in particular, that conditions (i) and (ii) in Proposition 2.6 are no longer equivalent when $\alpha > 1$ and $n = 1$. The equivalence does hold under certain additional assumptions, though; these are likewise discussed in [28, 29]. See also [9, 13] for an alternative study of this phenomenon.

(3) Given $\alpha \in (0, \infty) \setminus \mathbb{Z}$, suppose that $f \in A^\alpha$ and $\theta$ is an inner function. Comparing our Proposition 2.6 with Shirokov’s earlier results (see [27, 28, 29]), one infers that condition (2.3) holds if and only if

\begin{equation}
(2.12) \quad m(\sigma(\theta)) = 0 \quad \& \quad |f(\zeta)| = O\left(\frac{1}{|\theta'(\zeta)|^\alpha}\right) \quad \text{for} \quad \zeta \in \mathbb{T} \setminus \sigma(\theta),
\end{equation}

where $\sigma(\theta)$ is the set of boundary singularities for $\theta$. The equivalence between (2.3) and (2.12) was also verified directly in [7] Section 2].

(4) Theorem 2.1 and Proposition 2.6 remain valid in the case $\alpha = 0$ (with $n = 1$ and $1 < p = q < \infty$), provided that the spaces $\Lambda^0$ and $A^0$ are taken to be BMO and BMOA, respectively. This convention might be justified by the duality relations $A^\alpha = (H^{1/(1+\alpha)})^*$ and $BMOA = (H^1)^*$. The BMO versions of the above results are discussed in more detail in [7] Section 5].

(5) In [13], we also considered the algebra $H^\infty_n := \{f : f^{(n)} \in H^\infty\}, n \in \mathbb{N}$, in place of $A^\alpha$, and we came up with an analogue of Proposition 2.6 in that context.
3. Factorization in Dirichlet-type spaces

For a sequence \( w = \{w_k\}_{k=1}^{\infty} \) of nonnegative numbers, the corresponding Dirichlet-type space \( D_w \) is formed by those functions \( f \in H^2 \) for which the quantity

\[
\|f\|_w := \left( \sum_{k=1}^{\infty} w_k |\hat{f}(k)|^2 \right)^{1/2}
\]

is finite. The case \( w_k = k \) corresponds to the classical Dirichlet space \( D = D_{\{k\}} \), the set of all functions \( f \in H^2 \) with

\[
\|f\|_D := \left( \int_D |f'(z)|^2 dA(z) \right)^{1/2} < \infty
\]

(here \( A \) is the normalized area measure on \( \mathbb{D} \)), and we have \( \| \cdot \|_D = \| \cdot \|_{\{k\}} \).

We begin by establishing a certain orthogonality relation involving Toeplitz operators on Dirichlet-type spaces.

**Theorem 3.1.** Given numbers \( 0 \leq w_1 \leq w_2 \leq \ldots \), let \( w = \{w_k\}_{k=1}^{\infty} \) and let \( \gamma = \{\gamma_k\}_{k=1}^{\infty} \) be the sequence defined by

\[
\gamma_1 = w_1, \quad \gamma_k = w_k - w_{k-1} \quad (k = 2, 3, \ldots).
\]

Suppose that \( F \in H^2 \), \( \theta \) is an inner function, and \( \{g_n\} \) is an orthonormal basis in \( K_\theta \). If \( \Phi := zT_{\theta^{-1}}F \) and \( h_n := zT_{\theta^{-1}}(Fg_n) \), then

\[
\|F\|_w^2 = \|\Phi\|_w^2 + \sum_n \|h_n\|_{\gamma}^2
\]

(the definition of \( \| \cdot \|_{\gamma} \) being similar to (3.1) above).

To keep on the safe side, we remark that sequences with unspecified index sets, which we occasionally employ, are allowed to be finite (and sometimes empty). In particular, the orthonormal basis \( \{g_n\} \) in Theorem 3.1 will be finite if and only if \( \theta \) is a finite Blaschke product.

The proof will make use of the notion of a Hilbert–Schmidt operator. Recall that, given two separable Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), a linear operator \( T : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) is said to be Hilbert–Schmidt if the quantity

\[
\|T\|_{\mathcal{S}_2} := \left( \sum_n \|Te_n\|_{\mathcal{H}_2}^2 \right)^{1/2}
\]

is finite for some (or each) orthonormal basis \( \{e_n\} \) of \( \mathcal{H}_1 \). It is well known – and easily shown – that this quantity does not actually depend on the choice of \( \{e_n\} \) and is therefore well defined. The set of all Hilbert–Schmidt operators from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) is denoted by \( \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2) \).

Also, we need a lemma that relates Hilbert–Schmidt operators to Dirichlet-type spaces. We state and prove it now, before proceeding with the proof of Theorem 3.1.
Lemma 3.2. Let $F \in H^2$. Suppose that $w = \{w_k\}_{k=1}^\infty$ and $\gamma = \{\gamma_k\}_{k=1}^\infty$ are two sequences of nonnegative numbers related by

$$w_n = \sum_{k=1}^n \gamma_k \quad (n = 1, 2, \ldots).$$

Finally, consider the multiplier map $M_\gamma$ acting by the rule

$$M_\gamma \left( \sum_{k=1}^\infty a_k z^k \right) := \sum_{k=1}^\infty \sqrt{\gamma_k} a_k z^k, \quad z \in \mathbb{T}$$

(defined initially on the set of antianalytic trigonometric polynomials $\sum k a_k z^k$). Then the operator $M_\gamma H_F$ belongs (or has an extension belonging) to $\mathfrak{S}_2(H^2, H^2_0)$ if and only if $F \in D_w$. Moreover,

$$\|M_\gamma H_F\|_{\mathfrak{S}_2} = \|F\|_w.$$  

Proof. Since $\{z^n\}_{n=0}^\infty$ is an orthonormal basis in $H^2$, we have

$$\|M_\gamma H_F\|_{\mathfrak{S}_2}^2 = \sum_{n=0}^\infty \|M_\gamma H_F z^n\|_2^2,$$

where $\| \cdot \|_2$ is the usual $L^2$-norm. Letting $a_n := \hat{F}(n)$, we find that

$$H_F z^n = \sum_{k=1}^\infty \bar{a}_{n+k} z^k,$$

whence

$$M_\gamma H_F z^n = \sum_{k=1}^\infty \sqrt{\gamma_k} \bar{a}_{n+k} z^k,$$

and, by the Parseval identity,

$$\|M_\gamma H_F z^n\|_2^2 = \sum_{k=1}^\infty \gamma_k |a_{n+k}|^2.$$

Plugging this into (3.7) and recalling (3.4), we obtain

$$\|M_\gamma H_F\|_{\mathfrak{S}_2}^2 = \sum_{n=0}^\infty \sum_{k=1}^\infty \gamma_k |a_{n+k}|^2 = \sum_{j=1}^\infty \gamma_j |a_j|^2 \sum_{k=1}^j \gamma_k$$

$$= \sum_{j=1}^\infty w_j |a_j|^2 = \|F\|_w^2,$$

which proves (3.6) and the lemma. \qed

Proof of Theorem 3.1. Let $M_\gamma$ be the multiplier map defined by (3.5). From Lemma 3.2 we know that

$$\|F\|_w^2 = \|M_\gamma H_F\|_{\mathfrak{S}_2}^2.$$
Consider the functions $G_n$ defined (a.e. on $\mathbb{T}$) by $G_n := \bar{z}g_n\theta$. Since $\{g_n\}$ is an orthonormal basis in $K_\theta$, the same is true for $\{G_n\}$ (indeed, the map $f \mapsto \bar{z}f\theta$ is an antilinear isometry of $K_\theta$ onto itself). Furthermore, since $H^2 = \theta H^2 \oplus K_\theta$, the family $\{\theta z^n\}_{n=0}^\infty \cup \{G_n\}$ forms an orthonormal basis in $H^2$, and we may use it to compute the Hilbert–Schmidt norm in (3.8). In this way we obtain

$$(3.9) \quad \|M_\gamma H F\|_{\mathcal{S}_2}^2 = \sum_{n=0}^{\infty} \|M_\gamma H F(\theta z^n)\|_2^2 + \sum_n \|M_\gamma H F G_n\|_2^2 = S_1 + S_2,$$

where $S_1$ and $S_2$ denote the two preceding sums, in the same order. The elementary identity

$$(3.10) \quad P_- \varphi = \bar{z}P_+(\bar{z}\varphi), \quad \varphi \in L^2,$$

yields

$$S_1 = \sum_{n=0}^{\infty} \|M_\gamma H F(\theta z^n)\|_2^2 = \|M_\gamma H F\|_{\mathcal{S}_2}^2 = \|\Phi\|_w^2,$$

where the last equality relies on Lemma 3.2.

Another application of (3.10) gives

$$H F G_n = P_-(\bar{F}z^n) = P_-(\bar{F}\theta z^n) = P_-(\bar{F}\theta) \cdot z^n = P_-(\bar{F}z^n) = H F z^n.$$

Thus,

$$(3.11) \quad S_2 = \sum_n \|M_\gamma H F G_n\|_2^2 = \sum_n \|h_n\|_2^2 = \|\Phi\|_w^2,$$

where the last equality relies on Lemma 3.2.

As a consequence of Theorem 3.1, we now deduce a result of Korenb lum and Fa˘ıvyshevski˘ı concerning the action of certain Toeplitz operators on Dirichlet-type spaces. (In all fairness, their original theorem in [19] gives a bit more than our Corollary 3.3 below. Alternative routes to that result can be found in [18] and [22].) To state it, we need a minor modification of the $\|\cdot\|_w$ norm. Namely, given a sequence $\{\gamma_k\}$,
Let \( v = \{v_n\}_{n=0}^\infty \) be a nondecreasing sequence of positive numbers, and let \( \theta \) be an inner function. Then, for every \( f, g \in H^2 \), we have

\[
\|T_\theta f\|_{v,0} \leq \|f\|_{v,0}
\]

and

\[
\|g\|_{v,0} \leq \|g\theta\|_{v,0}.
\]

**Proof.** Put \( F := zf \) and define \( \Phi \) as in Theorem 3.1, so that

\[
\Phi = zT_\theta F = zT_\theta f.
\]

For \( n = 1, 2, \ldots \), let \( w_n = v_{n-1} \) and \( w = \{w_n\}_{n=1}^\infty \). Theorem 3.1 implies that \( \|\Phi\|_w \leq \|F\|_w \). Observing that \( \|\Phi\|_w = \|T_\theta f\|_{v,0} \) and \( \|F\|_w = \|f\|_{v,0} \), we arrive at (3.13). To prove (3.14), it suffices to apply (3.13) with \( f = g\theta \). \( \square \)

The next fact is likewise a straightforward consequence of Theorem 3.1.

**Theorem 3.4.** Let \( w = \{w_k\}_{k=1}^\infty \) be a nondecreasing sequence with \( w_1 \geq 0 \), and let \( \gamma = \{\gamma_k\}_{k=1}^\infty \) be defined by (3.2). If \( f \in H^2 \), \( \theta \) is an inner function, and \( \{g_n\} \) is an orthonormal basis in \( K_\theta \), then

\[
\|f\theta\|_w^2 = \|f\|_w^2 + \sum_n \|zf g_n\|_w^2.
\]

**Proof.** Put \( F := f\theta \), and define \( \Phi \) and \( h_n \) as in Theorem 3.1. We have then

\[
\Phi = zT_\theta(F\theta) = zT_\theta f = f - f(0),
\]

whence \( \|\Phi\|_w = \|f\|_w \). Also,

\[
h_n = zT_\theta(f\theta g_n) = zf g_n.
\]

The formula (3.3) therefore reduces to (3.15), and the proof is complete. \( \square \)

In some special cases, Theorem 3.4 can be used to derive a more explicit form of the (nonnegative) “discrepancy term”

\[
R_w(f, \theta) := \|f\theta\|_w^2 - \|f\|_w^2.
\]

One such case is pointed out in Theorem 3.5 below. Before stating the result, we need to recall some basic facts about angular derivatives.

Given a function \( \varphi \in H^\infty \) with \( \|\varphi\|_\infty = 1 \), we say that \( \varphi \) has an angular derivative (in the sense of Carathéodory) at a point \( \zeta \in \mathbb{T} \) if both \( \varphi \) and \( \varphi' \) have nontangential limits at \( \zeta \), the former of these being of modulus 1. (The two limits are then denoted by \( \varphi(\zeta) \) and \( \varphi'(\zeta) \), respectively.) The classical Julia–Carathéodory theorem (see [2]...
Chapter VI], [8 Chapter I] or [23 Chapter VI]) asserts that this happens if and only if

\[(3.17) \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty.\]

And if (3.17) holds, the theorem tells us also that \(\varphi'(\zeta)\) coincides with the limit of the difference quotient

\[\frac{\varphi(z) - \varphi(\zeta)}{z - \zeta}\]

as \(z \to \zeta\) nontangentially. Moreover, \(|\varphi'(\zeta)|\) will then agree with the value of the (unrestricted) \(\liminf\) in (3.17), and this remains true if \(\liminf\) is replaced by the corresponding nontangential limit.

Finally, if \(\theta = BS\) is an inner function (with \(B\) a Blaschke product and \(S\) singular), then

\[(3.18) \quad |\theta'(\zeta)| = \sum_j \frac{1 - |a_j|^2}{|\zeta - a_j|^2} + 2 \int_\mathbb{T} \frac{d\mu(\eta)}{|\zeta - \eta|^2}, \quad \zeta \in \mathbb{T},\]

where \(\{a_j\}\) is the zero sequence of \(B\) and \(\mu\) is the singular measure associated with \(S\). This formula can be found in [1]; it holds for every point \(\zeta\) of \(\mathbb{T}\), with the convention that \(|\theta'(\zeta)| = \infty\) whenever \(\theta\) fails to possess an angular derivative at \(\zeta\).

**Theorem 3.5.** Let \(\sigma\) be a positive Borel measure on \([0, 1]\) with \(\int_{[0,1]} x^2 d\sigma(x) < \infty\). Put

\[\gamma_k := \int_{[0,1]} x^{2k} d\sigma(x), \quad k = 1, 2, \ldots,\]

and define the sequence \(w = \{w_n\}_{n=1}^\infty\) by (3.4). If \(f \in H^2\) and \(\theta\) is an inner function, then

\[(3.19) \quad \|f\theta\|^2_w = \|f\|^2_w + \int_\mathbb{T} dm(\zeta) \int_{[0,1]} r^2 |f(r\zeta)|^2 \frac{1 - |\theta(r\zeta)|^2}{1 - r^2} d\sigma(r).\]

Here the value of \((1 - |\theta(r\zeta)|^2)/(1 - r^2)\) at \(r = 1\) is interpreted as \(|\theta'(\zeta)|\), the modulus of the angular derivative of \(\theta\) at \(\zeta\).

The proof will rely on Theorem 3.4 and on the following lemma.

**Lemma 3.6.** Let \(\theta\) be an inner function, and let \(\{g_n\}\) be an orthonormal basis in \(K_\theta\). Then

\[(3.20) \quad \sum_n |g_n(z)|^2 = \frac{1 - |\theta(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}.\]

Furthermore, if \(\zeta \in \mathbb{T}\) is a point at which the limits \(\lim_{r \to 1^-} g_n(r\zeta) =: g_n(\zeta)\) exist for all \(n\), then

\[(3.21) \quad \sum_n |g_n(\zeta)|^2 = |\theta'(\zeta)|.\]
To prove the lemma, consider the reproducing kernel
\[ k_z(w) = \frac{1 - \theta(z)\theta(w)}{1 - \bar{z}w} \]
of \( K_\theta \) and use Parseval’s identity to get
\[ \sum_n |g_n(z)|^2 = \sum_n |\langle g_n, k_z \rangle|^2 = \|k_z\|^2 = k_z(z) = \frac{1 - |\theta(z)|^2}{1 - |z|^2} \]
for \( z \in \mathbb{D} \). This yields (3.20), which in turn implies (3.21) upon putting \( z = r\zeta \) and passing to the limit as \( r \to 1^- \).

**Proof of Theorem 3.5.** We may assume that \( f \in \mathcal{D}_w \), since otherwise both sides of (3.19) equal \( \infty \). By Theorem 3.4, the “discrepancy term” (3.16) is given by
\[ R_w(f, \theta) = \sum_n \|zf g_n\|_{\gamma}^2, \]
where \( \gamma = \{\gamma_k\}_{k=1}^\infty \) and \( \{g_n\} \) is some (no matter which) orthonormal basis in \( K_\theta \). This said, we proceed by considering two special cases.

**Case 1:** \( \sigma \) has no atom at 1. We may think of the disk \( \mathbb{D} = \{r\zeta : r \in [0, 1), \zeta \in \mathbb{T}\} \) as of a measure space endowed with the product measure \( \sigma \times m =: \nu \). The monomials \( z^k (k = 1, 2, \ldots) \) are then mutually orthogonal in \( L^2(\mathbb{D}) \) and have norms \( \sqrt{\gamma_k} \).
Therefore, for a function \( h(z) = \sum_{k=1}^\infty \hat{h}(k) z^k \) in \( zH^1 \), we have
\[ \|h\|^2_{L^2(\mathbb{D}, \nu)} = \sum_{k=1}^\infty \gamma_k |\hat{h}(k)|^2 = \|h\|^2_{\gamma}. \]
Applying this to \( h_n := zf g_n \) gives
\[ \|h_n\|^2_{\gamma} = \|h_n\|^2_{L^2(\mathbb{D}, \nu)} = \int_{\mathbb{T}} \int_{[0,1]} r^2 |f(r\zeta)|^2 |g_n(r\zeta)|^2 d\sigma(r). \]
Consequently, in view of (3.22),
\[ R_w(f, \theta) = \sum_n \|h_n\|^2_{\gamma} = \int_{\mathbb{T}} \int_{[0,1]} r^2 |f(r\zeta)|^2 \sum_n |g_n(r\zeta)|^2 d\sigma(r). \]
By Lemma 3.6
\[ \sum_n |g_n(r\zeta)|^2 = \frac{1 - |\theta(r\zeta)|^2}{1 - r^2}, \]
and so (3.23) reduces to
\[ R_w(f, \theta) = \int_{\mathbb{T}} \int_{[0,1]} r^2 |f(r\zeta)|^2 \frac{1 - |\theta(r\zeta)|^2}{1 - r^2} d\sigma(r), \]
which proves (3.19).
Case 2: $\sigma$ is the unit point mass at 1. In this case, we have $\gamma_k = 1$ and $w_k = k$, so that $\| \cdot \|_{\gamma} = \| \cdot \|_2$ on $zH^2$, and $\| \cdot \|_w = \| \cdot \|_D$. Therefore, we can rewrite (3.22) in the form

$$\| f\theta \|_D^2 - \| f \|_D^2 = \sum_n \| zf_n \|_2^2 = \int_T |f(\zeta)|^2 \sum_n |g_n(\zeta)|^2 dm(\zeta).$$

Combining this with (3.21), we finally obtain

$$\| f\theta \|_D^2 - \| f \|_D^2 = \int_T |f(\zeta)|^2 \theta'(\zeta)dm(\zeta),$$

which coincides with (3.19) under the current hypothesis on $\sigma$.

The general case being a combination of Cases 1 and 2, the required result follows. \qed

Remark. Recalling the identity (3.18) and plugging it into (3.24), we find that

$$\| f\theta \|_D^2 = \| f \|_D^2 + \int_T |f(\zeta)|^2 \left( \sum_j \frac{1 - |a_j|^2}{|\zeta - a_j|^2} + 2 \int_T \frac{d\mu(\eta)}{|\zeta - \eta|^2} \right) dm(\zeta)$$

(here, as before, $\{a_j\}$ is the zero sequence of $\theta$, and $\mu$ is the associated singular measure). This was established by Carleson in [4]. In fact, the formula given there is a combination of (3.25) and an explicit expression for the Dirichlet integral $\| f \|_D^2$ of an outer function $f$.

4. Model subspaces in $\text{BMOA}$

It has been noticed that various smoothness properties of an inner function $\theta$, if available, tend to be inherited (typically, in a weaker form) by functions in $K^p_\theta$. This phenomenon becomes especially pronounced when passing from $\theta$ to $K^*_\theta := K^2_\theta \cap \text{BMOA}$, the star-invariant subspace of $\text{BMOA}$, in which case no loss of smoothness usually occurs. (Of course, the smoothness property in question should not be too strong – it should not even imply continuity – if we want a nontrivial inner function to have it.) A result to that effect will appear as Corollary 4.4 below; we shall deduce it from a more general theorem concerning the action of a coanalytic Toeplitz operator $T_g$, with $g \in H^1$, on $K^*_\theta$. However, the very meaning of the expression $T_g f$ (with $f \in K^*_\theta$) is not immediately clear, since the product $fg$ need not be integrable. The following proposition will clarify the situation.

**Proposition 4.1.** Given $f \in K^*_\theta$ and $g \in H^1$, there exists a function $\Phi \in \cap_{0<p<1} H^p$ such that

$$\| T_{\gamma_n} f - \Phi \|_p \to 0$$

for every $p \in (0,1)$ and every sequence $\{g_n\} \subset H^2$ with $\| g_n - g \|_1 \to 0$.

This (obviously unique) function $\Phi$ is then taken to be $T_g f$, the image of $f$ under the Toeplitz operator $T_g$.

The proof relies on the following lemma due to Cohn (see Lemma 3.2 in [5, p. 731]), which in turn results from an application of the ($H^1, \text{BMOA}$) duality.
Lemma 4.2. Let $\theta$ be inner, and let $f \in K_\theta$. Then $f = P_+(\bar{z}\psi\theta)$ for a function $\psi \in H^\infty$. Furthermore, $\psi$ may be chosen so that $\|f\|_* = \|\psi\|_\infty$.

Here and below, $\| \cdot \|_*$ is the dual space norm on BMOA induced by $H^1$.

Proof of Proposition 4.4. Let $f \in K_\theta$, $g \in H^1$, and suppose $\{g_n\}$ is a sequence of $H^2$-functions with $\|g_n - g\|_1 \to 0$. We have then

$$T_{\bar{g}_n}f = P_+((\bar{g}_n)P_+(\bar{z}\psi\theta)) = P_+\left(\bar{g}_n\bar{z}\psi\theta\right),$$

where $\psi$ is related to $f$ as in Lemma 4.2. Now put

$$\Phi := P_+\left(\bar{g}\bar{z}\psi\theta\right).$$

This definition makes sense, since $P_+$ is applied to an $L^1$-function; besides, it does not depend on the choice of $\psi$. (Indeed, if $\psi_1$ and $\psi_2$ are both eligible in the sense of Lemma 4.2 then $\psi_1 - \psi_2 \in \theta H^\infty$.) And since $P_+$ is a continuous mapping from $L^1$ to every $H^p$ with $0 < p < 1$ (cf. [16, p. 128]), we conclude that $\Phi \in H^p$ and $\|T_{\bar{g}_n}f - \Phi\|_p \to 0$ for any such $p$. □

Now suppose $X$ is a Banach space of analytic functions on the disk, with $X \subset H^1$. We say that $X$ is a $K$-space if, for each $\psi \in H^\infty$, the Toeplitz operator $T_\psi$ acts boundedly from $X$ to itself, with norm at most $\text{const} \cdot \|\psi\|_\infty$. (This is essentially equivalent to saying that $X$ enjoys the so-called $K$-property of Havin. The latter was defined in [17] by the formally weaker condition that $T_\psi(X) \subset X$, for all $\psi \in H^\infty$, but the norm estimate is usually automatic.)

Following [17], we remark that $X$ will be a $K$-space provided it is (isomorphic to) the dual of some Banach space $Y$, consisting of analytic functions on $\mathbb{D}$ and satisfying the conditions

(a) $H^\infty \cap Y$ is dense in $Y$, and

(b) for each $\psi \in H^\infty$, the multiplication operator $f \mapsto f\psi$ acts boundedly from $Y$ to itself, with norm at most $\text{const} \cdot \|\psi\|_\infty$.

(It is understood that the pairing between $X$ and $Y$ is given by $\langle f, g \rangle := \int_\mathbb{T} f\bar{g} \, dm$, which is meaningful at least for $f \in H^\infty \cap Y$ and $g \in X$.) The Toeplitz operator $T_\psi : X \to X$ is then the adjoint of the multiplication map in (b), which justifies our claim.

As examples of $K$-spaces, we list the following:

- $H^p$ with $1 < p < \infty$,
- the Hardy–Sobolev spaces $H^{p,n} := \{f \in H^p : f^{(n)} \in H^p\}$ with $1 \leq p < \infty$ and $n \geq 1$,
- BMOA, and more generally, $\text{BMOA}^{(n)} := \{f \in H^1 : f^{(n)} \in \text{BMOA}\}$ with $n \geq 0$,
- the Dirichlet-type spaces $D_w := \{f \in H^2 : \sum_{n \geq 1} w_n |\hat{f}(n)|^2 < \infty\}$ associated with nondecreasing sequences $w = \{w_n\}$ of positive numbers,
- the analytic Besov spaces $B^{s}_{p,q}$ with $s > 0$, $p \geq 1$, $q \geq 1$, and in particular
- the classical Lipschitz–Zygmund spaces $A^\alpha := B^{\alpha,\infty}_\infty$ with $0 < \alpha < \infty$. 
We recall that $B_{p,q}^s$ is defined as the set of those analytic $f$ on $D$ for which the function
\begin{equation}
(4.1) \quad r \mapsto (1-r)^{n-s} \| f_r^{(n)} \|_p
\end{equation}
is in $L^q$ over the interval $(0,1)$ with respect to the measure $dr/(1-r)$; here $n$ is some (any) fixed integer with $n > s$ and $f_r^{(n)}(\zeta) := f^{(n)}(r\zeta)$.

For most of the spaces considered, the $K$-property has been established by means of a duality argument, as outlined above. We refer to [17], where this is done for $A^\alpha$ and some special cases of Hardy–Sobolev and Besov spaces; to [25, 26] for general $H^{p,n}$ and $B_{p,q}^s$ classes, as well as for BMOA$(n)$; and finally to any of [18, 19, 22] in connection with $D_w$ spaces.

As further examples of $K$-spaces, we mention $K^p_\theta (1 < p < \infty)$ and $K^\infty_\theta$. Indeed, for $g \in H^\infty$, one verifies the inclusion $T_\bar{g}(K^p_\theta) \subset K^p_\theta$ by noting that $K^p_\theta$ is the kernel of the Toeplitz operator $T_\theta : H^p \to H^p$, which commutes with $T_\bar{g}$. Then one deduces that $T_\bar{g}(K^\infty_\theta) \subset K^\infty_\theta$, recalling that $K^\infty_\theta = K^2_\theta \cap \text{BMOA}$ and BMOA is a $K$-space.

And, of course, the two inclusions are accompanied by the natural norm estimates: the norm of $T_\bar{g}k$ is in both cases $O(\|g\|_\infty)$, just as it happens for the containing spaces $H^p (1 < p < \infty)$ and BMOA.

The main result of this section is as follows.

**Theorem 4.3.** Let $\theta$ be an inner function, $g \in H^1$, and let $X$ be a $K$-space. The following are equivalent.

(i) $T_\bar{g}$ acts boundedly from $K^\infty_\theta$ to $X$.

(ii) $T_\bar{g}$ acts boundedly from $K^p_\theta$ to $X$.

(iii) The function $$k(z) := \frac{\theta(z) - \theta(0)}{z}$$ satisfies $T_\bar{g}k \in X$.

Moreover, the operator norms $\|T_\bar{g}\|_{K^p_\theta \to X}$ and $\|T_\bar{g}\|_{K^\infty_\theta \to X}$ are comparable to each other and to $\|T_\bar{g}k\|_X$.

In most – perhaps all – cases of interest, condition (iii) above can be further rephrased by saying that $T_\bar{g}\theta \subset X$. In fact, since $k = T_\bar{z}\theta$ and $T_\bar{z}T_\bar{g} = T_\bar{g}T_\bar{z}$, the implication $T_\bar{g}\theta \subset X \implies T_\bar{g}k \subset X$ holds whenever $X$ is a $K$-space. The converse is true provided that $1 \in X$ and $zX \subset X$; indeed,

$$T_\bar{g}\theta = \text{const} + zT_\bar{g}k.$$

In particular, we certainly have $T_\bar{g}k \subset X \iff T_\bar{g}\theta \subset X$ when $X$ is one of our smoothness classes, such as $H^{p,n}$, $B_{p,q}^s$, $A^\alpha$ or BMOA$(n)$, let alone $H^p$ and BMOA. The theorem then states that the inclusion $T_\bar{g}f \subset X$ holds for all $f \in K^\alpha_\theta$ if and only if it holds for $f = \theta$.

The next fact is obtained by applying Theorem 4.3 with $g \equiv 1$, in which case $T_\bar{g}$ reduces to the identity map.
Corollary 4.4. Given an inner function $\theta$ and a $K$-space $X$, one has
\begin{equation}
K_{\theta} \subset X \iff K_{\theta}^\infty \subset X \iff k \in X.
\end{equation}

And since the latter condition, $k \in X$, is implied by (and is usually equivalent to) saying that $\theta \in X$, the nontrivial part of (4.2) amounts to the implication
\begin{equation}
\theta \in X \implies K_{*\theta} \subset X.
\end{equation}

Proof of Theorem 4.3. The part (i.1) $\implies$ (ii.1) is trivially true, as is the inequality
$$
\|T_{\bar{g}}\|_{K_{*\theta} \to X} \leq \|T_{\bar{g}}\|_{K_{\theta} \to X}.
$$
The part (ii.1) $\implies$ (iii.1), along with the estimate
$$
\|T_{\bar{g}}\|_{K_{*\theta} \to X} \geq \frac{1}{2}\|T_{\bar{g}}k\|_X,
$$
is also obvious, since $k \in K_{\theta}^\infty$ and $\|k\|_{\infty} \leq 2$.

What remains to be proved is the implication (iii.1) $\implies$ (i.1) and its quantitative version
\begin{equation}
\|T_{\bar{g}}\|_{K_{*\theta} \to X} \leq \text{const} \cdot \|T_{\bar{g}}k\|_X.
\end{equation}

To this end, we fix $f \in K_{*\theta}$ and then invoke Lemma 4.2 to find a function $\psi \in H_{\infty}$ such that
$$
f = T_{\bar{z}\psi}\theta, \quad \|f\|_{*} = \|\psi\|_{\infty}.
$$

Using the fact that coanalytic Toeplitz operators commute (and moreover, $T_{\bar{a}}T_{\bar{b}} = T_{\bar{ab}}$ whenever $a$, $b$ and $ab$ are $H^1$-functions such that the operators involved are all well-defined), we obtain
\begin{equation}
T_{\bar{g}}f = T_{\bar{g}}T_{\bar{z}\psi}\theta = T_{\bar{\psi}}T_{\bar{g}}T_{\bar{z}}\theta = T_{\bar{\psi}}T_{\bar{g}}k.
\end{equation}

Finally, we recall that $X$ is a $K$-space to get
$$
\|T_{\bar{g}}f\|_X \leq \|T_{\bar{\psi}}\|_{X \to X}\|T_{\bar{g}}k\|_X
\leq \text{const} \cdot \|\psi\|_{\infty}\|T_{\bar{g}}k\|_X
= \text{const} \cdot \|f\|_{*}\|T_{\bar{g}}k\|_X,
$$
which readily implies (4.4). $\square$

Finally, we supplement Theorem 4.3 with the following result.

Proposition 4.5. Let $\theta$, $g$ and $k$ be as above. The operator $T_{\bar{g}}$ acts boundedly from $K_{*\theta}$ to itself if and only if $T_{\bar{g}}k \in \text{BMOA}$. In this case we also have
$$
\|T_{\bar{g}}f\|_{p} \leq C_p\|T_{\bar{g}}k\|_{*}\|f\|_{p}, \quad 1 < p < \infty,
$$
for all $f \in K_{\theta}^\infty$, so that $T_{\bar{g}}$ extends to a bounded operator on $K_{\theta}^p$.

This might be compared to the “$T(1)$-” and/or “$T(b)$-theorem” of David, Journé and Semmes (cf. [14], Chapter 5 or [30], Chapter VII), results that provide boundedness criteria for certain singular integral operators on $L^p$. Just as in those theorems,
we only have to test the operator on a single function. We also remark that the assumption $T_\theta k \in \text{BMOA}$ can be rewritten as $\bar{T}_\theta k \in \text{BMOA}$, and a sufficient condition for this to happen is that
\[
\sup \{|g(z)| : z \in \Omega(\theta, \varepsilon)\} < \infty
\]
for some $\varepsilon \in (0, 1)$, where $\Omega(\theta, \varepsilon)$ is the sublevel set defined by \((2.2)\). A proof of this last assertion can be found in [8].

Proof of Proposition 4.5. The first statement, concerning the action of $\bar{T}_\theta$ on $K_\theta^*$, is obtained by applying Theorem 4.3 with $X = \text{BMOA}$ (or $X = K_\theta^*$).

Now suppose $\bar{T}_\theta k \in \text{BMOA}$, and let $1 < p < \infty$. Given a function $f \in K_\theta^\infty$, put $\psi := \bar{z} \bar{f} \theta (= \tilde{f})$ and note that $\psi \in H^\infty$. We have then
\[
f = \bar{z} \bar{f} \theta = P_+ (\bar{z} \bar{f} \theta) = T_{\bar{z} \bar{f}} \theta,
\]
and so \((4.5)\) remains in force. Setting $h := T_\theta k$ and making use of the elementary identity
\[
P_+ F = z P_- (\bar{z} F), \quad F \in L^1,
\]
we can rewrite the resulting equality from \((4.5)\) as
\[
T_{\bar{z} \bar{f}} f = T_{\bar{z} \bar{f}} h = \bar{z} H_{\bar{z} \bar{f}} \psi.
\]
In view of Nehari’s theorem (see, e.g., [20, Part B, Chapter 1]), the assumption that $h$, and hence $zh$, is in $\text{BMOA}$ implies that the Hankel operator $H_{\bar{z} \bar{f}}$ acts boundedly from $H^p$ to $\overline{H^p}$, with norm not exceeding $C_p \|h\|_s$. Consequently,
\[
\|T_{\bar{z} \bar{f}} f\|_p = \|H_{\bar{z} \bar{f}} \psi\|_p \leq C_p \|h\|_s \|\psi\|_p = C_p \|h\|_s \|f\|_p, \quad f \in K_\theta^\infty.
\]
Finally, since $K_\theta^\infty$ is dense in $K_\theta^p$ (indeed, $K_\theta^\infty$ contains the family of reproducing kernels for $K_\theta^2$), we conclude that $T_{\bar{z} \bar{f}}$ extends to a bounded operator on $K_\theta^p$, with the same norm. The proof is complete. □

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