Q-Ising neural network dynamics: 
a comparative review of various architectures

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Abstract. This contribution reviews the parallel dynamics of Q-Ising neural networks for various architectures: extremely diluted asymmetric, layered feedforward, extremely diluted symmetric, and fully connected. Using a probabilistic signal-to-noise ratio analysis, taking into account all feedback correlations, which are strongly dependent upon these architectures the evolution of the distribution of the local field is found. This leads to a recursive scheme determining the complete time evolution of the order parameters of the network. Arbitrary $Q$ and mainly zero temperature are considered. For the asymmetrically diluted and the layered feedforward network a closed-form solution is obtained while for the symmetrically diluted and fully connected architecture the feedback correlations prevent such a closed-form solution. For these symmetric networks equilibrium fixed-point equations can be derived under certain conditions on the noise in the system. They are the same as those obtained in a thermodynamic replica-symmetric mean-field theory approach.

1. Introduction

Artificial neural networks have been widely applied to memorize and retrieve information. During the last few years there has been considerable interest in neural networks with multistate neurons (see [1] and references cited therein) in order to function as associative memories for gray-toned or coloured patterns or to allow for a more complicated internal structure in the retrieval, e.g., a distinction between background and patterns.

Here we review the dynamics of so-called Q-Ising neural networks (see the references in [2]) for arbitrary $Q$. They are built from Q-Ising spin-glasses [3, 4] with couplings

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defined in terms of patterns through a learning rule. For $Q = 2$ one finds back the Hopfield model \[5, 6\], for $Q \to \infty$ one has an analog network (see \[7\] and references therein). One of the aims of these networks is to memorize a number of patterns and find them back as attractors of the retrieval process. Consequently these networks are also interesting from the point of view of dynamical systems.

Besides a learning rule one also needs to specify an architecture indicating how the spins (=neurons) are connected with each other. Several architectures have been studied in the literature for different purposes. From a practical point of view mostly perceptrons or, more general, feedforward layered networks are used since a very long time (see, e.g., \[8\] for a history). Hopfield \[5, 6\] studied a fully connected network with symmetric couplings because it satisfies the detailed balance principle and hence a Hamiltonian can be defined. Asymmetrically diluted models \[9\] were used because their dynamics can be solved exactly and because they can learn us something about the loss of information content when some of the synaptic couplings break down.

In this contribution we review the study of the parallel dynamics of these types of network using a probabilistic approach (see, e.g., \[10, 11\]). In more detail, employing a signal-to-noise ratio analysis based on the law of large numbers (LLN) and the central limit theorem (CLT) we derive the evolution of the distribution of the local field at every time step. This allows us to obtain a recursive scheme for the evolution of the relevant order parameters in the system being, in general, the main overlap for the condensed pattern, the mean of the neuron activities and the variance of the residual overlap responsible for the intrinsic noise in the dynamics of the main overlap (sometimes called the width parameter). The details of this approach depend in an essential way on the architecture because different temporal correlations are possible.

For extremely diluted asymmetric and layered feedforward architectures recursion relations have been obtained in closed form directly for the relevant order parameters \[8, 12, 13, 14\]. This has been possible because in these types of networks there are no feedback correlations as time progresses. As a technical consequence the local field contains only Gaussian noise leading to an explicit solution.

For the parallel dynamics of networks with symmetric connections, however, things are quite different \[1, 10, 11\]. Even for extremely diluted versions of these systems \[15, 16, 17\] feedback correlations become essential from the second time step onwards, complicating the dynamics in a nontrivial way. Therefore, explicit results concerning the time evolution of the order parameters for these models have to be obtained indirectly by starting from the distribution of the local field. Technically speaking, both for the symmetrically diluted and fully connected architectures the local field contains both a discrete and a normally distributed part. The difference between the diluted and fully connected models is that the discrete part at a certain time $t$ does not involve the spins at all previous times $t - 1, t - 2, \ldots$ up to 0 but only the spins at time step $t - 1$. 
But in both cases the discrete part prevents a closed-form solution of the dynamics for the relevant order parameters. Nevertheless, the development of a recursive scheme is possible in order to calculate their complete time evolution. In this way a comparative discussion of the parallel dynamics at zero temperature for the various architectures specified above is possible.

Finally, by requiring the local field to become time-independent implying that some correlations between its Gaussian and discrete noise parts are neglected we can obtain fixed-point equations for the order parameters. It turns out that they are equivalent to the fixed-point equations obtained through a thermodynamic replica-symmetric mean-field theory approach.

At this point we remark that we do not aim for complete rigour in our derivations. From the point of view of rigorous mathematics, the Hopfield model and, in general, spin-glass theory is recognized to be an extremely difficult, if not impossible, field. For a recent overview of the modest results obtained, mostly concerning thermodynamics, we refer to [18].

The rest of this contribution is organized as follows. In Section 2 we introduce the model, its dynamics and the Hamming distance as a macroscopic measure for the retrieval quality. In Section 3 we use the probabilistic approach in order to derive a recursive scheme for the evolution of the distribution of the local field, leading to recursion relations for the order parameters. The differences between the various architectures are outlined. We do not aim for complete rigour and mostly concentrate on zero temperature. In Section 4 we discuss the evolution of the system to fixed-point attractors. Some concluding remarks are given in Section 5.

2. Q-Ising neural networks

Consider a neural network Λ consisting of N neurons which can take values σ_i from a discrete set $\mathcal{S} = \{-1 = s_1 < s_2 < \ldots < s_Q = +1\}$. The p patterns to be stored in this network are supposed to be a collection of independent and identically distributed random variables (i.i.d.r.v.), $\{\xi^\mu_i \in \mathcal{S}\}, \mu \in \mathcal{P} = \{1, \ldots, p\}$ and $i \in \Lambda$, with zero mean, $E[\xi^\mu_i] = 0$, and variance $A = \text{Var}[\xi^\mu_i]$. The latter is a measure for the activity of the patterns. We remark that for simplicity we have taken the patterns and the neurons out of the same set of variables but this is no essential restriction. Given the configuration $\sigma_\Lambda(t) \equiv \{\sigma_j(t)\}, j \in \Lambda = \{1, \ldots, N\}$, the local field in neuron $i$ equals

$$h_i(\sigma_\Lambda(t)) = \sum_{j \in \Lambda} J_{ij}(t)\sigma_j(t)$$

with $J_{ij}$ the synaptic coupling from neuron $j$ to neuron $i$. In the sequel we write the shorthand notation $h_\Lambda,i(t) \equiv h_i(\sigma_\Lambda(t))$. 
It is clear that the $J_{ij}$ explicitly depend on the architecture. For the extremely
diluted (ED), both symmetric (SED) and asymmetric (AED), and the fully
closed (FC) architectures the couplings are time-independent and the diagonal
terms are absent, i.e. $J_{ii} = 0$. The configuration $\sigma_A(t = 0)$ is chosen as input. For the layered
feedforward (LF) model the time dependence of the couplings is relevant because the set-
up of the model is somewhat different. There each neuron in layer $t$ is unidirectionally
connected to all neurons on layer $t + 1$ and $J_{ij}(t)$ is the strength of the coupling from
neuron $j$ on layer $t$ to neuron $i$ on layer $t + 1$. The state $\sigma_A(t + 1)$ of layer $t + 1$ is
determined by the state $\sigma_A(t)$ of the previous layer $t$.

In all cases the couplings are chosen according to the Hebb rule such that we can
write

$$J_{ij}^{ED} = \frac{c_{ij}}{CA} \sum_{\mu \in P} \xi_i^\mu \xi_j^\mu \quad \text{for} \quad i \neq j, \quad (2)$$

$$J_{ij}^{FC} = \frac{1}{NA} \sum_{\mu \in P} \xi_i^\mu \xi_j^\mu \quad \text{for} \quad i \neq j, \quad (3)$$

$$J_{ij}^{LF}(t) = \frac{1}{NA} \sum_{\mu \in P} \xi_i^\mu (t + 1) \xi_j^\mu(t), \quad (4)$$

with the $\{c_{ij} = 0, 1\}, i, j \in \Lambda$ chosen to be i.i.d.r.v. with distribution $\Pr\{c_{ij} = x\} = (1 - C/N)\delta_{x,0} + (C/N)\delta_{x,1}$ and satisfying $c_{ij} = c_{ji}, \quad c_{ii} = 0$ for symmetric dilution, and
$c_{ij}$ and $c_{ji}$ statistically independent (with $c_{ii} = 0$) for asymmetric dilution.

At zero temperature all neurons are updated in parallel according to the rule

$$\sigma_i(t) \to \sigma_i(t + 1) = s_k : \min_{s \in S} \epsilon_i[s|\sigma_A(t)] = \epsilon_i[s_k|\sigma_A(t)]. \quad (5)$$

We remark that this rule is the zero temperature limit $T = \beta^{-1} \to 0$ of the stochastic
parallel spin-flip dynamics defined by the transition probabilities

$$\Pr\{\sigma_i(t + 1) = s_k \in S|\sigma_A(t)\} = \frac{\exp[-\beta \epsilon_i(s_k|\sigma_A(t))]}{\sum_{s \in S} \exp[-\beta \epsilon_i(s|\sigma_A(t))]}. \quad (6)$$

Here the energy potential $\epsilon_i[s|\sigma_A]$ is defined by

$$\epsilon_i[s|\sigma_A] = -\frac{1}{2}[h_i(\sigma_A)s - bs^2], \quad (7)$$

where $b > 0$ is the gain parameter of the system. The updating rule (5) is equivalent to
using a gain function $g_b(\cdot)$,

$$g_b(x) \equiv \sum_{k=1}^{Q} s_k \left[ \theta[b(s_{k+1} + s_k) - x] - \theta[b(s_k + s_{k-1}) - x] \right] \quad (8)$$

with $s_0 \equiv -\infty$ and $s_{Q+1} \equiv +\infty$. For finite $Q$, this gain function $g_b(\cdot)$ is a step function.
The gain parameter $b$ controls the average slope of $g_b(\cdot)$. 
In order to measure the retrieval quality of the system one can use the Hamming distance between a stored pattern and the microscopic state of the network

\[ d(\xi^\mu(t), \sigma_\Lambda(t)) = \frac{1}{N} \sum_{i \in \Lambda} [\xi^\mu_i(t) - \sigma_i(t)]^2. \]  

(9)

This introduces the main overlap and the arithmetic mean of the neuron activities

\[ m^\mu_\Lambda(t) = \frac{1}{NA} \sum_{i \in \Lambda} \xi^\mu_i(t)\sigma_i(t), \quad \mu \in \mathcal{P}; \quad a_\Lambda(t) = \frac{1}{N} \sum_{i \in \Lambda} [\sigma_i(t)]^2. \]  

(10)

We remark that for \( Q = 2 \) the variance of the patterns \( A = 1 \), and the neuron activity \( a_\Lambda(t) = 1 \).

3. Solving the dynamics

3.1. Correlations

We first discuss some of the geometric properties of the various architectures which are particularly relevant for the understanding of their long-time dynamic behaviour.

For a fully connected architecture there are two main sources of strong correlations between the neurons complicating the dynamical evolution: feedback loops and the common ancestor problem [20]. Feedback loops occur when in the course of the time evolution, e.g., the following string of connections is possible: \( i \to j \to k \to i \). We remark that architectures with symmetric connections always have these feedback loops. In the absence of these loops the network functions in fact as a layered system, i.e., only feedforward connections are possible. But in this layered architecture common ancestors are still present when, e.g., for the sites \( i \) and \( j \) there are sites in the foregoing time steps that have a connection with both \( i \) and \( j \).

In extremely diluted asymmetric architecture these sources of correlations are absent. This class of neural networks was introduced in connection with \( Q = 2 \)-Ising models [9]. We recall that the couplings are then given by eq. (2) and that in the limit \( N \to \infty \) two important properties of this network are essential [9, 21]. The first property is the high asymmetry of the connections, viz.

\[ \Pr\{c_{ij} = c_{ji}\} = \left(\frac{C}{N}\right)^2, \quad \Pr\{c_{ij} = 1 \land c_{ji} = 0\} = \frac{C}{N} \left(1 - \frac{C}{N}\right). \]  

(11)

Therefore, the number of symmetric connections in the infinite configuration \( c = \{c_{ij}\}, i, j \neq i \in \mathbb{N} \) is finite with probability one, i.e. almost all connections of the graph \( G_\mathbb{N}(c) = \{(i, j) : c_{ij} = 1, i, j \neq i \in \mathbb{N}\} \) are directed: \( c_{ij} \neq c_{ji} \).

The second property in the limit of extreme dilution is the directed local Cayley-tree structure of the graph \( G_\mathbb{N}(c) \). By the arguments above the probability \( F_k^{(\Lambda)}(c) \) that \( k \)
connections are directed towards a given site \( i \in \Lambda \) is

\[
F_k^{(\Lambda)}(C) \equiv \Pr\{k = |T^{(in)}_i|\} = \frac{N!}{k!(N-k)!} \left( \frac{C}{N} \right)^k \left( 1 - \frac{C}{N} \right)^{N-k}
\]  

(12)

where \( T^{(in)}_i = \{c_{ji} = 1, j \in \Lambda \setminus i\} \) is the in-tree for \( i \) and \( |T^{(in)}_i| \) its cardinality. This probability is equal to \( \Pr\{k = |T^{(out)}_i|\} = \{|\{c_{ij} = 1, j \in \Lambda \setminus i\}\|\} \) for connections directed outward a given site \( i \in \Lambda \). In the limit of extreme dilution we get a Poisson distribution:

\[
\lim_{N \to \infty} F_k^{(\Lambda)}(C) = \frac{C^k}{k!} e^{-C}.
\]  

(13)

Hence, the mean value of the number of in (out) connections for any site \( i \in \Lambda \) is \( E[|T^{(in)}_i|] = C \). The probability that two sites \( i \) and \( i' \) have site \( j \) as a common ancestor is obviously equal to \( C/N \). From \( E[|T^{(in)}_i|] = C \) it follows that after \( t \) time steps the cardinality of the cluster of ancestors for site \( i \) will be of the order of \( C^t \). The same is valid for site \( i' \). Therefore, the probability that the sites \( i \) and \( i' \) have disjoint clusters of ancestors approaches \( (1 - C^t/N)^C \approx \exp(-C^2t/N) \) for \( N \gg 1 \).

So we find that in the limit of extreme dilution: (i) Almost all (i.e. with probability 1) feedback loops in \( G_N(c) \) are eliminated. (ii) With probability 1 any finite number of neurons have disjoint clusters of ancestors. So we first dilute the system by taking \( N \to \infty \) and then we take the limit \( C \to \infty \) in order to get infinite average connectivity allowing to store infinitely many patterns \( p \).

This implies that for this asymmetrically diluted model at any given time step \( t \) all spins are uncorrelated and, hence, the first step dynamics describes the full time evolution of the network.

For the symmetrically diluted model the architecture is still a local Cayley-tree but no longer directed and in the limit \( N \to \infty \) the probability that the number of connections \( T_i = \{j \in \Lambda | c_{ij} = 1\} \) giving information to the the site \( i \in \Lambda \) is still a Poisson distribution with mean \( C = E[|T_i|] \). However, at time \( t \) the spins are no longer uncorrelated causing a feedback from \( t \geq 2 \) onwards [17].

In order to solve the dynamics we start with a discussion of the first time step dynamics, the form of which is independent of the architecture.

3.2. First time step

Consider a fully connected network. Suppose that the initial configuration of the network \{\( \sigma_i(0) \), \( i \in \Lambda \), is a collection of i.i.d.r.v. with mean \( E[\sigma_i(0)] = 0 \), variance \( \text{Var}[\sigma_i(0)] = a_0 \), and correlated with only one stored pattern, say the first one \{\( \xi_i^1 \)\}:

\[
E[\xi_i^m \sigma_j(0)] = \delta_{i,j} \delta_{m,1} m_0^1 A \quad m_0^1 > 0.
\]

(14)
This pattern is said to be condensed. By the law of large numbers (LLN) one gets for the main overlap and the activity at \( t = 0 \)

\[
m^1(0) \equiv \lim_{N \to \infty} m^1_N(0) \quad \text{Pr} \quad \frac{1}{A} E[\xi \sigma_i(0)] = m^1_0 \tag{15}
\]

\[
a(0) \equiv \lim_{N \to \infty} a^1_N(0) \quad \text{Pr} \quad E[\sigma^2_i(0)] = a_0 \tag{16}
\]

where the convergence is in probability \([22]\). In order to obtain the configuration at \( t = 1 \) we first have to calculate the local field \((1)\) at \( t = 0 \). To do this we employ the signal-to-noise ratio analysis (see, e.g., \([10, 12]\)). Recalling the learning rule \((3)\) we separate the part containing the condensed pattern, i.e., the signal, from the rest, i.e., the noise to arrive at

\[
h_i(\sigma_\Lambda(0)) = \xi_i \sum_{j \in \Lambda \setminus i} \xi_j \sigma_j(0) + \sqrt{\alpha} \sum_{\mu \in P \setminus 1} \xi^\mu_i \sum_{j \in \Lambda \setminus i} \xi^\mu_j \sigma_j(0) \tag{17}
\]

where \( \alpha = p/N \). The properties of the initial configurations \((14)-(16)\) assure us that the summation in the first term on the r.h.s of \((17)\) converges in the limit \( N \to \infty \) to

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j \in \Lambda \setminus i} \xi_j \sigma_j(0) \quad \text{Pr} \quad m^1(0) \tag{18}
\]

The first term \( \xi_i m^1(0) \) is independent of the second term on the r.h.s of \((17)\). This second term contains the influence of the non-condensed patterns causing the intrinsic noise in the dynamics of the main overlap. In view of this we define the residual overlap

\[
r^\mu(t) \equiv \lim_{N \to \infty} r^\mu_N(t) = \lim_{N \to \infty} \frac{1}{A \sqrt{N}} \sum_{j \in \Lambda} \xi^\mu_j \sigma_j(t) \quad \mu \in P \setminus \{1\}. \tag{19}
\]

Applying the CLT to this second term in \((17)\) we find

\[
\lim_{N \to \infty} \sqrt{\frac{\alpha}{p}} \sum_{\mu \in P \setminus 1} \xi^\mu_j r^\mu_{\Lambda \setminus i}(0) = \lim_{N \to \infty} \sqrt{\frac{\alpha}{p}} \sum_{\mu \in P \setminus 1} \xi^\mu_i \frac{1}{A \sqrt{N}} \sum_{j \in \Lambda \setminus i} \xi^\mu_j \sigma_j(0) \tag{20}
\]

\[
\overset{D}{=} \sqrt{\alpha} N(0, \text{var}(\xi_i m^1(0))) \tag{21}
\]

where the quantity \( N(0, \text{var}) \) represents a Gaussian random variable with mean 0 and variance \( V \) and where \( D(0) = \text{var}[r^\mu(0)] = a(0) \) Thus we see that in fact the variance of this residual overlap, i.e., \( D(t) \) is the relevant quantity characterising the intrinsic noise.

In conclusion, in the limit \( N \to \infty \) the local field is the sum of two independent random variables, i.e.

\[
h_i(0) \equiv \lim_{N \to \infty} h_{\Lambda, i}(0) \overset{D}{=} \xi_i m^1(0) + \sqrt{\alpha} N(0, a(0)) \tag{22}
\]

For a more rigorous discussion of the first time step for the underlying spin-glass model we refer to \([23]\). At this point we note that the structure \((22)\) of the distribution of the
local field at time zero – signal plus Gaussian noise – is typical for all architectures
discussed here because the correlations caused by the dynamics only appear for \( t \geq 1 \).
Some technical details are different for the various architectures. The first change in
details that has to be made is an adaptation of the sum over the sites \( j \) to \( \Lambda \) for the
layered feedforward architecture and to \( T_i \), the part of the tree connected to neuron \( i \),
in the diluted architectures. The second change is that for the diluted architectures
an additional limit \( C \to \infty \) has to be taken besides the \( N \to \infty \) limit. So in the
thermodynamic limit \( C, N \to \infty \) all averages will have to be taken over the treelike
structure, viz. \( \frac{1}{N} \sum_{i \in \Lambda} \to \frac{1}{C} \sum_{i \in T_i} \). Furthermore \( \alpha = p/N \) has to be replaced by
\( \alpha = p/C \).

### 3.3. Recursive dynamical scheme

The key question is then how these quantities evolve in time under the parallel dynamics
specified before. For a general time step we find from the LLN in the limit \( C, N \to \infty \)
for the main overlap and the activity (10)

\[
m_1(t + 1) \overset{Pr}{=} \frac{1}{A} \langle \xi_i^1 \langle \sigma_i(t + 1) \rangle \rangle, \quad a(t + 1) \overset{Pr}{=} \langle \langle \sigma_i(t + 1) \rangle \rangle^2
\]

with the thermal average defined as

\[
\langle f(\sigma_i(t + 1)) \rangle_\beta = \frac{\sum_{\sigma \in S} f(\sigma) \exp[\frac{1}{2} \beta(\sigma(h_i(t) - b\sigma))] \sum_{\sigma \in S} \exp[\frac{1}{2} \beta(\sigma(h_i(t) - b\sigma))]
\]

where \( h_i(t) \equiv \lim_{N \to \infty} h_{A,i}(t) \). In the above \( \langle \cdot \rangle \) denotes the average both over the
distribution of the embedded patterns \( \{\xi_i^\mu\} \) and the initial configurations \( \{\sigma_i(0)\} \). The
average over the latter is hidden in an average over the local field through the updating
rule (8). In the sequel we focus on zero temperature. Then, using eq. (8) these formula
reduce to

\[
m_1(t + 1) \overset{Pr}{=} \frac{1}{A} \langle \xi_i^1 g_b(h_i(t)) \rangle, \quad a(t + 1) \overset{Pr}{=} \langle g_b^2(h_i(t)) \rangle
\]

As seen already in the first time step, we have to study carefully the influence of the
non-condensed patterns causing the intrinsic noise in the dynamics of the main overlap.
The method used to obtain these order parameters is then to calculate the distribution
of the local field as a function of time. In order to determine the structure of the local
field we have to concentrate on the evolution of the residual overlap. The details of this
calculation are very technical and depend on the precise correlations in the system and
hence on the architecture of the network as discussed before. For these technical details
we refer to the relevant literature [1, 2, 12, 14, 15]. Here we give an extensive discussion
of the results obtained.

In general, the distribution of the local field at time \( t + 1 \) is given by

\[
h_i(t+1) = \xi_i^1 m_1(t+1) + \mathcal{N}(0, \alpha a(t+1)) + \chi(t) [F(h_i(t) - \xi_i^1 m_1(t)) + Boa_i(t)]
\]
where \( F \) and \( B \) are binary coefficients given below, which depend on the specific architecture. From this it is clear that the local field at time \( t \) consists out of a discrete part and a normally distributed part, viz.

\[
h_i(t) = M_i(t) + \mathcal{N}(0, V(t))
\]

where \( M_i(t) \) satisfies the recursion relation

\[
M_i(t + 1) = \chi(t)[F(M_i(t) - \xi_1^i(t)) + B \alpha \sigma_i(t)] + \xi_1^i(t + 1)
\]

and where \( V(t) = \alpha AD(t) \) with \( D(t) \) itself given by the recursion relation

\[
D(t + 1) = \frac{a(t + 1)}{A} + L \chi^2(t) D(t) + 2 F \chi(t) \text{Cov}[	ilde{r}^\mu(t), r^\mu(t)]
\]

where \( L \) is again a coefficient specified below. The quantity \( \chi(t) \) reads

\[
\chi(t) = \sum_{k=1}^{Q-1} f_{h_\mu}^\ast(s_{k+1} + s_k)(s_{k+1} - s_k)
\]

where \( f_{h_\mu}^\ast \) is the probability density of \( h_\mu = \lim_{N \to \infty} h_{\Lambda, i}^\mu(t) \) with

\[
\hat{h}_{\Lambda, i}^\mu(t) = h_{\Lambda, i}^\mu(t) - \frac{1}{\sqrt{N}} \xi^\mu \hat{r}^\mu(t).
\]

Furthermore, \( \tilde{r}^\mu(t) \) is defined as

\[
\tilde{r}^\mu(t) \equiv \lim_{N \to \infty} \frac{1}{A} \sum_{i \in \Lambda} \xi^\mu g_{\hat{r}^\mu}(h_{\Lambda, i}^\mu(t)).
\]

Finally, as can be read off from eq. (28) the quantity \( M_i(t) \) consists out of the signal term and a discrete noise term, viz.

\[
M_i(t) = \xi_1^i(t)L(t + 1) + B_1 \alpha \chi(t - 1) \sigma_i(t - 1) + B_2 \sum_{t'=0}^{t-2} \alpha \left[ \prod_{s=t'}^{t-1} \chi(s) \right] \sigma_i(t')
\]

Since different architectures contain different correlations not all terms in these final equations are present. In particular we have for the coefficients \( F, B, L, B_1 \) and \( B_2 \) introduced above

|     | \( F \) | \( B \) | \( L \) | \( B_1 \) | \( B_2 \) |
|-----|--------|--------|--------|--------|--------|
| FC  | 1      | 1      | 1      | 1      | 1      |
| SED | 0      | 1      | 0      | 1      | 0      |
| LF  | 0      | 0      | 1      | 0      | 0      |
| AED | 0      | 0      | 0      | 0      | 0      |

with \( B \) indicating the feedback caused by the symmetry in the architectures and \( L \) the common ancestors contribution.

At this point we remark that in the so-called theory of statistical neurodynamics \cite{24, 25} one starts from an approximate local field by leaving out any discrete noise (the
term in $\sigma_i(t)$. As a consequence the covariance in the recursion relation for $D(t)$ can be written down more explicitly since only Gaussian noise is involved. For more details we refer to [26].

We still have to determine the probability density of $f_{h_i(t)}$ in eq. (30), which in the thermodynamic limit equals the probability density of $f_{h_i(t)}$. This can be done by looking at the form of $M_i(t)$ given by eq. (33). The evolution equation tells us that $\sigma_i(t')$ can be replaced by $g_b(h_i(t' - 1))$ such that the second and third terms of $M_i(t)$ are the sums of stepfunctions of correlated variables. These are also correlated through the dynamics with the normally distributed part of $h_i(t)$. Therefore the local field can be considered as a transformation of a set of correlated normally distributed variables $x_s$, $s = 0, \ldots, t - 2, t$, which we choose to normalize. Defining the correlation matrix $W = (\rho(s, s') \equiv E[x_s x_s'])$ we arrive at the following expression for the probability density of the local field at time $t$

$$f_{h_i(t)}(y) = \int \prod_{s=0}^{t-2} dx_s dx_t \delta \left( y - M_i(t) - \sqrt{\alpha AD(t)} x_t \right)$$

$$\times \frac{1}{\sqrt{\det(2\pi W)}} \exp \left( -\frac{1}{2} x W^{-1} x^T \right)$$

(35)

with $x = (x_0, \ldots, x_{t-2}, x_t)$. For the symmetrically diluted case this expression simplifies to

$$f_{h_i(t)}(y) = \int \prod_{s=0}^{[t/2]} dx_{t-2s} \delta \left( y - \xi_1^1 m^1(t) - \alpha \chi(t) \sigma(t) - \sqrt{\alpha a(t)} x_t \right)$$

$$\times \frac{1}{\sqrt{\det(2\pi W)}} \exp \left( -\frac{1}{2} x W^{-1} x^T \right)$$

(36)

with $x = \{x_s\} = (x_{t-2[t/2]}, \ldots, x_{t-2}, x_t)$. The brackets $[t/2]$ denote the integer part of $t/2$.

So the local field at time $t$ consists out of a signal term, a discrete noise part and a normally distributed noise part. Furthermore, the discrete noise and the normally distributed noise are correlated and this prohibits us to derive a closed expression for the overlap and activity.

Together with the eqs. (25) for $m^1(t+1)$ and $a(t+1)$ the results above form a recursive scheme in order to obtain the order parameters of the system. The practical difficulty which remains is the explicit calculation of the correlations in the network at different time steps as present in eq. (29).

For AED and LF architectures this scheme leads to an explicit form for the recursion relations for the order parameters

$$m^\mu(t+1) = \delta_{\mu,1} \frac{1}{A} \left\langle \left\langle \xi^1(t+1) \int Dz g(\xi^1(t+1)m^1(t) + \sqrt{\alpha AD(t)} z) \right\rangle \right\rangle$$

(37)
\[ a(t+1) = \left\langle \left\langle \int Dz g^2(\xi^1(t+1)m^1(t) + \sqrt{\alpha AD(t)} z) \right\rangle \right\rangle \]  
(38)

\[ D(t+1) = \frac{a(t+1)}{A} + \frac{L}{\alpha A} \left[ \left\langle \left\langle \int Dz z g(\xi^1(t+1)m^1(t) + \sqrt{\alpha AD(t)} z) \right\rangle \right\rangle \right]^2. \]  
(39)

For the AED architecture \((L = 0)\) the second term on the r.h.s. of (39) coming from the correlations caused by the common ancestors is absent. For the LF architecture we remark that this explicit solution requires an independent choice of the representations of the patterns at different layers. At finite temperatures analogous recursion relations for the AED and LF networks can be derived \([12, 14]\) by introducing auxiliary thermal fields \([27]\) in order to express the stochastic dynamics within the gain function formulation of the deterministic dynamics. Furthermore, damage spreading \([14, 28, 29]\), i.e., the evolution of two network configurations which are initially close in Hamming distance can be studied \([12, 14]\). Finally, a complete self-control mechanism can be built in the dynamics of these systems by introducing a time-dependent threshold in the gain function improving, e.g., the basins of attraction of the memorized patterns \([31] - [32]\).

For explicit examples of this dynamical scheme with numerical results we refer to \([1, 15]\). By using the recursion relations the first few time steps are written out explicitly and studied numerically, e.g., for the \(Q = 2\) and \(Q = 3\) FC and SED models with equidistant states and a uniform distribution of patterns. These results are compared with the approximations studied in the literature \([10, 11, 16, 17, 24, 25, 33] - [38]\) by neglecting some feedback correlations for \(t \geq 2\). In the whole retrieval region of these networks we find that the first four or five time steps give us already a clear picture of their time evolution.

4. Fixed-point equations

Equilibrium results for the AED and LF \(Q\)-Ising models are obtained immediately by straightforwardly leaving out the time dependence in (37)-(39) \([see [12, 14]]\), since the evolution equations for the local field and the order parameters do not change their form as time progresses \([see eqs. (37)-(39)]\). This still allows small fluctuations in the configurations \(\{\sigma_i\}\). The difference between the fixed-point equations for these two architectures is that for the AED model the variance of the residual noise, \(D(t)\) is simply proportional to the activity of the neurons at time \(t\) while for the LF model a recursion is needed.

For the SED and FC architectures, however, the evolution equations for the order parameters do change their form by the explicit appearance of the \(\{\sigma_i(t')\}, t' = 1, \ldots, t\) term. Hence we can not use the simple procedure above to obtain the fixed-point equations. Instead we derive the equilibrium results of our dynamical scheme by
requiring through the recursion relations (26) that the distribution of the local field becomes time-independent. This is an approximation because fluctuations in the network configuration are no longer allowed. In fact, it means that out of the discrete part of this distribution, i.e., $M_i(t)$ (recall (33)), only the $\sigma_i(t-1)$ term is kept besides, of course, the signal term. This procedure implies that the main overlap and activity in the fixed-point are found from the definitions (10) and not from leaving out the time dependence in the recursion relation (25).

We start by eliminating the time-dependence in the evolution equations for the local field (26). This leads to

$$h_i = \xi^1_i m^1 + [\bar{\chi}^{ar}]^{-1}N(0, \alpha a) + [\bar{\chi}^{ar}]^{-1} \alpha \chi \sigma_i$$  \hspace{1cm} (40)

with $\bar{\chi}^{ar} \equiv 1 - F\chi$ being 1 for the SED and $1 - \chi$ for the FC model and $h_i \equiv \lim_{t \to \infty} h_i(t)$. This expression consists out of two parts: a normally distributed part $\tilde{h}_i = N(\xi^1_i m^1, \alpha a/[\bar{\chi}^{ar}]^2)$ and some discrete noise part. At this point some remarks are in order. First, the discrete noise coming from the correlations of the $\{\sigma_i(t)\}$ at different time steps (here only the preceding time step is considered) is inherent in the SED and FC dynamics. Second, the so-called self-consistent signal-to-noise ratio analysis of the FC network considered in the literature \[39, 40\] starts from such a type of equation by assuming the presence of a term proportional to the output in the local field without any reference or argumentation based upon the underlying dynamics of the network.

Employing this expression in the updating rule (8) one finds

$$\sigma_i = g_b(\tilde{h}_i + [\bar{\chi}^{ar}]^{-1} \alpha \chi \sigma_i)$$  \hspace{1cm} (41)

where $\tilde{h}_i = N(\xi^1_i m^1, \alpha a)$ is the normally distributed part of eq. (40). This is a self-consistent equation in $\sigma_i$ which in general admits more than one solution. These types of equation have been solved in the literature in the context of thermodynamics using a geometric Maxwell construction \[39, 40\]. We remark that for analog networks the geometric Maxwell construction is not necessary: the fixed-point equation (11) has only one solution. For more technical details we refer to \[26\].

This approach leads to a unique solution

$$\sigma_i = g_b(\tilde{h}_i), \quad \tilde{b} = b - [2 \bar{\chi}^{ar}]^{-1} \alpha \chi .$$  \hspace{1cm} (42)

We remark that plugging this result into the local field (10) tells us that the latter is the sum of two Gaussians with shifted mean (see also \[37\]).

Using the definition of the main overlap and activity (10) in the limit $N \to \infty$ for the FC model and limit $C, N \to \infty$ for the SED model, one finds in the fixed point

$$m^1 = \left\langle \left\langle \xi^1 \int Dz \ g_b \left( \xi^1 m^1 + \sqrt{\alpha AD} \ z \right) \right\rangle \right\rangle$$  \hspace{1cm} (43)

$$a = \left\langle \left\langle \int Dz \ g_b^2 \left( \xi^1 m^1 + \sqrt{\alpha AD} \ z \right) \right\rangle \right\rangle .$$  \hspace{1cm} (44)
From (29) and (30) one furthermore sees that

\[
D = \left[\chi_{ar}\right]^{-2} a / A
\]  

(45)

with

\[
\chi = \frac{1}{\sqrt{\alpha AD}} \left\langle \left\langle \int Dz \; z g_b \left( \xi^1 m^1 + \sqrt{\alpha AD} z \right) \right\rangle \right\rangle.
\]  

(46)

These resulting equations (43)-(45) are the same as the fixed-point equations derived from a replica-symmetric mean-field theory treatment in [41]–[44]. Their solution leads to capacity-gain parameter phase diagrams (see, e.g., [44]).

5. Concluding remarks

An evolution equation is derived for the distribution of the local field governing the parallel dynamics at zero temperature of extremely diluted symmetric and asymmetric, layered feedforward and fully connected \( Q \geq 2 \)-Ising networks. All feedback correlations are taken into account. In general, this distribution is not normally distributed but contains a discrete noise part.

Employing this evolution equation a general recursive scheme is developed allowing one to calculate the relevant order parameters of the system, i.e., the main overlap, the activity and the variance of the residual noise for any time step. For the extremely diluted asymmetric and the layered feedforward architectures this scheme immediately leads to explicit recursion relations for the order parameters because the discrete noise part in the local field is absent. For the extremely diluted and the fully connected architectures equilibrium fixed-point equations for the order parameters are obtained under the condition that the local field becomes time-independent, meaning that some of the discrete noise is neglected. The resulting equations are the same as those derived from a replica-symmetric mean-field theory approach.

Acknowledgments

This work has been supported in part by the Research Fund of the K.U.Leuven (Grant OT/94/9) and the Korea Science and Engineering Foundation through the SRC program. The authors are indebted to S. Amari, R. Kühn, G. Massolo, A. Patrick and V. Zagrebnov for constructive discussions. One of us (D.B.) thanks the Belgian National Fund for Scientific Research for financial support.

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