SPECTRAL INDEPENDENCE BEYOND UNIQUENESS
USING THE TOPOLOGICAL METHOD.

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ABSTRACT. We present novel results for fast mixing of Glauber dynamics using the newly introduced and powerful Spectral Independence method from [Anari, Liu, Oveis-Gharan: FOCS 2020]. In our results, the parameters of the Gibbs distribution are expressed in terms of the spectral radius of the adjacency matrix of G, or that of the Hashimoto non-backtracking matrix.

The analysis relies on new techniques that we introduce to bound the maximum eigenvalue of the pairwise influence matrix $I_{G}^{A,\tau}$ for the two spin Gibbs distribution $\mu$. There is a common framework that underlies these techniques which we call the topological method. The idea is to systematically exploit the well-known connections between $I_{G}^{A,\tau}$ and the topological construction called tree of self-avoiding walks.

Our approach is novel and gives new insights to the problem of establishing spectral independence for Gibbs distributions. More importantly, it allows us to derive new -improved- rapid mixing bounds for Glauber dynamics on distributions such as the Hard-core model and the Ising model for graphs that the spectral radius is smaller than the maximum degree.

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1. Introduction

The Markov Chain Monte Carlo method (MCMC) is a very simple, yet very powerful method for approximate sampling from Gibbs distributions on combinatorial structures. In the standard setting, we are given a very simple to describe, ergodic Markov chain and we need to analyse the speed of convergence to the equilibrium distribution. The challenge is to show that the chain mixes fast when the parameters of the equilibrium distribution belong to a certain region of values.

Here our focus is on combinatorial structures that are specified with respect to an underlying graph $G$, e.g., independent sets. For us the graph $G$ is always undirected, connected and finite and has maximum degree $\Delta$.

Recently, a new technique for analysing the speed of convergence of the Markov chain called Glauber dynamics has been introduced in [1]. This technique is called the Spectral Independence method. The authors in [1] use the Spectral Independence method (SI) to prove a long standing conjecture about the mixing time of Glauber dynamics for the so-called Hard-core model, improving on a series of result such as [28, 12]. Since then, it is not an exaggeration to claim that SI has revolutionised the study in the field. Using this method it has been possible to get positive results for approximate sampling from 2-spin Gibbs distributions that match the hardness ones, e.g., [11, 5, 26, 27].

In this work, our main focus is on the so-called pairwise influence matrix, denoted as $I_G^{\Lambda}$. This is a central concept as the rapid mixing bounds we derive using SI rely on showing that the maximum eigenvalue of this matrix is bounded. We propose novel techniques that allow us to derive more accurate estimations on the maximum eigenvalue of this matrix than what we have been getting from the previous approaches that are proposed in [1, 5]. In turn, we get new rapid mixing results for the Glauber dynamics on two spin Gibbs distribution such as the Hard-core model and the Ising model. Interestingly, in our results the parameters of the Gibbs distribution do not depend on the maximum degree $\Delta$ of the underlying graph $G$. They rather depend on the spectral radius of the adjacency matrix of $G$, denoted as $A_G$.

For concreteness, consider the Glauber dynamics on the Hard-core model for $G$ whose adjacency matrix has spectral radius $\rho$ which is a bounded number. Let $\lambda_c(k)$ by the critical value for the Gibbs uniqueness of the Hard-core model on the infinite $k$-ary tree. We prove that the Glauber dynamics mixes in $O(n \log n)$ steps for any fugacity $0 \leq \lambda < \lambda_c(\rho)$. For a comparison, recall that the max-degree-$\Delta$ bound for the Hard-core model comes from [1, 6] and requires fugacity $0 \leq \lambda < \lambda_c(\Delta - 1)$ in order to get $O(n \log n)$ mixing time. This implies that our approach gives better bounds when the spectral radius $\rho$ is smaller than $\Delta - 1$. As a reference, note that we always have that $\rho \leq \Delta$. On the other hand, the spectral radius can get much smaller than the maximum degree, e.g., for a planar graph we have that $\rho \leq \sqrt{8\Delta - 16 + 2\sqrt{3}}$, see further cases in Section 2.3. We get results of similar flavour for the Ising model, too.

We also consider the case where the spectral radius is unbounded. Even though we improve on results in the literature, our results are not as strong as the those we get for the bounded case and (most likely) they admit improvements. The case of unbounded spectral radius is considered in the Appendix, at the end of this paper.

The use of the spectral radius, or matrix norms of the adjacency matrix in order to derive rapid mixing bounds has also been studied in [16, 8]. As opposed to our approach here, that relies on SI, these results rely on the path coupling technique [4]. Our improvements on these works reflects the fact that SI is much stronger than path coupling. The techniques considered in these two paper do not seem to apply to the setting of analysis we have with SI. In that respect, our approach is orthogonal to the previous ones.

All the above shed some more light on the well-known hardness transition shown for approximate counting-sampling in the Hard-core model (and the antiferromagnetic Ising). Specifically, our results indicate that for graphs of maximum degree $\Delta$ the hardness transition occurs at $\lambda_c(\Delta - 1)$ only if the spectral radius of the adjacency matrix satisfies that $\rho(A_G) \geq \Delta - 1$. On the other hand, for $\rho(A_G) < \Delta - 1$ we show rapid mixing for Glauber dynamics for the parameters of the Gibbs distribution which are beyond the uniqueness
region, i.e., for $\lambda > \lambda_c(\Delta - 1)$. Interestingly, we manage to get access to this region of parameters by directly analysing the influence matrix.

When we deviate from the standard approach, i.e., the one that relies on the maximum degree $\Delta$, a natural challenge that arises is how to deal with the effect of high-degrees. That is, how to accommodate in the analysis the high degree vertices. This problem is common when the underlying structure is a random (hypergraph) graph $[2, 11, 13, 25]$. As far as SI is concerned, the work in $[2]$ deals with a version of this problem, however, it focuses only on random graphs and it takes advantage of properties that are special to the typical instances of this distribution. Unfortunately, there does not seem to be a way of exploiting them in the worst-case setting that we are focusing here.

Concluding, one might wonder what are the optimal bounds one can get using the assumption on the spectral radius of $A_G$, i.e., rather than the maximum degree $\Delta$. Note that the regions of the parameters of the Gibbs distribution we get here are quite natural and they are related to the spectral radius in the same way as the corresponding regions for Gibbs uniqueness are related to $\Delta - 1$. In the Gibbs uniqueness, the parameters of the distribution need to guarantee that the rate of correlation decay counterbalances the growth rate of the underlying graph $G$, which is at most $\Delta - 1$, hence their dependence on $\Delta$. In an analogy to uniqueness, here the correlation decay counterbalances the growth rate of the number walks between any two vertices in $G$, which is $\rho(A_G)$. Hence, the entries of the influence matrix do not grow too large for distant pairs of vertices. In that respect, our conjecture would be that under the assumption of $\rho(A_G)$ for $G$, our results cannot be further improved.

However, we believe that improvements are possible if we consider different matrices, i.e., other than $A_G$. Specifically, we believe that a more natural matrix to consider for the problem is the so-called (Hashimoto) non-backtracking matrix of $G$ introduced in $[15]$. The non-backtracking matrix is an object that has been studied extensively in mathematical physics. Getting results in terms of the spectrum of the non-backtracking matrix is quite desirable because it is considered to capture the structure of $G$ much better than that of the adjacency matrix.

Here, we further derive a rapid mixing result for the Ising model using the non-backtracking matrix of $G$, e.g., see Theorem $[2, 7]$. This result is only a preliminary one. Due to the intricacy of working with this matrix, we didn’t manage to show that the non-backtracking spectrum gives better bound than what we get with the adjacency one. We only show that the results that someone gets with the non-backtracking matrix are at least as good as those from the adjacency matrix. We believe that the direction of exploiting the spectrum of the non-backtracking matrix for the problem is worth further investigation.

The Topological Method for Spectral Independence. As mentioned earlier, a central notion in SI is the so-called pairwise influence matrix $I_G^{1,\tau}$. Given a set of parameters of the Gibbs distribution, the endeavour is to show that the maximum eigenvalue of $I_G^{1,\tau}$ is $O(1)$. Previously introduced approaches in $[1, 15]$ have been focusing on proving that either $\|I_G^{1,\tau}\|_1$, or $\|I_G^{1,\tau}\|_\infty$ is bounded, which in turn implies that the maximum eigenvalue is bounded. These approaches are quite elegant and provide, in a very natural way, bounds that depend on the maximum degree $\Delta$. However, for our purposes here they seem to be too crude.

The main contribution of our work amounts to introducing novel techniques to bound the maximum eigenvalue of $I_G^{1,\tau}$ which (for most of the cases) turn out to be more precise than the previous ones. We introduce a common framework that underlies these techniques that we call the topological method.

The basic idea for the topological method comes from the following, well-known, observation: each entry $I_G^{1,\tau}(w, v)$ can be expressed in terms of a topological construction called tree of self-avoiding walks (starting from $w$), together with a set of weights on the paths of this tree, which are called influences. The influences are specified by the parameters of the Gibbs distribution we consider. The entry of the matrix is nothing more than the sum of influences over an appropriately chosen set of paths in this tree.

In the topological method, we generalise the above concepts by introducing the notion of walk-matrix. For the entries in a walk-matrix, we don’t necessarily use trees of self-avoiding walks. We may use other topological constructions of $G$ such as path-trees, universal covers etc (see more about these constructions
in the excellent textbook \[14\]. Also, we have weights on the paths of this construction. The weight of each path can be chosen arbitrarily, i.e., it is not necessarily an influence. Each entry in the walk-matrix is a sum of weights of appropriately chosen paths in the topological construction. In that respect, one might regard the influence matrix \(I_G^{A,\tau}\) to be a special case of walk-matrix (e.g. see Lemma \[6.4\]).

Exploiting properties of these special matrices, i.e., walk-matrices, we introduce two techniques, orthogonal to each other, that we use to derive our results.

With the first technique we focus on comparing walk-matrices in terms of their corresponding spectral radii. That is, for two walk-matrices \(C\) and \(D\) the aim is to establish that the corresponding spectral radii satisfy \(\rho(C) \leq \rho(D)\). Since \(I_G^{A,\tau}\) is a special case of walk-matrix, we use this technique to establish an inequality which is similar to the aforementioned one, using the walk-matrix \(\sum_{\ell=0}^{n} \xi \cdot A_G^\ell\), where \(A_G\) is the adjacency matrix and \(\xi > 0\) is an appropriately chosen real number. Specifically, we show that

\[
\rho \left( I_G^{A,\tau} \right) \leq \rho \left( \sum_{\ell=0}^{n} \xi \cdot A_G^\ell \right).
\]

The value of \(\xi\) depends on the parameters of the Gibbs distribution that underlies \(I_G^{A,\tau}\). It is easy to see that adjusting these parameters such that \(\xi \leq (1-\epsilon)/\rho(A_G)\), for any fixed \(\epsilon > 0\), implies that the spectral radius of \(I_G^{A,\tau}\) - and hence the maximum eigenvalue - is bounded. Working as described above, allows us to derive new, very interesting results about the Ising model, see Theorem \[2.1\].

We establish another inequality, similar to the above, that is between \(I_G^{A,\tau}\) and a walk-matrix related to the non-backtracking matrix, e.g., see Corollary \[7.3\]. We exploit this inequality to get the rapid mixing result in Theorem \[2.7\] which is about the Ising model, too.

It turns out that for the Hard-core model the bounds we get from the previous approach are too crude. To this end, we follow a different one. We introduce a new matrix norm to bound the maximum eigenvalue of \(I_G^{A,\tau}\). Our norm provides better bounds compared to what we get from \(||I_G^{A,\tau}||_1\) and \(||I_G^{A,\tau}||_\infty\). Note that for the Hard-core model the problem becomes highly non-linear because of the fact that we use the potential method. In that respect, finding a matrix norm that allows us to derive the kind of bounds we are aiming for is a non-trivial task.

We derive our rapid mixing results for the Hard-core model in Theorem \[2.2\] by choosing an appropriate non-singular matrix \(D\) and showing that

\[
\left| \left| (D)^{-1} \cdot I_G^{A,\tau} \cdot D \right| \right|_\infty = O(1).
\]

More specifically, \(D\) is the diagonal matrix such that \(D^{(w,w)} = (f_1(w))^t\), for appropriate \(t \geq 1\), while \(f_1\) is the eigenvector that corresponds to the maximum eigenvalue of \(A_G\). Note that our assumptions about \(G\) imply the above norm is well-defined, i.e., \(D\) is non-singular. The parameter \(t \geq 1\), in the norm, needs to be specified in the context of the potential method.

In hindsight, the use of the above norm is, somehow, natural in the context of topological method. The related analysis gives rise to a further topological construction that we call walk-vector (see Section \[8.1\]).

2. Main Results

Consider the fixed graph \(G = (V, E)\) on \(n\) vertices. We assume that \(G\) is undirected, finite and connected.

The Gibbs distribution \(\mu\) on \(G\) with spins \(\{\pm 1\}\) is a distribution on the set of configurations \(\{\pm 1\}^V\). We use the parameters \(\beta \in \mathbb{R}_{\geq 0}\) and \(\gamma, \lambda \in \mathbb{R}_{>0}\) and specify that each configuration \(\sigma \in \{\pm 1\}^V\) gets a probability measure

\[
\mu(\sigma) \propto \lambda^{\text{# assignments } "1" \text{ in } \sigma} \times \beta^{\text{# edges with both ends } "1" \text{ in } \sigma} \times \gamma^{\text{# edges with both ends } "-1" \text{ in } \sigma},
\]

where the symbol \(\propto\) stands for “proportional to”. The above distribution is called ferromagnetic when \(\beta\gamma > 1\), while for \(\beta\gamma < 1\) it is called antiferromagnetic. Unless otherwise specified, we always assume that \(\mu\) is a two-spin Gibbs distribution.
Using the formalism in (1), one recovers the well-known Ising model by setting $\beta = \gamma$. In this case, the magnitude of $\beta$ specifies the strength of the interactions. The above, also, gives rise to the so-called Hard-core model if we choose $\beta = 0$ and $\gamma = 1$. Particularly, this distribution assigns to each independent set $\sigma$ probability measure which is proportional to $\lambda^{\vert \sigma \vert}$, where $\vert \sigma \vert$ is the size of the independent set. For the Hard-core model we use the term fugacity to refer to the parameter $\lambda$.

Given a Gibbs distribution $\mu$ we use the discrete time, (single site) Glauber dynamics $\{X_t\}_{t \geq 0}$ to approximately sample from $\mu$. Glauber dynamics is a very simple to describe Markov chain. The state space of the chain is the support of $\mu$. We assume that the chain starts from an arbitrary configuration $X_0$. For $t \geq 0$, the transition from the state $X_t$ to $X_{t+1}$ is according to the following rules: Choose uniformly at random a vertex $v$. For every vertex $w$ different than $v$, set $X_{t+1}(w) = X_t(w)$. Then, set $X_{t+1}(v)$ according to the marginal of $\mu$ at $v$, conditional on the neighbours of $v$ having the configuration specified by $X_t$.

For the cases we consider here, $\{X_t\}_{t \geq 0}$ satisfies a set of technical conditions that come with the name ergodicity. Ergodicity implies that $\{X_t\}_{t \geq 0}$ converges to a unique stationary distribution, which, in our case, is the Gibbs distribution $\mu$.

In this work, we study the rate that the chain converges to stationarity, when the parameters $\beta, \gamma$ and $\lambda$ of the Gibbs distribution vary in a range of parameters that depends on the spectral radius of the adjacency matrix $A$. Particularly, for the ferromagnetic case the measure is unique when $k > 1$. For the cases we consider here, $\{X_t\}_{t \geq 0}$ satisfies a set of technical conditions that come with the name ergodicity. Ergodicity implies that $\{X_t\}_{t \geq 0}$ converges to a unique stationary distribution, which, in our case, is the Gibbs distribution $\mu$.

2.1. The Ising Model. As mentioned earlier, the Ising model corresponds to the distribution in (1) such that $\beta = \gamma$. This implies that each configuration $\sigma \in \{\pm 1\}^V$ is assigned probability measure

$$
\mu(\sigma) \propto \lambda^{\sum_{x \in V} 1\{\sigma(x) = -1\}} \times \beta^{\sum_{\{x,y\} \in E} 1\{\sigma(x) = \sigma(y)\}}.
$$

When $\beta > 1$ we have the ferromagnetic Ising model, while when $\beta < 1$ we have the antiferromagnetic. Here, we always assume that $\lambda = 1$. This corresponds to what we call zero external field Ising model.

It is a well-known result that the uniqueness region of the Ising model on the infinite $k$-ary tree, where $k \geq 2$, corresponds to having

$$
\frac{k-1}{k+1} < \beta < \frac{k+1}{k-1}.
$$

Particularly, for the ferromagnetic case the measure is unique when $1 \leq \beta < \frac{k+1}{k-1}$, while for the antiferromagnetic case, the measure is unique when $\frac{k-1}{k+1} < \beta \leq 1$.

For $k \geq 2$ and $\delta \in (0, 1)$, we let the interval

$$
[\frac{k-1+\delta}{k+1-\delta}, \frac{k+1-\delta}{k-1+\delta}].
$$

We use the Spectral Independence method to get the following result about the Ising model.

**Theorem 2.1 (Adjacency Matrix).** For any $\delta \in (0, 1)$ and for bounded $\rho_G \geq 2$, consider the graph $G = (V, E)$ whose adjacency matrix $A_G$ has spectral radius $\rho_G$. Also, let $\mu_G$ be the Ising model on $G$ with zero external field and parameter $\beta \in M_{\text{Ising}}(\rho_G, \delta)$.

There is a constant $C = C(\delta)$ such that the mixing time of the Glauber dynamics on $\mu_G$ is at most $Cn \log n$.

Theorem 2.1 follows from a technical result we present later, i.e., Theorem 5.2, which we use to establish spectral independence for the zero external field Ising model $\mu_G$ with $\beta \in M_{\text{Ising}}(\rho_G, \delta)$. Once we establish spectral independence for $\mu_G$, we use results from [6] to derive the bound on the mixing time.

Theorem 2.1 improves on results in [16] for the Ising model by allowing a wider range for $\beta$. Specifically, in the ferromagnetic case the above bound allows for $\beta$ that is a constant factor larger than what we had.
before. The analogous holds for the antiferromagnetic Ising, i.e., Theorem 2.1 allows for \( \beta \) which is a constant factor smaller than what we had before.

2.2. The Hard-Core Model. Another distributions of interest is the so-called Hard-core model. The formalism in [10] gives rise to the Hard-core model with fugacity \( \lambda \) if we set \( \beta = 0 \) and \( \gamma = 1 \). This distribution assigns to each independent set \( \sigma \) of the graph \( G \), probability measure \( \mu(\sigma) \) such that

\[
\mu(\sigma) \propto \lambda^{|\sigma|},
\]

\( \text{(4)} \)

where \( |\sigma| \) is the size of the independent set.

We use \( \{\pm 1\}^V \) to encode the configurations of the the Hard-core model, i.e., the independent sets of \( G \). Particularly, the assignment \( +1 \) implies that the vertex is in the independent set, while \( -1 \) implies the opposite. We often use physics’ terminology where the vertices with assignment \( +1 \) are called “occupied”, whereas the vertices with \( -1 \) are the “unoccupied” ones.

For \( z > 1 \), we let the function \( \lambda_c(z) = \frac{z^2}{(z-1)(z+1)} \). It is a well-known result from [19] that the uniqueness region of the Hard-core model on the \( k \)-ary tree, where \( k \geq 2 \), holds for any \( \lambda \) such that

\[
\lambda < \lambda_c(k). \quad \text{(5)}
\]

As far as the Hard-core model is concerned we use the Spectral Independence method to derive the following result.

**Theorem 2.2** (Adjacency matrix). For any \( \epsilon \in (0, 1) \), \( \Delta \geq 2 \) and \( \rho_G > 1 \) which is bounded, consider the graph \( G = (V, E) \) of maxim degree \( \Delta \), whose adjacency matrix \( A_G \) has spectral radius \( \rho_G \). Also, let \( \mu_G \) be the Hard-core model on \( G \) with fugacity \( \lambda \leq (1 - \epsilon)\lambda_c(\rho_G) \).

There is a constants \( C = C(\epsilon) \) such that the mixing time of the Glauber dynamics on \( \mu_G \) is at most \( Cn \log n \).

Theorem 2.2 follows from a technical result, i.e., Theorem 5.5 which we use to establish spectral independence for the Hard-core model \( \mu_G \) with fugacity \( \lambda \leq (1 - \epsilon)\lambda_c(\rho_G) \). Once we establish spectral independence we use results from [6] to derive the bound on the mixing time.

The above improves on results in [16] for the Hard-core model by allowing a wider rage for \( \lambda \). The upper bound on \( \lambda \) here is by a constant factor larger than the previous one. Particularly, for large \( \rho_G \) this constant converges to \( \epsilon \), i.e., the base of natural logarithms.

**Notation.** For the graph \( G = (V, E) \) and the Gibbs distribution \( \mu \) on the set of configurations \( \{\pm 1\}^V \). For a configuration \( \sigma \), we let \( \sigma(A) \) denote the configuration that \( \sigma \) specifies on the set of vertices \( A \). We let \( \mu_A \) denote the marginal of \( \mu \) at the set \( A \). We let \( \mu(\cdot \mid A, \sigma) \), denote the distribution \( \mu \) conditional on the configuration at \( A \) being \( \sigma \). Also, we interpret the conditional marginal \( \mu_A(\cdot \mid A', \sigma) \), for \( A' \subseteq V \), in the natural way.

2.3. **Applications I.** There are a lot of interesting cases of graphs whose adjacency matrix has spectral radius much smaller than the maximum degree, and hence, our results give better rapid mixing bounds than the general one. A standard example is the planar graphs for which we have the following result from [10].

**Theorem 2.3** ([10]). Suppose that \( G = (V, E) \) is a planar graph of maximum degree \( \Delta \), then \( \rho(A_G) \leq \rho_{\text{planar}}(\Delta) \) where

\[
\rho_{\text{planar}}(\Delta) = \begin{cases} 
\Delta & \text{for } \Delta \leq 5, \\
\sqrt{12\Delta - 36} & \text{for } 6 \leq \Delta \leq 36, \\
8(\Delta - 2) + 2\sqrt{3} & \text{for } 37 \leq \Delta.
\end{cases}
\]

\( \text{(6)} \)

In what follows, we show the implications of the above theorem to the mixing time of Glauber dynamics for the Ising model and the Hard-core model. We focus on results for graphs of bounded maximum degree. The results for the unbounded case are straightforward so we omit their statement.

As far as the Ising model on planar graphs is concerned, we have the following result.
Corollary 2.4 (Planar Ising model). For \( \delta \in (0, 1) \), for fixed \( \Delta \geq 2 \), consider the planar graph \( G = (V, E) \) of maximum degree \( \Delta \). Let \( \mu_G \) be the zero external field Ising model on \( G \) with parameter \( \beta \) such that

\[
\beta \in \mathcal{M}_{\text{Ising}} (\rho_{\text{planar}} (\Delta), \delta),
\]

where \( \rho_{\text{planar}} (\Delta) \) is defined in (6). There is a constant \( C = C(\delta) \) such the Glauber dynamics on \( \mu_G \) exhibits mixing time which is at most \( C n \log n \).

As far as the Hard-core model on planar graphs is concerned, we have the following result.

Corollary 2.5 (Planar Hard-core model). For \( \epsilon \in (0, 1) \), for fixed \( \Delta \geq 2 \), consider the planar graph \( G = (V, E) \) of maximum degree \( \Delta \). Let \( \mu_G \) be the Hard-core model on \( G \) with fugacity \( \lambda \) such that

\[
\lambda \leq (1 - \epsilon) \lambda_c (\rho_{\text{planar}} (\Delta)),
\]

where \( \rho_{\text{planar}} (\Delta) \) is defined in (6). There is a constant \( C = C(\epsilon) \) such the Glauber dynamics on \( \mu_G \) exhibits mixing time which is at most \( C n \log n \).

There are further examples of graphs with spectral radius much smaller than the maximum degree. One very interesting case, which generalises the aforementioned one, is the graphs that can be embedded in a small Euler genus.

Theorem 2.6. Let the graph \( G = (V, E) \) be of maximum degree \( \Delta > 0 \). Suppose that \( G \) can be embedded in a surface of Euler genus \( g \geq 0 \). If \( \Delta \geq d(g) + 2 \), then

\[
\rho (A_G) \leq \sqrt{8(\Delta - d(g))} + d(g),
\]

where \( d(g) \) is such that

\[
d(g) = \begin{cases} 
10 & \text{if } g \leq 1, \\
12 & \text{if } 2 \leq g \leq 3 
\end{cases}
\quad \text{and} \quad
\begin{cases} 
2g + 6 & \text{if } 4 \leq g \leq 5, \\
2g + 4 & \text{if } 6 \geq g.
\end{cases}
\]

If, e.g., the Euler genus of \( G \) is much smaller than \( \Delta \), then, from the above theorem, it is immediate that \( \rho (A_G) \approx \sqrt{8 \Delta} \). It is straightforward to combine the above results with Theorems 2.2 and 2.1 and get results analogous to what we have in Corollaries 2.4 and 2.5. We omit the presentation of these results as their derivation is straightforward.

2.4. Applications II - Beyond the adjacency matrix. We further derive rapid mixing results for the Ising model using the spectral radius of the non-backtracking matrix of \( G \), denoted as \( H_G \). What motivates the use of this matrix instead of \( A_G \) is that in many cases the spectral radius of \( H_G \) is much smaller. This, potentially, could lead to even better rapid mixing bounds.

The result we present here is only a preliminary one, while its statement is not simple. The purpose of this small section is, also, to show that our techniques allow us to consider matrices other than \( A_G \).

For the graph \( G = (V, E) \), let \( \overrightarrow{E} \) be the set of oriented edges obtained by doubling each edge of \( E \) into two directed edge, i.e., one edge for each direction. The non-backtracking matrix \( H_G \) is an \( \overrightarrow{E} \times \overrightarrow{E} \) matrix with entries in \( \{0, 1\} \), such that for any pair of oriented edges \( e = (u, w) \) and \( f = (z, y) \) we have that

\[
H_G(e, f) = 1\{w = z\} \times 1\{u \neq y\}.
\]

That is, \( H_G(e, f) \) is equal to 1, if \( f \) follows the edge \( e \) without creating a loop, otherwise, it is equal to zero, e.g. see an example in Figure[1] As opposed to other matrices we have seen here, \( H_G \) is index by oriented edges. Also, note that \( H_G \) is not normal. In general, our understanding of \( H_G \) is not as good as that of \( A_G \).

In the following theorem, we use the Spectral Independence method to get a result for the Ising model where, rather than using \( A_G \), we use \( H_G \). Note that mixing time bound for Glauber dynamics we get from the theorem below does not have a simple expression.
Theorem 2.7 (Non-backtracking matrix). For any $\delta \in (0, 1)$, for bounded $\Delta, \nu_G > 1$, let $G = (V, E)$ be of maximum degree $\Delta$, while assume that the non-backtracking matrix $H_G$ has spectral radius $\nu_G$. Furthermore, let $\mu_G$ be the Ising model on $G$ with zero external field and parameter $\beta \in \mathbb{R}_{\text{Ising}}(\nu_G, \delta)$.

There is a constant $C = C(\delta)$ such that the Glauber dynamics on $\mu_G$ exhibits mixing time which is at most

$$\exp \left( C \cdot \left\| \left( I - \frac{1-\delta}{\nu_G} A_G - \left( \frac{1-\delta}{\nu_G} \right)^2 (D - I) \right)^{-1} \right\|_2 \right) n \log n$$

where $A_G$ is the adjacency matrix of $G$ and $D$ is the $V \times V$ diagonal matrix such that for every $u \in V$ we have $D(u, u) = \text{degree}(u)$.

Note that the matrix in the 2-norm is the same as the one that appears in Ihara’s theorem \[18\].

We need to remark that for the bound on the mixing time in Theorem 2.7 we don’t have guarantees that it is always polynomial in $n$. However, we can argue that it is $O(n \log n)$, at least for the same values of $\beta$ that Theorem 2.1 implies $O(n \log n)$ bound for the mixing time. Of course, this is not obvious at all from the statement of the theorem. For further discussion on this matter, the reader is referred to the end of Section 7.

3. OUR APPROACH - HIGH LEVEL OVERVIEW

Consider the graph $G = (V, E)$ and a two-spin Gibbs distribution $\mu$ on this graph. In the heart of Spectral Independence (SI) lies the notion of the pairwise influence matrix $I_G^{A,\tau}$. Let us give the description of this matrix since this is the main subject of our discussion here.

For a set of vertices $A \subset V$ and a configuration $\tau$ at $A$, we let the pairwise influence matrix $I_G^{A,\tau}$, indexed by the vertices in $V \setminus A$, be such that

$$I_G^{A,\tau}(w, u) = \mu_u(1 \mid (A, \tau), (\{w\}, 1)) - \mu_u(1 \mid (A, \tau), (\{w\}, -1)) \quad \forall v, w \in V \setminus A. \quad (7)$$

The Gibbs marginal $\mu_u(1 \mid (A, \tau), (\{w\}, 1))$ indicates the probability that vertex $u$ gets 1, conditional on the configuration at $A$ being $\tau$ and the configuration at $w$ being 1. We have the analogous for the marginal $\mu_u(1 \mid (A, \tau), (\{w\}, -1))$. Note that in some works, the entry $I_G^{A,\tau}(w, u)$ is denoted as $I_G^{A,\tau}(w \rightarrow u)$.

Our focus is on $\theta_{\text{max}}(I_G^{A,\tau})$, i.e., the maximum eigenvalue of $I_G^{A,\tau}$. If for any choice of $A, \tau$ we have that $\theta_{\text{max}}(I_G^{A,\tau}) = O(1)$, then we say that the Gibbs distribution $\mu$ exhibits spectral independence. Hence, the name of the method. Spectral independence for $\mu$ implies that the corresponding Glauber dynamics has polynomial mixing time \[1\]. The precise magnitude of the mixing time in this case is a subject of intense study. Here we don’t focus on this problem, i.e., we focus only on establishing spectral independence for $\mu$.

In \[15\] it was shown that $I_G^{A,\tau}$ has a remarkable property. That is, each entry $I_G^{A,\tau}(w, u)$ can be expressed in terms of a topological construction called tree of self-avoiding walks and a set influences on this tree. The influences are specified by the parameters of our problem. At this point, it is worth giving a high level (hence imprecise) description of the aforementioned relation. For for further details see Section 6.

A walk is called self-avoiding if it does not repeat vertices. For each vertex $w$ in $G$, we define $T_{\text{SAW}}(w)$, the tree of self-avoiding walks, starting from $w$, as follows: Consider the set consisting of every walk $v_0, \ldots, v_r$ in the graph $G$ that emanates from vertex $w$, i.e., $v_0 = w$, while one of the following two holds

1. $v_0, \ldots, v_r$ is a self-avoiding walk,
2. $v_0, \ldots, v_{r-1}$ is a self-avoiding walk, while there is $j \leq r - 3$ such that $v_r = v_j$. 

\[7\]
Each one of the walks in the set corresponds to a vertex in \( T_{SAW}(w) \). Two vertices in \( T_{SAW}(w) \) are adjacent if the corresponding walks are adjacent. Note that two walks in the graph \( G \) are considered to be adjacent if one extends the other by one vertex.\(^1\)

We also use the following terminology: for vertex \( u \) in \( T_{SAW}(w) \) that corresponds to the walk \( v_0, \ldots, v_r \) in \( G \) we say that “\( u \) is a copy of vertex \( v_r \) in \( T_{SAW}(w) \)”.\(^1\)

In Figures 2 and 3 we show an example of the above construction. Figure 2 shows the initial graph \( G \), while Figure 3 shows the tree of self-avoiding walks starting from vertex \( A \). Also, note that in Figure 3 the vertices of the tree which are indicated with letter \( A \) are exactly the copies of vertex \( A \) in the initial graph.

Each entry \( I_{\Lambda,\tau}^{A,r}(w,u) \) can be expressed in terms of a sum of influences over paths in \( T_{SAW}(w) \), i.e.,

\[
I_{G}^{A,r}(w,u) = \sum_{P} \text{Infl}(P),
\]

where \( P \) in the summation varies over the paths from the root to the copies of vertex \( u \) in \( T_{SAW}(w) \).

The above construction is key to establishing spectral independence for \( \mu \). This is for both the techniques from previous works, as well as the techniques we introduce here.

The approaches, prior to those we introduce here, establish spectral independence by bounding appropriately one of \( \| I_{G}^{A,r} \|_1 \) and \( \| I_{G}^{A,r} \|_\infty \). Recall that these two norms correspond to taking the maximum absolute column/row sum of the matrix, respectively. The above construction, with \( T_{SAW}(w) \) and the influences, allows the estimation of the aforementioned matrix norms using recursion.

We also use the above construction to establish our results. We give a high level overview of two alternative approaches to establishing spectral independence. For the sake of clarity, in the presentation of the results, as well as in the discussion that follows, we explicitly refer to bounding \( \rho(I_{G}^{A,r}) \), the spectral radius of the influence matrix, rather than the maximum eigenvalue. Note that bounding the spectral radius is a stronger notion.

The first approach does not rely on matrix norms at all. Actually, the argument is quite simple, yet, powerful. It is useful to start with a concrete example. We show that there is a real number \( \xi > 0 \), which depends on the parameters of the Gibbs distribution \( \mu \), such that

\[
\rho(I_{G}^{A,r}) \leq \rho(B_1),
\]

where

\[
B_1 = \sum_{\ell=0}^{n} (\xi \cdot A_G)^\ell.
\]

\(^*\)E.g. the walks \( P' = w_0, w_1, \ldots, w_r \) and \( P = w_0, w_1, \ldots, w_r, w_{r+1} \) are adjacent with each other.

\(^1\)In the related literature, influences are defined w.r.t. the vertices of the tree, not the edges. In that respect, the influence of an edge \( e = \{x, y\} \) here, corresponds to what is considered in other works as the influence at \( y \), where \( y \) is the child of vertex \( x \) in the tree.
The above holds without any assumptions about the underlying Gibbs distribution $\mu$, e.g., spatial mixing.

We can directly use the above inequality to establish spectral independence for the Gibbs distribution of interest. Particularly, we only need to adjust the parameters of the Gibbs distribution such that $\xi < (1 - \epsilon)/\rho(A_G)$ for some fixed $\epsilon > 0$. We rely on the above inequality in order to get Theorem 2.1.

Having seen the relation between $\mathcal{I}_G^{A,\tau}$ and the tree of self-avoiding walks, actually, it is not too hard to illustrate (at least on a high level) how we derive the above inequality. Essentially, we need to show that the matrix $B_1$ from (8) dominates $\mathcal{I}_G^{A,\tau}$ in the following sense: for any $u, w \in V \setminus A$, it holds that

$$|\mathcal{I}_G^{A,\tau}(w, u)| \leq B_1(w, u)$$

(9)

Once we have above, then it is not too hard to show that (8) is true.

Note that the matrices $B_1, \mathcal{I}_G^{A,\tau}$ are not necessarily of the same dimension, i.e., $B_1$ is indexed by the set $V$, while $\mathcal{I}_G^{A,\tau}$ is indexed by $V \setminus A$. For our purposes, we only need to focus on the vertices in $w, u \in V \setminus A$.

We choose $\xi$, the parameter for $B_1$, such that

$$\xi \geq \max_e \{|\text{Infl}(e)|\},$$

where $e$ varies over the edges in all self-avoiding trees. For such $\xi$, it is not hard to see that

$$|\mathcal{I}_G^{A,\tau}(w, u)| \leq \sum_{\ell \geq 0} (\xi)^{\ell} \times (\# \text{ length } \ell \text{ paths from the root to a copy of } u \text{ in } T_{\text{SAW}}(w)),$$

where “#” stands for “number of”. Then, we argue that

$$(\# \text{ length } \ell \text{ paths from root to a copy of } u \text{ in } T_{\text{SAW}}(w)) \leq (A_G)^\ell (w, u).$$

(10)

Combining the two above inequalities we immediately get (2) and consequently (8).

Perhaps, it is worth elaborating a bit more as to why the previous inequality holds, since we are extending it for our results with the non-backtracking matrix. Recalling the definition of $T_{\text{SAW}}(w)$, let us call SAW the set of paths $P$ in $G$ that we used earlier to build $T_{\text{SAW}}(w)$. It is not hard to see that the number of length $\ell$ paths in SAW from $w$ and $u$ is equal to the l.h.s. of the inequality above. Then, (10) follows from the observation that any set of length $\ell$ paths in $G$ from $w$ to $u$ does not have cardinality larger than the total number of length $\ell$ paths from $w$ to $u$, which is equal to $(A_G)^\ell (w, u)$. Recall that a walk of length $\ell$ in the graph $G$ is any sequence of vertices $w_0, \ldots, w_\ell$ such that each consecutive pair $(w_{i-1}, w_i)$ is an edge in $G$.

Our result for the non-backtracking matrix builds on the above by exploiting a further observation: It is not hard to see that every path in SAW is also a non-backtracking walk. Recall that the walk $w_0, \ldots, w_\tau$ is called non-backtracking, if we have that $w_{i-1} \neq w_i$ for all $i$. Then, one might argue that in (10) we could instead have used for the upper bound the number of length $\ell$ non-backtracking walks from $w$ to $u$. For a further discussion, see Section 7.

All the above are useful to prove our results for the Ising model. For the Hard-core model, we need to work differently, i.e., we need a more involved analysis. We typically study the Hard-core model, but also general two spin Gibbs distributions, by means of the so-called potential method. In that respect, it is somehow, natural to return back to using matrix norms.

We establish spectral independence for the Hard-core model by using the following matrix norm for $\mathcal{I}_G^{A,\tau}$:

$$\left\|D^{-1} \cdot \mathcal{I}_G^{A,\tau} \cdot D\right\|_\infty.$$
Showing that the above norm is bounded for any choice of $A$ and $\tau$, immediately implies spectral independence for $\mu$.

We give a high level overview of how we estimate the quantity $\|D^{-1}T_G^{A,\tau} \cdot D\|_\infty$. Note that the entries of the matrix $D^{-1} \cdot T_G^{A,\tau} \cdot D$ satisfy that

$$D^{-1} \cdot T_G^{A,\tau} \cdot D(w, u) = \left( \frac{f_1(u)}{f_1(w)} \right)^{1/t} \times \sum_P \text{Infl}(P),$$

where $P$ in the summation varies over the paths from the root to a copy of vertex $u$ in $T_{\text{SAW}}(w)$.

We use the above and a recursion to estimate the absolute row-sum $\sum_u |D^{-1} \cdot T_G^{A,\tau} \cdot D(w, u)|$. In what follows, we show how to estimate the contribution to the absolute row-sum above coming from paths of length $\ell \geq 1$ in $T_{\text{SAW}}(w)$. Note that the contribution coming from the path of length 0 is, trivially, 1.

For every vertex $x$ in $T_{\text{SAW}}(w)$ which is at level 0 < $h < \ell$, we estimate $Q_x$ the sum of absolute influences of the paths from $x$ to its decedents at level $\ell$ of the tree. Using the potential method we get the following recursive relation:

$$(Q_x)^\ell \leq \delta \cdot \sum_z (Q_z)^\ell,$$

where $z$ varies over the children of $x$ in $T_{\text{SAW}}(w)$. Similarly to $Q_x$, the quantity $Q_z$ is the sum of absolute influences of the paths from $z$ to its decedents at level $\ell$ of the tree. The parameter $\ell$ is specified in the setting of the potential method. The quantity $\delta$ depends on the parameters of the Gibbs distribution.

Note that the above relation excludes the cases where $h = 0$ and $h = \ell$ for vertex $x$. For $h = \ell$, i.e., $x$ is a vertex at level $\ell$ of $T_{\text{SAW}}(w)$, we have $(Q_x)^\ell = f_1(x)$. We leave the case where $h = 0$, i.e., $x$ is the root, for the end of this discussion.

It is standard to work with the above recurrence. Particularly, for vertex $x$ at level 0 < $h \leq \ell$ we get that

$$(Q_x)^\ell \leq (\rho(A_G) \cdot \delta)^{\ell-h} f_1(x).$$

In order to derive the above, we exploit that for any vertex $z$ in the graph $G$ and for $\Gamma$ the set of neighbours of $z$, we have

$$\sum_{y \in \Gamma} f_1(y) = f_1(z) \cdot \theta_{\max}(A_G) = f_1(z) \cdot \rho(A_G).$$

The first equality follows from the definition of eigenvector $f_1$. The second equality follows from our assumption that $G$ is connected, i.e., the Perron-Frobenius Theorem implies that $\theta_{\max}(A_G) = \rho(A_G)$.

For the case where $h = 0$, i.e., $x$ is root of $T_{\text{SAW}}(w)$, we have the following: there is $c > 0$ such that

$$Q_{\text{root}} \leq c \cdot (\delta \cdot \rho(A_G))^{\ell/t} \sum_{z \sim w} \left( \frac{f_1(z)}{f_1(w)} \right)^{1/t}. $$

One can simplify the above by noting that the rightmost sum is at most $\Delta^{1-\frac{t}{t}} (\rho(A_G))^{\frac{t}{t}}$.

Recall that the quantity $Q_{\text{root}}$ is the contribution of the paths of length $\ell$ into $\sum_u |D^{-1} \cdot T_G^{A,\tau} \cdot D(w, u)|$. We bound the absolute row-sum by summing the contributions for $\ell = 0, 1, \ldots, n$.

Subsequently, we choose the parameters of the Gibbs distributions such that $c = O(1)$ and $\delta = \frac{1 - \varepsilon}{\rho(A_G)}.$

For a graph with bounded maximum degree $\Delta$, this choice implies that the absolute row sum in $D^{-1} \cdot T_G^{A,\tau} \cdot D$ that corresponds to the row of vertex $w$ is bounded, for any $w$. Hence $\|D^{-1} \cdot T_G^{A,\tau} \cdot D\|_\infty$ is bounded, too.

3.1. Structure of the rest of the paper. In Section 4 we present some basic concepts from linear algebra, theory of Markov chains, spectral graph theory that we use for our results and the analysis. Our results start appearing from Section 5 where we present the basic set-up with Gibbs tree recursions and the basic theorems that establish spectral independence, i.e., Theorems 5.2, 5.3 and 5.5. The proofs of these results are grouped together and appear in Section 11. Section 6 is an introduction to the basic concepts and results
of what we call the topological method. The proofs of the results in Section 6 are, also, grouped together and appear in Section 12. In Section 7 we present results from the topological methods that we use to prove Theorems 5.2 and 5.3. The proof of any theorems in Section 7 appear in Section 13. In Section 8 we present results for the topological methods that we use to prove Theorem 5.5. The proofs of the results in Section 8 are grouped together and appear in Section 14. Finally, the proofs of our main results, i.e., Theorems 5.2 and 5.3. The proof of Theorem 2.2 for the Hard-core model appears in Section 10.

At the end of this paper we have an Appendix with supplementary results and material.

4. Preliminaries

4.1. Measuring the speed of convergence for Markov Chains. For measuring the distance between two distribution we use the notion of total variation distance. For two distributions \( \nu \) and \( \tilde{\nu} \) on the discrete set \( \Omega \), the total variation distance satisfies

\[
||\nu - \tilde{\nu}||_{tv} = \frac{1}{2} \sum_{x \in \Omega} |\nu(x) - \tilde{\nu}(x)|.
\]

We use the notion of mixing time as a measure for the rate that an ergodic Markov chain converges to equilibrium. More specifically, let \( P \) be the transition matrix of an ergodic Markov chain \( \{X_t\}_{t \geq 0} \) on a finite state space \( \Omega \) with stationary distribution \( \mu \). For \( t \geq 0 \) and \( \sigma \in \Omega \), we let \( P^t(\sigma, \cdot) \) be the distribution of \( \{X_t\}_{t \geq 0} \) when \( X_0 = \sigma \). Then, the mixing time of \( P \) is defined as

\[
T_{\text{mix}}(P) = \min\{t \geq 0 : \forall \sigma \in \Omega \ \ ||P^t(\sigma, \cdot) - \mu(\cdot)||_{tv} \leq 1/4\}.
\]

The transition matrix \( P \) with stationary distribution \( \mu \) is called time reversible if it satisfies the so-called detailed balance relation, i.e., for any \( x, y \in \Omega \) we have \( \mu(x)P(x, y) = P(y, x)\mu(y) \). For \( P \) that is time reversible the set of eigenvalues are real numbers and we denote them as \( 1 = \theta_1 \geq \theta_2 \geq \ldots \theta_{|\Omega|} \geq -1 \). Let \( \theta^* = \max\{ |\theta_2|, |\theta_{|\Omega|}| \} \), then from [9] we have that

\[
T_{\text{mix}}(P) \leq \frac{1}{1 - \theta^*} \log \left( \frac{4}{\min_{x \in \Omega} \mu(x)} \right).
\]

The quantity \( 1 - \theta^* \) is also known as the spectral gap of \( P \).

4.2. Spectral Independence. Consider a graph \( G = (V, E) \). Assume that we are given a Gibbs distribution \( \mu \) on the configuration space \( \{\pm 1\}^V \). In the heart of SI lies the notion of the pairwise influence matrix \( I_{G}^{A, \tau} \). Due to its use, let use write the definition for a second time.

For a given a set of vertices \( A \subset V \) and a configuration \( \tau \) at \( A \), we have that \( I_{G}^{A, \tau} \) is a matrix indexed by the vertices in \( V \setminus A \). Particularly, for any \( v, w \in V \setminus A \), the entry \( I_{G}^{A, \tau}(w, u) \) satisfies that

\[
I_{G}^{A, \tau}(w, u) = \mu_u(1 | (A, \tau), (\{w\}, 1)) - \mu_u(1 | (A, \tau), (\{w\}, -1)),
\]

where \( \mu_u(1 | (A, \tau), (\{w\}, 1)) \) is the Gibbs marginal that vertex \( u \) gets 1, conditional that the configuration at \( A \) is \( \tau \) and the configuration at \( w \) is 1. We have the analogous for \( \mu_u(1 | (A, \tau), (\{w\}, -1)) \).

As far as the influence matrix is concerned, the main focus is on \( \theta_{\text{max}}(I_{G}^{A, \tau}) \) the maximum eigenvalue. When, this is bounded for any choice of \( A \subset V \) and configuration \( \tau \in \{\pm 1\}^A \), then we say that the underlying Gibbs distribution \( \mu \) is spectral independent. Let us be more formal.

**Definition 4.1** (Spectral Independence). For a real \( \eta > 0 \), the Gibbs distribution \( \mu_G \) on \( G = (V, E) \) is \( \eta \)-spectrally independent, if for every \( 0 \leq k \leq |V| - 2 \), \( A \subseteq V \) of size \( k \) and \( \tau \in \{\pm 1\}^A \) we have that \( \theta_{\text{max}}(I_{G}^{A, \tau}) \leq 1 + \eta \).
Establishing spectral independence for $\mu$ implies that the corresponding Glauber dynamics mixes in polynomial time, i.e., polynomial in the size of the graph $n$.

**Theorem 4.2 ([1]).** For $\eta > 0$, if $\mu$ is an $\eta$-spectrally independent distribution, then Glauber dynamics for sampling from $\mu$ has spectral gap which is at least

$$n^{-1} \prod_{i=0}^{n-2} \left( 1 - \frac{\eta}{n - i - 1} \right).$$

(13)

One gets a close expression for the spectral gap we have in Theorem 4.2 by working as in [5] (Theorem 5 in arxiv version) to get the following result.

**Theorem 4.3 ([5]).** For $\eta > 0$, there is a constant $C \in [0, 1]$ such that if $\mu$ is an $\eta$-spectrally independent distribution, then Glauber dynamics for sampling from $\mu$ has spectral gap which is at least $C n^{-1(1+\eta)}$.

Note that Theorems 4.2 and 4.3 imply that with spectral independence the mixing time of Glauber dynamics is polynomial in $n$, i.e., see (11). However, this polynomial can be very large. There has been improvements on Theorem 4.2 since its introduction in [1], e.g., see [6, 7].

Here we mainly focus on establishing spectral independence for the Gibbs distributions of interest. Once this is established, we use results from other works to derive bounds on the mixing time. Specifically, for the bounded $\rho(A_G)$ case, which also implies that the maximum degree $\Delta$ is bounded, we use Theorem 1.9 from [6] (arxiv version). For the unbounded cases, see the discussion in Section C in the Appendix.

4.3. **Linear algebra.** For a square $N \times N$ matrix $M$, we let $\theta_i(M)$, for $i \in [N]$ denote the eigenvalues of $M$ such that $\theta_1(M) \geq \theta_2(M) \geq \ldots \geq \theta_N(M)$. Also, we let $\text{spect}(M)$ denote the set of distinct eigenvalues of $M$. We also refer to $\text{spect}(M)$ as the spectrum of $M$.

We define the **spectral radius** of $M$, denoted as $\rho(M)$, to be the real number such that

$$\rho(M) = \max\{|\theta| : \theta \in \text{spect}(M)\}.$$ It is a well-known result that the spectral radius of $M$ is the greatest lower bound for all of its matrix norms, e.g. see Theorem 6.5.9 in [17]. Letting $||\cdot||$ be a matrix norm on $N \times N$ matrices, we have that

$$\rho(M) \leq ||M||.$$ (14)

Perhaps, it is useful to mention that for the special case where $M$ is symmetric, i.e., $M(i,j) = M(j,i)$ for all $i,j \in [N]$, we have that $\rho(M) = ||M||_2$.

For $A, B, C \in \mathbb{R}^{N \times N}$, we let $|A|$ denote the matrix having entries $|A_{i,j}|$. For the matrices $B, C$ we define $B \leq C$ to mean that $B_{i,j} \leq C_{i,j}$ for each $i$ and $j$. The following is a folklore result in linear algebra (e.g. see [22, 17]).

**Lemma 4.4.** For integer $N > 0$, let $A, B \in \mathbb{R}^{N \times N}$. If $|A| \leq B$, then $\rho(A) \leq \rho(|A|) \leq \rho(B)$.

4.4. **Concepts from algebraic graph theory.** A **walk** in the graph $G$ is any sequence of vertices $w_0, \ldots, w_\ell$ such that each consecutive pair $(w_{i-1}, w_i)$ is an edge in $G$. The length of the walk is equal to the number of consecutive pairs $(w_{i-1}, w_i)$.

4.4.1. **The adjacency matrix.** For an undirected graph $G = (V,E)$ the adjacency matrix $A_G$ is $V \times V$ matrix with entries in $\{0, 1\}$ such that for every pair $u, w \in V$ we have that

$$A_G(u, w) = 1\{u, w \text{ are adjacent in } G\}.$$ A very natural property of the adjacency matrix is that for any two $u, w \in V$ and $\ell \geq 1$ we have that

$$\left(A_G^\ell\right)(u, w) = \# \text{ length } \ell \text{ walks from } u \text{ to } w.$$ (15)
Since we assume that the graph is undirected, we have that $A_G^\ell$ is symmetric, for any integer $\ell \geq 0$. Hence, $A_G$ has real eigenvalues, while the eigenvectors corresponding to distinct eigenvalues are orthogonal with each other.

We denote with $f_i \in \mathbb{R}^V$ the eigenvector of $A_G$ that corresponds to the eigenvalue $\theta_i (A_G)$, i.e., the $i$-th largest eigenvalue. Unless otherwise specified, we have $\|f_i\|_2 = 1$.

Our assumption that $G$ is always undirected, connected implies that $A_G$ is non-negative and irreducible. Hence, the Perron Frobenius Theorem (see Section A.2 in the Appendix) implies that

$$\rho(A_G) = \theta_1(A_G) \quad \text{and} \quad f_1(u) > 0 \quad \forall u \in V.$$  \hspace{1cm} (16)

Note that if $G$ is bipartite, then we also have $\rho(A_G) = |\theta_n(A_G)|$.

Furthermore, we let $S$ be the $V \times V$ diagonal matrix such that for any $u \in V$ we have that

$$S(v, v) = f_1(u).$$  \hspace{1cm} (17)

For a connected graph $G$, which is the case here, we have that $S$ is non-singular, i.e., since $f_1(u) > 0$.

4.4.2. The Hashimoto non-backtracking matrix. Here we define the well-known Hashimoto non-backtracking matrix, first introduced in \[15\] and has been studied in mathematical physics.

For the graph $G = (V, E)$, let $\overrightarrow{E}$ be the set of oriented edges obtained by doubling each edge of $E$ into two directed edge, i.e., one edge for each direction. The non-backtracking matrix, denoted as $H_G$, is an $\overrightarrow{E} \times \overrightarrow{E}$ matrix with entries in $\{0, 1\}$, such that for any pair of oriented edges $e = (u, w)$ and $f = (z, y)$ we have that

$$H_G(e, f) = 1\{w = z\} \times 1\{u \neq y\}.$$  \hspace{1cm} (18)

That is, $H(e, f)$ is equal to 1, if $f$ follows the edge $e$ without creating a loop, otherwise, it is equal to zero.

Note that $H_G$ is not normal, i.e., it does not commute with its transpose. However, it posses a certain kind of symmetry that we exploit in the analysis. Denoting with $e^{-1}$, the edge that has the opposite direction to the edge $e \in \overrightarrow{E}$, $H_G$ exhibits the following symmetry: for any $e, f \in \overrightarrow{E}$ and for any $\ell \geq 0$ we have that

$$H_G^\ell(e, f) = H_G^\ell(f^{-1}, e^{-1}).$$  \hspace{1cm} (18)

In mathematical physics, this type of symmetry is called PT-invariance, where PT stands for parity-time.

Drawing an analogy to \[15\], for any integer $\ell \geq 1$ and any $e, f \in \overrightarrow{E}$ we have that

$$(H_G^\ell)(e, f) = \# \text{ length } \ell \text{ non-backtracking walks that start from } e \text{ and end at } f.$$  \hspace{1cm} (19)

Recall that the walk $w_0, \ldots, w_r$ is called non-backtracking, if we have that $w_{i-1} \neq w_i$ for all $i$.

5. Spectral Bounds for $T_G^{\Lambda, \tau}$ Using Tree Recursions

We start, by considering the tree $T = (V_T, E_T)$, rooted at $r$, such that every vertex has at most $\Delta$ children, for some integer $\Delta > 0$. Also, let $\mu$ be a Gibbs distribution on $\{\pm 1\}^V$, specified as in \[1\] with respect to the parameters $\beta, \gamma$ and $\lambda$.

For the region $K \subseteq V_T \setminus \{r\}$ and $\tau \in \{\pm 1\}^K$, we consider the ratio of marginals at the root $R_r^{K, \tau}$ such that

$$R_r^{K, \tau} = \frac{\mu_r(+1 \mid K, \tau)}{\mu_r(-1 \mid K, \tau)}.$$  \hspace{1cm} (20)

Recall that $\mu_r(\cdot \mid K, \tau)$ denotes the marginal of the Gibbs distribution $\mu(\cdot \mid K, \tau)$ at the root $r$. Also, note that the above allows for $R_r^{K, \tau} = \infty$, e.g., when $\mu_r(-1 \mid K, \tau) = 0$ and $\mu_r(+1 \mid K, \tau) \neq 0$.

For a vertex $u \in V_T$, we let $T_u$ be the subtree of $T$ that includes $u$ and all its descendents. We always assume that the root of $T_u$ is the vertex $u$. With a slight abuse of notation, we let $R_u^{K, \tau}$ denote the ratio of marginals at the root for the subtree $T_u$, where the Gibbs distribution is, now, with respect to $T_u$, while we impose the boundary condition $\tau(K \cap T_u)$.  


Suppose that the root $r$ is of degree $d > 0$, while let the vertices $w_1, \ldots, w_d$ be its children. We express $R^K_{r,\tau}$ it terms of $R^K_{w_i,\tau}$s by having $R^K_{r,\tau} = F_d(R^K_{w_1,\tau}, R^K_{w_2,\tau}, \ldots, R^K_{w_d,\tau})$, for
\[
F_d : [0, +\infty]^d \to [0, +\infty] \quad \text{such that} \quad (x_1, \ldots, x_d) \mapsto \lambda \prod_{i=1}^d \frac{\beta x_i + 1}{x_i + \gamma}. \tag{21}
\]

For the analysis that follows, we get cleaner results by equivalently working with log-ratios rather than ratios of Gibbs marginals. Let $H_d = \log \circ F_d \circ \exp$, which means that
\[
H_d : [-\infty, +\infty]^d \to [-\infty, +\infty] \quad \text{s.t.} \quad (x_1, \ldots, x_d) \mapsto \log \lambda + \sum_{i=1}^d \log \left( \frac{\beta \exp(x_i) + 1}{\exp(x_i) + \gamma} \right). \tag{22}
\]

From [27], it is elementary to verify that $\log R^K_{r,\tau} = H_d(\log R^K_{w_1,\tau}, \ldots, \log R^K_{w_d,\tau})$.

Finally, we let the function
\[
h : [-\infty, +\infty] \to \mathbb{R} \quad \text{s.t.} \quad x \mapsto -\frac{(1 - \beta \gamma) \cdot \exp(x)}{(\beta \exp(x) + 1)(\exp(x) + \gamma)}. \tag{23}
\]

It is straightforward that for any $i \in [d]$, we have that $\frac{\partial}{\partial x_i} H_d(x_1, \ldots, x_d) = h(x_i)$. Note that, for any integer $N > 0$, we let the set $[N] = \{1, 2, \ldots, N\}$.

Furthermore, let the interval $J_d \subseteq \mathbb{R}$ be defined as follows:
\[
J_d = \begin{cases} 
[(\log \lambda \beta^d), \log(\lambda^d/\gamma^d)] & \text{if } \beta \gamma < 1, \\
[(\log \lambda^d/\gamma^d), \log(\lambda^d)] & \text{if } \beta \gamma > 1.
\end{cases}
\]

Standard algebra implies that $J_d$ contains all the log-ratios for a vertex with $d$ children. Also, let
\[
J = \bigcup_{d \in [\Delta]} J_d. \tag{24}
\]

Note that the set $J$ contains all log-ratios in the tree $T$.

5.1. A first attempt. Having introduced the notion of the (log-)ratio of Gibbs marginals and the related recursions we present the first set of results that we use to establish spectral independence. We utilise these results to prove Theorems [2.1] and [2.7] about the Ising model.

**Definition 5.1 (δ-contraction).** Let $\delta \geq 0$, the integer $\Delta \geq 1$ and $\beta, \gamma, \lambda \in \mathbb{R}$ are such that $0 \leq \beta \leq \gamma$, $\gamma > 0$ and $\lambda > 0$. We say that the set of functions $\{H_d\}_{d \in [\Delta]}$, defined in (22), exhibits $\delta$-contraction, with respect to $(\beta, \gamma, \lambda)$, if it satisfies the following condition:

For any $d \in [\Delta]$ and every $(y_1, \ldots, y_d) \in [-\infty, +\infty]^d$ we have that
\[
||\nabla H_d(y_1, \ldots, y_d)||_\infty \leq \delta. \tag{25}
\]

Clearly, the condition in (25) is equivalent to having $h(z) \leq \delta$, for any $z \in [-\infty, +\infty]$.

**Theorem 5.2.** Let $\Delta \geq 1$, $\rho_G \geq 1$, $\epsilon \in (0, 1)$ and $\beta, \gamma, \lambda \in \mathbb{R}$ be such that $0 \leq \beta \leq \gamma$, $\gamma > 0$ and $\lambda > 0$.

Let $G = (V, E)$ be of maximum degree $\Delta$, while $A_G$ is of spectral radius $\rho_G$. Also, consider $\mu_G$ the Gibbs distribution on $G$, specified by the parameters $(\beta, \gamma, \lambda)$.

For $\delta = \frac{\epsilon \tau}{\rho_G}$, suppose that the set of functions $\{H_d\}_{d \in [\Delta]}$ specified with respect to $(\beta, \gamma, \lambda)$ exhibits $\delta$-contraction. Then, for any $A \subseteq V$ and any $\tau \in \{\pm 1\}^A$, the pairwise influence matrix $I_G^{\tau A}$, induced by $\mu_G$, satisfies that
\[
\rho\left(I_G^{\tau A}\right) \leq \epsilon^{-1}
\]

We use the above result to prove Theorem [2.1] for the Ising model. Note that Theorem 5.2 applies to a general Gibbs distribution, i.e., not necessarily only on the Ising model. In Section [D] in the Appendix, we show how the above result implies rapid mixing for Glauber dynamics on general Gibbs distribution.
In order to prove Theorem 5.2, we use the following Theorem 5.3 which is similar in spirit to Theorem 5.2 but exploits the spectrum of the non-backtracking matrix. Note that the bound on the spectral radius of $\mathcal{I}^{A,\tau}_G$ we get from the theorem below does not have a simple expression.

**Theorem 5.3.** Let $\Delta \geq 1$, $\nu_G \geq 1$, $\epsilon \in (0,1)$ and $\beta, \gamma, \lambda \in \mathbb{R}$ be such that $0 \leq \beta \leq \gamma$, $\gamma > 0$ and $\lambda > 0$. Let $G = (V, E)$ be of maximum degree $\Delta$, while $H_G$ has spectral radius $\nu_G$. Also, consider $\mu_G$ the Gibbs distribution on $G$, specified by the parameters $(\beta, \gamma, \lambda)$.

For $\delta = \frac{1-\epsilon}{\nu_G}$, suppose that the set of functions $\{H_d\}_{d \in [\Delta]}$ specified with respect to $(\beta, \gamma, \lambda)$ exhibits $\delta$-contraction. Then, for any $\Lambda \subset V$ and any $\tau \in \{\pm 1\}^\Lambda$, the pairwise influence matrix $\mathcal{I}^{A,\tau}_G$, induced by $\mu_G$, satisfies that

$$\rho(\mathcal{I}^{A,\tau}_G) \leq \left\| I - \frac{1-\epsilon}{\nu_G} A_G + \left(\frac{1-\epsilon}{\nu_G}\right)^2 (D-I) \right\|^{-1}_2,$$  

(26)

where $A_G$ is the adjacency matrix of $G$ and $D$ is the $V \times V$ diagonal matrix such that for every $u \in V$ we have $D(u,u) = \text{degree}(u)$.

The proof of Theorem 5.3 appears in Section 11.3.

We need to remark that for the bound on $\rho(\mathcal{I}^{A,\tau}_G)$ we get from Theorem 5.3 we don’t have guarantees that is always bounded. The assumption of $\delta$-contraction for $\delta = \frac{1-\epsilon}{\nu_G}$ implies that the quantity on the r.h.s. of (26) is finite, however it might be increasing with $n$.

It is interesting to compare the bounds on spectral radius we get from Theorems 5.3 and 5.2. It is not obvious at all from its statement, but for the same parameters $\beta$ of the Ising model the bound we get from Theorem 5.3 is at most that we get for from Theorem 5.2. For further discussion on this matter, the reader is referred to the end of Section 7.

In light of all the above theorems, one might be tempted to use a condition which is weaker than $\delta$-contraction, e.g., consider norms of $\nabla H_d$ different than $\ell_\infty$. This is a perfectly reasonable idea and has been investigated in the literature in various different settings. Interestingly, with this approach it is natural to use the so-called potential method. In what follows, we introduce techniques that exploit the potential method to derive further results, somehow, stronger than those we have so far.

**5.2. A second attempt.** Perhaps it is interesting to mention that using Theorem 5.2 and working as in the proof of Theorem 2.1 one can retrieve the rapid mixing results for the Hard-core model in [16]. In order to get improved results for the Hard-core mode, we make use of potential functions, while we exploit results from [25].

Let $\Sigma$ be the set of functions $F : [-\infty, +\infty] \rightarrow (-\infty, +\infty)$ which are differentiable and increasing.

**Definition 5.4 ((s, $\delta, c$)-potential).** Let $s \geq 1$, allowing $s = \infty$, $\delta, c > 0$ and let the integer $\Delta \geq 1$. Also, let $\beta, \gamma, \lambda \in \mathbb{R}$ be such that $0 \leq \beta \leq \gamma$, $\gamma > 0$ and $\lambda > 0$.

Consider $\{H_d\}_{d \in [\Delta]}$, defined in (22) with respect to $(\beta, \gamma, \lambda)$. The function $\Psi \in \Sigma$, with image $S_{\Psi}$, is called $(s, \delta, c)$-potential if it satisfies the following two conditions:

**Contraction:** For $d \in [\Delta]$, for $(\tilde{y}_1, \ldots, \tilde{y}_d) \in (S_{\Psi})^d$, and $m = (m_1, \ldots, m_d) \in \mathbb{R}^d_{\geq 0}$ we have that

$$\chi(H_d(y_1, \ldots, y_d)) \cdot \sum_{j=1}^{d} \frac{|h(y_j)|}{\chi(y_j)} \cdot m_j \leq \delta^{\frac{s}{2}} \cdot ||m||_s,$$

(27)

where $\chi = \Psi'$, $y_j = \Psi^{-1}(\tilde{y}_j)$, while $h(\cdot)$ is the function defined in (23).

**Boundedness:** We have that

$$\max_{z_1, z_2 \in J} \left\{ \chi(z_1) \cdot \frac{|h(z_2)|}{\chi(z_2)} \right\} \leq c.$$  

(28)
Recall that the set $J$ in the index of $\max$ in (28), is defined in (24) and includes all the values of the log-ratios for a vertex with $d$ children, where $d \in [\Delta]$.

The notion of $(s, \delta, c)$-potential function we have above, is a generalisation of the so-called “$(\alpha, c)$-potential function” that was introduced in [5]. Note that the notion of $(\alpha, c)$-potential function implies the use of the $\ell_1$-norm in the analysis. The setting we consider here is more general. The condition in (27), somehow, implies that we are using the $\ell_r$-norm in our analysis, where $r$ is the Hölder conjugate of the parameter $s$ in the $(s, \delta, c)$-potential function.

**Theorem 5.5.** Let $\Delta \geq 1$, $\rho_G \geq 1$, $s \geq 1$, allowing $s = \infty$, let $\epsilon \in (0, 1)$ and $\zeta > 0$. Also, let $\beta, \gamma, \lambda \in \mathbb{R}$ be such that $\gamma > 0$, $0 \leq \beta \leq \gamma$ and $\lambda > 0$. Consider the graph $G = (V, E)$ of maximum degree $\Delta$, while $\Lambda_G$ is of spectral radius $\rho_G$. Consider, also, $\mu_G$ the Gibbs distribution on $G$ specified by the parameters $(\beta, \gamma, \lambda)$.

For $\delta = \frac{1-\epsilon}{\rho_G}$ and $c = \frac{\zeta}{\rho_G}$, suppose that there is a $(s, \delta, c)$-potential function $\Psi$ with respect to $(\beta, \gamma, \lambda)$.

Then, for any $\Lambda \subset V$, for any $\tau \in \{\pm 1\}^\Lambda$, the influence matrix $I_G^{A, \tau}$, induced by $\mu_G$, satisfies that

$$
\rho\left(I_G^{A, \tau}\right) \leq 1 + \zeta \cdot (1 - (1 - \epsilon)^s)^{-1} \cdot (\Delta/\rho_G)^{1-\frac{1}{2}}. \tag{29}
$$

Note that the bound in (29) includes the maximum degree $\Delta$. We can, easily, remove the dependency on $\Delta$ by using that $\sqrt{\Delta} \leq \rho_G$ and get

$$
\rho\left(I_G^{A, \tau}\right) \leq 1 + \zeta \cdot (1 - (1 - \epsilon)^s)^{-1} \cdot (\rho_G)^{1-\frac{1}{2}}.
$$

We use Theorem 5.5 in order to prove Theorem 2.2 for the Hard-core model.

Similar to Theorem 5.2, the above result applies to a general Gibbs distribution, i.e., not necessarily only to the Hard-core model. In Section D in the Appendix, we show how Theorem 5.5 implies rapid mixing for Glauber dynamics on general Gibbs distribution.

6. **THE TOPOLOGICAL METHOD - BASIC CONCEPTS**

6.1. **Walk-Trees.** In this section, we introduce the notion of walk-tree. Walk-trees are topological constructions which are defined with respect to the graph $G = (V, E)$ and a set of walks $\mathcal{P}$ in this graph. This notion generalises constructions from the algebraic graph theory and elsewhere such as the tree of self-avoiding walks, or the path-trees (introduced by Godsil, e.g., see [14, Chapter 6]), or the universal cover (e.g. see [23]) etc.

In the graph $G$, a walk $P$ is a sequence of vertices $w_0, \ldots, w_\ell$ such that each consecutive pair $(w_{i-1}, w_i)$ is an edge in $G$. The length of the walk $P$, denoted as $|P|$, is equal to the number of consecutive pairs $(w_{i-1}, w_i)$. Unless otherwise specified all the walks we consider will be of finite length. With the above definition, we consider the single vertex to be a walk of zero length.

Any two walks are considered to be adjacent with each other if and only if one of them extends the other, e.g. the walks $P' = w_0, w_1, \ldots, w_\ell$ and $P = w_0, w_1, \ldots, w_\ell, w_{\ell+1}$ are adjacent. Furthermore, the set $\mathcal{P}$ of walk in $G$ is called connected, if for every walk $P \in \mathcal{P}$ of length $\ell \geq 1$, there exists another walk $P' \in \mathcal{P}$ of length $\ell - 1$, such that $P$ and $P'$ are adjacent.

The above definition implies that if the non-empty set $\mathcal{P}$ is connected, then it should contain at least one walk of length zero. Particularly, the following holds for $\mathcal{P}$: if there is a vertex $r$ and a path in $\mathcal{P}$ that emanates from $r$, then $\mathcal{P}$ must include the path of length zero that consists only of the vertex $r$.

Let $\mathcal{P}$ be a connected set of walks in $G$ and let the vertex $r \in V$. Suppose that there is at least one walk in $\mathcal{P}$ that starts from $r$. We define $T_{\mathcal{P}}(r)$, the walk-tree induced by $(\mathcal{P}, r)$, as follows: the vertices of $T_{\mathcal{P}}(r)$ correspond to the walks in $\mathcal{P}$ that emanate from the vertex $r$. If two walks in $\mathcal{P}$ are adjacent, then their corresponding vertices in $T_{\mathcal{P}}(r)$ are adjacent, too. The root of $T_{\mathcal{P}}(r)$ is the vertex that corresponds to the walk that includes only the vertex $r$.

\footnote{That is, $r$ satisfies that $r^{-1} + s^{-1} = 1$.}
Since we always assume that \( P \) is a finite set, the corresponding walk-trees that are generated by \( P \) are finite graphs. When \( P \) does not include a path that starts from the vertex \( r \), then we follow the convention that \( T_P(r) \) is the empty graph.

There are natural extensions of the notion of walk-tree \( T_P(r) \) when the set of walks \( P \) is not connected. We don’t consider these cases here, as we will always deal with sets of walks that are connected. Unless otherwise specified when we refer to a set of walks \( P \) we always assume that it is connected.

For each \( u \in V \), we let \( A_{P,r}(u) \) be the subset of vertices in \( T_P(r) \) which correspond to walks \( w_0, \ldots, w_\ell \) in \( P \) such that \( w_\ell = u \). We refer to \( A_{P,r}(u) \) as the set of copies of vertex \( u \) in \( T_P(r) \). Note that it could be that \( A_{P,r}(u) = \emptyset \). Also, we let

\[
A_{P,r} = \bigcup_{u \in V} A_{P,r}(u). 
\]

That is, \( A_{P,r} \) corresponds to the vertex set of \( T_P(r) \). We also let \( E_{P,r} \) be the set of edges of \( T_P(r) \).

We consider various kinds of walk-trees in order to prove our results. In what follows we give examples of walk-trees which we use in the analysis. In the first example we have the tree of self-avoiding walks, encountered Section\[3\]. We present it now using the terminology of walk-trees.

**SAW walk-tree.** The walk \( w_0, \ldots, w_\ell \) in \( G \) is called self-avoiding, if we have that \( w_i \neq w_j \) for any \( i \neq j \). A single vertex is also considered to be a self-avoiding walk.

Let \( SAW \) be the set of the sequences of vertices \( w_0, \ldots, w_\ell \), for \( \ell \geq 0 \), such that one of the following two holds:

1. \( w_0, \ldots, w_\ell \) is a self-avoiding walk,
2. \( w_0, \ldots, w_{\ell-1} \) is a self-avoiding walk, while there is \( 0 \leq j \leq \ell - 3 \) such that \( w_\ell = w_j \).

It is straightforward that the longest sequence in \( SAW \) has length \( n \), i.e., the number of vertices in \( G \). In that respect \( SAW \) is a finite set. Furthermore, it is elementary to verify that for any \( \ell \geq 1 \) and \( P = w_0, \ldots, w_{\ell-1}, w_\ell \) such that \( P \in SAW \), the walk \( P' = w_0, \ldots, w_{\ell-1} \) is also in \( SAW \). This implies that \( SAW \) is connected.

The \( SAW \) walk-tree \( T_{SAW}(r) \), (or the tree of self-avoiding walks that starts from \( r \)) is the walk-tree that is induced by the set of walks \( SAW \) and vertex \( r \in V \).

**Non-backtracking walk-tree.** Recall that the walk \( w_0, \ldots, w_\ell \) in \( G \) is called non-backtracking, if we have that \( w_{i-1} \neq w_i \) for all \( i \). A single vertex is also considered to be a non-backtracking walk.

For integer \( k \geq 0 \), we let \( NB-k \) be the set of all non-backtracking walks in \( G \) which are of length at most \( k \). It is elementary to verify that \( NB-k \) is connected, i.e., for any \( \ell \geq 1 \) and \( P = w_0, \ldots, w_{\ell-1}, w_\ell \) such that \( P \in NB-k \), the walk \( P' = w_0, \ldots, w_{\ell-1} \) also belongs to \( NB-k \).

The \( NB-k \) walk-tree \( T_{NB-k}(r) \) corresponds to the walk-tree that is induced by the set of walks \( NB-k \) and vertex \( r \in V \).

A variant of the non-backtracking walk-tree that is commonly used in the algebraic graph theory is the tree \( T_{NB-k}(r) \), for \( k = \infty \). This is known as the universal cover of \( G \), e.g., see [14]. Note that the universal cover is an infinite walk-tree.

**MAX-k walk-tree.** Another kind of walk-tree that we consider here is what we call MAX-k walk-tree, where \( k \geq 0 \) is an integer parameter.

We let \( MAX-k \) be the set that contains all the walks in the graph \( G \) that have length at most \( k \). Arguing as in the previous two examples, we have that \( MAX-k \) is connected.

The \( MAX-k \) walk-tree \( T_{MAX-k}(r) \), corresponds to the walk-tree that is induced by the set \( MAX-k \) and the vertex \( r \in V \).

**Relations between walk-trees.** In our analysis it is common that we consider \( P \) and \( Q \) two sets of walks in the graph \( G = (V, E) \) and we need to deduce a relationship between the walk-trees \( T_Q(w) \) and \( T_P(w) \), for some vertex \( w \in V \).
Recall that we denote \( A_{P,w}, E_{P,w} \) the sets of vertices and edges, respectively, of the walk-tree \( T_P(w) \). Similarly, \( A_{Q,w}, E_{Q,w} \) for the walk-tree \( T_Q(w) \).

Each element in \( A_{P,w} \) and \( A_{Q,w} \) corresponds to a walk in the graph \( G \). If there are elements \( w_1 \in A_{P,w} \) and \( w_2 \in A_{Q,w} \) that correspond to the same walk \( P \) in the graph \( G \), then we consider that \( w_1 \) and \( w_2 \) are identical. This allows us to define the standard set relations for \( A_{P,w} \) and \( A_{Q,w} \), e.g., containment, intersection, etc.

We work similarly for \( E_{P,w} \) and \( E_{Q,w} \). Each element in \( E_{P,w} \) (and similarly \( E_{Q,w} \)) corresponds to an edge that extends walk \( P' \) to its adjacent walk \( P \), in the graph \( G \). If there are elements \( e_1 \in E_{P,w} \) and \( e_2 \in E_{Q,w} \) that both correspond to the same edge that extends walk \( P' \) to its adjacent walk \( P \), then we consider \( e_1 \) and \( e_2 \) to be identical. As before, this allows us to define the standard set relations for \( E_{P,w} \) and \( E_{Q,w} \).

**Lemma 6.1.** Let \( \mathcal{P} \) and \( \mathcal{Q} \) be two sets of walks in \( G = (V,E) \) such that \( \mathcal{Q} \subseteq \mathcal{P} \). Suppose that for the vertex \( w \in V \) the walk-trees \( T_P(w) \) and \( T_Q(w) \) are nonempty. Then, we have that \( T_Q(w) \) is a subtree of \( T_P(w) \).

The proof of Lemma 6.1 appears in Section 12.1.

### 6.2. Walk-Matrices

For a set of walks \( \mathcal{P} \) in \( G = (V,E) \), we let \( V_\mathcal{P} \subseteq V \) be the set of vertices \( r \in V \) for which there is at least one path \( P \in \mathcal{P} \) such that either \( P \) starts from vertex \( r \), or \( P \) ends at vertex \( r \).

For every \( r \in V_\mathcal{P} \), recall that \( E_{P,r} \) is the set of edges of the walk-tree \( T_P(r) \). We let

\[
E_\mathcal{P} = \bigcup_{r \in V_\mathcal{P}} E_{P,r}.
\]

Note that it is possible that an edge \( e \) appears in multiple walk trees. In \( E_\mathcal{P} \) we treat each occurrence of \( e \) in the sets \( E_{P,r} \) as a different element. That is, each element in \( E_\mathcal{P} \) is identified by the edge \( e \) and the tree that this edge belongs.

We let the set of weights \( \xi \in \mathbb{R}^{E_\mathcal{P}} \). That is, \( \xi \) assigns to each edge \( e \in E_\mathcal{P} \) weight \( \xi(e) \). Note that the definition of \( E_\mathcal{P} \) allows the same edge in different trees to take different weights.

We define the walk-matrix \( W_{\mathcal{P},\xi} \in \mathbb{R}^{V_\mathcal{P} \times V_\mathcal{P}} \) such that for every pair of vertices \( r, u \in V_\mathcal{P} \) we have that

\[
W_{\mathcal{P},\xi}(r,u) = \sum_M \text{weight}(M),
\]

where \( M \) varies over all paths in \( T_P(r) \) from the root to the set of copies of \( u \) in the tree, i.e., the set of vertices \( A_{P,r}(u) \), while

\[
\text{weight}(M) = \prod_{e \in M} \xi(e).
\]

That is, \( \text{weight}(M) \) is equal to the product of weights of the edges in \( M \). When \( M \) does not have any edges, i.e., \( M \) is a single vertex path, we follow the convention that \( \text{weight}(M) = 1 \).

For the sake of completeness we consider the following extreme cases. If there are \( r, u \in V_\mathcal{P} \) such that either \( A_{P,r}(u) = \emptyset \), then we have that \( W_{\mathcal{P},\xi}(r,u) = 0 \). Also, if \( T_P(r) \) is empty, then \( W_{\mathcal{P},\xi}(r,u) = 0 \) for all \( w \in V_\mathcal{P} \).

We remark that the walk-matrix \( W_{\mathcal{P},\xi} \) is a general matrix and does not necessarily have any special algebraic structure, e.g., symmetry, irreducibility, or being positive etc. In what follows, we show some interesting cases of walk-matrices, some of which we have already encountered in our discussion, while some others we will encounter in the analysis that follows.

For integer \( k \geq 0 \), consider \( W_{\text{MAX-}k,\psi} \), the walk-matrix that is induced by the set of walks \( \text{MAX-}k \) and the constant vector \( \psi \in \mathbb{R}^{E_{\text{MAX-}k}} \) such that for every edge \( e \) we have \( \psi(e) = \zeta \), where \( \zeta > 0 \).

**Lemma 6.2.** For the matrix \( W_{\text{MAX-}k,\psi} \) we define above we have that

\[
W_{\text{MAX-}k,\psi} = \sum_{\ell=0}^{k} (\zeta \cdot A_{G})^\ell.
\]

The above results implies that \( \sum_{\ell=0}^{k} (\zeta \cdot A_{G})^\ell \) can be regarded as a walk-matrix. The proof of Lemma 6.2 is standard. For the sake of completeness, the reader can find its proof in Section 12.2.

In the following example, we consider a walk-matrix that is related to the non-backtracking matrix \( H_G \). For integer \( k \geq 0 \), consider \( W_{\text{NB-}k,\psi} \), the walk-matrix induced by the set of walks \( \text{NB-}k \) and the constant vector \( \psi \in \mathbb{R}^{E_{\text{MAX-}k}} \) such that for every edge \( e \) we have \( \psi(e) = \zeta \), where \( \zeta > 0 \).
Lemma 6.3. For the walk-matrix $W_{NB-k,\psi}$ we define above, we have that

$$W_{NB-k,\psi} = I + K \cdot \left( \sum_{\ell=0}^{k-1} \zeta^{\ell+1} \cdot H_G^\ell \right) \cdot C,$$

(32)

where $H_G$ is the $\overrightarrow{E} \times \overrightarrow{E}$ non-backtracking matrix of $G$, while $K$, $C$ are $V \times \overrightarrow{E}$ and $\overrightarrow{E} \times V$ matrices, respectively, such that for any $v \in V$ and for any $(x, z) \in \overrightarrow{E}$ we have

$$K((x, z), v) = 1\{x = v\} \quad \text{and} \quad C((x, z), v) = 1\{z = v\}.$$  (33)

The proof of Lemma 6.3 appears in Section 12.3.

Note that the matrices $K$ and $C$ are used to address the discrepancy between $W_{NB-k,\psi}$ and $H_G$ that the first matrix is a $V \times V$ while the second one is $\overrightarrow{E} \times \overrightarrow{E}$.

In what follows, we show that the influence matrix $I_G^{A,\tau}$ can also be considered as a special case of walk-matrix.

6.3. $I_G^{A,\tau}$ viewed as a walk-matrix. Consider the Gibbs distribution $\mu_G$ on the graph $G = (V, E)$, defined as in 1 with parameters $\beta, \gamma, \lambda \geq 0$. For $A \subseteq V$ and $\tau \in \{\pm 1\}^A$, recall the definition of the influence matrix $I_G^{A,\tau}$ induced by $\mu$, from (12).

Consider the walk-tree $T = T_{SAW}(w)$. In what follows, we describe how the entry $I_G^{A,\tau}(w, v)$ can be expressed using an appropriately defined spin-system on $T$. The exposition relies on results from [1, 5].

Assume w.l.o.g. that there is a total ordering of the vertices in $V$, i.e., the vertex set of $G$. Recall that for every $u \in V$, $A_{SAW,w}(u)$ corresponds to the set of copies of $u$ in $T$. In what follows, we abbreviate $A_{SAW,w}(u)$ to $A(u)$.

Let $\mu_T$ be a Gibbs distribution on $T$ which has the same specification as $\mu_G$. That is, for $\mu_T$ we use the same parameters $\beta, \gamma$ and $\lambda$ as those we have for $\mu_G$. Each $z \in A(u)$ in the tree $T$, such that $u \in A$, is assigned fixed configuration equal to $\tau(u)$. Furthermore, if we have a vertex $z$ in $T$ which corresponds to a path $w_0, \ldots, w_\ell$ in $G$ such that $w_\ell = w_j$, for $0 \leq j \leq \ell - 3$, then we set a boundary condition at vertex $z$, as well. This boundary condition depends on the total ordering of the vertices. Particularly, we set at $z$

- (a) $-1$ if $w_\ell > w_{\ell-1}$,
- (b) $+1$ otherwise.

Let $\Gamma = \Gamma(G, A)$ be the set of vertices in $T$ which have a boundary condition in the above construction, while let $\sigma = \sigma(G, \tau)$ be the configuration we obtain at $\Gamma$.

Consider the set $E_{SAW,w}$, i.e., the set of edges in $T_{SAW}(w)$. For each $e \in E_{SAW,w}$ we specify weight $\beta(e)$ as follows: letting $e = \{x, z\}$ such that $x$ is the parent of $z$ in $T_{SAW}(w)$, we set

$$\beta(e) = \begin{cases} 0 & \text{if there is boundary condition at either } x, \text{ or } z, \\ h \left( \log R_z^{\Gamma,\sigma} \right) & \text{otherwise}, \end{cases}$$  (34)

The function $h(\cdot)$ is from 23, while $R_z^{\Gamma,\sigma}$ is a ratio of Gibbs marginals at $z$ (see definitions in Section 5).

It is easy to see that (34) specifies weights for every edge $e \in E_{SAW,w}$, for every $w \in V$. In that respect, all the above construction gives rise to the $V \times V$ walk-matrix $W_{SAW,\beta}$. As we will show later (see proof of Lemma 6.4) that

$$I_G^{A,\tau}(w, v) = \sum_M \prod_{e \in M} \beta(e),$$  (35)

where $M$ varies over all paths from the root of $T_{SAW}(w)$ to the set of vertices in $A(v)$.

In light of the above, it is immediate that $I_G^{A,\tau}$ is a principal submatrix of $W_{SAW,\beta}$. That is, removing the rows and the columns of $W_{SAW,\beta}$ that correspond to the vertices $u \in A$ we obtain $I_G^{A,\tau}$.
We can be more precise than the above. Consider walk-matrix $W_{S,\phi}$, where $S \subseteq \text{SAW}$ is the set of self-avoiding walks in $G$ that do not use vertices in $A$, while $\phi \in \mathbb{R}^{E_S}$ is a restriction $\beta$, i.e.,

$$\phi(e) = \beta(e), \quad \forall e \in E_S.$$  \hspace{1cm} (36)

Note that since $S \subseteq \text{SAW}$, we also have that $E_S \subseteq E_{\text{SAW}}$. Hence, $\phi$ is well defined. In what follows, we show that $W_{S,\phi}$ and $I_G^{A,\tau}$ are identical.

**Lemma 6.4.** For the graph $G = (V, E)$, for $\beta, \gamma, \lambda$ such that $\gamma > 0$, $0 \leq \beta \leq \gamma$ and $\lambda > 0$ consider $\mu_G$, the Gibbs distribution on $G$ with parameters $\beta, \gamma, \lambda$.

For $A \subseteq V$ and $\tau \in \{\pm 1\}^A$, consider the influence matrix $I_G^{A,\tau}$ induced by $\mu_G$ as well as the walk-matrix $W_{S,\phi}$, where $S$ and $\phi \in \mathbb{R}^{E_P}$ are defined above. Then, we have that $I_G^{A,\tau} = W_{S,\phi}$.

The proof of Lemma 6.4 appears in Section 12.4.

### 7. Spectral Independence Using Entry-Based Comparisons

In this section we present the first set of results that we use to prove Theorems 5.2 and 5.3. The objective here is to use the notion from the topological method to develop criteria which allow us to compare two walk-matrices in terms of their corresponding spectral radii. To this end we prove the following result.

**Theorem 7.1** (Monotonicity for walk-matrices). Let $P$ and $Q$ be sets of walks in the graph $G = (V, E)$ such that $Q \subseteq P$. Furthermore, suppose that we have $\xi_1 \in \mathbb{R}^{E_Q}$ and $\xi_2 \in \mathbb{R}^{E_P}$, such that

$$0 \leq \xi_1(e) \leq \xi_2(e), \quad \forall e \in E_Q \subseteq E_P.$$  \hspace{1cm} (37)

Then, for any $u, v \in V_Q \subseteq V_P$, we have that

$$W_{Q,\xi_1}(u, v) \leq W_{P,\xi_2}(u, v).$$  \hspace{1cm} (38)

Furthermore, if either $V_P = V_Q$, or the matrix $W_{P,\xi_2}$ is a symmetric matrix, then we also have that

$$\rho(W_{Q,\xi_1}) \leq \rho(W_{P,\xi_2}).$$  \hspace{1cm} (39)

The proof of Theorem 7.1 appears in Section 13.1.

For any set of walks $P$ in $G$ such every $M \in P$ is of length at most $k \geq 0$, we have that

$$P \subseteq \text{MAX}-k.$$  

This follows by noting that every $M \in P$ is a walk in $G$ of length $\leq k$, hence, we also have that $M \in \text{MAX}-k$.

Combining the above observation with Theorem 7.1 we get a natural upper bound for the spectral radius of any walk-matrix $W_{P,\xi}$.

**Corollary 7.2.** For integer $k \geq 0$, let the set of walks $P$ in $G = (V, E)$ be such that every $M \in P$ is of length $\leq k$. Let $\xi_1 \in \mathbb{R}^{E_P}$ and $1 \in \mathbb{R}^{E_{\text{MAX}}-k}$ be the all-ones vector.

For any $\zeta \geq \|\xi_1\|_{\infty}$ and $\xi_2 = \zeta \times 1$, we have that $\rho(W_{P,\xi_1}) \leq \rho(W_{\text{MAX}-k,\xi_2})$.

**Proof.** As argued before, our assumption that $P$ only contains paths of length $\leq k$ imply that $P \subseteq \text{MAX}-k$.

Then, the corollary follows from Theorem 7.1 by showing that $W_{\text{MAX}-k,\xi_2}$ is symmetric.

To see that $W_{\text{MAX}-k,\xi_2}$ is symmetric, note the following: since all the components of vector $\xi_2$ have the same value $\zeta > 0$, from Lemma 6.2 we have that $W_{\text{MAX}-k,\xi_2} = \sum_{\ell=0}^{k} (\zeta \cdot A_G)$. We conclude that $W_{\text{MAX}-k,\xi_2}$ is symmetric for any $\zeta \geq 0$. The claim follows.

We use Corollary 7.2, we prove Theorem 5.2. For the proof of Theorem 5.2 see Section 11.1.

Having $I_G^{A,\tau}$ in mind, it is not hard to see that each walk in SAW is also non-backtracking, and hence

$$\text{SAW} \subseteq \text{NB-n}.$$  \hspace{1cm} (40)
For the above, we also use the observation that SAW does not contain paths of length larger than $n$. We use the above observation to get the following corollary.

**Corollary 7.3** (Non-backtracking). For the graph $G = (V, E)$, for $\beta, \gamma, \lambda$ such that $\gamma > 0$, $0 \leq \beta \leq \gamma$, and $\lambda > 0$ consider $\mu_G$, the Gibbs distribution on $G$ with parameters $\beta, \gamma, \lambda$.

For $\Lambda \subseteq V$ and $\tau \subseteq \{\pm 1\}^\Lambda$, consider the influence matrix $T_G^{\Lambda, \tau}$ induced by $\mu_G$. Let $1 \in \mathbb{R}^{\text{NB-}n}$ be the all-ones vector, while consider the vector $\phi$ defined in (36).

For any $\zeta \geq \|\phi\|_\infty$ and $\xi = \zeta \times 1$, we have that $\rho(T_G^{\Lambda, \tau}) \leq \rho(W_{\text{NB-}n, \xi})$.

**Proof.** From Lemma 6.4 we have that $T_G^{\Lambda, \tau} = W_{\Lambda, \phi}$. Recall that $\mathcal{S} \subseteq \text{SAW}$ is the set of self-avoiding walks in $G$ that do not use vertices in $\Lambda$. Since $\mathcal{S} \subseteq \text{SAW}$, from (40) we also have that $\mathcal{S} \subseteq \text{NB-}n$.

Furthermore, from Lemma B.1 in the Appendix we have that $W_{\text{NB-}n, \xi}$ is symmetric. The corollary follows by making a standard application of Theorem 7.1, i.e., using (39). The claim follows. □

We use Corollary 7.3 to prove Theorem 5.3. For a proof of this result see Section 11.3.

Comparing the Corollaries 7.2 and 7.3 one might remark that the first one is the most general, while the second one is more specific to the influence matrix $T_G^{\Lambda, \tau}$. Combining the two corollaries we get that

$$\rho(T_G^{\Lambda, \tau}) \leq \rho(W_{\text{NB-}n, \psi_1}) \leq \rho(W_{\text{MAX-}n, \psi_2}),$$

where $\psi_1$ and $\psi_2$ are constant vectors such that each entry, in both vectors, is equal to $\max_e \{\phi(e)\}$ and $\phi(e)$ is defined in (36). That is, the first inequality follows from Corollary 7.3 and the second one from Corollary 7.2.

From (41), one might remark that the bound we get from Corollary 7.3 for the spectral radius of $T_G^{\Lambda, \tau}$ is at least as good as that we get from Corollary 7.2. Furthermore, note that for Theorem 5.2 we use the bound $\rho(T_G^{\Lambda, \tau}) \leq \rho(W_{\text{MAX-}n, \psi_2})$ while for Theorem 5.3 we use the bound $\rho(T_G^{\Lambda, \tau}) \leq \rho(W_{\text{NB-}n, \psi_1})$. In that respect, for the same parameters $\beta, \gamma$ and $\lambda$ of the Gibbs distribution the bound on the spectral radius of $T_G^{\Lambda, \tau}$ we get from Theorem 5.3 is at most that we get from Theorem 5.2.

In light of the above, a similar relation holds for the bounds on the mixing time of Glauber dynamics we get from Theorems 2.1 and 2.7. That is, for the same parameter $\beta$ in the zero external field Ising model, the bound on the mixing time of Glauber dynamics that we get from Theorem 2.7 is at most the corresponding bound we get from Theorem 2.1.

### 8. Spectral Independence Using Matrix Norms

In this section we consider the natural approach of bounding the spectral radius of a walk-matrix using norms. Consider $G = (V, E)$. For $K \subseteq V$ and the diagonal, non-singular matrix $D \in \mathbb{R}^{K \times K}$, for $p \geq 1$ (allowing $p = \infty$) and $t \geq 0$, we let the matrix norm $\|\cdot\|_{D,t,p}$ be such that for any $K \times K$ matrix $M$ we have

$$\|M\|_{D,t,p} = \left\| (D^{\otimes t})^{-1} \cdot M \cdot D^{\otimes t} \right\|_p,$$

where $D^{\otimes N}$ denotes the $N$-th *Hadamard power* of the matrix $D$, i.e., $D^{\otimes N}(w, u) = (D(w, u))^N$. It is elementary to verify that the norm $\|\cdot\|_{D,t,p}$ is a well-defined as long as $D$ is non-singular. Note that the matrix $D$ is assumed to be non-negative, i.e., all its entries are non-negative numbers.

It is a well-known that the spectral radius of a matrix $A$ is the greatest lower bound for the values of all matrix norms of $A$, e.g., see Theorem 5.6.9 in [17]. In that respect, we have the following result.

**Corollary 8.1.** For any $K \subseteq V$, for any non-singular, diagonal matrix $D \in \mathbb{R}^{K \times K}$, for real numbers $t \geq 0$ and $p \geq 1$, allowing $p = \infty$, the following is true: for any $K \times K$ matrix $M$, we have that

$$\rho(M) \leq \left\| M \right\|_{D,t,p}.$$
The idea here is to use the norm defined in \[42\] and Corollary \[8.1\] to bound the spectral radius of walk-matrices, with particular focus on \( I_{G_1}^{1,\tau} \). For our purposes we consider \( p = \infty \).

Note that the definition of the \((s, \delta, c)\)-potential function, i.e., Definition \[5.4\], is a bit standard and it is tailored to working with ratios of Gibbs marginals, e.g., it includes into its properties the function \( H_d(\cdot) \) from \[23\], etc. In the following definition we describe the related notion of “potential vector” which can be applied to general walk-matrices, i.e., dealing with general weights on the edges of the walk-trees rather than distributional recursions. For an example on how to directly relate the potential vectors with the potential functions, see the proof of Theorem \[5.5\] in Section \[11.2\].

**Definition 8.2.** Let \( s \geq 1 \), allowing \( s = \infty \), and \( \delta, c > 0 \). For a set of walks \( \mathcal{P} \) in \( G = (V, E) \) and \( \psi \in \mathbb{R}^{E_{\mathcal{P}}} \), we say that \( \gamma \in \mathbb{R}^{E_{\mathcal{P}}} \) is a \((s, \delta, c)\)-potential vector with respect to \( \mathcal{P} \) and \( \psi \) if the following holds:

For each \( r \in V_{\mathcal{P}} \), for any non-root vertex \( v \in T_{\mathcal{P}}(r) \) which has \( d > 0 \) children and for any \( z \in \mathbb{R}_{\geq 0}^d \) we have that

\[
\gamma(e) \cdot \sum_{i=1}^{d} \frac{|\psi(e_i)|}{\gamma(e_i)} \cdot z_i \leq \delta^{\frac{1}{s}} \cdot \|z\|_s,
\]

where \( e \) is the edge that connects \( v \) to its parent and \( e_i \), for \( i = 1, \ldots, d \), is the edge that connects \( v \) to its \( i\)-th child in \( T_{\mathcal{P}}(r) \) (e.g., see Figure \[4\]). Furthermore, we have that

\[
\max_{e_a, e_b \in E_{\mathcal{P}}} \left\{ \frac{\gamma(e_a) \cdot |\psi(e_b)|}{\gamma(e_b)} \right\} \leq c.
\]

In the standard setting, potential functions are introduced in the recursions of the ratio of Gibbs marginals by means of the mean value theorem of the real analysis. Here, instead, the potential vector arises naturally in our analysis by employing a simple telescopic trick, e.g., see the proof of Theorem \[8.3\].

The following theorem is the main result of this part of the paper.

**Theorem 8.3.** For \( \Delta \geq 1 \), \( \rho_G \geq 1 \), for \( s \geq 1 \), allowing \( s = \infty \), for \( \delta, c > 0 \) and integer \( k \geq 0 \), let \( G = (V, E) \) be a graph of maximum degree \( \Delta \), while \( A_G \) has spectral radius \( \rho_G \).

For any set of walks \( \mathcal{P} \) in \( G \) such that the longest walk in \( \mathcal{P} \) is of length \( \leq k \) and for any \( \xi \in \mathbb{R}^{E_{\mathcal{P}}} \) such that there is \( \gamma \in \mathbb{R}_{\geq 0}^{E_{\mathcal{P}}} \) which is \((s, \delta, c)\)-potential vector with respect to \( \mathcal{P} \) and \( \xi \) the following is true:

For the walk-matrix \( W_{\mathcal{P}, \xi} \) we have that

\[
\|W_{\mathcal{P}, \xi}\|_{S_{\mathcal{P}}, \infty} \leq 1 + c \cdot (\Delta)^{1-\frac{1}{s}} \cdot (\rho_G)^{\frac{s}{2}} \cdot \sum_{l=0}^{k-1} (\delta \cdot \rho_G)^{\frac{l}{s}},
\]

where \( S_{\mathcal{P}} \) is the \( V_{\mathcal{P}} \times V_{\mathcal{P}} \) diagonal matrix such that \( S_{\mathcal{P}}(u, v) = f_1(u) \), for \( u \in V_{\mathcal{P}} \), while \( f_1 \) is from \[16\].

Note that \( S_{\mathcal{P}} \) is a principal submatrix of the \( V \times V \) matrix \( S \) we define in \[17\]. Specifically, we obtain \( S_{\mathcal{P}} \) by removing all the rows and columns of \( S \) that correspond to vertices outside \( V_{\mathcal{P}} \).

Theorem \[5.5\] follows immediately from Theorem \[8.3\], Corollary \[8.1\] and Lemma \[6.4\]. For the full proof of Theorem \[5.5\] see Section \[11.2\].
8.1. Walk-vectors. In his section, as well as in the following one, we show how we derive Theorem 8.3. Consider the norm \( \| \cdot \|_{p,t,p} \) from (42) and set \( p = \infty \). Having \( p = \infty \), essential the norm corresponds to the maximum absolute row sum of the matrix \( (D^{st})^{-1} \cdot M \cdot D^{st} \).

Evaluating the above norm in the context of walk-matrices gives rise to another topological construction which we call walk-vector.

**Definition 8.4 (Walk-Vector).** For \( s \geq 1 \), allowing \( s = \infty \), and \( \delta, c > 0 \), for the set of walks \( P \) in the graph \( G = (V, E) \) and the invertible, diagonal matrix \( D \in \mathbb{R}^{K \times K}_{\geq 0} \), such that \( V_P \subseteq K \subseteq V \), we define the walk-vector \( q = q(P, D, s, \delta, c) \in \mathbb{R}^{V_P} \) as follows:

For \( r \in V_P \), suppose the root of \( T_P(r) \) has \( d > 0 \) children, while let \( T_i \) be the subtree that includes the \( i \)-th child of the root and its descendents. Then, for the component \( q(r) \) of the walk-vector we have that

\[
q(r) = 1 + \frac{c}{D(r,r)} \cdot \sum_{i=1}^{d} \sum_{s=0}^{\infty} \left( \delta^s \cdot \sum_{w \in V_P} |A_i(w)| \cdot (D(w,w))^s \right)^{1/2} > 0,
\]

where \( A_i(w) \) is the set of copies of vertex \( w \) in the subtree \( T_i \) that are at distance \( \ell \) from the root of \( T_i \).

Note that \( q \) is a positive vector. For us here, the parameter \( s \) is a bounded number, however, since the size of the walk-trees is assumed to always be finite, it is easy to see that the quantity in the r.h.s. of (43) is well-defined even for \( s = \infty \).

To get an intuition of what is the walk-vector \( q \) in Definition 8.4 consider the case where \( D \) is the identity matrix, i.e., \( D = I \) and \( s = 1 \). Then, \( q(r) \) is nothing more than the weighted sum over all paths of the tree \( T_P(r) \) that start from the root, such that each path of length \( \ell > 0 \) has weight \( c \cdot \delta^\ell \), while the path of length 0 has weight 1, that is

\[
q(r) = 1 + c \cdot \sum_{\ell} \delta^{\ell},
\]

where \( \ell \) varies over all paths of length greater than 0 in \( T_P(r) \) that emanate from the root.

In the general case, the walk-vectors arise when we are dealing with a walk-matrix \( W_{P, \xi} \) which admits a (\( s, \delta, c \))-potential vector \( \gamma \). Particularly, the component \( q(r) \) in (43) expresses a bound on the absolute row sum at the row \( r \) of \( (D^{os})^{-1} \cdot W_{P, \xi} \cdot D^{os} \). We derive this bound by using the potential vector \( \gamma \). In that respect, the following result comes naturally.

**Theorem 8.5.** For \( s \geq 1 \), allowing \( s = \infty \), for \( \delta, c > 0 \), let the set of walks \( P \) in \( G \) and \( \xi \in \mathbb{R}^{E_P} \), while assume that \( \gamma \in \mathbb{R}^{E_P} \) is a \( (s, \delta, c) \)-potential vector with respect to \( P \) and \( \xi \). For any diagonal, non-singular matrix \( D \in \mathbb{R}^{V_P \times V_P}_{\geq 0} \), the walk-vector \( q = q(P, D^{os}, s, \delta, c) \) satisfies that

\[
||W_{P, \xi}||_{D^{os}, \infty} \leq ||q||_{\infty}.
\]

The proof of Theorem 8.5 appears in Section 14.2.

In light of Theorem 8.5 Theorem 8.3 follows by bounding appropriately \( ||q||_{\infty} \), for the special case where \( D = S_P \), i.e., \( S_P \) is the matrix defined in the statement of Theorem 8.3. We study this problem in the following section by investigating the properties of walk-vectors.

8.2. Monotonicity properties for walk-vectors. In this section we prove monotonicity results for walk-vectors. These are similar in spirit to what we had in Theorem 7.1 for walk-matrices.

**Theorem 8.6 (Monotonicity for walk-vectors).** Let \( s \geq 1 \), allowing \( s = \infty \), and \( \delta, c \geq 0 \). For any \( P \) and \( S \), sets walks in the graph \( G = (V, E) \) such that \( S \subseteq P \) and for any invertible, diagonal matrix \( D \in \mathbb{R}^{V_P \times V_P}_{\geq 0} \), we consider the walk-vectors \( q_P = q_P(P, D, s, \delta, c) \in \mathbb{R}^{V_P} \) and \( q_S = q_S(S, D, s, \delta, c) \in \mathbb{R}^{V_S} \).

For any \( r \in V_S \subseteq V_P \) we have that

\[
0 < q_S(r) \leq q_P(r).
\]
Furthermore, for any \( t \geq 1 \), including \( t = \infty \), we have that
\[
\|q_S\|_t \leq \|q_P\|_t.
\] (47)

The proof of Theorem 8.6 appears in Section 14.3.

Note that in the above theorem all the parameters of the two walk-vectors \( q_P \) and \( q_S \) are the same, apart from the set of the corresponding walks.

At this point, we need to recall the observation we use for Corollary 7.2. That is, for any set of walks \( \mathcal{P} \) in \( G \) such that there is no walk in the set which is of length greater than \( k \), we have that \( \mathcal{P} \subseteq \text{MAX-} k \).

Combining this observation with Theorem 8.6 we get the following result.

**Corollary 8.7.** Let \( k \geq 1 \), let \( s \geq 1 \), allowing \( s = \infty \) and \( \delta, c \geq 0 \). For the set of walks \( \mathcal{P} \) in the graph \( G = (V, E) \) such that the length of the longest walk in \( \mathcal{P} \) is at most \( k \), for any invertible, diagonal matrix \( D \in \mathbb{R}_{\geq 0}^{V \times V} \), the following holds:

Consider the walk-vectors \( q_P = q_P(\mathcal{P}, D, s, \delta, c) \in \mathbb{R}_{\geq 0}^{V_P} \) and \( q_{\text{MAX-}k} = q_{\text{MAX-}k}(\text{MAX-}k, D, s, \delta, c) \in \mathbb{R}_{\geq 0}^V \). For any \( r \in V_P \) we have that
\[
q_P(r) \leq q_{\text{MAX-}k}(r).
\]

Furthermore, for any \( t \geq 1 \), allowing for \( t = \infty \), we have that \( \|q_P\|_t \leq \|q_{\text{MAX-}k}\|_t \).

Theorems 8.5, 8.6 and Corollary 8.7 imply that we can bound the quantity \( \|W_{P, \xi}\|_{S_{P, \frac{1}{2}, \infty}} \), in the statement of Theorem 8.3, by using the \( \ell_\infty \) norm of the walk-vector \( q_{\text{MAX-}k} = q_{\text{MAX-}k}(\text{MAX-}k, S^{\frac{1}{2}} \gamma, \lambda, s, \delta, c) \), where \( S \) is the \( V \times V \) matrix defined in (17) and the parameters \( s, \delta, c \) are specified in Theorem 8.3. Specifically, we have
\[
\|W_{P, \xi}\|_{S_{P, \frac{1}{2}, \infty}} \leq \|q_{\text{MAX-}k}\|_\infty.
\]

Theorem 8.3 follows by using the above and by bounding appropriately \( \|q_{\text{MAX-}k}\|_\infty \). For the full proof of Theorem 8.3, see Section 14.1.

### 9. Proof of Theorems 2.1 and 2.7

For \( d > 0 \), consider the functions \( H_d \) and \( h(\cdot) \) defined in (22) and (23), respectively. Recalling that the zero external field Ising model \( \mu_G \) corresponds to setting the parameters of \( \mu_G \) such that \( \beta = \gamma = 1 \), we have that
\[
H_d : [-\infty, +\infty]^d \to [-\infty, +\infty] \quad \text{s.t.} \quad (x_1, \ldots, x_d) \mapsto \sum_{i=1}^d \log \left( \frac{\beta \exp(x_i) + 1}{\exp(x_i) + \beta} \right).
\] (48)

Since \( \frac{\partial}{\partial x_i} H_d(x_1, \ldots, x_d) = h(x_i) \), we have that
\[
h(x) = -\frac{(1-\beta^2) \exp(x)}{(\beta \exp(x) + 1)(\exp(x) + \beta)}.
\] (49)

**Lemma 9.1.** For any \( d > 0 \), \( \zeta \in (0, 1) \), \( R \geq 1 \) and \( \beta \in \mathcal{M}_{\text{Ising}}(R, \zeta) \) we have the following: Consider the functions \( H_d \) defined in (22) with respect to the Ising model with parameter \( \beta \) and no external field. We have that
\[
\|\nabla H_d(y_1, y_2, \ldots, y_d)\|_\infty \leq (1 - \zeta)/R.
\] (50)

**Proof.** It suffices to show that any \( d > 0 \) and any \( (y_1, y_2, \ldots, y_d) \in [-\infty, +\infty]^d \) we have that
\[
\|\nabla H_d(y_1, y_2, \ldots, y_d)\|_\infty \leq \frac{|\beta - 1|}{\beta + 1}.
\] (51)

Before showing that (51) is true, let us show how it implies (50). That is, we show that for any \( \beta \in \mathcal{M}_{\text{Ising}}(R, \zeta) \), we have that \( \frac{|\beta - 1|}{\beta + 1} \leq \frac{1-\zeta}{R} \).
Consider the function \( f(x) = \frac{|x-1|}{x+1} \) defined on the closed interval \( \left[ \frac{R-1}{R+1}, \frac{R+1}{R+1} \right] \). Taking derivatives, it is elementary to verify that \( f(x) \) is increasing in the interval \( 1 < x \leq \frac{R-1}{R+1} \), while it is decreasing in the interval \( \frac{R-1}{R+1} \leq x < 1 \). Furthermore, noting that \( f(1) = 0 \), it is direct that
\[
\sup_{\beta \in M_{\text{Ising}}(r, \zeta)} f(\beta) = f \left( \frac{R-1+\zeta}{R+1-\zeta} \right) = f \left( \frac{R+1-\zeta}{R-1+\zeta} \right) = \frac{1-\zeta}{R}.
\]

It is immediate that indeed (51) implies (50). Hence, it remains to show that (51) is true.

Since we have that \( \frac{\partial}{\partial x_i} H_d(x_1, x_2, \ldots, x_d) = h(x_i) \), it suffices to show that for any \( x \in [-\infty, +\infty] \) we have that
\[
|h(x)| \leq \frac{1-\beta}{1+\beta}.
\]

For the distribution we consider here, the function \( h() \) is given from (49). From the above we get that
\[
|h(x)| = \frac{|1-\beta|^2 \exp(x)}{(\beta \exp(x)+1)(\beta \exp(x))} \leq \frac{|1-\beta|^2}{\beta^2+1+\beta(\exp(-x)+\exp(x))}.
\]

It is straightforward to verify that \( \phi(x) = e^{-x} + e^x \) is convex and for any \( x \in [-\infty, +\infty] \) the function \( \phi(x) \) attains its minimum at \( x = 0 \), i.e., we have that \( \phi(x) \geq 2 \). Consequently, we get that
\[
|h(x)| \leq \frac{|1-\beta|^2}{\beta^2+1+2\beta} = \frac{|1-\beta|^2}{(\beta+1)^2} = \frac{|1-\beta|}{1+\beta},
\]
for any \( x \in [-\infty, +\infty] \). The above proves that (52) is true and concludes our proof. \( \square \)

9.1. Proof of Theorem 2.1. Note that if \( \rho_G \), the spectral radius of \( A_G \), is bounded, then the same holds for the maximum degree \( \Delta \) of \( G \). This follows from the standard relation that \( \Delta \leq (\rho_G)^2 \leq \Delta^2 \). Similarly, if \( \rho_G \) is unbounded then \( \Delta \) is unbounded, too.

Using Lemma 9.1, we get the following: for any \( \beta \in M_{\text{Ising}}(\rho_G, \delta) \), the set of functions \( \{H_d\}_{d \in \Delta} \) defined in (48) with respect to the zero external field Ising model with parameter \( \beta \) exhibits \((1-\delta)/\rho_G\)-contraction, i.e.,
\[
||\nabla H_d(y_1, y_2, \ldots, y_d)||_\infty \leq (1-\delta)/\rho_G \quad \forall d \in \Delta.
\]

The above, combined with Theorem 5.2 imply that for \( A \subseteq V \) and \( \tau \in \{\pm1\}^A \), the pairwise influence matrix \( T_G^{A,\tau} \), induced by \( \mu_G \), satisfies that
\[
\rho(T_G^{A,\tau}) \leq \delta^{-1}.
\]

The theorem follows as a corollary from (53) and Theorem 1.9 in [6]. \( \square \)

9.2. Proof of Theorem 2.7. Using Lemma 9.1 we get the following: for any \( \beta \in M_{\text{Ising}}(\nu_G, \delta) \), the set of functions \( \{H_d\}_{d \in \Delta} \) defined in (48) with respect to the zero external field Ising model with parameter \( \beta \) exhibits \((1-\delta)/\nu_G\)-contraction, i.e.,
\[
||\nabla H_d(y_1, \ldots, y_d)||_\infty \leq (1-\delta)/\nu_G.
\]

The above, combined with Theorem 5.3 imply that for \( A \subseteq V \) and \( \tau \in \{\pm1\}^A \), the pairwise influence matrix \( T_G^{A,\tau} \), induced by \( \mu_G \), satisfies that
\[
\rho(T_G^{A,\tau}) \leq \left\| \left( I - \frac{1-\delta}{\nu_G} A_G + \frac{1-\delta}{\nu_G} (D - I) \right)^{-1} \right\|_2.
\]

The theorem follows by making a standard use of and Theorem 1.9 in [6]. \( \square \)
10. Proof of Theorem 10.1

We start our proof by deriving a bound on \( \rho(\mathcal{I}_G^{\lambda,\tau}) \), the spectral radius of the influence matrix.

**Theorem 10.1.** For any \( \epsilon \in (0, 1) \), for \( \Delta \geq 2 \) and \( \rho_G \geq 2 \), let \( G = (V, E) \) be of maximum degree \( \Delta \), while \( A_G \) has spectral radius \( \rho_G \). Also, let \( \mu_G \) be the Hard-core model on \( G \), with fugacity \( 0 < \lambda \leq (1 - \epsilon)\lambda_c(\rho_G) \).

There is a constant \( 0 < z < 1 \), that depends only on \( \epsilon \), such that for any \( \Delta \subseteq V \) and \( \tau \in \{-1, 1\}^A \), the pairwise influence matrix \( \mathcal{I}_G^{\lambda,\tau} \), induced by \( \mu_G \), satisfies that

\[
\rho(\mathcal{I}_G^{\lambda,\tau}) \leq 1 + e^3 \left( \frac{\Delta}{\rho_G} \right)^{1/2} z^{-1}.
\]

The proof of Theorem 10.1 appears in Section 10.1.

Note that if \( \rho_G \) is unbounded, then there are no guarantees from Theorem 10.1 that \( \rho(\mathcal{I}_G^{\lambda,\tau}) \) is bounded. This is due to the quantity \( \Delta/\rho_G \), on the r.h.s. of (29), which can be unbounded if \( \rho_G \) is unbounded.

**Proof of Theorem 10.2** As argued in the proof of Theorem 2.1, if the spectral radius \( \rho_G \) is bounded, then the same holds for the maximum degree \( \Delta \) as we always have that \( \Delta \leq (\rho_G)^2 \leq \Delta^2 \).

Since both \( \rho_G \) and \( \Delta \) are bounded, for fugacity \( 0 \leq \lambda \leq (1 - \epsilon)\lambda_c(\rho_G) \), Theorem 10.1 implies that

\[
\rho(\mathcal{I}_G^{\lambda,\tau}) = o(1) \quad \text{for any} \quad \Delta \subseteq V \quad \text{and} \quad \tau \in \{-1, 1\}^A.
\]

Then, Theorem 10.2 follows by a standard application of Theorem 1.9 in [6]. \( \square \)

10.1. Proof of Theorem 10.1

In order to prove Theorem 10.1 we introduce the potential function \( \Psi \) as follows: we define \( \Psi \) indirectly, i.e., in terms of \( \chi = \Psi' \). We have that

\[
\chi : \mathbb{R}_{>0} \to \mathbb{R} \quad \text{such that} \quad y \mapsto \sqrt{\frac{e^y}{1+e^y}}, \quad (54)
\]

while \( \Psi(0) = 0 \).

Note that the potential function \( \Psi \) was proposed in a more general form in [5]. It is standard to show that \( \Psi \) is well-defined, e.g., see [5]. Later in our analysis we need to use certain results from [25], which (essentially) use another, but related potential function from [21]. We postpone this discussion until later.

For any given \( \lambda > 0 \), we define, implicitly, the function \( \Delta_c(\lambda) \) to be the positive number \( z > 1 \) such that

\[
\frac{z^z}{(z-1)^{z-1}} = \lambda. \quad \text{Form its definition it is not hard to see that} \quad \Delta_c(\cdot) \quad \text{is the inverse map of} \quad \lambda_c(\cdot), \quad \text{i.e., we have that} \quad \Delta_c(x) = \lambda_c^{-1}(x). \quad \text{In that respect,} \quad \Delta_c(x) \quad \text{is well-defined as} \quad \lambda_c(x) \quad \text{is monotonically decreasing in} \quad x.
\]

**Theorem 10.2.** For \( \lambda > 0 \), let \( \Delta_c = \Delta_c(\lambda) \). The potential function \( \Psi \) defined in (54) is a \((s_0, \delta_0, c_0)\)-potential function (as in Definition 5.4) such that

\[
s_0^{-1} = 1 - \frac{\Delta_c^{-1}}{2} \log \left( 1 + \frac{1}{\Delta_c^{-1}} \right), \quad \delta_0 \leq \frac{1}{\Delta_c} \quad \text{and} \quad c_0 \leq \frac{\lambda}{1 + \lambda}. \quad (55)
\]

The proof of Theorem 10.2 appears in Section 10.2

**Claim 10.3.** For \( \epsilon \in (0, 1) \), \( R \geq 2 \) and \( 0 < \lambda < (1 - \epsilon)\lambda_c(R) \) the following is true: There is \( 0 < z < 1 \), which only depend on \( \epsilon \), such that for \( \Delta_c = \Delta_c(\lambda) \), we have

\[
\frac{1-z}{R} \geq \frac{1}{\Delta_c} \quad \text{and} \quad \frac{\lambda}{1 + \lambda} \leq e^3 \frac{R}{(1-z)} \quad (56)
\]

**Proof.** It elementary to verify that \( \Delta_c(z) \) is decreasing in \( z \). This implies that for \( \lambda \leq (1 - \epsilon)\lambda_c(R) \), \( \Delta_c(\lambda) \geq \Delta_c(\lambda_c(R)) = R \). Particularly, this implies that there is \( 0 < z < 1 \), which only depends on \( \epsilon \) such that \( \Delta_c(\lambda) \geq \frac{R}{(1-z)^2} \). This proves the leftmost inequality in (56).

As far as the rightmost inequality is concerned, we have that

\[
\frac{\lambda}{1 + \lambda} \leq \lambda < \lambda_c(R). \quad (57)
\]
The first inequality follows since \( \lambda > 0 \), while the second follows since \( \lambda < \lambda_c(R) \). From the definition of \( \lambda_c(\cdot) \), we have that
\[
\lambda_c(R) = \frac{R^e}{(R-1)^{(R-1)}} = \frac{1}{R} \left( 1 - \frac{1}{R} \right)^{(R-1)} = \frac{1}{R} \left( 1 + \frac{1}{R-1} \right)^{R-1} \leq \frac{1}{R} \exp \left( \frac{R+1}{R-1} \right) \leq e^{3/R}. \tag{58}
\]

For the one before the last inequality we use that \( 1 + x \leq e^x \). For the last inequality we note that \( \frac{R+1}{R-1} \) is decreasing in \( R \), hence, for \( R \geq 2 \), we have that \( \frac{R+1}{R-1} \leq 3 \). Plugging the above bound into \( \eqref{57} \), gives the rightmost inequality in \( \eqref{56} \). The claim follows.

Furthermore, using the standard inequality that \( \log(1 + y) \leq y \), for the quantity \( s_0 \) in Theorem \( \ref{10.2} \) we have that \( 1 \leq s_0 \leq 2 \).

Combining this observation with Theorem \( \ref{10.2} \) and Claim \( \ref{10.3} \) we get the following corollary.

**Corollary 10.4.** For \( 0 < \epsilon < 1 \), for \( \Delta \geq 2 \) and \( \rho_G \geq 2 \), let \( 0 < \lambda < (1 - \epsilon)\lambda_c(\rho_G) \). Let the graph \( G = (V, E) \) be of maximum degree \( \Delta \), whose adjacency matrix has spectral radius \( \rho_G \). Let, also the Hard-core model \( \mu_G \) on \( G \), with fugacity \( \lambda \). There is a constant \( 0 < z < 1 \), which depends on \( \epsilon \), such that the following is true:

The potential function \( \Psi \) defined in \( \eqref{54} \) is \((q, \gamma, g)\)-potential with respect to \( \mu_G \) where
\[
q \in [1, 2], \quad \gamma < (1 - z)/\rho_G \quad \text{and} \quad g \leq e^{3/\rho_G}.
\]

Theorem \( \ref{10.1} \) follows by combining Corollary \( \ref{10.4} \) and Theorem \( \ref{5.5} \).

### 10.2. Proof of Theorem 10.2

Recall that the ratio of Gibbs marginals \( R_x^{A,\tau} \), defined in Section 5, is possible to be equal to zero, or \( \infty \). Typically, this happens if the vertex \( x \) with respect to which we consider the ratio is a part of the boundary set \( \Lambda \), or has a neighbour in \( \Lambda \). When we are dealing with the Hard-core model, there is a standard way to avoid these infinities and zeros in our calculations.

Suppose that we have the Hard-core model with fugacity \( \lambda > 0 \) on a tree \( T \), while at the set of vertices \( \Lambda \) we have a boundary condition \( \tau \). Then, it is elementary to verify that this distribution is identical to the Hard-core model with the same fugacity on the tree (or forest) \( T' \) which is obtained from \( T \) by working as follows: we remove from \( T \) every vertex \( w \) which either belongs to \( \Lambda \), or has a neighbour \( u \in \Lambda \) such that \( \tau(u) = \text{"occupied"} \).

Perhaps it is useful to write down the functions that arise from the recursions in Section 5 for the Hard-core model with fugacity \( \lambda \). Recall that, in this case, we have \( \beta = 0 \) and \( \gamma = 1 \). In the following definitions, we take into consideration that boundary conditions have been removed as described above.

For integer \( d \geq 1 \), we have that
\[
F_d : \mathbb{R}_{>0}^d \to (0, \lambda) \quad \text{such that} \quad (x_1, \ldots, x_d) \mapsto \lambda \prod_{i=1}^d \frac{1}{x_i + 1}. \tag{59}
\]
We also define \( F_{d, \text{sym}} : \mathbb{R}_{>0}^d \to (0, \lambda) \) the symmetric version of the above function, that is
\[
x \mapsto F_d(x, x, \ldots, x). \tag{60}
\]
Recall, also, that \( H_d = \log \circ F_d \circ \exp \). For the Hard-core model with fugacity \( \lambda \), we have that
\[
H_d : \mathbb{R}^d \to \mathbb{R} \quad \text{s.t.} \quad (x_1, \ldots, x_d) \mapsto \log \lambda + \sum_{i=1}^d \log \left( \frac{1}{\exp(x_i) + 1} \right). \tag{61}
\]
For \( h(\cdot) \) such that \( \frac{\partial}{\partial x_i} H_d(x_1, \ldots, x_d) = h(x_i) \), we have
\[
h : \mathbb{R} \to \mathbb{R} \quad \text{such that} \quad x \mapsto -\frac{e^x}{e^x + 1}, \tag{62}
\]
Finally, the set of log-ratios \( J \), defined in \( \eqref{24} \), satisfies that
\[
J = (-\infty, \log(\lambda)). \tag{63}
\]
Note also, that the image of \( \Psi \), i.e., the set \( S_\Psi \), satisfies that \( S_\Psi = (-\infty, \infty) \).
With all the above, we proceed to prove the theorem. We need to show that $\Psi$ satisfies the contraction and the boundedness conditions, for appropriate parameters.

We start with the contraction. For any integer $d > 0$, we let $E_d : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be such that for $m = (m_1, \ldots, m_d) \in \mathbb{R}_{\geq 0}^d$, and $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ we have that

$$E_d(m, y) = \chi(H_d(y)) \sum_{j=1}^d \frac{|h(y_j)|}{\chi(y_j)} \times m_j.$$ 

**Proposition 10.5** (contraction). For $\lambda > 0$, let $\Delta_c = \Delta_c(\lambda)$. Let $q > 0$ be such that

$$q^{-1} = 1 - \frac{\Delta_c - 1}{2} \log \left(1 + \frac{1}{\Delta_c - 1}\right).$$

(64)

For $d > 0$, for $m \in \mathbb{R}_{\geq 0}^d$ we have that

$$\sup_{y \in (Q\lambda)^d} \{E_d(m, y)\} \leq \Delta_c^{-\frac{1}{q}} \cdot ||m||_q,$$

(65)

where $Q\lambda \subseteq \mathbb{R}$ contains every $y \in \mathbb{R}$ such that there is $\tilde{y} \in S\lambda$ for which we have $y = \psi^{-1}(\tilde{y})$.

The proof of Proposition 10.5 appears in Section 10.3. Note that Proposition 10.3 implies that $\Psi$ satisfies the contraction condition with the parameter we need in order to prove our theorem. We now focus on establishing the boundedness property of $\Psi$.

**Lemma 10.6** (boundedness). For $\lambda > 0$, we have that $\max_{y_1, y_2 \in J} \{\chi(y_2) \cdot \frac{|h(y_1)|}{\chi(y_1)}\} \leq \frac{\lambda}{1 + \lambda}$.

**Proof.** Using the definitions of the functions $\chi$ and $h$ from (67) and (62), respectively, we have that

$$\max_{y_1, y_2 \in J} \{\chi(y_2) \cdot \frac{|h(y_1)|}{\chi(y_1)}\} = \max_{y_1, y_2 \in J} \left\{\sqrt{h(y_1)h(y_2)}\right\} = \max_{y_1, y_2 \in J} \left\{\sqrt{\frac{e^{y_1}}{1 + e^{y_1}} \frac{e^{y_2}}{1 + e^{y_2}}}\right\} = \frac{\lambda}{1 + \lambda},$$

The last inequality follows from the observation that the function $g(x) = \frac{e^x}{1 + e^x}$ is increasing in $x$, while, from (63), we have that $e^{y_1}, e^{y_2} \leq \lambda$. The claim follows. 

The above lemma implies the contraction we need in order to prove our result. In light of Proposition 10.3 and Lemma 10.6, Theorem 10.2 follows.

10.3. **Proof of Proposition 10.5**. The proposition follows by using results from [25]. However, in order to apply these results, we need to bring $E_d(m, y)$ into an appropriate form.

For any $d > 0$, we let $J_d : \mathbb{R}_{\geq 0}^d \times \mathbb{R}_{\geq 0}^d \to \mathbb{R}$ be such that for $m = (m_1, \ldots, m_d) \in \mathbb{R}_{\geq 0}^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{R}_{\geq 0}^d$ we have

$$J_d(m, z) = \chi(\log F_d(z)) \sum_{j=1}^d \frac{|h(\log z_j)|}{\chi(\log z_j)} \times m_j.$$ 

Using the definitions in (59) and (61), it is elementary to verify that for any $d > 0$, for any $m \in \mathbb{R}_{\geq 0}^d$, $z \in \mathbb{R}_{\geq 0}^d$ and $y \in \mathbb{R}^d$ such that $z_j = e^{y_j}$, we have that

$$J_d(m, z) = E_d(m, y).$$

In light of the above, the proposition follows by showing that

$$\sup_{z \in \mathbb{R}_{\geq 0}^d} \{J_d(m, z)\} \leq \Delta_c^{-1/s} \cdot ||m||_s.$$ 

(66)

In order to prove (66), we let

$$\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$$

such that

$$y \mapsto \frac{1}{2} \sqrt{\frac{1}{y(1+y)}}.$$ 

(67)
Lemma 10.9. For any $m = (m_1, \ldots, m_d) \in \mathbb{R}_{\geq 0}^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{R}_{\geq 0}^d$ we have that
\[
\mathcal{J}_d(m, z) = \psi(F_d(z)) \times \sum_{i=1}^d \frac{m_i}{\psi(z_i)} \left| \frac{\partial}{\partial t_i} F_d(t) \right|_{t=z},
\] where $F_d$ and $\psi$ are defined in (59) and (67), respectively.

Proof. The claim follows by using simple rearrangements. We have that
\[
\mathcal{J}_d(m, z) = \chi \left( \log F_d(z) \right) \sum_{j=1}^d \frac{h(\log z_j)}{\chi(\log z_j)} \times m_j
\]
\[
= \sqrt{\frac{F_d(z)}{1+F_d(z)}} \sum_{j=1}^d \frac{z_j}{1+z_j} \times m_j
\]
\[
= \sqrt{\frac{1}{F_d(z)(1+F_d(z))}} \sum_{j=1}^d \frac{z_j(1+z_j)}{1+z_j} \times \frac{F_d(z)}{1+z_j} \times m_j.
\]

In (69), we substitute $\chi$ and $h$ according to (54) and (62), respectively. Using the definition of $\psi$ from (67), we get that
\[
\mathcal{J}_d(m, z) = \psi(F_d(z)) \sum_{j=1}^d \frac{1}{\psi(z_j)} \times \frac{F_d(z)}{1+z_j} \times m_j.
\]
The above implies (68). note that $\left| \frac{\partial}{\partial t_i} F_d(t) \right| = \frac{F_d(t)}{1+t_i}$, for any $i \in [d]$. The claim follows.

In light of Claim 10.7, (66) follows by showing that for any $m = (m_1, \ldots, m_d) \in \mathbb{R}_{\geq 0}^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{R}_{\geq 0}^d$ we have that
\[
\psi(F_d(z)) \times \sum_{i=1}^d \frac{m_i}{\psi(z_i)} \left| \frac{\partial}{\partial z_i} F_d(z) \right|_{z=x} \leq \Delta_c^{-\frac{1}{2}} \cdot ||m||_s.
\]
The above follows by using standard results form (25). For any $s \geq 1$, $d > 0$ and $x \geq 0$, we let the function
\[
\Xi(s, d, x) = \frac{1}{d} \left( \psi(F_{d, \text{sym}}(x)) \frac{F_{d, \text{sym}}'(x)}{\psi'(x)} \right)^s,
\]
where the functions $F_{d, \text{sym}}$, $\psi$ are defined in (60) and (67), respectively, while $F_{d, \text{sym}}'(x) = \frac{d}{ds} F_{d, \text{sym}}(x)$.

Lemma 10.8 (25). For any $\lambda > 0$, for integer $d \geq 1$, for $s \geq 1$, for $x \in \mathbb{R}_{\geq 0}^d$ and $m \in \mathbb{R}_{\geq 0}^d$, the following holds: there exists $\bar{x} > 0$ and integer $0 \leq k \leq d$ such that
\[
\psi(F_d(x)) \times \sum_{i=1}^d \frac{m_i}{\psi(x_i)} \left| \frac{\partial}{\partial z_i} F_d(z) \right|_{z=x} \leq (\Xi(s, k, \bar{x}))^{1/s} \times ||m||_s,
\]
where $x = (x_1, \ldots, x_d)$ and $m = (m_1, \ldots, m_d)$.

In light of the above lemma, our proposition follows as a corollary from the following result.

Lemma 10.9 (25). For $\lambda > 0$, consider $\Delta_c = \Delta_c(\lambda)$ and $F_{\Delta_c, \text{sym}}$ with fugacity $\lambda$. Let $q \geq 1$ be such that
\[
q^{-1} = 1 - \frac{\Delta_c^{-1}}{2} \log \left( 1 + \frac{1}{\Delta_c^{-1}} \right).
\]
For any $x > 0$, $d > 0$, we have that
\[
\Xi(q, d, x) \leq \Xi(q, \Delta_c, \bar{x}) = (\Delta_c)^{-1},
\]
where $\bar{x} \in [0, 1]$ is the unique fix-point of $F_{\Delta_c, \text{sym}}$, i.e., $\bar{x} = F_{\Delta_c, \text{sym}}(\bar{x})$.

By combining Lemmas 10.8 and 10.9 we get (70). This concludes the proof of Proposition 10.5. \qed
11. PROOFS OF THE RESULTS IN SECTION 5

11.1. Proof of Theorem 5.2
Firstly we note that for \( \phi \in \mathbb{R}^S \) as defined in (36), we have that
\[
\max_{e \in E_S} |\phi(e)| \leq \sup \{ h(x) \mid x \in [-\infty, +\infty] \}.
\]  
(71)

The assumption that the set of functions \( \{ H_d \}_{d \in \Delta} \) exhibits \( \delta \)-contraction implies that
\[
\sup \{ h(x) \mid x \in [-\infty, +\infty] \} \leq \delta.
\]  
(72)

Combining (71) and (72), we have that
\[
\max_{e \in E_S} |\phi(e)| \leq \delta.
\]  
(73)

Recall from Lemma 6.4 that \( T_{G}^{A, \tau} \) is a walk-matrix. Furthermore, using Corollary 7.2 we get that
\[
\rho \left( T_{G}^{A, \tau} \right) \leq \rho \left( W_{\text{MAX}, n, \psi} \right).
\]  
(74)

where the vector of weights \( \psi \) has all its components equal to \( \delta \). Note that we choose \( \text{MAX-}n \) because the length of any self-avoiding path is \( \leq n \). Since all the components of \( \psi \) have the same value \( \delta > 0 \), Lemma 6.2 implies that
\[
W_{\text{MAX}, n, \psi} = \sum_{\ell=0}^{n} (\delta \cdot A_G)^\ell.
\]

We have that
\[
\rho \left( W_{\text{MAX}, n, \psi} \right) \leq \||W_{\text{MAX}, n, \psi}||_2 = \||\sum_{\ell=0}^{n} (\delta A_G)^\ell||_2 \leq \sum_{\ell=0}^{n} |\delta|^\ell \cdot ||(A_G)^\ell||_2.
\]

Furthermore, since \( \delta = \frac{1-\epsilon}{\rho_G} > 0 \) and \( ||(A_G)^\ell||_2 = (\rho_G)^\ell \), because \( A_G \) is symmetric, we get that
\[
\rho \left( W_{\text{MAX}, n, \psi} \right) \leq \sum_{\ell=0}^{n} (1-\epsilon)^\ell \leq \sum_{\ell=0}^{\infty} (1-\epsilon)^\ell = e^{-\epsilon}.
\]

The theorem follows by plugging the above into (74). \( \square \)

11.2. Proof of Theorem 5.5
As we prove in Lemma 6.4 the matrix \( T_{G}^{A, \tau} \) is identical to the walk-matrix \( W_{S, \phi} \), where \( S \) is the set of self-avoiding walks in \( G \) that do not use vertices in \( \Lambda \), while \( \phi \in \mathbb{R}^S \) is defined in (36). In that respect, it suffices to prove that under the assumption of Theorem 5.5 we have that
\[
\rho \left( W_{S, \phi} \right) \leq 1 + \zeta \cdot (1 - (1 - e)^s)^{-1} \cdot (\Delta/\rho_G)^{-\frac{1}{2}},
\]

where \( \zeta, s \) and \( e \) are specified in the statement of Theorem 5.5.

In light of Corollary 8.1 the above follows by showing that
\[
||W_{S, \phi}||_{S_P, \frac{1}{2}, \infty} \leq 1 + \zeta \cdot (1 - (1 - e)^s)^{-1} \cdot (\Delta/\rho_G)^{-\frac{1}{2}},
\]  
(75)

where \( S_P \in \mathbb{R}_{\geq 0}^{V_P \times V_P} \) is the diagonal matrix whose entry \( S_P(u, u) = f_1(u) > 0 \).

In order to prove (75), we use Theorem 8.3. Particularly, (75) follows from Theorem 8.3 once we show that the assumptions of Theorem 5.5 imply that we can have \( \gamma \in \mathbb{R}^S \) which is a \((s', \delta', c')\)-potential with respect to \( S \) and \( \phi \) where \( s' = s, \delta' = \frac{1-\epsilon}{\rho_G} \) and \( c' = \frac{\zeta}{\rho_G} \). As mentioned above, \( s, e, \) and \( \zeta \) are specified in the statement of Theorem 5.5.

Note that Theorem 5.5 assumes the existence of a \((s, \delta, c)\)-potential \( \Psi \), for \( s \geq 1, \delta = \frac{1-\epsilon}{\rho_G} \) and \( c = \frac{\zeta}{\rho_G} \).

We let the function \( \chi = \Psi \). We define the weights \( \gamma \in \mathbb{R}^E \) as follows: for any edge \( e \in E_S \) such that \( e = \{ u, w \} \) and \( w \) is the child of \( u \), we have
\[
\gamma(e) = \chi(u).
\]

From the fact that \( \Psi \) is a \((s, \delta, c)\)-potential, it follows immediately (i.e., from Definitions 5.4, 8.2) that the weights \( \gamma \) are \((s', \delta', c')\)-potential with respect to \( S \) and \( \phi \). For the above it is useful to recall that the edge weights \( \phi \) are specified in (36). This concludes the proof of Theorem 5.5. \( \square \)
11.3. **Proof of Theorem 5.3.** Throughout the proof we use either \( \nu_G \), or \( \rho(H_G) \) to refer to the spectral radius of \( H_G \) and this should create no confusion.

Working as in the proof of Theorem 5.2, we have the following: for \( \phi \) defined in (36), we have that

\[
\max_e \{ |\phi(e)| \} \leq \delta.
\]

Furthermore, using the above and Corollary 7.3, we get that

\[
\rho \left( I_{G}^{A,\tau} \right) \leq \rho \left( W_{\text{NB}-n,\bar{\psi}} \right) \leq \rho \left( W_{\text{NB-}\infty,\bar{\psi}} \right).
\]

For the second inequality we use Lemma 4.4 and the fact that both \( W_{\text{NB}-n,\bar{\psi}} \) and \( W_{\text{NB-}\infty,\bar{\psi}} \) are non-negative matrices. Since \( \rho \left( W_{\text{NB-}\infty,\bar{\psi}} \right) \leq \| W_{\text{NB-}\infty,\bar{\psi}} \|_2 \), we have that

\[
\rho \left( I_{G}^{A,\tau} \right) \leq \| W_{\text{NB-}\infty,\bar{\psi}} \|_2.
\]  \( \tag{76} \)

The assumption that \( \delta = (1-\epsilon)/\rho(H_G) \), implies that \( \rho(\delta H_G) < 1 \), hence, it is standard that \( \sum_{\ell \geq 0} \delta^\ell H_G^\ell = (I - \delta \cdot H_G)^{-1} \). Hence, we get that

\[
W_{\text{NB-}\infty,\bar{\psi}} = I + \delta \cdot K \cdot (I - \delta \cdot H_G)^{-1} \cdot C.
\]  \( \tag{77} \)

The theorem follows by simplifying the above and showing that

\[
W_{\text{NB-}\infty,\bar{\psi}} = (I - \delta A_G + \delta^2 (D - I))^{-1}.
\]  \( \tag{78} \)

For \( k \geq 0 \), let \( W^{(k)} \) be the \( V \times V \) matrix, such that for every \( u, w \in V \) the entry \( W^{(k)}(u, w) \) is equal to the number of non-backtracking walks of length exactly \( k \) from vertex \( u \) to vertex \( w \). Note that

\[
W_{\text{NB-}\infty,\bar{\psi}} = \sum_{k=0}^{\infty} \delta^k \cdot W^{(k)}.
\]  \( \tag{79} \)

From (77) we have that the above summation is well defined. Furthermore, we have that \( W^{(0)} = I \) and \( W^{(1)} = A_G \), while for \( k \geq 2 \), we have the following recursive relation:

\[
W^{(k)} = A_G \cdot W^{(k-1)} - (D - I)W^{(k-2)}.
\]  \( \tag{80} \)

The above recursive relation is standard in the literature, e.g. see [20].

Consider the generating function \( F(x) = \sum_{k=0}^{\infty} x^k \cdot W^{(k)} \). Eq. (77) implies that the radius of convergence for \( F(x) \) is the open interval \((-\nu_G^{-1}, \nu_G^{-1})\). Specifically, from (80) it elementary to show that for any \( x \in (-\nu_G^{-1}, \nu_G^{-1}) \) we have that \( F(x) = (I - xA_G + x^2(D - I))^{-1} \). Then, (78) follows by noting that \( W_{\text{NB-}\infty,\bar{\psi}} = F(\delta) \). The theorem follows.

\[ \square \]

12. **Proof of results from Section 6**

12.1. **Proof of Lemma 6.1.** Consider the trees \( T_P(w) \) and \( T_Q(w) \). Using standard notation, for \( T_P(w) \) we have \( A_{P,w} \), \( E_{P,w} \) the set of vertices and edges. Similarly for \( T_Q(w) \) we have the corresponding sets \( A_{Q,w} \) and \( E_{Q,w} \). The lemma follows by showing that \( A_{Q,w} \subseteq A_{P,w} \) and \( E_{Q,w} \subseteq E_{P,w} \).

The relation that \( A_{Q,w} \subseteq A_{P,w} \) follows immediately from the fact that \( Q \subseteq P \). Specifically, every walk \( M \in Q \cap P \) corresponds to the same vertex \( z \) in both trees. Since \( Q \subseteq P \) we have that for \( M \in Q \) we have \( M \in Q \cap P \), as well. This implies that every vertex \( z \) in the tree \( T_Q(w) \) also belongs to the tree \( T_P(w) \).

In order to prove that \( E_{Q,w} \subseteq E_{P,w} \) we work as follows: Let the edge \( e = \{x, z\} \) in \( E_{Q,w} \), while assume that \( x \) is the parent of \( z \) in the tree \( T_Q(w) \). There are walks \( M_a = z_0, z_1, \ldots, z_{\ell} \) and \( M_b = z_0, z_1, \ldots, z_{\ell-1} \), for \( \ell \geq 1 \) such that \( M_a, M_b \in Q \) while \( M_a \) corresponds to the vertex \( z \) in \( T_Q(w) \) and \( M_b \) corresponds to the vertex \( x \). Since we have assumed that \( Q \subseteq P \), we have that \( M_a, M_b \in P \). Consequently vertices \( x, z \) appear in \( T_P(w) \), while they are adjacent and \( x \) is the parent of \( z \). This implies that the edge \( e \in E_{P,w} \). Hence, we conclude that \( E_{Q,w} \subseteq E_{P,w} \).

The above concludes the proof of the lemma. \[ \square \]
12.2. **Proof of Lemma 6.2** Let \( C = \sum_{\ell=0}^{k} (\zeta \cdot A_G)^\ell \). We prove the lemma by showing that for any \( k \geq 0 \) and any pair of vertices \( u, v \in V \), we have that
\[
W_{\text{MAX-}k,\psi}(u, v) = C(u, v).
\] (81)

Note that both \( W_{\text{MAX-}k,\psi}, C \in \mathbb{R}^{V \times V} \).

For any \( u, v \in V \) and any integer \( \ell \geq 1 \), let \( M_{u,v}^\ell \subseteq V^{\ell+1} \) contain every \( \ell + 1 \)-tuple of vertices \( w_0, w_1, \ldots, w_\ell \in V^{\ell+1} \) such that \( w_0 = u \) and \( w_\ell = v \). Note that we allow \( w_i = w_j \). We also define \( M_{u,v}^0 \) for \( \ell = 0 \). In this case, we have that \( M_{u,v}^0 = \{u\} \) only if \( u = v \). Otherwise, the set is empty.

For \( \ell \geq 0 \), let \( S_\ell \) be the set of all walks of length exactly \( \ell \) that start from \( u \) and end at \( v \) in the graph \( G \). From the definition of the set \( \text{MAX-}k \) we have that
\[
W_{\text{MAX-}k,\psi}(u, v) = \sum_{\ell=0}^{k} \sum_{w_0,\ldots,w_\ell \in M_{u,v}^\ell} \zeta^\ell \times 1\{w_0,\ldots, w_\ell \in S_\ell\}. \tag{82}
\]

Now, consider the adjacency matrix \( A_G \). It is standard that for any \( u, v \in V \) and any \( \ell \geq 0 \), we have that
\[
(A_G)^\ell (u, v) = \sum_{w_0,\ldots,w_\ell \in M_{u,v}^\ell} \prod_{i=0}^{\ell} A_G(w_{i-1}, w_i).
\]

Note that each summation on the r.h.s. is equal to 1 if \( w_0, w_1, \ldots, w_\ell \) is a walk from \( u \) to \( v \) in \( G \). Otherwise, the summation is zero. Using this observation, we conclude that
\[
(A_G)^\ell (u, v) = \sum_{w_0,\ldots,w_\ell \in M_{u,v}^\ell} 1\{w_0,\ldots, w_\ell \in S_\ell\}.
\]

The above implies that
\[
C(u, v) = \sum_{\ell=0}^{k} \sum_{w_0,\ldots,w_\ell \in M_{u,v}^\ell} \zeta^\ell \times 1\{w_0,\ldots, w_\ell \in S_\ell\}. \tag{83}
\]

Comparing (83) and (82) we immediately get (81). The lemma follows.

\[
\square
\]

12.3. **Proof of Lemma 6.3** Let \( M = \sum_{\ell=0}^{k-1} H^\ell \cdot G \) while let \( D = I + K \cdot M \cdot C \), where \( I \) is the \( V \times V \) identity matrix and \( K, C \) are defined in (33). Note that \( M \) is a \( \tilde{E} \times \tilde{E} \) matrix, while both \( W_{\text{NB-}k,\psi} \) and \( D \) are \( V \times V \).

We prove the lemma by showing that for any \( k \geq 0 \) and any pair of vertices \( u, v \in V \), we have that
\[
W_{\text{NB-}k,\psi}(u, v) = D(u, v). \tag{84}
\]

For \( (x, y), (r, s) \in V \times V \), allowing repetitions, for integer \( \ell \geq 3 \), let \( M_{(x,y),(r,s)}^\ell \subseteq V^{\ell+1} \) contain every \( \ell + 1 \)-tuple of vertices \( w_0, w_1, \ldots, w_\ell \in V^{\ell+1} \) such that \( w_0 = x, w_1 = y, w_{\ell-1} = r \) and \( w_\ell = s \). Note that we allow \( w_i = w_j \).

We also define \( M_{(x,y),(r,s)}^\ell \) for \( 1 \leq \ell \leq 2 \), but there are some restrictions on \( x, y, r, s \). For \( \ell = 1 \), we have that \( M_{(x,y),(r,s)}^1 = \{(x, y)\} \) only if \( x = r \) and \( y = s \). Otherwise, we have \( M_{(x,y),(r,s)}^1 = \emptyset \). Similarly, for \( \ell = 2 \), we have \( M_{(x,y),(r,s)}^2 = \{x, y, s\} \) only if \( y = r \). Otherwise, we have \( M_{(x,y),(r,s)}^2 = \emptyset \).

Furthermore, for \( \ell \geq 1 \), let \( S_\ell(e,f) \) be the set of all non-backtracking walks in \( G \) of length exactly \( \ell \) that start with edge \( e \), while the last edge in the path is \( f \). For \( e = (x, y) \) and \( f = (r, s) \), \( S_\ell(e,f) \) contains the non-backtracking walks such that the first vertex in the walk is \( x \), the second is \( y \) while the one before the last vertex is \( r \) and the last vertex is \( s \). Note that for the extreme case \( \ell = 1 \), \( S_\ell(e,f) \) is non-empty only if \( e = f \).
From the definition of the set NB-\(k\) we have that
\[
W_{NB-k,\psi}(u, v) = \mathbf{1}\{u = v\} + \sum_{e = (x,y) \in \bar{E}} \sum_{f = (r,s) \in \bar{E}} \sum_{\ell \in [k]} \sum_{w_0, \ldots, w_\ell \in M_{e,f}} \zeta^\ell \times \mathbf{1}\{w_0, \ldots, w_\ell \in S_{\ell}(e, f)\}. 
\] (85)

Working as in the proof of Lemma 6.2, we get the following: for any \(\ell \geq 0\) and \(e, f \in \bar{E}\), we have that
\[
H_G^\ell(e, f) = \sum_{w_0, \ldots, w_\ell \in M_{e,f}} \mathbf{1}\{w_0, \ldots, w_\ell \in S_{\ell+1}(e, f)\}. 
\]

Hence, we have that
\[
M(e, f) = \sum_{\ell=0}^{k-1} \zeta^\ell + \sum_{w_0, \ldots, w_\ell \in M_{e,f}} \mathbf{1}\{w_0, \ldots, w_\ell \in S_{\ell+1}(e, f)\}. 
\]

Combining the above with (85), we get that
\[
W_{NB-k,\psi}(u, v) = \mathbf{1}\{u = v\} + \sum_{e = (x,y) \in \bar{E}} \sum_{f = (r,s) \in \bar{E}} M(e, f) 
\] (86)

Furthermore, the definition of \(D, K\) and \(C\), implies that
\[
D(u, v) = (I + K \cdot M \cdot C)(u, v) = \mathbf{1}\{u = v\} + \sum_{(x,z)} K_{u,(x,z)} M_{(x,z),(y,q)} C_{(y,q),v} 
\]
\[
= \mathbf{1}\{u = v\} + \sum_{(x,z)} \sum_{(y,q)} \mathbf{1}\{u = x\} \times \mathbf{1}\{q = v\} \times M_{(x,z),(y,q)} 
\]
\[
= \mathbf{1}\{u = v\} + \sum_{e = (x,y) \in \bar{E}} \sum_{f = (r,s) \in \bar{E}} M_{e,f}. 
\] (87)

From (86) and (87) we conclude that (84) is true. The lemma follows.

12.4. Proof of Lemma 6.4. Consider \(W_{SAW,\beta}\) where \(\beta\) is defined in (34). From the definition of the matrix we have that \(W_{SAW,\beta}\) is indexed by the vertices in \(V\). Also, we have that both \(I_G^{A,T}\) and \(W_{P,\phi}\) are matrices indexed by the vertices in \(V \setminus \Lambda\).

The lemma will follow by showing the following two relations: for any \(w, v \in V \setminus \Lambda\), we have that
\[
I_G^{A,T}(w, v) = W_{SAW,\beta}(w, v), \quad (88)
\]
while, for the same \(w, v\) we also have that
\[
W_{S,\phi}(w, v) = W_{SAW,\beta}(w, v). \quad (89)
\]

Note that the lemma follows by combining the above with (88).

We start by proving that (88) is true. From the definition of walk-matrix, for \(\beta \in \mathbb{R}^{SAW}\), we have that \(W_{SAW,\beta} \in \mathbb{R}^{V \times V}\), while
\[
W_{SAW,\beta}(w, v) = \sum_M \prod_{e \in M} \beta(e), 
\] (90)
where \(M\) varies over the paths in \(T_{SAW}(w)\) from the root to the set \(A_{SAW,\beta}(v)\), while, as mentioned above, \(\beta\) is specified in (34). In light of the above, it is immediate that (88) is true by showing formally that (35) is true.

Consider \(T = T_{SAW}(w)\) and the Gibbs distribution \(\mu_T(\cdot | \Gamma, \sigma)\), where \(\Gamma\) and \(\sigma \in \{\pm 1\}^\Gamma\) are defined in Section 6.3. With respect to the aforementioned Gibbs distribution, let \(I_{T}^{A,T}\) be the corresponding pairwise influence matrix. The following result from [5] shows a useful relationship between the two influences matrices \(I_{G}^{A,T}\) and \(I_{T}^{A,T}\).
Lemma 12.1. Let vertex \( r \) be the root of the tree \( T_{SAW}(w) \). We have that
\[
\mathcal{I}_G^{\Lambda,\tau}(w, v) = \sum_{u \in \mathcal{A}(v)} \mathcal{I}_T^{\tau}(r, u),
\]
where for \( s \in V \), \( k(s) \) is the set of copies of vertex \( s \) in \( T_{SAW}(w) \) which do not belong in \( \Gamma \).

Combining results from \([1]\) and \([5]\), we have the following lemma, whose proof appears in Section 12.4.1.

**Lemma 12.2.** Let \( r \) be the root in \( T_{SAW}(w) \). For any vertex \( u \) in \( T_{SAW}(w) \), different than \( r \), the following holds: letting \( z_0, \ldots, z_e \) be the path in \( T_{SAW}(w) \) that connects the root of the tree with \( u \), i.e., \( z_0 = r \) and \( z_e = u \), we have that
\[
\mathcal{I}_T^{\tau}(r, u) = \prod_{i=1}^e \beta_i(\{z_{i-1}, z_i\}),
\]
where \( \beta \in \mathbb{R}^{E_{SAW}} \) is specified in \([34]\). Furthermore, we have \( \mathcal{I}_T^{\tau}(r, r) = 1 \).

In light of Lemmas 12.2 and 12.1, (35) follows as a simple corollary. We conclude that (88) is true.

Now, consider the matrix \( W_{S,\psi} \). Similarly to (90), we have that
\[
W_{S,\psi}(w, v) = \sum_M \prod_{e \in M} \psi(e),
\]
where \( M \) varies over the paths in \( T_S(w) \) from the root to the set \( A_{S,w}(v) \).

Comparing the trees \( T_S(w) \) and \( T_{SAW}(w) \), we have that the first one is a subtree of the second one. One obtains \( T_S(w) \) by removing the subtrees of \( T_{SAW}(w) \) which are rooted to vertices in \( \Gamma \). Due to (36), we have that \( W_{S,\psi}(w, v) \) and \( W_{SAW,\beta}(w, v) \) differ only on the sum of the weight of the paths that appear in \( T_{SAW}(w) \) but do not appear in \( T_S(w) \). However, it is immediate that the weight of these paths is equal to zero. Note that each one of these paths involves at least one vertex in \( \Gamma \), while all the edges \( e \) incident to such vertex have \( \beta(e) = 0 \), i.e., this is due to (34). Hence, (89) is true.

All the above, conclude the proof of Lemma 12.2.

**Proof of Lemma 12.2** In order to prove Lemma 12.2, we only need to use the following two results from [1] and [5], respectively.

**Lemma 12.3** ([1]). Suppose that \( x, y, z \) are three distinct vertices in \( T = T_{SAW}(w) \) such that \( y \) is on the unique path from \( x \) to \( z \). Then
\[
\mathcal{I}_T^{\tau}(x, z) = \mathcal{I}_T^{\tau}(x, y) \times \mathcal{I}_T^{\tau}(y, z).
\]

**Lemma 12.4** ([5]). Let \( v, u \) be two vertices in \( T = T_{SAW}(w) \), while suppose \( v \notin \Gamma \) and \( u \) is a child of \( v \). Then we have
\[
\mathcal{I}_T^{\Lambda,\tau}(v, u) = \begin{cases} h(\log R_{u,v}) & \text{if } u \notin \Gamma \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof of Lemma 12.2** Recall that \( z_0, z_1, \ldots, z_e \) is the path in \( T \) that starts from the root to the vertex \( u \).

If \( \ell = 0 \), then we immediately have \( \mathcal{I}_T^{\tau}(r, r) = 1 \). For what follows, we focus on the case \( \ell \geq 1 \).

Lemma 12.3 implies that
\[
\mathcal{I}_T^{\tau}(r, u) = \prod_{i=1}^\ell \mathcal{I}_T^{\Lambda,\tau}(z_{i-1}, z_i).
\]

Furthermore, Lemma 12.4 implies the following: If there is \( z_i \in \Gamma \), then
\[
\mathcal{I}_T^{\tau}(r, u) = 0,
\]
while if all \( z_i \)'s are outside \( \Gamma \), then we have
\[
\mathcal{I}_T^{\tau}(z_{i-1}, z_i) = h(\log R_{z_i}).
\]
From (34), it is immediate that

\[ T_T^\sigma(r, u) = \prod_{i=1}^\ell \xi(\{z_{i-1}, z_i\}). \]

The above concludes the proof. \(\square\)

### 13. Proofs for results from Section 7

#### 13.1. Proof of Theorem 7.1

W.l.o.g. assume that the sets \( \mathcal{P} \) and \( \mathcal{Q} \) are non-empty. For \( u, v \in V_Q \subseteq V_P \) consider the trees \( T_Q(u) \) and \( T_P(u) \). Let the sets \( A_Q(v) \) and \( A_P(v) \) be the set of copies of the vertex \( v \) in these two trees, respectively.

Let \( \mathcal{M}_Q \) be the set of paths in \( T_Q(u) \) that connect the root with the vertex set \( A_Q(v) \). Similarly, for the tree \( T_P(u) \) we define the set of paths \( \mathcal{M}_P \).

For each \( M \in \mathcal{M}_Q \) we consider the weight \( \text{weight}_Q(M) \) which is as follows:

\[ \text{weight}_Q(M) = \prod_{e \in M} \xi_1(e). \]

Similarly, for each \( M \in \mathcal{M}_P \) we consider the weight \( \text{weight}_P(M) \) and the weights of the edges are specified by \( \xi_2 \).

First we focus on proving (38). For this we use the following claim.

**Claim 13.1.** There is an injective map \( H : \mathcal{M}_Q \to \mathcal{M}_P \) such that for every \( M \in \mathcal{M}_Q \), we have that

\[ \text{weight}_Q(M) \leq \text{weight}_P(H(M)). \]  \hspace{1cm} (94)

**Proof.** From Lemma 6.1, we have the following: Since \( Q \subseteq P \), the tree \( T_Q(u) \) is a subtree of \( T_P(u) \). We construct the injective map \( H : \mathcal{M}_Q \to \mathcal{M}_P \) such that it maps every path \( M \) in the tree \( T_Q(u) \) to the same path in the tree \( T_P(u) \). The fact that \( H(\cdot) \) is injective follows from that \( T_Q(u) \) is a subtree of \( T_P(u) \).

As far as (13.1) is concerned, recall that

\[ \text{weight}_Q(M) = \prod_{e \in M} \xi_1(e) \quad \text{and} \quad \text{weight}_Q(H(M)) = \prod_{e \in H(M)} \xi_2(e). \]

However, from the definition of \( H(\cdot) \) we have that the set of edges in \( M \) is exactly the same as the set of edges in \( H(M) \). Then, we get (94) by recalling that for any edge \( e \) that appears in both \( T_Q(u), T_P(u) \) we have that \( \xi_1(e) \leq \xi_2(e) \). The claim follows. \( \square \)

Let \( S \subseteq \mathcal{M}_P \) be such that \( S = \mathcal{M}_P \setminus \mathcal{M}_Q \), that is \( S \) contains every \( M \in \mathcal{M}_P \) such that there is no \( M' \in \mathcal{M}_Q \) for which \( H(M') = M \). Using the above claim, we have that

\[
W_{P, \xi_2}(w, v) = \sum_{M \in \mathcal{M}_P} \text{weight}(M) = \sum_{M \in \mathcal{M}_Q} \text{weight}(H(M)) + \sum_{M \in S} \text{weight}(M) \\
\geq \sum_{M \in \mathcal{M}_Q} \text{weight}(H(M)) + \sum_{M \in S} \text{weight}(M) \\
\geq \sum_{M \in \mathcal{M}_Q} \text{weight}(M) = W_{Q, \xi_1}(w, v).
\]

The inequality in the last line follows by noting that for every \( M \in S \) we have that \( \text{weight}_P(M) \geq 0 \), i.e., since we have \( \xi_2 \in \mathbb{R}_{\geq 0}^P \). The above proves that (38) is true.

As far as (39) is concerned, let us first prove it by using the assumption that \( V_P = V_Q \). In this case, note that the two matrices \( W_{Q, \xi_1}, W_{P, \xi_2} \) are indexed by the same set of vertices, and hence, they are of the same dimension. Furthermore, from the assumption that \( \xi_2 \in \mathbb{R}_{\geq 0}^P \) we have that \( W_{P, \xi_2} \) is non-negative matrix, while (38) implies that

\[ |W_{Q, \xi_1}| \leq W_{P, \xi_2}, \]

where recall that for the matrices \( A, B, C \in \mathbb{R}^{N \times N} \), we let \( |A| \) denote the matrix having entries \( |A_{i,j}| \).

while we defined \( B \leq C \) to mean that \( B_{i,j} \leq C_{i,j} \) for each \( i \) and \( j \).

Then, (39) follows from the above by using Lemma 4.4.
We now proceed with (39) and assumeing that \(|V_P| > |V_Q|\), and \(W_{P,ξ_2}\) is symmetric. Consider the matrix \(W_{P,ξ_2}\), this is the principle submatrix of \(W_{P,ξ_2}\) obtained by removing the rows and columns that correspond to the vertices in \(V_P\) \(\setminus\) \(V_Q\).

Note, now, that \(W_{P,ξ_2}\) and \(W_{Q,ξ_1}\) are indexed by the same set of vertices. Additionally, we have that \(|W_{Q,ξ_1}| ≤ W_{P,ξ_2}\), which, together with Lemma 4.4, implies that

\[
ρ(W_{Q,ξ_1}) ≤ ρ(W_{P,ξ_2}).
\]

However, since we assumed that \(W_{P,ξ_2}\) is symmetric and \(W_{P,ξ_2}\) is a principal submatrix of \(W_{P,ξ_2}\), we also have that

\[
ρ(W_{P,ξ_2}) ≤ ρ(W_{P,ξ_2}).
\]

The above is a consequence of the well-known Cauchy’s interlacing theorem, e.g. see [17]. Combining the two inequalities above it is immediate to get (39). The theorem follows.

\[\square\]

14. PROOF OF RESULTS FROM SECTION 8

14.1. Proof of Theorem 8.3 We use Theorems 8.5, 8.6 and Corollary 8.7 to prove the theorem. Particularly, using these results, we obtain the following:

For the walk-vector \(q_{MAX-k} = q_{MAX-k}(MAX-k, S^{o(1/s)}, s, δ, c)\), where the \(V × V\) matrix \(S\) is defined in [17], we have that

\[
||W_{P,ξ}||_{S_P, \frac{1}{s}, \infty} ≤ ||q_{MAX-k}||_{∞}.
\]  \hspace{1cm} (95)

Recall that the diagonal matrix \(S_P\) at the index of the matrix norm, is obtained from \(S\) by removing the rows and columns that correspond to vertices outside \(V_P\). In light of (95), the theorem follows showing that

\[
||q_{MAX-k}||_{∞} ≤ 1 + c \cdot (Δ)^{1 - \frac{1}{s}} \cdot (ρ_G)^{\frac{k}{2}} \cdot ∑_{\ell=0}^{k-1} (δ \cdot ρ_G)^{\frac{\ell}{2}}.
\]  \hspace{1cm} (96)

According to Definition 8.4, for every \(r ∈ V\) the entry \(q_{MAX-k}(r)\) satisfies

\[
q_{MAX-k}(r) = 1 + \frac{c}{(S(r,r))^\frac{1}{2}} \times ∑_{i=1}^{d} ∑_{\ell=0}^{k-1} (δ^\ell \cdot ∑_{w∈V} |A_{i,ℓ}(w)| \cdot S(w, w))^{\frac{1}{2}}.
\]

In order to study the above quantity, note that the walk-tree of interest is \(T_{MAX-k}(r)\).

Let us recall the quantities in the above expression. The quantity \(d\) is the degree of the root of the walk-tree \(T_{MAX-k}(r)\). Letting \(T_i\) be the subtree that is induced by the \(i\)-th child of the root of \(T_{MAX-k}(r)\) and its decedents, \(A_{i,ℓ}(w)\) is the set of copies of vertex \(w\) in the subtree \(T_i\) that are at distance \(ℓ\) from the root of \(T_i\).

Using that \(S(w, w) = f_1(w)\) for every \(w ∈ V\), we have that

\[
q_{MAX-k}(r) = 1 + \frac{c}{(f_1(r))^\frac{1}{2}} \times ∑_{i=1}^{d} ∑_{\ell=0}^{k-1} (δ^\ell \cdot ∑_{w∈V} |A_{i,ℓ}(w)| \cdot f_1(w))^{\frac{1}{2}}.
\]  \hspace{1cm} (97)

For \(ℓ ≥ 1\), for every \(x, w ∈ V\), let \(A_{i,x,ℓ}(w) \subseteq A_{i,ℓ}(w)\) be the set which contains all vertices \(u\) in \(T_i\), copies of \(w\), such that the parent of \(u\) is in \(A_{i,(ℓ-1)}(x)\).

Since we assumed that the graph \(G\) is simple, it is straightforward that for all \(w ∈ V\), there are no two copies of \(w\) in \(T_{MAX-k}(r)\) that have the same parent. This implies that \(|A_{i,(ℓ-1)}(x)|\) is equal to \(|A_{i,x,ℓ}(w)|\),
for any \( w \) neighbour of \( x \) in \( G \). Using this observation, we have that
\[
\sum_{w \in V} |A_{i,\ell}(w)| \cdot f_1(w) = \sum_{w \in V} \sum_{x \in V} |A_{i,x,\ell}(w)| \cdot f_1(w) = \sum_{x \in V} \sum_{w \in V} |A_{i,x,\ell}(w)| \cdot f_1(w) = \sum_{x \in V} |A_{i,\ell-1}(x)| \sum_{w \in V} f_1(w),
\]
where in the second equation changed order of summation. Using the definition of \( f_1 \), note that the last summation is equal \( \theta_{\text{max}}(A_G) f_1(x) \). Hence, we have that
\[
\sum_{w \in V} |A_{i,\ell}(w)| \cdot f_1(w) = \theta_{\text{max}}(A_G) \cdot \sum_{x \in V} |A_{i,\ell-1}(x)| \cdot f_1(x) = \rho_G \cdot \sum_{x \in V} |A_{i,\ell-1}(x)| \cdot f_1(x).
\]
For the last equality we use that \( \theta_{\text{max}}(A_G) = \rho(A_G) \). This equality is a standard application of the Perron-Frobenius Theorem.

Repeating the above \( \ell \) times in total, we get that
\[
\sum_{w \in V} |A_{i,\ell}(w)| \cdot f_1(w) = (\rho_G)^\ell \cdot f_1(k_i),
\]
where \( k_i \in V \) is such that the root of \( T_i \) belongs in \( \Lambda(k_i) \). Note that the above applies for \( \ell \geq 1 \). For \( \ell = 0 \), it is immediate that \( \sum_{w \in V} |A_{i,\ell}(w)| \cdot f_1(w) = f_1(k_i) \).

Plugging the above into (97) and rearranging, we get that
\[
q_{\text{MAX-k}}(r) = 1 + c \cdot \sum_{\ell=0}^{k-1} (\delta \cdot \rho_G)^{\ell} \cdot \sum_{i=1}^{d} \left( \frac{f_1(k_i)}{f_1(r)} \right)^{\frac{1}{2}}.
\]
Note that the vertices \( k_1, \ldots, k_d \) are the neighbours of \( r \) in the graph \( G \).

We need to bound the rightmost sum in the equation above. Recall that we have \( \theta_{\text{max}}(A_G) = \rho(A_G) \), which implies that \( \sum_{i=1}^{d} f_1(k_i) = \rho_G \cdot f_1(r) \). Using this observation, we get that
\[
\sum_{i=1}^{d} \left( \frac{f_1(k_i)}{f_1(r)} \right)^{\frac{1}{2}} \leq \max_{z_1,\ldots,z_d \in (0,\rho_G)} \sum_{i=1}^{d} (z_i)^{\frac{1}{2}} \leq \sum_{i=1}^{d} (\rho_G)^{\frac{1}{2}} = d^{1-\frac{1}{2}} (\rho_G)^{\frac{1}{2}}.
\]
In the above series of inequalities, we use the following observations: Since we assumed that \( s \geq 1 \), it is elementary to show that for \( z_1, \ldots, z_d > 0 \), the function \( f(z_1, \ldots, z_d) = \sum_{i=1}^{d} (z_i)^{\frac{1}{2}} \) is concave. In the interval specified by the restrictions \( z_1, \ldots, z_d \in (0, \rho_G) \) and \( \sum_{i=1}^{d} z_i = \rho_G \), due to concavity, the function \( f(z_1, \ldots, z_d) \) attains its maximum when all \( z_i \)'s are equal with each other, i.e., \( z_i = \frac{\rho_G}{d} \), for \( i = 1, \ldots, d \).

Plugging (100) into (99) we get that
\[
q_{\text{MAX-k}}(r) \leq 1 + c \cdot d^{1-\frac{1}{2}} \cdot (\rho_G)^{\frac{1}{2}} \cdot \sum_{\ell=0}^{k-1} (\delta \cdot \rho_G)^{\ell} \leq 1 + c \cdot \Delta^{1-\frac{1}{2}} (\rho_G)^{\frac{1}{2}} \cdot \sum_{\ell=0}^{k-1} (\delta \cdot \rho_G)^{\ell}.
\]
For the last inequality we use that \( d \leq \Delta \).

Noting that the above bound holds for any \( r \in V \), it is immediate to get (96). The theorem follows.

14.2. **Proof of Theorem 8.5** Let \( C = (D^{\frac{1}{2}})^{-1} \cdot W_{P,\xi} \cdot D^{\frac{1}{2}} \). The theorem follows by showing that
\[
\sum_{w \in V_p} |C(r, u)| \leq q(r) \quad \forall r \in V_p,
\]
for the walk-vector \( q = q(P, D^{\frac{1}{2}}, s, \delta, c) \) specified in the statement of Theorem 8.5.

Before showing that (101) is indeed true, let us show how we can use it to prove the theorem. From the definition of the norm \( ||| \cdot |||_{D^{\frac{1}{2}},\infty} \), it is immediate that
\[
|||W_{P,\xi}|||_{D^{\frac{1}{2}},\infty} = |||C|||_{\infty} = \max_r \left\{ \sum_{w \in V_p} |C(r, w)| \right\}.
\]

\( ^{8} \)Note that the assumption that \( G \) is connected, implies that \( A_G \) is a non-negative, irreducible matrix.
Recalling that \( q \) is a strictly positive vector, using (102) and (101) we conclude that
\[
\| W_p, \xi \|_{D^\frac{1}{2}, \infty} \leq \max_r q(r) = \| q \|_\infty.
\]

It remains to show that (101) is true. For \( r \in V_p \), consider the walk-tree \( T = T_p(r) \). Also, consider a path \( M \) of length \( \ell \) in \( T \) that emanates from the root, while let \( e_1, e_2, \ldots, e_\ell \) be the edges on this path. In order to define \( W_p, \xi \), we specify that \( M \) has weight such that
\[
\text{weight}(M) = \prod_{i=1}^{\ell} \xi(e_i),
\]
where the weights \( \xi \) are specified in the statement of Theorem 8.5.

We use a simple telescopic trick to write the weight of \( M \) slightly differently than what we have above, i.e., involve the potential vector \( \gamma \).

**Claim 14.1.** We have that
\[
\text{weight}(M) = \frac{\gamma(e_\ell)}{\gamma(e_1)} \prod_{i=2}^{\ell} \frac{\gamma(e_{i-1})}{\gamma(e_i)} \xi(e_i).
\]

**Proof.** Since, for every \( e_i \in E_{p,r} \), we have that \( \gamma(e_i) > 0 \), it holds that
\[
\text{weight}(M) = \prod_{i=1}^{\ell} \frac{\gamma(e_{i-1})}{\gamma(e_i)} \frac{\xi(e_i)}{\xi(e_1)} = \frac{\gamma(e_\ell)}{\gamma(e_1)} \prod_{i=2}^{\ell} \frac{\gamma(e_{i-1})}{\gamma(e_i)} \xi(e_i).
\]
The claim follows. \( \square \)

For any \( u \in V_p \) and any integer \( \ell \geq 0 \), let \( M(\ell, u) \) be the set of paths of length \( \ell \) in \( T \) that connect the root of the tree with a vertex \( v \in A(u) \).

From the definition of \( W_p, \xi \), recall that
\[
W_p, \xi(r, u) = \sum_{\ell \geq 0} \sum_{M \in M(\ell, u)} \text{weight}(M).
\]

Since \( C = (D^\frac{1}{2})^{-1} \cdot W_p, \xi \cdot D^\frac{1}{2} \) and \( D^\frac{1}{2} \) is diagonal, for any \( u \in V_p \) we have that
\[
C(r, u) = \frac{D^\frac{1}{2}(u, u)}{D^\frac{1}{2}(r, r)} \cdot W_p, \xi(r, u) = \frac{D^\frac{1}{2}(u, u)}{D^\frac{1}{2}(r, r)} \sum_{\ell \geq 0} \sum_{M \in M(\ell, u)} \text{weight}(M). \tag{103}
\]

For every integer \( \ell \geq 0 \), we let
\[
C^{(\ell)}(r, u) = \frac{1}{D^\frac{1}{2}(r, r)} \left| \sum_{u \in V_p} \sum_{M \in M(\ell, u)} \text{weight}(M) \cdot D^\frac{1}{2}(u, u) \right|.
\]

It is easy to see that \( C^{(0)} = 1 \). From the definition of \( C^{(\ell)} \) and (103), it is immediate that
\[
\sum_{u \in V_p} |C(r, u)| \leq \sum_{\ell \geq 0} C^{(\ell)} = 1 + \sum_{\ell \geq 1} C^{(\ell)}. \tag{104}
\]

Given some fixed \( \ell \geq 1 \), for \( m = 0, \ldots, \ell \) consider the vertex \( z \) at distance \( m \) from the root of \( T \). Suppose that \( d_z \) is the number of children of \( z \) in \( T_z \), while let \( x_1, x_2, \ldots, x_{d_z} \) be these children. Recall that \( T_z \) is the subtree that is induced by \( z \) and all its descendant in \( T \).

With respect to vertex \( z \), we define the quantity \( J_z(\ell - m) \) as follows: For \( m = \ell \), we have that
\[
J_z(0) = \sum_{u \in V_p} 1\{z \in A(u)\} \times D^\frac{1}{2}(u, u). \tag{105}
\]
For \( 0 < m < \ell \), the quantity \( J_z(\ell - m) \) satisfies the following recursive relation:
\[
J_z(\ell - m) = \gamma(e_z) \sum_{j=1}^{d_z} \frac{\xi(e_j)}{\gamma(e_j)} \times J_{x_j}(\ell - m - 1),
\]
where \( e_z \) is the edge that connect \( z \) with its parent, while \( e_j \) is the edge that connects \( z \) with its child \( x_j \). Note that since we assumed that \( z \) is at level \( m > 0 \) it has a parent, i.e., \( z \) is not the root of \( T \).
Finally, for \( m = 0 \), i.e., \( z \) and the root of \( T \) are identical, we have that
\[
\mathcal{J}_z(\ell) = \frac{1}{D^{\frac{1}{2}}(r,r)} \max_{e_1,e_2 \in E_{r,r}} \left\{ \gamma(e_1) \cdot \frac{|\ell(e_2)|}{\gamma(e_2)} \right\} \sum_{j=1}^{d_z} \mathcal{J}_{x_j}(\ell - 1). \tag{106}
\]
Claim 14.1 and an elementary induction imply that for any \( \ell \geq 1 \), we have
\[
\mathcal{C}(\ell) \leq \mathcal{J}_z(\ell). \tag{107}
\]
Furthermore, the assumption that \( \gamma \in \mathbb{R}^E \) is a \((s, \delta, c)\)-potential vector with respect to \( \xi \) and \( \mathcal{P} \), together with (106) imply that
\[
\mathcal{J}_z(\ell) \leq \frac{c}{D^{\frac{1}{2}}(r,r)} \sum_{j=1}^{d_z} \mathcal{J}_{x_j}(\ell - 1). \tag{108}
\]
The same assumption about \( \gamma \) implies that for any \( 0 < m < \ell \) we have that
\[
\left[ \mathcal{J}_z(\ell - m) \right]^s \leq \delta \times \sum_{j=1}^{d_z} \left[ \mathcal{J}_{x_j}(\ell - m - 1) \right]^s. \tag{109}
\]
Suppose that \( k_i \) is the \( i \)-th child of the root, while the subtree \( T_i \) includes vertex \( k_i \) and all of its descendants. Then, from (109) and (105) it is elementary to get the that
\[
\left[ \mathcal{J}_{k_i}(\ell - 1) \right]^s \leq (\delta)^{\ell - 1} \times \sum_{u \in \mathcal{P}} [A_i,\ell - 1](u) \cdot \left( D^{\frac{1}{2}}(u,u) \right)^s,
\]
where, recall that, \( A_i,\ell - 1)(u) \) is the set of copies of vertex \( u \) in the subtree \( T_i \) that are at distance \( \ell - 1 \) from the root of \( T_i \). Plugging the above into (108) yields
\[
\mathcal{J}_z(\ell) \leq \frac{c}{D^{\frac{1}{2}}(r,r)} \sum_{i=1}^{d_z} (\delta^{\ell - 1} \sum_{u \in \mathcal{P}} |A_i,\ell - 1(u)| \cdot \left( D^{\frac{1}{2}}(u,u) \right)^s)^{\frac{1}{2}}.
\]
Combining the above with (107) and (104) we get that
\[
\sum_{u \in \mathcal{P}} |C(r,u)| \leq 1 + \frac{c}{D^{\frac{1}{2}}(r,r)} \sum_{i=1}^{d_z} \sum_{\ell \geq 0} (\delta^\ell \sum_{u \in \mathcal{P}} |A_i,\ell(u)| \cdot \left( D^{\frac{1}{2}}(u,u) \right)^s)^{\frac{1}{2}}
\]
\[
= 1 + \frac{c}{D^{\frac{1}{2}}(r,r)} \sum_{i=1}^{d_z} \sum_{\ell \geq 0} (\delta^\ell \sum_{u \in \mathcal{P}} |A_i,\ell(u)| \cdot \left( D^{\frac{1}{2}}(u,u) \right)^s)^{\frac{1}{2}},
\]
where, in the last equality we change variable. From the above and the definition of walk-vector \( q = q(\mathcal{P}, D^{\frac{1}{2}}, s, \delta, c) \), it is immediate that (101) is true. The theorem follows.

14.3. Proof of Theorem 8.6. Note that (47) follows immediately from (46) since, by definition, both \( q_S \) and \( q_P \) have positive entries. The rest of the proof focuses on proving (46).

Firstly, we note for any \( r \in V_S \subset V_P \), we have that that \( T_S(r) \) is a subtree of \( T_P(r) \). This follows from Lemma 6.1.

Suppose that the root of \( T_S(r) \) has degree \( d_S \), while let \( T_{S,i} \) be the subtree that is induced by the \( i \)-th child of the root of \( T_S(r) \) and its descendants. Also, for \( \ell \geq 0 \) and any vertex \( v \in V_S \), let \( A_{S,i,\ell}(v) \) be the set of copies of vertex \( v \) in the subtree \( T_{S,i} \) which is at distance \( \ell \) from the root of \( T_{S,i} \).

Following the definition of walk-vector, i.e., Definition 8.4, we have that
\[
q_S(r) = 1 + \frac{c}{D(r,r)} \times \sum_{i=1}^{d_S} \sum_{\ell \geq 0} \left( \delta^\ell \cdot \sum_{u \in V_S} |A_{S,i,\ell}(v)| \cdot (D(w,w)^s)^{\frac{1}{2}} \right).
\tag{110}
\]
Suppose that the root of \( T_P(r) \) has degree \( d_P \). In the same way as above, we define the subtrees \( T_{P,i} \) and the set of copies \( A_{P,i,\ell}(v) \) for every \( i \in [d_P] \) and \( \ell \geq 0 \). We also have that
\[
q_P(r) = 1 + \frac{c}{D(r,r)} \times \sum_{i=1}^{d_P} \sum_{\ell \geq 0} \left( \delta^\ell \cdot \sum_{u \in V_P} |A_{P,i,\ell}(v)| \cdot (D(w,w)^s)^{\frac{1}{2}} \right).
\tag{111}
\]
Note that, since \( \delta, D(w,w) > 0 \), the summands in both (110) and (111) are non-negative quantities.

The theorem follows by observing that \( d_S \leq d_P, V_S \subset V_P \) and \( |A_{S,i,\ell}(v)| \leq |A_{P,i,\ell}(w)| \), for any \( w \in V_S, i \in [d_S] \) and \( \ell \geq 0 \). This is due to the fact that \( T_S(r) \) is a subtree of \( T_P(r) \). The theorem follows. \( \square \)
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**APPENDIX A. BASICS IN ALGEBRA**

A.1. **Matrix and Vector Norms.** For what follows, we let $N, M$ be positive integers. Furthermore, we denote with $\mathbb{R}$ the set of real numbers, while $\mathbb{C}$ is the set of complex numbers.

For $p \geq 1$, the $p$-norm of the vector $x \in \mathbb{C}^N$, denoted as $||x||_p$, is defined such that

$$||x||_p = \left( \sum_{i=1}^{N} |x_i|^p \right)^{1/p}.$$ 

A matrix norms is a function $||\cdot||$ from the set of all complex matrices (of all finite orders) into $\mathbb{R}$ that satisfies the following properties:

- $P.1$: $||A|| \geq 0$, while $||A|| = 0 \iff A = 0$.
- $P.2$: $||\alpha A|| = |\alpha| \cdot ||A||$, for any scalar $\alpha$.
- $P.3$: $||A + B|| \leq ||A|| + ||B||$, for matrices of the same size.
- $P.4$: $||AB|| \leq ||A|| \cdot ||B||$, for all conformable matrices.

For each norm $||\cdot||$ on $\mathbb{R}^r$, where $r \in \{N, M\}$ there is a matrix norm that is “induced” by $||\cdot||$ on $\mathbb{R}^{N \times M}$ by setting

$$||A|| = \max_{||x||=1} ||Ax|| \quad \text{for } A \in \mathbb{R}^{N \times M} \text{ and } x \in \mathbb{R}^M.$$ 

It is standard that $||A||_\infty$ corresponds to the maximum absolute row sum in $A$, while $||A||_1$ corresponds to the maximum absolute column sum, i.e.,

$$||A||_\infty = \max \{ \sum_j |A_{i,j}| \} \quad \text{and} \quad ||A||_1 = \max \{ \sum_i |A_{i,j}| \}.$$ 

A.2. **Perron-Frobenius Theorem.** Let the matrix $M \in \mathbb{R}^{N \times N}$ be non-negative. That is, every entry $M_{i,j} \geq 0$. We say that $M$ is irreducible if and only if $(I + M)^{N-1}$ is a positive matrix, i.e., all its entries are positive numbers.

We can associate $M$ with the directed graph $G_M$ on the vertex set $[N]$, while the edge $(i, j)$ is in $G_M$ iff $M_{i,j} > 0$. Then, $M$ is irreducible, if the resulting graph $G_M$ is strongly connected.

In this work, it is common to use the so-called Perron-Frobenius Theorem. For the sake of keeping this paper self-contained, we state this theorem below.

**Theorem A.1** (Perron-Frobenius Theorem). Let $A \in \mathbb{R}^{N \times N}$ be irreducible and non-negative matrix and suppose that $N \geq 2$. Then,

1. $\rho(A) > 0$
2. $\rho(A)$ is an algebraically simple eigenvalue of $A$
3. there is a unique real vector $x = (x_1, \ldots, x_N)$ such that $A \cdot x = \rho(A)x$ and $x_1 + \cdots + x_N = 1$, while $x_j > 0$ for all $j \in N$
4. there is a unique real vector $y = (y_1, \ldots, y_N)$ such that $y^T A = \rho(A)y^T$ and $x_1 y_1 + \cdots + x_N y_N = 1$, while $y_j > 0$ for all $j \in N$.

**APPENDIX B. BASIC PROPERTIES OF $W_{NB-k,\psi}$**

Consider the graph $G = (V, E)$. For integer $k \geq 1$ and $\zeta \in \mathbb{R}_{\geq 0}$, consider the walk matrix $W_{NB-k,\psi}$, where $\psi \in \mathbb{R}_{\geq 0}$ is such that $\psi(e) = \zeta$, for all $e \in E_{NB-k}$. That is, $W_{NB-k,\psi}$ is induced by all the non-backpacking walks in $G$ which are of length $\leq k$, while for every walk-tree $T_{NB-k}(r)$ every edge $e$ has weight $\psi(e) = \zeta$. Recall from Lemma 6.3 that we have

$$W_{NB-k,\psi} = I + K \cdot \left( \sum_{\ell=0}^{k-1} \zeta^{\ell+1} \cdot H_G^{\ell} \right) \cdot C,$$  \hspace{1cm} (112)
where $H_G$ is the non-backtracking matrix of $G$, while $K, C$ are $V \times \overline{E}$ and $\overline{E} \times V$ matrices, respectively, such that for any $v \in V$ and for any $(x, z) \in \overline{E}$ we have
\[
K((v, (x, z))) = 1\{x = v\} \quad \text{and} \quad C((x, z), v) = 1\{z = v\}
\]

In what follows, we present we show that $W_{NB-k, \psi}$ is symmetric.

**Lemma B.1.** For any $k \geq 0$ and any $\zeta > 0$, the matrix $W_{NB-k, \psi}$ is symmetric.

**Proof.** From the definition of $W_{NB-k, \psi}$ in (112), we note that, since the identity matrix $I$ is symmetric, the lemma follows by showing that for any $\ell \geq 0$ the matrix
\[
X = K \cdot H_G^\ell \cdot C,
\]
is symmetric. Note that $X$ is a $V \times V$ matrix.

Elementary calculations, imply that for any $u, w \in V$, different with each other, we have that
\[
X(u, w) = \sum_{z \in V} \sum_{y \in V} H_y(u, z, y, w).
\]
From the PT invariance of $H^\ell$, i.e., (18), we get the following:
\[
X(u, w) = \sum_{z} \sum_{y} H_y(u, z, y, w) = \sum_{z} \sum_{y} H_y, y, z, u, w = X(u, w).
\]
It is in the second equality that we use the PT invariance of $(H_G)^\ell$.

The above implies that $X$ is a symmetric matrix. The lemma follows. \qed

**Appendix C. Results for Unbounded Spectral Radius**

In this section we present results which are analogous to Theorems 2.1 and 2.2 for graphs whose adjacency matrix $A_G$ has unbounded spectral radius, i.e., $\rho(A_G)$ grows with the size of the graph $n$. We start with the Ising model.

**Theorem C.1.** For any $\delta \in (0, 1)$ and for $\rho_G > 1$, consider the graph $G = (V, E)$ whose adjacency matrix $A_G$ has spectral radius $\rho_G$. Also, let $\mu_G$ be the Ising model on $G$ with zero external field and parameter $\beta \in M_{\text{Ising}}(\rho_G, \delta)$.

If $\rho_G$ is unbounded, there is a constant $C = C(\delta)$ such that the mixing time of the Glauber dynamics on $\mu_G$ is at most $Cn^\frac{1+\frac{1}{4}}{2}$.

Note that there is a discrepancy between the two mixing times we get from Theorems 2.1 and C.1. This has to do with how we use spectral independence, i.e., once this has been established, to bound the mixing time of Glauber dynamics. In that respect, we make a direct use of results from [5, 6] and hence, the discrepancy comes from the fact that these two papers derive different bounds on the mixing time. In the related literature there is also the work in [7] which can be considered for the unbounded case. However, it seems that the result in [7] requires from the Gibbs distribution to satisfy a strong condition called Gibbs uniqueness. The range of the parameters we consider for the Gibbs distribution in Theorems C.1 and 2.1 is beyond this region.

Theorem 2.1 improves on results in [16] for the Ising model by allowing a wider rage for $\beta$. For $\rho_G = \omega(1)$, the range of $\beta$ is wider than that in [16] but the increase is with regard to smaller order terms. Note that the mixing time there is $O(n \log n)$.

**Proof of Theorem C.1**. The proof of Theorem C.1 is almost identical to that of Theorem 2.1. Specifically, using Lemma 9.1 and Theorem 5.2 in the same way as in the proof of Theorem 2.1 we get the following: for $A \subseteq V$ and $\tau \in \{\pm 1\}^A$, the pairwise influence matrix $\mathcal{I}_{A,\tau}^G$, induced by $\mu_G$, satisfies that
\[
\rho(\mathcal{I}_{A,\tau}^G) \leq \delta^{-1}.
\]

The theorem follows as a corollary from (113), Theorem 4.3 and (11). \qed
We proceed with the Hard-core model on a graph with unbounded spectral radius. We prove the following result.

**Theorem C.2.** For any \( \epsilon \in (0, 1) \), \( \Delta \geq 2 \) and \( \rho_G \geq 2 \), consider the graph \( G = (V, E) \) of maxim degree \( \Delta \), whose adjacency matrix \( A_G \) has spectral radius \( \rho_G \). Also, let \( \mu_G \) be the Hard-core model on \( G \) with fugacity \( \lambda \leq (1 - \epsilon) \lambda_c(\rho_G) \).

If \( \rho_G \) is unbounded, there are constants \( C = C(\epsilon) \) and \( C' = C'(\epsilon) \) such that the mixing time of the Glauber dynamics on \( \mu_G \) is at most \( C n^{2+\epsilon} \sqrt{\Delta/\rho_G} \).

Note that the above result implies a polynomial bound for the mixing time only for the case where the ratio \( \sqrt{\Delta/\rho_G} \) is bounded.

The above improves on \cite{16} by allowing a wider rage for \( \lambda \). The improvement is the same as in the bounded case, i.e., get an extra factor \( \epsilon \). Note, though, the extra condition that the ratio of maximum degree \( \Delta \) over \( \rho_G(A_G) \) must be bounded.

**Proof of Theorem C.2**. The proof of Theorem C.2 is almost identical to that of Theorem 2.2. Specifically, using Theorem 10.1 we get the following: There is a constant \( z > 0 \) which depend on \( \epsilon \) such that for any \( \Lambda \subseteq V \) and \( \tau \in \{\pm 1\}^V \) we have that

\[
\rho\left( I_G^{A,\tau}\right) \leq 1 + e^3 (\Delta/\rho_G)^{1/2} z^{-1}.
\]

The theorem follows from the above inequality, Theorem 4.3 and (11). \( \square \)

**APPENDIX D. RAPID MIXING BOUND FOR GENERAL GIBBS DISTRIBUTIONS**

Note that Theorems 5.2 and 5.5 do not provide bounds for the spectral radius of the influence matrix only for the Ising model and the Hard-core model. They are general results and apply to any Gibbs distribution on \( G \). In that respect, it might be interesting to write the corresponding bounds we get from these two theorems on the mixing time of Glauber dynamics on a general two-spin Gibbs distribution.

Consider a graph \( G = (V, E) \) and assume that \( \rho(A_G) \) is bounded. Recall that this assumption implies that the maximum degree \( \Delta \) is also bounded.

For what follows, we need to introduce few useful concepts from \cite{6}. For \( S \subseteq V \), let the Hamming graph \( H_S \) be the graph whose vertices correspond to the configurations \( \{\pm 1\}^S \), while two configurations are adjacent iff they differ at the assignment of a single vertex, i.e., their Hamming distance is one. Similarly, any subset \( \Omega_0 \subseteq \{\pm 1\}^S \) is considered to be connected if the subgraph induced by \( \Omega_0 \) is connected.

A distribution \( \mu \) over \( \{\pm 1\}^V \) is considered to be **totally connected** if for every nonempty \( \Lambda \subseteq V \) and every boundary condition \( \tau \) at \( \Lambda \) the set of configurations in the support of \( \mu(A, \tau) \) is connected.

We need to remark here that all Gibbs distribution with soft-constraints such as the Ising model are totally connected in a trivial way. The same holds for the Hard-core model and this follows with standard arguments.

**Definition D.1.** For some number \( b \geq 0 \), we say that a distribution \( \mu \) over \( \{\pm 1\}^V \) is \( b \)-marginally bounded if for every \( \Lambda \subseteq V \) and any configuration \( \tau \) at \( \Lambda \) we have the following: for any \( V \setminus \Lambda \) and for any \( x \in \{\pm 1\} \) which is in the support of \( \mu(A, \tau) \), we have that

\[
\mu_u(x \mid \Lambda, \tau) \geq b.
\]

The following result is a part of Theorem 1.9 from \cite{6} (arxiv version). This is one of the results in the literature that improves on Theorem 4.2

**Theorem D.2** \cite{6}. Let the integer \( \Delta \geq 3 \) and \( b, \eta \in \mathbb{R}_{>0} \). Consider \( G = (V, E) \) a graph with \( n \) vertices and maximum degree \( \Delta \). Also, let \( \mu \) be a totally connected Gibbs distribution on \( \{\pm 1\}^V \).
If $\mu$ is both $b$-marginally bounded and $\eta$-spectrally independent, then there are constants $C_1, C_2 > 0$ such the Glauber dynamics for $\mu$ exhibits mixing time

$$T_{\text{mix}} \leq \left( \frac{\Delta}{b} \right)^{C_1 \left( \frac{\eta b}{b^2} + 1 \right)} \times C_2 \left( n \log n \right).$$

We combine Theorems 5.2 and 5.5 with the above to get the following rapid mixing results for the Glauber dynamics on a general Gibbs distributions on $G$ with bounded $\rho(A_G)$.

**Theorem D.3 ($\delta$-contraction).** Let the integer $\Delta \geq 3$, $b, \epsilon \in (0, 1]$, $\rho_G > 1$ and $\beta, \gamma, \lambda \in \mathbb{R}$ such that $0 \leq \beta \leq \gamma$, $\gamma > 0$ and $\lambda > 0$.

Consider $G = (V, E)$ a graph with $n$ vertices and maximum degree at most $\Delta$, while $A_G$ is of spectral radius $\rho_G$. Also, let $\mu$ be a totally connected Gibbs distribution on $\{\pm 1\}^V$ specified by the parameters $\beta, \gamma$ and $\lambda$ as in (1).

Setting $\delta = \frac{1 - \epsilon}{\rho_G}$, suppose that $\mu$ is $b$-marginally bounded while the set of functions $\{H_d\}_{d \in [\Delta]}$ exhibits $\delta$-contraction.

There are constants $C_1, C_2 > 0$ such the Glauber dynamics on $\mu$ exhibits mixing time $T_{\text{mix}}$ such that

$$T_{\text{mix}} = \left( \frac{\Delta}{b} \right)^{C_1 \left( \frac{\eta b}{b^2} + 1 \right)} \times C_2 \left( n \log n \right).$$

**Theorem D.3** is straightforward from Theorems D.2 and 5.2. For this reason, we omit its proof.

**Theorem D.4** ($(s, \delta, c)$-potential function). Let the integer $\Delta \geq 3$, $b, \epsilon \in (0, 1]$, $\zeta > 0$, $\rho_G > 1$ and $\beta, \gamma, \lambda \in \mathbb{R}$ such that $0 \leq \beta \leq \gamma$, $\gamma > 0$ and $\lambda > 0$.

Consider $G = (V, E)$ a graph with $n$ vertices and maximum degree at most $\Delta$, while $A_G$ is of spectral radius $\rho_G$. Also, let $\mu$ be a totally connected Gibbs distribution on $\{\pm 1\}^V$ specified by the parameters $\beta, \gamma$ and $\lambda$ as in (1).

Setting $\delta = \frac{1 - \epsilon}{\rho_G}$ and $c = \frac{\zeta}{\rho_G}$, suppose that $\mu$ is $b$-marginally bounded, while there is a $(s, \delta, c)$-potential function $\Psi$ with respect to $(\beta, \gamma, \lambda)$.

There is a constant $C > 0$ such the Glauber dynamics on $\mu$ exhibits mixing time $T_{\text{mix}}$ such that

$$T_{\text{mix}} = \left( \frac{\Delta}{b} \right)^{C \left( \frac{\eta b}{b^2} + 1 \right)} \times (n \log n),$$

where $\eta = 1 + \zeta \cdot \left( 1 - (1 - \epsilon)^s \right)^{-1} \cdot (\Delta/\rho_G)^{1-\frac{1}{s}}$.

**Theorem D.4** is straightforward from Theorems D.2 and 5.5. For this reason, we omit its proof.

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