Superposition rules for higher order systems and their applications

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Abstract
Superposition rules form a class of functions that describe general solutions of systems of first-order ordinary differential equations in terms of generic families of particular solutions and certain constants. In this work, we extend this notion and other related ones to systems of higher order differential equations and analyse their properties. Several results concerning the existence of various types of superposition rules for higher order systems are proved and illustrated with examples extracted from the physics and mathematics literature. In particular, two new superposition rules for the second- and third-order Kummer–Schwarz equations are derived.

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1. Introduction
The study of superposition rules can be traced back to the end of the 19th century, when Lie, Vessiot and Guldberg [1–5] characterized and analysed the properties of systems of first-order differential equations admitting this property, the so-called Lie systems [6–9]. Although the linear superposition rule for homogeneous linear systems of first-order differential equations admits a natural analogue for homogeneous linear systems of higher order differential equations (HODEs), the generalization of their nonlinear counterpart is not so evident and it has hardly been investigated so far [4, 10].

Recently, the necessity of a theory of (linear and nonlinear) superposition rules for systems of HODEs became even more evident, as this concept repeatedly came up in the study of certain systems of second-order differential equations (SODEs) with multiple applications in physics and mathematics [10–15].
In an attempt to fill this gap in the mathematics literature, this work aims to formalize the superposition rule notion for systems of HODEs and to analyse its properties. Since superposition rules for systems of SODEs represent one of the most relevant types of superposition rules appearing in the literature, special attention is paid to this case.

A notion of the superposition rule for systems of SODEs was introduced in [10]. Nevertheless, that work was more focused on the practical use of the concept than on studying its properties. That is why we start here by motivating this definition in detail and analysing some of its properties.

The fundamental problem on the analysis of superposition rules for systems of HODEs is to find coordinate-free geometric conditions ensuring their existence. This problem, solved by the Lie–Scheffers theorem for systems of first-order differential equations, is explicitly solved here for systems of SODEs. Our new result provides not only a new insight into the study of superposition rules for SODEs, but also shows the existence of new and more powerful types of superposition rules for such equations. These new notions can be regarded as generalizations of other concepts already defined for systems of first-order differential equations (see [9]). In addition, most of our achievements can be directly generalized to all systems of HODEs and they are also employed to review previous notions dedicated to the study of such systems, e.g., SODE Lie systems.

Apart from their mathematical interest, our results are also relevant for the study of all physical systems and problems, like nonquadratic Hamiltonians or Berry phases (see [16] and references therein), related to differential equations admitting a superposition rule, such as second-order Riccati equations [10] or Milne–Pinney equations [13].

To highlight the interest of our methods, they are illustrated by the analysis of examples extracted from the physics and mathematics literature. Special attention is paid to second- and third-order Kummer–Schwarz equations, whose mathematical interest is due, for instance, to their appearance in Kummer’s problem, the study of Schwarzian derivatives and other related topics [17, 18]. Furthermore, Kummer–Schwarz equations occur in the analysis of non-stationary two body problems [19, 20] and, via their relation to Riccati and Milne–Pinney equations [21, 22], can be employed to study several problems appearing in cosmology, quantum mechanics and other branches of physics [16, 19, 21, 22]. We derive here superposition rules for the analysis of such equations, which provide us with several advantages with respect to previous methods of studying these, and other related, equations [21–23]. As a byproduct, we find a new property of Kummer–Schwarz equations: their dynamics is determined by a curve in a Lie algebra of vector fields isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \).

The content of the paper is structured as follows. In section 2, we describe some notions and results of the theory of Lie systems to be used throughout the paper. Section 3 concerns the motivation and analysis of the definition of a superposition rule for SODEs as well as several particular types of it found in the literature. In section 4, we provide a characterization of systems of SODEs admitting certain types of superposition rules and we describe a new kind of superposition rules for SODEs. In addition, several properties of superposition rules for SODEs are analysed. The relation of our new results and the so-called SODE Lie systems is studied in section 5. The results of the previous sections lead to the definition and analysis, in section 6, of a general notion of a superposition rule for systems of first-order and higher order differential equations. Subsequently, we illustrate in sections 7 and 8 some of the theoretical results derived throughout our work by the investigation of several remarkable HODEs. Finally, section 9 summarizes our achievements and details some work to be accomplished in the future.
2. Fundamentals on Lie systems

We hereafter assume all geometrical objects and mappings, like vector fields or superposition rules, to be real, smooth and globally defined. In this way, we highlight the key points of our presentation by omitting the analysis of certain minor technical problems. For additional information, we refer to [9, 10].

**Definition 2.1.** A superposition rule for a system of first-order ordinary differential equations

\[ \frac{dx^i}{dt} = X^i(t, x), \quad i = 1, \ldots, n, \quad (2.1) \]

is a map \( \Phi : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) of the form

\[ x = \Phi(x_1(t), \ldots, x_m(t); k_1, \ldots, k_n), \quad (2.2) \]

allowing us to write the general solution of system (2.1) as

\[ x(t) = \Phi(x_1(t), \ldots, x_m(t); k_1, \ldots, k_n), \quad (2.3) \]

with \( x_1(t), \ldots, x_m(t) \) being a ‘generic’ family of particular solutions and \( k_1, \ldots, k_n \) being the constants related to the initial conditions of each particular solution.

**Note 2.2.** We shall not define rigorously what ‘generic’ means in the above definition, as it is not essential for our purposes and depends on the particular case. It shall be sufficient to bear in mind that, in the case of linear superposition rules for homogeneous linear systems of first-order differential equations, ‘generic’ means that the elements of the chosen finite family of particular solutions must be linearly independent.

The uppermost achievement of the theory of Lie systems was obtained by Lie [1], who succeeded in characterizing systems of first-order differential equations that admit a superposition rule.

**Theorem 2.3** (The Lie–Scheffers theorem). A system (2.1) admits a superposition rule (2.2) if and only if its right-hand side can be written as

\[ \frac{dx^i}{dt} = Z_1(t)\xi_1^i(x) + \cdots + Z_r(t)\xi_r^i(x), \quad i = 1, \ldots, n, \quad (2.4) \]

so that the vector fields

\[ X_\alpha(x) = \sum_{i=1}^n \xi_\alpha^i(x) \frac{\partial}{\partial x^i}, \quad \alpha = 1, \ldots, r, \quad (2.5) \]

with \( r \leq m \cdot n, \) span an \( r \)-dimensional real Lie algebra.

The following definition and lemma, whose proof is a straightforward consequence of the Jacobi identity, notably simplify several statements and proofs of various results concerning the theory of Lie systems.

**Definition 2.4.** Given a (finite or infinite) family \( \mathcal{A} \) of vector fields on \( \mathbb{R}^n \), we denote by \( \text{Lie}(\mathcal{A}) \) the smallest Lie algebra \( V \) of vector fields on \( \mathbb{R}^n \) containing \( \mathcal{A} \).

**Lemma 2.5.** Given a family of vector fields \( \mathcal{A} \), the linear space \( \text{Lie}(\mathcal{A}) \) is spanned by the vector fields of

\[ \mathcal{A}, [\mathcal{A}, \mathcal{A}], [\mathcal{A}, [\mathcal{A}, \mathcal{A}]], [\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{A}]]], \ldots, \]

where \([\mathcal{A}, \mathcal{B}]\), with \( \mathcal{B} = \mathcal{A}, [\mathcal{A}, \mathcal{A}], \ldots \), denotes the set of Lie brackets between the elements of the families \( \mathcal{A} \) and \( \mathcal{B} \) of vector fields.
Recall that if \( \tau : T\mathbb{R}^n \to \mathbb{R}^n \) denotes the tangent bundle projection and \( \tau_2 \) stands for the projection \( \tau_2 : (t, x) \in \mathbb{R} \times \mathbb{R}^n \mapsto x \in \mathbb{R}^n \), a \textit{time-dependent vector field} \( X \) on \( \mathbb{R}^n \) is a map \( X : (t, x) \in \mathbb{R} \times \mathbb{R}^n \mapsto X(t, x) \in T\mathbb{R}^n \), such that \( \tau \circ X = \tau_2 \). Observe that every time-dependent vector field \( X \) on \( \mathbb{R}^n \) can be regarded as a family \( \{X_r\}_{r \in \mathbb{R}} \) of vector fields on \( \mathbb{R}^n \), where \( X_r : x \in \mathbb{R}^n \mapsto X_r(x) = X(t, x) \in T_r \mathbb{R}^n \).

Similarly to the standard vector fields, time-dependent vector fields also admit integral curves [24, 25]. We hereafter call an \textit{integral curve} of \( X \) passing through \((t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n\) any integral curve \( \gamma_{s_0} : s \in \mathbb{R} \mapsto (t(s), \gamma(s)) \in \mathbb{R} \times \mathbb{R}^n \) of the one-dimensional distribution on \( \mathbb{R} \times \mathbb{R}^n \) spanned by the suspension of \( X \), i.e. the vector field \( \partial/\partial t + X(t, x) \) [25], satisfying that \((t_0, x_0) \in \text{Im} \gamma_{s_0}\).

From a modern geometric point of view, every system of first-order differential equations of the form (2.1) is described by the unique time-dependent vector field on \( \mathbb{R}^n \), namely \( X(t, x) = \sum_{i=1}^n X^i(t, x) \partial/\partial x^i \), whose integral curves are (up to an appropriate reparametrization) of the form \((t, x(t))\), with \( x(t) \) being a solution of system (2.1). For simplicity, we use the symbol \( X \) to refer to both a time-dependent vector field and the system of differential equations describing its integral curves.

In such geometric terms, the Lie–Scheffers theorem states that a system \( X \) admits a superposition rule if and only if there exists a finite-dimensional Lie algebra of vector fields \( V \), the so-called \textit{Vessiot–Guldberg Lie algebra}, such that \( \{X_r\}_{r \in \mathbb{R}} \subset V \). In consequence, \( X \) is a Lie system if and only if the Lie algebra \( \text{Lie}(\{X_r\}_{r \in \mathbb{R}}) \) is finite dimensional.

The geometrical interpretation of superposition rules as well as one of the techniques for their determination is based on the notion of \textit{diagonal prolongation} [9].

**Definition 2.6.** Given a time-dependent vector field \( X(t, x) = \sum_{i=1}^n X^i(t, x) \partial/\partial x^i \) on \( \mathbb{R}^n \), the time-dependent vector field \( \tilde{X} \) on \( \mathbb{R}^{n(m+1)} \) of the form

\[
\tilde{X} = \sum_{a=0}^m \sum_{i=1}^n X^i(t, x_{(a)}) \frac{\partial}{\partial x_{(a)}^i}
\]

is called the diagonal prolongation to \( \mathbb{R}^{n(m+1)} \) of \( X \).

A method for determining superposition rules is briefly described as follows (see [9, 10] for details and examples).

(i) Take a basis \( X_1, \ldots, X_r \) of a finite-dimensional Lie algebra (2.5) associated with the Lie system under study.

(ii) Choose the smallest positive integer \( m \), so that the diagonal prolongations of the elements of the previous basis to \( (\mathbb{R}^n)^{m+1} \) are linearly independent at a generic point.

(iii) Take the global coordinates \( x^1, \ldots, x^m \) on \( \mathbb{R}^m \). By defining this coordinate system on each copy of \( \mathbb{R}^n \) within \( (\mathbb{R}^n)^{m+1} \), we obtain a coordinate system \( \{x_{(a)}^i \mid i = 1, \ldots, n, \ a = 0, \ldots, m \} \) on \( (\mathbb{R}^n)^{m+1} \). Obtain \( n \) functionally independent first integrals \( F_1, \ldots, F_n \) common to all diagonal prolongations \( \tilde{X}_1, \ldots, \tilde{X}_r \) of \( X_1, \ldots, X_r \) to \( (\mathbb{R}^n)^{m+1} \), such that \( \partial(F_1, \ldots, F_n)/\partial(x_{(1)}^1, \ldots, x_{(m)}^n) \neq 0 \). This can be performed, for instance, by means of the well-known method of characteristics.

(iv) Assume the above first integrals to take certain real constant values, i.e. \( F_i = k_i \) for \( i = 1, \ldots, n \). By means of these equations, calculate the expressions of the variables \( x_{(1)}^1, \ldots, x_{(m)}^n \) in terms of \( x_1^1, \ldots, x_r^m \), with \( a = 1, \ldots, m \), and \( k_1, \ldots, k_n \).

(v) The obtained expressions give rise to a superposition rule in terms of any generic family of \( m \) particular solutions and the constants \( k_1, \ldots, k_n \).

Given two vector fields \( X \) and \( Y \), we have that \( [X, Y] = [\tilde{X}, \tilde{Y}] \), i.e. the Lie bracket of two diagonal prolongations is a diagonal prolongation. Another, much less evident, property
of diagonal prolongations is described in the following lemma, whose proof can be found in [9, lemma 1].

**Lemma 2.7.** Consider a family of vector fields $X_1, \ldots, X_r$ on $\mathbb{R}^n$ whose diagonal prolongations to $\mathbb{R}^{nm}$ are linearly independent at a generic point. Then, given their diagonal prolongations $\tilde{X}_1, \ldots, \tilde{X}_r$ to $\mathbb{R}^{n(m+1)}$, a vector field $\tilde{X} = \sum_{a=1}^r b_a \tilde{X}_a$, with $b_a \in C^\infty(\mathbb{R}^{n(m+1)})$, is again a diagonal prolongation if and only if the functions $b_a$ are constant.

It is worth noting that one can relate superposition rules to zero-curvature connections on a bundle $\text{pr} : (x(0), \ldots, x(m)) \in \mathbb{R}^{n(m+1)} \mapsto (x(1), \ldots, x(m)) \in \mathbb{R}^{nm}$ as follows (cf [9]).

**Proposition 2.8.** Each superposition rule (2.3) for a system $X$ is equivalent to a local $n$-codimensional foliation on $\mathbb{R}^{n(m+1)}$ whose leaves project, by $\text{pr}$, diffeomorphically onto $\mathbb{R}^{nm}$ and such that the vector fields $\{\tilde{X}_r\}_{r \in \mathbb{R}}$ are tangent to its leaves.

The above result can be used to easily prove the following new result, which is used posteriorly in order to analyse the existence of a particular class of superposition rules for systems of SODEs.

**Proposition 2.9.** A family of Lie systems admits a common superposition rule if and only if they admit a common Vessiot–Guldberg Lie algebra.

### 3. On the general definition of a superposition rule for SODEs

To motivate the general definition of a superposition rule for SODEs, let us start by analysing a particular property of standard superposition rules. It is well known that every homogeneous linear system on $\mathbb{R}^n$ of the form

$$\frac{dx^i}{dt} = \sum_{j=1}^n A_{ij}(t)x^j, \quad i = 1, \ldots, n,$$

(3.1)

where $A_{ij}$ are real $t$-dependent functions, admits its general solution $x(t)$ to be written as

$$x(t) = k_1 x_{1(t)}(t) + \cdots + k_n x_{n(t)}(t),$$

(3.2)

with $x_{1(t)}(t), \ldots, x_{n(t)}(t)$ being a family of linearly independent particular solutions of (3.1) and $k_1, \ldots, k_n$ a set of real constants. In other words, the system (3.1) admits a linear superposition rule. This leads to the existence of nonlinear systems admitting general superposition rules [8]. Indeed, every diffeomorphism $\phi : \mathbb{R}^n \ni x \mapsto z \in \mathbb{R}^n$ transforms the system (3.1) into

$$\frac{dz^i}{dt} = F^i(t, z), \quad i = 1, \ldots, n,$$

(3.3)

where the functions $F^i : \mathbb{R}^{n+1} \to \mathbb{R}$ are generally nonlinear in the variables $z^1, \ldots, z^n$, and, what is more important, whose general solution $z(t)$ can be expressed (maybe nonlinearly) as

$$z(t) = \phi(k_1 \phi^{-1}(z_{1(t)}(t)) + \cdots + k_n \phi^{-1}(z_{n(t)}(t))),$$

in terms of certain families of particular solutions $z_{1(t)}(t), \ldots, z_{n(t)}(t)$ of (3.3) and the constants $k_1, \ldots, k_n$. That is, since linearity depends on coordinate systems and the existence of superposition rules does not (recall the Lie–Scheffers theorem), the mere existence of linear superposition rules for homogeneous linear systems of first-order differential equations leads to the existence of nonlinear systems admitting superposition rules. In addition, it is worth noting that not every system admitting a nonlinear superposition rule is of this form. For instance, Riccati equations admit a superposition rule, but they cannot always be transformed diffeomorphically into linear homogeneous systems [26].
The aforementioned properties have an analogue for systems of SODEs. In fact, it can easily be proved that every homogeneous linear system of SODEs
\[
\frac{d^2x^i}{dt^2} = \sum_{j=1}^{n} \left( A^i_j(t) \frac{dx^j}{dt} + B^i_j(t)x^j \right), \quad i = 1, \ldots, n, \tag{3.4}
\]
with \(A^i_j\) and \(B^i_j\) being any set of \(2n^2\) time-dependent functions, admits its general solution to be written as
\[
x(t) = k_1x_{1(1)}(t) + \cdots + k_{2n}x_{2n(2n)}(t), \tag{3.5}
\]
in terms of some arbitrary constants \(k_1, \ldots, k_{2n}\) and a set of solutions \(\{x_{a(i)}(t) \mid a = 1, \ldots, 2n\}\), such that the vectors \((x_{a(i)}(t), \frac{dx_{a(i)}}{dt}) \in T\mathbb{R}^n\) are linearly independent at every \(t \in \mathbb{R}\). Now, a change of variables \(z = \phi(x)\) transforms the above system into a (generally nonlinear) new one
\[
\frac{d^2z^i}{dt^2} = H^i(t, z, \frac{dz}{dt}), \quad i = 1, \ldots, n, \tag{3.6}
\]
for certain functions \(H^i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}\), and, moreover, such a change enables us, in view of (3.5), to write its general solution \(z(t)\) in the form
\[
z(t) = \phi(k_1\phi^{-1}(z_{1(1)}(t))) + \cdots + k_{2n}\phi^{-1}(z_{2n}(t))), \tag{3.7}
\]
in terms of a generic family of particular solutions \(z_{1(1)}(t), \ldots, z_{2n}(t)\) for (3.6) and constants \(k_1, \ldots, k_{2n}\). Consequently, linear superposition rules for systems (3.4) give rise to the existence of ‘superposition rule-like’ expressions for systems of SODEs. Expressions of this type frequently appear in the literature, e.g., in the study of linear inhomogeneous systems of SODEs. This suggests us the following definition that was proposed and briefly analysed in [10] and that includes the previous expressions as particular cases.

**Definition 3.1.** A base-superposition rule for a system
\[
\frac{d^2x^i}{dt^2} = F^i(t, x, \frac{dx}{dt}), \quad i = 1, \ldots, n, \tag{3.8}
\]
is a map \(\Upsilon : (\mathbb{R}^n)^m \times \mathbb{R}^{2m} \to \mathbb{R}^n\) allowing us to write its general solution \(x(t)\) as
\[
x(t) = \Upsilon(x_{1(1)}(t), \ldots, x_{m(1)}(t); k_1, \ldots, k_{2n}), \tag{3.9}
\]
where \(x_{1(1)}(t), \ldots, x_{m(1)}(t)\) is a generic family of particular solutions and \(k_1, \ldots, k_{2n}\) are constants.

The above concept does not cover many other expressions found in the literature for describing systems of SODEs [10, 12, 13, 15]. For instance, consider a Milne–Pinney equation
\[
\frac{d^2x}{dt^2} = -\omega^2(t)x + \frac{c}{x^3}, \tag{3.10}
\]
with \(x > 0\) and \(\omega(t)\) being any time-dependent real function [27–29]. This equation is relevant due to its applications in quantum mechanics, cosmology, Bose–Einstein condensates and other physical topics [13, 15, 16]. Recently, it was proved (see [13]) that its general solution can be written as
\[
x(t) = \left[ k_1x_{1(1)}^2(t) + k_2x_{2(2)}^2(t) \pm 2\left[ \lambda_{12}\left[ I_2x_{1(1)}^2(t)x_{1(2)}^2(t) - c(x_{1(1)}^4(t) + x_{1(2)}^4(t)) \right]\right]^{1/2} \right]^{1/2}, \tag{3.11}
\]
in terms of a pair \(x_{1(1)}(t), x_{2(2)}(t)\) of generic particular solutions, the function \(\lambda_{12} = \lambda_{12}(k_1, k_2, c, I_1)\), the constant of motion
\[
I_2 = \left( \frac{dx_{1(1)}}{dt}x_{2(2)}(t) - \frac{dx_{2(2)}}{dt}x_{1(1)}(t) \right)^2 + c \left[ \left( \frac{x_{1(1)}}{x_{2(2)}}(t) \right)^2 + \left( \frac{x_{2(2)}}{x_{1(1)}}(t) \right)^2 \right], \tag{3.12}
\]
and two constants $k_1$ and $k_2$ related to initial conditions. Observe that expression (3.11) cannot be described by means of any base-superposition rule notion. Indeed, while $k_1$ and $k_2$ take different values to describe the different particular solutions of (3.11), the constant $I_2$, whose value is fixed by the chosen particular solutions and their time derivatives, does not appear in base-superposition rules. The same will happen for other new relevant expressions to be presented in this work. This motivates us to generalize the base-superposition rule as follows.

**Definition 3.2.** A quasi-base-superposition rule for a system of SODEs in $\mathbb{R}^n$ of the form (3.8) is a function $G : \mathbb{R}^{mn} \times \mathbb{R}^n \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ allowing us to cast its general solution $x(t)$ in the form

$$x(t) = G(x_{(1)}(t), \ldots, x_{(m)}(t), I_1, \ldots, I_q; k_1, \ldots, k_{2n}),$$  

(3.12)

in terms of any generic family $x_{(1)}(t), \ldots, x_{(m)}(t)$ of particular solutions of (3.8), a set of time-independent constants of motion $I_1, \ldots, I_q$, whose values are determined by the choice of the previous family and their derivatives with respect to the time, and a set of constants $k_1, \ldots, k_{2n}$.

Although almost every example of ‘superposition-rule like’ expression for SODEs is a particular instance of a quasi-base-superposition rule, this notion still fails to cover several expressions found in the literature. That is the case of the very recently discovered expression for second-order Riccati equations, presented in [10], which describes the general solution of such equations in terms of a generic family of particular solutions, their derivatives and several constants. This motivates us to generalize the the concept of a quasi-base-superposition rule as follows.

**Definition 3.3.** A system of second-order ordinary differential equations on $\mathbb{R}^n$ given by (3.8) admits a superposition rule if there exists a map $\Upsilon : (\mathbb{T}\mathbb{R}^n)^m \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ of the form

$$x = \Upsilon(x_{(1)}, v_{(1)}, \ldots, x_{(m)}, v_{(m)}; k_1, \ldots, k_{2n}),$$  

(3.13)

with $(x_{(a)}, v_{(a)}) \in T_{x_{(a)}} \mathbb{R}^n$ for $a = 1, \ldots, m$, such that the general solution $x(t)$ of (3.8) can be written as

$$x(t) = \Upsilon \left( x_{(1)}(t), \frac{dx_{(1)}}{dt}(t), \ldots, x_{(m)}(t), \frac{dx_{(m)}}{dt}(t); k_1, \ldots, k_{2n} \right),$$  

(3.14)

with $x_{(1)}(t), \ldots, x_{(m)}(t)$ being any generic family of $m$ particular solutions of the system, and $k_1, \ldots, k_{2n}$ being a set of constants related to the initial conditions of each particular solution.

As every constant of motion involved in a quasi-base-superposition rule can be considered as a function on $\mathbb{T}\mathbb{R}^{mn}$, quasi-base-superposition rules can be easily regarded as a particular type of superposition rules for SODEs. Base-superposition rules can also be regarded as superposition rules that depend only on the base variables of $\mathbb{T}\mathbb{R}^{mn}$ that justifies their name.

Let us now turn to analysing several properties of superposition rules for systems of SODEs. Similarly to superposition rules for systems of first-order differential equations, expression (3.14) cannot be applied for each family of $m$ particular solutions. Recall that even in the simple case of a homogeneous linear system of SODEs, expression (3.2) just remains valid for certain families of particular solutions. Consequently, in order to establish when a system (3.8) admits a superposition rule, it is essential to establish what ‘generic’ means in this new context. From now on, we say that (3.14) is satisfied by a generic set of $m$ particular solutions if there exists an open and dense subset $U$ of $(\mathbb{T}\mathbb{R}^n)^m$, such that expression (3.14) is valid for every family $x_{(1)}(t), \ldots, x_{(m)}(t)$ that satisfies

$$\left( x_{(1)}(0), \frac{dx_{(1)}}{dt}(0), \ldots, x_{(m)}(0), \frac{dx_{(m)}}{dt}(0) \right) \in U \subset (\mathbb{T}\mathbb{R}^n)^m.$$
Every family of particular solutions satisfying the above condition is called a fundamental system of particular solutions of the system (3.8).

Superposition rules for systems of SODEs possess properties different from those of superposition rules for systems of first-order ones. Let us illustrate this fact by means of a particular remarkable difference. Consider again (2.1) as a Lie system admitting superposition rule (2.3). A time reparametrization \( \tau = \tau(t) \), with the inverse \( t = t(\tau) \), transforms this system into

\[
\frac{dx^i}{d\tau} = \frac{dt}{d\tau} F^i(t(\tau), x), \quad i = 1, \ldots, n. \tag{3.15}
\]

As we assume (2.1) to be a Lie system, formula (2.4) applies and the right-hand term of the above expression can be brought into the form

\[
\frac{dx^i}{d\tau} = \frac{dr}{d\tau} (Z_1(t(\tau))\xi_1^i(x) + \cdots + Z_m(t(\tau))\xi_m^i(x)), \quad i = 1, \ldots, n. \tag{3.16}
\]

Consequently, in view of the Lie–Scheffers theorem, the system (3.15) becomes a Lie system. Moreover, as the general solution \( x(t) \) of (2.1) and the general solution \( x(\tau) \) of (3.15) satisfy \( x(t(\tau)) = x(\tau) \), then the superposition rule (2.3) for (2.1) allows one to write

\[
x(\tau) = \Phi(x_{(1)}(\tau), \ldots, x_{(m)}(\tau); k_1, \ldots, k_n),
\]

in terms of a generic family \( x_{(1)}(\tau), \ldots, x_{(m)}(\tau) \) of particular solutions of system (3.15) and \( k_1, \ldots, k_n \). In summary, Lie’s characterization of systems of first-order ordinary differential equations admitting a superposition rule is invariant under time reparametrizations and Lie systems related in this way share a common superposition rule. Indeed, note that this follows trivially from the form of (3.16) and proposition 2.9.

The above property is no longer valid for superposition rules of systems of SODEs. Given a system of SODEs admitting a superposition rule, the systems obtained from it by time reparametrizations do not necessarily possess the same superposition rule. For instance, consider a system of SODEs (3.8) admitting a superposition rule (3.14). A time reparametrization \( \tau = \tau(t) \), with the inverse \( t = t(\tau) \), transforms (3.8) into

\[
\frac{d^2x^i}{d\tau^2} = \frac{d^2t}{d\tau^2} \frac{dx^i}{dt} \frac{dt}{d\tau} + \left( \frac{dr}{d\tau} \right)^2 F^i(t(\tau), x, \frac{dx^i}{dt} \frac{dt}{d\tau} \frac{dr}{d\tau}), \quad i = 1, \ldots, n. \tag{3.17}
\]

whose general solution \( x(\tau) \) satisfies \( x(\tau) = x(t(\tau)) \), where \( x(t) \) is the general solution of (3.8). Hence, from the superposition rule (3.14), we obtain that \( x(\tau) \) can be expressed as

\[
x(\tau) = \Upsilon \left( x_{(1)}(\tau), \frac{dt}{d\tau}(t(\tau)) \frac{dx_{(1)}}{dt}(\tau), \ldots, x_{(m)}(\tau), \frac{dt}{d\tau}(t(\tau)) \frac{dx_{(m)}}{dt}(\tau); k_1, \ldots, k_{2n} \right).
\]

The above expression is not necessarily a superposition rule, as it may admit an explicit dependence on the new time variable \( \tau \). A simple example illustrating this fact can be found in section 7.

Obviously, we could have also required the superposition rule concept for systems of SODEs to be invariant under time reparametrizations, but this would exclude several important examples like second-order Riccati or Milne–Pinney equations [10, 15].

### 4. On the existence of superposition rules for SODEs

The following theorem characterizes systems of SODEs admitting a superposition rule. We hereafter use the canonical global coordinates \( (x^1, \ldots, x^n, v^1, \ldots, v^n) \) on \( T^n \mathbb{R}^n \). By defining this coordinate system on each copy of \( T^n \mathbb{R}^n \) within \( (T^n \mathbb{R}^n)^m \), we obtain a coordinate system \( \{ x_{(i)}^a, v_{(i)}^a \mid i = 1, \ldots, n, \ a = 1, \ldots, m \} \) on \( (T^n \mathbb{R}^n)^m \).
Theorem 4.1. A mapping \( \Upsilon : (\mathbb{T}^{n})^{m} \times \mathbb{R}^{2n} \to \mathbb{R}^{n} \) is a superposition rule for a system of SODEs (3.8) if and only if

(i) the functions \( u_k : (\mathbb{T}^{n})^{m} \ni p \mapsto u_k(p) = \Upsilon(p; k) \in \mathbb{R}^{n} \), with \( k \in \mathbb{R}^{2n} \), are the common solutions for the \( t \)-parametrized family of systems of PDEs on \((\mathbb{T}^{n})^{m}\) given by

\[
\left( X_{D}^{(m)} \right)_{i}^{j} u_k^i = F^j(t, u_k, \left( X_{D}^{(m)} \right)_{i}^{j} u_k), \quad i = 1, \ldots, n, \tag{4.1}
\]

where \( u_k = (u_k^1, \ldots, u_k^n) \in \mathbb{R}^n \), and \( X_{D}^{(m)} \) and \( X_{L}^{(m)} \) are the diagonal prolongations to \((\mathbb{T}^{n})^{m}\) of the time-dependent vector fields

\[
X_D = \sum_{i=1}^{n} \left( v^j \frac{\partial}{\partial x^i} + F^j(t, x, v) \frac{\partial}{\partial v^i} \right), \quad X_L = \sum_{i=1}^{n} \partial_t F^j(t, x, v) \frac{\partial}{\partial v^i} \tag{4.2}
\]

and

(ii) the map \( \varphi : (\mathbb{T}^{n})^{m} \times \mathbb{R}^{2n} \to \mathbb{T}^{n} \) of the form

\[
\varphi(p; k) = \left( u_k(p), \left( \left( X_{D}^{(m)} \right)_{i}^{j} u_k \right)(p) \right) \in T_{\varphi(p)} \mathbb{T}^{n} \tag{4.3}
\]

gives rise to a family of bijections \( \varphi_{p} : k \in \mathbb{R}^{2n} \mapsto \varphi(p; k) \in \mathbb{T}^{n} \), with \( p \) being a generic point of \( (\mathbb{T}^{n})^{m} \).

Proof. Assume that the SODE Lie system (3.8) has a superposition rule (3.13). One can define the function \( u_k : p \in (\mathbb{T}^{n})^{m} \mapsto \Upsilon(p; k) \in \mathbb{R}^{n} \), for each \( k \in \mathbb{R}^{2n} \), which leads, for every fundamental system of solutions \( x_{i1}(t), \ldots, x_{in}(t) \) of (3.8), to a new particular solution of this system:

\[
\tilde{x}(t) = u_k \left( x_{i1}(t), \frac{dx_{i1}}{dt}(t), \ldots, x_{in}(t), \frac{dx_{in}}{dt}(t) \right). \tag{4.4}
\]

On the other hand,

\[
\frac{d^2 \tilde{x}}{dt^2} = \sum_{a=1}^{m} \sum_{j=1}^{n} \left( v^j_{a(b)} \frac{\partial^2 u_k^j}{\partial x^a_{(a)} \partial x^b_{(b)}} + 2 v^j_{a(b)} F^j_{(a)} \frac{\partial^2 u_k^j}{\partial x^a_{(a)} \partial v^b_{(b)}} + F^j_{(a)} F^j_{(b)} \frac{\partial^2 u_k^j}{\partial v^a_{(a)} \partial v^b_{(b)}}, \quad i = 1, \ldots, n, \tag{4.5}
\]

where, for shortening the notation, we have denoted \( F^j_{(a)} = F^j/t, x_{(a)}, v_{(a)} \) and

\[
p(t) = \left( x_{i1}(t), \frac{dx_{i1}}{dt}(t), \ldots, x_{in}(t), \frac{dx_{in}}{dt}(t) \right). \tag{4.6}
\]

From the expression of \( X_D \) given in (4.2), we have

\[
\frac{d^2 \tilde{x}}{dt^2} = \left( \left( X_{D}^{(m)} \right)_{i}^{j} u_k \right)(p), \quad i = 1, \ldots, n. \tag{4.7}
\]

By differentiating expression (4.5) with respect to the time, we obtain

\[
\frac{d^3 \tilde{x}}{dt^3} = \left[ \sum_{i=1}^{n} \sum_{a=1}^{m} \sum_{j=1}^{n} \left( v^j_{a(b)} F^j_{(b)} \frac{\partial^2 u_k^j}{\partial x^a_{(a)} \partial x^b_{(b)}} + 2 v^j_{a(b)} F^j_{(a)} \frac{\partial^2 u_k^j}{\partial x^a_{(a)} \partial v^b_{(b)}} + F^j_{(a)} F^j_{(b)} \frac{\partial^2 u_k^j}{\partial v^a_{(a)} \partial v^b_{(b)}} \right) \right. \tag{5.1}
\]

\[
\left. + \sum_{a=1}^{m} \sum_{j=1}^{n} \left( F^j_{(a)} \frac{\partial u_k^j}{\partial x^a_{(a)}} + \frac{\partial F^j_{(a)}}{\partial t} \frac{\partial u_k^j}{\partial v^a_{(a)}} \right) \right] \frac{d^2 \tilde{x}}{dt^2} + \left[ \sum_{a=1}^{m} \sum_{j=1}^{n} \left( v^j_{a(b)} \frac{\partial F^j_{(a)}}{\partial x^a_{(a)}} + F^j_{(a)} \frac{\partial F^j_{(b)}}{\partial v^a_{(a)}} \frac{\partial u_k^j}{\partial v^b_{(b)}} \right) \right](p(t)).
\]
If we compare the above expression with
\[
\left( X_{D}^{(m)} \right)^{2}_{t} u_{k}^{i} = \sum_{j,l=1}^{m} \sum_{a,b=1}^{n} \left( v_{a}^{j} \frac{\partial}{\partial x_{a}^{j}} F_{a}^{j} + F_{a}^{j} \frac{\partial}{\partial v_{a}^{j}} \right) \left( v_{b}^{l} \frac{\partial}{\partial x_{b}^{l}} + F_{b}^{l} \frac{\partial}{\partial v_{b}^{l}} \right) u_{k}^{i}
\]
\[
= \sum_{j,l=1}^{m} \sum_{a,b=1}^{n} \left( v_{a}^{j} v_{b}^{l} \frac{\partial^{2} u_{k}^{i}}{\partial x_{a}^{j} \partial x_{b}^{l}} + 2 v_{a}^{j} v_{b}^{l} \frac{\partial^{2} u_{k}^{i}}{\partial x_{a}^{j} \partial v_{b}^{l}} + F_{a}^{j} F_{b}^{l} \frac{\partial^{2} u_{k}^{i}}{\partial x_{a}^{j} \partial v_{b}^{l}} \right)
\]
\[+ \sum_{a=1}^{n} \sum_{j=1}^{m} \left( v_{a}^{j} \frac{\partial F_{a}^{j}}{\partial x_{a}^{j}} + F_{a}^{j} \frac{\partial v_{a}^{j}}{\partial v_{a}^{j}} \right) \frac{\partial u_{k}^{i}}{\partial v_{a}^{j}},
\]
we obtain
\[
\frac{d^{2} \bar{\psi}}{dt^{2}}(t) = \left( \left( X_{D}^{(m)} \right)^{2}_{t} u_{k}^{i} + \left( X_{L}^{(m)} \right)^{2}_{t} u_{k}^{i} \right)(p(t)), \quad i = 1, \ldots, n. \tag{4.8}
\]

As \( \bar{x}(t) \) is a solution of system (3.8), and in view of expressions (4.7) and (4.8), it turns out that
\[
\left( \left( X_{D}^{(m)} \right)^{2}_{t} u_{k}^{i} + \left( X_{L}^{(m)} \right)^{2}_{t} u_{k}^{i} \right)(p(t)) = F^{i}(t, u_{k}(p(t)) + \left( X_{D}^{(m)} \right)^{2}_{t} u_{k}^{i})(p(t)). \tag{4.9}
\]

Now, equation (4.9) holds for every fundamental system. This implies that, for each \( t \in \mathbb{R} \), the above equation remains valid for a generic open and dense subset of \((\mathbb{T}^{m})^{n}\). Hence,
\[
\left( X_{D}^{(m)} \right)^{2}_{t} u_{k}^{i} + \left( X_{L}^{(m)} \right)^{2}_{t} u_{k}^{i} = F^{i}(t, u_{k} + \left( X_{D}^{(m)} \right)^{2}_{t} u_{k}^{i}), \quad i = 1, \ldots, n.
\]
for every \( t \in \mathbb{R} \). Additionally, as the above procedure is still valid for every \( k \in \mathbb{R}^{2n} \), every superposition rule provides us with a family of \( 2n \)-parametrized solutions \( u_{k}(\cdot) = \Upsilon(\cdot; k) \) of the \( t \)-parametrized family of systems of PDEs (4.1).

Consider now a fundamental system \( x_{1}(t), \ldots, x_{m}(t) \) and denote \( p = p(0) \). For an arbitrary \( (x_{0}, v_{0}) \in \mathbb{T}_{x_{0}} \mathbb{R}^{n} \), the theorem of the existence and uniqueness of solutions for systems of first-order differential equations shows that there exists a solution \( x(t) \) of system (3.8) with the initial conditions \( x(0) = x_{0} \) and \( dx(0)/dt = v_{0} \). In view of the properties of superposition rules, there exists a single \( k \in \mathbb{R}^{2n} \), such that \( x(t) = \Upsilon(p(t); k) = u_{k}(p(t)) \). Consequently, in view of expression (4.7), one has
\[
\begin{aligned}
x_{0}^{i} &= u_{k}^{i}(p), \\
v_{0}^{i} &= \frac{d u_{k}^{i}(p(t))}{dt} \bigg|_{t=0} = \left( X_{D}^{(m)} \right)^{2}_{t} u_{k}^{i}(p), \quad i = 1, \ldots, n.
\end{aligned}
\]
In other words, for a generic \( p \in \left( \mathbb{T}^{m} \right)^{n} \), there exists a single \( k \in \mathbb{R}^{2n} \), such that \( \varphi(p, k) = (x_{0}, v_{0}) \). It follows that \( \varphi \) is a bijection that concludes the 'if' part of our demonstration.

Let us now prove that a map \( \Upsilon : \left( \mathbb{T}^{m} \right)^{n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n} \) satisfying conditions (i) and (ii) is a superposition rule for (3.8). Consider any solution \( x(t) \) of (3.8). Given a generic family of \( m \) particular solutions \( x_{1}(t), \ldots, x_{m}(t) \) of (3.8), condition (ii) ensures that there exists a unique \( k \in \mathbb{R}^{2n} \), such that \( \varphi(p, 0) = (x(0), dx(0)/dt) \), where we took again \( p(t) \) to be of the form (4.6). In view of condition (i), the function \( u_{k}(\cdot) = \Upsilon(\cdot; k) \) is a solution for the family of systems of PDEs (4.1). Defining now \( \bar{x}(t) = u_{k}(p(t)) \) and using that expressions (4.7) and (4.8) are valid again, we obtain
\[
\frac{d^{2} \bar{x}}{dt^{2}}(t) = F^{i} \left( t, \bar{x}(t), \frac{d \bar{x}}{dt}(t) \right), \quad i = 1, \ldots, n. \tag{4.10}
\]
That is, \( \bar{x}(t) \) is a solution to (3.8). Moreover, in view of condition (ii) and formula (4.7), \( \bar{x}(0) = x(0) \) and \( d \bar{x}/dt(0) = dx/dt(0) \). Consequently, \( \bar{x}(t) \) and \( x(t) \) are both the solutions
of (3.8) with the same initial conditions and they hence coincide. In summary, for every solution \( x(t) \) of system (3.8) and a generic family of \( m \) particular solutions, there exists a unique \( k \in \mathbb{R}^{2n} \), such that \( x(t) = \Upsilon(p(t); k) \), so \( \Upsilon \) is a superposition rule. \( \square \)

Roughly speaking, theorem 4.1 states that the existence of a superposition rule for a system of SODEs (3.8) is determined by the existence of an ‘appropriate’ \( 2n \)-parametric family of particular solutions of the family of systems of PDEs (4.1). The interest of this result is obvious: it characterizes not only the existence of superposition rules for systems of SODEs, but also provides us with a tool, namely the family of systems (4.1), to determine them.

**Note 4.2.** Note that \( X_p \) and \( X_\ell \) are the properly defined \( t \)-dependent vector fields over \((\mathbb{T} \mathbb{R}^n)^m\) and they maintain the form (4.2) for every coordinate system on \((\mathbb{T} \mathbb{R}^n)^m\) induced by a coordinate system on \( \mathbb{R}^n \).

**Note 4.3.** Denote by \( S_{ij} \) a permutation of variables \( x(i) \leftrightarrow x(j) \), with \( i, j = 1, \ldots, m \). As (4.1) and (4.3) are invariant under such permutations, it can be easily inferred that if \( \Upsilon \) is a superposition rule for (3.8), then \( S_{ij} \Upsilon \) is also, which provides an analogue for systems of SODEs of a known result about standard superposition rules [9].

Apart from the main result of theorem 4.1, a careful analysis of its proof suggests new types of superposition rules for systems of SODEs generalizing previous notions used in the study of first-order differential equations [9, 30]. Indeed, given a particular solution of the systems of PDEs (4.1) and a family of particular solutions \( x_{(1)}(t), \ldots, x_{(m)}(t) \) of system (3.8), we can define

\[
\tilde{x}(t) = u_k \left( x_{(1)}(t), \frac{dx_{(1)}}{dt}(t), \ldots, x_{(m)}(t), \frac{dx_{(m)}}{dt}(t) \right).
\]

The above expression has the same form as that of (4.4). Following the calculations carried out in the ‘if’ part of theorem 4.1, we obtain that the first and the second derivatives of the above curve satisfy relations (4.7) and (4.8). From here, as \( u_k \) is a solution of (4.1), it follows that \( \tilde{x}(t) \) is a new solution of (3.8). In other words, a particular solution of the systems of PDEs (4.1) allows us to generate new solutions of system (3.8) from any set of \( m \) particular solutions for this same system. This fact enables us to define a new type of superposition rule for systems of SODEs as follows.

**Definition 4.4.** A partial superposition rule for a system of SODEs (3.8) is a mapping \( \mathcal{P} : (\mathbb{T} \mathbb{R}^n)^m \times \mathbb{R}^p \to \mathbb{R}^n \), with \( p < 2n \), such that

- for a generic set \( x_{(1)}(t), \ldots, x_{(m)}(t) \) of particular solutions of system (3.8),

\[
\tilde{x}(t) = \mathcal{P} \left( x_{(1)}(t), \frac{dx_{(1)}}{dt}(t), \ldots, x_{(m)}(t), \frac{dx_{(m)}}{dt}(t); k_1, \ldots, k_p \right)
\]

is a new solution of (3.8) and
- for a generic \( p \in (\mathbb{T} \mathbb{R}^n)^m \), the map \( \mathcal{P} \circ \tilde{k} : \tilde{k} \in \mathbb{R}^p \mapsto \mathcal{P}(p; \tilde{k}) \in \mathbb{R}^n \) is an immersion.

Obviously, for every fixed \( \tilde{k} = (k_1, \ldots, k_p) \), the map \( u_{\tilde{k}}(\cdot) = \mathcal{P}(\cdot; \tilde{k}) \) is a solution of the system (4.1). In view of this, it is easy to generalize theorem 4.1 in order to characterize systems of SODEs admitting partial superposition rules. Moreover, the above notion extends to systems of SODEs the notion of partial superposition rule for systems of first-order differential equations defined in [9].

Let us now illustrate how the above results and definitions work. Consider the SODE

\[
\frac{d^2 x}{dt^2} = t^2
\]

(4.11)
and look for a superposition rule depending on a single particular solution. Following the terminology used in theorem 4.1, we have \( m = 1 \) (one particular solution) and \( n = 1 \) (system defined on \( \mathbb{R} \)). Consequently, the corresponding family of systems of PDEs (4.1) reads

\[
\begin{align*}
\frac{\partial^2 u}{\partial x_1^2} + 2v_1\frac{\partial^2 u}{\partial x_1 \partial v_1} + \frac{\partial t^2 u}{\partial x_1 \partial v_1} + \frac{\partial^2 u}{\partial x_1 \partial v_1} + 2t \frac{\partial u}{\partial v_1} &= t^2, \\
\frac{\partial^2 u}{\partial x_1^2} &= 0,
\end{align*}
\]

whose common solutions, which do not depend on \( t \), are the solutions of the system

\[
\begin{align*}
\frac{\partial^2 u}{\partial x_1^2} &= 0, \\
\frac{\partial u}{\partial v_1} &= 0, \\
2v_1\left(\frac{\partial^2 u}{\partial x_1 \partial v_1} + \frac{\partial u}{\partial x_1 \partial v_1}\right) + \left(2\frac{\partial u}{\partial v_1} + \frac{\partial u}{\partial v_1}\right) &= t^2.
\end{align*}
\]

The solutions of the above system are of the form \( u = x_1 + k_1 \), with \( k_1 \) being an arbitrary constant. Obviously, the family of systems (4.12) does not give rise to a two parametric family of solutions and (4.11) does not admit any superposition rule in terms of one particular solution. Nevertheless, it is interesting to point out that the solutions \( u = x_1 + k_1 \) exemplify that, for every particular solution \( x_1(t) \) of (4.12), the new function \( u(x_1(t)) = x_1(t) + k_1 \) is a new solution of the system that gives rise to a partial superposition rule \( \mathcal{P} : (x_1(1), v_1(1); k_1) \in \mathbb{R} \times \mathbb{R} \mapsto (x_1(1) + k_1) \in \mathbb{R} \) for equation (4.11).

Let us now turn to determining all possible superposition rules for (4.11) involving two particular solutions. So, we have \( m = 2 \), \( n = 1 \), and the family (4.1) reads

\[
\sum_{a,b=1}^{2} \left( v_{(a)} v_{(b)} \frac{\partial^2 u}{\partial x_{(a)} \partial x_{(b)}} + 2v_{(a)} t^2 \frac{\partial^2 u}{\partial x_{(a)} \partial v_{(b)}} + t^4 \frac{\partial^2 u}{\partial v_{(a)} \partial v_{(b)}} \right) + \sum_{a=1}^{2} \left( t^2 \frac{\partial u}{\partial x_{(a)}} + 2t \frac{\partial u}{\partial v_{(a)}} \right) = t^2.
\]

Proceeding as before, we obtain that solutions of this \( t \)-parametrized family of PDEs are solutions of the system

\[
\sum_{a=1}^{2} \frac{\partial u}{\partial v_{(a)}} = 0, \\
\sum_{a,b=1}^{2} \frac{\partial^2 u}{\partial x_{(a)} \partial v_{(b)}} = 0, \\
\sum_{a,b=1}^{2} v_{(a)} v_{(b)} \frac{\partial^2 u}{\partial x_{(a)} \partial x_{(b)}} = 0,
\]

\[
\sum_{a=1}^{2} 2v_{(a)} \frac{\partial^2 u}{\partial x_{(a)} \partial v_{(b)}} + \sum_{a=1}^{2} \frac{\partial u}{\partial x_{(a)}} = 1.
\]

Putting the first equation of the above system into the others, we obtain that the above system is equivalent to

\[
\frac{\partial u}{\partial v_{(1)}} = -\frac{\partial u}{\partial v_{(2)}}, \\
\frac{\partial u}{\partial x_{(1)}} = 1 - \frac{\partial u}{\partial x_{(2)}}, \\
(v_{(1)} - v_{(2)})^2 \frac{\partial^2 u}{\partial x_{(1)} \partial x_{(2)}} = 0,
\]

whose solutions take the form \( u = (x_1 - x_2) f_1 + x_1 + f_2 \), with \( f_1 \) and \( f_2 \) being two arbitrary functions depending on \( v_{(1)} - v_{(2)} \). Now, by choosing appropriate one-parametric families of solutions of the above form, we can obtain partial superposition rules. For example, setting \( f_1 = k \) and \( f_2 = 0 \), with \( k \in \mathbb{R} \), we obtain the family of solutions \( u_k = (x_1 - x_2) k + x_1 \), which results in the partial superposition rule \( \mathcal{P} : (x_1(1), x_2(1); k) \mapsto (x_1(1) + x_2(k) + x_1, k) \), which generates new particular solutions out of two known ones and one constant. Moreover, theorem 4.1 shows that the determination of a superposition rule for the system (4.11) amounts to obtaining a two-parametric family of solutions \( u(k_1,k_2) \) of the above form such that condition (ii) of theorem 4.1 holds. This can be done in several ways. For instance, by setting \( f_1 = k_1 \) and \( f_2 = k_2 \), with \( k_1, k_2 \in \mathbb{R} \), we obtain the superposition rule

\[
\mathcal{P}(x_1, v_1, x_2, v_2; k_1, k_2) = u(k_1,k_2)(x_1, x_2) = k_1(x_1 - x_2) + x_1 + k_2.
\]
Definition 5.1. A system of second-order ordinary differential equations (3.8) is a SODE Lie system if the following two conditions hold.

1. The functions $\frac{\partial^2}{\partial x^2} = 0$, $\frac{\partial}{\partial x} = t(x)$, and $\frac{\partial^2}{\partial x^2} = t^2(x)$ are given by $v_{i1}(k_i x_{i1} + k_2)$, $x_{i1} + k_1$, and $k_1 x_{i1} + k_2$, respectively.

Using our methods, we can easily derive the results of Table 1. Special attention must be paid to the first example, illustrating that partial superposition rules may exist when superposition rules depending on the same number of particular solutions do not. In addition, this particular example is not a SODE Lie system (see definition 5.1), which was almost the only tool to study superposition rules for systems of SODEs so far.

Table 1. Superposition and partial superposition rules depending on a particular solution.

| SODE | Superposition rule | Partial superposition rule |
|------|-------------------|---------------------------|
| $\frac{d^2}{dt^2} = 0$ | $v_{i1}(k_i x_{i1} + k_2)$ | $x_{i1} + k_1$ |
| $\frac{d}{dt} = t(x)$ | Nonexistent | $x_{i1} + k_1$ |
| $\frac{d^2}{dt^2} = t^2(x)$ | $k_1 x_{i1} + k_2$ | $k_1 x_{i1}$ |

and if we choose $f_1 = k_1(v_{i1} - v_{21})$ and $f_2 = k_2(v_{i1} - v_{22})$, we arrive at $\mathcal{Y}(x_1, v_1, x_2, v_2; k_1, k_2) = u_{(k_1, k_2)}(x_{i1}, x_{i2}) = k_1(v_{i1} - v_{21})(x_{i1} - x_{i2}) + x_{i1} + k_2(v_{i1} - v_{22})$.

Using our methods, we can easily derive the results of Table 1. Special attention must be paid to the first example, illustrating that partial superposition rules may exist when superposition rules depending on the same number of particular solutions do not. In addition, this particular example is not a SODE Lie system (see definition 5.1), which was almost the only tool to study superposition rules for systems of SODEs so far.

Theorem 4.1 allows us to characterize systems of SODEs admitting a base-superposition rule as follows.

Corollary 4.5. A mapping $\mathcal{Y} : (\mathbb{R}^n)^m \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ is a base-superposition rule for a system of SODEs (3.8) if and only if the following two conditions hold.

1. The functions $u_k : p \in (\mathbb{R}^n)^m \rightarrow (\mathcal{Y}(p; k) \in \mathbb{R}^n$, with $k \in \mathbb{R}^{2n}$, are the solutions of the $t$-parametrized family of systems of PDEs on $(\mathbb{T}^{\mathbb{R}^n})^m$ given by $\{X_1^{(m)}, \frac{\partial}{\partial x^i}\}, X_2^{(m)} = F(t, u, (X_1^{(m)}), u_k), i = 1, \ldots, n$, where $X_1^{(m)}$ and $X_2^{(m)}$ are the diagonal prolongations to $(\mathbb{T}^{\mathbb{R}^n})^m$ of the time-dependent vector fields $X_1 = \sum_{i=1}^n v_i \frac{\partial}{\partial x^i}, X_2 = \sum_{i=1}^n F(t, x, v) \frac{\partial}{\partial x^i}$.

2. The map $\varphi : (\mathbb{T}^{\mathbb{R}^n})^m \times \mathbb{R}^{2n} \rightarrow \mathbb{T}^{\mathbb{R}^n}$ of the form $\varphi(p; k) = (u_k(p), [(X_1^{(m)}), u_k](p)) \in T_{u_k(p)} \mathbb{R}^n$ gives rise to a family of bijections $\varphi_p : k \in \mathbb{R}^{2n} \mapsto \varphi(p; k) \in \mathbb{T}^{\mathbb{R}^n},$ with $p$ being a generic point of $(\mathbb{T}^{\mathbb{R}^n})^m$.

5. Superposition rules and SODE Lie systems

Recently, the theory of Lie systems was employed to obtain a few results about superposition rules for systems of SODEs [11–15]. All these achievements were based on the notion of a SODE Lie system. We now describe this concept and provide several new results about the use of Lie systems to analyse different types of superposition rules for systems of SODEs.

Definition 5.1. A system of second-order ordinary differential equations (3.8) is a SODE Lie system if the first-order system

\[
\begin{align*}
\frac{dx}{dt} &= v_i, \\
\frac{dv_i}{dt} &= F(t, x, v),
\end{align*}
\]

obtained from (3.8) by adding the variables $v_i = dx_i/dt$, $i = 1, \ldots, n$, is a Lie system.
The Lie–Scheffers theorem is an effective tool to determine whether the system (3.8) is a SODE Lie system or not. Nevertheless, this method is based on analysing properties of the time-dependent vector field associated with the corresponding system (5.1) and does not provide any straightforward information about the superposition rules for these systems. In order to overcome this drawback, we provide the following characterization of SODE Lie systems in terms of properties of superposition rules.

Proposition 5.2. A system of SODEs (3.8) is a SODE Lie system if and only if it admits a superposition rule \( \Upsilon : (\mathbb{R}^n)^m \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n \), such that \( X_L^{(m)} \Upsilon = 0 \), where \( X_L \) is given by (4.2).

Proof. Consider a second-order system of the form (3.8) admitting a superposition rule (3.13). The general solution \( x(t) \) of this system can be put in the form (3.14). Differentiating this expression with respect to the time, we obtain

\[
\frac{dx_i}{dt}(t) = \sum_{a=1}^{m} \sum_{j=1}^{n} \frac{dv_{(a)}(t)}{dt} \frac{\partial \Upsilon^i}{\partial x_{(a)}^j}(t, x_{(a)}(t), \frac{dx_{(a)}(t)}{dt}),
\]

where \( i = 1, \ldots, n \) and \( p(t) \) is given by (4.6). Therefore, by defining

\[
\hat{\Upsilon}^i(x_{(1)}(t), v_{(1)}(t), \ldots, x_{(m)}(t), v_{(m)}(t)) = \sum_{a=1}^{m} \sum_{j=1}^{n} \frac{dv_{(a)}(t)}{dt} \frac{\partial \Upsilon^i}{\partial x_{(a)}^j}(t, x_{(a)}(t), v_{(a)}(t)),
\]

expressions (3.14) and (5.2) can be brought into the form

\[
\begin{align*}
\frac{dx_i}{dt}(t) & = \hat{\Upsilon}^i(x_{(1)}(t), v_{(1)}(t), \ldots, x_{(m)}(t)), \\
\frac{dx_j}{dt}(t) & = \hat{\Upsilon}^j(t, x_{(1)}(t), v_{(1)}(t), \ldots, x_{(m)}(t), v_{(m)}(t)),
\end{align*}
\]

Taking into account that the general solution \( (x(t), v(t)) \) of the first-order system (5.1) is obtained by adding the variables \( v^i = \frac{dx^i}{dt} \) to the system (3.8), we see that expressions (5.4) define a map \( \Phi : (t, p, k) \in \mathbb{R} \times (\mathbb{R}^n)^m \rightarrow (\hat{\Upsilon}(p, k), \hat{\Upsilon}(t, p, k)) \in \mathbb{R}^n \), which allows us to write the general solution of this first-order system in terms of a generic set of particular solutions \( x_{(a)}(t), v_{(a)}(t) \), with \( a = 1, \ldots, m \). In view of expression (5.3),

\[
\frac{\partial \hat{\Upsilon}^i}{\partial t}(t, x_{(1)}(t), v_{(1)}(t), \ldots, x_{(m)}(t), v_{(m)}(t)) = \sum_{a=1}^{m} \sum_{j=1}^{n} \frac{\partial F^i}{\partial t}(t, x_{(a)}(t), v_{(a)}(t)) \frac{\partial \Upsilon^j}{\partial v_{(a)}^j} = X_L^{(m)} \Upsilon.\]

Therefore, if \( X_L^{(m)} \Upsilon = 0 \), the mapping \( \hat{\Upsilon} \) and, in consequence, \( \Phi \) are time independent. This shows that the function \( \Phi \) is a superposition rule for the system (5.1), which is therefore a Lie system. Hence, the system (3.8) is a SODE Lie system.

Let us assume now that the system (3.8) is a SODE Lie system, i.e. the first-order system (5.1) is a Lie system and there exists a superposition rule \( \Phi : (p, k) \in \mathbb{R}^m \times \mathbb{R}^{2n} \rightarrow (\hat{\Upsilon}(p, k), \Phi_k(p, k)) \in \mathbb{R}^n \) such that its general solution \( (x(t), v(t)) \) can be written as

\[
\begin{align*}
x(t) & = \hat{\Upsilon}(x_{(1)}(t), v_{(1)}(t), \ldots, x_{(m)}(t), v_{(m)}(t); k_1, \ldots, k_{2n}), \\
v(t) & = \Phi_k(x_{(1)}(t), v_{(1)}(t), \ldots, x_{(m)}(t), v_{(m)}(t); k_1, \ldots, k_{2n}),
\end{align*}
\]

where \( (x_{(a)}(t), v_{(a)}(t)) \), with \( a = 1, \ldots, m \), is a generic family of particular solutions of system (5.1). Since \( \frac{dx_{(a)}(t)}{dt} = v_{(a)}(t) \), the function \( \Upsilon : \mathbb{R}^m \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n \) enables us to write the general solution \( x(t) \) of system (3.8) in the form

\[
x(t) = \hat{\Upsilon}(x_{(1)}(t), \frac{dx_{(1)}(t)}{dt}, \ldots, x_{(m)}(t), \frac{dx_{(m)}(t)}{dt}; k_1, \ldots, k_{2n}).
\]
where $x$ is a set of $2n$-constants. In other words, our system of SODEs admits a superposition rule. Consequently, differentiating the above expression with respect to $t$, we obtain, in virtue of (5.3), that $\Phi_x(p(t); k) = \tilde{\Upsilon}(t, p(t); k)$ for a generic $p(t)$, which is given by (4.6) and constructed from a family of particular solutions $x_1(t), \ldots, x_m(t)$. Hence, $\Phi_x(x_1(t), v_1), \ldots, x_m(t), v_m; k) = \tilde{\Upsilon}(t, x_1(t), v_1), \ldots, x_m(t), v_m; k) = X^{\text{cm}}_L \Upsilon_i = 0, \quad i = 1, \ldots, n$. 

The above proposition improves the results of [10], where it is only stated that SODE Lie systems admit superposition rules. Indeed, our new result also supplies additional information about such superposition rules, namely $X^{\text{cm}}_L \Upsilon = 0$. This property is going to be used next to easily retrieve some previous results found in [10] and various new ones.

**Proposition 5.3.** Every system of SODEs (3.8) admitting a quasi-base-superposition rule is a SODE Lie system.

**Proof.** Assume that the system (3.8) admits a quasi-base-superposition rule. Let us now prove that this quasi-base-superposition rule gives rise to a superposition rule for (3.8) such that $X^{\text{cm}}_L \Upsilon = 0$, which, in view of proposition 5.2, proves that the system (3.8) is a SODE Lie system.

The general solution $x(t)$ of (3.8) can be cast in the form (3.12). As the functions $I_j : \mathbb{R}^m \to \mathbb{R}$, with $j = 1, \ldots, q$, are constant along the $m$-tuples $(x_{(a)}(t), \frac{dx_{(a)}(t)}{dt})$ obtained from $m$ particular solutions $x_{(a)}(t)$ of system (3.8), i.e.

$$I_j \left( x_{(a)}(t), \frac{dx_{(a)}(t)}{dt}, \ldots, x_{(m)}(t), \frac{dx_{(m)}(t)}{dt} \right) = \text{const.},$$

we obtain

$$\frac{dI_j}{dt} = \sum_{i=1}^n \sum_{a=1}^m \left[ \frac{dx_{(a)}^i}{dt} \frac{\partial I_j}{\partial x_{(a)}^i} (p(t)) + F^i \left( t, x_{(a)}(t), \frac{dx_{(a)}(t)}{dt} \right) \frac{\partial I_j}{\partial v_{(a)}(p(t))} \right] = 0,$$

where $p(t)$ is given by expression (4.6). The above holds for every generic family of particular solutions. Then, $X^{\text{cm}}_L I_j = 0$, for $j = 1, \ldots, q$. Substituting expression (5.5) into (3.12), it turns out that there exists a superposition rule $\Upsilon : (\mathbb{R}^n)^m \times \mathbb{R}^m \to \mathbb{R}^n$ for the system (3.8) of the form $\Upsilon(p; k) = G(x_{1}(t), \ldots, x_m(t), I_1(p), \ldots, I_q(p); k)$, where $p = (x_1(t), v_1(t), \ldots, x_m(t), v_m(t))$. Indeed, in view of the definition of $\Upsilon$ and the properties of the quasi-base-superposition rule $G$,

$$x(t) = G(x_{1}(t), \ldots, x_m(t), I_1(p(t)), \ldots, I_q(p(t)); k) = \Upsilon(p(t); k),$$

where $x_1(t), \ldots, x_m(t)$ is any generic family of particular solutions of system (3.8). Then,

$$X^{\text{cm}}_L \Upsilon = \sum_{j=1}^n \sum_{a=1}^m \frac{\partial G}{\partial I_j} \left( t, x_{(a)}, \frac{dx_{(a)}}{dt} \right) \frac{\partial I_j}{\partial v_{(a)}} = \sum_{j=1}^n \frac{\partial G}{\partial I_j} (X^{\text{cm}}_L I_j) \frac{\partial G}{\partial I_j} = 0.$$

**Corollary 5.4.** Every system of SODEs admitting a base-superposition rule is a SODE Lie system.

The implication of the above corollary cannot be reversed, i.e. not every SODE Lie system admits a base-superposition rule. Indeed, the following results can be easily used to prove the existence of SODE Lie systems admitting no base-superposition rule.
Lemma 5.5. Given a system of SODEs (3.8) admitting a base-superposition rule, the systems
\[ \frac{d^2 x_i}{dt^2} = \frac{d}{dr} \left( \frac{d^2 t}{dr^2} dx_i + \left( \frac{dT}{dr} \right)^2 F^i \left( t(\tau), x, \frac{dt}{dr} \right) \right), \quad i = 1, \ldots, n, \] (5.6)
with \( t = t(\tau) \) being any time-reparametrization, are SODE Lie systems admitting a common base-superposition rule.

Proof. A time reparametrization \( t = t(\tau) \) maps the system (3.8) to a system of SODEs of the form (5.6) with the general solution \( x(\tau) = x(t(\tau)) \). If \( \Upsilon : \mathbb{R}^{mn} \times \mathbb{R}^{2n} \to \mathbb{R}^n \) is a base-superposition rule for (3.8), then \( x(\tau) = \Upsilon(x_m(\tau), \ldots, x_m(\tau); k_1, \ldots, k_2n) \), where \( x_m(\tau) \) is any generic family of particular solutions of (5.6). Consequently, all the SODEs of the family (5.6) admit a common base-superposition rule and, according to corollary 5.4, all systems (5.6) are SODE Lie systems.

Although only few SODE Lie systems admit them, base-superposition rules are the main superposition rules treated in the literature. The next proposition shows that SODE Lie systems must satisfy various restrictive conditions to admit a base-superposition rule.

Proposition 5.6. Given a system of SODEs (3.8) admitting a base-superposition rule, the associated first-order system (5.1) is a Lie system related to a Vessiot–Guldberg Lie algebra containing the Liouville vector field \( \Delta_1 \) of the tangent bundle \( T\mathbb{R}^n \) and the vector fields
\[ X^\lambda_p(x, v) = \sum_{i=1}^n \frac{d^p}{d\lambda^p} (\lambda^2 F^i(t_1, x, \lambda^{-1}v)) \frac{\partial}{\partial v^i}, \quad p = 1, 2, \ldots, \] (5.7)
where \( t_1 \in \mathbb{R} \) and \( \lambda \in \mathbb{R}^* \equiv \mathbb{R} - \{0\} \).

Proof. In view of lemma 5.5, time reparametrizations \( \tau = \tau(t) \), with inverses \( t = t(\tau) \), transform the system (3.8) into the family of SODE Lie systems (5.6), whose associated first-order systems
\[
\begin{align*}
\frac{dx^i}{d\tau} &= v^i, \\
\frac{dv^i}{d\tau} &= \frac{d}{dr} \left( \frac{d^2 t}{dr^2} v^i + \left( \frac{dT}{dr} \right)^2 F^i \left( t(\tau), x, \frac{dt}{dr} \right) \right),
\end{align*}
\]
with \( i = 1, \ldots, n \), admit a common base-superposition rule. According to proposition 2.9, this implies that there exists a finite-dimensional Lie algebra of vector fields \( V \) containing all the vector fields
\[ X^\tau_p(x, v) = \sum_{i=1}^n \left( v^i \frac{\partial}{\partial x^i} + F^i(\tau, x, v) \frac{\partial}{\partial v^i} \right) \left( \frac{d}{d\tau} + \left( \frac{dT}{d\tau} \right)^2 F^i \left( t(\tau), x, \frac{dt}{d\tau} \right) \right) \frac{\partial}{\partial v^i}, \]
with \( t = t(\tau) \) being any time reparametrization and \( \tau \in \mathbb{R} \). In particular, if we take \( t(\tau) = \tau \), we obtain that
\[ X_\tau = \sum_{i=1}^n \left( v^i \frac{\partial}{\partial x^i} + F^i(\tau, x, v) \frac{\partial}{\partial v^i} \right) \in V, \quad \forall \tau \in \mathbb{R}. \]
Now, if we take a time reparametrization \( t = \tilde{t}(\tau) \), such that
\[ \tilde{t}(\tau_1) = t_1, \quad \frac{d\tilde{t}}{dr}(\tau_1) = 1, \quad \frac{d^2\tilde{t}}{dr^2}(\tau_1) = 1, \]
...
it follows that $X_{t_1}^{(r)}(x, v) = X_t^r = X_t^r(0)$. Consequently, $X_{t_1}^{(r)} - X_{t_1} = 0$. Taking into account that
\[
X_{t_1}^{(r)}(x, v) = \sum_{i=1}^n \left( v^i \frac{\partial}{\partial x^i} + (v^j + F^i(t_1, x, v)) \frac{\partial}{\partial v^i} \right),
\]
we have $X_{t_1}^{(r)} - X_{t_1} = \sum_{i=1}^n v^i \frac{\partial}{\partial v^i} = \Delta_L \in V$.

On the other hand, consider a family of parameterizations $t = \tau(t_1, \lambda)$, with $\lambda \in \mathbb{R}^n$ and $t_1 \in \mathbb{R}$, satisfying the conditions
\[
t_\tau(t_1, \lambda) = t_1, \quad \frac{dT_{t_1}}{dt}(\lambda) = \lambda, \quad \frac{d^2T_{t_1}}{dt^2}(\lambda) = 0.
\]
Consequently, the family of vector fields
\[
X_{t_1}^\lambda(x, v) \equiv X_{t_1}^{\tau(t_1, \lambda)}(x, v) = \sum_{i=1}^n \left( v^i \frac{\partial}{\partial x^i} + \lambda^2 F^i(t_1, x, \lambda^{-1} v) \frac{\partial}{\partial v^i} \right), \quad \lambda \in \mathbb{R}^n,
\]
is included in $V$. Note that, for every $t_1$, the above family of vector fields can be considered as a curve in $V$. As $V$ is a vector space, all the derivatives of such a curve, i.e. the vector fields
\[
\frac{d^n}{dt^n}[X_{t_1}^\lambda(x, v)] = \sum_{i=1}^n \frac{d^n}{dt^n} \left( \lambda^2 F^i(t_1, x, \lambda^{-1} v) \right) \frac{\partial}{\partial v^i},
\]
are included in $V$.

\section{Superposition rules and systems of HODEs}

In order to introduce a general theory of superposition rules for systems of HODEs, let us recall some basic concepts of the theory of higher order tangent bundles [31].

Given two curves $\rho, \sigma : \mathbb{R} \to \mathbb{R}^n$, such that $\rho(0) = \sigma(0) = x_0 \in \mathbb{R}^n$, we say that they have a contact of order $s$ at $x_0$, with $s \in \mathbb{N}$, if they satisfy
\[
\frac{d^j(f \circ \rho)}{dt^j}(0) = \frac{d^j(f \circ \sigma)}{dt^j}(0), \quad j = 1, \ldots, s,
\]
for every function $f \in C^\infty(\mathbb{R}^n)$. The relation ‘to have a contact of order $s$ at $x_0$’ is an equivalence relation. Each equivalence class, say $t^s_{x_0}$, is called an $s$-tangent vector at $x_0$. Now, we define $T^s_{x_0} \mathbb{R}^n$ as the set of all $s$-tangent vectors at $x_0$ and we put
\[
T^s \mathbb{R}^n = \bigcup_{x_0 \in \mathbb{R}^n} T^s_{x_0} \mathbb{R}^n.
\]

It can be proved that $(T^s \mathbb{R}^n, \pi, \mathbb{R}^n)$, with $\pi : t^s_{x_0} \in T^s \mathbb{R}^n \mapsto x_0 \in \mathbb{R}^n$, can be endowed with a differential structure of fibre bundle. Let us briefly analyse this fact.

Every global coordinate system $\{x^1, \ldots, x^n\}$ on $\mathbb{R}^n$ induces a natural coordinate system on the space $T^s \mathbb{R}^n$. Indeed, consider again a curve $\rho$. The $s$-tangent vector $t^s_{x_0}$ associated with this curve admits a representative
\[
\rho^i(0) + \frac{t^i}{1!} \frac{d\rho^i}{dt}(0) + \cdots + \frac{t^i}{s!} \frac{d^s\rho^i}{dt^s}(0), \quad i = 1, \ldots, n,
\]
which can be characterized by its coefficients
\[
x^0_0 = \rho^0(0), \quad y^{(1)i}_0 = \frac{1}{1!} \frac{d\rho^i}{dt}(0), \ldots, y^{(s)i}_0 = \frac{1}{s!} \frac{d^s\rho^i}{dt^s}(0), \quad i = 1, \ldots, n.
\]

In consequence, the mapping $\varphi : t^s_{x_0} \in T^s \mathbb{R}^n \mapsto (x^0_0, y^{(1)i}_0, \ldots, y^{(s)i}_0) \in \mathbb{R}^{(s+1)n}$ gives a canonical global coordinate for $T^s \mathbb{R}^n$. Obviously, the map $\pi$ becomes a smooth submersion.
Definition 6.1. We say that a system of $s$-order ordinary differential equations on $\mathbb{R}^n$ if the first-order system

\begin{equation}
\frac{d}{dt}x(t) = \mathbf{F}(t, x(t), \frac{dx}{dt}, \ldots, \frac{d^{s-1}x}{dt^{s-1}}),
\end{equation}

admits a superposition rule if there exists a map $\Upsilon : (\mathbb{T}^{s-1}(\mathbb{R}^n))^m \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ of the form

\begin{equation}
x(t) = \Upsilon \left( t^{-1}x_1(t), \ldots, t^{-1}x_m(t); k_1, \ldots, k_m \right),
\end{equation}

such that the general solution $x(t)$ of system (6.1) can be written as

\begin{equation}
x(t) = \Upsilon \left( t^{-1}x_1(t), \ldots, t^{-1}x_m(t); k_1, \ldots, k_m \right),
\end{equation}

for any generic family of particular solutions $x_1(t), \ldots, x_m(t)$ of (6.1) and $k_1, \ldots, k_m$ being a set of constants related to the initial conditions of each particular solution.

Note 6.2. Observe that, according to the above definitions, we have $\mathbb{T}^1\mathbb{R}^n \simeq T\mathbb{R}^n$ and $t^1x_{i\alpha}(t) = \{x_{i\alpha}(t), dx_{i\alpha}(t)/dt\}$. Hence, definition 3.3 describing superposition rules for systems of SODEs turns out to be a particular case of the above definition. Moreover, if we put $T^0\mathbb{R}^n = \mathbb{R}^n$, definition 6.1 reduces to the standard superposition rule notion.

Definition 6.3. We say that a system of ordinary differential equations (6.1) is a HODE Lie system if the first-order system

\begin{equation}
\begin{cases}
\frac{dx}{dt} = y^{(1)i}, \\
\frac{dy^{(1)i}}{dt} = y^{(2)i}, \\
\vdots \\
\frac{dy^{(s-1)i}}{dt} = F^i(t, x, y^{(1)}, \ldots, y^{(s-1)}),
\end{cases}
\end{equation}

obtained from (6.1) by adding the new variables $y^{(j)i} = \frac{d^jx}{dt^j}$, with $i = 1, \ldots, n$ and $j = 1, \ldots, s-1$, is a Lie system.

Observe that the results and definitions described in previous sections can be directly generalized to systems of HODEs. This is why, instead of detailing such generalizations, we shall merely describe a simple but relevant result ensuring the existence of superposition rules for HODE Lie systems. In next sections, this result is used to determine a superposition rule for second- and third-order Kummer–Schwarz equations.

Proposition 6.4. Every HODE Lie system admits a superposition rule.

Proof. Note that every solution of system (6.3) is of the form $t^{-1}x_p(t)$ for a particular solution $x_p(t)$ of system (6.1) and vice versa. Consequently, the superposition rule
\[ \Phi : (T^{e^{-1}} \mathbb{R}^n)^m \times \mathbb{R}^m \to T^{e^{-1}} \mathbb{R}^n \] for (6.3) allows us to write the general solution \( t^{e^{-1}} x(t) \) of (6.3) in the form
\[
t^{e^{-1}} x(t) = \Phi(t^{e^{-1}} x_{(1)}(t), \ldots, t^{e^{-1}} x_{(m)}(t); k_1, \ldots, k_m).
\]
in terms of generic families of particular solutions \( x_{(1)}(t), \ldots, x_{(m)}(t) \) of (6.1), their derivatives up to \( s-1 \) order and the constants \( k_1, \ldots, k_m \). Applying the projection \( \pi : T^{e^{-1}} \mathbb{R}^n \to \mathbb{R}^n \) to both sides of the above relation, it follows that the general solution \( x(t) \) of system (6.1) can be written as
\[
x(t) = (\pi \circ \Phi)(t^{e^{-1}} x_{(1)}(t), \ldots, t^{e^{-1}} x_{(m)}(t); k_1, \ldots, k_m).
\]
In other words, \( \Upsilon = \pi \circ \Phi : (T^{e^{-1}} \mathbb{R}^n)^m \times \mathbb{R}^m \to \mathbb{R}^n \) is a superposition rule for (6.1).

7. Examples of superposition rules for systems of SODEs

Let us illustrate now the results described in the previous sections by means of various examples extracted from the physics and mathematics literature. As the first simple instance, consider the \( n \)-dimensional isotropic harmonic oscillator
\[
\frac{d^2 x_i}{dt^2} = -\omega^2(t) x_i, \quad i = 1, \ldots, n,
\]
with a time-dependent frequency \( \omega(t) \). This system appears widely in the physics literature. For instance, it occurs in the study of the fluctuations of the tachyon field obtained by using effective Lagrangians [32, 33], in the description of the movement of a particle on a heated spring [34] and in the analysis of the properties of diverse interesting nonlinear differential equations, like Milne–Pinney equations, with many applications in physics [14, 16, 35].

As shown in section 3, systems (7.1) admit a base-superposition rule. Consequently, proposition 5.4 ensures that the systems of the form (7.1) must be SODE Lie systems. Actually, this can be proved easily. In view of definition 5.1, demonstrating that each system of the form (7.1) is a SODE Lie system reduces to proving that every first-order system
\[
\begin{align*}
\frac{dx_i}{dt} &= v_i, \\
\frac{dv_i}{dt} &= -\omega^2(t) x_i,
\end{align*}
\]
is a Lie system. Any system of the above type describes integral curves of the time-dependent vector field \( X = X_1 + \omega^2(t) X_2 \), with
\[
X_1 = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \quad X_2 = \frac{1}{2} \sum_{i=1}^n \left( x_i' \frac{\partial}{\partial x_i} - v_i' \frac{\partial}{\partial v_i} \right), \quad X_3 = -\sum_{i=1}^n x_i' \frac{\partial}{\partial v_i},
\]
spanning a Lie algebra \( V_{\text{HIO}} \) of vector fields isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \) [15]. It turns out that every system (7.2) is a Lie system, and therefore, the isotropic harmonic oscillators with time-dependent frequency (7.1) are SODE Lie systems, as proposition 7.10 states.

As equations (7.1) admit the same base-superposition rule, proposition 5.6 ensures the existence of a finite-dimensional Lie algebra including \( V_{\text{HIO}} \) and \( \Delta_L = \sum_{i=1}^n v_i' \partial / \partial v_i \). Indeed, it is a straightforward computation to check that \( \Delta_L, X_1, X_2 \) and \( X_3 \) generate a Lie algebra of vector fields isomorphic to \( \mathfrak{gl}(2, \mathbb{R}) \).

Let us now turn to exemplifying that superposition rules for systems of SODEs need not be invariant under time reparametrizations, as pointed out in section 3. Recall that Milne–Pinney equations (3.10) admit a quasi-base-superposition rule depending on a constant of motion \( L_1 \).
A time reparametrization $\tau = \int_0^t e^{\xi(t')} dt'$ transforms (3.10) into the dissipative Milne–Pinney equation [36–39]

$$\frac{d^2x}{dt^2} = -\frac{dF}{dt}(t(\tau)) e^{-F(t(\tau))} \frac{dx}{d\tau} - \omega^2(t(\tau)) e^{-2F(t(\tau))} x + \frac{ce^{-2F(t(\tau))}}{x^3}, \tag{7.3}$$

and the superposition rule (3.11) for Milne–Pinney equations yields that the general solution of the above equation can be cast in the form (3.11) again, but in terms of a new constant $I_3$ reading

$$I_3 = e^{2F(t)} \left[ \frac{dx_1(t)}{d\tau} x_2(t) - \frac{dx_2(t)}{d\tau} (t) x_1(t) \right]^2 + c \left[ \left( \frac{x_1(t)}{x_2(t)} \right)^2 + \left( \frac{x_2(t)}{x_1(t)} \right)^2 \right].$$

This proves that the (quasi-base) superposition rule for Milne–Pinney equations is not invariant under time reparametrizations. Moreover, as SODEs (7.3) admit, generically, a time-dependent superposition rule (see [10] for details), they need not to be SODE Lie systems. Indeed, it was proved in [14] that systems (7.3) are not SODE Lie systems. Then, from lemma 5.5, Milne–Pinney equations cannot admit a base-superposition rule.

Let us now derive a new superposition rule for a relevant type of (nonautonomous) SODE: the second-order Kummer–Schwarz equation

$$\frac{d^2x}{dt^2} = \frac{3}{2x} \left( \frac{dx}{dt} \right)^2 - 2b_0 x^3 + 2a_0(t)x, \tag{7.4}$$

where $b_0$ is a constant and $a_0(t)$ is an arbitrary time-dependent function. The study of these, hereafter KS-2 equations, is motivated by their appearance in the theory of superposition rules [17] and in the analysis of SODEs, where they appear related to the so-called Kummer problem [40]. These equations also appear associated with the so-called second-order Gambier equation [22] and can be used to describe certain cosmological problems [41]. Moreover, the solution of several cases of KS-2 equations amounts to solving certain Milne–Pinney and Riccati equations [21, 22, 42]. As these equations are ubiquitous in the physics literature, e.g., they appear in cosmology, quantum mechanics and classical mechanics [41, 43], the study of KS-2 equations can be considered as a useful approach to the analysis of these equations and their respective related physical problems.

In order to describe a superposition rule for KS-2 equations, we shall first prove that these equations are SODE Lie systems, which ensures, by proposition 5.2, that they admit a superposition rule and indicates how to derive it.

Recall that demonstrating that KS-2 equations are SODE Lie systems relies on proving that the first-order system

$$\begin{align*}
\frac{dx}{dt} &= v, \\
\frac{dv}{dt} &= \frac{3}{2} \frac{v^2}{x} - 2b_0 x^3 + 2a_0(t)x,
\end{align*} \tag{7.5}$$

is a Lie system. To do this, consider the vector fields

$$X_1 = x \frac{\partial}{\partial v}, \quad X_2 = x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}, \quad X_3 = v \frac{\partial}{\partial x} + \left( \frac{3}{2} \frac{v^2}{x} - 2b_0 x^3 \right) \frac{\partial}{\partial v}. \tag{7.6}$$

Since

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_1,$$

they span a Lie algebra of vector fields isomorphic to $\mathfrak{s}(2, \mathbb{R})$, and as the system (7.5) is determined by the time-dependent vector field

$$X_i = v \frac{\partial}{\partial x} + \left( \frac{3}{2} \frac{v^2}{x} - 2b_0 x^3 + 2a_0(t)x \right) \frac{\partial}{\partial v} = X_3 + a_0(t)X_1,$$

the second-order Kummer–Schwarz equations are SODE Lie systems.
It is interesting that, like time-dependent frequency harmonic oscillators, Kummer–Schwarz equations are SODE Lie systems related to a Vessiot–Guldberg Lie algebra isomorphic to sl(2, R). This can be used to establish interesting relations between these equations and other (SODE) Lie systems associated with the same Lie algebra (cf [15]).

Once it has been proved that KS-2 equations are SODE Lie systems, the following step towards deriving their superposition rule is, in view of proposition 5.2, to determine the part of the standard superposition rule for the system (7.5) describing the $x$ coordinate of its general solution. To do this, let us apply the method described in section 2.

The vector fields $X_1, X_2, X_3$ form a basis for a Vessiot–Guldberg Lie algebra of (7.5) and their prolongations to $(\mathbb{R}^3)^2$ are linearly independent at a generic point. Let $X_1, X_2$ and $X_3$ be diagonal prolongations to $(\mathbb{R}^3)^3$. As $[X_1, X_2] = [X_1, X_3] = 2X_2$, if a function $F : (\mathbb{R}^3)^3 \to \mathbb{R}$ satisfies $\tilde{X}_1 F = \tilde{X}_3 F = 0$, then $\tilde{X}_2 F = 0$. Thus, obtaining a common first integral for $\tilde{X}_1, \tilde{X}_2$ and $\tilde{X}_3$ reduces to finding a common first integral for $\tilde{X}_1$ and $\tilde{X}_3$.

Consider the canonical coordinates $\{x_0, v_0, x_1, v_1, x_2, v_2\}$ in $(\mathbb{R}^3)^3$ and suppose that the common first integral $F$ for $\tilde{X}_1$ and $\tilde{X}_3$ depends only on the variables $x_0, x_1, v_0$ and $v_1$. As $\tilde{X}_3 F = 0$, the method of characteristics yields that $F$ must be constant along the solutions of the characteristic system

$$\frac{dv_0}{x_0} = \frac{dv_1}{x_1}, \quad dx_0 = dx_1 = 0. \quad (7.7)$$

Integrating the above system, we find that there exists a certain function $F_2 : \mathbb{R}^3 \to \mathbb{R}$, such that $F (x_0, x_1, v_0, v_1) = F_2 (x_0, x_1, \xi = x_1 v_0 - x_0 v_1)$. In terms of the variables $x_0, x_1, \xi, v_1$, the condition $\tilde{X}_1 F = \tilde{X}_3 F = 0$ reads

$$\left(\frac{\xi + v_1 x_0}{x_1} \frac{\partial F_2}{\partial x_0} + v_1 \frac{\partial F_2}{\partial x_1} + \frac{3}{2} \left(\frac{\xi^2 + 2\xi v_1 x_0}{x_1 x_0}\right) + 2b_0 (x_1^2 x_0 - x_0^2 x_1)\right) \frac{\partial F_2}{\partial \xi} = 0.$$

As $F_2$ does not depend on $v_1$, the above equation implies

$$\frac{\xi}{x_1} \frac{\partial F_2}{\partial x_0} + \frac{3 \xi^2}{2x_1 x_0} + 2b_0 (x_1^2 x_0 - x_0^2 x_1) \frac{\partial F_2}{\partial \xi} = 0, \quad \frac{x_0}{x_1} \frac{\partial F_2}{\partial x_0} + \frac{\partial F_2}{\partial x_1} + \frac{3 \xi}{x_1} \frac{\partial F_2}{\partial \xi} = 0.$$

Applying again the method of characteristics to the second equation, we see that there exists a function $F_3 : \mathbb{R}^2 \to \mathbb{R}$, such that $F_2 (x_0, x_1, \xi) = F_3 (K_1 = x_0 / x_1, K_2 = x_1^2 / \xi)$. Let us express the first of the above equations using the variables $K_1, K_2$ and $\xi$. As a result, it turns out that

$$\tilde{X}_1 F_3 = \frac{\xi}{x_1} \left(\frac{\partial F_3}{\partial K_1} - \left[\frac{3}{2K_1} + 2b_0 K_2^2 (K_1 - K_1^2)\right] \frac{\partial F_3}{\partial K_2}\right) = 0.$$

The characteristic system for the preceding equation is

$$dK_1 = -\frac{dK_2}{K_2^2 \left[2b_0 K_2^2 (K_1^2 - K_1)\right]}, \quad (7.8)$$

whose solution is

$$\Gamma_1 = \frac{(v_0 x_1 - v_1 x_0)^2}{x_0^2 x_1^2} + 4b_0 \frac{x_0^2 + x_1^2}{x_0 x_1}.$$

Hence, $\Gamma_1$ is a first integral common to $\tilde{X}_1$ and $\tilde{X}_3$. Similarly, if we suppose that $F$ depends only on $x_0, x_2, v_0$ and $v_2$, or, alternatively, on $x_1, x_2, v_1$ and $v_2$, two new first integrals appear:

$$\Gamma_2 = \frac{(v_0 x_2 - v_2 x_0)^2}{x_0^2 x_2^2} + 4b_0 \frac{x_0^2 + x_2^2}{x_0 x_2}, \quad \Gamma_3 = \frac{(v_1 x_2 - v_2 x_1)^2}{x_1^2 x_2^2} + 4b_0 \frac{x_1^2 + x_2^2}{x_1 x_2}. \quad (7.9)$$

Since $\frac{\partial (\Gamma_1, \Gamma_2)}{\partial (x_0, v_0)} \neq 0$ at a generic point of $(\mathbb{R}^3)^3$, the procedure described in section 2 allows us to determine the values of $x_0$ and $v_0$ in terms of $x_1, x_2, v_1, v_2$, and two
constants giving rise to a superposition rule for the system (7.5). Indeed, fixing \( \Gamma_1 = k_1 \)
enables us to determine the value of \( v_0 \) in terms of \( x_0, x_1, v_1 \) and \( k_1 \). Substituting this value
into the equation \( k_2 = \Gamma_2 \) and with the aid of \( \Gamma_3 \), we can express \( x_0 \) in terms of \( x_1, x_2, k_1, k_2 \)
and \( \Gamma_3 \) as
\[
x_0 = \frac{(\Gamma_3 k_1 - 8b_0 k_2) x_1 + (\Gamma_3 k_2 - 8b_0 k_1) x_2 \pm 2\lambda_{k_1,k_2}(\Gamma_3) [I x_1 x_2 - 4b_0 (x_1^2 + x_2^2)]^{1/2}}{16b_0 \Gamma_3 + x_1^{-1} x_2^{-1} [(k_1 x_1 - k_2 x_2)^2 - 64b_0^2 (x_1^2 + x_2^2)]},
\]
(7.10)
where
\[
\lambda_{k_1,k_2}(\Gamma_3) = \left[ \frac{256b_0^3 + k_1 k_2 \Gamma_3 - 4b_0 (k_1^2 + k_2^2 + \Gamma_3)}{2} \right]^{1/2}.
\]
Expressions (7.9) ensure that \( [I x_1 x_2 - 4b_0 (x_1^2 + x_2^2)]^{1/2} \) is real. Meanwhile, for each pair of
solutions \( x_1(t), x_2(t), k_1 \) and \( k_2 \) must be chosen so that \( \lambda_{k_1,k_2}(\Gamma_3) \) is real.

Since \( \Gamma_1 \) depends on the variables \( x_1, x_2, v_1 \) and \( v_2 \), it is clear that expression (7.10)
constitutes a part of a superposition rule for any system of the form (7.5), describing the component \( x \) of its general solution in terms of two particular solutions \( x_1(t) \) and \( x_2(t) \) of
(7.4), their derivatives \( v_1(t) \) and \( v_2(t) \), and two constants \( k_1 \) and \( k_2 \). Therefore, in view of
lemma 5.4, this allows us to write the general solution \( x(t) \) of equation (7.4) in terms of two particular
solutions, their derivatives and two constants. This provides us with a superposition rule \( \Upsilon : (t_1^i, t_2^j; k_1, k_2) \in (T \mathbb{R})^2 \times \mathbb{R}^2 \mapsto x(t_1^i, t_2^j; k_1, k_2) \in \mathbb{R} \) for KS-2 equations of the form
\[
x = \frac{(Ik_1 - 8b_0 k_2) x_1 + (Ik_2 - 8b_0 k_1) x_2 \pm 2\lambda_{k_1,k_2}(t) [I x_1 x_2 - 4b_0 (x_1^2 + x_2^2)]^{1/2}}{16b_0 I + x_1^{-1} x_2^{-1} [(k_1 x_1 - k_2 x_2)^2 - 64b_0^2 (x_1^2 + x_2^2)]},
\]
where \( I = \Gamma_3 \) is regarded as a function of the variables of \( (T \mathbb{R})^2 \). Note in addition that the above
expression can also be naturally considered as a quasi-base-superposition rule of the form \( G(x_1, x_2, I; k_1, k_2) \) for KS-2 equations.

8. A superposition rule for third-order Kummer–Schwarz equations

This section is devoted to the study of third-order Kummer–Schwarz equations [17, 44, 45] of
the form
\[
\frac{d^3 x}{dt^3} = \frac{3}{2} \left( \frac{dx}{dt} \right)^{-1} \left( \frac{d^2 x}{dt^2} \right)^2 - 2b_0 \left( \frac{dx}{dt} \right)^3 + 2a_0(t) \frac{dx}{dt},
\]
(8.1)
with \( b_0 \) being a constant and \( a_0(t) \) being any time-dependent function. Our aim is to exemplify
how the results of section 6 can be applied to investigate a relevant third-order differential
equation. As a result, it is shown that third-order Kummer–Schwarz equations, hereby
KS-3, are HODE Lie systems, and an interesting relation to KS-2, Riccati and Milne–Pinney
equations is pointed out. Finally, a new superposition rule for KS-3 equations depending on a
single particular solution and three constants is derived.

The relevance of the study of KS-3 equations relies, for instance, on their relation to the
so-called Kummer problem [17, 44, 45], Milne–Pinney equations [16] and Riccati equations
[16, 21, 46]. These relations can be used to study multiple physical systems described by
the latter equations through KS-3 equations, e.g., the case of quantum non-equilibrium dynamics
of many-body systems [47]. Furthermore, Kummer–Schwarz equations with \( b_0 = 0 \) can be
rewritten as \( x(t) = 2a_0(t) \), where \( x(t) \) is the so-called Schwarzian derivative of the function
\( x(t) \) with respect to \( t \) [48].
In order to study KS-3 equations, let us define \( y^{(1)} = \frac{dx}{dt}, y^{(2)} = \frac{d^2x}{dt^2} \), and write equation (8.1) in the form
\[
\begin{align*}
\frac{dx}{dr} &= y^{(1)}, \\
\frac{dy^{(1)}}{dr} &= y^{(2)}, \\
\frac{dy^{(2)}}{dr} &= \frac{3}{2} y^{(2)} + \frac{3}{2} y^{(4/3)} - 2 b_0 y^{(4/3)} + 2 a_0(t) y^{(1)}.
\end{align*}
\]

(8.2)

Consider the vector fields \( X_1, X_2 \) and \( X_3 \) on \( T^2 \mathbb{R} \),
\[
X_1 = 2 y^{(1)} \frac{\partial}{\partial y^{(2)}}, \quad X_2 = y^{(1)} \frac{\partial}{\partial y^{(2)}} + 2 y^{(2)} \frac{\partial}{\partial y^{(4/3)}}, \quad X_3 = y^{(1)} \frac{\partial}{\partial x} + y^{(2)} \frac{\partial}{\partial y^{(1)}} + \left( \frac{3}{2} y^{(4/3)} - 2 b_0 y^{(4/3)} \right) \frac{\partial}{\partial y^{(4/3)}},
\]
satisfying the commutation relations
\[
[X_1, X_3] = 2 X_2, \quad [X_2, X_3] = X_2, \quad [X_1, X_2] = X_1.
\]

Obviously, these vector fields span a Lie algebra of vector fields isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \).
Additionally, the system (8.2) describes integral curves of the time-dependent vector field
\( X_t = X_3 + a_0(t) X_1 \). Thus, KS-3 equations are HODE Lie systems. Let us derive a superposition rule for them.

The vector fields \( X_1, X_2 \) and \( X_3 \) are linearly independent at a generic point of \( T^2 \mathbb{R} \).
Therefore, the diagonal prolongations \( \tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \) of \( X_1, X_2, X_3 \) to \( (T^2 \mathbb{R})^2 \) span a generalized distribution \( D \). Such a generalized distribution is three dimensional in a neighbourhood of a generic point, where the distribution becomes regular. Hence, the vector fields of the distribution admit, at least locally, three common first integrals. As \( [X_1, X_1] = 2 X_2 \), we have \( [\tilde{X}_1, \tilde{X}_3] = 2 \tilde{X}_2 \), and obtaining first integrals common for all the vector fields of \( D \) reduces to determining first integrals common for \( \tilde{X}_1 \) and \( \tilde{X}_3 \).

Let us first analyse first integrals of the vector field \( \tilde{X}_1 \) on \( (T^2 \mathbb{R})^2 \), i.e. solutions \( F : (T^2 \mathbb{R})^2 \to \mathbb{R} \) of the equation
\[
\tilde{X}_1 F = 2 y_0^{(1)} \frac{\partial F}{\partial y_0^{(2)}} + 2 y_1^{(1)} \frac{\partial F}{\partial y_1^{(2)}} = 0.
\]
The method of characteristics shows that the first integrals of the above vector field are the functions constant along the solutions of the so-called characteristic system of \( \tilde{X}_1 \):
\[
\frac{dy_0^{(2)}}{y_1^{(1)}} = \frac{dy_1^{(2)}}{y_1^{(1)}}, \quad dy_0^{(1)} = dy_1^{(1)} = dx_0 = dx_1 = 0.
\]

Such solutions are given, in an implicit form, by the algebraic equations \( \xi = y_0^{(1)} y_1^{(2)} - y_1^{(1)} y_0^{(2)}, \quad v_0 = y_0^{(1)}, \quad v_1 = y_1^{(1)}, \quad y_0 = x_0, \quad y_1 = x_1 \), where \( \xi, v_0, v_1, y_0 \) and \( y_1 \) are certain real constants. In other words, any first integral \( F : (T^2 \mathbb{R})^2 \to \mathbb{R} \) of the vector field \( \tilde{X}_1 \) depends only on the previous variables. Hence, there exists a function \( F_2 : \mathbb{R}^5 \to \mathbb{R} \) such that \( F(x_0, x_1, y_0^{(1)}, y_1^{(1)}, y_0^{(2)}, y_1^{(2)}) = F_2(y_0, y_1, v_0, v_1, \xi) \).

Remember that we are interested in determining a common first integral for the vector fields \( \tilde{X}_1 \) and \( \tilde{X}_3 \). In view of the above result, \( \tilde{X}_3 F_2 = 0 \) amounts to
\[
\sum_{a=0,1} v_a \frac{\partial F_2}{\partial y_a} + \left( v_0 a_1 - \xi \right) \frac{\partial F_2}{\partial v_0} + a_1 \frac{\partial F_2}{\partial v_1} + \left( 3 \xi a_1 \frac{3}{2} v_1 - \frac{3}{2} v_1^3 - b_0 (v_0^3 - v_1^3) v_0 \right) \frac{\partial F_2}{\partial \xi} = 0,
\]
where we defined \(a_1 \equiv y_1^{(2)}\). In terms of the vector fields

\[
Z_1 = v_0 \frac{\partial}{\partial y_0} + v_1 \frac{\partial}{\partial y_1} - \xi \frac{\partial}{\partial v_0} - \left( -\frac{3\xi^2}{2v_1 v_0} + 2b_0 (v_0^3 v_1 - v_1^3 v_0) \right) \frac{\partial}{\partial \xi},
\]

\[
Z_2 = v_0 \frac{\partial}{\partial v_0} + v_1 \frac{\partial}{\partial v_1} + 3\xi \frac{\partial}{\partial \xi},
\]

we can write \(\tilde{X}_1 F_2 = Z_1 F_2 + \frac{\partial}{\partial Y} Z_2 F_2 = 0\). Since \(F_2\) does not depend on \(a_1\), the previous decomposition implies \(Z_1 F_2 = Z_2 F_2 = 0\). Applying the method of characteristics to \(Z_2 F_2 = 0\), we find that \(F_2\) must be constant along solutions of the characteristic system

\[
\frac{dv_0}{v_0} = \frac{dv_1}{v_1} = \frac{d\xi}{3\xi}, \quad dy_0 = dy_1 = 0.
\]

Therefore, \(F_2\) depends only on the variables \(K_1 = v_1/v_0, K_2 = v_1^2/\xi, y_0, y_1\), i.e. there exists a function \(F_3 : \mathbb{R}^4 \rightarrow \mathbb{R}\), such that \(F(x_0, x_1, v_0, v_1, a_0, a_1) = F_3(y_0, y_1, K_1, K_2)\).

To obtain a common first integral for all the vector fields in \(D\), it remains to impose \(Z_1 F = Z_1 F_3 = 0\). In the coordinate system \(\{y_0, y_1, K_1, K_2, \xi, a_1\}\), this equation reads

\[
\xi^{1/3} K_2^{1/3} \left( \frac{1}{K_1} \frac{\partial F_3}{\partial y_0} + \frac{\partial F_3}{\partial y_1} + K_2 \frac{\partial F_3}{\partial K_2} + \frac{3K_1}{2} - 2b_0 K_2^2 (K_1^{-3} - K_1^{-1}) \right) \frac{\partial F_3}{\partial K_1} = 0.
\]

The characteristic system corresponding to the above equation is

\[
dy_0 = \frac{dy_1}{K_1} = \frac{K_2 dK_2}{K_1^K_1} = \frac{3K_2^2 dK_2}{2K_1}.
\]

From the last equality of the above system, we obtain

\[
\frac{dK_2}{dK_1} = \frac{3K_2^2}{2K_1} - \frac{2b_0}{K_1^K_1} (1 - K_1^K_1) K_2^3 \rightarrow K_2(K_1) = \pm \frac{K_1^K_1}{\sqrt{K_1^K_1 - 4b_0 (1 + K_1^K_1)}},
\]

for a certain real constant \(\Gamma_1\). Consequently, a common first integral of the vector fields of \(D\) reads

\[
\Gamma_1 \equiv \frac{K_1^K_1 + 4b_0 K_2^2 (1 + K_1^K_1)}{K_1 K_2^K_2}.
\]

Now, \(dy_1 = K_2 dK_2/K_1^K_1\) and the above expression yields

\[
\frac{dy_1}{dK_1} = \text{sg}(K_2) (\Gamma_1 K_1 - 4b_0 (1 + K_1^K_1))^{-1/2}, \quad (8.4)
\]

where \(\text{sg}\) stands for the sign function. Assume, for simplicity, \(\text{sg}(K_2) = 1\) and \(b_0 < 0\). Hence,

\[
\Gamma_2 = ( -8b_0 K_1 + \Gamma_1 + 4\sqrt{b_0^2 (1 + K_1^K_1)} - 2b_0 K_1 \Gamma_1 ) e^{-2\sqrt{-b_0} \sqrt{\Gamma_1}}
\]

is a second first-integral. Likewise, from \(\Gamma_1\) and expression \(dy_0 = K_2 dK_1/K_2^K_2\), one obtains

\[
\frac{dy_0}{dK_1} = \left( K_1 \Gamma_1 - 4b_0 (1 + K_1^K_1) \right)^{-1},
\]

i.e.

\[
\Gamma_3 \equiv y_0 = \frac{1}{2\sqrt{-b_0}} \ln \left[ \frac{2\sqrt{-b_0} K_1}{-8b_0 + K_1 \Gamma_1 + 4\sqrt{b_0^2 (1 + K_1^K_1)} - 2b_0 K_1 \Gamma_1} \right].
\]
is another first integral. As \( \partial (\Gamma_1, \Gamma_2, \Gamma_3) / \partial (x_0, y_1^{(1)}, y_2^{(2)}) \not= 0 \) at a generic point of \((T^2 \mathbb{R})^2\), fixing \( k_1 = \Gamma_1 \) and \( k_2 = \Gamma_2 \), we can easily express \( K_1 \) in terms of \( k_1, k_2 \) and \( x_1 \). Using this and putting \( k_3 = \Gamma_3 \), we obtain

\[
x_0 = k_3 + \ln \left[ \frac{2 \sqrt{-b_0} (64b_0^2 - f_{k_1,k_2}^2(x_1))}{64b_0^2 (k_1 - 2 e^{2\sqrt{-b_0}k_2} k_2) - k_1 f_{k_1,k_2}^2(x_1) + 8b_0 (64b_0^2 - k_1^2 + e^{4\sqrt{-b_0}k_2^2})} \right]^{1/2}\sqrt{-b_0},
\]

where \( f_{k_1,k_2}(x_1) = k_1 - e^{2\sqrt{-b_0}k_2} k_2 \). Note that the above expression is a superposition rule for third-order Kummer–Schwarz equations, which provides the general solution \( x_0(t) \) of any instance of such equations in terms of a generic particular solution \( x_1(t) \) and the constants \( k_1, k_2 \) and \( k_3 \). Obviously, this represents an improvement with respect to other similar expressions for KS-3 equations, which allows us to describe their general solutions in terms of two particular solutions of a time-dependent frequency harmonic oscillator [17]. In addition, this expression is an instance of a quasi-base-superposition rule for a third-order differential equation.

### 9. Conclusions and outlook

We have proposed and analysed a general concept of a superposition rule for systems of HODEs. Some specific types of such superposition rules that appear in the literature have been studied and other new types have been introduced and investigated. All our results have been illustrated with examples extracted from the mathematics and physics literature. In particular, two new superposition rules for second- and third-order Kummer–Schwarz equations have been derived.

There are still many open questions concerning the properties of superposition rules for systems of HODEs. For instance, it would be interesting to find methods for analysing the existence of solutions of system (4.1), which would facilitate the determination of the existence of superposition rules for systems of SODEs. Additionally, it would be interesting to apply the methods developed here to analyse first-order systems from a new perspective.

In the future, we intend to study the whole Riccati hierarchy [49], some of whose members, like second-order Riccati equations, have already been analysed by means of the theory of Lie systems [10]. Furthermore, we aim to apply our results in the analysis of soliton solutions of PDEs described by the Riccati hierarchy [49]. Additionally, we plan to employ the theory of Lie systems so as to geometrically explain the relation of Kummer–Schwarz equations to Lie systems associated with a Vessiot–Guldberg Lie algebra isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \). This may be used to clarify their known connections with time-dependent harmonic oscillators or Riccati equations [21, 44] as well as to establish new ones. These and other topics will be analysed in forthcoming works.

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