Quark Condensation, Induced Symmetry Breaking and Color Superconductivity at High Density

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Abstract

The phase structure of hadronic matter at high density relevant to the physics of compact stars and relativistic heavy-ion collisions is studied in a low-energy effective quark theory. The relevant phases that figure are (1) chiral condensation, (2) diquark color condensation (color superconductivity) and (3) induced Lorentz-symmetry breaking (“ISB”). For a reasonable strength for the effective four-Fermi current interaction implied by the low energy effective quark theory for systems with a Fermi surface we find that the “ISB” phase sets in together with chiral symmetry restoration (with the vanishing quark condensate) at a moderate density while color superconductivity associated with scalar diquark condensation is pushed up to an asymptotic density. Consequently color superconductivity seems rather unlikely in heavy-ion collisions although it may play a role in compact stars. Lack of confinement in the model makes the result of this analysis only qualitative but the hierarchy of the transitions we find seems to be quite robust.

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1 Introduction

The dynamics of nucleons in dilute environment at very low energy can be approached via effective field theories such as chiral perturbation theory. For instance, the properties of two nucleons, both bound and scattering, can be very accurately calculated within the framework of well-defined strategy [1]. The situation with denser systems such as heavy nuclei and nuclear matter is an entirely different issue. While the traditional phenomenological methods have been quite successful in correlating a wide range of phenomena, there is very little predictive power in these techniques in the density regime where new physics can set in. The question one would like to raise is what happens to a hadronic (nuclear) matter system when it is compressed to a high density as in relativistic heavy ion collisions and in neutron stars. This requires going, perhaps much, beyond normal nuclear matter density, $\rho_0 \approx 0.16$ fm$^{-3}$. At some high density, it is plausible that nuclear matter undergoes one or more phase transitions. For instance, neutron matter found in neutron stars could be unstable at a density a few times $\rho_0$ against the condensation of kaons with an intriguing consequence on the structure of neutron stars and the formation of black holes [2, 3]. At about the same density or higher, chiral symmetry restoration with a possible induced Lorentz symmetry breaking could take place [4, 5].

Even more recently the old idea of color superconductivity proposed by Bailin and Love [6] drew a revived attention in a modern context and became a focus of wide speculations and debates [7, 8]. If this phenomenon were to take place, it could do so in the regime where either kaon condensation and/or chiral phase transition and/or induced Lorentz symmetry breaking might set in and the question is how and in what language, macroscopic (effective) or microscopic (quark-gluon), adapted to QCD one should address this problem. So far the most economical way of describing dense hadronic matter up to and near the normal nuclear matter density is to treat the hadrons that enter as quasi-particles, that is, fully dressed particles, with certain symmetries (such as chiral symmetry and scale invariance) assumed to be relevant. This is the basis of Brown-Rho scaling [9, 10] which is found to provide one of the simplest explanations for the low-mass lepton pairs observed in the CERES experiments [11]. The CERES experiments probe densities a few times the normal matter density, so at least for these data, the hadronic quasi-particle picture seems to work. However in dealing with the phase transitions that go over to QCD’s microscopic degrees of freedom, the hadronic quasi-particle picture should cede to one in which quasi-particles are quarks and gluons instead. The proper strategy must then be a Fermi-liquid theory of quarks and the phase transition of the color superconductivity type could be analyzed in terms of Landau Fermi-liquid theory and instability from a quark Fermi liquid.
Such an analysis involves the decimation of energy scales relative to the Fermi momentum and a thorough study of the renormalization group flow of the couplings corresponding to operators which are marginal at the given energy scale. If these coupling constants grow to order unity, one would have to resort to intuition to set up the operators corresponding to the new emerging physics. In the deconfined phase of QCD at high density, the new physics which is controlled by one-gluon exchange is most likely the formation of a color superconducting gap reminiscent of diquark condensation. For a quantitative investigation of this effect see [13]. On the other hand, we believe that the (possible) phase transition due to what we called “induced Lorentz symmetry breaking (or ISB in short)” in refs.[4, 5] occurs at rather low densities at which QCD is still strongly coupled. In this case, an educated guess of the relevant operators as required by the renormalization group analysis is difficult, and, if the induced symmetry breaking phenomenon interferes with quark confinement, impossible at the present stage.

In this paper we shall therefore adopt a more coarse-grained point of view and analyze the quark state at finite density resorting to a low-energy effective quark model. The model we shall consider consists of a gauge-invariant four-quark contact interaction with the quantum numbers of one-gluon exchange supplemented in certain channels (e.g., the flavor-singlet vector channel) with collective modes associated with the presence of a Fermi sea. Such an interaction represents the lowest-dimension operator that would result when the gluon is integrated out from the QCD partition function and would be dominant under the renormalization group flow as high energy states, both elementary and collective, are suitably “decimated.” The constraint we shall take into account in the current-current interaction – which controls the overall strength – is that it be consistent with the properties of pseudo-Goldstone bosons (pions) in matter-free space. The presence of Lorentz non-invariant collective excitations could affect significantly the flavor-singlet vector channel (that we shall frequently refer to as $V_0$ channel for reasons to be specified below) relevant to the ISB phase discussed in refs.[4, 5]. In particular, we shall take into account the possibility that the attraction in this channel could be enhanced by, and intimately correlated with, such collective modes as $N^*N^{-1}$ excitation considered in [14] for “dropping” vector masses in dense medium.

We admit that the model is plagued with certain shortcomings (e.g. lack of quark confinement). Nonetheless we believe it yields a comprehensive view of all three possible phases, namely, chiral symmetry phase, the ISB phase and color superconductivity phase. Given that lattice QCD measurements in the presence of chemical potential are totally lacking, qualitative studies of the sort we are presenting here seem to be the best one could do at the moment.

The plan of this paper is as follows. We shall briefly review in section 2 the scenario of
chiral phase restoration at finite chemical potential with an emphasis of the induced
breaking of Lorentz symmetry developed in [4, 5]. The model we shall use is
defined in section 3 wherein various condensates concerned are precisely defined.
The effective gluon-exchange interaction that sets the order of magnitude of the
parameters of the model is also given there. The phase structure that is expected to
emerge as density and/or temperature is increased is described in section 4. Section
5 contains discussions. Notations and definitions – some of which are quite standard
– are given in the appendices to make our discussions self-contained.

2 Phase transitions at finite chemical potential

To render the paper self-contained and to clear up some of the confusion on the key
points of refs.[4, 5], we briefly contrast the ISB scenario of chiral restoration
1 to the
more standard mechanism as provided by a constituent quark model.

In the last decade, numerous investigations [16] of constituent quark models have
shown that the constituent quark mass $M(\mu)$ rapidly decreases if the chemical po-
tential $\mu$ exceeds a certain value which we denote by $\mu_c$. In the constituent particle
picture the density of states at a given momentum $p$, i.e.

$$
n(p) = \frac{1}{\exp \left[(E(p) - \mu)/T \right] + 1}, \quad E(p) = \sqrt{M(\mu)^2 + p^2}, \quad (1)
$$

is governed by a sharp Fermi surface at zero temperature. In this case, the density
of constituent quarks remains zero for $\mu \leq \mu_{\text{onset}} = M(\mu)$. For $M(\mu) < \mu < \mu_c$, the
density is a monotonic increasing function of $\mu$. At the chiral phase transition at
$\mu = \mu_c$, the constituent mass drops. An inspection of (1) shows that one expects a
rapid increase of density.

1There seems to be some confusion caused by the term “ISB” which to the best of our knowledge
has not been used before (apart from us) in the literature. Since we do not know any other way of
calling it, we shall continue using this term although it could very well be a misnomer. The state
of matter involved here was defined in ref.[4] and clearly illustrated in a toy condensed-matter
model in [5]. One obvious question to ask is whether or not by “ISB” we are talking about a
phase transition which is completely different from the chiral phase transition that is believed to
be present in QCD. We should stress that the answer is no. That is, what we call “ISB” would be
implicitly present if QCD were solved reliably (see e.g., fig. 2 in [15]). What we are doing here –
which we believe is novel – is exploiting a characteristic of chiral phase transition that has not
been identified before in the modeling of the QCD phase structure. As defined below, the “ISB”
is characterized by the “jump” in baryon density as one goes across the critical chemical potential
$\mu_c$ corresponding to chiral phase transition and our point is that this is driven by an attraction in
the vector channel caused by certain collective modes, in analogy to the potential introduced in
the toy model of [5] (see figures 1 and 2). We will argue that this is related to the mechanism that
may be responsible for “dropping” vector meson masses.
At finite density, our assertion is that collective modes due to vibrations of the Fermi surface – which must be present on a general ground when the renormalization group flow to the Fermi surface is properly studied in the effective Lagrangian or in hadronic models of collective excitations\cite{14} – must be taken into account for a correct description of finite density hadron matter. In \cite{5, 4}, these modes were considered in terms of a local (effective) vector field $V_\mu$, the vacuum expectation value (VEV) of the zeroth component of which measures the departure from the constant chemical potential $\mu$ and the discontinuity in the Fermion number density. If the chemical potential reaches a certain strength $\mu_{ci}$, which is assumed to be smaller than $\mu_c$, an expectation value $\langle V_0 \rangle$ forms and signals an ISB (on top of the explicit breaking via the chemical potential). In the constituent quark model, the density of states is given in this case by

$$n(p) = \frac{1}{\exp\left[(E(p) - \langle V_0 \rangle - \mu)/T\right] + 1}. \quad (2)$$

With the assumption $\langle V_0 \rangle + \mu_{ci} > \mu_c$ the latter equation implies that the rise of the condensate $\langle V_0 \rangle$ catalyzes the onset of the chiral phase transition. The state of matter after chiral restoration is, in the present case, considerably richer than what is given by the naive (standard) scenario: as explained in \cite{4}, due to the dynamical origin of the condensate $\langle V_0 \rangle$, light (non-relativistic) vector excitations are predicted to exist as pseudo-Goldstone-type modes which dominantly decay into dilepton. An intriguing possibility is that this light mode is just the lower branch of the two-level model of \cite{14}.

Due to asymptotic freedom, there are two important deviations in our effective quark theory from the naive constituent quark picture which was used above. Note that the low energy effective quark model which we shall employ below implements asymptotic freedom by a sharp O(4)-invariant momentum cutoff. This analogy is of course only of qualitative nature and may not be quite realistic on a quantitative level.

Firstly, the finite range of momentum integration implies a smeared Fermi surface even at zero temperature. Including a self-consistent treatment of the field $V_0$, small values of density might be observed also for $\mu < \mu_{onset}$. Technically, this rise of density as function of $\mu < \mu_{onset}$ is a consequence of the smeared Fermi surface\footnote{The behavior of density vs. chemical potential for a low chemical potential $\mu << \mu_{ci}$ is irrelevant to the issue in question. For a dilute system, the correct degrees of freedom are nucleons or rather quasi-nucleons and not quarks or quasi-quarks. There is no reason to believe that an effective quark field theory without proper confinement mechanism can say anything about that regime which is of no concern to this work.}.

From a physical point of view, this rise emerges from a polarization of the Dirac
sea. Similar effects were observed before in NJL-type quark models when describing nucleons as solitonic excitations [17].

Secondly, the next question one might raise is whether this quark system suffers from an instability signaled by a steady increase of $\langle V_0 \rangle$. This instability is avoided by the shutdown of the quark interaction at large momentum transfer as expected from asymptotic freedom. This can be easily anticipated from the effective one-loop potential for $V_0$ when one resorts to a quark model with sharp momentum cutoff $\Lambda$. At least for values $V_0 > \Lambda$, one observes a monotonic increase of the potential as function of $V_0$ telling us that the formation of large values of $V_0$ is unlikely to take place.

Finally, the quark interaction mediated by the “collective” field $V_\mu$ is attractive (as suggested in [14]) and thus favors the formation of the $V_0$ condensate at sufficiently strong values of the chemical potential. One might question whether such an interaction is not an ad hoc input foreign to the expected properties of low-energy QCD merely to induce an ISB mechanism. The answer to this question is this: firstly, at least for $|V_0| < \mu$, the collective field describes perturbations of the Fermi surface. These fluctuations account for new important physics which comes into play at finite densities, and which is present besides the microscopic interaction mediated by gluons. At the hadronic level, the schematic model of [14] suggests that these collective modes could be excitations of the $N^*$-hole type and other varieties – analogous to phonons in solids and giant resonances in nuclei – in the isoscalar vector channel. The exchange of such collective modes between quarks could contribute an additional attraction. It may be possible to determine the size of the contribution from the collective modes using a model of the type studied in ref. [14]. But this program has not yet been fully worked out. In what follows, since we do not really know how to pin down the strength of the net attraction, we shall vary it within the range indicated by the one-gluon exchange plus that of collective modes and explore the qualitative structure of the possible phases.

3 From the model to the quark ground state

3.1 The model

We shall use a path-integral technique, so we define our model in Euclidean space in order to ensure a proper convergence of the functional integrals. In particular, the quark fields will be Euclidean spinors. Our notation and conventions can be found in appendix A. The partition function which we will use throughout this paper is
given by

\[
Z = \int Dq \, D\bar{q} \, Dq_c \, D\bar{q}_c \, D\sigma \, D\sigma \, D\Delta \, D\Delta^\dagger \, DV_\mu \, \exp \left\{ \int d^4x \, L \right\}, \quad (3)
\]

\[
L = -\left( \frac{\bar{q}}{q_c} \right) \left( \frac{S_0^{-1}}{\Delta} \frac{\kappa \Delta^\dagger}{\tilde{S}_0^{-1}} \right) \left( \frac{\bar{q}}{q_c} \right) - g_\Delta \Delta^\dagger \Delta
\]

\[
- V_\mu(x) \left( -\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu \right) V_\nu(x) - m_v^2 V_\mu V_\mu - g_\chi \left[ (\sigma - m)^2 + \pi^2 \right], \quad (4)
\]

where

\[
S_0^{-1} = \left[ i\not{\partial} - \sigma(x) + i\gamma_5 \bar{\pi}(x) \gamma^5 + iV_\mu(x) \gamma^\mu + i\mu \gamma^0 \right] 1_{\text{color}}, \quad (5)
\]

\[
\tilde{S}_0^{-1} := \left[ i\not{\partial} - \sigma(x) + i\gamma_5 \bar{\pi}(x) \gamma^5 - iV_\mu(x) \gamma^\mu - i\mu \gamma^0 \right] 1_{\text{color}}, \quad (6)
\]

where \( m \) is the current quark mass, \( \mu \) the chemical potential and \( g_\Delta, g_\chi \) and \( m_v \) are (necessarily positive) coupling constants which control the strength of the four-quark interactions that result when the bosonic fields – that we shall simply refer to as “mesonic” fields – \( \sigma, \bar{\pi}, V_\mu \) and \( \Delta, \Delta^\dagger \) are integrated out. The role these couplings play will be discussed below. The parameter \( \kappa \in \{1, i\} \) will be chosen to reproduce the correct sign of the four-quark current-current interaction which is well known for a correct description of the low energy physics of the light mesons \[18\] (see subsection 3.3). The charge conjugated quark fields possess the same kinetic energy as the quark fields and couple to the flavor singlet vector fields with opposite sign \[8\]. The quark fields \( q, \bar{q} \) and \( q_c, \bar{q}_c \) are treated as independent fields. At the classical level, the equations of motion yield the familiar relations \( \bar{q} = q^\dagger \) and \( q_c = C(q^\dagger)^T \) for Euclidean quark spinors (see appendix \[3\]), where \( C \) is the charge conjugation matrix.

Equation (4) is the simplest Lagrangian that contains all relevant degrees of freedom for the physics we are interested in. The quarks form flavor doublets and belong to the fundamental representation of the SU(3) color group. The “mesonic” fields \( \sigma, \bar{\pi} \) and \( V_\mu \) are color singlets, \( \sigma \) and \( V_\mu \) are flavor singlets while the pion fields form a flavor triplet. The quantum numbers of the \( \Delta \) field will be given below. The fields \( \sigma \) and \( \sigma^a \) are related to scalar and pionic degrees of freedom, respectively. The (collective) vector field \( V_\mu \) describes vibrations of the Fermi surfaces \[3\] and can be interpreted as the relativistic analog of the zero sound or the giant dipole resonance \[3\]. Finally, \( \Delta \) is the conjugate variable coupled to the diquark composite field and can therefore be viewed as the color superconductivity gap.

\[3\] As defined in \[3\] and \[3\], the collective field \( V_\mu \sim (V_0, \bar{0}) \) carries constants that reflect the (possibly density-dependent) strength of the coupling. They are buried in the constant \( m_\pi^2 \) in \[3\]. One could think of the factor \( m_\pi^2 \) that we shall vary later reflecting, say, the \( N^*N^{-1} \)-vector coupling strength of \[3\].
Let us study the symmetry of the model (3) in the case \( \Delta, \Delta^\dagger = 0 \). In the chiral limit \( m \to 0 \), the Lagrangian \( L \) for the two light-quark system, (4), is invariant under an \( SU(2)_A \times SU(2)_V \) flavor transformation of the quark fields \( q \) and \( q_c \), respectively (see appendix C). For a vanishing chemical potential \( \mu \), the model is O(4) invariant corresponding to the Lorentz symmetry in Minkowski space. For \( \mu \neq 0 \), the O(4) symmetry is explicitly broken to a residual O(3) invariance.

In order to compare the meson-induced interaction of the model (3) with the low energy effective quark interactions, it is instructive to integrate out all meson fields. Defining

\[
\Delta =: \Gamma \delta
\]

where \( \delta \) is a c-number and \( \Gamma \) specifies the Lorentz-, flavor- and color-structure of the superconducting gap, the following quark interactions (in derivative expansion and at low momentum transfer) are obtained

\[
\begin{align*}
+ m \bar{q}q + \frac{1}{4g_\chi} \left[ \bar{q}q \bar{q}q - \bar{q}\gamma_5\tau^a q \bar{q}\gamma_5\tau^a q \right] - \frac{1}{4m_v^2} \bar{q}\gamma^\mu q \bar{q}\gamma^\mu q \\
+ m \bar{q}_c q_c + \frac{1}{4g_\chi} \left[ \bar{q}_c q_c \bar{q}_c q_c - \bar{q}_c\gamma_5\tau^a q_c \bar{q}_c\gamma_5\tau^a q_c \right] - \frac{1}{4m_v^2} \bar{q}_c\gamma^\mu q_c \bar{q}_c\gamma^\mu q_c \\
+ \kappa^2 \frac{1}{g_\Delta} \bar{q}_c \Gamma q \bar{q}\Gamma^\dagger q_c.
\end{align*}
\]

3.2 From the effective potential to the quark condensates

The effective potential is a convenient tool for studying the ground state properties of an effective quark theory. Its global minima determine the condensates or ground-state expectation values of the bilinear quark fields that dictate the symmetries at low energies. The procedure is quite general and, in the case we are interested in, is equivalent to the BCS or Nambu-Gorkov formalism.

Let us be more specific. Firstly we derive a non-linear meson theory by integrating out the quark fields. Note that there is an intrinsic relation between quark condensates and ground state expectation values of the mesonic fields. Exploiting the fact that the partition function (3) is invariant under a shift of the mesonic integration variables yields the desired relations

\[
\begin{align*}
2g_\chi \langle \sigma - m \rangle &= \langle \bar{q}q \rangle + \langle \bar{q}_c q_c \rangle \\
2m_v^2 \langle V_0 \rangle &= \langle i\bar{q}\gamma^0 q \rangle - \langle i\bar{q}_c\gamma^0 q_c \rangle \\
- 2g_\Delta \langle \delta \rangle &= \langle \bar{q}_c \kappa \Gamma q \rangle + \langle \bar{q} \kappa \Gamma^\dagger q_c \rangle.
\end{align*}
\]

As expected, the quark and conjugate quark fields contribute to the scalar condensate with equal signs while they come with opposite signs in the case of the density.
\( \langle \sigma \rangle \) serves as an order parameter for the spontaneous breaking of chiral symmetry, while \( \langle \delta \rangle \) can be directly interpreted as the gap of a color superconducting state. \( \langle V_0 \rangle \) is proportional to the quark density which is always non-zero for non-vanishing chemical potential.

To define a convenient indicator – say, a “litmus paper” – \( v_{\text{ISB}} \) which signals the onset of the ISB phase transition, we first note that, in the limit \( m_v^2 \to \infty \), the four-quark interaction in the vector channel vanishes, and \( V_\mu \) is constrained to zero. The collective modes corresponding to the fluctuations of the field \( V_\mu \) decouple. In this case, the density is entirely generated by the chemical potential. For finite values of \( m_v^2 \), the interactions cause \( V_0 \) to depart from zero. We define the indicator or signal for the ISB phase by

\[
\rho_{\text{ISB}} := \rho\left(\langle \sigma \rangle, \langle \delta \rangle, \langle V_0 \rangle \right) - \rho\left(\langle \sigma \rangle, \langle \delta \rangle, 0 \right), \tag{10}
\]

i.e. we subtract from the full ground state density \( \rho \left(\langle \sigma \rangle, \langle \delta \rangle, \langle V_0 \rangle \right) \) the density \( \rho \left(\langle \sigma \rangle, \langle \delta \rangle, 0 \right) \) which is induced solely by the chemical potential. \( \rho_{\text{ISB}} \) therefore directly measures the contribution of the collective dynamics to the density and is equal to zero for \( m_v^2 \to \infty \). While \( \rho \left(\langle \sigma \rangle, \langle \delta \rangle, 0 \right) \) is expected to behave smoothly under increase of the chemical potential, \( \rho_{\text{ISB}} \) summarizes the non-trivial behavior due to the ISB phase transition.

To access the meson expectation values (in some approximation), one first introduces external sources that linearly couple to the meson fields in which we are interested. The generating functional in the case of our model (3) is therefore

\[
Z[j] = \int \mathcal{D}q \mathcal{D}\bar{q} \mathcal{D}q_c \mathcal{D}\bar{q}_c \mathcal{D}\sigma \mathcal{D}\pi^a \mathcal{D}\Delta \mathcal{D}\Delta^\dagger \mathcal{D}V_\mu \exp \left\{ \int d^4x \left[ L + L_{\text{source}} \right] \right\}, \tag{11}
\]

\[
L_{\text{source}} = j_\sigma(x) \sigma(x) + j_0(x) V_0(x) + j_\Delta(x) \delta(x). \tag{12}
\]

The meson expectation value \( \langle V_0 \rangle \), and therefore the condensate \( \langle \bar{q} \gamma^0 q \rangle \) will be non-trivial for non-vanishing values of the chemical potential \( \mu \). For convenience, we introduce the compact notation \( j = (j_\sigma, j_0, j_\Delta) \) and \( \phi = (\sigma, V_0, \delta) \). The effective action is defined by a Legendre transformation with respect to the external fields

\[
\mathcal{A}[\phi_c] = -\ln Z[j] + \int d^4x j(x) \phi_c(x), \tag{13}
\]

where

\[
\phi_{cl}(x) := \frac{\delta \ln Z[j]}{\delta j(x)}. \tag{14}
\]

The effective potential \( U_{\text{eff}}(\phi_{cl}) \) is obtained from \( \mathcal{A}[\phi_{cl}] \) by resorting to space-time constant classical fields.
In the case of many quark degrees of freedom (e.g. large number of colors), the functional integral \((11)\) can be approximately calculated neglecting mesonic fluctuations of the mesons (including \(\Delta\)) around their mean-field values. Within this approximation, we find

\[
U_{\text{eff}}(\phi_{cl}) = -\int \frac{d^4k}{(2\pi)^4} \text{tr} \ln \left( \frac{k - \sigma_{cl} + i(\mu + v_{cl})\gamma^0}{\kappa \Gamma \delta_{cl}} \right) k - \kappa \Gamma^\dagger \delta_{cl} - i(\mu + v_{cl})\gamma^0 \right) \\
+ g_\chi (\sigma_{cl} - m)^2 + m_v^2 v_{cl}^2 + \Delta \delta_{cl}^2.
\]

(15)

It can be proven that the fermion determinant is real if the diquark vertex \(\Gamma\) is either hermitian or anti-hermitian. This will be always the case for the examples studied below. The classical fields at which the effective potential \(U_{\text{eff}}\) takes its global minimum yield the desired condensates. In the following, we will interpret \(\sigma_{\min}^{\text{cl}}\) as quark condensate and \(v_{\min}^{\text{cl}}\) as quark density, and we will drop the index \(\text{cl}\), since \(U_{\text{eff}}\) consists only of classical fields. We will study the phase structure of our model \((3)\) by referring to different types of Lorentz- and color structures \(\Gamma\) of the diquark condensate.

### 3.3 Gluon-induced quark interactions

In this subsection, we will estimate the size of the parameters that figure in our model by comparing \((8,9)\) with the low-energy quark theory which may be obtained from effective one-gluon exchange. Note however that the resulting four-Fermi interaction represents a lot more physics than the exchange of one bare or dressed gluon and the effective size of the parameters will be fixed only when the additional corrections that arise in the decimation toward the Fermi surface (coming from, e.g., significant collective modes including the \(N^*N^{-1}\) excitation of \([14]\)) are suitably accounted for.

Quark interactions which are mediated by the exchange of a dressed gluon can be described in terms of the action

\[
S_{1\text{gluon}} = -\int d^4x \bar{q} \left[ i\gamma^\mu - A_\mu^a T^a \gamma^\mu \right] q - \int d^4x \bar{q}_c \left[ i\gamma^\mu + A_\mu^a (T^a)^T \gamma^\mu \right] q_c \\
- \int d^4x d^4y \ A^a_\mu(x) D_{\mu\nu}^{ab}(x - y) A^b_\nu(y),
\]

(16)

where \(T^a\) are the eight generators of the SU(3) color group. If \(\Gamma\) denotes the quark gluon vertex, the charge conjugate quark fields couple to the gluons via \(CT^T C^{-1}\) \([8]\).

Much work has been done on investigating the properties of light hadrons resorting to a wide variety of ansätze for the dressed effective gluon propagator \(D_{\mu\nu}^{ab}(x - y)\) \([21, 22, 23]\). Note that \(D_{\mu\nu}^{ab}\) is assumed to be a positive definite operator to ensure convergence of the gluonic functional integral. When employed in the Dyson-Schwinger equation for ground state properties and in the Bethe-Salpeter equation
for description of light mesons, these effective interactions are known to provide a successful description of the spontaneous breakdown of chiral symmetry and the corresponding low-energy pion physics. Over the last decade, research has extended these effective 1-gluon-exchange models by using non-trivial quark-gluon vertices that follow from Taylor-Slavnov identities. These investigations have met with a remarkable success in low-energy hadron phenomenology [24, 26].

Here, we shall not resort to such Dyson-Schwinger-type approaches to color superconductivity, but will employ the corresponding low energy effective quark theory which emerges from an elimination of the high energy modes by means of a renormalization group approach. In such an approach, the effective four-quark contact term provided by the current-current quark interaction naturally emerges, since this interaction respects all symmetries of low energy QCD. Moreover, it was observed in strongly coupled QED that such an interaction plays an important role in low energy physics due to a large anomalous dimension [25] of the corresponding operator. In the context of low energy QCD, the current-current interaction dictates the Lorentz, color and flavor structure while its strength must be chosen to correctly describe the physics of the light mesons [18]. Such an effective low-energy quark theory coming from a renormalizable QCD-inspired quark model generically yields a correct phase structure although it may fail in the description of certain dynamical processes (for details see [27]).

When applying the renormalization group decimation of energy scales to the model (16), the current-current interaction which is compatible with the low energy QCD symmetries is given by

\[ L = \ldots + G \left[ \bar{q}_1 F T^A \gamma^\mu q \bar{q}_1 F T^A \gamma^\mu q + \bar{q}_c 1_F \left( T^A \right)^T \gamma^\mu q_c \bar{q}_c 1_F \left( T^A \right)^T \gamma^\mu q_c \right. \]
\[ \left. - 2 \bar{q}_c 1_F \left( T^A \right)^T \gamma^\mu q_c \bar{q} 1_F T^A \gamma^\mu q \right], \tag{17} \]

where its absolute strength \( G \) must be chosen for e.g. reproducing the correct pion physics in free space.

We follow the ideas first presented in [7] for the diquark phase, and assume that a diquark condensate respects chiral symmetry while it breaks the residual global color symmetry from SU(3) down to SU(2). The color, flavor and Dirac structure of the diquark vertices which we will investigate below are (see appendix H)

- \( \Gamma = \epsilon_{ik} \epsilon^{\alpha \beta 3} \gamma_5 \) (scalar)
- \( \Gamma = \epsilon_{ik} \epsilon^{\alpha \beta 3} \) (pseudo-scalar)
- \( \Gamma = \epsilon_{ik} \epsilon^{\alpha \beta 3} \gamma^\mu \gamma_5 \) (vector)
- \( \Gamma = \epsilon_{ik} \epsilon^{\alpha \beta 3} \gamma^\mu \) (axial-vector),

where \( i, k = 1, 2 \) act in flavor space, and \( \alpha, \beta = 1, \ldots, 3 \) are color indices. Equation
will be our starting point for estimating the coupling strengths of our model (3). For this purpose, we rearrange the four-quark contact interaction (17) by means of a Fierz transformation so as to make sure that the classical equations of motion, i.e. the mean-field level, of our model equal the Dyson-Schwinger equations with the interaction (17). We relegate the details of this calculation to appendix D.1. The result is

\[ L = \ldots + \frac{G}{4} \left( \bar{q} 1_F 1_c q \bar{q} 1_F 1_c q - \bar{q} 1_c \gamma_5 \tau^a q \bar{q} 1_c \gamma_5 \tau^a q \right) \]  
\[ - G \left[ \frac{1}{2N} - \frac{1}{8} \right] \bar{q} 1_F 1_c \gamma^\mu q \bar{q} 1_F 1_c \gamma^\mu q \]  
\[ + \frac{G}{4N} (\bar{q}_c)_i^\alpha \epsilon_{ik} \epsilon^{\gamma \alpha \beta} q_k^\beta q_l^\gamma \epsilon_{lm} \epsilon^{\gamma_5 \omega} \gamma^\mu (q_c)_m^\omega \]  
\[ + \frac{G}{4N} (\bar{q}_c)_i^\alpha \epsilon_{ik} \epsilon^{\gamma \alpha \beta} q_k^\beta q_l^\gamma \epsilon_{lm} \epsilon^{\gamma_5 \omega} \gamma^\mu (q_c)_m^\omega \]  
\[ + \frac{G}{2N} (\bar{q}_c)_i^\alpha \epsilon_{ik} \epsilon^{\gamma \alpha \beta} q_k^\beta q_l^\gamma \epsilon_{lm} \epsilon^{\gamma \omega} (q_c)_m^\omega \]  
\[ - \frac{G}{2N} (\bar{q}_c)_i^\alpha \epsilon_{ik} \epsilon^{\gamma \alpha \beta} q_k^\beta q_l^\gamma \epsilon_{lm} \epsilon^{\gamma_5 \omega} \gamma^\mu (q_c)_m^\omega , \]

where \(1_F, 1_c\) are unit matrices in flavor and color spaces, respectively, and \(G\) sets the scale of the interaction strength. \(N = 3\) is the number of colors. Comparing this result with (8,9), we first deduce the “sign” of the meson quark couplings, i.e.

\[ \kappa_{\text{scalar}} = i, \quad \kappa_{\text{ps}} = 1, \quad \kappa_{\text{vector}} = i, \quad \kappa_{\text{axial}} = 1, \]  

and then read off the corresponding strengths, i.e.

\[ g_\chi = \frac{1}{2G}, \quad m_v^2 = \frac{N}{G} \frac{1}{1 - N/4}, \]  
\[ g_\Delta^{\text{scalar}} = g_\Delta^{\text{ps}} = \frac{2N}{G}, \quad g_\Delta^{\text{vector}} = g_\Delta^{\text{axial}} = \frac{4N}{G} . \]  

In particular, we find the following ratios for \(N = 3\)

\[ g_\chi : m_v^2 : g_\Delta^{\text{scalar}} : g_\Delta^{\text{vector}} = 1 : 6 : 6 : 12 . \]  

In the next section, we will study either case of diquark condensation and will tacitly assume that a mixed phase, e.g. with the scalar and vector diquark condensates simultaneously present, will not occur.

The parameter ratios (22) represent the quark interactions carrying the quantum numbers of one gluon exchange. As explained in section 3, the attraction already present in the current-current interaction for the \(V_0\) channel with three colors and
two flavors (see Appendix D for a caveat that may follow from a different Fierz rearrangement) is likely to be further enhanced by collective modes. In our calculation, this effect will be duly accounted for by lowering the mass $m^2_v$. Since little is known at present about the size of this additional attraction, we will explore in the next section the phase structure for a wide span of parameters while the order of magnitude of the parameters will be kept at the values given by the effective one-gluon exchange.

4 Phase structure

At finite baryon density, a rich phase structure is expected to appear in QCD. Whereas at zero density a quark condensate is formed in the QCD quark sector, two possible phases might exist at finite values of the chemical potential: the ISB phase in which Lorentz symmetry is “spontaneously” broken in the sense defined in [4, 5] and the phase in which a diquark condensate is formed which breaks the residual global color symmetry of QCD [7]. In the following subsections, we will explore the phase structure which arises from our low-energy effective quark theory when the chemical potential is increased.

4.1 Scalar and pseudo-scalar diquark condensations

We follow the ideas first presented in [7] for the diquark phase. There, it was conjectured that the color superconductor instability occurs in the scalar diquark. In this section, we will extend the analysis of [7] to the study of possible scalar, pseudo-scalar, vector and axial-vector diquark condensations as well.

In this subsection, we will study the possible emergence of scalar as well as pseudo-scalar diquark condensates, i.e.

$$\langle \bar{q}^\alpha_i \: \gamma_5 \: \epsilon_{1k} \: \epsilon^{\alpha\beta} \: q^\beta_k \rangle \quad \text{(scalar)} \quad (23)$$

$$\langle \bar{q}^\alpha_i \: \epsilon_{1k} \: \epsilon^{\alpha\beta} \: q^\beta_k \rangle \quad \text{(pseudo-scalar)} \quad (24)$$

Here $i,k = 1\ldots2$ are flavor indices, while $\alpha,\beta = 1\ldots3$ are color indices. Dirac indices are not explicitly shown. The behavior of the Euclidean quark spinors under Lorentz transformations can be found in appendix B. If one of these condensates is realized in a finite-density quark matter while the quark condensate is zero, chiral symmetry is restored (see appendix C.2).

We first consider the case of the pseudo-scalar superconducting gap, $\Delta = \epsilon_{1k} \epsilon^{\alpha\beta} 1 \delta$ in conjunction with the ISB phase transition extensively discussed in [4, 5]. The
fermion determinant in (15) can be calculated in a closed form (see appendix E), i.e.

\[
\text{Det}_{ps} = \prod_k \left( k_+^2 + \sigma^2 \right) \left( k_-^2 + \sigma^2 \right) \left( (w_+ k_0 - i(\mu + v) w_-)^2 + w_+^2 k^2 + w_-^2 \sigma^2 \right) \]

(25)

where

\[
k_\pm := (k_0 \pm i(\mu + v), \vec{k})^T, \quad w_\pm := 1 \pm \frac{\delta^2}{k_+^2 + \sigma^2}.
\]

The effective potential is

\[
U_{eff}^{ps}(\sigma, \delta, v) = -\ln \text{Det}_{ps} + g_\chi (\sigma - m)^2 + m_v^2 v^2 + g_\Delta \delta^2.
\]

(26)

The model (3) is considered to be an effective low-energy quark theory in that the momentum integration implicit in (26) is cut off at the scale \( \Lambda \). Note that for a reliable investigation of the model at finite values of the chemical potential it is crucial that the momentum cutoff leave the Lorentz, i.e. O(4), symmetry intact. To assure this, we used a sharp O(4) invariant momentum cutoff.

In the case of the scalar diquark composite, we choose \( \Delta = \gamma_5 \epsilon \epsilon_{\alpha\beta}^3 \delta \). The functional determinant in (13) is given by (see appendix E)

\[
\text{Det}_{scalar} = \prod_k \left( k_+^2 + \sigma^2 \right) \left( k_-^2 + \sigma^2 \right) \left( (w_+ k_0 - i(\mu + v) w_-)^2 + w_+^2 k^2 + w_-^2 \sigma^2 \right) \]

(27)

Comparing (27) with (23), one immediately sees that the determinants with pseudo-scalar and scalar diquark entries, respectively, coincide if chiral symmetry is respected, i.e. \( \sigma = 0 \). In the following, we will assume that we will not encounter a strong CP-problem, but that the scalar diquark condensation takes place.

We explored the parameter space, looking for the global minimum of the effective potential \( U_{eff} \), by resorting to Monte Carlo techniques. We find generically three different minima of the effective potential. One minimum shows a strong chiral condensate while the superconducting gap and \( v \) are close to zero. We call this minimum \( \text{min} - \sigma \). The second minimum, so-called \( \text{min} - v \), is located at strong values of \( v \) with the order parameters \( \sigma, \delta \) being small. Finally, the third minimum is dominated by a large value of the superconducting gap \( \delta \), and we shall call this \( \text{min} - \Delta \). All these minima are possible candidates for the true ground state which is characterized by the global minimum of \( U_{eff} \).
To establish the instability due to color superconductivity at high density as proposed in [7], we turn off the interaction strength in the vector channel by tuning $m_v^2 \to \infty$ and study the competition between the chiral phase and the superconductor phase as a function of the chemical potential. The numerical result of our model is shown in figure 1 (left panel). It indicates that the scalar diquark condensation sets in at some large values of the chemical potential $\mu = O(\Lambda)$, where the use of a four-Fermi contact interaction is most likely unjustified. Since we are here only interested in the qualitative phase structure, we shall not consider the non-local interaction due to the full effective one-gluon exchange.

The situation becomes more dramatic if we use a moderate value for $m_v^2$. For instance for $g_\Delta = g_\chi = m_v^2 = \Lambda^2/4\pi^2$, the effective potential at the minima comes out to be as shown in figure 1 (right panel). It turns out that ISB phase transition takes place at a small value ($\mu \approx 0.4\Lambda$) of the chemical potential, whereas the minimum which is characterized by the scalar diquark condensate, i.e. $\text{min} - \Delta$, turns into a saddle point.
In the case of finite $m^2$, we resorted to a full Monte-Carlo treatment to calculate the strength of the order parameters $\sigma$, $v$ and $\delta_{vee}$. It turns out that the scalar diquark condensate is not important for ground state properties if we choose the typical interaction strength of order unity, i.e. $g_\chi = g_\Delta = m^2_v = \Lambda^2/4\pi^2$ (see figure 2).

At zero chemical potential, the quark condensate ($\sigma \neq 0$) signals the spontaneous breakdown of chiral symmetry. The ISB phase ($\min v$) and the superconducting phase ($\min \Delta$) appear as meta-stable states. When the chemical potential is increased above a certain critical value $\mu$, a first-order phase transition takes place from the chiral broken phase to the ISB phase. An “induced” breaking of Lorentz symmetry takes place while chiral symmetry gets restored. This mechanism as well as its consequences for the light particle spectrum were extensively discussed in [4, 5].

We then reduced the strength of the quark interactions in the vector channel by a factor of 4, i.e. $g_\chi = g_\Delta = \Lambda^2/4\pi^2$, $m^2_v = 4 \Lambda^2/4\pi^2$, as this is the channel that gives
rise to the ISB phase. In fact, we searched for a region of parameter space where the superconducting state constitutes the state with the lowest energy density. In this case, one indeed observes a superconducting state at large values of the chemical potential (see figure 3). It appears, however, that if one takes the interaction strength to be of order unity – which is a typical strength one expects as before, the color superconductivity is unlikely to play a role in the density regime relevant for the physics we are interested in. This is because the ISB phase has a lower vacuum energy density.

### 4.2 Vector diquark condensation

One might wonder whether a vector diquark condensate and/or an axial-vector diquark condensate, i.e.

$$\langle \bar{q}^\alpha_i \epsilon_{ik} \epsilon^{\alpha 3} \gamma^\mu \gamma_5 q^\beta_k \rangle, \quad \text{(vector)}$$

(28)
\[
\left\langle \tilde{q}_C^\alpha \gamma^\mu q_k^\beta \right\rangle, \quad \text{(axial-vector)} \quad , \quad (29)
\]
might not compete with the ISB phase transition if the interaction is given by the effective one-gluon exchange. Since both diquark condensates transform as vectors, one might suspect that they could be more sensitive to the chemical potential which transforms as the fourth component of the vector.

The fermion determinant in (15) can be readily calculated for the vector diquark \((\Delta = \epsilon_{\alpha \beta} \epsilon^{033} \gamma^0 \gamma_5 \delta)\) and the axial-vector \((\Delta = \epsilon_{\alpha \beta} \epsilon^{033} \gamma^0 \gamma_5 \delta)\) diquark (see appendix E),

\[
\text{Det}_{\text{vec}} = \prod_k \left( k_+^2 + \sigma^2 \right)^{12} \left( k_-^2 + \sigma^2 \right)^4 \left[ \left( w_k - i \mu \bar{w} + \frac{1}{2} \bar{k}^2 + \frac{1}{2} \sigma^2 \right)^8 \right] \quad (30)
\]

\[
\text{Det}_{\text{axial}} = \prod_k \left( k_+^2 + \sigma^2 \right)^{12} \left( k_-^2 + \sigma^2 \right)^4 \left[ \left( w_k - i \mu \bar{w} + \frac{1}{2} \bar{k}^2 + \frac{1}{2} \sigma^2 \right)^8 \right] \quad (31)
\]

In order to gain some insight into the competition between the chiral phase and the phases exhibiting the various diquark condensates, we studied the limit \(m_0^2 \to \infty\) which amounts to the suppression of the ISB mode. The effective potential of the various local minima – which are the possible candidates for the true ground state – is shown in figure 3 as function of the chemical potential. It turns out that the scalar diquark condensation sets in at \(\mu = \mathcal{O}(\Lambda)\) whereas the other diquark condensates merely correspond to meta-stable states.

The reason why the vector diquark condensates have no significant influence on the ground state properties is that the four-quark interaction in the vector diquark channel has the “wrong” sign. If for the purpose of illustration one changes this sign by setting \(\kappa_{\text{vector}}^2 = 1\) (compare with (19)), the vector diquark order parameter has a much stronger effect (see figure 3), but still cannot prevent diquarks from condensing in the scalar channel.

### 4.3 Finite temperatures

Finally, we study the impact of temperature on the magnitude of the order parameters \(\sigma, \Delta\) and the ISB indicator \(\rho_{\text{ISB}}\). In view of the results of the previous subsections, we will only focus on the emergence of the scalar diquark condensate.

In taking into account temperature effects, we use the imaginary time formalism by introducing Matsubara frequencies and by replacing the \(k_0\) momentum integration

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in (30) by a discrete sum, i.e.

\[ k_0 = (2n + 1)\pi T \quad \text{and} \quad \int dk_0 \rightarrow 2\pi T \sum_n , \]

where \( T \) is the temperature. Note that the momentum integration is truncated by an O(4) symmetric momentum cutoff \( \Lambda \), i.e. \( k^2 + \vec{k}^2 \leq \Lambda^2 \). This implies that the momentum integral vanishes for \( \pi T > \Lambda \). From a physical point of view, this amounts to assuming that the non-local (e.g. gluon-induced) interactions at large momentum transfer \( (k^2 > \Lambda^2) \) that match the four-Fermi interaction of our effective quark theory at the scale \( \Lambda \) contributes only perturbatively to the order parameters \( \sigma, v \) and \( \Delta \) (see [19]). We therefore expect the melting of all condensates at \( T_c \approx \Lambda/\pi \). Using a generic energy scale of \( \Lambda \approx 600 \text{ MeV} \), one finds roughly \( T_c \approx 200 \text{ MeV} \) (at zero density), which is the right order of magnitude as revealed by lattice QCD studies [20].

Using Monte Carlo techniques, we searched for a given chemical potential and a given temperature for the global minimum of the effective potential \( U_{\text{eff}} \) which occurs for the parameters \( \sigma, v \) and \( \delta \). We then calculated \( \rho_{\text{ISB}} \) for the signal of the ISB phase transition. Even at a medium scale of the chemical potential (\( \mu < \Lambda \)) one observes a significant jump of \( \rho_{\text{ISB}} \). Although, strictly speaking, there is no phase transition but a crossover, one can define rather sharp phase boundaries in parameter space by

\[
\sigma > v_{\text{ISB}}, \sigma > \delta \quad \text{chiral phase} \\
v_{\text{ISB}} > \sigma, v_{\text{ISB}} > \delta \quad \text{ISB phase} \\
\delta > \sigma, \delta > v_{\text{ISB}} \quad \text{color superconductor phase},
\]

where \( v_{\text{ISB}} := \rho_{\text{ISB}}/2m^2 \). The numerical result of our model is sketched in figure 4.

Figure 4: Phase diagram for a medium size \( m^2_v = \Lambda^2/4\pi^2 \) interaction strength in the vector channel (left) and for a small size interaction \( m^2_v = 8\Lambda^2/4\pi^2 \) (right).
One finds that for the interaction strengths $g_\chi$, $g_\Delta^{\text{scalar}}$, $m_v^2$ all of order unity, diquark condensation plays no role. It is the ISB phase $\rho_{\text{ISB}} \neq 0$ that characterizes the quark state. At large values of the chemical potential, we expect the contribution from the quark interactions to the density to become small, and that there is a smooth transition from the ISB phase at some medium values of the chemical potential to the phase of perturbative QCD where the density is largely induced by the chemical potential.

If one were to choose an unnaturally weak vector interaction strength $m_v^2 \gg g_\chi, g_\Delta^{\text{scalar}}$, one might observe “shells” built up from the ISB phase and the color superconducting phase, respectively. We note, however, that the parameter range $\mu \approx \Lambda$ is beyond the scope of the present model. In order to access these ranges of the chemical potential, a non-local and asymptotically free quark interaction (as proposed e.g. in [29]) might be required.

5 Discussions

A detailed analysis of the possible phase structure of finite density quark matter based on an effective low-energy quark theory is presented. The four-quark contact interaction used in the study is provided by the effective quark current-current interaction which naturally emerges from a renormalization group decimation of high energy scales in a many-body system with a Fermi sea. From a phenomenological point of view, the effective current-current interaction is well known for a correct description of the physics of the light mesons at least at low densities. The simplest way to understand what we have done in this paper is that our strategy corresponds to interpolating the effective bottom-up partition function (3) with the explicit collective fields $V_\mu$ and $\Delta$ (as well as the pseudoscalars $\pi$ and the scalar $\sigma$) to that with the four-Fermi interactions (17) that must arise from a top-down approach starting from the QCD partition function. Although the effective theory might be incomplete in some aspects of nonperturbative structure of QCD, such as lack of quark confinement and additional many-body effects, we believe that it qualitatively describes the phases of quark matter when the chemical potential is varied. Hopefully such an approximate description of quark matter phases at high density will serve as an important input for a thorough renormalization group study (based on a decimation of energy scales relative to the Fermi momentum [12]) which will then yield more quantitative results.

Our finding is that the generic interaction strength provided by the current-current interaction in the diquark channel is much too weak to trigger a scalar diquark condensate. Pseudo-scalar, vector and axial-vector diquark condensates are even less favored. Instead of the color superconductivity, we find an induced Lorentz
symmetry breaking to occur at medium values of the chemical potential leading to chiral symmetry restoration with color symmetry remaining unbroken.

In sum, we are led to the following conjecture for the quark phase structure: at zero density, strong QCD interactions break chiral symmetry spontaneously leading to quark condensation. At small finite values of the chemical potential, the quark condensation weakens in the presence of Lorentz-symmetry-breaking terms, and a “roll-over” from $\langle \bar{q}q \rangle$ to $\langle \bar{q}\gamma_0 q \rangle$ is observed (called “ISB phase transition”). One of our main points is that this phenomenon in quark language is related to the hadronic description of the chiral phase transition and the “dropping mass” of the $\omega$ meson in [14]. As the chemical potential is further increased, the quark interaction is weakened due to asymptotic freedom. At a certain (second) critical value of the chemical potential, the interaction is no longer strong enough to sustain quark condensation in one of the Lorentz channels. In this case, it is most likely that color superconductivity occurs since an arbitrarily weak (but attractive) interaction at the Fermi surface is sufficient for (color) Cooper pair formation.

We should mention that our description may be much too simplistic to be realistic. It has been shown by Ilieva and Thirring [28] that even in a simple quantum mechanical system involving both the ISB-like (called “mean field” by the authors of [28]) phase and superconducting phase, the phase structure can be quite intricate and complicated. It is perhaps significant to note in particular that as in the Ilieva-Thirring model, an attractive “mean field” potential could even trigger (instead of prevent) superconductivity in contrast to what we have found in this paper. It is fair to say that the situation in QCD could be immensely more intricate and complex.

Finally let us comment on the apparent breaking of the local gauge invariance by the formation of the diquark condensate. In a recent important work, Schaden et al. [29] exploited a topological field theory to obtain a complete gauge fixing while avoiding the Gribov problem. In the so-called equivariant gauge, gauge invariance was fixed up to a global symmetry in color space. The authors then showed that quark confinement (in these gauges only) could be induced by a spontaneous breakdown of this residual global color symmetry. As a consequence, owing to Goldstone’s theorem, light colored fields (which arise in these gauges) decouple from local gauge invariant operators but contribute to the Wilson loop (due to its non-local nature) [29]. These Goldstone bosons then could produce an area law (and therefore confinement) due to inherent infra-red singularities [29].

Except on a lattice, any approach to color superconductivity, be that via an effective quark theory or via an renormalization group analysis, will have to be based on a completely gauge fixed QCD. In this respect, diquark condensation does not break local gauge invariance, but breaks instead the residual global color symmetry. In
view of the results presented in [29], one encounters the interesting possibility that
a hypothetical phase transition to a color superconducting state at high density
restores chiral symmetry, but does not imply quark deconfinement. However, we
should recall that, at least in certain supersymmetric Yang-Mills theories, chiral
restoration and deconfinement occur simultaneously [30].

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A Notation and conventions
The metric tensor in Minkowski space is
\[ g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \, . \] (A.1)
We define Euclidean tensors \( T_{(E)} \) from the tensors in Minkowski space \( T_{(M)} \) by
\[ T_{(E)}^{\mu_1 \ldots \mu_N}_{\nu_1 \ldots \nu_n} = (i)^r (-i)^s T_{(M)}^{\mu_1 \ldots \mu_N}_{\nu_1 \ldots \nu_n} \, , \] (A.2)
where \( r \) and \( s \) are the numbers of zeros within \( \{\mu_1 \ldots \mu_N\} \) and \( \{\nu_1 \ldots \nu_n\} \), respectively. In particular, we have for the Euclidean time and the Euclidean metric
\[ x^0_{(E)} = i x^0_{(M)} \, , \quad g^{\mu\nu}_{(E)} = \text{diag}(-1, -1, -1, -1) \, . \] (A.3)
Covariant and contra-variant vectors in Euclidean space differ by an overall sign. For
a consistent treatment of the symmetries, one is forced to consider the \( \gamma^\mu \) matrices
as vectors. Therefore, one is naturally led to anti-hermitian Euclidean matrices via
(A.2),
\[ \gamma^0_{(E)} = i \gamma^0_{(M)} \, , \quad \gamma^k_{(E)} = \gamma^k_{(M)} \, . \] (A.4)
In particular, one finds
\[ \left( \gamma^\mu_{(E)} \right)^\dagger = - \gamma^\mu_{(E)} \, , \quad \{ \gamma^\mu_{(E)}, \gamma^\nu_{(E)} \} = 2 g^{\mu\nu}_{(E)} = -2 \delta_{\mu\nu} \, . \] (A.5)
The so-called Wick rotation is performed by considering the Euclidean tensors (A.2)
as real fields.
In addition, we define the square of an Euclidean vector field, e.g. $V_\mu$, by

$$V^2 := V_\mu V_\mu = -V_\mu V^\mu. \quad (A.6)$$

This implies that $V^2$ is always a positive quantity (after the wick rotation to Euclidean space).

The Euclidean action $S_E$ is defined from the action $S_M$ in Minkowski space by

$$\exp\{iS_M\} = \exp\{S_E\}. \quad (A.7)$$

Using (A.2), it is obvious that the Euclidean Lagrangian $L_E$ is obtained from the Lagrangian $L_M$ in Minkowski space by replacing the fields in Minkowski space by Euclidean fields, i.e. $L_E = L_M$.

Let the tensor $\Lambda^\mu_\nu$ denote a Lorentz transformation in Minkowski space, i.e.

$$\Lambda^\mu_\alpha \Lambda^\nu_\beta g^{\alpha\beta} = g^{\mu\nu}. \quad (A.8)$$

Using the definition (A.2), one easily verifies that $\Lambda^\mu_{(E)\nu}$ are elements of an O(4) group, i.e. $\Lambda^T_{(E)\nu} \Lambda_{(E)\mu} = 1$, which is the counterpart of the Lorentz group in Euclidean space.

In order to define the Euclidean quark fields, we exploit the spinor transformation of the Euclidean quark field

$$q_{(E)}(x_E) \rightarrow q'_{(E)}(x'_E) = S(\Lambda_{(E)}) q_{(E)}(x_E), \quad x_E \rightarrow x'_E = \Lambda x_E, \quad (A.9)$$

$$S(\Lambda_{(E)}) \gamma^\mu_{(E)} S^\dagger(\Lambda_{(E)}) = \left(\Lambda^{-1}_{(E)}\right)^\mu_\nu \gamma^\nu_{(E)}, \quad (A.10)$$

where the matrices are given by

$$S(\Lambda_{(E)}) = \exp\left\{-i \frac{\omega_{\mu\nu}}{4} \sigma^\mu_{(E)\nu}\right\} \quad (A.11)$$

for $\det(\Lambda) = 1$ only. It is obvious that one must interpret

$$\bar{q}_{(E)} = q^\dagger_{(E)} \quad (A.12)$$

in order to ensure that e.g. the quantity $\bar{q}_{(E)} \gamma^\mu_{(E)} q_{(E)}$ transform as an Euclidean vector. We suppress the index $E$ throughout the paper and mark tensors with an index $M$, if they are Minkowskian.
B Charge conjugation in Euclidean space

The equation of motion of a Dirac spinor which couples to a U(1) gauge field is given by

\[-i\partial_\mu q^\dagger(x) \gamma^\mu - q^\dagger(x) eA = 0.\]  \hspace{1cm} (B.1)

Defining the charge conjugated spinor by

\[q_c(x) := C (q^\dagger(x))^T\]  \hspace{1cm} (B.2)

with some Dirac matrix \(C\) which will be specified below, equation (B.1) can be cast into

\[C i\partial^\dagger T C^{-1} q_c(x) + e C A^T C^{-1} q_c(x) = 0.\]  \hspace{1cm} (B.3)

One finds that \(q_c(x)\) satisfies a Dirac equation with reversed charge \(e \rightarrow -e\) if \(C\) satisfies \(C\gamma^\mu C^{-1} = -\gamma^\mu\). Let us choose a standard representation of the Dirac matrices, i.e.

\[
\gamma^0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -\tau_k \\ \tau_k & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]  \hspace{1cm} (B.4)

with \(\tau_k\) the SU(2) Pauli matrices. Note that

\[\{\gamma^\mu, \gamma_5\} = 0, \quad \gamma_5^\dagger = \gamma_5^T = \gamma_5.\]  \hspace{1cm} (B.5)

Using this representation, one finds

\[C = \gamma^0 \gamma^2, \quad C^\dagger = C^{-1} = C^T = -C.\]  \hspace{1cm} (B.6)

Let us investigate the behavior of the charge conjugated spinor \(q_c\) under Lorentz, i.e. O(4), transformations (A.11). Replacing \(S(\Lambda)q\) for \(q\) in the Dirac equation (B.1) and re-deriving \(q_c\) from this equation, one finds that the charge conjugated spinors transform as

\[q_c' = C (S^{-1}(\Lambda))^T C^{-1} q_c.\]  \hspace{1cm} (B.7)

We now confine ourselves to Lorentz transformations (A.11) with \(\det(\Lambda) = 1\) and will check the behavior of \(q_c\) under parity transformations later. Firstly note that

\[(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)^T = \gamma_\nu T \gamma_\mu T - \gamma_\mu T \gamma_\nu T = -C (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) C^{-1},\]

and therefore

\[S^T(\Lambda) = C S^\dagger(\Lambda) C^{-1}, \quad (\det\Lambda = 1).\]  \hspace{1cm} (B.8)

Using the last equation, one readily verifies for a solution \(q_c\) of the equation of motion that

\[q_c' = C (q^\dagger S(\Lambda))^T = C (S^\dagger)^T (q^\dagger)^T = S(\Lambda) q_c.\]
An inspection of (A.10) shows that $S(\Lambda) = \gamma_0$ generates a reflection of spatial coordinates and therefore serves as an example for a Lorentz transformation with $\det \Lambda = -1$. We learn from (B.7) that the conjugated spinor behaves as $q_c \rightarrow -q_c$ under this reflection transformation. We therefore conclude that the quark spinor $q_c$ transforms as

$$q'_c = \det(\Lambda) S(\Lambda) q_c.$$  \hfill (B.9)

In particular, one finds that the diquark condensates $\langle \bar{q}_c\gamma_5 q \rangle$ and $\langle \bar{q}_c\gamma_\mu\gamma_5 q \rangle$ transform as a scalar and a vector, respectively, as in Minkowski-space.

## C Chiral symmetry in Euclidean space

### C.1 Generalities

For simplicity, we will illustrate the emergence of chiral symmetry in effective meson theories for the case of two quark flavors. Generalization to $N_f$ quark flavors is obvious. One observes that the quark kinetic energy, i.e.

$$\bar{q} \left( i\partial - \sigma + i\gamma_5 \vec{\pi}\vec{\tau} \right) q$$  \hfill (C.1)

is invariant under a simultaneous transformation of meson and quark fields. Let $V$ be an SU(2) matrix which acts only on flavor indices, and $V$ a matrix which exists in flavor and Dirac space, i.e.

$$V = \exp \{ i\vec{\alpha}\vec{\tau} \} \, , \quad V = \exp \{ i\vec{\alpha}\vec{\tau} 1_D \} \, .$$  \hfill (C.2)

For convenience, we introduce the flavor matrix

$$M = \sigma - i\vec{\pi}\vec{\tau} \, .$$  \hfill (C.3)

Then the invariance under $V \in SU_V(2)$ transformation is observed for

$$q' = V q \, , \quad \bar{q}' = \bar{q} V^\dagger \, , \quad M' = V M V^\dagger \, .$$  \hfill (C.4)

To display the axial-vector symmetry, we choose $U$ to be another SU(2) matrix acting in flavor space and define

$$U = \exp \{ i\vec{\beta}\vec{\tau} \} \, , \quad U = \exp \{ i\vec{\beta}\vec{\tau} \gamma_5 \} \, .$$  \hfill (C.5)

Invariance under axial-vector chiral rotations, $U \in SU_A(2)$, is generated by

$$q' = U q \, , \quad \bar{q}' = \bar{q} U \, , \quad M' = U M U \, .$$  \hfill (C.6)
The so-called chiral circle is invariant separately under both transformations, i.e.

$$\sigma^2 + \vec{\pi}^2 = \frac{1}{2} \text{tr} \left( MM^\dagger \right). \quad (C.7)$$

Note, that $q$ and $\bar{q}$ are independent fields. This makes it possible to assign different flavor matrices to either field. Equations (C.4) and (C.6) establish the well known $SU_A(2) \times SU_V(2)$ chiral symmetry.

There is a subtle difference between the Euclidean formulation and the formulation in Minkowski space. Let $\psi$ and $\bar{\psi}$ denote the spinor solutions of the equations of motion. In Minkowski space, one finds

$$\bar{\psi}_{(M)} = \psi_{(M)}^\dagger \gamma_0^{(M)}, \quad \bar{\psi}'_{(M)} = (U \psi_{(M)})^\dagger \gamma_0^{(M)} = \bar{\psi}_{(M)} U \quad (C.8)$$

Equation (C.8) implies that the saddle points are degenerate by axial chiral rotations. This situation is different in Euclidean space. There one finds

$$\bar{\psi}_{(E)} = \psi_{(E)}^\dagger, \quad \bar{\psi}'_{(E)} = (U \psi_{(E)})^\dagger \neq \bar{\psi}_{(E)} U \quad (C.9)$$

implying that the degeneracy of the saddle points is lifted. Since we are interested in the non-linear meson theory that results from integrating out the quark fields, the resulting fermion determinant depends only on the chiral circle (C.7) in either formulation.

To illustrate this point for the Euclidean case, we calculate the fermionic determinant for the space-time constant fields $\sigma, \vec{\pi}$. The eigenvalues of the Dirac operator in momentum space, i.e. $\slashed{k} - \sigma + i\vec{\pi} \gamma_5$, are given by $\lambda_\pm = -\sigma \pm i\sqrt{k^2 + \vec{\pi}^2}$. One therefore obtains the desired result

$$\text{Det} = \prod_k (\lambda_+ \lambda_-)^2 = \prod_k \left(k^2 + \sigma^2 + \vec{\pi}^2\right)^2. \quad (C.10)$$

Finally, we must specify the behavior of the other independent quark fields $q_c, \bar{q}_c$ under chiral rotations. We define

$$q_c' = (\mathcal{V} \dagger)^T q_c, \quad \bar{q}_c' = \bar{q}_c \mathcal{V}^T, \quad \mathcal{V} \in SU_V(2), \quad (C.11)$$

$$q_c' = U^T q_c, \quad \bar{q}_c' = \bar{q}_c U^T, \quad U \in SU_A(2) \quad (C.12)$$

Using (C.4) and (C.11), one readily shows that the kinetic energy for the charge conjugate quark fields, i.e. $\bar{q}_c \tilde{S}_0^{-1} q_c$, is chiral invariant.
C.2 Chiral behavior of diquark fields

In this subsection, we study the behavior of the composite field $\bar{q}_c \epsilon \Gamma q$ under chiral transformations (C.4,C.6) and (C.11,C.12), respectively, where $\Gamma$ is a Dirac matrix which will be specified when needed, and $\epsilon_{ik}$ is the totally anti-symmetric tensor acting in flavor space only. Note that here $q, \bar{q}, q_c, \bar{q}_c$ are treated as independent fields with the transformation properties as given above, and that one must not use the classical equation of motion, which e.g. yields the relation $q_c = C(q^\dagger)^T$. In the Minkowskian formulation there would be no difference, since chiral symmetry is manifest at classical level. As explained above, this is not the case for the Euclidean formulation.

It is convenient to introduce left- and right-handed projectors by

$$P_L = \frac{1}{2}(1 + \gamma_5), \quad P_R = \frac{1}{2}(1 - \gamma_5), \quad P_L^T = P_L, \quad P_R^T = P_R. \quad (C.13)$$

The chiral matrices $V \in SU_V(2)$ (C.2) and $U \in SU_A(2)$ (C.5) can be written as

$$V = V \times 1_D, \quad U = U \times P_L + U^\dagger \times P_R, \quad (C.14)$$

where $U$ and $V$ are matrices with iso-spin indices only.

Let us study chiral vector transformations first. In this case, we find

$$(\bar{q}_c \epsilon \Gamma q)' = \bar{q}_c V^T \epsilon \Gamma V q = \bar{q}_c \epsilon \Gamma q, \quad (C.15)$$

where we have used

$$\epsilon_{ik} V_{il} V_{km} = \epsilon_{lm}. \quad (C.16)$$

The diquark composite is invariant under chiral vector transformations.

In the case of chiral axial-vector transformations, one obtains

$$(\bar{q}_c \epsilon \Gamma q)' = \bar{q}_c U^T \epsilon \Gamma U q = \bar{q}_c \left(U^T P_L + \left(U^\dagger\right)^T P_R\right) \epsilon \Gamma \left(U P_L + U^\dagger P_R\right) q. \quad (C.17)$$

In the case of an axial ($\Gamma = 1$) or scalar ($\Gamma = \gamma_5$) diquark composite, we find that the axial-vector chiral symmetry is respected, i.e.

$$(\bar{q}_c \epsilon \Gamma q)' = \bar{q}_c U^T \epsilon U P_L q + \bar{q}_c \left(U^\dagger\right)^T \epsilon U^\dagger P_R q. \quad (C.18)$$

Using (C.16), the invariance under axial-vector chiral transformations becomes obvious. In the case of an axial- vector ($\Gamma = \gamma^\mu$), the diquark composite is not invariant under chiral axial-vector transformations, i.e.

$$(\bar{q}_c \epsilon \gamma^\mu q)' = \bar{q}_c U^T \epsilon U^\dagger \gamma^\mu P_R q + \bar{q}_c \left(U^\dagger\right)^T \epsilon \gamma^\mu U P_L q. \quad (C.19)$$

The same is true for vector diquark composites ($\Gamma = \gamma^\mu \gamma_5$).
D Collective field approach to quark dynamics

D.1 Fierz transformation of the current-current interaction

As emphasized in numerous places in the main text, the dynamics in the $V_0$ channel associated with the ISB effect is not dominantly controlled by the exchange of one dressed gluon. However in the literature, analysis of the QCD phase structure involving color superconductivity are made with one-gluon exchange four-Fermi interactions. In this appendix, we write down the various terms that result when the current-current interaction coming from one-gluon exchange is Fierzed. In doing this, we shall make use of the relations

\[
T_{\alpha \beta}^A T_{\gamma \delta}^A = \frac{1}{2} \left( \delta_{\beta \gamma} \delta_{\alpha \delta} - \frac{1}{N} \delta_{\alpha \beta} \delta_{\gamma \delta} \right), \tag{D.1}
\]

\[
\delta_{ik} \delta_{lm} = \frac{1}{N_f} \delta_{lk} \delta_{im} + \frac{1}{N_f} \tau_{a}^{ik} \tau_{a}^{lm}, \tag{D.2}
\]

where $T^A$ are $SU(N = 3)$ color matrices with the proper normalization, i.e. $\text{tr}\{T^A T^B\} = \delta^{AB}/2$, and $\tau^a$ are the $SU(N_f)$ matrices that act in flavorspace. These matrices are normalized according $\text{tr}\{\tau^a \tau^b\} = N_f \delta^{ab}$ and coincide with the Pauli matrices in the case of a $SU(2)$ flavor group. The effective quark interaction in the soft momentum limit gives rise to the contact interaction (17). Let us concentrate on the first term of the right-hand side, i.e.

\[
\bar{q}^{a} i \delta_{ik} T_{\alpha \beta}^a \gamma^{\mu} q_{k} \bar{q}^{\gamma} \delta_{lm} T_{\gamma \delta}^a \gamma^{\mu} q_{m}. \tag{D.3}
\]

Lorentz indices are not explicitly presented. Replacing $T^A \times T^A$ in (D.3) with the help of (D.1), the second expression on the right hand side of (D.1) gives rise to the quark interaction in the vector channel, i.e.

\[
- \frac{1}{2N} \bar{q} 1_F 1_c \gamma^\mu q \bar{q} 1_F 1_c \gamma^\mu q. \tag{D.4}
\]

Rearranging the product of the $\gamma$ matrices, i.e.

\[
\gamma^\mu_{ik} \gamma^\mu_{lm} = - l_{ik} 1_{im} + (\gamma_5)_{ik} (\gamma_5)_{im} \tag{D.5}
\]

\[
- \frac{1}{2} \gamma^\mu_{lk} \gamma^\mu_{im} - \frac{1}{2} (\gamma^\mu \gamma_5)_{lk} (\gamma^\mu \gamma_5)_{im},
\]

and using (D.2), the first expression on the right hand side of (D.1) gives rise to an additional interaction in the vector channel. Taking into account a factor $(-1)$ which arises from the anti-commuting Grassmann fields $q$ and $\bar{q}$, the vector interaction is given by

\[
\left[ \frac{1}{2} \frac{1}{N_f} \frac{1}{2} - \frac{1}{2N} \right] \bar{q} 1_F 1_c \gamma^\mu q \bar{q} 1_F 1_c \gamma^\mu q. \tag{D.6}
\]
One easily verifies that (D.6) also holds for the one flavor case $N_f = 1$. We observe that the interaction in this channel changes sign if the number of flavors is increased from one to two: It is repulsive for one flavor but becomes attractive for two or more flavors if one takes three colors. It is interesting to note that it is repulsive for any number of flavors in the large $N$ limit. Since the large $N$ limit is believed to give a qualitatively correct description of hadron interactions, it is in this limit that the *known* repulsion between two nucleons in the isoscalar vector channel (i.e., the $\omega$ exchange channel) could be understood from the point of view of QCD variables. In our work we are exploiting the fact that the gluonic four-quark interaction is attractive for $N_f = 2$ and $N = 3$. But as stressed this is not the whole story because of the possible contribution coming from the collective degrees of freedom as discussed in [14]. Even if the gluonic four-quark interaction were repulsive to start with, as long as it is not too repulsive, it could be overpowered at some density by the attraction due to the “dropping” $V_0$ mode as the chiral phase transition is approached. This would be the scenario if the chiral transition mechanism in terms of hadronic variables discussed in [14] were realized.

The first expression on the right hand side of (D.1) also induces interactions in the scalar and the pionic channel, respectively, i.e.

$$
\frac{1}{2N_f} \bar{q} \gamma^\mu F_{1c} q \bar{q} \gamma^\mu F_{1c} q - \frac{1}{2N_f} \bar{q} \gamma^5 \tau^a F_{1c} q \bar{q} \gamma^5 \tau^a F_{1c} q .
$$

(D.7)

Note that this sign of the scalar interaction is crucial for quark condensation.

In the following, we specialize to the two flavor case, i.e. $N_f = 2$. The gluon induced interaction that involves charge conjugated quark fields is

$$
(\bar{q}_c)^{\alpha} \delta_{rl} \left( T^A_{\alpha\beta} \right)^T \gamma^\mu (q_c)^\beta \bar{q}^\gamma_i \delta_{ik} T^A_{\gamma\delta} \gamma^\mu q^\delta_k .
$$

(D.8)

We wish to estimate the strength this contact interaction assigns to the quark interaction (9) in the (axial) vector diquark channel. For this purpose, first note that

$$
\left( T^A \right)^T_{\alpha\beta} T^A_{\gamma\delta} = T^A_{\beta\alpha} T^A_{\gamma\delta} = \frac{1}{2N} \epsilon_{A\alpha\delta} \epsilon_{A\gamma\beta} + \ldots ,
$$

(D.9)

$$
\hat{\gamma}^\mu_{rl} \hat{\gamma}^\mu_{ik} = - \frac{1}{2} \gamma^\nu_{il} \hat{\gamma}^\nu_{rk} - \frac{1}{2} \left( \gamma^\nu \gamma_5 \right)_{il} \left( \gamma^\nu \gamma_5 \right)_{rk} - 1_{il} 1_{rk} + \left( \gamma^5 \right)_{il} \left( \gamma^5 \right)_{rk} + \ldots ,
$$

(D.10)

where we used (D.1) to derive (D.9). The (axial) vector diquark composite field which we investigated in section 4 is completely anti-symmetric in flavor space. Noting that $\epsilon_{ik} = -i \tau^{a=2}_{ik}$ and using (D.2), one finds

$$
\delta_{rl} \delta_{ik} = - \frac{1}{2} \epsilon_{il} \epsilon_{rk} + \ldots .
$$

(D.12)
Combining (D.9, D.10, D.12) and taking into account the additional factor \((-1)\) from the interchange of Grassmann fields, we finally obtain the desired result

\[- \frac{1}{8N} (\bar{q}_c)^\alpha_i \epsilon_{ik} \epsilon^{\gamma\alpha\beta} \gamma^\mu (\gamma_5) q_k^\beta \bar{q}_l^\kappa \epsilon_{lm} \epsilon^{\gamma\kappa\omega} \gamma^\mu (\gamma_5) (q_c)^\omega_m \]  

\[- \frac{1}{4N} (\bar{q}_c)^\alpha_i \epsilon_{ik} \epsilon^{\gamma\alpha\beta} q_k^\beta \bar{q}_l^\kappa \epsilon_{lm} \epsilon^{\gamma\kappa\omega} (q_c)^\omega_m \]  

\[+ \frac{1}{4N} (\bar{q}_c)^\alpha_i \epsilon_{ik} \epsilon^{\gamma\alpha\beta} \gamma_5 q_k^\beta \bar{q}_l^\kappa \epsilon_{lm} \epsilon^{\gamma\kappa\omega} \gamma_5 (q_c)^\omega_m . \]

D.2 The Fierz rearrangements and approximations

There are a variety of Fierz rearrangements that can be made on eq.(D.3) that could lead to different results depending on approximations made in the calculations. The expressions given above (eqs. (D.6) and (D.7)) represent one such rearrangement and are exact as they stand. Note that they involve no color generators and hence can in certain approximations be completely saturated by color singlet meson fields. An alternative rearrangement was used by Vogel et al [31] wherein even the second term of (D.1) is “Fierzed,” thereby obtaining an expression that contains terms involving the color generators. This is also exact. Thus in the operator form, it is identical to our Fierzed form. However the complete saturation of this form in the same approximation as with our expressions must require colored states in contrast to our formulas.

An exact field theoretic treatment of the four-Fermi interaction (D.3) should of course yield the same result whether one uses our expressions or those of ref.[31]. However if one does approximations in treating this effective four-Fermi interaction, there is no reason why one should expect the same result. Indeed if one does the conventional mean-field approximation on the two forms, one sees immediately that the results would be different, even qualitatively in some cases. Thus a comparison of the operators of the same form (e.g., same quantum numbers) would be completely meaningless even though in perturbation theory, one should get the same result (e.g., the quark propagator) order by order. In the next subsection, we describe the strategy – or approximations – that we adopt in handling the effective four-Fermi interaction that arises when gluon-mediated interactions flow to the Fermi surface. We admit that we have no rigorous proof that our approximation is the best for the problem but we believe that it catches the essence of the physics we are developing in this paper.

D.3 The approximation of weakly coupled mesons

Since the four-Fermi interaction (D.3) is an effective local Lagrangian, we would like to work with physical meson fields and approach the phase transition in terms
of hadronic variables. For this, we find it convenient to set up an approximation scheme by introducing auxiliary fields representing quark anti-quark collective fields (bosonization). In order to make our argument as general as possible, consider the general case of the nonlocal form

\[ S_I := \bar{q}_i \Gamma^A_{ik} q_k G^{AB} \bar{q}_l \Gamma^B_{lm} q_m , \]  

where \( k \) is a master index representing space-time as well as internal degrees of freedom. Calculating e.g. the quark self energy in perturbation theory with respect to \( G^{AB} \), the \( s \)- and \( t \)-channel contractions contribute with equal weight and give rise (at leading order) to the Hartree and Fock contribution as depicted in figure 5a.

Now such nonperturbative phenomena as quark condensation or dynamical mass generation take place in a strong-coupling regime which cannot be accessed by perturbation theory. To describe such phenomena, it is more advantageous to rewrite the interaction (D.14) by means of auxiliary meson fields, i.e.

\[ \exp\{-S_I\} = \int \mathcal{D}\Sigma \exp\{-S_M\} , \]  

\[ S_M = \frac{1}{2} \sum_{ik} \left( M^{-1} \right)_{ik,lm} \Sigma_{lm} + i \bar{q}_l q_m \Sigma_{lm} , \]  

\[ M_{ik,lm} = 2 \Gamma^A_{ik} G^{AB} \Gamma^B_{lm} \ldots \]  

The complete partition function can then be written in the form

\[ Z = \int \mathcal{D}q \mathcal{D}\bar{q} \mathcal{D}\Sigma \exp\left\{ -\bar{q}_i \left( S_0^{-1} \right)_{ik} q_k - S_M \right\} , \]
where \( S_0^{-1} = i\partial - m \) is the tree-level inverse quark propagator. Non-perturbative effects can be addressed, by integrating out the quark fields and treating the functional integral over the “meson” field \( \Sigma \) in a semi-classical approximation. Expanding the action of the effective meson theory up to second order in the meson field \( \Sigma = \sigma + \Sigma_0 \) yields

\[
Z = \int \mathcal{D}\sigma \exp\{-S_{\text{eff}}\}, \tag{D.19}
\]

\[
S_{\text{eff}} = \frac{1}{2} \sigma_{ik} \left( M^{-1} + \Pi \right)_{ik,lm} \sigma_{lm} - S_{ml} \sigma_{lm} - \ln S^{-1} + O(\sigma^3), \tag{D.20}
\]

where \( S_{ml} \) is the “constituent” quark propagator with \( \Sigma_0 \) as self-energy, i.e.

\[
S^{-1} = i\partial - m + i\Sigma_0, \tag{D.21}
\]

and

\[
\Pi_{ik,lm} = S_{kl} S_{mi}. \tag{D.22}
\]

If one chooses in the bosonized action \( \Sigma_0 \) in such a way that the linear terms in \( \sigma \) vanish, the quark self-energy will be given by a ladder summation of the Hartree terms as shown in the upper graph of figure 5b. The Fock terms of the perturbative expansion are generated, if Gaussian fluctuations \( \sigma \) around the mean field \( \Sigma_0 \) are included as in the lower graph of figure 5b.

There are several options available depending on how one bosonizes the interaction (D.3).

1. One may choose to bosonize eq. (D.3) directly and then do a semi-classical approximation in the mesonic functional integral. This procedure would however be a poor approximation for the physics we are interested in. The reason is that due to vanishing Hartree terms in this case, at the classical level, it would yield a trivial result for the quark self-energy, whereas the nontrivial phenomena (such as quark condensation, constituent quark mass generation etc.) would occur at semi-classical level or higher.

2. Suppose one Fierz transforms (partially), thereby rearranging the interaction terms (partially). In this case, the role of the Fock and Hartree terms involved will be (partially) interchanged. Subsequent bosonization and semi-classical approximation of the mesonic functional integral would correspond to a particular summation of selected perturbative diagrams. Now the crucial point is that different choices of the Fierz rearrangement would differ in the subset of Fock and Hartree diagrams which are summed up. Such different choices would correspond to doing different physics. The examples are the choice adopted in this paper and that adopted in [31].
• The first case to consider is the complete Fierz rearrangement of the interaction (D.3) studied in [31]. This choice would reproduce the important Fock terms at tree level. Doing (weak coupling) perturbation theory would bring minor corrections from the meson fluctuations. Although the semi-classical expansion induced by this choice of Fierz rearrangement may be rapidly converging in weak-coupling regime, this must not be the case for the coupling strength where quark condensation effects are expected to come into play. Indeed, an evidence for this shortcoming can be found in [18]: the Fierz rearrangement of [31] implies a ratio of the interaction strengths of scalar and vector channel, respectively, of $G_s/G_v = 2$ whereas a ratio of $G_s/G_v \ll 1$ seems to be needed to reproduce the $\pi$ and $\rho$ meson masses [18].

• Consider next the Fierz rearrangement adopted in this paper. We argue that doing mean-field approximation with this form corresponds to doing the large $N_c, N_f$ approximation, i.e.

$$N_f = \nu N_c, \quad \nu = \text{fixed const}, \quad N_c \to \infty. \quad \text{(D.23)}$$

In this limit, the effect of meson fluctuations will be suppressed\(^4\). Note that the limit (D.23) does not spoil the property of asymptotic freedom of Yang-Mills theory for $\nu < 11/2$ (note that one has $\nu = 1$ for QCD with three quark flavors). In the particular case (D.23), the four quark interaction strength is solely proportional to $1/N_c$. Bosonization of these interaction terms gives rise to large meson masses, i.e. proportional to $N_c$. Mesonic kinetic terms are produced by the logarithm of the quark determinant. These kinetic terms therefore also acquire a large pre-factor of order $N_c$. These considerations tell us that, in our case of the Fierz rearrangement, meson fluctuations are suppressed by a factor of $1/N_c$ thereby justifying our semi-classical expansion in the particular limit (D.23).

For practical applications ($N_c = 2$), it is not clear whether a truncation of the large $N_c, N_f$ expansion (D.23) at classical level is a truly good approximation. In particular at zero density, a higher-order approximation may be needed to generate the masses for the vector mesons. Rather than justifying the large $N_c, N_f$ expansion for the case of interest, e.g. $N_c = 3$, $N_f = 2$, we here adopt a more coarse-grained point of view and argue that in view of the results in [14] (see discussion in section 5) it seems worthwhile from a phenomenological point of view to introduce in the

\[\text{To those who are familiar with Walecka mean field theory of nuclear matter, this procedure is easily comprehensible. One could interpret this in terms of Landau's Fermi-liquid fixed point theory with the instability associated with the ISB and color superconductivity interpreted as the break-down of the Fermi-liquid ground state. See [32] for a discussion on this matter.}\]
case of finite densities the mean-field $V_0$ and leave the strength of the $V_0$ fluctuations as a variable parameter. The goal of our paper is to explore the behavior of the finite density quark matter by varying the strength in this particular channel rather than relying on the strength provided by the effective four-Fermi interaction of one-gluon exchange quantum numbers (D.3).

**E Fermion determinants**

Let us first quote two useful results, which will be used below, i.e.

\[
\det_D (\alpha \vec{k} + \beta) = \left[ \alpha^2 k^2 + \beta^2 \right]^2, \\
\det_D \left( \epsilon \gamma^0 \vec{k}^\gamma + \alpha k^- + \beta \right) = \left[ (\epsilon k_0^+ - \alpha k_0^-)^2 + (\epsilon + \alpha)^2 \vec{k}^2 + \beta^2 \right]^2,
\]

where $\alpha, \beta, \epsilon$ are constants, and $k^\pm := (k_0^\pm, \vec{k})^T$, $k_0^\pm = k_0 \pm iv$. The subscript $D$ indicates that the determinant is over Dirac indices. The equations (E.1,E.2) can be readily checked e.g. by resorting to the explicit representation of the $\gamma$–matrices (B.4).

The determinants of interest possess the structure

\[
T := \det_{cDCF} \left( \begin{array}{cc} (\vec{k}^+ - \sigma) 1_C 1_F & \kappa \Delta \hat{\gamma} \\
\kappa \Delta & (\vec{k}^- - \sigma) 1_C 1_F \end{array} \right)
\]

where the building blocks of the $2 \times 2$ matrix in the $q$ and $q_c$ space carry Dirac, flavor and color indices. The subscript $cDCF$ indicates the determinant extends over either space, i.e. $c$ charge conjugated space, $D$ space of Dirac indices, $C$ color and $F$ flavor space. Exploiting the simple structure of the diagonal entries, $T$ can be reduced to

\[
T = (k_1^2 + \sigma^2)^{12} \det_{DCF} \left[ \vec{k}^- - \sigma - \kappa^2 \Delta \left( \vec{k}^+ - \sigma \right)^{-1} \Delta \hat{\gamma} \right]
\]

\[
= (k_1^2 + \sigma^2)^{12} \det_{DCF} \left[ \vec{k}^- - \sigma + \frac{\kappa^2 \delta^2}{k_1^2 + \sigma^2} \Gamma (\vec{k}^+ + \sigma) \hat{\Gamma} \right]
\]

where we have used $\Delta = \delta \Gamma$. Let us specify the color, Dirac and flavor structure of the diquark vertex, i.e.

\[
\Gamma_{ik}^{\alpha \beta} = \epsilon^{\alpha \beta \gamma} \epsilon_{ik} \gamma_D,
\]

where $\alpha, \beta = 1 \ldots 3$ are color indices, and $i, k = 1, 2$ are flavor indices, respectively. Dirac indices are not explicitly shown. The Dirac structure will be specified below. Since the propagator $(\vec{k}^+ - \sigma)^{-1}$ is diagonal in color and flavor space, one obtains

\[
\Gamma (\vec{k}^+ - \sigma)^{-1} \hat{\Gamma} = \gamma_D (\vec{k}^+ - \sigma)^{-1} \gamma_D^\dagger \times 1_F \times P_C,
\]

\[
34
\]
where $1_F$ is the unit matrix in $2 \times 2$ flavor space, and $P_C = \text{diag}(1, 1, 0)$ is a projector in color space. We therefore obtain

$$T = \left( k_+^2 + \sigma^2 \right)^{12} \left( k_-^2 + \sigma^2 \right)^4 \left\{ \det_D \left[ k^- - \sigma + \frac{k^2 \delta^2}{k_+^2 + \sigma^2} \gamma_D \left( k^+ + \sigma \right) \gamma_D^\dagger \right] \right\}^4 .$$ (E.7)

Investigating the various types diquark composites, i.e.

- $\gamma_D = \gamma_5$ (scalar)
- $\gamma_D = 1_D$ (pseudo-scalar)
- $\gamma_D = \gamma^0 \gamma_5$ (vector)
- $\gamma_D = \gamma^0$ (axial-vector),

it is straightforward with the help of (E.1,E.2) to calculate the final result for the fermion determinants (25), (27), (30), (31).
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