Rabi oscillation in a damped rotating magnetic field: 
A path integral approach

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Abstract. In this paper we consider a spin 1/2 particle interacting with a damped rotating magnetic field using path integral formalism. The propagator is first of all written in the standard form by replacing the spin by two fermionic oscillators via the Schwinger’s model; then it is determined exactly thanks to the introduction of a particular rotations in coherent state space which has eliminated the rotation angle of the magnetic field and has simplified the Hamiltonian of the considered system. Thus, the Rabi formula are deduced.

1. Introduction
Up to now, a whole class of potentials have been treated successfully within the path-integral formalism, thanks to the use of certain transformations [1]. However, it is known that the most relativistic interactions are those where the spin is taken into account which is a very useful and very important notion in physics. From a practical point of view, the explicit calculus of propagators for such interactions by the path-integral formalism, has been discussed by many authors previously ([2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]).

In this paper we are devoted to this type of interaction; by considering a problem treats according to usual quantum mechanics[16]. It acts of a spin 1/2 which interacts with a damped rotating magnetic field.

\[
\mathbf{B}(t) = \left( B_1 e^{-\lambda t} \sin \omega t, \left( \frac{\omega}{g} - B_0 \right) e^{-\lambda t} - \frac{\omega}{g}, B_1 e^{-\lambda t} \cos \omega t \right)
\]

Its dynamics is described by the Hamiltonian

\[
H = -\frac{g}{2} \sigma \mathbf{B}
\]

where \( g \) is the gyromagnetic ratio. Then the Hamiltonian become

\[
H = -\omega_1 e^{-\lambda t} \sin \omega t \sigma_x + \left( \left( \frac{\omega}{g} - B_0 \right) e^{-\lambda t} - \frac{\omega}{g} \right) \sigma_y - \omega_1 e^{-\lambda t} \cos \omega t \sigma_z
\]

where we have put \( B_0 = \frac{\omega_0}{g} \) and \( B_1 = \frac{2\omega_1}{g} \). The Pauli matrices are the following:

\[
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]
Considering this problem by the path integral approach, our motivation is the following. We show that for interaction with the coupling of spin-field type, the propagator is first, by construction, written in the standard form
\[ \sum_{\text{path}} \exp \left( iS(\text{path}) / \hbar \right), \]
where \( S \) is the action that describes the system, where the discrete variable relative to spin being inserted as the (continuous) path using fermionic coherent states. The knowledge of the propagator is essential to the determination of physical quantities such as the Rabi formula which is the aim of this paper.

The paper is organized as follows. In section 2, we give some notations and the necessary spin coherent state path integral for spin \( \frac{1}{2} \) system for our further computations. In section 3, after setting up a path integral formalism for the propagator, we perform the direct calculations. The integration over the spin variables is easy to carry out thanks to simple transformations. The explicit result of the propagator is directly computed and the Rabi formula is then deduced. Finally, in section 4, we present our conclusions.

2. Coherent states formalism

Now, let us focus on some definitions, properties and notations needed for the further developments. As we are interested by the spin field interaction, we shall replace the Pauli matrices \( \sigma_i \) by a pair of fermionic operators \((u, d)\) known as Schwinger fermionic model of spin following the recipe:

\[
\sigma \rightarrow \left( u^\dagger, d^\dagger \right) \sigma \left( \begin{array}{c} u \\ d \end{array} \right) \tag{5}
\]

where the pair \((u, d)\) describes a two-dimensional fermionic oscillators.

Incidentally, the spin eigenstates \(|\uparrow\rangle\) and \(|\downarrow\rangle\) are generated from the fermionic vacuum state \(|0, 0\rangle\) by the action of the fermionic oscillators \(u^+\) and \(d^+\) following the relations

\[
u^+ |0, 0\rangle = |\uparrow\rangle \quad \text{and} \quad d^+ |0, 0\rangle = |\downarrow\rangle \tag{6}
\]

where the action of \(u\) and \(d\) on this vacuum state is given by the vanishing results

\[
u |0, 0\rangle = 0 \quad \text{and} \quad d |0, 0\rangle = 0 \tag{7}
\]

The pair of the fermionic oscillators \((u, d)\) and its adjoint \((u^+, d^+)\) satisfy the usual fermionic algebra defined by the following anticommutator relations:

\[
[u, u^+]_+ = 1, \quad [d, d^+]_+ = 1 \tag{8}
\]

where all other anticommutators vanish.

The notation \([A, B]_+\) stands for

\[
[A, B]_+ = AB + BA.
\]

Let us now introduce coherent states relative to this fermionic oscillators algebra. These states are generally defined as eigenvectors of the fermionic oscillators \(u\) and \(d\):

\[
u |\alpha, \beta\rangle = \alpha |\alpha, \beta\rangle, \quad d |\alpha, \beta\rangle = \beta |\alpha, \beta\rangle \tag{9}
\]

where \((\alpha, \beta)\) is a pair of Grassmann variables which are anticommuting with fermionic oscillators and with themselves, namely

\[
\begin{align*}
[\alpha, u]_+ &= [\alpha, u^+]_+ = [\alpha, d]_+ = [\alpha, d^+]_+ = 0 \\
[\beta, u]_+ &= [\beta, u^+]_+ = [\beta, d]_+ = [\beta, d^+]_+ = 0
\end{align*} \tag{10}
\]
and are commuting with vacuum states \(|0, 0\rangle\), \langle 0, 0|:

\[
\begin{align*}
\{ \alpha \mid 0, 0 \rangle = \mid 0, 0 \rangle \alpha, & \quad \langle 0, 0 \mid \alpha = \alpha \langle 0, 0 \mid \\
\beta \mid 0, 0 \rangle = \mid 0, 0 \rangle \beta, & \quad \langle 0, 0 \mid \beta = \beta \langle 0, 0 \mid
\end{align*}
\]

(11)

The above definitions are equivalent to the fact that these states are generated from the vacuum state according to the following relation

\[
|\alpha, \beta\rangle = \exp\left(-\alpha u^+ - \beta d^+\right) \mid 0, 0 \rangle
\]

(12)

The main properties of these states are:

- the completeness relation

\[
\int d\alpha d\alpha' |\alpha\rangle \langle \alpha'\mid = 1
\]

(13)

- non-orthogonality

\[
\langle \alpha, \beta | \alpha', \beta'\rangle = e^{\alpha \alpha' + \beta \beta'}.
\]

(14)

3. Path integral formulation

At this stage we shall provide a path integral expression for the propagator for the Hamiltonian given by the expression (3). This can be readily done by exploiting the above model of the spin by which this Hamiltonian converts to the following fermionic form:

\[
H = -\omega_1 e^{-\lambda t} \sin \omega t(u^d d + d^u u) + \left(\left(\frac{\omega}{g} - B_0\right) e^{-\lambda t} - \frac{\omega}{g}\right) (\text{+iu}d + \text{id}\text{u})
\]

\[
-\omega_1 e^{-\lambda t} \cos \omega t(u^\dagger u - d^\dagger d)
\]

(15)

Moreover, it is convenient to choose the quantum state as \(|\alpha, \beta\rangle\) where \((\alpha, \beta)\) describes the spin variables. According to the habitual construction procedure of the path integral, we define the propagator as the matrix element of the evolution operator between the initial state \(|\alpha_i, \beta_i\rangle\) and final state \(|\alpha_f, \beta_f\rangle\)

\[
\mathbf{K}(\alpha_f, \beta_f; \alpha_i, \beta_i; T) = \langle \alpha_f, \beta_f | U(T) | \alpha_i, \beta_i \rangle
\]

(16)

where

\[
U(T) = \mathbf{T}_D \exp\left(-\frac{i}{\hbar} \int_0^T H(t) dt\right)
\]

(17)

and \(\mathbf{T}_D\) is the Dyson chronological operator.

To move to path integral representation, we first subdivide the time interval \([0, T]\) into \(N + 1\) intervals of length \(\varepsilon\), intermediate moments, by using the Trotter’s formula and we then introduce the projectors according to these intermediate instants \(N\), which are regularly distributes between 0 and \(T\) in (17), we obtain the discretized path integral form of the propagator

\[
\mathbf{K}(\alpha_f, \beta_f; \alpha_i, \beta_i; T) = \lim_{N \to \infty} \prod_{n=1}^N \int d\bar{\alpha}_n d\alpha_n d\bar{\beta}_n d\beta_n e^{-\bar{\alpha}_n \alpha_n - \bar{\beta}_n \beta_n}
\]

\[
\times \prod_{n=1}^{n=N+1} \exp\left\{\bar{\alpha}_n \alpha_{n-1} + \bar{\beta}_n \beta_{n-1} - i\varepsilon \left(\frac{\omega}{g} - B_0\right) e^{-\lambda t} \left(-i\bar{\alpha}_n \beta_{n-1} + i\bar{\beta}_n \alpha_{n-1}\right)\right\}
\]

\[
+ i\frac{\varepsilon}{2} \left(\frac{\omega}{g} - B_0\right) e^{-\lambda t} \left(-i\bar{\alpha}_n \beta_{n-1} + i\bar{\beta}_n \alpha_{n-1}\right)
\]

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\[ +i\omega_1 e^{-\lambda t_n} \left[ \cos \omega t_n \left( \alpha_n \alpha_{n-1} - \bar{\beta}_n \beta_{n-1} \right) + \sin \omega t_n \left( \alpha_n \beta_{n-1} + \bar{\beta}_n \alpha_{n-1} \right) \right] \] (18)

with \((\alpha_0, \beta_0) = (\alpha_t, \beta_t)\) and \((\bar{\alpha}_N, \bar{\beta}_N) = (\bar{\alpha}_f, \bar{\beta}_f)\). This last expression represents the path integral of the propagator which has been the purpose subject of previous papers ([9],[10]), and has the advantage that it permits us to perform explicitly some concrete calculations.

4. Calculation of the propagator

To begin, we first introduce new Grassmann variables via an unitary transformation in spin coherent space which eliminates the angle \(\omega t\) present in the expression of the magnetic field:

\[
\begin{align*}
\begin{cases}
(\alpha_n, \beta_n) &\mapsto (\eta_n, \xi_n) \\
(\alpha_n, \beta_n) &\mapsto (\eta_n, \xi_n) \quad \text{and} \quad (\alpha_n, \beta_n) = (\bar{\eta}_n, \bar{\xi}_n) e^{i\frac{\lambda t_n}{2} \sigma_y}
\end{cases}
\end{align*}
\] (19)

Then, it is easy to show that the measure and the infinitesimal action become respectively

\[
\prod_{n=1}^{N} \left( d\bar{\alpha}_n d\alpha_n d\beta_n d\bar{\beta}_n e^{-\alpha_n \alpha_{n-1} - \bar{\beta}_n \beta_{n-1}} \right) = \prod_{n=1}^{N} \left( d\bar{\eta}_n d\eta_n d\xi_n d\bar{\xi}_n e^{-\eta_n \eta_{n-1} - \xi_n \xi_{n-1}} \right)
\] (20)

\[
\bar{\alpha}_n \alpha_{n-1} + \bar{\beta}_n \beta_{n-1} = (\bar{\eta}_n, \bar{\xi}_n) \left[ 1 + i \frac{\omega}{2} \epsilon \sigma_y + O(\epsilon^2) \right] \left( \eta_{n-1}, \xi_{n-1} \right)
\] (21)

and

\[
i\epsilon \omega_1 e^{-\lambda t_n} \left[ \sin \omega t_n \left( \alpha_n \beta_{n-1} + \bar{\beta}_n \alpha_{n-1} \right) + \cos \omega t_n \left( \alpha_n \alpha_{n-1} - \bar{\beta}_n \beta_{n-1} \right) \right]
= i\epsilon \omega_1 e^{-\lambda t_n} \left( \bar{\eta}_n \eta_{n-1} - \bar{\xi}_n \xi_{n-1} \right).
\] (22)

The propagator in function of the new Grassmann variables \(\eta\) and \(\xi\), becomes

\[
K(\alpha_f, \beta_f; \alpha_t, \beta_t; T) = \lim_{N \to \infty} \prod_{n=1}^{N} \left( d\bar{\eta}_n d\eta_n d\xi_n d\bar{\xi}_n e^{-\bar{\eta}_n \eta_{n-1} - \bar{\xi}_n \xi_{n-1}} \right)
\]

\[
\prod_{n=1}^{N+1} \exp \left[ \bar{\eta}_n \eta_n + \bar{\xi}_n \xi_n + i \epsilon \omega_1 e^{-\lambda t_n} \left( \bar{\eta}_n \eta_{n-1} - \bar{\xi}_n \xi_{n-1} \right) \right]
\]

\[
+ i \frac{\epsilon}{2} e^{-\lambda t_n} (\omega - \omega_0) \left( \bar{\eta}_n \eta_{n-1} + i \xi_n \eta_{n-1} \right)
\] (23)

Now using the following transformation:

\[
\epsilon = e^{\lambda t_n} \tau \quad \text{with} \quad \tau = s_n - s_{n-1}
\] (24)

The propagator becomes

\[
K(\alpha_f, \beta_f; \alpha_t, \beta_t; T) = \lim_{N \to \infty} \prod_{n=1}^{N} \left( d\bar{\eta}_n d\eta_n d\xi_n d\bar{\xi}_n e^{-\bar{\eta}_n \eta_{n-1} - \bar{\xi}_n \xi_{n-1}} \right)
\]

\[
\times \prod_{n=1}^{N+1} \exp \left\{ \bar{\eta}_n \eta_n + \bar{\xi}_n \xi_n + i \tau \left( \bar{\eta}_n, \bar{\xi}_n \right) \left( \omega_1 \sigma_z + \frac{1}{2} (\omega - \omega_0) \sigma_y \right) \right\}
\] (25)
which can be rewritten in the following form

\[
K(\alpha_f, \beta_f; \alpha_i, \beta_i; T) = \lim_{N \to \infty} \prod_{n=1}^{N} \left( d\eta_n d\xi_n e^{-\eta_n \theta - \xi_n \bar{\eta}_n} \right) \prod_{n=1}^{N+1} \exp \left[ \bar{\eta}_n \eta_{n-1} \right] \\
+ \bar{\xi}_n \xi_{n-1} + i\tau \sqrt{\omega_1^2 + \frac{1}{4} (\omega - \omega_0)^2 \left( \bar{\eta}_n, \bar{\xi}_n \right)} \cos \theta \sigma_z + \sin \theta \sigma_y \left( \eta_{n-1} \bar{\xi}_{n-1} \right)
\]

with

\[
\cos \theta = \frac{2\omega_1}{\sqrt{4\omega_1^2 + (\omega - \omega_0)^2}}
\]

then we introduce new Grassmann variables \(\gamma, \delta\) via an unitary transformation in spin coherent state space defined by

\[
\begin{align*}
(\eta_n, \xi_n) &\quad \mapsto \left( \gamma_n, \delta_n \right) = e^{\gamma_n \sigma_z} \left( \eta_n \delta_n \right) \quad \text{and} \quad
(\bar{\eta}_n, \bar{\xi}_n) &\quad \mapsto \left( \bar{\gamma}_n, \bar{\delta}_n \right) = (\gamma_n, \delta_n) e^{-i\frac{\eta}{2} \sigma_x} 
\end{align*}
\]

The role of this unitary transformation is twofold. First, it eliminates the angle \(\theta\) present in the expression of the action, and secondly it diagonalizes the Hamiltonian. In effect the measure and the infinitesimal action become respectively

\[
\prod_{n=1}^{N} \left( d\eta_n d\xi_n e^{-\eta_n \theta - \xi_n \bar{\eta}_n} \right) = \prod_{n=1}^{N} \left( d\gamma_n d\delta_n e^{-\gamma_n \bar{\gamma}_n - \delta_n \bar{\delta}_n} \right)
\]

and

\[
ie \sqrt{\omega_1^2 + \frac{1}{4} (\omega - \omega_0)^2 \left( \bar{\eta}_n, \bar{\xi}_n \right)} \cos \theta \sigma_z + \sin \theta \sigma_y \left( \eta_{n-1} \bar{\xi}_{n-1} \right) = \\
ie \sqrt{\omega_1^2 + \frac{1}{4} (\omega - \omega_0)^2 \left( \bar{\gamma}_n, \bar{\delta}_n \right)} \sigma_z \left( \gamma_{n-1} \delta_{n-1} \right)
\]

which modify the expression (27) to the following form

\[
K(\gamma_f, \delta_f; \gamma_i, \delta_i; T) = \lim_{N \to \infty} \prod_{n=1}^{N} d\gamma_n d\delta_n e^{-\gamma_n \bar{\gamma}_n - \delta_n \bar{\delta}_n}
\]

Having finished with the diagonalization, we are left with the task of integrating over the Grassmann variables \(\gamma, \delta\). For this, it is useful to write again the propagator in an appropriate form.

We note by \(q_n = \left( \begin{array}{c} \gamma_n \\ \delta_n \end{array} \right)\) and \(\bar{q}_n = \left( \bar{\gamma}_n, \bar{\delta}_n \right),\) so the propagator (33) becomes

\[
K(\gamma_f, \delta_f; \gamma_i, \delta_i; T) = \lim_{N \to \infty} \int \prod_{n=1}^{N} d\bar{q}_n dq_n \exp \left\{ \sum_{n=1}^{N} -\bar{q}_n q_n + \right\}
\]
where $R(n)$ is a diagonal matrix

$$R(n) = \begin{pmatrix}
1 + i\epsilon\sqrt{\omega^2 + \frac{1}{4} (\omega - \omega_0)^2} & 0 \\
0 & 1 - i\epsilon\sqrt{\omega^2 + \frac{1}{4} (\omega - \omega_0)^2}
\end{pmatrix}$$

(34)

we note that the action can be re-arranged as follows

$$\sum_{n=1}^{n=N} -\bar{q}_n q_n + \sum_{n=1}^{n=N+1} \bar{q}_n R(n) q_{n-1} =$$

$$= \sum_{j,n=1}^{n=N} \left[ \bar{q}_j (\delta_{j,n} - R(n) \delta_{j,n+1}) q_n \right] + \bar{q}_{N+1} R(N+1) q_N + \bar{q}_1 R(1) q_0.$$  

(35)

In order to rewrite the propagator in an adequate form, we introduce the vectors

$$P = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}, \quad V = \begin{pmatrix} R(1) q_0 \\ \vdots \\ 0 \end{pmatrix}, \quad W^\dagger = \begin{pmatrix} 0 & \ldots & 0 & \bar{q}_{N+1} R(N+1) \end{pmatrix},$$

(36)

the propagator, thus, takes the following form:

$$K(\gamma_f, \delta_f; \gamma_i, \delta_i; T) = \int dP dP^\dagger \exp\left( -P^\dagger \mathcal{M} P + P^\dagger V + W^\dagger P \right)$$

(37)

where $\mathcal{M}$ is a matrix defined by

$$\mathcal{M}_{i,n} = \delta_{j,n} - R(n) \delta_{j,n+1} \quad n, j \in [1, N]$$

(38)

With the help of the following change

$$P \rightarrow P + \mathcal{M}^{-1} V \quad \text{and} \quad P^\dagger \rightarrow P^\dagger + W^\dagger \mathcal{M}^{-1}$$

(39)

the integral over the Grassmann variables is reduced to

$$K(\gamma_f, \delta_f; \gamma_i, \delta_i; T) = \det \mathcal{M} \cdot \exp(W^\dagger \mathcal{M}^{-1} V)$$

As $\det \mathcal{M}$ equals 1 and

$$W^\dagger \mathcal{M}^{-1} V = \lim_{N \to \infty} \left[ \bar{q}_j R(N+1) R(N) \cdots R(1) q_0 \right] = \bar{q}_f R(T) q_0,$$

(40)

thus the propagator reduce to following form

$$K(\gamma_f, \delta_f; \gamma_i, \delta_i; T) = \left( \gamma_i, \gamma_f; T \right) \mathcal{R}(T) \left( \begin{array}{c} \gamma_i \\ \delta_i \end{array} \right)$$

(41)

with

$$R(S) = e^{\frac{i}{2} S \sqrt{\omega^2 + (\omega - \omega_0)^2} \sigma_z}.$$  

(42)

Now we come back to the old Grassmann variables $(\alpha, \beta)$. So, the exact expression of the propagator concerning to our problem is the following

$$K(\alpha_f, \beta_f; \alpha_i, \beta_i; T) = \exp\left( \alpha_f, \beta_f \right) M(S) \left( \begin{array}{c} \alpha_i \\ \beta_i \end{array} \right)$$

(43)

where $M$ is a matrix defined by

$$M(S) = e^{-i\frac{\sigma_s}{2} \sigma_y} e^{i\frac{\epsilon}{2} \sigma_z} R(S) e^{-i\frac{\epsilon}{2} \sigma_z}$$

(44)
5. The transition probability

If, for convenience, we specify the initial and final states as the eigenstates of \( \sigma_y \), the transition amplitude from the initial state \( |m_i⟩ = |↑⟩_y = \frac{1}{\sqrt{2}} (|↑⟩ + i|↓⟩) \) to the final state \( |m_f⟩ = |↓⟩_y = \frac{1}{\sqrt{2}} (|↑⟩ - i|↓⟩) \) is related to \( K(f,i;T) \) by:

\[
K(\downarrow, ↑; T) = y \langle \downarrow | U(T) | ↑ \rangle_y
\]  

(45)

With the help of the completeness relations, this amplitude becomes

\[
K(\downarrow, ↑; T) = \int d\alpha_f d\beta_f d\alpha_i d\beta_i e^{-\pi_f \alpha_f - \pi_i \alpha_i - \beta_f \beta_i}
\]

\[
\times y \langle \downarrow | \alpha_f, \beta_f⟩ \langle \alpha_i, \beta_i | ↑ \rangle_y K(\alpha_f, \beta_f, \alpha_i, \beta_i; T).
\]

(46)

We substitute \( |↓⟩_y \) and \( |↑⟩_y \) in (46) then

\[
K(\downarrow, ↑; T) = \frac{1}{2} \int d\alpha_f d\beta_f d\alpha_i d\beta_i e^{-\alpha_i \alpha_f - \beta_i \beta_f}
\]

\[
\left\{ \langle \uparrow | \alpha_f, \beta_f⟩ \langle \alpha_i, \beta_i | \uparrow \rangle - \langle \downarrow | \alpha_f, \beta_f⟩ \langle \alpha_i, \beta_i | \downarrow \rangle +
\right.
\]

\[
i \left\{ \langle \downarrow | \alpha_f, \beta_f⟩ \langle \alpha_i, \beta_i | ↑ \rangle + \langle \downarrow | \alpha_f, \beta_f⟩ \langle \alpha_i, \beta_i | ↓ \rangle \right\}
\]

\[
\times \exp \left( \bar{\alpha}_f − \bar{\alpha}_i \right) \left( \begin{array}{cc}
M_{11}(T) & M_{12}(T) \\
M_{21}(T) & M_{22}(T)
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\alpha_i \\
\beta_i
\end{array} \right)
\]

(47)

where \( M_{ij}(T) \) are the elements of the matrix \( M(T) \) which can be written as sum of four terms

\[
K(\downarrow, ↑; T) = K_1(\downarrow, ↑; T) + K_2(\downarrow, ↑; T) + K_3(\downarrow, ↑; T) + K_3(\downarrow, ↑; T)
\]

(48)

We calculate for instance the first term as [17]:

\[
\langle \uparrow | \alpha_f, \beta_f⟩ = \alpha_f \quad , \quad \langle \alpha_i, \beta_i | \uparrow \rangle = \bar{\alpha}_i \quad \text{and} \quad \alpha_f \bar{\alpha}_i = e^{-\bar{\alpha}_i \alpha_f} - 1
\]

(49)

Thus

\[
K_1(\downarrow, ↑; T) = \frac{1}{2} \int d\alpha_f d\beta_f d\alpha_i d\beta_i e^{\pi_f \alpha_f + \beta_f \beta_f} \exp \left[ \alpha_i M_{11} + \bar{\alpha}_f M_{12} + \bar{\beta}_f M_{21} + \bar{\beta}_i M_{22} - \bar{\alpha}_i \alpha_f \right]
\]

\[
- \exp \left[ \alpha_i M_{11} + \bar{\alpha}_f M_{12} + \bar{\beta}_f M_{21} + \bar{\beta}_i M_{22} - \bar{\alpha}_i \alpha_f \right]
\]

(50)

and with the help of the four-component vector \( \nu \) is defined by

\[
\nu = \left( \begin{array}{c}
\alpha_i \\
\alpha_f \\
\beta_i \\
\beta_f
\end{array} \right), \quad \nu^\dagger = \left( \begin{array}{c}
\nu_i \\
\nu_f \\
\bar{\beta}_i \\
\bar{\beta}_f
\end{array} \right)
\]

(51)

Accordingly, the expression (50) is reduced to

\[
K_1(\downarrow, ↑; T) = \int d\nu^\dagger d\nu \left[ \exp \nu^\dagger G^\prime \nu - \exp \nu^\dagger G \nu \right]
\]

\[
= [\text{det} G^\prime - \text{det} G],
\]

(52)
where $G$ and $G'$ are the two following $4 \times 4$ matrices

$$
G = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
M_{11} & -1 & M_{11} & 0 \\
0 & 0 & -1 & 0 \\
M_{21} & 0 & M_{22} & -1
\end{pmatrix} \quad \text{and} \quad
G' = \begin{pmatrix}
-1 & -1 & 0 & 0 \\
M_{11} & -1 & M_{11} & 0 \\
0 & 0 & -1 & 0 \\
M_{21} & 0 & M_{22} & -1
\end{pmatrix}.
$$

A simple computation shows that $\det G' = 1 + M_{11}$ and $\det G = 1$. Consequently,

$$
K_1(\downarrow, \uparrow; T) = \frac{1}{2} M_{11}.
$$

In the same manner, we proceed for the remaining elements. The result takes the following expression

$$
K(\downarrow, \uparrow; T) = \frac{1}{2} [M_{11}(T) - M_{22}(T) + i (M_{12}(T) - M_{21}(T))]
$$

By simple calculus the transition probability is then given by

$$
P_{\downarrow \uparrow} = |K(\downarrow, \uparrow; T)|^2
$$

$$
P_{\downarrow \uparrow} = \frac{4\omega_1^2}{4\omega_1^2 + (\omega - \omega_0)^2} \sin^2 \frac{S(T)}{2} \sqrt{4\omega_1^2 + (\omega - \omega_0)^2}, \quad \text{with} \quad S(T) = \frac{1 - e^{-\lambda T}}{\lambda}
$$

Thus

$$
P_{\downarrow \uparrow} = \frac{4\omega_1^2}{4\omega_1^2 + (\omega - \omega_0)^2} \sin^2 \frac{1 - e^{-\lambda T}}{2\lambda} \sqrt{4\omega_1^2 + (\omega - \omega_0)^2}
$$

This formula is exactly the well-known Rabi formula given in the literature[16].

6. Conclusion

By using the formalism of the path integral and the fermionic coherent states approach, we are able to calculate the explicit expression of the propagator relative to spin $1/2$ interacting with damped rotating magnetic field. To treat the spin dynamics, we have used the Schwinger’s recipe which replaces the Pauli matrices by a pair of fermionic oscillators. The introduction of a particular rotations in coherent state space has eliminated the rotation angle of the magnetic field and has then been simplified somewhat the Hamiltonian of the considered system. As a consequence, we are able to integrate over the spin variables described by fermionic oscillators. The exactness of the result is displayed in the evaluation of the Rabi formula.

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