A Multilevel Approach for Stability Conditions in Fractional Time Diffusion Problems

I. K. Youssef and Adel Rashed A. Ali

1 Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt. E-mail (kaoud22@hotmail.com).
2 Department of Mathematics, College of Education for Pure Science (Ibn al-Haitham), University of Baghdad, Baghdad, Iraq.

*Corresponding author: adil.r.a@ihcoedu.uobaghdad.edu.iq

Abstract. The Caputo definition of fractional derivatives introduces solution to the difficulties appears in the numerical treatment of differential equations due its consistency in differentiating constant functions. In the same time the memory and hereditary behaviors of the time fractional order derivatives (TFODE) still common in all definitions of fractional derivatives. The use of properties of companion matrices appears in reformulating multilevel schemes as generalized two level schemes is employed with the Gerschgorin disc theorems to prove stability condition. Caputo fractional derivatives with finite difference representations is considered. Moreover the effect of using the inverse operator which transmit the memory and hereditary effects to other terms is examined. The theoretical results is applied to a numerical example. The calculated solution has a good agreement with the exact solution.

1. Introduction

The numerical treatment of the standard parabolic equation

\[ \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \]

with initial condition \( u(x,0) = g(x) \) and Dirichlet boundary conditions of the form \( u(0,t) = u(1,t) = 0 \), defined on the domain \( 0 < x < 1, 0 < t < T \), is the cornerstone in the numerical treatment of PDE's in general. Most of the characteristics as well as the difficulties of finite difference method and its common properties appear in this simple form.

The basic idea of the finite difference method depends on the replacement of the derivatives by functional values at different arguments. Accordingly, replacing the functional differential equations by an algebraic relation. The accuracy of the solutions obtained by the use of the finite difference method depends on the convergence, consistency and stability requirements of the corresponding discrete problem. Studying the stability of implicit as well as explicit schemes for equation (1) was the main topic in many publications. Lax equivalence theorem states that satisfaction of only two among the convergence, the consistency and the stability will guarantee the satisfaction of the third. In this work we focus on studying the stability. The importance of proving stability conditions appears in many scientific and economic situations rather than the reliability of solutions. Choosing large steps within the admissible range well reduce the storage requirements as well as the running time.
are different methods used in the stability treatment, Von Neumann, energy and matrix methods are standard techniques [1, 2].

Our main task is to obtain with simple straightforward, easy and realistic method the stability conditions of the explicit scheme of the fractional time counter part equation (1)

\[
\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x, t), \quad 0 < \alpha < 1
\]  

with initial condition \( u(x,0) = u_0(x) \) and Dirichlet boundary conditions of the form \( u(0,t) = u(1,t) = 0 \), defined on the domain \( 0 < x < 1, 0 < t < T \), where the fractional order time derivative is understood in the Caputo sense.

The correspondence with the classical multilevel schemes treated in Richtmyer and Morton [2] with the relations on the norm of Frobenius matrices (appears in the reformulation of multilevel schemes as block two level schemes) and moreover the well-known Gerschgorin disc theorems have been reemployed to introduce systematic treatment.

**Definition 1.1** The Caputo time fractional derivative of order \( \alpha > 0 \) of the function \( u(x,t) \) is defined by [3, 4]:

\[
\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n}u(x,s)ds
\]  

where \( \alpha \in (n - 1, n), n \in \mathbb{N} \). If \( \alpha \in \mathbb{N} \) then this will coincide with the classical partial derivative. Equation (2) have appeared in many applications in physics, continuum mechanics, signal processing, and electromagnetic. Also, many publications have mentioned in biology, chemistry and biochemistry, hydrology, medicine, and finance [3, 4]. The fractional order partial differential equations (FOPDEs) are used to model anomalous diffusion and Hamiltonian Chaos. These equations describe the asymptotic behavior of continuous time random walks. Stochastic solutions to the simplest governing equations are Levy motions, generalizing the Brownian motion solution to the classical diffusion equation. Fractional kinetic equations have proved particularly useful in the context of anomalous subdiffusion [5, 6].

The fractional derivative considers the memory and hereditary effects which is not the case of the classical integer derivative which considers only the local behavior. In this work we are interested in this point and its effects on the stability conditions of the explicit schemes. Moreover, the corresponding between the treatment in the stability of multilevel schemes in the integer case and the explicit schemes in the fractional order case have been considered.

Models described in the form of FOPDEs, tend to be more appropriate for the description of memorial and hereditary properties of various materials and processes than the traditional integer order models [7].

It is interesting to note that the FOPDEs is a generalization of the classical partial differential equations and the limiting prosses as the fractional order approaches the classical integer order must introduce the classical case \( 0 < \alpha < 1 \), [8].

It is well known that there is no analytical method that can be considered as a master method for solving PDEs the situation in FOPDEs is more complicated. Laplace and Fourier transform methods [9] have their limitation. Semianalytic methods like the series solution method, the Adomian decomposition method [10] suffer from the complicated integrations. Numerical methods became the most reliable treatment in solving many problems in PDEs due to the development in computer devices. The finite difference method is considered as one of the simplest numerical methods that can treat many different problems [1, 11].

A number of numerical methods have been developed to solve the time fractional diffusion equation with Dirichlet boundary conditions. Yuste and Acedo [12] proposed a procedure with a new Von Neumann-type stability analysis in one dimension using Grünwald approximation for time fractional derivative. Liu et al [8] proposed another stability analysis procedure using discrete non-Markovian random walk approximation for time fractional derivative. LI and XU propose a spectral method in both temporal and spatial discretization [13]. Meerschaert et al. [14] use finite difference
approximations for fractional advection-dispersion flow equations and other numerical methods with finite difference approximation to fractional derivative [15, 16, 17, 18] with Von Neumann and matrix methods to study the stability analysis and convergence of the methods.

In the finite difference method, the continuous domain is replaced by a discrete grid superimpose the domain under consideration and the derivatives are replaced by the corresponding differences of functional values obtaining algebraic equation at each grid point. Solutions obtained by the finite difference method must satisfy some tests of consistency, stability and convergence to be reliable.

Some authors prefer to write the time fractional diffusion equation in the form [5, 12]:

\[
\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) + \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} f(x, t)
\]  

(4)

This form appears to have many difficulties in the finite difference approximation because it includes the time derivatives in both sides even the derivatives in the left-hand side is of integer order. Any algorithm using a finite difference discretization of the time fractional derivative has to take into account its nonlocal structure, i.e. the computation of the solution at a time level requires information about the solution at all previous time levels, which means high storage requirement.

To deal with this issue, Ford and Simpson [19] and Diethelm and Freed [20], developed a numerical technique to reduce the computational cost of the solution using the so called “fixed memory principle” as described in Podlubny [4]. We will discuss and compare equation (2) and equation (4) with discretization of time fractional derivative by Caputo definition, formula (7), for both equations with use the Multilevel method to derive the stability conditions.

2. The Finite Difference Method

In the Finite difference method (FDM) every differential equation is approximated by a corresponding finite differences scheme. The domain \([0,1] \times [0,T]\) of the given parabolic equation is superimposed with a grid. The interval \([a, b]\) is divided into \(J\) subintervals with length \(\Delta x = h = \frac{1}{j}, x_j = jh,\) for \(j = 0,1,2,\cdots , J\) and define the time step \(\Delta t = \tau\) and \(t_n = n\tau\).

The explicit scheme corresponding to equation (1) can be written in the form [2, 23]

\[
u_{j}^{n+1} = ru_{j}^{n} + (1 - 2r)u_{j}^{n} + ru_{j+1}^{n+1}
\]  

(5)

this scheme is consistent and stable for \(r = \frac{\tau}{h^2} \leq \frac{1}{2}\). To obtain the corresponding scheme for the fractional order equation (2) one must use the discretization of fractional order derivative, the inverse operator form equation (4) is also considered.

2.1. Discretization of Caputo Fractional derivatives

The time fractional derivative replaced by Caputo fractional derivative of order \(\alpha\), definition 1.1, and we use the following formulation [8]

\[
\frac{\partial^\alpha u(x_j,t_{n+1})}{\partial t^\alpha} = w_{\alpha}^{\tau} \sum_{k=0}^{n} b_k^{\alpha} (u_{j}^{n-k+1} - u_{j}^{n-k})
\]  

(6)

where \(w_{\alpha}^{\tau} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\) and \(b_k^{\alpha} = (k + 1)^{1-\alpha} - k^{1-\alpha}\), for \(k = 0,1,2,\cdots,n\), which can be rearranged in the form

\[
\frac{\partial^\alpha u(x_j,t_{n+1})}{\partial t^\alpha} = w_{\alpha}^{\tau} [u_{j}^{n+1} - \sum_{k=1}^{n} c_k^{\alpha} u_{j}^{n-k+1} - b_k^{\alpha} u_{j}^{0}]
\]  

(7)

with \(c_k^{\alpha} = b_k^{\alpha} - b_{k-1}^{\alpha}\).

Properties 1: the coefficients \(b_k^{\alpha}\) and \(c_k^{\alpha}\) having the following properties:

- \(c_k^{\alpha} = 2k^{1-\alpha} - (k - 1)^{1-\alpha} - (k + 1)^{1-\alpha}\), \(k = 1,2,3,\cdots\) And \(\sum_{k=1}^{\infty} c_k^{\alpha} = 1\).
- \(1 > 2 - 2^{1-\alpha} = c_1^{\alpha} > c_2^{\alpha} > c_3^{\alpha} > \cdots\), with \(\lim_{k \to \infty} c_k^{\alpha} = 0\).
• $1 = b_0^a > b_1^a > b_2^a > b_3^a > \ldots$, with $\lim_{k \to \infty} b_k^a = 0$.

Replacing the time derivative using equation (7) at the grid point $(x_i, t_n)$ and the space derivatives with the central difference approximation at the same grid point $(x_i, t_n)$, then the explicit scheme for the solution of equation (2) have the following difference equation

$$u_j^{n+1} = b_n^a u_j^0 + \sum_{k=1}^n c_k^a u_j^{n+1-k} + r_\alpha \Gamma(2-\alpha)\delta_x^2 u_j^n + \frac{1}{w_\alpha} f_j^n$$

### 3. Stability in Multilevel schemes

The Von Neumann technique for stability analysis uses for a two-time level finite difference scheme but for more than two-time level schemes we need to use the multilevel technique to check the stability conditions, for more details about this technique see [21, 22].

### 4. Discretization of Time Fractional Derivatives

Replacing the derivatives appears in differential equation by their finite difference approximations one obtains a corresponding scheme. The scheme properties (consistency, stability and convergence) should be examined to obtain reliable results. The same approach is used in case of fractional derivatives. We consider the fractional time derivative in Caputo definition and study its finite difference approximations, also we use this approximation in the diffusion like equations (2) and (4). The amplification matrix described above can be obtained with the Von Neumann method and Multilevel finite difference technique to study the stability conditions of the fractional time finite difference scheme. Putting

$$n_j^x = \xi^n e^{i\beta jh}$$

where $i = \sqrt{-1}$ and $\beta$ is a real spatial wave number.

The explicit scheme (8) is conditionally stable and the stability condition is $r_\alpha \leq \frac{1-2^{-\alpha}}{\Gamma(2-\alpha)}$, Liu et al [8].

We use the multilevel approach and obtain the same stability condition in the next theorem 4.1. The condition is depending on $\alpha$, figure 1 (a).

**Figure 1**: the stability condition on $r_\alpha$, left (a) for equation (8) and right (b) for equation (22)

**Theorem 4.1**: The fractional explicit scheme (8) is conditionally stable and the stability condition is

$$r_\alpha \leq \frac{1-2^{-\alpha}}{\Gamma(2-\alpha)}$$

**Proof**: The scheme (8) is a multilevel scheme and can be rewritten

$$u_j^{n+1} = [c_1^a + r_\alpha \Gamma(2-\alpha)\delta_x^2]u_j^0 + \sum_{k=2}^n c_k^a u_j^{n+1-k} + b_n^a u_j^0 + \frac{1}{w_\alpha} f_j^n$$

(10)
then the multilevel amplification matrix \( C \) can be defined by square block matrix of order \((n + 1)\) and every element of \( C \) is of order \(1\):

\[
C = \begin{bmatrix}
(c_1^a + r_a \Gamma(2 - \alpha) \delta^2)I & (c_2^a)I & \ldots & (c_n^a)I & (b_n^a)I \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{bmatrix}
\]

for the amplification matrix Insert expression (9) in equation (7) then we get

\[
\xi^{n+1}e^{i\beta jh} = \xi^0 b_n^a e^{i\beta jh} + \sum_{k=1}^{n} c_k^a \xi^{n+1-k} e^{i\beta jh} + r_a \Gamma(2 - \alpha) \xi^n e^{i\beta(j+1)h} - 2 e^{i\beta jh} + e^{i\beta(j-1)h}
\]

Divided by \(e^{i\beta jh}\) and using the formula \((e^{i\theta} - 2 + e^{-i\theta}) = -4\sin^2\frac{\theta}{2}\) to get

\[
\xi^{n+1} = \xi^0 b_n^a + \sum_{k=1}^{n} c_k^a \xi^{n+1-k} + r_a \Gamma(2 - \alpha) \xi^n \left[-4\sin^2\frac{\beta h}{2}\right]
\]

From Gerschgorin theorem for estimating the eigenvalues of any matrix \[24\], all rows of matrix \(M\) gives eigenvalues lies in the union of unit discs centered at \((0,0)\) in the complex plane except those corresponding to the first block. For each row of the first block one can see that the corresponding eigenvalue satisfies

\[
|\lambda| \leq \left|c_1^a - 4r_a \Gamma(2 - \alpha) \sin^2\frac{\beta h}{2}\right| + \sum_{k=2}^{n} |c_k^a| + |b_n^a|
\]

by the properties we have \(c_k^a > 0, b_n^a > 0\), and \(\sum_{k=2}^{n} c_k^a = 2^{1-\alpha} - 1 - b_n^a\), this lead to

\[
|\lambda| \leq \left|2 - 2^{1-\alpha} - 4r_a \Gamma(2 - \alpha) \sin^2\frac{\beta h}{2}\right| + 2^{1-\alpha} - 1
\]

if the right-hand inequality is less than or equal to one then \(|\lambda| \leq 1\), then we have

\[
-(2 - 2^{1-\alpha}) \leq \left(2 - 2^{1-\alpha} - 4r_a \Gamma(2 - \alpha) \sin^2\frac{\beta h}{2}\right) \leq 2 - 2^{1-\alpha}
\]

the right-hand inequality is satisfied and we need to calculate the condition on \(r_a\) to make the left-hand inequality satisfied, this lead to

\[
4r_a \Gamma(2 - \alpha) \sin^2\frac{\beta h}{2} \leq 4 - 2^{2-\alpha}
\]

then the stability condition is

\[
r_a \leq \frac{1-2^{1-\alpha}}{\Gamma(2-\alpha)}
\]
For equation (4) the time derivatives appears in both sides makes the finite difference representation is implicit and to obtain the explicit scheme and moreover the implicit, we introduce the weighted average approach to the time derivatives in the right hand side i.e we replace the term \( \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \) by its weighted approximation at the preceding time levels.

\[
\theta \left[ \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left( \frac{\partial^2 u(x,t_{n})}{\partial x^2} \right) \right] + (1 - \theta) \left[ \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left( \frac{\partial^2 u(x,t_{n-1})}{\partial x^2} \right) \right]
\]

Thus for \( \theta = 0 \), one obtains the explicit scheme obtained for equation (4) in the form

\[
\frac{u_j^{n+1} - u_j^n}{\tau} = \frac{1}{h^2} \Gamma(2(1-\alpha)) \left[ \delta_x^2 u_j^n - \sum_{k=1}^{n-1} c_k^{1-\alpha} \delta_x^2 u_j^{n-k} - b_{n-1}^{1-\alpha} \delta_x^2 u_j^n \right]
+ \frac{1}{\Gamma(2(1-\alpha))} \sum_{k=0}^{n-1} b_k^{1-\alpha} \left[ f(x_j, t_{n-k}) - f(x_j, t_{n-1-k}) \right]
\]

This fractional explicit scheme is conditionally stable and the stability condition is \( r_\alpha \leq \frac{\Gamma(1+\alpha)}{4} \), theorem 4.2. It is apparent that the condition is depend on \( \alpha \) the fractional order of the time derivative as shown in figure 1 (b), this is more convenient and includes the integer case.

**Theorem 4.2:** The fractional explicit scheme (22) is conditionally stable and the stability condition is \( r_\alpha \leq \frac{\Gamma(1+\alpha)}{4} \).

**Proof.** The scheme (22) is a multilevel scheme and can be rewritten

\[
u_j^{n+1} = u_j^n + \mu \left[ \delta_x^2 u_j^n - \sum_{k=1}^{n-1} c_k^{1-\alpha} \delta_x^2 u_j^{n-k} - b_{n-1}^{1-\alpha} \delta_x^2 u_j^n \right] + \frac{r_\alpha}{\Gamma(2-\gamma)} \left[ f_j^n - \sum_{k=1}^{n-1} c_k^{\gamma} f_j^{n-k} - b_{n-1}^{\gamma} f_j^0 \right]
\]

where \( \gamma = 1 - \alpha \), and \( \mu = \frac{r_\alpha}{\Gamma(1+\alpha)} \), then the multilevel amplification matrix \( C \) can be defined by square block matrix of order \( (n + 1) \) and every element of \( C \) is of order \( I - 1 \):

\[
C = \begin{bmatrix}
(1 + \mu \delta_x^2)I & (-\mu c_1^\gamma \delta_x^2)I & (-\mu c_2^\gamma \delta_x^2)I & \cdots & (-\mu c_{n-1}^\gamma \delta_x^2)I & (b_{n-1}^\gamma \delta_x^2)I \\
I & 0 & 0 & \cdots & 0 & 0 \\
0 & I & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I & 0
\end{bmatrix}
\]

for the amplification matrix insert expression (9) in equation (23) and divide by \( e^{i\beta h} \) to get

\[
\xi^{n+1} = \xi^n + \mu \left[ (-4\sin^2 \frac{\beta h}{2}) \xi^n + 4\sin^2 \frac{\beta h}{2} \sum_{k=1}^{n-1} c_k^\gamma \xi^{n-k} + 4b_{n-1}^\gamma \sin^2 \frac{\beta h}{2} \xi^0 \right]
\]

can be rewritten

\[
\xi^{n+1} = (1 - 4\mu \sigma)\xi^n + 4\mu \sigma \sum_{k=1}^{n-1} c_k^\gamma \xi^{n-k} + 4\mu \sigma b_{n-1}^\gamma \xi^0
\]

then the amplification matrix \( M \) can be defined by square block matrix of order \( (n + 1) \) and every element of \( M \) is of order \( I - 1 \):

\[
M = \begin{bmatrix}
(1 - 4\mu \sigma)I & (4\mu \sigma c_1^\gamma)I & (4\mu \sigma c_2^\gamma)I & \cdots & (4\mu \sigma c_{n-1}^\gamma)I & (4\mu \sigma b_{n-1}^\gamma)I \\
0 & I & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I & 0
\end{bmatrix}
\]

Employing the same procedure of using Gerschgorin theorem [24], as used in theorem 4.1 we find
\[ |\lambda| \leq |1 - 4 \mu \sigma| + \sum_{k=1}^{n-1} |4 \mu \sigma c_k^{\gamma}| + |4 \mu \sigma b_{n-1}^{\gamma}| \]  

(28)

by the properties we have \( c_k^{\gamma} > 0, b_{n-1}^{\gamma} > 0, \) and \( \sum_{k=1}^{n-1} c_k^{\gamma} = 1 + (n - 1)^{1-\gamma} - n^{1-\gamma} = 1 - b_{n-1}^{\gamma} \), this lead to

\[ |\lambda| \leq |1 - 4 \mu \sigma| + 4 \mu \sigma (1 - b_{n-1}^{\gamma}) + 4 \mu \sigma b_{n-1}^{\gamma} \]  

(29)

if the right hand inequality is less than or equal to one then \(|\lambda| \leq 1\), then we have

\[ |1 - 4 \mu \sigma| + 4 \mu \sigma \leq 1 \]  

(30)

then one can write

\[ -(1 - 4 \mu \sigma) \leq 1 - 4 \mu \sigma \leq 1 - 4 \mu \sigma \]  

(31)

the right hand inequality is satisfied and we need to calculate the condition on \( r_\alpha \) to make the left hand inequality satisfied, this lead to

\[ 8 \mu \sigma \leq 2 \]  

(32)

then the stability condition is

\[ r_\alpha \leq \frac{\Gamma(1+\sigma)}{4} \]  

(33)

5. Consistency of Time Fractional Finite Difference Schemes

The Caputo fractional derivative of \( O(\tau) \) [8], and from the Taylor’s expansion, we have

\[ \frac{1}{h^2} \delta^2_x u^n_j = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{2} h^2 \frac{\partial^3 u(x,t)}{\partial x^4} + O(h^4) \]  

(34)

Therefore, the difference schemes (8) and (22) for TFODE are consistent. The truncation error can be calculated and it is of the form \([O(\tau) + O(h^2)]\).

**Figure 2:** Comparison between the two schemes with the exact solution where \( \alpha = 0.9, \Delta x = 0.1, \Delta t = 0.00125 \) and \( T = 0.025 \). The absolute Errors in the right and the solutions in the left where \( N1 \) and \( N2 \) are the numerical solutions by schemes (8) and (22) respectively.

**Example 5.1** To test the two explicit formulas (8) and (22) consider equation (2) with \( f(x,t) = \left[ \frac{\partial u}{\partial t} e^{-t} \right] + \pi^2 e^{-t} \sin(\pi x) \), with initial condition \( u(x,0) = \sin(\pi x) \) and Dirichlet boundary conditions in the form \( u(0,t) = u(1,t) = 0 \) the exact solution is \( u(x,t) = e^{-t} \sin(\pi x) \).
Figure 3: Comparison between the two schemes with the exact solution where $\alpha = 0.8$, $\Delta x = 0.1$, $\Delta t = 0.0005$ and $T = 0.01$. The absolute Errors in the right and the solutions in the left where $N1$ and $N2$ are the numerical solutions by schemes (8) and (22) respectively.

Figure 4: Comparison between the two schemes with the exact solution where $\alpha = 0.7$, $\Delta x = 0.1$, $\Delta t = 0.00025$ and $T = 0.005$. The absolute Errors in the right and the solutions in the left where $N1$ and $N2$ are the numerical solutions by schemes (8) and (22) respectively.

6. Discussion and Conclusion
The implicit schemes are generally unconditional stable and the explicit schemes are conditionally stable and. In explicit schemes one obtains the solutions easily but the conditions on time steps restrict and increase the computational work. In the implicit schemes one has to solve coupled large algebraic systems in each time level. There are many methods to establish stability conditions among them the Von Neumann and the matrix methods are easily used. Consistency is a simple property and its prove is a reversible process to see the original differential equation from its finite difference representation. There are many problems in describing and establishing the properties of the finite difference schemes in the fractional order cases in comparison with the classical integer cases some of them due to the memory and hereditary effects. Simple stability proves through using the techniques of classical multilevel schemes were introduced. The theorems of Gerschgorin’s discs are applied to the amplification matrices. We have used the technique of multilevel in proving the condition of stability for two schemes for the time fractional diffusion equation. The method of prove is straightforward and more convenient and contains memory effects implicitly. we examined the conditions on numerical example.
In conclusion the explicit schemes still require small time steps in comparison with implicit schemes. The use of inverse operator has improved the calculated solutions and this is acceptable as illustrated because it extended the memory effects to the spatial terms.

It should be pointed that, the suggested methods can be possibly extended to finite difference schemes for variable order TFODE [25], anomalous order TFODE [26] and fractional advection diffusion equations [27].

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