An Elementary Proof of the Signature of Satellite Knots

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Abstract

We present a proof of Litherland’s formula for the Tristram-Levine signature of a satellite knot in terms of its constituents. Litherland’s original proof used more advanced algebraic techniques, while ours uses only linear algebra and some basic results in knot theory.

1 Background

First, recall the definition of a satellite knot and the Tristram-Levine signature. A (nontrivial) satellite knot is obtained as follows:

1. Embed a knot $K$ in the solid torus $T = S^1 \times D^2$. For nontrivial satellite knots we require that there is no simply-connected subset of the solid torus containing the knot and $K$ is not isotopic (in the solid torus) to the central $S^1$ of the solid torus.

2. Take the image of $K$ under a homeomorphism taking $T$ to the (solid) tubular neighborhood of another knot $J$. We require that this homeomorphism is “untwisted,” that is, linking numbers between any two closed curves in $T$ are preserved in the image.

Here, $K$ together with the embedding into $T$ is called the pattern and $J$ is called the companion knot.

A special class of satellites are those with $K = \text{the } (p, q) \text{ torus knot embedded in the standard way onto the surface of the torus. This is called the } (p, q) \text{ cable of } J (see Chapter 1 in [Lic97]).$

The Tristram-Levine signature is a knot invariant defined as the signature of the matrix $(1 - \omega)M - (1 - \overline{\omega})M^T$, where $M$ is any Seifert matrix of the knot and $\omega$ is a complex number with $|\omega| = 1$. The fact that this signature is the same for all Seifert surfaces of a knot follows by considering the effect from performing surgery along an arc to transform one Seifert surface to another; see Theorem 8.9 in [Lic97].

The signature of a knot $K$ is denoted by $\sigma_\omega(K)$.

We prove the following useful formula relating the Tristram-Levine signature of a satellite knot to the signatures of the constituent knots:

**Theorem 1.** If $K'$ is a satellite of $J$ by $K$ and $n$ is the winding number of the embedding of $K$ in the solid torus, then

$$\sigma_\omega(K') = \sigma_\omega(K) + \sigma_\omega(n)(J).$$

This formula was proven by Litherland in 1979 ([Lit79]) in order to study algebraic knots, which are a subset of the set of cables of cables of ... of torus knots. This was spurred by Rudolph’s question about the independence of algebraic knots in the concordance group [Rud76]. Litherland’s proof of the formula for the signature of a satellite knot uses algebraic techniques. We provide a proof which uses only linear algebra and some basic results of knot theory.

2 Lemmas

We will need the following four lemmas. First, a fact of linear algebra:

**Lemma 1.** Suppose $M$ is a Hermitian matrix and $M'$ is obtained from $M$ by one of the following operations:

- Add $z$ times row $r_i$ to $r_j$, then $\overline{z}$ times column $c_i$ to $c_j$ for some $z \in \mathbb{C}$.
- Replace row $r_i$ with $zr_i$, then column $c_i$ with $\overline{z}c_i$ for some $z \in \mathbb{C} \setminus \{0\}$.

Then $M'$ is also Hermitian, and if $\det(M) \neq 0$ then $\det(M') \neq 0$ and $\text{sgn}(M) = \text{sgn}(M')$.

**Proof.** This is a special case of Sylvester’s law of inertia for complex matrices; note that both of the operations above are congruences. \qed
Then we will need the following three facts from knot theory. All of these may be found in [Lic97].

**Lemma 2** (Part of the proof of Theorem 6.10(ii) in [Lic97]). For an appropriate choice of generators of the Seifert surface homology, if $A$ is the Seifert matrix associated to those generators, then

$$A - A^T = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
0 & 0 & -1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & -1 & 0 \\
\end{bmatrix}$$

where the number of $\begin{bmatrix}0 & 1 \\ -1 & 0 \end{bmatrix}$ blocks on the diagonal is equal to the genus of the Seifert surface.

**Lemma 3** (Part of the proof of Theorem 6.15 in [Lic97]). If $K'$ is a satellite of $J$ by $K$ and $n$ is the winding number of the embedding of $K$ in the solid torus, then a Seifert matrix $A$ for $K'$ is given by the block matrix

$$A = \begin{bmatrix}
M & 0 & 0 & \ldots & 0 \\
0 & N & N & \ldots & N \\
0 & N^T & N & \ldots & N \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & N^T & N & \ldots & N \\
\end{bmatrix}$$

where $M$ is any Seifert matrix for $K$ and $N$ any Seifert matrix for $J$, and there are $n \times n$ copies of $N$ and $N^T$.

**Lemma 4** (Theorem 6.15 in [Lic97]). If $K'$ is a satellite of $J$ by $K$ and $n$ is the winding number of the embedding of $K$ in the solid torus, then

$$\Delta_{K'}(t) \equiv \Delta_K(t) \Delta_J(t^n)$$

where $\Delta$ is the Alexander polynomial and $\equiv$ means equal up to a factor of $t^k$.

## 3 Proof of Theorem [1]

The proof of Lemma [1] essentially amounts to three steps after pulling out the $\det(tM - M^T)$ term from the matrix obtained by Lemma [3].

1. Obtain $t^nN - N^T$ in the matrix by row operations.
2. Zero out the other blocks in the same row as the $t^nN - N^T$ by column operations.
3. Pull out the $\det(t^nN - N^T)$ term and compute that the determinant of what remains is a unit.

Our proof is similar, though it is more difficult because the signature is only preserved under congruences (as opposed to the determinant, which changes predictably with arbitrary row/column operations).

In particular, we do the following after pulling out a $\sgn((1 - \omega)M + (1 - \overline{\omega})M^T)$ term:

1. Obtain (a multiple of) $(1 - \omega^n)N + (1 - \overline{\omega^n})N^T$ in the matrix by congruence.
2. Zero out the other blocks in the same row and column as $(1 - \omega^n)N + (1 - \overline{\omega^n})N^T$ by congruence.
3. Pull out the $\sgn((1 - \omega^n)N + (1 - \overline{\omega^n})N^T)$ term and compute that the signature of the remaining matrix is zero.

In the first and third steps, there are some additional subtleties not found in the proof of Lemma [4]. For the first step, there are some exceptional $\omega$ which make the determinant vanish; these are dealt with by applying Lemma [3]. For the third step, we involve an additional congruence to get the matrix into a form where the signature may be readily calculated using Lemma [2].

Now here is the proof of Theorem [1]...
Proof. Let $A$ be the Seifert matrix for $K'$ described in Lemma 3. Now consider

$$(1 - \omega)A + (1 - \overline{\omega})A^T = ((1 - \omega)M + (1 - \overline{\omega})M^T)$$

$$+ \begin{bmatrix}
(1 - \omega)N + (1 - \overline{\omega})N^T & (1 - \omega)N + (1 - \overline{\omega})N & \cdots & (1 - \omega)N + (1 - \overline{\omega})N \\
(1 - \omega)N^T + (1 - \overline{\omega})N^T & (1 - \omega)N + (1 - \overline{\omega})N^T & \cdots & (1 - \omega)N + (1 - \overline{\omega})N^T \\
\vdots & \vdots & \ddots & \vdots \\
(1 - \omega)N^T + (1 - \overline{\omega})N^T & (1 - \omega)N^T + (1 - \overline{\omega})N^T & \cdots & (1 - \omega)N + (1 - \overline{\omega})N^T
\end{bmatrix},$$

with $\oplus$ the direct sum of matrices. The signature of this is the signature of $(1 - \omega)M + (1 - \overline{\omega})M^T$ plus the signature of the second block matrix, which we will call $B$. We will perform some carefully chosen row/column operations on $B$ so that the signature does not change, guaranteed by Lemma 1.

Let $X$ be the (block) row matrix given by $\sum_{k=1}^n (\omega^{1-k} + \cdots + \omega^{n-k}) \times$ (block row $k$ of $B$). Let $Y$ be the (block) column matrix given by $\sum_{i=1}^n (\omega^{1-i} + \cdots + \omega^{n-i}) \times$ (block column $i$ of $B$). Observe by a simple telescoping argument that all the blocks of $X$ and $Y$ are just $(1 - \omega^n)N + (1 - \overline{\omega^n})N^T$. Replace the first row and column of $B$ with $X$ and $Y$, except the top-left block, which is twice $(1 - \omega^n)N + (1 - \overline{\omega^n})N^T$; this corresponds to doing all the row operations necessary to make the first row $X$ and all the column operations to make the first column $Y$. Order these operations so you first replace block row 1 with

$$1 + \omega + \cdots + \omega^{n-1}$$
times itself and block column 1 with

$$1 + \overline{\omega} + \cdots + \overline{\omega^{n-1}}$$
times itself. This corresponds to a matrix operation of the second type from Lemma 1 as long as $1 + \omega + \cdots + \omega^{n-1} \neq 0$, in which case the signature is unchanged. This value is zero exactly when $\omega$ is an $n$th root of unity other than 1. But if $\omega$ is an $n$th root of unity, note by factoring $(1 - \omega)A + (1 - \overline{\omega})A^T = (1 - \omega)(A - \overline{\omega}A^T)$ that the value of $\sigma$ as a function of $\omega$ can only change at zeros of the Alexander polynomial, and by Lemma 1 $\Delta_K(t) \leftarrow \Delta_K(t)\Delta_J(t^n)$. Set $t = \omega$ and note that $\Delta_J(\omega^n) = \Delta_J(1) \neq 0$ so $\Delta_J(t^n)$ does not have a zero at any $n$th root of unity, which is sufficient for this case.

The other row/column operations are the first type from Lemma 1, so they also do not change the signature.

Our matrix now looks like

$$\begin{bmatrix}
2((1 - \omega^n)N + (1 - \overline{\omega^n})N^T) & (1 - \omega^n)N + (1 - \overline{\omega^n})N^T & \cdots & (1 - \omega^n)N + (1 - \overline{\omega^n})N^T \\
(1 - \omega^n)N + (1 - \overline{\omega^n})N^T & (1 - \omega^n)N + (1 - \overline{\omega^n})N^T & \cdots & (1 - \omega^n)N + (1 - \overline{\omega^n})N^T \\
\vdots & \vdots & \ddots & \vdots \\
(1 - \omega^n)N + (1 - \overline{\omega^n})N^T & (1 - \omega^n)N^T + (1 - \overline{\omega^n})N^T & \cdots & (1 - \omega^n)N + (1 - \overline{\omega^n})N^T
\end{bmatrix}.$$ 

Subtract half of row 1 and half of column 1 from the rest of the matrix. The result is

$$(2((1 - \omega^n)N + (1 - \overline{\omega^n})N^T)) \oplus \begin{bmatrix}
D & U & U & \cdots & U \\
L & D & U & \cdots & U \\
L & L & D & \cdots & U \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
L & L & L & \cdots & D
\end{bmatrix}$$

with

$$D = (\omega^n - \omega)N + (\overline{\omega^n} - \overline{\omega})N^T$$
$$U = (\omega^n - \omega - \overline{\omega} + 1)N + (\overline{\omega^n} - 1)N^T$$
$$L = (\omega^n - 1)N + (\overline{\omega^n} - \overline{\omega} - 1)N^T.$$ 

The signature is unchanged, and the latter matrix in the direct sum above is now $(n - 1) \times (n - 1)$ blocks; e.g. the first block row is one $D$ and $(n - 2)$ $U$'s.

So the signature of the matrix in question is the signature of $2((1 - \omega^n)N + (1 - \overline{\omega^n})N^T)$ plus the signature of the rest, which we will call $C$. Subtract $\frac{1}{2}$ times each block row and column of $C$ from every other block row and column. This does not change the signature. The new diagonal blocks are equal to

$$\frac{1}{2}(-\omega + \overline{\omega})(N - N^T).$$
The blocks immediately above the diagonal are
\[
\begin{pmatrix}
\frac{n-3}{2(n-2)} \omega & \frac{n-1}{2(n-2)} + \frac{n-2}{n-2} \\
\frac{n-4}{2(n-2)} \omega & \frac{n-2}{2(n-2)} + \frac{n-3}{n-2}
\end{pmatrix} (N - N^T).
\]
The blocks above that are
\[
\begin{pmatrix}
\frac{n-2-d}{2(n-2)} \omega & \frac{n-d}{2(n-2)} + \frac{n-1-d}{n-2} \\
\frac{n-2}{2(n-2)} + \frac{n-d}{2(n-2)} \omega & \frac{n-2}{n-2}
\end{pmatrix} (N - N^T).
\]
since they incorporate one more \(U\) term and one less \(L\) term. This pattern continues, so that a block \(d\) above the diagonal is now
\[
\begin{pmatrix}
\frac{n-2-d}{2(n-2)} \omega & \frac{n-d}{2(n-2)} + \frac{n-1-d}{n-2} \\
\frac{n-2}{2(n-2)} + \frac{n-d}{2(n-2)} \omega & \frac{n-2}{n-2}
\end{pmatrix} (N - N^T).
\]
The matrix is still Hermitian, so a block \(d\) below the diagonal is
\[
\begin{pmatrix}
\frac{n-2-d}{2(n-2)} + \frac{n-d}{2(n-2)} \omega & \frac{n-2}{n-2} \\
\frac{n-2}{2(n-2)} \omega & \frac{n-2}{n-2}
\end{pmatrix} (N - N^T).
\]
This means the new matrix is equal to the Kronecker product
\[
\begin{pmatrix}
\frac{n-2}{2(n-2)} + \frac{n-2}{n-2} \\
\frac{n-2}{2(n-2)} \omega & \frac{n-2}{n-2}
\end{pmatrix} \otimes (N - N^T).
\]
The eigenvalues of this are equal to the pairwise products of eigenvalues of the two matrices. Notice that the first matrix is skew-Hermitian, so its eigenvalues are all pure imaginary. By Lemma 2, the latter matrix is of the form
\[
\begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
0 & 0 & -1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & -1 & 0
\end{bmatrix}
\]
for appropriate choice of generators of the Seifert surface homology. This has eigenvalues \(i\) and \(-i\) with equal multiplicity. Hence the eigenvalues of the Kronecker product come in \(+/-\) pairs with equal multiplicity, so the signature of \(C\) is 0.

We have shown
\[
\text{sgn}(1 - \omega)A + (1 - \overline{\omega})A^T = \text{sgn}(1 - \omega)M + (1 - \overline{\omega}M^T) + \text{sgn}(2((1 - \omega)^n)N + (1 - \overline{\omega}^n)N^T))
\]
\[
= \text{sgn}(1 - \omega)M + (1 - \overline{\omega}M^T) + \text{sgn}(1 - \omega^n)N + (1 - \overline{\omega}^n)N^T
\]
\[
\sigma_\omega(K') = \sigma_\omega(K) + \sigma_\omega(J).
\]

\[\square\]

4 Closing Remarks

One might compare this proof to [Shi71], in which the special case \(\omega = -1\) is proven also using essentially only linear algebra. It appears, however, that Shinohara’s proof does not generalize; in particular, the structure of the matrix obtained after the congruence fundamentally depends on the parity of \(n\), so Shinohara obtains specifically the formula
\[
\sigma_{-1}(K') = \begin{cases} 
\sigma_{-1}(K) & \text{if } n \text{ is even}, \\
\sigma_{-1}(K) + \sigma_{-1}(J) & \text{if } n \text{ is odd}
\end{cases}
\]
as a result. For other \(\omega\), the signature formula depends on more than just the parity of \(n\), so a different matrix congruence must be used.

Litherland also remarks that the result holds if \(K\) is a link as well, with the exception when \(\omega\) is an \(n\)th root of unity, where the formula may be off by up to \(\pm 2(m-1)\) where \(m\) is the number of components of \(K\). This is clear in our proof as well; the exception occurs when \(1 + \omega + \cdots + \omega^{n-1}\) is zero (that is, \(\omega\) is an \(n\)th root of unity other than 1) because the use of Theorem 6.15 from [Lic97] is only justified when \(K\) is a knot, not a link.
5 Acknowledgements

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6 Addendum

In Lemma 2.2 in [CK02], it is shown that what we call matrix $B$ is congruent to (in our notation)

$$\frac{\omega^n N - N^T}{\omega^n - 1} \oplus \sum_{k=1}^{n} \frac{\omega^{k+1} - 1}{(\omega - 1)(\omega - 1)} (N - N^T).$$

The first term here is $\frac{1}{\omega^n - 1}$ (a real number) times $(1 - \omega^n)N + (1 - \omega^n)N^T$, so they have same signature. The signature of all the other summands is 0. Cha and Ko find this congruence iteratively, one term at a time. Contrast this with our method, where we do our whole congruence in a few large steps.

Note that the large direct sum above is equal to the Kronecker product

$$\Delta \otimes (N - N^T) = \begin{bmatrix} 0 & 0 & \cdots \\ \omega^2 - 1 & 0 & \cdots \\ \cdots & \cdots & \ddots \end{bmatrix},$$

where $\Delta$ is a pure imaginary diagonal matrix. Our matrix $C$ was itself congruent to a Kronecker product of a skew-Hermitian matrix $S$ and $(N - N^T)$. Hence diagonalizing $iS$ (a Hermitian matrix) and scaling the diagonal entries appropriately should result in $i\Delta$ (a real diagonal matrix). If you do this diagonalization process entry-by-entry and compose that process with our congruence, you should get exactly Cha and Ko’s method.

The result in [CK02] is actually more general in two main ways:

- Let $\varepsilon \in \{-1, 1\}$. Replace $N^T$ by $\varepsilon N^T$ everywhere. The convention that the signature of a skew-Hermitian matrix is equal to the signature of $i$ times the matrix. The resulting signature formula includes a term involving $\text{sgn}(N - \varepsilon N^T)$; in the $\varepsilon = 1$ case (our case) this signature is zero so this term does not appear.

- Replace some $n - u$ diagonal blocks of $B$ with $(1 - \omega)\varepsilon N^T + (1 - \overline{\omega})N$. By rearranging the rows and columns, these can be taken to be the last $n - u$ blocks, so blocks 1 through $u$ are unchanged.

The resulting signature formula replaces $n$ with $2u - n$.

The iterative method in [CK02] lends itself well to these generalizations. Our proof can also handle these generalizations to an extent. In particular:

- If $\varepsilon = -1$ then exactly the same congruences work, just everywhere replacing $N^T$ with $-N^T$. The resulting formula is $\text{sgn}(B) = \text{sgn}((1 - \omega)N - (1 - \overline{\omega})N^T) + \text{sgn}(S)\text{sgn}(N + N^T)$. It is easy to verify that $\text{sgn}(A \otimes B) = \pm \text{sgn}(A)\text{sgn}(B)$ where $A$ and $B$ are (skew-)Hermitian, taking the $+$ sign if at least one is Hermitian and the $-$ sign if both are skew-Hermitian. Unfortunately, it does not seem there is an easy way to compute $\text{sgn}(S)$ other than by diagonalization. The answer, in [CK02], is $\text{sgn}(S) = n - 1 - \frac{1}{2} [\frac{n-1}{2}]$ where $\omega = e^{ix}$ with $0 \leq x \leq 2\pi$, unless $\frac{n-1}{2}$ is an integer in which case it is this expression minus 1. At $x = 0$, $\text{sgn}(S) = 0$. This may be seen fairly easily from $\Delta$, but given only $S$ it is somewhat surprising the signature has such a simple expression.

- If the last $n - u$ diagonal blocks are replaced with $\varepsilon((1 - \omega)N^T + (1 - \overline{\omega})N)$, then replace $X$ in the first step with

$$\sum_{k=1}^{u} (\omega^1 - k + \cdots + \omega^{2u - n - k}) \times (\text{block row } k \text{ of } B) - \sum_{k=u+1}^{n} (\omega^k - 2u + \cdots + \omega^{k-n-1}) \times (\text{block row } k \text{ of } B)$$

5
and modify $Y$ similarly. After this, perform the same step of subtracting half of row 1 and half of column 1 from the rest. The result is now

$$
(2(1 - \omega^{2u-n})N + (1 - \overline{\omega}^{2u-n})N^T) \oplus
\begin{bmatrix}
D_2 & U & U & \cdots & U \\
L & D_3 & U & \cdots & U \\
L & L & D_4 & \cdots & U \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
L & L & L & \cdots & D_n
\end{bmatrix}
$$

with

$$
D_k = \begin{cases}
(\omega^n - \omega)N + (\overline{\omega}^n - \overline{\omega})N^T, & k \leq u \\
(\omega^n - \overline{\omega})N + (\overline{\omega}^n - \omega)N^T, & k > u
\end{cases}
$$

$$
U = (\omega^{2u-n} - \omega - \overline{\omega} + 1)N + (\overline{\omega}^{2u-n} - 1)N^T
$$

$$
L = (\omega^{2u-n} - 1)N + (\overline{\omega}^{2u-n} - \omega + 1)N^T.
$$

As before, subtracting $\frac{1}{2(n-2)}$ times each block row and column from every other block row and column will result in a Kronecker product of a matrix $S$ and $(N - N^T)$. Now the first $u - 1$ diagonal entries of $S$ are $\frac{1}{d}(\omega - \overline{\omega})$ and the last $n - u$ are $\frac{1}{d}(\omega - \overline{\omega})$. The entries $d$ above a diagonal have three different forms. If the block was right of $D_i$ and above $D_j$ prior to this last step, then the corresponding entry of $S$ is now

$$
\begin{cases}
\frac{n-2-d}{2(n-2)} \omega - \frac{n-1-d}{n-2}, & i \leq u \text{ and } j \leq u \\
\frac{n-1-d}{2(n-2)} \omega - \frac{n-1-d}{n-2}, & i \leq u \text{ and } j > u \\
\frac{n-d}{2(n-2)} \omega - \frac{n-1-d}{n-2}, & i > u \text{ and } j > u.
\end{cases}
$$

The entries below the diagonal take a similar form; note $S$ is skew-Hermitian. Thus we find that the signature of $B$ is equal to $\text{sgn}((1 - \omega^{2u-n})N + (1 - \overline{\omega}^{2u-n})N^T)$.

In the case $\varepsilon = -1$, the same steps give that the signature is $\text{sgn}((1 - \omega^n)N - (1 - \overline{\omega}^n)N^T) + \text{sgn}(S)\text{sgn}(N + N^T)$ as expected, but once again it does not seem there is an easy way to compute $\text{sgn}(S)$. It turns out (from [CK02]) that $\text{sgn}(S)$ has the same formula as before, just replacing $n$ with $2u - n$.

References

[CK02] Jae Choon Cha and Ki Hyoung Ko. Signatures of links in rational homology spheres. *Topology*, 41:1161–1182, 2002.

[Lic97] W. B. Lickorish. An *Introduction to Knot Theory*. Graduate Texts in Mathematics. Springer New York, 1997.

[Lit79] R. A. Litherland. Signatures of iterated torus knots. In Roger Fenn, editor, *Topology of Low-Dimensional Manifolds*, pages 71–84. Springer, Berlin, Heidelberg, 1979.

[Rud76] Lee Rudolph. How independent are the knot-cobordism classes of links of plane curve singularities? In *Notices of the American Mathematical Society*, volume 23, page 410, 1976.

[Shi71] Yaichi Shinozaki. On the signature of knots and links. *Transactions of the American Mathematical Society*, 156:273–285, 1971.