The Cosmological Spacetime

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We present here the transformations required to recast the Robertson-Walker metric and Friedmann-Robertson-Walker equations in terms of observer-dependent coordinates for several commonly assumed cosmologies. The overriding motivation is the derivation of explicit expressions for the radius $R_h$ of our cosmic horizon in terms of measurable quantities for each of the cases we consider. We show that the cosmological time $dt$ diverges for any finite interval $ds$ associated with a process at $R \to R_h$, which therefore represents a physical limit to our observations. This is a key component required for a complete interpretation of the data, particularly as they pertain to the nature of dark energy. With these results, we affirm the conclusion drawn in our earlier work that the identification of dark energy as a cosmological constant does not appear to be consistent with the data.

Keywords: cosmology; dark energy; gravitation.

1. Introduction

Standard cosmology is based on the Robertson-Walker (RW) metric for a spatially homogeneous and isotropic three-dimensional space, expanding or contracting as a function of time:

$$ds^2 = c^2 dt^2 - a^2(t)[dr^2(1 - kr^2)^{-1} + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (1)$$

In the coordinates used for this metric, $t$ is the cosmic time, measured by a comoving observer (and is the same everywhere), $a(t)$ is the expansion factor, and $r$ is an appropriately scaled radial coordinate in the comoving frame. The geometric factor $k$ is $+1$ for a closed universe, $0$ for a flat universe, and $-1$ for an open universe.

The expansion of the universe is calculated from the Friedmann-Robertson-Walker (FRW) differential equations of motion,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \rho - \frac{kc^2}{a^2}, \quad (2)$$

The expansion of the universe is calculated from the Friedmann-Robertson-Walker (FRW) differential equations of motion,
\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\rho + 3p) , \]  

(3)

\[ \dot{\rho} = -3H(\rho + p) , \]  

(4)

derived from an application of the R W metric to the Einstein field equations. Here, an overdot denotes a derivative with respect to \( t \), and \( \rho \) and \( p \) represent, respectively, the total energy density and total pressure. Often, the latter is written as \( p = w\rho \), and \( w \) is then used to characterize the expansion properties of the medium. For example, \( w \approx 0 \) for (visible and dark) matter, \( +1/3 \) for radiation, and \( -1 \) for a pure cosmological constant (though the actual value of \( w \) is not restricted to just these values).

In previous papers,\(^1\)\(^2\) we demonstrated the usefulness of expressing the R W metric in terms of an observer-dependent coordinate \( R = a(t)r \), which explicitly reveals the dependence of the observed intervals of distance, \( dR \), and time on the curvature induced by the mass-energy content between the observer and \( R \); in the metric, this effect is represented by the proximity of the physical radius \( R \) to the cosmic horizon \( R_h \), defined by the relation

\[ \frac{2GM(R_h)}{c^2} = R_h . \]  

(5)

In this expression, \( M(R_h) \) is the mass enclosed within \( R_h \). In terms of \( \rho \), we may also write \( R_h = (3c^4/8\pi G\rho)^{1/2} \) or, more simply, \( R_h = c/H(t) \) in a flat universe. This is the radius at which a sphere encloses sufficient mass-energy to create a significant time dilation for an observer at the surface relative to the origin of the coordinates.

When the RW metric is written in terms of \( R \), the presence of \( R_h \) alters the intervals of time we measure progressively more and more as \( R \to R_h \). And since the gravitational time dilation becomes infinite near \( R_h \), it is physically impossible for us to see any process occurring beyond this distance. What this means, of course, is that light emitted beyond \( R_h \) is infinitely redshifted and therefore carries no signal.

The motivation for the introduction of these new coordinates was the set of recent observations pointing to a cosmic-horizon radius \( R_h(t_0) \approx ct_0 \), where \( t_0 \) is the inferred current age of the universe.\(^3\) (Throughout this paper, a subscript 0 denotes quantities measured at the present time.) This inference is based on precision measurements\(^3\) of the CMB radiation, which indicate that the universe is extremely flat (i.e., \( k = 0 \)). Thus \( \rho \) is at (or very near) the “critical” value \( \rho_0 \equiv 3c^2H_0^2/8\pi G \). The Hubble Space Telescope Key Project\(^4\) on the extragalactic distance scale has measured \( H \) with unprecedented accuracy, finding a current value \( H_0 \equiv H(t_0) = 71 \pm 6 \text{ km s}^{-1} \text{ Mpc}^{-1} \). It is straightforward to show that with this \( H_0, R_0 \approx ct_0 \) (specifically, 13.4 versus 13.7 billion lightyears). But this empirical result is very peculiar because the FRW equations predict that \( \dot{R}_h = c \) only for the very special equation of state \( w = -1/3 \) which, however, does not appear to be consistent with any of the known constituents of the universe, including a cosmological constant with \( w = w_\Lambda \equiv -1 \).
As a brief aside, it is interesting to note that de Sitter’s own metric (for a universe containing only a cosmological constant) was first written in terms of observer-dependent coordinates, though this is no longer widely known, and everyone now uses the form of the RW metric written in terms of r and t only. Of course, de Sitter’s work was completed prior to the introduction of the comoving coordinates \((ct, r, \theta, \phi)\) several years later. A physical motivation for how one might arrive at de Sitter’s metric from an application of Schwarzschild’s solution to a uniform infinite medium may be understood in terms of Birkhoff’s theorem and its corollary.

Given the significance of the fact that \(R_h(t_0) \approx ct_0\), our earlier discussion on this topic was largely based on a simple cosmological spacetime with \(w = -1/3\). The condition \(\dot{R}_h = c\) is then always true, guaranteeing the equality (or near equality) of \(R_h\) and \(ct\). However, if the equation of state is time dependent, \(w\) need not be exactly \(-1/3\) in order to achieve the result \(R_h(t_0) = ct_0\) in the current universe. Indeed, Type Ia supernova data seem to suggest a value of \(w\) smaller than this, which motivates us to consider a broader range of constituents in the Universe in order to fully interrogate the available data.

The purpose of this paper is to present the transformations required to recast the RW metric and FRW equations for a flat universe \((k = 0)\) into observer-dependent coordinates for several commonly assumed cosmologies. The overriding motivation is the derivation of an explicit expression for \(R_h\) in terms of measurable quantities for each of the cases we consider. As we shall see, this is a key component required for a complete interpretation of the observations, particularly as they pertain to the nature of dark energy. In the process, we shall affirm the conclusion drawn earlier that dark energy is probably not a cosmological constant.

2. General Coordinate Transformation

In order to write the RW metric in terms of observer-dependent coordinates, we begin with the physical radius

\[ R \equiv a(t) r. \tag{6} \]

However, it will become evident later that to simplify the mathematical transformations, it is convenient to recast Equation (1) using a new function \(f(t)\), where

\[ a(t) = e^{f(t)}. \tag{7} \]

In that case,

\[ ds^2 = c^2 dt^2 - e^{2f(t)} \left[ dr^2 + r^2 d\Omega^2 \right], \tag{8} \]

where

\[ d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2. \tag{9} \]

Now,

\[ r = Re^{-f}. \tag{10} \]
and therefore
\[ dr = e^{-f} \left[ dR - R \dot{f} \, dt \right], \] (11)
so that
\[ dr^2 = e^{-2f} \left[ dR^2 + \left( \frac{R \dot{f}}{c} \right)^2 c^2 \, dt^2 - 2 \left( \frac{R \dot{f}}{c} \right) c \, dt \, dR \right]. \] (12)

Collecting terms, we can now write the metric as
\[ ds^2 = \left[ 1 - \left( \frac{R \dot{f}}{c} \right)^2 \right] c^2 \, dt^2 + 2 \left( \frac{R \dot{f}}{c} \right) c \, dt \, dR - dR^2 - R^2 \, d\Omega^2, \] (13)
whereupon, completing the square, we see that
\[ ds^2 = \Phi \left[ c \, dt + \left( \frac{R \dot{f}}{c} \right) \Phi^{-1} \, dR \right]^2 - \Phi^{-1} \, dR^2 - R^2 \, d\Omega^2 . \] (14)
For convenience we have also defined the quantity
\[ \Phi \equiv 1 - \left( \frac{R \dot{f}}{c} \right)^2, \] (15)
which appears frequently in the metric coefficients. We can easily see that the radius of the cosmic horizon for the observer at the origin is
\[ R_h \equiv c/\dot{f} \] (16)
so that, in effect, the function
\[ \Phi \equiv 1 - \left( \frac{R}{R_h} \right)^2 \] (17)
signals the dependence of the metric on the proximity of the observation radius \( R \) to the maximum observable distance \( R_h \). Equation (14) thus becomes
\[ ds^2 = \Phi \left[ c \, dt + \left( \frac{R}{R_h} \right) \Phi^{-1} \, dR \right]^2 - \Phi^{-1} \, dR^2 - R^2 \, d\Omega^2 . \] (18)

This metric is already in a form we can use to examine the behavior of the cosmological spacetime at any radius \( R \). However, sometimes (e.g., as in the de Sitter metric) it may be useful to “complete” the transformation by introducing a new time coordinate \( T \), such that
\[ \Phi^{-1/2} \frac{c \, dT}{\eta(t, R)} \equiv \Phi^{1/2} c \, dt + \left( \frac{R}{R_h} \right) \Phi^{-1/2} \, dR , \] (19)
where \( \eta(t, R) \) is an integrating factor selected to guarantee that \( dT \) is an exact differential. There are possibly an infinite number of solutions for \( \eta(t, R) \), but only
one is dimensionless (ensuring that \( T \) has dimensions of time), while completely diagonalizing the metric. With this substitution,

\[
\text{ds}^2 = \Phi^{-1} \frac{c^2}{\eta^2(t, R)} \; dT^2 - \Phi^{-1} \; dR^2 - R^2 \; d\Omega^2 .
\]

(20)

But in order for \( dT \) to be an exact differential, \( T \) must satisfy the following condition:

\[
\frac{\partial^2 T}{\partial R \partial t} = \frac{\partial^2 T}{\partial t \partial R} ,
\]

(21)

which means that \( \eta(t, R) \) must be a solution to the equation

\[
\frac{\partial}{\partial R} \left[ \Phi \; \eta(t, R) \right] = \frac{\partial}{\partial t} \left[ \left( \frac{R}{cR_h} \right) \; \eta(t, R) \right] .
\]

(22)

As we shall learn shortly, however, the coordinate \( T \) has only limited applicability because it is rarely possible to integrate \( dT \) starting from the big bang at \( t = 0 \) to the present.

Let us now examine the behavior of the interval \( ds \) connecting any arbitrary pair of spacetime events at \( R \). For an interval produced at \( R \) by the advancement of time only (with \( dR = d\Omega = 0 \)), Equation (18) gives

\[
\text{ds}^2 = \Phi \; c^2 \; dt^2 .
\]

(23)

Thus, if we were to make a measurement a fixed distance \( R \) away from us, the time interval \( dt \) corresponding to any measurable (non-zero) value of \( ds \) must go to infinity as \( R \to R_h \). (In the context of black-hole physics, we recognize this effect as the divergent gravitational redshift measured by a static observer outside the event horizon.) This result is generic to all cosmologies though, as we shall see, in some cases \( R_h \) lies beyond the distance \( ct \) light has traveled since the big bang and is therefore not an observable quantity. What does change from case to case, however, are the constituents of the Universe, which directly determine \( w \), and therefore the function \( f \) and the horizon’s radius \( R_h \). Finding these quantities will be the goal of the next section.

But we can already see directly from the FRW equations how \( R_h \) should behave for any given value of \( w \). It is straightforward to demonstrate\(^{[2]}\) from Equations (2)–(4) that

\[
\dot{R}_h = \frac{3}{2} (1 + w) c .
\]

(24)

Thus, \( R_h \) is an increasing function of cosmic time \( t \) for any cosmology with \( w > -1 \). It is fixed only for de Sitter, in which \( \rho \) is a cosmological constant and \( w = -1 \). In addition, there is clearly a demarcation at \( w = -1/3 \). When \( w < -1/3 \), \( R_h \) increases more slowly than lightspeed, and therefore our universe would be delimited by this horizon because light would have traveled a distance \( ct \) greater than \( R_h(t) \) since the big bang. On the other hand, \( R_h \) is always greater than \( ct \) when \( w > -1/3 \), and our observational limit would then simply be set by the light travel distance \( ct \).
3. Specific Cosmologies

3.1. The De Sitter Universe \((w = -1)\)

As we pointed out above, de Sitter himself published the earliest form of his metric using the coordinate \(R\), though he was apparently not aware of its full significance. But as a simple, initial application of the method we derived in § 2 above, let us repeat this calculation by considering a universe containing only a cosmological constant. In this case,

\[
a(t) = e^{H_0 t},
\]

for which

\[
ds^2 = c^2 dt^2 - e^{2H_0 t} [dr^2 + r^2 d\Omega^2].
\]

Clearly,

\[
f(t) = \ln[a(t)] = H_0 t,
\]

so

\[
\dot{f} = H_0.
\]

Thus, inserting this form of \(\dot{f}\) into Equation (16), we obtain

\[
R_h = c/H_0.
\]

As we saw in Equation (24), the de Sitter metric is unique among the various cosmologies in that the density \(\rho\) is constant, and therefore the radius \(R_h\) is fixed for all cosmic time \(t\). However, for all the other cosmologies we will consider below, \(w > -1\), so \(\dot{R}_h > 0\). This means, of course, that whereas an observer can choose radii \(R\) that are always smaller than \(R_h\) in de Sitter, the same is not true when \(\dot{R}_h > 0\), since for any given \(R\), \(R_h < R\) when \(t \to 0\).

An important consequence of this distinction is that the interval \(ds\) is then imaginary at early times (since \(R_h\) is independent of time) that

\[
\Phi^{1/2} c dT \equiv \Phi^{1/2} \left[ c dt + \left( R/R_h \right) \Phi^{-1} dR \right].
\]

A comparison with Equation (19) tells immediately that the integrating factor \(\eta(t, R)\) in the case of de Sitter is simply equal to \(\Phi^{-1}\), and a straightforward integration of Equation (30) thus gives

\[
T(t, R) = t - \frac{1}{2H_0} \ln \Phi.
\]

At the origin, \(T\) is of course equal to \(t\), but since the gravitationally-induced dilation increases with \(R\), \(T\) also includes the additional redshift seen at progressively greater distances from the observer. Note, however, that for \(dR = d\Omega = 0\), \(dT = dt\), so both of these time intervals diverge for a finite \(ds\) as \(R \to R_h\).
Written in terms of $T$ and $R$, the de Sitter metric becomes
\[ ds^2 = \Phi c^2 dT^2 - \Phi^{-1} dR^2 - R^2 d\Omega^2 , \] (32)
which is the form originally presented by de Sitter. 5

3.2. A Cosmology with $R_h(t) = ct$ (i.e., $w = -1/3$)

This is the case in which $\dot{R}_h = c$ in Equation (24). Aside from being one of the simplest models of the universe that one can construct from the FRW equations, it is also motivated by the observed fact that $R_h(t) \approx ct_0$ at the present time. If $w$ is always $-1/3$, then it would not be surprising to see $R_h(t_0) = ct_0$ now, since this condition would have been true from the beginning.

But though this may seem like a highly idealized model that bears minimal relevance to the real universe, it is actually singularly significant[12] because an equation of state $w = -1/3$ is the only one for which the current age, $t_0$, of the universe can equal the light-crossing time, $t_h \equiv R_h/c$. For any other cosmology with $w < -1/3$, $t_0$ must be greater than $R_h/c$. We shall return to this shortly. For now, let us consider the transformation of coordinates when $w = -1/3$, for which
\[ a(t) = H_0 t . \] (33)

Then,
\[ f(t) = \ln[a(t)] = \ln(H_0 t) , \] (34)
and
\[ \dot{f} = \frac{1}{t} . \] (35)

We therefore confirm that $R_h = ct$ in this case, and from Equation (18), we obtain the metric
\[ ds^2 = \Phi \left[ c dt + \left( \frac{R}{ct} \right) \Phi^{-1} dR \right]^2 - \Phi^{-1} dR^2 - R^2 d\Omega^2 , \] (36)
with
\[ \Phi = 1 - \left( \frac{R}{ct} \right)^2 . \] (37)

As was the case with de Sitter, the cosmic time $dt$ diverges for a measurable line element as $R \to R_h$ (which, however, is equal to $ct$ here).

We may also write the metric in terms of $R$ and $T$. A dimensionless solution to Equation (22) is
\[ \eta(t, R) = \exp \left\{ \frac{1}{2} \left( \frac{R}{ct} \right)^2 \right\} , \] (38)
which leads to the metric
\[ ds^2 = e^{-(R/R_h)^2} \Phi^{-1} c^2 dT^2 - \Phi^{-1} dR^2 - R^2 d\Omega^2 . \] (39)
But as we pointed out above, this form of the metric has only a limited applicability, given that $ds^2 < 0$ when $R_h < R$ and $dR = d\Omega = 0$.

### 3.3. Radiation Dominated Universe ($w = +1/3$)

The real Universe may indeed contain a cosmological constant, in which case it approaches a de Sitter spacetime asymptotically, given that its other constituents have a density $\rho$ that drops off as $a(t)$ increases. (We will revisit this below.) But let us now move beyond the simple de Sitter application, and the special case with $w = -1/3$, and consider other cosmological phases (as we currently understand them) that may have emerged since the big bang.

In the very beginning, when radiation dominated the equation of state (with $w = w_{\text{rad}} \equiv +1/3$), the expansion parameter was given as

$$a(t) = (2H_0 t)^{1/2}.$$  \hfill (40)

In that case,

$$f(t) = \ln[a(t)] = \frac{1}{2}\ln(2H_0 t),$$  \hfill (41)

and

$$\dot{f} = \frac{1}{2t}.$$  \hfill (42)

Therefore $R_h = 2ct$. In this case, the visible Universe (extending out to $ct$) never reaches $R_h$, and the metric is

$$ds^2 = \Phi \left[c\, dt + \left(\frac{R}{2ct}\right)\Phi^{-1}\, dR\right]^2 - \Phi^{-1} dR^2 - R^2 d\Omega^2,$$  \hfill (43)

with

$$\Phi = 1 - \left(\frac{R}{2ct}\right)^2.$$  \hfill (44)

Thus, measurements made at a fixed $R$ and $\Omega$ still produce a gravitationally-induced dilation of $dt$ as $R$ increases, but this effect never becomes divergent within that portion of the Universe (i.e., within $ct_0$) that remains observable since the big bang.

For completeness, we note that a dimensionless solution for $\eta(t, R)$ from Equation (22) is

$$\eta(t, R) = 1,$$  \hfill (45)

and the metric written in terms of $R$ and $T$ is therefore

$$ds^2 = \Phi^{-1} c^2 dT^2 - \Phi^{-1} dR^2 - R^2 d\Omega^2.$$  \hfill (46)

The same restrictions as before pertain to $T$ since here also $R_h < R$ at early times, which would make $ds$ imaginary for $dR = d\Omega = 0$ back then.
3.4. Matter Dominated Universe \((w = 0)\)

When \(\rho\) is dominated by matter with \(w \approx w_{\text{matter}} \equiv 0\), the expansion factor grows according to

\[
a(t) = \left(\frac{3}{2} H_0 t\right)^{2/3}.
\]

Therefore,

\[
f(t) = \ln[a(t)] = \frac{2}{3} \ln \left(\frac{3}{2} H_0 t\right),
\]

and

\[
\dot{f} = \frac{2}{3t}.
\]

In this case, the radius of the cosmic horizon is \(R_h = (3/2)ct\), and the metric becomes

\[
ds^2 = \Phi \left[c dt + \left(\frac{R}{3ct/2}\right) \Phi^{-1} dR\right]^2 - \Phi^{-1} dR^2 - R^2 d\Omega^2,
\]

with

\[
\Phi = 1 - \left(\frac{R}{3ct/2}\right)^2.
\]

The situation is similar to that for a radiation dominated universe, in that \(R_h\) always recedes from us faster than lightspeed (see Equation 24). Although dilation is evident with increasing \(R\), curvature alone does not produce a divergent redshift.

A dimensionless integrating factor \(\eta(t, R)\) is

\[
\eta(t, R) = \sqrt{1 + \frac{1}{2} \left(\frac{R}{R_h}\right)^2}.
\]

Thus, in terms of \(R\) and \(T\),

\[
ds^2 = \Phi^{-1} \left[1 + \frac{1}{2} \left(\frac{R}{R_h}\right)^2\right]^{-1} c^2 dT^2 - \Phi^{-1} dR^2 - R^2 d\Omega^2.
\]

3.5. Universe Delimited by the Cosmic Horizon \((-1 < w < -1/3)\)

In general, \(R_h = (3/2)(1 + w)ct\) for a universe with constant \(w\). Except for the special cases \(w = -1\) and \(w = -1/3\), the metric may therefore be written

\[
ds^2 = \Phi \left[c dt + \left(\frac{2R}{3(1 + w)ct}\right) \Phi^{-1} dR\right]^2 - \Phi^{-1} dR^2 - R^2 d\Omega^2,
\]

with

\[
\Phi = 1 - \left(\frac{2R}{3(1 + w)ct}\right)^2.
\]
(Actually, both this form of the metric, as well as that given below in Equation 57, are valid for constant values of \( w \) greater than \(-1/3\). However, in this section we are primarily interested in equations of state for which \( \dot{R} < c \), i.e., \( w < -1/3 \).) A cosmology with \( w < -1/3 \) is therefore delimited by the horizon in the sense that our measurements—extending over a distance \( ct_0 \)—would have probed regions of spacetime that include \( R_h(t_0) \) today.

For example, in what may be a close approximation to the Universe we have currently (see next subsection), the line element for \( w = -2/3 \) is

\[
\begin{align*}
ds^2 &= \Phi \left[ c \, dt + \left( \frac{2R}{ct} \right) \Phi^{-1} \, dR \right]^2 - \Phi^{-1} \, dR^2 - R^2 \, d\Omega^2, \\
&= \Phi^{-1} \left[ 1 + \frac{1}{2} (1 + 3w) \left( \frac{R}{R_h} \right)^2 \right]^{(3w-1)/(3w+1)} \, c^2 \, dT^2 - \Phi^{-1} \, dR^2 - R^2 \, d\Omega^2.
\end{align*}
\]

\( \Box(56) \)

3.6. The ΛCDM Model

Let us now examine the role played by \( R_h \) in the “standard” model of cosmology. Following the radiation-dominated era, the universal expansion is believed to have been driven by a combination of matter plus a cosmological constant, the latter emerging at later times since \( \rho_{\text{matter}} \propto a(t)^{-3} \), whereas \( \rho_{\Lambda} = \text{constant} \). Based on the solutions we have examined thus far, we anticipate that this kind of Universe does become \( R_h \)-delimited, but only after the cosmological constant dominates.

From the FRW equations with \( \rho = \rho_{\text{matter}} + \rho_{\Lambda} \) and \( w = w_{\text{matter}} + w_{\Lambda} = -1 \), where \( w_{\text{matter}} = 0 \) and \( w_{\Lambda} = -1 \), we infer that

\[
a(t) = A \sinh^{2/3} \left( \frac{t}{t_\Lambda} \right).
\]

\( \Box(58) \)

The ratio \( \rho_{\text{matter}}/\rho_{\Lambda} \) is not known a priori; its possible range of values is subsumed into the (time) constant \( t_\Lambda \). The greater the value of \( \rho_{\text{matter}}/\rho_{\Lambda} \), the greater the value of \( t_\Lambda \), pushing the transition from a matter-dominated universe to a \( \Lambda \)-dominated one farther into the future. Notice from Equation (58) that \( a(t) \to \text{constant} \times t^{2/3} \) for \( t \ll t_\Lambda \), in agreement with Equation (47) for a matter-dominated universe. At the other extreme, \( a(t) \to \text{constant} \times \exp(2t/3t_\Lambda) \), the correct behavior exhibited in Equation (25) for a de Sitter universe. Clearly then,

\[
t_\Lambda = \frac{2}{3H_\infty},
\]

\( \Box(59) \)
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where $H_\infty \equiv \lim_{t \to \infty} H(t)$ is the asymptotic (constant) value of the Hubble constant describing a universe settled into its cosmological-constant driven expansion.

Thus, in $\Lambda$CDM cosmology,

$$f(t) = \ln[a(t)] = \ln(A) + \frac{2}{3} \ln[\sinh(t/t_\Lambda)] ,$$

and

$$\dot{f} = \frac{2}{3t_\Lambda \tanh(t/t_\Lambda)} = \frac{H_\infty}{\tanh(3tH_\infty/2)} .$$

It is trivial to check that $\dot{f}$ has the correct limiting forms given in Equations (28) (when $t \to \infty$) and (49) (when $t \to 0$). In this case,

$$R_h = \left(\frac{c}{H_\infty}\right) \tanh\left(\frac{3}{2}tH_\infty\right) ,$$

and therefore

$$\dot{R}_h = \frac{3}{2}\left[1 - \tanh^2\left(\frac{3}{2}tH_\infty\right)\right]c .$$

A direct comparison of this equation for $\dot{R}_h$ with that given in Equation (24) is not quite legitimate since $w$ was assumed to be constant in the earlier expression. However, for simple analysis over short intervals of time $\Delta t \ll t$, it can still be useful to approximate $w$ as follows:

$$w \approx -\tanh^2\left(\frac{3}{2}tH_\infty\right) ,$$

which again has the correct behavior in the appropriate time limits, i.e., $w \to 0$ when $t \to 0$ and $w \to -1$ when $t \to \infty$.

In this cosmology, the observer does not experience a divergent redshift with increasing $R$ at early times, since $\dot{R}_h > c$ then, but he will begin to encounter an observational limit at a finite radius $\sim R_h$ after a “transition” time $t_{\text{trans}}$ estimated from the condition

$$ct_{\text{trans}} = R_h(t_{\text{trans}}) .$$

Equation (65) has two solutions, the trivial value $t_{\text{trans}} = 0$ (which is probably irrelevant since the Universe is radiation dominated at the beginning), and $t_{\text{trans}} \sim 0.86/H_\infty$ (which is also approximately $t_\Lambda$). In a sense, $t_{\text{trans}}$ is roughly the point at which the Universe transitions from being matter-dominated to $\Lambda$-dominated. Eventually, the Universe becomes de Sitter and therefore $\dot{R}_h \to 0$, with $R_h$ settling at the fixed value $c/H_\infty$. 


4. Conclusions

An observer measuring intervals of time at progressively greater distances from his origin of coordinates sees a gravitationally-induced time dilation due to the mass-energy content of the Universe between himself and the radius at which he is making the observation. This gravitational redshift is manifested via the appearance of the radius $R_h$ (of our cosmic horizon) in the $g_{00}$ coefficient of the metric for all the cases we considered in this paper. Its modification to the intervals we measure at radii $R > 0$ is a generic feature of any universe with $\rho \neq 0$.

However, the actual form of $R_h$, and its time derivative $\dot{R}_h$, depend on the equation of state $w \equiv p/\rho$. The correspondence between $R_h$ and the event horizon introduced earlier was discussed at length previously. Briefly, $R_h$ is an instantaneous horizon that (except in the special case $w = -1$) increases with cosmic time $t$. It asymptotically approaches Rindler’s (fixed) event horizon in situations where $\dot{R}_h < c$, and is equal to the latter in the case of de Sitter, for which $R_h$ is a constant. In cosmologies with $w > -1/3$, $R_h$ is always greater than the distance traveled by light since the big bang, and therefore does not represent a limit to our range of observations. Of course, in these cases Rindler’s event horizon also does not exist.

During the radiation- and matter-dominated eras, $\dot{R}_h > c$, and though measured time intervals $dt$ are gravitationally dilated relative to the values they would otherwise have, they never diverge. In these cases, there is therefore no physical limitation to how far we can see except, of course, as restricted by the distance $ct$ light has traveled since $t = 0$.

But for any universe with an equation of state $w < -1/3$, we find that $\dot{R}_h < c$, so there exists a finite radius $\sim R_h(t_0) < ct_0$ (where $t_0$ is the current age of the Universe) at which $dt \to \infty$ for any finite interval $ds$. Thus, although $R_h$ increases with time $t$ (except in the special case $w = -1$), our past lightcone is always truncated by gravitational curvature.

In earlier work, we explored several observational consequences of this phenomenon. Of particular interest is the peculiar observation that $R_h(t_0)$ is currently equal (or nearly equal) to $ct_0$. As we have seen through the various cosmological spacetimes we considered in this paper, one would expect the condition $R_h = ct$ to be met only if $w = -1/3$, in which case it would always be true. Alternatively, it would be met just once, at a time $t_{\text{trans}} \sim 0.86/H_\infty$ in the context of $\Lambda$CDM. For all other times $t > t_{\text{trans}}$, $R_h$ must be less than $ct$. A more extended discussion of the physical interpretation of these results has appeared elsewhere.

Insofar as the nature of dark energy is concerned, however, the observed equality (or near equality) of $R_h(t_0)$ and $ct_0$ presents a problem for models in which $w \approx -1$. Through the various spacetimes we have considered in this paper, we see that the condition $R_h = ct$ can only be attained for $w < -1/3$; otherwise, $R_h$ always exceeds $ct$. But the fact that $R_h$ would then equal $ct$ just once in the entire history of the universe if $w \neq -1/3$ constitutes an unacceptably improbable coincidence that we should be seeing this transition occurring right now. We therefore affirm our earlier
conclusion that dark energy is probably not a cosmological constant.

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