An anisotropic cosmological model in a modified Brans–Dicke theory

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Abstract
Recently, it has been shown that a four-dimensional (4D) Brans–Dicke (BD) theory with an effective matter field and a self-interacting potential can be achieved from the vacuum 5D BD field equations, where we refer to as a modified Brans–Dicke theory (MBDT). We investigate a generalized Bianchi type I anisotropic cosmology in 5D BD theory, and by employing the obtained formalism, we derive the induced matter on any 4D hypersurface in the context of the MBDT. We illustrate that if the usual spatial scale factors are functions of the time while the scale factor of extra dimension is constant, and the scalar field depends on the time and the fifth coordinate, then, in general, one will encounter inconsistencies in the field equations. Then, we assume that the scale factors and the scalar field depend on the time and the extra coordinate as separated variables in the power-law forms. Hence, we find a few classes of solutions in 5D spacetime through which we probe the one which leads to a generalized Kasner relation among the Kasner parameters. The induced scalar potential is found to be in the power law or in the logarithmic form; however, for a constant scalar field and even when the scalar field only depends on the fifth coordinate, it vanishes. The conservation law is indeed valid in this MBDT approach for the derived induced energy–momentum tensor (EMT). We proceed our investigations for a few cosmological quantities, where for simplicity we assume that the metric and the scalar field are functions of the time. Hence, the EMT satisfies the barotropic equation of state, and the model indicates that the constant mean Hubble parameter is not allowed. Thus, by appealing to the variation of the Hubble parameter, we assume a fixed deceleration parameter, and set the evolution of the quantities with respect to the fixed deceleration, the BD coupling and the state parameters. The WEC allows a shrinking extra dimension for a decelerating expanding universe that, in the constant scalar field, evolves the same way as the flat FRW spacetime in GR. The quantities for the stiff fluid and the radiation-dominated universe indicate an expanding universe commenced with a big bang. There is a horizon for each of the fluids, and the rate of expansion slows down by the time. The
allowed ranges of the deceleration and the BD coupling parameters have been obtained, and the model gives an empty universe when the time goes to infinity.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The original version of the Kaluza–Klein theory [1, 2] assumes, as a postulate, that the fifth dimension is compact. However, a particular variety of the Kaluza–Klein theory which is non-compact has also been proposed by Wesson [3–6], known as the induced-matter theory (IMT) and also as the spacetime-matter theory. This theory is based on the Campbell–Magaard theorem [7–9], which states that any analytic $D$-dimensional Riemannian manifold can be locally embedded in a $(D + 1)$-dimensional Ricci-flat spacetime. The core of IMT applied to general relativity yields the 4D Einstein equation as a subset of a 5D Einstein vacuum equation with an effective or induced energy–momentum tensor (EMT) which contains the classical properties of matter. The route usually taken in the IMT is to adapt appropriate coordinates that yield the induced matter consisted of the observations. A generalized approach of the IMT, which relates a vacuum $(D + 1)$-dimensional spacetime to a $D$-dimensional theory with source, has also been proposed [10].

Many exact solutions have been found for 5D Einstein equations in the IMT by allowing the metric to depend on an extra coordinate and by regarding effects of this extra dimension as an induced matter in a 4D spacetime, see, e.g., [11]. Some of these solutions have remarkable mathematical and physical properties. For example, the solutions which are flat in 5D spacetime but curved in four dimensions provide suitable descriptions of ordinary matter in the present and the early universe [12, 13], and of waves in the inflationary phase [14]. The new solutions in 5D spacetime have significant applications and relatively more complicated geometry than 5D Riemannian one which may be needed to explain the gauge properties of elementary particles.

Recently, another generalization of the IMT has been proposed for the Brans–Dicke (BD) theory as a fundamental underlying theory [15–17], though the effective equations and quantities in 4D hypersurfaces obtained in [15] are different from those in [16, 17]. Further application of the former approach has been performed in [18], though in this work we employ the latter approach and also pursue a $D$-dimensional version of it in another work [19]. In [17], the modified Brans–Dicke theory (MBDT) equations are obtained from 5D BD equations in which the induced matter can still be viewed as being generated by a fifth dimension. The effective induced matter1, $T_{\mu\nu}^{\text{(EMT)}}$, is composed of two parts. One part, which we present by $T_{\mu\nu}^{(\varphi)}$, depends on the BD scalar field, $\varphi$, and its derivatives with respect to the fifth coordinate. Another part, which we denote by $T_{\mu\nu}^{(\text{IMT})}$, is induced from the extra dimensional part of the metric and corresponds to the related one in the IMT. An induced scalar potential is also produced geometrically from the 5D spacetime. As the BD theory can be considered as a modified version of general relativity, one may also regard the MBDT as a modified version of the IMT.

We present and discuss the exact solutions of a generalized Bianchi type I model in the MBDT. The Bianchi models are cosmological models that have spatial homogeneous sections invariant under the action of a three-dimensional Lie group, but not necessarily isotropic ones.

1 In this work, we assume that there is no matter in the 5D spacetime.
The Bianchi type I model corresponds to the flat spatial sections as a homogeneous universe. It generalizes the flat Friedmann–Robertson–Walker model (the simplest spatial homogeneous one), which allows for anisotropy.

The Bianchi type I model for the vacuum solutions is called the Kasner solutions due to Kasner who first found such solutions in 1921 [20]. When the EMT vanishes, the Einstein tensor, due to the Einstein equation, has to vanish as well, but it is not zero in the BD theory. The BD theory for the various Bianchi type I–IX models has been considered in the literature, see, e.g., [21–24]. Also, the Bianchi type I models with non-vanishing cosmological constant as a function of the scalar field, \( \Lambda(\phi) \), have been investigated in the framework of the BD theory [25]. In addition, the Kasner cosmologies have been considered in the BD theory in four dimensions [26, 27]. Various vacuum solutions as well as some dust solutions of the Bianchi type I models with cosmological constant have been found in the standard BD theory [22]. Furthermore, the Bianchi type I models have been studied in the context of the IMT [28, 29] and in the quantum cosmology [30, 31].

In the next section, we give a brief review of the MBDT. In section 3, we investigate a generalized Bianchi type I model in 5D spacetime, derive the corresponding field equations and, meanwhile, point to an inconsistency that appears due to the assumption made in the literature. In the rest of this section, we solve the equations in a general form under the assumption of separation of variables, and obtain five classes of solutions in five dimensions. Among these five classes, we are interested in a case (case II) which is a generalized Kasner model in 5D spacetime. Then, solving the corresponding equations leads to a few relations among the generalized Kasner parameters. In section 4, by applying the effective equations achieved in section 2, we derive the scalar potential, the energy density and pressure of the induced matter and also discuss the properties of some cosmological quantities. Finally, we draw conclusions in section 5.

### 2. Modified BD theory from 5D vacuum one

In this section, we launch a brief review on the MBDT, the core idea of [15–17] while focusing on the approach of the last two references. The action of 4D BD theory [32] with a self-interacting potential\(^2\) and a matter field is

\[
S^{(4)}[g_{\alpha \beta}, \phi] = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi} \left[ \phi R^{(4)} - \frac{\omega}{\phi} g^{\alpha \beta} (D_\alpha \phi)(D_\beta \phi) - V(\phi) \right] + L^{(4)}_{\text{matter}} \right\},
\tag{1}
\]

where\(^3\) \(\omega > -3/2\), \(c = 1\), \(D_\alpha\) is the covariant derivative in 4D spacetime and the Greek indices run from 0 to 3. The BD scalar field plays a role analogous to the inverse of the Newton gravitational constant, \(\omega\) is a dimensionless coupling parameter, the Lagrangian \(L^{(4)}_{\text{matter}}\) depends on the metric and other matter fields except \(\phi\) and we have chosen \(c = 1\). The variation of action (1) with respect to the metric and the scalar field gives the field equations

\[
G^{(4)}_{\alpha \beta} = \frac{8\pi}{\phi} T^{(4)}_{\alpha \beta} + \frac{\omega}{\phi^2} \left[ (D_\alpha \phi)(D_\beta \phi) - \frac{1}{2} g_{\alpha \beta} (D_\gamma \phi)(D^\gamma \phi) \right]
\]

\[
+ \frac{1}{\phi} \left[ D_\alpha D_\beta \phi - g_{\alpha \beta} \left( D^2 \phi + \frac{1}{2} V(\phi) \right) \right]
\]

\(^2\) In the literature, in order to obtain accelerating universes, inclusion of such potentials has been considered \textit{a priori} by hand.

\(^3\) In order to have a non-ghost scalar field in the conformally related Einstein frame, i.e. a field with a positive kinetic energy term in that frame, the BD coupling parameter must be restricted to \(\omega > -(D - 1)/(D - 2)\), where \(D\) is the dimension of spacetime [33–35].
and

\[ D^2 \phi = \frac{8\pi}{2\omega + 3} T^{(4)} + \frac{1}{2\omega + 3} \left[ \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \right]. \]  

(3)

where \( \delta(L) \equiv -\frac{1}{2} \sqrt{-g} \epsilon^{AB} \delta g^{AB} \) and \( D^2 \equiv D^\alpha D_\alpha \). In the standard 4D BD theory, one is led to the accelerating cosmos in a number of ways, for example, assuming the coupling parameter \( \omega \) as a function of the time \[36\], or introducing a time-dependent cosmological term \[37\] and/or adding a scalar potential to the Lagrangian by means of an ad hoc assumption \[38\]. However, by employing the Wesson idea for the BD theory, it has been shown \[16, 17\] that such a scalar potential can geometrically be induced due to the extra fifth dimension in the MBDT. In this approach, the scalar potential has the desired properties and leads to an accelerating universe. In what follows we review the approach of \[17\] and then we employ this formalism in the Bianchi type I anisotropic model.

The action for a BD theory in 5D spacetime, without a self-interacting potential, can analogously be written as \[39\]

\[
S^{(5)}[g_{AB} , \phi] = \int d^5x \sqrt{|g^{(5)}|} \left\{ \frac{1}{16\pi} \left[ \phi R^{(5)} - \frac{\omega}{\phi} g^{AB}(\nabla_A \phi)(\nabla_B \phi) + \frac{1}{2} (\nabla_A \nabla_B \phi - g_{AB} \nabla^2 \phi) \right] + T^{(5)}_{\text{mat}} \right\},
\]

(4)

where \( \omega > -\frac{4}{3} \), the Latin indices run from 0 to 4, \( \nabla_A \) is the covariant derivative in 5D spacetime, \( R^{(5)} \), \( g^{(5)} \) and \( L^{(5)}_{\text{mat}} \) are analogs of the 4D quantities. The field equations from action \( (4) \) obviously are

\[ G^{(5)}_{AB} = \frac{8\pi}{\phi} T^{(5)}_{AB} + \frac{\omega}{\phi^2} \left[ (\nabla_A \phi)(\nabla_B \phi) - \frac{1}{2} g_{AB}(\nabla_C \phi)(\nabla_C \phi) \right] + \frac{1}{\phi} (\nabla_A \nabla_B \phi - g_{AB} \nabla^2 \phi) \]

(5)

and

\[ \nabla^2 \phi = \frac{8\pi}{3\omega + 4} T^{(5)}, \]

(6)

where \( \nabla^2 \equiv \nabla_A \nabla^A \). The vacuum case is defined as a configuration where there is no matter, except the scalar field, in 5D spacetime. Thus, by setting \( T^{(5)}_{AB} = 0 \) and consequently \( T^{(5)} \equiv g^{AB} T^{(5)}_{AB} = 0 \), equations \( (5) \) and \( (6) \) read

\[ G^{(5)}_{AB} = \frac{\omega}{\phi^2} \left[ (\nabla_A \phi)(\nabla_B \phi) - \frac{1}{2} g_{AB}(\nabla_C \phi)(\nabla_C \phi) \right] + \frac{1}{\phi} (\nabla_A \nabla_B \phi - g_{AB} \nabla^2 \phi) \]

(7)

and

\[ \nabla^2 \phi = 0. \]

(8)

Now, following the idea of the IMT approach, one can extract the effective 4D equations. Suppose that the 5D line element can be written in local coordinates \( x^A = (x^\mu, l) \) as

\[ ds^2 = g_{AB}(x^C) dx^A dx^B = g_{\mu\nu}(x^C) dx^\mu dx^\nu + g_{44}(x^C) dl^2, \]

(9)

where \( l \) represents the fifth coordinate. This metric is restrictive, but one limits oneself to it for reasons of simplicity. It is also favorable to write the fifth component of the metric as \( g_{44}(x^C) = \epsilon h^2(x^C) \), where \( h(x^C) \) is a scalar field and \( \epsilon = \pm 1 \) indicates the sign of the fifth dimension. Assume that the 5D spacetime is foliated by a family of hypersurfaces \( \Sigma \) defined by the fixed values of the fifth coordinate, where the induced metric on each hypersurface, e.g. \( \Sigma_0(l = l_0) \), is given by

\[ ds^2 = g^{(4)}_{\mu\nu}(x^C, l_0) dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu. \]

(10)

The 4D orthogonal unit vector along the extra dimension is written as

\[ n^A = \frac{\delta^A_4}{h} \quad \text{where} \quad n_A n^A = 1. \]
Dimensional reduction of equations (7) and (8) yields the effective field equations in 4D hypersurfaces. Actually, reduction of the fundamental field equation (7) and the second field equation (8) on the hypersurface $\Sigma_0$ can be written exactly the same way as the usual 4D BD field equations (2) and (3). However, now, the term $T_{\alpha\beta}^{(\text{BD})}$, that for clarity is indicated by $T_{\alpha\beta}^{(\text{EMT})}$, is the induced EMT of the effective 4D modified BD theory and has the form

$$T_{\alpha\beta}^{(\text{EMT})} = T_{\alpha\beta}^{(\omega)} + T_{\alpha\beta}^{(\text{IMT})},$$

where

$$8\pi \frac{T^{(\omega)}_{\alpha\beta}}{\varphi} = \frac{\epsilon \varphi'}{2\varphi h^2} \left[ g_{\alpha\beta}' + g_{\alpha\beta} \left( \frac{\omega \varphi'}{\varphi} - g^{\mu\nu} g_{\mu\nu}' \right) \right] + \frac{1}{2\varphi} g_{\alpha\beta} V(\varphi),$$

and

$$8\pi \frac{T^{(\text{IMT})}_{\alpha\beta}}{\varphi} = \frac{D_\alpha D_\beta h}{h} - \frac{\epsilon}{2h^2} \left\{ h' g_{\alpha\beta}' - g_{\alpha\beta} + g^{\mu\nu} g_{\mu\nu} g_{\alpha\beta}' \right\} + \frac{g_{\mu\nu} g_{\mu\nu} h_{\alpha\beta}'}{2} + \frac{g_{\alpha\beta}}{4} \left[ g^{\mu\nu} g_{\mu\nu}' + (g^{\mu\nu} g_{\mu\nu}')^2 \right].$$

The prime denotes partial derivative with respect to the extra coordinate $l$. The first part of the induced EMT comes from the scalar field and its derivatives with respect to the fifth coordinate. The second part of $T_{\alpha\beta}^{(\text{EMT})}$, which is also induced geometrically from the extra dimensional part of the metric, is a modified version of the corresponding one in the IMT approach of [3].

Furthermore, in this formalism, the induced equations (2) and (3) can also be derived from an action similar to action (1). Hence, the induced EMT satisfies the conservation law $D\alpha T_{\alpha\beta}^{(\text{EMT})} = 0$ as in the standard BD theory in the Jordan frame. Also, $V(\varphi)$ is an induced scalar potential which is defined (or fixed) by the procedure rather than to be introduced a priori by hand. That is, on the hypersurface $\Sigma_0$ one has [17]

$$\varphi \frac{dV(\varphi)}{d\varphi} \bigg|_{\Sigma_0} = -2(\omega + 1) \left[ \frac{h' \varphi'}{h} + \frac{\epsilon}{h^2} \left( \varphi'' - \frac{\varphi'}{h'} \right) \right] - \frac{\epsilon \omega \varphi'}{h^2} \left( \frac{\varphi'}{\varphi} + g^{\mu\nu} g_{\mu\nu}' \right) + \frac{\epsilon \varphi'}{4h^2} [g^{\mu\nu} g_{\mu\nu}' + (g^{\mu\nu} g_{\mu\nu}')^2].$$

Obviously, in the particular case $\omega = -1$ and $l$ as a cyclic coordinate, the induced potential, without loss of generality, vanishes. Thus, to investigate a general version of 4D BD theory with a cyclic extra coordinate, one should assume that $\omega \neq -1$. However, when the metric depends on the extra coordinate, one has, in general, a non-vanishing scalar potential.

In the following we delineate a generalized Bianchi type I model, as an application of the above formalism in the cosmological context.

### 3. Generalized Bianchi type I metric in 5D BD theory

We apply a generalization of the Bianchi type I anisotropic model for a 5D spacetime (with space-like extra dimension) in a co-moving reference frame as

$$dS^2 = -dt^2 + a^2(t, l) \, dx^2 + b^2(t, l) \, dy^2 + c^2(t, l) \, dz^2 + h^2(t, l) \, dl^2,$$

where $t$ is the cosmic time, $(x, y, z)$ are the Cartesian coordinates and $a(t, l)$, $b(t, l)$, $c(t, l)$ and $h(t, l)$ are different cosmological scale factors in each of the four directions. When one
assumes that there is no matter in 5D spacetime and $\varphi = \varphi(t, l)$, the field equations (7) and (8), with a little manipulation, give

$$\frac{1}{\hbar^2} \left( -\frac{a'}{a} \frac{b'}{b} - \frac{a'}{a} \frac{c'}{c} + \frac{b'}{b} \frac{c'}{c} + \frac{a'}{a} \frac{h'}{h} + \frac{b'}{b} \frac{h'}{h} + \frac{c'}{c} \frac{h'}{h} + \frac{a''}{a} - \frac{b''}{b} - \frac{c''}{c} \right) + H_1 H_2 + H_1 H_3$$

$$+ H_1 H_4 + H_2 H_3 + H_2 H_4 + H_3 H_4 = \frac{\omega}{2} \left( \frac{\dot{\varphi}}{\varphi} \right)^2 + \frac{\omega}{2\hbar^2} \left( \frac{\varphi'}{\varphi} \right)^2 + \frac{\ddot{\varphi}}{\varphi},$$

where $\dot{\varphi}$ and $\ddot{\varphi}$ denote derivatives with respect to the cosmic time and $H_1, H_2, H_3, H_4$ have been used in [15] when solving the field equations (7) and (8) with inconsistencies when the field equations (7) and (8) are solved. For example, these conditions have been used in [15] when $a = b = c$. Indeed in this case, for $\varphi(t, l) = \varphi(t) \chi(l)$ one has, from equation (23),

$$\ddot{\varphi} + 3H(t)\varphi + |\lambda|\varphi = 0$$

and

$$\frac{d^2\chi}{dl^2} + |\lambda|\chi(l) = 0,$$

where the dot denotes derivative with respect to the cosmic time and $H_1, H_2, H_3, H_4$ are the corresponding Hubble parameters in each of the four directions, respectively.

Meanwhile, let us revisit a specific case that has been employed in the literature. When $h(t, l)$ is a constant and the other scale factors are functions of the cosmic time only, then the Einstein tensor, $G_{AB}$, will be a function of time as well. Then, if one insists that the scalar field, $\varphi$, remains a function of the fifth coordinate in addition to the time, one will encounter inconsistencies when the field equations (7) and (8) are solved. For example, these conditions have been used in [15] when $a = b = c$. Indeed in this case, for $\varphi(t, l) = \varphi(t) \chi(l)$ one has, from equation (23),

$$\ddot{\varphi} + 3H(t)\varphi + |\lambda|\varphi = 0$$

and

$$\frac{d^2\chi}{dl^2} + |\lambda|\chi(l) = 0,$$
where, without loss of generality, we have chosen the constant $h$ as $h = 1$. The solutions of the above equations are
\[ \phi(t) = \phi_0 e^{-\frac{3}{2}H_o \pm \frac{1}{2} \sqrt{9H_o^2 - 4|\lambda|}}t \]  
(26)
and
\[ \chi(l) = \chi_0 e^{\pm i \sqrt{3}|\lambda|}l, \]  
(27)
where $H_o$ is a constant Hubble parameter, $\lambda$ is a negative separation constant and $\phi_0$ and $\chi_0$ are integration constants. However, from equation (22) one has
\[ G_{04} = \omega \frac{\phi''}{\phi'} + \frac{(\dot{\phi})'}{\phi} = 0, \]  
(28)
where the last step comes from the fact that the $G_{04}$ component vanishes for the assumed metric. Substituting the separation form of $\phi$ with solutions (26) and (27) into equation (28) yields
\[ (\omega + 1) \left( -\frac{3}{2}H_o \pm \frac{1}{2} \sqrt{9H_o^2 - 4|\lambda|} \right) (\pm i |\lambda|) = 0. \]  
(29)
Relation (29) for $\omega \neq -1$ gives $|\lambda| = 0$. Therefore, $\chi(l)$ will be a constant; that is, $\phi$ does not depend on the fifth coordinate, and thus equations (12) and (15) indicate that $T_{\alpha\beta}^{(EMT)}$ and the induced scalar potential are null contrary to the results obtained in the literature [15]. The exception case is $\omega = -1$ which is precisely the value predicted when the BD theory is derived as the low energy limit of some string theories [40, 41].

Also note that there is a unique solution when all the metric components (including the extra component, contrary to the above condition) depend only on the time, while the scalar field is a function of the time and the extra coordinate [17].

Based on the usual spatial homogeneity, let us now solve the field equations (17)–(23) using separation of variables as
\[ \psi(t, l) = \psi_0 t^{p_0} l^{s_0}, \quad a(t, l) \propto t^{p_1} l^{s_1}, \quad b(t, l) \propto t^{p_2} l^{s_2}, \quad c(t, l) \propto t^{p_3} l^{s_3}, \quad h(t, l) = h_0 t^{p_4} l^{s_4}, \]  
(30)
where the $h_0$ and $\psi_0$ are constants, and the $p_A$'s and $s_A$'s are parameters satisfying field equations (17)–(23). Actually, it turns out to be easier to work with this ansatz. First substituting ansatz (30) into equation (23) gives
\[ p_0 \left( \sum_{A=0}^{4} p_A - 1 \right) h_0^2 t^{p_0 - 2p_4} l^{s_0} + s_0 \left[ 1 + s_4 - \left( \sum_{\mu=0}^{3} s_\mu \right) \right] t^{p_0 - 2p_4} l^{s_0 - 2} = 0. \]  
(31)
This relation is satisfied for the following five conditions:

I: $p_0 = 0$, \quad $s_0 = 0$,

II: $p_0 = 1 - \sum_{i=1}^{4} p_i$, \quad $s_0 = 1 + s_4 - \sum_{i=1}^{3} s_i$,

III: $p_0 = 0$, \quad $s_0 = 1 + s_4 - \sum_{i=1}^{3} s_i$,

IV: $p_0 = 1 - \sum_{i=1}^{4} p_i$, \quad $s_0 = 0$,

V: $p_0 = 1$, \quad $s_0 = -1$, \quad $h_0^2 p_0 \left[ \left( \sum_{A=0}^{4} p_A \right) - 1 \right] + s_0 \left[ 1 + s_4 - \left( \sum_{\mu=0}^{3} s_\mu \right) \right] = 0.$
Clearly, mixtures of these conditions may also occur. The solution case I implies that $\phi$ is a constant which corresponds to general relativity in five dimensions. Such a solution for the spatially flat FRW metric has been investigated in [17]. In this work, we purpose to investigate case II which leads to a generalized Kasner spacetime in five dimensions. Actually, in what follows we solve its corresponding equations and, in the next section by applying the formalism presented in the previous section, we investigate the properties of such a solution on each 4D hypersurface.

Now, substituting ansatz (30) into the rest of the field equations, namely equations (17)–(22), yields

\[
h_5^2 \left[ p_1 p_2 + p_1 p_3 + p_1 p_4 + p_2 p_3 + p_2 p_4 + p_3 p_4 - \left( \frac{\omega + 2}{2} \right) p_0^2 + p_0 \right] t^{2 p_0 - 2} = \left[ s_1^2 + s_2^2 + s_3^2 + \frac{\omega}{2} s_0^2 - (s_1 + s_2 + s_3) + s_1 s_2 + s_1 s_3 + s_2 s_3 - s_4(s_1 + s_2 + s_3) \right] t^{(-2 s_4 - 2)},
\]

(33)

\[
h_5^2 \left[ p_2^3 + p_2^3 + p_2^3 + \frac{\omega}{2} p_0^2 - (p_2 + p_3 + p_4) - p_0 p_1 + p_1 p_3 + p_1 p_4 + p_3 p_4 \right] t^{2 p_1 - 2} = \left[ s_1^2 + s_2^2 + s_3^2 + \frac{\omega}{2} s_0^2 - s_0 s_1 + s_2 s_1 - s_2 s_4 - s_3 s_4 - s_2 - s_3 \right] t^{(-2 s_4 - 2)},
\]

(34)

\[
h_5^2 \left[ p_1^3 + p_1^3 + p_1^3 + \frac{\omega}{2} p_0^2 - (p_1 + p_3 + p_4) - p_0 p_2 + p_2 p_3 + p_2 p_4 + p_3 p_4 \right] t^{2 p_2 - 2} = \left[ s_1^2 + s_2^2 + s_3^2 + \frac{\omega}{2} s_0^2 - s_0 s_2 + s_1 s_3 - s_1 s_4 - s_3 s_4 - s_1 - s_3 \right] t^{(-2 s_4 - 2)},
\]

(35)

\[
h_5^2 \left[ p_3^3 + p_3^3 + p_3^3 + \frac{\omega}{2} p_0^2 - (p_1 + p_2 + p_4) - p_0 p_3 + p_1 p_2 + p_1 p_4 + p_2 p_4 \right] t^{2 p_3 - 2} = \left[ s_1^2 + s_2^2 + s_3^2 + \frac{\omega}{2} s_0^2 - s_0 s_3 + s_1 s_2 - s_1 s_4 - s_2 s_4 - s_1 - s_2 \right] t^{(-2 s_4 - 2)},
\]

(36)

\[
h_5^2 \left[ p_1^3 + p_1^3 + p_1^3 + \frac{\omega}{2} p_0^2 - (p_1 + p_2 + p_3) - p_0 p_4 + p_1 p_2 + p_1 p_3 + p_2 p_3 \right] t^{2 p_4 - 2} = \left[ - \left( \frac{\omega + 2}{2} \right) s_0^2 + s_0 + s_0 s_4 + s_1 s_2 + s_1 s_3 + s_2 s_3 \right] t^{(-2 s_4 - 2)}
\]

(37)

and

\[
(\omega + 1) p_0 s_0 + p_1 s_1 + p_2 s_2 + p_3 s_3 = p_4(s_0 + s_1 + s_2 + s_3).
\]

(38)

For later convenience, we will set $h_5 = 1$ in ansatz (30). Also, in order to compare the solutions with the corresponding ones in the IMT [28, 29], one must set the BD scalar field to be a constant. By assuming $p_3 \neq 1$ and $s_4 \neq -1$, one can match the coefficients on the left- and right-hand sides of relations (33)–(37). Hence, after a little manipulation and considering relation (38), one finds the following constraints among the generalized Kasner parameters$^4$:

\[
\sum_{A=0}^{4} p_A = 1,
\]

(39)

\[
(\omega + 1) p_0^2 + \sum_{m=1}^{4} p_m^2 = 1,
\]

(40)

$^4$ The relations in the second line of (32), i.e. case II, have been rewritten as relations (39) and (41), respectively.
$$\sum_{\mu=0}^{3} s_{\mu} = 1 + s_{4},$$  \hspace{1cm} (41)$$

$$(\omega + 1) s_{0}^{2} + \sum_{i=1}^{3} s_{i}^{2} = (1 + s_{4})^{2},$$  \hspace{1cm} (42)$$

$$(\omega + 1) p_{0} s_{0} + \sum_{i=1}^{3} p_{i} s_{i} = p_{4} (1 + s_{4}),$$  \hspace{1cm} (43)$$

$$\sum_{m,n=1 \atop m < n}^{4} p_{m} p_{n} = \frac{(\omega + 2)}{2} p_{0}^{2} - p_{0}$$  \hspace{1cm} (44)$$

and

$$\sum_{i,j=1 \atop i < j}^{3} s_{i} s_{j} = \frac{(\omega + 2)}{2} s_{0}^{2} - s_{0} (1 + s_{4}).$$  \hspace{1cm} (45)$$

Constraints (39)–(43) are the generalized Kasner relations in 5D BD theory, though there are only five independent relations among the Kasner parameters, and relations (44) and (45) can actually be achieved from the others. As will be shown, the case $p_{0} = 0$ has particular properties, and hence we will highlight this case in all the following parts of the work.

- In the special case $p_{0} = 0 = s_{0}$, the BD scalar field is a constant regardless of any specific value of the BD coupling parameter. Thus, constraints (39)–(43) lead to the corresponding relations in 5D general relativity, [28, 29], which in turn gives the 5D analog of the well-known Kasner solution.

- When the scale factors and the scalar field only depend on time, for the particular case of spatial isotropy, by assuming $p_{1} = p_{2} = p_{3} \equiv p$, from (16), (30), (39) and (44), one obtains

\[ dS^2 = -dt^2 + A^2 t^2 p(dx^2 + dy^2 + dz^2) + B^2 t^2 p_4 dl^2, \]
\[ \phi = \phi_0 t^{-3p - p_4} \quad \text{and} \quad \omega = \frac{2(3p + p_4)(1 - p_4) - 12p^2}{(1 - 3p - p_4)^2}, \]

where the $A$ and $B$ are constants. These results are the same as the corresponding ones investigated for the spatially flat FRW solutions in 5D BD theory [17].

In the limit $p_4 \rightarrow 1 - 3p$, one obtains again a constant $\phi$ and, in general, an infinite $\omega$. In this limit\(^5\), only for $p = 1/2$, i.e. $p_4 = -1/2$, the field equations are satisfied, namely one has

\[ dS^2 = -dt^2 + A^2 t (dx^2 + dy^2 + dz^2) + B^2 t^{-1} dl^2. \]

Solution (47) is a unique solution in 5D Ricci-flat spacetime for metric (16) in the particular case where there is no isotropy in all dimensions.

In the next section, by employing the formalism obtained in the preceding section for the field equations, we derive the induced scalar potential, the energy density and pressure of the induced matter, and hence discuss the properties of the model.

\(^5\) We disregard the static case $p = 0$. 
4. Reduced BD cosmology in 4D spacetime

For metric (16), the induced metric on the hypersurface $\Sigma_0$ is just a part of it, and the non-vanishing components of the induced EMT (12) are

$$8\pi T^0_0(\text{EMT}) = -\frac{\dot{h}}{h} + \frac{1}{h^2} \left[ \frac{a'b'}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc} - \frac{\omega}{2} \left( \frac{\varphi'}{\varphi} \right)^2 + \left( \frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} \right) \frac{\varphi'}{\varphi} \right] + V(\varphi) \frac{2}{2\varphi}$$

(48)

and

$$8\pi T^i_1(\text{EMT}) = -\frac{\dot{h}}{ah} + \frac{1}{h^2} \left[ \frac{a'h'}{ah} + \frac{a''}{a} - \frac{b'c'}{bc} + \frac{\omega}{2} \left( \frac{\varphi'}{\varphi} \right)^2 - \left( \frac{b'}{b} + \frac{c'}{c} \right) \frac{\varphi'}{\varphi} \right] + V(\varphi) \frac{2}{2\varphi},$$

(49)

where the induced potential $V(\varphi)$ should be determined from (15). Also, by interchanging $a \leftrightarrow b$ and $a \leftrightarrow c$ in relation (49), one obtains $(8\pi/\varphi)T^0_2(\text{EMT})$ and $(8\pi/\varphi)T^3_3(\text{EMT})$, respectively. As, in general, the three components $T^i(\text{EMT})$ (where $i = 1, 2, 3$ with no sum) are not equal, one cannot consider the induced matter as a perfect fluid.

The induced scalar potential is obtained by substituting the power-law relations (30) into definition (15) as

$$\left. \frac{dV}{d\varphi} \right|_{\Sigma_0} = 2(\omega + 1) p_0 p_4 t^{-2} + \zeta h_0^{-2} l_0^{-2} \ln(\varphi)$$

(50)

where

$$\zeta \equiv -\omega s_0 [3s_0 + 2(s_1 + s_2 + s_3 - s_4 - 1)] + 2[s_0(s_4 - s_0 + 1) + s_1 s_2 + s_1 s_3 + s_2 s_3].$$

(51)

However, constraints (39)–(45) make $\zeta = 0$ for the model. Therefore, by integrating (50) while using the generalized Kasner relations, one obtains

$$V(\varphi) = \begin{cases} 
\left[ \frac{2(\omega + 1) p_0^2 p_4^2 \varphi^{\omega - 2}}{p_0 - 2} l_0^{\omega - 2} \varphi^{\omega - 2} \right] & \text{for } p_0 \neq 0, 2 \\
4\varphi_0(\omega + 1) p_4 l_0^{\omega} \ln(\varphi) & \text{for } p_0 = 2
\end{cases}$$

(52)

on the hypersurface, where we have set the constants of integration equal to zero. Note that, when $\omega = -1$ and or $p_4 = 0$, the effective potential vanishes.

• For the special case $p_0 = 0$, $\varphi \frac{dV}{d\varphi}|_{\Sigma_0}$ identically vanishes by (50). Therefore, without loss of generality, one can set $V = 0$ in this case.

In the following subsections we probe the properties of the induced EMT.

4.1. Conservation of induced energy–momentum tensor

As mentioned, in the MBDT approach, the induced EMT obeys the conservation law as in the Einstein theory. Let us check it for the anisotropic induced EMT.

The conservation law for the induced EMT in the case of the Bianchi type I line element reduces to

$$\dot{\rho} + \sum_{i=1}^{3} (\rho + p_i) H_i = 0,$$

(53)

where $\rho \equiv -T^0_0(\text{EMT})$ and $p_i \equiv T^i_1(\text{EMT})$ (no sum on $i$) are the energy density and the directional pressure of the induced EMT, respectively.
Now, from relations (30), (41), (45), (48), (49) and (52) one obtains
\[
\rho = \begin{cases} 
\frac{p_4 \psi_0^0}{8\pi} [ (p_0 - 2)(p_4 - 1) - (\omega + 1)p_0^2 ] t^{p_0 - 2} & \text{for } p_0 \neq 0, 2 \\
\frac{p_4 \psi_0^0}{8\pi} [ (p_4 - 1) - 2(\omega + 1)\ln(\psi_0^0 t^2) ] & \text{for } p_0 = 2
\end{cases}
\]
and
\[
P_i = \begin{cases} 
- \frac{p_4 \psi_0^0}{8\pi} [ (p_0 - 2)p_i - (\omega + 1)p_0^2 ] t^{p_0 - 2} & \text{for } p_0 \neq 0, 2 \\
- \frac{p_4 \psi_0^0}{8\pi} [ p_i - 2(\omega + 1)\ln(\psi_0^0 t^2) ] & \text{for } p_0 = 2.
\end{cases}
\]

Obviously, for \(i \neq j\), one has \(P_i \neq P_j\), which means the induced EMT is not a perfect fluid. By substituting the corresponding induced energy density and pressure from relations (54) and (55) for the power law and the logarithmic forms into (53) and using relations (39) and (40), one can explicitly show that the conservation law is satisfied. Indeed as mentioned, this result is expected as a straightforward consequent of the formalism presented in section 2.

- When \(p_0 = 0\), using the power-law relations (30), equations (48) and (49), and owing to the fact that the induced potential vanishes for this case, one easily obtains
\[
\rho = \frac{\psi_0^0}{8\pi} \frac{p_4 (p_4 - 1)}{t^2} 
\]
and
\[
P_i = - \frac{\psi_0^0}{8\pi} \frac{p_4 p_i}{t^2}.
\]

These results indicate that the conservation law is satisfied for this case as well.

### 4.2. Properties of the physical quantities

In the rest of this work, for simplicity, we assume that the components of metric (16) and the scalar field only depend on the cosmic time; namely, ansatz (30) reads
\[
\psi(t) = \kappa t^{p_0}, \quad a(t) \propto t^{p_1}, \quad b(t) \propto t^{p_2}, \quad c(t) \propto t^{p_3}, \quad h(t) = mt^{p_4},
\]
where the \(\kappa\) and \(m\) are constants. Now, equations (48) and (49) lead to
\[
\frac{8\pi}{\psi} \dot{\rho}_{\text{EEMT}} = - \frac{\dot{h}}{h} + \frac{V(\psi)}{2\psi}
\]
and
\[
\frac{8\pi}{\psi} \dot{\sigma}_{i\text{EMT}} = - H_i H_4 + \frac{V(\psi)}{2\psi} \quad (\text{no sum on } i).
\]

Also, from definition (15) and relations (58), one has
\[
\frac{dV}{d\psi} \bigg|_{\Sigma_0} = 2(\omega + 1)\kappa^{2/p_0} p_0 p_4 \psi^{(p_0 - 2)/p_0}.
\]

Integrating the above relation gives
\[
V(\psi) = 2(\omega + 1)\kappa^{2/p_0} \frac{p_0^2 p_4}{p_0 - 2} \psi^{(p_0 - 2)/p_0}
\]

Nevertheless, we have confined ourselves to a fixed hypersurface.
on the hypersurface, where we have set the constant of integration equal to zero, and have assumed \( p_0 \neq 0, 2 \). For the case \( p_0 = 0 \), one again has \( V = 0 \). Substituting relation (62) into relations (59) and (60) gives

\[
\rho = -\frac{T^0(E_{\text{MT}})}{0} = \frac{k}{8\pi} \left( \frac{p_0^4}{p_0 - 2} \right) \left[ (p_0 - 2)(p_4 - 1) - (\omega + 1)p_0^2 \right] t^{p_0 - 2} \tag{63}
\]

and

\[
P_i = T_i(E_{\text{MT}}) = -\frac{k}{8\pi} \left( \frac{p_0^4}{p_0 - 2} \right) \left[ (p_0 - 2)p_i - (\omega + 1)p_0^2 \right] t^{p_0 - 2}, \tag{64}
\]

where we have used relations (39) and (40).

For later convenience, one usually defines the spatial volume \( \bar{V} \), the average scale factor \( \alpha \) and the mean Hubble parameter \( H \) as [42–45]

\[
\bar{V} \equiv abc, \quad \alpha \equiv \sqrt[3]{abc} = \bar{V}, \quad \text{and} \quad H \equiv \frac{1}{3} \sum_{i=1}^3 H_i = \frac{\dot{V}}{3 \bar{V}} = \frac{\dot{\alpha}}{\alpha}. \tag{65}
\]

Hence, by relations (39) and (58) one obtains

\[
\alpha(t) \propto t^\beta \quad \text{with} \quad \beta \equiv \frac{(p_1 + p_2 + p_3)}{3} = \frac{(1 - p_0 - p_4)}{3}, \tag{66}
\]

where we exclude \( \beta = 0 \), for it gives a static universe. Let us also define the effective pressure on the hypersurface as

\[
P_{\text{eff}} \equiv \frac{1}{3H} \sum_{i=1}^3 H_i P_i, \tag{67}
\]

where this definition yields the conservation law for the isotropic case. Substituting relation (64) into definition (67) gives

\[
P_{\text{eff}} = -\frac{k}{8\pi} \frac{p_0^4(1 + p_4)}{(p_0 - 2)} \left[ \frac{(p_0 - 2)(1 - p_4) + (\omega + 1)p_0^2}{1 - p_0 - p_4} \right] t^{p_0 - 2}, \tag{68}
\]

where we have used relation (39). Hence, relations (63) and (68) yield a barotropic equation of state on the hypersurface as \( P_{\text{eff}} = w\rho \) with

\[
w = \frac{1 + p_4}{1 - p_0 - p_4}. \tag{69}
\]

Note that, as \( p_0 \neq 2 \), then \( w \neq -1 \).

Furthermore, in what follows we present the properties of a few important physical quantities such as the expansion scalar \( \Theta \), the shear scalar \( \sigma^2 \) and the deceleration parameter \( q \) defined in terms of the mean Hubble parameter respectively as

\[
\Theta = 3H, \tag{70}
\]

\[
\sigma^2 = \frac{1}{2} \sigma_{ij}\sigma^{ij} = \frac{1}{2} \left( \sum_{i=1}^3 H_i^2 - 3H^2 \right) \tag{71}
\]

and

\[
q = -\frac{d}{dt} \left( \frac{1}{H} \right) - 1 = -\frac{\dot{\alpha}}{\alpha^2}. \tag{72}
\]

\(^7\) From now on, we do not investigate the logarithmic potential with \( p_0 = 2 \), for it leads to some difficulties when the weak energy condition (WEC) is applied. For example, in order to satisfy the WEC, one can set \( p_4 = 0 \) when \( p_0 = 2 \), which in turn gives no induced EMT on the hypersurface.
These physical quantities are important in the observational cosmology. Substituting ansatz (30) into relations (70)–(72) gives

$$\Theta = \frac{1 - p_0 - p_4}{t},$$

(73)

$$\sigma^2 = \frac{1}{6t^2} \left[ 2p_4(1 - p_0 - 2p_4) - (3\omega + 4)p_0^2 + 2(1 + p_0) \right]$$

(74)

and

$$q = \frac{2 + p_0 + p_4}{1 - p_0 - p_4},$$

(75)

where again we have used relations (39) and (40). Note that relation (75) indicates that $q \neq -1$, i.e. the mean Hubble parameter cannot be constant in this model.

On the other hand, the special law of variation of the Hubble parameter introduced by Bermann [46] gives models with constant values of the deceleration parameter. Such a type of assumption has been applied in the literature, see, e.g., [47–50]. This type of variation of the Hubble parameter is consistent with the observation. By appealing to this law and hence by considering the deceleration parameter to be a constant, i.e. $q_0$, one obtains $\alpha(t) = (C_1 t + C_2)^{1/(q_0 + 1)}$, where $C_1 \neq 0$ and $C_2$ are the integration constants. With a proper choice of coordinates and constants of integration, the average scale factor can be written as\(^8\) $\alpha(t) \propto t^{1/(q_0 + 1)}$ that, comparing with relation (66), means

$$\beta = \frac{1}{q + 1}.$$  

(76)

For an expanding universe, one must set $q + 1 > 0$. Thus, any positive value of $q$ corresponds to a decelerating universe, while the negative values of $q$, restricted to $-1 < q < 0$, yield accelerating ones which are consistent with the recent observed data [51]. Now, we can set the most useful characters with respect to a fixed $q$ (as an observational quantity), the state parameter $w$ and the BD coupling parameter. Indeed, from relations (39), (66), (69) and (76), one obtains

$$p_0 = \frac{-1 + 2q - 3w}{q + 1} \quad \text{and} \quad p_4 = \frac{-(1 + q) + 3w}{q + 1}.$$  

(77)

Substituting $p_0$ and $p_4$ from (77) into relation (63) gives

$$\rho = \left( \frac{\kappa}{24\pi} \right) \left[ \frac{3w - (q + 1)}{(1 + w)(q + 1)^2} \right] \left[ 18w^2 + 9w(1 - 2q) - 5(1 + 2q) \right.$$  

$$+ 4q^2 + (1 + 3w - 2q^2)\omega t^{\frac{1-w}{q+1}},$$

(78)

• In the special case $p_0 = 0$, the scalar field is constant and, regardless of any specific value of $\omega$, the BD theory reduces to general relativity. Also, we recover the generalized Kasner relations of [28]. In this case, by substituting (57) into (67) and using relation (39), one obtains

$$P_{\text{eff}} = -\frac{\kappa}{8\pi} \frac{p_4(1 + p_4)}{t^2}.$$  

(79)

Thus, the effective equation of state holds with

$$w = \frac{1 + p_4}{1 - p_4},$$  

(80)

\(^8\) For convenience, we will drop the index zero from the constant $q_0$. 

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where (as mentioned before) \( p_4 \neq 1 \), and again \( w \neq -1 \) (which, in this special case, is also consistent with \( q \neq -1 \)).

In order to satisfy the WEC, one must have \( p_4 > 1 \) or \( p_4 < 0 \), where the latter choice indicates that the extra dimension shrinks by the time. Note that when \(-2 < p_4 < 1\), the deceleration parameter is positive, and when \((p_4 < -2 \text{ or } p_4 > 1)\), it is negative. For the radiation-dominated universe, by setting \( w = 1/3 \) one obtains \( p_4 = -1/2 \) which by substituting it into relations (66) and (56) and using relation (39) gives

\[
\alpha(t) \propto t^{1/2}, \quad \rho(t) \propto t^{-2} \quad \text{for} \quad P_{\text{eff}} = \frac{1}{3} \rho. \tag{81}
\]

For the matter-dominated universe, we set \( w = 0 \) which gives \( p_4 = -1 \). Then, from (66) and (56), one has

\[
\alpha(t) \propto t^{2/3}, \quad \rho(t) \propto t^{-2} \quad \text{for} \quad P_{\text{eff}} = 0. \tag{82}
\]

In both of the above eras, one has \( p_4 < 0 \); thus, the WEC is satisfied and the extra dimension shrinks by the cosmic time. Also for this case, regardless of any specific value of \( \omega \), the matter- and the radiation-dominated epoches evolve the same way as the corresponding ones of the flat FRW spacetime in general relativity, as expected.

In the following subsections, we probe the properties of two interesting cases of the radiation-dominated era and \( w = 1 \) (known as the stiff fluid), for \( p_0 \neq 0, 2 \).

### 4.2.1. Radiation-dominated universe

The main motivation of the IMT in deriving the field equations with matter sources on any 4D hypersurface is to interpret the classical properties of matter (viewed as a manifestation of the extra dimensions) rather than solely accepting them as a gift. In particular, when the components of the metric are independent of the extra coordinate, one obtains the induced EMT to be a traceless one [52], in other words, a radiation-like equation of state. Thus, in this subsection, we propose to investigate such an equation of state in the MBDT.

Substituting \( w = 1/3 \) into relations (77) and (78) gives

\[
p_0 = \frac{2(q - 1)}{q + 1}, \quad p_4 = -\frac{q}{q + 1} \tag{83}
\]

and

\[
\rho = 3P_{\text{eff}} = \frac{\kappa q}{8\pi(q + 1)^2}[2q(\omega + 2) - q^2(\omega + 1) - \omega]t^{-\frac{q}{q+1}}. \tag{84}
\]

Thus, relations (73), (74) and (62) give

\[
\Theta = \frac{3}{(q + 1)t}, \tag{85}
\]

\[
\sigma^2 = \frac{-4q^2(\omega + 1) + 2q(4\omega + 5) - (4\omega + 6)}{2(q + 1)^2t^2} \tag{86}
\]

and

\[
V(\phi) = \frac{2(\omega + 1)\kappa q(q - 1)^2}{(q + 1)^2} \phi^{\frac{q}{q+1}}. \tag{87}
\]

The \( \rho > 0 \) in relation (84) gives \(-3/2 < \omega < 0\) and positive values of \( q \) (which is the case for the radiation-dominated epoch)\(^{10}\). When the deceleration parameter takes positive values, note that, there are other ranges of \( \omega \) and \( q \) that satisfy the WEC, but the mentioned range is the only one in which the induced energy density behaves smoothly at the whole of the range.

\(^9\) In obtaining the generalized Kasner relations, we assumed this constraint.

\(^{10}\) Note that, there are other ranges of \( \omega \) and \( q \) that satisfy the WEC.
relation (83) implies that the extra dimension shrinks as the time increases. The general behavior of the induced energy density at an arbitrary fixed time is shown versus $q$ and $\omega$ in figure 1. Also, we have plotted the induced energy density versus time in figure 2 for an allowed value of $\omega$ and $q$.

By assuming the mentioned range that satisfies the WEC for the radiation-dominated universe, relations (84)–(87) show that at $t = 0$ the physical quantities $\rho$, $P_{\text{eff}}$, $\Theta$, $\sigma^2$, and $V(\varphi)$ take infinite values. However, when $t \to 0$ the BD scalar field takes finite values when $q \geq 1$ and diverges when $-1 < q < 1$. As in the limit $t \to 0$, the spatial volume goes to zero and the expansion scalar tends to infinity; thus, the evolution of universe starts with zero volume and the rate of the expansion is infinite. These results indicate that the model is in accordance with the big bang model. As $t$ increases, the expansion scalar decreases but the spatial volume increases; this shows that the rate of the expansion slows down by time. When $t \to \infty$, the spatial volume goes to infinity as well, whereas the other physical quantities, i.e. the expansion scalar, the scalar potential, the shear scalar, the induced energy density and the effective pressure, tend to zero. However, in this limit, the scalar field has two limits; it
4.2.2. Stiff fluid distribution. The high energy density regime of the early universe is usually described with the stiff cosmological fluid. This kind of fluid, which is specified by the equation of state $P_{\text{eff}} = \rho$, has been applied for the stellar and cosmological models with utter dense matter [53]. For the stiff fluid, by substituting $w = 1$ into relations (77) and (78), one gains

\begin{align*}
 p_0 &= \frac{2(q - 2)}{q + 1}, \\
p_4 &= \frac{2 - q}{q + 1} \\
P_{\text{eff}} &= \rho = \frac{\kappa(2 - q)}{24\pi(q + 1)^2} [2q^2(\omega + 1) - 2q(4\omega + 7) + (8\omega + 11)]t^{-\frac{2}{2}}.
\end{align*}

Substituting these values into relations (73), (74) and (62) gives the physical quantities as

\begin{align*}
 \Theta &= \frac{3}{(q + 1)t}, \\
\sigma^2 &= \frac{-2q^2(\omega + 1) + (8\omega + 11)(q - 1)}{(q + 1)^2t^2} \\
V(\psi) &= \frac{4\kappa\frac{\sqrt{4\omega + 7} - 3\sqrt{2\omega + 3}}{2(\omega + 1)}}{3(q + 1)^2} \psi^\frac{q}{2}.
\end{align*}

For any value of $\omega > -1$ and $0 < q < (4\omega + 7 - 3\sqrt{2\omega + 3})/[2(\omega + 1)]$, the WEC is satisfied for the stiff fluid. Figure 3 depicts the variation of the induced energy density at an arbitrary fixed time versus the BD coupling and the deceleration parameters. Also, in figure 4, the behavior of $\rho$ versus $q$ is plotted at an arbitrary fixed time and a constant value of the BD coupling parameter. The evolution of $\rho$, in this case, is similar to the one depicted in figure 2.

For this model, one observes that at $t = 0$, the spatial volume is zero while the expansion scalar is infinite, which shows that the universe starts its evolution with zero volume at the beginning where the rate of the expansion is infinite. At the initial point $t = 0$, the induced energy density, the effective pressure, the shear scalar and the induced scalar potential diverge. However, the BD scalar field has no initial singularity for $q \geq 2$ and diverges for $-1 < q < 2$. By increasing $t$, the expansion scalar decreases while the spatial volume increases, which indicates that the rate of the expansion decreases by the time. When $t$ goes to infinity, the physical quantities $\rho, P_{\text{eff}}, \sigma^2, \psi$ (for $-1 < q < 2$), $V(\psi)$ and $\Theta$ tend to zero, but the spatial volume and the BD scalar field (for $q > 2$) become infinitely large. Thus, in the case of the stiff fluid, one has an empty universe for large values of the time. Again, as the ratio $\sigma^2/\Theta^2$ does not depend on the time, the model, in general, does not approach isotropy for large values of the cosmic time. Integral (88) also shows that there is a horizon for the stiff fluid distribution, and relation (89) indicates that, for $q > 2$, its fifth dimension shrinks by the time.
Figure 3. The induced energy density at an arbitrary fixed time versus $\omega$ and $q$ for the stiff fluid, where the BD coupling parameter is restricted to $-1 < \omega < 100$.

Figure 4. The induced energy density at an arbitrary fixed time versus $q$ with $\omega = 0.5$ for the stiff fluid.

5. Conclusions

As the BD theory is a modified version of general relativity, one can view the MBDT as a modified version of the IMT in which general relativity is replaced by the BD theory as the fundamental underlying theory. We purpose to investigate a generalized Bianchi type I anisotropic cosmology in 5D BD theory. Then, by employing the obtained formalism, we derive the induced scalar potential, the energy density and pressure of the induced matter on any 4D hypersurface in the context of the MBDT. Hence, we probe the properties of the induced quantities for the specified cosmological model as the quantities fixed by the procedure rather than introduced a priori by hand. Indeed, the core of the IMT idea, and hence the MBDT, for inducing 5D equations on any 4D hypersurface has been motivated to interpret the classical properties of 4D matter rather than solely accepting them. Hence,
in the cosmological applications, what one obtains from vacuum 5D equations on any 4D hypersurface is actually the 5D quantities viewed as the ordinary matter.

Meanwhile, we illustrate that if a diagonalized metric, in the usual spatial dimensional isotropic case and in a gauge with a constant $g_{44}$, does not depend on the fifth coordinate while the scalar field does, it will lead, in general, to inconsistencies when the field equations are solved. Indeed, the induced scalar potential and the induced EMT vanish contrary to the results obtained in the literature. The exception case is $\omega = -1$ which is precisely the value predicted when the BD theory is derived as the low energy limit of some string theories. Also, there is a unique solution when all the metric components (including the extra component, contrary to the above condition) depend only on the cosmic time while the scalar field is a function of the cosmic time and the extra coordinate.

At first and based on the usual spatial homogeneity, we assume, in general, that the scale factors of the specified 5D metric and the BD scalar field depend on the time, and also on the extra coordinate as separated variables in the power-law forms. Thus, we find five classes of solutions for the generalized Bianchi type I geometry in 5D spacetime through which besides general investigations, for the sake of compactness, we probe only one interesting class of them that leads to a generalized Kasner cosmology in 5D BD theory. This class gives a generalized Kasner relation among the Kasner parameters. One important property of these relations occurs when the BD scalar field is a constant. That is, regardless of any specific value of the BD coupling parameter, the relations reduce to the corresponding relations in 5D general relativity, which in turn give the 5D analog of the well-known Kasner solution, as expected. Also, when the scale factors and the scalar field only depend on the time, in the particular case of spatial isotropy, the results are the same as the corresponding ones investigated before for the spatially flat FRW solutions in 5D BD theory.

We then derive the pressure and energy density of the specified induced matter on any 4D hypersurface. These results indicate that one cannot consider it as a perfect fluid; however, we discuss the effective 4D universe generated by the specified class of the solutions. The induced scalar potential is found to be either in the power law or in the logarithmic form. However, in the particular case of a constant scalar field and even when the scalar field only depends on the fifth dimension, the induced scalar potential, without loss of generality, vanishes. Then, the conservation law (which is supposed to be valid in this MBDT approach) has been checked explicitly for the derived induced EMT. Though, as the logarithmic form of the induced EMT has some difficulties in satisfying the WEC, we do not investigate it further.

We proceed our investigations for a few cosmological quantities where for simplicity (and somehow without loss of generality) we assume that the metric and the BD scalar field are the only functions of the cosmic time. As usual, in the Bianchi type I theories, we define a few (mean) parameters and important physical quantities; however, we have defined the effective pressure in a way to yield the conservation law in the case of isotropy. The other quantities under consideration are the spatial volume, the average scale factor, the mean Hubble parameter, the expansion scalar, the shear scalar and the deceleration parameter. First we obtain these quantities in terms of the generalized Kasner parameters. Then, we find that the induced EMT satisfies the barotropic equation of state $P_{\text{eff}} = w\rho$, where $w$ is a function of the Kasner parameters. Also, the model indicates that a constant mean Hubble parameter is not allowed. Hence, by appealing to the special law of variation of the Hubble parameter, we assume the deceleration parameter to be constant. And thus we set the evolution of all the quantities with respect to the fixed deceleration parameter $q$, the state parameter $w$ and the BD coupling parameter. In general, the average scale factor indicates that any positive value of $q$ corresponds to a decelerating expanding universe, while the negative values of $q$, restricted to $-1 < q < 0$, yield accelerating expanding ones.
In the special case of a constant scalar field, regardless of any specific value of the BD coupling parameter, the BD theory reduces to general relativity, and we recover the generalized Kasner relations achieved before in the literature. The model, by satisfying the WEC, allows a shrinking extra dimension for a decelerating expanding universe which includes the radiation-dominated and the matter-dominated epochs that evolve the same way as the corresponding ones of the flat FRW spacetime in general relativity.

We then probe the quantities, in the general case, versus $q$, $t$ and $\omega$ for the stiff fluid and the radiation-dominated universe. The results show that in the both fluids, one has an expanding universe commenced with a big bang, and there is a horizon for each of them. The rate of expansion slows down by time. Also, by applying the WEC, the allowed (or the well-behaved) ranges of the deceleration and the BD coupling parameters have been obtained for each of the fluids. The behavior of the quantities, in the very early universe and the very large time, has been discussed. In particular, the general behavior of the induced energy density, at an arbitrary fixed time and also its evolution, has been depicted in a few figures. The models give empty universes when the cosmic time goes to infinity. However, as the ratio $\sigma^2/\Theta^2$ does not depend on the cosmic time, the models, in general, do not approach isotropy for large values of the cosmic time.

As presented, the modified 4D BD theory with a non-vanishing scalar potential and the induced EMT is derived from the 5D BD equations with vacuum configuration, contrary to the standard 4D BD theory where the scalar potential is introduced by hand. Note that, though for the Bianchi type I model interpreted by the standard approach, the introduction of a scalar potential by hand may provide more freedom in obtaining interesting solutions, but we have investigated the response of the MBDT approach for the specified cosmological model.

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