The two-dimensional Coulomb plasma: quasi-free approximation and central limit theorem

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Abstract

For the two-dimensional one-component Coulomb plasma, we derive an asymptotic expansion of the free energy up to order \( N \), the number of particles of the gas, with an effective error bound \( N^{1-\kappa} \) for some constant \( \kappa > 0 \). This expansion is based on approximating the Coulomb gas by a quasi-free Yukawa gas. Further, we prove that the fluctuations of the linear statistics are given by a Gaussian free field at any positive temperature. Our proof of this central limit theorem uses a loop equation for the Coulomb gas, the free energy asymptotics, and rigidity bounds on the local density fluctuations of the Coulomb gas, which we obtained in a previous paper.

1 Introduction and main results

1.1. One-component plasma. The two-dimensional one-component Coulomb plasma (OCP) is a Gibbs measure on the configurations of \( N \) charges \( z = (z_1, \ldots, z_N) \in \mathbb{C}^N \). The Hamiltonian of this measure, in external potential \( V : \mathbb{C} \to \mathbb{R} \cup \{+\infty\} \), is given by

\[
H_{N,V}^G(z) = N \sum_j V(z_j) + \sum_{j \neq k} G(z_j, z_k),
\tag{1.1}
\]

where \( G(z_j, z_k) = C(z_j - z_k) \) is the two-dimensional Coulomb potential,

\[
C(z_j - z_k) = \log \frac{1}{|z_j - z_k|},
\tag{1.2}
\]

characterized by \( \Delta \log |\cdot| = 2\pi \delta_0 \) as distributions. The Coulomb plasma is our main interest, but throughout the paper we will also consider other symmetric interactions \( G(z_j, z_k) \). The Gibbs measure of this plasma (OCP) at the inverse temperature \( \beta > 0 \) is defined by

\[
P_{N,V,\beta}^G(dz) = \frac{1}{Z_{N,V,\beta}^G} e^{-\beta H_{N,V}^G(z)} m^\otimes N(dz),
\tag{1.3}
\]
where \( m \) denotes the Lebesgue measure on \( \mathbb{C} \) and \( Z_{N,V,\beta}^G \) the normalization constant (assuming that \( V \) has sufficient growth at infinity, so that the latter is well-defined). We will follow the convention that when \( G = \mathcal{C} \) then we will omit the superscript \( \mathcal{C} \) whenever there is no confusion. Similar conventions apply to other subscripts, and we will also often omit \( N \) and \( \beta \).

The two-dimensional Coulomb gas \( P_{N,V,\beta}^C \) has connections with a variety of models in mathematical physics and probability theory. For \( \beta = 1 \), it describes the eigenvalues density for some measures on non Hermitian random matrices \([16,22]\). In particular for quadratic \( V \) the complex vector \( \mathbf{z} \) is distributed like the spectrum of a matrix with complex Gaussian entries. Moreover, the properties of this two-dimensional gas are known to be related to the fractional quantum Hall effect: for \( \beta = 2s + 1 \), with \( s \) integer, \( P_{N,V,\beta}^C \) is the density obtained from Laughlin’s guess for wave functions of fractional fillings of type \((2s + 1)^{-1}\) \([30]\). Finally, an important problem is the crystallization of the two-dimensional Coulomb gas for small temperature \([2, 15]\).

For potentials \( V \) that are lower semicontinuous and satisfy the growth condition
\[
\liminf_{|z| \to \infty} (V(z) - (2 + \varepsilon) \log |z|) > -\infty \tag{1.4}
\]
for some \( \varepsilon > 0 \), it is well known (see e.g. \([40]\)) that there exists a compactly supported equilibrium measure \( \mu_V \) that is the unique minimizer of the convex energy functional
\[
\mathcal{I}_V(\mu) = \int \int \log \frac{1}{|z - w|} \mu(dz) \mu(dw) + \int V(z) \mu(dz) \tag{1.5}
\]
over the set of probability measures on \( \mathbb{C} \). The unique minimizer, denoted by \( \mu_V \), is supported on a compact set \( S_V \) and, assuming that \( V \) is smooth, it has the density
\[
\rho_V = \frac{1}{4\pi} \Delta V 1_{S_V} \tag{1.6}
\]
with respect to the Lebesgue measure \( m \). We write \( I_V = \mathcal{I}_V(\mu_V) \) for the minimum of \( \mathcal{I}_V \). For \( \mathbf{z} \in \mathbb{C}^N \), the empirical measure is defined by
\[
\hat{\mu} = \frac{1}{N} \sum_j \delta_{z_j}.
\]
For arbitrary \( \beta \in (0, \infty) \), it is well-known that \( \hat{\mu} \to \mu_V \) vaguely in probability as \( N \to \infty \), with \( \hat{\mu} \) distributed under \( P_{N,V} \). In \([6]\), we have proved two stronger estimates for the Coulomb gas. The first one is a local law that asserts that for any smooth \( f \) supported in a disk of radius \( b = N^{-s} \) \((s \in [0,1/2])\) centred at \( z_0 \) in the bulk of \( S_V \) (and the \( f \) supported in the bulk when \( s = 0 \)), we have
\[
\frac{1}{N} \sum_{j=1}^{N} f(z_j) - \int f(z) \mu_V(dz) = O\left( \left( 1 + \frac{1}{\beta} \right) \log N \right) \left( N^{-1-2s} \|\Delta f\|_\infty + N^{-\frac{1}{2} - s} \|\nabla f\|_2 \right), \tag{1.7}
\]
with probability at least \( 1 - e^{-(1+\beta)N^{1-2s}} \) for sufficiently large \( N \). A stronger estimate, which we shall call rigidity, asserting that
\[
\sum_{j=1}^{N} f(z_j) - N \int f(z) \mu_V(dz) = O(N^\varepsilon) \left( \sum_{l=1}^{4} N^{-l\varepsilon} \|\nabla^l f\|_\infty \right), \tag{1.8}
\]
with probability at least \( 1 - e^{-\beta N^\varepsilon} \) for sufficiently large \( N \) also holds under the same assumptions.

The main result of this paper is the identification of the random error term in the above rigidity estimate. It is given by the Gaussian free field with a nonzero mean.
1.2. Main results. Our main results are the following two theorems. The global potential $V$ is always assumed to satisfy

$$V \in C^4 \text{ on a neighborhood of } S_V = \text{supp } \mu_V, \quad \alpha_0 \leq \Delta V(z) \leq \alpha_0^{-1} \text{ for all } z \in S_V \quad (1.9)$$
for some $\alpha_0 > 0$, and we also assume that the boundary of $S_V$ is piecewise $C^1$.

**Theorem 1.1.** Suppose that the external potential $V$ satisfies the conditions (1.4) and (1.9). Then there exist a positive constant $\kappa > 0$ and $\zeta^{C,\beta} \in \mathbb{R}$ such that

$$\frac{1}{\beta N} \log \int e^{-\beta H_V} m^\otimes N(dz) = -NI_V + \frac{1}{2} \log N + \zeta^{C,\beta} + \left(1 - \frac{1}{\beta}\right) \int \rho_V \log \rho_V + O(N^{-\kappa}). \quad (1.10)$$

A similar result was already proved in [33], without a quantitative error bound. For our application to the proof of Theorem 1.2 below, a quantitative error bound is essential.

For the statement of Theorem 1.2, we need the following additional notations. For $f$ supported in a disk of radius $b = N^{-s}$, we introduce the norms

$$\|f\|_{k,b} = \sum_{j=0}^{k} b^j \|\nabla^j f\|_\infty. \quad (1.11)$$

When $b = 1$, we denote it by $\|f\|_k$. Moreover, for any compactly supported $f$, let

$$X^f_V = \sum_j f(z_j) - N \int f \, d\mu_V, \quad (1.12)$$

$$Y^f_V = \frac{1}{4\pi} \int \Delta f \log \Delta V \, dm = \frac{1}{4\pi} \int \Delta f(z) \log \rho_V(z) \, dm(z). \quad (1.13)$$

In the following theorem, $f : \mathbb{C} \to \mathbb{R}$ is supported on a disk with radius $b = N^{-s}$ for a fixed scale $s \in [0, 1/2)$, and $\|f\|_{4,b} < \infty$ uniformly in $N$. We also assume that the support of $f$ satisfies $\text{dist}(\text{supp}(f), S_V^c) > \varepsilon$ for some $\varepsilon > 0$ uniformly in $N$. The last condition can be relaxed to $\varepsilon = N^{-1/2+c}$ for arbitrarily small $c$ (i.e. $f$ still supported in the bulk), if a more detailed analysis is performed near the boundary; we will not pursue this direction in this paper.

**Theorem 1.2.** Suppose $V$ satisfies the condition (1.4) and (1.9), and $f$ satisfies the above conditions. There exists a small positive constant $\kappa > 0$ such that for any $\varepsilon > 0$ and $0 < \lambda \ll (Nb^2)^{1-2\kappa}$, we have

$$\frac{1}{\beta \lambda} \log \left(\mathbb{E} e^{-\beta \lambda \left(X^f_V - \left(\frac{3}{\pi} - \frac{1}{2}\right) Y^f_V\right)}\right) = \frac{\lambda}{8\pi} \int dm(z) |\nabla f(z)|^2 + O((N b^2)^{-\kappa + \varepsilon}).$$

Here the expectation is with respect to $P^C_{N,V,\beta}$.

Note that $\lambda$ is allowed to be very large in this theorem; this provides strong error estimates for the Gaussian convergence. This central limit theorem is noteworthy due to the absence of normalization: fluctuations of $X^f_V$ are only of order one, due to repulsion, but still Gaussian.

For the purpose of establishing the central limit theorem for $X^f_V$, it suffices to take $\lambda$ to be of order one (independent of $N$).

Finally, a result similar to Theorem 1.2 was obtained simultaneously and independently in [32].
1.3. Related results. The study of one- and two-dimensional Coulomb and log-gases has attracted considerable attention recently, see e.g. [21] for many aspects of these probability measures in connection with statistical physics. The subject of our work, abnormally small Gaussian charge fluctuations of the one-component plasma, was first predicted in the late 1970s (see [26] and the references therein).

In dimension two, in the special case $\beta = 1$, the central limit theorem was first proved for the Ginibre ensemble, i.e. for quadratic external potential $V$ [36, 37]. These results were extended to more general $V$ by combining tools from determinantal point processes and the loop equation approach [4,5]. In particular, in the latter works the determinantal structure was used to prove local isotropy of the point process, an important a priori estimate necessary to the loop equation approach. For general inverse temperature $\beta$, the determinantal structure does not hold; nevertheless an expansion of the partition function and correlation functions was predicted in [45–47]. The expansion of the partition function up to order $N$ was rigorously obtained in [33] (along with a corresponding large deviation principle for a tagged point process); see also the related earlier works [48, 49, 50]; in addition, see also [23]. Still for the two-dimensional Coulomb gas at any temperature, a local density [31, 44] was recently proved, together with abnormally small charge fluctuations in the sense of rigidity [31], see (1.8). Other recent results in this direction include [3, 34, 35, 39].

For the log-gas on the line, much more is known. Indeed, in dimension one the Selberg integrals are often a good starting point to evaluate partition functions, and anisotropy does not cause any trouble in the analysis of loop equations. For general $\beta$ and $V$, full expansions of the partition function and correlators were predicted in [19], proved at first orders in [43] and at all orders in [8,9]. A natural analogue of the rigidity (1.8) is also known to hold for log-gases on the real line [10]. Still for the log-gas in dimension 1, the central limit theorem was first discovered on the circle for $\beta = 2$ in [27], and on the real line for any $\beta$ in [28]. For test functions supported on a mesoscopic scale, the local central limit theorem was proved on the circle for some compact groups in [44], for general $\beta$ ensembles with quadratic $V$ in [11] and for general $V$ in [7].

For expansions at high temperatures, and exponential decay of microscopic correlations, in closely related models of Coulomb gases, see [13, 25]. For results on crystallization in the one-dimensional one-component Coulomb plasma, see [11, 12, 29]. Further results on Coulomb systems in statistical mechanics are reviewed in [14, 21].

1.4. Strategy. In Section 2 we establish an expansion of the free energy of the Coulomb gas up to order $N$, with quantitative error bound. The main idea is to approximate the Coulomb gas first by a short-range Yukawa gas, and then by a quasi-free Yukawa gas. Roughly speaking, a Yukawa gas with range $\ell \ll 1$ can be viewed, for the purpose of computing free energy, as an ideal gas consisting of independent squares of size $b$ satisfying $1 \gg b \gg \ell$ and with the gas inside each square being a Yukawa gas with range $\ell$. Since this gas is an ideal gas over a distance longer than a mesoscopic scale $b$, we call it a quasi-free approximation. The Yukawa approximation to the Coulomb gas is a well-known tool in the study of the quantum Coulomb gas, see, e.g., [17, 18]. However, the precision needed here is far beyond the previous results. Due to the rigidity estimate established in [6], we are able to show that the Yukawa approximation yields very mild errors.

In Section 3 we first prove that the central limit theorem holds after adding a local angle (random) term. From this result and the asymptotic expansion of the free energy for the Coulomb gas, Theorem 1.1 we obtain that the angle term in fact vanishes in a large deviation sense. We thus prove Theorem 1.2 for a test function $f$ with macroscopic support.
In Section 4, we prove the central limit theorem for local functions. We proceed via conditioning to a square of size $2b$ if the original function $f$ is supported in a square of size $b$. This conditioning procedure was used in [6]; it has the advantage of reformulating the question into a problem on the natural scale $b$. The price to pay is a boundary layer problem in the asymptotic expansion of the free energy for such a local gas. Since boundary effects are of order of the boundary, it is not hard to show that it is of lower order in free energy. This section therefore proves Theorem 1.2 for a test function $f$ with any mesoscopic support.

Throughout the paper, we will extensively use the local density and rigidity estimates for the Yukawa gas and Coulomb gas with additional angular interaction, in the sense of (1.7) and (1.8). In Appendices A and B, we therefore extend the estimates in [6] to the Yukawa gas and the Coulomb gas with angle term.

In [33], a version of Theorem 1.1 was proved without an error bound. In this version, the constant $\zeta^{C,\beta}$ was characterized via a large deviation principle with the rate function given by a minimization over a class of point processes. In the following proof of Theorem 1.1 we will characterize this constant through the “residual free energy” of a Yukawas gas on the unit torus. The residual free energy, in a nutshell, is the part of free energy of a Yukawa gas up to order $N$, the number of particles, which is not accounted for by the energy functional. We then show that the specific residual free energy is independent of the range parameter and has a limit as $N \to \infty$. This physical construction allows us to compute the free energy asymptotic with an effective error bound and prove Theorem 1.1.

### 2 Proof of Theorem 1.1: quasi-free approximation

In this section, we prove Theorem 1.1. As a preliminary technical step for our analysis, it is convenient to replace the Coulomb interaction by a Yukawa interaction (defined below) with range $R \gg 1$. We will then compute the asymptotic for the free energy of this Yukawa gas by approximation with a quasi-free Yukawa gas with a short range $\ell = N^{-1/2+\delta} \ll 1$. Roughly speaking, the quasi-free Yukawa gas is defined as follows. We devide $\mathbb{C}$ into squares of side length $b$ with $\ell \ll b = N^{-1/2+\delta+\alpha} \ll 1$, and declare that particles in different squares do not interact, while particles in the same square interact via the Yukawa potential with range $\ell$.

Using the rigidity estimates for the Coulomb gas [6] and their extension to the Yukawa gas (see Appendix A), we will show that, for the free energy, this approximation is accurate up to order $N^{1-\kappa}$. We start with this program by defining the Yukawa potential.

#### 2.1. The two-dimensional Yukawa gas.

For $z \in \mathbb{C} \cong \mathbb{R}^2$, the two-dimensional Yukawa potential with range $m^{-1}$ (where $m$ is the mass) is defined by the formula

$$
Y^{m^{-1}}(z) := (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-ip \cdot z} (p^2 + m^2)^{-1} dp
$$

$$
= \frac{1}{2} (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-ip \cdot z} \int_0^\infty e^{-t(p^2 + m^2)/2} dt dp
$$

$$
= \int_1^\infty e^{-a(s+1/2)/s} \frac{ds}{s} := g(a), \quad a = \frac{rm}{2}, \quad r = |z|,
$$

where $p \cdot z$ denotes the Euclidean inner product on $\mathbb{R}^2$. Thus $(-\Delta + m^2)Y^{m^{-1}} = 2\pi \delta_0$ as distributions, and $Y^{m^{-1}}(z)$ is both pointwise positive and positive definite. Moreover, there is
a constant $C_0$ such that
\[
Y^{m^{-1}}(z) \begin{cases} 
\sim -\log r - \log m + C_0 + O(mr) & \text{if } rm \ll 1 \\
\leq C_1 e^{-C_2 rm} & \text{if } rm \geq 1.
\end{cases}
\] (2.2)

In particular, up to the explicit constant $C_0 - \log m$, the two-dimensional Coulomb potential $-\log |z|$ is the limit $m \downarrow 0$ of $Y^{m^{-1}}$.

For points $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, we define the Yukawa energy with range $R$ in external potential $V$ by
\[
H_V^R(z) = N \sum_j V(z_j) + \sum_{j \neq k} Y_R^R(z_j - z_k).
\] (2.3)

Correspondingly, a probability measure is defined as in (1.3). Moreover, for probability measures $\mu$ on $\mathbb{C}$, the closely related energy functional on probability measures of the Yukawa gas is given by
\[
\mathcal{I}_V^R(\mu) := \int V(z) \mu(dz) + \int Y_R^R(z - w) \mu(dz) \mu(dw).
\] (2.4)

We denote by $\mu_V^R$ its unique minimizer, and by $I_V^R = \inf_{\mu \in \mathcal{I}_V^R} \mathcal{I}_V^R(\mu)$ its minimizing energy.

The existence of the minimizer (equilibrium measure) and general properties are summarized in Appendix A. There we also extend the estimates on the local density that we established in [6] for the Coulomb gas to the Yukawa gas; these will be used in this section.

Throughout this section, we make the standing assumption that $V$ satisfies the asymptotic condition (1.4) and (1.9).

### 2.2. Long-range Yukawa approximation of Coulomb gas

Theorem 1.1 is stated as an estimate for the Coulomb gas. However, it is convenient to approximate the Coulomb gas by the Yukawa gas with range $R$ tending to infinity. This will be justified by the following crude lemmas, which are sufficient for our purposes.

**Lemma 2.1.** Let $C_0$ be the constant in (2.2). There is a constant $C$ such that, as $R \to \infty$,
\[
|I_V^R(\mu_V^R) - (\log R + C_0) - I_V^R| \leq C/R,
\] (2.5)

**Proof.** By Theorem A.2, $\mu_V^R$ has compact support uniformly in $R$, i.e., there exists a constant $K > 0$ such that $\text{supp} \mu_V^R \subset B_K(0)$ for all $R$. For any probability measure $\omega$ supported in $B_K(0)$, the asymptotic behavior of $\mathcal{I}_V^R(\omega)$ as $R \to \infty$ is given by
\[
\left| \int (Y^R - C)(z - w)\omega(dz)\omega(dw) - \log R + C_0 \right| \leq CR^{-1},
\] (2.6)

with constant $C$ depending on $K$. This proves that
\[
I_V^R(\mu_V^R) = I_V(\mu_V^R) + O(1/R), \quad I_V^R(\mu_V) = I_V(\mu_V) + O(1/R).
\] (2.7)

Since $\mu_V$ minimizes $I_V$, it follows that $I_V \leq I_V^R + O(1/R)$. Since $\mu_V^R$ minimizes $I_V^R$, it follows that $I_V^R \leq I_V + O(1/R)$. Therefore $I_V = I_V^R + O(1/R)$, which completes the proof. 

From now on, we adopt the convention that an integration symbol $\int$ without specification of the measure always refers to integration with respect to the Lebesgue measure. Moreover, in the following, we will repeatedly use the Jensen inequality in the form
\[
\log \int e^{-B} + E^B(B - A) \leq \log \int e^{-A} \leq \log \int e^{-B} + E^A(B - A),
\] (2.8)
where $E^A X = \int e^{-A X}$. 

**Lemma 2.2.** There is a constant $C$ such that, as $R \to \infty$,

$$\left| \beta^{-1} \log \int e^{-\beta H^C_V} - \beta^{-1} \log \int e^{-\beta H^R_V} - (\log R + C_0)N(N-1) \right| \leq CN^2/R. \quad (2.9)$$

**Proof.** By the Jensen inequality in the form (2.8), we have

$$E_{V^R}(H_{V^R} - H_{V^C}) \leq \beta^{-1} \log \int e^{-\beta H^C_V} - \beta^{-1} \log \int e^{-\beta H^R_V} \leq E_V(H_{V^R} - H_{V^C}). \quad (2.10)$$

By (2.2),

$$H_{V^R} - H_{V^C} = \sum_{j \neq k} \left(\log R + C_0 + O(|z_j - z_k|)/R\right)$$

$$= N(N-1)(\log R + C_0) + \sum_{j \neq k} O(|z_j - z_k|)/R. \quad (2.11)$$

The claim follows since $E_V |z_j - z_k| \leq 2E_V |z_1| = O(1)$ uniformly in $R$ (Appendix A). This completes the proof of the proposition. \hfill \Box

### 2.3. Residual free energy.

The goal of Section 2 is to compute the free energy of the Yukawa gas with an effective error bound $N^{1-\kappa}$, as stated in (1.10). To define the constant $\zeta_{C,\beta}$ in (1.10), we need the concept of residual free energy on the torus, which we now define. We consider the Yukawa gas with range $\ell$ on a torus of side length $b$, with interaction

$$S^\ell_{b}(z - w) = Y^\ell(d_b(z - w)),$$

where $d_b(z - w)$ is the torus distance between $z$ and $w$. The energy of the corresponding mean-field variational functional on probability measures is given by

$$I^\ell_b = I^{(\gamma)}_{b} = \inf_{\mu, \gamma} \int S^\ell_{b}(z - w)\mu(\mu)\mu(\mu), \quad (2.12)$$

where $\gamma = \ell/b$ and all integrations and measures are on the torus $T_b$ of side length $b$. Here and below we use the convention that a superscript $\ell$ indicates the range of the interaction, while $\gamma$ in $(\gamma)$ indicates the ratio $\ell/b$. By translation invariance, the unique minimizer in (2.12) is the uniform measure on the torus, and thus

$$I^{(\gamma)}_{b} = b^{-2} \int S^\ell_{b}(z - w) m(\mu). \quad (2.13)$$

**Definition 2.3.** For $b > 0$ and $\gamma < 1$, we define the residual free energy for the Yukawa gas on the torus of side length $b$ with range $\ell = b\gamma$ by

$$\xi^{(\gamma)}_{b}(n) = F^{(\gamma)}_{b}(n) + n^2 I_{b}^{(\gamma)} - n \log \ell, \quad (2.14)$$

where

$$F^{(\gamma)}_{b}(n) = \beta^{-1} \log Z^{(\gamma)}_{n,b} = \beta^{-1} \log \int_{T_b} e^{-\beta \sum_{i \neq j} S^\ell_{b}(z_i - z_j)} m^N(dz).$$
The specific residual free energy of the Yukawa gas with range $\gamma$ on the unit torus is defined by

$$\zeta^{(\gamma)}(n) := \frac{\xi^{(\gamma)}_1(n)}{n} - \frac{1}{2} \log n. \quad (2.15)$$

The following two theorems are the main results of Section 2.

**Theorem 2.4.** There exists $\zeta_\infty$ independent of $\gamma$ and $n$ (but depending on $\beta > 0$) such that for any $c > 0$ there exists a constant $\tau > 0$ such that $\zeta^{(\gamma)}(n) = \zeta_\infty + O(n^{-\tau})$, provided that $\gamma \geq n^{-1/2+c}$.

**Theorem 2.5.** Let $R > 0$. Then there exists a positive number $\kappa > 0$ such that

$$\langle \beta N \rangle^{-1} \log \int e^{-\beta H_\nu} m^{\otimes N}(dz) = -N I_\nu^R + \left( \frac{1}{2} - \frac{1}{3} \right) \int \rho_\nu^R \log \rho_\nu^R + \frac{1}{2} \log N + \zeta_\infty + \log R + \frac{1}{2} \log N + O(N^{-\kappa}). \quad (2.16)$$

Theorem 2.5 immediately implies Theorem 1.1 concerning the free energy of the Coulomb gas, as we observe below. The remainder of Section 2 is devoted to the proof of Theorem 2.4 and 2.5, which is concluded in Section 2.9.

**Proof of Theorem 1.1.** We choose $R = N^A$ for large enough $A$. Then, by (2.9) and (2.16),

$$\langle \beta N \rangle^{-1} \log \int e^{-\beta H_\nu} m^{\otimes N}(dz) = -N I_\nu^R + \left( \frac{1}{2} - \frac{1}{3} \right) \int \rho_\nu^R \log \rho_\nu^R + \frac{1}{2} \log N + \zeta_\infty + \log R + \frac{1}{2} \log N + O(N^{-\kappa}).$$

By (2.5), the sum of $-N I_\nu^R$ and the first three terms on the second line give $-N I_\nu^C + C_\beta$ where we define $\zeta^{C,\beta} = \zeta_\infty - C_0$. Using (2.5) and $\int \rho_\nu^R \log \rho_\nu^R \to \int \rho_\nu \log \rho_\nu$, the proof of Theorem 1.1 is complete.

### 2.4. Short-range Yukawa approximation.

For any $\ell < R$, we decompose the Yukawa potential as $Y^R = Y^\ell(z) + L_R^\ell(z)$. The explicit formula (2.1) immediately implies that the Fourier transform of $L_R^\ell$ is nonnegative and thus that $L_R^\ell$ is positive definite. Recall that we denote the empirical measure by $\hat{\mu}(dz) = N^{-1} \sum_j \delta(z - z_j)$ and that we write $\hat{\mu}_V^R(dz) = \hat{\mu}(dz) - \mu^R_V(dz)$, where $\mu^R_V$ is the equilibrium measure for Yukawa gas with range $R$. We have the identity

$$\sum_{j \neq k} L_R^\ell(z_j - z_k) + N \sum_j V(z_j) \quad \begin{aligned} &= N^2 \left[ \int_{z \neq w} L_R^\ell(z-w)\hat{\mu}(dw)\hat{\mu}(dz) - \int L_R^\ell(z-w)\hat{\mu}(dw)\mu^R_V(dz) \\
- \int L_R^\ell(z-w)\mu^R_V(dw)\hat{\mu}(dz) + \int L_R^\ell(z-w)\mu^R_V(dw)\mu^R_V(dz) \\
+ \int V(z)\hat{\mu}(dz) + 2 \int L_R^\ell(z-w)\mu^R_V(dw)\hat{\mu}(dz) - \int L_R^\ell(z-w)\mu^R_V(dw)\mu^R_V(dz) \right] \\
= N^2(\Omega_1 + \Omega_2 - K_1), \quad (2.17) \end{aligned}$$
where
\[ K_1 = \int L_R^\ell(z-w) \mu^R_V(dw) \mu^R_V(dz) \]
\[ \Omega_1 = \int_{z \neq w} L_R^\ell(z-w) \bar{\mu}_V^R(dw) \bar{\mu}_V^R(dz) \]
\[ \Omega_2 = \int Q(z) \bar{\mu}(dz), \quad Q(z) = V(z) + 2 \int L_R^\ell(z-w) \bar{\mu}_V^R(dw). \quad (2.18) \]

Our goal is to approximate the Yukawa gas with range \( R \) and potential \( V \) by the Yukawa gas with range \( \ell \) and potential \( Q \), i.e., with Hamiltonian
\[ H_Q^\ell(z) = N^2 \int Q(z) \bar{\mu}(dz) + \sum_{i \neq j} Y^\ell(z_i - z_j) - N^2 K_1 = H_V^R(z) - N^2 \Omega_1. \quad (2.19) \]

As in the above equation, we will use the convention that the short-range Hamiltonian \( H_\ell^Q \) carries the extra constant term \(-N^2K_1\), and similarly, we include the constant \( K_1 \) in the corresponding energy functional on probability measures; namely, we use
\[ I_Q^\ell(\mu) = \int Q\mu + \int Y^\ell(z-w)\mu(dz)\mu(dw) - K_1, \]
while we define \( I_V^R \) without the constant \( K_1 \).

**Lemma 2.6.** With the conventions above, the minimizers of \( I_Q^\ell, I_V^R \) and their energies coincide:
\[ \mu_Q^\ell = \mu_V^R, \quad I_Q^\ell = I_V^R. \]

**Proof.** The equilibrium measures (minimizers) of \( I_V^R \) and \( I_Q^\ell \) are characterized by the Euler–Lagrange equations (A.5), which state that in their supports
\[ V + 2Y^R * \mu_V^R = c_V^R, \quad Q + 2Y^\ell * \mu_Q^\ell = c_Q^\ell \]
and that inequality holds outside the supports. By definition of \( Q \), the solution \( \mu_V^R \) satisfies
\[ Q + 2Y^\ell * \mu_V^R = V + 2Y^R * \mu_V^R = c_V^R. \]

By the uniqueness of the minimizers, we therefore conclude that \( \mu_Q^\ell = \mu_V^R \) and \( S_V^R = S_Q^\ell \), i.e., the two minimizers coincide. Moreover, a simple computation yields that \( I_V^R = I_Q^\ell \). \( \square \)

By Jensen’s inequality and positive definiteness of \( L_R^\ell \), we have the following upper bound on the free energy.

**Proposition 2.7.** For any \( \ell < R \), we have the upper bound on the free energy
\[ \beta^{-1} \log \int e^{-\beta H_R^\ell(z)} m^\otimes N(z) \leq \beta^{-1} \log \int e^{-\beta H_Q^\ell(z)} m^\otimes N(z) + N \log(R/\ell). \quad (2.21) \]

**Proof.** By Jensen’s inequality,
\[ \beta^{-1} \log \int e^{-\beta H_R^\ell(z)} m^\otimes N(dz) \leq -N^2 \mathbb{E}_V^\ell \Omega_1 + \beta^{-1} \log \int e^{-\beta H_Q^\ell(z)} m^\otimes N(dz), \quad (2.22) \]
where \( \mathbb{E}_Q^\ell \) is the expectation with respect to the Yukawa gas with Hamiltonian \( H_Q^\ell \). Since \( L_R^\ell \) is positive definite, we have
\[ -N^2 \mathbb{E}_V^\ell \Omega_1 = -N^2 \mathbb{E}_V^R \int L_R(z-w) \bar{\mu}_V^R(dw) \bar{\mu}_V^R(dz) + NL_R^\ell(0) \leq NL_R^\ell(0) = N \log(R/\ell). \]

By (2.22), this implies the claim. \( \square \)
2.5. Quasi-free approximation. In this section, we derive a formula for the partition function of the (long-range) Yukawa gas in terms of what we will call the quasi-free Yukawa gas, and which we now define.

Given parameters $\ell \ll b$, we divide $\mathbb{C}$ into a grid of squares of side length $b$ with centers $\alpha \in (b\mathbb{Z})^2 \subset \mathbb{C}$; sometimes we will also write $\alpha$ to mean the square with center $\alpha$. It will be useful to also consider the shifted grid, in which all squares are translated by $u \in \mathbb{C}$ so that their centers are $u + \alpha$. Given $b$ and $u$, we consider the gas obtained from $H^\ell$ by removing the interaction between particles in different squares, and whose interaction is made periodic inside each square. To be precise, the interaction in the square with center $u + \alpha$ is

$$S^\ell_{u}(z_i - z_j) = S^\ell_b(z_i - z_j) \mathbf{1}(z_i \in \alpha + u) \mathbf{1}(z_j \in \alpha + u),$$

where $S^\ell_b$ is the periodic Yukawa interaction of range $\ell$ defined by

$$S^\ell_b(z_i - z_j) = Y^\ell(\delta_b(z_i, z_j)),$$

and where $\delta_b$ is the periodic distance on a torus of size $b$, and the Hamiltonian of all particles is given by

$$\hat{H}^\ell_V(z) = N^2 \int Q(z) \mu(dz) + \sum_{\alpha_{i} \neq j} S^\ell_{\alpha_{i},u}(z_i - z_j) - N^2 K_1. \tag{2.23}$$

Since the particles in different squares are independent, we call the corresponding gas quasi-free. The following proposition asserts that the free energy of a Yukawa gas with Hamiltonian $H^\ell_V$ can be approximated by that of the quasi-free Yukawa gas.

**Proposition 2.8.** There exists $u$ with $|u| \leq b/2$ such that

$$\beta^{-1} \log \int e^{-\beta \hat{H}^\ell_V(z)} m^\otimes N(dz) \leq \beta^{-1} \log \int e^{-\beta \hat{H}^\ell_u} dz + N \log(R/\ell) + O(\log N \delta^3/b) + O(N^{1-2\delta+\epsilon}), \tag{2.24}$$

$$\beta^{-1} \log \int e^{-\beta \hat{H}^\ell_V(z)} m^\otimes N(dz) \geq \beta^{-1} \log \int e^{-\beta \hat{H}^\ell_u} dz + N \log(R/\ell) + O(\log N \delta^3/b) + O(N^{2b^4}) + O(N^{1-2\delta+\epsilon}). \tag{2.25}$$

**Remark 2.9.** We abbreviate $\gamma = \ell/b \ll 1$, and introduce the error parameter

$$\Xi(\ell, \gamma) = N^2 \ell^3 b^{-1} + \ell^{-2} + N^2 b^4 + N^2 \ell^2 b, \tag{2.27}$$

so that the error terms in (2.24), (2.25) are both bounded by $\Xi(\ell, \gamma)$. Usually, we use the choice

$$\ell = N^{-1/2+\delta}, \quad b = N^{-1/2+a+\delta} = \gamma^{-1} \ell, \quad \gamma = N^{-a}, \tag{2.28}$$

where, for some fixed $\kappa < 1/20$,

$$\delta = \kappa, \quad a = 4\kappa. \tag{2.29}$$

With this choice, $\Xi(\ell, \gamma)$ is bounded by $N^{1-\kappa}$.

For the proof of Proposition 2.8, we define the average of the quasi-free Yukawa interaction by

$$g(z - w) = \sum_\alpha \frac{1}{b^2} \int_0^b dp \int_0^b dq Y^\ell(d_{\alpha+u}(z, w)), \quad u = (p, q). \tag{2.30}$$
The proof of Proposition 2.8 depends on the following lemma, whose proof is a simple calculus exercise. The factor \((b - x)/b\) in this lemma is the probability that two points 0 and \(x\) in \(\mathbb{R}\) are in a random interval of size \(b\).

**Lemma 2.10.** The function \(g\) is a function of \(z - w\) and vanishes if \(|z - w| > \sqrt{2}b\). For \(w = 0\) and \(0 < x \leq b, 0 \leq y \leq b\) for \(z = (x, y)\), we have

\[
g(z) = \frac{(b - x)(b - y)}{b^2} \times \begin{cases} 
Y^\ell((b - x)^2 + (b - y)^2) & \text{if } b/2 < x \leq b, b/2 \leq y \leq b \\
Y^\ell((b - x)^2 + y^2) & \text{if } b/2 < x \leq b, 0 \leq y \leq b/2 \\
Y^\ell(\sqrt{x^2 + y^2}) & \text{if } 0 < x \leq b, 0 \leq y \leq b/2.
\end{cases}
\] (2.31)

In particular, when \(0 < x \leq b/2, 0 \leq y \leq b/2,\)

\[
g(z) - Y^\ell(z) = -\frac{x + y}{b} Y^\ell(z) + O(\frac{|z|^2}{b} Y^\ell(z)),
\] (2.32)

and for \(b/2 < x \leq b, 0 \leq y \leq b/2,\)

\[
g(z) \lesssim \frac{\ell}{b} 1(|b - x| \leq \ell).
\] (2.33)

**Proof of Proposition 2.8 upper bound.** By Jensen’s inequality,

\[
\beta^{-1} \log \int e^{-\beta H_v^R} m \otimes N(dz) \leq \beta^{-1} \log \int e^{-\beta H_u^\ell} m \otimes N(dz) + \mathbb{E}_v^R(\tilde{H}_u^\ell - H_v^R).
\] (2.34)

The last term can be rewritten as

\[
\mathbb{E}_v^R(\tilde{H}_u^\ell - H_v^R) = \mathbb{E}_v^R(\tilde{H}_u^\ell - H_v^\ell) + \mathbb{E}_v^R(H_v^\ell - H_v^R).
\] (2.35)

Using that \(L_R^\ell = Y^R - Y^\ell\) is positive definite, the last term is bounded above by

\[
\mathbb{E}_v^R(\tilde{H}_u^\ell - H_v^R) = -N^2 \mathbb{E}_v^R(\mu_{v}^R) = -N^2 \mathbb{E}_v^R \int L_R^\ell(z - w) \mu_{v}^R(dw) + N L_R^\ell(0)
\leq NL_R^\ell(0) = N \log(R/\ell).
\]

To bound the first term in (2.35), we now average (2.34) over \(u = (p, q)\) and use the definition of \(g\) in (2.30) to have

\[
\frac{1}{b^2} \int_0^b dp \int_0^b dq \mathbb{E}_v^R(\tilde{H}_u^\ell - H_v^\ell) = \mathbb{E}_v^R \sum_{i \neq j} [g(z_i - z_j) - Y^\ell(z_i - z_j)].
\]

We now bound \(g(z) - Y^\ell(z)\) using Lemma 2.10. We claim that, using that the local density of the Yukawa gas with range \(R\) is bounded by Theorem A.1 and that \(Y^\ell(r)\) decays exponentially at scale \(\ell\), we have

\[
\mathbb{E}_v^R \sum_{i,j} [g(z_i - z_j) - Y^\ell(z_i - z_j)] \leq O(\log NN(\ell^2)\ell/b).
\]

Indeed, to verify this inequality, consider one upper bound for \(g(z) - Y^\ell(z)\) in Lemma 2.10 so that we need to estimate

\[
\mathbb{E}_v^R \sum_{i \neq j} \frac{|z_i - z_j|}{b} Y^\ell(z_i - z_j) \leq O(\log NN(\ell^2)\ell/b).
\]
Here we have used that the probability of a particle to be near the boundary between squares so that the potential needs to be modified is $\ell/b$ and the typical number of particles in a radius of size $\ell$ is $N\ell^2$. The cases of Lemma 2.10 can be checked by an analogous argument. This proves the upper bound.

**Proof of Proposition 2.11.** To derive a lower bound, we can first restrict the integral in the partition function to have the correct number of particles in each square. Then, by Jensen’s inequality, we have

$$
\log\int dz e^{-\beta H_Q^\ell(z)} m^\otimes N(dz) \geq \log \int e^{-\beta H_Q^\ell(z)} \prod_{\alpha \cap \mathcal{S} \neq \emptyset} 1(n_\alpha = n_\mathcal{V}(\alpha)) m^\otimes N(dz) \\
\geq \log \int e^{-\beta H_Q^\ell(z)} m^\otimes N(dz) + \beta \mathbb{E} \tilde{H}_Q^\ell - H_Q^R(z).$$

We decompose $\tilde{H}_Q^\ell - H_V^R(z) = \tilde{H}_Q^\ell - H_Q^\ell + H_Q^\ell - H_V^R(z)$. Then

$$
\mathbb{E} \tilde{H}_Q^\ell - H_Q^\ell(z) = O(N \log N(N\ell^2)\ell/b),
$$

analogously to the previous proof of the upper bound using that the local density of the Yukawa gas is bounded (Theorem A.1). By (2.19), the other two terms are equal to

$$
\mathbb{E} \tilde{H}_Q^\ell [H_Q^\ell - H_V^R(z)] = -N^2 \mathbb{E} \tilde{H}_Q^\ell \Omega_1 + N \log (R/\ell) = -N^2 \mathbb{E} \tilde{H}_Q^\ell \sum_{\alpha,\beta} \Omega_{\alpha\beta} + N \log (R/\ell),
$$

where

$$
\Omega_{\alpha\beta} = \int_{z \in \alpha} \int_{w \in \beta} L_R^\ell(z - w) \tilde{\mu}_V(dz) \tilde{\mu}_V(dw),
$$

and where we have removed the restriction $z \neq w$ and compensated it by adding the diagonal term $N \log (R/\ell)$.

It remains to bound the terms $\mathbb{E} \tilde{H}_Q^\ell \Omega_{\alpha\beta}$. We start with the diagonal term $\alpha = \beta$. Recall that $\mathbb{E} \tilde{H}_Q^\ell$ is the expectation w.r.t. independent Yukawa gases on tori of size $b$ with no external potential. For each $\alpha$ fixed, define the local measure

$$
\tilde{\mu}_\alpha^\ell(dz) = \frac{1}{n_\mathcal{V}(\alpha)} \sum_j \delta_{z_j}(dz) - \frac{1}{b^2} 1(z \in \alpha) m(dz), \quad n_\mathcal{V}(\alpha) = Nb^2 \rho_\mathcal{V}(\alpha).
$$

(2.36)

Then

$$
N 1(z \in \alpha) \tilde{\mu}_V(dz) = n_\mathcal{V}(\alpha) \tilde{\mu}_\alpha^\ell(dz) + N[\rho_\mathcal{V}(z) - \rho_\mathcal{V}(\alpha)] m(dz),
$$

(2.37)

and hence we have

$$
N^2 \sum_{\alpha} \int_{z,w \in \alpha} L_R^\ell(z - w) \tilde{\mu}_V(dz) \tilde{\mu}_V(dw) = \sum_{\alpha} n_\mathcal{V}(\alpha)^2 \int_{z,w \in \alpha} L_R^\ell(z - w) \tilde{\mu}_\alpha^\ell(dz) \tilde{\mu}_\alpha^\ell(dw)
$$

$$
- \sum_{\alpha} Nn_\mathcal{V}(\alpha) \int_{z,w \in \alpha} L_R^\ell(z - w) \tilde{\mu}_\alpha^\ell(dz) [\rho_\mathcal{V}(w) - \rho(\alpha)] m(dw) - (z \leftrightarrow w)
$$

$$
+ N^2 \sum_{\alpha} \int_{z,w \in \alpha} L_R^\ell(z - w)[\rho_\mathcal{V}(z) - \rho(\alpha)] m(dz)[\rho_\mathcal{V}(w) - \rho(\alpha)] m(dw).
$$

(2.38)
Since the density on each square is a constant, for any function $g$ on the square indexed by $\alpha$,
\[
\int g(z) \tilde{\mu}_\alpha^\beta(dz) = 0. \tag{2.39}
\]
Hence the middle two terms on the right-hand side of (2.38) vanish. Moreover, using $\sum_\alpha 1 \leq Cb^{-2}$, the last term is explicit and it is bounded by
\[
b^{-2}N^2b^4\|\nabla \rho_V\|_\infty^2 = N^2b^4\|\nabla \rho_V\|_\infty^2, \tag{2.40}
\]
where the factor $b^4$ is from the integration of $z$ and $w$ and the factor $b^2\|\nabla \rho_V\|_\infty^2$ is from the size of $[\rho_V(z) - \rho(\alpha)][\rho_V(w) - \rho(\alpha)]$. Finally, by Proposition 2.11, which is stated below the proof and proved in Section A.6, the first term on the right-hand side of (2.38) is bounded by $n^\ell \ell^{-2}b^2$. Putting all these estimates together, we have bound the diagonal terms by
\[
N^2\sum_\alpha |E^{R\Delta}_\Omega_{\alpha\alpha}| \leq C\ell^{-2} + CN^2b^2\ell^2\|\nabla \rho_V\|_\infty^2, \tag{2.41}
\]
which is smaller than the error bound claimed in (2.24).

Finally, we need to estimate the off-diagonal terms. We have
\[
N^2 \int_{z \in \alpha, w \in \beta} L_R^\ell(z - w) \bar{\mu}_V(dz) \bar{\mu}_V(dw) \tag{2.42}
\]
\[
= N^2 \int_{z \in \alpha, w \in \beta} L_R^\ell(z - w) \bar{\mu}_\alpha^\beta(dz) \bar{\mu}_\beta^\alpha(dw)
- N \int_{z \in \alpha, w \in \beta} L_R^\ell(z - w) \mu^\ell_\alpha(dz)m(dz)[\rho_V(w) - \rho(\beta)] m(dw) - (z \leftrightarrow w)
+ N^2 \int_{z \in \alpha, w \in \beta} L_R^\ell(z - w)[\rho_V(z) - \rho(\alpha)] m(dz)[\rho_V(w) - \rho(\beta)] m(dw)
\]
Again, the second term on the right-hand side vanishes. Moreover, since $\alpha \neq \beta$ and particles in different squares are independently distributed with constant densities, the first term on the right-hand side also vanishes. To estimate the last term on the right-hand side, we use a Taylor expansion to find that the sum of these terms, after summation over $\alpha, \beta$, is bounded by
\[
N^2 \sum_{\alpha \neq \beta} \int_{z \in \alpha, w \in \beta} \left[ \nabla \rho(\alpha)(z - \alpha) + \nabla^2 \rho(\alpha)(z - \alpha)^2 \right] \left[ \nabla \rho(\beta)(w - \beta) + \nabla^2 \rho(\beta)(w - \beta)^2 \right]
\times \left[ L^\ell(\alpha - \beta) + \nabla L^\ell(\alpha - \beta)(z - \alpha - w + \beta) + \nabla^2 L^\ell(\alpha - \beta)(z - \alpha - w + \beta)^2 \right] m(dz)m(dw),
\]
where we have neglected the higher order terms which can be bounded easily. By symmetry, the cubic terms (in $(z - \alpha)$ and $(w - \beta)$) do not contribute, and the leading terms are therefore the quartic terms. Since $L$ is an explicit function, we can check that these terms are bounded by $N^2b^4$. The factor $b^4$ comes from $b^{-4}b^4b^4$ with the factor $b^{-4}$ coming from the summation, one $b^4$ factor coming from the integration of $z$ and $w$, and the last $b^4$ factor coming from the size of products of $(z - \alpha)$ and $(w - \beta)$ in the formula. \hfill \qed

The following estimate is proved in Appendix A.6.
Proposition 2.11. For the Yukawa gas with \(n\) particles with range \(\ell = n^{-1/2+\delta}\) on a torus of side length \(b\), with high probability, we have

\[
n^2 \int \int L_R^\ell(z - w) \, \hat{\mu}_Q(dz) \, \hat{\mu}_Q(dw) = O(b^2 \ell^{-2} n^\varepsilon). \tag{2.43}
\]

(Notice that there is no restriction \(z \neq w\) on the integral, and thus that there is no contribution of the form \(-nL_R^\ell(0)\) on the right-hand side.)

2.6. Free energy of the quasi-free Yukawa gas. In the previous subsection, we reduced the computation of the free energy of the (long-range) Yukawa gas to the (short-range) quasi-free Yukawa gas. In this subsection, we derive a formula for the free energy of the quasi-free Yukawa gas in terms of the specific residual free energy. The result is stated in Proposition 2.13 below.

First, the shift \(u\) in Proposition 2.8 plays no further role, and we therefore now assume \(u = 0\). For \(z \in S_R \cap \alpha\), we have

\[
Q(z) = 2cV - 2q_V(z), \quad q_V(z) = \int Y^\ell(d_b(z, w)) \mu_V^R(dw) \sim \ell^2 \tag{2.44}
\]

where \(d_b(z, w)\) is the periodic distance between \(z\) and \(w\). By assumption, the equilibrium density \(\rho_V^R\) is \(C^1\) in the interior of its support, which has a piecewise \(C^1\) boundary. Therefore, since \(Y^\ell\) has effective range \(\ell\), we have

\[
\int Y^\ell(z - w) \, \mu_V^R(dw) = \rho_V^R(z) \int Y^\ell(z - w) \, m(dw) + O(\ell N^\varepsilon). \tag{2.45}
\]

The definition of the Yukawa potential (2.1) implies the scaling property \(Y_{K\ell}(Kr) - Y_\ell(r) = 0\), from which we obtain that

\[
\eta := \ell^{-2} \int Y^\ell(z - w) \, m(dw) = 2\pi \tag{2.46}
\]

is independent of \(z\) and \(\ell\). As a consequence, the external potential \(Q\) in a square is essentially a constant, namely,

\[
Q(z) = 2c - 2\rho_V^R(\alpha)\ell^2 \eta(1 + O(b)) = Q(\alpha) + O(\ell^2 b). \tag{2.47}
\]

For the square \(\alpha\) with \(\alpha \in W\) where \(W\) is the square of size \(N^\varepsilon\) centered at the origin, we have

\[
Q(z) = V(z) + 2 \int L_R^\ell(z - w) \mu_V^R(dw) \geq V(z) + 2 \int Y_R^\ell(z - w) \mu_V^R(dw) \geq 2cV. \tag{2.48}
\]

For \(\alpha \notin W\), by the growth condition (1.4) on \(V\), we have

\[
Q(z) = V(z) + 2 \int L^\ell(z - w) \mu_V^R(dw) \geq C \log |\alpha| \tag{2.49}
\]

for some constant \(C\).

As in the previous section, we divide the integration defining the partition function of the Yukawa gas into squares of side length \(b\). For each number profile \(n = (n_\alpha)\) of particles in the squares, the number of ways to distribute \(N\) particles into groups of sizes \(n_1, n_2 \ldots\) is given by

\[
\binom{N}{\mathbf{n}} = \frac{N!}{\prod \alpha n_\alpha!}. \tag{2.50}
\]
The integration over the positions of the particles in a torus of size $b$ with a fixed number of particles yields the free energy $F_b^{(\gamma)}$ defined in Definition 2.3.

In terms of $\Xi(\ell, \gamma)$ introduced in (2.27), the upper bound (2.24) can be restated as

$$\beta^{-1} \log \int e^{-\beta H_R^V} \leq N^2 K_1 + \beta^{-1} \log \sum_{n: \sum_{\alpha} n_{\alpha}=N} \left( \begin{array}{c} N \\ n \end{array} \right) G(n) + N \log (R/\ell) + N^\varepsilon \Xi(\ell, \gamma), \quad (2.50)$$

where the energy functional $G$ is defined by

$$G(n) = \exp \left[ -\beta \sum_{\alpha \in W} \left\{ N n_{\alpha} \left[ 2c_V - 2\rho_V(\alpha) \ell^2 \eta \right] - F_b^{(\gamma)}(n_{\alpha}) \right\} - \beta C \sum_{\alpha \notin W} n_{\alpha} \log |\alpha| \right]. \quad (2.51)$$

It is easy to check that the contribution from $\alpha \notin W$ is negligible, and from now on, we will therefore drop such terms from $G$; namely we can multiply by the characteristic function

$$\chi_2(n) = 1 \left( n_{\alpha} = 0, \forall \alpha \notin W \right). \quad (2.52)$$

For the lower bound, we can restrict the numbers of particles in the squares $\alpha$ to be given by the rounded prediction $n_V(\alpha) = \lfloor N \rho_V(\alpha) \rfloor$ of the equilibrium measure. We denote this restriction by the characteristic function

$$\chi_1(n) = \prod_{\alpha} 1 \left( n_{\alpha} = n_V(\alpha) \right).$$

Thus we have

$$\beta^{-1} \log \int e^{-\beta H_R^V} \geq N^2 K_1 + \beta^{-1} \log \sum_{n: \sum_{\alpha} n_{\alpha}=N} \left( \begin{array}{c} N \\ n \end{array} \right) \chi_1(n) G(n) + N \log (R/\ell) - N^\varepsilon \Xi(\ell, \gamma). \quad (2.53)$$

Next we rewrite $G(n)$.

**Lemma 2.12.** Let $n_V^R(\alpha) = N \rho_V^R(\alpha) \ell^2$ be the classical number of particles in the square $\alpha$ predicted by the equilibrium measure at $\alpha$, and set

$$h_{\alpha}(n) = \eta \gamma^2 [n_{\alpha} - n_V^R(\alpha)]^2 - \xi_b^{(\gamma)}(n_{\alpha}).$$

Then

$$\log G(n) = -N^2 I_V^R - \beta \sum_{\alpha} h_{\alpha}(n) + O(N \ell^2 b) + O(e^{-N^\varepsilon}).$$

**Proof.** First, we write $F_b^{(\gamma)}(n_{\alpha})$ in the definition of $G(n)$ in terms of $\xi_b^{(\gamma)}(n_{\alpha})$ from (2.14), and note that the terms $n_{\alpha} \log \ell$ in this definition, summed over $\alpha$, cancel the term $N \log \ell$ in $N \log (R/\ell)$.

Moreover, by definition of $I_b^{(\gamma)}$ and of $\eta$, in (2.46),

$$I_b^{(\gamma)} = b^{-2} \int S_b^{(\gamma)}(z-w) m(dz) := \gamma^2 \eta + O(e^{-N^\varepsilon}), \quad (2.54)$$

where the error term comes from the fact that $S_b^{(\gamma)}$ is defined using the periodic distance while $\eta$ was defined with respect to the Lebesgue measure.
Define
\[ N^2 \tilde{I}(n) := \sum_{\alpha} N n_\alpha [2c_V - 2\rho_Y^R(\alpha)\ell^2 \eta] + n_\alpha^2 \eta \gamma^2. \quad (2.55) \]

Then we have
\[ \tilde{I}(n) = \eta N^{-2} \gamma^2 \sum_{\alpha} [n_\alpha - n_\alpha^R(\alpha)]^2 - C_1 + 2c_V, \quad C_1 = \eta \ell^2 b^2 \sum_{\alpha} \rho_Y^R(\alpha)^2. \quad (2.56) \]

From the definition of \( \eta \) in (2.46), and using that \( \rho_Y^R \) is \( C^1 \) in the interior of its support, which has a piecewise \( C^1 \) boundary, we can rewrite
\[ C_1 = b^2 \sum_{\alpha} \rho_Y^R(\alpha) \int Y^\ell(\alpha - w) \rho_Y^R(\alpha) \, dw = \int Y^\ell(z - w) \mu_Y^R(\alpha)(dz) \mu_Y^R(\alpha)(dw)(1 + O(b)). \quad (2.57) \]

By the Euler–Lagrange equation (A.5), we have \( Y^R \ast \mu_Y^R + \frac{1}{2} V = c_V \) in \( S_V \), and
\[-K_1 - C_1 + 2c_V = -\int L_R(z - w) \mu_Y^R(\alpha)(dw) \mu_Y^R(\alpha)(dz) - \int Y^\ell(z - w) \mu_Y^R(\alpha)(dz) \mu_Y^R(\alpha)(dw) + \int [V(z) + 2 \int Y^R(z - w) \mu_Y^R(\alpha)(dw)] \mu_Y^R(\alpha)(dz) + O(\ell^2 b) = \mathcal{I}_Y^R(\mu_Y^R) + O(\ell^2 b). \]

This completes the proof. \( \square \)

We summarize the main result we have proved so far as the following proposition.

**Proposition 2.13.** Define
\[ h_\alpha(n) = \eta \gamma^2 [n_\alpha - n_\alpha^R(\alpha)]^2 - \xi_b^{(\gamma)}(n_\alpha). \quad (2.58) \]

Then
\[ \beta^{-1} \log \int e^{-\beta H_Y^R(z)} \, m^\otimes N(dz) \geq -N^2 I_Y^R + N \log R + \beta^{-1} \log \sum_{n, \sum_{\alpha} n_\alpha = N} \left( \binom{N}{n} \right) \chi_1(n) e^{-\beta \sum_{\alpha} h_\alpha(n)} - N \varepsilon \Xi(\ell, \gamma), \quad (2.59) \]
\[ \beta^{-1} \log \int e^{-\beta H_Y^R(z)} \, m^\otimes N(dz) \leq -N^2 I_Y^R + N \log R + \beta^{-1} \log \sum_{n, \sum_{\alpha} n_\alpha = N} \left( \binom{N}{n} \right) \chi_2(n) e^{-\beta \sum_{\alpha} h_\alpha(n)} + N \varepsilon \Xi(\ell, \gamma), \quad (2.60) \]

where the \( \chi_i \) are defined in (2.53), (2.52), \( \Xi(\ell, \gamma) \) was defined in (2.27), \( \xi \) was defined in (2.14), and we recall that \( \alpha \) indexes squares of size \( b \).

### 2.7. Scaling relation and continuity for the residual free energy
To evaluate the sums on the right-hand sides of (2.59), (2.60), we need some properties of the residual free energy, which we derive in this section. We first show the following scaling relation for the residual free energy.
Lemma 2.14.

\[ \xi_{Kb}^{(\gamma)}(n) = \left( \frac{1}{\beta} - \frac{1}{2} \right) n \log K^2 + \xi_b^{(\gamma)}(n). \]  
(2.61)

In particular, by choosing \( K = b^{-1} \),

\[ \xi_b^{(\gamma)}(n) = n \zeta^{(\gamma)}(n) + \frac{n}{2} \log n + n \left( \frac{1}{2} - \frac{1}{b} \right) \log b^{-2}, \]
(2.62)

where we recall the definition of \( \zeta \) from (2.15).

**Proof.** To see (2.61), consider the Yukawa gas of \( n \) particles in a torus of size \( K b \) with \( K > 0 \). By definition of the Yukawa potential (2.1), \( Y_{K\ell}(Kr) - Y_{\ell}(r) = 0 \). Therefore, by changing variables to \( z = w K \),

\[
\beta^{-1} \log \int_{|z_i| \leq K b / 2} e^{-\beta \sum_{i \neq j} Y_{K\ell}(z_i - z_j)} \prod_i m(dz_i) = \beta^{-1} \log \int_{|w_i| \leq b / 2} e^{-\beta \sum_{i \neq j} Y_{\ell}(w_i - w_j)} \prod_i m(dw_i),
\]

where the term with \( \log K^2 \) comes from the scaling factor in the Jacobian. With \( \gamma = \ell / b \) and using the definition (2.14) of \( \xi \), we have the rescaling identity

\[
\xi_{Kb}^{(\gamma)}(n) = n^2 I_{Kb}^{(\gamma)} - n \log K \ell + \beta^{-1} n \log K^2 + \beta^{-1} \log \int e^{-\beta \sum_{i \neq j} Y_{\ell}(w_i - w_j) - \beta H_i} \prod_i m(dw_i)
\]

\[
= \left( \frac{1}{\beta} - \frac{1}{2} \right) n \log K^2 + \xi_b^{(\gamma)}(n) + n^2 [I_{Kb}^{(\gamma)} - I_b^{(\gamma)}].
\]

Since, by definition (2.12) and scaling, we have \( I_{Kb}^{(\gamma)} = I_b^{(\gamma)} \), this proof of (2.61) is complete. \[ \square \]

In view of the previous proposition, it suffices to study the (specific) residual free energy on the unit torus. The next lemma shows that it only depends weakly on \( n \).

**Lemma 2.15.** The residual free energy on the unit torus satisfies

\[ \xi_1^{(\gamma)}(n) \leq C n \log n, \]  
(2.63)

\[ \xi_1^{(\gamma)}(n) + I_1^{(\gamma)} \leq \xi_1^{(\gamma)}(n + 1) \leq \xi_1^{(\gamma)}(n) + O(n^{\varepsilon}). \]
(2.64)

In particular, in terms of \( \zeta \) we have

\[ \zeta^{(\gamma)}(n) - \zeta^{(\gamma)}(m) - \frac{1}{2} \log(n/m) = \frac{\xi_1^{(\gamma)}(n)}{n} - \frac{\xi_1^{(\gamma)}(m)}{m} = O((n + m)^{\varepsilon - 1}|m - n|). \]  
(2.65)

**Proof.** The bound (2.63) follows exactly as in (6) Proposition 4.1 or (A.12), by smearing out the point charges into densities and positive definiteness (for the upper bound) and by Jensen’s inequality (for the lower bound). We now prove that \( \zeta \) depends weakly on \( n \). On the unit torus, by Jensen’s inequality,

\[
\log \frac{\int e^{-\beta \sum_{i \neq j, i, j = 1}^{n+1} Y^{\gamma}(z_i - z_j)} \prod_{i=1}^{n} m(dz_i)}{\int e^{-\beta \sum_{i \neq j, i, j = 1}^{n} Y^{\gamma}(z_i - z_j)} \prod_{i=1}^{n} m(dz_i)} \geq -2\beta \mathbb{E}_n \sum_{j=1}^{n} Y^{\gamma}(z_{n+1} - z_j).
\]
Integrating both side over \( z_{n+1} \), we get
\[
\log Z_{n+1} \geq \int m(dz_{n+1}) \log \int e^{-\beta \sum_{i=1}^{n+1}(Y^\gamma(z_i-z_j))} \prod_{i=1}^{n} m(dz_i) \geq \log Z_n - 2nI_1^\gamma.
\]

Recall \( \xi_1^\gamma(n) \) is the residual free energy at length scale one. Hence
\[
\xi_1^\gamma(n+1) = (n+1)^2 I_1^\gamma - (n+1) \log \gamma + \beta^{-1} \log Z_{n+1}(\beta) \\
\geq (n+1)^2 I_1^\gamma - 2nI_1^\gamma + \beta^{-1} \log Z_n(\beta) - (n+1) \log \gamma \\
= \xi_1^\gamma(n) + I_1^\gamma - \log \gamma.
\]

On the other hand, set \( \hat{H}_k = \sum_{i \neq j, i, j \neq k} Y^\gamma(z_i - z_j) \). Then by Hölder’s inequality,
\[
Z_{n+1}(\beta) = \int \exp \left[ -\frac{\beta}{n-1} \sum_{k=1}^{n+1} \hat{H}_k \prod_{i=1}^{n+1} m(dz_i) \right] \\
\leq \int e^{-\beta \frac{n+1}{n-1} \hat{H}_k} \prod_{i=1}^{n+1} m(dz_i) = Z_n(\beta \frac{n+1}{n-1}).
\]

By \( \ref{2.63} \), the free energy of the Yukawa gas on the unit torus can be estimated by
\[
\beta^{-1} \log \int e^{-\beta H_n} \prod_{i=1}^{n} m(dz_i) = -n^2 I_1^\gamma + O(n^{1+\varepsilon}), \quad H_n = \sum_{i \neq j} Y^\gamma(z_i - z_j).
\]

By convexity of the function \( t \to \log \int e^{-tH_n} \), we have
\[
-E_n^{\gamma, \beta} H_n \leq \log \int e^{-(\beta+1)H_n} - \log \int e^{-\beta H_n} \leq -n^2 I_1^\gamma + O(n^{1+\varepsilon}).
\]

Integrating the relation \( \partial_\beta \log Z_n(\beta) = -E_n^{\gamma, \beta} H_n \), we therefore get
\[
\log Z_{n+1}(\beta) \leq \log Z_n(\beta \frac{n+1}{n-1}) = \log Z_n(\beta) - \int_{\beta}^{\beta \frac{n+1}{n-1}} E_n^{\gamma, \beta} H_n ds \\
\leq \log Z_n(\beta) - 2\beta \frac{n^2}{n-1} I_1^\gamma + O(n^\varepsilon).
\]

In summary, we have proved that
\[
\xi_1^\gamma(n+1) = (n+1)^2 I_1^\gamma - (n+1) \log \gamma + \beta^{-1} \log Z_{n+1}(\beta) \\
\leq (n+1)^2 I_1^\gamma + \beta^{-1} \log Z_n(\beta) - 2\beta \frac{n^2}{n-1} I_1^\gamma - n \log \gamma - \log \gamma + O(n^\varepsilon) \\
= \xi_1^\gamma(n) + O(n^\varepsilon).
\]

This concludes the proof of the lemma.

The following lemma shows that the residual free energy \( \zeta^\gamma \) defined in \( \ref{2.15} \) is essentially independent of the range parameter \( \gamma \) as long as \( \gamma \) is not too small. In view of this result, we will drop the superscript \( (\gamma) \) in \( \zeta \) after this lemma.
Lemma 2.16. For interaction ranges \( \omega \geq \nu \), the specific residual free energy obeys

\[
\zeta^{(\omega)}(n) - \zeta^{(\nu)}(n) = O(n^{\epsilon-1}/\nu^2).
\] (2.66)

In particular, with \( \nu = \gamma = N^{-q}, q > 0 \), we have

\[
\zeta^{(\omega)}(n) - \zeta^{(\gamma)}(n) = O(n^{\epsilon-1}/\gamma^2) = O(n^{\epsilon-1+2q}).
\] (2.67)

Proof. By Jensen’s inequality

\[
\zeta^{(\omega)}(n) = \zeta^{(\nu)}(n) + \int L_{\omega}(z)\tilde{\mu}(dz)\tilde{\nu}(dw) + \int \log(e^{-\beta H^\omega_n} - 1),
\]

where \( \tilde{\mu}(dz) = \mu(dz) - \rho(dz) \) with \( \mu(dz) \) the flat measure on the torus. Applying (2.21) on the unit torus, we get

\[
\beta^{-1}\log\int e^{-\beta H_\omega^\nu} \geq -n^2E \int \int L_{\omega}(z-w)\tilde{\mu}(dz)\tilde{\nu}(dw) + \beta^{-1}\log\int e^{-\beta H^\omega_n},
\]

where \( \tilde{\mu}(dz) = \rho(dz) - \mu(dz) \) with \( \mu(dz) \) the flat measure on the torus. Applying (2.43) on the unit torus, we get

\[
\beta^{-1}\log\int e^{-\beta H_\omega^\nu} = O(n^{-2}\nu^2).
\]

By definitions of \( \xi \) and \( \zeta \) in the Definition 2.3, these equations imply the lemma. \( \square \)

2.8. Density approximation. The main ingredient in the proof of Theorem 2.5 is the following proposition, which is a consequence of Proposition 2.14. As previously, we divide \( \mathbb{C} \) into squares of size \( b \), indexed by their centers, which we denote by \( \alpha \).

Proposition 2.17. There is a positive number \( \kappa > 0 \) such that

\[
(\beta N)^{-1}\log\int e^{-\beta H^\nu_\alpha}(\mathbf{z}) m^{\otimes N}(\mathbf{z}) = -NI_\nu^R + \frac{1}{2}\log N + \log R
\]

\[
+ \left(\frac{1}{2} - \frac{1}{\beta}\right)\rho_\nu^R \log\rho_\nu^R + N^{-1}\sum_\alpha n_\nu(\alpha)\zeta(\nu_\alpha) + O(N^\kappa\Xi(\ell, \gamma)),
\] (2.69)

where \( \zeta = \zeta^{(R)} \) and \( n_\nu(\alpha) = [N\rho_\nu^R(\alpha)b^2] \) with \( \rho_\nu^R \) being the equilibrium density.

Proof. We need to compute the sums on the right-hand sides of (2.59), (2.60). By Stirling’s formula, we have the asymptotic

\[
\log\left(\frac{N}{n}\right) = N\log N - \sum_\alpha n_\alpha \log n_\alpha + O(\log N).
\]

We absorb the \( O(\log N) \) term in the Stirling approximation into the error terms of (2.59), (2.60), and use (2.62) to express \( \xi^{(\nu)}_b \) in terms of \( \zeta \), to rewrite (2.59), (2.60) as

\[
\beta^{-1}\log\int e^{-\beta H^\nu_\alpha}(\mathbf{z}) m^{\otimes N}(\mathbf{z}) + N^2I_\nu^R + O(N^\kappa\Xi(\ell, \gamma)) = F_V + N\log R + \beta^{-1}N\log N,
\] (2.70)

where

\[
F_V = \beta^{-1}\log\sum_{n: \sum_\alpha n_\alpha = N} \chi(n)e^{\beta\xi(n)},
\]

\[
\xi(n) = \sum_\alpha \left[\left(\frac{1}{2} - \frac{1}{\beta}\right)n_\alpha \log(n_\alpha b^2) + n_\alpha \zeta(n_\alpha) - \eta\gamma^2[n_\alpha - n_\nu(\alpha)]^2\right].
\] (2.71)
Lemma 2.18 below shows that we can replace \( n_\alpha \) by its classical value \( n_V(\alpha) = [np_{V}(\alpha)b^2] \) in (2.71), with sufficiently small error. This replacement, inserted into (2.70), yields

\[
\sum_{\alpha} n_V(\alpha) \log(n_V(\alpha)b^{-2}) = N \int \rho_V(x) \log \rho_V(x) dx + N \log N + O(Nb),
\]

and we have proved Proposition 2.17.

Given that \( \xi_b^{(\gamma)}(n) = O(n \log n) \) by (2.63), in (2.59), (2.60) we only have to sum over \( n_\alpha \) satisfying \( \gamma^2 [n_\alpha - n_V(\alpha)]^2 \leq Cn_\alpha \) which implies that

\[
|n_\alpha - n_V(\alpha)| \leq C\sqrt{n_V(\alpha)b/\ell} \ll n_V(\alpha).
\]

Together with the continuity of \( \xi_b^{(\gamma)}(n) \) established in the previous subsection, this determines the density up to lower order corrections.

Lemma 2.18. \( F_V \) defined in (2.71) is bounded from above and below by

\[
F_V^0 \leq F_V \leq F_V^0 + O(N^{1+\varepsilon - 2\delta}),
\]

where

\[
F_V^0 = \sum_{\alpha} E_1(n_V(\alpha)), \quad E_1(n) = \left( \frac{1}{2} - \frac{1}{\beta} \right) n \log(nb^{-2}) + n\zeta(n).
\]

Proof. We only need to prove the upper bound. By definition,

\[
F_V - F_V^0 \leq \beta^{-1} \log \sum_{n: \sum_{\alpha} n_\alpha = N, \alpha \in W} \exp \left[ \sum_{\alpha} \beta \left\{ [E_1(n_\alpha) - E_1(n_V(\alpha))] - \eta \gamma^2 [n_\alpha - n_V(\alpha)]^2 \right\} \right].
\]

By (2.63), we have \( |E_1(n) - E_1(m)| \leq |n - m|(n + m)^\varepsilon \). One can check the following elementary inequality that for any positive fixed number \( A > 0 \) and any integer \( m \geq 0 \),

\[
\sum_{n=0}^{\infty} \exp \left[ |n - m|(n + m)^\varepsilon - A\gamma^2(n - m)^2 \right] \leq \gamma^{-1}(m + \gamma^{-2})^{2\varepsilon} e^{C\gamma^{-2}(m + \gamma^{-2})^{2\varepsilon}}.
\]

To get an upper bound, we can now drop the constraint \( \sum_{\alpha} n_\alpha = N \) in (2.72). Without this constraint, the summation of \( n_\alpha \) in each \( \alpha \) can be performed independently, and using the previous bound, we thus have

\[
F_V - F_V^0 \leq C \log N \sum_{\alpha \in W} \gamma^{-2}(n_V(\alpha) + \gamma^{-2})^{2\varepsilon} \leq C(Nb^2)^{2\varepsilon} b^{-2}\gamma^{-2} = O(N^{1+\varepsilon - 2\delta}).
\]

This completes the proof of the lemma.

2.9. Proofs of Theorems 2.4 and 2.5. In preparation of the proof of Theorem 2.4, we discuss the following conditions for \( \Xi(\ell, \gamma) \) to be bounded by \( N^{1-\kappa} \).

Remark 2.19. From (2.27), recall the definition

\[
\Xi(\ell, \gamma) = N^2 \ell^2 b + N^2 \ell^3/b + \ell^{-2} + N^2 b^4 = N^2 \ell^2 b^3 + N^2 \gamma^3 b^2 + \gamma^{-2} b^{-2} + N^2 b^4.
\]
Choosing
\[ \gamma \geq N^{-2/5}, \quad b = N^{-1/2} \gamma^{-5/4} \leq 1, \] (2.73)
therefore
\[ \Xi(\ell, \gamma) \leq O(\sqrt{N} \gamma^{-7/4}) + O(N \sqrt{\gamma}). \] (2.74)

Hence we get
\[ \Xi(\ell, \gamma) = O(N^{1-\kappa}), \quad \text{if} \quad N^{-2/7+\kappa} \leq \gamma \leq N^{-2\kappa}. \] (2.75)
The condition \( \gamma \geq N^{-2/5} \) automatically holds by our assumptions. The constraints on \( \gamma \) to have a solution can be satisfied if \( \kappa \leq 1/11 \), and with the choice of \( b \) above, we have \( Nb^2 = \gamma^{-5/2} =: n \) and the condition on \( \gamma \) is equivalent to
\[ N^{5/7-5\kappa/2} \geq n \geq N^{5\kappa}. \] (2.76)

**Proof of Theorem 2.4.** Applying (2.69) to the unit torus with \( R = \gamma, N = m, mb^2 = n \), we have
\[
\xi_1^{(\gamma)}(m) = m^2 I_1^{(\gamma)} - m \log \gamma + \beta^{-1} \log \int e^{-\beta H_0^\dagger(x)} m^{\infty N}(dx) \\
= \frac{1}{2} m \log m + m \left( \frac{1}{2} - \frac{1}{\beta} \right) \int \rho(z) \log \rho(z) + \sum_\alpha \tilde{n}(\alpha) \zeta(\tilde{n}(\alpha)) + O(m^{1-\kappa}),
\] (2.77)
where \( \tilde{n}(\alpha) \) denotes the rounded number of particles in the square \( \alpha \) according to the equilibrium measure, and we have assumed that the parameters satisfy (2.76) so that (2.75) holds. Since its density is \( \rho = 1 \) on the torus, we have \( \tilde{n}(\alpha) = n \), and the term \( \int \rho \log \rho \) vanishes. Denoting \( 1/b = B \) and \( m = B^2 n \), we obtain
\[ \zeta^{(\gamma)}(B^2 n) = \frac{\xi_1^{(\gamma)}(m)}{m} - \frac{1}{2} \log m = \zeta(n) + O((B^2 n)^{-\kappa}), \] (2.78)
provided that \( B \) is an integer and (2.76) holds with \( N \) replaced by \( nB^2 \).

Note that for any \( n \) and \( \kappa < 1/11 \), if we denote \( B = n^\varepsilon \), there is a range of \( x \) so that (2.76) holds with \( N \) replaced by \( nB^2 \). Similarly, we can consider the last equation with \( n \) replaced by \( \tilde{n} = nM \) with \( 1 \leq M \leq n^\varepsilon \) for small enough \( \varepsilon > 0 \), namely,
\[ \zeta^{(\gamma)}(A^2 \tilde{n}) = \zeta(\tilde{n}) + O((A^2 \tilde{n})^{-\kappa}), \]
provided that (2.76) holds with \( N \) replaced by \( nA^2 \). By (2.65),
\[ |\zeta^{(\gamma)}(B^2 n) - \zeta^{(\gamma)}(A^2 \tilde{n})| \leq \frac{|A^2 \tilde{n} - B^2 n|}{|A^2 n + B^2 n|^{1-\varepsilon}}. \] (2.79)
Now we can choose \( A \) and \( B \) such that \( |A \sqrt{M} - B| \leq 1 \) so that \( |A^2 M - B^2| \leq O(B) \). Thus we have
\[ |\zeta^{(\gamma)}(B^2 n) - \zeta^{(\gamma)}(A^2 \tilde{n})| \leq \frac{n^\varepsilon}{B^{1-2\varepsilon}}. \] (2.80)
Since \( \varepsilon \) is arbitrarily small, there exists \( \tilde{c}, A, B, c \) so that all previous requirements can be satisfied and \( B^{1-2\varepsilon} \leq n^{-c\kappa} \). This proves that
\[ |\zeta(n) - \zeta(\tilde{n})| \leq n^{-c\kappa} \] (2.81)
provided that \( \tilde{n} = n^{1+\varepsilon} \) and \( \tilde{c} \) is small enough. Applying this estimate to all scales, we get

\[
|\zeta(n) - \zeta(nA^2K)| \leq n^{-\tau} \sum_{j=0}^{K-1} A^{-2j\tau} = O(n^{-\tau}).
\]

This implies the existence of the limit and the estimate \( \zeta(n) = \zeta_\infty + O(n^{-\tau}) \) as well.

**Proof of Theorem 2.4.** By Theorem 2.4, we have \( \zeta(nV(\alpha)) = \zeta_\infty + O(n^{-\tau}) \). Substituting this estimate into (2.69), and using that, with the choice (2.28), (2.29), we have \( N^\varepsilon \Xi(\ell, \gamma) = O(N^{1-k}) \) for some \( k > 0 \), we obtain

\[
(t\beta N)^{-1} \log \mathbb{E}_{V} e^{-t\beta H_{V}^{R}} = \frac{tN}{8\pi} \int |\nabla f(z)|^2 m(dz) - \frac{1}{\beta} Y_{V}^{f} + O(tN^{2\sigma}) + O(N^{-2\sigma + \varepsilon}).
\]

(3.1)

for some \( \kappa > 0 \). This proves Theorem 2.5.

**3 Proof of Theorem 1.2: the central limit theorem**

This section proves Theorem 1.2, conditionally on Theorem 1.1. The main tool is the rigidity for the Coulomb gas, equation (1.8), proved in [6], and its generalization to a Coulomb gas with additional interaction given by a local angle term (Appendix B). In this section we treat the case of a test function \( f \) on a macroscopic scale, i.e. \( s = 0 \), the necessary adjustments for \( s \in (0, 1/2) \) being straightforward as explained in Section 4.

### 3.1. The central limit theorem with local angle terms.

We first prove that a version of Theorem 1.2 holds where we add some local angular terms. 

**Theorem 3.1.** Suppose \( V \) satisfies the conditions (1.4) and (1.9), and \( f \) satisfies the same assumptions as in Theorem 1.2. There is a random variable \( \hat{A}_h \), the local angle term (see 3.13), such that for any small \( \sigma, \varepsilon \), and for \( tN^{2\sigma} \ll 1 \), we have

\[
(t\beta N)^{-1} \log \mathbb{E}_{V} e^{-t\beta N(X_{f}^{V} - \text{Re} \hat{A}_h)} = \frac{tN}{8\pi} \int |\nabla f(z)|^2 m(dz) - \frac{1}{\beta} Y_{V}^{f} + O(tN^{2\sigma}) + O(N^{-2\sigma + \varepsilon}).
\]

We start the proof with an integration by parts formula. Consider a smooth bounded function \( v : \mathbb{C} \rightarrow \mathbb{C} \), and \( G \) smooth, defined on \( z_1 \neq z_2 \) such that \( G(z_1, z_2) = G(z_2, z_1) \), and

\[
\lim sup_{|z_2| \rightarrow \infty} |G(z_1, z_2)| / \log |z_2| = 1.
\]

(3.2)

for any fixed \( z_1 \). For any \( z \in \mathbb{C}^N \) we define

\[
W_{V}^{G, v}(z) = - \sum_{j \neq k} (v(z_j) - v(z_k)) \partial_{z_j} G(z_j, z_k) + \frac{1}{\beta} \sum_{j} \partial_j v(z_j) - N \sum_{j} v(z_j) \partial V(z_j).
\]

(3.3)

The following elementary lemma is often referred to as Ward identity or loop equation. For example it was used in [5] for fluctuations of the empirical measure when \( \beta = 1 \), and in [6] for rigidity when \( \beta > 0 \), with in both cases the interaction being \( C \). In this work we will need a perturbation \( G \) of the Coulomb interaction by angle terms.
Lemma 3.2. Under the above assumptions, we have (the expectation is with respect to $P_{N,V}^G$ defined in (1.3))

$$
\mathbb{E}
\left(\frac{W_{V}^{G,v}}{2}\right) = \frac{1}{2}\mathbb{E}
\left(\sum_{j\neq k}(v(z_j) + v(z_k)) (\partial_{z_k} + \partial_{z_j} G(z_j, z_k))\right).
$$

Proof. The proof is a classical simple integration by parts: for any $j \in [1, N]$, we have

$$
\mathbb{E}
\left(\partial_{z_j} v(z_j)\right) = \beta \mathbb{E}
\left(v(z_j) \partial_{z_j} H(z)\right),
$$

where both terms are absolutely summable and the boundary terms vanishes because (i) with probability 1, no two $z_i$’s have the same real or imaginary part, (ii) $v$ is bounded, $G$ satisfies the growth condition (3.2), $V$ satisfies the growth condition (1.4). Summation of the above equation over all $j \in [1, N]$ therefore gives

$$
\frac{1}{\beta N} \sum_{j=1}^{N} \mathbb{E}(\partial_{z_j} v(z_j)) = \mathbb{E} \left(\sum_{j=1}^{N} v(z_j) \left(\partial_{z_j} V(z_j) + \sum_{k \neq j} (\partial_{z_j} G(z_j, z_k) + \partial_{z_j} G(z_k, z_j))\right)\right)
$$

$$
= \mathbb{E} \left(\sum_{j=1}^{N} v(z_j) \left(\partial_{z_j} V(z_j) + \sum_{k \neq j} (\partial_{z_j} - \partial_{z_k}) G(z_j, z_k)\right)\right) + \mathbb{E} \left(\sum_{j=1}^{N} v(z_j) \partial_{z_j} G(z_j, z_k)\right)
$$

Using $G(z_j, z_k) = G(z_k, z_j)$, we can continue the equation with

$$
= \mathbb{E} \left(\sum_{j=1}^{N} v(z_j) \partial_{z_j} V(z_j) + \frac{1}{2} \sum_{j \neq k} \left(v(z_j) (\partial_{z_j} - \partial_{z_k}) G(z_j, z_k) + v(z_k) (\partial_{z_k} - \partial_{z_j}) G(z_k, z_j)\right)\right)
$$

$$
+ \frac{1}{2} \mathbb{E} \left(\sum_{j \neq k} (v(z_j) + v(z_k)) (\partial_{z_j} + \partial_{z_k}) G(z_j, z_k)\right)
$$

$$
= \mathbb{E} \left(\sum_{j=1}^{N} v(z_j) \partial_{z_j} V(z_j) + \frac{1}{2} \sum_{j \neq k} \left(v(z_j) - v(z_k)\right) \partial_{z_j} G(z_j, z_k)\right)
$$

$$
+ \frac{1}{2} \mathbb{E} \left(\sum_{j \neq k} (v(z_j) + v(z_k)) (\partial_{z_j} + \partial_{z_k}) G(z_j, z_k)\right).
$$

This concludes the proof. \(\square\)

Before we consider $G$ including angular terms, we temporarily restrict our attention to the Coulomb case: $\partial_{z_j} \mathcal{C}(z_j - z_k) = -\frac{1}{2}(z_j - z_k)^{-1}$. We will need the following notation,

$$
h(z) = \frac{\partial f(z)}{\partial \partial V(z)},
$$

(3.4)

which is used in the following identity.

Lemma 3.3. For any $f : \mathbb{C} \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$, supported on $S_V$, and $z \in \mathbb{C}^N$, we have

$$
X_V^f = -\frac{1}{N} W_V^{h}(z) + \frac{1}{\beta} \sum_{k} \partial h(z_k) + \frac{N}{2} \iint_{z \neq w} h(z) - h(w) \frac{\bar{\mu}_V(\mathrm{d}z)\mu_V(\mathrm{d}w)}{z - w},
$$

(3.5)

where we used the notations (1.12) and $\bar{\mu}_V = \bar{\mu} - \mu_V$. 

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Proof. First remember the following two identities:

\[ \int \frac{\mu_V(dw)}{z-w} = \partial V(z), \quad (3.6) \]

\[ f(z) = \frac{1}{\pi} \int \frac{\partial f(w)}{z-w} m(dw). \quad (3.7) \]

The first equation holds for \( z \in S_V \) and is obtained by the Euler-Lagrange equation, the second equation is a simple integration by parts. We therefore can write

\[
X^f_V = \sum_j \int \frac{h(w)}{z_j - w} \mu_V(dw) - N \int \frac{h(w)}{z-w} \mu_V(dw) \mu_V(dz)
\]

\[
= N \int \frac{h(w) - h(z)}{z-w} \bar{\mu}_V(dz) \mu_V(dw) + \sum_j h(z_j) \partial V(z_j) - \frac{N}{2} \int \frac{h(w) - h(z)}{z-w} \bar{\mu}_V(dz) \mu_V(dw)
\]

\[
= -\frac{1}{2N} \sum_{j \neq k} \int \frac{h(z_j) - h(z_k)}{z_j - z_k} + \sum_j h(z_j) \partial V(z_j) + \frac{N}{2} \int \frac{h(z) - h(w)}{z-w} \bar{\mu}_V(dz) \mu_V(dw),
\]

which is equivalent to (3.5). In the first equation we used (1.10), (3.4) and (3.7), and in the second equation we used (3.6).

We want to decompose the last term in (3.5) into a sum of the long range and short range terms. For this purpose, let \( \varphi(z) = e^{-|z|^2} \) and, given a mesoscopic scale

\[ \theta = N^{-\frac{1}{2}+\sigma}, \]

we define

\[ \Phi(z-w,r) = \frac{2}{\pi} \int \varphi \left( \frac{|z-\xi|}{r} \right) \varphi \left( \frac{|\xi-w|}{r} \right) d\mu(\xi) = r^2 e^{-\frac{|z-w|^2}{2r^2}}, \]

\[ \Phi^-_\theta(z-w) = \int_0^\theta \Phi(z-w,r) \frac{dr}{r^5} = e^{-\frac{|z-w|^2}{2r^2}}, \]

\[ \Phi^+_\theta(z-w) = \int_\theta^\infty \Phi(z-w,r) \frac{dr}{r^5} = 1 - e^{-\frac{|z-w|^2}{2r^2}}, \]

We also define

\[ \Psi^\pm_h(z,w) = \Phi^\pm_h(z-w)(\bar{z}-\bar{w})(h(z) - h(w)), \]

\[ \Psi^h(z,w) = \Psi^+_h(z,w) + \Psi^-_h(z,w). \]

Following an idea from [20] that we used in [6], we then have decomposed the last term in (3.5) into a relatively long range part and, essentially, a local angle term:

\[ \frac{N}{2} \int \frac{h(z) - h(w)}{z-w} \bar{\mu}_V(dz) \bar{\mu}_V(dw) = A^{h+}_V + A^{h-}_V, \]

where

\[ A^{h+}_V = \frac{N}{2} \int \Psi^+_h(z,w) \bar{\mu}_V(dz) \bar{\mu}_V(dw), \]

\[ A^{h-}_V = \frac{N}{2} \int \Psi^-_h(z,w) \bar{\mu}_V(dz) \bar{\mu}_V(dw). \]
Note that, in the above decomposition, we could have considered any fixed nonnegative function $\varphi \in C^\infty(\mathbb{C})$ with fast decay at infinity, as in [4 Section 7]. We chose the Gaussian scale function for the sake of concreteness and some convenient simplifications.

In the following, we will abbreviate
\[
\hat{A}^h = \frac{1}{2N} \sum_{i \neq j} \Psi^{-}_h(z_i, z_j).
\]  

(3.13)

We now state the central limit theorem up to the angle term. Its proof is an application of a loop equation and a rigidity estimate for the Coulomb gas with additional local angle interaction.

**Proof of Theorem 3.1.** For $0 \leq t \ll 1$, we define
\[
h_t(z) = \frac{\bar{\partial} f(z)}{\partial \bar{\partial}(V(z) + tf(z))},
\]

and abbreviate $h = h_0$. We also denote the new pairwise interaction $G_t = C - t \Re \Psi^{-}_t$, and a new Hamiltonian with angle term,
\[
H_{V+tf}^G = H_{V+tf}^C - tN \Re \hat{A}^h.
\]  

(3.14)

As $t \ll 1$, the supports $S_V$ and $S_{V+tf}$ coincide so that we can apply (3.5) to obtain
\[
\begin{align*}
\partial_t \frac{1}{\beta N} \log Z_{V+tf}^G &= \Re \mathbb{E}_{V+tf}^G \left( - X_{V+tf}^f + N \int f(\mathrm{d}\mu_V - \mathrm{d}\mu_{V+tf}) + \hat{A}^h \right) \\
&= \frac{Nt}{4\pi} \int |\nabla f|^2 \mathrm{d}m + \Re \mathbb{E}_{V+tf}^G \left( \frac{1}{N} W_{V+tf}^{h_t} - \frac{1}{N \beta} \sum_k \partial h_t(z_k) - A_{V+tf}^{h_t^+,} - A_{V+tf}^{h_t^-} + \hat{A}^h \right),
\end{align*}
\]  

(3.15)

where we used (1.6) and (3.3) with the choice $V + tf$ for the external potential. Moreover note that the new interaction $G_t$ satisfies $G_t(z_j, z_k) = G_t(z_k, z_j)$ and the growth assumption (3.2), so that Lemma 3.2 applies and we have
\[
\begin{align*}
\mathbb{E}_{V+tf}^G \left( W_{V+tf}^{h_t} + t \sum_{j \neq k} (h_t(z_j) - h_t(z_k)) \partial z_j \Re \Psi^{-}_{h_t}(z_j, z_k) \right) &= \mathbb{E}_{V+tf}^G \left( W_{V+tf}^{G_t, h_t} \right) \\
&= \frac{1}{2} \mathbb{E}_{V+tf}^G \left( \sum_{j \neq k} (h_t(z_j) + h_t(z_k))(\partial z_k + \partial z_j) G_t(z_j, z_k) \right).
\end{align*}
\]  

(3.16)

Equations (3.15) and (3.16) give
\[
\begin{align*}
\partial_t \frac{1}{\beta N} \log Z_{V+tf}^G &= \frac{Nt}{4\pi} \int |\nabla f|^2 \mathrm{d}m + \Re \mathbb{E}_{V+tf}^G \left( - \frac{t}{N} \sum_{j \neq k} (h_t(z_j) - h_t(z_k)) \partial z_j \Re \Psi^{-}_{h_t}(z_j, z_k) \\
&\quad - \frac{1}{N \beta} \sum_k \partial h_t(z_k) - A_{V+tf}^{h_t^+,} - A_{V+tf}^{h_t^-} + \hat{A}^h + \frac{1}{2N} \sum_{j \neq k} (h_t(z_j) + h_t(z_k))(\partial z_k + \partial z_j) G_t(z_j, z_k) \right).
\end{align*}
\]  

(3.17)

We now evaluate all terms in the above expectation.
First, from (3.26) and an integration by parts, we have
\[
\text{Re} \mathbb{E}_{V+tf}^G \left( \frac{1}{N} \sum_k \partial h_t(z_k) \right) = -\frac{1}{N} \text{Re} \int \partial h_t \mu_{V+tf} + O(N^{-1+\varepsilon})
\]
\[
= -\frac{1}{4\pi} \text{Re} \int \frac{\partial f}{\partial \partial(V+tf)} \Delta(V+mf) \partial h_t \mu_{V+tf} + O(N^{-1+\varepsilon})
\]
\[
= \frac{1}{\pi} \text{Re} \int \frac{\partial f}{\partial \partial(V+tf)} \log \Delta(V+mf) \partial h_t \mu_{V+tf} + O(N^{-1+\varepsilon})
\]
= \frac{1}{\pi} \int f \frac{\partial f}{\partial h_t} \Delta(V+mf) \partial h_t \mu_{V+tf} + O(N^{-1+\varepsilon})
\]
\[
= \frac{1}{\pi} \int f |\Delta f|^2 \partial h_t \mu_{V+tf} + O(N^{-1+\varepsilon})
\]
(3.18)

Second, from (3.27) we have
\[
\mathbb{E}_{V}^{C-t} \text{Re} \Psi_a^-(A_{V+tf}^{-1}) = O_e(N^{-2\sigma+\varepsilon}).
\]
(3.19)

Third, a simple differentiation using (3.8) gives
\[
\left| \frac{t}{N} \sum_{j \neq k} (h_t(z_j) - h_t(z_k)) \partial_{z_j} \text{Re} \Psi_a^-(z_j, z_k) \right|
\]
\[
\leq C \frac{t}{N} \|\nabla h_t\|_\infty \sum_{j \neq k: z_j \in \Omega} e^{-\frac{|z_j - z_k|^2}{2\sigma^2}} \left( 1 + \frac{|z_j - z_k|^2}{\theta^2} \right) + O(e^{-N^2}).
\]
where $\Omega$ is the $N^\varepsilon \theta$-neighborhood of the support of $f$. Using (3.20), we therefore have
\[
\text{Re} \mathbb{E}_{V+tf}^G \left( \frac{t}{N} \sum_{j \neq k} (h_t(z_j) - h_t(z_k)) \partial_{z_j} \text{Re} \Psi_a^-(z_j, z_k) \right)
\]
= \( O(tN^{2\sigma} \|\nabla h_t\|_\infty^2 N^{-2s}) \) .
(3.20)

Similarly, (3.3) yields
\[
\frac{t}{N} \sum_{j \neq k} (h_t(z_j) + h_t(z_k)) \partial_{z_j} G_t(z_j, z_k) = \frac{t}{N} \sum_{j \neq k} (h_t(z_j) + h_t(z_k)) \frac{\partial h_t(z_j) - \partial h_t(z_k)}{z_j - z_k} e^{-\frac{|z_j - z_k|^2}{2\sigma^2}}
\]
\[
= O(tN^{2\sigma} \|h_t\|_\infty^2 \|\nabla^2 h_t\|_\infty N^{-2s}) + O(e^{-N^2}).
\]
The rigidity estimate (3.20) then yields
\[
\text{Re} \mathbb{E}_{V+tf}^G \left( \frac{t}{N} \sum_{j \neq k} (h_t(z_j) + h_t(z_k)) \partial_{z_j} \partial_{z_k} G_t(z_j, z_k) \right)
\]
= \( O(tN^{2\sigma} \|h_t\|_\infty \|\nabla^2 h_t\|_\infty N^{-2s}) \) .
(3.21)

Finally, we evaluate $A_{V+tf}^{h_t} - \hat{A}^h$. To simplify $A_{V+tf}^{h_t}$, note that a simple calculation using (3.8) gives, for some constant $c$ of order 1,
\[
\int \Psi_{h_t}(z, w) \mu_{V+tf}(dw) = c \theta^2 \partial h_t(z) + O(\theta^3 \|\nabla^2 h_t\|_\infty),
\]
(3.22)
\[
\int \Psi_{h_t}(z, w) \mu_{V+tf}(dz) \mu_{V+tf}(dw) = O(\theta^3 \|\nabla^2 h_t\|_\infty).
\]
(3.23)
Moreover, from (3.26) we have
\[ \sum_{k=1}^{N} \partial_{h_{t}}(z_{k}) = O(N^{\varepsilon} \| \nabla h_{t} \|_{\infty}). \]
We therefore proved
\[ A^{h_{t}}_{V+ tf} = \hat{A}^{h_{t}} + O(N^{\varepsilon} \theta^{2} \| \nabla h_{t} \|_{\infty}) + O(N\theta^{3} \| \nabla^{2} h_{t} \|_{\infty}). \] (3.24)
Moreover, as \( \Delta V + tf > c > 0 \) in the support of \( f \), we have
\[ \partial_{t} \partial_{h_{t}}(z) = O(\| \nabla f \|_{\infty} \| \nabla^{3} f \|_{\infty} + \| \nabla^{2} f \|_{\infty}^{2}), \]
so that
\[ \hat{A}^{h_{t}} = \hat{A}^{h} + O(t N \theta^{2}), \] (3.25)
which concludes the proof.

The following rigidity estimates were used in the previous proof of Theorem 3.1. They are corollaries of Theorem B.1 and Theorem B.8.

**Corollary 3.4.** Consider the Coulomb gas with Hamiltonian (3.14), with \( t \sigma \ll 1 \), \( V, f \in C^{4} \).
Suppose \( f \) is supported in a cube of size \( N^{-s} \). Then for any \( \varepsilon > 0 \) we have
\[ X_{f} = O\left( (\log N) \sqrt{N} b \left( \| f \|_{\infty} + \| b \nabla f \|_{\infty} + \| b^{2} \nabla^{2} f \|_{\infty} \right) \right) \] (3.26)
with probability at least \( 1 - e^{-(1 + \beta)(N b^{2} + N \theta^{2}) + O(\log N)} \).
Moreover, uniformly in such \( t \), for any \( \varepsilon > 0 \) we have
\[ E_{V}^{C - t \Re \Psi_{\kappa}} \left( A_{V+ t}^{(-h_{t})} \right) = O_{\varepsilon}(N^{-2\sigma + \varepsilon}) \left( \sum_{\ell=1}^{4} N^{-\ell s} \| \nabla^{\ell} f \|_{\infty} \right). \] (3.27)

### 3.2. The central limit theorem for the two-dimensional Coulomb gas.
In this subsection, we prove Theorem 1.2 for global test functions \( f \), i.e. \( \| f \|_{4} < \infty \). The general case will be proved in Section 4. For \( f \) nearly global, the removal of the angle term relies on the following estimates of large deviations type, which is a direct consequence of Theorem 1.1.

**Corollary 3.5.** Suppose that \( V \in C^{3} \), and \( 0 < t \ll 1 \) are such that \( \sup_{z} |t \Delta f| \ll 1 \). Then there exists \( \kappa > 0 \) such that for any \( f \in C^{4} \),
\[ (\beta t)^{-1} \log E_{V} e^{-\beta t N X_{f}^{t}} = \frac{t N}{8 \pi} \int |\nabla f|^{2} + \left( \frac{1}{2} - \frac{1}{\beta} \right) \frac{1}{4 \pi} \int \Delta f \log \rho_{V} + O(t^{-1} N^{-\kappa} + t N^{2}). \] (3.28)

**Proof.** By Theorem 1.1 we have
\[ (\beta t)^{-1} \log E_{V} e^{-\beta t N X_{f}^{t}} = N \int f d\mu_{V} \]
\[ - \frac{N}{t} (I_{V+ tf} - I_{V}) + \left( \frac{1}{2} - \frac{1}{\beta} \right) \frac{1}{t} \left( \int \rho_{V+ tf} \log \rho_{V+ tf} - \int \rho_{V} \log \rho_{V} \right) + O(t^{-1} N^{-\kappa}). \] (3.29)
It is easy to verify (see, e.g., [6, Proposition 3.1]) that
\[ I_{V+ tf} - I_{V} = t \int f d\mu_{V} - \frac{t^{2}}{8 \pi} \int \| \nabla f \|^{2} dm. \] (3.30)
Since \( d\mu_V = \frac{1}{4\pi} \Delta V 1_{S_v} \, dm \),
\[
\frac{1}{t} \left( \int \rho_{V+tf} \log \rho_{V+tf} - \int \rho_V \log \rho_V \right) = \frac{1}{4\pi} \int \Delta f \log \rho_V + \frac{1}{t} \int \rho_{V+tf} \log \left( \frac{\rho_{V+tf}}{\rho_V} \right) \\
= \frac{1}{4\pi} \int \Delta f \log \rho_V + O \left( t \int (\Delta f)^2 \right), \tag{3.31}
\]
where for the last equality we expanded \( \log(1 + t\Delta f/\Delta V) \) at first order and used \( \int \Delta f = 0 \). Equations \((3.30)\) and \((3.31)\) in \((3.29)\) imply \((3.28)\). \(\square\)

The following concentration result states that prove the local angle term is negligible.

**Corollary 3.6.** Consider a Coulomb gas with external potential \( V \) satisfying \((1.9)\), let \( \kappa \) be given by Corollary \((3.5)\) and choose the local angles cutoff distance \( \theta = N^{-1/2+\sigma} \) with \( \sigma = \kappa/6 \). For any \( 0 \leq t \leq N^{-2\kappa/3} \), we have
\[
(\beta Nt)^{-1} \log E_V e^{t\beta N(\text{Re} \hat{A}^h + \frac{1}{2}Y_f)} = O\left( N^{-\kappa/3} \right). \tag{3.32}
\]
In particular, for any \( \varepsilon > 0 \), for large enough \( N \) we have, for all \( x > N^\varepsilon \),
\[
\mathbb{P} \left( \left| \text{Re} \hat{A}^h + \frac{1}{2}Y_f \right| \geq N^{-\frac{\varepsilon}{2}} \right) \leq \exp \left( -c_3 N^{1-\kappa} x \right). \tag{3.33}
\]

**Proof.** Combining Theorem \((3.1)\) and Corollary \((3.5)\), we have
\[
\frac{1}{\beta Nt} \log E_V e^{t\beta N\hat{A}^h} = \frac{1}{\beta Nt} \left( \log E_V e^{-\beta Nt(X_f^\dagger - \hat{A}^h)} - \log E_V e^{-\beta NtX_f^\dagger - tf} \right) \tag{3.34}
\]
\[
= -\frac{1}{2} Y_f + O(tN^{2\sigma} + N^{-2\sigma+\varepsilon}) + O(t^{-1}N^{-\kappa} + tN^{2\sigma}) \\
= -\frac{1}{2} Y_f + O(tN^{2\sigma} + N^{-2\sigma+\varepsilon}) + O(t^{-1}N^{-\kappa} + tN^{2\sigma}),
\]
where we used the easy estimate \( Y_f^\dagger - tf = Y_f + O(t \int |\Delta f|^2 \, dm) \), a direct consequence of the definition \((1.3)\). For concentration away from large positive values in \((4.10)\), the result follows by choosing \( t = N^{-4\sigma} \) and using Markov’s inequality. For concentration away from large negative values in \((4.10)\), note that changing \( f \) into \(-f\) changes \( \hat{A}^h \) into its negative.

Finally, we already proved \((3.32)\) when \( t = N^{-4\sigma} = N^{-2/3\kappa} \), and the result holds for smaller \( t \) by monotonicity of \( \nu \mapsto \nu^{-1} \log E(e^{\nu X}) \). \(\square\)

As the local angle term is now shown to be negligible, we can complete the proof of Theorem \((1.2)\).

**Proof of Theorem \((1.2)\) for global test functions.** By assumption, we have \( \|f\|_4 < \infty \). We choose the parameter \( \kappa \) from Corollary \((3.5)\) and the cutoff parameter \( \sigma = \kappa/6 \) in the definition of \( \hat{A} \) \((5.24)\). Then Theorem \((1.2)\) is an elementary consequence of our previous results. Indeed, one just needs to substitute \( \lambda = Nt \) in the following analogue of \((3.34)\),
\[
\frac{1}{\beta Nt} \log E_V e^{-\beta NtX_f^\dagger} = \frac{1}{\beta Nt} \left( \log E_V e^{-\beta Nt(X_f^\dagger - \hat{A}^h)} - \log E_V e^{-\beta NtX_f^\dagger + tf} \right),
\]
and use the estimates \((3.1)\) for the first term, and \((3.32)\) for the second term. The result follows when substituting \( \kappa \) with \( 3\kappa \). \(\square\)
4 Central limit theorem for the local Coulomb gas

The results of Sections 2–3 prove Theorem 1.2 for test functions \( f \) on the macroscopic scale. In this section we extend the proofs to the case of test functions with mesoscopic support, using the approach of local conditioning that we introduced in [6].

4.1. Central limit theorem for conditioned measures. We will in fact prove a (stronger) local version of Theorem 1.2, which we next describe. Let \( J \) be a square of size \( b \) contained in \( S \).

Consider the Coulomb gas obtained by conditioning on all the particles outside \( J \); call the effective external potential of this system \( W \). We recall from [6] that the potential \( W \) may be written down as follows. Let \( M \) denote the number of particles in \( J \) and let \((\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_M)\) denote the collection of particles inside \( J \). Correspondingly we write \((\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_{N-M})\) the particles outside \( J \). The Hamiltonian \( H_{N,V} \) may then be written as

\[
H_{N,V}(z) = \sum_{j \neq k} \log \frac{1}{|\tilde{z}_j - \tilde{z}_k|} + N \sum_j \left( V(\tilde{z}_j) - V_o(\tilde{z}_j | \hat{z}) \right) + E(\hat{z}),
\]

where

\[
V_o(w | \hat{z}) = \frac{2}{N} \sum_k \log \frac{1}{|w - \hat{z}_k|}, \quad E(\hat{z}) = \sum_{j \neq k} \log \frac{1}{|\hat{z}_j - \hat{z}_k|} + N \sum_j V(\hat{z}_j).
\]

The term \( E(\hat{z}) \) is independent of the particles in \( J \) and is thus irrelevant for the conditioned measure. For any configuration of external particles \( \hat{z} \in (\mathbb{C} \setminus B)^{N-M} \) and \( z \in \mathbb{C} \), we write

\[
W(w | \hat{z}) = \begin{cases} \frac{N}{M} (V(w) - V_o(w | \hat{z})) & (w \in B), \\ +\infty & (w \notin B) \end{cases}
\]

\[
P_{N,V,\beta}(dw | \hat{z}) = P_{M(\hat{z}),W(\cdot | \hat{z}),\beta}(dw).
\]

The Coulomb gas given by the potential \( W(\cdot | \hat{z}) \) is the conditional gas inside \( J \), given the external configuration \( \hat{z} \).

It was proven in [6] that under our assumptions on \( V \), the conditional potential satisfies the following properties. First, since \( V_o(\cdot | \hat{z}) \) is harmonic in \( J \) we have

\[
\mu_{W}(dz) = \frac{\Delta W(z)}{4\pi} m(dz) = \frac{N}{M} \mu_V(dz)
\]

in the interior of the support \( S_W \subset J \). Especially,

\[
M \int f \mu_W = N \int f \mu_V
\]

for \( f \) that have compact support in \( S_W \). Moreover, in our setting now, in which the external potential comes from the conditioning, so that definitions (1.12), (1.13) translate to

\[
X_W^f = \sum_j f(\tilde{z}_j) - M \int f \, d\mu_W, \quad Y_W^f = \frac{1}{4\pi} \int \Delta f \log \rho_W \, dm.
\]

Finally, from [6] we know that the measure \( \mu_W \) may be expressed as \( \frac{N}{M} \mu_V + v ds \), where \( ds \) is the length measure on \( \partial J \), \( v \in L^\infty(\partial J) \), and that the following properties hold.
Lemma 4.1. There exists some $\kappa > 0$ such that the following statements hold with probability at least $1 - e^{-\beta N^2}$:

(i) $S_W \supset \{ z \in J : d(z, \partial J) > M^{-\kappa/2}b \}$,

(ii) $\mu_W(\partial J) = \int v \, ds \leq M^{-\kappa}$,

and $M \sim Nb^2$.

Furthermore, with high probability, the conditioned measure satisfies the rigidity estimate (1.8). Namely for any fixed (large) $C > 0$, (small) $\varepsilon > 0$, $c \in (0,1)$, $s < s' < 1/2$ and $b' = N^{-s'}$, the outside configuration $\hat{z}$ is such that

$$\sum_{j=1}^{M} f(\hat{z}_j) - M \int f(z) \, d\mu_W = O(M^\varepsilon \| f \|_{4,4'}) \tag{4.5}$$

simultaneously for all $f$ supported on any square of width $b'$ at a distance $bc$ from $\partial J$, and $\| f \|_{4,b} < C$, with probability at least $1 - e^{-\beta N^\varepsilon}$ for sufficiently large $N$.

We call $A$ the set of $\hat{z}$ such that (i), (ii) in Lemma 4.1 and (4.5) hold. From Lemma 4.1 and (1.8), we know that

$$\mathbb{P}(A) \geq 1 - e^{-\delta N^\varepsilon} \tag{4.6}$$

for some small $\delta > 0$. Our local central limit theorem is the following result.

Theorem 4.2. Suppose $W$ is the conditional potential defined above. For any $\beta > 0$, $c \in (0,1)$ and large $C > 0$, there a positive constant $\tau > 0$ such that the following holds. For any $\hat{z} \in A$, $f : C \rightarrow \mathbb{R}$ supported in the square with same center as $J$ and width $b(1 - c)$ and satisfying $\| f \|_{4,b} < C$, and for any $0 \leq \lambda \leq M^{1-2\tau}$, we have

$$(\beta \lambda)^{-1} \log \left( \mathbb{E}_{M,W,\beta}^\varepsilon e^{-\lambda \beta \left( X_{W} - \left( 1 - \frac{1}{2} \right) Y_{W} \right) } \right) = \frac{\lambda}{8\pi} \int \rho_W(z) \frac{1}{2} \left( \nabla f(z) \right)^2 + O(M^{-\tau}). \tag{4.7}$$

Note that the measure associated to the external potential $W + \frac{1}{4\pi} f$ is a perturbation of the original measure provided that

$$|\lambda \Delta f| \ll |M \Delta W| = |N \Delta V|.$$ 

Our assumptions $\| f \|_{4,b} < C$ and $\lambda \leq M^{1-2\tau}$ guarantee this condition.

Remark 4.3. Note that in the context of the above Theorem 4.2, our test function has shrinking support so that

$$Y_{W}^f = \frac{1}{4\pi} \int \Delta f(z) \log \rho_W(z) \, dm(z) = \frac{1}{4\pi} \int \Delta f(z) \log \frac{V(z)}{V(z_0)} \, dm(z) = O(b \| f \|_{b,2} \| V \|_3) = o(1),$$

where we used (4.3) and the center of $J$ is denoted by $z_0$. Choosing $\lambda$ of order $1$, Theorem 4.2 implies there is no shift of the mean in the convergence to the Gaussian free field for mesoscopic observables:

$$X_{W}^f \overset{(d)}{\rightarrow} N \left( 0, \frac{1}{4\pi \beta} \int |\nabla f|^2 \right).$$
To prove Theorem 4.2 we follow the same strategy as in the global case by invoking the following two results. The first one is a local version of Theorem 3.1 stated as Theorem 4.4 below; the second one is a local version of Theorem 1.1. Theorem 4.4 can be proved by the same argument used in the proof of Theorem 3.1 and we will omit its proof. The proof of Theorem 4.5 needs some modifications from the one used in the proof of Theorem 1.1.

For the statement of the analogue of Theorem 3.1, we define the angular term with a different normalization:

\[
\hat{A}^i_h = \frac{1}{2M} \sum_{i \neq j} \Psi^i_h(z_i, z_j).
\]

(4.8)

where in this section \( h = \frac{\partial f}{\partial W(z)} \) in this subsection.

**Theorem 4.4.** Suppose \( W \) is the conditional potential defined above. For any \( \beta > 0, c \in (0, 1), \) large \( C > 0, \hat{z} \in \mathbb{A}, f : \mathbb{C} \to \mathbb{R} \) supported in the square with same center as \( \mathcal{J} \) and width \( b(1 - c) \) and satisfying \( \| f \|_{3,b} < C \), the following holds. For \( \sigma \) small enough and for \( tM^{2\sigma} \ll 1 \),

\[
(\beta tM)^{-1} \log \mathbb{E}_W e^{-t\beta M(X_W - \text{Re} \hat{A}^i_h)} = \frac{tM}{8\pi} \int dm(z) |\nabla f(z)|^2 - \frac{1}{\beta} Y^f_W + O(tM^{2\sigma}) + O(M^{-2\sigma + \varepsilon}).
\]

**Proof.** The same proof as the one for Theorem 3.1 applies readily for the following reasons. First, we need no detailed behavior of \( W \) near the boundary because \( f \) is supported strictly inside \( \mathcal{J} \). Second, all estimates in the proof of Theorem 3.1 have been made explicit in terms of \( b = N^{-s} \), the size of the support of \( f \), so that one can directly check that

\[
(\beta uN)^{-1} \log \mathbb{E}_W e^{-uN\beta(X^W - \text{Re} \hat{A}^i_h)} = \frac{uN}{8\pi} \int dm(z) |\nabla f(z)|^2 - \frac{1}{\beta} Y^f_W + O(uN^{2\sigma}N^{2s}) + O(N^{-2\sigma + \varepsilon}),
\]

provided \( uN^{2\sigma}N^{2s} \ll 1 \). Taking \( u = tM \) in the above equation gives exactly the expected result, under the new constraint \( tN^{2\sigma} \ll 1 \), which is equivalent to our statement as \( \sigma \) is an arbitrary small constant.

The local version of Theorem 1.1 is as follows. Recall that \( \mu_W \) denotes the unique minimizer of the energy functional

\[
\mathcal{I}_W(\mu) = \int \int \log \frac{1}{|z - w|} \mu(dz)\mu(dw) + \int W(z)\mu(dz),
\]

defined for probability measures supported in \( \mathcal{J} \), and that we denote its minimum value by \( I_W = \mathcal{I}_W(\mu_W) \).

**Theorem 4.5.** Under the assumptions of the previous theorem, there exists a positive constant \( \kappa > 0 \) such that with \( \zeta^{c,\beta} \in \mathbb{R} \) defined in Theorem 1.1

\[
(\beta M)^{-1} \log \int_{\mathcal{J}} e^{-\beta H^W} = -MI_W + \zeta^{c,\beta} + \frac{1}{2} \log M + \left( \frac{1}{2} - \frac{1}{\beta} \right) \int_{\mathcal{J}} \rho_W \log \rho_W + O(M^{-\kappa}).
\]

To prove Theorem 4.5 we will proceed in the same way as in the proof of Theorem 1.1 by using a short-range Yukawa approximation. This will be done in the Section 4.3. The last ingredient to prove Theorem 4.2 is a local version of Corollary 3.6 which we now state.
Corollary 4.6. Under the assumptions of the previous theorems in the local setting, and choosing the local angles cutoff distance \( \theta = M^{-1/2+\sigma} \) with \( \sigma = \kappa/6 \), for any \( 0 \leq t \leq M^{-2\kappa/3} \) we have

\[
(\beta t M)^{-1} \log \mathbb{E}_W e^{\beta t M (\text{Re} \hat{A}_b^h + \frac{1}{2} Y^f_W)} = O(M^{-\kappa/3}).
\] (4.9)

In particular, for any \( \varepsilon > 0 \), for large enough \( M \) we have, for all \( x > M^\varepsilon \),

\[
P \left( \left| \text{Re} \hat{A}_b^h + \frac{1}{2} Y^f_W \right| \geq M^{-\frac{7}{2}} x \right) \leq \exp \left( -c_{\beta} M^{1-\kappa} x \right).
\] (4.10)

Corollary 4.6 follows from Theorem 4.4 and Theorem 4.5 just like Corollary 3.6 followed from Theorem 3.1 and Theorem 1.1. The proof of Theorem 4.2 is then identical to our proof of Theorem 1.2 in the case of global test functions.

4.2. Proof of Theorem 1.2 for mesoscopic test functions. We now work with the global, original measure, not the conditioned one. Similarly to equation (3.15), we start with the loop equation except that we only consider the interaction \( C \) with no angle term. Thus we have

\[
\partial_t \frac{1}{\beta N} \log Z_{V+tf} = E_{V+tf} \left( -X^f_{V+tf} + N \int f(d\mu_V - d\mu_{V+tf}) \right)
= \frac{N t}{4\pi} \int |\nabla f|^2 dm + E_{V+tf} \left( \frac{1}{N} W^h_{V+tf} - \frac{1}{\beta N} \sum_k \partial h_t(z_k) - A^{h_t,+}_{V+tf} - A^{h_t,-}_{V+tf} \right),
\] (4.11)

where \( h_s(z) = \frac{\delta f(z)}{\delta u(V(z)+sf(z))} \). The rigidity of the global Coulomb gas implies, similarly to (3.19),

\[
E^C_V \left( A^{h_t,+}_{V+tf} \right) = O(\varepsilon N^{-2\sigma+\varepsilon}).
\] (4.12)

Moreover, similarly to (3.24), we have

\[
A^{h_t,-}_{V+tf} = \hat{A}^{h_t} + O(M^\varepsilon \|\nabla h_t\|_\infty + O(M^\theta^2 \|\nabla^2 h\|_\infty),
\] (4.13)

so that we can replace \( A^{h_t,-}_{V+tf} \) with \( \hat{A}^{h_t} \) in (4.11), up to a negligible error, and then with \( \hat{A}^h \) thanks to the following elementary analogue of (3.25):

\[
\hat{A}^{h_t} = \hat{A}^h + O(t M \theta^2 b^{-4}).
\]

Corollary 4.6 together with (4.6) implies then that the angle term \( \hat{A}^h = \hat{A}^h_0 \) can be replaced by \( \frac{1}{2} Y^f_W + O(M^{-\kappa/3+\varepsilon}) \) in the right hand side of (4.11). We then integrate (4.11) to prove Theorem 1.2 for mesoscopic test functions.

4.3. Proof of Theorem 4.5. We first consider the upper bound and divide the proof into the following main steps. Following the proof of Theorem 1.1, we will from now on switch to a Yukawa gas with a range \( R \) large as the starting point.

Step 1: Yukawa approximation. From (2.17), recall the identity

\[
\sum_{j \neq k} L^f_R(z_j - z_k) + N \sum_j V(z_j) = M^2 (\Omega_1 + \Omega_2 - K_1),
\]

\[
\Omega_1 = \int_{z \neq w} L^f_R(z - w)\tilde{\mu}^R_W(dw)\tilde{\mu}^R_W(dz), \quad K_1 = \int L^f_R(z - w)\mu^R_W(dw)\mu^R_W(dz)
\]

\[
\Omega_2 = \int Q(z)\tilde{\mu}(dz), \quad Q(z) = W(z) + 2 \int_{\mathcal{J}} L^f_R(z - w)\mu^R_V(dw),
\] (4.14)
where $K_1$ is a constant and all integrations are over $\mathcal{J}$. Moreover, recall that the short-range and long-range Yukawa Hamiltonians are related by

$$H_Q^\ell = M^2 \int Q(z) \mu(dz) + \sum_{i \neq j} Y^\ell(z_i - z_j) - M^2 K_1 = H - M^2 \Omega_1,$$  \hfill (4.15)

and we have the following bound parallel to (2.21):

$$\beta^{-1} \log \int e^{-\beta H^R_{M,W}} \leq \beta^{-1} \log \int e^{-\beta H^\ell_Q} + M \log(R/\ell).$$  \hfill (4.16)

Since the proof of (2.21) used only that the interaction $L$ is positive definite, it is valid on $\mathcal{J}$ as well.

**Step 2: quasi-free approximation.** Divide the original square $\mathcal{J}$ into a grid of mesh size $b_1$ with the grid center $u$. We index these squares of size $b_1$ by $\alpha + u$. We assume that $b = (2K + 1)b_1$ so that when $u = 0$, the squares next to the boundary have width $b_1 \times b_1/2$. All other squares have size $b_1 \times b_1$ except for the corners. We average over $u = (p,q)$ with $|p| \vee |q| \leq b_1/4$ and maintain the size of all squares to be the same except that those near the boundary can change from $b_1/4$ to $3b_1/4$. For the quasi-free approximation, we again declare that particles in different squares do not interact while particles in the same square interact by the periodic Yukawa potential, except the for the boundary squares where we use the Euclidean distance. From now on, we denote by $\ell = \ell/b$, $\bar{b} = b_1/b$ the rescaled length scales. Since all measures are normalized to be probability measures on $\mathcal{J}$, all estimates will be in terms of these rescaled quantities.

Next, we need a parallel of Lemma 2.10 in the above setup. For $z, w$ away from the boundary by a distance at least $3b_1/4$, the same estimate holds (with adjustment on the size of the average). For particles within distance $b_1/4$ to the boundary, the interactions do not change. Finally, for all other cases the interactions between particles will be smaller than the original since we declare some cases to have no interaction. Therefore, the local version of Proposition 2.8 holds, i.e.,

**Proposition 4.7.** There exists $u$ with $|u| \leq b_1/4$ such that

$$\beta^{-1} \log \int e^{-\beta H^R_{M,W}} \leq \beta^{-1} \log \int e^{-\beta H^\ell_Q} + M \log(R/\ell)$$

$$+ O(\log MM^2 \bar{b}^3/b) + O(M^\kappa \ell^{-2}).$$  \hfill (4.17)

We again assume $u = 0$ and divide the square $\mathcal{J}$ into $B \cup D$ with $D$ the union of the interior squares and $B$ the union of the boundary ones. The equilibrium measure restricted to $D$ has a smooth density $\rho^R_W$ which is a rescaling of $\rho^R_V$, and by Lemma 4.1 we have

$$M \int_B \rho^R_W \leq M^{1-\kappa}$$  \hfill (4.18)

for some $\kappa > 0$. (The straightforward adaptation to the Yukawa case is done in Appendix A.) Clearly, we can modify the definition of the quasi-free approximation so that the particles in the different squares in $B$ do interact between each other, and so we will from now on view $B$ as a special “square”. Hence we can replace (2.30) by

$$\beta^{-1} \log \int e^{-\beta H^R_W} \leq M^2 K_1 + \beta^{-1} \log \sum_{n \sum_{n_a = M}} \binom{M}{n} G(n) + M \log(R/\ell)$$

$$+ O(M^2 \bar{b}^2) + O(\log MM^2 \bar{b}^3/b) + O(M^\kappa \ell^{-2}),$$  \hfill (4.19)
where the energy functional $G$ is defined by

$$G(n) = \exp \left[ -\beta \sum_{\alpha \in D} \left\{ M n_\alpha \left[ 2c_W - 2\rho^R_W(\alpha)\ell^2 \eta \right] - F^\ell(\eta) \right\} - \beta E_B(n_B) + Cn_B \log M \right]$$

$$E_B(n) = \inf_{\omega = n} M \int_B Q(z) \omega(\omega) + \int_B Y^\ell(z - w) \omega(\omega) \omega(dw). \quad (4.20)$$

Here $n_B$ denotes the number of particles in $B$.

The contribution to the partition function from the interior squares $\alpha \in D$ can be estimated in the same way as in the global case. Define

$$\sum_{\alpha \in D} M n_\alpha \left[ 2c_W - 2\rho^R_W(\alpha)\ell^2 \eta \right] + n_\alpha^2 \eta^2 := M^2 I_D(n). \quad (4.21)$$

Let $n^R_W(\alpha) = N \rho^R_W(\alpha) b_1^2$ be the number of particles according to the equilibrium measure at $\alpha$. We can now express

$$I_D(n) = \eta N^{-2} \sum_{\alpha \in D} \left[ n_\alpha - n^R_W(\alpha) \right]^2 - C_1^D + 2c_W \frac{n_D}{M},$$

$$C_1^D = \eta \ell^2 b_1^2 \sum_{\alpha \in D} \rho^R_W(\alpha)^2 \int_D Y^\ell(z - w) \mu^R_W(dz) \mu^R_W(dw)(1 + O(b_1)),$$

where $n_D$ is the number of particles in $D$. As in the global case, we need to recombine several terms in the free energy. We start with

$$-K_1 - C_1^D = -\int_J L^\ell_R(z - w) \mu^R_W(dw) \mu^R_W(dz) - \int_J Y^\ell(z - w) \mu^R_W(dz) \mu^R_W(dw)$$

$$= -\int_J Y^R(z - w) \mu^R_W(dw) \mu^R_W(dz) + \int_J Y^\ell(z - w) \mu^R_W(dz) \mu^R_W(dw) + O(\ell^2 b) \quad (4.22)$$

Recall that $n_B$ is the number of particles in $B$ (which is a variable) and $n_W(B)$ the equilibrium number of particles in $B$ (which is a number). By the Euler-Lagrange equation \([A.5]\), which states that $Y^R * \mu^R_W + \frac{1}{2} W = c_W$ in the interior of $S_W$,

$$2 \frac{n_D}{M} c_W + 2 \frac{n_B - n_W(B)}{M} c_W = \int_D \left[ W(z) + 2 \int_J Y^R(z - w) \mu^R_W(dw) \right] \mu^R_W(dz)$$

$$= \int_J \left[ W(z) + 2 \int_J Y^R(z - w) \mu^R_W(dw) \right] \mu^R_W(dz) - 2c_W \int_B \mu^R_W(dz). \quad (4.23)$$

The two integrals over $J$ in \((4.22), (4.23)\) can be combined into $I^R_{W,J}(\mu^R_W)$ where

$$I^R_{W,J}(\mu) = \int_J W \mu + \int_J Y^R(z - w) \mu(dw). \quad (4.24)$$

Recall from Lemma \([2.6]\) that we have defined $Q$ so that $Y^R * \mu^R_W + \frac{1}{2} W = c_W = c_Q = Y^\ell * \mu^Q + \frac{1}{2} Q$ and the two equilibrium measures are equal, i.e., $\mu^R_W = \mu^Q$. Moreover, since the equilibrium density is bounded near on boundary, we have

$$\int_B \mu^R_W(dz) \int_D Y^\ell(z - w) \mu^R_W(dw) = O(\ell^3).$$
Thus we have
\[
\int_B Y^\ell(z - w)\mu_W^R(dz)\mu_W^L(dw) - 2c_W \int_B \mu_W^R(dz)
\]
\[
= \int_B Y^\ell(z - w)\mu_Q^L(dz)\mu_Q^L(dw) - \int_B \left[ Q(z) + 2 \int \ell(z - w)\mu_Q^L(dw) \right] \mu_Q^L(dz)
\]
\[
= -\mathcal{I}^L_{Q,B}(\mu_Q^L) + O(\tilde{\ell}^3),
\]
where
\[
\mathcal{I}^L_{Q,B}(\mu) = \int_B Q(z)\mu(dz) + \int_B Y^\ell(z - w)\mu(dw)\mu(dz).
\]
Collecting all these bounds, we have proved that
\[
-K_1 - C_1^D + 2\frac{n_B}{M}c_W = -2\frac{n_B - n_W(B)}{M}c_W + \mathcal{I}^L_{W,J}(\mu_W) - \mathcal{I}^L_{Q,B}(\mu_Q^L) + O(\tilde{\ell}^2b).
\]
Collecting all the error terms so far, as in Remark 2.13 and Proposition 2.13, there is a choice of parameters \( \tilde{\ell}, \tilde{b} \) so that for some positive constant \( \kappa > 0 \)
\[
M^2\tilde{\ell}^2b + \log MM^2\tilde{\ell}^3/\tilde{b} + M^2\tilde{\ell}^{-2} + \tilde{b}^2 \leq M^{1-\kappa},
\]
and we have thus proved the following free energy bound, parallel to (2.60),

\[
\beta^{-1} \log \left( \sum_{n} \frac{M}{n} \right) e^{-\beta H_W^L} \leq -M^2J_{W}^R + O(M^{1-\kappa}) + N \log R
\]

\[
+ \beta^{-1} \log \sum_{n} \frac{M}{n} \left( \frac{M}{n} \right) e^{-\beta \sum_{a \in D} h_a(n) + \beta \left( 2M (n_B - n_W(B))c_W + E_B(n_B) - M^2I^L_{Q,B}(\mu_W) \right)},
\]

\[
h_a(n) = \eta \gamma^2 [n_a - n_W^R(\alpha)]^2 - \xi_b(\gamma)(n_a).
\]

Note that
\[
E_B(n_B) - M^2I^L_{Q,B}(\mu_W) - 2M(n_B - n_W(B))c_W
\]
\[
= \inf_{\omega = n_B} \left[ M \int_B Q(z)\omega(dz) + \int_B Y^\ell(z - w)\omega(dz)\omega(dw) \right]
\]
\[
- M^2 \int_B \mu_W^R(dz)Q(z) - M^2 \int_B Y^\ell(z - w)\mu_W^R(dz)\mu_W^L(dw) - 2M(n_B - n_W(B))c_W
\]
\[
= \inf_{\omega = n_B} \left[ M \int_B \tilde{\omega}(dz) \left[ Q(z) + 2M \int_B Y^\ell(z - w)\mu_W^R(dw) \right] + \int_B Y^\ell(z - w)\tilde{\omega}(dz)\tilde{\omega}(dw) \right]
\]
\[
- 2M(n_B - n_W(B))c_W
\]
where \( \tilde{\omega} = \omega - M\mu_W^R \). The second to last term on the right-hand side of the previous equation is nonnegative, and recall that \( Q(z) + 2M \int_B Y^\ell(z - w)\mu_W^R(dw) = 2c_W \) on the support and \( Q(z) + 2M \int_B Y^\ell(z - w)\mu_W^R(dw) \geq 2c_W \) outside the support. Hence
\[
M \int_B \tilde{\omega}(dz) \left[ Q(z) + 2M \int_B Y^\ell(z - w)\mu_W^R(dw) \right] - 2M(n_B - n_W(B))c_W
\]
\[
\geq M \int_B \tilde{\omega}(dz) \left[ Q(z) + 2M \int_B Y^\ell(z - w)\mu_W^R(dw) - 2c_W \right] \geq 0
\]
We have thus proved that
\[ E_B(n_B) - M^2 I_{Q,B}^f(\mu_W) - 2M(n_B - n_W(B))c_W \geq 0. \]

Therefore, we can improve (4.27) to
\[ \beta^{-1} \log \int e^{-\beta H_W^R} \leq -M^2 I_W^R + O(M^{1-\kappa}) + N \log R + \beta^{-1} \log \sum_{n} \left( \frac{M}{n} \right) e^{-\beta \sum \epsilon_D \alpha(n)}. \]

From this inequality, the rest of the argument is essentially identical to the global case; we only have to carry out all computations in the square \( D \) which is slightly smaller than the original domain \( J \). We have thus proved that (with \( n_D = n_W(D) \))
\[ \beta^{-1} \log \int e^{-\beta \alpha W^R} = -M^2 I_W^R + n_D \epsilon^C + n_D^2 \frac{1}{2} \log n_D + M \left( \frac{1}{2} - \frac{1}{7} \right) \int_D \rho_{\alpha W}^R \log \rho_{\alpha W}^R + O(M^{1-\kappa}). \]

Finally, using the estimate on the boundary charges, (4.18), this proves the upper bound needed to prove Theorem 4.5.

**Lower bound.** Suppose we have two measures \( \nu \) and \( \omega \) on particle configurations in \( B \cup D \), with particle numbers \( n_\nu \) and \( n_\omega \) respectively, and that \( \nu \) is supported on configurations entirely in \( B \). Suppose that the density of \( \nu \) with respect to \( m^{\otimes n_\nu} \) is given by \( F \) and that of \( \omega \) w.r.t. \( m^{\otimes n_\omega} \) by \( e^{-H_\omega}/Z_\omega \). Then, by Jensen’s inequality,
\[ \log \int e^{-H} \geq \log Z_\omega + E^\nu \otimes \omega \left[ -\log F + H_\omega - H \right]. \]

We apply this idea to our setting. The measure \( \nu \) is taken as the product measure \( \mu_W^{\otimes n_\nu} \) with the number of particles given by \( n_\nu = n_W(B) = \int_B \rho_W \). The measure \( \omega \) is the quasi-free Yukawa gas with density in each square given by \( n_W(\alpha) \). We restrict the integration in the partition function for the Coulomb gas to have the correct number of particles in each square to get a lower bound. By Jensen’s inequality as above, we have
\[ \beta^{-1} \log \int e^{-\beta H_{Q,C}(z)} \geq \beta^{-1} \log \int e^{-\beta H_{Q,C}(z)} \prod_{\alpha} 1(n_\alpha = n_V(\alpha)) \]
\[ \geq \beta^{-1} \log \int_D e^{-\beta \tilde{H}_B^f} + E \tilde{H}_B^f \left[ \tilde{H}_D^f - H_{V,C}(z) \right] + O(n_W(\alpha) \log N), \]
where we have used \( E^\nu \otimes \omega \log F \leq n_W(B) \log N \). We only have to estimate \( E \tilde{H}_B^f \left[ \tilde{H}_D^f - H_{V,C}(z) \right] \) correctly. The quasi-free gas we are dealing with now is a product of two measures: on the boundary \( B \) it is a product measure; on \( D \) it is the same as in the upper bound except that the number of particles in each square is fixed by its equilibrium density. The fact that the number of particles in each square is fixed by its equilibrium density makes the computation of free energy even easier, hence we will not repeat it here. Recall that the product measure approximation yields the correct free energy asymptotic up to error \( N \log N \) for a gas with \( N \) particles (see, for example, [6, Proposition 4.1]). In our case this error is of order \( n_W(B) \log N \). Once again, using that \( n_W(B) \leq M^{1-\kappa} \), we have thus proved the lower bound.
A  Local density and rigidity for Yukawa gas

This section is an adaption of our results in [6] to the one-component Yukawa gas, in which the two-dimensional Coulomb potential \( \log |z| \) is replaced by the Yukawa potential \( Y_\ell(z) \) defined in (2.1). Our presentation here follows closely that of [6] and we will mainly present the differences. Throughout this section, we denote the range of the Yukawa potential by \( \ell \) and write \( m = 1/\ell \).

A.1. Yukawa gas on the plane. Given \( Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\} \) with sufficient growth, for \( z = (z_1, \ldots, z_N) \in \mathbb{C}^N \), we define the energy

\[
H_{N,Q}^\ell(z) = \sum_{j \neq k} Y_\ell(z_j - z_k) + N \sum_{j} Q(z_j),
\]

and the corresponding probability measure on \( \mathbb{C}^N \), for \( \beta > 0 \), by

\[
P_{N,Q}^\ell(dz) = \frac{1}{Z_{N,Q}^\ell} e^{-\beta H_{N,Q}^\ell(z)} m \otimes N(dz).
\]

The energy (A.1) is closely related to the Yukawa variational functional, defined for probability measures \( \mu \) on \( \mathbb{C} \), by

\[
\mathcal{I}_Q^\ell(\mu) = Q \mu + \int Y_\ell(z - w) \mu(dz) \mu(dw), \quad I_Q^\ell = \inf_{\mu} \mathcal{I}_Q^\ell(\mu).
\]

Under technical assumptions, standard arguments imply that the convex variational functional has a unique minimizer denoted \( \mu_Q^\ell \) over the set of probability measures on \( \mathbb{C} \); see Theorem A.2 below. Given a test function \( f : \mathbb{C} \to \mathbb{R} \), we denote the linear statistic centered by the equilibrium measure by

\[
X_f := X_{Q,f} := \sum f(z_k) - N \int f(z) \mu_Q^\ell(dz).
\]

We always make the following assumptions throughout this section:

(i) The set \( \Sigma_Q = \{ z : Q(z) < \infty \} \) has positive logarithmic capacity.

(ii) The potential \( Q \) is locally in \( C^{1,1} \) and satisfies for some \( \varepsilon > 0 \) the growth condition

\[
\liminf_{|z| \to \infty} (Q(z) - \varepsilon \log |z|) > -\infty.
\]

(iii) The density of \( \mu_Q^\ell \) is bounded below in a neighborhood of the area of interest.

In our application in Section 2, the potential \( Q \) is defined in terms of the long-range interaction of a Coulomb gas. In particular, \( \mu_Q^\ell \) then equals the minimizer \( \mu_V \) of the Coulomb variational functional, for which the density is explicitly given by \( \frac{1}{4\pi} \Delta V \) in its support. We also remark that in this application, the condition (A.3) corresponds to the condition (1.4) assumed of the original Coulomb gas. Moreover, for the purpose of using the results of this section in Section 2.2 we naturally assume that \( Q \) satisfies the stronger growth condition (1.4) with \( V \) replaced by \( Q \).

The first result of this appendix is the following local density estimate, which is an adaptation of Theorems 1.1 in [6].
Theorem A.1. Let \( s \in (0, \frac{1}{2}) \). Then for any \( f : \mathbb{C} \rightarrow \mathbb{R} \) supported in a ball of radius \( b = N^{-s} \) contained in the support of \( \mu_{\mathcal{V}}^\ell \),

\[
|X_f| \leq (\log N)^{O(1)} \sqrt{N b + \ell} \left[ \|f\|_\infty + \|b \nabla f\|_\infty + \|b^2 \nabla^2 f\| \right],
\]

(A.4)

with probability at least \( 1 - e^{-Nb^2} \).

To prove Theorems A.1, we require some properties of the equilibrium measure of the Yukawa gas. For a probability measure \( \mu \) on \( \mathbb{C} \), define the Yukawa potential by

\[
U^\ell_{\mu}(z) = \int Y^\ell(z - w) \mu(dw).
\]

The following standard result gives the existence and uniqueness for the equilibrium for the Yukawa gas. Its statement and proof are identical to that for the Coulomb case. Denote \( \Sigma_Q = \{z : Q(z) < \infty\} \) and let \( P(\Sigma_Q) \) be the set of probability measures supported in \( \Sigma_Q \).

Theorem A.2. Suppose \( Q \) satisfies (A.3) and that \( \Sigma_Q \) has positive capacity. Then there exists a unique \( \mu_{\mathcal{V}}^\ell \in P(\Sigma_Q) \) such that

\[
I^\ell_{\mu_{\mathcal{V}}^\ell}(\mu_{\mathcal{V}}^\ell) = \inf\{I^\ell_{\mu}(\mu) : \mu \in P(\Sigma_Q)\}.
\]

The support \( S_{\mathcal{V}}^\ell = \text{supp} \mu_{\mathcal{V}}^\ell \) is compact and of positive capacity, and \( I^\ell_{\mu_{\mathcal{V}}^\ell} \) is bounded. Furthermore, the energy-minimizing measure \( \mu_{\mathcal{V}}^\ell \) may be characterized as the unique element of \( P(\Sigma_V) \) for which there exists a constant \( c_Q \in \mathbb{R} \) such that Euler-Lagrange equation

\[
U^\ell_{\mu_{\mathcal{V}}^\ell} + \frac{1}{2} Q = c_Q \quad \text{a.e. in } S_{\mathcal{V}}^\ell \quad \text{and}
\]

\[
U^\ell_{\mu_{\mathcal{V}}^\ell} + \frac{1}{2} Q \geq c_Q \quad \text{a.e. in } \mathbb{C}
\]

(A.5)

holds. The equilibrium measure \( \mu_{\mathcal{V}}^\ell \) in the set \( S_{\mathcal{V}}^\ell \) is given by

\[
\mu_Q = \frac{1}{4\pi} (\Delta Q + m^2(2c_Q - Q)) = \frac{1}{4\pi} \left[(\Delta - m^2)Q + 2m^2c_Q\right],
\]

(A.6)

where the Laplacian is understood in the distributional sense. We will drop the subscript \( \ell \) when it is understood.

Proof. The proof is identical to that of the Coulomb case; see e.g. [10].

Remark A.3. By the same argument, under the additional assumption that \( Q \) satisfies (1.4), the support of \( \mu_{\mathcal{V}}^\ell \) is compact uniformly in \( \ell \).

The next theorem characterizes the Yukawa potential of the equilibrium measure in terms of an obstacle problem. Again, the theorem is similar to the Coulomb case, but requires a slightly different characterization of the admissible potentials than the one stated for the Coulomb case in [24], for example. We give a proof for completeness, as we were unable to locate a suitable reference.

Theorem A.4. Under the assumptions of Theorem A.2, the following holds. Define

\[
u_Q,\ell(z) = \sup\{-U^\ell_{\mu}(z) + c : -U^\ell_{\mu} + c \leq \frac{1}{2} Q, \nu \geq 0, \nu(\mathbb{C}) \leq 1\}.
\]

Then \( u_{Q,\ell} = -U^\ell_{\mu_{\mathcal{V}}} + c_Q \) where \( c_Q \) is as in (A.5).
Proof. By definition, \( u_{Q, \ell} \geq -U_{\bar{\eta}Q}^\ell + c_Q \) since the right-hand side is a subsolution of the same form as inside the supremum in (A.7). To prove that in fact equality holds, suppose otherwise that \( u_{Q, \ell}(z_0) > -U_{\bar{\eta}Q}^\ell(z_0) + c_Q \) for some \( z_0 \in \mathbb{C} \). Then there exists some positive measure \( \tilde{\eta} \) with \( \tilde{\eta}(\mathbb{C}) \leq 1 \) and constant \( c \in \mathbb{R} \) for which \( -U_{\tilde{\eta}Q}^\ell(z_0) + c > -U_{\bar{\eta}Q}^\ell(z_0) + c_Q \). By considering \( \tilde{\eta}|_{B_R} \) for \( R > 0 \) large enough we may suppose that \( \tilde{\eta} \) is compactly supported, and by convolving with a smooth mollifier we may suppose \( \tilde{\eta} \) has a smooth density. Consider the function

\[
g(z) = \max(-U_{\tilde{\eta}Q}^\ell(z) + \tilde{\epsilon}, -U_{\bar{\eta}Q}^\ell(z) + c_Q).
\]

By writing \( \max(a, b) = \frac{a+b}{2} + \frac{|a-b|}{2} \) and convolving the absolute value by a smooth, compactly supported, symmetric mollifier, we may check that \( g(z) = -U_{\tilde{\eta}Q}^\ell(z) + c \) for some positive measure \( \eta \), and necessarily \( c = \max(\tilde{\epsilon}, c_Q) \). To show that \( g \) is a subsolution of the form in (A.7) we need to show that \( \eta(\mathbb{C}) \leq 1 \). For this, suppose without loss of generality that \( c = \tilde{\epsilon} \). Denote \( D = \{ z : -U_{\tilde{\eta}Q}^\ell(z) + \tilde{\epsilon} < -U_{\bar{\eta}Q}^\ell(z) + c_Q \} \). Then

\[
\eta(\partial D) = \int_{\partial D} \partial_a (-U_{\tilde{\eta}Q}^\ell(z) + U_{\bar{\eta}Q}^\ell(z)) = \int_D \Delta(-U_{\tilde{\eta}Q}^\ell(z) + U_{\bar{\eta}Q}^\ell(z)) \]
\[
= \int_D (\tilde{\eta} - \mu_Q) + m^2 \int_D (-U_{\tilde{\eta}Q}^\ell(z) + U_{\bar{\eta}Q}^\ell(z)) \]
\[
= \int_D (\tilde{\eta} - \mu_Q) + m^2 \int_D (-U_{\tilde{\eta}Q}^\ell(z) + \tilde{\epsilon} - U_{\bar{\eta}Q}^\ell(z) + c_Q) + m^2 \int_D (c_Q - \tilde{\epsilon}) \leq \tilde{\eta}(D) - \mu_Q(D).
\]

Thus \( \eta(D \cup \partial D) = \eta(\partial D) + \mu_Q(D) \leq \tilde{\eta}(D) \). Since clearly \( \eta(\mathbb{C} \setminus (\partial D \cup D)) = \tilde{\eta}(\mathbb{C} \setminus (\partial D \cup D)) \), we have \( \eta(\mathbb{C}) \leq \tilde{\eta}(C) \leq 1 \). Now,

\[
g - \mu_Q = 0,
\]
\[
(\Delta - m^2)(-U_{\tilde{\eta}Q}^\ell(z) + U_{\bar{\eta}Q}^\ell(z)) = \eta - \mu \geq m^2(c - c_Q) \geq 0.
\]

Since strict inequality holds in the first inequality for \( z_0 \) and the functions involved are continuous, equality (as distributions) cannot hold on the second line. But this implies \( \eta(\mathbb{C}) > \mu_Q(\mathbb{C}) = 1 \), a contradiction.

To deal with conditioning on particles outside a small ball we next state adaptations of the results of [6] Section 3.3] to the Yukawa case. As in [6], we assume here that \( S_Q = \rho \mathbb{D} \) for some \( \rho > 0 \), where \( \mathbb{D} \) is the open unit disk. Furthermore, we assume that \( \frac{1}{2\rho}(\Delta - m^2)U_{\mu_Q}^\ell \geq \alpha \) in \( \rho \mathbb{D} \) for some parameter \( \alpha > 0 \). The class of perturbed potentials \( W \) is as follows. Let \( \nu \) be a positive measure with \( \text{supp} \nu \cap \rho \mathbb{D} = \emptyset \), \( t > 0 \) and let \( R \in C(\rho \mathbb{D}) \) satisfy \( (\Delta - m^2)R = 0 \) in \( \rho \mathbb{D} \). Then \( W \) is given by

\[
W(z) = \begin{cases} tV(z) + 2U_{\mu_Q}^\ell(z) + 2R(z), & z \in \rho \mathbb{D} \setminus B_R^*, \\ \infty, & z \in \rho \mathbb{D}^*. \end{cases}
\]

(A.8)

Both perturbations \( U_{\mu_Q}^\ell \) and \( R \) are \( m \)-harmonic inside \( \rho \mathbb{D} \). Especially, by (A.5) this means that the density of \( \mu_W \) is equal to \( t\mu_Q + \text{const.} \) in \( \mathbb{S}_W \).

We write \( \mathbb{D}^* = \mathbb{C} \setminus \mathbb{D} \) for the open complement of the unit disk. For \( z \in \partial \rho \mathbb{D} \) we write \( \bar{n} = \bar{n}(z) = z/|z| \) for the outer unit normal, and \( \partial_n f(z) = \lim_{\epsilon \downarrow 0} \frac{f(z) - f(z - \epsilon \bar{n})}{\epsilon} \) for the derivative in the direction \( \bar{n} \) taken from inside \( \rho \mathbb{D} \).

The next two propositions show that the bulk of the equilibrium measure \( \mu_Q \) is stable under suitable perturbations \( W \) of the form (A.8), and that the density of \( \mu_W \) on the boundary remains
bounded. For the stability of the bulk we use the obstacle problem characterization (A.7) of the support.

**Proposition A.5.** Suppose that $Q$ and $W$ are as above [A.6], and moreover that $\ell/\rho \geq A$ for a large enough absolute constant $A$. Then

$$S_W \supset \{ z \in \rho \mathbb{D} : \text{dist}(z, \rho \mathbb{D}^*) \geq \kappa \}, \quad \text{where } \kappa = C \sqrt{\max(\|\nu\|, 2\rho)\|\partial_n R\|_{\infty, \partial \rho \mathbb{D}} + (t - 1)}. \quad (A.9)$$

**Proof.** Choosing the constant $A$ large enough, this is a straightforward adaptation of [4, Proposition 3.3].

**Proposition A.6.** Suppose that $Q$ and $W$ are as above [A.6] and assume in addition that $\mu_Q$ is absolutely continuous with respect to the 2-dimensional Lebesgue measure. Then $\mu_W = \mu + \eta$, where $\mu$ is absolutely continuous with respect to $\mu_Q$, and $\eta$ absolutely continuous with respect to the arclength measure $s$ on $\partial \rho \mathbb{D}$ with the Radon–Nikodym derivative bounded by

$$\rho \| \frac{d\eta}{ds} \|_{\infty} \leq C \left( \|\eta\| + \|\nu\| + 2\rho \|\partial_n R\|_{\infty, \partial \rho \mathbb{D}} + \|1 - t|\rho|\partial_n Q\|_{\infty, \partial \rho \mathbb{D}} \right). \quad (A.10)$$

**Proof.** The only change in the proof of Proposition [A.6] as compared to [4] is changing the logarithmic potentials to the Yukawa potential. Especially, since $\nabla Y^\ell(z) \sim \nabla \log \frac{1}{|z|}$ for $z \to 0$, the following formula can be proven as in [4]. Let $\gamma \subset \mathbb{C}$ be a $C^1$ curve and $\eta$ a measure supported on $\gamma$ for which the potential $U^\ell_\eta$ is continuous on $\mathbb{C}$. Then for $z \in \gamma$ we have

$$\partial_n U^\ell_\eta(z) = \pi \lim_{r \to 0^+} \frac{\eta(B_r(z))}{s(B_r(z))} - \int_\gamma \nabla Y^\ell(z - w) \cdot \bar{n} \eta(dw), \quad (A.11)$$

where $\partial_n$ denotes a one-sided derivative in the normal direction $\bar{n} = \bar{n}(z)$ and $s$ denotes the arclength measure of $\gamma$, if the limit on the right-hand side exists.

**A.2. One-step estimate.** As in [4, Proposition 4.1], we use a simple mean-field partition function estimate to obtain a bound on the fluctuations of smooth linear statistics.

**Proposition A.7.** Let $\Sigma = \Sigma_W$ be a smooth domain. Given a potential $W \in C^{1,1}_{\text{loc}}(\Sigma_W)$ possibly depending on the number of particles $M$, assume that there exist $u : \Sigma_W \to \mathbb{R}_+$ and $v : \partial \Sigma_W \to \mathbb{R}_+$ such that $d\mu_W = u \text{dm} + v \text{ds}$, where $\text{dm}$ is the 2-dimensional Lebesgue measure and $\text{ds}$ is the arclength measure on $\partial \Sigma_W$. Assume the conditions (i)-(iv) as in [4] but replace the bounds on $\Delta W$ by the same bound on $u$, and also modify the assumption (iv) by replacing $\zeta$ by $\zeta^\ell = U^\ell_{\mu_W} + \frac{1}{\beta}Q - c_Q$, where $c_Q$ is as in [A.5]. Then, for any bounded $f \in C^2(\mathbb{C})$ with compactly supported $(\Delta - m^2) f$,

$$\log \int e^{-\beta H_{M,V} + \sum f(z_j)} \leq -\beta M^2 I^\ell_V(\mu_Q) + M(f, \mu_W) + \frac{1}{8\pi \beta}(f, -((\Delta - m^2)f)) + O(M \log M), \quad (A.12)$$

$$\log \int e^{-\beta H_{M,V} \geq -\beta M^2 I^\ell_V(\mu_Q) + O(M \log M)} \quad (A.13)$$
and consequently for any $\xi \geq 1 + 1/\beta$,

$$\left| \frac{1}{M} \sum_j f(z_j) - \int f \, d\mu_W \right| = O(\xi) \left( \frac{M \log M}{\alpha M^2} \| \Delta f \|_\infty + \sqrt{\frac{M \log M}{M^2}} (f, (-\Delta + m^2)f)^{1/2} \right),$$

(A.14)

with probability at least $1 - e^{-\xi \beta M \log M}$, with the implicit constant depending only on the numbers $A$ in the assumptions (i)-(iv).

**Proof.** The probability estimate is obtained as in [6] from the partition function bounds (A.12) and (A.13), which are analogous to [6] Lemmas 4.3 and 4.4 except that $\| \nabla f \|_2 = (f, (-\Delta)^{1/2})$ is replaced by $(f, (-\Delta + m^2)f)^{1/2}$. The lower estimate can be proved exactly the same way; for the upper estimate we bound the energy slightly differently from below, as follows.

All the properties of the Coulomb potential used in the proof of [6, Lemmas 4.3] also hold for the Yukawa potential. Replacing the point charges by charged disks of radius $\varepsilon$, we get the bound

$$H^\ell_M(z) - \frac{1}{\beta M} \sum_j f(z_j)$$

$$\geq M^2 D(f(\mu), \mu) + M^2 (W - \frac{1}{\beta M} f, \mu) + O(M \log \frac{1}{\varepsilon})$$

$$= M^2 \left( D(f(\mu), \mu) + (W, \mu) - (\frac{1}{\beta M} f, \mu) \right) + M^2 (\frac{1}{\beta M} f, \mu) + O(M \log \frac{1}{\varepsilon}).$$

Writing

$$D(f(\mu), \mu) = D(f(\mu_W, \mu_W) + 2D(f(\mu_W, \mu_W) + D(f(\mu_{\mu_W}, \mu_W, \mu_W)$$

and further using the Euler–Lagrange equation (A.5) to write

$$2D(f(\mu_W, \mu_W) + (W, \mu) = (W, \mu_W) + 2(\hat{\xi}, \mu - \mu_W) + 2(U_{\mu_W}^\ell, \mu_W)$$

where $\hat{\xi} = U_{\mu_W}^\ell + \frac{1}{2} W - c_W = 0$ on $S_W$, we arrive at the bound

$$H^\ell_M(z) - \frac{1}{\beta M} \sum_j f(z_j)$$

$$\geq M^2 \left( I(\mu_W) + D(f(\mu), \mu_W, \mu_W) - (\frac{1}{\beta M} f, \mu_W) \right) + 2M^2 \hat{\xi}_W + \mu_W$$

$$+ M^2 (\frac{1}{\beta M} f, \mu_W) + 2M^2 (U_{\mu_W}^\ell, \mu_W) + O(M \log \frac{1}{\varepsilon}).$$

Write

$$- (\frac{1}{\beta M} f, \mu_W) + D(f(\mu_W, \mu_W) + (\frac{1}{\beta M} f + U_{\mu_W}^\ell, \mu_W) - (\Delta - m^2)U_{\mu_W}^\ell$$

Yukawa potentials decay exponentially at infinity, so we may integrate by parts and use the elementary inequality $ab + b^2 \geq -a^2/4$ to get

$$\frac{1}{2\pi} (\frac{1}{\beta M} f + U_{\mu_W}^\ell, \mu_W), (-\Delta)U_{\mu_W}^\ell) = \frac{1}{2\pi} (\frac{1}{\beta M} f + \nabla U_{\mu_W}^\ell, \nabla U_{\mu_W}^\ell)$$

$$\geq -\frac{1}{8\pi \beta^2 M^2} (\nabla f, \nabla f) = -\frac{1}{8\pi \beta^2 M^2} (f, (-\Delta)f).$$
By the same inequality
\[
\frac{1}{2\pi} \left( \frac{1}{\beta M} f + U^\ell_{\mu^\varepsilon - \mu_W}, m^2 U^\ell_{\mu^\varepsilon - \mu_W} \right) \geq \frac{m^2}{8\pi\beta^2 M^2} (f, f).
\]

In conclusion,
\[
M^2 D^\ell(\hat{\mu}^\varepsilon, \hat{\mu}) + M^2 (W - \frac{1}{\beta M} f, \hat{\mu}) \\
\geq M^2 \left( I^\ell_W(\mu_W) - \frac{1}{\beta M} (f, \mu_W) - \frac{m^2}{8\pi\beta^2 M^2} (f, -(\Delta - m^2)f) \right) + 2M^2 (\zeta^\ell, \hat{\mu} - \mu_W) \\
+ M^2 (\frac{1}{\beta M} f, \hat{\mu}^\varepsilon - \hat{\mu}) + 2M^2 (U^\ell_{\mu_W}, \hat{\mu}^\varepsilon - \hat{\mu}) + O(M \log \frac{1}{\varepsilon}).
\]

The error terms on the last line and the term \(2M^2(\zeta^\ell, \hat{\mu} - \mu_W)\) are then handled the same way as in [6]. \(\square\)

**Remark A.8.** For test functions \(f\) supported in \(S^\ell_Q\) and satisfying the condition \(\int f \, dm = 0\),
\[
\int f \, d\mu^\ell_Q = \int f \, \frac{1}{4\pi}(\Delta Q - m^2 Q) \, dm.
\]
Consequently, if \(Q\) is replaced by \(Q + R\) with \((\Delta - m^2)R = 0\), and assuming that \(f\) is supported in the intersection of the supports of the equilibrium measures of \(Q\) and \(Q + R\), and that \(\int f \, dm = 0\), we have
\[
\int f \, d\mu^\ell_Q = \int f \, d\mu^\ell_{Q + R}.
\]

Since we are ultimately interested in test functions without the condition \(\int f \, dm = 0\), some additional care is required. (The condition was not necessary in the Coulomb case in [6].)

**A.3. Proof of Proposition A.1: local density.** To improve the estimate of Proposition A.7 to that asserted by Theorem A.1, we use the strategy of local conditioning introduced in [6].

First, we note that [6, Section 5] applies without changes except that the Coulomb potential \(\log 1/|z|\) is replaced by the Yukawa potential \(Y^\ell(z)\) in all expressions, and with the additional condition that \(\int f \, dm = 0\) in the assumption of [6, Proposition 5.3]. This condition is necessary because, with the \(m\)-harmonic perturbation \(V_0\), inside the support of \(\mu_W\) we now have
\[
\mu_W(dw) = \frac{N}{M} \mu_Q(dw) + \text{const.}
\]

by (A.6). As explained in Remark A.8 the additional constant has no effect if both sides are integrated against a test function \(f\) with support in the support of \(\mu_W\) that satisfies \(\int f \, dm = 0\).

Next, we adapt [6, Section 6] to the Yukawa case. Here two modifications are required. First, the scaling of the Yukawa gas is different, which leads to a different recursion of scales. Second, in the case of the Yukawa gas, as noted above, the density of the equilibrium is only stable under \(m\)-harmonic perturbations up to a constant, and thus an small extra argument is required to remove the mean zero condition.

As previously, we write \(\ell = N^{-1/2+\delta}\) for the range of the Yukawa potential. Given \(\varepsilon > 0\) (and assuming \(\varepsilon < \delta\)), we set \(s_0 = 0\) and
\[
s_{j+1} = \left( \frac{1}{4} + \frac{s_j}{2} \right) \wedge (s_j + \delta) - \varepsilon,
\]

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for \( \varepsilon > 0 \) fixed sufficiently small. As long as the second term in the minimum dominates, the sequence \( s_j \) grows linearly as \( j(\delta - \varepsilon) \) until the scale \( s = \frac{1}{2} - 2\delta \) is reached. After that, the first term dominates. Then \( s_j \) evolves according to \( \frac{1}{2} - \delta - \varepsilon \); then \( \frac{1}{2} - \frac{1}{2}\delta \); then \( \frac{1}{2} - \frac{1}{2}\delta - \frac{1}{2}\varepsilon \) and converges geometrically to \( \frac{1}{2} - 2\varepsilon \). In particular, given \( s \in (0, \frac{1}{2}) \) as in Theorem A.1, we can fix \( \varepsilon > 0 \) and \( n < \infty \) such that \( s_n = s \), and we will assume such a choice from now on. We also recall from [6] that by replacing \( Q(z) \) by \( Q(z - z_0) \) we consider functions supported in the balls \( B^0_s = B(0, \frac{1}{2}N^{-s}) \subset B_s = B(0, N^{-s}) \), and recall the notion of \( t \)-HP introduced in [6, Section 6].

The induction assumption \( (A_t) \) is modified as follows.

**Assumption** \( (A_t) \). For any bounded \( f \in C^2(\mathbb{C}) \) with \( \text{supp}(\Delta - m^2)f \subset B^0_s \cap S_Q \), with \( t \)-HP, we have

\[
\left| \frac{1}{N} \sum_j f(z_j) - \int f \, d\mu_Q \right| \leq \xi \left( N^{-1-2t}\|\Delta f\|_{\infty} + N^{-\frac{1}{2}t}(f, (-\Delta + m^2)f)^{\frac{1}{2}} \right), \tag{A.15}
\]

where \( \xi = (1 + \frac{1}{2})(\log N)^B \) for some constant \( B \).

**Proposition A.9.** For arbitrary \( \varepsilon > 0 \), \( (A_t) \) implies \( (A_s) \) for any \( 0 \leq t \leq s \leq \left( \frac{1}{2} + \frac{1}{2}t \right) \wedge (t + \delta) - \varepsilon \), with \( B \) replaced by \( B + 2 \).

**Proof.** First, we show that, for any \( s \) as asserted in the proposition, it suffices to prove that \( (A_t) \) implies \( (A'_s) \), where \( (A'_s) \) is defined exactly as \( (A_s) \) except that the test functions \( f \) are required to obey the additional assumption \( \int f \, dm = 0 \). Indeed, assume \( (A_t) \) and that \( (A'_s) \) for all \( s \) as in the statement of the proposition. For a function \( f \) supported on \( B^0_s \) we define \( f_i(z) = 2^{-2i}f(2^{-1}z) \), and write

\[
f = f_k + \sum_{i=0}^{k-1} (f_i - f_{i+1}),
\]

where \( k \) is the largest integer such that \( 2^k N^{-s} \leq N^{-t} \). We observe that

\[
\|\Delta f_j\|_{\infty} = 2^{-4j}\|\Delta f\|_{\infty}, \quad (f_i, (-\Delta + m^2)f_i) \leq 2^{-2j}(f, (-\Delta + m^2)f). \tag{A.16}
\]

Therefore, with \( s_i = s - (i + 1)/\log_2 N \) for \( i = 0, 1, \ldots, k - 1 \), applying \( (A'_s) \) to the function \( f_i - f_{i+1} \), we obtain with \( s_i \)-HP

\[
\left| \frac{1}{N} \sum_j (f_i(z_j) - f_{i+1}(z_j)) - \int (f_i - f_{i+1}) \, d\mu_Q \right| \\
\leq \xi \left( 2^{2i}N^{-1-2s}\|\Delta f_i - f_{i+1}\|_{\infty} + 2^{2i}N^{-\frac{1}{2}s}(f_i - f_{i+1}, (-\Delta + m^2)(f_i - f_{i+1}))^{1/2} \right) \\
\leq \xi \left( 2^{-2i}sN^{-1-2s}\|\Delta f\|_{\infty} + N^{-\frac{1}{2}s}(f, (-\Delta + m^2)f)^{1/2} \right).
\]

Similarly, applying \( (A_t) \) to \( f_k \), we have with \( t \)-HP

\[
\left| \frac{1}{N} \sum_j f_k(z_j) - \int f_k \, d\mu_Q \right| \\
\leq \xi \left( N^{-1-2t}\|\Delta f_k\|_{\infty} + N^{-\frac{1}{2}t}(f_k, (-\Delta + m^2)f_k)^{1/2} \right) \\
\leq \xi \left( 2^{-4k}sN^{-1-2s}\|\Delta f\|_{\infty} + 2^{-4k}N^{-\frac{1}{2}s}(f, (-\Delta + m^2)f)^{1/2} \right) \\
\leq C\xi \left( 2^{-4k}sN^{-1-2s}\|\Delta f\|_{\infty} + 2^{-4k}N^{-\frac{1}{2}s}(f, (-\Delta + m^2)f)^{1/2} \right).
\]

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Proof of Theorem A.1. Applying Proposition A.7 with $W = Q$ and $M = N$ verifies assumption (A$_0$). By inductive application of Proposition A.9 the assumption (A$_s$) is verified for all $s \in (0, \frac{1}{2})$, implying Theorem A.1.
A.4. Yukawa gas on the torus. Next, we adapt Theorem A.1 to the torus. Let $T$ be the unit torus, i.e., the unit square $[0,1]^2$ with periodic boundary conditions. Given $Q : T \to \mathbb{R}$, for $z = (z_1, \ldots, z_N) \in T^N$, we define the energy

$$H_{N,Q}^\ell(z) = \sum_{j \neq k} U_\ell^\ell(z_j - z_k) + N \sum_j Q(z_j),$$

(A.17)

where $U_\ell^\ell(z) = \sum_{n \in \mathbb{Z}^2} Y^\ell(z + n)$ is the Yukawa potential of range $\ell$ on $T$, i.e., the Green’s function of $-\Delta + m^2$, and we assume that $\ell \leq N^{-c}$ for some constant $c > 0$. We are mainly interested in $Q = 0$, but allow nonzero $Q$ for technical reasons. For $\beta > 0$, we define the corresponding probability measure on $T^N$ by

$$P_{N,Q}^\ell(dz) = \frac{1}{Z_N^\ell} e^{-\beta H^\ell(z)} m^\otimes N(dz),$$

where $m$ is the restriction of the Lebesgue measure to $T$. The energy (A.17) is closely related to the variational functional, defined for probability measures $\mu$ on $T^N$ by

$$I_{Q}^\ell(\mu) = \int U^\ell(z - w) \mu(dz) \mu(dw) + \int Q(z) \mu(dz)$$

(A.18)

For $Q = 0$, by translational invariance the unique minimizer of $I^\ell_0$ is $m$, and

$$\inf_{\mu} I_{Q}^\ell(\mu) = I_{Q}^\ell(m) = \int_C Y^\ell(z)m(dz) = \ell^2.$$

The local law Theorem A.1 on the full plane has the following analogue on the torus.

**Theorem A.10.** Let $Q = 0$. Let $s \in (0, \frac{1}{2})$. For any $f : T \to \mathbb{R}$ supported in a ball of radius $b = N^{-s}$,

$$|X_f| \leq (\log N)^{O(1)} \sqrt{N} b^\ell \left[\|f\|_\infty + \|b\nabla f\|_\infty + \|b^2 \nabla^2 f\|\right],$$

(A.19)

with probability at least $1 - e^{-N b_2}$.

We sketch the required changes. First, the following lemma allows us to replace the torus Yukawa potential by the full plane Yukawa potential.

**Lemma A.11.**

$$Y^\ell(d(z, w)) = U^\ell(z - w) + O(\ell e^{-c/\ell}).$$

**Proof.** By definition, $Y^\ell(d(z, w))$ is the term in the sum in the definition of $U$ such that $|z - w - n|_\infty < \frac{1}{2}$. For the other terms, therefore $|z - w + n|_\infty > \frac{1}{2}$. Without loss of generality, assume that $n = 0$. Then

$$U^\ell(z - w) = Y^\ell(d(z, w)) + \sum_{n \neq 0} Y^\ell(z - w + n) = Y^\ell(d(z, w)) + \sum_{n \neq 0} O(e^{-c|z-w+n|/\ell})$$

$$= Y^\ell(d(z, w)) + O(\ell e^{-c/\ell})$$

as claimed. \qed

**Proof of Theorem A.10.** The proof of Theorem A.10 is a straightforward adaption of that of Theorem A.1. We only mention the required changes here.
(i) As in the proof of Theorem A.10, we condition on the particles outside a ball. As in the full plane case, the equilibrium measure given by replacing the charges outside by the equilibrium measure is still the restricted measure.

(ii) Using Lemma A.11, we replace the torus Yukawa potential by the full plane Yukawa potential. Namely, by Lemma A.11 and since $m = N^a$ for some $a > 0$, we have

$$H_{N,0}(z) = \sum_{j \neq k} Y^\ell(d(z_j, z_k)) + O(N^2N^ae^{-cN^a}) = \sum_{j \neq k} Y^\ell(d(z_j, z_k)) + O(N^{-A}),$$

for any $A > 0$, and where the error bound is uniform in $z \in \mathbb{T}^N$.

(iii) Using (i), in the conditioned domain, we now replace the torus interaction by the full plane interaction. From this point on, everything is as in the proof of Theorem A.1.

This concludes the sketch of the proof.

### A.5. Rigidity for the Yukawa gas on the torus.

Next we derive the following analogue of [6, Theorem 1.2] for the Yukawa gas on the torus.

**Theorem A.12.** Let $s \in (0, \frac{1}{2})$ and $\varepsilon > 0$. For any $f$ supported in a ball of radius $b = N^{-s}$,

$$|X_f| = O(N^s)[b\ell^{-1} + 1][\|f\|_{k,b} + \|f\|_{2,h}^2],$$

(A.20)

with probability at least $1 - e^{-N^\varepsilon}$, for some $k$ large enough depending on $\varepsilon$.

The proof uses the same ideas as the proof of [6, Theorem 1.2]. As a first step, we state the loop equation for the Yukawa gas on the torus. As in (33), given a function $v : \mathbb{T} \to \mathbb{R}$, set

$$W^v_Q(z) = -\sum_{j \neq k} (v(z_j) - v(z_k))\partial U^\ell(z_j - z_k) + \frac{1}{\beta} \sum_j \partial \psi v(z_j) - N \sum_j v(z_j)\partial Q(z_j),$$

(A.21)

and recall that by Lemma A.22 $\mathbb{E}_Q W^v_Q = 0$. Given $q : \mathbb{C} \to \mathbb{R}$ supported in $S_Q$, further abbreviate

$$h(z) = \frac{1}{\pi \rho_Q(z)}, \quad g(z) = \frac{1}{\pi \rho_Q(z)},$$

(A.22)

**Lemma A.13.** For any $q : \mathbb{T} \to \mathbb{R}$ of class $\mathcal{C}^2$, supported on $S_Q$, and $z \in \mathbb{T}^N$, we have

$$X^q_Q = \frac{1}{N} W^h_Q(z) + \frac{1}{N\beta} \sum_k \partial h(z_k)$$

$$+ N \int_{z \neq w} (h(z) - h(w))\partial U^\ell(z - w)\tilde{\mu}_Q(dz)\tilde{\mu}_Q(dw)$$

$$+ \frac{Nm^2}{2} \int_{z \neq w} g(w)U^\ell(z - w)\tilde{\mu}_Q(dz)\mu_Q(dw),$$

(A.23)

where $m = \ell^{-1}$. Thus, for $f : \mathbb{T} \to \mathbb{R}$ satisfying $\int_{\mathbb{T}} f(z)m(dz) = 0$ and the condition that $q = f - m^2\Delta^{-1}f$ has support in $S_Q$,

$$X^f_Q = \frac{1}{N} W^h_Q(z) + \frac{1}{N\beta} \sum_k \partial h(z_k)$$

$$+ N \int_{z \neq w} (h(z) - h(w))\partial U^\ell(z - w)\tilde{\mu}_Q(dz)\tilde{\mu}_Q(dw).$$

(A.24)
Proof. As in the proof of Lemma 3.3, we have

\[ 2 \int \partial U^\ell(z - w) \mu_Q(dw) = \partial Q(z), \tag{A.25} \]

\[ q(z) = \frac{1}{2\pi} \int (-4\partial \bar{\partial}q(w) + m^2 q(w)) U^\ell(z - w)m(dw) \]

\[ = \frac{1}{2\pi} \int (4\partial q(w) \partial U^\ell(z - w) + m^2 q(w) U^\ell(z - w)m(dw), \tag{A.26} \]

where again the first equation holds for \( z \in S_Q \) by the Euler-Lagrange equation, the second equation holds by the definition of the Yukawa potential as the Green's function of \(-\Delta + m^2\) and integration by parts – the boundary term vanishes by periodicity. We therefore have

\[ X_Q^\ell = 2 \sum_j \int h(w) \partial U^\ell(z_j - w) \mu_Q(dw) + \frac{m^2}{2} \sum_j \int g(w) U^\ell(z_j - w) \mu_Q(dw) \]

\[ - 2N \int \int h(w) \partial U^\ell(z - w) \mu_Q(dw) \mu_Q(dz) - \frac{Nm^2}{2} \int \int g(w) U^\ell(z - w) \mu_Q(dw) \mu_Q(dz) \]

\[ = 2N \int \int (h(w) - h(z)) \partial U^\ell(z - w) \mu_Q(dz) \mu_Q(dw) + \sum_j h(z_j) \partial Q(z_j) \]

\[ - N \int \int (h(w) - h(z)) \partial U^\ell(z - w) \mu_Q(dw) \mu_Q(dz) \]

\[ + \frac{Nm^2}{2} \int \int g(w) U^\ell(z - w) \mu_Q(dz) \mu_Q(dw). \]

In the first equation we used (A.22) and (A.26), and in the second equation we used (A.25). Since the integrands in the double integrals are symmetric, we arrive at

\[ X_Q^\ell = -\frac{1}{N} \sum_{j \neq k} (h(z_j) - h(z_k)) \partial U^\ell(z_j - z_k) + \sum_j h(z_j) \partial Q(z_j) \]

\[ + N \int \int_{z \neq w} (h(z) - h(w)) \partial U^\ell(z - w) \mu_Q(dz) \mu_Q(dw) \]

\[ + \frac{Nm^2}{2} \int \int_{z \neq w} g(w) U^\ell(z - w) \mu_Q(dz) \mu_Q(dw), \]

which is equivalent to (A.23).

For the consequence, note that moving the last term on the right-hand side to the left-hand side, the left-hand side is \( X^\ell \) with

\[ f(z) = q(z) - \frac{m^2}{2\pi} \int q(w) U^\ell(z - w) m(dw) = [(1 - K)q](z), \]

where

\[ 1 - K = 1 - (1 - \ell^2 \Delta)^{-1} = \frac{\ell^2 \Delta}{\ell^2 \Delta - 1}, \quad (1 - K)^{-1} = 1 - m^2 \Delta^{-1}. \]

Therefore, given \( f \) as in the assumption, we can choose \( q = f - m^2 \Delta^{-1} f. \)

The following lemma expresses \( h \) explicitly in terms of \( f. \)
Lemma A.14. There exists a smooth function \( \eta : \mathbb{R} \to \mathbb{R} \) with support in \([-1, 1]\) such that, for any \( f : \mathbb{T} \to \mathbb{R} \) with \( \int f \, dm = 0 \), we have
\[
 h(z) = \partial [1 - m^2 \Delta^{-1}] f(z) \\
= \frac{1}{2\pi} \int_0^\infty ds \partial f_s(z), \\
f_s(z) = \frac{1}{2\pi} \int_C \eta(|z - v|^2/s^2) f(v) \, dv. \tag{A.27}
\]

Moreover, if \( f \) is supported in a ball of radius \( b \), then \( f_s \) has support in a ball of radius \( s + b \), and
\[
 |\nabla^k f_s(z)| = (s \wedge b)^{-k} \|f\|_{k,b} \begin{cases} 
 O(b \wedge s)^2 & (s \leq 1), \\
 O(b^k) & (s \geq 1).
\end{cases}
\]

Proof. For any \( \eta \in C^1_c(\mathbb{R}) \) with \( \eta(0) = 1 \), and \( z \neq 0 \),
\[
 - \log |z| = \int_0^\infty (\eta(|z|^2/s^2) - \eta(1/s^2)) \frac{ds}{s}.
\]
Since \([-\Delta]^{-1} f(z) = \frac{1}{2\pi} [-\log |\cdot| \ast f](z) \), the rest of the proof follows from a direct computation.

Note that \( U^\ell \) in Lemma A.13 can be replaced by \( Y^\ell \) with negligible error by Lemma A.14. In next lemma, we estimate the last term in \( (A.24) \). Let
\[
 U_Q^h = N^2 \int_{z \neq w} (h(z) - h(w)) \partial Y(z - w) \hat{\mu}_Q(dz) \hat{\mu}_Q(dw). \tag{A.28}
\]

Lemma A.15. Given \( f : \mathbb{T} \to \mathbb{R} \), for the Yukawa gas on the torus, we have
\[
 \frac{1}{N} \mathbb{E}_{0 + f/N}(U_{0 + sf/N}^h) = O\left(N^\varepsilon (1 + b^2/\ell^2)\right) \|f\|_{3,b}.
\]

Proof. Similarly as in \( (2.2) \), we can write \( \partial Y(z) = -\frac{1}{2\pi} \chi(z) \) where \( \chi \) is a smooth cutoff function with \( \chi(z) \sim 1 \) as \( z \to 0 \), and vanishing exponentially for \( |z| \geq \ell \). As in the proof of \( [6, \text{Lemma 7.5}] \), we decompose the singularity using a compactly supported \( \varphi : [0, \infty) \to \mathbb{R} \) as
\[
 \frac{h(z) - h(w)}{z - w} = C \int_0^\infty \int_C \varphi(|z - \zeta|/t) \varphi(|w - \zeta|/t)(\bar{z} - \bar{w})(h(z) - h(w)) m(d\zeta) \frac{dt}{t^5}, \tag{A.29}
\]
and assume that \( \varphi \) is smooth, that \( 0 \leq \varphi \leq 1 \), and that \( \varphi \) is supported in \([\frac{1}{2}, \frac{1}{2}]\). Moreover, using the exponential decay of \( \chi \), we may restrict the integration range of \( t \) to \( t \leq \ell \).

The singular contribution from \( t \leq N^{-1/2+\alpha} \) is estimated in the same way as in \( [6, \text{Lemma 7.5}] \), and we obtain the contribution
\[
 N \iint H_0(z, w) \hat{\mu}(dz) \hat{\mu}(dw) = O(N^{2\alpha} N^{-2s}) \|\nabla h\|_{\infty} = O(N^{2\alpha}) \|f\|_{2, N^{-s}}, \tag{A.30}
\]
with \( \alpha \)-HIP as needed. For the remaining part of the integral \( (A.29) \), \( \ell \geq t \geq N^{-\frac{1}{2}+\alpha} \), we now distinguish two cases. As in \( [6, \text{Lemma 7.5}] \), it suffices to estimate the integrand for fixed such \( t \) and any \( \zeta \). We write
\[
 (\bar{z} - \bar{w})(h(z) - h(w)) = (\bar{z} - \bar{\zeta})(h(z) - h(\zeta)) - (\bar{w} - \bar{\zeta})(h(z) - h(\zeta)) \\
- (\bar{z} - \bar{\zeta})(h(w) - h(\zeta)) + (\bar{w} - \bar{\zeta})(h(w) - h(\zeta)), \tag{A.31}
\]
and since the four terms are analogous, we only consider the first one. We use the decomposition (A.27) for $h$. By linearity, we may estimate the contributions $\partial f$ and $\ell^2 \partial f_s$ separately. The contribution due to $\partial f$ is estimated exactly as in [6, Lemma 7.5]. For $\ell^2 \partial f_s$, we distinguish between $s \leq \frac{1}{2}$ and $s > \frac{1}{2}$.

**Case $s \leq 1/2$.** Set $h_s = \partial f_s$. We have

$$h_s(z) = \int_C \eta(|z - v|^2/s^2) \partial f(v) dv,$$

$$\text{diam supp } h_s \approx b + s \quad \text{and} \quad \|\nabla^k h_s\|_\infty \lesssim (s^2 + b^2)\|\nabla^{k+1} f\|_\infty, \quad k = 0, 1, 2.$$  

Then with $u(z) = \varphi((|z - \zeta|/t)(\tilde{z} - \tilde{\zeta})(h_s(z) - h_s(\zeta)))$ and $v(w) = \varphi(|w - \zeta|/t)$ we have

$$\|u\|_\infty \lesssim Ct\|h_s\|_\infty, \quad \|\nabla u\|_\infty \lesssim Ct\|\nabla h_s\|_\infty, \quad \|\nabla^2 u\|_\infty \lesssim Ct\|\nabla^2 h_s\|_\infty + C\|\nabla h_s\|_\infty,$$

$$\|v\|_\infty \lesssim C, \quad \|\nabla v\|_\infty \lesssim C/t, \quad \|\nabla^2 v\|_\infty \lesssim C/t^2.$$  

We also have $r = \text{diam supp } u = (b + s) \wedge t =: \tilde{r}$ if $h_s(\zeta) = 0$ and $r = t$ if $h_s(\zeta) \neq 0$, and diam supp $v = t$. Then

$$N \int u(z) v(w) \tilde{\mu}_Q (dz) \tilde{\mu}_Q (dw) \lesssim \tilde{r} (\|u\|_\infty + \|r \partial u\|_\infty + \|r^2 \nabla^2 u\|_\infty) \times t (\|v\|_\infty + \|t \nabla v\|_\infty + \|t^2 \nabla^2 v\|_\infty)$$

$$\lesssim (s^2 + b^2) t \tilde{r} t (\|\nabla f\|_\infty + \|t \nabla f\|_\infty + \|t^2 \nabla f\|_\infty + \|\nabla^3 f\|_\infty)$$

Integrating over $\zeta$, the part where $h_s(\zeta) \neq 0$ contributes $\tilde{r}^2$ from volume with $r = t$, and the part where $h_s(\zeta) = 0$ contributes volume $\approx t^2$ with $r = \tilde{r}$. Thus

$$N \int m(d\zeta) \int u(z) v(w) \tilde{\mu}_Q (dz) \tilde{\mu}_Q (dw) \lesssim (s^2 + b^2) \tilde{r} t (\|\nabla f\|_\infty + \|t \nabla f\|_\infty + \|t^2 \nabla f\|_\infty + \|\nabla^3 f\|_\infty),$$

Performing the integrals over $t$ and $s \leq 1$, we obtain the claimed bound.

**Case $s \geq 1/2$.** The support of $h_s$ is of size $s$ and we have $\|\nabla^k f_s\| \lesssim b^k s^{-k} \|f\|_\infty$. This gives

$$N \int u(z) v(w) \tilde{\mu}_Q (dz) \tilde{\mu}_Q (dw) \lesssim \tilde{r} t (\|h_s\|_\infty + \tilde{r} t \|\nabla h_s\|_\infty + \tilde{r}^2 \|\nabla h_s\|_\infty + \tilde{r}^2 t \|\nabla^2 h_s\|_\infty),$$

The $\zeta$-integral gives $O(t^2)$, and performing the integrals over $t$ and $s$, we again obtain the claimed bound. 

**Proof of Theorem A.12.** We again employ the loop equation and calculate

$$\frac{1}{\beta} \log E_0^{t\ell} e^{-t\beta X_0^f} = \frac{1}{\beta} \int_{0}^{t} ds \frac{\partial}{\partial s} \log E_0^{t\ell} e^{-s\beta X_0^f} = \int_{0}^{t} ds ( - E_0^{t\ell} X_s^f - N \int f(\mu_0 - \mu_{s/f/N}) )$$

$$= \int_{0}^{t} ds E_0^{t\ell} \left( - \frac{1}{N} W_{s/f/N}^h - \frac{1}{N \beta} \sum_j \partial h_s(z_j) + \frac{1}{\beta} \sum_j \partial h_s(w_j) - N \int (h_s(z) - h_s(w)) \partial U^\ell(z - w) \tilde{\mu}_{s/f/N} (dz) \tilde{\mu}_{s/f/N} (dw) + N \int f(\mu_0 - \mu_{s/f/N}) \right),$$

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where \( h_s \) is associated to \( f \) and \( \mu_{sf/N} \) as in (A.22). By Lemma A.15

\[
N \mathbb{E}^{U_f}_{sf/N} \int (h_s(z) - h_s(w)) \partial U_f(z - w) \mu_{sf/N}(dz) \mu_{sf/N}(dw) = O(N^\varepsilon (1 + \frac{b^2}{\ell^2})) \|f\|_{2,b}.
\]

By Lemma A.32 \( \mathbb{E}^{U_f}_{sf/N} W_{sf/N}^{h_s} = 0 \). Next, since \( \text{supp} \mu_0 = T \) and on the flat torus we may without loss of generality assume that \( f \) has mean zero, we have by \( m = 1/\ell \)

\[
\int_0^\ell ds N \int f(\mu_0 - \mu_{sf/N}) = - N \int_0^\ell ds \int f(z) \frac{s}{4\pi N}(\Delta - m^2)f(z) m(dz)
\]

\[
= \int_0^\ell ds \frac{s}{4\pi} \int f(-\Delta + m^2)f = O(t^2(1 + b^2\ell^{-2})) \|f\|_{2,b}^2.
\]

Finally,

\[
- \int_0^\ell ds \frac{1}{N\beta} \mathbb{E}^{U_f}_{sf/N} \sum_j \partial h_s(z_j) = O(t) \|f\|_{3,b}.
\]

Collecting these estimates gives

\[
\frac{1}{\beta} \log \mathbb{E}_0^{U_f} e^{-t\beta X_0^f} = O(t) \|f\|_{2,b} + O\left(tN^\varepsilon (1 + \frac{b^2}{\ell^2}) \|f\|_{3,b} + O(t^2(1 + b^2\ell^{-2})) \|f\|_{2,b}^2 \right)
\]

\[
= O\left(N^\varepsilon (1 + b^2/\ell^2)(t + t^2)(\|f\|_{3,b} + \|f\|_{2,b}^2) \right).
\]

(A.32)

This implies the estimate

\[
|X_0^f| \leq O(N^\varepsilon(1 + b^2/\ell^2)(\|f\|_{3,b} + \|f\|_{2,b}^2) = O\left(N^\varepsilon \frac{b + \ell}{\ell b \sqrt{N}} \right) \sqrt{N} \frac{b + \ell}{\ell} (\|f\|_{3,b} + \|f\|_{2,b}^2)
\]

with probability at least \( 1 - e^{-N^\varepsilon} \). For \( b \leq \ell \) we have obtained the optimal estimate \( |X_0^f| = O(N^\varepsilon) \|f\|_{3,b} + \|f\|_{2,b}^2 \).

For \( b \geq \ell \) we have improved on the estimate provided by Theorem A.10 by a factor \( N^{\varepsilon}/\ell \sqrt{N} = N^{\varepsilon-\delta} \). Inserting this improved estimate into the proof of Lemma A.15 and repeating the calculation above, we gain an additional factor \( N^{\varepsilon - \delta} \) front in the bound of Theorem A.32 at the cost of one more derivative. This leads into the estimate

\[
|X_0^f| \leq O(N^{\varepsilon + \varepsilon_2 - \delta}(1 + b^2/\ell^2)(\|f\|_{4,b} + \|f\|_{2,b}^2)
\]

Thus for \( b \geq \ell \), we may iterate this procedure finitely many times to reach the bound

\[
\frac{1}{\beta} \log \mathbb{E}_0^{U_f} e^{-t\beta X_0^f} = O\left(N^\varepsilon(1 + b^2/\ell^2)(\frac{\ell^2}{b^2}t + t^2)(\|f\|_{k,b} + \|f\|_{2,b}^2) \right)
\]

\[
= O\left(N^\varepsilon(1 + \frac{b^2}{\ell^2}t^2)(\|f\|_{k,b} + \|f\|_{2,b}^2) \right),
\]

which implies the claimed bound of Theorem A.12 by setting \( t = \ell/b \). \( \square \)
A.6. Proof of Proposition 2.11. As an application of Theorem A.12, we now prove Proposition 2.11. For convenience, we restate it here, and here write $N$ instead of $n$ for the particle number, as in the remainder of this appendix.

**Proposition A.16.** Let $R = N^A$ for some constant $A$. Then for the Yukawa gas with $N$ particles with range $\ell = N^{-1/2+\delta}$ in a torus of size $b$, we have with high probability

$$N^2 \int \int L_R^\ell (z-w) \tilde{\mu}_Q (dz) \tilde{\mu}_Q (dw) = O(N^c b^2 \ell^{-2}). \quad \text{(A.33)}$$

**Proof.** We can assume that $R = 2^K \ell$ for some large integer $K$. We decompose

$$L_R^\ell = Y^R - Y^\ell = \sum_{k=1}^K R_k^\ell, \quad R_k^\ell = Y^{2^k \ell} - Y^{2^{k-1} \ell}.$$ 

Let $g_a$ be a nonnegative radial symmetric mollifier at the scale $a$ with $\int g_a(z) \, dz = 1$. By definition and Taylor expansion, we write

$$\int \int R_k^\ell (z-w) \tilde{\mu}(dz) \tilde{\mu}(dw)$$

$$= \int \int R_k^\ell (x-y)g_a(z-x)g_a(w-y) \, dx \, dy \tilde{\mu}(dz) \tilde{\mu}(dw) + \sum_{j=1}^{M-1} F_j + G_M,$$

where

$$F_j = \int \int (\nabla)^{2j} R_k^\ell (x-y)(z-w - (x-y))^{2j} g_a(z-x)g_a(w-y) \tilde{\mu}(dz) \tilde{\mu}(dw) \, dx \, dy$$

$$G_M = \int \int (\nabla)^{2M} R_k^\ell [(1-s)(z-w) + s(x-y)](z-w - (x-y))^{2M}$$

$$\times g_a(z-x)g_a(w-y) \tilde{\mu}(dz) \tilde{\mu}(dw) \, dx \, dy.$$

Since

$$(\nabla)^{2M} R_k^\ell (z) \sim \ell_k^{-2M} 1(|z| \sim \ell_k)$$

and while $|z-x| + |w-y| \leq a \leq \ell_k$ we have

$$N^2 G_M \leq (a^2 / \ell_k)^{2M} N^2$$

Let $a = \ell_k N^{-c}$ and choose $M$ large enough so that $(a^2 / \ell_k)^{2M} N^2 \leq 1$, so that we may neglect the error term $G_M$. Next, from the rigidity estimate Theorem A.12, we find

$$N^2 \int \int R_k^\ell (x-y)g_a(z-x)g_a(w-y) \tilde{\mu}(dz) \tilde{\mu}(dw) \, dx \, dy$$

$$= N^2 \int \int R_k^\ell (x-y) (\tilde{\mu} * g_a)(x) (\tilde{\mu} * g_a)(y) \, dx \, dy \leq O(N^c) \ell_k^2 \frac{(a + \ell)^2}{\ell^2 a^4} b^2, \quad \ell_k = 2^k \ell,$$

where we used that

$$\int \int R_k^\ell (x-y) \, dx \, dy \leq C \ell_k^2 b^2,$$

and finally.

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∥g_a∥_∞ ≤ C a^{-2}, and that, with high probability,
\[ N(\tilde{\mu} * g_a)(x) = N \int g_a(z - x)\tilde{\mu}(dz) \leq O(N^\varepsilon) a^{-2} \frac{a + \ell}{\ell}. \]

Here the factor \( b^2 \) comes from integration of the variable \( x + y \) over a torus of size \( b \). Finally, we bound the terms \( F_j \). Notice that \( F_j = \sum_{\alpha=0}^j \int (\nabla^2)^j R_k^\ell z \cdot \psi_a(z) \tilde{\mu}(dz) \]
\[ \times \left[ \int (w - y)^{2\alpha} g_a(w - y) \tilde{\mu}(dw) \right] dx dy \]
Since each derivative \( R_k^\ell \) produces a factor \( \ell_k^{-1} \) while \( |z - x| + |w - y| \leq a \leq \ell_k \), we can follow the previous argument to prove a similar bound
\[ N^2 F_j \leq O(N^\varepsilon) \ell_k^2 a^{-2} + \ell_k^2 a^{-2} \cdot b^2. \]
Recalling \( a = \ell_k N^{-c} \), we have thus proved that
\[ \int \int R_k^\ell(z - w)\tilde{\mu}(dz) \tilde{\mu}(dw) \leq O(N^\varepsilon) \left[ \frac{N^{2c}}{\ell^2} + \frac{N^{4c}}{\ell_k^2} + 1 \right] b^2 \]
Summing over \( k \) and using that \( K = O(\log N) \), we have proved that
\[ \int \int L^\ell(z - w)\tilde{\mu}(dz) \tilde{\mu}(dw) \leq O(N^\varepsilon) \left[ \log N \frac{N^{4c}}{\ell^2} + \log N \right] b^2, \]
and this completes the proof of the proposition.

### B Local density and rigidity for Coulomb gas with angle term

In this section, we study the Coulomb gas with an additional small interaction with mesoscopic range. More specifically, we deal with the Hamiltonian
\[ H_V^{C \Psi h} = H_V^{C - \Psi h} = H_V^C - \ell N \text{Re} \hat{A}^h, \]
where \( H_V^C \) is the Coulomb Hamiltonian and
\[ \hat{A}^h = \frac{1}{2N} \sum_{i \neq j} \Psi_h(z_i, z_j) = \frac{1}{2N} \sum_{i \neq j} e^{-\frac{|z_i - z_j|^2}{2\sigma^2}} \frac{h(z_i) - h(z_j)}{z_i - z_j} \]
is the local angle term of Theorem B.1. We will suppose here, without loss of generality, that \( \|\nabla h\| \leq C \) for some absolute constant.
We make the same assumption on \( V \) as in Section I, denote by \( \mu_V \) the equilibrium measure of the one-component Coulomb gas with potential \( V \) (defined without contribution from \( \hat{A}^h \)), and we also recall the definition
\[ X_f = \sum_{i \neq j} f(z_j) - N \int f \mu_V. \]
The results of this section are the following theorems, which are analogous to [6] Theorem 1.1 and 1.2].
Theorem B.1. Let \( s \in (0, \frac{1}{2}) \) and suppose \( t \theta^2 N \leq 1 \). Then for any \( f \) supported in a square of size \( b = N^{-s} \), we have
\[
|X_f| \leq O(\log N)\sqrt{N}b\left[\|f\|_\infty + \|b\nabla f\|_\infty + \|b^2 \nabla^2 f\|_\infty\right]
\] (B.2)
with probability at least \( 1 - e^{-(1+\beta)(Nb^2 + N\theta^2) + O(\log N)} \).

Theorem B.2. Let \( s \in (0, \frac{1}{2}) \) and suppose \( t \theta^2 N \leq 1 \). Then for any \( f \) supported in a square of size \( b = N^{-s} \), we have for any \( \varepsilon > 0 \)
\[
|X_f| \leq O(N^\varepsilon)\|f\|_{4,b}
\] (B.3)
with probability at least \( 1 - e^{-N^\varepsilon} \).

B.1. Local density. As for the Yukawa gas in Appendix A to prove the local density estimate Theorem B.1, we follow the strategy originally used in [6]. However, we will see that by the standing assumption that \( tN \theta^2 \leq 1 \) the modifications needed to prove the local law for the gas with Hamiltonian (B.1) are much smaller than for the Yukawa gas. Especially, we will not need to consider variational functionals or potential theory other than the one used in [6] in the case of the Coulomb gas. This being the case, we will only sketch the differences from the proof of the local law in [6].

The first statement for local density, the one from the induction setup below, is

Proposition B.3. Let \( t = N^{-2a} \) and suppose \( \text{supp } f \) has diameter \( N^{-s} \). Then
\[
\frac{1}{N}|X_f| = O((1 + 1/\beta)\log N)\left((N^{-\frac{1}{2}} - s + N^{-a-2s})\|\nabla f\|_2 + (N^{-1-2s} + N^{-2a-4s})\|\Delta f\|_\infty\right)
\]
with probability at least \( 1 - e^{-(1+\beta)(N^{-1-2s} + N^{-2a-4s}) + O(\log N)} \).

The one-step estimate corresponding to [6] Proposition 4.1, based on mean-field partition function bounds, is the following.

Proposition B.4. Let \( \Sigma = \Sigma_W \) be a smooth domain. Given a potential \( W \in C_{\text{locl}}^{1,1}(\Sigma_W) \) possibly depending on the number of particles \( M \), assume that there exist \( u : \Sigma_W \rightarrow \mathbb{R}_+ \) and \( v : \partial \Sigma_W \rightarrow \mathbb{R}_+ \) such that \( d\mu_W = u\,dm + v\,ds \), where \( dm \) is the 2-dimensional Lebesgue measure and \( ds \) is the arclength measure on \( \partial \Sigma_W \). Assume the conditions (i)-(iv) as in [6], and in addition that for every \( j = 1, 2, \ldots, M \),
\[
\sum_{i \neq j} e^{-\frac{|z_i - z_j|^2}{2b^2}} \frac{h(z_i) - h(z_j)}{z_i - z_j} \leq K.
\] (B.4)

Then for any bounded \( f \in C^2(\mathbb{C}) \) with compact \( \text{supp } \Delta f \subset S_W \), and for any \( \xi \geq 1 + 1/\beta \),
\[
\left| \frac{1}{M} \sum_j f(z_j) - \int f \,d\mu_W \right| = O(\xi) \left( \frac{tMK + M\log M}{\alpha M^2} \|\Delta f\|_\infty + \frac{(tMK + M\log M)^{1/2}}{M} \|\nabla f\|_2 \right)
\] (B.5)
with probability at least \( 1 - e^{-\xi(\alpha M^2 \log M)/2} \). Recall that \( \alpha \) stands for a lower bound for \( \Delta W \) in \( \text{supp } S_W \).
Note that if $tK \leq 1$ this is exactly the same as in [6]. It is thus enough to prove the local density down to scale $\theta$, so that we may estimate $K \lesssim N^{2\sigma}$. Then with the assumption $tN^{2\sigma} \leq 1$ the above estimate gives the same fluctuation bound for the linear statistic as in [6].

**Proof.** By (B.4) we may trivially estimate

$$\log \int e^{-\beta H_V^C} - tMK \leq \log \int e^{-\beta H_V^{C_t}} \leq \log \int e^{-\beta H_V^C} + tMK.$$ 

Estimating the partition function with the Coulomb Hamiltonian exactly as in [6], Lemmas 4.3 and 4.4, the error term $O(M \log M)$ is replaced by $tMK + O(M \log M)$ and we arrive at the probability estimate (B.3).

**B.2. Induction.** Write $t = N^{-2\alpha}$. The induction proceeds almost exactly as in [6], with the additional element that at each stage we improve also on the bound $K$. In the first step, we take the trivial bound $K = N^2$ and get

$$\frac{1}{N} |X_f| \lesssim O(\log N) \left( \left( \frac{t}{\alpha} + N^{-1} \right) \| \Delta f \|_\infty + \left( \frac{t^2}{2} + N^{-\frac{1}{2}} \right) \| \nabla f \|_2 \right)$$

$$= O(\log N) \left( (N^{-2\alpha} + N^{-1}) \| \Delta f \|_\infty + (N^{-\alpha} + N^{-\frac{1}{2}}) \| \nabla f \|_2 \right).$$

This gives effective estimates on numbers of particles on scales $N^{-s}$, $0 \leq s < a/2$. Then, supposing we can control particle numbers on the distance scale $N^{-t}$, in Proposition [B.4] applied to the conditional measure in a ball of this scale we have $M \approx N^{1-2t}$, $\alpha = N/M \approx N^{2t}$ and $K \lesssim M \vee N^{2\sigma}$, which results in the estimate

$$\frac{1}{N} |X_f| \lesssim O(\log N) \left( \left( \frac{t \vee N^{2\sigma}}{\alpha N} + \alpha^{-1} N^{-1} \right) \| \Delta f \|_\infty + \left( \frac{t^2}{2} \vee \frac{(MN^{2\sigma})^{\frac{1}{2}}}{N} + M^{\frac{3}{2}} N^{-1} \right) \| \nabla f \|_2 \right)$$

$$\lesssim O(\log N) \left( (N^{-2\alpha - 4t} + N^{-1-2t}) \| \Delta f \|_\infty + (N^{-a-2t} + N^{-\frac{3}{4} - \delta}) \| \nabla f \|_2 \right).$$

This is an effective estimate on particle numbers on scales $N^{-s}$, $s < (t + \frac{\alpha}{2}) \vee (\frac{1}{4} + \frac{t}{2})$. We remark that as far as the scales are concerned, this is the same recursion as in the case of the Yukawa gas, with $\delta$ replaced by $a/2$.

We proceed to set up the induction formally. Redefine $t$-HP to mean with probability at least $1 - e^{-(1+\beta)(N^{1-2t} + N^{2\alpha - 4t}) + O(\log N)}$. The assumption (A_1) of [6] is replaced by the following.

**Assumption (A'_1).** For any bounded $f \in C^2(\mathbb{C})$ with supp $\Delta f \subset B_t^c \cap S_V$, with $t$-HP, we have

$$\left| \frac{1}{N} \sum_j f(z_j) - \int f \, d\mu_V \right|$$

$$= O \left( \left( 1 + \frac{1}{\beta} \right) \log N \right) \left( (N^{-1-2t} + N^{-2\alpha - 4t}) \| \Delta f \|_\infty + (N^{-\frac{3}{4} - \delta} + N^{-a - 2t}) \| \nabla f \|_2 \right). \quad (B.6)$$

For $t = 0$ this is (B.5). It is thus enough to prove the next proposition.

**Proposition B.5.** For arbitrary $\varepsilon > 0$, (A'_1) implies (A'_s) for any

$$0 \leq t < s \leq \left( \frac{1}{4} + \frac{t}{2} \right) \wedge \left( \frac{a}{2} + t \right) - \varepsilon,$$

with the implicit constant in (B.6) depending only on $\varepsilon$. 

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To prove this, as in [6], we condition on the outside of a ball $B_s$ on scale $s$ and replace the Coulomb potential of the outside charges with the Coulomb potential of the equilibrium measure. To ensure that the equilibrium measure of the conditional system inside $B_s$ does not move much under this replacement, we use [6] Propositions 3.3 and 3.4 and the analogues of [6] Lemmas 6.2 and 6.3 where the input assumption is replaced by our current assumption $(A_\gamma)$; the Lemmas are then checked exactly as in the case of the Yukawa gas.

The only new element is that we need to give the bound $K \lesssim N^{1-2t} \vee N\theta^2$. For any fixed $i = 1, 2, \ldots, M$ we write

$$N[\hat{A}^h] = \left| \sum_{i \neq j} e^{-\frac{|x_i-x_j|^2}{2s^2}} \frac{h(z_j) - h(z_i)}{z_j - z_i} \right| \leq C \sum_{i \neq j} e^{-\frac{|x_i-x_j|^2}{2s^2}} \leq C \int_0^\infty \frac{r}{\theta^2} e^{-\frac{r^2}{2s^2}} |\{j : |z_j - z_i| \leq r\}|dr. \quad \text{(B.8)}$$

The claimed estimate then follows from estimating the particle numbers in (B.8) by using the induction assumption, just as in [6, Lemma 6.2].

**Proof of Theorem B.1.** All that remains is to apply the one-step bound (B.5) on an arbitrary scale $b$, using the optimal estimate $K \lesssim N\theta^2$ provided by Proposition B.3.\hfill\Box

### B.3. Rigidity with angle terms.

The following lemma is the loop equation for Coulomb gases with the angle term $tN\hat{A}^h$. Notice that the error term is bounded by $\tau N$ when we impose the usual condition $N^{2\sigma} \leq 1$.

**Lemma B.6.** Suppose the local density of the gas with potential $V + sg$ and interaction $C - t\Psi_h$ is bounded for all scales $\geq N^{-1/2+\varepsilon}$ for all $0 \leq s \leq \tau$ as in Theorem B.2. Then with $k_s(z) = \frac{\partial g(z)}{\partial (V(z)+sg(z))}$,

$$\frac{1}{\beta} \log Z_{V+sg,C-t\Psi_h} = -N^2 \int_0^\tau ds \int g(z)\mu_{V+sg}(dz) + \tau t O(N^{1+2\sigma})$$

$$\quad - \int_0^\tau ds E_{V+sg}^{C-t\Psi_h} \left[U_{k_s}^{V+sg} + \frac{1}{\beta} \sum \partial k_s(z_k)\right] ds \quad \text{(B.9)}$$

where $U$ is defined as

$$U_{k_s}^{V+sg} = \frac{N^2}{2} \int_{z \neq w} \frac{k_s(z) - k_s(w)}{z - w} \bar{\mu}_{V+sg}(dz)\bar{\mu}_{V+sg}(dw).$$

We will only need this for $\tau \sim N^{-1+a}$ for some small $a$.

**Proof.** The proof is similar to the proof of the loop equation in Section B.1. The simple calculation yields that

$$\frac{\partial}{\partial s} \frac{1}{\beta} \log Z_{V+sg,C-tN \Re A} = E_{V+sg}^{C-t\Psi_h} \left(-N \sum g(z_j)\right)$$

$$= \Re E_{V+sg}^{C-t\Psi_h} \left[W_{k_s}^{V+sg} - U_{k_s}^{V+sg} - N^2 \int g(z)\mu_{V+sg}(dz) - \frac{1}{\beta} \sum \partial k_s(z_j)\right]$$
Since we have changed the measure, we readjust \( W^{V+sg}_v \) to

\[
W^{V+sg}_v = W^{V+sg}_v C^{-t\Psi_h^{-}} - \tilde{t} \sum_{j \neq k} v(z_j) \partial_j \frac{1}{2} \Re e \frac{|z_j - z_k|^2}{2\theta^2} h(z_j) - h(z_k).
\]

Since \( E^{C-t\Psi_h^{-}} W^{V+sg}_v C^{-t\Psi_h^{-}} = 0 \) for any function \( v \), especially \( E^{C-t\Psi_h^{-}} W^{V+tf}_k C^{-t\Psi_h^{-}} = 0 \). Hence we have

\[
\frac{\partial}{\partial s} \frac{1}{\beta} \log Z_{V^{sg},C-t\Psi_h^{-}} = \Re E^{C-t\Psi_h^{-}} \left[ - U^{V+sg}_{k_s} + \tilde{t} \sum_{j \neq k} k_s(z_j) \partial_j \frac{1}{2} \Re e \frac{|z_j - z_k|^2}{2\theta^2} h(z_j) - h(z_k) \right] - N^2 \int g(z) \mu_{V+sg}(dz) - \frac{1}{\beta} \sum \partial k_s(z_j)
\]

(B.10)

The second term on the first line is given by

\[
\sum_{j \neq k} k_s(z_j) \partial_j e^{-\frac{|z_j - z_k|^2}{2\theta^2} h(z_j) - h(z_k)}
\]

\[
= -\frac{1}{2} \sum_{j \neq k} e^{-\frac{|z_j - z_k|^2}{2\theta^2} (k_s(z_j) - k_s(z_k))(\bar{z}_j - \bar{z}_k) h(z_j) - h(z_k)} + \frac{1}{2} \sum_{j \neq k} e^{-\frac{|z_j - z_k|^2}{2\theta^2} (k_s(z_j) - k_s(z_k))(h(z_j) - h(z_k))}
\]

and the corresponding terms for the conjugates; by Theorem [B.2] all these sums may be bounded by \( O(N^{1+2\sigma}) \), which proves the claim. \( \square \)

**Lemma B.7.** For any function \( k \) supported in a square of size \( b \),

\[
N^2 E^{C-t\Psi_h^{-}} \int \frac{k(z) - k(w)}{z - w} 1_{z \neq w} \tilde{\mu}(dz) \tilde{\mu}(dw) \leq N^{2\epsilon + 1} b ||k||_{2,b}.
\]

(B.11)

**Proof.** Using Theorem [B.1] as the input estimate, the proof is exactly the same as that of [6, Lemma 7.5]. \( \square \)

These two lemmas imply the main theorem of this appendix.

**Theorem B.8.** Theorem [B.2] holds. Furthermore, for all \( r \geq N^{-1/2+\epsilon} \), and functions \( k \) supported in a square of size \( b \),

\[
E^{C-t\Psi_h^{-}} N^2 \int \int_{z \neq w} \Psi_r^+(z, w)(\bar{z} - \bar{w})(k(z) - k(w)) \tilde{\mu}(dz) \tilde{\mu}(dw) \leq N^{2\epsilon} r^{-2} b ||k||_{2,b}.
\]

(B.12)

**Proof.** Using the estimate [B.11], we have

\[
E^{C-t\Psi_h^{-}} \sum_j (U_k^{V+sg}) = O(N^{1+\epsilon}).
\]

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Now we take $\tau \sim N^{-1+\varepsilon}$ with $a > 0$. By assumption, $tN^2 \leq 1$. Thus we get
\[ \log E^{c-\tau V_{\tau}} e^{-\beta \tau N X_{\tau}} = N \int_0^\tau ds s E^{c-\tau V_{\tau+s}} \sum_j k_s(z_j) \partial g(z_j) \]
\[ - N^2 \int_0^\tau ds \int g(z) \mu_{\tau+s} (dz) + \tau O(N^{1+\varepsilon}) \]
Following the argument of \[6\] Theorem 1.2, we have proved (B.3). Once (B.3) is proved, (B.12) follows by the standard argument.

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