Exact solutions for E. Verlinde emergent gravity and generalized G. Perelman entropy for geometric flows

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Abstract

We develop an approach to the theory of relativistic geometric flows and emergent gravity defined by entropy functionals and related statistical / geometric thermodynamics models. There are considered nonholonomic (subjected to non-integrable constraints) deformations of G. Perelman's functionals when the W-entropy is used for deriving relativistic geometric evolution flow equations, which (for self-similar configurations) describe generalized Ricci solitons via solutions being equivalent to entropic modified Einstein equations. We analyse possible connections between relativistic modifications of the Poincaré–Thurston conjecture for nonholonomic Ricci flows and the E. Verlinde conjecture that gravity links to an entropic force as a spacetime 'elasticity'. We prove that corresponding systems of nonlinear partial differential equations, PDEs, for entropic flows and modified gravity possess certain general decoupling and integration properties. There are constructed new classes of exact solutions for stationary and nonstationary configurations, generalized black hole and locally anisotropic cosmological metrics in (entropic) modified gravity theories, MGTs, and general relativity, GR. Such solutions describe scenarios of nonlinear geometric evolution and/or gravitational and matter field dynamics with pattern-forming and quasiperiodic structure and various space quasicrystal like and deformed spacetime crystal models. We analyse new classes of generic off-diagonal solutions for entropic gravity theories and speculate how certain physically important properties of such solutions can be used for explaining structure formation in dark energy and dark matter physics. Finally, we speculate why the approaches with Perelman–Lyapunov like functionals are more general or complementary to the constructions elaborated using the concept of Bekenstein–Hawking entropy.

Keywords: Relativistic geometric flows; generalized Perelman F- and W-entropy; entropic gravity and nonholonomic Ricci solitons; off-diagonal solutions with quasi-periodic structure.

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1 Introduction

The most inspiring ideas in recent development of the gravity theory and cosmology are on the emergent thermodynamic nature of the spacetime geometry, when the Einstein equations can be derived using area-entropy formulas for horizons of black holes, BH, [1, 2, 3, 4] and from supposed elastic properties of gravity [5, 7, 8, 9]. A substantial progress includes the research on the microscopic origin of Bekenstein–Hawking entropy in string theory [10], the holographic principle [11, 12], BH complementarity [13], and gauge/gravity correspondence [14, 15, 16]. Here we note a subsequent development of the (anti) de Sitter, (A)dS, and conformal field theories, CFTs, and AdS/CFT, correspondence [17] and formulating laws of the thermodynamics of ‘apparent’ horizons [18].

Later, it was proposed that gravity theories and generalized/modified/linearized Einstein equations are consequences of the quantum entanglement connecting nearby spacetime regions [19, 20, 21, 22, 23]. Recent theoretical activities are devoted to proofs that the entanglement first law and dual gravity are derived from the CFT and/or reveal a deep connection to ideas on emergence of spacetime and gravity from general quantum information principles [24, 25, 26, 27, 28]. It was pointed out that in a model of dual gravity with entanglement is equivalent to the full (nonlinear) equations [28]. An intriguing conjecture that gravity links to an entropic force as a spacetime elasticity was proposed by E. Verlinde [8, 9]. It was based on the idea that gravitational interactions result from information regarding the positions of material bodies. Emergent phenomena for gravity were investigated by many other authors and their efforts involve the holographic principle in particle physics and the information theory. There were studied geometric models and possible applications related to gravity and quantum computers; quantum gravity; cosmological inflation and acceleration; and dark energy and dark matter physics etc., see [29, 30, 31, 32, 33] and references therein.

In the quest to explore the connection between the models of emergent gravity and in modified gravity theories, MGTs, or general relativity, GR, one involve a strict area law for the BH, (A)dS, or entanglement entropy and further developments for holographic models. To derive gravitational field
equations was considered that a small but nonzero volume law entropy would compete with, and at large distances involves, the area law. In certain models this is due to thermalization, elastic spacetime properties, quantum entanglement, holographic effects etc. It was proposed that such a phenomenon occurs in the dS space being responsible for the presence of a cosmological horizon. Nevertheless, in this series of two works, see [34] as a partner letter variant and certain complementary results, we deal with quite different issues on geometric flow modifications of gravity and spacetime thermodynamics. It is just shown that relativistic generalizations with a corresponding choice of evolution and thermodynamic functionals support E. Verlinde’s ideas on the origin of gravity as an effect of the entropic force but with a new type of geometric thermodynamics entropy. Such an approach with Perelman–Lyapunov entropy type functionals was developed in our works on entropic nonholonomic geometric flow evolution, nonlinear dynamics and thermodynamics for relativistic, noncommutative, fractional, supersymmetric, Finsler-Lagrange-Hamilton etc. generalizations of the theory of Ricci flow evolution and applications in modern gravity and cosmology [35, 36, 37, 38, 39, 40, 41, 42, 43, 44].

The goal of this paper is to elaborate on geometric and physical theories relating relativistic generalizations of the Poincaré–Thurston conjecture (originally formulated and proved, respectively, due to R. Hamilton and G. Perelman, for Ricci flows of Riemannian metrics) and E. Verlinde conjecture that gravity results from an entropic force as a spacetime elasticity which explain fundamental properties of dark matter, DM, and dark energy, DE, in modern cosmology [8, 9]. On topology and geometry of Ricci flows, we refer to classical works [45, 46, 47] and [48, 49, 50] and reviews of mathematical results in monographs [51, 52, 53]). Here we note that D. Friedan published a series of works on nonlinear sigma models, σ–models, in 2+ε dimensions, see [54, 55, 56] where geometric flow equations were introduced for the renorm group, RG, theories, see recent results in [57, 58, 59].

The other goal of this article is to develop the anholonomic frame deformation method, AFDM, (on early works see [60, 61, 62, 63] and references therein), for constructing exact and parametric quasiperiodic solutions of geometric entropic flow and modified gravity equations. For reviews of recent results on black hole solutions in MGTs [64, 65], space and time like (quasi) crystals, pattern forming and nonlinear gravitational wave structures and applications in modern cosmology see [66, 67, 68, 69, 70] and references therein. We elaborate on new classes of generic off-diagonal stationary and cosmological solutions with entropic geometric flows which for self similar Ricci soliton configurations result in equations considered in E. Verlinde works [8, 9] and a covariant generalization due to S. Hossenfelder [71], see also critics Refs. [72, 73].

Three lines of evidence motivate this article and the partner letter [34]. First, we use our former results and nonholonomic geometric methods [35, 36, 37, 38, 39, 40, 41, 42, 43, 44] that generalized/modified relativistic flow equations and Einstein equations in GR and MGTs can be derived as systems of PDEs for modified Ricci solitons for respective nonholonomic modifications of G. Perelman’s F- and W-entropy functionals. On modified gravity and applications in modern cosmology and astrophysics, see reviews [74, 75, 76, 69]. Second, a number of recent works invoke ideas on origin of gravity as an emergent effect of the entropic force, entanglement etc., see [8, 24, 25, 26, 27, 9, 19, 20, 21, 22, 23, 28]. We argue that this can be grounded and explained, at a deeper level, through modifications of the Poincaré–Thurston conjecture on geometric flows, when the F- and W-functionals are generalized for metrics and generalized connections on Lorentz manifolds and/or certain supersymmetric/ noncommutative / fractional/ stochastic generalizations. In our approach, the spacetime evolution and gravity are treated via geometric entropy values which allows to formulate respective statistical thermodynamics models. Third, new advanced methods for constructing exact solutions in MGTs and GR allows us to proceed directly toward definition of gravitational entropy and thermodynamic values making no use of holography, area-entropy relation, CFT duality etc. Due to the competition between area and volume law of generalized W-entropy, we can charac-
terize thermodynamically new classes of BH and cosmological solutions with quasi-periodic structure, locally anisotropic inflation and accelerating scenarios, exhibiting memory effects in the form of an entropy displacement caused by matter etc.

This article is organized as follows: In Sec. 2 we provide an introduction to the geometry of double, 2+2 and 3+1 dimensional, spacetime fibrations defining elastic and quasiperiodic configurations both for gravitational and (effective) matter fields. Main concepts and most important results on nonlinear connection geometry on nonholonomic Lorentz manifolds, hypersurface geometric objects, and modified emergent/ elastic gravity theories are outlined. There are studied the geometry of distributions, and respective Lagrange densities and geometric evolution or dynamical fields defining elastic and quasiperiodic structures.

Sec. 3 is devoted to the theory of geometric flows and modified entropic gravity. We postulate canonical nonholonomic deformations of Perelman’s F- and W–functionals encoding geometric flow evolution scenarios of entropic spacetimes with quasiperiodic structure. Such values are defined in relativistic 4-d form and for 3-d hypersurface projections. There are derived respective (generalized R. Hamilton) geometric flow equations for entropic quasiperiodic flows. The concept of nonholonomic Ricci solitons as self-similar configurations is elaborated and related modifications of the Einstein equations are analyzed. We speculate on connection between relativistic generalizations of the Poincaré–Thurston conjecture for Ricci flows and geometric proofs of E. Verlinde’s conjecture.

In Sec. 4, we develop and apply the anholonomic frame deformation method, AFDM, [34, 69, 41, 64, 43, 70] in order to prove general decoupling properties and integrability of nonholonomic geometric flow and Ricci soliton equations encoding elastic and quasiperiodic spacetime and (effective) matter fields properties. Such solutions are described by generic off-diagonal metrics, and generalized connections, depending on all spacetime coordinates and temperature like parameters via general classes of generating functions and (effective) sources of entropic gravity and matter fields.

Sec. 5 is devoted to a study of explicit examples of exact and parametric solutions for geometric flows with/to stationary elastic and quasiperiodic configurations. We provide Table 2 summarizing the AFDM for constructing stationary solutions in entropic geometric flow and MGTs. This class of solutions describes, for instance, black hole (BH) nonholonomic / ellipsoid deformations by spacetime elastic forces or embedding of BH in Ricci solitonic entropic vacuum and nonvacuum backgrounds with quasiperiodic structure. Example of entropic deformation of BHs into quasiperiodic stationary solutions are considered.

Then, in sec. 6, we summarize the AFDM for constructing cosmological solutions for entropic quasiperiodic flow and MGTs, see Table 3. We emphasize that there are certain duality properties of such locally anisotropic entropic stationary and/or cosmological type solutions and certain nonlinear symmetries relate possible classes of generating functions and (effective) sources all encoding entropic, quasiperiodic, pattern forming, space and time quasicrystal, solitonic and other type structures. There are studied cosmological configurations generated by entropic quasiperiodic sources, nonstationary generating functions, cosmological metrics evolving in (off-) diagonal elastic and/or quasiperiodic media.

Finally, we conclude our work and discuss certain perspectives of the theory of G. Perelman and E. Verlinde entropic geometric flow and emergent gravity theories in Sec. 7. Possible applications in modern cosmology and dark energy and dark matter physics are analyzed. Appendix A summarizes necessary (for the main part of this article) results on the theory of nonholonomic hypersurface and relativistic geometric flows elaborated in our previous works. We emphasize that in those approaches the relativistic flows were studied for time like parameters. In sections 2-6 of this paper, we elaborate on entropic MGTs derived for temperature like geometric evolution scenarios.
2 Spacetime 2+2 & 3+1 fibrations with elastic and quasiperiodic structures

In this section, we summarize necessary results on the geometry of Lorentz manifolds enabled with nonholonomic (i.e., non-integrable, equivalently, anholonomic) distributions defining double 2+2 and 3+1 fibrations. There are developed nonholonomic geometric methods which are important for elaborating theories of relativistic Ricci flows and possible applications in modern cosmology and astrophysics, see details in [41]. As explicit examples, we shall consider nonholonomic distributions modelling elastic and/or quasiperiodic space and time structures (for instance, quasicrystal or solitonic like configurations) [8, 9, 71, 70, 65, 67, 68]. It should be noted that the 2+2 nonholonomic splitting is important for proofs of general decoupling and integration properties of the relativistic geometric and entropic flow evolution, nonholonomic Ricci soliton and (entropic modified) Einstein equations, see section 4. Additional 3+1 decompositions adapted to 2+2 splitting will be used for defining and computing entropic and thermodynamic like values for various classes of solutions of physically important systems of nonlinear partial differential equations, PDEs, see section 6.

2.1 Nonlinear connections with 2+2 splitting of Lorentz manifolds

Let us consider a four dimensional, 4-d, Lorentzian manifold \( V \), \( \dim V = 4 \), with local pseudo-Euclidean signature \((++--)\) for a metric field \( g = (hg, vg) \). The conventional horizontal, h, and vertical, v, nonholonomic decomposition is defined by a nonlinear connection, N-connection, structure \( N \). Such a geometric object can be always introduced as a Whitney sum

\[ N : TV = hV \oplus vV, \tag{1} \]

where \( TV \) is the tangent bundle on \( V \). The concept of nonholonomic manifold is used for a manifold enabled with a nonholonomic distribution. In this work, this refers to a Lorentz spacetime \( V := (V, N) \) enabled with a N-connection structure of type \((1)\). In local coordinates, \( N = N_i^a(u)dx^i \otimes \partial_a \), where \( N_i^a \) are N-connection coefficients. Any set \( \{ N_i^a \} \) defines subclasses of N-linear (co) frames which allows N-adapted diadic decompositions of geometric and physical objects.

On any nonholonomic manifold \( V \), we can consider covariant derivatives determined by affine (linear) connections which are, or not, adapted to a N-connection structure. A distinguished connection, \( d\)-connection, is a linear connection \( D = (hD, vD) \) which preserves under parallel transport a h-v-decompositions \((1)\). For any \( d\)-connection \( D \), we can define and compute in standard form the \( d\)-torsion, \( T \), the nonmetricity, \( Q \), and the \( d\)-curvature, \( R \), tensors

\[ T(X, Y) := D_X Y - D_Y X - [X, Y], Q(X) := D_X g, R(X, Y) := D_X D_Y - D_Y D_X - D_{[X,Y]} \]

We can parameterize the local coordinates in the form \( u^a = (x^i, y^a) \), (in brief, \( u = (x, y) \)), where indices respectively take values \( i, j, ... = 1, 2 \) and \( a, b, ... = 3, 4 \), considering that \( u^4 = y^4 = t \) is the time like coordinate. The Einstein convention on summation on "up-low" repeating indices will be applied if contrary will not be stated for some special cases. We use boldface symbols for spaces and geometric objects adapted to a N-connection splitting.

Such N-adapted local bases, \( e_\nu = (e_i, e_a) \), and cobases, \( e^\mu = (e^i, e^a) \), are defined by formulas

\[ e_i = \partial/\partial x^i - N_i^a(u)\partial/\partial y^a, \quad e_a = \partial_a, \quad e^i = dx^i, \quad e^a = dy^a + N_i^a(u)dx^i \]

and their arbitrary frame/coordinate transforms. The term nonholonomic used for a Lorentz manifold \( V \) comes from the fact that a basis (tetrad, equivalently, vierbeind) \( e_\nu = (e_i, e_a) \) satisfies certain relations \( [e_a, e_\beta] = e_\alpha e_{\beta} - e_\beta e_{\alpha} = W_{a\nu}^\beta e_\gamma \), with nontrivial anholonomy coefficients \( W_{\alpha\nu}^\beta = \partial_\alpha N^\beta_\nu - N^\beta_{\alpha\gamma} N^{\gamma}_\nu \). Holonomic (integrable) configurations are obtained if and only if \( W_{a\nu}^\beta = 0 \).

In general, a linear connection \( D \) is not adapted to a prescribed N-connection structure, i.e., it is not a \( d\)-connection. In such a case, one should be not used a boldface symbols for respective geometric objects determined by \( D \).
where \( X \) and \( Y \) are vector fields (i.e. d-vectors) on \( TV \).

Any metric tensor \( g = (hg, vg) \), on a nonholonomic \( V \) can be written as a distinguished tensor, d–tensor (d–metric), with respective splitting into h- and v-indices,

\[
g = g_\alpha(u)e^\alpha \otimes e^\beta = g_i(x)dx^i \otimes dx^i + g_\alpha(x, y)e^\alpha \otimes e^\alpha. \tag{2}
\]

With respect to a dual local coordinate basis \( du^\alpha \) the same metric field is expressed

\[
g = g_{\alpha\beta}du^\alpha \otimes du^\beta, \quad \text{where} \quad g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N^a_iN^b_jg_{ab} & N^a_jg_{ae} \\ N^b_ig_{be} & g_{ab} \end{bmatrix}.
\]

Using frame transforms (in general, not N–adapted), we can transform any metric into a d–metric\(^2\) an off-diagonal form with N–coefficients. For nontrivial anholonomy coefficients, such a metric is generic off-diagonal.

For our geometric constructions, there are two important linear connections determined by the same metric structure:

\[
\begin{align*}
\nabla : & \quad \nabla g = 0; \quad \nabla T = 0, \quad \text{the Levi–Civita, LC, connection;} \\
D : & \quad D g = 0; \quad hT = 0, \quad vT = 0. \quad \text{the canonical d–connection.} \tag{3}
\end{align*}
\]

The LC–connection \( \nabla \) can be defined uniquely by a metric \( g \) without any N–connection structure but \( \nabla \) can be distorted always to a necessary type d–connection allowing a general decoupling and integrability of certain important physically important systems of nonlinear PDEs. In our previous works \([11, 12, 13]\), we used a "hat" symbol (like \( \hat{D} \)) for the canonical d–connection in (3). In this paper, we shall work only with \( \nabla \) and \( D \) and omit "hats" on respective geometric objects. We note that all constructions performed for \( \nabla \) and \( D \) are related by a distortion relation, \( D[g, N] = \nabla[g, N] + Z[g, N] \), where \( Z \) is the distortion tensor determined in standard algebraic form by the torsion tensor \( T \); all values are completely defined by the metric tensor \( g \) adapted to \( N \).

The Ricci tensors of \( D \) and \( \nabla \) are defined and computed in standard forms for different linear connection structures but defined by the same metric tensor by contracting respective indices. We denote them, respectively, \( Ric = \{R_{\beta\gamma} := R^\gamma_{\alpha\beta\gamma}\} \) and \( Ric = \{R_{\beta\gamma} := R^\gamma_{\alpha\beta\gamma}\} \). Any (pseudo) Riemannian geometry can be equivalently described by both geometric data \( (g, \nabla) \) and \( (g, N, D) \), when the canonical distortion relations \( R = \nabla R + \nabla Z \) and \( Ric = Ric + Zic \), with respective distortion d-tensors \( \nabla Z \) and \( Zic \), are computed for the canonical distortion relations \( D = \nabla + Z \), see details in \([6, 6', 11, 12, 13]\) (in those works, there are used different systems of notations).

Using N–adapted coefficients of the canonical Ricci d-tensor,

\[
R_{\alpha\beta} = \{R_{ij} := R^k_{ijk}, R_{ia} := -R_k^{ik}a, R_{ai} := R_{abi}b, R_{ab} := R_{abc}c\}, \tag{4}
\]

we can compute the scalar of canonical d–curvature, \( R := g^{\alpha\beta}R_{\alpha\beta} \equiv g^{ij}R_{ij} + g^{ab}R_{ab} \). This geometric object is different from the scalar curvature of the LC-connection, \( R := g^{\alpha\beta}R_{\alpha\beta} \).

\(^4\)We can compute in N–adapted form the coefficients of any d–connection \( D = \{T^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^b_{ic})\} \). The coefficients of torsion, nonmetricity and curvature d–tensors are parameterized by \( h \)- and \( v \)-indices, respectively, \( T = \{T^\gamma_{\alpha\beta} = \{T^i_{jk}, T^a_{ja}, T^j_{ai}, T^b_{hi}, T^h_{bi}\}\} \), \( Q = \{Q^\gamma_{\alpha\beta}\} \), \( R = \{R^\gamma_{\beta\gamma\delta} = \{R^i_{hjk}, R^a_{hjk}, R^j_{hja}, R^b_{hja}, R^h_{hba}, R^e_{bec}\}\} \), when the coefficients formulas for such values determined by using \( T^\gamma_{\alpha\beta} \) and their partial derivatives.

\(^5\)The values \( hT \) and \( vT \) are respective torsion components which vanish on conventional h- and v–subspaces, but there are nontrivial components \( hvT \) defined by certain anholonomy (equivalently, nonholonomic/ non-integrable) relations. Such a d-torsion is induced by nonholonomic configurations.
Using $\nabla$, the Einstein equations in GR are written in standard form,

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \kappa mT_{\alpha\beta}. \tag{5}$$

In these formulas, $mT_{\alpha\beta}$ is the energy–momentum tensor of matter fields $A\phi$ determined by a general Lagrangian $m\mathcal{L}(g, \nabla, A\phi)$, where $\kappa$ is the gravitational coupling constant for GR.

We can define nonholonomic gravitational field equations using the Ricci d-tensor (4) for a canonical d-connection $\mathbf{D}$

$$R_{\alpha\beta} = \Upsilon_{\alpha\beta}. \tag{6}$$

Such equations are equivalent to (5) if there are imposed additional nonholonomic constraints, or found some smooth limits, for extracting LC–configurations, $\mathbf{D}|\hat{\tau}=0 = \nabla$, for instance, of type $T^\gamma_{\alpha\beta} = 0. \tag{7}$

In (6), a matter fields source $\Upsilon_{\mu\nu}$ can be constructed using a N–adapted variational calculus for $m\mathcal{L}(g, \hat{\mathbf{D}}, A\phi)$, when $\Upsilon_{\mu\nu} = \kappa(mT_{\mu\nu} - \frac{1}{2}g_{\mu\nu} mT) \rightarrow \kappa(mT_{\mu\nu} - \frac{1}{2}g_{\mu\nu} mT) \text{ for [coefficients of } \mathbf{D}] \rightarrow \text{[coefficients of } \nabla]\text{ even, in general, } \mathbf{D} \neq \nabla.$ In such formulas, we consider $mT = g_{\mu\nu} mT_{\mu\nu}$ for

$$mT_{\alpha\beta} := -\frac{2}{\sqrt{|g_{\mu\nu}|}} \frac{\delta(\sqrt{|g_{\mu\nu}|} \ m\mathcal{L})}{\delta g^{\alpha\beta}}. \tag{8}$$

We note that any (pseudo) Riemannian geometry and gravity theory, and various metric-affine modifications (for instance, $F(R)$-modified theories [74, 69]), can be formulated equivalently using geometric data $\mathbf{(g, } \nabla)$ and/or $\mathbf{(g, D)}$. There is an important motivation to use nonholonomic variables of type $\mathbf{(g, D)}$ because that they allow to decouple and integrate in general form various modified and standard Einstein equation. Such solutions can be with generic off-diagonal metrics and coefficients depending on all spacetime coordinates [60, 61, 62, 63]. A recent review of the so-called anholonomic frame deformation method, AFDM, of constructing exact solutions in GR and MGTs, geometric flow theory, and applications in modern cosmology and astrophysics can be found in [69]. In this work, we shall develop the AFDM for constructing exact solutions in entropic geometric flow and gravity theories.

2.2 Nonholonomic 3+1 splitting adapted to 2+2 structures

We outline some basic concepts on the geometry of 3+1 foliations of a nonholonomic Lorentzian manifold $\mathbf{(V, g, N)}$ of signature $(+++−)$ into a family of non-intersecting space like 3-d hypersurfaces $\Xi_t$ parameterized by a "time function", $t(u^a)$, stated as a scalar field as described as follows. Such spacetime decompositions are useful for elaborating various thermodynamic, locally anisotropic kinetic [77] and geometric evolution or hydrodynamic flow models [78] when a conventional splitting into time and space like coordinates is necessary. This allows definition of physical important values (for instance, entropy, effective energy etc.) and deriving fundamental geometric evolution equations. In our approach, we generalize the well–known geometric 3+1 formalism in GR (see, for instance, [78]) to the case of nonholonomic manifolds [11, 12, 13].

For a 3–d manifold $\Xi$, we consider an one-to-one image to a hypersurface $\Xi = \zeta(\Xi) \subset \mathbf{V}$ constructed as an homeomorphism with both continuous maps $\zeta$ and $\zeta^{-1}$, when $\Xi$ does not intersect
Let "up" or "low" labels by a vertical bar "\", " will be used in order to emphasize that certain geometric objects refer to 3–d manifolds / hypersurfaces. Such a 3–d space is supposed to be locally defined as a set of points for which a scalar field \( t \) on \( V \) is constant (for instance, i.e. \( t(p) = 0 \) for any point \( p \in \Xi \)). It is assumed also that \( t \) spans the real line \( \mathbb{R} \) and that any \( \Xi \) is a connected submanifold of \( V \) with the topology of \( \mathbb{R}^3 \).

We can also use the push–forward mapping \( \zeta_* : T_u \Xi \longrightarrow T_v V \); \( v = (v^i) \longrightarrow \zeta_* v = (v^i, 0) \) which allows us to transport geometric objects from \( \Xi \) to \( \tilde{\Xi} \), and inversely. In dual form, the pull–back mapping is defined by actions of type

\[
\zeta^* : T^*_v V \longrightarrow T^*_u \Xi,
\]

\[
\tilde{\zeta}^* \tilde{\mathbf{A}} : T_u \Xi \longrightarrow \mathbb{R}; \quad \tilde{\mathbf{A}} \longrightarrow < \tilde{\zeta}^*, \zeta^*, \mathbf{A} >,
\]

for \( \langle \ldots \rangle \) denoting the scalar product for geometric objects and maps \( T^*_v V \ni \tilde{\zeta} \mathbf{A} = (A_1, A_4) \longrightarrow \zeta^* \tilde{\zeta} \mathbf{A} = (A_i) \in T^*_u \Xi. \) For simplicity, we identify \( \Xi \) and \( \Xi = \zeta( \Xi) \) and write the formulas for a d–vector \( \mathbf{v} \) instead of \( \zeta( \mathbf{v} ) \). The same maps and objects are labelled in non-boldface forms for holonomic configurations.

An induced N-adapted 3-d hypersurface metric, i.e. the first fundamental form (the induced 3–metric) on \( \Xi \) is defined by

\[
\mathbf{q} := \zeta^* \mathbf{g},
\]

\[
\forall (1^a, 2^a) \in T_u \Xi \times T_u \Xi, \quad 1^a \cdot 2^a = \mathbf{g}(1^a, 2^a) = \mathbf{q}(1^a, 2^a).
\]

The are following the types of induced 3–metric (for a nontrivial N–connection, this can be represented as an induced d–metric)

\[
\begin{cases}
\text{spacelike } \Xi, & \mathbf{q} \text{ is positive definite with signature } (+,+,+); \\
\text{timelike } \Xi, & \mathbf{q} \text{ is Lorentzian with signature } (+,+,+); \\
\text{null } \Xi, & \mathbf{q} \text{ is degenerate with signature } (+,+,0).
\end{cases}
\]

We shall work with continuous sets of space like hypersurfaces \( \Xi_t, t \in \mathbb{R} \), covering some finite, or infinite, regions on a nonholonomic 4-d Lorentz manifolds \( V \). To study geometric flows and related thermodynamic models we shall consider only spacelike hypersurfaces \( \Xi \) (chosen to be closed and compact if necessary) endowed with Riemannian 3–metric \( \mathbf{q} \) if other conditions will be not stated.

Let us define the concept of unite normal d–vector, \( \mathbf{n} \), to a \( \Xi \). We consider a scalar field \( t(u^\alpha) \) on an open region \( U \subset V \) such as the level surface is identified to \( \Xi \). First, we use in N-adapted form the gradient 1–form \( dt \) and it dual d-vector \( \overline{\mathbf{e}} t = \{ e^\mu = g^{\mu \nu} e_\nu, t = g^{\mu \nu} (dt)_\nu \}. \) For any d–vector \( \mathbf{v} \) being tangent to \( \Xi \), the conditions \( dt, \mathbf{v} > 0 \) are satisfied. In this case, \( \overline{\mathbf{e}} t \) allows to define the unique direction normal to a not null hypersurface \( \Xi \). We can normalize such a d–vector and define

\[
\mathbf{n} := \pm \overline{\mathbf{e}} t / |\overline{\mathbf{e}} t|, \quad \begin{cases}
\mathbf{n} \cdot \mathbf{n} = -1, & \text{for spacelike } \Xi; \\
\mathbf{n} \cdot \mathbf{n} = 1, & \text{for timelike } \Xi.
\end{cases}
\]

We can label local coordinates for a 3+1 splitting in \( u^\alpha = (x^l, t) \), where indices \( \alpha, \beta, \ldots = 1, 2, 3, 4 \) and \( i, j, \ldots = 1, 2, 3 \) are related to a 2+2 splitting as in previous subsection (in brief, we shall write \( u = (\tilde{u}, t) \)). The continuous maps \( \zeta \) can be parameterized to "carry along" curves / vectors in \( \Xi \) to curves / vectors in \( V \), for \( \zeta : (x^l) \longrightarrow (x^l, 0) \). This way, it is possible to define and relate respective local bases \( \partial_i := \partial / \partial x^i \in T(\Xi) \) and \( \partial_\alpha := \partial / \partial u^\alpha \in TV \). The coefficients of 3–vectors and 4–vectors are expressed correspondingly, \( \mathbf{a} = a^i \partial_i \) and \( \mathbf{a} = a^\alpha \partial_\alpha \) (we shall use also capital letters, for instance, \( \mathbf{A} = A^i \partial_i \) and \( \mathbf{A} = A^\alpha \partial_\alpha \)). Similar formulas are considered for dual forms to vectors, 1–forms, when the dual bases \( dx^i \in T^*(\Xi) \) and \( du^\alpha \in T^* V \). The 1–forms will be parameterized for respective 3 and 4 dimensions, \( \mathbf{A} = A_i dx^i \) and \( \tilde{\mathbf{A}} = A_\alpha du^\alpha \). We shall omit the left/up label by a tilde ~ (writing \( \mathbf{A} \) and \( \mathbf{A} \)) if that will not result in ambiguities.
Using the unit normal vector to hypersurfaces, \( \mathbf{n}_\alpha \propto \partial_\alpha t \), with \( \partial_\alpha := \partial/\partial u^\alpha \), one can be constructed a future–directed time–like vector field. The value \( t \) can be used as a parameter for a congruence of curves \( \chi(t) \subset \mathbf{V} \) intersecting \( \Xi_t \) and when the vector \( \mathbf{t}^\alpha := du^\alpha/dt \) is tangent to the curves and \( \mathbf{t}^\alpha \partial_\alpha t = 1 \). We can consider arbitrary frames and systems of coordinates \( u^\alpha = u^\alpha(x^3, t) \) but always there is the possibility to define a vector \( \mathbf{t}^\alpha := \partial u^\alpha/\partial x^3 \) and the (tangent) vectors \( e_\alpha := \partial u^\alpha/\partial x^3 \) and the Lie derivative along \( \mathbf{t}^\alpha \) results in \( \mathcal{L}_{\mathbf{t}e_\alpha} = 0 \).

It should be noted that any 2+2 splitting by a nonholonomic distribution \( \mathbf{N} \) induces a N–connection structure for a hypersurface \( \Xi \), i.e. an induced N-connections \( \mathbf{N} : T\Xi = h \Xi \oplus v \Xi \). Using the coefficients of such an induced N-connection, any induced 3–metric tensor \( \mathbf{q} \) can be written in N–adapted frames as a d–tensor (d–metric) in the form

\[
\mathbf{q} = (h\mathbf{q}, v\mathbf{q}) = q_i(u)e^i \otimes e^3 = q_i(x^k)dx^i \otimes dx^3 + q_3(x^k, y^3) \cdot e^3 \otimes e^3
\]

for \( e^3 = du^3 + N_i^3(u)dx^i \),

where \( N_i^3(u) \) can be identified with \( N_i^3(u) \) choosing common frame and coordinate systems for \( \Xi \subset \mathbf{V} \). We can extend naturally such a 3-d metric \( \mathbf{q} \) to a 4-d d–metric \( \mathbf{g} \) re-parameterized in a form adapted both to 2+2 and 3+1 nonholonomic splitting,

\[
\mathbf{g} = (hg, vg) = \hat{y}_{ij}e^i \otimes e^j + g_4e^4 \otimes e^4 = q_i(u)e^i \otimes e^3 - N^2e^4 \otimes e^4
\]

\[
e^3 = du^3 + N_i^3(u)dx^i, \quad e^4 = \delta t = dt + N_i(u)dx^i.
\]

For \( \mathbf{g} \) \([9]\), the lapse function \( \hat{N}(u) > 0 \) is defined as a positive scalar field which ensues that the d–vector \( \mathbf{n} \) is a unite one. An "inverse hat" symbol is used in order to distinguish such a symbol from \( N \) is used traditionally in literature on GR \([78]\). Here we note that in another turn, the symbol \( N_i^\alpha \) is used traditionally for the N–connection and this also motivates a new symbol \( \hat{N} \). We can write \( \mathbf{n} = -\hat{N} \mathbf{e} t \) and/or \( \mathbf{n} := -\hat{N} dt \), with \( \hat{N} := 1/\sqrt{\mathbf{e} \cdot \mathbf{e} t} = 1/\sqrt{|<\mathbf{d}t, \mathbf{e} t>|} \), and use also the normal evolution d–vector \( \hat{\mathbf{m}} := \hat{N} \mathbf{n} \) subjected to the condition \( \mathbf{m} \cdot \mathbf{m} = -\hat{N}^2 \). In result, one can introduce the concept of the Lie N-adapted derivative when \( \mathcal{L}_{\mathbf{m}a} \in (h\Xi \oplus v\Xi), \forall a \in T\Xi_t \). In \([9]\), it is considered also the shift 3–vector field \( \hat{N}_i(u) \) defined as a d–vector \( \hat{N}_i(u) \).

For geometric constructions with 3+1 splitting, it is useful to define the unit normal \( \hat{n}_\alpha \) to the hypersurfaces when \( \hat{n}_\alpha = -\hat{N} \partial_\alpha t \) and \( \hat{n}_\alpha e^\alpha = 0 \). In N–adapted form, we can use \( \hat{n}_\alpha \) as a normalized version of \( \mathbf{n} \) considered above and define the decompositions

\[
\mathbf{t}^\alpha = \hat{N}^i e^i + \hat{N}\hat{n}^\alpha \quad \text{and} \quad du^\alpha = e^\alpha dx^3 + \mathbf{t}^\alpha dt = (dx^3 + \hat{N}^i dt)e^i + (\hat{N}dt)\hat{n}^\alpha.
\]

Here we also note that for any quadratic line element \( ds^2 = g_{\alpha\beta} du^\alpha du^\beta \) of a metric tensor \( \mathbf{g} \) there are such frame transforms to parameterizations when \( \hat{y}_{ij} = q_{ij} = g_{\alpha\beta}e^\alpha e^\beta \) is just the induced metric on \( \Xi_t \). In result, the determinants of 4-d and 3-d metrics are computed \( \sqrt{|g|} = \hat{N}\sqrt{|g|} = \hat{N}\sqrt{|q|} \). Using certain coordinates \( (x^3, t) \) being N-adapted on respective hypersurfaces, the time partial derivatives are computed \( \mathcal{L}_{\mathbf{t}q} = \partial_t q = q^\alpha \) and the spacial derivatives are computed \( q_i := e^i, q_\alpha \).

There are two types of induced linear connections completely determined by an induced 3–d hypersurface metric \( \mathbf{q} \).

\[
\mathbf{q} \rightarrow \begin{cases} \nabla & : \nabla \mathbf{q} = 0; \quad \nabla \mathcal{T} = 0, \\
\mathbf{D} & : \quad \mathbf{D} \mathbf{q} = 0; \quad h, \mathcal{T} = 0, \quad v, \mathcal{T} = 0, 
\end{cases}
\]

where \( h, \mathcal{T} \) are LC–connection and canonical d–connection. (10)

Such formulas are related to 4-d similar ones \([3]\). Both linear connections, \( \nabla \) and \( \mathbf{D} \), are subjected also to a distortion relation \( \mathbf{D}[\mathbf{q}, \mathbf{N}] = \nabla[\mathbf{q}] + \mathbf{Z}[\mathcal{T}(\mathbf{q}, \mathbf{N})] \).
Similarly, on 3-d hypersurfaces, there are two classes of trivial or nontrivial intrinsic torsions, nonmetricity and curvature fields for any data \((\Xi, q, N)\). They are defined and computed using corresponding hypersurface linear connections

\[
\begin{align*}
T(a, b) & := \nabla_a Y - \nabla_Y a - [a, Y] = 0, \quad Q(a) := \nabla_a q = 0, \\
R(a, b) & := \nabla_a \nabla_b - \nabla_b \nabla_a - \nabla_{[a, b]},
\end{align*}
\]

and, for canonical d-connections,

\[
\begin{align*}
\mathcal{T}(a, b) & := D_a b - D_Y a - [a, b] = 0, \quad Q(a) := D_a q = 0, \\
\mathcal{R}(a, b) & := D_a D_b - D_b D_a - D_{[a, b]},
\end{align*}
\]

for any \(a, b \in T\Xi\).

For 3-d configurations, we can compute the N–adapted coefficient formulas for nonholonomically induced torsion structure \(T = \{ T^i_{jk} \}\), determined by \(D\), and for the Riemannian tensors \(R = \{ R^i_{jk} \}\) and \(\mathcal{R} = \{ \mathcal{R}^i_{jk} \}\), determined respectively by \(\nabla\) and \(D\). Using 3–d subsect of coefficient formulas, we can compute respective N–adapted hypersurface coefficients of the Ricci d–tensor, \(R_{jk}\), and the Einstein d–tensor, \(\mathcal{E}_{jk}\). Contracting indices, we obtain the Gaussian curvature, \(\mathcal{K} = \mathcal{R}^i_{jk} R^i_{jk}\), and the Gaussian canonical curvature, \(\mathcal{K} = \mathcal{R}^i_{jk} R^i_{jk}\), of \((\Xi, q, N)\). It should be noted that all this types of N-adapted and not N-adapted geometric objects can be defined in abstract form which do not depend on the type of embedding of a nonholonomic 3-d manifold \((\Xi, q, N)\) into a 4-d one \((V, g, N)\).

For 3+1 decompositions, there are another type curvatures describing the (non) holonomic bending of \(\Xi\) and these formulas characterize the type of embedding. Respective N–adapted endomorphisms are stated by definition

\[
\begin{align*}
\varpi : T_p \Xi & \rightarrow T_q \Xi; \quad \text{and} \quad \varpi : T \Xi & \rightarrow T \Xi = h \Xi \oplus v \Xi \\
a & \rightarrow \nabla_a n; \quad \text{a} & \rightarrow D_a n = h \mathcal{D}_a n \oplus v \mathcal{D}_a n.
\end{align*}
\]

These maps are self–adjoint with respect to the induced 3–metric \(q\), when for any \(a, b \in T_p \Xi \times T_q \Xi\), \(a \cdot \varpi (b) = \varpi (a) \cdot b\) and \(a \cdot \varpi (b) = \varpi (a) \cdot b\), where dot means the scalar product on respective hypersurfaces defined by \(q\). Using such \(\varpi\) and \(\varpi\), we construct two second fundamental forms (i.e. corresponding extrinsic curvature d–tensors) of hypersurface \(\Xi\), which are stated by formulas

\[
\begin{align*}
K : \quad T_p \Xi \times T_q \Xi & \rightarrow \mathbb{R} \text{ LC –configurations}, \\
(a, b) & \rightarrow -a \cdot \varpi (b); \\
\hat{K} : \quad (h \Xi \oplus v \Xi) \times (h \Xi \oplus v \Xi) & \rightarrow \mathbb{R} \oplus \mathbb{R} \text{ canonical configurations}, \\
(a, b) & \rightarrow -a \cdot \varpi (b).
\end{align*}
\]

We can consider operators \(K(a, b) = -a \cdot \nabla_b n\) and \(\hat{K}(a, b) = -a \cdot \mathcal{D}_b n\) and compute their respective coefficients \(K_{ij}\) and \(\hat{K}_{ij}\). Any pseudo–Riemanian geometry on a 3-d hypersurface in a nonholonomic \(V\) can be described equivalently in terms of \((\mathcal{K}, \mathcal{K})\), or \((q, K)\), when there are stated certain geometric data \((\Xi, q, N, N^i, N)\) for 4-d configurations. Intuitively, we can work on 3–d spacelike hypersurfaces as in Riemannian geometry but using N-adapted geometric objects. Such 3+1 splitting nonholonomic variables are not convenient for a general decoupling of the geometric flow evolution and/or the gravitational field equations but are very important for elaborating respective entropic and geometric thermodynamic models.

### 2.3 Quasiperiodic space & time QC configurations

Let us consider two examples of space and time quasiperiodic structures defined in a curved spacetime following our works on quasicrystal, QC, models in modern cosmology [70, 69, 67] (alter-
In this formula, one). Such a QC structure can be defined by a generating function

\[ L(\varsigma) = \frac{1}{48}(g^{\alpha\beta}(e_\alpha \varsigma)(e_\beta \varsigma))^2 - \frac{1}{4}g^{\alpha\beta}(e_\alpha \varsigma)(e_\beta \varsigma) - \dot{V}(\varsigma). \]

In this formula, \( \dot{V}(\varsigma) \) is a nonlinear potential and \( e_\alpha \) are \( N \)-adapted partial derivatives. Corresponding \( N \)-adapted variational motion equations are \( \int \left[ \frac{1}{2}g^{\alpha\beta}(e_\alpha \varsigma)(e_\beta \varsigma) - 1 \right](D^\gamma D_\gamma \varsigma) = 2\dot{\varsigma}^i. \) The field \( \varsigma \) defines a 1-d time QC structure, 1-TQC, if it is a solution of these motion equations.

### 2.3.2 3-d QC structures on curved spaces

QC structures and analogous dynamic phase field crystal models can be elaborated as flow evolution theories on real parameter \( \tau \) (in next Section, this parameter will be identified with a geometric flows one). Such a QC structure can be defined by a generating function \( \mathcal{F} = \mathcal{F}(x^i, y^3, \tau) \) subjected to the condition that it is a solution of an evolution equation with conserved dynamics,

\[ \frac{\partial \mathcal{F}}{\partial \tau} = \Delta \left( \frac{\delta \mathcal{F}}{\delta \tilde{b}} \right) = - \Delta(\Theta \tilde{b} + Q\tilde{b}^2 - \tilde{b}^3). \]

Such evolution is considered on 3-d spacelike hypersurface \( \Xi_t \) when the canonically nonholonomically deformed hypersurface Laplace operator \( \tilde{\Delta} := (\nabla^2 - \varsigma \cdot \gamma) \partial_{\varsigma} \gamma \) constructed in 3-d Riemannian geometry, see previous subsection. The functional \( \mathcal{F} \) in (12) is characterized by an effective free energy

\[ \mathcal{F}[\tilde{q}] = \int \left[ -\frac{1}{2} \tilde{\Theta} \tilde{b} - \frac{Q\tilde{b}^3}{3} + \frac{1}{4} \tilde{b} \right] \sqrt{g} dx^1 dx^2 \delta y^3, \]

where \( q = \text{det} |q_{ij}|, \delta y^3 = e^3 \) and the operators \( \Theta \) and \( Q \) are defined and explained in [69, 70]. Such nonlinear interactions are stabilized by the cubic term with \( Q \) and the second order resonant interactions are varied by setting observable values of such constants (they are different for cosmological models, in astrophysics or condensed matter physics). The average value \( \langle \tilde{b} \rangle \) is conserved for any fixed time variable \( t \) and/or evolution parameter \( \tau_0 \). We can fix \( \langle \tilde{b} \rangle |_{\tau=\tau_0} = 0 \) when other values are accommodated by redefining values \( \Theta \) and \( Q \).

---

8For non-relativistic limits with \( g_{\alpha\beta} = [1, 1, 1, -1] \) and \( \varsigma \to \varsigma(t), \dot{L} \to \frac{1}{12}(\varsigma^\star)^4 - \frac{1}{2}(\varsigma^\star)^2 - \dot{V}(\varsigma) \), which leads to an effective energy \( E = \frac{1}{4}(\varsigma^\star)^2 - 1)^2 + \dot{V}(\varsigma) - \frac{1}{4} \) and motion equations \([[(\varsigma^\star)^2 - 1]\varsigma^\star^\star = -\frac{\partial \dot{V}}{\partial \varsigma} \] introduced in [83]. The Lagrange density (11) provides a generalization for 1-TQCs modeled on a curved spacetime which can be also modeled in entropic gravity theories.
2.4 Distributions defining spacetime elastic configurations

In letter [34], we shown that models of entropic gravity can be derived from nonholonomic modifications of the W-functional when the (modified) Einstein equations are equivalent to certain nonholonomic Ricci soliton equations. Here, we shall study the conditions when entropic elastic scenarios can be modelled as nonholonomic Ricci solitons in subsection 3.3. We shall consider certain examples of nonholonomic distributions and related Lagrange densities on a Lorentz manifold $V$ which are used in entropic gravity theories [8, 9, 71, 34]. Using such geometric constructions, we shall elaborate in next section on elastic flow evolution models and their self-similar nonholonomic Ricci soliton configurations. There are three important values:

$$\varepsilon_{\alpha\beta} = D_\alpha u_\beta - D_\beta u_\alpha - \text{the elastic strain tensor}; \phi = u/\sqrt{|\Lambda|} - \text{a dimensionless scalar};$$

$$\chi = \alpha(D_\nu u^\nu)(D_\rho u_\rho) + \beta(D_\rho u_\nu)(D^\mu u^\nu) + \gamma(D_\mu u_\nu)(D^\nu u^\mu) - \text{a general kinetic term for } u^\mu.$$

These geometric/physical objects are determined by a conventional displacement vector field $u^\alpha$, cosmological constant $\Lambda$ and some constants $\alpha, \beta, \gamma$; there are used short hand notations: $u := \sqrt{|u_\alpha u^\alpha|}$, $\varepsilon = \varepsilon^\beta_{\beta}$, and $n^\alpha := u^\alpha/u$.

On $V$, there are considered nonholonomic distributions for corresponding total, effective gravitational, usual matter, interaction and kinetic terms of Lagrangians postulated in the form

$$\text{tot} \mathcal{L} = gL + mL + \text{int} \mathcal{L} + \chi \mathcal{L}, \text{ for}$$

$$gL = M^2 F(sR), \text{ int} \mathcal{L} = -\sqrt{|\Lambda|} mT_{\mu\nu} u^\nu u^\mu /u, \ \chi \mathcal{L} = M^2 |\Lambda| (\chi^{3/2} + |\Lambda| |u[\varsigma, \bar{B}]|^2 u).$$

In these formulas, the Plank gravitational mass is denoted $M_P$ and the gravitational Lagrangian $gL$ is taken as in modified gravity [74, 75, 76, 69]. We can fix $z = 1$ if we search for compatibility with [71], or $z = 2$ if we search for a limit to the standard de Sitter space solution [72, 73] (as we use in [34]). To model STQC structures in entropic gravity and related geometric flow theories we can consider that the displacement vector field $u^\alpha[\varsigma, \bar{B}]$ is a functional of functions $\varsigma, \bar{B}$ subjected to certain conditions of type (11) and/or (12) [in principle, we can consider functionals for pattern forming, nonlinear wave soliton structures, fractional and diffusion processes etc.].

The energy-momentum tensors considered in above formulas and/or derived from respective Lagrangians in (13) and computed using variations on $g^{\mu\nu}$ similarly to $mT_{\mu\nu}$ [5] (in N-adapted form, details of such computations are provided in [64, 42, 65, 69]). For the full system, the effective energy-momentum tensor is computed

$$\text{tot} T_{\mu\nu} = (\frac{\partial F}{\partial sR})^{-1} mT_{\mu\nu} + F T_{\mu\nu} + \text{int} T_{\mu\nu} + \chi T_{\mu\nu}, \text{ where}$$

$$\text{int} T_{\gamma\beta} = \left[ \frac{1}{2} \left( F - \frac{\partial F}{\partial sR} \right) g_{\beta\gamma} - (g_{\beta\gamma} D_\alpha D_\alpha - D_\beta D_\gamma) \frac{\partial F}{\partial sR} \right] \frac{\partial F}{\partial sR}.\]\n
We can model "pure" elastic spacetime modifications of the Einstein gravity if we fix $F(sR) = sR$ and consider restrictions to the Levi-Civita connection $D = \nabla$. For such conditions, we obtain respective formulas for $\text{int} T_{\mu\nu}$ and $\chi T_{\mu\nu}$ which are similar to formulas (10)-(13) in [72].

In this work, the generalized (effective) source for MGT [5] splits into four components,

$$\text{tot} \Upsilon_{\mu\nu} := \chi \left( \text{tot} T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \text{tot} T \right) = (\frac{\partial F}{\partial sR})^{-1} m \Upsilon_{\mu\nu} + F \Upsilon_{\mu\nu} + \text{int} \Upsilon_{\mu\nu} + \chi \Upsilon_{\mu\nu}, \quad (14)$$

---

9 We use a system of notations which is similar (but without "hats") to [69, 37, 41, 42, 43]; such notations are different from those used in [71, 72, 73].
where \( \kappa \) is determined in standard form by the Newton gravitational constant \( G \). We need additional terms and parameterizations in order to describe structure formation in modern cosmology and to model dark energy and dark matter properties.

3 Relativistic geometric flows and modified entropic gravity

Grigory Perelman’s proof of the Poincaré conjecture \([48, 49, 50]\) on geometric flow evolution of 3-d Riemannian metrics \([45, 46, 47]\) provided fundamental results in geometric analysis and topology. There were also studied possible applications in modern mathematical and particle physics. We cite \([51, 52, 53]\) for reviews of rigorous mathematical results. For early applications, we refer to D. Friedan works \([54, 55, 56]\) (he considered geometric evolution equations related to renorm group equations before the Hamilton–Poincaré theory was elaborated). Further developments and applications were performed in \([57, 58, 59]\) and a series of works \([35, 36, 37, 38, 39, 40, 41, 42, 43, 44]\), see also references therein. In those works on theories of nonholonomic/ noncommutative/ supersymmetrics, fractional, diffusion etc. geometric flows, there were studied statistical and thermodynamic evolution models derived from certain Lyapunov type functionals. Such F- and W-entropy functionals are called in literature the Perelman functionals. The W-entropy has properties of "minus entropy" of statistical thermodynamics systems. In \([34]\), we elaborate on the idea that such a W-entropy can be considered for formulating E. Verlinde type entropic gravity theories \([8, 9]\). We study self-similar configurations of nonholonomic geometric flows resulting in entropic Ricci solitons (see subsection 3.3).

We note that G. Perelman suggested in his first preprint \([34]\) that the geometric flow theory may have certain implications in black hole physics and string theory. Nevertheless, the original theory of Ricci evolution flows was formulated in a non–relativistic form. To consider further generalizations and applications in modern physics and cosmology we elaborated on relativistic models of geometric flow theories \([41, 42, 43]\). Such constructions can be re-defined for nonholonomic configurations modeling elastic and quasiperiodic spacetime structure as in subsection 2.4 and allows rigorous geometric motivations for emergent entropic theories of type \([8, 9, 71, 34]\).

The goal of this section is to study generalizations of the hypersurface 3-d and relativistic 4-d F- and W-functionals and elaborate on respective geometric evolution scenarios supporting the E. Verlinde entropic gravity conjecture \([8, 9]\). Such constructions can be considered in the framework of a modified relativistic variant of the Poincaré–Thurston conjecture which was proven only for certain classes of Riemannian and Kähler metrics, see details in \([48, 49, 50, 51, 52, 53]\). For relativistic configurations, we can only elaborate on geometric evolution of certain 3-d hypersurface configurations subjected to the conditions that such 3-metrics can be extended to certain classes of 4-d metric and (non) linear connection structures derived as exact/ parametric solutions of certain nonholonomic/ entropic geometric flow equations. There will be considered also generalizations of the Hamilton equations for the entropic flow theory. The conditions for generating entropic modified Einstein equations as nonholonomic Ricci solitons will be also analysed. In Appendix A we summarize some important results on 3-d geometric flows and extensions on time like parameters for evolution on 4-d nonholonomic Lorentz manifolds. We emphasize that in the main part of this article there are studied relativistic geometric flow models with a temperature like evolution parameter.

3.1 Modified spacetime and hypersurface Perelman’s functionals

Let us consider families of nonholonomic 4-d manifolds \( V(\tau) \) parameterized by a positive parameter \( \tau, 0 \leq \tau \leq \tau_0 \) (it can be considered as a temperature like parameter) and enabled with a double nonholonomic 2+2 and 3+1 splitting \([41, 42, 43]\). Such manifolds are determined by respective families
of metrics $g(\tau) = g(\tau, u)$ and $N$-connections $N(\tau) = N(\tau, u)$ (we shall write only the parametric dependence if that will not result in ambiguities) for which canonically corresponding $d$-connection structures can be constructed $D(\tau) = D(\tau, u)$. We also suppose that on $V(\tau)$ there are defined corresponding families of Lagrange densities $\mathcal{L}(\tau)$, for gravitational fields in a MGT or GR, and $\mathcal{L}(\tau)$, as total Lagrangians for effective and matter fields [13]. For a double $2+2$ and $3+1$ splitting, we can consider local coordinates labeled as $u^a = (x^i, y^a) = (x^i, u^i = t)$ for $i, j, k, \ldots = 1, 2; a, b, c, \ldots = 3, 4$; and $i, j, k = 1, 2, 3$. The nonholonomic distributions for $N$-connections can be parameterized always in such forms that any open region $U \subset V$ is covered by a family of $3$-d spacelike hypersurfaces $\Xi_t$ parameterized by a time like parameter $t$.

### 3.1.1 Generalized Perelman functionals for entropic geometric flows and MGTs

For this class of theories, we postulate the modified Perelman’s functionals in the form

$$F(\tau) = \int_{t_1}^{t_2} \int_{\Xi_t} e^{-f} \sqrt{|g|} d^4 u \left[ F(\mathcal{S} R) + \mathcal{L} + |Df|^2 \right]$$

and

$$\mathcal{W}(\tau) = \int_{t_1}^{t_2} \int_{\Xi_t} (4\pi\tau)^{-3} e^{-f} \sqrt{|g|} d^4 u \left[ \tau \left( F(\mathcal{S} R) + \mathcal{L} + |hDf| + |vDf| \right)^2 + f - 8 \right].$$

The condition $\int_{t_1}^{t_2} \int_{\Xi_t} (4\pi\tau)^{-3} e^{-f} \sqrt{|g|} d^4 u = 1$ is imposed on the normalizing function $f(\tau, u)$.

Let us explain and motivate the difference of (15) and (16), introduced in the first partner work [34], from the original Grisha Perelman F- and W-functionals [48] postulated for the Ricci flows of $3$-d Riemannian metrics, see details in monographs [51], [52], [53]. In this work, we study geometric entropic flows of canonical geometric data $(g(\tau), N(\tau), D(\tau))$ for nonholonomic Lorentz manifolds and various generalizations for MGTs following the program elaborated in [69], [37], [42], [43], where possible connections to emergent gravity were not analyzed. In formulas (15) and (16), we consider the gravitational Lagrangian $\mathcal{S} \mathcal{L} = F(\mathcal{S} R)$ as a functional of the scalar curvature for $D$, or $\mathcal{S} \mathcal{L} = R[\nabla]$ for considering as particular cases models of geometric evolution of exact solutions in GR. The key difference from previous works is that in such relativistic functionals the term $\mathcal{L}$ is introduced, which is responsible for geometric evolution of configurations with elasticity and quasiperiodicity. Here we note that such an (effective) matter is not contained in (A.11) and/or (A.12). Those functionals can be generalized on a temperature like parameter $\tau$ and used as certain alternative geometric functionals, for instance, for W-entropy. Nevertheless, only nonholonomic elastic quasiperiodic functionals of type (15) and (16) result for self-similar configurations (see next subsections) in entropic gravity equations of E. Verlinde type [8], [9], [71], [34] and/or with quasiperiodic structure [70], [69], [67].

In this and partner [34] papers, we work with generalized geometric flow and entropy functionals determined by $F(\mathcal{S} R) + \mathcal{L}$ and $D$, respectively, instead of the Riemannian values $R$ and $\nabla$ used in the former mathematical works. In our nonholonomic approaches, above F- and W–functionals characterize relativistic thermodynamic models with analogous nonlinear hydrodynamic flows of families of entropic values, metrics and generalized connections, encoding interactions of gravitational and matter fields as it is motivated in [11], [42], [13]. In general, it is possible to work with any class of normalizing functions $f(\tau, u)$ which can be redefined in order to include geometric and matter Lagrange terms and certain constant values and parameters. In many cases, such a function is chosen in a non–explicit form. This allows us to study non–normalized geometric flows but with nonholonomic constraints. For such conditions, there found various general decoupling and integration properties of respective physically important systems of nonlinear PDEs. In result, generic off-diagonal solutions can be constructed in explicit form as in [34], [69], [37], [11], [42], [13], but with entropic and quasiperiodic modifications. The existence of such solutions validates our nonholonomic geometric flow entropic
3.1.2 Nonholonomic 3-d space like hypersurface F- and W-functionals

We can redefine and compute relativistic entropies (15) and (16) for any 3+1 splitting with 3-d closed hypersurface fibrations $\hat{\Sigma}_t$ as we described above in section 2.2.

Let us denote by $\mathcal{D} = D|_{\hat{\Sigma}_t}$ the canonical $d$–connection $D$ defined on a 3-d hypersurface $\hat{\Sigma}_t$, when all values depend on a temperature like parameter $\tau(\tau')$ with possible scale re-definitions for another parameter $\tau'$ etc. We define also $^sR := ^sR|_{\hat{\Sigma}_t}$. Using $q_i(\tau) = [q_i(\tau), q_3(\tau)]$ in a family of d-metrics ($9$), the Perelman’s functionals parameterized in $N$–adapted form are constructed in the form:

$$\mathcal{F} = \int_{\hat{\Sigma}_t} e^{-\frac{f}{\mu}} \sqrt{|q_{ij}|} d\hat{x}^3 \left[ (\mathcal{F}(^sR) + ^{tot}\mathcal{L}) + |D^s f|^2 \right], \text{ and}$$

$$\mathcal{W} = \int_{\hat{\Sigma}_t} \mu \sqrt{|q_{ij}|} d\hat{x}^3 \left[ \tau ((\mathcal{F}(^sR) + ^{tot}\mathcal{L}) + |^sD^s f|^2 + |^sD^s f|) + |D^s f|^2 - 6 \right].$$

These functionals are derived respectively from the previous 4-d elastic ones when the values $\mathcal{F}(^sR)$ and $^{tot}\mathcal{L}$ are computed as projections on a 3-d hypersurface for a redefined normalization function $f$. Using frame-coordinate transform and re-definition of the temperature like parameter, we can always chose a necessary type scaling function $f$ which satisfies normalization conditions $\int_{\hat{\Sigma}_t} \sqrt{|q_{ij}|} d\hat{x}^3 = 1$ for $\mu = (4\pi\tau)^{-3} e^{-\frac{f}{\mu}}$. For topological considerations, such normalizations are not important. Nevertheless, such normalisation conditions impose certain nonholonomic constraints on geometric objects which do not allow to solve derived geometric flow evolution equations in explicit form. For applications to entropic gravity and associated thermodynamic models, we can consider $f$ as an undetermined scalar function which can be related to certain conformal transforms or re-parameterizations. In result, we can prove certain general decoupling and integration properties of corresponding systems of nonlinear PDEs. Fixing for certain classes of solutions, we can chose certain integration functions and constants which reproduce/ predict certain experimental and/or observational data. Corresponding values of $f$ depend on chosen systems of reference.

The functionals (17) and (18) transform into standard Perelman functionals [48] for 3-d Riemannian metrics on $\hat{\Sigma}_t$ if $\mathcal{D} \rightarrow \nabla$, $\mathcal{F}(^sR) = ^sR$ and $^{tot}\mathcal{L} = 0$. In order to describe possible contributions on 3-d hypersurfaces of spacetime elasticity and quasiperiodic structure in entropic gravity, it is necessary to analyze physical effects of such nonholonomic deformations.

3.2 Geometric flow equations for modified gravitational & matter fields

Applying a variational procedure for a corresponding F-functional for geometric flows of 3-d Riemannian metrics, G. Perelman [48] provided a proof for R. Hamilton’s equations (A.1). For self-consistent configurations with a fixed flow parameter $\tau_0$ and respective normalization $\dot{\tau} = \text{const}$ in (A.4), one obtains 3-d Ricci soliton equations which are equivalent to the vacuum Einstein equations for $\nabla$ with (effective) cosmological constant $\dot{\tau}/5$. In similar forms to rigorous mathematical proofs in [48, 51, 52, 53] but elaborating on N-adapted variational procedures, for instance, for the functional $\mathcal{F}(\tau)$ (15) with a canonical $D$ used instead of $\nabla$ (see details in [41, 42, 43, 44]), we obtain a system of
nonlinear PDEs generalizing the R. Hamilton equations for entropic and quasiperiodic geometric flow evolution determined by canonical data \( (g = \{g_{\mu\nu} = [g_{ij}, g_{ab}]\}, N = \{N^i\}, D, \ \text{tot} \mathcal{L}) \),

\[
\begin{align*}
\partial_\tau g_{ij} &= -2(R_{ij} - \text{tot} \gamma_{ij}); \\
\partial_\tau g_{ab} &= -2(R_{ab} - \text{tot} \gamma_{ab}); \\
R_{ia} &= R_{ai} = 0; R_{ij} = R_{ji}; R_{ab} = R_{ba}; \\
\partial_\tau f &= -\hat{\Box}_f + |Df|^2 - sR + \text{tot} \gamma^\alpha_{\alpha}.
\end{align*}
\]

In these formulas, \( \hat{\Box}(\tau) = D^\alpha(\tau)D_\alpha(\tau) \) and \( \text{tot} \gamma_{\alpha\beta}(\tau) \) is chosen for geometric flows of (effective) sources of entropic gravity (14) (if we fix any \( \tau = \tau_0 \)). We note that the dependence on a flow parameter \( \tau \) for such (effective) matter sources is determined by certain evolutions of \( g(\tau) \) and \( D(\tau) \).

In such theories, we do not consider nonholonomic deformations and evolution of classical matter fields. For instance, we do not consider geometric flow evolution equations for the electromagnetic potential \( A_\alpha(\tau) \) with evolution terms of type \( \partial_\tau A_\alpha \) even such theories were studied in our previous works [35, 36, 37, 38, 39, 40, 41, 42, 43, 44], see references therein.

The conditions \( R_{ia} = 0 \) and \( R_{ai} = 0 \) for the Ricci tensor \( \text{Ric}[D] = \{R_{\alpha\beta} = [R_{ij}, R_{ia}, R_{ai}, R_{ab}]\} \) are necessary if we want to keep the metric \( g(\tau) \) to be symmetric under nonholonomic Ricci flow evolution determined by (19). Geometric flow evolution and nonholonomic gravity of theories with nonsymmetric metrics were studied in [61], see references therein. In principle, we can work with any type normalization function \( f \) which allows a general decoupling and integration of such systems of nonlinear PDEs. Such a normalization depends on frame and coordinate transforms and may encode (effective) cosmological constants, matter sources etc. We note that similar variational and/or geometric methods allows to derive from \( W(\tau) \) (16) certain types nonlinear evolution equations which are equivalent to (19). It is more difficult to solve explicitly such PDEs but a \( W \)-functional allows to elaborate directly on certain classes of thermodynamic models, see section 6.

### 3.3 Entropic gravity and gravitational field equations as Ricci solitons

For self-similar point \( \tau = \tau_0 \) configurations when \( \partial_\tau g_{\mu\nu} = 0 \), with a corresponding choice of the normalizing geometric flow function \( f \), the equations (19) transform into relativistic nonholonomic Ricci soliton equations

\[
R_{ij} = \text{tot} \gamma_{ij}, \ R_{ab} = \text{tot} \gamma_{ab}, \ R_{ia} = R_{ai} = 0
\]

which are equivalent to (modified) Einstein equations in (MGT) GR for corresponding definitions of \( \text{tot} \gamma_{\alpha\beta} \). A class of MG Ts and GR can be formulated as geometric theories of entropic elastic origin which is similar to the idea of emergent gravity put forward by E. Verlinde [8, 9], i.e. in the form (6) with (effective) entropic and quasiperiodic source \( \text{tot} \gamma_{\alpha\beta} \) (14).

We conclude that an emergent gravity model in the E. Verlinde sense [8, 9, 71], can be constructed for Lagrange distributions (13) and respective sources (14) introduced as generating data for the nonholonomic Hamilton equations (19) and respective relativistic Ricci solitons. Such geometric flow evolution theories and their spacetime elastic, quasiperiodic and thermodynamic properties are determined by the generalized \( W \)-entropy (16).

### 4 Decoupling and integrability of entropic flow equations

In this section, we prove that the system of nonlinear PDEs (19) describing spacetime elastic and quasiperiodic flows and entropic gravity theories can be formally integrated in very general forms for
generic off-diagonal metrics and canonical d-connections (in particular, for LC-configurations). The coefficients of geometric objects for such solutions depend on all spacetime coordinates via generating and integration functions and (effective) matter sources. The anholonomic frame deformation method, AFDM, for constructing exact solutions in MGTs and GR is developed for generating new classes of solutions encoding entropic quasiperiodic modifications in $g(\tau)$ (2), $D(\tau)$ (3), and $^{tot}T_{\alpha\beta}(\tau)$ (14). For similar details and mathematical proofs, we refer readers to our previous works [60, 61, 62, 63, 64, 65, 66, 69, 70], on exact solutions in MGTs, and [35, 36, 37, 38, 39, 40, 41, 42, 43, 44], for solutions with nonholonomic entropic Ricci flows, and citations therein.

4.1 Geometric flows with parametric modified Einstein equations

Introducing effective sources, entropic geometric flow equations can written as modified Einstein equations with dependence on a temperature like parameter $\tau$. We show that such systems of nonlinear PDEs can be decoupled in general forms.

4.1.1 Entropic quasiperiodic flow modifications of gravitational field equations

Using nonholonomic frame transforms and tetradic (vierbein) fields, we introduce effective sources which in N–adapted form are parameterized

$$\text{eff} \, \Upsilon_{\mu\nu}(\tau) = e^\nu_{\mu}(\tau) e^\mu_{\nu}(\tau) \left[ \, ^{tot} \Upsilon_{\mu\nu}(\tau) + \frac{1}{2} \partial_{\tau} g_{\mu\nu}(\tau) \right] = \left[ \, \hbar \Upsilon(\tau, x^k) \delta^i_j, \Upsilon(\tau, x^k, y^b) \delta^b_c \right]. \quad (21)$$

Such families of vielbein transforms $e^\nu_{\mu}(\tau) = e^\mu_{\nu}(\tau, u^\gamma)$ and their dual $e^\nu_{\mu}(\tau, u^\gamma)$, when $e^{\mu} = e^\mu_{\nu} \, du^\nu$ can be chosen in the form (A.15) and/or any frame/coordinate transforms of a N-splitting structure (1). In result, the system of nonholonomic entropic R. Hamilton equations (19) can be written in the form (6) but with geometric objects depending additionally on a temperature like parameter $\tau$ and for effective source (21),

$$R_{\alpha\beta}(\tau) = \text{eff} \, \Upsilon_{\alpha\beta}(\tau). \quad (22)$$

We note that such geometric evolution equations are for an undetermined normalization function $f(\tau) = f(\tau, (\tau, u^\gamma))$ which can be defined explicitly for respective classes of exact or parametric solutions. For self-similar point $\tau = \tau_0$ configurations with $\partial_{\tau} g_{\mu\nu}(\tau_0) = 0$, this system of nonlinear PDEs transforms into the nonholonomic entropic Ricci soliton equations (20).

4.1.2 Effective entropic sources for stationary and/or cosmological configurations

The values $\hbar \Upsilon(\tau, x)$ and $\Upsilon(\tau, x, y)$ in (21) can be considered as generating data for (effective) matter sources. Prescribing such data, we impose certain nonholonomic frame constraints on geometric evolution and self-similar configurations of entropic and quasiperiodic structures. This type of $\Upsilon$–generating functions allows formal integrations of the system (22) in certain general forms.

Using frame transforms, the $\tau$-evolution of d–metric $g(\tau)$ (2) can be parameterized for respective spherical symmetric coordinates $u^a = (r, \theta, y^3 = \varphi, t)$ or some cosmological coordinates $(x^k, y^4 = t)$,

$$g_1(\tau) = e^{\psi(\tau, r, \theta)}, \quad g_a(\tau) = \omega(\tau, r, \theta, y^b) h_a(\tau, r, \theta, \varphi), \quad (23)$$

$$N^3_i(\tau) = w_i(\tau, r, \theta, \varphi), \quad N^4_i(\tau) = n_i(\tau, r, \theta, \varphi), \quad \text{for } \omega = 1, \text{stationary configurations ;}$$

$$g_1(\tau) = e^{\psi(\tau, x^k)}, \quad g_a(\tau) = \omega(\tau, x^k, y^b) p_a(\tau, x^k, t), \quad (24)$$

$$N^3_i(\tau) = p_i(\tau, x^k, t), \quad N^4_i(\tau) = p_i(\tau, x^k, t), \quad \text{for } \omega = 1, \text{cosmological configurations}$$
The AFDM results in more simple and explicit (still very general classes) of solutions if we work with nonholonomic configurations possessing at least one Killing symmetry, for instance, on \( \partial_4 = \partial_t \) for stationary solution or on \( \partial_3 = \partial_\varphi \), locally anisotropic solutions.\(^{10}\)

We shall use brief notations of partial derivatives \( \partial_\alpha q = \partial q/\partial x^\alpha \) when a function \( q(x^k, y^a) \),

\[
\begin{align*}
\partial_\alpha q &= q^\alpha = \partial q/\partial x^1, \\
\partial_\beta q &= q^\beta = \partial q/\partial x^2, \\
\partial_\gamma q &= q^\gamma = \partial q/\partial y^3 = \partial q/\partial \varphi = q^\varphi, \\
\partial_q q &= \partial q/\partial t = \partial_q q = q^\alpha ,
\end{align*}
\]

\( \partial^2 q/\partial \varphi^2 = q^{\varphi^2}, \partial^2 q/\partial t^2 = \partial^2 q/\partial t^2 = q^{\alpha^2} \).

For respective Killing symmetries, the effective sources \( \Psi(\tau, x, y) \) in (21) can be parameterized

\[
\text{eff} \Psi_{\mu \nu}(\tau) = \left\{ \begin{array}{ll}
\eta(\tau, r, \theta)\delta^\tau_j \Psi(\tau, r, \theta, \varphi)\delta_{\varphi^b}^\tau, & \text{stationary configurations ;} \\
\eta(\tau, x^i)\delta^\tau_j \Psi(\tau, x^i, t)\delta_{\varphi^b}^\tau, & \text{cosmological configurations} .
\end{array} \right.
\tag{25}
\]

Considering as typical examples two types of a Killing space symmetry or time like Killing symmetry for effective generating sources, we shall construct and study properties of two general classes of exact solutions (the first one will be for stationary configurations which may contain BH solutions and the second one will be for cosmological type solutions).

### 4.2 Nontrivial Ricci d-tensors and decoupling of entropic flow equations

In this subsection, we outline the key steps for proofs of general decoupling and integrability of (modified) Einstein equations with effective sources (25).

#### 4.2.1 Off–diagonal metric ansatz, (non) holonomic variables, and ODEs and PDEs

Let us summarize in Table 1 below the data on nonholonomic 3+1 and 2+2 variables and corresponding ansatz which allows to transform geometric and entropic flow equations and, a nonholonomic Ricci solitons, gravitational field equations in entropic MGTs and GR into respective systems of nonlinear ordinary differential equations, ODEs, and partial differential equations, PDEs. All formulas will be proven in next subsections. We model a nonholonomic deformation with \( \eta \)-polarization functions, \( \hat{g} \rightarrow g(\tau) \), of a 'prime' metric, \( \hat{g} \), into a family 'target' d-metrics \( \mathbf{g}(\tau) \) (2), if

\[
\mathbf{g}(\tau) = \eta_t(\tau, x^k)\hat{g}_i dx^i \otimes dx^i + \eta_a(\tau, x^k, y^b)\hat{h}_a e^{a}[\eta] \otimes e^a[\eta],
\tag{26}
\]

where the target N-elongated basis is determined by \( N_i^\alpha(\tau, u) = \eta_i^\beta(\tau, x^k, y^b)\hat{N}_i^\alpha(\tau, x^k, y^b) \) in the form\(^{11}\)

\[
e^{a}[\eta] = (dx^i, e^-a = dy^a + \eta_a^\beta \hat{N}_a^\beta dx^i). \]

The values \( \eta_t(\tau) = \eta_t(\tau, x^k) \), \( \eta_a(\tau) = \eta_a(\tau, x^k, y^b) \) and \( \eta_b(\tau) = \eta_b^a(\tau, x^k, y^b) \) are called respectively geometric/entropic flow or gravitational polarization functions, or \( \eta \)-polarizations. Any \( \mathbf{g}(\tau) \) is subjected to the condition that it defines a solution of modified Einstein equations resulting in entropic quasiperiodic geometric flows and/or via nonholonomic deformations. A general prime metric in a coordinate parametrization is of type \( \hat{g} = \hat{g}_{\alpha\beta}(x^i, y^a)du^\alpha \otimes du^\beta \), which can be also represented equivalently in N-adapted form

\[
\begin{align*}
\hat{g} &= \hat{g}_{\alpha}(u)\hat{e}^\alpha \otimes \hat{e}^\beta = \hat{g}_i(x)dx^i \otimes dx^i + \hat{g}_a(x, y)\hat{e}^a \otimes \hat{e}^a, \\
\text{for } \hat{e}^\alpha &= (dx^i, e^-a = dy^a + \hat{N}_a^\beta(\tau, u)dx^i), \text{ and } \hat{e}_a = (\hat{e}_i = \partial/\partial y^a - \hat{N}_b^a(\tau, u)\partial/\partial y^b, e^-a = \partial/\partial y^a).
\end{align*}
\tag{27}
\]

\(^{10}\)In principle, we can construct for (22) certain classes of exact and parametric off-diagonal solutions generically depending on all spacetime coordinates \((x^k, y^a)\) but that would result in hundreds of pages with a cumbersome formulas for respective geometric techniques, see [60, 61, 62, 63, 64, 65] and references therein.

\(^{11}\)we do not consider summation on repeating indices if they are not written as contraction of "up-low" ones.
For certain subclasses of solutions, we can consider that \( \epsilon \) exists. Such a d-metric can be, or not, a solution of some gravitational field equations in a MGT or GR but for diagonalizable prime metrics (the off-diagonal structure of the Kerr metric is determined by rotation Friedman–Lemaître–Robertson–Walker (FLRW) type metric, or any Bianchi anisotropic metrics. For deformations with singular coordinates is convenient to construct exact solutions with nontrivial functional properties of some target and/or prime solutions, for instance, when \( \eta \) constructions are for effective matter sources encoding entropic nonholonomic flows and deformations.

\[
\begin{align*}
\text{Table 1: Entropic quasiperiodic flow modified Einstein eqs as systems of nonlinear PDEs} & \quad \text{and the Anholonomic Frame Deformation Method, AFDM, for constructing generic off-diagonal exact, parametric, and physically important solutions} \\
\begin{array}{ll}
\text{diagonal ansatz: PDEs } & \text{ODEs} \\
\text{radial coordinates } u^\alpha = (\tau, \theta, \phi, t) & u = (x, y) ; \\
\text{LC-connection } \hat{\nabla} & \text{connections} \\
\text{diagonal ansatz } \hat{g}_{\alpha \beta}(u) & \Rightarrow \hat{g}(\tau) \\
\hat{g}_{\alpha \beta} = \left\{ \begin{array}{ll}
\hat{g}_{\alpha \beta}(\tau) & \text{for BHs} \\
\hat{g}_{\alpha \beta}(t) & \text{for FLRW} \\
\end{array} \right.
\end{array} \\
\text{coord. transforms } e_\alpha = e_\alpha^\alpha \partial_{\tau}, e^\beta = e^\beta_\alpha du^\alpha, \hat{g}_{\alpha \beta} = \hat{g}_{\alpha \beta}^\alpha e^\alpha e^\beta & \Rightarrow \hat{g}(\tau, \theta, \phi) \\
\hat{g}(x^1, y^1) \rightarrow \hat{g}(\tau) \text{ or } \hat{g}(t), \hat{N}^\alpha (x^1, y^1) \rightarrow 0. \\
\hat{\nabla}, \hat{R}, \hat{Ric} = \hat{R}_{\beta \gamma} & \text{Ric tensors} \\
\text{d-metrics} & \text{sources} \\
\text{[N-adapt. fr.]} & \text{[diag]} \\
\text{circ. conf.} & \text{cosm. conf.} \\
\text{trivial equations for } \hat{\nabla}\text{-torsion} & \text{D. } \hat{R}_{\beta \gamma}(\tau) = \hat{g}^{\alpha \beta} \hat{g}^{\gamma \delta} \hat{R}_{\alpha \beta} \\
\text{LC-conditions} & \text{D. } \hat{R}_{\beta \gamma} = \hat{\nabla} \text{ extracting new classes of solutions in GR}
\end{align*}
\]

In our works, we are interested usually in two physically important cases when \( \hat{g}(\tau) \) defines a BH solution (for instance, a vacuum Kerr, or Schwarzschild, Kerr-(anti) de Sitter metric), or a Friedman–Lemaître–Robertson–Walker (FLRW) type metric, or any Bianchi anisotropic metrics. For diagonalizable prime metrics (the off-diagonal structure of the Kerr metric is determined by rotation frames and coordinates), we can always find a coordinate system when \( N^b_i = 0 \). To avoid nonholonomic deformations with singular coordinates is convenient to construct exact solutions with nontrivial functions \( \eta_\alpha = (\eta_\tau, \eta_\alpha), \eta^\alpha_i \), and nonzero coefficients \( \hat{N}^b_i(u) \). We have to consider necessary type frame/coordinate transforms. For a d-metric \( \hat{g}(\tau) \), we can analyze the conditions of existence and geometric/physical properties of some target and/or prime solutions, for instance, when \( \eta_\alpha \rightarrow 1 \) and \( \hat{N}^a_i = 0 \) can be imposed as some special nonholonomic constraints. In brief, we shall denote certain nonholonomic entropic deformations of a prime d-metrics into a target one as \( \hat{g} \rightarrow g = [\eta_\alpha \hat{g}_\alpha, \eta^\alpha_i \hat{N}^a_i] \).

Table 1 outlines the key steps for developing the AFDM to theories of entropic quasiperiodic geometric flows. In this work, the formulas depend on a flow temperature like parameter \( \tau \) and the constructions are for effective matter sources encoding entropic nonholonomic flows and deformations.

\[\text{We can consider flow evolution of a physical important target metric } g(\tau) \text{ with generic off-diagonal terms as an "almost" BH, or FLRW cosmological, like metric. Such parametric solutions are constructed for small nonholonomic deformations on some constant parameters } \eta_\alpha = (\eta_\tau, \eta_\alpha), \text{ for } 0 \leq \epsilon_\alpha, \epsilon^i_\alpha \ll 1, \text{ when } \eta_\alpha \approx \eta_\alpha(\tau, x^k, y^b) \approx 1 + \epsilon_\alpha \chi_\alpha(\tau, x^k, y^b) \approx 1 + \epsilon_\alpha \chi_\alpha(\tau, x^k, y^b) \approx 1 + \epsilon_\alpha \chi_\alpha(\tau, x^k, y^b) \approx 1 + \epsilon_\alpha \chi_\alpha(\tau, x^k, y^b). \text{ Parametric } \epsilon\text{-decompositions can be performed in a self-consistent form by omitting quadratic and higher terms after a class of solutions have been found for some evolution or nonholonomic deformation data } (\eta_\alpha, \eta^\alpha_i). \text{ For certain subclasses of solutions, we can consider that } \epsilon_\alpha, \eta^\alpha_i, \eta_\alpha \sim \epsilon, \text{ when only one small parameter is considered for all coefficients of nonholonomic deformations. We can work with mixed types of solutions and model only small diagonal deformations } \epsilon_\alpha, \eta^\alpha_i, \sim \epsilon \text{ of } g(\tau), \text{ for some general } \eta^\alpha_i. \text{ Alternatively, we consider nontrivial } \eta_\alpha \text{ but } \epsilon^i_\alpha \sim \epsilon.\]
4.2.2 Stationary Ricci d-tensors and modified Einstein equations

For stationary configurations, we can choose certain systems of reference/coordinates when coefficients of the geometric objects do not depend on $y^4 = t$ with respect to a class of N–adapted frames. Using parameterizations for a d-metric (23) with $\omega = 1$ and a source $[hY, Y]$ (23), we obtain such nontrivial N–adapted coefficients of the Ricci d-tensor, which allow to write the entropic flow modified Einstein equations (22) in the form

\begin{align*}
R_1^1(\tau) &= \frac{g_1'^2}{2g_1} + \frac{(g_2')^2}{2g_2} - g_2^\ast + \frac{g_1'g_2'}{2g_1} + \frac{g_2'}{2g_1} - g_1'' = -2g_1g_2 \ hY; \quad (28) \\
R_3^3(\tau) &= -Y(\tau) i.e. \frac{h_1^2}{2h_4} + \frac{h_3^2 h_4^2}{2h_3} - h_4^\ast = -2h_3h_4 \ Y; \quad (29) \\
2h_4R_{3k}(\tau) &= -w_k\frac{(h_4')^2}{2h_4} + \frac{h_3^2 h_4^2}{2h_3} - h_4^\ast + \frac{h_4^0}{2} \left[ \frac{\partial_k h_4}{h_4} + \frac{\partial_4 h_4}{h_4} \right] - \partial_k h_4^0 = 0, \quad (30) \\
R_{4k}(\tau) &= \frac{h_4^0}{2h_4} + \left[ \frac{3}{2} \frac{h_4^0}{h_3} - \frac{h_3^0}{h_4} \right] \frac{n_k^0}{2h_3} = 0. \quad (31)
\end{align*}

This system of nonlinear PDEs possesses a very important decoupling property: The equations (28) allow us to find $g_1$ (or, inversely, $g_2$) for any prescribed entropic flow and quasiperiodic h-source $hY(\tau, r, \theta)$ and any given coefficient $g_2(\tau, r, \theta)$ (or, inversely, $g_1(\tau, r, \theta)$). Integrating on variable $y^3$ in (29), we can define $h_3(\tau, r, \theta, \varphi)$ as a solution of first order PDE for any prescribed v-source $Y(\tau, r, \theta, \varphi)$ and given coefficient $h_4(\tau, r, \theta, \varphi)$. We can define $h_4(\tau, r, \theta, \varphi)$ if, inversely, $h_3(\tau, r, \theta, \varphi)$ is given but in such cases we have to solve a second order PDE.

In principle, we can construct nontrivial solutions if such conditions are not satisfied; we omit in this work considerations for such more special geometric evolution models.

In principle, we can construct nontrivial solutions if such conditions are not satisfied; we omit in this work considerations for such more special geometric evolution models.

Let us introduce the coefficients $\alpha_i = (\partial_3 h_4) (\partial_3 \varphi)$, $\beta = (\partial_3 h_4) (\partial_3 \varphi)$, $\gamma = (\partial_3 h_4) (\partial_3 \varphi)$, $\psi = (\partial_3 h_4) (\partial_3 \varphi)$, where $\varphi = \ln |\partial_3 h_4|/\sqrt{|h_3 h_4|}$. Using nonsingular values for $\partial_3 h_4 \neq 0$ and $\partial_1 \varphi \neq 0$, we obtain

\begin{align*}
\psi'' + \psi'' = 2 \ hY, \quad \varphi^\omega h_4^\ast = 2h_3h_4 \ Y, \quad \beta w_i - \alpha_0 = 0, \quad n_k^\ast + \gamma n_k = 0. \quad (32)
\end{align*}

This system can be integrated in explicit form (see next subsection) if there are prescribed a generating function $\Psi(\tau, r, \theta, \varphi) := e^{\varphi}$ and generating sources $hY(\tau, r, \theta)$ and $Y(\tau, r, \theta, \varphi)$.

Nonlinear symmetries for stationary generating functions and effective cosmological constant: The system of two equations for $\varphi$ and (22) relates four functions $(h_3, h_4, Y, \Psi)$ and posses an important nonlinear symmetry, $(\Psi(\tau), Y(\tau)) \leftrightarrow (\Phi(\tau), \Lambda(\tau))$, when

\begin{align*}
\Lambda(\Psi^2) = |Y|(|\Phi^2)|, \quad or \quad \Lambda \Psi = \Phi^2 |Y| - \int d^2 \varphi \Phi^2 |Y|^\omega. \quad (33)
\end{align*}

\begin{align*}
\text{in principle, we can construct nontrivial solutions if such conditions are not satisfied; we omit in this work considerations for such more special geometric evolution models.}
\end{align*}

\begin{align*}
\text{The LC-conditions (7) for stationary configurations transform into a system of 1st order PDEs,}
\partial_\varphi w_i = (\partial_i - w_i \partial_\varphi) \ln |h_3|, (\partial_i - w_i \partial_\varphi) \ln \sqrt{|h_4|} = 0, \partial_\varphi w_i = \partial_i w_k, \partial_\varphi n_i = 0, \partial_\varphi n_k = \partial_k n_i, \text{ imposing additional constraints on off-diagonal coefficients of metrics of type (23).}
\end{align*}
These formulas allow us to introduce a new generating function \( \Phi(\tau, r, \theta, \varphi) \) and an (effective) cosmological constant \( \Lambda \neq 0 \). The value \( \Lambda(\tau) \) can be chosen from certain physical considerations. It can be positive or negative and depending, in principle, on a Ricci flow parameter \( \tau \). For entropic gravity theories, such a value can be determined by the cosmological constant \( \Lambda \) in [13]. Solutions with \( \Lambda = 0 \) have to be studied by applying special methods, see details and examples in [60, 61, 62, 63, 64, 65, 66, 69, 70]. Using nonlinear symmetries, we can describe nonlinear systems of PDEs by two equivalent sets of generating data \((\Psi, \Upsilon)\) or \((\Phi, \Lambda)\). For some classes of solutions, we may work with effective cosmological constants but for another ones we can consider generating sources. Such alternatives are convenient for constructing more general classes of exact solutions or to elaborate on realistic physical models. Modules in formulas (33) should be taken in certain forms resulting in real, causal, compatible with observational data.

### 4.2.3 Cosmological Ricci d-tensors, LC-conditions, and nonlinear symmetries

For locally anisotropic cosmological configurations, we can consider geometric data when coefficients of the geometric objects do not depend on a space like \( y^3 \) with respect to certain classes of N-adapted frames. Using \( d \)-metric data (24) with \( \omega = 1 \) and a source \( \frac{\epsilon}{c^4}\bar{\Psi}(\tau, x^i), \bar{\Upsilon}(\tau, x^i, t) \) (25), we write the entropic flow modified Einstein equations (22) in the form (34):

\[
R^1_1(\tau) = R^2_2(\tau) = -h\bar{\Psi}(\tau) \text{ i.e. } \frac{g_{i}g_{j}}{2g_1} + \frac{(g_{j})^2}{2g_2} - g_{jj} + \frac{g_{i}g_{j}}{2g_1} - \frac{(g_{i})^2}{2g_1} = -2g_1g_2 \ h\bar{\Psi};
\]

\[
R^3_3(\tau) = R^4_4(\tau) = -\bar{\Upsilon}(\tau) \text{ i.e. } \frac{(h_3)^2}{2h_3} + \frac{h_3h_4}{2h_4} = 0;
\]

\[
2h_3R^3_4(\tau) = -\bar{w}_k\left[\frac{(h_3)^2}{2h_3} + \frac{h_3h_4}{2h_4}\right] - \frac{h_3(h_3) + \frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}}{h_3} = 0.
\]

This system of nonlinear PDEs can be transformed, respectively, into the system (28)-(31) if

\[
h\bar{\Psi}(\tau, x^i) \rightarrow h\bar{\Psi}(\tau, x^i), \bar{\Upsilon}(\tau, x^i, t) \rightarrow \bar{\Upsilon}(\tau, x^i, y^3 = \varphi),
\]

\[
h_3(\tau, x^i, \varphi), \bar{h}_3(\tau, x^i, t) \rightarrow h_3(\tau, x^i, \varphi), \bar{h}_3(\tau, x^i, t) \rightarrow h_3(\tau, x^i, \varphi),
\]

\[
\bar{h}_4(\tau, x^i, \varphi), \bar{w}_k(\tau, x^i, t) \rightarrow \bar{h}_4(\tau, x^i, \varphi), \bar{w}_k(\tau, x^i, t) \rightarrow \bar{w}_k(\tau, x^i, \varphi) \text{ etc.}
\]

Such a duality exists for Lorentz manifolds with a Killing symmetry when \( y^3 \) is a space like coordinate and \( y^4 = t \) is a time like coordinated. This duality simplifies various applications of the AFDM when we can redefine the procedure considered in the previous subsection for stationary nonholonomic configurations to certain time dependent ones. It allows use to prove decoupling properties and generate cosmological like solutions if there are known certain stationary configurations, or inversely to find certain stationary metrics as analogs of corresponding cosmological ones.

---

15 There are considered partial derivatives \( \partial_t q = \partial_t q = q^* \) and \( \partial_t q = (\partial_t q = q^*, \partial_t q = q^* \); we use over-lined symbols in order to emphasize that certain values are not stationary but depend on a time like coordinate \( t \).

16 The LC-conditions (7) for stationary configurations transform into equations with coefficients depending on \( t \),

\[
\partial_t \bar{w}_i = (\partial_i - \bar{\omega}_i \partial_t) \ln \sqrt{|h_3|}, (\partial_i - \bar{\omega}_i \partial_t) \ln \sqrt{|h_3|} = 0, \partial_k \bar{w}_i = \partial_i \bar{w}_k, \partial_i \bar{n}_i = 0, \partial_i \bar{n}_k = \partial_k \bar{n}_i.
\]

Such nonlinear first order PDEs containing \( \partial_t \) can be solved in explicit form for certain classes of additional nonholonomic constraints on cosmological \( d \)-metrics and \( N \)-coefficients, see (24).
We can rewrite the nonlinear PDE (34)–(37) in an explicit decoupled form if we introduce the coefficients \( \Psi_1 = (\partial_t \bar{h}_3) \), \( \Psi_2 = (\partial_t \bar{h}_4) \), \( \Psi_3 = \frac{\partial}{\partial t} (\ln |\bar{h}_3|^3/|\bar{h}_4|) \), where \( \bar{\omega} = \ln |\partial_t \bar{h}_3|/\sqrt{\bar{h}_3 \bar{h}_4} | \). For \( \partial_t h_a \neq 0 \) and \( \partial_t \bar{\omega} \neq 0 \), we obtain such equations
\[
\psi_{**} + \psi'' = 2 h \Psi; \bar{\omega} \bar{h}_3 = 2 \bar{h}_3 \bar{h}_4 \Psi; \bar{\omega} \bar{\eta} = 0; \beta \bar{\omega} \bar{\eta} - \bar{\omega} = 0.
\]
(38)

We can integrate such equations "step by step" for any generating function \( \Psi(x^i, t) := e^{\bar{\omega}} \) and sources \( h \Psi(x^i) \) and \( \Psi(x^k, t) \), see next subsection.

**Nonlinear symmetries for generating functions and sources with effective cosmological constant:** The system (38) with respective coefficients relates four functions \( (\bar{h}_3, \bar{h}_4, \Psi, \bar{\omega}) \) when a very important nonlinear symmetry for locally anisotropic cosmological solutions and respective generating functions, \( (\Psi, \bar{\omega}) \leftrightarrow (\Phi, \Lambda) \), can be found,
\[
\Lambda(\Psi^*) = |\Psi| (\Phi^*), \text{ or } \Lambda^2 = \Phi^2 |\Psi| - \int dt \Phi^2 |\Psi|^*.
\]
(39)

This nonlinear symmetry allows us to introduce a new generating function \( \Phi(x^i, t) \) and an (effective) cosmological constant \( \Lambda(\tau) \neq 0 \), which can be applied both for generating exact off-diagonal solutions in explicit forms and elaborating on locally anisotropic cosmological scenarios with cosmological constants, for instance, considered in entropic gravity.

### 4.3 Integrability of entropic quasiperiodic geometric flow equations

We generate and study geometric properties of two classes of generic off-diagonal solutions with elasticity and quasiperiodic structures of the system of nonlinear PDEs (22). The first one is for stationary configurations and the second one is considered for locally anisotropic cosmological models.

#### 4.3.1 Stationary solutions for off-diagonal metrics and N–coefficients

We can prove that integrating "step by step" the system (28)–(31) represented in the form (32) (see similar details in [64, 65, 66, 69]) one generates exact stationary solutions of modified Einstein equations with nonlinear symmetries (33) if the d–metric coefficients are computed
\[
g_1(\tau) = e^{\psi(\tau, x^k)} \text{ as a solution of 2-d Poisson eqs. } \psi_{**} + \psi'' = 2 h \Psi(\tau);
\]
\[
g_3(\tau) = h_3(\tau, r, \theta, \varphi) = - \frac{(\Psi^2)^2}{4Y^2 h_4} = - \frac{\partial_\varphi \Psi^2}{4Y^2 \left( h_4^{[0]}(\tau, x^k) - \int dy^3 (\Psi^2)^* / 4Y \right)}
\]
\[
= - \frac{(\Phi^2)^*}{h_4 \Lambda(\tau) \int dy^3 Y (\Phi^2)^*} = - \frac{\partial_\varphi (\Phi^2)^*}{4 \left[ h_4^{[0]}(\tau, x^k) - \Phi^2 / 4 \Lambda(\tau) \right] \int dy \partial_\varphi (\Phi^2)^*};
\]
\[
g_4(\tau) = h_4(\tau, r, \theta, \varphi) = h_4^{[0]}(\tau, x^k) - \int dy^3 (\Psi^2)^* / 4Y = h_4^{[0]}(\tau, x^k) - \Phi^2 / 4 \Lambda(\tau);
\]
and the N–connection coefficients are taken in the form
\[
N_i^3 = w_i(\tau, r, \theta, \varphi) = \frac{\partial_i \Psi}{\partial \varphi \Psi}; \quad N_i^4 = n_k(\tau, r, \theta, \varphi) = 1n_k(\tau, x^i) + 2n_k(\tau, x^i) \int dy^3 \frac{(\Psi^2)^2}{Y^2 h_4^{[0]}(\tau, x^k) - \int dy^3 (\Psi^2)^* / 4Y^{5/2}}
\]
\[
= 1n_k(\tau, x^i) + 2n_k(\tau, x^i) \int dy^3 \frac{(\Phi^2)^2}{4 \Lambda(\tau) \int dy^3 Y (\Phi^2)^* |h_4|^5 / 2}.
\]
In formulas (40) and (41), there are considered also integration functions \( h_3^0(\tau, x^k), n_k(\tau, x^i) \) encoding possible sets of (non) commutative parameters and integration constants but also nonlinear evolution scenarios on \( \tau \). Such values, together with generating geometric evolution data \((\Psi, \Upsilon)\), or \((\Phi, \Lambda)\), related by nonlinear differential / integral transforms \( \Phi \) can be stated in explicit form following certain topology / symmetry / asymptotic conditions for some classes of exact / parametric solutions of generalized Hamilton and/or gravitational field equations. The coefficients (40) and (41) define generic off-diagonal stationary solutions if the corresponding anholonomy coefficients are not trivial. Such geometric flow solutions are with nontrivial nonholonomically induced d-torsion determined by evolution of N-adapted coefficients of d-metric structures which can be computed in explicit form. We can impose additional nonholonomic constraints in order to extract LC-configurations under geometric flow evolution.

Let us study the nonholonomic evolution properties of off-diagonal stationary solutions using the formulas for effective sources (21). We obtain a system of equations with first order evolution derivatives \( \partial_\tau \) and N-adapted frames \( e^\mu_\nu(\tau) \) with arbitrary spacial coordinates \((x^k, y^3)\), \( \text{eff} \mathcal{Y}_a(\tau) = [e^\mu_a(\tau)]^2 \left[ \text{tot} \mathcal{Y}_{a'c'}(\tau) + \frac{1}{2} \partial_\tau g_{a'c'}(\tau) \right] = h(\tau, x^k, y^3). \)

So, prescribing any values for the matter sources \( \text{tot} \mathcal{Y}_{\mu\nu}(\tau) \) (for simplicity, we can consider N-adapted diagonal configurations), we can always integrate on \( \tau \) and determine evolution behaviour of \( g_{a'c'}(\tau) \) as a nonholonomic and nonlinear geometric diffusion process. In general form, this depends on a system of reference determined by \( e^\mu_\nu(\tau, x^k, y^3) \). We have to prescribe such generating values \([e^\mu_\nu, \text{tot} \mathcal{Y}_{\mu\nu}]\) which are compatible with certain observational data, for instance, in modern astrophysics or gravity.

### Line elements for off-diagonal stationary elastic quasiperiodic configurations:

It is important another property for re-defining the generating functions and sources for above constructed families of stationary solutions. We can consider as a generating function any coefficient \( h_4(\tau) = h_4^0 - \Phi^2/4\Lambda, h_4^0(\tau) \neq 0 \). Let us write \( \Phi^2(\tau) = -4\Lambda h_4(\tau) \) when \( \Phi^2 = \int d\varphi \Phi^2 \). Introducing this functional \( \Psi[h_4^0, h_4, \Upsilon] \) into respective formulas for \( h_4, N_i^a \) and \( \Upsilon \) in (40) and (41), we express possible generating functions and the d–metric (2) with stationary data (23) in terms of \( h_4 \), integration functions and effective entropic and quasiperiodic sources for geometric evolutions. We express the respective quadratic elements in three equivalent forms:

\[
\begin{align*}
\text{ds}^2 &= e^{\psi(\tau, x^k)}[(dx^1)^2 + (dx^2)^2] + \\
&- \frac{\partial_{\dot{\psi}} (\Phi^2) |^2}{4\Upsilon^2(\Psi^2 - \int dy^i (\Phi^2)^{1/2})} \left[ dy^3 + \frac{\partial_{\dot{\psi}} \Psi}{\Upsilon} dx^1 \right] - \\
h_4 \left[ dt + (1 + 2n_k \int f d\varphi \left[ \frac{\partial_{\dot{\psi}} (\Phi^2) |^2}{4\Upsilon^2(\Psi^2 - \int dy^i (\Phi^2)^{1/2})} dx^3 \right] \right],
\end{align*}
\]

\[
\begin{align*}
&\frac{\partial_{\dot{\psi}} (\Phi^2)}{4\Upsilon^2 |\Psi| (\Phi^2)^{1/2}} \left[ dy^3 + \frac{\partial_{\dot{\psi}} \Psi}{\Upsilon} dx^1 \right] - (h_4^0 - \int d\varphi |(\Phi^2)^{1/2}|) \left[ dt + (1 + 2n_k \int f d\varphi \left[ \frac{\partial_{\dot{\psi}} (\Phi^2) |^2}{4\Upsilon^2 |\Psi| (\Phi^2)^{1/2}} \right] \right],
\end{align*}
\]

\[
\begin{align*}
&\frac{\partial_{\dot{\psi}} (\Phi^2) |^2}{4\Lambda |\Psi| (\Phi^2)^{1/2}} \left[ dy^3 + \frac{\partial_{\dot{\psi}} \Psi}{\Upsilon} dx^1 \right] - (h_4^0 - \Phi^2/\Lambda) \left[ dt + (1 + 2n_k \int f d\varphi \left[ \frac{\partial_{\dot{\psi}} (\Phi^2) |^2}{4\Lambda |\Psi| (\Phi^2)^{1/2}} \right] \right],
\end{align*}
\]

Above formulas (40), (41), (42) and (43) encode entropic quasiperiodic data via generating source \( \Upsilon \) and effective cosmological constant \( \Lambda \). Nonlinear symmetries (33) mix such the structure of generating
functions with that of generating source. So, the emergent gravity scenarios are also determined by the type of generating functions ($\Psi$, or $\Phi$, or $h_4$) is chosen for constructing a class of stationary solutions.

**Entropic gravitational polarization functions:** We can consider nonholonomic deformations of a primary $d$-metric $\mathbf{g}$ into a target stationary one $\mathbf{g}(\tau)=[g_\alpha(\tau) = \eta_\alpha(\tau)\hat{g}_\alpha, \eta^\alpha_i(\tau)\bar{N}_i^\alpha]$ with Killing symmetry on $\partial_t$. All above formulas can be re-written in terms of $\eta$–polarization functions, $\eta_\alpha$ and $\eta^\alpha_i$, determined by generation and integration functions and respective sources and encoding primary data $[\hat{g}_\alpha, \bar{N}_i^\alpha]$. For instance, we can consider $\hat{g}$ as a BH in GR and study entropic quasiperiodic deformations by geometric flows or for certain nonholonomic Ricci soliton configurations which result in a stationary $\mathbf{g}(\tau)$. Off-diagonal nonholonomic deformations of the metric and (non) linear connection structures and sources may preserve the singular structure of a primary metric with certain possible deformations of the horizons, for certain classes of solutions. For more general classes of solutions, one can elaborate on scenarios eliminating the singular structure, or changing the topology in the resulting target solutions.

**Off-diagonal Levi-Civita stationary configurations:** We can impose additional constraints on generating functions and sources in order to extract solutions with zero torsion. The equations, see footnote 14 can be solved for a special class of generating functions and sources. For instance, we can consider a $\Psi(\tau) = \bar{\Psi}(\tau, x', \varphi)$ for which $(\partial_i \bar{\Psi})^\circ = \partial_i (\bar{\Psi}^\circ)$ and fix $\Upsilon(\tau, x', \varphi) = \Upsilon[\bar{\Psi}] = \bar{\Upsilon}(\tau)$, or $\Upsilon = \text{const}$. In result, the nonlinear symmetries (33) are restricted for certain LC-configurations, $\Lambda \hat{\Psi}^2 = \hat{\Phi}^2[\bar{\Upsilon}] - \int d\varphi \Phi^2[\bar{\Upsilon}]^\circ, \hat{\Phi}^2 = -4\Lambda \hat{h}_4(\tau, r, \theta, \varphi), \hat{\Psi}^2 = \int d\varphi \Upsilon(\tau, r, \theta, \varphi) \hat{h}_4^2(\tau, r, \theta, \varphi)$. Using such formulas, we conclude that $h_4(\tau) = \tilde{h}_4(\tau, r, \theta, \varphi)$ can be considered also as generating function when $h_3$ and $N$-connection coefficients are computed using certain nonlinear symmetries and nonholonomic constraints. For zero torsion but off-diagonal entropic and quasiperiodic stationary configurations, we find some functions $\tilde{A}(\tau) = \tilde{A}(\tau, r, \theta, \varphi)$ and $n(\tau) = n(\tau, r, \theta)$ when the $N$–coefficients are

$$w_i(\tau) = \partial_i \tilde{A}(\tau) = \frac{\partial_i (\int d\varphi \bar{\Psi}^\circ \hat{h}_4^\circ)}{\bar{\Psi}^\circ \hat{h}_4^\circ} = \frac{\partial_i \bar{\Psi}^\circ}{\bar{\Psi}^\circ} = \frac{\partial_i [\int d\varphi \bar{\Psi} (\hat{\Phi}^2)^\circ]}{\bar{\Psi} (\hat{\Phi}^2)^\circ} \text{ and } n_k(\tau) = \bar{n}_k(\tau) = \partial_k n(\tau, x').$$

In result, we can construct new classes of off-diagonal stationary solutions defined as subclasses of solutions (43) with zero torsion,

$$ds^2 = e^{\psi(\tau, x^k)}[(dx^1)^2 + (dx^2)^2] +$$

\[
\begin{align*}
&\left\{ \begin{array}{l}
-\frac{(h_4^2)^2}{\int d\varphi \bar{\Psi} h_4} [dy^3 + (\partial_i \tilde{A}) dx^i] + \tilde{h}_4 \left[ dt + (\partial_k n) dx^k \right], \quad \text{gener. funct. } \tilde{h}_4, \\
-\frac{\partial_\varphi (\bar{\Psi}^2)}{4 \bar{\Psi}^2 (h_4^2 - \int d\varphi \bar{\Psi} (\Phi^2)^2)} [dy^3 + (\partial_i \tilde{A}) dx^i] \\
+ (h_4^2 - \int d\varphi \bar{\Psi} (\Phi^2)^2) [dt + (\partial_k n) dx^k], \quad \text{source } \bar{\Psi}, \text{ or } \Lambda;
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
&\left\{ \begin{array}{l}
-\frac{(\Phi^2)^2}{4 \Lambda \int d\varphi \bar{\Psi} (\Phi^2)^2} [dy^3 + (\partial_i \tilde{A}) dx^i] \\
+ (h_4^2 - \frac{\Phi^2}{\Lambda}) \left[ dt + (\partial_k n) dx^k \right], \quad \text{effective } \Lambda \text{ for } \bar{\Psi}.
\end{array} \right.
\]

We can consider that this class of solutions describes geometric evolution scenarios determined by entropic quasiperiodic effective sources and nontrivial vacuum structure of certain classes of solutions in GR. For any value of flow parameter $\tau$, such stationary metrics are generic off-diagonal and define new classes of solutions which different, for instance, from the Kerr metric (defined by rotation coordinates, or other equivalent ones). See similar details (but not for elastic spacetime configurations) in Refs.
We can check if the anholonomy coefficients $C_{\alpha \beta}^\gamma = \{C^h_{\alpha \beta} = \partial_\alpha N^h_\beta, C^a_{\alpha \beta} = e_\alpha N^a_\beta - e_\beta N^a_\alpha\}$ are not zero for $N^3_i = \partial_i \bar{A}$ and $N^4_k = \partial_k n$ and understand if certain metrics are or not generic off-diagonal. It is possible to can fix and analyze certain nonholonomic configurations determined, for instance, by data $(\bar{\Psi}, h^0_4, \bar{n}_k)$, with $w_i = \partial_i \bar{A} \to 0$ and $\partial_k n \to 0$.

### 4.3.2 Off–diagonal cosmological solutions with elastic quasiperiodic structures

Applying the AFDM, we can construct cosmological solutions (which, in general, are locally anisotropic) of the entropic flow modified Einstein equations \[(22)\] with N-adapted sources $h\bar{\Psi}(\tau) = h\bar{\Psi}(\tau, x^k)$ and $\bar{\Psi}(\tau) = \bar{\Psi}(\tau, x^k, t)$, see parameterizations for cosmological configurations in \[(23)\]. Integrating "step by step" the system of the nonlinear PDEs \[(34)-(37)\] decoupled in the form \[(38)\], we obtain such d–metric coefficients for \[(2)\],

\[g_i(\tau) = e^\psi(\tau, x^k)\text{ as a solution of 2-d Poisson eqs. } \psi^{**} + \psi'' = 2 h\bar{\Psi}(\tau);
\]

\[g_3(\tau) = \bar{h}_3(\tau, x^i, t) = h^0_3(\tau, x^k) - \int dt \left(\frac{\bar{\Psi}^2}{4\bar{\Psi}}\right)^* = h^0_3(\tau, x^k) - \Phi^2/4\bar{\Lambda}(\tau);
\]

\[g_4(\tau) = \bar{h}_4(\tau, x^i, t) = -\left(\frac{\bar{\Psi}^2}{4\bar{\Psi}}\right)^* = -\frac{\left(\bar{\Psi}^2\right)^*}{4\bar{\Psi}^2(h^0_3(\tau, x^k) - \int dt \left(\bar{\Psi}^*\right)^*/4\bar{\Psi})};
\]

\[= -\frac{\left[\left(\bar{\Psi}^2\right)^*\right]^2}{4\bar{\Psi}_3[\bar{\Lambda}(\tau)] \int dt \left[\bar{\Psi}^2\right]^*} = \frac{\left[\left(\bar{\Psi}^2\right)^*\right]^2}{4[\bar{h}_3^0(\tau, x^k) - \bar{\Psi}^2/4\bar{\Lambda}(\tau)] \int dt \left[\bar{\Psi}^2\right]^*}.
\]

The N–connection coefficients are computed,

\[N^3_k(\tau) = \bar{\pi}_k(\tau, x^i, t) = 1 n_k(\tau, x^i) + 2 n_k(\tau, x^i) \int dt \frac{\left(\bar{\Psi}^2\right)^*}{4\bar{\Psi}_3[\bar{\Lambda}(\tau)] \int dt \left[\bar{\Psi}^2\right]^*}.
\]

\[= 1 n_k(\tau, x^i) + 2 n_k(\tau, x^i) \int dt \frac{\left(\bar{\Psi}^2\right)^*}{4\bar{\Psi}_3[\bar{\Lambda}(\tau)] \int dt \left[\bar{\Psi}^2\right]^*} = \frac{n_k(\tau, x^i) - \int dt \left[\bar{\Psi}^2\right]^*}{2\bar{\Psi}_3[\bar{\Lambda}(\tau)] \int dt \left[\bar{\Psi}^2\right]^*} = \frac{\partial_i \left[\int dt \left(\bar{\Psi}^2\right)^*\right]}{(\bar{\Psi}^2)^*}.
\]

In these formulas, $h^0_3(\tau, x^k)$, $1 n_k(\tau, x^i)$, and $2 n_k(\tau, x^i)$ are integration functions encoding various possible sets of (non) commutative parameters and integration constants running on $\tau$ for geometric evolution flows. We can chose different generating data $(\bar{\Psi}(\tau, x^i, t), \bar{\Psi}(\tau, x^i, t), \bar{\Lambda}(\tau))$ which are related by nonlinear differential / integral transforms \[(39)\], and respective integration functions. Such values should be chosen in explicit form following certain topology / symmetry / asymptotic conditions for some classes of exact / parametric cosmological solutions. The coefficients \[(45)\] and \[(46)\] define generic off-diagonal cosmological solutions if the corresponding anholonomy coefficients are not trivial. Such locally cosmological solutions are with nontrivial nonholonomically induced d-torsion and N-adapted coefficients which can be computed in explicit form. In order to generate as particular cases some well-known cosmological FLRW, or Bianchi, type metrics, we have to consider data of type $(\bar{\Psi}(\tau, t), \bar{\Psi}(\tau, t))$, $(\bar{\Psi}(\tau, t), \bar{\Lambda}(\tau))$, with integration functions which allow frame/ coordinate transforms to respective (off–) diagonal configurations $g_{\alpha \beta}(\tau, t)$.

Let us analyze certain important nonholonomic evolution properties of above locally anisotropic cosmological solutions using the formulas for effective sources \[(21)\] with cosmological parameterizations \[(25)\]. In N-adapted form, we obtain a system of equations with first order evolution derivatives $\partial_\tau$ when the v-part of vierbeinds depend on a time like coordinate $y^i = t$, $e^\nu_\mu(\tau) = [e^\nu_1(\tau, x^k), e^\nu_2(\tau, x^k)]$, \[(42)\].
there are considered coordinates \((x^k, t)\), when the dependence on \(y^3\) can be omitted because of Killing symmetry on \(\partial_3\). We can consider frame transforms for generating effective sources, 
\[
eff Y_i(\tau) = [e''_{(i)}(\tau)]^2[\, \text{tot } \Upsilon_{i}^{\nu}(\tau) + \frac{1}{2} \partial_{\nu} g_{\nu}(\tau)] = h Y(\tau, x^k),
\]
\[
eff Y_a(\tau) = [e''_{(a)}(\tau)]^2[\, \text{tot } \Upsilon_{a}^{\alpha}(\tau) + \frac{1}{2} \partial_{\tau} g_{\alpha}(\tau)] = Y(\tau, x^k, t).
\]

In these formulas, we can prescribe any values for the matter sources \(\text{tot } \Upsilon_{\mu\nu}(\tau)\) in a cosmological or spacetime QC model. Then, for simplicity, we can consider \(N\)-adapted diagonal configurations and integrate on \(\tau\) and determine a cosmological evolution flow of \(g_{\tau}(\tau, x^k, t)\) modelled as a nonholonomic and nonlinear geometric diffusion process. All geometric constructions are performed with respect to a new system of reference determined by \(e''_{(i)}(\tau, x^k, t)\). We have to prescribe some locally anisotropic generating values \([e''_{(i)}(\tau, x^k, t), \text{tot } \Upsilon_{\mu\nu}(\tau, x^k, t)]\) which are compatible with certain observational data, for instance, in modern cosmology and dark matter and dark energy physics.

**Line elements for off-diagonal cosmological configurations with elastic flows:** Any coefficient \(h_3(\tau) = h_3(\tau, x^k, t) = h_3^{[0]}(\tau, x^k) - \overline{\Phi}^2/4\Lambda(\tau)\), \(h_3 \neq 0\), can be considered also as a generating function, for instance, for entropic quasiperiodic configurations. Using formulas (45), we find \(\overline{\Phi}^2 = -4\Lambda(\tau)h_3(\tau, r, \theta, t)\) transforming (39) in \(\overline{\Psi}^* = \int dt \, \overline{h}_3^\alpha\). Introducing such values into the formulas for \(\overline{h}_\alpha\) and \(\overline{\Psi}\) in (45) and (46), we construct locally anisotropic cosmological solutions parameterized by \(d\)-metrics (2) with \(N\)-adapted coefficients (24),

\[
ds^2 = e^\psi(\tau, x^k)[(dx^3)^2 + (dx^2)^2] + \begin{cases}
\overline{h}_3[dy^3 + (1 + t_n + 2n) \int dt \frac{(\overline{h}^4)^2}{\overline{h}_3^{[0]} - \overline{\Phi}^2/4\Lambda(\tau)}dx^k] \\
-\int dt \frac{(\overline{y}^3)^2}{\overline{h}_3^{[0]} - \overline{\Phi}^2/4\Lambda(\tau)}dx^k, \\
\end{cases}
\]

\[
(h_3^{[0]} - \overline{\Phi}^2/4\Lambda(\tau))dy^3 + (1 + t_n + 2n) \int dt \frac{(\overline{\Phi}^2)^2}{4\Lambda(\tau)h_3^{[0]} - \overline{\Phi}^2/4\Lambda(\tau)}dx^k] \\
- \int dt \overline{\Phi}^2 dx^3, \\
\]

\[
(h_3^{[0]} - \overline{\Phi}^2/4\Lambda(\tau))dy^3 + (1 + t_n + 2n) \int dt \frac{(\overline{\Phi}^2)^2}{4\Lambda(\tau)h_3^{[0]} - \overline{\Phi}^2/4\Lambda(\tau)}dx^k] \\
- \int dt \overline{\Phi}^2 dx^3, \\
\]

\[
\overline{\Phi}^2 \overline{h}_3(\tau, r, \theta, t)h_3(\tau, r, \theta, t) \overline{\Phi}^2 \overline{h}_3(\tau, r, \theta, t).
\]

Such solutions posses a Killing symmetry on \(\partial_3\) and can be re-written in terms of \(\eta\)-polarization function functions for target locally anistropic cosmological metrics \(\overline{g} = [g_\alpha = \eta_\alpha \overline{g}_\alpha, \eta_\alpha^2 \overline{N}_\alpha]\) encoding primary cosmological data \([\eta_\alpha, \overline{N}_\alpha]\).

**Off-diagonal Levi-Civita entropic and quasiperiodic cosmological configurations:** We can extract and model entropic flow evolution of cosmological spacetimes in GR. To satisfy the zero torsion conditions (37), see equations in footnote (19) let us consider a special class of generating functions and sources when, for instance, \(\overline{\Psi}(\tau) = \overline{Y}(\tau, x^i, t)\), when \((\partial_\tau \overline{\Psi})^* = \partial_\tau(\overline{\Psi})\) and \(\overline{Y}(\tau, x^i, t) = \overline{Y}(\overline{\Psi}) = \overline{\Psi}(\tau)\), or \(\overline{Y} = \text{const}\). For such classes of entropic quasiperiodic generating functions and sources, the nonlinear symmetries (39) are written \(\overline{\Lambda}(\tau) \overline{\Psi} = \overline{\Phi}^2 \overline{Y} - \int dt \overline{\Phi}^2 \overline{Y}^*, \overline{\Phi}^2 = -4\Lambda(\tau)h_3(\tau, r, \theta, t), \overline{\Psi} = \int dt \overline{Y}(\tau, r, \theta, t)h_3(\tau, r, \theta, t)\). Using these formulas, we conclude that the coefficient \(\overline{h}_4(\tau) =

\( \overline{h}_4(\tau, x^i, t) \) can be considered also as generating function for entropic cosmological solutions. For such LC–configurations, there are some parametric on \( \tau \) functions \( \overline{A}(\tau, x^i, t) \) and \( n(\tau, x^i) \) when the N–connection coefficients are computed

\[
\overline{\pi}_k(\tau) = \overline{\pi}_k(\tau) = \partial_k \overline{\pi}(\tau, x^i) \quad \text{and} \quad \overline{w}_i(\tau) = \partial_i \overline{A}(\tau) = \frac{\partial_i\left(\int dt \overline{Y}(\overline{h}_3)\right)}{\overline{Y}(\overline{h}_3)} = \frac{\partial_i\overline{\Psi}}{\overline{Y}} = \frac{\partial_i\left[\int dt \overline{Y}(\overline{\Phi})^*\right]}{\overline{Y}(\overline{\Phi})^*}.
\]

Summarizing above formulas, we construct new classes of locally anisotropic cosmological solutions as ub GR defined as subclasses of solutions [43] with zero torsion but with entropic quasiperiodic geometric flow evolution,

\[
ds^2 = e^{\psi(\tau,x^i)}[(dx^1)^2 + (dx^2)^2] + \begin{cases} \overline{h}_3 [dy^3 + (\partial_k \pi)dx^k] - \frac{(\overline{h}_3)[\int dt \overline{Y}(\overline{h}_3)]}{\overline{Y}}[dt + (\partial_i \overline{A})dx^i], & \text{gener. funct.} \overline{h}_3, \\
(\overline{h}_3^{[0]} - \int dt \overline{Y}(\overline{h}_3)^*)[dy^3 + (\partial_k \pi)dx^k] - \frac{(\overline{\Phi})^*}{4\overline{Y}(\overline{h}_3^{[0]} - \int dt \overline{Y}(\overline{h}_3)^*)}[dt + (\partial_i \overline{A})dx^i], & \text{gener. funct.} \overline{\Psi}, \\
(\overline{h}_3^{[0]} - \frac{\overline{\pi}}{4\overline{\Lambda}})[dy^3 + (\partial_k \pi)dx^k] - \frac{(\overline{\Phi})^*}{4\overline{Y}(\overline{h}_3^{[0]} - \int dt \overline{Y}(\overline{h}_3)^*)}[dt + (\partial_i \overline{A})dx^i], & \text{effective} \overline{\Phi} \text{ for } \overline{Y}.
\end{cases}
\]

Such cosmological metrics are generic off-diagonal and define new classes of solutions if the anholonomy coefficients are not zero for \( N^3_k(\tau = \partial_k \pi) \) and \( N^4_k(\tau = \partial_i \overline{A}) \). They encode entropic quasiperiodic structures. We can analyze certain nonholonomic cosmological configurations determined, for instance, by data \((\overline{Y}, \overline{\Psi}, \overline{h}_3^{[0]}, \overline{n}_k)\), when \( \partial_k \pi \rightarrow 0 \) and \( \overline{w}_i = \partial_i \overline{A} \rightarrow 0 \). Zero values can be fixed also by certain additional nonholonomic constraints. Choosing data \((\overline{Y}(\tau, t), \overline{\Psi}(\tau, t), h_3^{[0]} = \text{const, } \overline{n}_k = \text{const})\), we can generate (off-) diagonal entropic metrics of Bianchi, or FLRW, types and generalizations to other type configurations \( g_{\alpha\beta}(\tau, t) \) in GR modified under geometric flow evolution.

## 5 Entropic flow & quasiperiodic stationary BH deformations

Generic off-diagonal nonholonomic deformations of BH like solutions in MGTs were studied in [77, 86, 63, 64, 65, 69] and, for relativistic flow theories, in [41, 42] by applying the AFDM. The goal of this section is to analyze possible physical implications of (non) stationary generic off-diagonal solutions describing entropic quasiperiodic deformations of some prime BH metrics. The geometric formalism elaborated in subsections [4.2.2] and [4.2.3] is summarized in Table 2 below.

### 5.1 Table 2: AFDM for constructing entropic flow stationary solutions

Considering a nonholonomic deformation procedure for a generating function \( h_4(\tau) = h_4(\tau, x^i, y^3) = h_4(\tau, \tau, \theta, \varphi) \) [43], we construct a class of off–diagonal stationary solutions with Killing symmetry on \( \partial_i \) determined by entropic flow sources \((A(Y(\tau), \overline{Y}(\tau))\) and running cosmological constant \( \Lambda(\tau)\),

\[
ds^2 = e^{\psi(\tau,x^k)}[(dx^1)^2 + (dx^2)^2] - \frac{[h_4^2(\tau)]^2}{\int dy^3 Y(\tau)[h_4^2(\tau)]} [dy^3 + \partial_i\int d\varphi Y(\tau)h_4^2(\tau) - \overline{h}_4(\tau)^2] dx^i
\]

\[
+ h_4(\tau)[dt + (4n_k + 2n_k) \int d\varphi [\int dy^3 Y(\tau)[h_4^2(\tau)] (h_4(\tau))^{5/2}] dx^k].
\]
Such solutions are, in general, with nontrivial nonholonomically induced torsion which can be nonholonomically constrained to LC-configurations \([14]\). Using nonlinear symmetries, it is possible to re-define such stationary metrics in terms of generating functions \(\Psi(t, r, \theta, \varphi)\) or \(\Phi(t, r, \theta, \varphi)\).

In terms of \(\eta\)-polarization functions, the stationary d-metrics and N-connections can be parameterized to describe nonholonomic deformations of a primary (for instance, BH) d-metric \(\tilde{g}\) into target generic off diagonal stationary solutions \(g\), see \([20]\), as \(\tilde{g} \rightarrow g = [g_{\alpha} = \eta_\alpha \tilde{g}_{\alpha}, \eta_i^a N_i^a]\).

| Table 2: Off-diagonal stationary elastic quasiperiodic solutions |
|---------------------------------------------------------------|
| **Exact solutions of** \(R_{\mu\nu}(t) = c^2 f_Y Y_{\mu\nu}(t)\) transformed into a system of nonlinear PDEs \([25], [51]\) |

| d-metric ansatz with \(\partial_4 = \partial_t\) |
|---------------------------------------------|
| \(ds^2 = g_4(\tau)(dx^4)^2 + g_{\alpha}(\tau)(dy^\alpha + N^\alpha(t)dx^4)^2\), for |
| \(g_4 = e^{\phi(\tau,x^4)}\), \(g_{\alpha} = h_\alpha(\tau, r, \theta, \varphi)\), \(N^\alpha = w_\alpha(\tau, r, \theta, \varphi)\), \(N^4 = n_4(\tau, r, \theta, \varphi)\), |
| \(Y_{\alpha\beta}(\tau) = [\eta Y(\tau, r, \theta, \varphi)Y(\tau, r, \theta, \varphi)\partial_4]\) |

| Nonlinear PDEs \([32]\) |
|-------------------------|
| \(\psi^{**} + \psi'' = 2 \ h_Y(\tau)\); |
| \(\omega = \ln |h_4 h_\alpha|/\sqrt|h_{44}|\); |
| \(\alpha_4 = (\partial_4 h_\alpha)/(\partial_4 \omega)\), |
| \(\beta_4 = (\partial_4 h_4)/(\partial_4 \omega)\), |
| \(\gamma = \partial_4 \ln |h_{44}/4h_4|^2/|h_4|^3\), |
| \(\partial_4 q = q, \partial_2 q = q', \partial_3 q = \partial q/\partial \varphi = q^o\) |

| Generating functions: \(h_4(\tau, r, \theta, \varphi)\), \(\Psi(\tau, r, \theta, \varphi) = e^{\omega} \Phi(\tau, r, \theta, \varphi)\); |
|---------------------------------------------|
| integration functions: \(h_{\alpha 4}(\tau, x^4)\), \(1 n_{\alpha}(\tau, x^4)\), \(2 n_{\alpha}(\tau, x^4)\); |

| \(\phi^{**} = -4 \Lambda(\tau) h_4\), see \([34]\). |

| Off-diag. solutions, d-metric N-connec. |
|----------------------------------------|
| \(g_4(\tau) = e^{\phi(\tau,x^4)}\) as a solution of 2-d Poison eqns. \(\psi^{**} + \psi'' = 2 \ h_Y(\tau)\); |
| \(h_4(\tau) = (-\Psi^2)/4\Psi^2 h_4\), see \([10]\). |
| \(h_4(\tau) = h_4^{[0]} - \int d\psi^2/4\Psi^2/4\Psi h_4^{[0]} - \Phi^2/4\Lambda(\tau)\); |
| \(w_\alpha(\tau) = \partial_\alpha \Psi/\partial_\psi \Psi = \partial_\alpha \Psi^2/\partial_\psi \Psi^2\); |
| \(n_\alpha(\tau) = \nu n_\alpha + 2 n_\alpha \int d\psi^2/\Psi^2 h_4^{[0]} - \int \Phi^2/4\Psi^2/4\Psi^{[2]}/2\); |

| \(\partial_\psi w_\alpha(\tau) = (\partial_\alpha - w_\alpha \partial_\psi) \ln \sqrt|h_4(\tau)|; (\partial_\alpha - w_\alpha \partial_\psi) \ln \sqrt|h_4(\tau)| = 0, \phi^{**} = \partial_\psi \phi/\partial_\psi \phi; \) |
| \(\Psi(\tau, x^4, \varphi) = \Psi[\psi] = \psi, \) or \(\psi = \text{const}\.\) |

| LC-configurations \([14]\) |
|------------------------|
| \(h_4(\tau) = h_4^{[0]} - \Phi^2/4\Lambda(\tau), h_4^{[2]} \neq 0, \Lambda(\tau) \neq 0 \) |

| N-connections, zero torsion |
|----------------------------|
| \(w_\alpha(\tau) = \partial_\alpha \Lambda(\tau) = \frac{\partial_\alpha (\int d\psi^2 \Psi^2 h_4^{[0]})/\Psi h_4^{[0]};}{\Psi^2/\partial_\psi \Psi}; \) |
| \(\partial_\psi \psi/\partial_\psi \psi; \) |
| \(\partial_\psi (\int d\psi^2 (\Phi^2)/\phi^{**} \Psi)\); |
| \(\eta_i(\tau) = \eta_i(\tau) = \partial_\psi n_i(\tau, x^4)\). \) |

| polarization functions |
|-----------------------|
| \(\tilde{g} \rightarrow \tilde{g} = [\eta_\alpha = \eta_\alpha, \eta_4^a N_i^a]\) |

| Prime metric defines a BH |
|---------------------------|
| \([\tilde{g}_4(\tau, \theta) = \tilde{h}_4(\tau, \theta); \tilde{N}_3^3 = \tilde{w}_3(\tau, \theta), \tilde{N}_4^4 = \tilde{h}_4(\tau, \theta)]\) |
| diagonalizable by frame/coordinate transforms. |

| Example of a prime metric |
|---------------------------|
| \(\tilde{g}_4 = (1 - r_\psi/r^4)^{-1}; \tilde{g}_4 = r^2; \tilde{h}_3 = r^2 \sin^2 \theta; \tilde{h}_4 = (1 - r_\psi/r^4), r_\psi = \text{const}\.\) |

| Solutions for polarization funct. |
|----------------------------------|
| \(\eta_4(\tau) = \eta_4(\tau, \theta, \varphi)\) as a generating function; \(\eta_4(\tau) = (\int d\psi^2/\Psi^2 h_{44}^{[1]} h_4^{[2]}/|h_{44}^{[2]}|)^{1/2}\); |
| \(\eta_4(\tau) = \eta_4(\tau, \theta, \varphi)\) as a generating function; \(\eta_4(\tau) = (\int d\psi^2/\Psi^2 h_{44}^{[1]} h_4^{[2]}/|h_{44}^{[2]}|)^{1/2}\); |
| \(\eta_4(\tau) = \eta_4(\tau, \theta, \varphi)\) as a generating function; \(\eta_4(\tau) = (\int d\psi^2/\Psi^2 h_{44}^{[1]} h_4^{[2]}/|h_{44}^{[2]}|)^{1/2}\); |

| Polariz. funct. with zero torsion |
|----------------------------------|
| \(\eta_4(\tau) = \eta_4(\tau, \theta, \varphi)\) as a generating function; \(\eta_4(\tau) = (\int d\psi^2/\Psi^2 h_{44}^{[1]} h_4^{[2]}/|h_{44}^{[2]}|)^{1/2}\); |
| \(\eta_4(\tau) = \eta_4(\tau, \theta, \varphi)\) as a generating function; \(\eta_4(\tau) = (\int d\psi^2/\Psi^2 h_{44}^{[1]} h_4^{[2]}/|h_{44}^{[2]}|)^{1/2}\); |
5.2 Nonlinear PDEs for entropic quasiperiodic stationary configurations

The goal of this subsection is to analyse two possibilities for constructing explicit examples of quasiperiodic stationary solutions of the entropic flow modified Einstein equations (52) transformed into systems of nonlinear PDEs with decoupling (32). The first one is to consider quasiperiodic sources of type (25) [ \( h \bar{\Psi}, \bar{\Psi} \) for entropic \( \text{tot} \, \Upsilon_{\mu\nu} \) (14)]. Such solutions can be constructed as additive or nonlinear functionals of a displacement vector field \( u^a[\varsigma, \vec{b}] \), which (in it turn) is a functional of functions \( \varsigma, \vec{b} \) subjected to certain quasiperiodic conditions of type (11) and/or (12). The second possibility is to consider nonlinear symmetries for generating functions \( h_4(\tau, [\varsigma, \vec{b}]) \), constructed as functionals of \( u^a[\varsigma, \vec{b}] \) and related via nonlinear symmetries (33).

5.2.1 Parametric stationary solutions for entropic quasiperiodic sources

We shall write that in an entropic flow source \( \Psi(\tau) \) there is, for instance, a term with left label "0" written \( \int_0^\infty \Psi(\tau) = \int_0^\infty \Psi(\tau, [\varsigma, \vec{b}]) \) if the corresponding term in \( \text{tot} \, \Upsilon_{\mu\nu} \) (11) is defined as a quasiperiodic functional on a displacement vector field \( u^a[\varsigma, \vec{b}] \). We say that the entropic force encodes a quasiperiodic structure under evolution. If it is written \( \int_0^\infty \Psi(\tau) \) without a left label "0", such a term correspond to a general \( \text{int} \, \Upsilon_{\mu\nu} \) (without any quasiperiodicity specification) in \( \text{tot} \, \Upsilon_{\mu\nu} \). Similarly, we shall write \( \int_0^\infty \Psi(\tau) = \Psi(\tau, [\varsigma, \vec{b}]) \), or \( \Psi(\tau) \). For simplicity, we shall not consider quasiperiodic functionals for \( \mu_{\nu} \) and/or \( \text{tot} \, \Upsilon_{\mu\nu} \) (i.e. we do not introduce terms of type \( \int_0^\infty \Psi(\tau, [\varsigma, \vec{b}]) \) because such constructions are similar to those for non-entropic MGTs, see details in [69]. In this work, an effective source term \( \text{tot} \, \Upsilon \) determined by geometric flows of the d-metric, \( \partial_t g_{\alpha\beta}(\tau) \), in (12) is introduced. It is quasiperiodic if the d-metric coefficients are quasiperiodic. We can consider entropic Ricci soliton configurations with \( \text{tot} \, \Upsilon = 0 \).

Stationary solutions with additive entropic quasiperiodic sources: For this class of solutions, we consider a source of type (25) (the left label is used for "additive stationary")

\[
\alpha \Psi(\tau) = \alpha \Psi(\tau, x^i, y^3) = \Psi(\tau, x^i, y^3) + m_{\nu}(\tau, x^i, y^3) + \text{int} \Psi(\tau, x^i, y^3) + \lambda \Psi(\tau, x^i, y^3).
\]

The second equation into (32) can be written \( \tau^\alpha \, h_1^\alpha = 2 h_3 h_4 \, \alpha \Psi \) and integrated on \( y^3 = \varphi \). In result, we generate entropic quasiperiodic off-diagonal metrics and generalized connections determined, for instance, by a generating function \( h_4(\tau, r, \theta, \varphi) \) with Killing symmetry on \( \partial_t \) and by effective sources \( (h, \alpha \Psi) \) and effective cosmolomical constant

\[
\alpha \Lambda(\tau) = \Psi(\tau, x^i, y^3) + m_{\nu}(\tau, x^i, y^3) + \text{int} \Lambda(\tau) + \lambda \Lambda(\tau)
\]

related to \( \alpha \Psi(\tau) \) (50) via nonlinear symmetry transforms (33).

Applying the AFDM following the procedure summarized in Table 2, we construct such a class of quadratic elements defining stationary entropic quasistationary solutions

\[
ds^2 = e^{\psi(\tau, x^k)} [(dx^1)^2 + (dx^2)^2] - \int d\varphi \, \frac{\alpha \Psi(h_4^\alpha(\tau))}{h_4(\tau)} \left[ dy^3 + \int d\varphi \, \frac{\alpha \Psi(h_4^\alpha(\tau))}{h_4(\tau)} \right] dx^i + h_4(\tau) \left[ dt + \left( 1 n_k(\tau) + 4 2 n_k(\tau) \right) \int d\varphi \, \frac{\alpha \Psi(h_4^\alpha(\tau))}{h_4(\tau)} \right] dx^k.
\]

Such solutions are, in general, with nontrivial nonholonomically induced torsion which can be constrained LC-conditions. The formulas (52) can be re-defined equivalently in terms of generating functions \( \Phi(\tau, r, \theta, \varphi) \) or \( \Phi(\tau, r, \theta, \varphi) \).
Nonlinear functionals for entropic quasiperiodic sources: We can consider entropic sources \( q^p Y(\tau, r, \theta, \varphi) = Y[\tau, \zeta, \b] \) as general nonlinear functionals on some entropic quasiperiodic data \([\zeta, \b]\). Such a source is subjected to nonlinear symmetries \( \Psi \), \( q^p \Lambda \) \( \Psi^2 = \Phi^2 \big| q^p Y \big| - \int d\varphi \Phi^2 \big| q^p Y \big| \), which allows us to introduce and effective cosmological constant \( q^p \Lambda \). We consider not a quasiperiodic generating function \( \Phi \) but the respective \( \Psi \) and \( q^p \Lambda \) will be quasiperiodic because of \( q^p Y \). The values \( q^p \Lambda \) can be different from \( \Lambda \) used for entropic configurations. The quasiperiodic functional \( q^p Y \) imposes certain nonholonomic constraints on \( mY \), and/or \( F^Y \) even such sources are not considered for quasiperiodic matter.

Stationary solutions of the nonlinear system PDEs \( (28) - (31) \) with entropic quasiperiodic source of type \( q^p Y \) are derived using the Table 2,

\[
ds^2 = e^{\psi(\tau,x^k)} [(dx^1)^2 + (dx^2)^2] - \left[ \frac{[h_4^\circ(\tau)]^2}{\int d\varphi \q Y(\tau)h_4^\circ(\tau)} \right] \left[ dy^3 + \partial_i \int d\varphi \q Y(\tau)h_4^\circ(\tau) dx^i \right] + h_4(\tau) \left[ dt + \left( 1n_k + 4n_k \int d\varphi \q Y(\tau)h_4^\circ(\tau) \right) dx^k \right].
\]

This formula is similar to that for the quadratic linear element \( \# \) but with another type of nonlinear generation functions for (effective) sources for dark and/or usual matter sources constructed as nonlinear functionals, when additive sources transform into a nonlinear functional for and effective source, \( q^p Y(\tau) \rightarrow \q Y(\tau) \) and \( q^p \Lambda(\tau) \rightarrow \q \Lambda(\tau) \). Similar re-definitions of additive quasiperiodic sources and cosmological constants into nonlinear functionals generate LC-configurations

\[
ds^2 = e^{\psi(\tau,x^k)} [(dx^1)^2 + (dx^2)^2] - \left[ \frac{[h_4^\circ(\tau)]^2}{\int d\varphi \q Y(\tau)h_4^\circ(\tau)} \right] \left[ dy^3 + \partial_i \Lambda(\tau) dx^i \right] + h_4(\tau) \left[ dt + (\partial_k n(\tau)) dx^k \right].
\]

We note that formulas \( \# \) provide respective generalizations of some classes of solutions \( \# \) considering general "functionals of functionals" with entropic quasiperiodic (effective) real and dark matter structures. Using corresponding functionals and parameters of nonlinear interactions, nonholonomic constraints, we can elaborate on stationary solutions imbedded into cosmic webs, some filaments branches, with quasiperiodic distributions etc.

5.2.2 Emergent gravity from stationary entropic quasiperiodic generating functions

We can generate generic off-diagonal stationary solutions using generalized for nonlinear quasiperiodic generating functionals \( q^p F(\tau) = q^p F[\tau, \zeta, \b] \) characterized by nonlinear symmetries of type \( \# \). The second equation into \( \# \) transforms into a functional equation

\[
\omega^\circ(\tau) \big[ q^p F(\tau), \Lambda(\tau) \big] h_4^\circ(\tau) \big[ q^p F(\tau), \Lambda(\tau) \big] = 2h_3(\tau) \big[ q^p F(\tau), \Lambda(\tau) \big] h_4(\tau) \big[ q^p F(\tau), \Lambda(\tau) \big] Y(\tau),
\]

which can be solved together with other equations form the system \( (28) - (31) \) for an arbitrary effective source \( Y(\tau) \) (it may not contain entropic fields) following AFDM summarized in Table 2.

The solutions for such stationary configurations in entropic quasiperiodic gravity determined by general nonlinear functionals for generating functions can be written in all forms \( \# \). For simplicity, we present here only the third type parametrization

\[
ds^2 = e^{\psi(\tau,x^k)} [(dx^1)^2 + (dx^2)^2] - \frac{\left( q^p F^2(\tau) \right)^\circ \left[ \left( q^p F^2(\tau) \right)^\circ \right]}{\Lambda(\tau) \int dy^3 Y(\tau) \left[ \left( q^p F(\tau) \right)^2 \right]^\circ \left( h_4[0] - \frac{q^p F^2(\tau)}{4\Lambda(\tau)} \right)} \left[ dy^3 + \partial_i \int dy^3 Y(\tau) \left[ \left( q^p F(\tau) \right)^2 \right]^\circ \left( h_4[0] - \frac{q^p F^2(\tau)}{4\Lambda(\tau)} \right) \right] dx^i + \left( h_4[0] - \frac{q^p F^2(\tau)}{4\Lambda(\tau)} \right) \left[ dt + (1n_k)(\tau,x^k) + 2n_k(\tau,x^k) \right] \int dy^3 \left[ \left( q^p F^2(\tau) \right)^\circ \left[ \left( q^p F^2(\tau) \right)^\circ \right] \left( h_4[0] - \frac{q^p F^2(\tau)}{4\Lambda(\tau)} \right) \left[ dt + (1n_k)(\tau,x^k) + 2n_k(\tau,x^k) \right] \int dy^3 \right].
\]
Imposing additional zero torsion constraints, there are extracted LC-configurations,

\[ ds^2 = e^{\psi(\tau)}[(dx^1)^2 + (dx^2)^2] - \frac{(\varphi \Phi^2(\tau))^\circ[\varphi \Phi^2(\tau)]}{|\varphi \Lambda(\tau)\int dy^3 \varphi \Psi(\tau)[(\varphi \Phi(\tau))^2]^\circ| (h_4^{[0]}(\tau, x^k) - \frac{\varphi \Phi^2(\tau)}{4 \varphi \Lambda(\tau)})^3 [dy^3 + (\partial_i \bar{A}(\tau))dx^i]}

\]

for certain generating functions \( \bar{A}(\tau) \) and \( n(\tau) \).

For the classes of solutions (55) and (56), the gravitational fields interactions are modelled by elastic terms contained in generating functionals. Such quasiperiodic structures encode certain dark energy nonlinear distributions with a more rich nonholonomic geometric structure and generalized nonlinear symmetries. For fixed values \( \tau_0 \), we can identify \( \Lambda \) for elastic models of gravity. The coefficients of this class of d–metrics can be chosen to be of necessary smooth class (for instance, nonsingular ones) when there are modelled certain stochastic sources, spacetime elastic configuration, fractional derivative processes, quantum corrections etc.

### 5.2.3 Nonlinear functionals for entropic quasiperiodic generating functions & sources

We can generate singular or nonsingular stationary off-diagonal generalized quasiperiodic solutions of the entropic flow modified Einstein equations determined both by nonlinear functionals for generating functions, \( \varphi \Phi(\tau) = \varphi \Phi[\tau, \varsigma, \bar{b}] \), and nonlinear functionals for (effective) sources, \( \varphi \Psi(\tau, \rho, \phi, \varphi) = \Psi[\tau, \varsigma, \bar{b}] \), as we considered above. The quasiperiodic data (for instance, scales, interaction constants and associated free energies) for the generating functions with nonlinear elastic properties are different from the quasiperiodic data for entropic flow sources. Nevertheless, such data can not be arbitrary independent ones but connected via nonlinear symmetries of type (53), \( \varphi \Lambda(\tau) \varphi \Phi^2(\tau) = \varphi \Phi^2(\tau)| \varphi \Psi(\tau)| - \int d\phi \varphi \Phi^2(\tau)| \varphi \Psi(\tau)|^\circ \).

Using the AFDM summarized in Table 2 but with generating functions and sources of type (\( \varphi \Phi(\tau), \varphi \Psi(\tau) \)), and/or, equivalently, (\( \varphi \Phi(\tau), \varphi \Lambda(\tau) \)), a multi-functional nonlinear generalization of stationary solutions (53) and (55) is constructed in the form

\[ ds^2 = e^{\psi(\tau, x^k)}[(dx^1)^2 + (dx^2)^2] - \frac{(\varphi \Phi^2(\tau))^\circ[\varphi \Phi^2(\tau)]}{\varphi \Lambda(\tau)\int dy^3 \varphi \Psi(\tau)[(\varphi \Phi(\tau))^2]^\circ| (h_4^{[0]}(\tau, x^k) - \frac{\varphi \Phi^2(\tau)}{4 \varphi \Lambda(\tau)})^3 [dy^3 + (\partial_i n(\tau))dx^i]}

\]

The class of solutions (55) describes off-diagonal stationary entropic configurations determined by multi-functional nonlinear quasiperiodic structures both in relativistic geometric flow theories and for nonholonomic Ricci soliton, equivalent, entropic MGTS. Such exact and parametric solutions can be used for modeling flow evolution and interactions of dark energy (nonlinear gravitational distributions) and dark (and standard) matter fields. In explicit form, such functional data for emergent gravity and geometric flows generating functions and sources can be stated to be compatible with observations in modern astrophysics and cosmology.

### 5.3 BHs in (off-) diagonal stationary entropic quasiperiodic media

New classes of classes of generic off-diagonal stationary solutions can be described in terms of \( \eta \)–polarization functions introduced in formulas (20) and following the AFDM summarized in Tables
1 and 2. To follow such an approach is useful if we are interested to compute and study physical implications of certain nonholonomic deformations of known metric \( \tilde{g} \) into a target \( g \) ones encoding certain effects, for instance, of entropic geometric flows and/or certain quasiperiodic structures. As a primary metric we consider a primary BH \( d \)-metric (for instance, it can be a Schwarzschild or Kerr metric) defined by data \( \tilde{g} = [\tilde{g}_i(r, \theta, \varphi), \tilde{g}_{ab} = \tilde{h}_a(r, \theta, \varphi); \tilde{N}^3_k = \tilde{w}_k(r, \theta, \varphi), \tilde{N}^4_k = \tilde{n}_k(r, \theta, \varphi)] \) \((27)\) which can be diagonalized by frame/coordinate transforms. The stationary target metrics \( g \) are generated by nonholonomic \( \eta \)-deformations, \( \tilde{g} \to g(\tau) = [\eta_i(\tau, x^k) = \eta_i(\tau)\tilde{g}_i, \eta_0(\tau, x^k, y^3) = \eta_0(\tau)\tilde{g}_0, \eta^a_i(\tau, x^k, y^3) = \eta^a_i(\tau)\tilde{N}^a_i] \), and constrained to the conditions to define exact and parametric solutions of the system of nonlinear PDEs with decoupling \((32)\). The quadratic line elements corresponding to by \( d \)-metrics are parameterized in some forms similar to \((26)\).

\[
ds^2 = \eta(\tau, r, \theta, \varphi)\tilde{g}_i(r, \theta, \varphi)[dx^i(r, \theta, \varphi)]^2 + \eta(\tau, r, \theta, \varphi)\tilde{g}_a(r, \theta, \varphi)[d\varphi + \eta^a_k(\tau, r, \theta, \varphi)\tilde{N}^a_k(r, \theta, \varphi)dx^k(r, \theta, \varphi)]^2,
\]

with summation on repeating contracted low-up indices. The values \( \eta(\tau) \) and \( \eta^a(\tau) \) are determined by entropic quasiperiodic flows and nonlinear interactions.

5.3.1 Singular stationary solutions generated by entropic quasiperiodic sources

Considering entropic quasiperiodic sources of type \( \psi^qY(\tau, r, \theta, \varphi) = Y[\tau, \zeta, \tilde{b}] \) \((25)\) as in \((53)\), we compute the coefficients for solutions of type \((58)\) following formulas (see also Table 2).

\[
\eta_1(\tau) = \frac{e^\psi(\tau, x^k)}{g_i}; \eta_3(\tau) = -\frac{4(1(\eta_1(\tau)\tilde{h}_4)^{1/2})^2}{\tilde{h}_3}\int dy^3 \; \psi^qY(\tau)\eta(\tau)\tilde{h}_4^\circ; \\
\eta_4(\tau) = \eta_4(\tau, r, \theta, \varphi) \text{ as a generating function}; \\
\eta^3_k(\tau) = \frac{\partial_i}{\tilde{w}_i} \psi^qY(\tau)(\eta_1(\tau)\tilde{h}_4)^\circ; \eta^4_k(\tau) = \frac{1\eta_k}{\tilde{n}_k} + 16 \frac{\tilde{w}_i}{\tilde{n}_k} \int dy^3 \psi^qY(\tau)(\eta_1(\tau)\tilde{h}_4)^\circ, \\
\]

for integration functions \( 1\eta_k(\tau, r, \theta) \) and \( 2\eta_k(\tau, r, \theta) \).

In formulas \((59)\) the value \( \eta_4(\tau, r, \theta) \) is taken as a (non) singular generating function which using nonlinear symmetries \((33)\) can be related to other type generating functions,

\[
\Phi^q(\tau) = -4 \psi^q\Lambda(\tau)\tilde{h}_4(\tau) = -4 \psi^q\Lambda[\eta_4(\tau, r, \theta, \varphi)\tilde{h}_4(\tau, r, \theta, \varphi)], \\
(\Psi^q)^\circ(\tau) = -\int d\varphi \psi^qY(\tau)\eta_4(\tau, r, \theta, \varphi)\tilde{h}_4(\tau, r, \theta, \varphi)^\circ.
\]

We can constrain the coefficients \((59)\) to a subclass of data generating target stationary off-diagonal metrics of type \((44)\) with zero torsion.

The nonlinear functionals for the entropic quasiperiodic \( \psi \)-source and (effective) cosmological constant considered above can be changed into additive functionals \( \psi^qY \to a^\circ Y \) and \( \psi^q\Lambda \to a^\circ \Lambda \) as in \( a^\circ Y \) \((50)\) and \( a^\circ \Lambda \) \((51)\). The singular behaviour of such solutions is generated by the prime BH data which can be preserved or changed for different classes of generating and integration functions. For certain classes of generating functions and sources and small nonholonomic deformations, the same type of singularity is preserved. Similar stationary configurations can be computed for general entropic quasiperiodic structures. The constructions depend on the type of explicit geometric evolution or dynamical model we construct (for instance, with certain web / filament / solitonic stationary distributions). Such generic off-diagonal stationary entropic solutions can be considered as certain conventional nonholonomically deformed BH configurations imbedded into some elastic (non) singular media with flows and off-diagonal interactions determined by stationary entropic fields modeling dark and usual matter quasiperiodic distributions.
5.3.2 BH solutions deformed by entropic quasiperiodic generating functions

Solutions with entropic $\eta$-polarizations can be constructed with coefficients of the d-metrics determined by nonlinear generating functionals $\Phi[\tau, \zeta, \overline{\eta}]$, or any additive functionals $\Phi[\tau, \zeta, \overline{\eta}]$, including terms with possible integration functions $h_4^0(\tau, r, \theta)$ for $h_4[\tau, \zeta, \overline{\eta}]$. Such configurations are defined also by some prescribed data $\Upsilon(\tau, r, \theta, \varphi)$ and $\Lambda(\tau)$, which are not obligatory of entropic or quasiperiodic nature. For formulas for nonlinear symmetries, we can compute (recurrently) corresponding nonlinear functionals, $\Phi \eta_4(\tau, r, \theta, \varphi)$ (for simplicity, we omit here similar formulas for additive functionals $\eta_4(\tau, r, \theta, \varphi)$) and related polarization functions,

\[ \Phi \eta_4(\tau) = -\frac{\Phi^2(\tau, \zeta, \overline{\eta})}{4\Lambda(\tau)}h_4(\tau, r, \theta), \]

\[ (\Phi \Psi^2(\tau))^\circ = -\int d\varphi \ U(\tau, r, \theta, \varphi)h_4^2(\tau) = -\int d\varphi \ U(\tau, r, \theta, \varphi)[\Phi \eta_4(\tau, r, \theta, \varphi)h_4(\tau, r, \varphi)]^\circ. \]

Using such above formulas for Table 2, the coefficients of d-metric are computed

\[ \eta_1(\tau) = e^{-\psi(\tau,x^k)}; \eta_3 = -\frac{4[|\Phi \eta_4(\tau)\hat{h}_4|^{1/2}]^2}{h_3|\int d\varphi \ U(\tau)\Phi \eta_4(\tau)h_4|^2}; \eta_4(\tau) = \Phi \eta_4(\tau, r, \theta, \varphi) \text{ as a generating function}; \]

\[ \eta^3_3(\tau) = \frac{1}{n_k} \eta_k(\tau) = \frac{1}{n_k} + 16 \frac{n_k}{\tilde{n}_k} \int d\varphi \ \left(\frac{\Phi \eta_4(\tau)\hat{h}_4}{\int d\varphi \ U(\tau)\Phi \eta_4(\tau)h_4}\right)^2, \]

for integrating functions $\frac{1}{n_k}(\tau, r, \theta)$ and $\frac{2}{n_k}(\tau, r, \theta)$.

Using $\eta(\tau)$, target stationary off-diagonal metrics with zero torsion can be generated by polarization functions subjected to additional nonholonomic constraints and integrability conditions,

\[ \eta_1(\tau) = e^{-\psi(\tau,x^k)}; \eta_3(\tau) = -\frac{4[|\Phi \eta_4(\tau)\hat{h}_4|^{1/2}]^2}{h_3|\int d\varphi \ U(\tau)\Phi \eta_4(\tau)h_4|^2}; \eta_4(\tau) = \tilde{\eta}_4(\tau, r, \theta, \varphi) \text{ as a generating function}; \eta_3^3(\tau) = \frac{1}{\tilde{n}_k} \eta_k(\tau) = \frac{1}{\tilde{n}_k} + 16 \frac{n_k}{\tilde{n}_k} \int d\varphi \ \left(\frac{\Phi \tilde{A}(\tau)\hat{h}_4}{\int d\varphi \ U(\tau)\Phi \tilde{A}(\tau)h_4}\right)^2, \]

for an integrating functions $\frac{1}{n_k}(\tau, r, \theta)$ and a generating function $\Phi \tilde{A}(\tau, r, \theta, \varphi)$.

The solutions constructed in this subsection describe certain nonholonomically deformed BH configurations self-consistently imbedded into entropic quasiperiodic gravitational media modeling certain elastic properties for nonholonomic dark energy distributions.

5.3.3 Stationary BH deformations by entropic sources & generating functions

More general classes of stationary entropic quasiperiodic deformations of BHs can be constructed using nonlinear functionals both for the generating functions and sources. Nonlinear superpositions of solutions of type and can be performed if the coefficients of d-metric are computed

\[ \eta_1(\tau) = e^{-\psi(\tau,x^k)}; \eta_3(\tau) = -\frac{4[|\Phi \eta_4(\tau)\hat{h}_4|^{1/2}]^2}{h_3|\int d\varphi \ U(\tau)\Phi \eta_4(\tau)h_4|^2}; \]

\[ \eta_4(\tau) = \Phi \eta_4(\tau, r, \theta, \varphi) \text{ as a generating function}; \eta_3^3(\tau) = \frac{1}{\tilde{n}_k} \eta_k(\tau) = \frac{1}{\tilde{n}_k} + 16 \frac{n_k}{\tilde{n}_k} \int d\varphi \ \left(\frac{\Phi \tilde{A}(\tau)\hat{h}_4}{\int d\varphi \ U(\tau)\Phi \tilde{A}(\tau)h_4}\right)^2, \]
where $n_k(\tau, x^k)$ and $2n_k(\tau, x^k)$ are integration functions.

In (61), we consider a nonlinear generating functional $\mathcal{F}(\tau, \xi, \delta)$ and some prescribed nonlinear functionals $\mathcal{Y}(\tau, r, \theta, \varphi)$ and $\mathcal{A}(\tau)$ related via nonlinear symmetries generalizing (33). This allows us to compute corresponding nonlinear functionals $\mathcal{H}_k(\tau, r, \theta, \varphi)$ and polarization functions,

$$\mathcal{F}\eta_4(\tau) = -\mathcal{F}^2(\tau, r, \theta, \varphi)/4 \mathcal{A}(\tau)\hat{h}_4(\tau, r, \theta, \varphi),$$

$$(\mathcal{F}\mathcal{Y}(\tau))^\circ = -\int d\varphi \mathcal{F}(\tau)\hat{h}_4(\tau) = -\int d\varphi \mathcal{Y}(\tau, r, \theta, \varphi)[\mathcal{F}\eta_4(\tau, r, \theta, \varphi)]^\circ.$$

Imposing additional conditions for a zero torsion, target stationary metrics (44) are generated.

The class of stationary solutions (61) describes nonholonomic entropic deformations of a BH self-consistently imbedded into quasiperiodic gravitational (elastic dark energy) backgrounds and entropic quasiperiodic dark and/or standard matter.

5.4 Off–diagonal deformations of Kerr metrics by entropic flow sources

In this section, we study how effective sources for entropic quasiperiodic flows result in generic off–diagonal deformations and generalizations of the 4-d Kerr metric and construct such new classes of exact solutions of (32), see appendices A.6 and A.5

5.4.1 Nonholonomic evolution of Kerr metrics with induced (or zero) torsion

Let us consider the coefficients (A.22) a Kerr solution as a prime metric $\hat{g}$ when $\hat{g}_1 = 1, \hat{g}_2 = 1, \hat{g}_3 = \hat{C}/\hat{A}$, but (for certain coordinate transforms) $g_{11} = \hat{C}/\hat{A}, g_{33} = \hat{C}/\hat{A}, \hat{N}_i = \hat{N}_i = -\partial_\nu(\hat{y}^3 + \varphi\hat{B}/\hat{A}) \to 0$. If such conditions are satisfied by a local coordinate system, we have $\hat{g}_3^3 \neq 0$ which allows us to construct nonholonomic deformations following the geometric formalism outlined in appendix A.5

For general $\eta$–deformations as in (A.17) and constraints $n_i = 0$, the entropic quasiperiodic flow modifications of the Kerr metric are computed (see Table 2)

$$ds^2 = e^{\psi(\tau, x^k)}[(dx^1)^2 + (dx^2)^2] - 4[|\eta_4(\tau)^{\hat{A}}|^2]^2 |\int dy^3 \mathcal{Y}(\tau)|\hat{A}|^{1/2}|\left(C - \frac{B^2}{A}\right)(\mathcal{E}^3)^2 + \eta_4(\tau)^{\hat{A}}(\mathcal{E}^4)^2,$$

$$\mathcal{E}^3 = dy^3 + \frac{\partial_\nu}{\mathcal{Y}(\tau)} \int dy^3 \mathcal{Y}(\tau)|\eta_4(\tau)^{\hat{A}}|dx', \mathcal{E}^4 = dt,$$

(62)

where $\eta_4(\tau) = \eta_4(\tau, x^k, y^3)$ is a generating function, $\mathcal{Y}(\tau) = \mathcal{Y}(\tau, x^k, y^3)$ is an entropic quasistationary flow generating source (42) and $\psi(\tau, x^k)$ is a solution of a 2-d Poisson equation (40)

Deformations of the Kerr and other type BH solutions [62] can be constructed in various theories with noncommutative and commutative variables, for warped and trapped brane type configurations in string, Finsler like and/or Hořava–Lifshitz MGTs, see [77, 86, 63, 64, 65, 69] and references therein, when nonholonomically induced torsion effects play a substantial role. We can extract LC-configurations for certain integrable generating functions $\eta_4(\tau)$ and a function $\hat{A}(\tau, x^k, y^3)$ defined by the conditions $\partial_\nu\hat{A} = \partial_{\nu}\int dy^3 \mathcal{Y}(\tau)|\eta_4(\tau)^{\hat{A}}|/\mathcal{Y}(\tau)|\eta_4(\tau)^{\hat{A}}|^\circ$. By straightforward computations

17In principle, we can consider nonholonomic deformations with $\hat{g}_3^3 = 0$ and/or $g_3^3 \neq 0$ when the solutions are constructed on certain hypersurfaces and certain models are with singular geometric evolution as we considered in our previous works [63, 64, 65]; for simplicity (in this work), we study elastic spacetime deformations in N-adapted coordinates which do not induce additional singularities with respect to a prime metric.
of the anholonomy coefficients of solutions of type (62) (even certain LC-conditions are satisfied), we find that such metrics are generic off-diagonal. They describe self-consistent embedding of the Kerr metric into elastic quasiperiodic flow spacetime media. In general, such target metrics may be not of BH type. If the \( \eta \)-polarizations are smooth functions and the coefficients preserve the character of the prime metric with \( \bar{C} - \bar{B}/\bar{A} \) and \( \bar{A} \), we can model geometric flow evolution with small deformations of the horizons and polarizations of the physical constants of Kerr solutions as in next subsections.

### 5.4.2 Small parametric modifications of Kerr BHs and effective entropic flow sources

We investigate models of entropic quasi-periodic geometric flows for nonholonomic distributions describing \( \varepsilon \)-deformations described by formulas (A.13). Such deformations of a prime Kerr metric (A.22) with \( \bar{A}(r, \theta) \rightarrow \bar{A}[x^{i}(r, \theta, y^{3})] \) for \( \bar{g}^{i}_{i} \neq 0 \) result in stationary target metrics (A.17) with coefficients (A.19). The corresponding quadratic line elements with d-metric and N-connection coefficients depending on parameters \( \varepsilon, \tau \) and respective h- and v-coordinates are written in the form

\[
\begin{align*}
    ds^{2} &= [1 + \varepsilon e^{0 \psi} \frac{1}{g_{i}} \frac{\partial}{\partial x^{i}} \frac{\partial}{\partial y^{3}}]^{2} + \left[ 1 + \varepsilon [v_{4}^{i} - \frac{\partial}{\partial y^{4}} \frac{\partial}{\partial y^{3}}]^{2} + \left[ 1 + \varepsilon v_{4}^{i}]^{2} = dx^{4} = dt.
\end{align*}
\]

In these formulas, \( 0 \psi(\tau) = 0 \psi(\tau, x^{k}) \) and \( 1 \psi(\tau) = 1 \psi(\tau, x^{k}) \) are solutions of 2-d Poisson equations with a generating h-source \( h \psi(\tau) = h \psi(\tau, x^{k}) = 0 \psi(\tau, x^{k}) + \varepsilon 1 \psi(\tau, x^{k}) \) as described in appendix A.5. \( \psi(\tau) = \psi(\tau, x^{k}, y^{3}) \) is a generating v-source for which a \( \varepsilon \)-decomposition is possible (we omit such cumbersome formulas in this work); \( v_{4}^{i} = v(\tau) = v(\tau, x^{k}, y^{3}) \) is a generating function. The parameterizations in (63) are for N-adapted systems of references and space coordinates \( [x^{i}(r, \theta, \phi), y^{3}(r, \theta, \phi)] \) for which \( \hat{g}_{i}^{i} = \frac{\partial}{\partial y^{4}} \frac{\partial}{\partial y^{3}} \) allow to compute \( \hat{g}_{i}^{i} \) and \( \hat{w} = \frac{\partial}{\partial y^{4}} [d^{4} \frac{\partial}{\partial y^{3}} / \psi(\tau) \hat{g}_{i}^{i}] \) is computed for some prescribed \( \psi \) and \( \hat{g}_{i}^{i} \neq 0 \). To simplify the formulas, we chosen integration functions \( 1n_{k}(\tau) = 0 \) and \( 2n_{k}(\tau) = 0 \) for which \( N_{1} = n_{1} = 0 \) but \( N_{3} = w_{i}(\varepsilon, \tau, x^{k}, y^{3}) \) are not trivial and result in nonzero nonholonomic torsion and anholonomy coefficients. We can impose additional constraints on \( \psi(\tau) \) and sources which allow us to extract LC-configurations as described in footnote 14.

### 5.4.3 Ellipsoidal entropic quasi-periodic deformations of Kerr metrics

We study an explicit example when the Kerr primary data (A.22) are nonholonomically deformed by entropic quasi-periodic flows into target as ellipsoidal configurations with a small eccentricity parameter \( \varepsilon \). Such new classes of generic off-diagonal solutions contain more degrees of freedom (with nontrivial anholonomic \( w \)-coefficients) are different from the standard Kerr metric even that also has ellipsoidal type ergo spheres. It should be emphasized here that BH uniqueness theorems cannot be directly applied to MGTs even certain stability properties can be proved for so-called black ellipsoids configurations, see discussion and references in [77, 86, 63, 64, 65, 69].

Let us construct solutions for \( \varepsilon \)-deformations of type (63) resulting in ellipsoidal configurations. We choose a generating function \( v(\tau, x^{k}, y^{3}) \), when the constraint

\[
g_{4}(\tau, x^{k}, y^{3}) = [1 + \varepsilon v(\tau, x^{k}, y^{3})]g_{4}(x^{k}, y^{3}) = 0 \tag{64}
\]

defines a stationary roto-dilatation configuration which, in general, is different from the equations for ergosphere for a standard Kerr BH. We consider that \( \hat{g}_{4}(x^{k}, y^{3}) \) is defined by any coordinate transforms \( r = r(x^{1}), \vartheta = \vartheta(x^{2}, y^{3}), \varphi = \varphi(y^{3}) \), when \( \partial \hat{g}_{4}/\partial \varphi = \varphi(\varphi) \), then \( \partial \hat{g}_{4}/\partial \varphi = \varphi(\varphi) \).
the Kerr metric in some systems of coordinates. We write \( \tilde{A}(r, x^2, \varphi) = \tilde{A}(r, \vartheta(x^2, y^3)) = \tilde{g}_4(x^k, y^3) \) for \( \tilde{A} = -\Xi^{-1}(\Delta - a^2 \sin^2 \vartheta) \) and \( \Delta = r^2 - 2m_0 + a^2, \) \( \Xi = r^2 + a^2 \cos^2 \vartheta \) stated in appendix A.6 when the standard Kerr metric is re-written in respective nonholonomic variables. Then we prescribe

\[
v(\tau, \varphi) = 2\zeta(\tau) \sin[\omega_0(\tau) \varphi + \varphi_0(\tau)],
\]

for possible flow parameter dependence of \( \zeta, \omega_0 \) and \( \varphi_0 \) (i.e. we suppose that in our models with entropic quasiperiodic flows there are possible flow and anisotropic modifications of some effective constants). For such assumptions, the formula (64) results in

\[
g_4(\tau) = \tilde{A}(r, x^2, \varphi)[1 + \varepsilon v(\tau, \varphi)]) = \tilde{A}(r, \vartheta, \varphi) = -\Xi^{-1}(\tilde{\Delta} - a^2 \sin^2 \vartheta), \tilde{\Delta}(\tau, r, \varphi) = r^2 - 2m(\tau, \varphi) + a^2,
\]

are considered as \( \varepsilon \)-deformations of the coefficients of the Kerr metric (A.22). For \( \varepsilon \rightarrow 0 \), we obtain

\[
m(\tau, \varphi) = m_0(1 + \varepsilon \zeta(\tau) \sin[\omega_0(\tau) \varphi + \varphi_0(\tau)]).
\]

We get an effective anisotropically polarized and entropic flow running mass \( m = m_0(1 + \varepsilon \zeta(\tau) \sin[\omega_0(\tau) \varphi + \varphi_0(\tau)]) \). In result, the horizon condition \( g_4 = 0, \) states an ellipsoidal "deformed horizon" \( r(\tau, \vartheta, \varphi) = m(\tau, \varphi) + (m^2(\tau, \varphi) - a^2 \sin^2 \vartheta)^{1/2} \). If \( a = 0 \), we obtain the parametric formula for a family of ellipses parameterized by a running \( \tau \) and with eccentricity \( \varepsilon \) and radial coordinates \( r_+ = 2m_0/1 + \varepsilon \zeta(\tau) \sin[\omega_0(\tau) \varphi + \varphi_0(\tau)] \). Such configurations can be described alternatively using generating functions \( \Psi(\tau, r, x^2, \varphi) \) and/or \( \Phi(\tau, r, x^2, \varphi) \) related via nonlinear symmetries (33), when

\[
g_4(\tau) = \tilde{A}(r, x^2, \varphi)(1 + \varepsilon 2\zeta(\tau) \sin[\omega_0(\tau) \varphi + \varphi_0(\tau)])) = h_i^0(\tau, r, x^2) - \int dy^3(\Psi^2(\tau))(\Phi(\tau) = h_i^0(\tau) - \Phi^2(\tau)/4\Lambda(\tau).\]

Formulas with \( \Psi \) and \( \Phi \) encode contributions from entropic quasiperiodic flow effective sources and/or running cosmological constants. If \( g_4(\tau) \) or \( \eta_4(\tau) \), or \( v(\tau) \), are stated as generating functions, the entropic data are encoded in values of \( g_4(\tau) \) and off-diagonal N-connection coefficients like \( w_i(\tau) \).

The corresponding entropic and quasiperiodic deformed Kerr spacetimes have one Killing symmetry on \( \partial/\partial y^3 \). For small \( \varepsilon \), the singularity at \( \Xi = 0 \) is "hidden" under a family of ellipsoidal deformed horizons if \( m_0 \geq a \). Fixing a \( \tau_0 \), we obtain nonholonomic stationary Ricci solitons which are similar to the Kerr solution and possess \( \varphi \)-deformed both an event horizon and a Cauchy horizon, respectively, \( r_+ = m(\varphi) + (m^2(\varphi) - a^2 \sin^2 \vartheta)^{1/2} \) and \( r_- = m(\varphi) - (m^2(\varphi) - a^2 \sin^2 \vartheta)^{1/2} \). These entropic quasiperiodic structure is effectively imbedded into an off–diagonal background.

Using an ellipsoid type generating function (65) in (63), we construct a class of generic off–diagonal stationary solutions of modified Einstein equations which being N-adapted as \( \varepsilon \)-deformations describe ellipsoid like entropic evolution and nonholonomic deformations of the Kerr metric. The corresponding families of quadratic line elements are

\[
ds^2(\tau, \varepsilon) = (1 + \varepsilon e^{\psi} [\psi]_g)^g_i(x_i(x_j) + [1 + 2\varepsilon(\tilde{\zeta} \tilde{\sin}(\omega_0 \varphi + \varphi_0) \tilde{g}_4)] - \int \frac{dy^3(\tilde{\Psi}^2(\tau)}{\int \frac{dy^3(\tilde{\Psi} \tilde{g}_i)}} (\tilde{g}_3(\varepsilon)^2 + [1 + 2\varepsilon \tilde{\zeta} \tilde{\sin}(\omega_0 \varphi + \varphi_0) \tilde{g}_4(\varepsilon)^2}\]

with respect to a N-adapted basis \( \varepsilon^i = dy^3 + [1 + 2\varepsilon(\partial_i \int dy^3(\tilde{g}_i) \tilde{g}_j) - \tilde{\zeta} \tilde{\sin}(\omega_0 \varphi + \varphi_0) \tilde{g}_4(\varepsilon)^2] \tilde{w}_i dx^i, \)

\( e^4 = dx^4 = dt, \) and coordinate conditions stated in the previous subsection for generating stationary configurations with singular coefficients as in the Kerr metric. Fixing in above formulas \( a = 0 \) for a \( \varepsilon \neq 0 \), we get ellipsoidal entropic quasiperiodic flow deformations of the Schwarzschild BHs.
6 Entropic quasiperiodic flows and cosmological solutions

The goal of this section is to consider physical implications of models with entropic and quasiperiodic flow evolution of locally anisotropic and inhomogeneous cosmological spacetimes. The geometric formalism and technical results are summarized in Table 3 and appendix [A.3].

6.1 Table 3 on the AFDM for entropic flow cosmological solutions

We outline the key steps on the AFDM for generating cosmological solutions with geometric flows and Killing symmetry on \( \partial_3 \). Considering a nonholonomic deformation procedure for a generating function \( g_3(\tau) = \gamma_3(\tau, x^i, y^3) \) \(^{[45]}\), cosmological constants \( \Lambda(\tau) \) and sources \( h\gamma(\tau) = h\gamma(\tau, x^k) \) and \( \gamma(\tau, x^k) \), see parameterizations \(^{[25]}\) and nonlinear symmetries \(^{[39]}\), we construct exact solutions of the system of nonlinear PDEs for emergent cosmology \(^{[38]}\).

Typical cosmological solutions of this class are parameterised

\[
\int dt \left[ \frac{[h_3^*(\tau)]^2}{[\int dt \gamma(\tau) h_3^*(\tau) [\gamma(\tau)]^{3/2} dx^k]} \right] - \int dt \left[ \frac{\partial_i (\int dt \gamma(\tau) h_3^*(\tau))}{\gamma(\tau) h_3^*(\tau)} dx^i \right].
\]

Such quadratic line elements are time dual to the stationary ones considered for Table 2.

6.2 Nonlinear PDEs for entropic quasiperiodic cosmology

We analyse two possibilities to transform the entropic flow modified Einstein equations \(^{[22]}\) into systems of nonlinear PDEs \(^{[34]}\) \(^{[37]}\) with generic off-diagonal or diagonal solutions depending in explicit form on a evolution parameter, a time like variable and two space like coordinates. In the first case, there are considered entropic quasiperiodic sources determined by some additive or general nonlinear functionals for effective matter fields. In the second case, respective nonlinear functionals determining quasiperiodic solutions for entropic configurations are prescribed for generating functions subjected to nonlinear symmetries \(^{[39]}\). We also note that is is possible to construct certain classes of locally anisotropic and inhomogeneous cosmological solutions using nonlinear / additive functionals both for generating functions and (effective) sources.

6.2.1 Cosmological solutions for entropic quasiperiodic sources

Cosmological configurations generated by additive entropic functionals for sources: For this class of cosmological solutions, we consider an additive functional for an entropic quasiperiodic source of type \( \gamma(\tau, x^i, t) \) \(^{[25]}\),

\[
\Lambda^{\gamma}(\tau) = \Lambda^{\gamma}(\tau, x^i, t) = \int dt \gamma(\tau, x^i, t) + \gamma(\tau, x^i, t) + \gamma(\tau, x^i, t) + \gamma(\tau, x^i, t) + \gamma(\tau, x^i, t). \quad (67)
\]

There is also an associated additive cosmological constant \( \Lambda^{\gamma}(\tau) \) \(^{[49]}\) related to different types of generating functions via nonlinear symmetries \(^{[39]}\) when the equation \(^{[35]}\) transforms into \( \gamma^{\gamma} \gamma^{\gamma} = 2\gamma^{\gamma} \gamma^{\gamma} \) \( \Lambda^{\gamma}(\tau) \) and can be integrated on time like variable \( y^1 = t \). The systems of nonlinear PDEs \(^{[38]}\) can be integrated following the procedure summarized in Table 3. Such generic off-diagonal cosmological solutions are parameterized in the form

\[
\int dt \left[ \frac{[h_3^*(\tau)]^2}{[\int dt \gamma(\tau) h_3^*(\tau) [\gamma(\tau)]^{3/2} dx^k]} \right] - \int dt \left[ \frac{\partial_i (\int dt \gamma(\tau) h_3^*(\tau))}{\gamma(\tau) h_3^*(\tau)} dx^i \right].
\]

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In such quadratic linear elements, we have to fix a sign of the coefficient \( \mathbf{\mathcal{C}}_3(\tau, x^k, t) \) which describes relativistic flow evolution with a generating function with Killing symmetry on \( \partial_3 \) determined by sources (\( \mathcal{h}(\tau), \mathcal{a}(\mathbf{¥})(\tau)) \). Such entropic and quasiperiodic flow solutions are of type \([18]\) and can be re-written equivalently with coefficients stated as functionals of \( \mathcal{a}(\mathbf{¥})(\tau, x^i, t) \) and \( \mathcal{a}(\mathbf{¥})(\tau, x^i, t) \).

### Table 3: Entropic flows of locally anisotropic cosmological solutions

| Exact solutions of \( R_{\mu\nu}(\tau) = \frac{\epsilon^{\mu\nu}}{\mathcal{Y}(\tau)} \) transformed into a system of nonlinear PDEs (35–37) |
|---|---|
| d-metric ansatz with Killing symmetry \( \partial_3 = \partial_\nu \) | \( ds^2 = g_1(\tau) \mathbf{d}x^2 + g_2(\tau) \mathbf{d}y^2 + N^\nu_1(\tau) \mathbf{d}z^2 \), for \( \mathcal{h}(\tau, x^k, t) \) \n
#### Nonlinear PDEs

- Generating functions: \( h_3(\tau, x^k, t), \mathbf{h}(\tau, x^k, t); \mathbf{\mathcal{Y}}(\tau, x^k, t) \)
- Integration functions: \( h_3(\tau, x^k, t), 1\mathbf{n}_k(\tau, x^k, t), 2\mathbf{n}_k(\tau, x^k, t) \)
- \& nonlinear symmetries

- Off-diag. solutions, d-metric N-connec.

| N-connections, zero torsion | \( \mathbf{\mathcal{Y}}(\tau, x^k, t) = \mathbf{\mathcal{Y}}(\tau, x^k, t) \), \( \mathbf{\mathcal{Y}}(\tau, x^k, t) \)
|---|---|
| polarization functions | \( \mathbf{\mathcal{Y}}(\tau, x^k, t) = \mathbf{\mathcal{Y}}(\tau, x^k, t) \), \( \mathbf{\mathcal{Y}}(\tau, x^k, t) \)
| Prime metric defines a cosmological solution | \( \mathbf{\mathcal{Y}}(\tau, x^k, t) \), \( \mathbf{\mathcal{Y}}(\tau, x^k, t) \), \( \mathbf{\mathcal{Y}}(\tau, x^k, t) \)

We can extract from off-diagonal d-metrics (68) certain cosmological LC-configurations determined by entropic quasiperiodic sources by imposing additional zero torsion constraints. Such anholonomy conditions restrict the respective classes of generating functions \( \mathbf{\mathcal{Y}}(\tau, x^i, t), \mathbf{\mathcal{Y}}(\tau, x^i, t) \) and/or
\[ \Phi(\tau, x^i, t) \) for \( \Phi(\tau, x^i, A(\tau, x^i, t)) \) and sources \( a_s \Phi(\tau) \) (61) and \( a_s \Lambda(\tau) \) (49),

\[
d s^2 = e^{\psi(\tau)}(d x^2 + (d x^2)^2) + h_3(\tau) \left[ d y^3 + (\partial_5 \Phi(\tau)) d x^k \right] - \frac{[h_3^s(\tau)]^2}{\int dt \ a_s \Phi(\tau)} d t + (\partial_t A(\tau)) d x^i \]. \tag{69}

The d-metrics (68) and/or (69) define off-diagonal cosmological solutions generated by entropic quasiperiodic additive sources \( a_s \Phi(\tau) \) and/or \( a_s \Lambda(\tau) \). The terms (67) encode and model respectively contributions of standard and/or dark matter fields and effective entropic evolution sources. Such values can be can be prescribed in certain forms being compatible to observational data of cosmological (and geometric/entropic) evolution for dark matter distributions with possible quasiperiodic, aperiodic, pattern forming, solitonic nonlinear wave interactions.

**Cosmological solutions for nonlinear entropic quasiperiodic functionals for sources:** Such classes of exact cosmological solutions can be generated by nonlinear quasiperiodic functionals for effective quasiperiodic additive sources \( a_s \Phi(\tau) \) and/or \( a_s \Lambda(\tau) \). The terms (67) encode and model respectively contributions of standard and/or dark matter fields and effective entropic evolution sources. Such values can be can be prescribed in certain forms being compatible to observational data of cosmological (and geometric/entropic) evolution for dark matter distributions with possible quasiperiodic, aperiodic, pattern forming, solitonic nonlinear wave interactions.

\[
d s^2 = e^{\psi(\tau)}(d x^2 + (d x^2)^2) + h_3(\tau) [d y^3 + (\partial_5 \Phi(\tau)) d x^k] - \frac{[h_3^s(\tau)]^2}{\int dt \ a_s \Phi(\tau) + \partial_t A(\tau)] d x^i}]. \tag{70}

This formula is similar to (52) involving a duality between different classes of stationary and cosmological solutions. This duality is encoded also in another type of nonlinear generation functions for (effective) dark and/or usual matter sources, which can be additional or with nonlinear functionals, when \( a_s \Phi \rightarrow \psi \Phi \) and \( a_s \Lambda \rightarrow \psi \Lambda \) and, respectively, \( a_s \Phi \rightarrow \psi \Phi \) and \( a_s \Lambda \rightarrow \psi \Lambda \).

For LC-configurations, we obtain

\[
d s^2 = e^{\psi(\tau)}(d x^2 + (d x^2)^2) + h_3(\tau) [d y^3 + (\partial_5 \Phi(\tau)) d x^k] - \frac{[h_3^s(\tau)]^2}{\int dt \ a_s \Phi(\tau) + \partial_t A(\tau) - d x^i}]. \tag{71}

For additive functionals for cosmological entropic and quasiperiodic sources, the formulas (70) and (71) transforms respectively into quadratic linear elements (68) and (69). Fixing a value \( \tau_0 \), we obtain cosmological solutions for Ricci solitons and MGTs.

### 6.2.2 Cosmological configurations with nonstationary entropic generating functions

In this section, the sources for (effective) matter fields and geometric flows are defined by arbitrary functions \( \Phi(\tau, x^k, \tau, x k, t) \). The quasiperiodic structure will be stated for some additive or general nonlinear functionals for the generating functions.

**Cosmological metrics with additive entropic generating functions:** Such functionals are

\[
a_s \Phi(\tau) = a_s \Phi(\tau, x^i, t) + f_s \Phi(\tau, x^i, t) + m_s \Phi(\tau, x^i, t) + \Phi_s (\tau, x^i, t) + \Phi_s (\tau, x^i, t) \tag{72}
\]

The values \( \Phi_s (\tau, x^i, t) = \Phi_s (\tau, x^i, t) \) and \( \Phi_s (\tau, x^i, t) = \Phi_s (\tau, x^i, t) \) are functionals on certain quasiperiodic (space time like QC or other type aperiodic, solitonic structures) given
by functions $\zeta$ and/or $\bar{\zeta}$ subjected to conditions of type \((11)\) and/or \((12)\). The terms in the sum \((72)\) correspond to the Lagrange densities \((13)\) and energy-momentum tensors \((14)\) nonholonomically parameterised by sources \((25)\). Changing systems of references and coordinates, we can compute sums of functionals for sources of type \((67)\) using nonlinear symmetries \((39)\) considering an associated additive cosmological constant $a_2\Lambda(\tau)$ \((49)\). For $\Phi(\tau)$ \((72)\), the equation \((35)\) transforms into a functional equation \(\overline{F}^\tau(\tau)[ a_2\Phi(\tau), \Lambda(\tau)] = \overline{h}_3(\tau)[ a_2\Phi(\tau), \Lambda(\tau)] = 2\overline{h}_3(\tau)[ a_2\Phi(\tau), \Lambda(\tau)]\overline{h}_4(\tau)[ a_2\Phi(\tau), \Lambda(\tau)] \overline{\Psi}(\tau)\), and related analogs of \((36)\) and \((37)\) in the decoupled system of nonlinear PDEs \((38)\). Such equations and their solutions can be written equivalently in different forms with additive functionals of type $a_2\overline{h}_{3,4}$ and/or $a_2\overline{\Psi}$ and respective nonlinear functionals for the coefficients in respective equations. Applying the AFDM as in the subsection \((1.3.2)\) (see a summary in Table 3), we generate a class of parametric solutions of \((38)\) parameterized in a form similarly to \((48)\),

$$
\begin{align*}
 ds^2 &= e^{\psi(\tau,x^k)}[(dx^1)^2 + (dx^2)^2] + [h_3^{0}\tau, \tau, x^i] - \frac{a\Phi^2}{4\Lambda(\tau)}

 \left[ dy^3 + \left( 1n_k(\tau, x^i) + 2n_k(\tau, x^i) \right) \int dt \frac{[a\Psi^2(\tau)]^2}{4\Lambda(\tau)} \right] dx^k

 - \frac{[a\Phi^2(\tau)]^2}{4\Lambda(\tau)} \left[ [dt] + \left( \frac{\partial_k a(\tau, x^1)}{\Lambda(\tau)} \right) dx^k \right],
\end{align*}
$$

(73)

for integration functions $h_3^{0}\tau, \tau, x^i$, $1n_k(\tau, x^i)$ and $2n_k(\tau, x^i)$.

LC-configurations can be extracted from \((73)\) imposing additional integrability conditions on coefficients resulting in zero nonholonomic torsion. Such solutions can be considered both for relativistic entropic Ricci solitons and in GR, being parameterized as in \((19)\),

$$
\begin{align*}
 ds^2 &= e^{\psi(\tau,x^k)}[(dx^1)^2 + (dx^2)^2] + [h_3^{0}\tau, \tau, x^i] - \frac{\Phi^2}{4\Lambda(\tau)}[dy^3 + (\partial_k a(\tau, x^1)) dx^k]

 - \frac{[a\Phi^2(\tau)]^2}{4\Lambda(\tau)}[\partial_t \overline{\Lambda}(\tau, x^1, t)] dx^k.
\end{align*}
$$

(74)

We can consider small parametric decompositions and frame/coordinate transforms (in a more general context, we can elaborate a formalism of $\eta$- and $\varepsilon$-polarization functions as for BH solutions but with time like dependence for cosmological configurations) in order to relate new classes of solutions \((73)\) and/or \((74)\) to some well known (off-) diagonal cosmological metrics.

**Cosmological configurations with nonlinear entropic functionals for generating functions:** Instead of additive generating functionals $a_2\overline{F}(\tau)$ \((72)\), we can work with nonlinear quasiperiodic generating functionals $\Phi_\tau(\tau) = \phi(\tau, x^i, t) = \phi(\Phi) f(\Phi), F(\Phi), \overline{\Phi}, \overline{\Phi}, \dot{\Phi}, \dot{\Phi}$ characterized by nonlinear symmetries of type \((39)\). The equation \((35)\) transforms into a nonlinear functional equation, \(\overline{F}^\tau(\tau)[ \Phi(\tau), \Lambda] \overline{h}_4(\tau)[ \Phi(\tau), \Lambda(\tau)] = 2\overline{h}_3(\tau)[ \Phi(\tau), \Lambda(\tau)]\overline{h}_4(\tau)[ \Phi(\tau), \Lambda(\tau)] \overline{\Psi}(\tau)\), which can be solved together with other equations with decoupling \((38)\). We obtain such solutions:

$$
\begin{align*}
 ds^2 &= e^{\psi(\tau,x^k)}[(dx^1)^2 + (dx^2)^2] + [h_3^{0}\tau, \tau, x^i] - \frac{\Phi^2}{4\Lambda(\tau)}

 \left[ dy^3 + \left( 1n_k(\tau, x^i) + 2n_k(\tau, x^i) \right) \int dt \frac{[\Phi^2(\tau)]^2}{4\Lambda(\tau)} \right] dx^k

 - \frac{[\Phi^2(\tau)]^2}{4\Lambda(\tau)} \left[ [dt] + (\partial_t \Lambda(\tau, x^1, t)) dx^k \right].
\end{align*}
$$

(75)
For zero torsion constraints, we extract LC-configurations,

\[
ds^2 = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + (h_3^{[0]}(\tau, x^i) - \frac{\psi \Phi(\tau)}{4 \Lambda(\tau)})[dy^3 + (\partial_k \mathbf{n}(\tau, x^i))dx^k]
\]

where \( h_3^{[0]}(\tau, x^i) \) and \( 1n_k(\tau, x^i) \) are integration functions.

The coefficients of d–metrics (75) and (76) can be chosen to be of necessary smooth class and involve certain entropic, quasiperiodic, stochastic, topological sources and fractional derivative processes. Such nonholonomic deformation and generalized transforms can be constructing with changing the topological spacetime structure and modeling certain dark energy and dark matter effects as results of entropic quasiperiodic flows or nonholonomic deformations of certain prime cosmological solutions.

6.2.3 Emergent quasiperiodic cosmology from both generating functionals & sources

We can generate entropic flow cosmological solutions using generalized quasiperiodic nonlinear functionals both for generating functions, \( \psi(\Phi)(\tau) \), and nonlinear functionals for (effective) sources, \( \psi(\Phi) (\tau) \), see above formulas. Such data are connected via nonlinear symmetries of type (39), when \( \psi(\Lambda)(\tau) \psi(\Phi)(\tau) = \psi(\Phi(\tau)) | \psi(\Phi)(\tau) - \int dt \psi(\Phi(\tau)) | \psi(\Phi)(\tau) \). Similar nonlinear symmetries exist for additive functions both for the gravitational fields and (effective) sources, with (72) and (67) and can be written in a particular form \( \Lambda(\tau) \psi(\Phi)(\tau) = \psi(\Phi(\tau)) | \psi(\Phi(\tau) - \int dt \psi(\Phi(\tau)) | \psi(\Phi)(\tau) \). Applying the AFDM summarized in Table 3 (using data ( \( \psi(\Phi)(\tau), \psi(\Phi)(\tau) \)), and/or, equivalently, ( \( \psi(\Phi)(\tau), \psi(\Phi)(\tau) \)), we construct multi-functional nonlinear entropic quasiperiodic cosmological configurations,

\[
ds^2 = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + (h_3^{[0]}(\tau, x^i) - \frac{\psi \Phi(\tau)}{4 \Lambda(\tau)})
\]

where for integration functions \( h_3^{[0]}(\tau, x^i) \), \( 1n_k(\tau, x^i) \) and \( 2n_k(\tau, x^i) \).

For LC-configurations, we obtain multi-functional nonlinear generalizations of (75) and (76) modelling locally anisotropic and inhomogeneous solutions in entropic flow gravity and GR,

\[
ds^2 = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + (h_3^{[0]}(\tau, x^i) - \frac{\psi \Phi(\tau)}{4 \Lambda(\tau)})[dy^3 + (\partial_k \mathbf{n}(\tau, x^i, t))dx^k]
\]

where integration functions \( h_3^{[0]}(\tau, x^i) \), \( 1n_k(\tau, x^i) \) and \( 2n_k(\tau, x^i) \).
The classes of cosmological solutions (77) and (78) describe off-diagonal entropic non-stationary configurations determined by multi-functional nonlinear quasiperiodic structures.

### 6.3 Cosmological metrics evolving in entropic quasiperiodic media

Generic off-diagonal entropic and quasiperiodic cosmological solutions are constructed in terms of \( \eta \)-polarization functions as in appendix [A.5] but following the AFDM method for parametric \( \tau \) and time like depending evolution summarized in Table 3. We consider a primary cosmological d-metric \( \tilde{\mathbf{g}} (27) \) defined by data \([\dot{g}_{i}(x^{i},t), \dot{g}_{a} = \dot{h}_{a}(x^{i},t); \tilde{N}_{k}^{3} = \tilde{n}_{k}(x^{i},t), \tilde{N}_{k}^{4} = \tilde{w}_{k}(x^{i},t)]\) which can be diagonalized for a FLRW cosmological metric by frame/coordinate transforms\(^{18}\). The cosmological entropic quasiperiodic \( \eta \) target metrics \( \bar{\mathbf{g}}(\tau) \) of type (26) are generated by nonholonomic deformations

\[
\bar{\mathbf{g}} \rightarrow \bar{\mathbf{g}}(\tau) = \bar{\mathbf{g}}(\tau, x^{k}) = \bar{\mathbf{g}}_{i}(\tau, x^{k}, t) \dot{g}_{i}, \bar{\mathbf{g}}_{a}(\tau, x^{k}, t) = \bar{\mathbf{g}}_{a}(\tau, x^{k}, t) \dot{g}_{a}, \bar{\mathbf{g}}_{i}^{3}(\tau, x^{k}, t) = \bar{\mathbf{g}}_{i}^{3}(\tau, x^{k}, t) \dot{N}_{i}^{3} \]

when overlined symbols are used for distinguishing cosmological d-metrics from stationary ones studied in previous sections. The quadratic line elements corresponding to target locally anisotropic and inhomogeneous cosmological metrics \( \bar{\mathbf{g}}(\tau) = \bar{\mathbf{g}}(\tau, x^{k}, t) \) can be written in terms of gravitational polarization functions,

\[
ds^{2} = \bar{\mathbf{g}}_{i}(\tau, x^{i}, t) \dot{g}_{i}(x^{i}, t)|dx^{i}|^{2} + \bar{\mathbf{g}}_{a}(\tau, x^{i}, t) \dot{g}_{a}(x^{i}, t)|dx^{i}|^{2}
+ \bar{\mathbf{g}}_{a}(\tau, x^{i}, t) \dot{h}_{3}(x^{i}, t)|dy^{3}|^{2} + \bar{\mathbf{g}}_{i}^{3}(\tau, x^{k}, t) \dot{N}_{i}^{3}(x^{k}, t)|dx^{i}|^{2} + \bar{\mathbf{g}}_{a}^{3}(\tau, x^{i}, t) \dot{h}_{4}(x^{i}, t)|dt + \bar{\mathbf{g}}_{i}^{3}(\tau, x^{k}, t) \dot{N}_{i}^{4}(x^{k}, t)|dx^{i}|^{2}.
\]

The \( \eta \)-coefficients will be constructed in some explicit forms determined by entropic quasiperiodics generation functions and/or effective sources as solutions of the system of nonlinear PDEs (38).

#### 6.3.1 Cosmological evolutions generated by nonstationary entropic sources

We consider entropic sources of type \( \varphi \mathbf{Y}(\tau) = \varphi \mathbf{Y}(\tau, x^{i}, t) = \varphi \mathbf{Y}(\varphi, m, \tilde{\mathbf{X}}, \tilde{\mathbf{Y}}, \varphi, \tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) \) as in (70) and (71) and compute the \( \eta \)-polarization functions following formulas

\[
\bar{\eta}_{i}(\tau) = e^{\psi(\tau, x^{i})} / \dot{g}_{i}; \text{ generating function} \bar{\eta}_{3}(\tau) = \bar{\eta}_{3}(\tau, x^{i}, t); \bar{\eta}_{4}(\tau) = -\frac{4(|(\bar{\mathbf{g}}_{3}(\tau) \dot{h}_{3})^{1/2}|^{2}}{h_{4}| \int dt \varphi \mathbf{Y}(\tau) (\bar{\mathbf{g}}_{3} (\tau) h_{3}^{*})^{2} |} \quad (79)
\]

\[
\bar{\eta}_{3}(\tau) = \frac{1n_{k}(\tau, x^{i})}{n_{k}} + 4 \frac{2n_{k}(\tau, x^{i})}{n_{k}} \int dt \frac{(|(\bar{\mathbf{g}}_{3}(\tau) \dot{h}_{3})^{-1/4}|^{2}}{\int dt \varphi \mathbf{Y}(\tau) (\bar{\mathbf{g}}_{3}(\tau) h_{3}^{*})^{2}}, \quad \bar{\eta}_{4}(\tau) = \frac{2h_{3} \int dt \varphi \mathbf{Y}(\tau) (\bar{\mathbf{g}}_{3}(\tau) h_{3}^{*})^{2}}{\varphi \mathbf{Y}(\tau) (\bar{\mathbf{g}}_{3}(\tau) h_{3}^{*})^{2}},
\]

where \( 1n_{k}(\tau, x^{i}) \) and \( 2n_{k}(\tau, x^{i}) \) are integration functions.

Using \( \bar{\eta}_{3}(\tau, x^{i}, t) \) as a generating function, we can compute other types of generating functions of the same target cosmological d-metric subjected to nonlinear symmetries (39),

\[
\mathbf{\Omega}^{2}(\tau) = 4 | \varphi \mathbf{Y}(\tau) [h_{3}^{(0)}(\tau, x^{k}) - \bar{\mathbf{g}}_{3}(\tau, x^{i}, t) \dot{h}_{3}(x^{k}, t)] |, \quad (\mathbf{Y}_{3}^{2})^{*}(\tau) = -\int dt \varphi \mathbf{Y}(\tau) [\bar{\mathbf{g}}_{3}(\tau, x^{i}, t) \dot{h}_{3}(x^{i}, t)]^{*}.
\]

For integrable generating functionals and sources, when the constructions (16) are subjected to target off-diagonal cosmological metrics (49) with zero torsion, we obtain

\[
\bar{\eta}_{3}(\tau) = e^{\psi(\tau, x^{i})} / \dot{g}_{3}; \bar{\eta}_{3}(\tau) = \bar{\eta}_{3}(\tau, x^{i}, t) \text{ as a generating function;}
\]

\[
\bar{\eta}_{4}(\tau) = -\frac{4(|(\bar{\mathbf{g}}_{3}(\tau) \dot{h}_{3})^{1/2}|^{2}}{h_{4}| \int dt \varphi \mathbf{Y}(\tau) (\bar{\mathbf{g}}_{3} (\tau) h_{3}^{*})^{2} |} \quad \bar{\eta}_{3}(\tau) = \frac{\partial k \bar{\mathbf{g}}(\tau, x^{i})}{\dot{w}_{k}} \quad \bar{\eta}_{4}(\tau) = \frac{\partial k \bar{A}(\tau, x^{i})}{\dot{w}_{k}}.
\]

\(^{18}\)In general, we can consider off-diagonal Bianchi anisotropic cosmological metrics or any cosmological solution in GR or MGTs.
In (79) and (80), the nonlinear functionals for the entropic quasiperiodic v-source and (effective) cosmological constant can be changed into additive functionals \( q^p \Psi(\tau) \rightarrow a^s \Psi(\tau) \) and \( q^p \Lambda(\tau) \rightarrow a^s \Lambda(\tau) \) which generates another classes of cosmological solutions.

### 6.3.2 Cosmology from nonstationary entropic generating functions

We can construct locally anisotropic and inhomogeneous cosmological solutions as nonholonomic deformations of some prime cosmological metrics when the coefficients of the d-metrics are determined nonlinear generating functionals \( q^p \Psi(\tau) = q^p \Psi(\tau, x^i, t) = q^p \Psi_i F^i, q^p \Lambda_i F_i, q^p \Lambda_0 F_0 \) as for (75). It is possible also to generate similar cosmological metrics by additive functionals \( a^s \Psi(\tau) \) (72) for prescribed families of effective sources \( \Psi(\tau) = \Psi(\tau, x^i, t) \) and cosmological constants \( \Lambda(\tau) \). Using formulas for nonlinear symmetries (39), we compute (recurrently) corresponding nonlinear functionals, \( q^p \Psi_3(\tau, x^i, t) \), or additive functionals, \( a^s \Psi_3(\tau, x^i, t) \), and related polarization functions,

\[
q^p \Psi^2(\tau) = 4| \Lambda(\tau)[h_3^{[0]}(\tau, x^k) - q^p \Psi_3(\tau, x^i, t)h_3(x^k, t)]|, \quad (q^p \Psi^2)^*(\tau) = - \int dt \ \Psi(\tau) \left[ (q^p \Psi_3(\tau, x^i, t)h_3(x^i, t))^* \right].
\]

The coefficients of quadratic elements of type (48) are recurrently computed,

\[
\eta_4(\tau) = e^{\psi(\tau, x^i)/\hat{g}_3}; \eta_3(\tau) = q^p \eta_3(\tau, x^i, t) \quad \text{as a generating function}; \quad (81)
\]

\[
\eta_4(\tau) = - \frac{4([q^p \eta_3(\tau)h_3|1/2]^*)_2}{h_4 | dt \Psi(\tau)(q^p \eta_3(\tau)h_3^*)|};
\]

\[
\eta_4(\tau) = \frac{1n_k(\tau, x^i)}{n_k} + 4 \frac{2n_k(\tau, x^i)}{n_k} \int dt \frac{\left( \left( q^p \eta_3(\tau)h_3 - 1/4 \right)^*_2 \right)}{| \int dt \Psi(\tau)(q^p \eta_3(\tau)h_3^*)|}; \quad \eta_4(\tau) = \partial_k \int dt \Psi(\tau)(q^p \eta_3(\tau)h_3^*) \frac{n_k}{\hat{w}_i \Psi(\tau)(q^p \eta_3(\tau)h_3^*)},
\]

where \( n_k(\tau, x^i) \) and \( 2n_k(\tau, x^i) \) are integration functions.

Target off-diagonal cosmological metrics (49) with zero torsion extracted from (81) can be generated by polarization functions

\[
\eta_6(\tau) = e^{\psi(\tau, x^i)/\hat{g}_3}; \quad \text{generating function } \eta_6(\tau) = q^p \eta_6(\tau, x^i, t); \quad (82)
\]

\[
\eta_4(\tau) = - \frac{4([q^p \eta_6(\tau)h_3|1/2]^*)_2}{h_4 | dt \Psi(\tau)(q^p \eta_6(\tau)h_3^*)|}; \quad \eta_4(\tau) = \partial_k \int dt \Psi(\tau)(q^p \eta_6(\tau)h_3^*) \frac{n_k}{\hat{w}_i \Psi(\tau)(q^p \eta_6(\tau)h_3^*)},
\]

The cosmological solutions generated in this subsection describe entropic flow nonholonomic deformations of prime cosmological configurations (for instance, a FLRW, or Bianchi, type metric, and various modifications in accelerating cosmology) self-consistently imbedded into a quasiperiodic gravitational (dark energy) media.

### 6.3.3 Cosmological configurations for entropic sources & generating functions

We can construct more general classes of nonholonomic deformations of prime cosmological metrics generated by entropic quasiperiodic flow nonlinear quasiperiodic functionals both for the generating functions and (effective) sources. For such locally anisotropic and inhomogeneous cosmological models defined by nonlinear superpositions of cosmological solutions (79) and (81) when the coefficients of (70) are computed,

\[
\eta_4(\tau) = e^{\psi(\tau, x^i)/\hat{g}_3}; \quad \text{generating function } q^p \eta_4(\tau) = q^p \eta_3(\tau, x^i, t); \quad q^p \eta_4(\tau) = - \frac{4([q^p \eta_3(\tau)h_3|1/2]^*)_2}{h_4 | dt \Psi(\tau)(q^p \eta_3(\tau)h_3^*)|}.
\]

44
\[ \eta_i^2(\tau) = \frac{1}{n_k} \eta_k + 4 \frac{2n_k}{n_k} \int dt \frac{\left( \left( q^p h_3(\tau) h_3 \right)^{-1/4} \right)^2}{\left( \left( q^p h_3(\tau) h_3 \right)^{-1/4} \right)} \eta_i(\tau) = \frac{\partial_i}{w_i} \left( q^p h_3(\tau) h_3 \right)^* \tag{82} \]

where \( n_k(\tau, x_i) \) and \( 2 n_k(\tau, x_i) \) are integration functions. In formulas (82), there are considered nonlinear generating functionals \( q^p h_3(\tau) = q^p h_3(\tau, x_i, t) = q^p h_3(\tau, f_i, m_i, F_i, 0 m_i, 0 F_i, 0 h_3 \overline{\overline{h_3}}) \) characterized by nonlinear symmetries \( \left( \overline{\overline{h_3}} \right) = 4 \left( \left( q^p h_3(\tau) h_3 \right)^{-1/4} \right)^2 \) for some prescribed families of nonlinear functionals \( q^p h_3(\tau) = q^p h_3(\tau, x_i, t), q^p h_3(\tau, x_i, t), \) or additive functions, \( q^p h_3(\tau, x_i, t), \) for other types polarization functions,

\[ \eta_i(\tau) = e^{\psi(\tau, x_i) / g_i} \text{ generating function } \eta_i(\tau) = q^p h_3(\tau, x_i, t); \]
\[ \eta_i(\tau) = - \frac{4}{h_4} \left( \left( q^p h_3(\tau) h_3 \right)^{1/2} \right)^2 \eta_i(\tau) = (\partial_k q^p h_3(\tau, x_i)) / \eta_k; \eta_k(\tau) = \partial_k A(\tau, x_i, t) / \psi_k. \]

Finally, it should be noted that there is duality on \( y^3 \) and \( y^4 \) coordinates and respective N-connection coefficients for the class of cosmological solutions \( \left[ 82 \right] \) and the stationary solutions \( \left[ 61 \right] \).
evolution theories with nonholonomic constraints, we have to develop an unified geometric formalism for metric-affine spaces, generalized Finsler-Lagrange-Hamilton geometry, almost Kähler and noncommutative geometries etc. Such constructions were performed for Hořava-Lifshits, \( f(R) \), \( R^2 \), and other types MGTs (see reviews [74, 75, 76, 88, 89]) with developments for models of thermodynamic / entropic / entanglement and emergent gravity [5, 6, 7, 8, 9, 71, 72, 73, 19, 21, 23, 22, 24, 25, 26, 27, 83]; and for locally anisotropic kinetic and thermodynamic theories on curved spaces (see [77, 86, 41, 84, 85, 87] and references therein) etc.

We have found that using certain classes of nonholonomic variables the (relativistic) geometric flow evolution and Ricci soliton equations, and related motion equations in entropic and other type MGTs, can be decoupled and integrated in some very general forms. This allows us to construct various classes of exact and parametric solutions with generic off-diagonal metrics and generalized connections. The coefficients of new classes of stationary and (in general, locally anisotropic and inhomogeneous) cosmological solutions depend on all spacetime and associated phase space (kinetic and/or thermodynamic) coordinates via generating and integration functions and various types of commutative and noncommutative parameters and integration and physical constants. Such geometric and analytic techniques of constructing exact solutions in geometric flow evolution and MGTs have been developed in the framework of the so-called anholonomic frame deformation method, AFDM [60, 61, 62, 63]. For details, examples of exact solutions, and various applications, we cite [44, 69, 64, 65, 66, 35, 41, 42, 43, 44] and references therein.

It should be noted that there were not elaborated corresponding topological methods and a well-defined analytic formalism for investigating geometric evolution equations of metrics of pseudo-Euclidean signature. A number of such conceptual and fundamental issues in nonlinear functional analysis and the geometry of Lorentz manifolds have not addressed or solved by mathematicians. The standard geometric flow paradigm was proposed as the Hamilton-Perleman program for Riemannian metrics defining entropy type functionals and deriving nonlinear evolution equations. To elaborate realistic relativistic physical models we deal with Ricci tensors which in the limit of weak gravitational/elastic flows approximate to the d’Alambert (wave) operator and not to the Laplace (diffusion) one used for Euclidean signatures. So, the original approach to the topology and geometric flows of Riemannian metrics has to be generalized in certain relativistic and nonholonomic forms which are compatible with modern experimental particle physics data and observational cosmology. The Poincaré–Thurston conjecture can be formulated and proven for non-relativistic evolution of any 3-d space like hypersurface. In this and partner [34] works, we advocate that using E. Verlinde conjecture on elastic emergent gravity, we can elaborate on generalizations of nonholonomic Ricci flow theories as certain models of relativistic flow evolution. Such models are determined by extensions of Perelman’s functionals for 3-d Riemannian metrics to certain modified 4-d F- and W-entropy nonholonomic analogs which are extended on a time like coordinate and/or a temperature like evolution parameter. For certain approaches with rich gravitational vacuum structure, we work with geometric relativistic kinetic/ hydrodynamic/ thermodynamic models (see details in [41]) or (for instance, in this work) with a \( \tau \)-parametric theory describing geometric entropic flows determined by relativistic and elastically spacetime modified nonholonomic F- and W-functionals.

As it was mentioned above, there is not yet formulated a rigorous mathematical approach to the theory relativistic of geometric/entropic flows of metrics with Lorentz signature and generalized connections. Nevertheless, we shown that such theories are characterized by certain classes of generalized R. Hamilton equations with effective parametric sources which may encode entropic and quasiperiodic structures (these are necessary for explaining, for instance, the complex structure of dark matter and energy in modern cosmology). Using the AFDM, we proved that such geometric flow evolution equations and their Ricci soliton variants can be decoupled and integrated in very general forms. In
sections 5 and 6 we constructed and analyzed possible physical implications of respective entropic stationary, and BH solutions, and locally anisotropic cosmological solutions. A series of our former results on astrophysical and cosmological models with quasiperiodic, pattern forming, quasicrystal time like structures, see [66, 70, 65, 67, 68, 69] and references therein, were used for developing in this work similar models for the entropic geometric flow and gravity theories. Such exact and parametric solutions provide also explicit examples of new physical important solutions for entropic gravity models developed in phenomenological theoretic manner in [8, 9, 71]. So, we conclude that the E. Verlinde conjecture on entropic character of gravity can be related to a relativistic extension of the Poincaré–Thurston conjecture. Even such geometric ideas have not been proven as explicit theorems in modern geometric analysis, there are rigorous exact solutions of respective systems of nonlinear PDES which support such entropic gravity and flow evolution ideas.

In sections 4-6 we shown how to construct in explicit form entropic and quaisiperiodic solutions for relativistic geometric flows, nonholonomic Ricci solitons and generalized gravitational field equations. Such a techniques was elaborated similarly to MGTs with quasiperiodic structure (constructed and studied in this and our partner works [34, 69, 70]) and involves generic off-diagonal metric and nonholonomically deformed non-Riemannian linear and nonlinear connections. We emphasized, see also [44], that such configurations are not characterized, in general, by certain entropy-area, holographic or duality conditions. As a consequence, it is not possible to elaborate on thermodynamic models of entropic MGTs and physical properties of their exact or parametric solutions using only the concepts related to the Bekenstein-Hawking entropy. We consider that there is an alternative and more general way when stationary and cosmological solutions in geometric/ entropic flow evolution theories, MGTs and GR, can be defined and characterized by nonholonomic deformations of Perelman’s W-entropy. Such constructions are similar to the well-known results on relativistic locally anisotropic thermodynamics and kinetics [41, 77] and can be generalized for emergent classical and quantum gravity theories.

Finally, we note that our geometric/ entropic flow approach to MGTs provides new mathematical methods and applications in the theory of classical and quantum informatics, for research of quantum systems with entanglement, models of quantum and emergent gravity, and accelerating cosmology and dark energy/ matter interactions etc. Our further research programs are related to developments in such directions.

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A Nonholonomic hypersurface and relativistic geometric flows

We summarize some results on nonholonomic 3-d hypersurface and relativistic 4-d geometric flows [37, 38, 39, 40, 41, 42, 43, 44] which are necessary for elaborating the theory of entropic flows and gravity considered in the main part of the paper. The (nonrelativistinc) Ricci flow evolution equations were postulated heuristically by R. Hamilton
We write such equations in the form
\[ \frac{\partial g_{ij}}{\partial \xi} = -2 R_{ij}. \] (A.1)

In these formulas, \( \xi \) is an evolution real parameter and the local coordinates \( u^i \) with indices \( i, j = 1, 2, 3 \) are considered for a real 3-d Riemannian manifold. The equations (A.1) describe a nonlinear diffusion process for geometric flow evolution of 3-d Riemannian metrics. For small deformations of a 3-d Euclidean metric \( g_{ij} \approx \delta_{ij} + h_{ij} \), with \( \delta_{ij} = \text{diag}[1, 1, 1] \) and \( |h_{ij}| \ll 1 \), the Ricci tensor approximates the Laplace operator \( \Delta = \frac{\partial^2}{(\partial u^1)^2} + \frac{\partial^2}{(\partial u^2)^2} + \frac{\partial^2}{(\partial u^3)^2} \). We obtain a linear diffusion equation if \( R_{ij} \sim \Delta h_{ij} \).

In modified and normalized forms (see explanations below, related to formulas (A.7) and (A.8)), equations of type (A.1) can be proven following a corresponding variational calculus for Perelman’s W- and F-functionals [51, 52, 53].

There were studied generalized Perelman’s functionals for various models of non–Riemannian geometries and MGTs, see [35, 36, 37, 38, 39, 40, 41, 42, 43, 44] and references therein. Unfortunately, in general, modifications of R. Hamilton’s equations (A.1) do not describe in a self-consistent manner certain evolution processes if additional assumptions on geometric and physical properties of evolution flows are not considered. Nevertheless, for the geometric flow evolution of 4–d metrics with Lorentz signature and non–stationary solutions in MGTs, it is not possible to formulate a statistical thermodynamic interpretation similarly to the case of 3-d Riemannian ones. For relativistic models, we have to elaborate on hydrodynamic anisotropic like transports of entropic fields and derived geometric objects.

For self-similar configurations in fixed points, the geometric flows (A.1) are described by Ricci soliton equations
\[ R_{ij} - \lambda g_{ij} = \nabla_i v_j + \nabla_j v_i, \] (A.2)
for \( \lambda = \pm 1, 0 \) and a vector field \( v_j \). Equations of type (A.2) are considered in various MGTs as gravitational field equations for pseudo–Riemannian metrics and generalized connections.

The main goal of our papers on geometric flows and gravity (cited above) is to study how the concept of W–entropy can be generalized in order to characterize 3-d hypersurface gravitational thermodynamic configurations and their relativistic evolution determined by exact solutions in (modified) gravity. The approach involves two other less established (modified) relativistic theories. The first one is on relativistic statistical thermodynamics and the nonlinear diffusion theory on curved spacetimes and in gravity. We note that historically the first relativistic generalizations of thermodynamics by M. Plank and A. Einstein. Latter, those works were subjected to critics and modifications. There are

\[ ^{19} \text{There are two general ways for elaborating models of 3–d Ricci flow evolution for 4–d spacetimes with pseudo–Euclidean signature. The first one is for theories of stochastic / diffusion and kinetic processes with local anisotropy, fractional geometric evolution etc. In this approach, one elaborates thermofield models of Ricci flow evolution on imaginary time } \zeta = -it(0 \leq \zeta \leq 1/\kappa T), \text{ where } \kappa \text{ is Boltzmann's constant and the symbol } T \text{ is used for the temperature (do not confuse similar notations with respective indices for torsion and energy-momentum tensors). In result, the pseudo–Riemannian spacetime is transformed into a Riemannian configuration space as one elaborates in thermal and/or finite temperature quantum field theory. It should be noted here that G. Perelman treated } \tau = \zeta^{-1} \text{ as a temperature parameter. The concept of W–entropy was elaborated following and analogy to formulas for the entropy in statistical mechanics } \text{[45]. We reproduce here the Remark 5.3 and next paragraph, just before section 6 in that paper: "An entropy formula for the Ricci flow in dimension two was found by Chow [49]; there seems to be no relation between his formula and ours. .... The interplay of statistical physics and (pseudo)-riemannian geometry occurs in the subject of Black Hole Thermodynamics, developed by Hawking et al. Unfortunately, this subject is beyond my understanding at the moment." It should be also emphasized that G. Perelman had not specified what type of underlying microstates and their energy should be taken in order to explain the geometric flows corresponding to certain thermodynamical and gravity models.} \]
provide main references and discussed such issues related to geometric flows in [41], see also alternative results in [77, 84, 85, 86, 87]. The second approach is to consider the flow parameter \( \xi \) as a time like coordinate when the 3-d configurations are subjected to certain condition of time evolution on a 4-d Lorentz manifold. In such an approach, the 3-d flow evolution is driven by solutions of certain (modified) Einstein equations in 4-d. This way, we obtain nonlinear systems of PDEs describing correlated geometric flow evolution and spacetime dynamical processes. In the Appendix, we provide formulas for relativistic time like evolution parameters, but the the main part of this work it is elaborated for the models with a temperature like parameter \( \tau \).

A.1 Geometric evolution of gravitational fields

To study relativistic geometric flows and elaborate on thermodynamical models we use a N–adapted 3+1 decomposition for the canonical d–connection, \( \mathbf{D} = (\mathbf{D}_i, \mathbf{i}D) \) and d–metric \( \mathbf{g} := (\mathbf{q}, \mathbf{N}) \) of a 4–d nonholonomic Lorentz manifold \( \mathbf{V} \). Such a connection is important for decoupling and integrating geometric evolution and dynamical systems of equations when at the end we can always extract LC-configurations imposing additional nonholonomic constraints.

On closed 3-d spacelike hypersurfaces, both the geometric flow and MGTs can be formulated in two equivalent forms using the connections \( \nabla \) and/or \( \mathbf{D} \) when the evolution of geometric objects are determined by the evolution of the hypersurface metric \( \mathbf{q} \) and an extension to \( \mathbf{g} \). We introduce the canonical Laplacian d-operator, \( \hat{\Delta} := \mathbf{D}_\alpha \mathbf{D}^\alpha \) and consider the canonical distortion tensor \( \mathbf{Z} \). Using distortions \( \nabla = \mathbf{D} - \mathbf{Z}, \) we compute

\[
\hat{\Delta} = \mathbf{D}_\alpha \mathbf{D}^\alpha = \Delta + \hat{Z} \hat{\Delta}
\]

and \( \Delta = \nabla_i \nabla^i = \nabla_\alpha \nabla^\alpha \),

\[
\hat{Z} \hat{\Delta} = \mathbf{Z}_i \nabla^i - [\mathbf{D}_i (\mathbf{Z}^i) + \mathbf{Z}_i (\mathbf{D}^i)] = \mathbf{Z}_\alpha \mathbf{Z}^\alpha - \mathbf{Z}_\alpha (\mathbf{D}_\alpha ) + \mathbf{Z}_\alpha (\mathbf{D}^\alpha )] ;
\]

\[
\mathbf{R}_{ij} = \mathbf{R}_{ij} - \mathbf{Z} \mathbf{c}_{ij} ; \quad \mathbf{R}_{\beta\gamma} = \mathbf{R}_{\beta\gamma} - \mathbf{Z} \mathbf{c}_{\beta\gamma} ;
\]

\[
\hat{\mathbf{R}} = \mathbf{R} - \mathbf{g}^{\beta\gamma} \mathbf{Z} \mathbf{c}_{\beta\gamma} = \mathbf{R} - \mathbf{q}^{ij} \mathbf{Z} \mathbf{c}_{ij} = \mathbf{R} - \mathbf{Z} ;
\]

\[
\hat{\mathbf{Z}} = \mathbf{g}^{\beta\gamma} \mathbf{Z} \mathbf{c}_{\beta\gamma} = \mathbf{q}^{ij} \mathbf{Z} \mathbf{c}_{ij} = \mathbf{h} \hat{Z} + \mathbf{v} \hat{\mathbf{Z}} ; \quad \hat{\mathbf{Z}} = \mathbf{g}^{ij} \mathbf{Z} \mathbf{c}_{ij} ; \quad \mathbf{v} \hat{\mathbf{Z}} = \mathbf{h}^{ab} \mathbf{Z} \mathbf{c}_{ab} ;
\]

\[
\mathbf{R} = \mathbf{h} \mathbf{R} + \mathbf{v} \mathbf{R} ; \quad \mathbf{h} \mathbf{R} := \mathbf{g}^{ij} \mathbf{R}_{ij} ; \quad \mathbf{v} \mathbf{R} = \mathbf{h}^{ab} \mathbf{R}_{ab} .
\]

Such values can be computed in explicit form for any class of exact solutions of modified Einstein equations when a double 2+2 and 3+1 splitting is prescribed and the LC–conditions can be imposed additionally.

A.2 Nonholonomic 3–d hypersurface Perelman’s functionals

On a normalized 3-d spacelike closed hypersurface \( \mathbf{c} \hat{\Sigma} \subset \mathbf{V} \), the normalized Hamilton equations can be written in a coordinate basis,

\[
\partial_\xi q_{ij} = -2 \mathbf{R}_{ij} + \frac{2\mathbf{v}}{5} q_{ij}, \quad (A.4)
\]

\[
q_{ij}|_{\xi = 0} = q_{ij}^0 [x^j] .
\]

The left label "c" is used for "compact and closed". We shall write explicitly the dependence on space coordinates (writing in brief \( q_{ij}(x^i, \xi) = q_{ij}(\xi) \)) if this do not result in ambiguities. The Ricci
tensor $R_{ij}$ is computed for the Levi–Civita connection $\nabla$ of $q_{ij}(\xi)$ parameterized by a real variable \( \xi \), \( 0 \leq \xi < \xi_0 \), for a differentiable function $\xi(t)$. The boundary conditions in (A.4) are stated for $\xi = 0$ and the normalizing factor

$$\hat{r} = \int_{\tilde{\Xi}_t} |R\sqrt{|q_{ij}|d x^3|}/\int_{\tilde{\Xi}_t} \sqrt{|q_{ij}|d x^3}$$

is introduced in the form to preserve the volume of $c_{\tilde{\Xi}}$, i.e. $\int_{\tilde{\Xi}} \sqrt{|q_{ij}|d x^3}$. For certain models, we can find solutions of (A.1) with $\hat{r} = 0$. In physical theories, such a factor can be related to a cosmological constant $\lambda$.

To find explicit solutions of (A.4) for $q_{ij} \subset g_{\alpha\beta}$ with $g_{\alpha\beta}$ considered as a solution of the 4-d (modified) Einstein equations we have to consider a nontrivial normalizing factor $\hat{r}$ which can be related to a cosmological constant. We can re-write (A.1) in any nonholonomic basis using respective formulas for geometric evolution of frame fields, $\partial_{\xi} e_i^k = q^i_j, R_{i\alpha k} e_i^k$, when $q_{ij}(\xi) = q_{ij}(\xi) e_i^k(\xi) e_j^l(\xi)$ for $e_i(\xi) = e_i^k(\xi)\partial_{\xi}^k$ and $e^j(\xi) = e^j(\xi)dx^j$. There is a unique solution for such systems of linear ordinary differential equations, ODEs, for any $\xi \in [0, \xi_0)$.

In nonholonomic variables and for the hypersurface canonical linear d–connection $\mathcal{D}$, the Perelman’s functionals can be written

$$\hat{F} = \int_{\Xi_t} e^{-f}\sqrt{|q_{ij}|d x^3}(\sqrt{\mathcal{R}} + |\mathcal{D}f|^2),$$

and

$$\hat{W} = \int_{\Xi_t} M\sqrt{|q_{ij}|d x^3}[\xi(\sqrt{\mathcal{R}} + |\mathcal{D}f| + |\mathcal{D}f|)^2 + f - 6],$$

where the scaling function $f$ satisfies $\int_{\Xi_t} M\sqrt{|q_{ij}|d x^3} = 1$ for $M = (4\pi \xi)^{-3} e^{-f}$. The functionals $\hat{F}$ and $\hat{W}$ transform into standard Perelman functionals on a hypersurface $\tilde{\Xi}_t$ if $\mathcal{D} \rightarrow \nabla$. The $W$–entropy $\hat{W}$ is a Lyapunov type non–decreasing functional and can be considered as an alternative to the Hawking-Bekenstein entropy. Such entropy type functionals can be used for elaborating hypersurface thermodynamic models.

### A.3 Nonholonomic Ricci flow equations for 3–d hypersurface metrics

Let us consider dependencies of geometric objects in formulas (A.5) and (A.6) on a smooth parameter $v(\xi)$ for which $\partial v/\partial \xi = -1$ (or simplicity, the normalizing terms can be omitted) Applying a variational N-adapted calculus, we prove the nonholonomic geometric evolution (modified Hamilton) equations for any induced 3-d metric $\mathbf{q}$ and canonical d–connection $\mathcal{D}$. We obtain

$$\partial_v q_{ij} = -2(\mathbf{R}_{ij} + \mathbf{Z} c_{ij}),$$

$$\mathbf{R}_{ia} = -\mathbf{Z} c_{ia},$$

$$\partial_v f = -(\hat{\Delta} - \frac{Z}{\hat{\Delta}})f + (|\mathcal{D} - \mathbf{Z}| f^2 - \sqrt{\mathcal{R}} - \mathbf{Z}).$$

The distortion tensors in these equations are completely determined by $q_{ij}$, see formulas (A.3), when

$$\partial_v \hat{F}(q, \mathcal{D}, f) = 2\int_{\Xi_t} e^{-f}\sqrt{|q_{ij}|d x^3}[\mathcal{D}_{ij} + \mathbf{Z} c_{ij} + (\mathcal{D}_i - \mathbf{Z}_i)(\mathcal{D}_j - \mathbf{Z}_j)f^2],$$

when the normalizing function $f$ is subjected to the condition that $\int_{\Xi_t} e^{-f}\sqrt{|q_{ij}|d x^3}$ is constant for a fixed $\xi$ and $f(v(\xi)) = f(\xi)$.
The system of equations (A.7) and (A.8) is equivalent to (A.4) (for self-similar conditions when \( \partial_t q_{ij} = 0 \), we obtain 3-d Ricci soliton equations) up to certain re-definition of nonholonomic frames and variables. We have to consider (A.8) as additional constraints because in nonholonomic variables the Ricci d–tensor is (in general) nonsymmetric. If we do not impose such constraints, the geometric evolution is with nonsymmetric metrics which is were studied in [61]. Additional nonholonomic parameterizations allow to generate entropic/ quasiperiodic Ricci soliton configurations.

A.4 Geometric evolution to nonholonomic 4–d Lorentz configurations

The geometric evolution of d-metrics (9) \( q(\xi(t)) \subset g(\xi(t)) := (q(\xi(t)),\tilde{N}(\xi(t))) \) is described by respective generalizations of functionals (A.3) and (A.6). For foliations \( \tilde{\Sigma}_t \) adapted to a N-connection structures and parameterized by a spacetime coordinate \( t \), we can introduce such 4–d functionals

\[
\hat{F}(q, D, f) = \int_{t_1}^{t_2} dt \, \tilde{N}(\xi) \cdot \hat{F}(q, D, f), \tag{A.9}
\]

and

\[
\hat{W}(q, D, f) = \int_{t_1}^{t_2} dt \, \tilde{N}(\xi) \cdot \hat{W}(q, D, f(\xi)). \tag{A.10}
\]

In these formulas, the parameter \( \xi \) for 3-d Ricci flows is extended as a time like coordinates on a open region in a 4-d \( \mathbf{V} \) determined by a Lorentzian d-metric \( g(\xi) \). For alternative models (see the main part of this work), we can consider additional dependencies on a temperature parameter \( \tau \) both for the 3-d and 4-d configurations. Using frame transforms on 4-d nonholonomic Lorentz manifolds, such values can be re–defined respectively in terms of data \((g(\tau), D(\tau))\). We obtain

\[
\hat{F}(\tau) = \int_{t_1}^{t_2} \int_{\tilde{\Sigma}_t} e^{-\tilde{f}} \sqrt{|g_{\alpha\beta}|} d^4 u ( s R + |D \tilde{f}|^2), \tag{A.11}
\]

and

\[
\hat{W}(\tau) = \int_{t_1}^{t_2} \int_{\tilde{\Sigma}_t} \tilde{M} \sqrt{|g_{\alpha\beta}|} d^4 u [\tau ( s R + | h D \tilde{f}| + | v D \tilde{f}|)^2 + \tilde{f} - 8], \tag{A.12}
\]

where the scaling function \( \tilde{f} \) satisfies \( \int_{t_1}^{t_2} \int_{\tilde{\Sigma}_t} \tilde{M} \sqrt{|g_{\alpha\beta}|} d^4 u = 1 \) for \( \tilde{M} = (4\pi \tau)^{-3} e^{-\tilde{f}} \). In these formulas \( \tau \) is a temperature like parameter describing relativistic evolution of geometric objects and associated thermodynamic values.

The functionals \( \hat{F}(\tau) \) and \( \hat{W}(\tau) \) for 4-d pseudo-Riemannian metrics should not be interpreted as static thermodynamic entropies like in the 3-d Riemannian case. We consider that they determine certain relativistic thermodynamic models with flows of entropic values on respective temperature like parameter \( \tau \) and a time like coordinate \( \xi \). Nevertheless, they describe nonlinear general relativistic diffusion type processes if \( q \subset g \) and \( D \subset D \) are determined by certain lapse and shift functions as certain solutions of 4–d gravitational equations. This diffusion can be modified by elastic processes, interactions with modified gravitational and matter field equations.

The formulas (A.7) and (A.8) can be generalized to 4–d configurations when the coefficients are determined by the Ricci d–tensors and distortions contain the left label "l". Using the hypersurface d-metric \( q_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta \), we write

\[
\partial_q g_{\alpha\beta} = -2( R_{\alpha\beta} + Z i c_{\alpha\beta} ) - \partial_u ( n_\alpha n_\beta ), \tag{A.13}
\]

\[
R_{\alpha\alpha} = - Z i c_{\alpha\alpha}, \text{ for } R_{\alpha\beta} \text{ with } \alpha \neq \beta.
\]
The term \( \partial_v (n_a n_\beta) \) can be computed in explicit form using formulas for the geometric evolution of N–adapted frames, see below \((A.16)\). Here we note that \( \partial_v f, \partial_v (\tilde{F}) \) and \( \partial_v \tilde{W} \) can be re–written for values with 4–d indices when the geometric objects are defined with respect to N-adapted or coordinate frames (we omit such formulas in this work).

For 4-d configurations with a corresponding re–definition of the scaling function, \( f \rightarrow \tilde{f} \), and using necessary type \( N \)–adapted distributions, we can construct models of geometric evolution of vacuum gravitational fields with \( h \)– and \( v \)–splitting for \( D \),

\[
\partial_v g_{ij} = -2R_{ij}, \quad \partial_v g_{ab} = -2R_{ab}, \quad (A.14)
\]

\[
\partial_v \tilde{f} = -\tilde{\Box} \tilde{f} + |D \tilde{f}|^2 - \ast R.
\]

These formulas can be derived from the modified Perelman functional \((A.9)\), or \((A.10)\), and any variant of type \((A.11)\) and/or \((A.12)\) following a similar calculus to that presented in the proof of Proposition 1.5.3 of \([51]\) but in \( N \)–adapted form. We have to impose the metric symmetry conditions \( R_{ia} = 0 \) and \( R_{ai} = 0 \) if we want to keep the total metric to be symmetric under the Ricci flow evolution and work with the canonical d-connection \( D \).

We present some important formulas on flow evolution on a parameter \( \upsilon \in [0, \upsilon_0] \) of \( N \)–adapted frames in a 4–d nonholonomic Lorentz manifold. Such flows can be computed as \( e_a(\upsilon) = e_a^\alpha(\upsilon, \upsilon) \partial_\upsilon \),

\[
e_a^\alpha(\upsilon, \upsilon) = \begin{bmatrix} e^i_\upsilon(\upsilon, \upsilon) - N^h_\upsilon(\upsilon, \upsilon) & e^b_\upsilon(\upsilon, \upsilon) \\ 0 & e^a_\upsilon(\upsilon, \upsilon) \end{bmatrix}, \quad \gamma_a(\upsilon, \upsilon) = \begin{bmatrix} e^i_\upsilon = \delta^i_\upsilon & e^h_\upsilon = N^h_\upsilon(\upsilon, \upsilon) & \delta^k_\upsilon \\ e^a_\upsilon = 0 & e^a_\upsilon = \delta^a_\upsilon & \delta^a_\upsilon \end{bmatrix},
\]

\[(A.15)\]

when the \( h \)– and \( v \)– components of a d-metric evolve as \( g_{ij}(\upsilon) = e^i_\upsilon(\upsilon, \upsilon) e^j_\upsilon(\upsilon, \upsilon) \eta_{ij} \) and \( g_{ab}(\upsilon) = e^a_\upsilon(\upsilon, \upsilon) e^b_\upsilon(\upsilon, \upsilon) \eta_{ab} \). For local parameterizations of Minkowski type with \( \eta_{ij} = \text{diag}[+, +] \) and \( \eta_{ab} = \text{diag}[1, -1] \) corresponding to the chosen signature of \( \tilde{g}_{[0]}^\alpha(\upsilon) \), we have the evolution equations

\[
\frac{\partial}{\partial \upsilon} e^\alpha_a = g^{\alpha \beta} R_{\beta \gamma} e^\gamma_a.
\]

\[(A.16)\]

Such equations are prescribed for models with geometric evolution determined by the canonical d–connection \( D \) and can be redefined using distortion relations or necessary type frame transforms. The equations \((A.16)\) can be written in terms of the LC-connection if distortions of type \((A.3)\) are considered for 4–d values determined by exact solutions in \( GR \), or certain modified elastic type gravity theories.

A.5 Small N-adapted stationary entropic flow deformations

We can construct various classes of generic off–diagonal stationary solutions for geometric flow \([35, 36, 37, 38, 39, 40, 41, 42, 43, 44]\) and MGTs \([60, 61, 62, 63, 64, 65, 66, 69, 70]\). In all such theories, it is not clear what physical meaning may have certain general nonholonomic configurations determined by generating and integration functions with arbitrary data. Nevertheless, we can understand certain important physical properties of some classes of solutions if there are considered small deformations from certain well-known and physically important solutions (for instance, from a black hole, BH, configuration of Kerr or Schwarzschild type). In this work, we study exact and parametric solutions characterized by locally anisotropic polarization/ running of constants and nonlinear off-diagonal gravitational interactions and geometric flows determined by entropic and quasiperiodic sources and generating functions.
Let us consider a prime pseudo–Riemannian metric of type \( \hat{g} = [\hat{g}_i, \hat{g}_a, \hat{N}_a] \) \([27]\) when \( \partial_3 \hat{g}_4 = \hat{g}_4^2 \neq 0 \), which is not diagonal but can be diagonalized via coordinate transforms, for instance, with rotation systems of reference and other type frame/coordinate transforms. Our goal is to formulate a geometric formalism for small generic off–diagonal parametric deformations of \( \hat{g} \), see footnote \([12]\) into certain target stationary metrics of type \( g \) \([26]\)

\[
\begin{align*}
\text{ds}^2 &= \eta_1(\varepsilon, \tau) \hat{g}_4(dx^i)^2 + \eta_2(\varepsilon, \tau) \hat{g}_a(e^a)^2, \\
\text{e}^3 &= dy^3 + w_1(\varepsilon, \tau) \hat{w}_i dx^i, \text{e}^4 = dt + \eta_3(\varepsilon, \tau) \hat{n}_i dx^i,
\end{align*}
\]

where the coefficients \([g_{\alpha} = \eta_{\alpha} \hat{g}_{\alpha}, w_1 \hat{w}_i, n_\eta n_i]\) depend on a small parameter \( \varepsilon \), \( 0 \leq \varepsilon \ll 1 \), and on evolution parameters \( \tau \) and define a solution of entropic flow evolution equations reduced to the system of nonlinear PDEs with decoupling \([32]\). We express the d-metric and N–connection \( \varepsilon \) –deformations in the form:

\[
\eta_1(\varepsilon, \tau) = 1 + \varepsilon \nu_1(\tau, x^k), \eta_2 = 1 + \varepsilon \nu_2(\tau, x^k, y^3)\text{for the coefficients of d-metrics ;}
\]

\[
w_1(\varepsilon, \tau) = 1 + \varepsilon w_1(\tau, x^k, y^3), \quad n_1(\tau, x^k, y^3) = 1 + \varepsilon \quad n_1(\tau, x^k, y^3)
\]

for the coefficients of d-metrics ,

considering \( g_4(\tau) = \eta_4(\tau) \hat{g}_4 = \eta_4(\tau, r, \theta, \varphi) \hat{g}_4(\tau, \theta, \varphi) = [1 + \varepsilon \nu(\tau, r, \theta, \varphi)] \hat{g}_4 \), for \( \nu = \nu_4(\tau, r, \theta, \varphi) \) and \( \hat{g}_4^2(\tau) \neq 0 \), as a generating function, see a similar \( \eta \)-decomposition in \([60]\).

Deformations of \( h \)-components of a stationary are characterized by \( \varepsilon g_1 = \hat{g}_1(1 + \varepsilon \nu_1) = e^{\psi(\tau, x^k)} \) as a solution of the 2-d Laplace equation in \([32]\). For \( \psi(\tau) = \hat{1}_y(\tau, x^k) + \varepsilon \hat{1}_y(\tau, x^k) \) and \( h \hat{Y}(\tau) = \hat{0}_h \hat{Y}(\tau, x^k) + \varepsilon \hat{1}_h \hat{Y}(\tau, x^k) \), we compute the deformation polarization functions in the form

\[
u_1 = e^{\psi^{(\tau)}(\tau, x^k)} \hat{g}_3 \text{h and} \quad \hat{1}_y = \hat{0}_h \hat{Y} \text{. In these formulas, there are used certain generating and source functions as solutions of} \quad \hat{0}_y \hat{y}^{(1)} + \hat{1}_y \hat{y}^{(0)} = \hat{0}_h \hat{Y} \text{and} \quad \hat{1}_y \hat{y}^{(1)} + \hat{1}_y \hat{y}^{(0)} = \hat{1}_h \hat{Y} \text{. Using such} \varepsilon \text{-decomposition of polarization functions of type} \quad \text{[(A.18)]}, \text{we obtain} \varepsilon \text{-decomposition of the target stationary d-metric and N-connection coefficients} \quad \text{[20]} \text{. For the d-metric, we compute}
\]

\[
\hat{g}_4 \varepsilon = e^{\psi(\tau, x^k)} \text{as a solution of 2-d Poisson equations}
\]

\[
\varepsilon g_1(\tau) = [1 + \varepsilon e^{\psi^{(1)}}] \hat{g}_3 \text{h for} \quad \hat{1}_y \hat{y}^{(1)} \text{, also constructed as a solution of 2-d Poisson equations for} \quad \hat{1}_y
\]

\[
\begin{align*}
\hat{g}_3 \varepsilon = \frac{-4[\left[\eta_4(\tau) \hat{g}_4(\tau)^{1/2}\right]^2}{\int dy^3 \hat{Y}(\tau)[\eta_4(\tau) \hat{g}_4(\tau)^2]} \text{i.e. } \\
\varepsilon g_3(\tau) = [1 + \varepsilon \nu_3] \hat{g}_3 \text{ for} \quad \hat{0}_y \hat{y}_3 = 2 \left[\frac{\hat{0}_y \hat{g}_4(\tau)^{\nu_3}}{\hat{g}_3(\tau)} - \int dy^3 \hat{Y}(\tau)(\hat{0}_y \hat{g}_4(\tau)^{\nu_3}) \int dy^3 \hat{Y}(\tau) \hat{g}_3(\tau)
\end{align*}
\]

where a new spacial system of coordinate \([x^i(r, \theta, \varphi), y^3(r, \theta, \varphi)]\) is used to satisfy the condition \( (\hat{g}_3(\tau)^2 = \hat{g}_3 \int dy^3 \hat{Y}(\tau) \hat{g}_3(\tau)\), which allow to compute \( \hat{g}_3 \) for any prescribed values \( \hat{g}_4 \) and \( \hat{Y}(\tau) \).

The N-connection coefficients are computed

\[
\begin{align*}
\eta_3(\tau) \hat{w}_i &= \frac{\partial_i \int dy^3 \hat{Y}(\tau)[\eta_4(\tau) \hat{g}_4(\tau)^{nu_3}]}{\hat{Y}(\tau) [\eta_4(\tau) \hat{g}_4(\tau)^{nu_3}]} \text{i.e. } \\
\varepsilon w_1(\tau) &= [1 + \varepsilon w_1(\tau)] \hat{w}_i \text{ for} \quad w_1(\tau, x^i, y^3) = \frac{\partial_i \int dy^3 \hat{Y}(\tau)(\hat{0}_y \hat{g}_4(\tau)^{nu_3}) - (\hat{0}_y \hat{g}_4(\tau)^{nu_3})}{\hat{g}_3(\tau)}
\end{align*}
\]

\[\text{[20]} \text{we omit details on computing vertical components for} \varepsilon \text{-decompositions generating solutions of the} \nu \text{-equations} \quad \text{[32]} \]

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when \( \dot{\omega}_i = \partial_i \int dy^3 \Psi(\tau) \dot{g}_i^0 / \Psi(\tau) \dot{g}_i^0 \) is computed for some prescribed \( \Psi(\tau) \) and \( \dot{g}_i^0 \):

\[
\eta_k^0(\tau) \dot{n}_k = 1 n_k(\tau) + 16 n_k(\tau) \int dy^3 \frac{((\eta_4(\tau) \dot{g}_4)^{-1/4})^2}{\int dy^3 \Psi(\tau)(\eta_4(\tau) \dot{g}_4)^{c} } \text{ i.e.,}
\]

\[
\varepsilon n_3(\tau) = [1 + \varepsilon n_3(\tau)] \dot{n}_k = 0 \text{ for } n_3(\tau, x^i, y^3) = 0,
\]

if we chose the integration functions \( n_k(\tau) = 0 \) and \( \dot{n}_k(\tau) = 0 \). For simplicity, we can always impose such conditions for solutions with Killing symmetry on \( \partial_i \) in order to not work with cumbersome formulas for \( n_3(\tau, x^i, y^3) \) and related nonholonomic torsion coefficient. Such configurations with nontrivial \( n_k \) are studied in \([64, 65]\) for certain models with string and extra dimension corrections (for 4-d entropic configurations, those constructions can be redefined for both nonholonomic geometric flows and Ricci solitons.)

The values with a "circle" are prescribed by a chosen prime solution (in our case, we can chose the 4-d Kerr metric but subjected to some additional frame and coordinated transform to satisfy the conditions \( \dot{g}_i^0 \neq 0 \) and relations of \( \dot{g}_i^0 \) to \( \dot{g}_3 \) and \( \dot{w}_i \) as we considered above). Fixing a small value \( \varepsilon \), we can compute such deformations for stationary configurations and prove their stability if the prime metric is stable. We conclude that \( \varepsilon \)-deformed quadratic elements can be written

\[
d s_{\varepsilon t}^2 = \varepsilon g_{\alpha\beta}(\tau, x^k, y^3) du^\alpha du^\beta
\]

or

\[
d s_{\varepsilon t}^2 = \varepsilon g_i(\tau, x^k) [(dx^1)^2 + (dx^2)^2] + \varepsilon h_3(\tau, x^k, y^3) [dy^3 + \varepsilon w_1(\tau, x^k, y^3)dx^1]^2 + \varepsilon g_4(\tau, x^k, y^3)dt^2.
\]

We can impose additional constraints in order to extract LC–configurations with zero torsion.

### A.6 The Kerr BH solution in nonholonomic variables

To apply the AFDM and study possible entropic quasi-periodic flows of BH solutions is necessary to define certain classes of nonholonomic variables which allow future N-adapted nonholonomic deformations, for instance, of Kerr metrics, see \([78]\) as a standard monograph on GR, and \([64, 65]\) for examples of nonholonomic deformations of BH solutions in MGTs.

Let consider a 4-d ansatz for a prime metric, \( ds_{\varepsilon 0}^2 = Y^{-1}e^{2h}(d\rho^2 + dz^2) + Y(d\varphi + Adt)^2 - \rho^2 Y^{-1}dt^2 \). This quadratic line element written in terms of three functions (\( h, Y, A \)) on coordinates \( x^i = (\rho, z) \), defines the Kerr solution of the vacuum Einstein equations (for rotating black holes) if the coefficients are chosen

\[
Y = 1 - (p\hat{x}_1)^2 - (q\hat{x}_2)^2, \quad A = 2Mq \big(1 - \hat{x}_2)(1 + p\hat{x}_1) \big) \big(1 + (p\hat{x}_1)^2 + (q\hat{x}_2)^2 \big),
\]

\[
e^{2h} = \frac{1 - (p\hat{x}_1)^2 - (q\hat{x}_2)^2}{p^2[(\hat{x}_1)^2 + (\hat{x}_2)^2]}, \quad \rho^2 = M^2(\hat{x}_1^2 - 1)(1 - \hat{x}_2^2), \quad z = M\hat{x}_1\hat{x}_2.
\]

If we fix \( M = \text{const} \) and \( \rho = 0 \), we obtain result a horizon \( \hat{x}_1 = 0 \) and the "north / south" segments of the rotation axis, \( \hat{x}_2 = +1 / -1 \). For further applications of the AFDM, it is convenient to write this Kerr metric in the form

\[
ds_{\varepsilon 0}^2 = (dx^1)^2 + (dx^2)^2 + Y(e^3)^2 - \rho^2 Y^{-1}(e^4)^2,
\]

where the coordinates \( x^1(\hat{x}_1, \hat{x}_2) \) and \( x^2(\hat{x}_1, \hat{x}_2) \) are defined for any

\[
(dx^1)^2 + (dx^2)^2 = M^2 e^{2h}(\hat{x}_1^2 - \hat{x}_2^2) Y^{-1} \left( \frac{d\hat{x}_1}{\hat{x}_1^2 - 1} + \frac{d\hat{x}_2}{1 - \hat{x}_2^2} \right) = \frac{M e^{2h}}{\hat{x}_1^2 - \hat{x}_2^2}.
\]
and, \( y^3 = \varphi + \hat{y}^3(x^1, x^2, t) \), \( y^4 = t + \hat{y}^4(x^1, x^2) \). We can consider an N-adapted basis \( e^3 = dy^3 + (\partial_i \hat{y}^3) dx^i \) and \( e^4 = dt + (\partial_i \hat{y}^4) dx^i \), for some functions \( \hat{y}^3 \), \( a = 3, 4 \), with \( \partial_i \hat{y}^3 = -A(x^k) \).

The Kerr metric was studied in the so-called Boyer–Linquist coordinates \((r, \vartheta, \varphi, t)\), for \( r = m_0(1 + p\hat{x}_1) \), \( \hat{x}_2 = \cos \vartheta \), which are convenient for applying of the AFDM. These coordinates can be related to parameters \( p, q \) for the total black hole mass, \( m_0 \) and the total angular momentum, \( am_0 \), for the asymptotically flat, stationary and anti-symmetric Kerr spacetime. Using formulas \( m_0 = Mp^{-1} \) and \( a = Mp^{-1} \), we can write (A.20) in the form

\[
d s^2_{(0)} = (dx^1)^2 + (dx^2)^2 + \left( \frac{\overline{C} - \overline{B}^2}{\overline{A}} \right) (e^3)^2 + \overline{A}(e^4)^2, \tag{A.21}
\]

for coordinate functions \( x^1(r, \vartheta), x^2(r, \vartheta), y^3 = \varphi, y^4 = t + \hat{y}^4(r, \vartheta, \varphi) + \varphi \overline{B} / \overline{A}, \partial_{\varphi} \hat{y}^4 = -\overline{B} / \overline{A}. \) In (A.21), \( (dx^1)^2 + (dx^2)^2 = \Xi (\Delta^{-1} dr^2 + d\vartheta^2) \), and the coefficients are defined in the form

\[
\begin{align*}
\overline{A} &= -\Xi^{-1}(\Delta - a^2 \sin^2 \vartheta) \overline{B} = \Xi^{-1} a \sin^2 \vartheta \left[ \Delta - (r^2 + a^2) \right], \\
\overline{C} &= \Xi^{-1} \sin^2 \vartheta \left[ (r^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta \right], \text{ and } \Delta = r^2 - 2m_0 + a^2, \Xi = r^2 + a^2 \cos^2 \vartheta.
\end{align*}
\]

The primed quadratic linear elements (A.20) and/or (A.21) can be written as stationary prime metric (27) with N-adapted coefficients

\[
\begin{align*}
\hat{g}_1 &= 1, \hat{g}_2 = 1, \hat{g}_3 = Y, \hat{g}_4 = -\rho^2 Y^{-1}, \hat{N}^a_i = \partial_i \hat{y}^a, \text{ or} \\
\hat{g}_1' &= 1, \hat{g}_2' = 1, \hat{g}_3' = \overline{C} - \overline{B}^2 / \overline{A}, \hat{g}_4' = \overline{A}, \hat{N}^a_i = \hat{w}_i = 0, \hat{N}^4_i = \hat{n}_i = -\partial_i (\hat{y}^3 + \varphi \overline{B} / \overline{A}),
\end{align*}
\]

define solutions of vacuum Einstein equations with zero sources.

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