The physics of space and time III: Classification of space-time experiments and the twin paradox

J.H. Field

Département de Physique Nucléaire et Corpusculaire Université de Genève. 24, quai Ernest-Ansermet CH-1211 Genève 4.

E-mail: john.field@cern.ch

Abstract

A nomenclature for inertial frames and a notation for space and time coordinates is proposed to give an unambiguous description of space-time experiments in special relativity. Of particular importance are the concepts of ‘base’ and ‘traveling’ frames and ‘primary’ and ‘reciprocal’ experiments. A detailed discussion of the twin paradox is presented. The physical basis of the differential aging effect is found to be a relativistic relative-velocity transformation relation, not, as hitherto supposed, the spurious ‘length contraction’ effect.

PACS 03.30.+p
1 Introduction

The present article is the third in a series of long papers attempting to give a comprehensive description of the physics of flat space-time (i.e., in the absence of gravitational effects) by correcting certain errors in the application of the space-time Lorentz transformation (LT) originating in Einstein’s first special relativity paper [1]. The error was the neglect of certain additive constants in the transformation equations, necessary to correctly describe synchronised clocks at different spatial positions. Although Einstein pointed out the necessity to include such constants in Ref. [1] this was not done by Einstein himself or by subsequent authors. As shown by the present author in Ref [2], correction of this error renders spurious the ‘length contraction’ (LC) and ‘relativity of simultaneity’ (RS) effects of conventional special relativity theory. For further discussion of this point see Refs. [3, 4, 5]. Although there is ample experimental verification of the time dilatation (TD) effect, predicted using the space-time LT in Ref [1], there is, to date, no published experimental test of the LC or RS effects [2]. The present author has proposed a test of RS using two satellites in low Earth orbit [2, 6], or GPS satellites in combination with a single satellite in low Earth orbit [6].

The first of the long papers [7] discussed the axiomatic basis of special relativity theory, recalling in an appendix two previously-published derivations of the LT that do not use Einstein’s second postulate concerning the constancy of the speed of light. This paper introduces a calculus of pointer-mark coincidences with a view to giving a mathematically rigorous description of experimental space and time measurements. Also presented are several different clock synchronisation procedures, not employing light signals and so not affected by considerations of ‘conventionality’ inherent in the the assumption of light-speed isotropy [8].

The second long paper [9] is a detailed critique of Einstein’s original special relativity paper [1]. As well as containing concepts and predictions that revolutionised the understanding of physics, Ref. [1] also contains many mistakes, both of a mathematical nature, and of physical interpretation, for example, the spurious RS and LC effects. In view of the importance of classical electromagnetism in the arguments presented in Ref. [1], the modern reassessment of this paper in Ref. [9] uses results from work by the present author on relativistic classical electrodynamics [10, 11, 12, 13] derived as the classical limit of quantum electrodynamics.

The subject of the present paper is a study of the precise operational meaning of the coordinate symbols which appear in the space-time LT, with the aim of understanding the physical basis of the differential aging effect (DAE) that underlies the ‘twin paradox’ as orginally introduced by Langevin [14]. This paradox arises from the following considerations. A travelling twin, T, makes a round trip to a distant star while her brother, R, remains on Earth. Assuming that the periods of uniform motion are much larger than those of acceleration or deceleration during the journey, then, due to TD, R will see a clock carried by T and measuring the increment of her age, to run slow, throughout the journey, compared to a similar clock at rest in his own frame and measuring the increment of his own age during T’s journey. R will then see that T is younger than he is on her return to Earth. However, from T’s point of view, assuming that that the TD
effect depends only on the magnitude of the relative velocity, it might be concluded that
the clock carried by R will appear to T to be running slow, throughout the journey, as
compared to the clock at rest in T’s frame. Therefore, on her return, T will find that she
is older than, not younger than, R. Since T cannot be both younger than, and older than,
R, the above description leads to a paradox.

The physical meaning of the time symbols occurring in the space-time LT is crucial
for the resolution of the above paradox — in fact it is the travelling twin, T, who is the
youngest. Particularly important is the distinction, explained in Section 4 below, between
a primary experiment and its reciprocal as compared to a transformation of space-time
coordinates and its inverse. Before the work presented in the present paper, its author had
not appreciated this important distinction and its relation to the ‘Reciprocity Principle’
(RP) [15] which states that: ‘If the velocity of an inertial frame S’ relative to another
such frame S is \( \vec{v} \), then the velocity of S relative to S’ is \(-\vec{v}\).’ As will be seen below,
the RP, although valid in Galilean relativity, no longer holds in special relativity as a
consequence of TD and the invariance of spatial intervals when space-time as opposed to
kinematical (energy-momentum) LTs are considered. Earlier versions of several papers
written recently by the present author, assuming the correctness of the RP in applications
of the space-time LT in special relativity [16, 17, 18, 19, 20] therefore contain erroneous
arguments and must be extensively revised. To date, the general validity of the RP in
special relativity has been universally (although usually tacitly) assumed. As will be
seen it is the basis of the ‘standard solution’ of the twin paradox in which the physical
origin of the DAE is traced to LC of spatial intervals in the travelling twin’s frame. As
will be seen, the RP must be replaced by a related but distinct ‘Kinematical Reciprocity
Principle’ (KRP) that, unlike the RP, is valid in both Galilean and special relativity.

The structure of this paper is as follows: In the following section as an instructive
preamble, five different ways to measure the length of a train are discussed. Reflecting
on the assumptions underlying these methods gives the key to understanding the correct
solution of the twin paradox. In Section 3 a nomenclature and notation to describe differ-
ent space-time experiments is introduced. Inertial frames are classified according to three
different criteria: subject or object, source or target and base or travelling. The notation
specifies each of these six attributes for any given coordinate. Several examples are given
of applications of the notation: time dilatation, simultaneity of events in two inertial
frames, Lorentz invariance of length intervals, relativistic reciprocity relations, velocity
addition formulae, transverse and longitudinal photon clocks, Einstein’s 1905 discussion
of ‘relativity of simultaneity’ and Einstein’s train-embankment thought experiment. Sec-
tion 4 discusses the differing concepts of inverse space-time transformations and reciprocal
experiments. The twin paradox and its solution is discussed, in the light of the analyses
of several different thought experiments, in Section 5. In Section 6 the standard solu-
tion of the twin paradox is applied to the ‘space billiard’ thought experiment introduced
in Section 5 and contrasted with the correct solution presented in the same section. In
Section 7 the twin paradox is discussed in relation to the Minkowski space-time plot. In
this connection, the recent paper [23] by the present author pointing out an angular sign
error in drawing the directions of the space and time coordinate axes of of the moving
frame, originating in Minkowski’s original work [24], is important. Correcting this error
invalidates the standard solution of the twin paradox as derived from the space-time plot.
Section 8 contains a brief summary and some closing comments.
2 Prologue: Five ways to measure the length of a train

In space-time physics there is a close connection between the concepts of physical space and time, and that of uniform motion, that is encapsulated in Galileo’s equation

\[ \Delta x = v \Delta t \]

where \( \Delta x \) and \( \Delta t \) are space and time intervals respectively defined in a common reference frame and \( v \) is a constant velocity describing, for example, the motion of the origin, \( O' \), of an inertial frame, \( S' \), relative to another one, \( S \), in the direction of a common \( x-x' \) axis. \( \text{Eqn}(2.1) \) can be used to convert space measurements into time measurements and \textit{vice versa} when the value of \( v \) is known. For example an analogue clock is based on a formula similar to \( (2.1) \) where the spatial interval is replaced by an angular one and the spatial velocity by a constant angular velocity \( \omega \).

The length of a physical object (say a train) can be measured either by direct comparison with some other physical object of known spatial extent — a ruler, or by making use of \( \text{Eqn}(2.1) \). The simplest way to measure the length of an object is by direct comparison of its ends with marks on a ruler in the rest frame of the former, the ruler also being at rest in the same frame. Time plays no role in such a measurement. The spatial coincidences of the front and back ends of the train with the ruler marks can be observed at any times convenient for the experimenter. The length of the train is defined as the difference between the numbers associated with the ruler marks that coincide with the front and back ends of the train [7].

The second method is similar, except that the measured object is in arbitrary motion (i.e., in uniform or accelerated motion). The length of the train is still measured by observing the spatial coincidences of the front and back of the train with ruler marks, but it is now essential that that the observations are performed simultaneously in the rest frame of the ruler. Alternatively, if the observers at different spatial positions are equipped with synchronised clocks, and it is known that the motion of the train is uniform, but not the value of its speed, the length of the train and its speed may be determined from observations of spatial coincidences of the front and back of the train with ruler marks at different, but known, times [7].

The third method, restricted to the case of uniformly moving object with a known speed, is for single observer in the rest frame of the ruler, equipped with a clock, to measure the time difference, \( \Delta t \), between the passages of the front and back ends of the moving train past a fixed ruler mark. The length of the train, \( \Delta x \), is then given by \( \text{Eqn}(2.1) \).

The fourth method is a variant of the third in which the train remains at rest, say in the frame \( S \), and the observer, equipped with a clock, passes by in another train, say with rest frame \( S' \), moving at a known uniform speed. The observer records the times at which he passes by the two ends of the train and calculates its length with \( \text{Eqn}(2.1) \). This method is also applicable when both the observer and the measured train are in motion with known speeds, when an appropriate velocity addition formula is used to calculate
the speed, \( v \), to be substituted in (2.1).

The fifth method involves observation, in the frame \( S \), of the time difference of spatial coincidences of the front end of the moving train, with rest frame \( S' \), used in the fourth method, with the back and front ends of the stationary to-be-measured train. Again the length of the train is given by Eq. \( (2.1) \).

Suppose now that two independent measurements of the length of a train are made, one using the fourth and the other the fifth of the methods just described. If different results are obtained for the length of the train, three different causes, in one-to-one correspondence with the quantities in Eqn\( (2.1) \), may be cited:

(i) The train changed length between the two measurements.

(ii) The observer’s train, was moving at a different speed during the two measurements.

(iii) The clocks employed in the two measurements, one in the frame \( S \), the other in the frame \( S' \), are running relatively fast (or slow) with respect to each other.

If it now happens that one clock is running slower than the other, in the comparison of the measurements, the observer, \( O(\text{Slow}) \), with the slower running clock, but unaware of this fact, has two possible ways to interpret the different measurement results on the basis of Eqn\( (2.1) \):

(a) The assumed speed is correct but the train was shorter during his measurement, than during that performed by the other observer, \( O(\text{Fast}) \).

(b) The train has the same length but appears to move faster for \( O(\text{Slow}) \) than for \( O(\text{Fast}) \).

Now if it is known, independently, both that the clock of \( O(\text{Slow}) \) does run slower than that of \( O(\text{Fast}) \) and that the train is the same length, only the interpretation (b) is possible. As will be seen in the following, this is the correct prediction in special relativity for the case that the clock in the frame \( S' \) runs slow, according to an observer at rest in the frame \( S \), due to the time dilatation effect. Hitherto the solution (a) relativistic ‘length contraction’ has been taken as the prediction of special relativity. How this spurious result arises from misinterpretation of the space-time Lorentz transformation will also be explained.

3 Nomenclature and notation for space-time experiments and some applications

The space-time Lorentz transformation relates space and time measurements recorded, predicted or specified in different inertial frames. In its application to any given experiment, care must be taken to define the exact operational meanings of the space and time
coordinates appearing in the transformation equations. In this connection, to avoid confusion, the mathematical symbols representing the coordinates must be such as to encode clearly the following essential information:

(I) The inertial frame in which the space or time measurements are actually performed, i.e. the frame of the observer in the experiment.

(II) The frame in which the primary space-time events, that undergo Lorentz transformation, are defined.

(III) Whether, in the experiment in which an object is moving between fixed points in an inertial frame, the observer is in the rest frame of such a ‘travelling object’, or in some other inertial frame.

The distinction made in (III) will be found to be of crucial importance for a correct understanding of the physics underlying time dilatation and the closely related ‘twin paradox’, as well as the *modus operandi* of ‘photon clocks’.

In order to furnish the information mentioned in the points (I)-(III) above, the following nomenclature and notation for space and time coordinates is suggested:

**Subject frame**

\((\mathbf{X}, T)\) coordinates in observer’s rest frame.

**Object frame**

\((\vec{x}, t)\) coordinates in any inertial frame in motion relative to that of the observer.

**Source frame**

\((\vec{X}, T); (\vec{x}, t)\) frame in which primary space-time events are specified i.e. events subjected to Lorentz transformation.

**Target frame**

\((\mathbf{X}, T); (\mathbf{x}, t)\) Lorentz-transformed coordinates.

**Base frame**

\((\vec{X}_B, T_B); (\vec{x}_B, T_B)\) coordinates specifying the position of an object (‘travelling object’) moving between fixed points in an observer’s frame.

**Travelling frame**

\((\vec{X}_T, T_T); (\vec{x}_T, T_T)\) coordinates specifying outset and arrival events (corresponding to fixed points in a base frame) in the subject frame of a travelling object.

It is important to note that physical predictions which change completely when base and travelling frames are exchanged, are, as will be seen, invariant with respect to exchange of source and target frames, i.e. to whether the LT or its inverse is used to perform
the transformation. Also physical predictions do not depend on whether a frame is considered to be a subject frame (capitalised symbols) or an object frame (uncapitalised symbols). This is because the predictions do not depend on whether or not an observer is present to perceive them. The use of capitalised symbols is then optional, but may improve the clarity of equations. An example given below is the measurement of the lifetime of an unstable particle where measurements are performed uniquely in the laboratory (subject) frame in order to obtain a quantity defined in the rest frame (object frame) of the decaying particle.

In the following, some applications are given to illustrate the use of the notation.

**Time dilatation**

Consider an experiment where clocks $C_0$ and $C'_0$ are placed at the origins, $O$ and $O'$ of inertial frames $S$ and $S'$, where $S'$ moves in the frame $S$ with speed $v_B$ along the common $x-x'$ axis of the two frames. The subscript ‘$B$’ indicates that $S$ is the base frame for the travelling clock $C'_0$ so that $\Delta X_B(C'_0) = v_B \Delta T_B(C'_0)$. Two, distinct, experiments are possible: $C'_0$ is observed from $S$ or $C_0$ is observed from $S'$. If the observations are made in the frame $S$ then the latter is the subject and base frame of the experiment and may be either the source or target frame for the LT. The former possibility is treated first, the other one will be considered below. For this the appropriate LT, relating the elapsed time $t'_0(C'_0)$ to the corresponding time, $T_0(C'_0)$ recorded by $C_0$, is:

\[
\begin{align*}
\tau'(C'_0) &= \gamma_B[X(C'_0)_B - v_B T(C'_0)_B] = 0 \\
\bar{\tau}'(C'_0) &= \gamma_B[T(C'_0)_B - \frac{v_B X(C'_0)_B}{c^2}]
\end{align*}
\]

where $\gamma_B \equiv 1/\sqrt{1-(v_B/c)^2}$. These transformation equations require that when $O$ and $O'$ are in spatial coincidence, $X(C'_0)_B = 0$, then $\tau'(C'_0) = T(C'_0)_B = 0$. Using (3.1) to eliminate $X(C'_0)_B$ from (3.2) gives immediately the time dilatation (TD) relation:

\[
T(C'_0)_B = \gamma_B \tau'(C'_0)
\] (3.3)

If, in the experiment with a reciprocal configuration, $C_0$ moves with speed $-v'_B$ relative to $C'_0$ along the positive $x'$ axis, so that $\Delta X'_B(C_0) = -v'_B \Delta T'_0(C_0)_B$ a similar calculation gives the TD relation:

\[
T'(C'_0)_B = \gamma'_B \bar{\tau}(C_0)
\] (3.4)

where $\gamma'_B \equiv 1/\sqrt{1-(v'_B/c)^2}$. The TD relation for the experiment with a reciprocal configuration is obtained by exchange of primed and unprimed quantities throughout Eqn(3.3).

An important feature of the TD relations (3.3) and (3.4) is their manifest translational invariance —no spatial coordinates appear in these equations. The physical meaning of TD is that a uniformly moving clock ($C'_0$ in (3.3)) is observed to run slower by the factor $1/\gamma_B$ relative to an identical clock at rest ($C_0$ in (3.3)), and this independantly of the spatial position of the clock. In an identical manner the clock $C_0$ is observed to run slower than $C'_0$ by the factor $1/\gamma_B$ when observed from $S'$. This apparently contradictory behaviour shows that TD is a subjective (observer-dependent) effect. The ‘twin paradox’,
to be discussed below, arises from the attempt to relate, by the LT, the independent events in an experiment and the different experiment with a reciprocal configuration.

**Simultaneity of events in two inertial frames**

If a second clock $C'_{L'}$ is at rest at $x' = L'$ in $S'$ its equation of motion in the frame $S$ is:

$$X(C'_{L'})_B = v_B T(C_0) + L \tag{3.5}$$

where $L$ is the separation of $C'_0$ and $C'_{L'}$ in $S$. The space transformation equation for $C'_{L'}$ is given by:

$$\vec{x}'(C'_{L'}) - L' = \gamma_B X(C'_{L'})_B - L - v_B T(C_0)_B = 0 \tag{3.6}$$

This equation satisfies simultaneously the initial condition $\vec{x}'(C'_{L'}) = L'$ and the equation of motion (3.5). It also reduces to (3.1) in the case $L = L' = 0$. The time transformation equation corresponding to (3.6) is

$$t'(C'_{L'}) = \gamma_B [T(C_0)_B - \frac{v_B (X(C'_{L'})_B - L)}{c^2}] \tag{3.7}$$

When $T(C_0)_B = 0$, then (3.6) gives $X(C'_{L'})_B = L$, independently of the value of $v_B$; (3.1) gives $X(C'_0)_B = 0$ and (3.2) and (3.7) give $\vec{t}'(C'_0) = \vec{t}'(C'_{L'}) = 0$. $C'_0$ and $C'_{L'}$ are therefore synchronised when $T(C_0)_B = 0$.

Using (3.5) to eliminate $X(C'_{L'})_B - L$ from (3.7) gives, analogously to (3.3), the TD relation:

$$T(C_0)_B = \gamma_B \vec{t}'(C'_{L'}) \tag{3.8}$$

Comparing (3.3) and (3.8) it can be seen that $C'_0$ and $C'_{L'}$ remain synchronous for all values of $T(C_0)_B$:

$$\vec{t}'(C'_0) = \vec{t}'(C'_{L'}) \equiv \vec{t} \tag{3.9}$$

There is therefore no ‘relativity of simultaneity’ effect for two synchronised clocks at different positions in the same inertial frame. That this must be so is already evident from the manifest translational invariance of the TD relations (3.3) and (3.8), that have no dependence on spatial coordinates. Using a simplified notation, if clock 1 in $S'$ is compared with clock 2 in $S$, and clock 3 in $S'$ is compared with clock 4 in $S$, where the spatial positions of all the clocks are arbitrary, the corresponding TD relations are:

$$T_2 = \gamma_B \vec{t}'_1 \tag{3.10}$$

$$T_4 = \gamma_B \vec{t}'_3 \tag{3.11}$$

It follows from these equations that if $\vec{t}'_1 = \vec{t}'_3$ then $T_2 = T_4$ and *vice versa* — clocks which are synchronous in $S(S')$ are also synchronous in $S'(S)$.

**Lorentz invariance of length intervals**

Combining (3.1) with (3.3) and (3.6) with (3.8) gives the relations

$$X(C'_0)_B = \gamma_B \beta_B c \vec{t}'(C'_0) \tag{3.12}$$

$$X(C'_{L'})_B = L + \gamma_B \beta_B c \vec{t}'(C'_{L'}) \tag{3.13}$$

where $\beta_B \equiv v_B/c$. 

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Figure 1: a) World lines of the clocks, $C'_0$ and $C'_L$, and simultaneous events, $E_1'$ and $E_2'$, in the frame $S'$. b) The same world lines and events as observed in the frame $S$, showing the invariance of the spatial separation of the clocks and the absence of any `relativity of simultaneity' effect. See text for discussion.
In virtue of the identity $\gamma_B^2 - \beta_B^2 \gamma_B^2 \equiv 1$ (3.3), (3.12) and (3.8), (3.13) are two pairs of parametric equations that specify hyperbolae in the Cartesian space-time plot in the frame S. Introducing a simplified notation: $X_1 \equiv X(C_0')_B$, $X_2 \equiv X(C_{L'})_B$, $\bar{t}_1 \equiv \bar{t}'(C_0')$, $\bar{t}_2 \equiv \bar{t}'(C_{L'})$ and $T \equiv T(C_0)_B$, the equations of these hyperbolae are

$$c^2 T_1^2 - X_1^2 = c^2 (\bar{t}_1)^2$$

$$c^2 T_2^2 - (X_2 - L)^2 = c^2 (\bar{t}_2)^2$$

(3.14)

(3.15)

Consider now simultaneous events in $S'$, $E_1'$ and $E_2'$, on the world lines of $C'_0$ and $C'_{L'}$ such that:

$$\bar{t}_1 = \bar{t}_2 = \bar{t}, \quad T_1 = T_2 = T$$

(3.16)

The world lines of $C'_0$ and $C'_{L'}$ in S' and the events $E_1'$ and $E_2'$ are shown in Fig. 1b. The transformed events in the frame S are shown in Fig. 1b, where the world lines of $C'_0$ and $C'_{L'}$ for $\beta_B = 1/3$ are drawn. Also shown are are the hyperbolae (3.14) and (3.15) corresponding to the simultaneity condition (3.16) together with their intersections with the world lines of $C'_0$ and $C'_{L'}$ for $\beta_B = 0$ and $\beta_B = 0.88$. For $\beta_B = 0$ the world lines are identical to those in Fig. 1a, lying along the $\bar{t}_1$ and $\bar{t}_2$ axes. Noting that with simultaneity condition (3.16), $X_1$ and $X_2$ are functions only of $\beta_B$, (3.14)-(3.16) give the relation:

$$X_2(\beta_B) - X_1(\beta_B) = L$$

(3.17)

where $L = X_2(T = 0)$ is independent of $\beta_B$. Eqn (3.17) holds for all values of $\beta_B$, in particular for $\beta_B \to 0$ when $X_1 \to X'_1$ and $X_2 \to X'_2$ so that:

$$X_2(0) - X_1(0) = X'_2 - X'_1 \equiv L' = L$$

(3.18)

As is also clear from inspection of Fig. 1b, the spatial separation of $C'_0$ and $C'_{L'}$ is a Lorentz-invariant quantity —there is no ‘relativistic length contraction’. How this spurious effect arises from misuse of the space-time LT is explained in Section 6 below.

**Relativistic reciprocity relations**

The TD effect and the invariance of length intervals imply modification of the velocity reciprocity relation of Galilean relativity, which may be stated as:

$$\left. \frac{dX_B}{dT_B} \right|_{X_T} \equiv v_B = - \left. \frac{dX_T}{dT_T} \right|_{X_B} \equiv \bar{v}'_T \quad \text{(Galilean Relativity)}$$

(3.19)

The bar in the symbol $\bar{v}'_T$ denotes that, unlike $v_B$, which is a fixed input parameter, the quantity is a derived one analogous the the transformed coordinates $\bar{x}$, $\bar{t}$. A similar notation is used below to classify other kinematical parameters. In Galilean relativity $\Delta X(O')_B = -\Delta X(O)_T$, $\Delta T_B = \Delta \bar{t} = \Delta X'_{T'}$. For an observer at rest in $S'$ (a ‘travelling’ observer if S is the base frame), $\Delta \bar{t}'$ becomes a subject frame interval. Because of the invariance of length intervals, the relation $\Delta X(O')_B = -\Delta X(O)_T$ holds also in special relativity, whereas, due to TD, time intervals in S and S’ are related in a different manner than in Galilean relativity: $\Delta T_B = \gamma_B \Delta \bar{t}' = \gamma_B \Delta \bar{t}'_T$. Combining these relations between infinitesimal intervals, the reciprocity relation in special relativity therefore differs from (3.19):

$$\left. \frac{dX_B}{dT_B} \right|_{X_T} \equiv v_B = - \left. \frac{dX_T}{\gamma_B dT_T} \right|_{X_B} \equiv \frac{\bar{v}'_T}{\gamma_B} \quad \text{(Special Relativity)}$$

(3.20)
So that
\[ \vec{v}_T' = \gamma' B v_B \] (3.21)

As will be seen below, the relation (3.21) is crucial for understanding the correct solution of the twin paradox.

Consideration of the experiment, with a reciprocal configuration, in which S’ is the base frame and C\(_0\) is the travelling object yields the following reciprocity relation:

\[ \frac{dX'}{dT'} \bigg|_{X_T} \equiv -v'_B = - \frac{dX_T}{\gamma'_B dT_t} \bigg|_{X'_B} \equiv -\vec{v}_T \] (3.22)
giving
\[ \vec{v}_T = \gamma'_B v'_B \] (3.23)

The relation (3.22) is obtained from (3.20) by exchange of primed and unprimed quantities followed by the operations \( v'_B \to -v'_B, \vec{v}_T \to -\vec{v}_T \). Eqn (3.23) is related to (3.21) simply by exchange of primed and unprimed quantities.

### Velocity addition formulae

The addition of base frame velocities is first considered. Suppose that the frame S’ moves with speed \( v_B \) in the positive \( x\)-direction and that an object, O, moves with speed \( u_B \) in the frame S’ in the same direction. Having fixed the values of \( v_B \) and \( u_B \), what is the speed, \( \vec{w}_B \), of O in the frame S? The differential form of the LT between S and S’ is:

\[ d\vec{x}'(O)_B = \gamma_B [dX(O)_B - v_B dT(O)_B] \] (3.24)
\[ d\vec{t}'(O)_B = \gamma_B [dT(O)_B - \frac{v_B dX(O)_B}{c^2}] \] (3.25)

Since both S and S’ are base frames for the travelling object O,

\[ \frac{d\vec{x}'(O)_B}{d\vec{t}'(O)_B} \equiv \vec{w}_B \] (3.26)
\[ \frac{dX_B(O)_B}{dT_B(O)_B} \equiv u_B \] (3.27)

the base frame velocity addition formula given by combining (3.24)-(3.27) is the conventional parallel velocity addition formula of special relativity as derived by Einstein [1]:

\[ \vec{w}_B = \frac{u_B - v_B}{1 - \frac{v_B u_B}{c^2}} \] (3.28)

Notice that in this derivation no travelling frame coordinates appear, only base frame or transformed base frame coordinates of the travelling object O in the frames S or S’.

Consider now a different experiment in which the object O is initially defined to move with speed \( w_B \) along the positive \( x\)-axis in S, and is subsequently observed in the travelling frame S’. If the origin of S, the origin of S’ and O coincide at time \( T_B = 0 \), then at the later time, \( T_B \), the separation, in S, of O from the origin of S’ is:

\[ \Delta X(O)_B = (w_B - v_B)T_B \] (3.29)
The corresponding travelling frame (S’) separation (c.f. Eqn (3.20)) is:

\[ \Delta \bar{X}'(O)_T = \bar{u}'_T \bar{T}'_T \]  

(3.30)

where \( \bar{u}'_T \) is the speed of O in S’. The Lorentz invariance of length intervals \( \Delta X(O)_B = \Delta \bar{X}'(O)_T \) and the TD relation \( T_B = \gamma_B \bar{T}'_T \) then give the velocity transformation formula:

\[ \bar{u}'_T = \gamma_B (w_B - v_B) \]  

(3.31)

When \( w_B = 0 \), \( \bar{u}'_T \to -\bar{v}'_T \) and the relation (3.21) is recovered.

It is interesting, in view of the discussion of ‘photon clocks’ in the following section, to also consider the velocity transformations for an object moving in the x-y plane. Introducing the \( x \)- and \( y \)-components of \( u_B \) and \( \bar{w}_B \), the following formulae are obtained from (3.24), (3.25) and the invariance of transverse coordinates under the LT:

\[ \bar{w}^{(x)}_B = \frac{u^{(x)}_B - v_B}{1 - \frac{v_B u^{(x)}_B}{c^2}} \]  

(3.32)

\[ \bar{w}^{(y)}_B = \frac{u^{(y)}_B}{\gamma_B (1 - \frac{v_B u^{(y)}_B}{c^2})} \]  

(3.33)

These are the usual relativistic formulae for the transformation of these velocity components. When \( u^{(x)}_B = v_B \) then (3.32) gives \( \bar{w}^{(x)}_B = 0 \) and (3.33) gives:

\[ \bar{w}_B = \bar{w}^{(y)}_B = \frac{u^{(y)}_B}{\gamma_B (1 - \frac{v_B u^{(y)}_B}{c^2})} = \gamma_B u^{(y)}_B \equiv (\bar{u}^{(y)}_T)' \]  

(3.34)

which is similar to the the transformation laws (3.21) and (3.31) of longitudinal relative velocity between the base frame S and the travelling frame S’. Notice that in (3.34) \( \bar{w}^{(y)}_B \) is the total velocity of O relative to the travelling frame S’.

**Transverse and longitudinal photon clocks**

In the discussion of TD above, the frame S was subject, source and base frame. In the case of ‘photon clocks’ the primary space time events in the problem are defined by spatial coincidences between light signals and stationary objects in the rest frame of the ‘clock’, which are then observed from a frame in uniform motion relative to the clock frame. Events on the world lines of the clocks \( C'_0 \) and \( C'_L \) introduced above, corresponding to light signal coincidences, are then the primary events in the source and travelling frame S’, while S remains the subject and base frame of the experiment, but is now the target frame of the LT. The appropriate LT for the problem is then the inverse of Eqns(3.1) and (3.2) with the replacements: \( X, T \to \bar{X}, \bar{T}, \bar{x}, \bar{t} \to x, t \):

\[ \bar{X}(C'_0)_B = \gamma_B [x'(C'_0) + v_B t'(C'_0)] \]  

(3.35)

\[ \bar{T}(C'_0)_B = \gamma_B [t'(C'_0) + \frac{v_B x'(C'_0)}{c^2}] \]  

(3.36)

Since \( x'(C'_0) = 0 \), the TD relation is given directly by Eqn(3.36):

\[ \bar{T}(C'_0)_B = \gamma_B t'(C'_0) \]  

(3.37)
which is related to (3.3) above simply by exchange of source and target frames.

The use of (3.37) to analyse transverse and longitudinal ‘photon clocks’, without further explicit use of the LT, is illustrated in Fig. 2 and Fig. 3 respectively. In order to render the photon coincidence events, defined in the frame S', visible to an observer in S, the photon sources and mirrors are equipped with light detectors (not shown) and lamps that flash when light signals are emitted, reflected or return to their source. In the figures and related text an evident abbreviated notation is used. For example $T(C_0)_B \rightarrow T$, $t'(C'_0), t'_L(C'_L) \rightarrow t'$, denoting the times recorded by a synchronised clock at any position in the frames S and S’ respectively, and $v_B \rightarrow v, \gamma_B \rightarrow \gamma$. In Figs. 2 and 3, $v = (\sqrt{3}/2)c$, $\gamma = 2$.

In Fig. 2a a directed light signal (for example a laser beam) is emitted at $t_0' = 0$ by the source So at rest in the frame S’, and the associated lamp L$_{So}$ flashes. The signal from L$_{So}$ is observed in the frame S at time $T_0 = 0$ (Fig. 2d). In the frame S, the directed light signal propagates at an angle $\theta = \arccos[v/c]$ relative to the positive x-axis. At time $t' = L/c$ in S’ the light signal arrives at the mirror, M, where it is detected and partially reflected back towards the source (Fig. 2b); the lamp L$_M$ fires after the reflection event with a negligibly small delay. According to Eqn(3.37) the signal from L$_M$ is observed in S at time $T_1 = \gamma L/c$ when the source is at $X = v\gamma L/c$ (Fig. 2e). As shown in Fig. 2c, the reflected part of the light signal arrives back at the source at time $t'_2 = 2L/c$ and the lamp L$_{So}$ flashes a second time. This flash is observed in S at time $T_2 = 2\gamma L/c$ when the source is at $X = 2v\gamma L/c$ (Fig. 2f). The apparent speed of the directed light signal in S as measured from the time interval between observation of L$_M$ and either the first or second observation of L$_{So}$, is:

$$\bar{v}_{\perp}^{app} = \frac{cD}{\gamma L} = \frac{c}{\gamma L} \left( \frac{v\gamma L}{c \cos \theta} \right) = c$$

(3.38)

where (see Fig. 2) $D = PQ = PR$. Note that (3.38) is consistent with the transformation formula (3.34) for the relative transverse velocity. In this case:

$$u_B^{(y)} = c \sin \theta = c \sqrt{1 - \frac{v^2}{c^2}} = \frac{c}{\gamma}$$

(3.39)

so that

$$(u_T^{(y)})' = \gamma u_B^{(y)} = c$$

(3.40)

The corresponding sequence of events for a longitudinal photon clock is shown in Fig. 3. Figs. 3a, 3c and 3e show the emission, reflection and return-to-source events in S’, while Figs. 3b, 3d and 3f show the same sequence of events in S, as observed via the prompt signals of the lamps L$_{So}$ and L$_M$, in a similar manner to that described above for the transverse photon clock. Note that, due to the relation (3.18), and contrary to text-book special relativity, the separation of So and M is the same in the frames S and S’. The geometry of Fig. 3d shows that the apparent speed of the light signal in S’, as viewed from S, obtained by observation of the time difference between the first signal from L$_{So}$ and the signal from L$_M$ is:

$$\bar{v}_{+}^{app} = \frac{L(1 + \beta \gamma)}{\gamma L/c} = c \left( \frac{1}{\gamma} + \beta \right)$$

(3.41)
where \( \beta \equiv v/c \), Similarly the apparent light speed for the return path given by time difference between the observations in S of the signal from \( L_M \) and the second signal from \( L_{So} \) is given by the geometry of Fig. 3f as:

\[
\bar{v}_{-app} = \frac{L(1 - \beta \gamma)}{\gamma L/c} = c \left( \frac{1}{\gamma} - \beta \right)
\]  

(3.42)

These formulae are consistent with the longitudinal relative velocity transformation since formula (3.41) can be written as:

\[
c = u'_+ = \gamma(\bar{v}_{+app} - v)
\]  

(3.43)

and (3.42) as

\[
c = u'_- = \gamma(\bar{v}_{-app} + v)
\]  

(3.44)

The relative velocity as, observed in S, of the photon and the moving photon clock is \( \bar{v}_{+app} - v \) on the outward transit and \( \bar{v}_{-app} + v \) on the return one.

---

**Figure 2:** A transverse ‘photon clock’. a) a directed light signal is emitted from the source \( So \) at rest in \( S' \) towards the mirror \( M \); the lamp \( L_{So} \) flashes. b) the light signal arrives at \( M \); the lamp \( L_M \) flashes. c) the reflected signal arrives back at \( So \) and \( L_{So} \) flashes a second time. d), e) and f) show the same sequence of events as observed in the frame \( S \), where \( S' \) moves with speed \( v \) along the positive \( \bar{X} \) axis. \( v = (\sqrt{3}/2)c \), \( \gamma = 2 \).

In text books and the pedagogical literature, following the premises of Einstein’s 1905 paper [1] it is universally, but incorrectly, assumed that the apparent speed of the light
Figure 3: A longitudinal ‘photon clock’. The event sequence in $S'$ is similar to that in Fig. 1. a), c) and e) show event configurations in the frame $S'$. b), d) and f) the same events as observed in $S$. $v = (\sqrt{3}/2)c$, $\gamma = 2$. Note that, with this choice of $v$, $\vec{v}^{app}$ is negative; i.e. the apparent velocity of the photon on the return path is directed in the positive $\vec{X}$ direction.
signals in the longitudinal photon clock, as viewed from S, is \( c \), as is indeed the case, as Eqn(3.38) shows, for the transverse photon clock. Also, events defined uniquely in the frame S are considered. In fact S is both the subject and source frame, but no object frame coordinates appear in the calculation since no LT is performed. These assumptions lead to the following S-frame times for the reflection and return-to-source events:

\[
T_{\text{refl}} = \frac{L}{c-v} \quad (3.45)
\]

\[
T_{\text{retn}} = \left[ \frac{L}{c-v} + \frac{L}{c+v} \right] \quad (3.46)
\]

Note that, unlike for the case of the correct application of the LT in Fig. 3, \( T_{\text{retn}} \neq 2T_{\text{refl}} \). The translational invariance of the TD relation then shows that \( T_{\text{refl}} \) and \( T_{\text{retn}} \) cannot be related by the LT with the corresponding events defined in the frame S’—equal time intervals between events in the frame S’ must correspond to equal time intervals between the same events viewed in the frame S. This is a necessary consequence of the TD relations (3.3) and (3.5). Making the further assumption, in contradiction with the invariant relation, (3.18), that the spatial separation of So and M is different in S and S’ and applying the TD relation (3.37) to the return-to-source event gives:

\[
T_{\text{retn}} = \frac{2\gamma L'}{c} = \left[ \frac{L}{c-v} + \frac{L}{c+v} \right] = \frac{2Lc}{c^2 - v^2} \quad (3.47)
\]

from which follows the spurious ‘length contraction’ relation:

\[
L = L' / \gamma \quad (3.48)
\]

The crucial error in this calculation is that (3.47) has been obtained, not by Lorentz transformation of the light signal-source coincidence event in the frame S’ into the frame S, but by considering only events in the frame S obtained on the incorrect assumptions that: Firstly, the apparent speed of light signal defined is S’ is \( c \) in the frame S, and secondly, that these events are related to the corresponding events in S’ by the LT (the first member of Eqn(3.47)). Correct application of the LT shows, viz Eqn(3.41) and (3.42), that the both of these assumptions are untrue. As will now be discussed, Similar mistakes occur in the discussion of ‘relativity of simultaneity’ in Einstein’s 1905 special relativity paper [1], as well as in Einstein’s interpretation of the train embankment thought experiment [25].

**Einstein’s 1905 discussion of ‘relativity of simultaneity’**

In the original special relativity paper [1], a ‘photon clock’ was used as the basis for Einstein’s light signal clock synchronisation procedure’ (LSCSP). Refering to the longitudinal photon clock shown in Fig. 3, and associating clocks \( C'_0 \) and \( C'_L \) with So and M respectively, the LSCSP states that that these clocks are synchronous in the frame S’ provided that:

\[
t1' - t0' = t2' - t1' \quad (3.49)
\]

\[
\frac{2L}{t2' - t0'} = c \quad (3.50)
\]

Einstein then considered the times of spatial coincidences of light signals specified in the frame S with the moving clocks \( C'_0 \) and \( C'_L \) in this frame, as observed in S, as discussed
above in connection with Eqns (3.45) and (3.46). S is thus both subject and source frame —
no space time transformations are considered. Introducing a similar notation for sucesive
coincidence events as in Figs. 2 and 3: \( T_0 = 0, T_{refl} = T_1 \) and \( T_{refn} = T_2 \) equations
similar to those of §2 of Ref. [1] are obtained:

\[
T_1 - T_0 = \frac{L}{c-v} \quad (3.51)
\]

\[
T_2 - T_1 = \frac{L}{c+v} \quad (3.52)
\]

Since \( T_1 - T_0 \neq T_2 - T_1 \), Einstein concluded that the clocks that are synchronous
according to the LSCSP in S’, are not so in S because the times in this frame do not
satisfy the condition (3.49). The tacit assumption made here (though at this stage of
the presentation in Ref. [1] the LT has not yet been introduced) is that \( T_0, T_1 \) and \( T_2 \)
are related to \( t_1', t_2' \) and \( t_3' \) respectively by the space-time transformation equations of
special relativity. The explicit results given above show that this is not the case. In fact
the correctly transformed times \( \bar{T}_0, \bar{T}_1 \) and \( \bar{T}_2 \) do satisfy the LSCSP condition (3.49):

\[
\bar{T}_1 - \bar{T}_0 = \bar{T}_2 - \bar{T}_1 \quad (3.53)
\]

— the clocks are indeed also synchronous in S according the LSCSP. However the times
in S respect not the condition (3.50) but instead

\[
\frac{2L}{T_2 - T_0} = \frac{c}{\gamma} \quad (3.54)
\]

The average speed of the light signals in S’, as viewed from S, is reduced by the factor
\( 1/\gamma \), consistent with a universal TD effect: all observed physical processes in the travelling
frame S’ — including light propagation — being slowed down in the same manner.

Einstein’s conclusion is therefore based on two false assumptions:

1. The times \( T_1 \) and \( T_2 \) are the Lorentz transforms of \( t_1' \) and \( t_2' \).
2. The apparent speed in S of light signals defined in S’ is \( c \).

In fact, the times \( T_1 \) and \( T_2 \) are specified by spatial coincidences of light signals with
M and So uniquely in the frame S. They are not connected by the LT with the similarly
defined coincidence times \( t_1' \) and \( t_2' \) in the frame S’. As shown by Eqs.\( (3.38),(3.41) \) and
\( (3.42) \), the assumption (2) is correct for a transverse photon clock but not for a longitudinal
one relevant to the present problem.

**Einstein’s train-embankment thought experiment**

In the popular book ‘Relativity, the Special and General Theory’ Einstein introduced
a thought experiment [25] intended to illustrate, in a simple way, ‘relativity of simultane-
ity’, without consideration of the LT, that is frequently discussed in text books and the
pedagogical literature [26, 27, 28]. Light signals are produced by lightning strikes which
simultaneously hit an embankment at positions coincident with the front and back ends
of a moving train. The signals are seen by an observer, \( O_T \), at the middle of the train and an
observer, \(O_B\), on the embankment, aligned with \(O_T\) at the instant of the lightning strikes. The light signals are observed simultaneously by \(O_B\) who concludes that the lightning strikes are simultaneous. Because of the relative motion of \(O_T\) and the light signals, the latter are not observed by \(O_T\) at the same time. Invoking the constancy of the speed of light in the train frame, Einstein concludes that \(O_T\) would not judge the strikes to be simultaneous, giving rise to a ‘relativity of simultaneity’ effect between the train and embankment frames.

Figure 4: Analysis of Einstein’s train-embankment thought experiment. Configurations a), b) and c) in the embankment frame (S); c), d) and e) in the train frame (S’). \(v = c/2\), \(\gamma = 2/\sqrt{3}\). See text for discussion.

This train-embankment thought experiment (TETE) is now analysed in terms of the concepts and nomenclature introduced above. The observer \(O_T\) is replaced by a two-sided light detector, \(D\), at the middle of the train. The latter moves to the right with speed \(v\). The embankment frame, \(S\), is the base and source frame of the experiment, the train frame, \(S’\), is the travelling, target and subject frame, since the aim is to calculate the times of emission of the light signals received by \(D\), i.e. in the train frame. At time \(t_0\) in \(S\) (Fig. 4a) light signals moving at speed \(c\) in the embankment frame are emitted, and move towards \(D\). The light signals are also ‘travelling objects’ in the source frame \(S\). The essential input parameters of the problem, \(v\) and \(c\) are therefore fixed in the frame \(S\). In accordance with Eqn(3.18) the length of the train, \(L\), is invariant. At time in \(S\) \(t_1 = L/[2(c+v)] + t_0\) (Fig. 4b) the left-moving light signal strikes \(D\), and at time in \(S\) \(t_2 = L/[2(c-v)] + t_0\) (Fig. 4c) the right-moving light signal strikes \(D\). The configurations in \(S’\) corresponding to those in \(S\) in Figs. 4a,b,c are shown in Figs. 4d,e,f respectively. The velocity transformation formula (3.31) implies that the speed in \(S’\) of the right-moving light signal relative to \(D\) is \(\bar{v}_T(+) = \gamma(c+v)\) while that of the left-moving light signal is
\( \bar{v}_{T}(+) = \gamma(c + v) \). The pattern of events in S and S’ is then the same, the only difference being that that the velocities of the light signals relative to D are greater in S’ by the factor \( \gamma \)—a necessary consequence of time dilatation and the invariance of length intervals. The left-moving light signal is then observed in S’ at the time:

\[
\bar{T}'_1 = \frac{L}{2\gamma(c + v)} + \bar{T}'0 = \frac{t1 - t0}{\gamma} + \bar{T}'0
\]  

(3.55)

and the right-moving one at the time:

\[
\bar{T}'_2 = \frac{L}{2\gamma(c - v)} + \bar{T}'0 = \frac{t2 - t0}{\gamma} + \bar{T}'0
\]  

(3.56)

The time dilatation effect for the travelling frame S’ is manifest in these equations.

On the assumption that an experimenter analysing the signals received by D knows the essential parameters of the problem, \( L, v, \) and \( c \), the measured times \( \bar{T}'1 \) and \( \bar{T}'2 \) in the train frame can be used to decide whether the left and right moving light signals were emitted simultaneously in this frame or not. If the right-moving and left-moving signals are emitted at times \( \bar{T}'0(−) \) and \( \bar{T}'0(+) \) respectively then (3.55) and (3.56) are modified to:

\[
\bar{T}'1 = \frac{L}{2\gamma(c + v)} + \bar{T}'0(+) = \frac{t1 - t0}{\gamma} + \bar{T}'0(+) 
\]  

(3.57)

and

\[
\bar{T}'2 = \frac{L}{2\gamma(c - v)} + \bar{T}'0(−) = \frac{t2 - t0}{\gamma} + \bar{T}'0(−) 
\]  

(3.58)

Subtracting (3.57) from (3.58) and rearranging:

\[
\bar{T}'2 - \bar{T}'1 = \bar{T}'0(−) - \bar{T}'0(+) + \frac{\gamma \beta L}{c} 
\]  

(3.59)

The observed time difference \( \bar{T}'2 - \bar{T}'1 \) and knowledge of the value of \( \gamma \beta L/c \) then enables determination of \( \bar{T}'0(−) - \bar{T}'0(+) \) so the simultaneity of emission of the light signals can be tested. For the event configurations shown in Fig. 4 it would be indeed concluded that \( \bar{T}'0(−) = \bar{T}'0(+) \), so the emission of the signals is found to be simultaneous in the train frame, contrary to Einstein’s assertion in Ref. [25]. The essential flaw in Einstein’s argument was the failure to distinguish between the speed of light, relative to some fixed object in an inertial frame, and the speed of light relative to some moving object in the same frame, which is what is relevant for the analysis of the TETE. Einstein’s interpretation corresponds to replacing \( \bar{T}'2 \) and \( \bar{T}'1 \) by \( t2 \) and \( t1 \), so that only events in the embankment frame are considered, and making the replacements, (confusing the speed of light, with the relative speed of light and a moving object): \( \gamma(c \pm v) \rightarrow c \) in (3.57) and (3.58), giving:

\[
\bar{T}'0(−) - \bar{T}'0(+) = t2 - t1 = \frac{\gamma^2 \beta L}{c}
\]  

(3.60)

This gives Einstein’s false conclusion that the light signal emission events would be found to be non-simultaneous in the train frame.
4 Inverse space-time transformations and reciprocal experiments

In this section the physical concept of *reciprocal space-time experiments*, crucial for a correct understanding of the ‘twin paradox’ of special relativity will be developed and contrasted with the mathematical concepts of a space-time coordinate transformation and its inverse.

The generic space-time LT is:

\[
x' = \gamma(x - vt) \tag{4.1}
\]

\[
t' = \gamma(t - \frac{vx}{c^2}) \tag{4.2}
\]

while its inverse is

\[
x = \gamma(x' + vt') \tag{4.3}
\]

\[
t = \gamma(t' + \frac{vx'}{c^2}) \tag{4.4}
\]

Eqns(4.3) and (4.4) are a necessary algebraic consequence of (4.1) and (4.2) and the definition \(\gamma \equiv 1/\sqrt{1 - (v/c)^2}\). It is implicit in these equations that clocks in the frame S (coordinates \(x, t\)) and \(S' (coordinates \ x', t')\) are synchronised so that \(t = t' = 0\) when \(x = x' = 0\). Roman symbols are used here and in Section 6 below to denote generic space and time coordinates, in the absence of the qualifying labels and symbols introduced in the previous section, and additional additive constants necessary to describe correctly synchronous clocks for which \(x' \neq 0\) [2, 3, 4, 5, 7].

The LT (4.1),(4.2) and its inverse (4.3),(4.4) are now used to discuss in detail a particular space-time experiment —observation of the TD effect— employing the nomenclature and notation introduced in the previous section. Clocks C and C’ are introduced at the origins O and O’ of S and S’, respectively and the case where the moving clock C’ is viewed by an observer at rest in S is first considered. In this case the corresponding reciprocal experiment is one in which the moving clock C is viewed by an observer at rest in S’.

The experiment is exemplified by the physically interesting case of observation of decay in flight of an unstable particle in which the TD relation is used to derive the lifetime \(T_D\) of the particle in its rest frame. This is done by making use of observations of its flight path \(X_B\) and lifetime \(T_B\) in the frame S. In practice, \(T_B\) is found from the relation \(T_B = X_B/v_B\), where \(v_B\) is deduced from the relativistic momentum (p) and energy (E) of the decay products of the unstable particle via the relation (see Eqn(4.26) below) \(v_B = pc^2/E\).

The appropriate version of the LT to describe such an experiment is the same as Eqns(3.1) and (3.2) above:

\[
\mathbf{\mathbf{\ell}}'(C') = \gamma_B[X(C')_B - v_BT(C)_B] = 0 \tag{4.5}
\]

\[
\mathbf{\mathbf{\ell}}(C') = \gamma_B[T(C)_B - \frac{v_BX(C')_B}{c^2}] \tag{4.6}
\]
yielding the TD relation

\[ T(C)_B = \gamma_B \bar{T}(C') \]  

(4.7)

In the specific case of the decay of an unstable particle, the rest frame decay lifetime is given by (4.7) as:

\[ t_D' = \frac{T_B}{\gamma_B} = \frac{X_B}{\gamma_B} \left( \frac{E}{p c^2} \right) = \frac{X_B m}{p} \]  

(4.8)

where \( m \) is the mass of the decaying particle. In (4.5)-(4.8) \( S \) is the subject, source and base frame while \( S' \) is the object, target and travelling frame.

For an observer at rest in \( S' \) during the experiment specified by the LT (4.5) and (4.6), \( T'(C') = \bar{T}'(C'_T) \) where \( \bar{T}'(C'_T) \) is the time recorded by a clock a rest in his/her own (travelling) frame. The TD relation may then be written, for small time increments, as:

\[ \delta T(C)_B = \gamma_B \delta T'(C'_T) \]  

(4.9)

From the Lorentz invariance of length intervals in the frames \( S \) and \( S' \), Eqn(3.18), and the fact that the direction of motion along the common \( x-x' \) axis of \( S' \) relative to \( S \), in \( S \) is opposite to that of \( S \) relative to \( S' \) in \( S' \), it follows that:

\[ \delta X(C'_B) = -\delta X'(C)_T \]  

(4.10)

where \( X'(C)_T \) is the coordinate in \( S' \) of the origin of \( S \). Differentiating (4.5), using (4.9) and (4.10) and taking the limit of vanishing intervals gives the same relation between derivatives as (3.20):

\[ \frac{dX(C'_B)}{dT(C)_B} = v_B = \frac{-dX'(C)_T}{\gamma_B d\bar{T}'(C'_T)} = \frac{\hat{v}'_T}{\gamma} \]  

(4.11)

so that

\[ \hat{v}'_T = \gamma_B v_B \]  

(4.12)

which is the same as Eqn(3.21) above.

As shown by the derivation of Eqn(3.37) above, identical predictions are obtained if the source and target frames are exchanged so that the inverse LT (4.5) and (4.6) may be used to obtain the TD relation:

\[ \bar{T}(C)_B = \gamma_B t'(C) \]  

(4.13)

The TD predictions are therefore invariant with respect to exchange of source and target frames, that is, to the use of a transformation or of its inverse. However this is not the case when experiments with reciprocal configurations are performed, i.e. when both subject and object and base and travelling frames are exchanged. This point is crucial for the correct understanding of the ‘twin paradox’ Predictions for an experiment with a configuration reciprocal to the one just described are given by exchange of primed for unprimed quantities in the above equations and reversing the signs of all velocity parameters. Hence, in the reciprocal configuration, the TD relation (4.7) is replaced by

\[ T'(C'_B) = \gamma'_B \bar{T}(C') \]  

(4.14)

and the velocity relation (4.12) by

\[ \hat{v}_T = \gamma'_B \hat{v}'_B \]  

(4.15)
The relations (4.12) and (4.15) show that the ‘Reciprocity Principle’ (RP) [15], mentioned in the Introduction, which has hitherto been assumed to hold in both Galilean and special relativity requires re-interpretation in special relativity. The RP states that:

\[(RP) \text{ If the velocity of an inertial frame } S' \text{ relative to another such frame } S \text{ is } \vec{v}, \text{ then the velocity of } S \text{ relative to } S' \text{ is } -\vec{v}.\]

The RP has hitherto, following Einstein in Ref. [1], been assumed to describe velocity observations in the frames S and S’ of a experiment in which events in the two frames are connected by the space-time LT. Such a interpretation is evidently at variance with Eqns,(4.12) and (4.15). The corresponding principle in special relativity may be called the ‘Kinematical Reciprocity Principle’ (KRP):

\[(KRP) \text{ The velocity of an inertial frame } S' \text{ relative to an inertial frame } S \text{ in primary space-time experiment is equal and opposite to the velocity of } S \text{ relative to } S' \text{ in the reciprocal experiment.}\]

Unlike the RP, the KRP describes a relation between the kinematical configurations of two physically distinct experiments, not that between observations in two different frames in the same experiment, as in (4.12) and (4.15). As will be seen below, although the events of a primary experiment are physically distinct (not related by the space-time LT) from those of the reciprocal experiment, the base frame kinematical configurations of a primary experiment and its reciprocal are related by a kinematical LT or the base frame velocity transformation formulae (5.6) and (5.7) above.

The experiments yielding the TD relations (4.7) and (4.14) may be described as having reciprocal configurations. In the case that the initial conditions of the experiment and the experiment with a reciprocal configuration are the same: \(v_B = v'_B\) and \(\gamma_B = \gamma'_B\), identical predictions (after exchange of primed and unprimed quantities) are obtained for an experiment and its reciprocal, i.e. the clock C’ as viewed from S is slowed down by the same ratio in comparison to a clock at rest as when C when viewed from S’. Then the two experiments may be called reciprocal ones. It can now be seen that the KRP is actually the definition of a reciprocal experiment, given some primary one, not some relation between observations in S and S’ for the primary experiment. Conversely, if an experiment and its reciprocal give (after exchange of primed and unprimed quantities) identical results then necessarily \(v_B = v'_B\). These predictions are summarised in the ‘measurement reciprocity postulate’ (MRP) [7, 29]:

\[(MRP) \text{ Reciprocal space-time measurements of similar rulers and clocks at rest in two different inertial frames } S, S', \text{ by observers at rest in } S', S \text{ respectively, yield identical results.}\]

As shown in Refs. [7, 29] the MRP, together with the requirement that the space-time transformation equations are single-valued functions of their arguments is sufficient to derive the LT (4.1)and(4.2) without consideration of any ‘light signals’ i.e. independently of Einstein’s second postulate. This type of derivation of the LT was first performed by Ignatowsky in 1910 [31].

The crucial point for the understanding of the twin paradox is that the experiment
and its reciprocal are completely independent of each other. Thus the TD relation (4.7)
where the clock C' is seen to run slower than C is an experiment performed by an observer
at rest in S, while the TD relation (4.14) where the clock C is seen to run slower than C'
is a different experiment performed by an observer at rest in S'. Because the experiments
are different the question of any contradiction between the predictions for them cannot,
even in principle, arise.

In special relativity then, just as in quantum mechanics, the results of an experiment
depend both on the a priori initial conditions and how, and by whom, the experiment is
performed. The predictions of special relativity describe therefore not the properties
of some abstract mathematical ‘space-time’ as envisaged by Minkowski [24] but the expected
space-time measurements of some particular observer. In this respect they are very similar
to those of linear perspective for the perception of three-dimensional space, which is also
intrinsically observer-dependent and based on the mathematics of projective geometry.
In fact the TD effect corresponds to a $\Delta x' = 0$ projection of the LT [32].

Without any further discussion, as in the following section, of the differential aging
implicit in (4.7) and (4.12) on the one hand, or (4.14) and (4.15) on the other, it can
already be seen that no paradox of self-contradiction can arise by comparison of the
predictions for an experiment with those of a reciprocal one even if the same clocks with
the same initial settings are used in both experiments. The physical meaning of $t'(C')$ in
(4.13) and $T'(C')_B$ in (4.14), although both are time intervals recorded by the same clock,
are completely different. Indeed, setting $t' = t'(C') = T'(C')_B$ and $t = T(C) = T(C)_B$ as in
the generic LT equations (4.1) and (4.2) gives, instead of (4.13) and (4.14), respectively:

$$\begin{align*}
t &= \gamma_B t' \\
t' &= \gamma'_B t
\end{align*}$$

These equations require either that $t = t' = 0$ or that $\gamma_B \gamma'_B = 1$, i.e. $\gamma_B = \gamma'_B = 1,
v_B = v'_B = 0$, in contradiction to the assumed initial condition of the experiment $v_B = v'_B > 0$. In order to avoid this antimony it is essential that the base frame and travelling
frame of both the experiment and its reciprocal be properly specified for the time intervals
measured in each subject frame. However, as previously pointed out, the predictions for
either an experiment or its reciprocal are invariant with respect to exchange of source and
target frames.

Although an experiment and its reciprocal are completely independent concerning the
space-time events occurring in either one or the other of them, it is important to note that
the initial kinematical configurations\footnote{A ‘kinematical configuration’ in a given inertial frame is specified by the vectorial velocities and spatial separations of physical objects in the frame at any instant} of an experiment and its reciprocal are related by
a kinematical (momentum-energy) LT\footnote{For a massive physical object an energy-momentum LT is equivalent to a LT of the 4-vector velocity of the object. The latter is however undefined (all components are infinite) for a massless object, whereas the energy-momentum LT is well-defined for both massive and massless objects (See below).}. Denoting by $d\tau(O)$ an element of proper time of the object O considered in Eqns(3.24) and (3.25) above, these equations give, on dividing
throughout by $d\tau(O)$ and multiplying throughout by the Newtonian mass $m$ of the object,
the LT of its momentum and energy:

$$p'(O) = \gamma_B [p(O) - v_B E(O)/c^2]$$ (4.18)
\[ E'(O) = \gamma_B[E(O) - v_B p(O)] \] (4.19)

where

\[ p'(O) \equiv m \frac{dx'(O)_B}{d\tau(O)} = \gamma_{w_B} m \frac{dx'(O)_B}{d\bar{t}'(O)_B} = \gamma_{w_B} \bar{w}_B m \] (4.20)

\[ p(O) \equiv m \frac{dX(O)_B}{d\tau(O)} = \gamma_{u_B} m \frac{dX(O)_B}{dT(O)_B} = \gamma_{u_B} u_B m \] (4.21)

\[ E'(O) \equiv mc^2 \frac{d\bar{t}'(O)_B}{d\tau(O)} = \gamma_{w_B} mc^2 \] (4.22)

\[ E(O) \equiv mc^2 \frac{dT(O)_B}{d\tau(O)} = \gamma_{u_B} mc^2 \] (4.23)

and \( \gamma_{u_B} \equiv 1/\sqrt{1 - (u_B/c)^2} \). Multiplying (4.18) on both sides by \( c^2 \) and dividing by (4.19) gives:

\[
\frac{c^2 p'(O)}{E'(O)} = \frac{c^2 p(O)/E(O) - v_B}{1 - v_B p(O)/E(O)} = \bar{w}_B = \frac{u_B - v_B}{1 - \frac{v_B u_B}{c^2}}
\] (4.24)

where the relations: \( u_B = c^2 p(O)/E(O) \) following from (4.21) and (4.23) and \( \bar{w}_B = c^2 p'(O)/E'(O) \) from (4.20) and (4.22) have been used. Eqn(4.24) is the parallel velocity addition formula (3.28). If \( -u_B = v_B = v'_B \) then \( \bar{w}_B = 0 \) so that the object is at rest in S. The kinematical configurations of a primary experiment with O at rest in S and its reciprocal where O moves with speed \( v'_B = v_B \) along the negative \( x' \)-axis in \( S' \) are therefore related by the kinematical LT (4.18) and (4.19).

For a massive object, the energy-momentum transformation equations (4.18) and (4.19) are equivalent to those:

\[ U' = \gamma_B[U - \frac{v_B}{c} U_0] \] (4.25)

\[ U'_0 = \gamma_B[U_0 - \frac{v_B}{c}] \] (4.26)

of its 4-vector velocity:

\[ U \equiv (\gamma_u c; \gamma_u \vec{u}) \equiv (U_0; \vec{U}) \]

For a massless object (for example a photon) the 4-vector velocity is physically meaningless, as all its components are infinite, but the the momentum-energy transformations (4.18) and (4.19) and the base frame velocity addition formula (3.28) or (4.24) remain valid. This is because both \( E = \gamma mc^2 \) and \( p = \gamma mv \) remain finite in the limit \( m \to 0, v \to c \gamma \to \infty \) as a consequence of the relation \( E^2 = m^2 c^4 + p^2 c^2 \) [29].

5 The twin paradox

Before entering into a more detailed discussion of the differential aging effect (DAE) that is the basis for the correct understanding of the twin paradox a simple space-time
The experiment is described in which the DAE is both manifest and experimentally verified. This is not merely a thought experiment, impossible to realise in practice, but a description of the space-time physics that actually governs the use of high energy secondary beams of unstable particles. The busy reader need read only this account to be convinced of the physical reality of the DAE effect.\(^3\)

The ‘clocks’ in the experiment are \(\pi^+\) mesons which are created at the same time, in the same event by the collision of a high energy (several GeV) extracted proton beam with a proton at rest in a hydrogen target. One charged pion, \(\pi_B^+\), (analogous to a clock at rest in the base (B) frame), produced by fragmentation of the target proton, is produced with a very low momentum, loses energy rapidly by ionisation of the liquid hydrogen and comes to rest in the target within a short time \(\leq 1\text{ns}\), that may be neglected in the following considerations. The other charged pion \(\pi_T^+\), (analogous to a clock at rest in the travelling (T) frame) produced by fragmentation of the beam proton has an energy of 2.8 GeV, corresponding to a speed in the laboratory (base) frame of 0.9988c or \(\gamma = 20\). It travels almost parallel to the initial proton beam over a distance of 100m where 334ns later, in the laboratory frame, it collides elastically, head-on, with another \(\pi^+\) at rest in a stopping target (unlikely, but certainly physically possible) and itself comes to rest in the laboratory frame. According to the nomenclature and notation of Section 3, the subject, source and base frame, S, of the experiment is that of the laboratory where \(\pi_B^+\) represents a clock at rest, while the object, target and travelling frame, S’, is the rest frame of \(\pi_T^+\). According to the TD relation (4.7) the elapsed time in the rest frame of \(\pi_T^+\) is:

\[ t'(\pi_T^+) = \frac{T_B}{\gamma} = \frac{334\text{ns}}{20} = 16.7\text{ns} \]  

(5.1)

Since the mean decay lifetime of a charged pion is 26ns the probability that \(\pi_T^+\) will survive to be scattered back into the laboratory system is large: \(\exp[-16.7/26] = 0.53\). It may be said that \(\pi_T^+\) is only ‘middle aged’ at the time it scatters into the laboratory frame. However the probability that \(\pi_B^+\) —a twin of \(\pi_T^+\), born at the same instant— is still in existence \(\simeq 334\text{ns}\) after it is created is only \(\exp[-334/26] = 2.7 \times 10^{-6}\). If the 26ns mean lifetime of a charged pion is compared to a human lifetime of 70yr, then \(\pi_T^+\) is only 45 years old on arriving in the laboratory frame after its travels, whereas \(\pi_B^+\) will, at this instant, have been dead and buried for 827yr! Since high energy pion beams have been used to perform experiments hundreds of metres from their production point in particle physics laboratories all over the world there can be no doubt of the correctness of the above account, i.e. of the reality of the DAE. Indeed, the TD effect has been quantitatively confirmed by observations of the decay of beam pions \([33]\).

To give more insight into the DAE and its relation to discussions of the twin paradox that have appeared in text books and the pedagogical literature, a more elaborate thought experiment, inspired by the simple one involving a charged pion just described, is now presented. To understand the role (if any) of acceleration in the DAE and the twin paradox

\(^3\)In spite of having followed the career of an experimental particle physicist, it is only very recently that consideration of this experiment convinced the present author of the correctness of the DAE that renders younger the travelling twin. I had earlier realised that the conventional explanation of the twin paradox, to be discussed in the following section, was erroneous, and so had thought it possible, that there was no DAE, in spite of the apparent demonstration of the effect in the experiment of Hafele and Keating \([34]\). I have since found that similar experiments involving charged pions or muons were discussed long ago by Adair \([35]\) and Williams \([36]\).
it is convenient to introduce macroscopic objects that can, by analogy with the charged pion in the above experiment, undergo head-on elastic collisions and so be transfered, with negligible time delay, between two inertial frames. These objects may be called SBs for ‘Space Billiards’. Like ideal billiard balls undergoing perfectly elastic collisions, they are reduced to a state of rest when colliding frontally with an identical object initially at rest. The SBs are assumed to move freely in field-free space, without rotation, according to Newton’s First Law of motion.

It will also be found convenient, following the original suggestion of Langevin [14], to dope certain of the SBs with a radioactive substance with a suitably chosen mean decay lifetime, $\tau_D$. The age, $a_{SB}$ of such an SB is then in one-to-one correspondence with the number $N$ of undecayed radioactive nuclei remaining at any instant, according to the relation, expressing the exponential decay law:

$$a_{SB} = \tau_D \ln \left( \frac{N_0}{N} \right)$$

where $N_0$ is the initial number of radioactive nuclei. The age difference of two such identically-prepared SBs is then given by the ratio of their activities, $A$, (number of radioactive decays per unit time) as measured at the same instant by two identical detectors:

$$a_1 - a_2 = \tau_D \ln \left( \frac{A_2}{A_1} \right)$$

In the proposed experiment, identical doped SBs are prepared at rest in the subject, source, and base frame, $S$. By elastic collisions with other, identical, moving, SBs, they are then projected into the object, target and travelling frame $S'$. Finally by further elastic collisions with identical SBs at rest in $S$ they are projected back into this frame, and their ages are compared to those of a similar SB that remained at rest in $S$ during their journey. Thus, instead of the round-trip scenario of the classical twin paradox, only the outward journey is considered. Because of the symmetry of outward and return journeys, both with respect to accelerations, decelerations, and times of passage, this simplification does not affect in any way the connection between the DAE observed and the resolution of the round-trip twin paradox. Also, the analysis of the acceleration-free sections of the journey is identical to that in the conventional twin paradox experiment where the travelling twin may be accelerated and decelerated during non-negligible proper time intervals. Since the stay-at-home twin (see below) undergoes identical acceleration and deceleration as the travelling twin, but has a negligibly short interval of uniform motion, the effect of acceleration and deceleration cancels completely in the calculation of the DAE, and so cannot be responsible for it.

As shown in Fig. 5a, initially, six SBs B0,B1,B2,T0,T1 and T2, each similarly, and simultaneously, doped with a radioactive substance, and so with the same age according to Eq. (5.2), are at rest in the frame $S$. The SBs B1 and B2 are separated from T1 and T2 respectively by the same distance, $L$, and B1 and T2 have the same $x$-coordinate. The SB B0, which will constitute the clock at rest in the frame $S$, (the analogue of the stay-at-home twin) is separated from T0 by a short distance (not shown) that is much less than $L$, but sufficiently large that B0 attains a uniform speed after its collision (see below) with P0. The SBs T0, T1 and T2 are set in motion by synchronous frontal elastic collisions with P0, P1 and P2 which each move with uniform speed $v_B$ parallel to the
Figure 5: The space billiard thought experiment.  

a) Initially, the SBs $B_0, B_1, B_2, T_0, T_1$ and $T_2$ are at rest in the base frame $S$ at the positions shown.  
b) At time $T_B = 0$, $T_0, T_1$ and $T_2$ are simultaneously set in motion by head-on elastic collisions with $P_0, P_1$ and $P_2$. SBs at rest are marked by double horizontal lines and SBs in motion by arrows.  
c) At time $T_B = L/v_B$, $T_1$ and $T_2$ are brought to rest by collisions with $B_1$ and $B_2$. Event configurations in the travelling frame $S'$ corresponding to a), b) and c) in $S$ are shown in d), e) and f) respectively.
x-axis (see Fig. 5b). Whereas T1 and T2 (the analogues of the travelling twin) are set in motion directly by collisions with P1 and P2, P0 first collides with B0, that after a negligibly small displacement collides with T0, setting it in motion and itself coming to rest. Since T1 and T2 are later brought to rest in S by elastic collisions with B1 and B2 respectively, identical accelerations and decelerations are undergone by B0, T1 and T2 so completely cancelling their contributions to the DAE when comparing the age of B0 with that of T1 or T2. The collisions of T1 with B1 and T2 with B2 occur, as shown Fig. 5c, at \( T_B = \frac{L}{v_B} \) in the frame S. The \( S' \) frame configurations corresponding to Figs. 5a, 5b and 5c, calculated using Eqns. (3.3),(3.9) and (3.12) are shown in Figs. 5d, 5e and 5f, respectively. The spatial configurations in Figs. 5d, 5e and 5f are identical to those in Figs. 5a, 5b and 5c respectively. However, after the collisions with P0, P1 and P2, B0, B1 and B2 all move to the left with the speed \( v'_T = \gamma_B v_B \). It can be seen that the collisions of T1 with B1 and of T2 with B2 that project T1 and T2 back into the frame S, where their ages may be compared with that of B0, occur simultaneously in both S and S consistent with Eqs. (3.9)-(3.11) above.

If now the ages of the travelling SBs T1 and T2 are compared with that of the stay-at-home SB T0, by measuring their activities in similar detectors at some later fixed time, special relativity predicts the DAE:

\[
a(B0) - a(T1) = a(B0) - t(T2) = \frac{L}{v_B} \left( 1 - \frac{1}{\gamma_B} \right) = \tau_D \ln \left( \frac{A(T1)}{A(B0)} \right) = \tau_D \ln \left( \frac{A(T2)}{A(B0)} \right)
\]

Note that the reduced the age increments of T1 and T2 in the frame \( S' \), as compared to that of B0 in the frame S, are due to the increase of the apparent velocity of T1 relative to B1 and T2 relative to B2 in the frame \( S' \): \( v'_T = \gamma_B v_B \), as compared to \( v_B \) in the frame S, the journey being of equal length in the two frames. In the hitherto conventional interpretation, to be discussed below, the relative velocity is assumed to be \( v_B \) in both S and \( S' \), and the spatial separations in the frame \( S' \) are scaled down by a factor \( 1/\gamma_B \), so that the reduced age increments of T1 and T2 in \( S' \) are explained by ‘length contraction’ — a shorter journey than in S at the same speed. Thus the solution (a) corresponding to measurement of the length of a train, as discussed in Section 2, is adopted, rather than solution (b) corresponding to the configurations shown in Fig. 5. It will be shown that the conventional type (a) interpretation of the SB thought experiment, arises from a misinterpretation of the space-time LT that implies a breakdown of translational invariance — an unphysical dependence of predictions for times of events in the frame \( S' \) on an arbitrary choice of spatial coordinate system— as well as coordinate-system-dependent and manifestly unphysical ‘rapid’ and ‘reverse’ aging effects.

To illustrate the crucial importance of the concepts of base and travelling frames, and the independent nature of a primary space-time experiment and its reciprocal, a variation of the charged pion thought experiment mentioned above is now considered. A negative pion \( \pi^-_T \) is produced in a similar manner to the \( \pi^+_T \) of the above experiment. All other physical parameters remain unchanged, but instead of scattering on another charged pion at rest, \( \pi^-_T \) undergoes the charge-exchange reaction \( \pi^-_T p \rightarrow \pi^0 n \). Observers in all inertial frames must agree on the occurrence of this interaction. However, as will now be shown, an interaction of \( \pi^-_T \) can occur, with an appreciable probability, only in the primary experiment as specified above, not in the reciprocal one. The primary experiment with
Figure 6: A $\pi^-$ with $\gamma = 20$ in the base frame $S$ and initially at 100m from a proton [a], collides with it [c], and undergoes the charge-exchange reaction $\pi^- p \rightarrow n\pi^0$. b) and d) show the corresponding event configurations in the travelling frame, $S'$, (the rest frame of the $\pi^-$). See text for discussion.
Figure 7: The experiment reciprocal to the one shown in Fig. 6. b) A proton with $\gamma = 20$ in the base frame $S'$, and 100m from it, is directed towards a $\pi^-$ at rest in this frame. d) At time $T^\prime_B(\pi) = \tau_\pi = 26$ ns the $\pi^-$ decays. f) At $T^\prime_B(\pi) = 334$ ns the $p$ is coincident with, $\tilde{\pi}(\mu\nu)$, the centre of mass the $\pi^-$ decay products. The corresponding event configurations in the travelling frame (the rest frame of the proton) are shown in a), c) and e). See text for discussion.
the laboratory frame, S, as subject, source and base frame and the rest frame of $\pi_T$, $S'$, as object, target and travelling frame is shown in Fig. 6, with configurations in the frame S on the left and in $S'$ on the right. The time interval between the production and interaction events is $334\text{ns}$ in S and $16.7\text{ns}$ in $S'$. As the mean lifetime is the same for positive and negative pions, there is a 53% probability that $\pi_T$ remains undecayed and is able to initiate the charge exchange reaction. The reciprocal experiment is shown in Fig. 7. In this case, $S'$, the charged pion rest frame is the subject, source and base frame while the laboratory frame S (the proton rest frame) is object, target and travelling frame. In this case the proton with which $\pi_T$ can potentially interact is the travelling object. In spite of the kinematical and spatial similarity of the initial states in the primary and reciprocal experiments it is essentially impossible that the charge exchange reaction can occur in the reciprocal experiment. Typically the pion will decay after a time $\tau_\pi = 26\text{ns}$, (Fig. 7d). In the travelling frame (proton rest frame) such a decay occurs after only $1.3\text{ns}$ when the pion is $7.8\text{m}$ from its production point. Denoting by $\bar{\pi}(\mu\nu)$ the physical system comprised of the pion decay products, it is seen in Fig. 5f that the proton is in coincidence with $\bar{\pi}(\mu\nu)$ in $S'$ after a flight time of $334\text{ns}$ or $12.8\tau_\pi$. The probability that the proton arrives at the charged pion before it decays, and so can interact with it, is then only $2.5\times10^{-6}$. It can be seen from Figs. 6 and 7 that the possible outcomes of two similar experiments, with initial states differing only by a Lorentz boost, depends crucially on the choice of base and travelling frames, i.e. in which inertial frame the initial conditions of the experiment are defined. For example, the potential experience of an observer in the proton rest frame is completely different if it is the base frame as in Fig. 6, or the travelling frame as in Fig. 7.

6 The standard interpretation of the space billiard experiment: with ‘relativity of simultaneity’ and ‘length contraction’

The interpretation of the space billiard thought experiment presented in the previous section is summarised in Fig. 8 where, for simplicity, only T1,T1,B1 and B2 are shown. The encounters T1,B1 and T2,B2 are simultaneous in both S and $S'$. The DAE is a consequence of the Lorentz invariance of spatial intervals and the relativistic reciprocity relation (3.21) which states that the magnitude of the velocity of B1,B2 relative to T1,T2 in the frame $S'$ is a factor $\gamma_B$ larger than that of T1,T2 relative to B1,B2 in the frame S.

In the standard interpretation based on the generic LT (4.1) and (4.2) the space-time configurations in the frame S are identical to those shown in Fig. 8a and Fig. 8b. However, the space-time configurations in $S'$ are found to be much more complicated than those shown in Fig. 8c and Fig. 8d. The positions and times of the SBs in $S'$, for the case where the origin of S is initially at B1 and that of $S'$ initially at T1, calculated with the aid of (4.1) and (4.2), are presented in Table 1. These calculated $S'$ frame times are shown on circular analogue clocks in correspondence with the positions of the different SBs at $t = 0$, when T1 and T2 have just been set in motion in Fig. 9b. Also shown for each SB on analogue clocks with square dials are the ages of the SBs, corresponding to a zero reading of the square clocks at $t = 0$. This age is essentially the same as that at time $t = -\delta$,
Figure 8: Simplified version of Fig.3 for comparison with the calculation using the generic LT (4.1) and (4.2) shown in Figs.8-10. a) T1 and T2 are set in motion, in the base frame S, by P1 and P2 (not shown). b) T1 and T2 are brought to rest in S by collisions with B1 and B2 respectively. c) and d) show the corresponding configurations in the travelling frame, S', (rest frame of T1 and T2) See text for discussion.
Table 1: Space and time coordinates of the SBs in the frame S’ as calculated using the generic LT (4.1) and (4.2). The coordinate origins in S [S’] are aligned with B1 [T2].

| t   | T1 | T2 | B1 | B2 |
|-----|----|----|----|----|
| 0   | -γL | γβL/c | 0 | γL | -γβL/c |
| L/v | -γL | γL/v | L/γv | -γL | γL/v | 0 | L/γv |

Table 1: Space and time coordinates of the SBs in the frame S’ as calculated using the generic LT (4.1) and (4.2). The coordinate origins in S [S’] are aligned with B1 [T2].

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where δ ≪ L/v, as in Fig. 9a where all the SBs shown are still at rest in S. The DAE is given by comparing the settings of these age clocks at t = 0 in Fig. 9b with those at time t = L/v in Fig. 9d where T1 and T2 are again at rest in S. The calculated S’ frame times and the ages of the SBs corresponding to their positions at the time t = L/v − δ, just before T1 and T2 come to rest in S are shown in Fig. 9c. In drawing Fig. 9 and the subsequent figures in this section it is assumed that β = v/c = √3/2, γ = 2. The times of passage L/v = 2L/√3c of T1 and T2 in S correspond to a rotation of sixty degrees of the hands of the clocks. Inspection of Fig. 9 shows that, apart from T2, situated at the origin of S’, indicated by the large inclined arrow, and B1 at t = 0, there is no correspondence between the ages of the SBs as given by the TD effect, and the clock settings in S’ as calculated with the generic LT (4.1) and (4.2).

The pattern of collision events in S’ corresponding to the S’ frame positions and times presented in Table 1, is shown in Fig. 10. At t’ = -γβL/c, (Fig. 10a) before any collisions have taken place, and where all objects move to the left with speed v, the separations of T1 and B1 and T2 and B2 are found to be ‘length contracted’ by the factor 1/γ. At t’ = 0 (Fig. 10b) P1 (not shown) collides with T2 and brings it to rest in the frame S’. At time t = L/γv (Fig. 10c) B2 collides with T2, projecting it back into the frame S, and itself comes to rest in S’. The age-increment of T2 while it is at rest in S’ is therefore L/γv, consistent with the TD effect. The smaller time interval experienced by T2 in the frame S’ is explained here as a consequence of the ‘length contraction’ of the separation of T2 and B2 in this frame, as compared the their separation, L, in the frame S, their relative velocity being the same in the two frames. At t’ = γβL/c (Fig. 10d) P1 (not shown) collides with T1 setting it into motion in S and bringing it to rest in S’. At t’ = γL/v (Fig. 10e) B1 collides with T1, projecting it back into the frame S. The elapsed time in S’ for which T1 is at rest in this frame: γL/v − γβL/c = L/γv gives the age increment of T1 in S’, again consistent with the TD formula. It may be remarked that the collisions between B2 and T2 (at t = L/γv) and B1 and T1 (t’ = γL/v) that are simultaneous in the frame S are separated by the time interval γβ^2L/v in S’, so that in this case there is an infinite DAE when comparing the time interval between the collisions as experienced by observers at rest in S and S’. In contrast, in Fig. 8d, the collisions of T1 with B1 and T2 with B2 are simultaneous and only the TD effect is experienced by observers in S and S’.

With a different choice of spatial coordinate origins —O’ at T1 and O such that x = L for B1— the different set of transformed positions and times presented in Table 2 is obtained by the use of the generic LT (4.1) and (4.2). The times are displayed in Fig. 11, which is similar to Fig. 9. In this case (indicated by the large inclined arrows) the calculated S’ frame clock settings are equal to the age of the moving SB only for T1, which is again situated at the origin of S’. The corresponding sequence of collision
Figure 9: Analysis of the space billiard thought experiment using the generic LT (4.1) and (4.2). Event configurations in the base frame S are shown. The ages of the SBs are shown on the analogue clocks with square dials. The analogue clocks with round dials show S’ (travelling) frame times as calculated using (4.1) and (4.2). a) $t = -\delta$; all SBs are at rest and have age $a_0 - \delta$. b) $T1$ and $T2$ are set in motion by collisions with $P1$ and $P2$ (not shown). c) $t = L/v - \delta$; ages and times just before collisions with $B1$ and $B2$. d) $t = L/v$; $T1$ and $T2$ are brought to rest by collisions with $B1$ and $B2$ respectively. The coordinate origins in $S[S']$ are aligned with $B1[T2]$. $v = (\sqrt{3}/2)c$, $\gamma = 2$. See text for discussion.

| $t$     | $x'$ | $t'$ | $x'$ | $t'$ | $x'$ | $t'$ |
|--------|------|------|------|------|------|------|
| 0      | 0    | 0    | $\gamma L$ | $-\gamma \beta L/c$ | $\gamma L$ | $-\gamma \beta L/c$ |
| $L/v$  | 0    | $L/\gamma v$ | $\gamma L$ | $-(\gamma^2 - 2)L/(\gamma v)$ | 0 | $L/\gamma v$ |

Table 2: Space and time coordinates of the SBs in the frame S’ as calculated using the generic LT (4.1) and (4.2). $B1$ is at $x = L$ in S and the origin of $S'$ is aligned with $T1$. 

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Figure 10: Event configurations in the travelling frame $S'$ as calculated using (4.1) and (4.2), corresponding to the sequence of $S$ (base frame) events shown in Fig. 7. $v = (\sqrt{3}/2)c$, $\gamma = 2$. See text for discussion.
events is shown in Fig. 12. The same pattern of events occurs as in Fig. 10, but shifted backward in time by $\gamma \beta L/c$. This implies that an identical set of events in the frame $S$, when transformed into the frame $S'$ using the generic LT (4.1) and (4.2) leads to a different sequence of events (events with different times) when different spatial coordinate systems are chosen. There is therefore a manifest breakdown of translational invariance, which requires that physical predictions in isotropic and homogeneous space must be independent of the choice of spatial coordinate system. Indeed, for example, the entries in Table 1 for $T_1$ at the time $t = L/v$ are in direct contradiction with the translationally-invariant TD relation (3.8) that is valid for an arbitrary value of $L'$. In this case, where $x(T_1) = 0$, Eq. (4.2) gives:

$$t'(T_1) = \gamma [t(T_1) - \frac{vx(T_1)}{c^2}] = \frac{\gamma L}{v}$$  \hfill (6.1)

To be compared with correct, translationally invariant, formula (3.8) which gives instead for this case,

$$a(T_1) - a_0 = t'(T_1) = \frac{t(T_1)}{\gamma} = \frac{L}{\gamma v}$$  \hfill (6.2)

Clearly, the formula (6.1) is inconsistent with the TD effect of (6.2) and with the DAE that must be the same for $T_1$ and $T_2$ that undergo identical motions in the frame $S$. In contrast, applying Eq. (4.2) to $T_2$, for which $x(T_2) = L$ at $t = L/v$ gives:

$$t'(T_2) = \gamma [t(T_2) - \frac{vx(T_2)}{c^2}] = \frac{L}{\gamma v}$$  \hfill (6.3)

in agreement with the TD relation for $T_2$:

$$a(T_2) - a_0 = t'(T_2) = \frac{t(T_2)}{\gamma} = \frac{L}{\gamma v}$$  \hfill (6.4)

The conventional calculation using (4.1) and (4.2) describes correctly the age increments of $T_1$ and $T_2$ while they are at rest in $S'$, but as is evident, in general, from inspection of Figs. 9 and 11, and for the special case of $T_1$ by comparing Eqs. (6.1) and (6.2), that the generic LT does not always calculate correctly the ages of the SBs, i.e. the values of $t$ calculated may be in manifest disagreement with the ages of the SBs. The exceptions to this are are $T_2$ in Fig. 9 and $T_1$ in Fig. 11—the SB that, in each case, are at the origin of $S'$. These correspond to the previously-discussed use of a ’local’ LT [2] where the coordinate origin in $S'$ lies at the position of the transformed event.

As previously remarked, the age of $T_2$ is correctly described by (4.1) and (4.2) with the choice of spatial coordinate systems shown in Fig. 9 and 10:

$$x'(T_2) = \gamma [x(T_2) - vt] = 0$$  \hfill (6.5)

$$t' = \gamma [t - \frac{vx(T_2)}{c^2}]$$  \hfill (6.6)

where $t$ is the time of a synchronised clock at any position in $S$. These equations may be rewritten in terms of the spatial coordinates of $T_1$ by making use of the identities:

$$x'(T_2) \equiv x'(T_1) + [x'(T_2) - x'(T_1)] = x'(T_1) + L'$$  \hfill (6.7)

$$x(T_2) \equiv x(T_1) + [x(T_2) - x(T_1)] = x(T_1) + L$$  \hfill (6.8)
Figure 11: As Fig. 8 except that B1 is at \( x = L \) in S and the origin of S' is aligned with T1. \( v = (\sqrt{3}/2)c \), \( \gamma = 2 \). See text for discussion.
Figure 12: Event configurations in the travelling frame \( S' \) as calculated using (4.1) and (4.2), corresponding to the sequence of \( S \) (base frame) events shown in Fig. 10. \( v = (\sqrt{3}/2)c, \gamma = 2 \). See text for discussion.
to give
\[ x'(T1) + L = \gamma [x(T1) + L - vt] = 0 \quad (6.9) \]
\[ t' = \gamma [t - \frac{v(x(T1) + L)}{c^2}] \quad (6.10) \]
where the relation (3.18), \( L = L' \) has been used. Setting \( t = L/v, \) (6.9) and (6.10) then give, on eliminating \( x(T1) + L: \)
\[ a(T1) - a_0 = t'(T1) = \frac{t(T1)}{\gamma} = \frac{L}{\gamma v} \quad (6.11) \]
in contrast to (6.1) and in agreement with the TD relation (6.2). Eq. (6.9) also gives correctly the age of T1: \( a(T1) - a_0 = t'(T1) = 0 \) at \( t = 0 \) when \( x(T1) = -L. \) In Table 1 and Fig. 9 it is seen instead that at \( t = 0, \) \( t'(T1) = \gamma \beta L/c \neq a(T1) - a_0 = 0 \)

Similarly, with the choice of spatial coordinate systems of Figs. 11 and 12, the age of T2 is correctly given by the equations:
\[ x'(T2) - L = \gamma [x(T2) - L - vt] = 0 \quad (6.12) \]
\[ a(T2) - a_0 = t'(T2) = \gamma [t - \frac{v(x(T2) - L)}{c^2}] \quad (6.13) \]
Introducing ‘local’ coordinate systems at the position of T1 in (6.9) and (6.10) or T2 in (6.12) and (6.13):
\[ \tilde{x}(T1) \equiv x(T1) + L, \quad \tilde{x}'(T1) \equiv x'(T1) + L \quad (6.14) \]
\[ \tilde{x}(T2) \equiv x(T2) - L, \quad \tilde{x}'(T2) \equiv x'(T2) - L \quad (6.15) \]
the form of the generic LT (4.1) and 4.2) is recovered:
\[ \tilde{x}'(T) = \gamma [\tilde{x}(T) - vt] = 0 \quad (6.16) \]
\[ t' = \gamma [t - \frac{v\tilde{x}(T)}{c^2}] \quad (6.17) \]
\[ T = T1, \quad T1 \quad (6.18) \]

The simplest manner to express the LT (6.16) and (6.17), which describes a synchronised clock at an arbitrary position in S' is:
\[ \tilde{x}'(T) = 0 \quad (6.19) \]
\[ \tilde{x}(T) = vt \quad (6.20) \]
\[ t = \gamma t' \quad (6.21) \]
Eqs.(6.19) and (6.20) are equivalent to the space transformation equation (6.16). They are the same as in Galilean relativity. The translationally invariant TD relation (6.21) obtained by eliminating \( \tilde{x}(T) \) between (6.16) and (6.17) may be contrasted with the corresponding equation: \( t = t' \) of Galilean relativity, that is the limit of Eq.(6.21) as \( c \to \infty. \) In fact, the full physical content of the the space-time LT concerning the transformation
of events on the world line of an object at rest in the frame $S'$, is contained in Eqs.(6.19)-(6.21). Galilean relativity is recovered on replacing the TD relation (6.21) by the absolute time: $t_N = t = t'$ of Newtonian mechanics.

It is important to stress the unphysical nature of the $S'$ frame times at $t = l/v - \delta$ of $T_1$, $B_1$ and $B_2$ in Fig. 9 and of $T_2$, $B_1$ and $B_2$ in Fig. 11 as calculated using the generic LT. These clearly do not correspond to the actual ages of the SBs, as given by the translationally-invariant TD relation, and shown on the clocks with square dials.

Consider, as another example of such a mismatch, $T_1$ at $t = 0$ in Fig. 9 where the generic prediction is $t'(T_1) = \gamma \beta L/c$. If this were really the age of the SB, then during the negligibly short collision time of $P_1$ with $T_1$ a fraction $1 - \exp[\gamma \beta l/(c \tau_D)]$ of the radioactive atoms doping the SB would have to decay. In the case of $T_2$ at $t = 0$ in Fig. 11 where $t'(T_2) = -\gamma \beta L/c$ a fraction $1 + \exp[\gamma \beta l/(c \tau_D)]$ of the atoms would have to ‘undecay’ or equivalently this fraction of undecayed atoms would have to be created during the collision of $P_2$ with $T_2$! Although this unphysical, indeed nonsensical, character of the predictions of the generic LT is particularly clear for the case of radioactive clocks, similar considerations apply equally to any type of clock. For example in the case of a spring-driven mechanical clock, the prediction of $t'(T_2)$ at $t = 0$ in Fig. 11, would require the spring to spontaneously rewind during the collision of $P_2$ with $T_2$! During more than a century such absurd predictions of generally accepted special relativity theory have appeared, without any adverse comment, in textbooks and the pedagogical literature.

7 The twin paradox and the Minkowski space-time plot

In this section the full outward and return journey of a travelling twin, $T$, is considered. As before, it is assumed that all time intervals during acceleration may be neglected as compared to those corresponding to periods of uniform motion. Initially, $T$ and the stay-at-home twin $R$ are at rest at the origin of the base frame $S$. $T$ is projected into the frame $S'$ moving with uniform speed $v_B$ relative to $S$. After travelling a distance $L$, $T$ is projected into the frame $S''$ moving with uniform speed $v_B$ in the direction opposite to that of $S'$. On arriving back at the origin of $S$, $T$ is projected into this frame. During $T$’s journey $R$ remains at rest at the origin of $S$.

As discussed in Ref. [23], the space-time LT is geometrically equivalent to the product of an orthogonal projection and a scale transformation. The corresponding equations representing $T$’s journey are as follows:

**Outward journey**

\[
\Delta \bar{X}'_T = f(\beta_B)(\Delta X_B \cos \theta - c \Delta T_B \sin \theta) \quad (7.1)
\]

\[
c\Delta \bar{T}'_T = f(\beta_B)(c \Delta T_B \cos \theta - \Delta X_B \sin \theta) \quad (7.2)
\]

---

4See, for example, Refs[21, 22].
Return journey

\[\Delta \bar{X}''_T = f(\beta_B)(\Delta X_B \cos \theta + c \Delta T_B \sin \theta) \quad (7.3)\]
\[c \Delta \bar{T}''_T = f(\beta_B)(c \Delta T_B \cos \theta + \Delta X_B \sin \theta) \quad (7.4)\]

where \(\beta_B \equiv v_B/c\) and

\[\cos \theta \equiv \frac{1}{\sqrt{1 + \beta^2_B}}, \quad \sin \theta \equiv \frac{\beta_B}{\sqrt{1 + \beta^2_B}}, \quad f(\beta_B) \equiv \frac{1 + \beta^2_B}{1 - \beta^2_B} \quad (7.5)\]

The world lines of R and T are shown with orthogonal \(X_B\) and \(cT_B\) axes in Fig. 13 with \(\beta_B = 1/3\). The two-dimensional rotations in the above equations correspond to orthogonal projections from a point on the world line of T on to the axes of the travelling frame coordinates. Note that for the transformations (7.1) and (7.2), the \(c \bar{T}''_T\) axis is obtained from the \(T_B\) axis by an anti-clockwise rotation through the angle \(\theta\), and the \(\bar{X}''_T\) axis from the \(X_B\) one by clockwise rotation through the angle \(\theta\). As pointed out in Ref. [23], in Minkowski's original paper a sign error was made in plotting the directions of the travelling frame axes so that, for example, the \(T'_T\) axis is superimposed on the corresponding world line. This error has persisted since in text books and the pedagogical literature, including, as will be seen below, many discussions of the twin paradox. As shown in Fig. 13, the combination of orthogonal projection and scale transformation is equivalent to projecting at the angle \(\phi_t\) onto the \(c \bar{T}''_T\) axis where [23]:

\[\tan \phi_t = \frac{2\beta_B}{\sqrt{1 - \beta^2_B(\sqrt{1 + \beta^2_B} - \sqrt{1 - \beta^2_B})}} \quad (7.6)\]

The world lines of T and R in the rest frame of T (\(S'\) on the outward journey, \(S''\) on the return one) are shown on orthogonal \(c \bar{T}''_T\) versus \(\bar{X}''_T\) and \(c \bar{T}''_T\) versus \(\bar{X}''_T\) plots in Fig. 14, also for \(\beta_B = 1/3\). The world lines of T are inclined at an angle \(\theta' = \arctan \gamma_B \beta_B\) to the \(c \bar{T}'_T\) or \(c \bar{T}''_T\) axes.

The DAE is read off from the different time intervals shown in Fig. 13. Denoting the age increments of T and R by \(\Delta a_T\) and \(\Delta a_R\) respectively then

\[\Delta a_T = \Delta \bar{T}'_T + \Delta \bar{T}''_T = 2\Delta \bar{T}'_T = f(\beta_B)\frac{\cos 2\theta}{\cos \theta} \Delta T_B\]
\[= \frac{1 + \beta^2_B}{\sqrt{1 - \beta^2_B}} \frac{1 - \beta^2_B}{\sqrt{1 + \beta^2_B}} \Delta T_B = \frac{\Delta T_B}{\gamma_B} = \frac{\Delta a_R}{\gamma_B} \quad (7.7)\]

The space-time description of Figs. 13 and 14, in the base and travelling frames respectively, is now compared with the standard ‘resolution’ of the twin paradox as presented in text books and the pedagogical literature. This can be found for example in Taylor and Wheeler’s book ‘Spacetime Physics’ [37] or in Chapter 3 of Marder’s book on the twin paradox [38]. As before, time intervals in which the travelling twin is accelerated or decelerated are neglected as compared to time intervals corresponding to uniform motion. It will now also be assumed that T remains at rest for a short period after completing the outward journey and before starting the return one. The space-time plot on which the
Figure 13: World lines of $R$ (WL(R)) and $T$ (WL(T)) in the frame $S$. Coordinate axes of the frames $S'$ and $S''$ as predicted by the LT (7.1)-(7.4) are also show. See text for discussion.
Figure 14: World lines of $R$ (WL(R)) and $T$ (WL(T)) in the frames $S'$ and $S''$ as predicted by the LT (7.1)-(7.4). See text for discussion.
discussion is based is that shown in Fig.15 (c.f. Fig.71 of Ref [37] or Fig. 17 of Ref [38]). Usually the vertical axis is labelled as $ct$ where $t$ is the age increment of the stay-at-home twin $R$, however detailed geometrical analysis of the plot (not performed in Refs [37, 38]) shows that this is not possible, so that the symbol $\tilde{t}$ is used in Fig. 15 to label the vertical axis. The relation connecting $t$ to $\tilde{t}$ is found below. Notice that the error in the original Minkowski paper [24] of drawing the $ct'$ axis (where $t_T = t'$ is $T$'s age increment during the outward journey) along the world line of $T$, and the $ct_T = ct''$ axis along the world line of $T$, during the return journey, is made. Compare with Fig. 13 where the correct directions of these axes as predicted by the LT (7.1) and (7.2) or (7.3) and (7.4) are shown. The $x'$ and $x''$ axes are also incorrectly drawn (they are reflected in the $x$ axis as compared to the correct directions shown in Fig. 13). In order to ‘resolve’ the paradox it is now assumed the the age increment of $R$, as observed from $T$, is subject to the TD effect for a moving clock in $R$’s frame as compared to a clock at rest in $T$’s frame. If $\Delta t_T$ is the total age increment for $T$ while in uniform motion relative to $R$, then the segment OU in Fig. 15 represents $c\Delta t_T/2$. The next assumption is that the line $PU$ parallel to the $x'$ axis is a ‘line of simultaneity’ in the frame $S'$ and that $OP$ represents an age increment in the frame $S$ given by the TD effect as $\Delta t_1 = \Delta t_T/(2\gamma)$. Since the geometry of Fig. 15 gives:

$$\frac{OU}{OP} = \frac{\sqrt{1 + \beta^2}}{1 - \beta^2}$$

(7.8)

it follows that the $\tilde{t}$ segment $OP$ must be multiplied by the factor $F \equiv \sqrt{1 + \beta^2}/\sqrt{1 - \beta^2}$ in order the represent correctly the age increment $\Delta t_1$. This means that in Fig. 15:

$$t = \sqrt{\frac{1 + \beta^2}{1 - \beta^2}} \tilde{t} \equiv F\tilde{t}$$

(7.9)

On the return journey the ‘line of simultaneity’ is SV so that $ST$ also represents the age increment $\Delta t_T/(2\gamma)$. According to the geometry of Fig. 15, the total age increment for $R$, $\Delta t$, as seen by $T$ during her two intervals of uniform motion, is then given by:

$$\Delta t = F\Delta \tilde{t} = F(OQ + RT)$$

$$= 2F(OU \cos \theta) = 2\sqrt{\frac{1 + \beta^2}{1 - \beta^2}} \frac{c\Delta t_T}{2} \frac{1}{\sqrt{1 + \beta^2}}$$

$$= \gamma \Delta t_T$$

(7.10)

which corresponds to the TD effect for $T$ in motion as observed from $R$ at rest. Since this agrees with $R$’s calculation of the DAE it is presented as the ‘solution’ of the paradox. However, this implies that the age of $R$ (as observed by $T$) increases by the time interval $\Delta t_2$ where:

$$c\Delta t_2 = Fc\Delta \tilde{t} = F(PQ) = \sqrt{\frac{1 + \beta^2}{1 - \beta^2}} \frac{c\Delta t}{2} \sin \theta \tan \theta = \frac{\gamma \beta^2 c\Delta t_T}{2}$$

(7.11)

during $T$’s negligibly short period of deceleration before coming to rest in $S$ at the end of the outward journey. Similarly, the age of $R$ (as observed by $T$) increases by the same amount, corresponding to the $ct$ interval $RS$ in Fig. 15 during the negligibly short period during which $T$ is accelerated from the point $V$ at the beginning of her return journey.
A similar interpretation was obtained by Muller [39] by direct use of the generic inverse LT (4.3) and (4.4). Substituting $x' = -vt'$ (corresponding to $x = 0$) in (4.4) gives:

$$t_O = \gamma(t_T + \frac{vx'}{c^2}) = \frac{t_T}{\gamma}$$

(7.12)

where $t_T$ is the traveller's proper time (here $t_T = t'$) and $t_O$ is the age increment of R at the end of the outward journey. Neglecting the time interval corresponding to UV in Fig. 15, Muller assumes that the relation between $t_T$ and $t$ at the beginning of the return journey is given by the first member of (7.12) with the replacement $v \rightarrow -v$:

$$t_R = \gamma(t_T - \frac{vx'}{c^2}) = \frac{t_T}{\gamma}$$

(7.13)

where now $t_T = t''$ and $t_R$ is interpreted as the age of R (as observed by T) at the beginning of the return journey. Combining (7.12) and (7.13) gives:

$$t_R - t_O = 2\Delta t_2 = 2\gamma\beta^2 t_T = \gamma\beta^2 \Delta t_T$$

(7.14)

in agreement with (7.11) above.

If $L$ and $L'$ are the separations of R and T, in the frames S and S' respectively, at the end of the outward journey just before T’s deceleration phase, use of the relations:

$$-x' = L' = \frac{v \Delta t_T}{2}, \quad L = \frac{v \Delta t}{2}$$

to eliminate $\Delta t_T$ and $\Delta t$ from the TD relation (7.10) gives the ‘length contraction’ relation:

$$L' = \frac{L}{\gamma}$$

(7.15)

As discussed in the previous section, the DAE of R and T is then explained as the result of the smaller distance covered by T on the assumption (contrary to the reciprocity relation (3.21)) that the magnitude of the velocity of R relative to T in the frame S’ is the same as the initially fixed velocity, $v$, of T relative to R in the frame S.

The ‘standard solution’ of the twin paradox is summarised in Fig. 15, previously discussed and in Figs. 16, 17 and 18. In Fig. 16 are shown the world lines of R as viewed from the proper frame of T, which is successively S,S’ S, S” and S. The remarkable feature of this plot is the change of the separation of T and R from $L/\gamma$ to $L$ during the deceleration of T at the end of the outward journey and from $L$ to $L/\gamma$ during the acceleration phase at the beginning of the return journey. For short acceleration and deceleration phases the displacement of R relative to T occurs with a superluminal velocity, that tends to infinity as the ratio of the periods of acceleration or deceleration to the periods of uniform motion tends to zero.

In Fig. 17, taken from Ref [39] is shown a plot of $ct$ versus $ct_T$ where $t(R)$ (solid line) is R’s age as observed by R and $t(T)$ (dashed line) is R’s age as observed by T, according to the calculation in Eqns (7.12)-(7.14). During the infinitesimally short deceleration period of T at the end of the outward journey R ages (according to T) by an amount $\gamma^2 \beta L/c$. 

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Figure 15: Space-time plot used in the conventional discussion of the twin paradox based on the generic LT (4.1)-(4.2). Note that the $t'$ and $t''$ axes are incorrectly drawn along the world lines. (Compare with Fig. 14). See text for discussion.
Figure 16: Space time plot or $R$ in the traveller’s rest frame in the conventional discussion of the twin paradox as in Ref. [40]. See text for discussion.
Figure 17: Plots of times in the frame $S$ as observed by $R$: $t(R)$, and by $T$: $t(T)$, as a function of $t_T$ according to the calculation of Ref. [39]. See text for discussion.
Figure 18: Plot of the separation of R from T in the frame S, as observed by T: $x(T)$, according to the calculation of Ref. [39]. See text for discussion.
R also ages by the same amount (according to T) during the acceleration period at the beginning of the return journey.

In Fig. 18 is shown the x-coordinate of T, $x(T)$, as a function of $t_T$ which corresponds to the dot-dashed line of Fig. 15. Near the end of the outward journey, while T is still in motion, the separation of R and T is $L/\gamma^2$. During the short deceleration period the separation increases to $L$. During acceleration at the beginning of the return journey the separation shrinks again to $L/\gamma^2$. The angle $\theta''$ in Fig. 16 is given by $\tan \theta'' = \beta/\gamma$ corresponding to a slowed-down relative velocity of R and T, as seen by T, of $v(T) = x(T)/t_T = c\tan \theta'' = v/\gamma$.

The scenario shown in Figs. 15 and 16 has been graphically described by McRea [40] (the travelling twin is here denoted by M):

'At the event of M completing the part of his outward journey, performed with speed $V$, observer R plots M as at a distance $X (= VT)$ from himself. (We say “plot” instead of “see” to avoid complications about times of travel of light in the process of seeing) Owing to the Lorentz contraction, M at the same event plots R as at distance $\alpha X$ from himself where $\alpha = \sqrt{1 - (V/c)^2}$. During the interval, $t$, when R,M are both at rest in F, they must plot each other as at distance $X$. But as soon as M has again acquired uniform speed $V$, this time towards R, he again plots R as at distance $\alpha X$.

R’s account of M’s journey relative to himself is, still neglecting quantities small compared with $X$ and $T$: M moves out and back through distance $X$ at speed $V$; total time $2T$. On the other hand, M’s account of R’s journey relative to himself is: R moves out the distance $\alpha X$ at speed $V$ then R “cheats” by moving out to the distance $X$ and back again to the distance $\alpha X$ in negligible time and finally completes his return at speed $V$; total time $2\alpha T$. This is less than $2T$ and is the usual result.'

This standard solution of the paradox will now be examined for mathematical and logical consistency, before analysing it according to the nomenclature and notation for space and time coordinates introduced in Section 3 above.

Qualitative arguments based on the incorrectly drawn Minkowski plot of Fig. 15 are to be found in many places in the the literature [37, 38, 41], but nowhere, to the present writer’s best knowledge, is the geometrical problem worked out in full detail. If it had been, the geometrical inconsistency of assuming that $OP/OU = 1/\gamma$, according to TD, when the vertical axis represents $ct$ would have been noticed. If the vertical axis is used to represent “time in S as observed from S’” as in the derivation of Eqn.(7.10), it cannot also represent “time in S as observed from S” as is usually assumed in drawing the Minkowski plot. Indeed identification of the line segment OU with $c\Delta t_T/2$ instead of $\sqrt{(c\Delta t_T/2)^2 + L^2}$ means the interpretation of the plot must be changed. In fact, the angle $\theta$ is now completely arbitrary. If it is assumed that $OP/OU = 1/\gamma$ to correctly describe the TD effect, any change in $\theta$ will be automatically compensated by the scale factor $F$ that

\[\text{For example, In Ref. [37] the solution is presented according to Eqns. (7.12)-(7.14) above as in Ref. [39] without any reference to the detailed geometry of the Minkowski plot.}\]
must be applied to the time coordinate, since the geometry of Fig. 15 does not correctly predict the TD effect if \( \tan \theta = \beta \).

As in the analogous case discussed in the previous section, the sudden ‘age increases’ of R as seen by T, during deceleration and acceleration, and associated with PQ and RS in Fig. 15 are spurious, resulting from incorrect use of the space-time LT. If there is a local clock at rest in the frame S, synchronised with R’s clock, at the position U in Fig. 15, at the end of the outward journey, this clock will be in advance of T’s clock by an amount \( L(1 - 1/\gamma)/v \), and both T and local observer in S at U will agree that this is the case. If the periods of deceleration at the end of the outward journey and acceleration at the beginning of the return journey are very short, T’s clock will still be retarded by essentially the same amount compared to the local clock in S at the start of the phase of uniform motion at the beginning of the return journey when T is in the frame S”.

Since the times recorded by the local clock at U and R’s clock are identical there is, contrary to the standard solution, no change of R’s age ‘as seen by T’ during the deceleration and acceleration phase. In order to correctly describe the TD effect during the return journey, additive constants must be included in the generic LT (4.1) and (4.2) in order to correctly represent the initial conditions of the return journey as just described. Thus the generic LT must be modified by including time offsets \( t_0 \) and \( t''_0 \) in order to correctly describe the transformation between the frames S and S”:

\[
x'' = \gamma [x + v(t - t_0)] \quad (7.16)
\]

\[
t'' - t''_0 = \gamma [t - t_0 + \frac{vx}{c^2}] \quad (7.17)
\]

The time offsets must be chosen so that \( t'' = L/(\gamma v) \) and \( t = L/v \) when \( x'' = 0 \) and \( x = L \). This requires that:

\[
t_0 = \frac{2L}{v} \quad (7.18)
\]

\[
t''_0 = \frac{2L}{\gamma v} \quad (7.19)
\]

so that (7.16) and (7.17) may be written as:

\[
x'' = \gamma [x - 2L + vt] \quad (7.20)
\]

\[
t'' = \gamma [t + \frac{v(x - 2L)}{c^2}] \quad (7.21)
\]

Eliminating \( x - 2L \) between (7.20) and (7.21) gives:

\[
t_T = t'' = \frac{t}{\gamma} \quad (7.22)
\]

which describes the TD effect during the return journey. Since the local clock at U is synchronised with the local clock at R and therefore measures R’s age, there are only two times in the problem: \( t \), which is the proper time of R, equal to his age increment during T’s journey, and \( t_T \) the proper time of T, equal to his age increment during his journey, which is equal to \( t' \) on the outward and equal to \( t'' \) on the return journey. As

\[\text{As this is no longer the generic LT of conventional text-book special relativity the coordinates no longer use a roman font.}\]
is also clear from inspection of the correct space time plots in Figs. 13 and 14, there are no sudden changes of ages and spatial separations as in the standard solution shown in Figs. 15-18. In particular there is no distinction between ‘the age of R’ and ‘the age of R as observed by T’. The latter quantity is a spurious mathematical artifact, which, as will now be seen, results from incorrectly identifying space-time coordinates in the physically distinct reciprocal experiment with transformed coordinates of the primary experiment.

The standard solution is now considered in the light of the classification of space-time experiments introduced in Section 3 above. In the twin paradox problem, S, the rest frame of R, is the base and source frame and S’ or S” is the travelling and target frame. S is the subject frame considering R as an observer and S’ or S” are subject frames considering T as an observer. The initial conditions of the problem (L and v) are therefore specified in the frame S. A single LT (or its inverse) describes the problem on the outward journey, another one on the return journey. On the outward journey it is the LT for which the spatial coordinates of T are \(x' = 0\) in S’ and \(x = vt\) in S, and \(t_T = t'\), whereas on the return journey it is one for which the spatial coordinates of T are \(x'' = 0\) in S” and \(x = 2L - vt\) in S, and \(t_T = t''\). The quantity \(t(T)\), ‘the time in S as observed from T’ corresponding to the dashed line in Fig. 16, does not occur in the description just given. How then, is it introduced in the standard solution? It is done by introducing the LT with \(x = 0\) and \(x' = -vt'\) into the problem. In the language introduced in Section 4, this is the LT appropriate to the experiment which is reciprocal to the one specified above. In the reciprocal experiment S’ is the base and source frame in which the initial conditions are specified, and S is the travelling and target frame. The LT is therefore one which applies to an experiment which is physically quite distinct, having nothing to do, as is assumed in the standard solution, with observation of R by T in the primary experiment. Indeed, if S is the base frame and S’ the travelling frame the equation of motion of R in S’ is given by Eqn(3.21) as \(x' = -\bar{v}t' = -\gamma vt'\) not \(x' = -vt'\) as in Eqn(7.12) above.

Considering the outward journey, the following space-time LTs (and their inverses) describe the primary experiment, in which S is the base and source frame and S’, the travelling frame and object frame, and the reciprocal experiment in which S’ is the base and source frame and S is the travelling and object frame\(^7\):

| Primary experiment |
| --- |
| **Transformation** |
| \(x'(T) = \gamma [x(T) - vt(R)] = 0 \rightarrow x(T) = vt(R)\) (7.23) |
| \(t'(T) = \gamma [t(R) - \frac{vx(T)}{c^2}] \rightarrow t'(T) = \frac{t(R)}{\gamma}\) (7.24) |

| Inverse Transformation |
| --- |
| \(x(T) = \gamma [\bar{x}'(T) + \bar{v}T'(T)] = \gamma \bar{v}T'(T) = vt(R)\) (7.25) |
| \(t(R) = \gamma [\bar{t}'(T) + \frac{v\bar{x}'(T)}{c^2}] = \gamma \bar{t}'(T)\) (7.26) |

\(^7\)Uncapitalised symbols are used throughout for simplicity since the equations are invariant under replacements such as \(x \rightarrow X, x' \rightarrow X'\) etc corresponding to the introduction of observers in the frames S, S’ respectively.
Reciprocal experiment

Transformation

\[
    \begin{align*}
    \bar{x}(R) &= \gamma [x'(R) + vt'(T)] = 0 \rightarrow x'(R) = -vt'(T) \quad (7.27) \\
    \bar{t}(R) &= \gamma [t'(T) + \frac{vx'(R)}{c^2}] \rightarrow \bar{t}(R) = \frac{t'(T)}{\gamma} \quad (7.28)
    \end{align*}
\]

Inverse Transformation

\[
    \begin{align*}
    x'(R) &= \gamma [\bar{x}(R) - v\bar{t}(R)] = -\gamma v\bar{t}(R) = -vt'(T) \quad (7.29) \\
    t'(T) &= \gamma [\bar{t}(R) - \frac{v\bar{x}(R)}{c^2}] = \gamma \bar{t}(R) \quad (7.30)
    \end{align*}
\]

The transformation equations for the reciprocal experiment are obtained from those of primary one by the exchanging primed and unprimed quantities and the labels R and T, and by making the replacement \( v \rightarrow -v \). It is important to note that identical predictions are obtained by use of a transformation and its inverse, and, as previously pointed out in Section 4, these predictions do not depend on the choice of source and target frames (i.e. they are invariant with respect to exchange of barred and unbarred quantities). Notice that the spatial coordinates of the base frame object, R in S in the primary experiment, and T in S’ in the reciprocal experiment, do not occur in the corresponding transformation equations.

The error made in the standard solution is now quite clear. The LT (7.27) and (7.28) of the reciprocal experiment is mistakenly identified with the inverse transformation (which is actually (7.25) and (7.26)) of the primary experiment. Notice that disregarding completely the labels and subscripts in the coordinates, as in the generic inverse LT (4.3) And (4.4), these transformation equations become identical. The quantities \( t(T) \) and \( x(T) \) plotted in Figs. 17 and 18 respectively are therefore mathematically meaningless and have no physical significance. The spurious LC effect of Fig. 16 results, as explained above, from incorrect use of the generic LT (4.1) and (4.2), that correctly describes a synchronised clock in S’ only at \( x' = 0 \), to describe a synchronised clock for which \( x' \neq 0 \). The correct world lines of R and T and S and S’ or S” are shown in Figs. 13 and 14 respectively. No age changes or sudden spatial displacements occur during the short periods of acceleration or deceleration.

8 Summary and closing remarks

The twin paradox of special relativity is defined in the Introduction. In the ‘prologue’ of Section 2 some different methods to measure the length of a train are considered: either statically or by using in various ways the formula \( \Delta x = v \Delta t \) relating constant velocity to space and time intervals. Of particular importance for the correct resolution of the twin
paradox is the case where observers in two different inertial frames, with clocks running at different rates, compare their estimations of the length of the train.

In Section 3 a general nomenclature and notation for an unambiguous description of different space-time experiments is introduced. The most important novel concepts are those of a ‘base frame’ and a ‘travelling frame’. The former is an inertial frame (say S) relative to which another inertial frame (say S’) is measured (or defined) to move with speed $v_B$. In this case S’ is the ‘travelling frame’. Important related concepts are those of a ‘primary experiment’, another experiment with a ‘reciprocal configuration’, and a ‘reciprocal experiment’. In the above example the primary experiment is that in which S is the base frame and S’ the travelling frame. Assuming that S’ moves along the $x, x’$-axis in S, then in an experiment with a reciprocal configuration, S’ is the base frame and S is a travelling frame moving with speed $v_B’$ along the negative $x, x’$-axis in S’. If $v_B = v_B’$ the latter experiment is said to be the reciprocal of the former and vice versa. Crucial for the correct interpretation of the twin paradox is that the primary experiment and its reciprocal are physically independent—in particular the base frame space-time coordinates in the primary experiment and its reciprocal are not connected by the space-time LT. It is the false assumption that this is the case which underlies the spurious ‘standard resolution’ of the paradox discussed in Section 6. However, base frame velocity, or energy-momentum configurations in an experiment and its reciprocal are connected by the kinematical (energy-momentum) LT or the base frame velocity addition formula (3.28).

Other less important attributes of inertial frames introduced in Section 3 are ‘subject’ and ‘object’ frames specifying in which frames observations or measurements are made or predicted and ‘source’ or ‘target’ frames i.e. whether a LT or an inverse LT is used to relate space and time coordinates in different inertial frames. All physical predictions are independent of whether a frame is classified as ‘subject’ or ‘object’ or by exchange of ‘source’ and ‘target’ frames. The use of this nomenclature is illustrated in Section 3 by working out several examples—Time dilatation, Lorentz invariance of length intervals, velocity addition formulae, transverse and longitudinal photon clocks, Einstein’s 1905 discussion [1] of ‘relativity of simultaneity’ and Einstein’s train-embankment thought experiment [25].

In Section 4 the important distinction between inverse transformations and reciprocal experiments is discussed in detail. Also discussed in Section 4 is the related ‘Measurement Reciprocity Postulate’ that provides a simple axiomatic derivation of the LT [7, 29] without any consideration of light signals or electromagnetism.

The twin paradox is discussed at length in Section 5 by analysing three different, but related, thought experiments in each of which the periods of acceleration or deceleration are negligibly short in comparison with periods of uniform motion. In these analyses the nomenclature and notation introduced in Section 3 is systematically applied. In the first experiment, the stay-at-home twin is identified with a charged pion, created in a high energy interaction, that is slow and comes rapidly to rest in the production target. The travelling twin is identified with another charged pion, created in the same event, but with high energy, that travels a distance of 100m before colliding head-on elastically with another charged pion at rest, which projects it into the same rest-frame as its stay-
at-home sibling. In this example the decay lifetimes of the pions measure their age and make manifest the differential aging effect—the age of the travelling pion when it is projected into the lab frame is much less than that of its stay-at-home sibling. The second experiment is similar to the first except that the pions are replaced by macroscopic objects: ‘Space Billiards’ doped with a radioactive substance serving as an internal clock to measure their age. They are projected into and out of the travelling frame S’ by head-on elastic collisions with identical space billiards. The analysis of the experiment shows that the physical phenomenon underlying the experiment is the relative velocity transformation formula derived in Section 3—if the speed of the space billiard is \( v_B \) in the base frame S, it is \( \gamma_Bv_B \) in the travelling frame S’. The third experiment is a variation of the first in which a travelling \( \pi^- \) induces the process: \( \pi^-p \rightarrow \pi^0n \). The comparison of this primary experiment with its reciprocal, in which the \( \pi^- \) remains at rest and the proton is in motion, makes clear the physical independence of the two experiments.

In Section 6 the ‘Space Billiard’ experiment of Section 5 is analysed according to the standard solution of the twin paradox where the DAE is found to be a consequence of a spurious ‘length contraction’ effect resulting from misinterpretation of the LT as explained in Refs [2, 3, 4, 5] and recalled in Section 7 of the present paper. Absurd consequences of the standard solution—times of events in the travelling frame changing when a different arbitrary choice of a spatial coordinate system is made in the base frame, and abrupt (positive or negative) age changes of an object during rapid acceleration or deceleration—are pointed out.

In Section 7 the DAE in the twin paradox experiment is discussed in relation the Minkowski space-time plot. In this connection Minkowski’s original sign error when drawing the direction of the axes of space and time coordinates in the travelling frame S’, pointed out in a recent paper by the present author [23], is important. This error remained uncorrected in all discussions of the twin paradox employing the Minkowski plot. Apart from this geometrical error, the fallacy in the standard Minkowski plot analysis of the twin paradox is the incorrect assumption that the TD effect of the physically independent reciprocal experiment is applicable in the primary experiment when the travelling twin observes the clock of her stay-at-home brother. Actually, as is obvious from inspection of the inverse LT for the primary experiment, the clock of the stay-at-home twin is observed to run faster (not slower, as in the reciprocal experiment) by the travelling twin. That this is the case is made perfectly clear by inspection of Eqns.(7.23)-(7.30).

The twin paradox arises because of an apparent symmetry between configurations where the twin R is at rest and the twin T is in motion or the twin T is at rest and the twin R is in motion. If only the relative motion of T and R is of physical significance the existence of such a symmetry appears to be well-founded. In the standard solution, the symmetry is broken by assuming the same relative velocity in the base and travelling frames but introducing ‘length contraction’ of the spatial separation of the two twins (only while they are in relative motion) in the frame of the travelling twin. In the correct solution the spatial separation between the twins is the same at all corresponding instants in the base and travelling frames, but their relative velocity is different in the two frames. The physical basis of the symmetry breaking is the TD effect itself—a moving clock is seen to run slow in comparison with an identical clock at rest in the observer’s frame. The erroneous nature of the standard solution, in which R’s clock is calculated to run
slow when observed by T, results from the false identification of a configuration from
the physically independent reciprocal experiment with an inverse transformation of the
primary experiment.

That the primary and reciprocal experiments are physically independent has been
previously pointed out in the literature. Builder [42] made the essential point that the LT
in the primary experiment where \( x' = 0 \) and \( x = vt \) is not the same (i.e. the space and
time coordinate symbols have a different physical meaning) as the LT of the reciprocal
experiment where \( x = 0 \) and \( x' = -vt' \). Builder also correctly stated that identical results
are obtained by using (in either the primary or reciprocal experiment) the LT or its inverse
—that is, in the primary experiment R’s clock will be observed by T to run faster than
her own, not slower, as in the standard solution. In spite of this, Builder subsequently
erroneously confused the base frame configuration of the reciprocal experiment with the
travelling frame configuration of the primary experiment as described abov [42, 43].

Another author who pointed out clearly the independence of the primary and recip-
rocal experiments was Max Born. Without considering the operational meaning of the
coordinate symbols in the LT, Dingle [44] had noticed the antinomy similar to that of
Eqns(4.16) and (4.17) above and concluded that the whole of special relativity theory
is excluded by reductio ad absurdum. In his response [45] to Dingle’s direct request for
comments, Born made the same correct observation as Builder—in one case \( x' = 0 \) and
\( x = vt \), in the other \( x = 0 \) and \( x' = -vt' \) so the symbols in the LT refer in the two cases
to physically distinct experiments.

To the present writer’s best knowledge, only one author [46] stated clearly in the
published literature that the standard application of the generic LT (as the entries of
Tables 1 and 2 and shown on the clocks with round dials in Figs. 9 and 11 above) does
not correctly give the age of the travelling twin, as calculated by the TD relation, and
shown on the clocks with square dials in Figs. 9 and 11. A clear distinction was drawn
between the age of the travelling twins and the times indicated by ‘synchronised’ clocks
but it was not pointed out that the latter are devoid of any physical meaning—indeed as
shown in Section 6 above correspond to patently absurd predictions. Qualitatively similar
remarks to those in Ref. [46] are found at [47].

Many authors state, incorrectly, that the source if the asymmetry between the stay-
at-home and travelling twins is that the travelling twin is accelerated and decelerated,
and therefore changes inertial frames, whereas the other stays always in the same iner-
tial frame. In the space-billiard experiment described in Section 5 above, the objects
equivalent to R and T undergo identical acceleration and decleration programs, but the
travelling object still ages less than the stay-at-home one. Langevin [14] stated, correctly,
that an inertial path linking two time-like separated space time points corresponds to a
maximum of the proper time integral between the two points, but incorrectly that this
fact explains the DAE in the twin paradox experiment. Indeed in the ‘three brothers’
version of the twin paradox due to Lord Halsbury [48], described by Marder [49], where
the clock setting of the brother moving away from the Earth is transfered at the distant
star to his brother moving towards the Earth at the same speed, the same global DAE
effect occurs and no acceleration of deceleration occurs in the problem.

Other author’s [50, 51, 52, 53, 54, 55] following Tolman [56] have used the Equiv-
alance Principle of general relativity to replace a constant proper frame acceleration of the travelling twin by an artificial ‘gravitational field’ in the instantaneous rest frame of this twin. However all these authors made the same error of confusing the base frame of the reciprocal experiment with the travelling frame of the primary experiment, explained above in Section 7, when considering the phases of uniform motion of the round trip.

Applying Occam’s razor —retaining only what is essential— the present paper has not addressed the case of non-impulsive accelerations and decelerations, or the interesting analysis initiated by Langevin [14] of exchange of light signals between the the twins during the voyage. Since, in the explanation of the DAE in the twin paradox given above, the acceleration and decleration phases of the motion play no role, the first restriction is of no importance for understanding the resolution of the twin paradox. The second subject is being addressed by a paper in preparation [57] containing a modern re-assessment of Langevin’s original thought experiment [14].
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