RESOLUTIONS OF IDEALS ASSOCIATED TO SUBSPACE ARRANGEMENTS

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1. INTRODUCTION

Suppose that $W_1, W_2, \ldots, W_d$ are subspaces of an $n$-dimensional $\mathbb{K}$-vector space $W \cong \mathbb{K}^n$ and let $I_1, I_2, \ldots, I_d \subseteq \mathbb{K}[x_1, x_2, \ldots, x_n]$ be the vanishing ideals of $W_1, W_2, \ldots, W_d$. Conca and Herzog showed that the Castelnuovo-Mumford regularity of the product ideal $I_1 I_2 \cdots I_d$ is equal to $d$ (see [1]). Derksen and Sidman showed that the Castelnuovo-Mumford regularity of the intersection ideal $I_1 \cap I_2 \cap \cdots \cap I_d$ is at most $d$ (see [2]) and similar results hold for more general ideals constructed from linear ideals (see [3]). In this paper we show that analogous results hold when we replace the polynomial ring with the exterior algebra and work over a field of characteristic 0. The proofs of aforementioned theorems rely on the existence of non-zero divisors, so this approach fails for the exterior algebra. Instead, we rely on the functoriality of free resolutions and construct a functor $\Omega$ from the category of polynomial functors to itself. The functor $\Omega$ transforms resolutions of ideals in the polynomial ring to resolutions of ideals in the exterior algebra.

2. PRELIMINARIES

Let us fix a field $\mathbb{K}$ of characteristic 0.

2.1. Polynomial functors. In this section we will define a polynomial functor and highlight some features that we will need in the rest of the paper. Let us denote by $\textbf{Vec}$ the category of finite dimensional $\mathbb{K}$-vector spaces whose morphisms are the $\mathbb{K}$-linear maps. This abelian category also has a tensor product, which makes $\textbf{Vec}$ into a symmetric monoidal category.

Definition 1. A functor $\mathcal{F}$ from $\textbf{Vec}$ to $\textbf{Vec}$ is a polynomial functor if the map

$$\mathcal{F} : \text{Hom}(X, Y) \to \text{Hom}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a polynomial mapping for all finite dimensional $\mathbb{K}$-vector spaces $X, Y$. We say that $\mathcal{F}$ is homogeneous of degree $d$ if $\mathcal{F}(\lambda h) = \lambda^d \mathcal{F}(h)$ for every linear map $h$ and $\lambda \in \mathbb{K}$.

Let $\mathcal{F}$ be a polynomial functor. We will consider the category of polynomial functors $\textbf{Poly}$. The morphism in $\textbf{Poly}$ are natural transformations of functors. If $\mathcal{F}$ and $\mathcal{G}$ are polynomial functors, we define the direct sum functor $\mathcal{F} \oplus \mathcal{G} : \textbf{Vec} \to \textbf{Vec}$ by $(\mathcal{F} \oplus \mathcal{G})(X) = \mathcal{F}(X) \oplus \mathcal{G}(X)$ for every finite dimensional vector space, and

$$(\mathcal{F} \oplus \mathcal{G})(h) = \begin{pmatrix} \mathcal{F}(h) & 0 \\ 0 & \mathcal{G}(h) \end{pmatrix} \in \text{Hom}(\mathcal{F}(X) \oplus \mathcal{G}(X), \mathcal{F}(Y) \oplus \mathcal{G}(Y))$$

for every linear map $h : X \to Y$. We can also define the tensor product of two polynomial functors $\mathcal{F}$ and $\mathcal{G}$ by $(\mathcal{F} \otimes \mathcal{G})(X) = \mathcal{F}(X) \otimes \mathcal{G}(X)$ for every finite dimensional vector space.
and \((\mathcal{F} \otimes \mathcal{G})(h) = \mathcal{F}(h) \otimes \mathcal{G}(h) : \mathcal{F}(X) \otimes \mathcal{G}(X) \to \mathcal{F}(Y) \otimes \mathcal{G}(Y)\) for any linear map \(h : X \to Y\). This makes \textbf{Poly} into a abelian, symmetric monoidal category. If \(\mathcal{F}\) and \(\mathcal{G}\) are homogeneous polynomial functors of degree \(d\) and \(e\) respectively, then \(\mathcal{F} \otimes \mathcal{G}\) is homogeneous of degree \(d + e\).

For categories \(A\) and \(B\) we denote the category of all functors from \(A\) to \(B\) by \(\text{Fun}(A, B)\). Morphisms in \(\text{Fun}(A, B)\) are natural transformations. We can view \textbf{Poly} as a subcategory of \(\text{Fun}(\text{Vec}, \text{Vec})\).

For an \(n\)-dimensional vector space \(V\), let \(\text{GL}(V) \subseteq \text{Hom}(V, V)\) be the group of invertible linear maps. A polynomial functor \(\mathcal{F}\) gives a polynomial map \(\text{Hom}(\mathcal{F}(V), \mathcal{F}(V))\) that restricts to a group homomorphism \(\rho : \text{GL}(V) \to \text{GL}(\mathcal{F}(V))\). This means that \(\mathcal{F}(V)\) is a polynomial representation of \(\text{GL}(V)\).

A partition is a sequence \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)\) of positive integers with \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r\). For each partition \(\lambda\) one can define a polynomial functor \(\mathcal{S}_\lambda : \text{Vec} \to \text{Vec}\) that is homogeneous of degree \(|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_r\). For a finite dimensional vector space \(V\), the representation \(\mathcal{S}_\lambda(V)\) is an irreducible representation of \(\text{GL}(V)\). The space \(\mathcal{S}_{(d)}(V) = S^d(V)\) is the \(d\)-th symmetric power and the \(\mathcal{S}_{(1,1,\ldots,1)}(V) = \mathcal{S}_{(d)}(V) = \wedge^d(V)\) is the \(d\)-th exterior power of \(V\). It follows from Schur’s lemma that

\[
\text{Hom}(\mathcal{S}_\lambda, \mathcal{S}_\mu) = \begin{cases} 
\mathbb{K} & \text{if } \lambda = \mu; \\
0 & \text{if } \lambda \neq \mu. 
\end{cases}
\]

Every polynomial functor is naturally equivalent to a finite direct sum of \(\mathcal{S}_\lambda\)’s. By grouping the \(\mathcal{S}_\lambda\)’s together we see that every polynomial functor \(\mathcal{P} \in \textbf{Poly}\) is naturally equivalent to a finite direct sum \(\mathcal{P} = \bigoplus_d \mathcal{P}_d\), where \(\mathcal{P}_d\) is a homogeneous polynomial functor of degree \(d\). We will denote the full subcategory of homogeneous polynomial functors of degree \(d\) by \(\text{Poly}_d\). For more details, the interested reader can consult [5, p. 150].

Let \(\text{Rep}_V\) denote the category of finite dimensional rational representations of \(\text{GL}(V)\) where the morphism are \(\text{GL}(V)\)-equivariant linear maps.

**Lemma 2.** A polynomial functor \(\mathcal{P}\) on the category of finite dimensional vector spaces \(\text{Vec}\) induces a functor \(\mathcal{P}_V\) on the category of \(\text{GL}(V)\)-representations \(\text{Rep}_V\).

**Proof.** Let us consider a \(\text{GL}(V)\)-representation \(\rho_U : \text{GL}(V) \to \text{GL}(U)\). The polynomial functor \(\mathcal{P}\) gives a polynomial map \(\text{Hom}(U, U) \to \text{Hom}(\mathcal{P}(U), \mathcal{P}(U))\) which restricts to a representation \(\text{GL}(U) \to \text{GL}(\mathcal{P}(U))\). The composition \(\text{GL}(V) \to \text{GL}(U) \to \text{GL}(\mathcal{P}(U))\) makes \(\mathcal{P}(U)\) into a representation of \(\text{GL}(V)\).

Let \(\phi\) be a \(\text{GL}(V)\)-equivariant map from \(U\) to \(U'\), so that for all \(g \in \text{GL}(V)\) the following diagram commutes:

\[
\begin{array}{ccc}
U & \xrightarrow{\phi} & U' \\
\downarrow{\rho_U(g)} & & \downarrow{\rho_{U'}(g)} \\
U & \xrightarrow{\phi} & U'
\end{array}
\]

Applying \(\mathcal{P}\) to this diagram, we notice that the resulting diagram also commutes as

\[
\mathcal{P}(\rho_U(g))\mathcal{P}(\phi) = \mathcal{P}(\rho_{U'}(g)) = \mathcal{P}(\rho_{U'}(g)) = \mathcal{P}(\phi)\mathcal{P}(\rho_U(g)),
\]

by functoriality of \(\mathcal{P}\) and our assumptions on \(\phi\). This shows that \(\mathcal{P}(\phi) : \mathcal{P}(U) \to \mathcal{P}(U')\) is \(\text{GL}(V)\)-equivariant.
We conclude that $P$ induces a functor from $\text{Rep}_V$ to itself.

We can consider the category $\text{Poly}_V$ of polynomial functors from $\text{Rep}_V$ to itself. Morphisms in the category $\text{Poly}_V$ are $\text{GL}(V)$-equivariant natural transformations. An object $P$ is the category $\text{Poly}$ induces an object $P_V$ in $\text{Poly}_V$ by Lemma 2.

2.2. Category $\text{GPoly}$. We also consider the category $\text{GVec}$ of graded vector spaces. The objects of $\text{GVec}$ are graded vector spaces $V = \bigoplus_{d=0}^{\infty} V_d$ such that $V_d$ is finite dimensional for all $d$. A morphism $\phi : V \to W$ in the category $\text{GVec}$ is a linear map that respects the grading, i.e., $\phi(V_d) \subseteq W_d$ for all $d$. The tensor product of two graded vector spaces $V,W$ in $\text{GVec}$ is defined by $(V \otimes W)_d = \bigoplus_{e=0}^{d} V_e \otimes W_{d-e}$. This makes $\text{GVec}$ into a symmetric monoidal category.

Next we describe the full subcategory $\text{GPoly}$ in the functor category $\text{Fun}(\text{Vec}, \text{GVec})$.

**Definition 3.** An object $F$ in $\text{GPoly}$ is a functor in $\text{Fun}(\text{Vec}, \text{GVec})$ with the property that $V \mapsto F(V)_d$ is a homogeneous polynomial functor of degree $d$. Morphisms in $\text{GPoly}$ are natural transformations.

An example of a functor in $\text{GPoly}$ is the functor $S := \text{Sym}$ mapping a vector space $V$ to the symmetric algebra $S(V) = \text{Sym}(V)$ on $V$. Similarly, the exterior functor $\wedge$ is a functor in $\text{GPoly}$ that maps a vector space $V$ to its exterior algebra $\wedge(V)$ is another such functor.

The category $\text{GPoly}$ is a symmetric monoidal category via the tensor structure inherited from $\text{GVec}$. In $\text{GPoly}$ we have that $$(F \otimes G)(V)_d = (F(V) \otimes G(V))_d = \bigoplus_{e=0}^{d} F(V)_e \otimes G(V)_{d-e}.$$ We will also view $\mathbb{K}$ as an object in $\text{GPoly}$ as the functor that sends every vector space to the graded vector space $\mathbb{K}$ that is concentrated at degree 0. This object $\mathbb{K}$ in $\text{GPoly}$ is the identity in the monoidal category $\text{GPoly}$. This means that we have a natural equivalence $\kappa : \mathbb{K} \otimes F \to F$ for every object $F$ in $\text{GPoly}$.

2.3. Algebras and modules in $\text{GPoly}$. We will define algebra functors and module functors in $\text{GPoly}$ These are objects in $\text{GPoly}$ that satisfy axioms analog to the axioms of rings and modules respectively.

**Definition 4.** An object $R$ in $\text{GPoly}$ is called an algebra functor if it comes equipped with a multiplication $\mu : R \otimes R \to R$ (i.e., a natural transformation of the functor $R \otimes R$ to the functor $R$) and an identity $1 : \mathbb{K} \to R$ that satisfy the following axioms.

- **Connected:** $1_0 : \mathbb{K}_0 \to R_0$ is a natural equivalence (so we assume $R_0(V) \cong \mathbb{K}$ for all vector spaces $V$);
- **Identity:** the following diagram commutes

$$
\begin{array}{ccc}
\mathbb{K} \otimes R & \xrightarrow{\kappa} & R \\
1 \otimes \text{Id}_R \downarrow & & \downarrow \text{Id}_R \\
R \otimes R & \xrightarrow{\mu} & R
\end{array}
$$
**Associative:** the following diagram commutes

\[
\begin{array}{c}
\mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R} \\
\downarrow \mu \otimes \text{Id}_{\mathcal{R}} \\
\mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R}
\end{array}
\xrightarrow{\cong} \begin{array}{c}
\mathcal{R} \otimes (\mathcal{R} \otimes \mathcal{R}) \\
\downarrow \text{Id}_{\mathcal{R}} \otimes \mu \\
\mathcal{R} \otimes (\mathcal{R} \otimes \mathcal{R})
\end{array}
\]

We will define a left module in a similar fashion.

**Definition 5.** Given an algebra functor \((\mathcal{R}, \mu, 1)\), a left module functor \(\mathcal{M}\) over \(\mathcal{R}\) is an object \(\mathcal{M}\) in \(\text{GPoly}\) equipped with a natural transformation \(\nu : \mathcal{R} \otimes \mathcal{M} \to \mathcal{M}\) that satisfies the following axioms.

**Identity:** the following diagram commutes

\[
\begin{array}{c}
\mathbb{K} \otimes \mathcal{M} \xrightarrow{1 \otimes \text{Id}_\mathcal{M}} \mathcal{R} \otimes \mathcal{M} \\
\downarrow \kappa \\
\mathcal{M}
\end{array}
\xrightarrow{\nu} \begin{array}{c}
\mathcal{M}
\end{array}
\]

**Associative:** the following diagram commutes

\[
\begin{array}{c}
(\mathcal{R} \otimes \mathcal{R}) \otimes \mathcal{M} \xrightarrow{\cong} \mathcal{R} \otimes (\mathcal{R} \otimes \mathcal{M}) \\
\downarrow \mu \otimes \text{Id}_\mathcal{M} \\
\mathcal{R} \otimes \mathcal{M} \xrightarrow{\nu} \mathcal{R} \otimes \mathcal{M}
\end{array}
\xrightarrow{\mu} \begin{array}{c}
\mathcal{R} \otimes (\mathcal{R} \otimes \mathcal{M}) \\
\downarrow \text{Id}_{\mathcal{R}} \otimes \nu \\
\mathcal{R} \otimes (\mathcal{R} \otimes \mathcal{M})
\end{array}
\]

Notice that for any vector space \(V\) and for every algebra functor \(\mathcal{R}\), we have that \(\mathcal{R}(V)\) is a \(\mathbb{K}\)-algebra. Similarly, \(\mathcal{M}(V)\) is a left module over \(\mathcal{S}(V)\). Moreover, the above axioms give us that for every \(f \in \text{Hom}(\text{Vec})\), we have that \(\mathcal{R}(f)\) is a homomorphism of \(\mathbb{K}\)-algebras. The symmetric algebra functor \(\mathcal{S} := \text{Sym}\) is an example of an algebra functor in \(\text{GPoly}\). If \(\mathcal{M}\) is a left module functor over \(\mathcal{S}\), then for every \(V \in \text{Obj}(\text{Vec})\) we have that \(\mathcal{M}(V)\) is a left module over \(\mathcal{S}(V)\).

In the next sections we will construct a functor \(\Omega\) from \(\text{GPoly}\) to itself such that \(\Omega(\mathcal{S}) = \bigwedge\).

In general, for \(\mathcal{F}_d\) any homogeneous polynomial functor of degree \(d\), we will have that \(\Omega(\mathcal{F}_d)\) is another homogeneous polynomial functor of degree \(d\). In fact, we will first construct \(\Omega_d\), the \(d\)th graded piece of \(\Omega\), a functor from the category of homogeneous polynomial functors of degree \(d\) to itself. The functor \(\Omega\) can be found in the literature in the context of \(\text{GL}_\infty\) representations [6, p. 1102]. In their context \(\Omega\) is called the transpose functor and it is defined for the infinite symmetric group and transferred to \(\text{GL}_\infty\)-representations via Schur-Weyl duality. For the convenience of the reader, we present a construction which does not require previous knowledge of the structure theory of \(\text{GL}_\infty\)-representations.

3. **Definition of \(\Omega_d\) on the Category \text{Poly}_d**

Let \(\text{Poly}_d\) be the full subcategory of \(\text{Poly}\) consisting of homogeneous polynomial functors of degree \(d\). To be able to define \(\Omega_d\), we will need to go through a multi-step process. The first sections will aim to define a functor \(\Omega_{V,d}\) on \(\text{Poly}_d\) for any fixed vector space \(V\). Then we will define \(\Omega_d\) as a direct limit of functors \(\Omega_{V,d}\).
3.1. **Definition of \( \Omega_{V,d} \) on the category \( \text{Poly}_d \).** We start with a construction from category theory. For a functor \( F : A \to B \), we can define the functor \( F^* : \text{Fun}(B, C) \to \text{Fun}(A, C) \) by \( F^*(G) = G \circ F \), for any functor \( G : B \to C \). Similarly, for any fixed \( G \in \text{Fun}(B, C) \), we can define \( G_* : \text{Fun}(A, B) \to \text{Fun}(A, C) \) by \( G_*(F) = G \circ F \), for any functor \( F : A \to B \).

We fix a vector space \( V \) of dimension \( n \). Let us consider the category \( \text{Fun}(\text{Vec}, \text{Rep}_V) \). We define the functor \( T_V : \text{Vec} \to \text{Rep}_V \) as the functor \( - \otimes V \) mapping \( W \in \text{Vec} \) to \( W \otimes V \). Notice that \( \text{GL}(V) \) acts on \( W \otimes V \) by trivial action on \( W \) and left multiplication on \( V \). Let us fix a degree \( d \) such that \( d \geq n \), where \( n \) is the dimension of the fixed vector space \( V \). In the category \( \text{Fun}(\text{Rep}_V, \text{Rep}_V) \) we consider the full subcategory \( \text{Poly}_{V,d} \) of homogeneous polynomial functors of degree.

An object \( \mathcal{P}_d \) in \( \text{Poly}_d \) induces an object \( \mathcal{P}_{V,d} \) in \( \text{Poly}_{V,d} \). Finally let us consider the functor \( \mathcal{H}_{V,d} : \text{Rep}_V \to \text{Vec} \), defined as \( \text{Hom}_{\text{GL}(V)}(\bigwedge^d(V),-) \) so that \( \mathcal{H}_{V,d} \) maps a \( \text{GL}(V) \)-representation \( U \) to its \( \bigwedge^d(V) \)-isotopic component.

**Definition 6.** For a polynomial functor \( \mathcal{P}_d \) of degree \( c \), the functor \( \Omega_{V,d}(\mathcal{P}_d) : \text{Vec} \to \text{Vec} \) is defined by

\[
\Omega_{V,d}(\mathcal{P}_d) = (\mathcal{H}_{V,d})_* T_V^*(\mathcal{P}_{V,d}) = \mathcal{H}_{V,d} \circ \mathcal{P}_{V,d} \circ T_V
\]

The following commuting diagram illustrates the effect of \( \Omega_{V,d}(\mathcal{P}_d) \) on objects in the category \( \text{Vec} \).

\[
\begin{array}{ccc}
W & \xrightarrow{T_V} & W \otimes V \\
\downarrow{\Omega_{V,d}(\mathcal{P}_d)} & & \downarrow{\mathcal{P}_{V,d}} \\
\Omega_{V,d}(\mathcal{P}_d)(W) & \xleftarrow{\mathcal{H}_{V,d}} & \mathcal{P}_{V,d}(W \otimes V)
\end{array}
\]

From our definition of \( \Omega_{V,d}(\mathcal{P}_d) \), it is clear that this functor depends on the choice of the polynomial functor \( \mathcal{P}_d \) and the choice of a vector space \( V \). Our goal is to be able to define a new functor, \( \Omega_d : \text{Poly}_d \to \text{Poly}_d \). To be able to do so, we consider the following lemma.

**Lemma 7.** The functor \( \Omega_{V,d}(\mathcal{P}_d) \) on \( \text{Vec} \) is a homogeneous polynomial functor of degree \( d \).

**Proof.** Recall the assumption that \( \mathcal{P}_d \) was itself a homogeneous polynomial functor of degree \( d \). We have defined \( \Omega_{V,d}(\mathcal{P}_d) = \mathcal{H}_{V,d} \circ \mathcal{P}_{V,d} \circ T_V \) so that to establish the claim we need to analyze the three functors used here. First, notice that \( T_V \) is a polynomial functor, being in particular a homogeneous linear functor. Moreover, we are given that \( \mathcal{P}_d \) is a homogeneous polynomial functor of degree \( d \) and the induced functor \( \mathcal{P}_{V,d} \) still retains this property. Finally, \( \mathcal{H}_{V,d} \) is a homogeneous linear functor being the restriction to the \( \text{GL}(V) \)-invariant component of the homogeneous linear functor \( \mathcal{H}_{V,d} = \text{Hom}(\bigwedge^d V, -) \). Thus, the composition of these three functor is a homogeneous polynomial functor of overall degree \( d \).

\[\square\]

The above lemma allows us to define for every \( \mathcal{P}_d \in \text{Poly}_d \) a new object in \( \text{Poly}_d \), namely \( \Omega_{V,d}(\mathcal{P}_d) \). Notice that since we defined \( \Omega_{V,d} \) as a composition of functors, its effect on morphisms in \( \text{Poly}_d \) (which are natural transformations between polynomial functors) is just the composition of the functors \( (\mathcal{H}_{V,d})_* \) and \( T_V^* \).
3.1.1. The functor $\Omega_{V,d}$ on Schur functors. To understand the effect of the functor $\Omega_{V,d}$ on $\text{Poly}$, we will first study the polynomial functor $\Omega_{V,d}(S_\lambda)$ in $\text{Poly}$, for $S_\lambda$ the Schur functor associated to $\lambda$, a partition of $d$.

**Lemma 8.** Let $\lambda$ be a partition of $n$ and $d \geq \dim V$. The polynomial functor $\Omega_{V,d}(S_\lambda)$ is naturally equivalent to $S'_\lambda$.

**Proof.** We have already showed that $\Omega_{V,d}(S_\lambda)$ is a homogeneous polynomial functor of degree $d$. Notice that the functor $S_\lambda T_V = S_\lambda(\otimes V)$ can be decomposed using the following formula

$$S_\lambda(\otimes V) = \bigoplus (S_{\mu}(\otimes S_\nu(V)))^{a_{\lambda,\mu,\nu}} = \ldots \oplus S_{\lambda'}(\otimes \wedge^d(V) \oplus \ldots,$$

where $a_{\lambda,\mu,\nu}$ is the Kronecker coefficient (the tensor product multiplicity for the corresponding representations of the symmetric group). We notice that in this decomposition the isotypic component of $\wedge^d(V)$ is given by $S'_{\lambda}(\otimes \wedge^d(V))$ corresponding to the Kronecker coefficient $a_{\lambda,\lambda',(1^d)} = 1$. Consider now the effect of the functor $H_{V,d}$. Only the image of the isotypic component of $\wedge^d(V)$ will be non-zero. In particular,

$$H_{V,d}(S_\lambda(\otimes \wedge^n(V))) \cong (\wedge^n(V)^* \otimes S_{\lambda'}(\otimes \wedge^n(V)))_{\text{GL}(V)} \cong S'_{\lambda}(\otimes V),$$

where all the isomorphisms are natural equivalences. $\square$

Notice that in the above proof, we studied the image of $\Omega_{V,d}(S_\lambda)$ by examining the isotypic component of $\wedge^d(V)$ in $S_\lambda(\otimes V)$. When we use the functor $\Omega_{V,d}$, we will often use this computational approach to understand its effect on polynomial functors. In particular, in the next result we use it to show that $\Omega_{V,d}$ behaves well with respect to direct sums. The proof of the following lemma is left to the reader.

**Lemma 9.** For polynomial functors $P_d$ and $P'_d$ the functors $\Omega_{V,d}(P_d \oplus P'_d)$ and $\Omega_{V,d}(P_d) \oplus \Omega_{V,d}(P'_d)$ are naturally equivalent.

Moreover, we have that $\Omega_{V,d}$ behaves well with respect to tensor products.

**Lemma 10.** Let $\lambda$ and $\mu$ be partitions of $d$ and $e$ respectively. We have that $\Omega_{V,d+e}(S_\mu \otimes S_\nu)$ is naturally equivalent to $\Omega_{V,d}(S_\mu) \otimes \Omega_{V,e}(S_\nu)$ if $\dim V \geq d + e$.

**Proof.** Recall that an application of the Littlewood Richardson rule gives that:

$$S_\mu \otimes S_\nu = \bigoplus \lambda S^{\lambda}_{\mu,\nu},$$

$$= \bigoplus \lambda S^{\lambda}_{\mu',\nu'},$$

by the properties of the Littlewood Richardson coefficients. Applying $\Omega_{V,d+e}$ to this equations, we get that

$$\Omega_{V,d+e}(S_\mu \otimes S_\nu) \cong \bigoplus \lambda \Omega_{V,d+e}(S_{\lambda})^{\lambda_{\mu',\nu'}} \cong \bigoplus \lambda S^{\lambda}_{\mu',\nu'}.$$
On the other hand, we have that

$$\Omega_V^d(S_\mu) \otimes \Omega_V^e(S_\nu) \cong S_\mu' \otimes S_\nu' = \bigoplus_{\lambda} S_\lambda^{\mu' \nu'}.$$  

As $\Omega_{d+e}(S_\mu \otimes S_\nu)$ and $\Omega_d(S_\mu) \otimes \Omega_e(S_\nu)$ have the same direct sum decomposition in terms of Schur functors, they are naturally equivalent polynomial functors.

3.2. The functor $\Omega_V$ on Poly. So far we have seen the effect of $\Omega_{V,d}$ on Schur functors, their direct sums, and on $\text{Hom}(\text{Poly})$. However, for any homogeneous polynomial functor of degree $d$, we can choose a natural equivalence so that

$$P_d \cong \bigoplus S_{\lambda}^{m_\lambda},$$

for some integers $m_\lambda$. Then, using the previous results, we obtain that

$$\Omega_V^d(P_d) \cong \Omega_V^d\left(\bigoplus S_{\lambda}^{m_\lambda}\right) \cong \bigoplus S_{\lambda}^{m_\lambda}.$$  

In fact, we will actually often just think of the functor $\Omega_V^d$ on $\text{Poly}_d$ in terms of its effect on Schur functors. Moreover, we can use this to define $\Omega_V$ on $\text{Poly}$.

**Definition 11.** Let $P$ be an object in the category $\text{Poly}$. We can decompose $P$ in its graded pieces i.e., $P = \bigoplus P_d$, where $P_d$ is a homogeneous polynomial functor of degree $d$. We define

$$\Omega_V(P) = \bigoplus_{d} \Omega_V^d(P_d)$$

Notice that after choosing some natural equivalence $P_d \cong \bigoplus_{\lambda \vdash d} S_{\lambda}^{m_\lambda}$ for each $d$ in the decomposition of $P$, we have that:

$$\Omega_V(P) \cong \bigoplus_{d \lambda \vdash d} S_{\lambda}^{m_\lambda}.$$  

Finally, we want to show that our definition is independent of the choice of $V$ i.e., if $V'$ is another vector space of dimension $m \geq d$, then for any $P_d \in \text{Poly}_d$ the functors $\Omega_{V,d}(P_d)$ and $\Omega_{V',d}(P_d)$ are naturally equivalent.

**Lemma 12.** Let $V, V'$ be two vector spaces of dimensions $n, m$, respectively, such that $n, m \geq d$. For every $P_d \in \text{Poly}_d$ we have that $\Omega_{V,d}(P_d)$ and $\Omega_{V',d}(P_d)$ are naturally equivalent functors.

**Proof.** Let $P_d \in \text{Poly}_d$. Then there exists a natural equivalence $\psi : P_d \to \bigoplus S_{\lambda}^{m_\lambda}$. By our definition of $\Omega_{V,d}, \Omega_{V',d}$ on $\text{Hom}(\text{Poly}_d)$, we have that $\Omega_{V,d}(\psi), \Omega_{V',d}(\psi)$ are natural equivalences in $\text{Hom}(\text{Poly}_d)$. Moreover, recall that $\Omega_{V,d}(\bigoplus S_{\lambda}^{m_\lambda}) \cong \bigoplus S_{\lambda}^{m_\lambda}$, by our results on the effect of $\Omega_{V,d}$ on Schur functors. Let us call this natural equivalence $\phi$. Similarly, there exist a natural equivalence $\phi' : \Omega_{V',d}(\bigoplus S_{\lambda}^{m_\lambda}) \to \bigoplus S_{\lambda}^{m_\lambda}$. We will define $\eta_{P_d} : \Omega_{V,d}(P_d) \to \Omega_{V',d}(\bigoplus S_{\lambda}^{m_\lambda})$ to be

$$\eta_{P_d} = \Omega_{V',d}(\psi)^{-1} \circ \phi'^{-1} \circ \text{Id} \circ \phi \circ \Omega_{V,d}(\psi),$$

then $\eta_{P_d}$ is a natural equivalence. Finally, we have that $\eta_{P_d} \circ \Omega_{V,d}(\psi) = \Omega_{V',d}(\psi)$ by our definition of $\Omega_{V,d}$ and $\Omega_{V',d}$, and $\eta_{P_d} \circ \Omega_{V',d}(\psi) = \Omega_{V,d}(\psi)$ by our definition of $\Omega_{V,d}$ and $\Omega_{V',d}$. Therefore, $\eta_{P_d}$ is a natural equivalence. 


or the top horizontal map in the following commuting diagram:

\[
\begin{array}{ccc}
\Omega_{V,d}(\mathcal{P}_d) & \xrightarrow{\eta_{\mathcal{P}_d}} & \Omega_{V',d}(\mathcal{P}_d) \\
\downarrow \Omega_{V,d}(\psi) & & \uparrow \Omega_{V',d}(\psi)^{-1} \\
\Omega_{V,d}(\bigoplus S^m_\lambda) & \xrightarrow{\phi} & \Omega_{V',d}(\bigoplus S^m_\lambda) \\
\downarrow \phi & & \uparrow \phi'^{-1} \\
\bigoplus S^m_\lambda & \xrightarrow{Id} & \bigoplus S^m_\lambda
\end{array}
\]

In conclusion, notice that since \(\eta_{\mathcal{P}_d}\) is a composition of natural equivalences in \(\text{Hom}(\text{Poly}_d)\), it is itself a natural equivalence in \(\text{Hom}(\text{Poly}_d)\). \qed

As two vector spaces of dimension greater than \(d\) yield naturally equivalent functors \(\Omega_{V,d}(\mathcal{P}_d)\) and \(\Omega_{V',d}(\mathcal{P}_d)\) in \(\text{Poly}_d\), for any functor \(\mathcal{P}\) in \(\text{Poly}\) of degree \(d\) (not necessarily homogeneous), we have that choosing vector spaces \(V, V'\) of dimension greater than \(d\) will yield the naturally equivalent functors \(\Omega_V(\mathcal{P})\) and \(\Omega_{V'}(\mathcal{P})\).

4. THE FUNCTOR \(\Omega_V\) AND THE TENSOR STRUCTURE OF Poly

Consider two Schur functors \(S_\mu, S_\nu\), where \(\mu, \nu\) are partitions of \(d\) and \(e\), respectively. We have previously noted that:

\[
\Omega_V(S_\mu \otimes S_\nu) = \Omega_{d+e,V}(S_\mu \otimes S_\nu) \cong \Omega_{d,V}(S_\mu) \otimes \Omega_{e,V}(S_\nu) = \Omega_V(S_\mu) \otimes \Omega_V(S_\nu)
\]

where \(V\) is a vector space of dimension greater or equal to \(d + e\).

However, we have not explicitly produced a natural equivalence between them. In particular, we have not studied how the functor \(\Omega_V\) interacts with the symmetric tensor structure of \(\text{Poly}\). Recall that for every \(\mathcal{P}, \mathcal{P}' \in \text{Poly}\), there are natural equivalence \(s_{\mathcal{P}, \mathcal{P}'}\), \(s_{\mathcal{P}', \mathcal{P}}\), where

\[
\mathcal{P} \otimes \mathcal{P}' \xrightarrow{s_{\mathcal{P}, \mathcal{P}'}, \mathcal{P} \otimes \mathcal{P}' \xrightarrow{s_{\mathcal{P}', \mathcal{P}}} \mathcal{P} \otimes \mathcal{P}',
\]

such that \(s_{\mathcal{P}', \mathcal{P}} \circ s_{\mathcal{P}, \mathcal{P}'} = Id_{\mathcal{P} \otimes \mathcal{P}'}\). Similarly, there are natural equivalences \(s_{\Omega_V(\mathcal{P}), \Omega_V(\mathcal{P}')}\), \(s_{\Omega_V(\mathcal{P}'), \Omega_V(\mathcal{P})}\) such that

\[
s_{\Omega_V(\mathcal{P}'), \Omega_V(\mathcal{P})} \circ s_{\Omega_V(\mathcal{P}), \Omega_V(\mathcal{P}')} = Id_{\Omega_V(\mathcal{P}) \otimes \Omega_V(\mathcal{P}')}.\]

The question arising from this set up is whether we can produce a natural equivalence \(\psi_{\mathcal{P}, \mathcal{P}'} : \Omega_V(\mathcal{P}) \otimes \Omega_V(\mathcal{P}') \to \Omega_V(\mathcal{P} \otimes \mathcal{P}')\) compatible with the tensor structure. In practice, we want to determine if the following diagram commutes:

\[
\begin{array}{ccc}
\Omega_V(\mathcal{P} \otimes \mathcal{P}') & \xrightarrow{\Omega_V(s_{\mathcal{P}, \mathcal{P}'})} & \Omega_V(\mathcal{P}' \otimes \mathcal{P}) \\
\uparrow \psi_{\mathcal{P}, \mathcal{P}'} & & \uparrow \psi_{\mathcal{P}', \mathcal{P}} \\
\Omega_V(\mathcal{P}) \otimes \Omega_V(\mathcal{P}') & \xrightarrow{s_{\Omega_V(\mathcal{P}), \Omega_V(\mathcal{P}')}} & \Omega_V(\mathcal{P}') \otimes \Omega_V(\mathcal{P})
\end{array}
\]

As every polynomial functor is a direct sum of Schur functors, it will be enough to study this diagram for \(\mathcal{P} = S_\lambda\) and \(\mathcal{P}' = S_\mu\), for \(\lambda\) a partition of \(d\) and \(\mu\) a partition of \(e\).
Proposition 13. We can define $\psi_{S_\lambda, S_\mu}$ so that the above diagram commutes up to sign $(-1)^{de}$.

Proof. First we will define $\psi_{S_\lambda, S_\mu}$ and from our definition we will conclude that the diagram only commutes up to a sign.

As remarked before, $\Omega_{V,d}(S_\lambda)(W)$ is the multiplicity space of the isotypic component of the irreducible $GL(V)$-representation $\Lambda^d(V)$ inside $S_\lambda(W \otimes V)$. Thus, to find a natural equivalence from $\Omega_{V,d}(S_\lambda) \otimes \Omega_{V,e}(S_\mu)$ to $\Omega_{V,d+e}(S_\lambda \otimes S_\mu)$ we need to exhibit for any finite dimensional vector space $W$ an isomorphism from the tensor product of the isotypic component of $\Lambda^d(V)$ in $\Omega_{V,d}(S_\lambda)(W)$ and the isotypic component of $\Lambda^e(V)$ in $\Omega_{V,e}(S_\mu)(W)$ to the isotypic component of $\Lambda^{d+e}(V)$ in $\Omega_{V}(S_\lambda \otimes S_\mu)$.

Going back to the definition of $\Omega_{V,d}(P_d)(W)$, we notice that this is equivalent to producing an isomorphism $\phi = \psi_{S_\lambda, S_\mu}(W)$:

$$
\begin{array}{ccc}
\text{Hom}_{GL(V)}(\Lambda^d(V), S_\lambda(W \otimes V)) \otimes \text{Hom}_{GL(V)}(\Lambda^e(V), S_\mu(W \otimes V)) & \xrightarrow{\phi} & \text{Hom}_{GL(V)}(\Lambda^{d+e}(V), S_\lambda(W \otimes V) \otimes S_\mu(W \otimes V))
\end{array}
$$

for every $W$. Note that $\Lambda^d(V) \otimes \Lambda^e(V)$ has a unique subrepresentation isomorphic to $\Lambda^{d+e}(V)$. Now we define $\phi$ as follows. If $f : \Lambda^d(V) \rightarrow S_\lambda(W \otimes V)$ and $g : \Lambda^e(V) \rightarrow S_\mu(W \otimes V)$ are $GL(V)$-equivariant linear maps, then we define $\phi(f \otimes g)$ as the restriction of $f \otimes g : \Lambda^d(V) \otimes \Lambda^e(V) \rightarrow S_\lambda(W \otimes V) \otimes S_\mu(W \otimes V)$ to the subrepresentation $\Lambda^{d+e}(V) \subseteq \Lambda^d(V) \otimes \Lambda^e(V)$. We can extend $\phi$ to a linear map.

By Schur’s Lemma any $GL(V)$-equivariant map in $f : \Lambda^d(V) \rightarrow S_\lambda(W \otimes V)$ can be written as $f = Id \otimes a : \Lambda^d(V) \rightarrow \Lambda^d(V) \otimes S_{\lambda'}(W) \subseteq S_\lambda(W \otimes V)$, for a some constant map from $\Lambda^d(V)$ to $S_{\lambda'}(W)$, as $S_{\lambda'}(W)$ is the multiplicity space of the isotypic component of $\Lambda^d(V)$ in $S_{\lambda'}(W \otimes V)$. Similarly, letting $b$ be a constant map to $S_{\mu'}(W)$, we will have that $g \in \text{Hom}_{GL(V)}(\Lambda^e(V), S_\mu(W \otimes V))$ can be written as $g = Id \otimes b$. As the multiplicity space of $\Lambda^{d+e}(V)$ in $S_{\lambda'}(W \otimes V) \otimes S_{\mu'}(W \otimes V)$ is precisely $S_{\lambda'}(W \otimes V) \otimes S_{\mu'}(W \otimes V)$, we have that $\phi$ sends $\Sigma_i Id \otimes a_i \otimes Id \otimes b_i$ to $\Sigma_i Id \otimes a_i \otimes b_i$. As this map is an injective map between isomorphic spaces, it is an isomorphism.

In the diagram below:

$$
\begin{array}{ccc}
\Lambda^d(V) \otimes \Lambda^e(V) & \xrightarrow{s_{\Lambda^d(V), \Lambda^e(V)}} & \Lambda^{d+e}(V) \\
\Lambda^e(V) \otimes \Lambda^d(V) & & \\
\end{array}
$$

the map $s_{\Lambda^d(V), \Lambda^e(V)}$ takes the pure tensor $a \otimes b$ to $(-1)^{de} b \otimes a$. Thus, the diagram only commutes up to sign $(-1)^{de}$.

So applying $\text{Hom}(\ldots, S_\lambda \otimes S_\mu(W \otimes V))$ to the diagram above, we obtain a new diagram that only commutes up to sign $(-1)^{de}$.
\[
\text{Hom}(\bigwedge^d(V) \otimes \bigwedge^e(V), S_\lambda \otimes S_\mu(W \otimes V)) \longrightarrow \text{Hom}(\bigwedge^{d+e}(V), S_\lambda \otimes S_\mu)(W)
\]
\[
\text{Hom}(\bigwedge^e(V) \otimes \bigwedge^d(V), S_\lambda \otimes S_\mu(W \otimes V))
\]

Notice that restricting the spaces above to the relevant invariant subspaces does not affect the sign in the above diagram.

The map
\[
\Omega_V(s_{S_\mu,S_\lambda})(W) : \Omega_V(S_\mu \otimes S_\lambda)(W) \to \Omega_V(S_\lambda \otimes S_\mu)(W)
\]
is just given by \( \Sigma_i \text{Id} \otimes b_i \otimes a_i \mapsto \Sigma_i \text{Id} \otimes a_i \otimes b_i \), so that on pure tensors we have that \( \text{Id} \otimes b \otimes a \mapsto \text{Id} \otimes a \otimes b \). Thus we conclude that the diagram
\[
\begin{align*}
\Omega_V(S_\lambda)(W) \otimes \Omega_V(S_\mu)(W) &\xrightarrow{\phi} \Omega_V(S_\lambda \otimes S_\mu)(W) \\
\Omega_V(S_\mu)(W) \otimes \Omega_V(S_\lambda)(W) &\xrightarrow{\phi} \Omega_V(S_\mu \otimes S_\lambda)(W)
\end{align*}
\]
only commutes up to sign \((-1)^{de}\) as first going right and then up sends \( \text{Id} \otimes b \otimes \text{Id} \otimes a \mapsto \text{Id} \otimes a \otimes b \), whilst first going up then right sends \( \text{Id} \otimes b \otimes \text{Id} \otimes a \mapsto (-1)^{de} \text{Id} \otimes a \otimes b \)

5. Definition of \( \Omega(P) \)

After examining the effect of \( \Omega_{d,V} \) on the tensor structure of \( \textbf{Poly} \), we want to be able to work with a functor defined independently from the choice of the vector space \( V \). For every \( i \geq 0 \), define \( V_i \cong \mathbb{K}^i \) as the vector space of all sequences \((a_1, a_2, a_3, \ldots) \in \mathbb{K}^\infty\) with \( a_j = 0 \) for all \( j > i \). Let \( \rho_{ji} \) be the inclusion \( \rho_{ji} : V_i \to V_j \). Suppose that \( i \leq j \). Consider \( \Omega_{d,V_j}(\mathcal{P}_d) \in \text{Hom}_{\text{GL}(V_j)}(\bigwedge^d_j, \mathcal{P}_d(V_j \otimes -)) \). The restriction of \( \Omega_{d,V_j}(\mathcal{P}_d) \) to \( \bigwedge^d_j \) is \( \text{GL}(V_j) \)-equivariant and the image is contained in \( \mathcal{P}_d(V_i \otimes -) \). So we have a natural transformation \( g_{i,j} : \Omega_{d,V_j}(\mathcal{P}_d) \to \Omega_{d,V_i}(\mathcal{P}_d) \). The kernel of this natural transformation consists exactly of all isotypic components \( S_\lambda \) where \( \lambda \) has \( > i \) parts. There is a unique splitting \( f_{j,i} : \Omega_{d,V_i}(\mathcal{P}_d) \to \Omega_{d,V_j}(\mathcal{P}_d) \) such that \( g_{i,j} \circ f_{j,i} \) is the identity. We have a direct system
\[
\begin{array}{cccccc}
\Omega_{d,V_0} & \xrightarrow{f_0} & \Omega_{d,V_1} & \xrightarrow{f_1} & \Omega_{d,V_2} & \xrightarrow{f_2} & \cdots \\
\end{array}
\]
where \( f_i = f_{i+1,i} \).

**Definition 14.** We define \( \Omega_d(\mathcal{P}_d) \) to be the following direct limit of the maps \( f_{j,i} \):
\[
\Omega_d(\mathcal{P}_d) = \lim_{\longrightarrow} \Omega_{d,V_i}(\mathcal{P}_d).
\]

In conclusion, we notice that this functor is well-defined as direct limits exist because the abelian category of polynomial functors is cocomplete. Finally, recall our definition of \( \Omega_V \) for \( \mathcal{P} \in \textbf{Poly} \) decomposed as \( \mathcal{P} = \bigoplus_d \mathcal{P}_d \):
\[
\Omega_V(\mathcal{P}) = \bigoplus_{d} \Omega_{V,d}(\mathcal{P}_d).
\]
Similarly, we have the following definition for $\Omega$.

**Definition 15.** For $\mathcal{P} \in \text{Poly}$ decomposed as $\mathcal{P} = \bigoplus_d \mathcal{P}_d$. We define $\Omega(\mathcal{P})$ to be:

$$
\Omega(\mathcal{P}) = \bigoplus_d \Omega_d(\mathcal{P}_d).
$$

We can notice that our definition of $\Omega_V$ is compatible with our definition of $\Omega$:

$$
\Omega(\mathcal{P}) = \bigoplus_d \Omega_d(\mathcal{P}_d) = \bigoplus_d \lim_{\rightarrow} \Omega_{d,V_i}(\mathcal{P}_d) = \lim_{\rightarrow} \bigoplus_d \Omega_{d,V_i}(\mathcal{P}),
$$

as direct sums and direct limits commute.

6. **The functor $\Omega$ on $\text{GPoly}$**

Recall that the category $\text{GPoly}$ is a functor category where each functor $\mathcal{F} \in \text{GPoly}$ can be decomposed as

$$
\mathcal{F} = \bigoplus \mathcal{F}_d
$$

with $\mathcal{F}_d$ a homogeneous polynomial functor of degree $d$. Moreover, as each polynomial functor is naturally equivalent to a direct sum of $\mathcal{S}_\lambda$'s. Thus we can also notice that

$$
\mathcal{F} \cong \bigoplus \mathcal{S}_\lambda^{m_\lambda},
$$

for $\lambda$'s of any length, but with the stipulation that for each polynomial functor $\mathcal{F}_d$ and any $W \in \text{Vec}$, we have that $\mathcal{F}_d(W) \cong \bigoplus \mathcal{S}_\lambda^{m_\lambda}(W)$ is a finite dimensional vector space.

In the previous section we have seen how to define $\Omega$ on any polynomial functors and actually we can extend this definition to any direct sum of polynomial functors, even though the direct sum itself may not be a polynomial functor. In particular, for any $W \in \text{Vec}$ we have that $\mathcal{S}(W)$, the symmetric algebra on $W$, is not a polynomial functor as it is infinite dimensional. However, it is a direct sum of polynomial functors as each graded piece $\mathcal{S}^d(W)$ is a finite dimensional vector space.

**Definition 16.** For any functor $\mathcal{F} \in \text{GPoly}$, where $\mathcal{F} = \bigoplus \mathcal{F}_d$, we let

$$
\Omega(\mathcal{F}) = \bigoplus \Omega_d(\mathcal{F}_d).
$$

Considering $\mathcal{S} = \bigoplus \mathcal{S}^d$, we have that

$$
\Omega(\mathcal{S}) = \bigoplus \Omega_d(\mathcal{S}^d) \cong \bigoplus \bigwedge^d = \bigwedge
$$

from our results on the effect of $\Omega$ on Schur functors.

7. **Resolutions and $\Omega$**

7.1. **Equivariant resolutions.** Let us fix a vector space $U$ of dimension $t$. The ring of polynomial functions on $U$ can be identified with the symmetric algebra $S(U^*)$. Let $R = S(U^*)$ and let $\mathfrak{m}$ be the homogeneous maximal ideal in $R$. Given a module $M$ over $R$ we can construct a minimal resolution by defining $D_0 := M$ and $E_0 := D_0/\mathfrak{m}D_0$. We can then extend in a unique way the homogeneous section $\phi_0 : E_0 \to D_0$ of the homogeneous quotient map $\pi_0 : D_0 \to E_0$ to a $R$-module homomorphism $\phi_0 : R \otimes E_0 \to D_0$. The tensor product $R \otimes E_0$ is naturally graded as a tensor product of graded vector spaces and $\phi_0$ is homogeneous with respect to this grading. Letting $D_1$ be the kernel of $\phi_0$, we see how to
proceed inductively to construct a free resolution of $M$. The resolution is finite by Hilbert’s syzygy theorem which states that $D_i = 0$ for $i > t$. In the resulting minimal free resolution:

$$0 \rightarrow R \otimes E_t \rightarrow \cdots \rightarrow R \otimes E_0 \rightarrow M \rightarrow 0,$$

we can naturally identify $E_i$ with $\text{Tor}_i(M, \mathbb{K})$.

Moreover, suppose that $U$ is a representation of a linearly reductive algebraic group $G$. If $G$ also acts on the module $M$ and the multiplication map $m : R \times M \rightarrow M$ is $G$-equivariant, then each graded piece $M_d$ of $M = \bigoplus M_d$ is a $G$-module. By linear reductivity, we can choose the maps $\phi_i$ to be $G$-equivariant giving each $E_i$ the structure of a graded $G$-module. In particular, we can choose a decomposition of each $E_i$ into irreducible $G$-representations. In the case of $G = \text{GL}(V)$ and $M$ a polynomial representation of $G$, we will have a decomposition of each $E_i$ in the equivariant resolution of $M$ in terms of the Schur functors $S^\lambda$’s. As a result we have the following equivariant resolution of the $G$-module $M$:

$$0 \rightarrow R \otimes \bigoplus S^\lambda(V)^{m^\lambda_0} \rightarrow \cdots \rightarrow R \otimes \bigoplus S^\lambda(V)^{m^\lambda_1} \rightarrow M \rightarrow 0.$$

In particular, we have that $M/\mathfrak{m}M = \text{Tor}_0(M, \mathbb{K})$ can be decomposed in irreducible $\text{GL}(V)$-representations as $\bigoplus S^\lambda(V)^{m^\lambda_0}$.

### 7.2. Resolutions in $\text{GPoly}$ and $\Omega$

Consider a module functor $\mathcal{M} \in \text{Obj}(\text{GPoly})$ over the algebra functor $\mathcal{R} \in \text{Obj}(\text{GPoly})$. A resolution for $\mathcal{M}$ is constructed analogously to a resolution for a $G_{\lambda}(\mathbb{K})$-module. In particular, the construction relies on the fact that $\text{GPoly}$ is a semisimple category as each object is naturally equivalent to a direct sum of simple objects: the irreducible polynomial functors $S^\lambda$’s.

The object $\mathcal{M}$ in $\text{GPoly}$ is equipped with a natural equivalence

$$\mathcal{M} \cong \bigoplus S^\lambda \otimes A_\lambda,$$

where $A_\lambda$ is just a vector space functioning as a multiplicity space for the polynomial functor $S^\lambda$. Let $\mathcal{N}$ be another $\mathcal{R}$-module functor, where $\mathcal{N} \cong \bigoplus S^\lambda \otimes B_\lambda$. We have that the map $\psi : \mathcal{M} \rightarrow \mathcal{N}$ can be viewed as a map

$$\bigoplus S^\lambda \otimes A_\lambda \rightarrow \bigoplus S^\lambda \otimes B_\lambda$$

and being a homomorphism in $\text{GPoly}$, we have that $\psi = \bigoplus \text{Id} \otimes \psi_\lambda$, where each $\psi_\lambda : A_\lambda \rightarrow B_\lambda$ is just a linear map. Thus each $\psi_\lambda$ has a section $\phi_\lambda : B_\lambda \rightarrow A_\lambda$ allowing us to construct $\phi := \bigoplus \text{Id} \otimes \phi_\lambda : \mathcal{N} \rightarrow \mathcal{M}$, a section of $\psi$. As $\text{GPoly}$ is an abelian category for each $\phi \in \text{GPoly}$, there exists an object $K \in \text{GPoly}$ which is the kernel of $\phi : \mathcal{N} \rightarrow \mathcal{M}$.

To construct the minimal resolution of $\mathcal{M}$, we will consider the map $\psi_0 : (\mathcal{M}/\mathfrak{m}\mathcal{M}) \otimes \mathcal{R} \rightarrow \mathcal{M}$, where $\mathfrak{m}$ is the homogeneous maximal module functor of $\mathcal{R}$ defined as the positively graded part of $\mathcal{R} \in \text{GPoly}$. We will define $D_0 := \mathcal{M}$ and $E_0 := \mathcal{M}/\mathfrak{m}\mathcal{M}$. Then $D_1 := \ker(\phi_0)$, where $\phi_0$ is the $\mathcal{R}$-module functor morphism arising from a section of $\psi_0$ as discussed above. Inductively, we define $D_i = \ker(\phi_{i-1})$ and $E_i = D_i/\mathfrak{m}D_i$. Then we will let $\psi_i : E_i \rightarrow D_i$ be the section of the quotient map $\psi_i : D_i \rightarrow E_i$ and we will extend $\phi_i$ to a $\mathcal{R}$-module functors map $\phi_i : \mathcal{R} \otimes E_i \rightarrow D_i$. As a result we obtain the following minimal resolution for $\mathcal{M}$:

$$\cdots \rightarrow \mathcal{R} \otimes \bigoplus S^\lambda_m \rightarrow \cdots \rightarrow \mathcal{R} \otimes \bigoplus S^\lambda_{m^\lambda_1} \rightarrow \mathcal{M} \rightarrow 0,$$

where $E_i \cong \bigoplus S^\lambda_{m^\lambda_1}$. 

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Notice that for \( W \in \text{Obj}(\text{Vec}) \) and \( \mathcal{R} = \mathcal{S}(W) \), the resolution for \( \mathcal{M}(W) \) will eventually terminate as by Hilbert’s Syzygy theorem \( \mathcal{D}_i = 0 \) when \( i > \dim(W) \). However, the categorical construction of the resolution of \( \mathcal{M} \) may be infinite.

Given a minimal resolution of \( \mathcal{M} \) constructed as above, we can apply \( \Omega \) to the resolution to obtain
\[
\cdots \to \Omega(\mathcal{R}) \otimes \bigoplus_s S_{\lambda}^{m_s} \to \cdots \to \Omega(\mathcal{R}) \otimes \bigoplus_s S_{\lambda}^{m_0} \to \Omega(\mathcal{M}) \to 0.
\]
In particular, notice that if \( \mathcal{R} = \mathcal{S} \), the symmetric algebra functor, then applying \( \Omega \) to a resolution of the \( \mathcal{S} \)-module functor \( \mathcal{M} \) will result in a resolution of \( \Omega(\mathcal{M}) \), a module over the algebra functor \( \Omega(\mathcal{R}) \cong \bigwedge \).

### 7.3. Castelnuovo-Mumford regularity of modules in \( \text{GPoly} \).

For a finite dimensional graded \( \mathbb{K} \)-vector space \( E = \bigoplus E_d \), we define
\[
\deg(E) := \max\{d : E_d \neq 0\}.
\]
If \( E = \{0\} \), then we define \( \deg(E) = -\infty \). Let \( \mathcal{M} \) be a module functor over \( \mathcal{R} \) for \( \mathcal{M}, \mathcal{R} \) in \( \text{Obj}(\text{GPoly}) \). For every \( V \) in \( \text{Vec} \) we have that \( \mathcal{M}(V) \) is a module, in the usual sense, over the \( \mathbb{K} \)-algebra \( \mathcal{R}(V) \). We have that \( \mathcal{M}(V) \) is \( s \)-regular if \( \deg(\text{Tor}^i(\mathcal{M}(V), \mathbb{K})) \leq s + i \), for all \( i \). In particular, notice that using the minimal resolution constructed above, we have that \( \text{Tor}^i(\mathcal{M}(V), \mathbb{K}) = \mathcal{E}_i(V) \). Thus, we have that \( \mathcal{M}(V) \) is \( s \)-regular if
\[
\deg(\mathcal{E}_i(V)) \leq s + i,
\]
for all \( i \). The *Castelnuovo-Mumford regularity* \( \text{reg}(\mathcal{M}(V)) \) of \( \mathcal{M}(V) \) is the smallest integer \( s \) such that \( \mathcal{M}(V) \) is \( s \)-regular. We define the *regularity* of \( \mathcal{M} \in \text{Obj}(\text{GPoly}) \) to be
\[
\lim_{\dim(V) \to \infty} \text{reg}(\mathcal{M}(V))
\]
if the limit exists.

**Proposition 17.** Let \( \mathcal{M} \) be a module over \( \mathcal{R} \) in \( \text{GPoly} \) with regularity \( d \). Then \( \Omega(\mathcal{M}) \) is a module over \( \Omega(\mathcal{R}) \) with regularity \( d \).

**Proof.** As the module \( \mathcal{M} \) in \( \text{GPoly} \) comes equipped with a multiplication map \( \nu : \mathcal{R} \otimes \mathcal{M} \to \mathcal{M} \), we have that \( \Omega(\nu) : \Omega(\mathcal{R}) \otimes \Omega(\mathcal{M}) \to \Omega(\mathcal{M}) \) equips \( \Omega(\mathcal{M}) \) with the structure of a \( \Omega(\mathcal{R}) \)-module. Consider a minimal resolution for \( \mathcal{M} \) as constructed above:
\[
\cdots \to \mathcal{R} \otimes \mathcal{E}_i \to \cdots \to \mathcal{R} \otimes \mathcal{E}_0 \to \mathcal{M} \to 0.
\]
We can apply \( \Omega \) to the resolution to obtain
\[
\cdots \to \Omega(\mathcal{R}) \otimes \Omega(\mathcal{E}_i) \to \cdots \to \Omega(\mathcal{R}) \otimes \Omega(\mathcal{E}_0) \to \Omega(\mathcal{M}) \to 0.
\]
Notice that for every vector space \( V \), we have that \( \deg(\Omega(\mathcal{E}_i)(V)) \leq \deg(\mathcal{E}_i(V)) \). In fact, it is possible that \( \mathcal{E}_i(V) \neq 0 \) but \( \Omega(\mathcal{E}_i)(V) = 0 \) for \( \dim V \leq i \). However, for \( \dim(V) \) large enough, we have that \( \deg(\Omega(\mathcal{E}_i)(V)) = \deg(\mathcal{E}_i(V)) \). Hence for \( \dim(V) \) large enough, we also have that \( \Omega(\mathcal{E}_i)(V) = \text{Tor}^i(\Omega(\mathcal{M})(V), \mathbb{K}) \). We have that \( \mathcal{M}(V) \) is \( s \)-regular if \( \max_i\{\deg(\mathcal{E}_i(V)) - i\} \leq s \). Thus, for \( \dim(V) \) large enough, we have that
\[
\max_i\{\deg(\mathcal{E}_i(V)) - i\} = \max_i\{\deg(\mathcal{E}_i(V)) - i\} \leq s,
\]
so \( \Omega(\mathcal{M})(V) \) is \( s \)-regular whenever \( \mathcal{M}(V) \) is \( s \)-regular.
Therefore,
\[ d = \text{reg}(\mathcal{M}) = \lim_{\text{dim}(V) \to \infty} \text{reg}(\mathcal{M}(V)) = \lim_{\text{dim}(V) \to \infty} \text{reg}(\Omega(\mathcal{M}))(V) = \text{reg} \Omega(\mathcal{M}). \]

\[ \square \]

8. The Module Functors of a Subspace Arrangement

For \( Y \) a \( \mathbb{K} \)-vector space, a subspace arrangement \( \mathcal{A} = \{ Y_1, \ldots, Y_t \} \) is a collection of linear subspaces in \( Y \). The ideal associated to \( \mathcal{A} \) is the vanishing ideal of the subspace arrangement: \( I_\mathcal{A} = \mathcal{I}(\mathcal{A}) = \mathcal{I}(Y_1 \cup \cdots \cup Y_t) \). Moreover, we can define \( J_\mathcal{A} = \prod_{i} \mathcal{I}(Y_i) = \mathcal{I}(Y_1)\mathcal{I}(Y_2) \cdots \mathcal{I}(Y_t) \)

Let \( W = Y^* \). Then we have that \( I_\mathcal{A}, J_\mathcal{A} \) are ideal in \( \mathbb{K}[Y] = S(W) \).

**Definition 18.** Let \( V \) be any object in \( \text{Vec} \) and let \( Z = V^* \). In the polynomial ring \( S(W \otimes V) \) we define \( \mathcal{I}_\mathcal{A}(V) \) to be the vanishing ideal of the subspace arrangement \( \mathcal{A} \otimes Z \), i.e.,

\[ \mathcal{I}_\mathcal{A}(V) = \mathcal{I}(Y_1 \otimes Z \cup \cdots \cup Y_t \otimes Z). \]

Moreover, we define \( \mathcal{J}_\mathcal{A}(V) \) to be the product ideal

\[ \mathcal{J}_\mathcal{A}(V) = \mathcal{I}(Y_1 \otimes Z)\mathcal{I}(Y_2 \otimes Z) \cdots \mathcal{I}(Y_t \otimes Z). \]

For any subspace arrangement \( \mathcal{A} \), we can now construct the module functors \( \mathcal{I}_\mathcal{A}, \mathcal{J}_\mathcal{A} \) in \( \text{GPoly} \) for the algebra functor \( S(W \otimes -) \). For any object \( V \) in \( \text{Vec} \), we have already defined \( \mathcal{I}_\mathcal{A}(V) \) and \( \mathcal{J}_\mathcal{A}(V) \). Notice that for every vector space \( V \), we have that \( \mathcal{I}_\mathcal{A}(V), \mathcal{J}_\mathcal{A}(V) \) are homogeneous ideals in \( S(W \otimes V) \) so that can define monomorphisms \( \mathcal{I}_\mathcal{A}, \mathcal{J}_\mathcal{A} : S(W \otimes -) \rightarrow S(W \otimes V) \). Thus for every \( d \), we have that \( (\mathcal{I}_\mathcal{A})_d, (\mathcal{J}_\mathcal{A})_d \) are polynomial functors of degree \( d \) giving \( \mathcal{I}_\mathcal{A} \) and \( \mathcal{J}_\mathcal{A} \) the structure of objects in \( \text{GPoly} \).

To show that \( \mathcal{I}_\mathcal{A} \) and \( \mathcal{J}_\mathcal{A} \) are module functors in \( \text{GPoly} \), as they inherit a multiplication map from \( S(W \otimes -) \), we only need to define \( \mathcal{I}_\mathcal{A}, \mathcal{J}_\mathcal{A} \) on \( \text{Hom}(\text{Vec}) \). For ease of notation, will proceed with the definition for \( \mathcal{I}_\mathcal{A} \), but the same construction works for \( \mathcal{J}_\mathcal{A} \). In fact, the reader may substitute \( \mathcal{J}_\mathcal{A} \) for \( \mathcal{I}_\mathcal{A} \) in the following paragraphs without affecting the results. Let \( f : V_1 \rightarrow V_2 \) be in \( \text{Hom}(\text{Vec}) \). Then also \( \text{Id} \otimes f : W \otimes V_1 \rightarrow W \otimes V_2 \) is in \( \text{Hom}(\text{Vec}) \). As \( S(W \otimes -) \) is an object in \( \text{GPoly} \), we have \( S(\text{Id} \otimes f) : S(W \otimes V_1) \rightarrow S(W \otimes V_2) \) in \( \text{Hom} (\text{GVec}) \).

**Proposition 19.** For any \( f \in \text{Hom}(\text{Vec}) \), we have that \( \mathcal{I}_\mathcal{A}(f) \) defined as \( S(\text{Id} \otimes f) |_{\mathcal{I}_\mathcal{A}(V_1)} \) is an element in \( \text{Hom}(\text{GVec}) \) such that

\[ S(\text{Id} \otimes f) |_{\mathcal{I}_\mathcal{A}(V_1)} : \mathcal{I}_\mathcal{A}(V_1) \rightarrow \mathcal{I}_\mathcal{A}(V_2). \]

Moreover, this shows that \( \mathcal{I}_\mathcal{A} \) is a module functor in \( \text{GPoly} \).

**Proof.** Notice that since \( \mathcal{I}_\mathcal{A}(V_1) \) is an ideal in \( S(W \otimes V_1) \), we can restrict \( S(\text{Id} \otimes f) \) to \( \mathcal{I}_\mathcal{A}(V_1) \). Moreover, we have that in degree one \( S(\text{Id} \otimes f)_1 = \text{Id} \otimes f \) and that \( \mathcal{I}_\mathcal{A}(V) \) is a linear ideal for any vector space \( V \) meaning that \( \mathcal{I}_\mathcal{A}(V) \) is generated in degree one. Thus, we only need to show that the linear generators of \( \mathcal{I}_\mathcal{A}(V_1) \) get mapped by \( S(\text{Id} \otimes f)_1 = \text{Id} \otimes f \) to the linear generators of \( \mathcal{I}_\mathcal{A}(V_2) \) and that \( S(\text{Id} \otimes f) \) is an algebra homomorphism, so that it maps ideals to ideals.

In general, for any \( g \in \text{Hom}(\text{Vec}) \) by our definition of an algebra functor \( \mathcal{R} \) in \( \text{GPoly} \) we have that \( \mathcal{R}(g) \) is an algebra homomorphism. In particular, \( S(\text{Id} \otimes f) \) is algebra homomorphism.
With regards to the generators of $I_A(V_1)$, let $w \in I_A$, so that $w = 0$ on $A$. Then for any $v \in V$ we have that $w \otimes v$ is a linear generator of $I_A(V)$. Moreover, if $w \otimes v_1 \in I_A(V_1)$ then $I_A(f)(w \otimes v_1) = w \otimes f(v_1)$ is a linear generator of $I_A(V_2)$ as $v_2 = f(v_1) \in V_2$ and $w \in I_A$.

Finally, as $I_A(f)$ is the restriction of the functorial map $S(Id \otimes f)$, we have that $I_A(f)$ is itself functorial.

Using the construction above, we can establish our main result.

**Theorem 20.** For any subspace arrangement $A$ of size $t$ consider the module functor $I_A$ in $\text{GPoly}$. For any vector space $V$, we have that $\Omega(I_A)(V)$ is a $t$-regular $\text{GL}(V)$-equivariant ideal in $\Omega(S(W \otimes V)) = \bigwedge(W \otimes V)$.

*Proof.* Derksen and Sidman proved in [2] that $I_A$ is $t$-regular and this implies that $I_A(V)$ is also $t$-regular. We have previously seen that $\Omega(S(W \otimes V)) = \bigwedge(W \otimes V)$ and that the image under $\Omega$ of a module functor is a module functor. Furthermore, $\Omega$ is an exact functor on $\text{GPoly}$ so that monomorphisms are sent under $\Omega$ to monomorphisms. Thus $\Omega(I_A)(V)$ is an ideal in $\bigwedge(W \otimes V)$. Finally, we have already established that $\Omega$ preserves the regularity of a module functor. Therefore, for every vector space $V$, we have that $\Omega(I_A)(V)$ is $t$-regular. \[\blacksquare\]

The same result holds for $\Omega(J_A)(V)$ as Conca and Herzog showed in [1] that the product ideal $J_A$ is $t$-regular and this implies that $J_A(V)$ is also $t$-regular for all $V$. Moreover, Theorem 20 also applies to a more general class of ideals constructed from linear ideals. See [3] for a description of this class of ideals in the symmetric algebra. Using Theorem 20 we can establish that the same class of ideals in the exterior algebra has the same regularity bound.

Let us characterize the ideal $\Omega(J_A)(V)$ in the exterior algebra $\bigwedge(W \otimes V)$.

**Proposition 21.** Let $A$ be the a subspace arrangement of cardinality $t$ and let $J_i$ be the vanishing ideal of the $i$-th subspace in $A$. We have that for every finite dimensional vector space $V$

$$\Omega(J_A)(V) = J_1(V) \wedge J_2(V) \wedge \cdots \wedge J_t(V).$$

*Proof.* Notice that $J_A(V) = J_1(V)J_2(V) \cdots J_t(V)$, using the multiplication structure of the symmetric algebra $S(W \otimes V)$. The functor $\Omega$ maps the multiplication map of $S$ to a multiplication map in $\Omega(S)$. Up to scalars, there is a unique $\text{GL}(V)$-equivariant multiplication in $\bigwedge$, namely the multiplication given by $\wedge$. \[\blacksquare\]

Using the proposition above, we can reformulate Theorem 20 in a more general statement.

**Theorem 22.** The wedge product of $t$ linear ideals in the exterior algebra is $t$-regular.

*Proof.* Every linear ideal $J_i$ is a vanishing ideal of a subspace $W_i$. Consider the subspace arrangement $A$ given by a set of linear ideals. Applying Theorem 20 we conclude that the associate module functor $\Omega(J_A)(V)$ in $\bigwedge(W \otimes V)$ is $t$-regular. Using Proposition 21 for $V$ a 1-dimensional vector space, we conclude that the wedge product of the linear ideals is $t$-regular. \[\blacksquare\]

9. **Equivariant Hilbert series and examples**

Let $s_\lambda(x_1, \ldots, x_n)$ be the symmetric function which is the character of the irreducible representation $S_\lambda(V)$ where $V$ is an $n$-dimensional vector space. As a result of the properties
of $\Omega$, we have that the character of $\Omega(S(V))$ is $s_{\lambda}(x_1, \ldots, x_n)$. For a polynomial functor $F$, we consider the symmetric function $H^e(F)$ such that $H^e(F)(V)$ is the character of $F(V)$ as a representation of $GL(V)$. We refer to $H^e(F)$ as the equivariant Hilbert series of the polynomial functor $F$. We have the following result.

**Theorem 23.** Let $A$ be a subspace arrangement, $J_A(V)$ be its associated $GL(V)$-equivariant ideal in the symmetric algebra, and $\Omega(J_A)(V)$ be the respective ideal in the exterior algebra. We have that

$$H^e(\Omega(J_A)(V)) = \omega(H^e(J_A(V))),$$

where $\omega$ is the involution on the ring of symmetric functions sending $s_{\lambda}$ to $s_{\lambda'}$.

**Proof.** Consider an equivariant resolution of $J_A$. We can read off $H^e(J_A)$ from the resolution as discussed by Derksen in [4]. Apply $\Omega$ to the resolution. Each Schur functor $S_{\lambda}$ is mapped to $S_{\lambda'}$. The effect on $H^e(J_A)$ is to change each $s_{\lambda}$ to $s_{\lambda'}$. Thus the new equivariant Hilbert series is $\omega(H^e(J_A))$. However we now have a resolution of $\Omega(J_A)$, so $\omega(H^e(J_A))$ is its equivariant Hilbert series. □

Consequently, an equivariant Hilbert series of $\Omega(J_A)$ can immediately be obtained from an equivariant Hilbert series of $J_A$. As we have a recursive combinatorial for $H^e(J_A)$ from [4], we can find write down a resolution for $J_A$ and consequently a resolution for $\Omega(J_A)$ from its equivariant Hilbert series $\omega(H^e(J_A))$.

**9.1. Example: powers of the maximal ideal.** Consider the subspace arrangement $A = \{Y_1, \ldots, Y_t\} \subset \mathbb{K}$ with all $Y_i = \{0\}$. This subspace arrangement is $t$ copies of the zero dimensional subspace in a vector space $Y$ of dimension one. The equivariant product ideal of this subspace arrangement is the $t$-th power of maximal ideal $\mathcal{M}(V) = (x_1, \ldots, x_n)$ for $V$ an $n$-dimensional vector space. So we have $\mathcal{M}^t(V)$ in $S(W \otimes V) \cong S(V)$. From [2] we know that $\mathcal{M}^t(V)$ is $t$-regular and from Theorem [20] we can establish that $\Omega(\mathcal{M}^t)(V)$ is also $t$-regular.

Using the combinatorial formula in [4] for $H^e(\mathcal{M}^t)$ we get a minimal resolution for $\mathcal{M}^t$. Let $\sigma = \sum_i s_i = 1 + s_1 + s_2 + \ldots$ and notice that $H^e(S) = \sigma$. For $t = 1$, we get that $H^e(\mathcal{M}) = \sigma - 1$, yielding the Koszul resolution for $\mathcal{M}$:

$$\cdots \to S \otimes S_{(1,1,1)} \to S \otimes S_{(1,1)} \to S \otimes S_{(1)} \to \mathcal{M} \to 0,$$

and the following Koszul resolution for $\Omega(\mathcal{M})(V)$:

$$\cdots \to \wedge \otimes S_{(3)} \to \wedge \otimes S_{(2)} \to \wedge \otimes S_{(1)} \to \Omega(\mathcal{M}) \to 0.$$

For $t = 2$, we get that $H^e(\mathcal{M}^2) = \sigma - (1 + s_1)$. This gives the following resolution for $\mathcal{M}^2$:

$$\cdots \to S \otimes S_{(2,1,1)} \to S \otimes S_{(2,1)} \to S \otimes S_{(2)} \to \mathcal{M}^2 \to 0,$$

yielding the following resolution for $\Omega(\mathcal{M}^2)$:

$$\cdots \to \wedge \otimes S_{(3,1)} \to \wedge \otimes S_{(2,1)} \to \wedge \otimes S_{(1,1)} \to \Omega(\mathcal{M}^2) \to 0.$$
9.2. Example: generic lines. Consider the subspace arrangement \( A = \{Y_1, \ldots, Y_t\} \subset \mathbb{K}^2 \) with \( Y_i = l_i \), a generic line in \( \mathbb{K}^2 \), for all \( i \). This subspace arrangement is given by \( t \) distinct lines in a vector space \( Y \) of dimension two. We get that

\[
H^e(J_A) = \sigma^2 - t\sigma - p[A],
\]

where \( p[A] \) is a symmetric polynomial of degree less than \( t \) depending on the combinatorics of \( A \).

In particular, for \( t = 2 \) we have the following resolution for \( J_A \):

\[
\cdots \to S \otimes (S_{(2,1)}^2 \oplus S_{(1,1,1)}^2) \to S \otimes (S_{(2)} \oplus S_{(1,1)}) \to J_A \to 0,
\]

yielding the following equivariant resolution for \( \Omega(J_A) \):

\[
\cdots \to \bigwedge \otimes (S_{(2,1)}^2 \oplus S_{(3)}^2) \to \bigwedge \otimes (S_{(1,1)} \oplus S_{(2)}) \to \Omega(J_A) \to 0.
\]

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