MacWilliams-type equivalence relations

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Abstract

Let \( \mathcal{P} \) be a poset on \([n] \), \( \mathcal{I}(\mathcal{P}) \) the set of order ideals of \( \mathcal{P} \) and \( E \) an equivalence relation on \( \mathcal{I}(\mathcal{P}) \). The concepts of the dual relation \( E^* \) of an equivalence relation \( E \), the \( E \)-weight (resp. \( E^* \)-weight) distribution of a linear poset code (resp. its dual poset code) and a MacWilliams-type equivalence relation are introduced. We give a characterization for a MacWilliams-type equivalence relation in terms of MacWilliams-type identities for a linear poset code. Three kinds of equivalence relations on \( \mathcal{I}(\mathcal{P}) \) which are of MacWilliams-type are found, i.e., (i) we show that every equivalence re-

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lation defined by the automorphism of $\mathcal{P}$ is a MacWilliams-type; (ii) we provide a new characterization for poset structures when the equivalence relation defined by the same cardinality on $\mathcal{I}(\mathcal{P})$ becomes a MacWilliams-type; (iii) we also give necessary and sufficient conditions for poset structures in which the equivalence relation defined by the order-isomorphism on $\mathcal{I}(\mathcal{P})$ is a MacWilliams-type.

**Keywords:** MacWilliams identity, poset codes, $\mathcal{P}$-weight distribution

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1. Introduction

Let $\mathbb{F}_q^n$ be the vector space of $n$-tuples over a finite field $\mathbb{F}_q$. The space $\mathbb{F}_q^n$ endowed with the Hamming metric is called the Hamming space. Coding theory may be considered as the study of the Hamming space. There are several possible metrics that can be defined on $\mathbb{F}_q^n$ [3, 5, 10, 20, 19]. The ordered Hamming space was first introduced by Niederreiter [16] to study uniform distributions of points in the unit cube, and developed by Rosenbloom and Tsfasman [19]; so the order distance in the ordered Hamming space is sometimes called the NRT-distance. The ordered Hamming space was further generalized by Brualdi et al. [3] to poset spaces on $\mathbb{F}_q^n$ by assigning the coordinate positions of $\mathbb{F}_q^n$ to arbitrary partially ordered sets. The Hamming space and ordered Hamming space are special cases of poset spaces given by anti-chain and the disjoint union of chains with the same length, respectively. The poset spaces have been extensively studied; for instances, the MacWilliams-type identity [2, 4, 9, 12, 14, 17, 21], perfect poset-codes [5, 11], the group of (linear) isometries of the full space [6, 18], and (near) MDS poset codes [1, 7].

One of the most fundamental results in coding theory is the MacWilliams identity on the Hamming space which states that the Hamming weight enumerator of a linear code is uniquely determined by that of its dual code. The MacWilliams identity is contributed to find the maximal subsets of $\mathbb{F}_q^n$ with the given minimum Hamming distance.

There are a number of attempts to derive the MacWilliams-type identity on $\mathbb{F}_q^n$ endowed with poset metrics; for instances, the ordered Hamming space [2, 4, 10, 15] and more generally poset space [9, 12, 17, 21]. Skriganov [21] derived the MacWilliams-type identity on chains with respect a poset weight enumerator. Martin and Stinson [15], and Dougherty and Skriganov [4] derived in different ways the MacWilliams-type identity on ordered
Hamming spaces with respect to a shape enumerator. Kim and Oh \cite{12} classified all poset structures which admit the MacWilliams-type identity on poset spaces and derived the MacWilliams-type identity to such posets with respect a poset weight enumerator.

The preceding discussions lead us to the following natural question:

**Question 1.1.** Is there a unifying way for the known results of MacWilliams-type identities on poset spaces?

The paper is organized to settle Question 1.1 as follows. In Section 2, we introduce some basic concepts and notations on poset codes; the dual relation $E^*$ of an equivalence relation $E$ (Definition 2.1), the $E$-weight (resp. $E^*$-weight) distribution of a poset code (resp. its dual poset) and a MacWilliams-type equivalence relation (Definition 2.7). In Section 3, we give necessary and sufficient conditions for an equivalence relation to be a MacWilliams-type equivalence relation (Theorem 3.3). We also derive the connection between the $E$-weight distribution of a linear poset code and the $E^*$-weight distribution of its dual poset code, called the MacWilliams-type identity. It is in the matrix form whose entries are explicitly formulated (Proposition 3.8). In Section 4, we find equivalence relations of MacWilliams-type (Theorem 4.1), that is, (i) we show that every equivalence relation defined by the automorphism of a poset is a MacWilliams-type; (ii) we provide a new characterization for poset structures established in \cite{12} with the equivalence relation defined by the cardinality on the set of order ideals of a poset; (iii) we show that every equivalence relation defined by the order isomorphism on the set of order ideals of a complement isomorphism poset is a MacWilliams-type and vice versa.

2. Preliminaries: Notations and concepts

In this section, we review on basic definitions and notations for poset spaces, and then define a MacWilliams-type equivalence relation on a poset space.

Let $\mathcal{P}$ be a poset on $[n] := \{1, 2, \ldots, n\}$ with a partial order $\preceq$. An anti-chain is a poset whose any two elements are incomparable. A chain is a poset whose any two elements are comparable. A subset $I$ of $\mathcal{P}$ is an order ideal if $a \in I$ and $b \preceq a$, then $b \in I$. Given a nonempty subset $X$ of $[n]$, we denote $\langle X \rangle_{\mathcal{P}}$ the smallest order ideal containing $X$. 

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Let \( I(\mathcal{P}) \) denote the set of order ideals of \( \mathcal{P} \) and let \( E \) be an equivalence relation on \( I(\mathcal{P}) \). Define the dual poset \( \mathcal{P}^* \) of \( \mathcal{P} \) as follows; \( \mathcal{P} \) and \( \mathcal{P}^* \) have the same underlying set and \( x \preceq y \) in \( \mathcal{P} \) if and only if \( y \preceq x \) in \( \mathcal{P}^* \). It is obvious that the complement \( I^c \) of \( I \) in \( I(\mathcal{P}) \) is also an order ideal of \( \mathcal{P}^* \). Thus there is a one-to-one correspondence between \( I(\mathcal{P}) \) and \( I(\mathcal{P}^*) \). We denote \( I \) (resp. \( I^c \)) the equivalence class of \( I(\mathcal{P}) \) (resp. \( I(\mathcal{P}^*) \)) containing \( I \) (resp. \( I^c \)) with respect to \( E \) (resp. \( E^* \)).

By \( M(I) \) and \( I_M \) for \( I \in I(\mathcal{P}) \) we mean the set of maximal elements of \( I \) and nonmaximal elements of \( I \), respectively. It is obvious that \( I_M \) is also an order ideal of \( \mathcal{P} \).

A permutation \( \sigma \) of \( \mathcal{P} \) is called an automorphism if \( \sigma \) and \( \sigma^{-1} \) preserves the order relation of \( \mathcal{P} \), i.e. \( x \preceq y \) if and only if \( \sigma(x) \preceq \sigma(y) \) for all \( x, y \) in \( \mathcal{P} \). It is easy to see that the set \( \text{Aut}(\mathcal{P}) \) of all automorphisms of \( \mathcal{P} \) forms a group which is called the automorphism group of \( \mathcal{P} \).

The support \( \text{supp}(x) \) and \( \mathcal{P} \)-weight \( w_\mathcal{P}(x) \) of a vector \( x \) in \( F_q^n \) are defined as
\[
\text{supp}(x) = \{i \mid x_i \neq 0\} \text{ and } w_\mathcal{P}(x) = |\langle \text{supp}(x) \rangle_\mathcal{P}|.
\]
The \( \mathcal{P} \)-distance between \( x \) and \( y \) in \( F_q^n \) is defined as
\[
d_\mathcal{P}(x, y) = w_\mathcal{P}(x - y).
\]
It is known [3] that \( d_\mathcal{P} \) is a metric on \( F_q^n \), called a poset metric or a \( \mathcal{P} \)-metric. If \( F_q^n \) is endowed with the \( \mathcal{P} \)-metric, then a (linear) code of \( F_q^n \) is called a (linear) \( \mathcal{P} \)-code.

The following definition plays an important role for deriving the MacWilliams-type identity.

**Definition 2.1.** Let \( \mathcal{P} \) be a poset on \([n]\) and \( E \) an equivalence relation on \( I(\mathcal{P}) \). We say that \( E^* \) is the dual relation on \( I(\mathcal{P}^*) \) of \( E \) if it is satisfied the following property: If \( (I, J) \in E \) is defined by property (A) on \( I(\mathcal{P}) \), then \( (I^c, J^c) \in E^* \) is also defined by property (A) on \( I(\mathcal{P}^*) \).

Definition 2.1 is well-defined because \( E^* \) is an equivalence relation on \( I(\mathcal{P}^*) \) and \( E^{**} = E \).

We now introduce three kinds of equivalence relations on the set of order ideals of a poset. Two of them induce naturally the dual relation but the other does not. See Examples 2.3 and 2.5.
Lemma 2.2. Let \( P \) be a poset on \([n]\) and \( I, J \) in \( \mathcal{I}(P) \).

(i) The relation \( E_C \) on \( \mathcal{I}(P) \) is defined by the rule

\[(I, J) \in E_C \text{ if and only if } |I| = |J|,\]

Then \( E_C \) is an equivalence relation on \( \mathcal{I}(P) \) and the dual relation \( E_C^* \) on \( \mathcal{I}(P^*) \) of \( E_C \) is automatically determined by \(|I^c| = |J^c|\).

(ii) Let \( H \) be a subgroup of \( \text{Aut}(P) \). The relation \( E_H \) on \( \mathcal{I}(P) \) is defined by the rule

\[(I, J) \in E_H \text{ if and only if } \sigma(I) = J \text{ for some } \sigma \in H.\]

Then \( E_H \) is an equivalence relation on \( \mathcal{I}(P) \) and the dual relation \( E_H^* \) on \( \mathcal{I}(P^*) \) of \( E_H \) is automatically determined by \( \sigma(I^c) = J^c \).

(iii) The relation \( E_S \) on \( \mathcal{I}(P) \) is defined by the rule

\[(I, J) \in E_S \text{ if and only if } I \simeq J \text{ as a poset.}\]

Then \( E_S \) is an equivalence relation on \( \mathcal{I}(P) \).

Proof. The proofs are straightforward. \( \square \)

Example 2.3. Let \( P \) be a poset on \([5]\) with the order relation: \( 1 \prec 2 \prec 3 \) and \( 4 \prec 5 \). We see that the set \( \mathcal{I}(P) \) becomes

\[
\begin{array}{ccc}
P & \prec & P \\
\end{array}
\]

\[
\{\emptyset, \{1\}, \{4\}, \{1, 2\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}, \{1, 2, 4, 5\}, P\}.
\]

So,

\[
\mathcal{I}(P)/E_C = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, P\},
\]

\[
\mathcal{I}(P^*)/E_C^* = \{\emptyset^c, \{1\}^c, \{1, 2\}^c, \{1, 2, 3\}^c, \{1, 2, 3, 4\}^c, \overline{P}\}.
\]

Notice here that \( \{1\} = \{\{1\}, \{4\}\} \), \( \{1, 2\} = \{\{1, 2\}, \{1, 4\}, \{4, 5\}\} \), \( \{1, 2, 3\} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 4, 5\}\} \), and \( \{1, 2, 3, 4\} = \{\{1, 2, 3, 4\}, \{1, 2, 4, 5\}\} \). On the other hand, we have

\[
\mathcal{I}(P)/E_S = \{\emptyset, \{1\}, \{1, 2\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}, \{1, 2, 4, 5\}, P\},
\]

\[5\]
\[I(P^*)/E^*_S = \{\emptyset, \{1\}^c, \{1,2\}^c, \{1,4\}^c, \{1,2,3\}^c, \{1,2,4\}^c, \{1,2,3,4\}^c, \{1,2,4,5\}^c, P^c\},\]

where \(\{1\} = \{\{1\}, \{4\}\}, \{1,2\} = \{\{1,2\}, \{4,5\}\}, \text{and} \{1,2,4\} = \{\{1,2,4\}, \{1,4,5\}\}.

We notice that in this dual relation \(E^*_S\) on \(I(P^*)\), every element \((I^c, J^c)\) in \(E^*_S\) is not defined by \(I^c \simeq J^c\) as a poset in \(P^*\) because \((\{1,2\}^c, \{4,5\}^c) \in E^*_S\), but \(\{1,2\}^c\) and \(\{4,5\}^c\) are not isomorphic as a poset in \(P^*\). Thus the dual relation \(E^*_S\) on \(I(P^*)\) does not exist for this poset.

Motivated by Example 2.3, we modify the relation \(E_S\) to give the following definition.

**Definition 2.4.** A poset \(P\) is a complement isomorphism poset if the following condition holds: for any \(I\) and \(J\) in \(I(P)\),

\[I \simeq J \text{ if and only if } I^c \simeq J^c.\]

An example of complement isomorphism posets is given in Figure 1.

![Figure 1](image-url)

**Example 2.5.** Let \(P\) be a poset on \([4]\) with order relation: \(1 < 3\) and \(2 < 4\). Then \(\text{Aut}(P) = \{(1), (12)(34)\}\). We see that

\[I(P)/E_{\text{Aut}(P)} = \{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}, P\},\]

\[I(P^*)/E^*_{\text{Aut}(P)} = \{\emptyset, \{1\}^c, \{1,2\}^c, \{1,3\}^c, \{1,2,3\}^c, P^c\},\]

where \(\{1\} = \{\{1\}, \{2\}\}, \{1,2\} = \{\{1,2\}\}, \{1,3\} = \{\{1,3\}, \{2,4\}\}, \{1,2,3\} = \{\{1,2,3\}, \{1,2,4\}\}, \text{and} P = \{P\}. \text{In this dual relation } E^*_{\text{Aut}(P)} \text{ on } I(P^*), \text{every element } (I^c, J^c) \text{ in } E^*_{\text{Aut}(P)} \text{ is also automatically determined by } \sigma(I^c) = J^c \text{ for some } \sigma \in \text{Aut}(P^*).\]

**Remark 2.6.** Let \(P\) be a poset on \([n]\). Then

(i) \(E_{\text{Aut}(P)} \subseteq E_S \subseteq E_C\).

(ii) If \(P\) is hierarchical (ordinal sum of anti chains), then the equalities hold.

(iii) The equalities do not hold in general.

To see (iii), let \(P\) be the poset defined in Example 2.3. Put \(G = \text{Aut}(P)\). It follows from \(\text{Aut}(P) = \{1_P\}\) that \(E_G = \{(I, I) \mid I \in I(P)\}\). Since \((\{1\}, \{4\}) \in E_S \text{ and } (I, I) \in E_S \text{ for } I \in I(P), \text{ we have } E_G \subseteq E_S. \text{ Since } ((1,2), (1,4)) \notin E_S \text{ and } |I| = |J| \text{ for } (I, J) \in E_S, \text{ we have } E_S \subseteq E_C.\)
Let $I$ be an order ideal of a poset $\mathcal{P}$ on $[n]$. We define the $I$-sphere $S_I(x)$ and the $I^\perp$-sphere $S_{I^\perp}(x)$ of $\mathbb{F}_q^n$ centered at $x$ in $\mathbb{F}_q^n$ as follows:

\[
S_I(x) = \{ y \in \mathbb{F}_q^n \mid \langle \mathrm{supp}(x-y) \rangle_{\mathcal{P}} = I \}, \\
S_{I^\perp}(x) = \{ y \in \mathbb{F}_q^n \mid \langle \mathrm{supp}(x-y) \rangle_{\mathcal{P}^*} = I^\perp \}.
\]

We also define the $\mathcal{I}$-sphere $S_{\mathcal{I},E}(x)$ and $\mathcal{I}^\perp$-sphere $S_{\mathcal{I}^\perp,E^*}(x)$ centered at $x$ with respect to $E$ and $E^*$ as follows:

\[
S_{\mathcal{I},E}(x) = \{ y \in \mathbb{F}_q^n \mid \langle \mathrm{supp}(x-y) \rangle_{\mathcal{P}} \cap E \in \mathcal{I} \}, \\
S_{\mathcal{I}^\perp,E^*}(x) = \{ y \in \mathbb{F}_q^n \mid \langle \mathrm{supp}(x-y) \rangle_{\mathcal{P}^*} \cap E^* \in \mathcal{I}^\perp \}.
\]

One can easily verify that

\[
S_{\mathcal{I},E}(x) = \bigcup_{J \in \mathcal{I}} S_J(x) \text{ and } S_{\mathcal{I}^\perp,E^*}(x) = \bigcup_{J \in \mathcal{I}^\perp} S_J(x),
\]

where the union is disjoint. For the sake of simplicity, we will write $S_I$ and $S_{I^\perp}$ (resp. $S_{\mathcal{I},E}$ and $S_{\mathcal{I}^\perp,E^*}$) instead of $S_I(0)$ and $S_{I^\perp}(0)$ (resp. $S_{\mathcal{I},E}(0)$ and $S_{\mathcal{I}^\perp,E^*}(0)$), where $0$ is the zero vector.

Let $\mathcal{C}$ be a $\mathcal{P}$-code in $\mathbb{F}_q^n$. We define

\[
A_{\mathcal{I},E}(\mathcal{C}) := |S_{\mathcal{I},E} \cap \mathcal{C}| = \sum_{J \in \mathcal{I}} |S_J \cap \mathcal{C}| \text{ and } W(\mathcal{C}, \mathcal{P}, E) := [A_{\mathcal{I},E}(\mathcal{C})]_{\mathcal{I} \in \mathcal{I}(\mathcal{P})/E}.
\]

We call $W(\mathcal{C}, \mathcal{P}, E)$ the weight distribution of $\mathcal{C}$ with respect to $E$ (or the $E$-weight distribution of $\mathcal{C}$). In particular, if $\mathcal{P}$ is an anti-chain on $[n]$, then $S_{\mathcal{I},E_{\mathcal{C}}}$ (resp. $S_{\mathcal{I}^\perp,E^*_{\mathcal{C}}}$) is the set of vectors of $\mathbb{F}_q^n$ of Hamming weight $|I|$ (resp. $n - |I|$), and the $E_{\mathcal{C}}$-weight distribution $W(\mathcal{C}, \mathcal{P}, E_{\mathcal{C}})$ of $\mathcal{C}$ is just the Hamming weight distribution of $\mathcal{C}$.

**Definition 2.7.** Let $\mathcal{P}$ be a poset on $[n]$, $E$ an equivalence relation on $\mathcal{I}(\mathcal{P})$ and $E^*$ the dual relation on $\mathcal{I}(\mathcal{P}^*)$ of $E$. An equivalence relation $E$ on $\mathcal{I}(\mathcal{P})$ is a MacWilliams-type if for any linear $\mathcal{P}$-codes $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbb{F}_q^n$.

\[
W(\mathcal{C}_1, \mathcal{P}, E) = W(\mathcal{C}_2, \mathcal{P}, E) \text{ implies } W(\mathcal{C}_1^\perp, \mathcal{P}^*, E^*) = W(\mathcal{C}_2^\perp, \mathcal{P}^*, E^*).
\]

We notice that Definition 2.7 is well-defined because $E$ is a MacWilliams-type equivalence relation on $\mathcal{I}(\mathcal{P})$ if and only if $E^*$ is a MacWilliams-type equivalence relation on $\mathcal{I}(\mathcal{P}^*)$ using the fact that $E^{**} = E$. 


3. Equivalent conditions for a MacWilliams-type equivalence relation

In this section, we give necessary and sufficient conditions for an equivalence relation to be a MacWilliams-type equivalence relation. The MacWilliams-type identities derived by our characterization are presented in the matrix forms, say $P_E$ and $Q_{E^*}$. The entries of $P_E$ and $Q_{E^*}$ are explicitly presented. Moreover, we prove that $P_E$ is a uniquely determined by $Q_{E^*}$, and vice versa.

An additive character $\chi$ of $\mathbb{F}_q$ is a homomorphism from the additive group of $\mathbb{F}_q$ into the multiplicative group of complex numbers of absolute value one \(^\text{(14)}\). Throughout all sections, we denote $\chi$ a nontrivial additive character of $\mathbb{F}_q$.

**Lemma 3.1.** Let $\mathcal{P}$ be a poset on $[n]$, $E$ an equivalence relation on $I(\mathcal{P})$ and $E^*$ the dual relation of $E$. Then for any linear $\mathcal{P}$-code $C$ of $\mathbb{F}_q^n$,

(i) \[ A_{T,E}(C) = \frac{1}{|C^\perp|} \sum_{T' \in I(\mathcal{P}^*)/E^*} \sum_{u \in C^\perp \cap S_{T',E^*}} \sum_{v \in S_{T,E}} \chi(u \cdot v) \text{ for } T \in I(\mathcal{P})/E, \]

(ii) \[ A_{T^*,E^*}(C^\perp) = \frac{1}{|C|} \sum_{T \in I(\mathcal{P})/E} \sum_{u \in C \cap S_{T,E}} \sum_{v \in S_{T^*,E^*}} \chi(u \cdot v) \text{ for } T^* \in I(\mathcal{P}^*)/E^*. \]

Proof. For a linear $\mathcal{P}$-code $C$ in $\mathbb{F}_q^n$, we see that

\[ C = \bigcup_{T \in I(\mathcal{P})/E} C \cap S_{T,E}, \]

where the union is disjoint. It is well-known \(^\text{(14)}\) that for any linear $\mathcal{P}$-code $C$ over $\mathbb{F}_q$,

\[ \sum_{v \in C} \chi(u \cdot v) = \begin{cases} |C| & \text{if } u \in C^\perp, \\ 0 & \text{if } u \notin C^\perp. \end{cases} \quad (1) \]
It follows that for $\overline{I} \in \mathcal{I}(\mathcal{P}^*)/\mathcal{E}^*$,

$$A_{\overline{I}, \mathcal{E}^*}(C^\perp) = \sum_{v \in C^\perp \cap S_{\overline{I}, \mathcal{E}^*}} 1 = \sum_{v \in S_{\overline{I}, \mathcal{E}^*}} \frac{1}{|C|} \sum_{u \in C} \chi(u \cdot v) \quad \text{(by (1))}$$

$$= \frac{1}{|C|} \sum_{u \in C} \sum_{v \in S_{\overline{I}, \mathcal{E}^*}} \chi(u \cdot v)$$

$$= \frac{1}{|C|} \sum_{T \in \mathcal{I}(\mathcal{P})/\mathcal{E}} \sum_{u \in C \cap S_T, \mathcal{E}^*} \sum_{v \in S_{\overline{I}, \mathcal{E}^*}} \chi(u \cdot v).$$

This proves (ii). In the same way, we can obtain (i). \qed

**Corollary 3.2.** Let $\mathcal{P}$ be a poset on $[n]$, $\mathcal{E}$ an equivalence relation on $\mathcal{I}(\mathcal{P})$ and $\mathcal{E}^*$ the dual relation of $\mathcal{E}$. Then for any 1-dimensional linear $\mathcal{P}$-code $C$ of $\mathbb{F}_q^n$ generated by a nonzero vector $u$,

(i) $A_{\overline{I}, \mathcal{E}}(C) = \begin{cases} 1 & \text{if } I = \emptyset, \\ q - 1 & \text{if } u \in S_{\overline{I}, \mathcal{E}}, \\ 0 & \text{otherwise.} \end{cases}$ for $\overline{I} \in \mathcal{I}(\mathcal{P})/\mathcal{E}$,

(ii) $A_{\overline{I}, \mathcal{E}^*}(C^\perp) = \frac{1}{q} \left( |S_{\overline{I}, \mathcal{E}^*}| + (q - 1) \sum_{u \in S_{\overline{I}, \mathcal{E}^*}} \chi(u \cdot v) \right)$ for $\overline{I} \in \mathcal{I}(\mathcal{P}^*)/\mathcal{E}^*$.

**Proof.** Since $C$ is generated by $u$, $C = \{\alpha u \mid \alpha \in \mathbb{F}_q\}$. If $u \in S_{\overline{I}, \mathcal{E}}$, then $\alpha u \in S_{\overline{I}, \mathcal{E}}$ for $\alpha \in \mathbb{F}_q^*$. This proves (i). It follows from Lemma 3.1 that for $\overline{I} \in \mathcal{I}(\mathcal{P}^*)/\mathcal{E}^*$,

$$A_{\overline{I}, \mathcal{E}^*}(C^\perp) = \frac{1}{|C|} \sum_{T \in \mathcal{I}(\mathcal{P})/\mathcal{E}} \sum_{u \in C \cap S_T, \mathcal{E}^*} \sum_{v \in S_{\overline{I}, \mathcal{E}^*}} \chi(u \cdot v)$$

$$= \frac{1}{q} \sum_{\alpha \in \mathbb{F}_q^*} \sum_{v \in S_{\overline{I}, \mathcal{E}^*}} \chi((\alpha u) \cdot v)$$

$$= \frac{1}{q} \left( \sum_{v \in S_{\overline{I}, \mathcal{E}^*}} \chi(0 \cdot v) + \sum_{\alpha \in \mathbb{F}_q^*} \sum_{v \in S_{\overline{I}, \mathcal{E}^*}} \chi((\alpha u) \cdot v) \right)$$

$$= \frac{1}{q} \left( |S_{\overline{I}, \mathcal{E}^*}| + \sum_{\alpha \in \mathbb{F}_q^*} \sum_{v \in S_{\overline{I}, \mathcal{E}^*}} \chi(u \cdot (\alpha v)) \right).$$
Since \( S_{\overline{\mathcal{I}}, \overline{E}^*} = \{ \alpha v \mid v \in S_{\overline{\mathcal{I}}, \overline{E}^*} \} \) for \( \alpha \in \mathbb{F}_q^* \), we have

\[
A_{\overline{\mathcal{I}}, \overline{E}^*}(C^\perp) = \frac{1}{q} \left( |S_{\overline{\mathcal{I}}, \overline{E}^*}| + (q - 1) \sum_{v \in S_{\overline{\mathcal{I}}, \overline{E}^*}} \chi(u \cdot v) \right).
\]

This proves (ii). \( \square \)

We are ready to state equivalent conditions for the MacWilliams-type equivalence relation.

**Theorem 3.3.** Let \( \mathcal{P} \) be a poset on \([n]\), \( E \) an equivalence relation on \( \mathcal{I}(\mathcal{P}) \) and \( E^* \) the dual relation of \( E \). The following statements are equivalent.

(i) \( E \) is a MacWilliams-type equivalence relation on \( \mathcal{I}(\mathcal{P}) \).

(ii) For \( \overline{\mathcal{I}} \in \mathcal{I}(\mathcal{P})/E \) and \( \overline{\mathcal{I}}^* \in \mathcal{I}(\mathcal{P}^*)/E^* \), we have

(a) If \( u \) and \( u' \) are in \( S_{\overline{\mathcal{I}}, \overline{E}^*} \), then

\[
\sum_{v \in S_{\overline{\mathcal{I}}, \overline{E}^*}} \chi(u \cdot v) = \sum_{v \in S_{\overline{\mathcal{I}}, \overline{E}^*}} \chi(u' \cdot v).
\]

(b) If \( v \) and \( v' \) are in \( S_{\overline{\mathcal{I}}, \overline{E}^*} \), then

\[
\sum_{u \in S_{\overline{\mathcal{I}}, \overline{E}^*}} \chi(u \cdot v) = \sum_{u \in S_{\overline{\mathcal{I}}, \overline{E}^*}} \chi(u \cdot v').
\]

(iii) There are matrices \( Q_{E^*} \) and \( P_{E} \) over \( \mathbb{F}_q \) such that for any linear \( \mathcal{P} \)-code \( \mathcal{C} \) in \( \mathbb{F}_q^n \), we have

(a) \( W(\mathcal{C}^\perp, \mathcal{P}^*, E^*) = \frac{1}{|\mathcal{I}|} W(\mathcal{C}, \mathcal{P}, E)Q_{E^*} \).

(b) \( W(\mathcal{C}, \mathcal{P}, E) = \frac{1}{|\mathcal{I}|} W(\mathcal{C}^\perp, \mathcal{P}^*, E^*)P_{E} \).

**Proof.** (i) \( \Rightarrow \) (ii) Suppose an equivalence relation \( E \) on \( \mathcal{I}(\mathcal{P}) \) doesn’t admit either (a) in (iii) or (b) in (ii). Without loss of generality, we assume that there are \( u \) and \( u' \) in \( S_{\overline{\mathcal{I}}, \overline{E}^*} \) such that \( \sum_{v \in S_{\overline{\mathcal{I}}, \overline{E}^*}} \chi(u \cdot v) \neq \sum_{v \in S_{\overline{\mathcal{I}}, \overline{E}^*}} \chi(u' \cdot v) \). Let \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) be 1-dimensional codes of \( \mathbb{F}_q^n \) generated by \( u \) and \( u' \), respectively. It follows from Corollary 3.2 that \( W(\mathcal{C}_1, \mathcal{P}, E) = W(\mathcal{C}_2, \mathcal{P}, E) \) and \( W(\mathcal{C}_1^\perp, \mathcal{P}^*, E^*) \neq W(\mathcal{C}_2^\perp, \mathcal{P}^*, E^*) \). So \( E \) is not a MacWilliams-type equivalence relation on \( \mathcal{I}(\mathcal{P}) \).

(ii) \( \Rightarrow \) (i) Suppose an equivalence relation \( E \) on \( \mathcal{I}(\mathcal{P}) \) admits (a) and (b) in (ii). We claim that for linear \( \mathcal{P} \)-codes \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) in \( \mathbb{F}_q^n \),

\[
W(\mathcal{C}_1, \mathcal{P}, E) = W(\mathcal{C}_2, \mathcal{P}, E)
\]

if and only if \( W(\mathcal{C}_1^\perp, \mathcal{P}^*, E^*) = W(\mathcal{C}_2^\perp, \mathcal{P}^*, E^*) \). Assume that \( W(\mathcal{C}_1, \mathcal{P}, E) = W(\mathcal{C}_2, \mathcal{P}, E) \). Since the equivalence relation \( E \) admits (a) in (ii), the summation \( \sum_{v \in S_{\overline{\mathcal{I}}, \overline{E}^*}} \chi(u \cdot v) \) is a constant for any \( u \in S_{\overline{\mathcal{I}}, \overline{E}^*} \). Put \( p_{E, \overline{\mathcal{I}}} = \sum_{v \in S_{\overline{\mathcal{I}}, \overline{E}^*}} \chi(u \cdot v) \) for \( u \in S_{\overline{\mathcal{I}}, \overline{E}^*} \). If follows from Lemma
that for $j = 1, 2$,

$$A_{\overline{T}, E^*}(C_j) = \frac{1}{|C_j|} \sum_{I \in \mathcal{I}(P)/E^*} \sum_{w \in C_j \cap S_{\overline{T}, E}} \sum_{v \in S_{\overline{T}, E^*}} \chi(w \cdot v)$$

$$= \frac{1}{|C_j|} \sum_{I \in \mathcal{I}(P)/E^*} A_{I, E}(C_j) P_{\overline{T}, \mathcal{T}},$$

(2)

which implies that $W(C_1^+, P^*, E^*) = W(C_2^+, P^*, E^*)$.

By the same argument as above, we can prove the other direction. Thus $E$ is a MacWilliams-type equivalence relation on $\mathcal{I}(P)$.

(ii) $\Rightarrow$ (iii) Suppose an equivalence relation $E$ on $\mathcal{I}(P)$ admits (a) and (b) in (ii). For $\overline{T} \in \mathcal{I}(P)/E$ and $\overline{T}' \in \mathcal{I}(P^*)/E^*$, the summations $\sum_{v \in S_{\overline{T}, E^*}} \chi(u \cdot v)$ and $\sum_{u \in S_{\overline{T'}, E^*}} \chi(u \cdot v)$ are constants for $u \in S_{\overline{T}, E}$ and $v \in S_{\overline{T'}, E^*}$. Define the matrix $P_E$ and $Q_{E^*}$ as follows:

$$P_E = [p_{\overline{T}, \mathcal{T}}] \quad \text{and} \quad Q_{E^*} = [q_{\overline{T'}, \mathcal{T}'}],$$

(3)

where $p_{\overline{T}, \mathcal{T}} = \sum_{v \in S_{\overline{T}, E^*}} \chi(u \cdot v)$ for $u \in S_{\overline{T}, E}$ and $q_{\overline{T'}, \mathcal{T}'} = \sum_{u \in S_{\overline{T'}, E^*}} \chi(u \cdot v)$ for $v \in S_{\overline{T'}, E^*}$. Here $P_E$ is an $|\mathcal{I}(P^*)/E^*| \times |\mathcal{I}(P)/E|$ matrix with rows and columns labelled by the elements of $\mathcal{I}(P^*)/E^*$ and of $\mathcal{I}(P)/E$, respectively, and $Q_E$ is an $|\mathcal{I}(P)/E| \times |\mathcal{I}(P)/E^*|$ matrix with rows and columns labelled by the elements of $\mathcal{I}(P)/E$ and of $\mathcal{I}(P)/E^*$, respectively. If follows from [2] that $W(C_1^+, P^*, E^*) = \frac{1}{|E^*|} W(C, P, E) Q_{E^*}$ for any linear $P$-code $C$ in $\mathbb{F}_q^n$.

In the same way, we can obtain $W(C, P, E) = \frac{1}{|E|} W(C_1^+, P^*, E^*) P_E$ for any linear $P$-code $C$ in $\mathbb{F}_q^n$.

(iii) $\Rightarrow$ (i) Suppose an equivalence relation $E$ on $\mathcal{I}(P)$ admits (a) and (b) in (iii). We claim that for linear $P$-codes $C_1$ and $C_2$ in $\mathbb{F}_q^n$,

$$W(C_1, P, E) = W(C_2, P, E) \quad \text{if and only if} \quad W(C_1^+, P^*, E^*) = W(C_2^+, P^*, E^*).$$

Assume that $W(C_1, P, E) = W(C_2, P, E)$. Since the equivalence relation $E$ admits (a) in (iii), we have

$$W(C_1^+, P^*, E^*) = \frac{1}{|C_1|} W(C_1, P, E) Q_{E^*} = \frac{1}{|C_2|} W(C_2, P, E) Q_{E^*} = W(C_2^+, P^*, E^*).$$

By the same argument as above, we can prove the other direction. Thus $E$ is a MacWilliams-type equivalence relation on $\mathcal{I}(P)$.

\[\square\]
Definition 3.4. Let $\mathcal{P}$ be a poset on $[n]$ and $E$ a MacWilliams-type equivalence relation on $I(\mathcal{P})$. We call the matrix $P_E$ defined in (3) the $P$-matrix with respect to $E$ and call the matrix $Q_{E^*}$ defined in (3) the $Q$-matrix with respect to $E^*$.

From now on, we try to find out formulae for the entries of $P_E$ and $Q_{E^*}$.

Lemma 3.5. Let $\mathcal{P}$ be a poset on $[n]$. For $I \in I(\mathcal{P})$, we have

$$S_I = \left\{ (v_1, v_2, \ldots, v_n) \in \mathbb{F}_q^n \mid v_i \in \begin{cases} \mathbb{F}_q^* & \text{if } i \in M(I), \\ \mathbb{F}_q & \text{if } i \in I_M, \\ \{0\} & \text{if } i \in I^c. \end{cases} \right\}.$$

Proof. From the definition of the $I$-sphere $S_I$, we have

$$S_I = \{ v \in \mathbb{F}_q^n \mid \langle \text{supp}(v) \rangle_P = I \}$$

Since $\langle \text{supp}(v) \rangle_P = I$ if and only if $M(I) \subseteq \text{supp}(v) \subseteq I$, we have the result.

Lemma 3.6. Let $\mathcal{P}$ be a poset on $[n]$. For $I$ and $J$ in $I(\mathcal{P})$, the following statements are equivalent.

(i) $\text{supp}(u) \cap (J^c)_M = \emptyset$ for $u \in S_I$.
(ii) $M(I) \cap (J^c)_M = \emptyset$.
(iii) $I \cap (J^c)_M = \emptyset$.
(iv) $I_M \cap J^c = \emptyset$.

Proof. (i) $\Rightarrow$ (ii) For $u \in S_I$, $M(I) \subseteq \text{supp}(u)$. It follows that $M(I) \cap (J^c)_M \subseteq \text{supp}(u) \cap (J^c)_M$. Hence (i) implies (ii).

(ii) $\Rightarrow$ (iii) Note that $\{ z \in M(I) \mid x \preceq z \text{ in } \mathcal{P} \} \neq \emptyset$ for $x \in I_M$ and $(J^c)_M$ is an order ideal of $\mathcal{P}^*$. If $x \in I_M \cap (J^c)_M$, then $y \in M(I) \cap (J^c)_M$ for $y \in \{ z \in M(I) \mid x \preceq z \text{ in } \mathcal{P} \}$. Hence (ii) implies (iii).

(iii) $\Rightarrow$ (iv) If $x \in I_M \cap J^c$, then $y \in I \cap (J^c)_M$ for $y \in \{ z \in M(I) \mid x \preceq z \text{ in } \mathcal{P} \}$. Hence (iii) implies (iv).

(iv) $\Rightarrow$ (i) Note that $\{ z \in M(J^c) \mid x \preceq z \text{ in } \mathcal{P}^* \} \neq \emptyset$ for $x \in (J^c)_M$. If $x \in \text{supp}(u) \cap (J^c)_M$, then $y \in I_M \cap J^c$ for $y \in \{ z \in M(J^c) \mid x \preceq z \text{ in } \mathcal{P}^* \}$. Hence (iv) implies (i).

We evaluate the sum of characters on the sphere of an order ideal.
Lemma 3.7. Let \( P \) be a poset on \([n]\). For \( I, J \in \mathcal{I}(P) \) and \( u \in S_I \), we have

\[
\sum_{v \in S_{J^c}} \chi(u \cdot v) = \begin{cases} 
(-1)^{|I \cap J^c|}(q-1)^{|M(J^c)| - |I \cap J^c|}q^{|(J^c)_M|} & \text{if } I_M \cap J^c = \emptyset, \\
0 & \text{if } I_M \cap J^c \neq \emptyset.
\end{cases}
\]

Proof. It follows from Lemma 3.5 that

\[
\sum_{v \in S_{J^c}} \chi(u \cdot v) = \sum_{v \in S_{J^c}} \prod_{i=1}^n \chi(u_i v_i)
= \sum_{v \in S_{J^c}} \prod_{i \in M(J^c)} \chi(u_i v_i) \prod_{i \in J} \chi(u_i v_i)
= \prod_{i \in M(J^c)} \sum_{\alpha \in F_q^*} \chi(u_i \alpha) \prod_{i \in J} \chi(\alpha)
= \prod_{i \in M(J^c)} \sum_{\alpha \in F_q^*} \chi(u_i \alpha) \prod_{i \in J} \chi(\alpha)
= \prod_{i \in M(J^c)} \sum_{\alpha \in F_q^*} \chi(u_i \alpha) \prod_{i \in J} \chi(\alpha)
\]

Since \( \sum_{\beta \in F_q} \chi(\alpha \beta) = \begin{cases} q & \text{if } \alpha = 0, \\
0 & \text{if } \alpha \neq 0, \end{cases} \) we have

\[
\sum_{v \in S_{J^c}} \chi(u \cdot v)
= (-1)^{|\supp(u) \cap M(J^c)|}(q-1)^{|\supp(u) \cap M(J^c)|}q^{|\supp(u) \cap (J^c)_M|}
= \begin{cases} 
(-1)^{|\supp(u) \cap M(J^c)|}(q-1)^{|\supp(u) \cap M(J^c)|}q^{|(J^c)_M|} & \text{if } \supp(u) \cap (J^c)_M = \emptyset, \\
0 & \text{if } \supp(u) \cap (J^c)_M \neq \emptyset.
\end{cases}
\]

The result follows from Lemma 3.6. \( \square \)

In the following proposition, the entries of \( P_E \) and \( Q_{E^*} \) are explicitly described.

Proposition 3.8. Let \( P \) be a poset on \([n]\), \( E \) an equivalence relation on \( \mathcal{I}(P) \) and \( E^* \) the dual relation of \( E \). Then the entries of \( P_E \) and \( Q_{E^*} \) are presented as follows:

For \( I, J \in \mathcal{I}(P) \) and \( u \in S_I \), we have

(i) \( p_{I,J} = (q-1)^{|M(J^c)|}q^{|(J^c)_M|} \sum_{K^c \in \mathcal{T}_I, J_M \cap K^c = \emptyset} \left( \frac{-1}{q-1} \right)^{|J \cap K^c|} \) for \( u \in S_{I,E}, \)

(ii) \( q_{I,J} = (q-1)^{|M(J)|}q^{|M(J)|} \sum_{K \in \mathcal{T}_I, J^c_M \cap K = \emptyset} \left( \frac{-1}{q-1} \right)^{|J^c \cap K|} \) for \( v \in S_{I,E^*}, \)
Proof. It follows from Lemma 3.7 that
\[
\sum_{v \in S_{\mathcal{T}^c, E^*}} \chi(u \cdot v) = \sum_{K^c \in \mathcal{T}^c} \sum_{v \in S_{K^c}} \chi(u \cdot v)
\]
\[
= \sum_{K^c \in \mathcal{T}^c, I_M \cap K^c = \emptyset} (-1)^{|I \cap K^c|} (q - 1)^{|M(K^c)| - |I \cap K^c|} q^{|K^c| M}.
\]
From \(|M(K^c)| = |M(J^c)|\) and \(|(K^c)_M| = |(J^c)_M|\) for \(K^c \in \mathcal{T}^c\), we obtain (i). In the same way, we can obtain (ii).

We prove that \(P_E\) is uniquely determined by \(Q_{E^*}\) and vice versa.

**Proposition 3.9.** Let \(\mathcal{P}\) be a poset on \([n]\), \(E\) an equivalence relation on \(\mathcal{I}(\mathcal{P})\) and \(E^*\) the dual relation of \(E\). If \(E\) is a MacWilliams-type equivalence relation on \(\mathcal{I}(\mathcal{P})\), then
\[
\frac{|\mathcal{I}|}{(q - 1)^{|M(J^c)|} q^{|J^c| M} |\mathcal{T}^c, E^*|} = \frac{|\mathcal{J}^c|}{(q - 1)^{|M(J^c)|} q^{|J^c| M} |\mathcal{T}, E^*|}
\]
for \(\mathcal{I} \in \mathcal{I}(\mathcal{P})/E\) and \(\mathcal{J}^c \in \mathcal{I}(\mathcal{P}^*)/E^*\).

Proof. Since \(E\) is a MacWilliams-type equivalence relation on \(\mathcal{I}(\mathcal{P})\), we have
\[
p_{\mathcal{T}^c, E^*} = \sum_{v \in S_{\mathcal{T}^c, E^*}} \chi(u \cdot v) \text{ for } u \in S_{\mathcal{T}, E}\] and \(q_{\mathcal{T}, E^*} = \sum_{u \in S_{\mathcal{T}^c, E^*}} \chi(u \cdot v) \text{ for } v \in S_{\mathcal{T}^c, E^*}\).

It follows from Lemmas 3.6 and Proposition 3.8 that
\[
\frac{p_{\mathcal{T}, \mathcal{J}}}{{(q - 1)^{|M(J^c)|} q^{|J^c| M}}} = \sum_{K^c \in \mathcal{T}^c, I_M \cap K^c = \emptyset} \left( \frac{-1}{q - 1} \right)^{|I \cap K^c|}
\]
\[
= \frac{1}{|\mathcal{I}|} \sum_{L \in \mathcal{I}} \sum_{K^c \in \mathcal{T}^c} \sum_{L \cap (K^c)_M = \emptyset} \left( \frac{-1}{q - 1} \right)^{|I \cap K^c|}
\]
\[
= \frac{|\mathcal{J}^c|}{|\mathcal{I}|} \sum_{L \in \mathcal{I}, L \cap (J^c)_M = \emptyset} \left( \frac{-1}{q - 1} \right)^{|L \cap K^c|}
\]
\[
= \frac{|J^c|}{|\mathcal{I}|} q_{\mathcal{T}, \mathcal{J}^c} \frac{|\mathcal{J}^c|}{|\mathcal{I}|} \frac{q_{\mathcal{T}, \mathcal{J}^c}}{(q - 1)^{|M(J^c)|} q^{|J^c| M}}.
\]
Multiplying \(|\mathcal{I}|\) on both sides, the result follows. □
Example 3.10. Let $\mathcal{P}$ be an antichain on $[n]$ and $E_{\mathcal{C}}$ an equivalence relation on $\mathcal{I}(\mathcal{P})$ defined by the cardinality. We see that

$$S_{\mathcal{T},E_{\mathcal{C}}} = \{ u \in \mathbb{F}_q^n \mid w_\mathcal{P}(u) = |I| \}$$

and

$$S_{\mathcal{J}^c,E_{\mathcal{C}}} = \{ v \in \mathbb{F}_q^n \mid w_\mathcal{P}(v) = |J^c| \},$$

for $\mathcal{T} \in \mathcal{I}(\mathcal{P})/E_{\mathcal{C}}$ and $\mathcal{J}^c \in \mathcal{I}(\mathcal{P}^*)/E^*_{\mathcal{C}}$. It follows from Lemma 6.17 in [14] that

$$p_{\mathcal{T},\mathcal{T}} = P_{|J^c|}(|I|; n)$$

and

$$q_{\mathcal{T},\mathcal{T}} = P_{|I|}(|J^c|; n),$$

where $P_k(x; n) := \sum_{j=0}^k (-1)^j (q-1)^{(n-x)} (q-1)^{(n-x)}$, $k = 0, 1, \ldots, n$, is the Krawtchouk polynomial. Therefore, we have $P_{E_{\mathcal{C}}} = Q_{E^*_{\mathcal{C}}}$. It follows from Proposition 3.9 that

$$\left( \frac{\mathcal{T}}{(q-1)|M(J^c)|q^{(|J^c|)M}|\mathcal{T},\mathcal{T} = \left( \frac{\mathcal{J}^c}{(q-1)|M(I)|q^{(|I|)M}|\mathcal{T},\mathcal{T} \right. \right.$$

Since $|\mathcal{T}| = \binom{n}{|I|}$, $|\mathcal{J}^c| = \binom{n}{|J^c|}$, $M(I) = I$, $M(J^c) = J^c$, and $I_M = (J^c)_M = \emptyset$, we see that

$$\left. \frac{(n)}{(q-1)|J^c|^p_{\mathcal{T},\mathcal{T}} = \left( \frac{(|J^c|)}{(q-1)|I|p_{\mathcal{T},\mathcal{T}}. \right. \right.$$

This coincides with Theorem 5.17 in [14].

4. Three sources of MacWilliams-type equivalence relations

In this section, We provide three kinds of equivalence relations of a MacWilliams-type, that is, equivalence relations defined by the cardinality on the set of order ideals of a poset, the automorphism of a poset and the order isomorphism on $\mathcal{I}(\mathcal{P})$ of a complement isomorphism poset. Moreover we classify posets admitting such equivalence relations to be a MacWilliams-type on $\mathcal{I}(\mathcal{P})$.

Let $f$ and $g$ be functions on the subsets of a finite set $X$. It is known [22] that

$$f(A) = \sum_{B \subseteq A} g(B)$$

for $A \subseteq X$ if and only if $g(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} f(B)$ for $A \subseteq X$,
which is called the Möbius inversion formula.

Now we are ready to state our main result of this section for classifying posets admitting a MacWilliams-type equivalence relation.

**Theorem 4.1.** Let $\mathcal{P}$ be a poset on $[n]$ and $H$ a subgroup of $\text{Aut}(\mathcal{P})$.

(i) $E_H$ is a MacWilliams-type equivalence relation on $\mathcal{I}(\mathcal{P})$.

(ii) The following statements are equivalent.

(a) $\mathcal{P}$ is a hierarchical poset.

(b) $E_C$ is a MacWilliams-type equivalence relation on $\mathcal{I}(\mathcal{P})$.

(c) Two equivalence relations $E_C$ and $E_{\text{Aut}(\mathcal{P})}$ are the same.

(iii) The following statements are equivalent.

(a) $\mathcal{P}$ is a complement isomorphism poset.

(b) $E_S$ is a MacWilliams-type equivalence relation on $\mathcal{I}(\mathcal{P})$.

Proof. (i) Note that $(I, J) \in E_H$ if and only if $(I^c, J^c) \in E_H^*$. For $u$ and $u'$ in $S_{\mathcal{I}, E_H}$, let $I_1 = \langle \text{supp}(u) \rangle_P$ and $I_2 = \langle \text{supp}(u') \rangle_P$. There is an automorphism $\sigma$ in $H$ such that $\sigma(I_1) = I_2$. Let $J \in \mathcal{I}(\mathcal{P})$. It follows from Proposition 3.8 that

$$\sum_{v \in S_{\mathcal{I}, E_H}} \chi(u \cdot v) = (q - 1)^{|M(J^c)|} q^{|J^c|} M \sum_{K^c \in \overline{J^c}, (I_1)^c \cap K^c = \emptyset} \left( \frac{-1}{q - 1} \right)^{|I_1 \cap K^c|}.$$

It can be easily checked that for $A, B \subseteq \mathcal{P}$, we obtain $\sigma(A \cap B) = \sigma(A) \cap \sigma(B)$ for all $\sigma \in \text{Aut}(\mathcal{P})$ and $\text{Aut}(\mathcal{P}^*) = \text{Aut}(\mathcal{P})$. It then follows that

$$\sum_{v \in S_{\mathcal{I}, E_H}} \chi(u \cdot v) = (q - 1)^{|M(J^c)|} q^{|J^c|} \sum_{\sigma(K^c) \in \overline{J^c}, \sigma((I_1)^c \cap K^c) = \emptyset} \left( \frac{-1}{q - 1} \right)^{|\sigma(I_1 \cap K^c)|}$$

$$= (q - 1)^{|M(J^c)|} q^{|J^c|} \sum_{\sigma(K^c) \in \overline{I_2^c}, \sigma(K^c) = \emptyset} \left( \frac{-1}{q - 1} \right)^{|I_2 \cap \sigma(K^c)|}$$

$$= \sum_{v \in S_{\mathcal{I}, E_H}^*} \chi(u' \cdot v).$$

This proves Theorem 3.3 (ii) (a). Note that $(I^c, J^c) \in E_H^*$ if and only if $\sigma(I^c) = J^c$ for some $\sigma \in H$. Theorem 3.3 (ii) (b) is proved in the same argument as above. This proves part (i)

(ii) (a) $\Rightarrow$ (b) Since $\mathcal{P}$ is a hierarchical poset, there is an automorphism $\sigma$ in
Let \( P \) be a hierarchical poset. From the structure of \( P \), it is easily shown that for \( I \) and \( J \in \mathcal{I}(P) \), \(|I| = |J|\) if and only if there is an element \( \sigma \) in \( \text{Aut}(\mathcal{P}) \) satisfying \( \sigma(I) = J \). Hence (a) implies (c). (b) \( \Rightarrow \) (a) It follows from Theorem 2.5 in [12]. (c) \( \Rightarrow \) (b) It follows from Theorem 4.1

(iii) (a) \( \Rightarrow \) (b) Suppose \( P \) is a complement isomorphism poset. For \( u, u' \in S_\mathcal{I,E} \), there exists an order isomorphism satisfying \( \sigma((\text{supp}(u))_P) = (\text{supp}(u'))_P \).

Put \( I_1 = (\text{supp}(u))_P \) and \( I_2 = (\text{supp}(u'))_P \). It follows from Lemma 3.8 and Proposition 3.8 that for \( J \in \mathcal{I}(\mathcal{P}^*)/E_S^* \),

\[
\sum_{v \in S_{Jc,E}^*} \chi(u \cdot v) = (q-1)^{|M(J^c)|} q^{|(J^c)_M|} \sum_{K^c \in J^c,I_1 \cap K^c \subseteq M(I_1)} \left( -\frac{1}{q-1} \right)^{|I_1 \cap K^c|}.
\]

Replacing \( I_1 \cap K^c \) by \( A \), we have

\[
\sum_{v \in S_{Jc,E}^*} \chi(u \cdot v) = (q-1)^{|M(J^c)|} q^{|(J^c)_M|} \sum_{A \subseteq M(I_1)} \left( -\frac{1}{q-1} \right)^{|A|} \sum_{K^c \in J^c,I_1 \cap K^c = A} 1.
\]

Applying the Möbius inversion formula, we obtain

\[
\frac{\sum_{v \in S_{Jc,E}^*} \chi(u \cdot v)}{(q-1)^{|M(J^c)|} q^{|(J^c)_M|}} = \sum_{A \subseteq M(I_1)} \left( -\frac{1}{q-1} \right)^{|A|} \sum_{B \subseteq A} (-1)^{|A \setminus B|} \sum_{K^c \in J^c,I_1 \cap K^c \subseteq B} 1.
\]

Let \( B \subseteq A \subseteq M(I_1) \). One can easily check that \(|A| = |\sigma(A)|\), \(|A \setminus B| = |\sigma(A) \setminus \sigma(B)|\), and \((I_1 \setminus B,I_2 \setminus \sigma(B)) \in E_S\) since \( \sigma : I_1 \to I_2 \) is an order isomorphism. Since \( P \) is a complement isomorphism poset, \((I_1 \setminus B)^c \simeq (I_2 \setminus \sigma(B))^c\). Hence we have

\[
\sum_{K^c \in J^c,I_1 \cap K^c \subseteq B} 1 = \sum_{K^c \in J^c,K^c \subseteq (I_1 \setminus B)^c} 1 = \sum_{K^c \in J^c,K^c \subseteq (I_2 \setminus \sigma(B))^c} 1 = \sum_{K^c \in J^c,I_2 \cap K^c \subseteq \sigma(B)} 1.
\]
It follows that

\[
\frac{1}{(q - 1)^{|M(J^c)| q^{|J^c| \cdot M|}}} \sum_{u \in \mathcal{J}^c, E_S^*} \chi(u \cdot v)
\]

\[=
\sum_{A \subseteq M(I_1)} \left( \frac{-1}{q - 1} \right)^{|A|} \sum_{B \subseteq A} (-1)^{|A \setminus B|} \sum_{K^c \in \mathcal{J}^c, I_1 \cap K^c \subseteq B} 1
\]

\[=
\sum_{\sigma(A) \subseteq M(I_2)} \left( \frac{-1}{q - 1} \right)^{|\sigma(A)|} \sum_{\sigma(B) \subseteq \sigma(A)} (-1)^{|\sigma(A) \setminus \sigma(B)|} \sum_{K^c \in \mathcal{J}^c, I_2 \cap K^c \subseteq \sigma(B)} 1
\]

\[=
\frac{1}{(q - 1)^{|M(J^c)| q^{|J^c| \cdot M|}}} \sum_{u' \in \mathcal{J}^c, E_S^*} \chi(u' \cdot v).
\]

This proves Theorem 3.3 (ii) (a). Since Theorem 3.3 (ii) (b) can be proved in the same way, the result follows.

(b) ⇒ (a) Suppose \( \mathcal{P} \) is not a complement isomorphism poset. Then there are \( I_1 \) and \( I_2 \) on \( \mathcal{I}(\mathcal{P}) \) such that \( I_1 \simeq I_2 \) and \( I_1^c \neq I_2^c \). Let \( C_1 \) and \( C_2 \) be linear codes of \( \mathbb{F}_q^n \) such that

\[ C_i = \{ x \in \mathbb{F}_q^n \mid \text{supp}(x) \subseteq I_i \}, \quad i = 1, 2. \]

It follows that \( W(C_1, \mathcal{P}, E_S) = W(C_2, \mathcal{P}, E_S) \). The dual codes \( C_1^\perp \) and \( C_2^\perp \) are given by

\[ C_i^\perp = \{ x \in \mathbb{F}_q^n \mid \text{supp}(x) \subseteq I_i^c \}, \quad i = 1, 2. \]

From Lemma 3.3 we have \( A_{\mathcal{I}(\mathcal{P}), E_S^*}(C_i^\perp) = (q - 1)^{|M(I_i^c)| q^{|I_i^c| \cdot M|}} \). Note that \( |I_1^c| = |J^c| \) because \( I_1^c \simeq J^c \). If \( x \in C_2^\perp \) such that \( \langle \text{supp}(x) \rangle_{\mathcal{P}^*} \simeq I_1^c \), then \( |\langle \text{supp}(x) \rangle_{\mathcal{P}^*}| = |I_1^c| = |I_1^c| \). It then follows from \( \langle \text{supp}(x) \rangle_{\mathcal{P}^*} \subseteq I_2^c \) that \( I_2^c = \langle \text{supp}(x) \rangle_{\mathcal{P}^*} \simeq I_1^c \), which is a contradiction to the fact that \( I_1^c \neq I_2^c \). It follows that \( W(C_1^\perp, \mathcal{P}^*, E_S^*) \neq W(C_2^\perp, \mathcal{P}^*, E_S^*) \). Therefore, we have the result.

\[\square\]

**Corollary 4.2.** Let \( \mathcal{P} \) be a poset on \( [n] \) and \( H \) a subgroup of \( \text{Aut}(\mathcal{P}) \). For \( \mathcal{T} \in \mathcal{I}(\mathcal{P})/E_H \) and \( J^c \in \mathcal{I}(\mathcal{P}^*)/E_{H^*}^c \), we have

\[ \frac{|\{ \sigma \in H \mid \sigma(J^c) = J^c \}|}{(q - 1)^{|M(J^c)| q^{|J^c| \cdot M|}}} = \frac{|\{ \sigma \in H \mid \sigma(I) = I \}|}{(q - 1)^{|M(I)| q^{|I| \cdot M|}}} \]
Proof. From Theorem 4.1, the equivalence relation $E_H$ on $\mathcal{I}(\mathcal{P})$ is a MacWilliams-type. It follows from Proposition 3.9 that

\[
\frac{|\overline{I}|}{(q - 1)^{|M(J^c)|}q^{|I^c|}_M} = \frac{|\overline{J^c}|}{(q - 1)^{|M(I^c)|}q^{|J^c|}_M},
\]

for $\overline{I} \in \mathcal{I}(\mathcal{P})/E_H$ and $\overline{J^c} \in \mathcal{I}(\mathcal{P}^*)/E_{H^*}$. Since $H = |\overline{I}| \{\sigma \in H \mid \sigma(I) = I\} = |\overline{J^c}| \{\sigma \in H \mid \sigma(J^c) = J^c\}$, the result follows.

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