Archimedean theory and $\epsilon$-factors for the Asai Rankin-Selberg integrals

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Abstract

In this paper, we partially complete the local Rankin-Selberg theory of Asai $L$-functions and $\epsilon$-factors as introduced by Flicker and Kable. In particular, we establish the relevant local functional equation at Archimedean places and prove the equality between Rankin-Selberg’s and Langlands-Shahidi’s $\epsilon$-factors at every place. Our proofs work uniformly for any characteristic zero local field and use as only input the global functional equation and a globalization result for a dense subset of tempered representations that we infer from work of Finis-Lapid-Müller. The results of this paper are used in [Beu] to establish an explicit Plancherel decomposition for $\text{GL}_n(F) \backslash \text{GL}_n(E)$, $E/F$ a quadratic extension of local fields, with applications to the Ichino-Ikeda and formal degree conjecture for unitary groups.

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1 Introduction

The goal of this paper is to partially complete the local Rankin-Selberg theory of Asai L-functions and ε-factors as introduced by Flicker [Fli] and Kable [Kab]. In particular we establish the relevant local functional equation and prove the equality between Rankin-Selberg’s and Artin’s ε-factors (which are the same as Langlands-Shahidi’s ε-factors by the recent paper [Shank]) in full generality (both for Archimedean and non-Archimedean fields). This will be used in the paper [Beu] to establish an explicit Plancherel decomposition for $GL_n(F) \backslash GL_n(E)$, $E/F$ a quadratic extension of local fields, with applications to the Ichino-Ikeda and formal degree conjecture for unitary groups.

Recall that the now classical Rankin-Selberg local theory for tensor L-functions and ε-factors has been developed by Jacquet-Piatetskii-Shapiro-Shalika, Jacquet-Shalika and Jacquet in [JPSS], [JS2] and [Jac]. The main results of those references roughly say that we can define local L-functions of pairs as the "greatest common divisor" (in some loose sense) of certain families of Zeta integrals and local ε-factors of pairs through certain functional equations satisfied by those families. Moreover, it is one of the characterizing properties of the local Langlands correspondence of [HT], [Hen] and [Sch] that in the p-adic case these so defined local L- and ε-factors match the Artin L- and ε-factors on the Galois side. In the Archimedean case, the Langlands correspondence is characterized by other means [La] and it is a result of Jacquet and Shalika [JS2], [Jac] that Artin L-functions of pairs can indeed be considered as the “greatest common divisor" of the relevant family of Zeta integrals and moreover that those satisfy the correct functional equation with respect to Artin ε-factors of pairs.

In [Fli], Flicker has introduced in the non-Archimedean case a family of Zeta integrals that ought to represent the Asai L-function of a given generic irreducible (smooth) representation $\pi$ of $GL_n(E)$ where $E$ is a quadratic extension of a non-Archimedean field $F$ (and the Asai L-function is taken with respect to this extension). In particular, he was able to define a
Rankin-Selberg type Asai $L$-function $L^{RS}(s, \pi, As)$ as the greatest common divisor of his family of Zeta integrals as well as a $\epsilon$-factor $\epsilon^{RS}(s, \pi, As, \psi')$ (where $\psi' : F \rightarrow S^1$ is a non-trivial character) through the existence of a functional equation satisfied by the same Zeta integrals. Similar results have been obtained independently by Kable in [Kab]. To be more specific, let $\psi$ be a nontrivial additive character of $E$ which is trivial on $F$ and let $\mathcal{W}(\pi, \psi)$ be the Whittaker model of $\pi$ with respect to the corresponding standard character of the standard maximal unipotent subgroup $N_u(E)$ of $GL_n(E)$. The Zeta integrals defined by Flicker and Kable are associated to functions $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in C_c^\infty(F^n)$ and defined by

$$Z(s, W, \phi) = \int_{N_u(F) \backslash GL_n(F)} W(h)\phi(\epsilon_n h)|\det h|^n \frac{dh}{n!}$$

where $s$ is a complex parameter, $\epsilon_n = (0, \ldots, 0, 1)$ and $|.|_F$ the normalized absolute value on $F$. Flicker and Kable show that this integral converges when the real part of $s$ is large enough and that it is represented by a rational function in $q^{-s}$, $q$ being the order of the residue field of $F$. Moreover, they also prove that the vector space spanned by $\{Z(s, W, \phi) | W \in \mathcal{W}(\pi, \psi), \phi \in C_c^\infty(F^n)\}$ is a fractional ideal for $\mathbb{C}[q^s, q^{-s}]$ generated by an unique element $L^{RS}(s, \pi, As)$ of the form $P(q^{-s})^{-1}$ where $P \in \mathbb{C}[T]$ is such that $P(0) = 1$. The next result of [Fli] and [Kab] is the functional equation. For $W \in \mathcal{W}(\pi, \psi)$ set $\tilde{W}(g) = W(w_n g^{-1})$ for every $g \in GL_n(E)$ where $w_n = \begin{pmatrix} 1 & \cdot & \\ \cdot & \cdot & \cdot \\ 1 & \end{pmatrix}$. Notice that $\tilde{W} \in \mathcal{W}(\tilde{\pi}, \psi^{-1})$ where $\tilde{\pi}$ stands for the contragredient of the representation $\pi$. Let $\psi' : F \rightarrow S^1$ be a nontrivial additive character and $\phi \mapsto \hat{\phi}$ be the usual Fourier transform on $C_c^\infty(F^n)$ defined using the character $\psi'$ and the corresponding autodual Haar measure. Then the functional equation reads as follows: there exists a unique monomial $\epsilon^{RS}(s, \pi, As, \psi')$ in $q^{-s}$ such that

$$\frac{Z(1-s, \tilde{W}, \hat{\phi})}{L^{RS}(1-s, \tilde{\pi}, As)} = \omega_\pi(\tau)^{n-1} |\tau|^\frac{n(n-1)}{2} \lambda_{E/F}(\psi') \cdot \epsilon^{RS}(s, \pi, As, \psi') Z(s, W, \phi)$$

for every $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in C_c^\infty(F^n)$ where $\omega_\pi$ stands for the central character of $\pi$, $\tau \in E$ is the unique element so that $\psi(z) = \psi'(Tr_{E/F}(\tau z))$ for every $z \in E$ (by the assumption made on $\psi$, we have $Tr_{E/F}(\tau) = 0$) and $\lambda_{E/F}(\psi')$ is the Langlands constant of the extension $E/F$ (see Section 3.1 for a reminder). The attentive reader will have noticed that the $\epsilon$-factor $\epsilon^{RS}(s, \pi, As, \psi')$ is normalized differently as in [Fli] and [Kab] (due to the appearance of the factor $\omega_\pi(\tau)^{n-1} |\tau|^\frac{n(n-1)}{2} \lambda_{E/F}(\psi') \cdot \epsilon^{RS}(s, \pi, As, \psi')$). The present normalization will be justified a posteriori by the equality between $\epsilon^{RS}(s, \pi, As, \psi')$ and the Artin $\epsilon$-factor $\epsilon(s, \pi, As, \psi')$.

By the work of Anandavardhanan-Rajan [AR] (for $\pi$ square-integrable) and Matringe [Mat] (for general $\pi$) the $L$-function $L^{RS}(s, \pi, As)$ matches Shahidi’s Asai $L$-function $L^{Sh}(s, \pi, As)$ ([Sha3], [Gold]) and hence by [Hen2] also the corresponding Artin $L$-function $L(s, \pi, As)$ (defined through the local Langlands correspondence). Recently, Anandavardhanan-Kurinczuk-Matringe-Sécherre-Stevens [AKMSS] have also established the equality between $\epsilon^{RS}(s, \pi, As, \psi')$ with the Shahidi’s $\epsilon$-factor $\epsilon^{Sh}(s, \pi, As, \psi')$ when $\pi$ is supercuspidal. By the recent preprint
of Shankman this also gives the equality $\epsilon^{RS}(s, \pi, As, \phi') = \epsilon(s, \pi, As, \psi')$ with the Artin $\epsilon$-factor when $\pi$ is supercuspidal. In this paper, we will complete those results in the characteristic zero case by showing that the previous equality between $\epsilon$-factors holds in general and also by working out the Archimedean theory. Moreover, we will also reproved most of the previous results in the $p$-adic case since our methods can treat uniformly the Archimedean and non-Archimedean case. Our main inputs will be the global functional equation satisfied by the corresponding global Zeta integrals (already established in [FL], [Kab]) as well as a globalization result allowing to realize a dense subset of the tempered dual of $GL_n(E)$ as local constituents of global cuspidal automorphic representations of $GL_n$ with a control on the ramification. We deduce this globalization result from the recent work of Finis-Lapid-Müller [FLM] on limit multiplicities for cuspidal automorphic representations of $GL_n$.

We now describe the main result of this paper. Let $E/F$ be a quadratic extension of local fields of characteristic zero. In the Archimedean case, by a smooth representation of $GL_n(E)$ we will mean a smooth admissible Fréchet representation of moderate growth in the sense of Casselman-W allach [Cas], [Wall2 Sect. 11] or, which is the same, an admissible SF of $GL_n$.

Let $\psi$ be a generic irreducible smooth representation of $GL_n(E)$ and $W(\pi, \psi)$ be its Whittaker model with respect to a fixed nontrivial additive character $\psi : E \to S^1$ which we again take to be trivial on $F$. To $W \in W(\pi, \psi)$ we associate $\widehat{W} \in W(\bar{\pi}, \psi^{-1})$ as before. Let $S(F^n)$ be $C_c^\infty(F^n)$ in the $p$-adic case and the usual Schwartz space in the Archimedean case. We let $\phi \mapsto \tilde{\phi}$ be the usual Fourier transform on $S(F^n)$ defined using a nontrivial additive character $\psi' : F \to S^1$ as before. Let $\tau \in E$ be the unique element so that $\psi(z) = \psi'(\operatorname{Tr}_{E/F}(\tau z))$ for every $z \in E$. Then, for $W \in W(\pi, \psi)$ and $\phi \in S(F^n)$ we define as above, whenever convergent, a Zeta integral $Z(s, W, \phi)$. The main result of this paper can now be stated as follows (see Theorems 3.4.1 and 3.4.2):

**Theorem 1** Let $W \in W(\pi, \psi)$ and $\phi \in S(F^n)$. Then:

(i) The integral defining $Z(s, W, \phi)$ is convergent when the real part of $s$ is sufficiently large and moreover it extends to a meromorphic function on $\mathbb{C}$.

(ii) We have the functional equation

$$Z(1-s, \widehat{W}, \phi) = \omega_{\pi}(\tau)^{-n-1} |\tau|_E^{-\frac{n(n-1)}{2}} \lambda_{E/F}(\psi')^{-\frac{n(n-1)}{2}} \epsilon(s, \pi, As, \psi') Z(s, W, \phi) L(s, \pi, As)$$

where $L(s, \pi, As), L(s, \pi, As)$ and $\epsilon(s, \pi, As, \psi')$ stand for the Artin $L$- and $\epsilon$-Asai factors.

(iii) The function $s \mapsto \frac{Z(s, W, \phi)}{L(s, \pi, As)}$ is holomorphic. Moreover, if $\pi$ is nearly tempered (see Section 3), for every $s_0 \in \mathbb{C}$ we can choose $W \in W(\pi, \psi)$ and $\phi \in S(F^n)$ such that this function does not vanish at $s_0$. 

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One comment is in order: if we expect $L(s, \pi, As)$ to be the “greatest common divisor" of all the Zeta integrals $Z(s, W, \phi)$, then of course the second part of (iii) should be true generally. This is known in the $p$-adic case by the aforementioned result of Matringe but remains to be seen in the Archimedean case for irreducible generic representations $\pi$ which are not nearly tempered in the sense of Section 3.

We now briefly describe the content of each section of this paper. In Part 2 we have gather general results which are not specific to $GL_n$. In Section 2.1 we set up most of our notation for a general connected reductive group. Section 2.2 contains a reminder on properties of topological vector spaces and functions valued in them that we shall use repeatedly in the paper. In Section 2.3 we set up some notation and conventions related to representations of local reductive groups. Section 2.4 contains a probably well-known result giving uniform bounds for matrix coefficients of a family of parabolically induced representations. In lack of a proper reference, we provide a proof. In Section 2.5 we introduce various “extended" Harish-Chandra Schwartz spaces associated to a generic character on a quasi-split local reductive group. These spaces will be the natural receptacle for the Whittaker models of a family of induced representations. Section 2.6 is devoted to establishing locally uniform bounds for analytic family of Whittaker functions living in the Whittaker models of a family of parabolically induced representations. The proof proceeds by a reduction to the case of usual matrix coefficients by “smoothing" the Whittaker functional and is far more technical in the Archimedean case. Similar (point-wise) bounds have been obtained by Wallach [Wall2, Theorem 15.2.5]. Using the result of Section 2.6 and the theory of the Jacquet’s functional, we construct in Section 2.7 certain “good" sections for Whittaker models of a family of parabolically induced representations. The existence of such families will be a crucial ingredient to extend Theorem 1 from a dense subset of the tempered dual to the set of all generic representations. In Section 2.8 we state a result on the automatic holomorphic continuation of certain functions in several complex variables. The proof of this result occupies Sections 2.9 and 2.10. The theory of local Asai Zeta integrals and their functional equations is the object of Part 3 Section 3.1 is a reminder on basic properties of local Asai $L$- and $\epsilon$-factors of Artin type. In Section 3.2 we introduce the relevant Zeta integrals and establish basic convergence results on them. In Section 3.3 we recall the meromorphic continuation and functional equation of these integrals in the split case (i.e. when $E = F \times F$) which is due to Jacquet, Piatetski-Shapiro and Shalika [JPSS], [JS2], [Jac]. In Section 3.4 we state the main theorems of this paper pertaining to the same meromorphic continuation and functional equation but in the inert case (i.e. when $E/F$ is a field extension). Section 3.5 is devoted to the unramified computation of these Zeta integrals which is already in the literature in all but one case (i.e. when $E/F$ is a ramified quadratic extension but $\pi$, $\psi'$ and $\psi$ are unramified). In Section 3.6 we recall the definition of the global Zeta integrals and their functional equation. Section 3.7 contains the aforementioned globalization result due to Finis-Lapid-Müller. The deduction from [FLM] is carefully explained. We also sketch how a similar result (in a slightly weaker form) can be proved for general reductive groups using part of loc. cit.. Finally, Sections 3.8 and 3.9 contain the proof of the main results.
1.1 General notation

In this paper $F$ will always be a local field of characteristic zero. We will denote by $|.|_F$ the normalized absolute value of $F$. In the non-Archimedean case, we let $O_F$ be the ring of integers of $F$ and $q=q_F$ be the cardinality of the residue field of $F$.

For two positive functions $f_1$, $f_2$ on a set $X$ a sentence like

$$f_1(x) \ll f_2(x) \text{ for all } x \in X$$

means that there exists a constant $C > 0$ such that $f_1(x) \leq Cf_2(x)$ for every $x \in X$. When we want to emphasize that the implicit constant depends on some auxiliary parameters $a, b, c...$ we will write “$f_1(x) \ll_{a,b,c...} f_2(x)$ for all $x \in X$”.

For every complex number $z \in \mathbb{C}$, we write $\Re(z)$ and $\Im(z)$ for the real and imaginary parts of $z$ respectively.

2 Preliminaries

2.1 Groups

Let $G$ be a connected reductive group over $F$. We denote by $A_G$ the maximal split torus in the center of $G$ and by $X^*(G)$, $X^*(A_G)$ the groups of algebraic characters of $G$ and $A_G$ respectively. Set

$$A_G^* = X^*(G) \otimes \mathbb{R} = X^*(A_G) \otimes \mathbb{R}, \quad A^*_{G,\mathbb{C}} = X^*(G) \otimes \mathbb{C} = X^*(A_G) \otimes \mathbb{C}$$

and

$$A_G = \text{Hom}(X^*(G), \mathbb{R}), \quad A^*_{G,\mathbb{C}} = \text{Hom}(X^*(G), \mathbb{C})$$

for their duals. We denote by $\langle ., . \rangle$ the natural pairing between $A_G^*$ and $A_G$ (resp. between $A^*_{G,\mathbb{C}}$ and $A^*_{G,\mathbb{C}}$). Let $H_G : G(F) \to A_G$ be the homomorphism characterized by $\langle \chi, H_G(g) \rangle = \log |\chi(g)|_F$ for every $g \in G(F)$ and $\chi \in X^*(G)$. For $\lambda \in A^*_{G,\mathbb{C}}$ we denote by $g \mapsto g^\lambda$ the unramified character of $G(F)$ given by $g^\lambda = e^{\langle \lambda, H_G(g) \rangle}$. Moreover, we let $\Re(\lambda) \in A^*_G$ be the real part of $\lambda$ (i.e. its projection to $A_G^*$ relative to the decomposition $A^*_{G,\mathbb{C}} = A^*_G \oplus i A^*_G$).

Let $P = MN$ be a parabolic subgroup of $G$. Then the restriction map $X^*(A_M) \to X^*(A_G)$ induces surjections $A^*_M \to A^*_G$, $A^*_{M,\mathbb{C}} \to A^*_{G,\mathbb{C}}$ whose kernels will be denoted by $(A^*_M)^*$ and $(A^*_{M,\mathbb{C}})^*$ respectively.

We fix a minimal parabolic subgroup $P_0$ of $G$ with a Levi decomposition $P_0 = M_0N_0$ and we set $A_0 = A_{M_0}$, $A^*_0 = A^*_{M_0}$, $(A^*_0)^* = (A^*_{M_0})^*$, $A_0 = A_{M_0}$, $H_0 = H_{M_0}$. We also choose a maximal compact subgroup $K$ of $G(F)$ which is special in the $p$-adic case and in good position relative to $M_0$. We endow $K$ with its unique Haar measure of total mass one. For every parabolic subgroup $P$ of $G$ we have the Iwasawa decomposition $G(F) = P(F)K$. As usual, by a standard parabolic subgroup we mean a parabolic subgroup of $G$ containing $P_0$. If $P$ is such a standard parabolic subgroup we will always write $P = MN$ for its unique Levi decomposition with $M_0 \subseteq M$ and $\overline{P} = M\overline{N}$ for the opposite parabolic subgroup. The
restriction map \( X^*(M) \to X^*(M_0) \) then induces an embedding \( A^*_M \to A^*_0 \) through which we will always consider \( A^*_M \) as a subspace of \( A^*_0 \). We shall also denote by \( \delta_P \) the modular character of \( P(F) \). By the choice of \( K \), for every standard parabolic subgroup \( P = MN \) we have \( K_P = K_M K_N \) where \( K_P = K \cap P(F), K_M = K \cap M(F) \) and \( K_N = K \cap N(F) \).

Let \( \Delta \subseteq X^*(A_0) \) be the set of simple roots of \( A_0 \) in \( N_0 \) and \( \Delta^0 \subseteq A_0 \) the corresponding subset of simple coroots. We set
\[
\overline{A^*_0} = \{ X \in A_0 \mid \langle X, \alpha \rangle \leq 0 \ \forall \alpha \in \Delta \}
\]
\[
(A^*_0)^\circ = \{ \lambda \in A^*_0 \mid \langle \lambda, \alpha \rangle \leq 0 \ \forall \alpha \in \Delta \}
\]
for the closed negative Weyl chambers in \( A_0 \) and \( A^*_0 \) respectively. We also let
\[
M^+_0 = H_0^{-1}(\overline{A^*_0}) = \{ m_0 \in M_0(F) \mid m_0^\alpha \leq 1, \ \forall \alpha \in \Delta \}.
\]
Let \( W^G = \text{Norm}_{G(F)}(M_0)/M_0(F) \) be the Weyl group of \( M_0 \). Then, \( W^G \) acts on \( A^*_0 \) and \( (A^*_0)^\circ \) is a fundamental domain for this action. For every \( \lambda \in A^*_0 \) we denote by \( |\lambda| \) the unique element in the intersection \( W^G \lambda \cap (A^*_0)^\circ \). We equip \( A^*_0 \) with the strict partial order \( < \) defined by
\[
\lambda < \mu \text{ if and only if } \mu - \lambda = \sum_{\alpha \in \Delta} x_\alpha \alpha \text{ where } x_\alpha > 0 \text{ for every } \alpha \in \Delta.
\]

We fix an algebraic group embedding \( \iota : G/A_G \to GL_N \) for some \( N \geq 1 \) and for every \( g \in G(F) \) we set
\[
\varpi(g) = \sup \{ \{1\} \cup \{ \log |\iota(g)_{i,j}|_F \mid 1 \leq i, j \leq N \} \}
\]
where the \( \iota(g)_{i,j} \)'s denote the entries of the matrix \( \iota(g) \).

We denote by \( \mathfrak{g} \) the (algebraic) Lie algebra of \( G \). Similar notations will be used for other algebraic groups (i.e. we will denote the Lie algebra of an algebraic group by the corresponding gothic letter). In the Archimedean case, we will also write \( \mathfrak{t} \) for the (real) Lie algebra of \( K \) and \( \mathcal{U}(\mathfrak{t}), \mathcal{U}(\mathfrak{g}) \) for the enveloping algebras of the complexifications of \( \mathfrak{t} \) and \( \mathfrak{g}(F) \) (considered as a real Lie algebra) respectively. We identify every element of \( \mathcal{U}(\mathfrak{g}) \) with the distribution supported at 1 that it defines.

We will assume that all the locally compact topological groups that we encounter have been equipped with Haar measures (bi-invariant Haar measures as we will always integrate, with one exception, over unimodular groups). The precise choices of these Haar measures will always be irrelevant. We denote by \( * \) the convolution product on a locally compact topological group \( H \). In the Archimedean case, this convolution product extends to distributions of compact support including elements of \( \mathcal{U}(\mathfrak{g}) \) and continuous compactly supported functions on closed subgroups (seen as distributions through the choice of a Haar measure on that subgroup).

### 2.2 Topological vector spaces

In this paper, by a topological vector space (TVS) we always mean a Hausdorff locally convex topological vector space over \( \mathbb{C} \). If \( F \) is a TVS, we shall denote by \( F' \) its continuous dual.
The TVS to be considered in this paper will all be LF spaces that is countable direct limit of Fréchet spaces. We will even only encounter strict LF spaces i.e. TVS $F$ that can be written as the direct limit of a sequence $(F_n)_n$ of Fréchet spaces where the transition maps $F_n \to F_{n+1}$ are closed embeddings. Strict LF spaces are complete. Moreover, since LF spaces are barreled [Th Corollary 3 of Proposition 33.2], they satisfy the uniform boundedness principle (aka Banach-Steinhaus theorem).

We refer the reader to [Bour] for the notions of smooth and holomorphic functions valued in topological vector spaces (TVS) that will be used thoroughly in this paper. Actually, we will only consider smooth and holomorphic functions valued in Fréchet or strict LF spaces for which the following criterion can be applied ([Bour 3.3.1 (v)]):

(2.2.1) Let $F$ be a quasi-complete TVS and $M$ a complex analytic manifold. Then, a function $\varphi : M \to F$ is holomorphic if and only if it is continuous and for some total subspace $H \subset V'$ the scalar-valued functions $m \in M \mapsto \langle \varphi(m), \lambda \rangle$ are holomorphic for every $\lambda \in H$.

Also when we say that a function on a totally disconnected locally compact topological space (e.g. $G(F)$ in the $p$-adic case) is smooth we always mean that it is locally constant.

2.3 Representations

By a representation of $G(F)$ we will always mean a smooth representation of finite length with complex coefficients. Here smooth has the usual meaning in the $p$-adic case (i.e. every vector has an open stabilizer) whereas in the Archimedean case it means a smooth admissible Fréchet representation of moderate growth in the sense of Casselman-Wallach [Cas, Wall2 Sect. 11] or, which is the same, an admissible SF representation in the sense of [BK]. We shall always abuse notation and denote by the same letter a representation and the space on which it acts. In the Archimedean case this space is always coming with a topology (it is a Fréchet space) whereas in the $p$-adic case it will sometimes be convenient, in order to make uniform statements, to equip this space with its finest locally convex topology (it then becomes a strict LF space). For $\pi$ a representation of $G(F)$ and $\lambda \in \mathcal{A}_{G,\mathbb{C}}^\ast$, we denote by $\pi_\lambda$ the twist of $\pi$ by the character $g \in G(F) \mapsto g^\lambda$. We let $\text{Irr}(G)$, $\text{Temp}(G)$ and $\Pi_2(G)$ be the sets of isomorphism classes of all irreducible representations, irreducible tempered representations and irreducible square-integrable representations of $G(F)$ respectively. We denote by $\tilde{\pi}$ the contragredient representation of $\pi$ (aka smooth dual) and by $\langle \cdot, \cdot \rangle$ the natural pairing between $\pi$ and $\tilde{\pi}$. In the Archimedean case, $\tilde{\pi}$ can be defined as the Casselman-Wallach globalization of the contragredient of the Harish-Chandra module underlying $\pi$. An alternative description of $\tilde{\pi}$, which is more suitable in practice, is as the space of linear forms on $\pi$ which are continuous with respect to any $G(F)$-continuous norm on $\pi$ together with the natural $G(F)$-action on it (a norm on $\pi$ is said to be $G(F)$-continuous if the action of $G(F)$ on $\pi$ is continuous for this norm). In the Archimedean case, we denote by $\pi'$ the topological dual of $\pi$ (i.e. the space of all continuous linear forms on $\pi$). There is a natural action $g \mapsto \pi'(g)$ of $G(F)$ on $\pi'$ and if we equip $\pi'$ with the topology of uniform convergence on compact subsets of $\pi$ then
this action of $G(F)$ on $p'$ is continuous and this allows to define $p'(\varphi)$ for every function $\varphi \in C_c(G(F))$ by integration.

Let $P = MU$ be a parabolic subgroup of $G$. For $\sigma$ a representation of $M(F)$, we denote by $i^G_P(\sigma)$ the smooth normalized parabolic induction of $\sigma$ from $P(F)$ to $G(F)$. The space of $i^G_P(\sigma)$ consists of smooth functions $e : G(F) \to \sigma$ satisfying $e(mug) = \delta_P(m)^{1/2} \sigma(m) e(g)$ for every $(m, u, g) \in M(F) \times U(F) \times G(F)$ (with its natural structure of Fréchet space in the Archimedean case) and the group $G(F)$ acts by right translation. Assume that $P$ is standard. Then, for every $\lambda \in \mathcal{A}^*_{M, C}$ restriction to $K$ induces a topological isomorphism between $\pi_\lambda = i^G_P(\sigma_\lambda)$ and $\pi_K = i^K_{K_P}(\sigma_{K_M})$ where $K_P = K \cap P(F)$, $K_M = K \cap M(F)$ and $i^K_{K_P}(\sigma_{K_M})$ denotes the spaces of smooth functions $e : K \to \sigma$ satisfying $e(muk) = \sigma(m) e(k)$ for every $(m, u, k) \in K_M \times (K \cap U(F)) \times K$. These identifications allows to define a notion of holomorphic sections $\lambda \in \mathcal{A}^*_{M, C} \mapsto e_\lambda \in \pi_\lambda$: we call such a map holomorphic if its composition with the isomorphisms $\pi_\lambda \simeq \pi_K$ is holomorphic. It turns out that this notion is actually independent of the choice of $K$. We also say that an assignment $\lambda \in \mathcal{A}^*_{M, C} \mapsto T_\lambda \in \pi_\lambda$ is holomorphic if for every holomorphic section $\lambda \mapsto e_\lambda \in \pi_\lambda$ the map $\lambda \in \mathcal{A}^*_{M, C} \mapsto T_\lambda(e_\lambda)$ is holomorphic. It is equivalent to ask that, identifying $T_\lambda$ with a linear form on $\pi_K$ for every $\lambda \in \mathcal{A}^*_{M, C}$, for every $e \in \pi_K$ the map $\lambda \in \mathcal{A}^*_{M, C} \mapsto T_\lambda(e)$ be holomorphic: Indeed as $\pi_K$ is Fréchet hence barreled, if this is so by the Banach-Steinhaus theorem for every compact $K \subseteq \mathcal{A}^*_{M, C}$ the subset $\{T_\lambda \mid \lambda \in \mathcal{K}\}$ of $\pi_K'$ is equicontinuous hence $\mathcal{A}^*_{M, C} \times \pi_K \to \mathbb{C}$, $(\lambda, e) \mapsto T_\lambda(e)$ is continuous and then we can apply [Jac2, Lemma 1].

**Lemma 2.3.1** Assume that $F$ is Archimedean. Let $\pi \in \Irr(G)$ and let $C_K$ be a Casimir element for $K$ (i.e. the element of $\mathcal{U}(\mathfrak{k})$ associated to a negative definite $K$-invariant real symmetric bilinear form on the Lie algebra of $K$). Then,

(i) $\pi(1 + C_K)$ is a topological isomorphism of $\pi$ onto itself.

Let $p_0$ be a $G(F)$-continuous Hilbert norm (i.e. associated to a scalar product) on $\pi$ which is $K$-invariant and set $p_\ell = p_0 \circ \pi(1 + C_K)^\ell$ for every integer $\ell \in \mathbb{Z}$. Then,

(ii) The map $\ell \mapsto p_\ell$ is increasing i.e. $p_\ell(v) \leq p_{\ell+1}(v)$ for every $\ell \in \mathbb{Z}$ and $v \in \pi$.

(iii) The family of norms $(p_\ell)_{\ell \geq 0}$ generates the topology on $\pi$.

(iv) Let $\pi^{(\ell)}$ be the completion of $\pi$ for $p_\ell$. Then, the natural pairing $\pi \times \hat{\pi} \to \mathbb{C}$ extends to a continuous bilinear form $\pi^{(\ell)} \times \hat{\pi} \to \mathbb{C}$ giving an embedding $\pi^{(\ell)} \subseteq (\hat{\pi})'$ for every $\ell$ and any equicontinuous subset of $(\hat{\pi})'$ is contained and bounded in $\pi^{(\ell)}$ for some $\ell$.

**Proof:** For each $\gamma \in \hat{K}$ the element $1 + C_K$ acts on $\gamma$ by a scalar $N(\gamma)$ which is greater or equal to 1 (Indeed, $C_K$ being in $\mathcal{U}(\mathfrak{t})^K$ the corresponding operator on $\gamma$ is scalar and as $C_K = -\sum_i X_i^2$ for a certain basis $(X_i)$ of $\mathfrak{t}$ it is also positive hermitian with respect to any $K$-invariant scalar product on the space of $\gamma$). Let $p_0$ be a $G(F)$-continuous Hilbert norm on $\pi$ which is $K$-invariant (such norm exists as the Harish-Chandra module underlying $\pi$ admits at least one Hilbert globalization). Set $p_\ell = p_0 \circ \pi(1 + C_K)^\ell$ for every integer $\ell \geq 0$. Then, by [BK, Proposition 3.9] the family of norms $(p_\ell)_{\ell \geq 0}$ generates the topology on $\pi$. This gives
(iii). For every vector $v \in \pi$ write $v = \sum_{\gamma \in \mathcal{K}} v_{\gamma}$ for its decomposition in $K$-isotypic components (the corresponding series is absolutely convergent in $\pi$). Then, by the $K$-invariance of $p_0$ for every $\ell \geq 0$ and $v \in \pi$ we have

$$p_{\ell+1}(v) = \left( \sum_{\gamma \in \mathcal{K}} p_0(\pi(1 + C_K)^{\ell+1} v_{\gamma}) \right)^{1/2} = \left( \sum_{\gamma \in \mathcal{K}} N(\gamma)^{2\ell+2} p_0(v_{\gamma}) \right)^{1/2} \geq \left( \sum_{\gamma \in \mathcal{K}} N(\gamma)^{2\ell} p_0(v_{\gamma}) \right)^{1/2} = p_\ell(v).$$

As $p_\ell(\pi(1 + C_K)v) = p_{\ell+1}(v)$ this shows that $\pi(1 + C_K)$ realizes a topological isomorphism between $\pi$ and one of its closed subspace. Since the subspace of $K$-finite vectors of $\pi$ is obviously contained in the image of $\pi(1 + C_K)$ and is dense in $\pi$ this shows (i) and therefore the definition of $p_\ell$ now also makes sense for $\ell \leq 0$. Moreover, the above inequality still holds for every $\ell \in \mathbb{Z}$ (with the same proof). This gives (ii). Finally, let $\tilde{p}_0$ be the norm on $\bar{\pi}$ dual to $p_0$ i.e.

$$\tilde{p}_0(v^\vee) = \sup_{v \in \bar{\pi}} |\langle v, v^\vee \rangle|, \quad v^\vee \in \bar{\pi}.$$

Set $\tilde{p}_\ell = \tilde{p}_0 \circ \pi(1 + C_K)^\ell$ for every $\ell \in \mathbb{Z}$. Then, we easily check that $p_{-\ell}$ is the norm dual to $\tilde{p}_\ell$ for every $\ell \in \mathbb{Z}$. This gives (iv) since the family of norms $(\tilde{p}_\ell)_{\ell \geq 0}$ generates the topology on $\bar{\pi}$.

### 2.4 Uniform bounds for matrix coefficients

For every $\lambda \in \mathcal{A}_0^*$, set $\pi_0^\lambda = i_{\mathcal{K}^0}(\lambda)$ where we have identified $\lambda$ with the character it defines on $M_0(F)$. We identify the space of $\pi_0^\lambda$ with $\pi_0^\lambda : = i_{\mathcal{K}^0}(1)$ by restriction to $K$ and we equip it with the $K$-invariant pairing given by

$$\langle e, e' \rangle_0 = \int_K e(k)e'(k) dk, \quad e, e' \in \pi_0^\lambda.$$  

By the previous identifications, $\langle \cdot, \cdot \rangle_0$ induces a $G(F)$-invariant continuous pairing between $\pi_0^\lambda$ and $\pi_0^{-\lambda}$ for every $\lambda \in \mathcal{A}_0^*$. Let $e_0 \in \pi_0^\lambda$ be the vector defined by $e_0(k) = 1$ for every $k \in K$. Then, we let

$$\Xi_\lambda^G(g) = \langle \pi_0^\lambda(g)e_0, e_0 \rangle_0, \quad \lambda \in \mathcal{A}_0^*, g \in G(F).$$

When $\lambda = 0$, we simply set $\Xi^G = \Xi_0^G$ (it is the usual Harish-Chandra’s Xi function see [Var Sect. II.8.5] and [Wal Sect. II.1]). We summarize the basic properties of the functions $\Xi_\lambda^G$ in the next proposition. In the Archimedean case, most of this is contained in [Wal Sect. 3.6].

**Proposition 2.4.1**

(i) $\Xi_{w\lambda}^G = \Xi_\lambda^G$ for every $\lambda \in \mathcal{A}_0^*$ and every $w \in W^G$.  

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(ii) For every $\lambda \in A_{\ast}^{0}$, $\mu \in A_{\ast}^{G}$ and $g \in G(F)$, we have $\Xi_{\lambda+\mu}^{G}(g) = \Xi_{\lambda}^{G}(g)\mu$.

(iii) Let $P = MU$ be a parabolic subgroup containing $P_{0}$ (with the Levi component chosen so that $M \supset M_{0}$). Choose for every $g \in G(F)$ a decomposition $g = m_{P}(g)u_{P}(g)k_{P}(g)$ with $(m_{P}(g), u_{P}(g), k_{P}(g)) \in M(F) \times U(F) \times K$. Then, we have

$$\Xi_{\lambda}^{G}(g) = \int_{K} \delta_{P}(m_{P}(kg))^{1/2} \Xi_{\lambda}^{M}(m_{P}(kg)) \, dk$$

for every $\lambda \in A_{\ast}^{0}$ and $g \in G(F)$.

(iv) There exists $d > 0$ such that

$$\delta_{0}(m_{0})^{1/2} m_{0}^{\lambda} \ll \Xi_{\lambda}^{G}(m_{0}) \ll \delta_{0}(m_{0})^{1/2} m_{0}^{\lambda} \sigma(m_{0})^{d}$$

for every $\lambda \in A_{\ast}^{0}$ and $m_{0} \in M_{0}^{+}$ where we have set $\delta_{0} = \delta_{P_{0}}$.

Proof:

(i) This is [Wald1, Proposition 3.6.2] in the Archimedean case and [Cas2, Proposition 4.1] in the non-Archimedean case.

(ii) is straightforward.

(iii) follows readily from the isomorphism of “induction by stages” $\pi_{\lambda}^{0} \simeq i_{P}^{G}(i_{P_{0}\cap M}^{M}(\lambda))$ (see [Wald1, Lemme II.1.6] for the case $\lambda = 0$).

(iv) First we prove the lower bound. By (i), we may assume $\lambda = |\lambda|$. Let $P_{0} = M_{0}N_{0}$ be the parabolic subgroup opposite to $P_{0}$ and $C_{0} \subseteq N_{0}(F)M_{0}(F)N_{0}(F)$ a compact neighborhood of $1$. We can find a compact subset $C_{0} \subseteq M_{0}(F)$ such that $m_{P_{0}}(km_{0}) \in m_{0}C_{0}$ for every $k \in C$ and $m_{0} \in M_{0}^{+}$. By (iii) and (ii), it follows that

$$\Xi_{\lambda}^{G}(m_{0}) = \int_{K} \delta_{0}(m_{P_{0}}(km_{0}))^{1/2} m_{P_{0}}(km_{0})^{\lambda} \, dk \geq \int_{K \cap C} \delta_{0}(m_{P_{0}}(km_{0}))^{1/2} m_{P_{0}}(km_{0})^{\lambda} \, dk$$

$$\geq \delta_{0}(m_{0})^{1/2} m_{0}^{\lambda} \delta_{0}(m_{0})^{1/2} m_{0}^{|\lambda|}$$

for all $m_{0} \in M_{0}^{+}$. We now prove the upper bound. By [Kos] and [BT, Proposition 4.4.4] (see also [Sil]), for every $k \in K$ and $m_{0} \in M_{0}(F)$, $H_{0}(m_{P_{0}}(km_{0}))$ belongs to the convex hull of $\{ \omega H_{0}(m_{0}) \mid \omega \in W^{G} \}$. It follows that $m_{P_{0}}(km_{0})^{\lambda} \leq m_{0}^{\lambda}$ for every $k \in K$ and $m_{0} \in M_{0}^{+}$. Therefore, using (ii) and (iii), we obtain

$$\Xi_{\lambda}^{G}(m_{0}) = \int_{K} \delta_{0}(m_{P_{0}}(km_{0}))^{1/2} m_{P_{0}}(km_{0})^{\lambda} \, dk \leq m_{0}^{\lambda} \int_{K} \delta_{0}(m_{P_{0}}(km_{0}))^{1/2} \, dk = m_{0}^{\lambda} \Xi_{\lambda}^{G}(m_{0})$$

for all $m_{0} \in M_{0}^{+}$. On the other hand, by [Var, Theorem 30 p.339] and [Wald1, Lemme II.1.1] there exists $d > 0$ such that $\Xi_{\lambda}^{G}(m_{0}) \ll \delta_{0}(m_{0})^{1/2} \sigma(m_{0})^{d}$ for all $m_{0} \in M_{0}^{+}$. The upper bound follows. ■
Let $P = MU$ be a standard parabolic subgroup and $\sigma \in \text{Temp}(M)$. For every $\lambda \in A^*_M, C$, we set $\pi_\lambda = i_G^\lambda(\sigma_\lambda)$ and identify the underlying space with $\pi_K := i_{K_P}^K(\sigma | K_M)$ where we have set $K_P = K \cap P(F)$ and $K_M = K \cap M(F)$ as before. Similarly, we let $\tilde{\pi}_\lambda = i_G^\lambda(\tilde{\sigma}_\lambda)$ and identify its underlying space with $\tilde{\pi}_K := i_{K_P}^K(\tilde{\sigma} | K_M)$ for every $\lambda \in A^*_M, C$. We define a bilinear pairing between $\pi_K$ and $\tilde{\pi}_K$ by

$$\langle e, e' \rangle = \int_K \langle e(k), e'(k) \rangle_\sigma dk, \quad e \in \pi_K, e' \in \tilde{\pi}_K$$

where $\langle ., . \rangle_\sigma$ denotes the natural pairing between $\sigma$ and $\tilde{\sigma}$. By the previous identifications, $\langle ., . \rangle$ induces a continuous $G(F)$-invariant pairing between $\pi_\lambda$ and $\tilde{\pi}_-\lambda$ which identifies the latter with the smooth contragredient of $\pi_\lambda$ for every $\lambda \in A^*_M, C$. The following proposition gives uniform bounds for the matrix coefficients of the $\pi_\lambda$. It is most probably well-known but in lack of a proper reference, we include a proof (see however [Knapp, Proposition 7.14] for the case of $K$-finite coefficients in the Archimedean case).

**Proposition 2.4.2** There exist continuous semi-norms $p$ and $\tilde{p}$ on $\pi_K$ and $\tilde{\pi}_K$ such that

$$|\langle \pi_\lambda(g)e, e' \rangle| \leq \Xi^G_{\Re(\lambda)}(g)p(e)\tilde{p}(e')$$

for every $\lambda \in A^*_M, C$, $g \in G(F)$ and $(e, e') \in \pi_K \times \tilde{\pi}_K$.

**Proof:** We have

$$\langle \pi_\lambda(g)e, e' \rangle = \int_K \delta_P(m_P(kg))^{1/2}m_P(kg)^\lambda \langle \sigma(m_P(kg))e(k_P(kg)), e'(k) \rangle_\sigma dk$$

and therefore

$$|\langle \pi_\lambda(g)e, e' \rangle| \leq \int_K \delta_P(m_P(kg))^{1/2}m_P(kg)^\Re(\lambda)\langle \sigma(m_P(kg))e(k_P(kg)), e'(k) \rangle_\sigma |dk$$

for every $\lambda \in A^*_M, C$, $g \in G(F)$ and $(e, e') \in \pi_K \times \tilde{\pi}_K$. By [CHH, Theorem 2] and [Sun], there exists continuous semi-norms $q$, $\tilde{q}$ on $\sigma$ and $\tilde{\sigma}$ such that $|\langle \sigma(m)v, v' \rangle| \leq \Xi^M(m)q(v)\tilde{q}(v')$ for every $m \in M(F)$ and $(v, v') \in \sigma \times \tilde{\sigma}$ (Here we emphasize that any semi-norm on $\sigma$ or $\tilde{\sigma}$ is continuous in the $p$-adic case). Consequently,

$$|\langle \pi_\lambda(g)e, e' \rangle| \leq p(e)\tilde{p}(e') \int_K \delta_P(m_P(kg))^{1/2}m_P(kg)^\Re(\lambda)\Xi^M(m_P(kg))dk$$

for every $\lambda \in A^*_M, C$, $g \in G(F)$, $(e, e') \in \pi_K \times \tilde{\pi}_K$ where we have set $p(e) = \sup_{k \in K} q(e(k))$ and $\tilde{p}(e') = \sup_{k \in K} \tilde{q}(e'(k))$. By Proposition 2.4.1, the above integral is equal to $\Xi^G_{\Re(\lambda)}(g)$ whereas $p$ and $\tilde{p}$ clearly define continuous semi-norms on $\pi_K$ and $\tilde{\pi}_K$ respectively.
2.5 Harish-Chandra Schwartz spaces of Whittaker functions

From this section and until the end of Section 2.7 we assume that $G$ is quasi-split. Thus $P_0 = B$ is a Borel subgroup, $T = M_0$ is a maximal torus and $A_0$ is the maximal split torus in $T$. Let $\xi : N_0(F) \to \mathbb{S}^1$ a generic character. For every $\lambda \in A_0^+$, we denote by $\mathcal{C}_\lambda(N_0(F)\backslash G(F), \xi)$ the space of functions $W : G(F) \to \mathbb{C}$ satisfying:

- $W(ug) = \xi(u)W(g)$ for every $(u, g) \in N_0(F) \times G(F)$;
- If $F$ is $p$-adic: $W$ is right-invariant by a compact-open subgroup $J \subseteq G(F)$ and there exists $R > 0$ such that for every $d > 0$ we have an inequality
  $$|W(tk)| \ll \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^R \right) \Xi^G(t)\sigma(t)^{-d}, \quad t \in T(F), k \in K;$$
- If $F$ is Archimedean: $W$ is smooth and there exists $R > 0$ such that for every $u \in \mathcal{U}(\mathfrak{g})$ and $d > 0$ we have an inequality
  $$|(R(u)W)(tk)| \ll \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^R \right) \Xi^G(t)\sigma(t)^{-d}, \quad t \in T(F), k \in K.$$

Then, $\mathcal{C}_\lambda(N_0(F)\backslash G(F), \xi)$ has a natural locally convex topology making it into a LF space. Notice that we have $\mathcal{C}_\lambda(N_0(F)\backslash G(F), \xi) = \mathcal{C}_{w\lambda}(N_0(F)\backslash G(F), \xi)$ for every $w \in W^G$ and $\lambda \in A_0^+$ (by Proposition 2.4.1(i)). The next lemma shows that $\mathcal{C}_\lambda(N_0(F)\backslash G(F), \xi)$ is actually a Fréchet space in the Archimedean case and a strict LF space in the $p$-adic case (in particular it is complete). Although it is probably well-known (see [Jac, Proposition 3.1] for a similar result) for the sake of completeness we provide a full proof.

**Lemma 2.5.1** Let $\lambda \in A_0^+$. Then, for every $R > 0$ and $d > 0$ there exists a continuous semi-norm $p_{R,d}$ on $\mathcal{C}_\lambda(N_0(F)\backslash G(F), \xi)$ such that

$$|W(tk)| \leq p_{R,d}(W) \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^{-R} \right) \delta_0(t)^{1/2}t^{\lambda|\ell_\sigma(t)|}$$

for every $W \in \mathcal{C}_\lambda(N_0(F)\backslash G(F), \xi)$, $t \in T(F)$ and $k \in K$.

**Proof:** By the uniform boundedness principle it suffices to show that for every $R > 0$, $d > 0$ and $W \in \mathcal{C}_\lambda(N_0(F)\backslash G(F), \xi)$ we have

$$|W(tk)| \ll \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^{-R} \right) \delta_0(t)^{1/2}t^{\lambda|\ell_\sigma(t)|}$$

for every $W \in \mathcal{C}_\lambda(N_0(F)\backslash G(F), \xi)$, $t \in T(F)$ and $k \in K$.

Notice that by Proposition 2.4.1(iv), there exist $R_0 > 0$ and $d_0 > 0$ such that

$$\Xi^G(t) \ll \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^{-R_0} \right) \delta_0(t)^{1/2}t^{\lambda|\ell_\sigma(t)|}$$

for every $W \in \mathcal{C}_\lambda(N_0(F)\backslash G(F), \xi)$, $t \in T(F)$. 

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Indeed, for every \( t \in T(F) \) there exists \( w \in W^G \) such that \( wt^{-1}w^{-1} \in T^+ \) and since \( w \) admits a lift in \( K \), by Proposition 2.4.1(iv) there exists \( d_0 > 0 \) such that

\[
\Xi^G_\lambda(t) = \Xi^G_\lambda(wt^{-1}) \ll \delta_0(t)^{1/2} (wt^{-1})^{|\lambda| |\sigma(t)|} = t^{w^{-1}|\lambda|} \delta_0(t)^{1/2} t^{|\lambda|} \delta_0(t)^{d_0}.
\]

As \( |\lambda| \in (A_0^\times) \), \( w^{-1}|\lambda| - |\lambda| \) is a nonnegative linear combination of simple roots. Hence, there exists \( R_0 > 0 \) (which we can of course choose independently of \( w \)) so that

\[
t^{w^{-1}|\lambda|-|\lambda|} \leq \prod_{\alpha \in \Delta} (1 + t^\alpha)^{R_0}.
\]

Therefore, we are left with showing that for every \( R > 0 \), \( d > 0 \) and \( W \in \mathcal{C}_\lambda(N_0(F) \setminus G(F), \xi) \) we have

\[
|W(tk)| \ll \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^{-R} \right) \Xi^G_\lambda(t) \sigma(t)^{-d}, \quad t \in T(F), k \in K.
\]

In the \( p \)-adic case we can prove the following stronger inequality by essentially the same argument as in the proof of \([\text{Wal2}, \text{Lemme 3.7}]\): for every compact-open subgroup \( J \) of \( G(F) \), there exists \( c = c_J > 0 \) such that for every \( W \in \mathcal{C}_\lambda(N_0(F) \setminus G(F), \xi) \) and \( d > 0 \) we have

\[
|W(tk)| \ll \left( \prod_{\alpha \in \Delta} 1_{[0,c]}(t^\alpha) \right) \Xi^G_\lambda(t) \sigma(t)^{-d}, \quad t \in T(F), k \in K
\]

where \( 1_{[0,c]} \) stands for the characteristic function of the interval \([0,c]\). Thus, we only consider the Archimedean case. Let \( W \in \mathcal{C}_\lambda(N_0(F) \setminus G(F), \xi) \) and \( d > 0 \). Clearly, it suffices to show the existence of \( R > 0 \) such that for every \( \Delta \)-tuple \( \underline{N} = (N_\alpha)_{\alpha \in \Delta} \) of nonnegative integers we have

\[
(2.5.1) \quad \left( \prod_{\alpha \in \Delta} t^{N_\alpha} \right) |W(tk)| \ll \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^{-R} \right) \Xi^G_\lambda(t) \sigma(t)^{-d}, \quad t \in T(F), k \in K.
\]

Let \( d\xi : n_0(F) \to \mathbb{C} \) denote the differential of \( \xi \) at the origin. Since \( \xi \) is generic for every \( \alpha \in \Delta \) there exists \( X_\alpha \in n_0(F) \) such that \( d\xi(X_\alpha) = 1 \) and \( \text{Ad}(a)X_\alpha = \alpha(a)X_\alpha \) for every \( a \in A_0(F) \). We make such a choice. Fix a norm \( \|\cdot\| \) on \( n_0(F) \). As \( T(F)/A_0(F) \) is compact, there exists \( c > 0 \) such that \( \|\text{Ad}(tk)^{-1}X_\alpha\| \leq c t^{-\alpha} \) for every \( t \in T(F), k \in K \) and \( \alpha \in \Delta \). Then, for every \( \Delta \)-tuple \( \underline{N} = (N_\alpha)_{\alpha \in \Delta} \) of nonnegative integers, setting \( u_{\underline{N}} = \prod_{\alpha \in \Delta} X_N^\alpha \in \mathcal{U}(n_0) \) (the product being taken in some fixed order), we have

\[
|W(tk)| = \prod_{\alpha \in \Delta} d\xi(X_\alpha)^{N_\alpha} W(tk) = \|(L(-\epsilon u_{\underline{N}})W)(tk)\| = \|(R(\text{Ad}(tk)^{-1}u_{\underline{N}})W)(tk)\| \leq \left( \prod_{\alpha \in \Delta} t^{-N_\alpha} \right) \sup_{u \in \mathcal{K}_N} |(R(u)W)(tk)|
\]

for every \( t \in T(F), k \in K \) where \( \epsilon \) is a certain sign and \( \mathcal{K}_N \) is the compact subset of \( \mathcal{U}(n_0) \) consisting of products of \( N = \sum_{\alpha \in \Delta} N_\alpha \) elements of \( n_0(F) \) of norm smaller than \( c \). By
definition of \( C_\lambda(N_0(F)\backslash G(F), \xi) \), there exists \( R > 0 \) such that for every integer \( N \geq 0 \) we have

\[
\sup_{u \in \mathbb{K}_N} |(R(u)W)(tk)| \ll \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^R \right)^{-\frac{d}{2}} \xi_\lambda(t)^{-d}, \quad t \in T(F), k \in K.
\]

This shows \( 2.5.1 \) and ends the proof of the lemma. ■

### 2.6 Uniform bounds for families of Whittaker functions

We continue with the setting of the previous sections, still assuming that \( G \) is quasi-split and fixing a generic character \( \xi : N_0(F) \to \mathbb{S}^1 \). Let \( P = MN \) be a standard parabolic subgroup, \( \sigma \in \text{Temp}(M) \) and set \( \pi_\lambda = i^K_G(\sigma_\lambda) \) for every \( \lambda \in \mathcal{A}_{M,C}^\ast \). We identify as before the space of \( \pi_\lambda \) with \( \pi_K := i^K_{K_P}(\sigma|_{K_M}) \). We assume given a family \( J_\lambda \in \text{Hom}_{N_0}(\pi_\lambda, \xi) \) of Whittaker functionals on \( \pi_\lambda = i^K_G(\sigma_\lambda) \) for \( \lambda \in \mathcal{A}_{M,C}^\ast \) i.e. a family of continuous linear forms \( J_\lambda : \pi_\lambda \to \mathbb{C} \) satisfying \( J_\lambda \circ \pi_\lambda(u) = \xi(u)J_\lambda \) for every \( u \in N_0(F) \) and \( \lambda \in \mathcal{A}_{M,C}^\ast \). We moreover suppose that the family \( \lambda \mapsto J_\lambda \in (\pi_\lambda)' \) is holomorphic in the sense of Section \( 2.3 \) By Frobenius reciprocity, for every \( \lambda \in \mathcal{A}_{M,C}^\ast \), \( J_\lambda \) induces a continuous \( G(F) \)-equivariant linear map

\[
\tilde{J}_\lambda : \pi_\lambda \to C^\infty(N_0(F)\backslash G(F), \xi)
\]

where \( C^\infty(N_0(F)\backslash G(F), \xi) \) stands for the space of all smooth functions \( W : G(F) \to \mathbb{C} \) such that \( W(ug) = \xi(u)W(g) \) for every \( (u, g) \in N_0(F) \times G(F) \). Recall that in Section \( 2.4 \) we have defined a strict partial order \( < \) on \( \mathcal{A}_{\gamma}^\ast \).

**Proposition 2.6.1**  
(i) For every \( \lambda \in \mathcal{A}_{M,C}^\ast \) and \( \mu \in \mathcal{A}_{\gamma}^\ast \) such that \( |\Re(\lambda)| < \mu \), the image of \( \tilde{J}_\lambda \) is included in \( C_\mu(N_0(F)\backslash G(F), \xi) \) and the resulting linear map (that we will still denote by \( \tilde{J}_\lambda \))

\[
\pi_\lambda \to C_\mu(N_0(F)\backslash G(F), \xi)
\]

is continuous.

(ii) Let \( \mu \in (\mathcal{A}_{\gamma}^G)^\ast \) and set \( \mathcal{U}[< \mu] = \{ \lambda \in (\mathcal{A}_{M,C}^G)^\ast \mid |\Re(\lambda)| < \mu \} \) (an open subset of \( (\mathcal{A}_{M,C}^G)^\ast ) \). Then, the family of continuous linear maps

\[
\lambda \in \mathcal{U}[< \mu] \mapsto \tilde{J}_\lambda \in \text{Hom}_{G(F)}(\pi_\lambda, C_\mu(N_0(F)\backslash G(F), \xi))
\]

is analytic in the sense that for every analytic section \( \lambda \mapsto e_\lambda \in \pi_\lambda \) the resulting map \( \lambda \in \mathcal{U}[< \mu] \mapsto \tilde{J}_\lambda(e_\lambda) \in C_\mu(N_0(F)\backslash G(F), \xi) \) is analytic.

**Proof:** We will show the following:

(2.6.1) For every compact subset \( K \subset \mathcal{A}_{M,C}^\ast \), there exists \( R > 0 \) and a continuous semi-norm \( p \) on \( \pi_K \) such that

\[
|J_\lambda(\pi_\lambda(t)e)| \leq p(e)\Xi_{\Re(\lambda)}(t)\left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^R \right)
\]

for every \( \lambda \in K, t \in T(F) \) and \( e \in \pi_K \).
Before proving \(2.6.1\), we explain how the proposition can be deduced from this. The first part can readily be inferred from \(2.6.1\) together with the following inequality which is a consequence of Proposition 2.4.1 (iv) and the Cartan decomposition \(G(F) = KT+K\): If \(|\Re(\lambda)| < \mu\) then for every \(d > 0\) we have

\[
(2.6.2) \quad \Xi_{\Re(\lambda)}(g) \ll \Xi_{\mu}(g)\sigma(g)^{-d}, \quad g \in G(F).
\]

For (ii), by the criterion 2.2.1 and since for every \(e \in \pi_K\) and \(g \in G(F)\) the map \(\lambda \in A^\ast_{M,C} \mapsto \widetilde{J}_\lambda(e)(g) = J_\lambda(\pi_\lambda(e)g)\) is analytic, it suffices to check that for every \(e \in \pi_K\) the map \(\lambda \in \mathcal{U}[< \mu'] \mapsto \widetilde{J}_\lambda(e) \in C^\ast(\mathcal{N}_0(F)\backslash G(F), \xi)\) is continuous. Let \(\lambda_0 \in \mathcal{U}[< \mu]\) and \(e \in \pi_K\). Then, by definition of the topology on \(C^\ast(\mathcal{N}_0(F)\backslash G(F), \xi)\), we need to show the following: there exists \(R > 0\) such that for every \(\epsilon > 0\) and \(d > 0\) if \(\lambda \in \mathcal{U}[< \mu]\) is sufficiently close to \(\lambda_0\) then

\[
(2.6.3) \quad \sup_{t \in T(F)} \Xi_{\mu}(t)^{-1} \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^{-R} \right) \sigma(t)^d |J_\lambda(\pi_\lambda(tk)e) - J_{\lambda_0}(\pi_{\lambda_0}(tk)e)| < \epsilon.
\]

Let \(R > 0\) and \(p\) be a continuous semi-norm on \(\pi_K\) so that \(2.6.1\) is satisfied on a compact neighborhood \(\mathcal{K} \subseteq \mathcal{U}[< \mu]\) of \(\lambda_0\). Then, there exists \(C > 0\) such that

\[
\sup_{t \in T(F)} \Xi_{\mu}(t)^{-1} \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^{-R} \right) \sigma(t)^d |J_\lambda(\pi_\lambda(tk)e)| < C
\]

for every \(\lambda \in \mathcal{K}\). Indeed, this follows from the (easily checked) fact that the inequality \(2.6.2\) can be made uniform on \(\mathcal{K}\). Set \(M = 2\epsilon^{-1}C\). Let \(T[> M]\) denote the subset of elements \(t \in T(F)\) such that \(\sigma(t) > M\) and \(T[\leq M]\) be its complement. Then, the above inequality easily implies

\[
(2.6.4) \quad \sup_{t \in T[> M]} \Xi_{\mu}(t)^{-1} \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^{-R} \right) \sigma(t)^d |J_\lambda(\pi_\lambda(tk)e)| < \frac{\epsilon}{2}
\]

for every \(\lambda \in \mathcal{K}\). On the other hand, \(T[\leq M]\) is compact modulo \(A^G_{G}(F)\) so that by continuity of the map \((\lambda, g) \mapsto J_\lambda(\pi_\lambda(g)e)\) for \(\lambda \in (A^G_{M,C})^\ast\) sufficiently close to \(\lambda_0\) we have

\[
\sup_{t \in T[\leq M]} \Xi_{\mu}(t)^{-1} \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^{-R} \right) \sigma(t)^d |J_\lambda(\pi_\lambda(tk)e) - J_{\lambda_0}(\pi_{\lambda_0}(tk)e)| < \epsilon.
\]

Together with \(2.6.4\), this readily implies \(2.6.3\) thus showing (ii) provided \(2.6.1\) is satisfied.

We now prove \(2.6.1\). Set \(\tilde{\pi}_\lambda = i_P(\tilde{\sigma}_\lambda)\) that we identify with \(\tilde{\pi}_K := i^K_{K_P}(\tilde{\sigma}|_{K_M})\) for every \(\lambda \in A^\ast_{M,C}\). We fix a pairing \(\langle \cdot, \cdot \rangle\) between \(\pi_K\) and \(\tilde{\pi}_K\) as in Section 2.3 that identifies \(\tilde{\pi}_-\lambda\) with the smooth contragredient of \(\pi_\lambda\) for every \(\lambda \in A^\ast_{M,C}\). This pairing induces an embedding \(\tilde{\pi}_K \subseteq \pi'_K\).
First, we treat the $p$-adic case. Since $\xi$ is generic, we can choose for every $\alpha \in \Delta$ a vector $X_\alpha \in n_0(F)$ so that $\xi(\exp(X_\alpha)) \neq 1$ and $\text{Ad}(a)X_\alpha = \alpha(a)X_\alpha$ for all $a \in A_0(F)$. Let $J \subseteq K$ be a compact-open subgroup. Then, there exists $C = C_J > 0$ such that for every $\alpha \in \Delta$ and $t \in T(F)$ with $t^\alpha \geq C$ we have $t^{-1}\exp(X_\alpha)t \in J$. Let $e \in (\pi_K)^J$, $\lambda \in \mathcal{A}^N_{M,C}$ and $t \in T(F)$. Then, if there exists $\alpha \in \Delta$ such that $t^\alpha \geq C$ we have

$$\xi(\exp(X_\alpha))J_\lambda(\pi_\alpha(t)e) = J_\lambda(\pi_\lambda(t)\pi_\lambda(t^{-1}\exp(X_\alpha)t)e) = J_\lambda(\pi_\lambda(t)e)$$

hence (as $\xi(\exp(X_\alpha)) \neq 1$) $J_\lambda(\pi_\lambda(t)e) = 0$. Thus,

$$J_\lambda(\pi_\lambda(t)e) = 0$$

unless $t^\alpha < C$ for every $\alpha \in \Delta$. On the other hand, there exists a compact-open subgroup $J' \subseteq K$ such that $J' \subseteq \text{Ker}(\xi)tJt^{-1}$ as soon as $t^\alpha < C$ for every $\alpha \in \Delta$ where we have denoted by $\text{Ker}(\xi)$ the kernel of $\xi$. Hence, if $t^\alpha < C$ for every $\alpha \in \Delta$ we have

$$J_\lambda(\pi_\lambda(t)e) = J_\lambda(\pi(e,J')\pi_\lambda(t)e)$$

where $e,J' := \text{vol}(J')^{-1}1_{J'}$. The family of functionals $\lambda \mapsto J_\lambda \circ \pi(e,J')$ is represented by an analytic section $\lambda \in \mathcal{A}^N_{M,C} \mapsto e_\lambda' \in \pi_\lambda = i_{K}^{\pi}(\pi_\lambda)$ and by what we just saw for every $t \in T(F)$ we have

$$J_\lambda(\pi_\lambda(t)e) = \begin{cases} \langle \pi_\lambda(t)e, e_\lambda' \rangle & \text{if } t^\alpha < C \text{ for every } \alpha \in \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

The inequality [2.6.1] (with $R = 0$) is now a consequence of Proposition [2.4.2].

Next, we treat the Archimedean case. For this, we need to introduce another set of notation. Fix a $M(F)$-continuous Hilbert norm $\tilde{q}_0$ on $\tilde{F}$ which is $K_M$-invariant and let $\tilde{p}_0$ be the $K$-invariant Hilbert norm on $\tilde{\pi}_K$ defined by

$$\tilde{p}_0(e^\varphi) = \left(\int_K \tilde{q}_0(e^\varphi(k))^2\right)^{1/2}, \quad e^\varphi \in \tilde{\pi}_K.$$

Then, for every $\lambda \in \mathcal{A}^N_{M,C} - \tilde{p}_0$ induces a $G(F)$-continuous norm on $\tilde{\pi}_\lambda$ through the identification $\tilde{\pi}_\lambda \simeq \tilde{\pi}_K$. Let $C_K$ be a Casimir operator for $K$. Set, as in Lemma [2.3.1], $\tilde{p}_\ell = \tilde{p}_0 \circ \tilde{\pi}_K(1 + C_K)^\ell$ for every integer $\ell \in \mathbb{Z}$ and $\tilde{\pi}_K^{(\ell)}$ for the completion of $\tilde{\pi}_K$ with respect to $\tilde{p}_\ell$ (a Banach space). Then, by Lemma [2.3.1] we have natural inclusions $\tilde{\pi}_K^{(\ell)} \subseteq \tilde{\pi}_K^{(\ell+m)}$ for every $\ell$, every equicontinuous subset of $\pi_\lambda'$ is contained and bounded in some $\tilde{\pi}_K^{(\ell)}$ where $\ell \leq 0$ and the family of norms $(\tilde{p}_\ell)_{\ell \geq 0}$ generates the topology on $\tilde{\pi}_K$. Let $\ell \leq 0$ and $m \geq -\ell$ be integers and $\varphi \in C^{2m}(G(F))$ then we have

$$\pi_\lambda'(\varphi)^\Lambda \in \tilde{\pi}_K^{(\ell+m)}$$

for every $\lambda \in \mathcal{A}^N_{M,C}$ and $\Lambda \in \tilde{\pi}_K^{(\ell)}$. Indeed, we have

$$\tilde{p}_{\ell+m}(\pi_\lambda(\varphi)^e) = \tilde{p}_0 \left(\pi_\lambda(L(1 + C_K)^{\ell+m}R(1 + C_K)^{-\ell}\varphi)e^\varphi K(1 + C_K)^{\ell}e^\varphi\right) \leq \left\|L(1 + C_K)^{\ell+m}R(1 + C_K)^{-\ell}\varphi\right\|_{L^1} \times \sup_{g \in \text{Supp}(\varphi)} \left\|\pi_\lambda(g)e^\varphi\right\|_1 \times \tilde{p}_0 \times \tilde{p}_\ell(e^\varphi)$$
for every $\lambda \in \mathcal{A}_{M,C}^*$ and $e^\gamma \in \widehat{\pi}_K$ where $\|L(1 + C_K)^{\ell + m}R(1 + C_K)^{-\ell} \varphi\|_{L^1}$ denotes the $L^1$-norm of $L(1 + C_K)^{\ell + m}R(1 + C_K)^{-\ell} \varphi \in C_c(G(F))$ and for every $g \in G(F)$ we have denoted by $\|\widehat{\pi}_\lambda(g)\|_{\widehat{p}_0}$ the operator norm of $\widehat{\pi}_\lambda(g)$ with respect to the norm $\widehat{p}_0$. This proves (2.6.5) by density of $\widehat{\pi}_K$ in $\widehat{\pi}_K^{(\ell)}$ for every $\ell$. Moreover, since the function $\lambda \in \mathcal{A}_{M,C}^* \mapsto \sup_{\varphi \in \text{Supp}(\varphi)}\|\widehat{\pi}_\lambda(g)\|_{\widehat{p}_0}$ is easily seen to be locally bounded, we have that the operator norm of $\pi'_\lambda(\varphi)$ seen as a continuous linear map from $\widehat{\pi}_K^{(\ell)}$ to $\widehat{\pi}_K^{(\ell + m)}$ is locally bounded in $\lambda$.

Let $K \subseteq \mathcal{A}_{M,C}^*$ be a compact subset. By continuity of $\lambda \mapsto J_\lambda \in \pi'_K$ for the weak-* topology and the uniform boundedness principle, the family $\{J_\lambda \mid \lambda \in K\}$ is equicontinuous. Hence, there exists $\ell \leq 0$ such that this family is included and bounded in $\widehat{\pi}_K^{(\ell)}$ and, by what we just seen, for every $m \geq -\ell$ and $\varphi \in C_c^{2m}(G(F))$ the family

$$\{J_\lambda \circ \pi_\lambda(\varphi) \mid \lambda \in K\}$$

is included and bounded in $\widehat{\pi}_K^{(\ell + m)}$. Let $p$ and $\widehat{p}$ be continuous semi-norms on $\pi_K$ and $\widehat{\pi}_K$ as in Proposition 2.4.2. Then, for $m$ sufficiently large $\widehat{p}$ extends (uniquely) to a continuous semi-norm on $\hat{\pi}_K^{(m+\ell)}$ and the inequality of Proposition 2.4.2 still holds for every $(e, e^\gamma) \in \pi_K \times \hat{\pi}_K^{(m+\ell)}$ (by density of $\pi_K$ in $\hat{\pi}_K^{(m+\ell)}$). Therefore, we have obtained:

(2.6.6) For $m$ sufficiently large and every $\varphi \in C_c^{2m}(G(F))$ there exists a constant $C > 0$ such that

$$|J_\lambda(\pi_\lambda(\varphi)\pi_\lambda(g)e)| \leq Cp(e)\Xi_{\mathcal{R}(\lambda)}(g)$$

for all $\lambda \in K$, $g \in G(F)$ and $e \in \pi_K$.

Let $\overline{B} = T\overline{N}_0$ be the Borel subgroup opposite to $B$ and let $Y_1, \ldots, Y_6$ be a $\mathbb{R}$-basis of $\overline{B}(F)$. Set $\Delta_{\overline{B}} = Y_1^2 + \ldots + Y_6^2 \in \mathcal{U}(\overline{B})$ and let $m$ be a positive integer that we assume sufficiently large in what follows. By elliptic regularity ([BK, Lemma 3.7]) there exist $\varphi_{\overline{B}}^1 \in C_c^{m'}(\overline{B}(F))$ and $\varphi_{\overline{B}}^2 \in C_c^{\infty}(\overline{B}(F))$ where $m' = 2m - \text{dim}(\overline{B}) - 1$ such that

$$\varphi_{\overline{B}}^1 \ast \Delta_{\overline{B}}^{m} + \varphi_{\overline{B}}^2 = \delta_{\overline{B}}^1$$

in the sense of distributions where $\delta_{\overline{B}}^1$ denotes the Dirac distribution at 1 on $\overline{B}(F)$. Let $\varphi_{N} \in C_c^{\infty}(\overline{N}_0(F))$ be such that

$$\int_{\overline{N}_0(F)} \varphi_{N}(u)\xi(u)du = 1.$$  

Then, setting $\varphi^i = \varphi_{N} \ast \varphi_{\overline{B}}^i$ for $i = 1, 2$, for every $\lambda \in \mathcal{A}_{M,C}^*$, $e \in \pi_K$ and $t \in T(F)$, we have

(2.6.7)  

$$J_\lambda(\pi_\lambda(t)e) = J_\lambda(\pi_\lambda(\varphi_{\overline{B}}^1)\pi_\lambda(\Delta_{\overline{B}}^{m})\pi_\lambda(t)e) + J_\lambda(\pi_\lambda(\varphi_{\overline{B}}^2)\pi_\lambda(t)e)$$

$$= J_\lambda(\pi_\lambda(\varphi^1)\pi_\lambda(\Delta_{\overline{B}}^{m})\pi_\lambda(t)e) + J_\lambda(\pi_\lambda(\varphi^2)\pi_\lambda(t)e).$$

Noticing that $\varphi^i \in C_c^{m'}(G(F))$ for $i = 1, 2$, from (2.6.6) we deduce that for $m$ sufficiently large there exists $C > 0$ such that

(2.6.8)  

$$|J_\lambda(\pi_\lambda(t)e)| \leq C(p(\pi_\lambda(\text{Ad}(t)^{-1}\Delta_{\overline{B}}^{m})e) + p(e))\Xi_{\mathcal{R}(\lambda)}(t)$$

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for every \( \lambda \in \mathcal{K}, \ t \in T(F) \) and \( e \in \pi_K \). To get \( \text{Proposition 2.6.1} \) it only remains to notice that there exist \( R > 0 \) and a continuous semi-norm \( q \) on \( \pi_K \) such that

\[
p(\pi_\lambda(\text{Ad}(t)^{-1} \Delta_{\pi}^m) e) \leq \left( \prod_{\alpha \in \Delta} (1 + t^\alpha)^R \right) q(e)
\]

for every \( \lambda \in \mathcal{K}, \ t \in T(F) \) and \( e \in \pi_K \). This follows from the fact that the function \( (\lambda, t) \in \mathcal{A}_{M,C} \times T(F) \mapsto \pi_\lambda(\text{Ad}(t)^{-1} \Delta_{\pi}^m) \) is polynomial (see [Del, Proposition 1]). □

### 2.7 Application to the existence of good sections for Whittaker models

We continue with the setting of the previous section: \( G \) is quasi-split, \( \xi : N_0(F) \to S^1 \) is a generic character, \( P = MN \) is a standard parabolic subgroup, \( \sigma \in \text{Temp}(M) \) and we set \( \pi_\lambda = i^G_\sigma(\sigma_\lambda) \) for every \( \lambda \in \mathcal{A}_{M,C}^* \) whose space is identified as before with \( \pi_K := i^K_K(\sigma|_{K_M}) \). Let \( w_0 \in W^G \) be the longest element of the Weyl group and choose a lift \( \tilde{w}_0 \in G(F) \). Let \( \xi^- \) be the generic character of \( \overline{N}_0(F) = w_0 N_0(F) w_0^{-1} \) defined by \( \xi^-(\overline{\pi}) = \xi(\widetilde{w}_0 \overline{\pi}^{-1} \widetilde{w}_0) \), \( \overline{\pi} \in \overline{N}_0(F) \).

Assume that \( \sigma \) is generic with respect to the restriction of \( \xi^- \) to \( \overline{N}_0(F) \cap M(F) \) i.e. there exists a continuous nonzero linear form \( \ell : \sigma \to \mathbb{C} \) such that \( \ell \circ \sigma(\overline{\pi}) = \xi^- (\overline{\pi}) \ell \) for every \( \overline{\pi} \in \overline{N}_0(F) \cap M(F) \). Then, the construction of the Jacquet’s integral ([Wall2, Sect. 15.4], [CS]) provides us with a holomorphic family of Whittaker functionals

\[
\lambda \in \mathcal{A}_{M,C}^* \mapsto J_\lambda \in \text{Hom}_{N_0}(\pi_\lambda, \xi)
\]

as in the previous section which is everywhere non-vanishing. For every \( \lambda \in \mathcal{A}_{M,C}^* \) we denote by \( \mathcal{W}(\pi_\lambda, \xi) \) the corresponding Whittaker model i.e. the space of functions of the form \( g \in G(F) \mapsto J_\lambda(\pi_\lambda(g)e) \) for \( e \in \pi_K \).

**Corollary 2.7.1** For every \( \lambda_0 \in (\mathcal{A}_{M,C}^G)^* \) and \( W_0 \in \mathcal{W}(\pi_{\lambda_0}, \xi) \) there exists a map

\[
\lambda \in (\mathcal{A}_{M,C}^G)^* \mapsto W_\lambda \in \mathcal{W}(\pi_\lambda, \xi)
\]

such that:

- for every \( \mu \in (\mathcal{A}_{M,C}^G)^* \) and \( \lambda \in \mathcal{U}[< \mu] = \{ \lambda \in (\mathcal{A}_{M,C}^G)^* \mid |\Re(\lambda)| < \mu \} \) we have \( W_\lambda \in \mathcal{C}_\mu(N_0(F) \setminus G(F), \xi) \) and the resulting map

\[
\lambda \in \mathcal{U}[< \mu] \mapsto W_\lambda \in \mathcal{C}_\mu(N_0(F) \setminus G(F), \xi)
\]

is analytic;

- \( W_{\lambda_0} = W_0 \).

**Proof:** Indeed, there exists \( e \in \pi_K \) such that (with the notation of Section 2.6, \( W_0 = \tilde{J}_{\lambda_0}(e) \) and then it suffices to set \( W_\lambda = \tilde{J}_\lambda(e) \) for every \( \lambda \in (\mathcal{A}_{M,C}^G)^* \) the required properties immediately follows from Proposition 2.6.1 □
2.8 Holomorphic continuation of certain functions

For every $C \in \mathbb{R} \cup \{-\infty\}$ we set $\mathcal{H}_C = \{s \in \mathbb{C} \mid \Re(s) > C\}$. Let $M$ be a complex analytic manifold and $C \in \mathbb{R} \cup \{-\infty\}$. Then, we say that a holomorphic function $F : \mathcal{H}_C \times M \to \mathbb{C}$ is rapidly decreasing in vertical strips in the first variable locally uniformly in the second variable if for every vertical strip $V \subset \mathcal{H}_C$, every $N > 0$ and every compact subset $K_M \subseteq M$ we have

$$\sup_{(s,t) \in V \times K_M} (1 + |\Im(s)|)^N |F(s,t)| < \infty.$$ 

The object of this section is the statement of the following proposition whose proof will be given in Section 2.10.

**Proposition 2.8.1** Let $M$ be a connected complex analytic variety, $U \subseteq M$ a nonempty connected relatively compact open subset and

$$Z_1, Z_2 : \mathbb{C} \times U \to \mathbb{C}$$

be two holomorphic functions satisfying

$$Z_1(-s,t) = Z_2(s,t) \quad (2.8.1)$$

for all $(s,t) \in \mathbb{C} \times U$. Moreover, we assume that one of the two points below is satisfied:

(1) For every $k \in \{1,2\}$, $Z_k$ is rapidly decreasing in vertical strips in the first variable locally uniformly in the second variable and moreover for every connected relatively compact open subset $U' \subseteq M$ containing $U$, there exists $C > 0$ such that $Z_k$ admits a (necessarily unique) holomorphic continuation $\mathcal{H}_C \times U' \to \mathbb{C}$ which is again rapidly decreasing in vertical strips in the first variable locally uniformly in the second variable.

(2) There exists $q > 1$ such that for every $k \in \{1,2\}$, $Z_k$ is periodic of period $\log(q)^{-1}2i\pi$ in the first variable and moreover for every connected relatively compact open subset $U' \subseteq M$ containing $U$, there exists $C > 0$ such that $Z_k$ admits a (necessarily unique) holomorphic continuation $\mathcal{H}_C \times U' \to \mathbb{C}$.

Then, $Z_1$ and $Z_2$ extend analytically to all of $\mathbb{C} \times M$. Moreover, these extensions still satisfy (2.8.1) and in case (1) they are again rapidly decreasing in vertical strips in the first variable locally uniformly in the second variable.

2.9 A détour: extended Schwartz spaces and holomorphic functions valued in generalized Paley-Wiener spaces

Let $C \in \mathbb{R} \cup \{-\infty\}$ and $q > 1$. Let $A \in \{\mathbb{R}_+^*, q^\mathbb{Z}\}$. Set $D = \frac{d}{dx}$ if $A = \mathbb{R}_+^*$ and $D = 0$ if $A = q^\mathbb{Z}$. Then, we define an "extended" Schwartz space $S_{C}^\infty(A)$ as the space of smooth functions $\phi : A \to \mathbb{C}$ such that for every $R > C$ and every integer $k \geq 0$ we have an inequality

$$|(D^k \phi)(x)| \ll_{R,k} x^R, \quad x \in A.$$
Then, \( S_C^+(A) \) has a natural structures of Fréchet spaces and we can recover the usual Schwartz space as

\[
S(A) = S_{-\infty}^+(A).
\]

Notice that, in some imprecise sense, an element of \( S_C^+(A) \) is “Schwartz in a neighborhood of 0” and of “moderate growth controlled by \( C \) in a neighborhood of \( +\infty \)” (Remark that if \( C < 0 \) this notion of moderate growth implies that the function actually converges to 0 at \( +\infty \)). Let \( \iota \) be the endomorphism of \( (S(A) \oplus S(A))^\iota \) which sends \((\phi_1, \phi_2)\) to \((\phi_1^\gamma, \phi_2^\gamma)\) where, as usual, \( \phi^\gamma \) denotes the function \( \phi^\gamma(x) = \phi(x^{-1}) \). Let \((S(A) \oplus S(A))^\iota\) be the subspace of \( \iota \)-fixed vectors. Then we easily check that

\[
(S(A) \oplus S(A))^\iota \text{ is a closed subspace of } S_C^+(A) \oplus S_C^+(A).
\]

We define \( PW_A(H_C) \) as the space of holomorphic functions \( f : H_C \to \mathbb{C} \) satisfying:

- If \( A = \mathbb{R}^*_+ \), \( f \) is rapidly decreasing in vertical strips: for every vertical strip \( V \subset H_C \) and every \( N > 0 \) we have
  \[
  \sup_{s \in V} (1 + |\Im(s)|)^N |f(s)| < \infty.
  \]

- If \( A = q^\mathbb{Z} \), \( f \) is periodic of period \( \log(q)^{-1}2i\pi \).

Then, \( PW_A(H_C) \) has a natural topology making it into a Fréchet space (if \( A = \mathbb{Z} \), we equip it with the topology of uniform convergence on compact subsets). To every \( \phi \in S_C^+(A) \) we associate a function \( \mathcal{M} \phi \in PW_A(H_C) \), its Mellin transform, given by

\[
(\mathcal{M} \phi)(s) = \int_{\mathbb{R}^*_+} \phi(x)x^{-s} dx \text{ if } A = \mathbb{R}^*_+,
\]

\[
(\mathcal{M} \phi)(s) = \sum_{x \in q \mathbb{Z}} \phi(x)x^{-s} \text{ if } A = q^\mathbb{Z}.
\]

Then \( \phi \mapsto \mathcal{M} \phi \) is a topological isomorphism

\[
S_C^+(A) \simeq PW_A(H_C)
\]

whose inverse \( f \mapsto \mathcal{M}^{-1} f \) is given by

\[
(\mathcal{M}^{-1} f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(s + i\xi)x^{s+i\xi} d\xi \text{ for any } s \in H_C \text{ if } A = \mathbb{R}^*_+,
\]

\[
(\mathcal{M}^{-1} f)(x) = \frac{\log(q)}{2\pi} \int_{\mathbb{R}/\log(q)^{-1}2\pi \mathbb{Z}} f(s + i\xi)x^{(s+i\xi)} d\xi \text{ for any } s \in H_C \text{ if } A = q^\mathbb{Z}.
\]

(see [Sak, Theorem 3.1.1] for the case where \( A = \mathbb{R}^*_+ \) and \( C = -\infty \). The proof, which is standard, extends without difficulty to the general case although we will not write the details here).
Let $M$ be a connected complex analytic variety. For any TVS $F$, we denote by $\text{Hol}(M, F)$ the space of holomorphic functions $M \to F$. By what precedes we have an isomorphism of vector spaces

$$\text{Hol}(M, S^+_C(A)) \simeq \text{Hol}(M, PW_A(H_C))$$

$$\mathcal{Z} \mapsto \hat{\mathcal{Z}} := \mathcal{M} \circ \mathcal{Z}.$$ Moreover, we can naturally identify $\text{Hol}(M, PW_A(H_C))$ with a certain space of functions $H_C \times M \to \mathbb{C}$. The next lemma characterizes the space of such functions.

**Lemma 2.9.1** A function $\mathcal{Z} : H_C \times M \to \mathbb{C}$ defines an element of $\text{Hol}(M, PW_A(H_C))$ if and only if it satisfies the following conditions:

- $\mathcal{Z}$ is holomorphic;
- If $A = \mathbb{R}^*_+$, $\mathcal{Z}$ is rapidly decreasing in vertical strips in the first variable locally uniformly in the second variable;
- If $A = q^{-z}$, $\mathcal{Z}$ is periodic of period $\log(q)^{-1}2i\pi$ in the first variable.

**Proof:** If $\mathcal{Z}$ is a holomorphic function $M \to PW_A(H_C)$ then the associated function $H_C \times M \to \mathbb{C}$ is analytic in each variable hence jointly analytic by Hartog’s theorem. If $A = q^{-z}$ then obviously the associated function is periodic of period $\log(q)^{-1}2i\pi$ in the first variable. If $A = \mathbb{R}^*_+$ then, $\mathcal{Z}$ being continuous, the image by $\mathcal{Z}$ of any compact subset $K_M \subseteq M$ is a compact subset of $PW_A(H_C)$ and this readily implies that the associated function $H_C \times M \to \mathbb{C}$ is rapidly decreasing in vertical strips in the first variable locally uniformly in the second variable.

Let now $\mathcal{Z} : H_C \times M \to \mathbb{C}$ be a function satisfying the conditions of the lemma. Clearly, for every $t \in M$ the function $s \in H_C \mapsto \mathcal{Z}(s, t)$ belongs to $PW_A(H_C)$ and we want to show that the induced map $M \to PW_A(H_C)$, $t \mapsto \mathcal{Z}(., t)$, is holomorphic. We first show that this map is continuous. If $A = q^{-z}$ this amounts to the following: for every compact subset $K_C \subseteq H_C$, every $t_0 \in M$ and every $\epsilon > 0$ if $t \in M$ is sufficiently close to $t_0$ then

$$\sup_{s \in K_C} |\mathcal{Z}(s, t) - \mathcal{Z}(s, t_0)| < \epsilon.$$ But this follows immediately from the joint continuity of $\mathcal{Z}$. If $A = \mathbb{R}^*_+$, we need to show the following: for every vertical strip $V \subseteq H_C$, every $N > 0$, every $t_0 \in M$ and every $\epsilon > 0$ if $t \in M$ is sufficiently close to $t_0$ then

$$\sup_{s \in V} (1 + |\Im(s)|)^N |\mathcal{Z}(s, t) - \mathcal{Z}(s, t_0)| < \epsilon.$$ Let $K_M \subseteq M$ be a compact neighborhood of $t_0$. Then, by the assumption that $\mathcal{Z}$ is rapidly decreasing in vertical strips in the first variable locally uniformly in the second variable, for $T > 0$ sufficiently large we have

$$\sup_{s \in V} \frac{(1 + |\Im(s)|)^N |\mathcal{Z}(s, t)|}{|\Im(s)| > T} \leq (1 + T)^{-1} \sup_{s \in V} (1 + |\Im(s)|)^{N+1} |\mathcal{Z}(s, t)| < \frac{\epsilon}{2}.$$
for every $t \in \mathcal{K}_M$. On the other hand, by continuity of $\mathcal{Z}$, for $t$ sufficiently close to $t_0$ we have

$$
(2.9.5) \quad \sup_{s \in Y} (1 + |\Im(s)|)^N |\mathcal{Z}(s,t) - \mathcal{Z}(s,t_0)| < \epsilon.
$$

From (2.9.4) and (2.9.5) we deduce (2.9.3) and thus the continuity of $t \in M \mapsto \mathcal{Z}(.,t) \in PW_A(\mathcal{H}_C)$ when $A = \mathbb{R}_+^*$. As $\mathcal{Z}$ is holomorphic, for every $s \in \mathcal{H}_C$ the function $t \in M \mapsto \mathcal{Z}(s,t)$ is holomorphic. To conclude we apply the criterion (2.2.1) to the total subset $H$ of $PW_A(\mathcal{H}_C)'$ corresponding to “evaluations at a point $s \in \mathcal{H}_C$”.

2.10 Proof of Proposition [2.8.1]

Set $A = \mathbb{R}_+^*$ if $Z_1$, $Z_2$ satisfy condition (1) of Proposition [2.8.1] and $A = q\mathbb{R}$ if they satisfy condition (2). Then, by Lemma [2.9.1] $\mathcal{Z} = (Z_1, Z_2)$ defines an holomorphic function $U \mapsto PW_A(\mathbb{C}) \oplus PW_A(\mathbb{C})$ i.e., by the isomorphism (2.9.2) an element $\hat{\mathcal{Z}}$ of $\text{Hol}(U, S(A) \oplus S(A))$. Let $\iota$ be the endomorphism of $S(A) \oplus S(A)$ defined in Section 2.9. Then, the functional equation implies $\hat{\mathcal{Z}}$ takes its values in $(S(A) \oplus S(A))^\iota$. Still by Lemma [2.9.1], the assumption of the proposition translates into: for every connected relatively compact open subset $U' \subseteq M$ containing $U$, there exists $C > 0$ such that after composing by the continuous inclusion $S(A) \subseteq S_C^+(A)$, $\hat{\mathcal{Z}}$ extends to a holomorphic function $U' \rightarrow S_C^+(A) \oplus S_C^+(A)$. Since $(S(A) \oplus S(A))^\iota$ is a closed subspace of $S_C^+(A) \oplus S_C^+(A)$ (by 2.9.1) and the image of $U$ by $\hat{\mathcal{Z}}$ lands into that subspace so does its extension to $U'$ (by connectedness). Therefore $\hat{\mathcal{Z}}$ extends to a holomorphic map $U' \rightarrow (S(A) \oplus S(A))^\iota$ for every connected relatively compact open subset $U' \subseteq M$ containing $U$. It follows that $\hat{\mathcal{Z}}$ actually extends to a holomorphic map $M \rightarrow S(A) \oplus S(A)$ hence $\mathcal{Z}$ extends to a holomorphic map $M \rightarrow PW_A(\mathbb{C}) \oplus PW_A(\mathbb{C})$. Using again Lemma [2.9.1] this gives exactly the conclusion of Proposition 2.8.1.

3 Zeta integrals and statement of the main theorems

Let $F$ be a local field of characteristic zero and $E$ be either a quadratic extension of $F$ (inert case) or $F \times F$ (split case). We write $|.|_F$ and $|.|_E$ for the normalized absolute values of $F$ and $E$ respectively. Thus, in the split case we have $|(\lambda, \mu)|_E = |\lambda|_F |\mu|_F$ for every $(\lambda, \mu) \in E$ and in both cases we have $|x|_E = |x|_F^2$ for every $x \in F$. In the non-Archimedean case, we let $\mathcal{O}_F$ and $\mathcal{O}_E$ be the rings of integers of $F$ and $E$ respectively. We fix non-trivial additive characters $\psi' : F \rightarrow S^1$ and $\psi : E \rightarrow S^1$ and we assume that $\psi$ is trivial on $F$. We denote by $\tau$ the unique element of $E$ such that $\psi(z) = \psi'(\text{Tr}_{E/F}(\tau z))$ for every $z \in E$ where $\text{Tr}_{E/F}$ stands for the trace of the extension $E/F$.

Let $n \geq 1$ be an integer. We write $G_n$ for the algebraic group $\text{GL}_n$, $\mathfrak{g}_n$ for its Lie algebra and we denote by $Z_n$, $B_n$, $A_n$ and $N_n$ the subgroups of scalar, resp. upper triangular, resp. diagonal, resp. unipotent upper triangular matrices in $G_n$. By a standard parabolic subgroup $P$ of $G_n$ we mean a parabolic subgroup containing $B_n$. We then write $P = MU$.
for its unique Levi decomposition with $A_n \subseteq M$. We denote by $\delta_n$ and $\delta_{n,E}$ the modular characters of $B_n(F)$ and $B_n(E)$ respectively. We will denote by $K_n$ the standard maximal compact subgroup of $G_n(F)$ that is $K_n = G_n(O_F)$ in the $p$-adic case and $K_n = O(n)$ or $U(n)$ in the cases where $F = \mathbb{R}$ or $\mathbb{C}$ respectively. For $g \in G_n$, we write $^t g$ for its transpose and $g_{i,j}$, $1 \leq i, j \leq n$, for its entries. Also, for $a \in A_n$ we simply write $a_i$ for $a_{i,i}$ ($1 \leq i \leq n$). Let $\mathcal{A}^\times = \mathcal{A}_{A_n}^\times$, $\mathcal{A}_C^\times = \mathcal{A}_{A_n,C}^\times$ and $(\mathcal{A}^{G_n})^\times = (\mathcal{A}_{A_n}^{G_n})^\times$. We identify $\mathcal{A}^\times$ with $\mathbb{R}^n$ through the choice of the basis $\chi_1, \ldots, \chi_n$ of $X^*(A_n)$ where $\chi_i (1 \leq i \leq n)$ is defined by $\chi_i(a) = a_i$ for every $a \in A_n$. Then, the closed negative Weyl chamber $(\mathcal{A}^\times)^+$ with respect to $B_n$ is the cone of $n$-uples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that $\lambda_1 \leq \ldots \leq \lambda_n$ and the subspace $(\mathcal{A}^{G_n})^\times$ consists of vectors $\lambda \in \mathbb{R}^n$ with $\lambda_1 + \ldots + \lambda_n = 0$. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{A}^\times$. Then, defining $|\lambda|$ as in Section 2.1 we have $|\lambda| = (\lambda_{w(1)}, \ldots, \lambda_{w(n)})$ where $w$ is the unique permutation of $\{1, \ldots, n\}$ such that $\lambda_{w(1)} \leq \ldots \leq \lambda_{w(n)}$. Moreover, the character $a \mapsto a^\lambda$ of $A_n(E)$ is given by

$$a^\lambda = |a_1|_{E}^{\lambda_1} \ldots |a_n|_{E}^{\lambda_n}.$$  

We set $\min(\lambda) := \min(\lambda_1, \ldots, \lambda_n)$. Notice that if $\lambda \in (\mathcal{A}^\times)^+$ then $\min(\lambda) = \lambda_1$.

For $\pi \in \text{Irr}(G_n(E))$ we will denote by $\omega_p \iota$ its central character seen as a character on $E^\times$ through the natural identification $Z_n(E) \cong E^\times$. Let $P = MU$ be a standard parabolic subgroup of $G_n$. Then $M$ is of the form

$$M = G_{n_1} \times \ldots \times G_{n_k}$$

for some integers $n_1, \ldots, n_k$ with $n_1 + \ldots + n_k = n$. Let $\tau_i \in \text{Irr}(G_{n_i}(E))$ for every $1 \leq i \leq k$. Then, $\sigma = \tau_1 \boxtimes \ldots \boxtimes \tau_k$ is an irreducible representation of $M(E)$ and we will use the notation

$$\tau_1 \times \ldots \times \tau_k$$

to denote the induced representation $^i_{\iota P(E)}(\sigma)$. A representation $\pi \in \text{Irr}(G_n(E))$ is \textit{generic} if it admits a nonzero Whittaker functional with respect to any (or equivalently one) generic character of $N_n(E)$. We will denote by $\text{Irr}_{\text{gen}}(G_n(E))$ the subset of generic representations in $\text{Irr}(G_n(E))$. By [Zel] Theorem 9.7 and [Vog] Theorem 6.2f, every $\pi \in \text{Irr}_{\text{gen}}(G_n(E))$ is isomorphic to a representation of the form $\tau_1 \times \ldots \tau_k$ where for each $1 \leq i \leq k$, $\tau_i$ is an essentially square-integrable (i.e. an unramified twist of a square-integrable) representation of some $G_{n_i}(E)$. We will say that a representation $\pi$ of $G_n(E)$ is \textit{nearly tempered} if it is isomorphic to an induced representation of the form

$$(\tau_1 \boxtimes |\det|_{E}^{\lambda_1}) \times \ldots \times (\tau_k \boxtimes |\det|_{E}^{\lambda_k})$$

where for each $1 \leq i \leq k$, $\tau_i \in \Pi_2(G_{n_i}(E))$ for some $n_i \geq 1$ and $\lambda_i$ is a real number with $|\lambda_i| < \frac{1}{2}$. Again by [Zel] Theorem 9.7 and also [VS], every nearly tempered representation is irreducible and generic. We will denote by $\text{Irr}_{\text{n temp}}(G_n(E)) \subseteq \text{Irr}_{\text{gen}}(G_n(E))$ the subset of nearly tempered representations.

We equip $\text{Temp}(G_n(E))$ with a topology as follows. For each standard parabolic subgroup $P = MU$ of $G_n$ and $\sigma \in \Pi_2(M(E))$ the map $\lambda \in iA^*_M \mapsto \iota_{P(E)}^G(\sigma_\lambda) \in \text{Temp}(G_n(E))$ identifies
a certain quotient of $i\mathcal{A}_M^*$ with a subset of Temp($G_n(E)$). Then we endow Temp($G_n(E)$) with the unique topology whose connected components are precisely these subsets equipped with the quotient topology from $i\mathcal{A}_M^*$.

We define generic characters $\psi_n : N_n(E) \to \mathbb{S}^1$, $\psi_n' : N_n(F) \to \mathbb{S}^1$ by

$$\psi_n'(u) = \psi'\left((-1)^n \sum_{i=1}^{n-1} u_{i,i+1}\right) \text{ and } \psi_n(u) = \psi\left((-1)^n \sum_{i=1}^{n-1} u_{i,i+1}\right).$$

If $\pi$ is an irreducible generic representation of $G_n(E)$, we write $\mathcal{W}(\pi, \psi_n)$ for its Whittaker model (with respect to $\psi_n$). For every Whittaker function $W \in C^\infty(N_n(E)\backslash G_n(E), \psi_n)$, we define $\widehat{W} \in C^\infty(N_n(E)\backslash G_n(E), \psi_n^{-1})$ by

$$\widehat{W}(g) = W(w_n^t g^{-1}) \quad (g \in G_n(E))$$

where $w_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. Then, if $\pi$ is an irreducible generic representation of $G_n(E)$ the map $W \mapsto \widehat{W}$ induces a (topological) isomorphism $\mathcal{W}(\pi, \psi_n) \simeq \mathcal{W}(\widehat{\pi}, \psi_n^{-1})$.

We let $\mathcal{S}(F^n) = C^\infty_c(F^n)$ be the space of all locally constant and compactly supported complex-valued functions on $F^n$ in the $p$-adic case and $\mathcal{S}(F^n)$ be the usual Schwartz space on $F^n$ in the Archimedean case. We denote by $\phi \mapsto \hat{\phi}$ be the usual Fourier transform on $F^n$ defined using the additive character $\psi'$ and the corresponding autodual measure i.e. for every $\phi \in \mathcal{S}(F^n)$ we have

$$\hat{\phi}(x_1, \ldots, x_n) = \int_{F^n} \phi(y_1, \ldots, y_n) \psi'(x_1 y_1 + \ldots + x_n y_n) dy_1 \ldots dy_n, \quad (x_1, \ldots, x_n) \in F^n$$

where the measure of integration is chosen so that $\hat{\phi}(v) = \phi(-v)$. Finally, we write $e_n$ for the vector $e_n = (0, \ldots, 0, 1) \in F^n$.

### 3.1 Asai $L$-functions and epsilon factors

Let $W_F$ be the Weil group of $F$ and set

$$W_F' = \begin{cases} W_F \times SL_2(\mathbb{C}) & \text{if } F \text{ is } p \text{-adic} \\ W_F & \text{if } F \text{ is Archimedean} \end{cases}$$

for the Weil-Deligne group of $F$. In the inert case, we define similarly the Weil-Deligne group $W_E'$ of $E$. An admissible complex representation of $W_F'$ (resp. $W_E'$) is by definition a continuous morphism $\phi : W_F' \to GL(M)$ (resp. $\phi : W_E' \to GL(M)$) where $M$ is a finite dimensional complex vector space which is semi-simple and algebraic when restricted to $SL_2(\mathbb{C})$ (in the $p$-adic case). To any admissible complex representation $\phi : W_F' \to GL(M)$ we associate a local $L$-factor $L(s, \phi)$ and a local $\epsilon$-factor $\epsilon(s, \phi, \psi')$ as in [Ta §3] and [GR].
§2.2. In the $p$-adic case, $L(s, \phi)$ is of the form $P(q^{-s})$ where $P \in \mathbb{C}[T]$ is such that $P(0) = 1$ whereas $\epsilon(s, \phi, \psi')$ is of the form $c q^{n(s-1/2)}$ where $n \in \mathbb{Z}$ and $c = \epsilon(1/2, \phi, \psi') \in \mathbb{C}^\times$. In the Archimedean case, $L(s, \phi)$ is a product of $\dim(M)$ factors of the form $\pi^{-s(s_0)/2} \Gamma((s + s_0)/2)$ for some $s_0 \in \mathbb{C}$ where $\Gamma$ denotes the usual gamma function and $\epsilon(s, \varphi, \psi')$ is of the form $c Q^{s-1/2}$ where $Q \in \mathbb{R}_+^*$ and $c = \epsilon(1/2, \varphi, \psi') \in \mathbb{C}^\times$. Using these invariants, we define the local $\gamma$-factor of $\phi$ as

$$\gamma(s, \phi, \psi') = \epsilon(s, \phi, \psi') \frac{L(1-s, \bar{\phi})}{L(s, \phi)}$$

where $\bar{\phi}$ stands for the contragredient of $\phi$. Setting $\psi'_\lambda(x) = \psi'(\lambda x)$ for every $(\lambda, x) \in F^\times \times F$ we have

$$\epsilon(s, \phi, \psi'_\lambda) = (\det(\phi)(\lambda)|\lambda|_F^{\dim(M)(s-\frac{1}{2})}) \epsilon(s, \phi, \psi')$$

for every $\lambda \in F^\times$ where we have identified $\det \phi$ with a character of $F^\times$ through class field theory. Moreover, if we identify the absolute value $||_F$ to a character $W'_F \to \mathbb{R}_+^*$ (again through class field theory) we have $L(s, \phi \otimes |.|_F^{s_0}) = L(s + s_0, \phi)$ and $\epsilon(s, \phi \otimes |.|_F^{s_0}, \psi') = \epsilon(s + s_0, \phi, \psi')$ for every $s_0 \in \mathbb{C}$. Let $\eta_{E/F}$ be the quadratic character of $W'_E$ associated to the extension $E/F$ and set

$$\lambda_{E/F}(\psi') = \epsilon\left(\frac{1}{2}, \eta_{E/F}, \psi'\right).$$

This is sometimes called the Langlands constant of the extension $E/F$. It is a fourth root of unity which is trivial in the split case. Moreover, in the inert case if $\phi$ is an admissible complex representation of $W'_E$ of dimension $n$, by the inductivity of $\epsilon$-factors in degree 0 ([Ta, Theorem 3.4.1]) we have

$$(3.1.1) \quad \epsilon(s, \phi, \psi'_E) = \lambda_{E/F}(\psi')^{-n} \gamma(s, \text{Ind}_{W'_E}^W(\phi), \psi')$$

where we have set $\psi'_E := \psi' \circ \text{Tr}_{E/F}$ and $\text{Ind}_{W'_E}^W$ denotes the functor of induction from $W'_E$ to $W_F$ (which sends admissible representations to admissible representations).

In the proofs, it will be convenient to modify slightly the definition of the local $L$-functions in the Archimedean case. For every $s \in \mathbb{C}$, we let $K_s$ be the modified $K$-Bessel function of the second kind (see e.g. [W, Appendix B.4]). It is a function of a complex variable $z \in \mathbb{C}$ and for fixed $z$ the function $s \mapsto K_s(z)$ is holomorphic. Then, for $\phi : W'_F \to \text{GL}(M)$ an admissible complex representation and $z \in \mathbb{R}_+^*$ we set

$$L_{K,z}(s, \phi) = \begin{cases} L(s, \phi) & \text{if } F \text{ is } p \text{-adic} \\ K_{\frac{s}{2} - \frac{1}{4}}(z)^{-\dim(M)} L(s, \phi) & \text{if } F \text{ is Archimedean}. \end{cases}$$

The basic properties of this modified $L$-function are summarized in the next lemma, the crucial point being property (iii) which is not satisfied by local $L$-functions in the Archimedean case.

**Lemma 3.1.1** Let $\phi : W'_F \to \text{GL}(M)$ be an admissible complex representation and $z \in \mathbb{R}_+^*$. Then:
(i) \( L_{K,z}(s, \phi) \) is meromorphic and does not vanish;

(ii) In the p-adic case, \( s \mapsto L_{K,z}(s, \phi) \) is periodic of period \( \log(q)^{-1}2i\pi \);

(iii) In the Archimedean case, for every vertical strip \( V \subseteq \mathbb{C} \) there exists \( d_V > 0 \) such that
\[
|L_{K,z}(s, \phi)^{-1}| \ll |s|^{d_V}
\]
for every \( s \in V \);

(iv) We have the equality
\[
\gamma(s, \phi, \psi) = \epsilon(s, \phi, \psi') L_{K,z}(1 - s, \tilde{\phi}) L_{K,z}(s, \phi).
\]

Proof: (i) follows from the fact that \( s \mapsto L(s, \phi) \) does not vanish and \( s \mapsto K_s(z) \) is holomorphic (this follows e.g. from the fact that \( K_s(z) \) is the Mellin transform of \( t \mapsto \frac{1}{2}e^{-\frac{1}{2}(t+1)} \) see the 4th equality on the top of [Iw, p.205]). (ii) is clear given the form of the local L-function in the p-adic case. (iv) is a consequence of the fact that \( s \mapsto K_s(z) \) is even. It only remains to show (iii). By the form of the local L-functions in the Archimedean case, it boils down to showing that for every \( s_0 \in \mathbb{C} \) and every vertical strip \( V \subseteq \mathbb{C} \) there exists \( d = d_{V,s_0} \) such that
\[
|\frac{K_s(z)}{\Gamma(s + s_0)}| \ll |s|^d, \quad s \in V.
\]

By [Wall2, Lemma 10.A.2.2], for any \( s_0 \in \mathbb{C} \) the function \( \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+s_0)} \) is essentially bounded in absolute value in any vertical strip by a power of \( |s| \) away from its (finitely many) poles. Therefore, we are reduced to showing the above inequality for \( s_0 = \frac{1}{2} \). By the second equality on the top of [Iw, p.205], for \( \Re(s) > 0 \) we have
\[
\frac{K_s(z)}{\Gamma(s + \frac{1}{2})} = \pi^{-1/2} (\frac{z}{2})^{-s} \int_0^{+\infty} (t^2 + 1)^{-s-\frac{1}{2}} \cos(tz) dt.
\]

By integration by parts and an easy induction for every integer \( k \geq 0 \) there exists polynomials \( P_{z,k,i} \in \mathbb{C}[T] \), \( k \leq i \leq 2k \) such that
\[
\int_0^{+\infty} (t^2 + 1)^{-s-\frac{1}{2}} \cos(tz) dt = \sum_{i=k}^{2k} P_{z,k,i}(s) \int_0^{+\infty} (t^2 + 1)^{-s-\frac{1}{2}-i} \cos(tz) dt.
\]

Clearly, the integral \( \int_0^{+\infty} (t^2 + 1)^{-s-\frac{1}{2}-i} \cos(tz) dt \) is absolutely convergent in \( \{\Re(s) > -i\} \) and defines a holomorphic function there which is bounded in vertical strips. The claim follows.

Assume that we are in the inert case. Fix \( s \in W'_F \setminus W'_E \). Let \( \phi : W'_E \to \text{GL}(M) \) be an admissible representation. We defined \( \text{As}(\phi) : W'_F \to \text{GL}(M \otimes M) \) by \( \text{As}(\phi)(w) = \phi(w) \otimes \ldots \)

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\( \phi(sw^{-1}) \) for \( w \in W'_E \) and \( \text{As}(\phi)(s) = (\text{Id}_M \otimes \phi(s^2)) \circ \iota \) where \( \iota \) is the linear automorphism of \( M \otimes M \) sending \( u \otimes v \) to \( v \otimes u \). Then, \( \text{As}(\phi) \) is an admissible representation of \( W'_E \) and we set \( L(s, \phi, As) = L(s, \text{As}(\phi)) \), \( \epsilon(s, \phi, As, \psi') = \epsilon(s, \phi, As, \psi') \), \( \gamma(s, \phi, As, \psi') = \gamma(s, \phi, As, \psi') \) and call them the Asai \( L \)-function, \( \epsilon \)-factor and \( \gamma \)-factor of \( \phi \) respectively. If \( \phi \) decomposes as a direct sum
\[
\phi = \phi_1 \oplus \ldots \oplus \phi_k
\]
of admissible representations of \( W'_E \) then we have
\[
(3.1.2) \quad \text{As}(\phi) = \bigoplus_{i=1}^{k} \text{As}(\phi_i) \oplus \bigoplus_{1 \leq i < j \leq k} \text{Ind}_{W'_E}^{W'_E}(\phi_i \otimes \phi_j^c)
\]
where \( \phi_j^c \) stands for the representation \( w \mapsto \phi_j(sw^{-1}) \). We have \( \text{As}(\phi \otimes \cdot |_{F^0}) = \text{As}(\phi \otimes \cdot |_{F^2}) \) for every \( s_0 \in \mathbb{C} \). Moreover, if \( \phi = \chi \) is a character of \( W'_E \) that we identify to a character of \( E^\times \) through class field theory then \( \text{As}(\chi) \) is the character of \( W'_E \) corresponding to the restriction of \( \chi \) to \( F^\times \) via class field theory.

Let \( \pi \in \text{Irr}(G_n(E)) \). Then, the local Langlands correspondence for \( \text{GL}_n \) (\La, 
HT, 
Hen, 
Sch) associates to \( \pi \) an admissible complex representation \( \phi_\pi : W'_E \rightarrow \text{GL}(M) \) of dimension \( n \). We set \( L(s, \pi, As) = L(s, \phi_\pi, As) \), \( \epsilon(s, \pi, As, \psi') = \epsilon(s, \phi_\pi, As, \psi') \) and \( \gamma(s, \pi, As, \psi') = \gamma(s, \phi_\pi, As, \psi') \) and call them the Asai \( L \)-function, \( \epsilon \)-factor and \( \gamma \)-factor of \( \pi \) respectively. We have \( \det \text{As}(\phi_\pi) = (\omega_\pi)^n \eta_{E/F}(n^{n-1}/2) \) and therefore
\[
(3.1.3) \quad \epsilon(s, \pi, As, \psi') = \omega_\pi(\lambda)^n |\lambda|_{F^2}^{n^2(s-\frac{1}{2})} \eta_{E/F}(\lambda)^{n(n-1)/2} \epsilon(s, \pi, As, \psi')
\]
for every \( \lambda \in F^\times \). Let \( z \in \mathbb{R}^+ \). We also set
\[
L_{K,z}(s, \pi, As) = L_{K,z}(s, \text{As}(\phi_\pi)) = \begin{cases} L(s, \pi, As) & \text{if } F \text{ is } p \text{-adic} \\ K_{\frac{2}{d} - \frac{1}{2}}(z)^{-n^2} L(s, \pi, As) & \text{if } F \text{ is Archimedean} \end{cases}
\]
Assume now that we are in the split case. Let \( \pi = \pi_1 \boxtimes \pi_2 \) be an irreducible representation of \( G_n(E) = G_n(F) \times G_n(F) \). Let \( \phi_1, \phi_2 : W'_E \rightarrow \text{GL}_n(\mathbb{C}) \) be the admissible complex representations associated to \( \pi_1 \) and \( \pi_2 \) respectively by the local Langlands correspondence. Then, we set
\[
L(s, \pi, As) = L(s, \pi_1 \times \pi_2) = L(s, \phi_1 \otimes \phi_2), \quad \epsilon(s, \pi, As, \psi') = \epsilon(s, \pi_1 \times \pi_2, \psi') = \epsilon(s, \phi_1 \otimes \phi_2, \psi')
\]
and
\[
\gamma(s, \pi, As, \psi') = \gamma(s, \pi_1 \times \pi_2, \psi') = \gamma(s, \phi_1 \otimes \phi_2, \psi').
\]
We now return to the inert case. Let \( P = MU \) be a standard parabolic subgroup of \( G_n \) and \( n_1, \ldots, n_k \in \mathbb{N}^* \) so that
\[
M(E) = G_{n_1}(E) \times \ldots \times G_{n_k}(E).
\]
For each $1 \leq i \leq k$, let $\sigma_i \in \Pi_2(G_{n_i}(E))$ and set $\sigma = \sigma_1 \boxtimes \cdots \boxtimes \sigma_k \in \Pi_2(M(E))$. We identify $A_{M,C}^*$ with $C^k$ so that for every $(\lambda_1, \ldots, \lambda_k) \in C^k$, we have
\[
\sigma_\lambda = \sigma_1|\det|_E^{\lambda_1} \boxtimes \cdots \boxtimes \sigma_k|\det|_E^{\lambda_k}.
\]
We set $\pi_\lambda = \pi_{G_{n}(E)}^{\Gamma_{\lambda}}(\sigma_\lambda)$ for every $\lambda \in A_{M,C}^*$. Then, $\pi_\lambda$ is irreducible for almost every $\lambda$. For such $\lambda$, $L(s, \pi_\lambda, As)$, $\epsilon(s, \pi_\lambda, As, \psi')$ and $\gamma(s, \pi_\lambda, As, \psi')$ are defined as above. For the remaining $\lambda$’s, we define these factors as the one associated to the unique irreducible subquotient $\pi_\lambda^0$ of $\pi_\lambda$ which is the Langlands quotient of $\pi_{Q(E)}^{G_{n}(E)}(\sigma_\lambda)$ for any parabolic $Q$ with Levi $M$ for which $\lambda$ is positive (in the large sense).

**Lemma 3.1.2**

(i) For every $\lambda = (\lambda_1, \ldots, \lambda_k) \in A_{M,C}^*$ we have
\[
L(s, \pi_\lambda, As) = \prod_{i=1}^{k} L(s + 2\lambda_i, \sigma_i, As) \prod_{1 \leq i < j \leq k} L(s + \lambda_i + \lambda_j, \sigma_i \times \sigma_j)
\]
and
\[
\epsilon(s, \pi_\lambda, As, \psi') = \lambda_{E/F}(\psi') \frac{k(k-1)}{2} \prod_{i=1}^{k} \epsilon(s + 2\lambda_i, \sigma_i, As, \psi') \prod_{1 \leq i < j \leq k} \epsilon(s + \lambda_i + \lambda_j, \sigma_i \times \sigma_j, \psi'_{E}).
\]

(ii) Set $U = \{ (\lambda_1, \ldots, \lambda_k) \in A_{M,C}^* \mid |\Re(\lambda_i)| < \frac{1}{4} \}$. Then, for every $s \in C$ with $\Re(s) = \frac{1}{2}$ the function $\lambda \in U \mapsto \gamma(s, \pi_\lambda, As, \psi')$ is holomorphic. Moreover, for $\pi$ nearly tempered, $L(s, \pi, As)$ has no poles in $\{ \Re(s) \geq \frac{1}{2} \}$.

(iii) For any $z \in \mathbb{R}_+^*$ the function $(s, \lambda) \mapsto L_{K,z}(s, \pi_\lambda, As)^{-1}$ is holomorphic. Moreover, in the Archimedean case it is of moderate growth in vertical strips in the first variable locally uniformly in the second variable i.e. for every vertical strip $V \subseteq C$ and compact subset $K \subseteq \mathbb{C}^*$ there exists $d > 0$ such that
\[
|L_{K,z}(s, \pi_\lambda, As)^{-1}| \ll |s|^d
\]
for every $s \in V$ and $\lambda \in K$.

**Proof:**

(i) By compatibility of the Langlands correspondence with parabolic induction and unramified twists, for every $\lambda = (\lambda_1, \ldots, \lambda_k) \in A_{M,C}^*$ the Langlands parameter of $\pi_\lambda$ (or rather $\pi_\lambda^0$) is given by
\[
\phi_{\sigma_\lambda} = \phi_{\sigma_1} \boxtimes \phi_{\lambda_1} |_{E} \oplus \cdots \oplus \phi_{\sigma_k} \boxtimes \phi_{\lambda_k} |_{E}.
\]
Therefore, the identities of the lemma follow directly from 3.1.1 and 3.1.2.

(ii) is a consequence of (i) and the fact that for $\pi_1$, $\pi_2$ tempered the $L$-functions $L(s, \pi_1, As)$ and $L(s, \pi_1 \times \pi_2)$ have no poles in $\{ \Re(s) > 0 \}$.

(iii) The first part is again a direct consequence of (i). The second part on the other hand follows from (i) and Lemma 3.1.1(iii). ■

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3.2 Definition and convergence of the local Zeta integrals

Let $\pi \in \text{Irr}_{\text{gen}}(G_n(E))$. For every $W \in W(\pi, \psi_n)$, $\phi \in S(F^n)$ and $s \in \mathbb{C}$ we define, whenever convergent, a Zeta integral

$$Z(s, W, \phi) = \int_{N_n(F) \backslash G_n(F)} W(h)\phi(e_nh)|\det h|^{s}_F dh.$$ 

Set $A = q^Z$ in the non-Archimedean case and $A = \mathbb{R}^*_+$ in the Archimedean case. Recall that for every $C \in \mathbb{R}$ we have set $H_C = \{ z \in \mathbb{C} \mid \mathfrak{R}(z) > C \}$ and in Section 2.3 we have defined a certain space $PW_A(H_C)$ of holomorphic functions on $H_C$. By Corollary 2.7.1 there exists $\mu \in \mathcal{A}^*$ such that $W(\pi, \psi_n) \subseteq C_\mu(N_n(E) \backslash G_n(E), \psi_n)$. Thus, the next lemma will show that the above Zeta integrals are at least convergent in some right half-plane.

**Lemma 3.2.1** Let $\mu \in \mathcal{A}^*$ and $\phi \in S(F^n)$. Then, for every $W \in C_\mu(N_n(E) \backslash G_n(E), \psi_n)$ the integral

$$Z(s, W, \phi) = \int_{N_n(F) \backslash G_n(F)} W(h)\phi(e_nh)|\det h|^{s}_F dh$$

is absolutely convergent for all $s \in H_{-\min(\mu)}$. Moreover, the function $s \mapsto Z(s, W, \phi)$ belongs to $PW_A(H_{-2\min(\mu)})$ and the linear map

$$C_\mu(N_n(E) \backslash G_n(E), \psi_n) \to PW_A(H_{-2\min(\mu)})$$

$$W \mapsto (s \mapsto Z(s, W, \phi))$$

is continuous.

**Proof:** Let $W \in C_\mu(N_n(E) \backslash G_n(E), \psi_n)$. By the Iwasawa decomposition $G_n(F) = N_n(F)A_n(F)K_n$, we need to show the convergence of

$$\int_{K_n} \int_{A_n(F)} |W(ak)||\phi(e_nak)||\det a|^{t}_F \delta_n(a)^{-1} da dk$$

for $t > -2\min(\mu)$. Since for each $R > 0$ we have $|\phi(e_nak)| \ll (1 + |a_n|)^{-R}$ for all $a \in A_n(F)$ and $k \in K_n$, by Lemma 2.5.1 we are reduced to show the convergence of

$$\int_{A_n(F)} \prod_{i=1}^{n-1} (1 + |a_i|)^{-R} (1 + |a_n|)^{-R} \delta_{n,E}(a)^{1/2} |a|^{\mu} |\det a|^{t}_F \delta_n(a)^{-1} da =$$

$$\int_{A_n(F)} \prod_{i=1}^{n-1} (1 + |a_i|)^{-R} (1 + |a_n|)^{-R} \prod_{i=1}^{n} |a_i|^{t+2|\mu|} da$$

where $|\mu|_1, \ldots, |\mu|_n$ denote the coordinates of $|\mu|$. We easily check that the above integral is absolutely convergent when $t + 2|\mu| > 0$ for every $1 \leq i \leq n$ i.e. when $t > -2\min(\mu)$ and moreover that the convergence is uniform on any compact subset of this open interval. Therefore, the function $s \in H_{-2\min(\mu)} \mapsto Z(s, W, \phi)$ is well-defined and holomorphic. In
the non-Archimedean case it is clear that it is also periodic of period \( \log(q)^{-1}2i\pi \) hence belongs to \( PW_A(\mathcal{H}_{-2\min(\mu)}) \). In the Archimedean case, we at least have that the function \( s \mapsto Z(s, W, \phi) \) is bounded in vertical strips. To show that it is actually rapidly decreasing on such strips, it suffices to show that \( s \mapsto sZ(s, W, \phi) \) is a sum of Zeta integrals of the same kind (since then by an easy induction we will have that for every \( N \geq 0 \) the function \( s \mapsto s^NZ(s, W, \phi) \) is bounded in vertical strips). Let \( Z = I_n \in \mathfrak{gl}_n(F) \). Then, for every \( s \in \mathbb{C} \) we have

\[
R(Z) (h \mapsto |\det h|^s) = ns|\det h|^s.
\]

Hence, by integration by part,

\[
nsZ(s, W, \phi) = \int_{N_n(F)/G_n(F)} W(h)\phi(e_n h)R(Z) (h \mapsto |\det h|^s) \, dh
\]

\[
= -\int_{N_n(F)/G_n(F)} R(Z) (h \mapsto W(h)\phi(e_n h)) |\det h|^s \, dh
\]

\[
= -\int_{N_n(F)/G_n(F)} (R(Z)W)(h)\phi(e_n h)|\det h|^s \, dh - \int_{N_n(F)/G_n(F)} W(h)(D\phi)(e_n h)|\det h|^s \, dh
\]

\[
= -Z(s, R(Z)W, \phi) - Z(s, W, D\phi)
\]

for every \( s \in \mathcal{H}_{-2\min(\mu)} \) where \( D \) stands for the Euler vector field \( v \mapsto \partial_v \) on \( F^n \) (where \( \partial_v \) stands for the operator of differentiation in the \( v \)-direction). This shows the claim and thus \( (s \mapsto Z(s, W, \phi)) \in PW_A(\mathcal{H}_{-2\min(\mu)}) \) in all cases. Finally, the fact that the linear map

\[
\mathcal{C}_n(N_n(E)/G_n(E), \psi_n) \rightarrow PW_A(\mathcal{H}_{-2\min(\mu)})
\]

\[
W \mapsto (s \mapsto Z(s, W, \phi))
\]

is continuous can be easily inferred from the closed graph theorem.

We now introduce the following convenient terminology: we say that a generic irreducible representation \( \pi \) of \( G_n(E) \) is \textit{nearly tempered} if it can be written as a parabolically induced representation

\[
\pi = \tau_{1, \lambda_1} \times \ldots \times \tau_{k, \lambda_k}
\]

where, for each \( 1 \leq i \leq k \), \( \tau_i \) is a discrete series of some \( G_{k_i}(E) \) and \( \lambda_1, \ldots, \lambda_k \in \mathbb{C} \) are such that

\[
|\Re(\lambda_i)| < \frac{1}{4}
\]

for every \( 1 \leq i \leq k \).

\textbf{Lemma 3.2.2} Assume that \( \pi \) is a generic irreducible representation of \( G_n(E) \) which is nearly tempered. Then, there exists \( \epsilon > 0 \) such that for every \( W \in \mathcal{W}(\pi, \psi_n) \) and \( \phi \in \mathcal{S}(F^n) \) the integral defining \( Z(s, W, \phi) \) converges absolutely on \( \mathcal{H}_{\frac{1}{2} - \epsilon} \) and defines a holomorphic function there.
Proof: Let $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$. By assumption, there exists a parabolic subgroup $P = MU$ of $G_n$, a discrete series $\sigma$ of $M(E)$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{A}_{M,C} \subseteq \mathcal{A}^*_C = \mathbb{C}^n$ satisfying $|\mathbb{R}(\lambda_i)| < \frac{1}{2}$ for every $1 \leq i \leq n$ such that $\pi \simeq i_{P(E)}(\sigma_\lambda)$. Let $\rho = (\frac{1}{n-1}, \ldots, \frac{1}{n-1}) \in \mathcal{A}^*$ be half the sum of the roots of $A_n$ in $B_n$. Then, for every $\eta > 0$ we have $|\mathbb{R}(\lambda)| < |\mathbb{R}(\lambda)| + \eta$. Therefore, by Proposition 2.6.1(i) (and standard properties of the Jacquet’s functional [Wall2 Sect. 15.4]) we have

$$\mathcal{W}(\pi, \psi_n) \subseteq C_{|\mathbb{R}(\lambda)|+\eta}(N_n(E) \backslash G_n(E), \psi_n)$$

for any $\eta > 0$. By the previous lemma it follows that for any $\eta > 0$, $Z(s, W, \phi)$ converges absolutely on $H_{-2 \min(|\mathbb{R}(\lambda)|+\eta\rho})$ and defines a holomorphic function there. Since

$$H_{-2 \min(|\mathbb{R}(\lambda)|)} = \bigcup_{\eta > 0} H_{-2 \min(|\mathbb{R}(\lambda)|)+\eta\rho}$$

we deduce that $Z(s, W, \phi)$ converges absolutely on $H_{-2 \min(|\mathbb{R}(\lambda)|)}$ and defines a holomorphic function there. Finally, the assumption made on $\lambda$ implies that $H_{-2 \min(|\mathbb{R}(\lambda)|)} = \mathcal{H}_{\frac{1}{2} - \epsilon}$ for some $\epsilon > 0$ and the lemma follows. $\blacksquare$

We have the following non-vanishing result.

**Lemma 3.2.3** For every $s_0 \in \mathbb{C}$, there exist finite families $W_i \in \mathcal{W}(\pi, \psi_n)$ and $\phi_i \in \mathcal{S}(F^n)$ indexed by $i \in I$ such that the function $s \mapsto \sum_{i \in I} Z(s, W_i, \phi_i)$ (which is defined on some right-half plane) admits a holomorphic continuation to $\mathbb{C}$ which is non-vanishing at $s_0$.

Proof: Let $P_n$ be the mirabolic subgroup of $G_n$ (i.e. the subgroup of elements $g \in G_n$ with last row $(0, \ldots, 0, 1)$), $U_n$ be the unipotent radical of $P_n$ and $U_n = {}^tU_n$. Then, we have the decomposition $G_n(F) = P_n(F)Z_n(F)\overline{U}_n(F)$ and correspondingly for Haar measures $dh = |\det p|^{-1}_F d_r p d\mathbf{w}$ where $d_r p$ denotes a right Haar measure on $P_n(F)$. Thus, by Lemma 3.2.1 for $\mathbb{R}(s) \gg 1$ the expression defining $Z(s, W, \phi)$ is absolutely convergent and we have

$$Z(s, W, \phi) = \int_{Z_n(F) \times U_n(F)} \int_{N_n(F) \backslash P_n(F)} W(pz\mathbf{w}) |\det p|^{-1}_F d_r p \phi(e_n z\mathbf{w}) |\det z|^\ast_F d z d\mathbf{w}$$

$$= \int_{Z_n(F) \times U_n(F)} \int_{N_n(F) \backslash P_n(F)} W(p\mathbf{w}) |\det p|^{-1}_F d_r p \phi(e_n z\mathbf{w}) \omega(z) |\det z|^\ast_F d z d\mathbf{w}$$

for every $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$. Let $\varphi_Z \in C^\infty_c(Z_n(F))$ and $\varphi_\mathbf{w} \in C^\infty_c(\overline{U}_n(F))$. Then, there exists a unique $\phi = \phi_{\varphi_Z, \varphi_\mathbf{w}} \in C^\infty_c(F^n)$ such that $\phi(e_n z\mathbf{w}) = \varphi_Z(z) \varphi_\mathbf{w}(\mathbf{w})$ for every $(z, u) \in Z_n(F) \times \overline{U}_n(F)$. For such a $\phi$, the above identity becomes

$$Z(s, W, \phi) = \int_{N_n(F) \backslash P_n(F)} (R(\varphi_\mathbf{w}) W)(p) |\det p|^{-1}_F d_r p \int_{Z_n(F)} \varphi_Z(z) \omega(z) |\det z|^\ast_F d z, \quad \Re(s) \gg 1$$

for every $W \in \mathcal{W}(\pi, \psi_n)$. Let $f \in C^\infty_c(N_n(E) \backslash P_n(E), \psi_n)$. By [GK], [Jac3 Proposition 5] and [Kem], there exists $W_0 \in \mathcal{W}(\pi, \psi_n)$ whose restriction to $P_n(E)$ coincides with $f$. By
Dixmier-Malliavin \[\text{DM}\] (in the Archimedean case), the exists finite families \((W_i)_{i \in I}\) and \((\varphi_{U,i})_{i \in I}\) of elements in \(\mathcal{W}(\pi, \psi_n)\) and \(C_c^\infty(U_n(F))\) respectively such that
\[
W_0 = \sum_{i \in I} R(\varphi_{U,i})W_i.
\]
Choose \(\varphi_Z \in C_c^\infty(Z_n(F))\) such that
\[
\int_{Z_n(F)} \varphi_Z(z)\omega_\pi(z)|\det z|^{s_0}dz \neq 0
\]
(Notice that the above integral is absolutely convergent for any complex value of \(s_0\)). Set \(\phi_i = \phi_{\varphi_Z, U,i}\) for every \(i \in I\). Then, by the above, for \(\Re(s) \gg 1\) we have
\[
\sum_{i \in I} Z(s, W_i, \phi_i) = \sum_{i \in I} \int_{N_n(F) \setminus P_n(F)} (R(\varphi_{U,i})W_i)(p)|\det p|^{-s-1}dp \int_{Z_n(F)} \varphi_Z(z)\omega_\pi(z)|\det z|^{s}dz
\]
\[= \int_{N_n(F) \setminus P_n(F)} W_0(p)|\det p|^{-s-1}dp \int_{Z_n(F)} \varphi_Z(z)\omega_\pi(z)|\det z|^{s}dz
\]
\[= \int_{N_n(F) \setminus P_n(F)} f(p)|\det p|^{-s-1}dp \int_{Z_n(F)} \varphi_Z(z)\omega_\pi(z)|\det z|^{s}dz.
\]
The above integrals are convergent for any \(s \in \mathbb{C}\) uniformly on compacta and therefore the resulting expression defines a holomorphic function on \(\mathbb{C}\). Moreover, we can certainly choose \(f\) so that
\[
\int_{N_n(F) \setminus P_n(F)} f(p)|\det p|^{-s-1}dp \neq 0.
\]
By our choice of \(\varphi_Z\) this implies that the holomorphic continuation of \(\sum_{i \in I} Z(s, W_i, \phi_i)\) does not vanish at \(s_0\). ■

### 3.3 Local functional equation: the split case

In this section we assume that we are in the split case i.e. \(E = F \times F\). Let \(\pi \in \text{Irr}_\text{gen}(G_n(E))\). Then \(\pi = \pi_1 \boxtimes \pi_2\) for some \(\pi_1, \pi_2 \in \text{Irr}(G_n(F))\) and in the case where \(\psi(x, y) = \psi'(x)\psi'(-y)\) for \((x, y) \in E\) (i.e. \(\tau = (1, -1)\)) and \(W = W_1 \otimes W_2\) for some \(W_1 \in \mathcal{W}(\pi_1, \psi_n')\), \(W_2 \in \mathcal{W}(\pi_2, \psi_n'^{-1})\), for any \(\phi \in \mathcal{S}(F^n)\) the Zeta integral \(Z(s, W, \phi)\) belongs to a family of expressions studied by Jacquet-Piatetskii-Shapiro-Shalika and Jacquet in [IPSS] and [Jac]. By the main results of these references together with properties of the local Langlands correspondence for \(\text{GL}_n\) ([HT], [Hen], [Sch]), in this case \(Z(s, W, \phi)\) admits a meromorphic continuation to \(\mathbb{C}\) and satisfies the functional equation
\[
Z(1-s, \hat{W}, \hat{\phi}) = \omega_{\pi_1}(-1)^{n-1}|\gamma(s, \pi_1 \times \pi_2, \psi')Z(s, W, \phi)
\]
\[= \omega_{\pi}(\tau)^{n-1}|\tau|_E^{n(n-1)/2}|_{s-1/2}\lambda_{E/F}(\psi')^{n(n-1)/2}\gamma(s, \pi, \text{As, } \psi')Z(s, W, \phi).
\]
Moreover, in the $p$-adic case the meromorphic continuation of $s \mapsto Z(s, W, \phi)$ and the above equality extend to any $W \in \mathcal{W}(\pi, \psi)$ by linearity whereas in the Archimedean case this follows from the remark after Theorem 2.3 of [Jac]. We record this as a theorem by removing the assumption on $\psi$.

**Theorem 3.3.1** Assume that $E = F \times F$. Let $\pi \in \text{ Irr}_{\text{gen}}(G_n(E))$. Then, for every $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$ the function $s \mapsto Z(s, W, \phi)$ has a meromorphic extension to $\mathbb{C}$ and satisfies the functional equation

$$(3.3.1) \quad Z(1-s, \hat{W}, \hat{\phi}) = \omega_{\pi}(\tau)^{n-1} |\tau|^\frac{n(n-1)}{2} \lambda_{E/F}(\psi') - \frac{n(n-1)}{2} \gamma(s, \pi, As, \psi') Z(s, W, \phi).$$

Proof: By the above, the theorem is satisfied when $\psi(x, y) = \psi'(x)\psi'(-y)$ for every $(x, y) \in E$. We therefore just need to study the effect of replacing $\psi$ by $\psi_{\lambda}$ for some $\lambda \in F^\times$ where $\psi_{\lambda}(z) = \psi(\lambda z)$ for every $z \in E$. Doing so amounts to replacing $\tau$ by $\lambda \tau$. Define the generic character $\psi_{n, \lambda}$ as $\psi_n$ using $\psi_\lambda$ instead of $\psi$. Then, there is an isomorphism $\mathcal{W}(\pi, \psi_n) \to \mathcal{W}(\pi, \psi_{n, \lambda})$, $W \mapsto \hat{W}_\lambda$ given by

$$W_\lambda(g) = W(a(\lambda)g), \quad g \in G_n(E)$$

where $a(\lambda) = \begin{pmatrix} \lambda^{n-1} & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$. By the change of variable $h \mapsto a(\lambda)^{-1}h$ we have

$$(3.3.2) \quad Z(s, W_\lambda, \phi) = \delta_n(a(\lambda)) |\det a(\lambda)|^{-1} \hat{Z}(s, W, \phi), \quad \Re(s) \gg 1.$$

This already shows that $Z(s, W_\lambda, \phi)$ has a meromorphic continuation if and only if $Z(s, W, \phi)$ has one. On the other hand, for all $g \in G_n(E)$ we have

$$\hat{(W_\lambda)}(g) = W(a(\lambda)w_n^{t}g^{-1}) = W(w_n^{-t}(w_n a(\lambda)^{-1} w_n^{-1} g)^{-1}) = \hat{W}(w_n a(\lambda)^{-1} w_n^{-1} g) = \omega_{\pi}(\lambda)^{n-1} \hat{W}_\lambda(g)$$

so that $\hat{W}_\lambda = \omega_{\pi}(\lambda)^{n-1} \hat{W}$ and finally (by 3.3.2)

$$(3.3.3) \quad Z(1-s, \hat{W}_\lambda, \hat{\phi}) = \omega_{\pi}(\lambda)^{n-1} \delta_n(a(\lambda)) |\det a(\lambda)|^{-1} \hat{Z}(1-s, \hat{W}, \hat{\phi}).$$

From 3.3.2 and 3.3.3 it follows that if $Z(s, W, \phi)$ and $Z(s, \hat{W}, \hat{\phi})$ satisfy 3.3.1 then

$$Z(1-s, \hat{W}_\lambda, \hat{\phi}) = \omega_{\pi}(\lambda)^{n-1} |\det a(\lambda)|^{-1} |\hat{W}_\lambda|^\frac{n(n-1)}{2} \lambda_{E/F}(\psi') - \frac{n(n-1)}{2} \gamma(s, \pi_1 \times \pi_2, \psi') \times Z(s, W_\lambda, \phi)$$

$$= \omega_{\pi}(\lambda \tau)^{n-1} |\tau|^\frac{n(n-1)}{2} \lambda_{E/F}(\psi') - \frac{n(n-1)}{2} \gamma(s, \pi_1 \times \pi_2, \psi') Z(s, W_\lambda, \phi)$$

which is precisely the functional equation for $\psi$ replaced by $\psi_{\lambda}$. ■
3.4 Local functional equation: the inert case

In this Section we assume that $E/F$ is a quadratic field extension. We are going to state two theorems which are the main results of this paper. These theorems will be proved in Sections 3.8 and 3.9. The first result is the exact analog of Theorem 3.3.1 in the inert case:

**Theorem 3.4.1** Assume that $E/F$ is a quadratic field extension. Let $\pi \in \text{Irr}_\text{gen}(G_n(E))$. Then, for every $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$ the function $s \mapsto Z(s, W, \phi)$ has a meromorphic extension to $\mathbb{C}$ and satisfies the functional equation

$$Z(1 - s, \hat{W}, \hat{\phi}) = \omega_\pi(\tau)^{n-1} |\tau|^{-\frac{n(n-1)}{2}(s-1/2)} \lambda_{E/F}(\psi')^{-\frac{n(n-1)}{2}} \gamma(s, \pi, \text{As}, \psi') Z(s, W, \phi). \quad (3.4.1)$$

**Theorem 3.4.2** Assume that $E/F$ is a quadratic field extension. Let $\pi \in \text{Irr}_\text{gen}(G_n(E))$. Then, for every $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$ the function

$$s \mapsto \frac{Z(s, W, \phi)}{L(s, \pi, \text{As})}$$

is holomorphic. Moreover, if $\pi$ is nearly tempered for every $s_0 \in \mathbb{C}$ there exist $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$ for which the above function does not vanish at $s_0$.

The aim of the next lemma is to check that Theorems 3.4.1 and 3.4.2 do not depend on the choices of $\psi$ and $\psi'$.

**Lemma 3.4.3** If Theorems 3.4.1 and 3.4.2 hold for one pair $(\psi, \psi')$ of nontrivial additive characters of $E$, $F$ respectively with $\psi$ trivial on $F$ then they hold for any such pair.

**Proof:** The independence on the choice of $\psi$ can be proved exactly the same way as in the proof of Theorem 3.3.1. Clearly Theorem 3.4.2 and the meromorphic extension of $s \mapsto Z(s, W, \phi)$ for every $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$ do not depend on the choice of $\psi'$. Thus, it only remains to show the independence of the functional equation 3.4.1 on the choice of $\psi'$. If we replace $\psi'$ by $\psi'_\lambda$ defined by $\psi'_\lambda(x) = \psi'(\lambda x)$ for every $x \in F$ for some $\lambda \in F^\times$ but keep $\psi$ fixed then we have to replace $\tau$ by $\lambda^{-1} \tau$. Therefore, the functional equation 3.4.1 for $\psi'$ replaced by $\psi'_\lambda$ reads

$$Z(1 - s, \hat{W}, \hat{\psi'}_\lambda) = \omega_\pi(\lambda^{-1} \tau)^{n-1} |\lambda^{-1} \tau|^{-\frac{n(n-1)}{2}(s-1/2)} \lambda_{E/F}(\psi'_\lambda)^{-\frac{n(n-1)}{2}} \gamma(s, \pi, \text{As}, \psi'_\lambda) Z(s, W, \phi) \quad (3.4.2)$$

for every $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$ where $\hat{\psi'}_\lambda$ stands for the Fourier transform of $\phi$ with respect to $\psi'_\lambda$ (rather than $\psi'$) and the corresponding autodual Haar measure on $F^n$. We have the relation $\hat{\psi'}_\lambda(v) = |\lambda|^{n/2} \hat{\phi}(\lambda v) \ (v \in F^n)$. Therefore, by the change of variable $h \mapsto \lambda^{-1} h$, we have

$$Z(1 - s, \hat{W}, \hat{\psi'}_\lambda) = |\lambda|^{n(s-1/2)} \omega_\pi(\lambda) Z(1 - s, \hat{W}, \hat{\phi}) \ .$$

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On the other hand,
\[(3.4.4) \quad \lambda_{E/F}(\psi') = \eta_{E/F}(\lambda)\lambda_{E/F}(\psi')\]
whereas by (3.1.3).

\[(3.4.5) \quad \gamma(s, \pi, \psi, \psi') = \omega(\lambda)^n|\lambda_n^{(s-1/2)}|\eta_{E/F}(\lambda)^{n-1/2}\gamma(s, \pi, \psi, \psi').\]

Combining (3.4.3) (3.4.4) and (3.4.5) we see that the functional equation (3.4.2) for \((\psi, \psi')\) reduces to the one for \((\psi, \psi')\) (that is (3.4.1)).

The next lemma is straightforward and allows to restrict to representations with unitary central character.

**Lemma 3.4.4** Let \(\pi\) be a generic irreducible representation of \(G_n(E)\) and \(\lambda \in A^*_{G_n, \mathbb{C}}\). Then if Theorems 3.4.1 and 3.4.2 hold for \(\pi\), they also hold for \(\pi_\lambda\).

### 3.5 Unramified computation

In this Section, we consider the case where \(E\) is non-Archimedean and the extension \(E/F\) is either inert or split. Recall that an irreducible representation \(\pi\) of \(G_n(E)\) is said to be unramified if it admits a nonzero \(G_n(O_E)\)-fixed vector in which case \(\pi^{G_n(O_E)}\) is a line. We also say that the characters \(\psi'\) and \(\psi\) are unramified if the maximal fractional ideals on which they are trivial are \(O_F\) and \(O_E\) respectively. If \(\pi\) and \(\psi\) are unramified there exists a unique \(W \in \mathcal{W}(\pi, \psi_n)^{G_n(O_E)}\) such that \(W(1) = 1\). The following unramified computation is standard and already in the literature in all but one case. We provide a proof in this missing case.

**Lemma 3.5.1** Assume that \(E\) is non-Archimedean and that \(\pi \in \operatorname{Irr}_{\text{gen}}(G_n(E)), \psi'\) and \(\psi\) are all unramified. Let \(W \in \mathcal{W}(\pi, \psi_n)^{G_n(O_E)}\) be normalized by \(W(1) = 1\) and \(\phi \in S(F^n)\) be the characteristic function of \(O_F^n\). Then, for \(\Re(s) > 1\) we have
\[Z(s, W, \phi) = \operatorname{vol}(N_n(O_F)\backslash G_n(O_F))L(s, \pi, \psi_n)\]

Proof: In the split case this is JS Proposition 2.3 and in the inert case when the extension \(E/F\) is unramified this is FL Proposition 3]. Thus, it only remains to deal with the case where \(E/F\) is a ramified field extension. Let \(\varpi_F\) be an uniformizer of \(F\). For any \(n\)-uple \(\lambda = (\lambda_1, \ldots, \lambda_n)\) of integers set
\[a(\lambda) = \left(\begin{array}{ccc} \varpi_F^{\lambda_1} \\ \vdots \\ \varpi_F^{\lambda_n} \end{array}\right)\]

Then, by the Iwasawa decomposition we have (for \(\Re(s) > 1\))
\[(3.5.1) \quad Z(s, W, \phi) = \operatorname{vol}(N_n(O_F)\backslash G_n(O_F)) \sum_{\lambda \in \mathbb{Z}^n} W(a(\lambda))\phi(\varpi_F^{\lambda_n}e_n)|\det a(\lambda)|_F^{s}\delta_n(a(\lambda))^{-1}\]
For every $n$-uple $\lambda = (\lambda_1, \ldots, \lambda_n)$ of decreasing integers let $s_\lambda$ be the Schur function as defined in [Fli2 §3]. Let $t = (t_1, \ldots, t_n) \in (\mathbb{C}^\times)^n / \mathfrak{S}_n$ be the Satake parameter of $\pi$. Then, by [CS, Shin], $W(a(\lambda))$ is zero unless $\lambda_1 \geq \ldots \geq \lambda_n$ in which case it equals $\delta_{t,E}(a(\lambda))^{1/2}s_\lambda(t)$. Moreover as $\phi = 1_{F_n}$ we have $\phi(\mathcal{W}_F^E e_n) = 0$ if $\lambda_n < 0$ and 1 otherwise. Therefore, by 3.5.1 and since $\delta_{t,E} = \delta^2_n$ on $A_n(F)$ we get

$$\text{vol}(N_n(O_F) \backslash G_n(O_F))^{-1} Z(s, W, \phi) = \sum_{\lambda \in \mathbb{Z}^n, \lambda_1 \geq \ldots \geq \lambda_n \geq 0} s_{2\lambda}(t) |\det(a(\lambda))|_F^s = \sum_{\lambda \in \mathbb{Z}^n, \lambda_1 \geq \ldots \geq \lambda_n \geq 0} s_{2\lambda}(t) q_F^{-s(\lambda_1 + \ldots + \lambda_n)} = \sum_{\lambda \in \mathbb{Z}^n, \lambda_1 \geq \ldots \geq \lambda_n \geq 0} s_{2\lambda}(q_F^{-s/2}t_1, \ldots, q_F^{-s/2}t_n).$$

But by [M] equality 5.(a) p.77 we have

$$\sum_{\lambda \in \mathbb{Z}^n, \lambda_1 \geq \ldots \geq \lambda_n \geq 0} s_{2\lambda}(q_F^{-s/2}t_1, \ldots, q_F^{-s/2}t_n) = \prod_{1 \leq i < j \leq n} (1 - q_F^{-s}t_i^2)^{-1} \prod_{1 \leq i \leq n} (1 - q_F^{-s}t_i t_j)^{-1}.$$  

On the other hand, since $\pi$ is unramified it is of the form $\chi_1 \times \ldots \times \chi_n$ where the $\chi_i$ are unramified character of $E^\times$. Thus, by Lemma 3.1.2(i), we have

$$L(s, \pi, As) = \prod_{i=1}^n L(s, \chi_i, As) \prod_{1 \leq i < j \leq n} L(s, \chi_i \chi_j) = \prod_{i=1}^n L(s, \chi_i |_{F^s}) \prod_{1 \leq i < j \leq n} L(s, \chi_i \chi_j) = \prod_{1 \leq i \leq n} (1 - q_F^{-s}t_i)^{-1} \prod_{1 \leq i < j \leq n} (1 - q_F^{-s}t_i t_j)^{-1}$$

and this ends the proof of the lemma. ■

**Corollary 3.5.2** Assume that $E/F$ is a quadratic extension of $p$-adic fields and that $\pi \in \text{Irr}_{\text{gen}}(G_n(E))$ is an unramified generic representation of $G_n(E)$. Then, there exist $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$ such that $s \mapsto Z(s, W, \phi)$ and $s \mapsto Z(s, \widehat{W}, \widehat{\phi})$ are not identically zero and admit meromorphic continuation to $\mathbb{C}$ satisfying

$$Z(1 - s, \widehat{W}, \widehat{\phi}) = \omega_{n}(\tau)^{-1-n} |\tau|_E^{-\frac{n(n-1)}{2}} \lambda_{E/F}(\psi')^{-\frac{n(n-1)}{2}} \gamma(s, \pi, As, \psi')Z(s, W, \phi).$$

**Proof:** By Lemma 3.4.3 (or rather its proof) the statement of the corollary does not depend on the choice of $(\psi, \psi')$. Therefore, we may assume that both $\psi'$ and $\psi$ are unramified. Then, taking for $W$ the unique element of $\mathcal{W}(\pi, \psi_n)^{G_n(O_E)}$ normalized by $W(1) = 1$ and for $\phi \in \mathcal{S}(F^n)$ the characteristic function of $O^n_F$, by the previous lemma we have

$$Z(s, W, \phi) = \text{vol}(N_n(O_F) \backslash G_n(O_F)) L(s, \pi, As),$$

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\[ Z(s, \tilde{W}, \hat{\phi}) = \text{vol}(N_n(O_F) \backslash G_n(O_F)) L(s, \tilde{\pi}, \text{As}) \]

for \( \Re(s) \gg 1 \). This shows that both \( s \mapsto Z(s, W, \phi) \) and \( s \mapsto Z(s, \tilde{W}, \hat{\phi}) \) are not identically zero and admit meromorphic continuation to \( \mathbb{C} \) satisfying

\[ Z(1 - s, \tilde{W}, \hat{\phi}) = \frac{L(1 - s, \tilde{\pi}, \text{As})}{L(s, \pi, \text{As})} Z(s, W, \phi). \]

Thus, it only remains to check that

\[ \epsilon(s, \pi, \text{As}, \psi') = \omega_x(\tau)^{1 - n} \left| \tau \right|^{\frac{n(n - 1)}{2} (1/2 - s)} \lambda_{E/F}(\psi')^{\frac{n(n - 1)}{2}}. \]

Since \( \pi \) is unramified, it is an induced representation of the form \( \chi_1 \times \ldots \chi_n \) where the \( \chi_i \)'s are unramified characters of \( E^\times \). By Lemma 3.1.2(i) we have

\begin{equation}
(3.5.2) \quad \epsilon(s, \pi, \text{As}, \psi') = \lambda_{E/F}(\psi')^{\frac{n(n - 1)}{2}} \prod_{1 \leq i < j \leq n} \epsilon(s, \chi_i \chi_j, \psi'_E) \times \prod_{i = 1}^n \epsilon(s, \chi_i | F^\times, \psi')
\end{equation}

Since \( \chi_i | F^\times \) and \( \psi' \) are unramified we have \( \epsilon(s, \chi_i | F^\times, \psi') = 1 \) for all \( 1 \leq i \leq n \). On the other hand, since \( \chi_i, \chi_j \) and \( \psi = \psi'_{E, \tau} \) are unramified we have

\[ \epsilon(s, \chi_i \chi_j, \psi'_E) = \left| \tau \right|^{1/2 - s} \chi_i(\tau) \chi_j(\tau)^{-1} \epsilon(s, \chi_i \chi_j, \psi) = \left| \tau \right|^{1/2 - s} \chi_i(\tau) \chi_j(\tau)^{-1} \]

for every \( 1 \leq i < j \leq n \). Therefore 3.5.2 becomes

\[ \epsilon(s, \pi, \text{As}, \psi') = \lambda_{E/F}(\psi')^{\frac{n(n - 1)}{2}} \prod_{1 \leq i < j \leq n} \chi_i(\tau)^{-1} \chi_j(\tau)^{-1} \]

\[ = \lambda_{E/F}(\psi')^{\frac{n(n - 1)}{2}} \left| \tau \right|^{\frac{n(n - 1)}{2} (1/2 - s)} \omega_\pi(\tau)^{1 - n} \]

which is exactly what we wanted. \( \square \)

### 3.6 Global Zeta integrals and their functional equation

In this section we let \( k'/k \) be a quadratic extension of number fields. For every place \( v \) of \( k \) we denote by \( k_v \) the corresponding completion of \( k \) and set \( k'_v = k_v \otimes_k k' \). Also if \( v \) is non-Archimedean, we let \( O_v, O_{k', v} \) be the ring of integers of \( k_v \) and \( k'_v \) respectively. Let \( A = \prod_v k_v \) and \( A_{k'} = A \otimes_k k' = \prod_v k'_v \) be the adeles rings of \( k \) and \( k' \) respectively and \( \left| . \right| \) be the normalized absolute value on \( A \). We will also denote by \( A_{k', \infty} = \prod_{v \mid \infty} k'_v \) the finite adeles of \( k' \) and by \( k'_{\infty} = \prod_{v \mid \infty} k'_v \) the product of the Archimedean completions of \( k' \) (so that \( A_{k'} = A_{k', \infty} \times k'_{\infty} \)). Let \( \Psi' \) and \( \Psi \) be nontrivial additive characters of \( k' \setminus A \) and \( k' \setminus A_{k'} \) respectively with \( \Psi \) trivial on \( k \setminus A \). For every place \( v \) we let \( \Psi_v \) be the local component of \( \Psi \) at \( v \) to which we associate a generic character \( \Psi_{n,v} : N_n(k'_v) \to S^1 \) as before. Then \( \Psi_{n,v} = \prod_v \Psi_{n,v} \) define a character of \( N_n(A_{k'}) \) which is at the same time trivial on \( N_n(k') \) and \( N_n(A) \). Let \( Z_{\infty} \) be the connected component (for the usual topology) of \( Z_n(\mathbb{R}) \). By the diagonal embedding
For $\omega$ a continuous unitary character of $Z_{\infty}$ we denote by $A_{\mathrm{cusp}}(Z_{\infty}G_n(k')\backslash G_n(\mathbb{A}_{k'}),\omega)$ the space of smooth functions $\varphi : G_n(k')\backslash G_n(\mathbb{A}_{k'}) \to \mathbb{C}$ (here smooth means that $\varphi$ is right invariant by an open subgroup of $G_n(\mathbb{A}_{k',f})$ and $C^\infty$ in the $G(k'_{\infty})$ component) with all their derivatives of moderate growth in the sense of $[MW, \S I.2.3]$, having central character $\omega$ (i.e. $\varphi(zg) = \omega(z)\varphi(g)$ for every $(z,g) \in Z_{\infty} \times G_n(\mathbb{A}_{k'}))$ and satisfying

$$\int_{N(k') \backslash N(\mathbb{A}_{k'})} \varphi(ug)du = 0, \quad g \in G_n(\mathbb{A}_{k'})$$

for every parabolic subgroup $P = MN$ of $G_n$. By a cuspidal automorphic representation of $G_n(\mathbb{A}_{k'})$ we mean a closed topologically irreducible subrepresentation of the space $A_{\mathrm{cusp}}(Z_{\infty}G_n(k')\backslash G_n(\mathbb{A}_{k'}),\omega)$ for some character $\omega$. If $\Pi$ is such a cuspidal automorphic representation of $G_n(\mathbb{A}_{k'})$ then there exists an isomorphism

$$\Pi \simeq \bigotimes_{v \mid \infty} \Pi_v \otimes \bigotimes_{v \mid \infty} \pi_v$$

where for every place $v$ of $k$, $\Pi_v$ is an irreducible representation of $G_n(k'_v)$ in the sense of Section 2.3. Moreover, the representation $\Pi_v$ is generic for every place $v$, unramified for almost all place $v$ and the restricted tensor product above is taken with respect to a family of (almost everywhere defined) $G_n(\mathcal{O}_{k',v})$-fixed vectors.

Let $\Pi$ be a cuspidal automorphic representation of $G_n(\mathbb{A}_{k'})$ and let, for every place $v$ of $k$, $W_v$ be a Whittaker function in $\mathcal{W}(\Pi_v,\Psi_{n,v})$ such that $W_v$ is $G_n(\mathcal{O}_{k',v})$-invariant and satisfies $W_v(1) = 1$ for almost all place $v$. Set

$$W = \prod_v W_v.$$ 

Then, $W$ is a well-defined function on $G_n(\mathbb{A}_{k'})$ satisfying $W(ug) = \Psi_n(u)W(g)$ for every $(u,g) \in N_n(\mathbb{A}_{k'}) \times G_n(\mathbb{A}_{k'})$. We define similarly $\hat{W} = \prod_v \hat{W}_v$. For every place $v$, choose a function $\phi_v \in S(k'_v)$ such that $\phi_v = 1_{G_n(\mathcal{O}_{k',v})}$ for almost all places $v$ and set $\hat{\phi} = \prod_v \hat{\phi}_v$ (a function on $\mathbb{A}^n_{k'}$). We define similarly $\hat{\phi} = \prod_v \hat{\phi}_v$, where the local Fourier transforms are defined with respect to the local components $\Psi'_{\ell}$ of $\Psi'$. For $s \in \mathbb{C}$, we define, whenever convergent, the following global analog of the Zeta integrals discussed in the previous sections:

$$Z(s,W,\phi) = \int_{N_n(\mathbb{A}) \backslash G_n(\mathbb{A})} W(h)\phi(e_nh)|\det h|^s dh.$$ 

Then, the following result is a consequence of $[Kab]$ Propositions 5 & 6] (notice that although strictly speaking the discussion of loc. cit. assume that $k'/k$ splits at all Archimedean places this assumption is not used in the proof of the result below).
Theorem 3.6.1 When $\Re(s)$ is sufficiently large, the integral defining $Z(s, W, \phi)$ is absolutely convergent and we have

$$Z(s, W, \phi) = \prod_v Z(s, W_v, \phi_v).$$

Moreover, the function $s \mapsto Z(s, W, \phi)$ has a meromorphic continuation to $\mathbb{C}$ that satisfies the functional equation

$$Z(s, W, \phi) = Z(1 - s, \hat{W}, \hat{\phi}).$$

3.7 A globalization result

We continue with the setting fixed in Section 3.6. Let $v_0, v_1$ be two distinct places of $k$ with $v_1$ non-Archimedean. Recall that in Section 2.3 we have equipped the set $\text{Temp}(G_n(k_{v_0}'))$ of tempered representations of $G_n(k_{v_0}')$ with a topology. We have the following globalization result which is a simple consequence of the main result of [FLM]. For convenience, we explain the deduction.

Theorem 3.7.1 Let $U$ be an open subset of $\text{Temp}(G_n(k_{v_0}'))$. Then, there exists a cuspidal automorphic representation $\Pi$ of $G_n(\mathbb{A}_{k'})$ such that $\Pi_{v_0} \in U$ and $\Pi_v$ is unramified for every non-Archimedean place $v \not\in \{v_0, v_1\}$.

Proof: Let $S_\infty$ be the set of all Archimedean places of $k$ and set $S = S_\infty \cup \{v_0\}$, $k_S' = \prod_{v \in S} k_v'$. Denote by $\text{Temp}(G_n(k_S'))$ the set of all (isomorphism classes of) irreducible tempered representations of $G_n(k_S')$ i.e. representations which can be written as (completed) tensor products of tempered irreducible representations of $G_n(k_v')$ for every $v \in S$. Of course, we have an identification $\text{Temp}(G_n(k_S')) = \prod_{v \in S} \text{Temp}(G_n(k_v'))$ and we equip this set with the product topology. Clearly, it suffices to show that for every open $U$ of $\text{Temp}(G_n(k_S'))$ there exists a cuspidal automorphic representation $\Pi$ of $G_n(\mathbb{A}_{k'})$ such that $\Pi_S := \bigotimes_{v \in S} \Pi_v$ belongs to $U$ and $\Pi_v$ is unramified for every $v \not\in S \cup \{v_1\}$. Let $G_n(\mathbb{A}_{k'})^1$ and $G_n(k_S')^1$ be the subgroups of elements $g$ in $G_n(\mathbb{A}_{k'})$ and $G(k_S')$ respectively satisfying $|\det g|_{\mathbb{A}_{k'}} = 1$ where $|\cdot|_{\mathbb{A}_{k'}}$ denotes the normalized absolute value on $\mathbb{A}_{k'}$. Then, we have decompositions $G_n(\mathbb{A}_{k'}) = G_n(\mathbb{A}_{k'})^1 \times \hat{Z}_\infty$ and $G_n(k_S') = G_n(k_S')^1 \times \hat{Z}_\infty$. Let $\text{Temp}(G_n(k_S')^1)$ be the set of isomorphism classes of tempered representations of $G_n(k_S')^1$ i.e. representations obtained by restriction of representations in $\text{Temp}(G_n(k_S'))$ that we equip with the quotient topology. Then, we have an isomorphism $\text{Temp}(G_n(k_S')) \cong \text{Temp}(G_n(k_S')^1) \times \hat{Z}_\infty$. Let $U$ be an open subset of $\text{Temp}(G_n(k_S')^1)$ that we may assume to be of the form $U_1 \times V$ where $U_1 \subset \text{Temp}(G_n(k_S')^1)$ and $V \subset \hat{Z}_\infty$ are open. Let $p$ be the prime ideal in $\mathcal{O}_{k'}$ corresponding to $v_1$ and for all $m \geq 1$ let $K^S(p^m)$ be the open subgroup of $K_{v_1}(m) \times \prod_{v \not\in S} \mathcal{O}_{k_v}(p^m)$ of $\text{Temp}(G_n(k_v'))$ where $K_{v_1}(m) := \{ k \in G_n(\mathcal{O}_{k', v_1}) | k \equiv I_n \mod p^m \}$. Let $\mu_{p^m}$ be the Plancherel measure for $G_n(k_v')$. Since $\mu_{p^m}(U_1)$ is nonzero, by [FLM, Theorem 2] there exists an irreducible representation $\Pi'$ of $G_n(\mathbb{A}_{k'})$ such that $\Pi' |_{G_n(\mathbb{A}_{k'})^1} \hookrightarrow L^2(G_n(k_v') \backslash G_n(\mathbb{A}_{k'})^1)$, $(\Pi')|_{G_n(k_v')^1} \in U_1$ and $(\Pi')|_{K^S(p^m)} = 0$ for some $m \geq 1$. Let $\omega \in V$ and denote by $\Pi'$ the unique irreducible representation of $G_n(\mathbb{A}_{k'})$ whose restriction to $G_n(\mathbb{A}_{k'}^1)$ coincides with $\Pi' |_{G_n(\mathbb{A}_{k'})^1}$.
and whose central character equals \( \omega \) on \( \mathbb{R} \). Then \( \Pi \leftrightarrow L^2(\mathbb{Z}_k G_n(k') \backslash G_n(\mathbb{A}_k'), \omega) \), \( \Pi_S \in U \) and \( \Pi^{K_S(p^n)} \neq 0 \). Moreover, since \( \Pi \) is tempered at every Archimedean place by \cite{WALB} we have that \( \Pi \) is actually a cuspidal (rather than just square-integrable) automorphic representation of \( G_n(\mathbb{A}_k') \). Obviously, \( \Pi \) has all of the desired properties.

**Remark 3.7.2** The proof of the above result uses the full strength of \cite{FLM} but in fact we can get a slightly weaker result which is however sufficient for the application we have in mind without appealing to the analysis of the continuous part of the spectral side of Arthur’s weighted trace formula which is the hard part of \cite{FLM}. More precisely, the following ought to be provable (although the author hasn’t checked all the details) using the limit property \cite{FLM, Proposition 3} of the geometric side of Arthur’s weighted trace formula together with Sauvageot’s density principle \cite{Sau}:

Let \( G \) be any connected reductive group over \( k \). Let \( v_0, v_1, v_2 \) be three distinct places of \( k \) with \( v_1, v_2 \) non-Archimedean. Let \( U \) be an open subset of \( \text{Temp}(G(k_{v_0})) \) and \( \pi_1 \) be a supercuspidal representation of \( G(k_{v_1}) \). Then, there exists a cuspidal automorphic representation \( \Pi \) of \( G(\mathbb{A}) \) such that \( \Pi_{v_0} \in U \), \( \Pi_{v_1} \) is isomorphic to an unramified twist of \( \pi_1 \) and \( \Pi_{v_2} \) is unramified for every non-Archimedean place \( v \notin \{v_0, v_1, v_2\} \).

### 3.8 Proof of Theorem 3.4.1 and Theorem 3.4.2 in the nearly tempered case

Let the setting be as in Section 3.4. We choose a quadratic extension \( k'/k \) of number fields together with a place \( v_0 \) of \( k \) such that:

- there exists an isomorphism \( k_{v_0}'/k_{v_0} \simeq E/F \) that we fix from now on;
- every Archimedean place of \( k \) different from \( v_0 \) splits in \( k' \).

Given a place \( v \) of \( k \), a nontrivial additive character \( \psi_v' \) of \( k_v \) and a generic irreducible representation \( \pi_v \) of \( G_n(k_v) \) we let \( \gamma^{Sh}(s, \pi_v, As, \psi_v') \) be the local Asai \( \gamma \)-factor defined by Shahidi (see \cite{Sha3, Gold}). The goal of this section is to show the following:

**Theorem 3.8.1** For every \( \pi \in \text{Irr}_{\text{ntemp}}(G_n(E)) \), \( W \in \mathcal{W}(\pi, \psi_n) \), \( \phi \in S(F^n) \) and \( s \in \mathbb{C} \) with \( \Re(s) = \frac{1}{2} \) we have

\[
Z(1 - s, \hat{W}, \hat{\phi}) = \omega_\tau(\tau)^{n-1} |\tau|^\frac{n(n-1)}{2} \lambda_{E/F}(\psi')^{-\frac{n(n-1)}{2}} e^{\frac{n(n-1)}{2}} \gamma^{Sh}(s, \pi, As, \psi') Z(s, W, \phi).
\]

First we explain how to deduce from it Theorems 3.4.1 and 3.4.2 in the nearly tempered case. Let \( \pi \in \text{Irr}_{\text{ntemp}}(G_n(E)) \), \( W \in \mathcal{W}(\pi, \psi_n) \) and \( \phi \in S(F^n) \). By the main results of \cite{Sha2} (in the Archimedean case) and \cite{Shank} (in the \( p \)-adic case) we have

\[
\gamma^{Sh}(s, \pi, As, \psi') = \gamma(s, \pi, As, \psi').
\]
Therefore, as \( s \mapsto \gamma(s, \pi, As, \psi') \) is a meromorphic function on \( \mathbb{C} \), the functional equation of Theorem 3.8.1 together with Lemma 3.2.2 implies that \( s \mapsto Z(s, W, \phi) \) admits a meromorphic continuation to \( \mathbb{C} \) and that the functional equation is satisfied for every \( s \in \mathbb{C} \). This already shows Theorem 3.4.1. For Theorem 3.4.2 we rewrite the functional equation as

\[
(3.8.2) \quad \frac{Z(1-s, \tilde{W}, \hat{\phi})}{L(1-s, \tilde{\pi}, As)} = \omega(s)(\tau)^{n-1}|\tau|^{-\frac{n(n-1)}{2}}(s-1/2)^{\frac{n(n-1)}{2}}E_{s/F}(\psi')^{-\frac{n(n-1)}{2}}\lambda_{s/F}(\psi')Z(s, W, \phi) \frac{L(s, \pi, As)}{L(s, \pi, As)}.
\]

By Lemma 3.2.2 and since the functions \( L(s, \pi, As) \) and \( L(1-s, \tilde{\pi}, As) \) have no zeros, the functions \( s \mapsto \frac{Z(s, W, \phi)}{L(s, \pi, As)} \) and \( s \mapsto \frac{Z(1-s, \tilde{W}, \hat{\phi})}{L(1-s, \tilde{\pi}, As)} \) are holomorphic on \( \{ \Re(s) \geq \frac{1}{2} \} \) and \( \{ \Re(s) \leq \frac{1}{2} \} \) respectively. Since the epsilon factor \( \epsilon(s, \pi, As, \psi') \) has no zeros, this implies together with 3.8.2 that \( s \mapsto \frac{Z(s, W, \phi)}{L(s, \pi, As)} \) is holomorphic on \( \{ \Re(s) \geq \frac{1}{2} \} \cup \{ \Re(s) \leq \frac{1}{2} \} = \mathbb{C} \). Moreover, as \( \pi \) is nearly tempered the \( L \)-functions \( L(s, \pi, As) \) and \( L(s, \tilde{\pi}, As) \) are holomorphic in \( \{ \Re(s) \geq \frac{1}{2} \} \) (Lemma 3.1.2(ii)). Hence, by Lemma 3.2.3 for every \( s_0 \in \mathbb{C} \) with \( \Re(s_0) \geq 1/2 \) (resp. \( \Re(s_0) \leq 1/2 \)) there exist \( W \in \mathcal{W}(\pi, \psi_n) \) and \( \phi \in S(F^n) \) with \( \frac{Z(s_0, W, \phi)}{L(s_0, \pi, As)} \neq 0 \) (resp. \( \frac{Z(1-s_0, \tilde{W}, \hat{\phi})}{L(1-s_0, \tilde{\pi}, As)} \neq 0 \)). By 3.8.2 this implies that for every \( s_0 \in \mathbb{C} \) there exist \( W \in \mathcal{W}(\pi, \psi_n) \) and \( \phi \in S(F^n) \) with \( \frac{Z(s_0, W, \phi)}{L(s_0, \pi, As)} \neq 0 \) and this shows Theorem 3.4.2.

We now proceed to the proof of Theorem 3.8.1. Let \( \Psi \) and \( \Psi' \) be nontrivial characters of \( k \setminus A \) and \( k' \setminus A_{k'} \) with \( \Psi \) trivial on \( A \). By Lemma 3.4.3 and the identity 3.8.1 we may assume that \( \psi' = \Psi_{v_0} \) and \( \psi = \Psi_{v_0} \). We now show that it suffices to establish Theorem 3.8.1 for a dense subset \( D \subseteq \text{Temp}(G_n(E)) \). Indeed, let \( \pi \in \text{Irr}_{ntemp}(G_n(E)) \) that we write as \( \pi = i^{G_n(E)}(\sigma, \lambda) \) where \( P = MU \) is a standard parabolic subgroup of \( G_n \), \( \sigma \in \Pi_2(M(E)) \) and \( \lambda = (\lambda_0, \ldots, \lambda_n) \in A_{M, \mathbb{C}}^n \subseteq A_{E}^n = \mathbb{C}^n \) satisfies

\[
|\Re(\lambda_{0, i})| < 1/4, \quad 1 \leq i \leq n.
\]

By Lemma 3.4.4. we may assume that \( \lambda_0 \in (A_{M, \mathbb{C}}^G)^* \). Set \( \pi_\lambda = i_{G_n(E)}^{G_n}(\sigma, \lambda) \) for every \( \lambda \in A_{M, \mathbb{C}}^* \) (so that \( \pi = \pi_{\lambda_0} \)). Let \( W \in \mathcal{W}(\pi, \psi_n) \) and \( \phi \in S(F^n) \). Let \( \rho = (\frac{1-n}{2}, \ldots, \frac{1-n}{2}) \in (A_{G_n}^E)^* \) be half the sum of the roots of \( A_n \) in \( B_n \). Then \( |\Re(\lambda)| < |\Re(\lambda_0)| + \epsilon \rho \) for any \( \epsilon > 0 \) and by 3.8.3 we can choose \( \epsilon > 0 \) such that \( -2 \min(\Re(\lambda_0) + \epsilon \rho, \frac{1}{2}) < \frac{1}{2} \). Set \( \mu = |\Re(\lambda_0)| + \epsilon \rho \) and \( U[< \mu] = \{ \lambda \in (A_{M, \mathbb{C}}^G)^* | |\Re(\lambda)| < \mu \} \). By Corollary 2.7.1 there exists a map

\[
\lambda \in (A_{M, \mathbb{C}}^G)^* \mapsto W_{\lambda} \in \mathcal{W}(\pi_\lambda, \psi_n)
\]

such that \( W_{\lambda_0} = W, \ W_{\lambda} \in C_\mu(N_n(E) \setminus G_n(E), \psi_n) \) for every \( \lambda \in U[< \mu] \) and the induced map

\[
\lambda \in U[< \mu] \mapsto W_{\lambda} \in C_\mu(N_n(E) \setminus G_n(E), \psi_n)
\]

is analytic. Assume that Theorem 3.8.1 holds for a dense subset \( D \subseteq \text{Temp}(G_n(E)) \). Then, since this theorem is insensitive to unramified twists (Lemma 3.4.4), there exists a dense subset \( D_\sigma \subseteq i(A_{M, \mathbb{C}}^G)^* \) such that Theorem 3.8.1 holds for \( \pi_\lambda \) for every \( \lambda \in D_\sigma \). In particular, we have

\[
(3.8.4) \quad Z(1-s, \tilde{W}, \hat{\phi}) = \omega(s)(\tau)^{n-1}|\tau|^{-\frac{n(n-1)}{2}}(s-1/2)^{-\frac{n(n-1)}{2}}\epsilon_{E/F}(\psi')^{-\frac{n(n-1)}{2}}\gamma_{s/F}(s, \pi, As, \psi')Z(s, W, \phi)
\]
for every \( \lambda \in D_\sigma \) and \( s \in \mathbb{C} \) with \( \Re(s) = \frac{1}{2} \). By the choice of \( \mu \) and Lemma 3.2.1 for any \( s \in \mathbb{C} \) with \( \Re(s) = \frac{1}{2} \) the maps \( \lambda \in \mathcal{U}[< \mu] \mapsto Z(s, W_\lambda, \phi) \) and \( \lambda \in \mathcal{U}[< \mu] \mapsto Z(1 - s, \widehat{W}_\lambda, \widehat{\phi}) \) are holomorphic. By 3.8.1 and Lemma 3.1.1(ii), the map \( \lambda \in \mathcal{U}[< \mu] \mapsto \gamma^{Sh}(s, \pi_\lambda, As, \psi') \) is also holomorphic. Therefore, by density of \( D_\sigma \) in \( i(A^G_M)^* \), 3.8.4 is also satisfied for every \( \lambda \in i(A^G_M)^* \) (by continuity) hence for every \( \lambda \in \mathcal{U}[< \mu] \) (by analyticity) and in particular for \( \lambda = \lambda_0 \). This shows that Theorem 3.8.1 holds for \( \pi \) and ends the proof of the reduction.

Let \( U \subset \text{Temp}(G_n(E)) \) be an open subset. We are now going to show that there exists \( \pi \in U \) for which Theorem 3.8.1 holds. Let \( v_1 \) be a non-Archimedean place of \( k \) which splits in \( k' \). By Theorem 3.7.1 there exists a cuspidal automorphic representation \( \Pi \) of \( G_n(A_{k'}) \) such that \( \Pi_{v_0} \in U \) and \( \Pi_v \) is unramified for every non-Archimedean place \( v \neq \{v_0, v_1\} \). Let \( S = S_x \cup \{v_0, v_1\} \) where \( S_x \) stands for the set of all Archimedean places of \( k \) and let \( T \) be the (finite) set of places of \( k \) outside \( S \) which ramify in \( k' \) or where \( \Psi_v \) or \( \Psi_v \) is not unramified. Then, by [Sha3, Theorem 3.5(4)] the product

\[
L^{S \cup T}(s, \Pi, As) = \prod_{v \notin S \cup T} L(s, \Pi_v, As),
\]

which converges for \( \Re(s) \) sufficiently large, admits a meromorphic continuation to \( \mathbb{C} \) and we have

\[
L^{S \cup T}(s, \Pi, As) = \prod_{v \in S \cup T} \gamma^{Sh}(s, \Pi_v, As, \Psi'_v) L^{S \cup T}(1 - s, \Pi^\vee, As).
\]

On the other hand, combining Theorem 3.6.1 with Lemma 3.5.1 we find that

\[
\prod_{v \in S \cup T} Z(s, W_v, \phi_v) L^{S \cup T}(s, \Pi, As) = \prod_{v \in S \cup T} Z(1 - s, \widehat{W}_v, \widehat{\phi}_v) L^{S \cup T}(1 - s, \Pi^\vee, As)
\]

for every \( (W_v)_v \in \prod_{v \in S \cup T} \mathcal{W}(\Pi_v, \Psi_{n,v}) \) and \( (\phi_v)_v \in \prod_{v \in S \cup T} S(k_v^n) \). Notice that we haven’t prove the meromorphic continuation of \( Z(s, W_v, \phi_v) \) yet but by Theorem 3.6.1 and the aforementioned result of Shahidi at least the products \( \prod_{v \in S \cup T} Z(s, W_v, \phi_v) \) and \( \prod_{v \in S \cup T} Z(1 - s, \widehat{W}_v, \widehat{\phi}_v) \) admit meromorphic continuations. As for every place \( v \in (S \cup T) \setminus \{v_0\} \), either \( v \) splits in \( k' \) or \( v \) is non-Archimedean and \( \Pi_v \) is unramified, by Theorem 3.3.1 Corollary 3.5.2 and Lemma 3.2.3 we can choose the local data so that \( Z(s, W_v, \phi_v) \), \( Z(1 - s, \widehat{W}_v, \widehat{\phi}_v) \) admit meromorphic continuation, are not identically zero and satisfy

\[
Z(1 - s, \widehat{W}_v, \widehat{\phi}_v) = \omega_{H_v}(\tau)^{-n-1} |\tau|_{k'_v}^{\frac{n(n-1)}{2}(s-1/2)} \lambda_{k'_v/k_v}(\Psi'_v)^{-\frac{n(n-1)}{2}} \gamma(s, \Pi_v, As, \Psi'_v) Z(s, W_v, \phi_v)
\]

for every \( v \in (S \cup T) \setminus \{v_0\} \) where \( \tau \in k' \) is such that \( \Psi(z) = \Psi'(\text{Tr}_{k'_v/k_v}(\tau z)) \) for all \( z \in A_{k'} \). This already implies that the local Zeta integrals \( Z(s, W, \phi) \) and \( Z(1 - s, \widehat{W}, \widehat{\phi}) \) admit meromorphic continuation for every \( W \in \mathcal{W}(\Pi_0, \psi_n) \) and \( \phi \in \mathcal{S}(F^n) \). Moreover, combining the above identities with the product formula

\[
\prod_{v \in S \cup T} \omega_{H_v}(\tau)^{-n-1} |\tau|_{k'_v}^{\frac{n(n-1)}{2}(s-1/2)} \lambda_{k'_v/k_v}(\Psi'_v)^{-\frac{n(n-1)}{2}} = \prod_v \omega_{H_v}(\tau)^{-n-1} |\tau|_{k'_v}^{\frac{n(n-1)}{2}(s-1/2)} \lambda_{k'_v/k_v}(\Psi'_v)^{-\frac{n(n-1)}{2}} = 1
\]

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(which is a consequence of the fact that $\Pi_v$, $k'_v/k_v$, $\Psi_v$ and $\Psi'_v$ are all unramified outside $S \cup T$) identity 3.8.6 becomes

$$
\omega_{\Pi_{v_0}}(\tau)^{n-1}|\tau|^{-\frac{n(n-1)}{2}(s-1/2)} E/F(\psi')^{-\frac{n(n-1)}{2}} Z(s, W, \phi)L^S_{\Pi_{v_0}}(s, \Pi, \lambda, \phi) = Z\left(1-s, \tilde{W}, \tilde{\phi}\right) \prod_{v \in S \cup T \setminus \{v_0\}} \gamma(s, \Pi_v, \lambda, \phi)L^S_{\Pi_v}(1-s, \Pi^\vee, \phi)
$$

for every $W \in \mathcal{W}(\Pi_{v_0}, \psi_n)$ and $\phi \in S(F^n)$. Taking the quotient with 3.8.5 and using 3.8.1 we obtain

$$
\omega_{\Pi_{v_0}}(\tau)^{n-1}|\tau|^{-\frac{n(n-1)}{2}(s-1/2)} E/F(\psi')^{-\frac{n(n-1)}{2}} \gamma^{S\Pi_{v_0}}(s, \Pi_{v_0}, \lambda, \phi)Z(s, W, \phi)
$$

$$
= Z\left(1-s, \tilde{W}, \tilde{\phi}\right) \prod_{v \in S \cup T \setminus \{v_0\}} \gamma(s, \Pi_v, \lambda, \phi)L^S_{\Pi_v}(1-s, \Pi^\vee, \phi) = Z\left(1-s, \tilde{W}, \tilde{\phi}\right)
$$

for every $W \in \mathcal{W}(\Pi_{v_0}, \psi_n)$ and $\phi \in S(F^n)$. This shows that Theorem 3.8.1 holds for $\Pi_{v_0}$ hence, as $U$ was arbitrary, for a dense subset of Temp$(G_n(E))$. By the previous reduction, this proves Theorem 3.8.1 in all cases. ■

### 3.9 Proof of Theorem 3.4.1 and Theorem 3.4.2 in the general case

Let $\pi$ be a generic irreducible representation of $G_n(E)$. There exists a standard parabolic subgroup $P = MU$ of $G_n$, $\sigma \in \Pi_2(M(F))$ and $\lambda_0 \in A^*_{M, \lambda}$ such that $\pi = i^{G_n(E)}(\sigma_{\lambda_0})$. By Lemma 3.4.4 we may assume that $\lambda_0 \in (A^*_{M, \lambda})$. Set $\pi_{\lambda} = i^{G_n(E)}(\sigma_{\lambda})$ for every $\lambda \in A^*_{M, \lambda}$. Let $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in S(F^n)$. Choose a section

$$
\lambda \in (A^*_{M, \lambda}) \mapsto W_{\lambda} \in \mathcal{W}(\pi_{\lambda}, \psi_n)
$$

as in Corollary 2.7.4 with $W_{\lambda_0} = W$. For $\mu \in (A^*_{M, \lambda})$ we define the open subset $U[< \mu]$ of $(A^*_{M, \lambda})$ as in Corollary 2.7.4. In particular, for every $\mu \in (A^*_{M, \lambda})$ the above section induces an analytic map

$$
\lambda \in U[< \mu] \mapsto W_{\lambda} \in C_{\mu}(N_n(E) \backslash G_n(E), \psi_n).
$$

Decomposing the $\varepsilon$-factor $\varepsilon(s, \pi_{\lambda}, \psi')$ according to Lemma 3.1.2(1) we see that there exists $C > 0$, a linear form $L$ on $(A^*_{M, \lambda})$ and an element $u \in \mathbb{C}$ of norm 1 such that

$$
\omega_{\pi}(\tau)^{n-1}|\tau|^{\frac{n(n-1)}{2}(s-1/2)} E/F(\psi')^{-\frac{n(n-1)}{2}} \varepsilon(s, \pi_{\lambda}, \psi') = uC^{L(\lambda)+s-\frac{1}{2}}
$$

for every $\lambda \in (A^*_{M, \lambda})$ and $s \in \mathbb{C}$. Let $u$ be a square root of $u$ and set

$$
\varepsilon_{1/2}(s, \pi_{\lambda}, \psi') := v(\sqrt{C})^{L(\lambda)+s-\frac{1}{2}}, \quad \lambda \in (A^*_{M, \lambda}), s \in \mathbb{C}
$$

so that by the above

$$
(3.9.1) \quad \omega_{\pi}(\tau)^{n-1}|\tau|^{\frac{n(n-1)}{2}(s-1/2)} E/F(\psi')^{-\frac{n(n-1)}{2}} \varepsilon(s, \pi_{\lambda}, \psi') = \varepsilon_{1/2}(s, \pi_{\lambda}, \psi')^2
$$

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for every \( \lambda \in (\mathcal{A}_{M,C}^G)^* \) and \( s \in \mathbb{C} \). Let \( z \in \mathbb{R}_+^* \). We are now going to apply Proposition 2.8.1 to the two following (for the moment partially defined) functions on \( \mathbb{C} \times (\mathcal{A}_{M,C}^G)^* \):

\[
Z_1(s, \lambda) = \epsilon_1(\frac{1}{2} + s, \pi_\lambda, \psi') \frac{Z(\frac{1}{2} + s, W_\lambda, \phi)}{L_{K,z}(\frac{1}{2} + s, \pi_\lambda, As)}
\]

and

\[
Z_2(s, \lambda) = \epsilon_1(\frac{1}{2} + s, \pi_\lambda, \psi')^{-1} \frac{Z(\frac{1}{2} + s, \tilde{W}_\lambda, \hat{\phi})}{L_{K,z}(\frac{1}{2} + s, \tilde{\pi}_\lambda, As)}.
\]

Let \( U \subseteq (\mathcal{A}_{M,C}^G)^* \) be the nonempty relatively compact connected open subset of vectors \( \lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathcal{A}_{M,C}^G)^* \subseteq \mathcal{A}_C^* \) satisfying \( |\lambda_i| < \frac{1}{4} \) for every \( 1 \leq i \leq n \). Then, for every \( \lambda \in U \), the representation \( \pi_\lambda \) is nearly tempered. Set as usual \( \rho = (\frac{n-1}{2}, \ldots, \frac{n-1}{2}) \in (\mathcal{A}^G)^* \) for half the sum of the roots of \( A_n \) in \( B_n \). By Lemma 3.2.2 the functions \( Z_1, Z_2 \) are at least well-defined on \( \{ \Re(s) \geq \frac{1}{2} \} \times U \). Moreover, by Lemma 3.2.1, Lemma 2.9.1 and Lemma 3.1.2(iii), for every \( \lambda \in U \) there exists \( \epsilon > 0 \) and \( \eta > 0 \) such that, setting \( \mu = |\Re(\lambda)| + \epsilon \rho \), \( Z_1 \) and \( Z_2 \) admit holomorphic continuation to \( \mathcal{H}_{-\eta} \times U[^{<}\mu] \) which are either periodic of period \( \log(q)^{-12i\pi} \) (in the \( p \)-adic case) or rapidly decreasing in vertical strips in the first variable locally uniformly in the second variable (in the Archimedean case). Furthermore by the results of the previous section, these extensions satisfy the functional equation

\[(3.9.2) \quad Z_1(s, \lambda) = Z_2(-s, \lambda)\]

for every \( \lambda \in U[^{<}\mu] \) and \( s \in \mathbb{C} \) such that \( |\Re(s)| < \eta \). Since \( U[^{<}\mu] \) is an open neighborhood of \( \lambda \), it follows that \( Z_1 \) and \( Z_2 \) admit holomorphic continuation to \( \mathbb{C} \times U \) which are again either periodic of period \( \log(q)^{-12i\pi} \) (in the \( p \)-adic case) or rapidly decreasing in vertical strips in the first variable locally uniformly in the second variable (in the Archimedean case) and satisfy 3.9.2 for every \( (s, \lambda) \in \mathbb{C} \times U \). Let \( U' \subseteq (\mathcal{A}_{M,C}^G)^* \) be a relatively compact connected open subset containing \( U \). Then, there exists \( \mu \in (\mathcal{A}^G)^* \) such that \( U' \subseteq U[^{<}\mu] \). Hence, again by Lemma 3.2.1, Lemma 2.9.1 and Lemma 3.1.2(iii), there exists \( C \) such that \( Z_1 \) and \( Z_2 \) both admit holomorphic continuation to \( \mathcal{H}_{C} \times U' \) which are either periodic of period \( \log(q)^{-12i\pi} \) (in the \( p \)-adic case) or rapidly decreasing in vertical strips in the first variable locally uniformly in the second variable (in the Archimedean case). Therefore, \( Z_1 \) and \( Z_2 \) satisfy the hypothesis of Proposition 2.8.1 and it follows that these two functions extend holomorphically to \( \mathbb{C} \times (\mathcal{A}_{M,C}^G)^* \) and that the extensions satisfy 3.9.2 for every \( (s, \lambda) \in \mathbb{C} \times (\mathcal{A}_{M,C}^G)^* \). Specializing to \( \lambda = \lambda_0 \), by definition of \( Z_1, Z_2, 3.9.1 \) and Lemma 3.1.1(iv), we deduce that \( Z(s, W, \phi) \) and \( Z(s, \tilde{W}, \hat{\phi}) \) admit meromorphic continuation to \( \mathbb{C} \) satisfying the functional equation

\[
Z(1 - s, \tilde{W}, \hat{\phi}) = \omega(\tau)^{n-1} |\tau|^{\frac{n(n-1)}{2}} \lambda_{E/F}(\psi')^{-\frac{n(n-1)}{2}} \gamma(s, \pi, As, \psi') Z(s, W, \phi).
\]

This already proves Theorem 3.1.1 for \( \pi \). Moreover, we have also obtained that \( \frac{Z(s, W_\lambda, \phi)}{L_{K,z}(s, \pi_\lambda, As)} \) is holomorphic. In the \( p \)-adic case, as \( L_{K,z}(s, \pi_\lambda, As) = L(s, \pi_\lambda, As) \) by definition, this gives
Theorem 3.4.2 for \( \pi \). In the Archimedean case we have

\[
\frac{Z(s, W_\lambda, \phi)}{L_{K, z}(s, \pi_\lambda, As)} = K_{\frac{s}{2} - \frac{1}{4}}(z) n^2 \frac{Z(s, W_\lambda, \phi)}{L(s, \pi_\lambda, As)}
\]

and therefore

\[
\frac{Z(s, W_\lambda, \phi)}{L(s, \pi_\lambda, As)}
\]

is holomorphic outside the vanishing set of \( s \mapsto K_{\frac{s}{2} - \frac{1}{4}}(z) \). Since \( z \) was an arbitrary positive real number and for every \( s \in \mathbb{C} \) there exists \( z \in \mathbb{R}_+^* \) so that \( K_s(z) \neq 0 \), it follows that

\[
\frac{Z(s, W_\lambda, \phi)}{L(s, \pi_\lambda, As)}
\]

is holomorphic everywhere and this shows Theorem 3.4.2 for \( \pi \) in this case too.

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