Long range integrable oscillator chains from quantum algebras

ANGEL BALLESTEROS and FRANCISCO J. HERRANZ

Departamento de Física, Universidad de Burgos
Pza. Misael Bañuelos s.n., 09001-Burgos, Spain

Abstract

Completely integrable Hamiltonians defining classical mechanical systems of \( N \) coupled oscillators are obtained from Poisson realizations of Heisenberg–Weyl, harmonic oscillator and \( sl(2, \mathbb{R}) \) coalgebras. Various completely integrable deformations of such systems are constructed by considering quantum deformations of these algebras. Explicit expressions for all the deformed Hamiltonians and constants of motion are given, and the long-range nature of the interactions is shown to be linked to the underlying coalgebra structure. The relationship between oscillator systems induced from the \( sl(2, \mathbb{R}) \) coalgebra and angular momentum chains is presented, and a non-standard integrable deformation of the hyperbolic Gaudin system is obtained.
1 Introduction

The construction of integrable systems is an outstanding application of Lie algebras in both classical and quantum mechanics \[1, 2\]. In fact, the very definition of integrability is based on the concept of involutivity of the conserved quantities with respect to a (either Poisson or commutator) Lie bracket. During last years, many new results concerning deformations of Lie algebras and groups have been obtained \[3\], and the question concerning the connection between these new algebraic structures and integrability properties arises as a keystone for future developments in the subject. From our point of view, such analysis should be based on the basic role that the coalgebra structure (i.e., the existence of an homomorphism \(\Delta\) between the algebra \(A\) and \(A \otimes A\), called the coproduct) plays in quantum deformations.

In this paper, we follow this idea and deal with a general and systematic construction of integrable systems from coalgebras that has been introduced very recently \[4, 5\]. Within it, the coproduct map plays an essential role: it propagates integrability from the one-particle Hamiltonian (which is defined as a certain realization of a given function of the algebra generators) to the \(N\)-particle one. Several classical realizations of the Gaudin system have been presented as the first examples of this coalgebra-induced construction. In this framework, quantum algebras (which are just coalgebra deformations) can be interpreted as dynamical symmetries that generate in a direct way a large class of integrable deformations. This has been the case for the Gaudin–Calogero systems \[6\] with respect to the (standard) quantum deformation \(U_z(\text{sl}(2, \mathbb{R}))\), that can be used to derive explicitly a family of completely integrable Hamiltonians that reduce to the Gaudin–Calogero one in the “classical” limit \(z \to 0\) \[4\].

In the same way, a sort of Ruijsenaars–Schneider Hamiltonian was derived from a quantum deformation of \((1+1)\) Poincaré algebra \[7\], and also a simple example of oscillator chain was obtained from the non-deformed oscillator algebra \(h_4\) \[5\]. However, in order to get a deeper insight into this general method and its consequences, it seems appropriate to construct some new models. This is the aim of the present paper, in which we shall concentrate on the explicit construction and analysis of classical mechanical oscillator chains obtained from (deformed and non-deformed) Heisenberg–Weyl \(h_3\), harmonic oscillator \(h_4\) and \(\text{sl}(2, \mathbb{R})\) coalgebras, all of them realized in terms of functions on the classical phase space endowed with the canonical Poisson bracket. We recall that quantum algebra deformations of \(h_3\) and \(h_4\) have been fully classified in \[8, 9\], and results concerning \(\text{sl}(2, \mathbb{R})\) are basic in quantum algebra theory and can be found, for instance, in \[4\].

In the next section we briefly recall the general construction of \[5\] and fix the notation. It is important to stress that the complete integrability of the resulting Hamiltonians by no means depends on the explicit form of the coproduct map, that can be either deformed or classical. Therefore, in the following sections we will pay attention to both types of situations, and we shall emphasize the fact that the underlying algebraic structure provides explicit expressions for the constants of motion in a straightforward way. However, we shall not address here the study of the dynamical contents of the (to our knowledge) new oscillator chains that will be introduced.

The simplest coalgebra structures are provided by the Heisenberg–Weyl algebra, and are analysed in section 3. In the non-deformed case, the coalgebra leads to an oscillator chain whose integrability cannot be immediately proven due to the “degeneracy” of
the Heisenberg phase space realization. On the contrary, a two-parameter quantum deformation of \( h_3 \) gives rise in a straightforward way to the corresponding integrable deformation, including the explicit form for the constants of motion. Therefore, degeneracies of the integrals of motion appearing in the non-deformed model are removed.

The same procedure is carried out for the oscillator algebra \( h_4 \) in section 4. The Hamiltonian there induced from the non-deformed coalgebra is the same as the one derived in the Heisenberg case, but now its complete integrability is easily demonstrated by the existence of an extra Casimir function (therefore, the degeneracy problem can be circumvented by taking into account that \( h_3 \subset h_4 \)). The non-standard quantum deformation of \( h_4 \) is also realized in terms of canonical coordinates and gives rise to another integrable chain.

Section 5 deals with oscillator chains obtained from \( \mathfrak{sl}(2, \mathbb{R}) \) coalgebras. In contrast with the previous cases, the non-deformed \( \mathfrak{sl}(2, \mathbb{R}) \) structure provides a chain of non-interacting oscillators through a linear Hamiltonian of the type \( J_+ + \alpha J_- \). In this way, the inclusion of the non-standard deformation of \( \mathfrak{sl}(2, \mathbb{R}) \) can be interpreted as a direct algebraic implementation of a certain type of long-range interaction involving the momenta. In this section we also present the deep relationship between these \( \mathfrak{sl}(2, \mathbb{R}) \) oscillator chains and classical spin models, and we explicitly construct the non-standard deformation of the hyperbolic Gaudin system [11]. As it happened with the standard deformation [5], the non-standard one is shown to generate a sort of variable range exchange [12]. The construction of (deformed and non-deformed) anharmonic chains is also studied, thus showing the number of integrable systems that can be easily derived by following the present approach.

2 From coalgebras to integrable Hamiltonians

The main result of [5] can be summarized as follows: any coalgebra \((A, \Delta)\) with Casimir element \( C \) can be considered as the generating symmetry that, after choosing a non-trivial representation, gives rise in a systematic way to a large family of integrable systems. We shall consider here classical mechanical systems only and, consequently, we shall make use of Poisson realizations \( D \) of Lie and quantum algebras of the form \( D : A \to C^\infty(q, p) \). However, we recall that the formalism is also directly applicable to quantum mechanical systems.

Let \((A, \Delta)\) be a (Poisson) coalgebra with generators \( X_i \ (i = 1, \ldots, l) \) and Casimir element \( C(X_1, \ldots, X_l) \). Therefore, the coproduct \( \Delta : A \to A \otimes A \) is a Poisson map. Let us consider the \( N \)-th coproduct \( \Delta^{(N)}(X_i) \) of the generators

\[
\Delta^{(N)} : A \to A \otimes A \otimes \ldots \otimes A
\]

which is obtained (see [3]) by applying recursively the two-coproduct \( \Delta^{(2)} \equiv \Delta \) in the form

\[
\Delta^{(N)} := (id \otimes id \otimes \ldots \otimes id) \otimes \Delta^{(2)} \circ \Delta^{(N-1)}. \tag{2.2}
\]

By taking into account that the \( m \)-th coproduct \((m \leq N)\) of the Casimir \( \Delta^{(m)}(C) \) can be embedded into the tensor product of \( N \) copies of \( A \) as

\[
\Delta^{(m)} : A \rightarrow \{A \otimes A \otimes \ldots \otimes A\} \otimes \{1 \otimes 1 \otimes \ldots \otimes 1\}, \tag{2.3}
\]
it can be shown that,
\[ \{ \Delta^{(m)}(C), \Delta^{(N)}(X_i) \} = 0 \quad i = 1, \ldots, l \quad 1 \leq m \leq N. \] (2.4)

With this in mind it can be proven [5] that, if \( H \) is an arbitrary (smooth) function of the generators of \( A \), the \( N \)-particle Hamiltonian defined on \( A \otimes A \otimes \ldots \otimes A \) as the \( N \)-th coproduct of \( H \)
\[ H^{(N)} := \Delta^{(N)}(H(X_1, \ldots, X_l)) = H(\Delta^{(N)}(X_1), \ldots, \Delta^{(N)}(X_l)), \] (2.5)
fulfils
\[ \{ C^{(m)}, H^{(N)} \} = 0 \quad 1 \leq m \leq N \] (2.6)
where the \( N \) functions \( C^{(m)} \) \((m = 1, \ldots, N)\) are defined through the coproducts of the Casimir
\[ C^{(m)} := \Delta^{(m)}(C(X_1, \ldots, X_l)) = C(\Delta^{(m)}(X_1), \ldots, \Delta^{(m)}(X_l)) \] (2.7)
and all the integrals of motion \( C^{(m)} \) are in involution
\[ \{ C^{(m)}, C^{(n)} \} = 0 \quad \forall m, n = 1, \ldots, N. \] (2.8)

Therefore, provided a realization of \( A \) on a one-particle phase space is given, the \( N \)-particle Hamiltonian \( H^{(N)} \) will be a function of \( N \) canonical pairs \((q_i, p_i)\) and is, by construction, completely integrable with respect to the ordinary Poisson bracket
\[ \{ f, g \} = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right). \] (2.9)
Moreover, its constants of motion will be given by the \( C^{(m)} \) functions, all of them functionally independent since each of them depends on the first \( m \) pairs \((q_i, p_i)\) of canonical coordinates. Note that with such one-particle realizations the first Casimir \( C^{(1)} \) will be a number, and we are left with \( N - 1 \) constants of motion with respect to \( H^{(N)} \).

In particular, this result can be applied to universal enveloping algebras of Lie algebras \( U(g) \), since they are always endowed with a natural (primitive) Hopf algebra structure of the form \( \Delta^{(2)}(X_i) = X_i \otimes 1 + 1 \otimes X_i \), being \( X_i \) any generator of \( g \). Moreover, since quantum algebras are also (deformed) coalgebras \((A_z, \Delta_z)\), any function of the generators of a given quantum algebra with Casimir element \( C_z \) will provide, under a chosen deformed representation, a completely integrable Hamiltonian.

### 3 Oscillator chains from Heisenberg–Weyl coalgebras

The obtention of integrable oscillator chains by using the previous approach can be achieved by selecting Poisson coalgebras \((A, \Delta)\) such that the one-dimensional harmonic oscillator Hamiltonian with angular frequency \( \omega \) (and unit mass) could be written as the phase space representation \( D \) of a certain function \( \mathcal{H} \) of the generators of \( A \):
\[ H = D(\mathcal{H}) = p^2 + \omega^2 q^2. \] (3.1)
It seems natural to consider the non-deformed Heisenberg–Weyl coalgebra (whose generators are essentially the canonical coordinates and a central element) as the first “dynamical coalgebra” for oscillator chains.
3.1 Non-deformed $U(h_3)$ coalgebra

The $h_3$ Lie–Poisson algebra

$$\{A_-, A_+\} = M \quad \{A_-, M\} = 0 \quad \{A_+, M\} = 0$$  \hspace{1cm} (3.2)

is endowed with a Poisson coalgebra structure by means of the usual primitive coproduct

$$\Delta(X) = X \otimes 1 + 1 \otimes X \quad X \in \{A_-, A_+, M\}$$  \hspace{1cm} (3.3)

which is a Poisson algebra homomorphism between $h_3$ and $h_3 \otimes h_3$. The Casimir element $C$ for this algebra is the central generator $M$.

A natural one-particle phase space realization $D$ of $h_3$ is given by:

$$f^{(1)}_[-] = D(A_-) = \omega_1 q_1 \quad f^{(1)}_+] = D(A_+) = p_1 \quad f^{(1)}_M = D(M) = \omega_1.$$  \hspace{1cm} (3.4)

Obviously, if we consider the quadratic function $H$ of the $h_3$ generators

$$H = A_+^2 + A_-^2,$$  \hspace{1cm} (3.5)

the associated one-particle Hamiltonian will be

$$H^{(1)} = D(H) = (f^{(1)}_+)^2 + (f^{(1)}_-)^2 = p_1^2 + \omega_1^2 q_1^2,$$  \hspace{1cm} (3.6)

and the Casimir function is just the angular frecuency

$$C^{(1)} = D(C) = \omega_1.$$  \hspace{1cm} (3.7)

Now, the coproduct map (3.3) will give us, under a $D \otimes D$ realization, three two-particle phase space functions that close again $h_3$ under the usual Poisson bracket $\{q_i, p_j\} = \delta_{ij}$:

$$f^{(2)}_- = (D \otimes D)(\Delta(A_-)) = \omega_1 q_1 + \omega_2 q_2$$
$$f^{(2)}_+ = (D \otimes D)(\Delta(A_+)) = p_1 + p_2$$
$$f^{(2)}_M = (D \otimes D)(\Delta(M)) = \omega_1 + \omega_2.$$  \hspace{1cm} (3.8)

By folowing (2.3), the two-particle Hamiltonian $H^{(2)}$ will be given by the realization of the coproduct of $H$ and reads

$$H^{(2)} = (D \otimes D)(\Delta(H)) = (f^{(2)}_+)^2 + (f^{(2)}_-)^2$$
$$= p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2 + 2(p_1 p_2 + \omega_1 \omega_2 q_1 q_2),$$  \hspace{1cm} (3.9)

which is just the Hamiltonian defining a pair of coupled oscillators with frecuencies $\omega_1$ and $\omega_2$ and whose interaction depends on the momenta. The constant of motion $C^{(2)}$ would be given by (2.7), i.e. the representation of the coproduct of the Casimir

$$C^{(2)} = (D \otimes D)(\Delta(C)) = \omega_1 + \omega_2.$$  \hspace{1cm} (3.10)

In this case, since the Casimir is a single generator whose representation is a real constant, its coproduct is just the sum of he two frecuencies.
The method sketched in section 2 allows us to generalize this result to the \( N \)-dimensional case in a straightforward way. Since, by using (2.2), the \( m \)-th coproduct of a primitive generator reads

\[
\Delta^{(m)}(X_i) = X_i \otimes 1 \otimes 1 \otimes \ldots \otimes 1 \otimes \ldots \otimes 1 \otimes X_i,
\]

the \( m \)-dimensional particle phase space realization of \((h_3, \Delta)\) will be:

\[
f^{(m)}_+ = (D \otimes \ldots \otimes D)(\Delta^{(m)}(A_-)) = \sum_{i=1}^{m} \omega_i q_i
\]

\[
f^{(m)}_- = (D \otimes \ldots \otimes D)(\Delta^{(m)}(A_+)) = \sum_{i=1}^{m} p_i
\]

\[
f^{(m)}_M = (D \otimes \ldots \otimes D)(\Delta^{(m)}(M)) = \sum_{i=1}^{m} \omega_i.
\]

Note that these three functions generate again \( h_3 \). Therefore, the corresponding \( N \)-dimensional Hamiltonian is:

\[
H^{(N)} = (D \otimes \ldots \otimes D)(\Delta^{(N)}(H)) = (f^{(N)}_+)^2 + (f^{(N)}_-)^2
\]

\[
= \sum_{i=1}^{N} (p_i^2 + \omega_i^2 q_i^2) + 2 \sum_{i<j} (p_ip_j + \omega_i \omega_j q_i q_j).
\]

In the same manner, the \( N \) constants of motion provided by the formalism would be

\[
C^{(m)} = (D \otimes \ldots \otimes D)(\Delta^{(m)}(C)) = f^{(m)}_M = \sum_{i=1}^{m} \omega_i \quad m = 1, \ldots, N.
\]

Apparently, we could conclude that in this case the formalism provides trivial constants of motion (3.14) only. This problem is due to the extreme simplicity of the chosen algebra and representation, and the Hamiltonian (3.13) will be proven to be completely integrable in section 4 by making use of the embedding of \( h_3 \) into the harmonic oscillator algebra \( h_4 \), which is the natural framework to construct this kind of non-deformed chains. However, the \( h_3 \) case provides an interesting benchmark for the use of quantum algebras to obtain integrable deformations. As we shall see in the sequel, the deformation will provide non-trivial Casimirs, thus breaking the degeneracy in the representation and turning the analogues of (3.14) into non-trivial functions.

### 3.2 Two-parameter quantum deformation \( U_{z,\lambda}(h_3) \)

Quantum deformations of the Heisenberg–Weyl algebra have been fully classified in \[8\]. We shall now make use of the two-parameter quantum algebra \( U_{z,\lambda}(h_3) \) with deformed coproduct

\[
\Delta(A_-) = A_- \otimes 1 + 1 \otimes A_- \quad \Delta(M) = M \otimes e^{zA_-} + e^{-zA_-} \otimes M
\]

\[
\Delta(A_+) = A_+ \otimes e^{zA_-} + e^{-zA_-} \otimes A_+ + \lambda M \otimes A_- e^{zA_-}.
\]
where we define the auxiliary quantities

\[ \{ A_-, A_+ \} = M \quad \{ A_-, M \} = 0 \quad \{ A_+, M \} = 0. \]  

(3.16)

As a consequence, the Casimir coincides again with the \( M \) generator and the one-particle representation \( D \) (3.4) is also valid for the deformed case.

In fact, the deformation arises with the coproduct and, consequently, it will become apparent in the two-particle system. Although the phase space realizations on each space are classical ones, their composition law (3.15) is not, and leads to the following functions

\[
\begin{align*}
 f_-^{(2)} &= (D \otimes D)(\Delta(A_-)) = \omega_1 q_1 + \omega_2 q_2 \\
 f_+^{(2)} &= (D \otimes D)(\Delta(A_+)) = (p_1 + \lambda \omega_1 \omega_2 q_2)e^{z \omega_2 q_2} + p_2 e^{-z \omega_1 q_1} \\
 f_M^{(2)} &= (D \otimes D)(\Delta(M)) = \omega_1 e^{z \omega_2 q_2} + \omega_2 e^{-z \omega_1 q_1},
\end{align*}
\]

(3.17)

whose Poisson brackets will close again an \( h_3 \) algebra. If the function \( H \) given in (3.3) is considered, these expressions will lead to a deformed two-particle Hamiltonian

\[
H_{z, \lambda}^{(2)} = (D \otimes D)(\Delta(H)) = (f_+^{(2)})^2 + (f_-^{(2)})^2 \\
= (p_1 + \lambda \omega_1 \omega_2 q_2)^2 e^{2z \omega_2 q_2} + p_2^2 e^{-2z \omega_1 q_1} + \omega_1^2 q_1^2 + \omega_2^2 q_2^2 + 2 \left( \{ p_1 + \lambda \omega_1 \omega_2 q_2 \} p_2 e^{-z(\omega_1 q_1 - \omega_2 q_2)} + \omega_1 \omega_2 q_1 q_2 \right),
\]

(3.18)

whose power series expansion around zero values for the deformation parameters is

\[
H_{z, \lambda}^{(2)} = H^{(2)} + 2z(p_1 + p_2)(\omega_2 q_2 p_1 - \omega_1 q_1 p_2) + 2\lambda \omega_1 \omega_2 q_2(p_1 + p_2) \\
+ 2z \omega_1 \omega_2 q_2(\omega_2 q_2 p_1 - \omega_1 q_1 p_2 + \omega_2 q_2(p_1 + p_2)) + o(z^2, \lambda^2)
\]

(3.19)

where \( H^{(2)} \) is the non-deformed Hamiltonian (3.3).

Deformed Casimirs are now

\[
C_{z, \lambda}^{(1)} = D(C) = \omega_1 \quad C_{z, \lambda}^{(2)} = (D \otimes D)(\Delta(C)) = \omega_1 e^{z \omega_2 q_2} + \omega_2 e^{-z \omega_1 q_1}.
\]

(3.20)

In particular, due to the deformed coproduct (3.15) for the central generator \( M \), the two-particle Casimir is no longer a constant, and the degeneracy problem is solved.

The construction of the \( N \)-dimensional system is based onto the general \( m \)-dimensional particle phase space realization given by:

\[
\begin{align*}
 f_-^{(m)} &= (D \otimes \ldots \otimes D)(\Delta(m)(A_-)) = \sum_{i=1}^{m} \omega_i q_i \\
 f_+^{(m)} &= (D \otimes \ldots \otimes D)(\Delta(m)(A_+)) = \sum_{i=1}^{m} \pi_i^{(m)} e^{z Q_i^{(m)}(q)} \\
 f_M^{(m)} &= (D \otimes \ldots \otimes D)(\Delta(m)(M)) = \sum_{i=1}^{m} \omega_i e^{z Q_i^{(m)}(q)}
\end{align*}
\]

(3.21)

where we define the auxiliary quantities

\[
\pi_i^{(m)} = p_i + \lambda \omega_i Q_i^{(m)}(q).
\]

(3.22)

The various \( Q \)-functions, that we shall often use from now on, are introduced in the appendix; within all of them, any sum defined on an empty set of indices will be assumed to be zero.
The final expression for the $N$-dimensional deformed chain is:

$$
H_{z,\lambda}^{(N)} = (D \otimes \ldots \otimes D)(\Delta^{(N)}(\mathcal{H})) = (f_+^{(N)})^2 + (f_-^{(N)})^2
$$

$$
= \sum_{i=1}^N \left( (\pi_i^{(N)})^2 e^{zQ_i^{(N)}(q_i)} + \omega_i^2 q_i^2 \right) + 2 \sum_{i<j}^N \left( \pi_i^{(N)} \pi_j^{(N)} e^{zQ_{ij}^{(N)}(q_i + q_j)} + \omega_i \omega_j q_i q_j \right). \quad (3.23)
$$

As it was expected, the $N - 1$ constants of motion depend on the canonical coordinates $q_i$ and read $(m = 2, \ldots , N)$:

$$
C_{z,\lambda}^{(m)} = (D \otimes \ldots \otimes D)(\Delta^{(m)}(\mathcal{C})) = f_M^{(m)} = \sum_{i=1}^m \omega_i e^{zQ_i^{(m)}(q_i)}. \quad (3.24)
$$

4 Oscillator chains from harmonic oscillator coalgebras

As we shall see now, the harmonic oscillator algebra $h_4$ will provide a natural setting for the construction of non-deformed chains. We shall also analyse a new system obtained from the (uniparametric) non-standard deformation of $h_4$ introduced in [9].

4.1 Non-deformed $U(h_4)$ coalgebra

We consider the Poisson oscillator algebra defined by

$$
\{A, A_+\} = A_+ \quad \{A, A_-\} = -A_- \quad \{A_-, A_+\} = M \quad \{M, \cdot \} = 0. \quad (4.1)
$$

As usual, the primitive coproduct

$$
\Delta(X) = X \otimes 1 + 1 \otimes X \quad X \in \{A, A_-, A_+, M\} \quad (4.2)
$$

is compatible with (4.1) and we have two Casimirs: $M$ and

$$
\mathcal{C} = AM - A_- A_+. \quad (4.3)
$$

The latter will play a relevant role in what follows. Obviously, $h_3$ is a subalgebra of $h_4$, and a natural phase space realization for $h_4$ is given by

$$
f_-^{(1)} = D(A_-) = \omega_1 q_1 \quad f_+^{(1)} = D(A_+) = p_1 \quad f_M^{(1)} = D(M) = \omega_1 \quad f^{(1)} = D(A) = q_1 p_1, \quad (4.4)
$$

and it is characterized by the values of the Casimirs $D(M) = \omega_1$ and $C^{(1)} = D(\mathcal{C}) = 0$.

By following our construction, we shall consider again the quadratic function

$$
\mathcal{H} = A_+^2 + A_-^2, \quad (4.5)
$$

that gives rise to the one dimensional harmonic oscillator through $D$:

$$
H^{(1)} = D(\mathcal{H}) = p_1^2 + \omega_1^2 q_1^2. \quad (4.6)
$$

Note that (4.5) and (4.6) are just the same expressions we had in the $h_3$ construction.
The coproduct (4.2) induces the two-particle phase space realization:

\[
\begin{align*}
 f_{-}^{(2)} &= \omega_1 q_1 + \omega_2 q_2 \\
 f_{+}^{(2)} &= p_1 + p_2 \\
 f_{M}^{(2)} &= \omega_1 + \omega_2 \\
 f^{(2)} &= q_1 p_1 + q_2 p_2.
\end{align*}
\] (4.7)

From (4.7) and (4.5) we are just lead to the two-dimensional Hamiltonian (3.9). But there exists a crucial difference with respect to the \( h_3 \) construction: the \((D \otimes D)\) realization of the coproduct of \( C \) is no longer a constant: namely,

\[
C^{(2)} = (\omega_1 p_2 - \omega_2 p_1)(q_2 - q_1).
\] (4.8)

Therefore, the complete integrability of (3.9) is easily proven through the use of the new Casimir function \( C \) of the \( h_4 \) coalgebra.

Similarly, the construction of the \( m \)-dimensional particle phase space realization in terms of \( C^{(1)} = 0 \) representations is guided by (3.11):

\[
\begin{align*}
 f_{-}^{(m)} &= \sum_{i=1}^{m} \omega_i q_i \\
 f_{+}^{(m)} &= \sum_{i=1}^{m} p_i \\
 f_{M}^{(m)} &= \sum_{i=1}^{m} \omega_i \\
 f^{(m)} &= \sum_{i=1}^{m} q_i p_i.
\end{align*}
\] (4.9)

The \( N \)-dimensional particle integrable system linked to (4.5) is clearly (3.13). Explicit integrability is thus provided by the constants given through the different coproducts \((m = 2, \ldots, N)\) for the Casimir (4.3), that can be explicity written as:

\[
C^{(m)} = \sum_{i=1}^{m} q_i p_i \{ \sum_{j=1}^{i-1} \omega_j + \sum_{l=i+1}^{m} \omega_l \} - \sum_{i=1}^{m} p_i \{ Q_{+i}^{(m)}(q) + Q_{-i}^{(m)}(q) \}.
\] (4.10)

Its Poisson coalgebra expressions are found to be

4.2 Non-standard quantum deformation \( U_z(h_4) \)

This quantum algebra, firstly introduced in [9], is characterized by a deformation of both the coproduct and the commutation rules. The former is given by

\[
\begin{align*}
\Delta(A_-) &= A_- \otimes 1 + 1 \otimes A_- \\
\Delta(M) &= M \otimes 1 + 1 \otimes M \\
\Delta(A_+) &= A_+ \otimes e^{zA_-} + e^{-zA_-} \otimes A_+ + 2zA \otimes Me^{zA_-} \\
\Delta(A) &= A \otimes e^{zA_-} + e^{-zA_-} \otimes A
\end{align*}
\] (4.11)

and the latter are translated in Poisson terms as

\[
\begin{align*}
\{ A, A_+ \} &= A_+ \cosh zA_- - zAMe^{zA_-} \\
\{ A, A_- \} &= -\frac{\sinh zA_-}{z} \{ A_-, A_+ \} = Me^{zA_-}.
\end{align*}
\] (4.12)

It is straightforward to check that (4.11) is a Poisson map with respect to (4.12), and that \( M \) and

\[
C = AMe^{zA_-} - \frac{\sinh zA_-}{z} A_+
\] (4.13)

are Casimir functions for this Poisson algebra.
The main consequence of the presence of the deformation in the Poisson brackets \( \{1.12\} \) is that the phase space realization \( \{4.4\} \) has to be deformed:

\[
\begin{align*}
  f_{-}^{(1)} &= D(A_{-}) = \omega_1 q_1 & f_{+}^{(1)} &= D(A_{+}) = p_1 e^{z\omega_1 q_1} \\
  f_{M}^{(1)} &= D(M) = \omega_1 & f^{(1)} &= D(A) = \frac{\sinh z\omega_1 q_1}{z\omega_1} p_1. \\
\end{align*}
\]

Therefore, if we preserve the function \( \mathcal{H} = A_{+}^2 + A_{-}^2 \), the one-particle Hamiltonian becomes a deformed oscillator:

\[
H_{z}^{(1)} = D(\mathcal{H}) = p_1^2 e^{2z\omega_1 q_1} + \omega_1^2 q_1^2,
\]

and the phase space realization of the Casimir is again zero \( C_{z}^{(1)} = D(\mathcal{C}) = 0 \). In the spirit of this method, \( \{4.13\} \) would be the basic non-standard “integrable” deformation of the harmonic oscillator linked to \( U_z(h_4) \).

The \( N = 2 \) integrable deformation is, as usual, a consequence of the two-particle phase space realization \( D \otimes D \) acting on the deformed coproduct of the generators. Namely,

\[
\begin{align*}
  f_{-}^{(2)} &= \omega_1 q_1 + \omega_2 q_2 & f_{+}^{(2)} &= \omega_1 + \omega_2 \\
  f_{M}^{(2)} &= p_1 \left(e^{z\omega_1 q_1} + 2 z\omega_2 \frac{\sinh z\omega_1 q_1}{z\omega_1} \right) e^{z\omega_2 q_2} + p_2 e^{z\omega_2 q_2} e^{-z\omega_1 q_1} \\
  f^{(2)} &= \frac{\sinh z\omega_1 q_1}{z\omega_1} p_1 e^{z\omega_2 q_2} + \frac{\sinh z\omega_2 q_2}{z\omega_2} p_2 e^{-z\omega_1 q_1}. \\
\end{align*}
\]

Now, from \( \mathcal{H} \), the deformed two-particle Hamiltonian reads:

\[
H_{z}^{(2)} = p_1^2 \left(e^{z\omega_1 q_1} + 2 z\omega_2 \frac{\sinh z\omega_1 q_1}{z\omega_1} \right)^2 e^{2z\omega_2 q_2} + p_2^2 e^{2z\omega_2 q_2} e^{-2z\omega_1 q_1} + \omega_1^2 q_1^2 + \omega_2^2 q_2^2 + 2 \left\{ p_1 p_2 \left(e^{z\omega_1 q_1} + 2 z\omega_2 \frac{\sinh z\omega_1 q_1}{z\omega_1} \right) e^{z\omega_2 q_2} e^{-z\omega_1 q_1} + \omega_1 \omega_2 q_1 q_2 \right\}.
\]

A power series expansion of this Hamiltonian around \( z = 0 \) will give us an idea of the perturbation generated by the non-standard deformation:

\[
H_{z}^{(2)} = H^{(2)} + 2 z (p_1 + p_2) \left\{ (p_1 + p_2) (\omega_1 q_1 + \omega_2 q_2) + 2 q_1 (\omega_2 p_1 - \omega_1 p_2) \right\} + o(z^2)
\]

where \( H^{(2)} \) is the Hamiltonian \( \{3.9\} \). A constant of motion is given by the coproduct of the Casimir:

\[
C_{z}^{(2)} = \left( \omega_1 p_2 - \omega_2 p_1 \right) \left\{ \frac{\sinh z\omega_2 q_2}{z\omega_2} e^{z\omega_2 q_2} - \frac{\sinh z\omega_1 q_1}{z\omega_1} e^{-z\omega_1 q_1} \right\}.
\]

Its power series expansion around \( z = 0 \) is

\[
C_{z}^{(2)} = C^{(2)} + z (\omega_1 p_2 - \omega_2 p_1) (\omega_1 q_1^2 + \omega_2 q_2^2) + o(z^2)
\]

where \( C^{(2)} \) is the non-deformed Casimir \( \{1.5\} \).

The \( N \)-dimensional result is now a matter of long but straightforward computations. Firstly, we have to deduce the \( m \)-th coproduct of any generator by applying the recurrence \( \{2.2\} \). From it, by considering a \( (D \otimes D \otimes \ldots \otimes D) \) representation, the \( m \)-particle phase space realization of \( U_z(h_4) \) is obtained in closed form:

\[
\begin{align*}
  f_{-}^{(m)} &= \sum_{i=1}^{m} \omega_i q_i & f_{+}^{(m)} &= \sum_{i=1}^{m} p_i \gamma_i^{(m)} e^{z Q_i^{(m)} (q)} \\
\end{align*}
\]
\[ f_M^{(m)} = \sum_{i=1}^{m} \omega_i \quad f^{(m)} = \sum_{i=1}^{m} \frac{\sinh z \omega_i q_i}{z \omega_i} p_i e^{zQ_i^{(m)}(q)} \]  

(4.21)

where the functions \( \gamma_i^{(m)} \) are defined as

\[ \gamma_i^{(m)} = e^{z \omega_i q_i} + 2z \sum_{l=i+1}^{m} \omega_l. \]  

(4.22)

From these expressions, we get the \( N \)-dimensional oscillator chain induced by \( U_z(h_4) \):

\[ H_z^{(N)} = \sum_{i=1}^{N} \left( p_i^2 (\gamma_i^{(N)})^2 e^{2zQ_i^{(N)}(q)} + \omega_i^2 q_i^2 \right) + 2 \sum_{i<j}^{N} \left( p_i p_j \gamma_i^{(N)} \gamma_j^{(N)} e^{zQ_{ij}(q)} + \omega_i \omega_j q_i q_j \right) \]  

(4.23)

whose integrals of motion are obtained from the coproduct of the Casimir. They are

\[ C_z^{(m)} = \sum_{i=1}^{m} p_i \frac{\sinh z \omega_i q_i}{z \omega_i} \left\{ e^{z \omega_i q_i} e^{2zQ_{+i,j}(q)} \sum_{j=1}^{i-1} \omega_j + e^{-z \omega_i q_i} e^{-2zQ_{-i,j}(q)} \sum_{l=i+1}^{m} \omega_l \right\} \]

\[ - \sum_{i=1}^{m} p_i \frac{e^{2zQ_{+i,j}(q)} - e^{-2zQ_{-i,j}(q)}}{2z}. \]  

(4.24)

In conclusion, by using \( h_4 \) we have been able to derive the explicit form for the constants of motion corresponding to the non-deformed Heisenberg–Weyl Hamiltonian (3.13). This is not surprising, since \( h_3 \) is a sub-coalgebra of \( h_4 \), and the latter contains the full information concerning the integrability of the system. However, the situation is completely different at a deformed level, where quantum \( h_3 \) algebras give by themselves integrable deformations of the same system that differ from the ones that can be deduced by using quantum \( h_4 \) algebras. This result can be understood if we recall that \( h_3 \) quantum algebras cannot be obtained as sub-quantum \( h_4 \) algebras (a comparison between the classification of quantum deformations for both algebras [8, 9] clearly shows this fact). Although we have chosen the closest representatives (\( U_z(\lambda(h_3)) \) and \( U_z(h_4) \)) of both families, results here presented show that the integrable deformations linked to both coalgebras lead to significantly distinct structures. As a consequence, the obtention of all quantum algebra deformations of a given \( U(g) \) could be meaningful in the search for new completely integrable systems.

5 Oscillator chains from \( sl(2, \mathbb{R}) \) coalgebras

It is well known that \( sl(2, \mathbb{R}) \) can be considered as a dynamical algebra for the one-dimensional harmonic oscillator. If \( sl(2, \mathbb{R}) \) coalgebras and their deformations are considered, a big class of new integrable oscillator chains can be obtained. Among them, a set of systems with anharmonic interactions arises in a very simple way.

5.1 Non-deformed \( U(sl(2, \mathbb{R})) \) coalgebra

In the case of \( sl(2, \mathbb{R}) \), we write the following non-deformed Poisson coalgebra

\[ \Delta(X) = X \otimes 1 + 1 \otimes X \quad X \in \{ J_3, J_-, J_+ \} \]  

(5.1)
\[\{J_3, J_+\} = 2J_+ \quad \{J_3, J_-\} = -2J_- \quad \{J_-, J_+\} = 4J_3 \quad (5.2)\]

whose Casimir element is
\[C = J_3^2 - J_- J_. \quad (5.3)\]

A phase space realization with vanishing Casimir is given by:
\[f^{(1)}_\pm = D(J_\pm) = q_1^2 \quad f^{(1)}_3 = D(J_3) = q_1 p_1. \quad (5.4)\]

From it, the harmonic oscillator Hamiltonian is recovered if the following linear function
of the generators of \(sl(2, \mathbb{R})\)
\[H = J_+ + \omega^2 J_- \quad (5.5)\]
is represented through \((5.4)\)
\[H^{(1)} = D(H) = p_1^2 + \omega^2 q_1^2. \quad (5.6)\]

Now, the use of \(D \otimes D\) onto the primitive coproduct \((5.1)\) leads to the two-particle phase space realization of \(sl(2, \mathbb{R})\)
\[f^{(2)}_- = q_1^2 + q_2^2 \quad f^{(2)}_+ = p_1^2 + p_2^2 \quad f^{(2)}_3 = q_1 p_1 + q_2 p_2 \quad (5.7)\]

that, in turn, gives rise to the uncoupled oscillator Hamiltonian:
\[H^{(2)} = f^{(2)}_+ + \omega^2 f^{(2)}_- = p_1^2 + p_2^2 + \omega^2 (q_1^2 + q_2^2). \quad (5.8)\]

Note that now the frequency of both oscillators is the same. The coproduct of the Casimir will give us the corresponding integral of motion
\[C^{(2)} = -(q_1 p_2 - q_2 p_1)^2 \quad (5.9)\]
that turns out to be the square of the angular momentum. (Note that, in this particular case, we know that two more functionally independent integrals exist, since the system \((5.8)\) is known to be superintegrable).

The construction of the \(m\)-dimensional particle phase space realization is straightforward:
\[f^{(m)}_- = \sum_{i=1}^{m} q_i^2 \quad f^{(m)}_+ = \sum_{i=1}^{m} p_i^2 \quad f^{(m)}_3 = \sum_{i=1}^{m} q_i p_i. \quad (5.10)\]

From it, the uncoupled chain of \(N\) harmonic oscillators (all of them with the same frequency) is obtained:
\[H^{(N)} = f^{(N)}_+ + \omega^2 f^{(N)}_- = \sum_{i=1}^{N} (p_i^2 + \omega^2 q_i^2) \quad (5.11)\]

together with the Casimirs \((m = 2, \ldots, N)\)
\[C^{(m)} = -\sum_{i < j} (q_i p_j - q_j p_i)^2. \quad (5.12)\]
5.2 Non-standard quantum deformation $U_z(sl(2, \mathbb{R}))$

Now, we introduce a suitable Poisson realization of the non-standard deformation of $sl(2, \mathbb{R})$ [10] as follows:

$$\Delta(J_-) = J_- \otimes 1 + 1 \otimes J_-$$
$$\Delta(J_+) = J_+ \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_+$$
$$\Delta(J_3) = J_3 \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_3$$

(5.13)

$$\{J_3, J_+\} = 2J_+ \cosh zJ_- \quad \{J_3, J_-\} = -2\sinh \frac{zJ_-}{z} \quad \{J_-, J_+\} = 4J_3.$$  \hspace{1cm} (5.14)

The deformed Casimir reads

$$C_z = J_3^2 - \frac{\sinh zJ_-}{z} J_+$$

which vanishes under the following one-particle deformed phase space realization of (5.14):

$$f^{(1)}_-(1) = D(J_-) = q_1^2 \quad f^{(1)}_+(1) = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 \quad f^{(1)}_3(1) = D(J_3) = \frac{\sinh zq_1^2}{zq_1^2} p_1.$$  \hspace{1cm} (5.15)

(5.16)

Let us now consider again the dynamical generator $\mathcal{H} = J_+ + \omega^2 J_-$. Under (5.16), we obtain a new deformed oscillator

$$H_z^{(1)} = D(\mathcal{H}) = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \omega^2 q_1^2.$$  \hspace{1cm} (5.17)

We follow the usual constructive method and derive the corresponding two-particle phase space realization from the coproduct (5.13):

$$f^{(2)}_- = q_1^2 + q_2^2 \quad f^{(2)}_+ = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 e^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2} p_2^2 e^{-zq_1^2} \quad f^{(2)}_3 = \frac{\sinh zq_1^2}{zq_1^2} p_1 e^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2} p_2 e^{-zq_1^2}.$$  \hspace{1cm} (5.18)

The associated two-particle Hamiltonian is

$$H_z^{(2)} = \frac{\sinh zq_1^2}{zq_1^2} p_1 e^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2} p_2 e^{-zq_1^2} + \omega^2 (q_1^2 + q_2^2)$$

(5.19)

and the deformed coproduct for the Casimir leads to

$$C_z^{(2)} = -\frac{\sinh zq_1^2 e^{zq_2^2}}{zq_1^2 q_2^2} (q_1 p_2 - q_2 p_1)^2 e^{-zq_1^2} e^{zq_2^2}.$$  \hspace{1cm} (5.20)

The $N$-dimensional generalization for this system is derived from the phase space realization of arbitrary dimension:

$$f^{(m)}_- = \sum_{i=1}^{m} q_i^2 \quad f^{(m)}_+ = \sum_{i=1}^{m} \frac{\sinh zq_i^2}{zq_i^2} p_i e^{zK^{(m)}_i(q^2)} \quad f^{(m)}_3 = \sum_{i=1}^{m} \frac{\sinh zq_i^2}{zq_i^2} p_i e^{zK^{(m)}_i(q^2)}$$

(5.21)

where the $K$-functions we use in this section are defined in the appendix.

As a consequence, the $N$-dimensional Hamiltonian is just $H_z^{(N)} = f^{(N)}_+ + \omega^2 f^{(N)}_-$ and the following constants of motion are deduced:

$$C^{(m)} = -\sum_{i<j}^{m} \frac{\sinh zq_i^2}{zq_i^2 q_j^2} \frac{\sinh zq_j^2}{zq_j^2} (q_i p_j - q_j p_i)^2 e^{zK^{(m)}_{ij}(q^2)}.$$  \hspace{1cm} (5.22)
5.3 A class of integrable anharmonic chains

Let us go back to the undeformed construction, and consider a more general dynamical Hamiltonian $\mathcal{H}$ of the form
\begin{equation}
\mathcal{H} = J_+ + \mathcal{F}(J_-),
\end{equation}
where $\mathcal{F}(J_-)$ is an arbitrary smooth function of $J_-$. The formalism ensures that the corresponding system constructed from (5.23) is also integrable, since $\mathcal{H}$ could be any function of the coalgebra generators. Explicitly, this means that any $N$-particle Hamiltonian of the form
\begin{equation}
H^{(N)} = f^{(N)}_+ + \mathcal{F}(f^{(N)}_-) = \sum_{i=1}^{N} p_i^2 + \mathcal{F} \left( \sum_{i=1}^{N} q_i^2 \right)
\end{equation}
is completely integrable, being (5.12) its constants of motion. Obviously, the linear case $\mathcal{F}(J_-) = \omega^2 J_-$ leads to the previous harmonic case, and the quadratic one $\mathcal{F}(J_-) = J_-^2$ would give us an interacting chain of quartic oscillators. Further definitions of the function $\mathcal{F}$ would give us many other anharmonic chains, all of them sharing the same dynamical symmetry and the same integrals of the motion.

Moreover, the corresponding integrable deformation of (5.24) is provided by a realization of (5.23) in terms of (5.21):
\begin{equation}
H^{(N)}_z = \sum_{i=1}^{N} \frac{\sinh z q_i^2}{z q_i^2} p_i^2 e^{z K^{(N)}_i(q^2)} + \mathcal{F} \left( \sum_{i=1}^{N} q_i^2 \right)
\end{equation}
and (5.24) are again the associated integrals. This example shows clearly the number of different systems that can be obtained through the same coalgebra, and the need for a careful inspection of known integrable systems in order to investigate their possible coalgebra symmetries.

5.4 Angular momentum chains and $sl(2, \mathbb{R})$ coalgebras

Finally, we would like to establish a connection between the previous $sl(2, \mathbb{R})$ oscillator chains and “classical spin” systems. Such a link is provided by the underlying coalgebra structure. If we substitute the canonical realizations used until now in terms of angular momentum realizations of the same abstract $sl(2, \mathbb{R})$ Poisson coalgebra, the very same construction will lead us to a long-range interacting “spin chain” of the Gaudin type on which the quantum deformation can be easily implemented.

In particular, let us consider the realization $S$
\begin{equation}
g^{(1)}_3 = S(J_3) = \sigma_3 \quad g^{(1)}_+ = S(J_+) = \sigma_+ \quad g^{(1)}_- = S(J_-) = \sigma_-
\end{equation}
where the classical angular momentum variables $\sigma_i$ fulfill
\begin{align}
\{\sigma_3, \sigma_+\} &= 2\sigma_+ \quad \{\sigma_3, \sigma_-\} = -2\sigma_- \quad \{\sigma_-, \sigma_+\} = 4\sigma_3
\end{align}
and are constrained by a given constant value of the Casimir function (5.3) in the form $c_1 = (\sigma_3^2)^2 - 2\sigma_+ \sigma_-$.

As usual, $m$ different copies of (5.26) (that, in principle, could have different values $c_i$ of the Casimir) can be distinguished with the aid of a superscript $\sigma_i$. Then, the $m$-th
coproduct \((3.11)\) provides the following realization of the non-deformed \(sl(2, \mathbb{R})\) Poisson coalgebra:

\[
g_l^{(m)} = (S \otimes \ldots \otimes S)(\Delta^{(m)}(\sigma_l)) = \sum_{i=1}^{m} \sigma_i^l \quad l = +, -, 3. \tag{5.28}
\]

Now, we can apply the usual construction and take \(\mathcal{H}\) from \((5.5)\). As a consequence, the uncoupled oscillator chain \((5.11)\) is equivalent to

\[
H^{(N)} = g_+^{(N)} + \omega^2 g_-^{(N)} = \sum_{i=1}^{m} \sigma_i^+ + \omega^2 \sigma_i^- 
\]

and the Casimirs \(C^{(m)}\) read \((m = 2, \ldots, N)\):

\[
C^{(m)} = (g_3)^2 - g_-^{(m)} g_+^{(m)} = \sum_{i=1}^{m} c_i + \sum_{i<j}^{m} (\sigma_i^j \sigma_j^i - \sigma_i^+ \sigma_j^- - \sigma_i^- \sigma_j^+).
\]

Note that these are just Gaudin Hamiltonians of the hyperbolic type \([1, 13]\).

A non-standard deformation of Gaudin system can be now obtained. The deformed angular momentum realization corresponding to \(U_c(sl(2, \mathbb{R}))\) is:

\[
g_3^{(1)} = S(J_3) = \frac{\sinh \sigma_3^+}{\sigma_3^-} \quad g_+^{(1)} = S(J_+) = \frac{\sinh \sigma_1^+}{\sigma_1^-} \quad g_-^{(1)} = S(J_-) = \sigma_1^- \tag{5.31}
\]

where the classical coordinates \(\sigma_i^l\) are defined on the cone \(c_1 = (\sigma_3^1)^2 - \sigma_1^+ \sigma_1^- = 0\), that is, we are considering the zero realization.

It is easy to check that the \(m\)-th order of the coproduct \((5.13)\) in the above representation leads to the following functions

\[
g_-^{(m)} = \sum_{i=1}^{m} \sigma_i^- \quad g_+^{(m)} = \sum_{i=1}^{m} \frac{\sinh \sigma_i^+}{\sigma_i^-} \sigma_i^+ e^{z K_3^{(m)}(\sigma_-)} \\
g_3^{(m)} = \sum_{i=1}^{m} \frac{\sinh \sigma_i^-}{\sigma_i^-} \sigma_i^- e^{z K_3^{(m)}(\sigma_-)} \tag{5.32}
\]

that define the non-standard deformation of \((5.28)\). Therefore, any function of these three objects (with \(m = N\)) can be taken as the deformed Hamiltonian, that will Poisson-commute with the following “non-standard Gaudin Hamiltonians”

\[
C_z^{(m)} = (g_3^{(m)})^2 - \frac{\sinh \sigma_-^{g_3^{(m)}} g_+^{(m)}}{g_-^{(m)}} \\
= \sum_{i=1}^{m} \left( \frac{\sinh \sigma_i^+}{\sigma_i^-} \right)^2 e^{2z K_3^{(m)}(\sigma_-)} \left\{ (\sigma_3^1)^2 - \sigma_i^+ \sigma_i^- \right\} \\
+ \sum_{i<j}^{m} \frac{\sinh \sigma_i^- \sinh \sigma_j^-}{z^2 \sigma_i^- \sigma_j^-} e^{z K_3^{(m)}(\sigma_-)} (\sigma_3^1 \sigma_3^j - \sigma_i^- \sigma_j^- - \sigma_i^+ \sigma_j^+). \tag{5.33}
\]

Since we are working in the zero representation with \((\sigma_3^1)^2 - \sigma_i^+ \sigma_i^- = 0\), \((5.33)\) can be simplified

\[
C_z^{(m)} = \sum_{i<j}^{m} \frac{\sinh \sigma_i^- \sinh \sigma_j^-}{z^2 \sigma_i^- \sigma_j^-} e^{z K_3^{(m)}(\sigma_-)} (\sigma_3^1 \sigma_3^j - \sigma_i^- \sigma_j^- - \sigma_i^+ \sigma_j^+). \tag{5.34}
\]
These integrals are the angular momentum counterparts to (5.22). Note that, as it was found for the standard case in [3], the deformation can be interpreted as the introduction of a variable range exchange in the model (compare (5.34) with (5.30)).

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**Appendix**

The $Q$-functions are defined as:

$$Q^{(m)}_{-i,i}(x) = \sum_{l=1}^{i-1} \omega_l x_l \quad Q^{(m)}_{+i,i}(x) = \sum_{l=i+1}^{m} \omega_l x_l \quad \text{(A.1)}$$

$$Q^{(m)}_i(x) = -Q^{(m)}_{-i,i}(x) + Q^{(m)}_{+i,i}(x) = -\sum_{k=1}^{i-1} \omega_k x_k + \sum_{l=i+1}^{m} \omega_l x_l \quad \text{(A.2)}$$

$$Q^{(m)}_{ij}(x) = Q^{(m)}_i(x) + Q^{(m)}_j(x) = -\omega_i x_i - \omega_j x_j - \sum_{k=1}^{i-1} \omega_k x_k + 2 \sum_{l=j+1}^{m} \omega_l x_l \quad i < j. \quad \text{(A.3)}$$

In the same way, the $K$-functions will be:

$$K^{(m)}_i(x) = -\sum_{k=1}^{i-1} x_k + \sum_{l=i+1}^{m} x_l \quad \text{(A.4)}$$

$$K^{(m)}_{ij}(x) = K^{(m)}_i(x) + K^{(m)}_j(x) = -(x_i - x_j) - \sum_{k=1}^{i-1} x_k + 2 \sum_{l=j+1}^{m} x_l \quad i < j. \quad \text{(A.5)}$$

Note that the following property is useful for computations involving these functions

$$\sinh(z \sum_{i=1}^{m} x_i) = \sum_{i=1}^{m} \sinh(z x_i) e^{z K^{(m)}_i(x)}. \quad \text{(A.6)}$$

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