Constraining the dark energy models using baryon acoustic oscillations: An approach independent of $H_0 \cdot r_d$

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ABSTRACT

The $H_0$ tension and the accompanying $r_d$ tension are a hot topic in current cosmology. In order to remove the degeneracy between the Hubble parameter $H_0$ and the sound horizon scale $r_d$ from the baryon acoustic oscillation (BAO) datasets, we redefined the likelihood by marginalizing over the $H_0 \cdot r_d$ parameter and then we performed a full Bayesian analysis for different models of dark energy (DE). We find that our datasets that are uncalibrated by early or late physics cannot constrain the DE models properly without further assumptions. By adding the type Ia supernova (SNIa) dataset, the models are constrained better with smaller errors on the DE parameters. The two BAO datasets we used – one with angular measurements and one with angular and radial ones, with their covariance – show statistical preferences for different models, with the ~cold dark matter ($\Lambda$CDM) model being the best model for one of them. Adding the Pantheon SNIa dataset with its covariance matrix boosts the statistical preference for the $\Lambda$CDM model.

Key words. cosmological parameters – methods: data analysis – astronomical databases: miscellaneous

1. Introduction

A turning point in modern cosmology is the measurement of the Hubble constant $H_0$, revealing the current accelerated expansion of the Universe (Riess et al. 1998; Freedman & Madore 2010). The estimation of $H_0$ from the late Universe can be obtained from direct measurements such as distance ladders, strong lensing, and gravitational wave standard sirens (Freedman et al. 2001; Perlmutter et al. 1999; Riess et al. 2016, 2021). The latest SH0ES measurement based on the supernovae calibrated by Cepheids is $H_0 = 73.04 \pm 1.04\,\text{km}\,\text{s}^{-1}\text{Mpc}^{-1}$ at a confidence level of 68% (Riess et al. 2022b). Further improvement comes from a SH0ES measurement of the distance ladder calibrated by parallaxes of Cepheids in open clusters, which combined with all anchors yields $H_0 = 73.01 \pm 0.99\,\text{km}\,\text{s}^{-1}\text{Mpc}^{-1}$ (Riess et al. 2022a).

Another type of measurement is provided by the Planck collaboration, which uses temperature and polarization anisotropies in the cosmic microwave background (CMB) to obtain $H_0 = 67.27^{+0.66}_{-0.56}\,\text{km}\,\text{s}^{-1}\text{Mpc}^{-1}$. The discrepancy between local model-independent measurements of $H_0$ and the early Universe CMB values can reach 5.3σ and it is one of the fundamental problems in cosmology (Schöneberg et al. 2019; Di Valentin 2017; Di Valentin et al. 2021a,b; Perivalopoulou & Skara 2022; Lucca 2021; Verde et al. 2019; Knox & Millea 2020; Jedamzik et al. 2021; Shah et al. 2021; Abdalla et al. 2022).

Baryon acoustic oscillations (BAOs) are sound waves in the baryon-photon plasma comprising the visible matter in the post-inflationary Universe, which froze at a recombination epoch. Today, they are observed in the clustering of large-scale structures by numerous galactic surveys (such as SDSS, DES, WiggleZ, BOSS). Due to rather simple physics of the plasma waves, BAOs can be considered as a standard ruler evolving with the Universe, thus providing another window into studying cosmological models (Dunkley et al. 2011; Addison et al. 2013; Aubourg & Bailey 2015; Cuesta et al. 2015; Ade et al. 2014a,b, 2016; Story et al. 2015; Alam et al. 2017; Troxel et al. 2018; Planck Collaboration VI 2020; Cuceu et al. 2019; Dainotti et al. 2021). A scale important for BAO measurements is set by the sound horizon at drag epoch. As it is known, at recombination epoch, the photons decouple from the baryons first, at $z_s \approx 1090$, which gives rise to the CMB. The baryons stop feeling the drag of photons at the drag epoch, $z_d \approx 1059$, which sets the standard ruler for the BAOs. The Planck Collaboration value of the sound horizon is $r_d^{\text{Pl18}} = 147.09 \pm 0.26\text{ Mpc}$ Planck Collaboration VI (2020), and the late-time estimation for it is $r_d^{\text{H0LiCOW+SN+BAO+SH0ES}} = 136.1 \pm 2.7\text{ Mpc}$ Arendse et al. (2020). Other estimations give numbers in this range, depending on the datasets in use, for example (e.g., Verde et al. 2017; Planck Collaboration VI 2020; Alam et al. 2021; Nunes & Bernui 2020; Nunes et al. 2020).

Many papers discuss the relation between $H_0$ and the sound horizon scale $r_d$ for different models (Aylor et al. 2019; Knox & Millea 2020; Pogosian et al. 2020; Aizpuru et al. 2021). Some claim that resolving the $H_0$ tension is not enough, since one has to also take into account the model’s effect on the sound horizon. This means that one should rule out models that resolve the $H_0$ tension without resolving the $r_d$ tension simultaneously (Jedamzik et al. 2021; Aizpuru et al. 2021; de la Macorra et al. 2022). Since $H_0$ and $r_d$ are strongly connected, it seems hard to disentangle them without making any assumptions. In order to have an independent crosscheck on dark energy (DE) models’ constraints, we removed the dependence on $H_0 \cdot r_d$ by marginalizing over it using a $\chi^2$ redefinition. Such an approach has already
been used to different extents in the literature. In Lazkoz et al. (2005), it was performed on the type Ia supernovae (SNIa) Gold dataset to compare different parametrizations of \( \Lambda(z) \). Basilakos & Nesseris (2016) studied the growth index by comparing the \( \Lambda \) cold dark matter (ACDM) model to several DE models by marginalizing over \( M_B \) and \( \sigma_8 \). Anagnostopoulos & Basilakos (2018) studied different cosmological models by marginalizing over \( H_0 \) and find that one cannot rule out non-flat models or dynamical DE. They observe that the time-varying equation of state parameter \( w(z) \) cannot be constrained by the current expansion data. Finally, Camarena & Marra (2021) used marginalization over \( H_0 \) and \( M_B \) in different datasets to show that a hockey-stick DE cannot solve the Hubble tension.

One possible way to resolve the tension is by changing the DE model. The question whether the DE is a constant energy density or with a dynamical parametrization, both in different works (Benisty et al. 2021; Capozziello & De Laurentis 2011; Bull et al. 2016; Di Valentin et al. 2021a; Yang et al. 2021). This motivates a host of DE parametrizations (Wang et al. 2018; Reyes & Escamilla-Rivera 2021; Colgán et al. 2021; Liu et al. 2022) to be used in the search for deviations from the cosmological constant, \( \Lambda \), in observational data. A justification for this can be found in numerous papers claiming that DE models may resolve the Hubble tension, particularly for the early DE models (Gogoi et al. 2021; Poulin et al. 2019; Sakstein & Trodden 2020; Tian & Zhu 2021; Nojiri et al. 2021; Seto & Toda 2021; Hill et al. 2022).

In this work, we used two types of BAO datasets and we combined them with the Pantheon SNIa dataset. Then we marginalized over \( H_0 \), \( r_d \) and \( H_0 \), and \( M_B \), respectively. This allowed us to remove the need to take priors on these quantities, and thus it removed some of the implied assumptions on the models. Using this method, we studied ACDM, wCDM, the Chevallier–Polarski–Linder (CPL) parametrization of wwaCDM, and also two emergent DE models: pEDE and gEDE. We show that even with this more extensive marginalization, one can see differences in the predictions of the different models inferred from the different datasets. The latter is particularly interesting in view of the growing sensitivity toward the implied assumptions in processing the data. We then performed a statistical analysis on the obtained results using four well-established measures. We confirm that constraining \( w(z) \) seems impossible from this method, while the errors on \( w_0 \) improve significantly when we add SNIa. Surprisingly, the different BAO datasets show different preferences for the flatness of the universe.

The plan of the work is as follows: Section 2 formulates the relevant theory. Section 3 describes the method. Section 4 shows the results with a model comparison. Finally, Sect. 5 summarizes the results.

2. Theory

A Friedmann–Lemaître–Robertson–Walker metric with the scale parameter \( a = 1/(1 + z) \) is considered, where \( z \) is the redshift. The evolution of the Universe for it is governed by the Friedmann equation, which connects the equation of state for the ACDM background:

\[
E(z)^2 = \Omega_m(1 + z)^3 + \Omega_k(1 + z)^2 + \Omega_{\Lambda}(z),
\]

with the expansion of the Universe \( E(z)^2 = H(z)/H_0 \), where \( H(z) \equiv \dot{a}/a \) is the Hubble parameter at redshift \( z \) and \( H_0 \) is the Hubble parameter today. \( \Omega_m \), \( \Omega_{\Lambda} \), and \( \Omega_k \) are the fractional densities of matter, DE, and the spatial curvature at redshift \( z = 0 \).

We ignored radiation, since we are looking at the late Universe. The spatial curvature is expected to be zero for a flat Universe, \( \Omega_k = 0 \). We can expand this simple model by considering a DE component depending on \( z \). This can be done with a generalization of the CPL parametrization (Chevallier & Polarski 2001; Linder 2003; Linder & Huterer 2005; Barger et al. 2006) of the wwaCDM model:

\[
\Omega_{\Lambda}(z) = \Omega_{\Lambda}^{(0)} \exp \left[ \int_0^z \frac{3(1 + w(z'))dz'}{1 + z'} \right]
\]

in which we considered three possible models:

\[
w(z) = \begin{cases} 
    w_0 + w_0'z & \text{Linear} \\
    w_0 + w_0'\frac{z}{1+z} & \text{CPL} \\
    w_0 - w_0' \log(z + 1) & \text{Log}
\end{cases}
\]

which recover the ACDM for \( w_0 = -1, w_0' = 0 \).

To this parametrization, we added another model, namely the phenomenologically Emergent Dark Energy (pEDE) model (Li & Shafieloo 2019, 2020) and its generalization (gEDE). gEDE is described by:

\[
\Omega_{\text{DE}}(z) = \Omega_{\Lambda} - 1 - \text{tanh} \left( \frac{\Delta z - 2}{\Delta z_0} \right) \frac{1 + z}{1 + \text{tanh} \left( \frac{\Delta z_0}{1 + z_0} \right)}
\]

with pEDE-CDM recovered for \( \Lambda = 1 \), and ACDM for \( \Lambda = 0 \). The parameter \( z_0 \) here is the transitional redshift, where \( \Omega_{\text{DE}}(z) = \Omega_m(1 + z)^3 \). It should be noted that \( z_0 \) is obtained as a solution of this equation, and thus it is not a free parameter, but a calculated one. The analytical form of \( w(z) \) could then be obtained from the integral Eq. (2), see Li & Shafieloo (2020).

The BAO measurements provide different directions. The radial projection \( D_R(z) \) gives:

\[
\frac{D_R}{r_d} = \frac{c}{H_0 r_d E(z)},
\]

which includes the parameter \( H_0 r_d \). The tangential BAO measurements are given in terms of the angular diameter distance \( D_A \):

\[
D_A = \frac{c}{H_0 (1 + z) \sqrt{|\Omega_k|}} \sinh \left[ \frac{|\Omega_k|^{1/2} \Gamma(z)}{1 + z} \right],
\]

where \( \sinh(x) \equiv \sin(x), x, \sinh(x) \) for \( \Omega_k < 0, \Omega_k = 0, \Omega_k > 0 \) respectively. The \( \Gamma \) function is defined as:

\[
\Gamma(z) = \int \frac{dz'}{E(z')}
\]

where \( E(z) \) is related to the equation of state of the Universe as defined above. Thus, the measurement \( D_A/r_d \) can expressed as:

\[
\frac{D_A}{r_d} = \frac{c}{H_0 r_d} f(z),
\]

where:

\[
f(z) = \frac{1}{(1 + z) \sqrt{|\Omega_k|}} \sinh \left[ \frac{|\Omega_k|^{1/2} \Gamma(z)}{1 + z} \right].
\]

A related quantity used in the radial BAO measurements is the comoving angular diameter distance \( D_M = D_A(1 + z) \).

Furthermore, we used the dataset featuring the BAO angular scale measurement \( \theta_{\text{BAO}}(z) \). It gives the angular diameter distance \( D_A \) at the redshift \( z \):

\[
\frac{r_d}{(1 + z) D_A(z)} = \frac{H_0 r_d}{c} \theta_{\text{BAO}}(z).
\]
with:

$$h(z) = \frac{1}{(1 + z) f(z)}.$$  \hfill (10)

We see that both $I_d / r_d$ and $\theta_{BAO}$ and $D_I / r_d$ depend on the quantity $H_0 \cdot r_d$, which can be eliminated from the corresponding $\chi^2$, as we demonstrate in the next section.

Finally, we added the SNIa measurements, described by the luminosity distance $\mu(z)$. It is related to the Hubble parameter through the angular diameter distance as $D_A = d_A(z)/(1 + z)$. For the SNIa standard candles, the distance modulus $\mu(z)$ is related to the luminosity distance through

$$\mu_B(z) - M_B = 5 \log_{10} (d_A(z)) + 25,$$  \hfill (11)

where $d_A$ is measured in units of Mpc, and $M_B$ is the absolute magnitude. There is a degeneracy between $H_0$ and $M_B$, in such a way that total absolute magnitude reads: $M_B + 25 + 5 \log_{10} (d_A(z)/H_0)$. This degeneracy, can also be used to remove the dependence on $H_0$ and $M_B$ in the $\chi^2$.

3. Method

In order to infer the parameters of a certain model from the observations, one needs to define the appropriate $\chi^2$. The goal of our analysis is to redefine the corresponding $\chi^2$ in all datasets, in a way that eliminates the dependence on degenerate parameters, such as $H_0 \cdot r_d$ (or $H_0$ and $M_B$ for SNIa), but maintains the dependence on the equation of state that enters into $\Gamma(z)$.

3.1. BAO redefinition

A DE model includes $n$-free parameters (i.e., $\Omega_m, \Omega_k, w_0, w_a, \ldots$), constrained by minimizing the $\chi^2$:

$$\chi^2 = \sum_i \left[ \theta_{\text{obs}} - \theta_{\text{model}} \right]^T C_{i,j}^{-1} \left[ \theta_{\text{obs}} - \theta_{\text{model}} \right],$$  \hfill (12)

where $\theta_{\text{obs}}$ is a vector of the observed points at each $z$ (i.e., $D_A / r_d, D_I / r_d, D_H / r_d$ or $\theta_{BAO}$) and $\theta_{\text{model}}$ is the theoretical prediction of the model. It is possible to rewrite the vector as the dimensionless function multiplied by the $\theta_{\text{model}}$ parameter:

$$\theta_{\text{model}} = \frac{c}{H_0 r_d} \left( f(z), E(z)^{-1} \right) = \frac{c}{H_0 r_d} \theta_{\text{model}}^\prime.$$  \hfill (13)

$C_{i,j}$ is the covariance matrix. For uncorrelated points the covariance matrix is a diagonal matrix, and its elements are the inverse errors $\sigma_i^{-2}$. The statistics of the BAO are not fully Gaussian but we consider this as an approximation. Following the approach in Lazkoz et al. (2005), Basilakos & Nessersis (2016), Anagnostopoulos & Basilakos (2018), Camarena & Marra (2021), one can isolate $H_0 r_d$ in the $\chi^2$ by writing it as:

$$\chi^2 = \left( \frac{c}{H_0 r_d} \right)^2 A - B \left( \frac{c}{H_0 r_d} \right) + C,$$  \hfill (14)

where:

$$A = f'(z_i) C_{i,j} f'(z_j),$$  \hfill (15a)

$$B = \frac{f'(z_i) C_{i,j} v_{\text{model}}^i(z_j) + v_{\text{model}}^j(z_i) C_{j,i} f'(z_i)}{2},$$  \hfill (15b)

$$C = v_{\text{model}}^i C_{i,j} v_{\text{model}}^j.$$  \hfill (15c)

Using Bayes’s theorem and marginalizing over $c / (H_0 r_d)$, we arrive at:

$$p(D, M) = \frac{1}{p(D|M)} \int \exp \left[ -\frac{1}{2} \chi^2 \right] d \frac{c}{H_0 r_d},$$

where $D$ is the data we used, and the $M$ is the model. Consequently, using $\tilde{\chi}^2_{BAO} = -2 \ln p(D, M)$, we get the marginalized $\chi^2$:

$$\tilde{\chi}^2 = C - \frac{B^2}{A} + \log \left( \frac{A}{2\pi} \right).$$

This last equation is the final form of $\chi^2$. Due to the marginalization procedure, this $\chi^2$ depends only on $f(z)$ and $h(z)$, which do not include $H_0$ and $r_d$ inside.

3.2. $\theta_{BAO}$ data

We used the same approach for the $\theta_{BAO}$ ($z$) measurements:

$$\tilde{\chi}^2_{\theta_{BAO}} = \left( \frac{H_0 r_d}{c} \right)^2 A_B - 2 B_\theta \left( \frac{H_0 r_d}{c} \right) + C,$$

where:

$$A_B = \sum_i \left[ \theta_i(z) - \theta_D^\prime \right]^2 / \sigma_i^2,$$  \hfill (20a)

$$B_\theta = \sum_i \frac{\theta_D^\prime(z_i)}{\sigma_i^2},$$  \hfill (20b)

$$C_\theta = \sum_i \left[ \theta_D^\prime \right]^2 / \sigma_i^2.$$  \hfill (20c)

Using Bayes’s theorem and marginalizing over $H_0 r_d / c$, we arrived at the marginalized $\chi^2$, which is the same as in Eq. (17), only with $A$, $B$, and $C$ now as functions of $0$. This $\tilde{\chi}^2_{\theta_{BAO}}$ also depends only on $h(z)$, without any dependence on $H_0 \cdot r_d / c$.

3.3. Supernova redefinition

Following the approach used in Di Pietro & Claeskens (2003), Nessersis & Perivolaropoulos (2004), Perivolaropoulos (2005), Lazkoz et al. (2005), we assumed no prior constraint on $M_B$, which is just some constant, and we integrated the probabilities over $M_B$. The integrated $\chi^2$ yields:

$$\tilde{\chi}^2_{SN} = D - \frac{E^2}{2 F} + \ln \frac{F}{2\pi},$$

where:

$$D = \sum_i \left( \frac{\mu_i - 5 \log_{10} |d_i(z_i)|}{\sigma_i} \right)^2,$$  \hfill (22a)

$$E = \sum_i \frac{\mu_i - 5 \log_{10} |d_i(z_i)|}{\sigma_i^2},$$  \hfill (22b)

$$F = \sum_i \frac{1}{\sigma_i^2}.$$

\hfill (22c)
Here $\mu$ is the observed luminosity, $\sigma_i$ is its error, and the $d_i(z)$ is the luminosity distance. The values of $M$ and $H_0$ don’t change the marginalized $\chi^2_{\Delta \mu}$. In order to use the covariance matrix provided for the Pantheon dataset, one needs to transform $D, E, F$ as follows:

\[
D = \sum \left( \Delta \mu C^{-1}_{\Delta \mu} \Delta \mu \right)^2,
\]

\[
E = \sum \left( \Delta \mu C^{-1}_{\Delta \mu} E \right),
\]

\[
F = \sum C^{-1}_{\Delta \mu},
\]

where $\Delta \mu = \mu - 5 \log_{10} d_L(z)$, $E$ is the unit matrix, and $C^{-1}_{\Delta \mu}$ is the inverse covariance matrix of the dataset. The total covariance matrix is given by $C_{\text{cov}} = D_{\text{stat}} + C_{\text{sys}}$, where $D_{\text{stat}} = \sigma^2_{\Delta \mu}$ comes from the measurement and $C_{\text{sys}}$ is provided separately (Deng & Wei 2018). We notice that the forms of $\chi^2_{\text{BAO}}$ and $\chi^2_{\text{SN}}$ are a bit different, since for the $\chi^2_{\text{BAO}}$ we removed the dependence of $\sigma/H_0 r_d$, which multiply the $f(z)$ and, in the case of $\chi^2_{\text{SN}}$, the parameter, $M$ is added to the value of $\mu$.

In our analysis, we also consider the combined likelihood:

\[
\chi^2 = \chi^2_{\text{BAO}} + \chi^2_{\text{SN}}.
\]

Here $\chi^2_{\text{BAO}}$ stands for the BAO or for the BAO$_6$ datasets independently. The distinction between the hyper-parameters quantifying uncertainties in a dataset and the free parameters of the cosmological model is purely conceptual. It is important to note that the so-defined $\chi^2$ is not normalized, and thus its absolute value is not a useful measure of the quality of a given fit. Moreover, it is biased toward a larger number of parameters and is not very good for small datasets, such as the ones we used (Lazkoz et al. 2005). For this reason, we used it only to calculate the more balanced statistical measures, as is discussed in the following section.

### Table 1. Compilation of BAO measurements from diverse releases of surveys such as SDSS, WiggleZ, and DES.

| $z$  | $D_{\text{A}}/r_d$ | $\sigma_{\text{Data}}$ | Year     | Ref.                       |
|------|--------------------|-------------------------|----------|----------------------------|
| 0.11 | 2.607              | 0.138                   | 2021     | de Carvalho et al. (2021)  |
| 0.24 | 5.594              | 0.305                   | 2016     | Chuang et al. (2017)       |
| 0.32 | 6.636              | 0.11                    | 2016     | Alam et al. (2017)         |
| 0.38 | 7.389              | 0.122                   | 2019     | Beutler et al. (2017)      |
| 0.44 | 8.19               | 0.77                    | 2012     | Blake et al. (2012)        |
| 0.51 | 7.893              | 0.279                   | 2015     | Carvalho et al. (2016)     |
| 0.54 | 9.212              | 0.41                    | 2012     | Seo et al. (2012)          |
| 0.6  | 9.37               | 0.65                    | 2012     | Blake et al. (2012)        |
| 0.697| 10.18              | 0.52                    | 2020     | Sridhar et al. (2020), Gil-Marín et al. (2020) |
| 0.73 | 10.42              | 0.73                    | 2012     | Blake et al. (2012)        |
| 0.81 | 10.75              | 0.43                    | 2017     | Abbott et al. (2019)       |
| 0.85 | 10.76              | 0.54                    | 2020     | Tamone et al. (2020)       |
| 0.874| 11.41              | 0.74                    | 2020     | Sridhar et al. (2020)      |
| 1.00 | 11.521             | 1.032                   | 2019     | Zhu et al. (2018)          |
| 1.480| 12.18              | 0.32                    | 2020     | Hou et al. (2020), Gil-Marín et al. (2020), Bautista et al. (2020) |
| 2.00 | 12.011             | 0.562                   | 2019     | Zhu et al. (2018)          |
| 2.35 | 10.83              | 0.54                    | 2019     | Blomqvist et al. (2019), du Mas des Bourboux et al. (2020) |
| 2.4  | 10.5               | 0.34                    | 2017     | du Mas des Bourboux et al. (2017) |

Notes. The values marked with (*) are calculated through their covariance matrices relating $D_{\text{M}}$ and $D_{\text{H}}$.

### Table 2. Compilation of angular BAO measurements from luminous red and blue galaxies, and quasars from diverse releases of the SDSS.

| $z$  | $\theta$ | $\sigma_\theta$ | Ref.                     |
|------|----------|-----------------|--------------------------|
| 0.11 | 19.80    | 3.26            | de Carvalho et al. (2020) |
| 0.235| 9.06     | 0.23            | Alcaniz et al. (2017)    |
| 0.365| 6.33     | 0.22            | Alcaniz et al. (2017)    |
| 0.450| 4.77     | 0.17            | Carvalho et al. (2016)   |
| 0.470| 5.02     | 0.25            | Carvalho et al. (2016)   |
| 0.490| 4.99     | 0.21            | Carvalho et al. (2016)   |
| 0.510| 4.81     | 0.17            | Carvalho et al. (2016)   |
| 0.530| 4.29     | 0.30            | Carvalho et al. (2016)   |
| 0.550| 4.25     | 0.25            | Carvalho et al. (2016)   |
| 0.570| 4.59     | 0.36            | Carvalho et al. (2020)   |
| 0.590| 4.39     | 0.330           | Carvalho et al. (2020)   |
| 0.610| 3.85     | 0.31            | Carvalho et al. (2020)   |
| 0.630| 3.90     | 0.43            | Carvalho et al. (2020)   |
| 0.650| 3.55     | 0.16            | Carvalho et al. (2020)   |
| 2.225| 1.77     | 0.31            | de Carvalho et al. (2018) |

Notes. This BAO$_6$ dataset was taken from Nunes et al. (2020).

### 3.4. Datasets and priors

In this work, we consider two different BAO datasets, to which we added the binned Pantheon supernovae dataset with its covariance matrix. The BAO datasets can be found summarized in Tables 1 and 2.

The first BAO dataset, shown on Table 1 and denoted as BAO, contains a combination of various angular measurements, to which we added points from the most recent to date eBOSS data release (DR16), which come as angular ($D_{\text{M}}$) and radial ($D_{\text{H}}$) measurements and their covariance. The points and the covariance matrices can be found in Cao & Ratra (2022). This choice of points allowed us to integrate the quantity $H_0 \cdot r_d$ by summing the corresponding $\chi^2$ of the two types of measurements. While the covariance for some points is known and we
include it in such cases, for the rest, we have to additionally test for possible correlations. To do so, we used the approach from Kazantzidis & Perivolaropoulos (2018), which we also used in Benisty & Staicova (2021). It consists of adding random correlation terms in the covariance matrix and testing the effect on the final result. Explicitly, we used

$$\sigma_{ij} \rightarrow \sigma_{ij} + \sigma_i \sigma_j / 2,$$

where $\sigma_i$ is the 1σ error of the points. Applying the procedure shows that the points can be considered “effectively uncorrelated,” which allowed us to use them to infer the cosmological parameters. Even if there are small correlations, the procedure shows that the small correlations do not affect the final result considerably.

The second dataset shown in Table 2, denoted as BAO_{\text{trans}}, consists of 15 points, coming from transversal BAO measurements (Nunes et al. 2020). Importantly, the transversal BAO analysis does not need to assume a fiducial cosmology, particularly on the $\Omega_X$ parameter, which is included in the standard BAO analysis (Nunes et al. 2020). These points are claimed to be uncorrelated, but using this cosmology-independent methodology means that their errors are larger than the errors obtained using the standard fiducial cosmology approach. One should note that using a fiducial cosmology is accounted for by the Alcock-Paczynski distortion Lepori et al. (2017), so it does not compromise the integrity of the first dataset. However, we would like to investigate the overall effect of intrinsic assumptions in the final results and check if the two datasets are equivalent in this respect.

Finally, we added the Pantheon dataset, which contains 1048 supernovae luminosity measurements in the redshift range $z \in (0.01, 2.3)$ Scdlnic et al. (2018) binned into 40 points. To the statistical error, we also added the systematic errors as provided by the binned covariance matrix\(^1\).

We performed the $H_0 \cdot r_E$-integration procedure, outlined in previous sections, first on the two different BAO datasets alone, and then on the combination of the appropriate BAO dataset plus the Pantheon dataset. The priors we used were: $\Omega_m \in (0.2, 0.4)$, $w_0 \in (-2, -0)$, $w_a \in (-2, 1)$, and $\Omega_K \in (-0.3, 0.3)$. We set $\Omega_\Lambda = 1 - \Omega_m - \Omega_K$. For gEDE we used the redefinition $\Delta = -\Lambda, w_a = z$, so that it could be plotted on the same plots as the other models. As mentioned before, $z_0$ is not a free parameter, and thus it is not a parameter in the Markov chain Monte Carlo (MCMC), and it is found by solving the appropriate transcendental equation using the package sympy. Regarding the problem of likelihood maximization, we used an affine-invariant MCMC nested sampler, as it is implemented within the open-source package Polychord (Handley et al. 2015) with the GetDist package (Lewis 2019) to present the results. In Polychord, convergence is defined as when the posterior mass contained in the live points is $p = 10^{-2}$ of the total calculated evidence. We checked that our chains were stable with respect to changes in the parameter $p$, and furthermore by checking the Geweke score and the Gelman-Rubin diagnostic with the package pymcmcstat.

4. Results

4.1. Posterior distributions

Figures 1–5, A.1 and A.2 in the appendix show the final values obtained by running the MCMC on the selected priors for the two different datasets, with the numerical values in the Tables II–VI. Since we integrated $H_0$ and $r_E$, the only physically measured parameter that remained was $\Omega_m$. We see that in all the cases, $\Omega_m$ is rather well constrained, even from the BAO-only datasets. The BAO_{\text{trans}} as expected, gives larger errors that the inclusion of supernova data improves. The closest to the Planck measurement of $\Omega_m = 0.315 \pm 0.007$ (Planck Collaboration VI 2020) is the Log model for BAO_{\text{trans}} and the LCDM model for BAO, and $\Lambda$CDM/OkCDM for BAO + SN and BAO_{\text{trans}} + SN, with the Log model being very close for the latter.

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\(^1\) https://github.com/dscolnic/Pantheon/
When we consider the other parameters, we see that the BAO-only datasets are not able to limit them properly. While the BAO0 dataset values contain $w_0 = -1$ within 1σ, the values for BAO infer $w_0 > -1$. Adding the SNIa dataset (which we mark on the plots and tables as “SN”) improves the constraints significantly. With respect to the parameter $w$, the inferred values have very big errors. When it comes to $w_0$, BAO points to $w_0 < 0$ (a closed universe), while BAO + SN datasets seem to give values closer to a flat universe.

The conclusion from our results is that the BAO-alone dataset values contain $\Omega_m$ within 1σ. Adding the SNIa dataset (which we mark as “SN”) improves the constraints significantly. With respect to the parameter $\Omega_m$, the inferred values have very big errors. When it comes to $\Omega_m$, BAO points to $\Omega_m < 0$ (a closed universe), while BAO + SN datasets seem to give values closer to a flat universe, while BAO points to $\Omega_m < 0$ (a closed universe).

From the Gaussians we see that some DE models have multipler peaks, hinting at some degeneracy. The results do not seem to change with increasing numbers of live points, hinting that this is a property of the models themselves or of the selected datasets.

### 4.2. Model selection

To compare the different models, we used different well-known statistical measures. We used the Akaike information criterion (AIC), the Bayesian information criterion (BIC), the deviance information criterion (DIC), and the Bayes factor (BF; Liddle 2007).

The AIC criterion is defined as

$$\text{AIC} = -2 \ln(L_{max}) + 2k + \frac{2k(k+1)}{N_{tot} - k - 1},$$

where $L_{max}$ is the maximum likelihood of the data under consideration, $N_{tot}$ is the total number of data points, and $k$ is the number of parameters. For large $N_{tot}$, this expression reduces to $\text{AIC} \approx -2 \ln(L_{max}) + 2k$, which is the standard form of the AIC criterion (Liddle 2007).

The BIC criterion is an estimator of the Bayesian evidence, (e.g., Liddle 2007), and is given as

$$\text{BIC} = -2 \ln(L_{max}) + k \log(N_{tot}).$$

The AIC and BIC criteria employ only the likelihood value at maximum. Since we evaluate this $L_{max}$ numerically, from the Bayesian analysis, one needs to use sufficiently long chains to ensure the accuracy of $L_{max}$ when evaluating AIC and BIC. The DIC (Liddle 2007) provides all the information obtained from the likelihood calls during the maximization procedure. The DIC estimator is defined as

$$\text{DIC} = 2(\overline{D(\theta)}) - D(\overline{\theta}),$$

where $\overline{\theta}$ is the vector of parameters being varied in the model, the overline denotes the usual mean value, and $D(\overline{\theta}) = -2 \ln(L(\overline{\theta})) + C$, where $C$ is a constant. We used these definitions to form the difference in the IC values of the default model (ΛCDM) and the other suggested models (namely, we calculated ΔIC|model = IC|ΛCDM − IC|model). The model with the minimal AIC is considered best, (Jeffreys 1939), so a positive ΔIC points to a preference toward the DE model, negative toward ΛCDM with $|\Delta IC| \geq 6$ signifying a possible tension, $|\Delta IC| \geq 6$ a medium tension, and $|\Delta IC| \geq 10$ a strong tension. Finally, we used the

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**Fig. 2.** Posterior distribution for $\Omega_m$ and $w_0$ for the wCDM model, with the BAO data in the upper panel and the BAO0 data in the lower panel.

**Fig. 3.** Posterior distribution for $\Omega_m$ and $\Omega_k$ for the Ω_k-LCDM model with the BAO data in the upper panel and the BAO0 data in the lower panel.
Bayes factor, defined as
\[ B_{ij} = \frac{p(d|M_i)}{p(d|M_j)}, \]
where \( p(d|M_i) \) is the Bayesian evidence for model \( M_i \). The evidence is difficult to calculate analytically, but in polychord, it is calculated numerically by the algorithm. In the tables below, we use the \( \ln(B_{ij}) \), where “0” is \( \Lambda \)CDM, which we compare with all the other models (denoted by the index “i”). According to the Jeffreys’ scale (Jeffreys 1939), \( \ln(B_{ij}) < 1 \) is inconclusive for any of the models, 1–2.5 gives weak support for the model “i”, 2.5–5 is moderate and >5 is strong evidence for the model “i”. A minus sign gives the same for model “j”.

The so-defined statistical measures for the two datasets are presented in Tables 3 and 4. In summary, the model comparison of the different datasets gives:

- For the BAO dataset: the best model from the AIC, BIC, and DIC is \( \Lambda \)CDM, followed closely (within <1 IC units) by pEDE. The BF agrees with that, with pEDE and gEDE being close to it. OkLCDM is comparable to LCDM.
- For the BAO + SN dataset: the best model is \( \Lambda \)CDM from all IC measures. BF agrees with that for most models, with an inconclusive preference for wCDM (\( \ln(BF) < -1 \)).
- For the BAO\(_b\) dataset, the best model for AIC and BIC is pEDE, followed by CPLDM. For DIC the best model is CPL, with all wCDM and wwaCDM models being better than \( \Lambda \)CDM. The IC difference, however, is too small to signify any tension. The BF agrees with the DIC, with the CPL model being best and \( \Lambda \)CDM the worst. Again, this is inconclusive.
- For the BAO\(_b\) + SN dataset, with respect to the AIC and BIC, the best model is \( \Lambda \)CDM, but pEDE is very close to it. With respect to DIC, all the models give better results than \( \Lambda \)CDM, with CPL being best, but the statistical significance is extremely low. With respect to the BF, however, the three parametrizations of wwaCDM give the best results, with values representing a weak but non-negligible support.

From this comparison we see that first, the use of statistical measures does not give an entirely consistent view on selecting the best model. This can be due to a number of factors, such as slow convergence of some of the models, or priors not having the similar weight.

Second, the two BAO datasets have preferences for different models. This may be due to different intrinsic assumptions with which the measurements were made. The BAO\(_b\) dataset, despite the larger errors, seems to give consistent results, with some weak support for DE models in the different measures. The more standard AIC and BIC, however, are always in favor of \( \Lambda \)CDM, with pEDE being close behind. The BAO dataset seems to always prefer \( \Lambda \)CDM in most measures.

We can conclude that from the two datasets of BAO points, only the BAO dataset has a strong preference for \( \Lambda \)CDM. Adding the Pantheon dataset to it boosts this preference to statistical significance. The fact that \( \Lambda \)CDM is not the best model statistically in all of the cases for the BAO-only datasets may be due to the big uncertainty related to the BAO measurements or the specific of the chosen dataset. While including the Pantheon dataset decreases the deviation in general, it does not eliminate it entirely for BAO\(_b\). This could be due to the different redshift distributions of BAO and Pantheon affecting the model fit: the maximum redshift for the binned Pantheon is \( z_{\text{SN}}^{\text{max}} = 1.6 \) vs. \( z_{\text{BAO}}^{\text{max}} = 2.4 \) for BAO, and the median redshifts are accordingly \( z_{\text{SN}} = 0.2 \) vs. \( z_{\text{BAO}} = 0.6 \). Taking into consideration the big errors of the DE parameters for the different models and that all the evidence against \( \Lambda \)CDM is weak, we see that one needs much better BAO data to get a statistically strong preference, if indeed it exists.
Table 3. Constraints at 68% confidence-level errors on the cosmological parameters for the different tested models for the two BAO-only datasets: BAO and BAOo.

| Model          | $\Omega_m$  | $\Omega_K$ | $w_0$   | $w_a$   | $\Delta\text{AIC}$ | $\Delta\text{BIC}$ | $\Delta\text{DIC}$ | ln(BF)   |
|----------------|-------------|------------|---------|---------|---------------------|---------------------|---------------------|----------|
| BAO            |             |            |         |         |                     |                     |                     |          |
| LCDM           | 0.314 ± 0.014 | –          | –       | –       | 0                   | 0                   | 0                   | 0        |
| wCDM           | 0.292 ± 0.027 | –          | –0.658 ± 0.119 | –       | –1.184              | −2.228              | 0.810               | 1.693    |
| wwaCDM         | 0.314 ± 0.053 | –          | −0.644 ± 0.135 | −0.181 ± 0.3 | −3.062             | −5.151              | 0.869               | 0.375    |
| OkLCDM         | 0.321 ± 0.015 | −0.061 ± 0.053 | –       | –       | 0                   | −0.4052             | −1.216              | −0.835   |
| Linear         | 0.315 ± 0.051 | –          | −0.63 ± 0.127 | −0.196 ± 0.293 | −2.995              | −5.084              | 0.930               | 0.241    |
| CPL            | 0.293 ± 0.054 | –          | −0.662 ± 0.173 | −0.061 ± 0.606 | −3.299             | −5.388              | 0.600               | 1.267    |
| Log            | 0.308 ± 0.046 | –          | −0.651 ± 0.149 | 0.153 ± 0.376 | −3.142              | −5.231              | 0.786               | 0.639    |
| pEDE           | 0.31 ± 0.015  | –          | –        | 0.717   | 0.717               | 0.486               | −3.796              |          |
| gEDE           | 0.311 ± 0.016 | –          | −0.278 ± 0.209 | 0.290   | −1.728              | −2.773              | 0.161               | −1.900   |
| BAOo           |             |            |         |         |                     |                     |                     |          |
| LCDM           | 0.325 ± 0.057 | –          | –        | –0.892 ± 0.356 | −1.837              | −2.545              | 0.113               | −0.545   |
| wCDM           | 0.324 ± 0.064 | –          | −0.929 ± 0.356 | −0.314 ± 0.73 | −3.916             | −5.332              | 0.067               | −0.538   |
| wwaCDM         | 0.319 ± 0.058 | –          | −0.89 ± 0.43   | −0.314 ± 0.73 | −3.916             | −5.332              | 0.067               | −0.538   |
| OkLCDM         | 0.327 ± 0.053 | 0.038 ± 0.181 | –       | –       | −2.075              | −2.783              | −0.084              | 0.092    |
| Linear         | 0.324 ± 0.064 | –          | −0.872 ± 0.449 | −0.337 ± 0.963 | −3.821             | −5.237              | 0.118               | −0.467   |
| CPL            | 0.327 ± 0.063 | –          | −0.854 ± 0.438 | −0.448 ± 0.932 | −3.791              | −5.207              | 0.141               | −0.628   |
| Log            | 0.316 ± 0.066 | –          | −1.07 ± 0.432 | −0.527 ± 0.97  | −3.853             | −5.269              | 0.094               | −0.597   |
| pEDE           | 0.341 ± 0.051 | –          | –        | 0.165   | 0.165               | 0.114               | −0.332              |          |
| gEDE           | 0.343 ± 0.044 | –          | −0.877 ± 0.693 | 0.214   | −1.916              | −2.624              | 0.060               | −0.184   |

Table 4. Constraints at 68% confidence-level errors on the cosmological parameters for the different tested models for the two BAO + SN datasets: BAO+SN and BAOo + SN.

| Model          | $\Omega_m$  | $\Omega_K$ | $w_0$   | $w_a$   | $\Delta\text{AIC}$ | $\Delta\text{BIC}$ | $\Delta\text{DIC}$ | ln(BF)   |
|----------------|-------------|------------|---------|---------|---------------------|---------------------|---------------------|----------|
| BAO            |             |            |         |         |                     |                     |                     |          |
| LCDM           |             |            |         |         |                     |                     |                     |          |
| wCDM           |             |            |         |         |                     |                     |                     |          |
| wwaCDM         |             |            |         |         |                     |                     |                     |          |
| OkLCDM         |             |            |         |         |                     |                     |                     |          |
| Linear         |             |            |         |         |                     |                     |                     |          |
| CPL            |             |            |         |         |                     |                     |                     |          |
| Log            |             |            |         |         |                     |                     |                     |          |
| pEDE           |             |            |         |         |                     |                     |                     |          |
| gEDE           |             |            |         |         |                     |                     |                     |          |
| BAOo + SN      | 0.3 ± 0.014  | –          | –        | 0.176   | −20.3               | −22.4               | −18.4               | 18.3     |
| LCDM           | 0.305 ± 0.011 | –          | –         | 0.163   | −2.777              | 0.240               | 0.0     |          |
| wCDM           | 0.302 ± 0.012 | –          | −0.986 ± 0.045 | −22.4 | −26.6               | −18.6               | 16.9     |          |
| wwaCDM         | 0.361 ± 0.034 | –          | −1.18 ± 0.139 | −0.376 ± 0.672 | −22.4 | −26.6               | −18.6               | 16.9     |
| OkLCDM         | 0.336 ± 0.018 | −0.211 ± 0.066 | −22.4 | −26.6 | −18.6               | 16.9     |          |
| Linear         | 0.333 ± 0.084 | −1.128 ± 0.118 | −0.125 | −22.4 | −26.6               | −18.6               | 16.9     |          |
| CPL            | 0.369 ± 0.021 | −1.166 ± 0.134 | −0.569 ± 0.889 | −22.2 | −26.4               | −18.4               | 16.4     |          |
| Log            | 0.335 ± 0.055 | −1.183 ± 0.111 | −0.2 ± 0.156 | −22.0 | −26.2               | −18.2               | 16.3     |          |
| pEDE           | 0.353 ± 0.014 | –          | –        | −18.1   | −18.1               | −18.2               | 18.4     |          |
| gEDE           | 0.36 ± 0.022  | −1.319 ± 0.478 | 0.176   | −20.3   | −22.4               | −18.4               | 18.3     |          |

5. Discussion

In order to avoid the problem of the degeneracy between $H_0$ and $r_d$ in the BAO measurements, and the assumptions on the data it imposes, this paper removes the combination $H_0 \cdot r_d$ entirely by marginalizing over it in the $\chi^2$. We used two different BAO datasets to test our approach. The first one, named BAO, comes from different measurements provided by surveys such as SDSS, WiggleZ, and DES, in addition to radial measurements coming from the eBOSS data release DR16 with their covariances. The other dataset is the BAOo compilation that measures $\theta(\zeta)$, which is based on angular BAO measurements obtained from analyses of luminous red galaxies, blue galaxies, and quasars. These transversal BAO data have the advantage of being weakly dependent on the cosmological model. Both $D_L/r_d$, $D_M/r_d$ and $D_H/r_d$ provided from the first dataset, and $\theta(\zeta)$ provided from the second one, depend only on the combination $H_0 \cdot r_d$, which we integrate out. In a similar way, one can integrate out the dependence on $H_0$ and $M_B$ in the Pantheon SNIa dataset, leaving all the likelihoods depending purely on the equation of state, namely...
we find $\Omega_m$ and the DE parameters $\Omega_{\Lambda}$, $w_0$, and $w_a$, which allows us to use these datasets to infer the corresponding cosmological parameters.

We find that the BAO-only datasets infer $\Omega_m$ very well, close to the expected values and with a small error, but they are not sufficient to significantly constrain the parameters of the DE models. The errors on $w_0$ and particularly on $w_a$ are significant within the rather wide priors that we use. The errors of the BAO-only dataset are larger than the errors of the BAO dataset, as expected.

Adding the SNIa dataset reduces the errors, especially for the $w_0$ parameter. For the BAO + SN dataset, we find $w = -0.986 \pm 0.045$. For $w_0$, CDM, we find $w_0 = -1.18 \pm 0.139$, $w_a = -0.376 \pm 0.672$. From the BAO + SN dataset, we find $w = -1.08 \pm 0.14$ for the wCDM model. For $w$, we find $w = -0.19 \pm 0.09$, $w_a = -0.31 \pm 0.74$. As for the curvature, the BAO + SN dataset prefers a closed, almost flat, universe ($\Omega_k \approx -0.21 \pm 0.07$, while the BAO-only dataset prefers a flat one ($\Omega_k = -0.09 \pm 0.15$). In both cases, the gEDE model is closer to pEDE than to aCDM.

Comparing to the SDSS-IV results (Alam et al. 2021), we see that they predict $w_0 = -0.939 \pm 0.073$, $w_a = -0.31 \pm 0.3$ when one considers BAO+SN+CMB, but $w_0 = -0.69 \pm 0.15$ only when the BAO dataset is used. Thus, our results are consistent in both cases, with the BAO+SN value for $w_0$ a little lower and the BAO-only value, very close to theirs. The mean value for $w_0$ is close, but with a much larger error. However, we see that in the SDSS-IV results, the error on $w_0$ is also rather large. Our results also predict a negative $\Omega_k$, with a larger error. One should note, however, that while we include some of the most recent BAO measurements, we include only the angular part of SDSS-III DR12, due to its inter-redshift covariance. Also, the BAO-only dataset has larger inherent errors, and thus it is be expected to lead to larger errors in the inferred parameters. Finally, under the procedure we applied, some precision was lost due to the marginalization itself. Taking into account all this, we see that the procedure we employed still gives results close to those expected.

We performed a number of statistical tests for model comparison. The two BAO datasets show small statistical preferences for different models: aCDM for the BAO dataset and DE (wCDM, but also pEDE/gEDE) for the BAO-only dataset. When we added the SN dataset, aCDM remained the best model for the BAO+SN dataset, but the BAO-only dataset shows a weak but non-negligible preference for DE models.

Our conclusion is that one cannot sufficiently constrain the DE models from the chosen uncalibrated, mostly angular, BAO datasets alone. Adding the SNIa dataset to further reduce the errors and to remove some possible degeneracy helps, but it only helps to constrain $w_0$ and not so much $w_a$. However, the results on $\Omega_m$ and $w_0$ seem constrained enough to confirm the usefulness of this new approach. A downside is that, for the moment, it is not possible to include all correlated $D_m-D_{\Omega_m}$ measurements, since it is not possible to integrate out $H_0 \cdot r_0$ for a covariance matrix over different $z$. For this reason we did not use all known correlations in the BAO data, which will improve on the errors and, thus, could lead to better constraints. We predict that future measurements of the BAO would increase the efficiency of the approach, as long as the correlation between some redshifts was not large. In any case, the marginalization approach offers a new perspective on the degeneracy $H_0 - r_0 - \Omega_{\Delta m}$ since, in this case, the only varying parameter is $\Omega_m$ and it could be a tool for an independent crosscheck of DE models.

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Appendix A: Corner plots for the different datasets

Fig. A.1. Posterior distribution for $\Omega_m$ and $w_0, w_a$ for different parametrizations of DE, with the BAO data only in the upper panel and the combined BAO + Pantheon data in the lower panel.

Fig. A.2. Posterior distribution for for $\Omega_m, w_0, w_a$ for different parametrizations of DE, with the BAO data only in the upper and the combined BAO + Pantheon data in the lower panel.