The phase diagram of a half-filled hard core boson two-leg ladder in a flux is investigated by means of numerical simulations based on the Density Matrix Renormalization Group (DMRG) algorithm and bosonization. We calculate experimentally accessible observables such as the momentum distribution and of the rung current, showing that the transition from Mott-Meissner (MM) to Mott-Vortex (MV) state falls in the universality class of the C-IC transition\cite{14}. For fluxes close to $\pi$, we observe an other incommensuration, whose origin is discussed within the non–interacting case: above a critical value of the interchain hopping\cite{30} the system remains in the Mott- Meissner (MM) state for any flux (see Fig. 1). Below the critical interchain hopping, both the behavior of the momentum distribution and of the rung current, show that the transition from Mott-Meissner (MM) to Mott-Vortex (MV) state falls in the universality class of the C-IC transition\cite{14}. For fluxes close to $\pi$, we observe other incommensuration, whose origin is discussed within bosonization.

We consider\cite{14} a two-component system of hard core bosons on two leg ladder, with a flux per plaquette $\lambda$ and interchain hopping $\Omega$:

$$H_{\lambda} = -t \sum_{j,\sigma} \left( b_{j,\sigma}^{\dagger} e^{i\lambda \sigma} b_{j+1,\sigma} + H.c. \right) + \Omega \sum_{j} \left( b_{j,\uparrow}^{\dagger} b_{j,\downarrow} + H.c. \right),$$

with $b_{j,\sigma}^{\dagger}$ ($b_{j,\sigma}$) bosonic creation (annihilation) operator at site $j$, $\sigma = \pm 1/2$ the chain index, and $te^{i\lambda \sigma}$ the hopping amplitude along the chain $\sigma$. This Hamiltonian can be
mapped onto a system of spin-1/2 bosons with spin-orbit coupling in a transverse magnetic field with each spinor state corresponding to one leg of the ladder. For half-filling, i.e. for one boson per rung, at λ = 0 and Ω ≠ 0 the ground state of (1) is a rung-Mott Insulator. For λ > 0, according to the bosonization treatment [13, 14, 15], the Mott-Meissner (MM) and the Mott-Vortex (MV) state. In the MM state, for 0 < λ < Λ_{c}, two currents of opposite sign flow along the legs [30], the interchain current

\[ J_r(l) = iΩ \left( b_{1,1}^\dagger b_{1,↓} - b_{1,↓}^\dagger b_{1,↑} \right) \]  

has zero expectation value and exponentially decaying correlations, and the screening current, i.e. the difference between the currents of the two legs

\[ J_s = -i t \sum_{j,σ} \left( σ e^{iλσ} b_{j,σ}^\dagger b_{j+1,σ} - σ e^{-iλσ} b_{j+1,σ}^\dagger b_{j,σ} \right), \]  

is a smooth function of the applied flux (increasing linearly at small flux). On increasing the flux λ > Λ_{c}(Ω), the system enters the MV state, there is a sudden drop \[ \mathcal{B} \] of the screening current \[ J_s \] and simultaneously the rung current correlations decay becomes algebraic \[ \mathcal{B} \] with an incommensurate modulation of wavevector \( q(λ) \). Close to the transition point \( Λ_{c}(Ω) \), the wavevector \( q(λ) \) \( \sim \sqrt{λ^2 - Λ_{c}^2} \). In the non-interacting case, the Hamiltonian Eq. (1) can be readily diagonalized \[ \mathcal{B} \] and \( Λ_{c}(Ω) = 2 \arctan[Ω/(4t)] \). The occurrence of the MV phase can be seen out also in the total, as well as in, the spin resolved momentum distribution \[ \mathcal{B} \] of the system:

\[ n(k) = \sum_σ n_σ(k) = \frac{1}{L} \sum_σ \sum_{i,j} e^{i k (r_i - r_j)} (b_{i,σ}^\dagger b_{j,σ}). \]  

In the MM phase \( n(k) \) has a single maximum at \( k = 0 \), whereas in the MV phase it exhibits a pair of maxima \( k = \pm q(λ)/2 \). We have obtained the ground state phase-diagram of (1) by computing various observables like the momentum distribution and the screening current \( J_s \) together with the Fourier Transform (FT) \( C(k) = \sum_l e^{-i k l} (J_r(l), J_r(0)) \) of the rung current correlation function.

While performing simulations with both periodic (PBC) and open (OBC) boundary conditions, we found the former to be more suitable for our system, despite the well-known computationally more demanding convergence properties typical of PBC. As such we run simulations employing PBC for system sizes ranging from \( L = 16 \) to \( L = 64 \), keeping up to \( m = 1256 \) states during the renormalization procedure. In this way the truncation error i.e. the weight of the discarded states, is at most of order \( 10^{-6} \), while the maximum error on the ground-state energy is of order \( 5 \times 10^{-5} \) at its most. We further extrapolate in the limit \( m → ∞ \) all the quantities calculated to characterize the phase diagram. In Fig. 1, we summarize our findings for the phase diagram at half-filling. At variance with the non-interacting case where there is a critical \( Λ_{c}^{(0)}(Ω) \) for all \( Ω \), in the presence of the hard-core interaction, for interchain hoppings \( Ω > Ω_c \), the commensurate-incommensurate transition disappears \[ \mathcal{B} \] and the MV phase is stable for all fluxes. Another effect of the hard-core interaction, as we will discuss below, is that in the Vortex phase, at λ = π and λ close to π, a commensurate peak appears in \( C(k \approx π) \), along with an incommensuration in the density correlations. At λ = π, and for Ω > Ω_c a fully rung localized phase is obtained. Such rung localized ground state was discussed in the limit Ω ≫ t in \[ \mathcal{B} \].

We have characterized the nature of the Mott-Meissner and Mott-Vortex phases by examining \( C(k) \), the staggered boson density wave \( S(k) \) and the symmetric bond-order wave \( S^{BOW}_{Bk} \) static structure factors which bring information on the spin density and bond-order waves.
respectively:

\[ S(k) = \frac{1}{L} \sum_{j,l=0}^{L-1} e^{ikj_{j-l}} \text{sign}(\sigma') \langle n_{j,\sigma} n_{l,\sigma'} \rangle \quad (5) \]

\[ S_{BOW}(k) = \frac{1}{L} \sum_{j,l=0}^{L-1} e^{ikj_{j-l}} \langle \delta B_j \delta B_l \rangle \quad (6) \]

where \( B_j = \sum_b b^d_{j+1,\sigma} b_{j,\sigma} + H.c. \) and \( \delta B_j = B_j - \langle B_j \rangle \).

In Fig. 2 we follow the the MM-MV phase transition at small \( \lambda \) and \( \Omega \) (see cut one in Fig.1). As predicted from bosonization [32] the vortex phase is signalled by the appearance in \( C(k) \) of two cusp-like peaks respectively at \( k = q(\lambda) \) and \( k = 2\pi - q(\lambda) \) (see panel a) of Fig. 2 whose heights do not scale with the size of the system (see Fig. 1 of [32]). In MV phase, the spin resolved momentum distribution \( n_{\sigma}(k) \) shows a symmetric peak centered at \( k = \sigma q(\lambda) \), as predicted by bosonization. In this region of parameter space the correlation length associated with the Mott gap is comparable to the system size, and the peak takes a cusp-like shape as in a Tomonaga-Luttinger liquid [31], instead of the typical Lorentzian-shape expected for a Mott-insulator. Also \( S(k) \) shows the expected low momentum behaviour according to bosonization approach: in the MM phase \( S(k) = S(0) + ak^2 + o(k^2) \), with \( S(0) \gg 0 \), while in the MV phase \( S(k) = K^* \frac{k}{k} + o(k) \), with \( K^* = 1 \) (as expected for a hard-core boson system) a signature of a TLL of vortices. The transition is also seen in \( S(k \sim \pi) \). In the MM phase, \( S(k \sim \pi) \) shows a Lorentzian-shaped peak while in the MV phase this peak takes a cusp-like shape. A similar change across the MM-MV transition is also seen in the correlation function \( S_{BOW}(k \sim \pi) \) (see Supplemental Material [32]). This description breaks down when \( \lambda \) is no longer a small quantity as \( q(\lambda) \) would be comparable to the momentum cutoff.

At \( \lambda = \pi \) the major changes from the conventional C-IC transition at small flux are observed. To derive the low energy Hamiltonian it becomes necessary to choose the gauge with the vector potential along the rungs of the ladder, so that the interchain hopping reads:

\[ H_{hop.} = \Omega \sum_{j,\sigma} \langle \tilde{\phi}_{-j,\sigma}^{\dagger} \tilde{\phi}_{-j,\sigma} \rangle \quad (7) \]

After applying bosonization, the hopping Hamiltonian can be rewritten in terms of a free boson \( \phi_c \) describing the total density fluctuations coupled to SU(2) Wess-Zumino-Novikov-Witten (WZNW) currents \( J_{R,L} \) describing the chain antisymmetric density fluctuations by a term \( \propto \Omega \cos \sqrt{2} \phi_c (J^R_H + J^L_H) \) (see [32] for details). Such a term can be treated in mean-field theory [20, 22]. This procedure leads to an effective Hamiltonian with a gap \( \Delta_c \sim \Omega^2 \) for the total density excitations, while the antisymmetric density modes remain gapless and develop an incommensuration of wavevector \( p(\Omega) \propto \Omega^2 \) (see Fig. 2 in [32]). The presence of this predicted incommensuration is visible in the low momentum behaviour of \( S(k) \)

![FIG. 2: We show FT of correlation functions as from DMRG simulation for \( L = 64 \) at \( \lambda = \pi/4 \) for two different values of the \( \Omega/t = 0.0625 \) and \( \Omega/t = 1.0 \) and 1.5, respectively in the Vortex (black solid line) and Meissner phase (red solid line). Panel a) shows the FT of the rung-current correlation function \( C(k) \), panel b) the spin correlation functions \( S(k) \) and panel c) the charge bond-order correlation function \( S_{BOW} \). In panel d) the spin resolved momentum distribution is shown, with \( n_{-\sigma}(k) = n_{\sigma}(-k) \).

![FIG. 3: We show FT of correlation functions as from DMRG simulations for \( L = 64 \) at \( \lambda = \pi \) at various \( \Omega/t \). Panel a) shows the FT of the rung-current correlation function \( C(k) \), panel b) the spin correlation function \( S(k) \) and panel c) the charge bond-order correlation function \( S_{BOW} \). In panel d) the spin resolved momentum distribution is shown. Red solid curves are for \( \Omega/t = 1.75 \) in the fully localised state, while black, blue, green and magenta solid lines are respectively for \( \Omega/t = 1.5, 1, 1.25, 1 \) and 0.5.](image-url)
and $C(k)$ (panel a) and panel b) of Fig. 3 that become
$$\propto \frac{K_c}{\pi^2} |p(\Omega)|^2 |1 + p(\Omega)|, \text{ i.e. constant for } |k| < p(\Omega) \text{ and linear in } k \text{ for } |k| > p(\Omega).$$
In the $S^c_{\text{BOW}}(k)$ we observe a cusp at the same vectors $p(\Omega)$ (panel c) of Fig. 3.
As expected, all these correlation functions also develop a peak at $k = \pi/a$. A sign of the incommensurability at $\lambda = \pi$ should be visible also in the momentum distribution $n_\sigma(k)$ (see Fig. 3 panel d). In this case, a calculation based on non-abelian bosonization and operator product expansion, would lead to three Lorentzian-like peaks centered in $\pi/(2a)$ and $\pi/(2a) \pm p(\Omega)/2$. However, these peaks cannot be separated if the correlation length in real space $u_c \sim \Omega^{-2}$ is shorter than the wavelength $2\pi/p(\Omega) \sim \Omega^{-2}$. In the numerical simulations, at $L = 64$ in PBC, (see Fig. 3) a broad peak is observed for $k = \frac{\pi}{a}$.

When $\lambda \lesssim \pi$ (second cut in Fig.1) we can proceed analogously to the previous case and choose a gauge such that:

$$H = -t \sum_{j,\sigma} \left( b_j^{\dagger} c_{j+1,\sigma} + H.c. \right) + \Omega \sum_{j,\sigma} (-1)^j b_j^{\dagger} b_{j-1,\sigma},$$

and define $\delta\lambda = (\lambda - \pi)$, so that the bosonized Hamiltonian contains the extra term $\delta\lambda(J_{k} - J_{k-1})$. For this case, the Fourier transform of the rung current correlation will present peaks at $k = \frac{\pi}{a}$ and $k = \frac{\pi}{a} \pm \sqrt{p(\Omega)^2 + (\delta\lambda/a)^2}$. When $\delta\lambda$ is increased, these last two peaks become dominant, and we crossover to the behavior already discussed for weak $\lambda$. At $\lambda < \pi$ the $C(k)$ (see Fig. 3) shows, beside the peak at $k = \frac{\pi}{a}$, two peaks symmetric around $k = \frac{\pi}{a}$, in real space these last two oscillations exhibit an exponential decay for $\Omega/t > 1$ and a power law for $\Omega/t \leq 1$ (region III and II in Fig. 1). The situation is reversed for the oscillation at $k = \frac{\pi}{a}$. At $\Omega/t = 1$ all oscillations, for systems with $L = 64$ in PBC, exhibit power law decay.

The effect of this incommensurability can also be followed in the behaviour at small $k$ of $S(k)$ that instead of being a constant value for $k < \sqrt{p(\Omega)^2 + (\delta\lambda/a)^2}$ shows a linear behaviour. In $S^c_{\text{BOW}}(k)$, for $\Omega/t \leq 1$ two symmetric peaks are present at $k = \pm q(\lambda)$. We checked that the phase is a single component Tomonaga-Luttinger liquid by computing the Von Neumann entropy at $\Omega/t = 1.5$ for $\lambda = 0.75\pi$ and $0.8125\pi$ and obtaining the expected logarithmic dependence with system size [30], ruling out a a Chiral Mott insulator [13, 15] for $\lambda \lesssim \pi$. At general commensurability filling $n$ and flux $\lambda = 2\pi n$, the incommensurability generating term becomes $-i\Omega e^{i\sqrt{2} q_0脑袋 = 1 + p(\Omega)}$ and the incommensurate density wave phases with incommensurability. Let us finish by noting that such incommensurability is specific of hard core boson systems. With less repulsive interactions, the term that gives rise to the vortex lattice state would be relevant [13, 12, 42], while stronger repulsion would make the term stabilizing the checkerboard density wave relevant. Adding a nearest neighbor intra-

chain interaction $V$ to the hard core repulsion, a first phase transition at $V < 0$ will separate the vortex lattice from the incommensurate state, and a second transition at $V > 0$ will separate the incommensurate state from the density wave state.

In conclusion, we have studied a two-leg hard core boson ladder in an artificial gauge field. In contrast to the non-interacting case, the vortex phase is suppressed when the interchain hopping exceeds a threshold value, as found in [30]. At flux $\pi$ per plaquette and $\Omega/t > 1.5$ the ground state becomes a tensor product of singly occupied rungs, as was expected [30] in the $\Omega/t \to \infty$ limit. For $\Omega/t < 1.5$, we have obtained an incommensurate insulating state similar to the spin-nematic state of frustrated XXZ spin chains [21, 22]. In the case of a system of weakly coupled ladders, a long range ordered phase could form in which density wave or rung current would possess a long range commensurate order, but exponentially damped incommensurate correlations would still be present. The presented results could be detectable in current experiments with cold atoms [23] and the evidence of a persisting Meissner state could be relevant for quantum computing purposes in defining a stable flux qubit [43, 44].

We thank F. Ortolani for the DMRG code. Simulations were run at Università di Salerno and Università di Pisa local computing facilities. M.D.D. and M.L.C. acknowledge partial support from PRIN-2011 "Collective
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Supplemental Material for “Persisting Meissner state and incommensurate phases of hard-core boson ladders in a flux”

Supplementary material for “Incommensurate phases of a hard-core boson two-leg ladder in a flux”

CORRELATION FUNCTIONS AND OBSERVABLES IN BOSONIZATION

Bosonized Hamiltonian and observables

Let us first consider a single leg \( \sigma \) in the case of \( \Omega = 0, \lambda = 0 \). Hard core bosons are mapped on non-interacting spinless fermions by the Jordan-Wigner transformation \([S1]\). These non-interacting spinless fermions are then bosonized\([S2]\). The bosonized form of the Hamiltonian \( H_{||} \) reads:

\[
H_{||} = \sum_{\sigma} \int \frac{dx}{2\pi} \left[ uK(\pi \Pi_{\sigma})^2 + \frac{u}{K} (\partial_x \phi_{\sigma})^2 \right], \tag{S1}
\]

where \([\phi_{\sigma}(x), \Pi_{\sigma}(x') = i\delta_{x-x'} \) and \( \pi \int \Pi_{\sigma} = \theta_{\sigma} \). in Eq. \( S3 \), \( u \) is the velocity of excitations, while \( K \) is the Tomonaga-Luttinger (TL) parameter. For non-interacting spinless fermions, \( K = 1 \) and \( u = 2ta\sin(\pi n/2) \), where \( n = \langle n_\uparrow + n_\downarrow \rangle \) is the average number of bosons per site and \( a \) is the lattice spacing. At half-filling \( (i. e. \ n = 1) \) \( u = 2t \). The boson annihilation operators are represented as \([S2]\):

\[
b_{j,\sigma} = e^{i\theta_{\sigma}(ja)} \left[ A_0 + \sum_{m \neq 0} A_m e^{2im(\phi_{\sigma})} \right], \tag{S2}
\]

In the presence of a vector potential along the legs of the ladder, the lattice Hamiltonian can be brought back to the case of \( \lambda = 0 \) by the canonical transformation \( b_{j,\sigma} = e^{-i\lambda x_{\sigma}} \tilde{b}_{j,\sigma} \). The bosonization technique can then be applied to the Hamiltonian written in terms of the \( \tilde{b}_{j,\sigma} \) bosons. One finds a Hamiltonian of the form \([S1]\), with \( \tilde{\theta}_{\sigma}, \tilde{\phi}_{\sigma} \) replacing \( \theta_{\sigma}, \phi_{\sigma} \). Using Eq. \( S2 \), one obtains \( \theta(x) = \theta(x) - \lambda a x/a, \) and \( \phi(x) = \phi(x) \) giving the bosonized Hamiltonian:

\[
H_{||} = \sum_{\sigma} \int \frac{dx}{2\pi} \left[ uK \left( \Pi_{\sigma} + \frac{\lambda}{a} \right)^2 + \frac{u}{K} (\partial_x \phi_{\sigma})^2 \right], \tag{S3}
\]

Actually, it is convenient to turn to symmetric \((c)\) and antisymmetric \((s)\) representation, \( \phi_{c,s} = \frac{\phi_{\uparrow} \pm \phi_{\downarrow}}{\sqrt{2}}, \) \( \theta_{c,s} = \frac{\theta_{\uparrow} \pm \theta_{\downarrow}}{\sqrt{2}} \), so that the full Hamiltonian reads:

\[
H = H_c + H_s^\lambda \tag{S4}
\]

\[
H_c = \int \frac{dx}{2\pi} \left[ u_c K_c(\pi \Pi_c)^2 + \frac{u_c}{K_c} (\partial_x \phi_c)^2 \right] \tag{S5}
\]

\[
H_s^\lambda = \int \frac{dx}{2\pi} \left[ u_s K_s \left( \Pi_s + \frac{\lambda}{a\sqrt{2}} \right)^2 + \frac{u_s}{K_s} (\partial_x \phi_s)^2 \right] \tag{S6}
\]

where \( u_c K_c = u_s K_s = u K, u_c/K_c = u_s/K_s = u/K, \) and we have used \( \sigma = \pm 1/2. \)

The boson annihilation operators become:

\[
b_{j,\sigma} = e^{i\sqrt{a}(\theta_c + 2\sigma \theta_s)} \left[ \sum_{m} A_m e^{im(\sqrt{2}\phi_c + 2\sigma \sqrt{2}\phi_s - 2\pi \langle n_\sigma \rangle x/a)} \right] \tag{S7}
\]

Therefore,

\[
\langle b_{\sigma} b_{\sigma}^\dagger \rangle \sim \langle e^{i\sqrt{a}(\theta_c(ja) e^{-i\sqrt{a}(\phi_0)})} e^{i\sqrt{a}(\theta_s(ja) e^{-i\sqrt{a}(\phi_0))}} \rangle + \ldots \tag{S8}
\]

Using instead the bosonized expression for the density \( \rho_{\sigma} = \frac{\langle n_\sigma \rangle}{a} - \partial_x \phi_{\sigma} + B_1 \sin(2\phi_{\sigma} - 2\pi \langle n_\sigma \rangle x/a) \) one obtains the total boson density \( \rho_{\text{tot}} = \rho_{\uparrow} + \rho_{\downarrow} \) as:

\[
\rho_{\text{tot}}(x) = \frac{1}{a} - \frac{\sqrt{2}}{\pi} \partial_x \phi_c + 2B_1 e^{i\pi/2} \sin \sqrt{2}\phi_c \cos \sqrt{2}\phi_s, \tag{S9}
\]
and the antisymmetric density $\sigma^z = (\rho^\uparrow - \rho^\downarrow)/2$ as:

$$\sigma^z(x) = -\frac{1}{\pi \sqrt{2}} \partial_x \phi_s + B_0 e^{i\pi \sigma^z} \cos \sqrt{2} \phi_s \sin \sqrt{2} \phi_s,$$

where we have assumed $\langle n_s \rangle = 1/2$. Now, if we turn on the interchain hopping $\Omega$, we will obtain with the help of $\text{S2}$ the following bosonized form:

$$H_{\text{trans.}} = \frac{2\Omega}{\alpha} \int dx \cos \sqrt{2} \theta_s \left[ A_0^2 + 2A_0^2 \cos \sqrt{2} \phi_c + 2A_1^2 \cos \sqrt{2} \phi_s + \ldots \right],$$

where ... stands for less relevant operators. It is important to note that in a renormalization group calculation, operators $\cos \sqrt{2} \phi_c$ and $\cos \sqrt{2} \phi_s$ are generated in the flow. Since we are in the half-filled case, $\langle n_\uparrow + n_\downarrow \rangle = 1$, umklapp scattering $\cos \sqrt{2} \phi_c$ is present $\text{S4, S5}$ and opens of a gap in the $c$ modes.

For $\langle n_\uparrow \rangle = \langle n_\downarrow \rangle$, the term $\cos \sqrt{2} \phi_s$ is present, but is marginally irrelevant for repulsive interactions.

We have in the ground state $\langle \phi_c \rangle = 0$. Using Eqs. $\text{S6, S7}$, the staggered and uniform components of the total density have exponentially decaying correlations. Meanwhile, the antisymmetric density $\text{S10}$ has a simplified expression for distances much larger than $u_c/\Delta_c$ where one can replace $\cos \sqrt{2} \phi_c$ by its average.

Besides density waves, the system can also present bond ordering. The bond order wave order parameter $O_{\text{BOW}}^s$ for spin $\sigma$ boson is defined by:

$$b_{j+1,\sigma}^\dagger b_{j,\sigma} + b_{j,\sigma}^\dagger b_{j+1,\sigma} = T(ja) + (-)^j O_{\text{BOW}}^s(ja),$$

where $T$ is the kinetic energy density, and in bosonization

$$O_{\text{BOW}}^s = C_0 \cos(2\phi_c)$$

We can define the two order parameters:

$$O_{\text{BOW}}^c = \sum_\sigma O_{\text{BOW}}^\sigma = 2C_0 \cos \sqrt{2} \phi_c \cos \sqrt{2} \phi_s$$

$$O_{\text{BOW}}^s = \sum_\sigma \text{sign}(\sigma) O_{\text{BOW}}^\sigma = 2C_0 \sin \sqrt{2} \phi_c \sin \sqrt{2} \phi_s$$

In a Mott phase, $O_{\text{BOW}}^c$ is always short range ordered, while $O_{\text{BOW}}^s \sim \cos \sqrt{2} \phi_s$. If we consider the real space Green’s functions for the bosons, due to the long range order of $\phi_c$, the exponentials $e^{i\alpha \phi_c}$ of the dual field are short range ordered, and the boson Green’s functions decay exponentially. Excitations of the total density are solitons and antisolitons of topological charge $\phi_c(+\infty) - \phi_c(-\infty) = \pi/\sqrt{2}$. Such an excitation corresponds to a change of particle number $\pm 1$, and for fixed particle number density excitations are formed of soliton/antisoliton pairs.

For $\lambda = 0$, the term $2\Omega A_0^2 \cos \sqrt{2} \phi_c$ is a relevant perturbation with scaling dimension $1/(2K_s)$, opening a gap $\Delta_s \sim \Omega^2 K_s/(4K_s - 1)$ in the antisymmetric sector. We have in the ground state $\langle \theta_s \rangle = \pi/\sqrt{2}$ as a result all the fields $e^{i\alpha \phi_s}$ are short range ordered. Therefore, in the Mott-Meissner phase, all the density wave and bond order wave parameters are short range ordered. Because of the long range order of $\theta_s$ we expect that the correlation functions $\langle b_{j,\sigma} b_{k,\sigma}^\dagger \rangle$ are non-zero. A more precise estimate of the gap (albeit without log corrections) can be obtained from the results of $\text{S6, S7}$. Using these results, we predict that the soliton mass $\Delta_s$ behaves as:

$$\Delta_s = \frac{u_s}{\alpha} \frac{2\Gamma\left(\frac{1}{8K_s - 2}\right)}{\sqrt{\pi} \Gamma\left(\frac{2K_s}{4K_s - 1}\right)} \left(\frac{1}{\Gamma\left(\frac{1}{4K_s}\right)} \frac{\Omega A_0}{u_s} \right)^{2K_s/(4K_s - 1)},$$

and the soliton/antisoliton dispersion is $E_s(k) = \sqrt{(u_s k)^2 + \Delta_s^2}$. In the case of hard core bosons, we have to set $K_s = 1$ in Eq. $\text{S16}$. In such case, besides the solitons and antisolitons, there are two breathers $\text{S8, S10}$, a light breather of mass $\Delta_s$ and a heavy breather of mass $\sqrt{3} \Delta_s$. The topological charge of the solitons/antisolitons is $\theta_s(+\infty) - \theta_s(-\infty) = \pm \pi/\sqrt{2}$. The solitons therefore carry a spin current $u_s K_s$. In the case where one is considering the gap between the ground state and an excited state of total spin current zero (i.e. containing at least one soliton and one antisoliton), the measured gap will be $2\Delta_s$. The amplitude $A_0$ can be estimated for hard core bosons in the case of half-filling $\text{S11}$:

$$A_0^2 = 2^{1/6} e^{1/2} A^{-6},$$

$$\lambda = 0$$
where \( A \approx 1.282427 \) is Glaisher's constant. Using [S10] with \( K_s = 1 \) and [S17], we find:

\[
\Delta_s = \frac{u_s}{a} \frac{2\Gamma(1/6)}{\sqrt{\pi} \Gamma(2/3)} \left( \frac{2^{1/6} e^{1/2} A^{-6} \Gamma(3/4)}{\Gamma(1/4)} \frac{\pi \Omega}{u_s} \right)^{2/3},
\]  

(S18)

for half-filling. Using \( u_s = 2ta \), we finally have \( \Delta_s/t = 3.3896(\Omega/t)^{2/3} \). The marginally irrelevant operator \( \cos \sqrt{\delta_s} \) can give rise to logarithmic corrections to that scaling [S12, S13] of the form \( \Delta_s \sim \Omega^{2/3} \ln |\Omega|^{1/6} \).

**Commensurate Incommensurate transition**

Neglecting the marginally irrelevant term \( \cos \sqrt{\delta_s} \) as in [S14], the Hamiltonian [S6] describes the C-IC transition [S10, S17]. When \( \lambda \) exceeds the threshold \( \lambda_c \sim (\Omega A_0^2 a/u_s)^{2-1/(2K_s)} \), it becomes energetically favorable to populate the ground state with a finite density of solitons of the field \( \theta_s \) to form a Tomonaga-Luttinger liquid (TLL) of solitons. The low energy properties of that TLL are described by the effective Hamiltonian:

\[
H^* = \int \frac{dx}{2\pi} \left[ u_s(\lambda) K_s^*(\lambda)(\pi \Pi^*_s)^2 + \frac{u_s(\lambda)}{K^*_s(\lambda)} (\partial_x \phi)^2 \right],
\]  

(S19)

and we have \( \pi \Pi_s(x) = \pi \Pi^*_s(x) - q(\lambda)/\sqrt{2} \); \( \theta_s(x) = \theta^*_s(x) - q(\lambda)x/\sqrt{2} \). Near the transition point \( \lambda_c \), we have \( q(\lambda) \sim C \sqrt{\lambda - \lambda_c} \). Moreover, as \( \lambda \to \lambda_c + 0 \), \( K^*_s(\lambda) \) goes to a limiting value \( K^*_s(0) \) such that [S17, S18] the scaling dimension of \( \cos \sqrt{\beta(\lambda)} \) becomes 1. Since the scaling dimension of \( \cos \sqrt{\beta(\lambda)} \) with a Hamiltonian of the form [S19] is \( 1/[2K^*_s(\lambda)] \) we have \( K^*_s(\lambda \to \lambda_c + 0) = 1/2 \). Using a fermionization method [S14], an explicit form of \( q(\lambda) \) can be obtained for \( K_s = 1/2 \).

The antisymmetric leg current (or screening current) operator

\[
J_s(j) = -it \sum_\sigma \left( \sigma e^{i\lambda \sigma} b^\dagger_{j+1,\sigma} b_{j,\sigma} - \sigma e^{-i\lambda \sigma} b^\dagger_{j,\sigma} b_{j+1,\sigma} \right),
\]  

(S20)

is obtained by differentiating the Hamiltonian with respect to the parameter \( \lambda \). One finds:

\[
J_s(x) = \frac{u_s K_s}{\pi \sqrt{2}} \left( \pi \Pi_s + \frac{\lambda}{a \sqrt{2}} \right)
\]  

(S21)

In the Meissner phase, \( \langle \Pi_s \rangle = 0 \) so that:

\[
\langle J_s \rangle = \frac{u_s K_s}{2\pi} \lambda
\]  

(S22)

In the vortex phase, \( J_s = u_s K_s (\lambda - \text{sign}(\lambda) q(\lambda))/(2\pi) \). By fermionization [S14], \( q(\lambda) = \sqrt{\lambda^2 - \lambda_c^2} \) is obtained for \( K_s = 1/2 \).

In the Mott-Vortex state, we obtain the rung current as:

\[
j_{\perp}(x) = \Omega A_0^2 \sin[\sqrt{2}\theta^*_s(x) - q(\lambda)x].
\]  

(S23)

In both the Meissner and the vortex phase, its expectation value \( \langle j_{\perp} \rangle = 0 \) vanishes. In the Meissner phase, the conversion current correlation function \( C(x) = \langle j_{\perp}(x) j_{\perp}(0) \rangle \) decays exponentially whereas in the vortex phase:

\[
C(x) = \frac{1}{2} (\Omega A_0^2)^2 \left( \frac{a^2}{x^2 + a^2} \right)^{\frac{1}{\sqrt{2}}} \cos[q(\lambda)x]
\]  

(S24)

For \( K_s^* = 1/2 \), the Fourier transform:

\[
C(k) = \frac{a (\Omega A_0^2)^2}{4} (e^{-|k-q(\lambda)a|} + e^{-|k+q(\lambda)a|}),
\]  

(S25)

has two cusps at \( k = \pm q(\lambda) \). Peaks divergent with system size appear in \( C(k) \) for \( K^*_s > 1 \).
We can also obtain the spin-spin and bond order correlation functions. In the Mott-Vortex phase, the fields \( e^{i\beta \phi} \) have quasi-long range order. As a result, we find that:

\[
\langle O_{BOW}^c(x)O_{BOW}^c(0) \rangle = C_0^2 (\cos \sqrt{2} |\phi_c|) \left( \frac{a^2}{x^2 + a^2} \right)^{K_s^*/2} \tag{S26}
\]

\[
\langle \sigma^x(x)\sigma^x(0) \rangle = \frac{K_s}{4\pi^2 (x^2 + a^2)^2} + (-\frac{\pi}{2}) B_0^2 (\cos \sqrt{2} |\phi_c|) \left( \frac{a^2}{x^2 + a^2} \right)^{K_s^*/2} \tag{S27}
\]

If we turn to the Fourier transforms, we find that for \( k \approx 0 \),

\[
S(k) = \frac{K_s^*|k|}{4\pi} e^{-|k|a}, \tag{S28}
\]

by using the integral \( \int_{-\infty}^{\infty} \frac{dxe^{ikx}}{x^2 + a^2} = \frac{\pi}{a} e^{-|k|a} \). For \( k \approx \frac{\pi}{a} \) and \( K_s^* < 1 \), we find that the correlation functions \( S(k) \) and \( S_{BOW}(k) \) are divergent as:

\[
S(k) \sim S_{bow}(k) \sim \left| k - \frac{\pi}{a} \right|^{K_s^* - 1}, \tag{S29}
\]

and as a result a divergence going as \( |ka|^{-1/2} \) is expected at the transition, while far from the transition \( K_s^* \approx 1 \), giving only a weak power law or logarithmic divergence. For \( K_s^* > 1 \), both \( S_s(k) \) and for \( S_{bow}(k) \) remain finite in the vicinity of \( \pi/a \).

If we turn to the momentum distribution, we will find:

\[
\langle b_{j,\sigma}^\dagger b_{j',\sigma'} \rangle = \delta_{\sigma\sigma'} e^{-i\sigma q(\lambda)(j-l)} \left( e^{i\theta_c(\lambda)a/\sqrt{2}} e^{-i\theta_c(\lambda)a/\sqrt{2}} \right) \left( e^{i\theta_c(\lambda)a/\sqrt{2}} e^{-i\theta_c(\lambda)a/\sqrt{2}} \right) \tag{S30}
\]

\[
= \delta_{\sigma\sigma'} e^{-i\sigma q(\lambda)(j-l)} \left( e^{i\theta_c(\lambda)a/\sqrt{2}} e^{-i\theta_c(\lambda)a/\sqrt{2}} \right) \left( 1 + (x/a)^2 \right)^{1/(8K_s^*)}. \tag{S31}
\]

because of the exponential decay of the charge correlator, we expect that the momentum distribution of spin \( \sigma \) particles will be centered around \( k = -\sigma q(\lambda) \). The total momentum distribution will thus have two peaks for \( k = \pm q(\lambda)/2 \).

**INCOMMENSURATION FOR \( \lambda \approx \pi \)**

For \( \lambda \) close to \( \pi \), the form \[S50\] for the Hamiltonian cannot be used as \( \lambda u/\alpha \) is not a small quantity compared with the energy cutoff \( u_\alpha/\alpha \). To describe the low energy physics at \( \lambda = \pi \), it is necessary to choose a gauge with the vector

---

**FIG. S1:** Size dependence of \( S(k = \pi) \) (red dots) and \( S_{BOW}(k = \pi) \) (blue dots), for system sizes \( L = 16, 32, 48, 64 \) and 96. Data are for \( \Omega = 0.25 \) and \( \lambda = \pi/2 \), well inside the vortex phase region. Both quantities don’t show a visible size dependence in agreement with \[S29\] for \( K_s^* = 1 \). Solid black dots are \( C(k = q(\lambda))\alpha \) with \( \alpha = \Omega^2/16 \) to be on the same scale on the graph. The size-scaling is compatible with a logarithmic dependence (red dotted line) as expected far from the transition region.
potential along the rungs of the ladder, so that the interchain hopping reads:

$$H_{\text{hop.}} = \Omega \sum_j (-)^j b^\dagger_{j,\sigma} b_{j,\sigma},$$  \hspace{1cm} (S32)

Applying bosonization to (S32), we obtain the following representation for interchain hopping:

$$H_{\text{hop.}} = \frac{\Omega}{2\pi a} \int dx \cos \sqrt{2} \phi_c \left[ e^{-i\sqrt{2}(\theta_{\sigma} + \phi_{\sigma})} + e^{-i\sqrt{2}(\theta_{\sigma} - \phi_{\sigma})} + e^{i\sqrt{2}(\theta_{\sigma} + \phi_{\sigma})} + e^{i\sqrt{2}(\theta_{\sigma} - \phi_{\sigma})} \right],$$  \hspace{1cm} (S33)

which can be rewritten in terms of SU(2)\textsubscript{1} Wess-Zumino-Novikov-Witten (WZNW) currents \cite{S14}:

$$H_{\text{hop.}} = \Omega \int dx \cos \sqrt{2} \phi_c (J^x_R + J^x_L).$$  \hspace{1cm} (S34)

The resulting Hamiltonian $H_c + H_s + H_{\text{hop.}}$ can be treated in mean-field theory \cite{S24 S23}, giving:

$$H_{\text{MF}} = H_c^{\text{MF}} + H_s^{\text{MF}},$$  \hspace{1cm} (S35)

$$H_c^{\text{MF}} = \frac{dx}{2\pi} \left[ \frac{g_c}{K_c} (\pi \Pi_c)^2 + \frac{g_c}{K_c} (\partial_\phi_c)^2 \right] + \frac{g_c}{\pi a} \int dx \cos \sqrt{2} \phi_c,$$  \hspace{1cm} (S36)

$$H_s^{\text{MF}} = \frac{2\pi u_s}{3} \int dx (J_R \cdot J_R + J_L \cdot J_L) + h_s \int dx (J^x_R + J^x_L),$$  \hspace{1cm} (S37)

where:

$$\frac{g_c}{\pi a} = 8\Omega (J^y_R + J^y_L)_{s,MF},$$

$$h_s = 8\Omega (\cos \sqrt{2} \phi_c)_{c,MF}.$$  \hspace{1cm} (S38)

Using a $\pi/2$ rotation around the $x$ axis, $J^y_R = J^y_R^x$, $J^y_L = -J^y_L^x$, and applying abelian bosonization \cite{S24}, we rewrite:

$$H_s^{\text{MF}} = \frac{dx}{2\pi u_s} \left[ (\pi \Pi_s)^2 + (\partial_\phi_s)^2 \right] - \frac{h_s}{\pi \sqrt{2}} \int \partial_x \tilde{\phi}_s dx,$$  \hspace{1cm} (S39)

which allows us to write:

$$- \frac{1}{\pi \sqrt{2}} (\partial_x \tilde{\phi}_s) = \sum_{\nu=R,L} \langle \tilde{j}^x_{\nu} \rangle = \langle J^x_R + J^x_L \rangle = - \frac{h_s}{2\pi u_s},$$  \hspace{1cm} (S40)

and allows us to solve \cite{S33} with $h_s \sim \Omega^2$ and $g_c \sim \Omega^3$. We obtain a gap in the total density excitations, $\Delta_c \sim \Omega^2$, while the antisymmetric modes remain gapless and develop an incommensuration. To characterize the incommensuration, we need to detail the rotation of the SU(2)\textsubscript{1} WZNW currents and primary fields. After shifting $\phi_s \rightarrow \phi_s + \frac{h_s x}{u_s \sqrt{2}}$ we find:

$$- \frac{1}{\pi \sqrt{2}} \partial_x \tilde{\phi}_s = - \frac{1}{2\pi a} \sum_{r,r'=\pm} e^{ir\sqrt{2}(\theta_{\sigma} + r\tilde{\phi}_s) + ir'\frac{h_s x}{u_s}},$$  \hspace{1cm} (S41)

$$\sum_{r=\pm} \sin \sqrt{2}(\theta_{\sigma} + r\phi_s) = \sum_{r=\pm} \sin \sqrt{2} \left( \theta_{\sigma} + r\tilde{\phi}_s + r\frac{h_s x}{u_s} \right),$$  \hspace{1cm} (S42)

$$\sum_{r=\pm} \cos \sqrt{2}(\theta_{\sigma} + r\phi_s) = - \frac{1}{\pi \sqrt{2}} \partial_x \tilde{\phi}_s - \frac{h_s}{2\pi u_s},$$  \hspace{1cm} (S43)

and:

$$\sin \sqrt{2}\theta_{\sigma} = \sin \sqrt{2}\tilde{\theta}_{\sigma},$$  \hspace{1cm} (S44)

$$\cos \sqrt{2}\theta_{\sigma} = \cos \left( \sqrt{2}\tilde{\phi}_s + \frac{h_s x}{u_s} \right),$$  \hspace{1cm} (S45)

$$\sin \sqrt{2}\phi_{\sigma} = - \cos \sqrt{2}\tilde{\theta}_{\sigma},$$  \hspace{1cm} (S46)

$$\cos \sqrt{2}\phi_{\sigma} = \cos \left( \sqrt{2}\tilde{\phi}_s + \frac{h_s x}{u_s} \right).$$  \hspace{1cm} (S47)
Since we have:

\[ j_\perp(j) = \frac{\Omega}{\pi a} \sum_{r=\pm} \sin \sqrt{2}(\theta_s + r\phi_s) + \frac{2\Omega(-)^{j}}{\pi a} \sin \sqrt{2}\theta_s, \]  
\[ \sigma^z(x) = -\frac{1}{\pi \sqrt{2}} \frac{1}{\pi a} \cos \frac{\sqrt{2} \phi_c}{\pi a} \sin \sqrt{2} \phi_s, \]  
\[ O_{BOW}^\prime = \frac{(\pi a)}{\pi a} \cos \sqrt{2} \phi_c \cos \sqrt{2} \phi_s \]

we find, after the rotation:

\[ j_\perp(j) = \frac{\Omega}{\pi a} \sum_{r=\pm} \sin \sqrt{2} \left( \tilde{\theta}_s + r\tilde{\phi}_s + r \frac{h_s x}{u_s} \right) + \frac{2\Omega(-)^j}{\pi a} \sin \sqrt{2}\tilde{\theta}_s, \]  
\[ \sigma^z(x) = \frac{1}{\pi a} \sum_{r=\pm} \cos \sqrt{2} \left( \tilde{\theta}_s + r\tilde{\phi}_s + r \frac{h_s x}{u_s} \right) - \frac{(\pi a)}{\pi a} \cos \sqrt{2} \phi_c \cos \sqrt{2} \tilde{\theta}_s, \]

\[ O_{BOW}^\prime = \frac{(\pi a)}{\pi a} \cos \sqrt{2} \phi_c \cos \left( \sqrt{2} \tilde{\phi}_s + \frac{h_s x}{u_s} \right) \]

so that:

\[ \langle j_\perp(j)j_\perp(j') \rangle \sim \frac{1}{2\pi^2 (j-j')^2} \cos \left( \frac{h_s (j-j')}{u_s} \right) + \frac{(-1)^{j-j'}}{|j-j'|}. \]
\[ \langle \sigma^z(x)\sigma^z(x') \rangle \sim \frac{1}{2\pi^2 (j-j')^2} \cos \left( \frac{h_s (j-j')}{u_s} \right) + \frac{(-1)^{j-j'}}{|j-j'|}. \]
\[ \langle O_{BOW}^\prime(j)O_{BOW}^\prime(j') \rangle \sim \frac{(\pi a)}{\pi a} \cos \left( \frac{h_s (j-j')}{u_s} \right) \]

We see that an incommensuration of wavevector \( p(\Omega) = h_s/u_s \) develops in the \( k \approx 0 \) component of the rung current and density wave correlations. Since \( p(\Omega) \sim \Omega^2 \) (see Fig. S2), the incommensuration increases with interchain hopping. The Fourier transform of the \( k \approx 0 \) component behaves as \( |k - p(\Omega)| + |k + p(\Omega)| \), i.e. it is constant for \( |k| < p(\Omega) \) and linear in \( k \) for \( |k| > p(\Omega) \).

![Graph](image-url)

**FIG. S2:** A graph of the incommensuration \( p(\Omega) = h_s/u_s \) at \( \lambda = \pi \) obtained from numerical data (L=64 in PBC). The blue dots are the value of \( S(k = 0) \), while the black dots correspond to the slope discontinuity \( k = p(\Omega) \). The red and green lines are quadratic fits. In the inset we show \( S(k = 0) \), open blue dots, together with the \( K* p(\Omega)/\pi \), black solid dots, with the choice \( K* = 1 \).

Since the boson annihilation operators do not correspond to primary fields of the \( SU(2) \) WZNW model, we cannot directly derive their expression from \( SU(2) \) symmetry. However, since \( e^{i\Omega} / \sqrt{2} \) has conformal dimensions \((1/16, 1/16)\)
its expression in terms of \( \tilde{\theta} \) and \( \tilde{\phi} \) has to be a sum of operators of conformal dimensions \((1/16, 1/16)\). A general expression is:

\[
e^{i \frac{\tilde{\theta}^2}{2}} = A_0 e^{i \frac{\tilde{\theta}^2}{2}} + A_1 e^{i \frac{\tilde{\phi}^2}{2}} + A_2 e^{-i \frac{\tilde{\theta}^2}{2}} + A_3 e^{-i \frac{\tilde{\phi}^2}{2}}.
\]  

(S57)

Moreover, the operator product expansion:

\[
e^{i \frac{\tilde{\phi}(x)}{\sqrt{s}}} e^{-i \frac{\tilde{\phi}(x')}{\sqrt{s}}} = \left| \frac{x - x'}{a} \right|^1 e^{i \sqrt{2} \eta_s(x)},
\]  

(S58)

has to be satisfied, so, for \( \lambda = \pi \), we must have \( \{A_k, A_j\} = \delta_{kj} e^{-i k \frac{\pi}{2}} \) which can be satisfied by writing \( A_k \) as the product of a \( 4 \times 4 \) Dirac matrix \( S \) by a phase \( e^{-i k \frac{\pi}{2}} \). In the case of \( \lambda = \pi \) we obtain the correlator:

\[
\langle e^{i \frac{\tilde{\phi}(x)}{\sqrt{s}}} e^{-i \frac{\tilde{\phi}(x')}{\sqrt{s}}} \rangle = \left( \frac{a}{|x|} \right)^2 \left[ 2 + 2 \cos \frac{h_s}{u_s} (x - x') \right],
\]  

(S59)

where the \( \cos \) results from the correlation of the \( e^{\pm i \tilde{\phi}/\sqrt{s}} \). This implies that the momentum distribution is:

\[
n_\sigma(k) = \int dx e^{i (k + \pi \sigma) x} \langle e^{-i \frac{\tilde{\phi}(x)}{\sqrt{s}}} e^{i \frac{\tilde{\phi}(0)}{\sqrt{s}}} \rangle \langle e^{i \frac{\tilde{\phi}(x)}{\sqrt{s}}} e^{-i \frac{\tilde{\phi}(0)}{\sqrt{s}}} \rangle
\]  

(S60)

\[
= \int dx e^{i (k + \pi \sigma) x} \langle e^{-i \frac{\tilde{\phi}(x)}{\sqrt{s}}} e^{i \frac{\tilde{\phi}(0)}{\sqrt{s}}} \rangle \left( \frac{a}{|x|} \right)^2 \left[ 2 + 2 \cos \frac{h_s}{u_s} \right]
\]  

(S61)

\[
= \nu(k + \pi \sigma) + \frac{1}{2} \nu \left( k + \pi \sigma + \frac{h_s}{u_s} \right) + \frac{1}{2} \nu \left( k + \pi \sigma - \frac{h_s}{u_s} \right),
\]  

(S62)

where:

\[
\nu(k) = \int dx e^{i k x} \langle e^{-i \frac{\tilde{\phi}(x)}{\sqrt{s}}} e^{i \frac{\tilde{\phi}(0)}{\sqrt{s}}} \rangle \left( \frac{a}{|x|} \right)^2 .
\]  

(S63)

Since the width at half maximum of the Lorentzian-shaped graph of the function \( \nu \) scales as \( \Delta_c / u_c \sim \Omega^2 \) and \( h_s / u_s \sim \Omega^2 \), the graph of \( n(k) \) can either comprise 3 peaks or a single broad peak centered in \( \pi \sigma \) depending on the dimensionless ratio \( h_s u_c / (u_s \Delta_c) \).

When \( \lambda < \pi \), we choose a gauge such that:

\[
H = -t \sum_{j, \sigma} \left( b_{j, \sigma} e^{i (\lambda - \pi) \sigma} b_{j+1, \sigma} + H.c. \right)
\]  

(S64)

\[
+ \Omega \sum_{j, \sigma} (-1)^j b_{j, \sigma} b_{j, \sigma} - \Delta, \quad \text{and we define } \delta \lambda = \lambda - \pi .
\]

The mean field Hamiltonian becomes \( H_{MF} = H_{c}^{MF} + H_{s}^{MF} \) with:

\[
H_{s}^{MF} = \frac{2 \pi u_s}{3} \int dx (J_R \cdot J_R + J_L \cdot J_L) + h_s \int dx (J_R^y + J_L^y) + \frac{u_s \delta \lambda}{a} \int dx (J_R^z - J_L^z),
\]  

(S65)

and \( H_{c}^{MF} \) unchanged. We now have to make a different rotation around \( x \) for the right moving and the left moving current:

\[
J_R^y = \sin \phi J_R^y + \cos \phi J_R^z
\]  

(S66)

\[
J_R^z = -\cos \phi J_R^y + \sin \phi J_R^z
\]  

(S67)

\[
J_L^y = -\sin \phi J_L^y + \cos \phi J_L^z
\]  

(S68)

\[
J_L^z = -\cos \phi J_L^y + \sin \phi J_L^z
\]  

(S69)

To find:

\[
H_{s}^{MF} = \int \frac{dx}{2 \pi u_s} \left[ (\pi \tilde{\phi}_s)^2 + (\partial \tilde{\phi}_s)^2 \right] - \frac{h_s(\lambda)}{\pi \sqrt{2}} \int \partial_x \tilde{\phi}_s dx,
\]  

(S70)
where $h_s(\lambda) = \sqrt{h_s^2 + \langle \delta \lambda \rangle^2}$. We still have $\langle J_R^x + J_L^x \rangle = -\frac{h}{2\pi a_x}$, so the mean-field equations remain the same. We also find:

\[
\sin \sqrt{2}\theta_s = \frac{h_s}{h_s(\lambda)} \sin \sqrt{2}\theta_s + \frac{u\delta \lambda / a}{h_s(\lambda)} \cos \left( \sqrt{2}\theta_s + \frac{h_s(\lambda)}{u_s} x \right) \quad \text{(S71)}
\]

\[
\cos \sqrt{2}\theta_s = \sin \left( \sqrt{2}\theta_s + \frac{h_s(\lambda)}{u_s} x \right) \quad \text{(S72)}
\]

\[
\sin \sqrt{2}\phi_s = -\cos \sqrt{2}\theta_s \quad \text{(S73)}
\]

\[
\cos \sqrt{2}\phi_s = \frac{h_s}{h_s(\lambda)} \cos \left( \sqrt{2}\phi_s + \frac{h_s(\lambda)}{u_s} x \right) - \frac{u\delta \lambda / a}{h_s(\lambda)} \sin \sqrt{2}\theta_s \quad \text{(S74)}
\]

The staggered part of the rung current correlations becomes:

\[
\langle j_\perp(x), j_\perp(0) \rangle \sim \left( \frac{-x/a^2 + \left( \frac{u\delta \lambda}{a} \right)^2}{|x|} \frac{h_s^2(\lambda)}{h_s^2(\lambda) + \left( \frac{u\delta \lambda}{a} \right)^2} \cos \left( \frac{h_s(\lambda) x}{u_s} \right) \right), \quad \text{(S75)}
\]

so that the Fourier transform will present peaks at $k = \frac{\pi}{2}$ and $k = \frac{\pi}{2} \pm h_s(\lambda)/u_s$. When $\delta \lambda$ is increased, the two peaks at $k = \frac{\pi}{2} \pm h_s(\lambda)/u_s$ become dominant, and we crossover to the behavior already discussed for weak $\lambda$. In the case of $S(k)$, the peak at $k = \pi$ is not split as $\lambda$ is reduced. If we look at the $BOW^c$ correlations, a peak at $k = \pi$ appears, and becomes the dominant peak when $\delta \lambda$ is increased. Using the rotation [S66] we can also obtain the antisymmetric density correlations as:

\[
\frac{1}{2\pi^2} \partial_x \phi_s(x) \partial_x \phi_s(0) = \frac{1}{2\pi^2 x^2} \left[ \frac{u\delta \lambda}{a} \right]^2 h_s^2(\lambda) + \frac{h_s^2(\lambda)}{h_s^2(\lambda) + \left( \frac{u\delta \lambda}{a} \right)^2} \cos \left( \frac{h_s(\lambda) x}{u_s} \right) \quad \text{(S76)}
\]

When $\delta \lambda$ increases, this expression crosses over to the $1/(2\pi^2 x^2)$ which was obtained at small $\lambda$. $S(k)$ now presents a change of slope at $|k| = h_s(\lambda)/u_s$.

As to the $k \perp 0$ component of the rung current, since it is proportional to $J_R^x + J_L^x$ it becomes $J_R^\perp + J_L^\perp$ under the rotation, and the correlator becomes:

\[
\frac{1}{2\pi^2 (j - j')}^2 \cos \left( \frac{h_s(\lambda)}{u_s} (j - j') \right), \quad \text{(S77)}
\]

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