A study of generalized second law of thermodynamics in modified $f(R)$ Horava-Lifshitz gravity

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(Dated: May 10, 2014)

This work investigates the validity of the generalized second law of thermodynamics in modified $f(R)$ Horava-Lifshitz gravity proposed by Chaichian et al (2010) [Class. Quantum Grav. 27 (2010) 185021], which is invariant under foliation-preserving diffeomorphisms. It has been observed that the equation of state parameter behaves like quintessence ($w > -1$). We study the thermodynamics of the apparent, event and particle horizons in this modified gravity. We observe that under this gravity, the time derivative of total entropy stays at positive level and hence the generalized second law is validated.

PACS numbers: 98.80.-k, 95.36.+x

I. INTRODUCTION

It is suggested in the modified gravity approach that the accelerated expansion of the universe $[1, 2]$ is caused by a modification of gravity at the early/late-time universe. For reviews of the modified gravity the readers are suggested to see $[3]$. Application of Horava-Lifshitz gravity as a cosmological framework has given rise to Horava-Lifshitz cosmology, which has proven to lead to interesting behaviors $[4]$. In a recent work, reference $[5]$ proposed a general modified Horava-Lifshitz gravity that could be easily related to $f(R)$, which is a traditional modified theory of gravity. This was termed as modified $f(R)$ Horava-Lifshitz gravity, that would be abbreviated as MFRHL in the remaining part of the paper. This has been shown in $[5]$ that for a special choice of parameters the MFRHL coincides with the traditional $f(R)$-gravity on the spatially flat Friedman-Robertson-Walker (FRW) background. For a standard $f(R)$ gravity, the action is given

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by

$$S_{f(R)} = \int d^4x \sqrt{-g} f(R)$$

(1)

where, \( f(R) \) is a function of the scalar curvature \( R \). Detailed discussion on the mathematical background of \( f(R) \) gravity is given in the references [6]. In the standard \( f(R) \) gravity, the metric is given by

$$ds^2 = -N^2 dt^2 + g_{ij}^\text{(3)} (dx^i + N^i dt)(dx^j + N^j dt), \quad i = 1, 2, 3$$

(2)

Here \( N \) is called the lapse variable and \( N^i \)'s are the shift variables. Then the scalar curvature \( R \) has the form

$$R = K^{ij} K_{ij} - K^2 + R^{\text{(3)}} + 2 \nabla_i (n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu)$$

(3)

and \( \sqrt{-g} = \sqrt{g^{\text{(3)}}} N \). Here \( R^{\text{(3)}} \) is the three-dimensional scalar curvature defined by the metric \( g^{\text{(3)}} \) and \( K_{ij} \) is the extrinsic curvature defined by

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i (N_j - \nabla_j^{\text{(3)}} N_i)) \quad K = K_j^j$$

(4)

\( n^\mu \) is a unit vector perpendicular to the three-dimensional hypersurface \( \Sigma_t \) defined by \( t = \text{constant} \) and \( \nabla_i^{\text{(3)}} \) expresses the covariant derivative on the hypersurface \( \Sigma_t \). Details of the mathematical background of \( f(R) \) gravity is available in the references [7] and [36].

II. MODIFIED \( f(R) \) HORAVA-LIFSHITZ GRAVITY

An exhaustive review on Horava-Lifshitz (HL) cosmology is available in the reference [14]. There are a number of cosmological implications of HL gravity. These are discussed in the references [15] and [16]. The basic quantities in HL gravity are: lapse \( N(t) \), shift \( N^i(t, \vec{x}) \) and 3D metric \( g_{ij}(t, \vec{x}) \).

All these terms are thoroughly explained in the review entitled “Horava-Lifshitz cosmology: a review” published in Class. Quantum Grav., 27, 223101 (2010) by Mukhoyama [14]. There is another review work by Kiritsis and Kofinas [17] [Nuclear Physics, 821, 467 (2009)], where HL cosmology has been discussed. For the dynamical variables mentioned above, the one can get a four-dimensional metric as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

(5)
Fig. 1 plots $\dot{S}_{\text{total}}$ for $\lambda = 3, \mu = 2$ under MFRHL considering the universe as a thermodynamical system with the apparent horizon surface being its boundary.

Fig. 2 plots $\dot{S}_{\text{total}}$ for $\lambda = 1, \mu = 1$ under MFRHL considering the universe as a thermodynamical system with the apparent horizon surface being its boundary. This choice of $\lambda$ and $\mu$ corresponds to the usual $f(R)$ gravity in spatially flat FRW cosmology.

The lapse $N$ as stated above, is a function of time only. However, the shift $N^i$ and 3D metric $g_{ij}$ depend both on time $t$ and spatial coordinate $\vec{x}$. The condition on lapse is called the projectability condition [14].

Recently an extension of the $f(R)$ gravity to a Horava-Lifshitz-type theory has been proposed by introducing the action [5]

$$S_{FHL} = \int d^4x \sqrt{g^{(3)}}NF(R_{HL}), \quad R_{HL} \equiv K^{ij}K_{ij} - \lambda K^2 - E^{ij}G_{ijkl}E^{kl}$$ (6)
Here $\lambda$ is a real constant in the ‘generalised De Witt metric’ or ‘super-metric’ (‘metric of the space of metric’) [5]:

$$G_{ijkl} = \frac{1}{2}(g^{(3)jk}g^{(3)jl} + g^{(3)il}g^{(3)jk}) - \lambda g^{(3)ij}g^{(3)kl}$$  \hspace{1cm} (7)

defined on three-dimensional hyperspace $\Sigma_t$, $E^{ij}$ can be defined by the so-called detailed balanced condition by using an action $W[g^{(3)}_{kl}]$ on the hypersurface $\Sigma_t$:

$$\sqrt{g^{(3)}}E^{ij} = \frac{\delta W[g^{(3)}_{kl}]}{\delta g_{ij}}$$  \hspace{1cm} (8)

and the inverse of $G^{ijkl}$ is written as

$$G^{ijkl} = \frac{1}{2}(g^{(3)ik}g^{(3)jl} + g^{(3)il}g^{(3)jk}) - \tilde{\lambda}g^{(3)ij}g^{(3)kl}, \quad \tilde{\lambda} = \frac{\lambda}{3\lambda - 1}$$  \hspace{1cm} (9)

In the HL-like $f(R)$ gravity proposed by [7] as stated above, the lapse $N$ was assumed to be a function of $t$ only, which is the projectability condition. Chaichian et al [5] proposed a new very general HL-like $f(R)$ gravity, which is a general approach for the construction of modified gravity which is invariant under foliation-preserving diffeomorphism. For the new form generalized gravity dubbed as “modified $f(R)$ Horava-Lifshitz gravity” (MFRHL), reference [5] proposed the following action

$$S_{f(\tilde{R})} = \int d^4\sqrt{g^{(3)}}Nf(\tilde{R})$$  \hspace{1cm} (10)

where,

$$\tilde{R} = K^{ij}K_{ij} - \lambda K^2 + 2\mu \nabla_\mu \left(n^\nu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\nu\right) - E^{ij}G_{ijkl}E^{kl}$$  \hspace{1cm} (11)

In flat FRW universe, the form of $\tilde{R}$ comes out to be

$$\tilde{R} = \frac{(3 - 9\lambda)H^2}{N^2} + \frac{6\mu}{a^3N} \frac{d}{dt} \left(Ha^3\right) = \frac{(3 - 9\lambda + 18\mu)H^2}{N^2} + \frac{6\mu}{N} \frac{d}{dt} \left(\frac{H}{N}\right)$$  \hspace{1cm} (12)

The $\tilde{R}$ of the above equation reduces to $R$ and consequently the gravity reduces to usual $f(R)$ gravity for $\lambda = \mu = 1$ in a spatially flat FRW universe. For the action in equation (10), we get by variation over $g^{(3)}_{ij}$ and by setting $N = 1$:

$$0 = f(\tilde{R}) - 2(1 - 3\lambda + 3\mu)(\dot{H} + 3H^2)f'(\tilde{R}) - 2(1 - 3\lambda)H \frac{df'(\tilde{R})}{dt} + 2\mu \frac{d^2f'(\tilde{R})}{dt^2} + p$$  \hspace{1cm} (13)
In the above equation, \( f' \) denotes the derivative with respect to its argument. In the above equation, the matter contribution is included by means of pressure \( p \). Considering \( \rho \) as the matter density the conservation equation can be written as

\[
\dot{\rho} + 3H(\rho + p) = 0
\]  

(14)

and subsequently, equation (13) produces

\[
0 = f(\tilde{R}) - 6 \left[ (1 - 3\lambda + 3\mu)H^2 + \mu \dot{H} \right] f'(\tilde{R}) + 6\mu H \frac{df'(\tilde{R})}{dt} - \rho - Ca^{-3}
\]  

(15)

where, \( Ca^{-3} \) is regarded as dark matter. In the present work we have considered \( C \neq 0 \). In this situation we have considered both \( \lambda = \mu = 1 \) as well as \( \lambda \neq 1, \mu \neq 1 \). It should be further noted that we have considered the scale factor \( a \) in the power law form i.e. \( a \propto t^n \) (\( n \) is a positive real number) as used earlier by the references like [18], [19], [20] and [21].

Purpose of the present work is to investigate the validity of the generalized second law of thermodynamics (GSL) in the modified gravity theory proposed by [5], i.e. in the MFRHL. For the field equation of the form (13) we shall investigate whether the total entropy of the universe i.e. sum of the time derivatives of the entropy on the horizon and inside the horizon stays at positive level. As the horizons for the enveloping surfaces we have considered the cases of event horizon, apparent horizon and particle horizons. Earlier [22] explored the thermodynamics of dark energy taking into account the existence of the observers event horizon in accelerated universes and showed that except for the initial stage of Chaplygin gas dominated expansion, the generalized second law of gravitational thermodynamics is fulfilled. Jamil et al [25] investigated the validity of the GSL of thermodynamics in a universe governed by Horava-Lifshitz gravity considering the universe as a thermodynamical system bounded by the apparent horizon. Bamba and Geng [11] investigated the GSL in \( f(R) \) gravity with realizing a crossing of the phantom divide in a universe enveloped by the apparent horizon. Debnath et al [26] investigated the GSL in various scenarios of the universe in the framework of fractional action cosmology for apparent, event and particle horizons. The issues associated with the GSL of thermodynamics are thoroughly discussed in the references mentioned above. However, for convenience we are giving a brief overview of the GSL of thermodynamics in the subsequent section.
III. GENERALIZED SECOND LAW

The connection between gravitation and thermodynamics was examined by following black hole thermodynamics (black hole entropy [8] and temperature [9]) and its application to the cosmological event horizon of de Sitter space [10]. Significant works are available on the study of the generalized second law of thermodynamics in cosmology. The studies include [30], [31], [35], [32], [33] and [34]. The basic necessity for the validity of GSL is that the time derivative of the total entropy

$$\dot{S}_{\text{Total}} = \dot{S}_H + \dot{S} \geq 0,$$

where \(\dot{S}\) indicates the time derivative of normal entropy and \(\dot{S}_H\) indicates the horizon entropy [22]. Thermodynamics under generalized gravity theories has been discussed in [23].

The first law of thermodynamics (Clausius relation) on the horizon is defined as

$$T_X dS_X = \delta Q = -dE_X.$$ 

From the unified first law, we may obtain the first law of thermodynamics as [24]

$$T_X dS_X = 4\pi R_X^3 H (\rho_{\text{eff}} + p_{\text{eff}}) dt \quad (16)$$

where, \(T_X\) and \(R_X\) are the temperature and radius of the horizons under consideration in the equilibrium thermodynamics. The suffix \(X\) will be replaced by \(E\), \(A\) and \(P\) for event, apparent and particle horizons respectively. Subsequently, the time derivative of the entropy on the horizon \((\dot{S}_X)\) and inside the horizon \((\dot{S}_{IX})\) can be derived as [24]

$$\dot{S}_X = \frac{4\pi R_X^3 H}{T_X} (\rho_{\text{eff}} + p_{\text{eff}}) ; \quad \dot{S}_{IX} = \frac{4\pi R_X^2}{T_X} (\rho_{\text{eff}} + p_{\text{eff}}) (\dot{R}_X - H R_X) \quad (17)$$

Finally, we can get the time derivative of total entropy as [27]

$$\dot{S}_{\text{Total}} = \dot{S}_X + \dot{S}_{IX} = \frac{R_X^2}{GT_X} \left( \frac{k}{a^2} - \dot{H} \right) \dot{R}_X \quad (18)$$

Our target is to investigate whether \(\dot{S}_X + \dot{S}_{IX} \geq 0\) holds. The radii of apparent, event and particle and event horizon are given by [26]

$$R_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}} ; \quad R_E = a \int_t^{\infty} \frac{dt}{a} ; \quad R_P = a \int_0^t \frac{dt}{a} \quad (19)$$

In equation (18), we replace \(X\) by \(A\), \(E\) and \(P\) respectively for investigating the GSL in MFRHL. It should be noted that as we are considering MFRHL in flat FRW universe, we shall take \(k = 0\),
Fig. 3 plots $\dot{S}_{\text{total}}$ for $\lambda = 3, \mu = 2$ under MFRHL considering the universe as a thermodynamical system with the event horizon surface being its boundary.

Fig. 4 plots $\dot{S}_{\text{total}}$ for $\lambda = 1, \mu = 1$ under MFRHL considering the universe as a thermodynamical system with the event horizon surface being its boundary. This choice of $\lambda$ and $\mu$ corresponds to the usual $f(R)$ gravity in spatially flat FRW cosmology.

for which apparent horizon reduces to Hubble horizon. To study the GSL of thermodynamics in MFRHL, we first consider $f(\tilde{R})$ as \[ f(\tilde{R}) = \tilde{R} + \xi \tilde{R}^\eta + \zeta \tilde{R}^\nu \] (20)

Here, $\xi, \eta, \zeta$ and $\nu$ are real constants. Using equations (13) and (15) we get the expressions for energy density and pressure. From (14) we get the solution for dark matter as $\rho = \rho_{m0}a^{-3}$ (i.e. $C = \rho_{m0}$ in (15)). Using the expression for $\tilde{R}$ from equation (12) in equation (20) and subsequently using (13), (15) and (18) we get the time derivatives for total entropy for different horizons. For
Fig. 5 plots $\dot{S}_{\text{total}}$ for $\lambda = 3, \mu = 2$ under MFRHL considering the universe as a thermodynamical system with the particle horizon surface being its boundary.

apparent horizon, the time derivative of total entropy becomes

$$
\dot{S}_{\text{total}} = -\frac{2\pi^{1-3n}}{3Gn^3(n-3n\lambda-2\mu+6n\mu)} \times
\left[ 3n^3t^{3n}(-1 + 8\pi G(-1 + 3\lambda))(-1 + 3\lambda - 6\mu) + 16\pi t^{2+3n}\mu(3^n\zeta(nt^{-2}(n - 3n\lambda - 2\mu + 6n\mu))^\nu) \\
\nu(1 - 3\nu + 2\nu^2) + 3^n\eta(1 - 3\eta + 2\eta^2)(nt^{-2}(n - 3n\lambda - 2\mu + 6n\mu))^\eta) \\
2n[3t^{3n}\mu - 4G\pi t^{2+3n}(3^n\zeta(nt^{-2}(n - 3n\lambda - 2\mu + 6n\mu))^\nu\nu(1 - 6\mu(-1 + \nu)- \\
2\nu + \lambda(-3 + 6\nu)) + 3^n\eta(1 - 3\lambda + \eta(-2 + 6\lambda - 6\mu) + 6\mu)(nt^{-2}(n - 3n\lambda - 2\mu + 6n\mu)^\eta) - 12G\pi t^2\mu\rho_{m0})+ \\
3n^2(\nu^{3n}(-1 - 8(1 + 2\pi G)\mu + \lambda(3 + 48\pi G\mu)) + 4\pi G\nu^2(1 - 3\lambda + 6\mu)\rho_{m0})] \\
\right]
$$

(21)

In figure 1, we plot $\dot{S}_{\text{total}}$ against cosmic time $t$ for $\lambda = 3$ and $\mu = 2$. For $\lambda = \mu = 1$ we plot $\dot{S}_{\text{total}}$ in figure 2. This case, as earlier mentioned, corresponds to the usual $f(R)$ gravity. In both of the cases we find the $\dot{S}_{\text{total}}$ to be in positive level. This confirms the validity of GSL of thermodynamics for usual $f(R)$ as well as modified $f(R)$ Horava-Lifshitz gravity proposed by [5], where $R$ is replaced by $\tilde{R}$. To get the $\dot{S}_{\text{total}}$ for event and particle horizons we adopt a little different procedure. For these two horizons we consider the derivatives

$$
\dot{R}_E = H R_E - 1 ; \quad \dot{R}_P = H R_P + 1
$$

(22)

Using (22) in (18) we get the plots for $\dot{S}_{\text{total}}$ corresponding to event and particle horizons. For both of the horizons, the $\dot{S}_{\text{total}}$ is found to stay at positive level irrespective of the values of $\lambda$ and $\mu$. Finally, before concluding we consider the equation of state parameter for the MFRHL.

For $\lambda = 3, \mu = 2$, we plot the equation of state parameter in figure 7, where it is observed that the equation of state parameter lies above $-1$. This indicates that the equation of state parameter
Fig. 6 plots $\dot{S}_{total}$ for $\lambda = 1, \mu = 1$ under MFRHL considering the universe as a thermodynamical system with the particle horizon surface being its boundary. This choice of $\lambda$ and $\mu$ corresponds to the usual $f(R)$ gravity in spatially flat FRW cosmology.

Fig. 7 plots equation of state parameter $w$ for $\lambda = 3, \mu = 2$ under MFRHL. Here $w > -1$ that indicates quintessence like behaviour.

behaves like quintessence. However, in figure 8, (where $\lambda = \mu = 1$) we plot the equation of state parameter is crossing the phantom divide $-1$. It has been already stated that for $\lambda = \mu = 1$ the modified $f(R)$ Horava-Lifshitz gravity reduces to usual $f(R)$ gravity and reference [11] proved that under usual $f(R)$ gravity the phantom divide is crossed.
Fig. 8 plots equation of state parameter $w$ for $\lambda = 1, \mu = 1$ under MFRHL. It crosses the phantom divide $w = -1$ and hence behaves like quintom $^{29}$. This choices of $\lambda$ and $\mu$ correspond to the usual $f(R)$ gravity in spatially flat FRW cosmology.

IV. CONCLUDING REMARKS

We have investigated the generalized second law of thermodynamics in modified $f(R)$ Horava-Lifshitz gravity proposed by $^{3}$. In reference $^{36}$ modified Horava-Lifshitz $f(R)$ gravity was considered for barotropic fluid and it was shown to have a quite rich cosmological structure: early/late-time cosmic acceleration of quintessence, as well as of phantom types. It has been shown in reference $^{5}$ that for $\lambda = \mu = 1$, the modified $f(R)$ Horava-Lifshitz gravity reduces to usual $f(R)$ gravity. We have examined the equation of state parameter $w$ for $\lambda = 3, \mu = 2$ i.e. for modified $f(R)$ Horava-Lifshitz gravity. It is observed that $w$ is staying above $-1$ that indicates quintessence $^{29}$. Thus, one notable difference between $f(R)$ and modified $f(R)$ Horava-Lifshitz gravity is that the second one does not realize the phantom divide, when the scale factor is taken in power law form ($a \propto t^n$). Rather its equation of state behaves like quintessence. The generalized second law of thermodynamics is satisfied not only in $f(R)$ gravity, but also in modified $f(R)$ Horava-Lifshitz gravity. Moreover, the radii of the enveloping horizon does not affect the validity of the generalized second law. Irrespective of the radius of the enveloping horizon, the time derivative of total entropy is staying at positive level and is increasing with the evolution of the universe.

While concluding the paper we would like to state that although we have considered the power-law form of scale factor in this work, other forms like logamediate, intermediate, emergent etc. Scale factor are possible also via using reconstruction scheme $^{37}$. We propose to investigate the
thermodynamic laws in the modified $f(R)$ Horava-Lifshitz gravity for such choices too in future works.

A. Acknowledgment

The first author dedicates this work to the loving memory of his mother Kamala Chattopadhyay, who was the source of all inspirations in his life. The first author sincerely acknowledges the Inter-University Center for Astronomy and Astrophysics (IUCAA), Pune, India, which provided Visiting Associateship to the author for the period of August 2011 to July 2014. The second author wishes to thank IUCAA, Pune for providing all facilities to carry out this research during a scientific visit in January, 2012. Sincere thanks are due to the anonymous reviewer for constructive comments on this work.

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