A TOPOLOGICAL STUDY OF PLANAR VECTOR FIELD SINGULARITIES

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Abstract. In this paper one extends results of Bendixson [1] and Dumortier [2] about the germs of vector fields at the origin of $\mathbb{R}^2$, which is assumed to be an singularity isolated from other singularities and periodic orbits as well. As a new tool, one uses minimal centred curves, which are curves surrounding the origin, with a minimal number of contact points with the vector field. A similar notion was introduced by Le Roux in [4]. It is noticeable that the arguments are essentially topological, with no use of a desingularization theory, as in [2] for instance.

1. Introduction. In this paper one considers a germ $(X,0)$ of a vector field at the origin $0 \in \mathbb{R}^2$. This germ is represented by a vector field $X$, defined on an arbitrarily small neighborhood $W$ of the origin. One will assume that the origin is an isolated singularity of $(X,0)$. This means that one can choose $W$ such that the origin is the unique singularity of $X$ on $W$.

Ivar Bendixson, in his seminal work [1] published in 1903, uses the ideas of Henri Poincaré [1], to give general topological properties of the phase portrait of such a germ, also assumed to be isolated from periodic orbits.

In the Part I of his article Bendixson proves general theorems for a germ of class $C^1$, introducing the notions of nodal regions and pair of orbits crossing the origin (these notions are recalled in Subsection 2.3). It is in this Part I that he states and proves that is known as Theorem of Poincaré-Bendixson, which in modern language may be formulated as follows:

Let $L$ be a half orbit of a $C^1$ vector field defined on an open subset $A$ of the plane. One assumes that the limit set $\Omega(L)$ of $L$ is a non-empty compact subset of $A$. Then $\Omega(L)$ is a singular point, a periodic orbit or a graphic (a closed invariant topological curve made by orbits joining singular points).

In the Part II of his article Bendixson proves general theorems for analytic germs with an algebraically isolated singularity (he says, with components $X,Y$ holomorphic without a common factor). He gives properties of orbits tending toward the origin (recalled in Subsection 3.2), and proves that there is a finite number of nodal regions and a finite number of pairs of orbits crossing the origin. He relates these numbers with the topological index of the vector field at the origin.

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In the present article I shall present some results in the spirit of the above mentioned results of Bendixson. In order to give a precise account of them, I want to introduce a more modern terminology. The following definitions are for germs of vector field at the origin, of class $C^1$.

**Definition 1.1.** A sector at the origin of $\mathbb{R}^2$, is the image by a homeomorphism sending 0 on 0, of an angular sector defined by $\{0 \leq \theta \leq \theta_0 \mid r \leq r_0\}$ in polar coordinates $(r,\theta)$. One assumes that a sector is equipped with a vector field of one of the 3 different types:

1. **Hyperbolic sector**: The sides are two orbits $L_+, L_-$ with $\omega(L_-) = \alpha(L_+) = \{0\}$. There are transverse sections $\sigma$ to $L_-$ and $\tau$ to $L_+$. The orbit $\gamma_p$ through $p \in \sigma$ reaches $\tau$ at a first point $q(p)$. When $p \to L_- \cap \sigma$ we have that $q(p) \to L_+ \cap \tau$ and that the arc of orbit between $p$ and $q(p)$ has a Hausdorff limit contained into $L_+ \cup L_-$. We can define as sector the union of these arcs and of their limit on $L_+ \cup L_-$.  

2. **Elliptic sector**: It is a disk $D(\Gamma)$, singular at the origin, (a pinched disk), bounded by an orbit $\Gamma$ such that $\omega(\Gamma) = \alpha(\Gamma) = \{0\}$. Each orbit $\gamma$ inside $D(\Gamma)$ has the same limit properties and also bounds a pinched disk $D(\gamma)$. Moreover, if $\gamma_1, \gamma_2$ are two such orbits, we have that $\gamma_1 \subset D(\gamma_2)$ or $\gamma_2 \subset D(\gamma_1)$.  

3. **Parabolic sector**: Every orbit has the origin as limit set. It is an attracting parabolic sector if each orbit $\gamma$ tends towards 0 for positive time ($\omega(\gamma) = \{0\}$). It is a repulsing parabolic sector if each orbit $\gamma$ tends towards 0 for negative time ($\alpha(\gamma) = \{0\}$).

The different sectors are represented in Figure 1.

![Figure 1. Sectors](image)

It is easy to find models for each of them: four hyperbolic sectors are found in a hyperbolic linear saddle; two elliptic sectors are found in the differential equation $\dot{x} = x^2 - y^2, \dot{y} = 2xy$, which, putting $z = x + iy$, corresponds to the holomorphic differential equation $\dot{z} = z^2$; two parabolic sectors are found in the previous example and one parabolic sector is found in a linear node (in fact a whole neighborhood of the origin may be considered as a unique generalized parabolic sector, taking $\theta_0 = 2\pi$).

**Definition 1.2.** A topological node (at the origin) is a germ of a $C^1$ vector field represented by a vector field $X$ on a neighborhood $W$ of the origin, such that each
orbit has the origin as $\omega$-set or such that each orbit has the origin as $\alpha$-set. In other words, if $(r(t), \theta(t))$ is the trajectory through any point $m \in W$, written in polar coordinates, we have that $r(t) \to 0 \in \mathbb{R}^2$ when $t \to +\infty$ for every $m \in W$ or that $r(t) \to 0 \in \mathbb{R}^2$ when $t \to -\infty$ for every $m \in W$ (nothing is asked for $\theta(t)$).

**Definition 1.3.** A $C^1$ vector field $X$ admits a sectoral decomposition on a neighborhood $W$ of the origin homeomorphic to a disk if $W$ is a finite union of sectors of the three types defined in Definition 1.1. When one turns around the origin, two successive elliptic or hyperbolic sectors are separated by a unique parabolic sector, which, between two successive hyperbolic sectors, may be reduced to a single orbit. One assumes that there does not exist adjacent parabolic sectors (a pair of adjacent parabolic sectors could be glued into a unique parabolic one). See Figure 2.

![Figure 2. A sectoral decomposition](image)

The principal difference between the present work and the one of Bendixson is that here it is made a systematic use of the contact points along closed curves embedded near the origin and surrounding it. More precisely one considers *minimal centred curves* and *contact indices* $c(X,W), c(X,0)$ associated to them (see Subsection 2.1). Orbits cutting minimal centred curves have remarkable properties given in Subsection 2.2: for instance, an orbit cannot cut a minimal centred curve in more than two points. Using these properties, one derives general results for $C^1$ germs of vector field, rather comparable to the results given in Part I of [B]. A new result (not existing in [1]) is proved in Subsection 2.5: a $C^1$ germ, with the origin as singularity isolated from other singularities and from periodic orbits, with finite contact index $c(X,0)$, is a topological node or admits a sectoral decomposition (Theorem 2.15 in Subsection 2.5).

In Part II of [1], Bendixson has proved results rather similar to the ones stated in Theorem 2.15, under the stronger hypothesis that the germ of vector field is analytic.
(he said holomorphic), without common factor for the components. In fact, these conditions imply that \( c(X,0) < \infty \), that allows to use Theorem 2.15. But in order to have this condition, one does not need the assumption of analyticity, as it is shown in Theorem 3.1.

Theorem 2.15 and Theorem 3.1 are used together in order to prove the main theorem below. In order to state it, we need some well known definitions concerning the orbits tending to the origin for positive or negative time: a characteristic orbit is an orbit tending toward the origin with a limit tangent direction and a spiral is an orbit which tends toward the origin, while turning indefinitely around it. A node is a germ of a vector field where all the orbits outside the origin are characteristic orbits and a focus, when the same orbits are spirals. Clearly, nodes and focuses are topological nodes, as defined above. Precise definitions are recalled in Subsection 3.2.

We can now state the main Theorem:

**Theorem 1.4. (Main Theorem)** For an integer \( k \geq 1 \), let \((X,0)\) be a germ of a \( C^{2k+3} \) vector field at the origin of \( \mathbb{R}^2 \), with a (topologically) isolated singularity at the origin, not accumulated by periodic orbits (i.e., the origin is isolated from periodic orbits). Let also assume that at the origin, the \((k-1)\)-jet of \((X,0)\) is zero but not the \(k\)-jet. Then, the germ \((X,0)\) is a focus, a node or admits a sectoral decomposition, where every orbit tending toward the origin, for positive or negative time, is a characteristic orbit.

Moreover, the number of hyperbolic and elliptic sectors is bounded by \(2(k+1)\). It follows that the topological index \(i(X,0)\) verifies that \(-k \leq i(X,0) \leq k+2\) and that the number of possible phase portraits for \((X,0)\), up time orientation, is roughly bounded by \(2^{4(k+1)}\).

**Remark 1.** 1. The conditions of Theorem 1.4 are fulfilled if \((X,0)\) is \( C^\infty \), with a non-zero infinite jet (one says that \((X,0)\) is not flat at the origin) and such that the origin is isolated from other singularities and periodic orbits. Theorem 1.4 implies that \((X,0)\) is a focus, a node or admits a sectoral decomposition. 

2. A sufficient condition for a germ \((X,0)\) with an isolated singularity at the origin, to be isolated from periodic orbits, is to have at least one characteristic orbit. In this case \((X,0)\) admits a sectoral decomposition (maybe reduced to a node). This condition may be more easy to check than the isolation from periodic orbits.

One can compare this result with the results obtained by Bendixson in Part II of [1]. As mentioned above, Bendixson supposed that the vector field is analytic with no common factor for the two components. But, looking at his proofs, it appears that Bendixson did not use the analyticity, but just that the germ has a non-zero jet and an isolated singularity at the origin, isolated from periodic orbits (exactly as in the statement of Theorem 1.4). Moreover, he did not introduced explicitly the notion of sectoral decomposition. In place of elliptic sector, he used the more general notion of closed nodal region. In a neighborhood of the origin where this point is the unique singularity, a closed nodal region is just a pinched disk \(D(\Gamma)\) bounded an orbit \(\Gamma\) such that \(\omega(\Gamma) = \alpha(\Gamma) = 0\). It is an easy consequence of the Bendixson-Poincaré theory that each other orbit contained into \(D(\Gamma)\) has the same property. Elliptic sectors are example of closed nodal region but the phase portrait inside a general closed nodal region \(D(\Gamma)\) may be much more complicated than in
the case of an elliptic sector. One can introduced a partial order between the orbits in a closed nodal region \(D(\Gamma)\), saying that \(\gamma\) is less than \(\gamma'\) if \(D(\gamma) \subset D(\gamma')\). This order is total precisely when \(D(\Gamma)\) is an elliptic sector. If not, the order is just partial and it may happen that there are infinitely many orbits not comparable. A part of the results of Bendixson may be also found in the Chapter X of Lefschetz’s book [3].

A more recent work about singularities of smooth planar vector fields, is the one of Dumortier [2] where the hypothesis that the singularity is topologically isolated is replaced by a strongest one: one considers a germ \((\text{of Dumortier}[2])\) where the hypothesis that the singularity is algebraically isolated, each of them with characteristic orbits through the origin. This result is stronger than Theorem 1.4 : if the germ \((X,0)\) verifies a Lojasiewicz inequality and has a characteristic orbit, then there is a \(k \in \mathbb{N}\), such that the \(k\)-jet of the germ \((X,0)\) determines its phase portrait. Moreover this phase portrait admits a sectoral decomposition (may be reduced to a node) which is determined by the \(k\)-jet of \((X,0)\). In contrast with the topological proof given in the present paper, the proof of Dumortier is constructive in the sense that the sectoral decomposition can be obtained from the singular points, all elementary, of the desingularized vector field.

As the existence of a characteristic orbit implies that the origin is isolated from periodic orbits and as the origin is isolated if it verifies a Lojasiewicz inequality, we can apply Theorem 1.4 under the conditions of the Dumortier’s result. Moreover, if \(z_k\) is any non-zero \(k\)-jet of vector field, there is always a smooth germ \((X,0)\) with the origin as algebraically isolated singularity and such that \(j_k X(0) = z_k\), and there are infinitely many such germs with 2 by 2 distinct jets of arbitrarily high degree. On the other side Theorem 1.4 says that there exists a finite collection \(C_k\) of sectoral decompositions, depending just on the number \(k\), such that if \((X,0)\) has a characteristic through the origin and a non-zero \(k\)-jet, then its phase portrait belongs to \(C_k\). As an example, let us consider the 1-jet \(z_1 = y \frac{\partial}{\partial y}\). The infinity of polynomial vector fields \(X_{l,\pm} = y \frac{\partial}{\partial y} \pm x l \frac{\partial}{\partial x}\), with \(l \geq 2\), have \(z_1\) as 1-jet and are algebraically isolated, each of them with characteristic orbits through the origin. They have just one of the three possible sectoral decompositions: a single node, two hyperbolic and one parabolic sectors or four hyperbolic sectors. This result is even true for a vector field as \(X_\infty = y \frac{\partial}{\partial y} + f(x) \frac{\partial}{\partial x}\), where \(f\) is a smooth germ with an isolated singularity at 0 and an infinite jet equal to zero, for instance \(f(x) = e^{-1/x^2}\). For such a vector field, the origin does not verify a Lojasiewicz inequality.
2. Phase portrait for a $C^1$ vector field near an isolated singularity. In this section we consider a germ of vector field at $0 \in \mathbb{R}^2$, of class $C^1$, such that the origin is an isolated singularity. We choose a representative for this germ: a $C^1$ vector field $X$ given on an arbitrary small neighborhood $W_0$ of $0 \in \mathbb{R}^2$, diffeomorphic to a Euclidean disk. We assume that $0$ is an isolated singularity of $X$ on $W_0$.

2.1. Minimal centred curves and contact indices.

**Definition 2.1.** In the text, a centred disk $W$ is an open neighborhood of the origin, interior of a $C^1$ closed disk $\overline{W}$ contained into $W_0$; this means that the boundary $\partial \overline{W} = \overline{W} \setminus W$ is a $C^1$ embedding of $S^1$ into $W_0 \setminus \{0\}$ (Euclidean disk centred at the origin and contained into $W_0$ are centred disks, but one will need more general centred disks). A centred curve is a $C^1$ embedding of $S^1$ into $W_0 \setminus \{0\}$, surrounding the origin (i.e. generating the 1-homology of $W_0 \setminus \{0\}$).

If $\gamma$ is a centred curve, there exists a centred disk $W = W(\gamma)$ such that $\gamma = \partial \overline{W}$. Inversely, if $W$ is a centred disk, the boundary $\partial \overline{W}$ of the closure of $W$ is a centred curve.

Let $\gamma$ be a centred curve. A contact point on $\gamma$ is a point where $X$ is tangent at $\gamma$. If $\gamma(s) : S^1 \to W_0$ is a $C^1$ parametrization of $\gamma$, the contact points on $\gamma$ correspond to the zeros of the continuous function $\begin{pmatrix} n(s) : s \to < X(\gamma(s)), N(\gamma(s)) > \end{pmatrix}$, where $< \cdot, \cdot >$ is the Euclidean scalar product and $N$ is a $C^0$ vector field along the curve $\gamma$, orthogonal to it.

**Definition 2.2.** An isolated contact point $m$ on a centred curve $\gamma$ is said even if the function $n(s)$ changes of sign at $m$ and odd if not. The trajectory through an even contact point is locally on one side of $\gamma$ (at an odd contact point the orbit is topologically transverse to $\gamma$). One says that a centred curve if generic if all its contact points are even. For a generic centred curve, these points are necessarily isolated and then are in finite number. One will call contact index $c(\gamma) \in \mathbb{N}$, the (finite) number of (even) contact points along a generic centred curve $\gamma$.

**Remark 2.** The notions of contact points, even or odd contact points, generic centred curve and then of of contact index are independent of the choice of the orthogonal vector field $N$. One can also notice that for smooth vector fields, the contact points of finite order with a smooth curve are even or odd if their order of contact are respectively even or odd.
curves, for the $C^0$ topology. For instance an odd point can be eliminated by a $C^0$ perturbation of the curve, located in its neighborhood. In particular there exist generic centred curves on any centred disk.

As the orbit at an even contact point is locally on one side of the centred curve, we can distinguish two types of even contact points on a generic centred curve:

**Definition 2.3.** Let $\gamma$ be a generic centred curve (there are just a finite number of contact points, all even). A contact point $m$ is said to be an interior contact point if locally the orbit of $m$ is inside $\bar{W}(\gamma)$. If not, the contact point is said to be exterior. Let $c_i(\gamma)$ be the number of interior contact points and $c_e(\gamma)$ be the number of exterior contact points. One has that $c(\gamma) = c_i(\gamma) + c_e(\gamma)$. The interlinking of the set of exterior contact points with the set of interior contact points along the curve $\gamma$ traveled in the direct sense, will be called the arrangement of contact points on $\gamma$. It may be represented, up to a cyclic permutation, by a list of $c(\gamma)$ symbols taken in the set $\{i,e\}$, with $c_i(\gamma)$ symbols $i$ and $c_e(\gamma)$ symbols $e$ (in Figure 2, the disk boundary $\gamma$ is a generic centred curve, with $c_i(\gamma) = 2$, $c_e(\gamma) = 2$ and the arrangement $(i,i,e,e)$).

This distinction between interior and exterior contact points will be crucial for the topological analysis of the phase portrait presented in this paper. For this moment, I just want to indicate the relation with the topological index of the germ $(X,0)$. I recall that if $(X,0)$ is a $C^1$-germ of a vector field with the origin as isolated singularity, one can define the topological index $i(X,0)$ in the following way. One chooses a representative $X$ of the germ on a neighborhood $W_0$, where $0$ is the unique singularity. Next, one considers any centred curve $\gamma \subset W_0$, and take a parametrization $\gamma(s)$ with the direct orientation. Then the topological index $i(X,0) \in \mathbb{Z}$ is the degree of the continuous map

$$s \in S^1 \rightarrow \frac{X}{||X||}(\gamma(s)) \in S^1.$$ 

Now, if $\gamma$ is a generic curve, it is easy to see that:

$$i(X,0) = \frac{1}{2}(c_i(\gamma) - c_e(\gamma)) + 1 \quad (1)$$

As $c(\gamma) = c_i(\gamma) + c_e(\gamma)$ is even, one has that $c_i(\gamma) - c_e(\gamma)$ is also even and then the formula (1) defines, as expected, an element of $\mathbb{Z}$.

One can now introduce the following contact index for the centred disks:

**Definition 2.4.** Let $W$ be a centred disk. One call contact index of $X$ in $W$, the minimum of the contact index $c(\gamma)$ among the generic curves $\gamma \subset W$ (as there exist generic centred curves in $W$, this index is finite and may even be equal to 0). One will write it : $c(X,W)$. A generic centred curve $\gamma$ in $W$, such that $c(\gamma) = c(X,W)$ will be called minimal centred curve in $W$ (such a minimal centred curve always exists!).

**Remark 3.** The concept of minimal centred curve has already been used by Le Roux in the appendix B of [4], where the contact index $c(X,W)$ is called “U-module”.

By definition, the number $c(\gamma) = c_i(\gamma) + c_e(\gamma)$ has the same value $c(X,W)$ for all the minimal centred curves in $W$. Using this remark and (1) we obtain the following:
Lemma 2.5. Let $\gamma$ be a minimal centred curve in a smooth disk $W$. Then,
\[
c_i(\gamma) = \frac{c(X,W)}{2} + i(X, 0) - 1 \quad \text{and} \quad c_e(\gamma) = \frac{c(X,W)}{2} - i(X, 0) + 1.
\] (2)

The expressions (2) show that the numbers $c_i(\gamma), c_e(\gamma)$ are the same for any choice of minimal centred curve $\gamma$ in $W$. We will see later that it is also true for the arrangement of contact points along a minimal centred curve $\gamma$.

Notation. Let $W$ be any centred disk (in $W_0$). As we have written $c(X,W)$ the number of contact points of any minimal centred curve in $W$, Lemma 2.5 allows to introduce also numbers $c_i(X,W)$ and $c_e(X,W)$ for the number of interior contact points and exterior contact points on any minimal centred curve in $W$.

If $W_1 \subset W_2$ are two centred disks as in Definition 2.4, we have that $c(X,W_1) \geq c(X,W_2)$. It follows that, if $(W_n)$, with $n \in \mathbb{N}$, is a nested sequence of centred disks in $W_0$, i.e. such that $W_{n+1} \subset W_n$ for all $n \in \mathbb{N}$, the sequence of numbers $(c(X,W_n))$ is increasing and has a limit in $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Call $\text{diam}(W)$ be the diameter of a centred disc $W$, for the Euclidean distance. If $(\text{diam}(W_n)) \to 0$ when $n \to \infty$, the limit of the sequence $(c(X,W_n))$ is clearly independent of the choice of the nested sequence of centred disks $(W_n)$. This allows the following definition:

Definition 2.6. The contact index $c(X,0)$ of the germ $(X,0)$ is the limit in $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ of the increasing sequence of numbers $(c(X,W_n))$ for any choice of a nested sequence $(W_n)$ of centred disks, whose sequence of diameters tends to zero. This index is independent of the choice of the sequence $(W_n)$ and depends just of the vector field germ $(X,0)$.

It is easy to find germs $(X,0)$ (even smooth) such that $c(X,0) = \infty$. But this property is exceptional for smooth germs (in a sense made precise in Section 3). For this reason, one will be more interested in germs $(X,0)$ with a finite contact index. The following proposition gives a characterization of this finiteness property:

Proposition 1. If $c(X,0) \in \mathbb{N}$ (i.e. is finite) one can choose a representative $X$ of the germ $(X,0)$ on an open Euclidean disk $W_0$ centred at the origin, such that $c(X,W_0) = c(X,0)$, and a sequence $(W_n)$ of centred disks with the following properties:

(a) $(W_n)$ is a nested sequence and $(\text{diam}(W_n)) \to 0$ for $n \to \infty$.
(b) For each $n$, the boundary $\gamma_n = \partial W_n$ is a minimal centred curve in $W_0$, i.e. $c(\gamma_n) = c(X,0)$.

Conversely, given any germ $(X,0)$, if there exists a nested sequence of centred disks $(W_n)$, with $(\text{diam}(W_n)) \to 0$ for $n \to \infty$, and such that each boundary $\gamma_n = \partial W_n$ is a minimal centred curve in $W_0$, we have that $c(X,0) = c(X,W_0)$ and then $c(X,0)$ is finite.

Proof. (1) One assumes that $c(X,0)$ is finite and choose a representative vector field $X$ on a open Euclidean disk $W$, centred at the origin (which is assumed to be the unique singularity of $X$ in $W$). Clearly, any finite subset in $\mathbb{N}$ contains its upper bound. As a consequence, there exists a open Euclidean disk $W_0 \subset W$, centred at the origin, such that $c(X,W_0) = c(X,0)$. Now, if $W$ is any centred disk contained into $W_0$, by definition of the contact index $c(X,0)$, we have also that that $c(X,W) =$
Assume that there are a finite constant $C$ and such that $(\gamma_{n+1}) = c(X,W_n) = c(X,0)$ (see above), the curve $\gamma_{n+1}$ is the boundary of a centred disk $W_{n+1} \subset W_n$, such that $\text{diam}(W_{n+1}) \leq \frac{1}{n+1}$.

(2) Conversely, let $(W_n)$ be a sequence of centred disks as in the statement. As the curve $\gamma_{n+1}$, boundary of $W_{n+1}$ is assumed to be a minimal centred curve in $W_0$, it is also a minimal centred curves in $W_n$.

Then, the sequence $c(X,W_n)$ is stationary at the value $c(X,W_0)$ and, as $(\text{diam}(W_n)) \to 0$ for $n \to \infty$, the proper definition of the index $c(X,0)$ implies directly that $c(X,0) = c(X,W_0)$.

Passing to subsequences of centred curves, it is easy to adapt the proof of Proposition 1 in order to obtain the following more practical sufficient condition for the finiteness of $c(X,0)$:

**Proposition 2.** Assume that there are a finite constant $C > 0$ and a sequence of generic centred curves $(\gamma_n)$, converging in the Hausdorff sense toward the origin and such that $c(\gamma_n) \leq C$ for any $n$. Then $c(X,0) \leq C$.

**Definition 2.7.** When $c(X,0)$ is finite, a generic centred curve $\gamma$ such that $c(\gamma) = c(X,0)$ (as in Proposition 1) will be called: absolute minimal centred curve. A minimal centred curve in a centred disk $W$ is absolute as soon as the diameter of $W$ is small enough. Apart from this condition, absolute minimal centred curves do not depend on the choice of a disk $W$, this is why we call them absolute.

It follows from Proposition 1 that the number of interior and exterior contact points is the same for any absolute minimal centred curve of a germ with a finite contact index. This allows to introduce the following:

**Notation.** Let $(X,0)$ a vector field germ such that $c(X,0)$ is finite. There exist integers $c_i(X,0)$ and $c_e(X,0)$ equal respectively to the number of interior contact points and exterior contact points of any absolute minimal centred curve.

**Remark 4.** As $c(\gamma)$ is an even number for any generic curve, so are the contact indices $c(X,W)$ and $c(X,0)$.

### 2.2. Relations between phase portrait and minimal centred curves.

In this subsection one will consider a generic centred curve $\gamma$, minimal in some centred disk $W \subset W_0$. One will prove some properties for the intersection of any orbit $\Gamma$ (inside $W$) with $\gamma$. In order to prove these properties one will use a very classical construction of curves that is recalled in the following lemma:

**Lemma 2.8.** Let $\Gamma$ be an orbit of a 2-dimensional $C^1$ vector field $X$. One assumes that $\Gamma$ is neither periodic nor reduced to a singular point and that there exists a transverse section $\Sigma$ such that $\Gamma \cap \Sigma = \{p, q\}$, with $p \neq q$. Let $[p, q]_{\Gamma}$, respectively $[p, q]_{\Sigma}$, be the arc of $\Gamma$, respectively the arc of $\Sigma$ between the points $p$ and $q$. Then, one can $C^0$-approximate the $C^1$-piecewise curve $\gamma_0 = [p, q]_{\Gamma} \cup [p, q]_{\Sigma}$ by a $C^1$-curve $\gamma$ transverse to $X$.

**Proof.** One chooses a flow box $T, C^1$-diffeomorphic to a rectangle, having $[p, q]_{\Gamma}$ as one $X$-tangent side and $[p, p'], [q, q']$, disjoints subintervals of $\Sigma$, as transverse sides, with $q'$ between $q$ and $p$ on $\Sigma$. See Figure 4.
Let be $\sigma$ the linear diagonal segment joining $p$ and $q'$ in $T$. This segment is transverse to $X$. It is easy to smooth $\gamma_1 = \sigma \cup [q', p]_\Sigma$ (where $[q', p]_\Sigma \subset \Sigma$) into a $C^1$-curve $\gamma$ transverse to $X$. We can construct $\gamma_1$, $C^0$ close to $\gamma_0$ and $\gamma$, $C^1$ close to $\gamma_1$, so that $\gamma$ is $C^0$ close to $\gamma_0$.

A first result for minimal centred curves is the following:

**Proposition 3.** Let $\gamma$ be a minimal centred curve of a centred disk $W \subset W_0$. Assume that an orbit $\Gamma$ in $W$ has a contact point $p$ with $\gamma$. Then, $\gamma \cap \Gamma = \{p\}$.

**Proof.** The proof for an exterior contact point is similar to the proof for interior contact point. Then, from now on, one will assume that $p$ is an interior contact point. One write $\varphi(t, m)$, for $m \in W$, be the trajectory of $X$ in $W$. Then $\Gamma$ is the image in $W$ of the map $t \to \varphi(t, p)$. In contradiction with the statement, one assumes that there exists a time $t_0 \neq 0$, such that $q = \varphi(t_0, p) \in \gamma$, and that $q \neq p$. One can assume that the point $q$ is a first intersection of $\Gamma$ with $\gamma$, i.e. that $\varphi([0, t_0], p) \cap \gamma = \emptyset$. Moreover, by taking a $C^1$-perturbation of $\gamma$ near $q$, one can also assume that $\gamma$ remains minimal and is such that the intersection of $\gamma$ with $\Gamma$ is transverse at $q$.

Let $[p, q]_\Gamma = \varphi([0, t_0], p)$ be the arc of orbit between $p$ and $q$. One notice that $[p, q]_\Gamma \subset \bar{W}(\gamma)$. One and only one of the two arcs on $\gamma$ with end points $p$ and $q$, arc that from now on is denoted $[p, q]_\gamma$, is such that $[p, q]_\gamma \cup [p, q]_\Gamma$ bounds a topological disk $B \subset \bar{W}(\gamma)$, which does not contain the origin. One will denote by $[p, q]_\gamma^C$ the complementary arc on $\gamma$. As the arcs $[p, q]_\gamma$ and $[p, q]_\Gamma$ are homotopic in $W \setminus \{0\}$, relatively to their end points, the $C^1$-piecewise curve $\gamma_1 = [p, q]_\gamma^C \cup [p, q]_\Gamma$ is also surrounding the origin (and is contained into $\bar{W}(\gamma)$)

Let $\Gamma_+$ be the half orbit from $p$ which contains the arc $[p, q]_\Gamma$, and $\Gamma_-$ the other half orbit. One will distinguish two cases : the good case when $\Gamma_+$ starts outside $B$ and the bad case when $\Gamma_-$ starts inside $B$. See Figure 5

**(a) One first considers the good case.** The idea is that one can $C^0$-approximate the curve $\gamma_1$ by a $C^1$ generic centred curve $\gamma_2$ contained in $W$, with at least two contact points less than $\gamma$. But this fact will be in contradiction with the minimality of $\gamma$, proving that this good case is impossible.

One constructs now the curve $\gamma_2$. Taking $\varepsilon > 0$ arbitrarily small, one considers a $\varepsilon$-thick $C^1$ flow box $T$, containing the orbit arc $[p, q]_\Gamma$. This flow box is a thin rectangle $[a, b, c, d]$ where the long sides $[ac]$ and $[bd]$ are tangent to the flow and the arcs of orbit inside $T$ are the segments parallel to these sides. The flow is transverse
to the short segments of length $\varepsilon$, parallel to the short sides $[ab]$ and $[cd]$. As $\gamma$ is contained in the interior of $W$, one can choose $\varepsilon$ small enough such that $T \subset W$.

One supposes that the arc $[p, q]_T$ is contained in the arc of orbit in the middle of $T$, with $p$ in the interior of $T$, at a distance of order $\varepsilon$ of the side $[ab]$ and $q$ on $[cd]$. One chooses a point $p'$ in $[p, q]^C$, outside $T$ and such that the flow is transverse to $\gamma$ along the half-closed arc $[p', p] \subset \gamma$. See Figure 6.

One can now construct $\gamma_2$ as an approximation of the $C^1$-piecewise curve $\tilde{\gamma}_2$, union of 3 arcs: the sub-arc $I$ of $[p, q]^C$ between $p'$ and $d$; a small arc $J$ joining $p'$ and $a$, transverse to $X$ and $\varepsilon$-near $\gamma$; an arc $K$ inside $T$, going in diagonal from $a$ to $d$, transverse to the flow in $T$. By choosing $\varepsilon$ small enough, one has that $\gamma_2$ is $C^0$ close to $\gamma_1$, and then is contained into $W$. Finally, one can $C^0$-approximate $\gamma_2$ by a $C^1$ curve $\gamma_2$, also contained into $W$.

If $\varepsilon$ is small enough, the curve $\gamma_2$ is surrounding the origin, as the curve $\gamma_1$. There is no contact point on the $J \cup K$ and the contact points on $I$ are the contact points of $\gamma$ which belong to $[p, q]^C \setminus \{p\}$. As $\gamma$ has at least 2 contact points along the arc $[p, q]_\gamma$ (the point $p$ and necessarily another contact point between $p$ and $q$), it follows that $\gamma_2$ is a generic centred curve in $W$ with at least 2 contact points less than $\gamma$, which is impossible by the minimality of $\gamma$. 

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**Figure 5.**

**Figure 6.** Elimination of contact points
(b) One considers now the bad case. Let us consider the half orbit $\Gamma_-$ starting inside $B$. One has the following alternative : $\Gamma_-$ is contained entirely in $B$ or $\Gamma_-$ leaves $B$ at a first point $q'$ in the interior of $[p,q]_\gamma$. One will consider the two terms of this alternative.

1. One first supposes that $\Gamma_- \subset B$. As $\Gamma_- \setminus \{p\}$ does not intersect $\partial B$, it has a limit-set in the interior of $B$. By the Poincaré-Bendixson Theorem, this limit-set must contain a singular point or is a periodic orbit, which must bound a disk in $B$ containing also a singular point. As $B$ is free of singular point, none of these possibilities can occur.

2. One now assumes that $\Gamma_-$ leaves $B$ at a first point $q'$ in the interior of $[p,q]_\gamma$. As above, one can assume (after a slight perturbation of $\gamma$), that this point is a transverse intersection of $\gamma$ with $\Gamma_-$. As the sub-arc between $p$ and $q'$ inside $[p,q]_\gamma$ bounds in union with $[p,q]'_\gamma$ a disk $B'$ contained into $B$, this arc is the arc $[p,q]'_\gamma$ as defined above. But, as $\Gamma_+$ is outside outside $B'$, the pair $(p,q')$ is of the good type. The impossibility follows from the part (a) above.

It may happen that the contact property $\gamma \cap \Gamma = \{p\}$ which is proved in Proposition 3 corresponds to a contact between a minimal centred curve $\gamma$ and a periodic orbit $\Gamma$ contained into $W$. This will imply a strong constraint on the contact index, as it appears in the following result :

**Lemma 2.9.** Suppose that a centred disk $W \subset W_0$ contains a periodic orbit. Then, $c(X,W) \leq 2$.

**Proof.** Let $\Gamma$ be a periodic orbit in the interior of $W$. As the origin is the unique singular point in $W$, this orbit must surround it. Choose a $C^1$ local transverse section $\Sigma$ to $\Gamma$ and consider the $C^1$ return map $P(u)$ on it ($u$ is a $C^1$ coordinate on $\Sigma$ such that $\Gamma \cap \Sigma = \{u = 0\}$). One has two possibilities :

1. If $P(u) \equiv u$ in a neighborhood of $u = 0$, one can find a generic curve $\gamma'$ which is $C^0$-near $\gamma$, with just 2 contact points. One can see this in the following way. One chooses a $C^1$ diffeomorphism $\psi(s, \rho)$ of the annulus $A_3 = \{(s, \rho) \in S^1 \times [-\delta, \delta] \}$ onto tubular neighborhood of $\Gamma$, sending each orbit of the smooth vector field $X_0 = \frac{\partial}{\partial s}$ on an orbit of $X$. For any $\epsilon$ such that $0 < \epsilon < \delta$, the curve of $A_3$ given by $\rho = \epsilon \sin s$ has two quadratic contact points with $X_0$. Its image by $\psi$ is a $C^1$ generic centred curve (as $\Gamma$ surrounds $0$), with two even contact points with $X$. This implies that $c(X,W) \leq 2$.

2. If $P(u) \neq u$ in a neighborhood of $u = 0$, one can find a $u_0$ arbitrarily near $0$ such that $P(u_0) \neq u_0$. In this case, using Lemma 2.8, one can $C^0$ approximate the $C^1$-piecewise curve $\gamma_1$, union of the interval of $\Sigma$ between $u_0$ and $P(u_0)$, and the arc or orbit between the same points, by a $C^1$ curve $\gamma_2$ transverse to $X$. As $\gamma_2$ can be constructed $C^0$ near $\Gamma$, it can be chosen to be a generic centred curve contained into $W$ (again, as $\Gamma$ surrounds the origin). This implies that $c(X,W) = 0$.

**Remark 5.** A direct consequence of Lemma 2.9 is that if the origin is accumulated by periodic orbits of $(X,0)$, then $c(X,0) \leq 2$.

In order to state the next result, one needs some definitions.

**Definition 2.10.** Let $\gamma$ be a generic centred curve in a centred disk $W \subset W_0$. An arch attached to $\gamma$ in $W$ is an arc of orbit $A = [p,q]'_\gamma \subset W$ cutting transversally $\gamma$ into two consecutive points $p,q$ on some orbit in $W$ (this means that $[p,q]'_\gamma \cap \gamma =$
Let \([p, q]_\gamma\), called the base of the arch, the single arc of \(\gamma\) with end points \(p, q\), such that \([p, q]_\gamma \cup [p, q]_\gamma\) is the boundary of a topological disk \(D_A\) which does not contain the origin. This disk will be called the arch-disk of \(A\).

The arch is interior if it is contained into \(\bar{W}(\gamma)\) and exterior if not. One will say that the arch is minimal if its base contains a unique contact point \(m_A\), with a contact exterior at \(D_A\) (i.e. an interior contact of \(\gamma\) for an exterior arch, and an exterior contact of \(\gamma\) for an interior arch) and if \(D_A \setminus \{m_A\}\) is filled by orbits joining any point between \(p\) and \(m_A\) to a point between \(m_A\) and \(q\), on the base \([p, q]_\gamma\).

**Figure 7.** General arch

**Figure 7.** Minimal arch

**Remark 6.** The foliation by the orbits in a minimal arch is \(C^1\) conjugate to the foliation by the intersections of parabolas \(y = -x^2 + \alpha\), for \(\alpha \in [0, 1]\) with the topological disk \(D = \{0 \leq y \leq 1 - x^2\} \cup \{1 \leq x \leq y\} \subset \mathbb{R}^2\), in coordinates \((x, y)\).

**Proposition 4.** Let \(\gamma\) be a minimal centred curve in a centred disk \(W\). Then, any arch attached to \(\gamma\) in \(W\) is a minimal arch.

**Proof.** One can restrict the study to interior arches, the proof for exterior arches being quite similar. Then, let \(A = [p, q]_\Gamma \subset W(\gamma)\) be an interior arch attached to \(\gamma\), with \([p, q]_\gamma\) and \(D_A\) as in Definition 2.10.

First, one can notice that there is no interior contact \(m\) to \(\gamma\) (also interior contact to \(D_A\)) on \([p, q]_\gamma\). On contrary, assume that there exists an interior contact point \(m\). One considers one half orbit \(\Gamma_+\) starting at \(m\). This half orbit cannot have a limit-set in \(D_A\), as this disk contains no singular point of \(X\) (see the proof of the case (b-1) in the proof of Proposition 3) Then, \(\Gamma_+\) must intersect \([p, q]_\gamma\) at a point \(m' \neq m\). But, as \(\gamma\) is minimal in \(W\), this is forbidden by Proposition 3.

Next, one observes that there exists one and only one exterior contact point \(m_A\) to \(\gamma\) on \([p, q]_\gamma\). In fact, if there exist \(c_e\) exterior contact points and no interior contact points on \([p, q]_\gamma\), it is easy to see that the topological index of \(X\) along \(\partial D_A\) is equal to \(i(X, \partial D_A) = \frac{1 - c_e}{2}\) (for a reason of orientability, \(c_e\) must be odd). As there is no singular point of \(X\) inside \(D_A\), this index is zero, as a consequence of the Hopf-Poincaré formula, and it follows that \(c_e = 1\).

Again, as \(D_A\) does not contain singular points of \(X\), it is a direct consequence of the Poincaré-Bendixson Theorem that each orbit in \(D_A \setminus \{m_A\}\) links on \([p, q]_\gamma\), a
point on the left of \( m_A \) to a point on the right of \( m_A \). This means that the arch \( A \) is a minimal one.

**Proposition 5.** Let \( \gamma \) be a minimal centred curve in a centred disk \( W \). Then, no orbit of \( X \) in \( W \) can cut \( \gamma \) into more than two distinct points.

**Proof.** If \( c(\gamma) = 0 \), i.e., if \( X \) is transverse to \( \gamma \), then each orbit \( \Gamma \) can cut \( \gamma \) at most one time. Then, the result is trivial in this case.

From now one will assume that \( c(\gamma) = c(X, W) > 0 \). One supposes that an orbit \( \Gamma \) cuts \( \gamma \) in three distinct points \( p, q, r \). One can also assume that these points are consecutive on \( \Gamma \), and by a slight \( C^1 \) perturbation of \( \gamma \), one can make that the intersection of \( \Gamma \) with \( \gamma \) is transverse at each of these points.

One considers the two arches \( A = [p, q]_{\Gamma} \) and \( B = [q, r]_{\Gamma} \). One will assume that \( A \) is exterior at \( \gamma \) and then, that \( B \) is interior (the other case is completely similar). By Proposition 4, there exists a single contact point on \( [p, q]_{\Gamma} \), which is an interior contact point \( m_1 \) and a single contact point on \( [q, r]_{\Gamma} \), which is an exterior contact point \( m_2 \). Taking into account the different possible arrangements for the bases, it appears that there are four possible cases which are shown in Figure 8, where one has indicated the different possible positions of the disks \( D_A \) and \( D_B \). One considers now these different cases.

![Figure 8](image-url)

**Figure 8.**

(1) **In the case 1,** one has that \( [pq]_{\gamma} \cap [q, r]_{\gamma} = \{q\} \). This means that \( [p, r]_{\gamma} = [pq]_{\gamma} \cup [q, r]_{\gamma} \) is a sub-arc of \( \gamma \). Let \( [p, r]_{\gamma}^C \) be the complementary sub-arc. One considers the topological curve \( \gamma_1 = [p, r]_{\gamma}^C \cup [p, q]_{\Gamma} \cup [q, r]_{\Gamma} \) contained in \( W \). This curve is homotopic to \( \gamma \) and then surrounds the origin. In similar way as in the proof of Proposition 3 for the good case, one can modify \( \gamma_1 \) in order to obtain a generic centred curve \( \gamma_2 \), which is \( C^0 \) near \( \gamma_1 \) (and then it belongs to \( W \)), and has two contact points less than \( \gamma \) (the points \( m_1 \) and \( m_2 \)). This contradicts the minimality of \( \gamma \) in \( W \).
(2) In the case 2, one has that \([p, q]_\gamma \subset [q, r]_\gamma\). But, as \(m_1 \neq m_2\) (one point is an exterior contact and the other one is interior contact), the base \([q, r]_\gamma\) would contain two distinct contact points. This is impossible by Proposition 4.

(3) In the case 3, one has that \([q, r]_\gamma \subset [p, q]_\gamma\). Then, the same proof as in case 2 shows that this case is impossible.

(4) In the case 4, at the point \(r\) which belongs to \([p, q]_\gamma\), the field \(X\) has the same transverse orientation as at the point \(p\). Then, the point \(r\) is between \(p\) and \(m_1\) on the base \([p, q]_\gamma\), and the field \(X\) is transverse to \(\gamma\) all along the arc \([p, r]_\gamma\). As in Lemma 2.8, we can \(C^0\)-approximate the piecewise \(C^1\)-curve \(\gamma_1 = [p, r]_\gamma \cup [p, r]_\gamma\), which is in the interior of \(W\) and surrounds the origin by a \(C^1\)-curve \(\gamma_2\) with the same properties and which is transverse to \(X\). But the existence of such a curve means that \(c(X, W) = 0\), in contradiction with the assumption that \(c(X, W) > 0\) made in this part of the proof.

2.3. Some topological results. As a consequence of Propositions 4 and 5 one wants to deduce now some global properties for a germ \((X, 0)\) of a \(C^1\) vector field with an isolated singularity at the origin. One will also assume that the origin is not accumulated by periodic orbits. But, one does not assume that \(c(X, 0)\) is finite.

One can represent the germ by a \(C^1\) vector field \(X\) on an open Euclidean disk \(W_0\), which contains no singular point other than the origin nor periodic orbit. Moreover one assumes that \(c(X, W_0) > 0\), but this contact index may be infinite: if a germ \((X, 0)\) is isolated from periodic orbits and such that \(c(X, 0) = 0\), one will see in subsection 2.5 that \((X, 0)\) is a topological node.

Let \(\gamma\) be a minimal centred generic curve in \(W_0\). As it is assumed that \(c(X, W_0) > 0\), this curve contains at least two contact points. One wants to give properties of the orbits of \(X\) on the disk \(\bar{W}(\gamma)\). These results are comparable (up the formulation) to the results obtained by Bendixson in the Part I of [1], but here the proofs are based on the notion of minimal centred curves, which was not the case in [1]. I shall recall and use the terminology introduced in [1].

**Proposition 6.** One considers a minimal centred curve in \(W_0\) for a field \(X\) as above. Let \(L\) be a half orbit of \(X\) through some point \(m\) of \(\gamma\) and assume that \(L \setminus \{m\}\) is contained into \(\bar{W}(\gamma)\). Then the limit set of \(L\) is the origin.

**Proof.** One assumes that \(L\) is a positive half orbit by \(m \in \Gamma\), i.e. defined for the positive time. We want to prove that \(\omega(L) = \{0\}\). The proof for a negative half orbit is exactly the same, replacing \(X\) by \(-X\), and will be omitted. Let \(\varphi(t)\) be the trajectory by \(m\). The positive half orbit \(L\) is the image of the trajectory for the positive times. As \(L\) is contained in the compact \(\bar{W}(\gamma)\), the trajectory is defined for all positive times in \(\mathbb{R}^+\) and its \(\omega\)-set is contained into \(\bar{W}(\gamma)\). It follows from the Poincaré-Bendixson Theorem that this limit set is reduced to a singular point (necessarily the origin) or must contain at least a regular point (which may belong to a periodic orbit or a graphic).

This second case is impossible. In fact, if \(\omega(L)\) contains a regular point \(r\) in \(\bar{W}(\gamma)\), the trajectory \(\varphi(t, m)\) accumulates on \(r\) for \(t \to +\infty\). It is easy to find a transverse section \(\Sigma\) in a neighborhood of \(r\), which is cut just in two points \(p, q\) by \(L\), i.e. such that \(L \cap \Sigma = \{p, q\}\). Using Lemma 2.8, one finds a curve \(\gamma'\) which is transverse to \(X\) and also contained into \(\bar{W}(\gamma)\). This curve must surround the origin, which is the unique singular point in \(\bar{W}(\gamma)\). Moreover, as \(\gamma'\) is transverse to \(X\), this curve is a generic centred curve which verifies that \(c(\gamma') = 0\). As \(c(X, \bar{W}(\gamma)) \geq c(X, W_0)\),
this contradicts the assumption that $c(X,W_0) > 0$ made at the beginning of this subsection 2.3. As a consequence, one has that $\omega(\Gamma) = \{0\}$. \hfill \Box

**Corollary 1.** Let $\gamma$ be a minimal centred curve in $W_0$ for a field $X$ as above and $\Gamma$ be the orbit of $X$ through some interior contact point $m$ of $\gamma$. Then $\omega(\Gamma) = \alpha(\Gamma) = \{0\}$.

*Proof.* It follows from Proposition 3 that each half orbit through the interior contact point $m$ is contained into $\tilde{W}(\gamma)$. Then one can apply Proposition 6 to obtain the result. \hfill \Box

An orbit $\Gamma$ as in Corollary 1 bounds a piecewise $C^1$ disk $D(\Gamma)$, singular at the origin: a *pinched disk*, in short. Each orbit $\tilde{\Gamma}$ inside $D(\Gamma)$ has the same limit property that $\Gamma$, i.e. $\omega(\tilde{\Gamma}) = \alpha(\tilde{\Gamma}) = \{0\}$. The proof is exactly the same as in Proposition 6 and follows from non-existence of singular point of $X$ in the interior of $D(\Gamma)$. Following Bendixson in [B], we define

**Definition 2.11** (Bendixson). A closed nodal region is a pinched disk $D(\Gamma)$ bounded by an orbit $\Gamma$ such that $\omega(\Gamma) = \alpha(\Gamma) = \{0\}$, filled by orbits with the same limit property.

Elliptic sectors introduced in Introduction are simple examples of closed nodal region. But the phase portrait inside a general closed nodal region may be much more complicated than inside an elliptic sector. One can introduce an order in the set of orbits contained into $D(\Gamma)$, putting that $\tilde{\Gamma}_1$ is less than $\tilde{\Gamma}_2$ if and only if $\tilde{\Gamma}_1 \subset D(\tilde{\Gamma}_2)$. This order is total for an elliptic sector but is just partial in general. As a consequence, the phase portrait inside a general closed nodal region may be much more complicated than the simple 1-parameter family of orbits that one finds inside an elliptic sector. See Figure 9 for an example.

Let $\gamma$ be a minimal centred curve in $W_0$. Inside $\tilde{W}(\gamma)$ one has $c_i(\gamma)$ disjoint closed nodal regions associated to the $c_i(\gamma)$ interior contact points of $\gamma$. One also associates to each exterior contact point of $\gamma$, a remarkable type of orbit pair, also introduced by Bendixson:

**Definition 2.12** (Bendixson). Let $(L,L')$ be a pair of orbits such that $\omega(L) = \alpha(L') = \{0\}$, at the boundary of a sector $S$. This pair $(L,L')$ is said to cross the origin (on the side of $S$) if one has the following. One considers a section $\sigma$, topologically transverse to $L$ and a section $\sigma'$, topologically transverse to $L'$, these two sections pointing in the direction of $S$ with a base point respectively $p_0$ on $L$ and $p_0'$ on $L'$. One assumes that each positive half orbit $\Gamma(p)$ through a point $p \in \sigma \setminus \{p_0\}$ cuts $\sigma' \setminus \{p_0'\}$ a first time at a point $p' = h(p)$, if $\sigma$ is chosen short enough. One also assumes that this transition map $h$, which is continuous on $\sigma \setminus \{p_0\}$, can be extended by continuity at $p_0$ by $h(p_0) = p_0'$, i.e. $h(p) \to p_0'$ when $p \to p_0$ on $\sigma \setminus \{p_0\}$.

**Remark 7.** If Definition 2.12 is verified for a choice of sections $\sigma, \sigma'$, it is also verified for any other choices. This is easily shown by using well-chosen tubular neighborhoods.

If $L$ and $L'$ are the sides of a hyperbolic sector $S$ at the origin, as defined in Introduction, it is clear that this pair of orbits is crossing the origin (on the side of $S$). But, as it was noticed by Bendixson that the converse is not always true. It may exist pair of orbits crossing the origin which bound a sector containing closed nodal regions, possibly in an infinite number. See Figure 10.

In order to state the next result we need the following definition:
Definition 2.13. Let $m$ be an exterior contact point on the minimal centred curve $\gamma$. Then, the maximal interval $[q_-(m), q_+(m)]$ of $m$, is the open interval of $\gamma$, union of the bases of all minimal arches associated to $m$ (the notation is chosen such that
X has an inward direction at points of \(|q_-(m), q_+(m)|\) near \(q_-(m)\), and the converse near \(q_+(m)\)).

**Remark 8.** As arch-disks associated to two different exterior contact points of \(\gamma\) are disjoint, it follows that maximal intervals associated to different exterior contact points are also disjoint.

The following result associates a pair of orbits crossing the origin at each exterior contact point:

**Proposition 7.** Let \(m\) be an exterior contact point and \(|q_-(m), q_+(m)|\) its maximal interval. Let \(L_-\) be the positive half orbit by \(q_-(m)\) and \(L_+\) be the negative half orbit by \(q_+(m)\). Then, \(L_-\) and \(L_+\) are contained into \(W(\gamma)\) and \(\omega(L_-) = \alpha(L_+) = \{0\}\). Let \(S\) be the sector with sides \(L_-, L_+\) and containing the arches associated to \(m\). Then, the pair \((L_-, L_+)\) is crossing the origin (on the side of \(S\)).

**Proof.**

(1) If a point \(q \in \gamma\) is an end point of an arch associated to an exterior contact \(m'\), it is the same for points in a whole neighborhood of \(q\) in \(\gamma\). It follows that \(q_-(m), q_+(m)\) cannot be end points of an arch associated to \(m\), by maximality, and cannot be end points of an arch associated to an exterior contact point \(m' \neq m\), because in this case there would exist an arch associated to two distinct exterior contact points \(m\) and \(m'\), that is not possible. As a consequence, the positive half orbit \(L_-\) through \(q_-(m)\) and the negative half orbit \(L_+\) through \(q_+(m)\) are contained into \(W(\gamma)\). It follows from Proposition 6 that \(\omega(L_-) = \alpha(L_+) = \{0\}\). Let be \(S\) the sector with sides \(L_-, L_+\), containing the arches associated to \(m\).

(2) Using notations of Definition 2.12, we take \(p_0 = q_-(m)\) and \(p'_0 = q_+(m)\). One takes sections \(\sigma = [p_0, q_0]\) and \(\sigma' = [p'_0, q'_0]\) for points \(q_0 \in [p_0, m[\subset[p_0, p'_0]\) and \(q'_0 \in [p'_0, m[\subset[p'_0, p_0]\), such that \(q_0, q'_0\) are the two end points of a same arch. The transition map \(h\) is defined by the condition that, for any \(p \in [p_0, q_0]\), \(h(p)\) is the second end point of the arch starting at \(p\). The map \(h\) is a homeomorphism from \([p_0, q_0]\) onto \([p'_0, q'_0]\), reversing the orientation. The continuous extension at \(p_0\) follows from the following observation : if \(p'\) is any point in \([p'_0, q'_0]\), then \(h\) sends the interval \([p_0, h^{-1}(p')]\) onto the interval \([p'_0, p']\). It follows that the pair \((L_-, L_+)\) is crossing the origin on the side of \(S\).

**Proposition 8.** Let \(m\) be an exterior contact point and \(|q_-(m), q_+(m)|\) be its maximal interval. Then, neither \(q_-(m)\) nor \(q_+(m)\) can be a contact point (i.e., at these points, \(X\) is transverse to \(\gamma\)). Moreover \(q_-(m) \neq q_+(m)\) and then \([q_-(m), q_+(m)]\) is a proper closed interval of \(\gamma\).

**Proof.** To simplify one writes \(q_-(m) = q_-\) and \(q_+(m) = q_+\). Clearly, neither \(q_-\) nor \(q_+\) can be an exterior contact point.

Let us assume that \(q_-\) an interior contact point and that \(q_+ \neq q_-\) (the symmetric case is completely similar). The orbit \(\Gamma\) in \(W_0\), starting at a point \(p\) of \([q_-, q_+[,\) sufficiently near \(q_+\), cuts \(\gamma\) at \(p\) and also at two transverse points near \(q_-\), and then distinct of \(p\). So, \(\Gamma\) has at least three intersections with \(\gamma\). As \(\gamma\) is minimal, this is excluded by Proposition 5.

One assumes now that \(q_- = q_+\). This can happens only in the case that \(q_-\) and \(q_+\) coincide with an interior contact point \(m'\). In this case, the orbit \(\Gamma\) starting at a point \(p\) near \(m' = q_- = q_+\) is recurrent. Then, \(c(X, W_0) = 0\) (by Lemma 2.8) or \(\Gamma\) is a periodic orbit. These two possibilities are discarded by hypothesis. See Figure 11.
Finally, $q_-(m)$ and $q_+(m)$ are distinct transverse points and $[q_-(m), q_+(m)]$ is a proper closed interval of $\gamma$.

A half orbit entering into $W(\gamma)$ and starting at a transverse point of $\gamma$ outside the maximal intervals of the exterior contact points, cannot return on $\gamma$. Then, this half orbit is contained into $\bar{W}(\gamma)$ and it follows of Proposition 6 that its limit set is the origin. It belongs to a parabolic sector (possibly reduced to a single orbit).

As a consequence one obtains that the disk $\bar{W}(\gamma)$ contains a finite union of $c_1(\gamma)$ closed nodal regions, $c_\infty(\gamma)$ sectors associated to pairs of orbits crossing the origin. These closed nodal regions and sectors are interlinked by $c(\gamma)$ parabolic sectors (possibly reduced to a single orbit located between two sectors associated to pairs of orbits crossing the origin). But, as the closed nodal regions or sectors are perhaps not elliptic or hyperbolic sectors, one has not necessarily obtained a sectorial decomposition.

2.4. Phase portrait inside a minimal centred annulus.

**Definition 2.14.** Let $W$ be a centred disk in $W_0$. A minimal centred annulus $C$ in $W$ is an annulus whose boundary is the union of two minimal curves $\gamma_1, \gamma_2$ in $W$ such that $\gamma_2$ is contained into $W(\gamma_1)$. The curve $\gamma_1$ is the exterior side and the curve $\gamma_2$ is the interior side of the boundary $\partial C$. In other words, $C = \bar{W}(\gamma_1) \setminus W(\gamma_2)$.

**Proposition 9.** Let $C$ be a minimal centred annulus with boundary $\gamma_1 \cup \gamma_2$ where $\gamma_1$ is the exterior boundary side and $\gamma_2$ the interior one, as defined above. One assumes that $A$ contains no periodic orbit.

Then, each interior contact of $\gamma_1$ is on a minimal arch with base on $\gamma_2$ and each exterior contact of $\gamma_2$ is on a minimal arch with base on $\gamma_1$. The arch-disks associated with these arches are contained in $C$ and are 2 by 2 disjoints. All the contact points of $\gamma_1$ and $\gamma_2$ belong to the union of these arch-disks. In the complement of the arch-disks the orbits are arcs transverse to $\partial C$, between a point on $\gamma_1$ and a point on $\gamma_2$. See Figure 12.

**Proof.** One considers any interior contact point $m$ on $\gamma_1$. Two half orbits $\Gamma_+$ and $\Gamma_-$ are starting from $m$, and enter into $C$. One wants to prove that each of them has a first intersection with $\gamma_2$, which is transverse. The proof is the same for the two half orbits, then we will just consider $\Gamma_+$, image of the trajectory $\varphi(t, m)$ in $C$, for $t \geq 0$. As there exists no singular point nor periodic orbit inside $C$, it follows from the Poincaré-Bendixson Theorem that $\varphi(t, m)$ cannot remain into the interior of $C$ for all $t > 0$. Then, there exists a first $t_0 > 0$ such that $p(m) = \varphi(t_0, m) \in \partial C$. 

![Figure 11.](image)

![Figure 12.](image)
One first assumes that \( p(m) \in \gamma_1 \). It could happen that \( p(m) = m \). But this would mean that the orbit by \( m \) in \( C \) is periodic, which is excluded by hypothesis. On the other side, as \( \gamma_1 \) is minimal in \( W \), the fact that \( p(m) \neq m \) is excluded by Proposition 3.

It follows that \( p(m) \in \gamma_2 \). It could happen that \( p(m) \) is an exterior contact point on \( \gamma_2 \). In this case the segment of \( \Gamma_+ \) between \( m \) and \( p(m) \) is contained into one of the two half orbits starting from \( p(m) \). Let \( \Gamma'_+ \) the other half orbit starting from \( p(m) \). By the same argument as above, one concludes that \( \Gamma'_+ \) must reach \( \gamma_1 \) at a point \( q(m) \). If \( q(m) = m \), one has again that the orbit through \( m \) is a periodic orbit inside \( C \), which is excluded by hypothesis. If \( q(m) \neq m \), this means that the orbit of \( m \) in \( C \) returns on \( \gamma_1 \) at a point distinct from \( m \), which is excluded by Proposition 3. The only possibility is that \( \Gamma_+ \) reaches \( \gamma_2 \) at a transverse point.

Then, there is an arch in \( C \) attached to \( \gamma_2 \), passing through the interior contact point \( m \in \gamma_1 \). Its base \([p, q] \subset \gamma_2 \) contains a unique contact point of \( \gamma_2 \), which is an interior contact and the arch is minimal, as it follows from Proposition 4. As the arch and its base are included in \( C \), its arch-disk also is also included in \( C \).

The same proof can be repeated for each interior contact point of \( \gamma_1 \) and for each exterior contact point of \( \gamma_2 \). We obtain \( c_\iota(X, W) \) arches attached to \( \gamma_2 \) and \( c_e(X, W) \) arches attached to \( \gamma_1 \). All these arches are minimal, with arch-disks in \( C \) and 2 by 2 disjoint precisely because they are minimal. Moreover, the \( 2c(X, W) \) contact points located on \( \partial A \) are contained in the union of the arch-disks (half of them on the arches and half of them on the bases of the arches).

One considers now any point \( m \in \gamma_1 \) and outside these arch-disks. The vector field \( X \) is transverse to \( \gamma_1 \) at \( m \) (no contact point is outside the union of arch-disks), and the same argument as above shows that the orbit by \( m \) is an arc joining \( m \) to a point \( p \in \gamma_2 \) where \( X \) is transverse to \( \gamma_2 \). Moreover, this arc lies outside the union of arch-disks. One has the same argument, starting at a point \( m \in \gamma_2 \). \( \square \)
Remark 9. 1. The result stated in Proposition 9 is trivial if $X$ is smooth and if we consider a smooth minimal centred curve $\gamma$ with quadratic contact points and another smooth curve $\gamma'$, sufficiently $C^2$ close to $\gamma$ and disjoint from it. The interest of the Proposition is that it is valid in class $C^1$, for any pair of disjoint minimal centred curves in $W$.

2. As follows from Lemma 2.9, a sufficient condition in order to have no periodic orbit in $C$, is that $c(X,W) \geq 3$.

Corollary 2. Consider two disjoint minimal centred curves in a disk $W \subset W_0$. Then, the arrangement of contact points in the same for the two curves.

Proof. The two minimal curves are the components $\gamma_1, \gamma_2$ of a minimal annulus, as above. If $c(X,W) \leq 2$, the result is obviously true (even if the annulus between the two curves contains a periodic orbit). Then one can assume that that $c(X,W) \geq 3$, and then that there exists no periodic orbit in $W$, and in particular between $\gamma_1$ and $\gamma_2$. Then, one can apply the Proposition 9.

Let $p_1, \ldots, p_k$ with $k = c(X,W)$ be the interior contact points on $\gamma_1$ and $q'_1, \ldots, q'_l$ with $l = c(X,W)$ be the exterior contact points on $\gamma_2$. At each $p_i$ is associated the base $I_i \subset \gamma_2$ of the arch through $p_i$. In the same way, at each $q'_j$ is associated the base $J_j \subset \gamma_j$ of the arch through $q'_j$. The $I_i$ are 2 by 2 disjoints and each $I_i$ contains a unique contact point of $\gamma_2$, which is the interior point $p'_i$. In the same way, the $J_j$ are 2 by 2 disjoints and each $J_j$ contains a unique contact point of $\gamma_1$, which is the exterior point $q_j$.

Now, as the arch-disks are 2 by 2 disjoint, it is clear that the arrangement of $\{p_1, \ldots, p_k, q_1, \ldots, q_l\}$ is the same as the arrangement of $\{p'_1, \ldots, p'_k, J_1, \ldots, J_l\}$, which is the same as the arrangement of $\{I_1, \ldots, I_k, q'_1, \ldots, q'_l\}$, which is the same as the arrangement of $\{p'_1, \ldots, p'_k, q'_1, \ldots, q'_l\}$. See Figure 13.

2.5. Vector field germs of class $C^1$ with finite contact index. In this part we suppose that $(X,0)$ is a $C^1$ germ at the origin, isolated from the periodic orbits and such that $c(X,0) \in \mathbb{N}$ (i.e. is finite).

The following theorem describes the phase portrait of $(X,0)$:
Theorem 2.15. Let \((X, 0)\) be a \(C^1\) germ of vector field at the origin of \(\mathbb{R}^2\), isolated from other singularities and periodic orbits, with a finite index \(c(X, 0)\). Let \(X\) be a representative of \((X, 0)\) on a Euclidean disk \(W_0\) centred at the origin, which does not contain any other singularity apart the origin nor periodic orbits, such that \(c(X, W_0) = c(X, 0)\). Let \(\gamma\) be an absolute minimal centred curve in \(W_0\), as above (i.e. \(\gamma\) is a minimal centred curve in \(W_0\)). Then, on \(W(\gamma)\), the vector field \(X\) is a topological node or admits a sectoral decomposition, as defined in Definition 1.3. There are \(c_i(X, 0)\) elliptic sectors, \(c_e(X, 0)\) hyperbolic sectors, which are separated by \(c(X, 0)\) parabolic sectors (some of them may be reduced to a single orbit).

Proof. The general idea of the proof will be to associate to \(\gamma\) a sequence of other absolute minimal centred curves \((\gamma_n)\) in \(W(\gamma)\), such that \((W(\gamma_n))\) is a nested sequence of centred disks, with a sequence of diameters tending to zero, as in Proposition 1. One begins by the simpler case \(c(X, 0) = 0\).

1) Case \(c(X, 0) = 0\). In this case the vector field \(X\) is transverse to the minimal centred curves \(\gamma\) and \(\gamma_n\), for all \(n\). Assume that the vector field is pointing inside along \(\gamma\) (for the converse case, one just replaces \(X\) by \(-X\)). As there exists no periodic orbit in the annulus bounded by \(\gamma\) and \(\gamma_n\), the positive half orbit \(\Gamma(m)\) starting at a point \(m \in \gamma\) arrives to a point \(m_n\) on \(\gamma_n\). When \(n \to +\infty\), one has that \(m_n \to 0 \in \mathbb{R}^2\). So, one obtains that \(\omega(\Gamma(m)) = \{0\}\) (to be precise, one has to use that the segment of orbit between \(m_n\) and \(m_{n+1}\) is contained into the annulus between \(\gamma_n\) and \(\gamma_{n+1}\) for any \(n\)). As the result is true for any \(m \in \gamma\), one obtains that \(X\) is a topological node on \(W(\gamma)\).

2) Case \(c(X, 0) > 0\). One will start with the general results of Subsection 2.3, taking into account the existence of the sequence of minimal centred curves \((\gamma_n)\). One first considers the interior contact points of \(\gamma\) and next the exterior contact points.

(a) Interior contact points. Let \(m\) be any interior contact point on \(\gamma\). It follows from Corollary 1 that the orbit \(\Gamma\) passing through \(m\) bounds a closed nodal region \(D(\Gamma)\) in \(W(\gamma)\). Recall that this means that any orbit \(\tilde{\Gamma}\) in \(D(\Gamma)\) verifies that \(\omega(\tilde{\Gamma}) = \alpha(\tilde{\Gamma}) = \{0\}\). One wants to prove that \(D(\Gamma)\) is in fact an elliptic sector, i.e. that if \(\tilde{\Gamma}_1, \tilde{\Gamma}_2\) are two orbits in \(D(\Gamma)\), then \(\tilde{\Gamma}_1 \subset D(\tilde{\Gamma}_2)\) or \(\tilde{\Gamma}_2 \subset D(\tilde{\Gamma}_1)\).

One first notices that there exists a number \(n_0\) such that if \(n \geq n_0\), the curve \(\gamma_n\) intersects \(\tilde{\Gamma}_1\) and \(\tilde{\Gamma}_2\), each at two transverse points ( if \(M_i > 0\) is the maximum of the distance between a point on \(\tilde{\Gamma}_i\) and the origin it suffices to take \(n_0\) such that \(\text{diam}(W(\gamma_{n_0})) < \text{Inf}(M_1, M_2)\)). One considers any \(n \geq n_0\) and calls \(C_n\) the minimal centred annulus between \(\gamma\) and \(\gamma_n\). One uses now Proposition 9 : there exists a minimal arch \(A_n\) passing through \(m\) and containing the two arches \(\alpha_1^n = \tilde{\Gamma}_1 \cap A_n\) and \(\alpha_2^n = \tilde{\Gamma}_2 \cap A_n\). Let \([q_1^n, - , q_1^n]\) and \([q_2^n, - , q_2^n]\) be the bases on \(\gamma_n\) of these two arches. See Figure 14.

As \(\alpha_1^n\) and \(\alpha_2^n\) are two arches contained into the arch-disk of a same minimal arch \(A_n\), the base one of them is included into the base of the other. Using again Proposition 9 for the annulus between any pair of curves \(\gamma_n\) one has that this order of inclusion is independent of \(n\). To fix the ideas, let us assume that \([q_1^n, - , q_1^n]\) \(\subset [q_2^n, - , q_2^n]\). This implies that \(\tilde{\Gamma}_1 \cap A_n \subset D(\tilde{\Gamma}_2) \cap A_n\) for any \(n \geq n_0\). Taking the limit for \(n \to +\infty\) one obtains the result : \(\tilde{\Gamma}_1 \subset D(\tilde{\Gamma}_2)\).

To conclude, inside \(W(\gamma)\) one has \(c_i(X, 0)\) elliptic sectors, which are 2 by 2 distinct, associated to the \(c_i(X, 0)\) interior contact points of \(\gamma\).
(b) Exterior contact points. Let \( m \) be any exterior contact point. Using Proposition 7, one can associate to \( m \) a sector \( S \) containing all arches associated to \( m \), with sides a pair \((L_-, L_+)\) of half orbits crossing the origin on the side of \( S \). Moreover \( L_- \) and \( L_+ \) cut \( \gamma \) respectively at points \( q_- \) and \( q_+ \), where \([q_-, q_+]_\gamma \) is the maximal interval associated to \( m \). Choose now any \( \varepsilon > 0 \) and a number \( n \) such that \( \text{diam}(W(\gamma_n)) < \varepsilon \). From now on, \( \varepsilon \) and \( n \) are maintained fixed.

For any point \( p \in [q_-, m] \subset [q_-, q_+]_\gamma \), there exists by definition an arch \( A(p) \) starting at the point \( p \) and ending at \( h(p) \in [q_+, m]_\gamma \). The map \( h \) is continuous and \( h(p) \to q_+ \) when \( p \to q_- \). One wants to prove, that if \( p \) is sufficiently near \( q_- \), then the Hausdorff distance between \( A(p) \) and \( \partial S = L_- \cup L_+ \) is less than \( \varepsilon \). Taking \( \varepsilon \to 0 \), this will imply that \( S \) is a hyperbolic sector.

Put \( q_n^a = L_- \cap \gamma_n \) and \( q_n^a = L_+ \cap \gamma_n \). Applying Proposition 9, one has that the annulus \( C_n \) between \( \gamma \) and \( \gamma_n \) is minimal. Then, one can associate to \( m \) a (unique) point \( \bar{p}_n \in [q_-, m]_\gamma \) such that the arch \( A(\bar{p}_n) \) passes through an exterior contact point \( m_n \) on \( \gamma_n \), located on the interval \( [q_n^a, q_n^a]_\gamma \) of \( \gamma_n \).

One first shows that \( m_n \) is the unique contact point of \( \gamma_n \), located on the interval \( [q_n^a(m), q_n^a(m)]_\gamma \). For this, one considers the topological rectangle \( T_n \subset C_n \), with sides: \([q_-, q_n^a]_L_-, [q_+, q_n^a]_L_+, [q_-, q_+]_\gamma \), and \([q_n^a, q_n^a]_\gamma \). The rectangle \( T_n \) is partitioned into three sub-regions: the arch-disk \( D_{A(\bar{p}_n)} \) between two sub-rectangles \( T_n^- \) and \( T_n^+ \). See Figure 15.

To begin with, using \( T_n^- \) one proves that there is no contact point on the interval \([q_n^a, m_n]_\gamma \). On the contrary, assume first that \( r \) is an interior contact point on \([q_n^a, m_n]_\gamma \). Using Proposition 9, one knows that there exists a minimal arch \( A \subset C_n \) associated with \( r \) and passing through an interior contact point \( r' \) point on \( \gamma \). As the sides \([q_-, q_n^a]_L_- \) and \([\bar{p}_n, m_n]_{A(\bar{p}_n)} \) of \( T_n^- \) are tangent to \( X \), the arch \( A \) would
be contained into $T_n^-$, and so the point $r'$ would belong to the interval $[q_-(m), m_{\gamma}]$. This is impossible. In a similar way one can prove that there is no exterior contact point on $[q_+^n, m_{\gamma}]$. The proof for the other interval $[q_-^n, m_{\gamma}]$ is completed similarly and will be omitted.

As there exist no contact point on $[q_-^n, q_+^n\setminus\{m_n\}]$, the vector field $X$ is transverse and pointing inside $W(\gamma)$ along $[q_-^n, m_{\gamma}]$ and transverse and pointing outside $W(\gamma)$ along $[q_+^n, m_{\gamma}]$. Applying the Poincaré-Bendixson Theorem in $T_n^{-}$, one has that each trajectory starting at a point $p \in [q_-, \bar{p}_n]$ reaches a point $p_n(p) \in [q_-, m_n]$. For a similar reason, there is a trajectory starting at a point $h_n(p) \in [q_+^n, m_n]$ which reaches the point $h(p) \in [q_+, h(\bar{p}_n)]$.

The four points $p, p_n(p), h_n(p)$ and $h(p)$ cut the arch $A(p)$ into three subintervals $[p, p_n(p)]_{A(p)}, [p_n(p), h_n(p)]_{A(p)}$ and $[h_n(p), h(p)]_{A(p)}$. See Figure 16. The interval in middle $[p_n(p), h_n(p)]_{A(p)}$ is contained into $W(\gamma)$. Then, its Hausdorff distance to $\partial S \cap W(\gamma)$ is less than $\varepsilon$ (by the choice of $n$). Next, one has that the interval $[p, p_n(p)]_{A(p)}$ tends toward the interval $[q_-, q_+^n]_{L_-}$ in the Hausdorff sense, when $p \to q_-$. Finally, as $h(p) \to q_+$ when $p \to q_-$, one has also that $[h(p), h_n(p)]_{A(p)}$ tends toward the interval $[q_+, q_+^n]_{L_+}$ in the Hausdorff sense, when $p \to q_-$. It follows that the arch $A(p)$ is at a Hausdorff distance of $\partial S$ less than $\varepsilon$, if $p$ is sufficiently near $q_-$. Taking $\varepsilon \to 0$, this shows that $S$ is a hyperbolic sector.

One has already observed at the end of Subsection 2.3 that the complement set in $W(\gamma)$ of the union of the $c_i(X, 0)$ elliptic sectors and the $c_r(X, 0)$ hyperbolic sectors, that we have found above, is filled up by parabolic sectors (some of them possibly reduced to a single orbit). This finishes the proof in the case $c(X, 0) > 0$.

If a germ of vector field is a topological node or admits a sectoral decomposition, then the origin is a singularity isolated from other singularities and periodic orbits. It is rather easy to construct curves, at arbitrarily small distance of the origin, which are generic with a number of contact points equal to the number of elliptic sectors plus the number of hyperbolic sectors. Clearly these curves are minimal.
centred curves and the contact index is finite, equal to the number of hyperbolic and elliptic sectors. This means that Theorem 2.15 admits a converse.

As the numbers of elliptic and hyperbolic sectors are equal respectively to $c_i(X,0)$ and $c_e(X,0)$ we obtain as consequence of Theorem 2.15, of the above observation and of formula (1), a formula given in [1] for the index $i(X,0)$:

**Corollary 3.** Let $(X,0)$ be a $C^1$ germ of vector field which is a topological node or admits a sectoral decomposition with $E(X,0)$ elliptic sectors and $H(X,0)$ hyperbolic sectors. Then the index of $(X,0)$ at the origin is given by:

$$i(X,0) = \frac{E(X,0) - H(X,0)}{2} + 1$$  \hspace{1cm} (3)

**Remark 10.** Mind that the convention is different in [1] and the index given by Bendixson has an opposite value than the one given here in (3).

One can associate to a sectoral decomposition, its arrangement of elliptic and hyperbolic sectors and also take into account that some parabolic sectors may be reduced to a single orbit. This is coded by a sequence of $c(X,0)$ symbols taken in the set $\{E,H\}$ (for the elliptic and hyperbolic sectors), interlinked by a sequence of $c(X,0)$ symbols in the set $\{P,S\}$: $P$ for true parabolic and $S$ for a single orbit. The symbol $S$ can just appear between two hyperbolic sectors. By this way one obtains a sequence of $4k$ symbols (with the above restrictions), defined up to a circular permutation. As an example one has the sequence $(E, P, E, P, H, S, H, P)$ for the sectoral decomposition in Figure 2. One can associate the trivial sequence $(P)$ to a topological node. It is rather trivial that two sectoral decompositions which are topologically equivalent (i.e., with the same phase portrait) have the same associated sequence. The reverse is true, but the proof, which can be founded in [2], is rather delicate.
One can use this sequence in order to obtain an estimation of the number of different phase portraits in terms of $c(X,0)$. It is also easy to obtain an estimation for the topological index $i(X,0)$:

**Proposition 10.** Let $(X,0)$ be a germ of $C^1$ vector field as in Theorem 2.15. Then, one has that $-c(X,0)/2 + 1 \leq i(X,0) \leq c(X,0)/2 + 1$. For any $c \in \mathbb{N}$, the number of different phase portraits such that $c(X,0) = c$, up to time orientation, is roughly bounded by $2^{2c}$.

**Proof.** To obtain the estimation on $i(X,0)$ one has just to notice $-c(X,0) \leq c_e(X,0) - c_c(X,0) \leq c(X,0)$, and uses (1).

To count the different possible phase portraits, we have just to count the number of sequences associated above to sectoral decompositions. The number of possible arrangements of $c_e(X,0)$ elliptic sectors among $c_c(X,0)$ hyperbolic sectors, up circular permutations is less than $2^{c_e(X,0)}$. The number of occurrences of a single orbit in place of a true parabolic sector is also surely less than $2^{c_e(X,0)}$. As a sequence corresponds to a phase portrait, up to time orientation, the number of distinct phase portraits with $c(X,0) = c$ and up to time orientation, is roughly bounded by $2^{2c}$.

3. **Differentiable germs with an isolated singularity.** In the Part II of [1], Bendixson considers what he calls “holomorphic vector field” germs at the origin of $\mathbb{R}^2$, whose components have no common factor at $\{0\}$. Clearly, he wants to speak of real analytic germs with an algebraically isolated singularity at the origin. But it appears that Bendixson did not use the analyticity, nor the fact that the origin is algebraically isolated. In fact, his arguments simply use that the germ is sufficiently differentiable, with a non-zero jet at the origin and has the origin as a (topologically) isolated singularity. For this reason, in this Section I shall consider germs with a finite class of differentiability and a non-zero finite jet at the origin.

3.1. **A sufficient condition for the finiteness of $c(X,0)$**. For an integer $k \geq 1$, one considers a $C^{2k+3}$ vector field germ $(X,0)$, represented by a $C^{2k+3}$ vector field $X$ on a disk $W_0$, neighborhood of the origin. One assumes that this vector field can be written:

$$\begin{align*}
\dot{x} &= P_k(x,y) + O(||(x,y)||^{k+1}) \\
\dot{y} &= Q_k(x,y) + O(||(x,y)||^{k+1}),
\end{align*}$$

(4) where $P_k, Q_k$ are homogeneous polynomials of degree $k$, such that at least one of the two is non-zero ($(X,0)$ has a zero $(k-1)$-jet but a non-zero $k$-jet). The remainders are functions of class $C^{2k+3}$.

The main result of this Section is the following one:

**Theorem 3.1.** For an integer $k \geq 1$, let $(X,0)$ be a $C^{2k+3}$ germ of vector field, such that the origin is an topological isolated singularity. Assume moreover that $j_{k-1}X(0) = 0$, but that $j_k X(0) \neq 0$, as expressed in (4). Then $c(X,0) \leq 2(k+1)$

**Proof.** For this proof one will use some ideas introduced by of Bendixson in [1], for another aim that is recalled in Subsection 3.2. The germ is represented by a vector field $X$ on $W_0$, as above.

(1) One first assumes that the polynomial $R(x,y) = xP_k(x,y) + yQ_k(x,y)$ is not zero. One considers the circles $\gamma^\rho$ given by $\gamma^\rho(\theta) = (\rho \cos \theta, \rho \sin \theta)$, which are contained into $W_0$ for $\rho$ small enough. Along these circles one chooses the orthogonal
vector field \( N = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \). Taking into account the homogeneity of \( P_k, Q_k \), the function \( n^\rho(\theta) = \langle X(\gamma^\rho(\theta)), N(\gamma^\rho(\theta)) \rangle \) is given by:

\[
n^\rho(\theta) = \rho^{k+1} \left( R(\cos(\theta), \sin(\theta)) + p h(\theta, \rho) \right),
\]

where the function \( h(\theta, \rho) \) is of class \( C^{k+1} \) (this function is obtained through the division of a remainder of class \( C^{2k+3} \) in \( (\theta, \rho) \) by the function \( \rho^{k+2} \)). The trigonometrical polynomial \( R(\cos(\theta), \sin(\theta)) \) is non-zero and homogeneous of degree \( k+1 \). Then, it has \( 2(k+1) \) zeros counted with their multiplicity. As the function \( \theta \to R(\cos(\theta), \sin(\theta)) + p h(\theta, \rho) \) converges in the \( C^{k+1} \)-topology towards \( R(\cos(\theta), \sin(\theta)) \) when \( \rho \to 0 \), we have that the function \( n^\rho(\theta) \) has also less than \( 2(k+1) \) zeros (counted with their multiplicity) for \( 0 < \rho < \rho_0 \), where \( \rho_0 > 0 \) is small enough.

For any \( \rho \), such that \( 0 < \rho < \rho_0 \), it is easy to approximate in the \( C^{k+1} \)-topology, the curve \( \gamma^\rho \) by a centred curve with \( 2(k+1) \) quadratic (and then even) contact points with \( X \). Then, such a curve is generic with less than \( 2(k+1) \) (even) contact points. By this way one has constructed a sequence \( (\gamma_n) \) of generic centred curves, converging toward the origin in the Hausdorff sense and such that \( c(\gamma_n) \leq 2(k+1) \).

It follows from Proposition 2, that \( c(X,0) \leq 2(k+1) \).

\(2\) One now assumes that \( R(x,y) \) is the zero polynomial. It follows from the polynomial identity \( x P_k(x,y) = -y Q_k(x,y) \) that there exists a non-zero homogeneous polynomial \( S_{k-1} \) of degree \( k-1 \) such that \( P_k(x,y) = -y S_{k-1}(x,y) \) and \( Q_k(x,y) = x S_{k-1}(x,y) \). Choosing constants \( a > 0 \) and \( b > 0 \), with \( a \neq b \), one replaces the circles used in case (1) by the ellipses \( \gamma^r \) given by:

\[
\gamma^r(\theta) = (a r \cos(\theta), b r \sin(\theta)),
\]

with cartesian equation:

\[
x^2 + \frac{y^2}{b^2} = r^2.
\]

The vector field \( N = b^2 x \frac{\partial}{\partial x} + a^2 y \frac{\partial}{\partial y} \) is orthogonal to these ellipses. The scalar product of \( X \) and \( N \) is the function:

\[
<X,N> = b^2 x P_k(x,y) + a^2 y Q_k(x,y) = (a^2 - b^2) x y S(x,y) + O(||(x,y)||^{k+2})
\]

Putting \( c = \cos(\theta) \) and \( s = \sin(\theta) \), the restriction of \( <X,N> \) along an ellipse \( \gamma^r \) is the function:

\[
n^r(\theta) = r^{k+1} \left( a b (a^2 - b^2) c s S_{k-1}(ac, bs) + r h(\theta, r) \right),
\]

where the function \( h(\theta, r) \) is of class \( C^{k+1} \). As the trigonometrical polynomial \( a b (a^2 - b^2) c s S_{k-1}(ac, bs) \) is non-zero and homogeneous of degree \( k+1 \), one can finish the proof as in case (1).

\[\square\]

Remark 11. In the proof of Theorem 3.1, one obtains the existence of a sequence \( (\gamma_n) \) of generic centred curves with a contact index bounded by \( C = 2(k+1) \), converging toward the origin. For \( n \to \infty \), their shape after rescaling, tends to be circular. But the curves \( \gamma_n \) are not absolute minimal centred curves and Proposition 2 gives no information about the asymptotic shape of absolute minimal curves converging toward the origin. This asymptotic shape could be obtained using the desingularization theory of [1], in case of a smooth vector field verifying a Lojasiewicz condition. In this case, absolute minimal curves converging toward the origin can be obtained as direct images through the desingularization mapping, of curves converging toward the critical locus of the desingularization. For instance, for a smooth germ \( \frac{1}{2} x^4 \frac{\partial}{\partial x} - \frac{1}{2} x^3 \frac{\partial}{\partial y} + \cdots \), with the origin as an algebraically isolated singularity, a \( \rho \)-family of absolute minimal centred curves is given by \( \{ax^4 + by^2 = \rho^2\} \), for \( a \neq b \) and \( r \to 0 \). They are associated to the desingularization mapping \( (x,y,u) \in S^1 \times IR^+ \to (ux, u^2y) \in IR^2 \).
3.2. Some results of Bendixson. Let $\Gamma$ be an orbit image of $\varphi(t)$, trajectory of a vector field $X$, which tends toward $0 \in \mathbb{R}^2$ for $t \to +\infty$ (if the limit is for $t \to -\infty$ we can make similar considerations, changing $X$ by $-X$). To study the limit, one can work in polar coordinates $(\rho, \theta)$ where the flow is given by $\varphi(t) = (\rho(t), \theta(t))$.

To say that $\varphi(t)$ tends to $0 \in \mathbb{R}^2$ is just to say that $r(t) \to 0$. Depending on the behavior of $\theta(t)$ we have different type of trajectory:

**Definition 3.2.** One considers a trajectory $\varphi(t) = (\rho(t), \theta(t))$ such that $\rho(t) \to 0$ when $t \to +\infty$. If $\theta(t) \to \theta_0 \in S^1$ the orbit $\gamma$ is said an (attracting) characteristic orbit. If $\theta(t) \to +\infty$ or $-\infty$ (in the universal covering of $S^1$) the orbit is called an (attracting) spiral in the direct or the clock-ward direction. If all points of a neighborhood of the origin the orbits are attracting spirals the field germ is said to be an attracting focus (spiraling in the direct or clock-ward direction). If the orbits by all the points of neighborhood of the origin are attracting characteristic orbits (with perhaps different angle limits) the field germ is said to be an attracting node.

One has the repulsing notions (repulsing spiral, characteristic orbit, focus, node) by changing $X$ in $-X$.

**Remark 12.** A characteristic orbit is an orbit with a limit tangent $\{\theta = \theta_0\}$. This terminology was not used by Bendixson in [1] but was used by Dumortier in [2]. In fact, for Bendixson, characteristic curve is just synonymous of orbit or sequence of orbits.

For a $C^1$ vector field, the behavior of an orbit which tends toward the origin may be much wilder than the behaviors described in Definition 3.2. But, we can found in [1] a result for analytic vector fields, which can be restated for differentiable vector fields as follows:

**Proposition 11.** For an integer $k \geq 1$, we consider a germ of a $C^{2k+3}$ vector field at the origin with a non-zero $k$-jet as in (4) and such that the origin is isolated. If $\varphi(t)$ is a trajectory tending toward the origin when $t$ tends toward $+\infty$ or toward $-\infty$, then the corresponding orbit is a spiral or a characteristic orbit.

In order to obtain this result, Bendixson proved, for an orbit tending toward the origin, that the angular component $\theta(t)$ must have a limit, finite or infinite. For proving this, he used for the tangential component $-yP + xQ$, the ideas that I have adapted to the radial component in the proof of Theorem 3.1 above. Also, as I have mentioned above, Bendixson supposed that the vector field is analytic and that the origin is algebraically isolated, but the assumptions made in Proposition 11 are sufficient. Another result of [1] can be restated as follows:

**Proposition 12.** One considers a germ of differentiable vector field as in Proposition 11. If there is at least one characteristic orbit, then every other orbit tending toward the origin (for positive or negative times) is also a characteristic orbit.

3.3. Proof of the main Theorem 1.4.

**Proof.** The proof is a direct consequence of the previous results. Applying Theorem 3.1 one has that $c(X,0) \leq 2(k + 1)$. As the origin is supposed to be isolated from other singularities and periodic orbits, one can also apply Theorem 2.15: the germ $(X,0)$ is a topological node or admits a sectoral decomposition with no more than $2(k + 1)$ sectors. It follows from Propositions 11 and 12 that a topological node for a differentiable germ with non-zero jet, as in the statement of Theorem 1.4, is a
focus or a node and that, in the case of a non trivial sectoral decomposition, each orbit tending toward the origin is a characteristic orbit.

To estimate the index \( i(X, 0) \) and the number of different phase portraits, we can apply Proposition 10. As \( c(X, 0) \leq 2(k + 1) \), we obtain that \( -k \leq i(X, 0) \leq k + 2 \) and that the number of different phase portraits is no more than \( 2^{4(k+1)} \). \( \square \)

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