Oracle-Efficient Online Learning for Beyond Worst-Case Adversaries

Nika Haghtalab, Yanjun Han, Abhishek Shetty, and Kunhe Yang

University of California, Berkeley
{nika,yjhan,shetty,kunheyang}@berkeley.edu

Abstract

In this paper, we study oracle-efficient algorithms for beyond worst-case analysis of online learning. We focus on two settings. First, the smoothed analysis setting of [RST11, HRS21] where an adversary is constrained to generating samples from distributions whose density is upper bounded by $1/\sigma$ times the uniform density. Second, the setting of $K$-hint transductive learning, where the learner is given access to $K$ hints per time step that are guaranteed to include the true instance. We give the first known oracle-efficient algorithms for both settings that depend only on the VC dimension of the class and parameters $\sigma$ and $K$ that capture the power of the adversary. In particular, we achieve oracle-efficient regret bounds of $O(\sqrt{Td\sigma^{-1/2}})$ and $O(\sqrt{Tk})$ respectively for these setting. For the smoothed analysis setting, our results give the first oracle-efficient algorithm for online learning with smoothed adversaries [HRS21]. This contrasts the computational separation between online learning with worst-case adversaries and offline learning established by [HK16]. Our algorithms also achieve improved bounds for worst-case setting with small domains. In particular, we give an oracle-efficient algorithm with regret of $O(\sqrt{Td|X|^{1/2}})$, which is a refinement of the earlier $O(\sqrt{T|X|})$ bound by [DS16].

1 Introduction

Online adversarial learning is a cornerstone of modern machine learning and has led to significant advances in theory of machine learning and computer science broadly. A recent line of work on “beyond the worst-case analysis” of online learning has brought into light the brittle (and overly pessimistic) nature of worst-case characterization of online learnability [RST11, GR17, HRS20, HRS21]. For example, under smoothed analysis, online learnability in presence of an adaptive adversary is statistically as easy as PAC learnability [HRS21], i.e., they are both characterized by the VC dimension of a hypothesis class as opposed to the much larger Littlestone dimension that characterizes online learnability in the worst-case [BDPSS09]. These works have collectively demonstrated the possibility of achieving regret guarantees in practical settings, where online learning was known to be impossible. But to deliver fully on this new possibility, we need to design algorithms that can achieve improved regret bounds via efficient computation.
In this work, we seek to reduce the computational gap between online learning and offline learning in presence of adversaries that are not worst-case. Our goal is to design algorithms whose regret are characterized by offline statistical complexity measures, such as the VC dimension of a hypothesis class $\mathcal{H}$, and whose runtimes are similarly characterized by how efficiently they can be implemented given access to offline optimization routines for $\mathcal{H}$. For the latter, we work within the oracle-efficient algorithm design framework (see e.g., [DHL+17, KV05, HK16]) where an online algorithm can make calls to an (ERM) oracle which computes an optimal hypothesis given a multi-set of the actions of the adversary. From the computational perspective, a similar impossibility plagues online learning, where a single call to the ERM oracle is a near-optimal offline learning algorithm while online algorithms have to run in time $\tilde{\Omega}(\sqrt{|H|})$ even when they are equipped with an ERM oracle [HK16].

We ask whether more practical adversarial settings allow for more efficient computation as well as improved regret bounds. We work within the smoothed analysis perspective of [ST04], formalized for online learning by [RSS12, HRS20]. From this perspective, we consider an adversary restricted to generating an instance at every round from a distribution that is not overly concentrated, i.e. a distribution whose density is upper bounded by $1/\sigma$ times that of the uniform distribution. As $\sigma$ decreases, the environment becomes more adversarial and as it increases the environment becomes more uniformly random. Smoothness of the adversary’s actions captures the noise and imprecision inherent in the real world. In this setting, [HRS21] showed a statistical upper bound of $\tilde{O}(\sqrt{Td \log(1/\sigma)})$ using an algorithm that maintains an exponentially sized net for the hypothesis class. We ask whether oracle-efficient algorithms exist whose regret are similarly characterized by the VC dimension.

Towards understanding the algorithmic aspects of smoothed online learning, in this work, we make a connection between different paradigms of beyond worst-case analysis of online learning, in particular, the literature on predictable adversaries [RS13b], transductive learning [KK06, CbS11], and learning with hints [DFHJ17, BCKP20]. We introduce a generalization of the transductive learning framework, called transductive learning with $K$ hints, in which the learner has access to a set of $K$ unlabeled instances per-round which is promised to include the true instance the learner will be tested on. As $K$ decreases, future instances becomes more predictable. In this setting, it is not difficult to create a computationally inefficient algorithm that plays Hedge with all the projections of hypothesis class $\mathcal{H}$ on the $KT$ hints and achieve a regret of $O\left(\sqrt{Td \log(KT)}\right)$. We ask whether there are oracle-efficient algorithms whose regret is characterized by the VC dimension and degrades gracefully as a function of $K$. As we shall see, this approach brings together several paradigms of beyond worst-case analysis by creating a connection between hints and smoothed analysis.

### 1.1 Main Results

Our work considers two settings in the beyond worst-case analysis of online learning: Smoothed analysis of online learning and $K$-hint transductive learning. In both cases, we consider large hypothesis classes $\mathcal{H}$ and seek regret bounds that are only depend on the VC dimension $\mathcal{H}$ and a measure of parameters of these models that capture the power of the adversary, i.e., $\sigma$ and $K$ respectively. We prove the following oracle-efficient regret bounds.

**Theorem 3.1 (Informal).** *In the setting of transductive learning with $K$-hints, there is an oracle-
efficient algorithm (Algorithm 3) with expected regret bounded by $O(\sqrt{dTK})$.

**Theorem 4.1** (Informal). In the setting of online learning with a $\sigma$-smooth adversary, there is a proper and oracle efficient algorithm (Algorithm 2) whose expected regret is upper bounded by $O\left(\min\left\{ \sqrt{dT\sigma^{-1/2}}, \sqrt{T(d|\mathcal{X}|)^{1/2}} \right\}\right)$.

We note that Theorem 4.1 establishes that, under smooth analysis, online learning is as computationally efficient as offline learning. This is in contrast with the results of [HK16] that showed a computational separation between offline learning and online learning with worst-case adversaries. While our main interest is on beyond the worst-case adversaries, our results improve upon existing results for worst-case analysis of online learning. For finite domain settings where worst-case adversaries are vacuously $\sigma$-smooth for $1/\sigma = |\mathcal{X}|$, Theorem 4.1 also achieves an oracle-efficient regret bound of $O(\sqrt{T(d|\mathcal{X}|)^{1/2}})$, which is a refinement of [DS16] bound of $O(\sqrt{T|\mathcal{X}|})$ because VC dimension $d$ is at most $|\mathcal{X}|$, and is usually much smaller.

**Corollary 1.1** (Regret for Small Domain). There is an oracle-efficient algorithm for online learning (in the worst-case) that achieves a regret of $O(\sqrt{T(d|\mathcal{X}|)^{1/2}})$ for any hypothesis class with VC dimension $d$ on domain $\mathcal{X}$.

In Section 5, we further discuss the computational-statistical gap between online learning algorithms in presence of beyond worst-case adversaries and highlight important directions for future research.

While we focus on the binary classification problem within the main body of this paper, we also extend our results to real-valued functions and achieve regret bounds that depend only on the *pseudo-dimension* of the class of functions and the time horizon. See Appendix G for more details.

### 1.2 Technical Overview

**Relaxations for Beyond Worst-Case Adversaries.** The design of our algorithms is based on the random playout technique and we will use the *admissible relaxation* framework of [RSS12] to analyze these algorithms. Each of our algorithms generates and randomly labels instances as a stand-in for the future. We show how this framework is especially useful for analyzing online learning algorithms in the beyond worst-case setting. In particular, the $K$-hint transductive setting, in contrast to the classical transductive setting, includes hints that are never realized as part of the sequence. To show that these hints have a small impact on the achievable regret, we prove that the *regularized Rademacher complexity* is monotone in the set of provided hints (Lemma 3.2). This monotonicity allows us to leverage existing relaxations that are admissible for highly predictable adversaries and to show that the regret gracefully degrades as a function of the power of the adversary.

**Self Generation of Hints.** In the smoothed analysis setting, the algorithm does not have access to hints, nevertheless, smoothness captures a level of predictability about the future. This is captured by a technique from [HRS21] that shows that any sequence of $T$ instances generated by adaptive smoothed adversaries can be seen being a subset of set of $T/\sigma$ uniformly random instances from
\(\mathcal{X}\) with high probability. We use this to self-generate hints and draw a parallel between \(K\)-hint transductive learning and smoothed analysis. Though the next instance given by the adversary is not guaranteed to be in this hint set, we show that this process accounts for the uncertainty in each step.

Another key aspect is the distribution from which hints are self-generated. We will crucially use the Poissonization technique in which we generate the number of hints from an appropriately chosen Poisson distribution. This allows us an additional degree of independence that is essential to controlling the loss from one step to the next. Both of these properties are key in our analyses of the admissibility of the corresponding relaxation.

**Generalization, Stability, and Admissibility.** In addition, we make use of a connection between the traditional notion of algorithmic stability and admissibility. Stability is a well-studied notion capturing how little the distribution of the actions of the learner changes across time steps and is used in the analyses of algorithms such as follow-the-perturbed-leader and follow-the-regularized-leader. We show that the admissibility of the relaxation in the smoothed online setting is implied by the stability. This allows us to use information theoretic techniques that crucially exploit the independence provided by Poissonization as discussed earlier.

As we show, the stability analysis of the algorithm also depends crucially on the generalization error of the ERM output when it is trained on uniformly self-generated hints and tested on smoothly distributed fresh instances. In order to bound the generalization error, we take advantage of a stronger property implied by the coupling lemma from [HRS21]. This states that there exists a coupling between uniform and adaptive smooth processes, such that when the inclusion property is satisfied, the distribution of the uniform variables realized in the inclusion conditional on the un-realized uniform variables is also identical to the smooth distributions given by the adversary. This will be instrumental for bounding the generalization error by allowing us to extract smooth variables from a set of uniform variables, which can then be used to for the purpose of symmetrization. This result may be of independent interest.

### 1.3 Related works

Our work relates to several paradigms and approaches to online learnability.

**Oracle-Efficient Online Learning.** Since the seminal work of [KV05], inspired by application domains such as game theory, there has been a long line of work elucidating the computational aspects of online learning. [KV05] proposed the influential follow-the-perturbed-leader algorithm. [KKL07] study notions of regret when the learner is given access to an approximate optimization oracle. [KK06] study the transductive learning setting and give an efficient algorithm that converts offline learnability to online learnability. [RSS12] propose a general admissible relaxation framework to develop efficient algorithms based on the upper bound of the value of the game. [DHL+17] present the computationally efficient Generalized-FTPL algorithm and provide conditions under which it achieves vanishing regret. On the flip size, [HK16] show that an \(\Omega(\sqrt{N})\) lower bound is unaviodable in general in order to obtain nontrivial regret where the \(N\) is the number of actions of the learner suggesting that one needs to look beyond the worst-case in order to get truly efficient algorithms.
Beyond Worst-case Approaches to Online Learning. Various notion of beyond worst-case behavior of online learning has been studied in the literature. [RST11] consider online learning where the adversary is constrained and build a framework based on minimax analysis and constrained sequential Rademacher complexity to analyze regret in these scenarios. These techniques have been applied to other constrained settings [KAH+19].

[GR17] consider smoothed online learning when looking at problems in online algorithm design. They prove that while optimizing parameterized greedy heuristics for combinatorial problems in presence of smoothing this problem can be learned with non-trivial sublinear regret. [CAK17] consider the same problem with an emphasis on the per-step runtime being logarithmic in $T$. [HRS20, HRS21] both study the notion of smoothed analysis with adaptive adversary and show that statistically the regret is bounded by $O(\sqrt{Td\log(1/\sigma)})$.

Smoothed analysis has also been studied in a number of other online settings, including linear contextual bandits, [KMR+18, RSWW18], but the focus has been on achieving improved regret bounds for the greedy algorithm that is not no-regret in the worst-case.

Another line of work has focused on the future sequences being predictable given the past instances. [RS13b] incorporate additional information available in terms of an estimator for future instances. They achieve regret bounds depending on the path length of these estimators and can beat the worst-case $\Omega(\sqrt{T})$ if the estimators are accurate. [HM07] models predictability as knowing the first coordinate of loss vectors, which is revealed to the learner before he chooses actions. Some other work model predictability through hints which are additive estimate of loss vectors [HK10, RS13a, SL14, MY16]. [DFHJ17, BCKP20] considers settings where the learner has access to hints in form of vectors that are weakly correlated with the future instances and show exponential improvement in the regret in some cases. The literature on hints represents an active and growing subarea of online learning (see [BCKP20] and references within).

Concurrent Work. In a concurrent and independent work, [BDGR22] also gives oracle-efficient algorithms for smoothed online learning. [BDGR22] obtains a regret bound of $\tilde{O}(\sqrt{Td\sigma^{-1}})$. In comparison, our main result (Theorem 4.1) concerning smoothed online learning demonstrates a regret bound of $\tilde{O}(\sqrt{Td\sigma^{-1/2}})$ with strictly better dependence on $\sigma$. Our regret bound’s improved dependence on parameter $\sigma$ can be attributed the following technical innovations of this work: 1) The relationship between stability and relaxation-based methods, 2) the careful analysis of modified generalization error and stability via a new coupling-based argument, and 3) a Poissonization approach for self-generating hints that allow us to leverage information theoretic arguments.

Interesting, both our work and [BDGR22] use a relaxation-based algorithms as a warmup achieving suboptimal regret bounds of $\tilde{O}(\sqrt{Td\sigma^{-1}})$ (our Theorem 4.2) and $\tilde{O}(\sigma^{-1}\sqrt{Td})$ [BDGR22, Theorem 8] before improving upon these using FTPL-based analysis. For the warmup, our regret bound’s improved dependence on parameter $\sigma$ is due to our different approach to self-generating hints that allow us to leverage the monotonicity of the Rademacher complexity.

## 2 Preliminaries

Let $\mathcal{X}$ be the space of instances, $\mathcal{Y} = \{-1, 1\}$ be the space of binary labels, and $\mathcal{H} : \mathcal{X} \rightarrow \mathcal{Y}$ be the hypothesis class. The analysis of continuous labels $\mathcal{Y} = [-1, 1]$ and general loss functions is deferred to Appendix G. An online binary classification game with an adaptive adversary takes
place over $T$ time steps. At each step $t \in [T]$, the adversary chooses a distribution $\mathcal{D}_t^Y \in \Delta(\mathcal{Y})$ and a random instance $x_t \sim \mathcal{D}_t^X$ is then drawn and presented to the learner. After receiving $x_t$, the learner predicts its label to be $\hat{y}_t \in \mathcal{Y}$, while the adversary simultaneously chooses $y_t \in \mathcal{Y}$ as its true label. The learner then suffers loss $l(\hat{y}_t, y_t)$. This is equivalent to a setting where the adversary chooses a distribution $\mathcal{D}_t \in \Delta(\mathcal{X} \times \mathcal{Y})$ over labeled instances $s_t = (x_t, y_t)$ and the learner simultaneously chooses a classifier $h_t \in \mathcal{Y}^\mathcal{X}$. We will abbreviate $\mathcal{D}_t^X \rightarrow \mathcal{D}_t$ when it is clear from the context.

We allow the adversary to be adaptive, i.e., it can choose each $\mathcal{D}_t$ based on the realizations of the inputs as well as the learner’s predictions in previous time steps. Let $\mathcal{D}$ denote the adaptive sequence of distributions $\mathcal{D}_1, \cdots, \mathcal{D}_T$. Accordingly, let $\mathcal{Q}_t \in \Delta(\mathcal{Y})$ denote the learner’s prediction rule on instance $x_t$, and let $\mathcal{Q}$ denote the adaptive sequence of distributions $\mathcal{Q}_1, \cdots, \mathcal{Q}_T$. We denote the expected regret of a learner with prediction rules $\mathcal{D}$ on the adaptive sequence $\mathcal{Q}$ by

$$
\mathbb{E}[\text{REGRET}(T, \mathcal{D}, \mathcal{Q})] = \mathbb{E}_{\mathcal{D}, \mathcal{Q}} \left[ \sum_{t=1}^{T} l(\hat{y}_t, y_t) - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} l(h(x_t), y_t) \right].
$$

We remove $\mathcal{D}$ and $\mathcal{Q}$ from this notation when they are clear from the context.

**Offline Optimization Oracle** Suppose the learner has access to an offline optimization oracle that performs empirical risk minimization (ERM) on any sequence of inputs.

**Definition 2.1 (ERM Oracle).** For a hypothesis class $\mathcal{H}$ and a loss function $l$, the oracle $\text{OPT}^{\text{opt}}$ takes a set $\mathcal{I}$ of inputs $S = \{(x_i, y_i)\}_{i \in [I]}$ where $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}$ for all $i \in [I]$ and returns $\text{OPT}_{\mathcal{H}, l}(S) = \inf_{h \in \mathcal{H}} \sum_{i=1}^{I} l(h(x_i), y_i)$ and $\text{opt}_{\mathcal{H}, l}(S) \in \arg \inf_{h \in \mathcal{H}} \sum_{i=1}^{I} l(h(x_i), y_i)$.

We consider each call to the ERM oracle as having unit cost plus the additional runtime needed for creating and inputting the set of inputs that is linear in the length of the said histories. We note that our approach and results directly extend to using ERM oracles with (arbitrarily small) additive approximation error, such as those guaranteed by FPTAS optimization algorithms, using standard techniques presented by [DHL+17, Section 6].

**Translation of Loss Functions** In order to analyze the regret, it is helpful to shift the original loss function by $-\frac{1}{2}$. In this way, we obtain a more convenient linear loss function to work with, which can be expressed as $l(\hat{y}, y) = 1\{\hat{y} \neq y\} - \frac{1}{2} = -\frac{\hat{y} \cdot y}{2}$. Importantly, this shifted loss function is equivalent with the original loss function in the sense of both the value of regret and the implementation of oracles. From now on, $l$ refer to the shifted loss function by default.

**Additional Notations** We also define $\circ$ to be the operator that assigns a sequence of labels to a sequence of inputs. Specifically, if $z_{1:T} \in \mathcal{X}^T$, $\epsilon_{1:T} \in \mathcal{Y}^T$ are two sequences of the same length, then $z_{1:T} \circ \epsilon_{1:T} = \{(z_i, \epsilon_i)\}_{i=1}^{T}$ returns the sequence of labeled instances. Additionally, we introduce

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1The inputs to the oracle are multisets. Unless specified otherwise, all the sets in this paper refer to multisets.
We work with the smoothed adaptive online adversarial setting from version of transductive learning. In this setting, the exact sequence of instances is replaced with a smooth adversaries, where a distribution is \( \sigma \)-smooth if its density is upper bounded by \( 1/\sigma \) times the density of the uniform distribution over the same domain.

**Definition 2.2 (\( \sigma \)-smoothness).** Let \( \mathcal{X} \) be a domain that supports a uniform distribution \( \mathcal{U} \). A measure \( \mu \) on \( \mathcal{X} \) is \( \sigma \)-smooth if for all measurable subsets \( A \subseteq \mathcal{X} \), \( \mu(A) \leq \frac{\mathcal{U}(A)}{\sigma} \). The set of all \( \sigma \)-smooth distributions on domain \( \mathcal{X} \) is denoted by \( \Delta_{\sigma}(\mathcal{X}) \).

In online learning with adaptive smoothed adversaries, at each time step \( t \), the adversary chooses a distribution \( \mathcal{D}_t \) whose marginal on \( \mathcal{X} \) is \( \sigma \)-smooth. The choice of \( \mathcal{D}_t \) can depend the previous instances \( \{(x_i, y_i)\}_{i=1}^{t-1} \) as well as the learner’s previous predictions. We denote with \( \mathcal{D}_\sigma \) the adaptive sequence of \( \sigma \)-smooth distributions on the instances. The corresponding definition of regret in this setting is given by Equation (1) with \( \mathcal{D}_\sigma \) as the set of distribution for the adversary.

An important property of smoothness is that it implies coupling between uniform and adaptive smooth processes. We will consider the original result from [HRS21] in Lemma D.1 and a slightly strengthened statement in Lemma 4.6. In Section 4, we will provide new insights on the properties of such couplings.

**2.2 Transductive Online Learning with \( K \) Hints**

In the traditional transductive setting, the adversary releases the sequence of unlabeled instances \( \{x_t\}_{t=1}^{T} \) to the learner before the game starts. We generalize this setting and introduce a \( K \)-hint version of transductive learning. In this setting, the exact sequence of instances is replaced with a sequence of \( K \) hints per time step such that the set of hints at each time step includes the instance at that time step. More formally, before the interaction starts, the adversary releases \( T \) sets (multisets) of size \( K \) to the learner. We denote these sets by \( \{Z_t = \{z_{t,1}, \ldots, z_{t,K}\}\}_{t=1}^{T} \). On releasing these sets, the adversary promises to always pick \( \mathcal{D}_t \) supported only on the elements of \( Z_t \). The regret is defined by Equation (1) with the appropriate restriction on the adversary.

**2.3 Relaxations and Admissibility**

**Definition 2.3 (Admissibility).** In an online learning setting where the adversary is restricted to playing \( \mathcal{D}_t \in \mathcal{D}_t \subseteq \Delta(\mathcal{X}) \) at each time \( t \), a relaxation \( \{\text{Rel}_T(\mathcal{H} \mid s_{1:t})\}_{t=0}^{T} \) is admissible if for any sequence of instances \( s_{1:T} \),

1. For all \( t \in [T] \),

\[
\sup_{\mathcal{D}_t \in \mathcal{D}_t \vDash X, Q_t \in \Delta(\mathcal{Y})} \inf_{y_t \in \mathcal{Y}} \sup_{y_{t-1} \sim Q_{t-1}} \left\{ \mathbb{E} [l(\hat{y}_t, y_t)] + \text{Rel}_T(\mathcal{H} \mid s_{1:t-1} \cup (x_t, y_t)) \right\} \leq \text{Rel}_T(\mathcal{H} \mid s_{1:t-1}),
\]

2. The final value satisfies \( \text{Rel}_T(\mathcal{H} \mid s_{1:T}) \geq -\inf_{h \in \mathcal{H}} L(h, s_{1:T}) \).
In the smoothed online learning setting, $\mathcal{D}_t = \Delta_\sigma(\mathcal{X})$ is the set of $\sigma$-smooth distributions on $\mathcal{X}$. In the $K$-hint transductive online learning setting, $\mathcal{D}_t$ is the set of distributions supported on $Z_t$.

Note that the reason we consider admissibility is that it points towards the existence of efficient algorithms. In particular, if we could exactly solve the sup-inf optimization in the first condition, then playing $\arg\inf_{Q_t}$ leads to an algorithm whose regret is upper bounded by $\text{Rel}_T(\mathcal{H} | \emptyset)$. However, when translating admissibility into practical algorithms, the optimization needs not be exact. In fact, any algorithm $\{Q_t\}_{t=1:T}$ that satisfies the first inequality is an admissible algorithm with respect to $\text{Rel}_T(\mathcal{H})$, and any admissible algorithm has regret upper bounded as in Proposition 2.1. The following proposition is the analog of the results of [RSS12] when the adversary is constrained. The full proof is presented in Appendix B.

**Proposition 2.1** (Regret Bound via Admissibility). In the online learning setting where the adversary’s choice is restricted to $\mathcal{D}_t$ ($t \in [T]$), let $\mathcal{D}$ be an algorithm that is admissible with respect to relaxations $\text{Rel}_T(\mathcal{H})$. Suppose $\hat{y}_t \sim Q_t$ ($t \in [T]$) are the strategies decided by $\mathcal{D}$, then the following bound on the expected regret holds regardless of the strategies $\mathcal{D}$ of the adversary:

$$
\mathbb{E}[\text{REGRET}(T)] = \mathbb{E}_{\mathcal{D}, \hat{\mathcal{D}}} \left[ \sum_{t=1}^{T} l(\hat{y}_t, y_t) - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} l(h(x_t), y_t) \right] \leq \text{Rel}_T(\mathcal{H} | \emptyset) + O(\sqrt{T}).
$$

## 3 $K$-hint Transductive Learning

As a prelude, we will first look at the $K$-hint transductive learning setting, which highlights some challenges that are present in the smoothed online setting while allowing us to discuss the required tools in a simpler scenario. The statistical upper bound given by the inefficient algorithm that simply plays Hedge on all the different labelings of the hint set is $O(\sqrt{dT \log(TK)})$. We will show an oracle-efficient regret upper bound of $O(\sqrt{dT K})$ by constructing an oracle-efficient algorithm based on the random playout technique.

Let us begin by describing our algorithm for the setting of $K$-hint transductive learning. At each time step $t$, our algorithm applies the ERM oracle to two madeup histories of the play: One where the real history $s_{1:t-1}$ is appended to a randomly labeled set of all hints corresponding to future time steps and the current instance is labeled $+1$, and another, where the current label is labeled $-1$. More specifically, with $\mathcal{E}^{(t)} = \{\epsilon_{i,k}^{(t)}\}_{i=t+1:T, k=1:K}$ denoting the set of random labels and $S^{(t)} = Z_{t+1:T} \circ \mathcal{E}^{(t)}$, the learner predicts $\hat{y}_t \in \{-1, +1\}$ such that

$$
\mathbb{E}[\hat{y}_t] = \text{OPT} \left( s_{1:t-1} \cup S^{(t)} \cup \{(x_t, -1)\} \right) - \text{OPT} \left( s_{1:t-1} \cup S^{(t)} \cup \{(x_t, +1)\} \right)
$$

$$
= \sup_{h \in \mathcal{H}} \left\{ \sum_{i=t+1:T} \sum_{k=1:K} \epsilon_{i,k}^{(t)} h(z_{i,k}) + \frac{\sum_{\tau=1}^{t-1} y_{\tau} h(x_{\tau})}{2} - \frac{h(x_t)}{2} \right\} - \sup_{h' \in \mathcal{H}} \left\{ \sum_{i=t+1:T} \sum_{k=1:K} \epsilon_{i,k}^{(t)} h'(z_{i,k}) + \frac{\sum_{\tau=1}^{t-1} y_{\tau} h'(x_{\tau})}{2} - \frac{h(x_t)}{2} \right\}.
$$

(2)

We formally present this as Algorithm 3 in the appendix. The following theorem provides an upper bound on the regret.
Theorem 3.1 (Regret Bound for Efficient K-Hint Transductive Learning). In the setting of transductive learning with K-hints, the above algorithm has expected regret bound of $O(\sqrt{dTK})$. The algorithm can be implemented using two calls to the optimization oracle per round.

The proof of Theorem 3.1 follows a similar approach to the work of [RSS12] and uses the admissible relaxation framework. In this section, we provide an overview of this approach and the modifications that allow us to incorporate a larger set of hints. Specifically, we will show that the algorithm is admissible with respect to the following relaxation:

$$
\text{Rel}_T(\mathcal{H} \mid s_{1:t}) = \mathbb{E}_{\xi(t)} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{i=t+1}^{T} \epsilon_{i,k} h(z_{i,k}) - \sum_{i=1}^{t} L(h, s_{i}) \right\} \right], \quad t = 0, \ldots, T. \tag{3}
$$

To simplify the notation, for a set of unlabeled instances $Z = \{z_i\}_{i=1}^{I}$ and a function $\Phi : \mathcal{H} \rightarrow \mathbb{R}$, we define $\mathfrak{R}(\Phi, Z)$ as the Rademacher complexity for set $Z$ regularized by $\Phi$, that is

$$
\mathfrak{R}(\Phi, Z) = \mathbb{E}_{\epsilon_1, \ldots, \epsilon_t \overset{i.i.d.}{\sim} \mathbb{U}(\pm 1)} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{i \leq t} \epsilon_i h(z_i) + \Phi(h) \right\} \right].
$$

Then the relaxation at the end of time step $t$ can be written as the Rademacher complexity for the union of future hints, regularized by the past total loss. That is,

$$
\text{Rel}_T(\mathcal{H} \mid s_{1:t}) = \mathfrak{R}(-L(\cdot, s_{1:t}), Z_{t+1:T}), \tag{4}
$$

where the first input to $L$ is a classifier $h \in \mathcal{H}$. We remark that while the original definition of relaxation keeps track of the sequence of inputs that are already known at the end of each time step $(s_{1:t})$, the regularized Rademacher complexity also explicitly emphasizes the instances $Z_{t+1:T}$ denoting unknown future. Between successive time steps, one extra data point $s_t$ will be added to the set of known inputs, but the set of unknown future instances will shrink by $K$.

To use the relaxation framework and Proposition 2.1, it suffices to establish two claims: 1) the relaxation in Equation (4) is admissible in the $K$-hint setting, 2) the value of this relaxation at the beginning of the game is not too large.

For the second claim, we notice that $\text{Rel}_T(\mathcal{H} \mid \emptyset)$ is equal to the unregularized Rademacher complexity for the dataset that includes all the hints. Since there are at most $TK$ hints, the Rademacher complexity is at most $O(\sqrt{dTK})$. That’s where we get the extra $\sqrt{K}$ in the bounds compared to the standard transductive setting.

The first claim is the more technically interesting one. For admissibility, here we will focus on proving the following bound

$$
\sup_{x_t \in Z_t} \inf_{Q_t, \in \Delta(Y)} \sup_{y_t \in \mathcal{Y}} \left[ \mathbb{E}_{\tilde{y}_t \sim Q_t} \left[ l(\hat{y}_t, y_t) \right] + \mathfrak{R}(-L_t, Z_{t+1:T}) \right] \leq \sup_{x_t \in Z_t} \mathfrak{R}(-L_{t-1}, Z_{t+1:T} \cup \{x_t\}) \leq \mathfrak{R}(-L_{t-1}, Z_{t:T}) \tag{5}
$$

Let us consider the L.H.S of the above inequality and note that for any fixed $x_t$, the term (a) captures the standard transductive learning setting with $Z_{t+1:T}$ being the set of unlabeled instances for the future. In this case, the min-max theorem can be used to show that the learner’s optimal
strategy, $Q_t$, is attained when the two values inside the supremum over $Y$ equalize as $y_t$ takes value $-1$ and $+1$. At a high level, this technique which is also used by [RSS12], gives rise to the algorithm in Equation (2) and proves the first inequality in inequality (5). A complete proof is presented in Appendix C.2.

The second transition in Inequality (5) can be establishing the fact that regularized Rademacher complexity is monotone in the dataset. See Appendix C.3 for a proof of lemma 3.2.

Lemma 3.2 (Monotonicity of Regularized Rademacher Complexity). For any dataset $z_{1:m} \in \mathcal{X}^m$ and any additional data point $x \in \mathcal{X}$, we have $\mathcal{R}(\Phi, z_{1:m}) \leq \mathcal{R}(\Phi, z_{1:m} \cup \{x\})$.

This monotonicity can be used recursively to add the extra set of hints $Z_t$ to the relaxation. This implies $\sup_{x_t \in Z_t} \mathcal{R}(-L_{t-1}, Z_{t+1:T} \cup \{x_t\}) \leq \mathcal{R}(-L_{t-1}, Z_{t+1:T} \cup Z_t)$.

4 Oracle-Efficient Smoothed Online Learning

In this section, we prove our main result for the case of smoothed analysis of online learning and give the first oracle-efficient algorithms for smoothed analysis of online learning whose regret are sublinear in $T$ and VC dimension.

Theorem 4.1 (Regret Bound for Efficient Smoothed Online Learning). In the setting of online learning with $\sigma$-smoothed adversaries, Algorithm 2 has regret that is at most

$$\tilde{O} \left( \min \left\{ \sqrt{\frac{Td}{\sigma^{1/2}}}, \sqrt{T(|\mathcal{X}|)^{1/2}} \right\} \right).$$

Furthermore, Algorithm 2 is a proper learning oracle-efficient: at every round $t$, this algorithm uses a single ERM oracle call a history that is of length $t + O(T / \sqrt{\sigma})$ with high probability.

We note that Theorem 4.1 establishes that under smoothed analysis online learning is as computationally efficient as offline learning. This is in contrast with the results of [HK16] that showed a computational separation between offline learning and online learning with worst-case adversaries. It is worth noting that Theorem 4.1 also matches or improves upon several existing results for worst-case online learning (without smoothed analysis). For example, for finite domains where worst-case adversaries are vacuously $\sigma$-smooth for $1/\sigma = |\mathcal{X}|$, Theorem 4.1 gives an oracle-efficient regret bound of $O(T^{1/2}(d|\mathcal{X}|)^{1/4})$ which is a refinement of [DS16] regret bound of $O(\sqrt{T|\mathcal{X}|})$ because $d \leq |\mathcal{X}|$. See Corollary 1.1 for a precise statement. Theorem 4.1 also implies the regret bound of $\tilde{O}(T^{3/4}d^{1/4})$ for the classical transductive learning setting, which corresponds to finite domains with $\sigma = 1/T$ parameter. This improves the $\tilde{O}(T^{3/4}d^{1/2})$ bound of [KK06].

The key challenge with adapting our previous approach is that, unlike the $K$-hint transductive setting, in smoothed analysis the learner no longer has access to any hints. For finite domains, such as the above examples, our (and existing) algorithms self-generate hints from every instance in $\mathcal{X}$, using uniform or geometric random variables. But these choices do not directly extend to large and infinite domains where most of the domain $\mathcal{X}$ will not be present in the hints. Therefore, for smoothed analysis with large or infinite domains, our algorithms must self-generate hints that leverage the anti-concentration properties of smooth distributions. In Section 4.1, we show
how this can be done by taking $K = \tilde{O}(1/\sigma)$ uniformly random instances from $\mathcal{X}$ and using the techniques we developed in Section 3 to achieve a regret bound of $\tilde{O}(\sqrt{Td/\sigma})$. Interestingly, the particular form of randomness used for producing hints is consequential. In Section 4.2, we show how the Poisson distribution over the size of the hint set is an appropriate choice to leverage properties implied by the smoothness of the distributions while leading to relaxations that enjoy better admissibility and stability guarantees.

Another important aspect of stability is the use of a modified definition of generalization error, i.e., relating the performance of ERM trained with uniform self-generated hints had we re-sampled the last smoothed adversarial instance. We will provide insights about the coupling approach that allows us to relate generalizability with respect to smooth distributions to that of the uniform distribution directly.

4.1 Re-imagining Smoothed Adversaries as $K$-hint Transductive Learning

In this section, we show how insights and techniques for the $K$-hint transductive setting in Section 3 transition to the setting of smoothed online setting and lead to $\tilde{O}(\sqrt{Td/\sigma})$ regret. We will further build upon this connection in Section 4.2 to improve on the regret bound of this section.

The key challenge with adapting the previous techniques is to self-generate hints that leverage the anti-concentration properties of smooth distributions. In particular, we will use the coupling technique introduced by [HRS21] (see Lemma D.1 for a complete description) to replace the sequence of $T$ random inputs $x_1:T$ generated by the adaptive adversary with $TK$ inputs $\{z_{t,k}\}_{t=1:T, k=1:K}$ that are generated i.i.d. from the uniform distribution over $\mathcal{X}$, such that with high probability $\{x_1, \cdots, x_T\} \subseteq \{z_{t,k}\}_{t=1:T, k=1:K}$. This property highlights the connection between a smoothed adaptive adversary and a transductive adversary with $K$ uniform random hints.

From a computational point of view, this observation implies that a learner can generate uniform random hints, denoted by $\{v_{i,k}^{(t)}\}_{i=t+1:T, k=1:K}$, and use them in place of $\{z_{i,k}\}_{i=t+1:T, k=1:K}$ in Algorithm 3. The resulting algorithm is summarized in Algorithm 1 and leads to the following regret upper bound.

**Theorem 4.2 (Warmup).** For any $\sigma$-smooth adversary $D_\sigma$, Algorithm 1 has expected regret upper bounded by $O(\sqrt{Td \log(T)/\sigma}) = \tilde{O}(\sqrt{Td/\sigma})$. Furthermore, Algorithm 1 is oracle-efficient: at every round $t$, this algorithm use two ERM oracle calls with histories of length $\tilde{O}(T/\sigma)$.

To prove Theorem 4.2, we adjust the relaxation we used in Section 3 and show that it is admissible with respect to the algorithm. We use the following relaxation in the smoothed learning setting,

$$
\text{Rel}_T(\mathcal{H}|x_{1:t}, y_{1:t}) = \mathbb{E}_{V^{(t)}, G^{(t)}} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{i=t+1:T, k=1:K} \varepsilon_{i,k}^{(t)} h(v_{i,k}^{(t)}) + \frac{1}{2} \sum_{i=1}^t y_i h(x_i) \right\} \right] + \beta(T - t),
$$

where $K = 100 \log T/\sigma$ and $\beta = 10TK(1 - \sigma)^K$. This relaxation incorporates two modifications to the relaxation in Equation (3). The first is an expectation over the randomness of future hints. A subtle point here is that this randomness crucially matches both random $V^{(t)}$ that the algorithm has access to and the coupled variables $\{z_{i,k}\}$ at a distribution level that are never revealed to the
Algorithm 1: Oracle-Efficient Smoothed Online Learning with a Fixed Number of Hints

Input: $T, \sigma$

1. $K \leftarrow 100 \log T/\sigma$.
2. for $t \leftarrow 1$ to $T$ do
3.   Receive $x_t$.
4.   for $i = t+1, \ldots, T$; $k = 1, \ldots, K$ do
5.     Draw new $v_{i,k}^{(t)} \sim \mathcal{U}(X)$.
6.     Draw new $\epsilon_{i,k}^{(t)} \sim \mathcal{U}\{-1, 1\}$.
7.   end
8.   $S_1(t) \leftarrow \{(v_{i,k}^{(t)}, \epsilon_{i,k}^{(t)}), (v_{i,k}^{(t)}, \epsilon_{i,k}^{(t)})\}_{i=t+1:T} \quad$ // Two copies of each tuple
9.   $S_2(t) \leftarrow \{(x_{\tau}, y_{\tau})\}_{\tau=1}^{t-1}$
10. $\tilde{y}_t \leftarrow \text{OPT}_{\mathcal{H}_i}(S_1^{(1)} \cup S_2^{(2)} \cup \{(x_t, -1)\}) - \text{OPT}_{\mathcal{H}_i}(S_1^{(1)} \cup S_2^{(2)} \cup \{(x_t, +1)\})$
11. With probability $q_t = \frac{1 - \tilde{y}_t}{2}$, predict $\hat{y}_t = -1$; otherwise predict $\hat{y}_t = +1$
12.   // $\mathbb{E}[\hat{y}_t] = \tilde{y}_t$.
13. Receive $y_t$, suffer loss $l(\hat{y}_t, y_t)$.
end

To ensure that our approach works with adaptive adversaries, it is essential that $V(t)$s are fresh samples per round. The second is an additional time dependent term $\beta(T - t)$, which accounts for the total failure probability of the coupling argument in the future $T - t$ time steps. Since the coupling argument shows that $K = O(\log(T)/\sigma)$, we use parameter $\beta = o(T)$ and achieve a regret upper bound of $O(\sqrt{dT K}) = \tilde{O}(\sqrt{T d/\sigma})$. See Appendix D.2 for more details about the proof.

4.2 Improved Algorithms via Poissonization

In this section we provide a new algorithm for oracle-efficient smoothed online learning that achieve the regret bound of $\tilde{O}(\sqrt{dT \sigma^{-1/2}})$ as stated in Theorem 4.1. This algorithm which is based on the the Follow-the-Perturbed-Leader (FTPL) framework differs from the hint-based algorithm in two ways. First, instead of accessing the ERM oracle twice and making a randomized decision at each round, the new algorithm only calls the ERM oracle once and follows the prediction of the output. Second, and importantly, the number of self-generated hints does not shrink over time and follows an appropriate probability distribution (such as Poisson). This Poissonization is the key to establishing improved stability of the algorithm.

A detailed description of the algorithm is illustrated below. At a high level, at each time step the learner generates $N \sim \text{Poi}(n)$ to represent the total number of self-generated hints it will be feeding itself, and then takes $N$ samples $\tilde{x}_i \sim \mathcal{U}(X)$, each labeled independently by a Rademacher variable $\tilde{y}_i$. The learner then uses the ERM oracle on the history of the play so far and the hints to compute a hypothesis $h_t$, which it uses for its prediction $\hat{y}_t = h_t(x_t)$ at time $t$.

In the remainder of this section, we present a proof of the regret upper bound $\tilde{O}(\sqrt{dT \sigma^{-1/2}})$ in Theorem 4.1 when $\sigma \geq d/|X|$. The proof of the other case $\sigma < d/|X|$ is slightly different and.
Algorithm 2: Smoothened Online Learning based on Poisson Number of Hints

**Input:** time horizon \( T \), smoothness parameter \( \sigma \), VC dimension \( d \)

1. \( n \leftarrow \min\{T/\sqrt{\sigma}, T\sqrt{|\mathcal{X}|/d}\} \)
2. for \( t \leftarrow 1 \) to \( T \) do
   3. generate \( N^{(t)} \sim \text{Poi}(n) \) fresh hallucinated samples \((\tilde{x}_1^{(t)}, \tilde{y}_1^{(t)}), \cdots, (\tilde{x}_N^{(t)}, \tilde{y}_N^{(t)})\), which are i.i.d. conditioned on \( N \) with \( \tilde{x}_i^{(t)} \sim \mathcal{U}(\mathcal{X}) \) and \( \tilde{y}_i^{(t)} \sim \mathcal{U}\{ \pm 1 \} \);
   4. call the ERM oracle to compute \( h_t \leftarrow \text{opt}_{h \in \mathcal{H}} \left\{ (\tilde{x}_i^{(t)}, \tilde{y}_i^{(t)}) \right\}_{i \in N^{(t)}} \cup \{x_{\tau}, y_{\tau}\}_{\tau \in [t-1]} \);
   5. observe \( x_t \), predict \( \hat{y}_t = h_t(x_t) \), and receive \( y_t \).
   6. end

will be presented in Section 4.2.3. Similar to Section 3, we will provide a relaxation-based upper bound for the regret of Algorithm 2. Writing \( s = (x,y) \) and \( L(h,s) = l(h(x),y) = -yh(x)/2 \), the relaxation is defined as

\[
\text{Rel}_{T}(\mathcal{H} \mid s_{1:t}) = \mathbb{E}_{R^{(t+1)}} \left[ \sup_{h \in \mathcal{H}} \left( - \sum_{i=1}^{N^{(t+1)}} L(h, \tilde{s}_i^{(t+1)}) - \sum_{\tau=1}^{t} L(h, s_{\tau}) \right) \right] + \eta(T-t), \tag{6}
\]

where

\[
\eta = \frac{1}{\sqrt{n} \sigma} + c \sqrt{\frac{d \log T}{n \sigma}} + \frac{n \sigma}{4T^2 \log T} + e^{-n/8} \in \tilde{O}\left( \sqrt{\frac{d}{n \sigma}} \right),
\]

with an absolute constant \( c > 0 \) given in Lemma 4.5 later, and \( R^{(t)} = (N^{(t)}, \{\tilde{s}_i^{(t)}\}_{i \in N^{(t)}}) \) is the fresh randomness generated at the beginning of time \( t \), which is independent of \( \{s_{\tau}\}_{\tau < t} \) generated by the adversary. The relaxation here is similar to Equation (3) in the transductive setting, where the key difference is a different generation process for the hint set and an additional term \( \eta(T-t) \) to account for the stability.

Let \( Q_t \) be the distribution of the learner’s action \( h_t \in \mathcal{H} \) in Algorithm 2, then the relaxation in Equation (6) is admissible with respect to Algorithm 2 if the following two conditions hold:

\[
\sup_{\mathcal{D}_t \in \Delta_{\mathcal{X}}(\mathcal{X})} \mathbb{E}_{x_t \sim \mathcal{D}_t} \sup_{y_t \sim Q_t} \left[ \mathbb{E}_{h_t \sim \mathcal{Q}_t} \left[ L(h_t,s_t) \right] + \text{Rel}_{T}(\mathcal{H} \mid s_{1:t}) \right] \leq \text{Rel}_{T}(\mathcal{H} \mid s_{1:t-1}), \quad \forall s_{1:t-1} \tag{7}
\]

\[
\text{Rel}_{T}(\mathcal{H} \mid s_{1:T}) \geq - \inf_{h \in \mathcal{H}} L(h, s_{1:T}). \tag{8}
\]

Note that Inequalities (7) and (8) are slightly different from those in Definition 2.3: we stick to the distribution \( Q_t \) used by Algorithm 2 instead of taking the infimum over \( Q_t \). This algorithm is admissible with respect to the relaxation defined above. See the remarks after Definition 2.3 for more details. Therefore, if both Inequalities (7) and (8) hold, the expected regret of Algorithm 2 will satisfy

\[
\mathbb{E}[\text{Regret}(T)] \leq \text{Rel}_{T}(\mathcal{H} \mid \emptyset) + O(\sqrt{T}) = \mathbb{E}_{R^{(1)}} \left[ \sup_{h \in \mathcal{H}} \left( - \sum_{i=1}^{N^{(1)}} L(h, \tilde{s}_i^{(1)}) \right) \right] + \eta T + O(\sqrt{T})
\]
\[ a \equiv O \left( \mathbb{E}_{N(1)} \left[ \sqrt{dN(1)} \right] + \eta T + \sqrt{T} \right) \]

and Theorem 4.1 follows from the choices \( n = T/\sqrt{\sigma} \) and \( \eta = O(\sqrt{d/n\sigma}) \). In the above inequality, step (a) follows from random labels and the upper bound \( O(\sqrt{nd}) \) on the Rademacher complexity of \( \mathcal{H} \) over \( n \) points, and step (b) is due to Jensen’s inequality and \( \mathbb{E}[N(1)] = n \).

Now it remains to verify Inequalities (7) and (8): this follows from the fact that for any random variable \( \lambda, \mathbb{E}[\sup_{\lambda} X_{\lambda}] \geq \sup_{\lambda} \mathbb{E}[X_{\lambda}] \) and \( \mathbb{E}[L(h, s_t)] = 0 \). The key technical difficulty is in the proof of Inequality (7). To overcome this challenge, we first draw a parallel between two types of analysis in online learning, by showing that the stability of learner’s distribution \( \mathcal{Q}_t \) implies the admissibility of the relaxation, where the stability is measured via

\[
\text{Stability} = \mathbb{E}_{s_t \sim \mathcal{D}_t} \left( \mathbb{E}_{h_t \sim \mathcal{Q}_t} [L(h_t, s_t)] - \mathbb{E}_{h_{t+1} \sim \mathcal{Q}_{t+1}} [L(h_{t+1}, s_t)] \right).
\]

Note that here \( s_t \sim \mathcal{D}_t \) denotes both the instance and its label and \( \mathcal{D}_t \)’s marginal over \( \mathcal{X} \) is \( \sigma \)-smooth. We further upper bound stability (see Lemma 4.3) as follows:

\[
\text{Stability} \leq \text{TV}(\mathcal{Q}_t, \mathcal{Q}_{t+1}) + \mathbb{E}_{s_t, s'_{t+1} \sim \mathcal{D}_t; \mathcal{R}(t+1)} [L(h_{t+1}, s'_t) - L(h_{t+1}, s_t)].
\]

Let us first describe the two terms in this bound further. The first term is the total variation (TV) distance between \( \mathcal{Q}_t \) and the mixture distribution \( \mathbb{E}_{s_t \sim \mathcal{D}_t} [\mathcal{Q}_{t+1}] \). Note that this TV distance would be an upper bound on the stability by itself, if neither of \( \mathcal{Q}_t \) and \( \mathcal{Q}_{t+1} \) depend on the new observation \( s_t = (x_t, y_t) \) at time \( t \). However, while \( \mathcal{Q}_t \) is independent of \( s_t, h_{t+1} \) is trained on \( s_t \) and thus distribution \( \mathcal{Q}_{t+1} \) does depend on \( s_t \). To overcome this dependence, we introduce a ghost sample \( s'_t \) that allows us to decouple \( h_{t+1} \sim \mathcal{Q}_{t+1} \) and the new observation. This gives rise to the second term which is a modified generalization error. In other words, let \( s'_t \) be an independent copy of \( s_t \) conditioned on \( s_{1:t-1} \). The expected loss of the classifier \( h_{t+1} \), which is trained on \( s_t \) but not \( s'_t \), on the ghost sample \( s'_t \) is denoted by \( \mathbb{E}_{s_t, s'_t \sim \mathcal{D}_t; \mathcal{R}(t+1)} [L(h_{t+1}, s'_t)] \). The expectation here is understood in the sense that ERM classifier \( h_{t+1} \) is determined by the self-generated samples, the current observation \( s_t \), and the history \( s_1, \ldots, s_{t-1} \) (which are held fixed). The generalization error is then defined to be the expected difference

\[
\text{Modified generalization error} := \mathbb{E}_{s_t, s'_t \sim \mathcal{D}_t; \mathcal{R}(t+1)} [L(h_{t+1}, s'_t) - L(h_{t+1}, s_t)].
\]

The following lemma formalizes this discussion and shows that a small TV distance and generalization error suffice to ensure the stability of the algorithm, which in turn implies the admissibility of the relaxation. This result could be of independent interest. The proof can be found in Appendix E.1.

**Lemma 4.3 (TV + Generalization ⇒ Stability ⇒ Admissibility).** Let \( \mathcal{Q}_t \) denote learner’s distribution over \( \mathcal{H} \) in Algorithm 2 at round \( t \), \( \mathcal{D}_t \) be adversary’s distribution at time \( t \) (given the history \( s_1, \ldots, s_{t-1} \), \( s_t \sim \mathcal{D}_t \) be the realized adversarial instance at time \( t \), and \( s'_t \) be an independent copy \( s'_t \sim \mathcal{D}_t \). It holds that

\[
\mathbb{E}_{s_t \sim \mathcal{D}_t} \left( \mathbb{E}_{h_t \sim \mathcal{Q}_t} [L(h_t, s_t)] + \text{Rel}_T(\mathcal{H} \mid s_{1:t}) \right) - \text{Rel}_T(\mathcal{H} \mid s_{1:t-1})
\]
Lemma 4.5 (Upper Bound of Generalization Error) for an absolute constant up to time $P$

Let $n$ and replacing the coupling and all details are presented in Section 4.2.2. remains to upper bound the TV distance and the generalization error, respectively.

For $t \in [T] \cup \{0\}$, let $r^t \in \mathbb{Z}^X$ be the $|X|$-dimensional random vector with $r^t(x)$ defined to be the difference between the number of $+1$ and $-1$ labels in the self-generated samples and the history up to time $t$ on instance $x$. Formally,

$$r^t(x) = \sum_{i=1}^{N(t+1)} \tilde{y}^{(t+1)}_i \cdot 1(\tilde{x}^{(t+1)}_i = x) + \sum_{\tau=1}^t y_\tau \cdot 1(x_\tau = x).$$

Let $P^t$ be the distribution of $r^t$. The reason why we introduce this notion is that $h_t$ in Algorithm 2 only depends on the vector $r^{t-1}$, so the ERM objective could be written as a quantity depending only on $r^{t-1}$ and $h \in \mathcal{H}$. We write $h_t = \text{opt}_{\mathcal{H},t}(r^t)$ in the sequel, and then $\text{opt}_{\mathcal{H},t}(r^t) \sim Q_t$ as $r^{t-1} \sim P^{t-1}$. Therefore, the data-processing inequality shows that

$$TV(Q_t, \mathbb{E}_{s_t \sim D_t} [Q_{t+1}]) \leq TV(P^{t-1}, \mathbb{E}_{s_t \sim D_t} [P^t]),$$

and the following lemma provides an upper bound on the TV distance $TV(P^{t-1}, \mathbb{E}_{s_t \sim D_t} [P^t])$.

**Lemma 4.4 (Upper Bound of TV Distance).** Let $P^t$ be the distribution over $r^t$ defined above. We have

$$\sup_{D_t \in \Delta_0(S)} TV(P^{t-1}, \mathbb{E}_{s_t \sim D_t} [P^t]) \leq \frac{1}{\sqrt{n\sigma}}.$$

The key ingredient in the proof of Lemma 4.4 is the Poissonization, which ensures the independence across the coordinates of $r^t$ and enables us to write down the mixture distribution $\mathbb{E}_{s_t \sim D_t} [P^t]$ in a compact form. The proof of Lemma 4.4 is presented in Section 4.2.1.

The following lemma upper bounds the generalization error for any smooth distribution $D_t$.

**Lemma 4.5 (Upper Bound of Generalization Error).** Under the notations of Lemma 4.3, it holds for an absolute constant $c > 0$ (independent of $(n, d, T, \sigma)$) that

$$\sup_{D_t \in \Delta_0(X)} \mathbb{E}_{s_t, s_t' \sim D_t, R^{t+1}} \left[ L(h_{t+1}, s_t') - L(h_{t+1}, s_t) \right] \leq c \sqrt{\frac{d \log T}{n\sigma}} + \frac{n\sigma}{4T^2 \log T} + e^{-n/8}.$$

The intuitive idea behind Lemma 4.5 is as follows. Consider the simpler setting of $t = 1$ (i.e. no history) and $D_t = \mathcal{U}(X \times \{\pm 1\})$ (i.e. the new observation $s_t$ follows the same distribution as the self-generated samples). In this case, the generalization error is precisely the difference between the test error and the training error with $N + 1$ iid training data, and classical Rademacher complexity gives an upper bound $O(\sqrt{d/n})$. For general $\sigma$-smooth $D_t$, a coupling argument essentially shows that $n$ iid training data from $\mathcal{U}(X \times \{\pm 1\})$ contain $n\sigma$ iid training data from $D_t$, and replacing $n$ by $n\sigma$ in the previous upper bound gives Lemma 4.5. The rigorous treatment of the coupling and all details are presented in Section 4.2.2.

Now the claimed result of Theorem 4.1 when $\sigma \geq d/|X|$ follows from Lemma 4.3, Lemma 4.4, and Lemma 4.5.
4.2.1 Upper Bounding TV Distance: Proof of Lemma 4.4

Let us first create a better understanding of the structures of the distributions $\mathcal{P}^{t-1}$ and $\mathcal{P}^t$. Without loss of generality we assume that $\mathcal{X}$ is discrete (the case of continuous $\mathcal{X}$ can be dealt by analyzing the appropriate Poisson point process). Let $n_+(x)$, $n_-(x)$ be the numbers of +1 and −1 labels, respectively, given instance $x$ in the self-generated samples:

$$n_+(x) = \sum_{i=1}^{N} 1(\tilde{x}_i = x, \tilde{y}_i = +1) \quad \text{and} \quad n_-(x) = \sum_{i=1}^{N} 1(\tilde{x}_i = x, \tilde{y}_i = -1).$$

As each $\tilde{x}_i$ is uniformly distributed on $\mathcal{X}$ and $\tilde{y}_i \sim \mathcal{U}(\{\pm 1\})$, by the subsampling property of the Poisson distribution, the $2|\mathcal{X}|$ random variables $\{n_{\pm}(x)\}_{x \in \mathcal{X}}$ are i.i.d. distributed as $\text{Poi}(n/2|\mathcal{X}|)$. This independence implied by the Poisson distribution plays a key role in the analysis. Moreover, $r^0(x) = n_+(x) - n_-(x)$, so $\mathcal{P}^0$ is determined by the joint distribution of $\{n_{\pm}(x)\}_{x \in \mathcal{X}}$.

As we move to general $t$, note that the only contribution of the historic data $\{s_\tau\}_{\tau < t}$ to both $\mathcal{P}^{t-1}$ and $\mathcal{P}^t$ is a common translation independent of $\mathcal{P}^0$. Since the TV distance is translation invariant, it suffices to upper bound $TV(\mathcal{P}^0, \mathbb{E}_{s_1}[\mathcal{P}^1])$. Let $n_{\pm}(x) = n_{\pm}(x) + 1(x_1 = x, y_1 = \pm 1)$, it holds that $r^1(x) = n_{\pm}(x) - n_{\pm}(x)$. Consequently, let $P$ and $Q$ be the probability distributions of $\{n_{\pm}(x)\}_{x \in \mathcal{X}}$ and $\{n_{\pm}(x)\}_{x \in \mathcal{X}}$, respectively, the data-processing inequality implies that $TV(\mathcal{P}^0, \mathbb{E}_{s_1}[\mathcal{P}^1]) \leq TV(P, Q)$.

As discussed above, the distribution $P$ is a product Poisson distribution:

$$P(\{n_{\pm}(x)\}) = \prod_{x \in \mathcal{X}} \prod_{y \in \{\pm 1\}} \mathbb{P}(\text{Poi}(n/2|\mathcal{X}|) = n_y(x)).$$

As for the distribution $Q$, it could be obtained from $P$ in the following way: the smooth adversary draws $x^* \sim D$, independent of $\{n_{\pm}(x)\}_{x \in \mathcal{X}} \sim P$, for some $\sigma$-smooth distribution $D \in \Delta_\sigma(\mathcal{X})$. He then chooses a label $y^* = y(x^*) \in \{\pm 1\}$ as a function of $x^*$, and sets

$$n_{y(x^*)}(x^*) = n_{y(x^*)}(x^*) + 1 \quad \text{and} \quad n_y(x) = n_y(x), \quad \forall (x, y) \neq (x^*, y(x^*)).$$

Consequently, given a $\sigma$-smooth distribution $D$ and a labeling function $y: \mathcal{X} \rightarrow \{\pm\}$ used by the adversary, the distribution $Q$ is a mixture distribution $Q = \mathbb{E}_{x^* \sim D}[Q_{x^*}]$, with

$$Q_{x^*}(\{n_{\pm}(x)\}) = \mathbb{P}(\text{Poi}(n/2|\mathcal{X}|) = n_{y(x^*)}(x^*) - 1) \times \prod_{(x, y) \neq (x^*, y(x^*))} \mathbb{P}(\text{Poi}(n/2|\mathcal{X}|) = n_y(x)).$$

To upper bound the TV distance between a mixture distribution $Q$ and a base distribution $P$, we will rely on the smoothness properties of $D$, in particular, that the probability of collision between two independent draws $x^*_1, x^*_2 \sim D$ is small. To formally address this, we make use of two technical lemmas, first to upperbound the TV distance in terms of the $\chi^2$ distance, and second to use the Ingster’s method for bounding the $\chi^2$ distance between a mixture distribution and a base distribution. See Lemma A.1 and Lemma A.2 in the Appendix A for more details. Let $x_1^*, x_2^*$ be an arbitrary pair of instance. Using the closed-form expressions of distributions $P$ and $Q_{x^*}$, it holds that

$$\frac{Q_{x_1^*}(\{n_{\pm}(x)\}) Q_{x_2^*}(\{n_{\pm}(x)\})}{P(\{n_{\pm}(x)\})^2} = \frac{2|\mathcal{X}| n_{y(x_1^*)}(x_1^*)}{n} \frac{2|\mathcal{X}| n_{y(x_2^*)}(x_2^*)}{n}. $$
Using the fact that \( \{n_\pm(x)\}_{x \in \mathcal{X}} \) are i.i.d. distributed as \( \text{Poi}(n/2|\mathcal{X}|) \) under \( P \), we have

\[
\mathbb{E}_{\{n_\pm(x)\} \sim P} \left( \frac{Q_{x_1}\{n_\pm(x)\}Q_{x_2}\{n_\pm(x)\}}{P\{n_\pm(x)\}^2} \right) = 1 + \frac{2|\mathcal{X}|}{n} \cdot 1(x_1^* = x_2^*).
\]

Now using the aforementioned lemmas (Lemma A.1 and Lemma A.2), we have

\[
\text{TV}(P, Q) \leq \sqrt{\frac{\chi^2(Q, P)}{2}} = \sqrt{\frac{\mathbb{E}[x \sim \text{D}(x), P] - \mathbb{E}[x \sim \text{D}(x), Q]}{2}} = \sqrt{\frac{|\mathcal{X}|}{n} \cdot \mathbb{E}_{x_1, x_2 \sim \text{D}}[1(x_1^* = x_2^*)]} = \frac{1}{\sqrt{\sigma n}},
\]

where (a) follows from the definition of a \( \sigma \)-smooth distribution. This completes the proof.

### 4.2.2 Upper Bounding Generalization Error: Proof of Lemma 4.5

In the proof of Lemma 4.5, we shall need the following property of smooth distributions which is a slightly strengthened version of the coupling lemma in Lemma D.1.

**Lemma 4.6.** Let \( X_1, \cdots, X_m \sim Q \) and \( P \) be another distribution with a bounded likelihood ratio: \( dP/dQ \leq 1/\sigma \). Then using external randomness \( R \), there exists an index \( I = I(X_1, \cdots, X_m, R) \in [m] \) and a success event \( E = E(X_1, \cdots, X_m, R) \) such that \( \Pr[E^c] \leq (1 - \sigma)^m \), and

\[
(X_I \mid E, X \setminus I) \sim P.
\]

Fix any realization of the Poissonized sample size \( N \sim \text{Poi}(n) \). Choose \( m = 4\sigma^{-1}\log T \) in Lemma 4.6, and without loss of generality assume that \( N \) is an integral multiple of \( m \). Since for any \( \sigma \)-smooth \( \mathcal{D}_t \), it holds that

\[
\frac{\mathcal{D}_t(s)}{\mathcal{U}(\mathcal{X} \times \{\pm 1\})(s)} = \frac{\mathcal{D}_t(x)}{\mathcal{U}(\mathcal{X})(x)} \cdot \frac{\mathcal{D}_t(y \mid x)}{\mathcal{U}(\{\pm 1\})(y)} \leq \frac{2}{\sigma},
\]

the premise of Lemma 4.6 holds with parameter \( \sigma/2 \) for \( P = \mathcal{D}_t, Q = \mathcal{U}(\mathcal{X} \times \{\pm 1\}) \). Consequently, dividing the self-generated samples \( \tilde{s}_1, \cdots, \tilde{s}_N \) into \( N/m \) groups each of size \( m \), and running the procedure in Lemma 4.6, we arrive at \( N/m \) independent events \( E_1, \cdots, E_{N/m} \), each with probability at least \( 1 - (1 - \sigma/2)^m \geq 1 - T^{-2} \). Moreover, conditioned on each \( E_j \), we can pick an element \( u_j \in \{\tilde{s}_{(j-1)m+1}, \cdots, \tilde{s}_{jm}\} \) such that

\[
(u_j \mid E_j, \{\tilde{s}_{(j-1)m+1}, \cdots, \tilde{s}_{jm}\}\setminus\{u_j\}) \sim \mathcal{D}_t.
\]

For notational simplicity we denote the set of unpicked samples \( \{\tilde{s}_{(j-1)m+1}, \cdots, \tilde{s}_{jm}\}\setminus\{u_j\} = v_j \). As a result, thanks to the mutual independence of different groups and \( s_t \sim \mathcal{D}_t \) conditioned on \( s_{1:t-1} \) (note that we draw fresh randomness at every round), for \( E \triangleq \cap_{j \in [N/m]} E_j \) we have

\[
(u_1, \cdots, u_{N/m}, s_t) \mid (E, s_{1:t-1}, v_1, \cdots, v_{N/m}) \overset{\text{iid}}{\sim} \mathcal{D}_t.
\]
Consequently, for each \( j \in [N/m] \) we have

\[
\mathbb{E}_{s_t \sim \mathcal{D}_t, R(t+1)} [L(h_{t+1, s_t} | E)] \\
= \mathbb{E}_{v, s_{1:t-1} \mid E} \mathbb{E}_{s_t} [L(\mathbb{opt}(s_1, \ldots, s_N, s_{1:t-1}, s_t), s_t) | E, s_{1:t-1}, v] \\
= \mathbb{E}_{v, s_{1:t-1} \mid E} \left( \mathbb{E}_{s_t, u_{1}, \ldots, u_{N/m}} [L(\mathbb{opt}(s_{1:t-1}, v, u_1, \ldots, u_{N/m}, s_t), s_t) | E, s_{1:t-1}, v] \right) \\
= (a) \mathbb{E}_{v, s_{1:t-1} \mid E} \left( \mathbb{E}_{s_t, u_{1}, \ldots, u_{N/m}} [L(\mathbb{opt}(s_{1:t-1}, v, u_1, \ldots, u_{j-1}, s_t, u_{j+1}, \ldots, u_{N/m}, u_j), u_j) | E, s_{1:t-1}, v] \right) \\
= (b) \mathbb{E}_{v, s_{1:t-1} \mid E} \left( \mathbb{E}_{s_t, u_{1}, \ldots, u_{N/m}} [L(\mathbb{opt}(s_{1:t-1}, v, u_1, \ldots, u_{N/m}, s_t), u_j) | E, s_{1:t-1}, v] \right) \\
= \mathbb{E}_{s_t \sim \mathcal{D}_t, R(t+1)} [L(h_{t+1, u_j} | E)],
\]

where (a) follows from the conditional iid (and thus exchangeable) property of \((u_1, \ldots, u_{N/m}, s_t)\) after the conditioning, and (b) is due to the invariance of the ERM output after any permutation of the inputs. On the other hand, if \( s_{t}', u_{1}', \ldots, u_{N/m}' \) are independent copies of \( s_t \sim \mathcal{D}_t \), by independence it is clear that

\[
\mathbb{E}_{s_t, s_{t}' \sim \mathcal{D}_t, R(t+1)} [L(h_{t+1, s_t'} | E)] = \mathbb{E}_{s_t, s_{t}' \sim \mathcal{D}_t, R(t+1)} [L(h_{t+1, u_{j}' } | E)], \quad \forall j \in [N/m].
\]

Consequently, using the shorthand \( u_0 = s_t, u_{0}' = s_{t}' \), we have

\[
\mathbb{E}_{s_t, s_{t}' \sim \mathcal{D}_t, R(t+1)} [L(h_{t+1, s_t'} - L(h_{t+1, s_t}) | E] \\
= \frac{1}{N/m + 1} \mathbb{E}_{s_t, s_{t}' \sim \mathcal{D}_t, R(t+1)} \left[ \sum_{j=0}^{N/m} (L(h_{t+1, u_{j}' }) - L(h_{t+1, u_j})) | E \right] \\
\leq \frac{1}{N/m + 1} \mathbb{E}_{u_0, \ldots, u_{N/m}, u_{0}', \ldots, u_{N/m}' \sim \mathcal{D}_t} \left[ \sup_{h \in \mathcal{H}} \sum_{j=0}^{N/m} (L(h, u_{j}') - L(h, u_j)) \right] \\
\leq \frac{2}{N/m + 1} \mathbb{E}_{u_0, \ldots, u_{N/m} \sim \mathcal{D}_t} \mathbb{E}_{\epsilon \sim \mathcal{N}/m} \left[ \sup_{h \in \mathcal{H}} \sum_{j=0}^{N/m} \epsilon_j h(u_j) \right] \leq c_0 \sqrt{\frac{d}{N/m + 1}},
\]

where the last inequality is due to the classical \( O(\sqrt{d/n}) \) upper bound on the Rademacher complexity, and \( c_0 > 0 \) in an absolute constant. Note that the union bound gives

\[
\Pr[E^n] \leq \sum_{j=1}^{N/m} \Pr[E_j^n] \leq \frac{N}{m T^2},
\]

the law of total expectation gives

\[
\mathbb{E}_{s_t, s_{t}' \sim \mathcal{D}_t, R(t+1)} [L(h_{t+1, s_t'} - L(h_{t+1, s_t})].
\]
\[ \leq \mathbb{E}_{s_t, s'_t \sim \mathcal{D}_t \cup R(t+1)} \left[ L(h_{t+1}, s'_t) - L(h_{t+1}, s_t) \mid E \right] + \Pr[E^c] \leq c_0 \sqrt{\frac{d}{N/m + 1} + \frac{N}{mT^2}}. \]

Finally, plugging the choice of \( m = 4\sigma^{-1} \log T \), taking the expectation of \( N \sim \text{Poi}(n) \), and using \( \Pr[N > n/2] \geq 1 - e^{-n/8} \) in the above inequality completes the proof of Lemma 4.5.

### 4.2.3 Completing the Proof of Theorem 4.1

In this section we complete the proof of the \( O(\sqrt{dT \mathcal{K}}) \) upper bound in Theorem 4.1 when \( \sigma < d/|X| \) (and thus \( n = T \sqrt{|X|/d} \)). The proof is still through the same relaxation in Equation (6), though we will choose a different parameter \( \eta \) and prove a slightly modified version of Lemma 4.3:

**Lemma 4.7** (Expected TV \( \Rightarrow \) Admissibility). Let \( Q_t \) denote learner’s distribution over \( \mathcal{H} \) in Algorithm 2 at round \( t \), and \( s_t \sim \mathcal{D}_t \) be the conditional distribution of \( s_t \) given the history \( s_1, \ldots, s_{t-1} \). It holds that

\[ \mathbb{E}_{s_t \sim \mathcal{D}_t} \left( \mathbb{E}_{h_t \sim Q_t} \left[ L(h_t, s_t) \right] + \text{Rel}_T(\mathcal{H} \mid s_{1:t}) \right) - \text{Rel}_T(\mathcal{H} \mid s_{1:t-1}) \leq \mathbb{E}_{s_t \sim \mathcal{D}_t} [\text{TV}(Q_t, Q_{t+1})] - \eta. \]

Note that in Lemma 4.7, the expectation is outside the TV distance and no smaller than the TV distance when the mixture distribution is inside the expectation compared with Lemma 4.3. We can simply upper bound this expected TV distance, with the worst case choice of \( s_t \) and apply the data processing inequality, i.e.,

\[ \mathbb{E}_{s_t \sim \mathcal{D}_t} [\text{TV}(Q_t, Q_{t+1})] \leq \sup_{s_t} \text{TV}(P_{t-1}, P^t). \]

Using the similar independence property of Poissonization in Section 4.2.1, the target TV distance is at most \( \text{TV}(P, Q) \), where \( P \sim \text{Poi}(n/2|X|) \), and \( Q \) is a right-translation of \( P \) by one. Consequently,

\[ \text{TV}(P, Q) \leq \sqrt{\frac{\chi^2(P, Q)}{2}} = \sqrt{\frac{1}{2} \left( \mathbb{E}_{X \sim P} \left[ \left( \frac{X}{n/2|X|} \right)^2 \right] - 1 \right)} = \sqrt{\frac{|X|}{n}}, \]

so the choice of \( \eta = \sqrt{|X|/n} \) and Lemma 4.7 again makes the relaxation in Equation (6) admissible, and we complete the proof of Theorem 4.1.

### 5 Discussion, Additional Results, and Open Problems

**Computational Lower Bounds.** Our main contribution is oracle-efficient algorithms that achieve an \( O(\sqrt{dT K}) \) regret upper bound for \( K \)-hint transductive learning, and an \( O(\sqrt{dT \sigma^{-1/2}}) \) upper bound for smoothed online learning. However, neither of these upper bounds is statistically optimal; the statistically optimal regrets for these scenarios are \( \tilde{\Theta}(\sqrt{dT \log K}) \) and \( \tilde{\Theta}(\sqrt{dT \log(1/\sigma)}) \) [HRS21], respectively. We ask the following question: is the above discrepancy an artifact of our regret analysis, or an intrinsic limitation of our or all oracle-efficient algorithms.
We show that no matter how we tune the parameters in Algorithm 2 and Algorithm 3, a lower bound $\Omega(\sqrt{dT \sigma^{-1/2}})$ or $\Omega(\sqrt{dTK^{1/2}})$ is unavoidable for these algorithms in the respective scenarios. In other words, our regret upper bound in Theorem 4.1 is tight, and there is an $O(K^{1/4})$ gap compared to the upper bound of Theorem 3.1; importantly, neither algorithm could achieve the logarithmic dependence on $1/\sigma$ or $K$ in the statistical upper bound. Formally, by parameter tuning, we mean any choice of the parameter $n$ in Algorithm 2, and any number $n$ of uniform samples from the $K$-hint set in Algorithm 3. The next theorem formally states the lower bounds.

**Theorem 5.1 (Limitations of Algorithms).** For any choice of the parameter $n$ in Algorithm 2, there exists a $\sigma$-smoothed online learning instance such that Algorithm 2 suffers from at least $\Omega(\min\{T, \sqrt{dT \sigma^{-1/2}}, \sqrt{T(d|\mathcal{X}|)^{1/2}}\})$ expected regret.

Similarly, for any choice of the parameter $n$ in Algorithm 3, there exists a $K$-hint transductive instance such that Algorithm 3 incurs at least $\Omega(\min\{T, \sqrt{dT K^{1/2}}\})$ expected regret.

For general efficient algorithms, we have the following computational lower bound for smoothed online learning following similar ideas to [HK16].

**Theorem 5.2 (Computational Lower Bound for Smoothed Online Learning).** For $1/\sigma \geq d$, any proper algorithm which only has access to the ERM oracle and achieves a regret $o(\min\{T, \sqrt{T(d/\sigma)^{1/2}}\})$ for any $\sigma$-smoothed online learning problem must have an $\omega(\sqrt{d/\sigma})$ total running time.

Theorem 5.2 implies an exponential statistical-computational gap in smoothed online learning: for exponentially small $\sigma$, achieving the statistical regret $\tilde{O}(\sqrt{td\log(1/\sigma)})$ requires an exponential running time. However, Theorem 5.2 still exhibits gaps to our computational upper bounds.

1. First, although both lower bounds of the regret and running time in Theorem 5.2 do not match the counterparts of Algorithm 2 in Theorem 4.1, the upper and lower bounds share the same $\Theta(\sigma^{-1/4})$ dependence on $\sigma$. This suggests that the improvement from $\Theta(\sigma^{-1/2})$ to $\Theta(\sigma^{-1/4})$ thanks to Poissonization is not superfluous and might be fundamental. We also conjecture that for all efficient algorithms with runtime poly$(T, d, 1/\sigma)$, the $\Theta(\sigma^{-1/4})$ dependence is the best one can hope for in the regret of such algorithms, as opposed to the $\Theta(\sqrt{\log(1/\sigma)})$ dependence in the statistical regret.

2. Second, Theorem 5.2 shows a poly$(d, 1/\sigma)$ computational lower bound to achieve the statistical regret $\tilde{O}(\sqrt{td\log(1/\sigma)})$, while the $\varepsilon$-net argument in [HRS21] requires a poly$(\sigma^{-d})$ computational time. One may wonder whether this exponential dependence on $d$ is in fact unavoidable, and this is a missing feature not covered in [HK16]. We ask the following open question:

**Open Question.** For $d/\sigma \gg T^2$ in the smoothed setting (or $dK \gg T$ in the $K$-hint transductive learning setting), does any algorithm achieving $o(T)$ regret require $\Omega(\text{poly}(T, 2^d, 1/\sigma (\text{or } K)))$ computational time given access to the ERM oracle?

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A Information Theoretic Lemmas

For two probability distributions $P$ and $Q$ over the same domain $X$, let $\chi^2(P, Q) = \sum_{x \in X} P(x)^2/Q(x) - 1$ be the $\chi^2$-divergence. The following lemma upper bounds the TV distance by the $\chi^2$-divergence; a proof could be found in [Tsy09, Chapter 2].

**Lemma A.1** (From TV to $\chi^2$). The following relations hold:

$$TV(P, Q) \leq \sqrt{\frac{1}{2} \log(1 + \chi^2(Q, P))} \leq \sqrt{\frac{\chi^2(Q, P)}{2}}.$$ 

The following statement is the well-known Ingster’s $\chi^2$ method, and we refer to the excellent book [IS03] for a general treatment.

**Lemma A.2** (Ingster’s $\chi^2$ method). For a mixture distribution $\mathbb{E}_{\theta \sim \pi} [Q_{\theta}]$ and a generic distribution $P$, the following identity holds:

$$\chi^2\left(\mathbb{E}_{\theta \sim \pi} [Q_{\theta}], P\right) = \mathbb{E}_{\theta, \theta' \sim \pi} \left[ \mathbb{E}_{x \sim P} \left( \frac{Q_{\theta}(x)Q_{\theta'}(x)}{P(x)^2} \right) \right] - 1,$$

where $\theta'$ is an independent copy of $\theta$.

B Proof of Proposition 2.1

**Proof.** To prove this lemma we break the expected regret into two parts:

$$\mathbb{E}[\text{REGRET}(T)] = \mathbb{E}_{\mathcal{F}, \mathcal{H}} \left[ \sum_{t=1}^{T} \mathbb{E}_{\tilde{y}_t \sim Q_t} [L(\tilde{y}_t, y_t)] - \inf_{h \in \mathcal{H}} L(h, s_1:T) \right] + \mathbb{E}_{\mathcal{F}, \mathcal{H}} \left[ \sum_{t=1}^{T} L(\tilde{y}_t, y_t) - \mathbb{E}_{\tilde{y}_t \sim Q_t} [L(\tilde{y}_t, y_t)] \right].$$
For the first part, we use an inductive argument to show that

\[
\mathbb{E}_{\mathcal{D}, \mathcal{Q}} \left[ \sum_{t=1}^{T} \mathbb{E}_{\hat{y}_t \sim \mathcal{Q}_t} [l(\hat{y}_t, y_t)] - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} l(h(x_t), y_t) \right] \leq \text{Rel}_T(\mathcal{H} \mid \emptyset). \tag{9}
\]

According to the definition of admissibility, we have

\[
\mathbb{E}_{\mathcal{D}, \mathcal{Q}} \left[ \sum_{t=1}^{T} \mathbb{E}_{\hat{y}_t \sim \mathcal{Q}_t} [l(\hat{y}_t, y_t)] - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} l(h(x_t), y_t) \right] \leq \text{Rel}_T(\mathcal{H} \mid s_{1:T}) \text{ by 2nd condition of admissibility}
\leq \mathbb{E}_{\mathcal{D}, \mathcal{Q}} \left[ \sum_{t=1}^{T-1} \mathbb{E}_{\hat{y}_t \sim \mathcal{Q}_t} [l(\hat{y}_t, y_t)] + \mathbb{E}_{y_T \sim \mathcal{D}_T} \left[ \mathbb{E}_{\hat{y}_T \sim \mathcal{Q}_T} [l(\hat{y}_T, y_T)] + \text{Rel}_T(\mathcal{H} \mid s_{1:T-1}) \right] \right] \leq \text{Rel}_T(\mathcal{H} \mid s_{1:T-1}) \text{ by 1st condition of admissibility}
\leq \mathbb{E}_{\mathcal{D}, \mathcal{Q}} \left[ \sum_{t=1}^{T-1} \mathbb{E}_{\hat{y}_t \sim \mathcal{Q}_t} [l(\hat{y}_t, y_t)] + \text{Rel}_T(\mathcal{H} \mid s_{1:T-1}) \right]
= \mathbb{E}_{\mathcal{D}, \mathcal{Q}} \left[ \sum_{t=1}^{T-1} \mathbb{E}_{\hat{y}_t \sim \mathcal{Q}_t} [l(\hat{y}_t, y_t)] + \text{Rel}_T(\mathcal{H} \mid s_{1:T-1}) \right],
\]

where the last step uses the tower property of conditional expectations.

Repeat this process for \((T - 1)\) times and note that \(\text{Rel}_T(\mathcal{H} \mid \emptyset)\) is a constant that does not dependent on \(\mathcal{D}\) proves Equation (9).

Since the second part is the expected sum of a martingale difference sequence, we apply the Azuma-Hoeffding inequality and obtain

\[
\mathbb{E}_{\mathcal{D}, \mathcal{Q}} \left[ \sum_{t=1}^{T} \mathbb{E}_{\hat{y}_t \sim \mathcal{Q}_t} [l(\hat{y}_t, y_t)] - \mathbb{E}_{\hat{y}_t \sim \mathcal{Q}_t} [l(\hat{y}_t, y_t)] \right] \leq \int_{0}^{\infty} \exp \left( -\frac{2t^2}{T} \right) dt \in O(\sqrt{T}). \tag{10}
\]

Combining Equation (9) and Equation (10) completes the proof.

\[\square\]

C Complementary Materials in Section 3

C.1 Efficient Algorithm for Transductive Online Learning with \(K\) Hints

See Algorithm 3 for a description of the oracle-efficient algorithm in the setting of transductive online learning with \(K\) hints.
The goal of this section is to show that the algorithm \( \mathcal{Z} \) equalized, that is, \( \hat{y}_t \) for end

### Algorithm 3: Oracle-Efficient Online Transductive Learning with \( K \) Hints

**Input:** \( T, K, \{ Z_t \}_{t=1}^T \)  

1. for \( t \leftarrow 1 \) to \( T \) do  
2. Receive \( x_t \). Assert that \( x_t \in Z_t \)  
3. for \( i = t+1, \ldots, T; k = 1, \ldots, K \) do  
4. Draw new \( \epsilon_{i,k} \sim \mathcal{U}(-1, 1) \).  
5. end  
6. \( S_1^{(t)} \leftarrow \{(z_{i,k}^{(t)}, \epsilon_{i,k}^{(t)}), (z_{i,k}^{(t)}, \epsilon_{i,k}^{(t)})\}_{i=t+1:T, k=1:K} \) // Two copies of each tuple  
7. \( S_2^{(t)} \leftarrow \{(x_t, y_t)\}_{t=1}^{T-1} \)  
8. \( p_t \leftarrow \frac{1}{2} + \frac{\text{OPT}_H(S_1^{(t)} \cup S_2^{(t)} \cup \{(x_t, +1)\}) - \text{OPT}_H(S_1^{(t)} \cup S_2^{(t)} \cup \{(x_t, -1)\})}{2} \).  
9. With probability \( p_t \), predict \( \hat{y}_t = -1 \); otherwise predict \( \hat{y}_t = +1 \)  
10. Receive \( y_t \), suffer loss \( l(\hat{y}_t, y_t) \).  
11. end

**C.2 More on the First Part of Inequality (5)**

The goal of this section is to show that the algorithm \( \mathcal{Q}_t \) given by Equation (2) satisfies that \( \forall x_t \in Z_t \),

\[
\sup_{y_t \in \mathcal{Y}} \left\{ \mathbb{E}_{\hat{y}_t \sim \mathcal{Q}_t} \left[ l(\hat{y}_t, y_t) \right] + \mathcal{R}(-L_t, Z_{t+1:T}) \right\} \leq \mathcal{R}(-L_{t-1}, Z_{t+1:T} \cup \{x_t\}).
\]

First of all, we verify that \( \mathcal{Q}_t \) is the min-max optimal strategy to label \( x_t \), i.e.,

\[
\mathcal{Q}_t = \arg \min_{q \in \Delta(\mathcal{Y})} \sup_{y_t \in \mathcal{Y}} \left\{ \mathbb{E}_{\hat{y}_t \sim q} \left[ l(\hat{y}_t, y_t) \right] + \mathcal{R}(-L, s_{1:t-1} \cup \{x_t, y_t\}, Z_{t+1:T}) \right\}
\]

This is because the optimal distribution \( q \) is attained when the two terms inside the supremum are equalized, that is

\[
\mathbb{E}_{\hat{y}_t \sim \mathcal{Q}_t} \left[ l(\hat{y}_t, +1) \right] + \mathcal{R}(-L, s_{1:t-1} \cup \{(x_t, +1)\}, Z_{t+1:T}) = \mathbb{E}_{\hat{y}_t \sim q} \left[ l(\hat{y}_t, -1) \right] + \mathcal{R}(-L, s_{1:t-1} \cup \{(x_t, -1)\}, Z_{t+1:T}).
\]

The above equation solves

\[
\mathbb{E}_{\hat{y}_t \sim \mathcal{Q}_t} \left[ l(\hat{y}_t) \right] = \mathcal{R}(-L, \{(x_t, y_t)\}_{i=1}^{t-1} \cup \{(x_t, +1)\}, Z_{t+1:T}) = \mathcal{R}(-L, \{(x_t, y_t)\}_{i=1}^{t-1} \cup \{(x_t, -1)\}, Z_{t+1:T})
\]

\[
= \mathbb{E} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{i=t+1:T, k=1:K} \epsilon_{i,k}^{(t)} h(z_{i,k}) + \frac{1}{2} \sum_{\tau=1}^{t-1} y_{\tau} h(x_{\tau}) + \frac{1}{2} h(x_t) \right\} \right]
\]

\[
- \sup_{h \in \mathcal{H}} \left\{ \sum_{i=t+1:T, k=1:K} \epsilon_{i,k}^{(t)} h(z_{i,k}) + \frac{1}{2} \sum_{\tau=1}^{t-1} y_{\tau} h(x_{\tau}) - \frac{1}{2} h(x_t) \right\}
\]

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where the last step is because the optimal distribution
\[ \hat{y} = \sup_{y \in \mathcal{Y}} \left\{ \mathbb{E} \left[ L(h, s_{1:t-1}) + 2L(h, Z_{t+1:T} \circ \mathcal{E}^{(t)}) + L(h, (x_t, -1)) \right] \right\} 
- \inf_{h \in \mathcal{H}} \left\{ L(h, s_{1:t-1}) + 2L(h, Z_{t+1:T} \circ \mathcal{E}^{(t)}) + L(h, (x_t, -1)) \right\} , \]

which is exactly the predicting rule outputted by Algorithm 1. So far, we have showed

\[
\sup_{y \in \mathcal{Y}} \left\{ \mathbb{E} \left[ l(\hat{y}_t, y_t) \right] + \mathfrak{R}(-L_t, Z_{t+1:T}) \right\} = \inf_{q \in \Delta(\mathcal{Y})} \sup_{y \in \mathcal{Y}} \left\{ \mathbb{E} \left[ l(\hat{y}_t, y_t) \right] + \mathfrak{R}(-L_t, Z_{t+1:T}) \right\} 
= \inf_{y \in \mathcal{Y}} \sup_{y \in \mathcal{Y}} \left\{ l(\hat{y}_t, y_t) + \mathfrak{R}(-L_t, Z_{t+1:T}) \right\} ,
\]

where the last step is because the optimal distribution \( q \) can only be supported on the inputs with same optimal value. The same reasoning applies to the adversary’s optimal choice \( y_t \in \mathcal{Y} \), so

\[
(11) = \inf_{\hat{y}_t \in \mathcal{Y}} \sup_{p \in \Delta(\mathcal{Y})} \mathbb{E} \left[ l(\hat{y}_t, y_t) + \mathfrak{R}(-L_t, Z_{t+1:T}) \right] .
\]

Furthermore, since the expression inside the supremum is linear in \( p \) and \( \hat{y}_t \), and the domain \( \mathcal{Y} \) is a finite set, we apply min-max theorem and obtain

\[
(11) = \inf_{\hat{y}_t \in \mathcal{Y}} \sup_{p \in \Delta(\mathcal{Y})} \mathbb{E} \left[ l(\hat{y}_t, y_t) + \mathfrak{R}(-L_t, Z_{t+1:T}) \right] = \sup_{p \in \Delta(\mathcal{Y})} \inf_{\hat{y}_t \in \mathcal{Y}} \mathbb{E} \left[ l(\hat{y}_t, y_t) + \mathfrak{R}(-L_t, Z_{t+1:T}) \right] .
\]

Substituting the definition of \( \mathfrak{R}(-L_t, Z_{t+1:T}) \) and loss function \( l(\hat{y}_t, y_t) \) into the above equation, and noticing that \( \hat{y}_t \) only shows in the loss term, we obtain

\[
(11) = \sup_{p \in \Delta(\mathcal{Y})} \inf_{\hat{y}_t \in \mathcal{Y}} \mathbb{E} \left[ \frac{-\hat{y}_t y_t}{2} + \mathbb{E} \left( \sum_{i=t+1:T} \epsilon^{(t)}_{i,k} h(z_{i,k}) + \frac{1}{2} \sum_{\tau=1}^{T} y_{\tau} h(x_{\tau}) \right) \right] 
= \sup_{p \in \Delta(\mathcal{Y})} \left\{ \inf_{\hat{y}_t \in \mathcal{Y}} \left[ -\frac{1}{2} \hat{y}_t \mathbb{E} \left[ y_t \right] \right] + \mathbb{E} \mathbb{E} \left( \sum_{i=t+1:T} \epsilon^{(t)}_{i,k} h(z_{i,k}) + \frac{1}{2} \sum_{\tau=1}^{T} y_{\tau} h(x_{\tau}) \right) \right\} ,
\]

where the second step is due to the linearity of expectations. Since the first term does not contain \( y_t, \mathcal{E}^{(t)} \) and \( h \), we further obtain

\[
(11) = \sup_{p \in \Delta(\mathcal{Y})} \mathbb{E} \mathbb{E} \left( \sum_{i=t+1:T} \epsilon^{(t)}_{i,k} h(z_{i,k}) + \frac{1}{2} \sum_{\tau=1}^{T} y_{\tau} h(x_{\tau}) \right) .
\]

Replacing the optimal \( \hat{y}_t \) by \( h(x_t) \), we further upper bound the above expression by

\[
(11) \leq \sup_{p \in \Delta(\mathcal{Y})} \mathbb{E} \mathbb{E} \left\{ \frac{-1}{2} h(x_t) \mathbb{E} \left[ y_t \right] + \sum_{i=t+1:T} \epsilon^{(t)}_{i,k} h(z_{i,k}) + \frac{1}{2} \sum_{\tau=1}^{T} y_{\tau} h(x_{\tau}) \right\} .
\]
Since \( \sup \) is a convex function, according to Jensen’s inequality, the above value is further upper bounded by

\[
\sup_{p \in \Delta(Y)} \mathbb{E} \mathbb{E} \left[ \sup_{h \in H} \left\{ -\frac{1}{2} h(x_t) y_t' + \sum_{i=t+1:T} \sum_{k=1:K} \epsilon_{i,k}^{(t)} h(z_{i,k}) + \frac{1}{2} \sum_{\tau=1}^{t} y_{\tau} h(x_{\tau}) \right\} \right] = \sup_{p \in \Delta(Y)} \mathbb{E} \mathbb{E} \left[ \sup_{h \in H} \left\{ \sum_{i=t+1:T} \sum_{k=1:K} \epsilon_{i,k}^{(t)} h(z_{i,k}) + \frac{1}{2} (y_t - y_t') h(x_t) + \frac{1}{2} \sum_{\tau=1}^{t-1} y_{\tau} h(x_{\tau}) \right\} \right].
\]

When \( \epsilon_t \) is an independently drawn Rademacher random variable, \((y_t - y_t')\) and \(\epsilon_t(y_t - y_t')\) are identically distributed, thus the above value is equal to

\[
\sup_{p \in \Delta(Y)} \mathbb{E} \mathbb{E} \left[ \sup_{h \in H} \left\{ \sum_{i=t+1:T} \sum_{k=1:K} \epsilon_{i,k}^{(t)} h(z_{i,k}) + \frac{1}{2} \epsilon_t(y_t - y_t') h(x_t) + \frac{1}{2} \sum_{\tau=1}^{t-1} y_{\tau} h(x_{\tau}) \right\} \right].
\]

Relaxing the constraint that \(y_t, y_t'\) are sampled from the same distribution, we obtain

\[
(11) \leq \sup_{y_t, y_t' \in \mathcal{Y}^{(t, t)}} \mathbb{E} \left[ \sup_{h \in H} \left\{ \sum_{i=t+1:T} \sum_{k=1:K} \epsilon_{i,k}^{(t)} h(z_{i,k}) + \frac{1}{2} \epsilon_t(y_t - y_t') h(x_t) + \frac{1}{2} \sum_{\tau=1}^{t-1} y_{\tau} h(x_{\tau}) \right\} \right]
\]

\[
\leq \sup_{y_t \in \mathcal{Y}^{(t, t)}} \mathbb{E} \left[ \sup_{h \in H} \left\{ \sum_{i=t+1:T} \sum_{k=1:K} \epsilon_{i,k}^{(t)} h(z_{i,k}) + \epsilon_t y_t h(x_t) + \frac{1}{2} \sum_{\tau=1}^{t-1} y_{\tau} h(x_{\tau}) \right\} \right],
\]

where the second step is because the optimization problem is symmetric for \(y_t\) and \(y_t'\), so the optimal solution should be the same. Finally, note that for any given \(y_t, \epsilon_t\) and \(\epsilon_t y_t\) has the same distribution, thus the above expression equals

\[
\mathbb{E}_{\mathcal{E}^{(t, t)}} \left[ \sup_{h \in H} \left\{ \sum_{i=t+1:T} \sum_{k=1:K} \epsilon_{i,k}^{(t)} h(z_{i,k}) + \epsilon_t h(x_t) + \frac{1}{2} \sum_{\tau=1}^{t-1} y_{\tau} h(x_{\tau}) \right\} \right] = \mathcal{R}(-L_{t-1}, Z_{t+1} \cup \{x_t\}),
\]

as desired. The proof is now complete.

**C.3 Proof of Lemma 3.2 (Monotonicity)**

**Proof.** Using \(\mathbb{E}[\sup_{\lambda} X_\lambda] \geq \sup_{\lambda} \mathbb{E}[X_\lambda]\), we have

\[
\mathcal{R}(\Phi, Z \cup \{x\}) = \mathbb{E}_{\epsilon_{1:m+1}} \left[ \sup_{h \in H} \left\{ \sum_{i=1}^{m} \epsilon_i h(z_i) + \epsilon_{m+1} h(x) + \Phi(h) \right\} \right]
\]

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\[
= \mathbb{E}_{\epsilon_{1:m}, \epsilon_{m+1}} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{i=1}^{m} \epsilon_i h(z_i) + \epsilon_{m+1} h(x) + \Phi(h) \right\} \right] \\
\geq \mathbb{E}_{\epsilon_{1:m}} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{i=1}^{m} \epsilon_i h(z_i) + \mathbb{E} \left[ \epsilon_{m+1} h(x) \right] + \Phi(h) \right\} \right] \\
= \mathcal{R}(\Phi, Z),
\]
as desired. \(\square\)

D Complementary Materials in Section 4.1

D.1 Coupling Lemma

Lemma D.1 (Coupling, [HRS21]). Let \(D_{\sigma}\) be an adaptive sequence of \(t\) \(\sigma\)-smooth distributions on \(\mathcal{X}\). Then, there is a coupling \(\Pi\) such that \((x_1, z_{1,1}, \ldots, z_{1,K}, \ldots, x_t, z_{t,1}, \ldots, z_{t,K}) \sim \Pi\) satisfy

a. \(x_1, \ldots, x_t\) is distributed according \(D_{\sigma}\).

b. For every \(j \leq t\), \(\{z_{i,k}\}_{i \geq j, k \in [K]}\) are uniformly and independently distributed on \(\mathcal{X}\), conditioned on \(x_1, \ldots, x_{j-1}\).

c. With probability at least \(1 - t(1 - \sigma)^K\), \(\{x_1, \ldots, x_T\} \subseteq \{z_{t,k}\}_{t=1:T, k=1:K}\).

D.2 Proof of Theorem 4.2

We first set some notation up. Following Section 3, we still use \(\mathcal{E}^{(t)} = \{\epsilon_{i,k}^{(t)}\}_{i=t+1:T, k=1:K}\) to denote the random labels generated at time step \(t\). But in this setting, the hints are random as well. We will denote the random hints drawn at time \(t\) with \(V_i^{(t)} = \{v_{i,k}^{(t)}\}_{k=1:K}\) and \(V^{(t)} = V_{t+1:T}^{(t)}\).

At a high level, the proof of Theorem 4.2 is also based on the relaxations framework. We formally state the relaxation used in this setting together with its admissibility result in Lemma D.2. The proof of Lemma D.2 is presented in Appendix D.3.

Lemma D.2 (Admissibility of Algorithm 1). Algorithm 1 is an admissible algorithm with respect to the following relaxation:

\[
\text{Rel}_T(\mathcal{H} \mid s_{1:t}) = \mathbb{E}_{V^{(t)}, \mathcal{E}^{(t)}} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{i=t+1:T} \epsilon_{i,k}^{(t)} h(v_{i,k}^{(t)}) + \frac{1}{2} \sum_{i=1}^{t} y_i h(x_i) \right\} \right] + \beta(T - t),
\]

where \(K = 100 \log T/\sigma\) and \(\beta = 10TK(1 - \sigma)^K\).
We will see a proof of Theorem 4.2 using the lemma. Recall that Proposition 2.1 shows that the regret can be upper bounded by the value of the relaxation at the start of the game. Thus, we have

\[
E[\text{REGRET}(T)] \leq \text{Rel}_T(\mathcal{H} \mid \emptyset) + O(\sqrt{T})
\]

\[
= \mathbb{E}_{V^{(0)},\epsilon^{(0)}} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{i=1}^{T} \sum_{k=1}^{K} \epsilon_{i,k}^{(0)} h(v_{i,k}^{(0)}) \right\} \right] + \beta T + O(\sqrt{T}).
\]

Note that the first term is the Rademacher complexity of the hypothesis class \( \mathcal{H} \) with respect to the uniform distribution for sample size \( KT \). Therefore, it can be upper bounded by \( O(\sqrt{dTK}) \) using the VC theorem. For the second term, since \( K = 100 \log T/\sigma \) and \( (1 - \sigma)^{\alpha/\sigma} \leq \exp(-\alpha) \) for \( \alpha = 100 \log T \), we have

\[
\beta T = 1000T^2K(1 - \sigma)^K \leq 1000T^2 \log T/\sigma \exp(-100 \log T) \in o(1).
\]

Combining the above bounds for the first and second term yields

\[
E[\text{REGRET}(T)] \leq O \left( \sqrt{\frac{Td}{\sigma} \log T} + \sqrt{T} \right) \in O \left( \sqrt{\frac{Td}{\sigma} \log T} \right)
\]

as claimed.

**D.3 Proof of Admissibility (Lemma D.2)**

To show admissibility, we need to verify both conditions stated in Definition 2.3.

For the second condition, at round \( T \), we have

\[
\text{Rel}_T(\mathcal{H} \mid s_{1:T}) = \sup_{h \in \mathcal{H}} \left\{ \frac{1}{2} \sum_{i=1}^{T} y_i h(x_i) \right\} = - \inf_{h \in \mathcal{H}} L(h, s_{1:T}),
\]

satisfying the second condition.

For the first condition, we can first rewrite the relaxation using the language of regularized Rademacher complexity introduced in Section 3,

\[
\text{Rel}_T(\mathcal{H} \mid s_{1:t}) = \mathbb{E}_{V^{(t)} \sim \mathcal{U}(X)} \left[ \mathcal{R}(\cdot, s_{1:t}, V^{(t)}) \right] + \beta(T - t).
\]

We first observe that the random instances \( V^{(t)} \) used in the algorithm exactly match the random instances used in defining the relaxation, so we can use the linearity of expectation together with Jensen’s inequality to bring \( \mathbb{E}_{V^{(t)}} \) to the most outer layer. This gives us

\[
\sup_{D_t \in \Delta_{\alpha}(X)} \mathbb{E} \sup_{\tilde{y}_t \sim D_t} \left\{ \mathbb{E}_{\hat{y}_t \sim Q_t} \left[ l(\hat{y}_t, y_t) \right] + \text{Rel}_T(\mathcal{H} \mid s_{1:t}) \right\} - \beta(T - t)
\]

\[
= \sup_{D_t \in \Delta_{\alpha}(X)} \mathbb{E} \sup_{\tilde{y}_t \sim D_t} \left\{ \mathbb{E}_{V^{(t)} \sim \mathcal{U}(X)} \mathbb{E}_{\hat{y}_t \sim Q_t(V^{(t)})} \left[ l(\hat{y}_t, y_t) \right] + \mathbb{E}_{V^{(t)} \sim \mathcal{U}(X)} \left[ \mathcal{R}(L_t, V^{(t)}) \right] \right\}
\]

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According to Lemma D.3, the above value is further upper bounded by
\[ Q \]
which shows that
\[ T \]
Taking the expectation and a supremum over smooth distributions to both sides of inequality (14),
to achieve this. We will first summarize the required property in lemma D.3 and prove it later.

Fixing \( x_t \) and \( V^{(t)} \), the prediction rule \( Q_t(V^{(t)}) \) coincides with the transductive setting with \( V^{(t)} \) being the union of hints for the future. This allows us to use the first inequality of inequality (5) in Section 3 (also see Appendix C.2 for a proof) to obtain
\[
\sup_{y_t \in Y} \left\{ \mathbb{E}_{\hat{y}_t \sim Q_t(V^{(t)})} \left[ l(\hat{y}_t, y_t) \right] + \mathcal{R}(-L_t, V^{(t)}) \right\} \leq \mathcal{R}(-L_{t-1}, V^{(t)} \cup \{x_t\}). \tag{14}
\]

Taking the expectation and a supremum over smooth distributions to both sides of inequality (14), and applying Jensen’s inequality again, we obtain
\[
(13) \leq \sup_{D_t \in \Delta_\sigma(X)} \mathbb{E} \left[ \mathbb{E}_{V^{(t)} \sim U(X)} \left[ \mathcal{R}(-L_{t-1}, V^{(t)} \cup \{x_t\}) \right] \right]
\leq \mathbb{E} \left[ \sup_{D_t \in \Delta_\sigma(X)} \mathbb{E}_{V^{(t)} \sim U(X)} \left[ \mathcal{R}(-L_{t-1}, V^{(t)} \cup \{x_t\}) \right] \right].
\]

The last step is an analog to the second inequality in inequality (5). We need to show that the above Rademacher value, which involves a supremum over all smooth distributions, can be bounded by an expectation over simply the uniform distribution. We use the coupling argument to achieve this. We will first summarize the required property in lemma D.3 and prove it later.

According to Lemma D.3, the above value is further upper bounded by
\[
\mathbb{E}_{V^{(t)} \sim U(X)} \left[ \mathbb{E}_{Z_t \sim U(X)} \left[ \mathcal{R}(-L_{t-1}, V^{(t)} \cup Z_t) \right] + \beta \right] = \text{Rel}_T(\mathcal{H} \mid s_{1:t-1}) - \beta(T - t). \tag{15}
\]

As a result, we have proved
\[
\sup_{D_t \in \mathcal{D}_t} \mathbb{E} \sup_{x_t \sim D_t} \mathbb{E}_{y_t \in Y} \left\{ \mathbb{E}_{\hat{y}_t \sim Q_t} \left[ l(\hat{y}_t, y_t) \right] + \text{Rel}_T(\mathcal{H} \mid s_{1:t-1} \cup (x_t, y_t)) \right\} \leq \text{Rel}_T(\mathcal{H} \mid s_{1:t-1}),
\]
which shows that \( Q_t \) satisfies the second condition in Definition 2.3. Thus completes the proof of admissibility.

**Lemma D.3 (Replacing Supremum by Expectation).** For any \( V^{(t)} \in \mathcal{X}^{K(T-t)} \), there exists a set of \( K \) variables \( Z_t = \{z_{t,k}\}_{k \in [K]} \), such that
\[
\sup_{D_t \in \mathcal{D}_t, x_t \sim D_t} \mathbb{E} \left[ \mathcal{R}(-L_{t-1}, V^{(t)} \cup \{x_t\}) \right] \leq \mathbb{E}_{Z_t \sim U(X)} \left[ \mathcal{R}(-L_{t-1}, V^{(t)} \cup Z_t) \right] + \beta
\]

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Proof. To establish the monotonicity property, we need to show that the random instance $x_t$ drawn from a smooth distribution belongs to a set of uniform i.i.d. hints with high probability. This is where the coupling lemma comes in. For the smooth distribution $D_t \in \Delta_\sigma(\mathcal{X})$ that achieves the supremum (assume the supremum is achievable), Lemma D.1 shows the existence of a coupling $\Pi$ on $(x_t, z_{t,1}, \cdots, z_{t,K})$ such that $x_t$ is distributed according to $D_t^X$ and $Z_t = \{z_{t,k}\}_{k \in [K]}$ are uniformly and independently distributed. We thus have

$$
\sup_{D_t \in \Delta_\sigma(\mathcal{X})} \mathbb{E}_{x_t \sim D_t} \left[ \mathcal{R}(-L_{t-1}, V(t) \cup \{x_t\}) \right] = \mathbb{E}_{\Pi} \left[ \mathcal{R}(-L_{t-1}, V(t) \cup \{x_t\}) \right]. \tag{16}
$$

This joint distribution $\Pi$ has the property that event the $E_t \equiv \{x_t \in Z_t\}$ happens with high probability. We now upper bound the expected value by conditioning on $E_t$ and $\bar{E}_t$ respectively.

Conditioned on $E_t$, this problem reduces to the $K$-hint transductive learning setting where $Z_t$ are hints for $x_t$. Using the techniques developed in Section 3, we apply the monotonicity of regularized Rademacher complexity (Lemma 3.2) recursively and obtain

$$
\mathcal{R}(-L_{t-1}, V(t) \cup \{x_t\}) \leq \mathcal{R}(-L_{t-1}, V(t) \cup Z_t). \tag{17}
$$

Conditioned on $\bar{E}_t$, we skirt the monotonicity issue by directly using upper and lower bounds on the regularized Rademacher complexity. To be more precise, we use Lemma D.4 in Appendix D.4 to show that

$$
\mathcal{R}(-L_{t-1}, V(t) \cup \{x_t\}) \leq TK \leq TK + \left( \mathcal{R}(-L_{t-1}, V(t) \cup Z_t) + \frac{T}{2} \right)
\leq \mathcal{R}(-L_{t-1}, V(t) \cup Z_t) + 2TK. \tag{18}
$$

Finally, we expand the right hand side of Equation (16) by conditioning on $E_t$ and $\bar{E}_t$ respectively. Putting Equations (17) and (18) together, we obtain

$$
(16) = \text{Pr}[E_t] \cdot \mathbb{E}_{\Pi} \left[ \mathcal{R}(-L_{t-1}, V(t) \cup \{x_t\}) \mid E_t \right] + \text{Pr}[\bar{E}_t] \cdot \mathbb{E}_{\Pi} \left[ \mathcal{R}(-L_{t-1}, V(t) \cup \{x_t\}) \mid \bar{E}_t \right]
\leq \text{Pr}[E_t] \cdot \mathbb{E}_{\Pi} \left[ \mathcal{R}(-L_{t-1}, V(t) \cup Z_t) \mid E_t \right] + \text{Pr}[\bar{E}_t] \cdot \mathbb{E}_{\Pi} \left[ \mathcal{R}(-L_{t-1}, V(t) \cup Z_t) + 2TK \mid \bar{E}_t \right]
= \mathbb{E}_{\Pi} \left[ \mathcal{R}(-L_{t-1}, V(t) \cup Z_t) \right] + \text{Pr}[\bar{E}_t] \cdot 2TK.
$$

Since $\Pi$ has uniform marginal distribution on $Z_t$, and that $\text{Pr}[\bar{E}_t] \cdot 2TK \leq (1-\sigma)^K \cdot 2TK \leq \beta$, we further obtain

$$
(16) \leq \mathbb{E}_{Z_t \sim D_t(X)} \left[ \mathcal{R}(-L_{t-1}, V(t) \cup Z_t) \right] + \beta,
$$

thus completes the proof.  \(\square\)
D.4 Upper and Lower Bounds on the Relaxation

**Lemma D.4 (Upper and Lower Bounds on the Relaxation).** For all $t \in [T]$, all sequence $s_{1:t}$, and all instance set $Z$ of size no larger than $(T - t)K$,

$$-\frac{T}{2} \leq \mathcal{R}(-L(\cdot, s_{1:t}), Z) \leq TK.$$

**Proof.** By convexity of the supremum,

$$\mathcal{R}(-L_t, Z) = \mathbb{E}_{E \sim \mathcal{U}(Y)} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{i=1}^{t} \epsilon_i h(z_i) - L_t(h) \right\} \right] = \sup_{h \in \mathcal{H}} \mathbb{E}_{E \sim \mathcal{U}(Y)} \left[ \sum_{i=1}^{t} \epsilon_i h(z_i) - L_t(h) \right] \geq -\frac{T}{2}.$$

For the upper bound, we notice that for all $E, h$,

$$\sum_{i=1}^{t} \epsilon_i h(z_i) - L_t(h) \leq I + \frac{t}{2} \leq (T - t)K + \frac{t}{2} \leq TK.$$

So the $\mathcal{R}(-L_t, Z)$ also has an upper bound of $TK$. $\square$

D.5 Remark on the Requirement of Fresh Dataset

In order to beat the adaptive adversary, the learner needs to sample fresh random hints in each round. Otherwise, the adversary can enforce high regret by correlating future labels with the history. More precisely, we will see that the matching randomness argument in Equation (13) uses the crucial fact that $V^{(t)}$ is a fresh dataset that is uniformly distributed independent of the interactions in the past. If $V^{(t)}$ is reused, then the adaptive adversary has the power to correlate $s_{1:t-1}$ with $V^{(t)}$ such that $V^{(t)}$ is no longer unbiased conditioned on the history. In this case, the algorithm fails to mimic the randomization in the relaxation, and the matching-randomness argument breaks down.

Another important property of the fresh self-generated hints $v_{i,k}^{(t)}$ is that they are identically distributed with the real hints $z_{i,k}$ in the coupling. Nevertheless, the analysis has to unite the fact that the learner can only access $v_{i,k}^{(t)}$s, and the monotonicity property (lemma D.3) is based on $z_{i,k}$. This point is subtle because it is impossible for the self-generated hints to really tell the future (i.e., ensure $x_t \in V_v^{(t-1)}$), since they are not controlled by the coupling $\Pi$. This issue is taken care of by Equation (15). We can see that it is sufficient for the uncoupled hints $V_v^{(t-1)}$ to resemble the coupled hints $Z_t$ at distribution level. This distributional resemblance is not achievable if $V^{(t-1)}$ were not independent with the past.
E.1 Proof of Lemma 4.3

Using the definitions of $Q_t$, $r^t$, and $P^t$, the following chain of inequalities holds for any fixed $s_t$:

$$
\begin{align*}
&\mathbb{E}_{h_t \sim Q_t} [L(h_t, s_t)] + \text{Rel}_T(\mathcal{H} \mid s_{1:t}) \\
= &\mathbb{E}_{h_t \sim Q_t} \left[ L(\text{opt}_{H,I}(r^{t-1}), s_t) \right] - \mathbb{E}_{r^{t+1}} \left[ \sum_{i=1}^{N(t+1)} L(\text{opt}_{H,I}(r^t), s_i^{(t+1)}) + \sum_{\tau=1}^{t} L(\text{opt}_{H,I}(r^t), s_\tau) \right] + \eta(T - t) \\
= &\mathbb{E}_{h_t \sim Q_t} \left[ L(\text{opt}_{H,I}(r^{t-1}), s_t) \right] - \mathbb{E}_{P^{t+1}} \left[ L(\text{opt}_{H,I}(r^t), s_t) \right] + \eta(T - t) \\
- &\mathbb{E}_{R^{t+1}} \left[ \sum_{i=1}^{N(t+1)} L(\text{opt}_{H,I}(r^t), s_i^{(t+1)}) + \sum_{\tau=1}^{t-1} L(\text{opt}_{H,I}(r^t), s_\tau) \right] \\
\leq &\mathbb{E}_{h_t \sim Q_t} [L(h_t, s_t)] - \mathbb{E}_{h_{t+1} \sim Q_{t+1}} [L(h_{t+1}, s_t)] + \eta(T - t) + \mathbb{E}_{R^{t+1}} \left[ \sup_{h \in \mathcal{H}} \left( -\sum_{i=1}^{N(t+1)} L(h, s_i^{(t+1)}) - \sum_{\tau=1}^{t-1} L(h, s_\tau) \right) \right] \\
= &\mathbb{E}_{h_t \sim Q_t} [L(h_t, s_t)] - \mathbb{E}_{h_{t+1} \sim Q_{t+1}} [L(h_{t+1}, s_t)] - \eta + \text{Rel}_T(\mathcal{H} \mid s_{1:t-1}),
\end{align*}
$$

where (a) uses the definition of $\text{opt}_{H,I}(r^t)$, and (b) is due to the fact that $R^{t+1}$ is an independent copy of $R^{t}$ conditioned on $\{s_\tau\}_{\tau < t}$. This implies the first inequality of Lemma 4.3.

For the second inequality, we further take the expectation with respect to $s_t \sim D_t$, and note that $Q_t$ and $\text{Rel}_T(\mathcal{H} \mid s_{1:t-1})$ are independent of $s_t$, while $Q_{t+1}$ depends on $s_t$:

$$
\begin{align*}
&\mathbb{E}_{s_t \sim D_t} \left( \mathbb{E}_{h_t \sim Q_t} [L(h_t, s_t)] + \text{Rel}_T(\mathcal{H} \mid s_{1:t}) \right) - \text{Rel}_T(\mathcal{H} \mid s_{1:t-1}) \\
\leq &\mathbb{E}_{s_t \sim D_t} \mathbb{E}_{h_t \sim Q_t} [L(h_t, s_t)] - \mathbb{E}_{s_t \sim D_t} \mathbb{E}_{h_{t+1} \sim Q_{t+1}} [L(h_{t+1}, s_t)] - \eta \\
\leq &\mathbb{E}_{s_t \sim D_t} \mathbb{E}_{h_t \sim Q_t} [L(h_t, s_t)] - \mathbb{E}_{s_t, s'_t \sim D_t} \mathbb{E}_{h_{t+1} \sim Q_{t+1}} [L(h_{t+1}, s_t)] \\
&\quad + \mathbb{E}_{s_t, s'_t \sim D_t} \mathbb{E}_{h_{t+1} \sim Q_{t+1}} [L(h_{t+1}, s'_t)] - \mathbb{E}_{s_t \sim D_t} \mathbb{E}_{h_{t+1} \sim Q_{t+1}} [L(h_{t+1}, s_t)] - \eta \\
= &\mathbb{E}_{s_t \sim D_t} \mathbb{E}_{h_t \sim Q_t} [L(h_t, s'_t)] - \mathbb{E}_{s_t, s'_t \sim D_t} \mathbb{E}_{h_{t+1} \sim Q_{t+1}} [L(h_{t+1}, s'_t)] \\
&\quad + \mathbb{E}_{s_t, s'_t \sim D_t} \mathbb{E}_{h_{t+1} \sim Q_{t+1}} [L(h_{t+1}, s'_t)] - \mathbb{E}_{s_t \sim D_t} \mathbb{E}_{h_{t+1} \sim Q_{t+1}} [L(h_{t+1}, s_t)] - \eta \\
= &\text{TV}(Q_t, \mathbb{E}_{s_t \sim D_t} [Q_{t+1}]) + \mathbb{E}_{s_t, s'_t \sim D_t; R^{t+1}} [L(h_{t+1}, s'_t) - L(h_{t+1}, s_t)] - \eta,
\end{align*}
$$

where (c) follows from the independence of $h_t \sim Q_t$ and $(s_t, s'_t)$, and (d) is due to $|\mathbb{E}_{X \sim P}[f(X)] - \mathbb{E}_{X \sim Q}[f(X)]| \leq \text{TV}(P, Q)$ for every measurable function $f$ with $\|f\|_\infty \leq 1$. 

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E.2 Proof of Lemma 4.6

The proof is essentially similar to [BDGR22, Lemma 12], and we include it here for completeness. For each \( i \in [m] \), compute the value \( p_i = \sigma \frac{dP}{dQ}(X_i) \), which lies in \([0, 1]\) due to the likelihood ratio upper bound. Now we draw an independent Bernoulli random variable \( Y_i \sim \text{Bern}(p_i) \), and define the random index \( I \) and success event \( E \) as follows:

\[
E \triangleq \cup_{i=1}^m \{ Y_i = 1 \},
I \triangleq \text{a uniformly random element of } \{ i \in [m] : Y_i = 1 \}.
\]

Note that \( Y_1, \ldots, Y_m \) are mutually independent, and for each \( i \in [m] \),

\[
\Pr[Y_i = 1] = \mathbb{E}_{X_i \sim Q}[p_i] = \mathbb{E}_{X_i \sim Q} \left[ \sigma \frac{dP}{dQ}(X_i) \right] = \sigma,
\]

we conclude that \( \Pr[E] = 1 - (1 - \sigma)^m \). For the second statement, we denote by \( r_i \) the external randomness used in drawing \( Y_i \sim \text{Bern}(p_i) \), and by \( r \) the external randomness used in the definition of \( I \). Then for any measurable set \( A \subseteq \mathcal{X} \),

\[
\Pr[X_I \in A \mid E, X_{\setminus I}] = \sum_{i,r,i,r} \Pr[X_i \in A \mid E, X_{\setminus i}, I = i, r, r] \cdot \Pr[I = i, r, r \mid E, X_{\setminus i}]
= \sum_{i,r,i,r} \Pr[X_i \in A \mid E, X_{\setminus i}, I = i, r, r] \cdot \Pr[I = i, r, r \mid E, X_{\setminus i}]\]
\[
\overset{(a)}{=} \sum_{i,r,i,r} \Pr[X_i \in A \mid Y_i = 1, X_{\setminus i}, r, r] \cdot \Pr[I = i, r, r \mid E, X_{\setminus i}]
= \sum_{i,r,i,r} P(A) \cdot \Pr[I = i, r, r \mid E, X_{\setminus i}]
= P(A),
\]

where (a) is due to the event \( \{ E, I = i, X_{\setminus i}, r, r \} \) is the same as \( \{ Y_i = 1, X_{\setminus i}, r, r \} \) as long as the former event \( \{ E, I = i, X_{\setminus i}, r, r \} \) is non-empty (note that empty events do not contribute to the sum), (b) follows from the mutual independence of \( (X_i, r_i, Y_i)_{i \in [m]} \) and \( r \), (c) is due to

\[
\Pr[X_i \in A \mid Y_i = 1] = \frac{\Pr[X_i \in A, Y_i = 1]}{\Pr[Y_i = 1]} = \frac{1}{\sigma} \mathbb{E}_{X_i \sim Q} \left[ 1(X_i \in A) \sigma \frac{dP}{dQ}(X_i) \right] = P(A).
\]

The above identity shows that the conditional distribution of \( X_I \) conditioned on \( (E, X_{\setminus I}) \) is always \( P \), as desired.
E.3 Proof of Lemma 4.7

The analysis is similar to the proof of Lemma 4.3. In fact, an intermediate step of Lemma 4.3 gives
\[\mathbb{E}_{h_t \sim Q_t} [L(h_t, s_t)] + \text{Rel}_T(H | s_{1:t}) - \text{Rel}_T(H | s_{1:t-1}) \leq \mathbb{E}_{h_t \sim Q_t} [L(h_t, s_t)] - \mathbb{E}_{h_{t+1} \sim Q_{t+1}} [L(h_{t+1}, s_t)] - \eta.\]

Now using \(|\mathbb{E}_{X \sim P}[f(X)] - \mathbb{E}_{X \sim Q}[f(X)]| \leq \text{TV}(P, Q)\) for every measurable function \(f\) with \(\|f\|_\infty \leq 1\), the RHS is further upper bounded by \(\text{TV}(Q_t, Q_{t+1}) - \eta\). The proof of Lemma 4.7 is completed by taking the expectation over \(s_t \sim D_t\).

F Proof of Lower Bounds (Theorem 5.1 and Theorem 5.2)

F.1 Proof of Theorem 5.1

This section proves the regret lower bounds for Algorithm 2 and Algorithm 3 stated in Theorem 5.1.

We split the analysis into two subsections, and in each subsection we prove a large regret both when the sample size parameter \(n\) is large and small.

F.1.1 Lower Bound Analysis for Algorithm 2

We shall only prove the regret lower bound \(\Omega(\sqrt{dT \sigma^{-1/2}})\) under the assumption \(\sigma \geq \max\{d/|X|, (d/T)^2\}\), for a smaller \(\sigma\) only makes the worst-case regret larger, and the other lower bounds follow from this case by taking \(\sigma = d/|X|\) and \(\sigma = (d/T)^2\), respectively. We split the analysis into two cases depending on the choice of parameter \(n\).

Case I: Large \(n\). When \(n\) is large, or more specifically, when \(n \geq T/\sqrt{\sigma}\), consider the behavior of Algorithm 2 on the following instance. Consider any domain \(X\) where \(|X|\) is an integral multiple of \(d\), and partition \(X = \bigcup_{j=1}^{d} X_j\) into \(d\) sets \(\{X_j\}_{j \in [d]}\) with an equal size. Consider the following hypothesis class:
\[\mathcal{H} = \{h : X \to \{\pm 1\} | h \text{ is a constant on } X_i, \forall i \in [d]\}.\]

Clearly \(\mathcal{H}\) has VC dimension \(d\). The adversary chooses a hypothesis \(h^* \in \mathcal{H}\) uniformly at random, and sets \(x_t\) to be uniformly distributed on \(X\). As for the label \(y_t\), the adversary sets \(y_t = h^*(x_t)\).

This adversary is 1-smooth, and the best expert in \(\mathcal{H}\) incurs a zero loss under this realizable setting. We claim that for each of the first \(\min\{T, c\sqrt{nd}\}\) time steps, for an absolute constant \(c > 0\) sufficiently small, Algorithm 2 makes a mistake with \(\Omega(1)\) probability. Summing over these steps, the expected regret of Algorithm 2 is then \(\Omega(\min\{T, c\sqrt{nd}\})\), which gives Theorem 5.1 by our assumption \(n \geq T/\sqrt{\sigma}\).

To prove this claim, we need the following lemma.

Lemma F.1 (Minimum Error on Hallucinated Samples). For \(N \sim \text{Poi}(n)\) hallucinated samples \((x_1, y_1), \ldots, (x_N, y_N)\), if \(n \geq d\), it holds that
\[\mathbb{P}\left(\sum_{i=1}^{N} y_i \cdot 1(x_i \in X_j) \geq \sqrt{\frac{n}{d}}\right) = \Omega(1), \quad \forall j \in [d].\]
Proof. For $j \in [d]$, let $n_{j,+}, n_{j,-}$ denote the number of hallucinate samples $(x_i, y_i)$ with $x_i \in \mathcal{X}_j$ and $y_i = \pm 1$, respectively. By the Poisson subsampling property, $\{n_{j,\pm}\}_{j \in [d]}$ are mutually independent $\text{Poi}(n/(2d))$ random variables. By definition of $\mathcal{H}$, we have

$$n_{j,+} - n_{j,-} = \sum_{i=1}^{N} y_i \cdot 1(x_i \in \mathcal{X}_j).$$

Consequently, the quantity of interest is $n_{j,+} - n_{j,-}$. As $n/d \geq 1$, by the Poisson tail property, both events $n_{j,+} \geq n/(2d) + \sqrt{n/d}/2$ and $n_{j,-} \leq n/(2d) - \sqrt{n/d}/2$ happen with $\Omega(1)$ probability, and their independence gives the claimed result.

Since $(d/T)^2 \leq \sigma \leq 1$, we have $T \geq d$ and thus $n \geq T/\sqrt{\sigma} \geq d$, the premise of Lemma F.1 holds. Consequently, at each time step $t \leq \min\{T, c\sqrt{n}d\}$ with $x_t \in \mathcal{X}_j$, with $\Omega(1)$ probability there are at least $\sqrt{n/d}$ net positive labels in the hallucinated samples, while the learner has only observed at most $\alpha c \sqrt{n/d}$ labels in the history with probability at least $1 - 1/\alpha$, by Markov’s inequality. By choosing constants $c > 0$ small and $\alpha > 0$ large, the perturbed leader will predict $+1$ depending only on the hallucination, and this prediction is independent of the choice of $h^*$ and thus incurs an error with probability $1/2$. This proves the claim that before time $\min\{T, c\sqrt{n}d\}$, there is always $\Omega(1)$ probability of error.

Case II: Small $n$. Now we turn to the scenario where $n < T/\sqrt{\sigma}$. Consider the following learning instance: choose $\mathcal{X}_0 \subseteq \mathcal{X}$ with $|\mathcal{X}_0| = \sigma|\mathcal{X}| \geq d$, the adversary always chooses $x_t \sim \mathcal{U}(\mathcal{X}_0)$, which is $\sigma$-smooth. Assuming that $|\mathcal{X}_0|$ is an integral multiple of $d$, we partition $\mathcal{X}_0 = \bigcup_{j=1}^{d} \mathcal{X}_j$ into $d$ subsets with equal size. Condition on each $\mathcal{X}_j$, consider an alternating label sequence:

$$(y_t : x_t \in \mathcal{X}_j)_{t=1}^{T} = (+1, -1, +1, -1, \cdots).$$

The hypothesis class $\mathcal{H}$ consists of $2^d$ functions:

$$\mathcal{H} = \{h : \mathcal{X} \to \{-1, 1\} \mid h \text{ is a constant on } \mathcal{X}_j, \forall j \in [d], \text{ and } h(x) \equiv 1, \forall x \in \mathcal{X} \setminus \mathcal{X}_0\}.$$  

Clearly $\mathcal{H}$ has VC dimension $d$, and the best hypothesis in $\mathcal{H}$ incurs a cumulative loss $T/2$.

Now we examine the performance of Algorithm 2. Let $r_j$ be the difference between the number of $+1$ and $-1$ labels in the hallucinated samples with feature in $\mathcal{X}_j$, similar to the proof of Lemma F.1 we have $r_j = n_{j,+} - n_{j,-}$ for independent Poisson random variables $n_{j,+}, n_{j,-} \sim \text{Poi}(n\sigma/2d)$. Suppose that ties are broken by always predicting $-1$ when calling the ERM oracle, we observe that Algorithm 2 always makes a mistake when $x \in \mathcal{X}_j$ and $r_j = 0$ – this is the same counterexample where Follow-The-Leader (FTL) makes a mistake at every step. Moreover, when $r_j \neq 0$, Algorithm 2 makes $T/2$ mistakes, same as the best expert in $\mathcal{H}$. Consequently, the expected regret of Algorithm 2 is at least $T \cdot \mathbb{P}(r_j = 0)$, where

$$\mathbb{P}(r_j = 0) = \mathbb{E}_{N \sim \text{Poi}(n\sigma/d)} \left[ \mathbb{P}\left( \text{Bin}(N, \frac{1}{2}) = \frac{N}{2} \right) \right] = \mathbb{E}_{N \sim \text{Poi}(n\sigma/d)} \left[ \Omega\left( \frac{1(N \text{ is even})}{\sqrt{N+1}} \right) \right]$$

$$\overset{(a)}{=} \Omega\left( \frac{\mathbb{P}_{N \sim \text{Poi}(n\sigma/d)}(N \text{ is even})}{\sqrt{n\sigma/d + 1}} \right) \overset{(b)}{=} \Omega\left( \frac{1}{\sqrt{n\sigma/d + 1}} \right) = \Omega\left( \min\left\{\frac{d}{n\sigma}, 1\right\} \right).$$

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In the above display, (a) follows from the conditional Jensen’s inequality, and (b) is due to
$$\mathbb{P}_{N \sim \text{Poi}(\lambda)}(N \text{ is even}) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^{2k}}{(2k)!} = e^{-\lambda} \cdot \frac{e^\lambda + e^{-\lambda}}{2} \geq \frac{1}{2}.$$  
This leads to the claimed regret lower bound in Theorem 5.1.

F.1.2 Lower Bound Analysis for Algorithm 3

Similar to the lower bound analysis for Algorithm 2, we also split into the cases where $n$ is large and $n$ is small, respectively. Recall that for Algorithm 3, the parameter $n$ is the number of random draws from $K$ hints at each future time.

Case I: Large $n$. We first focus on the case where $n \geq \sqrt{K}$. Consider the following learning instance: the domain $\mathcal{X}$ is $[dK]$, and we partition $\mathcal{X}$ into $\bigcup_{j=1}^{d} \mathcal{X}_j$ each of size $K$. At each time, the subsets $\mathcal{X}_j$ are given as the hint cyclically. Consider the same construction of $\mathcal{H}$ and the adversary in Section F.1.1, except that each $x_t$ is now uniformly distributed in the $K$-hint set.

The regret analysis is essentially the same as Section F.1.1. For every $t \leq T/2$, the learner in Algorithm 3 essentially generates $(T - t)n \geq T\sqrt{K}/4$ uniformly random samples (with replacement) in $\mathcal{X}$. A similar analysis to Lemma F.1 shows that for each $j \in [d]$, with $\Omega(1)$ probability there are $\Omega(\sqrt{K^{1/2}T/d})$ more +1 labels than −1 labels within $\mathcal{X}_j$ in the hallucinated samples. Consequently, for $t \leq \min\{T/2, \Omega((dT)^{1/2}K^{1/4})\}$, two calls of the ERM oracle in Algorithm 3 will return the same hypothesis, and the learner’s prediction is always +1. Similar to Section F.1.1, these time steps lead to an $\Omega(\min\{T, (dT)^{1/2}K^{1/4}\})$ regret.

Case II: Small $n$. Next we turn to the case where $n < \sqrt{K}$. Again, we construct $\mathcal{X}$ to be the disjoint union of $d$ sets $\{\mathcal{X}_j\}_{j \in [d]}$ each of size $K$, while construct $\mathcal{H}$ in a different way as follows: pick one element $x^*_j$ from each $\mathcal{X}_j$, and
$$\mathcal{H} = \{h : \mathcal{X} \rightarrow \{\pm 1\} \mid h(x) = 1, \forall x \notin \{x^*_1, \ldots, x^*_d\}\}.$$  
In other words, $\mathcal{H}$ shatters the set $\{x_1^*, \ldots, x_d^*\}$, while is always 1 on other inputs. Clearly the VC dimension of $\mathcal{H}$ is $d$.

The learning process is divided into $d$ epochs, each of length $T/d$. During the $j$-th epoch, the adversary chooses $x_t = x^*_j$, presents the set $\mathcal{X}_j$ to the learner as the hint, and sets the following alternating sequence of $y$:
$$(y_1, y_2, y_3, \cdots) = (+1, -1, +1, -1, \cdots).$$  
The best expert in $\mathcal{H}$ incurs a cumulative loss of $T/2$. For the performance of Algorithm 3, let $r^*_j$ be the difference between the number of +1 and −1 labels in the hallucinated samples with input $x^*_j$. One can check that if $r^*_j \neq 0$, the learner makes half of the mistakes along the alternating sequence; if $r^*_j = 0$, the fraction of mistakes becomes $3/4$ (Algorithm 3 cyclically predicts a wrong label and makes a random guess). Consequently, the expected regret of Algorithm 3 is lower bounded by
$$\Omega(T \cdot \mathbb{P}(r^*_j = 0)).$$  
To compute this probability, note that $r^*_j = 2M - N$, with $N \sim \text{Bin}(T/d, n/K)$ being the number of observations $x^*_j$ in the hallucinated data, and $M \mid N \sim \text{Bin}(N, 1/2)$. Using a similar argument to Section F.1.1, this probability is lower bounded by $\Omega(\min\{1, \sqrt{dK/(nT)}\})$, as desired.
F.2 Proof of Theorem 5.2

The proof of Theorem 5.2 uses a similar idea to [HK16]. There are two lower bound arguments in [HK16]: one reduces the problem to the Aldous’ problem, and the other is based on an explicit construction of the hard instance. Although both arguments could work for our problem, we adopt the latter which corresponds to Theorem 25 of [HK16]. In the sequel, we will always take the domain size \(|\mathcal{X}| = 1/\sigma|\) so that the smooth adversary becomes the usual adaptive adversary. In the next subsections, we first prove the theorem for the simpler case \(d = 1\), and then generalize our argument for any VC dimension \(d\).

F.2.1 The case \(d = 1\).

We first show how the argument in [HK16] proves the claimed \(\omega(\sqrt{|\mathcal{X}|})\) computational lower bound when \(T = \sqrt{|\mathcal{X}|} = 1/\sigma\). Assuming that \(N \triangleq \sqrt{|\mathcal{X}|}\) is an integer, we partition the domain \(\mathcal{X}\) into disjoint subsets \(\mathcal{X}_1, \ldots, \mathcal{X}_N\), each of size \(N\). For each \(x \in \mathcal{X}\), we associate two independent Rademacher variables \(\varepsilon(x)\) and \(\varepsilon^*(x)\), and they are mutually independent across different \(x \in \mathcal{X}\). For each \(i \in [N]\), the adversary chooses \(x^*_i \sim \mathcal{U}(\mathcal{X}_i)\), and sets the hypothesis class \(\mathcal{H} = \{h_x\}_{x \in \mathcal{X}}\) with

\[
h_x(x') = \begin{cases} 
\varepsilon^*(x') & \text{if } x = x^*_i, x' = x^*_j, \text{ and } i \geq j, \\
\varepsilon(x') & \text{otherwise.}
\end{cases}
\]

At each time \(t \in [N]\), the adversary sets \(x_t = x^*_t\), and \(y_t = h_{x^*_t}(x_t) = \varepsilon^*(x^*_t)\). Under this setting, [HK16] proved the following lower bound.

**Theorem F.2** (Theorem 25 of [HK16], restated). *Given access to the ERM oracle, any proper algorithm achieving an expected regret at most \(N/4\) requires \(\Omega(N) = \Omega(\sqrt{|\mathcal{X}|})\) running time.*

Here by running time, we assume that each oracle call takes unit time, and maintaining each element in the input \(\{(x_i, y_i)\}_{i \in I}\) to the oracle also takes unit time. We also sketch the proof idea of Theorem F.2 for completeness: the crucial observation is that, when the learner feeds the input \(\{(x_i, y_i)\}_{i \in I}\) to the ERM oracle, the oracle can always return some \(h \in \{h_0, h_{x^*_1}, \ldots, h_{x^*_t}\}\), where \(h_0\) is any hypothesis in \(\mathcal{H}\backslash\{h_{x^*_1}, \ldots, h_{x^*_t}\}\), and \(j \in [N]\) is the largest index such that \(x^*_j \in \{x_i\}_{i \in I}\). See Lemma 27 of [HK16] for a proof. Therefore, the label \(y_t = \varepsilon^*(x^*_t)\) at time \(t\) will look random to the learner unless the learner has seen a function \(h_{x^*_s}\) for some \(s \geq t\). By the above observation, this occurs only if the learner has set one (or more) of \(\{x^*_s\}_{s \geq t}\) as the input to the ERM oracle, but this requires one to find a random element in a size-\(N\) set and thus take \(\Omega(N)\) time (note that a proper algorithm only observes \(\{x^*_1, \ldots, x^*_{t-1}\}\) at time \(t\)). Consequently, with \(o(N)\) running time, the learner suffers from an \(\Omega(N)\) loss with high probability, while the best expert incurs zero loss - giving the \(\Omega(N)\) regret.

Since the restriction of \(\mathcal{H}\) on any two elements \(\{x, x'\}\) with \(x < x'\) could only be one of the three possibilities: \(\{(\varepsilon(x), \varepsilon(x')), (\varepsilon^*(x), \varepsilon^*(x'))\}\), the VC dimension of \(\mathcal{H}\) is 1. Therefore, Theorem F.2 gives a valid proof of Theorem 5.2 when \(d = 1\) and \(T = \sqrt{1/\sigma}\). For \(T < \sqrt{1/\sigma}\), the above construction still gives the \(\Omega(T)\) regret lower bound given \(o(\sqrt{|\mathcal{X}|})\) computational time. For general \(T > \sqrt{1/\sigma}\), we make the following modification to the adversary:
partition the time horizon $[T]$ into $N$ intervals $T_1, \ldots, T_N$, each of length $T/N$. For each $i \in [N]$ and $t \in T_i$, the adversary sets $x_t = x^*_i$, and

$$y_t = \begin{cases} h_{x^*_N}(x_t) & \text{with probability } \frac{1}{2} + \delta, \\ -h_{x^*_N}(x_t) & \text{with probability } \frac{1}{2} - \delta. \end{cases}$$

Consequently, the best expert $h_{x^*_N}$ incurs an expected cumulative loss $(1/2 - \delta)T$. Meanwhile, as long as the learner cannot distinguish the distributions $\operatorname{Bern}(1/2 + \delta)^{\otimes (T/N)}$ and $\operatorname{Bern}(1/2 - \delta)^{\otimes (T/N)}$, she is not able to estimate $\varepsilon^*(x^*_i)$ based on labels $\{y_t\}_{t \in T_i}$ in the $i$-th interval. This condition is fulfilled when $\delta \approx \sqrt{N/T}$. In addition, a similar argument for Theorem F.2 shows that with an $o(N)$ computational time, the learner cannot predict future $x^*_s$ either. Therefore, any proper learner with $o(N) = o(\sqrt{|X|})$ computational time must incur a regret $\Omega(\delta T) = \Omega(\sqrt{T|X|^{1/2}})$, which is precisely the statement of Theorem 5.2 for $d = 1$.

F.2.2 General $d$.

In this section we lift the hypothesis construction for $d = 1$ to general $d$. Since $1/\sigma \geq d$, we assume that $1/(\sigma d)$ is an integer. Partition $X = \bigcup_{j=1}^d X_j$ each of size $|X|/d$, we apply the hypothesis class $H$ in the previous section to each $X_j$, and set the entire hypothesis class as

$$H_d = \{h = (h_1, \ldots, h_d) \in H^d : h|_{X_j} = h_j, \forall j \in [d]\}.$$ 

Clearly the VC dimension of $H_d$ is $d$. The adversary is constructed as follows: partition $[T]$ into $d$ sub-intervals $T_1, \ldots, T_d$, each of size $T/d$. For the $i$-th sub-interval, we run the subroutine in the previous section independently on $X_i$. Now suppose that the total runtime is $o(\sqrt{d|X|})$, then for at least half of the sub-intervals, the runtime during each such interval is $o(\sqrt{|X|/d})$. By the lower bound for $d = 1$, the expected regret during each such sub-interval is

$$\Omega \left( \min \left\{ \frac{T}{d}, \sqrt{\frac{T}{d} \cdot \left( \frac{|X|}{d} \right)^{1/2}} \right\} \right) = \Omega \left( \min \left\{ \frac{T}{d}, \sqrt{T \cdot \left( \frac{|X|}{d^3} \right)^{1/2}} \right\} \right).$$

Summing over at least $d/2$ such independent sub-problems, the total regret lower bound is then $\Omega(\min\{T, \sqrt{T/(d|X|)^{1/2}}\})$, establishing the claim of Theorem 5.2.

G Smoothed Online Learning with Real-valued Labels

As an extension to the binary-label case, let us assume that the labels are real values ranging from $-1$ to $1$, i.e., $\mathcal{Y} = [-1, 1]$, and $H : X \to \mathcal{Y}$ is a real-valued hypothesis class with pseudo dimension $d$. Moreover, assume the loss function $l : \mathcal{Y} \times \mathcal{Y} \to [0, 1]$ is convex with Lipschitz constant $G$ in its first component.
G.1 Notions for Real-Valued Functions

In this section we introduce the notions that will be useful in analyzing real-valued hypothesis classes, including pseudo dimension and covering numbers.

**Definition G.1** (Pseudo-dimension, \[AB99\]). For every \( h \in \mathcal{H} \), let \( B_{h}(x, y) = \text{sgn}(h(x) - y) \) be the indicator of the region below or on the graph of \( h \). The pseudo-dimension of hypothesis class \( \mathcal{H} \) is defined as the VC dimension of the subgraph class \( B_{\mathcal{H}} = \{ B_{h} : h \in \mathcal{H} \} \).

We will see in the two following lemmas that pseudo dimension can be used to characterize the magnitude of covering numbers and Rademacher complexity.

**Lemma G.1** (\( d_{L_{1}(\mathcal{U})}\)-Covering Number Bound, \[AB99\]). The \( \epsilon \)-covering number of \( \mathcal{H} \) with respect to metric \( d_{L_{1}(\mathcal{U})} \), denoted by \( N(\epsilon, \mathcal{H}, L_{1}(\mathcal{U}(\mathcal{X}))) \), is the cardinality of the smallest subset \( \mathcal{H}' \) of \( \mathcal{H} \), such that for every \( h \in \mathcal{H} \), there exists \( h' \in \mathcal{H}' \) such that \( d_{L_{1}(\mathcal{U})}(h, h') \leq \epsilon \), where \( d_{L_{1}(\mathcal{U})}(f, g) = \mathbb{E}_{\mathcal{U}}[|f - g|] \). If \( d \) is the pseudo-dimension of \( \mathcal{H} \), then for any \( \epsilon > 0 \),

\[
\log N(\epsilon, \mathcal{H}, L_{1}(\mathcal{U}(\mathcal{X}))) \in \tilde{O}(d \log \frac{1}{\epsilon}).
\]

**Lemma G.2** (Rademacher Complexity Bound, \[Bar06\]). The Rademacher complexity of class \( \mathcal{H} \) for a set of \( n \) elements is upper bounded by \( O(\sqrt{dn \log n}) \), where \( d \) is the pseudo dimension of \( \mathcal{H} \).

The ERM oracles used in this section, are slightly different to those for binary classification. In particular, we consider oracles that can minimize a mixture of binary and real-valued loss values defined below.

**Definition G.2** (Real-valued optimization oracle). For a hypothesis class \( \mathcal{H} \) and two loss functions \( l^{r} \) and \( l^{b} \), the oracle \( \text{OPT} \) takes two sets of inputs \( S \) and \( S' \) over \( \mathcal{X} \times \mathcal{Y} \) and returns

\[
\text{OPT}_{\mathcal{H}, l^{r} l^{b}}(S, S') = \inf_{h \in \mathcal{H}} \left( \sum_{(x, y) \in S} l^{r}(h(x), y) + \sum_{(x', y') \in S'} l^{b}(h(x'), y') \right).
\]

We remark that these oracles are used in most previous works, including \[RSS12\]. They constitute a special form of regularized loss minimization oracles, with the regularization is given directly by a random process. Such oracles are indeed stronger than true ERM oracles, such as those used by \[DHL+17, DS16\] where regularization must happen through direct perturbations to the history of the play.

Throughout this section, we use real-valued loss functions \( l^{r}(\hat{y}, y) = \frac{1}{2G}l(\hat{y}, y) \), where \( G \) is the Lipschitz parameter of loss \( l \), and \( l^{b}(\hat{y}, y) = -\frac{1}{2}y\hat{y} \) to be the binary loss. We remove \( l^{r} \) and \( l^{b} \) from OPT notation when it is clear from the context.

G.2 Computationally-Efficient Upper Bound

In this section, we present a computational regret upper bound of order \( \tilde{O}(\sqrt{dT/\sigma}) \). We begin by describing our oracle-efficient algorithm. Just as in Algorithm 1, at every time step \( t \in [T] \), the
algorithm draws fresh new hints $V^{(t)}$ together with their random labels $\mathcal{E}^{(t)}$, and creates a dataset $S^{(t)} = V^{(t)} \circ \mathcal{E}^{(t)}$. Then it makes the following prediction through two oracle calls.

$$
\hat{y}_t = \text{OPT} \left( s_{1:t-1}; S^{(t)} \cup \{ (x_t, -1) \} \right) - \text{OPT} \left( s_{1:t-1}; S^{(t)} \cup \{ (x_t, +1) \} \right).
$$

(19)

The following theorem provides a regret upper bound on the above algorithm.

**Theorem G.3 (Computational Upper Bound).** For any $\sigma$-smooth adversary $\mathcal{Q}$, the algorithm $\mathcal{Q}$ with prediction rule (19) has expected regret upper bounded by $\tilde{O}(G\sqrt{dT}/\sigma)$, where $\tilde{O}$ hide factors that are polynomial in $\log(T)$ and $\log(1/\sigma)$. Furthermore, the algorithm is oracle-efficient: at every round $t$, this algorithm use two oracle calls with histories of length $\tilde{O}(T/\sigma)$.

**Proof.** To prove Theorem G.3 we use the following relaxation:

$$
\text{Rel}_T(\mathcal{H}|s_{1:t}) = 2G \mathbb{E}_{V^{(t)} \mathcal{E}^{(t)}} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{i=1:1-T, k=1:K} \epsilon_{i,k} h(v_{i,k}^{(t)}) - L^r(\cdot, s_{1:t}) \right\} \right] + 2G\beta(T-t),
$$

(20)

where $K = 100 \log T/\sigma$ and $\beta = 10TK(1-\sigma)^K$. We will show in Lemma G.4 that the above relaxation is admissible. Therefore, Proposition 2.1 gives us the following upper bound on the expected regret:

$$
\mathbb{E}[\text{REGRET}(T)] \leq \text{Rel}_T(\mathcal{H}|\emptyset) + O(\sqrt{T}) = 2G \mathbb{E}_{V^{(0)} \mathcal{E}^{(0)}} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{i=1:1-T, k=1:K} \epsilon_{i,k} h(v_{i,k}^{(0)}) \right\} \right] + 2G\beta T + O(\sqrt{T}).
$$

The first term (a) is the Rademacher complexity of the hypothesis class $\mathcal{H}$ with respect to the uniform distribution for sample size $TK$. By Lemma G.2, (a) $\leq O\left( \sqrt{dTK \log(TK)} \right)$. For the second term, we have $\beta T \in o(1)$ by Equation (12). Plugging in $K = O\left( \log(T)/\sigma \right)$, we have the following bound:

$$
\mathbb{E}[\text{REGRET}(T)] \leq O\left( G\sqrt{\frac{dT}{\sigma} \log T \log \left( \frac{T}{\sigma} \right)} \right) \in \tilde{O}(\sqrt{dT/\sigma}),
$$

where $\tilde{O}$ hide factors that are polynomial in $\log(T)$ and $\log(1/\sigma)$.

**Lemma G.4.** The relaxation defined by Equation (20) is admissible with respect to the algorithm $\mathcal{Q}$, that has prediction rule given by Equation (19).

**Proof.** Using the language of regularized Rademacher complexity, the above relaxation can be written as

$$
\text{Rel}_T(\mathcal{H}|s_{1:t}) = 2G \mathbb{E}_{V^{(t)} \mathcal{E}^{(t)}} \left[ \text{Rad}(\cdot, s_{1:t}, V^{(t)}) \right] + 2G\beta(T-t),
$$

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where \( L^t(\cdot, s_{1:t}) = \sum_{i=1}^{t-1} l(h(x_i), y_i) \). When \( t = T \), the relaxation becomes

\[
\text{Rel}_T(\mathcal{H} | s_{1:T}) = -2GL^t(h, s_{1:T}) = -\inf_{h \in \mathcal{H}} \sum_{i=1}^{T} l(h(x_i), y_i),
\]

thus it satisfies the second condition of Definition 2.3. For the first condition, we need to verify

\[
\sup_{D_t \in \mathcal{D}_t} \mathbb{E} \sup_{x_t \sim D_t, y_t \in \mathcal{Y}} \left\{ \mathbb{E} \left[ l(\hat{y}(x_t), y_t) \right] + \text{Rel}_T(\mathcal{H} | s_{1:t-1} \cup (x_t, y_t)) \right\} \leq \text{Rel}_T(\mathcal{H} | s_{1:t-1}).
\]

(21)

We first upper bound the LHS of Equation (21) by matching the randomness in \( V^{(t)} \) and applying Jensen’s inequality to the supremum function. This gives us

\[
\sup_{D_t \in \mathcal{D}_t} \mathbb{E} \sup_{x_t \sim D_t, y_t \in \mathcal{Y}} \left\{ \mathbb{E} \left[ l(\hat{y}(x_t), y_t) \right] + \text{Rel}_T(\mathcal{H} | s_{1:t-1} \cup (x_t, y_t)) \right\} \\
= \sup_{D_t \in \mathcal{D}_t} \mathbb{E} \sup_{x_t \sim D_t, y_t \in \mathcal{Y}} \mathbb{E} \left[ l(\hat{y}(x_t), y_t) \right] + \sup_{D_t \in \mathcal{D}_t} \mathbb{E} \mathbb{E} \sup_{x_t \sim D_t, y_t \in \mathcal{Y}} \left[ l(\hat{y}(x_t), y_t) \right] + 2G \beta (T - t) \\
\leq \sup_{D_t \in \mathcal{D}_t} \mathbb{E} \mathbb{E} \sup_{x_t \sim D_t, y_t \in \mathcal{Y}} \left[ l(\hat{y}(x_t), y_t) \right] + 2G \beta (T - t)
\]

(22)

For every fixed input \( x_t \) and hint set \( V^{(t)} \), our prediction rule in Equation (19), which we denote with \( Q_t(V^{(t)}) \), is the same as the transductive prediction rule in [RSS12, Equation (25)], with \( V^{(t)} \) being the set of unlabeled future instances and \( s_{1:t-1} \) being the historical data with labels.

According to [RSS12, Lemma 12], for all input \( x_t \) and unlabeled sequence \( \mathcal{X}^- \) (which plays the role of \( x_{t+1:T} \)), the decision rule \( Q_t(\mathcal{X}^-) \) satisfies

\[
\sup_{y_t \in \mathcal{Y}} \left\{ \mathbb{E} \left[ l(\hat{y}(x_t), y_t) \right] + 2G \mathbb{E} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{x \in \mathcal{X}^-} \epsilon_x h(x) - L^t(h, s_{1:t-1} \cup (x, y_t)) \right\} \right] \right\} \\
\leq 2G \mathbb{E} \left[ \sup_{h \in \mathcal{H}} \left\{ \sum_{x \in \mathcal{X}^- \cup \{x_t\}} \epsilon_x h(x) - L^t(h, s_{1:t-1}) \right\} \right].
\]

Therefore, if we choose the sequence \( \mathcal{X}^- \) to be \( V^{(t)} \), we obtain the following inequality which is written in the language of regularized Rademacher complexity:

\[
\sup_{y_t \in \mathcal{Y}} \left\{ \mathbb{E} \left[ l(\hat{y}(x_t), y_t) \right] + 2G \cdot \mathbb{R}(-L^t(\cdot, s_{1:t}), V^{(t)}) \right\} \leq 2G \cdot \mathbb{R}(-L^t(\cdot, s_{1:t-1}), V^{(t)} \cup \{x_t\}).
\]

By adding the expectations over \( V^{(t)} \) and \( x_t \) on both sides, we obtain the following upper bound:

\[
(22) \leq \sup_{D_t \in \mathcal{D}_t} \mathbb{E} \mathbb{E} \left[ 2G \cdot \mathbb{R}(-L^t(\cdot, s_{1:t-1}), V^{(t)} \cup \{x_t\}) \right] + 2G \beta (T - t)
\]
According to Lemma D.3, we can replace the $x_t$ sampled from the worst-case smooth distribution by $Z_t$ sampled independently from the uniform distribution, with the extra cost $\beta$. This gives

$$
(22) \leq \mathbb{E}_{V(t) \sim \mathcal{U}(X)} \mathbb{E}_{z_t \sim \mathcal{U}(X)} \left[ 2G \cdot \mathcal{R}(-L^t(\cdot, s_{1:t-1}), V^{(t)} \cup \{x_t\}) + 2G\beta(T-t) \right]
$$

which is precisely the RHS of Equation (21).

**G.3 Statistical Upper Bound**

In this section, we present a statistical upper bound achieved by a computationally inefficient algorithm. The $\mathcal{D}$ be the algorithm that runs Hedge on a finite subset $\mathcal{H}'$ on $\mathcal{H}$, where $\mathcal{H}'$ is an $\epsilon$-cover of $\mathcal{H}$ with respect to the uniform distribution $\mathcal{U}(X)$. The regret upper bound of this algorithm is bounded as follows.

**Theorem G.5** (Statistical Upper Bound). For any $\sigma$-smooth adversary $\mathcal{D}_\sigma$, the algorithm $\mathcal{D}$ described above has regret upper bound

$$
\mathbb{E}[\text{REGRET}(T, \mathcal{D}_\sigma, \mathcal{D})] \in \tilde{O}\left(\sqrt{T d \log \left(\frac{T}{d\sigma}\right)} + Gd \log \left(\frac{T}{d\sigma}\right)\right).
$$

**Proof.** Let $\mathcal{H}'$ be the smallest $\epsilon$-cover of $\mathcal{H}$ with respect to the uniform distribution, i.e., for any $h \in \mathcal{H}$, there exists a proxy $h' \in \mathcal{H}'$ such that $\mathbb{E}_{x \sim \mathcal{U}(X)}[|h(x) - h'(x)|] \leq \epsilon$. By lemma G.1, the size of $\mathcal{H}$ can be upper bounded in terms of the pseudo dimension $d$:

$$
\log(|\mathcal{H}'|) = \log \mathcal{N}(\epsilon, \mathcal{H}, L_1(\mathcal{U}(X))) \leq \tilde{O}\left(d \log \left(\frac{1}{\epsilon}\right)\right),
$$

where $\tilde{O}$ hide factors that are $\log \log(1/\epsilon)$. Based on the net $\mathcal{H}'$, we also define function class $\mathcal{G}$ as follows.

$$
\mathcal{G} = \{g_{h,h'}(x) = |h(x) - h'(x)| : h \in \mathcal{H}, h' \in \mathcal{H}' \text{ is its proxy.}\}
$$

Now consider the following regret decomposition:

$$
\mathbb{E}[\text{REGRET}(T)] = \mathbb{E}\left[\sum_{t=1}^{T} l(\hat{y}_t, y_t) - \inf_{h \in \mathcal{H}} L(h, s_{1:T})\right]
= \mathbb{E}\left[\sum_{t=1}^{T} l(\hat{y}_t, y_t) - \inf_{h' \in \mathcal{H}'} L(h', s_{1:T})\right] + \mathbb{E}\left[\inf_{h' \in \mathcal{H}'} L(h', s_{1:T}) - \inf_{h \in \mathcal{H}} L(h, s_{1:T})\right]
$$
Note that the first term is precisely the regret of Hedge on the cover $\mathcal{H}'$. It is thus bounded by

$$
\mathbb{E} \left[ \sum_{t=1}^{T} l(\hat{y}_t, y_t) - \inf_{h' \in \mathcal{H}'} L(h', s_{1:T}) \right] \leq O \left( \sqrt{T \log |\mathcal{H}'|} \right) \in \tilde{O} \left( \sqrt{T d \log \left( \frac{1}{\epsilon} \right)} \right).
$$

As for the second term, we reformulate it in terms of class $\mathcal{G}$:

$$
\mathbb{E} \left[ \inf_{h' \in \mathcal{H}'} L(h', s_{1:T}) - \inf_{h \in \mathcal{H}} L(h, s_{1:T}) \right] = \mathbb{E} \left[ \sup_{h \in \mathcal{H}} \inf_{h' \in \mathcal{H}'} \sum_{t=1}^{T} l(h'(x_t), y_t) - l(h(x_t), y_t) \right]
$$

$$
\leq \mathbb{E} \left[ \sup_{h \in \mathcal{H}} \inf_{h' \in \mathcal{H}'} \sum_{t=1}^{T} G|h(x_t) - h(x_t)| \right] = G \cdot \mathbb{E} \left[ \sup_{g \in \mathcal{G}} \sum_{t=1}^{T} g(x_t) \right], \quad (23)
$$

where (a) is because the loss function $l$ has Lipschitz constant $G$. Analogous to [HRS21, Claim 3.4], we apply the coupling argument in Lemma D.1 to replace the adaptive sequence $x_t$s by $z_{t,k}$s that are sampled independently from the uniform distribution. Thus we obtain

$$
\mathbb{E} \left[ \sup_{g \in \mathcal{G}} \sum_{t=1}^{T} g(x_t) \right] \leq T^2(1 - \sigma)^K + \mathbb{E} \left[ \sup_{g \in \mathcal{G}} \sum_{t=1}^{K} g(z_{t,k}) \right].
$$

The expected supremum can be further bounded in terms of the magnitude of $\mathcal{G}$ (i.e., $\epsilon$) as well as the pseudo dimension of the original hypothesis class $\mathcal{H}$. Using the bound in Lemma G.6, and together with Equation (23), we obtain

$$
\mathbb{E}[\text{REGRET}(T)] \leq \tilde{O} \left( \sqrt{T d \log \left( \frac{1}{\epsilon} \right)} + G \left( T^2 (1 - \sigma)^K + TK \epsilon + \sqrt{TKd \log \left( \frac{1}{\epsilon} \right)} \right) \right).
$$

In order to satisfy the condition on $n$ in Lemma G.6 and to make the failure probability of the coupling argument sufficiently small, we take $\alpha = 10 \log(T)$, $K = \frac{\alpha}{\sigma}$, $\epsilon = \Theta \left( \frac{1}{d \sigma} \log \left( \frac{T \log(T)}{d \sigma} \right) \right)$.

With this choice of parameters, we have $T^2 (1 - \sigma)^K = o(1)$ and

$$
\mathbb{E}[\text{REGRET}(T)] \leq O \left( \sqrt{T d \log \left( \frac{1}{\epsilon} \right)} + G \left( \frac{T \log(T)}{\sigma} \epsilon + \sqrt{\frac{T \log(T)}{d \sigma} \epsilon d \log \left( \frac{1}{\epsilon} \right)} \right) \right)
$$

$$
\leq \tilde{O} \left( \sqrt{T d \log \left( \frac{T}{d \sigma} \right)} + Gd \log \left( \frac{T}{d \sigma} \right) \right),
$$

as desired. \(\square\)

**Lemma G.6** (Concentration for the expected value of supreme). When $n \geq \Omega \left( \frac{d \log \left( \frac{1}{\epsilon} \right)}{\epsilon} \right)$, we have

$$
\mathbb{E}_{x_1,n \sim \mathcal{U}(\mathcal{X})} \left[ \sup_{g \in \mathcal{G}} \sum_{i=1}^{n} g(x_i) \right] \leq O \left( ne + \sqrt{n \epsilon d \log \left( \frac{1}{\epsilon} \right)} \right).
$$
Proof. We will use the bound on expected values of suprema of empirical processes in [GK06, Theorem 3.1]. To apply their result, the first step is to establish a bound on the $L_2(P)$-covering number of class $\mathcal{G}$. Let $P_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}$ be the empirical distribution based on independent samples $x_1, \cdots, x_n$. A similar argument to [BKP97, Lemma 2] gives us

$$
N(\epsilon, \mathcal{G}, L_2(P_n)) \leq N\left(\frac{\epsilon}{2}, \mathcal{H}, L_2(P_n)\right)^2.
$$

Thus we obtain

$$
\log N(\epsilon, \mathcal{G}, L_2(P_n)) \leq 2 \log N\left(\frac{\epsilon}{2}, \mathcal{H}, L_2(P_n)\right) \leq 2 \log \mathcal{M}\left(\frac{\epsilon}{2}, \mathcal{H}, L_2(P_n)\right) \leq O\left(d \log \left(\frac{1}{\epsilon}\right)\right),
$$

where $\mathcal{M}$ denotes the packing number and the last inequality is due to [Bar06, Theorem 3.1]. Therefore, for the function $H(x) = O(d \log x)$, we can guarantee that for any $\epsilon > 1$,

$$
\log N(\epsilon, \mathcal{G}, L_2(P_n)) \leq H(1/\epsilon),
$$

satisfying the condition of [GK06, Theorem 3.1]. Therefore, when $n \geq \Omega\left(\frac{H(1/\epsilon)}{\epsilon}\right) = \Omega\left(\frac{d}{\epsilon} \log \left(\frac{1}{\epsilon}\right)\right)$,

[GK06] gives us

$$
\mathbb{E}_{\mathcal{U}} \left[ \sup_{g \in \mathcal{G}} \sum_{i=1}^{n} (g(x_i) - \mathbb{E}[g(x_i)]) \right] \leq O\left(\sqrt{n \epsilon H(1/\epsilon)}\right) = O\left(\sqrt{n \epsilon d \log \left(\frac{1}{\epsilon}\right)}\right).
$$

Finally, since $\mathbb{E}_{\mathcal{U}} g(x) \leq \epsilon$ for any $g \in \mathcal{G}$, we obtain

$$
\mathbb{E} \left[ \sup_{g \in \mathcal{G}} \sum_{i=1}^{n} g(x_i) \right] \leq O\left(n \epsilon + \sqrt{n \epsilon d \log \left(\frac{1}{\epsilon}\right)}\right),
$$

and the proof is complete. \qed

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