Persistence exponent in superantiferromagnetic quenching

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Abstract

We measure the persistence exponent in a phase separating two-dimensional spin system with non-conserved dynamics quenched in a region with four coexisting stripe phases. The system is an Ising model with nearest neighbor, next-to-the-nearest neighbor and plaquette interactions. Due the particular nature of the ground states, the order parameter is defined in terms of blocks of spins. Our estimate of the persistence exponent, $\theta = 0.42$, differs from those of the two-dimensional Ising and four state Potts models. Our procedure allows the study of persistence properties also at finite temperature $T$: our results are compatible with the hypothesis that $\theta$ does not depend on $T$ below the critical point.

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I. INTRODUCTION

When a system is quenched from a disordered phase into a multiphase coexistence region, ordered domains form randomly and grow in a self-similar way [1]. The kinetics of coarsening domains is described by the algebraic growth $L(t) \sim t^z$ where $L(t)$ is the characteristic length of domains at time $t$ and $z$ is an universal exponent not depending on the dimensionality of the system nor on the final temperature of the quenching. In most models with non-conserved order parameter $z = 1/2$, while in scalar models with conserved order parameters $z = 1/3$ [2]. There is a general agreement on this scenario and the values of the exponent $z$ are used to classify the growth regimes in experimental systems.

More recently, new dynamical exponents have been considered also with the hope to better characterize the process of phase separation. However, the universal character of these new exponents is under debate and the determination of their values in particular cases can be useful to clarify their nature.

A new exponent introduced by Derrida et al. [3] is related to the persistence probability $p(t)$ defined as the probability that the local order parameter at a given point $x$ has never changed since the initial time. In spinodal decomposition problems this probability was first considered for the Ising model with Glauber dynamics at $T = 0$. Here $p(t)$ is the fraction of spins that have never flipped from the beginning of the process. This probability decays as $p(t) \sim t^{-\theta}$ and $\theta$ is called the persistence exponent. In $d = 1$ $\theta$ has been calculated exactly for the Potts model with non-conserved dynamics [4], for the Ising model $\theta = 3/8$. In higher dimension the evaluation of $\theta$ is based on Monte Carlo simulations [5]. For example in the two-dimensional Ising model $\theta = 0.22$ and different values have been calculated for other dimensions and Potts model [6, 7]. All these results concern the case $T = 0$.

At temperatures different from zero, fluctuations induce an exponential decay of the probability $p(t)$ and the definition of the persistence exponent is more subtle. A method has been proposed [8] to evaluate $\theta$ at $T \neq 0$ based on the comparison of two different systems, one starting from a random and the other from a ground state, evolving with the
same noise. The spin flips in the ordered state represent the effects of thermal fluctuations and are to be subtracted to the spin flips of the phase separating system in order to measure $\theta$. This method has been applied to Potts model with non-conserved dynamics \[9\].

In this paper we calculate the persistence exponent in a two-dimensional spin system quenched in a region with four coexisting stripe phases. The system is an Ising model with nearest neighbor, next-to-the-nearest neighbor and plaquette interactions. Due the particular nature of the ground states, the order parameter is defined in terms of blocks of spins. We study the persistence properties of these blocks and find a value of the persistence exponent different from that of Ising and Potts model. The method has been used also for calculating the persistence exponent at $T \neq 0$. Our results are compatible with the hypothesis that $\theta$ does not change with the temperature.

Section II in the paper is devoted to the description of the model and of the block variables used for describing the different phases. In Section III our results will be presented for various temperatures. Some conclusions will follow.

**II. MODEL AND METHODS**

The model we consider is the Ising version of the two-dimensional isotropic eight vertex model \[10\], with hamiltonian $H$ given by

$$-\beta H = J_1 \sum_{<ij>} s_i s_j + J_2 \sum_{<<ij>>} s_i s_j + J_3 \sum_{[i,j,k,l]} s_i s_j s_k s_l,$$

(1)

where $s_i$ are Ising spins on a two-dimensional square lattice and the sums are respectively on nearest neighbor pairs of spins, next to the nearest neighbor pairs and plaquettes. Periodic boundary conditions are always assumed. At $J_2 < 0, |J_1| < 2|J_2|$ and $J_3$ small a critical line separates the paramagnetic phase from a region where four phases corresponding to ground states with alternate plus and minus columns or rows coexist \[10\]. The system is prepared in a completely disordered configuration and the persistent behavior is studied in the stripe-ordered phase coexistence region by numerical simulations both at zero and finite temperature with heat-bath single-spin-flip dynamics \[11\].
The study of the evolution of the local order parameter in the phase separation process has to take into account the particular structure of the ground states. In Ising spin models the persistent fraction $p(t)$ is generally defined as the number of spins that never changed sign in the interval $[0, t]$ normalized with respect to the volume of the system. Here we have to consider the persistence of the local order parameter and so we must label each site as belonging to one of the four possible lamellar ground states.

The definition of the local order parameter is based on a block method that has already been used in [12, 13] for studying the growth of the stripe domains of model (1). Let us focus on the site $x$ of the lattice at time $t$: we consider a $(2l + 1) \times (2l + 1)$ square $S_l$ with $l$ a positive integer number and we compare the configuration of the system at time $t$ inside $S_l$ with the possible four ground states. If one of the four ground states agrees with the configuration in $S_l$ except a number of spins less than or equal to a threshold value $K$ we say that the site $x$ at time $t$ belongs to this ground state and we assign it the color $q = 1, ..., 4$; otherwise we say that the site is a defect and we do not assign it any color. All the simulations that will be discussed throughout the paper have been performed by choosing $K$ such that $K/|S_l| = K/(2l + 1)^2 \sim 0.1$, obviously we have checked that our results do not depend on the choice of $K$. In [12, 13] we have measured by this procedure the growth exponent and shown that the average size of domains growths as $R(t) \sim t^z$ where $z = 1/2$ or $z = 1/3$ in the non-conserved and conserved cases respectively.

We use the above definitions to measure the fraction $Q(t)$ of the number of sites not changing their color: in our procedure we let the system evolve in the interval $[0, t_0]$, so that domains form, we label all the spins at time $t_0$ and then we count the number of spins that do not change color during the further evolution. This method resembles the “block persistence” approach introduced in [14] to measure persistence exponents in the Ising model at finite temperature. We remark that in our problem the introduction of blocks is suggested by the character of the local order parameter not defined on a single site.

The definition of the persistence exponent at finite temperatures requires some observations. At zero temperature a spin flip occurs only when the site of the spin is reached by the
boundary of a growing domain and the number of persistent spins decreases as \( p(t) \sim t^{-\theta} \). At finite temperature a spin can flip randomly due to thermal disorder. That is, in the probability of one spin flip there is a constant term, not depending on time; this suggests the exponential correction \( p(t) \sim t^{-\theta} \times e^{-\lambda t} \) \([15]\).

In the block method the effect of spin flip due to thermal disorder can be eliminated in a natural way. When the temperature increases larger values of the side length \( 2l + 1 \) of the block used to label the spins can be used. Indeed, to destroy the local order inside a block \( S_l \) it is necessary that the number of thermal-due flips exceed \( K \). This means that the exponential correction due to thermal effect can be reduced by considering a sufficiently large block \( S_l \), so that a thermal destruction of the order inside \( S_l \) is unlikely.

Let us describe the following example: we initialise model (1) in an ordered phase and let it evolve at finite temperature and parameters \( J_1 = J_3 = 0 \) and \( J_2 = -0.5 \). After 1000 full sweeps of the grid the system is assumed to be in thermal equilibrium. Starting from the so obtained configuration, shown in Fig. 1, we apply our block procedure to label all the sites of the lattice. Since all defects are thermal due, all the lattice sites should be recognized as belonging to the same phase. In the four pictures of Fig. 2 black squares represent sites not recognized as belonging to the ordered phase for blocks \( S_l \ 2 \times 2 \) \([16]\), \( 3 \times 3 \), \( 5 \times 5 \) and \( 7 \times 7 \) from the left to the right and from the top to the bottom. All the spin flips are spurious and should be disregarded; from the pictures it is clear that by considering blocks of different increasing sizes, one is able to eliminate the effects of noise. We see that in this case a \( 7 \times 7 \) window is enough to label correctly almost all the sites.

III. RESULTS

Let us firstly consider the quenching at zero temperature for our coarsening system. The fraction of spins that never flipped from the initial time is found to decrease over time according to the same persistence exponent \( \theta = 0.22 \) as the one estimated for the two-dimensional Ising model \([3]\) (see Fig. 3, the lower curve). On the other hand one may
consider the fraction $Q(t)$ of spins that never changed phase since the beginning of the process, and the time evolution of this quantity is related to the probability that a given point is crossed for the first time by a domain wall. In order to label a site as belonging to a domain of one of the four phases, we compared the system’s configuration in a small window centered at that site with patterns corresponding to the four ground states, as described in Section II. The estimate for the persistence exponent was found to be $\theta = 0.42$. In Fig. 3 the center and top plots represent the logarithm of the number of persistent sites versus the logarithm of the number of iterations in a $600 \times 600$ and $1200 \times 1200$ system, respectively. We remark that this result differs from what it is observed in the case of the two-dimensional four state Potts model, which shares with the present model the feature of having a four-folded ground state: as described in [6], the log-log plot of the number of persistent spins versus the iteration number, in the Potts case, shows some curvature and an exponent $\theta = 0.36$ has been estimated in the case of the triangular lattice. Moreover we note that all results on persistence exponent do not depend on the choice of the $J's$ in the superantiferromagnetic phase, differently from what is found with respect to correlation properties [12, 13].

The independence from temperature of the persistence exponent has also been checked: in Fig. 4 we depict the time evolution of of the logarithm of the number of persistent sites after a quench with $J_1 = J_3 = 0$, $J_2 = -1$ and in the cases $l$ equal to 1 ($\bullet$), 2 ($\blacksquare$) and 3 ($\square$). As it has been explained in Section II, the procedure we used to label the sites is not sensitive to thermal noise provided that a wide enough window is chosen. As it appears from Fig. 4, the exponential decay of $Q(t)$ is manifest, in the time interval we considered, only in the case of $3 \times 3$ windows, while using $5 \times 5$ or $7 \times 7$ windows cancels the effects of temperature. The estimated value for $\theta$ is $\theta = 0.43$, consistent with the case $T = 0$.

When the temperature is increased, wider windows are needed. For $J_1 = J_3 = 0$ and $J_2 = -0.6$ the exponential decay is manifest even using a $7 \times 7$ window, but a $9 \times 9$ window is capable to cancel the effect of noise. In Fig. 3 Monte Carlo data are plotted in cases $l$ equals 2 ($\bullet$), 3 ($\blacksquare$), 4 ($\square$) and 5 ($\triangle$). In the case $l = 5$, corresponding to $11 \times 11$ windows, we
find $\theta = 0.47$. A similar variation of $\theta$ near the critical temperature has been numerically observed in the Ising and Potts models [8]; a possible mechanism for such dependence would be that the anisotropy of the surface tension in these lattice models depends on the temperature (see [8]). Better numerical simulations are needed to confirm that this variation of $\theta$ with temperature is real and due to the above described mechanism.

Finally we applied Derrida’s method to evaluate the persistence exponent for the present model at finite temperature. We evolved two systems, one starting from a disordered and the other from a ground state, subject to the same thermal noise. The $3 \times 3$ block has been used to label sites and color changes occurring simultaneously in both systems were not taken into account. Results similar to the ones described above are obtained. Indeed in Fig. 6 we plot the logarithm of the number of persistent sites, measured following Derrida’s scheme, in the case $J_1 = J_3 = 0$ and $J_2 = -1$, with lattice sizes 256 ($\bullet$), 600 ($\blacksquare$) and 1200 ($\square$). The estimate for the persistence exponent was found not to depend on the lattice sizes we used, and it was equal to $\theta = 0.43$.

IV. CONCLUSIONS

In this paper we have discussed persistence properties of a two-dimensional spin system with nearest neighbor, next-to-the-nearest neighbor and plaquette interactions, quenched from a disordered phase into the superantiferromagnetic phase.

Due the particular nature of the ground states, the order parameter is defined in terms of blocks of spins. A procedure has been introduced to label each site of the lattice as belonging to one of the four possible lamellar ground states. This method allows the definition of the persistence exponent both at zero and finite temperature, provided a large enough block $S_l$ is used.

Our estimate of the persistence exponent, $\theta = 0.42$, differs from those of the two-dimensional Ising and four state Potts models. Our results are compatible with the hypothesis that $\theta$ does not depend on temperature in the low-temperature phase of the quenching.
Some variation of $\theta$ is observed when the temperature is close to the critical temperature, but further investigations would be needed to better characterize the persistence properties of the separating system near the critical point.

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given site.
**Figure captions**

**Fig.1:** Configuration of model (1) after 1000 full updates of the lattice obtained starting from a completely ordered initial configuration, at finite temperature with $J_1 = J_3 = 0$ and $J_2 = -0.5$. Black and white squares represent, respectively, plus and minus spins.

**Fig.2:** Starting from the configuration depicted in Fig. 1 we have applied our block procedure to label all the sites of the lattice. Since all defects in Fig. 1 are thermal due, all the lattice sites should be recognized as belonging to the same phase. In the four pictures black squares represent sites not recognized as belonging to the ordered phase for blocks $S_l$ $2 \times 2$ [□], $3 \times 3$, $5 \times 5$ and $7 \times 7$ from the left to the right and from the top to the bottom.

**Fig.3:** The logarithm of the number of spins that never flipped from the initial time versus the logarithm of the number of iterations is depicted in the bottom plot. Monte Carlo data have been obtained by averaging over 51 different runs of the $600 \times 600$ system. The center (top) plot represents the logarithm of the number of sites that never changed phase versus the logarithm of time; data have been obtained by averaging over 37 (12) runs of the $600 \times 600$ ($1200 \times 1200$) system. All data refer to the zero temperature case with parameters $J_1 = J_3 = 0$ and $J_2 = -1$.

**Fig.4:** The logarithm of the number of sites that never changed phase is plotted versus the logarithm of time; data have been obtained on a $1200 \times 1200$ lattice by averaging over 20 runs, in the finite temperature case with parameters $J_1 = J_3 = 0$ and $J_2 = -1$. Different plots refer to different sizes of blocks $S_l$: $l$ equals 1 (●), 2 (■) and 3 (□).

**Fig.5:** The logarithm of the number of sites that never changed phase is plotted versus the logarithm of time; data have been obtained on a $1200 \times 1200$ lattice by averaging over 11 runs, in the finite temperature case with parameters $J_1 = J_3 = 0$ and $J_2 = -0.6$. Different plots refer to different sizes of blocks $S_l$: $l$ equals 2 (●), 3 (■), 4 (□) and 5 (△).
Fig.6: The logarithm of the number of sites that never changed phase is plotted versus the logarithm of time; data have been obtained by using Derrida’s method [8] in the finite temperature case with parameters $J_1 = J_3 = 0$, $J_2 = -1$ and $l = 1$ ($3 \times 3$ windows). Different plots refer to different lattice sizes, 256 (●), 600 (■) and 1200 (□), and to averages on more than 50 different runs.
FIGURES

FIG. 1.
FIG. 2.
FIG. 3.

The diagram shows the number of persistent spins over time. The x-axis represents time, while the y-axis represents the number of persistent spins. The graph displays three different lines, each indicating a different trend in the number of persistent spins as time progresses.
FIG. 4.
FIG. 5.
FIG. 6.