The Tracy-Widom distribution is not infinitely divisible.

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Abstract

The classical infinite divisibility of distributions related to eigenvalues of some random matrix ensembles is investigated. It is proved that the $\beta$-Tracy-Widom distribution, which is the limiting distribution of the largest eigenvalue of a $\beta$-Hermite ensemble, is not infinitely divisible. Furthermore, for each fixed $N \geq 2$ it is proved that the largest eigenvalue of a GOE/GUE random matrix is not infinitely divisible.

Keywords: beta Hermite ensembles; random matrices; largest eigenvalue, tail probabilities.

1 Introduction

Random matrix theory is an important field in probability, statistics and physics. One of the aims of random matrix theory is to derive limiting laws for the eigenvalues of ensembles of large random matrices. In this sense this note will focus in the study the behavior of eigenvalues of two types of matrix ensembles, the invariant Hermite and the tridiagonal $\beta$-Hermite.

The invariant Hermite ensembles consist of the Gaussian orthogonal, unitary, or symplectic ensembles, G(O/U/S)E, which are ensembles of $N \times N$ real symmetric, complex Hermitian or Hermitian real quaternion matrices, $H$, respectively, whose matrix elements are independently distributed random Gaussian variables with probability density function (PDF) proportional, modulo symmetries, to

$$\exp\left(-\frac{\beta}{4} \text{tr} H^2\right),$$

here, $\beta = 1, 2$ or 4 is used for the G(O/U/S)E ensembles, respectively. The joint PDF of their ordered eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ is given by

$$k_{N,\beta} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^{\beta} \exp\left(-\frac{\beta}{4} \sum_{i=1}^{N} \lambda_i^2\right), \quad (1)$$

where $k_{N,\beta}$ is a non negative constant and for $\beta = 1, 2$ or 4, it can be computed by Selberg’s Integral formula (see [2, Theorem 2.5.8]). The PDF (1) exhibits strong dependence of the
eigenvalues of the G(O/U/S)E ensembles. For more details related to these ensembles see [17], [9], [13], [2, sections 2.5 and 4.1]. The law (1) has a physical sense since it describes a one-dimensional Coulomb gas at inverse temperature $\beta$, [13, Section 1.4].

Each member of the G(O/U/S)E ensembles leads, by the Householder reduction, to a symmetric tridiagonal matrix $(H^\beta_N)_{N \geq 1}$ of the form

$$H^\beta_N := \frac{1}{\sqrt{\beta}} \begin{pmatrix} N(0, 2) & \chi_{(n-1)\beta} & \cdots & \chi_{(n-2)\beta} \\ \chi_{(n-1)\beta} & N(0, 2) & \cdots & \chi_{(n-3)\beta} \\ \cdots & \cdots & \ddots & \chi_{2\beta} \\ \chi_{(n-2)\beta} & \chi_{(n-3)\beta} & \cdots & N(0, 2) \end{pmatrix},$$

where $\chi_t$ is the $\chi$-distribution with $t$ degrees of freedom, whose probability density function is given by $f_t(x) = 2^{1-t/2}x^{t-1}e^{-x^2/2}/\Gamma(t/2)$. Here, $\Gamma(a) = \int_0^\infty v^{a-1}e^{-v}dv$ is Euler’s Gamma function. The matrix (2) has the important characteristic that all entries in the upper triangular part are independent. Trotter [35] apply the Householder reduction for the symmetric case ($\beta = 1$) while for the unitary and symplectic cases ($\beta = 2, 4$) the former reduction was applied by Dumitriu and Edelman [11]. Furthermore the latter considers the ensemble (2) for general $\beta > 0$ proving that in this case the PDF of the ordered eigenvalues of $H^\beta_N$ is still the PDF (1), see [1, Chapter 20] and [2, Section 4.1]. This matrix model it will named $\beta$-Hermite ensemble.

The classical Tracy-Widom distribution is defined as the limit distribution of the largest eigenvalue of a G(O/U/S)E random matrix ensemble. It is important due to its applications in probability, combinatorics, multivariate statistics, physics, among other applications. Tracy and Widom [33], [34] have written concise reviews for the situations where their distribution appears.

The $\beta$-Tracy-Widom distribution is defined as the limiting distribution of the largest eigenvalue of a $\beta$-Hermite ensemble. In the case $\beta = 1, 2, 4$ the $\beta$-Tracy-Widom distribution coincides with the classical Tracy-Widom, [21].

The main purpose of this paper is to determine the infinite divisibility of the classical Tracy-Widom and $\beta$-Tracy-Widom distributions as well as the infinite divisibility of the largest eigenvalue of the finite dimensional random matrix of GOE and GUE ensembles.

Recall that a random variable $X$ is said to be infinitely divisible if for each $n \geq 1$, there exist independent random variables $X_1, \ldots, X_n$ identically distributed such that $X$ is equal in distribution to $X_1 + \cdots + X_n$. This is an important property from the theoretical and applied point of view, since for any infinitely divisible distribution there is an associated Lévy process; Sato [24], Rocha-Arteaga and Sato [23]. These jump processes have been recently used for modelling purposes in a broad variety of different fields, including finance, insurance, physics among others; see Barndorff-Nielsen et al. [5], Cont and Tankov [8] and
Podolskij et al. [20] and for a physicists point of view see Paul and Baschnagel [19]. Other applications concern deconvolution problems in mathematical physics, Carasso [7].

This note is structured as follows: in section 2 it is presented preliminary results on the tail behavior of the classical and generalized $\beta$-Tracy-Widom distribution, useful to analyze infinite divisibility. The non infinite divisibility of the classical and generalized $\beta$-Tracy-Widom distribution is proved in Section 3. In Section 4 it is shown that for each $N \geq 2$ the largest eigenvalue of a G(O/U)E ensemble is not infinitely divisible. Finally in Section 5 it is presented more new results and some open problems.

2 Tracy-Widom distributions

2.1 Classical Tracy-Widom distribution

It is well known that the unique possible limit distributions for the maximum of independent random variables are the Gumbel, Fréchet and Weibull distributions. To classify the limit laws for the maximum of a large number of non independent random variables is still open problem. A possible strategy is to deal with particular models of non independent random variables.

The eigenvalues of random matrices provide a good example of such non independent random variables. For the Gaussian ensembles, i.e. for $N \times N$ matrices with independent Gaussian entries, the joint density function of their eigenvalues, $\lambda_1 \leq \cdots \leq \lambda_N$ is given by (1), and because the Vandermonde determinant $\prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^{\beta}$ they are strongly dependent. Due to the non independence of random variables with density function (1) it follows that the limit distribution of $\lambda_{\text{max}} = \lambda_N$ is not an usual extreme distributions. The distribution of $\lambda_{\text{max}}$ converges in the limit $N \to \infty$ to the Tracy–Widom laws.

Tracy-Widom, [31], [32] proved that the following limit, which is denoted by $F_\beta$, exists

$$F_\beta(x) := \lim_{N \to \infty} P \left[ N^{1/6} \left( \lambda_{\text{max}} - 2\sqrt{N} \right) \leq x \right], \beta = 1, 2, 4$$

and in this case

$$F_2(x) = \exp \left( - \int_x^\infty (s - x) [q(s)]^2 ds \right),$$

where $q$ is given in terms of the solution to a Painlevé type II equation, and

$$F_1(x) = \exp [E(x)] F_2^{1/2}(x), \quad F_4(x) = \cosh [E(x)] F_2^{1/2}(x),$$

where $E(x) = -\frac{1}{2} \int_x^\infty q(s) ds$. Furthermore it can be deduced from [31], [32], [34] and [4] (see also [2, Exercise 3.9.36]) that the asymptotics for $F_\beta(x)$ as $x \to \infty$, for $\beta = 1, 2$ or 4 is,

$$F_\beta(-x) = \exp \left\{ -\frac{1}{24} \beta x^3 [1 + o(1)] \right\}, \quad (3)$$
\[ 1 - F_\beta(x) = \exp \left\{ -\frac{2}{3} \beta x^{3/2} [1 + o(1)] \right\}, \tag{4} \]

where \( o(1) \) is the little-o of 1 which means that \( \lim_{x \to \infty} o(1) = 0 \).

With the tail probabilities \( (3) \) and \( (4) \) it is possible to conclude that the Tracy-Widom distribution is not infinitely divisible for \( \beta = 1, 2, 4 \) using the results of Section 4. Nevertheless it is convenient go to next section from which it will arrive at the non infinite divisibility of \( \beta \)-Tracy-Widom distribution for any \( \beta > 0 \), by considering the limit distribution of the largest eigenvalue of a matrix \( H_N^\beta \) defined in \( (2) \) for any \( \beta > 0 \).

### 2.2 \( \beta \)-Tracy-Widom distribution

The tridiagonal \( \beta \)-Hermite ensemble \( (2) \) can be considered as a discrete random Schrödinger operator. This stochastic operator approach to random matrix theory was conjectured by Edelman and Sutton \( [12] \), and was proved by Ramírez, Rider and Virág \( [21] \), who in particular established convergence of the largest eigenvalue of a \( \beta \)-Hermite ensemble for any \( \beta > 0 \).

Let \( \lambda_{\max} = \lambda_N(H_N^\beta) \), with \( H_N^\beta \) defined as in \( (2) \), in \( [21] \) it is shown the existence of a \( \beta \)-Tracy-Widom random variable \( TW_\beta \) such that

\[ N^{1/6} \left( \lambda_{\max} - 2\sqrt{N} \right) \xrightarrow{d} \frac{N \rightarrow \infty}{N} TW_\beta, \]

where the \( \beta \)-Tracy-Widom random variable is identified through a random variational principle:

\[ TW_\beta := \sup_{f \in L} \left\{ \frac{2}{3} \int_0^\infty f^2(x) \, db(x) - \int_0^\infty \left[ (f'(x))^2 + x f^2(x) \right] \, dx \right\}, \]

in which \( x \to b(x) \) is a standard Brownian Motion and \( L \) is the space of functions \( f \) which satisfy \( f(0) = 0 \), \( \int_0^\infty f^2(x) \, dx = 1 \), \( \int_0^\infty \left[ (f'(x))^2 + x f^2(x) \right] \, dx < \infty \).

The cases \( \beta = 1, 2, 4 \) coincide with the classical Tracy-Widom distribution \( F_\beta(x) = P(TW_\beta \leq x) \), \( [21] \). Ramírez, \textit{et al.} \( [21] \) also proved that the tails of \( TW_\beta \) are given by \( (3) \) and \( (4) \) for any \( \beta > 0 \).

### 3 Non infinite divisibility of Tracy-Widom distributions

First recall a well known result on a characterization of the Gaussian distribution in terms of the tail behavior; see \( [30] \), Corollary 4.9.9]: a non-degenerate infinitely divisible random variable \( X \) has a normal distribution if, and only if, it satisfies

\[ \limsup_{x \to \infty} - \log P(|X| > x) / x \log x = \infty. \tag{5} \]
Now, from (3) and (4) we get the following lemma, where, as usual, the expression \( f(s) \sim g(s) \) means that \( f(s)/g(s) \) tend to 1 when \( s \rightarrow \infty \).

**Lemma 1 (Two sided tails of the \( \beta \)-Tracy-Widom distribution)** Let \( \beta > 0 \), let \( TW_\beta \) be a random variable \( \beta \)-Tracy-Widom distributed. Then when \( x \rightarrow \infty \), it follows that

\[
P(\left| TW_\beta \right| > x) \sim P(TW_\beta > x) = \exp \left( -\frac{2}{3} \beta x^{3/2} \left[ 1 + o(1) \right] \right).
\]

**Proof.** Using (3) and (4) we get

\[
P(\left| TW_\beta \right| > x) = P(TW_\beta < -x) + 1 - P(TW_\beta < x)
\]

\[
= \exp \left( -\frac{1}{24} \beta x^3 (1 + o(1)) \right) + \exp \left( -\frac{2}{3} \beta x^{3/2} \left[ 1 + o(1) \right] \right),
\]

and hence

\[
\lim_{x \rightarrow \infty} \frac{P(\left| TW_\beta \right| > x)}{\exp \left( -\frac{2}{3} \beta x^{3/2} \left[ 1 + o(1) \right] \right)} = 1 + \lim_{x \rightarrow \infty} \exp \left( -\frac{1}{24} \beta x^3 (1 + o(1)) + \frac{2}{3} \beta x^{3/2} \left[ 1 + o(1) \right] \right)
\]

\[
= 1.
\]

Finally, using Lemma 1 it is possible to conclude that the \( \beta \)-Tracy-Widom distribution is not infinite divisibility:

**Theorem 2** For any \( \beta > 0 \) the \( \beta \)-Tracy Widom distribution is not infinitely divisible.

**Proof.** Let assume that \( X \) is infinitely divisible and given that neither it is normal nor degenerate we must have that (3) is false, that is

\[
\lim_{x \rightarrow \infty} \frac{\log \left( -\log P(\left| X \right| > x) \right)}{x \log x} < \infty.
\]

However,

\[
\lim_{x \rightarrow \infty} \frac{\log \left( -\log P(\left| X \right| > x) \right)}{x \log x} = \lim_{x \rightarrow \infty} \frac{1}{x \log x} \left\{ -\log \left( \frac{P(\left| X \right| > x) \exp \left( -\frac{2}{3} \beta x^{3/2} \left[ 1 + o(1) \right] \right)}{\exp \left( -\frac{2}{3} \beta x^{3/2} \left[ 1 + o(1) \right] \right)} \right) + \frac{2}{3} \beta x^{3/2} \left[ 1 + o(1) \right] \right\}
\]

\[
= \lim_{x \rightarrow \infty} \frac{2}{3} \beta \sqrt{x} \left[ 1 + o(1) \right]
\]

\[
= \infty,
\]

the third equality follows from Lemma 1.

**Remark 3** Taking \( \beta = 1, 2 \) or 4 in Theorem 2 the non infinite divisibility of the classical Tracy-Widom distribution is deduced.
4 Non infinite divisibility in the finite $N$ case

The following Lemma is necessary to determine the non infinite divisibility of the largest eigenvalue of a random matrix of a GOE/GUE ensemble.

**Lemma 4** If $X$ is a non Gaussian real random variable such that

$$P(|X| > x) \leq ae^{-bx^c} \text{ with } a, b > 0, \ c > 1,$$

then $X$ is not infinitely divisible.

**Proof.** As in the proof of Theorem 2 it is only necessary to prove that $X$ satisfy the limit (5). Indeed,

$$\lim_{x \to \infty} \frac{-\log P(|X| > x)}{x \log x} \geq \lim_{x \to \infty} \frac{-\log (ae^{-bx^c})}{x \log x} = \lim_{x \to \infty} \frac{-\log a + bx^c}{x \log x} = \infty.$$

Consider, $\lambda_{\text{max}}^N$ the largest eigenvalue of a GOE ensemble of dimension $N$. In [3, Lemma 6.3] is proved that the two sided tails of the largest eigenvalue of a GOE satisfy the following inequality

$$P (|\lambda_{\text{max}}^N| \geq x) \leq e^{-Nx^2/9}$$

and if $\lambda_{\text{max}}^N$ the largest eigenvalue of a GUE ensemble of dimension $N$. The following inequality is deduced in [15]

$$P (|\lambda_{\text{max}}^N - E(\lambda_{\text{max}}^N)| \geq x) \leq 2e^{-2Nx^2}.$$

from which, with help of Lemma 4 the next theorem follows:

**Theorem 5** Let $\lambda_{\text{max}}^N$ be the largest eigenvalue of a random matrix of a GOE/GUE ensemble of random matrices. For all $N \geq 2$, $\lambda_{\text{max}}^N$ is not infinitely divisible.

**Remark 6** The case $N = 2$ follows from Domínguez-Molina and Rocha-Arteaga [10].

5 Discussion

Recall that an infinitely divisible random variable, $X$, in $\mathbb{R}_+$ must comply that $-\log P(X > x) \leq ax \log x$, for some $a > 0$ and $x$ sufficiently large, [29]. With this result it is possible to deduce the non infinite divisibility of the following random variables:

I) Wigner surmise: $P(s) = \frac{\pi}{2} s \exp \left( -\frac{\pi}{4} s^2 \right)$.

II) The absolute value of $TW_\beta$, $Y_\beta = |TW_\beta|$.

III) The truncation to the left or to the right of $TW_\beta$.

Open problems
1. Free infinite divisibility of the classical Tracy-Widom distribution or the general $\beta$-Tracy-Widom distribution.

2. For each $N \geq 2$, the non infinite divisibility of the largest eigenvalue of random matrix of a GSE ensemble.

3. Determine if the Tracy-Widom distribution is indecomposable.

4. Investigate the infinite divisibility of $\lambda_{\text{max}}(H_{N}^{\beta})$ in the tridiagonal $\beta$-Hermite ensemble \cite{2}. The article \cite{16} it may be useful.

5. Look for an infinitely divisible interpolation between Tracy-Widom distribution and other distribution (except in the Tracy Widom case). Johanson \cite{14} discuss interpolation between the Gumbel distribution, and the Tracy-Widom distribution. Bohigas et al \cite{6} discuss a continuous transition from Tracy-Widom distribution to the Weibull distribution, and from Tracy-Widom distribution to Gaussian distribution. It is not known if these interpolations are infinitely divisible (except in the Tracy Widom case).

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