How to check universality of quantum gates?

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We provide two simple universality criteria. Our first criterion states that $S \subset G_d := U(d)$ is universal if and only if $S$ forms a $\delta$-approximate $t(d)$-design, where $t(2) = 6$ and $t(d) = 4$ for $d \geq 3$.

Our second universality criterion says that $S \subset G_d$ is universal if and only if the centralizer of $S^{t(d),t(d)} := \{U^{\otimes t(d)} \otimes \bar{U}^{\otimes t(d)} | U \in S \}$ is equal to the centralizer of $G_{d}^{t(d), t(d)} := \{U^{\otimes t(d)} \otimes \bar{U}^{\otimes t(d)} | U \in G_d \}$, where $t(2) = 3$, $t(3) = 2$ and $t(d) = 4$ for $d \geq 4$. The equality of the centralizers can be verified by comparing their dimensions.

Universal and efficient quantum gates play a central role in quantum computing.\textsuperscript{1} It is well known that in order to construct a universal set of gates for many qudits it is enough to take universal set for one qudit and extend it by a two-qudit entangling gate.\textsuperscript{2} It is a great challenge to find a time efficient procedure that enables deciding if a given set of gates $S \subset G_d := U(d)$ is universal. One can consider an analogous problem for Hamiltonians, i.e. given a set of anti-Hermitian traceless matrices $X \subset \mathfrak{su}(d)$, we want to check if they generate whole algebra of the special unitary group, $\mathfrak{su}(d)$. The answer turns out to be relatively simple and can be phrased in terms of the centralizer of the tensor squares of the elements belonging to $X$.\textsuperscript{3}

The approach through tensor powers can be extended from Hamiltonians to quantum gates. To this end one considers the centralizers of $S^{t, t} := \{U^{\otimes t} \otimes \bar{U}^{\otimes t} | U \in S \}$ and $G_{d}^{t, t} := \{U^{\otimes t} \otimes \bar{U}^{\otimes t} | U \in G_d \}$ for any $t$. In\textsuperscript{1,3,15} it was shown that the equality of the centralizers of $S^{1,1}$ and $G_{d}^{1,1}$ implies that $S$ is universal provided $S$ generates an infinite subgroup of $G_d$. We will call this necessary universality condition. For $S$ satisfying the necessary universality condition, in order to verify if the group generated by $S$ is universal it is enough to check if it forms an $\frac{3}{4\sqrt{2}}$-net, in the Hilbert-Schmidt distance.\textsuperscript{13,15,16}

Following this idea the authors of\textsuperscript{18} showed that $S$ is universal iff 1) $S \subset G_d$ satisfies the necessary universality condition, 2) $S$ is a $\delta$-approximate $t(d)$-design, with $\delta < 1$ and $t(d) = O(d^{3/2})$.

In this paper we show that actually the required $t(d)$ does not grow with $d$. Making use of the recent results regarding the so-called unitary $t$-groups\textsuperscript{19} we formulate our first universality criterion (Theorem \textsuperscript{1}). It states that $S \subset G_d$ is universal if and only if $S$ forms a $\delta$-approximate $t(d)$-design, where $t(2) = 6$ and $t(d) = 4$ for $d \geq 3$. Using additional knowledge regarding finite subgroups of $SU(2)$ and $SU(3)$\textsuperscript{22}, we then translate this criterion to the language of tensor products of representations. Our second universality criterion (Theorem \textsuperscript{2}) says that $S \subset G_d$ is universal if and only if the centralizer of $S^{t(d), t(d)}$ is equal to the centralizer $G_{d}^{t(d), t(d)}$, where $t(2) = 3$, $t(3) = 2$ and $t(d) = 4$ for $d \geq 4$. The equality of the centralizers can be verified by comparing their dimensions. We calculate the dimensions of the centralizers of $G_{d}^{t(d), t(d)}$ using representation theory methods, for all $t(d)$ listed above and use them in our final criterion for universality, i.e. Theorem \textsuperscript{3}.

Let $\{S, \nu_S\}$ be an ensemble of quantum gates, where $S$ is a finite subset of $G_d := U(d)$ and $\nu_S$ is the uniform measure on $S$. Such ensemble is called $\delta(t, \nu_S)$-approximate $t$-design if and only if

$$\delta(t, \nu_S) := \|T_{\nu_S,t} - T_{\mu,t}\|_\infty < 1,$$

where for any measure $\nu$ (in particular for the Haar measure $\mu$) we define a moment operator

$$T_{\nu,t} := \int_G d\nu(U)U^{\otimes t} \otimes \bar{U}^{\otimes t}.$$

One can easily show that $0 \leq \delta(t, \nu_S) \leq 1$\textsuperscript{18}. When $\delta(t, \nu_S) = 0$ we say that $S$ is an exact $t$-design and when $\delta(t, \nu_S) = 1$ we say that $S$ is not a $t$-design. To proceed we note that a map $U \mapsto U^{\otimes t} \otimes \bar{U}^{\otimes t}$ is a representation of the unitary group $G_d$. This representation turns out to be reducible and as such it decomposes into some irreducible representations $\pi$ of $G_d$

$$U^{\otimes t} \otimes \bar{U}^{\otimes t} \simeq \bigoplus_\pi \pi(U)^{\otimes m_\pi} \simeq (U \otimes \bar{U})^{\otimes t},$$

where $m_\pi$ is the multiplicity of $\pi$. The representations occurring in this decomposition are in fact irreducible representation of the projective unitary group, $PG_d = G_d/\sim$, where $U \sim V$ iff $U = e^{i\phi}V$. One can show that every irreducible representation of $PG_d$ arises this way for some, possibly large, $t$\textsuperscript{1}. For $t = 1$ the decomposition (3) is particularly simple and reads $U^{\otimes 1} \otimes \bar{U}^{\otimes 1} = Ad_G(1)$, where 1 stands for the trivial representation and $Ad_G$ is the adjoint representation of $G_d$ and $PG_d \simeq AdG_d$. For a set of $S \subset G_d$ we define

\textbf{Definition 1.} Let $S \subset G_d$ be a set of quantum gates. The group, $G_S$, generated by $S$ is

$$G_S := \langle S \rangle,$$

$$\langle S \rangle := \{S_t | t \in \mathbb{N}\}, \quad S_t := \{g_1g_2\ldots g_t | g_i \in S\}.$$

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The closure in \((1)\) corresponds to adding all the limiting points of the sequences of words from \(S_l\) when \(l\) goes to infinity.

**Definition 2.** A set \(S \subset G_d\) is universal if and only if \(PG_d \simeq G_S/\sim\), where \(U \sim V \iff U = e^{i\theta}V\).

Having any representation \(\pi\) of \(G_d\) we can consider its restriction to \(G_S\), which we denote by \(\text{Res}^{G_S}_{G_d}\). For any \(U \in G_S\) we simply have:

\[
\text{Res}^{G_S}_{G_d} \pi(U) = \pi(U). \tag{6}
\]

A crucial observation here is that \(\text{Res}^{G_S}_{G_d}\) can be a reducible representation of \(G_S\) even though \(\pi\) is an irreducible representation of \(G_d\). This observation plays a central role in representation theory (cf. branching rules) and turns out to be central in our context. For any irreducible representations occurring in the decomposition \((3)\) we can consider the restriction of \(T_{\nu_S, t}\) to \(\pi\) which is given by

\[
T_{\nu_S, t, \pi} = \int_{G_d} d\nu_S(U) \pi(U). \tag{7}
\]

It follows directly from the definitions and discussion above that

\[
T_{\nu_S, t} \simeq \bigoplus_\pi T_{\nu_S, t, \pi} \otimes_{\text{max}} \delta(t, \nu_S) = \sup_\pi \|T_{\nu_S, t, \pi}\|, \tag{8}
\]

where \(\pi\) goes over irreducible representations occurring in the decomposition \((3)\).

**Lemma 1.** Let \(S \subset G_d\) be an arbitrary set of quantum gates. Then \(\delta(t, \nu_S) = 1\) if and only if for some nontrivial irreducible representation \(\pi\) appearing in the decomposition \((3)\) of \(G_d\) we have \(\|T_{\nu_S, t, \pi}\| = 1\). Thus there exists normalized to one vector \(v \in \nu_S\) such that \(\sum_{g \in S} \pi(g)v = v\). Hence

\[
\sum_{g \in S} \langle v | \pi(g)v \rangle = |S|. \tag{9}
\]

By unitarity of \(\pi(g)\) we have \(|\langle v | \pi(g)v \rangle| \leq 1\), for any normalized \(v \in \nu_S\). Thus the only possibility to fulfill \((9)\) is \(\forall g \in S \pi(g)v = v\), which means \(v\) is a common eigenvector of all operators \(\pi(g), \ g \in S\). This in turn means that \(\text{Res}^{G_S}_{G_d}\) is reducible and contains a copy of the trivial representation. This ends the proof.

**Corollary 1.** Assume \(S\) is such that \(\delta(t, \nu_S) < 1\) for any finite \(t\).

We note that the converse of Corollary \((1)\) is in general false as \(\delta(t, \nu_S) < 1\) implies only that the restriction to \(G_S\) of any nontrivial irreducible representation of \(PG_d\) does not contain the trivial representation in its decomposition. Let us next define

\[
S^{t, t} = \{U^{\otimes t} \otimes U^{\otimes t}| U \in S\}, \quad G_d^{t, t} = \{U^{\otimes t} \otimes U^{\otimes t}| U \in G_d\}.
\]

For any set of matrices \(B \subset \text{Mat}(n, \mathbb{C})\) let

\[
\mathcal{C}(B) = \{X \in \text{Mat}(n, \mathbb{C}) | [X, Y] = 0, \forall Y \in B\}. \tag{10}
\]

Using \(U \otimes \bar{U} = \text{Ad}_U + 1\) we can rewrite Lemma 3.4 from \([19]\) as

**Lemma 2.** Assume \(S\) is such that

\[
\mathcal{C}(S^{t, t}) = \mathcal{C}(G_d^{t, t}). \tag{11}
\]

Then \(S\) is universal if and only if \(G_d\) is infinite.

In other words, whenever the condition \((11)\) is satisfied \(S\) is either universal or \(G_S\) is a finite subgroup of \(G_d\). In what follows we will call the condition \((11)\) the necessary universality condition. We note also that we always have \(\mathcal{C}(S^{t, t}) = \mathcal{C}(G_d^{t, t})\). Lemma \((4)\) implies that finite groups satisfying the necessary universality condition play a central role in deciding whether \(S\) is universal.

**Definition 3.** A finite subgroup \(G \subset G_d\) is a unitary \(t\)-group iff \(\delta(t, \nu_G) = 0\), where \(\nu_G\) is the uniform measure on \(G\).

The concept of unitary \(t\)-groups will play a central role in what follows.

**Lemma 3.** Assume that \(S\) is a generating set of a finite subgroup of \(G_d\). Then for any \(t\), either \(\delta(t, \nu_G) = 0\) or \(\delta(t, \nu_G) = 1\). Moreover, for any \(t\) (1) either \(\delta(t, \nu_G) < 1\) iff \(\delta(t, \nu_G) = 0\) or (2) \(\delta(t, \nu_G) = 1\) iff \(\delta(t, \nu_G) = 1\).

**Proof.** One can easily verify that for any measure \(\nu\)

\[
\delta(t, \nu^{*l}) = \delta(t, \nu)^l
\]

where \(\nu^{*l}\) is the \(l\)-fold convolution of \(\nu\). Under the assumption that \(G_S\) is finite there is \(l_0\) such that \(S_{l_0} = G_S\). On the other hand, it is known \([21]\) that for \(l \to \infty\) the measure \(\nu_S^{*l}\) converges to \(\nu_G\). Thus we have \(\delta(t, \nu_S)^l \to \delta(t, \nu_G)\). Note, however, that \(\nu_G = \nu_G^{*l}\). Thus \(\delta(t, \nu_G) = 0\) or \(\delta(t, \nu_G) = 1\). The result follows.

Recently there has been some development in the theory of unitary \(t\)-groups. The main result of \([19]\) states:

**Fact 1.** There are no unitary \(t\)-groups in \(G_d\) for \(t \geq 1\) and \(d \geq 2\). Moreover

1. When \(d = 2\) there is unitary 5-group but no unitary \(t\)-group with \(t \geq 6\).
2. When \(d \geq 3\) there is no unitary \(t\)-group with \(t \geq 4\).

This lead to our first main result.
Theorem 1. Let $S$ be a set of gates in $G_d$ such that $\mathcal{C}(S^{1:1}) = \mathcal{C}(G_d^{1:1})$. Then $S$ is universal if and only if

1. $\{S, \nu_S\}$ is $\delta$-approximate 6-design with $\delta < 1$, when $d = 2$.
2. $\{S, \nu_S\}$ is $\delta$-approximate 4-design with $\delta < 1$, when $d \geq 3$.

Proof. By Lemma 2 the set $S$ can be either universal or $G_S$ is a finite group. If $S$ is universal then by Corollary 1 we have that $\delta(t, \nu_S) < 1$ for any $t \geq 1$. On the other hand if $G_S$ is a finite group then by Lemma 3 and Fact 1 we can only have: (1) $\delta(t, \nu_S) = 1$ for $t = 6$ and $d = 2$, (2) $\delta(t, \nu_S) = 1$ for $t = 4$ and $d > 2$. This finishes the proof.

In order to state our second universality criterion we will need the following lemma.

Lemma 4. Assume that for some $t \geq 2$

$$\mathcal{C}(S^{1:1}) = \mathcal{C}(G_d^{1:1}).$$

Then any irreducible representation occurring in the decomposition $(3)$ remains irreducible when restricted to $G_S$. Moreover $\mathcal{C}(S^{1:1}) = \mathcal{C}(G_d^{1:1})$.

Proof. The fact that any irreducible representation occurring in the decomposition $(3)$ remains irreducible when restricted to $G_S$ is obvious. For the second part of the statement note that

$$U^{\otimes t} \otimes \tilde{U}^{\otimes t} \simeq (U^{\otimes t-1} \otimes \tilde{U}^{\otimes t-1}) \otimes (\text{Ad}_U \oplus 1) = (t^{\otimes t-1} \otimes \tilde{U}^{\otimes t-1}) \otimes \text{Ad}_U \oplus (U^{\otimes t-1} \otimes \tilde{U}^{\otimes t-1}).$$

Repeating this decomposition we get that $U \otimes \tilde{U}$ of the homomorphisms in the decomposition of $U^{\otimes t} \otimes \tilde{U}^{\otimes t}$. Hence, under condition $(12)$ we have $\text{Res}_{G_S}^{G_d} \text{Ad}$ is irreducible. Thus $\mathcal{C}(S^{1:1}) = \mathcal{C}(G_d^{1:1})$.

We can now formulate a sufficient condition for universality in terms of centralizers.

Corollary 2. Let $S$ be a set of gates in $G_d$. Then $S$ is universal if and only if:

$$\mathcal{C}(S^{t(d)}, t(d)) = \mathcal{C}(G_d^{t(d), t(d)}),$$

where $t(2) = 6$ and $t(d) = 4$ for $d > 2$.

Proof. The condition $(14)$ combined with Lemma 4 implies that the necessary condition for universality is satisfied and that all irreducible representations occurring in the decomposition $(3)$ remain irreducible when restricted to $G_S$, where $t(d)$ is as in the statement of the theorem. Hence by Lemma 4 we have $\delta(6, \nu_S) \neq 1$ for $d = 2$ and $\delta(4, \nu_S) \neq 1$ for $d > 2$. Assume $G_S$ is a finite group. Then by Lemma 3 and Fact 1 we have $\delta(6, \nu_S) = 1$ for $d = 2$ and $\delta(4, \nu_S) = 1$ for $d > 2$. Thus we get a contradiction and $S$ is universal.

Corollary 2 can be further improved if one knows the classification of all finite subgroups of $PG_d$. This is known for $PG_2$ and $PG_3$. In particular

Fact 2. Let $G$ be a finite subgroup of $SU(d)$ which satisfies the necessary universality condition. Then the maximal dimension of an irreducible representation of $G$ is 6 for $d = 2$ and 15 for $d = 3$.

Proof. The classification of the finite subgroups of $SU(d)$ is well known for $d = 2$, whereas provide self-contained references for $d = 3$. Assuming the necessary universality condition is satisfied, $G$ is isomorphic to one of the following finite groups:

| $SU(2)$ |
|-----------|
| $SU(3)$ |
| $\Sigma(168)$ | $1,2,3$ |
| $\Sigma(168) \times \mathbb{Z}_3$ | $1,2,3,\ldots,8$ |
| $\Sigma(72\phi)$ | $1,2,3,\ldots,8$ |
| $\Sigma(216\phi)$ | $1,2,3,\ldots,8,9$ |
| $\Sigma(360\phi)$ | $1,2,3,\ldots,8,9,10,15$ |

where the right column lists the dimensions of the irreducible representations.

Theorem 2. Let $S$ be a set of gates in $G_d$. Then $S$ is universal if and only if:

$$\mathcal{C}(S^{t(d)}, t(d)) = \mathcal{C}(G_d^{t(d), t(d)}),$$

where $t(2) = 3, t(3) = 2$ and $t(d) = 4$ for $d \geq 4$.

Proof. The case when $d \geq 4$ follows from Theorem 2. We are left with $d = 2$ and $d = 3$. Using representation theory of $PU(2)$ one gets that

$$U^{\otimes t} \otimes \tilde{U}^{\otimes t} = \bigoplus_{0 \leq \nu \leq t, \nu \text{even}} m_{\pi_{\nu}} \pi_{\nu}(U),$$

where $\dim \pi_{\nu} = \nu + 1$ and $m_{\pi_{\nu}} \neq 0$ for every even $\nu$ satisfying $0 \leq \nu \leq 2t$. Thus the highest dimensional representation appearing in $(16)$ is $\pi_{2t+1}$. Hence if $t = 3$ we have 7-dimensional irreducible representation of $PG_2$ and by Fact 2 its restriction to any finite subgroup of $PG_2$ is reducible. Thus for any finite subgroup of $PG_2$ we have $\mathcal{C}(S^{3:3}) \neq \mathcal{C}(G_2^{3:3})$. A decomposition similar to $(16)$ can be found for $PG_3$ [20, 21]. Recall first that $U \otimes \tilde{U} = \text{Ad}_U \oplus 1$, and thus the highest dimensional representation appearing in $U \otimes \tilde{U}$ is the adjoint representation, which for $PG_3$ has the dimension $d = 8 < 15$. Next, we know that irreducible representations of $PG_3$ are indexed by Young diagrams with at most $d - 1$ rows [20, 21]. For example the Young diagram corresponding to the adjoint representation of $PG_3$ is given by the partition $\lambda = (2,1)$. 


Using the rules of multiplication for Young diagrams one can directly check that for $PG_3$
\[ U^\otimes 2 \otimes \bar{U}^\otimes 2 \simeq \pi_{(4,2)} \oplus \pi_{(3,0)} \oplus \pi_{(3,3)} \oplus 4\pi_{(2,1)} \oplus 2, \]
where $\dim \pi_{(4,2)} = 27$, $\dim \pi_{(3,0)} = \dim \pi_{(3,3)} = 10$ and $\dim \pi_{(2,1)} = 8$. Hence if $t = 2$ and $d = 3$ we have $27$-dimensional irreducible representation, and by Fact 2 its restriction to any finite subgroup of $PG_3$ is reducible. Thus for any finite subgroup of $PG_3$ we have $C(S^{2,2}) \neq C(G_{d,3}^{2})$. This finishes the proof. 

One can calculate explicitly the dimension of the centralizer $C(G_{d,t}^{4})$ from the following formula [20]
\[ \dim C(G_{d,t}^{4}) = \sum_{\pi}(m_\pi)^2, \]
where $\pi$ are irreducible representations occurring in the decomposition [4]. Following Corollary 3.4 of [20] we know that
\[ \dim C(G_{d,t}^{4}) = (2t)! / d \geq 2t. \]

We can reformulate Theorem 2 to a more computationally friendly form:

**Theorem 3.** Let $S$ be a set of gates in $G_d$. Then $S$ is universal if and only if: (1) $\dim C(S^{3,3}) = 112$ for $d = 2$; (2) $\dim C(S^{2,2}) = 23$ for $d = 3$; (3) $\dim C(S^{4,4}) = 33324$ for $d = 4$; (4) $\dim C(S^{4,4}) = 39429$ for $d = 5$; (5) $\dim C(S^{4,4}) = 40270$ for $d = 6$; (6) $\dim C(S^{4,4}) = 40319$ for $d = 7$; (7) $\dim C(S^{4,4}) = 8! = 40320$ for $d \geq 8$.

**Proof.** Obviously, for any $S \subset G_d$ we have $\dim C(S^{t,d}) \geq \dim C(G_{d,t}^{4})$. The equality of these dimensions is possible if and only if the restrictions to $G_S$ of all irreducible representations occurring in the decomposition [3] are irreducible. Thus $\dim C(S^{t,d}) = \dim C(G_{d,t}^{4})$ if and only if $C(S^{t,d}) = C(G_{d,t}^{4})$. We are left with finding dimensions of $C(G_{d,t}^{4}, t(d))$, where $t(d)$ is as in the statement of Theorem 2. For $d \geq 8$ we use identity [19]. For $d = 2$ we have
\[ U^\otimes 3 \otimes \bar{U}^\otimes 3 \simeq (\pi_2(U) \oplus 1)^\otimes 3. \]

In addition we known that $\pi_1(U) \otimes \pi_k(U) \simeq \pi_{1+k} \oplus \pi_{l+k-2} \oplus \ldots \oplus \pi_{|l-k|}$. Using this identity we find that
\[ U^\otimes 3 \otimes \bar{U}^\otimes 3 \simeq \pi_6(U) \oplus 5 \pi_4(U) + 9 \pi_2(U) + 5. \]

Thus using formula [18] we get $\dim C(G_2^{3,3}) = 1 + 25 + 81 + 25 = 112$. Next, using [17] we get $\dim C(G_3^{2,2}) = 1 + 1 + 1 + 16 + 4 = 23$. For the other cases, to avoid tedious multiplication of Young diagrams we used LieArt package [22] to obtain the decomposition of $U^\otimes 4 \otimes U^\otimes 4$ into irreducible representations and then we used formula [18].

Finally note that calculating $\dim C(S^{t,d}, t(d))$ boils down to solving a set of $|S|d^{t(d)}$ linear equations, where $t(d)$ is as in Theorem 2, in particular $t(d) \leq 4$ for any $d$. Thus the number of linear equations to solve is bounded by $O(d^{t(d)})$.

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