ON THE CLOSE ENCOUNTERS BETWEEN PLUTINOS AND NEPTUNE TROJANS: 
I. STATISTIC ANALYSIS AND THEORETICAL ESTIMATIONS

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ABSTRACT

Close encounters (CEs) between celestial objects may exert significant influence on their orbits. The influence will be even enhanced when two groups of celestial objects are confined in stable orbital configurations, e.g. in adjacent mean motion resonances (MMRs). Plutinos and Neptune Trojans, trapped in the 2:3 and 1:1 MMRs with Neptune respectively, are such examples. As the first part of our investigation, this paper provides a detailed description of CEs between Plutinos and Trojans and their potential influences on the Trojans’ orbits. Statistical analyses of CE data from numerical simulations reveal the randomness lying in the CEs between the two planetesimals. The closest positions of CEs distribute symmetrically inside the given CE region and no particular bias is found between the positive and negative effects on the orbital elements of Trojans. Based on the Gaussian approximation on the distribution of the velocity orientation of Plutino, and the integral derivatives of Gaussian perturbation equations, a theoretical method is built to estimate the CE effects. To further verify the randomness of CEs, a Monte Carlo approach is applied, and it generates distribution features consistent with the numerical results. In summary, CEs brought by realistic Plutinos exert impartial effects and tiny total influence on the orbital elements of Trojans. However, driven by the random walk mechanism, tiny effects may accumulate to a prominent variation given sufficient CEs, which will be discussed in the accompanying paper.

Keywords: celestial mechanics — Kuiper belt: general — methods: miscellaneous — minor planets, asteroids: general

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1. INTRODUCTION

Neptune Trojans refer to a cluster of minor objects that orbit the Sun and locate around the stable Lagrangian points of Neptune, dynamically trapped in the 1:1 mean motion resonance (MMR) with Neptune. Among the so far discovered Neptune Trojans, more than half posses high inclinations. The dynamical stability and the driving mechanism of these highly inclined orbits have long become perplexing problems.

Secular perturbations from giant planets may significantly shape the dynamic structure of the resonant region, and indeed several works managed to locate stable regions of Neptune Trojans around high inclinations (Marzari et al. 2003; Dvorak et al. 2007, 2008; Zhou et al. 2009, 2011), but the specific mechanism which effectively elevates the inclination of Trojans remains ambiguous. The Nice Model (Morbidelli et al. 2005) proposes that the chaotic capture in Jovian Trojan region during the primordial migrations of giant planets will remarkably increase the inclination of Trojans. Similar mechanism is applicable to Neptune Trojans as well (Brasser et al. 2004; Li et al. 2007). However, as proposed by Nesvorný & Vokrouhlický (2009), the current capture efficiency is insufficient to explain the 4:1 ratio between Neptune Trojans with high inclinations and low inclinations (Sheppard & Trujillo 2006). Besides, it is rather unreliable that the early formed distribution structure can survive the lengthy evolution later. Certainly, the artificiality of the model is an insecure factor either.

Given the above considerations, we draw our attention to the interactions prompted by adjacent asteroid families like Plutinos. Plutinos are classified as a prime subpopulation of Trans-Neptunian objects (TNOs), and trapped in a 3:2 MMR with Neptune. With the semi-major axes locating around 39 AU and eccentricities extending up to 0.3, Plutinos can cross the orbit of Neptune and meet Trojans from time to time. Almeida et al. (2009) suggest that the significant orbital overlap will lead to frequent close encounters between Plutinos and Trojans. This event, bound by stable resonant configurations, is different from the interior close encounters among either population, thus may be an extra force to effectively shape the physical and orbital characteristics of the two interacting clusters. The idea then comes up that the frequent communications with Plutinos may play a significant role in the formation of high inclinations of Trojans. If standing up, this mechanism will be an efficient way to continually and inherently elevate the inclinations of Trojans, thus contributes to the present distributions.

Simply and directly, we test this idea by simulating the motion of a fictitious population of massless Trojans, using the state-of-art numerical integrator SyMBA (Levison & Duncan 2000). The Trojans are cloned from the known ones. As the minimum requirements to sustain Trojans, the standard model consists of the Sun and Neptune, while other models also include Pluto or several Plutinos. All models are integrated on the same population of Trojans simultaneously in order to reveal the differences. In Fig. 1, we show the orbital element distribution of Trojans perturbed by Pluto after 1Gyr evolution, which barely shows any difference from the standard case. The histograms of inclination almost coincide with each other. A further model including 10 fictitious Plutinos each with Pluto mass still brings minute effect. Nevertheless, when we introduce a fictitious Pluto with 100 times the realistic Pluto mass (about 0.2 Earth mass), the Trojans are scattered away from the Neptune orbit, indicating that Plutinos are indeed able to reach and influence the orbits of Trojans. Surely that under realistic situations, the total mass of Plutino population could never be as high as 100 Pluto mass. But it causes our major concern about the number of Plutinos. Currently there are about one hundred identified Plutinos. Could the cumulative effect of such number of Plutinos significantly perturb the orbits of Trojans through incessant close encounters, though their total mass is relatively low? Via pure numerical simulation, we then need to include all these known bodies simultaneously, which requires huge computing resources. And these bodies may still be just a small portion of the population in reality. It is hard to determine the exact number of bodies to be included in the simulation that is convincing enough to reflect the real-

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1 IAU: Minor Planet Center, http://www.minorplanetcenter.net/iau/lists/NeptuneTrojans.html
istic effect. Besides, a pure numerical method can not be
generalized for sure. When facing a new but similar problem
associated with the communication between overlapping res-
onances, we still need to cover all of the bodies and perform
numerical simulations, which is redundant effort but hardly
touches the physical essence.

Apart from the direct numerical integrations, statistical
analyses and theoretical estimations are always alternative
ways. With regard to the close encounters in the solar sys-
tem, numerous works focus on the circumstances that mi-
nor objects like asteroids and comets encounter giant planets,
and resolve to unravel the consequent effect on the orbital
elements or energy (Everhart 1969; Greenberg et al. 1988;
Carusi et al. 1990; Carruba et al. 2013). Statistical analyses
and semi-analytical methods are frequently applied in these
explorations.

More violent processes refer to collisions, and as a prereq-
usite for general analysis of collisional evolution, an esti-
mate of the collision probability has been studied in various
ways. Classical and practical theories include but not limited
to Wetherill (1967), Greenberg (1982), Bottke & Greenberg
(1993) and Dell’Oro & Paolicchi (1998). Pertinent meth-
ods were applied to the collisions among groups of minor
objects including planetary embryos (Weidenschilling et al.
1997), Main Belt asteroids (Farinella & Davis 1992), Hildas
(Dell’Oro et al. 2001), Jupiter Trojans (Marzari et al. 1996)
and near-Earth objects (Stuart & Binzel 2004). With the con-
tinuous discovery of TNOs, more interest was devoted to the
collisions among relevant subpopulations (Stern 1995; Davis
& Farinella 1997; Thébault & Doressoundiram 2003; Krivov
et al. 2005; Dell’Oro et al. 2013). Numerical methods includ-
ing Monte Carlo simulations are frequently applied in these
explorations.

Following the same route, in this paper we will deal with
the close encounters between Neptune Trojans and Plutinos,
which happen much more frequently than collisions, but have
been rarely studied beforehand. Since the pure numerical
model is incapable or not convincing enough to reveal the
effect of close encounters, we shall start from the simplest
case, including only one Plutino and one Trojan. By control-
ing key variables, this idea allows us to explore the important
relations between the geometries or effects of close encour-
ters, and the physical or orbital characteristics of either inter-
acting planetesimals. Thanks to the stable orbital configura-
tion offered by overlapping resonances, concise and straight-
forward theoretical estimations are available to explain these
relations, which gives us a better understanding of commu-
nications in such case. Furthermore, by introducing the ran-
dom walk model to combine the theoretical distributions of
the close encounter effect and the well-developed theoretical
formula on the encounter frequency, we can actually estimate
the cumulative effect contributed by single planetesimal or a
group of planetesimals with different characteristics. The ef-
effect of realistic population of Plutinos exerted on Trojans, or
at least the upper limit, can then be finally approached with
enough confidence. And obviously this analytical route can
be generalized to an arbitrary problem associated with over-
lapping resonances, such as the communication between the
Hilda group and Jupiter Trojans in the solar system, and simi-
lar cases in the extrasolar systems. The latter part is discussed
in the accompanying paper.

Therefore, the outline of this paper goes as follows. In
Sect. 2, numerical simulations will be implemented for cases
including only one Trojan and one Plutino. Statistical analy-
ses on the frequency and geometries of the close encounters,
along with available theoretical explanations and estimations
will be done at the same time. Those who are not interested in
the statistical tools or the derivation of the analytical formula
can choose to skip these details. In Sect. 3, Monte Carlo sim-
ulations are implemented to further justify the features of the
close encounters. Results and implications are summarized in
Sect. 4.

2. STATISTIC BEHAVIORS OF OBSERVED
PLANETESIMALS

In this section, we statistically explore the close encour-
ters (CEs) between Plutinos and Neptune Trojans. The first
question comes that whether or not there lies a bias in the in-
termittent influence on Trojan by Plutinos, which will prob-
ably result in the variation of the orbital elements of Trojan.
To address that, we first need to detect CEs in the numeri-
sical simulations. The number of detected CEs can indicate
the intensity of such events for different orbital characteris-
tics. Meanwhile, by measuring the relative distance and di-
rections in CEs between Plutino and Trojan, we can infer that
if there is any bias lying in the interaction. More directly, we
can collect the inclination and eccentricity change of Trojan
caused by each CE, whereby the influence bias will be re-
vealed clearly. In addition, the bias may concentrate within
certain timespan, whereupon the time that CE occurs matters
as well. Throughout these analyses, we are able to explore
the geometries and dynamic effect of the CEs, as well as their
crucial relations with each other, and more importantly, with
the orbital elements of Plutinos and Trojans. Such relations
will allow us to build a rigorous analytical formula to derive
the magnitude and distribution of CE effect from the orbital
characteristics of interacting planetesimals.

We pick several realistic planetesimals into our simul-
ations as examples. The orbital elements are listed in Table 1
for reference (from Almeida et al. 2009). These examples are
chosen in order to represent different inclination levels,
based on the supposition that the inclination affects CE effect
the most among all six orbital elements. The eccentricity of
Plutino example should be sufficiently large to allow its orbit
to approach Trojan, that is, about 0.25, which is a common
value among realistic Plutinos.

Note that, throughout this paper, the discussion will be
focused on these examples, including the theoretical deriva-
tions. Nevertheless, our theoretical method will not be lim-
ited to these particular cases. It will be demonstrated in the
accompanying paper (Dong & Zhou 2018) that this theoreti-
cal method is actually applicable to other conditions, despite
the specific initial physical or orbital characteristics of the
interacting planetesimals. Based on this, we may further es-
timate the CE effect of arbitrary interacting planetesimals or even a group of them, thus approaching our expectation.

Table 1. Orbital elements of several Plutinos (2004 UP10 and 2006 RJ103) and Neptune Trojans (1999 CE119 and 2001 FU172) involved (JD 2454200.5).

| Objects   | $a$ (au) | $e$  | $i$ (°) | $M$ (°) | $\omega$ (°) | $\Omega$ (°) |
|-----------|----------|------|---------|---------|--------------|-------------|
| 2004 UP10 | 30.099   | 0.025| 1.4     | 334.1   | 2.2          | 34.8        |
| 2006 RJ103| 29.973   | 0.028| 8.2     | 226.6   | 35.4         | 120.8       |
| 1999 CE119| 39.583   | 0.274| 1.473   | 352.711 | 34.967       | 171.553     |
| 2001 FU172| 39.636   | 0.272| 24.694  | 30.943  | 135.196      | 32.448      |

In our model, we will omit the perturbation of other giant planets like Jupiter or Saturn, in order to reveal the pure effect of CEs brought by Plutinos. Therefore, the model is extremely simplified leaving only the Sun, Neptune and the interacting planetesimals Plutino and Trojan. This simplification will not significantly impair the reliability of our model, because the secular perturbation from other planets is impossible to directly affect a transient event like CE, whose duration is as short as about 0.1yr. One may argue that the secular perturbation may modify the orbits of Plutino or Trojan in the long term, whereby CEs are influenced indirectly. But such mechanism is bound to be statistically covered in whole. Only when the secular perturbation persistently and unidirectionally pushes some orbital element of the whole population will it deflect the basic result of the realistic effect by our model, which is very unlikely. To merely explore the effects of orbital characteristics and also for simplification, all Trojans and Plutinos will be given the mass of Pluto, regardless of their realistic mass. As for the effect of mass, we will develop further analyses in the accompanying paper.

2.1. Theoretical evaluation of CE effects

2.1.1. The Gaussian formula

Before presenting the simulation results, we try to look into the scenario of close encounter theoretically and find out what the key elements contributing to the CE effects are.

As shown in Fig. 2, the reference frame adopted in this section originates at Trojan, with $z$-axis normal to the orbital plane of Trojan and the negative $x$-axis pointing to the Sun. This instantaneous reference frame co-moves with Trojan and is suitable for the Gaussian perturbation equation, where the part concerned with inclination can be expressed as (e.g. Murray & Dermott 2000)

$$\frac{dl}{dt} = \frac{r\bar{N} \cos (\omega + f)}{h},$$

which only contains the normal component of the disturbing acceleration $\bar{N}$. Here $I$, $\omega$, $f$ corresponds to the inclination, argument of pericentre, and true anomaly of Trojan respectively, and $h$ is the constant associated with the angular momentum of Trojan.

The total inclination change during one CE can be written as

$$\Delta I = \int_{t_b}^{t_e} \frac{r\bar{N} \cos (\omega + f)}{h} dt,$$

where $t_b$ and $t_e$ are the time that the integration begins and ends respectively. Here $r$, $\cos (\omega + f)$ and $h$ can be considered to be independent of time since the spatial scale of CE
is extremely small compared to the orbit circumference of Trojan. Specify $\bar{N}$ and we can get
\[
\Delta I = \frac{\mu r \cos (\omega + f)}{h} \int_0^\infty \frac{\mathbf{R} \cdot \mathbf{H}}{R^3} dt,
\]
(3)
where $\mathbf{R}$ is the relative position vector, $\mathbf{H}$ is the unit normal direction vector of the orbital plane of Trojan and $\mu = Gm_p$.\(^2\)

Due to the low mass of planetesimals and the small scale of the CE region, the motion of the Plutino in the vicinity of Trojan is assumed to be rectilinear (see Fig. 2). Such approximation has been often adopted in previous works, e.g., Wetherill (1967). Hence rewrite Eq. (3) as
\[
\Delta I = \frac{2\mu r \cos (\omega + f)}{h} \int_0^\infty \frac{(\mathbf{R}_0 + \mathbf{V}_0(t)) \cdot \mathbf{H}_0}{(R_0^2 + V_0^2(t)^3)^{1/2}} dt,
\]
(4)
where for simplicity, the relative velocity $\mathbf{V}$ is considered to be time-independent during CE and can be substituted by $\mathbf{V}_0$, which is the relative velocity vector at the closest position.\(^3\) $\mathbf{R}_0$ and $\mathbf{H}_0$ are the relative position vector and the unit direction vector at the closest position respectively. Given the rectilinear trajectory, we can integrate the perturbation from infinity, which means $t_b \rightarrow -\infty$ and $t_c \rightarrow \infty$. This leads to the total change as
\[
\Delta I = \frac{2\mu r \cos (\omega + f)}{h} \sin \theta_0 \frac{1}{V_0} \frac{R_0}{R_0},
\]
(5)
where $\sin \theta_0 = \mathbf{R}_0 \cdot \mathbf{H}/R_0$. Now if we write $r = a(1 - e^2)/(1 + e \cos f)$ and $h = \sqrt{\mu_\alpha a(1 - e^2)}$, where $\mu_\alpha \equiv G(m_T + m_S)$, and consider that $a$, $e$ varies in a narrow range during the evolution of Trojan, the theoretical expression for $\Delta I$ will merely contain $\omega$, $f$, $\theta_0$, $V_0$, $R_0$ as independent variables.

We can simplify the result even more by $V_0 = |\mathbf{V}_P - \mathbf{V}_T|$, where $\mathbf{V}_P$ is the velocity vector of Plutino at this moment as well as $\mathbf{V}_T$ is that of Trojan. Recall the vis viva equation
\[
V_\alpha^2 = \mu_\alpha \left(\frac{2}{r_\alpha} - \frac{1}{a_\alpha}\right).
\]
(6)
In our model, $r_\alpha \approx a_\alpha$ since all CEs take place close to the orbit of Trojan. Thus
\[
V_P^2 = \mu_P \left(\frac{2}{a_T} - \frac{1}{a_P}\right),
\]
\[
V_T^2 = \frac{\mu_T}{a_T},
\]
(7)
where $\mu_T \equiv G(m_T + m_S)$. The relative velocity can be then represented as
\[
V_0^2 = V_P^2 + V_T^2 - 2V_PV_T \cos \alpha_e,
\]
(8)
where $\alpha_e$ is the angle between $\mathbf{V}_T$ and $\mathbf{V}_P$. Considering both the mass of Trojan and Plutino are fairly small compared to the Sun, we have $\mu_T \approx \mu_P \approx \mu_S \equiv Gm_S$, which leads to
\[
V_0^2 = \mu_S \left[\frac{3}{a_T} - \frac{1}{a_P} - 2\left(\frac{2}{a_T} - \frac{1}{a_P}\right) \frac{1}{a_T} \cos \alpha_e\right].
\]
(9)
Hence the inclination change during one CE can be finally written as
\[
\Delta I = \frac{m_p a_T}{m_S R_0} \sqrt{2(1 - e_T^2)} \frac{\rho(\epsilon_T, f_T, \omega_T)}{\sqrt{A - B \cos \alpha_e}} \frac{1}{\gamma_R},
\]
(10)
where
\[
\gamma_R \equiv \frac{R_0}{R_{th}}, \quad A \equiv \frac{1}{2} \left(3 - \frac{a_T}{a_P}\right), \quad B \equiv \sqrt{2 - \frac{a_T}{a_P}}.
\]
(11)
and
\[
\rho(\epsilon_T, f_T, \omega_T) \equiv \frac{\cos \tilde{T}_T}{1 + e_T \cos f_T}, \quad \tilde{T}_T \equiv \omega_T + f_T.
\]
(12)
Here we use $R_{th}$, namely the CE threshold, to normalize $R_0$. The rightmost term $1/\gamma_R$ may need a little bit of modification in practice, which will be discussed later.

Expression (10) only consists of a few variables, namely the relative distance $R_0$ and four arguments $\tilde{T}_T$, $f_T$, $\omega_T$ and $\theta_0$. These variables can be further divided into two classes, $R_0$ and $\theta_0$ as microscopical since they are defined at the scale of CE, and $\tilde{T}_T$, $f_T$ and $\omega_T$ as macroscopical since they are concerned with the orbits of planetesimals. Technically, $R_0$ and $\theta_0$ are determined by $\tilde{T}_T$, $f_T$, $\omega_T$, and other macroscopical variables concerning the positions and velocities of planetesimals when CE happens, whereas they can yet be treated to be independent of each other statistically for the entirely different scale they reside in. Therefore we can basically hold that $\Delta I = \Delta I(R_0, \theta_0, \tilde{T}_T, f_T, \omega_T)$.

In addition, the term $\cos \tilde{T}_T$ in Eq. (10) implies that the disturbing force will have a stronger effect at the ascending or descending node. We can further infer that the relative distance $R_0$ will strongly affect the magnitude of $\Delta I$, while $\theta_0$ and $\tilde{T}_T$ will directly determine whether the effect is positive or negative.

Hereto we can verify the theoretical formula based on (5) by comparing it to the simulation results. In Fig. 3, we plot the locations of CEs within the CE region with the color representing the degree of CE effect, while the partial correlation $\Delta I = \Delta I(R_0, \theta_0)$ can be reflected using the contour line in the same figure, which fits the simulation data fairly well.

The CE effect on the eccentricity of Trojan can be evaluated in the same way. Gaussian perturbation equation gives (Murray & Dermott 2000)
\[
\frac{d\epsilon}{dt} = \frac{h}{\mu_T} \left[\bar{R} \sin f + \bar{T} \cos f + \bar{E} \sin E\right].
\]
(13)
By several simple derivations, we have
\[
\Delta \epsilon = \frac{m_p a_T}{m_S R_0} \sqrt{2(1 - e_T^2)} \frac{\rho(\epsilon_T, f_T, \omega_T)}{\sqrt{A - B \cos \alpha_e}} \frac{1}{\gamma_R},
\]
(14)
tions, we are aware that the classical Opik’s formula (Opik 1976) is more regularly applied in previous works (e.g. Greenberg et al. 1988; Carusi et al. 1990; Valsecchi et al. 2015) to handle similar cases. We now compare our above formula (hereinafter Gaussian formula) with Opik’s formula.

Opik’s formula is able to predict the entire set of post-encounter orbital elements, based on the pre-encounter elements of the two interacting planetesimals. While in this work, we only care about the CE effects, namely the change of inclination and eccentricity. To output the outcome orbits using Opik’s formula and then seek the minor differences seems rather complicated and overqualified. In contrast, the Gaussian formula is certainly more direct and convenient. One more difference lies in the input parameters. As shown in Eqs. (10) and (14), the input variables of Gaussian formula are mainly the CE parameters that denote the locations and orientations of CEs. Such information happens to be our main concern. Later in this section we will discuss these parameters one by one and derive the corresponding distributions for numerous CEs, and finally lead to the distribution of CE effects. This route gives us an explicit pattern about the relationship between CE parameters and dynamic effects, based on the physical connections offered by Gaussian formula. For Opik’s formula, the distributions of pre-encounter orbital elements are hard to obtain and also not very helpful in the understanding of CEs with regard to our concern.

\[ \Psi(e_T, f_T, \varphi_0) = \sin(f_T + \varphi_0) + \cos E_T \sin \varphi_0, \]
\[ \cos E_T = \frac{e_T + \cos f_T}{1 + e_T \cos f_T}. \]

and
\[ R_0 \cos \theta_0 \cos \varphi_0 = R_0 \cdot R, \]
\[ R_0 \sin \theta_0 \sin \varphi_0 = R_0 \cdot T. \]

Eq. (16) defines \( \varphi_0 \), where \( R_0 \) and \( \theta_0 \) have been previously defined while \( R \) and \( T \) are the unit radial and transversal vector in the reference frame illustrated in Fig. 2 respectively. Here we take the similar approximations as above and the expressions for \( \Delta I \) and \( \Delta e \) are basically similar.

2.1.2. Compared with Opik’s formula

Although the above theoretical evaluation presents a clear physical description of CE via Gaussian perturbation equations, we are aware that the classical Opik’s formula (Opik 1976) is more regularly applied in previous works (e.g. Greenberg et al. 1988; Carusi et al. 1990; Valsecchi et al. 2015) to handle similar cases. We now compare our above formula (hereinafter Gaussian formula) with Opik’s formula.

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\[ \Delta I (\text{DEG}) = \Delta I (R_0, \theta_0) \text{ based on Eq. (10)}, \]

\[ \Delta e = \Delta (R_0, \theta_0, e_0). \]

\[ R_0 \cos \theta_0 \cos \varphi_0 = R_0 \cdot R, \]
\[ R_0 \sin \theta_0 \sin \varphi_0 = R_0 \cdot T. \]

Nevertheless, for the sake of conciseness and understandability, the Gaussian formula introduces the rectilinear approximation, which is after all rough and may damage its performance in certain cases. In order to verify the reliability and practicability of Gaussian formula, we now compare its result with that of Opik’s formula, which is generally more sophisticated. The model will include a hypothetical planet with Pluto mass, situated at 30AU, moving in a circular orbit, and a particle at 40 AU, with \( e = 0.25 \) to make the CE possible.

Fig. 4 juxtaposes the CE effects calculated by both formulas. As the CE distance increases, the CE effects decrease...
sharply, which is consistent with Eq. (10). Generally speaking, the result of Gaussian equation is close to that of Ōpik’s formula, with minor underestimation because the rectilinear approximation makes the particle a little bit farther from the planet. The relative error curve further shows that deviation turns greater when $\gamma_R$ is smaller. This tendency is easy to understand for that when a CE is close to the surface of the planet, huge perturbation must bend the trajectory more and depreciate the accuracy of rectilinear approximation. However, this relative higher central deviation will not impair the practicability of Gaussian formula. As shown in Fig. 4, the absolute relative error between Gaussian and Ōpik will turn significant (1% generally) only when $\gamma_R \lesssim 5 \times 10^{-4}$, in which case, as will be later proven in Eq. (31), the probability of CE will be as low as $10^{-6}$. Considering that a general pair of planetesimals will merely generate a few thousand CEs in 1Gyr, a CE so close to the surface will rarely happen even for a large group of planetesimals.

In Fig. 4, one may notice another feature that the relative error for $\Delta e$ rises abnormally and turns positive in the far right end. This is due to the infinite integration in the derivation of Gaussian formula, which is acceptable when the CE scale is small, but clearly inappropriate when the relative distance is comparable to the length of the orbit.

![Figure 5. The CE effects calculated by Ōpik and Gaussian formulas as the mass of the planet varies. The detailed legends are the same as Fig. 4. The realistic masses of Pluto, Mercury and Earth are annotated on the top for reference.](image)

In Fig. 5, we test the performance of Gaussian formula with respect to the mass of the planet. $\gamma_R$ is fixed to be 0.01. The mass of the planet ranges from $10^{-9}$ to $10^{-5}$ times the Sun mass, which are the magnitude of Pluto mass and Earth mass respectively. As the mass increases, the CE effect and the relative error both become greater, which is due to the same reason mentioned before. At about $10^{-6}$ times the Sun mass, the relative error of $\Delta I$ exceeds the 1% critical value. This is slightly higher than the mass of Mercury. Note that on the bottom panel in Fig. 5, the relative error of $\Delta e$ bends abnormally at high mass, which leads to another deficiency of Gaussian equations. According to Eqs. (10) and (14), the CE effect will always increase proportionally with the mass of the planet, ceteris paribus. However, $\Delta I$ and $\Delta e$ have natural restrictions, i.e. $\pi$ for $I$ and 1 for $e$. This implies that when the mass of the planet is high, the result of Gaussian formula may not be reasonable.

In general, Gaussian formula is applicable to CEs that are not very close (farther than 0.0005 CE threshold), and disturbed by bodies comparable to asteroids in mass. Our problem meets this requirement precisely.

2.2. Distribution of CE location

As shown in Eq. (10) and Eq. (14), the CE effects are significantly influenced by $\theta_0$, $\varphi_0$ and $R_0$, which actually determine the location of CE in the CE sphere, as illustrated by Fig. 2. Therefore, in this section, we will focus on the distribution of these variables, while for convenience, the distributions of $\theta_0$, $\varphi_0$ and $R_0$ will be referred to as the azimuthal and radial distribution respectively.

Intended for more CE data to reveal a more clear and accurate distribution, 40 clones of the corresponding Plutino in each pair are introduced with arbitrary value of orbital elements. Concretely, we first carry out a simple simulation on the model including the Sun, Neptune and the realistic Plutino with Pluto mass, and record the upper and lower limits of the orbital elements of the Plutino during evolution, within which the orbital elements of clones are arbitrarily picked. Note that the angles like $M$, $\omega$ and $\Omega$ are picked within 0 and $2\pi$ in radians straightforwardly.

2.2.1. The azimuthal symmetry of vertical CE location

![Figure 6. The probability density function of the latitude of CE location $\theta_0$. The dotted curves are from numerical simulations. The positive and negative branches are quite symmetric, reflected by the small skewness annotated on the top. The mean value, and its upper and lower limits at the 5% significance level are annotated as well. The expected $\bar{\theta}_0 = 0$, denoted by a dashed line, is within the margin of error for each pair. The highlighted curves are the theoretical predictions (see text in Sect. 2.2.2).](image)
We have already shown the locations of CEs in Fig. 3, distributing quite symmetrically around \( z = 0 \), which corresponds to the instantaneous orbital plane of Trojan. To clearly reveal the symmetry between positive and negative branches, Fig. 6 shows the distribution of \( \theta_0 \) defined in Eq. (5), which can be alternatively expressed as

\[
\theta_0 \equiv \arctan \frac{z_r}{\|(x_r, y_r)\|_2},
\]

intuitively the latitude of the CE location in the comoving reference frame. \( x_r, y_r, \) and \( z_r \) here represent the three components of the spatial vector from Trojan to Plutino.

To further verify if the positive and negative branches share the same distribution, we introduce the two-sample Kuiper test (Fisher 1995). The null hypothesis is the two distributions are identical. At the 5% significance level, all pairs obey the null hypothesis, which points out a highly symmetric distribution of CE location.

2.2.2. The theoretical distribution of \( \theta_0 \)

Fig. 6 also shows that a Plutino or Trojan with high inclination will result in CEs gathering around \( \theta_0 = 0 \), i.e. \( z = 0 \) (two panels in the right column). Considering the particular situation during CE, this tendency should result from the orientation bias of the relative velocity vector due to different inclined level of the orbital plane of interacting planetesimals. Intuitively, one can imagine that when the inclination of Plutino is relatively large, the CE location, geometrically the tangent point between the trajectory of Plutino and the spherical surface of CE region centering on Trojan, is unlikely to locate at a high latitude.

To give a quantitative explanation and thus obtain the theoretical distribution of \( \theta_0 \), we will start from the theoretical estimations on the alignments of the velocity vectors of Plutino and Trojan, whereby \( \theta_0 \) can be derived through geometrical relations.

Concretely, in the first place, a von Mises distribution (Fisher 1995) is applied to the approximation of the distribution of \( \theta_{vp} \), defined as

\[
\theta_{vp} \equiv \arctan \frac{v_{pz}}{\sqrt{v_{px}^2 + v_{py}^2}},
\]

namely the latitude of the velocity vector of Plutino in the comoving coordinate. \( v_{px}, v_{py} \) and \( v_{pz} \) here represent the three components of the velocity vector of Plutino. Naturally, the value of \( \theta_{vp} \) is close to the inclination of Plutino. When the inclination of Trojan is further considered, the distribution of \( \theta_{vp} \) will split into four branches, as illustrated in Fig. 7. Therefore the probability density function of \( \theta_{vp} \) can be given as

\[
f(\theta_{vp}) = \frac{1}{8\pi I_0(\kappa_f)} \sum_{i=1}^{4} \exp \left[ \kappa_f \cos \left( \theta_{vp} - \overline{\theta}_{vp}^i \right) \right],
\]

where \( \kappa_f \) is a measure of concentration, \( I_0 \) is the modified Bessel function of order 0, and

\[
\overline{\theta}_{vp}^i = \pm I_p \pm I_T, \quad i = 1, 2, 3, 4.
\]

As illustrated in Fig. 8, \( \alpha_{CE} \) is the angle between the velocity directions of Plutino and Trojan in CE, namely the tangent separation at the intersection of two trajectories. Since the velocity direction of Trojan is perpendicular to the abscissa in the comoving coordinate (see Fig. 2), i.e. the radial direction of Trojan, and the velocity direction of Plutino can be either inward or outward with respect to the trajectory of

![Figure 7](image_url)

**Figure 7.** The illustration for the four different relative positions between the orbital planes of Plutino and Trojan, which result in the four branches of the distribution of \( \theta_{vp} \).

Similarly, we define

\[
\varphi_{vp} \equiv \arctan \frac{v_{px}}{v_{py}},
\]

namely the longitude of velocity vector of Plutino. The theoretical distribution of \( \varphi_{vp} \) can be estimated as the superposition of two von Mises distribution, namely

\[
f(\varphi_{vp}) = \frac{1}{4\pi I_0(\kappa_f)} \sum_{i=1}^{2} \exp \left[ \kappa_f \cos \left( \varphi_{vp} - \overline{\varphi}_{vp}^i \right) \right]
\]

where \( \kappa_f \) is the same parameter defined as above and

\[
\overline{\varphi}_{vp}^i = 90^\circ \pm \alpha_{CE}, \quad i = 1, 2.
\]
Trojan, the mean values of $\phi_v$ can be roughly linked to $\alpha_{CE}$ as in Eq. (23), which then contribute to the two centers in the distribution of $\phi_v$.

Note that when the orbits of Plutino and Trojan are inclined differently, the exact definition of $\alpha_{CE}$ is ambiguous, whereas it can still be approximated by hypothetically placing the two orbits on the same plane, since the realistic inclinations of Plutino and Trojan are low for most cases.

We now have the theoretical distribution of $\theta_v$ and $\phi_v$ in CE, by which the orientation of relative velocity vector can be obtained geometrically. Concretely, the relative velocity vector can be calculated as

$$\mathbf{v}_r = (V_P \cos \theta_v \cos \phi_v, V_P \cos \theta_v \sin \phi_v - V_T, V_P \sin \theta_v),$$

(24)

where $V_P$ and $V_T$ are the respective modules of the velocities of Plutino and Trojan mentioned in Eq. (7), which remain basically constant in CE. Therefore, we immediately derive the latitude and longitude of $\mathbf{v}_r$, respectively

$$\theta_v \equiv \arctan \frac{v_{r,z}}{\sqrt{v_{r,x}^2 + v_{r,y}^2}}, \quad \phi_v \equiv \arctan \frac{v_{r,y}}{v_{r,x}},$$

(25)

where $v_{r,x}, v_{r,y}$ and $v_{r,z}$ represent the three components of the relative velocity vector.

Now with the relative velocity vector determined, we can finally trace the CE picture and solve $\theta_0$. As illustrated in Fig. 9, given the specific orientation of $\mathbf{v}_r$, the CE location will reside at the orthodrome perpendicular to $\mathbf{v}_r$ on the sphere of CE region. To further determine the CE location, we require an auxiliary variable $\xi_0$, which measure the distance between the CE location and the equatorial plane (Trojan orbital plane). Consequently, with the help of spherical trigonometry, we have

$$\sin \theta_0 = \cos \theta_v \sin \xi_0.$$  

(26)

Overall, $\theta_0$ obeys a joint distribution of $\theta_v$ and $\xi_0$. The distribution of $\theta_v$ can be derived from the distributions of $\theta_v$ and $\phi_v$, which are given in Eq. (19) and Eq. (22) respectively. As for $\xi_0$, in consideration of the microscopical randomness of the relative positions when Plutino flies past Trojan, it can be simply treated as uniform.

Note that for all the von Mises distributions mentioned before, we use the identical parameter $\kappa_f$, making it the only artificial parameter in this derivation. Naturally, $\kappa_f$ involves the dispersion of the velocity orientation brought by randomness, which is not related to the orbital characteristics and should be constant. Nevertheless, due to the improper handling of $\alpha_{CE}$ when Plutino is highly inclined early in this section, we have to introduce an extra term regarding Plutino inclination in the determination of $\kappa_f$. Therefore, the expression of $\kappa_f$ can be empirically derived as

$$\left(\frac{1}{\sqrt{\kappa_f}}\right)^2 = (I_P^\circ)^{0.6} + 3.5.$$  

(27)

The left side is analogous to the standard deviation $\sigma$ in the normal distribution. We shall later show the low dependency of our analytical model on $\kappa_f$ in the accompanying paper.

In practice, due to the complex derivations above, to solve the explicit expression of the distribution of $\theta_0$ will be troublesome. Alternatively, we can obtain $\theta_0$ by generating a large sample of $\theta_v$ and $\phi_v$ distributing as defined by Eqs. (19) and (22), and then handle the data statistically. In Fig. 6, the theoretical distribution of $\theta_0$ is highlighted, adequately reflecting the statistical characteristics of the numerical result.

### 2.2.3. The horizontal distribution of CE location

Besides the vertical distribution of CEs along the z-axis, the horizontal distribution, namely how the projections of CE locations distribute on the $(x, y)$ plane also matters, which is thus shown in Fig. 10. The abscissa here indicates the $x$ component of the relative position vector from Trojan to Plutino in the reference frame introduced early while the ordinate corresponds to the $y$ component. Therefore the figure actually depicts the projections of CE locations on the orbital plane of Trojan. Note that $y = 0$ here is coincident with the heliocentric vector of Trojan. A pronounced “X” pattern resides within the CE threshold in the top left plot, whereas the pattern gradually fades away as the inclinations of the two interacting planetesimals increase, till a completely uniform distribution at a very high inclination. According to Fig. 8, one could picture that, given an appropriate eccentricity, the Plutino has two chances to cross the orbit of Trojan in one cycle, once inward and the other outward, thus creating the two strokes of figure-X. However, as the orbits become inclined, the crosses take place in all directions. Though a pattern may be observed in one specific direction, the projection tends to mix up the patterns and a uniform distribution appears in the end.
Figure 10. The horizontal location distribution of CEs between the Plutinos & Trojans listed in Table 1, which is similar to Fig. 3. The prominent “X” pattern fades away as inclinations of planetesimals increase, which is a phenomenon concerning the specific orientation of orbits when CEs happen.

Statistically, Fig. 11 presents the distribution of $\varphi_0$, defined in Eq. (16), which can be alternatively expressed as

$$\varphi_0 \equiv \arctan \frac{y_r}{x_r}. \quad (28)$$

One can find that the peaks, corresponding to figure-X in Fig. 10, fade away clearly as the inclination of planetesimals increases. Meanwhile, as illustrated by Fig. 9, $\varphi_0$ can be analytically obtained through

$$\varphi_0 = \varphi_v - \left(\frac{\pi}{2} + \xi_0\right). \quad (29)$$

where $\xi_0$ is the horizontal component of $\xi_0$ and geometrically

$$\tan \xi_0 = \sin \theta_v \tan \xi_0. \quad (30)$$

The resulting distributions of $\varphi_0$ are highlighted in Fig. 11.

2.2.4. The radial distribution of CE location

We are also interested in the radial distribution of the CE locations, namely the distribution of the minimum distances between two planetesimals in CEs. In Fig. 12, the distribution of the minimum distance $\gamma_R$ is presented, turning out to be a linear distribution with a high correlation coefficient. Though the interacting planetesimals differ from one pair to another, the fitting parameters basically remains the same.

In fact, the dependence of CE probability on the minimum distance has been analytically explored in many precedent works (Öpik 1951; Wetherill 1967), either through strict geometrical derivations, or in a classical “cross section” view. Therefore, here we directly mark down the correlation in a normalized way, since we are solely concerned with the distribution within the CE sphere. The probability of a CE locating within $R$ will be

$$P(\gamma_R < \gamma_R) = \gamma_R^2, \quad (31)$$

where the variables associated with $R$ are replaced by $\gamma_R$, after being normalized by the radius of CE sphere $R_{th}$. Thereupon the probability density function of $\gamma_R$ is

$$f(\gamma_R) = 2\gamma_R, \quad (32)$$

coinciding with the fitting parameters in Fig. 12 fairly well.

2.3. Distribution of CE time

We wonder if the CEs would concentrate on particular time and be absent in other period, worrying about a temporary swarm of CEs causing a strong interaction. Consequently we need to examine the distribution of CE time. Recall that previously we apply the model including the Sun, Neptune and the massless Trojan, where 40 clones of Plutino are introduced for more CE data. Nonetheless, the data concerning clones are not suitable here, since the dynamic instability of clones will always lead to a prominent drop of the quantity of CEs over time and thus contaminate the intrinsic distribution of CE time. Hence, here we will introduce only one Plutino into the model, with realistic initial value of orbital elements.

Fig. 13 shows the frequency histograms of CE time, as well as the empirical cumulative distribution function (ECDF) (Van der Vaart 2000) which indicates the portion of CEs before one particular time. The highlighted line, which is the
On the close encounters between Plutinos and Neptune Trojans

In the close encounters between Plutinos and Neptune Trojans,

Here we introduce the one-sample K-S test (Eadie et al. 1971) to check the uniformity. The null hypothesis is that the sample is drawn from the uniform distribution.

At the 5% significance level, all pairs obey the null hypothesis except Plutino 1999 CE119 and Trojan 2006 RJ103, which we attribute to the lack of sample points. After all, the K-S method is a rigorous test that technically ensures the compliance of the sample with the reference distribution.

2.4. Distribution of CE effects

2.4.1. The numerical distribution of $\Delta I$

Since we are trying to explore the interactions between Plutino and Trojan, the CE effect comes our main concern. Using the same data as in Sect. 2.2, we show the distribution of $\Delta I$, namely the inclination change of Trojan during each CE, in Fig. 14 in a logarithmic scale, with the positive and negative effect separated. The positive and negative CE effect conform to a nearly identical distribution, which implies that a CE has a completely even chance of increasing and decreasing the inclination of Trojan. Here we can introduce a two-sample K-S test to verify the consistency between positive and negative branches, with the result that the null hypothesis stands up for each pair of planetesimals.

Figure 14. The distribution of $\Delta I$ in a logarithmic scale. The ordinate indicates the proportion to total number of CEs. The red and blue curves depict the positive and negative $\Delta I$ from numerical simulations respectively, with their corresponding number of CEs annotated on top. For each pair, the two curves overlap each other pretty well, implying a great symmetry between positive and negative effects. On each plot, there is a pronounced gap between the mean value and the mean absolute value, indicated by dashed and dotted lines respectively, which suggests a counteraction between positive and negative branches. The dash-dotted line indicates the absolute sum of all $\Delta I$. The numerical data are blocked by the shadow region, within which the proportion is too low to allow one single CE. The cyan curve depicts the theoretical result.
On the other hand, we notice some differences between the distributions from different pairs. Other than our intuition, a highly inclined Plutino in the right column in Fig. 14 leads to a relatively low CE effect, which is reflected by the lower mean absolute value denoted by the thick dashed line. This is closely linked to the terms associated with $\theta_0$ and $\alpha_v$ in the analytical expression of $\Delta I$ in Eq. (10). Recall that in Fig. 6 the distribution of $\theta_0$ tends to concentrate on the center for Plutino with high inclination, which brings about relatively low values of the term $\sin \theta_0$. Meanwhile, the value of $\alpha_v$ will be conceivably closer to 0 for Plutino with low inclination than high, since the Plutino and Trojan are more likely to orbit on a same plane, which leads to high value of the term $\cos \alpha_v$. On the whole, the two terms both tend to be lower for a more inclined Plutino.

2.4.2. The theoretical distribution of $\Delta I$

Actually with several derivations above combined together, a theoretical distribution of $\Delta I$ is available approximately. As mentioned before, Eq. (10) only contains five variables, namely $\gamma$, $\theta_0$, $\alpha_v$, $\lambda_T$ and $f_T$, where the distributions of the first two are analytically solved already. Besides, given the velocity vectors of Plutino and Trojan, the angle between the two, namely $\alpha_v$, can be easily derived, which gives

$$\cos \alpha_v = \cos \theta_{\nu \nu} \sin \varphi_{\nu \nu}. \quad (33)$$

For the convenience of later analysis in the accompanying paper (Dong & Zhou 2018), here we set a simplified approximation for $\alpha_v$. As shown in Fig. 7, the orbits of Plutino and Trojan can lie on the same side or different sides of the reference plane, leading to $\alpha_v$ either close to the sum or difference of their inclinations. Thus an average value roughly gives $\alpha_v \approx (|I_P + I_T| + |I_P - I_T|) / 2 = \max(I_P, I_T)$.

We now solve the distribution of $\lambda_T$, which measures the argument of Trojan from its ascending node. This implies that $\lambda_T$ relates to the specific position of Trojan, thus cannot be derived from the orientation of velocities in the co-moving system as the variables before. Nevertheless, the distribution of $\lambda_T$ is still affected by the orbital characteristics of Plutino and Trojan.

Considering a model including a Trojan with high inclination and a Plutino orbiting on the ecliptic, i.e. with 0 inclination, the Trojan is possible to meet Plutino only when it comes back to the ecliptic, namely at its ascending or descending node, in which case, the distribution of $\lambda_T$ should concentrate on 0 or 180°. Another situation is that the Trojan has 0 inclination while the Plutino has high inclination. In this case, the concentration occurs in the argument of Plutino, but the distribution of $\lambda_T$ should be uniform due to the procession of Plutino. Other situations basically lie between the two.

We now again apply the von Mises distribution to analytically estimate the distribution of $\lambda_T$. Based on the above understanding, the distribution of $\lambda_T$ should consists of two von Mises distributions locating on 0 and 180° respectively, while the dispersion should be low when the inclination of Trojan is much higher than that of Plutino, and be high in reverse. Hence we can use $|I_T - I_F - I_C|$ to measure the dispersion, where $I_C$ is a preset bias. Using a linear correlation, we can empirically derive $\{1 / \sqrt{\lambda_T^2 / 8} - |I_T - I_F - I_C|\}$. The last involved variable is $f_T$, which we will simply treated as randomly distributed, since $f_T$ will not directly affect the location of CE given low eccentricity of Trojan. Fig. 16 supports this idea.

```
1999CE119&2004UP10
2001FU172&2004UP10
1999CE119&2006RJ103
2001FU172&2006RJ103

Figure 15. The probability density function of $\lambda_T$. The dashed lines indicate $\lambda_T = 0°$, which coincide with the ascending node of Trojan. The highlighted lines are the theoretical predictions.

Figure 16. The probability density function of $f_T$. The highlighted lines are the theoretical predictions.
```

The numerical and theoretical distributions of $\lambda_T$ are together shown in Fig. 15, which are consistent with each other.
Finally we can obtain the theoretical distribution of $\Delta I$. In order to be consistent with the results from numerical simulations, we have to make a minor modification on the derivation of $\Delta I$. Recall that in the beginning of this section, we mentioned our method to calculate CE effect in the numerical simulation, which is simply the difference of the orbital element between the CE entry and exit. This method naturally forces the effect to be 0 if an encounter happens outside the CE region. Likewise, in the integration of Gaussian perturbation equation, we need to cut off the passage outside the CE region and only count the passage from the entry to the exit. This will not impact the statistics of the distribution of CE effect much since the contribution from the CEs outside the CE region is basically negligible. The modified expression of Eq. (10) yields

$$\Delta I = \Delta I_0 \sqrt{\gamma_R^2 - 1}, \quad (34)$$

where only the last term associated with $\gamma_R$ is different. All other variables remains unchanged and are incorporated into a dimensional coefficient for simplicity. We can clearly see that now as a CE approaches the boundary, i.e. $\gamma_R \to 1$, $\Delta I \to 0$ as expected.

Now given the distributions of other variables derived above, we obtain the theoretical distribution of $\Delta I$ through Eq. (34), which coincides with the numerical results in Fig. 14 fairly well.

In fact, if we focus on the key term $\gamma_R$, we can approximately derive an explicit expression for the distribution of $\Delta I$. Simply treat the absolute value of $\Delta I_0$ in Eq. (34) as constant and we have

$$\gamma_I = \pm \sqrt{\frac{1}{\gamma_R^2} - 1}, \quad (35)$$

where $\gamma_I \equiv \Delta I/|\Delta I_0|$ is a dimensionless variable denoting the magnitude of inclination change.

Based on this relation, Eq. (31) can be rewritten in the form with $\gamma_I$ serving as the independent variable, namely

$$P(\gamma_I < \gamma_I) = \begin{cases} \gamma_I^2/(1 + \gamma_I^2), & \gamma_I \geq 0, \\ 1/(1 + \gamma_I^2), & \gamma_I < 0. \end{cases} \quad (36)$$

The probability density function of $\gamma_I$ can then be derived as

$$f(\gamma_I) = \frac{|\gamma_I|}{(1 + \gamma_I^2)^2}. \quad (37)$$

This functional form is quite close to Fig. 14, with the probability density both leading to 0 when $\gamma_I \to 0$ and $\gamma_I \to \infty$. The contributions from the angular terms are actually minor modifications to Eq. (37).

2.4.3. The distribution of positive and negative CEs over time
main interest is the comparison between the opposite effects, which barely shows any difference, pointing out a completely equal rate of occurrence of the two branches.

2.4.4. The distribution of $\Delta e$

The theoretical distribution of $\Delta e$ can be evaluated similarly, with no need for additional variables. As a reference, Fig. 18 shows the theoretical distribution of $\Delta e$ in cyan, which is consistent with the simulation data fairly well.

3. MONTE CARLO SIMULATION

We have paid great effort to demonstrate the randomness and impartiality lying in the close encounters between Plutinos and Trojans, which can be further verified by a Monte Carlo (M-C) simulation. The statistical distributions of previously discussed features will be reproduced here by the M-C method and juxtaposed with the numerical ones, to see if there is any difference between dynamic CEs and random ones.

3.1. Method

Here we develop a M-C strategy to simulate fictitious CEs, which is briefly outlined as:

Step 1 For each of the two planetesimals expected to CE, randomly choose a set of orbital elements within a given range, and convert into heliocentric coordinates. If the spatial distance between the two is less than a specific value, then a fictitious CE is obtained.

Step 2 Under the hypothesis that one body moves along a hyperbola against the other in a CE, the relative trajectory can be fitted based on the coordinates obtained above.

Step 3 Segment the hyperbolic trajectory within the CE sphere. In each segment use Gaussian perturbation equation to calculate the inclination or eccentricity change of Trojan, and add up to get the total effect of this CE.

Certainly the new strategy requires verification before being put in place. Apart from Step 1 that generates fictitious CEs, Step 2 - Step 3, summarized to be a Hyperbola-Perturbation method to calculate CE effect based on CE information, can be implemented to handle the CEs generated by a numerical simulation, with the results juxtaposed, to verify the practicability of this very method. Fig. 19 shows the relative error of CE effect calculated in the Hyperbola-Perturbation way to that from numerical simulation. $\eta_I = (|\Delta e_{\text{hyperbola}} - \Delta e_{\text{numeric}}|) / |\Delta e_{\text{numeric}}|$ and $\eta_e = (|\Delta e_{\text{hyperbola}} - \Delta e_{\text{numeric}}|) / |\Delta e_{\text{numeric}}|$. Subscripts “HP” and “NS” correspond to the Hyperbola-Perturbation way and the numerical simulation respectively. The abscissa indicates the distance between the interacting particles. The highlighted line denotes the value below which 90% of the dots reside.

Note that in Step 3, given the fitted orbital elements of Trojan in each segment, we introduce the perturbation equation to derive the inclination and eccentricity change, rather than directly subtracting the inclination and eccentricity change at the entry from that at the exit. In fact, the latter method, which seems more convenient, was also applied in our earliest attempts, but unfortunately deviating quite much from the numerical results. A possible explanation is that the prominent fitting error near the edge of the hyperbolic trajectory will be inherits or enlarged by the exit-entry method, while reduced by the perturbation method where the closest segments weigh the most.

3.2. Results and comparison

Now we can implement the M-C strategy to rapidly generate CE information. To present a direct comparison, here we will display simultaneously the results from numerical simulations and the M-C simulations on the statistical diagrams in Sect. 2. All the pairs of Plutinos and Trojans remain the same, as well as their initial conditions. The M-C simulations generate exactly the same number of CEs as produced by numerical simulation for each pair.

3.2.1. Distribution of CE location

Similar to Figs. 6 and 11, Figs. 20 and 21 give a direct comparison between $\theta_0$, and $\varphi_0$ in CEs generated by numerical simulations and M-C simulations. In each figure, the two methods agree with each other fairly well.
on the close encounters between Plutinos and Neptune Trojans

Fig. 20. The probability density function of the latitude of the CE location $\theta_0$, similar to Fig. 6. The black is produced by the numerical simulation while the red is generated by the M-C simulation. The dashed line indicates the respective mean value.

Fig. 21. The probability density function of the longitude of CE location $\phi_0$, similar to Fig. 11. The black is produced by the numerical simulation while the red is generated by the M-C simulation.

Fig. 22. The probability density function of the minimum CE distance $\gamma_R$, similar to Fig. 12. The black is produced by the numerical simulation while the red is generated by the M-C simulation.

Fig. 23. The distribution of absolute value of $\Delta I$ in a logarithmic scale. The ordinate indicates the proportion to total number of CEs. The black is produced by the numerical simulation while the red is generated by the M-C simulation. The dashed line indicates the absolute mean value.

3.2.2. Distribution of CE time

In a M-C simulation the concept of CE time is ambiguous since we are merely generating random numbers. However, the uniformity of the distribution of CEs over time in Sect. 2.3 definitely agrees with the result of a stochastic method.
3.2.3. Distribution of CE effects

The vital criterion that determines the feasibility of the M-C simulation should be the distribution of CE effects, which is our major concern. Fig. 23 compares the frequency histograms $\Delta T$ brought by numerical simulations and M-C simulations. The M-C productions basically coincide with the numerical data, and fully reproduce the main characteristics of the distribution curves, such as the correlation between high inclination and low CE effect. The mean value of the M-C productions is close to that of numerical data as well.

4. CONCLUSION AND DISCUSSION

In this work we implement a numerical method to efficiently detect close encounters (CEs) between Plutinos and Neptune Trojans. For typical planetesimals in reality, detailed statistical analyses are performed on the frequency and geometries of CEs, followed by consistent analytical estimations. We hereby present a better understanding of the geometries of CEs, followed by consistent analytical estimations. The vital criterion that determines the feasibility of the M-C simulation should be the distribution of CE effect. The mean value of the M-C productions is close to that of numerical data as well.

Specifically, as covered in Sect. 2.2, the CEs are found to distribute symmetrically against the orbital plane of Trojan inside the CE region, thus in whole exerting unbiased influence on Trojan. The minimum CE distance well conforms to a linear distribution, consistent with a “cross section” picture. Over a long timespan, the CEs take place uniformly, with no tendency of concentration. Particular orbital characteristics, typically the inclination, will directly affect the frequency of CE, as well as the angular distributions of CEs such as the longitude, latitude inside the CE region and the argument from the ascending node on the orbit of Trojan. Such features can be well explained by the approximate theory.

Besides, we investigate the CE effect, measured by the inclination and eccentricity change of Trojan. By detailedly comparing the distribution of positive and negative effects, along with their respective frequency over time, we conclude that the CE effect is impartial, thus little possible to observably alter the orbital elements of Trojan. In other words, the high inclination problem of Trojan is unrelated to the communications contributed by Plutinos. Furthermore, the theoretical distribution of CE effect is derived from the analytical estimations on associated variables according to Eq. (10) and Eq. (14), namely the integral formula of Gaussian perturbation function. Since the terms like $\cos \theta_{T}$ and $\sin \theta_{T}$, directly determine the sign of CE effect, are all unbiased due to the symmetrical distribution of $\lambda_{T}$ and $\theta_{T}$, the theoretical effect is certainly impartial, which again verify the numerical result.

Though Plutinos contribute little to the characteristics of Trojans by CEs, they may alternatively take effect by more violent interactions, namely collisions. Actually we can estimate the collision rate between Plutinos and Trojans for a regular case, using Eq. (31), namely the linear distribution of $R_{0}$. Taking Pluto as an example, $R_{Pluto} \approx 0.18$ AU, while naturally the minimum distance $R \leq 2R_{Pluto}$ for a collision caused by a sizable Trojan, where the radius of Pluto $R_{Pluto} \approx 1000$ km. As a result, the normalized collision probability is estimated as $P \approx 2 \times 10^{-9}$. Besides, in numerical simulations the number of CEs between one Plutino and one Trojan is generally the order of $10^{3}$, which implies that averagely only one collision will happen given $5 \times 10^{5}$ Trojans during 1 Gyr. Despite the large quantity of potential Neptune Trojans in reality, this is an extremely low probability, not to mention that typical planetesimals are far smaller than Pluto. Consequently, the collisions brought by Plutinos are too scarce to cause prominent effects on the overall distribution features of Trojans.

In the final part of this work, we implement a Monte Carlo (M-C) simulation to stochastically generate CEs, along with a Hyperbola-Perturbation method to calculate consequent effects. The consistency of statistical diagrams between numerical integrations and M-C simulations again proves the randomness and impartiality of CEs between Plutinos and Trojans. In addition, since the CE productions by M-C simulations are close to that by numerical integrations, we can use that to generate CEs in place of numerical approach under specific circumstances, with the benefit of high computational efficiency and no restriction of number of CEs.

We will further discuss the cumulative effect of CEs on the orbit of Trojan, for fear that tiny random effect may accumulate to be prominent, as long as there is a tremendous number of CEs or a sufficient number of Plutinos simultaneously interacting. That is where the theoretical distributions of CE effects come in handy. Though derived from the pattern of 2+2 planetesimals, we shall show in the accompanying paper that this analytical tool is actually applicable to a wide range of orbital elements. Therefore, with the help of the random walk theory, we can easily estimate the cumulative
effect of CEs contributed by an arbitrary Plutino, or even a group of different Plutinos. In this way, we are able to give the possible range of CE effects brought by realistic Plutinos, much more convincing than the result of pure numerical simulations including even hundreds of planetesimals.

In the end, it is necessary to point out that, our statistical methods and analytical estimations developed in this work can be directly applied to other circumstances associated with the overlap between mean motion resonances, e.g. Hilda group and Jupiter Trojans. Once the randomness of CEs is identified, further theoretical tools introduced in the subsequent paper can be implemented to quantitively determine the potential effect. Specific results may differ from what obtained for the case introduced in this work, due to distinct orbital and physical characteristics. Of course the CEs between planets and comets can be studied in the same way, but generally the low frequency and large effect in one single CE may impair the statistical significance.

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