Two Aspects of the Donoho-Stark Uncertainty Principle

Paolo Boggiatto, Evanthia Carypis and Alessandro Oliaro*
Department of Mathematics, University of Torino
Via Carlo Alberto, 10, I-10123 Torino (TO), Italy

Abstract
We present some forms of uncertainty principle which involve in a new way localization operators, the concept of \(\varepsilon\)-concentration and the standard deviation of \(L^2\) functions. We show how our results improve the classical Donoho-Stark estimate in two different aspects: a better general lower bound and a lower bound in dependence on the signal itself.

1 Introduction

An uncertainty principle (UP) is an inequality expressing limitations on the simultaneous concentration of a function, or distribution, and its Fourier transform. More in general UPs can express limitations on the concentration of any time-frequency representation of a signal. According to the meaning given to the term “concentration” many different formulations are possible and, starting from the classical works of Heisenberg, a vast literature is today available on these topics, see e.g. [3, 6, 7, 11, 13, 12, 16, 18].

In this paper we are concerned with the Donoho-Stark form of the UP of which we present an improvement in the form of a new general bound for the constant which is involved in the estimate, and a new type of estimation of the same constant in dependence on the signal.

The Donoho-Stark UP relies on the concept of \(\varepsilon\)-concentration of a function on a measurable set \(U \subseteq \mathbb{R}^d\). We start by recalling this definition followed by the statement of the classical theorem.

*E-mail addresses:
paolo.boggiatto@unito.it, evanthia.carypis@unito.it, alessandro.oliaro@unito.it
**Definition 1.** Given \( \varepsilon \geq 0 \), a function \( f \in L^2(\mathbb{R}^d) \) is \( \varepsilon \)-concentrated on a measurable set \( U \subseteq \mathbb{R}^d \) if
\[
\left( \int_{\mathbb{R}^d \setminus U} |f(x)|^2 \, dx \right)^{1/2} \leq \varepsilon \|f\|_2.
\]

**Theorem 2.** (Donoho-Stark) Suppose that \( f \in L^2(\mathbb{R}^d) \), \( f \neq 0 \), is \( \varepsilon_T \)-concentrated on \( T \subseteq \mathbb{R}^d \), and \( \hat{f} \) is \( \varepsilon_\Omega \)-concentrated on \( \Omega \subseteq \mathbb{R}^d \), with \( T, \Omega \) measurable sets in \( \mathbb{R}^d \) and \( \varepsilon_T, \varepsilon_\Omega \geq 0 \), \( \varepsilon_T + \varepsilon_\Omega < 1 \). Then
\[
|T| |\Omega| \geq (1 - \varepsilon_T - \varepsilon_\Omega)^2. \tag{1}
\]

(We use the convention \( \hat{f}(\omega) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \omega} f(x) \, dx \).)

Many variations and related information about this result can be found e.g. in \cite{12} and \cite{17}. Our investigations of the Donoho-Stark UP are based on two well-known fundamental results, namely the \( L^p \) boundedness result of Lieb for the Gabor transform, recalled in Theorem \ref{lieb} and the local UP of Price, which we recall in Theorem \ref{price}.

We present now in more details how the two aspects that we have mentioned are considered in the paper.

The first aspect we deal with is an improvement of the estimation of the Donoho-Stark constant which appears on the right-hand side of inequality (1). More precisely in section 2 we prove a new estimate, Lemma \ref{lemma4} for the norm of localization operators:
\[
f \rightarrow L^p_{\phi, \psi} f = \int_{\mathbb{R}^d} a(x, \omega) V_{\phi, f}(x, \omega) \mu_\omega \tau_x \psi \, dx \, d\omega \tag{2}
\]
acting on \( L^2(\mathbb{R}^d) \), with symbol \( a \in L^q(\mathbb{R}^{2d}) \), for \( q \in [1, \infty] \), and “window” functions \( \phi, \psi \in L^2(\mathbb{R}^d) \). Here, for \( \phi, \psi \in L^2(\mathbb{R}^d) \),
\[
V_{\phi, f}(x, \omega) = \int e^{-2\pi i t \cdot \omega} f(t) \overline{\phi(t - x)} \, dt
\]
is the Gabor transform of \( f \in L^2(\mathbb{R}^d) \), whereas \( \mu_\omega \tau_x \psi(t) = e^{2\pi i \omega \cdot t} \psi(t - x) \) are time-frequency shifts of \( \psi(t) \). See e.g. \cite{5, 9, 14, 15, 17} for references on this topic as well as for extensions to more general functional settings.

Although not new in its functional framework as boundedness result, as far as we know, the norm estimate of Lemma \ref{lemma4} does not appear before in literature. Our proof relies on Lieb’s estimate of the \( L^p \) norm of the Gabor transform (Theorem \ref{lieb}).
We then use Lemma 4 and some facts from the theory of pseudo-differential operators, in particular from Weyl calculus, to obtain our main result of this section, Theorem 6, which is an uncertainty principle for concentration operators, i.e. localization operators with characteristic functions as symbols.

The reason of the interest in these operators lies in the fact that the Donoho-Stark hypothesis of $\varepsilon$-concentration can be interpreted in terms of the action of concentration operators. This is used in section 3 to compare our results with the classical Donoho-Stark UP. It turns out that, as a limit case for window functions tending to the Dirac delta in the space of tempered distributions, this UP can be reobtained with a considerable improvement of the constant appearing in the estimate, see Proposition 10.

In section 4 we consider the second aspect of the Donoho-Stark UP starting from Price’s local UP, Theorem 11 (cf. [22], see also [2]). Qualitatively this theorem asserts that a highly concentrated signal $f$ cannot have Fourier transform which is too concentrated on any measurable set $\Omega$, the upper bound of the local concentration of $\hat{f}$ being given in terms of the Lebesgue measure of $\Omega$ itself and the standard deviation of $f$. We show that the concept of $\varepsilon$-concentration of $f$ and $\hat{f}$ respectively on sets $T$ and $\Omega$ can be used in this local context to get a version of the Donoho-Stark UP with an estimation whose constant depends on the signal $f$, Theorem 13.

Actually, as pointed out in Remark 14 the proof of Theorem 13 shows more than inequalities of Donoho-Stark type, as we get independent lower bounds for the measures of the sets $T$ and $\Omega$, whereas the Donoho-Stark estimate is a lower bound only for the product of the two measures.

Our final result in section 4, Theorem 15, concerns a “mixed” lower bound for the support of a signal and the standard deviation of its Fourier transform.

2 An uncertainty principle for localization operators

Localization operators of type (2) have been widely studied in literature (see e.g. [5, 17, 19, 26]). As first result of this section, Lemma 4 we obtain a new estimation for their $L^2$-boundedness constant by means of the classical Lieb’s $L^p$-boundedness result for the Gabor transform which we recall here for completeness (see e.g. [21, 17]).
Theorem 3. (Lieb) If $f, g \in L^2(\mathbb{R}^d)$ and $2 \leq p \leq \infty$, then

$$
\|V_g f\|_p \leq \left( \frac{2}{p} \right)^{\frac{d}{p}} \|f\|_2 \|g\|_2 \quad (\text{with } (\frac{2}{p})^{\frac{d}{p}} = 1 \text{ for } p = +\infty).
$$

Lemma 4. Let $\phi, \psi \in L^2(\mathbb{R}^d)$, $q \in [1, \infty]$ and consider the quantization (see (2)):

$$
L_{\phi, \psi} : a \in L^q(\mathbb{R}^{2d}) \rightarrow L^a_{\phi, \psi}(\mathbb{R}^{2d}) \in B(L^2(\mathbb{R}^{2d})).
$$

Then the following estimation holds

$$
\|L^a_{\phi, \psi}\|_{B(L^2)} \leq \left( \frac{1}{q} \right)^{d/q'} \|\phi\|_2 \|\psi\|_2 \|a\|_q,
$$

with $\frac{1}{q} + \frac{1}{q'} = 1$ and setting $(\frac{1}{q})^{\frac{1}{q'}} = 1$ for $q = 1$.

Proof. We indicate by $(\cdot, \cdot)$ the inner product in $L^2$ spaces. Recall (cf. for example [5]) that for every $f, g, \phi, \psi \in L^2(\mathbb{R}^d)$ and $a \in L^q(\mathbb{R}^{2d})$ we have $(L^a_{\phi, \psi} f, g) = (a, V_{\psi} g V_{\phi} f)$. Then, in view of Hölder’s inequality and Lieb’s UP, for $f, g \in L^2(\mathbb{R}^d)$ we have:

$$
|\langle L^a_{\phi, \psi} f, g \rangle| = |\langle a, V_{\psi} g V_{\phi} f \rangle| \\
\leq \|a\|_q \|V_{\psi} g V_{\phi} f\|_{q'} \\
\leq \|a\|_q \|V_{\psi} g\|_{q'k} \|V_{\phi} f\|_{q'k'} \\
\left( \text{for } k \in [1, \infty], \frac{1}{k} + \frac{1}{k'} = 1 \right)
$$

$$
\leq \|a\|_q \left( \frac{2}{q'k} \right)^{d/q'k} \|\psi\|_2 \|g\|_2 \left( \frac{2}{q'k'} \right)^{d/q'k'} \|\phi\|_2 \|f\|_2
$$

$$
= \|a\|_q \left( \frac{2}{q'} \right)^{\frac{1}{q'}} \left( \frac{1}{k} \right)^{1/k} \left( \frac{1}{k'} \right)^{1/k'} \|\psi\|_2 \|\phi\|_2 \|f\|_2 \|g\|_2,
$$

where we have set

$$
\alpha_k = \left( \frac{1}{k} \right)^{1/k} \left( \frac{1}{k'} \right)^{1/k'},
$$

and we have applied Theorem 3 supposing $k \in [1, \infty]$ such that $q'k \geq 2$ and $q'k' \geq 2$. It follows that

$$
\|L^a_{\phi, \psi}\|_{B(L^2)} \leq C \|a\|_q,
$$
with 
\[ C = \left( \frac{2}{q'} \right)^{d/q'} \| \psi \|_2 \| \phi \|_2 \left( \min \{ \alpha_k : k \in [1, \infty] \text{ such that } q'k \geq 2, q'k' \geq 2 \} \right)^{d/q'} . \]

Let us now study the function 
\[ \alpha_k = \left( \frac{1}{k} \right)^{1/k} \left( \frac{1}{k'} \right)^{1/k'} = f \left( \frac{1}{k} \right) . \]

Setting \( x = \frac{1}{k} \), we have \( f(x) = x^x (1 - x)^{(1-x)} \), which, for \( x \in [0,1] \) and setting \( \left( \frac{1}{\infty} \right)^\infty = 1 \), has an absolute minimum in \( x = 1/2 \). It follows that \( f(1/k) \) on the interval \( k \in [1,\infty] \) has an absolute minimum in \( k = 2 \).

Given \( q \in [1,\infty] \), we search now the minimum of the values \( \alpha_k \) with \( k \) satisfying the conditions:
\[
\begin{cases}
1 \leq k \leq \infty \\
q'k \geq 2 \\
q'k' \geq 2
\end{cases}
\]

For \( k \in [1,\infty] \), the condition \( q'k \geq 2 \) yields \( k \geq \frac{2q-2}{q-2} \); whereas the condition \( q'k' \geq 2 \) yields \( k \leq \frac{2q-2}{q-2} \) for \( q > 2 \), and \( k \geq \frac{2q-2}{q-2} \), for \( q < 2 \) (observe that \( q'k' \geq 2 \) is satisfied for every \( k \in [1,\infty] \) when \( q = 2 \)). Elementary considerations lead then to the conclusion that, for every \( q \in [1,\infty] \), the value \( k = 2 \) satisfies (3) and gives the absolute minimum for \( \alpha_k \).

As 
\[ \alpha_2 = \left( \frac{1}{2} \right)^{1/2} \left( \frac{1}{2} \right)^{1/2} = \frac{1}{2} \]

we have
\[ \| L^a_{\phi,\psi} \|_{B(L^2)} \leq \left( \frac{2}{q'} \right)^{d/q'} 2^{-d/q'} \| \phi \|_2 \| \psi \|_2 \| a \|_q \]
\[ \quad = \left( \frac{1}{q'} \right)^{d/q'} \| \phi \|_2 \| \psi \|_2 \| a \|_q \]

as desired.

We shall use the previous result to obtain an uncertainty principle involving localization operators in the special case where the symbol is the characteristic function of a set, expressing therefore concentration of energy.
on this set when applied to signals in $L^2(\mathbb{R}^d)$. In this case they are also known as concentration operators.

For the proof we shall need some tools from the pseudo-differential theory which we now recall in the $L^2$ functional framework, for more general settings and reference see e.g. [4], [20], [23], [24], [27].

**Definition 5.** The Wigner transform is the sesquilinear bounded map from $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$ defined by

$$(f, g) \rightarrow \text{Wig}(f, g)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi it\omega} f(x + t/2)g(x - t/2) dt.$$  

For short we shall write $\text{Wig}(f)$ instead of $\text{Wig}(f, f)$. In connection with the Wigner transform, Weyl pseudo-differential operators are defined by the formula

$$\left( W^a f, g \right)_{L^2(\mathbb{R}^d)} = \left( a, \text{Wig}(g, f) \right)_{L^2(\mathbb{R}^{2d})},$$

(4)

for $f, g \in L^2(\mathbb{R}^d), a \in L^2(\mathbb{R}^{2d})$. More explicitly they are maps of the type

$$f \in L^2(\mathbb{R}^d) \rightarrow W^a f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\omega} a\left(\frac{x+y}{2}, \omega\right) f(y) dy d\omega \in L^2(\mathbb{R}^d).$$

The fundamental connection between Weyl and localization operators is expressed by the formula which yields localization operators in terms of Weyl operators:

$$L^a_{\psi, \phi} = W^b,$$

(5)

with $\psi, \phi \in L^2(\mathbb{R}^d)$ and where, for a generic function $u(x)$, we use the notation $\bar{u}(x) = u(-x)$.

Of particular importance for our purpose will be the fact that Weyl operators with symbols $a(x, \omega)$ depending only on $x$, or only on $\omega$, are multiplication operators, or Fourier multiplier respectively. More precisely we have

$$a(x, \omega) = a(x) \implies W^a f(x) = a(x)f(x)$$

$$a(x, \omega) = a(\omega) \implies W^a f(x) = \mathcal{F}^{-1}[a(\omega)\hat{f}(\omega)](x).$$

(6)

Let us now fix some notations. Let $T \subseteq \mathbb{R}^d_x$, $\Omega \subseteq \mathbb{R}^d_\omega$ be measurable sets, and write for shortness $\chi_T = \chi_{T \times \mathbb{R}^d}$ and $\chi_\Omega = \chi_{\mathbb{R}^d \times \Omega}$, in such a way that $\chi_T = \chi_T(x)$ and $\chi_\Omega = \chi_\Omega(\omega)$. Moreover, for $j = 1, 2$, $\lambda_j > 0$, we set $\phi_j(x) = e^{-\pi\lambda_j x^2}$ and $\Phi_j(x) = c_j \phi_j(x)$, where $\Phi_j$ are normalized in $L^2(\mathbb{R}^d)$, i.e. $c_j = (2\lambda_j)^{d/4}$. Furthermore let

$$L_1 f = L^{\chi_T}_{\Phi_1} f = \int_{\mathbb{R}^{2d}} \chi_T(x)\Phi_1 f(x, \omega)\mu_{\omega, \tau_x} \Phi_1 dx d\omega$$

(7)
and
\[ L_2 f = L^{\chi_\Omega}_{\Phi_2} f = \int_{\mathbb{R}^d} \chi_\Omega(\omega) V_{\Phi_2} f(x,\omega) \mu_\omega \tau_x \Phi_2 dx d\omega \quad (8) \]
be the two localization operators with symbols $\chi_T, \chi_\Omega$ and windows $\Phi_1, \Phi_2$ respectively. We can state now the main result of this section which is an UP involving the $\varepsilon$-concentration of these two localization operators.

**Theorem 6.** With the previous assumptions on $T, \Omega, L_1, L_2$, suppose that $\varepsilon_T, \varepsilon_\Omega > 0$, $\varepsilon_T + \varepsilon_\Omega \leq 1$, and that $f \in L^2(\mathbb{R}^d)$ is such that
\[ \|L_1 f\|_2^2 \geq (1 - \varepsilon_T^2) \|f\|_2^2 \quad \text{and} \quad \|L_2 f\|_2^2 \geq (1 - \varepsilon_\Omega^2) \|f\|_2^2. \quad (9) \]
Then
\[ |T| \|\Omega| \geq \sup_{r \in [1, \infty)} (1 - \varepsilon_T - \varepsilon_\Omega) \left( \frac{r}{r - 1} \right)^{2d(r-1)}. \quad (10) \]

**Proof.** Writing the operators $L_j, j = 1, 2$, defined in (7) and (8) as Weyl operators we have:
\[
L_1 f = \mathcal{W}^{F_1} f, \quad \text{with} \quad F_1(x,\omega) = (\chi_T(x) \otimes 1_\omega) * \operatorname{Wig}(\Phi_1)(x,\omega)
\]
\[
L_2 f = \mathcal{W}^{F_2} f, \quad \text{with} \quad F_2(x,\omega) = (1_x \otimes \chi_\Omega(\omega)) * \operatorname{Wig}(\Phi_2)(x,\omega).
\]

An explicit calculation yields:
\[
\operatorname{Wig}(\Phi_j)(x,\omega) = c_j^2 \left( \frac{2}{\lambda_j} \right)^{d/2} e^{-2\pi \lambda_j x^2} e^{-\frac{\pi}{\lambda_j} \omega^2}, \quad j = 1, 2,
\]
therefore we have
\[
F_1(x,\omega) = c_1^2 \left( \frac{2}{\lambda_1} \right)^{d/2} e^{-2\pi \lambda_1 t^2} \chi_T(x - t) \left( \int e^{-\frac{\pi}{\lambda_1} s^2} ds \right)
\]
\[
= c_1^2 \int \chi_T(x - t) e^{-2\pi \lambda_1 t^2} dt
\]
\[
= c_1^2 \left( \chi_T * e^{-2\pi \lambda_1 (\cdot)^2} \right)(x)
\]
which shows that $F_1$ depends only on the time variable $x$. In a similar way we can prove that $F_2$ depends just on $\omega$. Precisely we have:
\[
F_2(x,\omega) = c_2^2 \left( \frac{2}{\lambda_2} \right)^{d/2} e^{-2\pi \lambda_2 s^2} \left( \int \chi_\Omega(\omega - s) e^{-\frac{\pi}{\lambda_2} s^2} ds \right)
\]
\[
= c_2^2 \left( \frac{2}{\lambda_2} \right)^{d/2} (2\lambda_2)^{-d/2} \left( \chi_\Omega * e^{-\frac{\pi}{\lambda_2} (\cdot)^2} \right)(\omega)
\]
\[
= c_2^2 \lambda_2^{-d} \left( \chi_\Omega * e^{-\frac{\pi}{\lambda_2} (\cdot)^2} \right)(\omega).
\]
It follows that
\[ L_1 f = W^{F_1} f = F_1 f, \]

i.e. \( L_1 \) is the multiplication operator by the function \( F_1 \) and
\[ L_2 f = W^{F_2} f = \mathcal{F}^{-1} F_2 \mathcal{F} f, \]
i.e. \( L_2 \) is the Fourier multiplier with symbol \( F_2 \). Now, for \( j = 1, 2 \), we compute
\[ \| f \|_2^2 = \| (f - L_j f) + L_j f \|_2^2 \]
\[ = \| f - L_j f \|_2^2 + \| L_j f \|_2^2 + (f - L_j f, L_j f) + (L_j f, f - L_j f) \]  
\[ = \| f - L_j f \|_2^2 + \| L_j f \|_2^2 + (f - L_j f, L_j f) + (L_j f, f - L_j f) \]  
\[ = \| f - L_j f \|_2^2 + \| L_j f \|_2^2 + (f - L_j f, L_j f) + (L_j f, f - L_j f) \]  
\[ \geq 0 \]  
(11)
Next we show that \( (f - L_j f, L_j f) \geq 0 \) if \( \Phi_j \) are normalized in \( L^2 \). For \( j = 1 \) we have
\[ (f - L_1 f, L_1 f) = (f, L_1 f) - (L_1 f, L_1 f) \]
\[ = \int \hat{f} \overline{F_1 f} - \int F_1 f \overline{\hat{f} f} \]
\[ = \int (1 - F_1 \overline{F_1 f}) |\hat{f}|^2 \geq 0, \]
as \( F_1 \) is real, non negative, and \( \| F_1 \|_\infty \leq 1 \); actually
\[ \| F_1 \|_\infty = c_1^2 \| \chi_T * e^{-2\pi \lambda_1 t^2} \|_\infty \]
\[ \leq c_1^2 \| \chi_T \|_\infty \| e^{-2\pi \lambda_1 t^2} \|_1 \]
\[ = c_1^2 (2\lambda_1)^{-d/2} \]
\[ = 1, \]
recalling that \( c_1 = (2\lambda_1)^{d/4} = \| \phi_1 \|_2^{-1} \).
Analogously, if \( j = 2 \) we have
\[ (f - L_2 f, L_2 f) = (f, L_2 f) - (L_2 f, L_2 f) \]
\[ = (f, \mathcal{F}^{-1} F_2 \mathcal{F} f) - (\mathcal{F}^{-1} F_2 \mathcal{F} f, \mathcal{F}^{-1} F_2 \mathcal{F} f) \]
\[ = (\hat{f}, F_2 \hat{f}) - (F_2 \hat{f}, F_2 \hat{f}) \]
\[ = \int \hat{f} F_2 f - \int F_2 f \overline{\hat{f} f} \]
\[ = \int (1 - F_2 \overline{F_2 f}) |\hat{f}|^2 \geq 0, \]
as $F_2$ is real, non negative, and $\|F_2\|_\infty \leq 1$, the last inequality following from

$$
\|F_2\|_\infty = c_2^2 \lambda_2^{-d} \|\chi_\Omega \ast e^{-\pi \frac{d}{\lambda_2^2}}\|_\infty
\leq c_2^2 \lambda_2^{-d} \|\chi_\Omega\|_\infty \|e^{-\pi \frac{d}{\lambda_2^2}}\|_1
= c_2^2 \lambda_2^{-d} \left(\frac{2}{\lambda_2}\right)^{-d/2}
= 1,
$$
as $c_2 = (2\lambda_2)^{d/4} = \|\phi_2\|_2^{-1}$.

Now, from (11), since $(f - L_jf, L_jf) \geq 0$, it follows

$$
\|f\|^2 = \|f - L_jf\|^2 + \|L_jf\|^2 + 2(f - L_jf, L_jf)
$$

and hence

$$
\|f - L_jf\|^2 \leq \|f\|^2 - \|L_jf\|^2. \quad (12)
$$

From the hypothesis and (12) we obtain

$$
\left\{\begin{array}{l}
\|f - L_1f\|^2 \leq \|f\|^2 - \|f - L_2L_1f\|^2,
\|f - L_2f\|^2 \leq \|f\|^2 - (\varepsilon_1 + \varepsilon_T)\|f\|^2.
\end{array}\right.
$$

Considering the composition of $L_1$ and $L_2$ we have

$$
\|f - L_2L_1f\|_2 \leq \|f - L_2f\|_2 + \|L_2f - L_2L_1f\|_2
\leq \varepsilon_\Omega \|f\|_2 + \|L_2\| \|f - L_1f\|_2
\leq \varepsilon_\Omega \|f\|_2 + 1 \cdot \varepsilon_T \|f\|_2
= (\varepsilon_\Omega + \varepsilon_T)\|f\|_2,
$$

where Lemma 4 has been used with $q = \infty$ in the estimation of the operator norm $\|L_2\|_{L^2(L^2)} \leq \|\Phi_2\|_2^2 \|\chi_\Omega\|_\infty = 1$. Then

$$
\|L_1L_2f\|_2 \geq \|f\|_2 - \|f - L_2L_1f\|_2
\geq \|f\|_2 - (\varepsilon_\Omega + \varepsilon_T)\|f\|_2
= (1 - \varepsilon_T - \varepsilon_\Omega)\|f\|_2,
$$
and, from this, it follows that for every $r \in [1, \infty)$

\[
1 - \varepsilon - \varepsilon_T \leq \frac{\|L_1 L_2 f\|_2}{\|f\|_2} \\
\leq \|L_1 L_2\| \\
\leq \|L_1\| \|L_2\| \\
\leq \|\chi_T\|_r \|\chi_\Omega\|_r \left(\frac{1}{r'}\right)^{2d/r'} \|\Phi_1\|_2^2 \|\Phi_2\|_2^2
\]

\[
= \left(\int_T dt\right)^{1/r} \left(\int_\Omega ds\right)^{1/r} \left(\frac{1}{r'}\right)^{2d/r'},
\]

where we have used again Lemma 4 with $q = r < +\infty$ in order to have norms involving the measures of the sets $T$ and $\Omega$. Hence, we finally have that

\[
|T| |\Omega| \geq \sup_{r \in [1, \infty)} (1 - \varepsilon_T - \varepsilon_\Omega)^r (r')^{2d/r},
\]

which proves the thesis.

**Remark 7.** From (10) we have in particular that

1. For $r \to 1^+$, then $|T| |\Omega| \geq 1 - \varepsilon_T - \varepsilon_\Omega$;

2. For $r = 2$, then $|T| |\Omega| \geq (1 - \varepsilon_T - \varepsilon_\Omega)^2 4^d$;

3. One can prove that for any fixed value of the parameter $1 - \varepsilon_T - \varepsilon_\Omega \in [0, 1)$ the supremum over $r \in [1, \infty)$ in the right-hand side of (10) is actually a maximum. For this maximum no explicit expression is available but a study of the function $f(r) = (1 - \varepsilon_T - \varepsilon_\Omega)^r (r')^{2d}$ can yield an approximation in dependence on $1 - \varepsilon_T - \varepsilon_\Omega$ which improves estimates (1) and (2).

4. Remark that in the case where the inequalities (9) of the hypothesis are strict, the same proof yields a strict estimate in the thesis (10).

We remark that whereas the case $\varepsilon_T = \varepsilon_\Omega = 0$ in the classical Donoho-Stark UP yields $|T| |\Omega| \geq 1$ (which is a trivial assertion since in this case either $|T| = +\infty$ or $|\Omega| = +\infty$, cf. [1]), the case $\varepsilon_T = \varepsilon_\Omega = 0$ in Theorem 6 is not trivial and actually yields the following result.

**Corollary 8.** Let $T$, $\Omega$, $L_1$, $L_2$ be as in Theorem 6 and suppose that there exists $f \in L^2(\mathbb{R}^d)$ such that

\[
\|L_1 f\|_2^2 = \|f\|_2^2 \quad \text{and} \quad \|L_2 f\|_2^2 = \|f\|_2^2
\]
then $\|T\|\Omega \geq e^{2d}$. 

**Proof.** Observe that for $\varepsilon_T = \varepsilon_{\Omega} = 0$ the hypotheses of Theorem 6 become $\|L_1 f\|_2 = \|L_2 f\|_2 = \|f\|_2$, since from Lemma 4 we have $\|L_1\|_{B(L^2)} = \|L_2\|_{B(L^2)} = 1$. Then the assertion is proved by taking $\varepsilon_T = \varepsilon_{\Omega} = 0$ in Theorem 6 and remarking that $\sup_{r \in [1, \infty)} \left( \frac{r}{r-1} \right)^{r-1} = \lim_{r \to +\infty} \left( \frac{r}{r-1} \right)^{r-1} = e$. \hfill \Box

Another consequence of Theorem 6 is an UP involving the marginal distributions of the spectrogram. We recall that the spectrogram is the time-frequency representation given by $\text{Sp}_\psi(f, g)(x, \omega) = V_{\psi} f(x, \omega) \overline{V_{\psi} g(x, \omega)}$, defined in terms of a Gabor transform with window $\psi \in L^2(\mathbb{R}^d)$. It is an important and widely used tool in signal analysis as well as in connection with the theory of pseudo-differential operators, see e.g. [5], [7], [8], [10], [14], [25]. We denote its marginal distributions with $\text{Sp}_{\psi}^{(1)}(f, g)(x) = \int_{\mathbb{R}^d} \text{Sp}_\psi(f, g)(x, \omega) \, d\omega$ and $\text{Sp}_\psi^{(2)}(f, g)(\omega) = \int_{\mathbb{R}^d} \text{Sp}_\psi(f, g)(x, \omega) \, dx$.

**Corollary 9.** Suppose that $f, g$ are functions in $L^2(\mathbb{R}^d)$ for which $\|f\|_2 = \|g\|_2 = 1$ and

$$\left| \int_{T} \text{Sp}_\psi^{(1)}(f, g)(x) \, dx \right| \geq \sqrt{1 - \varepsilon_T^2};$$
$$\left| \int_{\Omega} \text{Sp}_\psi^{(2)}(f, g)(\omega) \, d\omega \right| \geq \sqrt{1 - \varepsilon_{\Omega}^2}.$$

Then

$$\|T\|\Omega \geq \sup_{r \in [1, \infty)} (1 - \varepsilon_T - \varepsilon_{\Omega})^r \left( \frac{r}{r-1} \right)^{2d(r-1)}.$$ 

**Proof.** Using the fundamental connection between localization operators and spectrogram $(L_\psi^a f, g)_{L^2(\mathbb{R}^d)} = (a, \text{Sp}_\psi(g, f))_{L^2(\mathbb{R}^d)}$, which is a consequence of [13] and [5], we can rewrite the hypothesis $\|L_1 f\|_2 \geq (1 - \varepsilon_T^2)\|f\|_2$ of Theorem 6 as

$$\sqrt{1 - \varepsilon_T^2} \leq \sup_{\|g\|=1} |(L_1 f, g)| = \sup_{\|g\|=1} |(\chi_T, \text{Sp}_\Phi_1(g, f))| = \sup_{\|g\|=1} \left| \int_{T \times \mathbb{R}^d} \text{Sp}_\Phi_1(g, f) \, dx \, d\omega \right| = \sup_{\|g\|=1} \left| \int_{T} \text{Sp}_\Phi_1^{(1)}(f, g) \, dx \right| \tag{13}$$

In analogous way the hypothesis $\|L_2 f\|_2 \geq (1 - \varepsilon_{\Omega}^2)\|f\|_2$ reads

$$\sqrt{1 - \varepsilon_{\Omega}^2} \leq \sup_{\|g\|=1} \left| \int_{\Omega} \text{Sp}_\Phi_2^{(2)}(f, g) \, d\omega \right|. \tag{14}$$

In particular (13) and (14) are satisfied in our hypothesis and therefore the thesis follows from Theorem 6. \hfill \Box
3 Comparison with Donoho-Stark

This section is dedicated to the classical version of the Donoho-Stark theorem. We use the results of the previous section to prove a substantial improvement in constant $(1 - \varepsilon_T - \varepsilon_\Omega)^2$ appearing on the right-hand side of estimate (1). Our result is the following.

**Proposition 10.** Let $f \in L^2(\mathbb{R}^d)$, $T, \Omega \subset \mathbb{R}^d$, $\varepsilon_\Omega, \varepsilon_T > 0$ satisfy the hypotheses of Theorem 2 (Donoho-Stark), then

$$|T||\Omega| \geq \sup_{r \in [1, \infty)} (1 - \varepsilon_T - \varepsilon_\Omega)^r \left(\frac{r}{r - 1}\right)^{2d(r-1)},$$

and in particular

$$|T||\Omega| \geq (1 - \varepsilon_T - \varepsilon_\Omega)^2 4^d$$

**Proof.** Let $Pf = \chi_T f$ and $Qf = \mathcal{F}^{-1} \chi_\Omega \mathcal{F} f$, then the hypotheses of the Donoho-Stark UP (Thm. 2) can be rewritten as

$$\|Pf\|_2^2 \geq (1 - \varepsilon_T^2)\|f\|_2^2 \quad \text{and} \quad \|Qf\|_2^2 \geq (1 - \varepsilon_\Omega^2)\|f\|_2^2.$$ 

From the condition $\varepsilon_T + \varepsilon_\Omega < 1$ we can choose $\nu_T > \varepsilon_T$, $\nu_\Omega > \varepsilon_\Omega$, also satisfying $\nu_T + \nu_\Omega < 1$. For $\nu_T, \nu_\Omega$ the strict inequalities hold:

$$\|Pf\|_2^2 > (1 - \nu_T^2)\|f\|_2^2 \quad \text{and} \quad \|Qf\|_2^2 > (1 - \nu_\Omega^2)\|f\|_2^2.$$ (17)

Let us consider the operators $L_1$ and $L_2$ as defined in (17) and (18) respectively. We recall that $L_j = W^j$, $j = 1, 2$, as Weyl operators with $F_1 = c_1^2(\chi_T \ast e^{-\pi 2\lambda_1 x^2})(x)$ and $F_2 = c_2^2 \lambda_2 \lambda_2 \ast \delta(\chi_\Omega \ast e^{-\pi \frac{\lambda_2}{\lambda_2}} \rho)(\omega)$. Setting now $\varphi_\lambda(x) = \lambda^d/2 e^{-\pi \lambda x^2}$, we have $F_1 = \chi_T \ast \varphi_{2\lambda_1}$ and $F_2 = \chi_\Omega \ast \varphi_{2\lambda_2}$. Notice that $\|\varphi_{2\lambda_1}\|_1 = \|\varphi_{\frac{\lambda_2}{\lambda_2}}\|_1 = 1$ and that $\varphi_{2\lambda_1} \rightarrow \delta$ for $\lambda_1 \rightarrow +\infty$, and $\varphi_{\frac{\lambda_2}{\lambda_2}} \rightarrow \delta$ for $\lambda_2 \rightarrow 0^+$ in $S'(\mathbb{R}^d)$, so that $\{\varphi_{2\lambda_1}\}_{\lambda_1 \in \mathbb{R}}$ and $\{\varphi_{\frac{\lambda_2}{\lambda_2}}\}_{\lambda_2 \in \mathbb{R}}$ are approximate identities.

We prove now that if a function $f$ is suitably regular then:

(a) $\|(\chi_T \ast \varphi_{2\lambda_1}) f - \chi_T f\|_2 \rightarrow 0$, i.e. $L_1 f \rightarrow Pf$ in $L^2(\mathbb{R}^d)$, as $\lambda_1 \rightarrow +\infty$.

(b) $\|\mathcal{F}^{-1}[(\chi_\Omega \ast \varphi_{\frac{\lambda_2}{\lambda_2}}) \hat{f}] - \mathcal{F}^{-1}[\chi_\Omega \hat{f}]\|_2 \rightarrow 0$, i.e. $L_2 f \rightarrow Qf$ in $L^2(\mathbb{R}^d)$, as $\lambda_2 \rightarrow 0^+$. 

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Let us consider \((a)\):

\[
\| (\chi f * \varphi_{2\lambda_1}) f - \chi f \|_2 = \| (\chi f * \varphi_{2\lambda_1} - \chi f) f \|_2 \\
\leq \| \chi f * \varphi_{2\lambda_1} - \chi f \|_2 \| f \|_p
\]

for all \(p \in [1, \infty)\). From the properties of approximate identities the first norm in the last line goes to 0 as \(\lambda_1 \to \infty\), if \(p' < \infty\) and the second term is constant if \(f \in L^{2p}(\mathbb{R}^d)\). Therefore \((a)\) is valid for all \(f\) for which there exists \(p > 1\) such that \(f \in L^{2p}(\mathbb{R}^d)\). In particular, this is true for all functions in \(S(\mathbb{R}^d)\). In a similar way \((b)\) can be proven, indeed we have:

\[
\| \mathcal{F}^{-1} \left( (\chi \Omega * \varphi_{2/\lambda_2}) \hat{f} - \chi \Omega \hat{f} \right) \|_2 = \| \left( (\chi \Omega * \varphi_{2/\lambda_2}) - \chi \Omega \right) \hat{f} \|_2 \\
\leq \| \chi \Omega * \varphi_{2/\lambda_2} - \chi\Omega \|_{2p'} \| \hat{f} \|_{2p}
\]

where for \(p' < \infty\) the first norm in the last line goes to 0 as \(\lambda_2 \to 0^+\), and the second is constant if for instance \(f \in S(\mathbb{R}^d)\).

Suppose now that the function \(f \in L^2(\mathbb{R}^d)\) satisfies the Donoho-Stark hypotheses and let \(f_n \in S(\mathbb{R}^d)\) be such that \(f_n \to f\) in \(L^2(\mathbb{R}^d)\). Then \(P f_n \to P f\) in \(L^2(\mathbb{R}^d)\) and, therefore, \(\| Pf_n \|_2 \to \| Pf \|_2\). From the first inequality in \((17)\), \((1 - \nu_2^2)^{1/2} < \| Pf \|_2 / \| f \|_2\), and therefore there exists \(n_1\) such that for all \(n > n_1\) we have

\[
(1 - \nu_2^2)^{1/2} < \frac{\| Pf_n \|_2}{\| f_n \|_2}. \tag{18}
\]

On the other hand \(Q f_n \to Q f\) in \(L^2(\mathbb{R}^d)\) and hence \(\| Q f_n \|_2 \to \| Q f \|_2\). Similarly, by the second inequality in \((17)\), \((1 - \nu_2^2)^{1/2} < \| Q f \|_2 / \| f \|_2\), and it follows that there exists \(n_2\) such that \(\forall n > n_2\)

\[
(1 - \nu_2^2)^{1/2} < \frac{\| Q f_n \|_2}{\| f_n \|_2}. \tag{19}
\]

For \(n > \max\{n_1, n_2\}\) both \((18)\) and \((19)\) hold, i.e. the hypotheses of Donoho-Stark hold therefore on \(f_n\). As \(f_n \in S(\mathbb{R}^d)\), it follows that

\[
L_1 f_n \to Pf_n, \quad \text{as} \quad \lambda_1 \to +\infty
\]

and

\[
L_2 f_n \to Qf_n \quad \text{as} \quad \lambda_2 \to 0^+
\]

in \(L^2(\mathbb{R}^d)\). Then for \(\lambda_1\) sufficiently large and \(\lambda_2\) sufficiently small, from \((18)\) and \((19)\) we have:

\[
(1 - \nu_2^2)^{1/2} < \frac{\| L_1 f_n \|_2}{\| f_n \|_2} \quad \text{and} \quad (1 - \nu_2^2)^{1/2} < \frac{\| L_2 f_n \|_2}{\| f_n \|_2},
\]

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i.e. \( f_n \) satisfies the hypotheses of Theorem \([6]\) and the thesis follows from \([10]\) with \( \nu_T, \nu_\Omega \) in place of \( \varepsilon_T, \varepsilon_\Omega \) respectively, i.e.

\[
|T||\Omega| \geq \sup_{r \in [1,\infty)} (1 - \nu_T - \nu_\Omega)^r \left( \frac{r}{r-1} \right)^{2d(r-1)}.
\]  

(20)

Finally the thesis follows taking in \([20]\) the supremum over all \( \nu_T > \varepsilon_T \) and \( \nu_\Omega > \varepsilon_\Omega \).

4 Donoho-Stark Uncertainty principle and local uncertainty principle

The Donoho-Stark UP states that there are restrictions to the behavior of a function and its Fourier transform, from a local viewpoint. There are other results in this direction in the literature, see e.g. \([10]\); here we want to consider the local UP of Price, cf. \([22]\), and investigate some consequences as well as the connections between these two UPs. More precisely, under the hypotheses of Donoho and Stark we prove a different estimate of \(|T||\Omega|\), with a constant depending on the function \( f \). Moreover, we obtain a new UP involving the measure of the support of a function and the standard deviation of its Fourier transform.

We start by recalling the result of Price.

**Theorem 11** (Price \([22, \text{Theorem 1.1}]\)). Let \( \Omega \subset \mathbb{R}^d \) be a measurable set and \( \alpha > d/2 \). Then for every \( f \in L^2(\mathbb{R}^d) \) we have

\[
\int_\Omega |\hat{f}(\omega)|^2 d\omega < K_1 |\Omega| \| f \|_2^{2-d/\alpha} \| \hat{f} \|_2^{d/\alpha},
\]

(21)

where \( K_1 \) is a constant depending on \( d \) and \( \alpha \), given by

\[
K_1 = K_1(d, \alpha)
\]

\[
= \frac{\pi^{d/2}}{\alpha} \left( \Gamma \left( \frac{d}{2} \right) \right)^{-1} \Gamma \left( \frac{d}{2\alpha} \right) \Gamma \left( 1 - \frac{d}{2\alpha} \right) \left( \frac{2\alpha}{d} - 1 \right)^{\frac{d}{2\alpha}} \left( 1 - \frac{d}{2\alpha} \right)^{-1}
\]

(22)

and \( \Gamma \) is the Gamma function defined as \( \Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt \). Moreover, the constant \( K_1 \) is optimal, and equality in \( (21) \) is never attained.
At first, we observe that Theorem 11, stated in $L^2$ spaces in [22], can be easily generalized; in fact, it is proved in [22, Corollary 2.2] that for every $q \in (1, \infty]$, $\alpha > \frac{d}{q'}$, and $f \in L^q(\mathbb{R}^d)$, we have

$$\|\hat{f}\|_{\infty} \leq \tilde{K} \|f\|_q^{1-\frac{d}{\alpha q'}} \|t|^\alpha \|_{q'}^{\frac{d}{\alpha q}},$$  \hspace{1cm} (23)$$

where

$$\tilde{K} = \left[ \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{\alpha q} B \left( \frac{d}{\alpha q}, \frac{1}{q} - 1 - \frac{d}{\alpha q} \right) \right]^{\frac{q-1}{q}} \left( \frac{\alpha q'}{d} - 1 \right)^{\frac{d}{\alpha q'}} \left( 1 - \frac{d}{\alpha q} \right)^{-1/q}$$

and $B(\cdot, \cdot)$ is the Beta function, given by $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$. Then, by the same proof of [22, Theorem 1.1] we get the following result.

**Theorem 12.** Let $\Omega \subset \mathbb{R}^d$ be a measurable set, $q \in (1, \infty]$ and $\alpha > \frac{d}{q'}$. Then for every $f \in L^q(\mathbb{R}^d)$ we have

$$\int_{\Omega} |\hat{f}(\omega)|^2 d\omega \leq K(d, \alpha, q)|\Omega| \|f\|_q^{2-\frac{2d}{\alpha q'}} \|t|^\alpha \|_{q'}^{2d/\alpha q'},$$  \hspace{1cm} (25)$$

where $K(d, \alpha, q) = \tilde{K}^2$, and $\tilde{K}$ is given by (24).

**Proof.** We only have to prove the statement when the right-hand side of (25) is finite; in this case we have $f \in L^1(\mathbb{R}^d)$, and so $\hat{f}$ is a continuous bounded function; we then have

$$\int_{\Omega} |\hat{f}(\omega)|^2 d\omega \leq |\Omega| \|\hat{f}\|^2_{\infty},$$

and the conclusion is an application of (24). \hfill \Box

We can formulate our new version of the Donoho-Stark UP with constant depending on the signal $f$.

**Theorem 13.** Let $\Omega$ and $T$ be two measurable subsets of $\mathbb{R}^d$, $q_j \in (1, \infty]$, $\alpha_j > \frac{d}{q_j'}$, $j = 1, 2$ and $f \in L^1(\mathbb{R}^d)$ such that $\hat{f} \in L^1(\mathbb{R}^d)$, $f \neq 0$. Suppose that $f$ is $\varepsilon_T$-concentrated on $T$, and $\hat{f}$ is $\varepsilon_{\Omega}$-concentrated on $\Omega$, with $0 \leq \varepsilon_T, \varepsilon_{\Omega} \leq 1$ and $\varepsilon_T + \varepsilon_{\Omega} \leq 1$. Then

$$|T| |\Omega| \geq C_f (1 - \varepsilon_T - \varepsilon_{\Omega})^2,$$  \hspace{1cm} (26)$$
where \( C_f \) is the supremum over \( \mathbb{T}, \mathbb{W} \in \mathbb{R}^d \), \( q_j \in (1, \infty] \) and \( \alpha_j > d/q_j \), \( j = 1, 2 \), of the quantities

\[
\| f \|^2 \| \hat{f} \|_{q_1}^{2d/(\alpha_1 q_1')} \| f \|_{q_2}^{2d/(\alpha_2 q_2')}
\]

\[
K(d, \alpha_1, q_1)K(d, \alpha_2, q_2)\| f \|_{q_2}^2 \| \hat{f} \|_{q_1}^2 \| t - \mathbb{T} \omega f \|_{q_2}^{2d/(\alpha_2 q_2')}\| \omega - \mathbb{W} \omega f \|_{q_1}^{2d/(\alpha_1 q_1')},
\]

and \( K(d, \alpha_j, q_j) \), \( j = 1, 2 \), are the ones appearing in (25).

**Proof.** We can limit our attention to \( f \) such that \( C_f > 0 \), otherwise the result is trivial. The hypothesis \( f, \hat{f} \in \mathbb{L}^1(\mathbb{R}^d) \) implies that \( f, \hat{f} \in \mathbb{L}^\infty(\mathbb{R}^d) \), and so \( f, \hat{f} \in \mathbb{L}^q(\mathbb{R}^d) \) for every \( q \in [1, \infty] \).

Now, writing (25) for a translation by \( \mathbb{T} \) of \( f \), we have that the left-hand side does not change, since the Fourier transform turns translations into modulations. Moreover, in the right-hand side the only term that is affected by the translation is the last norm, and so we get the following more general estimate:

\[
\int_\Omega |\hat{f}(\omega)|^2 \, d\omega \leq K(d, \alpha, q) |\Omega| \| f \|_{q}^{2 - 2d/\alpha q'} \| t - \mathbb{T} \omega f \|_{q}^{2d/\alpha q'}. \tag{27}
\]

By interchanging the roles of \( f \) and \( \hat{f} \) in (27), we get

\[
\int_T |f(t)|^2 \, dt \leq K(d, \alpha, q) |\mathbb{T}| \| \hat{f} \|_{q}^{2 - 2d/\alpha q'} \| \omega - \mathbb{W} \omega \hat{f} \|_{q}^{2d/\alpha q'}. \tag{28}
\]

Observe now that, by the definition of \( \varepsilon_T \)-concentration of \( f \) on \( T \), we have

\[
\int_T |f(t)|^2 \, dt = \| f \|_{2}^2 - \int_{\mathbb{R}^d \setminus T} |f(t)|^2 \, dt \geq (1 - \varepsilon_T^2) \| f \|_{2}^2, \tag{29}
\]

and analogously the hypothesis that \( \hat{f} \) is \( \varepsilon_\Omega \)-concentrated on \( \Omega \) can be rewritten as

\[
\int_\Omega |\hat{f}(\omega)|^2 \, d\omega \geq (1 - \varepsilon_\Omega^2) \| f \|_{2}^2. \tag{30}
\]

Combining (29) with (28) (with \( \alpha_1 \) and \( q_1 \) instead of \( \alpha \) and \( q \), respectively) and (30) with (27) (with \( \alpha_2 \) and \( q_2 \) instead of \( \alpha \) and \( q \), respectively), we obtain

\[
|T| \geq (1 - \varepsilon_T^2) \frac{\| f \|_{2}^2 \| \hat{f} \|_{q_1}^{2d/(\alpha_1 q_1')}}{K(d, \alpha_1, q_1) \| \hat{f} \|_{q_1}^{2} \| \omega - \mathbb{W} \omega \hat{f} \|_{q_1}^{2d/(\alpha_1 q_1')}}, \tag{31}
\]

\[
|\Omega| \geq (1 - \varepsilon_\Omega^2) \frac{\| f \|_{2}^2 \| \hat{f} \|_{q_2}^{2d/(\alpha_2 q_2')}}{K(d, \alpha_2, q_2) \| \hat{f} \|_{q_2}^{2} \| t - \mathbb{T} \omega f \|_{q_2}^{2d/(\alpha_2 q_2')}}. \tag{32}
\]
Then, multiplying these last inequalities we get
\[ |T| |\Omega| \geq (1 - \varepsilon_T^2)(1 - \varepsilon_\Omega^2) \]
\[ \cdot \frac{\|f\|_q^2 \|\hat{f}\|_{q_1}^{2d/(\alpha_1 q'_1)} \|\hat{f}\|_{q_2}^{2d/(\alpha_2 q'_2)}}{K(d, \alpha_1, q_1)K(d, \alpha_2, q_2)\|f\|_{q_2}^2 \|\hat{f}\|_{q_1}^2 \|t - \bar{T}\|_{q_2}^{2d/(\alpha_2 q'_2)} \|\omega - \bar{\omega}\|_{\alpha_1}^{2d/(\alpha_1 q'_1)}}. \]
\[ (33) \]

Observe that, since \(0 \leq \varepsilon_T, \varepsilon_\Omega \leq 1\) and \(\varepsilon_T + \varepsilon_\Omega \leq 1\), we have \((1 - \varepsilon_T^2)(1 - \varepsilon_\Omega^2) \geq (1 - \varepsilon_T - \varepsilon_\Omega)^2\). Then the conclusion follows from (33), by taking the supremum over \(T, \Omega, \alpha_1, \alpha_2, q_1\) and \(q_2\) in the right-hand side.

We compare now the result of Theorem 13 with the classical formulation of the Donoho-Stark UP.

**Remark 14.** We observe at first that the statement of Theorem 13 is given in a parallel way to the classical one, but in the proof we have proved a stronger result. In fact, the local UP by Price gives estimates separately on the amount of energy of \(f\) and \(\hat{f}\) in \(T\) and \(\Omega\), respectively. So, under the hypotheses of Theorem 13 we have deduced (31) and (32), that contain lower bounds for the measures of \(T\) and \(\Omega\) separately. This gives more information than a lower bound of the product between them.

In order to make a further comparison between (26) and Donoho-Stark UP, we observe that in (26) we have a constant depending on the function \(f\). A very natural question is if for some \(f\) the result of Theorem 13 is stronger than Proposition 10 (and then stronger than the classical Donoho-Stark UP), or even stronger than any other estimate of the kind of (26) with a constant that does not depend on \(f\) (for example, this would happen if we find a sequence \(f_n\) for which the corresponding \(C_{f_n}\) tends to infinity). This seems a not trivial question, and is postponed to a further study; however, since Theorem 13 is based on the local estimates of Price, and such estimates are optimal, our feeling is that Theorem 13 can give better estimates for some functions \(f\).

As particular case of Theorem 13 if \(d = 1, q_1 = q_2 = 2, \alpha_1 = \alpha_2 = 1\) and \(f\) satisfies the hypotheses of Theorem 13 we have the inequality
\[ \Delta f \Delta \hat{f} \geq \frac{(1 - \varepsilon_T^2)(1 - \varepsilon_\Omega^2)}{4\pi^2 |T| |\Omega|} \|f\|_2^2 \]
\[ (34) \]
with \(\Delta f = \left( \int |t|^2 |f(t)|^2 \, dt \right)^{1/2}\) and analogous definition for \(\hat{f}\). This can be viewed as an \(\varepsilon\)-concentration version of the classical Heisenberg UP which
states that
\[ \Delta f \Delta \hat{f} \geq \frac{\|f\|_2^2}{4\pi} \]  
for every \( f \in L^2(\mathbb{R}) \). Inequality (35) is clearly of any interest only for the cases where the bound on its right-hand side exceeds the one of the classical Heisenberg UP (35). This happens if \( |T||\Omega| \leq \frac{1}{\pi}(1 - \varepsilon_T^2)(1 - \varepsilon_\Omega^2) \); On the other hand, from the improved lower bound (16) of the Donoho-Stark UP, we know that \( 4(1 - \varepsilon_T - \varepsilon_\Omega)^2 \leq |T||\Omega| \). An elementary calculation shows that actually there exists values \( \varepsilon_T, \varepsilon_\Omega \) and sets \( T, \Omega \) compatible with both conditions.

We end this section by presenting another consequence of the local UP of Price, in the form of a “mixed” UP that relates the measure of the support of a function with the concentration of its Fourier transform.

**Theorem 15.** Let \( f \in L^2(\mathbb{R}^d) \), \( \alpha > d/2 \), and \( \tau, \omega \in \mathbb{R}^d \). We have
\[ |\text{supp } f| \|\omega - \omega|^{\alpha} \hat{f}|_2^{d/\alpha} > \frac{1}{K} \|f\|_2^{d/\alpha}, \]  
and
\[ |\text{supp } \hat{f}| \|\tau - \tau|^{\alpha} f|_2^{d/\alpha} > \frac{1}{K} \|f\|_2^{d/\alpha}, \]  
where \( K = K(d, \alpha, 2) \).

**Proof.** The estimates are not trivial only for functions \( f \) such that one between \( f \) and \( \hat{f} \) has support with finite measure. Suppose that \( |\text{supp } f| \) is finite. By (28), with \( q = 2 \) and \( T = \text{supp } f \), we obtain
\[ \|f\|_2^2 = \int_{\text{supp } f} |f(t)|^2 dt \leq K |\text{supp } f| \|f\|_2^{2-d/\alpha} \|\omega - \omega|^{\alpha} \hat{f}|_2^{d/\alpha}, \]  
that is (36). The inequality (37) can be proved in the same way, by using (27) with \( \Omega = \text{supp } \hat{f} \) and \( q = 2 \).

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