Phantom-like dark energy from quantum gravity

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Abstract. We analyse the emergent cosmological dynamics corresponding to the mean field hydrodynamics of quantum gravity condensates, in the group field theory formalism. We focus in particular on the cosmological effects of fundamental interactions, and on the contributions from different quantum geometric modes. The general consequence of such interactions is to produce an accelerated expansion of the universe, which can happen both at early times, after the quantum bounce predicted by the model, and at late times. Our main result is that, while this fails to give a compelling inflationary scenario in the early universe, it produces naturally a phantom-like dark energy dynamics at late times, compatible with cosmological observations. By recasting the emergent cosmological dynamics in terms of an effective equation of state, we show that it can generically cross the phantom divide, purely out of quantum gravity effects without the need of any additional phantom matter. Furthermore, we show that the dynamics avoids any Big Rip singularity, approaching instead a de Sitter universe asymptotically.

Keywords: dark energy theory, quantum cosmology, quantum gravity phenomenology

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1 Introduction

A problem faced by many quantum gravity approaches is to recover the usual description of the universe in terms of a smooth spacetime and fields living on it, and their dynamics governed by (a possibly modified version of) General Relativity and effective quantum field theory. The task is simpler in approaches that in fact start from the same mathematical structures of effective field theory, like asymptotic safety, or some (even radical) generalization of them, like string theory, which can still make use to a large extent of the usual intuition and tools of spacetime physics (however, they may have then a harder time providing a precise description of the fundamental degrees of freedom underlying spacetime itself). Approaches trying to recover spacetime starting from more abstract, non-spatiotemporal entities find here, instead, a difficult challenge, which is harder the more distant their candidate fundamental entities are from usual spacetime-based fields. The set of such challenges is often referred to as the issue of the emergence of spacetime in quantum gravity [1].
A generalised quantum field theory formalism for candidate constituents of quantum spacetime, tensorial group field theories (TGFTs), belong to this second kind of quantum gravity approaches, with their fundamental interaction processes represented as simplicial complexes (of one dimension higher) [2–9]. In this respect, however, they have the advantage that, despite their fully background independent and non-spaciotemporal character, they can still rely on tools from quantum field theory (QFT) to investigate the emergence of spacetime from their quantum dynamics. This has been one important motivation in the study of renormalization group (RG) flows and critical behaviour of a large number of TGFT models [4, 6–8, 10, 11], having also in mind the way in which matrix models recover 2d continuum Liouville gravity. TGFTs can also rely, for solving the same issue, on the additional quantum geometric data labelling their fundamental quanta and enriching their quantum dynamics. Indeed, while this makes their quantum states and amplitudes more involved, it also provides a guideline for their spatiotemporal interpretation, and makes even the simplest types of approximation schemes geometrically rich enough to be interesting. From a phenomenological point of view, however, the “tensorial” nature of the field is not quite important and we will focus on a subclass of TGFTs, the group field theory (GFT).

GFT condensate cosmology [12–18] is a research programme aiming at the extraction of spacetime physics, in particular cosmology, from GFTs. It is based on the hypothesis that the emergent gravitational physics should be looked for in the hydrodynamic approximation of the full GFT quantum dynamics, and focuses in particular on condensate states, thus treating the universe as a peculiar quantum fluid, made out of the GFT quanta. A large number of recent analyses in this context have shown not only the general viability of this strategy, but also that physically interesting results can be obtained already in the mean field (or Gross-Pitaevskii) approximation. This is also the context of our present work.

Establishing a solid connection between fundamental quantum gravity formalisms and effective spacetime physics is the necessary ingredient to make them testable. This can happen by directly producing new testable predictions about modifications of established theories, like GR or the Standard Model. It can also happen by reproducing some of the existing phenomenological or otherwise simplified models incorporating hypothetical quantum gravity effects or specific features of existing fundamental formalisms (e.g. loop quantum cosmology [19, 20], whose dynamics can in fact be reproduced in a specific regime of GFT condensate cosmology). Most current quantum gravity phenomenology is of this type and it is thus waiting for a solid contact with fundamental quantum gravity formalisms. The same is true for existing semi-classical cosmological scenarios for the very early universe: inflationary models, bouncing or emergent universe scenarios. All of them, albeit to a different degree and in very different ways, rely on assumptions about the very early universe that only a more complete theory of quantum gravity can corroborate or replace.

Quantum gravity effects in cosmology, however, do not need to be confined to the very early universe. In particular, in any emergent spacetime context notions like separation of scales or locality, on which usual effective field theory reasoning is based, are by definition approximate, and one should rather expect that the whole spacetime structure and dynamics, including large scale features, could be discovered to be of direct quantum gravity origin.

One instance of such large scale cosmological issues that quantum gravity can be expected to resolve is dark energy [21] (or, closely related to it, the nature of the cosmological constant [22]). The nature of the observed cosmic acceleration, and the full characterization of its features, is a main theoretical challenge for modern cosmology. It can be tackled at a more phenomenological level, looking for the semi-classical field-theoretic model that best
fits cosmological observations, and indeed there exists a rich zoology of (at least partially) working models who do the job (we include in this category also modified gravity theories).

One such field-theoretic model is so-called phantom dark energy [23], based on a dynamical new component of the universe (‘phantom matter’) characterized by an equation of state \( w < -1 \) for a recent part of its history, before tending to the observed \( w \approx -1 \). This dark energy evolution, including such phantom crossing, is compatible with current observations and could even be favored by them, e.g. by the recent data on supernova [24–26].

However, a proper field theoretic modeling of phantom dark energy is challenging, if the phantom field is taken to be a fundamental component of the universe. In fact, the negative kinetic term needed to have \( w < -1 \) leads to vacuum instability, Lorentz violation or other pathology [23, 27]. Various solutions have been proposed, for example involving several scalar fields [28], but with no conclusive success. Another route to achieve phantom crossing without new fundamental scalar fields is to understand it as a consequence of modified gravity theories, rather than new matter, and this can be accomplished, for example, by suitable \( f(R) \) theories [29, 30]. For the current status of phantom dark energy, see [31].

An alternative route towards a resolution of the dark energy problem, and in particular for a top-down derivation of phantom dark energy, is to obtain it as an effective description of more fundamental quantum gravity dynamics. One can interpret in this way various attempts to model phantom dark energy in string-inspired scenarios that, although still semiclassical, incorporate features of string theory. For example, phantom-like dark energy can be obtained in braneworld models [32] and, even more in contact with the fundamental theory, in string gas cosmology [33] and in AdS/CFT scenarios [34]. Also, the late time acceleration of the universe can also be explained in asymptotically safe cosmology [35], with no need of dark energy or cosmological constants.

In this work we take this route as well, and show that phantom-like dark energy can be obtained naturally in GFT condensate cosmology. It arises as an effective description of the evolution of the universe at late times, in the hydrodynamic approximation of the fundamental quantum dynamics of spacetime constituents, without introducing any kind of special phantom matter, but purely as quantum gravity effect.

The presentation is organized as follows. We first set up the stage in section 2, by presenting a short review of the GFT formalism [2, 3] and of GFT condensate cosmology [14–16]. In section 3, we introduce the effective equation of state \( w \) whose dynamics is the central object of our analysis, summarize the main aspects of phantom dark energy, and recall the results of earlier work concerning the effect of GFT interactions in the emergent cosmological evolution in the single mode case. Then, we move on to our new results. In section 4, we consider the early universe dynamics right after the bounce, where the free part in the quantum gravity condensate dominates. We take all quantum geometric modes into account and show that the bounce is followed by a accelerated phase, but this phase is not long lasting in general. The role of GFT interactions is studied in section 5. We first consider the large volume behaviour of individual modes subject to one interaction term, in section 5.1. Then, in section 5.2, we study how the dynamics of two quantum geometric modes combine to determine the evolution of the universe volume, at late times. We show that the phantom divide can be naturally crossed. We then study the subsequent evolution and how a Big Rip singularity is avoided, leading instead to an asymptotically deSitter universe, in section 5.3. Finally, in section 5.4, we briefly consider also the case in which each quantum geometric mode is subject to two types of GFT interactions, showing how the late time acceleration phase can have an even richer dynamics, while maintaining the key features of phantom
crossing and deSitter asymptotics. Finally, in section 6 we will summarize our results and give a short outlook toward possible extensions of our work.

2 GFT condensate cosmology

In this section we present some basics of the GFT formalism and of quantum geometric models (GFTs) for 4d quantum gravity in particular, with a focus on the elements on which the extraction of cosmological dynamics is based. We only include the ingredients that are needed as immediate background of the work presented in this paper. For a more detailed introduction to GFT we refer to existing reviews [2, 3, 5, 36]. For the basics of GFT cosmology see instead the original work in [12, 13, 37, 38] and the reviews [14–16]. See also [17, 18] for the use of coherent peaked states for the study of relational observables and their dynamics, and for the discussion of their quantum fluctuations.

2.1 Group field theory formalism

GFTs are quantum field theories defined over several copies of a Lie group $G$, which replaces the usual spacetime manifold of standard field theories and does not have, to start with, any spatiotemporal interpretation. The lack of background spacetime leads to the problem of time, encountered in many quantum gravity theories as well, where there is no natural choice of clocks such that the evolution of universe can be tracked. A way out is to introduce an relational time using a free massless scalar field $\phi$, and recording the evolution of other quantities respect to it. With these considerations, the (usually complex) field in 4d quantum gravity models is a tensorial map

$$\varphi : G^{x4} \times \mathbb{R} \rightarrow \mathbb{C}, \varphi(g_v, \phi) = \varphi(g_1, \cdots, g_4, \phi),$$

where the rank of the tensor is chosen equal to the dimension of the spacetime one intends to reconstruct [2]. GFTs are understood, in fact, as field theory formulations of spacetime, more precisely of the kinematics and dynamics of its fundamental constituents, rather than on spacetime as it is the case for usual QFTs. The basic quanta of the theory can be depicted as combinatorial 3-simplices, i.e. tetrahedra, labelled by the group-theoretic data, which encode their quantum geometry (assumed to be spacelike). Quantum states and boundary data of such models will correspond to collections of such quanta. In the quantum geometric models proposed to date the relevant group manifold is $G = SL(2, \mathbb{C})$ or its rotation subgroup $SU(2)$ (for the Lorentzian signature), since the restrictions that the models impose on the group-theoretic data to ensure a proper geometric interpretation of the simplices allows (in most cases) to map the two formulations of their quantum geometry [39–42]. For details on the quantum geometric conditions, we refer to the cited literature. In the following we will take $G = SU(2)$.

Following the geometric restrictions, the field $\varphi(g_v; \phi)$ is required to be right invariant regards to the group arguments, $\varphi(g_v h, \phi) = \varphi(g_1 h, g_2 h, g_3 h, g_4 h, \phi) = \varphi(g_v, \phi), \forall h \in G$, and therefore $\varphi(g_v, \phi)$ can be projected into the complete and orthonormal basis of $L^2(G^{x4}/G)$, which is given in terms of $SU(2)$ Wigner representation functions contracted by group intertwiners; these are called spin network vertex functions. Using the Peter-Weyl decomposition, the second quantized field operator can be written as

$$\hat{\varphi}(g_v, \phi) = \sum_\vec{\lambda} \hat{c}_{\vec{\lambda}}(\phi) \nu_{\vec{\lambda}}(g_v), \quad \hat{\varphi}^\dagger(g_v, \phi) = \sum_\vec{\lambda} \hat{c}_{\vec{\lambda}}^\dagger(\phi) \nu_{\vec{\lambda}}(g_v), \quad (2.1)$$

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with $\hat{c}_x(\phi)$ and $\hat{c}^\dagger_x(\phi)$ are annihilation and creation operator respectively, satisfy the commutation relation
\[
[\hat{c}_x(\phi), \hat{c}^\dagger_x(\phi')] = \delta_{x,x'}\delta(\phi' - \phi), \quad [\hat{c}_x(\phi), \hat{c}_x(\phi')] = [\hat{c}^\dagger_x(\phi), \hat{c}^\dagger_x(\phi')] = 0. \tag{2.2}
\]

The basis $\kappa_x(g_v)$ satisfies the orthonormal relation under the normalized Haar measure $\int_C dg = 1$, i.e.,
\[
\int d^4 g \kappa_x(g_v) \kappa_x'(g_v) \equiv \int \prod_i dg_i \kappa_x(g_v) \kappa_x(g_v) = \delta_{x,x'}, \tag{2.3}
\]
and can be associated graphically to a spin network 4-vertex labelled by $\vec{x} = (\vec{j}, \vec{m}, \iota)$, i.e. a node with $d = 4$ open links associated with 4 spins $\vec{j} = (j_1, j_2, j_3, j_4)$, together with angular momentum projections $\vec{m}$, and the intertwiner quantum number $\iota$ associated instead to the node itself [43]. Geometrically, one can think the spin network vertex sitting inside the tetrahedron with the 4 links emanated from the node crossing its 4 triangular faces. Following the quantization of simplicial geometry for the tetrahedron (whose results are also consonant to the results obtained in the continuum canonical Loop Quantum Gravity context), such spin network states are eigenstates of relevant geometric operators, with the spin labels $j_i$ determining the areas of the four faces, while the intertwiner label specifying the volume of the tetrahedron.

In the following, we rely on this representation of GFT fields and quantum states.

### 2.2 GFT condensate cosmology

Coherent peaked states. To specify the universe at a given time, we need a large number of quanta (for the continuum limit) concentrated in a fixed relational time $\phi_0$ [13, 18], which lead us to introduce the coherent peaked state (CPS) [17]:
\[
|\sigma; \phi_0, \pi_0\rangle = N(\sigma) \exp \left( \int (dg)^4 \phi \eta_c(\phi - \phi_0, \pi_0) \hat{\sigma}(g_v, \phi) \right) |0\rangle, \tag{2.4}
\]
with $N(\sigma)$ is some normalization constant, $|0\rangle$ is the vacuum state, and $\eta_c(\phi - \phi_0, \pi_0)$ is a peaking function (usually taken as a Gaussian, see equation (52) in [18]) around $\phi_0$ with a typical width given by $\varepsilon$, and $\pi_0$ is a further parameter controlling the fluctuations of the operator corresponding to the conjugate momentum of the scalar field $\phi$. The reduced condensate function $\tilde{\sigma}(g_v, \phi)$, which is the actual dynamical variable in the hydrodynamic approximation, does not modify the peaking property of the CPS.

Taking into account the fact that our universe is homogeneous and isotropic, which becomes the requirement that the reduced condensate function $\tilde{\sigma}(g_v, \phi)$ only has support over equilateral tetrahedra\(^1\), and the fact that the geometrical conditions requires $\tilde{\sigma}(g_v, \phi)$ to be both left and right invariant under diagonal group actions, the reduced condensate function can be decomposed as
\[
\tilde{\sigma}(g_v, \phi) = \sum_j \tilde{\sigma}_j(\phi) \mathcal{I}_m^j \mathcal{I}_n^j d(j) 2^l \prod_{l=1}^4 D_{mn_\iota}^l(q_i), \tag{2.5}
\]
where we write $j$ for $\vec{j} = (j_1, j_2, j_3, j_4) = (j, j, j, j)$, and similarly for $\vec{m}, \vec{n}$; $\mathcal{I}_m^j$ is the intertwiner labeled by $\iota$, $d(j) = 2j + 1$ is the dimension of the spin $j$ representation and\(^1\) This corresponds to the restriction of all spin labels, i.e. the areas of its boundary triangles, being equal and the volume eigenvalue being the maximal one allowed by this choice of triangle areas [13].
be expressed as a total derivative, Using a global phase, then last equation gives two equations for real and imaginary parts respectively.

\[
\dot{x}(\phi) \left| \sigma_x ; \phi_0, \pi_0 \right. = \eta \left( \phi - \phi_0, \pi_0 \right) \pi \frac{\partial}{\partial \phi_0} \left( \frac{\partial}{\partial \pi_0} \right) \left. \right| \sigma_x ; \phi_0, \pi_0, \tag{2.6}
\]
i.e., \( \left| \sigma_x ; \phi_0, \pi_0 \right. \) is coherent state and only for \( j_1 = j_2 = j_3 = j_4 = j \) the action of \( \dot{x} \) with \( \ddot{x} = (j, \tilde{m}, \nu) \) is not vanishing.

**Effective dynamics.** Having fixed the peaking function \( \eta \left( \phi - \phi_0, \pi_0 \right) \), the dynamics of the condensate is encoded in the evolution of the reduced condensate function \( \tilde{\sigma}(\tilde{g}_v, \phi) \). And since the state is coherent, we see that at mean field level the dynamics can be extracted easily from the following effective action, which can be obtained from the Legendre transformation of the logarithm of partition function determined by the full quantum action [13, 18]:

\[
S(\tilde{\sigma}, \tilde{\sigma}) = \int d\phi_0 \left\{ \sum_j \left[ \tilde{\sigma}_j(\phi_0) \tilde{\sigma}_j''(\phi_0) - 2i\tilde{\pi}_0 \tilde{\sigma}_j(\phi_0) \tilde{\sigma}_j'(\phi_0) - \xi_j^2 \tilde{\sigma}_j(\phi_0) \tilde{\sigma}_j'(\phi_0) \right] + U(\tilde{\sigma}, \tilde{\sigma}) \right\},
\]
where \( \tilde{\pi}_0 = \frac{\pi_0}{\tilde{\sigma}_0^2 - 1} \), \( \xi_j \) is an effective parameter encoding the details of the kinetic term of the fundamental GFT action (in the isotropic restriction), and the derivatives \( \prime \) denote derivatives with respect to \( \phi_0 \). Finally, \( U(\tilde{\sigma}, \tilde{\sigma}) \) is the interaction kernel, and from a rather phenomenological approach, it can be modelled in a simple, rather general form, used also in previous work [44]:

\[
U(\tilde{\sigma}, \tilde{\sigma}) = \sum_j \left( \frac{2\lambda_j}{n_j} \left| \tilde{\sigma}_j(\phi_0) \right|^{n_j} + \frac{2\mu_j}{n_j} \left| \tilde{\sigma}_j(\phi_0) \right|^{n_j'} \right), \tag{2.8}
\]
where \( \lambda_j \) and \( \mu_j \) are interaction couplings correspond to each mode \( j \) satisfy that \( |\mu_j| \ll |\lambda_j| \ll |\xi_j^2 - \tilde{\sigma}_0^2| \) and we assume that \( n'_j > n_j > 2 \). Albeit definitely simpler than full-blown quantum geometric models, this choice still captures several relevant features of the same, and hopefully key aspects of what we may expect to be universal effective behaviour. We emphasis that at this stage our effective action is not derived from some underlying GFT model.

We choose the interaction kernel to be equation (2.8) as it is easy to handle and also has a similar structure of some microscopic GFT theories, such as the one corresponding to EPRL model [13]. In this sense, any GFT model that can reproduce such effective action (under mean-field approximation or with some quantum corrections) would lead to the same evolution of the universe that we will explore below.

To get the equation of motion of the condensate function, we vary the action (2.7) with respect to \( \dot{\tilde{\sigma}}_j \) [18, 45]

\[
\ddot{\tilde{\sigma}}_j - 2i\tilde{\pi}_0 \tilde{\sigma}_j - \xi_j^2 \tilde{\sigma}_j + 2\lambda_j |\tilde{\sigma}_j|^{n_j-2} \tilde{\sigma}_j + 2\mu_j |\tilde{\sigma}_j|^{n_j'}-2 \tilde{\sigma}_j = 0. \tag{2.9}
\]
Decomposing \( \tilde{\sigma}_j(\phi) = \rho_j(\phi) \exp[i\theta_j(\phi)] \) with real \( \rho_j \) (condensate density) and \( \theta_j \) (condensate phase), then last equation gives two equations for real and imaginary parts respectively. Using a global U(1) symmetry of our equation (and effective action), the imaginary part can be expressed as a total derivative, \( Q'_j = 0 \), with [13, 18]

\[
Q_j = (\theta'_j - \pi_0)\rho_j^2 \tag{2.10}
\]
being the corresponding conserved quantity. The remaining equation becomes \cite{18, 44}

$$\rho_j'' - \frac{Q_j^2}{\rho_j^2} - m_j^2 \rho_j + \lambda_j \rho_j^{n_j - 1} + \mu_j \rho_j^{n_j' - 1} = 0,$$  

(2.11)

where $m_j^2 = \xi_j^2 - \tilde{\pi}_0^2$ is now both a function of the fundamental parameters of the model (through $\xi_j$) and of the parameter $\epsilon$ characterizing the non-ideal nature of our clock. This equation can be directly integrated once, which gives another conserved quantity \cite{13}, as a result of the symmetry of the system under “clock-time translation” \cite{18, 44},

$$E_j = \frac{1}{2}(\rho_j')^2 - \frac{1}{2}m_j^2 \rho_j^2 + \frac{Q_j^2}{2\rho_j^2} + \lambda_j \rho_j^{n_j} + \mu_j \rho_j^{n_j'}.$$  

(2.12)

From this equation for the condensate density we will now derive an effective evolution equation for the volume of the universe in relational time.

### 2.3 Volume dynamics

For a homogeneous and isotropic universe, the evolution can be extracted from the behaviour of total volume. In GFT the volume operator can also be written in the second quantized form as

$$\hat{V} = \int d\phi \sum_{\vec{x}, \vec{x}'} V(t, t') \delta_{\vec{x} - \{\vec{t}\}} \phi(\vec{x}) \hat{c}_{\vec{x}}(\phi) \hat{c}_{\vec{x}'}(\phi)$$  

(2.13)

At the mean field level, the total volume is given by the expectation value of the volume operator $\hat{V}$ in the condensate state $|\sigma\rangle$ \cite{13, 17}

$$V(\phi_0) = \langle \sigma_{\vec{x}}; \phi_0, \pi_0 | \hat{V} | \sigma_{\vec{x}}; \phi_0, \pi_0 \rangle \approx \sum_j V_j \rho_j(\phi_0)^2,$$  

(2.14)

where $\rho_j = |\sigma_j|$ is the modulus of reduced condensate function $\tilde{\sigma}$, $V_j \propto |\tilde{\sigma}_j|^3/2$ is the volume contribution from each quantum (tetrahedron) in the spin $j$ representation, and we have used the intertwiner normalization condition $\sum_{\vec{m}} T_{\vec{m} + \vec{c}'} T_{\vec{m} - \vec{c}} = \delta_{\vec{c} - \vec{c}'}$. The approximation amounts to keeping only the dominant contribution to the saddle point approximation of the peaking function coming from our choice of state \cite{18}.

The dynamics of the universe volume can now be obtained by differentiating $V(\phi)$ respect to relational time and then substituting the equations (2.11) and (2.12) for $\rho_j$, writing them in the form of modified FLRW equations \cite{13}

$$\left(\frac{\mathcal{V}'}{3\mathcal{V}}\right)^2 = \left[\frac{2 \sum_j V_j \sqrt{2E_j \rho_j^2 - Q_j^2 + m_j^2 \rho_j^2} - \frac{2}{n_j} \lambda_j \rho_j^{n_j + 2} - \frac{2}{n_j'} \mu_j \rho_j^{n_j' + 2}}{3 \sum_k V_k \rho_k^2}\right]^2,$$  

(2.15)

$$\frac{\mathcal{V}''}{\mathcal{V}} = \frac{2 \sum_j V_j \left[2E_j + m_j^2 \rho_j^2 - \left(1 + \frac{2}{n_j}\right) \lambda_j \rho_j^{n_j} - \left(1 + \frac{2}{n_j'}\right) \mu_j \rho_j^{n_j'}\right]}{\sum_k V_k \rho_k^2}.$$  

(2.16)

Note that we only consider the expansion phase, so we chose the sector $\rho_j' \geq 0$ when we substituted equation (2.12).

It can be shown that for $\rho_j'$ to be real, $\rho_j$ can never vanishes under very general conditions, and since the total volume can be given by summing over modes, we see that the
volume can never vanish and the Big Bang singularity is replaced by a bounce [13]. Furthermore, as the volume grows, but before the GFT interactions become relevant, we reach a regime where the dynamics can be well approximated by the FLRW equation in the presence of a free massless field, for details see [13, 18].

We will focus on the two equations (2.15) and (2.16) in the following discussion, by writing them in the form of standard cosmological equations in terms of an effective equation of state in relational language, and analyzing its behaviour when the universe volume grows. The late time behaviour of the model, we will see, is particularly interesting and can naturally describe a dark energy-driven acceleration, of pure quantum gravity origin.

At this stage, we would like to emphasise again that our approach is phenomenological, where we choose the effective action in a general form, rather than derive it from some fundamental GFT model. GFT coherent states, as introduced above, might not provide a good approximation to the effective GFT dynamics when the interactions are strong [46]. But in the current work the use of CPS is to provide an easy way to explain the essential concepts in GFT cosmology, and to give a template of how to extract effective dynamics using mean-field approximation. The essential part is the condensation of a large number GFT quantas, i.e. the building blocks of spacetime, such that the continuum limit can be extracted. Suppose the true ground state is given by $|\Omega\rangle$, such that the expectation value of the field operator $\sigma(g_v,\phi) = \langle\Omega|\hat{\sigma}|\Omega\rangle$ does not vanish, then the full effective action, including all quantum corrections, can still be written in the form of $S = S(\bar{\sigma},\sigma)$, just as in our case where we used the coherent state. We see this as the form that the quantum effective action of some interesting model takes, after including (some) quantum corrections, rather than taking it literally as the classical mean field dynamics of a model, and hoping that it is not spoiled by quantum corrections, despite having strong interactions.

So, from this point of view, our analysis and results remain rather generic and hopefully robust. The main limitation comes, however, from the explicit form we use for the expectation value of the total volume. In using equation (2.14), we actually assumed that $\langle\Omega|\hat{c}_j^\dagger(\phi)\hat{c}_j(\phi)|\Omega\rangle = \langle\Omega|\hat{c}_j^\dagger(\phi)|\Omega\rangle \langle\Omega|\hat{c}_j(\phi)|\Omega\rangle$, which is only exactly true for coherent states. In general, there should be quantum fluctuations, marking the difference between the actual ground state and the coherent state, and hence the expectation value of the total volume would be given by

$$V(\phi_0) = \langle\Omega|\hat{V}|\Omega\rangle = \sum_j V_j \rho_j(\phi_0)^2 + \chi(\phi)^2,$$

(2.17)

with $\chi(\phi)^2$ specifies the fluctuations. Even though such fluctuations do not vanish for a general ground state, we can expect that they would be suppressed (at least in relative terms) when there are a large number of quanta, which is the case where we try to recover the continuum universe. Furthermore, for the evolution the ground state $|\Omega\rangle$ should be (relational) time-dependent, which means that the fluctuations $\chi(\phi)^2$ should also depend on the relational time $\phi$. When plugged into the equation of motion, we can expect that such fluctuations should also be suppressed over time, to give a stable ground state, such that the system remains in the condensate phase. Therefore, as a leading order approximation, we use the GFT field coherent state as our starting point, to be then improved by the effects of fluctuations on the expectation value of the universe volume. We leave a detailed analysis of fluctuations in the interacting case to future work.
3 Effective equation of state

A convenient way to capture the relevant features of the effective cosmological dynamics, that we can extract from the GFT condensate hydrodynamics, is to express it in terms of an effective matter component, in turn described entirely by its equation of state.

In a homogeneous universe, for example, the matter content is assumed to be a perfect fluid and can be characterized by its energy density $\rho$ and pressure $p$ in a comoving frame. The fluid then couples to the geometry, determining the cosmological evolution, through its equation of state $w = p/\rho$. For example, if the expanding universe is dominated by a fluid with $w < -1/3$, then the expansion will be accelerating. Current cosmological observations give a value $w \simeq -1$, thus indeed an accelerating expansion of the observable universe, while the usual matter content from the standard model would give $w = 1/3$ for relativistic particles and $w = 0$ for non-relativistic particles. Moreover, while a small positive cosmological constant could reproduce this value, one would be left to explain how the value of the cosmological constant is chosen, how it is affected by the quantum dynamics of matter and its interaction with (quantum) gravity, and, more important, how this value changes over time, since a simple constant value is not obviously compatible with what we know about cosmological evolution. This is, in summary, the problem of dark energy [47].

We now express our emergent cosmological dynamics in the same language, appropriately recast in terms of relational clock evolution.

For a homogeneous and isotropic metric with scale factor $a(t)$, the Hubble parameter can be given by $H = \dot{a}/a$ with the $\dot{}$ represents the derivative respect to comoving time $t$. Then the effective equation of state can be defined as $w = -1 - 2\dot{H}/(3H^2)$. In the GFT (and more generally, quantum gravity) context, we cannot rely at the fundamental level on any time coordinate or direction. We can use, instead, a relational definition of time in terms of a physical clock, for example a free massless scalar field $\phi$, as discussed in section 2. In appendix A we show that using this definition of relational time, the equation of state $w$ can be defined by

$$w = 3 - \frac{2V''}{V'}^2, \quad (3.1)$$

where $V$ is the total volume and the $'$ indicates the derivative with respect to the relational time $\phi$, and we chose the time gauge, in which the volume $V = a^3$ for scale factor $a$.

Using this effective equation of state, all the effects produced on the evolution of the universe by the underlying quantum gravity dynamics can be described as if they were due to some effective matter field $\psi$ satisfying $w_\psi \equiv p_\psi/\rho_\psi = w$, with $p_\psi$ and $\rho_\psi$ its pressure and energy density, respectively.

We emphasize that the field $\psi$ introduced this way is just a convenient rewriting of what remains due to the fundamental quantum gravity dynamics. As such, it is not required to possess the usual features of well-behaved matter field theories defined on cosmological backgrounds, nor the desiderata of effective field theory. For the same reason, we will not discuss possible Lagrangians for $\psi$, or dwell any further into its properties qua matter field.

One main advantage of introducing the fictitious field $\psi$, beside making the analysis of the volume evolution more practical, is that it helps to gain an intuitive understanding of quantum effects on geometry, or more precisely, on the scalar curvature, which is a rather tricky observable to define and compute in the fundamental quantum geometric GFT context. In fact, suppose the energy-momentum tensor of field $\psi$ is given by $T_{\mu\nu}$, then tracing the Einstein equation we see that the scalar curvature in a universe dominated by $\psi$ can be given
by $R = -T^\mu_\mu = -(1 + 3w)\rho$, where we used the fact that $T^\mu_\mu = \rho + 3p$ in the comoving frame. In particular, this helps identifying potentially singular regimes. For example, if $\rho \to \infty$, we see that the scalar curvature diverges as well (except for $w \neq -1/3$, which, as we can see in subsection 3.1, will not lead to a divergent energy density anyway); these correspond to Big Rip-like singularities, which is relevant for dark energy models [48–51], and on which we are going to have more to say in the following.

### 3.1 The evolution of $\psi$

Now we recall the evolution of an effective field $\psi$ endowed with the equation of state $w$. We stress once more that we intend this to be only an illustration of which properties a field of this type would have in the context of standard General Relativity and effective (quantum) field theory, making use of all the auxiliary structures (topological manifold, coordinates, gauge conditions, etc) that are useful tools in such context. It is not a determination of the physical properties of a physical field, corresponding to fundamental degrees of freedom and observables of our quantum gravity formalism, but only an effective rewriting of quantum ‘pregeometric’ gravity degrees of freedom, which are not described in terms of similar auxiliary structures. For example, we could define an energy density for the effective field $\psi$ from the equation of state $w$ and the universe volume $V$ and study its properties, but there is no independent fundamental observable corresponding to it, in the GFT algebra of (2nd quantized) observables.

Having clarified this important point, the energy density $\rho$ satisfies the conservation equation $\dot{\rho} + 3H(1 + w)\rho = 0$. Using the standard definition of Hubble parameter in time gauge $H = \dot{a}/a = \dot{V}/(3V)$, this equation can be rewritten as

$$
\frac{d\rho}{dV} + \frac{1 + w}{V} \rho = 0,
$$

which can indeed be taken as a definition of the energy density in terms of quantities corresponding to GFT observables. For constant $w$, equation (3.2) can be easily solved and the solution is given by

$$
\rho = \frac{\rho_0}{V^{1+w}},
$$

with the $\rho_0$ is the constant of integration. For $w > -1$, the energy density $\rho$ decreases as the volume grows, and tends to vanish when volume is large, i.e., we expect, at late times; for $w = -1$, the energy density is a constant, corresponding to a cosmological constant, and would tend to dominate over any other fluids with $w > -1$ at late times; for $w < -1$, on the other hand, $\rho$ increases as the volume becomes larger, and would tend to diverge for $V \to \infty$. Such a large energy density with $w < -1$ will tear apart every thing in the universe, leaving no bound system at all even when the size of the universe is finite, before the total volume diverges, as discussed in [51]. Furthermore, when $\rho$ diverges, the scalar curvature $R = -(1 + 3w)\rho$ would approach to infinity as well, lead to Big Rip-like singularities.

The above discussion gives a first intuition for the possible late time evolution of our universe, and of various issues constituting the dark energy problem. It should be clear, however, that things are so simple only under the assumption of constant equation of state $w$. Any dark energy model which is based on a dynamical equation of state would require a more detailed analysis.

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2 We thank Dr. Che-yu Chen for pointing out this reference.
A particularly interesting class of dark energy models is in fact based on fields with equation of state less than $-1$, producing a **phantom (dark) energy**, which is well compatible with present observational constraints.

**Phantom energy.** The mentioned feature of phantom energy compared to other field-theoretic models with $w > -1$, i.e. that its energy density increases as the universe volume grows, is the root of various difficulties in constructing a viable field theoretic model of phantom energy. In fact, $w < -1$ requires negative kinetic energy and leads to a violation of various energy conditions [23, 27, 52]. The negative kinetic energy is also unbounded from below, and straightforward introductions of a regularizing cutoff would lead, in general, to violations of Lorentz symmetry [53].

While these are serious difficulties for such field-theoretic phantom models, phantom energy cannot be ruled out based on cosmological data. On the contrary, several observations favor an equation of state less than $-1$ [24–26, 54]. In addition, it has been recently shown that the existence of phantom energy may alleviate the $H_0$ tension [55, 56], i.e. the fact that the value of the Hubble parameter when estimated from local experiments [57] is larger than what is deduced from CMB data [58].

Therefore we seem to be facing a situation in which a phantom-like evolution of the observed (late) universe struggles to find a compelling theoretical description. From our quantum gravity viewpoint, based on a formalism in which spacetime is naturally seen as emergent, the difficulties of a formulation of phantom energy in terms of a field theory framework is not particularly worrying. We expect the whole background cosmological dynamics, including its large-scale features, to be determined by the underlying quantum gravity dynamics, and no fundamental phantom field needs to be part of the story. On the other hand, our task is first of all to match cosmological observations, a difficult challenge for all fundamental quantum gravity approaches, and for this aim an effective phantom dark energy would be suitable. Indeed, we will show in the following how phantom-like dark energy can emerge from our GFT condensate model.

For completeness, we mention that one can also tackle the phantom energy problem in the context of modified gravity theories. That is, one can attribute the accelerated expansion of the universe to a modification of the underlying gravitational dynamics, with respect to GR, rather than to new exotic matter components, for example as a $f(R)$ theory [29]. In such a way, one can bypass the difficulties of constructing a well-defined matter field theory of phantom energy. This second approach is much closer in spirit to the one we take within our quantum gravity framework, and the emergent cosmological dynamics we extract from the fundamental quantum dynamics of ‘spacetime constituents’ could in principle be recast also in terms of some effective modified gravity theory.

**Big Rip singularity.** Quantum gravity effects can change the evolution of any matter content dramatically. For example, in the early universe, even ordinary matter with $w > -1$ can have phantom like behaviour due to discreteness of quantum geometry [59]. And at late times, quantum gravity effects can dissolve the Big Rip singularity in the presence of phantom matter with $w < -1$, as studied in the LQC context [60] and also in semi-classical analyses [61]. Here we will not consider the coupling between quantum gravity and matter, since the accelerated phase with effective equation of state $w < -1$ will emerge from pure quantum gravity in our model. But a Big Rip singularity is avoided due to a non-trivial time dependence of $w$. Indeed, as mentioned above, when $w$ is time dependent the evolution of (any effective) $\rho_w$ can be rather involved. In particular, if $w$ approaches to $-1$ fast enough, the
phantom energy density does not diverge but increases to a constant value, and the Big Rip singularity can be avoided. For example, consider $\rho_\psi$ as a cosmological constant plus some matter component with negative energy density, inversely proportional to the volume \cite{34}. This corresponds to a field $\psi$ with $w_\psi < -1$ but that approaches $-1$ at large volume, so that asymptotically we reach a de Sitter spacetime. This is referred to as \textit{phantom analogues of de Sitter space} in \cite{34}. In section 5.2 and section 5.3 we will see how exactly this kind of behaviour emerges from our quantum gravity model.

\subsection{3.2 $w$ from single-mode GFT condensates}

Before moving on to our new analysis of GFT cosmological dynamics in the presence of interactions, let us summarize earlier work on this issue. The result of \cite{44} is equivalent to a study of the behaviour of the effective $w$ under the assumption that a single GFT field mode $j$ contributes to the dynamics. While the asymptotic dominance of a single mode as the universe expands is expected also in the general case, the presence of other modes changes the way in which $w$ approaches the asymptotic value, which is of important physical relevance, as we explained, on top of making the dynamics much richer in any intermediate regime. But the analysis in \cite{44} is already important to show how GFT interactions can have very interesting consequences on the emergent cosmological dynamics, as we now discuss.

With only one $j$ mode, using (2.15) and (2.16) in the definition (3.1) we have

$$w = \frac{-3Q^2 + 4E\rho^2 + m^2\rho^4 + \left(1 - \frac{4}{n}\right)\lambda\rho^{n+2} + \left(1 - \frac{4}{m}\right)\mu\rho^{n+2}}{-Q^2 + 2E\rho^2 + m^2\rho^4 - \frac{2}{n}\lambda\rho^{n+2} - \frac{2}{m}\mu\rho^{n+2}}, \quad (3.3)$$

where we dropped the subscript denoting different modes $j$ for simplicity. Furthermore, for a single mode the total volume $V \propto \rho^2$, and we can get the evolution $w = w(V)$ even without solving the equation of motion. This greatly simplifies the analysis.

\textbf{Early time acceleration in the free case.} At early times, the module $\rho$ of the condensate is small, therefore the interaction terms can be ignored. Here we set $\lambda = \mu = 0$, then $w$ is simply

$$w = \frac{-3Q^2 + 4E\rho^2 + m^2\rho^4}{-Q^2 + 2E\rho^2 + m^2\rho^4}.$$ 

At the bounce, the denominator vanishes, $-Q^2 + 2E\rho^2 + m\rho^4 = 0$, which gives the value of $\rho$ at the bounce

$$\rho_b = \frac{1}{m} \sqrt{E^2 + m^2Q^2 - E}.$$ 

Put this back into $w$ we see that the numerator is negative, therefore $w \to -\infty$ near the bounce. This means that right after the bounce the expansion is accelerating, as we expect from a bouncing scenario.\footnote{The universe should expand which requires $V' > 0$ after the bounce, and at the bounce we have $V'_b = 0$, therefore we should also have $V''_b > 0$. Since the volume $V_b$ at the bounce is also positive, from the definition (3.1) of $w$ we see that $w \to -\infty$ at the bounce is a general feature.} However, we can show that this accelerating phase ends quickly, i.e., the volume at the end of acceleration is not large compared to the volume at the bounce \cite{44}. The situation is similar even if we consider the contributions from all modes, as we shall see in section 4.

It is worth mentioning that even if $w \to -\infty$ at the bounce, we do not run into singularities due to the quick end of the acceleration phase and the fact that the total volume has

\[\]
a minimum value $V_b > 0$. To see this we first note that for a single mode (assumed to be mode $j_0$), the total volume can be given by $V = V_{j_0} \rho^2$, therefore the equation of state can be rewritten as

$$w = \frac{-3Q^2 V_{j_0}^2 + 4E V_{j_0} V + m^2 V^2}{-Q^2 V_{j_0}^2 + 2E V_{j_0} V + m^2 V^2}.$$ 

Then we substitute this equation of state into the conservation equation (3.2) for the fictitious field $\psi$, we get the solution

$$\rho_\psi = \frac{\tilde{\rho}_\psi}{V^2} + \frac{2E_0 V_{j_0}}{V^2} \tilde{\rho}_\psi + \frac{Q^2 V_{j_0} \tilde{\rho}_\psi}{m^2}.$$ 

where $\tilde{\rho}_\psi$ is defined such that $\rho_\psi V^2 \to \tilde{\rho}_\psi$ as the total volume $V \to \infty$. Since the volume $V \geq V_b > 0$ is bounded, we see that the energy density $\rho_\psi$ remains finite, and there is no singularities. We also note that at the bounce we have $\rho_\psi(V_b) = 0$.

Emergence of the FLRW universe. As we have explained in section 2.3, the classical limit emerges already in the free case. It is obtained at large volume, where $\rho$ is also large. At leading order in $1/\rho$, we have $w = 1$ is a constant, corresponds to the equation of state of a free massless scalar field, the one we introduced as relational time. In fact, substituting $w = 1$ back into its definition (3.1), simple algebraic manipulation shows that

$$V'' = \left(\frac{V'}{V}\right)^2 - d \frac{d}{d\phi} \left(\frac{V'}{V}\right) = 0,$$

hence $V'/V = \text{const}$ which characterizes the FLRW equation using the relational language in the presence of a free massless field [13].

At the next order of $1/\rho$, we can approximate $w$ as

$$w = 1 + \frac{2E}{m^2 \rho^2},$$

confirming that the effective equation of state approaches 1 at large volume. Furthermore, for $E > 0$, $w$ approaches this asymptotic value from above; see figure 1. This is not the case when we consider more than one mode, as we shall see in section 4.3.

An emergent inflationary phase from quantum gravity. The next question is how the single-mode interactions change this picture, in particular concerning the early acceleration after the bounce. As showed in [44], one can indeed get a long lasting accelerated phase, in contrast to the free condensate. Furthermore, with two interaction terms this acceleration can end properly, and the time that the acceleration lasts can be adjusted by tuning couplings $\lambda$ and $\mu$ [44]. What is missing, however, is a subsequent FLRW phase, which is of course also crucial for a proper cosmological model. Let us see how this behaviour is reflected in the effective equation of state. Since we assumed that $|\mu| \ll |\lambda|$, there is an intermediate range, where $m^2 \rho^4$ and $\mu \rho^{n+2}$ are both small compared to $\lambda \rho^{n+2}$, and the behaviour of $w$ is determined by the $\lambda$ term. The $\lambda > 0$ case will give an additional root of the denominator of $w$, corresponds to the maximum value of $\rho$ and lead to a cyclic universe very quickly after the bounce. Hence we only consider the case $\lambda < 0$, where to leading order we have $w = 2 - n/2$. We see that for $n \geq 5$ we have $w < -1/3$ which corresponds to an accelerating phase. In absence of other interactions, this accelerated phase would simply not end. Otherwise, as $\rho$
increases further, the \( \mu \) term becomes important compared to the \( \lambda \) term. If \( \mu > 0 \), \( \rho' \) will vanish again (besides the point of minimal volume reached at the bounce), corresponding to the maximum value of \( \rho \) determined by

\[
\frac{2}{n} \lambda \rho^{n+2} = \frac{2}{n'} \mu \rho'^{n'+2},
\]

near which \( w \to \infty \). This means the accelerating phase dominated by the \( \lambda \) term stops. By adjusting the values of the couplings \( \lambda \) and \( \mu \) we can make this phase lasts long enough to account the observational constraints \([44]\). The magenta dash-dotted line in figure 4 shows the behaviour of \( w \) when \( \mu > 0 \) and we see that there is a nice inflationary phase with \( w = -1/2 \). However, as anticipated, this inflationary phase ends when the volume approaches its maximal value, being quickly followed by a contracting phase, with no FLRW phase in between. The important take home message, however, is that interesting large scale cosmological dynamics, like a long lasting inflationary (or more generally, accelerated) phase can be produced purely from fundamental quantum gravity dynamics, without the need of any exotic matter field (here, an inflaton).

**Phantom crossing.** Finally, in this simpler single-mode context, we can ask whether anything like a phantom-crossing can also be obtained as a result of the quantum gravity dynamics.

As we explained above, when \( w < -1 \) we have phantom energy. For a dynamical \( w \), it is possible for \( w \) to change from \( w > -1 \) to \( w < -1 \), a phenomenon called *phantom crossing* \([62]\). In our case, if \( \mu < 0 \), \( \rho \) can keep growing until the \( \mu \) term dominates, with the asymptotic behaviour of the equation of state given by

\[
w \to 2 - \frac{n'}{2} + \left(n' - n\right) \frac{n' \lambda}{2n\mu} \rho^{n-n'}.
\]

Since \( n' > n \), we see that for \( n \geq 5 \), we have \( w < -1/3 \) as \( \rho \) grows and the acceleration does not stop. And in contrast to the \( \mu > 0 \) case, where the volume has a maximum value after which the universe starts to collapse, when \( \mu < 0 \) the total volume can grow forever. Note that \( n' > n \) and that both \( \lambda \) and \( \mu \) are negative, thus we conclude that \( w \) approaches its asymptotic value from above. For \( n' = 6 \), we have \( w \to -1 \), which mimics the behaviour of a cosmological constant. Since \( w \) approaches this value from above, we have \( w > -1 \) after the end of early accelerating phase (which is dominated by the free parameters of the condensate). We conclude that for a single mode with \( n' \leq 6 \), \( w \) cannot cross the phantom divide \( w = -1 \). This is illustrated in figure 4 by the red dashed line.

On the other hand, for \( n' > 6 \), the asymptotic value of \( w \) would be less than \(-1\), so phantom crossing is possible. But now the energy density of the fictitious field \( \psi \) with effective equation of state \( w \) will diverge as the volume of the universe grows. When the volume is large enough, this energy density would produce a Big Rip singularity \([48]\). In section 5, we will show that when we consider two modes, we can get an equation of state \( w \) that crosses the phantom divide, and that, instead of a Big Rip singularity, the *phantom analogues of de Sitter space* \([34]\) is obtained.

Table 1 summarizes the influence of parameters in our model on the behaviour of the cosmological evolution. Note that we only consider an early accelerating phase as inflation when there is a graceful exit, which is not possible for the last row in the table, where \( \mu < 0 \).
Table 1. The influence of parameters on the cosmological evolution in single mode case. The blank cell indicates there is no such behaviour, the check mark ✓ means such behaviour would occur, while other non-empty cells suggest that the behaviour would show up for the given range for the parameters.

| λ > 0 | Inflationary phase | ✓ | Cyclic behaviour | ✓ | Late time acceleration | ✓ | Phantom crossing | ✓ | Big Rip |
|-------|---------------------|---|------------------|---|-----------------------|---|------------------|---|---------|
| λ < 0 | µ > 0               | n ≥ 5 | ✓ |               |   |                       |   |                   |   |         |
|       | µ < 0               | n' ≥ 5 | n' > 6 | n' > 6 |               |   |                   |   |         |

4 Acceleration in early time

We now start analyzing our emergent cosmological dynamics, in the case in which GFT interactions are taken into account and two spin modes contribute to it. We focus first on the early universe dynamics, right after the bounce, to see how the presence of two spin modes modifies the results obtained in [44].

In the last section, we have seen that for a single mode, the universe undergoes an accelerated expansion for a very short period after the bounce. But for the early universe, the volume is small, and in these conditions we have no reason to expect one mode to dominate over the others, so we should consider the contributions from several modes into account. Besides, the smallness of the condensate density $\rho_j$ means that the dynamics of each spin mode is dominated by the free part of the dynamics. Hence, we can consider the free condensate with $\lambda_j = \mu_j = 0$ for all $j$.

4.1 Accelerated expansion in the free condensate

We require that $\rho_j' ≥ 0$ in the region we considered, and then the condition $V' = 0$ for the volume at the bounce corresponds to requiring $\rho_j' = 0$, $\forall j$. The value of $\rho_j$ at the bounce can be obtained by solving the equation $\rho_j' = 0$, where $\rho_j'$ is obtained from the definition (2.12) of the GFT “energy” $E_j$ as

$$\rho_j'(\phi) = \frac{1}{\rho_j} \sqrt{2E_j \rho_j^2 - Q_j^2 + m_j^2 \rho_j^4 - \frac{2}{n_j} \lambda_j \rho_j^{n_j+2} - \frac{2}{n'_j} \mu_j \rho_j^{n'_j+2}}. \quad (4.1)$$

In the free case, at the bounce we have

$$\rho_{bj} = \frac{1}{m_j} \sqrt{E_j^2 + m_j^2 Q_j^2 - E_j}. \quad (4.2)$$

Given the initial value $\rho_j(0) = \rho_{bj}$, the differential equation (4.1) can be solved [45, 63] to give

$$\rho_j(\phi) = \frac{1}{m_j} \sqrt{E_j^2 + m_j^2 Q_j^2 \cosh(2m_j\phi) - E_j}. \quad (4.3)$$

Then the total volume (2.14) becomes

$$V = \sum_j V_j \rho_j^2 = \sum_j \frac{V_j \sqrt{E_j^2 + m_j^2 Q_j^2}}{m_j^2} \cosh(2m_j\phi) - \sum_j \frac{V_j E_j}{m_j^2}. \quad (4.4)$$
At the bounce $\phi = 0$, therefore the volume $V_b$ is simply $V = c_1 - c_2$, where $c_1$ and $c_2$ are given by
\[ c_1 = \sum_j \frac{\nu_j \sqrt{E_j^2 + m_j^2 Q_j^2}}{m_j}, \quad c_2 = \sum_j \frac{\nu_j E_j}{m_j^2}. \tag{4.4} \]

We can see that $c_1 > c_2 > 0$.

The volume should be convergent, in the sense that $V$ is finite at any given relational time $\phi$. In appendix B we show that this is equivalent to the requirement that $\sum \frac{\nu_j}{m_j^2} \sqrt{E_j^2 + m_j^2 Q_j^2}$ converges and all the $m_j$’s are bounded. A direct consequence is that at sufficiently large $\phi$, the volume is dominated by the mode with the largest value of $m_j = m$. This largest value defines, in this regime, the effective Newton’s constant $m^2 = 3\pi G$, and the dynamics reduces to the standard Friedmann equation with the matter content given by the free massless scalar field $[13]$. There are general arguments suggest that $m_j$ is monotonically decreasing with $j$, so that, at large volume, it is the smallest spin mode that eventually dominates $[63]$.

### 4.2 Upper bound of the number of e-folds

Now we are ready to check if the inclusion of all modes can make the acceleration phase after the bounce last long enough to be of phenomenological significance as a quantum gravity-induced inflation, even in the free case.

For simplicity, we introduce a function $\mathcal{P}(\phi)$ to characterize the acceleration
\[ P(\phi) = -\left(\frac{\nu'}{2}\right) \left(w + \frac{1}{3}\right), \tag{4.5} \]

taking in this free case the form
\[ P(\phi) = \sum_j 4\nu_j \sqrt{E_j^2 + m_j^2 Q_j^2} \cos(2m_j \phi) \sum_k \frac{\nu_k}{m_k^2} \left[\sqrt{E_k^2 + m_k^2 Q_k^2} \cos(2m_k \phi) - E_k\right] - \frac{5}{3} \sum_j \frac{2\nu_j}{m_j} \sqrt{E_j^2 + m_j^2 Q_j^2} \sin(2m_j \phi) \sum_k \frac{2\nu_k}{m_k} \sqrt{E_k^2 + m_k^2 Q_k^2} \sin(m_k \phi). \tag{4.6} \]

The accelerating expansion requires $w < -1/3$, i.e. $P(\phi) > 0$, while the decelerating phase corresponds to $P(\phi) < 0$.

At the bounce, where $\nu' = 0$, we have simply
\[ P(0) = \sum_j 4\nu_j \sqrt{E_j^2 + m_j^2 Q_j^2} (c_1 - c_2) > 0, \tag{4.7} \]

with $c_1$ and $c_2$ defined in (4.4), while for large $\phi$, the volume is dominated by a single mode, and equation (3.4) tells us that $w \to 1$ when volume is large. This implies $P(\phi) < 0$ at large volume. Therefore, there is a point where $P(\phi) = 0$ and the accelerating expansion stops. We now identify this point and show that the accelerating phase until then can not be long enough. More precisely, we get an upper bound on the ratio $\nu_e/\nu_b$, where $\nu_e$ is the volume when acceleration ends, and $\nu_b = c_1 - c_2$ is the volume at the bounce.

The time $\phi_e$ where the accelerating phase ends is determined by the requirement $P(\phi_e) = 0$. This equation is quite hard to solve for general $m_j$’s. If the acceleration is long lasting, $\phi_e$ would be large, and around this point $P(\phi)$ changes quickly. Therefore we can introduce
an approximated quantity $P_m(\phi)$, obtained by replacing $\cosh(2m_\phi \phi)$ and $\sinh(2m_\phi \phi)$ in (4.6) with $\cosh(2m_\phi \phi)$ and $\sinh(2m_\phi \phi)$ respectively, where $m$ is the maximum value of $m_j$'s. We can write $P_m(\phi)$ as

$$P_m(\phi) = -\frac{4m}{3} \left[ \cosh^2(2\sqrt{m} \phi) \left( 5c_1'^{2} - 3c_1 c_1'' \right) + \cosh(2\sqrt{m} \phi)(3c_1'' c_2 - 5c_1'^{2}) \right],$$

with $c_1$ and $c_2$ are given by equation (4.4) and the two new constants $c_1'$ and $c_2'$ are

$$c_1' = \sum_j \frac{\mathcal{V}_j}{m m_j} \sqrt{E_j^2 + m_j^2 Q_j^2}, \quad c_2' = \sum_j \frac{\mathcal{V}_j}{m^2} \sqrt{E_j^2 + m_j^2 Q_j^2}. \quad (4.8)$$

We see that $c_1 > c_1' > c_1'' > 0$. The equation $P_m(\phi) = 0$ has a root $\phi_0$ and one has

$$\cosh(2m_\phi \phi_0) = -3c_1'' c_2 + \sqrt{9c_1'^{2} c_2^2 + 20c_1' (5c_1'^{2} - 3c_1 c_1'')} \frac{2(5c_1'^{2} - 3c_1 c_1'')(c_1 - c_2)}{2(5c_1'^{2} - 3c_1 c_1'')} \quad (4.9)$$

Since $P(\phi)$ changes quickly near $\phi_e$, we have approximately $\phi_e \approx \phi_m$, which in turn leads to $\mathcal{V}_m = \mathcal{V}(\phi_e) \approx \mathcal{V}(\phi_m) < \mathcal{V}(\phi_m)$. Here we define $\mathcal{V}_m$ similarly as $P_m$, i.e., replacing $\cosh(2m_\phi \phi)$ in the volume (4.3) with $\cosh(2m_\phi \phi)$, and therefore, at $\phi = \phi_m$ we have

$$\mathcal{V}_m(\phi_m) = c_1 \cosh(2m_\phi \phi_m) - c_2.$$ 

Then the ratio between volume at the end of acceleration and the volume at the bounce satisfies

$$\frac{\mathcal{V}_e}{\mathcal{V}_b} < \frac{\mathcal{V}_m(\phi_m)}{\mathcal{V}_b} = \frac{-3c_1 c_2 c_2' + c_1 \sqrt{9c_1'^{2} c_2^2 + 20c_1 (5c_1'^{2} - 3c_1 c_1'')} \frac{2(5c_1'^{2} - 3c_1 c_1'')(c_1 - c_2)}{2(5c_1'^{2} - 3c_1 c_1'')}}{c_1 - c_2}. \quad (4.10)$$

with $c_1$ and $c_2$ are given by equation (4.4). Under the conditions $c_1 > c_2 > 0$ and $c_1 > c_1' > c_1'' > 0$, $\mathcal{V}_m(\phi_m)/\mathcal{V}_b$ has a maximum value

$$\left. \frac{\mathcal{V}_m(\phi_m)}{\mathcal{V}_b} \right|_{\max} = 1 + \frac{c_1}{c_2} + \frac{c_1}{c_2} \sqrt{\frac{c_1 + c_2}{c_1 - c_2}}.$$ 

Therefore, the original volume ratio with $j$ dependent $m_j$ has the upper bound

$$\frac{\mathcal{V}_e}{\mathcal{V}_b} < 1 + \frac{c_1}{c_2} + \frac{c_1}{c_2} \sqrt{\frac{c_1 + c_2}{c_1 - c_2}}, \quad (4.11)$$

with $c_1$ and $c_2$ are defined in equation (4.4).

The bound goes to infinity when $\frac{c_2}{c_1} \to 0$ or $\frac{c_2}{c_1} \to 1$. However, since the total volume $\mathcal{V}$ should be finite, both of $c_1$ and $c_2$ should be finite. Then, using their definition, we see that $\frac{c_2}{c_1} \to 0$ would require $E_j \to 0$ for all $j$ while $\frac{c_1}{c_2} \to 1$ would require $Q_j^2 \to 0$ for all $j$ (m cannot vanish otherwise $c_1$ and $c_2$ would diverge) for all $j$. Therefore, for general configurations corresponding to non-vanishing $Q_j$ and $E_j$ for some $j$, the bound on the number of e-folds would not be large. While for vanishing $Q_j$ and $E_j$ we need to find a different bound to reach a reliable conclusion, it is clear that this would correspond to a rather special case, thus of limited interest, especially in a phenomenological setting like this.
We conclude that the expansion of the universe becomes decelerating quickly after the bounce, confirming in this more general setting the results of [44].

We emphasize that this initial accelerating expansion is in fact a general feature of a bouncing universe, not necessarily linked to any inflationary-like scenario. Inflation as usually understood should instead start later, during the radiation dominating phase [64]. Such later inflationary acceleration can indeed be reproduced as it has been shown in the previous section, recalling the results of [44], when accounting for GFT interactions in our condensate. As we discussed, however, single-mode interactions which are strong enough to be relevant shortly after the bounce, and before a FLRW phase produced by the free GFT dynamics, end up preventing that such a FLRW phase is realized after the inflationary one, in contrast to a physically viable cosmological model. One may wonder if the contribution from multiple modes changes this picture. A moment of reflection, together with the analysis we present in the next section, would convince that this may only be possible in the presence of somewhat extreme fine-tuning of parameters and a very special behaviour of the condensate density, since in practice it would require that the contributions from the two interaction terms for the two modes approximately cancel for a long enough period of relational time, after the inflationary phase, so to effective reproduce the free dynamics and its FLRW phase. A situation of this type, even if possible in principle, would be of little interest, unless somehow governed by some symmetry principle or some other generic feature of the underlying quantum gravity model. Lacking this, we do not consider it further in the following.

We discuss instead in detail the role of GFT interactions in producing an accelerated expansion at even later times, in the next section. The important point to stress here is that, as long as the interaction couplings are small compared to ‘mass’ term \( m_j \), the behaviour of condensates can be well approximated by free solutions. Therefore, a very short-lived accelerated expansion after the bounce followed by a decelerating phase remains a general feature even in the presence of interactions. We are going to use this feature to ensure that, whatever the detailed late time evolution of the universe in our model is, an extended FLRW phase can be realized, before quantum gravity interactions become relevant, as required by observations.

### 4.3 Equation of state after the end of acceleration

More precisely, after the end of the post-bounce acceleration, the expansion itself does not stop and the volume of universe keeps growing. According to the free solution (4.2), for large \( \phi \) the module \( \rho_j \) increases exponentially. Therefore the mode with largest \( m_j \) dominates quickly as the volume growing, which means the equation of state will soon be dominated by this single mode as well. As we have already discussed, \( w \) will have the asymptotic value \( w = 1 \) as in the single mode case, corresponding to the equation of state of the free massless scalar field that we are using as relational time. However, the inclusion of other modes changes the precise way in which \( w \) approaches to the asymptotic value. Take the two-modes case as an example (for simplicity, we write \( \rho_{1,2} \equiv \rho_{j_1,j_2} \) etc). Assuming \( m_1 > m_2 \) and hence at large volume we have \( \rho_1 > \rho_2 \), then \( w \) can be expanded as

\[
 w \to 1 + \frac{2V_2 \rho_2^2}{V_1 \rho_1^2} \left( 2\sqrt{\frac{m_2}{m_1}} - 1 - \frac{m_2}{m_1} \right).
\]

Since \( 2\sqrt{m_1/m_2} < 1 + m_2/m_1 \), we see that \( w \) approaches the asymptotic value from below, in contrast with the single mode case. In figure 1 we compare the behaviour of \( w \) in the two-modes case and in single-mode case. At small volume near the bounce, \( w < 0 \) and its
absolute value is large; this corresponds to large acceleration right after the bounce. With the increase in volume, $w$ grows quickly and becomes larger than $-1/3$ soon, where the accelerated expansion stops. Then $w$ keeps growing and reaches its maximum value, after which $w$ starts to decrease. This behaviour is true for both the two-modes and single-mode cases. As the volume grows further, the evolution of $w$ starts to differ in the two cases. In the two-modes case, $w$ has a minimum value, which is smaller than 1, after which $w$ starts to increase again, and reaches $w = 1$ from below. In the single-mode case, instead, there is no local minimum, and $w$ keeps decreasing, and approaches the asymptotic value $w = 1$ from above. Something similar will happen in the interacting case. We will see that, for interactions of order 6, the asymptotic value will be the phantom divide $w = -1$. Therefore at large volume we have $w < -1$ and the phantom divide is crossed.

5 Late time accelerated expansion

We now turn to the main focus of our analysis, i.e. the emergent cosmological dynamics of interacting multi-mode condensates at late times.

In the last section, we have seen that for a free condensate, the accelerated expansion only lasts for a short while after the bounce. As volume increases, the quantum gravity condensate would then be described by a FLRW universe filled with a single massless scalar field. For large condensate densities (and thus volume), however, we expect the interactions to be relevant.

We first discuss how to solve the equation of motion for each mode, at least approximately. Then we extract the asymptotic behaviour of the effective equation of state $w$ in the two-modes case, showing that it is possible for the phantom divide to be crossed, thus producing a phantom-like dark energy purely from quantum gravity effects. In contrast to the single mode case, moreover, the phantom crossing does not lead to a Big Rip singularity. Finally, we also show that it is possible to produce at late times a more involved, if maybe less
phenomenologically interesting, combination of inflation-like and phantom-like dark energy in our model.

5.1 Large \( \rho \) behaviour of the interacting condensate

With interactions being included, the equation (4.1) is much harder to solve, and in general the solution cannot be written in close analytic form. Nevertheless, under our assumption that \( |\mu_j| \ll |\lambda_j| \ll m_j^2 \), the equation of motion can be solved piece-wisely. For simplicity, we first assume \( \mu_j = 0 \); then for \( \lambda_j < 0 \) and \( \rho_j \) is large, the equation (4.1) can be approximated as

\[
\rho_j'(\phi) = \sqrt{-2\lambda_j \rho_j(\phi)^{n_j}}. \tag{5.1}
\]

This equation can be easily solved and gives

\[
\rho_j(\phi) = \left( \frac{2}{n_j - 2} \right) \left( \frac{-2\lambda_j}{n_j} \right)^{-\frac{2}{n_j - 2}} \left( \frac{1}{(\phi_{j\infty} - \phi)^\frac{2}{n_j - 2}} \right), \tag{5.2}
\]

where \( \phi_{j\infty} \) is a constant of integration, determined by initial conditions. Its value can be fixed by matching with solutions in the free case (4.2). We choose the matching point \( \rho_{j0} \) to be where the ‘mass’ term equals to the interaction term, \( m_j^2 \rho_{j0} = -2\lambda_j \rho_{j0}^{n_j}/n_j \), i.e. the point where the two approximations we used to solve the dynamical equation reach their limit of validity. Assuming that the free solution (4.2) is valid up to \( \rho_{j0} \) for each individual \( j \), then \( \phi_{j0} \) can be determined inverting the solution (4.2). Taking \( (\phi_{j0}, \rho_{j0}) \) as an initial condition for the equation (5.1) and then inserting them into the solution (5.2), we can get an approximate value of the constant \( \phi_{j\infty} \) as

\[
\phi_{j\infty} = -\ln[-\lambda_j/(2m_j^2)]/(n_j - 2)m_j + \frac{1}{2m_j} \ln \left[ \frac{n_j^{\frac{2}{n_j - 2}}(2m_j^2)}{\sqrt{E_j^2 + m_j^2Q_j^2}} \right] - \ln 2 - 1 - \frac{2}{n_j - 2}. \tag{5.3}
\]

Furthermore, the accuracy of our approximate result of \( \phi_{j\infty} \) can be improved with the help of exact solutions in special cases. As showed in appendix C, for \( n_j = 4 \) the equation of motion (4.1) can be solved using elliptic functions. Then using the fact that \( |\lambda_j| \) is small, an expansion of \( \phi_{j\infty} \) can also be obtained. By comparing with the result in (5.3), we see that an additional term \( \frac{\ln 2 - 1 - \frac{2}{n_j - 2}}{m_j} \) should be added, and the corrected form of \( \phi_{j\infty} \) becomes

\[
\phi_{j\infty} = -\ln[-\lambda_j/(2m_j^2)]/(n_j - 2)m_j + \frac{1}{2m_j} \ln \left[ \frac{n_j^{\frac{2}{n_j - 2}}(2m_j^2)}{\sqrt{E_j^2 + m_j^2Q_j^2}} \right]. \tag{5.4}
\]

We can compare this form of \( \phi_{j\infty} \) for a given mode \( j \) with its numerical value, obtained by solving the equation of motion (4.1) numerically and substituting a large \( \rho_j \) (here taken to be \( \rho_j = 10^8 \)) into the solution. The result is shown in figure 2. We see that our formula also works for non-integer \( n_j \) and, despite various approximations, the result is quite accurate at the order of \( \lambda_j \). For comparison, we also plot the original \( \phi_{\infty} \), given by (5.3) without correction, which shows that the additional term indeed improves the accuracy of our result.

It is clear from equation (5.4) that, for each mode \( j \), the corresponding \( \phi_{j\infty} \) is different. Note that \( \rho_j(\phi) \) diverges when \( \phi = \phi_{j\infty} \), hence the total volume \( V = \sum_j V_j \rho_j^2 \) will diverge.
when $\phi$ reaches $\phi_\infty = \min\{\phi_j\infty\}$, the smallest one of the different $\phi_j\infty$’s corresponding to different modes. Moreover, when $V$ is large enough, the mode with $\phi_j\infty = \phi_\infty$ will dominate. To the leading order of $\lambda_j$, we have

$$\frac{\partial \phi_j\infty}{\partial m_j} = \frac{\ln[-\lambda_j/(2m_j^2)]}{(n_j - 2)m_j^2}.\]$$

For small $|\lambda_j|$, this derivative is less than 0, thus large $m_j$ will give small $\phi_j\infty$. Therefore, also with interactions the condensate dynamics tends to be dominated by the mode with largest $m_j$, which in general corresponds to small-$j$ modes as in the free case. We note here, anticipating the discussion in section 5.3, that the volume divergence at finite relational time $\phi$ does not necessarily imply the existence of Big Rip singularity. In fact, if we consider the fictitious field $\psi$ with equation of state equals to $w$, then for $n \leq 6$ its energy density $\rho_\psi$ will remain finite for $V \to \infty$; see section 5.3 for details.

We emphasize that the solution (5.2) only works for negative couplings. In fact, if we add another interaction term $\mu_j > 0$, even under the assumption that $|\mu_j| \ll |\lambda_j|$, so that the contribution of $\mu_j$ to the value of $\phi_j\infty$ can be ignored, the behaviour of $\rho_j$ at late times changes considerably. Explicitly, for $\mu_j > 0$, from equation (4.1) we see that, besides the bounce, $\rho_j'(\phi) = 0$ has an additional solution for some large $\rho_j$, determined by

$$\frac{2}{n_j} \lambda_j \rho_j^{n_j+2} = \frac{2}{n_j} \mu_j \rho_j^{n_j'+2},$$

which corresponds to the maximum value of $\rho_j$ (and thus of the volume) at late times. After that, to ensure $\rho_j'$ is real, we should require that $\rho_j$ starts to decrease, and it leads to a periodic evolution of $\rho_j$ and thus a cyclic universe (as in [44]). Since $|\mu_j| \ll |\lambda_j|$, we can take the value of $\phi$ approximately as $\phi \approx \phi_j\infty$ where $\rho_j$ first reaches its maximum. Therefore, in the case with $\mu_j > 0$, instead of being the largest value that $\phi$ can reach (as in the single interaction case), $\phi_j\infty$ now should be regarded as a half-period in the evolution of $\rho_j$, indicating that $\rho_j$ actually starts to decrease for $\phi > \phi_j$. On the other hand, for $\mu_j < 0$, $\rho_j$ can keep growing until $\phi$ reaches $\phi_j\infty$ where $\rho_j$ diverges.

We will see in the next section how the combination of two modes with opposite sign of $\mu_j$ makes it possible for the effective equation of state to cross the phantom divide $w = -1$. 

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**Figure 2.** Asymptotic value $\phi_\infty$ for different $n$. Black solid line is obtained from equation (5.4), the corrected value of $\phi_\infty$. Red dashed line is the uncorrected value of $\phi_\infty$, given by equation (5.3). Blue circles shows the numerical results obtained by solving the equation of motion (4.1) numerically (with $\mu_j = 0$) and set $\rho$ to be large. Parameters are $m^2 = 2$, $E = 9$, $Q^2 = 2.25$, $\lambda = -0.1$. 

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5.2 Phantom crossing in the two-modes case

In this section we consider how the presence of two interacting modes, each with an individual contribution to the cosmological dynamics of the type we have illustrated above, can shape it in very interesting ways at late times.

For simplicity, we use $\rho_{1,2}$ to indicate $\rho_{j_1,j_2}$ and similarly for other parameters. Although in previous sections we have seen that at sufficiently large volume there will be only one mode dominating also in the interacting case, we will see that the inclusion of a second mode does change the behaviour of the effective equation of state $w$, and in particular how the asymptotic value is approached, which is of direct cosmological relevance.

To begin with, we consider the case in which two modes both have a single interaction term, i.e. we set $\mu_1 = \mu_2 = 0$. Since the coupling $\lambda_1$ and $\lambda_2$ are small, $w$ will be dominated by the free part of condensate at small volume, and it will approach $w = 1$ from below as volume grows. This is the needed FLRW universe of the standard cosmological model, reached after the phase close to the big bang, here replaced by a quantum bounce. When the volume becomes larger still, the interaction term for both modes increasingly contributes to the condensate dynamics, until, for large enough values (of $\rho_j$ and thus of the volume), $w$ will be dominated by the interaction terms instead. If we further assume that $n_1 = n_2 = n$, considering only interaction terms in the expression for $w$ would suggest that $w$ only depends on the ratio $r = \rho_2/\rho_1$ (as it was the case also in the free case discussed above), and we have

$$w = 3 - \frac{(2 + n)(V_1 + r^2 V_2)(V_1 \lambda_1 + r^n V_2 \lambda_2)}{2 \left(V_1^2 \lambda_1 + r^{2+n} V_2^2 \lambda_2 - 2 r^{1+\frac{n}{2}} V_1 V_2 \sqrt{\lambda_1 \lambda_2} \right)} =$$

$$= 2 - \frac{n}{2} - \left(\frac{n}{2} + 1\right) \frac{V_1 V_2 r^2 \left(r^{n/2-1} - \sqrt{\lambda_1/\lambda_2} \right)^2}{\left(\sqrt{\lambda_1/\lambda_2} V_1 + V_2 r^{n/2+1}\right)^2}. \quad (5.5)$$

Since the parameters are all real and both couplings $\lambda_1$ and $\lambda_2$ are assumed to be negative, we see that $w < 2 - \frac{n}{2}$. Recall that when the volume is large, one of the two modes will dominate over the other, and then we have $r \to 0$ or $r \to \infty$. In either case $w$ will approach $2 - \frac{n}{2}$ from below, in contrast with the single mode case discussed in section 3.

There is a special case where $r = \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{2}{n+2}}$, then, within the approximation we have made, we see that $w = 2 - \frac{n}{2}$ is also a constant. From our solution (5.2) for each mode at large volume, we see that this indeed happens when $\phi_1 = \phi_2$. In fact, when $\rho_2 = r \rho_1$ is proportional to $\rho_1$, we have $V = V_1 \rho_1^2 + V_2 \rho_2^2 = (V_1 + r^2 V_2) \rho_1^2$, which is the same as the single mode case with a modified $V_1 = V_1 + r^2 V_2$. And therefore the equation of state is the same as in the single mode case, which indeed approaches the asymptotic value from above.

In figure 3 we plot the different behaviour of $w$ in the cases $\phi_1 < \phi_2$ and $\phi_1 = \phi_2$ using numerical solutions of equation of motion (2.11) in the single interaction case $\mu_j = 0$.

At small volume, the evolution is dominated by the free parameters, and the two case are identical. At a larger volume but when $\phi$ is still away from $\phi_1$, the ratio $r = \rho_2(\phi)/\rho_1(\phi)$ changes slowly, and the behaviour of $w$ in the two cases is still almost identical. As the volume grows further, $\phi$ approaches to $\phi_1$, then in the case $\phi_1 < \phi_2$, $\rho_1$ tends to $\infty$ and grows much fast than $\rho_2$, leads to $r \to 0$, and $w$ approaches to the phantom divide $w = -1$ from below. In the same regime, but for $\phi_1 = \phi_2$, we have $r = \lambda_1/\lambda_2$, so the last term in (5.5) vanishes, and $w$ will approach $w = -1$ from above, as in the single mode case.
Figure 3. The behaviour of $w$ in the two modes case, where both modes have only one interaction term. Blue solid line shows the case where $\phi_1^\infty < \phi_2^\infty$, while for red dashed line we have $\phi_1^\infty = \phi_2^\infty$. Two black dotted lines show $w = 1$ and the phantom divide $w = -1$, respectively. Parameters are same as in figure 1 with additional ones are $\lambda_1 = -10^{-8}$, $\mu_1 = 0$, $\mu_2 = 0$, $n_1 = n_2 = 6$ and $\lambda_2 = -9.5 \times 10^{-8}$ for $\phi_1^\infty < \phi_2^\infty$, $\lambda_2 = -9.5725 \times 10^{-8}$ for $\phi_1^\infty = \phi_2^\infty$.

Now we consider the case $n = 6$ and assume that $\phi_1^\infty < \phi_2^\infty$. Then at large volume the first mode will dominate and $r \to 0$. Expanding $w$ in equation (5.5) with respect to $r$ gives simply

$$w = -1 - \frac{4V_2}{V_1} r^2 = -1 - \frac{4V_2 \rho_2(\phi)^2}{V_1 \rho_1(\phi)^2}.$$

Therefore, when $n = 6$ the phantom divide $w = -1$ can be crossed at large volume and the corresponding effective field $\psi$ behaves just like a phantom energy, whose energy density increases as the volume of universe grows.

This is our main result, showing how a phantom-like dark energy dynamics at late times can be produced, under rather general conditions (albeit in a simplified model, and of course in a specific regime of the full theory) purely from quantum gravity effects, i.e. as an effective description of the underlying quantum dynamics of spacetime constituents.

One may then worry about whether this effective phantom energy, like in many field theoretic models, leads to a Big Rip singularity at later times also in our model. We will discuss this issue in the next section, showing that the effective energy density $\rho_\psi$, defined from the equation of state $w$, remains bounded in our model, tending towards to a finite value at asymptotically large volumes. To see this, we need some further approximation for the equation of state $w$, which we anticipate here.

Since $\phi_1^\infty < \phi_2^\infty$, and for large volume we have $\phi \to \phi_1^\infty$, we see that $\rho_2$ is nearly a constant given by $\rho_2(\phi_1^\infty)$. Using the solution (5.2), we get

$$\rho_2(\phi_1^\infty) = \left(\frac{1}{2} \sqrt{-\frac{\lambda_2}{3}}\right)^{-\frac{1}{2}} \frac{1}{(\phi_2^\infty - \phi_1^\infty)^\frac{1}{2}}.$$

Furthermore, when $\phi \to \phi_1^\infty$ the first mode $\rho_1$ would be much larger than $\rho_2$, hence in computing the total volume we can ignore $\rho_2$ and let $V = V_1 \rho_1^2$. Inserting this approximate expression back in the expression for $w$, we get

$$w = -1 - \frac{b}{V}, \quad (5.6)$$
where \( b = 4V \rho^2_{\phi_1\phi_2}(\phi_{1\phi_2}) \) is a constant. Notice again that \( b > 0 \), thus we have \( w < -1 \), and the phantom divide \( w = -1 \) is being crossed.

### 5.3 The Big Rip singularity

We pointed out that in the presence of interactions \( \rho_j \) and hence the volume will diverge at finite relation time \( \phi_\infty = \min\{\phi_j\infty\} \). Now we show why this does not necessarily mean that a Big Rip singularity is reached. Also, the phantom crossing \( w < -1 \) would raise the same worry, but, as we already mentioned, only for constant equation of state. We now see why such singularity does not occur in our setting.

Consider the fictitious field \( \psi \) we introduced with equation of state equals to \( w \). Its energy density \( \rho_\psi \), defined by the equation of state itself, satisfies the conservation equation (3.2). We can then substitute for \( w \) the approximate expression (5.6), to get

\[
\frac{d\rho_\psi}{dV} - \frac{b\rho_\psi}{V^2} = 0.
\]

We can then solve for \( \rho_\psi \) at large volume as

\[
\rho_\psi = \rho_{\psi 0} e^{-\frac{b}{V}} \approx \rho_{\psi 0} - \frac{\rho_{\psi 0} b}{V},
\]

where \( \rho_{\psi 0} \) is a constant of integration, representing the asymptotic value of \( \rho_\psi \) when \( V \to \infty \).

Thus we see that we obtain a constant asymptotic value for the energy density, which has the same effect as a cosmological constant. Therefore our model leads to a de Sitter spacetime asymptotically, with no Big Rip singularity. In fact, our model effectively belongs to the class of models considered in [34], where the Big Rip singularity is avoided even in presence of phantom matter by assuming that \( \rho_\psi \) can be obtained as a constant part plus some matter with negative energy density. Exactly this type of scenario is reproduced from the fundamental quantum gravity dynamics.

Let us stress that, in order to obtain a de Sitter spacetime asymptotically, the requirement that \( w \) approaches to the phantom divide \( w = -1 \) at large volume is a necessary but not sufficient condition. We need also that \( w \) approaches to \( w = -1 \) fast enough, as it happens naturally in our case. To see this, suppose that, when volume \( V \) is larger than some given \( V_0 \), the equation of state can be approximated by

\[
w = -1 - \frac{b}{\ln(V/V_0)}.
\]

Substituting this into the conservation equation (3.2), the evolution of the phantom energy density \( \rho_\psi \) now reads

\[
\rho_\psi = \rho_{\psi 0} [\ln(V/V_0)]^b,
\]

where \( \rho_{\psi 0} \) is again a constant, now given by the energy density at volume \( V = eV_0 \). In this case \( \rho_\psi \) diverges when \( V \to \infty \), and the asymptotic de Sitter spacetime can’t be obtained. Furthermore, as we discussed in section A, such a large energy density will destroy all bound system in the universe at a finite volume and hence should be avoided.

### 5.4 More involved late-time behaviour: combined inflation-like and phantom-like acceleration

We have seen that we can reproduce naturally the late time acceleration behaviour of our observed universe with a single interaction term for each mode. We also have reasons to
expect that the late-time cosmological dynamics is dominated by a single interaction (that of the highest order, if more than one is allowed with comparable weights by the parameters of the model). Thus, we can claim some degree of generality for our main results.

However, it is interesting to ask how the late-time dynamics, after a FLRW phase, is affected by the presence of multiple interactions, for each mode. This could be relevant for further cosmological applications, but it also has purely theoretical motivations. For example, although $n = 6$ interactions are needed to reproduce phantom crossing, most quantum geometric GFT models include $n = 5$ interactions because they come from the simplicial construction of their (lattice gravity and spin foam) amplitudes [2].

So, we conclude our present analysis by considering briefly the case in which two spin modes both have two interactions, with the new couplings being $\mu_1$ and $\mu_2$.

When both $\mu_1$ and $\mu_2$ are less than 0, both modes would produce a divergent condensate density eventually and lead to a similar result as the previous single interaction case. On the other hand, when both $\mu_1$ and $\mu_2$ are positive, there would be a turning point for the condensate density for each mode, after which $\rho_2$ starts to decrease, and the corresponding universe would become cyclic, as in the single mode case. The more interesting case, therefore, is when $\mu_1$ and $\mu_2$ have different signs.

We assume then that $\mu_1 < 0$ while $\mu_2 > 0$. As shown in [44], the mode $\rho_2$ alone can lead to a long-lasting inflationary-like phase. Now, with an additional mode $\rho_1$, we can have both the late time phantom-like acceleration as well as an inflationary-like phase before it.

With two interactions, we have three different cases according to the relative magnitude between $\phi_{1\infty}$ and $\phi_{2\infty}$. Since $\mu_2 > 0$, $\phi_{2\infty}$ would be the half-period of the $\rho_2$ mode rather than the maximum value that $\phi$ can reach as $\rho_2 \to \infty$. For $\phi_{1\infty} < \phi_{2\infty}$, the $\rho_1$ mode would increase faster than the $\rho_2$ mode, and dominate before inflation can end, leading to a similar dynamics as in the single interaction case. On the other hand, for $\phi_{1\infty} > \phi_{2\infty}$, $\rho_2$ will reach its maximum value before $\rho_1$ diverges. For large volume, but with $\phi < \phi_{2\infty}$, the $\rho_2$ mode would dominate and hence inflation can end. But since near $\rho_{2\infty}$, $\rho_2$ decreases very quickly, the total volume will also decrease for a while and then increase again when the $\rho_1$ mode takes over. Let us look at the resulting dynamics in more detail, considering the case where $\phi_{1\infty} = \phi_{2\infty}$ and assuming $n_1 = n_2 = 5$, $n'_1 = n'_2 = 6$.

Since the absolute value of the couplings $|\mu_{1,2}|$ is much less than $|\lambda_{1,2}|$, there would still be a region where the $\lambda$ interaction terms dominate. Furthermore, we can also ignore the influence of $\mu$ terms on the value of $\phi_{1\infty}$, and the solution of each modes can still be given by equation (5.2) in this region, with $\phi_{1\infty} = \phi_{2\infty}$. Then, as we discussed, in such case the ratio $\rho_1/\rho_2$ becomes a constant and the contribution from two modes cancels, leaves a constant equation of state $w = -\frac{1}{2}$ in this region, corresponding to an inflationary-like phase.

As the volume increases, the $\mu$ terms become important. In this region, the equation of state $w$ will increase first, and inflation will end after $w > -1/3$. Afterwards, $w$ decreases again to cross the phantom divide $w = -1$. At very large volume, the equation of state can still be approximated by $w = -1 - b/\mathcal{V}$, only this time with the constant $b$ given by $b = \frac{144 \mathcal{V}_2 \lambda_2^3}{25 \mu_2^2}$, which can be determined with the parameters of the second mode only.

We compared the behaviour of $w$ in two modes case and single mode case in figure 4.

As in the single mode case, table 2 summarizes the effects of parameters on the evolution behaviour in the two modes case. As we have explained, the inflationary phase should be accompanied by a proper exit, hence such phase won’t occur with only a single interaction.
Table 2. The influence of parameters on the cosmological evolution in two modes case. The meaning of symbols is the same as in table 1.

| Interaction | Inflationary phase | Cyclic behaviour | Late time acceleration | Phantom crossing | Big Rip |
|-------------|-------------------|-----------------|------------------------|-----------------|--------|
| Single      | $\phi_{1\infty} = \phi_{2\infty}$ |                | $n \geq 5$             | $n > 6$         | $n > 6$ |
|             | $\phi_{1\infty} < \phi_{2\infty}$ | $n \geq 5$     |                        | $n > 6$         | $n > 6$ |
| Two         | $\mu_1 > 0$, $\mu_2 > 0$ | $n \geq 5$     | $\checkmark$           | $n' \geq 6$    | $n' > 6$ |
| interactions| $\mu_1 < 0$, $\mu_2 > 0$ |                |                        | $n' \geq 6$    | $n' > 6$ |
|             | $\mu_1 < 0$, $\mu_2 > 0$ | $\phi_{1\infty} = \phi_{2\infty}$, $n \geq 5$ | $n' \geq 5$    | $n' \geq 6$    | $n' > 6$ |

Figure 4. The behaviour of $w$ in the interacting case. As in figure 1, the blue sold line shows $w$ in two modes case, red dashed line shows single mode case with $\rho_1$, and magenta dash-dotted line shows single mode case with $\rho_2$. At large volume $w$ for $\rho_1$ and $\rho_2$ differs significantly as the couplings $\mu_1$ and $\mu_2$ have different signs. The two black dotted lines represent $w = -0.5$ and $w = -1$ respectively. In 4(b) we plot the behaviour of $w$ with respect to redshift $z$ in the two modes case. The redshift is defined by $z = a_0/a - 1$, where scale factor $a = \sqrt[3]{V}$ and $a_0$ is its current value. Parameters are same as in figure 1 with additional ones are given by $\lambda_1 = -10^{-8}$, $\mu_1 = -1 \times 10^{-12}$, $\lambda_2 = -1.4757 \times 10^{-7}$, $\mu_2 = 1.2 \times 10^{-12}$ and $n_1 = n_2 = 5$, $n_1' = n_2' = 6$.

term in each mode. Since the case of positive $\lambda$ is already discussed in the single mode case, here we only consider the case $\lambda_1 < 0$ and $\lambda_2 < 0$. Furthermore, for simplicity we set $n_1 = n_2 = n$ and $n_1' = n_2' = n'$.

6 Summary and outlook

In this work, we have analysed the emergent cosmological dynamics corresponding to the mean field hydrodynamics of quantum gravity condensates, within the (tensorial) group field theory formalism.

In particular, we have extended previous analyses in the literature by studying the cosmological effects of fundamental interactions between GFT quanta, the candidate ‘quantum constituents of spacetime’, and on the contributions from different quantum geometric modes associated to them. The general consequence of such interactions is to produce an accelerated
expansion of the universe, which can happen both at early times, after the quantum bounce predicted by the model, and at late times.

We have analysed in detail the properties of such acceleration, by recasting the dynamics of the universe volume in terms of an effective equation of state, encoding the details of the quantum gravity dynamics.

In the early universe right after the quantum bounce replacing the classical big bang singularity, the total volume is small and interaction terms remain subdominant, while we need to take into account all quantum geometric modes. In this regime, we studied whether the acceleration experienced by the universe right after the bounce could be long-lasting enough to have interesting cosmological consequences as a replacement for a later inflationary expansion. We were able to get an upper bound of the ratio $V_e/V_b$ between the volume at the end of acceleration phase and the beginning of the acceleration phase or the bounce. This bound is small under natural assumptions, i.e. the acceleration phase ends quickly after the bounce.

Away from the bounce, as long as the universe volume grows but interactions remain subdominant, one has a standard FLRW phase, whose precise duration depends on the value of the interaction coupling constants (in relation to the free part of the GFT action). If instead interactions become relevant before such FLRW phase is reached, a long-lasting inflation-like expansion can be obtained, which however is not followed by a FLRW phase but by a collapsing phase (producing a cyclic universe).

Further away from bounce, at larger values of the total volume, after the FLRW phase, the interaction terms for each mode become relevant and then dominate over free terms.

In the case of a single interaction term, we solve the equation explicitly. If the effective equation of state is mostly determined by a single mode, one has an effective equation of state $w = 2 - n/2$ for the interaction of order $n$. Then for $n \geq 5$, the expansion of universe is accelerating, approaching its asymptotic value from above. For $n \leq 6$ this means that the phantom divide $w = -1$ cannot be crossed but only approached asymptotically, while for $n > 6$ we have negative finite $w + 1$, such that the reconstructed energy density of the fictitious field with such equation of state would diverge as volume grows, leading to a Big Rip singularity.

If two modes determine the effective equation of state, on the other hand, the resulting cosmological evolution is much more interesting, and provides already, in fact, an observationally viable scenario. The effective $w$ approaches its asymptotic value from below, and the phantom divide $w = -1$ can be crossed without the need of introducing interactions of order higher than 6. For $n = 6$, we get $w < -1$ at large volume, and hence the energy density of the fictitious field will increase as volume grows, just as we expect for a phantom field. On the other hand, $w + 1$ becomes infinitely small as $V \to \infty$, consequently the energy density of the fictitious field approaches to a finite value. Therefore, the Big Rip will not occur, rather, the universe will approach to a de Sitter spacetime asymptotically.

The inclusion of more interaction terms for each mode further complicates the detailed late time expansion, allowing for example for several accelerated phases of different type, but does not change this asymptotic phantom-like behaviour.

Therefore, our main result is that, the emergent cosmological dynamics for GFT condensates produces naturally a phantom-like dark energy dynamics at late times, compatible with cosmological observations and free of future singularities, purely out of quantum gravity effects without the need of any additional phantom matter.

Before turning to a broader outlook from our work, let us point out several aspects in which our analysis can be improved and its results sharpened.
The approximation methods we used to solve the interacting equations could certainly be improved, in particular in the case in which several interaction terms are present. Such improvement could give more details about the interplay of different interaction terms for different modes and provide a better understanding of the potential range of cosmological dynamics of this class of models. Most important, we need to develop numerical as well as analytical techniques to be able to take into account the multitude of quantum geometric modes entering the GFT quantum dynamics. While it is obviously true that we have barely scratched the potential richness of their emergent cosmological dynamics, we should also note that, even with two modes, the asymptotic value of \( w \) is still the same as in the single mode case, only the way that \( w \) approaches this value changed. Therefore we would expect that adding more modes would not change the fact that \( w \) approaches the asymptotic value from below, crossing the phantom divide and thus still producing a phantom-like dark energy dynamics for large volume. In this sense, it is the step we have taken in this work, i.e. from one to two modes, that encodes the main qualitative features of such models for what concerns the late time evolution of our universe, and our results can be expected to be rather general and solid.

Another technical point where more work is needed concerns the form we have used for the GFT interactions. As we noted, we have taken a rather phenomenological approach, by not working with any specific GFT model but with a rather general expression, incorporating some aspects of known models in the isotropic restriction (for example, the fact that different spin modes decouple, as in the EPRL model), but not their detailed expression. This has the advantage of ensuring a certain degree of generality for our results. It should be complemented by a careful analysis of specific GFT models (including the study of their renormalization group flow), to make sure that our expression captures their relevant features at this cosmological level, or to extract new ingredients that need to be added to the phenomenological expression, as potentially changing the resulting cosmological evolution.

As a basis for such effective phenomenological approach, we also used the mean-field approximation, which may not be trusted at late times, where the interactions become large (indeed, recent analyses confirm this worry [46]). But, we emphasize again, in our work the only truly relevant ingredients are encoded in the choice of effective action. We used the simplest (mean field) approximation to it for simplicity, and for a closer contact with previous work in the literature, but one can easily consider a more general setting. The main point of our results is that including more than one mode in such effective action can indeed change the evolution of the universe, especially at late times, where the single mode is expected to be dominating.

More precisely, in order to obtain the expression (2.14) for the total volume, we used the mean-field approximation based on field coherent states. However, for more general states, we do not expect that the template for the derivation of relational volume observable and its dynamics would be much different. Like in ordinary quantum field theory, the generic quantum effective action for GFTs is also a function of the effective mean field corresponding to the expectation value of the field operator in the true vacuum/ground state of the theory (rather than the simple coherent state we used), and a similar approximation in which such mean field is suitably peaked with respect to the relational clock would lead to the desired expression for corresponding observables as well. In this sense, as we have already pointed out in section 2, the effective action we used actually takes into account already the quantum corrections, and shouldn’t be viewed simply as obtaining from the mean-field approximation and subjecting modifications from quantum fluctuations.
Stepping into a more fundamental issue, our analysis, as well as the interpretation of its results, relied on the relational strategy for the definition of observables in a quantum gravity context (see [65, 66] and references cited therein), and in particular for a diffeomorphism invariant notion of temporal evolution. Most recent work on GFT cosmology has adopted the same strategy. For example, both the expression for the effective equation of state and its physical interpretation at different values depends on the interpretation of the scalar degree of freedom we used as a clock as a free massless scalar field. This is consistent with all we currently know about the coupling of such fields in a GFT (and discrete gravity or spin foam) formalism. However, much remains to be understood in this domain, i.e. matter coupling in this quantum gravity context, the construction of material reference frames and the detailed comparison with the corresponding constructions in classical gravitational physics. A more solid understanding of this issue at the interface between GFT quantum gravity and the foundations of spacetime/gravitational physics will provide an even more solid take on the cosmological results we have obtained.

From an even broader perspective, our universe is way too simple to be fully realistic. In our analysis we only considered isotropic and homogeneous universes spacetime, thus ignored the effects from anisotropies and inhomogeneities, even at a perturbative level, on the evolution of the universe. Interesting work in both these directions have been done, in the GFT cosmology literature [67–70]. The same is true, in fact, for the effects of thermal fluctuations of the GFT condensates on the emergent cosmological evolution [71]. With the same aim for a more realistic global picture of the universe evolution, even staying at the homogeneous level, we need to improve our analysis to include additional matter content, starting from general interacting scalar fields [72] but including then also the typical fluid components used in standard cosmological scenarios.

However, beyond their effects on global cosmological evolution, a proper description of cosmological inhomogeneities is what is needed to make solid contact with cosmological observations and truly embed physical cosmology within our quantum gravity framework. This remains our main goal.

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A The effective equation of state

We want to define the equation of state of the content in the universe using only geometrical quantities. From the FLRW equation in a universe filled with different matter contents (represented by $i$)

$$H^2 = \frac{1}{3} \sum_i \rho_i, \quad \dot{H} = -\frac{1}{2} \sum_i (\rho_i + p_i) = \frac{1}{2} \sum_i (1 + w_i) \rho_i,$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, $\dot{\cdot}$ represents derivative respect to comoving time, and $w_i = p_i/\rho_i$ is the equation of state for matter species. We can define an effective equation of
state as

\[ 1 + w = -\frac{2\dot{H}}{3H^2}. \]  

(A.1)

In the relational time \( \phi \), we have

\[ H = \frac{\dot{a}}{a} = \frac{1}{3} \frac{V'}{V} \dot{\phi}. \]

Using the fact that \( \pi_\phi = \dot{\phi}V \) is a conserved quantity, we have \( 0 = \ddot{\phi}V + \dot{\phi} \dot{V} = \ddot{\phi}V + \dot{\phi}^2 V' \), and \( \ddot{\phi} \) can be solved as

\[ \ddot{\phi} = -\frac{V''}{V} \dot{\phi}^2. \]

Therefore we have \[44\]

\[ \dot{H} = \frac{1}{3} \frac{d}{d\phi} \frac{V'}{V} \dot{\phi}^2 + \frac{1}{3} \frac{V'}{V} \ddot{\phi} = \frac{1}{3} \frac{V''}{V} \dot{\phi}^2 - \frac{1}{3} \left( \frac{V'}{V} \right)^2 - \frac{1}{3} \left( \frac{V'}{V} \right)^2 \dot{\phi}^2 = \frac{1}{3} \dot{\phi}^2 \left[ \frac{V''}{V} - 2 \left( \frac{V'}{V} \right)^2 \right]. \]

And the equation of state can be rewritten as

\[ w = 1 - \frac{2 \dot{H}}{3H^2} - 1 = -2 \left[ \frac{V V''}{(V')^2} - 2 \right] - 1 = 3 - \frac{2V V''}{(V')^2}. \]  

(A.2)

When the evolution of the equation of state is known, the evolution of volume of universe can be recovered. To do this, we first introduce the relational Hubble parameter\(^4\)

\[ G = \frac{V'}{V}. \]  

(A.3)

Then the effective equation of state (A.2) can be written by

\[ w = 1 - \frac{2G'}{G^2}. \]  

(A.4)

Suppose that \( w \) is known, the equation is an ordinary differential equation of \( G \) and can be solved by

\[ G = \left( \int_{\phi_0}^{\phi} \frac{w(\chi) - 1}{2} d\chi + \frac{1}{G_0} \right)^{-1}, \]  

(A.5)

where \( G_0 = G(\phi_0) \) is the initial value of \( G \). Then the definition (A.3) of \( G \) becomes a differential equation of volume \( V \), and can be solved as

\[ \ln V = \int_{\phi_0}^{\phi} d\kappa \left( \int_{\phi_0}^{\kappa} \frac{w(\chi) - 1}{2} d\chi + \frac{1}{G_0} \right)^{-1} + \ln V_0, \]  

(A.6)

with \( V_0 = V(\phi_0) \). Hence the evolution of volume respect to relational time \( \phi \) is recovered.

\(^4\)Using the relation between volume and scale factor \( V = a^3 \), we see that the relation between Hubble parameter \( H \) and the relational one \( G \) is \( H = \frac{\dot{a}}{a} = \frac{a'}{a} \phi = \frac{G}{3} \phi. \)
B The consequences of the convergence of total volume \( \mathcal{V} \)

In this appendix we consider how the convergence of \( \mathcal{V} \) will constrain parameters in our model. When \( \phi \) is large, we will have

\[
\sqrt{E_j^2 + m_j^2 Q_j^2} \cosh(2m_j \phi) - E_j > \sqrt{E_j^2 + m_j^2 Q_j^2}.
\]

Therefore, if \( \mathcal{V} \) is convergent, the series

\[
\sum_j \frac{\mathcal{V}_j}{m_j^2} \sqrt{E_j^2 + m_j^2 Q_j^2} \quad (B.1)
\]

must also be convergent. Then if \( m_j \) is unbounded in the sense that \( m_j \to \infty \) for \( j \to \infty \), \( \mathcal{V} \) would certainly be divergent cause in terms with sufficient large \( j \), we will have \( \cosh(2m_j \phi) \to \infty \) for non-zero \( \phi \). Therefore, the convergence of \( \mathcal{V} \) also requires bounded \( m_j \).

Conversely, if series (B.1) is convergent and \( \forall j, m_j \leq m \) with a given \( m \), then

\[
\cosh(2m_j \phi) \leq \cosh(2m \phi),
\]

which leads to the convergent of series

\[
\sum_j \left[ \frac{\mathcal{V}_j}{m_j^2} \sqrt{E_j^2 + m_j^2 Q_j^2} \cosh(2m \phi) \right] = \cosh(2m \phi) \sum_j \frac{\mathcal{V}_j}{m_j^2} \sqrt{E_j^2 + m_j^2 Q_j^2},
\]

since its right hand side is convergent according to our assumption. Therefore, series

\[
\sum_j \left[ \frac{\mathcal{V}_j}{m_j^2} \sqrt{E_j^2 + m_j^2 Q_j^2} \cosh(2m_j \phi) \right] \quad (B.2)
\]

converges as well. Furthermore, since \( E_j < \sqrt{E_j^2 + m_j^2 Q_j^2} \), we see that \( \sum_j \frac{\mathcal{V}_j E_j}{m_j^2} \) is also convergent.

In conclusion, if we require \( \rho_j = 0 \) at the bounce for all \( j \), then \( \mathcal{V} \) is convergent if and only if \( \sum_j \frac{\mathcal{V}_j}{m_j} \sqrt{E_j^2 + m_j Q_j^2} \) converges and \( m_j \)'s are bounded. Just as we referred in section 4.

C Behaviour of \( \phi_{j,\infty} \) in \( n_j = 4 \) case for small \( \lambda_j \)

Here we consider the large \( \rho_j \) behaviour for \( n_j = 4 \) case, where we have an exact solution. In fact, for \( n_j = 4 \), the solution of equation of motion (4.1) with \( \mu_j = 0 \) can be expressed using elliptic functions. With the convention that \( F(\phi, m) = \int_0^\phi \frac{1}{\sqrt{1-m \sin^2(\theta)}} \, d\theta \), we have the solution for a given mode \( j \) with \( \lambda_j < 0 \) [73]

\[
\phi = \sqrt{\frac{2}{-\lambda(\omega_3 - \omega_1)}} F \left( \sin^{-1} \left( \frac{\rho_j^2 - \omega_3}{\rho_j^2 - \omega_2} \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1} \right) \right), \quad (C.1)
\]

where \( \omega_3 > \omega_2 > \omega_1 \) are three real roots of the polynomial

\[
P(\chi) = \chi^3 - \frac{m^2}{2\lambda} \chi^2 - \frac{E}{\lambda} \chi + \frac{2Q^2}{\lambda}, \quad (C.2)
\]
and the solution valids for $\rho_j > \sqrt{\omega_3}$. Note that $|\lambda_j|$ should be small enough such that the three roots of the polynomial (C.2) are all real. Setting $\rho_j \to \infty$ in the solution (C.1), we get the exact asymptotic value $\phi_{j\infty}$ in $n_j = 4$ case

$$\phi_{j\infty} = \sqrt{\frac{2}{-\lambda_j(\omega_3 - \omega_1)}} K \left( \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1} \right). \quad (C.3)$$

Now we consider the behaviour of this $\phi_{j\infty}$ for small $|\lambda_j|$. To do this, we need first find the approximate roots for the polynomial (C.2). At the first order of $\lambda_j$, these roots are

$$\omega_1 = \frac{2m_j^2}{2\lambda_j} + \frac{2E_j}{m_j^2} + \frac{2E_j^2}{m_j^2} \lambda_j,$$

$$\omega_2 = \frac{Q_j^2}{E_j - \sqrt{E_j^2 + m_j^2Q_j^2}} + \frac{Q_j^4}{4m_j^2(E_j - \sqrt{E_j^2 + m_j^2Q_j^2})^2} \lambda_j,$$

$$\omega_3 = \frac{Q_j^2}{E_j + \sqrt{E_j^2 + m_j^2Q_j^2}} + \frac{Q_j^4}{4m_j^2(E_j + \sqrt{E_j^2 + m_j^2Q_j^2})^2} \lambda_j.$$

Then, putting these approximation of roots into equation (C.3), we can further expand $\phi_{j\infty}$ with respect to small $\lambda_j$ using the expansion $K(x) \to \ln \frac{4}{\sqrt{1-x}}$ for $x \to 1$, and we will obtain the same result as given by the corrected value (5.4) of $\phi_{j\infty}$.

References

[1] D. Oriti, *Levels of spacetime emergence in quantum gravity*, arXiv:1807.04875 [nSPIRE].

[2] D. Oriti, *The microscopic dynamics of quantum space as a group field theory*, in Foundations of Space and Time: reflections on quantum gravity, pp. 257–320 (2011) [arXiv:1110.5606] [nSPIRE].

[3] T. Krajewski, *Group field theories*, in Proc. 3rd Quantum Gravity Quantum Geom. Sch. PoS QGQGS 2011 (2013) 005.

[4] S. Carrozza, *Flowing in group field theory space: a review*, SIGMA 12 (2016) 070 [arXiv:1603.01902] [nSPIRE].

[5] D. Oriti, *Group field theory and loop quantum gravity*, (2014) [arXiv:1408.7112] [nSPIRE].

[6] V. Rivasseau, *The Tensor Track: an Update*, in 29th International Colloquium on Group-Theoretical Methods in Physics, (2012) [arXiv:1209.5284] [nSPIRE].

[7] V. Rivasseau, *Random tensors and quantum gravity*, SIGMA 12 (2016) 069 [arXiv:1603.07278] [nSPIRE].

[8] V. Rivasseau, *The Tensor Track, IV*, PoS CORFU2015 (2016) 106 [arXiv:1604.07860] [nSPIRE].

[9] N. Delporte and V. Rivasseau, *The Tensor Track V: Holographic Tensors*, in 17th Hellenic School and Workshops on Elementary Particle Physics and Gravity, (2018) [arXiv:1804.11101] [nSPIRE].

[10] M. Finocchiaro and D. Oriti, *Renormalization of Group Field Theories for Quantum Gravity: New Computations and Some Suggestions*, Front. in Phys. 8 (2021) 552354 [arXiv:2004.07361] [nSPIRE].
[11] A.G.A. Pithis and J. Thürigen, *Phase transitions in TGFT: functional renormalization group in the cyclic-melonic potential approximation and equivalence to O(N) models*, JHEP 12 (2020) 159 [arXiv:2009.13588] [insPIRE].
[12] S. Gielen, D. Oriti and L. Sindoni, *Homogeneous cosmologies as group field theory condensates*, JHEP 06 (2014) 013 [arXiv:1311.1238] [insPIRE].
[13] D. Oriti, L. Sindoni and E. Wilson-Ewing, *Emergent Friedmann dynamics with a quantum bounce from quantum gravity condensates*, Class. Quant. Grav. 33 (2016) 224001 [arXiv:1602.05881] [insPIRE].
[14] S. Gielen and L. Sindoni, *Quantum Cosmology from Group Field Theory Condensates: a Review*, SIGMA 12 (2016) 082 [arXiv:1602.08104] [insPIRE].
[15] D. Oriti, *The universe as a quantum gravity condensate*, Comptes Rendus Physique 18 (2017) 235 [arXiv:1612.09521] [insPIRE].
[16] A.G.A. Pithis and M. Sakellariadou, *Group field theory condensate cosmology: An appetizer*, Universe 5 (2019) 147 [arXiv:1904.00598] [insPIRE].
[17] L. Marchetti and D. Oriti, *Quantum fluctuations in the effective relational GFT cosmology*, Front. Astron. Space Sci. 8 (2021) 683649 [arXiv:2010.09700] [insPIRE].
[18] L. Marchetti and D. Oriti, *Effective relational cosmological dynamics from Quantum Gravity*, JHEP 05 (2021) 025 [arXiv:2008.02774] [insPIRE].
[19] A. Ashtekar and P. Singh, *Loop quantum cosmology: a status report*, Class. Quant. Grav. 28 (2011) 213001 [arXiv:1108.0893] [insPIRE].
[20] M. Bojowald, *Critical evaluation of common claims in loop quantum cosmology*, Universe 6 (2020) 36 [arXiv:2002.05703] [insPIRE].
[21] P. Brax, *What makes the Universe accelerate? A review on what dark energy could be and how to test it*, Rept. Prog. Phys. 81 (2018) 016902 [insPIRE].
[22] C.P. Burgess, *The cosmological constant problem: why it’s hard to get Dark Energy from Micro-physics*, in 100e Ecole d’Ete de Physique: Post-Planck Cosmology (2015) 147 [arXiv:1309.4133] [insPIRE].
[23] R.R. Caldwell, *A Phantom menace?*, Phys. Lett. B 545 (2002) 23 [astro-ph/9908168] [insPIRE].
[24] D.L. Shafer and D. Huterer, *Chasing the phantom: A closer look at Type Ia supernovae and the dark energy equation of state*, Phys. Rev. D 89 (2014) 063510 [arXiv:1312.1688] [insPIRE].
[25] G.-B. Zhao et al., *Dynamical dark energy in light of the latest observations*, Nature Astron. 1 (2017) 627 [arXiv:1701.08165] [insPIRE].
[26] Y. Wang, L. Pogosian, G.-B. Zhao and A. Zucca, *Evolution of dark energy reconstructed from the latest observations*, Astrophys. J. Lett. 869 (2018) L8 [arXiv:1807.03772] [insPIRE].
[27] S.M. Carroll, M. Hoffman and M. Trodden, *Can the dark energy equation-of-state parameter w be less than -1?*, Phys. Rev. D 68 (2003) 023509 [astro-ph/0301273] [insPIRE].
[28] R. Saitou and S. Nojiri, *Stable phantom-divide crossing in two scalar models with matter*, Eur. Phys. J. C 72 (2012) 1946 [arXiv:1203.1442] [insPIRE].
[29] S. Nojiri and S.D. Odintsov, *Modified gravity and its reconstruction from the universe expansion history*, J. Phys. Conf. Ser. 66 (2007) 012005 [hep-th/0611071] [insPIRE].
[30] K. Bamba, C.-Q. Geng, S. Nojiri and S.D. Odintsov, *Crossing of the phantom divide in modified gravity*, Phys. Rev. D 79 (2009) 083014 [arXiv:0810.4296] [insPIRE].
[31] K.J. Ludwick, *The viability of phantom dark energy: A review*, Mod. Phys. Lett. A 32 (2017) 1730025 [arXiv:1708.06981] [insPIRE].
[32] L.P. Chimento, R. Lazkoz, R. Maartens and I. Quiros, *Crossing the phantom divide without phantom matter*, JCAP 09 (2006) 004 [astro-ph/0605450] [insPIRE].
[33] B. McInnes, *The Phantom divide in string gas cosmology*, Nucl. Phys. B 718 (2005) 55 [hep-th/0502209] [insPIRE].
[56] G. Alestas, L. Kazantzidis and L. Perivolaropoulos, *H₀ tension, phantom dark energy, and cosmological parameter degeneracies*, *Phys. Rev. D* **101** (2020) 123516 [arXiv:2004.08363] [SPIRE].

[57] A.G. Riess, S. Casertano, W. Yuan, L.M. Macri and D. Scolnic, *Large Magellanic Cloud Cepheid Standards Provide a 1% Foundation for the Determination of the Hubble Constant and Stronger Evidence for Physics beyond ΛCDM*, *Astrophys. J.* **876** (2019) 85 [arXiv:1903.07603] [SPIRE].

[58] PLANCK collaboration, *Planck 2018 results. VI. Cosmological parameters*, *Astron. Astrophys.* **641** (2020) A6 [Erratum ibid. 652 (2021) C4] [arXiv:1807.06209] [SPIRE].

[59] P. Singh, *Effective state metamorphosis in semi-classical loop quantum cosmology*, *Class. Quant. Grav.* **22** (2005) 4203 [gr-qc/0502086] [SPIRE].

[60] D. Samart and B. Gumjudpai, *Phantom field dynamics in loop quantum cosmology*, *Phys. Rev. D* **76** (2007) 043514 [arXiv:0704.3414] [SPIRE].

[61] J. Haro, J. Amoros and E. Elizalde, *Fate of the phantom dark energy universe in semiclassical gravity*, *Phys. Rev. D* **83** (2011) 123528 [SPIRE].

[62] H. Zhang, *Crossing the phantom divide*, arXiv:0909.3013 [SPIRE].

[63] S. Gielen, *Emergence of a low spin phase in group field theory condensates*, *Class. Quant. Grav.* **33** (2016) 224002 [arXiv:1604.06023] [SPIRE].

[64] J. Martin, *Cosmic Inflation: Trick or Treat?*, arXiv:1902.05286 [SPIRE].

[65] J. Tambornino, *Relational Observables in Gravity: a Review*, *SIGMA* **8** (2012) 017 [arXiv:1109.0740] [SPIRE].

[66] P.A. Hoehn, A.R.H. Smith and M.P.E. Lock, *Trinity of relational quantum dynamics*, *Phys. Rev. D* **104** (2021) 066001 [arXiv:1912.00033] [SPIRE].

[67] S. Gielen, *Inhomogeneous universe from group field theory condensate*, *JCAP* **02** (2019) 013 [arXiv:1811.10639] [SPIRE].

[68] S. Gielen and D. Oriti, *Cosmological perturbations from full quantum gravity*, *Phys. Rev. D* **98** (2018) 106019 [arXiv:1709.01096] [SPIRE].

[69] F. Gerhardt, D. Oriti and E. Wilson-Ewing, *Separate universe framework in group field theory condensate cosmology*, *Phys. Rev. D* **98** (2018) 066011 [arXiv:1805.03099] [SPIRE].

[70] S. Gielen and D. Oriti, *Cosmological perturbations from full quantum gravity*, *Phys. Rev. D* **98** (2018) 106019 [arXiv:1709.01096] [SPIRE].

[71] F. Gerhardt, D. Oriti and E. Wilson-Ewing, *Separate universe framework in group field theory condensate cosmology*, *Phys. Rev. D* **98** (2018) 066011 [arXiv:1805.03099] [SPIRE].

[72] Y. Li, D. Oriti and M. Zhang, *Group field theory for quantum gravity minimally coupled to a scalar field*, *Class. Quant. Grav.* **34** (2017) 195001 [arXiv:1701.08719] [SPIRE].

[73] I.S. Gradshteyn, A. Jeffrey and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press (1996).