Operator estimates for non-periodically perforated domains: disappearance of cavities

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ABSTRACT
We consider a boundary value problem for a general second-order linear equation in a perforated domain. The perforation is made by small cavities, a minimal distance between the cavities is also small. We impose minimal natural geometric conditions on the shapes of the cavities and no conditions on their distribution in the domain. On the boundaries of the cavities, a nonlinear Robin condition is imposed. The sizes of the cavities and the minimal distance between them are supposed to satisfy a certain simple condition ensuring that under the homogenization the cavities disappear and we obtain a similar problem in a non-perforated domain. Our main results state the convergence of the solution of the perturbed problem to that of the homogenized one in $W^{1,2}$ and $L^{2}$-norms uniformly in $L^{2}$-norm of the right-hand side in the equation and provide the estimates for the convergence rates.

1. Introduction

Over past twenty years, a new direction devoted to so-called operator estimates appeared in the homogenization theory. Here the boundary value problems are treated in terms of the spectral theory, that is, the solutions are regarded as actions of the resolvents of the operators corresponding to the considered boundary value problems. The efforts are focused on proving the convergence of the resolvents in appropriate operator norms, that is, on proving the norm resolvent convergence. The operator estimates are ones for the convergence rates.

Elliptic boundary value problems in domains with fine perforation distributed along an entire domain are classical in the homogenization theory. There are a lot of works devoted to studying such problems and, not being able to cite all of them, we just mention a few books \textsuperscript{[1–5]}, as well as papers \textsuperscript{[6–8]}, see also the references therein. The classical results stated the convergence in a strong or weak sense in $L^{2}$ and $W^{1,2}$ for fixed right-hand sides in the equations and boundary conditions. Operator estimates for such problems were recently obtained in few papers \textsuperscript{[9–13]} for periodic and close-to-periodic perforations. In \textsuperscript{[11–13]}, the case of the Neumann condition was studied in the situation when the sizes of the cavities were of the same order as the distances between them; the perforation was pure periodic. The case of the Dirichlet condition on the boundaries of the cavities was addressed in \textsuperscript{[10]}; here the sizes of the cavities and the distances between them satisfied certain relation. The cavities were of the same shape but it was allowed to rotate them arbitrary and to locate almost arbitrary within the periodicity cell. In \textsuperscript{[9, 14]}, there was considered the case of a pure
periodic perforation by small balls with the Dirichlet or Neumann [14] or Robin [9] condition on the boundaries. In all cited papers, various operator estimates were obtained.

The case of a non-periodic perforation was addressed in [15] for a manifold perforated by arbitrary cavities with the Dirichlet or Neumann condition; as the operator, the Laplacian served. The main results were the operator estimates, which were proved under assuming the validity of certain local estimates for \( L_2 \)-norms in terms of \( W_2^1 \)-norms. These local estimates were the key ingredients in the proofs of the operator estimates. As an example, it was shown that these local estimates are valid for the perforation by small balls.

There is also a close direction on studying operator estimates for domain perforated along a given manifold, see [16–18]. Here the perforation was essentially non-periodic: the shapes of the cavities and their distribution were arbitrary and satisfied only minimal geometric assumptions. The main results of the cited papers were the operator estimates, which were shown to be order sharp in many cases.

In two very recent papers [19, 20], an elliptic boundary value problem in a perforated domain was considered. The equation was defined by a general second-order differential expression with variable coefficients, which was not formally symmetric. The perforation was arbitrary and made along the entire domain. Each cavity had its own shape and the distribution of the cavities in the domain was almost arbitrary. On the boundary of each cavity, the Dirichlet or a nonlinear Robin condition was imposed; both boundary conditions were allowed to be present on different boundaries at the same time. The sizes of the cavities and the mutual distances between them satisfied certain conditions, which determined then the form of the homogenized problems. In [19], these conditions were chosen so that a solution to the perturbed problem tended to zero uniformly in the right-hand side in the equation as the perforation became finer. In [20], similar conditions ensured the appearance of a strange term in the homogenized problem. In both papers, the estimates for the convergence rates were obtained; the order sharpness of these estimates was discussed.

In the present paper, we continue studying the general model proposed in [19, 20] but in a different situation. Namely, now we impose only the nonlinear Robin condition on the boundaries of the cavities and impose conditions ensuring that under the homogenization the cavities disappear and make no contribution to the homogenized problem. In the considered case, we succeed to make only minimal and very natural assumptions about the perforation, which are satisfied for a very wide class of non-periodic perforations: we have almost no conditions on the shapes of the cavities and on their distribution. Our main results are the operator estimates in the considered case. More precisely, we estimate the \( W_2^1 \)- and \( L_2 \) norms of the difference of the solutions to the perturbed and homogenized problems uniformly in \( L_2 \)-norm of the right-hand side in the equation. The established inequalities provide the estimates for the convergence rates.

2. Problem and main results

Let \( \Omega \) be an arbitrary domain in \( \mathbb{R}^d \), \( d \geq 2 \), which can be bounded or unbounded. If it has a boundary, its smoothness is supposed to be \( C^2 \). Let \( M_k^\varepsilon \in \Omega \), \( k \in \mathbb{N}^\varepsilon \), be a family of points, where \( \varepsilon \) is a small positive parameter and \( \mathbb{N}^\varepsilon \) is a set of indices, which is at most countable, and \( \omega_{k,\varepsilon} \subset \mathbb{R}^d \), \( k \in \mathbb{N}^\varepsilon \), be a family of bounded non-empty domains with \( C^1 \)-boundaries. We define

\[
\omega_k^\varepsilon := \{ x : \varepsilon^{-1} \eta^{-1}(\varepsilon)(x - M_k^\varepsilon) \in \omega_{k,\varepsilon} \}, \quad k \in \mathbb{N}^\varepsilon, \quad \theta^\varepsilon := \bigcup_{k \in \mathbb{N}^\varepsilon} \omega_k^\varepsilon,
\]

where \( \eta = \eta(\varepsilon) \) is a given function such that \( 0 < \eta(\varepsilon) \leq 1 \). A perforated domain is introduced as \( \Omega^\varepsilon := \Omega \setminus \theta^\varepsilon \).

In the vicinity of the boundaries \( \partial \omega_{k,\varepsilon} \) we introduce local variable \((\tau, s)\), where \( \tau \) is the distance measured along the normal vector to \( \partial \omega_{k,\varepsilon} \) directed inside \( \omega_{k,\varepsilon} \) and \( s \) are some local variables on \( \partial \omega_{k,\varepsilon} \). By \( B_r(M) \) we denote an open ball in \( \mathbb{R}^d \) of a radius \( r \) centered at a point \( M \). The cavities \( \omega_k^\varepsilon \) are supposed to satisfy the following geometric assumption.
Assumption 1: The points $M_k^\varepsilon$ and the domains $\omega_{k,\varepsilon}$ obey the conditions

$$B_{R_1}(y_{k,\varepsilon}) \subseteq \omega_{k,\varepsilon} \subseteq B_{R_2}(0), \quad B_{R_3}(M_k^\varepsilon) \cap B_{R_3}(M_j^\varepsilon) = \emptyset, \quad \text{dist}(M_k^\varepsilon, \partial\Omega) \geq R_3\varepsilon,$$

where $y_{k,\varepsilon}$ are some points, $k \neq j$, $k, j \in \mathbb{I}^\varepsilon$, and $R_1 < R_2 < R_3$ are some fixed constants independent of $\varepsilon$, $\eta$, $k$ and $j$. The sets $B_{R_1}(0) \setminus \omega_{k,\varepsilon}$ are connected. For each $k \in \mathbb{I}^\varepsilon$ there exist a local variable $s$ on $\partial\omega_{k,\varepsilon}$ such that the variables $(\tau, s)$ are well-defined at least on $\{x \in \mathbb{R}^d : \text{dist}(x, \partial\omega_{k,\varepsilon}) \leq \tau_0\} \subseteq B_{R_3}(0)$, where $\tau_0$ is a fixed constant independent of $k \in \mathbb{I}^\varepsilon$ and $\varepsilon$. The Jacobians corresponding to passing from variables $x$ to $(\tau, s)$ are separated from zero and bounded from above uniformly in $\varepsilon$, $k \in \mathbb{I}^\varepsilon$ and $x$ as $\text{dist}(x, \partial\omega_{k,\varepsilon}) \leq \tau_0$. The first derivatives of $x$ with respect to $(\tau, s)$ and of $(\tau, s)$ with respect to $x$ are bounded uniformly in $\varepsilon$, $k \in \mathbb{I}^\varepsilon$ and $x$ as $\text{dist}(x, \partial\omega_{k,\varepsilon}) \leq \tau_0$.

By $L_\infty(\Omega; \mathbb{C}^n)$, $n \geq 1$, we denote the space of vector function with values in $\mathbb{C}^n$, each component of which is an element of $L_\infty(\Omega)$. The space $L_\infty(\Omega; \mathbb{C}^n)$ is equipped with the standard norm

$$\|u\|_{L_\infty(\Omega; \mathbb{C}^n)} := \text{ess sup}_{x \in \Omega} |u(x)|.$$

By $\mathbb{M}_n$ we denote the linear space of all $n \times n$ matrices. The symbol $L_\infty(\Omega; \mathbb{M}_n)$ stands for the space of matrix functions with values in $\mathbb{M}_n$, each entry of which belongs to $L_\infty(\Omega)$. Throughout the work, we shall also employ similar Lebesgue and Sobolev spaces of vector- and matrix-valued functions; the symbols $\mathbb{C}^n$ and $\mathbb{M}_n$ will indicate a space, to which the values of these functions belong.

Let $A_{ij} = A_{ij}(x)$, $A_j = A_j(x)$, $A_0 = A_0(x)$ be matrix functions with complex entries defined on the domain $\Omega$ and obeying the conditions

$$A_{ij} \in W^1_\infty(\Omega; \mathbb{M}_n), \quad A_j, A_0 \in L_\infty(\Omega; \mathbb{M}_n),$$

$$\text{Re} \sum_{i,j=1}^d (A_{ij}(x)z_i, z_j)_{\mathbb{C}^n} \geq c_1 \sum_{i=1}^d |z_i|^2, \quad x \in \Omega, z_i \in \mathbb{C}^n,$$

where $c_1 > 0$ is some fixed constant independent of $x \in \Omega$ and $z_i \in \mathbb{C}^n$. Here the size $n$ of the considered matrix functions can be chosen arbitrarily and it is independent of the dimension $d$.

By $a^\varepsilon = a^\varepsilon(x, u)$, we denote a measurable vector function with values in $\mathbb{C}^n$ defined on $\partial\theta^\varepsilon \times \mathbb{C}^n$ and satisfying the following conditions:

$$|a^\varepsilon(x, u_1) - a^\varepsilon(x, u_2)| \leq \mu(\varepsilon)|u_1 - u_2|, \quad a^\varepsilon(x, 0) = 0,$$

where $\mu(\varepsilon)$ is some nonnegative function independent of $x \in \partial\theta^\varepsilon$ and $u_1, u_2 \in \mathbb{C}^n$. We assume that for the functions $\eta$ and $\mu$ the convergences hold:

$$\left( \varepsilon \eta(\varepsilon) \varkappa(\varepsilon) + \varepsilon^{-1} \eta^{d-1}(\varepsilon) \right) \mu(\varepsilon) \to +0, \quad \eta(\varepsilon) \to +0, \quad \varepsilon \to +0,$$

$$\varkappa(\varepsilon) := |\ln \eta(\varepsilon)| + 1 \text{ as } d = 2, \quad \varkappa(\varepsilon) := 1 \text{ as } d \geq 3.$$

In this paper, we consider the following boundary value problem:

$$(L - \lambda)u_\varepsilon = f \text{ in } \Omega^\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial\nu} + a^\varepsilon(x, u_\varepsilon) = 0 \text{ on } \partial\theta^\varepsilon, \quad u_\varepsilon = 0 \text{ on } \partial\Omega.$$

Here $L$ and $\frac{\partial}{\partial\nu}$ are a differential expression and a conormal derivative:

$$L := -\sum_{i,j=1}^d \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \sum_{j=1}^d A_j \frac{\partial}{\partial x_j} + A_0, \quad \frac{\partial}{\partial\nu} = \sum_{i,j=1}^d v_i A_{ij} \frac{\partial}{\partial x_j},$$

$f \in L_2(\Omega; \mathbb{C}^n)$ is an arbitrary vector function, $\lambda \in \mathbb{C}$ is a fixed constant, $\nu = (v_1, \ldots, v_d)$ is the unit normal to $\partial\theta^\varepsilon$ directed inside $\theta^\varepsilon$. Our aim is to prove that, under certain conditions, the solution of
this problem converges to that of a homogenized problem
\[(\mathcal{L} - \lambda)u_0 = f \text{ in } \Omega, \quad u_0 = 0 \text{ on } \partial \Omega,\] (7)
and to estimate $W^1_2$- and $L_2$-norms of $u_e - u_0$ uniformly in the $L_2$-norm of the right-hand side $f$.

Solutions to problems (6) and (7) are understood in the generalized sense. Namely, a generalized solution to problem (6) is a vector function $u_e \in W^1_2(\Omega^\varepsilon; \mathbb{C}^n)$ with the zero trace on $\partial \Omega$ such that
\[h^\varepsilon(u_\varepsilon, v) - \lambda(u_\varepsilon, v)_{L_2(\Omega^\varepsilon; \mathbb{C}^n)} = (f, v)_{L_2(\Omega^\varepsilon; \mathbb{C}^n)}\] (8)
for each $v \in W^1_2(\Omega^\varepsilon; \mathbb{C}^n)$ with the zero trace on $\partial \Omega$, where
\[
h(u, v) := h(u, v) + (d^\varepsilon(\cdot, u), v)_{L_2(\partial \Omega^\varepsilon; \mathbb{C}^n)},
\h(u, v) := \sum_{i,j=1}^d \left( A_{ij}^\varepsilon \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega^\varepsilon; \mathbb{C}^n)} + \sum_{j=1}^d \left( A_j^\varepsilon \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega^\varepsilon; \mathbb{C}^n)} + (A_0 u, v)_{L_2(\Omega^\varepsilon; \mathbb{C}^n)}.
\]

We denote:
\[\kappa(\varepsilon) := 0 \text{ as } d = 2, 3, \quad \kappa(\varepsilon) := |\ln \eta(\varepsilon)|^{\frac{1}{d}} + 1 \text{ as } d = 4, \quad \kappa(\varepsilon) := 1 \text{ as } d \geq 5.\]

Our main result is as follows.

**Theorem 2.1:** Let Assumption 1 and (5) be satisfied. Then there exists a fixed $\lambda_0 \in \mathbb{R}$ independent of $\varepsilon$ such that as $\text{Re} \lambda \leq \lambda_0$, problems (6), (7) are uniquely solvable for each $f \in L_2(\Omega; \mathbb{C}^n)$ and the solutions satisfy the estimate:
\[\|u_\varepsilon - u_0\|_{W_2^1(\Omega^\varepsilon; \mathbb{C}^n)} \leq C(\lambda) \left( (\varepsilon^2 \eta^2 \kappa + \eta^2 \varkappa^2 + \varepsilon^{-1} \eta^{d-1} \mu + \varepsilon \eta \varkappa^2 + \eta^2) \right) \|f\|_{L_2(\Omega; \mathbb{C}^n)},\] (9)
where $C$ is some constant independent of $\varepsilon$ and $f$. If, in addition, $A_j \in W_\infty^1(\Omega; \mathbb{C}^n)$, then
\[\|u_\varepsilon - u_0\|_{L^2(\Omega^\varepsilon; \mathbb{C}^n)} \leq C \left( (\varepsilon^2 \eta^2 \kappa + \eta^2 \varkappa^2 + \varepsilon^{-1} \eta^{d-1} \mu + \varepsilon \eta \varkappa^2 + \eta^2) \right) \|f\|_{L_2(\Omega; \mathbb{C}^n)} + \left( \varepsilon \eta \varkappa^2 + \eta^2 \right) \|f\|_{L_2(\Omega^\varepsilon; \mathbb{C}^n)},\] (10)
where $C$ is some constant independent of $\varepsilon$ and $f$.

Let us briefly discuss the problem and main results. We consider a general non-periodic perforation under minimal conditions both for the distribution of the cavities and for their shapes introduced in Assumption 1. The first two-sided relation in (1) says that all cavities are approximately of the same size (but not the shapes!), while two other relations in (1) just mean that all cavities are mutually disjoint and do not intersect the boundary of the domain $\Omega$. The minimal distance between the points $M_0^\varepsilon$ exceeds $2R_3\varepsilon$ and in view of the first relation in (1) the minimal distance between the cavities is $2(R_3 - R_2)\varepsilon$. At the same time, there is no a-priori upper bound for the mutual distances between the cavities. In particular, our model covers also the situations, when some of the distances between some cavities are much larger than $2(R_3 - R_2)\varepsilon$, for instance, these distances can be finite. The total number of the cavities can be also arbitrary including the case when this number is finite and the cavities are separated by finite non-small distances.

The regularity of the boundaries of the cavities postulated in Assumption 1 is rather natural and not very restrictive. The stated existence of local variables $(\tau, s)$ is an implicit condition for the uniform (in $k$) regularity of the shapes of the cavities and excludes, for instance, the situation, when the boundaries of the cavities have increasing oscillations on some sequence of the values of $k$. We stress that these...
conditions do not mean that all cavities are of the same shapes. Our conditions on the perforation are very weak and natural and we in fact deal with a very large class of non-periodic perforations.

On the boundaries of all cavities, we impose the nonlinear Robin condition, see (6). The growth of the nonlinearity is controlled by conditions (4). They mean that the nonlinearity is allowed to have at most a linear growth in $u$, while the parameter $\mu$ describes the strength of the nonlinearity. In view of usual restrictions imposed for the nonlinearity in the Robin condition in earlier works on perforated domains, our conditions (4), (5) are quite reasonable and not very strict. Of course, there is a very general framework for defining nonlinear term $a^\varepsilon$, which allows one to impose very general boundary conditions on the boundaries of the cavities, including Signorini condition. This approach is based on the notion of a maximal monotone operator, which maps a given Banach space into subsets of its adjoint space, see, for instance, [8], [1, Ch. 2, Sec. 2.1]. But once we deal only with the usual nonlinear Robin condition for linear elliptic equations, usual conditions imposed in earlier works included the identity $a^\varepsilon(x, 0) = 0$. Other assumptions for $a^\varepsilon(x, u)$ were either the monotonicity with respect to $u$ (see, for instance, [7]) or some kind of Lipschitz condition (see, for instance, [6]). From this point of view, our conditions (4), (5) are similar to ones in earlier works. Comparing these conditions with those in papers [19, 20], we can say that in the present paper we do not suppose the existence of the cavities, for which the nonlinearity in the Robin condition is positive and large in certain sense. For small $\varepsilon$ and $\eta$, such boundary condition behaved in [19, 20] similar to the Dirichlet condition and the cavities with such boundary condition played an important role in the cited papers. In this paper, such cavities are excluded and this is one of the reasons why the homogenized problem tracks no information about the cavities.

We consider the sizes of all cavities are of order $O(\varepsilon \eta)$ and the parameters $\eta$ and $\mu$ are to satisfy convergences (5). This is one of the main assumptions ensuring that under the homogenization the cavities disappear and make no contribution to the limiting (homogenized) problem. We also stress that the differential expression $\mathcal{L}$ involved in the equations in these problems is of a general form with variable complex-valued matrix coefficients and this expression is not formally symmetric. It is also important to say that since the coefficients are matrix functions, we in fact deal with a system of scalar equations.

Our main theorem states that under the above discussed conditions, the solution of problem (6) converges to that of the homogenized problem uniformly in $L_2(\Omega)$-norm of the right-hand side in the equation. In the case of the linear Robin condition, this means that the linear operator associated with the perturbed problem converges to that associated with the homogenized problem in the norm resolvent sense. Our inequality (9) also provides an estimate for the convergence rate in the case when the difference of the solutions to the perturbed and homogenized problems is estimated in $W^1_2(\Omega^\varepsilon)$-norm. In our second estimate (10), this difference is estimated in $L_2(\Omega^\varepsilon)$-norm and the convergence rate is again provided. Here we see that the coefficient at $\mu$ is the same as in (9), while the coefficients without $\mu$ at $\|f\|_{L_2(\Omega; C^n)}$ have the smallness order twice more than the similar coefficient in (9). This is due to the fact that in (10) we estimate a weaker norm of the difference $u_\varepsilon - u_0$ than in (9). There is also an additional term in (10) depending on $\|f\|_{L_2(\theta^\varepsilon; C^n)}$. The coefficient at this norm is the same as in (9) at $\|f\|_{L_2(\Omega; C^n)}$. At the same time, the values of the function $f$ in $\theta^\varepsilon$ make no influence on the solution of the perturbed problem. In view of this fact, we could a-priori suppose that $f$ vanishes on $\theta^\varepsilon$ for the considered values of $\varepsilon$ but in this case the right-hand side becomes $\varepsilon$-dependent and the same concerns $u_0$. Nevertheless, in this situation Theorem 2.1 still makes sense since it describes the error we make by replacing the perforated domain by the non-perforated one and this error is estimated uniformly in the right-hand side $f$. If we consider the case of some fixed $\varepsilon$-independent function $f$ then it is reasonable to suppose that it is non-zero on $\theta^\varepsilon$. In such situation, the presence of the additional term in (10) concerns estimating the contribution of the restriction of $f$ on $\theta^\varepsilon$ into the function $u_0$ and not to the difference $u_\varepsilon - u_0$.

The above described operator estimates, which are uniform in $L_2(\Omega)$-norm of the right-hand side in the equation in (6) and which are established for general non-periodic perforation, are the main advantage in comparison with earlier results. The first main feature is the non-periodicity of the
3. Operator estimates

In this section, we prove Theorem 2.1. We begin with auxiliary lemmata proved in [19, 20]. The first of them is Lemma 3.6 from [19].

Lemma 3.1: Under Assumption 1 for all $k \in \mathbb{I}^e$ and all $u \in W^1_2(B_{\varepsilon R_3}(M^e_k) \setminus \partial \omega^e_k)$ the estimate

$$\|u\|_{L^2(\partial \omega^e_k)}^2 \leq C \left( \varepsilon \|\nabla u\|_{L^2(B_{\varepsilon R_3}(M^e_k) \setminus \partial \omega^e_k)}^2 + \varepsilon^{-1} \eta^{d-1} \|u\|_{L^2(B_{\varepsilon R_3}(M^e_k) \setminus B_{\varepsilon R_2}(M^e_k))}^2 \right)$$

holds, where $C$ is a constant independent of the parameters $k, \varepsilon, \eta$ and the function $u$.

Taking $u \equiv 1$ in the above lemma, we immediately find that the $(d - 1)$-dimensional measures of $\partial \omega^e_{k, \varepsilon}$ are uniformly bounded:

$$\text{mes}_{d-1} \partial \omega^e_k \leq C \varepsilon^{d-1} \eta^{d-1},$$

(11)

where $C$ is a constant independent of $k, \varepsilon$ and $\eta$.

The next statement is Lemma 3.2 from [20]. Although in [20], it was assumed that the boundaries of the domains $\omega^e_{k, \varepsilon}$ had the smoothness $C^2$, this fact was not used in the proof of Lemma 3.2 and this is why it is valid also under our Assumption 1.

Lemma 3.2: Under Assumption 1 for all $k \in \mathbb{I}^e$ and all $u \in W^1_2(B_{\varepsilon R_3}(M^e_k) \setminus \partial \omega^e_k)$ the estimate

$$\|u\|_{L^2(B_{\varepsilon R_3}(M^e_k) \setminus \partial \omega^e_k)}^2 \leq C \left( \varepsilon^2 \eta^2 \|\nabla u\|_{L^2(B_{\varepsilon R_3}(M^e_k) \setminus \partial \omega^e_k)}^2 + \eta^d \|u\|_{L^2(B_{\varepsilon R_3}(M^e_k) \setminus \partial \omega^e_k)}^2 \right)$$

holds with a constant $C$ independent of $k, \varepsilon$ and $u$.

The next lemma is [19, Lm. 3.5].

Lemma 3.3: Under Assumption 1 for all $k \in \mathbb{I}^e$ and all $u \in W^1_2(B_{\varepsilon R_3}(M^e_k) \setminus \partial \omega^e_k)$ obeying the identity

$$\int_{B_{\varepsilon R_3}(M^e_k) \setminus \partial \omega^e_k} u(x) \, dx = 0$$

(12)

the estimate

$$\|u\|_{L^2(\partial \omega^e_k)}^2 \leq C \varepsilon \eta \|\nabla u\|_{L^2(B_{\varepsilon R_3}(M^e_k) \setminus \partial \omega^e_k)}^2,$$

(13)

holds, where $C$ is a constant independent of the parameters $k, \varepsilon, \eta$ and the function $u$.

Lemma 3.4: For all $k \in \mathbb{I}^e$ and all $u \in W^1_2(B_{\varepsilon R_3}(M^e_k))$ the inequality

$$\|u\|_{L^2(B_{\varepsilon R_3}(M^e_k))}^2 \leq C \left( \varepsilon^2 \eta^2 \|\nabla u\|_{L^2(B_{\varepsilon R_3}(M^e_k))}^2 + \eta^d \|u\|_{L^2(B_{\varepsilon R_3}(M^e_k))}^2 \right)$$

holds with a constant $C$ independent of $k, \varepsilon, \eta$ and $u$. 
Proof: Let $\chi_0 = \chi_0(t)$ be an infinitely differentiable cut-off function equaling to one as $|t| < \frac{R_3}{2}$ and vanishing as $|t| > R_3$. Then for an arbitrary $u \in W^2_2(B_{\varepsilon R_3}(M^\varepsilon_k))$ and $x \in B_{\varepsilon \eta R_3}(M^\varepsilon_k)$ we obviously have the identity

$$u(x) = \int_{x \in R_3}^{x - M^\varepsilon_k} \frac{\partial}{\partial t} u(M^\varepsilon_k + ty) \chi_0(t \varepsilon^{-1}) \, dt, \quad y := \frac{x - M^\varepsilon_k}{|x - M^\varepsilon_k|}.$$  

By the Cauchy–Schwarz inequality for $x \in B_{\varepsilon \eta R_3}(M^\varepsilon_k)$ we then obtain

$$|u(x)|^2 \leq C \int_{x - M^\varepsilon_k}^{x \in R_3} \frac{dt}{t \varepsilon^{-1}} \int_0^{x \in R_3} |\nabla u(M^\varepsilon_k + ty)|^2 t^d \, dt$$

$$+ C \varepsilon^{-2} \int_{\frac{R_3}{2} \leq \varepsilon}^{\varepsilon R_3} \frac{dt}{t^d - 1} \int_{\frac{R_3}{2} \leq \varepsilon}^{x \in R_3} |u(M^\varepsilon_k + ty)|^2 t^d \, dt$$

$$\leq C \int_{x - M^\varepsilon_k}^{x \in R_3} \frac{dt}{t \varepsilon^{-1}} \int_0^{x \in R_3} |\nabla u(M^\varepsilon_k + ty)|^2 t^d \, dt + C \varepsilon^{-d} \int_0^{x \in R_3} |u(M^\varepsilon_k + ty)|^2 t^d \, dt,$$

where $C$ are some constants independent of $x, \varepsilon, \eta, k$ and $u$. We integrate the obtained inequality over $B_{\varepsilon \eta R_3}(M^\varepsilon_k)$:

$$\|u\|_{L^2_2(B_{\varepsilon \eta R_3}(M^\varepsilon_k))}^2 \leq C \|\nabla u\|_{L^2_2(B_{\varepsilon \eta R_3}(M^\varepsilon_k))}^2 \int_0^{x \in R_3} \frac{dt}{t \varepsilon^{-1}} \int_0^{x \in R_3} \frac{dz}{\varepsilon^d - 1} + C \varepsilon^d \|u\|_{L^2_2(B_{\varepsilon \eta R_3}(M^\varepsilon_k))}^2$$

$$\leq C \varepsilon^d \eta^2 \varepsilon \|\nabla u\|_{L^2_2(B_{\varepsilon \eta R_3}(M^\varepsilon_k))}^2 + C \varepsilon^d \|u\|_{L^2_2(B_{\varepsilon \eta R_3}(M^\varepsilon_k))}^2,$$

where $C$ are some constants independent of $\varepsilon, \eta, k$ and $u$. The proof is complete. \(\blacksquare\)

Lemma 3.5: For all $u \in W^2_2(B_{\varepsilon R_3}(M^\varepsilon_k))$ the estimate

$$\|u\|_{L^2_2(\partial \omega_k)}^2 \leq C (\varepsilon^{-1} \eta^d + \kappa^2 \varepsilon^3 \eta^3) \|u\|_{W^2_2(B_{\varepsilon R_3}(M^\varepsilon_k))}^2$$

holds true, where $C$ is a constant independent of the parameters $k, \varepsilon, \eta$ and the function $u$.

Proof: Let $u \in W^2_2(B_{\varepsilon R_3}(M^\varepsilon_k))$ be a given function. We first consider the case $d = 2, 3$. We introduce one more function $\tilde{u}(\xi) := u(M^\varepsilon_k + \varepsilon \xi)$. It belongs to $W^2_2(B_{R_3}(0))$ and hence, due to the embedding of the latter space into $C(\bar{B}_{R_3}(0))$,

$$\|\tilde{u}\|_{C(\bar{B}_{R_3}(0))} \leq C \|\tilde{u}\|_{W^2_2(B_{R_3}(0))};$$

throughout the proof the symbol $C$ stands for various inessential constants independent of $k, \varepsilon, \eta, u$. Returning back to the function $u$, we immediately obtain one more estimate

$$\|u\|_{L^2_2(\partial \omega_k)}^2 \leq C \varepsilon^{-d} \varepsilon \|u\|_{W^2_2(B_{\varepsilon R_3}(M^\varepsilon_k))}^2.$$

Hence, in view of (11),

$$\|u\|_{L^2_2(\partial \omega_k)}^2 \leq C \varepsilon^{-1} \eta^d \|u\|_{W^2_2(B_{\varepsilon R_3}(M^\varepsilon_k))}^2,$$

and this proves the desired estimate for $d = 2, 3$. 

We proceed to the case \( d \geq 4 \). We denote
\[
\langle u \rangle := \frac{1}{\text{mes} B_\varepsilon \eta R_3(M_k^\varepsilon)} \int_{B_\varepsilon \eta R_3(M_k^\varepsilon)} u \, dx, \quad u_\perp := u - \langle u \rangle, \quad u_\perp^\varepsilon(\xi) := u_\perp(M_k^\varepsilon + \varepsilon \eta \xi).
\]

We obviously have
\[
\|u\|_{L^2(\partial \omega_\varepsilon)}^2 \leq C \varepsilon \eta \|u_\perp\|_{L^2(B_\varepsilon R_3)}^2 + C \|u_\perp\|_{L^2(\partial \omega_\varepsilon)}^2.
\]

The function \( u_\perp \) belongs to \( W^{1,2}_2(B_{R_3}(0)) \) and satisfies the identity
\[
\int_{B_{R_3}(0)} u_\perp(\xi) \, d\xi = 0.
\]

Hence, by the Poincaré inequality,
\[
\|u_\perp\|_{L^2(B_{R_3}(0))} \leq C \|\nabla u_\perp\|_{L^2(B_{R_3}(0))} = C \|\nabla u\|_{L^2(B_{R_3}(0))}.
\]

Using this inequality and proceeding as in the proof of Lemma 3.2 in [16], we easily find that
\[
\|u_\perp\|_{L^2(\partial \omega_{\varepsilon k})} \leq C \|u_\perp\|_{W^{1,2}_2(B_{R_3}(0))} \leq C \|\nabla u\|_{L^2(B_{R_3}(0))}.
\]

Returning back to the function \( u_\perp \) and using Lemma 3.4, we obtain
\[
\|u_\perp\|_{L^2(\partial \omega_{\varepsilon k})}^2 \leq C \varepsilon \eta \|\nabla u\|_{L^2(B_{\varepsilon \eta R_3}(M_k^\varepsilon))}^2 \leq C \left( \varepsilon^3 \eta^3 + \varepsilon \eta^{d+1} \right) \|u\|_{W^{1,2}_2(B_{R_3}(M_k^\varepsilon))}^2.
\]

We introduce an auxiliary function:
\[
X(x) := \begin{cases} \frac{|x|^2 - R_3^2 \varepsilon^2 \eta^2}{2d} + \frac{R_3^2 \varepsilon^2 \eta^d}{d(2 - d)} (\varepsilon^{-d+2} - 1) & \text{in } B_{R_3 \varepsilon \eta}(0), \\ \frac{R_3^2 \varepsilon^2 \eta^d}{d(2 - d)} \left( \frac{R_3 \varepsilon}{|x|} \right)^{d-2} - 1 & \text{in } B_{R_3 \varepsilon}(0) \setminus B_{R_3 \varepsilon \eta}(0). \end{cases}
\]

This function solves the following boundary value problem:
\[
\Delta X = 1 \text{ in } B_{\varepsilon \eta R_3}(0), \quad \Delta X = 0 \text{ in } B_{R_3 \varepsilon}(0) \setminus B_{R_3 \varepsilon \eta}(0), \quad X = 0 \text{ on } \partial B_{R_3 \varepsilon}(0).
\]

Using this problem, we integrate by parts as follows:
\[
\int_{B_{R_3 \varepsilon}(M_k^\varepsilon)} u(x) \, dx = \int_{B_{R_3 \varepsilon}(M_k^\varepsilon)} u(x) \Delta X(x - M_k^\varepsilon) \, dx
\]
\[
= \int_{\partial B_{R_3 \varepsilon}(M_k^\varepsilon)} u(x) \frac{\partial X}{|x|} (x - M_k^\varepsilon) \, ds + \int_{B_{R_3 \varepsilon}(M_k^\varepsilon)} X(x - M_k^\varepsilon) \Delta u(x) \, dx.
\]
The function \( \hat{u}(\xi) := u(M^\varepsilon_k + \varepsilon \xi) \) belongs to \( W^1_2(B_3(0)) \) and satisfies the estimate
\[
||\hat{u}||^2_{L_2(\partial B_3(0))} \leq C||\hat{u}||^2_{W^1_2(B_3(0))}.
\]

Returning back to the function \( u \), we then find
\[
||u||^2_{L_2(\partial B_{3\varepsilon}(0))} \leq C^{-1}||u||^2_{W^1_2(B_{3\varepsilon}(0))},
\]

Using the explicit formula for \( X \) once again, we calculate the \( L_2(B_{3\varepsilon}(M^\varepsilon_k)) \)-norm of \( X \) and we get
\[
||X||^2_{L_2(B_{3\varepsilon}(M^\varepsilon_k))} \leq C(\varepsilon \eta)^{d+4}, \quad d \geq 5, \quad ||X||^2_{L_2(B_{3\varepsilon}(M^\varepsilon_k))} \leq C(\varepsilon \eta)^{8}(\ln \eta + 1), \quad d = 4.
\]

We substitute this estimate and (17) into (16) and then the resulting inequality is combined with (15), (14). This gives the desired inequality and completes the proof.

\[\square\]

**Remark 3.1:** Local estimates established in above Lemmata 3.1, 3.2, 3.3, 3.4, 3.5 are key ingredients for proving our main results. Similar local estimates were also established and employed in many earlier papers on perforated domains, mostly in periodic setting. As an example, we mention Lemmata 1–3 in [21]. The main advantage made in our Lemmata 3.1, 3.2, 3.3, 3.4, 3.5 is that the local estimates are established in the general non-periodic case and the constants in these estimates are independent not only of the small parameter and the function but also of the shapes of the cavities and of their distribution.

Let us prove the unique solvability of problems (6), (7). To do this, we follow general results from the theory of monotone operators, see [22, Ch. 1, Sect. 1.20], [23, Ch. VI, Sect. 18.4]. According to these results, in our case the unique solvability is ensured by the following conditions:

1. For all \( u, v, w \in W^1_2(\Omega^\varepsilon; \mathbb{C}^n) \) the function \( t \mapsto \mathcal{h}^\varepsilon(u + tv, w) - \lambda(u + tv, w)_{L_2(\Omega^\varepsilon; \mathbb{C}^n)} \) is continuous;
2. For all \( u, v \in W^1_2(\Omega^\varepsilon; \mathbb{C}^n), \quad u \neq v \), the inequality \( \text{Re}(\mathcal{h}^\varepsilon(u, u - v) - \mathcal{h}^\varepsilon(v, u - v) - \lambda ||u - v||^2_{L_2(\Omega^\varepsilon; \mathbb{C}^n)}) > 0 \) holds;
3. The convergence is valid:
\[
\frac{\text{Re}(\mathcal{h}^\varepsilon(u, u) - \lambda ||u||^2_{L_2(\Omega^\varepsilon; \mathbb{C}^n)})}{||u||^2_{W^1_2(\Omega^\varepsilon; \mathbb{C}^n)}} \rightarrow +\infty \quad \text{as} \quad ||u||_{W^1_2(\Omega^\varepsilon; \mathbb{C}^n)} \rightarrow +\infty.
\]

We are going to check all these three conditions for the form \( \mathcal{h}^\varepsilon \) as well as the same conditions for the form
\[
\mathcal{h}^0(u, v) := \sum_{i,j=1}^{d} \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega; \mathbb{C}^n)} + \sum_{j=1}^{d} \left( A_{j} \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega; \mathbb{C}^n)} + (A_0 u, v)_{L_2(\Omega; \mathbb{C}^n)}.
\]

This will prove the unique solvability of problems (6), (7).
Condition 1 obviously holds true for both forms $f^e$ and $f^0$. By conditions (2), (3) and the Cauchy–Schwartz inequality we immediately get

\[
\text{Re}(u, u) \geq (c_1 - \delta) \|\nabla u\|^2_{L^2(\Omega; \mathbb{C}^n)} - C\delta^{-1} \|u\|^2_{L^2(\Omega; \mathbb{C}^n)} \quad \text{for all } u \in W^1_2(\Omega^e; \mathbb{C}^n),
\]

\[
\text{Re}^0(u, u) \geq (c_1 - \delta) \|\nabla u\|^2_{L^2(\Omega; \mathbb{C}^n)} - C\delta^{-1} \|u\|^2_{L^2(\Omega; \mathbb{C}^n)} \quad \text{for all } u \in W^1_2(\Omega; \mathbb{C}^n),
\]

(18)

where $\delta > 0$ is arbitrary and fixed, while $C > 0$ is some constant independent of $\delta$ and $u$. Choosing then $\delta := c_1/2$ and $\lambda_0 := -1 - 2C/c_1$, we obtain the estimate

\[
\text{Re}\left(f^0(u, u) - \lambda \|u\|^2_{L^2(\Omega; \mathbb{C}^n)}\right) \geq \frac{c_1}{2} \|\nabla u\|^2_{L^2(\Omega; \mathbb{C}^n)} + \|u\|^2_{L^2(\Omega; \mathbb{C}^n)}
\]

(19)

for $\lambda \in \mathbb{C}$ such that $\text{Re}\lambda \leq \lambda_0$. This estimate implies the validity of Conditions 2, 3 for the form $f^0$.

It follows from (4) that

\[
\left|\left(a^e(\cdot, u) - a^e(\cdot, v), u - v\right)_{L^2(\partial \Omega; \mathbb{C}^n)}\right| \leq \mu \|u - v\|^2_{L^2(\partial \Omega; \mathbb{C}^n)} = \mu \sum_{k \in \mathbb{I}^e} \|u - v\|^2_{L^2(\partial \Omega^e; \mathbb{C}^n)},
\]

\[
\left|\left(a^e(\cdot, u), u\right)_{L^2(\partial \Omega; \mathbb{C}^n)}\right| \leq \mu \|u\|^2_{L^2(\partial \Omega; \mathbb{C}^n)} = \mu \sum_{k \in \mathbb{I}^e} \|u\|^2_{L^2(\partial \Omega^e; \mathbb{C}^n)}.
\]

Applying Lemma 3.1, we find

\[
\left|\left(a^e(\cdot, u) - a^e(\cdot, v), u - v\right)_{L^2(\partial \Omega; \mathbb{C}^n)}\right| \leq C \left(\varepsilon \eta \varkappa + \varepsilon^{-1} \eta^{d-1}\right) \mu \|u - v\|^2_{W^1_2(\Omega^e; \mathbb{C}^n)},
\]

\[
\left|\left(a^e(\cdot, u), u\right)_{L^2(\partial \Omega; \mathbb{C}^n)}\right| \leq C \left(\varepsilon \eta \varkappa + \varepsilon^{-1} \eta^{d-1}\right) \mu \|u\|^2_{W^1_2(\Omega^e; \mathbb{C}^n)},
\]

where $C$ is some constant independent of $\varepsilon, \eta, \mu$ and $u$. Taking into consideration the first convergence in (5), by the above estimates and (18) with $\delta = c_1/4$ for sufficiently small $\varepsilon$ we get

\[
\text{Re}\left(f^e(u, u - v) - f^e(v, u - v) - \lambda \|u - v\|^2_{L^2(\Omega; \mathbb{C}^n)}\right) \geq \frac{c_1}{2} \|\nabla (u - v)\|^2_{L^2(\Omega; \mathbb{C}^n)} + \|u - v\|^2_{L^2(\Omega; \mathbb{C}^n)},
\]

\[
\text{Re}\left(f^e(u, u) - \lambda \|u\|^2_{L^2(\Omega; \mathbb{C}^n)}\right) \geq \frac{c_1}{2} \|\nabla u\|^2_{L^2(\Omega; \mathbb{C}^n)} + \|u\|^2_{L^2(\Omega; \mathbb{C}^n)},
\]

(20)

for all $u, v \in W^1_2(\Omega^e; \mathbb{C}^n)$ and $\lambda \in \mathbb{C}$ such that $\text{Re}\lambda \leq \lambda_0$ with some fixed $\lambda_0$ independent of $\varepsilon, \eta, u$ and $v$. These two estimates imply Conditions 2, 3 for the form $f^e$. Hence, the problems (6), (7) are uniquely solvable for all $f \in L^2(\Omega; \mathbb{C}^n)$. Estimate (20) also yields

\[
\|u^e\|_{W^1_2(\Omega^e; \mathbb{C}^n)} \leq C \|f\|_{L^2(\Omega; \mathbb{C}^n)},
\]

(21)

where $C$ is some constant independent of $\varepsilon$ and $f$. By (19) we get a similar estimate for $\|u^0\|_{W^1_2(\Omega; \mathbb{C}^n)}$. Using then standard smoothness improving theorems, we obtain

\[
\|u^0\|_{W^2_2(\Omega; \mathbb{C}^n)} \leq C(\lambda) \|f\|_{L^2(\Omega; \mathbb{C}^n)}
\]

(22)

with some constant $C$ independent of $f$.

We proceed to proving inequality (9). Given an arbitrary $f \in L^2(\Omega)$, we let $v^e := u^e - u_0$, where $u^e$ and $u_0$ are the solutions of problems (6), (7). The function $v^e$ is an element of $W^1_2(\Omega^e; \mathbb{C}^n)$ and has
the zero trace on \( \partial \Omega \). Integral identity \((8)\) with \( v_{\varepsilon} \) as the test function reads

\[
b^\varepsilon(u_{\varepsilon}, v_{\varepsilon}) - \lambda(u_{\varepsilon}, v_{\varepsilon}) \in L^2(\Omega^\varepsilon; \mathbb{C}^n) = (f, v_{\varepsilon})_{L^2(\Omega^\varepsilon; \mathbb{C}^n)}. \tag{23}\]

Since \( u_0 \) belongs to \( W^2_2(\Omega; \mathbb{C}^n) \), we can multiply the Equation in \((7)\) by \( v_{\varepsilon} \) and integrate by parts over \( \Omega^\varepsilon \). This gives

\[
b(u_0, v_{\varepsilon}) - \left( \frac{\partial u_0}{\partial v}, v_{\varepsilon} \right)_{L^2(\partial \Omega^\varepsilon; \mathbb{C}^n)} - \lambda(u_0, v_{\varepsilon}) \in L^2(\Omega^\varepsilon; \mathbb{C}^n) = (f, v_{\varepsilon})_{L^2(\Omega^\varepsilon; \mathbb{C}^n)}. \tag{24}\]

Deducting this identity from \((23)\), we obtain

\[
b^\varepsilon(v_{\varepsilon}, v_{\varepsilon}) - \lambda \|v_{\varepsilon}\|^2_{L^2(\Omega^\varepsilon; \mathbb{C}^n)} = g_{\varepsilon}, \]

\[
g_{\varepsilon} := (a^\varepsilon(\cdot, v_{\varepsilon}) - a^\varepsilon(\cdot, u_{\varepsilon}), v_{\varepsilon})_{L^2(\partial \Omega^\varepsilon; \mathbb{C}^n)} - \left( \frac{\partial u_0}{\partial v}, v_{\varepsilon} \right)_{L^2(\partial \Omega^\varepsilon; \mathbb{C}^n)}. \tag{25}\]

By inequality \((20)\), we have

\[
Re \left( b^\varepsilon(v_{\varepsilon}, v_{\varepsilon}) - \lambda \|v_{\varepsilon}\|^2_{L^2(\Omega^\varepsilon; \mathbb{C}^n)} \right) \geq C \|v_{\varepsilon}\|^2_{W^1_2(\Omega^\varepsilon; \mathbb{C}^n)}. \tag{26}\]

Hereinafter, by \( C \) we denote various inessential constants independent of \( f, u_0, u_{\varepsilon}, v_{\varepsilon}, \varepsilon, k \) and spatial variables but, in general, depending on \( \lambda \).

Our next step is to estimate the real part of the right-hand side in \((25)\). It follows from the inequality in \((4)\) that

\[
|a^\varepsilon(\cdot, v_{\varepsilon}) - a^\varepsilon(\cdot, u_{\varepsilon})| \leq \mu(\varepsilon) |v_{\varepsilon} - u_{\varepsilon}| = \mu(\varepsilon) |u_0|.
\]

Hence, by Lemmata 3.1, 3.5 and estimate \((22)\),

\[
\left| (a^\varepsilon(\cdot, v_{\varepsilon}) - a^\varepsilon(\cdot, u_{\varepsilon}), v_{\varepsilon})_{L^2(\partial \Omega^\varepsilon; \mathbb{C}^n)} \right| \leq C \mu \sum_{k \in \mathbb{Z}^n} \|u_0\|_{L^2(B_{1/2}(M_k^\varepsilon) \setminus \alpha_k^\varepsilon)} \|v_{\varepsilon}\|_{L^2(B_{1/2}(M_k^\varepsilon) \setminus \alpha_k^\varepsilon)} \]

\[
\leq C \left( \varepsilon \eta \varepsilon + \varepsilon^{-1} \varepsilon^d - 1 \right) \left( \varepsilon^2 \varepsilon^3 \varepsilon^3 + \varepsilon^{-1} \varepsilon^d - 1 \right) \frac{1}{2} \mu \]

\[
\cdot \sum_{k \in \mathbb{Z}^n} \|u_0\|_{W^2_2(B_{1/2}(M_k^\varepsilon) \setminus \alpha_k^\varepsilon)} \|v_{\varepsilon}\|_{W^2_2(B_{1/2}(M_k^\varepsilon) \setminus \alpha_k^\varepsilon)} \]

\[
\leq C \left( \varepsilon^2 \eta^2 \mu + \eta^2 \varepsilon^2 + \varepsilon^{-1} \varepsilon^d - 1 \right) \mu \|u_0\|_{W^2_2(\Omega^\varepsilon; \mathbb{C}^n)} \|v_{\varepsilon}\|_{W^2_2(\Omega^\varepsilon; \mathbb{C}^n)} \]

\[
\leq C \left( \varepsilon^2 \eta^2 \mu + \eta^2 \varepsilon^2 + \varepsilon^{-1} \varepsilon^d - 1 \right) \mu \|f\|_{L^2(\Omega^\varepsilon; \mathbb{C}^n)} \|v_{\varepsilon}\|_{W^2_2(\Omega^\varepsilon; \mathbb{C}^n)}. \tag{27}\]

For each \( k \in \mathbb{Z}^n \), we define

\[
\langle v_{\varepsilon}\rangle_k := \frac{1}{\text{mes}B_{1/2}(M_k^\varepsilon) \setminus \alpha_k^\varepsilon} \int_{B_{1/2}(M_k^\varepsilon) \setminus \alpha_k^\varepsilon} v_{\varepsilon} \, dx, \quad v_{\varepsilon,k} := v_{\varepsilon} - \langle v_{\varepsilon}\rangle_k.
\]

Then we can rewrite the second term in \( g_{\varepsilon} \) as

\[
\left( \frac{\partial u_0}{\partial v}, v_{\varepsilon} \right)_{L^2(\partial \Omega^\varepsilon; \mathbb{C}^n)} = \sum_{k \in \mathbb{Z}^n} \left( \frac{\partial u_0}{\partial v}, v_{\varepsilon} \right)_{L^2(\partial \Omega_k^\varepsilon; \mathbb{C}^n)}
\]

\[
= \sum_{k \in \mathbb{Z}^n} \left( \int_{\Omega_k^\varepsilon} \frac{\partial u_0}{\partial v} \, ds, \langle v_{\varepsilon}\rangle_k \right)_{\mathbb{C}^n} + \sum_{k \in \mathbb{Z}^n} \left( \frac{\partial u_0}{\partial v}, v_{\varepsilon,k} \right)_{L^2(\partial \Omega_k^\varepsilon; \mathbb{C}^n)}. \tag{28}\]
The functions \( v_{\epsilon,k} \) obey condition (12) and this is why by estimate (13), we have
\[
\| v_{\epsilon,k} \|_{L^2(\partial \omega_k^\epsilon; \mathbb{C}^n)} \leq C \epsilon \eta \| \nabla v_{\epsilon} \|_{L^2(\partial \beta_{\eta R_3}(M_k^\epsilon) \setminus \omega_k^\epsilon; \mathbb{C}^n)}.
\]

Hence, by Lemmata 3.1, 3.3 and inequality (22),
\[
\left| \sum_{k \in \mathbb{N}} \left( \frac{\partial u_0}{\partial \nu}, v_{\epsilon,k} \right)_{L^2(\partial \omega_k^\epsilon; \mathbb{C}^n)} \right| \leq C \left( \epsilon \eta \chi^2 + \epsilon^{-1} \eta^{d-1} \right)^{\frac{1}{2}} \left( \epsilon \eta \right)^{\frac{1}{2}} \| u_0 \|_{W^2_2(\Omega^\epsilon; \mathbb{C}^n)} \| v_{\epsilon} \|_{W^1_2(\Omega^\epsilon; \mathbb{C}^n)}
\]
\[
\leq C \left( \epsilon \eta \chi^2 + \eta^2 \right) \| f \|_{L^2(\Omega^\epsilon; \mathbb{C}^n)} \| v_{\epsilon} \|_{W^1_2(\Omega^\epsilon; \mathbb{C}^n)}.
\]

Since the function \( u_0 \) belongs to \( W^2_2(\Omega; \mathbb{C}^n) \), we can integrate by parts as follows:
\[
\int_{\partial \omega_k^\epsilon} \frac{\partial u_0}{\partial \nu} \, ds = - \int_{\omega_k^\epsilon} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} A_{ij} \frac{\partial u_0}{\partial x_j} \, dx.
\]

Therefore,
\[
\left| \int_{\partial \omega_k^\epsilon} \frac{\partial u_0}{\partial \nu} \, ds \right| \leq C \epsilon \chi^2 \| u_0 \|_{W^2_2(\omega_k^\epsilon; \mathbb{C}^n)}.
\]

We also observe that by the Cauchy–Schwartz inequality we have
\[
| (v_{\epsilon,k}) | \leq C \epsilon^{-\frac{d}{2}} \eta^{-\frac{d}{2}} \| v_{\epsilon} \|_{L^2(\beta_{\eta R_3}(M_k^\epsilon) \setminus \omega_k^\epsilon)}.
\]

Using this inequality, (31), (22) and Lemma 3.2, we estimate the first term in the right-hand side of (28):
\[
\left| \sum_{k \in \mathbb{N}} \left( \int_{\partial \omega_k^\epsilon} \frac{\partial u_0}{\partial \nu} \, ds, (v_{\epsilon,k}) \right)_{\mathbb{C}^n} \right| \leq C \sum_{k \in \mathbb{N}} \| u_0 \|_{W^2_2(\omega_k^\epsilon; \mathbb{C}^n)} \| v_{\epsilon} \|_{L^2(\beta_{\eta R_3}(M_k^\epsilon) \setminus \omega_k^\epsilon; \mathbb{C}^n)}
\]
\[
\leq C \left( \epsilon \eta \chi^2 + \eta^2 \right) \| f \|_{L^2(\Omega^\epsilon; \mathbb{C}^n)} \| v_{\epsilon} \|_{W^1_2(\Omega^\epsilon; \mathbb{C}^n)}.
\]

This estimate and (29), (28), (27) yield:
\[
| g_{\epsilon} | \leq C \left( \left( \epsilon^2 \eta^2 \chi + \eta^2 \chi^2 + \epsilon^{-1} \eta^{d-1} \right) \mu + \epsilon \eta \chi^2 + \eta^2 \right) \| f \|_{L^2(\Omega^\epsilon; \mathbb{C}^n)} \| v_{\epsilon} \|_{W^1_2(\Omega^\epsilon; \mathbb{C}^n)}.
\]

Having this inequality in mind, we take the real part of identity (25) and in view of (26) we obtain
\[
\| v_{\epsilon} \|_{W^1_2(\Omega^\epsilon; \mathbb{C}^n)} \leq C \left( \left( \epsilon^2 \eta^2 \chi + \eta^2 \chi^2 + \epsilon^{-1} \eta^{d-1} \right) \mu + \epsilon \eta \chi^2 + \eta^2 \right) \| f \|_{L^2(\Omega^\epsilon; \mathbb{C}^n)},
\]
which proves inequality (9).

We proceed to proving (10). We use an approach based on duality arguments, see [24–28], with a slight modification proposed recently in [29]. We first introduce a formally adjoint differential expression for \( L \):
\[
L^* := - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} - \sum_{j=1}^n \frac{\partial}{\partial x_j} A_j + A_0.
\]

In view of the condition \( A_j \in W^1_2(\Omega; \mathbb{C}^n) \), this differential expression is of the same structure as \( L \).
In particular, for \( u, v \in W^2_2(\Omega; \mathbb{C}^n) \) with the zero trace on \( \partial \Omega \) we have \( (u, L^* v)_{L^2(\Omega; \mathbb{C}^n)} = \mathcal{H}^0(u, v) \).
Using this identity, we can reproduce the proof of the solvability of problem (7) and we see that with the same \( \lambda_0 \) the problem
\[
(\mathcal{L}^* - \lambda_0)w = h \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega,
\]
is uniquely solvable in \( W^2_2(\Omega; \mathbb{C}^n) \) as \( \text{Re} \lambda \leq \lambda_0 \) for each \( h \in L_2(\Omega; \mathbb{C}^n) \) and
\[
\|w\|_{W^2_2(\Omega; \mathbb{C}^n)} \leq C(\lambda)\|h\|_{L_2(\Omega; \mathbb{C}^n)},
\]
where a constant \( C(\lambda) \) is independent of \( h \). We choose \( h \) in (34) as \( h := v_\varepsilon \) in \( \Omega^\varepsilon \), \( h = 0 \) in \( \theta^\varepsilon \); the corresponding solution is denoted by \( w_\varepsilon \). Then we multiply Equation in (34) by \( v_\varepsilon \) and integrate once by parts over \( \Omega^\varepsilon \). This gives
\[
\|v_\varepsilon\|_{L_2(\Omega^\varepsilon; \mathbb{C}^n)}^2 = \mathcal{H}(v_\varepsilon, w_\varepsilon) - \lambda(v_\varepsilon, w_\varepsilon)_{L_2(\Omega^\varepsilon; \mathbb{C}^n)} - \left( v_\varepsilon, \frac{\partial w_\varepsilon}{\partial \nu^*} \right)_{L_2(\partial \Omega^\varepsilon; \mathbb{C}^n)},
\]
where
\[
\frac{\partial}{\partial \nu^*} := \sum_{i,j=1}^d v_i A_{ji} \frac{\partial}{\partial x_j} + \sum_{i=1}^d v_i A_i.
\]
Then we write integral identity (8) with \( w_\varepsilon \) as the test function and rewrite identity (24) with \( v_\varepsilon \) replaced by \( w_\varepsilon \). The difference of two obtained relations gives an identity similar to (25):
\[
\mathcal{H}(v_\varepsilon, w_\varepsilon) - \lambda(v_\varepsilon, w_\varepsilon)_{L_2(\Omega^\varepsilon; \mathbb{C}^n)} = - \left( a^\varepsilon(\cdot, u_\varepsilon), w_\varepsilon \right)_{L_2(\partial \theta^\varepsilon; \mathbb{C}^n)} - \left( \frac{\partial u_0}{\partial \nu}, w_\varepsilon \right)_{L_2(\partial \theta^\varepsilon; \mathbb{C}^n)},
\]
It follows from this identity and (36) that
\[
\|v_\varepsilon\|_{L_2(\Omega^\varepsilon; \mathbb{C}^n)}^2 = - \left( a^\varepsilon(\cdot, u_\varepsilon), w_\varepsilon \right)_{L_2(\partial \theta^\varepsilon; \mathbb{C}^n)} - \left( \frac{\partial w_\varepsilon}{\partial \nu^*}, w_\varepsilon \right)_{L_2(\partial \theta^\varepsilon; \mathbb{C}^n)} - \left( \frac{\partial u_0}{\partial \nu}, w_\varepsilon \right)_{L_2(\partial \theta^\varepsilon; \mathbb{C}^n)}. \tag{37}
\]
Conditions (4) yield that \( |a^\varepsilon(x, u_\varepsilon)| \leq \mu(\varepsilon)|u_\varepsilon| \). Using this inequality, (21), Lemma 3.1 and the identity
\[
\int_{\partial \Omega^\varepsilon} \frac{\partial w_\varepsilon}{\partial \nu^*} \, ds = - \int_{\partial \Omega^\varepsilon} \left( \sum_{i,j=1}^d \frac{\partial}{\partial x_j} A_{ji} \frac{\partial w_\varepsilon}{\partial x_i} + \sum_{i=1}^d \frac{\partial}{\partial x_i} A_i \frac{\partial w_\varepsilon}{\partial x_j} \right) \, dx
\]
instead of (30), we estimate as in (27), (28), (29), (32):
\[
\left| \left( a^\varepsilon(\cdot, u_\varepsilon), w_\varepsilon \right)_{L_2(\partial \theta^\varepsilon; \mathbb{C}^n)} \right| \leq C \mu \|u_\varepsilon\|_{L_2(\partial \theta^\varepsilon; \mathbb{C}^n)} \|w_\varepsilon\|_{L_2(\partial \theta^\varepsilon; \mathbb{C}^n)},
\]
\[
\leq C \left( \varepsilon^2 \eta^2 \kappa + \eta^d \varepsilon^{1+d} + \varepsilon^{-1} \eta^{d-1} \right) \mu \|f\|_{L_2(\Omega; \mathbb{C}^n)} \|w_\varepsilon\|_{W^2_2(\Omega; \mathbb{C}^n)} \|w_\varepsilon\|_{W^2_2(\Omega; \mathbb{C}^n)}, \tag{38}
\]
\[
\left| \left( \frac{\partial w_\varepsilon}{\partial \nu^*}, w_\varepsilon \right)_{L_2(\partial \theta^\varepsilon; \mathbb{C}^n)} \right| \leq C \left( \varepsilon \eta \varepsilon^{1+d} + \eta^{d} \right) \|v_\varepsilon\|_{W^1_2(\Omega^\varepsilon; \mathbb{C}^n)} \|w_\varepsilon\|_{W^2_2(\Omega; \mathbb{C}^n)}.
\]
Let us estimate the third term in the right-hand side of (37). Since the function \( u_0 \) belongs to \( W^2_2(\Omega; \mathbb{C}^n) \), we can rewrite this term via the following integration by parts:
\[
\left( \frac{\partial u_0}{\partial \nu}, w_\varepsilon \right)_{L_2(\partial \theta^\varepsilon; \mathbb{C}^n)} = ((\mathcal{L} - \lambda)u_0, w_\varepsilon)_{L_2(\theta^\varepsilon; \mathbb{C}^n)} - \sum_{i,j=1}^d \left( A_{ji} \frac{\partial u_0}{\partial x_j}, \frac{\partial w_\varepsilon}{\partial x_i} \right)_{L_2(\theta^\varepsilon; \mathbb{C}^n)}.
\]
\[-\sum_{j=1}^{d} \left( A_j \frac{\partial u_0}{\partial x_j}, w_e \right)_{L^2_2(\Omega;\mathbb{C}^n)} - (A_0 - \lambda)u_0, w_e \right)_{L^2_2(\Omega;\mathbb{C}^n)}.

By Lemma 3.4 and Equation in (7), we then obtain
\[
\left| \left( \frac{\partial u_0}{\partial n}, w_e \right)_{L^2_2(\Omega;\mathbb{C}^n)} \right| \leq C \left( \varepsilon \eta \kappa + \eta \frac{d}{2} + \varepsilon^{-1} \eta^{d-1} \right) \| f \|_{L^2_2(\Omega;\mathbb{C}^n)} \| w_e \|_{W^1_2(\Omega;\mathbb{C}^n)} + C \left( \varepsilon \eta \kappa + \eta \frac{d}{2} \right) \| u_0 \|_{W^2_2(\Omega;\mathbb{C}^n)} \| w_e \|_{W^2_2(\Omega;\mathbb{C}^n)}.
\]

This inequality and (38), (37), (35), (33) imply
\[
\| v_e \|_{L^2_2(\Omega;\mathbb{C}^n)}^2 \leq C \left( \varepsilon^2 \eta^2 \kappa + \eta \frac{d}{2} + \varepsilon^{-1} \eta^{d-1} \right) \mu \| f \|_{L^2_2(\Omega;\mathbb{C}^n)} \| w_e \|_{W^1_2(\Omega;\mathbb{C}^n)} + C \left( \varepsilon \eta \kappa + \eta \frac{d}{2} \right) \| f \|_{L^2_2(\Omega;\mathbb{C}^n)} \| w_e \|_{W^2_2(\Omega;\mathbb{C}^n)} + C \left( \varepsilon \eta \kappa + \eta \frac{d}{2} \right) \| u_0 \|_{W^2_2(\Omega;\mathbb{C}^n)} \| w_e \|_{W^2_2(\Omega;\mathbb{C}^n)}
\]
\[
\leq C \left( \left( \varepsilon^2 \eta^2 \kappa + \eta \frac{d}{2} + \varepsilon^{-1} \eta^{d-1} \right) \mu + \varepsilon \eta \kappa + \eta \frac{d}{2} \right) \| f \|_{L^2_2(\Omega;\mathbb{C}^n)} \| v_e \|_{L^2_2(\Omega;\mathbb{C}^n)} + \left( \varepsilon \eta \kappa + \eta \frac{d}{2} \right) \| f \|_{L^2_2(\Omega;\mathbb{C}^n)} \| v_e \|_{L^2_2(\Omega;\mathbb{C}^n)}.
\]

The obtained inequality proves (10).

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**Data availability**

Not applicable in the manuscript as no datasets were generated or analysed during the current study.

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