The Relationship between Maximum Principle and Dynamic Programming Principle for Stochastic Recursive Control Problem with Random Coefficients*

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Summary. This paper aims to explore the relationship between maximum principle and dynamic programming principle for stochastic recursive control problem with random coefficients. Under certain regular conditions for the coefficients, the relationship between the Hamilton system with random coefficients and stochastic Hamilton-Jacobi-Bellman equation is obtained. It is very different from the deterministic coefficients case since stochastic Hamilton-Jacobi-Bellman equation is a backward stochastic partial differential equation with solution being a pair of random fields rather than a deterministic function. A linear quadratic recursive utility optimization problem is given as an explicitly illustrated example based on this kind of relationship.

1 Introduction

As we all know, Pontryagin maximum principle (MP) and Bellman dynamic programming principle (DPP) serve as the most two important methods in solving optimal control problems. Both of them aim to obtain some necessary conditions of optimal controls. Hence it is natural to think that they have some kind of relationship, although they have been developed separately and independently in literature to a great extent. In general, the MP gives a necessity condition of the optimal control by the Hamilton system which is a forward-backward equation consisting of the optimal state equation, the adjoint equation and optimality condition. On the other hand, the DPP characterizes the optimal control by the Hamilton-Jacobi-Bellman (HJB) equation, to which the value function is a solution. Therefore, the relationship between Hamiltonian system and HJB equation can be thought as a relationship between MP and DPP.

For the deterministic control system, the Hamiltonian system is an ordinary differential equation and the HJB equation is a first-order partial differential equation (PDE), whose connection was first given by Pontryagin, Boltyanski, Gamkrelidze and Mischenko [17] in 1962. Since the value function $V$ is not always smooth, some nonsmooth versions of the relationship were studied by using nonsmooth analysis and generalized derivatives. For example, an attempt to relate these two without assuming the smoothness of the value function was done by Barron and Jensen [1], where the viscosity solution was used to derive the MP from the DPP. The relationship in deterministic case is known as

$$
\Psi_t = -V_x(t, \bar{X}_t) \quad \text{and} \quad V_t(t, \bar{X}_t) = H(t, \bar{X}_t, \bar{u}_t, \Psi_t),
$$

where $\bar{u}$ is the optimal control, $\bar{X}$ is the optimal state, $\Psi$ is the adjoint variable, $H$ is the Hamiltonian function, and $V$ is the value function, respectively. For the stochastic control system whose state equation is a stochastic differential equation (SDE) with deterministic coefficients,

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the Hamiltonian system is a forward-backward stochastic differential equation (FBSDE) with deterministic coefficients and the HJB equation a second-order fully nonlinear PDE, their connection was given by Bismut [3] and Bensoussan [2]. As for nonsmooth value function, Zhou [29, 30] obtained the relationship between them in the viscosity sense of HJB equation. The relationship in this case can be summarized as

\[ p_t = -V_x(t, \bar{x}_t), \quad q_t = -V_{xx}(t, \bar{x}_t)\sigma(t, \bar{x}_t, \bar{u}_t), \]

and

\[ V_t(t, \bar{x}_t) = G(t, \bar{x}_t, \bar{u}_t, -V_x(t, \bar{x}_t), -V_{xx}(t, \bar{x}_t)), \]

where \( \sigma \) is the diffusion coefficient, \((p, q)\) is the adjoint pair and \( G \) is the generalized Hamiltonian function.

However, when the state equation is a SDE with random coefficients, things are much different. Bear in mind that HJB equation in this case is a backward stochastic partial differential equation (BSPDE) with a pair of adapted solution, rather than a deterministic PDE with a deterministic solution. There should be also a relationship between MP and DPP, as well as between FBSDE with random coefficients and stochastic HJB equation, but no existing literature is concerned with this issue as far as we know.

The relationship between MP and DPP not only demonstrates the connection between two main methods of control theory, but also plays a very important role in economic theory as pointed out in Yong and Zhou [27]. Moreover, the relationship can be regarded as an extension of Feynman-Kac formula to fully nonlinear PDE, if one notices that the Hamiltonian system is a stochastic forward-backward system and HJB equation is a fully nonlinear PDE in a stochastic control system with deterministic coefficients. For the random coefficients settings, Feynman-Kac formula is further extended to non-Markovian framework and fully nonlinear BSPDE. The reader can refer to [5, 8, 9, 23] for related studies.

The control system we consider to find the relationship between MP and DPP is the stochastic recursive control system with a general cost functional, which is governed by the following controlled FBSDE:

\[
\begin{aligned}
  dX_s &= b(s, X_s, u_s)ds + \sigma(s, X_s, u_s)dW_s, \\
  dY_s &= -f(s, X_s, Y_s, Z_s, u_s)ds + Z_sdW_s, \\
  X_0 &= x, \\
  Y_T &= h(X_T),
\end{aligned}
\]

and the following cost functional:

\[ J(0, x; u(\cdot)) \triangleq Y_0^0.x;u. \]

The above stochastic recursive control system was given by Peng [13] to establish DPP in the Lipschitz setting of the generator and explore the connection between its value function and HJB equation. On the other hand, Duffie and Epstein [6] studied such a control system from mathematical finance point of view, i.e. they put forward the stochastic (recursive) differential utility which can be regarded as the solution of FBSDE.

From MP point of view, Peng [14] also studied the above recursive control system and derived a local MP by representing the adjoint equation as a FBSDE, in which the control domain is convex. For the general settings that the control domain is nonconvex and the diffusion depends on control, the Ekeland variational principle was applied to obtain the MP in Wu [25] and Yong [26] by treating the second solution and the terminal condition in backward stochastic differential equation (BSDE) as a control and a constraint, respectively. By introducing new and general first-order and second-order adjoint equations, Hu [7] obtained the MP for the recursive stochastic optimal control problem without unknown parameters. These results, especially Hu [7], eventually solved the long-standing open problem put forward in Peng [16].
There has been results on the relationship between MP and DPP for stochastic recursive optimal control system with deterministic coefficients. With sufficiently regular assumptions on the coefficients, Shi [20] and Shi and Wu [21] first demonstrated this relationship. Nie, Shi and Wu [11, 12] studied the relationship between MP and DPP in the sense of viscosity solution of HJB equation. The relationship is summarized as follows:

\[
\begin{align*}
 p^* & = V_x(t, \bar{x}_t)^T q^*_t, \\
 k^*_t & = [V_{xx}(t, \bar{x}_t)\sigma(t, \bar{x}_t, \bar{u}_t) + V_x(t, \bar{x}_t) \\
 & \quad \times f_z(t, \bar{x}_t, -V(t, \bar{x}_t), -V_x(t, \bar{x}_t), -V_{xx}(t, \bar{x}_t), \bar{u}_t)] q^*_t
\end{align*}
\]

and

\[
V_t(t, \bar{x}_t) = G(t, \bar{x}_t, -V(t, \bar{x}_t), -V_x(t, \bar{x}_t), -V_{xx}(t, \bar{x}_t), \bar{u}_t),
\]

where \((p^*, q^*)\) is the adjoint pair of the forward part, \(k^*\) is the adjoint process of the backward part in stochastic recursive control system and \(G\) is the corresponding generalized Hamiltonian function.

In our paper, the most important feature is that the coefficients of the system we consider are random. We emphasize that this is an essential difference from existing literature. In 1992, Peng [43] studied the optimal control problem of non-Markovian stochastic systems using dynamic programming. Compared with the optimal control problem of Markov stochastic systems, the value function is no longer a deterministic function, but a random field. In other words, it is a family of semi-martingales. Furthermore, the HJB equation derived from Bellman’s principle of optimality is no longer a second-order fully nonlinear PDE, but a second-order fully nonlinear BSPDE, whose solution is a pair of random fields as BSDE’s. To distinguish it from the classical HJB equation, we call it the stochastic HJB equation. As in the deterministic case, the existence of the solution for stochastic HJB equation is a very hard problem. The solvability has only been proved for a few cases, see [22, 24, 18, 19, 28] for instance. One contribution of our paper is to show that the value function of the recursive optimal control problem will be the classical solution of stochastic HJB equation, if the needed regularity is satisfied. It can be seen as a general form of Feynman-Kac representation. In this sense, our work extends the result of Tang [23], in which the author used a forward-backward system to represent semilinear backward stochastic partial differential equation. In fact, our proof is partly inspired from that work, i.e. we also use the random field generated by the controlled SDE. Furthermore, we proved a verification theorem to show that the solution of stochastic HJB equation gives the optimal control. Another contribution of our paper is to show the connection between the MP and the DPP. Our result extends those for stochastic recursive optimal control system with deterministic coefficients. Note that we also assume that the value function is smooth to obtain the desired result, but how to deal with nonsmooth case is still unsolved. Actually, the solvability for the stochastic HJB equation in a general form is a long-existing open problem.

The rest of this article is organized as follows. In Section 2, we introduce some notations and the basic setup of our problem. We characterize the optimal control by DPP, i.e. the relation between the value function and stochastic HJB equation in Section 3. In Section 4, the optimal control is characterized by MP, i.e. the stochastic Hamiltonian system. In Section 5, we show the connection between the MP and the DPP. As an application we discuss a linear quadratic (LQ) recursive utility portfolio optimization problem with the random coefficients in Section 6, in which the state feedback optimal control is obtained by both MP and DPP methods, and the relations we obtained are demonstrated explicitly.
2 Notations & Statement of the problem

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space, and \(\{W_t, 0 \leq t \leq T\}\) is a one-dimensional standard Brownian motion on it generating a right-continuous filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\). Let \(E\) be an Euclidean space, and its inner product and norm are denoted by \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\), respectively. For a function \(\phi : \mathbb{R}^n \to \mathbb{R}\), we denote by \(\phi_x\) its gradient and by \(\phi_{xx}\) its Hessian (a symmetric matrix). If \(\phi : \mathbb{R}^n \to \mathbb{R}^k (k \geq 2)\), \(\phi_x = \left(\frac{\partial \phi}{\partial x}\right)\) is the corresponding \(k \times n\) Jacobian matrix.

Next we introduce some useful spaces of random variables and stochastic processes. For any \(\alpha \in [1, \infty)\) and \(\beta \in (0, \infty)\), we let:

- \(M^\beta_{\alpha}(0, T; E)\): the space of all \(\mathcal{F}_t\)-adapted processes \(f : \Omega \times [0, T] \to E\) satisfying
  \[
  \|f\|_{M^\beta_{\alpha}(0, T; E)} \triangleq \left( \mathbb{E} \int_0^T |f_t|^\beta dt \right)^{\frac{1}{\beta}} < \infty.
  \]

- \(S^\beta_{\alpha}(0, T; E)\): the space of all \(\mathcal{F}_t\)-adapted càdlàg processes \(f : \Omega \times [0, T] \to E\) satisfying
  \[
  \|f\|_{S^\beta_{\alpha}(0, T; E)} \triangleq \left( \mathbb{E} \sup_{t \in [0, T]} |f_t|^\beta dt \right)^{\frac{1}{\beta}} < +\infty.
  \]

- \(L^\beta(\Omega; E)\): the space of all random variables \(\xi : \Omega \to E\) satisfying \(\|\xi\|_{L^\beta(\Omega; E)} \triangleq (\mathbb{E} |\xi|^\beta)^{\frac{1}{\beta}} < \infty\).

- \(M^\beta_{\alpha, \beta}(L^\alpha([0, T]; E))\): the space of all \(\mathcal{F}_t\)-adapted processes \(f : \Omega \times [0, T] \to E\) satisfying
  \[
  \|f\|_{\alpha, \beta} \triangleq \left[ \mathbb{E} \left( \int_0^T |f_t|^\alpha dt \right)^{\frac{\beta}{\alpha}} \right]^{\frac{1}{\beta}} < \infty.
  \]

For any \(t, s \in [0, T]\) with \(t \leq s\), we define the admissible control set \(\mathcal{U}^2[t, s] = M^2_{\beta}(L^2([t, s]; \mathbb{R}^k))\) with \(U\) being a closed convex subset of \(\mathbb{R}^k\). Given \(x \in \mathbb{R}^n\) and \(u \in \mathcal{U}^2[t, T]\), we consider the following FBSDE

\[
\begin{align*}
  dX^0_{0,x;u} &= b(s, X^0_{s,x;u}, u_s) ds + \sigma(s, X^0_{s,x;u}, u_s) dW_s, \\
  dY^0_{0,x;u} &= -f(s, X^0_{s,x;u}, Y^0_{s,x;u}, Z^0_{s,x;u}, u_s) ds + Z^0_{s,x;u} dW_s, \\
  X^0_{0,x;u} &= x, \\
  Y^0_{T,x;u} &= h(X^0_{T,x;u}),
\end{align*}
\]

with the cost functional

\[
J(0, x; u) \triangleq Y^0_{0,x;u},
\]

where \(b : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}^n\), \(\sigma : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n\), \(f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}\), \(h : \Omega \times \mathbb{R}^n \to \mathbb{R}\).

We need the following assumptions on coefficients \((b, \sigma, f, h)\).

**Assumption 2.1** For any \((\omega, t, x, u) \in \Omega \times [0, T] \times \mathbb{R}^n \times U\), \(b(\cdot, x, u)\) and \(\sigma(\cdot, x, u)\) are \(\mathcal{F}_t\)-adapted processes; \(b(t, \cdot, u), \sigma(t, \cdot, u) \in C^2(\mathbb{R}^n, \mathbb{R}^n)\); \(b_x(t, x, u), \sigma_x(t, x, u), \sigma_u(t, x, u), \sigma_{uu}(t, x, u)\) are continuous in \((x, u)\); there exists a constant \(K\) such that

\[
|b(t, x, u)|, |\sigma(t, x, u)| \leq K(1 + |x| + |u|) \text{ and } |b_x|, |b_u|, |b_{xx}|, |\sigma_x|, |\sigma_u|, |\sigma_{xx}| \leq K.
\]

**Assumption 2.2** For any \((\omega, t, x_1, x_2, y, z, u_1, u_2) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times U \times U\), \(f(\cdot, x, y, z, u)\) is an \(\mathcal{F}_t\)-adapted process and \(h(x)\) an \(\mathcal{T}_T\)-measurable random variable; \(f\) is differentiable with respect to \((x, y, z, u)\) and \(h\) is differentiable with respect to \(x\); \(f_x(t, x, y, z, u), f_y(t, x, y, z, u), f_z(t, x, y, z, u)\) are continuous in \((x, y, z, u)\), \(f_u(t, x, y, z, u)\) is continuous in \(x\); there exists a constant \(K\) such that for \(\gamma \in [0, 1)\)
Under Assumption 2.1 and 2.2, by Theorem 6.3 in [4] again, FBSDE (3) admits a unique strong solution $(Y, Z)$ of the state equation (1) is called the optimal state process, and consequently $(u; X, Y, Z)$ is called an optimal pair of Problem 2.1.

Problem 2.1 Find an admissible control $\bar{u}$ such that
\[
J(0, x; \bar{u}) = \inf_{u \in \mathcal{U}^2[0, T]} J(0, x; u). \tag{2}
\]

Any $\bar{u} \in \mathcal{U}^2[0, T]$ satisfying (2) is called an optimal control process of Problem 2.1. With $\bar{u}$, the solution $(X, \bar{Y}, \bar{Z})$ of the state equation (1) is called the optimal state process, and consequently $(\bar{u}; X, \bar{Y}, \bar{Z})$ is called an optimal pair of Problem 2.1.

3 The Dynamic Programming Principle and Stochastic HJB Equation for Stochastic Recursive Control Problem

In this section, we are concerned with the dynamic programming principle and the corresponding stochastic HJB Equation for stochastic recursive control Problem 2.1. We shall show that, if the value function is a random field with some regularities, it will be the solution for the stochastic HJB equation. To this end, for $t \in [0, T]$ and $\zeta \in L^2(\Omega; \mathbb{R}^n)$ and $u \in \mathcal{U}^2[0, T]$, we consider the following parameterized FBSDE:

\[
\begin{align*}
&dX^t_{s, u} = b(s, X^t_{s, u}, \zeta_{s, u}, u_s)ds + \sigma(s, X^t_{s, u}, \zeta_{s, u}, u_s)dW_s, \\
&dY^t_{s, u} = -f(s, X^t_{s, u}, Y^t_{s, u}, Z^t_{s, u}, \zeta_{s, u}, u_s)ds + Z^t_{s, u}dW_s, \\
&X^t_T = \zeta, \\
&Y^t_T = h(X^t_T).
\end{align*}
\]

Under Assumption 2.1 and 2.2, by Theorem 6.3 in [4] again, FBSDE (3) admits a unique strong solution $\Theta^{t, \zeta_{u}} = (X^{t, \zeta_{u}}, Y^{t, \zeta_{u}}, Z^{t, \zeta_{u}}) \in S^2_{\beta}(t, T; \mathbb{R}^n) \times S^2_{\beta}(t, T; \mathbb{R}) \times M^2_{\beta}(t, T; \mathbb{R})$ for any $\beta \in (0, 1)$. We call $\Theta^{t, \zeta_{u}}$, or $\Theta = (X, Y, Z)$ whenever its dependence on $u$ and $(t, \zeta)$ is clear from context, the state process and $(u; \Theta)$ is the admissible pair.

For a given control process $u \in \mathcal{U}^2[0, T]$, we define the associated cost functional as follows.

\[
J(t, x; u) \triangleq J^{t, x; u}, \quad (t, x) \in [0, T] \times \mathbb{R}^n.
\]

From Theorem A.2 in [15], we get the following relation

\[
J(t, \zeta, u) = Y^{t, \zeta}_{t, u}. \tag{4}
\]

For $\zeta = x \in \mathbb{R}^n$, the value function we define in this part is

\[
V(t, x) \triangleq \inf_{u \in \mathcal{U}^2[0, T]} J(t, x; u), \quad (t, x) \in [0, T] \times \mathbb{R}^n.
\]
Now we discuss a generalized DPP for our stochastic optimal control problem. For this purpose, we define the family of (backward) semigroups associated with FBSDE (3), which was first introduced by Peng [15]. Given the initial data \((t, x)\), a positive number \(\delta \leq T - t\), an admissible control process \(u \in L^2[t, t + \delta]\) and a real-valued random variable \(\eta \in L^2(\Omega, \mathcal{F}_{t+\delta}; \mathbb{R})\), we put
\[
G_{t,t+\delta}^{t,x,u}(\eta) := \tilde{Y}_{s,s}^{t,x,u}, \quad s \in [t, t + \delta]
\]
where \((X_t^{t,x,u}, \tilde{Y}_t^{t,x,u}, \tilde{Z}_t^{t,x,u})\) is the solution of the following FBSDE with the time horizon \(t + \delta\),
\[
\begin{align*}
\frac{dX_s^{t,x,u}}{ds} &= b(s, X_s^{t,x,u}, u_s)ds + \sigma(s, X_s^{t,x,u}, u_s)dW_s, \\
\frac{d\tilde{Y}_s^{t,x,u}}{ds} &= -f(s, X_s^{t,x,u}, \tilde{Y}_s^{t,x,u}, \tilde{Z}_s^{t,x,u}, u_s)ds + \tilde{Z}_s^{t,x,u}dW_s, \\
X_t^{t,x,u} &= x, \\
Y_{t+\delta}^{t,x,u} &= \eta.
\end{align*}
\]
Obviously, for any admissible control pair \((X_t^{t,x,u}, \tilde{Y}_t^{t,x,u}, \tilde{Z}_t^{t,x,u}; u)\), we have
\[
G_{t,t+\delta}^{t,x,u}(h(X_t^{t,x,u})) = G_{t+\delta,t+\delta}^{t,x,u}(Y_{t+\delta}^{t,x,u}) = G_{t+\delta,t+\delta}^{t,x,u}(J(t+\delta, X_{t+\delta}^{t,x,u}; u)).
\]
Moreover, the following dynamic programming principle holds by a similar proof as in [15].

**Theorem 3.1** Under Assumption 2.1 and 2.2, the value function \(v(t,x)\) obeys the following DPP: for any \(0 \leq t < t + \delta \leq T\), \(x \in \mathbb{R}^n\),
\[
V(t,x) = \inf_{u \in U[t,t+\delta]} G_{t+\delta,t+\delta}^{t,x,u}(V(t+\delta, X_{t+\delta}^{t,x,u})).
\]

Next we shall show the relation between the value function and stochastic HJB equation. For this purpose, the following lemma in [23] is needed.

**Lemma 3.1** For any fixed admissible control \(u\), set \(X_s^x\) to be the solution of the following SDE:
\[
\begin{align*}
\frac{dX_s}{ds} &= b(s, X_s, u_s)ds + \sigma(s, X_s, u_s)dW_s, \\
X_0 &= x.
\end{align*}
\]
Then, almost surely, for each \(s \in [0, T]\), \(X_s^x\) is a diffeomorphism of \(C^1\). The gradient \(\partial X_s^x\) satisfies the following SDE:
\[
\begin{align*}
\frac{d\partial X_s^x}{ds} &= b_x(s, X_s^x, u_s)\partial X_s^x ds + \sigma_x(s, X_s^x, u_s)\partial X_s^x dW_s, \\
\partial X_0^x &= I.
\end{align*}
\]
Moreover, from the boundedness of the derivatives, classical estimation for SDE yields that
\[
\mathbb{E}\left[ \sup_{s \in [0,T]} |\partial X_s^x|^4 \right] \leq M,
\]
where \(M\) is a constant independent of \(x\).

Then main result of this section is presented below.

**Proposition 3.1** In additional to Assumptions 2.1 and 2.2, we also assume that the control region \(U \subset \mathbb{R}^k\) is bounded and, for each \(t \in [0, T]\) and \(x \in \mathbb{R}^n\), the infimum of the cost functional \(J(t, x; \cdot)\) is attained by an optimal control \(u^*_{t,x}\). Moreover, assume that the value function \(V(t, x)\) admits the following semimartingale decomposition:
\[
V(t,x) = h(x) + \int_t^T \Gamma(s,x)ds - \int_t^T \Psi(s,x)dW_s, \quad t \in [0, T],
\]
where the \(\mathbb{R}\)-valued function \(\Gamma(t, \cdot)\) and \(\Psi(t, \cdot)\) are \(\mathcal{F}_t \times B(\mathbb{R}^n)\) measurable for each \(t \in [0, T]\) and \(V, \Gamma, \Psi\) satisfy the following assumptions:
(i) \((t, x) \mapsto V(t, x)\) is continuous a.s.,

(ii) \(x \mapsto V(t, x)\) is \(C^2\) for each \(t \in [0, T]\) a.s.,

(iii) \(x \mapsto \Gamma(t, x)\) is continuous for each \(t \in [0, T]\) a.s.,

(iv) \(x \mapsto \Psi(t, x)\) is \(C^1\) for each \(t \in [0, T]\) a.s.,

(v) There exists \(K \in M_2^c(L^2(0, T; \mathbb{R}^+))\) such that
\[
|V(t, x)|, |h(t, x)|, |\Gamma(t, x)|, |\Psi(t, x)| \leq K_t(1 + |x|^2),
\]
\[
|\partial_x V(t, x)|, |\partial_x \Psi(t, x)| \leq K_t(1 + |x|),
\]
\[
|\partial_{xx} V(t, x)| \leq K_t,
\]
\[
|\Gamma(t, x) - \Gamma(t, y)| \leq K_t(1 + |x| + |y|)|x - y|.
\]

Then, the value function \(V\), together with \(\Psi\), constitutes a pair solution of the so-called backward HJB equation
\[
\begin{cases}
\frac{dV(t, x)}{dt} = -\inf_u G(t, x, V(t, x), \Psi(t, x), V_x(t, x), \Psi_x(t, x), V_{xx}(t, x), u)dt + \Psi(t, x)dW_t, \\
V(T, x) = h(x),
\end{cases}
\]
where
\[
G(t, x, y, z, p, q, A, u) = \langle p, b(t, x, u) \rangle + \langle q, \sigma(t, x, u) \rangle + \frac{1}{2}tr\left((\sigma^\ast)(t, x, u)A\right) + f(t, x, y, \sigma^\ast p + z, u).
\]

Proof. Let \(\{x_i\} = Q\). For a fixed admissible control \(u\) and \(x_i\), we abbreviate \(X\) for \(X^{0, x_i; u}\) for simplicity. Applying Itō-Ventzell formula to \(V(t, X_t)\), we have
\[
V(t, X_t) = V(t + \delta, X_{t+\delta}) + \int_t^{t+\delta} \Gamma(s, X_s) - G(s, X_s, V(s, X_s), \Psi(s, X_s), V_x(s, X_s), \Psi_x(s, X_s), V_{xx}(s, X_s), u_s) ds + f(s, X_s, V(s, X_s), Z'_s, u_s) ds - \int_t^{t+\delta} Z'_s dW_s,
\]
where \(Z'_s = \sigma^\ast V_x(s, X_s) + \Psi(s, X_s)\). From condition (v) in the theorem, it can be verified that \(Z' \in M_2^c(0, T)\). Consider the following BSDE
\[
Y_r = V(t + \delta, X_{t+\delta}) + \int_r^{t+\delta} f(s, X_s, Y_s, Z_s, u_s) ds - \int_r^{t+\delta} Z_s dW_s.
\]

From the DPP, we shall have that \(Y_t \geq V(t, X_t)\). After linearization, \(Y_t - V(t, X_t)\) can be written as
\[
V(t, X_t) - Y_t = \mathbb{E} \left[ \int_t^{t+\delta} \xi_s \Delta(s, X_s, u_s) ds \bigg| \mathcal{F}_t \right] \leq 0,
\]
where
\[
\Delta(s, x, u) = \Gamma(s, x) - G(s, x, V(s, x), \Psi(s, x), V_x(s, x), \Psi_x(s, x), V_{xx}(s, x), u)
\]
and \(\xi_s\) satisfies the following SDE:
\[
\begin{cases}
\frac{d\xi_s}{ds} = A_s \xi_s ds + B_s \xi_s dW_s, \\
\xi_t = 1
\end{cases}
\]
with
\[ A_s = \frac{f(s, X_s, V(s, X_s), Z'_s, u_s) - f(s, X_s, Y_s, Z'_s, u_s)}{V(s, X_s) - Y_s}, \]
and
\[ B_s = \frac{f(s, X_s, Y_s, Z'_s, u_s) - f(s, X_s, Y_s, Z_s, u_s)}{Z'_s - Z_s}. \]
Since \( f(t, x, y, z, u) \) is Lipschitz continuous with respect to \( y \) and \( z \), it is easy to see that \( A \) and \( B \) are uniformly bounded processes. Then, the classical estimation for linear SDEs yields that
\[
\mathbb{E} \left[ |\xi_s - 1|^2 |\mathcal{F}_t \right] \leq C \mathbb{E} \left[ \left( \int_t^s |A_u| du \right)^2 + \int_t^s |B_u|^2 du \right] \leq C(t - s + |t - s|^2). \tag{12}
\]
Here and throughout this paper, \( C \) is a generic constant whose values may change from line by line. To emphasize its dependence on \( t \) and \( \delta \), we also denote \( \xi \) as \( \xi^{t,\delta} \). Then, we claim that, for any \( t \) and \( \delta \),
\[
\mathbb{E} \left[ \int_t^{t+\delta} \Delta(s, X_s, u_s) ds \right] \mathcal{F}_t \leq 0, \quad a.s. \tag{13}
\]
To see this, for fixed \( t \) and \( \delta \), we obtain similarly that, for any \( n \) and \( k \leq n \),
\[
\mathbb{E} \left[ \int_t^{t+\frac{k+1}{n}\delta} \xi_s^{t+\frac{k+1}{n}\delta} \Delta(s, X_s, u_s) ds \right] \mathcal{F}_t \leq 0. \tag{14}
\]
Then, from (14), we have
\[
\mathbb{E} \left[ \int_t^{t+\frac{k+1}{n}\delta} \Delta(s, X_s, u_s) ds \right] \mathcal{F}_t \\
= \mathbb{E} \left[ \int_t^{t+\frac{k+1}{n}\delta} \xi_s^{t+\frac{k+1}{n}\delta} \Delta(s, X_s, u_s) ds \right] \mathcal{F}_t + \mathbb{E} \left[ \int_t^{t+\frac{k+1}{n}\delta} (1 - \xi_s^{t+\frac{k+1}{n}\delta}) \Delta(s, X_s, u_s) ds \right] \mathcal{F}_t \\
\leq \mathbb{E} \left[ \int_t^{t+\frac{k+1}{n}\delta} (1 - \xi_s^{t+\frac{k+1}{n}\delta}) \Delta(s, X_s, u_s) ds \right] \mathcal{F}_t \\
\leq \left( \mathbb{E} \left[ \int_t^{t+\frac{k+1}{n}\delta} (1 - \xi_s^{t+\frac{k+1}{n}\delta})^2 ds \right] \mathcal{F}_t \right)^{1/2} \left( \mathbb{E} \left[ \int_t^{t+\frac{k+1}{n}\delta} |\Delta(s, X_s, u_s)|^2 ds \right] \mathcal{F}_t \right)^{1/2}
\]
Summing over \( k \), we have
\[
\mathbb{E} \left[ \int_t^{t+\delta} \Delta(s, X_s, u_s) ds \right] \mathcal{F}_t \\
\leq \sum_{k=0}^{n-1} \left( \mathbb{E} \left[ \int_t^{t+\frac{k+1}{n}\delta} (1 - \xi_s^{t+\frac{k+1}{n}\delta})^2 ds \right] \mathcal{F}_t \right)^{1/2} \left( \mathbb{E} \left[ \int_t^{t+\frac{k+1}{n}\delta} |\Delta(s, X_s, u_s)|^2 ds \right] \mathcal{F}_t \right)^{1/2} \tag{15}
\]
\[
\leq \left( \sum_{k=0}^{n-1} \mathbb{E} \left[ \int_t^{t+\frac{k+1}{n}\delta} (1 - \xi_s^{t+\frac{k+1}{n}\delta})^2 ds \right] \mathcal{F}_t \right)^{1/2} \left( \mathbb{E} \left[ \int_t^{t+\delta} |\Delta(s, X_s, u_s)|^2 ds \right] \mathcal{F}_t \right)^{1/2},
\]
where the last inequality is obtained due to Hölder inequality. By (12), we have
Thus,

\[ E \left[ \int_{t+\frac{k+1}{n}}^{t+\frac{k}{n}} (1 - \xi_s^{t+\frac{k+1}{n}}) \, ds \bigg| \mathcal{F}_t \right] = \int_{t+\frac{k+1}{n}}^{t+\frac{k}{n}} E \left[ (1 - \xi_s^{t+\frac{k+1}{n}})^2 \bigg| \mathcal{F}_t \right] \, ds \]

\[ \leq C \int_{t+\frac{k}{n}}^{t+\frac{k+1}{n}} |t + \frac{k+1}{n} - s| + |t + \frac{k}{n} - s|^2 \, ds \]

\[ \leq C \frac{\delta^2}{n^2}. \]

Thus,

\[ \left( \sum_{k=0}^{n-1} E \left[ \int_{t+\frac{k}{n}}^{t+\frac{k+1}{n}} (1 - \xi_s^{t+\frac{k+1}{n}})^2 \, ds \bigg| \mathcal{F}_t \right] \right)^{1/2} \to 0, \quad \text{as } n \to \infty. \]

Due to Assumption 2.1 and boundedness of the control region \( U, X \in S^\beta_T(0, T; \mathbb{R}^n) \) for any \( \beta \geq 2 \), and thus \( E \left[ \int_t^{t+\delta} |\Delta(s, X_s, u_s)|^2 \, ds \bigg| \mathcal{F}_t \right] \) is bounded. Hence, letting \( n \to \infty \) in (15), we have that

\[ E \left[ \int_t^{t+\delta} \Delta(s, X_s, u_s) \, ds \bigg| \mathcal{F}_t \right] \leq 0. \]  

(16)

For fixed \( t \in [0, T] \) and any nonnegative \( \mathbb{R} \)-valued random variable \( \eta \in \mathcal{F}_t \), it follows from (16) that

\[ E \left[ \int_0^T \Delta(s, X_s, u_s) \eta I_{(t,t+\delta)}(s) \, ds \right] = E \left[ \eta E \left[ \int_t^{t+\delta} \Delta(s, X_s, u_s) \, ds \bigg| \mathcal{F}_t \right] \right] \leq 0. \]

Consequently, for any nonnegative simple progress \( \phi \in M^2_T(0, T; \mathbb{R}) \),

\[ E \left[ \int_0^T \Delta(s, X_s, u_s) \phi_s \, ds \right] \leq 0. \]

For any nonnegative progress \( \psi \in M^2_T(0, T; \mathbb{R}) \), there exists a sequence of nonnegative simple progresses \( \phi^n \in M^2_T(0, T; \mathbb{R}), n \in \mathbb{N} \), such that

\[ \lim_{n \to \infty} E \left[ \int_0^T |\phi^n_s - \psi_s|^2 \, ds \right] = 0. \]

Hence

\[ \lim_{n \to \infty} \left| E \left[ \int_0^T \Delta(s, X_s, u_s) \phi^n_s \, ds \right] - E \left[ \int_0^T \Delta(s, X_s, u_s) \psi_s \, ds \right] \right| \leq \lim_{n \to \infty} \left( E \left[ \int_0^T |\Delta(s, X_s, u_s)|^2 \, ds \right] \right)^{1/2} \left( E \left[ \int_0^T |\phi^n_s - \psi_s|^2 \, ds \right] \right)^{1/2} = 0, \]

which implies that

\[ E \left[ \int_0^T \Delta(s, X_s, u_s) \psi_s \, ds \right] \leq 0. \]
Noticing the arbitrariness of nonnegative process $\psi$, we have that
\[ \Delta(s, X_s, u_s) \leq 0 \quad \text{for a.e. } s \in [0, T], \text{ a.s.} \]

Let $X_s^x$ be the stochastic flow generated by the SDE (3.1). From Lemma 3.1, with probability 1, for each $s$, $X_s$ is a diffeomorphism of class $C^1$. For each $x$, we also have that
\[ \Delta(s, X_s^x, u_s) \leq 0 \quad \text{for a.e. } s \in [0, T], \text{ a.s.} \]

Since $\Delta(s, x)$ and $X_s^x$ is continuous with respect to $x$, we shall get that
\[ \Delta(s, X_s^x, u_s) \leq 0 \quad \text{for all } x \in \mathbb{R}^n, \text{ a.e. } s \in [0, T], \text{ a.s.} \]

From the growth condition of the coefficients and the value function, we see that
\[ |\Delta(t, X_t^x, u_t)|^2 \leq C(1 + K_t^2)(1 + |X_t^x|^4). \]

Then,
\[
E \left[ \int_0^T |\Delta(t, X_t^x, u_t)|^2 dt \right] \\
\leq C E \left[ \int_0^T (1 + K_t^2)(1 + |X_t^x|^4) dt \right] \\
\leq C E \left[ \sup_t (1 + |X_t^x|^4) \int_0^T (1 + K_t^2) dt \right] \\
\leq C \left( E \left[ \left( \sup_t (1 + |X_t^x|^4) \right)^2 \right] \right)^{1/2} \left( E \left[ \int_0^T (1 + K_t^2) dt \right] \right)^{1/2} \\
\leq C(1 + |x|^4).
\]

Now, let $\varphi$ be a smooth function such that
\[ \varphi(x) = \begin{cases} 1, & \text{for } |x| \leq 1; \\ 0, & \text{for } |x| \geq 2; \\ \in [0, 1], & \text{otherwise.} \end{cases} \]

For $s \in [0, T]$, define $\tilde{X}_s$ to be the inverse function of $X_s$ and consider a random function
\[ g(s, x) = \xi(X_s^x)\varphi \left( \frac{x}{N} \right) \det \partial \tilde{X}_s^y|_{y=X_s^x}^{-1} p_s, \]
where $N \in \mathbb{N}$, $p$ is an arbitrarily given bounded non-negative adapted process and $\xi$ is a smooth non-negative function with a compact support. Let us first prove $E \left[ \int_0^T \int_{\mathbb{R}^n} \Delta(s, X_s^x) g(s, x) dx ds \right] < \infty$. By Hölder inequality, it holds that
\[
E \left[ \int_0^T \int_{\mathbb{R}^n} |\Delta(s, X_s^x, u_s)| g(s, x) dx ds \right] \leq \left( E \left[ \int_0^T \int_{\mathbb{R}^n} |\Delta(s, X_s^x, u_s)|^2 \varphi \left( \frac{x}{N} \right) dx ds \right] \right)^{1/2} \\
\left( E \left[ \int_0^T \int_{\mathbb{R}^n} \xi^2(X_s^x) \varphi \left( \frac{x}{N} \right) \det \partial \tilde{X}_s^y|_{y=X_s^x}^{-2} p_s^2 dx ds \right] \right)^{1/2}
\]
For the first term on the right hand side, we have
\[
E \left[ \int_0^T \int_{\mathbb{R}^n} |\Delta(s, X_s^x, u_s)|^2 \varphi \left( \frac{x}{N} \right) dx ds \right] \leq \int_{|x| \leq N+2} E \left[ \int_0^T |\Delta(s, X_s^x, u_s)|^2 ds \right] dx < \infty.
\]

Note that \( \hat{X}_s^x = x \). Hence \( \partial_y \hat{X}_s^y |_{y=X_s^x} \partial_x X_s^x = I \), and thus \( |\det \partial_y \hat{X}_s^y |_{y=X_s^x} |^{-1} = |\det \partial_x X_s^x| \). For the second term, it holds that
\[
E \left[ \int_0^T \int_{\mathbb{R}^n} \xi(\hat{X}_s^x) \varphi \left( \frac{x}{N} \right) |\det \partial_x X_s^x|^2 dx ds \right] \leq C E \left[ \int_0^T \int_{\mathbb{R}^n} \varphi \left( \frac{x}{N} \right) |\det \partial_x X_s^x|^2 dx ds \right] \leq C \int_{|x| \leq N+2} E \left[ \int_0^T |\det \partial_x X_s^x|^2 ds \right] dx < \infty.
\]

Thus, we see that \( E \left[ \int_0^T \int_{\mathbb{R}^n} \Delta(s, X_s^x, u_s) g(s, x) dx ds \right] < \infty \). Then we have
\[
0 \geq E \left[ \int_0^T \int_{\mathbb{R}^n} \Delta(s, X_s^x, u_s) g(s, x) dx ds \right] = E \left[ \int_0^T \int_{\mathbb{R}^n} \Delta(s, X_s^x, u_s) \xi(\hat{X}_s^x) \varphi \left( \frac{x}{N} \right) |\det \partial_y \hat{X}_s^y |_{y=X_s^x} |^{-1} p_s dx ds \right] = E \left[ \int_0^T \int_{\mathbb{R}^n} \Delta(s, x, u_s) \xi(x) \varphi \left( \frac{X_s^x}{N} \right) p_s dx ds \right],
\]

where we apply the change of variable from the second to the third line in the above. As \( N \to +\infty \), it reduces to
\[
E \left[ \int_0^T \int_{\mathbb{R}^n} \Delta(s, x, u_s) \xi(x) p_s dx ds \right] \leq 0.
\]

From the arbitrariness of \( \xi, p \) and \( u_s \), we have that
\[
\sup_u \Delta(s, x, u) \leq 0 \quad \text{for all } x \in \mathbb{R}^n, \text{ a.e. } s \in [0, T], \text{ a.s.} \tag{17}
\]

Next, we show that the equality holds. Since the optimal control \( u_0^{*, t, x} \) and its corresponding state denoted by \( X_0^{u_0^{*, t, x}} \) exist. For simplicity, we abbreviate \( (X_0^{u_0^{*, t, x}}, u_0^{*, t, x}) \) as \( (X^*, u^*) \). It follows from (11) that
\[
\Delta(s, X_s^x; u_s^x) = 0, \text{ for a.e. } s \in [t, T], \text{ a.s.}
\]

Denote by \( \Delta(s, x) := \sup_u \Delta(s, x, u) \). Then, we see that
\[
\Delta(s, x, 0) \leq \Delta(s, x) \leq 0.
\]

This implies that
\[
|\Delta(s, x)| \leq |\Delta(s, x, 0)| \leq CK_t(1 + |x|^2),
\]

which further yields that \( \Delta(s, x) \in M_{\mathbb{R}}^2(L^2(0, T; \mathbb{R}^-)) \) for any \( x \). Let \( \zeta(t) \) be a mollifier defined on \([0, +\infty)\), i.e.
\[ \zeta(t) = \begin{cases} C \exp(-\frac{1}{1-t^2}), & \text{if } t \leq 1; \\ 0, & \text{otherwise}; \end{cases} \]

with the constant \( C \) selected so that \( \int_0^\infty \zeta(t) dt = 1 \) and \( \zeta_n(t) = n\zeta(nt) \). Define

\[ \Delta_n(s, x) = \int_0^\infty \zeta_n(u) \Delta(s + u, x) du. \]

We shall have that

\[ \mathbb{E} \left[ \int_0^T \Delta_n(s, x) ds \right] \to \mathbb{E} \left[ \int_0^T \Delta(s, x) ds \right] \tag{18} \]

as \( n \to +\infty \). Note that

\[ \Delta_n(s, x) = \int_0^\infty \zeta_n(u) \Delta(s + u, x) du \]
\[ \geq \int_0^\infty \zeta_n(u) \Delta(s + u, x, u^*_s + u) du \]
\[ = \int_0^\infty \zeta_n(u) (\Delta(s + u, x, u^*_s + u) - \Delta(s + u, X^*_s + u^*_s, u^*_s + u)) du \]

From the assumption of the theorem, we see that

\[ |\Delta(s + u, x, u^*_s + u) - \Delta(s + u, X^*_s + u^*_s, u^*_s + u)| \leq CK_{s+u}(1 + |x| + |X^*_s + u| + |u^*_s + u|)|X^*_s + u - x|. \]

Hence,

\[ \mathbb{E} \left| \int_0^\infty \zeta_n(u) (\Delta(s + u, x, u^*_s + u) - \Delta(s + u, X^*_s + u^*_s, u^*_s + u)) du \right| \]
\[ \leq C \left( \mathbb{E} \left[ \int_0^\infty \zeta_n(u) K_{s+u} (1 + |x| + |X^*_s + u| + |u^*_s + u|)^2 du \right] \right)^{1/2} \]
\[ \leq C \left( \mathbb{E} \left[ \int_0^\infty \zeta_n(u) K_{s+u} |X^*_s + u - x|^2 du \right] \right)^{1/2} \]
\[ \leq C \left( \mathbb{E} \left[ \int_0^\infty \zeta_n(u) K^2_{s+u} du \right] \right)^{1/2} \left( \mathbb{E} \left[ \int_0^\infty \zeta_n(u) (1 + |x| + |X^*_s + u| + |u^*_s + u|)^4 du \right] \right)^{1/4} \]
\[ \left( \mathbb{E} \left[ \int_0^\infty \zeta_n(u) |X^*_s + u - x|^4 du \right] \right)^{1/4} \]

Then, we see that, for all \( s \),

\[ \mathbb{E} \left[ \int_0^\infty \zeta_n(u) |X^*_s + u - x|^4 du \right] \to 0 \]

and

\[ \mathbb{E} \left[ \int_0^\infty \zeta_n(u) (1 + |x| + |X^*_s + u| + |u^*_s + u|)^4 du \right] \]

is uniformly bounded with respect to \( n \). Moreover, it holds that, for almost all \( s \),

\[ \mathbb{E} \left[ \int_0^\infty \zeta_n(u) K^2_{s+u} du \right] \to \mathbb{E} \left[ K^2_s \right]. \]
Hence, for almost $s$,

$$\liminf_n \mathbb{E}[\Delta_n(s,x)] \geq 0.$$  

From (18), we have

$$\mathbb{E}\left[\int_0^T \Delta(s,x)ds\right] \geq 0.$$  

Combining with the fact that $\Delta(s,x) \leq 0$, we obtain that

$$\Delta(s,x) = 0.$$  

$\square$

In above, we have proved that the value function is the solution of the stochastic HJB equation under suitable conditions. Next, we will prove a converse result.

**Proposition 3.2 [Stochastic Verification Theorem]** Let $(\Phi,\Psi)$ be the solution of stochastic HJB equation (10) and assume that they satisfy the regularity assumptions in Proposition 2.1. Then, for any $(t,x)$ and admissible control $u$, we have

$$V(t,x) \leq J(t,x;u).$$

Moreover, if there exists an admissible control $u$ such that, for almost all $s \in [t,T]$, 

$$G(s,X_t^{s,x,u},V(t,X_t^{t,x,u}),\Psi(t,X_t^{t,x,u}),\Psi_x(t,X_t^{t,x,u}),\Psi_{xx}(t,X_t^{t,x,u}),u_s)$$ 

$$= \inf_v G(s,X_t^{s,x,u},V(t,X_t^{t,x,u}),\Psi(t,X_t^{t,x,u}),\Psi_x(t,X_t^{t,x,u}),\Psi_{xx}(t,X_t^{t,x,u}),v), a.e.,$$

then $u$ is the optimal control.

**Proof.** The result is obtained by applying Itô formula to $V(s,X_t^{t,x,u})$ and comparing it with $Y_t^{t,x,u}$. Since the calculation is almost the same to previous proposition, we omit the proof here. $\square$

### 4 The Maximum Principle of Stochastic Recursive Control Problem

In this section, we derive the stochastic maximum principle of Problem 2.1. We first define the Hamiltonian function $H : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}$ by

$$H(t,x,y,z,p,q,k,u) = \langle p, b(t,x,u) \rangle + \langle q, \sigma(t,x,u) \rangle - kf(t,x,y,z,u).$$

To simplify our argument, we introduce some abbreviated notations. Now, let $(\bar{u}; \bar{X}, \bar{Y}, \bar{Z})$ be an optimal pair of Problem 2.1. For $\varphi = b, \sigma, b_x, b_u, \sigma_x, \sigma_u$, define

$$\bar{\varphi}(t) := \varphi(t, \bar{X}_t, \bar{u}_t),$$

for $\varphi = f, f_x, f_y, f_z, f_u$,

$$\bar{\varphi}(t) := \varphi(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t),$$

and for $h$,

$$\bar{h}(T) := h(\bar{X}_T), \quad \bar{h}_x(T) := h_x(\bar{X}_T).$$

Now we are ready to give the necessary conditions of optimality for the optimal control of Problem 2.1. Let $(\bar{u}; \bar{\Theta}) = (\bar{u}; \bar{X}, \bar{Y}, \bar{Z})$ be an optimal 4-tuple. Fix any admissible control...
It is easy to see that $u^\varepsilon$ is also an admissible control. Denote by $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$ the corresponding state equation and consider the following variational equations:

\[
\begin{cases}
    dX^1_t = \left[\tilde{b}_x(t)X^1_t + \tilde{b}_u(t)u^1_t\right]dt + \left[\tilde{\sigma}_x(t)X^1_t + \tilde{\sigma}_u(t)u^1_t\right]dW_t, \\
    dY^1(t) = -\left[\tilde{f}_x(t)X^1_t + \tilde{f}_y(t)Y^1_t + \tilde{f}_z(t)Z^1_t + \tilde{f}_u(t)u^1_t\right]dt + Z^1_t dW_t, \\
    X^1_0 = 0, \\
    Y_T = \tilde{h}_x(T)X^1_T.
\end{cases}
\] (19)

Since $h_x$ is of linear growth with respect to $x$, the terminal $\tilde{h}_x(T)X^1_T$ is not $L^2$-integrable in general. Thus, the solvability of (19) is not obvious. For that purpose, we shall introduce the following result for BSDE with $L^p$-terminal. It has been proved in [4].

**Lemma 4.1** Consider the following BSDE

\[
\begin{cases}
    dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t, \\
    Y_T = \xi,
\end{cases}
\]

with $f$ is uniformly Lipschitz continuous with respect to $(y, z)$ and $\xi$ is $L^p$-integrable with some $p > 1$. There exists a unique solution $(Y, Z)$, and for some constant $\tilde{C}$,

\[
\|Y\|_{\mathcal{S}_p}^p + \|Z\|_{M_p}^p \leq \tilde{C}E\left[|\xi|^p + \left(\int_0^T |f(t, 0, 0)| dt\right)^p\right].
\]

We shall have the following lemmas.

**Lemma 4.2** Under Assumptions 2.1, it holds that

\[
E\sup_{0 \leq t \leq T} |X^\varepsilon_t - \tilde{X}_t|^p = O(\varepsilon^p),
\] (20)

and

\[
E\sup_{0 \leq t \leq T} |X^\varepsilon_t - \tilde{X}_t - \varepsilon X^1_t|^p = o(\varepsilon^p),
\] (21)

for any $p > 1$.

**Proof.** The proof is rather standard. For (20), by the $L^p$ estimate for SDE (see Proposition 2.1 in [10]) and Assumptions 2.1, we have

\[
E\left(\sup_{0 \leq t \leq T} |X^\varepsilon_t - \tilde{X}_t|^p\right) \leq C E\left[\int_0^T |b(t, \tilde{X}_t, u^\varepsilon_t) - b(t, \tilde{X}_t, \tilde{u}_t)| dt\right]^p + E\left[\int_0^T |\sigma(t, \tilde{X}_t, u^\varepsilon_t) - \sigma(t, \tilde{X}_t, \tilde{u}_t)|^2 dt\right]^{p/2} \leq C E\left(\int_0^T |u^\varepsilon_t - \tilde{u}_t|^2 dt\right)^{p/2}
\]
From previous estimation for $X^\varepsilon - \bar{X}$ and $X^\varepsilon - X^1$, it is also easy to show that

$$\frac{dX^\varepsilon}{dt} = \tilde{b}_x(t)X^\varepsilon + \tilde{b}_x(t)X^1 dt + \tilde{\sigma}_x(t)\int_0^t (X^\varepsilon - \bar{X}_t) dW_t,$$

with

$$\tilde{b}_x(t) := \int_0^1 b_x(t, \tilde{X}_t + \lambda \bar{X}_t, \tilde{u}_t + \lambda \varepsilon u_1\lambda) d\lambda$$

and

$$\tilde{\sigma}_x(t) := \int_0^1 \sigma_x(t, \tilde{X}_t + \lambda \bar{X}_t, \tilde{u}_t + \lambda \varepsilon u_1\lambda) d\lambda.$$

From previous estimation for $X^\varepsilon - \bar{X}$ and the standard estimation for SDEs, we shall have (21).

The proof is completed.

\[\square\]

It is also easy to show that $X^1$ is $L^p$-integrable for any $p > 1$, which implies that the terminal $\tilde{h}_x(T)X^1$ is $L^p$-integrable for any $p \in (1, 2)$. Combining Lemma 4.1, we shall have

**Lemma 4.3** Under Assumptions 2.1 and 2.2, FBSDE (19) admits a unique solution $(X^1, Y^1, Z^1)$. Moreover, $X^1 \in S^{p_1}$ and $(Y^1, Z^1) \in S^{p_2} \times M^{p_2}$ for any $p_1 > 1$ and any $p_2 \in (1, 2)$.

Next, we prove the following expansion for $\bar{Y}$.

**Lemma 4.4** Under Assumptions 2.1 and 2.2, we have for any $p \in (1, 2)$

$$\lim_{\varepsilon \to 0} \mathbb{E} \sup_{0 \leq t \leq T} \left| \frac{Y^\varepsilon_t - \bar{Y}_t}{\varepsilon} - Y^1_t \right|^p = 0. \quad (22)$$

**Proof.** A direct calculation gives

$$\begin{align*}
Y^\varepsilon_t - \bar{Y}_t - \varepsilon Y^1_t &= \tilde{h}_x(T)(X^\varepsilon_T - \bar{X}_T) + \tilde{h}_x(T)\delta X(T)
\int_t^T \tilde{f}_x(s)(X^\varepsilon_s - \bar{X}_s - \varepsilon X^1_s) ds + \int_t^T \tilde{f}_y(s)(Y^\varepsilon_s - \bar{Y}_s - \varepsilon Y^1_s) ds \\
&+ \int_t^T \tilde{f}_z(s)(Z^\varepsilon_s - \bar{Z}_s - \varepsilon Z^1_s) ds + \int_t^T (\tilde{f}_x(s) - \tilde{f}_x(s))\varepsilon X^1_s ds \\
&+ \int_t^T (\tilde{f}_y(s) - \tilde{f}_y(s))\varepsilon Y^1_s ds + \int_t^T (\tilde{f}_z(s) - \tilde{f}_z(s))\varepsilon Z^1_s ds \\
&+ \varepsilon \int_t^T (\tilde{f}_u(s) - \tilde{f}_u(s))u_1 ds + \int_t^T (\bar{Z}(s) - \bar{Z}(s) - \varepsilon Z^1_s) dW_s,
\end{align*}$$

with

$$\tilde{f}_x(t) := \int_0^1 f_x(t, \tilde{X}_t + \lambda \bar{X}_t, \tilde{u}_t, \bar{Y}(t) + \lambda (Y^\varepsilon_t - \bar{Y}_t), \bar{Z}_t + \lambda \bar{Z}_t) d\lambda,$$
and \( \tilde{f}_y, \tilde{f}_z \) and \( \tilde{h}_x \) similarly defined. Combining Lemma 4.1 and Lemma 4.3, we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} |Y_t^\varepsilon - \bar{Y}_t - \varepsilon Y_t^1|^p = o(\varepsilon^p),
\]

which is equivalent to (22).

Finally, we shall have the following maximum principle.

**Theorem 4.1** Under Assumptions 2.1 and 2.2, set \((\bar{u}; \bar{\Theta}) = (\bar{u}; X, Y, Z)\) be an optimal 4-tuple of Problem 2.1. Then, we have, for a.e. \( t \in [0, T] \), almost surely

\[
H_u(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, p_t, q_t, k_t, \bar{u}_t)(u - \bar{u}_t) \geq 0, \text{ for any } u \in U,
\]

where \( \Lambda = (p, q, k) \) is the solution to the following FBSDE:

\[
\begin{cases}
  dp_t = -\bar{H}_x(t)dt + q_t dW_t, \\
  dk_t = -\bar{H}_p(t)dt - \bar{H}_z(t)dW_t, \\
  p_T = -\bar{h}_x(T)k_T, \\
  k_0 = -1, \quad 0 \leq t \leq T,
\end{cases}
\]

with \( \eta(t) = H, H_x, H_y, H_z, H_u \), defined as

\[ \eta(t) := \eta(t, \bar{\Theta}_t, \Lambda_t, \bar{u}_t). \]

**Proof.** Fix any admissible control \( u \in \mathcal{U}_1[0, T] \). For any \( \varepsilon \in [0, 1] \), we construct a perturbed admissible control

\[ u^\varepsilon = \bar{u} + \varepsilon u^1, \]

with \( u^1_t = \frac{u_t - \bar{u}_t}{|u_t - \bar{u}_t|} \) and the corresponding state equation is denoted by \((X^\varepsilon, Y^\varepsilon, Z^\varepsilon)\). Let \((X^1, Y^1, Z^1)\) be the solution of FBSDE (19). From Lemma 4.4, we have for any \( p \in (0, 1) \),

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \frac{Y_t^\varepsilon - \bar{Y}_t}{\varepsilon} - Y_t^1 \right|^p \right] = 0.
\]

Then, it holds that

\[
Y_t^1 = \lim_{\varepsilon \to 0^+} \frac{Y_t^\varepsilon - \bar{Y}_t}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{J(0, x, u^\varepsilon) - J(0, x, \bar{u})}{\varepsilon} \geq 0.
\]

Applying Itô formula to \( \langle Y_t^1, k_t \rangle + \langle X_t^1, p_t \rangle \), we have

\[ Y_0^1 = E \int_0^T H_u(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t, p_t, q_t, k_t)u^1_t dt. \]

Thus, by the variational inequality (26), we have

\[ E \int_0^T H_u(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t, p_t, q_t, k_t)u^1_t dt \geq 0, \]

which is equivalent to

\[ E \int_0^T H_u(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t, p_t, q_t, k_t) \frac{u_t - \bar{u}_t}{|u_t - \bar{u}_t| \vee 1} dt \geq 0, \]

for any \( u \in U^2[0, T] \). Due to the arbitrariness of \( u_1 \), we shall get that

\[ H_u(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t, p_t, q_t, k_t) \frac{u - \bar{u}_t}{|u - \bar{u}_t| \vee 1} \geq 0, \]

for any \( u \in U \). This will implies (23).
5 The Relationship between SMP and DPP

In this section, we will state the relation between SMP and DPP for the recursive utility setup.

**Theorem 5.1** We assume that the value function admits the following form

\[ V(t, x) = h(x) + \int_t^T \Gamma(s, x) ds - \int_t^T \Psi(s, x) dW_s, \quad t \in [0, T], \]

(27)

where for a.e. \( s \in [t, T] \) a.s. \( \omega \in \Omega \),

\[ \Gamma(s, \bar{X}_s^{t,x}) = G(s, \bar{X}_s^{t,x}, \bar{u}_s, V(s, \bar{X}_s^{t,x}), \Psi(s, \bar{X}_s^{t,x}), V_x(s, \bar{X}_s^{t,x}), V_{xx}(s, \bar{X}_s^{t,x})) \]

(28)

\[ = \inf_{\omega \in U} G(s, \bar{X}_s^{t,x}, \omega, V(s, \bar{X}_s^{t,x}), \Psi(s, \bar{X}_s^{t,x}), V_x(s, \bar{X}_s^{t,x}), V_{xx}(s, \bar{X}_s^{t,x})). \]

If \( V \in C^{1,3}([0, T] \times \mathbb{R}^n) \) and \( \Gamma_x, \Psi_x \in C^{0,0}([0, T] \times \mathbb{R}^n) \), we have

\[ p_s = -V_x(s, \bar{X}_s^{t,x})k_s, \]

(29)

\[ q_s = -V_{xx}(s, \bar{X}_s^{t,x})\sigma(s, \bar{X}_s^{t,x}, \bar{u}_s) + V_x(s, \bar{X}_s^{t,x}, \Psi(s, \bar{X}_s^{t,x}, \bar{u}_s) + V_x(s, \bar{X}_s^{t,x})) \]

for a.e. \( s \in [0, T] \) a.s., where \( k_s \) satisfies \( k_0 = -1 \) and

\[ d\bar{k}_s = f_s(s, \bar{X}_s^{t,x}, \bar{u}_s, V(s, \bar{X}_s^{t,x}), \Psi(s, \bar{X}_s^{t,x}), V_x(s, \bar{X}_s^{t,x}), V_{xx}(s, \bar{X}_s^{t,x})) \]

(30)

\[ f_s(s, \bar{X}_s^{t,x}, \bar{u}_s, V(s, \bar{X}_s^{t,x}), \sigma(s, \bar{X}_s^{t,x})V_x(s, \bar{X}_s^{t,x}) + \Psi(s, \bar{X}_s^{t,x}, \bar{u}_s) + \Psi_x(s, \bar{X}_s^{t,x})) \]

\[ = 0 = \inf_{u \in U} G(s, x, u, V(s, x), \Psi(s, x), V_x(s, x), \Psi_x(s, x), V_{xx}(s, x)) \]

Then, since \( V \) satisfies HJB equation (10) and has a form as (9), we conclude

\[ \Gamma(s, x) = \inf_{u \in U} G(s, x, u, V(s, x), \Psi(s, x), V_x(s, x), \Psi_x(s, x), V_{xx}(s, x)) \]

(30)

\[ \leq G(s, x, \bar{u}_s, V(s, x), \Psi(s, x), V_x(s, x), \Psi_x(s, x), V_{xx}(s, x)). \]

Thus

\[ 0 = G(s, \bar{X}_s^{t,x}, \bar{u}_s, V(s, \bar{X}_s^{t,x}), \Psi(s, \bar{X}_s^{t,x}), V_x(s, \bar{X}_s^{t,x}), \Psi_x(s, \bar{X}_s^{t,x}), V_{xx}(s, \bar{X}_s^{t,x})) \]

(30)

\[ \leq G(s, x, \bar{u}_s, V(s, x), \Psi(s, x), V_x(s, x), \Psi_x(s, x), V_{xx}(s, x)) - \Gamma(s, x). \]

Bearing in mind that \( V \in C^{1,3}([0, T] \times \mathbb{R}^n) \) and \( \Gamma_x \in C^{0,0}([0, T] \times \mathbb{R}^n) \), we have

\[ \frac{\partial}{\partial x} \bigg\{ G(s, x, \bar{u}_s, V(s, x), \Psi(s, x), V_x(s, x), \Psi_x(s, x), V_{xx}(s, x)) - \Gamma(s, x) \bigg\}_{x = \bar{X}_s^{t,x}} = 0. \]

This implies

\[ (\sigma_x)^*(s, \bar{X}_s^{t,x}, \bar{u}_s)(V_{xx}(s, \bar{X}_s^{t,x})) + 1 2 \frac{1}{2} \sigma_x^*(s, \bar{X}_s^{t,x}, \bar{u}_s)V_{xx}(s, \bar{X}_s^{t,x})) \]

\[ + b_x^*(s, \bar{X}_s^{t,x}, \bar{u}_s)V_x(s, \bar{X}_s^{t,x}) + V_{xx}(s, \bar{X}_s^{t,x})b_x(s, \bar{X}_s^{t,x}, \bar{u}_s) + \sigma_x^*(s, \bar{X}_s^{t,x}, \bar{u}_s) \Psi_x(s, \bar{X}_s^{t,x}) \]
Here and in the rest of this paper,
\[ \frac{1}{2} \text{tr}((\sigma\sigma^*)V_{xxx}) \triangleq \left( \text{tr}(\sigma\sigma^*V_{xxx}^1), \text{tr}(\sigma\sigma^*V_{xxx}^2), \ldots, \text{tr}(\sigma\sigma^*V_{xxx}^n) \right)^{\ast}. \]

On the other hand, from (27), we have
\[ V_x(t, x) = h_x(x) + \int_t^T \Gamma_x(s, x)ds - \int_t^T \Psi_x(s, x)dW_s, \quad t \in [0, T]. \]

Then by the application of Itô's formula to \(-V_x(s, \bar{X}^t,x)_k_s\), it turns out, from (31), that
\[ -V_x(s, \bar{X}^t,x)_k_s \]
\[ = -V_x(T, \bar{X}^t,T)_k_T + \int_s^T k_s dV_x(r, \bar{X}^t,x) + \int_s^T V_x(r, \bar{X}^t,x)dk_r + \int_s^T dV_x(r, \bar{X}^t,x) \cdot dk_r \]
\[ = -V_x(T, \bar{X}^t,T)_k_T + \int_s^T \left[ -\Gamma_x(r, \bar{X}^t,r) + \frac{1}{2} \text{tr}((\sigma\sigma^*)^r(r, \bar{\bar{X}}^t,r, \bar{\bar{u}}))V_{xxx}(r, \bar{X}^t,x) \right] k_r dr \]
\[ + \int_s^T [\Psi_x(r, \bar{X}^t,x) + V_{xx}(r, \bar{X}^t,x)] \sigma(r, \bar{X}^t,x, \bar{u})] k_r dr \]
\[ + \int_s^T V_x(r, \bar{X}^t,x)_f y(r, \bar{X}^t,x, \bar{Y}^t,x, \bar{Z}^t,x, \bar{u}) k_r dr + \int_s^T V_x(r, \bar{X}^t,x)_f y(r, \bar{X}^t,x, \bar{Y}^t,x, \bar{Z}^t,x, \bar{u}) k_r dW_r \]
\[ + \int_s^T \left[ \Psi_x(r, \bar{X}^t,x) + V_{xx}(r, \bar{X}^t,x)\sigma(r, \bar{X}^t,x, \bar{u}) \right] f_z(r, \bar{X}^t,x, \bar{Y}^t,x, \bar{Z}^t,x, \bar{u}) k_r dr \]
\[ = -V_x(T, \bar{X}^t,T)_k_T \]
\[ + \int_s^T \left[ -\sigma_x^x(r, \bar{X}^t,x, \bar{u}) \Psi_x(r, \bar{X}^t,x) - f_x(r, \bar{X}^t,x, \bar{Y}^t,x) \Psi_x(r, \bar{X}^t,x) + \sigma^x V_x(r, \bar{X}^t,x), \bar{u}) \right] k_r dr \]
\[ + \int_s^T \left[ -\sigma_x^x(r, \bar{X}^t,x, \bar{u}) \Psi_x(r, \bar{X}^t,x) - f_x(r, \bar{X}^t,x, \bar{Y}^t,x) \Psi_x(r, \bar{X}^t,x) + \sigma^x V_x(r, \bar{X}^t,x), \bar{u}) \right] k_r dW_r \]
\[ + \int_s^T V_x(r, \bar{X}^t,x)_f y(r, \bar{X}^t,x, \bar{Y}^t,x, \bar{Z}^t,x, \bar{u}) k_r dr + \int_s^T V_x(r, \bar{X}^t,x)_f y(r, \bar{X}^t,x, \bar{Y}^t,x, \bar{Z}^t,x, \bar{u}) k_r dW_r \]
From the maximum principle we proved in previous section, we shall have the following theorem.

The cost functional is defined as following:

\[ h_x(\bar{X}_T^{t,x}) = V_x(T, \bar{X}_T^{t,x}), \]

by the uniqueness of the solution to FBSDE (24), we obtain (29).

6 An Example: LQ Problem

In this section, we take the LQ problem as an example to show the relationship between stochastic maximum principle and stochastic dynamical programming. Consider the following forward-backward stochastic system:

\[
\begin{align*}
    dX_s &= \left[ A_s X_s + B_s u_s \right] ds + \left[ C_s X_s + D_s u_s \right] dW_s \\
    X_t &= x, \\
    dY_s &= -\left[ \lambda_s Y_s + Q_s X_s, X_s \right] ds + Z_s dW_s \\
    Y_T &= \langle GX_T, X_T \rangle.
\end{align*}
\]

The cost functional is defined as following:

\[ J(t, x, u) = Y_T^{t,x,u}. \]

We have the following assumptions for the coefficients.

**Assumption 6.1**

1. The coefficients $A, B, C, D, \lambda, Q,$ and $R$ are all bounded \( \{ \mathcal{F}_t \} \)-adapted processes;
2. The coefficients $Q$ and $R$ are uniformly positive definitive, i.e., there exists a constant $C$ such that $Q_s, R_s \geq CI,$ for all $s \in [t, T], a.s.$, where $I$ is the identity matrix.

For any admissible control $u$ and initial state $x$, we introduce the corresponding adjoint equation:

\[
\begin{align*}
    dp_s &= -\left[ A^*_s p_s + C^*_s q_s - 2k_s Q_s X_s \right] ds + q_s dW_s \\
    p_T &= -2k_T G X_T, \\
    dk_s &= \lambda_s k_s ds \\
    k_t &= -1.
\end{align*}
\]

From the maximum principle we proved in previous section, we shall have the following theorem.
Corollary 6.1 If an admissible pair \((u, X)\) is the optimal pair of LQ problem, \((u, X)\) satisfies
\[
-2k_sR_su_s + D_s^*q_s + B_s^*p_s = 0,
\]
where \((p, q, k)\) is the solution to the corresponding adjoint equation. Therefore, the optimal control has the dual presentation as below:
\[
u_s = \frac{1}{2}k_s^{-1}R_s^{-1}[D_s^*q_s + B_s^*p_s].
\]

If we give an explicit presentation to \((p, q, k)\), a further expression of optimal control can be demonstrated. For this, combining the adjoint system with the original controlled system, we have the following stochastic Hamilton system:
\[
\begin{align*}
\frac{dX_s}{ds} &= [A_sX_s + B_su_s]ds + [C_sX_s + D_su_s]dW_s, \\
X_0 &= x, \\
\frac{dY_s}{ds} &= [-LY_s + (Q_sX_s, X_s) + (R_su_s, u_s)]ds + Z_sdW_s, \\
Y_T &= \langle G_sX_T, X_T \rangle, \\
\frac{dp_s}{ds} &= [-A_s^*p_s + C_s^*q_s - 2k_sQ_sX_s]ds + q_sdW_s, \\
p_T &= -2k_TG_sX_T, \\
\frac{dk_s}{ds} &= \lambda_sK(ds), \\
k_0 &= -1, \\
-2k_sR_su_s + D_s^*q_s + B_s^*p_s &= 0.
\end{align*}
\]

In summary, the stochastic Hamilton system completely characterizes the optimal control in LQ problem. Therefore, solving LQ problem is equivalent to solving the stochastic Hamilton system. But this Hamilton system consists of coupled FBSDEs. Thus, this characterization is far from satisfactory. We then introduce the Riccati equation to give the state feedback representation of the optimal control and further discussion of stochastic Hamilton system.

Different from the Markovian case, the Riccati equation here is a BSDE due to the non-Markovian coefficients:
\[
\begin{align*}
\frac{dP_s}{ds} &= -\{A_s^*P_s + P_sA_s + C_s^*P_sC_s + \lambda_sP_s + C_s^*L_s + L_sC_s + Q_s \\
&\quad - [P_sB_s + C_s^*P_sD_s + L_sD_s] \\
&\quad \times [R_s + D_s^*P_sD_s]^{-1} [P_sB_s + C_s^*P_sD_s + L_sD_s]^* \}ds + L_sdW_s, \\
P_T &= G.
\end{align*}
\]

The solvability of (33) had been studied by Tang [24].

Theorem 6.1 Under Assumption 6.1, the stochastic Riccati equation (33) has a unique solution \((P, L)\), where \(P\) is a uniformly bounded and nonnegative matrix-valued process and \(L\) satisfies
\[
E \left( \int_0^T |L_s|^p ds \right) < \infty,
\]
for any \(p > 1\).

For the concerned LQ problem, we still define its value function as
\[
V(t, x) \triangleq \inf_{u \in \mathcal{A}} J(t, x; u) = \inf_{u \in \mathcal{A}} Y^{x,u}_t.
\]

Then, the corresponding stochastic HJB equation is
Proposition 6.1

With the help of stochastic Riccati equation, we can obtain a solution of above stochastic HJB equation.

Proof. Set

\[ v(s, x) = (P_s x, x), \quad \psi(s, x) = (L_s x, x). \]

First note that \( v_x(s, x) = (P_s + P_s^*) x = 2P_s x, \psi_x(s, x) = (L_s + L_s^*) x = 2L_s x, v_{xx}(s, x) = P_s + P_s^* = 2P_s. \) Then we have

\[
\begin{align*}
\inf_u \{ &v_x(s, x), A_s x + B_s u\} + \frac{1}{2} \text{tr}((C_s x + D_s u)(C_s x + D_s u)^* v_{xx}(s, x)) + \langle \psi_x(s, x), C_s x + D_s u \rangle \\
&\quad + \lambda_s v(s, x) + \langle Q_s x, x \rangle + \langle R_s u, u \rangle \\
&= \inf_u \{ (2P_s, A_s x + B_s u) + \frac{1}{2} \text{tr}((C_s x + D_s u)(C_s x + D_s u)^* 2P_s) + \langle 2L_s x, C_s x + D_s u \rangle \\
&\quad + \lambda_s (P_s, x) + \langle Q_s x, x \rangle + \langle R_s u, u \rangle \\
&= \inf_u \{ \langle x, P_s A_s + A_s^* P_s + Q_s + C_s^* P_s C_s + C_s^* L_s + L_s C_s + \lambda_s P_s \rangle x \} \\
&\quad + 2u \left[ P_s B_s + C_s^* P_s D_s + L_s D_s \right]^* x) + \langle u, (R_s + D_s^* P_s D_s) u \rangle \\
&= \left[ P_s A_s + A_s^* P_s + Q_s + C_s^* P_s C_s + C_s^* L_s + L_s C_s + \lambda_s P_s \right] x, x \}
\end{align*}
\]

Thus, noticing (33), we have

\[
\begin{align*}
d(P_s x, x)
&= -\{ [P_s A_s + A_s^* P_s + Q_s + C_s^* P_s C_s + C_s^* L_s + L_s C_s + \lambda_s P_s] x, x \} \\
&\quad - \langle [P_s B_s + C_s^* P_s D_s + L_s D_s] (R_s + D_s^* P_s D_s)^{-1} [P_s B_s + C_s^* P_s D_s + L_s D_s] x, x \} ds \\
&\quad + (L_s x, x) dW_s.
\end{align*}
\]

By the definition for \((v, \psi)\), together with (35), it turns out that

\[
\begin{align*}
dv(s, x)
&= -\inf_u \{ v_x(s, x), A_s x + B_s u\} + \frac{1}{2} \text{tr}((C_s x + D_s u)(C_s x + D_s u)^* v_{xx}(s, x)) + \langle \psi_x(s, x), C_s x + D_s u \rangle \\
&\quad + \lambda_s v(s, x) + \langle Q_s x, x \rangle + \langle R_s u, u \rangle ds + \psi(s, x) dW_s,
\end{align*}
\]

which demonstrates that \((v, \psi)\) is the classical solution of the stochastic HJB equation.
Having a classical solution of stochastic HJB equation. One can find the optimal control for the LQ problem.

**Proposition 6.2** The optimal control of LQ problem is given by

\[ u_s = -(R_s + D^*_s P_s D_s)^{-1} [P_s B_s + C^*_s P_s D_s + L_s D_s]^* X_s. \]

*Proof.* By Proposition 3.2, we see that the candidate \( u \) for the optimal control is of the following feedback form:

\[ u_s = -(R_s + D^*_s P_s D_s)^{-1} [P_s B_s + C^*_s P_s D_s + L_s D_s]^* X_s. \]

To show that it is indeed the optimal control, one only need to prove that it is a admissible control, which is proved in Tang [24].

Finally, applying Itô formula to \( dP_s X_s k_s \), we immediately have the desired relationship for LQ problem.

**Theorem 6.2** For LQ Problem, we have the relationship between stochastic maximum principle and stochastic dynamical programming below:

\[
\begin{align*}
    p_s &= -2P_s X_s k_s, \\
    q_s &= -2 [P_s (C_s X_s + D_s u_s) + L_s X_s] k_s.
\end{align*}
\]

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