On Continuum- and Bound-State $\ell^-$–Decay Rates of Pionic and Kaonic Hydrogen in the Ground State

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(Dated: February 10, 2009)

We calculate the continuum- and bound-state $\ell^-$–decay rates of pionic and kaonic hydrogen in the ground state, where $\ell^-$ is either the electron $e^-$ or the muon $\mu^-$.

PACS: 12.15.Ff, 13.15.+g, 23.40.Bw, 26.65.+t

INTRODUCTION

It is well-known that pionic and kaonic hydrogen in the ground state play an important role for the investigation of low–energy strong interactions. Pionic and kaonic hydrogen are the bound states, where the electron is replaced by a negatively charged $\pi^-$–meson and $K^-$–meson, respectively. Since the $\pi^-$ and $K^-$ mesons are unstable under weak decays into the leptonic pairs $\ell^-\bar{\nu}_\ell$, where $\ell^-$ is either the electron $e^-$ or muon $\mu^-$ and $\bar{\nu}_\ell$ is the antineutrino with the electron $\bar{\nu}_e$ or muon $\bar{\nu}_\mu$ flavour, the lifetimes of mesic hydrogen should be restricted by the lifetimes of mesons. The lifetimes of the free $\pi^-$ and $K^-$ mesons are equal to $\tau_{\pi^-} = 2.60 \times 10^{-8}$ s and $\tau_{K^-} = 1.24 \times 10^{-8}$ s, respectively. However, one can assume that these lifetimes do not define real lifetimes of pionic and kaonic hydrogen in the ground state. Since a negatively charged meson is bound, this can probably change the lifetime of mesic hydrogen. In addition mesic hydrogen can have also the decay channels different to the emission of the free leptonic pairs $\ell^-\bar{\nu}_\ell$ only. Indeed, mesic hydrogen, which we denote as $H^{1s}_m$ for $m = \pi^-$ or $K^-$, is unstable under the continuum–state $\ell^-$–decay $H^{1s}_m \to p + \ell^- + \bar{\nu}_\ell$ with the emission of the free leptonic pair $\ell^-\bar{\nu}_\ell$ and the bound-state $\ell^-$–decay $H^{1s}_m \to H_\ell + \ell^- + \bar{\nu}_\ell$, where $H_\ell$ is hydrogen with a bound lepton $\ell^-$. In this letter we calculate the continuum- and bound–state $\ell^-$–decay rates of mesic hydrogen in the ground state. According to the classification of the $\beta$–decays, these are allowed $\ell^-$–decays with the selection rule $\Delta J^P = 0^+$. The calculation of the continuum- and bound-state $\ell^-$–decay rates of mesic hydrogen in the ground state we carry out in the standard theory of weak interactions of hadrons using the technique developed in [6, 7] for the analysis of the decay rates of the $H$–like and $He$–like heavy ions. The weak interaction Hamilton density operator we take in the form [8]

$$\mathcal{H}_W(x) = \frac{G_F}{\sqrt{2}} V_{ud} J^{a+b+\text{ib}}(x) \times \left[ \bar{\psi}_\ell(x) \gamma^\mu (1 - \gamma^5) \psi_\nu(x) \right],$$

where $G_F = 1.166 \times 10^{-11}$ MeV$^{-2}$ is Fermi’s weak constant, $V_{ud}$ is an element of the CKM–matrix with $q = d$ or $s$ for the $\pi^-$ or $K^-$ meson, respectively, equal to $V_{ud} = 0.97377$ and $V_{us} = 0.22570$ [2]. $J^{a+b+\text{ib}}(x)$ is a hadronic current with $V - A$ structure and $SU_f(3)$–flavour indices $a = 1$ and $b = 2, 5$ for the $\pi^-$ and $K^-$ meson decays, respectively, $\psi_\ell(x)$ and $\psi_\nu(x)$ are the field operators of the $\ell^-$–lepton and the antineutrino (neutrino) with the $\ell^-$–lepton flavour.

The wave function of mesic hydrogen in the ground state we take in the following form [8]

$$|H^{1s}_m(\vec{K}, \sigma)\rangle = \frac{1}{(2\pi)^3} \sqrt{2E_{H^{1s}_m}(\vec{K})} \times \int \frac{d^3p}{\sqrt{2E_p(\vec{K})}} \frac{d^3k}{\sqrt{2E_k(\vec{K} - \vec{k} - \vec{p})}} \Phi_{1s}(\frac{m \vec{p} - m_p \vec{k}}{m_p + m_m}) c^\dagger_m(\vec{k}) a^\dagger_p(\vec{p}, \sigma)|0\rangle,$$

where $\Phi_{1s}(\vec{K})$ is the wave function of mesic hydrogen in the 1s ground state in the momentum representation [8], $c^\dagger_m(\vec{k})$ and $a^\dagger_p(\vec{p}, \sigma)$ are creation operators of the meson $m$ and the proton, respectively. The operators of creation and annihilation of mesons and protons obey standard relativistic covariant commutation and anti-commutation relations, respectively [8]. The energy of mesic hydrogen is defined by $E_{H^{1s}_m}(\vec{K}) = \sqrt{\vec{K}^2 + M^2_{H^{1s}_m}}$, where $M_{H^{1s}_m} = m_p + m_m + \epsilon_{1s}$.
and $\epsilon_{1s} = -\alpha/2a_B = -\alpha^2\mu_m/2$ are the mass of mesic hydrogen and the binding energy, $\mu_m = m_m m_p/(m_m + m_p)$ is the reduced mass of the $mp$ pair and $a_B = 1/\alpha \mu_m$ is the Bohr radius. The wave function $\Phi_{1s}(\vec{k})$ is equal to
\[
\Phi_{1s}(\vec{k}) = \frac{8\sqrt{\pi}a_B^3}{(1 + a_B^3k^2)^3}.
\]
Since the wave function $\Phi_{1s}(\vec{k})$ is normalised to unity, the wave function Eq. 2 has a standard relativistic invariant normalisation:
\[
\langle H_{1s}^{(1s)}(\vec{k}'', \sigma'')|H_{1s}^{(1s)}(\vec{k}, \sigma)\rangle = (2\pi)^32E_{H_{1s}}(\vec{k})\delta(\vec{k}' - \vec{k})\delta_{\sigma',\sigma}.
\]
Using these definitions we can proceed to calculating the continuum- and bound-state $\ell^-$-decay rates.

CONTINUUM-STATE $\ell^-$-DECAY RATE OF MESIC HYDROGEN IN THE GROUND STATE

The $\mathcal{T}$-matrix of weak interactions, taken to first order in perturbation theory, is equal to
\[
\mathcal{T} = -\int d^4x \mathcal{H}_W(x)
\]
For the $H_{1s}^{(1s)} \rightarrow p + \ell^+ + \bar{\nu}_\ell \rightarrow \bar{\nu}_\ell$ decay the matrix element of the $\mathcal{T}$-matrix is
\[
\langle \bar{\nu}_\ell \ell^- p | \mathcal{T} | H_{1s}^{(1s)} \rangle = (2\pi)^4\delta(4)(k_\nu + k_\ell + k_p - \vec{K}) \times M(H_{1s}^{(1s)} \rightarrow p\ell^- \bar{\nu}_\ell),
\]
where $k_\nu = (E_\nu, \vec{k}_\nu)$, $k_\ell = (E_\ell, \vec{k}_\ell)$, $k_p = (E_p, \vec{k}_p)$ and $K = (E_{H_{1s}}, \vec{K})$ are the 4-momenta of the anti-neutrino, the lepton $\ell^-$, the proton and mesic hydrogen, respectively. The lepton $\ell^-$, the proton and mesic hydrogen have the polarisations $\sigma_\ell$, $\sigma_\nu$, and $\sigma$, respectively, the anti-neutrino is polarised along its 3-momentum $\vec{k}_\nu$ with $\sigma_\nu = +\frac{1}{2}$. The amplitude of the $H_{1s}^{(1s)} \rightarrow p + \ell^+ + \bar{\nu}_\ell$ decay is defined by
\[
M(H_{1s}^{(1s)} \rightarrow p\ell^- \bar{\nu}_\ell) = \langle \bar{\nu}_\ell \ell^- p | \mathcal{H}_W(0) | H_{1s}^{(1s)} \rangle.
\]
The lepton current $[\bar{\nu}_\ell \gamma^\mu(1 - \gamma^5) \nu_\ell]$, where $\bar{\nu}_\ell$ and $\nu_\ell$ are Dirac spinors of the lepton $\ell^-$ and the anti-neutrino $\bar{\nu}_\ell$, respectively, and the matrix element of the hadronic current $(p|J_{\mu}^{a+ib}(0)|mp)$. The product $(p|J_{\mu}^{a+ib}(0)|mp)$ contains disconnected and connected parts
\[
\langle p(k_p, \sigma_p)|J_{\mu}^{a+ib}(0)|m(k)\rangle = \langle p(k_p, \sigma_p)|J_{\mu}^{a+ib}(0)|m(k)\rangle_{\text{disconn.}} + \langle p(k_p, \sigma_p)|J_{\mu}^{a+ib}(0)|m(k)\rangle_{\text{conn.}}.
\]
The disconnected part takes the form
\[
\langle p(k_p, \sigma_p)|J_{\mu}^{a+ib}(0)|m(k)\rangle_{\text{disconn.}} = -\delta_{\sigma,\sigma' - \mu} \times (2\pi)^32E_\nu(q)\delta(k - k_p)(0|A_{\mu}^{a+ib}(0)|m(k))
\]
and gives the main contribution to the continuum-state $\ell^-$-decay rate. The contribution of the connected part, which we define in Appendix A, is smaller compared to the disconnected one.

As a result the amplitude Eq. 3, defined by the disconnected part of the matrix element of the hadronic current, is
\[
M(H_{1s}^{(1s)} \rightarrow p\ell^- \bar{\nu}_\ell) = \frac{2M_{H_{1s}}^2 |J_{\mu}^{a+ib}(0)|^2}{2E_{H_{1s}}}
\]

\[
\times \Phi_{1s}(\vec{k}_p) \frac{G_F}{\sqrt{2}} V_{ud} (0|A_{\mu}^{a+ib}(0)|m(-\vec{k}_p))
\]

\[
\times [\bar{u}_\ell(k_\ell, \sigma_\ell)\gamma^\mu(1 - \gamma^5) \nu_\ell(k_\nu, +\frac{1}{2})].
\]
The matrix element part $\langle 0|A_{\mu}^{a+ib}(0)|m(-\vec{k}_p))\rangle$ we take in the standard form (2) (see also 3),
\[
\langle 0|A_{\mu}^{a+ib}(0)|m(-\vec{k}_p))\rangle = i \sqrt{2} F_m Q_{\mu}.
\]
The decay rate of the continuum-state $\ell^-$-decay is defined by
\[
\lambda_{\ell^-}^{(m)} = \frac{1}{2M_{H_{1s}}^2} \int d^4k_p \frac{d^4k_\ell}{(2\pi)^32E_\nu} \frac{d^4k_\nu}{(2\pi)^32E_\nu} \times (2\pi)^4\delta(4)(k_\nu + k_\ell + k_p - K)
\]

\[
\times |M(H_{1s}^{(1s)} \rightarrow p\ell^- \bar{\nu}_\ell)|^2,
\]
where $|M(H_{1s}^{(1s)} \rightarrow p\ell^- \bar{\nu}_\ell)|^2$ is given by
\[
|M(H_{1s}^{(1s)} \rightarrow p\ell^- \bar{\nu}_\ell)|^2 = \frac{1}{2} \sum_{\sigma} \sum_{\sigma_\ell} \sum_{\sigma_\nu} \langle p(k_p, \sigma_p)|J_{\mu}^{a+ib}(0)|m(k)\rangle_{\text{disconn.}} + \langle p(k_p, \sigma_p)|J_{\mu}^{a+ib}(0)|m(k)\rangle_{\text{conn.}}.
\]
\[ \frac{|M(H_{m}^{(1s)} \rightarrow p \ell^{-} \bar{\nu}_\ell)|^2}{2M_{H_{m}^{(1s)}}} = \frac{2M_{H_{m}^{(1s)}} E_p (\vec{k}_p)}{2E_m (\vec{k}_p)} \]
\[ \times 16 G_F^2 |V_{uq}|^2 F_m^2 |\Phi_{1s}(\vec{k}_p)|^2 \]
\[ \times \left( (Q \cdot k_\ell) (Q \cdot k_\nu) - \frac{1}{2} Q^2 (k_\ell \cdot k_\nu) \right). \]  

Substituting Eq. (14) into Eq. (13) we arrive at the decay rate

\[ \lambda_{\ell \ell}^{(m)} = \int \frac{d^3k_\ell}{(2\pi)^3} \frac{d^3k_\nu}{(2\pi)^3} \delta^{(4)}(k_\ell + k_\nu - k - K) 16 G_F^2 |V_{uq}|^2 F_m^2 \]
\[ \times \left( (Q \cdot k_\ell) (Q \cdot k_\nu) - \frac{1}{2} Q^2 (k_\ell \cdot k_\nu) \right). \]  

For the integration over the phase volume of the lepton \( \ell^- \bar{\nu}_\ell \) pair we use the formula

\[ T^{(3)}(P) = \frac{1}{2} \int \left( k_\ell k_\nu^{\dagger} + k_\nu k_\ell^{\dagger} - g^{(3)}(k_\ell \cdot k_\nu) \right) \]
\[ \times (2\pi)^4 \delta^{(4)}(P - k_\ell - k_\nu - k) \frac{d^3k_\ell}{(2\pi)^3} \frac{d^3k_\nu}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \]
\[ = \left[ \left( 1 + \frac{\mu^2}{P^2} \right) \frac{P^{(3)}}{P^{(3)}} \right] g^{(3)} \frac{P^2}{48\pi} \left( 1 - \frac{\mu^2}{P^2} \right)^2, \]  

where \( P = K - k_p \simeq Q \) and \( Q^2 \simeq M_n^2 \simeq m_n^2 \), as the main contribution to the integral over \( \vec{k}_p \) comes from the region \( |\vec{k}_p| \sim 1/a_B = \alpha_{n}\mu_{m} \).

Substituting Eq. (16) into Eq. (15) and integrating over \( \vec{k}_p \), we get

\[ \lambda_{\ell \ell}^{(m)} = G_F^2 |V_{uq}|^2 F_m^2 m_l^2 \frac{(M_n^2 - m_l^2)^2}{4\pi M_n^3}, \]  

where we have used the normalisation condition Eq. (4) for the wave function \( \Phi_{1s}(\vec{k}_p) \). At the neglect of the binding energy \( M_n = m_n + \epsilon_{1s} \simeq m_n \) the obtained continuum-state \( \ell^- \)-decay rate coincides with the \( \ell^- \)-decay rate of a free \( m^- \)–meson.

**BOUND-STATE \( \ell^- \)-DECAY RATE OF MESIC HYDROGEN IN THE GROUND STATE**

In the bound-state \( \ell^- \)-decay of mesic hydrogen \( H_{m}^{(1s)} \), \( H_{m}^{(1s)} \rightarrow H + \ell^- \bar{\nu}_\ell \), we get hydrogen \( H + \) with the lepton \( \ell^- \), which is practically in the bound (\( nL \)) state \( \left[ \ell \right] \), where \( n \) is the principal quantum number, and the anti-neutrino \( \bar{\nu}_\ell \). Due to the hyperfine interaction \( \left[ \ell \right] \) hydrogen \( H + \) can be in the hyperfine (\( nL \)) states with atomic spin \( F = 0 \) and \( F = 1 \). The contribution of the excited \( nL \)-states with \( L > 0 \), where \( L = 1, \ldots, n - 1 \) is the angular momentum, is negligible small. Nevertheless, for the calculation of the bound-state \( \ell^- \)-decay rate we will analyse the contribution of all excited \( nLM \)-states of hydrogen \( H + \), where \( M_L = 0, \pm 1, \ldots, \pm L \) is the magnetic quantum number. We will show that the main contribution comes from the \( nS \)-states only.

The wave function of hydrogen \( H_{\ell} \) in the \( nLM \)-state we take in the form [12]–[14]

\[ |H_{\ell}^{(nLM)}(q)\rangle = \frac{1}{(2\pi)^3} \sqrt{2E_{H_{\ell}^{(n)}}(q)} \]
\[ \times \int \frac{d^3k_\ell}{\sqrt{2E_\ell(\vec{k}_\ell)}} \frac{d^3k_\nu}{\sqrt{2E_\nu(\vec{k}_\nu)}} \]
\[ \times \phi_{nLM}(m_\ell \vec{k}_\ell - m_\ell \vec{k}_\nu) \]
\[ \times a_{nLML}^\dagger(\vec{k}_\ell, \sigma_\ell) a_{1}^\dagger(\vec{k}_\nu, \sigma_\nu)|0\rangle, \]  

where \( E_{H_{\ell}^{(n)}}(q) = \sqrt{M_{H_{\ell}^{(n)}}^2 + q^2} \) and \( q \) are the energy and the momentum of hydrogen, \( M_{H_{\ell}^{(n)}} = m_\ell + m_\ell + \epsilon_n \) is the mass of hydrogen and \( \epsilon_n \) is the binding energy of the \( nLM \)-state, \( \phi_{nLML}(\vec{k}) \) is the wave function of the \( nLM \)-state in the momentum representation [12] (see also [12]–[14]).

For the amplitude of the bound-state \( \ell^- \)-decay we obtain the following expression

\[ M(H_{m}^{(1s)} \rightarrow H_{\ell}^{(nLM)} + \ell^- \bar{\nu}_\ell) = \]
\[ = - \sqrt{2} G_F V_{uq} \sqrt{\frac{2M_{H_{m}^{(1s)}} E_{H_{\ell}^{(n)}}(q)}{2M_{m}}} \]
\[ \times \int \frac{d^3k_\ell}{(2\pi)^3} \frac{d^3k_\nu}{(2\pi)^3} \Phi_{1s}(\vec{k}) \]
\[ \times \langle 0|A_{\mu}^{\dagger}(0)|m(\vec{k})\rangle \]
\[ \times \phi_{nLM}(\vec{k}, \sigma_\ell) \chi_\ell, \]  

where \( \phi_{nLM}(\vec{k}, \sigma_\ell) \) and \( \chi_\ell \) are the spinorial wave functions of the lepton \( \ell^- \) and the anti-neutrino \( \bar{\nu}_\ell \) respectively, and \( \sigma_\mu = (1, -\sigma) \). The contribution of the matrix element of the hadronic current we define only by the disconnected part (see Appendix A).

Using the standard expression for the matrix element of the axial–vector current [2, 10]

\[ \langle 0|A_{\mu}^{\dagger}(0)|m(\vec{k})\rangle = i\sqrt{2} F_m Q_{\mu} \]  

with \( Q_{\mu} = (E_m, -\vec{k}) \), we get

\[ M(H_{m}^{(1s)} \rightarrow H_{\ell}^{(nLM)} + \ell^- \bar{\nu}_\ell) = \]
\[ = - 2i G_F V_{uq} \sqrt{\frac{2M_{H_{m}^{(1s)}} E_{H_{\ell}^{(n)}}(q)}{2M_{m}}} \]
\[ \times \left\{ M_m [\phi_{\ell}^{\dagger} \chi_\ell] \int d^3x \psi_{nLML}^{*}(\vec{r}) \Phi_{1s}(\vec{r}) e^{-i\vec{k}_\nu \cdot \vec{r}} \right\}, \]
\[-i\hbar \gamma^\mu \sigma_i \sigma_j \psi_{nL} \cdot (\vec{r}) \cdot \int d^3 x \psi_{nL} \times e^{i\vec{k}_\nu \cdot \vec{r}} \cdot \psi_{1s}(\vec{r}) \]

where we have proceeded to the coordinate representation \(\vec{r}\) (see also [2]) and used the obvious approximation \(E_n \approx M_m = m_m + \epsilon_{ns}\), as the main contribution to the integral over \(\vec{k}\) is defined by the region \(|\vec{k}| \approx \alpha_B \approx \alpha m_m\). Since de Broglie wave length of the anti-neutrino is smaller compared to the Bohr radius of mesic hydrogen, the main contribution to the spatial integral comes from the origin. This gives

\[
M(H^{(3s)}_{m} \rightarrow H^{(nLM)}_{\ell} + \nu_\ell) = -i G_F |V_{uq}| F_m \delta_{\sigma_\sigma},
\]

\[
\times \delta_{\sigma_\sigma}, \frac{1}{2} \delta_{M_{l}M_{l}} \sqrt{2M_{H_{l}^{(3s)}}} 2E_{H_{l}^{(3s)}}(q') 2M_{m}E_{\nu}
\]

\[
\times \psi_{n00}(0) \left(1 - \frac{E_{\nu}}{M_{m}} \right) \Phi_{1s}(E_{\nu}),
\]

(22)

where \(\psi_{n00}(0)\), the wave function of hydrogen \(H_{\ell}\) in the \(n\text{--state at the origin, is equal to } \psi_{n00}(0) = \sqrt{\alpha^3 m_e^2/\pi n^3}\) with the reduced mass of the \(\ell^{-}\)p pair \(\mu_{\ell} = m_m m_p/(m_m + m_p)\). This testifies that the main contributions to the bound-state \(\ell^{-}\)--decay rate of mesic hydrogen come from the \((n\text{--states, which are split into the hyperfine } (n\text{s})\ell\text{--states with } F = 0 \text{ and } F = 1, \text{ respectively.}

The bound-state \(\ell^{-}\)--decay rate of mesic hydrogen is defined by [3]

\[
\lambda_{\ell} = \sum_{n=1}^{\infty} \sum_{F=0,1} \lambda_{\ell}^{(n)} \left( \frac{1}{2M_{H_{l}^{(3s)}}} \sum_{n=1}^{\infty} \right.
\]

\[
\times \frac{d^3 q}{(2\pi)^3} \frac{d^3 k_{\nu}}{(2\pi)^3} \frac{d^3 k_{\nu}}{(2\pi)^3} (2\pi)^4 \delta(q + k_{\nu} - p)
\]

\[
\times \frac{1}{2} \sum_{\sigma_\sigma, \sigma_\sigma} |M(H_{l}^{(3s)} \rightarrow H_{l}^{(n\text{s})} + \nu_\ell)|^2.
\]

(23)

The calculation of the r.h.s. of Eq. (23) is rather straightforward and the result is

\[
\lambda_{\ell} = \zeta(3) G_F^2 |V_{uq}|^2 F_m^2 m_m \sqrt{(m_p + m_\ell)^2 + E_{\nu}^2}
\]

\[
\times \frac{\alpha^3 (m_p m_\ell)^3}{\pi^2 (m_p + m_\ell)^3} \left(1 - \frac{E_{\nu}}{m_m} \right)^2 |\Phi_{1s}(E_{\nu})|^2
\]

\[
\times \frac{E_{\nu}^2}{(m_p + m_\ell)^4},
\]

(24)

where \(\zeta(3) = 1.202\) is the Riemann function, appearing as a result of the summation over the principal quantum number \(n\), and \(E_{\nu}\) is equal to

\[
E_{\nu} = (m_m - m_\ell) \left(1 - \frac{1}{2} m_m/m_\ell \right).
\]

(25)

For the calculation of \(\lambda_{\ell}\), we have neglected the contribution of the binding energies. The numerical values of the lifetimes of mesic hydrogen, caused by the bound-state \(\ell^{-}\)--decays, are adjoined in Table 1.

| \(\lambda_{\ell}^{-}\) | \(\tau_{\ell}^{(3s)}\) | \(\tau_{\ell}^{(3s)}\) |
|-----------------|------------|------------|
| \(\lambda_{\ell}^{-}\) | \(15.83 \text{ min}\) | \(4.18 \text{ yr}\) |
| \(\lambda_{\ell}^{-}\) | \(2.29 \times 10^9 \text{ yr}\) | \(1.10 \times 10^9 \text{ yr}\) |

TABLE I: The lifetimes of pionic and kaonic hydrogen, caused by the bound-state \(\ell^{-}\)--decays. The lifetimes are related to the decay rates as \(\tau_{\ell}^{-} = 1/\lambda_{\ell}^{-}\).

CONCLUDING DISCUSSION

We have calculated the continuum- and bound-state \(\ell^{-}\)--decay rates of pionic and kaonic hydrogen in the \(1s\) ground state, where \(\ell^{-}\) is either the electron \(e^{-}\) or the muon \(\mu^{-}\). According to classification of \(\beta\)--decays [3, 4], these are allowed decays obeying the selection rule \(\Delta J^P = 0^+\). To calculate these decays we have applied the technique, which we have used for the analysis of the \(H^{-}\)-- and \(He^{-}\)--heavy \(^{140}\text{Pt}^{58+}\) and \(^{140}\text{Pt}^{57+}\) ions [6] and the continuum- and bound-state \(\beta^{-}\)--decays of bare \(^{207}\text{Tl}^{81+}\) and \(^{205}\text{Hg}^{80+}\) ions [7]. The advantage of the \(\ell^{-}\)--decays of pionic and kaonic hydrogen is that the hadronic matrix elements, equivalent to the nuclear matrix elements for the weak decays of heavy ions, can be calculated explicitly within current algebra and the PCAC hypothesis for the \(\pi^{-}\) and \(K^{-}\) mesons [10].

We have shown that the decay rates of the continuum-state \(\ell^{-}\)--decays of pionic and kaonic hydrogen practically coincide with the decay rates of the free \(\pi^{-}\) and \(K^{-}\) meson. This means that the Coulomb interaction, responsible for the existence of pionic and kaonic hydrogen, is not enough to influence considerable on the lifetime time of bound mesons. As a result the lifetimes of pionic and kaonic hydrogen should be restricted by the lifetimes of the free \(\pi^{-}\) and \(K^{-}\) mesons. The Coulomb corrections to the continuum--state \(\ell^{-}\)--decay rate, caused by the Coulomb interaction between the proton and lepton \(\ell^{-}\) in the final \(p\ell^{-}\) state, can be described by the Fermi function [3, 4]. However, for the continuum-state \(\ell^{-}\)--decay of mesic hydrogen the final--state Coulomb interaction between the proton and lepton \(\ell^{-}\) can be neglected, since the main contribution to the integrals over the phase volume of the final \(p\ell^{-}\) state comes
from the region \( E_\ell \sim M_m/2 \), where leptons are relativistic.

The lifetimes of pionic and kaonic hydrogen, caused by the bound-state \( \mu^- \)–decays, are \( \tau_{\mu^-}^{\pi^-} = 15.83 \text{ min} \) and \( \tau_{\mu^-}^{K^-} = 4.18 \text{ yr} \), respectively. In turn, the lifetimes of pionic and kaonic hydrogen, related to the bound-state \( \beta^- \)–decays, are of order of \( \tau_{\mu^-}^{\pi^-} \sim 10^8 \text{ yr} \) and \( \tau_{\mu^-}^{K^-} \sim 10^9 \text{ yr} \), respectively.

### Appendix A: Connected part of the matrix element of the hadronic current

The calculation of the connected part of the matrix element of hadronic current we perform by using the reduction technique \[10,13\] and the PCAC hypothesis \[10,13\]. This gives

\[
\langle \hat{k}_\mu, \sigma \rangle | J^{a+ib}_\mu(0) | m(\vec{k}) p(-\vec{k}, \sigma) \rangle_{\text{conn}} = \lim_{x \to 0} \frac{m^2 \vec{k}^2}{\sqrt{2} F_m m^2} (\hat{\sigma} - i) \int d^4 x e^{-i \vec{k} \cdot \vec{x}} \langle \hat{k}_\mu, \sigma \rangle | T(A^{a+ib}_\mu(0) \partial^\alpha A^{a-ib}_\alpha(x)) | m(\vec{k}) p(-\vec{k}, \sigma) \rangle,
\]

(A-1)

where \( T \) is a time–ordering operator \[13\]. The calculation of the r.h.s of Eq. (A-1) we carry out in the chiral limit \( k \to 0 \) (soft–meson limit) and current algebra technique \[10,13\]. Using the relation

\[
T(A^{a+ib}_\mu(0) \partial^\alpha A^{a-ib}_\alpha(x)) = \delta(\vec{x}) T(A^{a+ib}_\mu(0) A^{a-ib}_\alpha(x)) - \delta(\vec{x}) | A^{a+ib}_\mu(0), A^{a-ib}_\alpha(x) \rangle,
\]

(A-2)

taking the chiral limit and keeping only the leading contributions we arrive at the following expression \[10,13\]

\[
\langle \hat{k}_\mu, \sigma \rangle | J^{a+ib}_\mu(0) | m(\vec{k}) p(-\vec{k}, \sigma) \rangle_{\text{conn}} = \frac{i}{\sqrt{2} F_m} \int d^4 x e^{-i \vec{k} \cdot \vec{x}} \langle \hat{k}_\mu, \sigma \rangle | T(A^{a+ib}_\mu(0) A^{a-ib}_\alpha(x)) | m(\vec{k}) p(-\vec{k}, \sigma) \rangle,
\]

(A-3)

where \( Q^{a-ib}_5(0) \) is axial charge operator \[10,13\].

The result of the calculation of the matrix element is \[10,13,10\]

\[
\langle \hat{k}_\mu, \sigma \rangle | J^{a+ib}_\mu(0) | m(\vec{k}) p(-\vec{k}, \sigma) \rangle_{\text{conn}} = \frac{i (k_\mu + p)_\mu}{\sqrt{2} F_m} \mathcal{F}_m(\vec{k}_\mu^2),
\]

(A-4)

where \( k_\mu + p = (E_\mu(\vec{k}_\mu) + m_\mu, \vec{k}_\mu) \) and \( \mathcal{F}_m(\vec{k}_\mu^2) \) is the form factor defined by \( \mathcal{F}_{K^-(\vec{k}_\mu^2)} = 2 \mathcal{F}_{\pi^-}(\vec{k}_\mu^2) = 2 \mathcal{F}_V(\vec{k}_\mu^2) \[13,16\].

The connected part of the matrix element of the hadronic current is proportional to the amplitude of low–energy \( mp \) scattering. Indeed, taking the limit \( \vec{k}_\mu \to 0 \) and multiplying both sides of Eq. (A-4) by the factor \((-i) m_\mu/\sqrt{2} F_m\), related to the \( m \)–field in the final state \[10\], we arrive at the Weinberg–Tomozaawa term of \( m \) scattering at threshold \[13,17\]

\[
M(mp \to mp) = -i \frac{m_\mu}{\sqrt{2} F_m} | J^{a+ib}_\mu(0) | mp \rangle_{\text{conn}} = \\
= \frac{m_\mu m}{F_m^2} \mathcal{F}_m(0),
\]

(A-5)

where \( \mathcal{F}_{\pi^-}(0) = 1 \) and \( \mathcal{F}_{K^-(0)} = 2 \) for \( \pi^- \) and \( K^- \) scattering (see \[17\] and \[16\], respectively).

For the continuum-state \( \ell^- \)–decay rate, caused by the connected part of the matrix element of hadronic current, we get the following expression

\[
\lambda_{\ell^-}^{(\text{conn})} = |\Psi_{1\alpha}(0)|^2 \frac{G_F^2 |V_{ud}|^2 m_\mu^2 (M_m^2 - m_\mu^2)^2}{32 \pi^3 F_m^2 M_f^3} \times \int_0^\infty dk_p k_p^2 F^2 (k_p^2),
\]

(A-6)

where \( k_p = |\vec{k}_\mu| \) and \( \Psi_{1\alpha}(0) = \sqrt{\alpha^3 \mu_\mu/\pi} \) is the coordinate wave function of mesic hydrogen in the ground state at the origin.

The ratio \( R_{\ell^-} = \lambda_{\ell^-}^{(\text{conn})}/\lambda_{\ell^-}^{(m)} \) is equal to

\[
R_{\ell^-}^{(m)} = \frac{1}{8 \pi} \frac{\alpha^3 \mu_\mu^3}{F^4 M_f^2} \int_0^\infty dk_p k_p^2 F^2 (k_p^2) = \frac{\alpha^3 \mu_\mu^3 M_f^3}{256 \pi^2 M_m^2 F_m^2 \lambda_{\ell^-}^{(m)}},
\]

(A-7)

where \( M_f = 843 \text{ MeV} \) is a slope parameter \[16\].

The numerical values of the ratio for the \( \pi^- \) and \( K^- \) mesons is to

\[
R_{\ell^-}^{(m)} = \begin{cases} 1.2 \times 10^{-7} & , \ m = \pi^- \\ 3.2 \times 10^{-7} & , \ m = K^- \end{cases}
\]

(A-8)

This value gives a hint that the contribution of the connected part of the matrix element of the hadronic current to the continuum-state \( \ell^- \)–decay rate should be small. The correct contribution of the connected part to the continuum-state \( \ell^- \)–decay rate one can estimate from the interference of the connected and disconnected part of the matrix elements of the hadronic current. Using the result Eq. (A-7) one finds that the ratio of the continuum-state \( \ell^- \)–decay rate to the interference term should be of order 0.1 %. This shows that the main contribution to the continuum-state \( \ell^- \)–decay rate of mesic hydrogen is defined by the disconnected part of the matrix element of hadronic current. Of course, the same conclusion is valid for the bound-state \( \ell^- \)–decay rate.

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