DISCONTINUOUS GALERKIN METHODS FOR VLASOV-MAXWELL EQUATIONS

YINGDA CHENG *, IRENE M. GAMBA †, FENGYAN LI ‡, AND PHILIP J. MORRISON §

Abstract. In this paper, we propose to use discontinuous Galerkin methods to solve the Vlasov-Maxwell system. These methods are chosen because they can be designed systematically as accurate as one wants, meanwhile with provable conservation of mass and possibly also of the total energy. Such property in general is hard to achieve within other numerical method frameworks to simulate the Vlasov-Maxwell system. The proposed scheme employs discontinous Galerkin discretizations for both the Vlasov and the Maxwell’s equations, resulting in a consistent description of the probability density function and electromagnetic fields. We prove that up to that some boundary effect, the total particle number is conserved, and the total energy could be preserved upon a suitable choice of the numerical flux for the Maxwell’s equations and the underlying approximation spaces. Error estimates are further established based on several flux choices. We test the scheme on the Weibel instability and verify the accuracy order and the conservation of the proposed method.

Key words. Vlasov-Maxwell system, discontinuous Galerkin methods, energy conservation, error estimates, Weibel instability

AMS subject classifications. 65M60, 74S05

1. Introduction. In this paper, we consider the Vlasov-Maxwell (V-M) system, the most important equation for the modeling of collisionless magnetized plasmas. In particular, we study the evolution of a single species of nonrelativistic electrons under the self-consistent electromagnetic field while the ions are treated as uniform fixed background. Under the scaling of the characteristic time by the inverse of the plasma frequency $\omega_p^{-1}$ and length scaled by the Debye length $\lambda_D$, and characteristic electric and magnetic field as $E = B = -m\omega_p/e$, the dimensionless equations become

$$\begin{align*}
\partial_t f + \xi \cdot \nabla_x f + (E + \xi \times B) \cdot \nabla_\xi f &= 0, \\
\frac{\partial E}{\partial t} &= \nabla \times B - J, \\
\frac{\partial B}{\partial t} &= -\nabla \times E, \\
\nabla_x \cdot E &= \rho - \rho_i, \\
\nabla_x \cdot B &= 0,
\end{align*}$$

(1.1)

Here the equations are defined on $\Omega = \Omega_x \times \Omega_\xi$, where $x \in \Omega_x$ denotes the physical space, and $\xi \in \Omega_\xi$ is the velocity space. $f(x, \xi, t) \geq 0$ is the distribution function of electrons at position $x$ with velocity $\xi$ at time $t$. $E(x, t)$ is the electric field, $B(x, t)$ is the magnetic field, $\rho(x, t)$ is the electron charge density, and $J(x, t)$ is the current density. The charge density of background ions is denoted by $\rho_i$. In particular $\int_{\Omega_\xi} \rho(x, t) - \rho_i \, dx = 0$, due to the quasineutrality of plasmas. The problem is endowed with periodic boundary conditions in $x$-space and initial conditions denoted by $f_0 = f(x, \xi, 0)$, $E_0 = E(x, 0)$ and $B_0 = B(x, 0)$ respectively. We assume that the initial density mass function $f_0(v, x) \in H^m(\mathbb{R}_x \times \mathbb{R}_\xi) \cap L^1_2(\mathbb{R}_\xi)$, that is the initial state is in a Sobolev space or order $m$ and it is integrable, with finite energy in $\xi - space$. The initial fields $E_0(x)$ and $B_0(x)$ are also in $H^m(\mathbb{R})$.

The V-M system has wide applications in plasma physics, ranging from space and laboratory plasmas, fusion, to high-power microwave generation and large scale particle accelerators. The computation of the initial boundary value problem associated to the V-M system is quite challenging, mainly due to the high-dimensionality (6D+time) of the Vlasov equation, multiple temporal and spatial scales associated with various physical phenomena, and the conservation of the physical quantities due to the Hamiltonian structure of the system. Particle-in-cell (PIC) methods [5, 33] have long been very popular numerical tools, in which the particles are advanced in a Lagrangian framework, while the field equations are solved on a mesh. In recent

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years, there has been growing interest in computing the Vlasov equation in a deterministic framework. In the context of the Vlasov-Poisson system, semi-Lagrangian methods [10, 47], finite volume (flux balance) methods [6, 21, 22], Fourier-Fourier spectral methods [37, 38], and continuous finite element methods [50, 51] were proposed, among many others. In the context of V-M simulations, Califano et al. used a semi-Lagrangian approach to compute the Weibel instability [9], current filamentation instability [40], magnetic vortices [8], magnetic reconnection [7]. Various methods were also proposed for the relativistic V-M system [46, 4, 48, 34].

In this paper, we propose to use discontinuous Galerkin (DG) methods to solve the V-M system. What motivates us to choose DG methods, besides their many widely recognized properties, is that they can be designed systematically as accurate as one wants, meanwhile with provable conservation of mass and possibly also of the total energy. This is in general hard to achieve within other numerical method frameworks to simulate the Vlasov-Maxwell system. The proposed scheme employs DG discretizations for both the Vlasov and the Maxwell’s equations, resulting in a consistent description of the probability density function and electromagnetic fields. We will show that up to some boundary effect, depending on the size of the computational domain, the total particle number is conserved, and the total energy could be preserved upon a suitable choice of the numerical flux for the Maxwell’s equations and the underlying approximation spaces. Error estimates are further established based on several flux choices. The DG scheme can be implemented on both structured and unstructured meshes with provable accuracy and stability for many linear and nonlinear problems; it is advantageous in long time wave simulations with its low dispersive and dissipative errors [1], and it is very suitable for adaptive and parallel implementations. The original DG method was introduced by Reed and Hill [43] for neutron transport equation. Lesaint and Raviart [39] performed the first error estimate for the original DG method. Cockburn and Shu in a series of papers [17, 16, 15, 14, 18] developed the Runge-Kutta DG (RKDG) methods for hyperbolic equations. RKDG methods have been used to simulate the Vlasov-Poisson system in plasmas [32, 31, 12] and for the gravitational infinite homogeneous stellar system [11]. Some theoretical aspects about stability, accuracy and conservation of these methods in their semi-discrete form are discussed in [31, 3, 2]. Recently semi-Lagrangian DG methods [44, 42] were proposed for the Vlasov-Poisson system. In [35, 36], DG discretizations for the Maxwell’s equations were coupled with PIC methods to solve the V-M system.

The rest of the paper is organized as follows: in Section 2, we describe the numerical algorithm. In Section 3, the conservation and the stability are established for the method. In Section 4, we provide the error estimates of the scheme. Section 5 is devoted to the discussion of the simulation results. We conclude with a few remarks in Section 6.

2. Numerical Methods. In this section, we will introduce the DG algorithm for the V-M system. We consider an infinite homogeneous plasma, where all boundary conditions in \( \mathbf{x} \) are set to be periodic, and \( f(x, \xi, t) \) is assumed to be compactly supported in \( \xi \). This assumption is consistent with the fact that the solution of the V-M system is expected to decay at infinity in \( \xi \)-space, preserving integrability and its kinetic energy.

Without loss of generality, we assume \( \Omega_x = (-L_x, L_x]^d_x \) and \( \Omega_\xi = [-L_\xi, L_\xi)^d_\xi \). Here, the domain in the velocity space \( \Omega_\xi \) is chosen large enough so that \( f = 0 \) at and near the phase space boundaries. We take \( d_x = d_\xi = 3 \) in the following sections, although the method and its analysis can be extended directly to the cases when \( d_x \) and \( d_\xi \) take other values from \{1, 2, 3\}.

In our analysis, it is assumed that the solution \( f(x, \xi, t) \) has compact support in \( \xi \). In fact, it is an open question in general settings whether the solution is compacted supported in \( \xi \) if it is initially. The answer is important for one to show the existence of a globally defined classical solution, or whether shocks can form in the solutions of the V-M system. One major open problem is whether the three-dimensional Vlasov-Maxwell system is globally well-posed as a Cauchy problem. All that is known is, on one hand, the global existence but not uniqueness of weak solutions and, on the other hand, well-posedness and regularity of solutions assuming either some symmetry or almost neutrality [27, 28, 23, 20, 24, 26, 25].

2.1. Notations. Throughout the paper, the standard notations will be used for the Sobolev spaces. Given a bounded domain \( D \subseteq \mathbb{R}^n \) (with \( n = d_x, d_\xi, \) or \( d_x + d_\xi \) ) and any nonnegative integer \( m \), \( H^m(D) \) denotes the \( L^2 \)-Sobolev space of order \( m \) with the standard Sobolev norm \( \| \cdot \|_{m, D} \) and \( W^{m,\infty}(D) \) denotes the \( L^\infty \)-Sobolev space of order \( m \) with the standard Sobolev norm \( \| \cdot \|_{m,\infty, D} \) and the semi-norm \( | \cdot |_{m,\infty, D} \).

When \( m = 0 \), we also use \( H^0(D) = L^2(D) \) and \( W^{0,\infty}(D) = L^\infty(D) \).

Let \( T_k^x = \{ K_x \}, T_k^\xi = \{ K_\xi \} \) be a partition of \( \Omega_x \) and \( \Omega_\xi \), respectively, with \( K_x \) and \( K_\xi \) being (rotated)
Cartesian elements or simplices, then $\mathcal{T}_h = \{ K : K = K_x \times K_\xi, \forall K_x \in \mathcal{T}_h^x, \forall K_\xi \in \mathcal{T}_h^\xi \}$ defines a partition of $\Omega$. Let $\mathcal{E}_x$ be the set of the edges of $\mathcal{T}_h^x$, $\mathcal{E}_\xi$ be the set of the edges of $\mathcal{T}_h^\xi$, then the edges of $\mathcal{T}_h$ will be $\mathcal{E} = \{ K_x \times \xi : \forall K_x \in \mathcal{T}_h^x, \forall \xi \in \mathcal{E}_\xi \} \cup \{ \xi \times K : \forall \xi \in \mathcal{E}_x, \forall K_\xi \in \mathcal{T}_h^\xi \}$. Here we take into account the periodic boundary condition in $x$-direction when defining $\mathcal{E}_x$ and $\mathcal{E}$. Furthermore, $\mathcal{E}_h = \mathcal{E}_x \cup \mathcal{E}_\xi$ with $\mathcal{E}_x^\xi$ and $\mathcal{E}_\xi^\xi$ being the set of interior and boundary edges of $\mathcal{T}_h^\xi$, respectively. In addition, we denote the mesh size of $\mathcal{T}_h$ as $h = \max(h_x, h_\xi) = \max_{K \in \mathcal{T}_h} h_{K}$, where $h_x = \max_{K_x \in \mathcal{T}_h^x} h_{K_x}$ with $h_{K_x} = \text{diam}(K_x)$, $h_\xi = \max_{K_\xi \in \mathcal{T}_h^\xi} h_{K_\xi}$ with $h_{K_\xi} = \text{diam}(K_\xi)$, and $h_K = \max(h_{K_x}, h_{K_\xi})$ for $K = K_x \times K_\xi$. When the mesh is refined, we assume both $\frac{h_{K_\xi}}{\rho_{K_\xi}} \leq \sigma_*$ and $\frac{h_x}{h_{\text{min}}} = \min \{ \frac{h_x}{h_{\text{min}}} \}$. Here $\mathcal{K}_x, \mathcal{K}_\xi$ and $\mathcal{K}_x, \mathcal{K}_\xi$ are uniformly bounded from above by a positive constant $\sigma_0$. Here $h_{\text{min}} = \min_{K \in \mathcal{T}_h} h_K$. It is further assumed that $\{ \mathcal{T}_h \}$ is shape-regular with $\ast = x$ or $\xi$. That is, if $\rho_K$ denotes the diameter of the largest sphere included in $K_\ast$, there is

$$\frac{h_{K_\ast}}{\rho_{K_\ast}} \leq \sigma_*, \quad \forall K_\ast \in \mathcal{T}_h$$

for a positive constant $\sigma_*$ independent of $h_\ast$.

Next we define the discrete spaces

$$\mathcal{G}_h^k = \left\{ g \in L^2(\Omega) : g|_{K_x \times K_\xi} \in P^k(K_x \times K_\xi), \forall K_x \in \mathcal{T}_h^x, \forall K_\xi \in \mathcal{T}_h^\xi \right\}, \quad (2.1a)$$

$$\mathcal{U}_h^k = \left\{ U \in [L^2(\Omega)]^{d_x} : U|_{K_x} \in [P^r(K_x)]^{d_x}, \forall K_x \in \mathcal{T}_h^x \right\}, \quad (2.1b)$$

where $P^r(D)$ denotes the set of polynomials of the total degree at most $r$ on $D$, and $k$ and $r$ are nonnegative integers. Note the space $\mathcal{G}_h^k$ we use to approximate $f$ is of P-type, and it can be replaced by the tensor product of P-type spaces in $x$ and $\xi$,

$$\left\{ g \in L^2(\Omega) : g|_{K_x \times K_\xi} \in P^k(K_x) \times P^k(K_\xi), \forall K_x \in \mathcal{T}_h^x, \forall K_\xi \in \mathcal{T}_h^\xi \right\}, \quad (2.2)$$

or by the tensor product space in each variable, which is also called the Q-type space

$$\left\{ g \in L^2(\Omega) : g|_{K_x \times K_\xi} \in Q^k(K_x) \times Q^k(K_\xi), \forall K_x \in \mathcal{T}_h^x, \forall K_\xi \in \mathcal{T}_h^\xi \right\}. \quad (2.3)$$

Here $Q^r(D)$ denotes the set of polynomials of the degree at most $r$ in each variable on $D$. The numerical methods formulated in this paper as well as the conservation, stability, and error estimates hold when any of the spaces above is used to approximate $f$. In our simulation in Section 5, we choose to use the P-type in (2.1a) as it is the smallest and therefore renders most cost efficient algorithm. In fact the ratio of these three spaces defined in (2.1a), (2.2) and (2.3) are $\sum_{n=0}^{k} \binom{n+2d-1}{2d-1} : \binom{n+1}{d-1}^2 \leq 2d$ with $d_x = d_\xi = d$.

For piecewise defined functions with respect to $\mathcal{T}_h^x$ or $\mathcal{T}_h^\xi$, we further introduce the jumps and averages as follows. For any edge $e = \{ K_x^+ \cap K_x^- \} \in \mathcal{E}_x$, with $n_x^\pm$ as the outward unit normal to $\partial K_x^\pm$, $g^\pm = g|_{K_x^\pm}$, and $U^\pm = U|_{K_x^\pm}$, the jumps across $e$ are defined as

$$[g]|_e = g^+ n_x^+ + g^- n_x^-, \quad [U]|_e = U^+ \cdot n_x^+ + U^- \cdot n_x^-,$$

and the averages are

$$\{g\}|_e = \frac{1}{2}(g^+ + g^-), \quad \{U\}|_e = \frac{1}{2}(U^+ + U^-).$$

By replacing the subscript $x$ with $\xi$, one can define $[g]|_\xi$, $[U]|_\xi$, $\{g\}|_\xi$, and $\{U\}|_\xi$ for an interior edge of $\mathcal{T}_h^\xi$ in $\mathcal{E}_\xi$. For a boundary edge $e \in \mathcal{E}_h^\xi$ with $n_\xi$ being the outward unit normal, we use

$$[g]|_\xi = g n_\xi, \quad \{g\}|_\xi = \frac{1}{2} g, \quad \{U\}|_\xi = \frac{1}{2} U.$$

This is consistent with the fact that the exact solution $f$ is compactly supported in $\xi$. 

3
For convenience, we introduce some shorthand notations,
\[ \int_{\Omega^*} = \int_{T^*} = \sum_{K \in T^*} \int_K, \quad \int_{\Omega} = \int_{T} = \sum_{K \in T} \int_K, \quad \int_{E^*} = \sum_{e \in E^*} \int_e, \]
here * is \( x \) or \( \xi \). In addition, \( |g|_{0,E} = (|g|^2_{0,T_x \times T_\xi} + |g|^2_{0,T_x^* \times E_\xi})^{1/2} \) with
\[ ||g||_{0,E_x \times T^*_K} = \left( \int_{E_x} \int_{T^*_K} g^2 d\xi dx \right)^{1/2}, \quad ||g||_{0,T^*_x \times E_\xi} = \left( \int_{T^*_x} \int_{E_\xi} g^2 ds_x dx \right)^{1/2}, \]
and \( ||g||_{0,E_x} = \left( \int_{E_x} g^2 ds_x \right)^{1/2} \). There are several equalities which will be used later and can be easily verified based on the definitions of averages and jumps.
\[ \frac{1}{2} |g|^2_* = \{g\}_* \{g\}_*, \quad \text{with } * = x \text{ or } \xi, \quad (2.5a) \]
\[ [U \times V]_x + \{V\}_x \cdot [U]_\tau - \{U\}_x \cdot [V]_\tau = 0, \quad (2.5b) \]
\[ [U \times V]_x + V^+ \cdot [U]_\tau - U^- \cdot [V]_\tau = 0, \quad [U \times V]_x + V^- \cdot [U]_\tau - U^+ \cdot [V]_\tau = 0. \quad (2.5c) \]

In the end, we summarize some standard approximation properties of the discrete spaces as well as inverse inequalities [13]. For any nonnegative integer \( m \), let \( \Pi^m \) be the \( L^2 \) projection onto \( G^m_h \), and \( \Pi^m_x \) be the \( L^2 \) projection onto \( U^m_h \), then

**Lemma 2.1 (Approximation properties).** There exists a constant \( C > 0 \), such that for any \( g \in H^{m+1}(\Omega) \) and \( U \in [H^{m+1}(\Omega_x)]^{d_x} \), there are
\[ \|g - \Pi^m g\|_{0,K} + h^{1/2}_K \|g - \Pi^m g\|_{0,\partial K} \leq Ch^{m+1}_K \|g\|_{m+1,K}, \quad \forall K \in T_h, \]
\[ \|U - \Pi^m_x U\|_{0,K_x} + h^{1/2}_{K_x} \|U - \Pi^m_x U\|_{0,\partial K_x} \leq Ch^{m+1}_{K_x} \|U\|_{m+1,K_x}, \quad \forall K_x \in T_{h^x}, \]
\[ \|U - \Pi^m_x U\|_{0,\infty,K_x} \leq Ch^{m+1}_{K_x} \|U\|_{m+1,\infty,K_x}, \quad \forall K_x \in T_{h^x}. \]
The constant \( C \) is independent of the mesh sizes \( h_K \) and \( h_{K_x} \), it depends on \( m \) and the shape regularity parameters \( \sigma_x \) and \( \sigma_\xi \) of the mesh.

**Lemma 2.2 (Inverse inequality).** There exists a constant \( C > 0 \), such that for any \( g \in P^m(K) \) or \( P^m(K_x) \times P^m(K_\xi) \) with \( K = (K_x \times K_\xi) \in T_h \), and for any \( U \in [P^m(K_x)]^{d_x} \), there are
\[ \|\nabla_x g\|_{0,K} \leq Ch^{-1}_{K_x} \|g\|_{0,K}, \quad \|\nabla_\xi g\|_{0,K} \leq Ch^{-1}_{K_\xi} \|g\|_{0,K}, \]
\[ \|U\|_{0,\infty,K_x} \leq Ch^{-d_x/2}_{K_x} \|U\|_{0,K_x}, \quad \|U\|_{0,\partial K_x} \leq Ch^{-1/2}_{K_x} \|U\|_{0,K_x}. \]
The constant \( C \) is independent of the mesh sizes \( h_{K_x} \), \( h_{K_\xi} \), it depends on \( m \) and the shape regularity parameters \( \sigma_x \) and \( \sigma_\xi \) of the mesh.

**2.2. The Semi-Discrete DG Methods.** On the PDE level, the two equations in (1.1c) involving the divergence of the magnetic and electric fields can be derived from the remaining part of the V-M system, the numerical methods proposed in this section are therefore formulated for the V-M system without (1.1c). We want to stress that in certain circumstance, one may need to consider such divergence conditions in order to produce physically relevant numerical simulations [41].
Given \( k, r \geq 0 \), the semi-discrete DG methods for the V-M system are defined as follows. For any \( K = K_x \times K_\xi \in \mathcal{T}_h \), look for \( f_h \in \mathcal{G}^k_h, \ E_h, B_h \in \mathcal{U}^k_h \), such that for any \( g \in \mathcal{G}^k_h, \ U, V \in \mathcal{U}^k_h \),

\[
\int_K \partial_r f_h g d\xi dx - \int_K f_h \xi \cdot \nabla g d\xi dx - \int_K f_h (E_h + \xi \times B_h) \cdot \nabla g d\xi dx
\]

\[
+ \int_{K_x} \int_{\partial K_x} f_h \xi \cdot n_x g ds \, d\xi + \int_{K_\xi} \int_{\partial K_\xi} (f_h (E_h + \xi \times B_h) \cdot n_\xi) g ds \, dx = 0 , \tag{2.6a}
\]

\[
\int_{K_x} \partial_r E_h \cdot U dx = \int_{K_x} B_h \cdot \nabla \times U dx + \int_{\partial K_x} n_x \times B_h \cdot U ds - \int_{K_x} J_h \cdot U dx , \tag{2.6b}
\]

\[
\int_{K_x} \partial_r B_h \cdot V dx = - \int_{K_x} E_h \cdot \nabla \times V dx - \int_{\partial K_x} n_x \times E_h \cdot V ds , \tag{2.6c}
\]

with

\[
J_h(x, t) = \int_{T_h^k} f_h(x, \xi, t) \xi d\xi . \tag{2.7}
\]

Here \( n_x \) and \( n_\xi \) are outward unit normals of \( \partial K_x \) and \( \partial K_\xi \), respectively. All hat functions are numerical fluxes, and they are taken to be upwinding,

\[
f_h \hat{\xi} \cdot n_x := f_h \hat{\xi} \cdot n_x = \left( \{ f_h \hat{\xi} \} x + \frac{\xi \cdot n_x}{2} [f_h]_\xi \right) \cdot n_x , \tag{2.8a}
\]

\[
f_h (E_h + \xi \times B_h) \cdot n_\xi := f_h (E_h + \xi \times B_h) \cdot n_\xi
\]

\[
= \left( \{ f_h (E_h + \xi \times B_h) \} \xi + \frac{(E_h + \xi \times B_h) \cdot n_\xi}{2} [f_h]_\xi \right) \cdot n_\xi , \tag{2.8b}
\]

\[
n_x \times \hat{E}_h := n_x \times \hat{E}_h = n_x \times \left( \{ E_h \} x + \frac{1}{2} [B_h]_\tau \right) , \tag{2.8c}
\]

\[
n_x \times \hat{B}_h := n_x \times \hat{B}_h = n_x \times \left( \{ B_h \} x - \frac{1}{2} [E_h]_\tau \right) . \tag{2.8d}
\]

For the Maxwell part, we also consider two other numerical fluxes: central flux and alternating flux

Central flux: \( \hat{E}_h = \{ E_h \}, \ \hat{B}_h = \{ B_h \} \), \tag{2.9a}

Alternating flux: \( \hat{E}_h = E_h^+, \ \hat{B}_h = B_h^+, \) or \( \hat{E}_h = E_h^-, \ \hat{B}_h = B_h^- \). \tag{2.9b}

With (2.6a) being summed up with respect to \( K \in \mathcal{T}_h \), similarly to (2.6b) and (2.6c) with respect to \( K_x \in \mathcal{T}^k_h \), the numerical method becomes: look for \( f_h \in \mathcal{G}^k_h, \ E_h, B_h \in \mathcal{U}^k_h \), such that

\[
a_h(f_h, E_h, B_h; g) = 0 , \tag{2.10a}
\]

\[
b_h(E_h, B_h; U, V) = l_h(J_h; U) , \tag{2.10b}
\]

for any \( g \in \mathcal{G}^k_h, \ U, V \in \mathcal{U}^k_h \), where

\[
a_{h, 1}(f_h; g) = a_{h, 1}(f_h; g) + a_{h, 2}(f_h, E_h, B_h; g) , \quad l_h(J_h; U) = - \int_{T_h^k} J_h \cdot U dx
\]

\[
b_h(E_h, B_h; U, V) = \int_{T_h^k} \partial_r E_h \cdot U dx - \int_{T_h^k} B_h \cdot \nabla \times U dx - \int_{E_x} \hat{B}_h \cdot [U]_x ds , \tag{2.10b}
\]

\[
+ \int_{T_h^k} \partial_r B_h \cdot V dx + \int_{E_x} \hat{E}_h \cdot [V]_x ds ,
\]

and

\[
a_{h, 1}(f_h; g) = \int_{T_h^k} \partial_r f_h g d\xi d\xi - \int_{T_h} f_h \xi \cdot \nabla g d\xi d\xi + \int_{T_h^k} \int_{E_x} \frac{\tilde{f}_h \xi \cdot [g]_x ds \, d\xi}{} , \tag{2.6a}
\]

\[
a_{h, 2}(f_h, E_h, B_h; g) = - \int_{T_h} f_h (E_h + \xi \times B_h) \cdot \nabla g d\xi d\xi + \int_{T_h^k} \int_{E_x} f_h (\hat{E}_h + \tilde{\hat{\xi}} \hat{E}_h) \cdot [g]_\xi ds \, d\xi dx . \tag{2.6a}
\]
Note \( a_h \) is linear with respect to \( f_h \) and \( g \), yet it is in general nonlinear with respect to \( E_h \) and \( B_h \) due to (2.8b). Recall the exact solution \( f \) has compact support in \( \xi \), therefore the numerical fluxes in (2.8a)-(2.8d) and in (2.9a)-(2.9b) are consistent and so is the proposed method. That is, the exact solution \( (f, E, B) \) satisfies

\[
a_h(f, E, B; g) = 0, \quad \forall g \in \mathcal{G}^{k}_h,
\]

\[
b_h(E, B; U, V) = l_h(J; U), \quad \forall U, V \in \mathcal{U}_h.
\]

2.3. Temporal Discretizations. We use total variation diminishing (TVD) high-order Runge-Kutta methods to solve the method of lines ODE resulting from the semi-discrete DG scheme, \( \frac{d}{dt} G_h = R(G_h) \). Such time stepping methods are convex combinations of the Euler forward time discretization. The commonly used third-order TVD Runge-Kutta method is given by

\[
\begin{align*}
G^{(1)}_h &= G^n_h + \frac{\Delta t}{3} R(G^n_h), \\
G^{(2)}_h &= \frac{3}{4} G^n_h + \frac{1}{4} G^{(1)}_h + \frac{1}{4} \Delta t R(G^{(1)}_h), \\
G^{n+1}_h &= \frac{1}{3} G^n_h + \frac{2}{3} G^{(2)}_h + \frac{2}{3} \Delta t R(G^{(2)}_h).
\end{align*}
\]

(2.11)

here \( G^n_h \) represents a numerical approximation of the solution at discrete time \( t_n \). Detailed description of the TVD Runge-Kutta method can be found in [45], see also [29], and [30] for strong-stability-preserving methods.

3. Conservation and Stability. In this section, we will establish the conservation and stability results for the semi-discrete DG methods. In particular, we prove that subject to boundary conditions, the total particle number (mass) is always conserved. As for the total energy of the system, the conservation depends on the choice of numerical fluxes for the Maxwell’s equations. It was further shown that \( f_h \) is \( L^2 \) stable, and this will facilitate the error analysis in the next section.

**Lemma 3.1 (Mass conservation).** The numerical solution \( f_h \in \mathcal{G}^{k}_h \) with \( k \geq 0 \) satisfies

\[
\frac{d}{dt} \int_{T_h} f_h d\xi + \Theta_{h,1}(t) = 0,
\]

with

\[
\Theta_{h,1}(t) = \int_{T_h} \int_{\mathcal{E}_h^k} f_h \max((E_h + \xi \times B_h) \cdot n_\xi, 0) ds_\xi d\xi.
\]

Equivalently, with \( \rho_h(x, t) = \int_{T_h} f_h(x, \xi, t) d\xi \), for any \( T > 0 \), there is

\[
\int_{T_h} \rho_h(x, T) dx + \int_{0}^{T} \Theta_{h,1}(t) dt = \int_{T_h} \rho_h(x, 0) dx.
\]

(3.2)

**Proof.** Let \( g(x, t) = 1 \). Note that \( g \in \mathcal{G}^{k}_h \) for any \( k \geq 0 \), it is continuous, and \( \nabla_x g = 0 \). Take this \( g \) as the test function in (2.10a), one has

\[
\frac{d}{dt} \int_{T_h} f_h d\xi + \int_{T_h} \int_{\mathcal{E}_h^k} f_h (E_h + \xi \times B_h) \cdot [g]_\xi ds_\xi d\xi = 0.
\]

With the numerical flux in (2.8b), the average and jump on \( \mathcal{E}_h^k \) in (2.4), the second term above becomes

\[
\int_{T_h} \int_{\mathcal{E}_h^k} f_h (E_h + \xi \times B_h) \cdot n_\xi ds_\xi d\xi
\]

\[
= \int_{T_h} \int_{\mathcal{E}_h^k} \frac{f_h}{2} ((E_h + \xi \times B_h) \cdot n_\xi + |(E_h + \xi \times B_h) \cdot n_\xi|) ds_\xi d\xi = \Theta_{h,1}(t),
\]

(3.3)

(3.4)
and this gives (3.1). One can further apply an integration in time from 0 to $T$ to obtain (3.2).

Lemma 3.2 (Energy conservation 1). For $k \geq 2$, $r \geq 0$, the numerical solution $f_h \in G^k_h$, $E_h, B_h \in U^k_h$ with the upwind numerical fluxes (2.8a)-(2.8d) satisfies

$$\frac{d}{dt} \left( \int_{T_h} f_h |\xi|^2 d\xi + \int_{T_h} (|E_h|^2 + |B_h|^2) dx \right) + \Theta_{h,2}(t) + \Theta_{h,3}(t) = 0 ,$$  

with

$$\Theta_{h,2}(t) = \int_{E_x} \left( |E_h|^2 + |B_h|^2 \right) d-s_x , \quad \Theta_{h,3}(t) = \int_{T_h} \int_{E_x} f_h |\xi|^2 \max((E_h + \xi \times B_h) \cdot n_\xi, 0) ds_\xi dx .$$

Proof. Step 1: Let $g(x, \xi) = |\xi|^2$. Note that $g \in G^k_h$ for $k \geq 2$, and it is continuous. In addition, $\nabla_x g = 0$, $\nabla_\xi g = 2\xi$, and $\xi \times U \cdot \nabla_\xi g = 0$ for any function $U$. Take this $g$ as the test function in (2.10a), one has

$$\frac{d}{dt} \int_{T_h} f_h |\xi|^2 d\xi = 2 \int_{T_h} f_h E_h \cdot \xi d\xi - \int_{T_h} \int_{E_x} f_h (E_h + \xi \times B_h) \cdot (|\xi|^2) \xi ds_\xi dx$$

$$= 2 \int_{T_h} E_h \cdot \left( \int_{E_x} f_h \xi d\xi \right) dx - \int_{T_h} \int_{E_x} \left( \frac{1}{2} (E_h + \xi \times B_h) f_h + \frac{|E_h + \xi \times B_h| n_\xi}{2} f_h n_\xi \right) \cdot (|\xi|^2 n_\xi) ds_\xi dx$$

$$= 2 \int_{T_h} E_h \cdot J_h dx - \int_{T_h} \int_{E_x} f_h \left( (E_h + \xi \times B_h) \cdot n_\xi + |(E_h + \xi \times B_h) \cdot n_\xi| \right) |\xi|^2 ds_\xi dx$$

$$= 2 \int_{T_h} E_h \cdot J_h dx - \int_{T_h} \int_{E_x} f_h |\xi|^2 \max((E_h + \xi \times B_h) \cdot n_\xi, 0) ds_\xi dx$$

Step 2: With $U = E_h$ and $V = B_h$, (2.10b) becomes

$$- \int_{T_h} J_h \cdot E_h dx = \frac{1}{2} \frac{d}{dt} \int_{T_h} |E_h|^2 dx - \int_{T_h} B_h \cdot \nabla \times E_h dx - \int_{E_x} \tilde{B}_h \cdot [E_h]_r ds_x$$

$$+ \frac{1}{2} \frac{d}{dt} \int_{T_h} |B_h|^2 dx + \int_{T_h} E_h \cdot \nabla \times B_h dx + \int_{E_x} \tilde{E}_h \cdot [B_h]_r ds_x$$

$$= \frac{1}{2} \frac{d}{dt} \int_{T_h} (|E_h|^2 + |B_h|^2) dx - \int_{E_x} \left( [E_h \times B_h]_x + \tilde{B}_h \cdot [E_h]_r - \tilde{E}_h \cdot [B_h]_r \right) ds_x$$

$$= \frac{1}{2} \frac{d}{dt} \int_{T_h} (|E_h|^2 + |B_h|^2) dx + \frac{1}{2} \int_{E_x} ([E_h]_x^2 + |B_h|^2) ds_x$$

The last equality uses the formulas of the upwind fluxes (2.8c)-(2.8d) as well as (2.5b).

Combines the results in previous two steps, one can conclude (3.5).

Corollary 3.3 (Energy conservation 2). For $k \geq 2$, $r \geq 0$, the numerical solution $f_h \in G^k_h$, $E_h, B_h \in U^k_h$ with the upwind numerical flux (2.8a)-(2.8b) for the Vlasov part, either the central or alternating flux in (2.9a)-(2.9b) for the Maxwell part, satisfies

$$\frac{d}{dt} \left( \int_{T_h} f_h |\xi|^2 d\xi + \int_{T_h} (|E_h|^2 + |B_h|^2) dx \right) + \Theta_{h,3}(t) = 0 .$$

Proof. The proof proceeds the same way as for Lemma 3.2. The only difference is that with the equalities (2.5b)-(2.5c),

$$[E_h \times B_h]_x + \tilde{B}_h \cdot [E_h]_r - \tilde{E}_h \cdot [B_h]_r = 0$$

holds for $\tilde{E}_h$ and $\tilde{B}_h$ defined in the central or alternating flux in the Maxwell solver.
With either the central or alternating flux for the Maxwell solver, the energy does not change due to the tangential jump of the magnetic and electric fields as in Lemma 3.2. This on the other hand may have some effect on the accuracy of the methods, see next two sections, and also [1].

**Remark 3.4.** We note that in the conservation results in Lemmas 3.1-3.2 and Corollary 3.3, the conservation error terms \( \Theta_{h,i} \geq 0 \), and \( \Theta_{h,1} \) and \( \Theta_{h,3} \) depend on first two moments of the numerical solution \( f_h \) on the computational boundary in \( \xi \)-space and they are proportional to the averages in \( \mathcal{T}^\varepsilon \) of \( E_h \) and \( B_h \). It can be easily seen that the terms \( \Theta_{h,i} \approx 0 \) for \( i = 1, 2, 3 \) but choosing the computational domain in \( \xi \)-space large enough. Indeed, these errors are not just a numerical issue but rather an important component of computational properties of the kinetic problem at hand, and can be easily controlled due to the choice of spacial periodic boundary conditions on the field pair \((E, B)\) and on the probability density \( f \), as well as the decay conditions of \( f \) in \( \xi \)-space. In particular assuming \((E, B)\) uniformly bounded, i.e. in \( L^\infty(0,T,L^\infty(\Omega)) \), and \( f(x, \xi, t) \) integrable in \( \xi \)-space with bounded kinetic energy as well, then a zero cut-off as boundary conditions is adequate, not only initially but also at later times, just provided that the \( \max_x \{|E_h| + |B_h|\} \) is uniformly bounded. Therefore, assuming \( f_h(x, \xi, t) \in L^1_b(\Omega) \), then \( f_h \approx 0 \), \( |\xi| f_h \) and \( |\xi|^2 f_h \approx 0 \) on \( \partial \Omega_t \), with \( \text{diam}(\Omega_t) = \text{diam}(\Omega_{\text{initial}}) + 2 \max_x \{|E_h| + |B_h|\} \). Then \( \Theta_{h,i} \approx 0 \) in \( \Omega_t \).

**Remark 3.5.** The energy conservation holds as long as \( |\xi|^2 \in G^k_h \). Indeed, for \( k < 2 \), the energy conservation results in Lemma 3.2 and Corollary 3.3 can be obtained if one replaces \( G^k_h \) with \( G^k_h \equiv G^k_h \oplus \{ |\xi|^2 \} = \{ g + c|\xi|^2 \mid g \in G^k_h, c \in \mathbb{R} \} \).

Finally, we can get the \( L^2 \)-stability result for \( f_h \), which is independent of numerical fluxes in the Maxwell solver and will be used in the error analysis.

**Lemma 3.6 (\( L^2 \)-stability of \( f_h \)).** For \( k \geq 0 \), the numerical solution \( f_h \in G^k_h \) satisfies

\[
\frac{d}{dt} \left( \int_{T_h} |f_h|^2 \, dx \, d\xi \right) + \int_{E_x} \int_{T_h} [\xi \cdot n_x][|f_h|^2] \, ds_x \, d\xi + \int_{E_x} \int_{T_h} \{\xi \cdot n_x\} [|f_h|^2] \, ds_x \, d\xi = 0 .
\]  

(3.6)

**Proof.** Take \( g = f_h \) in (2.10a), one gets

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{T_h} |f_h|^2 \, dx \, d\xi \right) + R_1 + R_2 = 0 ,
\]  

(3.7)

with

\[
R_1 = - \int_{T_h} f_h \xi \cdot \nabla_x f_h \, dx \, d\xi + \int_{E_x} \int_{T_h} \tilde{f}_h \xi \cdot [f_h]_x \, ds_x \, d\xi ,
\]

\[
R_2 = a_{h,2}(f_h, E_h, B_h ; f_h) .
\]

First

\[
R_1 = - \int_{T_h} \sum_{K_e \in \mathcal{T}_h} \int_{K_e} \xi \cdot \nabla_x \left( \frac{f_h^2}{2} \right) \, dx \, d\xi + \int_{E_x} \int_{T_h} \tilde{f}_h \xi \cdot [f_h]_x \, ds_x \, d\xi ,
\]

\[
= - \int_{T_h} \sum_{K_e \in \mathcal{T}_h} \int_{\partial K_e} \xi \cdot n_e \left( \frac{f_h^2}{2} \right) \, ds_x \, d\xi + \int_{E_x} \int_{T_h} \tilde{f}_h \xi \cdot [f_h]_x \, ds_x \, d\xi ,
\]

\[
= - \int_{T_h} \int_{E_x} \left( \frac{1}{2} |\xi f_h|_x + \{ f_h \} x \cdot [f_h]_x + \frac{1}{2} |\xi| n_e \cdot [f_h]_x \cdot [f_h]_x \right) \, ds_x \, d\xi ,
\]

\[
= \int_{T_h} \int_{E_x} \left( \frac{1}{2} |f_h|_x + \{ f_h \} x \cdot [f_h]_x \right) \cdot \xi + \frac{1}{2} |\xi| n_e \cdot [f_h]_x^2 \right) \, ds_x \, d\xi ,
\]

\[
= \frac{1}{2} \int_{T_h} \int_{E_x} |\xi| n_e \cdot [f_h]_x^2 \, ds_x \, d\xi ,
\]
The fourth equality uses the definition of the numerical flux (2.8a), and the last one is due to (2.5a). Similarly,

\[
R_2 = -\int_{T^e} \sum_{K \in T^e} \int_{K} \left( \mathbf{E}_h + \xi \times \mathbf{B}_h \right) \cdot \nabla \xi \left( \frac{f^k}{2} \right) d\xi dx + \int_{T^e} \int_{E^e} f_h \left( \mathbf{E}_h \times \xi \times \mathbf{B}_h \right) \cdot [f_h]_\xi d\xi dx,
\]

\[
= \int_{T^e} \sum_{K \in T^e} \int_{K} \left( \frac{1}{2} \left( \mathbf{E}_h + \xi \times \mathbf{B}_h \right) f^k + \{ f_h \left( \mathbf{E}_h + \xi \times \mathbf{B}_h \right) \}_\xi + \frac{1}{2} (\mathbf{E}_h + \xi \times \mathbf{B}_h) \cdot \mathbf{n}_\xi \right) [f_h]_\xi d\xi dx,
\]

\[
= \int_{T^e} \sum_{K \in T^e} \int_{K} \left( \frac{1}{2} f^k + \{ f_h \}_\xi \right) \cdot (\mathbf{E}_h + \xi \times \mathbf{B}_h) + \frac{1}{2} (\mathbf{E}_h + \xi \times \mathbf{B}_h) \cdot \mathbf{n}_\xi \| [f_h]_\xi \|^2 d\xi dx,
\]

\[
= \frac{1}{2} \int_{T^e} \sum_{K \in T^e} \int_{K} \left| \mathbf{E}_h + \xi \times \mathbf{B}_h \right| \cdot \mathbf{n}_\xi \| [f_h]_\xi \|^2 d\xi dx.
\]

The second equality is due to \( \nabla \xi \cdot (\mathbf{E}_h + \xi \times \mathbf{B}_h) = 0 \) and the definition of the numerical flux in (2.8b). The third one uses (2.5a) and \( \mathbf{E}_h + \xi \times \mathbf{B}_h \) being continuous in \( \xi \). With (3.7), we can now conclude the \( L^2 \) stability (3.6).

4. Error Estimates. In this section, we will establish the error estimates at any given time \( T > 0 \) for the proposed semi-discrete DG methods in Section 2.2 with \( k = r \). It is assumed that the exact solution \( f \in C^1([0, T]; H^{k+1}(\Omega) \cap W^{1, \infty}(\Omega)) \) and \( \mathbf{E}, \mathbf{B} \in C^0([0, T]; H^{k+1}(\Omega)) \cap W^{1, \infty}(\Omega)) \). They are periodic in \( x \) and \( f \) has compact support in \( \Omega \). To prevent the proliferation of constants, we use \( A \leq (\text{constant})B \), where the positive constant is independent of the mesh size \( h, h_x, \) and \( h_\xi \), and it can depend on the polynomial degree \( k \), mesh parameters \( \sigma_0, \sigma_x \) and \( \sigma_\xi \), and domain parameters \( L_x \) and \( L_\xi \).

Let \( \zeta_h = \Pi^k f - f \) and \( \varepsilon_h = \Pi^k f - f_h \), then \( f - f_h = \varepsilon_h - \zeta_h \). Let \( \varepsilon^E_h = \Pi^k (\mathbf{E} - \mathbf{E}_h) \), \( \varepsilon^B_h = \Pi^k (\mathbf{B} - \mathbf{B}_h) \), \( \varepsilon^E_h = \Pi^k f \). Next, we will state Lemmas 4.1 and 4.2, with which the main error estimate is established in Theorem 4.3 for the proposed semi-discrete DG method with the upwind numerical fluxes. The proof of Lemmas 4.1 and 4.2 will be given in subsections 4.1 and 4.2. For the proposed method using the central or alternating flux in (2.9a)-(2.9b) for the Maxwell solver, the error estimates are given in Theorem 4.6.

**Lemma 4.1 (Estimate of \( \varepsilon_h \)).** Based on the semi-discrete DG discretization for the Vlasov equation in (2.10a) with the upwind flux (2.8a)-(2.8b), we have

\[
\frac{d}{dt} \left( \int_{T} |\varepsilon_h|^2 dxd\xi \right) + \int_{T} \int_{E} \xi \cdot \mathbf{n}_\xi ||\varepsilon_h||_x^2 ds_d\xi + \int_{T} \int_{E} \left( \left( \mathbf{E}_h + \xi \times \mathbf{B}_h \right) \cdot \mathbf{n}_\xi \right) ||\varepsilon_h||_\xi^2 d\xi dx
\]

\[
\lesssim \left( h^{k+1} \Lambda + h^k ||f||_{k+1, \Omega} \left( ||\varepsilon^E_h||_{0, \Omega_x} + ||\varepsilon^B_h||_{0, \Omega_x} \right) + f \right)_{1, \infty, \Omega} (||\varepsilon^E_h||_{0, \Omega_x} + ||\varepsilon^B_h||_{0, \Omega_x}) ||\varepsilon_h||_{0, \Omega}
\]

\[
+ h^{k+1} \left| f \right|_{k+1, \Omega} \left( ||\varepsilon^E_h||_{0, \Omega_x}^{1/2} + ||\varepsilon^B_h||_{0, \Omega_x}^{1/2} + ||\mathbf{B}||_{0, \Omega_x}^{1/2} + ||\mathbf{E}||_{0, \Omega_x}^{1/2} \right) \right)^{1/2}
\]

\[
\left( \int_{T} \sum_{K} \int_{K} \left( \left| \mathbf{E}_h + \xi \times \mathbf{B}_h \right| \cdot \mathbf{n}_\xi \right) ||\varepsilon_h||_\xi^2 ds_d\xi dx \right)^{1/2}
\]

\[
\lesssim h^{k+1} \left| f \right|_{k+1, \Omega} \left( \int_{T} \sum_{K} \int_{K} \left| \xi \cdot \mathbf{n}_\xi \right| ||\varepsilon_h||_x^2 ds_d\xi d\xi \right)^{1/2},
\]

with

\[
\Lambda = ||\partial_x f||_{k+1, \Omega} + (1 + ||\mathbf{E}||_{1, \infty, \Omega_x} + ||\mathbf{B}||_{1, \infty, \Omega_x}) ||f||_{k+1, \Omega}
\]

\[+ (||\mathbf{E}||_{k+1, \Omega_x} + ||\mathbf{B}||_{k+1, \Omega_x}) ||f||_{1, \infty, \Omega} .
\]
Lemma 4.2 (Estimate of $\varepsilon^E_h$ and $\varepsilon^B_h$). Based on the semi-discrete DG discretization for the Maxwell equations in (2.10b) with the upwind fluxes (2.8c)-(2.8d), we have

$$
\frac{d}{dt} \int_{T_h^k} (|\varepsilon^E_h|^2 + |\varepsilon^B_h|^2) \, dx + \int_{\mathcal{E}_x} (|\varepsilon^E_h|^2 + |\varepsilon^B_h|^2) \, ds_x
$$

$$
\lesssim (|\varepsilon_h|_{0,\Omega} + h^{k+1}||f||_{k+1,\Omega}) (|\varepsilon^E_h|_{0,\Omega_x} + h^{k+\frac{1}{2}} (||E||_{k+1,\Omega_x} + ||B||_{k+1,\Omega_x}) \left( \int_{\mathcal{E}_x} |\varepsilon^E_h|^2 + |\varepsilon^B_h|^2 \, ds_x \right)^{1/2}).
$$

Theorem 4.3 (Error estimate 1). For $k \geq 2$, the semi-discrete DG method (2.10a)-(2.10b) for Vlasov-Maxwell equations with the upwind fluxes (2.8a)-(2.8d) has the following error estimate

$$
|||f-f_h(t)|||_{0,\Omega_x}^2 + |||E-E_h|||_{0,\Omega_x}^2 + |||B-B_h|||_{0,\Omega_x}^2 \leq C h^{2k+1}, \quad \forall \, t \in [0,T].
$$

Here the constant $C$ depends on the upper bound of $||\partial_t f||_{k+1,\Omega}$, $||f||_{k+1,\Omega}$, $||f||_{1,\infty,\Omega}$, $||E||_{1,\infty,\Omega_x}$, $||B||_{1,\infty,\Omega_x}$, $||E||_{k+1,\Omega_x}$, $||B||_{k+1,\Omega_x}$ over the time interval $[0,T]$, it also depends on the polynomial degree $k$, mesh parameters $\sigma_0, \sigma_x$ and $\sigma_{\xi}$, and domain parameters $L_x$ and $L_{\xi}$.

Proof. With several applications of Cauchy-Schwarz inequality and

$$
\tilde{\Lambda} = h^{1/2} \tilde{\Lambda} + |||f|||_{k+1,\Omega} \left( 1 + |||E|||_{1,\infty,\Omega_x} + |||B|||_{1,\infty,\Omega_x} \right),
$$
equation (4.2) becomes

$$
\frac{d}{dt} \left( \int_{T_h} |\varepsilon_h|^2 \, d\xi \right)
\leq c \left( h^{2k+1} \tilde{\Lambda}^2 + (h^k ||f||_{k+1,\Omega} (||\varepsilon^E_h||_{0,\Omega_x} + ||\varepsilon^B_h||_{0,\Omega_x}) + |f|_{1,\infty,\Omega} (||\varepsilon^E_h||_{0,\Omega_x} + ||\varepsilon^B_h||_{0,\Omega_x}))^2
+ h^{2k+1} |f||_{k+1,\Omega} (||\varepsilon^E_h||_{0,\Omega_x} + ||\varepsilon^B_h||_{0,\Omega_x}) + ||\varepsilon_h||_{0,\Omega}^2
\right.
\leq c \left( h^{2k+1} \tilde{\Lambda}^2 + h^k (1+h) |f||_{k+1,\Omega}^2 (||\varepsilon^E_h||^2_{0,\Omega_x} + ||\varepsilon^B_h||^2_{0,\Omega_x}) + |f|_{1,\infty,\Omega}^2 (||\varepsilon^E_h||^2_{0,\Omega_x} + ||\varepsilon^B_h||^2_{0,\Omega_x})
+ ||\varepsilon_h||_{0,\Omega}^2 \right).
$$

Here and below, the constant $c > 0$ only depends on $k$, mesh parameters $\sigma_0, \sigma_x$ and $\sigma_{\xi}$, and domain parameters $L_x$ and $L_{\xi}$. Moreover, with the inverse inequality in Lemma 2.2, and $\frac{h}{h_{\sigma,\min}}$ being uniformly bounded by $\sigma_0$ when the mesh is refined, there is

$$
\tilde{\Lambda} = h^{1/2} \tilde{\Lambda} + |||f|||_{k+1,\Omega} \left( 1 + |||E|||_{1,\infty,\Omega_x} + |||B|||_{1,\infty,\Omega_x} \right),
$$
equation (4.5) and this leads to

$$
\frac{d}{dt} \left( \int_{T_h} |\varepsilon_h|^2 \, d\xi \right)
\leq c \left( h^{2k+1} \tilde{\Lambda}^2 + h^{2k-\delta} |f||_{k+1,\Omega}^2 (||\varepsilon^E_h||^2_{0,\Omega_x} + ||\varepsilon^B_h||^2_{0,\Omega_x})
+ ||\varepsilon_h||_{0,\Omega}^2 \right).
$$

Recall $\delta = 3$, then for $k \geq 2$, there is $2k-\delta \geq 0$ and therefore $h^{2k-\delta} < \infty$. Similarly, with Cauchy-Schwarz inequality, (4.3) becomes

$$
\frac{d}{dt} \int_{T_h} (|\varepsilon^E_h|^2 + |\varepsilon^B_h|^2) \, dx
\leq c \left( ||\varepsilon_h||^2_{0,\Omega_x} + h^{2k+2} |f||_{k+1,\Omega} + h^{2k+1} (||E||^2_{k+1,\Omega_x} + ||B||^2_{k+1,\Omega_x}) \right) + ||\varepsilon^E_h||^2_{0,\Omega_x}.
$$

Now we sum up (4.6) and (4.7), and get

$$
\frac{d}{dt} \left( \int_{T_h} |\varepsilon_h|^2 \, d\xi \right) + \int_{T_h} |\varepsilon^E_h|^2 + |\varepsilon^B_h|^2 \, dx
\leq \Lambda h^{2k+1} + \Theta \left( \int_{T_h} |\varepsilon_h|^2 \, d\xi + \int_{T_h} |\varepsilon^E_h|^2 + |\varepsilon^B_h|^2 \, dx \right) \cdot
$$
Here $\Lambda$ depends on $(f, E, B)$ in their Sobolev norms $||\partial_t f||_{k+1, \Omega}$, $||f||_{k+1, \Omega}$, $||f||_{1, \Omega}$, $||E||_{1, \Omega_x}$, $||B||_{1, \Omega_x}$, $||E||_{k+1, \Omega_x}$, $||B||_{k+1, \Omega_x}$ at time $t$, and $\Theta$ depends on $||f||_{k+1, \Omega}$ and $||f||_{1, \Omega}$ at time $t$. Both $\Lambda$ and $\Theta$ depend on the polynomial degree $k$, mesh parameters $\sigma_0, \sigma_x$ and $\sigma_\xi$, and domain parameters $L_x$ and $L_\xi$. Now with a standard application of the Gronwall’s inequality, a triangular inequality, and the approximation results in (4.1), we can conclude the error estimate (4.4).

**Remark 4.4.** Theorem 4.3 shows that the proposed methods are $(k + \frac{1}{2})$-th order accurate, and this is standard for upwind DG methods to solve hyperbolic problems on general meshes. The assumption on the polynomial degree $k \geq 2$ is due to the lack of the $L^\infty$ error estimate for the DG solutions to the Maxwell solver and the use of an inverse inequality in handling the nonlinear coupling (see (4.5)-(4.7) in the proof of Theorem 4.3). If the computational domain in $x$ is one- or two-dimensional with $d_x = 1$ or 2, Theorem 4.3 holds for $k \geq 1$.

If the upwind numerical flux for the Maxwell solver (2.10b) is replaced by either the central or alternating flux (2.9a)-(2.9b), we will have the estimates for $\varepsilon_h^E$ and $\varepsilon_h^B$ in Lemma 4.5 instead, provided an additional assumption is made for the mesh when it is refined. That is, we assume there is a positive constant $\delta < 1$ such that for any $K_x \in T_h^x$,

$$\delta \leq \frac{h_{K_x'}}{h_{K_x}} \leq \frac{1}{\delta}$$

(4.8)

where $K_x'$ is any element in $T_h^x$ satisfying $K_x' \cap K_x \neq \emptyset$.

**Lemma 4.5 (Estimate of $\varepsilon_h^E$ and $\varepsilon_h^B$ with the non-upwinding flux).** Based on the semi-discrete DG discretization for the Maxwell equations in (2.10b) with either the central or alternating flux in (2.9a)-(2.9b), we have

$$\frac{d}{dt} \int_{T_h^x} (|\varepsilon_h^E|^2 + |\varepsilon_h^B|^2) \, dx \lesssim (||\varepsilon_h||_{0, \Omega} + h^{k+1}||f||_{k+1, \Omega})||\varepsilon_h^E||_{0, \Omega_x}$$

$$+ c(\delta)h_x^k(||E||_{k+1, \Omega_x} + ||B||_{k+1, \Omega_x}) \left( \int_{T_h^x} (|\varepsilon_h^E|^2 + |\varepsilon_h^B|^2) \, dx \right)^{1/2} .$$

(4.9)

The proof of this Lemma is given in subsection 4.3. With Lemma 4.5 and a similar proof as Theorem 4.3, the following error estimates can be established, and the proof is omitted.

**Theorem 4.6 (Error estimate 2).** For $k \geq 2$, the semi-discrete DG method (2.10a)-(2.10b) for Vlasov-Maxwell equations, with the upwind numerical flux (2.8a)-(2.8b) for the Vlasov solver and either the central or alternating flux in (2.9a)-(2.9b) for the Maxwell solver, has the following error estimate

$$||(f - f_h)(t)||_{0, \Omega}^2 + (||E - E_h||_{0, \Omega_x}^2 + ||B - B_h||_{0, \Omega_x}^2 \leq Ch^{2k}, \quad \forall t \in [0, T] .$$

(4.10)

Besides the dependence as in Theorem 4.3, the constant $C$ also depends on $\delta$ in (4.8).

Theorem 4.6 indicates that with either the central or alternating numerical flux for the Maxwell solver, the proposed method will be $k$-th order accurate. One can also see easily that the accuracy can be improved to $(k + \frac{1}{2})$-th order as in Theorem 4.3 if the discrete space for Maxwell solver is one degree higher than that for the Vlasov equation, namely, $r = k + 1$. This improvement will require higher regularity for the exact solution $E$ and $B$.

In [2], optimal error estimates were established for some DG methods solving the multi-dimensional Vlasov-Poisson problem on Cartesian meshes with tensor-structure discrete space, defined in (2.3), and $k \geq 1$. Some of the techniques in [2] are used in our analysis. In the present work, we focus on the P-type space $G_h^k$ in (2.1a) in the numerical section, as it renders better cost efficiency and can be used on more general meshes. Our analysis is established only for $k \geq 2$ due to the lack of the $L^\infty$ error estimate of the DG solver for the Maxwell part which is of hyperbolic nature, as pointed out in Remark 4.4.

In the next three subsections, we will provide the proofs of Lemmas 4.1, 4.2 and 4.5.

**4.1. Proof of Lemma 4.1.** Since the proposed method is consistent, we have the error relation related to the Vlasov solver,

$$a_h(f, E, B; g_h) - a_h(f_h, E_h, B_h; g_h) = 0, \quad \forall g_h \in G_h^k .$$

(4.11)
Note $\varepsilon_h \in G_h^k$, by taking $g_h = \varepsilon_h$ in (4.11), one has
\[
a_h(\varepsilon_h, E_h, B_h; \varepsilon_h) = a_h(\Pi^k f, E_h, B_h; \varepsilon_h) - a_h(f, E, B; \varepsilon_h). \tag{4.12}
\]

Following the same lines as in the proof of Lemma 3.6, we get
\[
a_h(\varepsilon_h, E_h, B_h; \varepsilon_h) = \frac{1}{2} \frac{d}{dt} \left( \int_{T_h} |\varepsilon_h|^2 dx \right) + \frac{1}{2} \int_{T_h} \int_{E_x} |\xi \cdot n_x| \|\varepsilon_h\| dx ds d\xi
+ \frac{1}{2} \int_{T_h} \int_{E_x} |(E_h + \xi \times B_h) \cdot n_x| \|\varepsilon_h\| dx ds d\xi. \tag{4.13}
\]

Next we will estimate the remaining terms in (4.12). Note
\[
a_h(\Pi^k f, E_h, B_h; \varepsilon_h) - a_h(f, E, B; \varepsilon_h) = T_1 + T_2,
\]
where
\[
T_1 = a_{h,1}(\Pi^k f; \varepsilon_h) - a_{h,1}(f; \varepsilon_h) = a_{h,1}(\zeta_h; \varepsilon_h),
T_2 = a_{h,2}(\Pi^k f, E_h, B_h; \varepsilon_h) - a_{h,2}(f, E, B; \varepsilon_h).
\]

**Step 1: to estimate $T_1$.** We start with
\[
T_1 = \int_{T_h} (\partial_t \zeta_h) \varepsilon_h dx d\xi - \int_{T_h} \zeta_h \xi \cdot \nabla_x \varepsilon_h dx d\xi + \int_{T_h} \int_{E_x} \zeta_h \xi \cdot [\varepsilon_h]_x dx ds d\xi = T_{11} + T_{12} + T_{13}.
\]

It is easy to verify that $\partial_t \Pi^k = \Pi^k \partial_t$, and therefore $\partial_t \zeta_h = \Pi^k (\partial_t f) - (\partial_t f)$. With the approximation result in Lemma 2.1, we have
\[
|T_{11}| = \int_{T_h} |(\partial_t \zeta_h) \varepsilon_h| dx d\xi \leq \|\partial_t \zeta_h\|_{0, \Omega_i} \|\varepsilon_h\|_{0, \Omega_i} \lesssim h^{k+1} \|\partial_t f\|_{k+1, \Omega_i} \|\varepsilon_h\|_{0, \Omega_i}. \tag{4.14}
\]

Next let $\xi_0$ be the $L^2$ projection of the function $\xi$ onto the piecewise constant space with respect to $T_h^\xi$, then
\[
T_{12} = -\int_{T_h} \zeta_h (\xi - \xi_0) \cdot \nabla_x \varepsilon_h dx d\xi - \int_{T_h} \zeta_h \xi_0 \cdot \nabla_x \varepsilon_h dx d\xi. \tag{4.15}
\]

Since $\xi_0 \cdot \nabla_x \varepsilon_h \in G_h^k$ and $\zeta_h = \Pi^k f - f$ with $\Pi^k$ being the $L^2$ projection onto $G_h^k$, the second term in (4.15) vanishes. Hence
\[
|T_{12}| \leq \int_{T_h} |\zeta_h (\xi - \xi_0) \cdot \nabla_x \varepsilon_h| dx d\xi,
\]
\[
\leq \|\xi - \xi_0\|_{0, \Omega_i} \sum_{K \times K_t = T_h} (h^{-1}_{K_t} \|\zeta_h\|_{0, K_t}) (h_{K_t} \|\nabla_x \varepsilon_h\|_{0, K_t}),
\]
\[
\lesssim \|\xi - \xi_0\|_{0, \Omega_i} \sum_{K \times K_t = T_h} h^{k+1} h^{-1}_{K_t} \|f\|_{k+1, K_t} \|\varepsilon_h\|_{0, K_t},
\]
\[
\lesssim h_{K_t} \|\xi\|_{1, \Omega_i} h^k \|f\|_{k+1, \Omega_i} \|\varepsilon_h\|_{0, \Omega_i},
\]
\[
\lesssim h^{k+1} \|f\|_{k+1, \Omega_i} \|\varepsilon_h\|_{0, \Omega_i}. \tag{4.16}
\]

The third inequality above uses the approximating result in Lemma 2.1 and the inverse inequality in Lemma 2.2. The fourth inequality uses an approximating result similar to the last one in Lemma 2.1, and $\frac{h_{K_t}}{h_{\varepsilon, \min}}$ being uniformly bounded by $\sigma_0$ when the mesh is refined.

Next, \[
T_{13} = \int_{T_h} \int_{E_x} \left( [\zeta_h]_x \xi + \frac{[\xi \cdot n_x]}{2} [\zeta_h]_x \right) \cdot [\varepsilon_h]_x dx ds d\xi,\]
\[
= \int_{T_h} \int_{E_x} \left( [\zeta_h]_x (\xi \cdot \hat{n}_x) \hat{n}_x + \frac{[\xi \cdot n_x]}{2} [\zeta_h]_x \right) \cdot [\varepsilon_h]_x dx ds d\xi.
\]
Step 2: to estimate $T_2$. Note

$$T_2 = a_{h,2}(\Pi_k f, E_h, B_h; \varepsilon_h) - a_{h,2}(f, E, B; \varepsilon_h) = a_{h,2}(\zeta_h, E_h, B_h; \varepsilon_h) + a_{h,2}(f, E_h, B_h; \varepsilon_h) - a_{h,2}(f, E, B; \varepsilon_h) = T_{21} + T_{22} + T_{23},$$

with

$$T_{21} = -\int_{T_h} \zeta_h (E_h + \xi \times B_h) \cdot \nabla \xi \varepsilon_h dx d\xi, \quad T_{22} = \int_{T_h} \zeta_h (E_h + \xi \times B_h) \cdot [\varepsilon_h] dx d\xi,$$

$$T_{23} = a_{h,2}(f, E_h, B_h; \varepsilon_h) - a_{h,2}(f, E, B; \varepsilon_h).$$

For $T_{21}$, we will proceed as how $T_{12}$ is estimated. Let $E_0 = \Pi_x^0 E, B_0 = \Pi_x^0 B$ be the $L^2$ projection of $E, B$, respectively, onto the piecewise constant vector space with respect to $T_h^x$, then

$$\int_{T_h} \zeta_h (E_h + \xi \times B_h) \cdot \nabla \xi \varepsilon_h dx d\xi = \int_{T_h} \zeta_h (E_h - E_0 + \xi \times (B_h - B_0)) \cdot \nabla \xi \varepsilon_h dx d\xi + \int_{T_h} \zeta_h (E_0 + \xi \times B_0) \cdot \nabla \xi \varepsilon_h dx d\xi,$$

and the second term above vanishes due to $(E_0 + \xi \times B_0) \cdot \nabla \xi \varepsilon_h \in \mathcal{G}_h^k$, and therefore

$$\left| \int_{T_h} \zeta_h (E_h + \xi \times B_h) \cdot \nabla \xi \varepsilon_h dx d\xi \right| \leq \int_{T_h} \left| \zeta_h (E_h - E_0 + \xi \times (B_h - B_0)) \cdot \nabla \xi \varepsilon_h dx d\xi \right| \leq (||E_h - E_0 + \xi \times (B_h - B_0)||_{0, \Omega}) \sum_{K_x \times K_z = K \in T_h} (h_{K_x}^{-1} ||\zeta_h||_{0, K})(h_{K_z}^{-1} ||\nabla \xi \varepsilon_h||_{0, K}),$$

$$\leq (||E_h - E_0||_{0, \Omega_x} + ||B_h - B_0||_{0, \Omega_x}) \sum_{K_x \times K_z = K \in T_h} h_k^{k+1} h_{K_x}^{-1} ||f||_{k+1, \Omega} ||\varepsilon_h||_{0, \Omega},$$

$$\lesssim \varepsilon_h ||f||_{k+1, \Omega} (||E_h^E||_{0, \Omega_x} + ||E_h^B||_{0, \Omega_x} + ||\Pi_x^k E - E_0||_{0, \Omega_x} + ||\Pi_x^k B - B_0||_{0, \Omega_x}) ||\varepsilon_h||_{0, \Omega}.$$

Note that $\Pi_x^k E - E_0 = \Pi_x^{k,k}(E - E_0)$, and $\Pi_x^k$ is bounded in any $L^p$-norm ($1 \leq p \leq \infty$) [19, 2], then

$$||\Pi_x^k E - E_0||_{0, \Omega_x} \lesssim ||E - E_0||_{0, \Omega_x} \lesssim h_x ||E||_{1, \Omega_x},$$
and similarly \(|\Pi^k_\epsilon B - B_0||_{0,\infty,\Omega_x} \lesssim h_x||B||_{1,\infty,\Omega_x}\). Hence
\[
\int_{T_h} \zeta_h (E_h + \xi \times B_h) \cdot \nabla \zeta_h d\xi dx 
\leq h^k ||f||_{k+1,\Omega} (||\xi|^2||_{0,\infty,\Omega_x} + ||E_h^B||_{0,\infty,\Omega_x} + h_x(||E||_{1,\infty,\Omega_x} + ||B||_{1,\infty,\Omega_x}))(||\xi||_{0,\Omega}) .
\]

For \(T_{22}\), we will proceed as we estimate \(T_{13}\). Note that \(E_h\) and \(B_h\) only depends on \(x\), and \(\xi\) is continuous,
\[
\int_{T_h} \int_{E_\xi} \zeta_h (E_h + \xi \times B_h) \cdot [\epsilon_h \xi] d\xi dx 
= \int_{T_h} \int_{E_\xi} \left( \zeta_h (E_h + \xi \times B_h) \xi + \frac{|E_h + \xi \times B_h|}{2} \cdot n_\xi [\zeta_h \xi] \cdot [\epsilon_h \xi] d\xi dx \right) ,
\]
\[
= \int_{T_h} \int_{E_\xi} \left( \zeta_h ((E_h + \xi \times B_h) \cdot n_\xi) \bar{n}_\xi + \frac{|E_h + \xi \times B_h|}{2} \cdot n_\xi [\zeta_h \xi] \cdot [\epsilon_h \xi] d\xi dx \right) , \quad \bar{n}_\xi = n_\xi \text{ or } -n_\xi 
\leq \int_{T_h} \int_{E_\xi} \left( |(E_h + \xi \times B_h) \cdot n_\xi| \left( \zeta_h \xi \right) + \frac{|\zeta_h \xi|}{2} \right) [\epsilon_h \xi] d\xi dx ,
\]
\[
\leq \left( \int_{T_h} \int_{E_\xi} \left( |(E_h + \xi \times B_h) \cdot n_\xi| \left( \zeta_h \xi \right) d\xi dx \right)^{1/2} \right) \left( \int_{T_h} \int_{E_\xi} \left( |(E_h + \xi \times B_h) \cdot n_\xi| [\epsilon_h \xi]^2 d\xi dx \right)^{1/2} \right) ,
\]
in addition,
\[
\left( \int_{T_h} \int_{E_\xi} \left( |(E_h + \xi \times B_h) \cdot n_\xi| \left( \zeta_h \xi \right) d\xi dx \right)^{1/2} \right) 
\lesssim \left( \int_{T_h} \int_{E_\xi} \left( |(E_h + \xi \times B_h) \cdot n_\xi| \left( \zeta_h \xi \right) d\xi dx \right)^{1/2} \right) 
\lesssim \left( \int_{T_h} \int_{E_\xi} \left( |(E_h + \xi \times B_h) \cdot n_\xi| [\epsilon_h \xi]^2 d\xi dx \right)^{1/2} \right) ,
\]
and therefore
\[
T_{22} \lesssim h^{k+\frac{1}{2}} ||f||_{k+1,\Omega} (||E_h^B||_{0,\infty,\Omega_x} + ||E_h^B||_{1,\infty,\Omega_x} + ||E||_{1,\infty,\Omega_x} + ||B||_{1,\infty,\Omega_x}) 
\left( \int_{T_h} \int_{E_\xi} \left( |(E_h + \xi \times B_h) \cdot n_\xi| [\epsilon_h \xi]^2 d\xi dx \right)^{1/2} \right) .
\]

Finally, we will estimate \(T_{23}\). Since \(f\) is continuous in \(\xi\), and \(\nabla \xi \cdot (E_h - E + \xi \times (B_h - B)) = 0\),
\[
T_{23} = a_{h,2}(f, E_h, B_h; \xi) - a_{h,2}(f, E, B; \xi) 
= -\int_{T_h} f(E_h - E + \xi \times (B_h - B)) \cdot \nabla \xi \varepsilon_h dx d\xi + \int_{T_h} \int_{E_\xi} f(E_h - E + \xi \times (B_h - B)) \cdot [\varepsilon_h \xi] d\xi dx ,
\]
\[
= \int_{T_h} \nabla \xi f \cdot (E_h - E + \xi \times (B_h - B)) \varepsilon_h dx d\xi ,
\]
therefore
\[
|T_{23}| \leq ||E_h - E + \xi \times (B_h - B)||_{0,\Omega} ||f||_{1,\infty,\Omega} ||\varepsilon_h||_{0,\Omega} ,
\]
\[
\lesssim (||E_h - E||_{0,\Omega} + ||(B_h - B)||_{0,\Omega_x}) ||f||_{1,\infty,\Omega} ||\varepsilon_h||_{0,\Omega} ,
\]
\[
\lesssim (||E_h^E||_{0,\Omega_x} + ||E_h^B||_{0,\Omega_x} + ||C_h^E||_{0,\Omega_x} + ||C_h^B||_{0,\Omega_x}) ||f||_{1,\infty,\Omega_x} ||\varepsilon_h||_{0,\Omega} ,
\]
\[
\lesssim (||E_h^E||_{0,\Omega_x} + ||E_h^B||_{0,\Omega_x} + h_x^{k+1} ||f||_{k+1,\Omega_x} + ||B||_{k+1,\Omega_x}) ||f||_{1,\infty,\Omega} ||\varepsilon_h||_{0,\Omega} .
\]
Now we can combine the estimates in (4.14) and (4.16)-(4.20), and get the result in Lemma 4.1.
4.2. Proof of Lemma 4.2. Since the proposed method is consistent, we have the error equation related to the Maxwell solver,

\[ b_h(E - E_h, B - B_h; U, V) = l_h(J - J_h, U), \quad \forall U, V \in \mathcal{U}_h. \quad (4.21) \]

We further take the test functions in (4.21) as \( U = \varepsilon_h^E \) and \( V = \varepsilon_h^B \), and this gives

\[ b_h(\varepsilon_h^E, \varepsilon_h^B; \varepsilon_h^E, \varepsilon_h^B) = b_h(\zeta_h^E, \zeta_h^B; \varepsilon_h^E, \varepsilon_h^B) + l_h(J - J_h, \varepsilon_h^E). \quad (4.22) \]

Following the same lines of Step 2 in the proof of Lemma 3.2,

\[ b_h(\varepsilon_h^E, \varepsilon_h^B; \varepsilon_h^E, \varepsilon_h^B) = \frac{1}{2} \frac{d}{dt} \int_{T_h}(||\varepsilon_h^E||^2 + ||\varepsilon_h^B||^2) \, dx + \frac{1}{2} \int \left( ||\varepsilon_h^E||^2 + ||\varepsilon_h^B||^2 \right) \, ds. \quad (4.23) \]

What remained is to estimate the two terms on the right side of (4.22),

\[
\begin{align*}
&b_h(\zeta_h^E, \zeta_h^B; \varepsilon_h^E, \varepsilon_h^E) \\
&= \int_{T_h} \partial_t \zeta_h^E \cdot \varepsilon_h^E \, dx - \int_{T_h} \zeta_h^B \cdot \nabla \times \varepsilon_h^E \, dx - \int_{\mathcal{E}_x} \zeta_h^B \cdot [\varepsilon_h^E]_x \, ds \\
&+ \int_{T_h} \partial_t \varepsilon_h^B \cdot \zeta_h^E \, dx + \int_{T_h} \zeta_h^E \cdot \nabla \times \varepsilon_h^B \, dx + \int_{\mathcal{E}_x} \zeta_h^E \cdot [\varepsilon_h^B]_x \, ds \\
&= - \int_{\mathcal{E}_x} \zeta_h^E \cdot [\varepsilon_h^E]_x \, ds + \int_{\mathcal{E}_x} \zeta_h^E \cdot [\varepsilon_h^B]_x \, ds \\
&\leq \left( \int_{\mathcal{E}_x} |\zeta_h^E|^2 + |\zeta_h^B|^2 \, ds \right)^{1/2} \left( \int_{\mathcal{E}_x} ||\varepsilon_h^E||^2 + ||\varepsilon_h^B||_x^2 \, ds \right)^{1/2}, \\
&\lesssim \sum_{K_x \in T_h} \left( ||\zeta_h^E||_{0, \partial K_x} + ||\zeta_h^B||_{0, \partial K_x} \right) \left( \int_{\mathcal{E}_x} ||\varepsilon_h^E||^2 + ||\varepsilon_h^B||_x^2 \, ds \right)^{1/2}, \\
&\approx h_x^{k+2} \left( ||E||_{k+1, \Omega_x} + ||B||_{k+1, \Omega_x} \right) \left( \int_{\mathcal{E}_x} ||\varepsilon_h^E||^2 + ||\varepsilon_h^B||^2 \, ds \right)^{1/2}.
\end{align*}
\]

All volumes integrals in (4.24) vanish due to that \( \partial_t \Pi_x^k = 0 \), and \( \varepsilon_h^E, \varepsilon_h^B, \nabla \times \varepsilon_h^E, \nabla \times \varepsilon_h^B \in \mathcal{U}_h \). And for the last two inequalities, the definition of the numerical fluxes are used together with the approximation result in Lemma 2.1. Finally,

\[
\begin{align*}
&|l_h(J - J_h; \varepsilon_h^E)| = \left| \int_{T_h} (J - J_h) \cdot \varepsilon_h^E \, dx \right| \\
&\leq | J - J_h |_{0, \Omega_x} ||\varepsilon_h^E||_{0, \Omega_x} = \left| \int_{T_h} (f - f_h) \xi d\xi \right| ||\varepsilon_h^E||_{0, \Omega_x} \\
&\leq ||f - f_h||_{0, \Omega} ||\xi||_{0, \Omega_x} ||\varepsilon_h^E||_{0, \Omega_x} \\
&\approx (||\varepsilon_h||_{0, \Omega} + ||\varepsilon_h||_{0, \Omega}) ||\varepsilon_h^E||_{0, \Omega_x} \lesssim (||\varepsilon_h||_{0, \Omega} + h^{k+1} ||f||_{k+1, \Omega}) ||\varepsilon_h^E||_{0, \Omega_x}. \quad (4.25)
\end{align*}
\]

Combining (4.23)-(4.25), we can conclude Lemma 4.2.

4.3. Proof of Lemma 4.5. The proof proceeds similarly as for Lemma 4.2 in subsection 4.2. Based on the error equation (4.21) related to the Maxwell solver with some specific test functions, we get (4.22). With either the central or alternating flux in (2.9a)-(2.9b), there is

\[
b_h(\varepsilon_h^E, \varepsilon_h^B; \varepsilon_h^E, \varepsilon_h^B) = \frac{1}{2} \frac{d}{dt} \int_{T_h}(||\varepsilon_h^E||^2 + ||\varepsilon_h^B||^2) \, dx.
\]


The same estimate as (4.25) can be obtained for the second term on the right of (4.22). To estimate the first one,

\[
\begin{align*}
    & b_h(\zeta_h^E, \zeta_h^B, \epsilon_h^E, \epsilon_h^B) \\
    & = \int_{T_h^k} \partial_t \zeta_h^E \cdot \epsilon_h^E dx - \int_{T_h^k} \zeta_h^B \cdot \nabla \times \epsilon_h^E dx - \int_{E_x} \zeta_h^E \cdot [\epsilon_h^E]_\tau ds_x \\
    & \quad + \int_{T_h^k} \partial_t \zeta_h^B \cdot \epsilon_h^B dx + \int_{T_h^k} \zeta_h^E \cdot \nabla \times \epsilon_h^B dx + \int_{E_x} \zeta_h^E \cdot [\epsilon_h^B]_\tau ds_x \\
    & = -\int_{E_x} \zeta_h^B \cdot [\epsilon_h^E]_\tau ds_x + \int_{E_x} \zeta_h^E \cdot [\epsilon_h^B]_\tau ds_x
\end{align*}
\]

(4.26)

\[
\begin{align*}
    & \leq \left( \sum_{e \in E_x} \int_{e} h^{-1}_{K_x} (|\zeta_h^E|^2 + |\zeta_h^B|^2) ds_x \right)^{1/2} \left( \sum_{e \in E_x} \int_{e} h_{K_x} (|[\epsilon_h^E]_\tau|^2 + |[\epsilon_h^B]_\tau|^2) ds_x \right)^{1/2} \\
    & \lesssim c(\delta) \left( \sum_{K_x \in T_h^k} \int_{\partial K_x} h^{-1}_{K_x} (|\zeta_h^B|^2 + |\zeta_h^E|^2) ds_x \right)^{1/2} \left( \sum_{K_x \in T_h^k} \int_{\partial K_x} h_{K_x} (|[\epsilon_h^E]_\tau|^2 + |[\epsilon_h^B]_\tau|^2) ds_x \right)^{1/2} \\
    & \lesssim c(\delta) \left( \sum_{K_x \in T_h^k} h^k_{K_x} (||E||^2_{k+1,K_x} + ||B||^2_{k+1,K_x}) \right)^{1/2} \left( \sum_{K_x \in T_h^k} (||\epsilon_h^E||^2_{0,K_x} + ||\epsilon_h^B||^2_{0,K_x}) \right)^{1/2} \\
    & \lesssim c(\delta) h^k (||E||^{k+1,0} + ||B||^{k+1,0}) \left( \int_{T_h^k} (||\epsilon_h^E||^2 + ||\epsilon_h^B||^2) dx \right)^{1/2}
\end{align*}
\]

(4.27)

All volumes integrals in (4.26) vanish due to \( \partial_t \Pi^k \) \( \Pi^k \partial_t \), and \( \epsilon_h^E, \epsilon_h^B, \nabla \times \epsilon_h^E, \nabla \times \epsilon_h^B \in \mathcal{U}_h^k \). In (4.28), \( K_x \) is any element an edge \( e \) belongs to. To get (4.29), we use the definitions of the numerical fluxes, jumps, as well as the assumption (4.8) on the ratio of the neighboring mesh elements. Here \( c(\delta) \) is a positive constant dependent of \( \delta \). We obtain (4.30) by applying an approximation result in Lemma 2.1 and an inverse inequality in Lemma 2.2. With all above, one can now conclude Lemma 4.5.

5. Numerical result. In this section, we perform a detailed numerical study of the proposed scheme. In particular, we focus on the test example of the Weibel instability [49]. The Weibel instability is a plasma instability present in homogeneous or nearly homogeneous electromagnetic plasmas which possess an anisotropy in velocity space. This anisotropy is most generally understood as two temperatures in different directions, and the magnetic field in the Weibel instability will grow in time. We follow the setting of Califano et al. [9]. The variables under consideration are the distribution function \( f(x_2, \xi_1, \xi_2, t) \), a 2D electric field \( E = (E_1(x_2, t), E_2(x_2, t), 0) \) and a 1D magnetic field \( B = (0, 0, B_3(x_2, t)) \). The Vlasov-Maxwell system is reduced to

\[
\begin{align*}
    & f_t + \xi_2 f_{x_2} + (E_1 + \xi_2 B_3) f_{\xi_1} + (E_2 - \xi_1 B_3) f_{\xi_2} = 0, \\
    & \frac{\partial B_3}{\partial t} = \frac{\partial E_1}{\partial x_2}, \quad \frac{\partial E_1}{\partial t} = \frac{\partial B_3}{\partial x_2} - j_1, \quad \frac{\partial E_2}{\partial t} = -j_2,
\end{align*}
\]

(5.1)

(5.2)

where

\[
\begin{align*}
    & j_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_2, \xi_1, \xi_2, t) \xi_1 d\xi_1 d\xi_2, \quad j_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_2, \xi_1, \xi_2, t) \xi_2 d\xi_1 d\xi_2.
\end{align*}
\]

(5.3)

The initial conditions are given by

\[
\begin{align*}
    & f(x_2, \xi_1, \xi_2, 0) = \frac{1}{\pi \beta} e^{-\xi_2^2/\beta} [\delta e^{-\xi_1 - \nu_0,1}^2/\beta} + (1 - \delta) e^{-\xi_1 + \nu_0,2}^2/\beta}, \\
    & E_1(x_2, \xi_1, \xi_2, 0) = E_2(x_2, \xi_1, \xi_2, 0) = 0, \quad B_3(x_2, \xi_1, \xi_2, 0) = b \sin(k_0 x_2).
\end{align*}
\]

(5.4)

(5.5)
Following [9], we take $\beta = 0.01, b = 0.001$ (the amplitude of the initial perturbation to the magnetic field). $
abla_x = [0, L_y]$, where $L_y = 2\pi/k_b$, and $\Omega_\zeta = [-1.2, 1.2]^2$.

**Accuracy test:** The V-M system is time reversible, and this provides a way to test the accuracy of our scheme. In particular, let $f(x, \xi, 0), E(x, 0), B(x, 0)$ be the initial conditions of the V-M system, and $f(x, \xi, T), E(x, T), B(x, T)$ be the solution at $t = T$. If we enforce $f(x, -\xi, T), E(x, T), -B(x, T)$ be the initial condition of the V-M system at $t = 0$, then at $t = T$, one would recover $f(x, -\xi, 0), E(x, 0), -B(x, 0)$. In Tables 5.1, 5.2, 5.3, we provide the $L^2$ errors and orders of the numerical solutions with three flux choices for the Maxwell’s equations: the upwind flux, the central flux, and one of the alternating fluxes $\bar{E}_h = E_h^+$ and $\bar{B}_h = B_h^-$. The parameters are taken as $\delta = 0.5, v_{0,1} = v_{0,2} = 0.3$, and $k = 0.2$. In the numerical simulations, the third order TVD Runge Kutta time discretization is used, with the CFL number $C_{eff} = 0.19$ for the upwind and central fluxes, and $C_{eff} = 0.12$ for the alternating flux in $P^1$ and $P^2$ cases. For $P^3$, we take $\Delta t = O(\Delta x^{2/3})$ to ensure that the spatial and temporal accuracy is of the same order. From Tables 5.1, 5.2, 5.3, we observe that the schemes with the upwind and alternating fluxes achieve optimal $(k+1)$-th order of accuracy in approximating the solution, while for odd $k$, the central flux gives suboptimal approximations in some of the solution component.

**Table 5.1**

Upwind flux for Maxwell’s equation, $L^2$ errors and orders. Run to $T=5$ and back to $T = 10$.

| Space | Mesh=20⁴ | Mesh=40⁴ | Mesh=80⁴ |
|-------|----------|----------|----------|
|       | error    | error    | error    | order    | order    | order    |
| $G^3_h, \mathcal{U}^3_h$ | $f$ | 0.18E+00 | 0.50E-01 | 1.82 | 0.13E-01 | 1.96 |
|       | $B_3$ | 0.26E-05 | 0.66E-06 | 2.01 | 0.16E-06 | 2.01 |
|       | $E_1$ | 0.21E-05 | 0.68E-06 | 1.61 | 0.19E-06 | 1.81 |
|       | $E_2$ | 0.10E-05 | 0.22E-06 | 2.23 | 0.22E-07 | 3.29 |
| $G^2_h, \mathcal{U}^2_h$ | $f$ | 0.56E-01 | 0.77E-02 | 2.87 | 0.10E-02 | 2.92 |
|       | $B_3$ | 0.23E-06 | 0.20E-07 | 3.12 | 0.32E-08 | 3.06 |
|       | $E_1$ | 0.16E-06 | 0.16E-07 | 3.32 | 0.14E-08 | 3.54 |
|       | $E_2$ | 0.16E-06 | 0.22E-07 | 2.90 | 0.15E-08 | 3.91 |
| $G^1_h, \mathcal{U}^1_h$ | $f$ | 0.12E-01 | 0.10E-02 | 3.56 | 0.70E-04 | 3.90 |
|       | $B_3$ | 0.97E-07 | 0.23E-08 | 5.37 | 0.12E-09 | 4.34 |
|       | $E_1$ | 0.19E-07 | 0.27E-09 | 6.16 | 0.57E-11 | 5.54 |
|       | $E_2$ | 0.14E-07 | 0.79E-09 | 4.11 | 0.16E-10 | 5.64 |

**Table 5.2**

Central flux for Maxwell’s equation, $L^2$ errors and orders. Run to $T=5$ and back to $T = 10$.

| Space | Mesh=20⁴ | Mesh=40⁴ | Mesh=80⁴ |
|-------|----------|----------|----------|
|       | error    | error    | error    | order    | order    | order    |
| $G^3_h, \mathcal{U}^3_h$ | $f$ | 0.18E+00 | 0.50E-01 | 1.82 | 0.13E-01 | 1.96 |
|       | $B_3$ | 0.13E-04 | 0.85E-05 | 0.66 | 0.50E-05 | 0.75 |
|       | $E_1$ | 0.19E-05 | 0.13E-05 | 0.51 | 0.58E-06 | 1.17 |
|       | $E_2$ | 0.92E-06 | 0.19E-06 | 2.26 | 0.20E-07 | 3.24 |
| $G^2_h, \mathcal{U}^2_h$ | $f$ | 0.56E-01 | 0.77E-02 | 2.87 | 0.10E-02 | 2.92 |
|       | $B_3$ | 0.28E-06 | 0.28E-07 | 3.34 | 0.32E-08 | 3.15 |
|       | $E_1$ | 0.18E-07 | 0.56E-09 | 5.00 | 0.88E-11 | 5.99 |
|       | $E_2$ | 0.16E-06 | 0.22E-07 | 2.90 | 0.15E-08 | 3.91 |
| $G^1_h, \mathcal{U}^1_h$ | $f$ | 0.12E-01 | 0.10E-02 | 3.56 | 0.70E-04 | 3.90 |
|       | $B_3$ | 0.10E-06 | 0.44E-08 | 4.57 | 0.16E-09 | 4.81 |
|       | $E_1$ | 0.46E-07 | 0.82E-10 | 9.12 | 0.30E-10 | 1.45 |
|       | $E_2$ | 0.14E-07 | 0.79E-09 | 4.12 | 0.16E-10 | 5.65 |
Macroscopic quantities: The purpose here is to validate our theoretical result about conservation through two numerical examples, the symmetric case and the non-symmetric case. We first use parameter choice 1 as in the Califano et al. [9], the symmetric case, where \( \delta = 0.5, k_0 = 0.2, v_{0,1} = v_{0,2} = 0.3 \) with three different fluxes for the Maxwell’s equations. The results are illustrated in Figure 5.1. In all the plots, we have rescaled the macroscopic quantities by the physical domain size. For all three fluxes, the mass (total particle number) is well conserved. The largest relative error for the particle number for all three fluxes is smaller than \( 4 \times 10^{-4} \) for upwind flux, and bounded by \( 1 \times 10^{-7} \) for central and alternating fluxes. In Figure 5.2, we study the effect of enlarging the domain in the velocity direction. The growth in the total mass, as time approaches \( T = 200 \) when \( \Omega_\xi = [-1.2, 1.2]^2 \), implies that up to this time, a larger domain should be used in order for the assumption, \( f \) being compactly supported in \( \xi \), to still hold. This growth in relative error is not observed when \( \Omega_\xi = [-1.5, 1.5]^2 \). On the other hand, the decay in the total energy with the upwind fluxes is largely due to the tangential jump terms in the electric and magnetic field as derived in Lemma 3.2. Therefore, enlarging the domain has little effects on this. As for the decay in total energy with central and alternating fluxes, we can see that enlarging the domain roughly reduces the error by half. The other part of the error is coming from the dissipative nature of the TVD-RK scheme that we have used.

In Figure 5.3, we plot the time evolution of the kinetic, electric and magnetic energy. In particular, we have plotted the separated components for the kinetic and electric energy. K1 energy is defined as \( \frac{1}{2} \int f \xi_1^2 dx_1 dx_2 \), K2 energy is defined as \( \frac{1}{2} \int f \xi_2^2 dx_1 dx_2 \), E1 energy is defined as \( \frac{1}{2} \int E_1^2 dx_2 \), E2 energy is defined as \( \frac{1}{2} \int E_2^2 dx_2 \). We also consider the first four Log Fourier mode of the fields \( E_1, E_2, B_3 \) in Figure 5.4. Here, the \( n \)-th Log Fourier mode for a function \( W(x, t) \) [32] is defined as

\[
\log FM_n(t) = \log_{10} \left( \frac{1}{L} \left( \int_0^L W(x, t) \sin(knx) dx \right)^2 + \int_0^L W(x, t) \cos(knx) dx \right)^2. 
\]

In Figures 5.5, 5.6, we plot the 2D contour of \( f \) at selected location \( x_2 \) and time \( t \) when the upwind flux is applied in Maxwell’s solve. In Figure 5.7, the plot of density \( \rho_h \) is given at those times. For completeness, we also include the plot of the electric and magnetic field at the final time in Figure 5.8.

We further use parameter choice 2 in the Califano paper, the nonsymmetric case, where \( \delta = 1/6, k_0 = 0.2, v_{0,1} = 0.5, v_{0,2} = 0.1 \). The results are gathered in Figures 5.9, 5.10, 5.11, 5.12, 5.13, 5.14 and 5.15. Again we could observe relatively larger error in the total energy with the upwind flux used in the Maxwell’s equations. But overall, both mass and total energy are very well preserved.
Weibel instability with parameter choice 1 as in Califano et al. [9] (δ = 0.5, v_{0,1} = v_{0,2} = 0.3, k_0 = 0.2), the symmetric case. The mesh is 100^3 with piecewise quadratic polynomials. Time evolution of mass, total energy with three numerical fluxes for the Maxwell's equations.

6. Concluding Remarks. In the future, we will explore other time stepping methods to improve the efficiency of the overall algorithm. Note that Gauss laws are not considered in the present framework. We plan to investigate them together with some correction techniques for the continuity equation. The proposed methods will also be applied to study other important plasma physics examples, especially those of higher dimension.

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Weibel instability with parameter choice 1 as in Califano et al. [9] ($\delta = 0.5, v_{0,1} = v_{0,2} = 0.3, k_0 = 0.2$), the symmetric case. Effects of enlarging the domain. The runs are conducted on the same mesh size, but with different domains in the velocity space. The small domain is $\Omega_\xi = [-1.2, 1.2]^2$, the large domain is $\Omega_\xi = [-1.5, 1.5]^2$. 

Fig. 5.2.
Fig. 5.3. **Weibel instability with parameter choice 1 as in [9]** ($\delta = 0.5, v_{0,1} = v_{0,2} = 0.3, k_0 = 0.2$). The mesh is $100^3$ with piecewise quadratic polynomials. Time evolution of kinetic, electric and magnetic energy by alternating flux for the Maxwell’s equations.
Fig. 5.4. Weibel instability with parameter choice 1 as in [9] ($\delta = 0.5, v_{0.1} = v_{0.2} = 0.3, k_0 = 0.2$). The mesh is $100^3$ with piecewise quadratic polynomials. The first four Log Fourier modes of $E_1$, $E_2$, $B_3$ computed by the alternating flux for the Maxwell’s equations.
Fig. 5.5. 2D contour plots of the computed distribution function $f_h$ in Weibel instability, with parameter choice $l$ as in [9] ($\delta = 0.5, v_{0,1} = v_{0,2} = 0.3, k_0 = 0.2$), at selected locations $x_2$ and time $t$. The mesh is $100^3$ with piecewise quadratic polynomials. The upwind flux is applied.
Fig. 5.6. 2D contour plots of the computed distribution function $f_h$ in Weibel instability, with parameter choice 1 as in [9] ($\delta = 0.5, v_0_1 = v_0_2 = 0.3, k_0 = 0.2$), at selected locations $x_2$ and time $t$. The mesh is $100^3$ with piecewise quadratic polynomials. The upwind flux is applied.
Fig. 5.7. Plots of the computed density function $\rho_h$ in Weibel instability, with parameter choice 1 as in [9] ($\delta = 0.5, v_{0,1} = v_{0,2} = 0.3, k_0 = 0.2$), at selected time $t$. The mesh is $100^3$ with piecewise quadratic polynomials. The upwind flux is applied.
Fig. 5.8. Weibel instability with parameter choice 1 as in [9] ($\delta = 0.5, v_{0,1} = v_{0,2} = 0.3, k_0 = 0.2$). The mesh is $100^3$ with piecewise quadratic polynomials. The electric and magnetic fields at $T = 200$. 
Fig. 5.9. Weibel instability with parameter choice 2 as in Califano et al. [9] $\delta = 1/6, v_{0,1} = 0.5, v_{0,2} = 0.1, k_0 = 0.2$, the non-symmetric case. The mesh is $100^3$ with piecewise quadratic polynomials. Time evolution of mass, total energy with three numerical fluxes for the Maxwell’s equations.

Fig. 5.10. Weibel instability with parameter choice 2 as in [9] $(\delta = 1/6, v_{0,1} = 0.5, v_{0,2} = 0.1, k_0 = 0.2)$. The mesh is $100^3$ with piecewise quadratic polynomials. Time evolution of kinetic, electric and magnetic energy by alternating flux for the Maxwell’s equations.
Fig. 5.11. Weibel instability with parameter choice 2 as in [9] ($\delta = 1/6, v_{0,1} = 0.5, v_{0,2} = 0.1, k_0 = 0.2$). The mesh is $100^3$ with piecewise quadratic polynomials. The first four Log Fourier modes of $E_1, E_2, B_3$ computed by the alternating flux for the Maxwell’s equations.
Fig. 5.12. 2D contour plots of the computed distribution function $f_h$ in Weibel instability, with parameter choice 2 as in [9] ($\delta = 1/6, v_{0,1} = 0.5, v_{0,2} = 0.1, k_0 = 0.2$), at selected location $x_2$ and time $t$. The mesh is $100^3$ with piecewise quadratic polynomials. The upwind flux is applied.
Fig. 5.13. 2D contour plots of the computed distribution function $f_h$ in Weibel instability, with parameter choice 2 as in [9] ($\delta = 1/6, v_{0,1} = 0.5, v_{0,2} = 0.1, k_0 = 0.2$), at selected locations $x_2$ and time $t$. The mesh is $100^3$ with piecewise quadratic polynomials. The upwind flux is applied.
Fig. 5.14. Plots of the computed density function $\rho_h$ in Weibel instability, with parameter choice 2 as in [9] ($\delta = 1/6, v_{0,1} = 0.5, v_{0,2} = 0.1, k_0 = 0.2$), at selected time $t$. The mesh is $100^2$ with piecewise quadratic polynomials. The upwind flux is applied.
Weibel instability with parameter choice 2 as in [9] (δ = 1/6, \(v_{0,1} = 0.5, v_{0,2} = 0.1, k_0 = 0.2\)). The mesh is 100×7 with piecewise quadratic polynomials. The electric and magnetic fields at \(T = 200\).