The crossing number of pancake graph $P_4$ is six

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Abstract

The crossing number of a graph $G$ is the least number of pairwise crossings of edges among all the drawings of $G$ in the plane. The pancake graph is an important topology for interconnecting processors in parallel computers. In this paper, we prove the exact value of the crossing number of pancake graph $P_4$ is six.

Keywords: Crossing number; Drawings; Pancake graph.

1 Introduction

The notion of crossing number is a central one for Topological Graph Theory with long history, which means the minimum possible number of edge crossings in a drawing of graph $G$ in the plane. In recent years, because of its applications in various fields such as discrete and computational geometry, VLSI theory, wiring layout problems, and in several other areas of theoretical computer science, the crossing number problem has been studied extensively by mathematicians including Erdős, Guy, Turán and Tutte, et al. (see [9, 11, 14, 15]). However, the investigation on the crossing number of graphs is an extremely difficult problem. In 1973, Erdős and Guy wrote, “Almost all questions that one can ask about crossing numbers remain unsolved.” Actually, Garey and Johnson [10] proved that computing the crossing number is NP-complete. Also, it’s not surprising that the exact crossing numbers are known only for a few families of graphs (see [11, 13]). In most cases, to give the upper and lower bounds is a more practical way (see [12, 16, 17]). As to a nice drawing of a graph with the number of crossings that can hardly be decreased, it is very difficult to prove that the number of crossings in this drawing is indeed the crossing number of the graph we studied.

The pancake graph was proposed by Akers and Krishnamurthy in [8] as a special case of Cayley graphs. It not only possesses several attractive features just like hypercubes, such as symmetry properties and high fault tolerant, but also offers three significant advantages

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over hypercubes: a lower degree, a smaller diameter and average diameter. Therefore, there are more and more research about pancake graphs recently. In [1], Lin, Huang and Hsu proved that the n-dimensional pancake graph $P_n$ is super connected if and only if $n \neq 3$. In addition, Deng and Zhang proved that the automorphism group of the pancake graph $P_n$ is the left regular representation of the symmetric group $S_n$ for $n \geq 5$ in [3]. More research about pancake graph can be found in [2–6].

In [18], Sýkora and Vrt’o proved approximative values of crossing number of the pancake graphs. However, their results are valuable only when the dimension $n$ is large enough. Yet there is little study of the exact crossing number of pancake graphs when $n$ is small, which is of theoretical importance and practical value. In this paper, we prove that the crossing number of pancake graph $P_4$ is exactly six.

2 Notations and basic lemmas

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq E(G)$, let $[S]$ be the subgraph of $G$ induced by $S$. Let $P_{v_1 v_2 \cdots v_n}$ be the path traversing from $v_1$ to $v_n$ with $n$ vertices. Let $C_{v_1 v_2 \cdots v_n v_1}$ be the circle with $n$ vertices from $v_1$ to $v_n$.

A drawing of $G$ is said to be a good drawing, provided that no edge crosses itself, no adjacent edges cross each other, no two edges cross more than once, and no three edges cross in a point. It is well known that the crossing number of a graph is attained only in good drawings of the graph. So, we always assume that all drawings throughout this paper are good drawings.

For a drawing $D$ of a graph $G$, let $\nu(D)$ be the number of crossings in $D$. In a drawing $D$, if an edge is not crossed by any other edge, we say that it is clean in $D$.

For two disjoint subsets of an edge set $E$, say $A$ and $B$, the number of the crossings formed by an edge in $A$ and another edge in $B$ is denoted by $\nu_D(A, B)$ in a drawing $D$. The number of the crossings that involve a pair of edges in $A$ is denoted by $\nu_D(A)$. Then $\nu_D(A \cup B) = \nu_D(A) + \nu_D(B) + \nu_D(A, B)$ and $\nu(D) = \nu_D(E)$.

Definition 2.1. (Pancake Graphs) The n-dimensional pancake graph, denoted by $P_n$ and proposed by Akers and Krishnameurthy, is a graph consisting of $n!$ vertices labelled with $n!$ permutations on a set of the symbols $1, 2, \cdots, n$. There is an edge from vertex $i$ to vertex $j$ if and only if $j$ is a permutation of $i$ such that $i = i_1 i_2 \cdots i_k i_{k+1} \cdots i_n$ and $j = i_k \cdots i_2 i_1 i_{k+1} \cdots i_n$, where $2 \leq k \leq n$.

The pancake graphs $P_2, P_3$ and $P_4$ are shown in Figure 2.1 for illustration.

There are four 6-cycles $C_i (1 \leq i \leq 4)$ in $P_4$. For $1 \leq i \leq 4$, the subgraph of $P_4$ induced by $V(P_4) - V(C_i)$ is homeomorphic to graph $G_{12}$ shown in Figure 2.2.

Lemma 2.1. Let $D$ be an arbitrarily drawing of $G_{12}$, then $\nu(D) \geq 2$. 
Proof. Let $m$ be the smallest number of the edges of $G_{12}$ whose deletion from $G_{12}$ results in a planar subgraph $G_{12}^*$ of $G_{12}$. $G_{12}^*$ has 12 vertices, $18 - m$ edges. Let $D_{12}^*$ be a planar drawing of $G_{12}^*$ and $p$ denote the number of faces in $D_{12}^*$. Then, according to the Euler Polyhedron Formula,

$$12 - (18 - m) + p = 2,$$

$$p = 8 - m.$$ 

Since all cycles in $G_{12}$ have length at least six except for three 4-cycles, there are at most three 4-cycles in $G_{12}^*$ but no 3-cycles. By counting the number of edges of each face in $D_{12}^*$, we have

$$3 \times 4 + (8 - m - 3) \times 6 \leq |E(G_{12}^*)| = 2 \times (18 - m),$$

$$4m \geq 6.$$ 

It follows $m \geq 2$. Hence $\nu(D) \geq 2$. 

Lemma 2.2. Let $D$ be a drawing of $G_{12}$, where at least one pair of 4-cycles crosses each other, then $\nu(D) \geq 3$.

Proof. By contradiction. Suppose $\nu(D) \leq 2$. Then there is exact one pair of 4-cycles, say $C_1^4, C_2^4$, crosses each other. Since $\nu(D) \leq 2$, no 4-cycle crosses itself, and edges
Let \( v_1v_9, v_3v_{11}, v_6v_{10} \) and \( v_8v_{12} \) are all clean. Since vertices \( v_9, v_{10}, v_{11} \) and \( v_{12} \) lie in outside of \( C_4^1 \) and \( C_4^2 \), vertices \( v_1, v_3, v_6 \) and \( v_8 \) have to lie on the bounder of same area. It follows at least one edge of \( v_1v_9, v_3v_{11}, v_6v_{10} \) and \( v_8v_{12} \) is crossed, \( \nu(D) \geq 3 \), a contradiction (See Figure 2.3(1)).

![Figure 2.3: Some drawings of \( G_{12} \)](https://via.placeholder.com/150)

**Lemma 2.3.** Let \( D \) be a good drawing of \( G_{12} \), where any pair of 4-cycles does not cross each other and any two 4-cycles lie in the same side of the third 4-cycle, then \( \nu(D) \geq 3 \).

**Proof.** By contradiction. Suppose \( \nu(D) \leq 2 \).

**Case 1.** There is at least one 4-cycle, say \( C_4^1 \), crosses itself. Without loss of generality, we may assume that \( v_1v_4 \) crosses \( v_2v_3 \). We show this situation in Figure 2.3(2). Since \( \nu(D) \leq 2 \), at least one cycle of the disjoint cycles \( C_{v_1v_9v_{10}v_6v_7v_2v_1} \) and \( C_{v_3v_{11}v_{12}v_8v_5v_4v_3} \), say cycle \( C_{v_1v_9v_{10}v_6v_7v_2v_1} \), does not cross itself. And at least one cycle of cycles \( C_{v_4v_9v_{10}v_3v_1v_4} \) and \( C_{v_5v_4v_9v_{10}v_3v_4} \), say cycle \( C_{v_4v_9v_{10}v_3v_1v_4} \), does not cross itself, for they only have one common edge. Considering the possible locations for vertex \( v_{12} \), we find at least one edge of \( v_{12}v_{11}, v_{12}v_9, v_{12}v_8, v_{12}v_7 \) is crossed, since edges \( v_{11}v_{12}, v_9v_{12} \) and path \( P_{v_4v_9v_{10}v_7} \) can not be in the same region. Hence, cycle \( C_{v_4v_9v_{10}v_3v_1v_4} \) can not cross itself. Since \( \nu(D) \leq 2 \), path \( P_{v_4v_9v_{12}v_{11}} \) can be crossed at most one time, edge \( v_8v_{12} \) has to lie in outside of cycle \( C_{v_5v_4v_9v_{10}v_3v_1v_4} \). It follows edges \( v_{12}v_9 \) and \( v_8v_7 \) are both crossed, \( \nu(D) \geq 3 \), a contradiction.

**Case 2.** No 4-cycle crosses itself. Since \( \nu(D) \leq 2 \), at least one of all the three pairs of 4-cycles, say \( C_4^1 \) and \( C_4^2 \), satisfies the following conditions: the edges between that pair of 4-cycles do not cross each other, and they do not cross the pair of 4-cycles either. By symmetry, we may assume \( v_3, v_6 \) lie inside of cycle \( C_{v_1v_2v_7v_8v_5v_4v_1} \) (See Figure 2.3(3)). Since any pair of 4-cycles does not cross each other and any two 4-cycles lie on the same side of the third 4-cycle, 4-cycle \( C_4^1 \) has to lie in outside of cycle \( C_{v_2v_7v_6v_5v_4v_3v_2} \), or in inside of cycle \( C_{v_2v_7v_6v_5v_4v_3v_2} \). By symmetry, we may assume 4-cycle \( C_4^1 \) lies in outside of cycle \( C_{v_2v_7v_6v_5v_4v_3v_2} \). Then edges \( v_3v_{11} \) and \( v_6v_{10} \) are crossed. Since \( \nu(D) \leq 2 \), edges \( v_1v_9 \) and \( v_8v_{12} \) are clean. It follows at least one edges of \( v_3v_{11} \) and \( v_6v_{10} \) is crossed at least two times, \( \nu(D) \geq 3 \), a contradiction.
3 Crossing number of $P_4$

In Figure 3.1, we show a drawing of $P_4$ with 6 crossings. Hence, we have:

**Lemma 3.1.** $cr(P_4) \leq 6$.

![Figure 3.1: A good drawing of $P_4$ with 6 crossings](image)

In the rest of this section, we shall prove that the value of $cr(P_4)$ is exactly equal to 6. We rename the vertices of $P_4$ as shown in Figure 3.1.

For $i = 1, 2, 3, 4$, let

$$C_i = C_{P_4}^i, V_i = V_{P_4}^i, E_i = E_{P_4}^i,$$

$$E_{i,j} = E_{P_4}^{i,j}, E'_i = E_{P_4}^{i}, E''_i = E_{P_4}^{i}.$$

For convenience, we abbreviate

$$C_i = C_{P_4}^i, V_i = V_{P_4}^i, E_i = E_{P_4}^i,$$

$$E_{i,j} = E_{P_4}^{i,j}, E'_i = E_{P_4}^{i}, E''_i = E_{P_4}^{i}.$$

Then we have the following important observation.

**Observation 3.1.** For $1 \leq i \neq j \leq 4$, let $uav_au_b$ be $E_{i,j}$, $u_au_b \in V_i$ and the path between $u_a$ and $u_b$ is $P_{u_au_b}$ (or $P_{u_au_b}$) on $C_i$. Then $u_c$ and $u_d$ are connected to different 6-cycles except for $C_i$ and $C_j$ (See Figure 3.1).

Since $\overline{E_i}$ is homeomorphic to $G_{12}$ (See Figure 3.2). By Lemmas 2.1, 2.2 and 2.3, we have

**Lemma 3.2.** For $i = 1, 2, 3, 4$,

1) Let $D$ be an arbitrarily drawing of $\overline{E_i}$, then $\nu(D) \geq 2$.

2) Let $D$ be a drawing of $\overline{E_i}$, where at least one pair of 6-cycles crosses each other, then
3) Let \( D \) be a drawing of \([E_1']\), where any pair of 6-cycles does not cross each other and any two 6-cycles lie in the same side of the third 6-cycle, then \( \nu(D) \geq 3 \).

**Lemma 3.3.** Let \( D \) be a drawing of \( P_4 \), where at least two pairs of 6-cycles cross each other, then \( \nu(D) \geq 6 \).

*Proof.* By contradiction. Suppose \( \nu(D) \leq 5 \). Since each pair of 6-cycles crossing each other will produce at least two crossings, there are at most two pairs of 6-cycles crossing each other. By symmetry, there are two cases:

**Case 1.** \( C_1 \) crosses \( C_2 \) and \( C_3 \). There are two subcases depending on \( C_4 \)'s position.

**Case 1.1.** \( C_4 \) lies in outside of \( C_2 \) and \( C_3 \) (See Figure 3.3(1)). By Lemma 3.2, \( \nu_D([E_1']) \geq 3 \). Then, it follows \( \nu(D) \geq 3 + 4 = 7 \), a contradiction.

**Case 1.2.** \( C_4 \) lies in inside of \( C_3 \) (See Figure 3.3(2) and (3)). By Lemma 3.2, \( \nu_D([E_1']) \geq 2 \). Then, it follows \( \nu(D) \geq 2 + 4 = 6 \), a contradiction.

**Case 2.** \( C_1 \) crosses \( C_2 \), \( C_3 \) crosses \( C_4 \). By symmetry, we need only consider the case that \( C_3 \) and \( C_4 \) lie in outside of \( C_1 \) and \( C_2 \) (See Figure 3.3(4)). By Lemma 3.2, \( \nu_D([E_1']) \geq 3 \). Since \( \nu(D) \leq 5 \), \( C_1 \) does not cross itself, any edge of \( \bigcup_{j=2,3,4} E_{1,j} \) is clean. It follows edges \( v_1v_{13} \) and \( v_4v_{16} \) are clean. Now at least one edge of \( E_{1,4} \) is crossed, it contradicts any edge of \( \bigcup_{j=2,3,4} E_{1,j} \) is clean (See Figure 3.3(5)).

\[ \square \]

**Lemma 3.4.** Let \( D \) be a drawing of \( P_4 \), where just one pair of 6-cycles crosses each other, then \( \nu(D) \geq 6 \).

*Proof.* By contradiction. Suppose \( \nu(D) \leq 5 \). By symmetry, we need only consider the case that \( C_3 \) lie in outside of \( C_1 \) and \( C_2 \) (See Figure 3.4(1)). There are three cases depending on \( C_4 \)'s position:

**Case 1.** \( C_4 \) lies in inside of \( C_3 \) (See Figure 3.4(2)). Then each edge of \( E_{4,1} \) crosses the edges of \( E_3 \) at least one time, each edge of \( E_{4,2} \) crosses the edges of \( E_3 \) at least one time. It follows \( \nu(D) \geq 2 + 2 + 2 = 6 \), a contradiction.
Figure 3.3: Some Drawings of $P_4$, where just two pairs of 6-cycles cross each other

Case 2. $C_4$ lies inside $C_2$ (See Figure 3.4(3), (4)). Then each edge of $E_{4,3}$ crosses the edges of $E_2$ at least one time. By Lemma 3.2, $\nu_D(E'_2) \geq 2$. It follows $\nu(D) \geq 2 + 2 + 2 = 6$, a contradiction.

Case 3. $C_4$ lies in outside $C_1$, $C_2$ and $C_3$ (See Figure 3.4(5)). By Lemma 3.2, $\nu_D(E'_1) \geq 3$. Since $\nu(D) \leq 5$, $C_1$ does not cross itself, any edge of $\bigcup_{j=2,3,4} E_{1,j}$ is clean. It follows edges $v_1v_{13}$ and $v_1v_{16}$ are clean. Now at least one edge of $E_{1,4}$ is crossed, it contradicts any edge of $\bigcup_{j=2,3,4} E_{1,j}$ is clean.

Figure 3.4: Some Drawings of $P_4$, where just one pair of 6-cycles crosses each other

Lemma 3.5. Let $D$ be a drawing of $P_4$, where any pair of 6-cycles does not cross each
other, then \( \nu(D) \geq 6 \).

**Proof.** By contradiction. Suppose \( \nu(D) \leq 5 \).

**Case 1.** \( C_2 \) lies in inside of \( C_1, C_3 \) and \( C_4 \) lie in outside of \( C_1 \).

**Case 1.1.** \( C_4 \) lies in inside of \( C_3 \). Then each edge of \( E_{2,4} \) crosses the edges of \( E_1 \) at least one time, crosses the edges of \( E_3 \) at least one time. Meanwhile, each edge of \( E_{2,3} \) crosses the edges of \( E_1 \) at least one time, and each edge of \( E_{4,1} \) crosses the edges of \( E_3 \) at least one time. It follows \( \nu(D) \geq 4 + 2 + 2 = 8 \) (See Figure 3.5(1)).

**Case 1.2.** \( C_4 \) lies in outside of \( C_3 \). Then each edge of \( E_{2,3} \) crosses the edges of \( E_1 \) at least one time, while each edge of \( E_{2,4} \) crosses the edges of \( E_1 \) at least one time. By Lemma 3.2 \( \nu_D(E_1') \geq 3 \). Then, it follows \( \nu(D) \geq 3 + 4 = 7 \), a contradiction. (See Figure 3.5(2)).

**Case 2.** each \( C_i \) lies in outside of other three \( C_j \) \( (1 \leq j \leq 4, j \neq i) \).

**Case 2.1.** \( C_3 \) does not cross itself. Since \( \nu(D) \leq 5 \) and \( \nu_D(E_3') \geq 3 \), \( \nu_D(E_3') + \nu_D(E_3', E_3) \leq 2 \). It follows that \( \nu_D(E_1, 3) + \nu_D(E_3, E_1, 3) + \nu_D(E_{2,3}) + \nu_D(E_3, E_{2,3}) + \nu_D(E_{3,4}) + \nu_D(E_3, E_{3,4}) \leq 2 \). Without loss of generality, we may assume \( \nu_D(E_1, 3) + \nu_D(E_3, E_1, 3) = 0 \) (See Figure 3.5(3)). Then at least one edge of \( E_{3,4} \) crosses one edge of \( E_1 \cup E_3 \cup E_{1,3} \). After that, at least one edge of \( E_{3,4} \) crosses one edge of \( E_1 \cup E_3 \cup E_{1,3} \) and at least one edge of \( E_{3,4} \) crosses one edge of \( E_2 \cup E_3 \cup E_{2,3} \). It follows \( \nu_D(E_1') + \nu_D(E_3', E_3') \geq 3 \), which contradicts \( \nu_D(E_1') + \nu_D(E_3', E_3') \leq 2 \).

**Case 2.2.** \( C_3 \) crosses itself. Since \( \nu(D) \leq 5 \) and \( \nu_D(E_3') \geq 3 \), \( \nu_D(E_3') + \nu_D(E_3', E_3) \leq 2 \). It follows that \( \nu_D(E_1, 3) + \nu_D(E_3, E_1, 3) + \nu_D(E_{2,3}) + \nu_D(E_3, E_{2,3}) + \nu_D(E_{3,4}) + \nu_D(E_3, E_{3,4}) \leq 1 \) since \( C_3 \) crosses itself. Without loss of generality, we may assume \( \nu_D(E_1, 3) + \nu_D(E_3, E_1, 3) = \nu_D(E_{2,3}) + \nu_D(E_3, E_{2,3}) = 0 \). If \( C_1 \) does not cross itself, then the two edges of \( E_{1,4} \) cross at least three times in total (See Figure 3.5(4)). If \( C_1 \) crosses itself, then each edge of \( E_{1,4} \) crosses the edges of \( E_2 \cup E_3 \cup E_{2,3} \) at least one time (See Figure 3.5(5)). By Lemma 3.2 \( \nu_D(E_1') \geq 3 \). It follows \( \nu(D) \geq 3 + 3 = 6 \), a contradiction.

![Figure 3.5: Some Drawings of P₄, where any pair of 6-cycles does not cross each other](image)

By Lemmas 3.1, 3.3 - 3.5 we have
Theorem 3.1. $cr(P_4) = 6$.

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