Nonlocality tests of Bell Inequalities and of Hardy’s ”ladder theorem” without ”supplementary assumptions”

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Abstract

We have experimentally tested the non local properties of the states generated by a high brilliance source of entanglement which virtually allows the direct measurement of the full set of photon pairs created by the basic QED process implied by the parametric quantum scattering. Standard Bell measurements and Bell’s inequality violation test have been realized over the entire cone of emission of the degenerate pairs. By the same source we have verified the Hardy’s ladder theory up to the 20th step and the contradiction between the standard quantum theory and the local realism has been tested for 41% of entangled pairs.

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Entanglement, "the characteristic trait of quantum mechanics" according to Erwin Schroedinger, is playing an increasing role in nowadays physics [1]. Since the EPR discovery in 1935 followed by a many decades long endeavour ending with the emergence of the Bell’s inequalities and with the experiment by Alain Aspect, entanglement is considered as the irrevocable signature of quantum nonlocality, i.e. the scientific paradigm recognized as the fundamental cornerstone of our yet uncertain understanding of the Universe [2–4]. In recent years the violation of these inequalities has been successfully tested many times by optical experiments, mostly involving polarization entangled photons generated by Spontaneous Parametric Down Conversion (SPDC) in a nonlinear (NL) crystal. In addition, an other nonlocality test not involving inequalities was proposed years ago by Lucien Hardy’s [5] and soon realized experimentally by a SPDC process [6].

The present work reports yet another nonlocality test both of the standard Bell configuration and of the Hardy’s no-inequality scheme. The novelty of this experiment consists of the peculiar spatial properties of the output $k$–vector distribution generated by the SPDC source implied by the present scheme [7,8]. As shown in the paper, this source allows, at least in principle, the coupling to the output detectors of the full set of optical modes carrying the particle pairs involved in the EPR measurement. In other words, all entangled pairs created over the entire set of wavevectors allowed by phase matching can virtually be detected. Since then the detected emission process is entirely "quantum", i.e. not affected by any previous "classical" manipulation, such as wavelength ($\lambda$) of wavevector ($k$) filtering, e.g. by filters and/or limiting pinholes, the new scheme allows in principle the realization of the necessary premises underlying the original formulation of the "EPR Paradox" [9]. Indeed all nonlocality tests performed so far were affected by a quantum-efficiency ($QE$) "loophole" expressing the overall lack of detection of all couples of entangled photons generated by the EPR source [10]. This effect is ascribable either to the limited $QE$ of the detectors ("detection $QE$": $dQE$) [11] and to the loss of the pairs that, created by the underlying QED quantum process, could not reach the detectors for geometrical reasons ("collection $QE$": $cQE$). Note that, while $dQE$ can be of the order of $10^{-1}$ for normal detectors in the visible...
range, the \(e\)\(QE\) contribution has been always typically less than \(10^{-5}\). As shown later, this filtering truncation of the distribution of the emitted entangled pairs necessarily results in a mixed character of the detected state, at variance with the original EPR assumptions [9].

In principle our scheme relieves the need for the \textit{fair sampling} and \textit{no enhancement} "supplementary assumptions" in the analysis of the test outcomes, a condition long advocated by John Bell himself and never realized in practice [3,10–13].

A detailed description of the high brilliance source of entanglement was already given in previous papers [7,8]. Pairs of horizontally (\(H\)) polarized SPDC photons are emitted at wavelength \(\lambda\) over the surface of the phase matching cone of a thin (0.5\(mm\)) type I BBO crystal which is excited by a cw vertically (\(V\)) polarized \(Ar^+\) laser beam (\(\lambda_p = \lambda/2\)) (see Fig. 1). A spherical mirror \(M\) with radius \(R\), placed at a distance \(d = R\) from the crystal, reflects back both the pump and the photons. By a zero-order \(\lambda/4\) waveplate (wp) placed between \(M\) and the BBO the \(H \rightarrow V\) transformation for the \(\lambda\) photons polarization is performed while the pump beam is left in its original polarization state. This excites an identical SPDC process over a new radiation cone which is spatially and temporally indistinguishable from the previous one. The state of the overall radiation is then expressed by the entangled state:

\[
|\Phi\rangle = \frac{1}{\sqrt{2}} (|HH\rangle + e^{i\phi}|VV\rangle)
\]

(1)

with phase \(\phi\) (\(0 \leq \phi \leq \pi\)) reliably controlled by micrometric displacements of \(M\). By a positive lens the overall conical emission distribution is transformed into a cylindrical one whose transverse circular section, spatially selected by an annular mask, identifies the Entanglement-ring (E-ring) (Fig. 1). After division of the ring along a vertical axis, the two resulting equal portions are detected at sites \(A\) and \(B\) within a bandwidth \(\Delta\lambda = 6nm\). More than \(4 \times 10^3\) sec\(^{-1}\) coincidences are measured at a pump power \(P \simeq 100mW\) over the entire E-ring.

We may analyze the structural characteristics of the quantum state of any photon pair generated by our source by accounting first for the excited electromagnetic modes which, in our case are grouped in correlated pairs by the 3–wave SPDC interaction. Assume that
each SPDC $k$-cone is represented by a linear superposition of correlated pairs of e.m. modes $(k_1, k_2)$. Since only one pair of photons is detected at the output of the source, each mode corresponds to a Fock 2-mode product-state that can be either $|0, 0\rangle$ or $|1_H, 1_H\rangle$ or $|1_V, 1_V\rangle$. Accordingly, we can express the overall entangled-state by the quantum superposition:

$$|\Phi\rangle = \int dk_1dk_2 \left( |1_H, 1_H\rangle_{k_1k_2} |0_V, 0_V\rangle_{k_1k_2} + e^{i\phi} |0_H, 0_H\rangle_{k_1k_2} |1_V, 1_V\rangle_{k_1k_2} \right) \bigotimes_{(k'|k_2')\neq(k_1k_2)} |0_H, 0_H\rangle_{k_1'k_2'} |0_V, 0_V\rangle_{k_1'k_2'} \quad (2)$$

This state should indeed express the exact form of the single photon-pair output state of the source if the full set of mode pairs could be coupled to the detectors $\mathcal{A}$ and $\mathcal{B}$. Indeed, it is not difficult to conceive an ideal experiment (an approximate one is in fact in progress in our laboratory) by which the full set of modes at any wavelength, either degenerate or non-degenerate can be coupled to the detectors without any geometrical or frequency constraint, i.e. without any spatial or $\lambda$-filtering. In practice, in this case limitations for an overall full particle detection should come from the limited $\lambda$-extension of the photocathode $dQE$'s and of the performance of the optical components (mirrors, lenses etc.). Nevertheless, this should not affect in principle the structural character of the output entangled-state. As said this condition gets rid of the fair sampling and no enhancement "supplementary assumptions" in the analysis of the test outcomes [3,10].

Note that in any typical SPDC-based experiment the set of mode pairs coupled to the detectors are drastically reduced by the use of very narrow spatial-filtering pinholes [14,15] in order to realize the photodetection over a single pair of correlated $k$-vectors belonging to the distribution appearing in Eq.(2). However this operation cannot be realized but within a mode uncertainty $\Delta k$ because of the inescapable effect of diffraction. In these conditions there is a definite probability that only one photon in a pair passes through the spatial filter while the other one is intercepted. A similar effect can be ascribed to any frequency filtering operation as well. As a consequence, this drastic truncation implies necessarily a mixed character of the output entangled state [16]. These considerations lead to the quasi-purity of the generated output state. The state purity condition may be simply analyzed as follows.
The well known unitary character of the SPDC quantum operator $\hat{S}$ assures that the purity of the input state implies also the purity of the output state: $|\Phi\rangle_{out} = \hat{S}|\Phi\rangle_{in}$ [17]. Adopting the common hypothesis of a undepleted "classical" pump beam, the input pure state is expressed by the overall vacuum-state character of the full set of input modes acted upon by the SPDC process: $|\Phi\rangle_{in} \equiv |\text{vac}\rangle$. Within the single-pair emission approximation, the output pure state is found: $|\Phi\rangle_{out} \simeq |\Phi\rangle + |\text{vac}\rangle$, viz. consists of the sum of the state given by Eq.(2) and of the vacuum-state expressing the non realization of the QED scattering process. As a consequence, $|\Phi\rangle$ given by Eq.(2) is not, strictu sensu, a pure state but one out of a two components mixture. However, in the common case of a conditional experiment where the overall registration system is activated by a trigger pulse elicited by the source itself, the output state $|\Phi\rangle$ may be considered a "post-selected" pure state. This last condition is often referred to as expressing the "conditional purity" of the output state.

The experimental interference pattern, with coincidence visibility $V \geq 94\%$, shown in Fig. 2a gives a strong indication of the entangled nature of the Bell state $|\Phi^{-}\rangle$, ($\phi = \pi$) over the entire emission cone at $\lambda = 2\lambda_p$. In this condition it is possible to evaluate that $cQ\!E$ is enhanced of a factor $\geq 70$ with respect the standard pinhole configuration. The dotted line corresponds to the limit boundary between the quantum and the classical regimes [18] while the theoretical continuous curve expresses the ideal interferometric pattern with maximum visibility: $V = 1$. By performing the standard Bell-inequality test we have evaluated the non locality parameter $S$ [3]. The measured value $S = 2.5564 \pm 0.0026$ [7], obtained by integrating the data over 180s, corresponds to a violation as large as 213 standard deviations respect to the limit value $S = 2$ implied by local realistic theories.

Hardy's theorem represents an alternative proof of nonlocality [5,6]. It is obtained in the case of non maximally entangled states of two spin 1/2 particles:

$$|\Phi\rangle = \alpha |H, H\rangle - \beta |V, V\rangle, \quad (0 \leq \alpha \leq \beta; \ \alpha^2 + \beta^2 = 1).$$

We have realized these states by inserting a zero-order $\lambda_p/4$ wp between $M$ and the BBO, intercepting only the UV beam (Fig. 1). In our system, by rotating the UV wp by an angle
\( \theta_p \), the back-reflected UV pump beam experiences a polarization rotation of \( 2\theta_p \) respect to the optical axes of the NL crystal slab. As a consequence, the emission efficiency of the \(|H, H\rangle\) cone is decreased by a coefficient \( \propto \cos^2 2\theta_p \). By adjusting \( \theta_p \) in the range \( 0 - \pi/4 \), the degree of entanglement \( \gamma = \alpha/\beta \) can be continuously tuned between 0 and 1.

A full presentation of Hardy’s theorem can be found in Ref [6]. For two photons in the state (3), the polarization measurements performed along \( K + 1 \) possible directions at sites \( \mathcal{A} \) and \( \mathcal{B} \) of Fig. 1 give a corresponding set of propositions \( A_k, k = 0, \ldots, K \) which imply \( A_0 \iff A_1 \iff \ldots \iff A_K \), with the further condition \( A_0 \not\iff A_K \). It can be demonstrated that the fraction of pairs \( P_K \) with non local properties increases with \( K \) and is a function of the entanglement degree \( \gamma \). For each value of \( K \), a proper \( \gamma \) exists which maximizes \( P_K \).

Hardy’s ladder proof is purely logical and doesn’t involve inequalities. However inequalities are necessary as a quantitative test in a real experiment in order to avoid the conceptual problems associated to the realization of a nullum experiment [6]. A proper inequality, violated by quantum theory, can be derived by combining Hardy’s theorem with Clauser-Horne inequality [2,3]. It consists of the measurement of \( 2K + 2 \) joint detection probabilities \( P(\theta_A, \theta_B) \), where \( \theta_A, \theta_B \) are the angular settings of polarizers on sites \( \mathcal{A} \) and \( \mathcal{B} \) in Fig.1:

\[
P(\theta_K, \theta_K) \leq P(\theta_0, \theta_0) + \sum_{k=1}^{K} \left[ P(\theta_k, \theta_{k-1}^\perp) + P(\theta_{k-1}^\perp, \theta_k) \right] = \mathcal{P}
\]

where \( \theta_k = (-1)^k \arctan(\gamma^{k+\frac{1}{2}}) \), \( \theta_k^\perp = \theta_k + \frac{\pi}{2} \), with \( k = 0, \ldots, K \), and \( P(\theta_K, \theta_K) = P_K \).

The experimental observation of the inequality violation becomes more and more difficult as \( K \) increases because of an eventually unperfect definition of the state and of the experimental uncertainties associated to all the \( 2K + 2 \) measurements. The experiments realized so far were performed only for low values of \( K \), in particular for \( K \leq 3 \) [6,19,20]. The above described source possesses unique characteristics for this experiment. In fact, it allows the direct generation of non maximally entangled states without postselection. Moreover, the particular configuration of ”single arm” interferometer guarantees a very high phase stability for long periods (\( > 1hr \)). Finally, the high brilliance character of the source allows to accumulate large sets of statistical data in a short measurement time \( \Delta T \) also with a
relatively low UV pump power. By taking advantage from all these properties of our source, we could successfully test Hardy’s ladder proof for large values of $K$.

The experiment, realized for $K = 4, 5, 10, 20$, has given the following violations of the inequality (4):

- $K = 4$ ($\Delta T = 60$ sec): $P_4 = 0.2586 \pm 0.0041$; $P = 0.1213 \pm 0.0022$. Inequality violated for $30 \sigma$.
- $K = 5$ ($\Delta T = 60$ sec): $P_5 = 0.3152 \pm 0.0050$; $P = 0.1184 \pm 0.0022$. Inequality violated for $37 \sigma$.
- $K = 10$ ($\Delta T = 120$ sec): $P_{10} = 0.3402 \pm 0.0045$; $P = 0.2288 \pm 0.0015$. Inequality violated for $26 \sigma$.
- $K = 20$ ($\Delta T = 180$ sec): $P_{20} = 0.4132 \pm 0.0053$; $P = 0.2439 \pm 0.0016$. Inequality violated for $21 \sigma$.

The probabilities of each outcome for all the 42 polarization settings of $K = 20$ are reported in Table 1. These have been obtained by normalizing the coincidence measurements to the sum of coincidence rates measured in the basis $|HH\rangle$ and $|VV\rangle$.

The count rates for each value of $P_K$ are plotted in Fig. 2b as a function of $K$. We report for comparison the results obtained in the experiment of ref [6]. The theoretical curve shown in the same Figure indicates a very slow convergence to the asymptotic value $P_K = 0.5$.

Finally, additional measurements of $P_K$ are plotted as a function of $\gamma$ for $K = 4, 5, 10, 20$ in Fig. 3. The angle $\theta_K$ has been calculated for each value of $\gamma$ by using the above given expression. The agreement with the theory appears very good.

In summary, we have presented two different experimental tests of quantum nonlocality realized by a high brilliance source of polarization entanglement. The value of the collection Quantum Efficiency ($cQE$) realized by the present system is about 2 order of magnitude larger than for all previous experiments. We have obtained in these conditions a $213 \sigma$ Bell inequality violation. Furthermore, in virtue of the very large overall efficiency of the source, within the framework of the Hardy’s ladder theory, a contradiction between standard quantum theory and local realism has been attained by for a fraction as large as 41% of the
entangled photon pairs and as many as 20 steps of the ladder have been realized.

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The condition $cQE = 1$ was advocated by J. Bell (A. Aspect, private comm. to FDM; J. Bell, *Atomic-cascade photons and quantum mechanical nonlocality*, Comments on *Atomic and Molecular Physics*, 9, 121(1980) reprinted in *Speakable and Unspeakable in Quantum Mechanics* (Cambridge U. Press, 1987)). There the Aspect’s experiment, in which no momentum correlation between entangled particles exists, was considered. In the case of SPDC tests, in which the spatial correlation is imposed by phase matching condition, the diffraction effects are determined by the finite thickness of the NL crystal and by the finite pump beam transverse profile.

Here we deal with the supplementary assumptions which are necessary because of the limited value of $cQE$.

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**Figure Captions**

Fig. 1- Layout of the high brilliance source of polarization entanglement. The dimension of
the annular mask are $D = 1.5\text{cm}$, $\delta = .07\text{cm}$.

Fig. 2- (a) Measurement of the polarization entanglement for the state $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|H, H\rangle - |V, V\rangle)$ obtained by varying the angle $\theta_A$ on site $A$ in the range $(45^\circ - 135^\circ)$, having kept fixed the angle $\theta_B = 45^\circ$ on site $B$. (b) Plot of $P_K$ against $K$. Black circles: experimental results for $K = 4, 5, 10, 20$ (error bars are lower than the dimension of the corresponding experimental points). White circles: experimental results obtained in ref. [6].

Fig. 3- Plots of $P_4, P_5, P_{10}, P_{20}$ as a function of $\gamma$. The solid curves represent the theoretical predictions. The error bars are lower than the dimension of the corresponding experimental points.

Tab. 1- Experimental joint probabilities for $K = 20$. 

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For $K = 4$, $P_4$, $P_5$, $P_{10}$, and $P_{20}$ as functions of $\gamma$. Each graph shows the probability distribution for different values of $K$. The $y$-axis represents the probability, and the $x$-axis represents $\gamma$. The graphs illustrate how the probability varies with $\gamma$ for different $K$ values.
