ABELIAN SUBGROUPS OF TWO-DIMENSIONAL ARTIN GROUPS

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Abstract. We classify abelian subgroups of two-dimensional Artin groups.

1. Introduction

Let $S$ be a finite set and for all $s \neq t \in S$ let $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$. The associated Artin group $A_S$ is given by generators and relations:

$$A_S = \langle S \mid s_{\cdot}^{m_{st}} \cdots = t_{\cdot}^{m_{ts}} \rangle.$$

Assume that $A_S$ is two-dimensional, that is, for all $s, t, r \in S$, we have

$$\frac{1}{m_{st}} + \frac{1}{m_{tr}} + \frac{1}{m_{sr}} \leq 1.$$

We say that $A_S$ is of hyperbolic type if the inequality above is strict for all $s, t, r \in S$.

In [MP19, Thm D] we classified explicitly all virtually abelian subgroups of $A_S$ of hyperbolic type. In this article we extend this, using different techniques, to all two-dimensional $A_S$. Charney and Davis [CD95, Thm B] showed that two-dimensional Artin groups satisfy the $K(\pi, 1)$-conjecture, in particular they are torsion-free and of cohomological dimension 2. Thus all noncyclic virtually abelian subgroups of $A_S$ are virtually $\mathbb{Z}^2$. The first step of our classification involves the modified Deligne complex $\Phi$ (see Section 2) introduced by Charney and Davis [CD95] and generalising a construction of Deligne for Artin groups of spherical type [Del72].

Theorem A. Let $A_S$ be a two-dimensional Artin group, and let $H$ be a subgroup of $A_S$ that is virtually $\mathbb{Z}^2$. Then:

(i) $H$ is contained in the stabiliser of a vertex of $\Phi$, or
(ii) $H$ is contained in the stabiliser of a standard tree of $\Phi$, or
(iii) $H$ acts properly on a Euclidean plane isometrically embedded in $\Phi$.

The stabilisers of the vertices of $\Phi$, appearing in (i), are cyclic or conjugates of the dihedral Artin groups $A_{st}$ (see Section 2), which are virtually $\mathbb{Z} \times F$ for a free group $F$, see e.g. [HJP16, Lem 4.3(i)]. The stabilisers of the standard trees of $\Phi$, appearing in (ii), are also $\mathbb{Z} \times F$ [MP19 Lem 4.5] and were described more explicitly in [MP19 Rm 4.6].

In the second step of our classification, we will list all $H \cong \mathbb{Z}^2$ satisfying (iii). Note that the statement might seem daunting, but in fact it arises in a straightforward way from reading off the labels of the Euclidean planes obtained in the course of the proof.

We use the following notation. Let $S$ be an alphabet. If $s \in S$, then $s^*$ denotes the language (i.e. set of words) of form $s^n$ for $n \in \mathbb{Z} - \{0\}$. We treat a letter $s \in S^\pm$
as a language consisting of a single word. If \( \mathcal{L}, \mathcal{L}' \) are languages, then \( \mathcal{L} \mathcal{L}' \) denotes the language of words of the form \( ww' \) where \( w \in \mathcal{L}, w' \in \mathcal{L}' \). If \( \mathcal{L} \) is a language, then \( \mathcal{L}^n \) denotes the union of the languages \( \mathcal{L}^n \) for \( n \geq 1 \).

**Theorem B.** Let \( A_S \) be a two-dimensional Artin group. Suppose that \( \mathbb{Z}^2 \subset A_S \) acts properly on a Euclidean plane isometrically embedded in \( \Phi \). Then \( \mathbb{Z}^2 \) is conjugated into:

(a) \( \langle w, w' \rangle \), where \( w \in A_T, w' \in A_{T'} \) and \( m_{tt'} = 2 \) for all \( t \in T, t' \in T' \), for some disjoint \( T, T' \subset S \);

or \( \mathbb{Z}^2 \) is conjugated into one of the following, where \( s, t, r \in S \):

(b) \( \langle str str, w \rangle \), where \( w \in (t^s str)^* \) and \( m_{st} = m_{tr} = m_{sr} = 3 \).

(c) \( \langle str, w \rangle \), where \( w \in (r^s t^s t^{-1} s) t^s \) and \( m_{st} = m_{tr} = 4, m_{sr} = 2 \).

(d) \( \langle str str, w \rangle \), where \( w \in (t^s str)^* \) and \( m_{st} = m_{tr} = 4, m_{sr} = 2 \).

(e) \( \langle st str str, w \rangle \), where \( w \in (t^s str)^* \) and \( m_{st} = 6, m_{tr} = 3, m_{sr} = 2 \).

(f) \( \langle ts str, w \rangle \), where \( w \in (s^t str^{-1})^* \) and \( m_{st} = 6, m_{tr} = 3, m_{sr} = 2 \).

It is easy to check directly that the above groups are indeed abelian. Since \( A_S \) is torsion-free \( \text{CD}95 \) Thm B1, the only other subgroups of \( A_S \) that are virtually \( \mathbb{Z}^2 \) are isomorphic to the fundamental group of the Klein bottle. They can be also classified, see Remark 5.8.

In the proof of Theorem B we will describe in detail the Euclidean planes in \( \Phi \) stabilised by \( \mathbb{Z}^2 \subset A_S \). Huang and Osajda established properties of arbitrary quasiflats in the Cayley complex of \( A_S \), and one can find similarities between our results and \( \text{HO17} \; \text{§5.15.2 and Prop 8.3} \).

**Organisation of the article.** In Section 2 we describe the modified Deligne complex \( \Phi \) of Charney and Davis and we prove Theorem A. In Section 3 we prove a lemma on dihedral Artin groups fitting in the framework of \( \text{AS}83 \). In Section 4 we introduce a polarisation method for studying Euclidean planes in \( \Phi \). We finish with the classification of admissible polarisations and the proof of Theorem B in Section 5.

### 2. Modified Deligne complex

Let \( A_S \) be a two-dimensional Artin group. For \( s, t \in S \) satisfying \( m_{st} < \infty \), let \( A_{st} \) be the dihedral Artin group \( A_{S'} \) with \( S' = \{ s, t \} \) and exponent \( m_{st} \). For \( s \in S \), let \( A_s = \mathbb{Z} \).

Let \( K \) be the following simplicial complex. The vertices of \( K \) correspond to subsets \( T \subseteq S \) satisfying \( |T| \leq 2 \) and, in the case where \( |T| = 2 \) with \( T = \{ s, t \} \), satisfying \( m_{st} < \infty \). We call \( T \) the type of its corresponding vertex. Vertices of types \( T, T' \) are connected by an edge of \( K \), if we have \( T \subseteq T' \) or vice versa. Similarly, three vertices span a triangle of \( K \), if they have types \( \emptyset, \{ s \}, \{ s, t \} \) for some \( s, t \in S \).

We give \( K \) the following structure of a simple complex of groups \( K \) (see \( \text{BH}99 \) §II.12 for background). The vertex groups are trivial, \( A_v \), or \( A_{st} \), when the vertex is of type \( \emptyset, \{ s \}, \{ s, t \} \), respectively. For an edge joining a vertex of type \( \{ s \} \) to a vertex of type \( \{ s, t \} \), its edge group is \( A_s \); all other edge groups and all triangle groups are trivial. All inclusion maps are the obvious ones. It follows directly from the definitions that \( A_S \) is the fundamental group of \( K \).

We equip each triangle of \( K \) with the Moussong metric of an Euclidean triangle of angles \( \frac{\pi}{2m_{st}}, \frac{\pi}{2m_{st}}, \frac{(m_{st}-1)\pi}{2m_{st}} \) at the vertices of types \( \{ s, t \}, \{ s \}, \emptyset \), respectively. As explained in \( \text{MP19} \) §3, the local developments of \( K \) are \( \text{CAT}(0) \) and hence \( K \) is strictly developable and its development \( \Phi \) exists and is \( \text{CAT}(0) \). See \( \text{CD}95 \) for a detailed proof. We call \( \Phi \) with the Moussong metric the modified Deligne complex. In particular all \( A_s \) and \( A_{st} \) with \( m_{st} < \infty \) map injectively into \( A_S \) (which follows
also from [vdL83 Thm 4.13]). Vertices of $\Phi$ inherit types from the types of the
vertices of $K$.

Let $r \in S$ and let $T$ be the fixed-point set in $\Phi$ of $r$. Note that since $A_S$ acts on $\Phi$
without inversions, $T$ is a subcomplex of $\Phi$. Since the stabilisers of the triangles
of $\Phi$ are trivial, we have that $T$ is a graph. Since $\Phi$ is CAT(0), $T$ is convex and
thus it is a tree. Thus we call a standard tree the fixed-point set in $\Phi$ of a conjugate
of a generator $r \in S$ of $A_S$.

Remark 2.1 ([MP19 Rm 4.4]). Each edge of $\Phi$ belongs to at most one standard
tree.

Proof of Theorem A. Let $\Gamma \subset H$ be a finite index normal subgroup isomorphic
to $\mathbb{Z}^2$. By [Bri99], $\Gamma$ acts on $\Phi$ by semi-simple isometries. Let $\operatorname{Min}(\Gamma) = \bigcap_{\gamma \in \Gamma} \operatorname{Min}(\gamma)$, where $\operatorname{Min}(\gamma)$ is the Minset of $\gamma$ in $\Phi$. By a variant of the Flat Torus Theo-
rem not requiring properness [BH99 Thm II.7.20(1)], $\operatorname{Min}(\Gamma)$ is nonempty. By
[BH99 Thm II.7.20(4)] we have that $H$ stabilises $\operatorname{Min}(\Gamma)$.

Suppose first that each element of $\Gamma$ fixes a point of $\Phi$. Then $\Gamma$ acts trivially on $\operatorname{Min}(\Gamma)$. By the fixed-point theorem [BH99 Thm II.2.8(1)] the finite group $H/\Gamma$ fixes a point of $\operatorname{Min}(\Gamma)$, and since the action is without inversions, we can take this point to be a vertex as required in (i).

Secondly, suppose that $\Gamma$ has both an element $\gamma$ that fixes a point of $\Phi$ and an
element that is loxodromic. Then $\operatorname{Min}(\Gamma)$ is not a single point, so it contains an
edge $e$. Since $\operatorname{Min}(\Gamma) \subset \operatorname{Fix}(\gamma)$, we have that $\gamma$ fixes $e$. Thus $\gamma$ is a conjugate of an
element of $S$ and so $\operatorname{Min}(\Gamma)$ is contained in a standard tree $T$. For any $h \in H$ we
have that the intersection $h(T) \cap T$ contains $\operatorname{Min}(\Gamma) \supset e$ and thus by Remark 2.1
we have $h \in \operatorname{Stab}(T)$, as required in (ii).

Finally, suppose that all elements of $\Gamma$ are loxodromic. By [BH99 Thm II.7.20(1,4)]
we have $\operatorname{Min}(\Gamma) = Y \times \mathbb{R}^n$ with $H$ preserving the product structure and $\Gamma$ acting
trivially on $Y$. As before $H/\Gamma$ fixes a point of $Y$ and so $H$ stabilises $\mathbb{R}^n$ isomet-
rically embedded in $\Phi$. By [BH99 Thm II.7.20(2)], we have $n \leq 2$, but since $H$
acts by simplicial isometries, we have $n = 2$ and the action is proper, as required
in (iii). □

3. Girth lemma

Lemma 3.1. Let $S = \{s, t\}$ with $m_{st} \geq 3$. A word with $2m$ syllables (i.e. of
form $s^{i_1}t^{j_1} \ldots s^{i_m}t^{j_m}$ with all $i_k, j_k \in \mathbb{Z} - \{0\}$) is trivial in $A_S$ if and only if up to
interchanging $s$ with $t$, and a cyclic permutation, it is of the form:

\begin{itemize}
  \item $s^k t^{m-1} s^{-k} t^{m-1} \ldots t^{m-1}$ for $m$ odd,
  \item $s^k t^{m-1} s^{-k} t^{m-1} \ldots t^{m-1}$ for $m$ even,
\end{itemize}

where $k \in \mathbb{Z} - \{0\}$.

Proof. The ‘if’ part follows immediately from Figure 1. We prove the ‘only if’ part
by induction on the size of any reduced (van Kampen) diagram $M$ of the word $w$ in
question, where we prove the stronger assertion that, up to interchanging $s$ with $t,$
$M$ is as in Figure 1.
We use the vocabulary from [AS83], where the 2-cells of $M$ are called regions and the interior degree $i(D)$ of a region is the number of interior edges of $\partial D$ (after forgetting vertices of valence 2). For example the two extreme regions in Figure 1 have interior degree 1. A region $D$ is a simple boundary region if $\partial D \cap \partial M$ is nonempty, and $M - \overline{D}$ is connected. For example, the two extreme regions in Figure 1 are simple boundary regions, but the remaining ones are not. A singleton strip is a simple boundary region with $i(D) \leq 1$. A compound strip is a subdiagram $R$ of $M$ consisting of regions $D_1, \ldots, D_n$, with $n \geq 2$, with $D_{n-1} \cap D_k$ a single interior edge of $R$ (after forgetting vertices of valence 2), satisfying $i(D_1) = i(D_n) = 2, i(D_k) = 3$ for $1 < k < n$ and $M - R$ connected.

Let $R$ be a strip of $M$ with boundary labelled by $rb$, where $r$ labels $\partial R \cap \partial M$ and so $w = rw'$. Assume also that $R$ shares no regions with some other strip (such a pair of strips exists by [AS83, Lem 2]). By [AS83, Lem 5], we have that the syllable lengths satisfy $||r|| \geq ||b|| + 2$ and so by [AS83, Lem 6], we have $||r|| \geq m + 1$, hence $||w'|| \leq m + 1$. In fact, since the outside boundary of the other strip has also syllable length $\geq m + 1$, we have $||r|| = m + 1$, and hence $||b|| = m - 1$. Let $M'$ be the diagram with boundary labelled by $b^{-1}w'$ obtained from $M$ by removing $R$. By the induction hypothesis, $M'$ is as in Figure 1. If $R$ is a singleton strip, then there is only one way of gluing $R$ to $M'$ to obtain $||w'|| = 2m$ and it is as in Figure 1. If $R$ is a compound strip, then by the induction hypothesis $R$ is also as in Figure 1. Moreover, since all regions of $R$ share exactly one edge with $M'$, up to interchanging $s$ with $t$, and/or $b$ with $b^{-1}$, we have $b = b^k \cdot s \cdot \cdots \cdot s^l$, where $k \geq 2$.

Since $m > 2$, we have that $b$ cannot be a subword of the boundary word of $M'$, unless $M'$ is a mirror copy of $R$, contradiction.

**Remark 3.2.** Let $S = \{s, t\}$ with $m_{st} = 2$. A word with 4 syllables is trivial in $A_S = \mathbb{Z}^2$ if and only if up to interchanging $s$ with $t$ it is of the form $s^k \cdot t^l \cdot s^{-k} \cdot t^{-l}$, where $k, l \in \mathbb{Z} - \{0\}$.

4. **Polarisation**

**Definition 4.1.** Let $F$ be a Euclidean plane isometrically embedded in $\Phi$. Then for each vertex $v$ in $F$ of type $\{s, t\}$ there are exactly $4m_{st}$ triangles in $F$ incident to $v$. We assemble them into regular $2m_{st}$-gons, and call this complex the tiling of $F$. We say that a cell of this tiling has type $T$ if its barycentre in $\Phi$ has type $T$.

For a Coxeter group $W$, let $\Sigma$ denote its Davis complex, i.e. the complex obtained from the standard Cayley graph by adding $k$-cells corresponding to cosets of finite $(T)$ for $T \subset S$ of size $k$. For example for $W$ the triangle Coxeter group with exponents $\{3, 3, 3\}$, the complex $\Sigma$ is the tiling of the Euclidean plane by regular hexagons.
Lemma 4.2. Let $F$ be a Euclidean plane isometrically embedded in $\Phi$. Then the tiling of $F$ is either the standard square tiling, or the one of the Davis complex $\Sigma$ for $W$, where $W$ is the triangle Coxeter group with exponents $\{3,3,3\}, \{2,4,4\}$ or $\{2,3,6\}$.

Proof. The 2-cells of the tiling are regular polygons with even numbers of sides, hence their angles lie in $[\pi/2, \pi)$. If there is a vertex $v$ of $F$ incident to four 2-cells, then all these 2-cells are squares. Consequently, any vertex of $F$ adjacent to $v$ is incident to at least two squares, and thus to exactly four squares. Then, since the 1-skeleton of $F$ is connected, the tiling of $F$ is the standard square tiling.

If $v$ is incident to three 2-cells, which are $2m, 2m', 2m''$-gons, then since $\frac{1}{m} + \frac{1}{m'} + \frac{1}{m''} = 1$, we have $\{m, m', m''\} = \{3,3,3\}, \{2,4,4\}$ or $\{2,3,6\}$. Moreover, a vertex $u$ of $F$ adjacent to $v$ is incident to two of these three 2-cells, and this implies that the third 2-cell incident to $u$ has the same size as the one incident to $v$. This determines uniquely the tiling of $F$ as the one of $\Sigma$. □

Henceforth, let $\Sigma$ be the Davis complex for $W$, where $W$ is the triangle Coxeter group with exponents $\{3,3,3\}, \{2,4,4\}$ or $\{2,3,6\}$.

Lemma 4.3. Suppose $\Sigma$ is the tiling of a Euclidean plane isometrically embedded in $\Phi$. Then the natural action of $W$ on $\Sigma$ preserves the edge types coming from $\Phi$.

In particular, $W = W_T$ for some $T \subset S$ with $|T| = 3$.

Proof. Chose a vertex $v$ of $\Sigma$, and let $\{s\}, \{t\}, \{r\}$ be the types of edges incident to $v$. Let $u$ be a vertex of $\Sigma$ adjacent to $v$, say along an edge $e$ of type $\{r\}$. Hence the 2-cells in $\Sigma$ incident to $e$ have types $\{s,r\}$ and $\{t,r\}$. Consequently, the types of the remaining two edges incident to $u$ are also $\{s\}$ and $\{t\}$, and in such a way that the reflection of $\Sigma$ interchanging the endpoints of $e$ preserves the types of these edges. This determines uniquely the types of the edges of $\Sigma$, and guarantees that they are preserved by $W$. □

Definition 4.4. A polarisation of $\Sigma$ is a choice of a longest diagonal $l(\sigma)$ in each 2-cell $\sigma$ of $\Sigma$. A polarisation is admissible if every vertex of $\Sigma$ belongs to exactly one $l(\sigma)$.

Definition 4.5. Suppose $\mathbb{Z}^2 \subset A_S$ acts properly and cocompactly on $\Sigma \subset \Phi$. For an edge $e$ of type $\{s\}$ in $\Sigma$, its vertices correspond to elements $g, gs^k \in A_S$ for $k > 0$. We direct $e$ from $g$ to $gs^k$. By Lemma 3.1, the boundary of each 2-cell $\sigma$ is subdivided into two directed paths joining two opposite vertices. The induced polarisation of $\Sigma$ assigns to each $\sigma$ the longest diagonal $l(\sigma)$ joining these two vertices.

Lemma 4.6. An induced polarisation is admissible.

Proof. Step 1. For each vertex $v$ of $\Sigma$, there is at most one $l(\sigma)$ containing $v$.

Indeed, suppose that we have $v \in l(\sigma), l(\tau)$. Without loss of generality suppose that the edge $e = \sigma \cap \tau$ is directed from $v$. Then the other two edges incident to $v$ are also directed from $v$. We will now prove by induction on the distance from $v$ that each edge of $\Sigma$ is oriented from its vertex closer to $v$ to its vertex farther from $v$ in the 1-skeleton $\Sigma^1$ (they cannot be at equal distance since $\Sigma^1$ is bipartite).

For the induction step, suppose we have already proved the induction hypothesis for all edges closer to $v$ than an edge $uu'$, where $u'$ is closer to $v$ than $u$. Let $u''$ be the first vertex on a geodesic from $u'$ to $v$ in $\Sigma^1$. Let $\sigma$ be the 2-cell containing the path $uu''$. By [Ron89] Thms 2.10 and 2.16, $\sigma$ has two opposite vertices $u_0$ closest to $v$ and $u_{\max}$ farthest from $v$. By the induction hypothesis, the edge $u'u''$ is oriented from $u''$ to $u'$, and both edges of $\sigma$ incident to $u_0$ are oriented from $u_0$. [End of Proof]
Thus if the edge $uu'$ was oriented to $u'$ we would have that $u'$ is opposite to $u_0$, so $u' = u_{\text{max}}$, contradiction. This finishes the induction step.

As a consequence, $v$ is the unique vertex of $\Sigma$ with all edges incident to $v$ oriented from $v$. This contradicts the cocompactness of the action of $\mathbb{Z}^2$ on $\Sigma$ and proves Step 1.

**Step 2.** For each $v$ there is at least one $l(\sigma)$ containing $v$.

Among the edges incident to $v$ there are at least two edges directed from $v$ or at least two edges directed to $v$. The 2-cell $\sigma$ containing such two edges satisfies $l(\sigma) \ni v$. □

5. Classification

**Proposition 5.1.** Let $\Sigma$ be the Davis complex for $W_T$ the triangle Coxeter group with exponents $\{3,3,3\}$ with an admissible polarisation $l$. Then there is an edge $e$ such that each hexagon $\gamma$ of $\Sigma$ satisfies

\[\heartsuit:\] the diagonal $l(\gamma)$ has endpoints on edges of $\gamma$ parallel to $e$.

Note that if the conclusion of Proposition 5.1 holds, then the translation $\rho$ mapping one hexagon containing $e$ to the other preserves $l$.

**Remark 5.2.** It is easy to prove the converse, i.e. that if each $l(\gamma)$ has endpoints on edges of $\gamma$ parallel to $e$, and if $l$ is $\rho$-invariant, then $l$ is admissible. This can be used to classify all admissible polarisations, but we will not need it.

To prove Proposition 5.1 we need the following reduction.

**Lemma 5.3.** Let $e$ be an edge and $\rho$ a translation mapping one hexagon containing $e$ to the other. If $\heartsuit$ holds for all hexagons $\gamma$ in one $\rho$-orbit, then it holds for all $\gamma$.

**Proof.** Suppose that $\heartsuit$ holds for all hexagons $\gamma$ in the $\rho$-orbit of a hexagon $\sigma$. Let $\tau$ be a hexagon adjacent to two of them, say to $\sigma$ and $\rho(\sigma)$. Let $v = \sigma \cap \rho(\sigma) \cap \tau$. Since $\heartsuit$ holds for $\gamma = \sigma$ and $\gamma = \rho(\sigma)$, by the admissibility of $l$, $v$ belongs to one of $l(\sigma), l(\rho(\sigma))$. Thus $v \notin l(\tau)$ and hence $\heartsuit$ holds for $\gamma = \tau$. Proceeding inductively, by the connectivity of $\Sigma$, we obtain $\heartsuit$ for all $\gamma$. □

**Proof of Proposition 5.1** Case 1. There are adjacent hexagons $\sigma, \tau$ with non-parallel $l(\sigma), l(\tau)$.

Let $f = \sigma \cap \tau$. Without loss of generality $l(\sigma) \cap f = \emptyset, l(\tau) \cap f \neq \emptyset$. Let $v$ be the vertex of $f$ outside $l(\tau)$. By the admissibility of $l$, $v$ is contained in $l(\sigma')$ for the third hexagon $\sigma'$ incident to $v$. Hence $\heartsuit$ holds for $e = \sigma \cap \sigma'$ and $\gamma = \sigma, \sigma', \tau$ (see Figure 2). Let $\rho$ be the translation mapping $\sigma$ to $\sigma'$. Replacing the pair $\sigma, \tau$ with $\tau, \sigma'$ and repeating inductively the argument shows that $\heartsuit$ holds for $\gamma = p^n(\sigma), p^n(\tau)$ for all $n > 0$ (note that $e$ gets replaced by parallel edges in this procedure).
Furthermore, by the admissibility of $l$, since $l(\rho^{-1}(\tau))$ is disjoint from $l(\sigma)$ and $l(\tau)$, it leaves us only one choice for $l(\rho^{-1}(\tau))$, and it satisfies $\clubsuit$ for $\gamma = \rho^{-1}(\tau)$. Replacing the pair $\sigma, \tau$ with $\rho^{-1}(\tau), \sigma$ and repeating inductively the argument shows that $\clubsuit$ holds for $\gamma = \rho^{-n}(\sigma), \rho^{-n}(\tau)$ for all $n > 0$. It remains to apply Lemma 5.3.

**Case 2.** All the $l(\sigma)$ are parallel.

In this case it suffices to take any edge $e$ intersecting some $l(\sigma)$.

**Proposition 5.4.** Let $\Sigma$ be the Davis complex for $W_T$ the triangle Coxeter group with exponents $\{2, 4, 4\}$ with an admissible polarisation $l$. Then there is an edge $e$ such that each octagon $\gamma$ of $\Sigma$ satisfies

$\heartsuit$: the diagonal $l(\gamma)$ has endpoints on edges of $\gamma$ parallel to $e$.

An edge $e$ of $\Sigma$ lies either in two octagons $\sigma, \sigma'$ or there is a square with two parallel edges $e, e'$ in octagons $\sigma, \sigma'$. The translation of $\Sigma$ mapping $\sigma$ to $\sigma'$ is called an $e$-translation. Note that if the conclusion of Proposition 5.4 holds, then an $e$-translation preserves $l$.

**Lemma 5.5.** Let $e$ be an edge and $\rho$ an $e$-translation. If $\heartsuit$ holds for all octagons $\gamma$ in one $\rho$-orbit, then it holds for all $\gamma$.

**Proof.** Suppose that $\heartsuit$ holds for all octagons $\gamma$ in the $\rho$-orbit of an octagon $\sigma$. We can assume $e \subset \sigma$. Suppose first that $e$ lies in another octagon $\sigma'$. Then let $\tau$ be an octagon outside the $\rho$-orbit of $\sigma$ adjacent to some $\rho^k(\sigma)$, say $\sigma$. Let $\Box, \rho(\Box)$ be the two squares adjacent to both $\sigma$ and $\tau$ (see Figure 3). By the admissibility of $l$, we have that $l(\Box), l(\rho(\Box))$ contain the two vertices of $\sigma \cap \tau$. Consequently, $l(\tau)$ intersects the edge $\tau \cap \rho(\tau)$, and so $\gamma = \tau$ satisfies $\heartsuit$. It is easy to extend this to all the octagons $\gamma$. 

![Figure 2.](image-url)
It remains to consider the case where there is a square with two parallel edges $e, e'$ in octagons $\sigma, \sigma'$. Let $\tau$ be an octagon adjacent to two of them, say to $\sigma$ and $\rho(\tau)$. Let $v = e \cap \tau, x = e' \cap \tau$. Since $\diamondsuit$ holds for $\gamma = \sigma, \sigma'$, each of $v, x$ lies in one of $l(\sigma), l(\square), l(\sigma')$. Thus by the admissibility of $l$, we have $v, x \notin l(\tau)$. Any of the two remaining choices for $l(\tau)$ satisfy $\diamondsuit$ for $\gamma = \tau$. It is again easy to extend this to all the octagons $\gamma$.

Proof of Proposition 5.4. Note that we fall in one of the following two cases.

Case 1. There is an edge $f$ in octagons $\sigma, \tau$ with $l(\sigma) \cap f = \emptyset, l(\tau) \cap f \neq \emptyset$.

Let $v$ be the vertex of $f$ distinct from $u = l(\tau) \cap f$. By the admissibility of $l$, the vertex $v$ is contained in $l(\square)$ for the square $\square$ incident to $v$. Let $x$ be the vertex in $\tau \cap \square$ distinct from $v$, and let $\sigma'$ be the octagon incident to $x$ distinct from $\tau$. By the admissibility of $l$, the vertex $x$ is contained in $l(\sigma')$. Hence $\diamondsuit$ holds for $e = \sigma \cap \square$ and $\gamma = \sigma', \tau$ (see Figure 4). Let $\rho$ be the translation mapping $\sigma$ to $\sigma'$.

Now let $\square$ be the square incident to $u$, and let $z$ be the vertex in $\sigma \cap \square$ distinct from $u$. Note that by the admissibility of $l$, we have $z \in l(\square)$, and consequently $\rho^{-1}(x) \in l(\sigma)$ and $\rho^{-1}(u) \in l(\rho^{-1}(\square))$. Hence $l(\rho^{-1}(\tau))$ cannot contain neither $z$, nor $\rho^{-1}(u)$, nor $\rho^{-1}(x) \in l(\sigma)$. There is only one remaining choice for $l(\rho^{-1}(\tau))$, and it satisfies $\diamondsuit$ for $\gamma = \rho^{-1}(\tau)$.

We can now argue exactly as in the proof of Proposition 5.1 that that $\diamondsuit$ holds for $\gamma = \rho^n(\sigma), \rho^n(\tau)$ for all $n \in \mathbb{Z}$. It then remains to apply Lemma 5.5.

Case 2. For each edge $e$ in octagons $\sigma, \sigma'$ with $l(\sigma) \cap e \neq \emptyset$ we have $l(\sigma') \cap e \neq \emptyset$.

Let $\sigma$ be any octagon and $e$ an edge contained in another octagon $\sigma'$ and intersecting $l(\sigma)$. Let $\rho$ be the translation mapping $\sigma$ to $\sigma'$. One can show inductively
that ♦ holds for octagons $\gamma = \rho^n(\sigma)$ for all $n \in \mathbb{Z}$. It then remains to apply Lemma 5.5. □

Note that for $l$ satisfying ♦ for all octagons $\gamma$, the values of $l$ on octagons determine its values on squares.

**Proposition 5.6.** Let $\Sigma$ be the Davis complex for $W_T$ the triangle Coxeter group with exponents $\{2, 3, 6\}$ with an admissible polarisation $l$. Then there is an edge $e$ such that each 12-gon $\gamma$ of $\Sigma$ satisfies

♥: the diagonal $l(\gamma)$ has endpoints on edges of $\gamma$ parallel to $e$.

Let $e$ be an edge. An $e$-translation is the translation of $\Sigma$ mapping $\sigma$ to $\sigma'$ in of the two following configurations. In the first configuration we have a square with two parallel edges $e, e'$ in 12-gons $\sigma, \sigma'$. In the second configuration we have four parallel edges $e, e', e'', e'''$ such that $e', e''$ lie in a square, $e, e'$ in a hexagon $\phi$ and $e'', e'''$ in another hexagon, and we consider 12-gons $\sigma \supset e, \sigma' \supset e''$. Again, if the conclusion of Proposition 5.6 holds, then an $e$-translation preserves $l$. To see this in the configuration with hexagons it suffices to observe that $l(\phi)$ (and similarly for the other hexagon) is not parallel to $e$: otherwise $l(\phi)$ would intersect $l(\square)$ for $\square$ the square containing $e \cap l(\sigma)$.

**Lemma 5.7.** Let $e$ be an edge and $\rho$ an $e$-translation. If ♥ holds for all 12-gons $\gamma$ in one $\rho$-orbit, then it holds for all $\gamma$.

The proof is easy, it goes along the same lines as the proofs of Lemmas 5.3 and 5.5 and we omit it.

**Proof of Proposition 5.6.** We adopt the convention that if we label the vertices of an edge in a 12-gon $\sigma$ by $v_0v_1$, then all the other vertices of $\sigma$ get cyclically labelled by $v_2 \cdots v_{11}$.

Let $\tau$ be a 12-gon and suppose that $l(\tau)$ contains a vertex $v_1$ of an edge $v_0v_1 \subset \tau$ for a square $\square = v_0v_1v_1u_0$. Let $\sigma$ be the 12-gon containing $u_0u_1$. Then $l(\square) = u_1v_0$ and furthermore $l$ assigns to the hexagon and square containing $u_1u_2, u_2u_3$, respectively, the longest diagonal containing $u_2, u_3$, respectively. Thus the only three remaining options for $l(\sigma)$ are the diagonals $u_0u_6, u_5u_{11}$, and $u_4u_{10}$. Thus we fall in one of the following three cases.

**Case 0.** There is such a $\tau$ with $l(\sigma) = u_4u_{10}$.

Let $\phi$ be the hexagon containing $u_3u_4$. Then $l(\phi)$ is parallel to $u_3u_4$ and there is no admissible choice for $l$ in the square containing $u_4u_5$ (see Figure 5). This is a contradiction.
Case 1. There is such a $\tau$ with $l(\sigma) = u_5 u_{11}$.

It is easy to see that $l$ agrees with Figure 6 on the hexagon containing $v_0 v_{11}$ and the square containing $v_{11} v_{10}$. Thus the only 2-cell $\phi$ with $l(\phi)$ containing $v_{10}$ may be (and is) the hexagon containing $v_{10} v_9$. Consequently the only 2-cell $\blacksquare$ with $l(\blacksquare)$ containing $v_9$ may be (and is) the square containing $v_9 v_8$. Denote by $\sigma'$ the 12-gon adjacent to both $\phi$ and $\blacksquare$ at the vertex $x \neq v_9$. Then $x$ may lie (and lies) only in $l(\sigma')$. Denote by $\rho$ the translation mapping $\sigma$ to $\sigma'$.

![Figure 6](image)

It is also easy to see that the 2-cells surrounding $\sigma$ and $\rho^{-1}(\tau)$ depicted in Figure 7 have $l(\cdot)$ as indicated. This leaves only two choices for $l(\rho^{-1}(\tau))$, where one of them leads to Case 0, and the other satisfies $\heartsuit$.

![Figure 7](image)

Replacing repeatedly $\sigma$ and $\tau$ in the above argument by $\tau$ and $\sigma'$ or by $\rho^{-1}(\tau)$ and $\sigma$ gives that $\heartsuit$ holds for $\gamma = \rho^n(\sigma), \rho^n(\tau)$ for all $n \in \mathbb{Z}$. It remains to apply Lemma 5.7.

Case 2. There is no such $\tau$ as in Case 0 or 1.

It is then easy to see that $e = v_0 v_1$ and $\rho$ mapping $\sigma$ to $\tau$ satisfy the hypothesis of Lemma 5.7.

$$\Box$$

Note that for $l$ satisfying $\heartsuit$ for all 12-gons $\gamma$, the values of $l$ on 12-gons determine its values on squares and hexagons.

We are now ready for the following.
Proof of Theorem B. Let $F$ be a Euclidean plane isometrically embedded in $\Phi$ with a proper (and thus cocompact) action of $\mathbb{Z}^2$. By Lemma 4.2, the tiling of $F$ is either the standard square tiling, or the one of the Davis complex $\Sigma$ for $W$, where $W$ is the triangle Coxeter group with exponents $\{3,3,3\},\{2,4,4\}$ or $\{2,3,6\}$.

First consider the case where the tiling of $F$ is the standard square tiling. We can partition the set of edges into two classes horizontal and vertical of parallel edges. Let $T$ be the set of types of horizontal edges and $T'$ be the set of types of vertical edges. Since for each square the type of its two horizontal (respectively, vertical) edges is the same, we have $m_{tt'} = 2$ for all $t \in T, t' \in T'$. Moreover, by Remark 3.2 if one of the edges is of form $g, gt^k$, then the other is of form $h, ht^k$.

Thus, up to a conjugation, the stabiliser of $F$ in $A_S$ is generated by a horizontal translation $w \in A_T$ and a vertical translation $w' \in A_{T'}$. This brings us to Case (a) in Theorem B.

It remains to consider the case where the tiling of $F$ is the one of $\Sigma$. Consider its induced polarisation $l$ from Definition 4.5. By Lemma 4.6, $l$ is admissible. By Propositions 5.1, 5.4 and 5.6, there is an edge $e$ such that each for each $\gamma$ a maximal size $2$-cell, the diagonal $(\gamma)$ has endpoints on edges of $\gamma$ parallel to $e$, and there is a particular translation $\rho$ in the direction perpendicular to $e$ preserving $l$.

For an edge $f$ of type $r$ in $\Sigma$, its vertices are of form $g, gs^k$ for $k > 0$, directed from $g$ to $gs^k$. If $k > 0$, then we call $f$ $k$-long. By Lemma 3.1 and Remark 3.2 if $f$ is $k$-long, then so is its opposite edge in both of the $2$-cells that contain $f$. Consequently all the edges crossing the bisector of $f$ are $k$-long. Moreover, all such bisectors are parallel, since otherwise the $2$-cell $\sqcup$ where they crossed would have four long edges, so $\sqcup$ would be a square by Lemma 3.3. Analysing $l$ in the $2$-cells adjacent to $\sqcup$ leaves then no admissible choice for $l(\sqcup)$.

Furthermore, by Lemmas 5.1, 5.3, 5.5 and 5.7 if $f$ is a long edge, then we can assume that $f$ is parallel to $e$.

Suppose first that $W_T$ is the triangle Coxeter group with exponents $\{3,3,3\}$ and $T = \{s, t, r\}$. Let $\omega$ be a combinatorial axis for the action of $\rho$ on $\Sigma$. Since none of the edges of $\omega$ are parallel to $e$, by the definition of the induced polarisation we see that they are all directed consistently (see Figure 8).

Thus, up to replacing $F$ by its translate and interchanging $t$ with $r$, the element $strt$ preserves $\omega$, and coincides on it with $\rho^3$. In fact, $\rho^3$ not only preserves the types of edges, but also by Lemma 5.1, their direction and $k$-longness. Thus $strt$ preserves $F$. The second generator of the type preserving translation group of $\Sigma$ in $W_T$ is $tstr$. Note that the path representing it in $\Sigma \subset \Phi$ corresponds to a word in $t^*strt \subset A_T$. That word depends on whether the second edge of the path is long and on the polarisation. Since $\mathbb{Z}^2$ acts cocompactly, there is a power of $tstr$ such that its corresponding path in $\Sigma \subset \Phi$ reads off a word in $(t^*strt)^* \subset A_T$ that is the other generator of the orientation preserving stabiliser of $F$ in $A_S$. This brings us to Case (b) in Theorem B. One similarly obtains the characterisations of orientation preserving stabilisers of $F$ for the two other $W$ (see Figures 9 and 10).
Figure 9.

Figure 10.
Remark 5.8. Analysing the full stabilisers of $F$ in $A_S$ one can easily classify also the subgroups of $A_S$ acting properly on $\Phi$ isomorphic to the fundamental group of the Klein bottle. For example, suppose that the second generator of the $\mathbb{Z}^2$ in Case (b) of Theorem 4 has the form $g = (t^k \text{str})(t^{-k} \text{str})$. Then our $\mathbb{Z}^2$ is generated by $\text{str} \text{str}$ and $g' = g(\text{str} \text{str})^{-1} = t^k \text{sr}^{-k}$. Note that $\text{str}$ normalises our $\mathbb{Z}^2$ with $(\text{str})^{-1}g'(\text{str}) = (g')^{-1}$. Thus $\langle \text{str}, g' \rangle$ is isomorphic to the fundamental group of the Klein bottle. We do not include a full classification, since it is not particularly illuminating.

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