Pair correlation of sequences \( \{a_n\alpha\}_{n\in\mathbb{N}} \) with maximal order of additive energy

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Abstract
We show for sequences \( (a_n)_{n\in\mathbb{N}} \) of distinct positive integers with maximal order of additive energy, that the sequence \( \{a_n\alpha\}_{n\in\mathbb{N}} \) does not have Poissonian pair correlations for any \( \alpha \). This result essentially sharpens a result obtained by J. Bourgain on this topic.

1 Introduction and statement of main results

Definition 1 Let \( \| \cdot \| \) denote the distance to the nearest integer. A sequence \( (x_n)_{n\in\mathbb{N}} \) in \([0,1)\) is said to have (asymptotically) Poissonian pair correlations, if for each \( s > 0 \) the pair correlation function

\[
R_2([-s,s], (x_n), N) := \frac{1}{N} \# \left\{ 1 \leq i \neq j \leq N \big| \|x_i - x_j\| \leq \frac{s}{N} \right\}
\]

tends to \( 2s \) as \( N \to \infty \).

It is known that if a sequence \( (x_n)_{n\in\mathbb{N}} \) has Poissonian pair correlations, then it is uniformly distributed modulo 1, cf., [3, 7, 16]. The converse is not true in general.

The study of Poissonian pair correlations of sequences, especially of sequences of the form \( \{a_n\alpha\}_{n\in\mathbb{N}} \), where \( \alpha \) is irrational, and \( (a_n)_{n\in\mathbb{N}} \) is a sequence of distinct positive integers, is primarily motivated by certain questions in quantum physics, especially in connection with the Berry-Tabor conjecture in quantum mechanics, cf., [1, 12]. The investigation of Poissonian pair correlations was started by Rudnick, Sarnak and Zaharescu, cf., [13, 14, 15], and was continued by many authors in the subsequent, cf., [2] and the references given there.

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A quite general result which connects Poissonian pair correlations of sequences \( \{a_n\alpha\}_n \in \mathbb{N} \) to concepts from additive combinatorics was given in \cite{2}:

For a finite set \( A \) of reals the additive energy \( E(A) \) is defined as

\[
E(A) := \sum_{a+b=c+d} 1,
\]

where the sum is extended over all quadruples \( (a, b, c, d) \in A^4 \). Trivially one has the estimate \( |A|^2 \leq E(A) \leq |A|^3 \), assuming that the elements of \( A \) are distinct.

The additive energy of sequences has been extensively studied in the additive combinatorics literature, cf., \cite{17}. In \cite{2} the following was shown:

**Theorem A (in \cite{2})** Let \( (a_n)_{n \in \mathbb{N}} \) be a sequence of distinct integers, and let \( A_N \) denote the first \( N \) elements of this sequence. If there exists a fixed \( \varepsilon > 0 \) such that

\[
E(A_N) = O(N^{3-\varepsilon}),
\]

then for almost all \( \alpha \) the sequence \( \{a_n\alpha\}_{n \in \mathbb{N}} \) has Poissonian pair correlations.

On the other hand Bourgain showed in \cite{2} the following negative result:

**Theorem B (in \cite{2})** If \( E(A_N) = \Omega(N^3) \), then there exists a subset of \([0, 1]\) of positive measure such that for every \( \alpha \) from this set the pair correlations of \( \{a_n\alpha\}_{n \in \mathbb{N}} \) are not Poissonian.

In \cite{8}, the authors gave a sharper version of the result of Bourgain by showing that the set of exceptional values \( \alpha \) from Theorem B has full measure.

It is the aim of this paper to show the best possible version of a result in this direction, namely:

**Theorem 1** If \( E(A_N) = \Omega(N^3) \), then for every \( \alpha \) the pair correlations of \( \{a_n\alpha\}_{n \in \mathbb{N}} \) are not Poissonian.

In fact, we conjecture that even more is true:

**Conjecture 1** If for almost all \( \alpha \) the pair correlations of \( \{a_n\alpha\}_{n \in \mathbb{N}} \) are not Poissonian, then the pair correlations of this sequence are not Poissonian for any \( \alpha \).

In \cite{18} A. Walker proved for \( (a_n) = (p_n) \) the sequence of primes that for almost all \( \alpha \) the pair correlations of \( \{p_n\alpha\}_{n \in \mathbb{N}} \) are not Poissonian. Our conjecture would imply that there is no \( \alpha \) such that \( \{p_n\alpha\}_{n \in \mathbb{N}} \) is Poissonian.

To be able to prove our result we need an alternative classification of integer sequences \( (a_n)_{n \in \mathbb{N}} \) with \( E(A_N) = \Omega(N^3) \):

For \( v \in \mathbb{Z} \) let \( A_N(v) \) denote the cardinality of the set

\[
\left\{(x, y) \in \{1, \ldots, N\}^2, x \neq y : a_x - a_y = v \right\}.
\]
Then
\[
E(A_N) = \Omega(N^3) \tag{1}
\]
is equivalent to
\[
\sum_{v \in \mathbb{Z}} A_N^2(v) = \Omega(N^3), \tag{2}
\]
which implies that there is a \(\kappa > 0\) and positive integers \(N_1 < N_2 < N_3 < \ldots\) such that
\[
\sum_{v \in \mathbb{Z}} A_{N_i}^2(v) \geq \kappa N_i^3, \quad i = 1, 2, \ldots. \tag{3}
\]

It will turn out that sequences \((a_n)_{n \in \mathbb{N}}\) satisfying (11) have a strong linear substructure. From (3) we can deduce by the Balog–Szemerédi–Gowers-Theorem (see [4, 6]) that there exist constants \(c, C > 0\) depending only on \(\kappa\) such that for all \(i = 1, 2, 3, \ldots\) there is a subset \(A_0^{(i)} \subset (a_n)_{1 \leq n \leq N_i}\) such that
\[
|A_0^{(i)}| \geq c N_i \quad \text{and} \quad |A_0^{(i)} + A_0^{(i)}| \leq C |A_0^{(i)}| \leq C N_i.
\]
The converse is also true: If for all \(i\) for a set \(A_0^{(i)}\) with \(A_0^{(i)} \subset (a_n)_{1 \leq n \leq N_i}\) with \(|A_0^{(i)}| \geq c N_i\) we have \(|A_0^{(i)} + A_0^{(i)}| \leq C |A_0^{(i)}|\), then
\[
\sum_{v \in \mathbb{Z}} A_{N_i}^2(v) \geq \frac{1}{C} |A_0^{(i)}|^3 \geq \frac{c^3}{C} N_i^3
\]
and consequently \(\sum_{v \in \mathbb{Z}} A_{N_i}^2(v) = \Omega(N^3)\) (this an elementary fact, see for example Lemma 1 (iii) in (11)).

Consider now a subset \(A_0^{(i)}\) of \((a_n)_{1 \leq n \leq N_i}\) with
\[
|A_0^{(i)}| \geq c N_i \quad \text{and} \quad |A_0^{(i)} + A_0^{(i)}| \leq C |A_0^{(i)}|.
\]

By the theorem of Freiman (see [4]) there exist constants \(d\) and \(K\) depending only on \(c\) and \(C\), i.e., depending only on \(\kappa\) in our setting, such that there exists a \(d\)-dimensional arithmetic progression \(P_i\) of size at most \(K N_i\) such that \(A_0^{(i)} \subset P_i\). This means that \(P_i\) is a set of the form
\[
P_i := \left\{ h_1 + \sum_{j=1}^{d} r_j k_j^{(i)} \left| 0 \leq r_j < s_j^{(i)} \right. \right\}, \tag{4}
\]
with \(h_1, k_1^{(i)}, \ldots, k_d^{(i)}, s_1^{(i)}, \ldots, s_d^{(i)} \in \mathbb{Z}\) and such that \(s_1^{(i)} s_2^{(i)} \ldots s_d^{(i)} \leq K N_i\).

In the other direction again it is easy to see that for any such set \(A_0^{(i)}\)
\[
|A_0^{(i)} + A_0^{(i)}| \leq 2^d K N_i.
\]

Based on these observations we make the following definition:
Definition 2 Let \((a_n)_{n \in \mathbb{N}}\) be a strictly increasing sequence of positive integers. We call this sequence quasi-arithmetic of degree \(d\), where \(d\) is a positive integer, if there exist constants \(C, K > 0\) and a strictly increasing sequence \((N_i)_{i \geq 1}\) of positive integers such that for all \(i \geq 1\) there is a subset \(A^{(i)} \subset (an)_{1 \leq n \leq N_i}\) with \(|A^{(i)}| \geq CN_i\) such that \(A^{(i)}\) is contained in a \(d\)-dimensional arithmetic progression \(P^{(i)}\) of size at most \(KN_i\).

The above considerations show:

Proposition 1 For a strictly increasing sequence \((a_n)_{n \in \mathbb{N}}\) of positive integers we have \(E(A_N) = \Omega(N^3)\) if and only if \((a_n)_{n \in \mathbb{N}}\) is quasi-arithmetic of some degree \(d\).

Hence, our Theorem 1 stated above is equivalent to:

Proposition 2 If \((a_n)_{n \in \mathbb{N}}\) is quasi-arithmetic of degree \(d\), then there is no \(\alpha\) such that the pair correlations of \((\{a_n \alpha\})_{n \in \mathbb{N}}\) are Poissonian.

The result was already proven by the first author for \(d = 1\) in a previous work, see [9]. This case can also be recovered by Theorem 1 in [10].

2 Proof of Theorem 1

As noted above it is sufficient to prove Proposition 2. Let now \((a_n)_{n \in \mathbb{N}}\) be quasi-arithmetic of degree \(d\). That means (see Definition 2): There exists a strictly increasing subsequence \((N_i)_{i \in \mathbb{N}}\) of the positive integers and \(C, K > 0\) with the following property: For all \(i \geq 1\) there is a subset \(b_1 < b_2 < \ldots < b_{M_i}\) of \((a_n)_{n=1,...,N_i}\) with \(M_i \geq CN_i\), such that \((b_j)_{j=1,...,M_i}\) is a subset of

\[
P_i := \left\{ h_i + \sum_{j=1}^{d} r_j k_j^{(i)} \mid 0 \leq r_j < s_j^{(i)} \right\}
\]

with certain \(h_i, k_1^{(i)}, \ldots, k_d^{(i)} \in \mathbb{Z}, s_1^{(i)}, \ldots, s_d^{(i)} \in \mathbb{N}\) and \(s_1^{(i)} s_2^{(i)} \ldots s_d^{(i)} \leq KN_i\).

Fix now any \(i\), and for simplicity we omit the index \(i\) in the above notations, i.e., we put \(M := M_i, h := h_i\) and so on. In the sequel, we will put \(K = 1\) and \(h = 0\). The general case is treated similarly. Further, for \(k = 1, \ldots, M\), we set

\[
b_k = r_1^{(k)} k_1 + \ldots + r_d^{(k)} k_d
\]

and we identify \(b_k\) with the vector

\[
(r_1^{(k)}, \ldots, r_d^{(k)}) =: r_k.
\]

We have \(0 \leq r_j^{(k)} < s_j\) for all \(k = 1, \ldots, M\) and all \(j = 1, \ldots, d\). Consider the differences \(r_k - r_l\) for \(k, l = 1, \ldots, M\). This yields \(M^2 \geq C^2 N^2\) vectors (counted
with multiplicity)

\[ u := \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} , \]

with \(- (s_j - 1) \leq u_j \leq (s_j - 1)\) for \(j = 1, \ldots, d\). There exist at most \(2^d s_1 \ldots s_d \leq 2^d N\) different such vectors \(u\). For each such given vector \(u\) there exist at most \(M \leq N\) pairs \(r_k, r_l\) such that \(r_k - r_l = u\). Let \(\gamma := \frac{C^2}{1 + 2^d}\), then there exist at least \(\gamma N\) different such vectors \(u\) such that there exist at least \(\gamma N\) pairs \(r_k, r_l\) with \(r_k - r_l = u\). Otherwise we had:

\[
C^2 N^2 \leq M^2 \leq \gamma N M + (2^d N - \gamma N) \gamma N \\
\leq \gamma N^2 + (2^d - \gamma) \gamma N^2,
\]

hence

\[
C^2 \leq \gamma + (2^d - \gamma) \gamma < \gamma (1 + 2^d),
\]

i.e., \(\gamma > \frac{C^2}{1 + 2^d}\), a contradiction.

In the sequel, we will refer to this observation as Property 1. Take now \(\gamma N\) such \(d\)-tuples \(u\) having Property 1 and consider the corresponding \(\gamma N\) values

\[ \{(u_1 k_1 + \ldots + u_d k_d) \alpha\}, \quad \text{in } [0, 1). \tag{5} \]

Let \(L := \frac{2}{\gamma}\), then there is a \(\beta \in [0, 1)\), such that the interval \(\left[ \beta, \beta + \frac{L}{\gamma N} \right)\) contains at least \(L\) elements of the form \(5\), say the elements

\[ \{(u_1^x k_1 + \ldots + u_d^x k_d) \alpha\}, \quad \text{for } x = 1, \ldots, L. \]

We call this fact Property 2.

For every choice of \(x\), we now consider \(\gamma N\) pairs of \(d\)-tuples, say

\[ r_{i,x} := \begin{pmatrix} r_{1(i,x)} \\ \vdots \\ r_{d(i,x)} \end{pmatrix} , \text{ and } \tilde{r}_{i,x} := \begin{pmatrix} \tilde{r}_{1(i,x)} \\ \vdots \\ \tilde{r}_{d(i,x)} \end{pmatrix} , \]

for \(i = 1, \ldots, \gamma N\), such that

\[ r_{i,x} - \tilde{r}_{i,x} = \begin{pmatrix} u_{1}^{(x)} \\ \vdots \\ u_{d}^{(x)} \end{pmatrix} . \]

We will show that there exist \(x, y \in \{1, \ldots, L\}\) with \(x \neq y\) such that

\[ \# \left( \{r_{i,x} | i = 1, \ldots, \gamma N\} \cap \{r_{i,y} | i = 1, \ldots, \gamma N\} \right) \geq \frac{N}{L^2}. \tag{6} \]
Assume this were not the case and define
\[ \mathcal{M}_x := \{ r_{i,x} | i = 1, \ldots, \gamma N \} . \]

Then, we had
\[
N \geq \sum_{j=1}^{L} |M_j| - \sum_{x=1}^{L} |M_j \cap M_y| \\
> L \gamma N - L^2 \frac{N}{L^2} = N,
\]
which is a contradiction.

Let now \( x \) and \( y \) satisfying (6) be given. Let
\[
r_i, \quad i = 1, \ldots, L^2, \\
\hat{r}_{i,x}, \quad i = 1, \ldots, L^2, \\
\hat{r}_{i,y}, \quad i = 1, \ldots, L^2
\]
be such that
\[
r_i - \hat{r}_{i,x} = \begin{pmatrix} u_1^{(x)} \\ \vdots \\ u_d^{(x)} \end{pmatrix}, \quad \text{and} \quad r_i - \hat{r}_{i,y} = \begin{pmatrix} u_1^{(y)} \\ \vdots \\ u_d^{(y)} \end{pmatrix}.
\]

Then,
\[
\hat{r}_{i,y} - \hat{r}_{i,x} = \begin{pmatrix} u_1^{(x)} - u_1^{(y)} \\ \vdots \\ u_d^{(x)} - u_d^{(y)} \end{pmatrix} =: \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix},
\]
for \( i = 1, \ldots, \frac{N_i}{L^2} \). Due to Property 2, we have
\[
\frac{L}{\gamma N} \geq |\{(z_1k_1 + \ldots + z_dk_d)\alpha\}|.
\]

To sum up, we have shown that for all \( N_i \) there exist at least
\[
\frac{N_i}{L^2} = \frac{4(1+2^d)^2}{C^4} N_i =: \tau N_i,
\]
pairs \((k, l)\) with \(1 \leq k \neq l \leq N_i\), such that all expressions \(\|\{a_k \alpha\} - \{a_l \alpha\}\|\) have the same value and satisfy

\[
\|\{a_k \alpha\} - \{a_l \alpha\}\| \leq \frac{L}{\gamma N_i} = \frac{2(1 + 2^d)^2}{C^4} \frac{1}{N_i} =: \psi \frac{1}{N_i}.
\]

Note, that \(\tau\) and \(\psi\) only depend on \(d\) and \(C\) (and on \(K\) if \(K \neq 1\)). For every \(i\) choose now \(\psi_i\) minimal such that there exist at least \(\tau N_i\) pairs \((k, l)\) with \(1 \leq k \neq l \leq N_i\), such that

\[
\|\{a_k \alpha\} - \{a_l \alpha\}\| = \psi_i \frac{1}{N_i}.
\]

Of course, \(\psi_i \leq \psi\) for all \(i\). Let now \(\rho := \frac{\tau}{3}\) and assume that \(\psi_i < \rho\) for infinitely many \(i\). Therefore, we have for these \(i\)

\[
\frac{1}{N_i} \# \left\{ 1 \leq k \neq l \leq N_i \, \| \{a_k \alpha\} - \{a_l \alpha\}\| \leq \rho \frac{1}{N_i} \right\} \geq \tau = 3\rho,
\]

which is a contradiction and consequently the pair correlations are not Poissonian.

Assume now that \(\psi_i \geq \rho\) for infinitely many \(i\). Consequently, there exists an \(s_1 \in [\rho, \psi]\) such that

\[
\psi_i \in \left[s_1, s_1 + \frac{\tau}{3}\right)
\]

for infinitely many \(i\). In the following, we only consider these \(i\) and we will set

\[
s_2 := s_1 + \frac{2\tau}{3}.
\]

Then, we have

\[
\frac{1}{N_i} \# \left\{ 1 \leq k \neq l \leq N_i \, \| \{a_k \alpha\} - \{a_l \alpha\}\| \leq s_2 \frac{1}{N_i} \right\} - \frac{1}{N_i} \# \left\{ 1 \leq k \neq l \leq N_i \, \| \{a_k \alpha\} - \{a_l \alpha\}\| \leq s_1 \frac{1}{N_i} \right\} \geq \tau.
\]

If \((\{a_n \alpha\})_{n \in \mathbb{N}}\) were Poissonian, then the above difference should converge, as \(i \to \infty\), to \(2(s_2 - s_1) = \frac{2\tau}{3} < \tau\), which is a contradiction.

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