UNIFORMLY EXPANDING COUPLED MAPS:
SELF-CONSISTENT TRANSFER OPERATORS AND
PROPAGATION OF CHAOS

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Abstract. In this paper we study systems of $N$ uniformly expanding coupled maps when $N$ is finite but large. We introduce self-consistent transfer operators that approximate the evolution of measures under the dynamics, and quantify this approximation explicitly with respect to $N$. Using this result, we prove that uniformly expanding coupled maps satisfy propagation of chaos when $N \to \infty$, and characterize the absolutely continuous invariant measures for the finite dimensional system. The main working assumption is that the expansion is not too small and the strength of the interactions is not too large, although both can be of order one. In contrast with previous approaches, we do not require the coupled maps and the interactions to be identical. The technical advances that allow us to describe the system are: the introduction of a framework to study the evolution of conditional measures along some non-invariant foliations where the dependence of all estimates on the dimension is explicit; and the characterization of an invariant class of measures close to products that satisfy exponential concentration inequalities.

1. Introduction

The equations for a system of $N$ identical globally coupled maps with pairwise additive interactions have the form

$$(1) \quad x_i(t + 1) = f(x_i(t)) + \frac{1}{N} \sum_{j=1}^{N} h(x_i(t), x_j(t)) \quad i = 1, \ldots, N$$

where $x_i(t)$ denotes the state at time $t$ of the $i$-th map. Since the beginning of their study at the end of the ’80s (e.g. \[Kan90a, PK94, Kan90b, Jus95, EP95\]), numerical evidence immediately suggested that, despite the apparent simplicity of their equations, these systems present complex behavior: from coherence (e.g. synchronization and clustering), to turbulence and chaos (e.g. attractors with absolutely continuous invariant measures). Their rigorous mathematical study, however, is notoriously hard with a majority of the results in the literature coming from numerical experiments. Most of the rigorous analysis, especially in the chaotic regime, is restricted to the study of the thermodynamic limit obtained letting $N$ go to infinity.

In the thermodynamic limit, the state of the system is given by a probability measure describing the distribution of the states of the maps, and its time evolution is prescribed by a self-consistent transfer operator (STO) that acts non-linearly on measures. Fixed points of STOs can be interpreted as equilibrium states for the thermodynamic limit and one is concerned with establishing their existence, uniqueness, stability, stability under perturbations of the equations, linear response,... For uniformly expanding coupled maps, which are the topic of this

\[\text{See Section}\,2\,\text{for a more precise formulation.}\]
paper, these questions have been addressed in the case of small coupling by extending various results from perturbation theory of linear transfer operator to nonlinear STOs ([BKST18, Gal22, Ke00, SB16, ST21]). For the treatment of other types of maps see e.g. [BLS22, BKZ09, ST22].

The following question now arises: to which extent does the thermodynamic limit describe the finite dimensional system? In this paper we give a quantitative answer to this question for uniformly expanding globally coupled maps, and we show that, under certain assumptions on the expansion of the maps and strength of the interactions, one can define a STO that approximates the evolution of the finite dimensional system and we provide quantitative estimates showing that the approximation error decays polynomially with $N$.

The main feature of these globally coupled systems is their large number of degrees of freedom. On one hand, high-dimensional phenomena like concentration of measure help us approximate the system with a simplified mean-field version where the average of the interactions in (1) can be substituted by an expectation; on the other hand we incur in the dimensionality curse and for large $N$ it becomes unclear which measure should be used as reference, and which spaces of measures and distance between measures one should consider. These issues are exemplified by the fact that product measures, if not identical, tend to become singular with respect to each other when $N \to \infty$. Furthermore, the system converges to its mean-field approximation only when $N$ becomes large, thus we end up with a perturbation problem where the perturbation parameter is the dimension of the system. Existing perturbation results for uniformly expanding maps cannot deal with this scenario, and new frameworks that keep explicit account of the dimension are needed. To the best of our knowledge, such frameworks are lacking and what we present in this paper is the first instance where this problem is addressed in the context of coupled chaotic maps.

Our approach introduces a class of measures close to products (i.e. close to factorized measures) that are kept invariant by uniformly expanding maps. The main feature of these measures is that dependencies between different coordinates are of order $N^{-1}$, and that they satisfy exponential concentration inequalities analogous to those for product measures. To prove invariance of such a class we are going to study the evolution of conditional measures with respect to non-invariant low dimensional foliations whose leaves are obtained by fixing all coordinates but one. This requires geometric control on the evolution of the foliations, with estimates explicit on the dimension $N$.

As a byproduct of our main result, we show that the unique absolutely continuous invariant probability measure of the coupled system is close, in some sense that will be made precise, to an attracting fixed point of the STO. Furthermore, we prove that the system of coupled maps exhibits propagation of chaos, i.e. pushing forward a product measure under the coupled dynamics, the marginals over (fixed) finitely many coordinates converge to a product measure in the limit for $N \to \infty$.

Another achievement of our approach is that the results hold for systems lacking symmetry. Few exceptions aside ([Gal22, ST22]), full permutation symmetry has been assumed in the study of coupled maps, and more generally in the study of interacting particle systems. Generalizations to systems lacking symmetry are

\footnote{This is also related to the well known fact that in infinite dimensions, $N = \infty$, there is no natural reference measure on the phase space.}
especially important having applications to biology and artificial systems in mind, where the components making up a system are rarely identical. The rest of the paper is organized as follows. In Section 2 we present the system of coupled maps we are going to study, together with a definition for the self-consistent transfer operator. In Section 3 we define some relevant spaces of quasi-product measures, and state the main results. In Section 4 we proceed with the study of evolution of quasi-product measures under the dynamics of uniformly expanding coupled maps and give sufficient conditions for invariance of some classes of quasi-product measures. In Section 5 we provide a result on concentration of quasi-product measures. In Section 6 we prove the main results. Appendix A gathers some general results on cones and the projective Hilbert metric, while Appendix B and C contain some of the more technical and computationally demanding proofs for the statements in section 4.

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2. Setup

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the 1D torus and for every $i, j \in \mathbb{N}$ with $i < j$ define the set of indices $[i, j] := \{i, i + 1, \ldots, j\}$. We are going to consider a system of $N \in \mathbb{N}$ coupled maps where each map is described by a variable $x_i \in \mathbb{T}$. For every $i \in [1, N]$, let $f_i : \mathbb{T} \to \mathbb{T}$ be the $i$-th uncoupled map, and for every $i, j \in [1, N]$ let $h_{ij} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ prescribe the shape of the interactions between $i$-th and $j$-th maps. The regularity of these functions will be prescribed later. Given $f_i : \mathbb{R} \to \mathbb{R}$ and $h_{ij} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ lifts of $f_i$ and $h_{ij}$, define $F := (F_1, \ldots, F_N) : \mathbb{T}^N \to \mathbb{T}^N$ as

$$F_i(x_1, \ldots, x_N) := f_i(x_i) + \frac{1}{N} \sum_{j=i}^{N} h_{ij}(x_i, x_j) \mod 1$$

which is the map for the evolution equations of the system of globally coupled maps. Later on we are going to drop the bar from the notation as there is no risk of confusion.

2.1. Regularity assumptions on $F$. In our results we study maps as in (2) satisfying the following assumption.

Assumption 2.1. Given $(\kappa, K, E) \in (\mathbb{R}_0^+)^3$ with $\kappa - E > 1$, $f_i$, $h_{ij}$ are in $C^3$ and satisfy

$$\|h_{ij}\|_{C^3} \leq E$$

$$|\partial^2_i f_i|_{\infty} := \sup_{x \in \mathbb{T}} |\partial^2_i f_i(x)|, |\partial^3_i f_i|_{\infty} \leq K, \quad \inf_{x \in \mathbb{T}} |\partial_i f_i(x)| > \kappa$$

for all $j, i \in [1, N]$ and $j \neq i$.

The above assumption implies that $F$ is a local diffeomorphism. Notice that condition (3) implies that the influence of the $j$-th coordinate on the evolution of the $i$-th coordinate is of order at most $N^{-1}$. Furthermore, given $F$ as in (2) satisfying

3By Gershgorin circle theorem all the eigenvalues of the Jacobian matrix $DF$ have modulus bounded away from zero.
Assumption 2.1 for every fixed $i \in [1, N]$ and $\hat{x}_i \in \mathbb{T}^{N-1}$, $F_i(\cdot; \hat{x}_i) : \mathbb{T} \to \mathbb{T}$ is a $C^3$ uniformly expanding map with expansion lower bounded by $\kappa - E$ and distortion upper bounded by

$$D := \frac{K + E}{\kappa^2}.$$

### 2.2. Self-consistent transfer operator.

For $x = (x_1, \ldots, x_N) \in \mathbb{T}^N$ and $i \in [1, N]$ we adopt the notation

$$\hat{x}_i = (x_1, \ldots, x_{i-1}, x_i, \ldots, x_N) \in \mathbb{T}^{N-1} \text{ and } x = (x_i; \hat{x}_i).$$

We will denote by $\pi_i : \mathbb{T}^N \to \mathbb{T}$ the projection on the $i$-th coordinate

$$\pi_i(x_i; \hat{x}_i) = x_i,$$

and by $\hat{\pi}_i : \mathbb{T}^N \to \mathbb{T}^{N-1}$ the projection on all coordinates but the $i$-th one,

$$\hat{\pi}_i(x_i; \hat{x}_i) = \hat{x}_i.$$

Let also $\Pi_i := \pi_{i*}$ and $\hat{\Pi}_i := \hat{\pi}_{i*}$ denote the push-forwards of these projections that when applied to a measure return, respectively, the marginal on the $i$-th coordinate and the marginal on all but the $i$-th coordinate.

**Definition 2.1** (Mean-Field Approximation). Given $F : \mathbb{T}^N \to \mathbb{R}$ and $\mu \in \mathcal{M}_1(\mathbb{T}^N)$, for every $i \in [1, N]$ define $F_{\mu,i} : \mathbb{T} \to \mathbb{T}$

$$F_{\mu,i}(x_i) := \int_{\mathbb{T}^{N-1}} F_i(x_i; \hat{x}_i) \, d\hat{\Pi}_i \mu(\hat{x}_i) \mod 1$$

and the product map $F_\mu := (F_{\mu,1}, \ldots, F_{\mu,N})$ on $\mathbb{T}^N$, which we call the mean-field approximation of $F$ with respect to $\mu$.

The rationale for introducing the above mean-field maps is the following: Consider $\mu = \mu_1 \otimes \ldots \otimes \mu_N$ a product measure on $\mathbb{T}^N$ with $\mu_1, \ldots, \mu_N \in \mathcal{M}_1(\mathbb{T})$; then if $F$ satisfies Assumption 2.1 the dependence of $F_i$ on each coordinate $j \neq i$ is of order $N^{-1}$ and classical results on concentration of measure imply that the map $F_i(\cdot; \hat{x}_i) : \mathbb{T} \to \mathbb{T}$ is “close” to $F_{\mu,i}$ for $\hat{x}_i$ in a subset of $\mathbb{T}^{N-1}$ whose complement has $\Pi_i \mu$-measure exponentially small in the system’s dimension $N$. This suggests the heuristics that there is $\mathcal{G} \subset \mathbb{T}^N$ with $\mu(\mathcal{G}) \approx 1$ where $F$ can be approximated by the product map $F_\mu$. In practical terms, this means that for $N$ finite, but very large, if we were to investigate the evolution under $F$ with finite precision and drew an initial condition at random with respect to the measure $\mu$, with high probability, we would expect the evolution of the $i$-th coordinate to be indistinguishable from the map $F_{\mu,i}$. In turn, the measure $F_\mu \mu$ is expected to be approximately $(F_\mu)_* \mu$.

Guided by these heuristic arguments, we give the following definition.

**Definition 2.2** (Self-Consistent Transfer Operator). Given $F : \mathbb{T}^N \to \mathbb{T}^N$, let $\mathcal{F} : \mathcal{M}_1(\mathbb{T}^N) \to \mathcal{M}_1(\mathbb{T}^N)$ be defined as

$$\mathcal{F}_\mu := (F_\mu)_* \mu.$$

\(^4\)A “$*$” in place of $x_i$, emphasizes that $x_i$ is thought as the free variable for this function, while $\hat{x}_i$ is considered a fixed parameter.

\(^5\)E.g. McDirmind’s inequality [McD98].
In general, $\mathcal{F}$ is nonlinear, as the linear operator $(F_\mu)_*$ depends on the measure it is applied to. $\mathcal{F}$ has been given the name of self-consistent transfer operator (STO). Notice that if the uncoupled maps $f_i$ and the interaction functions $h_{ij}$ are all identical and $\mathcal{F}$ is restricted to product measures having identical factors, i.e. the system has full permutation symmetry, then we recover the standard definition of self-consistent operator previously appeared in the literature, therefore this definition is a generalization of the STO to cases without full permutation symmetry\(^6\).

The goal of this paper is to study to which extent $F_\mu$ can be approximated by $\mathcal{F}$.

3. Results

Roughly speaking, we find sufficient conditions on $F$ so that $F_\mu$ keeps invariant a class of measures close to product that satisfy concentration inequalities and whose evolution is well approximated by the self-consistent transfer operator $\mathcal{F}_\mu$, in a sense that will be made precise below.

3.1. Measures with Lipschitz disintegrations along coordinates. The first task to make the heuristic picture in Section 2.2 rigorous, is to characterize the measures that are close to a product. We are going to do so in terms of their disintegrations\(^7\) with respect to a natural class of foliations:

**Definition 3.1** (Disintegrations along Coordinates). Given $\mu \in \mathcal{M}(\mathbb{T}^N)$, for every $i \in [1, N]$, we denote by $\{\mu_{\hat{x}_i}\}_{\hat{x}_i \in \mathbb{T}^{N-1}}$ the disintegration with respect to the measurable foliation

$$\{T_{\hat{x}_i}\}_{\hat{x}_i \in \mathbb{T}^{N-1}} := \{(x_1, \ldots, x_{i-1}) \times \mathbb{T} \times (x_{i+1}, \ldots, x_N) : \hat{x}_i \in \mathbb{T}^{N-1}\}$$

where $\mu_{\hat{x}_i}$ is the conditional probability measure of $\mu$ on $T_{\hat{x}_i}\[^8^\]$. We are going to refer to $\{T_{\hat{x}_i}\}_{\hat{x}_i \in \mathbb{T}^{N-1}}$ as the foliation along the $i$-th coordinate, to $T_{\hat{x}_i}$ as the leaf or fiber over $\hat{x}_i$, and to $\{\mu_{\hat{x}_i}\}_{\hat{x}_i \in \mathbb{T}^{N-1}}$ as the disintegration of $\mu$ along the $i$-th coordinate.

In the following, we restrict to the case where $\mu \in \mathcal{M}(\mathbb{T}^N)$ is absolutely continuous with respect to Lebesgue and has density $\rho : \mathbb{T}^N \to \mathbb{R}^+$. Then $\mu_{\hat{x}_i}$ can be chosen to be the probability measure on $\mathbb{T}$ having density

$$\rho_{\hat{x}_i}(x) := \frac{\rho(x; \hat{x}_i)}{\int_{\mathbb{T}} \rho(s; \hat{x}_i) ds}. \quad (5)$$

Notice that $\mu \in \mathcal{M}^\otimes(\mathbb{T}^N)$, i.e. is a product or equivalently a factorized measure, if and only if all disintegrations of $\mu$ along coordinates can be chosen to be constant. This suggests that one possibility to control how far a measure is from being a product we should control, for every $i \in [1, N]$, how $\mu_{\hat{x}_i}$ varies with $\hat{x}_i$. This still leaves a lot of freedom on the metric space to adopt and on the regularity to impose on $\hat{x}_i \mapsto \mu_{\hat{x}_i}$. We are going to consider: $\mu_{\hat{x}_i}$ with density $\rho_{\hat{x}_i}$ in $(\mathcal{V}_a, \theta_a)$, the convex cone of twice continuously differentiable positive function with bounded log-Lipschitz constant:

$$\mathcal{V}_a := \left\{ \psi \in C^2(\mathbb{T}, \mathbb{R}^+) : \left| \frac{d}{dx} \log \psi(x) \right| < a \right\}; \quad (6)$$

\(^6\)Another subtle difference between the definition of STO given here and the one given in the literature, is that here $\mathcal{F}$ depends on $N$.

\(^7\)For a general treatment of disintegration of measures with respect to measurable foliations the reader can consult for example [Sim12].

\(^8\)We interchangeably see $\mu_{\hat{x}_i}$ as a measure on $T_{\hat{x}_i}$ and on $\mathbb{T}$.
and $\hat{x}_i \mapsto \mu_{\hat{x}_i}$ to be Lipschitz. More precisely

**Definition 3.2 (M$_{a,b,L}$-spaces).** Given $a \geq 0$, $b \geq a$, $L \geq 0$, $i \in [1,N]$ and $j \neq i$ define $M_{a,b,L}^{(i,j)} \subset M(T^N)$ the space of measures $\mu$ whose disintegration along the $i$-th coordinate satisfies

i) $\rho_{\hat{x}_i} \in V_a$ for every $\hat{x}_i \in T^{N-1}$,

ii) for every $\hat{x}_i, \hat{x}'_i \in T^{N-1}$ differing only for their $k$-th coordinates $x_k, x'_k \in T$

$$\theta_b (\rho_{\hat{x}_i}, \rho_{\hat{x}'_i}) \leq L |x_k - x'_k|.$$ 

Define also

$$M_{a,b,L}^{(i)} := \bigcap_{j \neq i} M_{a,b,L}^{(i,j)} \quad \text{and} \quad M_{a,b,L} := \bigcap_{i \in [1,N]} M_{a,b,L}^{(i)}.$$ 

**Remark 3.3.** Notice that if $b > a$ in the definition above, the conditional measures on the leaves belong to $V_a$, but when prescribing the Lipschitz constant $L$, we measure their distance with respect to the “weaker” Hilbert metric on $V_b \supset V_a$. This will play a crucial role in our arguments to control the Lipschitz constant under application of $F_*$. 

It is well known that there is an Hilbert projective metric $\theta_a : V_a \times V_a \rightarrow \mathbb{R}^+$ intrinsically defined on the convex cone $V_a$ and that linear transformations are contractions with respect to it (see Theorem A.1 in the Appendix). For this reason, cones of functions have been successfully used to study transfer operators of uniformly hyperbolic maps and, in particular, of uniformly expanding maps [Liv95]. Additional information on cones and the Hilbert metric can be found in Appendix A. 

One advantage of using cones is that linear transfer operators contract the Hilbert metric in one step. This is in contrast with the contraction observed on (suitably) normed linear spaces where operators have a spectral gap for which, in general, multiple iterates are needed before observing shrinking of the norms. This fact is often exploited when composing different maps that keep the same cone invariant, for example in the study of random and sequential dynamical systems [NTV18]. Another advantage is that being a projective metric, the Hilbert metric only distinguishes directions. This allows to compare conditional measures without worrying about normalization factors that add extra terms to already involved computations (e.g. the normalizing factor in (5) can be omitted when measuring the Hilbert distance between conditional densities, but not when comparing densities with respect to norms).

In Section 5 – whose content can be read independently from the rest of the paper – we are going to provide concentration estimates for measures in $M_{a,b,c,N^{-1}}$ analogous to estimates classically obtained for product measures. In virtue of this fact, measures having Lipschitz disintegrations along coordinates with Lipschitz constant scaling as $N^{-1}$ will be referred to as quasi-product measures. These estimates will be crucial to extend the heuristic argument in Section 2.2 from product measures, to measures in the image of $F_*$ that, due to the interactions, are bound to have dependencies among the coordinates.

**3.2. Main results.** Loosely speaking, the following theorem claims that one can find $E_0$ sufficiently small and $\kappa_0$ sufficiently large – independent of $N$ – such that for $N$ sufficiently large, a map $F$ satisfying Assumption A.1 with $E \leq E_0$ and $\kappa \geq \kappa_0$
leaves a space of quasi-product measures invariant and the evolution of the measures in this set can be well approximated by the self-consistent transfer operator.

To control the evolution of the measures under $F$, they will have to satisfy an additional uniform bound on their second derivatives.

**Definition 3.4** ($C_\alpha$). For $\alpha \geq 0$

$$C^2_\alpha(\mathbb{T}^N) := \left\{ \mu \in \mathcal{M}_1(\mathbb{T}^N) : \rho := \frac{d\mu}{d\text{Leb}_N} \in C^2(\mathbb{T}^N, \mathbb{R}^+), \left| \frac{\partial_i \partial_j \rho}{\rho} \right|_\infty \leq \alpha \ \forall i,j \right\}.$$ 

Now we are ready to state the main result.

**Theorem 3.5.** Fixing $K > 0$, there are $E_0 > 0$ and $\kappa_0 > 1$ such that for every $E \leq E_0$ and $\kappa \geq \kappa_0$ if Assumption 4.1 holds with datum $(\kappa, K, E)$, then

1. there are $a_0 \geq 0$, $b_0 > a_0$, $\alpha_0 \geq 0$, and $c > 0$ such that for any $N$ sufficiently large

   $$F_*(\mathcal{M}_{a_0,b_0,cN^{-1}} \cap C^2_\alpha) \subset \mathcal{M}_{a_0,b_0,cN^{-1}} \cap C^2_\alpha$$

2. For any $\gamma \in [0, \frac{1}{2}]$ there is $C_\gamma > 0$ independent of $N$ such that

   $$\sup_{i \in [1,N]} \theta_{b_0}(\Pi_i F_* \mu, \Pi_i \mathcal{F} \mu) \leq C_\gamma N^{-\gamma}$$

for every $\mu \in \mathcal{M}_{a_0,b_0,cN^{-1}} \cap C^2_\alpha$.

**Remark 3.6.** The values of $E_0$ and $\kappa_0$ depend on $K$. Keeping track of explicit estimates leading to $E_0$, $\kappa_0$, although possible, is a daunting task and we are going to avoid it. Nonetheless, the size of $E$ is expected to be of the same order of $\kappa$ and, assuming that $K$ is constant, larger values of $\kappa$ allow one to pick larger values of $E$.

Theorem 3.5 implies propagation of chaos with quantitative estimates.

**Corollary 3.7.** Let the assumptions of Theorem 3.5 hold. Then, for any fixed $k \in \mathbb{N}$ and $t > 0$, and measures $\{\mu_i\}_{i=1}^N \subset \mathcal{V}_{a_0} \cap C^2_\alpha(\mathbb{T})$, letting $\mu := \mu_1 \otimes \ldots \otimes \mu_N$

$$\|\Pi_{[1,k]} F_\mu - \Pi_1 \mathcal{F} \mu \otimes \ldots \otimes \Pi_k \mathcal{F} \mu\|_{TV} = O(N^{-\gamma}) \quad \forall \gamma \in [0,1/2]$$

where $\Pi_{[1,k]}$ denotes the projection to the marginal on the first $k$ coordinates and $\| \cdot \|_{TV}$ denotes the total variation norm.

We say that $\overline{\mu}$ is a fixed point for the self-consistent transfer operator if $\mathcal{F} \overline{\mu} = \overline{\mu}$. Under Assumption 4.1, $F_\overline{\mu}$ is an uncoupled map and $F_\overline{\mu}_i$ are maps with distortion and minimal expansion uniformly bounded with respect to $i$ and having a unique invariant measure with log-Lipschitz density in $\mathcal{V}_{a_0}$. This immediately implies, that if $\overline{\mu}$ is a fixed point for $\mathcal{F}$ with disintegrations along coordinates having densities in $\mathcal{V}_{a_0}$, then $\overline{\mu}$ is a product measure.

The following corollary states that if $\mathcal{F}$ has a fixed point with some stability properties, then $F_*$ leaves a neighborhood of this fixed point invariant and the size of the neighborhood decays polynomially in $N$.

**Corollary 3.8.** Let the assumptions of Theorem 3.5 stand, and assume that $\mathcal{F}$ has a fixed point $\overline{\mu} \in \mathcal{M}_{a_0,b_0}$, and there are $\delta > 0$ and $\lambda \in [0,1)$ such that for every $\mu$ in

$$B_\delta(\overline{\mu}) := \left\{ \mu \in \mathcal{M}_{a_0,b_0,cN^{-1}} \cap C^2_\alpha : \sup_{i \in [1,N]} \theta_{b_0}(\Pi_i \mu, \Pi_i \overline{\mu}) < \delta \right\}$$

$^{9}_i \pi_{[1,k]} : T^N \rightarrow T^k$ with $\pi_{[1,k]}(x_1, \ldots, x_N) = (x_1, \ldots, x_k)$ and $\Pi_{[1,k]} = (\pi_{[1,k]})_*$.
the following holds
\[ \theta b_0(\Pi, \mathcal{F} \mu, \Pi, \mathcal{F} \nu) \leq \lambda \theta b_0(\Pi, \mu, \nu) \quad \forall \mu, \nu \in B_\delta(\bar{\mu}), \forall i \in [1, N]. \]

Then for every \( \gamma \in [0, 1/2) \) there is \( c_\gamma > 0 \) such that for every \( N \) sufficiently large
\[ \mathcal{F}^* \left( B_{c_\gamma N^{-\gamma}}(\bar{\mu}) \right) \subset B_{c_\gamma N^{-\gamma}}(\bar{\mu}). \]

In particular, one expects the unique absolutely continuous invariant probability measure of \( \mathcal{F}^* \) to have marginals at distance \( O(N^{-\gamma}) \) from the marginals of the fixed point of \( \mathcal{F}^* \). For examples of STOs having unique attracting fixed points as above see e.g. [ST21].

Before proceeding with the proofs of the results above, which are given in Section 6.2, we first study, in Section 4, the evolution of quasi-product measures under the dynamics of globally coupled maps and provide a result on concentration for quasi-product measures, in Section 5.

4. Evolution of Measures under Coupled Maps

The objective of this section is to study what happens to disintegrations along coordinates under evolution with respect to \( H_* \) when \( H : \mathbb{T}^N \to \mathbb{T}^N \) is a local diffeomorphism satisfying assumptions analogous to those imposed on \( F \), but where the dependencies among coordinates are more general than pairwise interactions.

**Assumption 4.1.** With the datum \((E, \kappa) \in \mathbb{R}_0^+\) satisfying \( \kappa - E > 1 \), \( H \in C^1(\mathbb{T}^N, \mathbb{T}^N) \) and
\[ |\partial_i H_i| > \kappa, \quad |\partial_j H_i| < E N^{-1} \]
for all \( j, i \in [1, N] \) and \( j \neq i \).

**Assumption 4.2.** With the datum \((E, K) \in \mathbb{R}_0^+, H \in C^3(\mathbb{T}^N, \mathbb{T}^N) \) and for every \( i \in [1, N] \)
\[ |\partial_i^2 H_i| \leq K, \quad |\partial_k \partial_j H_i| \leq \begin{cases} EN^{-1} & j \neq i \\ E N^{-2} & i, j, k \text{ distinct} \end{cases} \]
and if \( \ell, j \) and \( k \) are distinct indices
\[ |\partial_\ell \partial_i^2 H_j| \leq E N^{-1}, \quad |\partial_\ell \partial_k \partial_j H_j| \leq E N^{-2}. \]

The main step of our analysis is the investigation of \( \{ (H_* \mu)_{T_\bar{\mu}} \}_{\bar{\mu} \in \mathbb{T}^N-1} \), the disintegration of the push-forward \( H_* \mu \) with respect to coordinate foliations. To this end we study \( \{ \mu_{H^{-1}(T_{\bar{\mu}})} \}_{\bar{\mu} \in \mathbb{T}^N-1} \), the disintegration of \( \mu \) with respect to the pullback foliation \( \{ H^{-1}(T_{\bar{\mu}}) \}_{\bar{\mu} \in \mathbb{T}^N-1} \). It is important to notice that the foliation \( \{ T_{\bar{\mu}} \}_{\bar{\mu} \in \mathbb{T}^N-1} \) is not invariant due to the interactions between coordinates, and that the generality of our setup does not allow to rely on discernible invariant foliations. Therefore, studying the relation between \( \{ \mu_{\bar{\mu}} \}_{\bar{\mu} \in \mathbb{T}^N-1} \) and \( \{ \mu_{H^{-1}(T_{\bar{\mu}})} \}_{\bar{\mu} \in \mathbb{T}^N-1} \) will be crucial for our analysis and represents one of the main technical advances of this paper. Since the proofs in this section are long and technical, they are postponed to the appendix.
4.1. Pull-back foliation: a global change of charts straightening the leaves.

The following proposition describes \( \{ H^{-1}(T_{\hat{x}_i}) \}_{\hat{x}_i \in T^{N-1}} \), the pull back of the foliation \( \{ T_{\hat{x}_i} \}_{\hat{x}_i \in T^{N-1}} \) under \( H \). It is immediate to show that if \( H \) is a local diffeomorphism, then \( \{ H^{-1}(T_{\hat{x}_i}) \}_{\hat{x}_i \in T^{N-1}} \) is a foliation with \( H^{-1}(T_{\hat{x}_i}) \) integrating the vector field \( X_i := H^* e_i \), where \( H^* \) denotes the pull-back of \( H \), and \( e_i \) is the \( i \)-th vector of the standard basis in \( \mathbb{R}^N \).

The following proposition states that under the assumptions above and when \( N \) is sufficiently large, each leaf \( H^{-1}(T_{\hat{x}_i}) \) can be written as the disjoint union of circles roughly aligned with the \( i \)-th coordinate, where the deviation from straight circles is carefully estimated with respect to the dimension \( N \). To this end we need careful estimates on the the size of the entries of the inverse Jacobian matrix, \( D H^{-1} \) with respect to the dimension \( N \) that are provided in Appendix B. Since keeping track of all constant dependencies on the parameters in assumptions 4.1 and 4.2 is a daunting task, from now on, we are going to denote by \( K_\# \) a generic constant that depends on the parameters \( E \), \( \kappa \), and \( K \) only – in particular is independent of \( N \) – and \( K_\# \to 0 \) when either \( E \to 0 \) or \( \kappa \to \infty \), with all the other constants fixed. Following the proofs it would be possible, in principle, to explicitly estimate \( K_\# \) once the parameters are known.

We will use the following notation: let \( H_i := \pi_i \circ H \) and \( \hat{H}_i := \hat{\pi}_i \circ H \), so that

\[
H(x) = (H_i(x_i; \hat{x}_i); \hat{H}_i(x_i; \hat{x}_i)),
\]

and let \( \hat{D}_i \) denote the differential with respect to all coordinates, but the \( i \)-th one.

**Proposition 4.1.** Let \( H : T^N \to T^N \) be a map satisfying Assumption 4.1. Then, for \( N \) sufficiently large – depending on \( E \) and \( \kappa \) – and every \( i \in [1, N] \), there is a diffeomorphism \( \Phi_i = (\Phi_{i,1}, \ldots, \Phi_{i,N}) : T \times T^{N-1} \to T^N \) implicitly defined by

\[
\Phi_{i,i}(y_i, \hat{y}_i) = y_i \quad \hat{H}_i(\Phi_i(y_i, \hat{y}_i)) = \hat{H}_i(0; \hat{y}_i)
\]

such that for every \( \hat{x}_i \in T^{N-1} \) there is a finite set \( \mathcal{Y}_{\hat{x}_i} \subset T^{N-1} \) satisfying

\[
H^{-1}(T_{\hat{x}_i}) = \bigcup_{\hat{y}_i \in \mathcal{Y}_{\hat{x}_i}} \Phi_i(T \times \{ \hat{y}_i \})
\]

and \( \Phi_i(T \times \{ \hat{y}_i \}) \) is homotopic to \( T \times \{ \hat{y}_i \} \).

Furthermore, if Assumption 4.2 is satisfied, we get the following estimates for the derivatives of \( \Phi_i \):

1. **First Derivatives:** For every \( m, k \in [1, N] \)

\[
|\partial_k \Phi_{i,m}|_{\infty} \begin{cases} 
= 1 & m = k = i \\
= 0 & m = i, k \neq i \\
\leq K_\# N^{-1} & m = k, k \neq i \\
\leq K_\# N^{-2} & k = i, m \neq k \\
\end{cases}
\]

where \( |O(N^{-1})| \leq K_\# N^{-1} \).

2. **Second Derivatives:**

\[
|\partial_i \partial_k \Phi_{i,m}|_{\infty} \begin{cases} 
= 0 & m = i \\
\leq K_\# N^{-1} & m \neq i, \ell = m \\
\leq K_\# N^{-1} & m \neq i, \ell = i \\
\leq K_\# N^{-2} & \text{otherwise}
\end{cases}
\]
and given

\[ i \]

with Definition 4.2. is given by a skew-product map as shown below.

The main implication of the above proposition is that, under the assumptions and for \( N \) large, one can find a global change of coordinates where the pullback foliation is made of straight vertical circles. Crucially, in this new coordinates, the dynamics is given by a skew-product map as shown below.

**Definition 4.2.** Given \( H : \mathbb{T}^N \to \mathbb{T}^N \) satisfying the assumptions of Proposition\(^{10}\) and given \( i \in [1, N] \), define the map \( G : \mathbb{T} \times \mathbb{T}^{N-1} \to \mathbb{T} \times \mathbb{T}^{N-1} \) as \( G := (G_i; \hat{G}_i) \) with

\[
G_i(y_i; \hat{y}_i) = H_i(\Phi_i(y_i; \hat{y}_i)) \quad \hat{G}_i(y_i; \hat{y}_i) = \hat{H}_i(0; \hat{y}_i).
\]

Notice that with the definition above,

\[
H = G \circ \Phi_i^{-1}
\]

and

**Lemma 4.3.** For any \( \mu \in \mathcal{M}(\mathbb{T}^N) \) absolutely continuous with respect to Lebesgue

\[
H_*\mu = (\text{Id}_\mathbb{T}; \hat{G}_i)_* (G_i; \text{Id}_{\mathbb{T}^{N-1}})_* \Phi_i^{-1/2}_i \mu.
\]

**Proof.** Immediately follows from the skew-product structure of \( G \).

Thus, changing coordinates through \( \Phi_i \) allowed us to express the evolution of the disintegration along the \( i \)-th coordinate as the contribution of two effects: evolution on each \( \mathbb{T}_{\hat{y}_i} \) fiber, which is given by the composition of the push-forward of \( G_i(\cdot; \hat{y}_i) : \mathbb{T} \to \mathbb{T} \), and the interactions among the other coordinates, given by the push-forward \( \hat{G}_i : \mathbb{T}^{N-1} \to \mathbb{T}^{N-1} \).

The following lemma shows that when \( N \) is large, for all \( \hat{x}_i \in \mathbb{T}^{N-1}, H_i(\cdot; \hat{x}_i) \) and \( G_i(\cdot; \hat{x}_i) \) are close in \( C^2(\mathbb{T}, \mathbb{T}) \) which suggests that, if the maps are uniformly expanding, statistical stability results imply that \( (G_i; \text{Id}_{\mathbb{T}^{N-1}})_* \) and \( (H_i; \text{Id}_{\mathbb{T}^{N-1}})_* \) share similar spectral properties.

**Lemma 4.4.** Let \( H \) satisfy Assumption\(^{11}\) and Assumption\(^{1.2}\). For \( N \) sufficiently large and any \( i \in [1, N] \), let \( G \) be as in Definition 4.2. Then

\[
d_{C^2}(G_i(\cdot; \hat{x}_i), H_i(\cdot; \hat{x}_i)) \leq K_i N^{-1}
\]

for all \( \hat{x}_i \in \mathbb{T}^{N-1} \).

**Proof.** Denoting \( \Phi_i = (\Phi_{i,i}; \hat{\Phi}_i) \), the definitions imply

\[
G_i(x_i; \hat{x}_i) = H_i(x_i; \hat{\Phi}_i(x_i; \hat{x}_i))
\]

\[
H_i(x_i; \hat{x}_i) = H_i(x_i; \hat{\Phi}_i(0; \hat{x}_i))
\]

\(^{10}\)Here we did not express explicitly the dependence of the map \( G \) from \( i \).
therefore, the mean-value theorem together with the estimates from assumptions 4.1 and 4.2, and from Proposition 4.1 give
\[ |G_i(x_i; \hat{x}_i) - H_i(x_i; \hat{x}_i)| \leq \sum_{\ell \neq i} |\partial_i H_i|_{\infty} |\partial_i \Phi_{i,\ell}|_{\infty} \leq K_# N^{-1} \]
\[ |\partial_i G_i(x_i; \hat{x}_i) - \partial_i H_i(x_i; \hat{x}_i)| \leq \sum_{\ell \neq i} |\partial_i H_i|_{\infty} |\partial_i \Phi_{i,\ell}|_{\infty} + \sum_{\ell \neq i} |\partial_i H_i|_{\infty} |\partial_i \Phi_{i,\ell}|_{\infty} \]
\[ \leq K_# N^{-1} \]
and analogous estimates for the second derivative.

4.2. Lipschitz disintegration along coordinates. Recall the class of measures \( \mathcal{M}_{a,b,L} \) from Definition 3.2. In this section we use the information obtained above to prove that under certain hypotheses – boiling down to the strength of the expansion “beating” the strength of the interactions – there is a set \( \mathcal{M}_{a,b,L} \) that is invariant under application of \( H_{\ast} \), with \( L \) of order \( N^{-1} \).

The strategy to control the Lipschitz constant \( L \) is to use expression (16) and look at the density of \( H_{\ast} \mu \) as the result of the composition of the following actions on \( \mu \):

Step 1. apply the global change of charts \( \Phi_{i}^{-1} \), i.e. change to coordinates where the leaves of \( \{ H^{-1}(T_{x_{i}}) \}_{x_{i} \in \mathbb{T}^{N-1}} \) straight;

Step 2. apply \( (G_{i}; \text{Id}_{T_{N-1}}) \), which gives the dynamics on the leaves;

Step 3. apply \( (\text{Id}_{T_{i}}; \hat{G}_{i}) \), which gives the dynamics of the leaves;

and keep track of how the Lipschitz constant is modified at each one of the steps above. We will prove that, with respect to a suitable projective Hilbert metric:

Step 1. the Lipschitz constant is multiplied by a factor possibly greater than one, depending on the strength of the interactions, and an \( O(N^{-1}) \) term is added, thus reducing the regularity;

Step 2. under suitable assumptions on \( H \), the Lipschitz constant is multiplied by a factor less than one, so there is a regularizing effect due to the uniform expansion, and \( O(N^{-1}) \) is added;

Step 3. multiplies the constant by a factor, possibly greater than one, that depends on a measure of distortion for \( \hat{G}_{i} \).

The main result of this section can be stated in an informal way as:

Claim 4.5. Under assumptions [4.1 and 4.2] and suitable assumptions on \( H \) ensuring that the regularizing effect of the expanding dynamics is stronger than the effect of the interactions, there are \( a \geq 0, b \geq 0, c \geq 0 \), and \( \alpha \geq 0 \) all independent of \( N \), such that

\[ H_{\ast} (\mathcal{M}_{a,b,cN^{-1}} \cap C_{\alpha}^{2}) \subset \mathcal{M}_{a,b,cN^{-1}} \cap C_{\alpha}^{2}. \]

A formal statement is given in Proposition 4.14 in subsection 4.2.4. Before that we give precise statements on how the regularity changes under application of each one of the steps above.

4.2.1. Step 1. Straightening of the foliation along the \( i \)-th coordinate. Since \( \Phi_{i} \) is a diffeomorphism, if \( \mu \in \mathcal{M}_{1}(\mathbb{T}^{N}) \) has density \( \eta : \mathbb{T}^{N} \rightarrow \mathbb{R}^{+} \), denoting by \( |D\Phi_{i}| \) the determinant of \( D\Phi_{i} \), the density of \( \Phi_{i}^{-1}\mu \) is given by

\[ \Phi_{i}^{-1}\eta(x_{i}; \hat{x}_{i}) := |D\Phi_{i}|(x_{i}; \hat{x}_{i}) \cdot \eta \circ \Phi_{i}(x_{i}; \hat{x}_{i}) \]
which is the Perron-Frobenius operator associated to $\Phi^{-1}_i$ applied to $\eta$. To study the regularity of the expression above, we first study the regularity of the two factors $|D\Phi_i|$ and $\eta \circ \Phi_i$ separately. The results are summarized in the lemmas below whose proofs are postponed to Appendix C.

The first lemma addresses the composition by $\Phi_i$ and determines the changes in regularity for disintegrations along the $i$-th coordinate, and for the marginal on $\hat{\pi}_i(\mathbb{T}^N)$ (which will be needed in later steps). It turns out that this composition produces a substantial change on disintegrations along the $i$-th coordinate, while changes along other coordinates are $O(N^{-1})$ and thus negligible when $N \to \infty$.

**Lemma 4.6.** Under assumptions 4.1 and 4.2 there are constants $L, K = O(1)$, and $\hat{L}, \hat{K} = 1 + O(N^{-1})$ – depending on $E, \kappa, K$ – such that for $N$ sufficiently large: If $\eta \in M_{a,b,L} \cap C^2$, then for every $i \in [1,N]$

i) for any $k \neq i$

$$\eta \circ \Phi_i \in M_{a,b,L}^{(i,k)}$$

with $L' := L + \frac{a+a}{Kb-Ka}\hat{K}#N^{-1}$;

ii) for any $j \neq i$ and $k \neq j, i$

$$\eta \circ \Phi_i \in M_{a,b,L}^{(j,k)}$$

with $\hat{L}' := \hat{L} + \frac{a+a}{Kb-Ka}\hat{K}#N^{-1}$.

**Remark 4.7.** Following the proofs one can see that the constant $L, K, \hat{L}, \hat{K}$ should be chosen to satisfy

\begin{align}
(17) & \quad K > \max \left\{ \sum_{i=1}^{N} |\partial_i \Phi_{i,\ell}|_{\infty} : i \in [1,N] \right\} \\
(18) & \quad L > \max \left\{ |\partial_i \Phi_{i,i}|_{\infty} \sum_{m=1}^{N} \sum_{\ell \neq m} |\partial_i \Phi_{i,\ell}|_{\infty} |\partial_k \Phi_{i,m}|_{\infty} : i \in [1,N], k \neq i \right\} \\
(19) & \quad \hat{K} > \max \left\{ \sum_{\ell=1}^{N} |\partial_k \Phi_{i,\ell}|_{\infty} : i \in [1,N], k \neq i \right\} \\
(20) & \quad \hat{L} > \max \left\{ |\partial_j \Phi_{i,j}|_{\infty} \sum_{m=1}^{N} \sum_{\ell \neq m} |\partial_j \Phi_{i,\ell}|_{\infty} |\partial_k \Phi_{i,m}|_{\infty} : i \in [1,N], j \neq i, k \neq i, j \right\}.
\end{align}

and, comparing with Proposition 4.1, one can check that the order of magnitudes claimed at the beginning of the lemma are achieved.

**Remark 4.8.** This lemma is the first instance where it is crucial for our estimates that the distance between the conditional measures on the leaves is measured with respect to a Hilbert metric “weaker” than the Hilbert metric of the cone to which the densities belong.

The second lemma characterizes the regularity of $|D\Phi_i|$.
Lemma 4.9. Under assumptions 4.1 and 4.2 there is \( a_\Phi(\kappa, E, K) \geq 0 \) independent of \( N \) such that for any \( i \in [1, N] \) and any \( b_\Phi > a_\Phi \),

\[
|D\Phi_i| \in \mathcal{M}^{(i)}_{a_\Phi, b_\Phi, L_\Phi}
\]

where \( L_\Phi := \frac{b_\Phi + 1}{b_\Phi - a_\Phi} K_\# N^{-1} \).

Combining the lemmas above, we obtain

Proposition 4.10. Under assumptions 4.1, 4.2 and for \( N \) sufficiently large, letting \( L, K, \hat{L}, \hat{K}, \) and \( a_\Phi \) be as in Lemma 4.6 and Lemma 4.9, for any \( i \in [1, N] \),

i) \[
\Phi^{-1}_{i*}(\mathcal{M}_{a,b,L} \cap C^2_\alpha) \subseteq \mathcal{M}^{(i)}_{a',b',L'} \cap C^2_{\alpha'}
\]

with

\[
a' := K a + a_\Phi
\]

\[
L' := \left( L + \frac{a + \alpha}{K b - K a} K_\# N^{-1} \right) \cdot L + \frac{b_\Phi + 1}{b_\Phi - a_\Phi} K_\# N^{-1}
\]

\[
\alpha' := K^2 \alpha + K_\#
\]

for any \( b_\Phi > a_\Phi \) and \( b' \geq K b + b_\Phi \);

ii) if \( j \neq i \) and \( k \neq j,i \)

\[
\hat{\Pi}_{i*} \Phi^{-1}_{i*}(\mathcal{M}_{a,b,L}(T^N) \cap C^2_\alpha) \subseteq \mathcal{M}^{(j,k)}_{a',b',L'}(T^{N-1})
\]

with

\[
\hat{a}' := K a + a_\Phi
\]

\[
\hat{b'} \geq K b_\Phi + b_\Phi
\]

\[
\hat{L}' := \left( \hat{L} + \frac{a + \alpha}{K b'_\Phi - K a} K_\# N^{-1} \right) \cdot L + \frac{b_\Phi + 1}{b_\Phi - a_\Phi} K_\# N^{-1}
\]

For a proof see Section C.4

4.2.2. Step 2. Evolution on the fibers. This is given by application of \((G_i; \text{Id}_{T^{N-1}})_*\).

Here there are two effects. On one hand the uniform expansion of \( G_i(\cdot; \hat{x}_i) \) and the contraction properties of its transfer operator, decrease the Lipschitz constant by a factor less than one. On the other hand, since a different map, \( G_i(\cdot; \hat{x}_i) \), is applied on each fiber, a term is added to the Lipschitz constant that depends on how \( G_i(\cdot; \hat{x}_i) \) varies with \( \hat{x}_i \) and, given the assumptions, this term is expected to be \( O(N^{-1}) \).

Proposition 4.11. Let \( H \) satisfy Assumption 4.1 and Assumption 4.2 with datum \((\kappa, K, E)\), with \( N \) sufficiently large. For \( i \in [1, N] \), let \( G \) be as in Definition 4.2.

Suppose that \( a > 0 \) satisfies

\[
\kappa^{-1}(a + D) < a.
\]

Then for any \( b > a \)

\[
(G_i; \text{Id}_{T^{N-1}})_* \left( \mathcal{M}^{(i)}_{a,b,L} \cap C^2_\alpha \right) \subseteq \mathcal{M}^{(i)}_{a',b',L'}
\]
where

\[ a' := \kappa^{-1}(a + D) \]
\[ b' > \kappa^{-1}(b + D) \]
\[ L' := \Lambda L + \kappa \frac{1 + \alpha}{b - a} N^{-1} \]

and where \( \Lambda := 1 - e^{\text{diam}(b',\kappa^{-1}(b + D))} \), with \( \text{diam}(b',\kappa^{-1}(b + D)) \) the diameter of \( \mathcal{V}_{\kappa^{-1}(b + D)} \) in \( \mathcal{V}_b \).

The proof of this proposition is given in Section C.5.

4.2.3. Step 3. Evolution of the fibers. Recall that the evolution of the fibers is prescribed by \( (\text{Id}_\mathbb{T}; \hat{G}_i)^* \).

Proposition 4.12. Under Assumption 4.1 and Assumption 4.2 with datum \( (\kappa, K, E) \), and for \( N \) sufficiently large, given \( \mu \in \mathcal{M}_{a,b,L}(\mathbb{T}^N) \) with density \( \eta \) and such that \( \hat{\Pi}_i \mu \in \mathcal{M}_{a,b,L}(\mathbb{T}^N) \), then

\[ (\text{Id}_\mathbb{T}; \hat{G}_i)^* \mu \in \mathcal{M}_{a',b',L'} \]

with

\[ L'' := (1 + a) \kappa^{-1} + D K L' + \kappa \frac{1 + \alpha}{b - a} N^{-1} \]

and \( K L' \to 1 \) as \( L' \to 0 \).

Remark 4.13. Recall that in the above \( D \to 0 \) as \( K, E \to 0 \).

The proof of this proposition is given in Appendix C.6.

4.2.4. Invariance of quasi-product measures.

Proposition 4.14. Assume that \( H : \mathbb{T}^N \to \mathbb{T}^N \) is a local diffeomorphism satisfying Assumption 4.1 and Assumption 4.2. Furthermore, assume that:

i) for \( K \) as in Lemma 4.6

\[ \max\{\kappa^{-1}K, \kappa^{-1}K^2\} < 1 \]

ii) there are \( a_0 > 0 \) satisfying\(^{11}\)

\[ \kappa^{-1}(Ka_0 + a \Phi + D) < a_0 \]

and \( b_0 > Ka_0 + a \Phi \) such that, calling \( \Lambda := 1 - e^{\text{diam}(b_0,\kappa^{-1}(b_0 + D))} \) where \( \text{diam}(b_0,\kappa^{-1}(b_0 + D)) \) is the diameter of \( \mathcal{V}_{b_0,\kappa^{-1}(b_0 + D)} \) in \( \mathcal{V}_{b_0} \),

\[ [(1 + a_0)\kappa^{-1} + D] \Lambda \mathcal{L} < 1. \]

Then there are \( C \geq 1, \alpha_0 \geq 0 \) independent of \( N \), such that for all \( N \) sufficiently large,

\[ H_*(\mathcal{M}_{a_0,b_0,CN^{-1}} \cap \mathcal{C}_{\alpha_0}^2) \subset \mathcal{M}_{a_0,b_0,CN^{-1}} \cap \mathcal{C}_{\alpha_0}^2. \]

\(^{11}\)The existence of such an \( a_0 \) is always guaranteed provided that \( \kappa^{-1}K < 1 \).
5. Concentration inequalities for quasi-product measures

The space $\mathcal{M}_{a,b,O(N-1)}(T^N)$, contains measures whose disintegrations with respect to coordinate foliations have log-Lipschitz densities and Lipschitz dependence on the leaves with constant of order $N^{-1}$. For these measures the dependence between the coordinates tends to zero when $N \to \infty$, from which the denomination quasi-product measures. It turns out that measures in $\mathcal{M}_{a,b,O(N-1)}$ satisfy concentration estimates, similar to those of product measures.

**Definition 5.1.** Given a function $\psi : T^k \to \mathbb{R}$, define its oscillation with respect to the $j$-th, coordinate as

$$\text{Osc}_j \psi := \sup \{|\psi(x_j; \mathbf{x}_j) - \psi(x'_j; \mathbf{x}_j)| : x_j, x'_j \in T, \mathbf{x}_j \in T^k\}.$$  

For $k \leq N$ and $\alpha \geq 0$, define

$$O_\alpha(T^k) := \{\psi : T^k \to \mathbb{R} : \forall j \in [1, k] \text{ Osc}_j(\psi) \leq \alpha, \psi \in L^1(T^k, \text{Leb})\}.$$  

We are going to prove the following theorem:

**Theorem 5.2.** Fix $a, b > 0$. Then, there is a constant $K > 0$ such that for every $N \in \mathbb{N}$, $\mu \in \mathcal{M}_{a,b,N-1}$, and $g \in O_{N-1}(T^N)$

$$\mu(\{|g - \mathbb{E}_\mu[g]| \geq \varepsilon\}) \leq 2 \exp \left[-\frac{\varepsilon^2 N}{1 + O(N^{-1})}\right].$$  

To prove this concentration result we are going to use the martingale approach (Azuma-Hoeffding) as in the classical proofs of results such as McDirmind’s inequality [M+89, Hoe94, Azu67] (for a general treatment to the concentration of measures phenomenon see [Tal95, Led01]). This approach relies on the following theorem

**Theorem 5.3** (Azuma-Hoeffding). Let $\{X_n\}_{n \geq 0}$ be a martingale with respect to the increasing sequence of $\sigma$-algebras $\{\mathcal{F}_n\}_{n \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_0 = 0$. Let $M_i := X_{i+1} - X_i$, if $|M_i| \leq \sigma_i$, then, for every $\varepsilon > 0$ and $n \in \mathbb{N}$

$$\mathbb{P}(X_n \geq \alpha) \leq \exp \left[-\frac{\alpha^2}{\sum_{i=0}^{n-1} \sigma_i^2}\right].$$  

We now construct a martingale that will allow us to prove Theorem 5.2 using Theorem 5.3

**Definition 5.4.** For any measure $\mu$ and observable $g : T^N \to \mathbb{R}$ define: $g_N := g$; and for $k \in [1, N], g_{N-k} : T^N \to \mathbb{R}$

$$g_{N-k}(\mathbf{x}) := \int_{T^N \setminus \sum_{i=0}^{n-1} \sigma_i^2} g(x_1, ..., x_k; x'_k, ..., x'_N) \, d\mu(x_1, ..., x_k)$$  

where $\mathbf{x} = (x_1, ..., x_N)$, and $\mu(x_1, ..., x_k)$ is the conditional of $\mu$ on $\{(x_1, ..., x_k)\} \times T^{N-k}$.

Notice that the values of $g_{N-k}$ depend only on the first $k$ variables $(x_1, ..., x_k)$. This implies the following proposition whose proof is omitted.

**Proposition 5.5.** For every $k \in [0, N]$, the random variable $g_k : T^N \to \mathbb{R}$ is measurable w.r.t. $\mathcal{B}_k := \mathcal{B}(T^{N-k}) \times \{0, T^k\} \subset \mathcal{B}(T^N)$ and the sequence of random variables $\{g_k\}_{k=0}^N$ is a martingale with respect to the decreasing sequence of $\sigma$-algebras $\{\mathcal{B}_k\}_{k=0}^N$.

In Section 5.1 below we are going to prove that
Proposition 5.6. Consider \( \mu \in \mathcal{M}_{a,b,CN}^{-1} \), \( g \in \mathcal{O}_{N-1}(\mathbb{T}^N) \), and for every \( k \in [0, N] \) let \( g_k : \mathbb{T}^N \to \mathbb{R} \) be as in Definition 5.4.

Then, for \( N \) sufficiently large,

\[
|g_{k+1} - g_k| \leq N^{-1} + O(N^{-2})
\]

for every \( k \in [1, N] \).

For the moment, let’s show how Theorem 5.2 follows from Proposition 5.6.

Proof of Theorem 5.2. With reference to (28), notice that \( g_0 = \mathbb{E}_\mu[g] \). On the probability space \( (\mathbb{T}^N, \mathcal{B}(\mathbb{T}^N), \mu) \), define the martingale \( X_k := g_k - \mathbb{E}_\mu[g] \). By Proposition 5.6, \( |X_k - X_{k-1}| \leq CN^{-1} + O(N^{-2}) \) with \( C := 1 + O(N^{-1}) \).

Therefore, \( \{X_k\} \) satisfies the assumptions of Theorem 5.3, and

\[
\mu(g_n - \mathbb{E}_\mu[g] > \varepsilon) \leq \exp\left[ -\frac{\varepsilon^2 N}{C^2} \right].
\]

Repeating the same argument for the martingale \( Y_k := -g_k + g_0 \), one gets

\[
\mu(-g_n + \mathbb{E}_\mu[g] > \varepsilon) \leq \exp\left[ -\frac{\varepsilon^2 N}{C^2} \right].
\]

and therefore

\[
\mu(|g_n - \mathbb{E}_\mu[g]| > \varepsilon) \leq 2 \exp\left[ -\frac{\varepsilon^2 N}{C^2} \right].
\]

\( \square \)

The next subsection is dedicated to the proof of Proposition 5.6.

5.1. Proof of Proposition 5.6. Fix a measure \( \mu \in \mathcal{M}_{a,b,CN}^{-1} \) on \( \mathbb{T}^N \). For every \( k \in [1, N] \), define

\[
B(k) := \sup \left\{ \int_{\mathbb{T}^{N-k}} \psi \, d\left[ \mu(x_{i_1}, \ldots, x_{i_j}, \ldots, x_{i_k}) - \mu(x_{i_1}', \ldots, x_{i_j}', \ldots, x_{i_k}') \right] : j \in [1, k]; \psi \in \mathcal{O}_1(\mathbb{T}^{N-k}); \right. 
\]

\[
\left. i_1, \ldots, i_j \in [1, N] \text{ distinct}; x_{i_1}, \ldots, x_{i_k}, x'_{i_j} \in \mathbb{T} \right\}
\]

(30)

These quantities are going to play a crucial role in obtaining the bound in equation (29).

The lemma below follows immediately.

Lemma 5.7. For any \( k \in [1, N] \), \( j \in [1, k] \), \( i_1, \ldots, i_j \in [1, N] \) distinct, \( x_{i_1}, \ldots, x_{i_k}, x'_{i_j} \in \mathbb{T} \), and \( \psi : \mathbb{T}^{N-k} \to \mathbb{R} \) with \( \text{Osc} \, \psi < \infty \),

\[
\int_{\mathbb{T}^{N-k}} \psi \, d\left[ \mu(x_{i_1}, \ldots, x_{i_j}, \ldots, x_{i_k}) - \mu(x'_{i_j}, \ldots, x_{i_k}) \right] \leq B(k) \, \text{Osc} \, \psi.
\]

Proposition 5.8. Consider \( \mu \in \mathcal{M}_{a,b,KN}^{-1} \) and define \( B(k) \) as in (30). Then, for \( N \) sufficiently large there is a constant \( A > 0 \) depending on \( b \) and \( K \) only such that

\[
B(k) \leq \frac{A}{N}
\]

for all \( k \in [1, N - 1] \).

Before proceeding with the proof of the proposition above, let’s show how it implies Proposition 5.6.
Proof of Proposition 5.6. Recalling Definition 5.4

\[ |g_{N-k}(x_1, \ldots, x_k) - g_{N-k-1}(x_1, \ldots, x_k, x_{k+1})| = \]

\[ = \left| \int_{\mathbb{T}} d\Pi_{k+1} \mu(x_1, \ldots, x_k) \left( \int_{\mathbb{T}^{N-k-1}} g(x_1, \ldots, x_k, x_{k+1}', y) \, d\mu(x_1, \ldots, x_k, x_{k+1}')(y) - \int_{\mathbb{T}^{N-k-1}} g(x_1, \ldots, x_k, y) \, d\mu(x_1, \ldots, x_k, y)'(y) \right) \right| \]

Now

\[ \left| \int_{\mathbb{T}^{N-k-1}} g(x_1, \ldots, x_k; x_{k+1}', y) \, d\mu(x_1, \ldots, x_k, x_{k+1}')(y) - \int_{\mathbb{T}^{N-k-1}} g(x_1, \ldots, x_k, y) \, d\mu(x_1, \ldots, x_k, y)'(y) \right| \leq \]

\[ \leq \left| \int_{\mathbb{T}^{N-k-1}} g(x_1, \ldots, x_k; x_{k+1}', y) \, d\mu(x_1, \ldots, x_k, x_{k+1}')(y) - \mu(x_1, \ldots, x_k, y) \right| + \]

\[ + \int_{\mathbb{T}^{N-k-1}} \left[ g(x_1, \ldots, x_k; x_{k+1}', y) - g(x_1, \ldots, x_k, y) \right] \, d\mu(x_1, \ldots, x_k')(y) \]

\[ \leq B(k + 1) \text{Osc}(g) + \text{Osc}_{k+1}(g) \]

\[ \leq AN^{-2} + N^{-1}. \]

To prove Proposition 5.8 we are going to need the two lemmas below.

If \( \nu \) is a measure in \( \mathbb{T}^N \), \( i_{k+1}, \ldots, i_N \in [1, N] \) are distinct, denote by \( \Pi_{[i_{k+1}, \ldots, i_N]} \nu \) the marginal of \( \nu \) on the torus \( \mathbb{T}^{N-k} \) relative to the coordinates \( i_{k+1}, \ldots, i_N \) coordinates of \( \mathbb{T}^N \).

**Lemma 5.9.** Let \( \mu \in \mathcal{M}_{a,b;KN-1} \) and \( B(k) \) be defined as above. For every permutation \( (i_1, \ldots, i_N) \) of the indices in \( [1, N] \), \( k \in [1, N] \), \( j \in [1, k-1] \), \( x_{i_1}, \ldots, x_{i_{k-1}}, x_{i_j}' \in \mathbb{T} \), and \( \psi \in L^1(\mathbb{T}^{N-k}, \text{Leb}) \) with \( \text{Osc} \psi < \infty \)

\[ \int_{\mathbb{T}^{N-k}} \psi \, d\Pi_{[i_{k+1}, \ldots, i_N]}[\mu(x_1, \ldots, x_j, \ldots, x_{i_{k-1}}) - \mu(x_1, \ldots, x_j', \ldots, x_{i_{k-1}})] \leq 2B(k) \text{Osc} \psi. \]

**Proof.** Modulo renaming the coordinates, one can assume that the permutation \( (i_1, ..., i_N) \) equals \( (1, ..., N) \).

\[ \int_{\mathbb{T}^{N-k}} \psi \, d\Pi_{[k+1, \ldots, N]}[\mu(x_1, \ldots, x_j, \ldots, x_{k-1}) - \mu(x_1, \ldots, x_j', \ldots, x_{k-1})] = \]

\[ = \int_{\mathbb{T}} d\Pi_{k+1} \mu(x_1, \ldots, x_j, x_{j+1}', \ldots, x_k)(y) \int_{\mathbb{T}^{N-k}} \psi(y) \, d\mu(x_1, \ldots, x_j, x_{j+1}', \ldots, x_k)(y). \]

For every \( x_j, x_j', x_{j+1}', x_k \in \mathbb{T} \)

\[ \left| \int_{\mathbb{T}^{N-k}} \psi(y) \, d\mu(x_1, \ldots, x_j, x_{j+1}', x_k')(y) - \int_{\mathbb{T}^{N-k}} \psi(y) \, d\mu(x_1, \ldots, x_j, x_{j+1}', x_k')(y) \right| \leq \]

\[ \leq \left| \int_{\mathbb{T}^{N-k}} \psi(y) \, d\mu(x_1, \ldots, x_j, x_{j+1}', x_k')(y) - \int_{\mathbb{T}^{N-k}} \psi(y) \, d\mu(x_1, \ldots, x_{j+1}', x_{j+1}', x_k')(y) \right| \]

\[ + \int_{\mathbb{T}^{N-k}} \psi(y) \, d\mu(x_1, \ldots, x_j, x_{j+1}', x_k')(y) \right| \leq 2B(k) \text{Osc} \psi \]

by definition of \( B(k) \), and this implies the lemma. \( \square \)
Lemma 5.10. Let $\mu \in \mathcal{M}_{a,b,K^{N-1}}$. For every $k \in [1,N]$, any distinct indices $i_1,\ldots, i_{k-1}, i_k \in [1,N]$, any $j \in [1,k-1]$, and $\psi : T \to \mathbb{R}$ with $\text{Osc} \, \psi < \infty$

\begin{equation}
\int_T \psi \, d\Pi_k \left[ \mu(x_{i_1},\ldots,x_{i_{k-1}}) - \mu(x_{i_1}',\ldots,x_{i_{k-1}}') \right] \leq B(N-1) \left[ 1 + 2B(k) \right] \text{Osc} \psi.
\end{equation}

Proof. Let’s assume without loss of generality that $(i_1,\ldots,i_k) = (1,\ldots,k)$ and $j = 1$. By Fubini

\begin{align*}
\int_T \psi \, d\Pi_k \mu(x_{i_1},\ldots,x_{i_{k-1}}) = \\
= \int_{T^{N-k}} d\Pi_{[k+1,\ldots,N]} \mu(x_{i_1},\ldots,x_{i_{k-1}}) (x_{k+1}',\ldots,x_N') \int_T \psi(x_k') \, d\mu(x_{i_1},\ldots,x_{i_{k-1}},x_{k+1}',\ldots,x_N')(x_k')
\end{align*}

therefore

\begin{align*}
\int_T \psi \, d\Pi_k [\mu(x_{i_1},\ldots,x_{i_{k-1}}) - \mu(x_{i_1}',\ldots,x_{i_{k-1}})] = \\
= \int_{T^{N-k}} d\Pi_{[k+1,\ldots,N]} \left[ \mu(x_{i_1},\ldots,x_{i_{k-1}}) - \mu(x_{i_1}',\ldots,x_{i_{k-1}}) \right] (y) \int_T \psi(x_k') \, d\mu(x_{i_1},\ldots,x_{i_{k-1}};y)(x_k') + \\
+ \int_{T^{N-k}} d\Pi_{[k+1,\ldots,N]} \mu(x_{i_1},\ldots,x_{i_{k-1}})(y) \int_T \psi(x_k') \, d \left[ \mu(x_{i_1},\ldots,x_N;y) - \mu(x_{i_1}',\ldots,x_N;\psi)(x_k') \right].
\end{align*}

The second term can be bounded by $B(N-1) \text{Osc} \psi$. For the first term, having fixed $x_1, x_1', x_2, \ldots, x_{k-1}$,

$$
\overline{\psi} := \int_T \psi(x_k') \, d\mu(x_{i_1},\ldots,x_{i_{k-1}};y)(x_k')
$$

is a function $\overline{\psi} : T^{N-k} \to \mathbb{R}$ of $y$ only. Applying Lemma 5.9 we get

\begin{align*}
\int_{T^{N-k}} d\Pi_{[k+1,\ldots,N]} \left[ \mu(x_{i_1},\ldots,x_{i_{k-1}}) - \mu(x_{i_1}',\ldots,x_{i_{k-1}}) \right] (y) \overline{\psi}(y) \leq 2B(k) \text{Osc} \overline{\psi}
\end{align*}

and by definition of $B(N-1)$,

$$
\text{Osc} \overline{\psi} \leq B(N-1) \text{Osc} \psi.
$$

Putting these bounds together the lemma is proved. \hfill \square

Proof of Proposition 5.8. For every $j \in [1,k-1]$, any $x_1,\ldots,x_{k-1}, x_j' \in T$ and any $\psi \in \mathcal{O}_1(T^{N-k+1})$, we want to estimate

\begin{equation}
\int_{T^{N-k+1}} \psi \, d[\mu(x_{i_1},\ldots,x_{i_{k-1}}) - \mu(x_{i_1},\ldots,x_{i_{k-1}},x_j')].
\end{equation}

Without loss of generality let’s put $j = 1$. By definition of disintegration

\begin{align*}
\int_{T^{N-k+1}} \psi \, d\mu(x_{i_1},\ldots,x_{i_{k-1}}) = \\
= \int_T d\Pi_k \mu(x_{i_1},\ldots,x_{i_{k-1}}) (x_k') \int_{T^{N-k}} \psi(x_k';y) \, d\mu(x_{i_1},\ldots,x_{i_{k-1}},x_k')(y)
\end{align*}
From which it follows that

\[
\int_{T^{N-k}} \psi(y) \, d[\mu(x_1,\ldots,x_{k-1}) - \mu(x_1',\ldots,x_{k-1})](y) = \\
= \int_T d\Pi_k \left[ \mu(x_1,\ldots,x_{k-1}) - \mu(x_1',\ldots,x_{k-1}) \right] (x_k') \int_{T^{N-k}} \psi(x_k'; y) \, d\mu(x_1,\ldots,x_{k-1},x_k')(y) \\
+ \int_T d\Pi_k \mu(x_1',\ldots,x_{k-1}) \int_{T^{N-k}} \psi(x_k'; y) \, d\left[ \mu(x_1,\ldots,x_{k-1},x_k') - \mu(x_1',\ldots,x_{k-1},x_k') \right] (y)
\]

(34)

\[
\leq B(N-1)[1 + 2B(k)] \text{Osc} \left( \int_{T^{N-k}} \psi(x_k; y) \, d\mu(x_1,\ldots,x_{k-1},x_k)(y) \right) + B(k)
\]

(35)

where the first equality is obtained by adding and subtracting the same term; inequality (34) follows by application of Lemma 5.10 and the definition of \( B(k) \); inequality (35) follows from

\[
\left| \int_{T^{N-k}} \psi(x_k; y) \, d\mu(x_1,\ldots,x_{k-1},x_k)(y) - \int_{T^{N-k}} \psi(x_k'; y) \, d\mu(x_1,\ldots,x_{k-1},x_k')(y) \right| \leq \\
\leq \int_{T^{N-k}} |\psi(x_k; y) - \psi(x_k'; y)| \, d\mu(x_1,\ldots,x_{k-1},x_k)(y) + \\
+ \left| \int_{T^{N-k}} \psi(x_k'; y) \, d\left[ \mu(x_1,\ldots,x_{k-1},x_k) - \mu(x_1,\ldots,x_{k-1},x_k)(y) \right) - d\mu(x_1,\ldots,x_{k-1},x_k')(y) \right| \\
\leq \text{Osc} \psi + B(k).
\]

The same bound holds starting with any permutation \((i_1,\ldots,i_N)\) of \((1,\ldots,N)\) and any \( j \in \{i_1,\ldots,i_k\} \), therefore

\[
B(k-1) \leq B(k)[1 + 3B(N-1)] + 2B(N-1)B(k)^2.
\]

Since \( \mu \in \mathcal{M}_{a,b,CN^{-1}} \), a computation gives

\[
B(N-1) \leq C_bN^{-1},
\]

which implies that there are \( C' > 0 \) and \( C'' > 0 \), depending on \( b \) and \( C \) only such that

\[
B(k-1) \leq B(k)[1 + C'N^{-1}] + C'N^{-1}B(k)^2 \\
\leq (1 + C'N^{-1})^{N-1}B(N-1) + C'N^{-1} \sum_{i=N-1}^k (1 + C'N^{-1})^{2(i-k)}B(i)^2 \\
\leq C''N^{-1} + C''N^{-1} \sum_{i=N-1}^k B(i)^2
\]
We guess\footnote{12} for some constant $A > 0$. Assuming the ansatz holds for $B(i)$ with $i \in [k, N - 1]$ we get

\begin{equation}
B(k - 1) \leq C'' N^{-1} + C'' N^{-1} \sum_{i=N-1}^{k} \frac{A^2}{N^2}
\leq C'' N^{-1} + C'' \frac{A^2}{N^2}
\end{equation}

and, picking $A$ such that\footnote{13}

\begin{equation}
A \geq C'' + \frac{C'' A^2}{N^2}
\end{equation}

is verified for $k - 1$. \hfill \Box

6. Proofs of the main results

6.1. The good set. We’re going to use the concentration of measure result in Theorem 5.2 to show that given a map $F : T^N \to T^N$ satisfying Assumption 2.1 and a measure $\mu \in \mathcal{M}_{a,b,CN^{-1}}$, and $\varepsilon > 0$, there is a set $G_{\varepsilon} \subset T^N$ whose complement has measure $\mu$ exponentially small in $N$, such that on $G_{\varepsilon}$, $F_i(\cdot; \hat{x}_i)$ is $C^2$-$\varepsilon$-close to the mean-field approximation $F_{\mu,i}$.

Proposition 6.1. Consider a map $F : T^N \to T^N$ as in (2) satisfying Assumption 2.1 with datum $(\kappa, K, E)$. Then for every $C > 0$, there is $K_# > 0$ depending on $C$ and the datum – in particular independent of $N$ – such that for every measure $\mu \in \mathcal{M}_{a,b,CN^{-1}}$, $\varepsilon > 0$, and $i \in [1, N]$, there is a set $G_{\mu,\varepsilon,i} \subset T^N$ such that

\begin{equation}
d_{C_2}(F_i(\cdot; \hat{x}_i), F_{\mu,i}(\cdot)) \leq \varepsilon \quad \forall (x_i; \hat{x}_i) \in G_{\mu,\varepsilon,i}
\end{equation}

with

\begin{equation}
\mu(G_{\mu,\varepsilon,i}) > 1 - \varepsilon^{-1} \exp\left[-K_#N\varepsilon^2\right].
\end{equation}

Proof. It follows from the assumptions that for every $i \in [1, N]$

\begin{equation}
|\partial^3_i F_i| \leq K + E
\end{equation}

i.e. $K' := K + E$ is an upper bound on the third derivative of $F_i(\cdot; \hat{x}_i)$.

Fix any $i \in [1, N]$. With an $n > 1 + K' \varepsilon^{-1}$, consider $\{z_1, \ldots, z_n\}$ points on $\mathbb{T}$ at distance at most $K' \varepsilon$. For every $\ell = 1, \ldots, n$ define $f_{k,\ell} : T^N \to \mathbb{R}$

\begin{align*}
f_{2,\ell}(x_i; \hat{x}_i) &:= \partial^2_i F_i(z_\ell, \hat{x}_i) \\
f_{1,\ell}(x_i; \hat{x}_i) &:= \partial_i F_i(z_\ell; \hat{x}_i) \\
f_{0,\ell}(x_i; \hat{x}_i) &:= F_i(z_\ell; \hat{x}_i).
\end{align*}
Notice that the dependence of \( f_{j,\ell} \) on \( x_i \) is mute. From the assumptions on \( F \), it follows that \( f_{j,\ell} \in \mathcal{O}_{EN^{-1}}(\mathbb{T}^N) \). Applying Theorem 5.2,
\[
\mu \left( \frac{|f_{k,\ell} - \mathbb{E}_\mu[f_{k,\ell}]|}{K} \geq \varepsilon \right) \leq 2 \cdot \exp \left[ -\frac{\varepsilon^2}{C^2} N \right]
\]
and the set
\[
\mathcal{G}_{\mu,\varepsilon,i} := \bigcap_{k=0}^{2n} \bigcap_{\ell=1}^{m} \{ |f_{1,\ell} - \mathbb{E}_\mu[f_{1,\ell}]| \leq \varepsilon \}
\]
has measure
\[
\mu(\mathcal{G}_{\mu,\varepsilon,i}) \geq 1 - 6(1 + \varepsilon^{-1} K') \cdot \exp \left[ -\frac{\varepsilon^2}{C^2} N \right]
\]
Since \( f_{k,\ell} \) is independent of \( x_i \), if \((x_i; \hat{x}_i) \in \mathcal{G}_{\mu,\varepsilon,i}\) then the whole fiber \( T_{\hat{x}_i} \) is contained in this set. For every \( \hat{x}_i \) such that \( T_{\hat{x}_i} \subset \mathcal{G}_{\mu,\varepsilon,i} \) and every \( z \in \mathbb{T} \), pick \( \ell \) with \(|z_\ell - z| \leq K' \varepsilon^{-1} \). Then
\[
|\partial^2_i F_i(z; \hat{x}_i) - F''_{\mu,i}(z)| \leq |\partial^2_i F_i(z; \hat{x}_i) - \partial^2_i F_i(z_\ell; \hat{x}_i)| + |\partial^2_i F_i(z_\ell; \hat{x}_i) - \mathbb{E}_\mu[f_{2,\ell}]| + \|E_{\mu}[f_{2,\ell} - F''_{\mu,i}(z)]\|
\leq |\partial^2_i F_i|_{\infty} |z - z_\ell| + \varepsilon + \int_{\mathbb{T}_N} |\partial^2_i F_i(z_\ell; \hat{x}_i) - \partial^2_i F_i(z; \hat{x}_i)| \, d\Pi_{\mu}(\hat{x}_i)
\leq 3 \varepsilon.
\]
By Taylor's theorem
\[
\partial_i F_i(z; \hat{x}_i) = \partial_i F_i(z_\ell; \hat{x}_i) + \partial^2_i F_i(z_\ell; \hat{x}_i) \cdot (z - z_\ell) + \frac{1}{2} \partial^3_i F_i(\bar{z}; \hat{x}_i) \cdot (z - z_\ell)^2
\]
\[
F''_{\mu,i}(z) = F''_{\mu,i}(z_\ell) + F''_{\mu,i}(z_\ell) \cdot (z - z_\ell) + \frac{1}{2} F''_{\mu,i}(\bar{z}) \cdot (z - z_\ell)^2
\]
for some points \( \bar{z}, \bar{z}' \in \mathbb{T} \). Then
\[
|\partial_i F_i(z; \hat{x}_i) - F''_{\mu,i}(z)| \leq |\partial_i F_i(z_\ell; \hat{x}_i) - F''_{\mu,i}(z_\ell)| + |\partial^2_i F_i(z_\ell; \hat{x}_i) - F''_{\mu,i}(z_\ell)| \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} |\partial^3_i F_i|_{\infty} (K' \varepsilon^{-1} \varepsilon) \frac{1}{2}
\leq \varepsilon + \frac{1}{2} \varepsilon + \frac{1}{2} \varepsilon
\leq 2 \varepsilon.
\]
where in the second inequality we upper bounded \((z - z_\ell)^2 \leq K' \varepsilon^{-1} \varepsilon \cdot \frac{1}{2} \).
Analogously
\[
F_i(z; \hat{x}_i) = F_i(z_\ell; \hat{x}_i) + \partial_i F_i(z_\ell; \hat{x}_i) \cdot (z - z_\ell) + \frac{1}{2} \partial^2_i F_i(\bar{z}; \hat{x}_i) \cdot (z - z_\ell)^2
\]
\[
F''_{\mu,i}(z) = F''_{\mu,i}(z_\ell) + F''_{\mu,i}(z_\ell) \cdot (z - z_\ell) + \frac{1}{2} F''_{\mu,i}(\bar{z}) \cdot (z - z_\ell)^2
\]
for some points \( \bar{z}, \bar{z}' \in \mathbb{T} \), and similar computations to the ones above yield
\[
|F_i(z; \hat{x}_i) - F''_{\mu,i}(z)| \leq 2 \varepsilon.
\]
Rescaling \( \varepsilon > 0 \), the proof follows. \( \square \)
\footnote{One factor is upper bounded by the diameter of \( \mathbb{T} \).}
Definition 6.2. With the $\mathcal{G}_{\mu,\varepsilon,i}$ provided by the theorem above, we define

$$\mathcal{G}_{\mu,\varepsilon} := \bigcap_{i \in [1,N]} \mathcal{G}_{\mu,\varepsilon,i}.$$

Given $F$ and $\mu \in \mathcal{M}_{a,b,CN^{-1}}$, this is the good portion of phase space where $F$ is close to the mean-field approximation $F_\mu$. Notice that

$$\mu(\mathcal{G}_{\mu,\varepsilon}) > 1 - \varepsilon^{-1}N \exp \left[ -K_\# \varepsilon^2 N \right].$$

6.2. Proof of the main results.

Proof of Theorem 3.5. Let’s start by noticing that if $F$ satisfies Assumption 2.1 with datum $(\kappa, K, E)$, then it satisfies assumptions 1.1 and 1.2 with the same datum. Notice that fixed $K$, increasing $\kappa$ and decreasing $E$, conditions (25) and (26) can be always ensured. For fixed $K$, there are $\kappa_0 > 0$ and $E_0 > 0$ such that for all $\kappa \geq \kappa_0$ and $E \leq E_0$, $F$ satisfies the assumptions of Proposition 4.14 and there are $a_0, b_0, c \geq 0$, and $\alpha_0 \geq 0$ independent of $N$ such that

$$F_* (\mathcal{M}_{a_0,b_0,cN^{-1}} \cap C_{\alpha_0}^2) \subset \mathcal{M}_{a_0,b_0,cN^{-1}} \cap C_{\alpha_0}^2$$

which proves the first point of the theorem.

Let $\mu \in \mathcal{M}_{a,b,cN^{-1}}$ and $G$ as in Definition 4.2 with $H = F$. Recalling (15), since $(\Id_\Gamma, G_i)_*$ acts on all coordinates but the $i$-th one, we have

$$\Pi_\iota F_\mu = \Pi_\iota (G_i; \Id_{\Gamma N^{-1}})_* \Phi_{i_\iota}^{-1} \mu.$$

Define $\nu := (G_i; \Id_{\Gamma N^{-1}})_* \Phi_{i_\iota}^{-1} \mu$. Denoting $g_{\hat{x}_i}(\cdot) := G_i(\cdot; \hat{x}_i)$

$$\nu_{\hat{x}_i} = g_{\hat{x}_i} \left[ \Phi_{i_\iota}^{-1} \mu \right]_{\hat{x}_i}$$

and therefore by (37)

$$\Pi_\iota F_\mu = \int_{\Gamma N^{-1}} \nu_{\hat{x}_i} \ d^\Pi \nu(\hat{x}_i).$$

Since by definition of $\Phi_i$, $\Pi_\iota \Phi_{i_\iota}^{-1} \mu = \Pi_\iota \mu$, we have

$$(F_{\mu,i})_* \Pi_\iota \mu = (F_{\mu,i})_* \Pi_\iota \Phi_{i_\iota}^{-1} \mu$$

$$= (F_{\mu,i})_* \int_{\Gamma N^{-1}} \left[ \Phi_{i_\iota}^{-1} \mu \right]_{\hat{x}_i} \ d^\Pi \Phi_{i_\iota}^{-1} \mu(\hat{x}_i)$$

$$= \int_{\Gamma N^{-1}} (F_{\mu,i})_* \left[ \Phi_{i_\iota}^{-1} \mu \right]_{\hat{x}_i} \ d^\Pi \nu(\hat{x}_i).$$

Therefore

$$\theta_a (\Pi_\iota F_\mu; (F_{\mu,i})_* \Pi_\iota \mu) =$$

$$\theta_a \left( \int_{\Gamma N^{-1}} g_{\hat{x}_i} \left[ \Phi_{i_\iota}^{-1} \mu \right]_{\hat{x}_i} \ d^\Pi \nu(\hat{x}_i); \int_{\Gamma N^{-1}} (F_{\mu,i})_* \left[ \Phi_{i_\iota}^{-1} \mu \right]_{\hat{x}_i} \ d^\Pi \nu(\hat{x}_i) \right).$$

By Proposition 4.10 point ii), $\hat{\Pi}_\iota \nu \in \mathcal{M}_{\alpha,N^0,C'} N^{-1}$, fixed any $\varepsilon > 0$, applying Proposition 6.11 and Proposition A.7, we can find $G$ such that

$$\sup_{\hat{x}_i \in G} \theta_a (\int_{\Gamma N^{-1}} g_{\hat{x}_i} \left[ \Phi_{i_\iota}^{-1} \mu \right]_{\hat{x}_i}; (F_{\mu,i})_* \left[ \Phi_{i_\iota}^{-1} \mu \right]_{\hat{x}_i}) \leq \varepsilon$$
and $\hat{\Pi}_i\nu(G^c) \leq \varepsilon^{-1}\exp[-K\#\varepsilon^2N]$. Then, rewriting the first entry of $\theta_a$ in (38) as the convex combination

$$
\int_{\mathcal{T}_N^{-1}} g_{x,1}[\Phi_i^{-1}\mu]_{\hat{x}_i} \, d\hat{\Pi}_i\nu(\hat{x}_i) = \hat{\Pi}_i\nu(G^c) \int_{G^c} g_{x,1}[\Phi_i^{-1}\mu]_{\hat{x}_i} \, d\hat{\Pi}_i\nu_G(\hat{x}_i) + [1 - \hat{\Pi}_i\nu(G^c)] \int_{G^c} g_{x,1}[\Phi_i^{-1}\mu]_{\hat{x}_i} \, d\hat{\Pi}_i\nu_G(\hat{x}_i)
$$

with $\hat{\Pi}_i\nu_G$ and $\hat{\Pi}_i\nu_G$ the (probability measures) restrictions of $\hat{\Pi}_i\nu$ to $G^c$ and $G$ respectively, and a similar convex combination for the second entry; applying Proposition $[\text{A.3}]$ point iii) we get

$$
\theta_a(\Pi_i,F^*_i\mu, (F_{\mu,i}), \Pi_i\mu) \leq \leq C_a \left[ \hat{\Pi}_i\nu(G^c) \cdot \theta_a \left( \int_{G^c} g_{x,1}[\Phi_i^{-1}\mu]_{\hat{x}_i} \, d\hat{\Pi}_i\nu_{G^c}(\hat{x}_i), \int_{G^c} (F_{\mu,i})_*[\Phi_i^{-1}\mu]_{\hat{x}_i} \, d\hat{\Pi}_i\nu_{G^c}(\hat{x}_i) \right) + \right. \\
+ (1 - \hat{\Pi}_i\nu(G^c)) \cdot \theta_a \left( \int_{G} g_{x,1}[\Phi_i^{-1}\mu]_{\hat{x}_i} \, d\hat{\Pi}_i\nu_G(\hat{x}_i), \int_{G} (F_{\mu,i})_*[\Phi_i^{-1}\mu]_{\hat{x}_i} \, d\hat{\Pi}_i\nu_G(\hat{x}_i) \right) \right]
$$

where we upper bounded the distance in the first term with $D_a$, the diameter of the cone. By triangle inequality

$$
\theta_a \left( g_{x,1}[\Phi_i^{-1}\mu]_{\hat{x}_i}, (F_{\mu,i})_*[\Phi_i^{-1}\mu]_{\hat{x}_i} \right) \leq \theta_a \left( g_{x,1}[\Phi_i^{-1}\mu]_{\hat{x}_i}, f_{x,1}[\Phi_i^{-1}\mu]_{\hat{x}_i} \right) + \theta_a \left( f_{x,1}[\Phi_i^{-1}\mu]_{\hat{x}_i}, (F_{\mu,i})_*[\Phi_i^{-1}\mu]_{\hat{x}_i} \right) \leq O(N^{-1}) + \varepsilon
$$

where we used that Lemma [4,3] and Proposition $[\text{A.7}]$ imply

$$
\theta_a \left( g_{x,1}[\Phi_i^{-1}\mu]_{\hat{x}_i}, f_{x,1}[\Phi_i^{-1}\mu]_{\hat{x}_i} \right) \leq K\#d_{C^2}(g_{x,1}, f_{x,1}) \leq O(N^{-1}),
$$

and (39). Putting all the estimates together

$$
\theta_a(\Pi_i,F^*_i\mu, (F_{\mu,i}), \Pi_i\mu) \leq K\# \varepsilon^{-1}\exp(-K\#\varepsilon^2N) + \varepsilon + N^{-1}
$$

and picking $\varepsilon = N^{-\gamma}$ for any $\gamma < 1/2$

$$
\theta_a(\Pi_i,F^*_i\mu, (F_{\mu,i}), \Pi_i\mu) \leq N^{-\gamma} \exp(-K\#N^{1-2\gamma}) + N^{-\gamma} + N^{-1} \leq C\gamma N^{-\gamma}
$$

for some $C_\gamma > 0$ that can be chosen uniformly in $N$. Since $F_\mu$ is an uncoupled map, 

$$(F_{\mu,i})_\Pi\mu = \Pi_i(F_\mu)_\mu = \Pi_iF_\mu
$$

and point 2. follows. \qed

Proof of Corollary $[\text{3.7}]$. Let’s start noticing that Proposition $[\text{A.7}]$ implies that $F$ restricted to $\mathcal{M}_{a_0,b_0,c,N^{-1}}$ is Lipschitz, and in particular, there is $C > 0$ such that, for any $\mu, \nu \in \mathcal{M}_{a_0,b_0,c,N^{-1}}$ and $i \in [1, N]$

$$
\theta_{b_0}(\Pi_i,F_\mu, \Pi_i(F_\nu)) = \theta_{b_0}(\Pi_i(F_\mu)_\mu, \Pi_i(F_\nu)_\nu) \leq \theta_{b_0}(\Pi_i(F_\mu)_\mu, \Pi_i(F_\mu)_\nu) + \theta_{b_0}(\Pi_i(F_\mu)_\nu, \Pi_i(F_\nu)_\nu) \\
= \theta_{b_0}(\Pi_i(F_{\mu,i})_\mu, \Pi_i(F_{\nu,i})_\nu) + \theta_{b_0}(\Pi_i(F_{\mu,i})_\nu, \Pi_i(F_{\nu,i})_\nu) \leq C \sup_{i \in [1,N]} \theta_{b_0}(\Pi_i\mu, \Pi_i\nu)
$$

(40)
where to bound the second term in (40) we used Proposition A.7 and that
\[ d_{C^2}(F_{t,i}, F_{0,i}) \leq C' \sup_{i \in [1,N]} \theta_{0b} (\mu_i, \nu_i) \]
for some \( C' > 0 \).

It follows from point 2. of Theorem 3.5 that
\[ \theta_{0b} (\Pi, F_s^t \mu, \Pi, F_s^{t-1} \mu) \leq \theta_{0b} (\Pi, F_s^t (F_s^{t-1} \mu), \Pi, F_s (F_s^{t-1} \mu)) \]
\[ + \theta_{0b} (\Pi, F_s^{t-1} \mu, \Pi, F_s^{t-1} \mu) \]
\[ \leq C \gamma N^{-\gamma} + C \sup_{i \in [1,N]} \theta_{0b} (\Pi, F_s^{t-1} \mu, \Pi, F_s^{t-1} \mu) \]
and by induction
\[ \sup_{i \in [1,N]} \theta_{0b} (\Pi, F_s^{t-1} \mu, \Pi, F_s^{t-1} \mu) \leq \left( \sum_{r=0}^{t-1} C'' \right) C \gamma N^{-\gamma} \]
that for fixed \( t > 0 \) goes to zero as \( N \to \infty \).

If \( \mu \in \mathcal{M}_{a,0,b,cN^{-1}}(\mathbb{T}^N) \) and
\[ \Pi_{[1,k]} \mu = \int_{\mathbb{T}^{N-k}} \mu(x_{k+1}, \ldots, x_N) \, d\Pi_{[k+1,N]} \mu(x_{k+1}, \ldots, x_N), \]
then \( \Pi_{[1,k]} \mu \in \mathcal{M}_{a,0,b,cN^{-1}}(\mathbb{T}^k) \). This, together with (41) implies that the density of \( \Pi_{[1,k]} F_s^t \mu \) converges to \( \Pi_{[1,k]} F_s^{t-1} \mu \) as shown below: First of all, since \( \Pi_{[1,k]} F_s^t \mu \in \mathcal{M}_{a,0,b,cN^{-1}}(\mathbb{T}^k) \),
\[ \theta_{0b} (\Pi_{[1,k]} F_s^{t-1} \mu, \Pi_{[1,k]} F_s^{t-1} \mu) = O(N^{-1}) \]
for all \( \hat{x}_1 = (x_2, \ldots, x_k) \in \mathbb{T}^{k-1} \) - (\( \Pi_{[1,k]} F_s^{t-1} \mu \))\( \hat{x}_1 \) is the conditional of \( \Pi_{[1,k]} F_s^t \mu \) on \( \mathbb{T} \times \{ \hat{x}_1 \} \) - and therefore
\[ \| \Pi_{[1,k]} F_s^t \mu - (\Pi_{[1,k]} F_s^{t-1} \mu) \otimes \Pi_{[2,k]} F_s^t \mu \|_{TV} \leq \]
\[ \leq \| \Pi_{[1,k]} F_s^t \mu - (\Pi_{[1,k]} F_s^{t-1} \mu) \otimes \Pi_{[2,k]} F_s^{t-1} \mu \|_{TV} + \]
\[ + \| (\Pi_{[1,k]} F_s^{t-1} \mu) \otimes \Pi_{[2,k]} F_s^{t-1} \mu - (\Pi_{[1,k]} F_s^{t-1} \mu) \otimes \Pi_{[2,k]} F_s^{t-1} \mu \|_{TV} \]
\[ = \left\| \int_{\mathbb{T}^{k-1}} d\Pi_{[2,k]} F_s^{t-1} \mu (\hat{x}_1) \left[ (\Pi_{[1,k]} F_s^{t-1} \mu ) \hat{x}_1 - \Pi_{[1,k]} F_s^{t-1} \mu \right] \right\|_{TV} + \]
\[ + \| (\Pi_{[1,k]} F_s^{t-1} \mu ) - (\Pi_{[1,k]} F_s^{t-1} \mu ) \|_{TV} \]
\[ \leq O(N^{-1}) + O(N^{-\gamma}) \]
where to bound the first term we used \( \| \cdot \|_{TV} \), and the second term is bounded with \( \| \cdot \|_{TV} \). Repeating the argument inductively \( (k \) times), one obtains
\[ \| \Pi_{[1,k]} F_s^t \mu - (\Pi_{[1,k]} F_s^{t-1} \mu) \otimes ... \otimes (\Pi_{k} F_s^{t} \mu) \|_{TV} \leq O(N^{-\gamma}) \].

\[ \Box \]

\(^{15}\)To go from the Hilbert metric \( \theta_{0b} \) to the Total Variation norm \( \| \cdot \|_{TV} \) we are using that the measures we compare are all probability measures with continuous density, and that a bound on \( \theta_{0b} \), implies a bound in \( C^0 \), which gives a bound with respect to the Total Variation norm.
Proof of Corollary 3.8. Consider $\mu \in B_\delta(\overline{\nu})$. From point 2. of Theorem 3.5 follows that

$$\theta_a(\Pi_1 F_* \mu, \Pi_0 \overline{\nu}) \leq \theta_a(\Pi_1 F_* \mu, \Pi_0 F_* \mu) + \theta_a(\Pi_0 F_* \mu, \Pi_0 \overline{\nu})$$

$$\leq C_\gamma N^{-\gamma} + \lambda \sup_{t \in [1,N]} \theta_a(\Pi_0 \mu, \Pi_0 \overline{\nu})$$

which implies that the neighborhood $B_{C_\gamma N^{-\gamma}}(\overline{\nu})$ with $C_\gamma' := C_\gamma/(1 - \lambda)$ is invariant under $F_*$ (provided that $N$ is sufficiently large so that $C_\gamma' N^{-\gamma} < \delta$).

APPENDIX A. CONES OF DIFFERENTIABLE LOG-LIPSCHITZ FUNCTIONS

A convex cone $\mathcal{V}$ is a subset of a linear vector space $\mathcal{V}$ such that: $t_1 v_1 + t_2 v_2 \in \mathcal{V}$ for every $v_1, v_2 \in \mathcal{V}$ and $t_1, t_2 > 0$; and $\overline{\mathcal{V}} \cap \mathcal{V} = \{0\}$ where

$$\overline{\mathcal{V}} := \{ w : \exists v \in \mathcal{V} \text{ and } t_n \to 0 \text{ s.t. } w + t_n v \in \mathcal{V} \}.$$ 

It is known that on convex cones is defined a Hilbert projective metric $\theta : \mathcal{V} \times \mathcal{V} \to \mathbb{R}_+^*$ in the following way:

$$\theta(v_1, v_2) := \log \beta(v_1, v_2) + \log \beta(v_2, v_1)$$

where

$$\beta(v, w) := \inf \{ t > 0 : tw - v \in \mathcal{V} \}.$$ 

The following theorem states that linear maps are contractions with respect to the Hilbert metric.

**Theorem A.1** ([Liv95, Via97]). Assume $\mathcal{V} \subset \mathcal{V}$ and $\mathcal{V}' \subset \mathcal{V}'$ are two convex cones with Hilbert metrics $\theta$ and $\theta'$ respectively. If $\mathcal{P} : \mathcal{V} \to \mathcal{V}'$ is a linear transformation, then

$$\theta' (\mathcal{P} \psi_1, \mathcal{P} \psi_2) \leq [1 - e^{-\text{diam}(\mathcal{P}(\mathcal{V}), \mathcal{V}')} ] \theta(\psi_1, \psi_2)$$

for all $\psi_1, \psi_2 \in \mathcal{V}$, where $\text{diam}(\mathcal{P}(\mathcal{V}), \mathcal{V})$ is the diameter of $\mathcal{P}(\mathcal{V})$ in $\mathcal{V}'$.

For a recent treatment of convex cones and the Hilbert metric see [LN12, LN13], for applications to dynamical systems see, among others, [Liv95, Via97].

A.1. Log-Lipschitz functions. Consider $\mathcal{V}_a$, the cone of functions defined in (6). The result below gives an explicit expression for $\beta_a$ defined as in (43).

**Proposition A.2.** Consider $\psi_1, \psi_2 \in \mathcal{V}_a$, then

$$\beta_a(\psi_1, \psi_2) = \sup_{x \in \mathbb{T}} \left\{ \frac{\psi_2(x)}{\psi_1(x)} - \frac{a \psi_2 - \psi_2'}{a \psi_1 - \psi_1'}(x), \frac{a \psi_2 + \psi_2'}{a \psi_1 + \psi_1'}(x) \right\}$$

**Proof.** For $t > 0$, let $\psi_t := t \psi_1 - \psi_2$. Notice that $\psi_t \in C^2$ for every $t$. Therefore $\psi_t \in \mathcal{V}_a$ is equivalent to: $\dot{\psi}_t > 0$ and

$$-a < \frac{d}{dx} \log \psi_t < a.$$ 

One can check that $\psi_t > 0$ if and only if

$$t > \sup_{x \in \mathbb{T}} \frac{\psi_2(x)}{\psi_1(x)}$$

which is the first condition we impose on $t$. Moreover

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16Is zero when evaluated on proportional vectors, but always nonzero otherwise.
\[
\frac{d}{dx} \log \psi = \frac{t\psi'_1 - \psi'_2}{t\psi_1 - \psi_2}
\]

Therefore
\[
\frac{t\psi'_1 - \psi'_2}{t\psi_1 - \psi_2} < a \quad \text{iff} \quad t > \frac{a\psi_2 - \psi'_2}{a\psi_1 - \psi'_1}
\]

and
\[
\frac{t\psi'_1 - \psi'_2}{t\psi_1 - \psi_2} > -a \quad \text{iff} \quad t > \frac{a\psi_2 + \psi'_2}{a\psi_1 + \psi'_1}
\]

where we used that \(a\psi_1 - \psi'_1, a\psi_1 + \psi'_1 \neq 0\). Taking the infimum of all \(t\) that satisfy all the conditions above we get the claim. \(\square\)

If we drop the differentiability requirement in the definition of \(\mathcal{V}_a\), we obtain the more “traditional” cone of log-Lipschitz functions
\[
\mathcal{V}'_a := \left\{ \psi : T \to \mathbb{R}^+ : \frac{\psi(x)}{\psi(y)} \leq e^{a|x-y|} \right\}.
\]

It is well known that for \(\psi_1, \psi_2 \in \mathcal{V}'_a\), then
\[
\beta'_a(\psi_1, \psi_2) = \sup_{x \in \mathbb{T}} \left\{ \frac{\psi_2(x)}{\psi_1(x)}, \frac{\psi_2(x)e^{a|x-y|} - \psi_2(y)}{\psi_1(x)e^{a|x-y|} - \psi_1(y)} \right\}.
\]

Since for any \(\psi_1, \psi_2 \in \mathcal{V}_a\), \(\psi_t := t\psi_1 - \psi_2 \in \mathcal{V}_a\) if and only if \(\psi_t := t\psi_1 - \psi_2 \in \mathcal{V}'_a\), then
\[
\beta_a(\psi_1, \psi_2) = \beta'_a(\psi_1, \psi_2).
\]

In other words, (45) gives an alternative expression for \(\beta_a\).

A.2. Distance between Convex Combinations of Densities.

**Proposition A.3.** Assume that \(\{w_i\}_{i=1}^n\) are real numbers \(w_i \geq 0\) such that \(\sum_i w_i = 1\), and \(\{\rho_{1,i}\}_{i=1}^n, \{\rho_{2,i}\}_{i=1}^n\) are two collections of probability densities on \(T\) from \(\mathcal{V}_a\). Then:

i) \(\bar{\rho}_1 := \sum_i w_i \rho_{1,i}, \bar{\rho}_2 := \sum_i w_i \rho_{2,i} \in \mathcal{V}_a\)

ii) \(\theta_a(\bar{\rho}_1, \bar{\rho}_2) \leq \max_{i=1,...,n} \theta_a(\rho_{1,i}, \rho_{2,i})\)

iii) There is a constant \(C_a > 0\) depending on \(a\) only such that
\[
\theta_a(\bar{\rho}_1, \bar{\rho}_2) \leq C_a \sum_{i=1}^n \alpha_i \theta_a(\rho_{1,i}, \rho_{2,i}).
\]

**Proof.** i) Is immediate from the definition of \(\mathcal{V}_a\).

ii) We are going to prove the statement when \(n = 2\). The case \(n > 2\) can be worked out by induction. Furthermore, we are going to use the expression for \(\beta_a\) given in (45).
Step 1 Since \( \overline{p}_1 \) and \( \overline{p}_2 \) are continuous probability densities on \( \mathbb{T} \), there is \( x_0 \in \mathbb{T} \) such that \( \overline{p}_1(x_0) = \overline{p}_2(x_0) \). Assume \( \overline{p}_1 \neq \overline{p}_2 \), and \( x \in \mathbb{T} \) realizes the maximum of \( \overline{p}_1 \). Then

\[
\frac{\overline{p}_1(x)}{\overline{p}_2(x)} = \frac{\overline{p}_1(x) e^{a|x-x_0|}}{\overline{p}_2(x) e^{a|x-x_0|}} \leq \frac{\overline{p}_1(x) e^{a|x-x_0|} - 1}{\overline{p}_2(x) e^{a|x-x_0|} - 1} = \frac{\overline{p}_1(x) e^{a|x-x_0|} - \overline{p}_1(x)}{\overline{p}_2(x) e^{a|x-x_0|} - \overline{p}_2(x)}
\]

where the inequality holds because

\[
\frac{\overline{p}_1(x)}{\overline{p}_1(x_0)} e^{a|x-x_0|} \geq \frac{\overline{p}_2(x)}{\overline{p}_2(x_0)} e^{a|x-x_0|} > 0.
\]

From this follow that to bound \( \beta_a(\overline{p}_1, \overline{p}_2) \) it is sufficient to obtain a bound on

\[
\frac{\overline{p}_1(x) e^{a|x-y|} - \overline{p}_1(y)}{\overline{p}_2(x) e^{a|x-y|} - \overline{p}_2(y)}
\]

for every \( x \neq y \).

Step 2 For every \( x \neq y \):

\[
\frac{\overline{p}_1(x) e^{a|x-x|} - \overline{p}_1(y)}{\overline{p}_2(x) e^{a|x-x|} - \overline{p}_2(y)} = \frac{\alpha_1 \rho_{1,1}(x) e^{a|x-y|} - \rho_{1,1}(y)}{\alpha_2 \rho_{2,2}(x) e^{a|x-y|} - \rho_{2,2}(y)} = \frac{\alpha_1 A_{11} + \alpha_2 A_{12}}{\alpha_1 A_{21} + \alpha_2 A_{22}} =: \ell(\alpha, x, y)
\]

where \( \alpha = \alpha_1 \) and \( \alpha_2 = 1 - \alpha \). We claim that fixed \( x \neq y \), the function \( \ell(\alpha) = \ell(\alpha, x, y) \) is monotonic as a function of \( \alpha \), in fact its derivative is

\[
\frac{(A_{11} - A_{12}) A_{22} - (A_{21} - A_{22}) A_{12}}{[\alpha(A_{21} - A_{22}) + A_{22}]^2} = \frac{A_{11} A_{22} - A_{21} A_{12}}{[\alpha(A_{21} - A_{22}) + A_{22}]^2}.
\]

and the above expression does not change sign changing \( \alpha \). Notice that, by definition of \( \beta_a \), \( \ell(0) = \frac{A_{11}}{A_{21}} \), \( \ell(1) = \frac{A_{12}}{A_{22}} \) and

\[
\beta_a(\rho_{1,1}, \rho_{2,1})^{-1} \leq \frac{A_{11}}{A_{21}} \leq \beta_a(\rho_{1,1}, \rho_{2,2}) \quad \text{and} \quad \beta_a(\rho_{1,2}, \rho_{2,2})^{-1} \leq \frac{A_{12}}{A_{22}} \leq \beta_a(\rho_{1,2}, \rho_{2,2}).
\]

Call

\[
a_1 := \beta_a(\rho_{1,1}, \rho_{2,1})^{-1}, \quad a_2 := \beta_a(\rho_{1,1}, \rho_{2,1}), \quad b_1 := \beta_a(\rho_{1,2}, \rho_{2,2})^{-1}, \quad b_2 := \beta_a(\rho_{1,2}, \rho_{2,2}).
\]

By monotonicity of \( \ell \), we have that for every \( x, y \in \mathbb{T} \)

\[
a_1 + (b_1 - a_1) \alpha \leq \ell(\alpha, x, y) \leq a_2 + (b_2 - a_2) \alpha.
\]

\[\text{If} \, \overline{p}_1 \neq \overline{p}_2 \, \text{or} \, \rho_2 > \rho_1, \, \text{then they cannot both have integral equal to one.}\]
Step 3 Notice that
\[
\beta_a(\overline{p}_1, \overline{p}_2)\beta_a(\overline{p}_2, \overline{p}_1) = \sup_{x \neq y} \sup_{x' \neq y'} \frac{\overline{p}_1(x)e^{\alpha(x-y)} - \overline{p}_1(y)\overline{p}_2(x')e^{\alpha(x'-y')} - \overline{p}_2(y')}{\overline{p}_1(x'e^{\alpha(x'-y')}) - \overline{p}_1(y')}
\]
\[
= \sup_{x \neq y} \ell(\alpha, x, y)[\ell(\alpha, x', y')]^{-1}
\]
\[
\leq \frac{a_2 + (b_2 - a_2)\alpha}{a_1 + (b_1 - a_1)\alpha}
\]
Arguing as for \(\ell(\alpha)\), \(\frac{a_2 + (b_2 - a_2)\alpha}{a_1 + (b_1 - a_1)\alpha}\) is monotonic with respect to \(\alpha\) meaning that it takes its maximum and minimum at the extrema. This implies that
\[
\beta_a(\overline{p}_1, \overline{p}_2)\beta_a(\overline{p}_2, \overline{p}_1) \leq \max \left( \frac{a_2}{a_1}, \frac{b_2}{b_1} \right)
\]
\[
= \max (\beta_a(\rho_{1,1}, \rho_{2,1})\beta_a(\rho_{1,1}, \rho_{2,1}), \beta_a(\rho_{1,2}, \rho_{2,2})\beta_a(\rho_{1,2}, \rho_{2,2}))
\]
and from this follows that
\[
\theta_a(\overline{p}_1, \overline{p}_2) \leq \max (\theta_a(\rho_{1,1}, \rho_{2,1}), \theta_a(\rho_{1,2}, \rho_{2,2})).
\]

iii) It follows from (15) that
\[
\beta_a \left( \sum_i \alpha_i \rho_{1,i}, \sum_i \alpha_i \rho_{2,i} \right) = \sup_{x,y \in \mathbb{T}} \left\{ \sum_i \alpha_i \rho_{1,i}(x), \sum_i \alpha_i \rho_{2,i}(x) \right\} \frac{\alpha_i[\rho_{1,i}(x)e^{\alpha(x-y)} - \rho_{1,i}(y)]}{\sum_i \alpha_i[\rho_{2,i}(x)e^{\alpha(x-y)} - \rho_{2,i}(y)]}
\]
and since
\[
\rho_{1,i}(x) \leq \rho_{2,i}(x) \cdot \beta_a(\rho_{1,i}, \rho_{2,i})
\]
\[
\rho_{1,i}(x)e^{\alpha(x-y)} - \rho_{1,i}(y) \leq [\rho_{2,i}(x)e^{\alpha(x-y)} - \rho_{2,i}(y)] \cdot \beta_a(\rho_{1,i}, \rho_{2,i})
\]
we have
\[
\beta_a \left( \sum_i \alpha_i \rho_{1,i}, \sum_i \alpha_i \rho_{2,i} \right) \leq \sup_{x,y \in \mathbb{T}} \left\{ \sum_{i=1}^n \alpha_i \rho_{2,i}(x) \beta_a(\rho_{1,i}, \rho_{2,i}), \sum_{i=1}^n \alpha_i \rho_{2,i}(x)e^{\alpha(x-y)} - \rho_{2,i}(y) \right\} \frac{\beta_a(\rho_{1,i}, \rho_{2,i})}{\beta_a(\rho_{1,i}, \rho_{2,i})}.
\]
Now
\[
\frac{\alpha_i \rho_{2,i}(x)}{\sum_i \alpha_j \rho_{2,j}(x)} \leq \alpha_i \sup_{x,y \in \mathbb{T}} \frac{\rho_{2,i}(x)}{\rho_{2,j}(x)} \leq \alpha_i \max_j \beta_a(\rho_{2,i}, \rho_{2,j}) \leq \alpha_i \max_j \theta_a(\rho_{2,i}, \rho_{2,j})
\]
and analogously
\[
\alpha_i' \leq \alpha_i \max_i \beta_a(\rho_{2,i}, \rho_{2,j}) \leq \alpha_i \max_i \theta_a(\rho_{2,i}, \rho_{2,j})
\]
which imply
\[
\beta_a \left( \sum_i \alpha_i \rho_{1,i}, \sum_i \alpha_i \rho_{2,i} \right) \leq \sum_{i=1}^n \alpha_i' \beta_a(\rho_{1,i}, \rho_{2,i}).
\]
Now since for every \(x_{\text{max}} > 0\) there is \(C(x_{\text{max}}) > 0\) such that when \(0 \leq x \leq x_{\text{max}}\)
\[
C^{-1}x \leq \log(1 + x) \leq x
\]
with \( \lim_{x_{\max} \to 0} C(x_{\max}) = 1 \), there is \( C > 0 \) such that
\[
\log \left[ 1 + \sum_i \alpha_i' \beta_a(\rho_{1,i}, \rho_{2,i}) - 1 \right] \leq \sum_i \alpha_i' \beta_a(\rho_{1,i}, \rho_{2,i}) - 1 \leq \sum_i \alpha_i' C \log \beta_a(\rho_{1,i}, \rho_{2,i})
\]
with \( C \to 1 \) as \( \max_i \beta_a(\rho_{1,i}, \rho_{2,i}) \to 0 \). Wlog, we can assume \( \theta_a(\overline{r}_1, \overline{r}_2) \leq 2 \log \beta_a(\overline{r}_1, \overline{r}_2) \), and therefore
\[
\theta_a(\overline{r}_1, \overline{r}_2) \leq 2C \sum_i \alpha_i' \log \beta_a(\rho_{1,i}, \rho_{2,i}) \leq 2C \sum_i \alpha_i' \theta_a(\rho_{1,i}, \rho_{2,i}) \leq 2Ce^{\text{diam}(V_a)} \sum_i \alpha_i \theta_a(\rho_{1,i}, \rho_{2,i}).
\]
where \( \text{diam}(V_a) \) is the diameter of \( V_a \).

**Proposition A.4.** Let \( \rho_1, ..., \rho_n \in V_a, \beta_1, ..., \beta_n > 0, \beta'_1, ..., \beta'_n > 0 \). Then
\[
\theta_a \left( \sum_i \beta_i \rho_i, \sum_i \beta'_i \rho_i \right) \leq \left[ 1 - \exp\left( 2 \min_j \max_i \theta_a(\rho_j, \rho_i) \right) \right] \cdot \log \left( \max_{i,j} \frac{\beta_i}{\beta'_i} \cdot \frac{\beta_j}{\beta'_j} \right).
\]

**Proof.** Consider the simplex
\[
V_{\{\rho_i\}} = \left\{ \sum_i \beta_i \rho_i : \beta_i > 0 \right\}.
\]
It follows from the definition that this set is a cone with \( V_{\{\rho_i\}} \subset V_a \), and we can estimate the diameter of \( V_{\{\rho_i\}} \) in \( V_a \): For all probability weights \( \beta_1, ..., \beta_n \) and any \( j = 1, ..., n \)
\[
\theta_a(\rho_j, \sum_i \beta_i \rho_i) = \theta_a(\sum_i \beta_i \rho_j, \sum_i \beta_i \rho_i) \leq \max_i \theta_a(\rho_j, \rho_i)
\]
where the last inequality follows by Proposition A.3. Therefore, by triangle inequality
\[
\theta_a \left( \sum_i \beta_i \rho_i, \sum_i \beta'_i \rho_i \right) \leq 2 \min_j \max_i \theta_a(\rho_j, \rho_i).
\]
for any \( \beta_1, ..., \beta_n, \beta'_1, ..., \beta'_n > 0 \). Setting \( D := 2 \min_j \max_i \theta_a(\rho_j, \rho_i) \) and calling \( \tilde{\theta} \) the projective Hilbert metric on \( V_{\{\rho_i\}} \), by Theorem A.1 applied to the inclusion map \( \iota : V_{\{\rho_i\}} \to V_a \)

\[
(47) \quad \theta_a \left( \sum_i \beta_i \rho_i, \sum_i \beta'_i \rho_i \right) \leq (1 - e^{-D}) \tilde{\theta} \left( \sum_i \beta_i \rho_i, \sum_i \beta'_i \rho_i \right).
\]

Now consider \( \Delta_n := \{(\beta_1, ..., \beta_n) : \beta_i > 0 \ \forall i = 1, ..., n\} \), and the map \( \mathcal{L} : \Delta_n \to V_{\{\rho_i\}} \)
\[
\mathcal{L}(\beta_1, ..., \beta_n) = \sum_i \beta_i \rho_i.
\]
By Theorem A.1 applied to $L$, for any $\beta = (\beta_1, ..., \beta_n)$ and $\beta' = (\beta'_1, ..., \beta'_n)$ in $\Delta_n$

$$\tilde{\theta} \left( \sum_i \beta_i \rho_i, \sum_i \beta'_i \rho_i \right) \leq \theta_{\Delta_n}(\beta, \beta')$$

where we denoted by $\theta_{\Delta_n}$ the projective Hilbert metric on $\Delta_n$. Combining this last equation with (47) we get

$$\theta_a \left( \sum_i \beta_i \rho_i, \sum_i \beta'_i \rho_i \right) \leq (1 - e^{-D}) \theta_{\Delta_n}(\beta, \beta').$$

One can easily find an expression for $\theta_{\Delta_n}$:

$$\theta_{\Delta_n}(\beta, \beta') = \log \max \left\{ \frac{\beta_i}{\beta'_i} \right\}_{i,j}, i = 1, ..., n \right\}$$

and this concludes the proof.

A.3. Distance between Products of Densities.

**Lemma A.5.** Given $\varphi \in \mathcal{V}_a$ and $b > 0$, consider the linear transformation $L_\varphi : \mathcal{V}_b \rightarrow \mathcal{V}_{a+b}$ defined as $L_\varphi(\psi) := \varphi \psi$. Then

$$\theta_{a+b}(L_\varphi \psi_1, L_\varphi \psi_2) \leq (1 - e^{-\text{diam}(\mathcal{V}_{a+b})}) \theta_b(\psi_1, \psi_2)$$

for all $\psi_1, \psi_2 \in \mathcal{V}_b$, where $\text{diam}(\mathcal{V}_{a+b})$ is the diameter of $\mathcal{V}_{a+b}$.

**Proof.** Multiplication by $\psi$ is a linear application mapping $\mathcal{V}_b$ to $\mathcal{V}_{a+b}$, and the lemma follows by Theorem A.1.

The following is a corollary to the lemma.

**Proposition A.6.** Consider functions $\{\varphi_{ij}\}_{i,j}$ with $i = 1, 2$ and $j = 1, ..., M$ such that $\varphi_{ij} \in \mathcal{V}_{a_j}$ for some $a_j > 0$.

Then, calling $a := \sum_{j=1}^M a_j$, for any $b > a$

$$\theta_b \left( \prod_{j=1}^M \varphi_{1,j}, \prod_{j=1}^M \varphi_{2,j} \right) \leq \sum_{j=1}^M \theta_{b_j}(\varphi_{1,j}, \varphi_{2,j})$$

where $b_j := b - \sum_{k \neq j} a_k$.

**Proof.** It is enough to prove it for $M = 2$. The general case follows by induction.

Notice that $\varphi_{i1} \varphi_{i2} \in \mathcal{V}_a$ for every $i_1, i_2$, therefore

$$\theta_b(\varphi_{11} \varphi_{12}, \varphi_{21} \varphi_{22}) \leq \theta_b(\varphi_{11} \varphi_{12}, \varphi_{11} \varphi_{22}) + \theta_b(\varphi_{11} \varphi_{22}, \varphi_{21} \varphi_{22})$$

$$\leq \theta_b(\varphi_{12}, \varphi_{22}) + \theta_b(\varphi_{11}, \varphi_{21})$$

where the last inequality follows by Lemma A.5 (e.g. multiplication by $\varphi_{11} \in \mathcal{V}_{a_1}$ sends functions of $\varphi_{12}, \varphi_{22} \in \mathcal{V}_{a_2} \subset \mathcal{V}_{b-a_1}$ inside $\mathcal{V}_b$).
**A.4. Distance of Operators in the Hilbert metric.**

**Proposition A.7.** Consider local diffeomorphisms $f, g \in C^3(\mathbb{T}, \mathbb{T})$ such that $\exists a, \varepsilon \geq 0$ and $\lambda \in [0, 1)$ such that

$$d_{C^2}(f, g) < \varepsilon$$

and $f_{t*}(\mathcal{V}_a), g_{t*}(\mathcal{V}_a) \subset \mathcal{V}_\lambda a$ where $\{f_\ell^{-1}\}_{\ell \in \mathcal{I}_{f}}$ and $\{g_\ell^{-1}\}_{\ell \in \mathcal{I}_{g}}$ are the inverse branches of $f$ and $g$. Then

$$\theta_a(f_{**}, g_{*}) \leq C\varepsilon$$

where $C$ depends on $\|f\|_{C^3}$, $\|g\|_{C^3}$, inf $|f'|$, inf $|g'|$, and $\|\psi\|_{C^2}$.

**Proof.** Assume that $\varepsilon > 0$ is sufficiently small and $f$ and $g$ have the same degree $(\mathcal{I}_{f} = \mathcal{I}_{g} = \mathcal{I})$. Since

$$f_{*}\psi(x) = \sum_{\ell \in \mathcal{I}} \frac{\psi \circ f_\ell^{-1}(x)}{|f'| \circ f_\ell^{-1}(x)} = \sum_{\ell \in \mathcal{I}} (f_\ell)_{*}\psi$$

$f_{*}\psi$ belongs to $\mathcal{V}_a$ and analogously for $g_{*}\psi$.

Now we estimate $\beta_{a}(f_{*}\psi(x), g_{*}\psi(x))$ using expression (43) for $\beta_{a}$, i.e. we find a lower bound on $t > 0$ ensuring that

$$tg_{*}\psi - f_{*}\psi = \sum_{\ell \in \mathcal{I}} \frac{t \psi \circ g_\ell^{-1}}{|g'| \circ g_\ell^{-1}} - \frac{\psi \circ f_\ell^{-1}}{|f'| \circ f_\ell^{-1}} \in \mathcal{V}_a.$$  

The above is implied by the conditions

(48) \[ t \frac{\psi \circ g_\ell^{-1}}{|g'| \circ g_\ell^{-1}} - \frac{\psi \circ f_\ell^{-1}}{|f'| \circ f_\ell^{-1}} > 0 \]

(49) \[ \frac{\psi \circ g_\ell^{-1}}{|g'| \circ g_\ell^{-1}}(x) - \frac{\psi \circ f_\ell^{-1}}{|f'| \circ f_\ell^{-1}}(x) \leq e^{a|x-g|} \]

for every $\ell \in \mathcal{I}$. Below we fix an $\ell \in \mathcal{I}$, and denote for brevity, $f^{-1} = f_\ell^{-1}$ and $g^{-1} = g_\ell^{-1}$.

Notice that

$$|f^{-1}(x) - g^{-1}(x)| \leq (\inf |f'|)^{-1}|f(g^{-1}(x)) - g(g^{-1}(x))| \leq (\inf |f'|)^{-1}\varepsilon.$$  

Call $C_1 := (\inf |f'|)^{-1}$.

**Condition** (43).

$$t > \frac{\psi \circ f^{-1}(x)}{|f'| \circ f^{-1}(x)} = \frac{\psi \circ f^{-1}(x)}{|g'| \circ g^{-1}(x)} \frac{|g'| \circ g^{-1}(x)}{|f'| \circ f^{-1}(x)}$$

Furthermore

$$\frac{\psi \circ f^{-1}(x)}{|g'| \circ g^{-1}(x)} \leq e^{a|f^{-1}(x) - g^{-1}(x)|} \leq e^{aC_1\varepsilon} \leq 1 + aC_1\varepsilon + O(\varepsilon^2)$$
Therefore
\[
\frac{|g'| \circ g^{-1}(x)}{|f'| \circ f^{-1}(x)} = 1 + \frac{|g'| \circ g^{-1}(x) - |f'| \circ f^{-1}(x)}{|f'| \circ f^{-1}(x)}
\]
\[
\leq 1 + \frac{|g' \circ g^{-1}(x) - f' \circ g^{-1}(x)|}{|f'| \circ f^{-1}(x)} + \frac{|f' \circ g^{-1}(x) - f' \circ f^{-1}(x)|}{|f'| \circ f^{-1}(x)}
\]
\[
\leq 1 + \varepsilon \kappa^{-1} + \kappa^{-1} |f''|_{\infty} |f^{-1}(x) - g^{-1}(x)|
\]
\[
\leq 1 + \varepsilon \kappa^{-1} + \kappa^{-1} |f''|_{\infty} C_1 \varepsilon.
\]

Putting together the above, condition (48) is implied by \( t > 1 + O(\varepsilon) \).

**Condition (49).** This condition can be rewritten as

\[
\frac{t - \varphi(x)}{t - \varphi(y)} < \frac{\psi \circ f^{-1}(x)}{|f'|_{\infty} |f^{-1}(x)|} \cdot e^{\alpha |x-y|}
\]

with
\[
\varphi(x) := \frac{\psi \circ f^{-1}(x)}{|f'|_{\infty} |f^{-1}(x)|}.
\]

The assumptions imply that
\[
\frac{\psi \circ f^{-1}(x)}{|f'|_{\infty} |f^{-1}(x)|} \geq e^{-\lambda |x-y|}
\]

and (50) is implied by
\[
\frac{t - \varphi(x)}{t - \varphi(y)} \leq e^{(1-\lambda) \alpha |x-y|}
\]

that after some computations becomes

\[
t \geq \varphi(x) \cdot \frac{\varphi(y) e^{(1-\lambda) \alpha |x-y|} - 1}{e^{(1-\lambda) \alpha |x-y|} - 1}.
\]

Let's estimate
\[
\frac{\varphi(x)}{\varphi(y)} = \frac{\psi \circ f^{-1}(x)}{\psi \circ g^{-1}(x)} \cdot \frac{\psi \circ g^{-1}(y)}{\psi \circ f^{-1}(y)} \cdot \frac{|g'| \circ g^{-1}(x)}{|f'| \circ f^{-1}(x)} \cdot \frac{|g'| \circ g^{-1}(y)}{|f'| \circ f^{-1}(y)}.
\]

Now
\[
\left| \frac{d}{dx} [\log g' \circ g^{-1}(x) - \log f' \circ f^{-1}(x)] \right| = \frac{|g''|}{(g')^2} \circ g^{-1}(x) - \frac{f''}{(f')^2} \circ f^{-1}(x)
\]
\[
\leq \left| \frac{g''}{(g')^2} \circ g^{-1}(x) - \frac{g''}{(g')^2} \circ f^{-1}(x) \right|
\]
\[
+ \left| \frac{f''}{(f')^2} \circ g^{-1}(x) - \frac{f''}{(f')^2} \circ f^{-1}(x) \right|
\]
\[
\leq \left[ \inf |g'|^{-2} \|g\|_{C^3} \left[ |g^{-1}(x) - f^{-1}(x)| + C_\# d_{C^2} (f, g) \right] \right]
\]
\[
\leq \|g\|_{C^3} C_\# \varepsilon.
\]
where $C_\#$ is a generic constant, and the above implies, by the mean-value theorem and exponentiation
\[
|g'| \circ g^{-1}(x) \cdot |f'| \circ f^{-1}(y) \leq \exp[C_\#\varepsilon|x - y|].
\]

Analogously
\[
\frac{d}{dx} \left[ \log \psi \circ f^{-1}(x) - \log \psi \circ g^{-1}(x) \right] \leq C_\#d_{C^2}(f, g)
\]
where here $C_\#$ depends also on $\|\psi\|_{C^2}$, and therefore
\[
\frac{\psi \circ f^{-1}(x)}{\psi \circ g^{-1}(x)} \cdot \frac{\psi \circ g^{-1}(y)}{\psi \circ f^{-1}(y)} \leq \exp[C_\#\varepsilon|x - y|].
\]

The condition on $t$ in (51) is implied by
\[
t \geq [1 + O(\varepsilon)] \cdot \frac{e^{[K_\#\varepsilon + (1 - \lambda)a]|x - y|} - 1}{e^{(1 - \lambda)a|x - y|} - 1}
\]
and since
\[
\frac{e^{K_\#\varepsilon|x - y|}e^{(1 - \lambda)a|x - y|} - 1}{e^{(1 - \lambda)a|x - y|} - 1} \leq \frac{[K_\#\varepsilon + (1 - \lambda)a]|x - y| + O(\varepsilon^2|x - y|^2)}{(1 - \lambda)a|x - y|} = 1 + O(\varepsilon)
\]
the condition in (51) is implied by
\[
t = 1 + O(\varepsilon),
\]
and therefore
\[
\beta_a(f_*, \psi, g_*\psi) \leq 1 + O(\varepsilon).
\]
The claim of the lemma then follows by definition of $\theta_a$. \hfill \Box

**APPENDIX B. SOME ESTIMATES ON COUPLED MAPS**

**B.1. Estimates on the entries of the inverse Jacobian matrix.** Consider $H : \mathbb{T}^N \to \mathbb{T}^N$ a differentiable map, and denote by $DH$ its differential. Throughout this section the main assumption we impose on $H$ is Assumption 4.1. In particular, recall the definitions of $E, \kappa > 0$.

In the following proposition we give sufficient conditions for $H$ to be a local diffeomorphism and such that the inverse of the Jacobian matrix has entries of order one on the diagonal while off diagonal entries of order $N^{-1}$, as for $DH$. These estimates will be crucial in the proof of Proposition 4.1.

**Proposition B.1.** Assume that $H : \mathbb{T}^N \to \mathbb{T}^N$ satisfies Assumption 4.1.

Then $H$ is a local diffeomorphism, and $DH^{-1}$ satisfies
\[
|(DH^{-1})_{ij}| < \kappa^{-1} \left[ E + \frac{E^2}{\kappa - E} \right] N^{-1},
\]
\[
|(DH^{-1})_{ii}| \leq \kappa^{-1} \left[ 1 + E \left( E + \frac{E^2}{\kappa - E} \right) N^{-1} \right]
\]
and
\[
|(DH^{-1})_{ij}| \geq |DH_{ij}|^{-1} \left[ 1 - E \left( E + \frac{E^2}{\kappa - E} \right) N^{-1} \right]
\]
for every $i \in [1, N]$ and $j \neq i$. 

Proof. Pick \( v = (v_1, \ldots, v_N) \in \mathbb{R}^N \) any vector such that \( |v_1| = \sum_i |v_i| = 1 \). Now

\[
|DHv_1| \geq \sum_i \left[ |DH_{ii}|v_i| - \sum_{j \neq i} |DH_{ij}|v_j| \right] \geq \kappa \sum_i |v_i| - E = \kappa - E > 1
\]

which proves that \( DH \) is invertible and \( H \) is a local diffeomorphism.

To prove that the off-diagonal terms of \( DH^{-1} \) are of order \( N^{-1} \), we employ a geometrical argument. Consider \( x = (x_1, \ldots, x_N) \in \mathbb{T}^N \) and \( y = (y_1, \ldots, y_N) \in \mathbb{T}^N \) such that \( H(y) = x \) and restrict to a small neighborhood of \( y \) where \( H \) is invertible. Abusing notation, denote by

\[
H^{-1} = (H_1^{-1}, \ldots, H_N^{-1}) = (H_1^{-1}, H_2^{-1})
\]

the inverse map restricted to the image of this neighborhood. Without loss of generality let's estimate \( (DH^{-1})_{12}(x) = \partial_1 H_2^{-1}(x) \). Estimates for \( (DH^{-1})_{ij}(x) \) for any \( i \neq j \) follow analogously. Define \( x' := (x_1 + h, x_2, x_3, \ldots, x_N) \). Our goal is to estimate \( H_1^{-1}(x') - H_1^{-1}(x) \). Consider \( x'' := (x_1, x_2', x_3, \ldots, x_N) \in \mathbb{T}^N \) such that

\[
H_1^{-1}(x'') = H_1^{-1}(x')
\]

which exists by continuity of the first derivative provided that \( h \) is sufficiently small, and that \( \partial_1 H_2 \) is not identically zero for every \( \ell \neq 1 \), in which case, \( \partial_1 H_2^{-1} = 0 \) for every \( \ell \). Notice that by definition of \( x' \) and \( x'' \)

\[
H_2^{-1}(x') - H_2^{-1}(x) = H_2^{-1}(x'') - H_2^{-1}(x).
\]

Considering now the curve \( \gamma : [0,1] \to \mathbb{T}^N \), \( \gamma(s) = (1-s)x + sx'' \ s \in [0,1] \), we have: \( \gamma_2(s) = x_2 \), \( \gamma'_2 \) is constant and

\[
|\gamma'_2| \leq hEN^{-1} \quad |\gamma'_1| \leq hE
\]

which follows from the fact that \( x' \) and \( x'' \) are on the image of the same vertical fiber and mean value theorem. These inequalities reflect the fact that the image of a vertical fiber will be almost vertical. Define \( \tilde{\gamma} : [0,1] \to \mathbb{T}^N \), \( \tilde{\gamma} = H^{-1}\gamma \). We have that

\[
\tilde{\gamma}'(s) = DH^{-1}_{\gamma(s)}\gamma'(s)
\]

therefore, by (57) and (55)

\[
|\tilde{\gamma}'| \leq \frac{hE}{\kappa - E}.
\]

Also

\[
\gamma'_2(s) = \sum_k DH_{2k}(\gamma(s)) \gamma'_k(s) = DH_{22}\gamma'_2 + \sum_{k \neq 2} DH_{2k}(\gamma(s)) \gamma'_k(s)
\]

so

\[
|\gamma'_2| \leq \frac{1}{|DH_{22}|} \left[ hEN^{-1} + \max_{k \neq 2} |DH_{2k}| |\gamma'_1| \right] \\
\leq \kappa^{-1} \left[ hEN^{-1} + \frac{hE^2}{\kappa - E}N^{-1} \right]
\]

The above implies that

\[
H_2^{-1}(x'') - H_2^{-1}(x) = \tilde{\gamma}_2(1) - \tilde{\gamma}_2(0) \leq h\kappa^{-1}N^{-1} \left[ E + \frac{E^2}{\kappa - E} \right]
\]

and recalling (56), the inequality in (52) is proved.
To prove (59), notice that
\[
1 = \sum_{j=1}^{N} D H_{ij} (D H^{-1})_{ji} \geq D H_{ii} (D H^{-1})_{ii} - \sum_{j \neq i} E \left( E + \frac{E^2}{\kappa - E} \right) N^{-2}
\]
from which
\[
|(D H^{-1})_{ji}| \leq \frac{1}{|D H_{ii}|} \left[ 1 + E \left( E + \frac{E^2}{\kappa - E} \right) N^{-1} \right] \leq \kappa^{-1} \left[ 1 + E \left( E + \frac{E^2}{\kappa - E} \right) N^{-1} \right].
\]
Furthermore
\[
1 = \sum_{j=1}^{N} D H_{ij} (D H^{-1})_{ji} \leq |D H_{ii}| |(D H^{-1})_{ii}| + \sum_{j \neq i} E \left( E + \frac{E^2}{\kappa - E} \right) N^{-2}
\]
which implies
\[
|(D H^{-1})_{ii}| \geq \frac{1}{|D H_{ii}|} \left( 1 - E \left( E + \frac{E^2}{\kappa - E} \right) N^{-1} \right)
\]
so for \( N \) sufficiently large, \( |(D H^{-1})_{ii}| \) is bounded away from zero. \( \square \)

B.2. Proof of Proposition 4.1

Proof of Proposition 4.1 Pull-back Foliation. Let Assumption 4.1 stand. We will first prove that for \( N \) large, \( H^{-1}(\mathbb{T}_{\tilde{x}_1}) \) can be written as disjoint union of circles, and then we will prove existence of \( \hat{\Phi}_i \) so that (11) is satisfied. Without loss of generality, we are going to assume throughout the proof that \( i = 1 \). Fix \( \tilde{x}_1 \in \mathbb{T}^{N-1} \).

The inverse function theorem readily implies that locally \( H^{-1}(\mathbb{T}_{\tilde{x}_1}) \) is the graph of a function. More precisely, pick any \((y_1; \tilde{y}_1) \in H^{-1}(\mathbb{T}_{\tilde{x}_1})\) such that \( H(y_1, \tilde{y}_1) = (0, \tilde{x}_1) \), then there is \( U \subset H^{-1}(\mathbb{T}_{\tilde{x}_1}) \) a neighborhood of \((y_1; \tilde{y}_1)\) on \( H^{-1}(\mathbb{T}_{\tilde{x}_1}) \) and \( \varphi : (y_1 - \delta, y_1 + \delta) \to \mathbb{T}^{N-1} \) such that \( U \cap \{ z_1 \in (y_1 - \delta, y_1 + \delta) \} \) is given by the graph of \( \varphi \) and
\[
\hat{H}_1(z_1; \varphi(z_1)) = \tilde{x}_1.
\]
This implicit relation implies that the derivative of \( \varphi \) is
\[
\varphi'(z) = -(\hat{D}_1 \hat{H}_1)^{-1}|_{(z, \varphi(z))} \nabla \hat{H}_1|_{(z, \varphi(z))}
\]
where \( \hat{D}_1 \) is the differential with respect to coordinates \((x_2, ..., x_N)\). It follows from Assumption 4.1 and Proposition B.1 applied to \( \tilde{x}_1 \mapsto \hat{H}_1(x_1; \tilde{x}_1) \) that
\[
(\hat{D}_1 \hat{H}_1^{-1})_{ii} < (\kappa')^{-1}, \quad (\hat{D}_1 \hat{H}_1^{-1})_{ij} < E'N^{-1} \quad \forall i \text{ and } \forall j \neq i
\]
with \( E' := \kappa^{-1} \left[ E + \frac{E^2}{\kappa - E} \right] \) and \( (\kappa')^{-1} = \kappa^{-1} + O(N^{-1}) \). Denoting \( \varphi(z) = (\varphi_2(z), ..., \varphi_N(z)) \) we get
\[
\varphi'_k(z) = -\sum_{m=2}^{N} (\hat{D}_1 \hat{H}_1^{-1})_{km} \partial_1 H_m
\]
so
\[
|\varphi'_k(z)| \leq |(\hat{D}_1 \hat{H}_1^{-1})_{kk}| |\partial_1 H_k| + \sum_{m \neq 1, k} E'N^{-1}EN^{-1} \leq (K' + E')EN^{-1}.
\]
If \(((\kappa')^{-1} + E')EN^{-1}\) is sufficiently small, i.e. the Euclidean norm of \(\varphi'(z)\) is sufficiently small, then the unit tangent vectors at \(U\) have strictly nonzero component along the \(x_1\)-direction. Let’s pick (for the moment)
\[
\mathcal{K}_\# := ((\kappa')^{-1} + E')E.
\]
Notice that \(\mathcal{K}_\#\) can be made arbitrarily small letting either \(\kappa^{-1} \to 0\) or \(E \to 0\).

Under this condition, \(H^{-1}(\mathbb{T}_{\mathbb{Z}_1})\) is transverse to \(\{x_1 = 0\} \subset \mathbb{T}^N\). Now, for some index set \(I\), the collection \(\{\mathbf{p}_i\}_{i \in I}\) of all the intersection points of \(H^{-1}(\mathbb{T}_{\mathbb{Z}_1})\) with \(\{x_1 = 0\}\). Recall that \(H^{-1}(\mathbb{T}_{\mathbb{Z}_1})\) integrates \(X_1 := H^*e_1\), and therefore \(X_1\) points along the same direction as \((1, \varphi'(z))\). Starting from any of the \(\mathbf{p}_i(0) = \mathbf{p}_i\) integrate the vector field to obtain \(\mathbf{p}_i(t)\) until \(\pi_1(\mathbf{p}_i(t_0)) = 1 \sim (\mathbb{I})\). We claim that the collection of curves \(\gamma_{\mathbb{Z}_1, \ell} := \{\mathbf{p}_j(t) : t \in [0, t_0)\}\) is a partition of \(H^{-1}(\mathbb{T}_{\mathbb{Z}_1})\) into disjoint circles. In fact, by contradiction, suppose that \(\exists \mathbf{p} \in H^{-1}(\mathbb{T}_{\mathbb{Z}_1}) \setminus \cup \gamma_{\mathbb{Z}_1, \ell},\) then starting from \(\mathbf{p}\) follow the vector field \(X_1\) either forward or backward and one reaches \(\{x_1 = 0\}\), so you will meet \(\mathbf{p}_i\) for some \(\ell\) and \(\mathbf{p} \in \gamma_{\mathbb{Z}_1, \ell}\) (this is once again because the first component of \(X_1\) is bounded away from zero). Clearly \(\gamma_{\mathbb{Z}_1, \ell} \cap \gamma_{\mathbb{Z}_1, \ell'} = \emptyset\) for \(\ell \neq \ell'\) as the segment integrates the vector field \(X_1\) and solutions are unique. To prove that \(\gamma_{\mathbb{Z}_1, \ell}\) is a circle, one has to show that \(\mathbf{p}_i(0) = \mathbf{p}_i(t_0)\) which, by the definition of \(\mathbf{p}_i(t)\), also implies that \(\gamma_{\mathbb{Z}_1, \ell}\) is homotopic to \(T \times \mathbf{p}_i(0)\). Arguing by contradiction, assume that the \(j\)-th component \(y_i(0)\) and \(y_j(t_0)\) of \(\mathbf{p}_i(0)\) and \(\mathbf{p}_i(t_0)\) are different, and \(|y_j(0) - y_j(t_0)| \geq |y_k(0) - y_k(t_0)|\) for all \(k \neq j\). Then, by the mean-value theorem for some \(\mathbf{p}' \in \mathbb{T}^N\)
\[
H_j(\mathbf{p}_i(0)) - H_j(\mathbf{p}_i(t_0)) = DH_j(\mathbf{p}')[\mathbf{p}_i(0) - \mathbf{p}_i(t_0)]
= \partial_j H_j(\mathbf{p}')(y_j(0) - y_j(t_0)) + \sum_{k \neq j} \partial_k H_j(\mathbf{p}') (y_k(0) - y_k(t_0))
\]
and since \((61)\) implies that \(|y_k(0) - y_k(t_0)| = O(N^{-1})\)
\[
|H_j(\mathbf{p}_i(0)) - H_j(\mathbf{p}_i(t_0))| \geq \kappa |y_j(0) - y_j(t_0)| - \sum_{k \neq j} E N^{-1} |y_k(0) - y_k(t_0)| > 0,
\]
\[
|H_j(\mathbf{p}_i(0)) - H_j(\mathbf{p}_i(t_0))| \leq K' |y_j(0) - y_j(t_0)| + \sum_{k \neq j} E N^{-1} |y_k(0) - y_k(t_0)|
\leq O(N^{-1})
\]
so for \(N\) large, \(H_j(\mathbf{p}_i(0))\), \(H_j(\mathbf{p}_i(t_0))\) cannot coincide on the circle, but by assumption \(H_1(\mathbf{p}_i(0)) = H_1(\mathbf{p}_i(t_0))\) and we get a contradiction.

With the above, one can define \(\Phi^{-1}_{1} : \mathbb{T}^N \to T \times \mathbb{T}^{N-1}\) in the following way: suppose that \((y_i; \hat{y}_1) \in \gamma_{\mathbb{Z}_1, \ell}\) and \(\gamma_{\mathbb{Z}_1, \ell} \cap \{0\} \times \mathbb{T}^{N-1} = \{(0, \hat{y}'_1)\},\) then
\[
\Phi^{-1}_{1}(y_i; \hat{y}_1) = (y_i; \hat{y}'_1).
\]
This defines a diffeomorphism \(\Phi_1\) satisfying the implicit relations \((10)\). To ease notation, we will call \(\Phi = (\Phi_1, ..., \Phi_N) := \Phi_1\).

First Derivatives of \(\Phi\). From the first equation in \((10)\) follows that \(\partial_1 \Phi_1 = 1\) and \(\partial_k \Phi_1 = 0\) while \((61)\) implies that for every \(k \neq 1\),
\[
|\partial_k \Phi_k| \leq \mathcal{K}_\# N^{-1}.
\]
From the second equation in (10) follows that
\[
D\hat{H}_1\Phi(y_1, \hat{y}_1)D\Phi(y_1, \hat{y}_1) = D\hat{H}_1(0, \hat{y}_1)
\]
which is equivalent to
\[
(63) \quad D\hat{H}_1\Phi(y_1, \hat{y}_1) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ D\Phi(y_1, \hat{y}_1) \end{pmatrix} = \begin{pmatrix} 0 & \cdots & \hat{D}_1\hat{H}_1(0, \hat{y}_1) \\ 0 \end{pmatrix}
\]
where \( \hat{\Phi}_1 : \mathbb{T}^N \to \mathbb{T}^N \) equals \( \hat{\pi}_1 \circ \Phi \) and, recalling that \( \hat{D}_1 \) denotes the differential with respect to coordinates \((x_2, \ldots, x_N)\), block multiplication implies:
\[
\hat{D}_1\hat{H}_1\Phi(y_1, \hat{y}_1)\hat{D}_1\hat{\Phi}_1(y_1, \hat{y}_1) = \hat{D}_1\hat{H}_1(0, \hat{y}_1).
\]
This can be written as
\[
(64) \quad \hat{D}_1\Phi(y_1, \hat{y}_1) = (\hat{D}_1\hat{H}_1\Phi(y_1, \hat{y}_1))^{-1}\hat{D}_1\hat{H}_1(0, \hat{y}_1)
\]
\[
(65) \quad = \text{Id}_{2N-1} + (\hat{D}_1\hat{H}_1\Phi(y_1, \hat{y}_1))^{-1}[\hat{D}_1\hat{H}_1\Phi(0, \hat{y}_1) - \hat{D}_1\hat{H}_1\Phi(y_1, \hat{y}_1)]
\]
By the mean-value theorem, for \( k \in [2, N] \),
\[
[\hat{D}_1\hat{H}_1\Phi(0, \hat{y}_1) - \hat{D}_1\hat{H}_1\Phi(y_1, \hat{y}_1)]_{kl} = \partial_k H_k(\Phi(0, \hat{y}_1)) - \partial_k H_k(\Phi(y_1, \hat{y}_1))
\]
\[
(66) \quad = \sum_{m=1}^{N} \partial_m \partial_k H_k(\Phi(z; \hat{y}_1)) \partial_l \Phi_m(z; \hat{y}_1)|_{y_1}
\]
and from Assumption 4.2 and equation (62), it follows that\[20\]
\[
(67) \quad \boxed{\left|[\hat{D}_1\hat{H}_1\Phi(0, \hat{y}_1) - \hat{D}_1\hat{H}_1\Phi(y_1, \hat{y}_1)]_{kl}\right| \leq \left\{ \begin{array}{ll} K_\# N^{-1} & k = \ell \\ K_\# N^{-2} & k \neq \ell \end{array} \right.}
\]
where we eventually update \( K_\# \) that now depends also on \( K \), i.e. the bound on the second derivative. Notice that as long as \( K \) is bounded, \( K_\# \) still goes to zero if either \( E \to 0 \) or \( \kappa \to \infty \).

Finally, recalling Proposition 3.1\[21\]
\[
\left| (\hat{D}_1\hat{H}_1\Phi(y_1, \hat{y}_1))^{-1}(\hat{D}_1\hat{H}_1\Phi(0, \hat{y}_1) - \hat{D}_1\hat{H}_1\Phi(y_1, \hat{y}_1)) \right|_{kl} \leq \left\{ \begin{array}{ll} K_\# N^{-1} & k = \ell \\ K_\# N^{-2} & k \neq \ell \end{array} \right.
\]
and (12) is proved.

\[19\]Here we use that for \( k = \ell \), the term with \( m = 1 \) in the sum (65) is
\[
|\partial_1 \partial_1 H_k(\Phi(z; \hat{y}_1))| \leq EN^{-1}
\]
for any \( m \neq 1 \)
\[
|\partial_m \partial_1 H_k(\Phi(z; \hat{y}_1))| \leq \left\{ \begin{array}{ll} K_2 & k = \ell = m \\ EN^{-1} & k = \ell \neq m \end{array} \right.
\]
while for \( \ell \neq k \)
\[
|\partial_1 \partial_1 H_k(\Phi(z; \hat{y}_1))| \leq EN^{-2}
\]
and
\[
|\partial_m \partial_1 H_k(\Phi(z; \hat{y}_1))| \leq \left\{ \begin{array}{ll} EN^{-1} & k = m \text{ or } \ell = m \\ EN^{-2} & k, \ell \text{ distinct} \end{array} \right.
\]
also, for \( m \neq 1 \)
\[
\partial_1 \Phi_1(z; \hat{y}_1) = 1 \quad |\partial_1 \Phi_m(z; \hat{y}_1)| \leq K_\# N^{-1}.
\]

\[20\]\( \hat{D}_1\hat{H}_1\Phi(y_1, \hat{y}_1) \) has diagonal entries bounded by a constant \( K' \), while constant off diagonal are bounded by \( E'N^{-1} \) with \( E' \) and \( K' \) depending on \( E \) and \( \kappa \) only. The result follows by matrix multiplication.
Second Order Partial Derivatives of $\Phi$ Containing $\partial_1$. One immediately gets that $\partial_k \partial_1 \Phi_1 = 0 \forall k \in [1, N]$. Looking at the first column of the matrix in equation (63) we obtain

$$D_1 \hat{H}_{1 \Phi(y_1; y_1)} \left( \begin{array}{c} 1 \\ \partial_1 \Phi_1(\Phi(y_1; \hat{y}_1)) \end{array} \right) = \partial_1 \hat{H}_{1}(\Phi(y_1; \hat{y}_1)) + D_1 \hat{H}_{1 \Phi(y_1; y_1)} \partial_1 \Phi_1(\Phi(y_1; \hat{y}_1))$$

and the above equals zero, therefore

$$\partial_1 \Phi_1(\Phi(y_1; \hat{y}_1)) = -(D_1 \hat{H}_{1 \Phi(y_1; y_1)})^{-1} \partial_1 \hat{H}_{1}(\Phi(y_1; \hat{y}_1))$$

Taking the partial derivative of the above along the $k$-th coordinate we get

$$\partial_k \partial_1 \Phi_1 = -(D_1 \hat{H}_{1 \Phi(y_1; y_1)})^{-1} \partial_k (\partial_1 \hat{H}_{1}(\Phi(y_1; \hat{y}_1)))$$

(68)

and

$$\partial_k (D_1 \hat{H}_{1 \Phi(y_1; y_1)}) \partial_1 \hat{H}_{1}(\Phi(y_1; \hat{y}_1)) \text{ for any } k \neq 1$$

(69)

For the RHS on (68), for any $\ell \in [2, N]$, we have

$$|[\partial_1(\partial_1 \hat{H}_{1} \circ \Phi)]|_{\ell} \leq \sum_{m=1}^{N} |\partial_m \partial_1 H_{\ell} \circ \Phi| |\partial_1 \Phi_m| \leq \left\{ \begin{array}{ll} K_\# N^{-1} & \ell = 1 \\ K_\# N^{-2} & \ell \neq 1 \end{array} \right.$$  

while for $k \neq 1$

$$|[\partial_k(\partial_1 \hat{H}_{1} \circ \Phi)]|_{\ell} \leq \sum_{m=1}^{N} |\partial_m \partial_1 H_{\ell} \circ \Phi| |\partial_k \Phi_m| \leq \left\{ \begin{array}{ll} K_\# N^{-1} & k = \ell \\ K_\# N^{-2} & k \neq \ell \end{array} \right.$$

and from the estimates on the entries of $(D_1 \hat{H}_{1 \Phi(y_1; y_1)})^{-1}$

$$|[\partial_k(D_1 \hat{H}_{1 \Phi(y_1; y_1)})^{-1} \partial_k(\partial_1 \hat{H}_{1}(\Phi(y_1; \hat{y}_1)))]|_{\ell} \leq \left\{ \begin{array}{ll} K_\# N^{-1} & k = \ell \\ K_\# N^{-2} & k \neq \ell. \end{array} \right.$$  

For the term in (69) let’s start by estimating the entries of $\partial_k(D_1 \hat{H}_{1}^{-1})$. Since $D_1 \hat{H}_{1}^{-1} D_1 \hat{H}_{1} = 1d_{N-1}$, we have

$$\partial_k(D_1 \hat{H}_{1}^{-1}) = -D_1 \hat{H}_{1}^{-1} \partial_k(D_1 \hat{H}_{1}) D_1 \hat{H}_{1}^{-1}$$

so using the estimates for the entries of the matrices above,

$$|\partial_1(D_1 \hat{H}_{1}^{-1})|_{\ell m} \leq \left\{ \begin{array}{ll} K_\# N^{-1} & \ell = m \\ K_\# N^{-2} & \ell \neq m \end{array} \right.$$  

(72)

and for $k \neq 1$

$$|\partial_k(D_1 \hat{H}_{1}^{-1})|_{\ell m} \leq \left\{ \begin{array}{ll} K_\# N^{-1} + K_\# N^{-1} & \ell = m = k \\ K_\# N^{-1} & \ell = k \text{ xor } m = k \\ K_\# N^{-2} & \ell, k, m \text{ distinct.} \end{array} \right.$$  

(73)

Now

$$|\partial_k(D_1 \hat{H}_{1 \Phi(y_1; y_1)}^{-1})|_{\ell m} = \sum_{j=1}^{N} (\partial_j D_1 \hat{H}_{1 \Phi(y_1; y_1)}^{-1})_{\ell m} \partial_k \Phi_j(y_1; \hat{y}_1)$$

For $k \neq 1$, since $\partial_k \Phi_j = O(N^{-2})$ whenever $j \neq k$, and $= 1 + O(N^{-1})$ otherwise, the entries of $\partial_k(D_1 \hat{H}_{1 \Phi(y_1; y_1)}^{-1})$ are of the same order than those of $\partial_k(D_1 \hat{H}_{1}^{-1})$. For $k = 1$, since $\partial_1 \Phi_j = O(N^{-1})$ whenever $j \neq 1$ and $= 1$ otherwise, one can see that

---

21 See computations in footnote 19
the entries on the diagonal are of \(O(1)\) while those off the diagonal are of \(O(N^{-2})\). Since the entries of \(\partial_1 \hat{H}_1\) are bounded in modulus by \(K\# N^{-1}\)

\[
(74) \quad [\partial_k (\hat{D}_1 \hat{H}_1^{-1} (y_1; \hat{y}_1)) \partial_1 \hat{H}_1 (\Phi (y_1; \hat{y}_1))]_m \leq \begin{cases} 
    K\# N^{-2} & k = 1 \\
    K\# N^{-1} & k = m \neq 1 \\
    K\# N^{-2} & k \neq m 
\end{cases}
\]

Combining (70) and (74) we obtain

\[
[\partial_k \partial_1 \hat{F}_1]_m = \begin{cases} 
    K\# N^{-1} & m = k \\
    K\# N^{-2} & m \neq k 
\end{cases}
\]

**Second Order Partial Derivatives of \(\Phi\) Not Containing \(\partial_1\).** From now on we put \(k \neq 1\). From equation (65) we get

\[
(75) \quad \partial_k \hat{D}_1 \hat{F}_1 (y_1; \hat{y}_1) = \partial_k [[\hat{D}_1 \hat{H}_1^{-1} (y_1; \hat{y}_1)] [\hat{D}_1 \hat{H}_1 (0; \hat{y}_1)] - \hat{D}_1 \hat{H}_1 (y_1; \hat{y}_1)] + (\hat{D}_1 \hat{H}_1 (y_1; \hat{y}_1))^{-1} \partial_k [\hat{D}_1 \hat{H}_1 (0; \hat{y}_1) - \hat{D}_1 \hat{H}_1 (y_1; \hat{y}_1)]
\]

For the term in (76), by repeated use of mean-value theorem

\[
\partial_k [\hat{D}_1 \hat{H}_1 (0; \hat{y}_1) - \hat{D}_1 \hat{H}_1 (y_1; \hat{y}_1)] = \sum_{j=1}^{N} \partial_j \hat{D}_1 \hat{H}_1 (0; \hat{y}_1) \partial_k \Phi_j (0; \hat{y}_1) -
\]

\[
- \sum_{j=1}^{N} \partial_j \hat{D}_1 \hat{H}_1 (y_1; \hat{y}_1) \partial_k \Phi_j (y_1; \hat{y}_1)
\]

\[
= \sum_{j=1}^{N} [\partial_j \hat{D}_1 \hat{H}_1 (0; \hat{y}_1) - \partial_j \hat{D}_1 \hat{H}_1 (y_1; \hat{y}_1)] \partial_k \Phi_j (0; \hat{y}_1) +
\]

\[
+ \partial_j \hat{D}_1 \hat{H}_1 (y_1; \hat{y}_1) [\partial_k \Phi_j (0; \hat{y}_1) - \partial_k \Phi_j (y_1; \hat{y}_1)]
\]

\[
= \sum_{j=1}^{N} \sum_{p=1}^{N} \partial_p \partial_j \hat{D}_1 \hat{H}_1 \Phi (z'; \hat{y}_1) \partial_1 \Phi_p \partial_k \Phi_j \cdot y_1 +
\]

\[
+ \sum_{j=1}^{N} \partial_j \hat{D}_1 \hat{H}_1 \Phi (y_1; \hat{y}_1) \partial_1 \partial_k \Phi_j (z'; \hat{y}_1) \cdot y_1
\]

and this implies

\[
\left| [\partial_k [\hat{D}_1 \hat{H}_1 (0; \hat{y}_1) - \hat{D}_1 \hat{H}_1 (y_1; \hat{y}_1)]_m \right| \leq \sum_{j=1}^{N} \sum_{p=1}^{N} |\partial_p \partial_j \partial_k H_m| |\partial_1 \Phi_p| |\partial_k \Phi_j| +
\]

\[
+ \sum_{j=1}^{N} |\partial_j \partial_k H_m| |\partial_1 \partial_k \Phi_j|
\]
From the above \(^2\)

\[
\left| \partial_h [\hat{D}_1 \hat{H}_1 \Phi(0, \mathbf{y}_1) - \hat{D}_1 \hat{H}_1 \Phi(y_1, \mathbf{y}_1)] \right| \leq \begin{cases} 
K_N^{-1} & \ell = m = k \\
K_N^{-2} & \ell = m \neq k \\
K_N^{-2} & \ell = k \text{ xor } m = k \\
K_N^{-3} & \text{otherwise}
\end{cases}
\]

and matrix multiplication implies that the entries of the term in (69) satisfy similar bounds. Estimates on entries for the RHS of (75) can be obtained combining (67) and the estimates on \(\partial_h [(\hat{D}_1 \hat{H}_1 \Phi(y_1, \mathbf{y}_1)]\) obtained above. Altogether they give (41).

\[\square\]

\section*{Appendix C. Proofs from Section 4.2.1}

\subsection*{C.1. Characterization of \(\mathcal{M}_{a,b,L}\)}

First of all we present a criterion to characterize measures in \(\mathcal{M}_{a,b,L}\) which will be used many times in the following.

\begin{definition}
Let \(\eta : \mathbb{T}^N \to \mathbb{R}^+\) be a \(C^2\) probability density such that for every \(i \in [1, N]\) and \(\mathbf{x}_i \in \mathbb{T}^{N-1}\), \(\eta(\cdot; \mathbf{x}_i) \in \mathcal{V}_a\). For any \(b \geq a\), every \(i \in [1, N]\) and \(k \in [1, N]\) with \(k \neq i\), define

\[
A_{\eta,b}^{(i,k)} (\mathbf{x}_i) := \sup_{x \in \mathbb{T}} \left\{ \frac{\partial_k \eta(x; \mathbf{x}_i)}{\eta(x; \mathbf{x}_i)}, \frac{b \partial_\kappa \eta(x; \mathbf{x}_i) - \partial_k \partial_\ell \eta(x; \mathbf{x}_i)}{b \eta(x; \mathbf{x}_i) - \partial_\ell \eta(x; \mathbf{x}_i)}, \frac{b \partial_k \eta(x; \mathbf{x}_i) + \partial_\ell \eta(x; \mathbf{x}_i)}{b \eta(x; \mathbf{x}_i) + \partial_\ell \eta(x; \mathbf{x}_i)} \right\}
\]

and

\[
B_{\eta,b}^{(i,k)} (\mathbf{x}_i) := \inf_{x \in \mathbb{T}} \left\{ \frac{\partial_k \eta(x; \mathbf{x}_i)}{\eta(x; \mathbf{x}_i)}, \frac{b \partial_\kappa \eta(x; \mathbf{x}_i) - \partial_k \partial_\ell \eta(x; \mathbf{x}_i)}{b \eta(x; \mathbf{x}_i) - \partial_\ell \eta(x; \mathbf{x}_i)}, \frac{b \partial_k \eta(x; \mathbf{x}_i) + \partial_\ell \eta(x; \mathbf{x}_i)}{b \eta(x; \mathbf{x}_i) + \partial_\ell \eta(x; \mathbf{x}_i)} \right\}
\]

In the two results below, we use the above definition to give necessary and sufficient conditions for a measure to belong to \(\mathcal{M}_{a,b,L}^{(i,k)}\).

\begin{proposition}
A \(C^2\) density \(\eta : \mathbb{T}^N \to \mathbb{R}^+\) belongs to \(\mathcal{M}_{a,b,L}^{(i,k)}\) if and only if the following two conditions are satisfied

\begin{enumerate}
\item \(\eta(\cdot; \mathbf{x}_i) \in \mathcal{V}_a\) for all \(i \in [1, N]\) and \(\mathbf{x}_i \in \mathbb{T}^{N-1}\)
\item for every \(\mathbf{x}_i \in \mathbb{T}^{N-1}\)
\end{enumerate}

\[
A_{\eta,b}^{(i,k)} (\mathbf{x}_i) - B_{\eta,b}^{(i,k)} (\mathbf{x}_i) \leq L.
\]

\[\text{When } \ell = m = k, \left| \partial_h [\hat{D}_1 \hat{H}_1 \Phi(0, \mathbf{y}_1) - \hat{D}_1 \hat{H}_1 \Phi(y_1, \mathbf{y}_1)] \right| \]

\[
\leq \sum_{j=1}^{N} \sum_{p=1}^{N} |\partial_p \partial_j \partial_k H_k||\partial_1 \Phi_p||\partial_k \Phi_j| + \sum_{j=1}^{N} |\partial_j \partial_k H_k||\partial_1 \Phi_j| \leq \\
K_N^{-1} + \sum_{j \neq k} K_N^{-3} + \sum_{j \neq k} K_N^{-1} - 1 + \sum_{j \neq k} K_N^{-1} - 1 - 1 \\
+ \sum_{j \neq k} K_N^{-1} - 1 + \sum_{j \neq k} K_N^{-2} \\
\leq K_N^{-1}
\]

and the other estimates follow from analogous computations.
Proof. Fix $i \in [1, N]$, $\hat{x}_i \in \mathbb{T}^{N-1}$, $\delta > 0$, and $k \neq i$. Recall the expression for $\beta_b$ given in (12). Applying it to estimate $\beta_b(\eta(\cdot; \hat{x}_i), \eta(\cdot; \hat{x}_i + \delta e_k))$, where $e_k$ denotes the $k$-th vector of the standard basis of $\mathbb{R}^N$, the first term in the expression is
$$\frac{\eta(x; \hat{x}_i + \delta e_k)}{\eta(x; \hat{x}_i)} = 1 + \delta \frac{\partial_k \eta(x; \hat{x}_i)}{\eta(x; \hat{x}_i)} + o(\delta)$$
the second:
$$\frac{b_n \eta(x; \hat{x}_i + \delta e_k) - \partial_i \eta(x; \hat{x}_i + \delta e_k)}{b_n \eta(x; \hat{x}_i) - \partial_i \eta(x; \hat{x}_i)} = 1 + \delta \frac{b \partial_k \eta(x; \hat{x}_i) - \partial_k \partial_i \eta(x; \hat{x}_i)}{b \eta(x; \hat{x}_i) - \partial_i \eta(x; \hat{x}_i)} + o(\delta)$$
and the third:
$$\frac{b_n \eta(x; \hat{x}_i + \delta e_k) + \partial_i \eta(x; \hat{x}_i + \delta e_k)}{b_n \eta(x; \hat{x}_i) + \partial_i \eta(x; \hat{x}_i)} = 1 + \delta \frac{b \partial_k \eta(x; \hat{x}_i) + \partial_k \partial_i \eta(x; \hat{x}_i)}{b \eta(x; \hat{x}_i) + \partial_i \eta(x; \hat{x}_i)} + o(\delta).$$
Analogously
$$\frac{\eta(x; \hat{x}_i)}{\eta(x; \hat{x}_i + \delta e_k)} = 1 - \delta \frac{\partial_k \eta(x; \hat{x}_i)}{\eta(x; \hat{x}_i)} + o(\delta)$$
$$\frac{b_n \eta(x; \hat{x}_i) - \partial_i \eta(x; \hat{x}_i)}{b_n \eta(x; \hat{x}_i + \delta e_k) - \partial_i \eta(x; \hat{x}_i + \delta e_k)} = 1 - \delta \frac{b \partial_k \eta(x; \hat{x}_i) - \partial_k \partial_i \eta(x; \hat{x}_i)}{b \eta(x; \hat{x}_i) - \partial_i \eta(x; \hat{x}_i)} + o(\delta)$$
$$\frac{b_n \eta(x; \hat{x}_i) + \partial_i \eta(x; \hat{x}_i)}{b_n \eta(x; \hat{x}_i + \delta e_k) + \partial_i \eta(x; \hat{x}_i + \delta e_k)} = 1 - \delta \frac{b \partial_k \eta(x; \hat{x}_i) + \partial_k \partial_i \eta(x; \hat{x}_i)}{b \eta(x; \hat{x}_i) + \partial_i \eta(x; \hat{x}_i)} + o(\delta).$$
From the above follows that for $\delta > 0$
$$\beta_b(\eta(\cdot; \hat{x}_i), \eta(\cdot; \hat{x}_i + \delta e_k)) = 1 + \delta A^{(i,k)}_{n,b}(\hat{x}_i) + o(\delta)$$
$$\beta_b(\eta(\cdot; \hat{x}_i + \delta e_k), \eta(\cdot; \hat{x}_i)) = 1 - \delta B^{(i,k)}_{n,b}(\hat{x}_i) + o(\delta)$$
while for $\delta < 0$
$$\beta_b(\eta(\cdot; \hat{x}_i), \eta(\cdot; \hat{x}_i + \delta e_k)) = 1 - \delta B^{(i,k)}_{n,b}(\hat{x}_i) + o(\delta)$$
$$\beta_b(\eta(\cdot; \hat{x}_i + \delta e_k), \eta(\cdot; \hat{x}_i)) = 1 + \delta A^{(i,k)}_{n,b}(\hat{x}_i) + o(\delta)$$
which implies
$$\lim_{\delta \to 0} \frac{1}{\delta} \beta_b(\eta(\cdot; \hat{x}_i), \eta(\cdot; \hat{x}_i + \delta e_k)) = A^{(i,k)}_{n,b}(\hat{x}_i) - B^{(i,k)}_{n,b}(\hat{x}_i).$$
Assuming $\eta \in \mathcal{M}_{a,b,L}$, for every $\hat{x}_i \in \mathbb{T}^{N-1}$ and $\delta > 0$
$$\theta_a(\eta(\cdot; \hat{x}_i), \eta(\cdot; \hat{x}_i + \delta e_k)) \leq L|\delta|$$
that with (77) implies
$$A^{(i,k)}_{n,b}(\hat{x}_i) - B^{(i,k)}_{n,b}(\hat{x}_i) \leq L.$$
Conversely, assume that $A^{(i,k)}_{n,b}(\hat{x}_i) - B^{(i,k)}_{n,b}(\hat{x}_i) \leq L$ for every $\hat{x}_i \in \mathbb{T}^{N-1}$. For any $\hat{x}_i$ and $\hat{x}'_i$ differing only on their $k$-th coordinates $x_k, x'_k \in \mathbb{T}$, for $m \in \mathbb{N}$ consider
$$x_k =: y_0 < y_1 < \ldots < y_m := x'_k$$
such that $|y_j - y_{j+1}| \leq |x_k - x'_k|/m$. Then, calling $\hat{x}^{(j)}_i \in \mathbb{T}^{N-1}$ having $k$-th coordinate equal to $y_j$, and all other coordinates equal to $\hat{x}_i$
$$\theta_b(\eta(\cdot; \hat{x}^{(j)}_i), \eta(\cdot; \hat{x}^{(j+1)}_i)) \leq \log \left(1 + \frac{|x_k - x'_k|}{m} (A^{(i,k)}_{n,b}(\hat{x}^{(j)}_i) - B^{(i,k)}_{n,b}(\hat{x}^{(j)}_i)) + o(m^{-1})\right).$$
From this and triangle inequality follows that
\[
\theta_0(\eta(\cdot; \hat{x}_i), \eta(\cdot; \hat{x}'_i)) \leq \sum_{j=0}^{m-1} \theta_0(\eta(\cdot; \hat{x}^{(j)}_i), \eta(\cdot; \hat{x}'^{(j+1)}_i)) \leq m \log \left( 1 + \frac{|x_k - x'_k|}{m} L + o(m^{-1}) \right)
\]
and since \( m \) is arbitrary
\[
\theta_0(\eta(\cdot; \hat{x}_i), \eta(\cdot; \hat{x}'_i)) \leq L|x_k - x'_k|.
\]

The following corollary is immediate

**Corollary C.3.** A \( C^2 \) density \( \eta : \mathbb{T}^N \to \mathbb{R}^+ \) belongs to \( \mathcal{M}_{\alpha}^{(i)} \) if and only if the following two conditions are satisfied:
\( i \) \( \eta(\cdot; \hat{x}_i) \in \mathcal{V}_a \) for all \( i \in [1,N] \) and \( \hat{x}_i \in \mathbb{T}^{N-1} \)
\( ii \) \( \sup_{\hat{x}_i \in \mathbb{T}^{N-1}} \sup_{k \neq i} A_{\eta,b}^{(i,k)}(\hat{x}_i) - B_{\eta,b}^{(i,k)}(\hat{x}_i) \leq L. \)

### C.2. Proof of Lemma 4.6

Now we use the results in the previous subsection to compute the effect on the Lipschitz constant of composing by \( \Phi_i \).

**Proposition C.4.** Let \( \mu \in \mathcal{M}_{a,b,L} \cap \mathcal{C}_{\alpha}^2 \) with density \( \eta \), and for \( i \in [1,N] \) consider \( \Phi_i \) as in Proposition 4.4. Then letting \( \tilde{\eta} := \eta \circ \Phi_i \), and defining \( K_j := \sum_{t=1}^{N} |\partial_j \Phi_i|_{\infty} \) for \( j = a, b \) \( i \) for any \( j \in [1,N] \) and \( \hat{x}_j \in \mathbb{T}^{N-1} \), \( \tilde{\eta}(\cdot; \hat{x}_j) \in \mathcal{V}_a \)
\( ii \) for any \( j \in [1,N] \), \( k \neq j \), \( \hat{x}_j \in \mathbb{T}^{N-1} \)
\[
A_{\tilde{\eta},b}^{(j,k)}(\hat{x}_j) - B_{\tilde{\eta},b}^{(j,k)}(\hat{x}_j) \leq |\partial_j \Phi_{i,j}^{-1}|_{\infty} \sum_{m=1}^{N} \sum_{t \neq m} |\partial_j \Phi_{i,\ell}|_{\infty} \left[ \sup_{x_j \in \mathbb{T}} (\partial_k \Phi_{i,m} C_k^{(j,m)}(x_j; \hat{x}_j) - \inf_{x_j \in \mathbb{T}} (\partial_k \Phi_{i,m} D_k^{(j,m)}(x_j; \hat{x}_j)) + \frac{a + \alpha}{b_j - a_j} K_j \right] N^{-1}
\]
where
\[
C_k^{(j,m)}(x; \hat{x}_j) := \begin{cases} A_{\tilde{\eta},b}^{(j,m)}(\hat{x}_j) & \text{if } \partial_k \Phi_{i,m}(x; \hat{x}_j) \geq 0 \\ B_{\tilde{\eta},b}^{(j,m)}(\hat{x}_j) & \text{if } \partial_k \Phi_{i,m}(x; \hat{x}_j) < 0 \end{cases}
\]
\[
D_k^{(j,m)}(x; \hat{x}_j) := \begin{cases} A_{\tilde{\eta},b}^{(j,m)}(\hat{x}_j) & \text{if } \partial_k \Phi_{i,m}(x; \hat{x}_j) \geq 0 \\ B_{\tilde{\eta},b}^{(j,m)}(\hat{x}_j) & \text{if } \partial_k \Phi_{i,m}(x; \hat{x}_j) < 0 \end{cases}
\]

Before proceeding with the proof, we show how to use this proposition to prove Lemma 4.6.

**Proof of Lemma 4.6.** Fix \( i \in [1,N] \) once and for all, call \( \Phi := \Phi_i \) to ease notation, and pick any \( j \in [1,N] \), \( k \neq i \), and \( \hat{x}_j \in \mathbb{T}^{N-1} \). Define \( \mathcal{I}_1 := \{ m \neq j : \exists x \in \mathbb{T} \text{ s.t. } \partial_k \Phi_m(x; \hat{x}_j) = 0 \}, \mathcal{I}_2 := [1,N] \setminus \mathcal{I}_1. \)

With this definitions:
and the result follows and point i) follows.

Since by Proposition 4.1, $\partial_j \partial_k \Phi_m = O(N^{-2})$, for every $j \neq i$ or $m \in I_1 \setminus \{k\}$, and $\partial_k \partial_k \Phi_k = O(N^{-1})$ by mean-value theorem $|\partial_j \Phi_m(x; \hat{x}_j)| = O(N^{-2})$ for every $j \neq i$ or $m \in I_1 \setminus \{k\}$ and $|\partial_k \Phi_k(x; \hat{x}_j)| = O(N^{-1})$;

- for every $m \in I_2$, $\partial_k \Phi_m(x; \hat{x}_j) \neq 0$ for every $x \in \mathbb{T}$, and by continuity has fixed sign for every $x \in \mathbb{T}$;

- since for $N$ sufficiently large, it follows from Proposition 4.1 that $|\partial_j \Phi_j(x; \hat{x}_j)| = 1 + O(N^{-1}) \neq 0$, $\partial_j \Phi_j(x; \hat{x}_j)$ is different from zero and $j \in I_2$.

Crucially, for $m \in I_2$, $C_k^{(j,m)}(x; \hat{x}_j)$ and $D_k^{(j,m)}(x; \hat{x}_j)$ defined as in (78) and (78) do not depend on $x$, and looking at all the possible cases,

$$C_k^{(j,m)}(\hat{x}_i) \sup_{x \in \mathbb{T}} \partial_k \Phi_m(x; \hat{x}_j) - D_k^{(j,m)}(\hat{x}_i) \inf_{x \in \mathbb{T}} \partial_k \Phi_m(x; \hat{x}_j) \leq [A_{\eta,b}^{(j,k)}(\hat{x}_j) - B_{\eta,b}^{(j,k)}(\hat{x}_j)] |\partial_k \Phi_m|_{\infty} \leq L |\partial_k \Phi_m|_{\infty}.$$

For $m \in I_1$, instead, if $j \neq i$ or $m \neq k$

$$\sup_{x \in \mathbb{T}} (C_k^{(j,m)} \partial_k \Phi_m)(x; \hat{x}_i) - \inf_{x \in \mathbb{T}} (D_k^{(j,m)} \partial_k \Phi_m)(x; \hat{x}_i) \leq 2L |\partial_k \Phi_m|_{\infty} = L O(N^{-2})$$

while if $j = i$ and $m = k$ the above quantity is bounded by $LO(N^{-1})$. Therefore

$$A_{\eta,b}^{(j,k)}(\hat{x}_j) - B_{\eta,b}^{(j,k)}(\hat{x}_j) + O(N^{-1}) \leq \left| \partial_j \Phi_j^{-1} \right|_{\infty} \sum_{m \in I_1} \sum_{\ell \neq m} \left| \partial_j \Phi_\ell \right|_{\infty} \left[ C_k^{(j,m)}(\hat{x}_i) \sup_{x \in \mathbb{T}} \partial_k \Phi_m(x; \hat{x}_i) - D_k^{(j,m)}(\hat{x}_i) \inf_{x \in \mathbb{T}} \partial_k \Phi_m(x; \hat{x}_i) \right] + \left| \partial_j \Phi_j^{-1} \right|_{\infty} \sum_{m \in I_2} \sum_{\ell \neq m} \left| \partial_j \Phi_\ell \right|_{\infty} \left[ \sup_{x \in \mathbb{T}} (C_k^{(j,m)} \partial_k \Phi_m)(x; \hat{x}_i) - \inf_{x \in \mathbb{T}} (D_k^{(j,m)} \partial_k \Phi_m)(x; \hat{x}_i) \right] \leq LO(N^{-1}) + L \sum_{m \in I_2} |\partial_j \Phi_m|_{\infty} \leq L \left[ O(N^{-1}) + \left| \partial_j \Phi_j^{-1} \right|_{\infty} \sum_{m = \ell \neq m}^N \left| \partial_j \Phi_\ell \right|_{\infty} \left| \partial_k \Phi_m \right|_{\infty} \right].$$

Proof of Proposition C.4. Without loss of generality let’s fix $i = 1$ and let’s call $\Phi := \Phi_1$ to shorten notation.

For any $j \in [1, N]$

$$\left| \frac{\partial_j (\eta \circ \Phi)}{\eta \circ \Phi} \right| \leq \sum_{\ell = 1}^N \left| \frac{\partial_j \eta \circ \Phi}{\eta \circ \Phi} \right| \left| \partial_j \Phi_\ell \right| \leq a \sum_{\ell = 1}^N |\partial_j \Phi_\ell|_{\infty}$$

and the result follows and point i) follows.

For what concerns point ii), let’s estimate the terms in the expressions for $A_{\eta,b}^{(j,k)}(\hat{x}_j)$ and $B_{\eta,b}^{(j,k)}(\hat{x}_j)$ given in Definition C.1. From now on fix $\hat{x}_{j,k} \in \mathbb{T}^{N-2}$, i.e. fix all coordinates bu the $j$-th and $k$-th one. Notice that for every $x_k \in \mathbb{T}$

$$\tilde{\eta}(\cdot; x_k; \hat{x}_{j,k}) \in \mathcal{V}_{a_{\eta,j,k}}$$
where

\[ a_{\hat{x}, t} := \alpha \sum_{j=1}^{N} \sup_{x_j, x_k \in \mathbb{R}} |\partial_j \Phi^j(x_j; x_k; \hat{x}_{j,k})|. \]

Denote \( |\partial_j \Phi^j|_{\infty} := \sup_{x_j, x_k \in \mathbb{R}} |\partial_j \Phi^j(x_j; x_k; \hat{x}_{j,k})| \) and

\[ K'_j := \sum_{\ell=1}^{N} |\partial_j \Phi^{j}_{\ell}|_{\infty}. \]

Now, since \( b'_j := K'_j b \leq K_j b =: b_j \)

\[
A^{(j,k)}_{\eta,b'_j}(\hat{x}_j) - B^{(j,k)}_{\eta,b'_j}(\hat{x}_j) \geq A^{(j,k)}_{\eta,b_j}(\hat{x}_j) - B^{(j,k)}_{\eta,b_j}(\hat{x}_j)
\]

for any \( \hat{x}_j = (x_k; \hat{x}_{j,k}) \). We proceed to estimate \( A^{(j,k)}_{\eta,b_j}(\hat{x}_j) \) and \( B^{(j,k)}_{\eta,b_j}(\hat{x}_j) \), by estimating the three terms appearing in their definition.

First Term:

\[
\frac{\partial_j (\eta \circ \Phi)}{\eta \circ \Phi} = \sum_{\ell=1}^{N} \frac{\partial_l \eta \circ \Phi \partial_j \Phi^{\ell}}{\eta \circ \Phi}
\]

from which

\[
\sum_{\ell=1}^{N} D^{(\ell,m)}_{k}(\eta \circ \Phi) \partial_j \Phi^{\ell}(x) \leq \frac{\partial_j (\eta \circ \Phi)}{\eta \circ \Phi}(x) \leq \sum_{\ell=1}^{N} C^{(\ell,m)}_{k}(x) \partial_j \Phi^{\ell}(x).
\]

Second Term:

\[
\frac{b'_j \partial_k (\eta \circ \Phi) - \partial_k \partial_j (\eta \circ \Phi)}{b'_j \eta \circ \Phi - \eta \circ \Phi} = \frac{b'_j \sum_{m=1}^{N} \partial_m \eta \circ \Phi \partial_k \Phi^{\ell} - \partial_k \left( \sum_{m=1}^{N} \partial_m \eta \circ \Phi \partial_j \Phi^{m}\right)}{b'_j \eta \circ \Phi - \eta \circ \Phi}
\]

\[
= \sum_{m=1}^{N} b'_j \partial_m \eta \circ \Phi - \sum_{\ell \neq m} \partial_m \partial_{\ell} \eta \circ \Phi \partial_j \Phi^{\ell} - \sum_{\ell=1}^{N} \partial_{\ell} \partial_m \eta \circ \Phi \partial_k \Phi^{\ell} - \sum_{\ell=1}^{N} \partial_{\ell} \partial_m \eta \circ \Phi \partial_k \Phi^{\ell} \partial_j \Phi^{\ell}
\]

\[
= \sum_{m=1}^{N} b'_j \partial_m \eta \circ \Phi - \sum_{\ell \neq m} \partial_m \partial_{\ell} \eta \circ \Phi \partial_j \Phi^{\ell} - \sum_{\ell=1}^{N} \partial_{\ell} \partial_m \eta \circ \Phi \partial_k \Phi^{\ell} \partial_j \Phi^{\ell}
\]

\[
(81)
\]

\[
= \sum_{m=1}^{N} \sum_{\ell \neq m} \frac{\partial_j \Phi^{\ell} \partial_m \eta \circ \Phi - \sum_{\ell \neq m} \partial_{\ell} \partial_m \partial_{\ell} \eta \circ \Phi \partial_j \Phi^{\ell}}{b'_j \eta \circ \Phi - \eta \circ \Phi} \partial_k \Phi^{m}
\]

\[
(82)
\]

\[
+ \sum_{m=1}^{N} \frac{|\partial_j \Phi^{m}|_{\infty} \partial_m \eta \circ \Phi \partial_k \Phi^{m}}{b'_j \eta \circ \Phi - \eta \circ \Phi} - \sum_{\ell=1}^{N} \frac{\partial_{\ell} \partial_m \eta \circ \Phi \partial_k \Phi^{\ell} \partial_j \Phi^{\ell}}{b'_j \eta \circ \Phi - \eta \circ \Phi}
\]

\[
(83)
\]
Now, for the sum in (81),
\[
\sum_{i \neq m} | \partial_j \Phi_{\ell} | \partial_m \eta \circ \Phi - \sum_{i \neq m} \partial_m \partial_i \eta \circ \Phi \partial_j \Phi_{\ell} =
\]
\[
= \sum_{i \neq m} | \partial_j \Phi_{\ell} | \frac{b \partial_m \eta \circ \Phi - s \partial_m \partial_i \eta \circ \Phi}{b'_j \eta \circ \Phi - \partial_j (\eta \circ \Phi)} + \sum_{i \neq m} \partial_m \partial_i \eta \circ \Phi \frac{\partial_j \Phi_{\ell} - \partial_j (\eta \circ \Phi)}{b'_j \eta \circ \Phi - \partial_j (\eta \circ \Phi)}
\]
where \( s \) is the sign of \( \partial_j \Phi_{\ell} \), the first sum in (81)
\[
| \partial_j \Phi_{\ell} | | \partial_j \Phi |^{-1} B_{\eta,b}^{(\ell,m)} \leq | \partial_j \Phi_{\ell} | | \partial_j \Phi |^{-1} A_{\eta,b}^{(\ell,m)}
\]
while since \( | \partial_j \Phi_{\ell} |^{-1} \partial_j \Phi | \leq | \partial \Phi | + | \partial \partial_j \Phi | \), recalling the estimates in Proposition 4.1, the second sum in (81) can be estimated as
\[
\left| \sum_{i \neq m} \frac{\partial_m \partial_i \eta \circ \Phi}{b'_j \eta \circ \Phi - \partial_j (\eta \circ \Phi)} \right| \leq (b'_j - a_j)^{-1} \alpha \kappa \eta N^{-1}.
\]

For the sum in (83)
\[
\left| \sum_{m=1}^{N} \frac{\partial_j \Phi_{m} | \partial_j \Phi_{m+1} |}{b'_j \eta \circ \Phi - \partial_j (\eta \circ \Phi)} \right| \leq \sum_{m=1}^{N} \frac{\partial_j \Phi_{m} | \partial_j \Phi_{m+1} |}{b'_j \eta \circ \Phi - \partial_j (\eta \circ \Phi)} \leq \frac{a}{b'_j - a_j} \kappa \eta N^{-1}
\]
where above we used that \( | \partial \eta \circ \Phi | \eta \circ \Phi \) \( \leq a \), and the estimates on the derivatives of \( \Phi \) given in Proposition 4.1. Analogously, for the terms in (83), using that \( | \partial \partial_j \eta \circ \Phi | \eta \circ \Phi \) \( \leq \alpha \)
\[
\left| \sum_{i \neq 1}^{N} \frac{\partial_j \Phi_{i+1} - \partial_j (\eta \circ \Phi)}{b'_j \eta \circ \Phi - \partial_j (\eta \circ \Phi)} \right| \leq \frac{a + \alpha}{b'_j - a_j} \kappa \eta N^{-1}.
\]
Putting all the estimates together we get
\[
\frac{b'_j \partial_k (\eta \circ \Phi) - \partial_k \partial_j (\eta \circ \Phi)}{b'_j \eta \circ \Phi - \partial_j (\eta \circ \Phi)} \leq \left| \partial_j \Phi_{\ell} \right|^{-1} \sum_{m=1}^{N} \frac{| \partial_j \Phi_{\ell} | | \partial_j \Phi_{m} \circ \Phi |}{b'_j \eta \circ \Phi - \partial_j (\eta \circ \Phi)} \leq \frac{a + \alpha}{b'_j - a_j} \kappa \eta N^{-1}
\]
\[
\frac{b'_j \partial_k (\eta \circ \Phi) - \partial_k \partial_j (\eta \circ \Phi)}{b'_j \eta \circ \Phi - \partial_j (\eta \circ \Phi)} \geq \left| \partial_j \Phi_{\ell} \right|^{-1} \sum_{m=1}^{N} \frac{| \partial_j \Phi_{\ell} | | \partial_j \Phi_{m} \circ \Phi |}{b'_j \eta \circ \Phi - \partial_j (\eta \circ \Phi)} \geq \frac{a + \alpha}{b'_j - a_j} \kappa \eta N^{-1}.
\]

Third Term:
\[
\frac{b'_j \partial_k (\eta \circ \Phi) + \partial_k \partial_j (\eta \circ \Phi)}{b'_j \eta \circ \Phi + \partial_j (\eta \circ \Phi)},
\]
the estimates are analogous.

\[^{23}\text{Here plays a crucial role the fact that we considered distances with respect to } b'_j \text{ and not } b_j.\]
C.3. **Proof of Lemma 4.9.** Below we are going to rely on Jacobi’s formula for the derivative of a determinant: If \( t \mapsto A(t) \) is a differentiable mapping where \( A(t) \) is an \( n \times n \) square matrix, then

\[
\frac{d|A(t)|}{dt} = |A(t)| \text{tr} \left( A(t)^{-1} \frac{dA(t)}{dt} \right).
\]  

**Proof of Lemma 4.9.** Fix \( i = 1 \), \( k \neq 1 \), and call \( \Phi := \Phi_1 \). First of all notice that \(|D\Phi| = |\hat{D}_1 \Phi_1|\). By (85),

\[
a_{\Phi} := \sup \left| \frac{\partial_1 |\hat{D}_1 \Phi_1|}{|\hat{D}_1 \Phi_1|} \right| = \sup \left| \text{tr} \left( \hat{D}_1 \Phi_1^{-1} \partial_1 \hat{D}_1 \Phi_1 \right) \right|
\]

and using the estimates in Proposition B.1 and applying Proposition B.1 to estimate the entries of \( \hat{D}_1 \Phi_1^{-1} \), one can see that the above is bounded by \( \mathcal{K}_{\#} \).

Fixing \( b > a_{\Phi} \), we proceed estimating \( A_{|\hat{D}_1 \Phi_1|b}^{(1, k)} \) and \( B_{|\hat{D}_1 \Phi_1|b}^{(1, k)} \) and the terms in their definition. Similarly to the above, the first term can be estimated as

\[
\left| \frac{\partial_1 |\hat{D}_1 \Phi_1|}{|\hat{D}_1 \Phi_1|} \right| = \left| \text{tr} \left( \hat{D}_1 \Phi_1^{-1} \partial_1 \hat{D}_1 \Phi_1 \right) \right| \leq \mathcal{K}_{\#} N^{-1}
\]

where we used that \( k \neq 1 \).

For the second term, consider

\[
\partial_k \partial_1 |\hat{D}_1 \Phi_1| = \partial_k \left[ |\hat{D}_1 \Phi_1| \text{tr} \left( \hat{D}_1 \Phi_1^{-1} \partial_1 \hat{D}_1 \Phi_1 \right) \right]
\]

\[
= |\hat{D}_1 \Phi_1| \text{tr} \left( \hat{D}_1 \Phi_1^{-1} \partial_1 \hat{D}_1 \Phi_1 \right) \text{tr} \left( \hat{D}_1 \Phi_1^{-1} \partial_1 \hat{D}_1 \Phi_1 \right) + |\hat{D}_1 \Phi_1| \text{tr} \left( \partial_k (\hat{D}_1 \Phi_1^{-1}) \partial_1 \hat{D}_1 \Phi_1 + \hat{D}_1 \Phi_1^{-1} \partial_k \partial_1 \hat{D}_1 \Phi_1 \right)
\]

It was estimated above that

\[
\left| \text{tr} \left( \hat{D}_1 \Phi_1^{-1} \partial_1 \hat{D}_1 \Phi_1 \right) \right| \leq \mathcal{K}_{\#} N^{-1}
\]

and

\[
\left| \text{tr} \left( \hat{D}_1 \Phi_1^{-1} \partial_1 \hat{D}_1 \Phi_1 \right) \right| \leq \mathcal{K}_{\#} N^{-2}.
\]

Also, since \( \partial_k (\hat{D}_1 \Phi_1)^{-1} = (\hat{D}_1 \Phi_1)^{-1} (\partial_k \hat{D}_1 \Phi_1) (\hat{D}_1 \Phi_1)^{-1} \),

\[
\left| \text{tr} \left( \partial_k (\hat{D}_1 \Phi_1^{-1}) \partial_1 \hat{D}_1 \Phi_1 \right) \right| \leq \mathcal{K}_{\#} N^{-2}.
\]

We now need to estimate the derivatives \( \partial_k \partial_1 \hat{D}_1 \Phi_1 \) which involve the third-order partial derivatives of \( \Phi_1 \) that we haven’t computed yet.

**Lemma C.5.**

\[
[\partial_k \partial_1 \hat{D}_1 \Phi_1]_{\ell m} \leq \begin{cases} \mathcal{K}_{\#} N^{-1} & \ell = m = k \\ \mathcal{K}_{\#} N^{-2} & \text{otherwise} \end{cases}
\]

**Proof.** From equation (64) follows that

\[
\partial_k \partial_1 \hat{D}_1 \Phi_1(g_1, \hat{y}_1) = \partial_k \partial_1 \left[ (\hat{D}_1 \hat{H}_{1 : 0}(g_1, \hat{y}_1))^{-1} \hat{D}_1 \hat{H}_{1 : 0}(g_1, \hat{y}_1) \right] + \partial_1 \left[ (\hat{D}_1 \hat{H}_{1 : 0}(g_1, \hat{y}_1))^{-1} \partial_k [\hat{D}_1 \hat{H}_{1 : 0}(g_1, \hat{y}_1)] \right]
\]

\[
+ \partial_1 \left[ (\hat{D}_1 \hat{H}_{1 : 0}(g_1, \hat{y}_1))^{-1} \partial_k \partial_1 [\hat{D}_1 \hat{H}_{1 : 0}(g_1, \hat{y}_1)] \right] + \partial_1 \left[ (\hat{D}_1 \hat{H}_{1 : 0}(g_1, \hat{y}_1))^{-1} \partial_k \partial_1 \partial_1 [\hat{D}_1 \hat{H}_{1 : 0}(g_1, \hat{y}_1)] \right]
\]
For the first term
\[ \partial_k \partial_1[(\hat{D}_1 \hat{H}_1 \Phi)\Phi(\mathbf{y}_1, \mathbf{y}_1)]^{-1} = \sum_{\ell=1}^{N} \sum_{n=1}^{N} \partial_n \partial_\ell (\hat{D}_1 \hat{H}_1)^{-1}_\Phi(\mathbf{y}_1, \mathbf{y}_1) \partial_k \Phi_n \partial_1 \Phi_\ell \]
\[ + \sum_{\ell=1}^{N} \partial_\ell (\hat{D}_1 \hat{H}_1)^{-1}_\Phi(\mathbf{y}_1, \mathbf{y}_1) \partial_1 \partial_\ell \Phi \]

and in the above, since \( \partial_n \partial_\ell [(\hat{D}_1 \hat{H}_1)^{-1}(\hat{D}_1 \hat{H}_1)] = 0 \),
\[ \partial_n \partial_\ell (\hat{D}_1 \hat{H}_1)^{-1} = -(\hat{D}_1 \hat{H}_1)^{-1}_n (\hat{D}_1 \hat{H}_1)^{-1}_\ell (\hat{D}_1 \hat{H}_1)^{-1}_n (\hat{D}_1 \hat{H}_1) + (\hat{D}_1 \hat{H}_1)^{-1}_\ell (\hat{D}_1 \hat{H}_1) \]
\[ + \hat{D}_1 \hat{H}_1^{-1} \partial_\ell \partial_\ell (\hat{D}_1 \hat{H}_1) \].
Combining (72) and (73)
\[ [\partial_\ell (\hat{D}_1 \hat{H}_1)^{-1}]_{\Phi(\mathbf{y}_1, \mathbf{y}_1)} \leq \left\{ \begin{array}{ll} K_\# N^{-1} & \ell = m = k \\
K_\# N^{-2} & \text{otherwise.} \end{array} \right. \]

For the second term
\[ \partial_1[(\hat{D}_1 \hat{H}_1 \Phi)\Phi(\mathbf{y}_1, \mathbf{y}_1)]^{-1} = \sum_{n=1}^{N} \partial_n[(\hat{D}_1 \hat{H}_1)^{-1}] \cdot \partial_1 \Phi_n \]
and again combining (72) and (73)
\[ \left| [\partial_1[(\hat{D}_1 \hat{H}_1 \Phi)\Phi(\mathbf{y}_1, \mathbf{y}_1)]^{-1}]_{\partial_\ell [\hat{D}_1 \hat{H}_1(\mathbf{y}_1, \mathbf{y}_1)]}\right| \leq \left\{ \begin{array}{ll} K_\# N^{-1} & \ell = m = k \\
K_\# N^{-2} & \text{otherwise.} \end{array} \right. \]

The above lemma implies that
\[ \left| tr \left( \hat{D}_1^{-1} \partial_\ell \partial_1 \hat{D}_1 \Phi \right) \right| \leq K_\# N^{-1}. \]
Combining all of the above we get that
\[ (88) \left| \frac{\partial_\ell \partial_1 [\hat{D}_1 \Phi]}{|D_1 \Phi_1|} \right| \leq K_\# N^{-1} \]
Putting together (86) and (88)
\[ \frac{b \partial_k |\hat{D}_1 \Phi_1| + |\partial_k \partial_1 \hat{D}_1 \Phi_1|}{b |D_1 \Phi_1| - \partial_1 |D_1 \Phi_1|} = \left( b - \frac{\partial_1 [\hat{D}_1 \Phi_1]}{|D_1 \Phi_1|} \right)^{-1} \frac{b |\hat{D}_1 \Phi_1| + |\partial_k \partial_1 \hat{D}_1 \Phi_1|}{|D_1 \Phi_1|} \]
\[ = \frac{1 + b}{b - a \Phi} K_\# N^{-1}. \]
For any \( b > a \Phi \)
\[ A_{|D_1 \Phi_1|, b}^{(i,k)} B_{|D_1 \Phi_1|, b}^{(i,k)} \leq \frac{1 + b}{b - a \Phi} K_\# N^{-1} \]
from which the result follows.
C.4. Proof of Proposition 4.10

Proof of Proposition 4.10. Without loss of generality assume \( i = 1 \), and denote \( \Phi := \Phi_1 \).

By Lemma 4.6 and Lemma 4.9, applying Lemma A.5 we get \( \Phi_*^{-1} \eta(\cdot ; \hat{x}_1) \in \mathcal{V}_\alpha \), and for \( j \neq 1 \), \( \Phi_*^{-1} \eta(\cdot ; \hat{x}_j) \in \mathcal{V}_\alpha \).

Let \( b_1 := \mathcal{K} b + \mathcal{B}_\Phi \) and \( \mathcal{L}_1 := \mathcal{L} \) while for \( j \neq 1 \), \( b_j := \hat{\mathcal{K}} b + \mathcal{B}_\Phi \) and \( \mathcal{L}_j := \hat{\mathcal{L}} \).

By Proposition A.6 for \( \hat{x}_j, \hat{x}_j' \in \mathbb{T}^{N-1} \) differing only for coordinates \( k \neq 1 \), \( x_k, x'_k \in \mathbb{T} \) one has

\[
\theta_{b_j} \left( \Phi_*^{-1} \eta(\cdot ; \hat{x}_j), \Phi_*^{-1} \eta(\cdot ; \hat{x}_j') \right) \leq \theta_{b_j-b_\Phi} \left( \eta \circ \Phi(\cdot ; \hat{x}_j), \eta \circ \Phi(\cdot ; \hat{x}_j') \right) + 
\theta_{b_\Phi} \left( |D\Phi(\cdot ; \hat{x}_j)|, |D\Phi(\cdot ; \hat{x}_j')| \right)
\]

\[
\leq \mathcal{L}_j \cdot |L|x_k - x'_k| + \frac{a + \alpha}{\mathcal{K} b - \mathcal{K} a} \mathcal{K}_\# N^{-1} \cdot |L|x_k - x'_k|
\]

where in the last inequality we used Lemma 4.6 and Lemma 4.9.

Now we turn to prove that \( \eta \in \mathcal{C}_1 \).

\[
\partial_k (\eta \circ \Phi_i \cdot |D\Phi_i|) = \partial_k (\eta \circ \Phi_i) \cdot |D\Phi_i| + \eta \circ \Phi_i \cdot \partial_k |D\Phi_i|
\]

\[
\partial_j \partial_k (\eta \circ \Phi_i \cdot |D\Phi_i|) = \partial_j \partial_k (\eta \circ \Phi_i) \cdot |D\Phi_i| + \partial_k (\eta \circ \Phi_i) \cdot \partial_j |D\Phi_i| + 
\partial_j \partial_k (\eta \circ \Phi_i) \cdot \eta \circ \Phi_i \cdot \partial_k |D\Phi_i|
\]

and therefore

\[
\frac{\partial_j \partial_k (\eta \circ \Phi_i \cdot |D\Phi_i|)}{\eta \circ \Phi_i \cdot |D\Phi_i|} = \frac{\partial_j \partial_k (\eta \circ \Phi_i)}{\eta \circ \Phi_i} + \frac{\partial_k (\eta \circ \Phi_i)}{\eta \circ \Phi_i} \frac{\partial_j |D\Phi_i|}{|D\Phi_i|} + 
\frac{\partial_j \partial_k (\eta \circ \Phi_i) \partial_k |D\Phi_i|}{\eta \circ \Phi_i} + \frac{\partial_j \partial_k |D\Phi_i|}{|D\Phi_i|}
\]

\[
= \sum_{\ell, m=1}^{N} \frac{\partial_{\ell} \partial_m \eta \circ \Phi_i \cdot \partial_k \Phi_{i,\ell} \cdot \partial_j \Phi_{i,m}}{\eta \circ \Phi_i} + 
\frac{\partial_k (\eta \circ \Phi_i) \partial_j |D\Phi_i|}{\eta \circ \Phi_i} + \frac{\partial_j \partial_k |D\Phi_i|}{|D\Phi_i|}.
\]

The above is in modulus less than

\[
\alpha \cdot \sum_{\ell, m=1}^{N} |\partial_k \Phi_{i,\ell}||\partial_j \Phi_{i,m}| + a_\Phi (a_k + a_j) + \mathcal{K}_\# N^{-1}
\]

where we used that \( \eta \in \mathcal{M}_{\alpha', b', L'} \), and the estimates from Lemma 4.9. Thus point i) is proven.

To prove point ii), recall that

\[
\Pi_1 \Phi_*^{-1} \eta(\hat{x}_1) = \int_{\mathbb{T}} ds_1 |D\Phi|(s_1; \hat{x}_1) \cdot \eta \circ \Phi(s_1; \hat{x}_1).
\]

is the marginal on \( \mathbb{T}^{N-1} \), where the first coordinate has been integrated out.
For $j \in [2, N]$ and any $\hat{x}_{1,j} \in T^{N-2}$,
\[
\Pi_j \Phi^{-1}_s(\cdot; \hat{x}_{1,j}) \in \mathcal{V}_{a_j}
\]
which follows from
\[
\partial_j \log \Pi_j \Phi^{-1}_s \eta \leq \frac{\int_T ds \partial_j [D \Phi \cdot \eta \circ \Phi](s, \hat{x}_1)}{\int_T ds [D \Phi \cdot \eta \circ \Phi](s, \hat{x}_1)} \leq a_j
\]
where we used that, by point i), $\partial_j [[D \Phi \cdot \eta \circ \Phi](s, \hat{x}_1) \leq a_j [[D \Phi \cdot \eta \circ \Phi](s, \hat{x}_1)$ for every $(s, \hat{x}_1) \in T^{N}$.

Fix any $j \in [2, N]$ and $k \neq 1$ and $j$, and call $\tilde{\eta} := \Phi^{-1}_s \eta$. We proceed by estimating $A^{(j,k)}(\hat{x}_{1,j})$ and $B^{(j,k)}(\hat{x}_{1,j})$. Recalling that by point i), $\tilde{\eta} \in \mathcal{M}_{\eta,b_j,L_j,k}$, we get
\[
\frac{\partial_k \Pi_j \Pi_j}{\Pi_j \tilde{\eta}}(\hat{x}_1) = \frac{\int_T dx_1 \partial_k \tilde{\eta}(x_1; \hat{x}_1)}{\int_T dx_1 \tilde{\eta}(x_1; \hat{x}_1)} \leq A^{(j,k)}(\tilde{x}_1)
\]
where we used that, for every $(x_1; \tilde{x}_1) \in T^{N}$, $\partial_k \tilde{\eta}(x_1; \tilde{x}_1) \leq A^{(j,k)}(\tilde{x}_1) \cdot \tilde{\eta}(x_1; \tilde{x}_1)$. Similarly,
\[
\frac{b_j \partial_k \Pi_j \Pi_j \partial_k \Pi_j \Pi_j \tilde{\eta}(\hat{x}_1)}{b_j \Pi_j \Pi_j \partial_k \Pi_j \Pi_j \tilde{\eta}(\hat{x}_1)} = \frac{\int_T dx_1 [b_j \partial_k \tilde{\eta}(x_1; \hat{x}_1) - \partial_k \tilde{\eta}(x_1; \hat{x}_1)]}{\int_T dx_1 [b_j \tilde{\eta}(x_1; \hat{x}_1) - \partial_k \tilde{\eta}(x_1; \hat{x}_1)]} \leq A^{(j,k)}(\tilde{x}_1)
\]
as $[b_j \partial_k \tilde{\eta}(x_1; \tilde{x}_1) - \partial_k \tilde{\eta}(x_1; \tilde{x}_1)] \leq A^{(j,k)}(\tilde{x}_1) \cdot [b_j \tilde{\eta}(x_1; \tilde{x}_1) - \partial_k \tilde{\eta}(x_1; \tilde{x}_1)]$ for all $x_1 \in T$ and $\tilde{x}_1 \in T^{N-1}$, and
\[
\frac{b_j \partial_k \Pi_j \Pi_j \partial_k \Pi_j \Pi_j \tilde{\eta}(\hat{x}_1)}{b_j \Pi_j \Pi_j \partial_k \Pi_j \Pi_j \tilde{\eta}(\hat{x}_1)} \leq A^{(j,k)}(\tilde{x}_1).
\]
Inequalities (89) - (90) imply that
\[
A^{(j,k)}(\tilde{x}_1) \leq A^{(j,k)}(\tilde{x}_1).
\]
Analogously, one can estimate
\[
B^{(j,k)}(\tilde{x}_1) \leq B^{(j,k)}(\tilde{x}_1).
\]
Inequalities (92) and (93) together with Proposition C.2 applied to $\Pi_j \tilde{\eta}$, imply that
\[
\Pi_j \tilde{\eta} \in \mathcal{M}_{\eta,b_j,L_j,k}.
\]
\[\square\]

C.5. Proof of Proposition 4.11

Proof of Proposition 4.11 Let’s begin by noticing that if $H$ satisfies Assumption 4.2 then for every $i \in [1, N]$ and $\hat{x}_i \in T^{N-1}$, $h_{\hat{x}_i} : T \to T$ defined as
\[
h_{\hat{x}_i}(\cdot) := H_i(\cdot; \hat{x}_i)
\]
is a local diffeomorphism. Let’s call $\{h^{-1}_{\hat{x}_i,\ell}\}_{\ell \in \mathbb{Z}}$ its inverse branches. It also follows that $h_{\hat{x}_i}$ are uniformly expanding with expansion bounded below by $\kappa > 1$ and distortion bounded above by $D = \frac{K\delta}{1-K\delta}$. Lemma 4.4 implies that, provided $N$ is sufficiently large, $g_{\hat{x}_i}(\cdot) = G_i(\cdot; \hat{x}_i)$ is also a uniformly expanding local diffeomorphism with minimal expansion lower bounded by $\kappa + O(N^{-1})$ and distortion bounded above by $D + O(N^{-1})$. Then $g_{\hat{x}_i,\hat{\eta}}(\cdot; \hat{x}_i) \in \mathcal{V}_{\kappa^{-1}(\alpha+\delta)} O(N^{-1})$ and (24) implies that
for $N$ sufficiently large $\kappa^{-1}(a+D) + O(N^{-1}) < a$. This implies that given a density $\eta: \mathbb{T}^N \to \mathbb{R}^+$, defining

$$\zeta(\cdot; \hat{x}_i) := (G_i; \text{Id})_* \eta(\cdot; \hat{x}_i) = g_{\hat{x}_i} \eta(\cdot; \hat{x}_i),$$

if $\eta(\cdot; \hat{x}_i) \in \mathcal{V}_a$, then $\zeta(\cdot; \hat{x}_i) \in \mathcal{V}_a$.

Now fix any $b > a$ and let

$$\Lambda' := 1 - e^{1-\text{diam}(b,\kappa^{-1}(b+D)+O(N^{-1}))} = \Lambda + O(N^{-1}).$$

Pick $\eta \in \mathcal{M}_{a,b,L}^{(i)}$ and for $j \neq i$ consider $\hat{x}_i, \hat{x}'_i \in \mathbb{T}^{N-1}$ differing only on their $j$-th coordinates $x_j, x'_j \in \mathbb{T}$. By triangle inequality

$$\theta_b \left( g_{\hat{x}_i} \eta(\cdot; \hat{x}_i), g_{\hat{x}'_i} \eta(\cdot; \hat{x}'_i) \right) \leq \theta_b \left( g_{\hat{x}_i} \eta(\cdot; \hat{x}_i), g_{\hat{x}_i} \eta(\cdot; \hat{x}'_i) \right) + \theta_b \left( g_{\hat{x}_i} \eta(\cdot; \hat{x}'_i), g_{\hat{x}'_i} \eta(\cdot; \hat{x}'_i) \right) \leq \Lambda' \theta_b(\eta(\cdot; \hat{x}_i), \eta(\cdot; \hat{x}'_i)) + \mathcal{K}_N^{-1} |x_j - x'_j| \leq (\Lambda' \mathcal{K}_N^{-1}) |x_j - x'_j|$$

where for the second inequality we used that $d_{C^2}(g_{\hat{x}_i}, g_{\hat{x}'_i}) \leq \mathcal{E}N^{-1}$ and Proposition A.7.

\[ \square \]

C.6. Proof of Proposition 4.12. Before proceeding with the proof, we give a lemma.

**Lemma C.6.** Let $\mu \in \mathcal{M}_{a,b,L}^{(i)}$ with density $\eta$, and for $j \neq i$ let $\{s_{\hat{x}_{i,j}^\ell}\}_{\hat{x}_{i,j}^\ell \in \mathbb{T}^{N-1}}$ be a family of local diffeomorphisms of $\mathbb{T}$ with $|s'_{\hat{x}_{i,j}^\ell}| \geq \kappa_\# > 0$ and $\left| \frac{s'_{\hat{x}_{i,j}^\ell}}{|s_{\hat{x}_{i,j}^\ell}|} \right| \leq \mathcal{D}_\#$. Then there is $K_L > 0$ satisfying $\lim_{L \to 0} K_L = 1$ such that, denoting by $\{s_{\hat{x}_{i,j}^\ell}\}_{\ell \in \mathbb{N}}$ the inverse branches of $s_{\hat{x}_{i,j}}$

$$\zeta(\cdot, x_j; \hat{x}_{i,j}) := \sum_{\ell \in \mathbb{N}} \eta(\cdot; s_{\hat{x}_{i,j}^\ell}(x_j); \hat{x}_{i,j}) \in \mathcal{M}_{a,b,\mathcal{L}'}^{(i,j)}$$

with

$$\mathcal{L}' := \kappa_\#^{-1} + a \kappa_\#^{-1} + K_L \mathcal{D}_\#.$$

**Proof.** Let $i = 1$ and $j = 2$, and consider $\zeta(x_1, x_2; \hat{x}_{1,2})$ as defined above.

Notice that $\zeta(\cdot, x_2; \hat{x}_{1,2}) \in \mathcal{V}_a$ and

$$\frac{\zeta(x_1, x_2; \hat{x}_{1,2})}{\int_T \zeta(y_1, x_2; \hat{x}_{1,2}) \, dy_1} = \sum_{\ell \in \mathbb{N}} p_\ell(\hat{x}_{1}) \frac{\eta(x_1, s_{\hat{x}_{1,2}^\ell}(x_2); \hat{x}_{1,2})}{\int_T \eta(y_1, s_{\hat{x}_{1,2}^\ell}(x_2); \hat{x}_{1,2}) \, dy_1}$$

where

$$p_\ell(\hat{x}_{1}) := |s'_{\hat{x}_{1,2}^\ell}|^{-1} \left( s^{-1}_{\hat{x}_{1,2}^\ell}(x_2) \right) \frac{\int_T \eta(y_1, s_{\hat{x}_{1,2}^\ell}(x_2); \hat{x}_{1,2}) \, dy_1}{\int_T \eta(y_1, s_{\hat{x}_{1,2}^\ell}(x_2); \hat{x}_{1,2}) \, dy_1} > 0.$$
Consider \( \hat{x}_1, \hat{x}_1' \in \mathbb{T}^{N-1} \) differing only for their coordinates \( x_2, x_2' \in \mathbb{T} \). Then

\[
\theta_b \left( \sum_{\ell \in T} p_\ell(\hat{x}_1) \eta(\cdot, s_{\hat{x}_1,\ell}, (x_2); \hat{x}_1, 2), \sum_{\ell \in T} p_\ell(\hat{x}_1') \eta(\cdot, s_{\hat{x}_1',\ell}, (x_2'); \hat{x}_1, 2) \right) \leq \theta_b \left( \sum_{\ell \in T} p_\ell(\hat{x}_1) \eta(\cdot, s_{\hat{x}_1,\ell}, (x_2); \hat{x}_1, 2), \sum_{\ell \in T} p_\ell(\hat{x}_1) \eta(\cdot, s_{\hat{x}_1,\ell}, (x_2); \hat{x}_1, 2) \right) + \theta_b \left( \sum_{\ell \in T} p_\ell(\hat{x}_1) \eta(\cdot, s_{\hat{x}_1,\ell}, (x_2); \hat{x}_1, 2), \sum_{\ell \in T} p_\ell(\hat{x}_1') \eta(\cdot, s_{\hat{x}_1',\ell}, (x_2'); \hat{x}_1, 2) \right)
\]

(94)

Term in (94). By Proposition A.3, this term can be upper bounded by

\[
\max_{\ell \in T} \theta_b \left( \eta(\cdot, s_{\hat{x}_1,\ell}, (x_2); \hat{x}_1, 2), \eta(\cdot, s_{\hat{x}_1,\ell}, (x_2'); \hat{x}_1, 2) \right) \leq L|s_{\hat{x}_1,\ell}^{-1}(x_2) - s_{\hat{x}_1,\ell}^{-1}(x_2')| \leq L \left| \frac{1}{s_{\hat{x}_1,\ell}^{-1}} \right| |x_2 - x_2'|
\]

(95)

Term in (95). Let’s apply Proposition A.4. To this end, let’s start by noticing that

\[
p_\ell(\hat{x}_1) \leq \left[ \max_{\ell} \left| s'_{\hat{x}_1,\ell}^{-1}(s_{\hat{x}_1,\ell}(x_2)) \right| \cdot \max_{\ell, y} \left| \eta(y_1, s_{\hat{x}_1,\ell}^{-1}(x_2); \hat{x}_1, 2) \right| \right]^2
\]

and by the mean-value theorem

\[
\log \left| \frac{s'_{\hat{x}_1,\ell}(s_{\hat{x}_1,\ell}^{-1}(x_2'))}{s'_{\hat{x}_1,\ell}(s_{\hat{x}_1,\ell}^{-1}(x_2))} \right| \leq \left| \frac{\partial y \eta}{\eta} \right| \left| \frac{1}{s_{\hat{x}_1,\ell}^{-1}} \right| |x_2 - x_2'|
\]

and

\[
\log \frac{\eta(y_1, s_{\hat{x}_1,\ell}^{-1}(x_2); \hat{x}_1, 2)}{\eta(y_1, s_{\hat{x}_1,\ell}^{-1}(x_2'); \hat{x}_1, 2)} \leq \left| \frac{\partial y \eta}{\eta} \right| \left| \frac{1}{s_{\hat{x}_1,\ell}^{-1}} \right| |x_2 - x_2'|
\]

which imply

\[
\log \max \left\{ \frac{p_\ell(\hat{x}_1)}{p_\ell(\hat{x}_1)}, \frac{p_\ell(\hat{x}_1')}{p_\ell(\hat{x}_1)} : \ell, \ell' \in T \right\} \leq 4 \left[ \left| \frac{s''_{\hat{x}_1,\ell}}{(s'_{\hat{x}_1,\ell})^2} \right| \left| \frac{1}{s_{\hat{x}_1,\ell}^{-1}} \right| |x_2 - x_2'| + a \left| \frac{1}{s_{\hat{x}_1,\ell}^{-1}} \right| |x_2 - x_2'| \right]
\]

where we used that \( \left| \frac{\partial y \eta}{\eta} \right|_\infty < a \) by assumption.

Also,

\[
\theta_b \left( \eta(\cdot, s_{\hat{x}_1,\ell}^{-1}(x_2'); \hat{x}_1, 2), \eta(\cdot, s_{\hat{x}_1,\ell'}^{-1}(x_2'); \hat{x}_1, 2) \right) \leq L \left| s_{\hat{x}_1,\ell}^{-1}(x_2') - s_{\hat{x}_1,\ell'}^{-1}(x_2') \right| \leq L \left| s_{\hat{x}_1,\ell}^{-1}(x_2') - s_{\hat{x}_1,\ell'}^{-1}(x_2') \right| \leq L \frac{1}{2}
\]

where we used that \( \left| s_{\hat{x}_1,\ell}^{-1}(x_2') - s_{\hat{x}_1,\ell'}^{-1}(x_2') \right| \) is upper bounded by the diameter of \( \mathbb{T} \), i.e. \( \frac{1}{2} \). The conclusion of Proposition A.4 states that the term in (95) is upper
bounded by
\[
4(1 - e^{-L/2}) \left[ \frac{s''_{\hat{z}_1,2}}{(s'_{\hat{z}_1,2})^2} + a \frac{1}{s'_{\hat{z}_1,2}} \right] |x_2 - x'_2| \leq 2K_L L \left[ \frac{s''_{\hat{z}_1,2}}{(s'_{\hat{z}_1,2})^2} + a \frac{1}{s'_{\hat{z}_1,2}} \right] |x_2 - x'_2|
\]
with \( K_L \to 1 \) as \( L \to 0 \).

Putting the estimates of (94) and (95) together we get the claim. \( \square \)

**Proof of Proposition 4.12.** It is a consequence of Lemma 4.4 that if \( H : T^N \to T^N \) satisfies Assumption 4.1 and Assumption 4.2 for \( N \) sufficiently large also \( \hat{G}_i \) does with datum differing from that of of \( H \) only by \( O(N^{-1}) \). Therefore one can apply Proposition 4.1 to \( \hat{G}_i : T^{N-1} \to T^{N-1} \) and, for any \( j \neq i \), consider the foliation of \( T^{N-1} \) given by circles \( \{ \gamma_{\hat{j},\ell} \}_{\hat{j},\ell} \in T^{N-2} \) such that \( \gamma_{\hat{j},\ell} \subset \hat{G}_i^{-1}(T_{\hat{z}_j}) \) and \( \gamma_{\hat{j},\ell} \) is the graph of \( \psi_{\ell}(\cdot; \hat{z}_j) \) for some \( \psi_{\ell} : T \times T^{N-2} \to T^{N-1} \). Then, analogously to Definition 4.2, we can consider the change of coordinates \( \Psi_{i,j} : T \times T^{N-2} \to T^{N-1} \) such that, denoting \( \hat{G}_i = (G_j; \hat{G}_{i,j}) \), for \( (z_j; \hat{z}_j) \in T \times T^{N-2} \)

\[
\Psi_{i,j}(z_j; \hat{z}_j) = (z_j; \psi_{\ell}(z_j; \hat{G}_{i,j}(0; \hat{z}_j))),
\]
and let \( S : T \times T^{N-2} \to T \times T^{N-2} \) be the skew-product map defined as

\[
S_j(z_j; \hat{z}_j) = G_j(z_j; \Psi_{i,j}(z_j; \hat{z}_j)) \quad \hat{S}_j(z_j; \hat{z}_j) = \hat{G}_{i,j}(0; \hat{z}_j).
\]
It follows that \( \hat{G}_i = S \circ \Psi_{i,j}^{-1} \), and therefore

\[
(Id_T; \hat{G}_i)_*\eta = (Id_T; S)_*(Id_T; \Psi_{i,j})^{-1}_*\eta.
\]
Without loss of generality let’s put \( i = 1 \) and pick any \( j \neq 1 \).

**Step 1.** Notice that

\[
\eta_1(x_1; \hat{\alpha}_1) := (Id_T; \Psi_{1,j}^{-1})_*\eta(x_1; \hat{\alpha}_1) = \eta(x_1; \Psi_{1,j}(\hat{\alpha}_1))|D\Psi_{1,j}|(\hat{\alpha}_1).
\]
Calling \( \zeta_1(x_1; \hat{\alpha}_1) := \eta(x_1; \Psi_{1,j}(\hat{\alpha}_1)) \), and arguing as in the proof of Lemma 4.6 \( \eta_1 \in M^{(1)}_{a, b, \ell'_1} \) with

\[
\ell'_1 \leq 1 + \frac{K}{b - a} N^{-1}
\]
where we used that \( |\partial_k \Psi_{i,j}|_\infty = 1 + O(N^{-1}) \) and for \( \ell \neq k \) \( |\partial_k \Psi_{i,j}|_\infty = O(N^{-2}) \) as implied by Proposition 4.1 applied to \( \hat{G}_1 : T^{N-1} \to T^{N-1} \).

**Step 2.** Call

\[
\eta_2 := (Id_T; S)_*\eta_1 = (Id_T; \hat{S}_j)_*(Id_T; S_j; Id_{T^{N-2}})_*\eta_1.
\]
Let’s first study

\[
\zeta_2(x_1; x_j; \hat{\alpha}_{1,j}) := (Id_T; S_j; Id_{T^{N-2}})_*\eta_1(x_1; x_j; \hat{\alpha}_{1,j})
\]
\footnote{Here \( T_{\hat{z}_j} := \{ \hat{z}_j \} \times T \) is the leaf over \( \hat{z}_j \in T^{N-2} \) of the foliation of \( T^{N-1} \) along coordinate \( j \).}
Calling \( \{s_{\hat{x},j}^{-1,}\ell\}_{\ell \in \mathcal{L}'} \) the inverse branches\(^{23}\) of \( s_{\hat{x},j}(\cdot) := S_j(\cdot; \hat{x},j) \),

\[
\zeta_2(\cdot; x_j; \hat{x},j) = \sum_{\ell \in \mathcal{L}'} \eta_2(\cdot; s_{\hat{x},j,\ell}^{-1}(x_j); \hat{x},j)
\]

and by Lemma \(^{C.6} \) \( \zeta_2 \in \mathcal{M}^{(1,j)}_{a,b,c,L' L} \) with

\[
\mathcal{L}' = \mathcal{L} + \mathcal{K} \mathcal{L} + O(N^{-1})
\]

where in applying the lemma we used that \( |s_{\hat{x},j}^{-1}| \geq \kappa + O(N^{-1}) \) and \( \frac{s_{\hat{x},j,\ell}^{-1}}{s_{\hat{x},j,\ell}^{-1}} \leq \mathcal{D} + O(N^{-1}) \)

Letting \( \{\hat{S}_{j,\ell}^{-1}\}_{\ell \in \mathcal{L}'} \) be the inverse branches of \( \hat{S}_{j,\ell} : \mathbb{T}^{N-2} \to \mathbb{T}^{N-2} \),

\[
\eta_2(x_1;j, \hat{x},j) := (\text{Id}_T; \hat{G}_1) \zeta_2(x_1;j, \hat{x},j)
\]

and it’s easy to check that \( \eta_2 \in \mathcal{M}^{(1,j)}_{a,b,c,L' L} \) with the same parameters as \( \zeta_2 \).

Since the above is true for any \( j \neq 1 \), and

\[
\eta_2 = (\text{Id}_T; \hat{G}_1) \eta \in \mathcal{M}^{(1)}_{a,b,c,L' L}
\]

the proposition is proved. \( \square \)

C.7. Proof of Proposition 4.14

Proof of Proposition 4.14 The proof of this proposition is obtained by successive application of propositions 4.10, 4.11 and 4.12. For some \( C > 0 \), pick \( \mu \in \mathcal{M}_{a,b,c,N-1} \cap \mathcal{C}_a^2 \) and fix \( i \in [1, N] \).

Applying Proposition 4.10

\[
\Phi_{\mu}^{-1} \in \mathcal{M}^{(i)}_{a_1,b_1,L_1} \quad \hat{\Pi}, \Phi_{\mu}^{-1} \in \mathcal{M}_{a',b',L'}
\]

with

\[
\begin{align*}
 a_1 &= \kappa a_0 + a \alpha \\
b_1 &= \kappa b_0 + a \alpha \\
 L_1 &= [\mathcal{L} + O(N^{-1})]L + O(N^{-1})
\end{align*}
\]

and

\[
\begin{align*}
 a'_1 &= \kappa a_0 + a \alpha' \\
b'_1 &= \kappa b_0 + a \alpha' \\
 L'_1 &= [\hat{\mathcal{L}} + O(N^{-1})]L + O(N^{-1})
\end{align*}
\]

and also \( \Phi_{\mu}^{-1} \in \mathcal{C}_a^2 \) with \( \alpha' = \kappa^2 \alpha + \kappa \) independent of \( N \).

\(^{23}\)By Lemma 4.3 applied to \( G, G_j(\cdot; \hat{x},j) \) and \( S_j(\cdot; \hat{x},j) \) are at distance bounded by \( O(N^{-1}) \) in \( \mathcal{C}^2(T, T) \), therefore, for \( N \) sufficiently large, \( |S_j| > 0 \) and \( x_j \to S_j(x_j; \hat{x},j) \) is a local diffeomorphism.

\(^{26}\)Applying Lemma 4.4 twice, once to \( S \) and once to \( \hat{G}_1, s_{\hat{x},j} \) is \( O(N^{-1}) \) \( C^2 \)-close to \( G_j(\cdot; \hat{x},j) \) that is \( O(N^{-1}) \) \( C^2 \)-close to \( F_j(\cdot; x_1, \hat{x},j) \) for any \( x_1 \in T \).
Applying Proposition 4.11,

\[(G_i; \text{Id}_{\mathbb{T}^{N-1}})_* \Phi_{i_*}^{-1} \mu \in \mathcal{M}_{a_2, b_2, L_2},\]

with

\[a_2 := \kappa^{-1}(a_1 + D) = \kappa^{-1}(K a_0 + a_\Phi + D) < a_0\]
\[b_2 := \kappa^{-1}(b_1 + D) < b_0\]
\[L_2 := \Lambda L_1 + O(N^{-1})\]

As the action on the coordinate different from \(i\) is the identity, the marginal on this coordinates does not change and

\[\hat{\Pi}_i(G_i; \text{Id}_{\mathbb{T}^{N-1}})_* \Phi_{i_*}^{-1} \mu \in \mathcal{M}_{a'_1, b'_1, L'_1}.\]

Applying Proposition 4.12,

\[(\text{Id}_{\mathbb{T}}; \hat{G}_i)_* (G_i; \text{Id}_{\mathbb{T}^{N-1}})_* \Phi_{i_*}^{-1} \mu \in \mathcal{M}_{a_3, b_3, L_3},\]

with \(a_3 := a_2, b_3 := b_2\) and

\[L_3 := [(1 + a'_1)\kappa^{-1} + D K_{L_2} + K_\# N^{-1}] L_2\]
\[= [(1 + \hat{K} a_0 + a_\Phi)\kappa^{-1} + D K_{L_2} + O(N^{-1})] \Lambda L_1 + O(N^{-1})\]
\[= [(1 + \hat{K} a_0)\kappa^{-1} + K_{L_2} D] \Lambda L + L O(N^{-1}) + O(N^{-1})\]

since \(L = C N^{-1}\), for \(N \to \infty\) \(K_{L_2} \to 1\), and this together with condition (26) implies that, for every \(N\) sufficiently large

\[[(1 + \hat{K} a_0)\kappa^{-1} + K_{L_2} D] \Lambda L + O(N^{-1}) < 1.\]

Picking

\[C > \frac{O(1)}{1 - (1 + \hat{K} a_0)\kappa^{-1} - D}\]

ensures that for every \(N\) sufficiently large

\[L_3 < C N^{-1}.\]

Now we proceed to estimate the second derivatives. Consider \(\mu \in \mathcal{M}_{a_0, b_0, C N^{-1}} \cap \mathcal{C}_\alpha^2\) with density \(\eta\) and let \(i \in [1, N]\).

**Step 1.** Let’s start by estimating

\[\left| \frac{\partial^2 H_* \rho}{\partial x_i} \right| .\]

To this end, recall that \(H_* \rho = (\text{Id}_T; \hat{G}_i)_* (G_i; \text{Id}_{\mathbb{T}^{N-1}})_* \Phi_{i_*}^{-1} \rho\). It follows from point i) of Proposition 4.10 that

\[\rho_1 := \Phi_{i_*}^{-1} \rho \in \mathcal{C}_{\alpha_1}^2, \quad \alpha_1 := \mathcal{K}^2 \alpha + \mathcal{K}_\#.\]

Now call

\[\rho_2(x_i; \hat{x}_i) := (G_i; \text{Id}_{\mathbb{T}^{N-1}})_* \rho_1(x_i; \hat{x}_i) = \sum_{\ell} \rho_1(g_{\hat{x}_i, \ell}(x_i); \hat{x}_i) / \left| g'_{\hat{x}_i, \ell}(x_i) \right| g_{\hat{x}_i, \ell}^{-1}(x_i)\]
\[
\frac{\partial^2}{\partial \rho_1^2} \left( \frac{\rho_1 (g^{-1}_{x, i, \ell} (x); \hat{x}_i) \cdot (g_{x, i, \ell} (x))}{|g_{x, i, \ell}^1|} \right) = \frac{\partial}{\partial \rho_1} \left( \frac{\partial \rho_1 (g_{x, i, \ell} (x); \hat{x}_i) \cdot (g_{x, i, \ell} (x))}{|g_{x, i, \ell}^1|} - \rho_1 (g_{x, i, \ell} (x); \hat{x}_i) \cdot (g_{x, i, \ell}^2 (x)) \right) \\
\qquad = \frac{\partial^2 \rho_1}{|g_{x, i, \ell}^1|} - 2 \frac{\partial \rho_1}{g_{x, i, \ell}^1} \frac{g''_{x, i, \ell}}{|g_{x, i, \ell}^1|} - \frac{\partial \rho_1}{g_{x, i, \ell}^1} \frac{g''_{x, i, \ell}}{|g_{x, i, \ell}^1|^2} - \rho_1 \frac{g''_{x, i, \ell}^2}{|g_{x, i, \ell}^1|^2} + 3 \rho_1 \frac{(g''_{x, i, \ell})^2}{|g_{x, i, \ell}^1|^5}.
\]

From which
\[
\left| \frac{\rho_1 (g_{x, i, \ell} (x); \hat{x}_i)}{|g_{x, i, \ell}^1|} \right|^{-1} \left| \frac{\rho_1 (g_{x, i, \ell} (x); \hat{x}_i)}{|g_{x, i, \ell}^1|} \right| \leq \alpha_1 \kappa^{-2} + 3a_1 \kappa^{-3}K + K \kappa^{-3} + 3K^2 \kappa^{-4} + K_{\#}N^{-1}
\]

therefore
\[
\left| \frac{\partial^2 \rho_2}{\rho_2} \right| \leq \kappa^{-2} K^2 \alpha + O(1).
\]

Notice that \(O(1)\) is uniformly bounded with \(N\) and goes to zero as \(\kappa \to \infty\).

Call
\[
\rho_3(x; \hat{x}_i) := (\text{Id}; \hat{G}_i)_* \rho_2 (x; \hat{x}_i) = \sum_{\ell} \rho_2(x; \hat{G}_{x, i, \ell} (\hat{x}_i)) \frac{|D\hat{G}_i| (\hat{G}_{x, i, \ell} (\hat{x}_i))}{|D\hat{G}_i| (\hat{G}_{x, i, \ell} (\hat{x}_i))}.
\]

It is immediate from the above expression that
\[
\left| \frac{\partial^2 \rho_3}{\rho_3} \right| \leq \left| \frac{\partial^2 \rho_2}{\rho_2} \right| \leq \kappa^{-2} K^2 \alpha + O(1).
\]

**Step 2.** We proceed to estimate
\[
\left| \frac{\partial \rho (H_{x, \rho})}{\rho} \right|.
\]

\(i, j \in [1, N]\) and \(j \neq i\).

Arguing as in the proof of Proposition 4.12, we can find a global change of charts \(\Psi_{i,j} : \mathbb{T} \times \mathbb{T}^{N-2} \to \mathbb{T}^{N-1}\) and a skew-product map \(S : \mathbb{T} \times \mathbb{T}^{N-2} \to \mathbb{T} \times \mathbb{T}^{N-2}\) such that
\[
H_{x, \rho} = (\text{Id}_T; S_j)_* (\text{Id}_T; S_j)_* (G_i; \text{Id}_T; \Psi^{-1}_{i,j} \Phi^{-1}_{i,j} \rho).
\]

Letting \(\rho_1 := \Psi^{-1}_{i,j} \Phi^{-1}_{i,j} \rho\), applying point i) of Proposition 4.10 twice, one obtains that \(\rho_1 \in C^2_{\alpha_1}\) with
\[
\alpha_1 := \mathcal{K}^4 \alpha + \mathcal{K}_{\#}.
\]

Call
\[
\rho_2 := (\text{Id}_T; S_j)_* (G_i; \text{Id}_T; \text{Id}_T)_* \rho_1 (x; x_j; \hat{x}_{i,j}) = \sum_{\ell, \ell'} \rho_1 (g^{-1}_{x, i, \ell} (x); s^{-1}_{x, i, \ell} \circ (x_j); \hat{x}_{i,j}) \frac{|g_{x, i, \ell}^1| (g_{x, i, \ell}^{-1} (x)) \cdot |s_{x, i, \ell}^1| (s_{x, i, \ell}^{-1} (x_j))}{|g_{x, i, \ell}^{-1} (x)| (s_{x, i, \ell}^{-1} (x_j))}.
\]
Then for every $\ell$ and $\ell'$

$$
\partial_j \partial_i \left( \frac{\rho_1 (g_{\ell,\ell}(x_i); s_{\ell,i}, t(x_j); \hat{x}_{i,j})}{g_{\ell,i}((g_{-1,\ell}(x_i), s_{\ell,i}, s_{\ell,i}, t(x_j)))} \right) = \\
= \partial_j \left( \frac{\partial_i \rho_1}{|g_{\ell,i}^2| s_{\ell,i}^2} - \partial_i \rho_1 \frac{g_{\ell,i}''}{|g_{\ell,i}^3| s_{\ell,i}^3} \right)
$$

and the above terms can be estimated as

$$
\left| \frac{\rho_1 (g_{\ell,\ell}(x_i); s_{\ell,i}, t(x_j); \hat{x}_{i,j})}{|g_{\ell,i}((g_{-1,\ell}(x_i), s_{\ell,i}, s_{\ell,i}, t(x_j)))|} \right| \leq \alpha_1 \kappa^{-2} + a_1 K \kappa^{-2} + a_1 \kappa^{-1} K_N^{-1} + a_1 K \kappa^{-1} N^{-1} + a_1 K \kappa^{-3} + a_1 \kappa^{-3} + K^2 \kappa^{-3} - K_N^{-1}.
$$

yielding

$$
\left| \frac{\partial_j \partial_i \rho_2}{\rho_2} \right| \leq (\kappa^{-2} + K_N^{-1}) \kappa^4 \alpha + O(1)
$$

where $O(1)$ goes to zero as $\kappa \to \infty$. So this proved that $\rho_2 \in C_{\alpha_2}$ with $\alpha_2 := (\kappa^{-2} + K_N^{-1}) \kappa^4 \alpha + O(1)$. It is easy to check that

$$
\rho_3 := (\Id_T; \Id_T; S_j)_{*} \rho_2 \in C_{\alpha_2}.
$$

The above estimate implies that provided $\kappa^{-2} \kappa^4 < 1$, then there is $\alpha_0 \geq 0$ for which the statement holds.

\[\square\]

**References**

[Azu67] Kazuoki Azuma, *Weighted sums of certain dependent random variables*, Tohoku Mathematical Journal, Second Series 19 (1967), no. 3, 357–367.

[BKST18] Péter Bálint, Gerhard Keller, Fanni M Sélley, and Imre Péter Tóth, *Synchronization versus stability of the invariant distribution for a class of globally coupled maps*, Nonlinearity 31 (2018), no. 8, 3770.

[BKZ09] Jean-Baptiste Bardet, Gerhard Keller, and Roland Zweimüller, *Stochastically stable globally coupled maps with bistable thermodynamic limit*, Communications in Mathematical Physics 292 (2009), no. 1, 237–270.

[BLS22] Wael Bahsoun, Carlangelo Liverani, and Fanni M Sélley, *Globally coupled anosov diffeomorphisms: Statistical properties*, arXiv preprint arXiv:2208.02517 (2022).

[EP95] Sergey V Ershov and Alexey B Potapov, *On mean field fluctuations in globally coupled maps*, Physica D: Nonlinear Phenomena 86 (1995), no. 4, 523–558.
[Gal22] Stefano Galatolo, Self-consistent transfer operators: Invariant measures, convergence to equilibrium, linear response and control of the statistical properties, Communications in Mathematical Physics (2022), 1–58.

[Hoe94] Wassily Hoeffding, Probability inequalities for sums of bounded random variables, The collected works of Wassily Hoeffding, Springer, 1994, pp. 409–426.

[Jus95] Wolfram Just, Globally coupled maps: phase transitions and synchronization, Physica D: Nonlinear Phenomena 81 (1995), no. 4, 317–340.

[Kan90a] Kunihiko Kaneko, Clustering, coding, switching, hierarchical ordering, and control in a network of chaotic elements, Physica D: Nonlinear Phenomena 41 (1990), no. 2, 137–172.

[Kan90b] Kunihiko Kaneko, Globally coupled chaos violates the law of large numbers but not the central-limit theorem, Physical review letters 65 (1990), no. 12, 1391.

[Kel00] Gerhard Keller, An ergodic theoretic approach to mean field coupled maps, Fractal geometry and stochastics II, Springer, 2000, pp. 183–208.

[Led01] Michel Ledoux, The concentration of measure phenomenon, no. 89, American Mathematical Soc., 2001.

[Liv95] Carlangelo Liverani, Decay of correlations, Annals of Mathematics (1995), 239–301.

[LN12] Bas Lemmens and Roger Nussbaum, Nonlinear perron-frobenius theory, vol. 189, Cambridge University Press, 2012.

[LN13] Bas Lemmens and Roger Nussbaum, Birkhoff’s version of hilbert’s metric and its applications in analysis, arXiv preprint arXiv:1304.7921 (2013).

[M+89] Colin McDiarmid et al., On the method of bounded differences, Surveys in combinatorics 141 (1989), no. 1, 148–188.

[McD98] Colin McDiarmid, Concentration, Probabilistic methods for algorithmic discrete mathematics, Springer, 1998, pp. 195–248.

[NTV18] Matthew Nicol, Andrew Török, and Sandro Vaienti, Central limit theorems for sequential and random intermittent dynamical systems, Ergodic Theory and Dynamical Systems 38 (2018), no. 3, 1127–1153.

[PK94] Arkady S Pikovsky and Jürgen Kurths, Do globally coupled maps really violate the law of large numbers?, Physical review letters 72 (1994), no. 11, 1644.

[SB16] Fanni Sélley and Péter Bálint, Mean-field coupling of identical expanding circle maps, Journal of Statistical Physics 164 (2016), no. 4, 858–889.

[Sim12] David Simmons, Conditional measures and conditional expectation; rohlin’s disintegration theorem, Discrete & Continuous Dynamical Systems 32 (2012), no. 7, 2565.

[ST21] Fanni M Sélley and Matteo Tanzi, Linear response for a family of self-consistent transfer operators, Communications in Mathematical Physics 382 (2021), no. 3, 1601–1624.

[ST22] Fanni M Sélley and Matteo Tanzi, Synchronization for networks of globally coupled maps in the thermodynamic limit, Journal of Statistical Physics 189 (2022), no. 1, 1–26.

[Tal95] Michel Talagrand, Concentration of measure and isoperimetric inequalities in product spaces, Publications Mathématiques de l’Institut des Hautes Études Scientifiques 81 (1995), no. 1, 73–205.
[Via97] M Viana, *Stochastic dynamics of deterministic systems, lect*, Notes XXI Braz. Math Colloq., IMPA, 1997.

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