ERROR ESTIMATES FOR A NONLINEAR LOCAL PROJECTION STABILIZATION OF TRANSIENT CONVECTION–DIFFUSION–REACTION EQUATIONS

Petr Knobloch

Department of Numerical Mathematics, Faculty of Mathematics and Physics
Charles University, Sokolovská 83, 18675 Praha 8, Czech Republic

Abstract. A recently proposed local projection stabilization (LPS) finite element method containing a nonlinear crosswind diffusion term is analyzed for a transient convection-diffusion-reaction equation using a one-step θ-scheme as temporal discretization. Both the fully nonlinear method and its semi-implicit variant are considered. Solvability of the discrete problem is established and a priori error estimates in the LPS norm are proved. Uniqueness of the discrete solution is proved for the semi-implicit approach or for sufficiently small time steps.

1. Introduction. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded polygonal (polyhedral) domain with a Lipschitz-continuous boundary $\partial \Omega$ and let $[0,T]$ be a finite time interval. Let us consider the transient convection-diffusion-reaction equation

$$
\begin{align*}
\frac{\partial u}{\partial t} - \varepsilon \Delta u + b \cdot \nabla u + cu &= f & \text{in } (0,T) \times \Omega, \\
\hat{u} &= u_b & \text{in } [0,T] \times \partial \Omega, \\
u(0,\cdot) &= u_0 & \text{in } \Omega.
\end{align*}
$$

(1)

It is assumed that $\varepsilon$ is a positive constant and $b \in L^\infty(0,T;W^{1,\infty}(\Omega)^d)$, $c \in L^\infty(0,T;L^\infty(\Omega))$, $f \in L^2(0,T;L^2(\Omega))$, $u_b \in L^2(0,T;H^{1/2}(\partial \Omega))$, and $u_0 \in H^1(\Omega)$ are given functions satisfying

$$\sigma := c - \frac{1}{2} \nabla \cdot b \geq \sigma_0 > 0 \quad \text{in } [0,T] \times \Omega,$$

where $\sigma_0$ is a constant.

The numerical solution of (1) is still a challenge if convection dominates diffusion. In the framework of the finite element method, the common approach is to apply a stabilized method, see [7] for a review. Linear stabilized methods typically provide approximate solutions that possess spurious oscillations in layer regions. These oscillations can be suppressed without smearing the layers significantly by adding an additional artificial diffusion term depending on the approximate solution in a nonlinear way, see [3] for a review of various approaches of this type that we call spurious oscillations at layers diminishing (SOLD) methods.

Here we concentrate on local projection stabilizations (LPS) [2, 4, 5]. In comparison with residual-based methods, the linear LPS has several advantages. In

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particular, it does not contain second order derivatives, which may be costly to implement, and if applied to systems of PDEs, it does not lead to additional couplings between various unknowns. To suppress oscillations in layer regions, we use a nonlinear stabilization that was recently introduced in [1] and is inspired by both the linear LPS and the above-mentioned nonlinear SOLD methods. Since it is assumed that the linear LPS adds enough artificial diffusion in the streamline direction, only crosswind diffusion is introduced through the nonlinear term. To preserve the above-mentioned advantages of the LPS, the residual usually appearing in SOLD terms is replaced by a fluctuation of the crosswind derivative of the approximate solution. In [1], existence of the discrete solution and an error estimate were proved for the resulting nonlinear method applied to a steady convection-diffusion-reaction equation. In the present work, we extend the analysis of [1] to the transient case by using a one-step \( \theta \)-scheme as temporal discretization. We consider both the fully nonlinear problem and its linearized (semi-implicit) variant that computes the stabilization parameter with the solution from the previous discrete time and hence significantly reduces the computational cost. For both approaches, we prove the existence of a solution, without any restriction on the time step and the multiplicative factor in the nonlinear term. The uniqueness is proved for the semi-implicit variant and, in the case of sufficiently small time steps, also for the fully nonlinear method. Furthermore, we establish a priori error estimates with respect to the standard LPS norm that hold simultaneously for both approaches. Numerical results for the steady case showing that the crosswind diffusion term leads to a reduction of spurious oscillations compared to the standard linear LPS method were presented in [1].

The plan of the paper is as follows. Section 2 will summarize the main abstract hypothesis imposed on the different partitions of \( \Omega \) and the finite element spaces considered. Section 3 presents the method and analyzes its well-posedness. A priori error estimates are derived in Section 4.

2. Assumptions on approximation in space. Given \( h > 0 \), let \( W_h \subset W^{1,\infty}(\Omega) \) be a finite-dimensional space approximating the space \( H^1(\Omega) \) and set \( V_h = W_h \cap H_0^1(\Omega) \). Next, let \( \mathcal{M} \) be a set consisting of a finite number of open subsets \( M \) of \( \Omega \) such that \( \Omega = \bigcup_{M \in \mathcal{M}} M \backslash \). It will be supposed that, for any \( M \in \mathcal{M} \),

\[
\begin{align*}
&\text{card}\{M' \in \mathcal{M} : M \cap M' \neq \emptyset\} \leq C, \\
&h_M := \text{diam}(M) \leq C h, \\
&h_M \leq C h_M', \quad \forall M' \in \mathcal{M}, M \cap M' \neq \emptyset.
\end{align*}
\]

The space \( W_h \) is assumed to satisfy the inverse inequality \( |v_h|_{1,M} \leq C h_M^{-1} \|v_h\|_{0,M} \) for any \( v_h \in W_h, M \in \mathcal{M} \). For any \( M \in \mathcal{M} \), a finite-dimensional space \( D_M \subset L^\infty(M) \) is introduced. It is assumed that there exists a positive constant \( \beta_{LP} \) independent of \( h \) such that

\[
\sup_{v \in V_M} \frac{(v, q)_M}{\|v\|_{0,M}} \geq \beta_{LP} \|q\|_{0,M} \quad \forall q \in D_M, M \in \mathcal{M},
\]

where \( V_M = \{ v_h \in V_h : v_h = 0 \text{ in } \Omega \setminus M \} \) and \((\cdot, \cdot)_M\) is the inner product in \( L^2(M) \). Furthermore, for any \( M \in \mathcal{M} \), a finite-dimensional space \( G_M \subset L^\infty(M) \) containing the space \( D_M \) is introduced such that \((\partial v_h / \partial x_i)|_M \in G_M \) for any \( v_h \in W_h, i = 1, \ldots, d \), and it is assumed that

\[
\|q\|_{0,\infty,M} \leq C h_M^{-\frac{d}{2}} \|q\|_{0,M} \quad \forall q \in G_M, M \in \mathcal{M}.
\]
To characterize the approximation properties of the spaces $W_h$ and $D_M$, it is assumed that there exist interpolation operators $i_h \in \mathcal{L}(H^2(\Omega), W_h) \cap \mathcal{L}(H^1(\Omega) \cap H_0^1(\Omega), V_h)$ and $j_M \in \mathcal{L}(H^1(\Omega), D_M)$, $M \in \mathcal{M}$, such that, for some constants $l \in \mathbb{N}$ and $C > 0$ and for any set $M \in \mathcal{M}$, it holds

$$
|v - i_h v|_{1,M} + h_M^{-1} ||v - i_h v||_{0,M} \leq C h_M^k |v|_{k+1,M} \quad \forall v \in H^{k+1}(M), \quad k = 1, \ldots, l,
$$

(7)

$$
\|q - j_M q\|_{0,M} \leq C h_M^k \|q\|_{k,M} \quad \forall q \in H^k(M), \quad k = 1, \ldots, l.
$$

(8)

Examples of spaces $W_h$ and $D_M$ and partitions $\mathcal{M}$ satisfying the hypotheses made in this section can be found in, e.g., [4, 6].

3. A local projection discretization of the transient problem. A weak form of problem (1) reads as follows: Find $u \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$ such that $u = u_b$ on $[0,T] \times \partial \Omega$, $u(0,\cdot) = u_0$ and

$$
(u_t, v) + a(u,v) = (f, v) \quad \forall v \in H_0^1(\Omega), \quad \text{for almost every } t \in [0,T],
$$

(9)

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$ and the bilinear form $a$ is given by

$$
a(u,v) = \varepsilon (\nabla u, \nabla v) + (b \cdot \nabla u, v) + (c u, v).
$$

To avoid technicalities in the analysis, it is assumed that the boundary condition does not depend on time, $u_b(t, \cdot) = u_b$. The initial condition $u_0$ is assumed to satisfy $u_0|_{\partial \Omega} = u_b$.

To perform the discretization of the time derivative, the time interval $[0,T]$ is divided into $N_T$ equidistant strips of length $\delta t = T/N_T$. The constant time step is used only for simplicity of presentation; for variable time steps the same techniques can be applied leading to essentially the same results. The nodes are denoted by $t^n = n \delta t$ for $n = 0, 1, \ldots, N_T$ and the abbreviations $u^n := u(t^n, \cdot)$, $f^n := f(t^n, \cdot)$, etc. are used. The superscript $n + \theta$ denotes for all functions which are defined in $[0,T]$ the values at time $t^{n+\theta} := \theta t^{n+1} + (1 - \theta) t^n$ with any $n \in \{0, \ldots, N_T - 1\}$ and $\theta \in [0,1]$, e.g. $b^{n+\theta} = b(t^{n+\theta}, \cdot)$. For functions, which are defined only at the discrete times $t^n$ and $t^{n+1}$, it denotes the linear interpolation, e.g. $u_h^{n+\theta} = \theta u_h^{n+1} + (1 - \theta) u_h^n$. To emphasize that the bilinear form $a$ depends on time, the notation $a^{n+\theta}$ will be used instead of $a$, i.e.,

$$
a^{n+\theta}(u,v) = \varepsilon (\nabla u, \nabla v) + (b^{n+\theta} \cdot \nabla u, v) + (c^{n+\theta} u, v).
$$

Now we introduce notation that will be needed to define local projection stabilization terms. First, for any $M \in \mathcal{M}$, a continuous linear projection operator $\pi_M$ is introduced which maps the space $L^2(M)$ onto the space $D_M$. It is assumed that $\|\pi_M\|_{\mathcal{L}(L^2(M), L^2(M))} \leq C$ for any $M \in \mathcal{M}$. Using this operator, the fluctuation operator $\kappa_M := id - \pi_M$ is defined, where $id$ is the identity operator on $L^2(M)$. Then, clearly

$$
\|\kappa_M\|_{\mathcal{L}(L^2(M), L^2(M))} \leq C \quad \forall M \in \mathcal{M}.
$$

(10)

An application of $\kappa_M$ to a vector-valued function means that $\kappa_M$ is applied componentwise.

For any $M \in \mathcal{M}$, $n \in \{0, \ldots, N_T - 1\}$, and $\theta \in [0,1]$, a constant $b_M^{n+\theta} \in \mathbb{R}^d$ is chosen such that

$$
|b_M^{n+\theta}| \leq \|b^{n+\theta}\|_{0,\infty,M}, \quad \|b^{n+\theta} - b_M^{n+\theta}\|_{0,\infty,M} \leq Ch_M|b^{n+\theta}|_{1,\infty,M}.
$$

(11)

A typical choice for $b_M^{n+\theta}$ is the value of $b^{n+\theta}$ at one point of $M$, or the integral mean value of $b^{n+\theta}$ over $M$. In addition, a function $u_{bh} \in W_h$ is introduced such
Lemma 1. For any $u_h, v_h \in W_h$ and $z_h := u_h - v_h$, one has

$$ |d_{h}^{n+\theta}(u_h, z_h) - d_{h}^{n+\theta}(v_h, z_h)| \leq C \sum_{M \in \mathcal{M}_h} h_M^{-1} \|b_{n+\theta}\|_{0,\infty,M} \|z_h\|_{0,M} $$

(13)

with any $n \in \{0, \ldots, N_T - 1\}$ and $\theta \in [0, 1]$.

**Proof.** For any $M \in \mathcal{M}_h$ set

$$ d_{h}^{n+\theta}(w; u, v) = (\tau_{M,n+\theta}(w) \kappa_{M}(P_{n+\theta}^{M}\nabla u), \kappa_{M}(P_{n+\theta}^{M}\nabla v))_M , $$

then

$$ d_{h}^{n+\theta}(w; u, v) = \sum_{M \in \mathcal{M}_h} d_{h}^{n+\theta}(w; u, v) . $$

Consider any $M \in \mathcal{M}_h$. Using (11), (10), and $\|P_{n+\theta}^{M}\|_2 = 1$, one obtains

$$ \|\tau_{M,n+\theta}(w)\|_{0,1,M} \leq C h_{M}^{1+d} \|b_{n+\theta}\|_{0,\infty,M} \quad \forall w \in H^1(M) . $$
Given \( u, v \in H^1(M) \) with \( |u|_{1,M} \neq 0, |v|_{1,M} \neq 0 \), one similarly derives

\[
\| \tau_M^{\text{sold},n+\theta}(u) - \tau_M^{\text{sold},n+\theta}(v) \|_{0,1,M} \leq \beta h_M^{1+d} \| b_M^{n+\theta} \|_{0,1,M} \left( \frac{1}{\| u \|_{1,M}^2} - \frac{1}{\| v \|_{1,M}^2} \right) + \beta h_M^{1+d} \| b_M^{n+\theta} \|_{0,1,M} \left( \frac{\| P_M^{n+\theta} \nabla u \|_{0,M}^2}{\| v \|_{1,M}^2} - \frac{\| P_M^{n+\theta} \nabla v \|_{0,M}^2}{\| v \|_{1,M}^2} \right) \leq C h_M^{1+d} \| b_M^{n+\theta} \|_{0,\infty,M} \frac{|u - v|_{1,M}}{|u|_{1,M} + |v|_{1,M}}.
\]

Without loss of generality, one may assume that \( |u|_{1,M} \leq |v|_{1,M} \), which gives

\[
\| \tau_M^{\text{sold},n+\theta}(u) - \tau_M^{\text{sold},n+\theta}(v) \|_{0,1,M} \leq C h_M^{1+d} \| b_M^{n+\theta} \|_{0,\infty,M} \frac{|u - v|_{1,M}}{|u|_{1,M} + |v|_{1,M}} \quad \forall u, v \in H^1(M).
\]

Furthermore, applying (6), (10), \( \| P_M^{n+\theta} \|_2 = 1 \), and the inverse inequality, one derives

\[
\| P_M^{n+\theta} \nabla v_h \|_{0,\infty,M} \leq C h_M^{-d/2} |v_h|_{1,M} \leq C h_M^{-1-d/2} \| v_h \|_{0,\infty,M} \quad \forall v_h \in W_h.
\]

Combining the above estimates, one arrives at the required estimate:

\[
\| P_M^{n+\theta} \nabla v_h \|_{0,\infty,M} \leq C h_M^{-d/2} |v_h|_{1,M} \leq C h_M^{-1-d/2} \| v_h \|_{0,\infty,M} \quad \forall v_h \in W_h, z_h = u_h - v_h.
\]

In the analysis, the error will be measured using the following mesh-dependent norm

\[
\| u \|_{\text{LPS},n+\theta} := \left\{ \varepsilon \| u \|_{1,\Omega}^2 + \| \sigma^{n+\theta} \|_0^{1/2} \| u \|_{0,\Omega}^2 + s_h^{n+\theta}(u,v) \right\}^{1/2}.
\]

Note that integrating by parts gives

\[
a^{n+\theta}(u,v) + s_h^{n+\theta}(u,v) = \| u \|_{\text{LPS},n+\theta}^2 \quad \forall u \in H_{0}^1(\Omega).
\]

To simplify the notation, we will write \( \| u \|_{\text{LPS}} \) instead of \( \| u \|_{\text{LPS},n+\theta} \) in the following. The time instant at which the functions \( b \) and \( \sigma \) in the definition of the norm \( \| \cdot \|_{\text{LPS}} \) are evaluated will be implicitly determined from the context or by the argument of the norm. Thus, if we write, e.g., \( \| u_h^{n+\theta} \|_{\text{LPS}} \), the norm \( \| \cdot \|_{\text{LPS}} \) is defined using \( b^{n+\theta} \) and \( \sigma^{n+\theta} \).

For \( \theta = 1/2 \), the discrete problem (12) corresponds to the Crank–Nicolson scheme and for \( \theta = 1 \), the implicit Euler scheme is obtained. For \( \omega = 1 \), a nonlinear problem has to be solved in each time step whereas, for \( \omega = 0 \), only one linear system needs to be solved per time step. We shall see that error estimates are not affected by the choice of \( \omega \) and hence the linearized scheme has to be preferred from the point of view of computational complexity. Moreover, the linearized problem is uniquely solvable for any non-negative integrable stabilization parameter \( \tau_M^{\text{sold},n+\theta} \), whereas the uniqueness for \( \omega = 1 \) is an open problem in general. Our results on the solvability and uniqueness of the approximate solution are summarized in the following theorem.
Theorem 2. Consider any $\theta \in (0, 1]$ and $\omega \in \{0, 1\}$. Let $n \in \{0, 1, \ldots, N_T - 1\}$ and $u^n_h \in W_h$ with $u^n_h|_{\partial \Omega} = \tilde{u}_h$ be given. Then the problem (12) possesses a solution $u^{n+1}_h$. If $\omega = 0$, then this solution is unique. Furthermore, if $\omega = 1$, then there is a constant $K > 0$ such that the solution of the scheme (12) is unique if $\delta t \| b^{n+\theta} \|_{0, \infty, M} \leq K h M$ for any $M \in \mathcal{M}$.

Proof. The discretization of the temporal derivative can be written in the form
\[
(\frac{u_h^{n+1} - u_h^n}{\delta t}, v_h) = \frac{1}{\theta} \left( \frac{u_h^{n+\theta} - u_h^n}{\delta t}, v_h \right).
\]
The bilinear form
\[
\tilde{a}^{n, \theta}(u_h, v_h) := \frac{1}{\theta \delta t} (u_h, v_h) + a^{n+\theta}(u_h, v_h) + s_h^{n+\theta}(u_h, v_h) + d^{n+\theta}(u_h^n; u_h, v_h)
\]
is clearly elliptic on $V_h$ and hence the discrete problem is uniquely solvable for $\omega = 0$. If $\omega = 1$, then the solvability immediately follows from the fact that, in each time step, the nonlinear problem is of the same type as the steady-case discretization investigated in [1]. To prove the uniqueness result, let us assume that there are two solutions $u^{n+\theta}$ and $\tilde{u}^{n+\theta}$ of (12). Then, setting $z_h = u^{n+\theta} - \tilde{u}^{n+\theta}$, one has
\[
\frac{1}{\theta \delta t} \| z_h \|^2_{0, \Omega} + \| z_h \|^2_{LPS} + d^{n+\theta}_h(u_h^{n+\theta}; u_h^{n+\theta}, z_h) - d^{n+\theta}_h(\tilde{u}_h^{n+\theta}; \tilde{u}_h^{n+\theta}, z_h) = 0.
\]
Thus, it follows from (13) and (2) that $z_h = 0$ if $K$ is sufficiently small. 

4. Error estimates. In this section, error estimates are derived for the solution of the discrete problem (12) with $\theta \in [1/2, 1]$.

It was shown in [5] and [1] that, due to the inf-sup condition (5), the interpolation operator $i_h$ can be modified in such a way that one obtains an operator $r_h \in \mathcal{L}(H^2(\Omega), W_h) \cap \mathcal{L}(H^2(\Omega) \cap H^1_0(\Omega), V_h)$ satisfying the usual estimate
\[
\| v - r_h v \|_{1, \Omega} + h^{-1} \| v - r_h v \|_{0, \Omega} \leq C h^k \| v \|_{k+1, \Omega} \quad \forall v \in H^{k+1}(\Omega), \; k = 1, \ldots, l,
\]
and, in addition,
\[
|\langle v - r_h v, w \rangle| \leq C \sum_{M \in \mathcal{M}_h} \| v - i_h v \|_{0, M} \| k_M w \|_{0, M} \quad \forall v \in H^2(\Omega), \; w \in L^2(\Omega).
\]
Moreover, $i_h - r_h \in \mathcal{L}(H^2(\Omega), V_h)$. This makes it possible to prove the following result.

Lemma 3. Let $u \in H^{k+1}(\Omega)$ for some $k \in \{1, \ldots, l\}$, and let $\eta := u - r_h u$. Then, for any $\eta \in V_h$ with $\{0\}$, the following estimate holds
\[
\| \eta \|_{LPS, n+\theta} + \frac{a^{n+\theta}(\eta, v_h) + s_h^{n+\theta}(\eta, v_h) - s_h^{n+\theta}(u, v_h)}{\| v_h \|_{LPS, n+\theta}} \leq C \left( \bar{\varepsilon} + h \| b^{n+\theta} \|_{0, \infty, \Omega} + h^2 \| \sigma^{n+\theta} \|_{0, \infty, \Omega} + h^2 \| b^{n+\theta} \|_{1, \infty, \Omega} \sigma_0^{-1} \right)^{1/2} h^k \| u \|_{k+1, \Omega}
\]
with any $n \in \{0, \ldots, N_T - 1\}$ and $\theta \in [0, 1]$.

Proof. See [5].

Furthermore, the following bound on the nonlinear form $d_h$ will be useful.
Lemma 4. For any \( w_h \in W_h \) and \( u, v \in H^{k+1}(\Omega) \) with \( k \in \{1, \ldots, l\} \), it holds
\[
d_h^{n+\theta}(w_h; r_h u, r_h v) \leq C \|b^{n+\theta}\|_{0, \infty, \Omega} H^{2k+1} \|u\|_{k+1, \Omega} |v|_{k+1, \Omega}
\]
with any \( n \in \{0, \ldots, N_T - 1\} \) and \( \theta \in [0, 1] \).

Proof. The proof is analogous as for Lemma 4 in [1].

To simplify the presentation of our results, we introduce the quantities
\[
Q^N = h \left( |u_0|_{k+1, \Omega} + |u^N|_{k+1, \Omega} + \sigma^{-1/2}_0 \|u_t\|_{L^2(0, T; H^{k+1}(\Omega))} \right) + \left( \delta t \sum_{n=0}^{N-1} \left( \varepsilon + h \|b^{n+\theta}\|_{0, \infty, \Omega} + h^2 \|\sigma^{n+\theta}\|_{0, \infty, \Omega} + h^2 \sigma_0^{-1} \|b^{n+\theta}\|_{L^2(\Omega)} \right) \right)^{1/2},
\]
\[
X^N = \max_{n=0, \ldots, N-1} \left( \varepsilon + h \|b^{n+\theta}\|_{0, \infty, \Omega} + \|\sigma^{n+\theta}\|_{0, \infty, \Omega} + \sigma_0^{-1} \|b^{n+\theta}\|_{L^2(\Omega)} \right)^{1/2},
\]
where \( N = 1, 2, \ldots, N_T \).

Theorem 5. Let \( \theta \in [1/2, 1] \) and \( \omega \in (0, 1) \) be given. Let the weak solution of (1) satisfy \( u, u_t \in L^2(0, T; H^{k+1}(\Omega)) \) for some \( k \in \{1, \ldots, l\} \) and assume \( u_{tt} \in L^2(0, T; L^2(\Omega)) \). Let \( \bar{u}_0 \in H^2(\Omega) \) be an extension of \( u_0 \) and let \( \bar{u}_{bh} = \bar{v}_0 \bar{u}_0 \). Assume \( u_0 \in H^{k+1}(\Omega) \) and let \( u_0^0 = \bar{v}_0 u_0 \). Let \( \{u_h^N\}_{n=0}^{N_T} \) be the solution of the local projection discretization (12). Then the following error estimate holds for \( N = 1, 2, \ldots, N_T \)
\[
\|u^N - u_h^N\|_{0, \infty, \Omega} + \left( \delta t \sum_{n=0}^{N-1} \|u^{n+\theta} - u_{h}^{n+\theta}\|_{L^2(\Omega)} \right)^{1/2} \leq C h^k Q^N
\]
\[
+ C \delta t X^N \|u_t\|_{L^2(0, T; H^{1}(\Omega))} + C \delta t \sigma_0^{-1/2} \|u_{tt}\|_{L^2(0, T; L^2(\Omega))}.
\]
Moreover, if \( \theta = 1/2 \), \( u_{tt} \in L^2(0, T; H^1(\Omega)) \), and \( u_{ttt} \in L^2(0, T; L^2(\Omega)) \), then, for \( N = 1, 2, \ldots, N_T \),
\[
\|u^N - u_h^N\|_{0, \infty, \Omega} + \left( \delta t \sum_{n=0}^{N-1} \|u^{n+\theta} - u_{h}^{n+\theta}\|_{L^2(\Omega)} \right)^{1/2} \leq C h^k Q^N
\]
\[
+ C (\delta t)^2 X^N \|u_t\|_{L^2(0, T; H^{1}(\Omega))} + C (\delta t)^2 \sigma_0^{-1/2} \|u_{tt}\|_{L^2(0, T; L^2(\Omega))}.
\]

Proof. Let us denote the error by \( e^{n+\theta} := u^{n+\theta} - u_{h}^{n+\theta} \). As usual, the error will be split into an interpolation error and a remainder which belongs to the finite element space. The decomposition of the error \( e^{n+\theta} \) has the form
\[
e^{n+\theta} = \eta^{n+\theta} - \tilde{e}_{h}^{n+\theta} \quad \text{with} \quad \eta^{n+\theta} := u^{n+\theta} - \bar{r}_h^{n+\theta}, \quad e_{h}^{n+\theta} := u_{h}^{n+\theta} - \tilde{r}_h^{n+\theta} \in V_h,
\]
where
\[
\tilde{r}_h^{n+\theta} = \theta r_h u^{n+1} + (1 - \theta) r_h u^n.
\]
Using this decomposition, one obtains with the triangle inequality

\[ \|e^N\|_{0,\Omega}^2 + \delta t \sum_{n=0}^{N-1} \|e^{n+\theta}\|^2_{LPS} \leq 2 \left[ \|\eta^N\|_{0,\Omega}^2 + \delta t \sum_{n=0}^{N-1} \|\eta^{n+\theta}\|^2_{LPS} \right] + 2 \left[ \|e^N_h\|_{0,\Omega}^2 + \delta t \sum_{n=0}^{N-1} \|e^{n+\theta}_h\|^2_{LPS} \right]. \] (17)

First let us estimate the interpolation errors. In view of (14), (10), (11), and the geometrical hypotheses (3) and (2), one has

\[ \|\eta^N\|_{0,\Omega} = |u^N - r_h u^N|_{0,\Omega} \leq C h^{k+1} |u^N|_{k+1,\Omega}, \]

\[ \|\eta^{n+\theta}\|_{LPS} \leq \left( \varepsilon + C h \|b^{n+\theta}\|_{0,\infty,\Omega} \right)^{1/2} \|\eta^{n+\theta}\|_{1,\Omega} + \|\kappa^{n+\theta}\|_{0,\infty,\Omega} \|\eta^{n+\theta}\|_{0,\Omega}. \]

The starting point for estimating \(\|\eta^{n+\theta}\|_{0,\Omega}\) and \(\|\eta^{n+\theta}\|_{1,\Omega}\) is the identity

\[ \eta^{n+\theta} = u^{n+\theta} - \theta u^{n+1} - (1 - \theta) u^n + \theta (u^{n+1} - r_h u^{n+1}) + (1 - \theta) (u^n - r_h u^n). \] (18)

One has

\[ u^{n+\theta} - \theta u^{n+1} - (1 - \theta) u^n = (1 - \theta) \int_{t^n}^{t^{n+\theta}} u_t(t) \, dt - \theta \int_{t^n+\theta}^{t^{n+1}} u_t(t) \, dt, \] (19)

which, in view of (14), leads to

\[ \|\eta^{n+\theta}\|_{0,\Omega} \leq C h^{k+1} \left( |u^n|_{k+1,\Omega} + |u^{n+1}|_{k+1,\Omega} \right) + \sqrt{\delta t} \|u_t\|_{L^2(t^n,t^{n+1};L^2(\Omega))}, \]

\[ |\eta^{n+\theta}|_{1,\Omega} \leq C h^k \left( |u^n|_{k+1,\Omega} + |u^{n+1}|_{k+1,\Omega} \right) + \sqrt{\delta t} \|u_t\|_{L^2(t^n,t^{n+1};H^1(\Omega))}. \]

Applying successively integration by parts gives

\[ u^n = u^{n+\theta} - \theta \delta t u^{n+\theta}_t + \int_{t^n}^{t^{n+\theta}} u_t(t) (t^n - t) \, dt, \] (20)

\[ u^{n+1} = u^{n+\theta} + (1 - \theta) \delta t u^{n+\theta}_t + \int_{t^n+\theta}^{t^{n+1}} u_t(t) (t^{n+1} - t) \, dt. \] (21)

This may be used to derive improved interpolation estimates with respect to the time step provided that \(u_t \in L^2(0, T; H^1(\Omega))\). Indeed,

\[ u^{n+\theta} - \theta u^{n+1} - (1 - \theta) u^n = -(1 - \theta) \int_{t^n}^{t^{n+\theta}} u_t(t) (t - t^n) \, dt - \theta \int_{t^n+\theta}^{t^{n+1}} u_t(t) (t^{n+1} - t) \, dt, \] (22)

which leads to

\[ \|\eta^{n+\theta}\|_{0,\Omega} \leq C h^{k+1} \left( |u^n|_{k+1,\Omega} + |u^{n+1}|_{k+1,\Omega} \right) + (\delta t)^{3/2} \|u_t\|_{L^2(t^n,t^{n+1};L^2(\Omega))}, \]

\[ |\eta^{n+\theta}|_{1,\Omega} \leq C h^k \left( |u^n|_{k+1,\Omega} + |u^{n+1}|_{k+1,\Omega} \right) + (\delta t)^{3/2} \|u_t\|_{L^2(t^n,t^{n+1};H^1(\Omega))}. \]

Now let us estimate the norms of the discrete part of the error on the right-hand side of (17). To derive an equation for this part of the error, the weak formulation (9) at \(t = t^{n+\theta}\) is subtracted from (12) with \(v = v_h = e^{n+\theta}_h\). Then, using the fact that \(u^{n+\theta}_t = e^{n+\theta}_h + \tilde{e}^{n+\theta}_h\), one deduces that

\[ (e^{n+1}_h - e^{n}_h, e^{n+\theta}_h) + \delta t \|e^{n+\theta}_h\|_{LPS}^2 + \delta t a^{n+\theta}_h (u^{n+\theta}_h, e^{n+\theta}_h) \]

\[ = \delta t \left[ (u_t^{n+\theta} - \frac{\tilde{e}^{n+1}_h - \tilde{e}^{n}_h}{\delta t}, e^{n+\theta}_h) + a^{n+\theta}_h (\eta^{n+\theta}, e^{n+\theta}_h) - s^{n+\theta}_h (\bar{e}^{n+\theta}_h, e^{n+\theta}_h) \right]. \]
A straightforward computation gives
\[
(e_h^{n+1} - e_h^n, e_h^{n+\theta}) = \frac{1}{2} \left( \|e_h^{n+1}\|_0^2 - \|e_h^n\|_0^2 \right) + \frac{2\theta - 1}{2} \|e_h^{n+1} - e_h^n\|_0^2.
\]

Furthermore, using Hölder’s and Young’s inequalities, one gets
\[
|d_h^{n+\theta} (u_h^{n+\omega} ; r_h^{n+\theta}, e_h^{n+\theta})| \leq \sqrt{d_h^{n+\theta} (u_h^{n+\omega} ; r_h^{n+\theta}, \overline{r_h^{n+\theta}})} \sqrt{d_h^{n+\theta} (u_h^{n+\omega} ; e_h^{n+\theta}, \overline{e_h^{n+\theta}})}
\]
\[
\leq \frac{1}{4} d_h^{n+\theta} (u_h^{n+\omega} ; r_h^{n+\theta}, e_h^{n+\theta}) + d_h^{n+\theta} (u_h^{n+\omega} ; r_h^{n+\theta}, e_h^{n+\theta})
\]
and hence
\[
d_h^{n+\theta} (u_h^{n+\omega} ; u_h^{n+\theta}, e_h^{n+\theta}) = d_h^{n+\theta} (u_h^{n+\omega} ; e_h^{n+\theta}, e_h^{n+\theta}) + d_h^{n+\theta} (u_h^{n+\omega} ; \overline{r_h^{n+\theta}}, \overline{e_h^{n+\theta}}).
\]

Therefore, one obtains the following upper bound for the discrete part of the estimate (17):
\[
\|e_h^n\|^2_{0,\Omega} + \delta t \sum_{n=0}^{N-1} \|e_h^{n+\theta}\|^2_{\text{LPS}}
\]
\[
\leq \|e_h^0\|^2_{0,\Omega} + 2\delta t \sum_{n=0}^{N-1} \left[ \left( u_t^{n+\theta} - \frac{\overline{r_h^{n+1}} - \overline{r_h^n}}{\delta t} , e_h^{n+\theta} \right) + a^{n+\theta} (\eta^{n+\theta}, e_h^{n+\theta})
\]
\[
- s^{n+\theta} (\overline{r_h^{n+\theta}}, e_h^{n+\theta}) + e_h^{n+\theta} (u_h^{n+\omega} ; \overline{r_h^{n+\theta}}, \overline{e_h^{n+\theta}}) \right].
\]

Using (14), (7), (3), and (2), one obtains
\[
\|e_h^0\|_{0,\Omega} = \|i_h u^0 - r_h u^0\|_{0,\Omega} \leq C h^{k+1} |u^0|_{k+1,\Omega}.
\]

Applying the Cauchy–Schwarz and Young inequalities gives
\[
\left( u_t^{n+\theta} - \frac{\overline{r_h^{n+1}} - \overline{r_h^n}}{\delta t} , e_h^{n+\theta} \right) \leq \frac{1}{\sigma_0} \left\| u_t^{n+\theta} - \frac{\overline{r_h^{n+1}} - \overline{r_h^n}}{\delta t} \right\|^2_{0,\Omega} + \frac{1}{4} \|e_h^{n+\theta}\|^2_{\text{LPS}}.
\]
The last term can be hidden in the left-hand side of (23). The first term is a mixture of discretization errors in time and space. Elimination of $u_t^{n+\theta}$ from (20) and (21) yields
\[
u^{n+\theta} = \frac{u^{n+1} - u^n}{\delta t} - \frac{1}{\delta t} \int_{t^n}^{t^{n+\theta}} u_{tt}(t) (t^n - t) \, dt - \frac{1}{\delta t} \int_{t^{n+\theta}}^{t^{n+1}} u_{tt}(t) (t^{n+1} - t) \, dt.
\]
Since interpolation in space and differentiation in time commute, one has
\[
u^{n+1} - \overline{r_h^{n+1}} - (u^n - \overline{r_h^n}) = \int_{t^n}^{t^{n+1}} (u_t - r_h u_t)(t) \, dt.
\]
Thus, applying the Cauchy–Schwarz inequality, one derives
\[
\left\| u_t^{n+\theta} - \frac{\overline{r_h^{n+1}} - \overline{r_h^n}}{\delta t} \right\|^2_{0,\Omega} \leq \frac{2}{\delta t} \left\| u_t - r_h u_t \right\|^2_{L^2(t^n,t^{n+1}; L^2(\Omega))} + 2\delta t \left\| u_{tt} \right\|^2_{L^2(t^n,t^{n+1}; L^2(\Omega))}.
\]
Assuming \( u_{ttt} \in L^2(0, T; L^2(\Omega)) \) and replacing (20) and (21) by

\[
\begin{align*}
  u^n &= u^{n+\theta} - \theta \delta t u^{n+\theta}_{tt} + \frac{\theta^2}{2} (\delta t)^2 u^{n+\theta}_{ttt} + \frac{1}{2} \int_{t^n}^{t_{n+\theta}} u_{ttt}(t) (t^n - t)^2 \, dt, \\
  u^{n+1} &= u^{n+\theta} + (1 - \theta) \delta t u^{n+\theta}_{tt} + \frac{(1 - \theta)^2}{2} (\delta t)^2 u^{n+\theta}_{ttt} \\
  &+ \frac{1}{2} \int_{t^n}^{t^{n+1}} u_{ttt}(t) (t^{n+1} - t)^2 \, dt,
\end{align*}
\]

one obtains

\[
\begin{align*}
  u^n &= u^{n+1} - \frac{\delta t}{2} \left[ \theta^2 - (1 - \theta)^2 \right] u^{n+\theta}_{tt} \\
  &- \frac{1}{2 \delta t} \int_{t^n}^{t^{n+\theta}} u_{ttt}(t) (t^n - t)^2 \, dt - \frac{1}{2 \delta t} \int_{t^n}^{t^{n+1}} u_{ttt}(t) (t^{n+1} - t)^2 \, dt.
\end{align*}
\]

Therefore, if \( \theta = 1/2 \), one gets

\[
\left\| u^{n+1/2} - \bar{r}^{n+1} + \bar{r}^n \right\|_{0, \Omega}^2 \leq \frac{2}{\delta t} \left\| u^t - r_h u_t \right\|_{L^2(t^n, t^{n+1}; L^2(\Omega))}^2 + (\delta t)^3 \left\| u_{ttt} \right\|_{L^2(t^n, t^{n+1}; L^2(\Omega))}^2.
\]

Now let us consider the remaining three terms on the right-hand side of (23). According to (18), one has

\[
a^{n+\theta}(\eta^{n+\theta}, e_h^{n+\theta}) - s_h^{n+\theta}(r_h^{n+\theta}, e_h^{n+\theta}) = a^{n+\theta}(u^{n+\theta} - \theta u^{n+1} - (1 - \theta) u^n, e_h^{n+\theta}) \\
+ \theta \left[ a^{n+\theta}(u^{n+1} - r_h u^n, e_h^{n+\theta}) - s_h^{n+\theta}(r_h u^n, e_h^{n+\theta}) \right] \\
+ (1 - \theta) \left[ a^{n+\theta}(u^n - \theta u^{n+1}, e_h^{n+\theta}) - s_h^{n+\theta}(r_h u^n, e_h^{n+\theta}) \right].
\]

The last two terms can be estimated by (15) and the estimation of the first term on the right-hand side is performed using

\[
\left\| u^{n+\theta} - \theta u^{n+1} - (1 - \theta) u^n \right\|_{1, \Omega}^2 \leq \delta t \left\| u^t \right\|_{L^2(t^n, t^{n+1}; H^1(\Omega))}^2,
\]

resp.

\[
\left\| u^{n+\theta} - \theta u^{n+1} - (1 - \theta) u^n \right\|_{1, \Omega}^2 \leq (\delta t)^3 \left\| u_{ttt} \right\|_{L^2(t^n, t^{n+1}; H^1(\Omega))}^2,
\]

which follows from (19), resp. (22). Finally, using (16), one obtains

\[
d_{h}^{n+\theta}(u_{h}^{n+\omega}, r_{h}^{n+\theta}, \bar{r}_{h}^{n+\theta}) \leq C \left( \left\| b^{n+\theta} \right\|_{0, \infty, \Omega} \right)^2 h^2 + \left( \left\| u^{n+1} \right\|_{k+1, \Omega} + \left\| u^n \right\|_{k+1, \Omega} \right)^2.
\]

Collecting all the above estimates proves the theorem.

\[\square\]

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E-mail address: knobloch@karlin.mff.cuni.cz