FINITENESS CONDITIONS IN COVERS OF POINCARÉ DUALITY SPACES

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Abstract. A closed 4-manifold (or, more generally, a finite PD4-space) has a finitely dominated infinite regular covering space if and only if either its universal covering space is finitely dominated or it is finitely covered by the mapping torus of a self homotopy equivalence of a PD3-complex.

A space $X$ is a Poincaré duality space if it has the homotopy type of a cell complex which satisfies Poincaré duality with local coefficients (with respect to some orientation character $w: \pi = \pi_1(X) \to \{\pm 1\}$). It is finite if the singular chain complex of the universal cover $\tilde{X}$ is chain homotopy equivalent to a finite free $\mathbb{Z}[\pi]$-complex. (The PD-space $X$ is homotopy equivalent to a Poincaré duality complex $\iff$ it is finitely dominated $\iff \pi$ is finitely presentable. See [2].) Closed manifolds are finite PD-complexes. The more general notion arises naturally in connection with Poincaré duality groups [4], and in considering covering spaces of manifolds [11].

In this note we show that finiteness hypotheses in two theorems about covering spaces of PD-complexes may be relaxed. Theorem 5 extends a criterion of Stark to all Poincaré duality groups. The main result is Theorem 6, which characterizes finite PD4-spaces with finitely dominated infinite regular covering spaces.

1. SOME LEMMAS

Let $X$ be a PD$_n$-space with fundamental group $\pi$. Let $\beta_i(X; \mathbb{Q}) = \dim_{\mathbb{Q}}H_i(X; \mathbb{Q})$ and $\beta_i^{(2)}(X) = \dim_{\mathbb{Q}\langle\pi\rangle}H_i(X; \mathbb{Q}\langle\pi\rangle)$ be the $i$th rational Betti number and $i$th $L^2$ Betti number of $X$, respectively.

**Lemma 1.** Let $X$ be a PD$_n$-space with fundamental group $\pi$. Then $\sum \beta_i(X; \mathbb{Q}) < \infty$ and $\sum \beta_i^{(2)}(X) < \infty$. If $X$ is finite then $\chi(X) = \Sigma(-1)^i\beta_i(X; \mathbb{Q}) = \Sigma(-1)^i\beta_i^{(2)}(X)$.

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Lemma 1 to the short exact sequences and $\text{Ext}$ now apply the long exact sequence of modules which are free of (the same) finite rank as abelian groups. We the kernel of the projection from $\mathbb{Z}$ a quotient of a finitely generated free $G$ of the action of $q$ finitely generated as an abelian group) for all $\text{Proof.}$ Let $H/\mathbb{Z}[G]$ be a group and $k$ be $\mathbb{Z}$ or a field, and let $A$ be a $k[G]$-module which is free of finite rank $m$ as a $k$-module. Then $\text{Ext}^q_{k[G]}(A, k[G]) \cong (H^q(G; k[G]))^m$ for all $q$.

Proof. Let $(g\phi)(a) = g . \phi(g^{-1}a)$ for all $g \in G$ and $\phi \in \text{Hom}_k(A, k[G])$. Let $\{\alpha_i\}_{1 \leq i \leq m}$ be a basis for $A$ as a free $k$-module, and define a map $f : \text{Hom}_k(A, k[G]) \to k[G]^m$ by $f(\phi) = (\phi(\alpha_1), \ldots, \phi(\alpha_m))$ for all $\phi \in \text{Hom}_k(A, k[G])$. Then $f$ is an isomorphism of left $k[G]$-modules. The lemma now follows, since $\text{Ext}^q_{k[G]}(A, k[G]) \cong H^q(G; \text{Hom}_k(A, k[G]))$. (See Proposition III.2.2 of \cite{1}.)

Lemma 3. If $H^q(G; \mathbb{Z}[G])$ is $0$ (respectively, finitely generated as an abelian group) for all $q \leq q_0$ and $B$ is a $\mathbb{Z}[G]$-module which is finitely generated as an abelian group then $\text{Ext}^q_{\mathbb{Z}[G]}(B, \mathbb{Z}[G])$ is $0$ (respectively, finitely generated as an abelian group) for all $q \leq q_0$.

Proof. Let $T$ be the $\mathbb{Z}$-torsion submodule of $B$, and let $H$ be the kernel of the action of $G$ on $T$. Then $T$ is a finite $\mathbb{Z}[G/H]$-module, and so is a quotient of a finitely generated free $\mathbb{Z}[G/H]$-module $A$. Let $A_1$ be the kernel of the projection from $A$ to $T$. Clearly $A$ and $A_1$ are $\mathbb{Z}[G]$-modules which are free of (the same) finite rank as abelian groups. We now apply the long exact sequence of $\text{Ext}^*_{\mathbb{Z}[G]}(-, \mathbb{Z}[G])$ together with Lemma 1 to the short exact sequences

$$0 \to A_1 \to A \to T \to 0$$

and

$$0 \to T \to B \to B/T \to 0.$$
2. Virtual Poincaré duality groups

Stark has shown that a finitely presentable group $G$ of finite virtual cohomological dimension is a virtual Poincaré duality group if and only if it is the fundamental group of a closed $PL$ manifold $M$ whose universal cover $\tilde{M}$ is homotopy finite [13]. The main step in showing the sufficiency of the latter condition involves showing first that $G$ is of type $vFP$, and is established in [14]. If $G_1$ is an $FP$ subgroup of finite index in $G$ then $B = K(G_1, 1)$ is finitely dominated. Hence on applying the Gottlieb-Quinn Theorem to the fibration $\tilde{M} \to M_1 \to B$ of the associated covering space $M_1$ it follows that $\tilde{M}$ and $B$ are Poincaré duality complexes. In particular, $G_1$ is a Poincaré duality group.

There are however Poincaré duality groups in every dimension $n \geq 4$ which are not finitely presentable. We shall give an analogue of Stark’s sufficiency result for such groups, using an algebraic criterion instead of the Gottlieb-Quinn Theorem. In the next two results we shall assume that $M$ is a $PD_n$-space with fundamental group $\pi$, $M_\nu$ is the covering space associated to a normal subgroup $\nu$ of $\pi$, $G = \pi/\nu$ and $k$ is $\mathbb{Z}$ or a field.

**Lemma 4.** Suppose that $H_p(M_\nu; k)$ is finitely generated for all $p \leq [n/2]$. Then $H_p(M_\nu; k)$ is finitely generated for all $p$ if and only if $H^q(G; k[G])$ is finitely generated as a $k$-module for $q \leq [(n - 1)/2]$, and then $H^q(G; k[G])$ is finitely generated as a $k$-module for all $q$. If $H^s(G; k[G]) = 0$ for $s < q$ then $H_{n-s}(M_\nu; k) = 0$ for $s < q$ and $H_{n-q}(M_\nu; k) \cong H^q(G; k[G])$.

**Proof.** Let $E_{pq}^2 = Ext_{k[G]}^q(H_p(M; k[G]), k[G]) \Rightarrow H^{p+q}(M; k[G])$ be the Universal Coefficient spectral sequence for the equivariant cohomology of $M$. Then $E_{pq}^2 = Ext_{k[G]}^q(H_p(M_\nu; k), k[G])$, while $H^{p+q}(M; k[G]) \cong H_{n-p-q}(M_\nu; k)$, by Poincaré duality for $M$.

If $H^q(G; k[G])$ is finitely generated for $q \leq [(n - 1)/2]$ then $E_{2q}^p$ is finitely generated for all $p + q \leq [(n - 1)/2]$, by Lemmas 2 and 3. Hence $H_p(M_\nu; k)$ is finitely generated for all $p \geq n - [(n - 1)/2]$, and hence for all $p$. Conversely, if this holds and $H^s(G; k[G])$ is finitely generated for $s < q$ then $E_{pq}^2$ is finitely generated for all $p \geq 0$, $r \geq 2$ and $s < q$. Since $H^q(M; k[G]) \cong H_{n-q}(M_\nu; k)$ is finitely generated as a $k$-module it follows that $H^q(G; k[G])$ is finitely generated as a $k$-module. Hence $H^q(G; k[G])$ is finitely generated for all $q$.

The final assertion is an immediate consequence of duality and the universal coefficient spectral sequence. $\square$
Theorem 5. If $H_p(M, k)$ is finitely generated for all $p$ then $G$ is $FP_\infty$ over $k$ and $H^s(G; k[G]) \neq 0$ for some $s \leq n$. If moreover $k = \mathbb{Z}$ and $v.c.d.G < \infty$ then $G$ is virtually a $PD_r$-group, for some $r \leq n$.

Proof. Let $C_*(\tilde{M})$ be the equivariant chain complex of the universal covering space $\tilde{M}$. Since $M$ is a $PD_n$-space $C_*(\tilde{M})$ is chain homotopy equivalent to a finite projective $\mathbb{Z}[\pi]$-complex. Hence $C_*(M, k) = k[G] \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{M})$ is chain homotopy equivalent to a finite projective $k[G]$-complex. The arguments of [14] apply equally well with coefficients $k$ a field (instead of $\mathbb{Z}$), and thus the hypotheses of Lemma 4 imply that $G$ is $FP_\infty$ over $k$.

If $v.c.d.G < \infty$ we may assume without loss of generality that $c.d.G < \infty$, and so $G$ is $FP$. Since $H_q(M, \mathbb{Z})$ is finitely generated for all $q$ the groups $H^s(G; \mathbb{Z}[G])$ are all finitely generated, and since $H_0(M, \mathbb{Z}) = \mathbb{Z}$ we must have $H^s(G; \mathbb{Z}[G]) \neq 0$ for some $s \leq n$, by Lemma 4. Then $G$ is a $PD_g$-group, by Theorem 3 of [7].

A finitely generated group $G$ is a weak $PD_r$-group if $H^r(G; \mathbb{Z}[G]) \cong \mathbb{Z}$ and $H^q(G; \mathbb{Z}[G]) = 0$ for $q \neq r$. Theorem 5 complements the main result of [11], in which it is shown that if the $\mathbb{Z}[^2\nu]$-chain complex $C_*(\tilde{M}) = C_*(\tilde{M})|_{\nu}$ has finite $[n/2]$-skeleton and $G$ is a weak $PD_g$-group then $M$ is a $PD_{n-r}$-space.

For each $n \geq 2$ and $k \geq \binom{n+1}{2}$ there are weak $PD_k$-groups which act freely and cocompactly on $S^{2n-1} \times \mathbb{R}^k$, but which are not virtually torsion-free [8]. Thus if $r \geq 6$ weak $PD_r$-groups need not be virtual $PD_r$-groups, and so the other conditions in Theorem 5 do not imply that $v.c.d.G < \infty$, in general. Weak $PD_1$-groups have two ends, and so are virtually $\mathbb{Z}$, while $FP_2$ weak $PD_2$-groups are virtual $PD_2$-groups [1]. Little is known about the intermediate cases $r = 3, 4$ or 5. In particular, it is not known whether a group $G$ of type $FP_\infty$ such that $H^3(G; \mathbb{Z}[G]) \cong \mathbb{Z}$ must be a virtual $PD_3$-group. (The fact that local homology manifolds which are homology 2-spheres are standard may be some slight evidence for this being true.)

Stark’s argument for realization in the finitely presentable case can be adapted to show that any virtual $PD_n$-group acts freely on a 1-connected homotopy finite complex, with quotient a $PD_m$-space for some $m \geq n$. However finite presentability is needed in order to obtain a free cocompact action on a 1-connected complex. A natural converse to Theorem 5 (analogous to Stark’s realization result) might be that every virtual $PD$ group $G$ acts freely and cocompactly on some connected manifold $X$ with $H_q(X; \mathbb{Z})$ finitely generated for all $q$. It would suffice to show that $G \cong \pi/\nu$ where $\pi$ is a finitely presentable
generated. For there is a closed PL manifold $M$ with $\pi_1(M) \cong \pi$ and $\bar{M}$ homotopy finite, by Stark’s result. The quotient group $G$ acts freely and cocompactly on $M_\nu$, and a spectral sequence argument shows that $H_*(M_\nu; \mathbb{Z})$ is finitely generated.

3. FINITELY DOMINATED COVERING SPACES

Let $M$ be a $PD_4$-space with fundamental group $\pi$, and suppose that $M$ has a finitely dominated infinite regular covering space $M_\nu$. Then $\nu = \pi_1(M_\nu)$ is finitely presentable and $\pi/\nu$ has one or two ends. In [9] we showed that if $\pi/\nu$ has two ends then $M$ is the mapping torus of a self homotopy equivalence of a $PD_3$-complex, while if $\pi/\nu$ has one end and $\nu$ is $FP_3$ then either the universal covering space $\bar{M}$ is contractible or homotopy equivalent to $S^2$. We shall show here that the hypothesis that $\nu$ be $FP_3$ is redundant if $M$ is a closed 4-manifold, or more generally if $M$ is a finite $PD_4$-space.

The results from [9] used in the next theorem were originally formulated in terms of $PD_4$-complexes. The arguments given in [9] apply equally well to $PD_4$-spaces, since they need only the $L^2$-Euler characteristic formula of Lemma 1 above.

**Theorem 6.** Let $M$ be a finite $PD_4$-space with fundamental group $\pi$, and let $\nu$ be an infinite normal subgroup of $\pi$ such that $G = \pi/\nu$ has one end and the associated covering space $M_\nu$ is finitely dominated. Then $G$ is of type $FP_\infty$ and $M$ is aspherical.

**Proof.** Let $k$ be $\mathbb{Z}$ or a field. Then $G$ is of type $FP_\infty$ and $H^q(G; k[G])$ is finitely generated as a $k$-module for all $q$, by Lemma 4 and Theorem 5. Moreover $Ext^q_{k[G]}(H_p(M_\nu; k), k[\pi]) = 0$ for $q \leq 1$ and all $p$, since $G$ has one end, and so $H_q(M_\nu; k) = 0$ for $q \geq 3$. In particular, $H^2(G; \mathbb{Z}[G]) \cong H_2(M_\nu; \mathbb{Z})$ is torsion-free, and so is a free abelian group of finite rank.

We may assume that $M_\nu$ is not acyclic and $G$ is not virtually a $PD_2$-group, by Theorem 3.9 of [9]. Therefore $H^2(G; k[G]) = 0$ for all $k$, by the main result of [11]. Hence $H_2(M_\nu; \mathbb{F}_p) = 0$ for all primes $p$, so $H_1(M_\nu; \mathbb{Z})$ is torsion-free and nonzero. Therefore $H^s(G; \mathbb{Z}[G]) = H_{4-s}(M_\nu; \mathbb{Z}) = 0$ for $s < 3$ and $H^3(G; \mathbb{Z}[G]) \cong H_1(M_\nu; \mathbb{Z}) = \nu/\nu'$ is a nontrivial finitely generated abelian group. Therefore $\nu/\nu' \cong H^3(G; \mathbb{Z}[G]) \cong \mathbb{Z} [7].$

Thus we may assume that $M_\nu$ is an homology circle. Let $\tilde{G} = \pi/\nu'$ and let $t \in \tilde{G}$ represent a generator of the infinite cyclic group $\nu/\nu'$. Let $M'_\nu$ be the covering space associated to the subgroup $\nu'$. Since $M_\nu$ is finitely dominated a Wang sequence argument shows that $H_q(M'_\nu; k)$
is a finitely generated \( k[t, t^{-1}] \)-module on which \( t - 1 \) acts invertibly, for all \( q > 0 \). Then \( H^q(M'_\nu; \mathbb{F}_p) \) is finitely generated for all primes \( p \) and all \( q > 0 \). Now \( H^*(G; k[G]) = 0 \) for all \( k \) and all \( s < 4 \), by a Lyndon-Hochschild-Serre spectral sequence argument. Therefore \( H^q(M'_\nu; \mathbb{F}_p) = 0 \) for all primes \( p \) and all \( q > 0 \), by Lemma 4. Nontrivial finitely generated \( \mathbb{Z}[t, t^{-1}] \)-modules have nontrivial finite quotients, and so we may conclude that \( M'_\nu \) is acyclic.

Since \( M \) is a \( PD_4 \)-space \( C_*(\tilde{M}) \) is chain homotopy equivalent to a finite projective \( \mathbb{Z}[\pi] \)-complex \( C_* \). Thus \( D_* = \mathbb{Z} \otimes_{\mathbb{Z}[\pi']} C_* \) is a finite projective \( \mathbb{Z}[\tilde{G}] \)-complex, and is a resolution of \( \mathbb{Z} \). Therefore \( \tilde{G} \) is a \( PD_4 \)-group. (In particular, we see again that \( G = \tilde{G}/(\nu/\nu') \) is \( FP_\infty \).)

Since \( \nu/\nu' \) is a torsion-free abelian normal subgroup of \( \tilde{G} \) the group ring \( \mathbb{Z}[\tilde{G}] \) has a flat extension \( R \), obtained by localising with respect to the nonzero elements of \( \mathbb{Z}[t, t^{-1}] \), such that \( R \otimes_{\mathbb{Z}[\tilde{G}]} \mathbb{Z} = 0 \). (See page 23 of [9] and the references there.) Hence \( R \otimes_{\mathbb{Z}[\tilde{G}]} D_* \) is a contractible complex of finitely generated projective \( R \)-modules.

We may in fact assume that \( C_* \) is a finite free \( \mathbb{Z}[\pi] \)-space, since \( M \) is a finite \( PD_4 \)-complex. It follows that \( \chi(M) = \chi(R \otimes_{\mathbb{Z}[\tilde{G}]} D_*) = 0 \). Since \( \nu \) is an infinite \( FP_2 \) normal subgroup of \( \pi \) and \( \pi/\nu \) has one end \( \beta_1^{(2)}(\pi) = 0 \) and \( H^s(\pi; \mathbb{Z}[\pi]) = 0 \) for \( s \leq 2 \). Therefore \( M \) is aspherical, by Corollary 3.5.2 of [9].

With this result we may now reformulate Theorem 3.9 of [9] as follows.

**Corollary.** A finite \( PD_4 \)-space \( M \) has a finitely dominated infinite regular covering space if and only if either \( M \) is aspherical, or \( \tilde{M} \simeq S^2 \), or \( M \) has a 2-fold cover which is homotopy equivalent to the mapping torus of a self-homotopy equivalence of a \( PD_3 \)-complex. If \( M \) has a finitely dominated regular covering space and is not aspherical it is a \( PD_4 \)-complex.

**Proof.** Only the final sentence needs any comment. If \( \tilde{M} \simeq S^2 \) then \( \pi_1(M) \) is virtually a \( PD_2 \)-group and so is finitely presentable. This is also clear if \( M \) has a 2-fold cover which is the mapping torus of a self-homotopy equivalence of a \( PD_3 \)-complex. Thus in each case \( M \) is a \( PD_4 \)-complex.

There are \( PD_n \) groups of type \( FF \) which are not finitely presentable, for each \( n \geq 4 [5] \). The corresponding \( K(G, 1) \) spaces are aspherical finite \( PD_n \)-spaces which are not \( PD_n \)-complexes.

The hypothesis that \( M \) be finite is used only in the final paragraph of the proof of Theorem 6, in the appeal to Corollary 3.5.2 of [9] and
in the calculation of $\chi(M)$. (If we assumed instead that $v.c.d.G < \infty$ then we could use multiplicativity of the Euler characteristic to show that $\chi(M) = 0$.)

A more substantial issue is that the argument for Theorem 6 does not appear to extend to the case when $\nu$ is an ascendant subgroup of $\pi$, as considered in [10] (where the $FP_3$ condition is also used). Is there an argument along the following lines? Let $C_\ast$ be a finite projective $\mathbb{Z}[\pi]$-complex with $H_0(C_\ast) \cong \mathbb{Z}$ and $H_1(C_\ast) = 0$. Show that $\text{Hom}_{\mathbb{Z}[\pi]}(H_2(C_\ast), \mathbb{Z}[\pi]) = 0$ if $[\pi : \nu] = \infty$ and $C_\ast|_\nu$ is chain homotopy equivalent to a finite projective $\mathbb{Z}[\nu]$-complex. If so, the proofs of Theorem 3.9 of [9] and Theorem 6 of [10] would apply, without needing to assume that $\nu$ is $FP_3$ or that $M$ is finite.
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