FasTer: Fast Tensor Completion with Nonconvex Regularization

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Abstract

Due to the easy optimization, the convex overlapping nuclear norm has been popularly used for tensor completion. However, it over-penalizes top singular values and lead to biased estimations. In this paper, we propose to use the nonconvex regularizer, which can less penalize large singular values, instead of the convex one for tensor completion. However, as the new regularizer is nonconvex and overlapped with each other, existing algorithms are either too slow or suffer from the huge memory cost. To address these issues, we develop an efficient algorithm, which is based on the proximal average (PA) algorithm, for real-world problems. Compared with the direct usage of PA algorithm, the proposed algorithm runs orders faster and needs orders less space. We further speed up the proposed algorithm with the acceleration technique, and show the convergence to critical points is still guaranteed. Experimental comparisons of the proposed approach are made with various other tensor completion approaches. Empirical results show that the proposed algorithm is very fast and can produce much better recovery performance.

1 Introduction

Tensors, which can be seen as high-order matrices, have been commonly used to describe the multilinear relationships inside the data. In this paper, we focus on 3-order tensors, since they naturally appear in many applications. For example, the color image can be seen as a 3-order tensor [Liu et al. 2013]; in remote sensing applications, a hyperspectral image with multiple bands can be naturally represented as a 3-order tensor [Signoretto et al. 2011]; in Youtube’s social network, users can follow each other and can have the same subscribed channels. By treating these attributes as another dimension, rating prediction becomes a problem on the 3-order 3-order tensor (Lei, Wang, and Liu 2009). Low-rank tensor completion is popularly used for filling and predicting these unseen values (Tomioka, Hayashi, and Kashima 2010; Acar et al. 2011; Song et al. 2017). Mathematically, let the observed elements indicating by 1’s in the binary tensor \( \Omega \in \{0, 1\}^{I_1 \times I_2 \times I_3} \) with values given by corresponding positions in \( \mathcal{O} \in \mathbb{R}^{I_1 \times I_2 \times I_3} \), the low-rank tensor completion problem is:

\[
\min_{X \in \mathbb{R}^{I_1 \times I_2 \times I_3}} \frac{1}{2} \|P_{\Omega}(X - \mathcal{O})\|_F^2 + \lambda r(X),
\]

where \( r \) is a regularizer encouraging \( X \) to be low-rank, and \( \|P_{\Omega}(X)\|_{I_1 \times I_2 \times I_3} = X_{I_1 \times I_2 \times I_3} \) if \( \Omega_{I_1 \times I_2 \times I_3} = 1 \) and 0 otherwise.

Recently, due to the success of the nuclear norm for low-rank matrix completion (Candes and Recht 2009), convex relaxation of tensor rank based on the nuclear norm are also introduced for tensor completion (Tomioka, Hayashi, and Kashima 2010; Gandy, Recht, and Yamada 2011). Among them, the overlapped nuclear norm is the most popular ones. Compared with the tensor trace norm, it can be computed exactly without the approximation (Cheng et al. 2016). Then, it can encourage tensor to be low-rank on every mode simultaneously, which can better approximate a low-rank tensor than the latent nuclear norm (Tomioka, Hayashi, and Kashima 2010). Besides, the theoretical guarantee has also been provided, which shows the exact recovery is possible (Tomioka et al. 2011; Tomioka and Suzuki 2013; Mu et al. 2014).

However, one potential problem of the nuclear norm is that it equally penalizes all singular values, while the larger singular values can be more informative (Mazumder, Hastie, and Tibshirani 2010). To less penalize large singular values, adaptive nonconvex regularizers, have been introduced for low-rank matrix learning. Examples are, capped-\( \ell_1 \) norm (Zhang 2010b), log-sum-penalty (LSP) (Candès, Wakin, and Boyd 2008), truncated nuclear norm (TNN) (Hu et al. 2013), smoothed-capped-absolute-deviation (SCAD) (Fan and Li 2001) and minimax concave penalty (MCP) (Zhang 2010a). They give both better empirical performance (Lu et al. 2016; Yao et al. 2018) and statistical guarantee (Gui, Han, and Gu 2016) than the convex nuclear norm. Besides, these regularizers have not been used for tensor completion yet.

Motivating by the problem of over-penalizing top singular values of the convex nuclear norm, we propose to use nonconvex penalties with overlapped nuclear norm as \( r \) in

\[1\]Without loss of generality, we assume that \( I_1 \geq I_2 \geq I_3 \).
Since the regularizers are nonconvex and overlap with each other, the resulting problem is hard to optimize. We first find that the proximal average (PA) algorithm (Zhong and Kwok 2014) becomes the only applicable algorithm which also has the convergence guarantee.

Unfortunately, due to the needs of folding and unfolding operations of tensor, PA algorithm suffers from the extremely large space requirement and expensive iteration complexity. These greatly limit the usage of the new model in real problems, especially when the data scale is large. To address these issues, we propose an important proposition avoiding such operations in iterations of PA algorithm. Moreover, we adopt the acceleration technique to speed up the convergence of PA algorithm, while still offers the convergence guarantee. This helps to significantly cut down the number of iterations required by PA algorithm. As a result, the proposed algorithm needs orders less memory but runs orders faster than the direct usage of PA algorithm. Finally, we perform experiments on both synthetic and real data sets. Results show that the proposed algorithm is very efficient and can produce much better empirical performance than other tensor regularization and decomposition methods.

### 2 Related Works

#### 2.1 Nonconvex Low-Rank Matrix Regularization

Nonconvex regularizers, such as the capped-$\ell_1$ norm, LSP and TNN, have been introduced in low-rank matrix learning. They can less penalize the larger, more informative singular values. Thus, they also have better empirical performance and theoretical guarantee than nuclear norm regularization (Lu et al. 2016; Gui, Han, and Gu 2016; Yao et al. 2018). The optimization problem is formulated as

$$\min_{X \in \mathbb{R}^{I_1 \times I_2}} f(X) + \lambda \phi(X),$$

where $f$ is a $\rho$-Lipschitz smooth loss (i.e., $\|\nabla f(X) - \nabla f(Y)\|^2_F \leq \rho \|X - Y\|^2_F$ for any $X$ and $Y$).

$$\phi(X) = \sum_{i=1}^{I_3} \kappa(\sigma_i(X))$$

is the nonconvex regularizer, and $\kappa$ is the nonconvex penalty satisfying Assumption 1. Some popular choices of $\kappa$ are, capped-$\ell_1$ $\kappa(\sigma_i(X)) = \log(1 + \sigma_i(X)) + 1$ and LSP $\kappa(\sigma_i(X)) = \min(\sigma_i(X), \theta)$, where $\theta > 0$ is a constant. More examples are in Appendix ??.

#### Assumption 1

$\kappa(\alpha)$ is a concave, nondecreasing and $L$-Lipschitz continuous function on $\alpha \geq 0$ with $\kappa(0) = 0$.

Recently, the proximal algorithm (Parikh and Boyd 2013) has been used to solve the nonconvex optimization problem in (2), with convergence to a critical point (Lu et al. 2016; Yao et al. 2018). At iteration $t$, it produces the iterate

$$X_{t+1} = \text{prox}_{\frac{1}{\tau} \phi}(Z_t),$$

where $Z_t = X_t - \frac{1}{\tau} \nabla f(X_t)$, $\tau > 0$ is the stepsize, and

$$\text{prox}_{\frac{1}{\tau} \phi}(Z) = \arg \min_X \frac{1}{2} \|X - Z\|^2_F + \frac{\lambda}{\tau} \phi(X),$$

is the proximal step. For these regularizers, we usually have closed-form solutions on their proximal steps (Lemma 2.1). Thus, a SVD is required to compute the proximal step.

**Lemma 2.1 (Lu et al. 2016).** Let the SVD of $Z \in \mathbb{R}^{I_1 \times I_2}$ be $U \Sigma V^\top$. Then $\text{prox}_{\frac{1}{\tau} \phi}(Z) = U \text{Diag}(y_1, \ldots, y_{I_3}) V^\top$, where $y_i = \arg \min_{y \geq 0} \frac{1}{2} \|y - \sigma_i(Z)\|^2 + \lambda \phi(y)$.

### Matrix Completion: “Sparse plus Low-rank” Structure

In matrix completion, $f(X) = \frac{1}{2} \|P_{I_1}(X - O)\|^2_F$, in (2) (Candès and Recht 2009). $Z_t$ then becomes:

$$Z_t = X_t - \frac{1}{\tau} P_{I_1}(X_t - O),$$

where $\frac{1}{\tau} P_{I_1}(X_t - O)$ is sparse and $X_t$ is low-rank. This “sparse plus low-rank” structure can enable the SVD computation to be much more efficient (Mazumder, Hastie, and Tibshirani 2010; Yao et al. 2018). Specifically, to compute the SVD of $Z_t$, with the power method, most of the efforts are on computing multiplications of the form $b^\top Z_t$ (where $b \in \mathbb{R}^{I_3}$) and $a^\top Z_t$ (where $a \in \mathbb{R}^{I_2}$). Let $X_t$ be low-rank factorized as $U_t V_t^\top$. Using (6),

$$Z_t b = U_t (V_t^\top b) - \frac{1}{\tau} P_{I_2}(Y_t - O)b.$$

This takes $O((I_1 + I_2)k_t + \|O\|_1)$ time, where $k_t$ is the rank of $X_t$. Usually, $k_t \ll I_2$ and $\|O\|_1 \ll I_1 I_2$. Thus, it is much faster than direct multiplying $Z_t$ and $b$, which takes $O(I_1 I_2)$ time. The same also holds for $a^\top Z_t$. Similarly, the proximal step takes $O((I_1 + I_2)k_t k_{t+1} + \|O\|_1 k_{t+1})$ time, while a direct computation without utilizing the “sparse plus low-rank” structure takes $O(I_1 I_2 k_{t+1})$ time. Besides, as only $P_{I_2}(X_t)$ and the factorized form of $X_t$ need to be kept, the space requirement is reduced from $O(I_1 I_2)$ to $O((I_1 + I_2)k_t + \|O\|_1)$.

### 2.2 Low-rank Tensor Learning

The nuclear norm has been popularly used in low-rank matrix learning. Recently, it has been extended to the learning of low-rank tensors (Tomiioka, Hayashi, and Kashima 2010; Signoretto et al. 2011). The most commonly used low-rank tensor regularizer is the (convex) overlapped nuclear norm, $\|X\|_{\text{ON}} = \sum_{i=1}^{I_3} \sigma_i(X)$, and the Frobenius norm of $X$ is $\|X\|_F = \sqrt{\langle X, X \rangle}$. The problem is formulated as

$$\min_{X \in \mathbb{R}^{I_1 \times I_2 \times I_3}} f(X) + \lambda \phi(X),$$

where $f$ is a $\rho$-Lipschitz smooth loss, $\|\nabla f(X) - \nabla f(Y)\|^2_F \leq \rho \|X - Y\|^2_F$ for any $X$ and $Y$.

$$\phi(X) = \sum_{i=1}^{I_3} \kappa(\sigma_i(X))$$

is the nonconvex regularizer, and $\kappa$ is the nonconvex penalty satisfying Assumption 1. Some popular choices of $\kappa$ are, capped-$\ell_1$ $\kappa(\sigma_i(X)) = \log(1 + \sigma_i(X)) + 1$ and LSP $\kappa(\sigma_i(X)) = \min(\sigma_i(X), \theta)$, where $\theta > 0$ is a constant. More examples are in Appendix ??.
For a $M$-order tensor $\mathcal{X}$, the overlapped nuclear norm is $||\mathcal{X}||_{\text{overlap}} = \sum_{m=1}^{M} \lambda_{m} ||\mathcal{X}_{(m)}||_{*}$, where $\lambda_{m} \geq 0$ are hyperparameters.

Factorization methods, such as the Tucker/CP (Kolda and Bader 2009) and tensor-train decompositions (Oseledets 2011), have also been used for low-rank tensor learning. Compared to nuclear norm regularization, their optimization problems are nonconvex and more difficult, and their empirical performance is often inferior (Tomioka et al. 2011; Yu, Micol, and Xing 2015). Has also been extended to nonconvex $\tau$-divergence. This is achieved by integrating a nonconvex regularizer with the overlapped nuclear norm.

Definition 1. (Tomioka, Hayashi, and Kashima 2010) For a $M$-order tensor $\mathcal{X}$, the overlapped nuclear norm is $||\mathcal{X}||_{\text{overlap}} = \sum_{m=1}^{M} \lambda_{m} ||\mathcal{X}_{(m)}||_{*}$, where $\lambda_{m} \geq 0$ are hyperparameters.

For $M$-order tensors, the completion problem becomes

\[
\min_{\mathcal{X} \in \mathcal{H}} f(\mathcal{X}) + \sum_{i=1}^{K} \lambda_{i} g_{i}(\mathcal{X}),
\]

where $g_{i}$s are regularizers. Sometimes, the proximal step for each individual $g_{i}$ can be easily computed, but not for $g(\mathcal{X}) = \sum_{i=1}^{K} g_{i}(\mathcal{X})$. Hence, the proximal algorithm cannot be efficiently used.

Instead, the proximal average (PA) algorithm (Bauschke et al. 2008) generates the next iterate $x_{t+1}$ as

\[
z_{t} = x_{t} - \frac{1}{\tau} \nabla f(x_{t}),
\]

\[
y_{i} = \text{prox}_{\frac{1}{\tau} g_{i}}(z_{t}), \quad i = 1, \ldots, K,
\]

\[
x_{t+1} = \sum_{i=1}^{K} \alpha_{i} y_{i}^{t+1},
\]

where $\alpha_{i} = \lambda_{i}/\sum_{i=1}^{K} \lambda_{i}$. As the proximal steps in (10) are easy, the PA algorithm can be significantly faster than the proximal algorithm. When both $f$ and $g$ are convex, the PA algorithm converges to an optimal solution of (8) with a proper choice of $\tau$ (Yu 2013). Recently, the PA algorithm has also been extended to nonconvex $f$ and $g_{i}$'s (Zhong and Kwok 2014, Yu, Micol, and Xing 2015).

3 Proposed Algorithm

In this section, we propose to improve tensor completion performance by reducing penalties for the tensor’s top singular values. This is achieved by integrating a nonconvex regularizer with the overlapped nuclear norm.

For $3$-order tensors, the completion problem becomes

\[
\min_{\mathcal{X}} F(\mathcal{X}) = \frac{1}{2} ||\mathcal{P}_{\Omega}(\mathcal{X} - \mathcal{O})||_{F}^{2} + \sum_{d=1}^{D} \lambda_{d} \phi(\mathcal{X}_{(d)}),\]

where $\phi$ is as defined in (3). Note that the regularizer above only sums over $D \leq M$ modes (instead of $M$ in Definition 1). This is because in some applications (e.g., those involving color images), the third mode (number of colors) is already small, and so does not need to be low-rank regularized.

When $D = 1$, (12) reduces to $\min_{\mathcal{X} \in \mathcal{H}} F(\mathcal{X}) = \frac{1}{2} ||\mathcal{P}_{\Omega}(\mathcal{X} - \mathcal{O}_{(1)})||_{F}^{2} + \lambda_{1} \phi(\mathcal{X})$. Thus, this is a matrix completion problem with nonconvex low-rank regularizer, and can be solved by the proximal algorithm as in (Lu et al. 2016; Yao et al. 2018). When $\kappa(\alpha)$ in (3) is $|\alpha|$, (12) reduces to (convex) overlapped nuclear norm regularization. While $D$ may not be equal to $M$, it can be easily shown that optimization solvers such as alternating direction of multiple multipliers (ADMM) (Tomioka, Hayashi, and Kashima 2010) and fast low-rank tensor completion (FaLRTC) (Liu et al. 2013) can still be used. However, with nonconvex regularization, ADMM no longer has convergence guarantee and the dual in FaLRTC cannot be derived. Factorization-based algorithms, such as gradient descent (Acar et al. 2011) and optimization on Riemannian manifold (Kasai and Mishra 2016), also cannot be used as $F$ in (12) is not smooth.

3.1 First Attempt: Using the PA algorithm

In this section, we show how the PA algorithm for nonconvex regularizers (Zhong and Kwok 2014) can be used to solve (12) when $D \neq 1$. First, note that it requires each component nonconvex regularizer of $g$ in (8) to be Lipschitz continuous and admits a difference of convex decomposition. This holds for our $\phi(\mathcal{X})$ (Yao et al. 2018). Moreover, $f(\mathcal{X})$ in (8) equals $\frac{1}{2} ||\mathcal{P}_{\Omega}(\mathcal{X} - \mathcal{O})||_{F}^{2}$. Given the current iterate $\mathcal{X}_{t}$, we have from (9)-(11).

\[
z_{t} = x_{t} - \frac{1}{\tau} \mathcal{P}_{\Omega}(\mathcal{X}_{t} - \mathcal{O}),
\]

\[
y_{i} = \text{prox}_{\frac{1}{\tau} \phi}(z_{t}), \quad i = 1, \ldots, D,
\]

\[
x_{t+1} = \frac{\sum_{i=1}^{D} \alpha_{i} y_{i}^{t+1}}{\sum_{i=1}^{D} \alpha_{i}},
\]

where $\alpha_{i} = \lambda_{i}/\sum_{i=1}^{D} \lambda_{i}$ as the proximal step in (14) requires SVD on the unfolded $z_{t}$, while (15) requires the folding of $y_{i}^{t+1}$. A direct implementation of (13)-(15) takes a space complexity of $O(3^{3})$ and an iteration time complexity of $O(3^{3} \sum_{i=1}^{D} I_{i})$, where $I_{\times} = \prod_{i=1}^{D} I_{i}$ and $I_{+} = \sum_{i=1}^{D} I_{i}$, and is expensive.

3.2 Cheap Iterations: Utilizing problem structure

As discussed in Section 2.1, the “sparse plus low-rank” structure can significantly speed up the proximal algorithm for matrix completion. Here, we attempt to use a similar structure, in the tensor setting.

Using the “Sparse Plus Low-Rank” Structure. Let $Y_{i}^{t}$ be factorized as $Y_{i}^{t} = \mathcal{U}_{i}^{t}(\mathcal{V}_{i}^{t})^{\top}$. From (13) and (15).

\[
z_{t} = \sum_{i=1}^{D} \alpha_{i} (\mathcal{U}_{i}^{t}(\mathcal{V}_{i}^{t})^{\top}) - \frac{1}{\tau} \mathcal{P}_{\Omega}(\mathcal{X}_{t} - \mathcal{O}),
\]

where $\sum_{i=1}^{D} \alpha_{i} (\mathcal{U}_{i}^{t}(\mathcal{V}_{i}^{t})^{\top})$ is a sum of low-rank matrices, and $\frac{1}{\tau} \mathcal{P}_{\Omega}(\mathcal{X}_{t} - \mathcal{O})$ is a sparse tensor. Thus, $z_{t}$ also has a “sparse plus low-rank” structure.

To compute the proximal step in (14), recall that we have to perform matrix multiplications of the form $[\mathcal{Z}_{t}]_{(i)} b$ (with
\( b \in \mathbb{R}^{I_x / I_t} \) and \( a^T (\mathbf{Z}_t)_{(i)} \) (with \( a \in \mathbb{R}^{I_x} \)). Using (16),
\[
(\mathbf{Z}_t)_{(i)} b = \alpha_i U_i^T (V_i^T)^T b + \sum_{j \neq i} \alpha_j \left( ([U_j (V_j^T)^T]^{(j)})_{(i)} b \right) - \frac{1}{\tau} [P_{3t}(\mathbf{X}_t - \mathbf{O})]_{(i)} b.  
\]
Similarly, for \( a^T (\mathbf{Z}_t)_{(i)} \),
\[
a^T (\mathbf{Z}_t)_{(i)} = \alpha_i a^T (U_i^T) (V_i^T)^T a^T (U_j^T) (V_j^T)^T b - \frac{1}{\tau} a^T [P_{3t}(\mathbf{X}_t - \mathbf{O})]_{(i)} b. 
\]
The first terms in (17) and (18) can be directly computed, and take \( O((I_x / I_t + I_t) k^2_j) \) space and time, where \( k^2_j \) is the rank of \( \mathbf{Y}_t^j \). However, the other terms need more careful consideration in order to be efficient.

**Computing** \([P_{3t}(\mathbf{X}_t - \mathbf{O})]_{(i)} b\) and \( a^T [P_{3t}(\mathbf{X}_t - \mathbf{O})]_{(i)} b\). A direct computation involves unfolding, which takes \( O(I_x / I_t) \) space and time. In the following, we avoid unfolding by computing only the observed elements from the low-rank factorizations of \( \mathbf{Y}_t^j \)’s in (14), and then use sparse tensor multiplications.

To efficiently process the sparse tensor \( P_{3t}(\mathbf{X}_t - \mathbf{O}) \), we use the popular coordinate format (Bader and Kolda 2007). Its \( p \)th observed element with value \( v_p \) is represented as \( \{i^1_p, i^2_p, i^3_p, v_p\} \), where \( \{i^1_p, i^2_p, i^3_p\} \) indexes positions in each mode. Using \( \mathbf{Y}_t^j = U_j^T (V_j^T)^T \), each observed element in \( P_{3t}(\mathbf{X}_t - \mathbf{O}) \) can be computed by Algorithm 1 Construction of the whole \( P_{3t}(\mathbf{X}_t - \mathbf{O}) \) then takes \( O(||\mathbf{X}||_1 \sum_{i=1}^{D} k^2_i) \) time. We then use sparse tensor packages (such as the Tensor Toolbox in (Bader and Kolda 2007)) to compute \([P_{3t}(\mathbf{X}_t - \mathbf{O})]_{(i)} b\) and \( a^T [P_{3t}(\mathbf{X}_t - \mathbf{O})]_{(i)} b\), which takes \( O(||\mathbf{X}||_1) \) space and time.

**Algorithm 1** Computing the \( p \)th element in \( P_{3t}(\mathbf{X}_t - \mathbf{O}) \).

**Require:** index \( \{i^1_p, i^2_p, i^3_p\} \), factorizations of \( \mathbf{Y}_t^1, \mathbf{Y}_t^2, \mathbf{Y}_t^3 \);
1: \( u_1 \leftarrow (i^1_p) \)th row of \( U_1^T \);
2: \( v_1 \leftarrow (i^2_p) \)th row of \( V_1^T \);
3: \( u_2 \leftarrow (i^2_p) \)th row of \( U_2^T \);
4: \( v_2 \leftarrow (i^3_p) \)th row of \( V_2^T \);
5: if \( D = 3 \) then
6: \( u_3 \leftarrow (i^3_p) \)th row of \( U_3^T \);
7: \( v_3 \leftarrow (i^1_p) \)th row of \( V_3^T \);
8: end if
9: \( o_p \leftarrow \) \( p \)th element in \( P_{3t}(\mathbf{O}) \);
10: return \( v_p = \sum_{i=1}^{D} \alpha_i u^T v - o_p \).

**Computing** \( a^T (U_j (V_j^T)^T)^{\{j\}}_{(i)} \) and \( ([U_j (V_j^T)^T]^{\{j\}}_{(i)} b \). These terms involve folding and unfolding operations. A direct computation will take \( O(I_x / I_t) \) space and \( O(k^2_j I_x) \) time. The following Proposition shows that these products can be computed more efficiently without folding and unfolding.

**Proposition 3.1.** Let \( \{\pi_1, \pi_2, \pi_3\} \) be any permutation of \( \{1, 2, 3\} \), \( U \in \mathbb{R}^{I_x \times k} \), \( V \in \mathbb{R}^{I_t \times k} \), and \( u_p \) (resp. \( v_p \)) be the \( p \)th column in \( U \) (resp. \( V \)). For any \( a \in \mathbb{R}^{I_t} \) and \( b \in \mathbb{R}^{I_t \times k} \), we have

(i) \( a^T (U V^T)^{\{j\}}_{(i)} b = \sum_{p=1}^{k} u^T_p \otimes [a^T \mathbf{mat}(v_p)] \) and

(ii) \( ([U V^T]^{\{j\}}_{(i)} b = \sum_{p=1}^{k} \mathbf{mat}(v_p)^T u_p \).

where \( \otimes \) is the Kronecker product, and \( \mathbf{mat}(c) \) reshapes vector \( c \in \mathbb{R}^{I_x \times I_t} \) into a \( I_x \times I_t \) matrix.

Taking \( \pi_1 = j \) and \( \pi_2 = i \) in Proposition 3.1 \( a^T (v_p) \) takes \( O(I_x / I_t) \) space and time, and the Kronecker product takes \( O(I_x / I_t) \) space and time. Thus, computation of \( a^T (U_j (V_j^T)^T)^{\{j\}}_{(i)} b \) takes a total of \( O((1 / I_t + 1 / I_t) k^2_j I_x) \) time and \( O(I_x / I_t + I_x / I_t) \) space. The same holds for the computation of \( ([U_j (V_j^T)^T]^{\{j\}}_{(i)} b \).

**Total Complexity.** Combining the above complexities, and noting that we have to keep the factorized form \( U_j (V_j^T)^T \) of \( \mathbf{Y}_t^j \), computing (17) and (18) take \( O(\sum_{p=1}^{k} (1 / I_t + 1 / I_t) k^2_j I_x + ||\mathbf{X}||_1) \) space and time. As (14) performs \( D \) proximal steps, plugging techniques here into (13), (15), the resulting algorithm takes a total of

\[
O(\sum_{i=1}^{D} \sum_{j \neq i} \left( \frac{1}{I_t} + \frac{1}{I_j} \right) k^2_j I_x + ||\mathbf{X}||_1),
\]
for each iteration. As \( k^2_j \), \( k^2_{j+1} \ll I_t \), these are much lower than those of a direct implementation in Section 3.1.

### 3.3 Speedup Convergence

**Using Acceleration.** The PA algorithm uses only first-order information, and may suffer slow convergence (Parikh and Boyd 2013). To address this problem, we adopt the popularly used Nesterov acceleration (Nesterov 1983). Note that it has not been used before with the PA algorithm on nonconvex optimization. We adopt the acceleration scheme in (Yao et al. 2017) here, and the resultant procedures are shown in Algorithm 2. Note that, techniques in Section 3.2 can still be used, we only need to keep sparse tensors \( P_{3t}(\mathbf{O}) \) and \( P_{3t}(\mathbf{V}_t) \), and factorized forms of \( \mathbf{Y}_{t+1}^j \) during the iteration. No dense tensors need to be explicitly formed. Thus, while \( \mathbf{Y}_t \) is more complex than \( \mathbf{X}_t \), it still has the special “sparse plus low-rank” structure (details are in Appendix B). The iteration time and complexity of Algorithm 2 are still in (19) and (20). As a result, “sparse plus low-rank” structure can still be preserved with acceleration, and we can achieve both fast convergence and cheap iterations.

**Convergence Analysis.** While the Nesterov acceleration (Nesterov 1983) has been popularly used, it has not been used with the PA algorithm on nonconvex optimization. Thus, previous proofs cannot be directly used here. The following from (Zhong and Kwok 2014) shows the proximal step in (14) implicitly produces a new regularizer \( g_r \).
All the elements in \(N\) are independently from Gaussian distribution.

Algorithm 2 Fast Tensor Completion with Nonconvex Regularization (FasTer) algorithm.

1: initialize \(X_0 = X_1 = 0\) and stepsize \(\tau > (\rho + DL)\);
2: for \(t = 1, \ldots, T\) do
3: \(Y_t = X_t + \frac{\tau}{t+1}(X_t - X_{t-1})\);
4: if \(F_t(Y_t) \leq F_t(X_t)\) then
5: \(V_t = Y_t\);
6: else
7: \(V_t = X_t\);
8: end if
9: \(Z_t = V_t - \frac{1}{\tau}P_t(V_t - \mathcal{O})\);
10: for \(i = 1, \ldots, T\) do
11: \(X^t_{i+1} = \text{prox}_{\frac{1}{\tau}g}(Z_t(i))\);
12: end for
13: \(V_{T+1} = \sum_{i=1}^D \alpha_i(X^t_{i+1})^{(i)}\); // construct implicitly
14: end for
15: return \(X_{T+1}\).

Proposition 3.2. There exists a function \(\bar{g}_\tau\) such that
\[\text{prox}_{\bar{g}_\tau}(Z_t) = \sum_{i=1}^D \alpha_i \text{prox}_{\frac{1}{\tau}g_i}(Z_t(i))\] for any \(\tau > 0\).

The following bounds the difference between the optimal values of the objectives \(F(X)\) in (12) and \(F_t(X) \equiv \frac{1}{2}\|P_t(X - \mathcal{O})\|_F^2 + \bar{g}_\tau(X)\).

Proposition 3.3. \(0 \leq \min F - \min F_t \leq \frac{1}{\tau} \sum_{i=1}^D \alpha_i L^2\).

Theorem 3.4. The sequence \(\{X_t\}\) generated from Algorithm 2 has at least one limit point, and all limit points are critical points of \(F_t(X)\).

Combining Theorem 3.4 with Proposition 3.3 we see that a larger \(\tau\) leads to better approximation to the original problem \(F\). However, it also leads to smaller steps, and slower convergence. In the experiments, we set \(\tau = 1\); besides, since \(F_t\) is hard to evaluate, and so we use \(F\) instead as in (Zhong and Kwok 2014). Finally, while Theorem 3.4 provides no convergence rate, empirically the accelerated algorithm is significantly faster than its non-accelerated version (Section 4.1).

4 Experiments

Experiments are performed on a PC with Intel i7 CPU and 32GB memory. All algorithms are implemented in Matlab, except operations on sparse tensor and matrix are implemented in C and linked to Matlab by mex files.

4.1 Synthetic Data

In the synthetic data, we consider two cases, i.e., \(D = 2\) and \(3\) in (12). The difference is that, when \(D = 2\), the low-rank regularizations are imposed on the first two unfolding matrices from \(X\) but not the third one.

We follow the setup suggested in (Song et al. 2017). The synthetic data is generated as \(\mathcal{O} = \sum_{i=1}^s s_i(a_i \circ b_i \circ c_i)\) where \(a_i \in \mathbb{R}^{f_1}\), \(b_i \in \mathbb{R}^{f_2}\) and \(c_i \in \mathbb{R}^{f_3}\). All the elements in \(a_i, s, b_i, c_i\) and \(s_i\) are sampled independently from Gaussian distribution \(N(0, 1)\). A total number of \(\|\Omega\|_1 = I_s I_3 \log(I_s)\) elements are uniform randomly observed from \(\mathcal{O} = \mathcal{O} + \mathcal{N}\), where \(\mathcal{N}\) is the noise sampled independently from \(N(0, 0.01)\). We use 50% of the observations for training, the remained 50% for validation, and the testing is performed on the remain unobserved elements in \(\mathcal{O}\).

Three nonconvex penalties as examples here, i.e., capped-\(\ell_1\) norm (Zhang 2010a), LSP (Candès, Wakin, and Boyd 2008) and TNN (Hu et al. 2013). Following (Lu et al. 2016; Yao et al. 2018; Yao et al. 2017), the recovery performance is measured by (i) root-mean-square-error (RMSE): \(\text{RMSE} = \|P_t(X - \mathcal{O})\|_F/\sqrt{\|\Omega\|_1}\), where \(\mathcal{X}\) is the low-rank tensor recovered from different algorithms, and \(\Omega\) denotes the unobserved elements in \(\mathcal{O}\); (ii) running CPU time.

We compare different variants of PA algorithm for the overlapped regularization here, as it is the only algorithm solving (12) with convergence guarantee. Thus, the following algorithms are compared (i). GDPAN (Zhong and Kwok 2014): plain PA algorithm for nonconvex problem (12); (ii). The proposed Algorithm 2 (denoted as FasTer), which utilizing acceleration, the “sparse plus low-rank” structure, and Proposition 3.3 in computing the proximal step; (iii). Variant of FasTer without acceleration (denoted as FasTer(na)); and (iv). Finally, PA-APG (Yu 2013): accelerated PA algorithm for the convex overlapped nuclear norm is used as a baseline.

Case (i): Small \(I_3\). We take \(I_1 = I_2 = 25c_1\), \(I_3 = 5\) and vary \(c_1 = \{50, 100\}\) here. Experiments are repeated five times. Recovery performance is in Table 1. We can see that PA-APG, which is based on the convex overlapped nuclear norm, has much higher testing RMSEs than, GDPAN, FasTer(na) and FasTer, which are based on the nonconvex regularization. Besides, three nonconvex penalties, i.e., capped-\(\ell_1\), LSP and TNN, have the similar empirical performance. This is also observed in (Lu et al. 2016; Yao et al. 2017; Yao et al. 2018).

The convergence of testing RMSE and objective value are in Figure 1. First, we can see that both PA-APG and GDPAN are very slow. Then, when measured by the number of iterations, FasTer(na) converges as fast as GDPAN. However, when measured by CPU time, FasTer(na) becomes much faster than GDPAN, as it utilizes the special “sparse plus low-rank” structure on computing the proximal steps. TNCR is even faster than FasTer(na) due to acceleration.

(a) objective v.s. iterations. (b) testing RMSE v.s. time.

Figure 1: Convergence of testing RMSE on synthetic data (Case (i)) and \(c = 1000\), and capped-\(\ell_1\) is used.

3The best and comparable performances (according to the pairwise t-test with 95% confidence) are highlighted.

We follow (Kolda and Bader 2009), and \(\circ\) denotes out products with \(a \circ b \circ c\).
Table 1: Testing RMSE (scaled by $10^{-2}$) and CPU time (in seconds) of various algorithms on synthetic data. The left part is for Case (i), and the right part is for Case (ii).

|                | $c_1 = 50$, nnz:5.64% | $c_2 = 100$, nnz:3.09% | $c_2 = 20$, nnz:4.77% | $c_2 = 40$, nnz:2.70% |
|----------------|------------------------|------------------------|------------------------|------------------------|
|                | RMSE (time (sec))      | RMSE (time (sec))      | RMSE (time (sec))      | RMSE (time (sec))      |
| convex         |                         |                        |                        |                        |
| PA-APG         | 1.03±0.01              | 257.2±34.8             | 1.49±0.11              | 2131.7±419.9           |
| nonconvex (capped-$\ell_1$) | 1.03±0.01              | 64.2±29.5              | 1.03±0.01              | 665.6±99.8             |
| nonconvex (LSP) | 1.03±0.01              | 7.1±4.5                | 1.03±0.01              | 27.9±3.1               |
| nonconvex (TNN) | 1.03±0.01              | 21.4±1.4               | 1.03±0.01              | 5.9±1.6                |
| FastTer(na)    | 1.03±0.01              | 69.2±26.4              | 1.04±0.01              | 654.1±214.7            |
| FastTer        | 1.04±0.01              | 6.6±1.6                | 1.03±0.01              | 6.5±0.7                |

Case (ii): Large $I_3$. Here we take $I_1 = I_2 = I_3 = 10c_2$, and vary $c_2 = \{20, 40\}$ here. Experiments are repeated five times. Results are in Table 1. Again, we can see capped-$\ell_1$ norm, LSP and TNN, have the same testing RMSE. GDPAN, FastTer(na) and FastTer, all have much lower testing RMSEs than PA-APG. Convergence of testing RMSE and objective value are in Figure 2. Again, FastTer(na) is much faster than GDPAN due to Section 3.2. Besides, FastTer is the fastest due to utilization of special structure and acceleration.

Table 2: Comparison of FastTer with different $D$ used in (12) on the synthetic data with case (i) and (ii). Testing RMSEs are scaled by $10^{-2}$ and CPU time is in seconds.

|                | Case (i), $c_1 = 100$ | Case (ii), $c_2 = 40$ |
|----------------|------------------------|------------------------|
|                | RMSE (time (sec))      | RMSE (time (sec))      |
| capped-$\ell_1$ |                         |                        |
|                | 1.03±0.01              | 95.9±3.9               |
|                | 1.04±0.01              | 916.8±500.9            |
| LSP            | 1.04±0.01              | 5.8±2.8                |
| TNN            | 1.03±0.01              | 899.7±404.4            |

Comparison of Case (i) and (ii). Here, we examine the difference of $D = 2$ and $3$ in (12) on above case (i) and (ii). Only FastTer is considered here due to its efficiency. The performance is compared in Table 2. For case (i), we can see that $D = 2$ and 3 can achieve the same testing RMSEs. However, $D = 3$ leads to much longer CPU time than $D = 2$. This due to $I_3$, small, the low-rank assumption cannot hold on the third mode and $k_3^2 < I_3$ does not hold. Thus, the proximal step for the third mode (i.e., $i = 3$ for step 11 in Algorithm 2) is much more expensive than these for the first two modes. For case (ii), CPU time, as $D = 3$ has one more low-rank regularization than the $D = 2$, it is slightly more expensive. However, on testing RMSE, the performance obtained from $D = 2$ is much worse than $D = 3$, which results from FastTer with $D = 2$ cannot capture the low-rank property on the third mode.

4.2 Real Data sets

In the sequel, based on above observations, we use $D = 2$ in (12) if $I_3$ is very small (i.e., $I_3 \leq 10$ empirically), otherwise we use $D = 3$; besides, as all three nonconvex penalty functions have similar performance, only LSP is considered. All experiments are repeated five times in the sequel.

Three types of algorithms are used for comparison here. (i) **Convex models**, including, ADMM (Tomioka, Hayashi, and Kashima 2010), FaLRTC (Liu et al. 2013), PA-APG (Yu 2013), FFW (Guo, Yao, and Kwok 2017) and TNN (Zhang and Aeron 2017); (ii). **Factorization approaches**, including, RP (Kasai and Mishra 2016), TMac (Xu et al. 2013), CP-WOPT (Acar et al. 2011) and TMac-TT (Bengua et al. 2017). (iii). **Finally, nonconvex regularization**, including GDPAN (Zhong and Kwok 2014) and the proposed FastTer (Algorithm 2). FastTer(na) in Section 4.1 is not used as it is a slower variant of FastTer. Due to lack of space, we put a more complete comparison of above algorithms in Appendix ??.

**Color Image.** We use the images, i.e., **windows**, tree and rice, from (Guo, Yao, and Kwok 2017) and (Hu et al. 2013). We resize size of all images to $1000 \times 1000 \times 3$ if here in order to make a decomposition for TMac-TT algorithm. Images are show in Appendix ???. We normalize each pixels into range $[0, 1]$. We randomly sample $10\%$ of all the pixels for training, and add Gaussian noise $N(0, 0.01)$ to them. Half of the ratings for training are used for validation. The rest unseen clean pixels are used for testing. Performance of various algorithms are measured by RMSE on the testing pixels and the CPU time.

Testing RMSEs are in Table 3. We can see that GDPAN and FastTer achieves similar testing RMSE, and are much lower than both convex regularization (ADMM, FaLRTC, PA-APG, FFW and TNN) and factorization approach (RP, TMac, CP-OPT, TMac-TT). This again verifies that nonconvex regularization offers better empirical performance than the other approaches.

Figure 2: Convergence of testing RMSE on synthetic data (Case (ii)) with $c = 400$, and capped-$\ell_1$ is used.
Table 4: Testing RMSEs of algorithms on color images.

|               | windows | tree | rice |
|---------------|---------|------|------|
| convex        | ADMM    | 0.661±0.006 | 0.615±0.005 | 0.526±0.003 |
|               | FaLRTC  | 0.559±0.014 | 0.576±0.004 | 0.489±0.006 |
|               | PA-APG  | 0.361±0.007 | 0.395±0.021 | 0.483±0.009 |
|               | TNN     | 0.779±0.006 | 0.629±0.005 | 0.598±0.004 |
| factorization | RP      | 0.450±0.021 | 0.518±0.010 | 0.568±0.028 |
|               | TMac    | 1.512±0.005 | 1.922±0.003 | 1.748±0.005 |
|               | CP-OPT  | 0.911±0.122 | 0.823±0.092 | 0.704±0.057 |
|               | TMac-TT | 1.118±0.104 | 0.740±0.055 | 0.682±0.086 |
| ncvx          | GDPAN   | 0.303±0.004 | 0.467±0.002 | 0.505±0.005 |
|               | FasTer  | 0.504±0.002 | 0.467±0.002 | 0.590±0.002 |

Table 3: Testing RMSEs of algorithms on color images.

|        | convex | FaLRTC | PA-APG | TNN | RP | TMac | CP-OPT | TMac-TT | FacTer |
|--------|--------|--------|--------|-----|----|------|--------|---------|--------|
| windows| 10.50±0.12 | 23.08±0.24 | 23.69±0.24 | 11.57±0.03 | 9.62±0.04 | 18.89±0.08 | 20.96±0.06 | 4.67±0.11 | 3.79±0.12 |
| tree   | 23.58±0.24 | 11.33±0.04 | 9.54±0.04  | 6.92±0.04  | 5.67±0.04  | 3.29±0.04  | 4.06±0.04  | 2.61±0.04  | 2.57±0.04  |
| rice   | 23.69±0.24 | 11.33±0.04 | 9.54±0.04  | 6.92±0.04  | 5.67±0.04  | 3.29±0.04  | 4.06±0.04  | 2.61±0.04  | 2.57±0.04  |

Hyper-Spectrum Image. Experiments are performed on three hyper-spectral images, i.e., Cabbage (1312×432×49), Scene (1312×951×49) and Female (592×409×148). Their details are in Appendix ??). The third dimension denotes the bands of images. We use the same setup as for the color image. ADMM, TNN, GDPAN are not compared as they are too slow; TMac-TT is also not compared as its initialization cannot be done in 12 hours (Appendix ??).

Performance is shown in Table ??). Again, we can see that, as nonconvergent regularizations can less penalize large singular values, FasTer can achieve much lower testing RMSE than both convex and factorization based approaches. Testing RMSE v.s. CPU time is shown in Figure ??). The lowest RMSE is offered by FasTer, and it is much faster than all other approaches except for TMac and FFW. While TMac and FFW are very fast, their testing RMSEs are much higher than that of FasTer. Thus, FasTer is very efficient and offers good empirical performance.

Table 4: Testing RMSEs (scaled by 10^{-3}) of various algorithms on hyperspectral images.

|        | Cabbage | Scene | Female |
|--------|---------|-------|--------|
| convex | FaLRTC  | 10.50±0.12 | 23.08±0.24 | 11.33±0.04 |
|        | PA-APG  | 9.04±0.04  | 23.69±0.24 | 11.57±0.03 |
|        | FFW     | 9.62±0.04  | 18.89±0.08 | 20.96±0.06 |
| factorization | RP    | 4.67±0.11  | 19.61±0.05 | 6.47±0.03  |
|        | TMac    | 48.73±0.59 | 60.06±0.06 | 198.9±0.06 |
|        | CP-OPT  | 24.17±4.24 | 49.54±3.01 | 19.75±1.44 |
| ncvx   | FasTer  | 3.79±0.12  | 16.40±0.01 | 5.93±0.14  |

Social Networks. Finally, we perform experiments on the YouTube data set (Lei, Wang, and Liu 2009). It contains 15,088 users, and describes five types of user interactions. Thus, it forms a 15088 × 15088 × 5 tensor, with a total of 27,257,790 nonzero elements. Following (Guo, Yao, and Kwok 2017), we formulate multi-relational link prediction as a tensor completion problem. Here, we perform experiments on both a small YouTube subset, which is obtained by random selecting 1000 users (leading to 12,101 observations), and the full YouTube dataset. We use 50% of the observations for training, another 25% for validation and the remaining for testing.

Experiments are repeated five times. Testing RMSEs are in Table ??) and testing RMSE v.s. CPU time is in Figure ??). In these datasets, we can see FasTer not only achieves lower testing RMSEs but also runs much faster than other algorithms. This again demonstrates that FasTer is very efficient and offers good empirical performance.

Table 5: Testing RMSEs on Youtube data set. FaLRTC, PA-APG and CP-OPT are too slow, thus are not run on full set.

|        | convex | FaLRTC | PA-APG | TNN | RP | TMac | CP-OPT | TMac-TT | FacTer |
|--------|--------|--------|--------|-----|----|------|--------|---------|--------|
| subset | 0.627±0.060 | 0.593±0.047 | 0.395±0.001 | 0.554±0.026 | 0.380±0.035 | 0.611±0.007 | 0.667±0.016 | 0.488±0.017 | 0.370±0.001 |
| full set | — | — | — | — | — | — | — | — | — |

5 Conclusion

In this paper, we propose a new low-rank tensor completion model with nonconvex regularizations. While the nonconvex proximal average (PA) algorithm is the only applicable algorithm, it has large space demand and expensive iteration complexity. To address these problems, we show how proximal steps in PA algorithm can be computed efficiently computed and how its slow convergence can be sped up with acceleration. The convergence to critical points is of the new algorithm still guaranteed. Finally, experimental comparison of the proposed approach with various other tensor completion approaches demonstrates...
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