An index theorem for higher orbital integrals. (English) Zbl 1492.19007 Math. Ann. 382, No. 1-2, 169-202 (2022).

If $G$ is a locally compact group acting properly, isometrically and cocompactly on a Riemannian manifold $X$, and $D$ is a $G$-equivariant elliptic differential operator on $X$, then $D$ has an equivariant index $\text{ind}_G(D)$ in the $K$-theory of the reduced $C^*$-algebra of $G$. To study this index, which plays a fundamental role in various areas in geometry, topology, representation theory and more, it is useful to calculate pairings of $\text{ind}_G(D)$ with various cyclic cocycles in order to extract numerical invariants. The main result of the paper at hand is an index formula for such a pairing, in the case when $G$ is a real linear reductive Lie group. The cocycles involved in the pairing are generalized orbital integrals, introduced by two of the authors in [Y. Song and X. Tang, “Higher orbit integrals, cyclic cocycles, and $K$-theory of reduced group $C^*$-algebra”, Preprint, arXiv:1910.00175]. Defined on the Harish-Chandra Schwartz algebra of $G$, these cocycles have the property that if all of their pairings with a given class in $K^*(C^*_r(G))$ vanish, then the class must be zero, making the expression for a general pairing given in the paper particularly meaningful from a topological standpoint. The main theorem is stated for Spin$^c$-Dirac operators. The authors also indicate how to extend it to arbitrary elliptic operators, along with other applications, including a generalization of Connes-Moscovici’s $L^2$-index theorem and a new proof of injectivity of Dirac induction.

Reviewer: Pierre Clare (Williamsburg)

MSC:

19K56 Index theory
58J20 Index theory and related fixed-point theorems on manifolds
22E46 Semisimple Lie groups and their representations
46L80 $K$-theory and operator algebras (including cyclic theory)

Keywords:
index theorems; orbital integrals; elliptic operators; proper actions

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