Relativistic Heavy Quark Effective Action

Norman H. Christ, Min Li and Huey-Wen Lin

1Physics Department, Columbia University, New York, NY 10027

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We study the fermion action needed to accurately describe the low energy physics of systems including heavy quarks in lattice QCD even when the heavy fermion mass \( m \) is on the order of, or larger than, the inverse lattice spacing: \( m \geq 1/a \). We carry out an expansion through first order in \( |\vec{p}|a \) (where \( \vec{p} \) is the heavy quark momentum) and all orders in \( ma \), refining the analysis of the Fermilab and Tsukuba groups. We demonstrate that the spectrum of heavy quark bound states can be determined accurately through \( |\vec{p}|a \) and \((ma)^n\) for arbitrary exponent \( n \) by using a lattice action containing only three unknown coefficients: \( m_0 \), \( \zeta \) and \( c_P \) (a generalization of \( c_{SW} \)), which are functions of \( ma \). In a companion paper, we show how these three coefficients can be precisely determined using non-perturbative techniques.

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The mass spectrum and decay properties of hadrons containing charm and bottom quarks provide some of the most precise information about the masses and weak-interaction couplings of the underlying quarks. In many cases, the methods of lattice QCD potentially allow the most accurate connection between these observed hadronic properties and the important, underlying Standard Model parameters. However, the relatively large masses of the charm and bottom quarks make it difficult to perform conventional lattice calculations since practical limitations often do not permit the use of a sufficiently small lattice spacing to properly control discretization errors of $O(ma)^{n}$.

Fortunately, in many cases those phenomena which occur at energies on the order of the heavy quark mass are “irrelevant” to the masses and matrix elements of interest, contributing only renormalization effects which can be accounted for by the proper choice of a small number of parameters. In the case of a lattice QCD calculation, this may mean that even the distortions implied by $ma \geq 1$ can be completely compensated by such a choice of a few parameters.

This philosophy underlies the static and non-relativistic approaches to lattice calculations of the properties of hadrons containing heavy quarks. In this paper we will study a more general approach to the treatment of heavy quarks introduced by the Fermilab group\[1\] and refined and studied in detail by the group at Tsukuba\[2\]. (For a recent review of these lattice QCD approaches to heavy quarks see Ref.\[3\].)

In the Fermilab approach, one studies the properties of hadrons containing heavy quarks in their rest system and argues that the spatial momenta $\vec{p}$ carried by the heavy quark(s) will be significantly smaller than the heavy quark mass: $|\vec{p}| \approx \Lambda_{QCD}$ for heavy-light systems and $|\vec{p}| \approx \alpha_s m$ for heavy-heavy system. Here $\alpha_s$ is the strong coupling constant evaluated at the energy scale appropriate for the heavy-heavy bound state. Because of the potentially large heavy quark mass, the temporal momentum may be very large and terms of all orders in $ma$ must be properly included. Thus, this approximation scheme naturally treats time and space differently, breaking the axis interchange symmetry of the usual lattice formulation.

As will be reviewed below, accurate masses for such states containing heavy quarks can be obtained by using a variant of the usual Wilson action in which axis interchange symmetry is broken and particular bare lattice parameters are chosen to be specific functions of $ma$. In contrast to previous work, we demonstrate that all errors of order $|\vec{p}| a$ and all orders in $ma$ may be removed by the choice of only three parameters in the lattice action, the
ratio $\zeta$ of the coefficients of the spatial and temporal derivatives, the coefficient $c_P$ of the axis-interchange-symmetric Pauli term and the bare fermion mass $m_0$.

If other on-shell quantities such as matrix elements are to be computed accurately to this order, one must also explicitly add $ma$-dependent improvement terms to the operator whose matrix element is being evaluated and to any interpolating field being used to create or destroy fermion states. For spin-1/2 states such improvement of the interpolating field will involve both an overall normalization factor $Z_q$ and a spinor transformation containing a further axis-exchange-asymmetric term with an additional mass-dependent coefficient.

Thus, apart from additional improvement coefficients needed to evaluate specific on-shell matrix elements, only three parameters need to be determined to carry out such $O(|\vec{p}|a)$, $O(ma)^n$ lattice calculations. As is demonstrated in the companion paper [4], these three parameters can be accurately determined from a finite-volume step-scaling procedure, suggesting that this approach to the lattice calculation of the properties of hadrons containing heavy fermions can be done with no reliance on lattice perturbation theory and good control of all systematic errors.

In the next section, Sec. II we describe this approach to heavy quark physics in greater detail, specifying the lattice and effective continuum actions and the field transformations that can be used to simplify the latter. Then in Sec. II we discuss a simplified example showing why only three parameters need to be tuned in the lattice action to achieve an accurate continuum result. Section III makes an explicit comparison with the results of the Fermilab and Tsukuba groups and analyzes the disparity between the number of parameters introduced in those treatments (four and five respectively).

Section IV contains a complete inductive proof, that finite lattice spacing errors of order $O(|\vec{p}|a)$ and $O(ma)^n$ can be removed by the proper choice of the three lattice parameters $m_0$, $\zeta$ and $c_P$. In Section V we examine the physical heavy quark mass, on-shell quark propagator and quark-gluon vertex at tree level to demonstrate explicitly that $O(|\vec{p}|a)$ and $O(ma)^n$ accuracy requires the choice of only three parameters and the use of an improved quark interpolating field. Some conclusions are presented in Sec. VI.
I. GENERAL DISCUSSION

Our objective is to describe hadrons containing heavy quarks using a lattice action which includes a number of improvement terms chosen to remove all finite lattice spacing errors to a given order in $|\vec{p}|a$ and all orders in $ma$. Further, as discussed earlier, we expect that a different treatment will be required for the spatial and temporal momenta of the heavy quark, implying a lattice action which is axis-exchange asymmetric. As we will demonstrate, the desired $O(|\vec{p}|a)$ and $O(ma)^n$ accuracy can be achieved if we begin with a lattice fermion action of the form

$$S_{\text{lat}} = \sum_{n',n} \bar{\psi}_{n'} \left( \gamma^0 D^0 + \zeta \vec{\gamma} \cdot \vec{D} + m_0 - \frac{r_t}{2} (D^0)^2 - \frac{r_s}{2} \vec{D}^2 \right)$$

$$+ \sum_{i,j} \frac{i}{4} c_B \sigma_{ij} F_{ij} + \sum_i \frac{i}{2} c_E \sigma_0 F_{0i} \right) \psi_n$$  \hspace{1cm} (1)

and a simple choice of the bare lattice parameters: $r_s = r_t = 1$, $c_E = c_B$. Here $\psi_n$ is the heavy quark field at the site $n$, $U_\mu(n)$ is the $SU(3)$ matrix providing gauge parallel transport from the site $n + \mu$ to the site $n$ and

$$\left( D_\mu \psi \right)_n = \frac{1}{2} \left[ U_\mu(n) \psi_{n+\hat{\mu}} - U_\mu(n-\hat{\mu}) \psi_{n-\hat{\mu}} \right]$$

$$\left( D^2_\mu \psi \right)_n = \left[ U_\mu(n) \psi_{n+\hat{\mu}} + U_\mu(n-\hat{\mu}) \psi_{n-\hat{\mu}} - 2 \psi_n \right]$$

$$\left( F_{\mu\nu} \psi \right)_n = \frac{1}{8} \sum_{s,s' = \pm 1} s s' \left[ U_{s\mu}(n) U_{s'\nu}(n + s \hat{\mu}) \right.$$

$$\times \left. U_{-s\mu}(n + s \hat{\mu} + s' \hat{\nu}) U_{-s'\nu}(n + s' \hat{\nu}) - \text{h.c.} \right] \psi_n.$$  \hspace{1cm} (4)

We are using Hermitian gamma matrices $\gamma_\mu$ obeying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and have defined the Yang-Mills field strength tensor $F_{\mu\nu}$ to be an anti-Hermitian color matrix.

A. Continuum effective action

In the limit that the lattice spacing becomes small we can analyze the resulting theory and enumerate the largest lattice spacing errors by constructing the Symanzik effective action describing a continuum theory which approximates the lattice theory, including the discretization errors through a given order. Including terms representing errors of order $a$
This effective action can be written:

\[
S_{\text{eff}} = \int d^4x \overline{\psi}(x) \left( \gamma^0 D^0 + \zeta \bar{\gamma} \cdot \vec{D} + m_r - a \frac{r_c^2}{2} (D^0)^2 - a \frac{r_c^2}{2} \vec{D}^2 \right) + \sum_{ij} \frac{i}{4} \epsilon^{\mu}_{\nu} \alpha \sigma_{ij} F_{ij} + \sum_{i} \frac{i}{2} \epsilon^{\mu}_{\nu} \alpha \sigma_{i0} F_{i0} + \sum_{i} \frac{1}{8} \delta^{\mu}_{\nu} \alpha \sigma_{i0} \{D^i, D^0\}. \right) \psi(x). \tag{5}
\]

The superscript label \(c\) representing “continuum” has been added to the parameters appearing in this effective continuum action to distinguish them from the similar parameters which enter the lattice action of Eq. [1]. The corresponding continuum mass has been written \(m_r\).

Here we are anticipating a choice of lattice parameters which violate axis-interchange symmetry and have therefore introduced all possible dimension 3, 4 and 5 terms which obey only the requirement of rotational symmetry. In Eq. [5] \(\psi(x)\) and \(\overline{\psi}(x)\) are the usual continuum fermion fields with normalization chosen to make the coefficient of the \(\gamma^0 D^0\) equal to 1. The derivatives \(\vec{D}\) and \(D^0\) are the usual gauge-covariant continuum derivatives and \(F_{\mu, \nu} = [D_{\mu}, D_{\nu}]\) is the Yang-Mills field strength tensor.

Since we are interested in treating the case where terms of the form \((m_q a)^n\) or \((D^0 a)^n\) may be large, we will generalize Eq. [5] to include correction terms containing arbitrary powers of these two quantities but, unless accompanied by such a factor of the heavy quark energy or mass, we neglect all other terms of order \(a^2\) or higher. The resulting general heavy quark Symanzik effective action might be written:

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff},-1} + \mathcal{L}_{\text{eff},0} + \mathcal{L}_{\text{eff},1} + \ldots. \tag{6}
\]

where

\[
\mathcal{L}_{\text{eff},-1} = \overline{\psi} \left( \frac{1}{a} B^{-1,1} + \gamma^0 D^0 C^{-1,1} \right) \psi \tag{7}
\]

\[
\mathcal{L}_{\text{eff},0} = \overline{\psi} \left( \{\gamma \vec{D}, B^{0,1}\} + a\{[\gamma \vec{D}, \gamma^0 D^0], C^{0,1}\}\right) \psi \tag{8}
\]

\[
\mathcal{L}_{\text{eff},1} = a \overline{\psi} \left( \vec{D}^2 B^{1,1} + a\{\vec{D}^2, \gamma^0 D^0\} C^{1,1} \right. \\
\left. + [\gamma^i, \gamma^j] [D^i, D^j] B^{1,2} + a\{[\gamma^i, \gamma^j] [D^i, D^j], \gamma^0 D^0\} C^{1,2} \right) \\
\left. + [\gamma^i, \gamma^0] [D^i, D^0] B^{1,3} + a\{[\gamma^i, \gamma^0] [D^i, D^0], \gamma^0 D^0\} C^{1,3} \right) \psi. \tag{9}
\]

Here the coefficient functions \(B^{i,j}\) and \(C^{i,j}\) are actually polynomials of arbitrary order in the
product $m_0 a$, the operator $(a D^0)^2$ and the gauge coupling $g^2$:

$$B^{i,j} = \sum_{k,l,n} b_{k,l,n}^{i,j} (m_0 a)^k \left((a D^0)^2\right)^l g^{2n}$$

$$C^{i,j} = \sum_{k,l,n} c_{k,l,n}^{i,j} (m_0 a)^k \left((a D^0)^2\right)^l g^{2n}.$$  \hspace{1cm} (10)

Because we will work to arbitrary order in $m_0 a$ and $a D^0$ it is natural to adopt an expansion in lattice spacing $a$ where we count only powers of $a$ which are not compensated by added powers of $m_0$ or $D^0$. We will refer to such an expansion as “relativistic heavy quark” or RHQ power counting. The subscripts appearing on the three terms in Eq. 6 refer to such a scheme.

**B. Discrete symmetries**

The coefficients in the Symanzik effective action appearing in Eqs. 5 and 9 can be constrained if, as is conventional, we work with an underlying lattice action which obeys various discrete symmetries and reality conditions. The simplest are the four symmetries corresponding to the change in sign of one of four Euclidean coordinates: $x_\mu \rightarrow (-1)^{\delta_{\nu \mu}} x_\mu$ where $\nu$ is the direction being inverted. In lattice coordinates we replace the fields at the site with coordinates $n_\mu$ with those at the site $n_\mu^{P(\nu)}$ where $n_\mu^{P(\nu)} = L_\mu - 1 - n_\mu$ for $\mu = \nu$ and $n_\mu^{P(\nu)} = n_\mu$ otherwise. Here we are assuming a general space-time volume of size $L_0 \times L_1 \times L_2 \times L_3$ with $0 \leq n_\mu < L_\mu$ and $L_\mu$ even. Our lattice fields transform as:

$$\psi_n \rightarrow \gamma^\nu \gamma^5 \psi_{n}^{P(\nu)}$$ \hspace{1cm} (11)

$$\bar{\psi}_n \rightarrow \bar{\psi}_n^{P(\nu)} \gamma^5 \gamma^\nu$$ \hspace{1cm} (12)

$$U_\nu(n) \rightarrow U_\nu^{\dagger}(n^{P(\nu)} - \hat{\epsilon}_\nu (1 - L \delta_{n_\nu,L_\nu-1}))$$ \hspace{1cm} (13)

$$U_\mu(n) \rightarrow U_\mu(n^{P(\nu)}) \text{ for } \mu \neq \nu.$$ \hspace{1cm} (14)

Here $\hat{\epsilon}_\nu$ is a vector extending one site in the $\nu$ direction.

For $m a \geq 1$ the mass shell condition $p_0 = +m$ will imply that the negative energy, anti-quark states are far outside the domain of validity of our approximation. Thus, it is important that the improved lattice action obey charge-conjugation symmetry so that both heavy quarks and heavy anti-quarks will be treated with the same accuracy. This can be
accomplished if we require that our improved lattice action and therefore the continuum Symanzik action are symmetric under the following change of variables:

\[ \psi_n \rightarrow C\overline{\psi}_n \]  

(15)

\[ \overline{\psi}_n \rightarrow -\psi_n^t C^{-1} \]  

(16)

\[ U_\mu(n) \rightarrow U_\mu(n)^* \]  

(17)

where the Dirac charge conjugation matrix \( C \) obeys

\[ C^{-1}\gamma_\mu C = -\gamma_\mu. \]  

(18)

Here we are treating the Grassmann variables \( \psi \) and \( \overline{\psi} \) as \( 4 \times 1 \) and \( 1 \times 4 \) spinor matrices respectively which requires the appearance of the transpose operation in Eqs. 15 and 16 indicated by the superscript \( t \).

The lattice action given in Eq. 1 already obeys the above axis reversal symmetry given our requirement that only even powers of the operator \( aD_0 \) appear. All of the terms in Eq. 1 are also charge conjugation even except for the terms containing the functions \( C^{0,1} \) and \( C^{1,3} \). These are odd under \( C \) and can be set to zero.

Finally we should determine the phases of the coefficients appearing in the effective action of Eq. 6. We begin with the lattice action given in Eq. 1. Here we have introduced factors of \( i \) in such a way that this action will yield a Hermitian, but possibly not positive, transfer matrix if the bare, lattice parameters \( m_0, \zeta, r_s, r_t, c_B \) and \( c_E \) are all chosen real.

The phases of the parameters appearing in the continuum effective action of Eq. 6 can then be easily constrained if we recognize that when the above 6 bare lattice parameters are real, the underlying lattice action obeys a simple symmetry under complex conjugation. Specifically, we consider the fermion path integral in a fixed gauge background:

\[ Z[\eta, \overline{\eta}] = \int d[\psi]d[\overline{\psi}] \exp \left\{ S[\psi, \overline{\psi}]_{\text{lat}} + \int d^4x \{ \overline{\psi}(x)\eta(x) + \overline{\eta}(x)\psi(x) \} \right\}, \]  

(19)

where we have introduced explicit sources \( \eta \) and \( \overline{\eta} \) so that arbitrary Green’s functions can be determined. The integral in Eq. 19 will evaluate to a polynomial in the Grassmann variables \( \eta \) and \( \overline{\eta} \) with complex coefficients. If we define \( Z[\eta, \overline{\eta}]^* \) as that same polynomial but with the coefficients replaced by their complex conjugates, then one can easily show by a standard change of variables in the path integral in Eq. 19

\[ \psi \rightarrow \gamma^5\overline{\psi} \quad \overline{\psi} \rightarrow -\psi^t \gamma^5, \]  

(20)
that when \( m_0, \zeta, r_s, r_t, c_B \) and \( c_E \) are real the following relation is obeyed:

\[
Z[\eta, \bar{\eta}]^* = Z[\gamma^5\bar{\eta}, -\eta^i \gamma^5].
\]  

(21)

Since the continuum effective action is determined directly from the lattice action, it also must obey this reality condition. This requires that each of the functions \( B^{i,j} \) and \( C^{i,j} \) appearing in Eq. 5 be polynomials in the three quantities \( m_0, (aD^0)^2 \) and \( g^2 \) with real coefficients.

C. Field transformations

As is well known, many of the unwanted terms in Eq. 6 have no effect on physical states or fermion Green’s functions evaluated on the mass shell and can be removed by a redefinition of the fermion fields \( \psi \) and \( \bar{\psi} \). We will therefore make a series of such transformations chosen to remove many of the terms that appear in the Symanzik effective action of Eq. 6. The coefficients of those terms that remain after these transformations are then presumed to be potentially important lattice artifacts that must be eliminated by an explicit choice of additional improvement terms in the underlying lattice action.

The removal of these redundant terms is most easily analyzed in a series of steps exploiting the ordering of the terms in Eq. 6 \( O(1/a), O(a^0), O(a), \) etc. in the RHQ expansion. The largest field transformation introduces terms of order \( a^0 \) in this RHQ expansion and can be written:

\[
\psi = (1 + R^{0,1} + a\gamma^0 D^0 S^{0,1})\psi'
\]

(22)

\[
\bar{\psi} = \bar{\psi}'(1 + \overline{R}^{0,1} - a\gamma^0 \overline{D}^0 \overline{S}^{0,1}),
\]

(23)

where \( R^{0,1}, S^{0,1}, \overline{R}^{0,1} \) and \( \overline{S}^{0,1} \) are arbitrary polynomials in \( m_0 a, (aD^0)^2 \) and \( g^2 \). We adopt the convention in the transformation equations above and the four to follow, that the \( aD^0 \) argument will always act to the right in the equations for \( \psi \) and to the left in the equations for \( \bar{\psi} \). (Note that as the covariant derivative, the operator \( D_\mu \) will have a different form when acting on \( \bar{\psi} \), a color vector whose gauge transformation properties are the hermitian conjugate of those of \( \psi \), see Appendix A, Eqs. A4 and A5.) This transformation will effect all three terms shown in Eq. 6 \( O(a^{-1}), O(a^0) \) and \( O(a^1) \) and will generate extra terms that can be used to simplify the resulting action.
Relevant to the order in $a$ to which we are working are two further transformations. The first transformation introduces terms of order $a^1$ in $\psi$ and $\overline{\psi}$ and takes the form:

$$
\psi = (1 + a\gamma^i\overline{D}R^{1,1} + a[\gamma^i\overline{D}, a\gamma^0D^0]S^{1,1})\psi' \tag{24}
$$

$$
\overline{\psi} = \overline{\psi'} (1 - a\overline{R}^{1,1}\gamma^i\overline{D} - a\overline{S}^{1,1}[\gamma^i\overline{D}, a\gamma^0\overline{D}^0]). \tag{25}
$$

This transformation will act on the $L_{\text{eff}, -1}$ and $L_{\text{eff}, 0}$ terms in Eq. 6 and produce terms of order $a^0$ and $a^1$ in the transformed action.

Finally, we must discuss a third transformation which is of order $a^2$:

$$
\psi = \left(1 + a^2\gamma^i\overline{D}^2R^{2,1} + a^2\{\gamma^i\overline{D}^2, a\gamma^0D^0\}S^{2,1}
+ a^2[\gamma^i, \gamma^j][D^i, D^j]R^{2,2} + a^2\left\{[\gamma^i, \gamma^j][D^i, D^j], a\gamma^0D^0\right\}S^{2,2}
+ a^2[\gamma^i, \gamma^0][D^i, D^0]R^{2,3} + a^2\left[[\gamma^i, \gamma^0][D^i, D^0], a\gamma^0D^0\right\}S^{2,3}\right)\psi' \tag{26}
$$

$$
\overline{\psi} = \overline{\psi'} \left(1 + a^2\gamma^i\overline{D}^2R^{2,1} - a^2\{\gamma^i\overline{D}^2, a\gamma^0\overline{D}^0\}S^{2,1}
+ a^2[\gamma^i, \gamma^j][\overline{D}^i, \overline{D}^j]\overline{R}^{2,2} - a^2\left\{[\gamma^i, \gamma^j][\overline{D}^i, \overline{D}^j], a\gamma^0\overline{D}^0\right\}S^{2,2}
+ a^2[\gamma^i, \gamma^0][\overline{D}^i, \overline{D}^0]\overline{R}^{2,3} - a^2\left[[\gamma^i, \gamma^0][\overline{D}^i, \overline{D}^0], a\gamma^0\overline{D}^0\right\}S^{2,3}\right). \tag{27}
$$

This order $a^2$ transformation was not investigated in Ref. [2] nor in Section III on redundant couplings in Ref. [1] although later in that paper this transformation is discussed, see Eq. 5.23.

This transformation acts on only the $L_{\text{eff}, -1}$ term in Eq. 6 to produce terms of order $a^1$ in the transformed action. The effects of these transformations will be considered below, first in a simplified context in Sec. II and then in generality in Sec. IV. Here we will specialize these three transformations to preserve the charge conjugation symmetry and reality properties discussed above.

In fact, with the choice of signs in Eqs. 22-27 charge conjugation requires, $R^{i,j} = \overline{R}^{i,j}$ and $S^{i,j} = \overline{S}^{i,j}$ while preservation of the form of the reality condition requires that all coefficients in the polynomials $R^{i,j}$ and $S^{i,j}$ be real.

This completes our general discussion of the lattice action, the resulting effective continuum action and the field transformations that can be applied to that effective action consistent with our charge conjugation and reality conditions.
II. SIMPLIFIED EXAMPLE

In Sec. IV we use induction to apply the field transformations discussed in the previous section to systematically eliminate all terms from the effective action of Eq. 6 except for three, mass-dependent coefficients. These field transformations will leave the effective action in the form given in Eq. 5 with only the three coefficients $m_r$, $\zeta$, and $c^c_{P} \equiv c^c_{B} = c^c_{E}$ non-zero functions of the quark mass times the lattice spacing, $ma$. However, in this section we will present this argument in a simplified case which should make the conclusion and the essential ingredients needed to reach it easier to understand.

We will consider the case that the effective continuum action is determined by the Lagrangian given in Eq. 5 which can be written:

$$S_{\text{eff}} = \sum_n \overline{\psi}(x) \left( \gamma^0 D^0 + \zeta \gamma^i \vec{D} + m_r - a \frac{r^c}{2} (D^0)^2 - a \frac{r^e}{2} \vec{D}^2 \right) \psi(x).$$

Here we are simplifying the general problem by dropping potentially large time derivative terms, $(aD^0)^{2n}$, beyond those appearing explicitly in Eq. 28. We have also omitted the final term proportional to $\delta$ in Eq. 5 since it violates charge conjugation symmetry.

Through a combination of tuning the bare lattice parameters and redefinition of the fields $\psi$ and $\overline{\psi}$ we will be able to put the Lagrangian above into the standard continuum form:

$$L_{\text{eff}} = \overline{\psi} \left\{ \gamma^0 D^0 + \gamma^i D^i + m_r \right\} \psi'.$$

As is conventional, we will work backward from Eq. 29 performing transformations on the fields $\psi'$ and $\overline{\psi}'$ in an attempt to generate as many as possible of the terms appearing in Eq. 28. We can then be guaranteed that if the other terms, not created by these transformations, are set to zero by tuning an improved lattice action, these remaining terms can then be eliminated by a field transformation.

Let us now extend the usual transformations $\psi' \rightarrow \psi$ and $\overline{\psi}' \rightarrow \overline{\psi}$ to demonstrate the redundancy of all but the three parameters listed above: $m_0 a$, $\zeta$, $c_P$. As in the more complete discussion of Sec. IV we will organize this discussion using RHQ power counting where the quantities $m$ and $D^0$ are treated as order $a^{-1}$ instead of $a^0$.

We begin by making transformations of $O(a^0)$ in the RHQ power counting sense and
$O(a^1)$ in the usual sense:

$$\psi' = (1 + a\gamma^0 D^0 S^{0,1})\psi$$  \hspace{1cm} (30)

$$\overline{\psi}' = \overline{\psi}(1 - a\gamma^0 \overline{D}^0 S^{0,1}),$$  \hspace{1cm} (31)

where the function $S^{0,1}$ is real and Eqs. 30 and Eqs. 31 are related by charge conjugation symmetry. We have adopted a somewhat cumbersome notation that will be useful later: the first integer in the superscript of $S^{i,j}$ identifies the RHQ power counting order of the transformation and the second enumerates the different terms of that order. This transformation generates two terms when acting on the action of Eq. 29:

$$\overline{\psi}\left\{2m_r a\gamma^0 D^0 S^{0,1} + 2S^{0,1} a(D^0)^2\right\}\psi.$$  \hspace{1cm} (32)

As is customary, we neglect terms quadratic in $S^{0,1}$ treating these terms as small. (This issue will be dealt with in a more systematic way in Sec. IV.) Since the quantity in Eq. 32 is generated by a change of Grassmann variables in the path integral, we can treat such a combination of terms as zero, were it to appear in the effective Lagrangian of our improved lattice theory. Of course, since by construction the expression in Eq. 32 is linear in the Dirac operator appearing in the final action, one can also describe the vanishing of these terms as a consequence of the equations of motion. These two styles of derivation are really one and the same.

The vanishing of the combination of terms in Eq. 32 implies we can adjust the function $S^{0,1}$ to set $r^c_t$ to zero. (Note, this gives us the freedom in the improved lattice Lagrangian to choose the conventional value of 1 for the bare version of $r_t$.) The only effect on the resulting action will be that of the first term, $2m_r a S^{0,1} \overline{\psi}\gamma^0 D^0 \psi$, which can be removed by a rescaling of $\psi$ and $\overline{\psi}$.

Next consider transformations of order $O(a^1)$ in the RHQ power counting sense and also $O(a^1)$ in the usual sense:

$$\psi' = (1 + a\gamma^0 \overline{D} R^{1,1})\psi$$  \hspace{1cm} (33)

$$\overline{\psi}' = \overline{\psi}(1 - a\gamma^0 \overline{D} R^{1,1}).$$  \hspace{1cm} (34)

Acting on the continuum Lagrangian in Eq. 29 These transformations will produce the terms

$$\overline{\psi}\left(2m_r a\gamma^0 \overline{D} + \frac{1}{2} a[\gamma^i, \gamma^0][D^i, D^0] + 2a(D^i)^2\right)R^{1,1}\psi.$$  \hspace{1cm} (35)
Hence with a proper choice for $R_{1,1}$ we can use the $a(D^i)^2$ in Eq. 35 to set $r_s^c = 0$ for any choice of $r_s$ in the bare lattice Lagrangian (including our conventional value $r_s = \zeta$).

Thus, using the set of two transformation considered so far we have been able to argue that an effective Lagrangian with any set of values of $r_s$ and $r_t$ can be transformed to the proper continuum form. This is the standard argument reducing the number of relevant parameters from six to four. However, there is one further transformation which is of $O(a^2)$ in the sense of both RHQ and conventional power counting that can remove one more parameter:

$$
\psi' = (1 + a^2[\gamma^i, \gamma^0][D^i, D^0]R^{2,3})\psi
$$

$$
\bar{\psi}' = \bar{\psi}(1 + a^2[\gamma^i, \gamma^0]D^i, D^0)R^{2,3}),
$$

where we using the label $R^{2,3}$ to maintain consistency with Eqs. 26 and 27. This transformation, when acting on the two $O(1/a)$ terms in the continuum action, will produce the following combination of terms of $O(a)$ according to RHQ power counting:

$$
a^2\bar{\psi}\left(2m_r[\gamma^i, \gamma^0][D^i, D^0] + \gamma^i\left[[D^i, D^0], D^0]\right)R^{2,3}\psi.
$$

As before, we can treat this combination of terms as vanishing either because they were generated by a transformation of path integration variables or as a result of the equations of motion since it was obtained as a sum of left and right multiplication by the continuum Dirac operator.

While the first term in Eq. 38 involves the usual $\sigma^{i0}F^{i0}$ associated with $c_E$ and is nominally of order $a$ in RHQ power counting, the second term in which both factors of $D^0$ appear in commutators has no compensating factor of $m$ and hence is $O(a^2)$. Thus, the vanishing of the sum of terms in Eq. 38 on-shell implies that the $c_E^c$ term in the effective action can be related to other terms that are explicitly of order $a^2$ in the sense of RHQ power counting. Because of the presence of the $m_r a$ factor appearing in this term, we cannot completely remove the $c_E$ term in the effective action since that term will contain contributions that are not proportional to the mass. However, the difference between $c_E$ and $c_B$ can be arranged to be proportional to the heavy quark mass. We must merely avoid a gratuitous violation of axis-interchange symmetry when choosing arbitrary parameters in the lattice Lagrangian, e.g. we must choose $r_t - r_s \propto (m_r a)^1$. That is, if only axis-interchange asymmetry proportional to $m_r a$ is introduced, the difference between $c_E$ and $c_B$ will also vanish as $m_r a \rightarrow 0$. Thus,
we can adjust the transformation parameter $R^{2,3}$ to set $c_E' = c_B' \equiv c_P$. Note, in the limit $m_r a \ll 1$, $c_P(m_r a) \to c_{SW}$, the usual Sheikholeslami and Wohlert coefficient of Ref. [6].

It is natural to consider also a transformation of $O(a^2)$ in which the coefficient of $c_B$ appears:

$$\psi' = (1 + a^2[\gamma^i, \gamma^j][D^i, D^j]R^{2,2})\psi \quad (39)$$
$$\bar{\psi}' = \bar{\psi}(1 + a^2[\gamma^i, \gamma^j][\bar{D}^i, \bar{D}^j]R^{2,2}). \quad (40)$$

However, in contrast to the previous transformation in Eqs. 36 and 37 this transformation results in a collection of terms which involves the combination: \{[$\gamma^i, \gamma^j$]$[D^i, D^j], \gamma^0 D^0$\}. This is a new term, not included in the simplified action of Eq. 28 which is nominally of order $a$ in our RHQ power counted scheme and hence potentially significant. Replacing the $c_B$ term with this one is merely trading one non-redundant term for another. Of course, the appearance of this new term indicates the limitations of our simplified example and motivates the complete discussion given in Sec. IV.

This result that the difference between $c_B'$ and $c_E'$ in the continuum effective Lagrangian contributes a term of order $(\vec{p} a)^2$ can be understood qualitatively as follows. In the case that $m_r < 1/a$ we are dealing with the standard $O(a)$ improvement of Sheikholeslami and Wohlert with $c_B' = c_E'$. To the extent that $m_r \approx 1/a$, asymmetries between space and time will be visible and we expect $c_B' - c_E' \propto m_r a$. However, for such a heavy quark case we expect the matrix elements of the correction terms $\bar{\psi}\sigma_{\mu\nu} F^{\mu\nu}\psi$ to be of order $1/m_r$. The resulting combination of an overall factor of $a$ present because this is a dimension-5 correction term, the factor $m_r a$ coming from $c_B' - c_E'$ and this $1/m_r$ estimate gives an over-all size of $O(a^2)$ with no compensating factors for $m_r$, demonstrating that their difference can be neglected to our intended order of accuracy.

Thus, to construct an improved lattice Lagrangian which will yield heavy quark spectral quantities which are accurate up to but not including $O(\vec{p} a)^2$ we need only tune 3 lattice parameters: $m_0$, $\zeta$, and $c_P$.

### III. ON-SHELL IMPROVEMENT AND EARLIER WORK

In the previous sections we have determined the number of parameters that must be tuned in the lattice action if the resulting effective continuum action is to be equivalent to
the standard continuum fermion action after a redefinition of the fermion fields. In this section we will consider the limitations of the resulting improved theory and its relation to the results of the Fermilab \(^1\) and Tsukuba \(^2\) groups.

As is well known, the physical masses determined by an effective theory are not changed by a change of field variables in the path integral defining the Green’s functions of that theory. This observation underlies the reduction of parameters that we have been investigating. Those parameters that can be removed by a redefinition of fields cannot effect the predicted masses. In the earlier work of the Fermilab group, the total number of parameters remaining after compensating for the redundancy implied by field transformations was given as four: \(m_r, \zeta^c, c^c_E\) and \(c_E\) in our notation. By considering the additional field transformation given in Eqs. 36 and 37 we have shown that the number of relevant parameters can be reduced to three: \(m_r, \zeta^c, c^c_p\).

Before comparing with the results of the Tsukuba group, we should discuss the question of computing on-shell Green’s functions with the effective actions under consideration. While our ability to remove redundant terms from the action is established by examining possible field transformations, such transformations are not actually made. Making these field transformations and casting the effective action in the desired continuum form would require knowing these extra, “redundant” parameters. Thus, the quark fields that appear in a lattice calculation with properly tuned values for the three relevant input parameters (here denoted \(\psi_0\) and \(\bar{\psi}_0\)) are un-transformed fields which correspond to an effective action which is not in the continuum form.

Thus, we will obtain appropriate, continuum on-shell Green’s functions only after we relate the un-transformed, interpolating fields appearing in a lattice calculation with the transformed fields corresponding to a proper, continuum-like effective theory (here labeled \(\psi^c\) and \(\bar{\psi}^c\)). While the fields \(\psi_0, \bar{\psi}_0\) and \(\psi^c, \bar{\psi}^c\) are related by a complicated transformation, non-linear in the gluon fields, we need to relate only their on-shell matrix elements. For such “pole” contributions all of the added powers of the gluon field present in the lattice fields \(\psi_0\) and \(\bar{\psi}_0\) must be contracted within field renormalization subdiagrams. (These are one-particle-irreducible subdiagrams with two external lines, which contain the external quark line and the internal quark line contributing to the single particle pole, illustrated in Fig. 1.) Thus, for the purposes of evaluating on-shell Green’s functions these two sets of fields are
related by a simple spinor renormalization factor:

\[
(\psi_0)_\alpha = \sum_\beta Z_{\alpha,\beta} (\psi^c)_{\beta} \quad (41)
\]

\[
(\overline{\psi}_0)_\alpha = \sum_\beta (\overline{\psi}^c)_{\beta} Z_{\beta,\alpha} \quad (42)
\]

Here \( Z_{\alpha,\beta} \) is a simple \( 4 \times 4 \) spinor matrix that can be written:

\[
Z = Z_1 + Z_2 a \gamma^0 \partial. \quad (43)
\]

Here the coefficients \( Z_i \) are arbitrary polynomials \( m_q a \) and \( (\partial_q a)^2 \). Imposing charge conjugation and reality constraints we find that

\[
\overline{Z} = Z_1 - Z_2 a \gamma^0 \partial. \quad (44)
\]

and that the polynomials \( \{Z_i\}_{i=1,2} \) have real coefficients. Note, we have used the equations of motion to remove a possible \( \gamma^0 \partial_0 \) term. Since these relations are only to be used on-shell, the argument \( (\partial_q a)^2 = (m_q a)^2 + (\overline{p} a)^2 \) and we can drop the final \( (\overline{p} a)^2 \) term. Thus, we will adopt the form

\[
Z = Z_q^{-1/2}(1 + \delta a \gamma^0 \partial) \quad (45)
\]

\[
\overline{Z} = Z_q^{-1/2}(1 - \delta a \gamma^0 \partial). \quad (46)
\]

where \( Z_q \) and \( \delta \) are functions of \( ma \) only.

Thus, our failure to actually transform to the proper continuum fields requires that on-shell Green’s functions in which the quark fields appear as interpolating fields must have the additional renormalization matrices \( Z \) and \( \overline{Z} \) applied to obtain the correct continuum form. Of course, such factors are not needed to extract the correct mass from the large-time limit of such Green’s functions.

With this background, we can now discuss the work of the Tsukuba group. They emphasizes the importance of working with five parameters, one more than the number determined in the Fermilab paper. By introducing a fifth parameter, they are able to include an additional field transformation which eliminates the parameter \( \delta \) above, insuring a lattice action which will yield on-shell quark propagators which take directly the continuum form. This is useful for lattice perturbative calculations where such on-shell quark propagators have meaning.
The Tsukuba group uses the field equations to derive their results, not the approach using field transformations taken in the Fermilab work and used in the present paper. However, there is no difference between these two methods because one typically justifies the use of the equations of motion when evaluating an on-shell amplitude by applying field transformations in the path integral. These two approaches are formally equivalent in this situation. The additional field transformation of Eq. 36 which permits us to use \( c_E = c_B \) can also be cast as a field equation implying the same result. Thus, we conclude that from both approaches only three parameters are needed if an improved lattice action is to yield continuum on-shell Green’s functions, up to the spinor transformations of Eqs. 45 and 46.

Since our objective is to use the improved lattice action to compute non-perturbative quantities, we do not benefit from simplifying on-shell quark Green’s functions. However, the field renormalization discussed above applies equally well to composite spin-1/2 operators that might be used to create, for example, a charmed baryon. Since such a composite operator will receive significant contributions from lattice-distorted short-distances, the quantities \( Z \) and \( \delta \) appropriate for such a physical heavy fermion will be different from a single quark field and the Tsukuba choice of a fifth (now fourth) parameter will not make \( \delta \) vanish for the case of a charmed baryon operator. Fortunately, from a non-perturbative perspective, the \( Z \)-factors above are relatively easy to deal with. They do not need to be known in advance and do not effect the action used in a simulation. Instead, they can be easily determined \textit{a posteriori} from the large time behavior of the heavy baryon propagator and then used elsewhere to accurately remove the lattice artifacts associated with using that heavy quark composite field.

IV. INDUCTIVE TRANSFORMATION OF THE EFFECTIVE ACTION

In this section we study the complete continuum effective action given in Eqs. 6-9 whose coefficients are polynomials of arbitrary order in \( m_0a \), \((aD^0)^2\) and \( g^2 \). We will use induction in the order of these polynomials to demonstrate that by applying the field transformations of Eqs. 22-27 this general effective action can be transformed to that given in Eq. 5 where only the coefficients \( m_r \), \( \zeta^c \) and \( c_B^c = c_E^c \) are non-zero and functions of \( m_0a \) and \( g^2 \) alone.

The coefficient functions \( B^{i,j} \) and \( C^{i,j} \) appearing in the original effective action of Eqs. 6-9 are polynomials of arbitrary order in the product \( m_0a \), the operator \((aD^0)^2\) and the gauge
coupling $g^2$:

\[ B^{i,j} = \sum_{k,l,n} b^{i,j}_{k,l,n} (m_0 a)^k (a D^0)^l g^{2n} \]
\[ C^{i,j} = \sum_{k,l,n} c^{i,j}_{k,l,n} (m_0 a)^k (a D^0)^l g^{2n}. \]  

For later purposes it is important to recognize that only the usual terms in the Dirac action will have non-zero coefficients in leading order:

\[ b_{0,0,0}^{-1,1} = 0, \quad b_{1,0,0}^{-1,1} = c_{0,0,0}^{-1,1} = 2b_{0,0,0}^{0,1} = 1. \]  

We will find it convenient to reorganize the sums in Eq. (47) collecting terms into homogeneous polynomials of degree $N$ in the three variables, $m_0 a$, $(a D^0)^2$ and $g^2$:

\[ B^{i,j} = \sum_N b^{i,j}_N \]
\[ C^{i,j} = \sum_N c^{i,j}_N. \]

where $b^{i,j}_N$ and $c^{i,j}_N$ are such homogeneous polynomials of degree $N$ in these three variables. In terms of these polynomials, the character of the tree-level, continuum limit of the standard Wilson action can be summarized by the requirement that $b_{0,0,0}^{-1,1} = 0$, $b_{1,0,0}^{-1,1} = m_0 a$, and $c_{0,0,0}^{-1,1} = 2b_{0,0,0}^{0,1} = 1$ (equivalent to Eq. (48)) and that all the other $N = 0$ coefficients $b^{i,j}_0$ and $c^{i,j}_0$ must vanish.

Equations 6, 7, 8 and 9 are organized in increasing powers of the lattice spacing where we treat $m$ and $D^0$ as order $1/a$ to accommodate the possibility that $m \sim 1/a$. However, it is important to bear in mind that the term $L_{\text{eff,n}}$ is characterized only by the lack of terms of lower order in $a$ than $a^n$. This term will necessarily contain terms that are of higher order in $a$. Commutators/anti-commutators have been introduced into the definitions in Eqs. 7, 8 and 9 in an attempt to organize these higher order terms. The polynomials $B^{i,j}$ and $C^{i,j}$ above are labeled so the left index indicates the order of the term in this scheme for RHQ power counting, e.g. $O(a^i)$ while the right index enumerates the various terms that can occur in that order. Note, we are using two separate expansions. One expansion is in powers of $a$, presuming that $m$ may be of order $1/a$. The second is the expansion in the over-all order of the three variables $m_0 a$, $(a D^0)^2$ and $g^2$ were the term $(m_0 a)^k (a D^0)^l g^{2n}$ is identified as of order $N = k + l + n$. 
A. Field Transformations

As is discussed above, many of the unwanted terms in Eq. 6 have no effect on physical states or fermion Green's functions evaluated on the mass shell and can be removed by a redefinition of the fermion fields $\psi$ and $\psi'$. We will now make a series of such transformations chosen to remove many of the terms that appear in the Symanzik effective action of Eq. 6. The coefficients of those terms that remain after these transformations are then presumed to be potentially important lattice artifacts that should be eliminated by an explicit choice of additional improvement terms in the underlying lattice action.

The removal of these redundant terms is most easily analyzed in a series of steps exploiting the ordering of the terms in Eq. 6: $O(1/a)$, $O(a^0)$, $O(a)$, etc. in the RHQ expansion. We will first make the large, $O(a^0)$, transformation of Eqs. 22 and 23 which we will be able to chose to return the $O(1/a)$ terms in $L_{\text{eff},-1}$ to the form found in the conventional continuum action:

$$L_{\text{sym}} = \overline{\psi} (\gamma^\mu D_\mu + m_r) \psi'.$$

We will then consider the effect of both this $O(a^0)$ transformation as well the most general $O(a)$ transformation given in Eqs. 24 and 25 on the $O(a^0)$ term $L_{\text{eff,0}}$. Finally, the effect of all three transformations, $O(a^0)$, $O(a)$ and the $O(a^2)$ given in Eqs. 26 and 27 will be studied on the final term of interest, $L_{\text{eff,1}}$.

1. Redundant terms in $L_{\text{eff},-1}$

In order to analyze the $O(1/a)$ terms in the Symanzik effective action, we must consider the effects of a field transformation of $O(a^0)$ on that action. The most general such field transformation are given in Eqs. 22 and 23 and repeated here for convenience, incorporating charge conjugation symmetry:

$$\psi = (1 + R^{0,1} + a\gamma^0 D^0 S^{0,1}) \psi'$$
$$\overline{\psi} = \overline{\psi} (1 + R^{0,1} - a\gamma^0 D^0 S^{0,1}).$$

The two functions $R^{0,1}$ and $S^{0,1}$ are polynomials of arbitrary order in $m_0 a$, $(aD^0)^2$ and $g^2$, similar to the coefficient functions $B^{i,j}$ and $C^{i,j}$ of Eqs. 7-9 and 47. These transformations are most easily analyzed if we proceed in a systematic fashion, removing sequentially terms
in $\mathcal{L}_{\text{eff},-1}$ of increasing order $m_0a$, $(aD^0)^2$ and $g^2$ where, as above, we identify a term of the form $(m_0a)^k(aD^0)^lg^{2n}$ as being of order $N = k + l + n$. Reliance on such a formal expansion is a standard approach to linearize the problem at hand, at the expense of requiring that an inductive argument be created to deal with polynomials of arbitrary order.

Specifically, we will achieve the general field transformation described in Eqs. 51 and 52 by performing a sequence of simpler transformations where each involves a homogenous polynomial of order $N$ in the three variables $m_0a$, $(aD^0)^2$ and $g^2$:

\begin{align*}
\psi &= (1 + r^{0,1}_N + a\gamma^0D^0s^{0,1}_N)\psi' \quad \text{(53)} \\
\bar{\psi} &= \bar{\psi}'(1 + r^{0,1}_N - a\gamma^0D^0s^{0,1}_N). \quad \text{(54)}
\end{align*}

Here the quantities $r^{i,j}_N$ and $s^{i,j}_N$ are homogenous polynomials of order $N$ in the three variables $m_0a$, $(aD^0)^2$ and $g^2$. The index $i$ identifies the order of the term in the RHQ expansion and the index $j$ labels the specific operator appearing in the transformation.

**Theorem** By proper choice of the transformation coefficients, $r^{0,1}_N$ and $s^{0,1}_N$ it is possible to transform $\mathcal{L}_{\text{eff},-1}$ into the form:

\begin{equation}
\mathcal{L}_{\text{eff},-1} = \bar{\psi}\{\gamma^0D^0 + m_r\}\psi' \quad \text{(55)}
\end{equation}

where $m_r$ is a polynomial in the variables $m_0a$ and $g^2$. This theorem can be proven by induction in $N$.

**Proof** To leading order in $N$, Eq. 55 is satisfied without any transformation. As observed above, the coefficient of $\gamma^0D^0$, $C^{-1,1} = c^{-1,1}_0 = 1$ to order $N = 0$. Likewise at order $N = 1$, the coefficient of $1/a$, $B^{-1,1} = b^{-1,1}_0 + b^{-1,1}_1 = m_0a$ so that through order $N = 1$, $m_r = m$. Thus, as the first step in our induction proof, we note that $c^{-1,1}_0 = 1$, $b^{-1,1}_0 = 0$ and $b^{-1,1}_1 = m_0a$.

Next we assume Eq. 55 is valid to order $N = N_0$ in the sense that after the previous $N_0$ steps, the resulting coefficients in the Lagrangian $\mathcal{L}_{\text{eff},-1}$ of Eq. 55 obey: $C^{-1,1} = c^{-1,1}_0 = 1$ and $B^{-1,1} = (m_r a)_{N_0+1}$. Thus, we must attempt to remove the next order terms in $\mathcal{L}_{\text{eff},-1}$:

\begin{equation}
\mathcal{L}_{\text{eff},-1} = \bar{\psi}\{\gamma^0D^0(1 + c^{-1,1}_{N_0+1}) + \frac{1}{a}(m_r a)_{N_0+1} + b^{-1,1}_{N_0+2}\}\psi. \quad \text{(56)}
\end{equation}

Here, by induction, $(m_r a)_{N_0+1}$ is assumed to be a polynomial of order $N \leq N_0 + 1$ in the variables $m_0a$ and $g^2$. The coefficients $b^{-1,1}_{N_0+2}$ and $c^{-1,1}_{N_0+1}$ are closely related to those appearing
in Eqs. 47 and 49 differing only by the effects of the field transformations made previously to achieve the form in Eq. 56:

\[
\psi = \prod_{N=0}^{N_0} \left( 1 + r_N^{0,1} + a\gamma^0 D^0 s_N^{0,1} \right) \psi'
\]  

(57)

\[
\bar{\psi} = \bar{\psi}' \prod_{N=0}^{N_0} \left( 1 + r_N^{0,1} - a\gamma^0 \bar{D}^0 s_N^{0,1} \right).
\]  

(58)

Performing the next transformation of order \(N_0 + 1\):

\[
\psi = (1 + r_{N_0+1}^{0,1} + a\gamma^0 D^0 s_{N_0+1}^{0,1}) \psi'
\]  

(59)

\[
\bar{\psi} = \bar{\psi}'(1 + r_{N_0+1}^{0,1} - a\gamma^0 \bar{D}^0 s_{N_0+1}^{0,1}).
\]  

(60)

\(\mathcal{L}_{\text{eff},-1}\) of Eq. 56 becomes:

\[
\mathcal{L}_{\text{eff},-1} = \bar{\psi}' \left\{ \gamma^0 D^0 [1 + c_{N_0+1}^{-1,1} + 2r_{N_0+1}^{0,1}]
\right.
\]

\[
+ \frac{1}{a} [(m_r a)_{N_0+1} + 2b_{N_0+2}^{-1,1} + 2(aD^0)^2 s_{N_0+1}^{0,1}] \} \psi'.
\]  

(61)

Thus, we can establish our theorem to order \(N_0 + 1\) if we require:

\[
c_{N_0+1}^{-1,1} + 2r_{N_0+1}^{0,1} = 0
\]  

(62)

and choose \(s_{N_0+1}^{0,1}\) to remove the \((aD^0)^{2N}\) terms for \(1 \leq N \leq N_0 + 2\) from the coefficient \(b_{N_0+2}^{-1,1}\) so that the definition

\[
(m_r a)_{N_0+2} = (m_r a)_{N_0+1} + b_{N_0+2}^{-1,1} + 2(aD^0)^2 s_{N_0+1}^{0,1}
\]  

(63)

will contain no \((aD^0)^2\) terms as required.

Following this inductive procedure, we are thus able to express the \(O(1/a)\) Symanzik Lagrangian in the standard continuum form. Only the mass parameter \(m_r\) must be tuned by an appropriate choice of lattice action to agree with the mass of the heavy quark which this Lagrangian is intended to describe.

2. Redundant terms in \(\mathcal{L}_{\text{eff},0}\)

The order \(a^0\), Symanzik effective Lagrangian, \(\mathcal{L}_{\text{eff},0}\) is altered by two sorts of field transformations. The first is the \(O(a^0)\) transformations discussed above. The second are the
O(a) transformations of Eqs. [24] and [25] which act on $L_{\text{eff},-1}$ and generate terms of the type which appear in $L_{\text{eff},0}$. Including the constraints of charge conjugation symmetry these $O(a)$ transformations can be written:

$$
\psi = \left(1 + a\gamma\tilde{D} R^{1,1} + a^2[\gamma\tilde{D}, \gamma^0 D^0] s^{1,1}\right) \psi' \quad (64)
$$

$$
\overline{\psi} = \overline{\psi}' \left(1 - a\gamma\tilde{D} R^{1,1} - a^2[\gamma\tilde{D}, \gamma^0 D^0] s^{1,1}\right). \quad (65)
$$

As in the previous discussion, it will be convenient to view the coefficient functions $R^{1,1}$ and $S^{1,1}$ as a sum of homogenous polynomials in the three variables $m_0 a$, $(aD^0)^2$ and $g^2$:

$$
R^{i,j} = \sum_N r_N^{i,j} \quad S^{i,j} = \sum_N s_N^{i,j} \quad (66)
$$

Again, we will proceed inductively to prove the following result:

**Theorem** By proper choice of the transformation coefficients, $r_N^{1,1}$ and $s_N^{1,1}$, it is possible to transform $L_{\text{eff}}$ so that $L_{\text{eff},0}$ takes the form:

$$
L_{\text{eff},0} = \overline{\psi}\gamma\tilde{D}\psi. \quad (67)
$$

**Proof** This result is automatically valid to order $N = 0$ which is the case of the tree-level Lagrangian with $b_0^{0,1} = 1/2$ and $c_0^{0,1} = 0$. Next, assume the inductive hypothesis that when working to order $N_0$ we are able to simplify $L_{\text{eff},0}$ so that all terms of order $N_0 + 1$ and lower take the form:

$$
L_{\text{eff},0} = \overline{\psi}\{\gamma\tilde{D}, (1/2 + b_0^{0,1}N_0 + 1)\} \psi. \quad (68)
$$

(Recall that the $C^{0,1}$ term in Eq. 8 vanishes when charge conjugation symmetry is imposed.)

We will now apply the transformations of order $N_0 + 1$ given in Eqs. [59] and [60] and those in Eqs. [64] and [65] specialized to the polynomials of order $N_0$,

$$
\psi = \left(1 + a\gamma\tilde{D} r_{N_0}^{1,1} + a^2[\gamma\tilde{D}, \gamma^0 D^0] s_{N_0}^{1,1}\right) \psi' \quad (69)
$$

$$
\overline{\psi} = \overline{\psi}' \left(1 - ar_{N_0}^{1,1} \gamma\tilde{D} - a^2 s_{N_0}^{1,1}[\gamma\tilde{D}, \gamma^0 D^0]\right). \quad (70)
$$

to $L_{\text{eff},-1} + L_{\text{eff},0}$.

These transformations yield $L_{\text{eff},0}$ of the following form:

$$
L_{\text{eff},0} = \overline{\psi}'\{\gamma\tilde{D}, (1/2 + b_0^{0,1}N_0 + 1 + r_{N_0}^{0,1} + m_0 a r_{s,N_0}^{1,1} - (aD^0)^2 s_{N_0}^{1,1})\} \psi'. \quad (71)
$$

Since the difference between the coefficient of $\gamma^0 D^0$ which has now been set to one and that of $\gamma\tilde{D}$ must vanish when the anisotropic effects of the special treatment of $m_0 a$ and $D^0$ are
absent, the combination \( b_{N_0+1}^{0,1} + r_{N_0+1}^{0,1} \) must be proportional to a linear combination of \( m_0a \) and \((aD^0)^2\) and can therefore be completely canceled by an appropriate choice of the terms \( m_0a r_{N_0}^{1,1} \) and \(-2(aD^0)^2 s_{N_0}^{1,1}\), completing our inductive proof.

As the preceding discussion reveals, our inductive approach to determining the redundant parameters in \( \mathcal{L}_{\text{eff}} \) requires that both the order \( a^0 \) and order \( a^1 \) field transformations are to be applied at the same time so that a common inductive step is taken to show that the desired form will hold at order \( N_0 + 1 \) provided it holds at order \( N_0 \). It is in this sense that we are combining the order \( a^0 \) and \( a^1 \) transformations in Eq. 71.

3. Redundant terms in \( \mathcal{L}_{\text{eff},1} \)

The last step in this discussion is an analysis of the freedom to simplify the terms of order \( a \) in \( \mathcal{L}_{\text{eff}} \), i.e. \( \mathcal{L}_{\text{eff},1} \). These can be effected by three different sorts of field transformations: transformations of order \( a^0 \) acting on \( \mathcal{L}_{\text{eff},1} \), transformations of order \( a \) acting on \( \mathcal{L}_{\text{eff},0} \) and transformations of order \( a^2 \) acting on \( \mathcal{L}_{\text{eff},-1} \). We will again state our result in the form of a theorem to be proven by induction in the order of the polynomials appearing in \( \mathcal{L}_{\text{eff},1} \):

**Theorem** By an appropriate field transformation \( \mathcal{L}_{\text{eff},1} \) can be cast in the form:

\[
\mathcal{L}_{\text{eff},1} = -\bar{\psi}c_P \left\{ \frac{1}{8}[\gamma^i, \gamma^j][D^i, D^j] + \frac{1}{4}[\gamma^i, \gamma^0][D^i, D^0] \right\} \psi \tag{72}
\]

where \( c_P = B^{1,2} = B^{1,3}/2 \) is a polynomial in \( m_0a \) and \( g^2 \) only.

**Proof** We begin by observing that to order \( N = 0 \) Eq. \( 72 \) is automatically obeyed with \( c_P = -8B_{N=0}^{1,2} = 1 \), the original, tree-level result of Sheikholeslami and Wohlert. Next we assume that this is true to order \( N_0 \) so that to order \( N_0 + 1 \), \( \mathcal{L}_{\text{eff},1} \) takes the form:

\[
\mathcal{L}_{\text{eff},1} = a\bar{\psi}\left\{ \bar{D}^2 b_{N_0+1}^{1,1} + a\{\bar{D}^2, \gamma^0 D^0\} c_{N_0+1}^{1,1} \right\} \psi \tag{73}
\]

We will now attempt to remove the redundant terms in Eq. \( 73 \) by the following three field transformations. The first is the \( O(a^0) \) transformations of Eqs. 53 and 54 that involve polynomials in \( m_0a \), \((aD^0)^2\) and \( g^2 \) of combined order \( N = N_0 + 1 \). These \( O(a^0) \) transformations
will have the following $O(a)$ effects. When acting on $\mathcal{L}_{\text{eff},-1}$, these $O(a^0)$ transformations produce only terms of $O(1/a)$. No terms of $O(a^0)$ or $O(a)$ are produced. If these $O(a^0)$ transformations act on $\mathcal{L}_{\text{eff},0}$ both terms of $O(a^0)$ and of $O(a)$ are created. Those of $O(a^0)$ appear in Eq. 74 and have been removed by the transformations of order $a^1$. The terms of $O(a)$ which will effect $\mathcal{L}_{\text{eff},1}$ take the following form:

$$\Delta \mathcal{L}_{\text{eff},1}^{0,0} = a\overline{\psi}\left\{\gamma^j \overrightarrow{D}, \gamma^0 D^0\right\}\left(s_{s,N_0+1}^{0,1} + 2(aD^0)^2 \frac{s_{s,N_0+1}^{0,1}}{\partial((aD^0)^2)}\right)\psi$$

(74)

Here the $i,j$ superscript on $\Delta \mathcal{L}_{\text{eff},1}^{i,j}$ identifies this expression as the change in $\mathcal{L}_{\text{eff},1}$ coming from applying a transformation of order $a^i$ to $\mathcal{L}_{\text{eff},j}$.

Next we should consider the effect of this $O(a^0)$ transformation on the $O(a)$ Lagrangian $\mathcal{L}_{\text{eff},1}$. However, since we will not need to use the effects of this transformation on $\mathcal{L}_{\text{eff},1}$, we will assume that its effects have already be taken into account in the coefficients $b_{N_0+1}^{1,j}$ and $c_{N_0+1}^{1,j}$ that appear in Eq. 73.

Having completely accounted for the effects on $\mathcal{L}_{\text{eff}}$ of the transformations of $O(a^0)$ given in Eqs. 59 and 60, we will now consider the transformations of $O(a)$ given in Eqs. 69 and 70 First as they act on $\mathcal{L}_{\text{eff},-1}$ they will produce the following changes in $\mathcal{L}_{\text{eff},1}$:

$$\Delta \mathcal{L}_{\text{eff},1}^{1,-1} = \frac{a}{2}\overline{\psi}[\gamma^i, \gamma^0][D^i, D^0]r_{N_0+1}^{1,1}\psi$$

(75)

Note, this term was generated from the $\gamma^0 D^0$ term in $\mathcal{L}_{\text{eff},-1}$. No terms of order $a$ are produced from the $m$ term.

The next case to consider is the effect of these transformations of $O(a)$ on $\mathcal{L}_{\text{eff},0}$. The resulting changes to $\mathcal{L}_{\text{eff},1}$ are:

$$\Delta \mathcal{L}_{\text{eff},1}^{1,0} = \overline{\psi}\left\{a \left(2\overrightarrow{D}^2 + \frac{1}{2} [\gamma^i, \gamma^j][D^i, D^j]\right) r_{N_0+1}^{1,1}
\right.\right.$$

$$\left.\left.+ a \left(2\overrightarrow{D}^2 + \frac{1}{2} [\gamma^i, \gamma^j][D^i, D^j]\right) , a \gamma^0 D^0\right)s_{N_0+1}^{1,1}\right}\psi$$

(76)

The final $O(a)$ effects to consider are those of transformations of $O(a^2)$ acting on $\mathcal{L}_{\text{eff},-1}$. If Eqs. 26 and 27 are specialized to respect charge conjugation symmetry, the relevant $O(a^2)$
field transformations can be written:

\[
\psi = \left(1 + a^2 \bar{D}^2 r_{N_0+1}^{2,1} + a^2 \{ \bar{D}^2, a\gamma^0 D^0 \} s_{N_0}^{2,1} \right)\psi
+ a^2 [\gamma^i, \gamma^j] [D^i, D^j] r_{N_0+1}^{2,2}
+ a^2 \left\{ [\gamma^i, \gamma^j] [D^i, D^j], a\gamma^0 D^0 \right\} s_{N_0}^{2,2}
+ a^2 [\gamma^i, \gamma^0] [D^i, D^0] r_{N_0}^{2,3}
+ a^2 \left[ [\gamma^i, \gamma^0] [D^i, D^0], a\gamma^0 D^0 \right] s_{N_0}^{2,3}) \psi'.
\]

\[
\bar{\psi} = \bar{\psi} \left(1 + a^2 \bar{D}^2 r_{N_0+1}^{2,1} - a^2 \{ \bar{D}^2, a\gamma^0 D^0 \} s_{N_0}^{2,1} \right)\psi
+ a^2 [\gamma^i, \gamma^j] [\bar{D}^i, \bar{D}^j] r_{N_0+1}^{2,2}
- a^2 \left\{ [\gamma^i, \gamma^j] [\bar{D}^i, \bar{D}^j], a\gamma^0 D^0 \right\} s_{N_0}^{2,2}
+ a^2 [\gamma^i, \gamma^0] [\bar{D}^i, \bar{D}^0] r_{N_0}^{2,3}
+ a^2 \left[ [\gamma^i, \gamma^0] [\bar{D}^i, \bar{D}^0], a\gamma^0 D^0 \right] s_{N_0}^{2,3}\right). \]

The resulting $O(a)$ terms are:

\[
\Delta L_{\text{eff},1}^{2,-1} = \bar{\psi} a\bar{D}^2 \left(2m_0 a r_{N_0+1}^{2,1} + 4(aD^0)^2 s_{N_0}^{2,1} \right)\psi
+ a\{ \bar{D}^2, a\gamma^0 D^0 \} (2m_0 a s_{N_0}^{2,1} + r_{N_0+1}^{2,1})
+ a[\gamma^i, \gamma^j] [D^i, D^j] \left(2m_0 a r_{N_0+1}^{2,2} + 4(aD^0)^2 s_{N_0}^{2,2} \right)
+ a\{ [\gamma^i, \gamma^j] [D^i, D^j], \gamma^0 D^0 \} \left(2m_0 a s_{N_0}^{2,2} + r_{N_0+1}^{2,2} \right)
+ a[\gamma^i, \gamma^0] [D^i, D^0] \left(2m_0 a r_{N_0+1}^{2,3} - 4(aD^0)^2 s_{N_0}^{2,3} \right)\psi.
\]

We can now combine the $O(a)$ terms created by these three field transformations with those already present in Eq. (73). We will do this by considering in turn each of the three types of operators appear in Eq. (73) with coefficients whose right hand superscript is $1 \leq j \leq 3$, which we will denote $L_{\text{eff},1}^{(j=1,2,3)}$.

We first examine $L_{\text{eff},1}^{(1)}$ constructed by collecting terms from Eqs. (73), (76) and (78):

\[
L_{\text{eff},1}^{(1)} = a\bar{\psi} \left(\bar{D}^2 \left(b_{N_0+1}^{1,1} + 2r_{N_0+1}^{1,1} + 2m_0 a r_{N_0+1}^{2,1} + 4(aD^0)^2 s_{N_0}^{2,1} \right)
+ a\{ \bar{D}^2, \gamma^0 D^0 \} \left(c_{N_0+1}^{1,1} + 2s_{N_0+1}^{1,1} + 2m_0 a s_{N_0}^{2,1} + r_{N_0+1}^{2,1} \right)\right)\psi.
\]
Since we will adopt the usual conventions of fixing the spatial Wilson term in the lattice action to have normalization 1, we will adjust $r_{N_{0}+1}^{1,1}$ appearing above to remove the corresponding $\bar{D}^2$ term. This implies that we cannot make the choice described in the previous discussion of $\mathcal{L}_{\text{eff,0}}$ to set the coefficient of $\gamma \bar{D}$ to one, which was also accomplished by a different choice of these same coefficients, see Eq. 71 and following. The second term of the form $a\{\bar{D}^2, a\gamma^0 D^0\}$ in Eq. 80 will be removed the choice of $r_{N_{0}+1}^{2,1}$.

We next examine $\mathcal{L}_{\text{eff,1}}^{(2)}$ constructed by collecting terms from Eqs. 73, 76 and 79

$$\mathcal{L}_{\text{eff,1}}^{(2)} = a\overline{\psi} \left\{ a[\gamma^i, \gamma^j][D^i, D^j] \left( -\frac{1}{8} (c_P)_{N_0} + b_{N_{0}+1}^{1,2} + r_{N_{0}+1}^{1,1} + 2m_0 a r_{N_{0}+1}^{2,2} + 4(aD^0)^2 s_{N_0}^{2,2} \right) \right. \\
+ a \left\{ [\gamma^i, \gamma^j][D^i, D^j], a\gamma^0 D^0 \right\} \left\{ \frac{1}{2} s_{N_{0}+1}^{1,1} + 2m_0 a s_{N_0}^{2,2} + r_{N_{0}+1}^{2,2} \right\} \left\{ \overline{\psi} \right. \right.$$ (81)

Since the coefficient $r_{N_{0}+1}^{1,1}$ has already been used to remove the $\bar{D}^2$ term and the coefficient $r_{N_{0}+1}^{2,2}$ will be used below, we have only the freedom to adjust the combination $s_{N_0}^{2,2}$ to remove the terms proportional to $(aD^0)^2$ for the coefficient of $a[\gamma^i, \gamma^j][D^i, D^j]$. Thus, the parameter $c_P$ will require mass-dependent tuning. However, the second term, $a\{[\gamma^i, \gamma^j][D^i, D^j], a\gamma^0 D^0\}$ can be entirely removed by a choice of $r_{N_{0}+1}^{2,2}$.

Finally we consider the term $\mathcal{L}_{\text{eff,1}}^{(3)}$ constructed by collecting terms from Eqs. 73, 76 and 79

$$\mathcal{L}_{\text{eff,1}}^{(3)} = a\overline{\psi} a[\gamma^i, \gamma^0][D^i, D^0] \left\{ -\frac{1}{4} (c_P)_{N_0} + b_{N_{0}+1}^{1,3} + s_{s, N_{0}+1}^{0,1} \right. \\
+ 2(aD^0)^2 \frac{s_{s, N_{0}+1}^{0,1}}{\partial((aD^0)^2)} + \frac{1}{2} r_{N_{0}+1} + 2m_0 a r_{s, N_{0}+1}^{2,3} - 4(aD^0)^2 s_{N_0}^{2,3} \right\} \overline{\psi}. \quad (82)$$

We can now exploit the freedom to choose the coefficient $s_{N_0}^{2,3}$ to remove the terms containing $(aD^0)^2$ from the coefficient of $[\gamma^i, \gamma^0][D^i, D^0]$ and can determine $m_0 a r_{N_{0}+1}^{2,3}$ to set this coefficient equal to that of $[\gamma^i, \gamma^0][D^i, D^0]/2$ since their difference must be proportional to $m_0 a$.

Thus, we have shown that with the proper choice of field transformations to remove redundant terms, the general Symanzik action, invariant under axis reversal and charge conjugation will contain only three independent parameters. This result is summarized in Table III where the various field transformations and the terms which they eliminate are listed.

We conclude that a lattice calculation accurate through order $|\overline{\psi}a|$ and to arbitrary order in $ma$ requires the determination of the three parameters $m_0$, $\zeta$ and $c_P$ appearing in the improved lattice action.
V. TREE-LEVEL RESULTS

In order to investigate the number of required parameters further, we have carried out a tree-level calculation of both the quark propagator and the quark gluon vertex for a general, heavy-quark lattice Lagrangian, but evaluated in the limit $|\vec{p}a| \ll 1$. We begin with the general lattice action given in Eq. 1 which depends on six parameters, $m_0$, $\zeta$, $r_s$, $r_t$, $c_B$ and $c_E$. We then demonstrate that a continuum result can be obtained on-shell, accurate through $O(|\vec{p}a|)$ and to all orders in $m_r a$ by adjusting only the expected three parameters $m_0$, $c_p \equiv c_B = c_E$ and $\zeta$ while at the same time performing a simple $4 \times 4$ matrix rotation on the Dirac spinors.

The presence of hyperbolic trigonometric functions in the Minkowski-space lattice propagator makes the algebra in this section somewhat complex. This complexity is compounded by the approximation $|\vec{p}a| \ll 1$ which is being made to functions of the two variables $|\vec{p}a|$ and $m_r a$. Depending on the size of the second variable $m_r a$, the treatment of the quantity $|\vec{p}a|$ can be quite different. It is natural to divide the possible values of $m_r a$ into two regions. In the first region $m_r a \ll 1$, and we have the kinematics of standard, light fermions. In this case we cannot neglect $|\vec{p}|/m_r$ but can treat $m_r a$ as a small parameter. In second region we assume $p \ll m_r$. Here we cannot neglect errors of order $m_r a$ but can treat $|\vec{p}|/m_r$ as small. Since these two regions have a non-vanishing overlap, $|\vec{p}| \ll m_r \ll 1/a$, we will be able to demonstrate that the tree level amplitudes are consistent with our treatment for all values of $m_r a$ if we are able to provide satisfactory bounds on the errors in both of these two regions.

We propose to do this as follows. First we introduce a small parameter $\epsilon = |\vec{p}a|$. Our objective is to show that at tree level, working with the improved, 3-parameter action and an appropriate $4 \times 4$ spinor transformation matrix, we can reproduce continuum results up to errors of order $\epsilon^2$. We will divide the range of values of $m_r a$ into two non-overlapping regions. In the first, Region I, we require $m_r a \leq \sqrt{\epsilon}$. Here we can Taylor expand in the parameter $m_r a$ but must control errors up to order $(m_r a)^4$. Region II corresponds to the remaining range of $m_r$: $\sqrt{\epsilon} < m_r a$. Now we can expand in $|\vec{p}|/m_r = |\vec{p}|a/(m_r a) \leq \sqrt{\epsilon}$ but must therefore work up to $O((|\vec{p}|/m_r)^4)$. 
A. Momentum-dependent energy

The quark wave function renormalization constant $Z_q$ and the parameters $m_0$, $\zeta$ and $r_s$ can be constrained by demanding that the lattice quark propagator $G_q(p)$ derived from Eq.(1) should reproduce the relativistic form

$$G_q(p_0,p_i) = \frac{1}{Z_q} \frac{-i\gamma^0 p_0 - i\vec{\gamma} \cdot \vec{p} + m_r}{p_0^2 + \vec{p}^2 + m_r^2} + \text{(non-pole terms)} + O\left((p_i a)^2\right)$$

(83)

at the heavy quark pole in the limit $|p_i a| \ll 1$. The location of the pole in the tree-level lattice propagator is that value of $p_0$ at which the inverse propagator vanishes:

$$aG_q^{-1}(p_0,p_i) = i\gamma^0 \sin(p_0 a) + i\zeta \sum_i \gamma^i \sin(p_i a) + m_0 a + r_t \left(1 - \cos(p_0 a)\right) + r_s \sum_i \left(1 - \cos(p_i a)\right) = 0.$$  

(84)

By first examining the simplest case of zero spatial momenta, $p_i = 0$, we can obtain equations for $m_0$ and $Z_q$:

$$m_r a = \ln \left(\frac{m_0 a + r_t + \sqrt{(m_0 a)^2 + 2r_t m_0 a + 1}}{1 + r_t}\right)$$

(85)

$$Z_q = \cosh(m_r a) + r_t \sinh(m_r a)$$

(86)

To obtain constraints on $\zeta$ and $r_s$ we need to examine the case of finite spatial momentum.

From the dispersion relation $p_0 = i\sqrt{m_r^2 + \vec{p}^2}$ which we would like to reproduce, we can get a relationship between $r_s$ and $\zeta$. Starting from Eq. 84 and defining a new variable $\tilde{p}_0 \equiv -ip_0$, we obtain:

$$(r_t^2 - 1) \cosh^2(\tilde{p}_0 a) - 2r_t B \cosh(\tilde{p}_0 a) + 1 + \zeta^2 \sin^2(p_i a) + B^2 = 0$$

(87)

where $B = r_t + m_0 a + r_s \sum_i (1 - \cos(p_i a))$. Neglecting quantities of order $O((p_i a)^2)$ and higher, the two roots of the quadratic equation for $\cosh(\tilde{p}_0 a)$ can be written:

$$R_{\pm} = \frac{r_t B \pm \sqrt{r_t^2 B^2 - (r_t^2 - 1)(1 + B^2 + \zeta^2 \vec{p}^2)}}{r_t^2 - 1}$$

(88)

Here we choose $R_-$ as the physical root since $R_+$ goes to infinity when $r_t \to 1$. After we substitute the expression for $B$ into the $R_-$ and expand to first order in the quantity $(\tilde{p}a)^2$, we find:
\[
cosh (\tilde{p}_0 a) = \frac{r_t (m_0 a + r_t)}{r_t^2 - 1} + \frac{r_t r_s (p_0 a)^2/2}{r_t^2 - 1} - \sqrt{[(m_0 a + r_t)^2 - (r_t^2 - 1)] + [r_s (m_0 + r_t) - \zeta^2 (r_t^2 - 1)] (p_0 a)^2} \\
= \frac{r_t (m_0 a + r_t) - \sqrt{(m_0 a + r_t)^2 - (r_t^2 - 1)}}{r_t^2 - 1} \\
+ \left\{ \frac{r_t r_s}{2(r_t^2 - 1)} - \frac{r_s (m_0 a + r_t) - \zeta^2 (r_t^2 - 1)}{2(r_t^2 - 1) \sqrt{(m_0 a + r_t)^2 - (r_t^2 - 1)}} \right\} (p_0 a)^2 \\
= \cosh (m_r a) + \frac{r_s \sinh (m_r a) + \zeta^2}{2(r_t \sinh (m_r a) + \cosh (m_r a))} (p_0 a)^2
\] (89)

where the last line is obtained using Eq. (85).

Equation (91) can be rewritten in the suggestive form:

\[
\tilde{p}_0 a = \sinh^{-1} \left\{ \sqrt{\sinh^2 (m_r a) + (\tilde{p} a)^2 \cosh (m_r a) - \frac{r_s \sinh (m_r a) + \zeta^2}{r_t \sinh (m_r a) + \cosh (m_r a)}} \right\}.
\] (92)

If \( m_r a \ll 1 \) then \( \sinh (z) \) and \( \sinh^{-1} (z) \) can both be replaced by \( z \) and Eq. (92) gives the usual relativistic dispersion relation if we set \( \zeta = 1 \). If \( m_r a \) is sufficiently large that this approximation to \( \sinh (z) \) and \( \sinh^{-1} (z) \) is a poor one, then we can expand the square root in Eq. (92) to first order in \( (\tilde{p} a)^2 \) and obtain the result:

\[
\tilde{p}_0 a = m_r a + \frac{(\tilde{p} a)^2}{2 \sinh m_r a} \frac{r_s \sinh (m_r a) + \zeta^2}{r_t \sinh (m_r a) + \cosh (m_r a)}.
\] (93)

Thus, we will obtain the correct dispersion relation in both cases if we require:

\[
r_s \sinh (m_r a) + \zeta^2 = \frac{\sinh (m_r a)}{m_r a} (r_t \sinh (m_r a) + \cosh (m_r a)).
\] (94)

As discussed above, we can establish the equivalence of Eq. (92) to the usual dispersion relation

\[
\tilde{p}_0 a = \sinh^{-1} \left\{ \sqrt{\sinh^2 (m_r a) + \frac{\sinh (m_r a) \cosh (m_r a)}{m_r a} (\tilde{p} a)^2} \right\} = \sqrt{(m_r a)^2 + (\tilde{p} a)^2}
\] (95)

up to relative errors of order \( (\tilde{p} a)^2 \equiv \epsilon^2 \) for all values of \( m_r a \) by showing it to holds to this accuracy in the two regions \( m_r a \leq \sqrt{\epsilon} \) (region I) and \( \sqrt{\epsilon} < m_r a \) (region II). Here the left-hand equality in Eq. (95) is simply Eq. (92) with the constraint in Eq. (94) imposed. Establishing that the right-hand equality holds without errors larger than \( O(\epsilon^2) \) requires in
Region I that we examine to next-leading order a Taylor series expansion in the variables \((m_r a)^4/((m_r a)^2 + (\vec{p} a)^2)\) and \((m_r a)^2\). In Region II, we need only continue the expansion in \((\vec{p} a/m_r a)^2\) begun in Eq. 33 to demonstrate that this second equality holds up to relative errors of order \((\vec{p} a/m_r a)^4 \sim \epsilon^2\). While straight-forward, some care is required to verify that the \(m_r a\)-dependent coefficient of \((\vec{p} a/m_r a)^4\) is bounded throughout the region \(\epsilon \leq m_r a\).

Thus, we can reproduce the correct momentum dependence of the heavy quark energy if and only if the parameters \(r_s\) and \(\zeta\) satisfy the relationship in Eq. 94.

**B. Propagator spinor structure**

Without a second constraint which then determines both \(r_s\) and \(\zeta\), the general action under consideration will not reproduce the correct spinor structure for the propagator. However, as discussed earlier, we can also achieve the conventional, on-shell spinor structure for the propagator by applying a simple matrix transformation to the on-shell spinor fields even if we have chosen an arbitrary \(r_s\) and an appropriate value for \(\zeta(r_s)\) so that Eq. 94 is obeyed. If we adopt this approach then we have freedom to choose \(r_s\) in the action for convenience, e.g. \(r_s = \zeta\), thereby reducing the number of parameters in the action by one.

We begin by examining the matrix form of the propagator as presently determined:

\[
aG_q(p_0, p_i) = \frac{-i\gamma^0 \sin(p_0 a) - i\zeta \vec{\gamma} \cdot \vec{p} a + F}{\sin^2(p_0 a) + \zeta^2 (\vec{p} a)^2 + F^2}
\]  

(96)

where \(F\) is given by

\[
F = m_0 a + r_t (1 - \cos(p_0 a)) + \frac{r_s}{2} (\vec{p} a)^2.
\]

(97)

We will now try to find a pair of \(4 \times 4\) spinor matrices \(U_L(\vec{p})\) and \(U_R(\vec{p})\) able to transform the matrix in the numerator of Eq. 96 to the correct one:

\[
U_L(\vec{p}) \frac{-i\gamma^0 \sin(p_0 a) - i\zeta \vec{\gamma} \cdot \vec{p} a + F}{\sin^2(p_0 a) + \zeta^2 (\vec{p} a)^2 + F^2} U_R(\vec{p}) \approx \frac{1}{Z_q} \frac{-i\gamma^0 p_0 - i \sum_i \gamma^i p_i + m_r}{p_0^2 + \sum_i p_i^2 + m_r^2},
\]

(98)

in the sense that both expressions should have the same residue at the heavy quark pole.

We begin by examining the numerator of the left-hand side of Eq. 98 and substitute \(p_0 \equiv i\vec{p}_0\)

\[
\gamma^0 \sinh(\vec{p}_0 a) - i\zeta \vec{\gamma} \cdot \vec{p} a + m_0 a + r_t (1 - \cosh(\vec{p}_0 a)) + \frac{r_s}{2} (\vec{p} a)^2.
\]

(99)
Two steps are needed to put this equation in a convenient form. First we replace the coefficient \( \sinh(\tilde{p}_0 a) \) multiplying the \( \gamma_0 \) in Eq. 99 by an expression closer to the continuum value:

\[
\sinh(\tilde{p}_0 a) \approx \frac{\sinh(m_r a)}{m_r a} \tag{100}
\]

This approximation can be justified by using Eqs. 92 and 95 obeyed by \( \tilde{p}_0 \) to write:

\[
\frac{\sinh(\tilde{p}_0 a)}{\tilde{p}_0} = \frac{\sinh(m_r a)}{m_r a} \left[ 1 + \frac{(\tilde{p}_a)^2 \cosh(m_r a)}{m_r a \sinh(m_r a)} \right]^{1/2} \tag{101}
\]

We can then evaluate the difference between the contents of the square bracket in this equation and one:

\[
[\ldots] - 1 = \frac{(\tilde{p}_a)^2}{m_r a} \left( \frac{\cosh(m_r a)}{\sinh(m_r a)} - \frac{1}{m_r a} \right) \leq (\tilde{p}_a)^2. \tag{102}
\]

Here the final inequality, showing that this difference can be neglected, follows from the relation \( x \coth(x) \leq (1 + x + x^2)/(1 + x) \).

The second relation that we need approximates:

\[
cosh(\tilde{p}_0 a) = \cosh(m_r a) \left[ 1 + \frac{(\tilde{p}_a)^2 \sinh(m_r a)}{m_r a \cosh(m_r a)} \right]^{1/2} \approx \cosh(m_r a) + \frac{(\tilde{p}_a)^2}{2m_r a} \sinh(m_r a). \tag{103}
\]

Here the equality follows directly from Eq. 92 while the inequality requires the neglect of a term of order \( (\tilde{p}_a)^4 \) whose coefficient can be shown to be bounded through use of the relation \( \tanh(x) \leq x \).

Next we substitute Eqs. 100 and 103 into Eq. 99 writing the numerator of the propagator as:

\[
\gamma_0 \tilde{p}_0 a \frac{\sinh(m_r a)}{m_r a} - i \tilde{\gamma} \cdot \tilde{p}_a + \sinh(m_r a) + \frac{1}{2} (\tilde{p}_a)^2 \left( r_s - r_t \frac{\sinh(m_r a)}{m_r a} \right). \tag{104}
\]

We must now find matrices \( U_L \) and \( U_R \) which will transform this expression into the desired continuum form. Thus, we must make the coefficient of \( \tilde{\gamma} \cdot \tilde{p} \) agree with that of \( \gamma_0 p^0 \) and remove the \( \tilde{p}^2 \) term. This can be accomplished by matrices of the form

\[
U_L = U_R = (1 + i \delta \tilde{\gamma} \cdot \tilde{p}_a) \quad \text{where} \quad \delta = \frac{\zeta}{2 \sinh(m_r a)} - \frac{1}{2m_r a} \tag{105}
\]
It is easy to see that when these transformations act on the $\sinh(m_r a)$ term in Eq. 104, a term is generated which precisely replaces $-i\zeta \vec{\gamma} \cdot \vec{p} a$ with the desired expression:

$$-i\vec{\gamma} \cdot \vec{p} \sinh(m_r a)/m_r.$$

However, the elimination of the $(\vec{p} a)^2$ term appearing in Eq. 104 is less direct. For this term the effect of our transformation generates a $(\vec{p} a)^2$ contribution which is only approximately zero:

$$(\vec{p} a)^2 \left\{ 2\zeta \left( \frac{\zeta}{2 \sinh(m_r a)} - \frac{1}{2 m_r a} \right) + \frac{1}{2} (r_s - r_t \frac{\sinh(m_r a)}{m_r a}) \right\} \approx 0. \quad (107)$$

To neglect the expression in Eq. 107 we must make two observations. First, we recognize that when expanded in a power series in $m_r a$ the expression in curly brackets in Eq. 107 begins at order $(m_r a)^1$ when $\zeta$ is determined by Eq. 94. This implies that for small $m_r a$, this unwanted $(\vec{p} a)^2$ term has the size $(\vec{p} a)^2 m_r a$ and is therefore $O(\vec{p} a)^2$ relative to the mass term, $m_r a$. Second, as $m_r a$ increases this expression grows no faster than the other $\sinh(m_r a)$ factors in Eq. 104. Thus, the unphysical $(\vec{p} a)^2$ term is actually of order $(\vec{p} a)^2$ relative to the continuum terms in the Dirac propagator for all values of $m_r a$.

The fact that a single choice of the transformation parameter $\delta$ is sufficient to both replace the coefficient of $\vec{\gamma} \cdot \vec{p}$ by its proper value and to remove the $(\vec{p} a)^2$ term is a result of the relationship in Eq. 94 between $\zeta$ and $r_s$, derived previously to insure the correct dispersion relation.

### C. Quark-gluon vertex

We will now determine the parameters $c_B$ and $c_E$ by computing the quark-gluon vertex after the transformation of Eq. 105 has been applied to the initial and final spinors. In particular, $c_E$ and $c_B$ should be chosen so that the tree-level lattice vertex agrees with the corresponding continuum expression, with errors no larger than $(\vec{p} a)^2$, $(\vec{p} a)^2$ and $\vec{p} \cdot \vec{p} a^2$. There should be no contribution of order $(m_r a)^n$, $|\vec{p} a|(m_r a)^n$ or $|\vec{p} a|(m_r a)^n$ for all values of $n$.

Following the conventions listed in Appendix A we can determine the quark-gluon vertex $\Lambda_\mu(p', p)$ and then impose the on-shell conditions:

$$\bar{u}(\vec{p}') \Lambda_\mu(p', p) u(\vec{p}) = Z_q \bar{u}(\vec{p}') \gamma_\mu u(\vec{p}). \quad (108)$$
Here all quantities are evaluated following our Euclidean space conventions with the exception of the time components of the on-shell fermion momenta $p'_0 = i\tilde{p}'_0 = i\sqrt{(\vec{p'})^2 + m_r^2}$ and $p_0 = i\tilde{p}_0 = i\sqrt{(\vec{p})^2 + m_r^2}$.

The tree-level lattice vertex matrices $\Lambda_\mu(p', p)$ can be derived from the lattice action of Eq. 1 and written without approximation as:

$$\Lambda^k(p', p) = \gamma^k \zeta \cos \left( \frac{(p'_k + p_k)a}{2} \right) - ir_s \sin \left( \frac{(p'_k + p_k)a}{2} \right) \sin \left( \frac{(p'_j - p_j)a}{2} \right)$$

$$+ \frac{c_B}{2} \sum_j \sigma_{kj} \cos \left( \frac{(p'_k - p_k)a}{2} \right) \sinh \left( \frac{(\vec{p}'_0 - \vec{p}_0)a}{2} \right)$$

$$+ i \frac{c_E}{2} \sigma_{k0} \cos \left( \frac{(p'_k - p_k)a}{2} \right) \sin \left( \frac{(p'_j - p_j)a}{2} \right)$$

$$\Lambda^0(p', p) = \gamma^0 \cosh \left( \frac{(\vec{p}'_0 + \vec{p}_0)a}{2} \right) + r_t \sinh \left( \frac{(\vec{p}'_0 + \vec{p}_0)a}{2} \right)$$

$$+ \frac{c_E}{2} \sum_j \sigma_{0j} \cosh \left( \frac{(p'_0 - p_0)a}{2} \right) \sin \left( \frac{(p'_j - p_j)a}{2} \right) + ic_E \sigma_{k0} \sinh \left( \frac{(\vec{p}'_0 - \vec{p}_0)a}{2} \right)$$

$$+ \frac{c_B}{2} \sum_j \sigma_{kj} \cosh \left( \frac{(p'_0 - p_0)a}{2} \right) \sin \left( \frac{(p'_j - p_j)a}{2} \right)$$

(109)

(110)

1. **Spatial component of the quark-gluon vertex**

We first examine the spatial quark gluon vertex $\Lambda^k$ transformed by the spinor matrices $U_L(\vec{p}')$ and $U_R(\vec{p})$:

$$[\Lambda_k(p', p)]_T = U_L(\vec{p}')^\dagger \Lambda_k(p', p) U_R(\vec{p})^\dagger$$

$$= \zeta \gamma_k - i \left( \frac{r_s}{2} + \delta \zeta \right) (p_k + p'_k)a + (\frac{c_B}{2} + \delta \zeta) \sum_j \sigma_{kj} (p'_j - p_j)a$$

$$+ i \frac{c_E}{2} \sigma_{k0} \sinh \left( \frac{(\vec{p}'_0 - \vec{p}_0)a}{2} \right)$$

(111)

(112)

where the subscript $T$ indicates that we have applied the spinor transformations $U_L(\vec{p}')$ and $U_R(\vec{p})$. In addition, some terms of relative order $(\vec{p}')^2$ and $(\vec{p}_0)^2$ have been neglected.

The expression in Eq. (112) can be simplified if we recognize that this matrix is to be evaluated between the spinors $\bar{u}(\vec{p}')$ and $u(\vec{p})$ so that we can use the relevant Dirac equation:

$$\left( \gamma^0 \vec{p}_0 - i \vec{\gamma} \cdot \vec{p} - m_r \right) u(\vec{p}) = 0$$

(113)

$$\bar{u}'(\vec{p}') \left( \gamma^0 \vec{p}'_0 - i \vec{\gamma} \cdot \vec{p}' - m_r \right) = 0.$$  

(114)

These two equations can be multiplied by $\gamma^k$ on the left and right respectively to derive an equation for $\sigma_{kj}(p'_j - p_j)$:

$$\sigma_{kj}(p'_j - p_j) = 2m_r \gamma^k + i(p'_k + p_k) - i\sigma_{k0}(\vec{p}'_0 - \vec{p}_0)$$

(115)
Substituting the relation above for the \(\sigma_{kj}\) term in Eq. 112, we find

\[
[\Lambda_k(p',p)]_T = \gamma^k (\zeta + m_r a(c_B + 2\delta \zeta) + \frac{c_B - r_s}{2}(\vec{p}'_k + \vec{p}_k) a + i_\sigma_{k0} \left( \frac{c_E}{2} \sinh[(\vec{p}'_0 - \vec{p}_0)a] - (\delta \zeta + \frac{c_B}{2})(\vec{p}'_0 - \vec{p}_0)a \right) \tag{116}
\]

The matrix \(\Lambda^k(p',p)\) will reduce to the desired continuum quantity \(Z_q\gamma^k\) provided the following conditions are obeyed:

\[
c_B = r_s \tag{117}
\]

\[
\zeta + m_r a(c_B + 2\delta \zeta) = Z_q \tag{118}
\]

\[
\bar{u}'(\vec{p}')\sigma_{k0} u(\vec{p}) \left( \frac{c_E}{2} \sinh[(\vec{p}'_0 - \vec{p}_0)a] - (\delta \zeta + \frac{c_B}{2})(\vec{p}'_0 - \vec{p}_0)a \right) = O(\vec{p}a)^2 Z_q \tag{119}
\]

We will treat the first of these conditions, Eq. 117, as determining the quantity \(c_B\). The second equation, Eq. 118, is then automatically obeyed as can be seen by using \(Z_q\) as determined by Eq. 106 and substituting the expressions given for \(\delta\) and \(\zeta\) in Eqs. 106 and 94 respectively.

Establishing the final condition, Eq. 119, requires a little more effort since it is not exact and must hold for the full range of a variety of values of \(r_s\) and \(r_t\). First we demonstrate the argument of the difference \((\vec{p}'_0 - \vec{p}_0)a\) is small so that an expansion of the \(\sinh(x)\) function is justified. We consider the square:

\[
(\vec{p}'_0 - \vec{p}_0)^2a^2 = \left( \frac{(\vec{p}'_0)^2 - (\vec{p}_0)^2}{\vec{p}'_0 + \vec{p}_0} \right)^2 a^2 \tag{120}
\]

\[
= \left( \frac{(\vec{p}')^2 - (\vec{p})^2}{(\vec{p}'_0 + \vec{p}_0)^2} \right) ((\vec{p}'a)^2 - (\vec{p}a)^2). \tag{121}
\]

Here the first factor on the right hand side of Eq. 121 is bounded for all values of \(\vec{p}'\) and \(\vec{p}\), while the second factor is \(O(\vec{p}a)^2\). This justifies keeping only the first term in an expansion of the \(\sinh(x)\) function and replacing the third condition by

\[
\bar{u}'(\vec{p}')\sigma_{k0} u(\vec{p}) \left( \frac{c_E - c_B}{2} - \delta \zeta \right) (\vec{p}'_0 - \vec{p}_0)a = O(\vec{p}a)^2 Z_q \tag{122}
\]

Next we make the choice \(c_E = c_B\), multiply and divide the left hand side of Eq. 122 by \(m_r a\) and divide by \(Z_q\), writing the resulting condition as:

\[
\frac{m_r \bar{u}'(\vec{p}')\sigma_{k0} u(\vec{p})}{\vec{p}'_0 + \vec{p}_0} \left( (\vec{p}'a)^2 - (\vec{p}a)^2 \right) \frac{\delta \zeta}{m_r a Z_q} = O(\vec{p}a)^2. \tag{123}
\]
The left-most ratio in Eq. 123 is a kinematic function, which is bounded for all values of \( m_r \). The central factor provides the desired \( O(\tilde{p} a)^2 \) suppression. We need to show that the final factor, \( \delta \zeta / (m_r a Z_q) \) is bounded for all \( m_r a \). To do this we must require that for small \( m_r a \), \( r_t - r_s \propto m_r a \). Without this requirement, \( \delta \zeta \) approaches a constant as \( m_r a \to 0 \) and this factor diverges for small \( m_r \) as \( 1/m_r a \). Were we to choose non-covariant values for \( r_t \) and \( r_s \) in the limit of small \( m_r a \), then the non-covariant choice \( c_E \neq c_B \) would also be required. For simplicity, we make the choice \( r_t = r_s \) for all values of \( m_r \). Under these circumstances, it is easy to see by direct numerical evaluation that the factor \( \delta \zeta / (m_r a Z_q) \leq 1/12 \), its value at \( m_r a = 0 \) for all values of \( r_s > 0 \) and \( m_r a \). Thus, condition 123 is also satisfied for the choice \( c_E = c_B \) and the spatial components of the quark gluon coupling agree with the expected continuum values to the claimed accuracy.

2. Temporal component of the quark-gluon vertex

Finally we examine the time component of the quark-gluon vertex given in Eq. 110. As a first step we will simplify this expression by recognizing that:

\[
\cosh[(\tilde{p}_0' + \tilde{p}_0)a/2] = \cosh(m_r a) + O(\tilde{p} a)^2
\]

\[
\sinh[(\tilde{p}_0' + \tilde{p}_0)a/2] = (\tilde{p}_0' + \tilde{p}_0)\frac{\sinh(m_r a)}{2m_r a} + O(\tilde{p} a)^2
\]

and neglecting the \( O(\tilde{p} a)^2 \) terms. These two equations are easy to derive from Eq. 100 using \( \cosh(x) = \sqrt{\sinh^2(x) + 1} \), the formula for the hyperbolic sine of the sum of two angles and the inequality \( \sinh(x) \leq x \cosh(x) \). With these simplifications \( \Lambda^0 \) becomes:

\[
\Lambda^0(p', p) = \gamma^0 \cosh(m_r a) + r_t a(\tilde{p}_0' + \tilde{p}_0)\frac{\sinh(m_r a)}{2m_r a} + \frac{r_s}{2} \sum_j \sigma_{0j}(p'_j - p_j)a
\]

where we have replaced \( c_E \) by the value determined earlier, \( c_E = c_B = r_s \).

Next, the spinor transformations \( U_L(p') \) and \( U_R(p) \) are made yielding

\[
\Lambda^0(p', p)_T = U_L(p')^\dagger \Lambda^0(p', p) U_R(p)^\dagger
\]

\[
= \gamma^0 \cosh(m_r a) + r_t a(\tilde{p}_0' + \tilde{p}_0)\frac{\sinh(m_r a)}{2m_r a}
+ \left(\frac{r_s}{2} + \delta \cosh(m_r a)\right) \sum_j \sigma_{0j}(p'_j - p_j)a.
\]
Here we have neglected the term:

\[
\bar{u}(\vec{p}')\gamma^j(p_j' + p_j)au(p)r_t(a(\vec{p}'_0 + \vec{p}_0)\sinh(m_r a) \over 2m_r a)
\]

(129)

because, as in the case of Eq. 123, the spinor structure mixes upper and lower spinor components implying that this expression of order \((\vec{p}a)^2\) and therefore negligible.

As a final step we use the time-component equivalent of Eq. 115 multiplied by \(r_t\sinh(m_r a)/(2m_r a)\),

\[
\frac{r_t\sinh(m_r a)}{2m_r a} a(\vec{p}'_0 + \vec{p}_0) = \frac{r_t\sinh(m_r a)}{2m_r a} \left\{ 2m_r a\gamma^0 - \sum_j \sigma_0 j (p'_j - p_j) a \right\},
\]

(130)

to eliminate the \(\vec{p}'_0 + \vec{p}_0\) term from Eq. 128. The resulting expression is

\[
\Lambda^0(p',p)_T = \gamma^0 \left\{ \cosh(m_r a) + r_t \sinh(m_r a) \right\} + \left\{ \frac{r_s}{2} + \delta \cosh(m_r a) - r_t r_t \sinh(m_r a) \over 2m_r a \right\} \sum_j \sigma_0 j (p'_j - p_j) a.
\]

(131)

The first term in this equation is precisely the desired matrix \(\gamma^0 Z_q\) while the second can be shown to be of order \((\vec{p}a)^2 Z_q\) using the same style of argument that permitted us to neglect the similar term in Eq. 123 and the expression 129.

In conclusion, we have verified at tree level that only three, mass-dependent parameters, \(m_0, \zeta\) and \(c_P = c_B = c_E\), are needed to realize a heavy quark action that is accurate through order \(|\vec{p}|a\) and to arbitrary order in \(m_r a\). We have the freedom to choose \(r_s\) and \(r_t\) as is convenient but must require that as \(m_r a\) approaches zero, \(r_s \to r_t\). The on-shell quark propagator and quark-gluon vertex take their continuum form after a simple \(4 \times 4\) transformation is performed on the two external spinors.

VI. CONCLUSION

It is presently impractical to study charm or bottom physics on a sufficiently fine lattice to control discretization errors of order \(ma\). However, as established in Refs. [1] and [2], such errors can be avoided even when \(ma \geq 1\) by using an improved heavy quark action. Such an action will accurately describe heavy quark states which are at rest or have small spatial momenta and, as the quark mass is made lighter or the lattice spacing finer, will smoothly approach the usual \(O(a)\)-improved fermion action of Sheikholeslami and Wohlert [6]. Here
we are referring to this improved action as the “relativistic heavy quark” action because of this smooth connection with relativistic fermions as $ma \rightarrow 0$ and to distinguish it from the non-relativistic and static approximations which do not have this property.

By carrying out a systematic expansion in powers of $a$ but working to all orders in the product $ma$, we have established that only three parameters, $m_0$, $\zeta$ and $c_P$, need to be tuned to remove all discretization errors of order $\Lambda_{\text{QCD}}$. It is interesting to point out that the possible overestimate of the number of relevant parameters in the Fermilab and Tsukuba results was actually suggested to us by the numerical work described in the companion paper [4].

In that paper we attempt to determine the relativistic heavy quark parameters by a process of step scaling, beginning with a very fine lattice where a direct use of the domain wall fermion formulation gives accurate results. We initially attempted to determine the four parameters, $m_0$, $\zeta$, $c_B$ and $c_E$, that could be used on at $16^3 \times 32$, $1/a = 3.6$ GeV lattice to reproduce the heavy-heavy and heavy-light spectra given by a domain wall fermion calculation on a $24^3 \times 48$, $1/a = 5.4$ GeV lattice. To our dismay, this was not a solvable problem, at least with masses measured on the one percent level. We found a one-dimensional subspace in this four-dimension parameter space along which all of the seven, finite-volume masses that we computed did not change. This surprising numerical result lead us to study more closely the underpinnings for the relativistic heavy quark formalism and to the 3-parameter result presented here.

As is explained in detail in the companion paper, the problem of determining three parameters by step-scaling is numerically very stable and determining these heavy quark parameters to a few percent is not difficult. Although this first exploratory numerical work is done within the quenched approximation, as discussed in Ref. [4], we believe that similar results will be possible in full QCD. Thus, this approach to heavy quark physics, especially in the charm region where only 2 or 3 step-scaling steps are needed, may provide a first-principles approach with no reliance on perturbation theory. The three parameters needed in the heavy quark action as well as those required for improved operators can be determined non-perturbatively by this step-scaling approach, with the three in the action requiring the most effort.

Further, as available resources increase, one can work at increasingly fine lattice spacing, minimizing the higher order errors that have not been explicitly removed. No change in
formalism is needed. However, extrapolation to the continuum limit is in general not possible with this approach. For example, the \( O(\Lambda_{QCD}a)^2 \) terms neglected in the treatment above are expected to enter with coefficients which are themselves functions of \( ma \). Thus, a simple \( a^2 \) behavior for small \( a \) will be seen only in the limit \( ma \approx 0 \), a region in which the improvements we have discussed are not needed.

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**APPENDIX A: CONVENTIONS**

As described in Sec. I, we use hermitian Dirac gamma matrices, appropriate for Euclidean lattice QCD calculations, which satisfy \( \{ \gamma^\mu, \gamma^\nu \} = 2\delta^\mu\nu \). In addition, we use the matrices \( \sigma_{\mu,\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \). Since the the methods discussed here are to be applied in the approximate rest system of the heavy quark, when an explicit choice for the gamma matrices is needed, we adopt conventions where \( \gamma^0 \) is diagonal. Specifically, in terms of standard \( 2 \times 2 \) blocks, we use:

\[
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}
\]

where \( I \) is the \( 2 \times 2 \) identity matrix and the \( \sigma^i \) are the standard Pauli matrices.

With this choice of gamma matrices, a spinor solution of the Dirac equation, describing an on-shell particle with 3-momentum \( \vec{p} \) and energy \( \tilde{p}_0 = \sqrt{\vec{p}^2 + m_r^2} \), can be written

\[
u_s(\vec{p}) = \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma} \cdot \vec{p}}{\tilde{p}_0 + m_r} \chi_s \end{pmatrix}
\]

where \( \chi_s \) is a two-component column vector describing the two possible spin-1/2 states labeled by \( s = \pm 1/2 \).

We determine the tree-level quark-vertex from the lattice action of Eq. II by replacing the link variables \( U_\mu(n) \) by the combination \( 1 - igt^a A_\mu^n(n + \hat{e}_\mu/2) \), and writing the external field \( A_\mu^n(n + \hat{e}_\mu/2) \) in terms of the Euclidean Fourier transform:

\[
A_\mu^n(n + \hat{e}_\mu/2) = (\frac{a}{2\pi})^2 \prod_{\nu=0}^3 \int_{\pi/a}^{-\pi/a} dq e^{i q \cdot (n + \hat{e}_\mu/2)} A_\mu^a(q).
\]
Here the matrices $t^a$ are hermitian generators of the gauge group. The matrix coefficient of the amplitude $A_\mu(p' - p)$ appearing in the tree-level evaluation of the action is identified as 

$$-ig\bar{u}(p')t^aA_\mu(p', p)u(p).$$

Note with the relationship between $U_\mu(n)$ and $A_\mu(n + \hat{e}_\mu/2)$ adopted above, our covariant derivatives become:

$$D_\mu \psi = (\partial_\mu -igt^a A^a_\mu)\psi \quad (A4)$$

$$\bar{\psi}D_\mu \psi = \bar{\psi}(\partial_\mu +igt^a A^a_\mu) \quad (A5)$$

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TABLE I: Enumeration of the transformation coefficients that have been chosen to remove redundant terms in $\mathcal{L}_{\text{eff}}$ and the result achieved.

| Coefficient | Term in $\mathcal{L}_{\text{eff}}$ that was improved | Order |
|-------------|-----------------------------------------------|-------|
| $r_{N_0+1}^{0,1}$ | Coef. of $\gamma^0 D^0$ set to 1 | $1/a$ |
| $s_{N_0+1}^{0,1}$ | Coefs. $\propto (aD^0)^2$ removed from the mass term | $1/a$ |
| $r_{N_0+1}^{1,1}$ | Coef. of $\vec{D}^2$ set to 0 | $a$ |
| $s_{N_0}^{1,1}$ | Coefs. $\propto (aD^0)^2$ removed from $\gamma \vec{D}$ term | $a^0$ |
| $r_{N_0+1}^{2,1}$ | Coef. of $a\{\vec{D}^2, a\gamma^0 D^0\}$ set to 0 | $a$ |
| $s_{N_0}^{2,2}$ | Coefs. $\propto (aD^0)^2$ removed from $[\gamma^i, \gamma^j][D^i, D^j]$ term | $a$ |
| $r_{N_0+1}^{2,2}$ | Coef. of $a\{[\gamma^i, \gamma^j][D^i, D^j], a\gamma^0 D^0\}$ set to 0 | $a$ |
| $s_{N_0}^{2,3}$ | Coefs. $\propto (aD^0)^2$ removed from $[\gamma^i, \gamma^0][D^i, D^0]$ term | $a$ |
| $r_{N_0}^{2,3}$ | Equate $[\gamma^i, \gamma^0][D^i, D^0]$ coef. to that of $[\gamma^i, \gamma^j][D^i, D^j]$ | $a$ |
FIG. 1: A class of diagrams contributing to the $4 \times 4$ spinor renormalization matrix $Z_{\alpha\beta}$ connecting the improved and un-improved fields, $\bar{\psi}^c$ and $\bar{\psi}^0$ respectively. Here the point-like vertex represented by the cross corresponds to the composite, improved operator $\psi^c$ which contains products of the quark and gluon fields. The graph contained within the shaded circle must be one-particle-irreducible.