The behaviour of the master field in “induced QCD” near the edge of its support is studied. An extended scaling domain, where the shape of the master field is a universal function, is found. This function is determined explicitly for the case of dimensions, close to one, and the $D - 1$-expansion is constructed. The problem of the meson spectrum corresponding to this solution is analyzed. As a byproduct of these calculations, a new, explicit equation for the meson spectrum in “induced QCD” with a general potential is derived.

April 1994
1. Introduction.

The “induced QCD” is a lattice gauge theory of a scalar matrix field $\Phi(x)$ defined by the functional integral \[ Z = \int \mathcal{D}\Phi(x)\mathcal{D}U_\mu(x) \exp \left\{ -N \sum_x \left[ \text{tr}U(\Phi(x)) - \text{tr} \sum_\mu \Phi(x)U_\mu(x)\Phi(x + \mu a)U_\mu^\dagger(x) \right] \right\}. \] (1.1)

In this formula $x$ marks the sites of the $D$-dimensional cubic lattice with lattice spacing $a$, $\Phi(x)$ are $N \times N$ Hermitian matrices and $U_\mu(x)$ is the gauge field ($\mu$ referring to the directions of lattice links, $\mu = 1, \ldots, D$).

The field $\Phi(x)$ interacts with itself through the potential $U(\Phi)$ and has the obvious kinetic term $\text{tr} \left[ \Phi(x)U_\mu(x)\Phi(x + \mu a)U_\mu^\dagger(x) \right]$. This term is the only place where the gauge field enters this theory. Indeed, no special action for the gauge field is included in the model.

Since $U_\mu(x)$ is incorporated in the action in such a simple way, the “induced QCD” is, in principle, exactly solvable in the large $N$ limit. In this respect (1.1) is one of the very few matrix theories which admit an exact solution not only in $D \leq 1$, but also in $D > 1$ dimensions. Needless to say, it would be extremely interesting to investigate such a solution, especially in the $D > 1$ case. In particular, the scaling properties of this model near its phase transition point must exhibit universal features, characteristic of other $D > 1$ models.

In the large $N$ limit the eigenvalue density $\rho(\phi)$ of the field $\Phi(x)$ obeys classical equations of motion. That is to say, $\rho(\phi)$ plays the role of the master field in “induced QCD”. Its classical dynamics can be formulated in two ways. One way is to use the Schwinger–Dyson equations for the functional integral (1.1), obtaining a nonlinear singular integral equation for $\rho(\phi)$ [2]. Quite remarkably, for the case of quadratic $U(\Phi)$ this equation can be solved exactly [3]. On the other hand, the same density $\rho(\phi)$ is related to solutions of a certain quasilinear partial differential equation (the Hopf equation) [4]. These two formulations may appear very different, but in fact they are equivalent. Under some general assumptions about the potential $U(\Phi)$ both of them reduce to the same functional equation [4], [5], [6].

In the vicinity of critical points, where $\rho(\phi)$ vanishes, the small fluctuations of the eigenvalue density become significant, leading to critical behaviour. The location of the
critical point depends on parameters of the potential, but the critical indices are, presumably, universal. Remarkably, in the critical domain the master field equation simplifies, which allows one to determine the critical indices without constructing the complete solution of the whole master field equation.

The critical point can be positioned either inside of the support of $\rho(\phi)$ or at the edge of this support. The critical indices, corresponding to these two cases, are different. If $\rho(\phi)$ vanishes inside of its support (for an even potential this would occur at $\phi = 0$) the eigenvalue density behaves at the critical point as

$$\rho(\phi) \propto |\phi|^{1+\gamma}, \quad \cos \pi\gamma = \frac{D}{3D-2}.$$  

This case is rather sophisticated. In particular, an important problem of the mass spectrum corresponding to this critical point is still unsolved.

The other, simpler situation was considered recently by Boulatov. He assumed that the master field vanishes at the endpoint of its support according to

$$\rho(\phi) \propto (\phi - a)^{1+\gamma} \theta(\phi - a).$$

He found two possible solutions for $\gamma$:

$$\cos \pi\gamma = D \quad \text{or} \quad \cos \pi\gamma = \frac{D}{2D-1}.$$  

However, it is still unclear how the Boulatov’s master field behaves away from the endpoint. The scalar field potential giving rise to this solution is not known either. Finally, although, by construction, this master field corresponds to an extremum of the action of the “induced QCD”, it may maximize, rather than minimize, the action.

In this paper we will take a closer look at Boulatov’s solution. In section 2 we determine the shape of the master field away from the endpoint. We find that there is a whole domain where this field is described by certain real universal functions, which can be represented in terms of a power series. In section 3 we consider the case of dimensions, close to 1, and construct the $D-1$-expansion of the master field. This allows us to sum the series and evaluate these universal functions explicitly.

Finally, we address the problem of spectrum in “induced QCD”. In section 4 we use the connection between the “induced QCD” and the Hopf boundary problem to obtain a new, simpler form of the wave equation, describing the meson spectrum of the theory. This derivation applies to “induced QCD” with any potential and is not in any way restricted to
Boulatov’s solution, the spectrum of which we shall determine in section 5. Our calculations show that in this case the meson spectrum contains tachyons. This implies that such solution does not correspond to a local minimum of the action. Nevertheless, the master field we are investigating exhibits some general features which are likely to be present in other, healthier solutions.

2. The Series Expansion of the Master Field.

The exact solution of the large-$N$ “induced QCD” is based on the saddle point method [1], [2]. First, one integrates out $U_\mu(x)$ in (1.1) using the Itzykson-Zuber integral [8]

$$I(\Phi, \Psi) = \int \mathcal{D}U \exp[N \text{tr} \Phi U \Psi U^\dagger] = \frac{\det[\exp(N\phi_i\psi_j)]}{\Delta(\phi)\Delta(\psi)},$$

where

$$\Delta(x) = \prod_{i < j}(\phi_i - \phi_j),$$

$\phi_i$ and $\psi_j$ being the eigenvalues of the $N \times N$ matrices $\Phi$ and $\Psi$. One is left then with an effective theory of the “eigenvalue fields” $\phi_i(x)$ which has the action

$$S_{\text{eff}}[\phi(x)] = N \sum_{x,i} U(\phi_i(x)) - \sum_{x,i \neq j} \ln|\phi_i(x) - \phi_j(x)|$$

$$+ \sum_{x,\mu} \ln[I(\phi(x), \phi(x + \mu a))].$$

In the large $N$ limit the remaining functional integral over $\phi_i(x)$ is dominated by the saddle point of $S_{\text{eff}}[\phi(x)]$, that is, by $x$-independent $\phi_i(x) = \phi_i$ satisfying

$$\frac{\partial}{\partial \phi_i} S_{\text{eff}}[\phi] = 0.$$

Since the effective action $S_{\text{eff}}$ is invariant with respect to any permutation of $\phi_i, i = 1, \ldots, N$, the saddle point can be fully described by specifying the density of eigenvalues $\phi_i$ on the real line, $\rho(\phi)$. This density plays the role of the “master field” in this theory. To solve the “induced QCD” means to find $\rho(\phi)$ for the given interaction potential $U(\Phi)$. This, in turn, allows one to evaluate other physical observables of the theory.

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1 The second term in $S_{\text{eff}}$ is due to the fact that the integration measure $\mathcal{D}\Phi(x)$ after the change of variables to $\phi_i(x)$ reduces to $\Delta^2(\phi(x)) \prod_i \mathcal{D}\phi_i(x)$. 
As a consequence of the saddle point condition (2.3), \( \rho(\phi) \) obeys a certain nonlinear singular integral equation \([2]\). However, for a generic potential \( U(\Phi) \) it is possible to derive a functional constraint fixing \( \rho(\phi) \) \([3]\). One introduces the two functions

\[
G_\pm(x) = \frac{1}{2D} U'(x) + \frac{D - 1}{D} \mathcal{P} \int \frac{\rho(y) dy}{x - y} \pm i \pi \rho(x). \tag{2.4}
\]

Then the saddle point of the “induced QCD” is determined by the equations

\[
\begin{align*}
G_+(G_-(x)) &= x, \\
G_-(G_+(x)) &= x. \tag{2.5}
\end{align*}
\]

That is to say, \( G_+(x) \), as a function of \( x \), is inverse with respect to \( G_-(x) \).

Generally speaking, the master field \( \rho(x) \) is not equal to zero only inside a finite interval (referred to as the support of \( \rho(x) \)), and vanishes at its endpoints. However, it is the behaviour of \( \rho(x) \) in the vicinity of endpoints that determines the universal properties of the theory. For example, if \( \rho(x) \simeq (a - x)^{\gamma+1} \) near the endpoint \( a \), then the exponent \( \gamma \) is universal.

The general solution of equations (2.5) is unknown. Therefore, there is no direct way to classify all possible values of \( \gamma \) that can be achieved in “induced QCD”. Nevertheless, in the vicinity of a point where \( \rho(x) \) vanishes, the equations should simplify. Boulatov \([4]\) has observed that (2.5) admits a very simple solution, \( G_+(x) = G_-(x) = -x \). Clearly, such solution by itself does not describe the “induced QCD” with any potential, since it contains no imaginary part at all. To correct this drawback, Boulatov perturbed \( G_\pm \) with a powerlike term,

\[
G_+(x) = -x + \alpha(-x)^{1+\gamma} + \ldots. \tag{2.6}
\]

If \( \gamma \) is not an integer, then for \( x > 0 \)

\[
\pi \rho(x) = \text{Im} G_+(x) = \alpha x^{1+\gamma} \sin(\pi \gamma) + \ldots \neq 0.
\]

Boulatov found that this ansatz is consistent with (2.4) only if \( \gamma \) assumes a certain value,

\[
\cos \pi \gamma = D \quad \text{or} \quad \cos \pi \gamma = \frac{D}{2D - 1}. \tag{2.7}
\]

2 This formula defines \( G_\pm(x) \) for real \( x \). For complex \( x \), \( G_\pm(x) \) are defined by means of analytic continuation from the real axis.

3 It is not known, however, whether or not some nonanalytic solutions of induced QCD can be described by this constraint.
In other words, the behaviour of the master field near the endpoint of its support is completely determined by the dimension $D$. Notice that, except for some special cases, $\gamma$ is not a rational number.

Since the constraint (2.5) is nonlinear, any perturbation like $(-x)^{1+\gamma}$ will inevitably generate perturbations of higher orders. One ends up with a whole series for $G_+(x)$,

$$G_+(x) = -x + \sum_{k=1}^{\infty} g_k(-x)^{\gamma_k+1}. \quad (2.8)$$

If $\gamma$ were rational, $\gamma_k + 1$ could become an integer for a sufficiently large $k$. In this case the $k$-th order term should be interpreted as induced by the (polynomial) potential $U(x)$ in (2.4). However, if $\gamma$ is not rational, such phenomenon can never occur, and all of the coefficients $g_k$ can be determined dynamically, from the master field equations (2.5) alone. This suggests that the whole infinite series (2.8), not only its first term $(-x)^{1+\gamma}$, has some universal properties. Let us show that this is indeed the case.

To begin with, let us note that the function

$$f(x) = \frac{1}{2(D-1)} U'(x) + \int \frac{\rho(y)dy}{x-y} \quad (2.9)$$

is an analytic function of $x$, which has a cut along the support of $\rho(x)$. Since on the cut $f(x \pm i0) = \frac{1}{2(D-1)} U'(x) + \mathcal{P} \int \frac{\rho(y)dy}{x-y} \mp i\pi \rho(x)$, it follows from (2.4) that

$$G_+(x) = -x + \frac{2D-1}{2D} f(x) - \frac{1}{2D} f(x),$$

$$G_-(x) = -x + \frac{2D-1}{2D} f(x) - \frac{1}{2D} f(x). \quad (2.10)$$

Boulatov’s ansatz (2.6) essentially means that $f(x)$ has a branch cut along the real axis from 0 to $+\infty$. Consequently, it can be expanded as

$$f(x) = x \sum_{k=1}^{\infty} f_k(-x)^{\gamma_k}. \quad (2.11)$$

This expansion makes sense by itself for real negative $x$. To define $f(x)$ for real positive $x$, we need to perform the analytic continuation, with the result

$$f(x) = x \sum_{k=1}^{\infty} f_k z^k x^{\gamma_k},$$
where
\[ z = e^{i\pi \gamma}. \]

Then
\[ \mathcal{F}(x) = x \sum_{k=1}^{\infty} f_k z^{-k} x^{\gamma k}. \]

Substituting this into (2.10) and introducing the notation
\[ A_k = \frac{z^k}{2D} - \frac{2D - 1}{2D} z^{-k}, \]
\[ B_k = \frac{1}{2D} - \frac{2D - 1}{2D} z^{2k}, \]

we easily derive the following expansions:
— for real positive \( x \) on the upper edge of the cut, \( \text{Im} x = +i0 \):
\[ G_+(x) = -x \left( 1 + \sum_{k=1}^{\infty} f_k A_k x^{\gamma k} \right) \]
\[ G_-(x) = -x \left( 1 + \sum_{k=1}^{\infty} f_k B_k z^{-k} x^{\gamma k} \right) \]

— for real negative \( x \) (there is no cut at \( x < 0 \)):
\[ G_+(x) = -x \left( 1 + \sum_{k=1}^{\infty} f_k A_k z^k (-x)^{\gamma k} \right) \]
\[ G_-(x) = -x \left( 1 + \sum_{k=1}^{\infty} f_k B_k (-x)^{\gamma k} \right) \]

Our goal is to determine \( f_k \). To this end, we must impose the constraint (2.5). This can be done in two different inequivalent ways. The first way is to impose \( G_+ G_- = x \) on the upper edge of the cut, where \( \text{Im} x = +i0 \).

In view of (2.13a) and (2.14d) this requirement can be written down as
\[ \left( 1 + \sum_{k=1}^{\infty} f_k A_k x^{\gamma k} \right) \left( 1 + \sum_{p=1}^{\infty} f_p B_p x^{\gamma p} \left( 1 + \sum_{q=1}^{\infty} f_q A_q x^{\gamma q} \right)^{\gamma p} \right) = 1. \]

Then one can check that the other constraint, \( G_- G_+ = x \), is fulfilled on the lower edge of the cut, where \( \text{Im} x = -i0 \). In addition, both of the constraints (2.5) are satisfied on the real axis outside of the cut, at \( x < 0 \) (see fig. 1).
The second way to impose (2.5) is to demand $G_+(G_+(x)) = x$ on the lower edge of the cut. This will lead to an equation, different from (2.13). As the treatment of these two cases is completely analogous, we will now concentrate on the first one. After that, we will briefly describe the second case.

Equation (2.13) allows us to find the coefficients $f_k$ recursively, one after another. To do this, we expand the right hand side of (2.13) in a power series of a formal parameter $t \equiv x^\gamma$:

$$
1 + (A_1 + B_1)f_1 t + ((A_2 + B_2)f_2 + (1 + \gamma)A_1 B_1 f_1^2) t^2 \\
+ [(A_3 + B_3)f_3 + ((1 + \gamma)A_2 B_1 + (1 + 2\gamma)A_1 B_2) f_1 f_2 \\
+ \frac{\gamma(1 + \gamma)}{2} A_1^2 B_1 f_1^3] t^3 + \ldots = 1.
$$

Since $f_1 \neq 0$, we obtain the consistency condition

$$
A_1 + B_1 = 0
$$

as well as

$$
f_2 = -(1 + \gamma) \frac{A_1 B_1}{A_2 + B_2} f_1^2,
$$

as well as
Using (2.12) and (2.16), we see that the consistency condition translates into
\[ z + 1 = \frac{2D}{2D - 1}, \tag{2.18} \]
which, in view of \( z = e^{i\pi \gamma} \), implies the Boulatov’s result (2.7). Since \( \gamma \) has to be real, this solution makes sense only for \( D > 1 \) or \( D < 1/3 \).

We can simplify (2.17), using (2.12) to express \( A_k \) and \( B_k \) in terms of \( z \), and eliminating \( D \) by virtue of (2.18):
\[
\begin{align*}
f_2 &= -\frac{1 + \gamma}{z^{-1} + 1 + z} f_1^2 = -(1 + \gamma) \frac{2D - 1}{4D - 1} f_1^2, \\
f_3 &= \frac{(1 + \gamma)(2 + 3\gamma)}{2(z^{-2} + z^{-1} + 2 + z + z^2)} f_1^3 = (1 + \gamma)(2 + 3\gamma) \frac{(2D - 1)^2}{4D(4D - 1)} f_1^3, \tag{2.19}
\end{align*}
\]
and so on. Note the remarkable feature of these expressions: they are all real. Indeed, since \( z \) is a complex number of absolute value 1, the denominators of \( f_k \) are invariant under complex conjugation. It is not at all obvious from (2.15) that this should be the case. Indeed, the reality of \( f_k \) is closely connected to the fact that outside of the cut the first of the two equations (2.5), \( G_+(G_-(x)) = x \), implies the second, \( G_-(G_+(x)) = x \), and vice versa. Let us now prove that all of the \( f_k \) are given by real numbers.

We will do this in two steps. The statements below are easy to infer by inspection of (2.17).

When viewed as functions of \( \{A_1, A_2, \ldots\}, \{B_1, B_2, \ldots\} \), the functions \( f_k(\{A_i\}, \{B_j\}) \) are symmetric with respect to the interchange of \( A \) and \( B \):
\[ f_k(\{A_i\}, \{B_j\}) = f_k(\{B_i\}, \{A_j\}), \]
provided that the consistency condition (2.16) is satisfied.

At first sight, this might appear false. Indeed, the expression for \( f_3 \) in (2.17b) is not at all symmetric under the interchange of \( A \) and \( B \). However, its asymmetric part equals
\[
f_3(\{A_i\}, \{B_j\}) - f_3(\{B_i\}, \{A_j\}) = \frac{\gamma(1 + \gamma)}{2} \frac{A_1 B_1}{A_3 + B_3} \frac{A_2 - B_2}{A_2 + B_2} (A_1 + B_1)
\]
and vanishes due to the consistency condition (2.16).

To prove the symmetry of $f_k$, we introduce an auxiliary variable

$$u = x \left(1 + \sum_{k=1}^{\infty} f_k x^{\gamma_k}\right). \quad (2.20)$$

Then it is a consequence of (2.13) that

$$x = u \left(1 + \sum_{p=1}^{\infty} f_p B_p u^{\gamma_p}\right). \quad (2.21)$$

Substituting (2.21) back into (2.20), we get

$$u = u \left(1 + \sum_{p=1}^{\infty} f_p B_p u^{\gamma_p}\right) \left(1 + \sum_{q=1}^{\infty} f_q A_q u^{\gamma_q} \left(1 + \sum_{l=1}^{\infty} f_l B_l u^{\gamma_l}\right)^{\gamma q}\right) \quad (2.22)$$

which is the same equation as (2.15), but with all $A_i$ replaced by $B_i$ and vice versa. Since the $f_k$ can be recursively determined from either of the equations (2.15) or (2.22) and since $f_k$ are unique, they have to be symmetric in $A$ and $B$.

If we assign to $A_j$ and $B_j$ a formal degree of $j$, then $f_k$ has degree zero for any $k$.

That is to say, if we scale

$$A_i^{(\lambda)} = \lambda^i A_i, \quad B_i^{(\lambda)} = \lambda^i B_i; \quad (2.23)$$

then

$$f_k \left(\{A_i^{(\lambda)}\}, \{B_j^{(\lambda)}\}\right) = f_k \left(\{A_i\}, \{B_j\}\right).$$

This statement follows immediately from the observation that such rescaling amounts to a redefinition of expansion parameter $x$ in (2.15), $x^\gamma \to \lambda x^\gamma$, which does not change the equations for $f$.

Now we are able to prove that $f_k$ is real for any $k$. Indeed, we can eliminate $D$ in (2.12) in terms of $z$, using (2.18). This gives

$$A_k(z) = z^k - \frac{z}{z^2+1} \left(z^k + \frac{1}{z^k}\right), \quad (2.24)$$

$$B_k(z) = 1 - \frac{z}{z^2+1} (1+z^{2k}).$$

Under complex conjugation,

$$A_k^*(z) = z^{-k} B_k(z),$$

$$B_k^*(z) = z^{-k} A_k(z).$$
Therefore,
\[ f^*_k(\{A_i\}, \{B_j\}) = f_k(\{z^{i-1}A_i\}, \{z^{-j}B_j\}) = f_k(\{z^{-i}B_i\}, \{z^{-j}A_j\}). \]
Setting \( \lambda = z^{-1} \) in (2.23),
\[ f_k(\{z^{-i}B_i\}, \{z^{-j}A_j\}) = f_k(\{B_i\}, \{A_j\}) \]
and, by the symmetry property,
\[ f_k(\{B_i\}, \{A_j\}) = f_k(\{A_i\}, \{B_j\}). \]
Hence
\[ f^*_k(\{A_i\}, \{B_j\}) = f_k(\{A_i\}, \{B_j\}), \]
so that all of the \( f_k \) are indeed real.

One can see from (2.19) that \( f_k / f^*_1 \) are universal numbers, which depend only on the dimension \( D \). The coefficient \( f_1 \) (which can be arbitrary) enters all formulas in the combination \( f_1 x^\gamma \), thus setting the scale for \( x \). We conclude that there is a whole scaling domain, where \( f_1 x^\gamma \sim 1 \). The (universal) behaviour of the master field in this domain is more complicated than the powerlike shape of the singularity at \( x \to 0 \). In the next section we will investigate this behaviour in more detail.

Finally, let us note that the condition on \( \gamma \), (2.18), admits not only a positive, but also a negative solution, \( \gamma = -(1/\pi) \arccos D/(2D - 1) \). In this case (2.14) is not a good approximation at all. However, if one takes into account the whole series for \( f(x^\gamma) \), it is possible to derive the behaviour of \( \rho(x) \) as \( x \to 0 \). In the next section we will see that in this case the master field develops a logarithmic singularity.

3. The \( D - 1 \)-expansion.

One might wonder how the universal function \( f \) behaves at finite values of its argument. Obviously, the first few terms of a Taylor series do not answer this question. One needs to evaluate all coefficients of the series and perform the summation.

This can be done explicitly when \( \gamma \to 0 \). Indeed, it is easy to see from (2.19) that the \( f_k(\gamma = 0) \) are not singular. We will see below that in fact they describe the first order in
the $D - 1$-expansion around $D = 1$. To calculate the numbers $f_k \equiv f_k(\gamma = 0)$, we return to the general formula (2.15), with $t \equiv x^\gamma$:

$$
\left(1 + \sum_{k=1}^{\infty} f_k A_k t^k\right)\left(1 + \sum_{p=1}^{\infty} f_p B_p t^p\left(1 + \sum_{q=1}^{\infty} f_{q} t^q\right)^p\right) = 1.
$$

(3.1)

Let us take the limit of this equation as $\gamma \to 0$, while keeping $t$ fixed. Using (2.24) we derive

$$
A_k(z) = i\pi \gamma k - \frac{\pi^2 \gamma^2}{2} + O(\gamma^3),
$$

$$
B_k(z) = -i\pi \gamma k + \pi^2 \gamma^2\left(k^2 - \frac{1}{2}\right) + O(\gamma^3).
$$

(3.2)

Expanding (3.1) up to $O(\gamma^3)$ and keeping in mind that $A_k(z) \sim B_k(z) \sim O(\gamma)$, we get

$$
\sum_{k=1}^{\infty} f_k(A_k + B_k) t^k + \sum_{k,p=1}^{\infty} f_k f_p A_k B_p t^{k+p} + O(\gamma^3) = 0.
$$

By virtue of (3.2) this means that the $f_k(\gamma = 0)$ satisfy the identity

$$
\sum_{k=1}^{\infty} (k^2 - 1) f_k t^k + \sum_{k,p=1}^{\infty} k p f_k f_p t^{k+p} = 0
$$

which translates into an ordinary differential equation for

$$
f(t) \equiv \sum_{k=1}^{\infty} f_k t^k,
$$

$$
t \frac{d}{dt} \frac{d}{dt} f(t) - f(t) + \left(t \frac{d}{dt} f(t)\right)^2 = 0.
$$

Introducing a new variable $\xi = \log t$ and denoting $\dot{f} \equiv df/d\xi$, we obtain

$$
\ddot{f} + \dot{f}^2 - f = 0.
$$

(3.3)

The solution of this equation is given by

$$
\log t = \int \sqrt{\frac{2}{2u + e^{-2u} - 1}} du.
$$

(3.4)

Taking into account that $f(t) \sim f_1 t$ as $t \to 0$, we finally obtain

$$
\log \frac{f_1 t}{f(t)} = \int_0^{f(t)} \left\{\sqrt{\frac{2u^2}{2u + e^{-2u} - 1}} - 1\right\} \frac{du}{u}.
$$

(3.5)
In the same approximation the master field equals
\[ \pi \rho(x) = \text{Im} G_+(x + i0) = -x \sum_{k=1}^{\infty} f_k t^k \text{Im} A^k \]
\[ = -\pi \gamma x \sum_{k=1}^{\infty} k f_k t^k = -\pi \gamma x \dot{f}(t = x^\gamma) \]
\[ = -\pi \gamma x \sqrt{f(x^\gamma) - \frac{1}{2} \left(1 - e^{-2f(x^\gamma)}\right)}. \]

(3.6)

Note that we have fixed \( x^\gamma \) to be finite, although \( \gamma \to 0 \). This means that in the scaling domain, where our expansion is valid, \( |\log x| \sim \mathcal{O}(1/\gamma) \).

Since \( \rho(x) \) has to be positive, (3.6) implies that \( f \gamma < 0 \). Therefore, there are two possibilities: either \( \gamma < 0 \) and \( f > 0 \), or \( \gamma > 0 \) and \( f < 0 \). In the second case the terms in the series expansion of \( f(t) \) get smaller and smaller, as \( x \to 0 \), and Boulavot’s result gives a good approximation. This picture fails, however, in the first case, when \( \gamma < 0 \). Then, as \( x \to 0 \), we have to use the \( t = x^\gamma \to \infty \) asymptotics for \( f(t) \). From (3.4) we derive, remembering that \( f > 0 \),
\[ \log t \simeq \int_0^{f(t)} \frac{du}{\sqrt{u}} = 2\sqrt{f(t)} \]
so that
\[ f(t) \simeq \frac{1}{4} \log^2 t, \]
and
\[ \rho(x) \simeq -\frac{\gamma^2}{2} x \log x, \quad x \to 0. \]

We see that in the scaling regime with \( \gamma < 0 \) the master field develops a logarithmic singularity at the endpoint of its support.

Since as \( D \to 1 \),
\[ \gamma = \pm \frac{1}{\pi} \sqrt{2(D-1)}, \]
(3.6) represents the leading order of the \( \sqrt{D-1} \)-expansion around \( D = 1 \). Note that the expansion parameter is \( \sqrt{D-1} \), rather than \( D-1 \). This is related to the fact that at \( D-1 \) the master field vanishes identically. The perturbation, therefore, is singular in \( D-1 \).

Higher orders of expansion can be obtained in a similar way. To evaluate the first correction, we set \( f_k = f_k(\gamma = 0) + \gamma f_k^{(1)} + \ldots \) and expand (2.15) to \( \mathcal{O}(\gamma^4) \). After some algebra one derives that the function
\[ f^{(1)}(t) \equiv \sum_{k=1}^{\infty} f_k^{(1)} t^k \]
satisfies an ordinary differential equation:

\[ \ddot{f}^{(1)} + 2 \dot{f} \dot{f}^{(1)} - f^{(1)} = - \ddot{f} \dot{f}, \]

where \( \dot{f}^{(1)} \equiv df^{(1)}/d\xi, \) \( \xi = \log t, \) and \( f \) is the leading order result given by (3.4).

In fact, this \( D-1 \)-expansion gives a quite accurate approximation. We have calculated the function \( f(t) \) numerically for \( D = 2 \) by the following extrapolation technique. We used (3.5) as a guide, and modified it as follows

\[
\log \frac{f_{11} t}{f(t)} = \int_0^{f(t)} \left\{ \sqrt{\frac{2u^2}{2u + e^{-2u} - 1}} g(u) - 1 \right\} \frac{du}{u}. \tag{3.7}
\]

The function \( g(u) \) with \( g(0) = 1 \) was computed as a Taylor series from exact equation at \( D \neq 1. \) Then, the Padé approximation was used. We checked the precision by comparing diagonal with nondiagonal approximants. The 15/15 and 14/16 approximant practically coincided for \(-2 < u < 2.\) This is where we numerically computed the corresponding integral, with 15/15 approximant for \( g(u).\)

The result, along with the plot of the function \( f \) for \( \gamma = 0 \) is presented in the figure. We checked numerically that the difference remains small even at \( D = 3.\)

![Plot](image)

**Fig. 2:** Plots of the function \( f(t) \) for \( D \to 1 (\gamma = 0) \) and for \( D = 2.\)
Analogous calculations can be carried out for the second solution, which is obtained by imposing the constraint $G_+(G_-(x)) = x$ on the lower edge of the branch cut (see the discussion after (2.15)). This leads to another solution, valid at $|D| < 1$,

$$z + \frac{1}{z} = 2D,$$

(3.8)

It is possible to construct $1 - D$-expansion (in contrast with $D - 1$-expansion we discussed above), with $\gamma = \pm \sqrt{2(1-D)}/\pi$, $\pi \rho(x) = -\pi \gamma \tilde{f}(x^{\gamma})$, and

$$\log t = \int \sqrt{\frac{2}{e^{2u} - 2u - 1}} du.$$  

(3.9)

In this case, if $\gamma < 0$, $\tilde{f}(t)$ is defined only for $f_1 t < 1.27635$, meaning that the ansatz (2.11) is internally inconsistent. The only self-consistent possibility in this case is $\gamma > 0$, $\tilde{f} < 0$.

In fact, the universal function defined by (3.9) is just a continuation of the function $\tilde{f}$ in (3.3) to $t < 0$. Indeed, using (3.3) and (3.9), and keeping in mind that $\tilde{f}(t) \sim f(t) \sim f_1 t$ as $t \to 0$, it is easy to check that

$$\tilde{f}(t) = -f(-t).$$

We see that the case $D > 1$ is described by the function $f(t)$ with $t > 0$, while its analytic continuation to $t < 0$ will give the solution for $D < 1$.

If any of these solutions is to describe the “induced QCD”, it has to minimize the action in (1.1). However, the master field equations only guarantee that the solutions above are the extremal points of the action. To find the type of extremum corresponding to these solutions one has to consider the quadratic variation of the action functional and determine the normal modes and eigenfrequencies of small oscillations around the master field.

4. General Theory of the Spectrum in “Induced QCD”.

In this section we will obtain an equation describing the normal modes of small fluctuations around the master field in the “induced QCD”. An integral equation answering this question has been derived before [7]. However, we will derive a new, simpler, equation in a different way. The result will be applicable to the “induced QCD” with any potential $U(\phi)$. We will apply it to the particular case of the Boulatov’s solution in the next section.
As one can see from (2.2), the effective action of “induced QCD” in the large $N$ limit is

$$
S_{\text{eff}} = N^2 \sum_x \left\{ \int \rho_x(\phi)U(\phi)d\phi - \sum_{\mu} F[\rho_x(\phi), \rho_{x+\mu a}(\phi)]
- \int \int d\phi_1 d\phi_2 \rho_x(\phi_1) \rho_x(\phi_2) \ln|\phi_1 - \phi_2| \right\}
$$

(4.1)

where $\rho_x(\phi)$ is the eigenvalue density at the lattice site $x$, and $F[\rho(\phi), \sigma(\psi)]$ is the large $N$ asymptotics of the Itzykson–Zuber integral:

$$
F[\rho(\phi), \sigma(\psi)] = \lim_{N \to \infty} \frac{1}{N^2} \ln I[\Phi, \Psi]
$$

$\rho(\phi)$ and $\sigma(\psi)$ being the eigenvalue distributions of matrices $\Phi$ and $\Psi$.

We would like to see whether or not the small perturbations of the master field are unstable. To this end, let us derive the equation of motion for these small perturbations (we shall refer to it as the wave equation).

Writing

$$
\rho_x(\phi) = \rho_0(\phi) + \delta \rho_x(\phi) = \rho_0(\phi) + \sum_{\vec{p}} e^{i \vec{p} \cdot \vec{x}} \delta \rho_{\vec{p}}(\phi)
$$

where $\rho_0(\phi)$ is the translationally invariant master field, determined from (2.3), we find that the part of the action, linear in $\delta \rho_{\vec{p}}(\phi)$, vanishes due to the master field equation (2.3), while the part quadratic in the fluctuations $\delta \rho_{\vec{p}}(\phi)$ is

$$
\delta^2 S_{\text{eff}} = - \sum_{\vec{p}} \int d\phi_1 d\phi_2 \delta \rho_{\vec{p}}(\phi_1) K_{\vec{p}}(\phi_1, \phi_2) \delta \rho_{-\vec{p}}(\phi_2)
$$

(4.2)

where

$$
K_{\vec{p}}(\phi_1, \phi_2) = \ln|\phi_1 - \phi_2| + D \frac{\delta^2 F[\rho, \sigma]}{\delta \rho(\phi_1) \delta \rho(\phi_2)} \bigg|_{\rho(x) = \sigma(x) = \rho_0(x)} + \Omega^2(\vec{p}) \frac{\delta^2 F[\rho, \sigma]}{\delta \sigma(\phi_1) \delta \rho(\phi_2)} \bigg|_{\rho(x) = \sigma(x) = \rho_0(x)}
$$

(4.3)

$$
\Omega^2(\vec{p}) \equiv \sum_{\mu=1}^D \cos a p_\mu.
$$

Note that since $\rho(x) = \sigma(x)$, the kernel $K_{\vec{p}}$ is symmetric in $\phi_1, \phi_2$.

Since $\int \rho(\phi)d\phi$ always equals 1, the perturbations $\delta \rho_{\vec{p}}(\phi)$ must satisfy the normalization condition

$$
\int \delta \rho_{\vec{p}}(\phi)d\phi = 0.
$$
Therefore, we can introduce the functions $\psi_\vec{p}(\phi)$ such that

$$\delta \rho_\vec{p}(\phi) = \frac{d}{d\phi} \psi_\vec{p}(\phi), \quad \psi_\vec{p}(-\infty) = \psi_\vec{p}(+\infty) = 0.$$  

The equations of motion for $\psi_\vec{p}(\phi)$ can be obtained by varying $\delta S_{\text{eff}}$ with respect to $\psi_\vec{p}(\phi)$:

$$\frac{\delta S_{\text{eff}}}{\delta \psi_\vec{p}(x)} = 2 \int \frac{\partial}{\partial x} K_\vec{p}(x, y) \delta \rho_\vec{p}(y) dy = 0. \quad (4.4)$$

This equation has solutions only for special discrete values of $\Omega^2_n$, which form the spectrum of “induced QCD”.

To proceed, we will need to evaluate the second derivatives of the Itzykson–Zuber integral, which are present in the kernel (4.3). Although $I(\Phi, \Psi)$ is in general given by a simple explicit formula (2.1), the function $F[\rho, \sigma]$ has a very complicated structure.

Let us first outline the calculation of the first functional derivatives of the Itzykson–Zuber integral. It is convenient to introduce the two functions $v(\phi)$ and $u(\psi)$, defined by

$$\frac{\partial}{\partial \phi} \frac{\delta F[\rho(\phi), \sigma(\psi)]}{\delta \rho(\phi)} = \phi + v(\phi) - \mathcal{P} \int \frac{\rho(\phi') d\phi'}{\phi - \phi'},$$

$$\frac{\partial}{\partial \psi} \frac{\delta F[\rho(\phi), \sigma(\psi)]}{\delta \sigma(\psi)} = \psi + u(\psi) - \mathcal{P} \int \frac{\sigma(\psi') d\psi'}{\psi - \psi'}. \quad (4.5)$$

Then $v(\phi)$ and $u(\psi)$ can be determined as follows [4]. One introduces an auxiliary complex function $\varphi(x, t)$ and sets up the boundary problem for the so-called Hopf equation:

$$\frac{\partial \varphi(x, t)}{\partial t} + \varphi(x, t) \frac{\partial \varphi(x, t)}{\partial x} = 0,$$

$$\text{Im} \varphi(x, t = 0) = \pi \rho(x),$$

$$\text{Im} \varphi(x, t = 1) = \pi \sigma(x). \quad (4.6)$$

Then it is possible to prove that

$$v(x) = +\text{Re} \varphi(x, t = 0),$$

$$u(x) = -\text{Re} \varphi(x, t = 1).$$

Let us emphasize that that we are considering the case when the two eigenvalue distributions entering the Itzykson–Zuber integral are not necessarily the same.

In terms of $v(x)$ and $u(x)$ the kernel of the wave equation becomes

$$\frac{\partial}{\partial \phi_1} K_\vec{p}(\phi_1, \phi_2) = (1 - D) \frac{1}{\phi_1 - \phi_2} + D \frac{\delta v(\phi_1)}{\delta \rho(\phi_2)} + \Omega^2(\vec{p}) \frac{\delta u(\phi_1)}{\delta \rho(\phi_2)}. \quad (4.7)$$
To find the functional derivatives $\delta v(\phi_1)/\delta \rho(\phi_2)$ and $\delta u(\phi_1)/\delta \rho(\phi_2)$, which correspond to the second variations of the Itzykson–Zuber integral, we have to see how the solution of the Hopf equation (4.6) changes when we vary the boundary conditions. This will allow us to derive two integral equations, constraining $\delta v(\phi_1)/\delta \rho(\phi_2)$ and $\delta u(\phi_1)/\delta \rho(\phi_2)$. The easiest way to do this is to use the fact that the Hopf equation has infinitely many integrals of motion. We can choose

$$H_q(t) = -\frac{1}{\pi t(q + 1)} \text{Im} \int dx (x - t\varphi(x,t))^{q+1}.$$  \hspace{1cm} (4.8)

It is a consequence of (4.4) that $dH_q/dt = 0$. Hence $H_q(t = 1) = H_q(t = 0)$, which, in view of the boundary conditions for $\varphi$ entails

$$-\frac{1}{\pi} \text{Im} \int dx (x - \varphi(x,1))^{q+1} = \int dx x^q \rho(x).$$

Considering a small variation of $\rho(x)$ and setting $\delta \sigma(x) = 0$ (that is, varying one of the distributions in the Itzykson–Zuber integral while keeping the other fixed), we get

$$-\frac{1}{\pi} \int dx \delta u(x)(x + u(x) - i\pi \sigma(x))^q = \int dx x^q \delta \rho(x).$$

Summing these equations for $q = 0, 1, \ldots$ with weight $z^{-q-1}$, we obtain, provided that $z$ is outside of the support of $\rho(x)$:

$$-\frac{1}{\pi} \text{Im} \int dx \frac{\delta u(x)}{z - (x + u(x)) + i\pi \sigma(x)} = \int dx \frac{\delta \rho(x)}{z - x}.$$

Now we can put $\sigma(x) = \rho(x) = \rho_0(x)$, $u(x) = v(x)$, and, denoting $R(x) = x + u(x) = x + v(x)$, we obtain the first integral equation, which determines $\delta u(x)/\delta \rho(y)$:

$$\frac{\partial}{\partial y} \int dx \left( \frac{\delta u(x)}{\delta \rho(y)} \right) \frac{\pi \rho(x)}{(z - R(x))^2 + \pi^2 \rho^2(x)} = \frac{\partial}{\partial y} \frac{\pi}{z - y}. \hspace{1cm} (4.9)$$

To obtain an equation for $\delta v(x)/\delta \rho(y)$, we use another set of conservation laws

$$I_q(t) = \frac{1}{\pi (1 - t)(q + 1)} \text{Im} \int dx (x + (1 - t)\varphi(x,t))^{q+1}.$$  

Since now

$$\delta I_q(t = 1) = \int dx x^q \delta \sigma(x) = 0,$$
the same procedure yields
\[ \delta I_q(t = 0) = \frac{1}{\pi} \text{Im} \int dx (\delta v(x) + i\pi \delta \rho(x)) (x + v(x) + i\pi \rho(x))^q = 0, \]
so that
\[ \text{Im} \int dx \frac{\delta v(x) + i\pi \delta \rho(x)}{z - R(x) - i\pi \rho(x)} = 0 \]
and
\[ \frac{\partial}{\partial y} \int dx \left( \frac{\delta v(x)}{\delta \rho(y)} \right) \frac{\pi \rho(x)}{(z - R(x))^2 + \pi^2 \rho^2(x)} = -\frac{\partial}{\partial y} \frac{z - R(y)}{(z - R(y))^2 + \pi^2 \rho^2(y)}. \quad (4.10) \]
Notice that the integration weight in (4.9) and (4.10) is the same,
\[ \frac{\pi \rho(x)}{(z - R(x))^2 + \pi^2 \rho^2(x)}. \]
So we can integrate the wave equation (4.4) over \( x \) with this weight to eliminate \( \delta v(\phi_1)/\delta \rho(\phi_2) \) and \( \delta u(\phi_1)/\delta \rho(\phi_2) \). Keeping in mind (4.7), we get the final form of the wave equation\[ \int \left[ -\frac{\Omega^2(p)}{z - y} + D \frac{z - R(y)}{(z - R(y))^2 + \pi^2 \rho^2(y)} \right. \]
\[ \left. + (D - 1) \mathcal{P} \int \frac{dx}{x - y} \frac{\pi \rho(x)}{(z - R(x))^2 + \pi^2 \rho^2(x)} \right] \delta \rho(y) dy = 0. \quad (4.11) \]
The functions \( R(x) \) and \( \rho(x) \) are known from the solution of the master field equation. In fact, using (4.5), (4.6) and (2.4), one can deduce that
\[ G_{\pm}(x) = R(x) \pm i\pi \rho(x). \]

The equation we derived here is very explicit. It contains no auxiliary functions and can be written down at once for any given solution of the master field equation. For example, it can be used to derive the spectrum of the “induced QCD” with quadratic potential \[ \text{[3]}, \]
recovering the result of Aoki and Gocksch \[ \text{[9]} \]. We will use this equation to analyze the Boulatov’s spectrum.

\[ \text{[4]} \] We remind that, by construction, \( z \) is positioned outside of the support of \( \rho(x) \).
5. The Spectrum of the Solution with the Endpoint Singularity.

The functions $R(x)$ and $\pi \rho(x)$ in the Boulatov’s case equal (see (2.6)):

$$R(x) = -x - \tilde{\alpha} \cos \pi \gamma x^{1+\gamma} + \ldots, \quad \pi \rho(x) = \tilde{\alpha} \sin \pi \gamma x^{1+\gamma} + \ldots.$$ 

As $x \to 0$, we can approximate the weight in (4.11):

$$\frac{\pi \rho(x)}{(z - R(x))^2 + \pi^2 \rho^2(x)} \to \delta(z - R(x)) \simeq \delta(z + x).$$

In this approximation the principal value integral, present in the wave equation, can be estimated as follows

$$\mathcal{P} \int \frac{dx}{x-y} \frac{\pi \rho(x)}{(z - R(x))^2 + \pi^2 \rho^2(x)} = \mathcal{P} \int \frac{dx}{x-y} \delta(z + x) = -\frac{1}{z+y}.$$ 

With the same accuracy,

$$\frac{z - R(y)}{(z - R(y))^2 + \pi^2 \rho^2(y)} \to \mathcal{P} \frac{1}{z - R(y)} \simeq \mathcal{P} \frac{1}{z + y},$$

and we see that (4.11) in the leading order becomes

$$\int \left[ -\frac{\Omega^2(\vec{p})}{z-y} + \mathcal{P} \frac{1}{z+y} \right] \delta \rho_{\vec{p}}(y) dy = 0. \quad (5.1)$$

This equation admits powerlike solutions, $\delta \rho_{\vec{p}}(y) = y^\alpha$. To find $\Omega^2(\vec{p})$, we use the formulas

$$\mathcal{P} \int \frac{y^\alpha dy}{z+y} = -\pi \cot \pi \alpha (-z)^\alpha,$$

$$\int \frac{y^\alpha dy}{z-y} = \frac{\pi}{\sin \pi \alpha} (-z)^\alpha. \quad (5.2)$$

This implies

$$\Omega^2(\vec{p}) = -\cos \pi \alpha.$$ 

At small lattice spacings

$$\Omega^2(\vec{p}) = \sum_{\mu=1}^{D} \cos a p_\mu = D - \frac{1}{2} a^2 \vec{F}^2.$$ 

---

5 We are keeping in mind that $z < 0$. 
We see that, if \( D > 1 \), whatever the index \( \alpha \) is, the particles of this theory have \( \vec{P}^2 = -m^2 > 0 \). Hence, \( m^2 < 0 \), that is, these particles are tachyons.

To study the \( D < 1 \) case we have to determine the index \( \alpha \). We shall argue that \( \alpha = 1 + n\gamma \), with \( n \) an integer.

Indeed, the wave equation (4.4) means that, as long as \( \rho_0(\phi) \) solves the master field equation, so does \( \rho_0(\phi) + \delta \rho(x) \). Since \( \rho_0(\phi) \) has an expansion in powers of \( \phi^{\gamma k} \), it is natural to look for small, coordinate-dependent, perturbations of the coefficients of this expansion. This suggests that the \( n \)-th normal mode of the wave equation arises if we perturb the \( n \)-th coefficient in, say, (2.8). Such mode will be given by

\[
\delta^{(n)} \rho_\vec{p}(x) = x^{1+\gamma n} \sum_{k=0}^{\infty} d_k^{(n)}(\vec{p}) x^{\gamma k}.
\]

The functions \( \delta^{(n)} \rho_\vec{p}(x) \) can be obtained by linear combinations from the infinite set \( \{x^{1+\gamma}, x^{1+2\gamma}, \ldots\} \). A very important property of this set is that it remains invariant under the action of the wave operator \( \partial \hat{K}/\partial x \).

Let us represent the action of the integral operator in (4.11) by an infinite-dimensional matrix \( \hat{M} \):

\[
-\int \frac{\pi \rho(x) dx}{(z - R(x))^2 + \pi^2 \rho^2(x)} \int dy \frac{\partial K(x,y)}{\partial x} \begin{pmatrix} y^{1+\gamma} \\ y^{1+2\gamma} \\ \vdots \end{pmatrix} = \hat{M}(\Omega^2) \begin{pmatrix} (-z)^{1+\gamma} \\ (-z)^{1+2\gamma} \\ \vdots \end{pmatrix}.
\]

The matrix \( \hat{M} \) has an upper-triangular form

\[
\hat{M} = \begin{pmatrix} 
\lambda_1(\Omega^2) & m_1^1 & m_1^2 & \ldots \\
0 & \lambda_2(\Omega^2) & m_2^2 & \ldots \\
0 & 0 & \lambda_3(\Omega^2) & \ldots \\
0 & 0 & 0 & \ldots 
\end{pmatrix}.
\]

The numbers \( \lambda_i \) are the exact eigenvalues of \( \hat{M} \). Remarkably, they are determined by the part of \( \partial \hat{K}/\partial x \), which does not increase the degree in \( x \), that is, by the operator on the left hand side of (5.1):

\[
\lambda_n(\Omega^2) = -\frac{\pi}{\sin \pi \alpha_n}(\Omega^2 + \cos \pi \alpha_n), \quad (5.3)
\]

where \( \alpha_n = 1 + n\gamma \). By adequately adjusting \( \Omega^2 \), we can set any given \( \lambda_n \) to zero, which would correspond to the solution of the wave equation. Therefore, the spectrum is given by

\[
\Omega_n^2 = -\cos \pi \alpha_n = \cos \pi n\gamma, \quad (5.4)
\]

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so that

\[ \vec{P}^2 \sim D - \cos \pi n \gamma. \]

If \( D < 1 \), then \( D = \cos \pi \gamma \), and \( \vec{P}^2 \sim \cos \pi \gamma - \cos \pi n \gamma \). We see that, for \( \gamma \) irrational, \(-\cos \pi n \gamma\) can get arbitrarily close to 1, thus again providing the evidence for tachyons in the spectrum.

The only possible loophole in this argument could be the locality of our analysis. We did not find the global solution, but rather expanded it near the edge singularity.

In fact, we can decide whether our solution corresponds to a minimum by checking that the eigenfrequencies of all normal modes are real and not imaginary for any value of \( \Omega^2 \). On the other hand, it is easy to see that the squares of eigenfrequencies of the operator \(-\partial \hat{K} / \partial x\) have the same sign as \( \lambda(\Omega^2) \). If \( \alpha_n = 1 + n \gamma \),

\[ \lambda_n(\Omega^2) = \frac{\pi}{\sin \pi n \gamma} (\Omega^2 - \cos \pi n \gamma). \]

Consequently, for small \( n \) these are always positive and do not cause any instability. However, at large enough \( n \gamma > 1 \), when \( \sin \pi n \gamma < 0 \) the sign in front of \( P^2 \) changes, which creates ghosts.

Although the higher \( n \) might indeed bring in ghosts and/or tachyons, one can imagine the situation where these modes are excluded by the boundary conditions. Strictly speaking, it is necessary to know the master field throughout the whole region of support to determine the spectrum fully.

6. Conclusions.

We have found that there is a special scaling domain where the shape of the master field in “induced QCD” is a universal function. We have demonstrated how to calculate this function in terms of a power series. Furthermore, we have constructed the \( D - 1 \)-expansion to investigate the global structure of this power series. Finally, we have evaluated the meson spectrum, corresponding to this master field.

Independently of the above, we have used the description of “induced QCD” in terms of the Hopf equation to derive a new, explicit, version of the wave equation for the meson spectrum of the theory. The equation we have found is valid for “induced QCD” with any potential and is not restricted to the particular application we have considered in this paper.
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