Γ-convergence of Integral Functionals
Depending on Vector-valued Functions over Parabolic Domains

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ABSTRACT: We study Γ-convergence for a sequence of parabolic functionals, \( F^\varepsilon(u) = \int_0^T \int_\Omega f(\varepsilon^\tau, t, \nabla u) dx dt \) as \( \varepsilon \to 0 \), where the integrand \( f \) is nonconvex, and periodic on the first variable. We obtain the representation formula of the Γ-limit. Our results in this paper support a conclusion which relates Γ-convergence of parabolic functionals to the associated gradient flows and confirms one of De Giorgi’s conjectures partially.

KEYWORDS: Γ-convergence, parabolic-minima, nonconvex functionals, parabolic equations.

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1. INTRODUCTION

We begin with the characterization of Γ-convergence in [1, 2].

DEFINITION 1.1. Let \((X, \tau)\) be a first countable topological space and \(\{F^h\}_{h=1}^\infty\) be a sequence of functionals from \(X\) to \(\mathbb{R} \cup \{-\infty, \infty\}\), \(u \in X, \lambda \in \mathbb{R}\). We call

\[ \lambda = \Gamma(\tau) \lim_{h \to \infty} F^h(u) \]

if and only if for every sequence \(\{u^h\}\) converging to \(u\) in \((X, \tau)\)

\[ \lambda \leq \liminf_{h \to \infty} F^h(u^h), \tag{1.1} \]

and there exists a sequence \(\{u^h\}\) converging to \(u\) in \((X, \tau)\) such that

\[ \lambda \geq \limsup_{h \to \infty} F^h(u^h). \tag{1.2} \]

We call \(\lambda = \Gamma(\tau) \lim_{\varepsilon \to 0} F^\varepsilon(u)\) if and only if for every \(\varepsilon_h \to 0\) \((h \to \infty)\)

\[ \lambda = \Gamma(\tau) \lim_{h \to \infty} F^{\varepsilon_h}(u). \]

Throughout this paper, we assume that \(\Omega\) is a bounded open set in \(\mathbb{R}^n\). Let \(p > 1, T > 0, \) and \(m\) be a positive integer. Denote

\[ \Omega_T = \Omega \times (0, T), \quad V_p(\Omega_T, m) = L^p([0, T], W^{1,p}(\Omega, \mathbb{R}^m)), \]

\[ V_p^0(\Omega_T, m) = L^p([0, T], W_0^{1,p}(\Omega, \mathbb{R}^m)). \]
and

\[ Du(x, t) = \nabla u(x, t) = \left( \frac{\partial u_i(x, t)}{\partial x_j} \right) \quad (1 \leq i \leq m, 1 \leq j \leq n) \]

for a vector valued function \( u \).

Consider the functionals

\[ F_\varepsilon^1(v, \Omega) = \int_\Omega f_1 \left( \frac{x}{\varepsilon}, Dv \right) dx, \quad v \in W^{1,p}(\Omega, R^m), (\varepsilon \to 0^+) \]  \hspace{1cm} (1.3)

and the corresponding parabolic functionals in the following form:

\[ F_\varepsilon^p(u, \Omega_T) = \int_{\Omega_T} f \left( \frac{x}{\varepsilon}, t, D\varepsilon u \right) dx dt, \quad u \in V_p(\Omega_T, m), (\varepsilon \to 0^+), \]  \hspace{1cm} (1.4)

where \( f: R^{n+1} \times R^{mn} \to R \) is a Caratheodory function satisfying

\[ C_1|\lambda|^p \leq f(x, t, \lambda) \leq C_2(1 + |\lambda|^p) \]  \hspace{1cm} (1.5)

for some positive constants \( C_2 > C_1 \).

In 1979, E. De Giorgi [3] conjectured that when a sequence of functionals, for instance, the one in (1.4) or in a more general form, converges in the sense of \( \Gamma \)-convergence to a limiting functional, the corresponding gradient flows will converge as well (maybe after an appropriate change of timescale). Also [4, p.216] and [5, p.507].

In [6], the author proved the De Giorgi’s conjecture for a rather wide kind of functionals. Thus, a natural question is that under what conditions, the functional sequence like (1.4) can be \( \Gamma \)-convergence.

A first result related to this question was appeared in [7]. Because the integrands in [7] have the same scale for the variables \( x \) and \( t \), the methods there can’t be applied to functionals (1.4) whose integrands are anisotropic in \( x \) and \( t \).

In this paper, we will cleverly combine the arguments in [8, 9, 10], all of which study the \( \Gamma \)-convergence of elliptic functionals like (1.3) with the weak-topology of \( W^{1,p}(\Omega, R^m) \), to prove that the \( \Gamma \)-convergence holds for the functional (1.4) under assumption (1.5) and a periodic hypothesis (see (1.8) below). For this purpose, we construct functionals as follows.

Let \( Y = (0, 1)^n = \{ 0 < y_i < 1, i = 1, 2, \cdots n \} \), \( kY = (0, k)^n \), and \( k_T = kY \times (0, T) \). For \( \lambda \in R^{mn} \) and a.e. \( t \in R \), define

\[ \bar{f}(t, \lambda) = \inf_{k \in N} \inf_{|kY|^{-1}} \int_{kY} f(y, t, \lambda + D\phi(y, t)) dy : \phi \in V^0_p(k_T, m), \]  \hspace{1cm} (1.6)

where and below \( |E| \overset{\text{def}}{=} L^n(E) \) and \( L^k \) is used to denote the \( k \)-dimensional Lebesque measure.
Obviously (1.5) implies that $\bar{f}(t, Du)$ is nonnegative and measurable, so we can define the homogenized functional

$$F(u, \Omega_T) = \int_{\Omega_T} \bar{f}(t, Du) dx dt, \quad u \in V_p(\Omega_T, m). \tag{1.7}$$

The main result of this paper is the following theorem.

**THEOREM 1.2.** If hypotheses (1.4) and (1.5) are satisfied, and suppose

$$f(y, t, \lambda) \text{ is } \bar{Y} - \text{periodic on the first variable } y, \tag{1.8}$$

then for every $T > 0$ and every bounded open set $\Omega \subset \mathbb{R}^n$ with $L^n(\partial \Omega) = 0$

$$\Gamma(\tau) \lim_{\varepsilon \to 0} F^\varepsilon(u, \Omega_T) = F(u, \Omega_T), \forall u \in V_p(\Omega_T, m),$$

where $\tau$ is taken as the sw-topology of $V_p(\Omega_T, m)$. (See the Def. 1.2 in [6] for the sw-topology.

The proof of this theorem will be given in section 4.

**2. PRELIMINARY LEMMAS**

We collect some properties of the $\Gamma$-limits in [1, 2] which are well-known but important for the coming arguments.

If the lim sup in (1.2) is replaced by lim inf, the definition 1.1 is turned to the definition of **low** $\Gamma$-limit. In this case, we denote it by

$$\lambda = \Gamma^-(\tau) \lim_{h \to \infty} F^h(u).$$

Similarly, we have **upper** $\Gamma$-limit and denote it by $\lambda = \Gamma^+(\tau) \lim_{h \to \infty} F^h(u)$.

Obviously, $\Gamma(\tau) \lim_{h \to \infty} F^h(u)$ exists if and only if $\Gamma^+(\tau) \lim_{h \to \infty} F^h(u) = \Gamma^-(\tau) \lim_{h \to \infty} F^h(u)$.

**LEMMA 2.1.** $F^-(u) = \Gamma^-(\tau) \lim_{h \to \infty} F^h(u)$ exists for every $u \in X$, and $F^-(u)$ is lower semicontinuous in $(X, \tau)$. If $F(u) = \Gamma(\tau) \lim_{h \to \infty} F^h(u)$ exists for every $u \in X$, then $F(u)$ is also lower semicontinuous in $(X, \tau)$.

**LEMMA 2.2.** For each sequence $\{F^h\}$ of functionals in $(X, \tau)$, there exists a subsequence $F^{h_k}$ and $F^\infty$ from $X$ to $\bar{R}$, such that

$$F^\infty(u) = \Gamma(\tau) \lim_{k \to \infty} F^{h_k}(u) \quad \forall u \in X.$$  

**LEMMA 2.3.** Suppose that $\lambda = \Gamma(\tau) \lim_{\varepsilon \to 0} F^\varepsilon(u)$ and $\varepsilon_h \to 0 (h \to \infty)$, then

$$\Gamma^-(\tau) \lim_{h \to \infty} F^{\varepsilon_h}(u) = \Gamma(\tau) \lim_{h \to \infty} F^{\varepsilon_h}(u) = \lambda$$
LEMMA 2.4. Suppose that $f: \mathbb{R} \times \mathbb{R} \rightarrow \bar{\mathbb{R}}$, then there exists a function $\delta: \varepsilon \rightarrow \delta(\varepsilon)$ such that $\varepsilon \rightarrow 0$ implies $\delta(\varepsilon) \rightarrow 0$ and

$$\limsup_{\varepsilon \rightarrow 0} f(\delta(\varepsilon), \varepsilon) \leq \limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} f(\delta, \varepsilon) \quad (2.1)$$

Moreover, the opposite inequality for low limits and the equality for limits hold true respectively.

From now on, we restrict ourselves to the sequence of functionals (1.4), or more general functionals:

$$F^{\varepsilon}(u, \Omega \times (a, b)) = \int_{a}^{b} \int_{\Omega} f(\frac{x}{\varepsilon}, t, Du) dx dt, \quad (\varepsilon \rightarrow 0^+) \quad (2.2)$$

We will fix $T > 0$ and allow $\Omega$ and $(a, b)$ to be arbitrary. Let $S = \mathbb{R}^n \times (0, T)$, $\beta_T$ be the $\sigma$-ring generated by the set

$$\{ \Omega \times (a, b): 0 \leq a < b \leq T, \Omega \subset \mathbb{R}^n \text{ are bounded open sets} \}.$$  

Then $(S, \beta_T, L^{n+1})$ is a measure space. Let

$$V_{p, \text{loc}} = L^p([0, T], W^{1,p}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)) \quad (2.3)$$

LEMMA 2.5. Assume that (1.4), (1.5) and (1.8) are satisfied. Then for every sequence $\varepsilon \rightarrow 0^+$, there exist a subsequence $\varepsilon_h \rightarrow 0^+$ ($h \rightarrow 0$) and a family of $\sigma$-finite and $\sigma$-additive measures $H(u, \Omega \times (a, b))$ on $\beta_T$, such that for every $u \in V_{p, \text{loc}}$, every finite interval $(a, b)$ and every bounded open set $\Omega \subset \mathbb{R}^n$ with $L^n(\partial \Omega) = 0$,

$$\Gamma(\tau) \lim_{h \rightarrow \infty} F^{\varepsilon_h}(u, \Omega \times (a, b)) = H(u, \Omega \times (a, b)) \quad (2.4)$$

and

$$0 \leq H(u, \Omega \times (a, b)) \leq C \int_{a}^{b} \int_{\Omega} (1 + |Du|^p) dx dt, \quad (2.5)$$

where $\tau$ is the sw-topology of $V_p(\Omega \times (a, b))$.

Proof. We follow the proof of in [9, Theorem 3.1]. $D$ is used to denote the algebra generated by all open cubes in $\mathbb{R}^{n+1}$ with rational vertices and $E$ the class of all bounded open sets in $\mathbb{R}^{n+1}$. Applying lemma 2.2 and a diagonalization argument, we can find a sequence $\varepsilon_h$ ($h \rightarrow \infty$) such that $\Gamma(\tau) \lim_{h \rightarrow \infty} F^{\varepsilon_h}(u, Q)$ exists for all $Q \in D$, i.e

$$H^{-}(u, Q) = H^{+}(u, Q), \quad \forall Q \in D,$$

where

$$H^{-}(u, Q) = \Gamma^{-}(\tau) \lim_{h \rightarrow \infty} F^{\varepsilon_h}(u, Q)$$
and
\[ H^+(u, Q) = \Gamma^+(\tau) \lim_{h \to \infty} F^\varepsilon_h(u, Q). \]

In the same way as in [9, p.738-739], by lemma B in [6], we can prove that \( H^- \) is (finitely) super-additive and \( H^+ \) is sub-additive over \( D \). For \( e \in E \), define
\[ H(u, e) = \sup_{Q \subset e} H^-(u, Q) = \sup_{Q \subset e} H^+(u, Q) \quad Q \in D, \]
then \( H(u, e) \) is an increasing, inner regular and finitely additive set function. Therefore, the routine methods implies that (2.4) holds and \( H(u, \Omega \times (a, b)) \) can be extended to a \( \sigma \)-finite and \( \sigma \)-additive measure on \( \beta_T \) (see [11, Prop. 5.5 and Theorem 5.6]). From (2.4) and (1.5), the estimate (2.5) follows immediately.

3. \( \Gamma \)-LIMITS OF LAYERED AFFINE FUNCTIONS

Throughout this section, suppose that (1.4), (1.5) and (1.8) are satisfied. \( \tau \) is used to denote the \( \text{sw} \)-topology of \( V_p(\Omega_T, m) \). For simplicity, \( V_p(\Omega T) \) denotes the space \( V_p(\Omega_T, m) \). We intend to determine the \( \Gamma \)-limits of \( F^\varepsilon(u, \Omega_T) \) for \( u = \lambda(t) \cdot x + a(t) \) with \( \lambda \in L^p([0, T], M(m \times n)) \) and \( a \in L^p([0, T], R^m) \), where we define the norm on \( M(m \times n) \), the space of all real \( m \times n \) matrices, as the same as on \( R^{mn} \).

**Lemma 3.1.** For each \( u_{\lambda, a} = \lambda(t) \cdot x + a(t) \) with
\[ \lambda \in L^p([0, T], M(m \times n)) \quad \text{and} \quad a \in L^p([0, T], R^m), \]
there exists a sequence of functions \( \{u^\varepsilon\} \subset V_p(\Omega_T) \) satisfying
\[ \{u^\varepsilon - u_{\lambda, a}\} \subset V_p^0(\Omega_T) \quad \text{and} \quad u^\varepsilon \xrightarrow{\tau} u_{\lambda, a} \quad \text{in} \quad V_p(\Omega_T) \quad \text{as} \quad \varepsilon \to 0^+ \]
such that
\[ \lim_{\varepsilon \to 0^+} F^\varepsilon(u^\varepsilon, \Omega_T) = \int_{\Omega_T} \bar{f}(t, \lambda)dxdt = F(u_{\lambda, a}, \Omega_T), \]
where \( \bar{f}(t, \lambda) \) is given by (1.6) and \( F \) by (1.7).

**Proof.** Fix \( \delta \in (0, 1) \), one can choose \( k \in N \) and \( \phi^\delta \in V^0_p(k_T, m) \) (see (1.6)) such that
\[ \bar{f}(t, \lambda(t)) \leq |kY|^{-1} \int_{kY} f(y, t, \lambda + D\phi^\delta)dy \leq \bar{f}(t, \lambda(t)) + \delta. \quad (3.1) \]
We use \( E^*_\eta \) to denote the extension of \( \eta\bar{Y} \) on the \( \eta Y \)-period, and let
\[ \Omega^*_\eta = \{e \in E^*_\eta, e \subset \Omega\}, \quad E_\eta = \bigcup_{e \in E^*_\eta} e, \quad \Omega_\eta = \bigcup_{e \in \Omega^*_\eta} e, \]
then $E_\eta = R^n$. As $\Omega$ is bounded, $\Omega^*_\eta$ is a finite set for each $\eta > 0$, and

$$\lim_{\eta \to 0^+} L^n(\Omega \setminus \Omega^*_\eta) = 0. \quad (3.2)$$

For every $t \in [0, T]$, extend $\phi^\delta(y, t)$ such that it is a $kY$-periodic function on the variable $y$, then define

$$v^{\varepsilon, \delta}(x, t) = \begin{cases} u_{\lambda, a}(x, t) + \varepsilon \phi^\delta\left(\frac{x}{\varepsilon}, t\right), & \Omega_{\varepsilon_k} \\ u_{\lambda, a}(x, t), & \Omega \setminus \Omega_{\varepsilon_k} \end{cases} \quad (3.3)$$

It is easy to know that $v^{\varepsilon, \delta} \in V_p(\Omega_T)$, $v^{\varepsilon, \delta} - u_{\lambda, a} \in V^0_p(\Omega_T)$. For each $D \in \Omega_{\varepsilon_k}$, by the periodicity of

$$g(y, t) = f(y, t, \lambda(t) + D\phi^\delta(y, t)),$$

we have

$$\int_{D \times (0, T)} f\left(\frac{x}{\varepsilon}, t, Dv^{\varepsilon, \delta}\right) dx dt = \int_0^T \left[\varepsilon^n \int_{D/\varepsilon} f(y, t, \lambda(t) + D\phi^\delta(y, t)) dy\right] dt$$

$$= L^n(D) \int_0^T dt |kY|^{-1} \int_{kY} f(y, t, \lambda(t) + D\phi^\delta) dy. \quad (3.4)$$

Summing up the both sides for all $D \in \Omega_{\varepsilon_k}$ and applying (3.1), we obtain that

$$L^n(\Omega_{\varepsilon_k}) \int_0^T \bar{f}(t, \lambda(t)) dt \leq \int_0^T dt \int_{\Omega_{\varepsilon_k}} f\left(\frac{x}{\varepsilon}, t, Dv^{\varepsilon, \delta}\right) dx$$

$$\leq L^n(\Omega_{\varepsilon_k}) \int_0^T (\bar{f}(t, \lambda(t)) + \delta) dt.$$

Thus, it follows from (1.5) and (3.3) that

$$L^n(\Omega_{\varepsilon_k}) \int_0^T \bar{f}(t, \lambda(t)) dt \leq \int_{\Omega_T} f\left(\frac{x}{\varepsilon}, t, Dv^{\varepsilon, \delta}\right) dx dt$$

$$\leq L^n(\Omega_{\varepsilon_k}) \int_0^T (\bar{f}(t, \lambda(t)) + \delta) dt + \int_0^T dt \int_{\Omega \setminus \Omega_{\varepsilon_k}} f\left(\frac{x}{\varepsilon}, t, \lambda(t)\right) dx$$

$$\leq L^n(\Omega_{\varepsilon_k}) \int_0^T (\bar{f}(t, \lambda(t)) + \delta) dt + CL^n(\Omega \setminus \Omega_{\varepsilon_k}) \int_0^T (1 + |\lambda|^p) dt. \quad (3.5)$$

By this estimate and (3.2), we see that

$$\lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \int_{\Omega_T} f\left(\frac{x}{\varepsilon}, t, Dv^{\varepsilon, \delta}\right) dx dt = \int_{\Omega_T} \bar{f}(t, \lambda(t)) dt dx. \quad (3.6)$$
Moreover, we have
\[ \|v^{\varepsilon,\delta} - u_{\lambda,a}\|_{L^p(\Omega_T)} = \varepsilon^p \sum_{D \in \Omega_{k\varepsilon}} |D| \int_0^T dt |kY|^{-1} \int_{kY} |\varphi^\delta|^p dy. \] 

(3.7)

Applying (3.6), (3.7), and lemma 2.4, one can find a sequence
\[ \delta(\varepsilon) \to 0^+ \quad \text{as} \quad \varepsilon \to 0^+ \]

such that \( \{u^{\varepsilon} = v^{\varepsilon,\delta(\varepsilon)}: \varepsilon > 0\} \) satisfy that
\[ \{u^{\varepsilon} - u_{\lambda,a}\} \subset V^0_p(\Omega_T), \quad \lim_{\varepsilon \to 0^+} \|u^{\varepsilon} - u_{\lambda,a}\|_{L^p(\Omega_T)} = 0 \]

and
\[ \lim_{\varepsilon \to 0^+} F^{\varepsilon}(u^{\varepsilon},\Omega_T) = F(u_{\lambda,a},\Omega_T). \]

On the other hand, the coercive condition in (1.5) and (3.5) imply that \( \{Du^{\varepsilon}\} \) is bounded in \( L^p(\Omega_T, R^{mn}). \) Thus, by lemma B in [6], we obtain that \( u^{\varepsilon} \rightharpoonup u_{\lambda,a}. \) This proves the desired result.

**LEMMA 3.2.** Let \( u_{\lambda,a}(x,t) = \lambda(t) \cdot x + a(t) \) be the same as in lemma 3.1. Then for each sequence \( u^{\varepsilon} \rightharpoonup u_{\lambda,a} \) in \( V^p_p(\Omega_T) \) \( (\varepsilon \to 0^+) \),
\[ \liminf_{\varepsilon \to 0^+} F^{\varepsilon}(u^{\varepsilon},\Omega_T) \geq F(u_{\lambda,a},\Omega_T) = \int_{\Omega_T} \bar{f}(t,\lambda(t))dt dx. \]

**Proof.** (1) Firstly, assume \( u^{\varepsilon} \rightharpoonup u_{\lambda,a} \) and \( u^{\varepsilon} - u_{\lambda,a} \in V^0_p(\Omega_T) \). As \( \Omega \) is bounded, we find an open cube \( D \) whose sides are parallel to axes and whose center coincides with the origin, such that \( \Omega \subset D \). The side length of \( D \) is denoted by \( 2d \), and let
\[ k_\varepsilon = \lfloor \frac{2d}{\varepsilon} \rfloor + 3, \quad a_\varepsilon = \lfloor -\frac{d}{\varepsilon} \rfloor, \]
\[ x_\varepsilon = (a_\varepsilon, \ldots, a_\varepsilon) \in R^n, D_\varepsilon = \varepsilon(x_\varepsilon + k_\varepsilon Y), \]
where \( [\kappa] \) denote the maximum integer not greater than \( \kappa \). It is not difficult to get
\[ D \subset D_\varepsilon, \quad \lim_{\varepsilon \to 0^+} L^n(D_\varepsilon) = L^n(D). \] 

(3.8)

Let
\[ Q = D \setminus \bar{\Omega}, \quad Q_T = Q \times (0,T). \]

(3.9)

Applying lemma 3.1 to the open set \( Q \), we can choose a sequence
\[ v^{\varepsilon} \to u_{\lambda,a} \quad \text{sw} \quad \text{in} \quad V^p_p(Q_T), \quad v^{\varepsilon} - u_{\lambda,a} \in V^0_p(Q_T) \]
such that
\[
\lim_{\varepsilon \to 0^+} F^\varepsilon (v^\varepsilon, Q_T) = \int_Q \bar{f}(t, \lambda)dxdt. \tag{3.10}
\]
For fixed \( t \in [0, T] \), define
\[
\phi^\varepsilon (x, t) = \begin{cases}
  u^\varepsilon - u_{\lambda, a}, & x \in \Omega \\
  v^\varepsilon - u_{\lambda, a}, & x \in D \backslash \Omega = Q \\
  0, & x \in D_\varepsilon \backslash D
\end{cases}. \tag{3.11}
\]
By the periodicity of \( f(y, t, \lambda) \), using the variable transformation, we obtain
\[
\int_{D_\varepsilon} f(\frac{x}{\varepsilon}, t, \lambda + D\phi^\varepsilon (x)) dx = |k_\varepsilon Y|^{-1} \int_{k_\varepsilon Y} f(y, t, \lambda + D\psi^\varepsilon (y)) dy, \tag{3.12}
\]
where \( \psi^\varepsilon (y, t) = \varepsilon^{-1} \phi^\varepsilon (\varepsilon(y + x_\varepsilon), t) \). Obviously, (3.11) gives us
\[
\psi^\varepsilon \in V^0_p((k_\varepsilon Y) \times (0, T)).
\]
Thus, we deduce, from (3.12) and (1.6) yield that for each \( t \in [0, T] \),
\[
|D_\varepsilon|^{-1} \int_{D_\varepsilon} f(\frac{x}{\varepsilon}, t, \lambda + D\phi^\varepsilon (x)) dx = |k_\varepsilon Y|^{-1} \int_{k_\varepsilon Y} f(y, t, \lambda + D\psi^\varepsilon (y)) dy \geq \bar{f}(t, \lambda).
\]
Therefore
\[
\int_0^T \int_{D_\varepsilon} f(\frac{x}{\varepsilon}, t, \lambda + D\phi^\varepsilon ) dxdt \geq \int_0^T \int_D \bar{f}(t, \lambda) dt.
\]
On the other hand, by (1.5) and (3.8), we have
\[
\liminf_{\varepsilon \to 0^+} \int_0^T dt \int_{D_\varepsilon} f(\frac{x}{\varepsilon}, t, \lambda + D\phi^\varepsilon ) dxdt = \liminf_{\varepsilon \to 0^+} \int_0^T dt \int_D f(\frac{x}{\varepsilon}, t, \lambda + D\phi^\varepsilon ) dx.
\]
This yields
\[
\liminf_{\varepsilon \to 0^+} F^\varepsilon (u_{\lambda, a} + \phi^\varepsilon, D \times (0, T)) \geq \int_0^T \int_D \bar{f}(t, \lambda) dxdt.
\]
Combing this estimate, (3.9), (3.10) with (3.11), we have
\[
\liminf_{\varepsilon \to 0^+} F^\varepsilon (u^\varepsilon, \Omega_T) = \liminf_{\varepsilon \to 0^+}[F^\varepsilon (u_{\lambda, a} + \phi^\varepsilon, D \times (0, T)) - F^\varepsilon (v^\varepsilon, Q \times (0, T))]
\geq \int_0^T \int_D \bar{f}(t, \lambda) dxdt - \int_0^T \int_Q \bar{f}(t, \lambda) dxdt
= \int_{\Omega_T} \bar{f}(t, \lambda) dxdt.
\]

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In order to remove the restriction \( u^\varepsilon - u_{\lambda,a} \in V_p^0(\Omega_T) \), it is sufficient to apply the De Giorgi’s arguments and the result of the case (1). See [11], or [8, p.197] for the details.

**DEFINITION 3.3.** Let \( \{\Omega_i; i = 1, 2, \cdots, h\} \) be a finite partition of \( \Omega \) into open sets (except for a set of measure zero ), \( \lambda_i \in L^p([0,T], M(m,n)) \), \( a_i \in L^p([0,T], R^m) \). We call the function

\[
W(x,t) = \begin{cases} 
\lambda_i(t) \cdot x + a_i(t), & x \in \Omega_i \\
0, & x \in \Omega \setminus \bigcup_{i=1}^h \Omega_i
\end{cases}
\]

a \( L^p \)-layered affine function on \( \Omega_T \).

To sum up lemmas 3.1 and 3.2 (observing that \( \Omega \) maybe arbitrary there ), lemmas 2.5 and 2.3, we obtain the following theorem.

**THEOREM 3.4.** Suppose that \( \Omega \) is a bounded open set in \( R^n \) with \( L^n(\partial\Omega) = 0 \), \( H(u,\Omega_T) \) is given by lemma 2.5, then

\[
\Gamma(\tau) \lim_{\varepsilon \to 0^+} F^\varepsilon(w,\Omega_T) = \int_{\Omega_T} \bar{f}(t,Dw)dxdt = H(w,\Omega_T)
\]

for any \( w \), a \( L^p \)-layered affine function on \( \Omega_T \).

### 4. A PROOF OF THEOREM 1.2

In this section, we suppose that all the hypotheses of Theorem 1.2 are satisfied. Applying the same argument as in [9, Section 5], we can prove that for almost \( t \in [0, T] \), \( \bar{f}(t, \lambda) \) is convex if \( n = 2 \); and convex with respect to each column vector if \( n > 2 \). This implies that

**LEMMA 4.1.** For a.e. \( t \in [0, T] \), \( \bar{f}(t, \lambda) \) is continuous in \( M(m,n) \).

**LEMMA 4.2.** Suppose \( v \in V_p(\Omega_T) \) \( (1 < p < \infty) \), then there exists a sequence of \( L^p \)-layered affine functions:

\[
v^k(x,t) = \begin{cases} 
\lambda_i^k(t) \cdot x + a_i^k(t), & x \in \Omega_i \\
0, & x \in \Omega \setminus \bigcup_{i=1}^h \Omega_i
\end{cases}
\]

such that \( \|v - v^k\|_{V_p(\Omega_T)} \to 0 \) as \( k \to \infty \).

**Proof.** (1) Suppose \( 1 < p \leq 2 \). Fix \( v \in V_p(\Omega_T) \). For any \( \varepsilon > 0 \) we can choose \( u \in V_2(\Omega_T) \) such that

\[
\|u - v\|_{V_p(\Omega_T)} < \varepsilon.
\]

Because \( H \overset{\text{def}}{=} W^{1,2}(\Omega) \) is a Hilbert space, one can assume that \( \{\psi_i\}_{i=1}^\infty \) is its complete orthonormal basis. Let

\[
C_i(t) = \langle u(t, \cdot), \psi_i \rangle_H,
\]

Because \( H \overset{\text{def}}{=} W^{1,2}(\Omega) \) is a Hilbert space, one can assume that \( \{\psi_i\}_{i=1}^\infty \) is its complete orthonormal basis. Let

\[
C_i(t) = \langle u(t, \cdot), \psi_i \rangle_H,
\]
then $C_l(t) \in L^2[0, T]$, and for a.e. $t \in [0, T]\)

$$I_k(t) = \|u - \sum_{l=1}^{k} C_l \psi_l(x)\|_H \to 0 \quad (k \to \infty).$$

Thus the dominated convergence theorem implies that for some integer $k$

$$\|u - \sum_{l=1}^{k} C_l \psi_l\|_{V_p(\Omega_T)} \leq \varepsilon. \quad (4.2)$$

It is well known that there exist piecewise affine functions $\omega_l(x)$ in $\Omega$ such that

$$\max_{1 \leq l \leq k} \|\psi_l - \omega_l\|_{W^{1,p}(\Omega)} \leq \varepsilon(1 + \sum_{l=1}^{k} \|C_l\|_{L^p(\Omega)}^{-1}).$$

Let

$$v^\varepsilon(x, t) = \sum_{l=1}^{k} C_l(t) \omega_l(x),$$

then

$$\|v^\varepsilon - \sum_{l=1}^{k} C_l \psi_l\|_{V_p(\Omega_T)} \leq C(p)\varepsilon. \quad (4.3)$$

Combining (4.1), (4.2) with (4.3), we get

$$\|v - v^\varepsilon\|_{V_p(\Omega_T)} \leq C(m, n, p)\varepsilon.$$ 

Observing that each $v^\varepsilon$ can be written as a layered function on $\Omega_T$, we have completed the proof.

(2) Suppose $2 < p < \infty$. Applying Sobolev embedding theorem we can find an integer $k$, \( \frac{k-1}{n} \geq \frac{1}{2} - \frac{1}{p} \), such that

$$H_1 \overset{\text{def}}{=} W^{k,2}(\Omega) \hookrightarrow W^{1,p}(\Omega).$$

Given $v \in V_p(\Omega_T)$. For $\varepsilon > 0$, one can find $u \in L^p([0, T], W^{k,2}(\Omega))$ such that

$$\|u - v\|_{V_p(\Omega_T)} < \varepsilon. \quad (4.4)$$

Let $\{\psi_l\}_{l=1}^{\infty}$ be the complete orthonormal basis of the Hilbert space $H_1$, then

$$C_l(t) \overset{\text{def}}{=} \langle u(\cdot, t), \psi_l \rangle_{H_1} \in L^p[0, T].$$

The remaining part is entirely the same as the case (1).
Now we are in the position to prove Theorem 1.5. We will use the idea of [9, p.750-751]. For \( u \in V_p(\Omega_T) \), we can extend \( u \) such that \( u \in V_{p,loc} \) (recall (2.3)). From lemma 4.2, choose a sequence of \( L^p \)-layered functions \( \omega^k(x,t) \), such that
\[
\|u - \omega^k\|_{V_p(\Omega_T)} \to 0 \quad (k \to \infty).
\] (4.5)
By taking a subsequence, one can assume that \( D\omega^k \to Du \) almost everywhere on \( \Omega_T \) and
\[
\bar{f}(t,D\omega^k) \to \bar{f}(t,Du) \quad \text{a.e in } \Omega_T
\]
by the virtue of the continuity of \( \bar{f}(t,\cdot) \) (see Lemma 4.1).

We deduce, from the absolute continuity of \( \int |Du|^pdxdt \), Egoroff theorem, theorem 3.4 and inequality (2.5), that
\[
\liminf_{k \to \infty} \int_{\Omega_T} \bar{f}(t,D\omega^k)dxdt \leq \int_{\Omega_T} \bar{f}(t,Du)dxdt.
\]

Therefore, by the semi-continuity of \( H(u,\Omega_T) \) (see lemma 2.1),
\[
H(u,\Omega_T) \leq \liminf_{k \to \infty} H(\omega^k,\Omega_T) = \liminf_{k \to \infty} \int_{\Omega_T} \bar{f}(t,D\omega^k)dxdt \leq \int_{\Omega_T} \bar{f}(t,Du)dxdt.
\]

On the other hand, according to lemma 2.5 and Lebesgue-Nikodym theorem (see §3 of Ch.3 in [12]), we have
\[
H(u,\Omega_T) = \int_{\Omega_T} h(x,t)dxdt
\]
for some \( h \in L^1_{loc}(R^n \times (0,T)) \) and all \( \Omega_T = \Omega \times (0,T) \). By approximation argument, one can easily prove that for a.e \((x,t) \in \Omega_T\), there exists \( r_k \to 0^+ \) such that
\[
\frac{u(x + r_k(y - x),t) - u(x,t)}{r_k} \to Du(x,t) \cdot (y - x) \quad \text{in } V_p(B(x,1) \times (0,T)).
\] (4.8)
Since
\[
\left| \int_0^T h(x,t)dt - \int_0^T |B(x,r_k)|^{-1} \int_{B(x,r_k)} h(y,t)dydt \right|
\]
\[
\leq |B(x,r_k)|^{-1} \int_{B(x,r_k)} \left| \int_0^T [h(x,t) - h(y,t)]dt \right|dy
\]

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and
\[ \int_0^T h(y, t) dt \in L^1_{loc}(\mathbb{R}^n), \]
so
\[ \int_0^T h(x, t) dt = \lim_{k \to \infty} \int_0^T |B(x, r_k)|^{-1} \int_{B(x, r_k)} h(y, t) dy dt \text{ for a.e. } x \in \Omega. \tag{4.9} \]

Fix \( k \) and \( x \), set
\[ r = r_k, \quad B_r = B(x, r), \quad B_{r,T} = B_r \times (0, T). \]

By (4.7) and lemma 2.5, we can find a sequence
\[ u^h \overset{\tau}{\to} u \text{ in } V_p(B_{r,T})(h \to 0) \]
such that
\[ \int_0^T |B_r|^{-1} \int_{B_r} h(y, t) dy dt = |B_r|^{-1} H(u, B_{r,T}) \]
\[ = \lim_{h \to \infty} \int_0^T |B_r|^{-1} \int_{B_r} f \left( \frac{y + \frac{\epsilon_h k_h}{\epsilon_h}, t, Du^h} {\epsilon_h}, t \right) dy dt, \quad \left( k_h \overset{\text{def}}{=} \left[ \frac{x(r-1)}{\epsilon_h} \right] \right) \]
\[ \geq \liminf_{h \to \infty} \int_0^T |B_{1/2}|^{-1} \int_{B_{1/2}} f \left( \frac{y + \frac{x(r-1)}{\epsilon_h}, t, Du^h(y + a_h, t)} {\epsilon_h}, t \right) dy dt \]
\[ \left( \text{note that } a_h \overset{\text{def}}{=} x(r-1) - \epsilon_h k_h \to 0^+ \right) \]
\[ = \liminf_{h \to \infty} \int_0^T |B_{1/2}|^{-1} \int_{B_{1/2}} f \left( \frac{y}{\epsilon_h}, t, D(u^h(x + r(y - x) + a_h, t)} {\epsilon_h}, t \right) \]
\[ - u(x, t)) \right) \right) \text{dy dt}. \tag{4.10} \]

Let \( u_{r,x}(y, t) = r^{-1} [u(x + r(y - x), t) - u(x, t)]. \) Obviously
\[ r^{-1} [u^h(x + r(y - x) + a_h, t) - u(x, t)] \overset{\tau}{\to} r^{-1} u_{r,x} \text{ in } V_p(B_{1/2,T}) \text{ as } h \to \infty. \]

Let
\[ \delta_h = r^{-1} \epsilon_h, \quad a = |B_{1/2}|^{-1}, \quad F^-(u, Q) = \Gamma^-(\tau) \lim_{h \to \infty} F^{\delta_h}(u, Q). \]

By (4.10), lemmas 2.1 and 2.3, (4.8) and theorem 3.4 in that order, we deduce that
\[ \lim_{k \to \infty} \int_0^T |B(x, r_k)|^{-1} \int_{B(x, r_k)} h(y, t) dy dt \geq a \cdot \liminf_{k \to \infty} F^-(u_{r_k,x}, B_{1/2,T}) \]
\[ \geq a \cdot F^-(Du(x, t)(y - x), B_{1/2,T}) \]
\[ = a \int_0^T \int_{B(x, 1/2)} \bar{f}(t, Du(x, t)) dt dy \]
\[ = \int_0^T \bar{f}(t, Du(x, t)) dt. \]
Combing this estimate with (4.9), we obtain
\[
\int_{\Omega_T} h(x,t)dxdt \geq \int_{\Omega_T} \bar{f}(x,Du)dxdt,
\]
which together with (4.7) implies the opposite inequality of (4.6). Hence
\[
H(u,\Omega_T) = \int_{\Omega_T} \bar{f}(t,Du)dxdt \quad \forall u \in V_p(\Omega_T).
\]
Observing that
\[
F(u,\Omega_T) = \int_{\Omega_T} \bar{f}(t,Du)dxdt
\]
is independent of \(\{\varepsilon_h\}\), we have completed the proof of Theorem 1.2 by lemma 2.5.

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