ON HARMONIC WEAK MAASS FORMS OF HALF INTEGRAL WEIGHT

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Abstract. Since Zweger [11] found a connection between mock theta functions and harmonic Maass forms this subject has been a vast research interest recently. Motivated by Zweger’s work harmonic Maass-Jacobi forms were introduced in [2], which include the classical Jacobi forms. We show the isomorphisms among the space $H_{k + \frac{1}{2}}^+(\Gamma_0(4m))$ of (scalar valued) harmonic weak Maass forms of half integral weight whose Fourier coefficients are supported on suitable progressions, the space $H_{k + \frac{1}{2}}, \rho_L$ of vector valued ones, and the space $\hat{J}_{k+1,m}$ of certain harmonic Maass-Jacobi forms of integral weight:

$H_{k + \frac{1}{2}}^+(\Gamma_0(4m)) \simeq H_{k + \frac{1}{2}}, \rho_L \simeq \hat{J}_{k+1,m}$

for $k$ odd and $m = 1$ or a prime. This is an extension of the result developed by Eichler and Zagier [7], which showed the isomorphisms among the Kohnen plus space $M_{k + \frac{1}{2}}^+(\Gamma_0(4m))$ of (scalar valued) modular forms of half integral weight, the space $M_{k + \frac{1}{2}}, \rho_L$ of vector valued ones, and the space $J_{k+1,m}$ of Jacobi forms of integral weight:

$M_{k + \frac{1}{2}}^+(\Gamma_0(4m)) \simeq M_{k + \frac{1}{2}}, \rho_L \simeq J_{k+1,m}$.

1. Introduction and statement of a result

Let $k$ be an integer, and $m$ a positive integer. We denote by $M_{k + \frac{1}{2}}^+(\Gamma_0(4m))$ and $M_{k + \frac{1}{2}}^1(\Gamma_0(4m))$ the space of holomorphic and weakly holomorphic modular forms, respectively, of weight $k + \frac{1}{2}$ for $\Gamma_0(4m)$. Further we define a subspace $M_{k + \frac{1}{2}}^1(\Gamma_0(4m))$ of $M_{k + \frac{1}{2}}^+(\Gamma_0(4m))$ by

$M_{k + \frac{1}{2}}^1(\Gamma_0(4m)) := \{ f \in M_{k + \frac{1}{2}}^+(\Gamma_0(4m)) \mid c_f(n) = 0 \text{ unless } (-1)^k n \equiv \square \mod 4m \}$.

Let $L'$ be the lattice $2m\mathbb{Z}$ equipped with the quadratic form $Q(x) = x^2/4m$. Then its dual $L'$ equals $\mathbb{Z}$. Let $\rho_L$ denote the Weil representation associated to the discriminant form $(L'/L, Q)$, and $\tilde{\rho}_L$ its dual representation. We denote by $M_{k + \frac{1}{2}, \rho_L}$ the space of $\mathbb{C}[L'/L]$-valued holomorphic modular forms of weight $k + \frac{1}{2}$ and type $\rho_L$. Then Eichler and Zagier

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Theorems 5.1, 5.4, and 5.6] proved the following isomorphisms: for $k$ odd and $m = 1$ or a prime,

$$M_{k+1}^+(\Gamma_0(4m)) \simeq M_{k+1,\rho_L} \simeq J_{k+1,m}.$$ 

Here $J_{k+1,m}$ is the space of Jacobi forms of weight $k + 1$ and index $m$ on the full modular group $\Gamma(1)$.

Our result extends this isomorphism to the spaces of harmonic weak Maass forms (Theorem 1 and Theorem 2): if $k$ is odd and $m = 1$ or a prime,

$$H_{k+1/2}^+(\Gamma_0(4m)) \simeq \hat{J}_{k+1,m}^{\text{cusp}}.$$ 

Here $H_{k+1/2}^+(\Gamma_0(4m))$, $H_{k+1/2,\rho_L}$, and $\hat{J}_{k+1,m}^{\text{cusp}}$ are the spaces consisting of corresponding harmonic ones (see Section 2).

In order to state our main results more precisely, we let $k$ be an integer and fix $m = 1$ or a prime. We denote by $H_{k+1/2}^+(\Gamma_0(4m))$ the space of harmonic weak Maass forms of weight $k + 1/2$ for $\Gamma_0(4m)$ (see Section 2.1). Then it is known (see, for instance, [5]) that $f \in H_{k+1/2}^+(\Gamma_0(4m))$ has a unique decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{n \gg -\infty} c_+^f(n)q^n,$$

$$f^-(\tau) = \sum_{n < 0} c_-^f(n)\Gamma\left(\frac{1}{2} - k, 4\pi|n|y\right)q^n.$$

Here $\Gamma(a,y) = \int_y^\infty e^{-t}t^{a-1}dt$ denotes the incomplete Gamma function. We define a subspace $H_{k+1/2}^+(\Gamma_0(4m))$ of $H_{k+1/2}^+(\Gamma_0(4m))$ by

$$H_{k+1/2}^+(\Gamma_0(4m)) := \{f \in H_{k+1/2}^+(\Gamma_0(4m)) | c_+^f(n) = 0 \text{ unless } (-1)^k n \equiv \square \mod 4m\}.$$ 

Let $H_{k+1/2,\rho_L}$ denote the space of $\mathbb{C}[L'/L]$-valued harmonic weak Maass forms of weight $k + 1/2$ and type $\rho_L$ (see Section 2.2). We denote the standard basis elements of the group algebra $\mathbb{C}[L'/L]$ by $e_\gamma$ for $\gamma \in L'/L$. Suppose that the discriminant form $(L'/L, Q)$ is given by $(\mathbb{Z}/2m\mathbb{Z}, Q)$, where $Q(\gamma) = \gamma^2/4m$ for $\gamma \in \mathbb{Z}/2m\mathbb{Z}$ with the signature $(b^+, b^-)$. Then the level of $L$ equals $4m$, and $b^+ - b^- \equiv 1 \mod 8$. For example if we take

$$L = \{X = \begin{pmatrix} b-a/m & \_ \\ c & -b \end{pmatrix} \in \text{Mat}_2(\mathbb{Q}) | a, b, c \in \mathbb{Z}\}$$

with $Q(X) = -m \det(X)$, then $(b^+, b^-) = (2, 1)$ and

$$L' = \{X = \begin{pmatrix} b/2m & -a/m \\ c & -b/2m \end{pmatrix} \in \text{Mat}_2(\mathbb{Q}) | a, b, c \in \mathbb{Z}\}.$$
For a given $f \in H_{k+\frac{1}{2}}^+(\Gamma_0(4m))$ we define a $\mathbb{C}[L'/L]$-valued function $F = \sum_{\gamma \in \mathbb{Z}/2m\mathbb{Z}} F_{\gamma} \epsilon_{\gamma}$ by

$$F_{\gamma}(\tau) := \frac{1}{s(\gamma)} \sum_{n \in \mathbb{Z}} (-1)^{k} c_f(n, y/4m) q^{n/4m}.$$ 

Here $c_f(n, y) := c_f^+(n) + c_f^-(n) \Gamma(\frac{1}{2} - k, 4\pi|n|y)$, and $s(\gamma) = 1$ if $\gamma \equiv 0, m \mod 2m$, and 2 otherwise.

**Theorem 1.** With the notation as above we have the following.

1. If $k$ is even, then the map $f \mapsto F$ defines an isomorphism of $H_{k+\frac{1}{2}}^+(\Gamma_0(4m))$ onto $H_{k+1,\rho, L}$.
2. If $k$ is odd, then the map $f \mapsto F$ defines an isomorphism of $H_{k+\frac{1}{2}}^+(\Gamma_0(4m))$ onto $H_{k+1,\bar{\rho}, L}$.

**Remark 1.** (1) For a given vector valued modular form $F = \sum_{\gamma} F_{\gamma} \epsilon_{\gamma}$ the map $F \mapsto f$ will be the inverse isomorphism where $f(\tau) := \sum_{\gamma} F_{\gamma}(4m\tau)$.

(2) If we restrict the domain on $M_{k+\frac{1}{2}}^+(\Gamma_0(4m))$, we get an isomorphism onto $M_{k+\frac{1}{2}, \rho, L}$ ($k$ even), and so on.

(3) This kind of result for weakly holomorphic modular forms of integral weight is proved by Bruinier and Bundschuh [4]. Let $p$ be an odd prime. We write $M_{k}^!(\Gamma_0(p), (\frac{\cdot}{p}))$ for the space of weakly holomorphic modular forms of weight $k$ for $\Gamma_0(p)$ with Nebentypus $(\frac{\cdot}{p})$. For $\epsilon \in \{\pm 1\}$ we define the subspace

$$M_{k}^!(\Gamma_0(p), (\frac{\cdot}{p})) := \{ f \in M_{k}^!(\Gamma_0(p), (\frac{\cdot}{p})) | c_f(n) = 0 \text{ if } (\frac{n}{p}) = -\epsilon \}.$$ 

Let $L$ be the lattice so that the discriminant group $L'/L$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. On $L'/L$ the quadratic form is equivalent to $Q(x) = \alpha x^2/p$ for some $\alpha \in \mathbb{Z}/p\mathbb{Z} - \{0\}$. Put $\epsilon = (\frac{\alpha}{p})$. Then Bruinier and Bundschuh [4, Theorem 5] showed that $M_{k}^!(\Gamma_0(p), (\frac{\cdot}{p}))$ is isomorphic to $M_{k, \rho, L}$.

For integral weight case Bruinier and Bundschuh’s argument can be applied to the spaces of harmonic weak Maass forms (see [6, Theorem 1.2]). However, for half integral weight case, we need another argument because Eichler and Zagier’s argument depends on the dimension formulas for the spaces of holomorphic modular forms (see the proof of [7, Theorem 5.6]). It is essential that our proof of Theorem 1 relies on some nontrivial properties of the Weil representation.

Next we show that the spaces in Theorem 1 (2) are isomorphic to the space of harmonic Maass-Jacobi forms recently developed by Bringmann and Richter [2].
Let $L$ be the lattice $2m\mathbb{Z}$ equipped with the positive definite quadratic form $Q(x) = x^2/4m$. Then the space $J_{k,m}$ of Jacobi forms of weight $k$ and index $m$ is isomorphic to the space $M_{k-1/2}^\rho(L')$ of $\mathbb{C}[L'/L]$-valued holomorphic modular forms of weight $k - \frac{1}{2}$ and type $\rho_L$ (see [7, Theorem 5.1]). Recently, Bringmann and Richter [2] introduced harmonic Maass-Jacobi forms, which include the classical Jacobi forms. Let $\hat{J}_{k,m}^{\cusp}$ be the space of certain harmonic Maass-Jacobi forms

$$\phi(\tau, z) = \sum_{n,r \in \mathbb{Z}} c^+(n, r)q^n \zeta^r + \sum_{n,r \in \mathbb{Z}, D > 0} c^-(n, r)\Gamma\left(\frac{3}{2} - k, \frac{\pi D'y}{m}\right) q^n \zeta^r$$

of weight $k$ and index $m$ (see Section 2.3). Here, $D = r^2 - 4nm$, $q = e^{2\pi i \tau}$, $\zeta = e^{2\pi iz}$. By the transformation property of harmonic Maass-Jacobi forms [2, Definition 3], one can deduce that if $r' \equiv r \mod 2m$ and $D' = D$ with $D' := r'^2 - 4n'm$, then

$$c^\pm(n', r') = c^\pm(n, r), \quad \Gamma\left(\frac{3}{2} - k, \frac{\pi D'y}{m}\right) = \Gamma\left(\frac{3}{2} - k, \frac{\pi D'y}{m}\right).$$

Hence, we can decompose $\phi(\tau, z)$ by a linear combination of the theta functions as

$$\phi(\tau, z) = \sum_{\mu \in \mathbb{Z}/2m\mathbb{Z}} h_\mu(\tau)\theta_{m, \mu}(\tau, z),$$

where

$$h_\mu(\tau) := \sum_{N \gg -\infty}^\infty c^+\left(\frac{N + r^2}{4m}, r\right) q^{N/4m} + \sum_{N < 0} c^-\left(\frac{N + r^2}{4m}, r\right) \Gamma\left(\frac{3}{2} - k, -\frac{\pi N'y}{m}\right) q^{N/4m}$$

with any $r \in \mathbb{Z}$, $r \equiv \mu \mod 2m$, and

$$\theta_{m, \mu}(\tau, z) := \sum_{r \in \mathbb{Z}, r \equiv \mu \mod 2m} q^{r^2/4m} \zeta^r.$$

Using the same argument in [7, Theorem 5.1], the $2m$-tuples $(h_\mu)_{\mu \in (2m)}$ satisfies the desired transformation formula for vector valued harmonic weak Maass forms. Now the remaining thing is to check $\Delta_{k-\frac{1}{2}} h_\mu = 0$. By definition, $\phi(\tau, z)$ vanishes under the action of the Casimir element $C_{k,m}$ (see [2, p. 2305]), and the action of $C_{k,m}$ on functions in $\hat{J}_{k,m}^{\cusp}$ agrees with that of

$$C_{k,m} = -2\Delta_{k-\frac{1}{2}} + \frac{(\tau - \bar{\tau})^2}{4\pi im} \partial_{\tau \bar{\tau}}$$

(see [2, Proof of Lemma 1]). Using the fact that $\theta_{m, \mu}(\tau, z)$ is in the heat kernel, that is,

$$\left(\partial_\tau - \frac{1}{8\pi im} \partial_{zz}\right)(\theta_{m, \mu}) = 0,$$
one can conclude by a direct computation that
\[ C^{k,m}(\phi) = 0 \implies \Delta_{k-\frac{1}{2}}(h_{\mu}) = 0 \]
for all \( \mu \in \mathbb{Z}/2m\mathbb{Z} \). In conclusion,
\[ \hat{J}_{k,m}^{\text{cusp}} \simeq H_{k-\frac{1}{2},\bar{\rho}_L} \]
Hence, we get the following theorem:

**Theorem 2.** Let \( k \) be even, and \( m = 1 \) or a prime. Then
\[ \hat{J}_{k,m}^{\text{cusp}} \simeq H_{k-\frac{1}{2}}(\Gamma_0(4m)). \]

**Remark 2.** (1) In the case of \( k \) odd, \( M_{k-\frac{1}{2}}^+(\Gamma_0(4m)) \) is isomorphic to the space of skew holomorphic Jacobi forms of weight \( k \) and index \( m \). So one can guess that there must be a similar isomorphism as above when \( k \) is odd. To do that, we need to introduce a new definition of Maass Jacobi forms which include skew holomorphic Jacobi forms. We hope this can be done by following and modifying the work of Bringmann and Richter [2].

(2) In the argument of the proof, the growth condition doesn’t matter. Namely, if we redefine \( H_{k-\frac{1}{2},\bar{\rho}_L} \) and \( H_{k-\frac{1}{2}}^+(\Gamma_0(4m)) \) so that they have at most linear exponential growth at cusps, then
\[ \hat{J}_{k,m} \simeq H_{k-\frac{1}{2},\bar{\rho}_L} \simeq H_{k-\frac{1}{2}}^+(\Gamma_0(4m)) \]
for \( k \) even and \( m = 1 \) or a prime. Here, \( \hat{J}_{k,m} \) is the corresponding bigger space (see Section 2.3).

2. Preliminaries

2.1. **Scalar valued modular forms.** Let \( \tau = x + iy \in \mathbb{H} \), the complex upper half plane, with \( x, y \in \mathbb{R} \). Let \( k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z} \), and \( m \) a positive integer. Put \( \varepsilon_d := (\frac{-1}{d})^{\frac{1}{4}} \).

Recall that **weakly holomorphic modular forms of weight \( k \) for \( \Gamma_0(4m) \)** are holomorphic functions \( f : \mathbb{H} \to \mathbb{C} \) which satisfy:

(i) For all \( (\frac{a}{c} b \frac{c}{d}) \in \Gamma_0(4m) \) we have
\[ f \left( \frac{a\tau + b}{c\tau + d} \right) = \left( \frac{c}{d} \right)^{-2k} (c\tau + d)^k f(\tau); \]

(ii) \( f \) has a Fourier expansion of the form
\[ f(\tau) = \sum_{n \in \mathbb{Z}} c_f(n)q^n; \]

and analogous conditions are required at all cusps.
A smooth function $f : \mathbb{H} \to \mathbb{C}$ is called a harmonic weak Maass form of weight $k$ for $\Gamma_0(4m)$ if it satisfies:

(i) For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4m)$ we have

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\frac{c}{d}\right)^{-2k + 1} (c\tau + d)^k f(\tau);$$

(ii) $\Delta_k f = 0$, where $\Delta_k$ is the weight $k$ hyperbolic Laplace operator defined by

$$\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right);$$

(iii) There is a Fourier polynomial $P_f(\tau) = \sum_{-\infty < n < 0} c_f^+(n) e^{\pi i \tau n^2} q^n \in \mathbb{C}[q^{-1}]$ such that $f(\tau) = P_f(\tau) + O(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

We denote the space of these harmonic weak Maass forms by $H_k(\Gamma_0(4m))$. This space can be denoted by $H^+_k(\Gamma_0(4m))$ in the context of [5], which is the inverse image of $S_{2-k}(\Gamma_0(4m))$ under the certain differential operator $\xi_k$. We have $M_k^+(\Gamma_0(4m)) \subset H_k(\Gamma_0(4m))$. The polynomial $P_f \in \mathbb{C}[q^{-1}]$ is called the principal part of $f$ at the corresponding cusps.

### 2.2. Vector valued modular forms.

We write $M_p^+(\mathbb{R})$ for the metaplectic two-fold cover of $\text{SL}_2(\mathbb{R})$. The elements are pairs $(M, \phi)$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi(\tau)^2 = c\tau + d$. The multiplication is defined by

$$(M, \phi(\tau))(M', \phi'(\tau)) = (MM', \phi(M'\tau)\phi'(\tau)).$$

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$ we use the notation $\tilde{M} := ((a b \cdot c d), \sqrt{cd}) \in M_p^+(\mathbb{R})$. We denote by $M_p(\mathbb{Z})$ the integral metaplectic group, that is the inverse image of $\text{SL}_2(\mathbb{Z})$ under the covering map $M_p^+(\mathbb{R}) \to \text{SL}_2(\mathbb{R})$. It is well known that $M_p(\mathbb{Z})$ is generated by $T := ((1 1 \cdot 0 1), 1)$ and $S := ((0 -1 \cdot 1 0), \sqrt{-1})$.

Let $(V, Q)$ be a non-degenerate rational quadratic space of signature $(b^+, b^-)$. Let $L \subset V$ be an even lattice with dual $L'$. We denote the standard basis elements of the group algebra $\mathbb{C}[L'/L]$ by $\mathbf{c}_\gamma$ for $\gamma \in L'/L$, and write $\langle \cdot, \cdot \rangle$ for the standard scalar product, anti-linear in the second entry, such that $\langle \mathbf{c}_\gamma, \mathbf{c}_{\gamma'} \rangle = \delta_{\gamma, \gamma'}$. There is a unitary representation $\rho_L$ of $M_p(\mathbb{Z})$ on $\mathbb{C}[L'/L]$, the so-called Weil representation, which is defined by

$$\rho_L(T)(\mathbf{c}_\gamma)^{} := e(Q(\gamma))\mathbf{c}_\gamma,$$

$$\rho_L(S)(\mathbf{c}_\gamma)^{} := e((b^- - b^+)/8 \sqrt{|L'/L|}) \sum_{\delta \in L'/L} e(- (\gamma, \delta))\mathbf{c}_\delta,$$
where $e(z) := e^{2\pi iz}$ and $(X, Y) := Q(X + Y) - Q(X) - Q(Y)$ is the associated bilinear form. We denote by $\overline{\rho}_L$ the dual representation of $\rho_L$.

Let $k \in \frac{1}{2}\mathbb{Z}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a weakly holomorphic modular form of weight $k$ and type $\rho_L$ for the group $\text{Mp}_2(\mathbb{Z})$ if it satisfies:

(i) $f(M\tau) = \phi(\tau)^{2k} \rho_L(M,\phi)f(\tau)$ for all $(M,\phi) \in \text{Mp}_2(\mathbb{Z})$;

(ii) $f$ is meromorphic at the cusp $\infty$.

Here condition (ii) means that $f$ has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}+Q(\gamma)} c_f(\gamma,n)e(n\tau)e_\gamma.$$ 

The space of these $\mathbb{C}[L'/L]$-valued weakly holomorphic modular forms is denoted by $M^!_{k,\rho_L}$.

Similarly we can define the space $M^!_{k,\overline{\rho}_L}$ of $\mathbb{C}[L'/L]$-valued weakly holomorphic modular forms of type $\overline{\rho}_L$.

A smooth function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a harmonic weak Maass form of weight $k$ and type $\rho_L$ for the group $\text{Mp}_2(\mathbb{Z})$ if it satisfies:

(i) $f(M\tau) = \phi(\tau)^{2k} \rho_L(M,\phi)f(\tau)$ for all $(M,\phi) \in \text{Mp}_2(\mathbb{Z})$;

(ii) $\Delta_k f = 0$;

(iii) There is a Fourier polynomial $P_f(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}+Q(\gamma)} c_f^+(\gamma,n)e(n\tau)e_\gamma$ such that

$$f(\tau) = P_f(\tau) + O(e^{-\varepsilon y})$$ as $y \to \infty$ for some $\varepsilon > 0$.

We denote by $H_{k,\rho_L}$ the space of these $\mathbb{C}[L'/L]$-valued harmonic weak Maass forms. This space is denoted by $H^!_{k,L}$ in [5], which is the inverse image of $S_{2-k,L}$ under $\xi_k$. We have $M^!_{k,\rho_L} \subset H_{k,\rho_L}$. Similarly we define the space $H_{k,\overline{\rho}_L}$. In particular $f \in H_{k,\rho_L}$ has a unique decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}+Q(\gamma)} c^+_f(\gamma,n)e(n\tau)e_\gamma,$$

$$f^-(\tau) = \sum_{\gamma \in L'/L} \sum_{n < 0} c^-_f(\gamma,n)\Gamma(1-k,4\pi|n|y)e(n\tau)e_\gamma.$$ 

2.3. Harmonic Maass-Jacobi forms. The most part of this session we follow the notation given in [2].

Definition 3. A function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is a harmonic Maass-Jacobi form of weight $k$ and index $m$ if $\phi$ is real-analytic in $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$, and satisfies the following conditions:
(1) For all \( A = [(a b),(c d)], (\lambda, \mu) \in \text{SL}_2(\mathbb{Z}) \rtimes \mathbb{Z}^2 \)
\[
\phi \left( \frac{a \tau + b}{c \tau + d}, \frac{z + \lambda \tau + \mu}{c \tau + d} \right) (c \tau + d)^{-k} e^{2\pi im(-\frac{c(z+\lambda \tau+\mu)}{c \tau + d}+\lambda^2 \tau + 2\lambda z)} = \phi(\tau, z).
\]

(2) \( C_{k,m}(\phi) = 0 \), where \( C_{k,m} \) is the Casimir element of the real Jacobi group (see p. 2305 in [2]).

(3) \( \phi(\tau, z) = O(e^{ay}e^{2\pi mv^2/y}) \) as \( y \to \infty \) for some \( a > 0 \).

Let \( \mathfrak{j}_{k,m} \) be the space of harmonic Maass-Jacobi forms of weight \( k \) and index \( m \), which are holomorphic in \( z \). In fact, we are interested in the subspace \( \mathfrak{j}_{k,m}^{\text{cusp}} \) consisting of the elements \( \phi \in \mathfrak{j}_{k,m} \) whose Fourier expansion is of the form

\[
\phi(\tau, z) = \sum_{n, r \in \mathbb{Z}} c^+(n, r) q^n \zeta^r + \sum_{n, r \in \mathbb{Z}} c^-(n, r) \Gamma \left( \frac{3}{2} - k, \frac{\pi D y}{m} \right) q^n \zeta^r.
\]

The space \( \mathfrak{j}_{k,m}^{\text{cusp}} \) is in fact the inverse image of \( J_{3-k,m}^{\text{sk,cusp}} \) under the certain differential operator \( \xi_{k,m} \) (see [2, Remarks (1) on p. 2307]).

3. Proof of Theorem

We will only give a proof of (1) because exactly the same argument can be applied. We first prove that for a given \( f \in H_{k+\frac{1}{2}}^{\omega}(\Gamma_0(4m)) \) the \( \mathbb{C}[L'/L] \)-valued function \( F \) as defined in Section 1 belongs to \( H_{k+\frac{1}{2},\rho_L} \). One has that \( f(\tau) = \sum_{\gamma \in L'/L} F_\gamma(4m \tau) \) by inspecting the Fourier expansion of \( f \). Since it is straightforward to check \( F(\tau + 1) = \rho_L(T) F(\tau) \), we show that

\[
F \left( \frac{1}{\tau} \right) = \tau^{k+\frac{1}{2}} \rho_L(S) F(\tau).
\]

In [8] Kim proved (3.1) for \( m \in \mathfrak{S} \), where

\[
\mathfrak{S} = \{ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71 \}.
\]

We will prove (3.1) for \( m = 1 \) or a prime by following his argument. For details we refer to [8, pp. 735-737]. For \( j \) prime to \( 4m \) put

\[
f_j := f \big|_{k+\frac{1}{2}}^{\omega} \left( \left( \begin{array}{cc} 1 & j \\ 0 & 4m \end{array} \right), (4m)^{1/4} \right) \big|_{k+\frac{1}{2}}^{\omega} W_{4m},
\]

where

\[
W_{4m} := \left( \left( \begin{array}{cc} 0 & -1 \\ 4m & 0 \end{array} \right), (4m)^{1/4} \sqrt{-i\tau} \right).
\]
We choose \( b, d \in \mathbb{Z} \) so that \( jd - 4mb = 1 \). Then one finds that

\[
((\begin{smallmatrix} 1 & j \\ 0 & 4m \end{smallmatrix}), (4m)^{1/4}) W_{4m} = ((\begin{smallmatrix} 4mj & -1 \\ 16m^2 & 0 \end{smallmatrix}), 2\sqrt{-mi\tau})
= (M, J(M, \tau))(\begin{smallmatrix} 4m & -d \\ 0 & 4m \end{smallmatrix}, \psi_j),
\]

where

\[
\psi_j := \left(\frac{4m}{j}\right) \sqrt{\left(\frac{-1}{j}\right)} e(-1/8)
\]

and \( M = (\begin{smallmatrix} j & b \\ 4m & d \end{smallmatrix}) \) and \( J(M, \tau) \) denotes the automorphy factor for the theta function \( \sum_{n \in \mathbb{Z}} q^{n^2} \), that is,

\[
J(M, \tau) := \left(\frac{c}{d}\right) \varepsilon^{-1} \left(\frac{c\tau + d}{4m}\right)^{1/2}, \quad M = (a \ b) \in \Gamma_0(4m).
\]

Here \( (\cdot) \) is the usual Jacobi symbol.

This implies that

\[
f_j = f|_{k+\frac{1}{2}}((\begin{smallmatrix} 1 & j \\ 0 & 4m \end{smallmatrix}), (4m)^{1/4}) W_{4m}
= \psi_j^{-2k-1} f\left(\tau - \frac{j-1}{4m}\right)
= \psi_j^{-2k-1} \sum_{\gamma(2m)} e\left(-\frac{j^{-1}\gamma^2}{4m}\right) F_{\gamma}(4m\tau),
\]

where \( j^{-1} \) denotes an integer which is the inverse of \( j \) in \((\mathbb{Z}/4m\mathbb{Z})^\times\). On the other hand we have by definition that

\[
f_j = (4m)^{(-2k-1)/4} \left(\sum_{\gamma(2m)} e\left(\frac{j\gamma^2}{4m}\right) F_{\gamma}\right)|_{k+\frac{1}{2}} W_{4m}
= (4m)^{(-2k-1)/2} \sqrt{-1}\tau^{-2k-1} \sum_{\gamma(2m)} e\left(\frac{j\gamma^2}{4m}\right) F_{\gamma}\left(-\frac{1}{4m\tau}\right).
\]

Replacing \( \tau \) by \( \tau/4m \) in (3.2) and (3.3) one has the following identity:

\[
\sum_{\gamma(2m)} e\left(\frac{j\gamma^2}{4m}\right) F_{\gamma}\left(-\frac{1}{\tau}\right) = \left(\frac{4m}{j}\right) \sqrt{\left(\frac{-1}{j}\right)} \tau^{k+\frac{1}{2}} \sum_{\gamma(2m)} e\left(-\frac{j^{-1}\gamma^2}{4m}\right) F_{\gamma}(\tau).
\]

Let \( R \) be a \( 2m \times 2m \) matrix defined by

\[
R := \frac{e(-1/8)}{\sqrt{2m}} \left( e\left(-\frac{l\gamma}{2m}\right) \right)_{l(2m), \gamma(2m)}.
\]
In order to show $F \in H_{k+\frac{1}{2}\varphi_2}$ we first need to prove (3.1), i.e.

$$
\begin{pmatrix}
\vdots \\
F_{\gamma} \\
\vdots
\end{pmatrix} (-1/\tau) = \tau^{k+\frac{1}{2}} R 
\begin{pmatrix}
\vdots \\
F_{\gamma} \\
\vdots
\end{pmatrix} (\tau).
$$

Since $F_\gamma = F_{-\gamma}$ the above identity is equivalent to

$$
B \begin{pmatrix}
\vdots \\
F_{\gamma} \\
\vdots
\end{pmatrix} (-1/\tau) = \tau^{k+\frac{1}{2}} BR \begin{pmatrix}
\vdots \\
F_{\gamma} \\
\vdots
\end{pmatrix} (\tau)
$$

for some matrix $B$ with $2m$ columns of which the first $m + 1$ ones are linearly independent. For instance, in [8] $B$ was chosen as

$$
A := \left( e\left( \frac{j_\ell \gamma^2}{4m} \right) \right)_{\ell \in \{\varphi(4m)\}, \gamma(2m)}
$$

and checked that its rank is $m + 1$ if $m \in \mathcal{G} - \{2\}$. Here $j_\ell$ is the $\ell$th largest element in \{\(j \mid 1 \leq j \leq 4m, (j, 4m) = 1\}\}.

In this paper we take $B$ as $CA$ where

$$
C := \left( e\left( -\frac{j_\ell \beta^2}{4m} \right) \right)_{\beta \in \{\varphi(4m)\}}.
$$

**Lemma 4.** The rank of $B := CA$ is $2\varphi(m)$, and the first $2\varphi(m)$ columns of $B$ are linearly independent.

**Proof.** The $\beta\gamma$-th entry $b_{\beta\gamma}$ of the $2m \times 2m$ matrix $B$ is given by

$$
b_{\beta\gamma} = \sum_{\ell \in \{\varphi(4m)\}} e\left( \frac{j_\ell (\gamma^2 - \beta^2)}{4m} \right)
= \begin{cases} 
2\varphi(m) & \text{if } \beta = \gamma \\
-2 & \text{if } \beta \neq \gamma \text{ and } \beta \equiv \gamma \mod 2 \\
0 & \text{otherwise}.
\end{cases}
$$

From this one can easily infer that $B$ has rank $2\varphi(m)$, and its first $2\varphi(m)$ columns are linearly independent. $\square$

Since

$$
AR = \left( \left( \frac{4m}{j_\ell} \right)^{-1} \frac{-1}{j_\ell} \right) e\left( -\frac{j_\ell^{-1}\gamma^2}{4m} \right)_{\ell \in \{\varphi(4m)\}, \gamma(2m)}
$$
from the Gauss sum formula (see [8, p. 736]), the identity (3.4) is equivalent to

\[
A \begin{pmatrix} 
\vdots \\
F_{\gamma} \\
\vdots 
\end{pmatrix} (-1/\tau) = \tau^{k+\frac{1}{2}} AR \begin{pmatrix} 
\vdots \\
F_{\gamma} \\
\vdots 
\end{pmatrix} (\tau).
\]

This implies that the identity

\[
B \begin{pmatrix} 
\vdots \\
F_{\gamma} \\
\vdots 
\end{pmatrix} (-1/\tau) = \tau^{k+\frac{1}{2}} BR \begin{pmatrix} 
\vdots \\
F_{\gamma} \\
\vdots 
\end{pmatrix} (\tau)
\]

holds true. Combining with Lemma 4, \( F \) satisfies the transformation property (3.1) for \( m \neq 2 \). From (3.2), (3.4), and Lemma 4 we can infer that each \( F_{\gamma} \) can be written as a linear combination of \( f_{j} := f|_{k+\frac{1}{2}}((1 \; j \; 0 \; 4m), (4m)^{1/4})|_{k+\frac{1}{2}} W_{4m} \) for all \((j, 4m) = 1\).

Since \( \Delta_{k+\frac{1}{2}} \) commutes with the Petersson slash operator (see [9]), each \( \Delta_{k+\frac{1}{2}} F_{\gamma} \) vanishes, i.e. \( F \in H_{k+\frac{1}{2}, \rho_{L}} \) for \( m \neq 2 \). The same procedure as given in [8, Remark 3.2] can be applied to the case when \( m = 2 \), so we omit the detailed proof.

Now we consider the converse. For a given \( F = \sum_{(2m)} F_{\gamma} e_{\gamma} \in H_{k+\frac{1}{2}, \rho_{L}} \) we define

\[
f(\tau) := \sum_{\gamma(2m)} F_{\gamma}(4m\tau).
\]

It is straightforward to verify that \( f \) satisfies the condition \( c_{\pm}(n) = 0 \) unless \((-1)^{k}n \equiv \square \mod 4m \) by inspecting the Fourier expansion of \( F \). Also \( \Delta_{k+\frac{1}{2}} f = 0 \) by (3.5). Thus it suffices to show that

\[
f \left( \frac{a\tau + b}{c\tau + d} \right) = \left( \frac{c}{d} \right) \sqrt{\left( \frac{-1}{d} \right)} (c\tau + d)^{k+\frac{1}{2}} f(\tau)
\]

for all \((a \; c \; b \; d) \in \Gamma_{0}(4m)\). We may assume that \( d > 0 \) by multiplying \((-1 \; 0 \; -1)\) if necessary.

**Remark 3.** In what follows, our results hold for arbitrary \( m > 0 \). In fact one may apply our argument even for somewhat general discriminant forms \((L'/L, Q)\).

**Lemma 5.** For any \( n \in \mathbb{Z} \) one has

\[
\rho_{L} \begin{pmatrix} 
1 & 0 \\
0 & 1 
\end{pmatrix} \sum_{\gamma \in \mathcal{L}/L} c_{\gamma} = \sum_{\gamma \in L'/L} c_{\gamma}.
\]
Proof. Since \( \left( \frac{1}{n} \right) = \left( \frac{1}{1} \right)^n \) it suffices to prove that

\[
\rho_L \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sum_{\gamma \in L'/L} e_{\gamma} = \sum_{\gamma \in L'/L} e_{\gamma}.
\]

For \( M \in \text{SL}_2(\mathbb{Z}) \) we define the coefficients \( \rho_{\beta\gamma}(\tilde{M}) \) of the representation \( \rho_L \) by

\[
\rho_{\beta\gamma}(\tilde{M}) = \langle \rho_L(\tilde{M}) e_{\gamma}, e_{\beta} \rangle.
\]

From Shintani’s result [3, Proposition 1.1] for \( \rho_{\beta\gamma}(\tilde{M}) \) one has

\[
\sum_{\gamma \in L'/L} \rho_{\beta\gamma} \left( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) e_{\gamma} = \sum_{\gamma \in L'/L} e_{\gamma}.
\]

The last equality is from Milgram’s formula (see [1]), that is,

\[
\sum_{\gamma \in L'/L} e(Q(\gamma)) = \sqrt{|L'/L|} e((b^+ - b^-)/8).
\]

First notice that

\[
f(\tau) = \langle F(4m\tau), \sum_{\gamma \in L'/L} e_{\gamma} \rangle.
\]

Since \( \left( \begin{smallmatrix} 4m & 0 \\ 0 & 1 \end{smallmatrix} \right) \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) = \left( \begin{smallmatrix} a/4m & 4mb \\ c/4m & d \end{smallmatrix} \right) \left( \begin{smallmatrix} 4m & 0 \\ 0 & 1 \end{smallmatrix} \right) \) we get for \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(4m) \) that

\[
f \left( \frac{a\tau + b}{c\tau + d} \right) = \langle F\left((\frac{a}{4m}) \left( \begin{smallmatrix} 4m & 0 \\ 0 & 1 \end{smallmatrix} \right)\right), \sum_{\gamma \in L'/L} e_{\gamma} \rangle
\]

\[
= \langle (c\tau + d)^k F(4m\tau), \rho_L\left( \begin{smallmatrix} a/4m \\ c/4m \end{smallmatrix} \right)^{-1} \sum_{\gamma \in L'/L} e_{\gamma} \rangle.
\]

Observe that \( \Gamma^0(4m) = (\Gamma_0(4m) \cap \Gamma^0(4m), \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)) \). More precisely one has

\[
\left( \begin{smallmatrix} a & 4mb \\ c/4m & d \end{smallmatrix} \right) = \left( \begin{smallmatrix} a(1-bc) & 4mb \\ ac/4m & d \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right).
\]

We first consider the case \( a > 0 \). The consistency condition implies that

\[
\left( \begin{smallmatrix} a & 4mb \\ c/4m & d \end{smallmatrix} \right) = \left( \begin{smallmatrix} a(1-bc) & 4mb \\ ac/4m & d \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right).
\]
Since $\rho_L\left(\frac{1}{ac/4m} \ 0 \ 1\right) \sum_{\gamma \in L'/L} \mathbf{e}_\gamma = \sum_{\gamma \in L'/L} \mathbf{e}_\gamma$ by Lemma 5, one finds from Borcherds’ result [1, Theorem 5.4] that

$$\rho_L\left(\frac{a}{c/4m} \ 4mb \ d\right) \sum_{\gamma \in L'/L} \mathbf{e}_\gamma = \rho_L\left(\frac{a(1-bc)}{-4mb(c/4m)^2} \frac{4mb}{d}\right) \sum_{\gamma \in L'/L} \mathbf{e}_\gamma$$

$$= \left(\left(\frac{-4mb}{d}\right) \sqrt{\frac{-1}{d}}\right)^1 \left(\frac{b^+-b^-+(\frac{ac}{d})}{4}\right) \sum_{\gamma \in L'/L} \mathbf{e}_\gamma$$

$$= \left(\frac{c}{d}\right) \sqrt{\frac{-1}{d}} \left(\frac{m}{d}\right) \sum_{\gamma \in L'/L} \mathbf{e}_\gamma.$$

Because $\sqrt{\left(\frac{-1}{d}\right)^{1-\left(\frac{ac}{d}\right)} \left(\frac{m}{d}\right)} = 1$ we get the following identity:

$$\rho_L\left(\frac{a}{c/4m} \ 4mb \ d\right) \sum_{\gamma \in L'/L} \mathbf{e}_\gamma = \left(\frac{c}{d}\right) \sqrt{\frac{-1}{d}} \sum_{\gamma \in L'/L} \mathbf{e}_\gamma. \tag{3.7}$$

We claim that (3.7) holds true for the case $a < 0$. If $c = 0$, then $a = d = -1$ and thereby it is straightforward to verify (3.7). So we assume that $c \neq 0$. If we choose $x \in \mathbb{Z}$ so that $a + xc > 0$, then from the elementary identity

$$\left(\frac{1}{0} \ \frac{4mx}{4m}\right) \left(\frac{a}{c/4m} \ 4mb \ d\right) = \left(\frac{a+xc}{c/4m} \ 4m(b+xd) \ d\right)$$

one can see that (3.7) holds true even for $a < 0$. Now if we insert (3.7) into (3.6), we obtain

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\frac{c}{d}\right) \sqrt{\frac{-1}{d}} \left(\frac{c\tau + d}{c\tau + d}\right)^{k+\frac{1}{2}} \left(\sum_{\gamma \in L'/L} F(4m\tau), \mathbf{e}_\gamma\right)$$

$$= \left(\frac{c}{d}\right) \sqrt{\frac{-1}{d}} \left(\frac{c\tau + d}{c\tau + d}\right)^{k+\frac{1}{2}} f(\tau).$$

This completes the proof of Theorem 1.

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