Thermodynamics of $d$-dimensional charged rotating black brane and AdS/CFT correspondence

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We compute the Euclidean actions of a $d$-dimensional charged rotating black brane both in the canonical and the grand-canonical ensemble through the use of the counterterms renormalization method, and show that the logarithmic divergencies associated with the Weyl anomalies and matter field vanish. We obtain a Smarr-type formula for the mass as a function of the entropy, the angular momenta, and the electric charge, and show that these quantities satisfy the first law of thermodynamics. Using the conserved quantities and the Euclidean actions, we calculate the thermodynamics potentials of the system in terms of the temperature, angular velocities, and electric potential both in the canonical and grand-canonical ensembles. We also perform a stability analysis in these two ensembles, and show that the system is thermally stable. This is commensurate with the fact that there is no Hawking-Page phase transition for a black object with zero curvature horizon. Finally, we obtain the logarithmic correction of the entropy due to the thermal fluctuation around the equilibrium.

I. INTRODUCTION

The thermodynamics of asymptotically anti de-Sitter (AAdS) black holes has attracted a great deal of attention in recent years. First this is due to the fact that it produces an aggregate of ideas from thermodynamics, quantum field theory and general relativity, and second...
because of its role in the AdS/CFT duality \cite{1}. According to this duality the low energy limit of string theory in AAdS (times some compact manifold) is equivalent to quantum field theory living on the boundary of AAdS. This equivalence between the two formulations means that, at least in principle, one can obtain complete information on one side of the duality by performing a computation on the other side. For example, one can gain some insight into the thermodynamic properties and phase structures of strong 't Hooft coupling conformal field theories by studying the thermodynamics of AAdS black holes. An interesting application of the AdS/CFT correspondence is the interpretation of Hawking-Page phase transition between thermal AdS and AAdS black hole as the confinement-deconfinement phases of the Yang-Mills (dual gauge) theory defined on the AdS boundary \cite{2}.

This conjecture is now a fundamental concept that furnishes a means for calculating the action and thermodynamic quantities intrinsically without reliance on any reference spacetime \cite{3, 4, 5}. It has extended to the case of asymptotically de Sitter spacetimes \cite{6, 7}. Although the (A)dS/CFT correspondence applies for the case of a specially infinite boundary, it was also employed for the computation of the conserved and thermodynamic quantities in the case of a finite boundary \cite{8}. This conjecture has also been applied for the case of black objects with constant negative or zero curvature horizons \cite{9, 10}.

For AAdS spacetimes, the presence of a negative cosmological constant makes it possible to have a large variety of black holes/branes, whose event horizons are hypersurfaces with positive, negative, or zero scalar curvatures \cite{11}. The AAdS rotating solution of Einstein’s equation with cylindrical and toroidal horizon and its extension to include the electromagnetic field have been considered in Ref. \cite{12}. The generalization of this AAdS charged rotating solution of Einstein-Maxwell’s equation to the higher dimensions has been done in Ref. \cite{13}. Many authors have been considered thermodynamics and stability conditions of these black holes \cite{9, 14}. In this paper, we study the phase behavior of the charged rotating black branes in \((n + 1)\) dimensions with zero curvature horizon and show that there is no Hawking-Page phase transition in spite of the angular momentum of the branes. This is in commensurable with the fact that there is no Hawking-Page phase transition for black object whose horizon is diffeomorphic to \(\mathbb{R}^p\) (p-brane solution) and therefore the system is always in the high temperature phase \cite{2}. According to the AdS/CFT dictionary, this means that the corresponding field theory, which now lives on \(S^1 \times \mathbb{R}^p\), has no phase transition as a function of temperature in the large \(\mathcal{N}\) limit.
The outline of our paper is as follows. We review the basic formalism in Sec. II. In Sec. III we consider the \((n+1)\)-dimensional AAdS charged rotating black brane. We also compute the Euclidean actions of the system both in the canonical and the grand-canonical ensemble, and obtain the logarithmic divergences associated to the Weyl anomalies and matter fields. In Sec. IV we study the thermodynamics of the brane, and perform a thermal stability analysis. Also, the logarithmic correction to the Bekenstein-Hawking entropy of the black brane is obtained. We finish our paper with some concluding remarks.

II. GENERAL FORMALISM

The gravitational action for Einstein-Maxwell theory in \((n + 1)\) dimensions for AAdS spacetimes is

\[
I_G = -\frac{1}{16\pi} \int_M d^{n+1}x \sqrt{-g} \left( R - 2\Lambda - F_{\mu\nu} F^{\mu\nu} \right) + \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{-\gamma} K(\gamma),
\]

where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the electromagnetic tensor field and \(A_\mu\) is the vector potential. The first term is the Einstein-Hilbert volume term with negative cosmological constant \(\Lambda = -\frac{n(n-1)}{2l^2}\) and the second term is the Gibbons Hawking boundary term which is chosen such that the variational principle is well defined. The manifold \(\mathcal{M}\) has metric \(g_{\mu\nu}\) and covariant derivative \(\nabla_\mu\). \(K\) is the trace of the extrinsic curvature \(K^{\mu\nu}\) of any boundary(ies) \(\partial\mathcal{M}\) of the manifold \(\mathcal{M}\), with induced metric(s) \(\gamma_{i,j}\). The AdS/CFT correspondence states that if the metric near the conformal boundary \((x \to 0)\) can be expanded in the AAdS form

\[
ds^2 = \frac{dx^2}{l^2 x^2} + \frac{1}{x^2} h_{ij} dx^i dx^j,
\]

with nondegenerate metric \(h_{ij}\), then one may remove the non logarithmic divergent terms in the action by adding a counterterm action \(I_{ct}\) which is a functional of the boundary curvature invariants. It is worthwhile to mention that the induced metric on the boundary is \(\gamma_{ij} = h_{ij}/\epsilon^2\) where \(\epsilon \ll 1\). The counterterm for asymptotically AdS spacetimes up to seven dimensions is

\[
I_{ct} = \frac{1}{8\pi} \int_{\partial \mathcal{M}_\infty} d^n x \sqrt{-\gamma} \left\{ \frac{n-1}{l} - \frac{l\Theta(n-3)}{2(n-2)} R \right. \\
- \frac{l^3\Theta(n-5)}{2(n-4)(n-2)} \left( R_{ab} R^{ab} - \frac{n}{4(n-1)} R^2 \right) + ... \right\},
\]

(3)
where $R$, $R_{abcd}$, and $R_{ab}$ are the Ricci scalar, Riemann, and Ricci tensors of the boundary metric $\gamma_{ab}$ and $\Theta(x)$ is the step function which is equal to one for $x \geq 0$ and zero otherwise. Although other counterterms (of higher mass dimension) may be added to $I_{ct}$, they will make no contribution to the evaluation of the action or Hamiltonian due to the rate at which they decrease toward infinity, and we shall not consider them in our analysis here. These counterterms have been used by many authors for a wide variety of the spacetimes, including Schwarzschild-AdS, topological Schwarzschild-AdS, Kerr-AdS, Taub-NUT-AdS, Taub-bolt-AdS, and Taub-bolt-Kerr-AdS [15].

Of course, for even $n$ one has logarithmic divergences in the partition function which can be related to the Weyl anomalies in the dual conformal field theory [3]. These logarithmic divergences associated with the Weyl anomalies of the dual field theory for $n = 4$ and $n = 6$ are [16]

$$I_{\log}^\text{em} = -\frac{\ln \epsilon}{64 \pi l^3} \int d^4x \sqrt{-h^0} \left[ (R^0_{ij}) F^{(0)ij} - \frac{1}{2} R^0_{ij} R^0_{ij} \right],$$

$$I_{\log}^\text{em} = -\frac{\ln \epsilon}{8 \pi l^3} \int d^6x \sqrt{-h^0} \left\{ \frac{3}{50} (R^0)^3 + R^{(0)ij} R^{(0)kl} R^0_{ij} - \frac{1}{2} R^0 R^{(0)ij} R^0_{ij} \right\} + \frac{1}{5} R^{(0)ij} D_i D_j R^0 - \frac{1}{2} R^{(0)ij} \Box R^0 - \frac{1}{20} R^0 R^0 \right\}. $$

In Eqs. (4) and (5) $R^0$ and $R^{(0)ij}$ are the Ricci scalar and Ricci tensor of the leading order metric $h^0$ in the following expansion:

$$h_{ij} = h^0_{ij} + x^2 h^2_{ij} + x^4 h^4_{ij} + ..., $$

and $D_i$ is the covariant derivative constructed by the leading order metric $h^0$. Also, one should note that the inclusion of matter fields in the gravitational action produces an additional logarithmic divergence in the action for even $n$. This logarithmic divergence for $n = 4$ and $n = 6$ are [17]

$$I_{\log}^\text{em} = -\frac{\ln \epsilon}{64 \pi l^3} \int d^4x \sqrt{-h^0} F^{(0)ij} F^0_{ij},$$

$$I_{\log}^\text{em} = -\frac{\ln \epsilon}{8 \pi l^3} \int d^6x \sqrt{-h^0} \left\{ \frac{1}{16} R^0 F^{(0)ij} F^0_{ij} - \frac{1}{8} R^{(0)ij} F^{(0)l}_{ij} F^0_{lj} \right\} + \frac{1}{64} F^{(0)ij} (D_j D^k F^0_{ki} - D_i D^k F^0_{kj}) \right\}.$$
the matter field will cause a power law divergence in the action which can be removed by a counterterm of the form \[17\]

\[I_{ct}^{em} = \frac{1}{256\pi} \int d^n x \sqrt{-\gamma} \frac{(n-8)}{(n-4)} \Theta(n-5) F^{ij} F_{ij}. \quad (9)\]

Thus, the total action can be written as a linear combination of the gravity term (1), the logarithmic divergences (43), (44), (7), and (8) and the counterterms (3) and (9). We will show that for the charged rotating black branes investigated in this paper, all the logarithmic divergences and the counterterm \[I_{ct}^{em}\] are zero and therefore the total renormalized action is

\[I = I_G + I_{ct}. \quad (10)\]

In order to obtain the Einstein-Maxwell equations by the variation of the volume integral with respect to the fields, one should impose the boundary condition \[\delta A^\mu = 0 \] on \[\partial \mathcal{M}.\] Thus the action (10) is appropriate to study the grand-canonical ensemble with fixed electric potential \[18\]. To study the canonical ensemble with fixed electric charge one should impose the boundary condition \[\delta(n^a F_{ab}) = 0\], and therefore the total action is \[19\]

\[\tilde{I} = I - \frac{1}{4\pi} \int_{\partial \mathcal{M}} d^n x \sqrt{-\gamma} n_a F^{ab} A_b. \quad (11)\]

Having the total finite action, one can use the Brown and York definition \[20\] to construct a divergence free stress-energy tensor as

\[T^{ab} = \frac{1}{8\pi} \left\{ (K^{ab} - K \gamma^{ab}) - \frac{n-1}{n-2} \gamma^{ab} + \frac{l}{n-2} (R^{ab} - \frac{1}{2} R \gamma^{ab}) \right. \]

\[+ \frac{l^2 \Theta(n-5)}{(n-4)(n-2)^2} \left[ \frac{1}{2} \gamma^{ab} (R^{cd} R_{cd} - \frac{n}{4(n-1)} R^2) - \frac{n}{(2n-2)} R R^{ab} \right. \]

\[+ 2 R_{cd} R^{abcd} - \frac{n-2}{2(n-1)} \nabla^a \nabla^b R + \nabla^2 R^{ab} - \frac{1}{2(n-1)} \gamma^{ab} \nabla^2 R \right] + \ldots. \quad (12)\]

The above stress-tensor is divergence free for \[n \leq 6\], but we can always add more counterterms to have a finite action in higher dimensions (see e.g. Ref. [21]).

The conserved charges associated to a Killing vector \(\xi^a\) is

\[Q(\xi) = \int_B d^n x \sqrt{\sigma} n^a T_{ab} \xi^b, \quad (13)\]

where \(\sigma\) is the determinant of the metric \(\sigma_{ij}\) and \(N\) is the lapse function, appearing in the ADM-like decomposition of the boundary metric

\[ds^2 = -N^2 dt^2 + \sigma_{ab} (dx^a + N^a dt)(dx^b + N^b dt). \quad (14)\]
For boundaries with timelike Killing vector \((\xi = \partial/\partial t)\) and rotational Killing vector field \((\zeta = \partial/\partial \phi)\) one obtains the conserved mass and angular momentum of the system enclosed by the boundary \(\mathcal{B}\). In the context of AdS/CFT correspondence, the limit in which the boundary \(\mathcal{B}\) becomes infinite \((\mathcal{B}_\infty)\) is taken, and the counterterm prescription ensures that the action and conserved charges are finite. No embedding of the surface \(\mathcal{B}\) in to a reference of spacetime is required and the quantities which are computed are intrinsic to the spacetimes.

III. THE ACTION AND THERMODYNAMIC QUANTITIES OF AADS CHARGED ROTATING BLACK BRANE

The metric of \((n + 1)\)-dimensional AAdS charged rotating black brane with \(k\) rotation parameters is \(^{[13]}\)

\[
d s^2 = -f(r) \left( \Xi dt - \sum_{i=1}^{k} a_i d\phi_i \right)^2 + \frac{r^2}{l^2} \sum_{i=1}^{k} (a_i dt - \Xi l^2 d\phi_i)^2 \\
+ \frac{dr^2}{f(r)} - \frac{r^2}{l^2} \sum_{i<j}^{k} (a_i d\phi_j - a_j d\phi_i)^2 + r^2 d\Omega^2, \tag{15}
\]

where \(\Xi = \sqrt{1 + \sum_i^k a_i^2 / l^2}\) and \(d\Omega^2\) is the Euclidean metric on the \((n - 1 - k)\)-dimensional submanifold. The maximum number of rotation parameters in \((n + 1)\) dimensions is \([(n + 1)/2]\), where \([x]\) denotes the integer part of \(x\). In Eq. (16) \(f(r)\) is

\[
f(r) = \frac{r^2}{l^2} - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2n-4}}, \tag{16}
\]

and the gauge potential is given by

\[
A_\mu = -\sqrt{\frac{n-1}{2n-4}} \frac{q}{r^{n-2}} \left( \Xi \delta_\mu^0 - \delta_\mu^i a_i \right), \quad \text{(no sum on } i) \tag{17}.
\]

As in the case of rotating black hole solutions of Einstein’s gravity, the above metric given by Eqs. (15)-(17) has two types of Killing and event horizons. The Killing horizon is a null surface whose null generators are tangent to a Killing field. It was proved that a stationary black hole event horizon should be a Killing horizon in the four-dimensional Einstein gravity \(^{[19]}\). This fact is also true for this \((n + 1)\)-dimensional metric and the Killing vector

\[
\chi = \partial_t + \sum_{i=1}^{k} \Omega_i \partial_{\phi_i}, \tag{18}
\]
is the null generator of the event horizon. The metric of Eqs. (15)–(17) has two inner and outer event horizons located at $r_-$ and $r_+$, if the metric parameters $m$ and $q$ are chosen to be suitable [13]. For later use in the thermodynamics of the black brane, it is better to present an expression for the critical value of the charge in term of the radius of the event horizon $r_+$. It is easy to show that the metric has two inner and outer horizons provided the charge parameter, $q$ is less than $q_{\text{crit}}$ given as

$$q_{\text{crit}} = \sqrt{\frac{n}{n-2}} \frac{r_+^{n-1}}{l}.$$  (19)

In the case that $q = q_{\text{crit}}$, we will have an extreme black brane.

The mass, the angular momenta, the Hawking temperature and the angular velocities of the outer event horizon have been calculated in Ref. [13]. We bring them here for later use

$$M = \frac{V_{n-1}}{16\pi} m \left[ n\Xi^2 - 1 \right],$$  (20)

$$J_i = \frac{V_{n-1}}{16\pi} n\Xi m a_i,$$  (21)

$$T = \frac{1}{\beta_+} = \frac{nr_+^{2(n-2)} - (n-2)q^2 l^2}{4\pi l^2 \Xi r_+^{(2n-3)}},$$  (22)

$$\Omega_j = \frac{a_j}{\Xi l^2},$$  (23)

where $V_{n-1}$ denotes the volume of the hypersurface boundary $B$ at constant $t$ and $r$, and $\beta_+$ is the inverse Hawking temperature. Equation (19) shows that the temperature $T$ in Eq. (22) is positive for the allowed values of the metric parameters and vanishes for the extremal solution.

To obtain the total action we first calculate the logarithmic divergences due to the Weyl anomaly and matter field given in Eqs. (4), (5), (7), and (8). The leading metric $h_{ij}^0$ can be obtained as

$$h_{ij}^0 dx^i dx^j = -\frac{1}{l^2} dt^2 + d\phi^2 + d\Omega^2.$$  (24)

Therefore the curvature scalar $R^0(h^0)$ and Ricci tensor $R_{ij}^0(h^0)$ are zero. Also it is easy to show that $F_{ij}^0$ in Eqs. (7) and (8) vanishes. Thus, all the logarithmic divergences for the $(n+1)$-dimensional charged rotating black brane are zero. It is also a matter of calculation to show that the counterterm action due to the electromagnetic field in Eq. (9) is zero. Thus, using Eqs. (1), (3), (10), and (11), the Euclidean actions in the grand-canonical and
the canonical ensemble can be calculated as
\[ I = -\frac{\beta + V_{n-1}}{16\pi} \frac{r_{+}^{(2n-2)} + q^2l^2}{r_{+}^{(n-2)} l^2}, \]  \hfill (25)

\[ \tilde{I} = -\frac{\beta + V_{n-1}}{16\pi} \frac{r_{+}^{(2n-2)} - (2n - 3)q^2l^2}{r_{+}^{(n-2)} l^2}. \]  \hfill (26)

The electric charge \( Q \), can be found by calculating the flux of the electromagnetic field at infinity, yielding
\[ Q = \frac{\Xi V_{n-1}}{4\pi} \sqrt{\frac{(n-1)(n-2)}{2}} q. \]  \hfill (27)

The electric potential \( \Phi \), measured at infinity with respect to the horizon, is defined by [18]
\[ \Phi = A_{\mu} \chi_{\mu} \bigg|_{r \to \infty} - A_{\mu} \chi_{\mu} \bigg|_{r = r_+}, \]
where \( \chi \) is the null generators of the event horizon given by Eq. [18]. One obtains
\[ \Phi = \sqrt{\frac{n - 1}{2(n - 2)}} \frac{q}{\Xi r_+^{(n-2)}}. \]  \hfill (28)

Since the area law of the entropy is universal, and applies to all kinds of black holes/branes [22], the entropy is
\[ S = \frac{\Xi V_{n-1}}{4} r_+^{(n-1)}. \]  \hfill (29)

For \( n = 3 \), these quantities given in Eqs. (20)-(29) reduce to those calculated in Ref. [9].

IV. THERMODYNAMICS OF BLACK BRANE

A. Energy as a function of entropy, angular momentum, and charge

We first obtain the mass as a function of the extensive quantities \( S, J, \) and \( Q \). Using the expression for the entropy, the mass, the angular momenta, and the charge given in Eqs. (20), (21), (27), (29), and the fact that \( f(r_+) = 0 \), one can obtain a Smarr-type formula as
\[ M(S, J, Q) = \frac{(nZ - 1)\sqrt{\sum_{i}^k J_i^2}}{nl\sqrt{Z(Z - 1)}}, \]  \hfill (30)

where \( Z = \Xi^2 \) is the positive real root of the following equation:
\[ (Z - 1)^{(d-2)} - \frac{Z}{16S^2} \left\{ \frac{4\pi(n-1)(n-2)lSJ}{n[(n-1)(n-2)S^2 + 2\pi^2Q^2l^2]} \right\}^{(2n-2)} = 0. \]  \hfill (31)
One may then regard the parameters $S$, $J$, and $Q$ as a complete set of extensive parameters for the mass $M(S, J, Q)$ and define the intensive parameters conjugate to $S$, $J$ and $Q$. These quantities are the temperature, the angular velocities, and the electric potential

$$
T = \left( \frac{\partial M}{\partial S} \right)_{J,Q}, \quad \Omega_i = \left( \frac{\partial M}{\partial J_i} \right)_{S,Q}, \quad \Phi = \left( \frac{\partial M}{\partial Q} \right)_{S,J}.
$$

(32)

It is a matter of straightforward calculation to show that the intensive quantities calculated by Eq. (32) coincide with Eqs. (22), (23), and (28) found in Sec. (III). Thus, the thermodynamic quantities calculated in Sec. (III) satisfy the first law of thermodynamics

$$
dM = TdS + \sum_{i=1}^{k} \Omega_i dJ_i + \Phi dQ.
$$

(33)

**B. Thermodynamic potentials**

We now obtain the thermodynamic potential in the grand-canonical and canonical ensembles. Using the definition of the Gibbs potential $G(T, \Omega, \Phi) = I/\beta$, we obtain

$$
G = -\frac{V_{n-1}}{16\pi} \left( \frac{2}{n^2(n-1)(1-\sum_i l^2\Omega_i^2)} \right)^{n/2} (\gamma^2 + n^2(n-2)\Phi^2) (\gamma l)^{(n-2)},
$$

(34)

where

$$
\gamma = \sqrt{2n-2}T\pi l + \sqrt{2(n-1)\pi^2T^2l^2 + n(n-2)^2\Phi^2}.
$$

(35)

Using the expressions (22), (23), and (28) for the inverse Hawking temperature, the angular velocities and the electric potential, one obtains

$$
G(T, \Omega, \Phi) = M - TS - \sum_i \Omega_i J_i - \Phi Q,
$$

(36)

which means that $G(T, \Omega, \Phi)$ is, indeed, the Legendre transformation of the $M(S, J_i, Q)$ with respect to $S$, $J_i$, and $Q$. It is a matter of straightforward calculation to show that the extensive quantities

$$
J_i = -\left( \frac{\partial G}{\partial \Omega_i} \right)_{T,\Phi}, \quad Q = -\left( \frac{\partial G}{\partial \Phi} \right)_{T,\Omega}, \quad S = -\left( \frac{\partial G}{\partial T} \right)_{\Omega,\Phi},
$$

(37)

turn out to coincide precisely with the expressions (21), (27), and (29).

For the canonical ensemble, the Helmholtz free energy $F(T, J, Q)$ is defined as
\[ F(T, J, Q) = \frac{\tilde{T}}{\beta} + \sum_{i}^{k} \Omega_i J_i, \]

(38)

where \( \tilde{T} \) is given by Eq. (26). One can verify that the conjugate quantities

\[ \Omega_i = \left( \frac{\partial F}{\partial J_i} \right)_{T,Q}, \quad \Phi = \left( \frac{\partial F}{\partial Q} \right)_{T,J}, \quad S = -\left( \frac{\partial F}{\partial T} \right)_{J,Q}, \]

(39)

agree with expressions (23), (28), and (29). Also it is worthwhile to mention that \( F(T, J, Q) \) is the Legendre transformation of the \( M(S, J_i, Q) \) with respect to \( S \), i.e

\[ F(T, J, Q) = M - TS. \]

(40)

C. Stability in the canonical and the grand-canonical ensemble

The stability of a thermodynamic system with respect to the small variations of the thermodynamic coordinates, is usually performed by analyzing the behavior of the entropy \( S(M, J, Q) \) around the equilibrium. The local stability in any ensemble requires that \( S(M, J, Q) \) be a convex function of their extensive variables or its Legendre transformation must be a concave function of their intensive variables. Thus, the local stability can in principle be carried out by finding the determinant of the Hessian matrix of \( S \) with respect to its extensive variables \( X_i, H^S_{X_i X_j} = [\partial^2 S/\partial X_i \partial X_j] \), or the determinant of the Hessian of the Gibbs function with respect to its intensive variables \( Y_i, H^G_{Y_i Y_j} = [\partial^2 G/\partial Y_i \partial Y_j] \) [18, 23]. Also, one can perform the stability analysis through the use of the Hessian matrix of the mass with respect to its extensive parameters [24]. In our case the entropy \( S \) is a function of the mass, angular momenta, the charge. The number of thermodynamic variables depends on the ensemble which is used. In the canonical ensemble, the charge and the angular momenta are fixed parameters, and therefore the positivity of the thermal capacity \( C_{J,Q} = T(\partial S/\partial T)_{J,Q} \) is sufficient to assure the local stability. The thermal capacity \( C_{J,Q} \) at constant charge and angular momenta is

\[ C_{J,Q} = \frac{\Xi V^{n-1}}{4} r^{(n-1)} [n r^{(2n-2)} - (n - 2) q^2 l^2] [r^{(2n-2)} + q^2 l^2] \]

\[ \times [(n - 2) \Xi^2 + 1] \{(n - 2) q^4 l^4 [(3n - 6) \Xi^2 - (n - 3)] \]
Figure 1 shows the behavior of the heat capacity as a function of the charge parameter. It shows that $C_{J,Q}$ is positive in various dimensions and goes to zero as $q$ approaches its critical value (extreme black brane). Thus, the $(n + 1)$-dimensional AAdS charged rotating black brane is locally stable in the canonical ensemble.

In the grand-canonical ensemble, we find it more convenient to work with the Gibbs potential $G(T, \Omega_i, \Phi)$. Here the thermodynamic variables are the temperature, the angular velocities, and the electric potential. After some algebraic manipulation, we obtain

$$|\mathbf{H}_{T,\Omega_i,\Phi}^G| = \left( \frac{n - 3}{n} \right) \left[ \frac{nm}{16\pi} \right] \left[ \frac{(n - 2)\Xi^2 + 1}{r_+^{n} + (\frac{n-2}{2})ml^2} \right] (l\Xi)^{2k+2}z^{3n-4}. \quad (42)$$

As one can see from Eq. (42), $|\mathbf{H}_{T,\Omega_i,\Phi}^G|$ is positive for all the phase space, and therefore the $(n + 1)$-dimensional AAdS charged rotating black brane is locally stable in the grand-canonical ensemble. As mentioned in the previous paragraph, one can also perform the local stability analysis through the use of the determinant of the Hessian matrix of $M$ with respect to $S$, $J$ and $Q$, which has the same result, since $|\mathbf{H}_{S,J,Q}^M| = |\mathbf{H}_{T,\Omega_i,\Phi}^G|^{-1}.$
D. Logarithmic correction to the Bekenstein-Hawking entropy

In recent years, there are several works in literature suggesting that for a large class of black holes, the area law of the entropy receives additive logarithmic corrections due to thermal fluctuation of the object around its equilibrium [25]. Typically, the corrected formula has the form

$$S = S_0 - K \ln(S_0) + ..., \quad (43)$$

where $S_0$ is the standard Bekenstein-Hawking term and $K$ is a number. In Ref. [26], an expression has been found for the leading-order correction of a generic thermodynamic system in terms of the heat capacity $C$ as

$$S = S_0 - K \ln(CT^2). \quad (44)$$

Equation (44) has been considered by many authors for Schwarzschild-AdS, Reissner-Nordstrom-AdS, BTZ, and slowly Kerr-AdS spacetimes [27]. Thus, it is worthwhile to investigate its application for the charged rotating black brane considered in this paper. Using Eqs. (22), (29), and (41) with $q = 0$, one obtains

$$S = S_0 - \frac{n + 1}{2(n - 1)} \ln(S_0) - \Gamma_n(\Xi), \quad (45)$$

where $\Gamma_n(\Xi)$ is a positive constant depending on $\Xi$ and $n$. Equation (45) shows that the correction of the entropy is proportional to the logarithm of the area of the horizon. For small values of $q$ the logarithmic correction in Eq. (44) can be expanded in terms of the power of $S_0$ as

$$S = S_0 - \frac{n + 1}{2(n - 1)} \ln(S_0) - \Gamma_n(\Xi) + \frac{l^2 \Xi^2 [(n - 4) \Xi^2 + 2]}{16[(n + 2) \Xi^2 - n - 1]} \frac{q^2}{S_0^2} + ... \quad (46)$$

Again the leading term is a logarithmic term of the area.

V. CLOSING REMARKS

In this paper, we calculated the conserved quantities and the Euclidean actions of the charged rotating black branes both in the canonical and the grand-canonical ensemble through the use of counterterms renormalization procedure. Also we obtained the charge and
electric potentials of the black brane in an arbitrary dimension. We found that the logarithmic divergencies associated with the Weyl anomalies and matter field are zero. We obtained a Smarr-type formula for the mass as a function of the extensive parameters $S$, $J$ and $Q$, calculated the temperature, angular velocity, and electric potential, and showed that these quantities satisfy the first law of thermodynamics. Using the conserved quantities and the Euclidean actions, the thermodynamics potentials of the system in the canonical and grand-canonical ensemble were calculated. We found that the Helmholtz free energy $F(T, J, Q)$ is a Legendre transformation of the mass with respect to $S$ and the Gibbs potential is a Legendre transformation of the mass with respect to $S$, $J$ and $Q$ in the grand-canonical ensemble.

Also, we studied the phase behavior of the charged rotating black branes in $(n + 1)$ dimensions and showed that there is no Hawking-Page phase transition in spite of the angular momentum of the branes. Indeed, we calculated the heat capacity and the determinant of the Hessian matrix of the Gibbs potential with respect to $S$, $J$, and $Q$ of the black brane and found that they are positive for all the phase space, which means that the brane is stable for all the allowed values of the metric parameters discussed in Sec. III. This analysis has also be done through the use of the determinant of the Hessian matrix of $M(S, J, Q)$ with respect its extensive variables and we got the same phase behavior. This phase behavior is commensurate with the fact that there is no Hawking-Page transition for a black object whose horizon is diffeomorphic to $\mathbb{R}^p$ and therefore the system is always in the high temperature phase.

Finally, we obtained the logarithmic correction of the entropy due to the thermal fluctuation around the thermal equilibrium. For the case of uncharged rotation black brane, we found that only a term which is proportional to $\ln(\text{area})$ will appear. But we found that for the charged rotating black brane, the correction contains other powers of the area including the logarithmic term. Verlinde drew a fundamental connection between the holographic principle, the entropy formula for conformal field theory, and the Friedman-Robertson-Walker equations for a closed radiation dominated universe. Therefore the investigation of the effect of the logarithmic corrections to the Cardy-Verlinde formula when thermal fluctuations of the AAdS black brane are taken into account and their influence on the braneworld
cosmology remains a subject for future.

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