AV-differential geometry and calculus of variations

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August 30, 2018

Supported by KBN, Grant 2PO3A 041 18, Grant 2PO3A 036 25

Abstract

The calculus of variations for lagrangians which are not functions on the tangent bundle, but sections certain affine bundles is developed. We follow a general approach to variational principles which admits boundary terms of variations.

MSC 2000: 70G45, 70H03, 70H05

Key words: affine spaces, Lagrangian formalism, Euler-Lagrange equation

1 Introduction

It is already commonly accepted that the gauge independent lagrangian for relativistic charged particle is not a function, but a section of a bundle of affine lines over the tangent bundle of the space-time manifold [9]. Also frame-independent lagrangian for Newtonian particle is not a function, but a section of an affine bundle [3]. In [1, 10] we have shown that the proper geometric tools for such a frame-independent formulation of Lagrangian systems are provided by the geometry of affine values (AV-differential geometry). We call the geometry of affine values the differential geometry which is built using sections of a principal $\mathbb{R}$-bundle over the manifold instead of functions on the manifold. Since the bundle we use is equipped with the fiber action of the group $(\mathbb{R}, +)$, we can add reals to elements of fibres and real functions to sections, but there is no distinguished "zero section".

In the present paper we apply these tools to develop the calculus of variations for lagrangians with affine values. First, we recognize the affine space of values of the action functional. Then, we find affine analogues of constructions, which lead to the proper representation of the differential of the action functional. Finally, we formulate variational principles for lagrangians with affine values. Note that the affine analogues of the Euler-Lagrange equations have been obtained purely geometrically in [4, 2].

*Research supported by the Polish Ministry of Scientific Research and Information Technology under the grant No. 2 P03A 036 25.
2 Bundles of affine values

An affine bundle \( \zeta : Z \to M \), modelled on the trivial vector bundle \( M \times \mathbb{R} \to M \), will be called a bundle of affine values (shortly, an AV-bundle). We can say equivalently that an AV-bundle is a principal bundle with the structural group \((\mathbb{R}, +)\).

An AV-bundle is canonically associated with any special affine bundle \( A = (A, v_A) \), i.e. an affine bundle \( A \) with a distinguished nowhere-vanishing section \( v_A \) of the model vector bundle \( V(A) \). The free \( \mathbb{R} \)-action induced from translations in the direction of \(-v_A\) makes \( A \) an into an \((\mathbb{R}, +)\)-principal bundle over \( A/\langle v_A \rangle \). This AV-bundle we will denote \( AV(A) \).

An affine covector is an equivalence class of the relation defined in the set of pairs of \((m, \varphi)\), where \( m \in M \) and \( \varphi \) is a section of \( Z \). We say that \((m, \varphi)\), \((m', \varphi')\) are equivalent if \( m = m' \) and \( d(\varphi - \varphi')(m) = 0 \), where we have identified the difference of sections of \( Z \) with a function on \( M \). The equivalence class of \((m, \varphi)\) is denoted by \( d\varphi(m) \). An affine analogue of the cotangent bundle is the union of all affine covectors. It is an affine bundle modelled on \( T^*M \) and equipped with a canonical symplectic form. We denote it by \( PZ \) and we call it the phase bundle for \( Z \).

Since in the space of affine 1-forms (sections of \( PZ \to M \)) we have a distinguished family of exact forms \( d\varphi \) and the de Rham differential \( d(\sigma - d\varphi) \) does not depend on \( \varphi \), there is a well-defined affine de Rham differential from affine 1-forms into affine 2-forms, which are ordinary 2-forms on \( M \). The de Rham complex for \( Z \) can be then continued as in the standard case.

3 Other examples of affine constructions

In the following, we need two simple constructions concerning affine spaces and affine forms.

Let \( A, B \) be two affine spaces with the same model vector space \( V \). In the product \( A \times B \), we define an equivalence relation. Two pairs \((a, b)\) and \((a', b')\) are equivalent if \( a - a' = b' - b \). The equivalence class of a pair \((a, b)\) is, by definition, the sum of \( a \) and \( b \). We denote it by \( a \boxplus b \). The family of all sums is, obviously, an affine space modelled on \( V \) with

\[
(a \boxplus b - a' \boxplus b') = (a - a') + (b - b')
\]

and we denote it by \( A \boxplus B \). Similarly, we define \( A \boxminus B \).

The same method can be used to define an affine space of values of integrals of affine forms along an oriented 1-dimensional cell in \( M \). The value of the integral is the equivalence class of pairs \((\sigma, I)\), where \( \sigma \) is an affine 1-form, i.e. a section of the phase bundle \( PZ \), and \( I \) is a cell in \( M \). Two pairs \((\sigma, I)\) and \((\sigma', I')\) are equivalent if \( I = I' \), and

\[
\int_I (\sigma - \sigma') = 0
\]

The equivalence class of a pair \((\sigma, I)\) will be denoted by \( \int_I \sigma \). It is obvious that the set of values of integrals along \( I \) is an affine space with the model vector space \( \mathbb{R} \). The linear part of \( \int_I \) is the standard integral of 1-forms.
Proposition 3.1. Let $\gamma: [a, b] \to M$ be a parameterization of a cell $I$. Then the space of values of the integral is canonically isomorphic to $Z_{\gamma(b)} \sqcup Z_{\gamma(a)}$.

Proof. For $\sigma = d\varphi$ we put $J(\sigma) = \varphi(\gamma(b)) \sqcup \varphi(\gamma(a)) \in Z_{\gamma(b)} \sqcup Z_{\gamma(a)}$. It is obvious that the mapping $J$ is well-defined and surjective. Moreover

$$J(d(\varphi + f)) = J(d\varphi) + (f(\gamma(b)) - f(\gamma(b))) = J(d\varphi) + \int_I df,$$

i.e. the linear parts of $J$ and $\int_I$ coincide on exact forms. It follows that the mapping

$$\int_I d\varphi \mapsto J(d\varphi)$$

is well defined and gives canonical isomorphism of $\in Z_{\gamma(b)} \sqcup Z_{\gamma(a)}$ and the AV-space for $\int_I$.

4 Special vector spaces and duality

Every finite-dimensional vector space $V$ can be considered as the space of linear functions on its dual $V^*$, i.e. the space of linear sections of the trivial bundle $V^* \times \mathbb{R}$. This bundle is an example of an AV-bundle associated with a special vector space, namely the space $V \times \mathbb{R}$ with the distinguished non-zero vector $(0, 1)$. Also elements of an affine space $A$ with the model vector space $V$ can be interpreted as linear sections of an AV-bundle $\tau: A^! \to V^*$. The canonical choice for $A^!$ is the special vector space of affine functions on $A$. The projection $\tau$ associates with an affine function its linear part.

Proposition 4.1. For an AV-bundle $Z = (Z, v_Z)$, we have a canonical isomorphism of special vector spaces $((P_mZ)^!, 1)$ and $(T_ZZ, v_0)$, where $m = \zeta(z)$ and $v_0$ is the fundamental vector for the group action on $Z$.

Proof. Let $a = d_m\varphi$ with $\varphi(m) = z$. The mapping

$$a: T_mM \to T_zZ: v \mapsto T_\varphi(v)$$

is a linear section of the AV-bundle, associated to special vector space $(T_zZ, v_0)$. By duality, we get canonical isomorphism of $(P_mZ)^!$ and $(T_zZ, v_0)$. □

The group action on $Z$ gives isomorphisms of tangent spaces along fibers of $\zeta$. It follows that the bundle $(PZ)^!$ can be identified with $\tilde{T}Z = TZ/\mathbb{R}$. We conclude that there is a one-to one correspondence between linear sections of $\tilde{T}\zeta: \tilde{T}Z \to TM$ and affine one-forms. In the following we use this isomorphism to identify affine covectors in terms of its affine values.
4.1 The action functional

It is known that the lagrangians for relativistic charged particles, lagrangians in Newtonian mechanics, etc., are not functions but sections of an AV-bundle of the form \( \tilde{T}_{\zeta} \colon \tilde{T} \to TM \) for certain bundle \( \zeta : Z \to M \). The action for such lagrangian can be defined in the following way. First, we observe that by Proposition 4.1 every affine 1-form, i.e. a section \( \sigma \) of the phase bundle \( \mathcal{P}Z \), defines a section of \( \tilde{T}_{\zeta} \). We denote it by \( i^a_T \varphi \). Let \( \gamma : [a, b] \to M \). We put

\[
\int_a^b i^a_T \varphi \circ \dot{\gamma} := \int_{\gamma([a,b])} \varphi \in Z_{\gamma(b)} \sqcup Z_{\gamma(a)},
\]

and for any lagrangian \( \lambda : TM \to \tilde{T}Z \),

\[
\int_a^b \lambda \circ \dot{\gamma} = \int_a^b (\lambda \circ \dot{\gamma} - i^a_T \varphi \circ \dot{\gamma}) + \int_a^b i^a_T \varphi \circ \dot{\gamma}.
\]

The action for a lagrangian \( \lambda \) can be defined in a different (but equivalent) way. Let \( \gamma^*Z \) be the pull-back of \( Z \) with respect to the curve \( \gamma \). We have mappings \( \tilde{T}_{\gamma} : \tilde{T} \gamma^*Z \to \tilde{T}Z \),

\[
\text{defined in an obvious way. It is clear that the image of } \lambda \circ \dot{\gamma} \text{ belongs to the image of } \tilde{T}_{\gamma}.
\]

Consequently, there is a unique section

\[
\lambda_\gamma : [a, b] \to \tilde{T} (\gamma^*Z)
\]

such that

\[
(\tilde{T}_{\gamma}) \circ \lambda_\gamma = \lambda \circ \dot{\gamma}.
\]

The section \( \lambda_\gamma \) defines an \( \mathbb{R} \)-invariant vector field on \( \gamma^*Z \) and its family of integral curves is also \( \mathbb{R} \)-invariant. It follows that for each integral curve \( \lambda_\gamma : [a, b] \to \gamma^*Z \) we can define the affine number \( \tilde{\lambda_\gamma}(b) \sqcup \tilde{\lambda_\gamma}(a) \) which does not depend on the choice of the curve.

**Proposition 4.2.** For each curve \( \gamma : [a, b] \to M \), we have

\[
\tilde{\lambda_\gamma}(b) \sqcup \tilde{\lambda_\gamma}(a) = \int_a^b \lambda \circ \dot{\gamma}.
\]

5 Operations on affine 1-forms

5.1 Pull-back of an affine 1-form

Let \( g : N \to M \) be a differentiable mapping and let \( \pi : P \to M \) be an affine bundle, modelled on \( T^*M \). The AV-bundle for \( P \) is \( \zeta_P : P^! \to TM \). The bundle \( \pi_g : P_g \to N \) we define as the dual to the pull-back of the bundle of affine values for \( P \), with respect to the tangent mapping \( Tg \):
\[(P_g)^\dagger = (\Gamma g)^* P^\dagger.\] (5.1)

The pull-back \(g^* \sigma\) of a section \(\sigma\) of \(\pi\) is defined by
\[i_T^g(g^* \sigma) = (\Gamma g)^* i_T^g(\sigma).\] (5.2)

If the bundle \(P\) is the phase bundle for an AV-bundle \(Z\), then we have the obvious isomorphism
\[(PZ)_g = P(g^* Z)\] (5.3)
and, as in the standard case,
\[g^* d\varphi = d(g^*(\varphi)),\] (5.4)
i.e. the pull-back commutes with the affine exterior differential.

### 5.2 The total derivative \(d_T^a\)

Derivations \(d_T\) and \(i_F\) play fundamental role in the geometric integration by parts - the central point in the calculus of variations. Both, \(d_T\) and \(i_F\) are derivations defined on the algebra of differential forms on a manifold \(N\), with values in differential forms on \(TN\). Here, we define their affine counter-parts, but for affine 1-forms only.

In the standard geometry, the total derivative \(d_T\) can be defined in two equivalent ways. First, as the commutator
\[d_T = i_T d + di_T,\]
where \(i_T\) is the derivation given by the identity mapping on \(TN\), interpreted as a vector-valued function on \(TN\), with values in \(TN\). Second, by the formula
\[i_T(d_T \omega) = (d_T i_T \omega) \circ \kappa_N,\] (5.5)
where \(\omega\) is a 1-form, and \(\kappa_N: TTN \rightarrow TTN\) is the canonical flip.

Both formulae have their counterparts in the affine case. Let \(Y\) be an AV-bundle over \(N\). The affine total derivative can be defined by the formula
\[d_T^a(\sigma) = i_T d\sigma + di_T^a \sigma\] (5.6)
for a section \(\sigma\) of \(PZ\). Since \(d\sigma\) is an ordinary 2-form, the first term in \(5.6\) is a 1-form on \(TN\). The second term is a section of \(PTY\) and the corresponding dual AV-bundle is \(\tilde{TY}\). We conclude that the AV-bundle for \(d_T\) is also \(\tilde{TY}\).

Another definition of the total derivative is by a formula, analogous to \(5.5\). First, we observe that the canonical flip \(\kappa_Y: TTY \rightarrow TTY\) reduces to an isomorphism
\[\tilde{\kappa}_Y: \tilde{TY} \rightarrow \tilde{TY} .\]

**Proposition 5.1.** Let \(\sigma\) be an affine 1-form on \(N\), i.e. a section of \(PY\). then
\[i_T^a(d_T \sigma) = \tilde{\kappa}_Y \circ (i_T^a d(i_T^{a} \sigma)) \circ \kappa_N\] (5.7)
Proof. The linear parts of [5.6] and [5.7] coincide, so it is enough to compare these formulae for \( \sigma = d\varphi \), where \( \varphi \) is a section of the AV-bundle \( Y \). The formula [5.6] gives in this case

\[
d_i^o d\varphi = d_i^o d\varphi
\]

and the corresponding section of the AV-bundle

\[
i_i^o (d_T d\varphi) = i_i^o d_i^o d\varphi.
\]  

(5.8)

It follows from [4.1] that \( i_i^o d(i_i^o d\varphi) \) is reduced \( TT\varphi \), i.e. \( \tilde{T}\tilde{T}\varphi \). The canonical flip is an equivalence of functors, hence

\[
\tilde{k}_Y \circ (\tilde{T}\tilde{T}\varphi) \circ \kappa_N \equiv \tilde{T}\tilde{T}\varphi.
\]

We obtain

\[
i_i^o (d_T d\varphi) = i_i^o d_i^o d\varphi = \tilde{k}_Y \circ (i_i^o d(i_i^o d\varphi)) \circ \kappa_N.
\]  

(5.9)

5.3 The operation \( i_i^o \)

\( i_F \) is a derivation in the algebra of forms on \( TN \) given by the vertical endomorphism \( F: TTN \rightarrow TTN \), interpreted as vector-valued 1-form. For a standard 1-form \( \omega \) on \( TN \) we have \( i_F \omega(w) = \omega(Fw) \). The same formula we use for affine 1-forms - sections of \( P\tilde{TY} \). Since the AV-bundle for \( P\tilde{TY} \) is \( \tilde{T}\tilde{T}Y \), the AV-bundle for \( i_i^o \sigma \) is the pull-back bundle \( F^*\tilde{T}\tilde{T}Y \).

Proposition 5.2. The AV-bundle \( F^*\tilde{T}\tilde{T}Y \) is canonically isomorphic to \( (\tau^0_1)^*\tilde{T}Y \), where \( \tau^0_1: TN \rightarrow N \) is the canonical projection.

Proof. An element \( \tilde{w} \) of \( (\tau^0_1)^*\tilde{T}Y \) is represented by a vector in \( \tilde{T}\tilde{T}Y \). If the vector \( w = \tilde{T}\tilde{T}(\tilde{w}) \in TTN \) is vertical, then also \( \tilde{w} \) is vertical with respect to the canonical projection \( \tilde{k}_Y: \tilde{T}Y \rightarrow N \). Since \( \tilde{T}Y \) is a vector bundle over \( N \), a vertical vector on \( \tilde{T}Y \) can be identified with an element of \( \tilde{T}Y \). It follows that AV-values fibre over \( w \in TTN \) for an affine covector \( i_i^o \sigma \) can be canonically identified with the fibre of \( \tilde{T}Y \) over \( \tau^0_1(w) \).

6 The Euler-Lagrange equation

Integration by parts in the calculus of variations for curves is based on the following decomposition of the differential of a lagrangian \( L: TM \rightarrow \mathbb{R} \)

\[
(\tau^1_1)^* dL = ((\tau^1_2)^* dL - d_T (i_F dL)) + d_T (i_F dL),
\]  

(6.1)

where \( \tau^1_2: T^2M \rightarrow T^1M = TM \) is the canonical projection. The first component in (6.1) is a 1-form on \( T^2M \), vertical with respect to projection \( T^2M \rightarrow M \). It can be considered as a mapping \( T^2M \rightarrow T^*M \)
Now, let $\lambda: TM \to \bar{T}$ be an affine lagrangian, i.e. a section of $\bar{T}\zeta$. We have the affine version of (6.1)

$$\tau_2^1 \ast d\lambda = (((\tau_2^1) \ast d\lambda \ominus d\eta (i_\eta^0 \ast d\lambda)) \ominus d\eta (i_\eta^0 \ast d\lambda)),$$

(6.2)

The AV-bundle for the pull-back $(\tau_2^1) \ast d\lambda$ is (see Section 5.1)

$$(T\tau_2^1) \ast (\bar{T}T\mathbf{Z}) = \bar{T}((\tau_2^0) \ast \bar{T}\mathbf{Z}) = \bar{T}((\tau_2^0) \ast \bar{T}\mathbf{Z}).$$

(6.3)

We have used that the pull-back commutes with the exterior derivative and $\tau_2^1$ coincides with $\tau_2^0$.

The AV-bundle for $i_\eta^0 \ast d\lambda$ is (see the previous section)

$$(T\tau_2^0) \ast \bar{T}\mathbf{Z} = \bar{T}((\tau_2^0) \ast \mathbf{Z}).$$

This, together with (6.3) implies that the AV-bundle for $((\tau_2^1) \ast d\lambda \ominus d\eta (i_\eta^0 \ast d\lambda))$ is trivial. We can write $(\tau_2^1) \ast d\lambda - d\eta (i_\eta^0 \ast d\lambda)$, which is ordinary 1-form on $T^2M$. Since for $\lambda = i_\eta^0 (d\phi)$ we have (arguing as in Proposition 5.2) that

$$i_\eta^0 (d\phi) = (\tau_2^0) \ast d\phi = d((\tau_2^0) \ast d\phi)$$

and

$$d\eta (i_\eta^0 (d\phi)) = d((\tau_2^0) \ast d\phi) = (\tau_2^1) \ast d((\tau_2^0) \ast d\phi).$$

It follows that the first term in the decomposition (6.2) equals zero. It is known that the first term in the decomposition (6.1) is vertical with respect to the projection $\tau_2^0$, i.e. it can be considered as a mapping $\mathcal{E}\lambda: T^2M \to T^*M$. We conclude, that the same remains valid for affine lagrangian $\lambda$. Similarly, we have that the affine 1-form $i_\eta^0 \ast d\lambda$ is vertical with respect to the projection $\tau_2^0$. It is meaningful because the AV-bundle for $i_\eta^0 \ast d\lambda$ is the pull-back of $\bar{T}\mathbf{Z}$ (Proposition 5.2). Therefore $i_\eta^0 \ast d\lambda$ defines a mapping $\mathcal{P}\lambda: TM \to \mathcal{P}\mathbf{Z}$, which is the affine Legendre map.

### 7 Variational principles

Variational principles are based on the proper representation of the differential of the action functional

$$\mathcal{L}: \mathcal{M}_{[a,b]} \to \mathbb{R} : \gamma \mapsto \int_a^b \lambda \circ \dot{\gamma},$$

where $\mathcal{M}_{[a,b]}$ is the space of smooth mappings from the interval $[a, b]$ to $M$. As we have seen in Section 4.1, the AV-space for $\mathcal{L}$ at $\gamma$ is $\mathcal{Z}_{\gamma(b)} \boxplus \mathcal{Z}_{\gamma(a)}$. A vector tangent to $\mathcal{M}_{[a,b]}$ is a curve in $TM$, i.e. a mapping $w: [a, b] \to TM$. The AV-bundle for the differential $d\mathcal{L}$ at $\gamma$ is then $\mathcal{T}_w M \boxplus \mathcal{T}_{\gamma(a)} M$. A convenient representation of the differential is suggested by the decomposition (6.2)

$$\langle d\mathcal{L}(\gamma), w \rangle = \langle (\mathcal{P}\lambda \circ \dot{\gamma}(b), w(b)) \boxplus (\mathcal{P}\lambda \circ \dot{\gamma}(a), w(a)) \rangle = \int_a^b \langle \mathcal{E}\lambda \circ \ddot{\gamma}(t), w(t) \rangle.$$  

The above equality suggests that a covector should be represented by a mapping $f: [a, b] \to T^*M$ and two affine covectors $p_a$ and $p_b$ in $\mathcal{P}\mathbf{Z}$. The variational principle
\[ \int_a^b \langle d\lambda, \dot{w} \rangle = \langle p_b, w(b) \rangle \square \langle p_a, w(a) \rangle - \int_a^b \langle f(t), w(t) \rangle dt \]

produces the Euler-Lagrange equation

\[ E \lambda \circ \dot{\gamma} = f \]

and the momentum-velocity relations

\[ (P \lambda \circ \dot{\gamma})(a) = p_a, \quad (P \lambda \circ \dot{\gamma})(b) = p_b. \]

References

[1] K. Grabowska, J. Grabowski and P. Urbański: AV-differential geometry: Poisson and Jacobi structures, J. Geom. Phys. 52 (2004) no. 4, 398–446.

[2] K. Grabowska, J. Grabowski and P. Urbański: AV-differential geometry: Euler-Lagrange equations, to appear.

[3] K. Grabowska and P. Urbański: AV-differential geometry and Newtonian mechanics, Rep. Math. Phys. 58 (2006), 21–40.

[4] D. Iglesias, J. C. Marrero, E. Padrón, D. Sosa: Lagrangian submanifolds and dynamics on Lie affgebroids, Rep. Math Phys. 57 (2006), 385–436.

[5] G. Marmo, W. M. Tulczyjew, P. Urbański: Dynamics of autonomous systems with external forces, Acta Physica Polonica B, 33 (2002), 1181–1240

[6] G. Pidello and W. Tulczyjew : Derivations of differential forms on jet bundles, Ann. Mat. Pura Appl. 147 (1987), 249–265.

[7] W. Tulczyjew: Sur la différentielle de Lagrange, C. R. Acad. Sci. Paris. 280, (1975), 1295–1298.

[8] W. Tulczyjew: The Origin of Variational Principles, in "Classical and Quantum Integrability", Grabowski, J., Marmo, G., Urbański, P. (eds.), Banach Center Publications, vol. 59 (2003), 41–76.

[9] W.M. Tulczyjew, P. Urbański, An affine framework for the dynamics of charged particles, Atti Accad. Sci. Torino Suppl. n. 2, 126 1992, 257–265.

[10] P. Urbański, Affine framework for analytical mechanics, in "Classical and Quantum Integrability", Grabowski, J., Marmo, G., Urbański, P. (eds.), Banach Center Publications, vol. 59 (2003), 257–279.