THE ASYMPTOTIC BEHAVIOR OF TEICHMÜLLER RAYS

MASANORI AMANO

Abstract. In this paper, we consider the asymptotic behavior of two Teichmüller geodesic rays determined by Jenkins-Strebel differentials, and we obtain a generalization of a theorem in [Ama14]. We also consider the infimum of the asymptotic distance in shifting base points of the rays along the geodesics. We show that the infimum is represented by two quantities. One is the detour metric between the end points of the rays on the Gardiner-Masur boundary of the Teichmüller space, and the other is the Teichmüller distance between the end points of the rays on the augmented Teichmüller space.

1. Introduction

Let $X$ be a Riemann surface of genus $g$ with $n$ punctures such that $3g-3+n>0$, and $T(X)$ be the Teichmüller space of $X$. Any Teichmüller geodesic ray on $T(X)$ is determined by a holomorphic quadratic differential on a starting point of the ray. A geodesic ray is called a Jenkins-Strebel ray if it is given by a Jenkins-Strebel differential. In [Ama14], we obtain a condition for two Jenkins-Strebel rays to be asymptotic (Corollary 1.2 in [Ama14]). To obtain this condition, we use Theorem 1.1 in [Ama14] which gives the explicit asymptotic value of the Teichmüller distance between two similar Jenkins-Strebel rays with the same end point in the augmented Teichmüller space. In this paper, we improve this theorem, and obtain the asymptotic value of the distance between any two Jenkins-Strebel rays.

Abstract. In this paper, we consider the asymptotic behavior of two Teichmüller geodesic rays determined by Jenkins-Strebel differentials, and we obtain a generalization of a theorem in [Ama14]. We also consider the infimum of the asymptotic distance in shifting base points of the rays along the geodesics. We show that the infimum is represented by two quantities. One is the detour metric between the end points of the rays on the Gardiner-Masur boundary of the Teichmüller space, and the other is the Teichmüller distance between the end points of the rays on the augmented Teichmüller space.

1. Introduction

Let $X$ be a Riemann surface of genus $g$ with $n$ punctures such that $3g-3+n>0$, and $T(X)$ be the Teichmüller space of $X$. Any Teichmüller geodesic ray on $T(X)$ is determined by a holomorphic quadratic differential on a starting point of the ray. A geodesic ray is called a Jenkins-Strebel ray if it is given by a Jenkins-Strebel differential. In [Ama14], we obtain a condition for two Jenkins-Strebel rays to be asymptotic (Corollary 1.2 in [Ama14]). To obtain this condition, we use Theorem 1.1 in [Ama14] which gives the explicit asymptotic value of the Teichmüller distance between two similar Jenkins-Strebel rays with the same end point in the augmented Teichmüller space. In this paper, we improve this theorem, and obtain the asymptotic value of the distance between any two Jenkins-Strebel rays.

Let $r, r'$ be Jenkins-Strebel rays on $T(X)$ from $r(0) = [Y, f], r(0)' = [Y', f']$ determined by Jenkins-Strebel differentials $q, q'$ with unit norm on $Y, Y'$ respectively. It is known (cf. [HS07]) that the Jenkins-Strebel rays $r, r'$ have limits, say $r(\infty), r'(\infty)$, on the boundary of the augmented Teichmüller space $\hat{T}(X)$. Suppose that $r, r'$ are similar, that is, there exist mutually disjoint simple closed curves $\gamma_1, \ldots, \gamma_k$ on $X$ such that the set of homotopy classes of core curves of the annuli corresponding to $q, q'$ are represented by $f(\gamma_1), \ldots, f(\gamma_k)$ on $Y$ and $f'(\gamma_1), \ldots, f'(\gamma_k)$ on $Y'$ respectively. We denote by $m_j, m'_j$ the moduli of the annuli on $Y, Y'$ with core curves homotopic to $f(\gamma_j), f'(\gamma_j)$ respectively. We can define the Teichmüller distance $d_{\hat{T}(X)}(r(\infty), r'(\infty))$ between the end points $r(\infty), r'(\infty)$.

Our main result is the following:

2000 Mathematics Subject Classification. Primary 32G15, Secondary 30F60.

Key words and phrases. Teichmüller space; Teichmüller distance; Teichmüller geodesic; augmented Teichmüller space.
**Theorem 1.1.** For any two Jenkins-Strebel rays \( r, r' \),

\[
\lim_{t \to \infty} d_{T(X)}(r(t), r'(t)) = \begin{cases} 
\max \left\{ \frac{1}{2} \log \max_{j=1,\ldots,k} \frac{m'_j}{m_j}, \d_T(X)(r(\infty), r'(\infty)) \right\} & \text{(if } r, r' \text{ are similar)} \\
+\infty & \text{(otherwise)}
\end{cases}
\]

**Corollary 1.2.** If \( r, r' \) are similar, the minimum value of the equation above when we shift the starting points of \( r, r' \) is given by

\[
\max \left\{ \frac{1}{2} \delta, d_T(X)(r(\infty), r'(\infty)) \right\},
\]

where \( \delta = \frac{1}{2} \log \max_{j=1,\ldots,k} \frac{m'_j}{m_j} + \frac{1}{2} \log \max_{j=1,\ldots,k} \frac{m_j}{m'_j} \).

**Remark.** The quantity \( \delta \) is known as the *detour metric* between end points of the rays \( r, r' \) in the Gardiner-Masur boundary of \( T(X) \). We refer to [Wal12], and also [Ama14].

## 2. Preliminaries

2.1. **Teichmüller spaces.** Let \( X \) be an analytically finite Riemann surface which has genus \( g \) and \( n \) punctures, briefly, we say it is of type \( (g, n) \). We assume that \( 3g - 3 + n > 0 \). Let \( T(X) \) be the *Teichmüller space* of \( X \). It is the set of equivalence classes of pairs of a Riemann surface \( Y \) and a quasiconformal mapping \( f : X \to Y \). Two pairs \( (Y, f) \) and \( (Y', f') \) are equivalent if there is a conformal mapping \( h : Y \to Y' \) such that \( h \circ f \) is homotopic to \( f' \). We denote by \([Y, f]\) the equivalence class of a pair \((Y, f)\). The *Teichmüller distance* \( d_T(X) \) is a complete distance on \( T(X) \) which is defined by the following. For any \([Y, f], [Y', f']\) in \( T(X) \),

\[
d_T(X)([Y, f], [Y', f']) = \frac{1}{2} \log \inf K(h),
\]

where the infimum ranges over all quasiconformal mappings \( h : Y \to Y' \) such that \( h \circ f \) is homotopic to \( f' \), and \( K(h) \) is the maximal quasiconformal dilatation of \( h \).

2.2. **Holomorphic quadratic differentials.** A *holomorphic quadratic differential* \( q \) on \( X \) is a tensor of the form \( q(z)dz^2 \) where \( q(z) \) is a holomorphic function of a local coordinate \( z \) on \( X \). For any \( q \neq 0 \), a zero of \( q \) or a puncture of \( X \) is called a *critical point* of \( q \). Then, \( q \) has finitely many critical points. We allow \( q \) to have poles of order 1 at punctures of \( X \). Then, the \( L^1 \)-norm \( \|q\| = \int_X |q| \) is finite, where \( |q| = |q(z)|dxdy \). If \( \|q\| = 1 \), we call \( q \) of unit norm.

Let \( p_0 \) be a non-critical point of \( q \) and \( U \) be a small neighborhood of \( p_0 \) which does not contain any other critical points of \( q \). For any point \( p \) in \( U \), we can define a new local coordinate \( \zeta(p) = \int_{z(p_0)}^z q(z)^2dz \) on \( X \) where \( z \) is a local coordinate on \( U \). The coordinate \( \zeta \) is called a *q-coordinate*. By \( q \)-coordinates, we see that \( q = d\zeta^2 \), and in a common neighborhood of two \( q \)-coordinates \( \zeta_1, \zeta_2 \), the equation \( \zeta_2 = \pm \zeta_1 + \text{constant} \) holds.
Suppose that $p_0$ is a critical point of $q$, and its order is $n \geq -1$. In a small neighborhood of $p_0$ which does not contain any other critical points of $q$, there exists a local coordinate $z$ on $X$ such that $z(p_0) = 0$ and $q = z^n dz^2$. For instance, we refer to [Str84]. For any non-critical point in the neighborhood of $p_0$, there exists a $q$-coordinate $\zeta$. By $d\zeta^2 = z^n dz^2$, the transformation $\zeta = \frac{1}{n+2} z^{\frac{n+2}{2}}$ holds.

For any $k = 0, \cdots, n+1$, the set \( \{ \frac{2\pi k}{n+2} \leq \arg z \leq \frac{2\pi (k+1)}{n+2} \} \) on the $z$-plane is mapped to the half plane \( \{ 0 \leq \arg \zeta \leq \pi \} \) or \( \{ \pi \leq \arg \zeta \leq 2\pi \} \) on the $\zeta$-plane. We can see the trajectory flow in the neighborhood of $p_0$ as the $n+2$ copies of the half plane with the gluing along each horizontal edge of the planes (Figure 1).

\[ \begin{array}{c}
\text{z-plane} \\
\text{n = 1} \\
\text{-----------} \\
\text{-----------} \\
\text{-----------} \\
\text{-----------} \\
\text{-----------} \\
\text{n = 0} \\
\text{-----------} \\
\text{-----------} \\
\text{n = -1} \\
\end{array} \]

\[ \text{z-plane} \quad \text{z-plane} \]

\text{Figure 1. The trajectory flow in the neighborhood of $p_0$ in the case of $n = 1, 0, -1$}

A horizontal trajectory of $q$ is a maximal smooth arc $z = \gamma(t)$ on $X$ which satisfies $q(\gamma(t))(\frac{d\gamma(t)}{dt})^2 > 0$. By definition, horizontal trajectories of $q$ do not contain critical points of $q$. All horizontal trajectories of $q$ are Euclidean horizontal arcs in $q$-coordinates, moreover, by the form of transformations of $q$-coordinates, “horizontal directions” are preserved. A saddle connection of $q$ is a horizontal trajectory which joins critical points of $q$. We denote by $\Gamma_q$ the set of all critical points of $q$ and all saddle connections of $q$. Any component of $X - \Gamma_q$ is classified to the following two cases.

- **Annulus**: It is an annulus which is swept out by simple closed horizontal trajectories of $q$. These are free homotopic to each other. We call the simple closed horizontal trajectories the core curves of the annulus.
- **Minimal domain**: This domain is generated by infinitely many recurrent horizontal trajectories which are dense in the domain.

Since $q$ has finitely many critical points, the number of components of $X - \Gamma_q$ is finite. If $X - \Gamma_q$ has only annuli, we call $q$ a Jenkins-Strebel differential.
2.3. **Teichmüller geodesic rays.** For any holomorphic quadratic differential \( q \neq 0 \) on \( X \), a quasiconformal mapping \( f : X \to Y \) whose Beltrami coefficient is of the form \( \mu_f = -\frac{K(f)-1}{K(f)+1} \frac{d}{d\zeta} \) is called the Teichmüller mapping. For any quasiconformal mapping \( g : X \to Y \), there exists a Teichmüller mapping \( f : X \to Y \) which is homotopic to \( g \). Furthermore, the Teichmüller mapping satisfies \( K(f) \leq K(g) \) where the equality holds if and only if \( f = g \).

**Remark.** More generally, if \( X \) and \( Y \) have same genus and punctures, for any orientation preserving homeomorphism \( g : X \to Y \), there exists a Teichmüller mapping \( f : X \to Y \) which is homotopic to \( g \) (Theorem 1 in §1.5 of Chapter II of [Ah83]).

Let \( f : X \to Y \) be a Teichmüller mapping and \( q \) be the associated unit norm holomorphic quadratic differential on \( X \). In this situation, there exists a unit norm holomorphic quadratic differential \( \varphi \) on \( Y \) such that \( f \) maps each zero of order \( n \) of \( q \) to a zero of order \( n \) of \( \varphi \), and is represented by \( w \circ f \circ z^{-1} = K(f)^{-\frac{1}{2}}x + iK(f)^{\frac{1}{2}}y \) where \( z = x + iy \) and \( w \) are \( q \) and \( \varphi \)-coordinates respectively. Such \( \varphi \) is uniquely determined. For more details of the discussion, we refer the reader to [IT92].

Let \( p = [Y,f], q \neq 0 \) be a unit norm holomorphic quadratic differential on \( Y \), and \( z \) be any \( q \)-coordinate. The mapping \( r : \mathbb{R}_{\geq 0} \to T(X) \) is called a **Teichmüller geodesic ray from \( p \) determined by \( q \)** if for any \( t \geq 0 \), we assign a point \([Y_t, g_t \circ f] \) in \( T(X) \) to \( r(t) \) where \( g_t \) is a Teichmüller mapping on \( Y \) which is of the form \( z = x + iy \to z_t = e^{-t}x + ie^{t}y \), and \( Y_t \) is a Riemann surface which is determined by the coordinates \( z_t \). We assume that \( g_0 = id_Y \) and \( Y_0 = Y \). By properties of Teichmüller mappings, we have \( d_{T(X)}(r(s), r(t)) = |s-t| \) for any \( s, t \geq 0 \). If \( q \) is Jenkins-Strebel, we call \( r \) a **Jenkins-Strebel ray**.

Let \( r, r' \) be any two Jenkins-Strebel rays on \( T(X) \) from \( r(0) = [Y,f], r'(0) = [Y', f'] \) determined by Jenkins-Strebel differentials \( q, q' \) with unit norm on \( Y, Y' \) respectively. The rays \( r, r' \) are **similar** if there exist mutually disjoint simple closed curves \( \gamma_1, \cdots, \gamma_k \) on \( X \) such that the set of homotopy classes of core curves of the annuli corresponding to \( q, q' \) are represented by \( f(\gamma_1), \cdots, f(\gamma_k) \) on \( Y \) and \( f'(\gamma_1), \cdots, f'(\gamma_k) \) on \( Y' \) respectively.

2.4. **Augmented Teichmüller spaces.** We refer to [HS07] and [IT92] for augmented Teichmüller spaces. Let \( R \) be a connected Hausdorff space which satisfies following conditions:

- Any \( p \in R \) has a neighborhood which is homeomorphic to the unit disk \( \mathbb{D} = \{ |z| < 1 \} \) or the set \( \{(z_1, z_2) \in \mathbb{C}^2 | \, |z_1| < 1, |z_2| < 1, z_1 \cdot z_2 = 0 \} \). (In the latter case, \( p \) is called a **node** of \( R \).)
- Let \( p_1, \cdots, p_k \) be nodes of \( R \). We denote by \( R_1, \cdots, R_r \) the connected components of \( R - \{p_1, \cdots, p_k\} \). For any \( i = 1, \cdots, r \), each \( R_i \) is a Riemann surface of type \((g_i, n_i)\) which satisfies \( 2g_i - 2 + n_i > 0 \), \( n = \sum_{i=1}^r n_i - 2k \) and \( g = \sum_{i=1}^r g_i - r + k + 1 \).

We call \( R \) the **Riemann surface of type \((g, n)\) with nodes**.

The **augmented Teichmüller space** \( \hat{T}(X) \) is the set of equivalence classes of pairs of a Riemann surface of type \((g, n)\) with or without nodes \( R \) and a deformation...
f : X → R. The deformation f is a continuous mapping such that some disjoint loops on X are contracted to nodes of R, and is homeomorphic except to these loops. Two pairs (R, f) and (R′, f′) are equivalent if there is a conformal mapping h : R → R′ such that h ∘ f is homotopic to f′, where the conformal mapping means that each restricted mapping of a component of R − {nodes of R} onto a component of R′ − {nodes of R′} is conformal. Obviously, T(X) is included in T(Y). A topology of T(Y) is induced by the following. Let [R, f] in T(Y). For any compact neighborhood V of the set of nodes of R and any ε > 0, a neighborhood U_{V,ε} of [R, f] is defined by the set of [S, g] in T(Y) such that there is a deformation h : S → R which is (1 + ε)-quasiconformal on h^{-1}(R - V) such that f is homotopic to h ∘ g.

2.5. The end points of Jenkins-Strebel rays. We consider the end points of Jenkins-Strebel rays. In the following discussion, we use the detailed description in §4.1 of [HS07]. Let r be a Jenkins-Strebel ray on T(X) from r(0) = [Y, f] determined by a Jenkins-Strebel differential q with unit norm on Y. All components of Y - Γ_q are represented by rectangles C_1, · · · , C_k with identifications of vertical edges of them in q-coordinates. Let m_1, · · · , m_k be moduli of C_1, · · · , C_k respectively. We cut off each rectangle in the half height, and the resulting half rectangle C'_j is mapped conformally to the annulus A'_j(0) = \{e^{-m_j}π ≤ |z| < 1\} for any j = 1, · · · , k and l = 1, 2. Then, we can assume that the original surface Y is constructed by \{A'_j(0)\}_{j=1}^{l=1,2} with gluing mappings which are determined naturally. Let r(t) = [Y_t, g_t ∘ f] be the representation of r for any t ≥ 0. The Teichmüller mapping g_t is represented by z = re^{iθ} → r^{2l}e^{iθ} on each A'_j(0). We set \{A'_j(t) = \{e^{-2l_{m_j}π ≤ |z| < 1}\} for any j = 1, · · · , k, l = 1, 2, and t ≥ 0, then Y_t is constructed by them as in the case of t = 0. In this representation, we can set A'_j(∞) as the unit disk D = \{|z| < 1\} for any j = 1, · · · , k and l = 1, 2. We obtain the Riemann surface with nodes Y∞ by \{A'_j(∞)\}_{j=1, · · · , k} with the similar gluing mappings as in the case of t ≥ 0. The deformation g∞ : Y → Y∞ is obtained by z = re^{iθ} → h_j(r)e^{iθ} on A'_j(∞) where h_j : [e^{-m_jπ, 1}] → [0, 1) is an arbitrary monotonously increasing diffeomorphism for any j = 1, · · · , k and l = 1, 2. The homotopy class of g∞ is independent of the choices of h_j for any j = 1, · · · , k.

Proposition 2.1. (cf. [HS07]). The Jenkins-Strebel ray r(t) = [Y_t, g_t ∘ f] on T(X) converges to a point r(∞) = [Y∞, g∞ ∘ f] in T(Y) as t → ∞.

Suppose that r, r′ are similar Jenkins-Strebel rays on T(X) from r(0) = [Y, f], r′(0) = [Y′, f′] determined by Jenkins-Strebel differentials q, q′ with unit norm on Y, Y′ respectively. Let γ_1, · · · , γ_k be as in the definition of “similar” in §2.3. There is a homeomorphism α : X - f^{-1}(Γ_q) → X - f′^{-1}(Γ_{q′}) which is homotopic to the identity such that the mapping f′ ∘ α ∘ f^{-1} maps the core curves of the annuli corresponding to f(γ_j) to the core curves of the annuli corresponding to f′(γ_j) for any j = 1, · · · , k. We set r(∞) = [Y∞, g∞ ∘ f], r′(∞) = [Y′∞, g′∞ ∘ f′] and let \{Y_{∞, λ}\}_{λ=1, · · · , Λ}, \{Y′_{∞, λ}\}_{λ=1, · · · , Λ} be the components of Y∞ - {nodes of Y∞}, Y′∞ - {nodes of Y′∞} respectively, such that (g′∞ ∘ f′) ∘ α ∘ (g∞ ∘ f)^{-1}(Y_{∞, λ}) = Y′_{∞, λ} for any λ = 1, · · · , Λ. We define the Teichmüller distance between r(∞), r′(∞) by
We recall our main theorem.

**Theorem 1.1.** For any two Jenkins-Strebel rays $r$, $r'$,

$$\lim_{t \to \infty} d_{T(X)}(r(t), r'(t)) = \max \left\{ \frac{1}{2} \log \max_{j=1, \ldots, k} \left( \frac{m_j'}{m_j} \right), d_{T(X)}(r(\infty), r'(\infty)) \right\}$$

We use the following lemma.

**Lemma 3.1.** Let $R$, $R'$ be Riemann surfaces with nodes and $f : R \to R'$ be a $K$-quasiconformal Teichmüller mapping. This means that $f$ is a homeomorphism, each restricted mapping of $f$ which maps a component of $R - \{\text{nodes of } R\}$ onto a component of $R' - \{\text{nodes of } R'\}$ is a Teichmüller mapping, and the maximum of maximal dilatations of such mappings is $K$. Then, for any sufficiently small $\varepsilon > 0$, there exists the $(K + O(\varepsilon))$-quasiconformal mapping $g : R \to R'$ such that $g$ is conformal on a neighborhood of the set of nodes of $R$, and is homotopic to $f$.

The lemma is proved in the paper of [FM10], however we give a new proof of the latter part of their proof.

**Proof of Lemma 3.1.** Let $\mu$ be the Beltrami coefficient of $f$. For any $\varepsilon > 0$, we consider a new Beltrami coefficient

$$\mu_\varepsilon = \begin{cases} 0 & (0 < |z| < \varepsilon) \\ \mu & (\text{otherwise}) \end{cases}$$

on $R$, where $z$ is each local coordinate near nodes of $R$ and the domain $\{|z| < \varepsilon\}$ represents a neighborhood of nodes $z = 0$. Then, there exist a Riemann surface with nodes $R_\varepsilon$ and a $K$-quasiconformal mapping $f_\varepsilon : R \to R_\varepsilon$ such that $f_\varepsilon$ is conformal on the neighborhood of nodes of $R$. For sufficiently small $\varepsilon$, $f_\varepsilon$ is close to $f$ and $R_\varepsilon$ is close to $R'$. The mapping $f \circ f_\varepsilon^{-1} : R_\varepsilon \to R'$ is $K$-quasiconformal in a small neighborhood of nodes of $R_\varepsilon$ and is conformal on the outside of the neighborhood. We use local coordinates such that nodes of $R_\varepsilon$ and $R'$ correspond to $0$, and $f \circ f_\varepsilon^{-1}(0) = 0$. Let $p_1, \ldots, p_k$ be all nodes of $R_\varepsilon$. For any $j = 1, \ldots, k$, we take small disks $N_j^1 = N_j^2 = \{|z| < \delta\}$ about $p_j$ in $R_\varepsilon$ where $f \circ f_\varepsilon^{-1}$ is $K$-quasiconformal in $N_j^1 \cup N_j^2$. For any $j = 1, \ldots, k$ and $l = 1, 2$, the image $f \circ f_\varepsilon^{-1}(N_j^l)$
Proof of Lemma 3.2. We notice that $f$ where the supremum ranges over all $x, t$ the case of $-R$ as $\varepsilon \to 0$. However, any point of $\{0 < |z| < \delta\}$ is a holomorphic point of $F_\varepsilon$ for sufficiently small $\varepsilon$, then $F_0$ is holomorphic in $\{0 < |z| < \delta\}$. Since $F_0$ fixes 0 and $\delta$, we can see that $F_0$ is an automorphism on $\{|z| < \delta\}$ and then it is the identity.

Now, we rescale $\{|z| < \delta\}$ to $\mathbb{D}$, and assume that $\{F_\varepsilon\}$ as mappings of $\mathbb{D}$ onto itself. We set a conformal mapping $\phi(z) := (z-i)/(z+i)$ of $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im} z > 0\}$ onto $\mathbb{D}$. We consider mappings $f_\varepsilon := \phi^{-1} \circ F_\varepsilon \circ \phi$ of $\mathbb{H}$ unto itself.

**Lemma 3.2.** We have

$$\lim_{\varepsilon \to 0} \sup_{x, t} \frac{f_\varepsilon(x+t) - f_\varepsilon(x)}{f_\varepsilon(x) - f_\varepsilon(x-t)} = 1,$$

where the supremum ranges over all $x, t \in \mathbb{R}$ such that $t \neq 0$.

**Proof of Lemma 3.2.** We notice that $\frac{f_\varepsilon(x+t) - f_\varepsilon(x)}{f_\varepsilon(x) - f_\varepsilon(x-t)}$ is positive, and its reciprocal is the case of $-t$. Then, it suffice to show that for any $t > 0$. Let

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_2}{z_1 - z_3}, \frac{z_3 - z_4}{z_2 - z_4}$$

be a cross ratio for any $z_1, z_2, z_3, z_4 \in \mathbb{C}$. We write $\phi(x) = e^{i\theta}, \phi(x+t) = e^{i(\theta+\varphi)}$ and $\phi(x-t) = e^{i(\theta-\psi)}$ where $0 \leq \theta < 2\pi$ and $\varphi, \psi > 0$. Since all Möbius transformation preserve cross ratios,

$$\frac{f_\varepsilon(x+t) - f_\varepsilon(x)}{f_\varepsilon(x) - f_\varepsilon(x-t)} = -(f_\varepsilon(x), f_\varepsilon(x+t), f_\varepsilon(x-t), \infty)$$

$$= -(F_\varepsilon \circ \phi(x), F_\varepsilon \circ \phi(x+t), F_\varepsilon \circ \phi(x-t), 1)$$

$$= \frac{|F_\varepsilon(e^{i(\theta+\varphi)}) - F_\varepsilon(e^{i\theta})|}{|F_\varepsilon(e^{i(\theta-\psi)}) - F_\varepsilon(e^{i(\theta+\varphi)})|} = 1.$$ (2)

We set $z = e^{iy}$. By log $F_\varepsilon(e^{iy}) = i(\text{arg } F_\varepsilon(e^{iy}) + 2n\pi)$, we have

$$\frac{d \arg F_\varepsilon(e^{iy})}{dy} = \frac{z \frac{dF_\varepsilon(z)}{dz}}{F_\varepsilon(z)}.$$

Since $F_\varepsilon(z)$ and $\frac{dF_\varepsilon(z)}{dz}$ converge to $z$ and 1 uniformly on $\partial \mathbb{D}$ respectively, then

$$\sup_{0 \leq y < 2\pi} \left| \frac{d \arg F_\varepsilon(e^{iy})}{dy} - 1 \right| = \sup \left| \frac{z \frac{dF_\varepsilon(z)}{dz}}{F_\varepsilon(z)} - 1 \right| \leq \sup \left( \left| \frac{dF_\varepsilon(z)}{dz} - 1 \right| + |z - F_\varepsilon(z)| \right) \to 0$$

as $\varepsilon \to 0$. For any $0 < E < 1$, we take sufficiently small $\varepsilon$ such that

$$\left| \frac{d \arg F_\varepsilon(e^{iy})}{dy} - 1 \right| < E.$$
holds for any $0 \leq y < 2\pi$. Now, we calculate and estimate each term of (2).

\[
|F_\epsilon(e^{i(\theta + \varphi)}) - F_\epsilon(e^{i\theta})| = 2 \sin \frac{\arg F_\epsilon(e^{i(\theta + \varphi)}) - \arg F_\epsilon(e^{i\theta})}{2}
\]

\[
= 2 \sin \int_\theta^{\theta + \varphi} \frac{d\arg F_\epsilon(e^{iy})}{dy} dy
\]

\[
< 2 \sin \frac{(1 + E)\varphi}{2}
\]

\[
= 2 \sin \frac{(1 + E)(\arg \phi(x + t) - \arg \phi(x))}{2}
\]

\[
= 2 \sin (1 + E) \int_x^{x + t} \frac{d\phi(y)}{dy} dy
\]

\[
= 2 \sin \frac{(1 + E) \int_x^{x + t} \frac{d\phi(y)}{dy} + \frac{\alpha}{E}}{2}
\]

\[
= 2 \sin \frac{(1 + E) \int_x^{x + t} \frac{d\phi(y)}{dy}}{2}
\]

\[
= 2 \sin \left\{ (1 + E)(\arctan(x + t) - \arctan x) \right\},
\]

and similarly,

\[
|F_\epsilon(e^{i(\theta + \varphi)}) - F_\epsilon(e^{i\theta})| > 2 \sin \left\{ (1 - E)(\arctan(x + t) - \arctan x) \right\}.
\]

For any $0 < \alpha \leq \pi/2$,

\[
\left| \frac{\sin\left\{ (1 \pm E)\alpha \right\}}{\sin\alpha} - 1 \right| = \left| \frac{\sin\alpha \cos(E\alpha) \pm \cos\alpha \sin(E\alpha)}{\sin\alpha} - 1 \right|
\]

\[
= \left| \cos(E\alpha) \pm \cos\alpha \frac{\sin(E\alpha)}{\sin\alpha} - 1 \right|
\]

\[
\leq \left| \cos(E\alpha) - 1 \right| + \left| \sin(E\alpha) \right| \frac{\alpha}{\sin\alpha} E
\]

\[
\leq E\alpha + \frac{\pi}{2} E \leq \pi E \to 0
\]

as $E \to 0$. This means that we can write $\sin\left\{ (1 \pm E)\alpha \right\} = (1 + O(E)) \sin \alpha$. Therefore,

\[
2 \sin \left\{ (1 \pm E)(\arctan(x + t) - \arctan x) \right\} = 2(1 + O(E)) \sin(\arctan(x + t) - \arctan x).
\]

We conclude that

\[
|F_\epsilon(e^{i(\theta + \varphi)}) - F_\epsilon(e^{i\theta})| < 2(1 + O(E)) \sin(\arctan(x + t) - \arctan x)
\]

and

\[
|F_\epsilon(e^{i(\theta + \varphi)}) - F_\epsilon(e^{i\theta})| > 2(1 + O(E)) \sin(\arctan(x + t) - \arctan x).
\]

Also we have similar estimates for
Finally, we can see that

\[
\frac{f_\varepsilon(x) - f_\varepsilon(x)}{f_\varepsilon(x) - f_\varepsilon(x)} < (1 + O(E)) \frac{\sin(\arctan(x + t) - \arctan x) \sin(\frac{\pi}{2} - \arctan(x - t))}{\sin(\arctan(x - t) - \arctan(x + t))} \sin(\arctan x - \arctan(x - t)) \sin(\frac{\pi}{2} - \arctan(x - t))
\]

\[
= (1 + O(E)) \frac{\sqrt{1 + (x + t)^2} \sqrt{1 + x^2}}{\sqrt{1 + (x - t)^2} \sqrt{1 + x^2} \sqrt{1 + (x + t)^2}} = 1 + O(E).
\]

The lower estimate is similar. \qed

Lemma 3.2 implies that the mapping \( f_{j,\varepsilon} \circ f \circ f_\varepsilon^{-1} \) is \((1 + O(\varepsilon))-\)quasisymmetric on the circle \( \partial N_j \) for sufficiently small \( \varepsilon \). We apply Lemma 5.1 in [Gup11]. Then, there exists a mapping \( \eta_{j,\varepsilon} : N_j \rightarrow \{|z| < \delta\} \) which is \((1 + O(\varepsilon))-\)quasiconformal, \( \eta_{j,\varepsilon}|_{\partial N_j} = f_{j,\varepsilon} \circ f \circ f_\varepsilon^{-1}|_{\partial N_j} \), and \( \eta_{j,\varepsilon} \) is the identity in a sufficiently small neighborhood of 0. The mapping \( (f_{j,\varepsilon}^{-1} \circ \eta_{j,\varepsilon}^{-1}) : N_j \rightarrow f_\varepsilon^{-1}(N_j) \) is \((1 + O(\varepsilon))-\)quasiconformal and satisfies \( f_{j,\varepsilon} \circ \eta_{j,\varepsilon} = f \circ f_\varepsilon^{-1}|_{\partial N_j} \) and \( (f_{j,\varepsilon}^{-1} \circ \eta_{j,\varepsilon})|_{\partial N_j} = 0 \). We consider the mapping of \( R_\varepsilon \) onto \( R' \) which is \((f_{j,\varepsilon}^{-1} \circ \eta_{j,\varepsilon})^{-1} \) in \( N_j \) for any \( j = 1, \ldots, k \) and \( l = 1, 2 \), and is \( f \circ f_\varepsilon^{-1} \) on \( R_\varepsilon - \bigcup_{j=1}^{k} \partial N_j \). This mapping is \((1 + O(\varepsilon))-\)quasiconformal and is clearly homotopic to \( f \circ f_\varepsilon^{-1} \).

Therefore, we conclude that there exists a \((1 + O(\varepsilon))-\)quasiconformal Teichmüller mapping \( h_\varepsilon : R_\varepsilon \rightarrow R' \) which is homotopic to \( f \circ f_\varepsilon^{-1} \).

We deform \( h_\varepsilon \) to a \((1 + O(\varepsilon))-\)quasiconformal mapping which is conformal on a neighborhood of nodes of \( R_\varepsilon \). We use \( \varepsilon' < 1 \) instead of \( O(\varepsilon) \). Let \( q \) be the holomorphic quadratic differential on \( R_\varepsilon - \{\text{nodes of } R_\varepsilon\} \) which is corresponding to the Teichmüller mapping \( h_\varepsilon \). Let \( z = x + iy \) be any \( q \)-coordinate, then \( h_\varepsilon \) is represented by \( z \mapsto x + i(1 + \varepsilon')y \). We consider the set \( D_\varepsilon = \{ -\varepsilon' \leq x \leq \varepsilon', 0 \leq y \leq \varepsilon' \} - \{0\} \) where 0 corresponds to a node of \( R_\varepsilon \). Let \( H_\varepsilon \) be a mapping on the half set \( \{0 \leq x, y \leq \varepsilon'\} - \{0\} \) which is defined the following:

\[
H_\varepsilon(z) = \begin{cases} 
  z & (0 \leq x, y \leq \varepsilon', z \neq 0) \\
  x + \frac{iy - \varepsilon'}{1 - \varepsilon'} & (0 \leq x, \varepsilon^2 \leq y \leq \varepsilon') \\
  x + \frac{y + i(1 + \varepsilon' - \varepsilon^2) - \varepsilon^2 \cdot y}{1 - e^2} & (\varepsilon^2 \leq x, 0 \leq y \leq \varepsilon^2) \\
  x + \frac{(1 - \varepsilon^2 - \varepsilon' + (x + 1 - \varepsilon' - \varepsilon^2))y + \varepsilon^2 (x - \varepsilon')}{(1 - e^2)} & (\varepsilon^2 \leq x, y \leq \varepsilon') 
\end{cases}
\]

An easy calculation shows that \( H_\varepsilon \) is \((1 + O(\varepsilon))-\)quasiconformal. We extend \( H_\varepsilon \) symmetrically to \( D_\varepsilon \). We assume that \( z = 0 \) is a critical point of \( q \) of order \( n \geq -1 \). We consider the \( n + 2 \) copies of \( D_\varepsilon \) around 0 whose horizontal segments of the boundary of \( D_\varepsilon \) lie on the horizontal trajectories of \( q \) which tend to \( 0 \), and vertical segments of the boundary of \( D_\varepsilon \) are joined adjacent to other \( D_\varepsilon \) (Figure 2).

Finally, we extend the mapping \( H_\varepsilon \) as the original mapping \( h_\varepsilon \) on the outside of all \( D_\varepsilon \), i.e., it is of the form \( z \mapsto x + i(1 + \varepsilon')y \). Then the mapping \( H_\varepsilon \) is defined on the whole surface \( R_\varepsilon \). The quasiconformal mapping \( H_\varepsilon \circ h_\varepsilon^{-1} : R' \rightarrow R' \)
tends to the identity as $\varepsilon \to 0$. Hence, for sufficiently small $\varepsilon$, the quasiconformal mapping $H_\varepsilon \circ h_\varepsilon^{-1}$ is homotopic to the identity because each mapping class group on components of $R' - \{\text{nodes of } R'\}$ is discrete. We conclude that the composition $H_\varepsilon \circ f_\varepsilon$ is homotopic to $f$, and it is our desired mapping $g$. \hfill \Box

**Proof of Theorem 1.1.** If $r$, $r'$ are not similar, the result is already known. We can see it in [Iva01, LM10] and also Ama14.

Let $r$, $r'$ be similar Jenkins-Strebel rays on $T(X)$ from $r(0) = [Y,f]$, $r(0') = [Y',f']$ determined by Jenkins-Strebel differentials $q$, $q'$ with unit norm on $Y$, $Y'$ respectively. By definition, there exist mutually disjoint simple closed curves $\gamma_1, \ldots, \gamma_k$ on $X$ such that the set of homotopy classes of core curves of the annuli corresponding to $q$, $q'$ are represented by $f(\gamma_1), \ldots, f(\gamma_k)$ on $Y$ and $f'(\gamma_1), \ldots, f'(\gamma_k)$ on $Y'$ respectively. Moreover, there is a homeomorphism $\alpha : X - f^{-1}(\Gamma_q) \to X - f'^{-1}(\Gamma_{q'})$ which is homotopic to the identity such that the mapping $f' \circ \alpha \circ f^{-1}$ maps the core curves of the annuli corresponding to $f(\gamma_j)$ to the core curves of the annuli corresponding to $f'(\gamma_j)$ for any $j = 1, \ldots, k$. We denote by $m_j, m'_j$ the moduli of the annuli on $Y$, $Y'$ with core curves homotopic to $f(\gamma_j)$, $f'(\gamma_j)$ respectively. For any $t \geq 0$, we set $r(t) = [Y_t, g_t \circ f]$, $r'(t) = [Y'_t, g'_t \circ f']$ where $g_t : Y \to Y_t$, $g'_t : Y' \to Y'_t$ are Teichmüller mappings. Let $r(\infty) = [Y_\infty, g_\infty \circ f]$, $r'(\infty) = [Y'_\infty, g'_\infty \circ f']$ be the end points of $r$, $r'$ in the augmented Teichmüller space $\hat{T}(X)$ respectively. Let $\{Y_{\infty,\lambda}\}_{\lambda = 1, \ldots, \Lambda}$, $\{Y'_{\infty,\lambda}\}_{\lambda = 1, \ldots, \Lambda}$ be the components of $Y_\infty - \{\text{nodes of } Y_\infty\}$, $Y'_\infty - \{\text{nodes of } Y'_\infty\}$ respectively, such that $(g_\infty \circ f')^{-1}(Y_{\infty,\lambda}) = Y'_{\infty,\lambda}$ for any $\lambda = 1, \ldots, \Lambda$.

First, we consider the upper estimate. Let $h_\lambda : Y_{\infty,\lambda} \to Y'_{\infty,\lambda}$ be the Teichmüller mapping which is homotopic to $(g_\infty \circ f')^{-1} \circ \alpha \circ (g_\infty \circ f)^{-1}$, and we set the mapping $h : Y_\infty \to Y'_\infty$ constructed by $\{h_\lambda\}_{\lambda = 1, \ldots, \Lambda}$. We set $K = \exp(2d_f(X)(r(\infty), r'(\infty))) = \max_{\lambda = 1, \ldots, \Lambda} K(h_\lambda)$. The Riemann surfaces with nodes $Y_{\infty,\lambda}$, $Y'_{\infty,\lambda}$ are represented by the unions of closed unit disks $\{A_j(\infty)\}_{j=1,\ldots,k_l}^l = \{A'_j(\infty)\}_{j=1,\ldots,k_l}^l$ respectively. Let $h^j_l := h|_{A_j^l(\infty)}$ be the restriction to $A_j^l(\infty)$ of $h$ for any $j = 1, \ldots, k$, and $l = 1, 2$, then there is $\lambda$ such that $h^j_l$ is equal to $h_{\lambda}|_{A_j^l(\infty)}$. We apply Lemma 3.1 to the mapping $h : Y_\infty \to Y'_\infty$. Hence, for any $j = 1, \ldots, k$ and $l = 1, 2$, we assume that $h^j_l$ is $(K + O(\varepsilon))$-quasiconformal such that it is conformal in a neighborhood of 0 in $A_j^l(\infty)$ and is homotopic to $(g_\infty \circ f')^{-1} \circ \alpha \circ (g_\infty \circ f)^{-1}$. Now, we can use the idea of

...
the proof of Theorem [11] in [Ama14]. In this neighborhood, \( h_j^i \) is represented by a power series, i.e., we can write \( h_j^i(z) = c_j^i z + c_j^{i,2} z^2 + \cdots = c_j^i + \psi_j^i(z) \) where \( c_j^i \neq 0 \), \(-\pi < \arg c_j^i \leq \pi \) and \(-\pi \leq \arg c_j^{i,2} < \pi \). For any \( j = 1, \ldots, k \), we set \( M_j = \frac{m_j^{X_j}}{m_j^j} \), and for any \( t \geq 0 \), \( \delta_j(t) = e^{-ct^j m_j^j} \), \( \delta_j'(t) = e^{-ct^j m_j^j} \), then \( \delta_j'(t) = \delta_j(t)^{M_j} \). We only consider the case of \( M_j > 1 \), so we fix such \( j \). For sufficiently small \( \varepsilon \), again, we use \( \varepsilon' < 1 \) instead of \( O(\varepsilon) \). We take \( X_j \) as

\[
X_j < \frac{\log \frac{\varepsilon'}{\log M_j}}{\log M_j} < 0.
\]

We take sufficiently large \( t \) such that an inequality \( \delta_j(t)^{M_j} < |c_j^i| \delta_j(t)^{M_j^{X_j}} \) holds, and set \( \Delta_j(t) = \delta_j(t)^{M_j^{X_j}} \). Also, we assume that a domain such that \( h_j^i \) can be represented by the power series contains \( \{ |z| \leq 2\Delta_j(t) \} \). We construct \( F_{j,t}^l : A_j^i(t) \to h(A_j^i(t)) - \{ |z| < \delta_j(t) \} \) by the following:

\[
F_{j,t}^l(z) = \begin{cases} 
  P_{j,t}^l(z) & (\delta_j(t) \leq |z| \leq \Delta_j(t)) \\
  Q_{j,t}^l(z) & (\Delta_j(t) \leq |z| \leq 2\Delta_j(t)) \\
  h_j^i(z) & (2\Delta_j(t) \leq |z| < 1) 
\end{cases}
\]

(i) In \( \delta_j(t) \leq |z| \leq \Delta_j(t) \), we set

\[
P_{j,t}^l(z) = \Delta_j(t)^{\frac{1-M_j}{1-M_j^{X_j}}} \cdot c_j^i \cdot e^{\frac{\log |z|}{\log M_j^{X_j}}} \cdot \frac{\log c_j^i}{2(M_j^{X_j}-1) \log \delta_j(t)} + \frac{\alpha_j}{2} + 1,
\]

which satisfies \( P_{j,t}^l(z) = \delta_j(t)^{M_j^{X_j}} \cdot z \) on \( |z| = \delta_j(t) \), \( P_{j,t}^l(z) = c_j^i z \) on \( |z| = \Delta_j(t) \).

The mapping \( P_{j,t}^l \) is a quasiconformal mapping because it is conjugate to a one-to-one affine mapping by log \( z \). The maximal dilatation of \( P_{j,t}^l \) is

\[
K(P_{j,t}^l) = \frac{\log c_j^i}{2(M_j^{X_j}-1) \log \delta_j(t)} + \frac{\alpha_j}{2} + 1 + \frac{\log c_j^i}{2(M_j^{X_j}-1) \log \delta_j(t)} + \frac{\alpha_j}{2},
\]

where \( \alpha_j = -\frac{1-M_j}{1-M_j^{X_j}} \). We see that

\[
K(P_{j,t}^l) \to \frac{M_j - M_j^{X_j}}{1 - M_j^{X_j}} < M_j + \varepsilon'
\]

as \( t \to \infty \).

(ii) In \( \Delta_j(t) \leq |z| \leq 2\Delta_j(t) \), we set

\[
Q_{j,t}^l(z) = c_j^i z + \phi_{\Delta_j(t)}(|z|) \psi_j^i(z),
\]

where \( \phi_{\Delta_j(t)} : [\Delta_j(t), 2\Delta_j(t)] \to [0, 1] \) is defined by

\[
\phi_{\Delta_j(t)}(|z|) = \frac{|z|}{\Delta_j(t)} - 1.
\]

Then \( Q_{j,t}^l(z) = c_j^i z \) on \( |z| = \Delta_j(t) \), \( Q_{j,t}^l(z) = h_j^i(z) \) on \( |z| = 2\Delta_j(t) \). We consider the partial derivatives of \( Q_{j,t}^l \).
\[
\partial_z Q_{j,t}^l = \frac{1}{2\Delta_j(t)} z^{-\frac{1}{2}} \bar{z}^{\frac{1}{2}} \psi_j^l(z), \\
\partial_z Q_{j,t}^l = c_j^l + \frac{1}{2\Delta_j(t)} z^{-\frac{1}{2}} \bar{z}^{\frac{1}{2}} \psi_j^l(z) + \phi_{\Delta(t)}(|z|) \frac{d\psi_j^l(z)}{dz}.
\]

These are continuous in \( \Delta_j(t) \leq |z| \leq 2\Delta_j(t) \). There is \( C > 0 \) such that \( |\psi_j^l(z)| \leq C \Delta_j(t)^2 \) for sufficiently large \( t \). We see that

\[
\left| \frac{1}{2\Delta_j(t)} z^{-\frac{1}{2}} \bar{z}^{\frac{1}{2}} \psi_j^l(z) \right| \leq \left| \frac{1}{2\Delta_j(t)} z^{-\frac{1}{2}} \bar{z}^{\frac{1}{2}} \psi_j^l(z) \right| = \frac{|\psi_j^l(z)|}{2\Delta_j(t)} \leq \frac{C \Delta_j(t)}{2} \to 0
\]

and then \( |\partial_z Q_{j,t}^l| \to 0, |\partial_z Q_{j,t}^l| \to |c_j^l| \neq 0 \) as \( t \to \infty \). Hence, for sufficiently large \( t \), \( \text{Jac} Q_{j,t}^l = |\partial_z Q_{j,t}^l|^2 - |\partial_z Q_{j,t}^l|^2 > 0 \), and we conclude that \( Q_{j,t}^l \) is a local \( C^1 \)-diffeomorphism. We denote by \( D \) the closed set whose fundamental group is \( \pi_1(D) = \mathbb{Z} \) and its boundary components are \( Q_{j,t}^l([[z] = \Delta_j(t)]) = \{|w| = |c_j^l|\Delta_j(t)\} \) and \( Q_{j,t}^l([[z] = 2\Delta_j(t)]) = h_j^l([[z] = 2\Delta_j(t)]) \). Since \( Q_{j,t}^l \) is a local \( C^1 \)-diffeomorphism, we have \( Q_{j,t}^l([[\Delta_j(t) \leq |z| \leq 2\Delta_j(t)]) = D \). Furthermore, by the compactness of \( \{\Delta_j(t) \leq |z| \leq 2\Delta_j(t)\} \), \( Q_{j,t}^l \) is proper. Then we can regard the mapping \( Q_{j,t}^l : \{\Delta_j(t) \leq |z| \leq 2\Delta_j(t)\} \to D \) as a covering. Let \( Q_{j,t}^l : \pi_1([\Delta_j(t) \leq |z| \leq 2\Delta_j(t)]) \to \pi_1(D) \) be the group homomorphism induced by \( Q_{j,t}^l \). We see that \( Q_{j,t}^l \pi_1([\Delta_j(t) \leq |z| \leq 2\Delta_j(t)]) = \mathbb{Z} \circ \pi_1(D) \) because \( Q_j^l(z) = c_j^l z \) on \( |z| = \Delta_j(t) \). Then, the covering \( Q_{j,t}^l \) is regular, and its covering transformation group is \( \mathbb{Z}/\mathbb{Z} = 1 \). Therefore, we conclude that \( Q_{j,t}^l \) is a \( C^1 \)-diffeomorphism. By the partial derivatives of \( Q_{j,t}^l \), for sufficiently large \( t \), it is a quasiconformal mapping and satisfies \( K(Q_{j,t}^l) \to 1 \) as \( t \to \infty \).

(iii) In \( 2\Delta_j(t) \leq |z| < 1 \), \( F_{j,t}(z) = h_j^l(z) \) and \( K(h_j^l) \leq K \).

By the above discussions, for sufficiently large \( t \), we obtain a quasiconformal mapping \( F_{j,t}^l \) such that

\[
K(F_{j,t}^l) = \max\{K(P_{j,t}^l), K(Q_{j,t}^l), K(h_j^l)\} \to \max\left\{ \frac{M_j - M_j^X_j}{1 - M_j^X_j}, K(h_j^l) \right\} < \max\{M_j, K\} + \varepsilon'
\]

as \( t \to \infty \).

In the cases of \( M_j < 1, M_j = 1 \), we also have

\[
\lim_{t \to \infty} K(F_{j,t}^l) < \max\left\{ \frac{1}{M_j}, K \right\} + \varepsilon'
\]

and

\[
\lim_{t \to \infty} K(F_{j,t}^l) = K
\]

by similar arguments.

Thus, for sufficiently large \( t \), we can construct the quasiconformal mapping \( F_t : Y_t \to Y_t' \) by gluing \( \{F_{j,t}^l\}_{j=1, \ldots, k} \). We obtain the inequality

\[
\lim_{t \to \infty} K(F_t) < \max\left\{ \max_{j=1, \ldots, k} \left\{ \frac{m_j^l}{m_j^l}, m_j^l \right\}, K \right\} + \varepsilon'.
\]
Next, we confirm that $F_t$ is homotopic to $(g_t' \circ f') \circ (g_t \circ f)^{-1}$. In any case, each $h^t_j$ is homotopic to $(g_t' \circ f') \circ (g_t \circ f)^{-1}$ in $\{2\Delta_j(t) < |z| < 1\}$. Each $Q^t_{j,t}$ satisfies $K(Q^t_{j,t}) \to 1$ as $t \to \infty$ and the domain $\{\Delta_j(t) < |z| < 2\Delta_j(t)\}$ has the constant modulus for any $t$. Finally, each $P^t_{j,t}$ produces a twist of angle $\arg c^t_j$ in $\{\delta_j(t) < |z| < \Delta_j(t)\}$ and satisfies $|\arg c^t_j + \arg c^t_j| < 2\pi$. Therefore, for sufficiently large $t$, the mapping $F_t$ is homotopic to $(g_t' \circ f') \circ (g_t \circ f)^{-1}$. Since $\alpha$ is homotopic to the identity on $X$, we are done. We conclude that

$$\limsup_{t \to \infty} d_{T(X)}(r(t), r'(t)) \leq \max \left\{ \frac{1}{2} \log \max_{j=1, \cdots, k} \left\{ \frac{m'_j}{m_j}, \frac{m_j}{m'_j} \right\}, d_{\tilde{T}(X)}(r(\infty), r'(\infty)) \right\}.$$ 

For the lower estimate, we can use the following inequality.

**Proposition 3.3.** ([Ama14]) We have

$$\liminf_{t \to \infty} d_{T(X)}(r(t), r'(t)) \geq \frac{1}{2} \log \max_{j=1, \cdots, k} \left\{ \frac{m'_j}{m_j}, \frac{m_j}{m'_j} \right\}.$$ 

Furthermore, we use the following fact.

**Proposition 3.4.** ([Mas75]) We have

$$\liminf_{t \to \infty} d_{T(X)}(r(t), r'(t)) \geq d_{\tilde{T}(X)}(r(\infty), r'(\infty)).$$ 

Combining the above two inequalities, we obtain the inequality

$$\liminf_{t \to \infty} d_{T(X)}(r(t), r'(t)) \geq \max \left\{ \frac{1}{2} \log \max_{j=1, \cdots, k} \left\{ \frac{m'_j}{m_j}, \frac{m_j}{m'_j} \right\}, d_{\tilde{T}(X)}(r(\infty), r'(\infty)) \right\}.$$ 

**Corollary 1.2.** If $r, r'$ are similar, the minimum value of the equation (11) when we shift the starting points of $r, r'$ is given by

$$\max \left\{ \frac{1}{2} \delta, d_{\tilde{T}(X)}(r(\infty), r'(\infty)) \right\},$$

where $\delta = \frac{1}{2} \log \max_{j=1, \cdots, k} \frac{m'_j}{m_j} + \frac{1}{2} \log \max_{j=1, \cdots, k} \frac{m_j}{m'_j}.$

**Proof of Corollary 1.2.** We see that

$$\frac{1}{2} \log \max_{j=1, \cdots, k} \left\{ \frac{m'_j}{m_j}, \frac{m_j}{m'_j} \right\} \geq \frac{1}{2} \delta.$$ 

The values $\frac{1}{2} \delta$ and $d_{\tilde{T}(X)}(r(\infty), r'(\infty))$ are invariant when we shift the starting points of the rays $r, r'$. Hence, by Theorem 1.1

$$\lim_{t \to \infty} d_{T(X)}(r(t), r'(t + \alpha)) \geq \max \left\{ \frac{1}{2} \delta, d_{\tilde{T}(X)}(r(\infty), r'(\infty)) \right\}$$

for any $\alpha \in \mathbb{R}$. The equality holds if
Indeed, we calculate that

\[
\max_{j=1,\ldots,k} e^{2\alpha m_j'} = \max_{j=1,\ldots,k} \left\{ \sqrt{\max_{j=1,\ldots,k} m_j' \cdot m_j} \right\} = \sqrt{\max_{j=1,\ldots,k} m_j'} \cdot \sqrt{\max_{j=1,\ldots,k} m_j'} = \max_{j=1,\ldots,k} e^{2\alpha m_j'}.
\]

Therefore, we conclude that

\[
\lim_{t \to \infty} d_{\bar{T}(X)}(r(t), r'(t + \alpha)) = \max \left\{ \frac{1}{2} \log \max_{j=1,\ldots,k} \left\{ \frac{e^{2\alpha m_j'}}{m_j} \right\}, d_{\bar{T}(X)}(r(\infty), r'(\infty)) \right\} = \max \left\{ \frac{1}{2} \delta, d_{\bar{T}(X)}(r(\infty), r'(\infty)) \right\}.
\]

\[\Box\]

**Acknowledgements**

I would like to express the deepest appreciation to Professor Hiroshige Shiga for his insightful comments and suggestions. This work is supported by Global COE Program “Computationism as a Foundation for the Sciences”.

**References**

[Abi80] William Abikoff. *The real analytic theory of Teichmüller space*, volume 820 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Ama14] Masanori Amano. On behavior of pairs of Teichmüller geodesic rays. *Conform. Geom. Dyn.*, 18:8–30, 2014.

[FM10] Benson Farb and Howard Masur. Teichmüller geometry of moduli space, I: distance minimizing rays and the Deligne-Mumford compactification. *J. Differential Geom.*, 85(2):187–227, 2010.

[Gup11] Subhojoy Gupta. Asymptoticity of grafting and Teichmüller rays I. *arXiv:1109.5365v1*, 2011.

[HS07] Frank Herrlich and Gabriela Schmithüsen. On the boundary of Teichmüller disks in Teichmüller and in Schottky space. In *Handbook of Teichmüller theory. Vol. I*, volume 11 of *IRMA Lect. Math. Theor. Phys.*, pages 293–349. Eur. Math. Soc., Zürich, 2007.

[IT92] Yoichi Imayoshi and Masahiko Taniguchi. *An introduction to Teichmüller spaces*. Springer-Verlag, Tokyo, 1992. Translated and revised from the Japanese by the authors.
[Iva01] Nikolai V. Ivanov. Isometries of Teichmüller spaces from the point of view of Mostow rigidity. In Topology, ergodic theory, real algebraic geometry, volume 202 of Amer. Math. Soc. Transl. Ser. 2, pages 131–149. Amer. Math. Soc., Providence, RI, 2001.

[LM10] Anna Lenzhen and Howard Masur. Criteria for the divergence of pairs of Teichmüller geodesics. Geom. Dedicata, 144:191–210, 2010.

[Maa75] Howard Masur. On a class of geodesics in Teichmüller space. Ann. of Math. (2), 102(2):205–221, 1975.

[Str84] Kurt Strebel. Quadratic differentials, volume 5 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1984.

[Wal12] Cormac Walsh. The asymptotic geometry of the Teichmüller metric. arXiv:1210.5565v1, 2012.

Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguroku, Tokyo 152-8551, JAPAN

E-mail address: amano.m.ab@m.titech.ac.jp