Modelling of dynamic processes in an elastic base with a heterogeneous coating

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Abstract. In this paper, we consider a mathematical model describing the dynamic properties of heterogeneous materials having a non-uniform layered structure in the form of a non-uniform elastic coating with a microstructure lying on an elastic base. The dispersion function for a two-layer composite medium is given, showing that the heterogeneous coating of an elastic half-space significantly changes the dynamics of the medium.

1. Introduction
Designing composite materials that have unique antifriction, acoustic, and optical properties opens up broad prospects for their use in tribology, medicine, industry, and space equipment and is an urgent task [1, 2]. One of the ways to create composites is to obtain heterogeneous materials by combining viscoelastic media and viscous liquids using nanoscale additives [3, 4]. It is very interesting to receive information about the properties of the new material “a priori”, before the implementation of the laborious experimental process [5]. Studying the physical and mechanical properties of the described composites is impossible without creating adequate mathematical models describing the wave processes in them. The creation of mathematical models that allow, based on the analysis of dynamic processes, to predict the properties of new artificially created materials, is one of the priorities of modern science. Solving such problems is of fundamental and applied engineering importance. Recently, when designing and calculating the stress-strain state of objects, software systems based on the finite element method, for example, ANSYS, COSMOS, ABAQUS NASTRAN, etc., are often used. It should be noted that the calculation of the problem in the formulation described below with these complexes is impracticable. This article is devoted to mathematical modeling of the dynamic properties of heterogeneous materials having a non-uniform layered structure in the form of a porous-elastic viscous-elastic base saturated with fluid.

2. Setting and solving the problem
We consider the model problem of the propagation of vibrations in a viscoelastic half-space \(-\infty < x < \infty, y \leq 0\) with heterogeneous coating layer \(-\infty < x < \infty, 0 \leq y \leq h\). The upper layer is rigidly coupled with the base. The load oscillating is applied on the surface of the layer. The load varies according to the harmonic law: \(q(x)e^{-\omega t}, |x| \leq a, q(x) = (q_1, q_2)\). Next, the time factor is separable from displacements and stresses. The presentation will be conducted for the amplitude values of the corresponding functions. The internal microstructure of the base consists of an elastic skeleton and a
filler. The skeleton has the properties of a viscoelastic body. The filler has the properties of a viscous amorphous liquid. Heterogeneous coating is described by the defining ratios of the two-phase Biot medium [6]. The Bio equation model for the heterogeneous medium in the case of stationary oscillations have the form:

\[
\begin{align*}
(q_{11}-2)\nabla \cdot \nabla u + 2\nabla \nabla \cdot v + q_{12} \nabla \nabla \cdot u + q_{22} \nabla \nabla \cdot v + g^2 (\gamma_{11} u + \gamma_{12} v) &= 0; \\
q_{12} \nabla \nabla \cdot u + q_{22} \nabla \nabla \cdot v + g^2 (\gamma_{12} u + \gamma_{22} v) &= 0; \\
\rho &= \rho_{11} + 2\rho_{12} + \rho_{22}; \\
\gamma_{11} &= \rho_{11} + ib/\omega; \\
\gamma_{12} &= \rho_{12} - ib/\omega; \\
\gamma_{22} &= \rho_{22} + 2ib/\omega; \\
\sigma'_y &= (q_{11}-2)e\delta_y + 2e_y + q_{12}e\delta_y, \\
\sigma' &= q_{12}e + q_{22}e, \quad i, j = 1, 2.
\end{align*}
\]

\(q_{ij}, i, j = 1, 2\) – dimensionless mechanical characteristics of a two-phase medium [7], \(e_y, e\) – strain tensors corresponding to the vector of displacement of the solid phase and the liquid phase \(u\{u_1(x, y), u_2(x, y)\}, v\{v_1(x, y), v_2(x, y)\}\):

\[
e_y = (u_{ij} + u_{ji})/2, \quad e = \nabla \cdot u, \quad \sigma' = \nabla \cdot v, \quad \Gamma_y = \sigma'_y + \delta_y \sigma', \quad \sigma' = \text{stress tensor acting on a viscoelastic skeleton}, \quad \Gamma_y = \text{pressure acting on the fluid in the pores}.
\]

Relations (1) are written in a dimensionless form, displacements are related to the characteristic linear size \(l\), stresses to the shear modulus \(N\) of the elastic skeleton of the coating.

The behavior of the amplitude values of displacements \(u^0\{u_1^0(x, y), u_2^0(x, y)\}\) in a viscoelastic half-space is determined by the Lamé equations with complex elastic moduli [6]. We assume that the boundary of the heterogeneous layer is impermeable to the fluid. The boundary conditions of the stated boundary value problem are written as:

\[
y = h; u_1 = v_2; \quad \sigma_{12} = \sigma_{22} = 0 \quad |x| > a; \\
y = 0; u_1 = v_2; \quad \sigma_{12} = \sigma_{22} = 0; \quad \Gamma_{12} = \sigma_{22} = 0.
\]

The stated boundary problem described by 6 partial differential equations and boundary conditions (1) – (2). In this case, it is quite time-consuming to find the Green matrix, which describes the relationship of the applied loads and displacements caused by them.

Applicable to formulas (1) – (2) Fourier transform:

\[
U(x, y) = \int_{-\infty}^{\infty} u(x, y)e^{i\alpha x}dx, \quad V(x, y) = \int_{-\infty}^{\infty} v(x, y)e^{i\alpha x}dx
\]

For the elastic half-plane, the connection of displacements and stresses \(Q^0\) on its boundary in Fourier images is written through the Green matrix:

\[
U^0 = B^0Q^0 / \Delta_0; \quad B^0 = (b^0_i), i, j = 1, 2
\]

Solutions of system (1) are sought as the sum of three potentials corresponding to the potential and vortex parts of the displacements, three types of waves propagating in a heterogeneous layer [9]. To reduce the laboriousness of the process, we will look for a general expression for displacements in the form of a sum of two parts, corresponding to symmetric and skew-symmetric tasks for an impenetrable heterogeneous layer, and the Green matrix, respectively, have the form:

\[
U^c = B^cQ^c / \Delta_c; U^k = B^kQ^k / \Delta_k; \quad B^c = (b^c_i), i, j = 1, 2, U = U^c + U^k,
\]

The expressions of the resulting matrices are cumbersome. We give an expression for the dispersion function of the heterogeneous layer, the roots of this function determine the wave propagation velocity:
\[ \Delta \epsilon = \alpha \Delta d \epsilon + e \left( s_0 \Delta d - \alpha^2 \gamma_{12} \right); \]
\[ d \epsilon = (m_{01} - m_{02}) s_0 s_1 e \epsilon e \epsilon; \]
\[ d \epsilon = m_{01} s_0 z_1 - m_{02} s_0 z_2; \]
\[ q_{01} = q_{11} + q_{12} + (q_{12} + q_{22}) m_1; \]
\[ q_{01} = 1 + \gamma_{12} / \gamma_{22}; \]
\[ s_0 = \alpha^2 - 0.5 \beta^2; \]
\[ s_0 = 2 \alpha^2 - q_{01} \beta^2; \]
\[ e = \left( 1 - e^{-2 \pi \alpha} \right) / 2; \]
\[ e = \left( 1 + e^{-2 \pi \alpha} \right) / 2; \]
\[ m_{01} = 1 - m_1. \]

It should be noted that in (6) all hyperbolic functions for improving the convergence of the further computational process are transformed by moving the growing exponential factors beyond its framework.

Expressions for \( B^\alpha \) and \( \Delta \alpha \) are obtained from formulas (6) by replacing \( e \leftrightarrow e' \), \( e \leftrightarrow e' \).

Using relations (3) – (6) after satisfying the boundary conditions, we arrive at an expression describing movements in a two-layer medium with a heterogeneous coating:
\[ U(\alpha, y) = G(\alpha)Q\alpha, \]
\[ G(\alpha) = \frac{1}{\Delta \alpha \Delta \alpha} G^\theta (\alpha); \]
\[ \Delta \alpha = \text{det} \left( B^\theta (\alpha) \right). \]

The determinant of the matrix \( B^\theta (\alpha) \) generates a dispersion function for a two-layer composite medium. It should be noted that the heterogeneous coating of the elastic half-space significantly changes the dynamics of the medium [10]. The representation of the Green matrix (7) shows that in a layered medium there are no waves that exist in a half-space without coating. The wave process in the composite medium forms a heterogeneous layer, and the presence of an elastic base leads to its quantitative modification.

We obtain an integral representation of the displacements taking into account the relations (7) – (8). On the surface of a heterogeneous layer, we have:
\[ u(x, h) = \frac{1}{2 \pi i} \int_{\gamma_0} \frac{1}{\Delta \alpha \Delta \alpha} G^\theta (\alpha) Q(\alpha)e^{-ix} d\alpha \]

Matrix elements \( G^\theta (\alpha) \) are analytic functions in the complex plane. The integration contour is located in the complex plane in accordance with the principles of wave radiation.

In fig. 1 shows graphs illustrating the behavior of the transforms of the normal displacements on the surface of a two-layer medium. Mechanical characteristics correspond to phenylone [3] containing an oil fraction in the pores, for different values \( \omega = 150 \) (left side), \( \omega = 50 \) (right side) Hz, the mechanical characteristics of the base matched to steel. Porosity equal to 0.15. h=1, integrated integration loop \( \text{Im} \alpha = -0.2 \). On the graphs, the solid and dashed line corresponds to the real and imaginary parts of the functions.
Figure 1. Transformants of the normal displacements on the surface of a two-layer medium, $\omega = 150$ (left side), $\omega = 50$ (right side), integrated integration loop $\text{Im } \alpha = -0.2$, porosity equal to 0.15, $h=1$.

In fig. 2 shows graphs illustrating the behavior of the transformants of the normal displacements on the surface of a two-layer medium. Mechanical characteristics correspond to phenylone [4] containing an oil fraction in the pores $\omega = 50$, porosity equal to 0.15, $h=2$ (left side), $h=3$ (right side), integrated integration loop $\text{Im } \alpha = -0.2$. On the graphs, the solid and dashed line corresponds to the real and imaginary parts of the functions. The graphs illustrate the substantial dependence of displacements in a non-uniform two-layer strip on the internal microstructure and thickness of a heterogeneous coating of a viscoelastic half-space.

Figure 2. Transformants of the normal displacements on the surface of a two-layer medium, $\omega = 50$, integrated integration loop $\text{Im } \alpha = -0.2$, porosity equal to 0.15, $h=2$ (left side), $h=3$ (right side).

3. Conclusion
Based on the constructed mathematical model that takes into account the internal microstructure and band heterogeneity, it is possible to predict the properties of the composite material, estimate the level of oscillations, find frequency ranges at which enhanced material wear occurs. The described method for modeling dynamic processes can be extended to two-phase materials used as coatings in highly loaded friction units.
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