REMARKS ON GENERATORS AND DIMENSIONS OF TRIANGULATED CATEGORIES

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Abstract. In this paper we prove that the dimension of the bounded derived category of coherent sheaves on a smooth quasi-projective curve is equal to one. We also discuss dimension spectrums of these categories.

Let $\mathcal{T}$ be a triangulated category. We say that an object $E \in \mathcal{T}$ is a classical generator for $\mathcal{T}$ if the category $\mathcal{T}$ coincides with the smallest triangulated subcategory of $\mathcal{T}$ which contains $E$ and is closed under direct summands.

If a classical generator generates the whole category for a finite number of steps then it called a strong generator. More precisely, let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two full subcategories of $\mathcal{T}$. We denote by $\mathcal{I}_1 \ast \mathcal{I}_2$ the full subcategory of $\mathcal{T}$ consisting of all objects such that there is a distinguished triangle $M_1 \to M \to M_2$ with $M_i \in \mathcal{I}_i$. For any subcategory $\mathcal{I} \subset \mathcal{T}$ we denote by $\langle \mathcal{I} \rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{I}$ and closed under finite direct sums, direct summands and shifts. We put $\mathcal{I}_1 \circ \mathcal{I}_2 = \langle \mathcal{I}_1 \ast \mathcal{I}_2 \rangle$ and we define by induction $\langle \mathcal{I} \rangle_k = \langle \mathcal{I} \rangle_{k-1} \circ \langle \mathcal{I} \rangle$. If $\mathcal{I}$ consists of an object $E$ we denote $\langle \mathcal{I} \rangle$ as $\langle E \rangle_1$ and put by induction $\langle E \rangle_k = \langle E \rangle_{k-1} \circ \langle E \rangle_1$.

Definition 1. Now we say that $E$ is a strong generator if $\langle E \rangle_n = \mathcal{T}$ for some $n \in \mathbb{N}$.

Note that $E$ is classical generator if and only if $\bigcup_{k \in \mathbb{Z}} \langle E \rangle_k = \mathcal{T}$. It is also easy to see that if a triangulated category $\mathcal{T}$ has a strong generator then any classical generator of $\mathcal{T}$ is strong as well.

Following to [2] we define the dimension of a triangulated category.

Definition 2. The dimension of a triangulated category $\mathcal{T}$, denoted by $\dim \mathcal{T}$, is the minimal integer $d \geq 0$ such that there is $E \in \mathcal{T}$ with $\langle E \rangle_{d+1} = \mathcal{T}$.

We also can define the dimension spectrum of a triangulated category as follows.

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Definition 3. The dimension spectrum of a triangulated category $T$, denoted by $\sigma(T)$, is a subset of $\mathbb{Z}$, which consists of all integer $d \geq 0$ such that there is $E \in T$ with $\langle E \rangle_{d+1} = T$ and $\langle E \rangle_d \neq T$.

A. Bondal and M. Van den Bergh showed in [1] that the triangulated category of perfect complexes $\text{Perf}(X)$ on a quasi-compact quasi-separated scheme $X$ has a classical generator. (Recall that a complex of $\mathcal{O}_X$-modules is called perfect if it is locally quasi-isomorphic to a bounded complex of vector bundles.)

For the triangulated category of perfect complexes on a quasi-projective scheme we can present a classical generator directly.

Theorem 4. Let $X$ be a quasi-projective scheme of dimension $d$ and let $\mathcal{L}$ be a very ample line bundle on $X$. Then the object $E = \bigoplus_{i=k-d}^k \mathcal{L}^i$ is a classical generator for the triangulated category of perfect complexes $\text{Perf}(X)$.

Proof. The scheme $X$ is an open subscheme of a projective scheme $X' \subset \mathbb{P}^N$ and $\mathcal{L}$ is the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ on $X$. Let us take $N+1$ linear independent hyperplanes $H_i \subset \mathbb{P}^N, i = 0, ..., N$. In this case the intersection $H_0 \cap \cdots \cap H_N$ is empty. The hyperplanes $H_i$ give a section $s$ of the vector bundle $U = O(1)^{\oplus(N+1)}$ which does not have zeros. This implies that the Koszul complex induced by $s$

$$0 \rightarrow \Lambda^{N+1}(U^*) \rightarrow \Lambda^N(U^*) \rightarrow \cdots \rightarrow \Lambda^2(U^*) \rightarrow U^* \rightarrow \mathcal{O}_{\mathbb{P}^N} \rightarrow 0$$

is exact on $\mathbb{P}^N$. Consider the restriction of the truncated complex on $X$

$$\Lambda^{d+1}(U_X^*) \rightarrow \cdots \rightarrow \Lambda^2(U_X^*) \rightarrow U_X^*.$$ 

It has two nontrivial cohomologies, one of which is $\mathcal{O}_X$. And, moreover, since the dimension of $X$ is equal to $d$ the sheaf $\mathcal{O}_X$ is a direct summand of this complex. Tensoring this complex with $\mathcal{L}^{k+1}$ we obtain that the triangulated subcategory which contains $\mathcal{L}^i$ for $i = k-d, \ldots, k$ also contains $\mathcal{L}^{k+1}$. Thus, it contains $\mathcal{L}^i$ for all $i \geq k-d$. By duality this category contains also all $\mathcal{L}^i$ for all $i \leq k$. Thus we have all powers $\mathcal{L}^i$, where $i \in \mathbb{Z}$.

Finally, it easy to see that $\{\mathcal{L}^i\}_{i \in \mathbb{Z}}$ classically generate the triangulated category of perfect complexes $\text{Perf}(X)$. Indeed, for any perfect complex $E$ we can construct a bounded above complex $P$, where all $P^k$ are direct sums of line bundles $\mathcal{L}^i$, together with a quasi-isomorphism $P \sim E$. Consider the brutal truncation $\sigma^{\geq-m}P$ for sufficiently large $m$ and the map $\sigma^{\geq-m}P \rightarrow E$. The cone of this map is isomorphic to $\mathcal{F}[m+1]$, where $\mathcal{F}$ is a vector bundle. And since the $\text{Hom}(E, \mathcal{F}[m+1]) = 0$ for sufficiently large $m$ we get that $E$ is a direct summand of $\sigma^{\geq-m}P$. $\blacksquare$
A. Bondal and M. Van den Bergh also proved that for any smooth separated scheme $X$ the triangulated category of perfect complexes $\text{Perf}(X)$ has a strong generator ([1], Th. 3.1.4). Furthermore, R. Rouquier showed that for quasi-projective scheme $X$ the property to be regular is equivalent to the property that the triangulated category of perfect complexes $\text{Perf}(X)$ has a strong generator (see [2], Prop 7.35). On the other hand, there is a remarkable result of R. Rouquier which says that under some general conditions the bounded derived category of coherent sheaves $D^b(\text{coh}(X))$ has a strong generator. More precisely it says

**Theorem 5.** (R. Rouquier, [2] Th. 7.39) Let $X$ be a separated scheme of finite type. Then there are an object $E \in D^b(\text{coh}(X))$ and an integer $d \in \mathbb{Z}$ such that $D^b(\text{coh}(X)) \cong \langle E \rangle_{d+1}$. In particular, $\dim D^b(\text{coh}(X)) < \infty$.

Keeping in mind this theorem we can ask about the dimension of the derived category of coherent sheaves on a separated scheme of finite type. It is proved in [2] that

- for a reduced separated scheme $X$ of finite type $\dim D^b(\text{coh}(X)) \geq \dim X$;
- for a smooth affine scheme $\dim D^b(\text{coh}(X)) = \dim X$;
- for a smooth quasi-projective scheme $\dim D^b(\text{coh}(X)) \leq 2 \dim X$.

In this paper we show that the dimension of the derived category of coherent sheaves on a smooth quasi-projective curve $C$ is equal to 1. For affine curve it is known and for $\mathbb{P}^1$ it is evident. Thus, it is sufficient to consider a smooth projective curve of genus $g \geq 1$.

**Theorem 6.** Let $C$ be a smooth projective curve of genus $g \geq 1$. Then $\dim D^b(\text{coh}(C)) = 1$.

At first, we should bring an object which generates $D^b(\text{coh}(C))$ for one step. Let $\mathcal{L}$ be a line bundle on $C$ such that $\deg \mathcal{L} \geq 8g$. Let us consider $E = \mathcal{L}^{-1} \oplus \mathcal{O}_C \oplus \mathcal{L} \oplus \mathcal{L}^2$. We are going to show that $E$ generates the bounded derived category of coherent sheaves on $C$ for one step, i.e. $\langle E \rangle_2 = D^b(\text{coh}(X))$.

Since any object of $D^b(\text{coh}(X))$ is a direct sum of its cohomologies it is sufficient to prove that any coherent sheaf $\mathcal{G}$ belongs to $\langle E \rangle_2$. Further, each coherent sheaf $\mathcal{G}$ on a curve is a direct sum of a torsion sheaf $T$ and a vector bundle $\mathcal{F}$.

**Lemma 7.** Let $C$ be a smooth projective curve of genus $g \geq 1$ and let $\mathcal{L}$ be a line bundle on $C$ as above. Then there is an exact sequence of the form

$$(\mathcal{L}^{-1})^{\oplus r_1} \longrightarrow \mathcal{O}_C^{\oplus r_0} \longrightarrow T \longrightarrow 0$$

for any torsion coherent sheaf $T$ on $C$. 
Let $\mathcal{F}$ be a vector bundle on the curve $C$. Consider the Harder-Narasimhan filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$. It is such filtration that every quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semi-stable and $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$ for all $0 < i < n$, where $\mu(\mathcal{G})$ is the slope of a vector bundle $\mathcal{G}$ and is equal to $c_1(\mathcal{G})/r(\mathcal{G})$.

**Main Lemma 8.** Let $\mathcal{L}$ be a line bundle with $\deg \mathcal{L} \geq 8g$. Let $\mathcal{F}$ be a vector bundle on $C$ and let $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$ be its Harder-Narasimhan filtration. Choose $0 \leq i \leq n$ such that $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) \geq 4g > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$. Then there are exact sequences of the form

\[
\begin{align*}
\text{(a)} & \quad (\mathcal{L}^{-1})^{\oplus r_1} \xrightarrow{\alpha} \mathcal{O}_C^{\oplus r_0} \rightarrow \mathcal{F}_i \rightarrow 0, \\
\text{(b)} & \quad 0 \rightarrow \mathcal{F}/\mathcal{F}_i \rightarrow \mathcal{L}^{\oplus s_0} \xrightarrow{\beta} (\mathcal{L}^2)^{\oplus s_1}.
\end{align*}
\]

To prove Lemma 7 and the Main Lemma 8 we need the following lemma which is well-known.

**Lemma 9.** Let $\mathcal{G}$ be a vector bundle on a smooth projective curve $C$ over a field $k$. Denote by $\overline{\mathcal{G}}$ its pullback on $\overline{C} = C \otimes_k \overline{k}$. Assume that for any line bundle $\mathcal{M}$ on $\overline{C}$ with $\deg \mathcal{M} = d$ we have $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \mathcal{M}) = 0$. Then

i) $H^1(C, \mathcal{G} \otimes \mathcal{N}) = 0$ for any $\mathcal{N}$ on $C$ with $\deg \mathcal{N} \geq d$;

ii) any sheaf $\mathcal{G} \otimes \mathcal{N}$ is generated by the global sections for all $\mathcal{N}$ with $\deg \mathcal{N} > d$.

**Proof.** i) Since any field extension is strictly flat it is sufficient to check that $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \mathcal{N}) = 0$. From an exact sequence

\[
0 \rightarrow \overline{\mathcal{G}} \otimes \mathcal{N}(-x) \rightarrow \overline{\mathcal{G}} \otimes \mathcal{N} \rightarrow (\overline{\mathcal{G}} \otimes \mathcal{N})_x \rightarrow 0
\]

on $\overline{C}$ we deduce that if $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \mathcal{N}(-x)) = 0$ then $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \mathcal{N}) = 0$. This implies i).

ii) By the same reason as above it is enough to show that the sheaf $\overline{\mathcal{G}} \otimes \mathcal{N}$ is generated by the global sections. Since $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \mathcal{N}(-x)) = 0$ the map

\[
H^0(\overline{C}, \overline{\mathcal{G}} \otimes \mathcal{N}) \rightarrow H^0(\overline{C}, (\overline{\mathcal{G}} \otimes \mathcal{N})_x)
\]

is surjective for any $x \in \overline{C}$. Hence, $\overline{\mathcal{G}} \otimes \mathcal{N}$ and $\mathcal{G} \otimes \mathcal{N}$ are generated by the global sections for all $\mathcal{N}$ of degree greater than $d$. $\square$

**Proof of Lemma 7.** Any torsion sheaf $T$ is generated by the global sections. Consider the surjective map $\mathcal{O}_C^{\oplus r_0} \rightarrow T$, where $r_0 = \dim H^0(T)$. Denote by $U$ the kernel of this map. Now it is evident that $H^1(U \otimes \mathcal{M}) = 0$ for any line bundle $\mathcal{M}$ on $\overline{C}$ with $\deg \mathcal{M} \geq 2g - 1$, because $H^1(\mathcal{M}) = 0$. Applying Lemma 9 we get that $U \otimes \mathcal{L}$ is generated by the global sections. Hence, there is an exact sequence of the form

\[
(\mathcal{L}^{-1})^{\oplus r_1} \rightarrow \mathcal{O}_C^{\oplus r_0} \rightarrow T \rightarrow 0.
\]

for any torsion sheaf $T$. $\square$
Proof of the Main Lemma. If \( G \) is a semi-stable vector bundle on \( C \) with \( \mu(G) \geq 2g \) then by Serre duality we have \( H^1(C, G \otimes \mathcal{M}) = 0 \) for all \( \mathcal{M} \) with \( \deg \mathcal{M} \geq -1 \). Therefore, by Lemma \( \# \) the bundle \( G \) is generated by the global sections.

Now \( F_i \subseteq F \) as an extension of semi-stable sheaves with \( \mu \geq 4g \) is generated by the global sections as well. Consider the short exact sequence

\[
0 \rightarrow U \rightarrow O_C^{\oplus r_0} \rightarrow F_i \rightarrow 0,
\]

where \( r_0 \) is the dimension of \( H^0(F_i) \). Take a line bundle \( \mathcal{M} \) on \( \overline{C} \) of degree \( 2g \) and consider the diagram

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \overline{U} \otimes \mathcal{M} & (\mathcal{M}^{-1})^{\oplus r_0} & \overline{F}_i \otimes \mathcal{M}^{-1} & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \overline{U}^{\oplus 2} & O_C^{\oplus 2r_0} & \overline{F}_i^{\oplus 2} & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \overline{U} \otimes \mathcal{M} & \mathcal{M}^{\oplus r_0} & \overline{F}_i \otimes \mathcal{M} & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

Since the sheaf \( \overline{F}_i \otimes \mathcal{M}^{-1} \) is the extension of semi-stable sheaves with \( \mu \geq 2g \) we have \( H^1(\overline{F}_i \otimes \mathcal{M}^{-1}) = 0 \). Hence, the map \( H^0(\overline{F}_i^{\oplus 2}) \rightarrow H^0(\overline{F}_i \otimes \mathcal{M}) \) is surjective. Further, we know that the map \( H^0(O_C^{\oplus 2r_0}) \rightarrow H^0(O_C^{\oplus 2r_0}) \) is surjective and the map \( H^0(O_C^{\oplus 2r_0}) \rightarrow H^0(\mathcal{M}^{\oplus r_0}) \) is injective. This implies that the map \( H^0(\mathcal{M}^{\oplus r_0}) \rightarrow H^0(\overline{F}_i \otimes \mathcal{M}) \) is surjective as well. Hence \( H^1(\overline{U} \otimes \mathcal{M}) = 0 \). Therefore, by Lemma \( \# \) the bundle \( \overline{U} \otimes \mathcal{M}' \) is generated by the global sections for all \( \mathcal{M}' \) with \( \deg \mathcal{M}' \geq 2g + 1 \). In particular, \( U \otimes \mathcal{L} \) is generated by the global sections. Thus, we get an exact sequence

\[
(L^{-1})^{\oplus r_1} \xrightarrow{\alpha} O_C^{\oplus r_0} \rightarrow F_i \rightarrow 0.
\]

Sequence b) can be obtained by dualizing of sequence a) applied for the sheaf \( F^* \otimes \mathcal{L} \). \( \square \)

Proof of Theorem \( \# \). At first, since the category of coherent sheaves on \( C \) has homological dimension one we see that any torsion sheaf \( T \) is a direct summand of the complex of the form \( (L^{-1})^{\oplus r_1} \rightarrow O_C^{\oplus r_0} \). Hence, it belongs to \( \langle E \rangle_2 \).

Now consider a vector bundle \( F \) on \( C \) with the Harder-Narasimhan filtration \( 0 = F_0 \subset F_1 \subset \cdots \subset F_n = F \). As above let us fix \( 0 \leq i \leq n \) such that \( \mu(F_i/F_{i-1}) \geq 4g > \mu(F_{i+1}/F_i) \).
Applying the Main Lemma we obtain the following long exact sequence

$$0 \longrightarrow \text{Ker } \alpha \longrightarrow (\mathcal{L}^{-1})^{\oplus r_1} \overset{\alpha}{\longrightarrow} \mathcal{O}_C^{\oplus r_0} \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^{\oplus s_0} \overset{\beta}{\longrightarrow} (\mathcal{L}^2)^{\oplus s_1} \longrightarrow \text{Coker } \beta \longrightarrow 0.$$ 

Furthermore, it is easy to see that the canonical map $\text{Ext}^1(\mathcal{L}^{\oplus s_0}, \mathcal{O}_C^{\oplus r_0}) \longrightarrow \text{Ext}^1(\mathcal{F}/\mathcal{F}_i, \mathcal{F}_i)$ is surjective. Let us fix $e \in \text{Ext}^1(\mathcal{F}/\mathcal{F}_i, \mathcal{F}_i)$ which defines $\mathcal{F}$ as the extension and choose some its pull back $e' \in \text{Ext}^1(\mathcal{L}^{\oplus s_0}, \mathcal{O}_C^{\oplus r_0})$.

Now let us consider the map

$$\phi : (\mathcal{L}^{-1})^{\oplus r_0} \oplus \mathcal{L}^{\oplus s_0}[-1] \longrightarrow \mathcal{O}_C^{\oplus r_0} \oplus (\mathcal{L}^2)^{\oplus s_1}[-1], \quad \text{where } \phi = \begin{pmatrix} \alpha & e' \\ 0 & \beta \end{pmatrix}$$

and take a cone $C(\phi)$ of $\phi$. The cone $C(\phi)$ is isomorphic to a complex that has three nontrivial cohomologies $H^{-1}(C(\phi)) \cong \text{Ker } \alpha$, $H^1(C(\phi)) \cong \text{Coker } \beta$ and, finally, $H^0(C(\phi)) \cong \mathcal{F}$. Thus, $\mathcal{F}$ is a direct summand of $C(\phi)$ and, consequently, it belongs to $\langle \mathcal{E} \rangle_2$. This implies that the whole bounded derived category of coherent sheaves on $C$ coincides with $\langle \mathcal{E} \rangle_2$ and the dimension of $\mathcal{D}^b(\text{coh}(C))$ is equal to 1.

Having in view of the given theorem we may assume, that the following conjecture can be true.

**Conjecture 10.** Let $X$ be a smooth quasi-projective scheme of dimension $n$. Then $\dim \mathcal{D}^b(\text{coh}(X)) = n$.

**Remark 11.** For a non regular scheme it is evidently not true. For example, the dimension of the bounded derived category of coherent sheaves on the zero-dimension scheme $\text{Spec}(k[x]/x^2)$ equals to 1.

It is also very interesting to understand what the spectrum $\sigma(\mathcal{D}^b(\text{coh}(X)))$ forms. In particular we can ask the following questions

**Question 12.** Is the spectrum of the bounded derived category of coherent sheaves on a smooth quasi-projective scheme bounded? Is it bounded for a non smooth scheme?

**Question 13.** Does the spectrum of the bounded derived category of coherent sheaves on a (smooth) quasi-projective scheme form an integer interval?

Let us try to calculate the dimension spectra of the derived categories of coherent sheaves on some smooth curves.

**Proposition 14.** Let $C$ be a smooth affine curve. Then the dimension spectrum $\sigma(\mathcal{D}^b(\text{coh } C))$ coincides with $\{1\}$. 

Proof. If \( \mathcal{E} \) is a strong generator then it has a some locally free sheaf \( \mathcal{F} \) as a direct summand. Now since \( C \) is affine then there is an exact sequence of the form
\[
\mathcal{F}^{r_1} \rightarrow \mathcal{F}^{r_0} \rightarrow \mathcal{G} \rightarrow 0
\]
for any coherent sheaf \( \mathcal{G} \) on \( C \). Hence, any coherent sheaf \( \mathcal{G} \) belongs to \( \langle \mathcal{E} \rangle_2 \). Since the global dimension of \( \text{coh} \ C \) is equal to 1 we obtain that \( \langle \mathcal{E} \rangle_2 = D^b(\text{coh} \ C) \). \( \square \)

We can also find the dimension spectrum of the projective line.

**Proposition 15.** The dimension spectrum \( \sigma(D^b(\text{coh} \mathbb{P}^1)) \) coincides with the set \( \{1, 2\} \).

**Proof.** Indeed, 1 is the dimension. And, for example, the object \( \mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O} \) generate the whole category \( D^b(\text{coh} \mathbb{P}^1) \) for one step. Now, the object \( \mathcal{E} = \mathcal{O}_p \oplus \mathcal{O}_p \), where \( p \) is a point, is a generator, because \( \mathcal{O}(-1) \) belongs to \( \langle \mathcal{E} \rangle_2 \). This also implies that \( \langle \mathcal{E} \rangle_3 \cong D^b(\text{coh} \mathbb{P}^1) \). On the other hand, \( \langle \mathcal{E} \rangle_2 \not\cong D^b(\text{coh} \mathbb{P}^1) \). To see it we can check that an object \( \mathcal{O}_q \), where \( q \neq p \), doesn’t belong to \( \langle \mathcal{E} \rangle_2 \). Indeed, \( \mathcal{O}_q \) is completely orthogonal to \( \mathcal{O}_p \) and doesn’t belong to subcategory generated by \( \mathcal{O} \). Finally, it easy to see that any object \( \mathcal{E} \), which generates the whole category, generates it at least for two steps, i.e. \( \langle \mathcal{E} \rangle_3 \cong D^b(\text{coh} \mathbb{P}^1) \). If \( \mathcal{E} \) contains as direct summands two different line bundles then it generates the whole category for one step. If \( \mathcal{E} \) has only one line bundle as a direct summand then it also has a torsion sheaf as a direct summand. This implies that \( \langle \mathcal{E} \rangle_2 \) has another line bundle. Therefore, \( \langle \mathcal{E} \rangle_3 \) is the whole category. \( \square \)

Another simple result says

**Proposition 16.** Let \( C \) be a smooth projective curve of genus \( g > 0 \) over a field \( k \). Assume that \( C \) has at least two different points over \( k \). Then the dimension spectrum \( \sigma(D^b(\text{coh} \ C)) \) contains \( \{1, 2\} \) as a proper subset, i.e. \( \{1, 2\} \) is strictly contained in the dimension spectrum.

**Proof.** The spectrum contains 1 as the dimension of the category. Let us now take a line bundle \( \mathcal{L} \) on \( C \) which satisfies the condition as in Theorem \( \S \) i.e. \( \deg \mathcal{L} \geq 8g \) and consider the object \( \mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}^2 \). It is easy to see that the line bundles \( \mathcal{L}^{-1} \) and \( \mathcal{L} \) belong to \( \langle \mathcal{E} \rangle_2 \), because there are exact sequences
\[
0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{L}^{-2} \rightarrow \mathcal{L}^2 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{L}^{-1} \rightarrow \mathcal{O}_C^3 \rightarrow \mathcal{L}^2 \rightarrow 0.
\]

The proof of Theorem \( \S \) (see the map \( \exists \)) implies that \( \langle \mathcal{E} \rangle_3 \cong D^b(\text{coh} \ C) \). On the other hand, the subcategory \( \langle \mathcal{E} \rangle_2 \) doesn’t coincide with the whole \( D^b(\text{coh} \ C) \). For example, a nontrivial line bundle \( \mathcal{M} \) from \( \text{Pic}^0 C \) doesn’t belong to \( \langle \mathcal{E} \rangle_2 \), because it is completely orthogonal to the structure sheaf \( \mathcal{O}_C \) and, evidently, could not be obtained from the line bundle \( \mathcal{L}^2 \).
Let us take a point \( p \in C \) and consider the object \( \mathcal{E} = \mathcal{O}_C \oplus \mathcal{O}_p \), where \( \mathcal{O}_p \) is the skyscraper in \( p \). This object is a strong generator and we can show that \( \langle \mathcal{E} \rangle_3 \neq D^b(\text{coh} C) \).

Take another point \( q \neq p \) and consider the skyscraper sheaf \( \mathcal{O}_q \). It is completely orthogonal to \( \mathcal{O}_p \) and have only one-dimensional 1-st Ext to \( \mathcal{O}_C \). Hence, if \( \mathcal{O}_q \) belongs to \( \langle \mathcal{E} \rangle_3 \) then it should be a direct summand of an object \( M \) which is included in an exact triangle of the form

\[
\mathcal{O}_C^{\oplus k} \rightarrow N \rightarrow M \rightarrow \mathcal{O}_C^{\oplus k}[1],
\]

where \( N \in \langle \mathcal{E} \rangle_2 \). Since the 1-st Ext from \( \mathcal{O}_q \) to \( \mathcal{O}_C \) is one-dimensional we can take \( k = 1 \). The composition of the map \( \mathcal{O}_q \rightarrow M \) with \( M \rightarrow \mathcal{O}_C \) should be the nontrivial 1-st Ext from \( \mathcal{O}_q \) to \( \mathcal{O}_C \). Now object \( N \) is a direct sum of indecomposable objects from \( \langle \mathcal{E} \rangle_2 \). It is easy to see that we can consider only objects for which there are nontrivial homomorphisms from \( \mathcal{O}_C \) and nontrivial homomorphisms to \( \mathcal{O}_q \). All other can be removed from \( N \). Thus \( N \) is a direct sum of \( \mathcal{O}(p) \) and objects \( U \) that are extensions

\[
0 \rightarrow \mathcal{O}_C^{\oplus r_1} \rightarrow U \rightarrow \mathcal{O}_C^{\oplus r_2} \rightarrow 0.
\]

Finally, split embedding \( \mathcal{O}_q \rightarrow M \) gives us a nontrivial map from \( \mathcal{O}(q) \) to \( N \). But there are no nontrivial maps from \( \mathcal{O}(q) \) to \( \mathcal{O}(p) \) and to \( U \) of the form (3). Therefore, \( \mathcal{O}_q \) can not belong to \( \langle \mathcal{E} \rangle_3 \).

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References

[1] A. Bondal and M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J., 3 (2003), pp. 1–36, arXiv:math/0204218.

[2] R. Rouquier, Dimension of triangulated categories (2003), arXiv:math/0310134. Will appear in Journal of K-theory.

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