REVERSE FABER-KRAHN INEQUALITIES FOR ZAREMBA PROBLEMS

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Abstract. Let Ω be a multiply-connected domain in \( \mathbb{R}^n \) (\( n \geq 2 \)) of the form \( \Omega = \Omega_{\text{out}} \setminus \Omega_{\text{in}} \). Set \( \Omega_D \) to be either \( \Omega_{\text{out}} \) or \( \Omega_{\text{in}} \). For \( p \in (1, \infty) \), and \( q \in [1, p] \), let \( \tau_{1,q}(\Omega) \) be the first eigenvalue of
\[
-\Delta_p u = \tau \left( \int_{\Omega} |u|^q dx \right)^{\frac{q-2}{q}} |u|^{q-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega_D, \quad \frac{\partial u}{\partial \eta} = 0 \text{ on } \partial \Omega \setminus \partial \Omega_D.
\]
Under the assumption that \( \Omega_D \) is convex, we establish the following reverse Faber-Krahn inequality
\[
\tau_{1,q}(\Omega) \leq \tau_{1,q}(\Omega^\star),
\]
where \( \Omega^\star = B_R \setminus B_r \) is a concentric annular region in \( \mathbb{R}^n \) having the same Lebesgue measure as \( \Omega \) and such that
(i) (when \( \Omega_D = \Omega_{\text{out}} \)) \( W_1(\Omega_D) = \omega_n R^{n-1} \), and \( (\Omega^\star)_D = B_R \),
(ii) (when \( \Omega_D = \Omega_{\text{in}} \)) \( W_{n-1}(\Omega_D) = \omega_n r \), and \( (\Omega^\star)_D = B_r \).
Here \( W_i(\Omega_D) \) is the \( i \)th quermassintegral of \( \Omega_D \). We also establish Sz. Nagy’s type inequalities for parallel sets of a convex domain in \( \mathbb{R}^n (n \geq 3) \) for our proof.

1. Introduction and statements of the main results

In the book The Theory of Sound [31], Lord Rayleigh conjectured that “among all the planar domains with the fixed area, disk minimizes the first Dirichlet eigenvalue of the Laplacian.” To state this result mathematically, consider the first eigenvalue \( \lambda_1(\Omega) \) of the following eigenvalue problem on a bounded domain \( \Omega \subset \mathbb{R}^n \):
\[
-\Delta u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\]
Let \( \Omega^\star \) be the open ball centered at the origin with the same volume as \( \Omega \). Then the Rayleigh’s conjecture reads as: for \( n = 2 \),
\[
\lambda_1(\Omega^\star) \leq \lambda_1(\Omega).
\]
Rayleigh’s conjecture was proved independently by Faber [12] and Krahn [22]. Later Krahn extended the result for \( n > 2 \) in [23]. The proof given by Faber is based on discretization and approximation, whereas Krahn’s proof [22, 23] is based on the classical isoperimetric inequality and the Coarea formula. In [8], it is shown that the equality in Rayleigh-Faber-Krahn inequality holds only if \( \Omega \) is a ball up to a set of capacity zero.

Now let us consider the first non-trivial Neumann eigenvalue \( \mu_2(\Omega) \) of the following Neumann eigenvalue problem on a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^n \):
\[
-\Delta u = \mu u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial \Omega.
\]

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2020 Mathematics Subject Classification. 35P15, 35P30, 49R05, 49Q10.
Key words and phrases. Zaremba problems, \( p \)-Laplacian, Reverse Faber-Krahn inequality, Quermassintegrals, Steiner Formula, Nagy’s inequality, Method of interior parallel sets.
where $\eta$ is the unit outward normal to $\partial \Omega$. For $n = 2$, Kornhauser and Stakgold [21] have shown that
\begin{equation}
\mu_2(\Omega) \leq \mu_2(\Omega^*),
\end{equation}
provided $\Omega$ is obtained by a small area-preserving perturbation of a disk. Further, they have established that a maximizing domain (if exists) for $\mu_2$, in the class of simply-connected planar domains with the fixed area, must be a disk. Later, Szegő [35] proved (3) for a simply-connected planar domain $\Omega$ which is bounded by an analytic curve. In 1956, Weinberger [36] extended (3) for general bounded Lipschitz domain in $\mathbb{R}^n$ without the simply connectedness assumption. The inequality (3) is known as the Szegő-Weinberger inequality in the literature.

The equality in Szegő-Weinberger inequality holds if and only if $\Omega$ is a ball up to a set of Lebesgue measure zero; see, for instance, [16, Theorem 7.1.1].

For $1 < p < \infty$, we study the similar inequalities for the first eigenvalue of the Zaremba problems (mixed boundary conditions) for the $p$-Laplace operator $\Delta_p$, defined by $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$, on multiply-connected domains. More precisely, we consider domains of the following form:
\begin{equation}
\begin{aligned}
\Omega &= \Omega_{\text{out}} \setminus \overline{\Omega_{\text{in}}}, \quad \Omega \text{ is Lipschitz} \\
\text{and } \Omega_{\text{out}}, \Omega_{\text{in}} \text{ are open sets in } \mathbb{R}^n \text{ such that } \overline{\Omega_{\text{in}}} \subset \Omega_{\text{out}}.
\end{aligned}
\end{equation}

Let $\Omega_D$ be either $\Omega_{\text{out}}$ or $\Omega_{\text{in}}$ and $\Gamma_D := \partial \Omega_D$. Now for $1 \leq q \leq p$, consider the following Zaremba eigenvalue problem for the $p$-Laplacian on $\Omega$:
\begin{equation}
\begin{aligned}
-\Delta_p u &= \tau \left( \int_\Omega |u|^q \, dx \right)^{\frac{p-q}{q}} |u|^{p-2}u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma_D, \\
\frac{\partial u}{\partial \eta} &= 0 \quad \text{on } \partial \Omega \setminus \Gamma_D.
\end{aligned}
\end{equation}

where $\tau \in \mathbb{R}$ and $\eta$ is the unit outward normal to the boundary of $\Omega$. For $q = p$, (P) coincides with the eigenvalue problem for the $p$-Laplace operator. Indeed, (P) admits a least positive eigenvalue $\tau_{1,q}(\Omega)$ (see Proposition 2.1) and it has the following variational characterization:
\begin{equation}
\tau_{1,q}(\Omega) = \inf \left\{ \mathcal{R}_q(u) : u \in W^{1,p}_{\Gamma_D}(\Omega) \setminus \{0\} \right\},
\end{equation}
where $\mathcal{R}_q(u) := \int_{\Omega_D} |\nabla u|^p + \int_{\partial \Omega_D} |u|^q \, d\sigma$ and $W^{1,p}_{\Gamma_D}(\Omega)$ is the space of all functions in $W^{1,p}(\Omega)$ that vanishes on $\Gamma_D$. In fact, $\frac{1}{\tau_{1,q}(\Omega)}$ is the best constant of the Sobolev embedding $W^{1,p}_{\Gamma_D}(\Omega) \hookrightarrow L^q(\Omega)$, for $q \in [1,p]$.

The shape optimization problems of such nonlinear eigenvalue are considered by Bucur and Giaconini [5] for the first Robin eigenvalue of the Laplacian. They have established a family of Faber-Krahn type inequalities (for $1 \leq q \leq \frac{2n}{n-2}$) for the first eigenvalue of the Robin Laplacian. In [3], Bobkov and Kolonitskii studied a monotonicity result (with respect to domain perturbation) for the first Dirichlet eigenvalue of the $p$-Laplacian with suitable nonlinearity; see [2,7] also for a similar result when $q = p$. The symmetries of the minimizers of $\mathcal{R}_q$ subject to different boundary conditions can be found in [18,26,27]. We also refer to the monographs [15,16] for further readings in this direction.

In this article, we consider the following two cases:

(i) Outer Dirichlet problem: $\Omega_D = \Omega_{\text{out}}$.

(ii) Inner Dirichlet problem: $\Omega_D = \Omega_{\text{in}}$. 
For a Lebesgue measurable set $A$ in $\mathbb{R}^n$, $|A|$ denotes the Lebesgue measure of $A$ and $P(A)$ denotes the perimeter ($(n-1)$-dimensional Hausdorff measure of $\partial A$) of $A$. For $a > 0$, $B_a$ denotes the open ball of radius $a$ centered at the origin. For each of the outer and inner Dirichlet problems, we choose $0 < r < R$ (whence two concentric annular regions) as follows:

(A1) For outer Dirichlet problem:

$$A_O(\Omega) = B_R \setminus \overline{B_r}$$

such that $|\Omega| = |A_O(\Omega)|$ and $P(\Omega_D) = P(B_R)$,

(A2) For inner Dirichlet problem:

$$A_I(\Omega) = B_R \setminus \overline{B_r}$$

such that $|\Omega| = |A_I(\Omega)|$ and $P(\Omega_D) = P(B_r)$.

1.1 Outer Dirichlet problem ($\Omega_D = \Omega_{\text{out}}$). In [29], for $n = 2$, $p = q = 2$, Payne and Weinberger established that

$$\tau_{1,2}(\Omega) \leq \tau_{1,2}(A_O(\Omega)).$$

(6)

In particular, by setting $\Omega_{\text{in}} = \emptyset$, it is easy to see that over the class of domains with fixed area and perimeter, $\lambda_1(\Omega)$ (first Dirichlet eigenvalue) is bounded above by $\tau_{1,2}(A_O(\Omega))$. This result sharpens the various upper bounds for $\lambda_1(\Omega)$ obtained earlier in [25, 30]. In [17], Hersch provided a different proof of (6). Both Hersch and Payne-Weinberger used Sz. Nagy’s inequality [34] for inner parallels sets. Now we recall the inner parallel set of a domain and the Sz. Nagy’s inequality for the inner parallel set. For a non-empty set $A \subset \mathbb{R}^n$, we define $d(x, A) = \inf\{d(x, y) : y \in A\}$, where $d$ is the euclidean distance function.

1.1.1 Inner parallel set: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\Gamma \subset \partial \Omega$. For $\delta > 0$, consider the set

$$\Omega_{-\delta} := \{x \in \Omega : d(x, \Gamma) \geq \delta\}.$$

The set $\partial \Omega_{-\delta}$ is known as the inner parallel set with respect to $\Gamma$ at a distance $\delta$. For $n = 2$, $\Omega$ simply connected and $\Gamma = \partial \Omega$, Sz. Nagy [34] proved that

$$P(\Omega_{-\delta}) \leq P(\Omega) - 2\pi \delta, \quad \forall \delta > 0.$$

For $\Omega$, we define $\Omega^\#$ in the following way:

$$\Omega^\#$$

is the open ball centered at the origin such that $P(\Omega^\#) = P(\Omega)$.

(7)

For $n = 2$, it is easy to observe that $P(\Omega_{-\delta}^\#) = P(\Omega) - 2\pi \delta$ and hence Sz. Nagy’s inequality can be restated as

$$P(\Omega_{-\delta}) \leq P(\Omega_{-\delta}^\#).$$

(8)

In [1], for $n \geq 3$ and $\Omega$ as given in (4), Anoop and Ashok observed that the equation (8) holds for $\Gamma = \Gamma_D$ when $\Omega_D$ is a ball in $\mathbb{R}^n$. Using (8), for $q = p$, they extended reverse Faber-Krahn inequality (6) for the first eigenvalue $\tau_{1,q}(\Omega)$ of (P) to higher dimensions under the assumption that $\Omega_D$ is a ball; cf [1, Theorem 1.1]. They named (6) as the reverse Faber-Krahn inequality. In [28], the authors recently extended this result (for $q = p$) for $\Omega$ with $\Omega_D$ as a convex domain. Their proof is based on constructing a web function using the first eigenfunction of (P) on $A_O(\Omega)$. In this article, we establish this result for $\tau_{1,q}$ using a different method explicitly based on the Sz. Nagy’s type inequality for inner parallel sets in higher dimensions. Furthermore, we prove that the concentric annulus is the unique maximizer (up to a translation) for $\tau_{1,q}$ in the admissible class of Lipschitz domains. First, we establish (8) for convex domains in higher dimensions using the Alexandrov-Fenchel inequality (Section 3).
Proposition 1.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain and $r_\Omega$ be the inradius of $\Omega$. Let $\Omega^\#$ be as defined in (7). Then

(i) $P(\Omega_{-\delta}) \leq P(\Omega^\#_{-\delta})$, for all $\delta \in (0, r_\Omega)$,

(ii) for $n \geq 3$, the equality holds in the above inequality for some $\delta$ if and only if $\Omega$ is a ball.

Next, as an application of Proposition 1.1, we extend (6) for the nonlinear eigenvalues $\tau_{1,q}(\Omega)$ of (P) provided that $\Omega_D$ is convex.

Theorem 1.2. Let $\Omega$ and $A_O(\Omega)$ be as given in (4) and (A1), respectively, with $\Omega_D = \Omega_{\text{out}}$. For $q \in [1, p]$, let $\tau_{1,q}(\Omega)$ be the first eigenvalue of (P) on $\Omega$. If $\Omega_D$ is convex, then

$$\tau_{1,q}(\Omega) \leq \tau_{1,q}(A_O(\Omega)).$$

Furthermore, for $n \geq 3$, the equality holds if and only if $\Omega = A_O(\Omega)$ (up to a translation).

1.2 Inner Dirichlet problem ($\Omega_D = \Omega_{\text{in}}$). In [17], Hersch consider (P) for the case $\Omega_D = \Omega_{\text{in}}$. For $n = 2$ and $p = q = 2$, Hersch established the following reverse Faber-Krahn inequality:

$$\tau_{1,2}(\Omega) \leq \tau_{1,2}(A_I(\Omega)), \tag{9}$$

where $A_I(\Omega)$ is as defined in (A2). The key step in proving (9) is Sz. Nagy’s inequality [34] for outer parallels for a bounded planar domain $\Omega$. Let us define the outer parallel sets and describe the Sz. Nagy’s inequality for them.

1.2.1 Outer parallel set: For a bounded domain $\Omega \subset \mathbb{R}^n$, the outer parallel body $\Omega_\delta$ of $\Omega$ with respect to $\Gamma \subset \partial \Omega$ at a distance $\delta > 0$ is defined by

$$\Omega_\delta := \{x \in \Omega^c : d(x, \Gamma) \geq \delta\}^c. \tag{10}$$

The boundary $\partial \Omega_\delta$ is called as the outer parallel set at a distance $\delta$. For $n = 2$, $\Omega$ simply connected and $\Gamma = \partial \Omega$, Sz. Nagy’s inequality [34] for outer parallel sets states that

$$P(\Omega_\delta) \leq P(\Omega^\#_\delta), \tag{11}$$

where $\Omega^\#$ is as given in (7).

Remark 1.3. Notice that, for a convex planar domain, Steiner formula gives the equality in (11); see [33] or [14, Theorem 10.1].

In [1], for $n \geq 3$ and $\Omega$ as given in (4), Anoop and Ashok observed that the inequality (11) holds for $\Gamma = \Gamma_D$ when $\Omega_D$ is a ball in $\mathbb{R}^n$. As a consequence, they obtained (9) for the first eigenvalue $\tau_{1,q}(\Omega)$ of (P) with $q = p$ in higher dimensions under the assumption that $\Omega_D$ is a ball; cf. [1, Theorem 1.2]. We observed that the inequality (11) with $\Gamma = \partial \Omega$ fails in higher dimensions, even for a general convex domain. More precisely, we establish the following reverse type of Sz. Nagy’s inequality for convex sets in higher dimensions.

Theorem 1.4. Let $\Omega \subset \mathbb{R}^n \ (n \geq 3)$ be a bounded, convex domain and $\delta > 0$. Let $\Omega^\#$ be as defined in (7). Then

$$P(\Omega_\delta) \geq P(\Omega^\#_\delta).$$

Furthermore, equality holds for some $\delta$ if and only if $\Omega$ is a ball.

Thus (11) fails for $\Omega^\#$, which is chosen with the perimeter constraint. In view of the above theorem, to extend (11) to higher dimensions, we need to come up with a constraint that
gives a ball $B$ for which the following inequality holds:

$$P(\Omega_\delta) \leq P(B), \forall \delta > 0. \quad (12)$$

For this, we consider the Steiner formula available for convex domains in higher dimensions.

### 1.2.2 Steiner Formula for a convex domain

Let $\Omega \subset \mathbb{R}^n$ be a non-empty bounded convex domain and $\delta > 0$. Then for $\Omega_\delta$ as given in (10) with $\Gamma = \partial \Omega$, the Steiner formula (cf. [32, Chapter 4]) provides an expression for the perimeter of $\Omega_\delta$ as a polynomial in $\delta$:

$$P(\Omega_\delta) = n \sum_{i=0}^{n-1} \binom{n-1}{i} W_{i+1}(\Omega) \delta^i,$$  

(13)

where the co-efficients $W_1(\Omega), \ldots, W_n(\Omega)$ are called Quermassintegrals or Minkowski functionals of $\Omega$, which are special cases of mixed volumes (cf. [6, Section 19.1] or [32, Section 4.2]). In particular,

$$W_0(\Omega) = |\Omega|, \quad W_1(\Omega) = \frac{P(\Omega)}{n}, \quad W_n(\Omega) = \omega_n,$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. For an open ball $B \subset \mathbb{R}^n (n \geq 2)$ of radius $r$, the quermassintegrals can be computed using [32, (4.2.28)] as below:

$$W_j(B) = \omega_n r^{n-j}, \quad \text{for } 0 \leq j \leq n-1. \quad (14)$$

Now, it is clear that if we choose $B$ such that $W_i(\Omega) \leq W_i(B)$ for $i = 1, 2, \ldots, n-1$, then (13) yields (12). So we choose a ball $\Omega^\circ$ in the following way:

$$\Omega^\circ \text{ is the open ball centered at the origin such that } W_{n-1}(\Omega^\circ) = W_{n-1}(\Omega). \quad (15)$$

Indeed, we establish that $W_i(\Omega) \leq W_i(\Omega^\circ)$ for $i = 1, 2, \ldots, n-1$ (see Proposition 3.3).

**Remark 1.5.** Notice that, for $n = 2$, $W_{n-1}(\Omega) = \frac{P(\Omega)}{2}$ and hence $\Omega^\circ$ coincides with $\Omega^\#$. However, for $n > 2$, $\Omega^\#$ is a smaller ball than $\Omega^\circ$ (see Proposition 3.3).

Let $\Omega$ be as defined in (4) with $\Omega_D = \Omega_\infty$. For $n > 2$ and $\Omega_D$ convex, we consider a concentric annulus as below:

$$\tilde{A}_f(\Omega) = B_R \setminus B_r \text{ such that } |\Omega| = |\tilde{A}_f(\Omega)| \text{ and } W_{n-1}(\Omega_D) = W_{n-1}(B_r) = \omega_n r. \quad (16)$$

Now we state the reverse Faber-Krahn type inequality for the first eigenvalue $\tau_{1,q}(\Omega)$ of (P) when $\Omega_D = \Omega_\infty$.

**Theorem 1.6.** Let $\Omega$ and $\tilde{A}_f(\Omega)$ be as given in (4) and (16), respectively, with $\Omega_D = \Omega_\infty$. For $q \in [1, p]$, let $\tau_{1,q}(\Omega)$ be the first eigenvalue of (P) on $\Omega$. If $\Omega_D$ is convex, then

$$\tau_{1,q}(\Omega) \leq \tau_{1,q}(\tilde{A}_f(\Omega)).$$

Furthermore, for $n \geq 3$, the equality holds if and only if $\Omega = \tilde{A}_f(\Omega)$ (up to a translation).

**Remark 1.7.** For $n = 2$, since $A_f(\Omega) = \tilde{A}_f(\Omega)$, the results obtained in [17, Section 1.5] for convex $\Omega_D$ will follow from the above theorem. If $\Omega_D$ is a ball, then $A_f(\Omega) = \tilde{A}_f(\Omega)$. Thus Theorem 1.2 of [1] is a special case of the above theorem.

In [9], Della Pietra and Piscitelli established the inequality part in Theorem 1.6 (see [9, Theorem 1.1, Remark 3.1]) using a web function and the co-area formula. Here we use the higher dimensional analogue (see Corollary 3.4) of Sz. Nagy’s inequality for outer parallel sets of a convex set. In addition, we prove the uniqueness of the maximizer in the admissible class of Lipschitz domains.
Remark 1.8. (Ω_D is non-convex) To the best of our knowledge, the analogous results of the Theorem 1.2 and Theorem 1.6 are not available in the literature for a non-convex domain Ω_D in \( \mathbb{R}^n \) (\( n \geq 3 \)). Here we provide various upper bounds for \( \tau_{1,q}(\Omega) \) that work for certain non-convex cases as well.

(i) **Outer Dirichlet** (\( \Omega_D = \Omega_{\text{out}} \)): Let \( \Omega = \Omega_D \setminus \overline{\Omega_{\text{in}}} \) be as defined in (4). Assume that \( \tilde{\Omega} \) is such that there exists an open ball \( B_{\text{out}} \) such that \( \Omega_{\text{in}} \subset B_{\text{out}} \subset \Omega_D \). Let \( \bar{\Omega} = B_{\text{out}} \setminus \overline{\Omega_{\text{in}}} \). Then Theorem 1.2 yields that

\[
\tau_{1,q}(\bar{\Omega}) \leq \tau_{1,q} \left( A_{\Omega}(\tilde{\Omega}) \right),
\]

where \( A_{\Omega}(\tilde{\Omega}) = B_{R} \setminus \overline{B_{r}} \) satisfying \( |\tilde{\Omega}| = |A_{\Omega}(\tilde{\Omega})| \) and \( P(B_{\text{out}}) = P(B_{R}) \). Since \( \tilde{\Omega} \subset \Omega \), using domain monotonicity, we get \( \tau_{1,q}(\Omega) \leq \tau_{1,q}(\tilde{\Omega}) \). Hence using (17), we obtain

\[
\tau_{1,q}(\Omega) \leq \tau_{1,q} \left( A_{\Omega}(\tilde{\Omega}) \right).
\]

If \( \Omega_D \) is a ball, then we can take \( B_{\text{out}} = \Omega_D \), then the above result coincides with [1, Theorem 1.1].

(ii) **Inner Dirichlet** (\( \Omega_D = \Omega_{\text{in}} \)): Suppose \( \Omega = \Omega_{\text{out}} \setminus \overline{\Omega_{\text{D}}} \) is as defined in (4). Let \( \text{Conv}(\Omega_D) \) be the convex hull of \( \Omega_D \). Assume that \( \text{Conv}(\Omega_D) \subset \Omega_{\text{in}} \). Let \( \tilde{\Omega} = \Omega_{\text{in}} \setminus \overline{\text{Conv}(\Omega_D)} \).

Now by Theorem 1.6, we have

\[
\tau_{1,q}(\tilde{\Omega}) \leq \tau_{1,q} \left( \widetilde{A_{I}}(\tilde{\Omega}) \right),
\]

where \( \widetilde{A_{I}}(\tilde{\Omega}) = B_{R} \setminus \overline{B_{r}} \) satisfying \( |\tilde{\Omega}| = |\widetilde{A_{I}}(\tilde{\Omega})| \) and \( W_{n-1}(\text{Conv}(\Omega_D)) = W_{n-1}(B_{r}) \).

Since \( \Omega_D \subset \text{Conv}(\Omega_D) \), we have \( \tilde{\Omega} \subset \Omega \) and hence using domain monotonicity, we get \( \tau_{1,q}(\Omega) \leq \tau_{1,q}(\tilde{\Omega}) \). Therefore from (18), we conclude

\[
\tau_{1,q}(\Omega) \leq \tau_{1,q} \left( \widetilde{A_{I}}(\tilde{\Omega}) \right).
\]

The rest of the paper is organized as follows. In Section 2, we state the existence results for the first eigenvalue of (P). We prove the Sz. Nagy’s type inequalities (Proposition 1.1, Theorem 1.4) in Section 3. In Section 4, we prove Theorem 1.2, and Section 5 contains the proof of Theorem 1.6. Finally, in Section 6, we point out some consequences of the main results of this article and state a few open problems.

2. Preliminaries

In this section, we prove the existence of the first eigenvalue \( \tau_{1,q} \) of (P) and state a few properties of the first eigenfunctions.

**Proposition 2.1.** For \( q \in [1,p) \), let \( \tau_{1,q}(\Omega) \) be as defined in (5). Then there exists \( u_{q} \in W_{1,p}^{1}(\Omega) \) such that

(i) \( \mathcal{R}_{q}(u_{q}) = \tau_{1,q}(\Omega) \) and \( u_{q} > 0 \) a.e. in \( \Omega \).

(ii) \( u_{q} \in C^{1,\gamma}(\Omega) \), where \( 0 < \gamma < 1 \).

(iii) if \( \mathcal{R}_{q}(u_{q}) = \tau_{1,q}(\Omega) \), for some \( v_{q} \in W_{1,p}^{1}(\Omega) \), then \( u = cu_{q} \) for some \( c \in \mathbb{R} \).

**Proof.** (i) For \( q \in [1,p) \), since the inclusion \( W_{1,p}^{1}(\Omega) \hookrightarrow L^{p}(\Omega) \) is compact, using the standard variational arguments (cf. [13] or [1, Proposition A.2]) we easily obtain that \( \tau_{1,q}(\Omega) > 0 \) and \( \exists u_{q} \in W_{1,p}^{1}(\Omega) \) such that \( \mathcal{R}_{q}(u_{q}) = \tau_{1,q}(\Omega) \). Using the strong maximum principle [19, Proposition 3.2], we also deduce that \( u_{q} > 0 \) a.e. in \( \Omega \).

(ii) For a proof, see [24, Theorem 1].
(ii) This result is known for the complete Dirichlet problem; cf. [27, Proposition 1.1]. The same set of arguments work for the mixed boundary cases as well.

□

Next, we state a monotonicity result for the first eigenfunction of (P) on concentric annular regions.

**Proposition 2.2.** Let \( \Omega = B_\beta \setminus \overline{B_\alpha} \), for some \( 0 < \alpha < \beta \), where \( B_\alpha \) and \( B_\beta \) are open balls centered at the origin of radius \( \alpha \) and \( \beta \), respectively. Suppose that \( q \in [1,p] \). Let \( u_q > 0 \) be an eigenfunction of (P) associated to \( \tau_{1,q}(\Omega) \). Then \( u_q \) is a radial function. Furthermore, the following holds:

(i) (Outer Dirichlet) \( u_q \) is strictly radially decreasing if \( \Gamma_D = \partial B_\beta \),
(ii) (inner Dirichlet) \( u_q \) is strictly radially increasing if \( \Gamma_D = \partial B_\alpha \).

**Proof.** By Proposition 2.1, \( \tau_{1,q} \) is simple, and hence radiality of \( u_q \) follows immediately.

(i) We adopt the ideas of proof of [1, Proposition A.4] here.

Let \( \Gamma_D = \partial B_\beta \). As \( u_q \) is radial, \( u_q(x) = f_q(|x|) \), for some function \( f_q : \mathbb{R} \rightarrow \mathbb{R} \). Therefore \( f_q \) satisfies the following ordinary differential equation associated to (P):

\[
- \left( |f_q'(r)|^{p-2} f_q'(r) r^{n-1} \right)' = \tau_{1,q}(\Omega) r^{n-1} f_q(r)^{q-1} \text{ in } (\alpha, \beta), \\
f_q'(\alpha) = 0; f_q(\beta) = 0.
\]

Since \( f_q(r) > 0 \) in \( (\alpha, \beta) \), we get \( \left( |f_q'(r)|^{p-2} f_q'(r) r^{n-1} \right)' < 0 \) in \( (\alpha, \beta) \). Therefore, \( |f_q'(r)|^{p-2} f_q'(r) r^{n-1} \) is strictly decreasing and hence by the boundary condition at \( \alpha \), we get \( f_q' < 0 \) as required.

(ii) Proof follows along the same line as in (i).

□

**Remark 2.3.** For \( q > p \), a first eigenfunction \( u_q \) of (P) need not be radial on an arbitrary annular region \( B_\beta \setminus \overline{B_\alpha} \), see [26, Proposition 1.2] for such symmetry-breaking phenomena. Thus the assumption \( q \in [1,p] \) in Proposition 2.2 can not be dropped.

### 3. Nagy’s Type Inequality in Higher Dimensions

In this section, we prove the Sz. Nagy’s type inequality for both outer parallel sets \( \partial \Omega_i \) and inner parallel sets \( \partial \Omega_{-i} \) for a convex domain \( \Omega \) in \( \mathbb{R}^n \). The following Alexandrov-Fenchel inequalities for quermassintegrals (cf. [32, Section 6.4, (6.4.7)]) of a convex domain in \( \mathbb{R}^n \) plays a vital role in our proofs.

**Proposition 3.1** (Alexandrov-Fenchel inequality). Let \( \Omega \subset \mathbb{R}^n \) be a bounded convex domain. Then,

\[
\left( \frac{W_i(\Omega)}{\omega_n} \right)^{\frac{1}{n-i}} \geq \left( \frac{W_j(\Omega)}{\omega_n} \right)^{\frac{1}{n-j}}, \quad \text{for } 0 \leq i < j < n,
\]

and the equality holds for some \( i \) and \( j \), if and only if \( \Omega \) is a ball.

Observe that, for \( i = 0 \), and \( j = 1 \), (19) gives the following classical isoperimetric inequality [20, Chapter 2]:

\[
P(\Omega) \geq n \omega_{n-1} |\Omega|^{\frac{1}{n-1}}.
\]

Let \( r_\Omega \) be the inradius of \( \Omega \), i.e., \( r_\Omega \) is the supremum of the radii of all spheres that are contained in \( \Omega \). We need the following lemma (cf. [4, Lemma 3.1]) to prove Proposition 1.1.
Lemma 3.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain. Then
\[ -\frac{d}{dt}P(\Omega_{-t}) \geq n(n-1)W_2(\Omega_{-t}), \text{ for a.e. } t \in (0, r_\Omega). \]
If $\Omega$ is a ball, then equality holds for every $t \in (0, r_\Omega)$.

Now we give a proof of Sz. Nagy’s type inequality for inner parallels of a convex set in any space dimensions.

Proof of Proposition 1.1. (i) By Lemma 3.2 and using (19) with $j = 2$ and $i = 1$, we get
\[ -\frac{d}{dt}P(\Omega_{-t}) \geq n(n-1)W_2(\Omega_{-t}) \geq n^{\frac{1}{n-1}}(n-1)\omega_n^{\frac{1}{n-1}} P(\Omega_{-t})^{\frac{n-2}{n-1}} \text{ for a.e. } t \in (0, r_\Omega). \]  
(20)
Thus
\[ -\frac{d}{dt}P(\Omega_{-t}) \geq n^{\frac{1}{n-1}}(n-1)\omega_n^{\frac{1}{n-1}}, \text{ for } t \in (0, r_\Omega) \text{ for a.e. } t \in (0, r_\Omega). \]
For the ball $\Omega^\#$, the same computations as above yields
\[ -\frac{d}{dt}P(\Omega_{-t}) = n^{\frac{1}{n-1}}(n-1)\omega_n^{\frac{1}{n-1}}, \text{ for every } t \in (0, r_\Omega). \]
Therefore,
\[ -\frac{d}{dt}P(\Omega_{-t}) \geq \frac{d}{dt}P(\Omega_{-t}^\#) = n^{\frac{1}{n-1}}(n-1)\omega_n^{\frac{1}{n-1}}, \text{ for a.e. } t \in (0, r_\Omega). \]
Now for $\delta \in (0, r_\Omega)$, we integrate the above inequality from 0 to $\delta$ to get
\[ P(\Omega)^{\frac{1}{n-1}} - P(\Omega_{-\delta})^{\frac{1}{n-1}} \geq P(\Omega^\#)^{\frac{1}{n-1}} - P(\Omega_{-\delta}^\#)^{\frac{1}{n-1}}, \text{ for } \delta \in (0, r_\Omega). \]
Since $P(\Omega) = P(\Omega^\#)$ and $\delta$ is arbitrary, we obtain
\[ P(\Omega_{-\delta}) \leq P(\Omega_{-\delta}^\#), \text{ for all } \delta \in (0, r_\Omega). \]  
(21)
(ii) If $\Omega$ is a ball, then $\Omega = \Omega^\#$ (upto translation) and hence equality holds in (21). Conversely, if $P(\Omega_{-\delta}) = P(\Omega_{-\delta}^\#)$, for some $\delta \in (0, r_\Omega)$, then by retracing the calculations that yield (21), we conclude that equality must happen in (20) for a.e. $t \in (0, \delta)$. In particular, for some $t \in (0, \delta)$, we have
\[ n(n-1)W_2(\Omega_{-t}) = n^{\frac{1}{n-1}}(n-1)\omega_n^{\frac{1}{n-1}} P(\Omega_{-t})^{\frac{n-2}{n-1}}. \]
This yields
\[ \left( \frac{W_2(\Omega_{-t})}{\omega_n} \right)^{\frac{n-2}{n-1}} = \left( \frac{W_1(\Omega_{-t})}{\omega_n} \right)^{\frac{1}{n-1}}. \]
Therefore, by Proposition 3.1 (for $j = 2$ and $i = 1$), $\Omega_{-t}$ must be a ball. Now $\Omega = (\Omega_{-t})_t$, and hence $\Omega$ must be an open ball. This completes the proof.

Next, we give a proof of Theorem 1.4.

Proof of Theorem 1.4. Let $R$ be the radius of $\Omega^\#$. Since $P(\Omega) = P(\Omega^\#)$, we have $R = \left( \frac{P(\Omega)}{n\omega_n} \right)^{\frac{1}{n-1}}$ and $W_1(\Omega) = W_1(\Omega^\#)$. Observe that
\[ W_j(\Omega^\#) = \omega_n^{\frac{j}{n-j+1}} \left( W_{j-1}(\Omega^\#) \right)^{\frac{n-j+1}{n-j+1}} \]
Now, by (19) and the above identity, we get
\[ W_2(\Omega) \geq \omega_n^{\frac{1}{n-1}} (W_1(\Omega))^{\frac{2}{n-1}} = W_2(\Omega^\#). \tag{22} \]
Proceeding in this way, we get
\[ W_j(\Omega) \geq \omega_n^{\frac{1}{j-1}} W_{j-1}(\Omega)^{\frac{n-j+1}{n-j+1}} \geq \omega_n^{\frac{1}{j-1}} (W_{j-1}(\Omega^\#))^{\frac{n-j+1}{n-j+1}} = W_j(\Omega^\#), \quad \text{for } 2 \leq j < n. \]
Thus
\[ W_j(\Omega) \geq W_j(\Omega^\#), \quad \text{for } 1 \leq j < n. \tag{23} \]

Now by Steiner formula (13), we get
\[
P(\Omega_\delta) = n \sum_{i=0}^{n-1} \binom{n-1}{i} W_{i+1}(\Omega)\delta^i \geq n \sum_{i=0}^{n-1} \binom{n-1}{i} W_{i+1}(\Omega^\#)\delta^i = P(\Omega^\#_\delta).
\]
Conversely, if \( \Omega \) is not a ball, then by Proposition 3.1, strict inequality occurs in (23) for all \( 2 \leq j < n \). Thus by (13), \( P(\Omega_\delta) > P(\Omega^\#_\delta) \).

First, we prove the following proposition.

**Proposition 3.3.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded, convex domain and \( \Omega^\# \) as defined in (15). Then
\[ W_i(\Omega) \leq W_i(\Omega^\#), \quad \text{for all } 0 \leq i < n - 1. \]
Moreover, equality holds if and only if \( \Omega \) is a ball.

**Proof.** Since \( W_{n-1}(\Omega) = W_{n-1}(\Omega^\#) \), by taking \( j = n - 1 \) in (19), for \( i = 0, 1, \ldots, n - 2 \), we get
\[
\left( \frac{W_i(\Omega)}{\omega_n} \right)^{\frac{1}{n-i}} \leq \frac{W_{n-1}(\Omega)}{\omega_n} = \frac{W_{n-1}(\Omega^\#)}{\omega_n} = \left( \frac{W_i(\Omega^\#)}{\omega_n} \right)^{\frac{1}{n-i}}.
\]
Therefore
\[ W_i(\Omega) \leq W_i(\Omega^\#), \quad \text{for } 0 \leq i < n - 1, \]
and equality occurs if and only if \( \Omega \) is a ball. This completes the proof. \( \square \)

Now, as a consequence of the above Proposition, we get the following result that gives a Sz. Nagy’s type inequality for the outer parallel sets of a convex domain in higher dimensions.

**Corollary 3.4.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded, convex domain and \( \Omega^\# \) as defined in (15). Then
\[ P(\Omega_\delta) \leq P(\Omega^\#_\delta), \quad \text{for every } \delta > 0. \]
Furthermore, if \( n \geq 3 \), then the equality holds in the above inequality if and only if \( \Omega \) is a ball.

**Proof.** Using Steiner formula (13) and Proposition 3.3, we have
\[
P(\Omega_\delta) = n \sum_{i=0}^{n-1} \binom{n-1}{i} W_{i+1}(\Omega)\delta^i \leq n \sum_{i=0}^{n-1} \binom{n-1}{i} W_{i+1}(\Omega^\#)\delta^i = P(\Omega^\#_\delta).
\]
If \( P(\Omega_\delta) = P(\Omega^\#_\delta) \), then \( W_i(\Omega) = W_i(\Omega^\#) \), for all \( 1 \leq i \leq n - 1 \). Thus by Proposition 3.3, \( \Omega \) must be a ball. \( \square \)

**Remark 3.5.** Notice that, if \( n = 2 \) and \( \Omega \) is convex, then by Steiner formula (see [33] or [14, Theorem 10.1]), we have \( P(\Omega_\delta) = P(\Omega^\#_\delta) \) for every \( \delta > 0 \).
4. Outer Dirichlet problem

In this section, we prove Theorem 1.2. Let \( \Omega \) be as stated in (4), \( \Omega_D = \Omega_{\text{out}} \) and \( A_O(\Omega) = B_R \setminus \overline{B_r} \) be the annulus having the same volume as \( \Omega \) such that \( P(\Omega_D) = P(B_R) \).

Define
\[
\begin{align*}
t^* &= \sup\{t > 0 : P(\Omega_{D-t} \cap \Omega) > 0\}, \\
s(\delta) &= P(\Omega_{D-\delta} \cap \Omega), \text{ for } \delta \in (0, t^*] \text{ and } S(\delta) = P(B_{R-\delta}), \text{ for } \delta \in (0, R - r].
\end{align*}
\]

**Remark 4.1.** Using Proposition 1.1, we obtain
\[
s(\delta) \leq P(\Omega_{D-\delta}) \leq P(B_{R-\delta}) = S(\delta), \text{ for all } \delta \in [0, R - r]. \tag{24}
\]

In [1, Lemma 2.3], the above inequality follows easily from the fact that \( \Omega_D \) is a ball.

Let \( v(\delta) \) be the volume of the portion of \( \Omega \) lying between \( \partial \Omega_{D-\delta} \) and \( \partial \Omega_D \), and \( V(\delta) \) be the volume of the portion of \( A_O(\Omega) \) lying between \( \partial B_{R-\delta} \) and \( \partial B_R \). Then
\[
v(\delta) = \int_0^\delta s(t) dt, \text{ where } \delta \in [0, t^*], \quad V(\delta) = \int_0^\delta S(t) dt, \text{ where } \delta \in [0, R - r]. \tag{25}
\]

Observe that, both \( v \) and \( V \) are strictly increasing, and
\[
v(\delta) \leq V(\delta), \forall \delta \in [0, R - r], \quad v(t^*) = |\Omega| = |A_O(\Omega)| = V(R - r).
\]

For \( \alpha \in [0, |\Omega|] \), define \( h \) and \( H \) as follows:
\[
h(\alpha) = s(v^{-1}(\alpha)), \quad H(\alpha) = S(V^{-1}(\alpha)).
\]

Next, we prove an auxiliary lemma that was proved in [1, Lemma 2.5] under the assumption that \( \Omega_D \) is a ball.

**Lemma 4.2.** Let \( \Omega \) and \( A_O(\Omega) \) be as mentioned in (4) and (A1), respectively, \( \Omega_D \) convex. Then the following holds:

(i) \( t^* \geq R - r \). Furthermore, for \( n \geq 3 \), equality occurs if and only if \( \Omega = A_O(\Omega) \) (up to a translation).

(ii) \( h(\alpha) \leq H(\alpha) \), for all \( \alpha \in [0, |\Omega|] \). Moreover, for \( n \geq 3 \), \( h(|\Omega|) = H(|\Omega|) \) if and only if \( \Omega = A_O(\Omega) \) (up to a translation). Furthermore, if \( \Omega \) is not a concentric annulus, then there exists \( \alpha_0 \in (0, |\Omega|) \) such that \( h(\alpha) < H(\alpha) \), for all \( \alpha \in [\alpha_0, |\Omega|] \).

**Proof.** (i) If possible, let \( t^* < R - r \). Since \( s(\delta) \leq S(\delta) \) (by (24)), we obtain
\[
|A_O(\Omega)| = \int_0^{R-r} S(\delta) d\delta \geq \int_0^{t^*} s(\delta) d\delta + \int_{t^*}^{R-r} S(\delta) d\delta = |\Omega| + \int_{t^*}^{R-r} S(\delta) d\delta > |\Omega|,
\]
a contradiction to the fact that \( |\Omega| = |A_O(\Omega)| \). Thus \( t^* \geq R - r \). If \( t^* = R - r \), then
\[
\int_0^{R-r} (S(\delta) - s(\delta)) d\delta = |A_O(\Omega)| - |\Omega| = 0.
\]

Since \( s \) and \( S \) are continuous, and \( s(\delta) \leq S(\delta) \), we deduce that \( s(\delta) = S(\delta) \), for all \( \delta \in [0, R - r] \). Thus by (24), we get \( P(\Omega_{D-\delta}) = P(B_{R-\delta}) \). Therefore, using Proposition 1.1, we conclude \( \Omega_D = B_R \) (up to a translation). Since \( t^* = R - r \), by the definition of \( t^* \), we must have \( P(B_{R-t} \cap \Omega) = 0 \), for all \( t > R - r \), i.e., \( P(B_s \cap (B_R \setminus \Omega_{in})) = 0 \), for all \( s < r \). Thus
\[
|B_r \setminus \Omega_{in}| = 0.
\]
Notice that,

\[ |B_R \setminus B_r| = |B_R \setminus (\Omega_{in} \cup B_r)| + |\Omega_{in} \setminus B_r|, \]

and

\[ |B_R \setminus \Omega_{in}| = |B_R \setminus (\Omega_{in} \cup B_r)| + |B_r \setminus \Omega_{in}|. \]

Since \(|B_R \setminus B_r| = |B_R \setminus \Omega_{in}|\) and \(|B_r \setminus \Omega_{in}| = 0\), we get \(|\Omega_{in} \setminus B_r| = 0\). Since \(\Omega_{in}\) is Lipschitz (as \(\Omega\) is), we can prove that either \(\Omega_{in} = B_r\) or \(|(\Omega_{in} \setminus B_r) \cup (B_r \setminus \Omega_{in})| > 0\). As \(|(\Omega_{in} \setminus B_r) \cup (B_r \setminus \Omega_{in})| = 0\), we conclude that \(\Omega = A_O(\Omega)\) (up to a translation).

(ii) Since \(v^{-1}(\alpha) \geq V^{-1}(\alpha)\) and \(S\) is a decreasing function, we obtain

\[ h(\alpha) = s(v^{-1}(\alpha)) \leq S(v^{-1}(\alpha)) \leq S(V^{-1}(\alpha)) = H(\alpha). \]

If \(h(|\Omega|) = H(|\Omega|)\), then \(s(t^*) = S(R-r)\). Thus we must have \(t^* = R-r\). Otherwise, if \(t^* > R-r\), then

\[ h(|\Omega|) = s(t^*) \leq S(t^*) < S(R-r) = H(|\Omega|), \]

a contradiction. Therefore the equality case follows from (i). Now if \(\Omega\) is not a concentric annulus, then \(t^* > R-r\). Set \(\delta_0 = R-r\). Clearly, \(v(\delta_0) < V(\delta_0)\). If not, then

\[ |A_O(\Omega)| = V(R-r) = v(R-r) < v(t^*) = |\Omega|, \]

a contradiction. Now from the definition of \(v\) and \(V\), we have

\[ v(\delta) = v(\delta_0) + \int_{\delta_0}^{\delta} s(t)dt < V(\delta) \quad \text{for all } \delta \in [\delta_0, t^*]. \]

Therefore taking \(\alpha_0 = V(\delta_0)\) and using the above inequality, we obtain \(v^{-1}(\alpha) > V^{-1}(\alpha)\) for all \(\alpha \in [\alpha_0, |\Omega|]\). Since \(S\) is strictly decreasing, we get

\[ h(\alpha) = s(v^{-1}(\alpha)) \leq S(v^{-1}(\alpha)) < S(V^{-1}(\alpha)) = H(\alpha) \quad \text{for all } \alpha \in [\alpha_0, |\Omega|]. \]

Now we give a proof of Theorem 1.2. Our proof for the inequality part is similar to the proof of Theorem 1.1 in [1] except at a few lines. For the sake of completeness, we supply a proof.

**Proof of Theorem 1.2.** For \(q \in [1, p]\), let \(w_q\) be a positive eigenfunction associated to \(\tau_{1,q}(A_O(\Omega))\). Now by Proposition 2.2, \(w_q\) is a radial function in \(A_O(\Omega)\). Also \(w_q \in C^{1,\gamma}(A_O(\Omega))\), for some \(0 < \gamma < 1\) (using Proposition 2.1-(ii)). Therefore, we can choose \(\phi_q \in C^1(\mathbb{R})\) with \(\phi_q(0) = 0\) such that \(w_q\) can be represented as \(w_q(x) = \phi_q(|x|)\), for all \(x \in A_O(\Omega)\). Set \(\psi_q = \phi_q \circ V^{-1}\), where \(V\) is as defined in (25). Notice that \(\psi_q(0) = 0\). Let \(\rho_1\) and \(\rho_2\) be the distance functions from \(\partial B_R\) and \(\Gamma_D\), respectively, defined as follows:

\[ \rho_1(x) = d(x, \partial B_R), \quad \rho_2(x) = d(x, \Gamma_D) \quad \text{for all } x \in \mathbb{R}^n. \]

Therefore \(w_q(x) = \psi_q(V(\rho_1(x)))\) for all \(x \in A_O(\Omega)\). Now we define a test function \(u_q\) on \(\Omega\) in the following way:

\[ u_q(x) = \psi_q(v(\rho_2(x))), \quad \text{for all } x \in \Omega, \]

where \(v\) is given by (25). Then \(u_q \in W^{1,p}(\Omega)\) and \(u_q\) vanishes on \(\Gamma_D\). Using the fact that \(|\nabla \rho_2(x)| = 1\), (cf. [11, Theorem 3.14]) and by the Coarea formula (cf. [10, Appendix C, Theorem 5]), we get

\[ \int_{\Omega} |\nabla u_q(x)|^p dx = \int_0^{t^*} \int_{(\partial \Omega_{\delta_x}) \cap \Omega} |\nabla u_q(x)|^p d\sigma d\delta. \]
Now make a change of variable $\alpha = v(\delta)$ in the above expression to get
\[
\int_{\Omega} |\nabla u_q(x)|^p dx = \int_0^{\|\cdot\|_{\theta}} |\psi_q'(\alpha)|^p h(\alpha)^p d\alpha.
\] (26)

In a similar manner, we obtain
\[
\int_{A_\theta(\Omega)} |\nabla w_q(x)|^p dx = \int_0^{\|\cdot\|_{\theta}} |\psi_q'(\alpha)|^p H(\alpha)^p d\alpha.
\] (27)

Therefore, Lemma 4.2-(ii), along with (26) and (27), yields
\[
\int_{\Omega} |\nabla u_q(x)|^p dx \leq \int_{A_\theta(\Omega)} |\nabla w_q(x)|^p dx.
\] (28)

Now by performing similar computations, we get
\[
\int_{\Omega} |u_q(x)|^q dx = \int_0^{\|\cdot\|_{\theta}} |\psi_q(\alpha)|^q d\alpha = \int_0^{\|\cdot\|_{\theta}} |\psi_q(\alpha)|^q d\alpha = \int_{A_\theta(\Omega)} |\psi_q(x)|^q dx.
\] (29)

Now using (28) and (29) in the variational characterization (5) of $\tau_{1,q}$, we conclude that
\[
\tau_{1,q}(\Omega) \leq \tau_{1,q}(A_\theta(\Omega)).
\]

If $\tau_{1,q}(\Omega) = \tau_{1,q}(A_\theta(\Omega))$, then we have the equality in (28) and hence from (26) and (27), we conclude that $h(\alpha) = H(\alpha)$ for all $\alpha \in [0, |\Omega|]$. Thus by Lemma 4.2-(ii), $\Omega = A_\theta(\Omega)$ (up to a translation). This completes the proof. \qed

5. Inner Dirichlet Problem

In this section, we prove Theorem 1.6. Let $\Omega$ be as stated in (4) with $\Omega_D = \Omega_{in}$ and $A_\theta(\Omega) = B_R \setminus \overline{B_r}$ is the concentric annulus with the same volume as $\Omega$ and $W_{n-1}(\Omega_D) = W_{n-1}(B_r)$.

Let us define
\[
\delta_e = \sup\{\delta > 0 : P(\Omega_{D\delta} \cap \Omega) > 0\}, \quad s(\delta) = P(\Omega_{D\delta} \cap \Omega), \quad \text{for } \delta \in (0, \delta_*)
\]
and $S(\delta) = P(B_{r_{\delta}})$, for $\delta \in (0, R - r]$.

**Remark 5.1.** By applying Corollary 3.4, we have
\[
s(\delta) \leq P(\Omega_{D\delta}) \leq P(B_{r_{\delta}}) = S(\delta), \quad \text{for all } \delta \in [0, R - r].
\] (30)

However, in [1, Section 2.2], authors obtained the above inequality easily by assuming that $\Omega_D$ is a ball.

For $p \in (1, \infty)$, consider the parametrizations $t$ and $T$, similar to Hersch [17] for $p = 2$ and [1] for $p \neq 2$, as below
\[
t(\delta) = \int_0^\delta \frac{1}{s(\rho)^{p-1}} d\rho, \quad T(\delta) = \int_0^\delta \frac{1}{S(\rho)^{p'-1}} d\rho,
\] (31)
where $p' = \frac{p}{p-1}$ is the holder conjugate of $p$. Let $t_* = t(\delta_*)$ and $T_# = T(R - r)$. Define
\[
g(\alpha) = s(t^{-1}(\alpha)), \quad \text{for } \alpha \in [0, t_*) \text{ and } G(\alpha) = S(T^{-1}(\alpha)), \quad \text{for } \alpha \in [0, T_#].
\]
Next, we state a technical lemma without proof that will be used to prove the main results. This lemma was proved in [1, Lemma 2.7] for the case when $\Omega_D$ is a ball. The similar proof can be carried out for $\Omega_D$ convex also with the help of (30).

**Lemma 5.2.** Let $\Omega$ and $\tilde{\Omega}_I(\Omega)$ be as stated in (4) and (16), respectively, with $\Omega_D$ convex. Then we have the following:

(i) $R - r \leq \delta_*$ and $T_# \leq t_*$. Furthermore, for $n \geq 3$, $R - r = \delta_*$ or $T_# = t_*$ if and only if $\Omega = \tilde{\Omega}_I(\Omega)$ (up to a translation).

(ii) $g(\alpha) \leq G(\alpha)$, for all $\alpha \in [0,T_#]$ and for $n \geq 3$, equality happens if and only if $\Omega = \tilde{\Omega}_I(\Omega)$ (up to a translation). Furthermore, if $\Omega$ is not a concentric annulus, then $g(\alpha) < G(\alpha)$ on $(\alpha',T_#]$, for some $\alpha' \in [0,T_#]$.

(iii) $\int_0^{t_*} g(\alpha)^p \ d\alpha = \int_0^{T_#} G(\alpha)^p \ d\alpha = |\Omega|$.

Now we give a proof of Theorem 1.6. We use the ideas of the proof of [1, Theorem 1.2]. For the sake of completeness, we prove it here.

**Proof of Theorem 1.6.** Let $q \in [1,p]$ and $v_q$ be a positive eigenfunction corresponding to $\tau_{1,q}(\tilde{\Omega}_I(\Omega))$. Then $v_q$ is radially symmetric and radially increasing in the concentric annular region $\tilde{\Omega}_I(\Omega)$ (by Proposition 2.2). Let $\rho_1$ and $\rho_2$ be the distance functions from $\partial B_1$ and $\Gamma_D$, respectively, defined as below

$$\rho_1(x) = d(x,\partial B_1), \quad \rho_2(x) = d(x,\Gamma_D) \quad \text{for all } x \in \mathbb{R}^n.$$

Now we can choose a $C^1$ function $\phi_q$ on $\mathbb{R}$ such that $\phi_q(0) = 0$ and $v_q$ can be expressed as follows:

$$v_q(x) = \phi_q(|x| - r) = \phi_q(\rho_1(x)) = (\phi_q \circ T^{-1})(T(\rho_1(x))) = \psi_q(T(\rho_1(x))),$$

where $T$ is as stated in (31) and $\psi_q := \phi_q \circ T^{-1}$. Since $v_q$ is radially increasing in $\tilde{\Omega}_I(\Omega)$, the maximum of $v_q$ will be on the outer boundary $\partial B_R$. Therefore, $\psi_q(T_#) = \max_{[0,T_#]} \psi_q$. Now we define a test function on $\Omega$ in the following way. Define

$$u_q(x) = \begin{cases} \psi_q(t(\rho_2(x))), & \text{if } t(\rho_2(x)) \in [0,T_#], \\ \psi_q(T_#), & \text{if } t(\rho_2(x)) \in (T_#,t_*], \end{cases}$$

where $t$ is as defined in (31). Observe that $u_q \in W^{1,p}(\Omega)$ and $u_q$ vanishes on $\Gamma_D$. Also $\nabla u_q(x) = 0$, if $t(\rho_2(x)) \in (T_#,t_*)$. Therefore, using the fact that $|\nabla \rho_2(x)| = 1$, $\forall x \in \Omega$ and by the Coarea formula, we get

\[
\int_{\Omega} |\nabla u_q(x)|^p \ dx = \int_0^{t_1(T_#)} \int_{(\partial \Omega_D) \cap \Omega} |\nabla u_q(t(\rho_2(x)))|^p \ d\sigma \ d\delta \\
= \int_0^{t_1(T_#)} |\psi_q'(t(\delta))|^p \ / s(\delta)^{p-1} \ d\delta \\
= \int_0^{T_#} |\psi_q'(\alpha)|^p \ d\alpha, \quad \text{by a change of variable } t(\delta) = \alpha.
\]

Hence

\[
\int_{\Omega} |\nabla u_q(x)|^p \ dx = \int_0^{T_#} |\psi_q'(\alpha)|^p \ d\alpha.
\]
Similarly, we have \( \int_{\tilde{A}_I(\Omega)} |\nabla v_q(x)|^p dx = \int_0^{T^\#} |\psi_q(\alpha)|^p d\alpha \). Thus,
\[
\int_\Omega |\nabla u_q(x)|^p dx = \int_{\tilde{A}_I(\Omega)} |\nabla v_q(x)|^p dx.
\] (32)

Also using the definition of \( u_q \), we obtain
\[
\int_\Omega |u_q(x)|^q dx = \int_0^{T^\#} |\psi_q(\alpha)|^q g(\alpha)^{\rho'} d\alpha
= \int_0^{T^\#} |\psi_q(\alpha)|^q g(\alpha)^{\rho'} d\alpha + |\psi_q(T^\#)|^q \int_{T^\#} g(\alpha)^{\rho'} d\alpha
\]
and
\[
\int_{\tilde{A}_I(\Omega)} |v_q(x)|^q dx = \int_0^{T^\#} |\psi_q(\alpha)|^q G(\alpha)^{\rho'} d\alpha.
\]

Therefore, by Lemma 5.2-(ii) and using the fact that \( \psi(T^\#) \geq \psi(\alpha), \forall \alpha \in [0, T^\#] \), we have
\[
\int_{\tilde{A}_I(\Omega)} |v_q(x)|^q dx - \int_\Omega |u_q(x)|^q dx \leq |\psi_q(T^\#)|^q \left\{ \int_0^{T^\#} (G(\alpha)^{\rho'} - g(\alpha)^{\rho'}) d\alpha - \int_{T^\#} g(\alpha)^{\rho'} d\alpha \right\} \leq 0.
\]

Thus
\[
\int_{\tilde{A}_I(\Omega)} |v_q(x)|^q dx \leq \int_\Omega |u_q(x)|^q dx.
\] (33)

Hence the conclusion follows using (32) and (33) in variational characterization (5) of \( \tau_{L,q} \).

The equality case follows using similar arguments as in the proof of Theorem 1.2. This concludes the proof. \( \square \)

6. Some remarks and open problems

**Torsional rigidity**: The arguments used in this article can be adapted to study the torsion problem on multiply-connected domains with mixed boundary conditions. Let \( \Omega, A_\Omega(\Omega) \) and \( \tilde{A}_I(\Omega) \) be as stated in (4), (A1), and (16), respectively. For \( p \in (1, \infty) \), consider the following problem:
\[
\begin{align*}
-\Delta_p u &= 1 & \text{in } \Omega, \\
\quad u &= 0 & \text{on } \Gamma_D, \\
\quad \partial u / \partial \eta &= 0 & \text{on } \partial \Omega \setminus \Gamma_D.
\end{align*}
\] (T)

Now the variational characterization of \( p \)-torsional rigidity \( T(\Omega) \) of (T) is given by
\[
\frac{1}{T(\Omega)} = \inf \left\{ \frac{\int_\Omega |\nabla u|^p dx}{\left( \int_\Omega |u|dx \right)^p} : u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\} \right\}.
\] (34)

Then using a similar test function as in the proof of Theorem 1.2 and Theorem 1.6 respectively, we can establish the following results.

**Theorem 6.1.** Let \( \Omega \) and \( A_\Omega(\Omega) \) be as in Theorem 1.2. Assume that \( \Omega_D = \Omega_{\text{out}} \) and \( \Omega_D \) is convex. Then \( T(A_\Omega(\Omega)) \leq T(\Omega) \) and for \( n \geq 3 \), equality holds if and only if \( \Omega = A_\Omega(\Omega) \) (up to a translation).
Theorem 6.2. Let $\Omega$ and $\tilde{A}_f(\Omega)$ be as in Theorem 1.6. Assume that $\Omega_D = \Omega_m$ and $\Omega_D$ is convex. Then $T(\tilde{A}_f(\Omega)) \leq T(\Omega)$ and for $n \geq 3$, equality holds if and only if $\Omega = \tilde{A}_f(\Omega)$ (up to a translation).

Open problems: Now we state a few open problems related to Sz. Nagy’s inequalities and the reverse Faber-Krahn inequalities discussed here.

(i) Sz. Nagy’s inequality for non-convex domains: We have extended Sz. Nagy’s inequalities (8) and (11) to higher dimensions (Proposition 1.1 and Corollary 3.4, respectively) for convex domains. The analogue of these results for the non-convex domains in higher dimensions are not known.

(ii) Let $p^* = \frac{np}{n-p}$ and $q \in (p, p^*)$. By the compactness of the Sobolev embedding $W^{1,p}_D(\Omega) \hookrightarrow L^q(\Omega)$, $\tau_{1,q}(\Omega)$ is attained for some $u \in W^{1,p}_D(\Omega)$. However, as we pointed out in Remark 2.3, $u$ need not be radial on an arbitrary concentric annular region. Therefore, our method of proof is not applicable for $q \in (p, p^*)$ and the analogue of Theorem 1.2 and 1.6 seems to be a challenging open problem in these cases.

(iii) We give an analogue of Hersch’s result (9) in higher dimensions using the Sz. Nagy’s type inequality (Corollary 3.4) for outer parallel sets with quermassintegral constraint. The extension of (9) to higher dimensions with respect to the perimeter or any other quermassintegral constraint on the Dirichlet boundary is entirely open.

(iv) Though Sz. Nagy’s inequality fails (see Theorem 1.4) with the perimeter constraints, the reverse Faber-Krahn inequality (9) for the inner Dirichlet problem still holds for a certain convex domain in higher dimensions. We provide a numerical example using COMSOLE MULTIPHYSICS (Version 4.3).

Let

$$\Omega_{\text{out}} = \{(x, y, z) \in \mathbb{R}^3 : |x| < 0.5, |y| < 0.75, |z| < 1\},$$

$$\Omega_{\text{in}} = \{(x, y, z) \in \mathbb{R}^3 : |x| < 0.4, |y| < 0.65, |z| < 0.9\}.$$

Suppose $\Omega = \Omega_{\text{out}} \setminus \Omega_{\text{in}}$ and $\Omega_D = \Omega_{\text{in}}$. Let $A_f(\Omega) = B_R \setminus \overline{B_r}$ be the concentric annular region such that $|A_f(\Omega)| = |\Omega|$ and $P(B_r) = P(\Omega_D)$. Then the following reverse Faber-Krahn inequality holds (approximately)

$$\tau_{1,2}(\Omega) \approx 0.23429 < 0.87586 \approx \tau_{1,2}(A_f(\Omega)).$$

However, Sz. Nagy’s inequality for the outer parallel sets of $\Omega_D$ fails with the perimeter constraint (see Theorem 1.4). Thus a different approach for proving the reverse Faber-Krahn inequality that applies to this kind of domain can be explored.

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