JUXTAPOSING $d^*$ AND $\bar{d}$

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Abstract. We take a close look at two versions of Furstenberg’s correspondence principle which establish a link between the combinatorial properties of large sets in an amenable group $G$ with the properties of measure-preserving $G$-actions. Our analysis reveals that an *"ergodic"* version of Furstenberg’s correspondence principle which uses the upper Banach density ($d^*$) is better suited for applications than the version which involves the upper density, $\bar{d}$. For example, we show that an ergodic version of Furstenberg’s correspondence principle allows one to obtain an easy proof of a general version of a result of Hindman, which states that if $E$ is a subset of $\mathbb{N}$ with $d^*(E) > 0$, then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d^*\left(\bigcup_{i=1}^{N}(E - i)\right) > 1 - \varepsilon$ (this result does not hold if one replaces the upper Banach density $d^*$ with the upper density $\bar{d}$). We also establish a new characterization of amenable minimally almost periodic groups.

1. Introduction

Many results in additive combinatorics are of the form: If $E \subseteq \mathbb{N} = \{1, 2, \ldots, N\}$ is a "large" set, then $E$ is "highly organized". For example, the celebrated Szemerédi theorem [Sz] states that if $E$ has positive upper density, $\bar{d}(E) := \limsup_{N \to \infty} \frac{|E \cap \{1, \ldots, N\}|}{N} > 0$, then $E$ is "AP-rich", meaning that $E$ contains arbitrarily long arithmetic progressions. An equivalent form of Szemerédi’s theorem is the following: if $E \subseteq \mathbb{N}$ has positive upper Banach density, i.e. $d^*(E) := \limsup_{N-M \to \infty} \frac{|E \cap \{M, \ldots, N-1\}|}{N-M} > 0$, then $E$ is AP-rich [B1].

The functions $\bar{d}$ and $d^*$ have very similar properties. For example, both $\bar{d}$ and $d^*$ satisfy $d^*(\mathbb{N}) = 1$ and $\bar{d}(\mathbb{N}) = 1$ and are shift-invariant (i.e. $\bar{d}(E - n) = \bar{d}(E)$ for all $n \in \mathbb{N}$ and $d^*(E - n) = d^*(E)$ for all $n \in \mathbb{N}$), which allows one to view $\mathbb{N}$ as a generalized probability space with either $\bar{d}$ or $d^*$ serving as an (admittedly vague) substitute for the probability measure.

Still, in some situations, $d^*$ allows for stronger/sharper results. Consider, for example, the following theorems:

Theorem 1.1. (Furstenberg’s correspondence principle for $\bar{d}$. (cf. [B1], Theorem 1.1)) Let $E \subseteq \mathbb{Z}$ be such that for some sequence of intervals $(I_N)$ given by $I_N = [a_N, b_N]$, where $b_N - a_N \to \infty$, we have $\bar{d}(I_N)(E) := \limsup_{N \to \infty} \frac{|E \cap I_N|}{|I_N|} > 0$. Then, there is an invertible measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) = \bar{d}(I_N)(E) > 0$ satisfying

\[
\bar{d}(I_N)(E \cap (E - h_1) \cap \ldots \cap (E - h_r)) \geq \mu(A \cap T^{-h_1}A \cap \ldots \cap T^{-h_r}A)
\]

1Indeed, one can show that both the $d^*$ and $\bar{d}$ versions of Szemerédi’s Theorem are equivalent to the original "finitistic" version in [Sz]. See also Theorem 1.5 in [B1].

2Observe that if we let $I_N = [1, N]$ we recover the usual notion of upper density, i.e. $\bar{d}([1,N])(E) = \bar{d}(E)$.
for all \( r \in \mathbb{N} \) and \( h_1, \ldots, h_r \in \mathbb{N} \).

All proofs of this result known to the authors are such that one can also establish a version of (1.1) for unions, i.e.
\[
\bar{d}_{(I_N)}(E) \geq \mu(A \cup T^{-h_1}A \cup \cdots \cup T^{-h_r}A)
\]
for all \( r \in \mathbb{N} \) and \( h_1, \ldots, h_r \in \mathbb{N} \). This observation was made and utilized in [BBF].

**Theorem 1.2. (Ergodic Furstenberg’s correspondence principle. (cf. [BHK], Proposition 3.1))** Let \( E \subseteq \mathbb{N} \) be such that \( d^*(E) > 0 \). Then, there is an ergodic measure preserving system \((X, \mathcal{B}, \mu, T)\) and a set \( A \in \mathcal{B} \) with \( \mu(A) = d^*(E) > 0 \) satisfying
\[
d^*(E \cap (E - h_1) \cap \cdots \cap (E - h_r)) \geq \mu(A \cap T^{-h_1}A \cap \cdots \cap T^{-h_r}A)
\]
for all \( r \in \mathbb{N} \) and \( h_1, \ldots, h_r \in \mathbb{N} \).

Similarly to the situation with Theorem 1.1 one can show that a version of (1.3) holds for unions, i.e.
\[
d^*(E \cup (E - h_1) \cup \cdots \cup (E - h_r)) \geq \mu(A \cup T^{-h_1}A \cup \cdots \cup T^{-h_r}A)
\]
for all \( r \in \mathbb{N} \) and \( h_1, \ldots, h_r \in \mathbb{N} \).

Observe that either Theorem 1.1 or Theorem 1.2 allows one to immediately derive Szemerédi’s theorem from Furstenberg’s ergodic Szemerédi theorem (EST), which states that for any probability measure preserving system \((X, \mathcal{B}, \mu, T)\), any \( A \in \mathcal{B} \) with \( \mu(A) > 0 \), and any \( k \in \mathbb{N} \), there exists \( n \in \mathbb{N} \) such that
\[
\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0.
\]

Furstenberg’s approach (see [F2]) to deriving Szemerédi’s theorem from the EST can be described as follows. Let \( I_N = [a_N, b_N] \) be a sequence of intervals such that \( b_N - a_N \to \infty \) for which \( \bar{d}_{(I_N)}(E) > 0 \). Viewing the 0-1 valued sequence \( \xi(m) = 1_E(m), m \in \mathbb{Z} \), as an element of the symbolic space \( \{0, 1\}^\mathbb{Z} \), and denoting by \( T \) the shift transformation \( T x(n) = x(n+1) \) for all \( n \in \mathbb{Z} \), let \( X := \{T^l \xi : l \in \mathbb{Z}\} \) be the orbit closure of \( \xi \) in \( \{0, 1\}^\mathbb{Z} \).

One can show that there exists a \( T \)-invariant Borel measure \( \mu \) on \( X \) with the property that \( \mu(\{x \in X : x(0) = 1\}) > 0 \). Let \( A := \{x \in X : x(0) = 1\} \) and let \( k \in \mathbb{N} \). Then, by the EST, there exists some \( n \in \mathbb{N} \) such that \( \mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0 \). Then, for any \( x \in A \cap T^{-n}A \cap \cdots \cap T^{-kn}A \) we have \( x(0) = 1, x(n) = 1, \ldots, x(kn) = 1 \). Since \( x \in X \), we can choose \( l \in \mathbb{N} \) such that \( T^l \xi \) and \( x \) are as close as we wish. This implies that for some \( l \in \mathbb{N} \)
\[
\mathbb{1}_E(l) = \mathbb{1}_E(l + n) = \cdots = \mathbb{1}_E(l + kn) = 1
\]
and hence \( E \) contains the arithmetic progression \( \{l, l+n, \ldots, l+kn\} \) of length \( k + 1 \).

To establish the existence of the aforementioned measure \( \mu \) on \( X \), Furstenberg uses the "averaging scheme" based on the intervals \((I_N)\) with the property \( \bar{d}_{(I_N)}(E) > 0 \) in order to get a normalized, positive, translation invariant linear functional \( L \) on \( C(X) \) such that \( L(\varphi) > 0 \), where \( \varphi(x) = x(0) \) for all \( x \in X \). (The last property follows from the fact that \( \bar{d}_{(I_N)}(E) > 0 \).

Now, by Riesz’s representation theorem, there is a Borel probability measure \( \mu \) on \( X \) such
that $L(f) = \int_X f \, d\mu$. This gives $\mu(A) = \int_X \varphi \, d\mu = L(\varphi) > 0$. Note that the inequality (1.1) (or (1.3)) is not needed to derive Szemerédi's theorem from EST. However, it is a natural artifact of the proof sketched above. Indeed, one can check that the functional $L$ above satisfies the identity

$$(1.5) \quad L(1_E \cdot 1_{E-n} \cdots 1_{E-kn}) = \int_X 1_A(x) \cdot 1_A(T^n x) \cdots 1_A(T^{kn} x) \, d\mu,$$ (see [F2] p. 210)

and from (1.5) it is not hard to derive (1.1) (Equation (1.5) also implies inequality (1.2) via the inclusion-exclusion principle).

A priori one could expect that, given $E$ with $\bar{d}(E) > 0$, one can judiciously choose the system $(X, B, \mu, T)$ in the above construction to be ergodic. Somewhat surprisingly, this is not always the case. To see this, we will invoke the following interesting result of Hindman:

**Theorem 1.3.** (Hindman's covering theorem [H2]) Let $E \subseteq \mathbb{N}$ be a set with $d^*(E) > 0$. Then, for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$(1.6) \quad d^* \left( \bigcup_{i=1}^N (E - i) \right) > 1 - \varepsilon.$$  

Curiously enough, Theorem [1.3] fails to be true if one replaces $d^*$ with $\bar{d}$. Consider, for example, the following set $E \subseteq \mathbb{N}$ provided by Hindman in [H2]:

$$(1.7) \quad E := \bigcup_{n \in \mathbb{N}} [2^{2n}, 2^{2n+1}).$$

Then $\bar{d}(E) = \frac{2}{3}$, and one can check that, moreover, $\bar{d}(\bigcup_{i=0}^N (E - i)) = \frac{2}{3}$ for all $N \in \mathbb{N}$. It follows that for this set $E$ any measure preserving system $(X, B, \mu, T)$ satisfying (1.2) cannot be ergodic. Otherwise, we would have that for all $N \in \mathbb{N}$, (see (1.2))

$$(1.8) \quad \bar{d} \left( \bigcup_{i=1}^N (E - i) \right) \geq \mu \left( \bigcup_{i=1}^N T^{-i} A \right),$$

where $\mu(A) = \bar{d}(E) > 0$, which would contradict ergodicity of the transformation $T$. Indeed, observe that if the transformation $T$ were ergodic, then we would have $\mu \left( \bigcup_{i \in \mathbb{N}} T^{-i} A \right) = 1$, since $\bigcup_{i \in \mathbb{N}} T^{-i} A$ is $T$-invariant and $\mu(A) > 0$. Continuity of $\mu$ would then contradict (1.3).

To obtain a proof of Theorem [1.2] (see [BHK]), one has to amplify the proof of Theorem [1.1] sketched above. This amplification, already hinted at in [F3], involves two additional tools: the ergodic decomposition, and the fact that for any continuous map $T : X \to X$, where $X$ is a compact metric space, for any $x_0 \in X$, any ergodic measure $\mu$ on the orbital closure $Y := \{ T^n x_0 : n \geq 0 \}$ has the property that $x_0$ is quasi-generic for $\mu$ (see Proposition 3.9 in [F3]). The term quasi-generic will be defined in Section 2 in a more general context.

The discussion above indicates that $d^*$ has, so to say, stronger ergodic properties. For example, one can easily see that Theorem [1.2] implies (via (1.1)) Theorem [1.3]. The ergodic approach to Theorem [1.3] has two additional advantages. First, it will allow us to characterize sequences $(n_k)_{k \in \mathbb{N}}$ with the "Hindman property", i.e., sequences $(n_k)_{k \in \mathbb{N}}$ such that for
any $E \subseteq \mathbb{Z}$ with $d^*(E) > 0$ one has that for all $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$
(1.9) \quad d^* \left( \bigcup_{k=1}^{N} (E - n_k) \right) > 1 - \varepsilon,
$$

(see (1.6)). Second, the robustness of the ergodic approach will allow us to formulate and prove a Hindman-like result for any amenable group. (Note that it is not clear how to adjust the combinatorial proof of Theorem 1.3 in [H2] to more general groups.)

We will also obtain a new characterization of countable amenable WM groups in terms of the "Hindman property". A group $G$ is called weakly mixing or WM if any measure preserving ergodic action of $G$ on a probability space is automatically weakly mixing. For example $A(\mathbb{N})$, the group of finite, even permutations of $\mathbb{N}$ is a countable amenable WM group. It is worth noting that in [BCRZ] a different characterization of WM groups is obtained via Hindman’s "finite sums" theorem [H1].

The structure of the paper is as follows. In Section 2 we give a proof of an enhanced version of Theorem 1.2, which works for any combination of unions, intersections and complements of shifts of $E$ for countable amenable groups. Moreover, we also give a proof of an enhanced version of Theorem 1.1 comparing the two methods of proof, and pinpointing what exactly in the proof allows us to have ergodicity. This is, as we will see, one manifestation of the "ergodic properties of $d^*$".

This enhanced correspondence principle allows us to give a quick proof of a general form of Hindman’s theorem 1.3 for countable cancellative amenable semigroups in Section 3. We also generalize the example 1.7 to countable abelian groups and to finitely generated virtually nilpotent groups. In Section 3 we prove a few more combinatorial results which make use of the "ergodic" nature of $d^*$.

For example, we show Theorem 3.8 which states

**Theorem 1.4.** Let $G$ be a countable amenable group and let $E \subseteq G$ be such that $d^*(E) > 0$. We denote by $A_E$ the algebra of sets generated by shifts of the set $E$, (i.e. sets of the form \( \{ g^{-1}E : g \in G \} \)) with the help of the operations of union, intersection and complement. Then, there exists a Følner sequence $(G_N)_{N \in \mathbb{N}}$ in $G$, such that for all Følner sequences $(F_N)_{N \in \mathbb{N}}$ and for all $E_1, E_2 \in A_E$, we have

$$
(1.10) \quad \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} d_{(G_K)}(E_1 \cap g^{-1}E_2) = d_{(G_K)}(E_1)d_{(G_K)}(E_2).
$$

Moreover, if we let $E_1 = E_2 = E$ in (1.10), we get

$$
(1.11) \quad \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} d_{(G_K)}(E \cap g^{-1}E) = d_{(G_K)}(E)^2 = d^*(E)^2.
$$

Section 4 is devoted to the characterization of sequences $(n_k)$ for which (1.9) holds and is followed by Section 5, which gives a characterization of WM groups via the Hindman

\[3\]Such groups also appear in the literature under the name minimally almost periodic groups.
property. Lastly, in Section 6 we establish a general form of an ergodic Furstenberg correspondence principle for discrete amenable semigroups and derive as a corollary a general form of Hindman's theorem.

**Remark 1.5.** The results discussed in Sections 2 and 3 can be effortlessly carried over to countable cancellative amenable semigroups. Indeed, if $S$ is such a semigroup, then one can embed it into $G := \{st^{-1} : s, t \in S\}$, which is now going to be a countable amenable group (see Proposition 1.17 in [P]). It is straightforward to check that a Følner sequence in $S$ becomes a Følner sequence in $G$, and this is all that is needed to push the results in these two sections to this more general context.

**Remark 1.6.** Throughout this paper, all the measures used are normalized so that $\mu(X) = 1$.

2. **An enhanced Furstenberg correspondence principle: $\bar{d}$ and $d^*$ versions**

The goal of this section is two-fold. First, we will formulate and prove "amenable" versions of Theorems 1.1 and 1.2 (Theorems 2.3 and 2.8) that (i) encompass not only intersections but unions of sets and their complements, and (ii) are valid for general discrete countable amenable groups. Second, we will pinpoint the distinction between $\bar{d}$ and $d^*$ which allows for the stronger, ergodic version of Furstenberg’s correspondence principle (see Theorem 2.8 below).

A definition of amenability which is convenient for our purposes uses the notion of Følner sequence. A sequence $(F_N)$ of finite non-empty subsets of a countable group $G$ is a (left) Følner sequence if

$$\lim_{N \to \infty} \frac{|F_N \triangle gF_N|}{|F_N|} = 0,$$

for all $g \in G$. A countable group $G$ is **amenable** if it admits a (left) Følner sequence.

To facilitate the discussion, we will present first the versions of Theorems 1.1 and 1.2 (see Theorems 2.3 and 2.8) for general countable amenable groups. The proofs are in essence the same as those of Theorems 1.1 and 1.2 but we will need this generality for the applications in the forthcoming sections.

We will then juxtapose the proofs of Theorems 2.3 and 2.8 which will allow us to explain what exactly in the proof of Theorem 2.8 leads to the ergodicity of the system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$.

To formulate Theorems 2.3 and 2.8 we need a few definitions:

**Definition 2.1.** Let $E \subseteq G$ and let $(F_N)$ be a Følner sequence. We denote by $\bar{d}_{(F_N)}(E)$ the upper density of the set $E$ along $(F_N)$. This notion of largeness is given by the formula

$$\bar{d}_{(F_N)}(E) := \limsup_{N \to \infty} \frac{|F_N \cap E|}{|F_N|}.$$
We are now in a position to define upper Banach density in this more general context:

**Definition 2.2.** Let $E \subseteq G$. We denote by $d^*(E)$ the upper Banach density of the set $E$, which is given by

$$d^*(E) := \sup \{ \bar{d}(F_N)(E) : (F_N) \text{ is a Følner sequence} \}.$$  \[5\]

We begin with a short proof (based on the idea of the proof of Theorem 1.1 in [F1] and [F2] (see also [FKO])) of a generalization of Theorem 1.1 for countable amenable groups. Note that Theorem 1.1 corresponds to the special case $G := \mathbb{Z}$ and $F_N := [a_N, b_N]$ with $b_N - a_N \to \infty$.

In what follows we will use the notation $A^1 = A$ and $A^0 = A^c$.

**Theorem 2.3.** [Enhanced Furstenberg Correspondence Principle for countable amenable groups, $\bar{d}(F_N)$ version] Let $(F_N)$ be a Følner sequence in a countable amenable group $G$ and let $E$ be a subset of $G$ with $\bar{d}(F_N)(E) > 0$. Then there exist a probability measure preserving system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ and a set $A \in \mathcal{B}$ with $\mu(A) = \bar{d}(F_N)(E)$ such that

$$(2.1) \quad \mu((T_{g_1})^{-1} A^{w_1} \ast \cdots \ast (T_{g_k})^{-1} A^{w_k}) \leq \bar{d}(F_N)(g_1^{-1} E^{w_1} \ast \cdots \ast g_k^{-1} E^{w_k}),$$

for all $k \in \mathbb{N}$, all $\{g_1, \ldots, g_k\} \subseteq G$ and all $\{w_1, \ldots, w_k\} \in \{0, 1\}^k$, and where each of the stars denotes either union or intersection with the understanding that

(i) for all $1 \leq i \leq k - 1$, the operation represented by $\ast$ which stands between $E^{w_i}$ and $E^{w_{i+1}}$ is the same as the operation appearing between $A^{w_i}$ and $A^{w_{i+1}}$.

(ii) the choices of parentheses which are needed to make the expressions on both sides of formula (2.1) well defined also match. \[6\]

**Proof.** Let $X = \{0, 1\}^G$ (viewed as a compact metric space with the usual product topology). Let $(T_g)_{g \in G}$ be the action of $G$ on $X$ by homeomorphisms defined by the formula $(T_g x)_{g_0} = x_{g g_0}$ for all $g, g_0 \in G$. Define $\omega \in X$ by setting $\omega(g) = 1$ if $g \in E$ and $\omega(g) = 0$ otherwise. Put $A = \{x \in X : x(e) = 1\}$ (Here and elsewhere, $e$ denotes the neutral element of the group $G$). Note that $A$ is a clopen set in $X$ (and hence $1_A$ is a continuous function). Moreover, we have that $T_g \omega \in A$ if and only if $g \in E$. Let $(F_{N_k})$ be a subsequence such that

$$\bar{d}(F_N)(E) = \lim_{k \to \infty} \frac{|E \cap F_{N_k}|}{|F_{N_k}|}.$$  \[7\]

Let $\mu$ be any weak* limit point of the sequence of measures

$$\frac{1}{|F_{N_k}|} \sum_{g \in F_{N_k}} \delta_{T_g \omega}.$$  \[7\]
Moreover, since $C(Y)$ is separable, there exists a further subsequence of $(F_{N_k})$, which we will, in order not to overload the notation, still denote by $(F_{N_k})$, such that, in the weak* topology, $\mu = \lim_{k \to \infty} \frac{1}{|F_{N_k}|} \sum_{g \in F_{N_k}} \delta_{T_g \omega}$. Clearly, $\mu$ is a $G$-invariant probability measure on $X$. We claim that $\mu(A) = \tilde{d}(F_N)(E)$. Indeed, since $1_A$ is a continuous function, we can write

$$
\mu(A) = \int_X 1_A \, d\mu = \lim_{k \to \infty} \frac{1}{|F_{N_k}|} \sum_{g \in F_{N_k}} 1_A(T_g \omega) = d(F_{N_k})(E) = \tilde{d}(F_N)(E). 
$$

Now let $g_1, \ldots, g_k \in G$ and consider the set $(T_{g_1})^{-1}A^{w_1} \ast \ldots \ast (T_{g_k})^{-1}A^{w_k}$, where the stars denote an arbitrary fixed choice of either union or intersection. This is a clopen set in $X$, so its indicator function is, again, continuous. We have

$$
\mu \left( (T_{g_1})^{-1}A^{w_1} \ast \ldots \ast (T_{g_k})^{-1}A^{w_k} \right) = \lim_{k \to \infty} \frac{1}{|F_{N_k}|} \sum_{g \in F_{N_k}} 1_{(T_{g_1})^{-1}A^{w_1} \ast \ldots \ast (T_{g_k})^{-1}A^{w_k}}(T_g \omega) = \lim_{k \to \infty} \frac{1}{|F_{N_k}|} |(g_1^{-1}E^{w_1} \ast \ldots \ast g_k^{-1}E^{w_k}) \cap F_{N_k}| \leq \tilde{d}(F_N)(g_1^{-1}E^{w_1} \ast \ldots \ast g_k^{-1}E^{w_k}).
$$

We are done. \hfill \square

**Remark 2.4.** Observe that when $G := \mathbb{Z}$ and $F_N := [a_N, b_N]$ (where $b_N - a_N \to \infty$) inequality (2.1) implies the inequalities (1.1) and (1.2). As we saw in the Introduction with the help of Theorem 1.3, for some sets $E \subseteq \mathbb{Z}$ no system $(X, \mathcal{B}, \mu, T)$ satisfying inequality (1.2) can be ergodic. The enhanced version of Theorem 1.1 which also involves complements of sets, allows us to arrive at the same conclusion without invoking Theorem 1.3. Indeed, we see that (for $G := \mathbb{Z}$ and $F_N := [1, N]$) inequality (2.1) implies

$$
\tilde{d}(E^c \cap (E - h)) \geq \mu(A^c \cap T^{-h}A)
$$

for all $h \in \mathbb{Z}$. Take $E$ as in the Introduction (see (1.7)). Then, $\mu(A) = \tilde{d}(E) = \frac{2}{3}$, so $\mu(A^c) > 0$. However, one can easily check (see also Section 3), that $\tilde{d}(E^c \cap (E - h)) = 0$ for all $h \in \mathbb{Z}$. This contradicts inequality (2.3).

**Remark 2.5.** Let $G$ be a countable amenable group. Call a set $E$ a Hindman set if there exists some Følner sequence $(F_N)$ such that

$$
0 < \tilde{d}(F_N)(E) < 1 \text{ and } \tilde{d}(F_N) \left( \bigcup_{g \in F} g^{-1}E \right) < \frac{3}{4}
$$

for all finite sets $F \subseteq G$. One can show (see Proposition 3.3) that any countable abelian group has a Hindman set.

It is interesting to observe that if our countable amenable group $G$ admits a Hindman set, then any measure preserving system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ satisfying inequality (2.1) cannot be ergodic. This can be seen in two ways. First, using the special case of inequality (2.3) for unions, i.e.

$$
\tilde{d}(F_N) \left( \bigcup_{g \in F} g^{-1}E \right) \geq \mu \left( \bigcup_{g \in F} g^{-1}A \right) \text{ for all finite sets } F \subseteq G,
$$

where
and arguing, as in the Introduction, that inequality (2.4) together with the fact that
\[ d(F_N) \left( \bigcup_{g \in F} g^{-1}E \right) = d(F_N)(E) < 1 \]
contradict ergodicity.

Alternatively, we can use another special case of inequality (2.3), namely:
\[ d(F_N)(E^c \cap g^{-1}E) \geq \mu(A^c \cap g^{-1}A) \]
for all \( g \in G \), together with the fact that if \( E \) is a Hindman set then \( d(F_N)(E^c \cap g^{-1}E) = 0 \) for all \( g \in G \).
(This discussion will be completed in Section 3).

We move now to an ergodic version of Theorem 1.2 for general countable amenable groups.

**Definition 2.6.** Let \( X \) be a compact metric space on which \( G \) acts by homeomorphisms. Let \( \mu \) be a \( G \)-invariant measure. We say that \( x_0 \in X \) is quasi-generic for \( \mu \) if there exists a Følner sequence \( (F_N) \) such that
\[ \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} f(T_g x_0) = \int_X f \, d\mu \]
for every \( f \in C(X) \).

The following Proposition is an amenable version of Proposition 3.9 in \[F3\]. We include the proof for the reader’s convenience.

**Proposition 2.7.** Let \( (T_g)_{g \in G} \) be an action of \( G \) by homeomorphisms on a compact metric space \( X \). Let \( x_0 \in X \), and let \( Y := \{ T_g x_0 : g \in G \} \). Suppose that \( \mu \in \mathcal{M}(Y) \) is an ergodic \( G \)-invariant measure. Then \( x_0 \) is quasi-generic for \( \mu \).

**Proof.** Since \( \mu \) is an ergodic measure, it follows by the mean ergodic theorem, that for any Følner sequence \( (F_N) \), and any \( f \in L^2(\mu) \)
\[ \lim_{N \to \infty} \frac{1}{|F_{N_k}|} \sum_{g \in F_{N_k}} f(T_g x) = \int_Y f \, d\mu, \]
where the convergence is in the \( L^2(\mu) \)-norm. Thus, there exists some subsequence \( (F_{N_k}) \) along which we have pointwise convergence for all \( f \) in a countable dense subset of \( C(Y) \), which in turn, by a simple triangle inequality argument, implies that, for any \( f \in C(Y) \) we have
\[ \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} f(T_g x) = \int_X f \, d\mu, \]
for a.e. \( x \in Y \) (and so a.e. \( x \in Y \) is quasi-generic for \( \mu \)).

Let \( x_1 \in Y \) be quasi-generic for \( \mu \) along the Følner sequence \( (F_{N_k}) \). Take a countable dense set \( \{ f_k : k \in \mathbb{N} \} \) in \( C(Y) \) such that
\[ \left| \frac{1}{|F_{N_k}|} \sum_{g \in F_{N_k}} f_j(T_g x_1) - \int_Y f_j \, d\mu \right| < \frac{1}{k}. \]
for all \(k \in \mathbb{N}\) and all \(j = 1, \ldots, k\). Since the functions \(f_j\) are continuous, we can pick \((g_k) \subseteq G\) such that the inequality (2.5) holds if we replace \(x_1\) by \(T_{g_k} x_0\), which after a change of variables in the sum yields

\[
\left| \frac{1}{|F_{N_k}|} \sum_{g \in F_{N_k} g_k} f_j(T_g x_0) - \int_Y f_j \, d\mu \right| < \frac{1}{k},
\]

which implies that

\[
\lim_{k \to \infty} \frac{1}{|F_{N_k} g_k|} \sum_{g \in F_{N_k} g_k} f(T_g x_0) = \int_Y f \, d\mu
\]

for all \(f \in C(Y)\). In other words, \(x_0\) is a quasi-generic point for \(\mu\) with respect to the Følner sequence \((F_{N_k} g_k)\). Indeed, observe that for all \(g \in G\) we have that \(|g F_{N_k} g_k \Delta F_{N_k} g_k| = |g F_{N_k} \Delta F_{N_k}|\), which implies that \((F_{N_k} g_k)\) is still a Følner sequence. \(\square\)

We are now ready to formulate the amenable ergodic Furstenberg’s Correspondence Principle, adapting arguments from both \([BHK]\) and \([BBF]\). We note that Theorem 1.2 corresponds to the special case \(G := \mathbb{Z}\) in Theorem 2.8.

**Theorem 2.8** (Enhanced Ergodic Furstenberg Correspondence Principle for countable amenable groups). Let \(E\) be a subset of a countable amenable group \(G\) with positive upper Banach density. Then there exists an ergodic probability measure preserving system \((X, \mathcal{B}, \mu, (T_g)_{g \in G})\) and a set \(A \in \mathcal{B}\) with \(\mu(A) = d^*(E)\) such that

\[
\mu((T_{g_1})^{-1} A_{w_1} \ast \cdots \ast (T_{g_k})^{-1} A_{w_k}) \leq d^*(g_1^{-1} E_{w_1} \ast \cdots \ast g_k^{-1} E_{w_k})
\]

for all \(k \in \mathbb{N}\), all \(\{g_1, \ldots, g_k\} \subseteq G\) and all \((w_1, \ldots, w_k) \in \{0, 1\}^k\), and where each of the stars denotes either union or intersection with the understanding that

(i) for all \(1 \leq i \leq k - 1\), the operation represented by \(\ast\) which stands between \(E_{w_i}\) and \(E_{w_{i+1}}\) is the same as the operation appearing between \(A_{w_i}\) and \(A_{w_{i+1}}\).

(ii) the choices of parentheses which are needed to make the expressions on both sides of formula (2.1) well defined also match.

**Proof.** We start as in the proof of Theorem 2.3. Let \(X = \{0, 1\}^G\) and \((T_g)_{g \in G}\) be the action of \(G\) on \(X\) by homeomorphisms defined by the formula \((T_g x)_{g_0} = x_{g g_0}\) for all \(g, g_0 \in G\). Define, as before, \(\omega \in X\) by setting \(\omega(g) = 1\) if \(g \in E\) and \(\omega(g) = 0\) otherwise.

Let \(Y = \{T_g \omega : g \in G\}\). Put \(A = \{x \in Y : x(e) = 1\}\). Then, \(A\) is a clopen set in \(Y\). Moreover, we have that \(T_g \omega \in A\) if and only if \(g \in E\). Let \((F_N)\) be a Følner sequence such that \(d^*(E) = d(F_N)(E)\). Let \(\nu\) be any weak* limit point of the sequence of measures

\[
\frac{1}{|F_N|} \sum_{g \in F_N} \delta_{T_g \omega}.
\]

Then, as in Theorem 2.3, \(\nu\) is a \(G\)-invariant probability measure on \(Y\) such that \(\nu(A) = d^*(E)\).

By the ergodic decomposition theorem (see Theorem 4.2 in \([V]\)), there is a probability measure \(\lambda\) on the set of ergodic normalized measures \(\mathcal{M}_G(X)\) such that

\[
\nu(C) = \int_{\mathcal{M}_G(X)} \mu_z(C) \, d\lambda(z)
\]
for all \( C \in \mathcal{B} \). It follows from the equality (2.7) that there exists some \( z \) such that 
\[
\mu_z(A) \geq \nu(A) = d^*(E).
\]

We show that the measure \( \mu_z \), which we will now denote by \( \mu \), works. Let \( g_1, \ldots, g_k \in G \) and observe that the set 
\[(T_{g_1})^{-1}A^{w_1} \ast \cdots \ast (T_{g_k})^{-1}A^{w_k}\]
is a clopen set in \( Y \), so its indicator function is continuous.

By Proposition 2.7 there exists a Følner sequence \( (G_N) \), with respect to which the point \( \omega \) is quasi-generic for the measure \( \mu \). This implies that
\[
\mu((T_{g_1})^{-1}A^{w_1} \ast \cdots \ast (T_{g_k})^{-1}A^{w_k}) = \lim_{N \to \infty} \frac{1}{|G_N|} \sum_{g \in G_N} \mathbb{1}_{T_{g_1}^{-1}A^{w_1} \ast \cdots \ast T_{g_k}^{-1}A^{w_k}}(T_g \omega)
\]
\[
= \lim_{N \to \infty} \frac{1}{|G_N|} \left| \left( g_1^{-1}E^{w_1} \ast \cdots \ast g_k^{-1}E^{w_k} \right) \right| \cap G_N \right| \leq d^*(g_1^{-1}E^{w_1} \ast \cdots \ast g_k^{-1}E^{w_k}).
\]

In particular, letting \( k = 1, w_1 = 1 \) and \( g_1 = e \) in inequality (2.8) we obtain \( \mu(A) \leq d^*(E) \).

Recalling the previous inequality \( \mu(A) \geq \nu(A) = d^*(E) \) we see that \( \mu(A) = d^*(E) \), so we are done.

We conclude this section with some comments on why the utilization of \( d^* \) allows us to achieve in Theorem 2.3 the goal of ergodicity (whereas, as we saw above, there are sets \( E \) for which Theorem 2.3 cannot guarantee it). In both proofs, we start with a Følner sequence (\( F_N \)) which satisfies \( d(F_N)(E) > 0 \) (in Theorem 2.3) or \( d^*(E) = d(F_N)(E) \) (in Theorem 2.8). Then we consider weak* limits of the sequences of measures 
\[
\frac{1}{|F_N|} \sum_{g \in F_N} \delta_{T_g\omega} (\omega = (1_E(g))_{g \in G})
\]
along a subsequence of \( (F_N) \). The advantage of \( d^* \) comes in handy when, after invoking the ergodic decomposition in the proof of Theorem 2.3, we start using shifts \( (F_{N_k}, g_k) \) of a relevant subsequence \( (F_{N_k}) \) in order to use quasi-genericity of \( \omega \), as guaranteed by Proposition 2.7.

Unlike the proof of Theorem 2.3 where we are, so to say, stuck with a given Følner sequence \( (F_N) \) and its subsequences, in the proof of Theorem 2.8 we are conveniently passing to a different Følner sequence without affecting the value of \( d^* \) for expressions of the form \( g_1^{-1}E^{w_1} \ast \cdots \ast g_k^{-1}E^{w_k} \).

3. Hindman’s Theorem via Ergodic Theory and Some Other Consequences of the Ergodic Version of Furstenberg’s Correspondence Principle

In this section we will give a short proof of a natural generalization of Hindman’s theorem to the context of countable cancellative amenable semigroups. We also provide some evidence that Theorem 3.1 is a manifestation of the ergodic properties of the upper Banach density \( d^* \).

We begin with showing that a general version of Hindman’s theorem (see Theorem 1.6 in the introduction) is an immediate corollary of Theorem 2.8.

**Theorem 3.1.** Let \( G \) be a countable cancellative amenable semigroup, and let \( E \) be a subset of \( G \) with \( d^*(E) > 0 \). Then, for every \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) and \( g_1, \ldots, g_k \in G \) such that 
\[
d^*(g_1^{-1}E \cup \cdots \cup g_k^{-1}E) > 1 - \varepsilon.
\]

---

8This property of \( d^* \), when applied to unions, is also behind Hindman’s proof of Theorem 1.3. However, the proof methods in [12], do not seem to easily generalize to amenable groups.
Proof. In light of Remark 1.5 we may and will assume that $G$ is in fact a group. By Theorem 2.8 there exists an ergodic system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ and a set $A \in \mathcal{B}$ such that $\mu(A) = d^*(E)$ and for which
\[ d^*(h_1^{-1}E \cup \cdots \cup h_k^{-1}E) \geq \mu((T_{h_1})^{-1}A \cup \cdots \cup (T_{h_k})^{-1}A) \]
for all $k \in \mathbb{N}$ and $h_1, \ldots, h_k \in G$. By ergodicity of the action $(T_g)_{g \in G}$ we have
\[ \mu \left( \bigcup_{g \in G} (T_g)^{-1}A \right) = 1, \]
and so since $G$ is countable, the result follows. \qed

Another corollary that can be obtained from Theorem 2.8 is an ergodicity-like statement for the group $G$. Namely:

Corollary 3.2. Let $G$ be a countable cancellative amenable semigroup, and let $E \subseteq G$ be such that $d^*(E) \in (0, 1)$. Then there exists some $g \in G$ such that $d^*(E^c \cap g^{-1}E) > 0$.

Proof. Let $E \subseteq G$ be such that $d^*(E) \in (0, 1)$. By Theorem 2.8 we can find an ergodic measure preserving system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ and a set $A \in \mathcal{B}$ such that $\mu(A) = d^*(E)$ and for all $h \in G$,
\[ d^*(E^c \cap h^{-1}E) \geq \mu(A^c \cap (T_h)^{-1}A). \]
Since $\mu(A) > 0$ and $\mu(A^c) > 0$ and the action $(T_g)_{g \in G}$ is ergodic, there is some $g \in G$ such that $\mu(A^c \cap (T_g)^{-1}A) > 0$, so we are done. \qed

We proceed to show that an example similar to 1.7 exists in the context of countable abelian groups.

Proposition 3.3. Let $G$ be a countably infinite abelian group. Let $(F_N) \subseteq G$ be a Følner sequence. Then, there exists a set $E \subseteq G$ such that
\[ (3.1) \quad \bar{d}(F_N)(E) > 0, \quad \text{and} \quad \bar{d}(F_N) \left( \bigcup_{g \in F} g^{-1}E \right) < \frac{3}{4}, \]
for all finite sets $F \subseteq G$.

Proof. We will show first that (3.1) holds for a particular Følner sequence, and then upgrade it to an arbitrary Følner sequence.

Assume first that $G$ is finitely generated and $\{g_1, \ldots, g_k\}$ generate $G$. Then, one of the generators of $G$, say $g_1$, has infinite order. Consider the Følner sequence $G_N := \{g_1^{a_1} \cdots g_k^{a_k} : 0 \leq a_i \leq 2^N \text{ for } 1 \leq i \leq N\}$. Define $A_N = \{g_1^{a_1} \cdots g_k^{a_k} : 2^{2N-1} \leq a_1 \leq 2^N \text{ and } 0 \leq a_i \leq 2^{2N} \text{ otherwise }\}$ and put $E = \bigcup_{N \in \mathbb{N}} A_N$. It is not hard to check that the set $E$ satisfies (3.1).

Assume now that $G$ is infinitely generated, and let $\{g_n : n \in \mathbb{N}\}$ be a set of generators for the group $G$. We distinguish two cases. In the first case, one of the generators, say $g_1$, has infinite order. Consider the Følner sequence $G_N := \{g_1^{a_1} g_2^{a_2} \cdots g_N^{a_N} : 0 \leq a_i \leq 2^N, \text{ for } 1 \leq i \leq N\}$,
and set

\[(3.2)\quad A_N := \{g_1^{a_1}g_2^{a_2}\cdots g_N^{a_N} : 2^N \leq a_1 \leq 2^{2N}, \text{ and } 0 \leq a_i \leq 2^{2N} \text{ for } 2 \leq i \leq N\}.\]

Letting \(E := \bigcup_{N \in \mathbb{N}} A_N\), we get

\[\bar{d}_{(G_N)}(E) = \bar{d}_{(G_N)} \left( \bigcup_{g \in F} g^{-1}E \right) < \frac{3}{4},\]

for all finite sets \(F \subseteq G\).

Now assume that all elements of \(G\) have finite order and that the enumeration of \((g_n)\) is such that \(\text{ord}(g_{n+1}) \geq \text{ord}(g_n)\) for all \(n \in \mathbb{N}\). Consider the Følner sequence

\[G_N := \{g_1^{a_1} \cdots g_{2^{2N}}^{a_{2^{2N}}} : 0 \leq a_i \leq \text{ord}(g_i) \text{ for all } 1 \leq i \leq 2^{2N}\}\]

and the sets

\[A_N := \{g_1^{a_1} \cdots g_{2^{2N}}^{a_{2^{2N}}} : 0 \leq a_i \leq b_i \text{ for all } 1 \leq i \leq 2^{2N}, \text{ with } i \text{ even}; a_i = 0 \text{ with } i \text{ odd}\},\]

where \(b_i\) is chosen so that \(\frac{|\bigcup_{j=1}^{N} A_j \cap G_i|}{|G_i|} \in (\frac{1}{4}, \frac{1}{2})\). Letting \(E := \bigcup_{N \in \mathbb{N}} A_N\), we get that

\[\bar{d}_{(G_N)}(E) \geq \frac{1}{4}, \quad \text{and} \quad \bar{d}_{(G_N)} \left( \bigcup_{g \in F} g^{-1}E \right) < \frac{3}{4},\]

for all finite subsets \(F \subseteq G\).

Let now \((F_N)\) be an arbitrary Følner sequence. We indicate next how to construct the set \(E\) that satisfies (3.1) for \((F_N)\). To do this, we need the following general fact, which follows from Lemma 4.1 in [DHZ] (we thank Professor Tomasz Downarowicz for providing this information).

**Fact 3.4.** Fix a Følner sequence \((F_N)\). Given another Følner sequence \((G_N)\), we can find a Følner sequence \((G'_N)\) that is equivalent to \((G_N)\) (i.e., satisfying \(\frac{|G_N \Delta G'_N|}{|G_N|} \to 0\) as \(N \to \infty\)) such that for each \(N \in \mathbb{N}\), \(G'_N\) is a union of \(\varepsilon_N\)-disjoint subsets of the form \(\{F_N g : g \in G\}\) (i.e., further subsets \(T\) of \(F_N g\) with \(|T| \geq (1 - \varepsilon_N)|F_N|\), where \(\varepsilon_N \to 0\) as \(N \to \infty\)).

It is not hard to see that Fact 3.4 implies that the set \(E\) in question can be constructed with the help of the same argument utilized above for the special Følner sequence \((G_N)\). □

**Remark 3.5.** It is worth noting that the set \(E\) constructed in Proposition 3.3 also satisfies (as in Hindman’s original example)

\[(3.3)\quad \bar{d}_{(F_N)}(E^c \cap g^{-1}E) = 0\]

for all \(g \in G\).

It is of interest to know whether the phenomenon exhibited in Proposition 3.3 takes place in non-commutative amenable groups. We cannot show this in complete generality, but for virtually nilpotent groups, we have the following theorem:
Theorem 3.6. Let $G$ be a finitely generated, countable virtually nilpotent group. Let $(F_N)$ be a Følner sequence. Then, there exists a set $E \subseteq G$ with

$$
\bar{d}_{(F_N)}(E) > 0 \text{ and } \bar{d}_{(F_N)} \left( \bigcup_{g \in B} g^{-1}E \right) \leq \frac{3}{4},
$$

for all finite subsets $B \subseteq G$.

Sketch of the proof. First, by Fact 3.4 it suffices to show the result for a particular Følner sequence $(F_N)$ of our choosing. Let $F := \{x_1, \ldots, x_k\}$ be a set of generators for $G$. Then, the sequence $\left( \bigcup_{j=1}^{n} F^j \right)$ (i.e. words of length at most $n$ generated by $F$) is a Følner sequence of polynomial growth.

Let $F_N := \bigcup_{j=1}^{2^N} F^j$, and let $A_N$ the set of words in $F$ of length between $2^{2N-1}$ and $2^{2N}$. Then the set $E := \bigcup_{N \in \mathbb{N}} A_N$ satisfies (3.4).

We will describe now another interesting application of Theorem 2.8.

Definition 3.7. Let $E \subseteq G$. We denote by $A_E$ the algebra of sets generated by shifts of the set $E$, i.e. sets of the form $\{g^{-1}E : g \in G\}$ with the help of the operations of union, intersection and complementation.

Theorem 3.8. Let $E \subseteq G$ be such that $d^*(E) > 0$. Then, there exists a Følner sequence $(G_N)_{N \in \mathbb{N}}$ in $G$, such that for all Følner sequences $(F_N)_{N \in \mathbb{N}}$, for all $E_1, E_2 \in A_E$, we have

$$
\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} d_{(G_K)}(E_1 \cap g^{-1}E_2) = d_{(G_K)}(E_1) d_{(G_K)}(E_2).
$$

Moreover, if we let $E_1 = E_2 = E$ in (3.5), we get

$$
\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} d_{(G_K)}(E \cap g^{-1}E) = d_{(G_K)}(E)^2 = d^*(E)^2.
$$

Proof. Let $E \subseteq G$ with $d^*(E) > 0$ and $E_1, E_2 \in A_E$. By Theorem 2.8 there exists an ergodic measure preserving system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ and a set $A \in \mathcal{B}$ with $\mu(A) = d^*(E)$ satisfying (2.6). Let $A_1, A_2 \in \mathcal{B}$ be the sets corresponding to the sets $E_1, E_2$. By the proof of Theorem 2.8 the functions $\mathbbm{1}_{A_1}$ and $\mathbbm{1}_{A_2}$ are continuous. Let $(G_K)$ be a Følner sequence such that $\mu = w^* \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} \delta_{T_g \omega}$, where $\omega = (\mathbbm{1}_E(g))_{g \in G}$. Thus, we can write

$$
\mu(A_1 \cap g^{-1}A_2) = \lim_{N \to \infty} \frac{1}{|G_N|} \sum_{g \in G_N} \mathbbm{1}_{A_1 \cap g^{-1}A_2}(T_g \omega) = d_{(G_K)}(E_1 \cap g^{-1}E_2).
$$

Taking an average over $g \in G$ in (3.7) along any left Følner sequence $(F_N)$ in $G$ immediately yields (3.5), given that the action $(T_g)_{g \in G}$ is ergodic. By construction of $\mu$, we have that $\mu(A) = d_{(G_K)}(E) = d^*(E)$ whence (3.6) also follows. \hfill \square

We remark that (3.5) and (3.6) do not hold for arbitrary sequences $(G_N)$: let $E$ be as in (1.7), take $E_1 = E_2 = E$ and put $G_N = [1, 2^{2N}]$. Nonetheless, we have the following Proposition:
Proposition 3.9. Let $E \subseteq G$. Let $(G_N)$ be a Følner sequence in $G$. Then, there exists a Følner subsequence $(G_{N_k})$ such that for all Følner sequences $(F_N)$ and all sets $F \in \mathcal{A}_E$ we have

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} d_{(G_{N_k})}(F \cap g^{-1}F) \geq d_{(G_{N_k})}(F)^2.$$  

If we let $F = E$ in (3.8) we have the following variant of (3.8):

$$\liminf_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} \bar{d}_{(G_N)}(E \cap g^{-1}E) \geq \bar{d}_{(G_N)}(E)^2.$$  

Proof. Given that $G$ is countable, the family of sets $\mathcal{A}_E$ is countable. Thus, via a diagonal procedure, we can take a subsequence $(G_{N_k})$ of our given Følner sequence $(G_N)$ so that

$$\mu := \text{w}-\lim_{k \to \infty} \frac{1}{|G_{N_k}|} \sum_{g \in G_{N_k}} \delta_{T_g \omega},$$

where $\omega = (1_E(g))_{g \in G}$. Letting $A := \{x : x(e) = 1\}$ we see that $F(x) = x(0) = 1_A(x) \in C(X)$, so the function $F_1$ representing the set $E$ is also in $C(X)$. Thus, we can write

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} d_{(G_{N_k})}(F \cap g^{-1}F) = \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} \int X F_1 \cdot T_g F_1 \, d\mu.$$  

Using the ergodic decomposition for $\mu$, we see that the last term in Equation (3.11) can be rewritten as

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} \int X \left( \int X F_1 \cdot T_g F_1 \, d\mu_t \right) d\lambda(t),$$

where each $\mu_t$ is an ergodic measure. Thus, by von Neumann’s Mean Ergodic Theorem and Jensen’s inequality, we have

$$\int \left( \int X F_1 \, d\mu_t \right)^2 d\lambda(t) \geq \left( \int X F_1 \, d\mu \right)^2.$$  

By construction, the right hand side in (3.13) is equal to $d_{(G_{N_k})}(F)^2$. To prove (3.9), we use Theorem 2.3 to obtain a measure preserving system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ and a set $A \in \mathcal{B}$ with $\mu(A) = \bar{d}_{(G_N)}(E)$ satisfying inequality (2.1). In particular, this means that for all $g \in G$ we have

$$\bar{d}_{(G_N)}(E \cap g^{-1}E) \geq \mu(A \cap (T_g)^{-1}A).$$

Using the same argument that leads to inequality (3.13), we get that for any Følner sequence $(F_N)$

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} \mu(A \cap (T_g)^{-1}A) \geq \mu(A)^2.$$  

Combining (3.14) and (3.15) we obtain
\[ \liminf_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} \bar{d}_{(G_N)}(E \cap g^{-1}E) \geq \mu^2(A) = \bar{d}_{(G_N)}(E)^2, \]

as desired. \( \square \)

4. Ergodic Sequences and Hindman’s Theorem

The goal of this section is to extend Hindman’s covering theorem (Theorem 1.3) to unions of the form \( \bigcup_{n=1}^{N} (E - k_n) \). In particular, we will characterize the sequences \( (k_n) \) for which the generalized version of Hindman’s theorem holds. We remark that it is not clear how to extend Hindman’s original combinatorial proof to this level of generality. On the other hand, the ergodic approach can easily be extended to amenable groups. We will discuss this after the proof of Theorem 4.6.

**Definition 4.1.** We say that a sequence of positive integers \( (k_n)_{n \in \mathbb{N}} \) has the sweeping out property if for every \( E \subseteq \mathbb{Z} \) with \( d^*(E) > 0 \) and every \( \varepsilon > 0 \) there exists \( N(\varepsilon) \) such that

\[ d^* \left( \bigcup_{n=1}^{N} (E - k_n) \right) > 1 - \varepsilon. \]

There class of sequences satisfying (4.1) is quite wide. For example, ergodic sequences have the sweeping out property.

**Definition 4.2.** We say that a sequence of positive integers \( (k_n) \) is an ergodic sequence if for every ergodic measure preserving system \( (X, \mathcal{B}, \mu, T) \) we have

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-k_n}B) = \mu(A)\mu(B). \]

for all \( A, B \in \mathcal{B} \).

**Lemma 4.3.** Let \( (k_n) \) be an ergodic sequence. Then \( (k_n) \) has the sweeping out property.

**Proof.** We first note that if \( (k_n) \) is an ergodic sequence, \( (X, \mathcal{B}, \mu, T) \) is an ergodic measure preserving system and \( A \in \mathcal{B} \) is such that \( \mu(A) > 0 \), then

\[ \mu \left( \bigcup_{n \in \mathbb{N}} T^{-k_n}A \right) = 1. \]

(Otherwise taking \( B = X \setminus \bigcup_{n \in \mathbb{N}} T^{-k_n}A \) we would get a contradiction with (4.2).)

Now let \( E \subseteq \mathbb{Z} \) be such that \( d^*(E) > 0 \) and take \( \varepsilon > 0 \). By Theorem 2.8 there exists

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One can show that \( (k_n) \) is an ergodic sequence if and only if for all \( f \in L^2(\mu) \) we have

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^{k_n}f = \int_X f \, d\mu, \]

where the convergence is with respect to the \( L^2(\mu) \) norm.
an ergodic measure preserving system \((X, \mathcal{B}, \mu, T)\) and a set \(A \in \mathcal{B}\) with \(\mu(A) = d^*(E)\) satisfying

\[
    d^* \left( \bigcup_{n=1}^{N} (E - k_n) \right) \geq \mu \left( \bigcup_{n=1}^{N} T^{-k_n} A \right)
\]

for all \(N \in \mathbb{N}\). Using (4.3) and continuity of \(\mu\) we see that \((k_n)\) satisfies (1.8). \(\square\)

There are sequences with the sweeping out property that are not ergodic. A rather cheap example is provided by the sequence \(k_n := \lfloor \log n \rfloor\). While this sequence is not good for (4.2), it takes on all nonnegative integer values and hence is sweeping out. A more interesting example is given by the sequence \(k_n := [n^2 + \log n]\). By [BKQW], \((k_n)\) is not ergodic. However, one can show that for any ergodic measure preserving system \((X, \mathcal{B}, \mu, T)\) and any sets \(A, B \in \mathcal{B}\) with \(\mu(A) > 0\) and \(\mu(B) > 0\) there is some \(n \in \mathbb{N}\) such that \(\mu(A \cap T^{-(n^2 + \log n)} B) > 0\). This implies that \(\mu \left( \bigcup_{n \in \mathbb{N}} T^{-(n^2 + \log n)} A \right) = 1\). This fact together with Theorem 4.6 below imply that \((k_n)\) is sweeping out.

**Definition 4.4.** We say that an invertible measure preserving system \((X, \mathcal{B}, \mu, T)\) has a topological model if there exists a measure-theoretically isomorphic system \((\hat{X}, \hat{\mathcal{B}}, \hat{\mu}, \hat{T})\), where \(\hat{X}\) is a compact metric space and \(T\) is a homeomorphism from \(\hat{X}\) to itself.

**Theorem 4.5 (Jewett-Krieger Theorem).** Every ergodic invertible measure preserving system \((X, \mathcal{B}, \mu, T)\) has a uniquely ergodic\(^{10}\) topological model.

The following theorem characterizes sequences which are "good" for Hindman's covering theorem (see Theorem 1.3):

**Theorem 4.6.** Let \((k_n)\) be a sequence of integers. Then, \((k_n)\) is sweeping out if and only if for every ergodic measure preserving system \((X, \mathcal{B}, \mu, T)\) and for every \(A \in \mathcal{B}\) with \(\mu(A) > 0\) we have

\[
    \mu \left( \bigcup_{n \in \mathbb{N}} T^{-k_n} A \right) = 1.
\]

**Proof.** We prove the backwards direction first. Let \(E \subseteq \mathbb{Z}\) with \(d^*(E) > 0\). Let \((X, \mathcal{B}, \mu, T)\) be a measure preserving system and \(A \in \mathcal{B}\) which are guaranteed by Theorem 2.8 and satisfy the following special case of (4.6):

\[
    d^* \left( \bigcup_{n=1}^{N} (E - k_n) \right) \geq \mu \left( \bigcup_{n=1}^{N} T^{-k_n} A \right)
\]

for all \(N \in \mathbb{N}\). Since \(\mu(A) = d^*(E)\), the result follows from (4.6) by continuity of the measure \(\mu\).

For the other direction, we will show the contrapositive. Assume that there exists an ergodic system \((X, \mathcal{B}, \mu, T)\) and a set \(A \in \mathcal{B}\) with \(\mu(A) > 0\) such that \(\mu \left( \bigcup_{n \in \mathbb{N}} T^{-k_n} A \right) = 1 - \delta\) for some \(\delta > 0\). Without loss of generality, one can assume that \((X, \mathcal{B}, \mu, T)\) is invertible. Indeed, otherwise we can work with the invertible extension of \((X, \mathcal{B}, \mu, T)\). This will not

\(^{10}\)A measure preserving system \((X, \mathcal{B}, \mu, T)\) is called uniquely ergodic if \(X\) is a compact metric space, \(T : X \to X\) is a homeomorphism and \(\mu\) is the unique \(T\)-invariant normalized Borel measure on \(X\).
affect the extra properties our set $A$ has, and we will be able to apply Theorem 4.5.

In view of the Jewett-Krieger theorem (Theorem 4.5), we can also assume that $(X, T)$ is a uniquely ergodic topological dynamical system.

Since $X$ is a compact metric space, the probability measure $\mu$ is regular. Let $K_1 \subseteq A$ be a compact set such that $\mu(K_1) > 0$ and $K_2 \subseteq X \setminus \bigcup_{n \in \mathbb{N}} T^{-k_n} A$ another compact subset such that $\mu(K_2) \geq \frac{\delta}{2}$. Since $T$ is ergodic, von Neumann’s mean ergodic theorem implies

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^n f = \int_{X} f \, d\mu,$$

where $f = 1_{K_1}$ and the convergence is in the $L^2(\mu)$-norm. Thus, there is a subsequence $(N_k)$ and $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that for all $x \in X_0$ we have

$$\lim_{k \to \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} f(T^n x) = \int_{X} f \, d\mu.$$

Let $x_0 \in X_0$ and consider the set

$$E := \{ n \in \mathbb{Z} : T^n x_0 \in K_1 \}.$$

Note that by the choice of $x_0$ and $E$

$$\lim_{k \to \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} 1_E(n) = \lim_{k \to \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} 1_{K_1}(T^n x_0) = \int_{X} 1_{K_1} \, d\mu = \mu(K_1) > 0$$

which implies that $d^*(E) > 0$. We claim that for all $N \in \mathbb{N}$, we have

$$d^* \left( \bigcup_{n=1}^{N} (E - k_n) \right) \leq 1 - \frac{\delta}{2}.$$

Indeed, let $N \in \mathbb{N}$ and $(I_N)_{N \in \mathbb{N}}$ be a Følner sequence. By our choice of $K_1$, the set $T^{-k_1} K_1 \cup \cdots \cup T^{-k_N} K_1$ is a compact set disjoint from $K_2$, so by Urysohn’s lemma there is some continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ if $x \in T^{-k_1} K_1 \cup \cdots \cup T^{-k_N} K_1$ and $f(x) = 0$ if $x \in K_2$. Thus,

$$d^* \left( \bigcup_{n=1}^{N} (E - k_n) \right) = \limsup_{N \to \infty} \frac{1}{|I_N|} \sum_{n \in I_N} 1_{T^{-k_1} K_1 \cup \cdots \cup T^{-k_N} K_1}(T^n x_0)$$

$$\leq \lim_{N \to \infty} \frac{1}{|I_N|} \sum_{n \in I_N} f(T^n x_0) = \int_{X} f \, d\mu \leq 1 - \frac{\delta}{2}.$$

(Note that we used the fact that for uniquely ergodic systems, $\lim_{N \to \infty} \frac{1}{|I_N|} \sum_{n \in I_N} f(T^n x_0)$ exists for any Følner sequence $(I_N)$, any continuous function $f$ and any $x_0 \in X$). Since the constant $1 - \frac{\delta}{2}$ does not depend on $N$ or on the Følner sequence $(I_N)$, we obtain the desired result. \qed
Remark 4.7. Let \((k_n)\) be a sequence of nonnegative integers. If \((X, \mathcal{B}, \mu, T)\) is an invertible measure preserving system such that for all \(A \in \mathcal{B}\) with \(\mu(A) > 0\) the condition (4.6) holds, then \(\mu\) is ergodic. This implies that the ergodicity assumption in Theorem 4.6 is only needed in one of the directions.

The characterization given in Theorem 4.6 generalizes without any significant changes to the setup of general discrete measure preserving actions of an amenable group. For the record we give a definition of sweeping out for discrete countable amenable groups:

Definition 4.8. Let \(G\) be a discrete countable amenable group. Let \(a : \mathbb{N} \to G\) be a sequence of elements of \(G\). We say that \(a\) is sweeping out if for any set \(E \subseteq G\) with \(d^*(E) > 0\) and for all \(\varepsilon > 0\) there exists \(N(\varepsilon)\) such that

\[
d^* \left( \bigcup_{n=1}^{N} a(n)^{-1} E \right) > 1 - \varepsilon
\]

(4.8)

To carry out the generalization of Theorem 4.6, one has to invoke a general form of Jewett-Krieger’s theorem due to Rosenthal (see [R]). We have chosen to stick to \(\mathbb{Z}\) for the sake of clarity.

We conclude this section with some examples of ergodic sequences both in \(\mathbb{Z}\) and \(\mathbb{Z}^d\). (See [BLes] and [BKQW].)

1. \(\{bn^c : n \in \mathbb{N}\}\), where \(c \notin \mathbb{Q}\), \(c > 1\) and \(b \neq 0\).
2. \(\{bn^c + dn^n : n \in \mathbb{N}\}\), where \(b, d \neq 0\), \(b/d \notin \mathbb{Q}\), \(c \geq 1\), \(a > 0\) and \(a \neq c\).
3. \(\{bn^c(\log n)^d : n \in \mathbb{N}\}\), where \(b \neq 0\), \(c \notin \mathbb{Q}\), \(c > 1\) and \(d\) is any number.
4. \(\{bn^c(\log n)^d : n \in \mathbb{N}\}\), where \(b \neq 0\), \(c \in \mathbb{Q}\), \(c > 1\) and \(d \neq 0\).
5. \(\{bn^c + d(\log n)^a : n \in \mathbb{N}\}\), where \(b, d \neq 0\), \(c \geq 1\) and \(a > 1\).

Another class of examples of ergodic sequences is provided by sequences of the form \([g(n)]\), where \(g\) is any tempered function\(^{11}\) (see [BK], Theorem 7.1)

We conclude this section with some examples of ergodic sequences involving primes for \(\mathbb{Z}^d\) actions. It is clear that these sequences will be sweeping out for \(\mathbb{Z}^d\) due to a straightforward generalization of Lemma 4.3. All these sequences come from the paper [BKS].

1. \(\{(\xi_1(p_n), \ldots, \xi_d(p_n)) : n \in \mathbb{N}\}\), where \(p_n\) denotes the \(n\)-th prime with the standard order, and where \(\xi_1, \ldots, \xi_d\) are functions in a Hardy field with subpolynomial growth such that either

\[
\lim_{x \to \infty} \frac{\xi(x)}{x^{l+1}} = \lim_{x \to \infty} \frac{x^l}{\xi(x)} = 0 \text{ for some } l \in \mathbb{N}, \text{ or } \lim_{x \to \infty} \frac{\xi(x)}{x} = \lim_{x \to \infty} \frac{\log x}{\xi(x)} = 0,
\]

and such that any combination of the form \(\sum_{i=1}^{d} b_i \xi_i\) also satisfies (4.9) for all \((b_1, \ldots, b_d) \in \mathbb{R}^d \setminus \{0\}\).

In particular, sequences of the form \(\{(p_{n_1}^{c_1}, \ldots, p_{n_d}^{c_d}) : n \in \mathbb{N}\}\), where \(p_n\) denotes the \(n\)-th prime, and where \(c_1, \ldots, c_d\) are distinct positive real numbers such that \(c_i \notin \mathbb{N}\) for all \(i\). (This special case was obtained in [BKMST]).

\(^{11}\)A real valued function \(g\) defined on a half line \([\alpha, +\infty)\) is called a tempered function if there exist \(k \in \mathbb{N}\) such that \(g\) is \(k\) times continuously differentiable, \(g^{(k)}\) tends monotonically to zero as \(x \to \infty\) and \(\lim_{x \to \infty} x |g^{(k)}(x)| = +\infty\).
5. A Characterization of Countable Amenable Weakly Mixing Groups

In this section, we will establish a characterization of countable weakly mixing groups with the help of the amenable version of Hindman’s theorem (Theorem 3.1). Recall that a group $G$ is weakly mixing (or minimally almost periodic) if any ergodic measure preserving action of $G$ on a probability space $(X, \mathcal{B}, \mu)$ is automatically weakly mixing (i.e., the product action on $X \times X$ is ergodic).

We begin with the following proposition:

**Proposition 5.1.** Let $G$ be a countable amenable WM group. Let $E \subseteq G$ with $d^*(E) > 0$ and let $k \in \mathbb{N}$. Then, for all $\varepsilon > 0$, there exist $g_1, \ldots, g_k \in G$ such that

\[ d^* \left( \bigcup_{i=1}^{k} (g_i, \ldots, g_i)^{-1}(E \times \cdots \times E) \right) > 1 - \varepsilon. \]

**Proof.** By Theorem 2.8, there exists an ergodic measure preserving system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ and a set $A \in \mathcal{B}$ with $\mu(A) = d^*(E)$ satisfying (2.6). Since $G$ is a WM group, the measure $\mu \otimes \cdots \otimes \mu$ will be ergodic for the product action on $X \times X$.

Let $(G_N)$ be a Følner sequence such that

\[ \mu = \operatorname{w}^*-\lim_{N \to \infty} \frac{1}{|G_N|} \sum_{g \in G_N} \delta_{T_g \omega}, \]

where $\omega = (1_E(g))_{g \in G}$, as in Section 2. Notice that for each $k \in \mathbb{N}$ we have

\[ d^* \left( \bigcup_{i=1}^{k} (g_i, \ldots, g_i)^{-1}(E \times \cdots \times E) \right) \geq d_{(G_N \times \cdots \times G_N)} \left( \bigcup_{i=1}^{k} (g_i, \ldots, g_i)^{-1}(E \times \cdots \times E) \right), \]

for all $g_1, \ldots, g_k \in G$. Applying the inclusion-exclusion principle and the definition of $\mu$ we see that the quantity on the right hand side in (5.2) equals

\[ \mu \otimes \cdots \otimes \mu \left( \bigcup_{i=1}^{k} (T_{g_i} \times \cdots \times T_{g_i})(A \times \cdots \times A) \right) \]

It is well known that if $G$ is a weakly mixing measure preserving group action, then so is the diagonal action $G \times \cdots \times G$ of an arbitrary finite product of $G$. Thus, the measure $\mu \otimes \cdots \otimes \mu$ is ergodic for the diagonal $G$-action. Finally, by choice of $A$ we have $(\mu \otimes \cdots \otimes \mu)(A \times \cdots \times A) > 0$, whence

\[ (\mu \otimes \cdots \otimes \mu) \left( \bigcup_{g \in G} (T_g \times \cdots \times T_g)^{-1}(A \times \cdots \times A) \right) = 1, \]

so the result follows by continuity of $\mu$. \hfill \Box

We will show next that the covering property (5.1) characterizes weakly mixing groups:

**Theorem 5.2.** Let $G$ be a countable amenable group that is not WM. Then there exists a set $E \subseteq G$ with $d^*(E) \in (0, 1)$ such that for all $r \in \mathbb{N}$ and any finite subset $\{h_1, \ldots, h_r\}$ of $G$ we have

\[ d^* \left( \bigcup_{i=1}^{r} (h_i, h_i)^{-1}(E \times E) \right) \leq C < 1, \]

where the constant $C$ in (5.3) is positive and independent of $\{h_1, \ldots, h_r\}$. 19
Proof. Before constructing the set $E$, we need to do some preparatory work. First, we observe that by a Corollary of Lemma 3.3 in [BBF] (see equation (5) in [BBF]), it suffices to show that for any pair of Følner sequences $(I_N)$ and $(J_N)$ we have

$$d_{(I_N \times J_N)} \left( \bigcup_{i=1}^{r}(h_i, h_i)^{-1}(E \times E) \right) \leq C < 1.$$  

for all $r \in \mathbb{N}$ and all finite sets $\{h_1, \ldots, h_r\}$ for some constant $0 < C < 1$.

Next, we observe that since $G$ is not WM it must admit a non-trivial finite dimensional representation $\pi : G \to U(k)$, where $U(k)$ is the unitary group of $k \times k$ complex matrices (see for example [S], Theorem 3.4). Thus, $H = \{\pi(g) : g \in G\}$ is a non-trivial subgroup of $U(k)$.

Let $v \in \mathbb{C}^k$ be a vector of norm 1, and take $X := \{h \cdot v : h \in H\}$, the closure of the orbit of $v$ under the action by matrices from the subgroup $H$. The subgroup $H$ induces a group structure on $X$ as follows. First, we can define a group operation on $Hv \times Hv$ via the formula $h_1v \cdot h_2v := (h_1 \cdot h_2)v$. Next, observe that since elements of $H$ are unitary matrices, the operation $\cdot$ is uniformly continuous on $Hv \times Hv$. Therefore, we can extend it to $X \times X$ by continuity using the fact that $Hv$ is dense in $X$. In this way, $X$ becomes a compact metric group (with the usual topology inherited from $\mathbb{C}^k$). As such, it carries a unique Haar probability measure (fully supported on $X$) which we denote by $\mu$.

Moreover, $G$ acts on $X$ by translations as follows: let $R_gx := \pi(g)v \cdot x$. Note that the action $(R_g)_{g \in G}$ is minimal because $\pi(G) = H$ and $Hv$ is dense in $X$. Clearly, $(X, \text{Borel}(X), \mu, (R_g)_{g \in G})$ is a uniquely ergodic measure preserving system.

Let $d$ be a bi-invariant metric on $X$ (such a metric exists by the Birkhoff-Kakutani Theorem, see [Bi] and [K]). Let $g_0 \in X$ be such that $d(e, g_0) = b = \max \{d(e, g) : g \in G\} > 0$ (since $X$ is compact and non-trivial, $0 < b < \infty$). Let $U := B(e, \frac{b}{16})$ (the open ball of radius $\frac{b}{16}$ centered at $e$). Clearly $\mu(U) > 0$ since $\mu$ is fully supported on $X$. Let $U' := B(e, r)$ where $0 < r < \frac{b}{16}$ is such that $\mu(\partial U') = 0$. Such an $r$ must exist (see for example Lemma 2.21 in [BCRZ]). At this point we are ready to define the set $E$. Namely, choose an arbitrary $x_0 \in X$ and put

$$E := \{g \in G : R_gx_0 \in U'\}.$$  

Since $(X, \mu, (R_g)_{g \in G})$ is uniquely ergodic and $\mu(\partial U') = 0$, the function $1_{U'}$ can be approximated by continuous functions. From these two facts, it $w^\ast$-lim$_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} \delta_{T_gx_0}$ converges to $\mu$.

Next, observe that for all $g \in X$, the open set $W := B(g_0, \frac{b}{16}) \times B(e, \frac{b}{16})$ satisfies

$$d^{-1}(U \times U) \cap W = \emptyset,$$  

because otherwise, by the triangle inequality we would get that

$$d(e, g) \leq d(e, g_1u) + d(g_1u_1, g_1u_2) + d(g_1u_2, g_0) < 2 \cdot \frac{b}{16} + d(u_1, u_2) < \frac{b}{8} + \frac{b}{8} < b,$$  

where $g_1 \in G$ and $u_1, u_2 \in U$ are such that $d(e, g_1u_1) < \frac{b}{16}$ and $d(g_0, g_1u_2) < \frac{b}{16}$, a contradiction with our
choice of $e, g_0$.

It follows from (5.6) and from the fact that $G$ acts minimally on $X$ that $\Delta \cdot (U \times U) \neq X \times X$, where $\Delta$ is the diagonal in $X^2$. Thus,

$$(\mu \otimes \mu)(\Delta \cdot (U \times U)) := C < 1.$$  

Note also that since $\mu$ is fully supported on $X$, $\mu \otimes \mu$ is fully supported on $X \times X$, and so we have $(\mu \otimes \mu)(\Delta \cdot (U \times U)) > 0$. Let $(I_N), (J_N)$ be two Følner sequences in $G$, and $\{h_1, \ldots, h_r\}$ a finite set of elements of $G$. Then,

$$(5.7) \quad \bar{d}_{(I_N \times J_N)} \left( \bigcup_{i=1}^r (h_i, h_i)^{-1}(E \times E) \right) = \limsup_{N \to \infty} \frac{1}{|I_N||J_N|} \sum_{g \in F_N} \sum_{h \in J_N} \|1_{\bigcup_{i=1}^r (h_i, h_i)^{-1}(E \times E)}(g, h) \|.$$ 

By (5.5), the last term in equation (5.7) is equal to

$$(5.8) \quad \limsup_{N \to \infty} \frac{1}{|I_N||J_N|} \sum_{g \in F_N} \sum_{h \in J_N} \|1_{\bigcup_{i=1}^r (T_{h_i} \times T_{h_i})^{-1}(U' \times U')} (T_g x_0, T_h x_0) \|.$$ 

We now use the inclusion-exclusion principle to separate the variables in each summand. Let $\prod_{i=1}^t 1_{T_{h_i} U'}$, $1 \leq t \leq r$, be a typical term obtained through this process. By unique ergodicity of the action $(R_g)_{g \in G}$ we see that for any Følner sequence $(F_N)$

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} \|1_{\bigcap_{i=1}^t T_{h_i} U'} (T_g x_0) \| = \mu \left( \bigcap_{i=1}^t T_{h_i} U' \right).$$

Indeed, since $U'$ is an open set with $\mu(\partial U') = 0$, we also have that $\mu(\partial(T_g U')) = 0$ for all $g \in G$ given that $T_g(\partial U') = \partial(T_g U')$ as $T_g$ is a measure preserving homeomorphism. Consequently, functions of the form $\prod_{i=1}^t 1_{T_{h_i} U'}$, $1 \leq t \leq r$, can be approximated by continuous functions. Thus, the limit in equation (5.8) in fact exists and (after performing the inclusion-exclusion principle backwards) is equal to

$$(5.9) \quad (\mu \otimes \mu) \left( \bigcup_{i=1}^r (T_{h_i} \times T_{h_i})^{-1}(U' \times U') \right) \leq (\mu \otimes \mu)(\Delta \cdot (U \times U)) = C < 1,$$

which completes the proof.

6. A GENERAL FORM OF HINDMAN’S COVERING THEOREM

One may wonder if a version of Hindman’s covering theorem (Theorem 3.1) is valid for discrete amenable semigroups which are not necessarily countable or cancellative. In this section we will show that this is indeed the case. Recall that a discrete semigroup $G$ is left amenable if there exists a left invariant mean $m : \ell^\infty(G) \to \mathbb{C}$ \footnote{We say that $m \in \ell^\infty(G)^*$ is a left invariant mean if it is a continuous linear functional from $\ell^\infty(G)$ to $\mathbb{C}$ such that (i) for every $f \in \ell^\infty(G)$ and for every $g \in G$ we have $m(gf) = m(f)$, where $gf(x) := f(gx)$ for all $x \in G$, (ii) $m(f) \geq 0$ for any non-negative function $f : G \to \mathbb{C}$, and (iii) $m(1) = 1$.} Note that, for discrete countable groups, this is equivalent to the definition of left amenability given in Section 2.

In the context of means, a notion of largeness presents itself. Let us denote by $\mathcal{M}(G)$ the space of left invariant means on $G$. We say a subset $E$ of a discrete left amenable semigroup
Proof. We proceed by contradiction, so assume that for all $G$ is large if $m(1_E) > 0$ for some mean $m \in \mathcal{M}(G)$. This leads to the following definition of a notion of upper Banach density that is valid in all amenable semigroups (see Definition 2.7 [BG]):

**Definition 6.1.** Let $G$ be an amenable semigroup, and let $E \subseteq G$. The upper Banach density of $E$ is

$$d^*(E) := \sup \{ m(1_E) : m \in \mathcal{M}(G) \}.$$ 

If $G$ is a discrete countable amenable group and $E \subseteq G$, then the upper Banach density of $E$ as given in Definition 2.2 agrees with the one in Definition 6.1. Moreover, in this case, we have $d^*(E) = \max \{ m(1_E) : m \in \mathcal{M}(G) \}$. We are now in a position to formulate a general version of Hindman’s theorem.

**Theorem 6.2.** Let $G$ be an amenable semigroup, and let $E$ be a subset of $G$ with $d^*(E) > 0$. Then, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ and $g_1, \ldots, g_k \in G$ such that

$$d^*(g_1^{-1}E \cup \cdots \cup g_k^{-1}E) > 1 - \varepsilon.$$ 

The proof of Theorem 6.2 requires a few preliminary results which will be given next.

**Lemma 6.3.** Let $G$ be an amenable semigroup $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ an ergodic measure preserving system. Let $A, B \in \mathcal{B}$ be such that $\mu(A) > 0$ and $\mu(B) > 0$. Then,

$$(6.1) \quad R_{A,B}(\varepsilon) := \left\{ g \in G : \mu(B \cap (T_g)^{-1}A) > \frac{\mu(A)\mu(B)}{2} \right\} \neq \emptyset$$

Proof. We proceed by contradiction, so assume that for all $g \in G$ we have $\mu(B \cap (T_g)^{-1}A) \leq \frac{\mu(A)\mu(B)}{2}$. Let $(F_\alpha)$ be a (left) Følner net in $G$. Let $A, B \in \mathcal{B}$ be such that $\mu(A) > 0$ and $\mu(B) > 0$. By von Neumann’s mean ergodic theorem, we have for any $f \in L^2(\mu)$

$$(6.2) \quad \lim_{\alpha} \frac{1}{|F_\alpha|} \sum_{g \in F_\alpha} T_gf = \int_X f \, d\mu,$$

where the convergence is in the $L^2(\mu)$-norm. From (6.2) we immediately obtain

$$(6.3) \quad \lim_{\alpha} \frac{1}{|F_\alpha|} \sum_{g \in F_\alpha} \mu(B \cap (T_g)^{-1}A) = \mu(A)\mu(B).$$

Equation (6.3) contradicts the assumption that, for all $g \in G$, $\mu(B \cap (T_g)^{-1}A) \leq \frac{\mu(A)\mu(B)}{2}$, so we are done.

**Lemma 6.4.** Let $G$ be an amenable semigroup and $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ an ergodic measure preserving system. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. Then, for all $\varepsilon > 0$, there are $g_1, \ldots, g_k \in G$ such that

$$\mu \left( \bigcup_{i=1}^{k} (T_{g_i})^{-1}A \right) > 1 - \varepsilon.$$ 

Proof. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. We claim that

$$(6.4) \quad \sup \left\{ \mu \left( \bigcup_{g \in B} (T_g)^{-1}A \right) : B \subseteq G \text{ and } |B| = |\mathbb{N}| \right\} = 1.$$ 

As is easily seen from the proof below, one can change the $\frac{1}{2}$ in the definition of $R_{A,B}$ to any $\lambda \in (0,1)$.
We proceed by contradiction. Let us assume that this is not the case. Then there exists \( 0 < \delta < 1 \) such that

\[
\sup \left\{ \mu \left( \bigcup_{g \in B} (T_g)^{-1}A \right) : B \subseteq G \text{ and } |B| = |N| \right\} = 1 - \delta.
\]

Let \( 0 < \varepsilon' < \frac{\mu(A)\delta}{2} \), and choose \( B \subseteq G \) with \( |B| = |N| \) such that

\[
\mu \left( \bigcup_{g \in B} (T_g)^{-1}A \right) \geq 1 - \delta - \varepsilon'.
\]

By assumption, there is some \( C \subseteq X \setminus \bigcup_{g \in B} (T_g)^{-1}A \) with \( \mu(C) \geq \delta \). Now, by Lemma 6.3, we can find \( g_0 \in G \) such that \( \mu(C \cap (T_{g_0})^{-1}A) \geq \frac{\mu(A)\delta}{2} \).

(Notice that, in particular, it follows that \( g_0 \not\in B \)). We have

\[
\mu \left( \bigcup_{g \in B \cup \{g_0\}} (T_g)^{-1}A \right) \geq 1 - \delta - \varepsilon' + \frac{\mu(A)\delta}{2} > 1 - \delta,
\]

by our choice of \( \varepsilon' \) and \( g_0 \). This contradicts the definition of supremum, since clearly \( B \cup \{g_0\} \) is still a countable subset of \( G \). Thus, (6.4) holds.

To complete the proof, let \( \varepsilon > 0 \) and choose \( B \subseteq G \) with \( |B| = |N| \) such that

\[
\mu \left( \bigcup_{g \in B} (T_g)^{-1}A \right) > 1 - \frac{\varepsilon}{2}.
\]

Since \( \mu \) is continuous and \( B \) is countable, there exist \( g_1, \ldots, g_k \in B \) such that

\[
\mu \left( \bigcup_{i=1}^{k} (T_{g_i})^{-1}A \right) > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2},
\]

so we are done. \( \square \)

In order to obtain a suitable enhanced version of Furstenberg’s correspondence principle we need two additional results. The first one is a version of Furstenberg’s correspondence principle for means.

**Theorem 6.5** (Furstenberg correspondence for means (cf. [BLc] and [BMc])). Let \( G \) be a discrete amenable semigroup. Let \( m \in \mathcal{M}(G), E \subseteq G \) with \( m(\mathds{1}_E) > 0 \). Then there exists a probability measure preserving system \((X, \mathcal{B}, \mu, (T_g)_{g \in G})\) such that \( X \) is compact and Hausdorff, \( \mathcal{B} = \text{Borel}(X) \) and \((T_g)_{g \in G}\) is a \( G \)-action on \( X \) by continuous self-maps of \( X \). Finally, \( A \) is a set in \( \mathcal{B} \) for which \( \mu(A) = m(\mathds{1}_B) \), and \( \mu \) is such that for all \( k \in \mathbb{N} \), \( g_1, \ldots, g_k \in G \), we have

\[
m \left( \mathds{1}_E \prod_{i=1}^{k} \mathds{1}_{g_i^{-1}E} \right) = \mu(A \cap T_{g_1}^{-1}A \cap \cdots \cap T_{g_k}^{-1}A).
\]

**Proof.** The proof for discrete amenable groups provided in [BLc] extends verbatim to our context without any major modification. \( \square \)
Remark 6.6. It is worth mentioning that the proof of Theorem 6.5 in [BLLei] can be easily adjusted to include unions and complements as in Theorem 2.8.

The other result we need can be found in [P]:

Theorem 6.7 (Proposition 0.1 [P]). The space of left invariant means $\mathcal{M}(G)$ is a weak*-compact, convex spanning subset of $\ell^\infty(G)^*$.

We are now able to formulate and prove an enhanced general version of the ergodic Furstenberg correspondence principle.

Theorem 6.8 (Enhanced Ergodic Furstenberg correspondence for means). Let $E \subseteq G$ be such that $m(1_E) > 0$ for some mean $m \in \mathcal{M}(G)$. Then there exists a mean $\tilde{m} \in \mathcal{M}(G)$, and an ergodic measure preserving system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ such that for all $k \in \mathbb{N}$ we have

$$m(1_{E^{w_k} \star g_{k-1} E^{w_1} \star \cdots \star g_1 E^{w_0}}) = \mu(A^{w_0} \star (T_{g_1})^{-1} A^{w_1} \star \cdots \star (T_{g_k})^{-1} A^{w_k})$$

where $A \in \mathcal{B}$ is such that $0 < \mu(A) \leq \tilde{m}(1_E)$, and each of the stars denotes either union or intersection with the understanding that

(i) for all $1 \leq i \leq k-1$, the operation represented by $\star$ which stands between $E^{w_i}$ and $E^{w_{i+1}}$ is the same as the operation appearing between $A^{w_i}$ and $A^{w_{i+1}}$.

(ii) the choices of parentheses which are needed to make the expressions on both sides of formula (2.1) well defined also match.

Proof. First, we remark that, as in the proof of Theorem 6.5, any invariant mean on $G$ is given by a $G$-invariant probability measure on $\beta G$ via the isomorphism $(\ell^\infty(G))^* \cong C(\beta G)^*$. It is easy to see that this isomorphism is behind the formula (6.9).

By Choquet’s theorem (which we can apply by Theorem 6.7), we can write

$$m = \int_{\text{Ext}(\mathcal{M}(G))} m_t \, d\lambda(t),$$

for some probability measure $\lambda$ supported on $\text{Ext}(\mathcal{M}(G))$, the set of extreme points of $\mathcal{M}(G)$.

Notice that extreme points of $\mathcal{M}(G)$ get mapped to extreme points of the set of probability measures on $\beta G$ via the isomorphism $(\ell^\infty(G))^* \cong C(\beta G)^*$. Observe also that the measures that are extreme points in the set of $G$-invariant probability measures on $\beta G$ are in fact ergodic.

Next, by formula (6.10), observe that since $m(1_E) > 0$, then, for a set of positive $\lambda$-measure of $t$, we have that $m_t(1_E) > 0$.

Thus we can choose $\tilde{m}$, an extreme point of $\mathcal{M}(G)$ for which $\tilde{m}(1_E) \geq m(1_E) > 0$. Using the aforementioned isomorphism, we obtain a measure $\mu$ for which (6.9) is easily checked, and for which the $G$-action $(T_g)_{g \in G}$ is ergodic.

We are now ready for a proof of Theorem 6.2.
Proof of Theorem 6.2. Let \( E \subseteq G \) with \( d^*(E) > 0 \). Then, there is some \( m_0 \in \mathcal{M}(G) \) such that \( m_0(1_E) > 0 \). Let \((X, \mathcal{B}, \mu, (T_g)_{g \in G})\) be an ergodic measure preserving system satisfying the equality (6.9) for some set \( A \in \mathcal{B} \) with \( \mu(A) > 0 \). (See Theorem 6.8 above). Let \( E \subseteq G \) with \( d^*(E) > 0 \). Then, there is some \( m_0 \in \mathcal{M}(G) \) such that \( m_0(1_E) > 0 \). Let \((X, \mathcal{B}, \mu, (T_g)_{g \in G})\) be an ergodic measure preserving system satisfying the equality (6.9) for some set \( A \in \mathcal{B} \) with \( \mu(A) > 0 \). (See Theorem 6.8 above).

Let \( E \subseteq G \) with \( d^*(E) > 0 \). Then, there is some \( m_0 \in \mathcal{M}(G) \) such that \( m_0(1_E) > 0 \). Let \((X, \mathcal{B}, \mu, (T_g)_{g \in G})\) be an ergodic measure preserving system satisfying the equality (6.9) for some set \( A \in \mathcal{B} \) with \( \mu(A) > 0 \). (See Theorem 6.8 above).

Let \( \varepsilon > 0 \). By Lemma 6.4, we can find \( g_1, \ldots, g_k \in G \) such that
\[
\mu \left( \bigcup_{i=1}^{k} (T_{g_i})^{-1} A \right) > 1 - \varepsilon,
\]
since \( \mu(A) > 0 \). Equality (6.9) then implies that for some \( m \in \mathcal{M}(G) \) we have
\[
m(\bigcup_{i=1}^{k} g_i^{-1} 1_E) > 1 - \varepsilon,
\]
whence the result follows from the definition of \( d^* \). \( \square \)

We conclude this section with brief remarks on yet another generalization of Hindman’s covering theorem. Assume that \( G \) is a locally compact amenable group (i.e. that \( G \) has a left invariant topological mean, see [Gre] for more details). The Furstenberg’s correspondence principle which was proved in [BCRZ] (see also [BF]) can be "upgraded" to an enhanced ergodic Furstenberg correspondence principle similar to Theorem 2.8 and Theorem 6.5. Based on this enhancement one can prove a version of Hindman’s Theorem for locally compact groups. Before providing the formulation we need two definitions.

Definition 6.9. Let \( G \) be a locally compact amenable group and let \( E \subseteq G \). We define the upper Banach density of \( E \) as follows:
\[
d^*(E) = \sup \{ m(1_E) : m \text{ is a left-invariant topological mean on } L^1(G, \mu) \},
\]
where \( \mu \) is a Haar measure on \( G \).

Definition 6.10. Let \( G \) be a locally compact amenable group. We say that a set \( E \subseteq G \) is substantial if \( E \supseteq UW \), where \( U \) is a non-empty open subset in \( G \) containing \( id_G \) and \( W \) is a measurable set with \( d^*(W) > 0 \).

Theorem 6.11. Let \( G \) be a locally compact amenable group. Let \( E \subseteq G \) be a substantial set. Let \( \varepsilon > 0 \). Then there exist \( g_1, \ldots, g_k \in G \) such that
\[
d^* \left( \bigcup_{i=1}^{k} g_i^{-1} E \right) > 1 - \varepsilon.
\]

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