APPROXIMATE BIPROJECTIVITY OF CERTAIN SEMIGROUP ALGEBRAS

A. SAHAMI AND A. POURABBAS

Abstract. In this paper, we investigate the notion of approximate biprojectivity for semigroup algebras and for some Banach algebras related to semigroup algebras. We show that $\ell^1(S)$ is approximately biprojective if and only if $\ell^1(S)$ is biprojective, provided that $S$ is a uniformly locally finite inverse semigroup. Also for a Clifford semigroup $S$, we show that approximate biprojectivity $\ell^1(S)^{**}$ gives pseudo amenability of $\ell^1(S)$. We give a class of Banach algebras related to semigroup algebras which is not approximately biprojective.

1. Introduction

Amenable Banach algebras were introduced by Johnson in [13]. In fact a Banach algebra $A$ is amenable, if every continuous linear derivation $D : A \to X^*$ is inner, for every Banach $A$-bimodule $X$. He showed that $A$ is amenable Banach algebra if and only if $A$ has an approximate diagonal, that is a bounded net $(m_{\alpha})_\alpha$ in $A \otimes_p A$ such that $\pi_A(m_{\alpha})a \to a$ and $a \cdot m_{\alpha} - m_{\alpha} \cdot a \to 0$, for every $a \in A$.

Most important notions related to amenability in the theory of homological Banach algebras are biflatness and by biprojectivity which introduced by Helenski in [11]. Indeed, $A$ is called biflat (biprojective), if there exists a bounded $A$-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ ($\rho : A \to A \otimes_p A$) such that $\pi^{**} \circ \rho$ is the canonical embedding of $A$ into $A^{**}$ ($\rho$ is a right inverse for $\pi_A$), respectively.

Recently some modified notions of amenability like approximate amenability and pseudo amenability introduced, see [7], [9] and [10]. In order to these new notions, approximate homological notions like approximate biprojective Banach algebras and approximate biflat Banach algebras introduced, for more information see [24] and [23].

Kamnith et al. in [14] introduced the notion of left $\phi$-amenable Banach algebras, where $\phi$ is a character on that Banach algebra. Later on the concepts of left $\phi$-contractible and character amenable Banach algebras were defined, see [22] and [16].

Semigroup algebras are very important Banach algebras. The amenability of these Banach algebras studied in many papers, common reference about the amenability of semigroup algebras is [3]. Recently modified notions like pseudo-amenability, pseudo-contractibility and approximate amenability of semigroup algebras have been investigated, see [6], [5] and [18]. Indeed

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they studied pseudo-amenability, pseudo-contractibility and approximate amenability of $\ell^1(S)$, where $S$ is an inverse group, band semigroup and etc. Biflatness and biprojectivity of semigroup algebras were another problem which investigated in [2] and [17]. In fact in [17] author showed that for an inverse semigroup $S$, $\ell^1(S)$ is biflat (biprojective) if and only if each maximal subgroup $S$ is amenable (finite) and $S$ is uniformly locally finite semigroup, respectively. The question is what will happen if semigroup algebra $\ell^1(S)$ is approximate biprojective?

In this paper we use left $\phi$-contractibility and left $\phi$-amenability to investigate approximate biprojectivity of semigroup algebras. We show that approximate biprojectivity of $\ell^1(S)$ implies the finiteness of $S$, for some classes of semigroups. We study approximate biprojectivity of the second dual of semigroup algebras. We show that for Clifford semigroup $S$, approximate biprojectivity of $\ell^1(S)**$ implies that $\ell^1(S)$ is pseudo-amenable. We give a criteria which shows that some triangular Banach algebras related to semigroup algebras are not approximate biprojective.

2. Preliminaries

Let $A$ be a Banach algebra. We recall that if $X$ is a Banach $A$-bimodule, then $X^*$ is also a Banach $A$-bimodule via the following actions

$$(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \quad (a \in A, x \in X, f \in X^*).$$

Throughout, the character space of $A$ is denoted by $\Delta(A)$, that is, all non-zero multiplicative linear functionals on $A$. Let $\phi \in \Delta(A)$. Then $\phi$ has a unique extension $\tilde{\phi} \in \Delta(A^{**})$ which is defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$.

Let $A$ and $B$ be Banach algebras. The projective tensor product of $A$ with $B$ is denoted by $A \otimes_p B$. The Banach algebra $A \otimes_p A$ is a Banach $A$-bimodule via the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

We recall that $\Delta(A \otimes_p B) = \{\phi \otimes \psi | \phi \in \Delta(A), \psi \in \Delta(B)\}$, where $\phi \otimes \psi (a \otimes b) = \phi(a) \psi(b)$, for every $a \in A$ and $b \in B$. We denote $\pi_A : A \otimes_p A \to A$ for the product morphism which specified by $\pi_A(a \otimes b) = ab$.

Let $\{A_\alpha\}_{\alpha \in \Gamma}$ be a collection of Banach algebras. Then we define the $\ell^1$-direct sum of $A_\alpha$ by

$$\ell^1 - \oplus_{\alpha \in \Gamma} A_\alpha = \{(a_\alpha) \in \prod_{\alpha \in \Gamma} A_\alpha : \sum_{\alpha \in \Gamma} ||a_\alpha|| < \infty\}.$$

It is easy to verify that

$$\Delta(\ell^1 - \oplus_{\alpha \in \Gamma} A_\alpha) = \{\oplus \phi_\beta : \phi_\beta \in \Delta(A_\beta), \beta \in \Gamma\},$$

where $\oplus \phi_\beta((a_\alpha)_{\alpha \in \Gamma}) = \phi_\beta(a_\beta)$ for every $(a_\alpha)_{\alpha \in \Gamma} \in \ell^1 - \oplus_{\alpha \in \Gamma} A_\alpha$ and every $\beta \in \Gamma$.

Let $A$ be a Banach algebra and let $\Lambda$ be a non-empty set. The set of all $\Lambda \times \Lambda$ matrixes $(a_{i,j})_{i,j}$ which entries come from $A$ is denoted by $M_\Lambda(A)$. With matrix multiplication and the
following norm

\[ \| (a_{i,j})_{i,j} \| = \sum_{i,j} |a_{i,j}| < \infty, \]

\( \mathbb{M}_\Lambda(A) \) is a Banach algebra. \( \mathbb{M}_\Lambda(A) \) belongs to the class of \( \ell^1 \)-Munn algebras. The map \( \theta : \mathbb{M}_\Lambda(A) \to A \otimes_p \mathbb{M}_\Lambda(\mathbb{C}) \) defined by \( \theta((a_{i,j})) = \sum_{i,j} a_{i,j} \otimes E_{i,j} \) is an isometric algebra isomorphism, where \( (E_{i,j}) \) denotes the matrix unit of \( \mathbb{M}_\Lambda(\mathbb{C}) \). Also it is well-known that \( \mathbb{M}_\Lambda(\mathbb{C}) \) is a biprojective Banach algebra [17, Proposition 2.7].

The main reference for the semigroup theory is [12]. Let \( S \) be a semigroup and let \( E(S) \) be the set of its idempotents. A partial order on \( E(S) \) is defined by

\[ s \leq t \iff s = st = ts \quad (s, t \in E(S)). \]

If \( S \) is an inverse semigroup, then there exists a partial order on \( S \) which is coincide with the partial order on \( E(S) \). Indeed

\[ s \leq t \iff s = ss^*t \quad (s, t \in S). \]

For every \( x \in S \), we denote \( (\overline{x}) = \{ y \in S \mid y \leq x \} \). \( S \) is called locally finite (uniformly locally finite) if for each \( x \in S \), \( \| x \| < \infty \) \( (\sup \{ \| x \| : x \in S \} < \infty) \), respectively.

Suppose that \( S \) is an inverse semigroup. Then the maximal subgroup of \( S \) at \( p \in E(S) \) is denoted by \( G_p = \{ s \in S \mid ss^* = s^*s = p \} \). For an inverse semigroup \( S \) there exists a relation \( \mathcal{D} \) such that \( s\mathcal{D}t \) if and only if there exists \( x \in S \) such that \( ss^* = xx^* \) and \( t^*t = x^*x \). We denote \( \{ \mathcal{D}_\lambda : \lambda \in \Lambda \} \) for the collection of \( \mathcal{D} \)-classes and \( E(\mathcal{D}_\lambda) = E(S) \cap \mathcal{D}_\lambda \). An inverse semigroup \( S \) is called Clifford if for each \( s \in S \), there exists \( s^* \) such that \( ss^* = s^*s \).

3. Approximate biprojectivity of semigroup algebras

We recall that a Banach algebra \( A \) is approximately biprojective, if there exists a net \( (\rho_\alpha)_{\alpha} \) of continuous \( A \)-bimodule morphism from \( A \) into \( A \otimes_p A \) such that \( \pi_A \circ \rho_\alpha(a) \to a \) for every \( a \in A \). For more details see [24].

A Banach algebra \( A \) is called left \( \phi \)-amenable (left \( \phi \)-contractible), where \( \phi \in \Delta(A) \), if there exists \( m \in A^{**} \) (\( m \in A \) such that \( am = \phi(a)m \) and \( \tilde{\phi}(m) = 1 \) \( (\phi(m) = 1) \), for every \( a \in A \), respectively, see [14] and [10].

A Banach algebra \( A \) is called pseudo-contractible if there exists a not necessarily bounded net \( (m_\alpha)_{\alpha} \) in \( A \otimes_p A \) such that \( a \cdot m_\alpha = m_\alpha \cdot a \) and \( \lim_\alpha \pi_A(m_\alpha)a = a \), for every \( a \in A \). For further details see [10].

We remind that \( S \) is a left amenable (a right amenable) semigroup if there exists an element \( m \in \ell^1(S)^{**} \) such that

\[ s \cdot m = m \ (m = m \cdot s), \quad \| m \| = m(\phi) = 1 \quad (s \in S), \]

where \( \phi \) is the augmentation character of \( \ell^1(S) \), respectively. The semigroup \( S \) is called amenable, if it is both left and right amenable.
Proposition 3.1. Let $S$ be a semigroup and let $Z(S)$ be a non-empty set. Then

(i) If $\ell^1(S)^{**}$ is approximately biprojective, then $S$ is amenable;

(ii) If $\ell^1(S)$ is approximately biprojective and $S$ has left or right unit, then $S$ is finite.

Proof. (i) Let $\ell^1(S)^{**}$ be approximately biprojective. Then there exists a net of bounded $\ell^1(S)^{**}$-bimodule morphisms $\rho_a : \ell^1(S)^{**} \to \ell^1(S)^{**} \otimes p \ell^1(S)^{**}$ such that $\pi_{\ell^1(S)^{**}} \circ \rho_a(a) \to a$ for every $a \in \ell^1(S)^{**}$. By [5, Lemma 1.7] there exists a bounded linear map $\psi : \ell^1(S)^{**} \otimes p \ell^1(S)^{**} \to (\ell^1(S) \otimes p \ell^1(S))^{**}$ such that for $a, b \in \ell^1(S)$ and $m \in \ell^1(S)^{**} \otimes p \ell^1(S)^{**}$, the following holds:

\[
\begin{align*}
(*) & \quad \psi(a \otimes b) = a \otimes b, \\
(**) & \quad \psi(m) \cdot a = \psi(m \cdot a), \\
(***) & \quad \pi_{\ell^1(S)}(\psi(m)) = \pi_{\ell^1(S)^{**}}(m).
\end{align*}
\]

It is easy to see that $\rho_0^\ell = \psi \circ \rho_a|_{\ell^1(S)} : \ell^1(S) \to (\ell^1(S) \otimes p \ell^1(S))^{**}$ is a net of bounded $\ell^1(S)$-bimodule morphisms.

Let $\phi$ be the augmentation character on $\ell^1(S)$ and $\tilde{\phi}$ be its extension to $\ell^1(S)^{**}$, for every $s_0 \in Z(S)$ we have $\phi(s_0) = 1$. Since $\tilde{\phi} \circ \pi_{\ell^1(S)^{**}} \circ \rho_0^\ell(a) \to \phi(a)$ for every $a \in \ell^1(S)$, by taking $m_a = \rho_0^\ell(s_0)$, one can easily see that $a \cdot m_a = m_a \cdot a$ and $\tilde{\phi} \circ \pi_{\ell^1(S)}(m_a) \to 1$. We can assume $\tilde{\phi} \circ \pi_{\ell^1(S)}(m_a) = 1$ by considering $\frac{m_a}{\phi \circ \pi_{\ell^1(S)}(m_a)}$ instead of $m_a$. So we have $a \cdot m_a = m_a \cdot a = \phi(a)m_a$ and $\tilde{\phi} \circ \pi_{\ell^1(S)}(m_a) = 1$.

Set $\mu_a = \pi_{\ell^1(S)}(m_a)$, so $\mu_a \in \ell^1(S)^{**}$, $\delta_s \mu_a = \mu_a \delta_s = \mu_a$ and $\tilde{\phi}(\mu_a) = \mu_a(\phi) = 1$, hence $S$ is an amenable semigroup, see the proof of [5, Corollary 2.10].

(ii) Suppose that $(\rho_a)_a$ is a net of continuous $\ell^1(S)$-bimodule morphism such that $\lim_a \pi_{\ell^1(S)^{**}} \circ \rho_a(a) = a$, for every $a \in A$. Set $M_a = \rho_a(s_0)$, where $s_0 \in Z(S)$, then it is easy to see that $a \cdot M_a = M_a \cdot a$ and $\phi(\pi_{\ell^1(S)}(M_a)) \to 1$, where $\phi$ is the augmentation character. Without loss of generality we may assume that $a \cdot M_a = M_a \cdot a = \phi(a)M_a$ and $\phi(\pi_{\ell^1(S)}(M_a)) = 1$. So $\ell^1(S)$ is left and right $\phi$-contractible. Now using the same arguments as in the [5, Corollary 2.10], we can find $m \in \ell^1(S)$ such that

$$
\delta_s m = m \delta_s = m.
$$

If $e_l$ is a left identity for $S$, then for every $s \in S$, we have

$$
m(s) = m(e_l s) = \delta_s m(e_l) = m(e_l),
$$

that is, $m \in \ell^1(S)$ is a constant function on $S$, so $S$ must be finite. \hfill \Box

Remark 3.2. Note that the converse of the previous Proposition (i) is not always true. To see this let $S = G$ be an infinite, discrete and amenable group. Suppose that $\ell^1(G)^{**}$ is approximately biprojective. Since $\ell^1(G)$ is unital, then $\ell^1(G)^{**}$ is unital too. Hence by [10, Proposition 3.8] $\ell^1(G)^{**}$ is a pseudo-contractible Banach algebra. So $\ell^1(G)^{**}$ is pseudo-amenable. Applying [10, Proposition 4.2] $G$ must be finite which is a contradiction.
Example 3.3. Let $S = \{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{C} \}$. With the matrix multiplication $S$ is a semigroup. We claim that $\ell^1(S)$ is not approximately biprojective. We go toward a contradiction and suppose that $\ell^1(S)$ is approximately biprojective. Let $s_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in Z(S)$ and let $s_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ be a right unit for $S$. Now the hypothesis of the previous Proposition(ii) holds. So $S$ is finite which is a contradiction.

Similarly for $S = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C} \}$ or $S = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \}$, $\ell^1(S)$ is not approximately biprojective.

Moreover if $S = \{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{C} \}$, then $S$ is amenable, but $\ell^1(S)$ is not approximately biprojective. To see this we suppose that $\ell^1(S)$ is approximately biprojective. Then there exists a net of $\ell^1(S)$-bimodule morphism $(\rho_a)_a : \ell^1(S) \to \ell^1(S) \otimes_p \ell^1(S)$ such that $\pi_{\ell^1(S)} \circ \rho_a(a) \to a$, for every $a \in \ell^1(S)$. Set $s_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $s_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Let $\rho_a(\delta_{s_1}) = \sum_{i=1}^\infty a_i^o \otimes b_i^o$, for some nets $\{a_i^o\}$ and $\{b_i^o\}$ in $\ell^1(S)$. Since $\delta_{s'} \delta'_s = \delta_{s_0}$ for every $s, s'$ in $S$, there exists a net $\{x_\alpha\}$ in $\mathbb{C}$ such that $\pi_{\ell^1(S)} \circ \rho_\alpha(\delta_{s_1}) = x_\alpha \delta_{s_0}$, then $\pi_{\ell^1(S)} \circ \rho_\alpha(\delta_{s_1}) = x_\alpha \delta_{s_0} \not\to \delta_{s_1}$, which is a contradiction.

Proposition 3.4. Let $S$ be a semigroup. If $\ell^1(S)^{**}$ is pseudo-contractible, then $S$ is amenable.

Proof. Let $\ell^1(S)^{**}$ be pseudo-contractible. By [1], Theorem 1.1, $\ell^1(S)^{**}$ is left and right $\phi$-contractible, for every $\phi \in \Delta(\ell^1(S))$. By [10], Proposition 3.5, $\ell^1(S)$ is left and right $\phi$-amenability, for every $\phi \in \Delta(\ell^1(S))$ including the augmentation character, so with similar argument as in the proof of Proposition 3.1, one can show that $S$ is an amenable semigroup.

The following lemma is similar to [17], Proposition 2.2 which we omit the proof.

Lemma 3.5. Let $A$ and $B$ be Banach algebras. Suppose that $A$ is unital and $B$ has a non-zero idempotent. If $A \otimes_p B$ is approximately biprojective, then $A$ is approximately biprojective.

Theorem 3.6. Let $S$ be an inverse semigroup which $(E(S), \leq)$ is uniformly locally finite. Then $\ell^1(S)$ is approximately biprojective if and only if $\ell^1(S)$ is biprojective.

Proof. Let $\ell^1(S)$ be approximate biprojective. Then there exists a net $(\rho_a)_a$ of continuous $\ell^1(S)$-bimodule morphism from $\ell^1(S)$ into $\ell^1(S) \otimes_p \ell^1(S)$ such that $\pi_{\ell^1(S)} \circ \rho_a(a) \to a$, for every
\(a \in \ell^1(S)\). Since \(S\) is a uniformly locally finite semigroup, by [17] Theorem 2.18 we have
\[
\ell^1(S) \cong \ell^1 - \bigoplus \{\mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}}))\},
\]
where \(\mathcal{D}_\lambda\) is a \(\mathcal{D}\)-class and \(G_{p_{\lambda}}\) is a maximal subgroup of \(S\) at \(p_{\lambda}\).

Let \(P_{p_{\lambda}} : \ell^1(S) \to \mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}}))\) be a homomorphism which is dense range. It is easy to see that \(P_{p_{\lambda}}\) is a bounded \(\ell^1(S)\)-bimodule morphism. Define
\[
\eta_{a} = P_{p_{\lambda}} \otimes P_{p_{\lambda}} \circ \rho_a |_{\mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}}))} : \mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}})) \to \mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}})) \otimes_p \mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}})).
\]

It is easy to see that \((\eta_{a})_{a} \in \mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}}))\)-bimodule morphism which satisfied
\[
(\pi_{\mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}}))} \circ \eta_{a}(a) = \pi_{\mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}}))} \circ P_{p_{\lambda}} \otimes P_{p_{\lambda}} \circ \rho_a |_{\mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}}))}(a) \rightarrow a,
\]
for every \(a \in \mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}}))\). Therefore \(\mathcal{M}_{E(\mathcal{D}_\lambda)}(\ell^1(G_{p_{\lambda}}))\) is an approximately biprojective Banach algebra. By Lemma 3.5 it is easy to see that \(\ell^1(G_{p_{\lambda}})\) is approximately biprojective. Then by [10] Proposition 3.8 \(\ell^1(G_e)\) is pseudo-contractible, hence \(G_{p_{\lambda}}\) is finite. Then \(\ell^1(S)\) is biprojective by the main result of [17].

Converse is clear.

\[
\square
\]

**Theorem 3.7.** Let \(S = \cup_{e \in E(S)} G_e\) be a Clifford semigroup such that \(E(S)\) is uniformly locally finite. If \(\ell^1(S)^{**}\) is approximately biprojective, then \(\ell^1(S)\) is pseudo-amenable.

**Proof.** Let \(\ell^1(S)^{**}\) be approximately biprojective. Then there exists a net \((\rho_a)_{a}\) of continuous \(\ell^1(S)^{**}\)-bimodule morphism from \(\ell^1(S)^{**}\) into \(\ell^1(S)^{**} \otimes_p \ell^1(S)^{**}\) such that \(\pi_{\ell^1(S)^{**}} \circ \rho_a(a) \rightarrow a\) for every \(a \in \ell^1(S)^{**}\). Since \(S\) is a uniformly locally finite semigroup, by [17] Theorem 2.16 \(\ell^1(S) \cong \ell^1 - \oplus_{e \in E(S)} \ell^1(G_e)\). Let \(x_e\) be a unit element of \(\ell^1(G_e)\) and let \(\phi \in \Delta(\ell^1(G_e))\). It is well-known that the maps \(b \mapsto x_e b\) and \(b \mapsto x_e b\) are \(w^* - w^*\)-continuous on \(\ell^1(S)^{**}\). Then for every \(a \in \ell^1(S)^{**}\), we have \(ax_e = x_e a\) and \(\tilde{\phi}(x_e) = 1\), where \(\tilde{\phi} \in \Delta(\ell^1(S)^{**})\) is the extension of \(\phi\). Define \(m_{\alpha}^e = \rho_a(x_e) \in \ell^1(S)^{**} \otimes \ell^1(S)^{**}\). Using [8] Lemma 1.7 we can consider \(m_{\alpha}^e\) in \((\ell^1(S) \otimes_p \ell^1(S))^{**}\). It is easy to see that \(a \cdot m_{\alpha}^e = m_{\alpha}^e \cdot a\) and \(\tilde{\phi} \circ \pi_{\ell^1(S)}^*(m_{\alpha}^e) = 1\), for every \(a \in \ell^1(S)^{**}\). Applying [20] Proposition 2.2 \(\ell^1(S)^{**}\) is left \(\tilde{\phi}\)-amenable. By [14] Proposition 3.4 \(\ell^1(S)\) is left \(\phi\)-amenable. Since \(\phi|_{\ell^1(G_e)}\) is non-zero, by [14] Lemma 3.1 \(\ell^1(G_e)\) is left \(\phi\)-amenable. Then by similar argument as in the proof of [19] Theorem 2.18, \(G_e\) is amenable, for every \(e \in E(S)\). To finish the proof apply [6] Theorem 3.7.

\[
\square
\]

**Example 3.8.** (i) There exists a pseudo-amenable Banach algebra which is not approximately biprojective. To see this, let \(G\) be an infinite amenable group. Then by [10] Proposition 4.1 \(\ell^1(G)\) is pseudo-amenable. Suppose that \(\ell^1(G)\) is approximately biprojective. Since \(\ell^1(G)\) is unital, using the same argument as in the proof of previous Proposition, one can show that \(\ell^1(G)\) is left \(\phi\)-contractible. Then by [16] Theorem 6.1 \(G\) is finite which is a contradiction. Hence \(\ell^1(G)\) is not approximately biprojective.
(ii) There exists an approximately biprojective Banach algebra which is not pseudo-contractible. To see this, let $A = M_{\Lambda}(\mathbb{C})$, where $\Lambda$ is an infinite set. By [17] Proposition 2.7 $A$ is biprojective, so $A$ is approximately biprojective. On the other hand if $A$ is pseudo-contractible, then $A$ has a central approximate identity. Therefore by [5] Theorem 2.2 $A$ is finite, which is a contradiction.

(iii) Now we give a semigroup algebra which is approximately biprojective but it is not pseudo-contractible. Let $S$ be a right zero semigroup, that is, $st = t$ for every $s, t \in S$, and let $|S| \geq 2$. Let $\phi$ be the augmentation character on $\ell^1(S)$, so for every $f, g \in \ell^1(S)$ we have $f * g = \phi(f)g$. One can see that $\ell^1(S)$ is biprojective, hence it is approximately biprojective, but if $\ell^1(S)$ is pseudo-contractible, then $\ell^1(S)$ has a right approximate identity $(e_\alpha)$. Consider $f_0 \in \ell^1(S)$ such that $\phi(f_0) = 1$, so

$$f_0 = \lim_\alpha f_0 \ast e_\alpha = \lim_\alpha \phi(f_0)e_\alpha = \lim_\alpha e_\alpha,$$

that is $f_0$ is a right unit for $\ell^1(S)$. On the other hand

$$g \ast f_0 = \lim_\alpha g \ast e_\alpha = \phi(g)f_0,$$

for every $g \in \ell^1(S)$. Let $s$ be an arbitrary element of $S$. Then by (3.2) and (3.3), we have $\delta_s = \delta_s \ast f_0 = f_0$ which implies that $|S| = 1$. Therefore a contradiction reveals.

Note that example (iii) shows that the hypothesis $Z(S) \neq \emptyset$ in the Proposition 3.1(ii) is necessary. Because if we consider a right zero semigroup $S$ with $|S| = \infty$, then $Z(S) = \emptyset$ and $S$ has a left identity. One can show that $\ell^1(S)$ is approximately biprojective but $S$ is not finite.

Zhang in [24] gives an example of approximately biprojective Banach algebra which is not biprojective.

**Theorem 3.9.** Let $A$ be an approximately biprojective Banach algebra with a left approximate identity (right approximate identity) and let $\phi \in \Delta(A)$. Then $A$ is left $\phi$-contractible(right $\phi$-contractible), respectively.

**Proof.** Suppose that $A$ is approximately biprojective. Then there exists a net of $A$-bimodule morphisms $(\rho_\alpha)_\alpha$ from $A$ into $A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(a) \to a$, for every $a \in A$. Let $L = \ker \phi$. Define $\eta_\alpha := id_A \otimes q \circ \rho_\alpha : A \to A \otimes_p \frac{A}{L}$, where $q$ is a quotient map. It is easy to see that $\eta_\alpha$ is a left $A$-module morphism, for every $\alpha$. Since $A$ has a left approximate identity, $\overline{AL} = L$, so for every $l \in L$, there exist $a \in A$ and $l' \in L$ such that $l = al'$. Also since for every $l \in L$, $q(l) = 0$ and $(\rho_\alpha)$ is a net of $A$-bimodule morphism, we have

$$\eta_\alpha(l) = id_A \otimes q \circ \rho_\alpha(l) = id_A \otimes q \circ \rho_\alpha(al') = id_A \otimes q \circ (\rho_\alpha(a) : l') = 0.$$

Thus $\eta_\alpha$ can be dropped on $\frac{A}{L}$, for every $\alpha$. So we can see that $\eta_\alpha : \frac{A}{L} \to A \otimes_p \frac{A}{L}$ is a left $A$-module morphism.
We define a character $\phi$ on $\mathbb{A}$ by $\phi(a + L) = \phi(a)$, for every $a \in A$. Consider $\gamma_{\alpha} = i d_{A} \otimes \phi \circ \eta_{\alpha} : \mathbb{A} \to A$. Since
\begin{equation}
\gamma_{\alpha}(a \cdot x + L) = id_{A} \otimes \phi \circ \eta_{\alpha}(ax + L) = id_{A} \otimes \phi \circ \eta_{\alpha}(ax) = a \cdot (id_{A} \otimes \phi \circ \eta_{\alpha})(x),
\end{equation}
(3.4)
and $\eta_{\alpha}$ is a left $A$-module morphism, $\gamma_{\alpha}$ is a left $A$-module morphism. Note that $(\gamma_{\alpha})$ is a net of non-zero maps. To see this consider
\begin{equation}
\phi \circ \gamma_{\alpha}(x + L) = \phi \circ \phi \circ \eta_{\alpha}(x + L) = \phi \circ \phi \circ \eta_{\alpha}(x)
\end{equation}
(3.5)
for every $x \in A$. Pick $x_{0}$ in $A$ such that $\phi(x_{0}) = 1$. Define $m_{\alpha} = \gamma_{\alpha}(x_{0} + L)$. Then we have
\begin{equation}
\phi(m_{\alpha}) = \phi \circ \gamma_{\alpha}(x_{0} + L) = \phi \circ \pi_{A} \circ \rho_{\alpha}(x_{0}) \to \phi(x_{0}) = 1.
\end{equation}
Consider
\begin{equation}
ax_{0} + L = (a - \phi(a)x_{0} + \phi(a)x_{0})x_{0} + L = ax_{0} - \phi(a)x_{0}^{2} + \phi(a)x_{0}^{2} + L
\end{equation}
(3.6)
\begin{align*}
&= \phi(a)x_{0}^{2} + L \\
&= \phi(a)x_{0} + L,
\end{align*}
since $x_{0}^{2} - x_{0} \in L$. Therefore
\begin{equation*}
am_{\alpha} = a \gamma_{\alpha}(x_{0} + L) = \gamma_{\alpha}(ax_{0} + L) = \phi(a)\gamma_{\alpha}(x_{0} + L) = \phi(a)m_{\alpha}.
\end{equation*}
Replacing $(m_{\alpha})$ with $(\frac{m_{\alpha}}{\phi(m_{\alpha})})$, we can assume that $am_{\alpha} = \phi(a)m_{\alpha}$ and $\phi(m_{\alpha}) = 1$. Then $A$ is left $\phi$-contractible, see [16 Theorem 2.1].

**Remark 3.10.** Existence of a left approximate identity is essential for previous Theorem, which we cannot omit it. To see this let $A = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$. With matrix operation $A$ becomes a Banach algebra. It is easy to see that $A$ is a biprojective Banach algebra. Then $A$ is approximately biprojective. If $A$ has a left approximate identity, then an easy calculation shows that $\dim A = 1$ which is impossible. On the other hand if we define $\phi(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}) = b$. It is easy to see that $\phi$ is a character on $A$. One can show that $A$ is left $\phi$-contractible if and only if $\dim A = 1$ which is impossible.

At the following result we extend [5 Corollary 2.10](ii), to the approximate biprojective case.

**Corollary 3.11.** Let $S$ be a semigroup with a left unit. If $\ell^{1}(S)$ is approximately biprojective with a right approximate identity, then $S$ is finite.

**Proof.** Let $\ell^{1}(S)$ be approximately biprojective. By Theorem [3.9] $\ell^{1}(S)$ is left and right $\phi$-contractible for every $\phi \in \Delta(\ell^{1}(S))$. Follow the same arguments as in the proof of [5 Corollary 2.10] to finish the proof. \qed
Remark 3.12. Let $S$ be a bicyclic semigroup, that is, $S$ is a semigroup, generated by two elements $p$ and $q$ which $pq = e$ for a unit element $e$. Then $\ell^1(S)$ is a unital Banach algebra. Using the previous Corollary, one can see that $\ell^1(S)$ is not approximately biprojective.

Consider the semigroup $S = \mathbb{N}_\lor$, with semigroup operation $m \lor n = \max\{m, n\}$, where $m$ and $n$ are in $\mathbb{N}$. It is easy to see that $\ell^1(S)$ is a unital Banach algebra with unit $\delta_1$. Since $S$ is an infinite semigroup, by previous Corollary $\ell^1(S)$ is not approximately biprojective.

Suppose that $A$ and $B$ are Banach algebras and $M$ is a Banach $(A, B)$-module. The matrix algebra $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is called a triangular Banach algebra which equipped with the norm $\| \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \|_T = \|a\|_A + \|m\|_M + \|b\|_B$ for $a \in A$, $m \in M$ and $b \in B$.

In [15, Corollary 3.3] the authors showed that some triangular Banach algebras are not biprojective at all. Here at the following theorem we are going to extend this result to the approximately biprojective case.

**Theorem 3.13.** Let $A$ be a Banach algebra with a left approximate identity and let $\phi \in \Delta(A)$. Then $T = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ is not approximately biprojective.

**Proof.** We are going toward a contradiction and suppose that $T$ is approximately biprojective.

Define a character $\psi_\phi$ on $T$ by $\psi_\phi\left( \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right) = \phi(b)$, for every $a, x$ and $b$ in $A$. Since $A$ has a left approximate identity, by Theorem 3.9 $T$ is a left $\psi_\phi$-contractible Banach algebra. Set $I = \begin{pmatrix} 0 & A \\ 0 & A \end{pmatrix}$. Clearly $I$ is a closed ideal in $T$ which $\psi_\phi|_I \neq 0$. Then by [16, Proposition 3.8] $I$ is left $\psi_\phi$-contractible too. Then there exists $\begin{pmatrix} 0 & i \\ 0 & j \end{pmatrix} \in I$ such that

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & i \\ 0 & j \end{pmatrix} = \psi_\phi\left( \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \right) \begin{pmatrix} 0 & i \\ 0 & j \end{pmatrix} = \phi(b) \begin{pmatrix} 0 & i \\ 0 & j \end{pmatrix}$$

and

$$\psi_\phi\left( \begin{pmatrix} 0 & i \\ 0 & j \end{pmatrix} \right) = \phi(j) = 1,$$

for every $a, b \in A$. Suppose that $(e_\alpha)_{\alpha}$ is the left approximate identity of $A$. Let $a \in \{e_\alpha\}_{\alpha}$ and $b$ be an arbitrary element of $\ker \phi$. Put $a$ and $b$ in (3.7) we have $aj = \phi(b)i = 0$. This implies that $e_\alpha j = 0$ for every $\alpha$. Since $e_\alpha$ is an approximate identity for $A$, we have $j = 0$. On the other hand $\phi(j) = 1$ which is a contradiction. \[\square\]

Consider the semigroup $\mathbb{N}_\land$, with the semigroup operation $m \land n = \min\{m, n\}$, where $m$ and $n$ are in $\mathbb{N}$. Let $w : \mathbb{N}_\land \rightarrow \mathbb{R}^+$ be a weight, that is a function which $w(st) \leq w(s)w(t)$, for
every $s,t \in S$, for the further details see \cite{4}. We recall that for every weight $\ell^1(N_\lambda)$ has an approximate identity, see \cite{4} Proposition 3.3.1. Also $\Delta(\ell^1(N_\lambda), w)$ consists precisely of the all functions $\phi_n : \ell^1(N_\lambda, w) \to \mathbb{C}$ defined by $\phi_n(\sum_{i=1}^\infty \alpha_i \delta_i) = \sum_{i=n}^\infty \alpha_i$ for every $n \in \mathbb{N}$.

**Corollary 3.14.** Let $S = N_\lambda$ and $w$ be a weight on $S$. Then $T = \begin{pmatrix} \ell^1(S, w) & \ell^1(S, w) \\ 0 & \ell^1(S, w) \end{pmatrix}$ is not approximately biprojective.

**Proof.** It is well-known that for every weight $\ell^1(S, w)$ has an approximate identity. Now apply previous Theorem, to finish the proof. \qed

**Corollary 3.15.** Let $S = N_\lambda$. Then $T = \begin{pmatrix} \ell^1(S)^** & \ell^1(S)^** \\ 0 & \ell^1(S)^** \end{pmatrix}$ is not approximately biprojective.

**Proof.** We go toward a contradiction and suppose that $T$ is approximately biprojective. Since $\ell^1(S)$ has a bounded approximate identity, see \cite{4} Proposition 3.3.1, $\ell^1(S)^**$ has a right unit. So $T$ has a right unit. Let $\phi \in \Delta(\ell^1(S)^**)$ and $\psi_\phi \left( \begin{array}{cc} a & x \\ 0 & b \end{array} \right) = \phi(a)$, for every $a,x$ and $b$ in $\ell^1(S)^**$. Apply Theorem 3.9 $T$ is right $\psi_\phi$-contractible. Set $I = \begin{pmatrix} \ell^1(S)^** & \ell^1(S)^** \\ 0 & 0 \end{pmatrix}$. Then $I$ is right $\psi_\phi|_I$-contractible. But follow the similar arguments as in the proof of Theorem 3.13 implies that $I$ is not right $\psi_\phi|_I$-contractible which is a contradiction. \qed

**Corollary 3.16.** Let $S$ be a right zero semigroup. Then $T = \begin{pmatrix} \ell^1(S) & \ell^1(S) \\ 0 & \ell^1(S) \end{pmatrix}$ is not approximately biprojective.

**Proof.** Since $S$ is a right zero semigroup, $\ell^1(S)$ has a left unit. Then $T$ has a left unit too. By Theorem 3.13 $T$ is not approximately biprojective. \qed

**Proposition 3.17.** Let $S = \cup_{e \in E(S)} G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite. Then $T = \begin{pmatrix} \ell^1(S)^** & \ell^1(S)^** \\ 0 & \ell^1(S)^** \end{pmatrix}$ is not approximately biprojective.

**Proof.** It is well-known that $\ell^1(S) \cong \ell^1 - \oplus_{e \in E(S)} \ell^1(G_e)$. Let $x_e$ denote for unit element of $\ell^1(G_e)$. It is easy to see that $\delta_{x_e}$ commutes with every elements of $\ell^1(S)$. Since two maps $b \mapsto \delta_{x_e} b$ and $b \mapsto b \delta_{x_e}$ are $w^* - w^*$-continuous on $\ell^1(S)^**$, where $b \in \ell^1(S)^*$, $\delta_{x_e}$ also commutes with every elements of $\ell^1(S)^**$. Consider the element $t = \begin{pmatrix} \delta_{x_e} & 0 \\ 0 & \delta_{x_e} \end{pmatrix} \in T$, it is easy to see that $t$ commutes with every element of $T$. Let $\phi$ be the augmentation character on $\ell^1(S)$ and $\widetilde{\phi}$ its extension to $\ell^1(S)^**$ and $\psi_{\widetilde{\phi}}$ be the character on $T$ which defined in the proof of Theorem 3.13 with respect to $\widetilde{\phi}$. Now go toward a contradiction and suppose that $T$ is approximate.
biprojective. Follow the same arguments as in the proof of Proposition 3.1. It is easy to see that $I$ is a closed ideal of $T$. Then by [16] Proposition 3.8, $I$ is left and right $\hat{\psi}$-contractible. Hence there exists $t_1$ and $t_2$ in $I$ such that $at_1 = \hat{\psi}(a)t_1$, $t_2a = \hat{\psi}(a)t_2$ and $\hat{\psi}(t_1) = \hat{\psi}(t_2) = 1$, for every $a \in I$. Define $m = t_1t_2 \in I$, then there exists element $i$ and $j$ in $\ell^1(S)^{**}$ such that $m = \begin{pmatrix} 0 & i \\ 0 & j \end{pmatrix}$. It is easy to see that 

\[(3.8) \quad am = ma, \quad \psi_\beta(m) = 1,\]

for every $a \in I$. Set $a = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$, where $x, y \in \ell^1(S)^{**}$ and put $a$ in (3.8). Then we have 

\[(3.9) \quad xj = iy, \quad \tilde{\phi}(j) = 1,\]

for every $x, y \in \ell^1(S)^{**}$. Set $x = \delta_{C^+}$ and $y$ be any element of $\ker \hat{\phi}$. Put these $x$ and $y$ in (3.9), and take $\tilde{\phi}$ on equation $xj = iy$ it implies that $\tilde{\phi}(j) = 0$ which is a contradiction. \hfill \Box

**Proposition 3.18.** Let $G$ be a locally compact group. Then $\ell^1(S) \otimes_p M(G)$ is approximately biprojective if and only if $G$ is finite, where $S$ is the semigroup which is defined in the Example 3.8.

**Proof.** Suppose that $\ell^1(S) \otimes_p M(G)$ is approximately biprojective. Since $M(G)$ is unital and $\ell^1(S)$ has a left identity. Then by Theorem 3.9 $\ell^1(S) \otimes_p M(G)$ is left $\phi$-contractible for every $\phi \in \Delta(\ell^1(S) \otimes_p M(G))$. Apply [16] Theorem 3.14] to show that $M(G)$ is character contractible, hence by [16] Corollary 6.2] $G$ is finite.

Converse holds By biprojectivity of $\ell^1(S)$ and [17] Proposition 2.4. \hfill \Box

**Remark 3.19.** We want to give some Banach algebras which is never approximately biprojective. Consider the semigroup $N_\lor$, with semigroup operation $m \lor n = \max\{m, n\}$, where $m$ and $n$ are in $\mathbb{N}$. Authors in [21] Example 3.5] showed that for every weight $w : N_\lor \to \mathbb{R}^+$, $\ell^1(N_\lor, w)$ is not pseudo-contractible, then it is not approximately biprojective, since $\ell^1(N_\lor, w)$ has a unit $\delta_1$, see [4] page 43]. Let $A$ be a Banach algebra with a bounded left approximate identity, with $\Delta(A) \neq \emptyset$. Then use the similar arguments as in the previous Proposition $A \otimes_p \ell^1(N_\lor, w)$ and $A \oplus \ell^1(N_\lor, w)$ are never approximately biprojective.

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Faculty of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Avenue, 15914 Tehran, Iran.

E-mail address: amir.sahami@aut.ac.ir

E-mail address: arpourabbas@aut.ac.ir