Quantum Entanglement Analysis Based on Abstract Interpretation

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Abstract. Entanglement is a non local property of quantum states which has no classical counterpart and plays a decisive role in quantum information theory. Several protocols, like the teleportation, are based on quantum entangled states. Moreover, any quantum algorithm which does not create entanglement can be efficiently simulated on a classical computer. The exact role of the entanglement is nevertheless not well understood. Since an exact analysis of entanglement evolution induces an exponential slowdown, we consider approximative analysis based on the framework of abstract interpretation. In this paper, a concrete quantum semantics based on superoperators is associated with a simple quantum programming language. The representation of entanglement, i.e. the design of the abstract domain is a key issue. A representation of entanglement as a partition of the memory is chosen. An abstract semantics is introduced, and the soundness of the approximation is proven.

1 Introduction

Quantum entanglement is a non local property of quantum mechanics. The entanglement reflects the ability of a quantum system composed of several subsystems, to be in a state which cannot be decomposed into the states of the subsystems. Entanglement is one of the properties of quantum mechanics which caused Einstein and others to dislike the theory. In 1935, Einstein, Podolsky, and Rosen formulated the EPR paradox [7].

On the other hand, quantum mechanics has been highly successful in producing correct experimental predictions, and the strong correlations associated with the phenomenon of quantum entanglement have been observed indeed [2].

Entanglement leads to correlations between subsystems that can be exploited in information theory (e.g., teleportation scheme [3]). The entanglement plays also a decisive, but not yet well-understood, role in quantum computation, since any quantum algorithm can be efficiently simulated on a classical computer when the quantum memory is not entangled during all the computation. As a consequence, interesting quantum algorithms, like Shor’s algorithm for factorisation [19], exploit this phenomenon.

In order to know what is the amount of entanglement of a quantum state, several measures of entanglement have been introduced (see for instance [13]). Recent works consist in characterising, in the framework of the one-way quantum
quantum computation \cite{20}, the amount of entanglement necessary for a universal model of quantum computation. Notice that all these techniques consist in analysing the entanglement of a given state, starting with its mathematical description.

In this paper, the entanglement evolution during the computation is analysed. The description of quantum evolutions is done via a simple quantum programming language. The development of such quantum programming languages is recent, see \cite{17,8} for a survey on this topic.

An exact analysis of entanglement evolution induces an exponential slowdown of the computation. Model checking techniques have been introduced \cite{9} including entanglement. Exponential slowdown of such analysis is avoided by reducing the domain to stabiliser states (i.e. a subset of quantum states that can be efficiently simulated on a classical computer). As a consequence, any quantum program that cannot be efficiently simulated on a classical computer cannot be analysed.

Prost and Zerrari \cite{16} have recently introduced a logical entanglement analysis for functional languages. This logical framework allows analysis of higher-order functions, but does not provide any static analysis for the quantum programs without annotation. Moreover, only pure quantum states are considered.

In this paper, we introduce a novel approach of entanglement analysis based on the framework of abstract interpretation \cite{5}. A concrete quantum semantics based on superoperators is associated with a simple quantum programming language. The representation of entanglement, i.e. the design of the abstract domain is a key issue. A representation of entanglement as a partition of the memory is chosen. An abstract semantics is introduced, and the soundness of the approximation is proved.

2 Basic Notions and Entanglement

2.1 Quantum Computing

We briefly recall the basic definitions of quantum computing; please refer to Nielsen and Chuang \cite{13} for a complete introduction to the subject.

The state of a quantum system can be described by a density matrix, i.e. a self adjoint \footnote{1} positive-semidefinite \footnote{2} complex matrix of trace \footnote{3} less than one. The set of density matrices of dimension $n$ is $D_n \subseteq \mathbb{C}^{n \times n}$.

The basic unit of information in quantum computation is a quantum bit or qubit. The state of a single qubit is described by a $2 \times 2$ density matrix $\rho \in D_2$. The state of a register composed of $n$ qubits is a $2^n \times 2^n$ density matrix. If two registers $A$ and $B$ are in states $\rho_A \in D_{2^n}$ and $\rho_B \in D_{2^m}$, the composed system $A, B$ is in state $\rho_A \otimes \rho_B \in D_{2^{n+m}}$.

The basic operations on quantum states are unitary operations and measurements. A unitary operation maps an $n$-qubit state to an $n$-qubit state, and is

\footnote{1} $M$ is self adjoint (or Hermitian) if and only if $M^\dagger = M$.
\footnote{2} $M$ is positive-semidefinite if all the eigenvalues of $M$ are non-negative.
\footnote{3} The trace of $M$ ($\text{tr}(M)$) is the sum of the diagonal elements of $M$.