TOTAL VARIATION CUTOFF FOR THE FLIP-TRANSPOSE TOP WITH RANDOM SHUFFLE

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Abstract. We consider a random walk on the hyperoctahedral group $B_n$ generated by the signed permutations of the forms $(i, n)$ and $(-i, n)$ for $1 \leq i \leq n$. We call this the flip-transpose top with random shuffle on $B_n$. We find the spectrum of the transition probability matrix for this shuffle. We prove that the mixing time for this shuffle is of order $n \log n$. We also show that this shuffle exhibits the cutoff phenomenon. In the appendix, we show that a similar random walk on the demihyperoctahedral group $D_n$ also has a cutoff at $(n - \frac{1}{2}) \log n$.

1. Introduction

Card shuffling problems are mathematically analysed by considering them as random walks on symmetric groups [6,7,11,17,19]. In this paper our main aim is to study the properties of a random walk on Coxeter groups of type B [1]. This work is a generalisation of the transpose top with random shuffle [7] on signed permutations. A signed permutation [1] is a bijection $\pi$ from $\{-1, \ldots, \pm n\}$ to itself satisfying $\pi(-i) = -\pi(i)$ for all $1 \leq i \leq n$. A signed permutation is completely determined by its image on the set $[n] := \{1, \ldots, n\}$. Given a signed permutation $\pi$, we write it in window notation by $[\pi_1, \ldots, \pi_n]$, where $\pi_i$ is the image of $i$ under $\pi$. The set of all signed permutations forms a group under composition and is known as the hyperoctahedral group and is denoted by $B_n$. The subset of $B_n$ consisting of those signed permutations having even number of negative entries in their window notation form a subgroup of $B_n$, called the demihyperoctahedral group and is denoted by $D_n$.

Suppose there are $n$ cards labelled from 1 to $n$ and each card has two orientations namely ‘face up’ and ‘face down’. Given an arrangements of these $n$ cards in a row we associate a signed permutation $[\pi_1, \pi_2, \ldots, \pi_n]$ to it in the following way: $\pi_i$ is the label of the $i$th card (counting started from left) with sign

$$
\begin{cases}
\text{positive}, & \text{if the orientation of the card is ‘face up’} \\
\text{negative}, & \text{if the orientation of the card is ‘face down’}.
\end{cases}
$$

Thus every arrangement of the $n$ cards in a row represents a signed permutation in its window notation. We consider the following shuffle on the set of all arrangements of these $n$ cards in a row: Given an arrangement, either interchange the last card with a random card, or interchange the last card with a random card and flip both of them, with equal probability.

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We call this shuffle the flip-transpose top with random shuffle. Formally, this shuffle is the random walk on $B_n$ driven by the probability measure $P$ on $B_n$ given by

\[
P(\pi) = \begin{cases} 
\frac{1}{2n}, & \text{if } \pi = \text{id}, \text{ the identity element of } B_n, \\
\frac{1}{2n}, & \text{if } \pi = (i, n) \text{ for } 1 \leq i \leq n - 1, \\
\frac{1}{2n}, & \text{if } \pi = (-i, n) \text{ for } 1 \leq i \leq n, \\
0, & \text{otherwise}.
\end{cases}
\]

In this paper, we will show that this random walk satisfies the cutoff phenomenon and determine the mixing time for this random walk. We will first recall some concepts and terminologies which we will use in this paper frequently.

1.1. Representation theoretic background. Let $V$ be a finite-dimensional complex vector space and $GL(V)$ be the group of all invertible linear operators from $V$ to itself under composition of linear mappings. Elements of $GL(V)$ can be thought of as invertible matrices over $\mathbb{C}$. Let $G$ be a finite group, a mapping $\rho : G \to GL(V)$ is said to be a linear representation of $G$ if $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$ for all $g_1, g_2$ in $G$. The dimension of the vector space $V$ is said to be the dimension of the representation $\rho$ and is denoted by $d_\rho$. $V$ is called the $G$-module corresponding to the representation $\rho$ in this case. If $\mathbb{C}[G] = \{ \sum_i c_i g_i \mid c_i \in \mathbb{C}, g_i \in G \}$, then we define the right regular representation $R : G \to GL(\mathbb{C}[G])$ of $G$ by

\[
R(g) \left( \sum_{h \in G} C_h h \right) = \sum_{h \in G} C_h h g, \text{ where } C_h \in \mathbb{C}.
\]

Let $H$ be a subgroup of $G$. The restriction of the representation $\rho$ to $H$ is denoted by $\rho \downarrow^G_H$ and is defined by $\rho \downarrow^G_H (h) := \rho(h)$ for all $h \in H$. The trace of the matrix $\rho(g)$ is said to be the character value of $\rho$ at $g$ and is denoted by $\chi^\rho(g)$. A vector subspace $W$ of $V$ is said to be stable (or ‘invariant’) under $\rho$ if $\rho(g)(W) \subseteq W$ for all $g \in G$. The representation $\rho$ is irreducible if $V$ is non-trivial and $V$ has no non-trivial proper stable subspace. Two representations $(\rho_1, V_1)$ and $(\rho_2, V_2)$ of $G$ are said to be isomorphic if there exists an invertible linear map $T : V_1 \to V_2$ such that the following diagram commutes for all $g \in G$:

\[
\begin{array}{ccc}
V_1 & \xrightarrow{\rho_1(g)} & V_1 \\
\downarrow T & & \downarrow T \\
V_2 & \xrightarrow{\rho_2(g)} & V_2
\end{array}
\]

If $V_1 \otimes V_2$ denotes the tensor product of the vector spaces $V_1$ and $V_2$, then the tensor product of two representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ is a representation denoted by $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$ and defined by,

\[
(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2) \text{ for } v_1 \in V_1, v_2 \in V_2 \text{ and } g \in G.
\]

We will state some results from representation theory of finite groups without proof. For more details, see [14, 16, 20].
1.2. Random walks on finite groups. We first recall some terminology. Let \( p \) and \( q \) be two probability measures on a finite group \( G \). The Fourier transform \( \hat{\rho} \) of \( p \) at the representation \( \rho \) is defined by the matrix \( \sum_{x \in G} p(x) \rho(x) \). We define the convolution \( p \ast q \) of \( p \) and \( q \) by

\[
(p \ast q)(x) := \sum_{\{u,v \in G | uv = x\}} p(u)q(v).
\]

It can be easily seen that \( \hat{p} \ast \hat{q} = \hat{p} \hat{q} \). For the right regular representation \( R \), the matrix \( \hat{p}(R) \) can be thought of as the action of the group algebra element \( \sum_{g \in G} p(g)g \) on \( \mathbb{C}[G] \) from the right.

A random walk on a finite group \( G \) driven by a probability measure \( p \) is a discrete time Markov chain with state space \( G \) and transition probabilities \( M_p(x, y) = p(x^{-1}y), x, y \in G \). The transition matrix \( M_p \) is the transpose of \( \hat{p}(R) \). If \( p^k \) denotes the \( k \)-fold convolution of \( p \) with itself, then the probability of reaching state \( y \) starting from state \( x \) using \( k \) transitions is \( p^k(x^{-1}y) \). The random walk is said to be irreducible if given any two states \( u \) and \( v \) there exists \( t \) (depending on \( u \) and \( v \)) such that \( p^{t\ell}(u^{-1}v) > 0 \). We now state the lemma regarding the irreducibility of the random walk on \( G \) driven by \( p \).

**Lemma 1.1** (*[17] Proposition 2.3*). Let \( G \) be a finite group and \( p \) be a probability measure on \( G \). The random walk on \( G \) driven by \( p \) is irreducible if and only if the support of \( p \) generates \( G \).

A probability vector (a row vector with non-negative components which sum to one) \( \Pi \) is said to be a stationary distribution of the random walk if \( \Pi \) is a left eigenvector of the transition matrix with eigenvalue 1. There exists a unique stationary distribution for each irreducible random walk. If the random walk on \( G \) driven by \( p \) is irreducible, then the stationary distribution for this random walk is the uniform distribution on \( G \) ([17] Section 2.2). From now on, we denote the uniform distribution on \( G \) by \( U_G \). Let us consider a random walk and fix one state \( x \in G \). The greatest common divisor of the set of all times when it is possible for the walk to return to the starting state \( x \) is said to be the period of the state \( x \). All the states of an irreducible random walk have the same period (see [10] Lemma 1.6). An irreducible random walk is said to be aperiodic if the common period for all its states is 1.

Let \( \mu \) and \( \nu \) be two probability distributions on \( \Omega \). The total variation distance between \( \mu \) and \( \nu \) is defined by

\[
||\mu - \nu||_{TV} := \sup_{A \subset \Omega} |\mu(A) - \nu(A)|.
\]

The total variation distance between two discrete distributions \( \mu \) and \( \nu \) is half the \( \ell_1 \) distance between them (see [10] Proposition 4.2). If the random walk on a finite group \( G \) driven by a probability measure \( p \) on \( G \) is irreducible and aperiodic, then the distribution after the \( k \)th transition converges to the uniform measure on \( G \) in total variation distance as \( k \to \infty \). We now define the total variation cutoff phenomenon.

**Definition 1.1.** Let \( \{G_n\}_0^\infty \) be a sequence of finite groups and \( p_n \) be probability measures on \( G_n, n \geq 0 \). For each \( n \geq 0 \), consider the irreducible and aperiodic random walks on \( G_n \) driven by \( p_n \). We say that the total variation cutoff phenomenon holds for the family \( \{(G_n, p_n)\}_0^\infty \) if there exists a sequence \( \{\tau_n\}_0^\infty \) of positive reals such that the following hold:
(1) \( \lim_{n \to \infty} \tau_n = \infty \).

(2) For any \( \epsilon \in (0, 1) \) and \( k_n = \lfloor (1 + \epsilon) \tau_n \rfloor \), \( \lim_{n \to \infty} ||P_n^{k_n} - U_{B_n}||_{TV} = 0 \) and

(3) For any \( \epsilon \in (0, 1) \) and \( k_n = \lfloor (1 - \epsilon) \tau_n \rfloor \), \( \lim_{n \to \infty} ||P_n^{k_n} - U_{B_n}||_{TV} = 1 \).

Here \( \lfloor x \rfloor \) denotes the floor of \( x \) (the largest integer less than or equal to \( x \)).

Informally, we will say that \( \{ (G_n, p_n) \}_{0}^{\infty} \) has a total variation cutoff at time \( \tau_n \). Roughly the cutoff phenomenon depends on the multiplicity of the second largest eigenvalue of the transition matrix [5].

**Proposition 1.2.** The flip-transpose top with random shuffle on \( B_n \) is irreducible and aperiodic.

**Proof.** We know that the set \( \{ (-1, 1), (1, 2), (2, 3), \ldots, (n - 1, n) \} \) generates \( B_n \). Let \( i \) be any integer from \( [n - 1] \). Then \((i, i + 1) = (i + 1, n)(i, n)(i + 1, n) \) and \((-1, 1) = (1, n)(-n, n)(1, n) \). Therefore the support of the measure \( P \) generates \( B_n \) and hence the chain is irreducible by Lemma [1.1]. Given any \( \pi \in B_n \), the set of all times when it is possible for the chain to return to the starting state \( \pi \) contains the integer 1 (\( \because \) the identity element of \( B_n \) is in the support of \( P \)). Therefore the period of the state \( \pi \) is 1 and hence from irreducibility all the states of this chain have period 1. Thus this chain is aperiodic. \( \square \)

Proposition 1.2 says that the flip-transpose top with random shuffles on \( B_n \) has unique stationary distribution \( U_{B_n} \) and the distribution after the \( k \)th transition will converge to its stationary distribution as \( k \to \infty \).

The plan of the rest of the paper is as follows: In Section 2 we will find the spectrum of the transition matrix \( \hat{P}(R) \). We will find an upper bound of \( ||P^k - U_{B_n}||_{TV} \) for \( k \geq n \log n + cn \), \( c > 0 \) in Section 3. Finally in Section 4, we will find a lower bound of \( ||P^k - U_{B_n}||_{TV} \) for \( k = n \log n + cn \), \( c \ll 0 \) (large negative number) and show that the total variation cutoff for the shuffle on \( B_n \) occurs at \( n \log n \).

In Appendix A, we give a outline of the irreducible representations of the demihyperoctahedral group \( D_n \). We also give an idea for the deduction of irreducible representations of \( D_n \) from that of \( B_n \). In Appendix B, we consider a random walk analogous to the flip-transpose top with random shuffles and show that this random walk exhibits the total variation cutoff phenomenon with cutoff at \((n - \frac{1}{2}) \log n \).

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### 2. Spectrum of The Transition Matrix \( \hat{P}(R) \)

In this section we find the eigenvalues of the transition matrix \( \hat{P}(R) \), the Fourier transform of \( P \) at the right regular representation \( R \) of \( B_n \). To find the eigenvalues of \( \hat{P}(R) \) we will use the representation theory of the hyperoctahedral group \( B_n \). We briefly discuss the representation theory of \( B_n \). For more details one can see [3, 12, 15].
**Definition 2.1.** A partition $\lambda$ of a positive integer $n$ is denoted by $\lambda \vdash n$ and is defined by a finite sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ and $\sum_{i=1}^{\ell} \lambda_i = n$. The Young diagram of shape $\lambda$ is an arrangement of $n$ boxes into $\ell$ rows in a left justified way such that the $i$th row contains $\lambda_i$ boxes for $1 \leq i \leq \ell$. The content of a box in row $i$ and column $j$ of a Young diagram is the integer $j - i$. A standard Young tableau of shape $\lambda$ is a filling of the boxes of the Young diagram of shape $\lambda$ with the numbers $1, \ldots, n$ such that the numbers are increasing along each row and each column. The set of all standard Young tableaux of shape $\lambda$ is denoted by $\text{tab}(\lambda)$ and the number of standard Young tableaux of shape $\lambda$ is denoted by $d_\lambda$.

![Figure 1. All elements of $D_3$.](image)

**Definition 2.2.** Let $n$ be a positive integer. A (Young) double-diagram with $n$ boxes $\mu$ is a pair of Young diagram such that the total number of boxes is $n$. We define $||\mu|| = n$. The set of all double-diagram with $n$ boxes is denoted by $D_n$. For example the double-diagrams with 3 boxes are listed in Figure 1. A standard (Young) double-tableau of shape $\mu$ is obtained by taking the double-diagram $\mu$ and filling its $||\mu||$ boxes (bijectively) with the numbers $1, 2, \ldots, ||\mu||$ such that the numbers in the boxes strictly increase along each row and each column of all Young diagrams occurring in $\mu$. Let $\text{tab}_D(n, \mu)$, where $\mu \in D_n$, denote the set of all standard double-tableaux of shape $\mu$ and let $\text{tab}_D(n) = \bigcup_{\mu \in D_n} \text{tab}_D(n, \mu)$. For example an element of $\text{tab}_D(8)$ is given in Figure 2. Let $T \in \text{tab}_D(n, \mu)$ and $i \in [n]$. Let $b_T(i)$ be the box of the Young diagram in $\mu$, in which the number $i$ resides. We denote the content of the box $b_T(i)$ by $c(b_T(i))$. For the standard double-tableau given in Figure 2 we have $c(b_T(1)) = 0$, $c(b_T(2)) = 1$, $c(b_T(3)) = 0, \ldots, c(b_T(8)) = 2$.

![Figure 2. An element of $\text{tab}_D(8)$.](image)

**Definition 2.3.** The Young-Jucys-Murphy elements $X_1, X_2, \ldots, X_n$ of $\mathbb{C}[B_n]$ are defined by $X_1 = 0$ and $X_i = \sum_{k=1}^{i-1} (k, i) + \sum_{k=1}^{i-1} (-k, i)$, for all $2 \leq i \leq n$. 


**Definition 2.4.** Let \( \mu \in \hat{B}_n \) and consider the \( B_n \)-module \( V^\mu \). Since the branching is simple [12, Section 3], the decomposition into irreducible \( B_{n-1} \)-modules is canonical and is given by
\[
V^\mu = \bigoplus_\lambda V^\lambda,
\]
where the sum is over all \( \lambda \in \hat{B}_{n-1} \), with \( \lambda \not\geq \mu \) (i.e. there is an edge from \( \lambda \) to \( \mu \) in the branching multi-graph). Iterating this decomposition of \( V^\mu \) into irreducible \( B_1 \)-submodules, we obtain
\[
V^\mu = \bigoplus_T v_T,
\]
where the sum is over all possible chains \( T = \mu_1 \not\geq \mu_2 \not\geq \cdots \not\geq \mu_n \) with \( \mu_i \in \hat{B}_i \) and \( \mu_n = \mu \). We call [2] the Gelfand-Tsetlin decomposition of \( V^\mu \) and each \( v_T \) in [2] a Gelfand-Tsetlin vector of \( V^\mu \). We note that if \( 0 \neq v_T \), then \( \mathbb{C}[B_n] v_T = V^\mu \). The Gelfand-Tsetlin vectors of \( V^\mu \) form a basis of \( V^\mu \).

The irreducible representations of \( B_n \) are parametrised by elements of \( D_n \) [12, Lemma 6.2, Theorem 6.4]. We may index the the Gelfand-Tsetlin vectors of \( V^\mu \) by standard double-tableaux of shape \( \mu \) for \( \mu \in D_n \) [12, Theorem 6.5] and write the Gelfand-Tsetlin decomposition as
\[
V^\mu = \bigoplus_{T \in \text{tab}_D(n, \mu)} v_T.
\]
Let \( \mu = (\mu^{(1)}, \mu^{(2)}) \in D_n \) and \( T \in \text{tab}_D(n, \mu) \). Then the action [12, Theorem 6.5] of the Young-Jucys-Murphy elements \( X_i \) and the signed permutation \( (i, -i) \) on \( v_T \) are given by
\[
X_i \ v_T = 2c(b_T(i)) \ v_T \quad \text{for all} \quad i \in [n],
\]
\[
(-i, i) \ v_T = \begin{cases} v_T & \text{if} \ b_T(i) \text{ is in } \mu^{(1)} \\ -v_T & \text{if} \ b_T(i) \text{ is in } \mu^{(2)} \end{cases} \quad \text{for all} \quad i \in [n].
\]
We now come to our main problem of finding the eigenvalues of the transition matrix \( \hat{P}(R) \). The eigenvalues of \( \hat{P}(R) \) are the eigenvalues of \( \frac{1}{n} (\text{id} + (-n, n) + X_n) \) acting on \( \mathbb{C}[B_n] \) by multiplication on the right. The following theorem gives the eigenvalues of \( \hat{P}(R) \).

**Theorem 2.1.** Corresponding to each integer \( m \) satisfying \( 0 \leq m \leq \lfloor \frac{n}{2} \rfloor \), let \( \mu = (\mu^{(1)}, \mu^{(2)}) \in D_n \) be such that \( |\mu^{(1)}| = m \) and \( |\mu^{(2)}| = n - m \). For each \( T \in \text{tab}_D(n, \mu) \), \( \frac{c(b_T(n)) + 1}{n} \) and \( \frac{c(b_T(n))}{n} \) are eigenvalues of \( \hat{P}(R) \) with multiplicity \( M(\mu) \) each, where
\[
M(\mu) = \begin{cases} \binom{n}{m} d_{\mu^{(1)}} d_{\mu^{(2)}}, & \text{if } 0 \leq m < \frac{n}{2}, \\ \frac{1}{2} \binom{n}{m} d_{\mu^{(1)}} d_{\mu^{(2)}}, & \text{if } m = \frac{n}{2} \text{ (when } n \text{ is even)}. \end{cases}
\]

**Proof.** For each \( \mu = (\mu^{(1)}, \mu^{(2)}) \in D_n \), we have another double-diagram \( \bar{\mu} \) with \( n \) boxes such that \( \bar{\mu} = (\mu^{(2)}, \mu^{(1)}) \). We first find the eigenvalues of the matrix \( \hat{P}(R) \) in the irreducible \( B_n \)-modules \( V^\mu \) and \( V^{\bar{\mu}} \). For each \( T = (T_1, T_2) \in \text{tab}_D(n, \mu) \), \( \bar{T} = (T_2, T_1) \in \text{tab}_D(n, \bar{\mu}) \). If \( b_T(n) \) is in \( \mu^{(1)} \), then \( b_{\bar{T}}(n) \) is in \( \mu^{(2)} \). Without loss of generality, let us assume that \( b_T(n) \) is in \( \mu^{(1)} \) and \( b_{\bar{T}}(n) \) is in \( \mu^{(2)} \). Let us recall \( v_T \) (respectively \( v_{\bar{T}} \)) is the Gelfand-Tsetlin vector of
$V^\mu$ (respectively $V^{\hat{\mu}}$). From \cite{3} we have $(-n, n) v_T = v_T$ and $X_n v_T = 2c(b_T(n)) v_T$, which implies the following:

$$\begin{align*}
(id + (-n, n) + X_n) v_T &= (1 + 1 + 2c(b_T(n))) v_T \\
&= (2c(b_T(n)) + 2) v_T.
\end{align*}$$

(5)

Since $\{v_T : T \in \text{tab}_D(n, \mu)\}$ form a basis of $V^\mu$, the eigenvalues of the action of $(id + (-n, n) + X_n)$ on $V^\mu$ can be obtained from \cite{5}. Now using (3) again we have $(-n, n) v_T = -v_T$ and $X_n v_T = 2c(b_T(n)) v_T$, thus

$$\begin{align*}
(id + (-n, n) + X_n) v_T &= (1 - 1 + 2c(b_T(n))) v_T \\
&= 2c(b_T(n)) v_T.
\end{align*}$$

(6)

Therefore the eigenvalues of the action of $(id + (-n, n) + X_n)$ on $V^{\hat{\mu}}$ are obtained from (3), as $\{v_T : T \in \text{tab}_D(n, \hat{\mu})\}$ form a basis of $V^{\hat{\mu}}$. Thus considering the action of $\frac{1}{n} (id + (-n, n) + X_n)$ on $V^\mu$ and $V^{\hat{\mu}}$ simultaneously, the eigenvalues of $\hat{P}(R)$ are given by $\frac{1}{2n} \left(\binom{n}{m} d_{\mu(1)} d_{\mu(2)}\right)$ and $\frac{1}{2n} \left(\binom{n}{m} d_{\hat{\mu}(1)} d_{\hat{\mu}(2)}\right)$ for each $T \in \text{tab}_D(n, \mu)$.

Now we know that the multiplicity of every irreducible representation in the right regular representation is equal to its dimension. Therefore the multiplicity of the eigenvalues are $\dim(V^\mu) = \binom{n}{m} d_{\mu(1)} d_{\mu(2)} = \dim(V^{\hat{\mu}})$ if $0 \leq m < \frac{n}{2}$ and the multiplicity of the eigenvalues are $\frac{1}{2n} \left(\binom{n}{m} d_{\mu(1)} d_{\mu(2)}\right)$ if $m = \frac{n}{2}$ (when $n$ is even). The multiplicity of the eigenvalues for the case of $m = \frac{n}{2}$ is half of the dimension of the corresponding $B_n$-module because of the following:

In this case $m = n - m$. Thus both $\mu = (\mu^{(1)}, \mu^{(2)})$ and $\hat{\mu} = (\mu^{(2)}, \mu^{(1)})$ are in $D_n$ such that their first component is a partition of $m$ and the second component is a partition of $n - m$. Therefore while computing the eigenvalues of $\hat{P}(R)$ by considering the irreducible $B_n$-modules $V^\mu$ and $V^{\hat{\mu}}$, each space is counted twice. Now the proof of the theorem follows from the fact that all the irreducible representations of $B_n$ are parameterised by $D_n$. \hfill \Box

3. Upper bound of total variation distance

In this section, we will prove the theorem giving an upper bound of the total variation distance $\|P^k - U_B\|_{TV}$ for $k \geq n \log n + cn$, $c > 0$. Given a positive integer $\ell$, throughout this section we write $\lambda \vdash \ell$ to denote $\lambda$ is a partition of $\ell$. Let us recall that $\text{tab}(\lambda)$ denote the set of all standard Young tableaux of shape $\lambda$.

**Lemma 3.1** (Upper bound lemma, \cite{3} Lemma 4.2]). Let $p$ be a probability measure on a finite group $G$ such that $p(x) = p(x^{-1})$ for all $x \in G$. Suppose the random walk on $G$ driven by $p$ is irreducible. Then we have the following

$$\|P^k - U_G\|_{TV}^2 \leq \frac{1}{4} \sum_{\rho} d_{\rho} \text{Tr} \left( (\hat{\rho}(\rho))^{2k} \right),$$

where the sum is over all non-trivial irreducible representations $D_\rho$ of $G$ and $d_\rho$ is the dimension of $D_\rho$. 

Lemma 3.2. Let $m$ be any positive integer satisfying $1 \leq m \leq \frac{n}{2}$ and $\mu = (\mu^{(1)}, \mu^{(2)}) \in \mathcal{D}_n$ be such that $|\mu^{(1)}| = m$, $|\mu^{(2)}| = n - m$. If $\mu^{(i)}_1$ denotes the largest part of the partition $\mu^{(i)}$ for $i = 1, 2$, then

$$\sum_{T \in \text{tab}_\mathcal{D}(n, \mu)} \left( \frac{c(b_T(n)) + 1}{n} \right)^{2k} \leq \binom{n}{m} \sum_{T \in \mathcal{T}_1} \left( \frac{c(b_T(n)) + 1}{n} \right)^{2k} + \sum_{T \in \mathcal{T}_2} \left( \frac{c(b_T(n)) + 1}{n} \right)^{2k}.$$

Proof. The set $\text{tab}_\mathcal{D}(n, \mu)$ is a disjoint union of the sets $\mathcal{T}_1 = \{(T_1, T_2) \in \text{tab}_\mathcal{D}(n, \mu) : b_T(n) \text{ is in } T_1\}$ and $\mathcal{T}_2 = \{(T_1, T_2) \in \text{tab}_\mathcal{D}(n, \mu) : b_T(n) \text{ is in } T_2\}$. Therefore we have

$$\sum_{T \in \text{tab}_\mathcal{D}(n, \mu)} \left( \frac{c(b_T(n)) + 1}{n} \right)^{2k} = \sum_{T \in \mathcal{T}_1} \left( \frac{c(b_T(n)) + 1}{n} \right)^{2k} + \sum_{T \in \mathcal{T}_2} \left( \frac{c(b_T(n)) + 1}{n} \right)^{2k}.$$

Now the right hand side of (7) is equal to

$$\binom{n-1}{m} d^{(2)}_{\mu^{(2)}} \sum_{T_1 \in \text{tab}(\mu^{(1)})} \left( \frac{c(b_{T_1}(m)) + 1}{n} \right)^{2k} + \binom{n-1}{m} d^{(1)}_{\mu^{(1)}} \sum_{T_2 \in \text{tab}(\mu^{(2)})} \left( \frac{c(b_{T_2}(n-m)) + 1}{n} \right)^{2k}$$

$$\leq \binom{n}{m} \left( d^{(2)}_{\mu^{(2)}} \sum_{T_1 \in \text{tab}(\mu^{(1)})} \left( \frac{c(b_{T_1}(m)) + 1}{n} \right)^{2k} + d^{(1)}_{\mu^{(1)}} \sum_{T_2 \in \text{tab}(\mu^{(2)})} \left( \frac{c(b_{T_2}(n-m)) + 1}{n} \right)^{2k} \right)$$

$$\leq \binom{n}{m} d^{(2)}_{\mu^{(2)}} d^{(1)}_{\mu^{(1)}} \left( \frac{\mu^{(1)}_1}{n} \right)^{2k} + \left( \frac{\mu^{(2)}_1}{n} \right)^{2k}.$$  \hfill \Box

Lemma 3.3. Let $\ell$ be a positive integer. For a partition $\lambda$ of $\ell$, if $\lambda_1$ denotes the largest part of $\lambda$, then

$$\sum_{\lambda \vdash \ell} d^2_{\lambda} \left( \frac{\lambda_1}{\ell} \right)^{2k} < e^{\ell^2 e^{-\frac{2k}{\ell}}}. $$

Proof. For any partition $\zeta$ of $\ell - \lambda_1$ with largest part $\zeta_1$ less than or equal to $\lambda_1$, we have $d_{\zeta} \leq \binom{\ell}{\lambda_1} d_{\zeta}$. Therefore $\sum_{\lambda \vdash \ell} d^2_{\lambda} \left( \frac{\lambda_1}{\ell} \right)^{2k}$ is less than or equal to

$$\sum_{\lambda_1 = 1}^{\ell} \sum_{\zeta \vdash (\ell - \lambda_1)} \left( \frac{\ell}{\lambda_1} \right)^2 d^2_{\zeta} \left( \frac{\lambda_1}{\ell} \right)^{2k} \leq \sum_{\lambda_1 = 1}^{\ell} \left( \frac{\ell}{\lambda_1} \right)^2 \left( \frac{\lambda_1}{\ell} \right)^{2k} \sum_{\zeta \vdash (\ell - \lambda_1)} d^2_{\zeta}$$

$$= \sum_{\lambda_1 = 1}^{\ell} \left( \frac{\ell}{\ell - \lambda_1} \right)^2 \frac{\ell}{(\ell - \lambda_1)!} \left( 1 - \frac{\ell - \lambda_1}{\ell} \right)^{2k}.$$ 

(8)
Now writing $t = \ell - \lambda_1$ and using $1 - x \leq e^{-x}$ for $x \geq 0$, the expression in (8) less than or equal to $\sum_{t=0}^{\ell-1} \binom{\ell}{t} t! e^{-\frac{2k t}{t}}$. Thus we have

$$\sum_{\lambda=\ell} d^2_{\lambda} \left( \frac{\lambda_1}{\ell} \right)^{2k} \leq \sum_{t=0}^{\ell-1} \binom{\ell}{t} t! e^{-\frac{2k t}{t}} = \sum_{t=0}^{\ell-1} \frac{(\ell(\ell-1)\ldots(\ell-t+1))^2}{t!} e^{-\frac{2k t}{t}} \leq \sum_{t=0}^{\ell-1} \frac{(\ell^2 e^{-\frac{2k}{t}})^t}{t!} < e^{2e^{-\frac{2k}{t}}}.$$  

**Theorem 3.4.** For the random walk on $B_n$ driven by $P$, we have the following:

1. $\|P^k - U_{B_n}\|_{TV} < \sqrt{e+1} e^{-c} + o(1)$, for $k \geq n \log n + cn$ and $c > 0$.
2. $\lim_{n \to \infty} \|P^{k_n} - U_{B_n}\|_{TV} = 0$, for any $\epsilon \in (0, 1)$ and $k_n = [(1 + \epsilon)n \log n]$.

**Proof.** We know that the trace of the $(2k)$th power of a matrix is the sum of the $(2k)$th powers of its eigenvalues. Therefore Lemma 3.1 implies $4\|P^k - U_{B_n}\|_{TV}$ is bounded above by the sum of $(2k)$th powers of the non-largest eigenvalues (which are strictly less the largest eigenvalue $1$) of $\hat{P}(\mathcal{R})$. Thus from Theorem 2.1 we have

\begin{equation}
4\|P^k - U_{B_n}\|_{TV} \leq \left(\frac{n-1}{n}\right)^{2k} + \sum_{\lambda \neq n} d_\lambda \left( \sum_{T \in \text{tab}_1(\lambda)} \left( \frac{c(b_T(n)) + 1}{n} \right)^{2k} + \left( \frac{c(b_T(n))}{n} \right)^{2k} \right) + \sum_{m=1}^{\left\lceil \frac{n}{2} \right\rceil} \sum_{\mu(1) \vdash (n-m) \atop \mu(1) = (1, \mu(2))} \sum_{\mu(2) \vdash (n-m) \atop \mu(2) = (\mu(1), \mu(2))} M(\mu) \left( \sum_{T \in \text{tab}_2(\mu, n \mu)} \left( \frac{c(b_T(n)) + 1}{n} \right)^{2k} + \left( \frac{c(b_T(n))}{n} \right)^{2k} \right).
\end{equation}

$M(\mu)$ is defined in (4) and can be written as $M(\mu) = I(n, m) \binom{n}{m} d_{\mu(1)} d_{\mu(2)}$, where

$$I(n, m) = \begin{cases} 1 & \text{if } 0 \leq m < \frac{n}{2}, \\ \frac{1}{2} & \text{if } m = \frac{n}{2} \text{ (when } n \text{ is even).} \end{cases}$$

The third term in the right hand side of (9) is less than the following expression

\begin{equation}
\sum_{m=1}^{\left\lceil \frac{n}{2} \right\rceil} \sum_{\mu(1) \vdash (n-m) \atop \mu(1) = (1, \mu(2))} 2M(\mu) \left( \sum_{T \in \text{tab}_2(\mu, n \mu)} \left( \frac{c(b_T(n)) + 1}{n} \right)^{2k} \right).
\end{equation}
Using Lemma 3.2, the expression in (10) is less than

\[
\sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{\mu^{(1)}=m}^{\mu^{(2)}=n-m} 2M(\mu) \binom{n}{m} d_{\mu^{(2)}} d_{\mu^{(1)}} \left( \left( \frac{\mu^{(1)}}{n} \right)^{2k} + \left( \frac{\mu^{(2)}}{n} \right)^{2k} \right)
\]

\[
= 2 \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \mathbb{I}(n, m) \binom{n}{m}^2 \sum_{\mu^{(1)}=m}^{\mu^{(2)}=n-m} d_{\mu^{(1)}}^2 d_{\mu^{(2)}}^2 \left( \left( \frac{\mu^{(1)}}{n} \right)^{2k} + \left( \frac{\mu^{(2)}}{n} \right)^{2k} \right)
\]

\[
(11) \quad = 2 \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \mathbb{I}(n, m) \binom{n}{m}^2 (n-m)! \sum_{\mu^{(1)}=m}^{\mu^{(2)}=n-m} d_{\mu^{(1)}}^2 \left( \frac{\mu^{(1)}}{n} \right)^{2k} + m! \sum_{\mu^{(2)}=n-m}^{\mu^{(1)}=m} d_{\mu^{(2)}}^2 \left( \frac{\mu^{(2)}}{n} \right)^{2k}
\]

and the definition of \( \mathbb{I}(n, m) \) implies that the expression \( (11) \) is equal to

\[
(12) \quad 2 \sum_{m=1}^{n-1} \binom{n}{m}^2 (n-m)! \sum_{\mu^{(1)}=m}^{\mu^{(2)}=n-m} d_{\mu^{(1)}}^2 \left( \frac{\mu^{(1)}}{n} \right)^{2k}
\]

Replacing \( \ell \) (respectively \( \lambda \)) by \( m \) (respectively \( \mu^{(1)} \)) in Lemma \( 3.3 \) we have

\[
\sum_{\mu^{(1)}=m}^{\mu^{(2)}=n-m} d_{\mu^{(1)}}^2 \left( \frac{\mu^{(1)}}{m} \right)^{2k} < e^{m^2 e^{-\frac{k}{m}}}.
\]

Thus \( \sum_{\mu^{(1)}=m}^{\mu^{(2)}=n-m} d_{\mu^{(1)}}^2 \left( \frac{\mu^{(1)}}{m} \right)^{2k} < e \), if \( k \geq m \log m \). Therefore when \( k \geq n \log n \) (which implies \( k \geq m \log m \)), the expression in \( (12) \) and hence the third term in the right hand side of \( (9) \) is less than

\[
2e \sum_{m=1}^{n-1} \binom{n}{m}^2 (n-m)! \left( \frac{m}{n} \right)^{2k} = 2e \sum_{t=1}^{n-1} \binom{n}{t}^2 \left( 1 - \frac{t}{n} \right)^{2k} < 2e \sum_{t=1}^{n-1} \frac{(n^2 e^{-\frac{2k}{n}})^t}{t!} < 2e \left( e^{n^2 e^{-\frac{2k}{n}}} - 1 \right).
\]

Now we consider the second term in the right hand side of \( (9) \). The second term in the right hand side of \( (9) \) is bounded above by

\[
2 \sum_{\lambda=1}^{\lambda=n} d_{\lambda} \sum_{T \in \text{tab}(\lambda)} \left( \frac{c(b_T(n)) + 1}{n} \right)^{2k}
\]

\[
= 2 \left( \sum_{\lambda=1}^{\lambda=n} d_{\lambda} \sum_{T \in \text{tab}(\lambda)} \left( \frac{c(b_T(n)) + 1}{n} \right)^{2k} - 1 \right)
\]

The expression in the right hand side of \( (14) \) is less than or equal to

\[
2 \left( \sum_{\lambda=1}^{\lambda=n} d_{\lambda} \sum_{T \in \text{tab}(\lambda)} \left( \frac{\lambda_1}{n} \right)^{2k} - 1 \right) \leq 2 \left( \sum_{\lambda=1}^{\lambda=n} d_{\lambda}^2 \left( \frac{\lambda_1}{n} \right)^{2k} - 1 \right)
\]

\[
\text{Subhajit Ghosh}
\]

(10) \quad \sum_{i=1}^{m} \sum_{\mu^{(1)}=m}^{\mu^{(2)}=n-m} 2M(\mu) \binom{n}{m} d_{\mu^{(2)}} d_{\mu^{(1)}} \left( \left( \frac{\mu^{(1)}}{n} \right)^{2k} + \left( \frac{\mu^{(2)}}{n} \right)^{2k} \right)
\]

\[
(13)
\]
The right hand side of the expression (15) and hence the second term in the right hand side of (9) is less than 2 \((e^{n^2e^{-2c}} - 1)\) by Lemma 3.3. Thus the inequality (9) becomes

\[
4||P^k - U_B_n||^2_{TV} \leq e^{-\frac{2k}{n}} + (2 + 2e) \left(e^{n^2e^{-2c}} - 1\right), \quad \text{for } k \geq n \log n.
\]

Now if \(k \geq n \log n + cn\) and \(c > 0\), then the right hand side of (16) becomes

\[
(2e + 2) \left(e^{-2c} - 1\right) + \frac{e^{-2c}}{n^2} < (4e + 4)e^{-2c} + \frac{e^{-2c}}{n^2} = 4(e + 1)e^{-2c} + o(1).
\]

This proves the first part of the theorem. Now for \(\epsilon \in (0, 1)\), \(k_n = \lfloor (1 + \epsilon)n \log n \rfloor\) implies, \(k_n \geq (1 + \epsilon)n \log n\). Thus the right hand side of (16) is bounded above by

\[
(2e + 2) \left(e^{\frac{1}{n^2}} - 1\right) + n^{-2(1+\epsilon)}.
\]

Therefore the proof of the second part follows from

\[
\lim_{n \to \infty} \left( (2e + 2) \left(e^{\frac{1}{n^2}} - 1\right) + \frac{1}{n^{2(1+\epsilon)}} \right) = 0. \quad \square
\]

4. LOWER BOUND OF TOTAL VARIATION DISTANCE

In this section, we will find lower bound of the total variation distance \(||P^k - U_B_n||_{TV}\) for \(k = \log n + cn, \ c \ll 0\). We also find the mixing time for the flip-transpose top with random shuffle on \(B_n\) driven by \(P\). At the end of this section we show that this random walk exhibits cutoff phenomenon. Throughout this section \(I_n\) denotes the identity matrix of order \(n \times n\). To start with, we define an random variable \(X\) on \(B_n\) as follows:

\[
X(\pi) = \text{number of fixed points of } \pi \text{ when } \pi \in B_n.
\]

**Remark 4.1.** For each \(i \in [n]\), the signed permutation which fixes \(i\) will automatically fix \((-i)\). Thus \(X\) takes values from the set of non-negative even integers.

Let \(E_k(X)\) be the expectation and \(\text{Var}_k(X)\) be the variance of \(X\) with respect to the probability measure \(P^k\) on \(B_n\). Now \(X\) can also be described as follows.

Let \(V = \mathbb{C}[\{-n, \ldots, -1, 1, \ldots, n\}]\) be the vector space of all formal \(\mathbb{C}\)-linear combinations of elements of the set \(\{-n, \ldots, -1, 1, \ldots, n\}\). Also let \(V^+ = \mathbb{C}[\{a_1, a_2, \ldots, a_n\}]\) and \(V^- = \mathbb{C}[\{b_1, b_2, \ldots, b_n\}]\) be two vector subspaces of \(V\), where \(a_i = i + (-i)\) and \(b_i = i - (-i)\) for all \(i \in [n]\). We note that \(a_i = a_i, b_i = b_i\) for all \(i \in [n]\). Let us define

\[
\rho^+ : B_n \rightarrow GL(V^+) \text{ by } \rho^+(\pi)(a_i) = a_{\pi(i)} \text{ on the basis elements of } V^+, \text{ for } \pi \in B_n;
\]

\[
\rho^- : B_n \rightarrow GL(V^-) \text{ by } \rho^-(\pi)(b_i) = b_{\pi(i)} \text{ on the basis elements of } V^-, \text{ for } \pi \in B_n.
\]

It can be easily seen that \(\rho^+(\pi)\) and \(\rho^-(\pi)\) are well defined for \(\pi \in B_n\). We note that \(\rho^+\) and \(\rho^-\) are two representations of \(B_n\). Using \(\rho^+\) and \(\rho^-\) we can interpret \(X\) as follows:

\[
X(\pi) = \text{Tr} \left( \rho^+(\pi) + \rho^-(\pi) \right), \text{ for } \pi \in B_n.
\]
Now we define the representation $\rho^{\text{def}}$ of the symmetric group $S_n$, known as the defining representation of $S_n$. Let $\mathbb{C}\{\{1, \ldots, n\}\}$ be the subspace of $V$ spanned by $\{1, \ldots, n\}$. Then $\rho^{\text{def}}: S_n \rightarrow GL(\mathbb{C}\{\{1, \ldots, n\}\})$ is defined by

$$
(\sum_{c_i \in \mathbb{C}} c_i 1) = \sum_{c_i \in \mathbb{C}} c_i \pi(i).
$$

$(t, n)$ is assumed to be the transposition interchanging $t$ and $n$ in $S_n$ when we write $\rho^{\text{def}}((t, n))$, otherwise $(t, n)$ is the signed permutation $[1, \ldots, t-1, n, t+1, \ldots, n-1, t]$, $1 \leq t < n$. From now on the matrices $\rho^+(\pi)$ (respectively $\rho^-\pi$(π)) are defined with respect to the ordered bases $(a_1, \ldots, a_n)$ (respectively $(b_1, \ldots, b_n)$) for $\pi \in B_n$ and $\rho^{\text{def}}(\pi)$ is defined with respect to the ordered basis $(1, \ldots, n)$ for $\pi \in S_n$. Before going to further details we first note that $\sum_{t=1}^{n-1} (t, n)$ is the $n$th Young-Jucys-Murphy element in $\mathbb{C}\{S_n\}$. Let $s_T$ be the Gelfand-Tsetlin basis vector of $S^\lambda$ corresponding to the standard Young tableau $T$ and $c(b_T(n))$ is the content of the box in $T$ containing $n$. For a partition $\lambda$ of $n$, the action of $\sum_{t=1}^{n-1} (t, n)$ on the irreducible $S_n$-module $S^\lambda$ is given by [2][13]:

$$(18) \quad \left(\sum_{t=1}^{n-1} (t, n)\right) s_T = c(b_T(n)) s_T.
$$

**Lemma 4.2.** The eigenvalues of $\sum_{t=1}^{n-1} \rho^{\text{def}}((t, n))$ are given below:

| Eigenvalues | Multiplicities |
|-------------|----------------|
| $n-1$       | 1              |
| $n-2$       | 2              |
| $-1$        | 1              |

**Proof.** The defining representation of $S_n$ splits into two irreducible $S_n$-modules $S^{(n)}$ (trivial) and $S^{(n-1,1)}$ with multiplicity one each [16] Example 2.1.8 and Theorem 2.11.2. Therefore the lemma follows from [13] and straightforward calculations. $\square$

**Lemma 4.3.** The eigenvalues of $\sum_{t=1}^{n-1} (\rho^{\text{def}}((t, n)) \otimes \rho^{\text{def}}((t, n)))$ are given as follows:

| Eigenvalues | Multiplicities |
|-------------|----------------|
| $n-1$       | $2$            |
| $n-2$       | $3(n-2)$       |
| $-1$        | $3$            |
| $0$         | $n-2$          |
| $-2$        | $n-2$          |
| $n-3$       | $n^2 - 5n + 5$ |

**Proof.** The decomposition of $\rho^{\text{def}} \otimes \rho^{\text{def}}$ into irreducible $S_n$-modules is given below (see [16] Example 2.1.8] and [9] Lemma 2.9.16]):

$$
\rho^{\text{def}} \otimes \rho^{\text{def}} = 2S^{(n)} \oplus 3S^{(n-1,1)} \oplus S^{(n-2,2)} \oplus S^{(n-2,1,1)}.
$$

Here the coefficient of irreducible $S_n$-module denotes its multiplicity in $\rho^{\text{def}} \otimes \rho^{\text{def}}$. In this case also the lemma follows from [13] and straightforward calculations. $\square$

**Lemma 4.4.** The matrices $\rho^+((t, n))$, $\rho^+((-t, n))$ and $\rho^{\text{def}}((t, n))$ are the same for all $t \in \{1, \ldots, n-1\}$ and $\rho^+((-n, n)) = I_n$. 
Proof. The lemma follows by looking at the action of each of these matrices on the basis vector. Now for each $i \in \{1, \ldots, n\}$ we have the following:

$$
\rho^+((-t,n))(a_i) = \begin{cases} 
  a_i & \text{if } i \neq t, n \\
  a_n & \text{if } i = t \\
  a_t & \text{if } i = n 
\end{cases} = \rho^+((t,n))(a_i).
$$

Also $\rho^+((-n,n)) = I_n$ follows trivially by looking at its action on the basis elements. □

**Lemma 4.5.** For each $i \in \{1, \ldots, n\}$, let $M_i$ denote the $n \times n$ matrix with $(i,i)$th entry 1 and 0 elsewhere. Then for all $t \in \{1, \ldots, n-1\}$ we have $\rho^-((t,n)) + \rho^-((-t,n)) = 2(I_n - M_t - M_n)$ and $\rho^-((-n,n)) = 2(I_n - M_n) - I_n$.

Proof. This lemma follows by looking at the action of the matrices on the basis elements. For $i \in \{1, \ldots, n\}$ we have the following:

$$
\left(\rho^-((t,n)) + \rho^-((-t,n))\right)(b_i) = \begin{cases} 
  2b_i & \text{if } i \neq t, n \\
  0 & \text{if } i = t \\
  0 & \text{if } i = n 
\end{cases}
$$

and

$$
\left(I_n + \rho^-((-n,n))\right)(b_i) = \begin{cases} 
  2b_i & \text{if } i \neq n \\
  0 & \text{if } i = n. 
\end{cases}
$$

□

**Lemma 4.6.** Let $\beta_i$ denote the matrix $\sum_{t=1}^{n-1} \rho^{\text{def}}((t,n)) - \rho^{\text{def}}((i,n))$ for all $1 \leq i < n$. Then we have the following:

1. The matrices $\beta_i$ and $\beta_j$ are similar for $i \neq j$ and $i, j \in \{1, \ldots, n-1\}$.
2. For each $i \in \{1, \ldots, n\}$, the eigenvalues of $\beta_i$ are the following:
   - Eigenvalues: $n - 2$ \hspace{1em} $n - 3$ \hspace{1em} $-1$
   - Multiplicities: $2$ \hspace{1em} $n - 3$ \hspace{1em} $1$

Proof. We have $\left(\rho^{\text{def}}((i,j))\right)^{-1} = \rho^{\text{def}}((i,j))$ as the transposition $(i,j) \in S_n$ is self-inverse. Thus the first part of the lemma follows from the following fact:

$$
\rho^{\text{def}}((i,j))\beta_i\left(\rho^{\text{def}}((i,j))\right)^{-1} = \rho^{\text{def}}((i,j))\beta_i\rho^{\text{def}}((i,j))
$$

$$
= \rho^{\text{def}}((i,j))\beta_i\rho^{\text{def}}((i,j))
$$

$$
= \rho^{\text{def}}((i,j))\left(\sum_{t=1}^{n-1} \rho^{\text{def}}((t,n))\right)\rho^{\text{def}}((i,j)) - \rho^{\text{def}}((i,j))\rho^{\text{def}}((i,n))\rho^{\text{def}}((i,j))
$$

$$
= \sum_{t=1}^{n-1} \rho^{\text{def}}((i,j)(t,n)(i,j)) - \rho^{\text{def}}((i,j)(i,n)(i,j)) = \sum_{t=1}^{n-1} \rho^{\text{def}}((t,n)) - \rho^{\text{def}}((j,n)) = \beta_j.
$$
The eigenvalues of $\beta_1, \beta_2, \ldots, \beta_{n-1}$ are same by the first part of this lemma. Therefore to prove the second part it is enough to find the eigenvalues of $\beta_1$. Let us consider the linearly independent vectors $v_1, v_2, v_3, \ldots, v_{n-1}, v_n$ of $\mathbb{C}[\{1, \ldots, n\}]$. The eigenvalues of $\beta_1$ are obtained from Table 1.

| Eigenvector | Action of $\beta_1$ on eigenvector | Eigenvalues |
|-------------|----------------------------------|-------------|
| $v_1 = 1 + 2 + \cdots + n$ | $\beta_1(v_1) = (n - 2)v_1$ | $n - 2$ |
| $v_2 = 1$ | $\beta_1(v_2) = (n - 2)v_2$ | $n - 2$ |
| $v_i = 1 - 2$ for $3 \leq i \leq n - 1$ | $\beta_1(v_i) = (n - 3)v_i$ | $n - 3$ |
| $v_n = v_1 - v_2 - (n - 1)n$ | $\beta_1(v_n) = (-1)v_n$ | $-1$ |

**Table 1.** Eigenvectors and eigenvalues of $\beta_1$

**Lemma 4.7.** Let $O_n$ be the zero matrix of size $n \times n$ and $\beta_1, \ldots, \beta_{n-1}$ be as defined in Lemma 4.6. If $\text{Blockdiag}(\beta_1, \beta_2, \ldots, \beta_{n-1}, O_n)$ denote the block diagonal matrix with $i$th block $\beta_i$ for all $i \in \{1, 2, \ldots, n-1\}$ and $n$th block $O_n$, then

$$\sum_{i=1}^{n-1} \left( \rho^-((t, n)) \otimes \rho^+((t, n)) + \rho^-((-t, n)) \otimes \rho^+((-t, n)) \right) = 2 \text{Blockdiag}(\beta_1, \beta_2, \ldots, \beta_{n-1}, O_n).$$

**Proof.** Using Lemma 4.4 and Lemma 4.5 the matrix in the statement can be written as

$$\sum_{i=1}^{n-1} \left( \rho^-((t, n)) + \rho^-((-t, n)) \right) \otimes \rho^\text{def}((t, n))$$

$$= 2 \sum_{i=1}^{n-1} (I_n - M_t - M_n) \otimes \rho^\text{def}((t, n))$$

$$= 2 \sum_{i=1}^{n-1} (I_n - M_n) \otimes \rho^\text{def}((t, n)) - 2 \sum_{i=1}^{n-1} M_t \otimes \rho^\text{def}((t, n))$$

$$= 2 \sum_{i=1}^{n-1} \text{Blockdiag} \left( \rho^\text{def}((t, n)), \rho^\text{def}((t, n)), \ldots, \rho^\text{def}((t, n)), O_n \right)$$

$$- 2 \text{Blockdiag} \left( \rho^\text{def}((1, n)), \rho^\text{def}((2, n)), \ldots, \rho^\text{def}((n - 1, n)), O_n \right)$$

$$= 2 \text{Blockdiag}(\beta_1, \beta_2, \ldots, \beta_{n-1}, O_n).$$

**Lemma 4.8.** The eigenvalues of $\sum_{i=1}^{n} (\rho^-((t, n)) \otimes \rho^-((t, n)) + \rho^-((-t, n)) \otimes \rho^-((-t, n)))$ are given below:

| Eigenvalues: | 2n | 2(n - 1) | 0 | 2 | -2 | 2(n - 2) |
|---------------|----|----------|---|---|----|---------|
| Multiplicities: | 1 | n - 2 | 1 | n - 1 | n - 1 | (n - 1)(n - 2) |
Proof. For simplicity let us call the matrix in the statement $R$. Now let us consider the following vectors of $V^- \otimes V^-$. 

\begin{align}
v_{1,1} &= b_1 \otimes b_1 + \cdots + b_n \otimes b_n \\
v_{i,i} &= b_i \otimes b_i - b_1 \otimes b_1 \quad \text{for } i \in \{2, \ldots, n-1\} \\
v_{n,n} &= v_{1,1} - n(b_n \otimes b_n) \\
v_{i,n}^+ &= b_i \otimes b_n + b_n \otimes b_i \quad \text{for } i \in \{1, \ldots, n-1\} \\
v_{i,n}^- &= b_i \otimes b_n - b_n \otimes b_i \quad \text{for } i \in \{1, \ldots, n-1\} \\
v_{i,j} &= b_i \otimes b_j \quad \text{for } i, j \in \{1, \ldots, n-1\} \text{ and } i \neq j.
\end{align}

(19)

It can be easily seen that the vectors in (19) are linearly independent. Now the lemma follows from the following:

\begin{align*}
R(v_{1,1}) &= (2n)v_{1,1} \\
R(v_{i,i}) &= (2n-2)v_{i,i} \quad \text{for } i \in \{2, \ldots, n-1\} \\
R(v_{n,n}) &= 0 \\
R(v_{i,n}^+) &= 2v_{i,n}^+ \quad \text{for } i \in \{1, \ldots, n-1\} \\
R(v_{i,n}^-) &= (-2)v_{i,n}^- \quad \text{for } i \in \{1, \ldots, n-1\} \\
R(v_{i,j}) &= (2n-4)v_{i,j} \quad \text{for } i, j \in \{1, \ldots, n-1\} \text{ and } i \neq j.
\end{align*}

$\square$

**Proposition 4.9.** Let $X$, $E_k(X)$ be defined as in the beginning of this section. Then we have, $E_k(X) = 1 + (2n-3)\left(1 - \frac{1}{n}\right)^k$.

**Proof.** Using (17) and the definition of expectation of a random variable we have the following:

\begin{align*}
E_k(X) &= \sum_{\pi \in B_n} X(\pi)P^{\pi k}(\pi) = \sum_{\pi \in B_n} \text{Tr} \left( \rho^+(\pi)P^{\pi k}(\pi) + \rho^-(-\pi)P^{\pi k}(\pi) \right) \\
&= \text{Tr} \left( \sum_{\pi \in B_n} \rho^+(\pi)P^{\pi k}(\pi) \right) + \text{Tr} \left( \sum_{\pi \in B_n} \rho^-(-\pi)P^{\pi k}(\pi) \right) \\
&= \text{Tr} \left( \tilde{P}^{\pi k}(\rho^+) \right) + \text{Tr} \left( \tilde{P}^{\pi k}(\rho^-) \right) \\
&= \text{Tr} \left( \left( \tilde{P}(\rho^+) \right)^k \right) + \text{Tr} \left( \left( \tilde{P}(\rho^-) \right)^k \right) .
\end{align*}

(20)

Now using Lemma 4.4 we have

\begin{align}
\tilde{P}(\rho^+) &= \frac{1}{2n} \sum_{t=1}^n \left( \rho^+(t,n) + \rho^+(-t,n) \right) = \frac{1}{n} \left( I_n + \sum_{t=1}^{n-1} \rho^{\text{def}}((t,n)) \right).
\end{align}

(21)
Proof. Therefore from (21) and Lemma 4.2, the eigenvalues of \( \hat{P}(\rho^+) \) are 1, \( (1 - \frac{1}{n}) \) and 0 with multiplicities 1, \( (n - 2) \) and 1 respectively. Again from Lemma 4.5, we have

\[
\hat{P}(\rho^-) = \frac{1}{2n} \sum_{t=1}^{n} \left( \rho^-(t, n) + \rho^-(t, -n) \right)
\]

Thus the eigenvalues of \( \hat{P}(\rho^-) \) are: \( (1 - \frac{1}{n}) \) with multiplicity \( (n - 1) \) and 0 with multiplicity 1. Hence the proposition follows from (20).

\[
\text{Proposition 4.10. Let } X 	ext{ and } \text{Var}_k(X) 	ext{ are defined as in the beginning of this section. Then we have}
\]

\[
\text{Var}_k(X) = 2 + (4n - 6) \left( 1 - \frac{1}{n} \right)^k + (4n^2 - 16n + 13) \left( 1 - \frac{2}{n} \right)^k
\]

\[
+ (2n - 3) \left( \frac{1 + (-1)^k}{n^k} \right) - (4n^2 - 12n + 9) \left( 1 - \frac{1}{n} \right)^{2k}.
\]

Proof. We first find \( E_k(X^2) \). Now using (17), for each \( \pi \in B_n \) we have the following:

\[
(X(\pi))^2 = \text{Tr} \left( \left( \rho^+(\pi) + \rho^-\pi \right) \otimes \left( \rho^+(\pi) + \rho^-\pi \right) \right)
\]
\[
= \text{Tr} \left( \rho^+(\pi) \otimes \rho^+(\pi) \right) + 2 \text{ Tr} \left( \rho^-(\pi) \otimes \rho^+(\pi) \right) + \text{ Tr} \left( \rho^-(\pi) \otimes \rho^-(\pi) \right)
\]
\[
= \text{ Tr} (\rho_1(\pi)) + 2 \text{ Tr} (\rho_2(\pi)) + \text{ Tr} (\rho_3(\pi)).
\]

Here \( \rho_1 : B_n \to GL(V^+ \otimes V^+) \), \( \rho_2 : B_n \to GL(V^- \otimes V^+) \) and \( \rho_3 : B_n \to GL(V^- \otimes V^-) \) be three representations of \( B_n \) defined below

\[
\rho_1(\pi) = (\rho^+ \otimes \rho^+) (\pi)(v_i \otimes v_j) = \rho^+ (\pi)(v_i) \otimes \rho^+ (\pi)(v_j) \text{ for } \pi \in B_n, v_i \in V^+, v_j \in V^+,
\]
\[
\rho_2(\pi) = (\rho^- \otimes \rho^+) (\pi)(v_i \otimes v_j) = \rho^- (\pi)(v_i) \otimes \rho^+ (\pi)(v_j) \text{ for } \pi \in B_n, v_i \in V^-, v_j \in V^+,
\]
\[
\rho_3(\pi) = (\rho^- \otimes \rho^-) (\pi)(v_i \otimes v_j) = \rho^- (\pi)(v_i) \otimes \rho^- (\pi)(v_j) \text{ for } \pi \in B_n, v_i \in V^-, v_j \in V^-.
\]
Now we have
\[ E_k(X^2) = \sum_{\pi \in B_n} (X(\pi))^2 P^k(\pi) \]
\[ = \sum_{\pi \in B_n} \left( \text{Tr} (\rho_1(\pi)) + 2 \text{Tr} (\rho_2(\pi)) + \text{Tr} (\rho_3(\pi)) \right) P^k(\pi) \]
\[ = \text{Tr} \left( \sum_{\pi \in B_n} \rho_1(\pi) P^k(\pi) \right) + 2 \text{Tr} \left( \sum_{\pi \in B_n} \rho_2(\pi) P^k(\pi) \right) + \text{Tr} \left( \sum_{\pi \in B_n} \rho_3(\pi) P^k(\pi) \right) \]
\[ = \text{Tr} \left( \hat{P}^k(\rho_1) \right) + 2 \text{Tr} \left( \hat{P}^k(\rho_2) \right) + \text{Tr} \left( \hat{P}^k(\rho_3) \right) \]
(24)
\[ = \text{Tr} \left( (\hat{P}(\rho_1))^k \right) + 2 \text{Tr} \left( (\hat{P}(\rho_2))^k \right) + \text{Tr} \left( (\hat{P}(\rho_3))^k \right). \]

Now from Lemma 4.4 we have
\[ \hat{P}(\rho_1) = \frac{1}{2n} \left( \sum_{t=1}^n (\rho^+(t,n) \otimes \rho^+(t,n)) + \rho^+((-t,n)) \otimes \rho^+((-t,n)) \right) \]
\[ = \frac{1}{n} \left( \sum_{t=1}^{n-1} (\rho^+(t,n) \otimes \rho^+(t,n)) + I_n \otimes I_n \right). \]
(25)

Therefore from Lemma 4.3 the eigenvalues of \( \hat{P}(\rho_1) \) are: 1 with multiplicity \( 2 \left( 1 - \frac{1}{n} \right) \) with multiplicity \( 3(n-2) \), 0 with multiplicity 3, \( \left( 1 - \frac{2}{n} \right) \) with multiplicity \( (n^2 - 5n + 5) \) and \( \pm \frac{1}{n} \) with multiplicities \( (n-2) \) to each. Again from Lemma 4.4 4.3 and 4.7 we have the following:
\[ \hat{P}(\rho_2) = \frac{1}{2n} \left( \sum_{t=1}^n (\rho^-(t,n) \otimes \rho^+(t,n)) + \rho^-((-t,n)) \otimes \rho^+((-t,n)) \right) \]
\[ = \frac{1}{2n} \left( 2 \text{Blockdiag} (\beta_1, \beta_2, \ldots, \beta_{n-1}, O_n) + (I_n + \rho^-((-n,n))) \otimes I_n \right) \]
\[ = \frac{1}{n} \left( \text{Blockdiag} (\beta_1, \beta_2, \ldots, \beta_{n-1}, O_n) + (I_n - M_n) \otimes I_n \right) \]
(26)
\[ = \frac{1}{n} \text{Blockdiag} (\beta_1 + I_n, \beta_2 + I_n, \ldots, \beta_{n-1} + I_n, O_n). \]

Therefore from (26) and Lemma 4.6 the eigenvalues of \( \hat{P}(\rho_2) \) are: \( \left( 1 - \frac{1}{n} \right) \) with multiplicity \( 2(n-1) \), \( \left( 1 - \frac{2}{n} \right) \) with multiplicity \( (n-3)(n-1) \) and 0 with multiplicity \( (2n-1) \). Also from Lemma 4.8 the eigenvalues of \( \hat{P}(\rho_3) \) are: 0, 1 with multiplicities 1 to each, \( \pm \frac{1}{n} \) with multiplicities \( (n-1) \) to each, \( \left( 1 - \frac{1}{n} \right) \) with multiplicity \( (n-2) \) and \( \left( 1 - \frac{2}{n} \right) \) with multiplicity \( (n-1)(n-2) \). Hence from (24) we have
\[ E_k(X^2) = 3 + (8n - 12) \left( 1 - \frac{1}{n} \right)^k + (4n^2 - 16n + 13) \left( 1 - \frac{2}{n} \right)^k + (2n - 3) \left( \frac{1 + (-1)^k}{n^k} \right). \]

Thus the proposition follows from Proposition 4.9 and straightforward calculations. \( \square \)

**Proposition 4.11.** Let \( E_{U_{B_n}}(X) \) denote the expectation of \( X \) with respect to the uniform distribution on \( B_n \), \( E_{U_{B_n}}(X) = 1 \).
Proof. We note that signed permutations which fix $i$ will automatically fix $(-i)$. Let $B_i$ be the set of sign permutations in $B_n$ which fix $i$. Basic combinatorial arguments imply $|B_i| = 2^{n-1}(n-1)!$ for all $i \in \{\pm 1, \pm 2, \ldots, \pm n\}$.

Let $R_{\text{def}} : B_n \to \text{GL}(V)$ be the defining representation on $B_n$. Then we have the following:

$$E_{U_{B_n}}(X) = \sum_{\pi \in B_n} X(\pi) U_{B_n}(\pi) = \frac{1}{|B_n|} \sum_{\pi \in B_n} \text{Tr} \left( R_{\text{def}}(\pi) \right) = \frac{1}{|B_n|} \sum_{i \neq -n} |B_i| = \frac{2n|B_i|}{2^nn!} = 1.$$

\[ \square \]

**Theorem 4.12.** Let $X$, $E_k(X)$ and $\text{Var}_k(X)$ be as defined above. Then we have

1. For large $n$, $\|P^k - U_{B_n}\|_{TV} \geq 1 - \frac{10(1+2e^{-c}+o(1))}{(1+2e^{-c}+o(1))}$, when $k = n \log n + cn$ and $c \ll 0$.
2. $\lim_{n \to \infty} ||P^{k_n} - U_{B_n}||_{TV} = 1$, for any $\epsilon \in (0, 1)$ and $k_n = \lceil (1 - \epsilon)n \log n \rceil$.

**Proof.** For any positive constant $a$, by Chebychev’s inequality, we have

$$P^k \left( \{ \pi \in B_n : |X(\pi) - E_k(X)| \leq a\sqrt{\text{Var}_k(X)} \} \right) \geq 1 - \frac{1}{a^2}.$$  

Now we choose a positive constant $a$ such that $E_k(X) - a\sqrt{\text{Var}_k(X)} > 0$. Then by Markov’s inequality and Proposition [1.11] we have

$$U_{B_n} \left( \{ \pi \in B_n : X(\pi) \geq E_k(X) - a\sqrt{\text{Var}_k(X)} \} \right) \leq \frac{E_{U_{B_n}}(X)}{E_k(X) - a\sqrt{\text{Var}_k(X)}} = \frac{1}{E_k(X) - a\sqrt{\text{Var}_k(X)}}.$$

Now from the definition of total variation distance, we have

$$\|P^k - U_{B_n}\|_{TV} = \sup_{A \subseteq B_n} |P^k(A) - U_{B_n}(A)|$$

$$\geq P^k \left( \{ \pi \in B_n : |X(\pi) - E_k(X)| \leq a\sqrt{\text{Var}_k(X)} \} \right)$$

$$- U_{B_n} \left( \{ \pi \in B_n : |X(\pi) - E_k(X)| \leq a\sqrt{\text{Var}_k(X)} \} \right)$$

$$\geq P^k \left( \{ \pi \in B_n : |X(\pi) - E_k(X)| \leq a\sqrt{\text{Var}_k(X)} \} \right)$$

$$- U_{B_n} \left( \{ \pi \in B_n : X(\pi) \geq E_k(X) - a\sqrt{\text{Var}_k(X)} \} \right)$$

$$\geq 1 - \frac{1}{a^2} - \frac{1}{E_k(X) - a\sqrt{\text{Var}_k(X)}}.$$  

The inequality (29) follows by using (27) and (28). In particular, if we take $a = \frac{E_k(X)}{2\sqrt{\text{Var}_k(X)}} > 0$ in the above inequality, we get

$$\|P^k - U_{B_n}\|_{TV} \geq 1 - \frac{4\text{Var}_k(X)}{(E_k(X))^2} - \frac{2}{E_k(X)}.$$  

(30)
Now if \( n \) is large, we have

\[
E_k(X) \approx 1 + (2n - 3)e^{-\frac{k}{n}}
\]

from Proposition 4.9 and

\[
\text{Var}_k(X) \approx 2 + (4n - 6)e^{-\frac{k}{n}} - 4(n - 1)e^{-\frac{2n}{n}} + (2n - 3)\left(\frac{1 + (1-k)^n}{n^k}\right)
\]

from Proposition 4.10. Here ‘\( \approx \)’ means ‘asymptotic to’ i.e. \( a_n \approx b_n \) means \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \).

Now if \( n \) is large, \( c \ll 0 \) and \( k = n \log n + cn \), then by (30), (31) and (32), we have the first part of this theorem. Again for any \( \epsilon \in (0,1) \) and \( k_n = [(1 - \epsilon)n \log n] \) from (30), (31) and (32), we have

\[
1 \geq \|P^{\ast k_n} - U_{B_n}\|_{\text{TV}} \geq 1 - \frac{10 + (20 + o(1))n^\epsilon + n^{2\epsilon}o(1) + o(1)}{(1 + (2 + o(1))n^\epsilon)^2}
\]

for large \( n \). Therefore, the second part of this theorem follows from (33) and the fact that

\[
\lim_{n \to \infty} \frac{10 + (20 + o(1))n^\epsilon + n^{2\epsilon}o(1) + o(1)}{(1 + (2 + o(1))n^\epsilon)^2} = 0.
\]

Therefore from the first part of Theorems 3.4 and 4.12 we can say that the mixing time for the flip-transpose top with random shuffle on \( B_n \) is \( O(n \log n) \) (i.e., order of \( n \log n \)). Furthermore, the second part of Theorems 3.4 and 4.12 implies that this shuffle satisfies the cutoff phenomenon and the total variation cutoff for this shuffle occurs at \( n \log n \).

**Appendix A. Representation theory of demihyperoctahedral group \( D_n \)**

In this section we briefly discuss the irreducible representations of \( D_n \) (detailed proofs are omitted). Our main aim is to look at the restriction of the irreducible representations of \( B_n \) to \( D_n \).

Let us consider the one-dimensional character (or representation) \( \xi : B_n \to (\{\pm 1\}, \cdot) \) of \( B_n \). The action of \( \xi \) on the generators of \( B_n \) is defined by

\[
\xi(\pi) = \begin{cases} -1, & \text{if } \pi = (-1,1), \\ 1, & \text{if } \pi = (i,i+1) \text{ for } 1 \leq i \leq n - 1. \end{cases}
\]

It can be easily seen that \( \ker(\xi) = D_n \) and the \( B_n \)-module \( V \otimes \xi \) is irreducible if and only if the \( B_n \)-module \( V \) is irreducible. We have already seen in Section 2 that the irreducible representations of \( B_n \) are indexed by \( D_n \). If \( \mu = (\mu^{(1)}, \mu^{(2)}) \in D_n \), then \( \bar{\mu} = (\mu^{(2)}, \mu^{(1)}) \in D_n \). Now from [8, Proposition II.1.(ii)] it follows that the irreducible \( B_n \)-modules \( V^\mu \otimes \xi \) and \( V^{\bar{\mu}} \) are isomorphic for \( \mu \in D_n \).

**Theorem A.1.** For the irreducible \( B_n \)-module \( V^\mu \) indexed by \( \mu = (\mu^{(1)}, \mu^{(2)}) \in D_n \), we have the following:
(1) If $\mu^{(1)} \neq \mu^{(2)}$, then the restriction $V^\mu \uparrow_{D_n}^{B_n}$ of $V^\mu$ to $D_n$ is irreducible as a $D_n$-module. We denote this irreducible $D_n$-module by the same notation $V^\mu$. Moreover if $\bar{\mu} = (\mu^{(2)}, \mu^{(1)})$, then $V^\mu$ and $V^{\bar{\mu}}$ are isomorphic as $D_n$-modules. If $\nu \in D_n$ be such that $\nu \neq \mu$ and $\nu \neq \bar{\mu}$, then $V^\nu$ and $V^{\mu}$ are non-isomorphic as $D_n$-modules.

(2) If $\mu^{(1)} = \mu^{(2)}$, then the restriction $V^\mu \uparrow_{D_n}^{B_n}$ of $V^\mu$ to $D_n$ is a direct sum of two irreducible $D_n$-modules with same dimension. We denote these irreducible $D_n$-modules by the $V^\mu_+$ and $V^\mu_-$. 

Proof. The proof follows by mimicking the steps of deducing the irreducible representations of $A_n$ from that of $S_n$ [14, Theorem 4.4.2, Theorem 4.6.5]. Here $A_n$ denotes the alternating group. For this proof $B_n$ (respectively $D_n$) will play the role of $S_n$ (respectively $A_n$) and $\xi$ will play the role of the one-dimensional sign character of $S_n$. □

Let $\mathcal{S}$ be the collection of subsets $\Gamma$ of $\mathcal{D}_n$ satisfying the following properties:

(1) $\mu^{(1)} \neq \mu^{(2)}$ for each $(\mu^{(1)}, \mu^{(2)}) \in \Gamma$,

(2) $(\mu^{(2)}, \mu^{(1)}) \notin \Gamma$ if and only if $(\mu^{(1)}, \mu^{(2)}) \in \Gamma$.

Let $\Gamma_1$ be the maximal element of the poset $(\mathcal{S}, \subseteq)$ and $\Gamma_2 = \{(\mu^{(1)}, \mu^{(2)}) \in \mathcal{D}_n : \mu^{(1)} = \mu^{(2)}\}$. Then from Theorem A.1 and the observation

$$
\sum_{\mu \in \Gamma_1} (\dim(V^\mu))^2 + \sum_{\mu \in \Gamma_2} \left( (\dim(V^\mu))^2 + (\dim(V^\bar{\mu}))^2 \right),
$$

$$
= \frac{1}{2} \left( 2 \sum_{\mu \in \Gamma_1} (\dim(V^\mu))^2 + \sum_{\mu \in \Gamma_2} (\dim(V^\mu))^2 \right) = |B_n| - |D_n|,
$$

all the irreducible $D_n$-modules are given by $\{V^\mu : \mu \in \Gamma_1 \} \cup \{V^\mu_+, V^\mu_- : \mu \in \Gamma_2 \}$.

**Appendix B. A random walk on $D_n$ analogous to the walk on $B_n$ driven by $P$**

Let us consider the random walk on the demihyperoctahedral group $D_n$ driven by the probability measure $Q$ on $D_n$ defined as follows:

$$(35) \quad Q(\pi) = \begin{cases} 
\frac{1}{2n-1}, & \text{if } \pi = \text{id}, \text{ the identity element of } D_n, \\
\frac{1}{2n-1}, & \text{if } \pi = (i, n) \text{ for } 1 \leq i \leq n-1, \\
\frac{1}{2n-1}, & \text{if } \pi = (-i, n) \text{ for } 1 \leq i \leq n-1, \\
0, & \text{otherwise}.
\end{cases}
$$

It can be easily seen that the support of $Q$ generates $D_n$ and hence this random walk is irreducible. Moreover this random walk is aperiodic too. Thus the distribution after $k$th transition for this random walks will converge to $U_{D_n}$ in total variation distance as $k \to \infty$. Let us recall that $\hat{Q}(R)$ is the Fourier transform of $Q$ at the right regular representation $R$ of $D_n$. The transition matrix for the random walk on $D_n$ driven by $Q$ is the transpose of $\hat{Q}(R)$. To find the eigenvalues of $\hat{Q}(R)$ we will use the representation theory of $D_n$.

**Theorem B.1.** The eigenvalues of $\hat{Q}(R)$ are given by

(1) If $\mu = (\mu^{(1)}, \mu^{(2)}) \in \Gamma_1$, then for each $T \in \text{tab}_D(n, \mu)$, $\frac{2\text{vol}(\tau(n)) + 1}{2n-1}$ is an eigenvalue of $\hat{Q}(R)$ with multiplicity $\dim(V^\mu)$. 
(2) If \( \mu = (\mu^{(1)}, \mu^{(2)}) \in \Gamma_2 \), then for each \( T \in \text{tab}_D(n, \mu) \), \( \frac{2c(b_T(n)) + 1}{2n-1} \) is an eigenvalue of \( \hat{Q}(R) \) with multiplicity \( \frac{1}{2} \dim(V^\mu) \).

Recall \( c(b_T(n)) \) is the content of the box containing \( n \) in \( T \).

Proof. We have \( \hat{Q}(R) = \frac{1}{2n-1} (X_n + \text{id}) \), where \( X_n \) is the \( n \)th Young-Jucys-Murphy element of \( B_n \) and \( \text{id} \) is the identity element of \( D_n \). Here we identify the elements of \( D_n (\subseteq B_n) \) by the elements of \( B_n \).

For \( \mu = (\mu^{(1)}, \mu^{(2)}) \in \Gamma_1 \), we have \( \mu^{(1)} \neq \mu^{(2)} \). Therefore the restriction of irreducible \( B_n \)-module \( V^\mu \) to \( D_n \) is irreducible (Theorem A.1). Now for each \( T \in \text{tab}_D(n, \mu) \), let \( v_T \) be the Gelfand-Tsetlin vector of \( V^\mu \) satisfying \( X_n v_T = 2c(b_T(n)) v_T \). Also we know that \( \{v_T : T \in \text{tab}_D(n, \mu)\} \) forms a basis of \( V^\mu \). Therefore the eigenvalues of \( \hat{Q}(R) \) on the irreducible \( D_n \)-module \( V^\mu \) are given by \( \frac{2c(b_T(n)) + 1}{2n-1} \) for each \( T \in \text{tab}_D(n, \mu) \). Since the multiplicity of every irreducible representation in the right regular representation is equal to its dimension, therefore the multiplicity of these eigenvalues are \( \dim(V^\mu) \).

Now for \( \mu = (\mu^{(1)}, \mu^{(2)}) \in \Gamma_2 \) we have \( \mu^{(1)} = \mu^{(2)} \). Then the restriction of the irreducible \( B_n \)-module \( V^\mu \) to \( D_n \) splits into two irreducible \( D_n \)-modules \( V_+^\mu \) and \( V_-^\mu \) (Theorem A.1). In this case also \( v_T \) is the Gelfand-Tsetlin vector of \( V^\mu \) and \( \{v_T : T \in \text{tab}_D(n, \mu)\} \) forms a basis of \( V_+^\mu \oplus V_-^\mu \). Therefore, by similar arguments in case of \( \mu^{(1)} \neq \mu^{(2)} \), the eigenvalues of \( \hat{Q}(R) \) on the irreducible \( D_n \)-modules \( V_+^\mu \) and \( V_-^\mu \) are given by \( \frac{2c(b_T(n)) + 1}{2n-1} \) for each \( T \in \text{tab}_D(n, \mu) \). The multiplicity of these eigenvalues are \( \frac{1}{2} \dim(V^\mu) \) (\( : \dim(V_+^\mu) = \dim(V_-^\mu) = \frac{1}{2} \dim(V^\mu) \)) \( \square \)

Theorem B.2. For the random walk on \( D_n \) driven by \( Q \), we have the following:

1. \( ||Q^{*k} - U_{D_n}||_{TV} < \sqrt{\frac{e^c}{2}} e^{-c} \), for \( k \geq \left( n - \frac{1}{2} \right) (\log n + c) \) and \( c > 0 \).
2. \( \lim_{n \to \infty} ||Q^{*k_n} - U_{D_n}||_{TV} = 0 \), for any \( \epsilon \in (0, 1) \) and \( k_n = \left( (1 + \epsilon) \left( n - \frac{1}{2} \right) \log n \right) \).

Proof. Using Lemma 3.1 and following similar steps of Theorem 3.3, we have

\[ 4 ||Q^{*k} - U_{D_n}||_{TV}^2 \leq (1 + e) \left( e^{n^2 e^{-\frac{4k}{2n-1}}} - 1 \right), \quad \text{for } k \geq \left( n - \frac{1}{2} \right) \log n. \]

Now if \( k \geq \left( n - \frac{1}{2} \right) (\log n + c) \) and \( c > 0 \), then the right hand side of (36) becomes

\[ (e + 1) \left( e^{e^{-2c}} - 1 \right) < (2e + 2) e^{-2c}. \]

This proves the first part of the theorem. Now for \( \epsilon \in (0, 1) \), \( k_n = \left( (1 + \epsilon) \left( n - \frac{1}{2} \right) \log n \right) \) implies, \( k_n \geq (1 + \epsilon) \left( n - \frac{1}{2} \right) \log n \). Thus the right hand side of (36) is bounded above by \( (e + 1) \left( e^{\frac{1}{10e}} - 1 \right) \). Therefore the proof of the second part follows from

\[ \lim_{n \to \infty} (e + 1) \left( e^{\frac{1}{10e}} - 1 \right) = 0. \]

To obtain the lower bound for the total variation distance \( ||Q^{*k} - U_{D_n}||_{TV} \) we define the random variable \( Y \) as in the case of the walk on \( B_n \) driven by \( P \) as follows:

\[ Y(\pi) = \text{number of fixed points of } \pi \text{ when } \pi \in D_n. \]
Now using the definitions of \( \rho^+ \), \( \rho^- \) and \( \rho^{\text{def}} \) and the conventions for the ordering of the bases to obtain the matrices as given in section 4 we have the following:

\[
Y(\pi) = \text{Tr} \left( \rho^+ \left| B_n \right\rangle \langle D_n \left| \rho^- \right| B_n \right\rangle \langle D_n \right) = \text{Tr} \left( \rho^+(\pi) + \rho^-(\pi) \right), \quad \text{for } \pi \in D_n.
\]

Now, if \( E_{U_{D_n}}(Y) \) denotes the expectation of \( Y \) with respect to the uniform distribution on \( D_n \), then from standard combinatorial arguments one can show that \( E_{U_{D_n}}(Y) = 1 \).

**Lemma B.3.** The eigenvalues of \( \sum_{t=1}^{n-1} (\rho^-((t, n)) \otimes \rho^-((t, n)) + \rho^-((-t, n)) \otimes \rho^-((-t, n))) \) are given as below:

| Eigenvalues: | 2n - 2 | 2(n - 2) | -2 | 2 | 2(n - 3) |
|-------------|--------|----------|----|---|---------|
| Multiplicities: | 1 | n - 2 | n | n - 1 | (n - 1)(n - 2) |

**Proof.** The \( n^2 \) independent vectors defined in (19) are the eigenvectors in this case also. \( \square \)

**Proposition B.4.** Let \( Y, E_k(Y) \) be as defined in the beginning of this section. Then we have \( E_k(Y) = 1 + (2n - 3) \left( 1 - \frac{2}{2n - 1} - \frac{1}{2} \right)^k + \frac{1 + (1 - 1)^k}{(2n - 1)^k} \).

**Proof.** Following similar steps we used in Proposition 4.9 to get (20) and using (17) we have the following:

\[
E_k(Y) = \sum_{\pi \in D_n} Y(\pi)Q^{\pi}(\pi) = \text{Tr} \left( (\hat{Q} \left| B_n \right\rangle \langle D_n \right| B_n \right) + \text{Tr} \left( (\hat{Q} \left| B_n \right\rangle \langle D_n \right) \right).
\]

Now from Lemmas 4.4, 4.5 and by the similar arguments given in the proof of Proposition 4.9 we have the following: The eigenvalues of \( \hat{Q} \left| B_n \right\rangle \langle D_n \right| B_n \) are: 1, \( -\frac{1}{2n - 1} \) with multiplicity 1 and \( (1 - \frac{2}{2n - 1}) \) with multiplicity \( (n - 2) \). The eigenvalues of \( \hat{Q} \left| B_n \right\rangle \langle D_n \right| B_n \) are: \( (1 - \frac{2}{2n - 1}) \) with multiplicity \( (n - 1) \) and 1 with multiplicity 1. The proposition follows from (37) and the fact that the trace of \( k \)th power of matrix is the sum of \( k \)th powers of its eigenvalues. \( \square \)

**Proposition B.5.** Let \( Y \) and \( \text{Var}_k(Y) \) be as defined in the beginning of this section. Then we have

\[
\text{Var}_k(Y) = 2 + (4n - 6) \left( 1 - \frac{2}{2n - 1} \right)^k + (4n^2 - 16n + 13) \left( 1 - \frac{4}{2n - 1} \right)^k + \frac{1 + (1 - 1)^k}{(2n - 1)^k} \left( 3n - 1 - \frac{2}{(2n - 1)} + 2 \left( 1 - \frac{2}{2n - 1} \right)^k \right) + (n - 2) \left( 3^k - (-3)^k \right) \left( 2n - 1 \right)^k.
\]

**Proof.** We first find \( E_k(Y^2) \). From (17) and by similar arguments given in the proof of Proposition 4.10 to obtain (23), we have

\[
(Y(\pi))^2 = \text{Tr} \left( \rho_1 \left| B_n \right\rangle \langle D_n \right| B_n \right) + 2 \text{Tr} \left( \rho_2 \left| B_n \right\rangle \langle D_n \right| B_n \right) + \text{Tr} \left( \rho_3 \left| B_n \right\rangle \langle D_n \right| B_n \right) \quad \text{for each } \pi \in D_n.
\]

Again following similar steps we used in the proof of Proposition 4.10 to get (24) we have

\[
E_k(Y^2) = \text{Tr} \left( (\hat{Q} \left| B_n \right\rangle \langle D_n \right) \right) + 2 \text{Tr} \left( (\hat{Q} \left| B_n \right\rangle \langle D_n \right) \right) + \text{Tr} \left( (\hat{Q} \left| B_n \right\rangle \langle D_n \right) \right).
\]
Now following Lemmas 4.3, 4.4 and similar steps in the proof of Proposition 4.10, we have the Eigenvalues of $\hat{Q} \left( \rho_1 \downarrow D_n \right)$ are:

- **Eigenvalues:** $1, \left(1 - \frac{2}{2n-1}\right), -\frac{1}{2n-1}, \left(1 - \frac{4}{2n-1}\right), \frac{1}{2n-1}, -\frac{3}{2n-1}$
- **Multiplicities:** $2, 3(n-2), 3, n^2 - 5n + 5, n - 2, n - 2$

From Lemmas 4.6 and 4.7, the eigenvalues of $\hat{Q} \left( \rho_2 \downarrow D_n \right)$ are:

- **Eigenvalues:** $\left(1 - \frac{2}{2n-1}\right), \left(1 - \frac{4}{2n-1}\right), -\frac{1}{2n-1}, \frac{1}{2n-1}$
- **Multiplicities:** $2(n-1), (n-3)(n-1), n-1, n$

Finally from Lemma B.3, the eigenvalues of $\hat{Q} \left( \rho_3 \downarrow D_n \right)$ are:

- **Eigenvalues:** $1, -\frac{1}{n}, \frac{3}{2n-1}, \left(1 - \frac{2}{2n-1}\right), \left(1 - \frac{4}{2n-1}\right)$
- **Multiplicities:** $1, n-1, n-2, (n-1)(n-2)$

Therefore this proposition follows from (39) and Proposition B.4.

**Theorem B.6.** Let $Y, E_k(Y)$ and $\text{Var}_k(Y)$ be as defined above. Then we have the following:

1. For large $n$, $||Q^{*k} - U_{D_n}||_\text{TV} \geq 1 - \frac{10(1+2e^{-c}+o(1))}{(1+2e^{-c}+o(1))^2}$, when $k = (n - \frac{1}{2})(\log n + c)$ and $c \ll 0$.

2. $\lim_{n \to \infty} ||Q^{*k_n} - U_{D_n}||_\text{TV} = 1$, for any $\epsilon \in (0, 1)$ and $k_n = \lfloor (1 - \epsilon)n \log n \rfloor$.

**Proof.** Repeating the same steps we used in the proof of Theorem 4.12, we have the following:

$$||Q^{*k} - U_{D_n}||_\text{TV} \geq 1 - \frac{4 \text{Var}_k(Y)}{(E_k(Y))^2} - \frac{2}{E_k(Y)}.$$  

For large enough $n$, we have

$$E_k(Y) \approx 1 + (2n - 3)e^{-\frac{2k}{2n-1}} + o(1)$$

from Proposition B.4 and

$$\text{Var}_k(Y) \approx 2 + 2 \left(2n - 3 - \frac{(1 + (-1)^k)}{(2n-1)^k}\right) e^{-\frac{2k}{2n-1}} - 4(n-1)e^{-\frac{4k}{2n-1}} + o(1)$$

from Proposition B.5. Now if $n$ is large, $c \ll 0$ and $k = (n - \frac{1}{2})(\log n + c)$, then by (40), (41) and (42), we have the first part of this theorem. Again for any $\epsilon \in (0, 1)$ and $k_n = \lfloor (1 - \epsilon)n \log n \rfloor$ from (40), (41) and (42), we have

$$1 \geq ||Q^{*k_n} - U_{D_n}||_\text{TV} \geq 1 - \frac{10 + (20 + o(1))n^\epsilon + n^{2\epsilon}o(1) + o(1)}{(1 + (2 + o(1))n^\epsilon)^2}$$

for large $n$. Therefore, the second part of this theorem follows from (43) and the fact that

$$\lim_{n \to \infty} \frac{10 + (20 + o(1))n^\epsilon + n^{2\epsilon}o(1) + o(1)}{(1 + (2 + o(1))n^\epsilon)^2} = 0.$$  

□
Therefore from the first part of Theorems B.2 and B.6 we can say that the mixing time for the random walk on $D_n$ driven by $Q$ is $O\left(\left(n - \frac{1}{2}\right) \log n\right)$. Furthermore, the second part of Theorems B.2 and B.6 implies that this shuffle satisfies the cutoff phenomenon and the total variation cutoff for this shuffle occurs at $\left(n - \frac{1}{2}\right) \log n$.

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