On Completeness of Sliced Spaces under the Alexandrov Topology

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Abstract: We show that in a sliced spacetime \((V, g)\), global hyperbolicity in \(V\) is equivalent to \(T_A\)-completeness of a slice, if and only if the product topology \(T_P\), on \(V\), is equivalent to \(T_A\), where \(T_A\) denotes the usual spacetime Alexandrov “interval” topology.

Keywords: sliced space; Alexandrov interval topology; global hyperbolicity; slice completeness

1. Preliminaries

Sliced spaces have attracted the attention of several authors in studies related to systems of Einstein equations (see [1]), completeness (see [2]), global hyperbolicity (see [3,4]), as well as in problems of a more geometric nature on quantum cosmology (see [5,6]).

Definition 1. Let \(V = M \times \mathbb{R}\), where \(M\) is an \(n\)-dimensional smooth manifold, such that \(V\) is equipped with an \(n + 1\)-dimensional Lorentz metric \(g\), which splits in the following way:

\[
g = -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j,
\]

where \(\theta^0 = dt\), \(\theta^i = dx^i + \beta^i dt\), \(N = N(t, x^i)\) is called lapse function, \(\beta^i(t, x^i)\) is called shift function and \(M_1 = M \times \{t\}\), called spatial slices of \(V\), are spacelike submanifolds equipped with the time-dependent spatial metric \(g_t = g_{ij} dx^i dx^j\). Such a product space \(V\) is called a sliced space.

Let \((V, g)\) be a sliced space. A base for the product topology \(T_P\), on \(V\), consists of all sets of the form \(A \times B\), where \(A \in T_M\) and \(B \in T_\mathbb{R}\). Here \(T_M\) denotes the natural topology of the manifold \(M\) where, for an appropriate Riemann metric \(h\), it has a base consisting of open balls \(B^h_r(x)\) and \(T_\mathbb{R}\) is the usual topology on the real line.

The Alexandrov topology (or “interval topology”) \(T_A\) on a spacetime \(V\) has a base consisting of open sets of the form \(< x, y > = I^+(x) \cap I^-(y)\), where \(I^+(x) = \{z \in V : x \ll z\}\) and \(I^-(y) = \{z \in V : z \ll y\}\), where \(\ll\) is the chronological order defined as \(x \ll y\) iff there exists a future oriented timelike curve, joining \(x\) with \(y\). By \(I^+(x)\) one denotes the topological closure of \(I^+(x)\) and by \(I^-(y)\) that one of \(I^-(y)\) (see [7]).

A spacetime \(V\) is strongly causal, if and only if it is strongly causal at every point, that is, for every point \(p \in V\), \(p\) has arbitrarily small causally convex neighbourhoods. We say that \(V\) is globally hyperbolic, if and only if \(V\) is strongly causal and every set \(I^+(x) \cap I^-(y)\) (called a “closed diamond”) is compact. Global hyperbolicity is considered the strongest causality condition in the causal hierarchy of spacetimes (see [8]) and is equivalent to the existence of a Cauchy hypersurface \(S\) for \(V\) (see Section 5, in [7]); this supplies us with the benefit to construct on \(V\) well-defined initial-value problems (see [9,10], Theorem 10.2.2). One can also view global hyperbolicity as a property on a spacetime...
which guarantees the absence of naked singularities in $V$ (for its role in the strong cosmic censorship, see [11]).

In the next section we will show that global hyperbolicity in a sliced spacetime $(V, g)$ is equivalent to completeness with respect to the Alexandrov topology of a slice $(M_t, g_t)$. Although completeness of the Alexandrov topology $T_A$, by itself, is not a criterion of nonsingularity (in the Schwarzschild space and the Friedmann–Robertson–Walker cosmologies, for example, $T_A$ is complete, but these spaces are singular; see [12]), it is interesting that in the particular case of sliced spacetimes that are equipped with their natural product topology, completeness of a slice with respect to $T_A$ can be considered as a criterion of global hyperbolicity for the entire space.

Thoughout our text, for topological terms like Hausdorff space and completeness, we refer to the seminal book of Engelking, [13].

2. A Topological Condition for the Completeness of a Sliced Space

In [3], sliced spaces are being considered to have uniformly bounded lapse, shift and spatial metric, in order to achieve the equivalence of global hyperbolicity of $(V, g)$ with the completeness of the slice $(M_t, g_t)$ (Theorem 2.1). Being motivated by this result, in the Theorem that follows, we consider global topological conditions, for showing the equivalence of global hyperbolicity of $(V, g)$ with a slice $(M_t, g_t)$ being $T_A$-complete. Our Theorem 2.1, below, differs from Theorem 2.1 of [3] in that the slices in [3] are complete Riemannian manifolds (with uniformly bounded spatial metric, lapse and shift functions) while in our case the slices are $T_A$-complete. We discuss this further in Section 3.

**Theorem 1.** Let $(V, g)$ be a sliced space, with respect to its natural product topology $T_p$, where $V = M \times \mathbb{R}$, $M$ is an $n$-dimensional manifold and $g$ the $n + 1$-Lorentz “metric” on $V$. Let also $T_A$ be the Alexandrov topology on $V$. Then, the following statements are equivalent:

1. $(V, g)$ is globally hyperbolic.
2. $T_p \equiv T_A$.
3. $(M_t, g_t)$ is complete with respect to $T_A$.

**Proof.** 1. has been shown to be equivalent to 2. in [4].

To show that 2. implies 3., we first notice that since $(V, g)$ is globally hyperbolic, it is also strongly causal. Since, also, $T_p \equiv T_A$, we have that for every $t \in \mathbb{R}$, $M_t$ is a subset of a spacetime $V$, with nondegenerate spacetime metric, with subspace topology $T_A$ inherited from $V$, such that $M_t$ is strongly causal. Hence, $T_A$, on $M_t$, is complete (see [12], Theorem 2).

For proving that 3. implies 1., for each $t \in \mathbb{R}$, we let $(M_t, g_t)$ to be complete with respect to $T_A$, where each $M_t$ is a spacelike submanifold with time dependent spatial metric $g_t \equiv g_j^i dx^i dx^j$. But since each $M_t$ is complete, the Alexandrov topology $T_A$, on $M_t$, is strongly causal (see, again, [12]). So, each point of $M_t$ is strongly causal, which means that for every point $P \in M_t$ there exists an arbitrarily small convex neighbourhood. But, $V = \bigcup_{t \in \mathbb{R}} M \times \{t\}$, so $P \in V$ if and only if there exists $M_t = M \times \{t\}$, such that $P \in M_t$, and hence $V$ is strongly causal with respect to $T_A$. That the closed $T_A$-diamonds, in $V$, are compact, has been shown in Theorem 3, from [4]. Thus, $(V, g)$ is globally hyperbolic.  

3. Discussion

Question 1: Can our Theorem 1 hold, if one substituted in 3. “$(M_t, g_t)$ is complete with respect to $T_A$” with the statement “$(M_t, g_t)$ is a complete Riemannian manifold”? The answer is negative, since in a spacetime manifold, $T_A$ is usually a coarser topology than the spacetime topology, and it is equivalent to the manifold topology only if it is Hausdorff (see [7], Theorem 4.24). So, in order for this question to have a positive answer, one would have to add in Theorem 1 the extra condition that $T_A$, on $V$, is Hausdorff. As a continuation of this question, we ask whether the spacetimes considered in [3] may well have their Alexandrov topology $T_A$ not being Hausdorff. In such a case, strong causality will fail due to this (see, for example, Remark 4.25 of [7]). Spacetimes where $T_A$ fails to be Hausdorff,
according to Penrose, admit a null geodesic along which strong causality fails and this is one aspect of a general result concerning the region of strong causality failure in a spacetime [7]. Given the above argument, we conjecture that for a physically reasonable spacetime, the statements of Theorem 2.1 of [3] and of Theorem 2.1 here should be equivalent. A rigorous proof showing the equivalence of a uniformly bounded spatial metric, lapse and shift functions with the condition of the topologies \( T_P \) and \( T_A \) to be equivalent, will be of a great interest.

In [3], there are conditions introduced, so that global hyperbolicity to be equivalent to geodesic completeness. In particular, in Theorem 3.1, the term \( \text{trivially sliced space} \) is introduced, so that a slice is a complete Riemannian manifold, if and only if the space \((V, g)\) is geodesically complete. The “disadvantage” of this condition is that the spatial metric \( g_{ij} \) is time-independent.

**Question 2:** Can one relate slice-completeness and geodesic completeness of \((V, g)\) with a time-dependent spatial metric \( g_{ij} \)?

Question 2 does not seem to have a trivial answer. In a possible variation of Theorem 1, towards an answer to this question, one could make use of the classical Hopf-Rinow Theorem (see [14]), which gives that metric completeness, in a spacetime, is equivalent to geodesic completeness. Again, \( T_A \) should be Hausdorff.

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