Thermodynamics of black plane solution

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Abstract

We obtain a new phantom black plane solution in 4D of the Einstein-Maxwell theory coupled with a cosmological constant. We analyse...
their basic properties, as well as its causal structure, and obtain the extensive and intensive thermodynamic variables, as well as the specific heat and the first law. Through the specific heat and the so-called geometric methods, we analyse in detail their thermodynamic properties, the extreme and phase transition limits, as well as the local and global stabilities of the system. The normal case is shown with an extreme limit and the phantom one with a phase transition only for null mass, which is physically inaccessible. The systems present local and global stabilities for certain values of the entropy density with respect to the electric charge, for the canonical and grand canonical ensembles.

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1 Introduction

It is well known that a black hole can radiate a black-body radiation when one takes into account the effects of classical gravitational field on quantized matter fields, i.e., a semi-classical analysis of the gravity \[1\]. So, we can make a study of the thermodynamic system of each new black hole solution. The most common method in the literature is the analysis made through the specific heat of the black hole \[2\], which informs us if the system is thermodynamically interacting, if there exists any case in which the black hole is extreme or it passes across a second order phase transition.

Recently, attention is attached to the methods for analysing the thermodynamic system through the geometry of the so-called thermodynamic space of the equilibrium states. The most common are the methods of Weinhold \[3\], Ruppeiner \[4\], geometrothermodynamics \[5\] and that of Liu-Lu-Luo-Shao \[6\]. These methods also notify if the system possesses thermodynamic interaction and if it undergoes a second order phase transition, in addition to the properties about the stability.

In this work, we desire to make a detailed analysis of the thermodynamic system of a well known class of solutions, with a particularly interesting symmetry, the planar. This class of solutions has been previously obtained for the case of planar and static symmetry in 4\(D\), by Cai and Zhang \[7\]. This symmetry was then applied to traversable wormholes \[8\], and later, generalized to topological black holes in \[9\], and its various applications. We
focus our attention to a class of solutions, called phantom [10], but now with a planar symmetry.

Before beginning the analysis of this new class of phantom black holes, we will present briefly our interest in obtaining and studying such exotic solutions. With the discovery of the acceleration of the universe, various observational programs of studying the evolution of our universe were deployed, including the relationship of the magnitude-versus-redshift types supernovae Ia and the spectrum of the anisotropy of the cosmic microwave background. These programs promote an accelerated expansion of our universe, which should be dominated by an exotic fluid and should have a negative pressure. Moreover, these observations show that this fluid can be phantom, i.e., with the contribution of the energy density of dark energy [11].

As the interest in obtaining these classes has increased, we also found ourselves wanting to analyse a specific phantom model. We can mention here some recent results in the literature, such as the wormhole solutions and conformal continuation [12], the black hole solutions of Einstein-Maxwell-Dilaton theory, [13], the higher-dimensional black holes by Gao and Zhang [14], and the higher-dimensional black branes by Grojean et al [15]. Analysis were also made in algebraic structures of this type of phantom system, as the case of the algebra generated by metrics depending on two temporal coordinates, with $D \geq 5$, which provides phantom fields in $4D$, fulfilled by Hull [16], and Sigma models by Clément et al [17]. Here, we will obtain and study the thermodynamic properties of a solution arising from the coupling of Einstein-Hilbert action with a field of spin 1, which can be Maxwell or anti-Maxwell (phantom), and a cosmological constant, where the spacetime possesses planar symmetry. The idea of using the ruse of negative electric energy density is quit old, Einstein and Rosen being the first to use it [28]. Recently, through the work of Babichev et al [29] and Bronnikov et al [30], we have seen a keen interest in phantom solutions [31].

The paper is organized as follows. In Section 2, we present a new phantom black plane solution. The causal structure of the solutions are studied and the thermodynamic variables are obtained. The first law of thermodynamics is established and the specific heat is calculated. In Section 3, we minutely study the thermodynamics of normal and phantom solutions, using the analysis through the specific heat, subsection 3.1, and through the geometric methods of Weinhold, subsection 3.2, the geometrothermodynamics, subsection 3.3, and that of Liu-Lu-Luo-Shao, subsection 3.4. We finish the section with the study of local and global stabilities in subsection 3.5. The
2 The field equations and the black holes solutions

The action of the theory is given by:

$$ S = \int d^4x \sqrt{-g} \left[ R + \eta F^{\mu\nu} F_{\mu\nu} + 2\Lambda \right], \quad (2.1) $$

where the first term is that of Einstein-Hilbert, the second is the coupling of (anti)Maxwell field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with the gravitation, and the third is the cosmological constant. Making the functional variation of the action $(2.1)$ with respect to the field $A_\mu$ and the inverse of the metric, $g^{\mu\nu}$, using $R = -4\Lambda$, we get the following equations of motion

$$ \nabla_\mu [F^{\mu\alpha}] = 0, \quad (2.2) $$

$$ R_{\mu\nu} = 2\eta \left( \frac{1}{4} g_{\mu\nu} F^2 - F^{\sigma}_\mu F^\sigma_{\nu} \right) - \Lambda g_{\mu\nu}. \quad (2.3) $$

Let us write the static and plane symmetric line element as

$$ ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)(dx^2 + dy^2), \quad (2.4) $$

with $r = |z|$. We will also assume that the Maxwell field is purely electric and only depends on $r$. With $(2.4)$, one can integrate $(2.2)$ and obtain

$$ F^{10}(r) = \frac{q}{C\sqrt{AB}}, \quad (F^2 = -2\frac{q^2}{C^2}), \quad (2.5) $$

with $q$ a real integration constant. Substituting $(2.5)$ into the equations of motion $(2.3)$, we obtain the equations

$$ \frac{A''}{A} - \frac{1}{2} \left( \frac{A'}{A} \right)^2 - \frac{A'B'}{2AB} + \frac{A'C'}{AC} = 2B \left( \frac{\eta q^2}{C^2} - \Lambda \right), \quad (2.6) $$

$$ \frac{A''}{A} - \frac{1}{2} \left( \frac{A'}{A} \right)^2 - \frac{A'B'}{2AB} + \frac{C''}{C} - \frac{B'C'}{2BC} - \left( \frac{C'}{C} \right)^2 = 2B \left( \frac{\eta q^2}{C^2} - \Lambda \right), \quad (2.7) $$

$$ -\frac{A'C'}{2AC} - \frac{C''}{C} + \frac{B'C'}{2BC} = 2B \left( \frac{\eta q^2}{C^2} + \Lambda \right), \quad (2.8) $$
where the “prime” denotes the derivative with respect to $r$. Choosing the coordinates such that

$$A(r) = B^{-1}(r), \quad C(r) = \alpha^2 r^2$$

with $\Lambda = -3\alpha^2$, the solution of the equations of motion (2.6)-(2.8) is given by

$$\begin{align*}
\{ & dS^2 = A(r) dt^2 - A^{-1}(r) dr^2 - C(r) (dx^2 + dy^2), \\
& F = -\frac{q}{C(r)} dr \wedge dt, \quad A(r) = \alpha^2 r^2 - \frac{m}{r} + \eta \frac{q^2}{\alpha^4 r^2}, \quad C(r) = \alpha^2 r^2,
\end{align*}$$

where $m$ is the mass and $q$ the electric charge of the (phantom) black plane. This is the same solution as that of [7], for $\eta = 1$, and phantom black plane solution for $\eta = -1$, obtained for the first time here.

We can rewrite the solution in terms of the densities of mass $M$ and electric charge $Q$, as calculated in [7], yielding

$$\begin{align*}
\{ & dS^2 = A(r) dt^2 - A^{-1}(r) dr^2 - C(r) (dx^2 + dy^2), \\
& F = -\frac{2\pi Q}{C(r)} dr \wedge dt, \quad A(r) = \alpha^2 r^2 - \frac{4\pi M}{\alpha^2 r} + \eta \frac{4\pi^2 Q^2}{\alpha^4 r^2}, \quad C(r) = \alpha^2 r^2.
\end{align*}$$

One can calculate the horizon of this solution, vanishing $A(r)$, obtaining

$$\alpha^2 r^2 - \frac{4\pi M}{\alpha^2 r} + \eta \frac{4\pi^2 Q^2}{\alpha^4 r^2} = 0.$$ (2.12)

This solution possesses two complex and two real roots. The real roots are given by

$$r_\pm = \frac{1}{2} \left[ \sqrt{2k} \pm \sqrt{\frac{8\pi M}{\alpha^4 \sqrt{2k}} - 2k} \right],$$ (2.13)

$$k = \sqrt{\frac{4\pi M}{\alpha^4}} + \sqrt{\frac{(\pi M)^4}{\alpha^4} - \eta \left( \frac{4\pi^2 Q^2}{3\alpha^6} \right)^3}$$

$$+ \sqrt{\frac{(\pi M)^4}{\alpha^4} - \eta \left( \frac{4\pi^2 Q^2}{3\alpha^6} \right)^3}.$$ (2.14)

For the normal solution, $\eta = 1$, one has $0 < r_- < r_+$, and for $\eta = -1$, the corresponding is $r_- < 0 < r_+$, with $r_+ > |r_-|$. We observe that in the
phantom solution, $r_-$ is in the negative part, but here something happens that we do not have in the spherical symmetry, because as $r_{\pm} = |z_{1,2}|$, one gets $z_{1(\pm)} = \pm r_+$ and $z_{2(\pm)} = \pm r_-$. As $r_- < 0$, one gets $z_{1(-)} < z_{2(+)} < 0 < z_{2(-)} < z_{1(+)}$. Then, the singular plan $z = r_s = z_s = 0$ is covered by the plans $z = z_{1(-)}, z = z_{2(+)}$, $z = z_{2(-)}$ and $z = z_{1(+)}$ (see Figure 1). In the case of spherical symmetry, the internal horizon $r_-$ could not be achieved, for a solution of non-degenerate horizon. Hence, here we have a drastic change in the causal structure of the phantom black plane solution, whose singularity is covered by two horizons in the positive part of $z$. This could not occur in the phantom solutions with spherical symmetry, where just one horizon covered the singularity. However, another unusual event happens, where we get two horizons but with the property of non existence of extreme case, i.e, these horizons can never be equal, when we consider only real values.

Figure 1: Structure of spacetime in $z$ direction, for the phantom solution (2.11).

The curvature scalar of the metric (2.4) is given by

$$R = \frac{2C''}{BC} - \frac{(C')^2}{2BC^2} - \frac{B' C''}{B^2 C} + \frac{A' C''}{ABC} - \frac{A' B''}{2AB^2} + \frac{A''}{2AB} - \frac{(A')^2}{2A^2B}. \quad (2.15)$$
The scalar of Kretschmann is given by

\[ K = R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} = C^2 (C')^2 + B^2 (C'')^2 - C (C')^2 C'' - \frac{B^2 (C''')^2 C''}{C} - \frac{B' (C'')^2 C''}{B} - \frac{B' (C''')^2 C'^2}{B} \]

\[ - BB'C' C'' + \frac{C^2 (C')^4}{4 B^2} + \frac{B^2 (C')^4}{4 C^2} + \frac{C r^4}{4} + \frac{B'C (C')^3}{2 B} + \frac{B B' (C')^3}{2 C} \]

\[ + \frac{(B')^2 C^3 (C'')^2}{4 B^2} + \frac{(A')^2 C^3 (C'')^2}{4 B^2} + \frac{(B')^2 (C'')^2}{4} + \frac{A^2 (A')^2 (C'')^2}{4 B^2} + \frac{A^2 (A')^2 (B')^2}{8 B^2} \]

\[ + \frac{(A')^2 (B')^2}{8} - \frac{A'A'' B B'}{2 A} + \frac{(A')^3 B B'}{4 A} - \frac{A^2 A'A'' B'}{2 B} + \frac{A (A')^3 B'}{4 B} + \frac{(A'')^2 B^2}{2} \]

\[ - \frac{(A')^2 A'' B^2}{2 A} + \frac{(A')^4 B^2}{8 A^2} - \frac{A^2 (A'')^2}{2} - \frac{A (A')^2 A''}{2} + \frac{(A')^4}{8}. \] (2.16)

By substituting \( A(r) = B^{-1}(r) \) and \( C(r) \) in (2.11), the curvature scalar \( (R = 12\alpha^2) \) and that of Kretschmann are finite throughout the space-time, except in the singular plane \( r_s = z = 0 \).

In order to construct the Penrose diagram of this solution, we define several new coordinates for getting a description (non-singular on the horizons) of this space-time of type Kruskal. So, the Eddington-Finkelstein coordinates are gives by

\[ u = t + r^*, v = t - r^*, \] (2.17)

where the tortoise coordinate is give by

\[ r^* = \int A^{-1}(r) dr = \frac{1}{\alpha^2} \left\{ \frac{1}{r_+ - r_-} \ln \left| \frac{r - r_+}{r - r_-} \right| - \frac{(r_+ + r_-)^2 + r^2}{(r_+ - r_-)[(r_+^2 + r_-^2) + 2r^2]} \right\} \times \]

\[ \times \left[ \frac{\ln \left| r - r_+ \right| + \frac{2r + r_+}{2(r_+^2 + r_-^2) + 2r_+ r_-(r_+^2 + r_-^2)}}{\sqrt{(r_+ + r_-)^2 + 2(r_+^2 + r_-^2)}} \right] \frac{\ln \left| r - r_- \right|}{(r_+ - r_-)[(r_+^2 + r_-^2) + 2r^2]} \]

\[ \times \frac{(r_+ + r_-)^3}{4(r_+ + r_-)^4 + 2(r_+^2 + r_-^2)^2} \ln \left| r^2 + (r_+ - r_-)r + (r_+ + r_-)^2 - r_+ r_- \right| \}. \] (2.18)

With these coordinates, we can rewrite the line element (2.11) as

\[ dS^2 = A(r) du^2 + 2 du dv - C(r) (dx^2 + dy^2) . \] (2.19)
Also defining the coordinates of type Kruskal

\[ U = \arctan \left\{ \pm k_0 \exp \left[ -\frac{\alpha^2}{2} (r_+ - r_-)[2 + (1 + k_1)^2]v \right] \right\}, \quad (2.20) \]

\[ V = \arctan \left\{ \pm k_0 \exp \left[ \frac{\alpha^2}{2} (r_+ - r_-)[2 + (1 + k_1)^2]u \right] \right\}, \quad (2.21) \]

\[ k_1 = \frac{r_+}{r_-}, \quad k_0 = \frac{r_+^{k_1}}{\sqrt{r_+}} (r_+^2 + r_+ r_- + r_-^2) - \left( \frac{1 - k_1}{2} \right) \frac{(1 + k_1)^2[2 + (1 + k_1)^2]}{4(1 + k_1)^4 + 2(1 + k_1^2)^2} \times \]

\[ \exp \left\{ - \left( \frac{1 - k_1}{2} \right) \frac{(1 + k_1)^4 + 2(1 + k_1^2)(1 + k_1 + k_1^2)}{[(1 + k_1)^2 + 2k_1^2] \sqrt{(1 + k_1)^2 + 2(1 + k_1^2)^2}} \right\} \times \]

\[ \arctan \left( \frac{1 + k_1}{\sqrt{(1 + k_1)^2 + 2(1 + k_1^2)}} \right) \quad (2.22) \]

we can rewrite (2.19) as

\[ dS^2 = \Omega(U, V) dUdV - C(r) \left( dx^2 + dy^2 \right). \quad (2.24) \]

With the use of these coordinates we can construct the causal structure of this solution, which is very similar to the Reissner-Nordstrom-AdS one (see Figure 2).

We can see in Figure 2 that if we think to follow the decreasing \( z \), starting from positive infinity, we have the region \( Z_1 \) \( (z_{1(+)} < z < +\infty) \), passing by the first horizon at \( z = z_{1(+)} \), for the second region \( Z_2 \) \( (z_{2(-)} < z < z_{1(+)}) \). After we passed the second horizon at \( z = z_{2(-)} \), for the third region \( Z_3 \) \( (0 \leq z < z_{2(-)}) \).

After arriving at the singular plane at \( z = 0 \). These regions \( z \geq 0 \) are causally disconnected from those for which \( z \leq 0 \). Regions from \( Z_4 \) to \( Z_6 \) are the exact reflection (symmetrical values of positive \( z \)) for positive values of \( z \). So, we can think alike to follow a direction of creasing values of \( z \), beginning at negative infinity. Thus, we perform the reflected route, and spent from \( Z_6 \) \( (-\infty < z < z_{1(-)}) \) to \( Z_5 \) \( (z_{1(-)} < z < z_{2(+)}), \) and then, to the region \( Z_4 \) \( (z_{2(+)} < z \leq 0), \) reaching the singular plane at \( z = 0 \).

Now, we are interested in the geometrical analysis representing semi-classical gravitational effects of the black hole solutions as mentioned before.

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5This singularity is timelike and can be avoided, depending on the particle energy applied in its path.
By semi-classical we mean quantize the called matter fields, while the background gravitational field is treated classically. Therefore, we will work with the semi-classic thermodynamics of black holes, studied first by Hawking [1], and further developed by many other authors [18].

There are several techniques to derive the Hawking temperature law. For example we can mention the Bogoliubov coefficients [19] and the energy-momentum tensor methods [2] [18], the euclidianization of the metric [20], the transmission and reflection coefficients [21] [22], the analysis of the anomaly term [23], and the black hole superficial gravity [24]. Since all these methods have been proved to be equivalent [25], then we opt, without loss of generality, to calculate the Hawking temperature by the superficial gravity method.

The surface gravity of a black plane is given by [7]:

\[
\kappa = \left[ \frac{g'_{\theta \theta}}{2\sqrt{-g_{\theta \theta}g_{\phi \phi}}} \right]_{r = r_+},
\]

(2.25)

where \( r_+ \) is the event horizon radius, and the Hawking temperature is related with the surface gravity through the relationship [1] [24]

\[
T = \frac{\kappa}{2\pi}.
\]

(2.26)

Then, for the black plane solution (2.11), we get the surface gravity (2.25).
as
\[ \kappa = \alpha^2 r_+ + \frac{2\pi M}{\alpha^2 r_+^2} - \eta \frac{4\pi^2 Q^2}{\alpha^4 r_+^3}, \] (2.27)
and the Hawking temperature (2.26) in this case is:
\[ T = \frac{1}{2\pi} \left[ \alpha^2 r_+ + \frac{2\pi M}{\alpha^2 r_+^2} - \eta \frac{4\pi^2 Q^2}{\alpha^4 r_+^3} \right]. \] (2.28)
We define the entropy per unit of area of the black plane as two times the quarter of the horizon area
\[ S = 2 \times \frac{1}{4} A = \frac{\alpha^2 r_+^2}{2}, \] (2.29)
where the factor 2 is due to the contribution of two planes \( z = \pm r_+. \)
From (2.11), we can calculate the electric potential scalar at the horizon
\[ A_0 = \int_{+\infty}^{r} F_{10}(r')dr' \bigg|_{r=r_+} = \frac{2\pi Q}{\alpha^2 r_+}. \] (2.30)
Let us check the first law for the solution (2.11). Taking the differential of the mass, isolated from (2.12), of the electric charge and of the entropy (2.29), we get
\[ dM = \left( \frac{3\alpha^4 r_+^2}{4\pi} - \eta \frac{\pi Q^2}{\alpha^2 r_+^2} \right) dr_+ + \eta \frac{2\pi Q}{\alpha^2 r_+} dQ, \quad dS = \alpha^2 r_+ dr_+, \] (2.31)
which satisfies the first law of thermodynamics
\[ dM = TdS + \eta A_0 dq. \] (2.32)
Note that we introduced a compensating sign \( \eta \) in (2.32) due to the contribution of the negative energy density, in the phantom case, the field of spin 1, \( F_{\mu\nu} \), which provides a work with an inverted sign in the first law.
As we need to study the thermodynamic system through the geometric methods, we must first write the mass in terms of the entropy and the electric charge. We can do this by isolating the mass in (2.12) and then replace \( r_+ \) in terms of the entropy \( \sqrt{2S} \), with the use of (2.29), which yields
\[ M(S, Q) = \frac{\alpha^2 S^2 + \eta \pi^2 Q^2}{\pi \alpha \sqrt{2S}}, \] (2.33)
\[ \text{We take } r_+ = \sqrt{2S}/\alpha, \text{ with the sign of } \alpha > 0. \text{ The negative sign of } \alpha \text{ can be considered taking } r_+ = -\sqrt{2S}/\alpha. \]
where we have the conditions $Q^2 \leq (3\alpha^6/4\pi^2)(\pi M/\alpha^4)^{4/3}$ for $\eta = 1$ (real horizon in (2.13)) and $Q^2 \leq (\alpha^2 S^2/\pi^2)$ for $\eta = -1$. We also write the temperature and the electric potential in terms of the entropy and the electric charge. Taking (2.28) and (2.30), for $r_+$ in terms of the entropy, we get

$$T(S, Q) = \frac{3\alpha^2 S^2 - \eta \pi^2 Q^2}{\pi \alpha (\sqrt{2} S)^3}, \quad A_0 = \frac{2\pi Q}{\alpha \sqrt{2} S}. \quad (2.34)$$

We can then calculate the specific heat by the expression

$$C_Q = \left( \frac{\partial M}{\partial T} \right)_Q = \left( \frac{\partial M}{\partial S} \right)_Q / \left( \frac{\partial^2 M}{\partial S^2} \right)_Q = \frac{2S}{3} \frac{(3\alpha^2 S^2 - \eta \pi^2 Q^2)}{(\alpha^2 S^2 + \eta \pi^2 Q^2)}. \quad (2.35)$$

We now have in hand the basic requirements to begin our analysis of the thermodynamic system of these solutions. In the next section we will study the specific heat (2.35) and through the four geometric methods, the thermodynamic properties of these planar solutions.

### 3 Thermodynamics of black plane

In this section we will study in detail the thermodynamic properties of the planar solutions (2.11), both for normal and phantom cases. Through the specific heat and the curvature scalar of the thermodynamic spaces of the equilibrium states, we will examine whether there is an extreme case (only by the usual method), phase transition and finally, the local and global stabilities of the thermodynamic system.

#### 3.1 Analysis of specific heat

Historically, the study of specific heat for revealing the thermodynamic properties was the first to be used [2] and has been called of usual method.

Here, we have the expression of the specific heat (2.35), which, equating to zero, reveals the value of the entropy for which the solution is extreme, i.e, for $S = S_e = \pi Q \sqrt{\eta/\alpha} \sqrt{3}$, which is real only for $\eta = 1$. Therefore, there does not exist an extreme case for the phantom solution with $\eta = -1$, as we had seen in its causal structure.

Similarly, we can find the value of the entropy for which the system undergoes a second order phase transition, i.e, when the specific heat diverges.
In this case the specific heat (2.35) diverges for $S = S_t = -i\sqrt{\eta}Q/\alpha$, which shows that the normal case $\eta = 1$ has no phase transition, while the phantom case possesses a phase transition in $S = S_t$. Note that this case is the specific value where the mass (2.33) vanishes. So, here, we have a mathematical chance of the system going from a locally stable phase ($C_Q > 0$ and positive mass), for an unstable phase, with $C_Q < 0$ and negative mass (2.33). The phase transition of second order is not physically possible because the energy of the phantom black plane should be reduced continuously such that it passes from the positive values to zero, and even reaching negative values. This will be well examined in the stability study of the system.

We plot the evolution of the specific heat (2.35) for a specific choice of the parameters, as shown in Figure 3.

![Figure 3](image_url)

**Figure 3:** The mass (2.33) (blue), the temperature (2.34) (purple) and specific heat (2.35) (green) for $Q = 0.5, \alpha = 2, \eta = -1$. The phase transition point is given by $S_t = 0.785398$.

We will take the results of the study of specific heat as the basis for comparing with a geometric analysis of the thermodynamic system, through the four most popular methods in the literature. All these methods have in common the definition of a metric for the thermodynamic space of the equilibrium states, where the calculation of the curvature scalar of this metric reveals the existence or not of thermodynamic interaction, phase transition points, among other thermodynamic properties. Let us calculate this object with the aid of a mathematical software.

In the next subsection we will analyse the thermodynamic system through the method of Weinhold.
3.2 The Weinhold method

Historically, Weinhold was one of the first to formulate a geometric description applicable to a thermodynamic system. The method of Weinhold [3], as it is known, aims to define a metric for the thermodynamic space of the equilibrium states, through the mass (2.33) as thermodynamic potential. The metric constructed in this way provides a curvature scalar $R_W$, which, for this method can be interpreted as a function of extensive variables that shows the points of phase transition, when there exists, where the thermodynamic system goes by. Then, we define the metric of Weinhold as being

$$d^2_W = \frac{\partial^2 M}{\partial S^2} dS^2 + 2 \frac{\partial^2 M}{\partial S \partial Q} dS dQ + \frac{\partial^2 M}{\partial Q^2} dQ^2$$

$$= \frac{3(\alpha^2 S^2 + \eta \pi^2 Q^2)}{4 \sqrt{2} \pi \alpha S^{5/2}} dS^2 - \frac{2\eta \pi Q}{\sqrt{2} \alpha S^{3/2}} dS dQ + \frac{\eta \sqrt{2} \pi}{\alpha \sqrt{S}} dQ^2. \quad (3.36)$$

Here we see that the curvature scalar $R_W$ of this metric is identically zero, which prevents us of doing an analysis of the phase transition of the thermodynamic system. This result does not agree with the study of the specific heat. In the next subsection we will study the thermodynamics through the method of geometrothermodynamics.

3.3 The Geometrothermodynamics method

The Geometrothermodynamics (GTD) [5] makes use of differential geometry as a tool to represent the thermodynamics of physical systems. Let us consider the $(2n + 1)$-dimensional space $\mathcal{T}$, whose coordinates are represented by the thermodynamic potential $\Phi$, the extensive variable $E^a$ and the intensive variables $I^a$, where $a = 1, ..., n$. If the space $\mathcal{T}$ has a non degenerate metric $G_{AB}(Z^C)$, where $Z^C = \{\Phi, E^a, I^a\}$, and the so called Gibbs 1-form $\Theta = d\Phi - \delta_{ab} I^a dE^b$, with $\delta_{ab}$ the delta Kronecker; then, the structure $(\mathcal{T}, \Theta, G)$ is said to be a contact riemannian manifold if $\Theta \wedge (d\Theta)^n \neq 0$ is satisfied [26]. The space $\mathcal{T}$ is known as the thermodynamic phase space. We can define a $n$-dimensional subspace $\mathcal{E} \subset \mathcal{T}$, with extensive coordinates $E^a$, by the map $\varphi : \mathcal{E} \to \mathcal{T}$, with $\Phi \equiv \Phi(E^a)$, such that $\varphi^*(\Theta) \equiv 0$. We call the space $\mathcal{E}$ the thermodynamic space of the equilibrium states.

We can then define the metric of the thermodynamic space of the equilibrium states $\mathcal{E}$, through the derivation of the thermodynamic potential and
its extensive variables as \[27\]

\[
dl^2_{G(\Phi)} = \left(E^c \frac{\partial \Phi}{\partial E^c}\right) \left(\eta_{ad} \delta^{di} \frac{\partial^2 \Phi}{\partial E^d E^i}\right) dE^a dE^b , \tag{3.37}\]

which, by definition, is invariant under Legendre transformations. Through the metric (3.37), we can calculate the curvature scalar of the space \(\mathbb{E}\), which informs if the system passes by a phase transition, when the scalar diverges for some value of extensive coordinates. If the scalar is not zero, the system possesses thermodynamic interaction, i.e, the Hawking temperature is non null.

Here, we will do the calculation of the metric of \(\mathbb{E}\), using the mass (2.33) as the thermodynamic potential, which provides

\[
dl^2_{G(M)} = -\frac{9(\alpha^2 S^2 + \eta \pi^2 Q^2)^2}{16 \alpha^2 \pi^2 S^3} dS^2 + \frac{3\eta(\alpha^2 S^2 + \eta \pi^2 Q^2)}{2 \alpha^2 S} dQ^2 . \tag{3.38}\]

The curvature scalar of this metric is given by

\[
R_G = \frac{8 \alpha^2 \pi^2 S^3 (-5 \alpha^2 S^2 + 7 \eta \pi^2 Q^2)}{(\alpha^2 S^2 + \eta \pi^2 Q^2)^4} . \tag{3.39}\]

We get the value for which the scalar (3.39) diverges, which is given by \(S_t = -i \sqrt{\eta \pi Q / \alpha}\), in agreement with the value obtained through the specific heat (2.35). This result is consistent with the specific heat, where we have found that the normal case has no phase transition and in the phantom case has one point of second order phase transition in \(S = S_t\).

In the next subsection we will see the analysis made by the geometric method of Liu-Lu-Luo-Shao.

### 3.4 The Liu-Lu-Luo-Shao method

The geometric method of the analysis of the more recent thermodynamic system is that of Liu-Lu-Luo-Shao \[6\], which defines a metric in the thermodynamic space of the equilibrium states, based on the Hessian matrix of several free energy, the Helmholtz’s one in our case, and which can be written as follows

\[
dl^2_{LLLS(F)} = -dT dS + \eta dA_0 dq = -\frac{\partial T}{\partial S} dS^2 + \left(\eta \frac{\partial A_0}{\partial S} - \frac{\partial T}{\partial q}\right) dS dq + \eta \frac{\partial A_0}{\partial q} dq^2
\]

\[
= -\frac{3(\alpha^2 S^2 + \eta \pi^2 Q^2)}{4 \sqrt{2} \alpha \pi S^{5/2}} dS^2 + \eta \frac{\sqrt{2} \pi}{\alpha \sqrt{S}} dQ^2 . \tag{3.40}\]

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The curvature scalar of this metric is given by

\[ R_{LLLS} = -\frac{\sqrt{2}\alpha^3\pi S^{5/2}}{3 (\alpha^2S^2 + \eta\pi^2Q^2)^2} . \] (3.41)

Then, the analysis by this method shows that the normal case does not possess phase transition and the phantom case possess a transition phase at \( S = S_t = -i\sqrt{\eta\pi Q}/\alpha \), which is in agreement with the specific heat.

In the next subsection we will study the local and global stabilities of the black plane solutions.

### 3.5 The local and global stability

Let us now study the local and global stabilities of these solutions. Through the specific heat (2.35)\(^7\) and the temperature (2.34), one can see that in the normal case, \( \eta = 1 \), the system is locally stable for \( 3\alpha^2S^2 > \pi^2Q^2 \), with \( C_q, T > 0 \), and unstable for the other values. In the phantom case, \( \eta = -1 \), the system presents a local stability for \( \alpha^2S^2 > \pi^2Q^2 \), with \( C_q, T, M > 0 \) (see Figure 3).

Defining the Gibbs’s potential

\[ G = M - TS - \eta A_0Q = -\left( \frac{\alpha^2S^2 + \eta\pi^2Q^2}{2\pi\alpha\sqrt{2S}} \right) = -\frac{M}{2} , \] (3.42)

we get that in the normal case, in the grand canonical ensemble, the system is globally stable for any values of \( S \) and \( Q \), with \( G < 0, \forall S, Q \). But in the phantom case, the system is globally stable only if \( \alpha^2S^2 > \pi^2Q^2 \), which agrees with the local stability of the specific heat.

Here it is clear that both the specific heat and the Gibbs potential are closely linked to the sign of the mass (2.33). We have already seen from the specific heat that the mass value, zero, is precisely the point of phase transition of the phantom case. Here, it is also clear from the Gibbs potential that, passing to the negative values of the energy (mass), the system is unstable, not only locally, but also globally. This shows that the system cannot move to that physically impossible stage. The explanation is that, when the system loses its energy, approaching zero, this should be treated by a more elaborated quantization, and not a simple semi-classical analysis, as we

\(^7\)we can do \( C_q = (4S/3)[T(S, Q)/M(S, Q)] \).
see here. Thus, we can conclude here that the phase transition presented by the phantom case, is nothing more than a purely mathematical transition, showing a divergence in the specific heat, but which is physically inaccessible to the states of the thermodynamic system.

In the canonical ensemble, we can define the Helmholtz free energy as

\[ F = M - TS = -\left(\frac{\alpha^2 S^2 - 3\eta\pi^2 Q^2}{2\pi\alpha\sqrt{2S}}\right), \quad (3.43) \]

which yields a globally stable system \((F < 0)\), for the normal case, when \(\alpha^2 S^2 > 3\pi^2 Q^2\), and for the phantom case \(F < 0, \forall S, Q\).

4 Conclusion

We obtained a new phantom black plane solutions in (2.11). We analysed their basic geometric properties, the causal structure, obtaining the thermodynamic variables, temperature (2.28), entropy density (2.29) and the electric potential (2.30). We established the first law of thermodynamics in (2.32) and calculated the specific heat (2.35).

We analysed the thermodynamic system through the study of the specific heat and the geometric methods called Weinhold, the geometrothermodynamics and that of Liu-Lu-Luo-Shao. In the Weinhold’s case, the space metric is not invariant under Legendre transformations, and thus cannot reconcile a good thermodynamic analysis, therefore, in general, this method cannot agree with that of specific heat. By the use of the geometrothermodynamics and the method of Liu-Luo-Shao, we obtain the same results as in the case of specific heat, which shows that these two geometric methods agree with the usual one.

The summarized results are that the normal case possesses an extreme limit for \(S = S_e = \pi Q\sqrt{\eta}/\alpha\sqrt{3}\), and the phantom case presents a phase transition point in \(S = S_t = -i\sqrt{\eta}\pi Q/\alpha\), which represents a solution with mass (2.33) identically null. The interpretation of massless solutions has been presented in [21], but without any conclusion about its thermodynamics.

The normal case presents locally stable thermodynamic system, for \(3\alpha^2 S^2 > \pi^2 Q^2\), and globally stable, in grand canonical ensemble, when \(G < 0, \forall S, Q\), and in canonical ensemble for \(\alpha^2 S^2 > 3\pi^2 Q^2\). On the other hand, the phantom case is locally stable when \(\alpha^2 S^2 > \pi^2 Q^2\), and globally stable, in grand
canonical ensemble, when $\alpha^2S^2 > \pi^2Q^2$, and in canonical ensemble, when $F < 0, \forall S, Q$.

We conclude with the most important result here, which is the demonstration that normal and phantom cases have no physical phase transition, and that the normal case is an extreme case but not the phantom one.

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