CHEBYSHEV SPECTRAL METHODS FOR COMPUTING CENTER MANIFOLDS

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Abstract. We propose a numerical approach for computing center manifolds of equilibria in ordinary differential equations. Near the equilibria, the center manifolds are represented as graphs of functions satisfying certain partial differential equations (PDEs). We use a Chebyshev spectral method for solving the PDEs numerically to compute the center manifolds. We illustrate our approach for three examples: A two-dimensional system, the Hénon-Heiles system (a two-degree-of-freedom Hamiltonian system) and a three-degree-of-freedom Hamiltonian system which have one-, two- and four-dimensional center manifolds, respectively. The obtained results are compared with polynomial approximations and other numerical computations.

1. Introduction. Stable, unstable and center manifolds are important for understanding the dynamics of nonlinear systems [9, 27]. Analytical expressions of these invariant manifolds are generally difficult to obtain. So numerical computation approaches of stable and unstable manifolds have been well established [5, 10, 15, 16]. However, for numerical computation of center manifolds, polynomial approximations have been used widely (see, e.g., [2, 7, 8, 11, 13]) except that iterative methods which are not suitable for higher-dimensional systems were used to compute one- and two-dimensional center manifolds for two- and three-dimensional systems, respectively, in [17, 18]. In the polynomial approximations, the Taylor series expansions of the center manifolds at equilibria, i.e., their local properties, are used, so that it is unclear how they expand globally, whatever high-order polynomials are manipulated. Actually, the use of higher-order polynomials may provide a worse result, as seen in Section 4.1. So it is an interesting question how much the use of their global properties improves the computation results.

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In this paper, we propose another approach for numerically computing center manifolds of equilibria in ordinary differential equations (ODEs). Near the equilibria, the center manifolds are represented as graphs of functions satisfying certain partial differential equations (PDEs). We use a Chebyshev spectral method [3, 4, 14, 24] for solving the PDEs numerically to compute the center manifolds. So global properties of center manifolds are used to some extent here. We illustrate our approach for three examples: A two-dimensional system, the Hénon-Heiles system [12] (a two-degree-of-freedom Hamiltonian system) and a three-degree-of-freedom Hamiltonian system which have one-, two- and four-dimensional center manifolds, respectively. The obtained results are compared with polynomial approximations and other numerical computations.

The outline of this paper is as follows: In Section 2 we recall a necessary part of the center manifold theory. In Section 3 we describe our numerical approach along with necessary information on the Chebyshev spectral method. Finally, we give the three examples in Section 4.

2. Center manifold theory. We first review the center manifold theory. Let $n_c, n_h \in \mathbb{N}$ and $n = n_c + n_h$. Consider $n$-dimensional systems of the form

$$
\begin{align*}
\dot{x} &= A_c x + f(x, y) =: a(x, y), \\
\dot{y} &= A_h y + g(x, y) =: b(x, y), \\
(x, y) &\in \mathbb{R}^{n_c} \times \mathbb{R}^{n_h},
\end{align*}
$$

where $A_c$ and $A_h$ are $n_c \times n_c$ and $n_h \times n_h$ matrices whose eigenvalues have, respectively, zero and nonzero real parts; $f : \mathbb{R}^{n_c} \times \mathbb{R}^{n_h} \to \mathbb{R}^{n_c}$ and $g : \mathbb{R}^{n_c} \times \mathbb{R}^{n_h} \to \mathbb{R}^{n_h}$ are $C^r$ ($r \geq 2$); and their first derivatives vanish at the origin. In this setting, we have the following result [9, 25, 27].

**Theorem 2.1 (Existence).** There exists a $C^r$ center manifold $W^c(0)$ which is tangent to the $x$-plane, $\{(x, y) \in \mathbb{R}^{n_c} \times \mathbb{R}^{n_h} | y = 0\}$, and invariant for the flow of (1).

We remark that the center manifold $W^c(0)$ is not necessarily unique [9, 25, 27]. See also Section 4.1.

We can represent the center manifold $W^c(0)$ near the origin as

$$
W^c(0) = \{(x, y) \in \mathbb{R}^{n_c} \times \mathbb{R}^{n_h} | y = h(x), x \in U\},
$$

where $U$ is a neighborhood of the origin in $\mathbb{R}^{n_c}$ and $h : U \to \mathbb{R}^{n_h}$ is $C^r$, and satisfies

$$
h_j(0) = \frac{\partial h_j}{\partial x_i}(0) = 0, \quad i = 1, \ldots, n_c, \quad j = 1, \ldots, n_h.
$$

Here we have written the $i$th components of $x$ and $h$ as $x_i$ and $h_i$, respectively. Substituting $y = h(x)$ into the second equation of (1) and using the chain rule, we obtain a system of PDEs

$$
\sum_{i=1}^{n_c} a_i(x, h) \frac{\partial h_j}{\partial x_i} = b_j(x, h), \quad j = 1, \ldots, n_h.
$$

Such a system of PDEs is also obtained for stable and unstable manifolds. See, e.g., [10]. The system of PDEs (4) is difficult to analytically solve in general under the condition (3) but a polynomial approximation of the solution can be obtained as follows [9, 25, 27].
Theorem 2.2 (Polynomial approximation). Let \( \ell (< r) \) be a positive integer. Suppose that an \( \ell \)-th-order polynomial \( h = \hat{h}_\ell(x) \) satisfies (3) and

\[
\sum_{i=1}^{n_c} a_i(x, h) \frac{\partial h_j}{\partial x_i} = b_j(x, h) + O(|x|^\ell+1), \quad j = 1, \ldots, n_h,
\]

as \( x \to 0 \). Then

\[
h(x) = \hat{h}_\ell(x) + O(|x|^\ell+1) \quad \text{as} \quad x \to 0.
\]

Using Theorem 2.2, we calculate a polynomial approximation for the center manifold \( W^c(0) \). See, e.g., Section 3.1 of [9] or Section 18.1 of [27]. Note that the polynomial approximation is unique even if \( W^c(0) \) is not. The calculation is simple for lower order approximations but it is so complicated for higher-order approximations that a computer algebra system such as Mathematica [20] is required. More sophisticated formal series/power matching schemes based on automatic differentiation were proposed in [8].

3. Numerical approaches. In this section we describe our numerical approach for computing the center manifold \( W^c(0) \) in (1). Especially, the Chebyshev spectral method [3, 4, 14, 24] is used for numerically solving the PDEs (4) under the condition (3). In [10] Eq. (4) was discretized and solved by the ordered upwind method [23] to compute stable and unstable manifolds although the approach is not appropriate for computing center manifolds.

3.1. Chebyshev spectral method. We first give necessary information on the Chebyshev spectral method [3, 4, 14, 24]. Fix \( i \in \{1, 2, \ldots, n_c\} \) and let \( \xi = x_i \). Consider the interval \([-1, 1]\) for the variable \( \xi \) and choose \( N + 1 \) points

\[
\xi_i = \cos(i\pi/N), \quad i = 0, 1, \ldots, N,
\]

which are called Chebyshev points, where \( N \) is a positive integer. Especially, we choose an even number as \( N \) since the point \( \xi = 0 \) should be taken as one of the Chebyshev points for our purpose, although a similar treatment is possible when \( N \) is odd. In the Chebyshev spectral method, the solution \( \varphi(\xi) \) is approximated as

\[
\varphi(\xi) = \sum_{i=0}^{N} \varphi(\xi_i)p_i(\xi),
\]

where

\[
p_i(\xi) = \frac{1}{p_{i0}} \prod_{j=0, j \neq i}^{N} (\xi - \xi_j), \quad p_{i0} = \prod_{j=0, j \neq i}^{N} (\xi_i - \xi_j).
\]

Under the approximation (6), the differentiation of \( \varphi(\xi) \) with respect to \( \xi \) at \( \xi_i, i = 0, 1, \ldots, N, \) are estimated as

\[
D_N \begin{pmatrix} \varphi(\xi_0) \\ \varphi(\xi_1) \\ \vdots \\ \varphi(\xi_N) \end{pmatrix},
\]

(7)
where

\[
D_N = \begin{pmatrix}
\frac{2N^2 + 1}{6} & \frac{2(-1)^j}{1 - \xi_j} & \frac{1}{2}(-1)^N \\
\frac{-1 (-1)^i}{2(1 - \xi_i)} & -\xi_i & \frac{(-1)^{i+j}}{\xi_i - \xi_j} \\
\frac{(-1)^{i+j}}{\xi_i - \xi_j} & \frac{2(-1)^{N+j}}{1 + \xi_j} & -\frac{2(-1)^N + 1}{6}
\end{pmatrix}.
\]

Especially,

\[
(D_N)_{ij} = \frac{(-1)^{i+j}}{\xi_i - \xi_j}, \quad i \neq j, \quad i, j \neq 0, N.
\]

The matrix \(D_N\) is called the Chebyshev differentiation matrix. If the interval \([-\delta, \delta]\) (\(\delta \neq 0\)) is used instead of \([-1, 1]\), then Eqs. (5) and (7) are replaced with

\[
\xi_i = \delta \cos(i\pi/N), \quad i = 0, 1, \ldots, N,
\]

and

\[
\frac{1}{\delta} D_N \begin{pmatrix}
\varphi(\xi_0) \\
\varphi(\xi_1) \\
\vdots \\
\varphi(\xi_N)
\end{pmatrix},
\]

respectively.

### 3.2. Computation of center manifolds

We now present our numerical approach for computing the center manifold \(W^c(0)\). For the sake of the reader's understanding the basic idea easily, we begin with the simplest case of \(n_c = 1\).

Consider the ODE (4) (with \(n_c = 1\)) on the interval \([-\delta_1, \delta_1]\) for some \(\delta_1 > 0\). Let \(x_{1i} = \delta_1 \cos(i\pi/N)\) and \(\hat{h}_{ji} = \hat{h}_j(x_{1i})\) for \(i = 0, \ldots, N\) and \(j = 1, \ldots, n_h\), and define

\[
A_1 = \text{diag}(a_1(x_{10}, \hat{h}_{10}, \ldots, \hat{h}_{n_0,0}), \ldots, a_1(x_{1N}, \hat{h}_{1N}, \ldots, \hat{h}_{n_0,N})),
\]

\[
\hat{b}_j = (b_j(x_{10}, \hat{h}_{10}, \ldots, \hat{h}_{n_0,0}), \ldots, b_j(x_{1N}, \hat{h}_{1N}, \ldots, \hat{h}_{n_0,N}))^T,
\]

\[
\hat{h}_j = (\hat{h}_{j0}, \ldots, \hat{h}_{jN})^T,
\]

where \(^T\) represents the transpose operator. Using the relation (9) for (4) with \(n_c = 1\), we have

\[
\frac{1}{\delta_1} A_1 D_N \hat{h}_j = \hat{b}_j, \quad j = 1, \ldots, n_h.
\]

We numerically solve (10) about \(\hat{h}_j, \ i = 0, \ldots, N\) and \(j = 1, \ldots, n_h\), along with \(\hat{h}_{j,N/2} = 0\) and

\[
\sum_{i=0}^{N} (D_N)_{N/2,i} \hat{h}_{ji} = 0,
\]
which follows from (3). Concretely, we set zero to all elements in the $N/2$-th row of $D_N$ and replace the $N/2$-th element of (10) with the relation (11) since $\hat{h}_{j,N/2} = 0$ and
\[
a_1(x_{1,N/2}, \hat{h}_{1,N/2}, \ldots, \hat{h}_{n_h,N/2}) = b_j(x_{1,N/2}, \hat{h}_{1,N/2}, \ldots, \hat{h}_{n_h,N/2}) = 0
\]
for $j = 1, \ldots, n_h$. Thus, we compute an approximate solution of the form (6) to (4) under the condition (3), and obtain an approximate expression of $W^c(0)$ via (2).

We consider the general case of $n_c \in \mathbb{N}$. Let $\prod_{k=1}^{n_c}[-\delta_k, \delta_k]$ be the domain of (4) and let $N_k > 0$, $k = 1, \ldots, n_c$, be even integers. Let $x_{ki} = \delta_k \cos(i\pi/N_k)$, $i = 0, \ldots, N_k$ and $k = 1, \ldots, n_c$, and let $\hat{x}_l = (x_{11}, \ldots, x_{n_t n_c})$ for
\[
l = \sum_{k=1}^{n_c-1} \left( \prod_{r=k+1}^{n_c} \left( N_r + 1 \right) \right) i_k + i_{n_c}.
\]
Let $\hat{h}_{jl} = h_j(\hat{x}_l)$, $j = 1, \ldots, n_h$ and $l = 0, \ldots, \hat{N} := \prod_{l=1}^{n_c} (N_l + 1) - 1$, and define
\[
A_k = \text{diag}(a_k(\hat{x}_0, \hat{h}_{10}, \ldots, \hat{h}_{n_00}), \ldots, a_k(\hat{x}_{N}, \hat{h}_{1N}, \ldots, \hat{h}_{n_NN}))
\]
\[
b_j = (b_j(\hat{x}_0, \hat{h}_{10}, \ldots, \hat{h}_{n_00}), \ldots, b_j(\hat{x}_{N}, \hat{h}_{1N}, \ldots, \hat{h}_{n_NN}))^T,
\]
\[
\hat{h}_j = (\hat{h}_{j0}, \ldots, \hat{h}_{jN})^T.
\]
Using the relation (9) for (4), we have
\[
\sum_{k=1}^{n_c} \frac{1}{\delta_k} A_k \hat{D}_k \hat{h}_j = b_j, \quad j = 1, \ldots, n_h,
\]
where
\[
\hat{D}_k = \begin{cases} 
D_{N_k} \otimes I_{\hat{N}_k^+} & \text{for } k = 1; \\
I_{\hat{N}_k^-} \otimes D_{N_k} \otimes I_{\hat{N}_k^+} & \text{for } k = 2, \ldots, n_c - 1; \\
I_{\hat{N}_n_c^-} \otimes D_{N_k} & \text{for } k = n_c
\end{cases}
\]
with
\[
\hat{N}_k^- = \prod_{l=1}^{k-1} (N_l + 1), \quad k > 1,
\]
\[
\hat{N}_k^+ = \prod_{l=k+1}^{n_c} (N_l + 1), \quad k < n_c.
\]
Here $I_N$ represents the $N \times N$ identity matrix and $A \otimes B$ denotes the Kronecker product
\[
A \otimes B = \begin{pmatrix} 
A_{11}B & A_{12}B & \cdots & A_{1q}B \\
A_{21}B & A_{22}B & \cdots & A_{2q}B \\
\vdots & \vdots & \ddots & \vdots \\
A_{pq1}B & A_{12B} & \cdots & A_{pqB}
\end{pmatrix},
\]
where $A$ and $B$ are $p \times q$ and $p' \times q'$ matrices and $A_{ij}$ is the $(i,j)$-element of $A$. Note that $A \otimes B$ is a $pp' \times qq'$ matrix. Especially, when $n_c = 2$, Eq. (12) becomes
\[
\frac{1}{\delta_1} A_1(D_{N_1} \otimes I_{N_2+1}) \hat{h}_j + \frac{1}{\delta_2} A_2(I_{N_1+1} \otimes D_{N_2}) \hat{h}_j = b_j, \quad j = 1, \ldots, n_h.
\]
We numerically solve (12) about \( \hat{h}_{jl} \), \( j = 1, \ldots, n_h \) and \( l = 0, \ldots, \hat{N} \), along with \( \hat{h}_{j,\hat{N}/2} = 0 \) and
\[
\sum_{i=0}^{N_k} (\hat{D}_{N_k})_{\hat{N}/2,i} \hat{h}_{ji} = 0, \quad k = 1, \ldots, n_c,
\]
which follows from (3). Concretely, we set zero to all elements in the \( \hat{N}/2 \)-th row of \( \hat{D}_{k} \), \( k = 1, \ldots, n_c \), and replace the \( \hat{N}/2 \)-th element of (12) with the relation
\[
\sum_{k=1}^{n_c} \left( \sum_{i=0}^{N_k} (\hat{D}_{N_k})_{\hat{N}/2,i} \hat{h}_{ji} \right)^2 = 0, \quad j = 1, \ldots, n_h,
\]
since \( \hat{h}_{j,\hat{N}/2} = 0 \) and
\[
a_k(\hat{x}_{\hat{N}/2}, \hat{h}_{1,\hat{N}/2}, \ldots, \hat{h}_{n_h,\hat{N}/2}) = b_j(\hat{x}_{\hat{N}/2}, \hat{h}_{1,\hat{N}/2}, \ldots, \hat{h}_{n_h,\hat{N}/2}) = 0
\]
for \( k = 1, \ldots, n_c \) and \( j = 1, \ldots, n_h \). Thus, we compute an approximate solution of the form (6) to (4) under the condition (3), and obtain an approximate expression of the \( n_c \)-dimensional center manifold \( W^c(0) \) via (2).

We remark that our approach is difficult to succeed for large constants \( \delta_k \), \( k = 1, \ldots, n_c \), but they should be large enough to obtain a precise computation. Moreover, in the above discussions, the smoothness of the vector field of (1), i.e., of \( f \) and \( g \), is more than \( C^N \) with \( N \geq N_k \), \( k = 1, \ldots, n_c \), since the function \( h(x) \) which expresses \( W^c(0) \) as its graph is approximated by an \( N \)-th order polynomial in our approach based on the Chebyshev spectral method. See also Theorems 2.1 and 2.2. In the examples of the next section, such large values that our approach succeeds are taken as the constants \( \delta_k \), \( k = 1, \ldots, n_c \), and the vector fields are \( C^\infty \).

4. Examples. In this section we present numerical results obtained by using the approach of Section 3 for the three examples stated in Section 1. The numerical computing system Matlab [19] was used to carry out necessary calculations of the Chebyshev spectral method through these examples, as in [24]. Especially, the function fsolve, returns optimal solutions with a priori given tolerances, was used to solve the modified version of (10) or (12). So our approach gives unique results as well as polynomial approximations even if center manifolds are not unique. The termination tolerance TolFun was set at \( 10^{-6} \) except when specifically stated, and the initial data of \( \hat{h}_{jl} \) was set at zero for \( j = 1, \ldots, n_h \) and \( l = 0, \ldots, \hat{N} \) except that the 2nd-order polynomial approximation was chosen when the problem was difficult as specifically stated below. A Matlab code named chebdif [26] was also used to calculate the Chebyshev differentiation matrix (8).

4.1. Two-dimensional system. Consider the two-dimensional system
\[
\dot{x} = xy, \quad \dot{y} = -y + \alpha x^2, \quad (x, y) \in \mathbb{R}^2,
\]
where \( \alpha \neq 0 \) is a constant. The origin \( (x, y) = (0, 0) \) is an equilibrium at which the Jacobian matrix of the right-hand side of (13) has eigenvalues 0 and \(-1\). So it has a one-dimensional center manifold \( W^c(0) \) which is tangent to the \( x \)-axis there. We numerically computed \( W^c(0) \) for \( \alpha = 1 \) and \(-1\). Here \( \delta_1 = 1 \) and 0.4 were taken for \( \alpha = 1 \) and \(-1\), respectively. We remark that the center manifold is not unique for \( \alpha = -1 \) since the origin is asymptotically stable.

Figure 1 shows numerical computations of \( W^c(0) \) obtained by the present approach with \( N_1 = 8 \), which is plotted as black lines there. For comparison, the
Figure 1. Numerically computed center manifold in (13): (a) \( \alpha = 1 \); (b) \( \alpha = -1 \). Black lines represent numerical results obtained by our approach with \( N_1 = 8 \). For comparison, the 2nd-, 4th-, 16th- and 32nd-order polynomial approximations are plotted as purple, green, blue and red lines, respectively. Numerically computed orbits are also plotted as orange lines with arrows representing their directions.

Figure 2. Numerically computed center manifold by our approach for \( N_1 = 4, 8 \) and 16 in (13): (a) \( \alpha = 1 \); (b) \( \alpha = -1 \). Red, black and blue lines, which agree almost completely in the computed ranges, represent the results for \( N_1 = 4, 8 \) and 16, respectively. 2nd-, 4th-, 16th- and 32nd-order polynomial approximations are plotted as purple, green, blue and red lines, respectively, and numerically computed orbits are plotted as orange lines. We see that the present approach gives fairly good results, especially in Fig. 1(a). On the other hand, polynomial approximations give different results for different degrees of the polynomials. In particular, higher order ones do not necessarily give better results, as seen in Fig. 1. We present numerical computations of \( W^c(0) \) obtained by the present approach for \( N_1 = 4, 8 \) and 16 in Fig. 2. The results agree almost completely for \( N_1 = 8 \) and 16 in both figure. Thus, our approach seems to converge to a certain result as \( N_1 \to \infty \), and \( N_1 = 8 \) was almost sufficient.
4.2. Hénon-Heiles system. Consider the two-degree-of-freedom Hamiltonian system

\[
\begin{align*}
\dot{x}_1 &= x_2, \quad \dot{x}_2 = -x_1 - 2cx_1 y_1, \\
\dot{y}_1 &= y_2, \quad \dot{y}_2 = -y_1 - cx_1^2 - dy_1^2
\end{align*}
\]

with the Hamiltonian

\[H(x, y) = \frac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2) + cx_1 y_1 + \frac{1}{3}dy_1^3,\]

where \(c, d\) are constants. The system (14) is called the Hénon-Heiles system and the case of \((c, d) = (1, -1)\) was originally studied by Hénon and Heiles [12]. It was shown that the system (14) is integrable if \(c/d = 0, \frac{1}{6}, 1\) and nonintegrable otherwise. Moreover, chaotic dynamics was proven to occur if \(c/d \neq 0, \frac{1}{6}, \frac{1}{2}, 1\). See [28] and references therein for more details. Especially, the original version of (14) studied in [12] was rigorously proved to be nonintegrable and exhibit chaos.

When \(c/d < 1/2\) (resp. \(c/d > 1/2\)), under the transformation

\[
y_1 \mapsto y_1 - 1/d
\]

\[
\begin{align*}
x_1 &\mapsto \frac{x_1 - \mu(y_1 - y_{10})}{\sqrt{1 + \mu^2}}, & x_2 &\mapsto \frac{x_2 - \mu y_2}{\sqrt{1 + \mu^2}}, \\
y_1 &\mapsto \frac{y_1 - y_{10} + \mu x_1}{\sqrt{1 + \mu^2}}, & y_2 &\mapsto \frac{y_2 + \mu x_2}{\sqrt{1 + \mu^2}},
\end{align*}
\]

the system (14) becomes

\[
\begin{align*}
\dot{x}_1 &= x_2, \quad \dot{x}_2 = -\omega^2 x_1 - \kappa_1 x_1^2 - \kappa_2 x_1 y_1, \\
\dot{y}_1 &= y_2, \quad \dot{y}_2 = y_1 - \frac{1}{2}\kappa_2 x_1^2 - \kappa_3 y_1^2,
\end{align*}
\]

where

\[
\kappa_1 = 0, \quad \kappa_2 = 2c, \quad \kappa_3 = d, \quad \omega = \sqrt{1 - \frac{2c}{d}},
\]

\[
\begin{align*}
\kappa_1 &= \frac{\mu(c + d)}{\sqrt{1 + \mu^2}}, & \kappa_2 &= \frac{2(d - c)}{\sqrt{1 + \mu^2}}, & \kappa_3 &= \frac{2c}{\sqrt{1 + \mu^2}}, \\
\omega &= \frac{\mu}{\sqrt{2}}, & y_{10} &= \frac{\sqrt{1 + \mu^2}}{2c}, & \mu &= \sqrt{\frac{2 - d}{c}}.
\end{align*}
\]

In (15), the origin \((x, y) = (0, 0) \in \mathbb{R}^2 \times \mathbb{R}^2\) is an equilibrium of saddle-center type at which the Jacobian matrix of the right-hand side of (15) has a pair of purely imaginary eigenvalues \(\pm i\omega\) and a pair of positive and negative eigenvalues \(\pm 1\). The saddle-center has a two-dimensional center manifold which is tangent to the \(x\)-plane there and contains a one-parameter family of periodic orbits whose existence is guaranteed via Lyapunov’s center theorem (see, e.g., Theorem 5.6.7 of [1] or Theorem 9.2.1 of [21]). The Hamiltonian for (15) is given by

\[H(x, y) = \frac{1}{2}(x_1^2 + y_1^2) + \frac{1}{2}(\omega^2 x_1^2 - y_1^2) + \frac{1}{2}\kappa_1 x_1^4 + \frac{1}{2}\kappa_2 x_1^2 y_1 + \frac{1}{2}\kappa_3 y_1^3.
\]

The polynomial approximation for \(W^c(0)\) up to second-order is given by

\[h_1(x) = \frac{(2\omega^2 + 1)\kappa_2}{2(4\omega^2 + 1)}x_1^2 + \frac{\kappa_2}{4\omega^2 + 1}x_2^2, \quad h_2(x) = \frac{\kappa_2}{4\omega^2 + 1}x_1 x_2.
\]

We computed the two-dimensional center manifold \(W^c(0)\) of the saddle-center at the origin in (15) for the following two cases:

\[
\begin{align*}
\text{(1)} & \quad \text{or} \quad \text{Theorem 9.2.1 of [21]). The Hamiltonian for (15) is given by}
\end{align*}
\]
(i) $\omega = \sqrt{\frac{2}{3}}$, $\kappa_1 = 0$, $\kappa_2 = 1$ and $\kappa_3 = 3$ \ ($c = \frac{1}{2}$, $d = 3$);
(ii) $\omega = \sqrt{3}$, $\kappa_1 = 0$, $\kappa_2 = 2$ and $\kappa_3 = -1$ \ ($c = 1$, $d = -1$).

Here $(\delta_1, \delta_2) = (0.45, 0.45)$ was taken for both cases (i) and (ii). The second-order polynomial approximation (16) was used as the initial data of $\hat{h}_{jl}$, $j = 1, \ldots, n_h$ and $l = 0, \ldots, N$ for $N_1 = N_2 = 8$ and 16 in case (i) and for $N_1 = N_2 = 16$ in case (ii). Moreover, the termination tolerance $\text{TolFun}$ was set at $10^{-5}$. Cases (i) and (ii) correspond to $(c, d) = \left(\frac{1}{2}, 3\right)$ and $(1, -1)$. The system (15) is integrable for case (i) but it is nonintegrable and exhibits chaos for case (ii), like (14). We also used the computer software AUTO [6] and computed the one-parameter family of periodic orbits for each case. See Section 4.3 of [22] for more details on the computations. The obtained results are displayed in Fig. 3. The one-parameter family of periodic orbits lies on $W^c(0)$ and gives its shape near the origin at least.
Figure 5. Polynomial approximations of the center manifold on the section $x_2 = 0$ for (15): (a) Case (i); (b) case (ii). Purple, green, blue and red lines represent the 2nd-, 4th-, 8th- and 16th-order polynomial approximations. For comparison, the one-parameter family of periodic orbits in Fig. 3 are plotted as black dashed lines.

Figure 6. Differences between our numerical results or polynomial approximations and the one-parameter family of periodic orbits in Fig. 3 on the section $x_2 = 0$ for (15): (a) Case (i); (b) case (ii). Green and blue lines, respectively, represent the 8th- and 16th-order polynomial approximations, and red and black lines, respectively, our numerical results with $N_1 = N_2 = 8$ and 16.

Figure 4 shows numerical computations of $W^c(0)$ obtained by the present approach with $N_1 = N_2 = 4$, 8 and 16, which are plotted as green, blue and red lines there, respectively, on the section $x_2 = 0$. Figure 5 shows the 2nd-, 4th-, 8th- and 16th-order polynomial approximations which are plotted as purple, green, blue and red lines, respectively, there. For comparison, the one-parameter families of periodic orbits computed by AUTO around the saddle-center are plotted as black dashed lines in both figures. All the results obtained by the present approach agree with one by AUTO in Fig. 4(b) while there are small differences between them for $N_1 = N_2 = 4$ and 8 in Fig. 4(a). On the other hand, the 8th- and 16th-order polynomial approximations give fairly good results in Fig. 5. Figure 6 displays the differences between
our numerical results or polynomial approximations and the one-parameter family of periodic orbits in Fig. 3. Here \( \Delta = |y_{1\text{app}} - y_{1p}| \) represents their differences, where \( y_{1\text{app}} \) and \( y_{1p} \) are, respectively, the values of \( y_1 \) obtained by each computation or approximation and the one-parameter family of periodic orbits. Although the periodic orbits were not computed so precisely, the figure supports the above observations further.

4.3. Three-degree-of-freedom Hamiltonian system. Consider the three-degree-of-freedom Hamiltonian system

\[
\begin{align*}
\dot{x}_1 &= x_3, & \dot{x}_2 &= -\omega_1^2 x_1 - \kappa_1 x_1^2 - \kappa_2 x_1 y_1 - \kappa_6 x_1 x_2 - \frac{1}{2} \kappa_7 x_2^2, \\
\dot{x}_2 &= x_4, & \dot{x}_3 &= \omega_3 x_2 - \kappa_4 x_2^2 - \kappa_5 x_2 y_1 - \frac{1}{2} \kappa_6 x_2^2 - \kappa_7 x_1 x_2, \\
\dot{y}_1 &= y_2, & \dot{y}_2 &= y_1 - \frac{1}{2} \kappa_2 x_1^2 - \kappa_3 y_1^2 - \frac{1}{2} \kappa_5 x_2^2
\end{align*}
\]

(17)

with the Hamiltonian

\[
H(x, y) = \frac{1}{2} (\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2 + \omega_4^2 y_1^2 + \omega_5^2 y_2^2) + \frac{1}{2} \kappa_1 x_1^4 + \frac{1}{2} \kappa_2 x_1^2 y_1 + \frac{1}{2} \kappa_3 y_1^2 + \frac{1}{2} \kappa_4 x_2^4 + \frac{1}{2} \kappa_5 x_2^4 y_1 + \frac{1}{2} \kappa_6 x_2^2 x_2 + \frac{1}{2} \kappa_7 x_1 x_2^2,
\]

where \( \omega_j, j = 1, 2, \) and \( \kappa_j, j = 1-7, \) are constants. The origin \((x, y) = (0, 0) \in \mathbb{R}^4 \times \mathbb{R}^2\) is an equilibrium of saddle-center type at which the Jacobian matrix of the right-hand side of (15) has two pairs of purely imaginary eigenvalues \( \pm i \omega_1, \pm i \omega_2 \) and a pair of positive and negative eigenvalues \( \pm 1. \) The saddle-center has a four-dimensional center manifold which is tangent to the \( x \)-plane there and contains two one-parameter families of periodic orbits via Lyapunov’s center theorem if \( \omega_1/\omega_2, \omega_2/\omega_1 \notin \mathbb{N}: \) One is tangent to the \((x_1, x_3)\)-plane and the other is tangent to the \((x_2, x_4)\)-plane at the origin. The polynomial approximation for \( W^c(0) \) up to second-order is given by

\[
\begin{align*}
h_1(x) &= \frac{(2\omega_1^2 + 1)\kappa_2}{2(4\omega_1^2 + 1)} x_1^2 + \frac{(2\omega_2^2 + 1)\kappa_5}{2(4\omega_2^2 + 1)} x_2^2 + \frac{\kappa_2}{4\omega_1^2 + 1} x_1^2 + \frac{\kappa_5}{4\omega_2^2 + 1} x_2^2, \\
h_2(x) &= \frac{\kappa_2}{4\omega_1^2 + 1} x_1 x_3 + \frac{\kappa_5}{4\omega_2^2 + 1} x_2 x_4.
\end{align*}
\]

(18)

We computed the four-dimensional center manifold \( W^c(0) \) of the saddle-center at the origin for the following two cases:

(i) \( \omega_1 = \sqrt{2}, \kappa_1 = 0, \kappa_2 = 1, \kappa_3 = 3, \kappa_4 = 1 \) and \( \kappa_5 = \kappa_6 = \kappa_7 = 0.5; \)

(ii) \( \omega_1 = \sqrt{3}, \kappa_1 = 0, \kappa_2 = 2, \kappa_3 = -1, \kappa_4 = 1 \) and \( \kappa_5 = \kappa_6 = \kappa_7 = 0.5. \)

For both cases, \( \omega_2 = \Delta, \omega_1 \) is taken, where \( \Delta = \frac{1}{2}(\sqrt{5} - 1) = 0.618 \ldots \) is the golden mean. Here \((\delta_1, \delta_2, \delta_3, \delta_4) = (0.4, 0.12, 0.2, 0.06) \) and \((0.5, 0.5, 0.8, 0.6) \) were taken for cases (i) and (ii), respectively. The second-order polynomial approximation (18) was used as the initial data of \( \tilde{h}_{ij}, j = 1, \ldots, n_h \text{ and } l = 0, \ldots, N \) and the termination tolerance TolFun was set at \( 10^{-4} \) for \( N_j = 8, j = 1-4, \) in both cases (i) and (ii). Note that the number of nodes is \( 9^4 = 6561 \) and already big since the number of elements of the coefficient matrix in (12) is \( 6561^2 = 43046721, \) when \( N_j = 8, j = 1-4. \) We also used the computer software AUTO and computed the two one-parameter families of periodic orbits for each case. The obtained results are displayed in Fig. 7. The two one-parameter families of periodic orbits are contained in \( W^c(0). \)

Figure 8 shows numerical computations of \( W^c(0) \) obtained by the present approach with \( N_j = 4 \) and 8, \( j = 1-4, \) which are plotted as green and red lines there,
Figure 7. One-parameter families of periodic orbits around the saddle-center in (17): (a) and (b) Case (i); (c) and (d) case (ii). The one-parameter families of periodic orbits which are tangent to the \((x_1, x_3)\)-plane (resp. the \((x_2, x_4)\)-plane) at the origin are plotted in Figs. (a) and (c) (resp. Figs. (b) and (d)).

respectively. Figure 9 shows the 2nd-, 4th- and 8th-order polynomial approximations which are plotted as green, blue and red lines, respectively, there. Here the data of \(x_j, j = 1-4\), for the one-parameter families of periodic orbits displayed in Figs. 7(a) and (b) (resp. in Figs. 7(c) and (d)) are, respectively, used in Figs. (a) and (b) (resp. Figs. (c) and (d)) of both figures. For comparison, the corresponding families of periodic orbits computed by AUTO around the saddle-center are plotted as black dashed lines in both figures. In Figs. 8(b) and 9(b), the reader may think that the computed center manifold is not a graph of a function of the \(x\) variable but this is not true: The dashed line is just a projection onto the two-dimensional \((x_2, y_1)\)-plane of a two-dimensional surface consisting of the family of periodic orbits on such a graph.

All the results obtained by the present approach agree with the one by AUTO except when \(N_j = 4, j = 1-4\), in Fig. 8(a), for which there is a small difference between them. On the other hand, the 8th-order polynomial approximations give fairly good results except in Fig. 9(a), for which there is a small difference from the one by AUTO, while the 4th-order polynomial approximations have a small difference from it except in Fig. 9(c). Figure 10 displays the differences between our
Figure 8. Numerically computed center manifold for (17): (a) and (b) Case (i); (c) and (d) case (ii). The data of $x_j$, $j = 1-4$, for the one-parameter families of periodic orbits displayed in Figs. 7(a) and (b) (resp. in Figs. 7(c) and (d)) are, respectively, used in Figs. (a) and (b) (resp. Figs. (c) and (d)). Blue and red lines represent numerical results obtained by the approach with $N_j = 4$ and 8, $j = 1-4$, respectively. For comparison, the one-parameter families of periodic orbits in Fig. 7 are plotted as black dashed lines.

Numerical results or polynomial approximations and the one-parameter family of periodic orbits in Fig. 7. Here $\Delta = |y_{1\text{app}} - y_{1p}|$ represents their differences, where $y_{1\text{app}}$ and $y_{1p}$ are, respectively, the values of $y_1$ obtained by each computation or approximation and the one-parameter family of periodic orbits, as in Fig. 6. In Fig. 10(a), the 4th- and 8th-order polynomial approximations almost coincide in the log scale. Although the periodic orbits were not computed so precisely again, the figure supports the above observations further.

Figure 11 shows comparisons between numerical computations of $W^c(0)$ obtained by the present approach with $N_j = 8$, $j = 1-4$, and the 8th-order polynomial approximations, which are plotted as red and blue lines, respectively. Here the section $x_2 = x_3 = x_4 = 0$ is used in Figs. 10(a) and (c) while the section $x_1 = x_3 = x_4 = 0$ is used in Figs. 10(b) and (d). We see that they agree almost completely except in Fig. 10(a), for which there is a small difference between them and the present approach is suspected to give a better one from comparing Figs 8(a) with
Figure 9. Polynomial approximations of the center manifold for (17): (a) and (b) Case (i); (c) and (d) case (ii). The data of $x_j$, $j = 1-4$, for the one-parameter families of periodic orbits displayed in Figs. 7(a) and (b) (resp. in Figs. 7(c) and (d)) are, respectively, used in Figs. (a) and (b) (resp. Figs. (c) and (d)). Green, blue and red lines represent the 2nd-, 4th- and 8th-order polynomial approximations. For comparison, the one-parameter families of periodic orbits in Fig. 7 are plotted as black dashed lines.

Thus, comparing them with the results by our approach, we confirm that the polynomial approximations give good results except in Figs. 9(a) and 11(a).

Thus, we succeed in computing four-dimensional center manifolds but still have a problem for computation of higher-dimensional center manifolds. For example, the dimension of the coefficient matrix in (12) is $n_2^2(N + 1)^{2nc} \approx 1.0 \times 10^{13}$ for six-dimensional center manifolds although it is about $6.9 \times 10^8$ for four-dimensional center manifolds when we take the size of $N_r = N = 8$, $r = 1, \ldots, n_c$. The number $1.0 \times 10^{13}$ is too large and it is almost the same as the array size limit of the Matlab system. So a new idea is expected to overcome this difficulty for computation of six- or higher-dimensional center manifolds.

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Figure 10. Differences between our numerical results or polynomial approximations and the one-parameter family of periodic orbits in Fig. 7 for (17): (a) and (b) Case (i); (c) and (d) case (ii). The section $x_2 = x_3 = x_4 = 0$ is taken in Figs. (a) and (c) while the section $x_1 = x_3 = x_4 = 0$ is taken in Figs. (b) and (d). Green and blue lines, respectively, represent the 4th- and 8th-order polynomial approximations, and red and black lines, respectively, our numerical results with $N_j = 4$ and 8, $j = 1-4$.

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Figure 11. Numerically computed center manifold for (17): (a) and (b) Case (i); (c) and (d) Case (ii). The section $x_2 = x_3 = x_4 = 0$ is taken in Figs. (a) and (c) while the section $x_1 = x_3 = x_4 = 0$ is taken in Figs. (b) and (d). Red and blue lines, respectively, represent numerical results obtained by the approach with $N_j = 8$, $j = 1 - 4$, and the 8th-order polynomial approximations.

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