A Quantum Generalized Mittag-Leffler Function Via Caputo q-Fractional Equations

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Abstract
Some Caputo q-fractional difference equations are solved. The solutions are expressed by means of a new introduced generalized type of q-Mittag-Leffler functions. The method of successive approximation is used to obtain the solutions. The obtained q-version of Mittag-Leffler function is thought as the q-analogue of the one introduced previously by Kilbas and Saigo.

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1 Introduction and Preliminaries

The concept of fractional calculus is not new. However, it has gained its popularity and importance during the last three decades or so. This is due to its distinguished applications in numerous diverse fields of science and engineering ([16], [15], [17]). The q-calculus is also not of recent appearance. It was initiated in twenties of the last century. For the basic concepts in q-calculus we refer the reader to [9]. Starting from the q-analogue of Cauchy formula [13], Al-Salam started the fitting of the concept of q-fractional calculus. After that he ([12], [11]) and Agarwal R. [10] continued on by studying certain q-fractional integrals and derivatives, where they proved the semigroup properties for left and right (Riemann)type fractional integrals but without variable lower limit and variable upper limit, respectively. Recently, the authors in [14] generalized the notion of the (left)fractional q-integral and q-derivative by introducing variable lower limit and proved the semigroup properties.

Very recently and after the appearance of time scale calculus (see for example [7]), some authors started to pay attention and apply the techniques of time
scale to discrete fractional calculus ([4], [5], [6], [2]) benefitting from the results announced before in [8]. All of these results are mainly about fractional calculus on the time scales $T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $h\mathbb{Z}$ [3]. As a contribution in this direction and being motivated by all above, in this article we introduce the q-analogue of a generalized type Mittag-Leffler function used before by Kilbas and Saigo in [18]. Such functions are obtained by solving linear q-Caputo initial value problems. The results obtained in this article generalize also the results of [1].

For the theory of q-calculus we refer the reader to the survey [9] and for the basic definitions and results for the q-fractional calculus we refer to [6]. Here we shall summarize some of those basics.

For $0 < q < 1$, let $T_q$ be the time scale

$$T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}.$$ 

where $\mathbb{Z}$ is the set of integers. More generally, if $\alpha$ is a nonnegative real number then we define the time scale

$$T_q^\alpha = \{q^{n+\alpha} : n \in \mathbb{Z}\} \cup \{0\},$$

we write $T_q^0 = T_q$.

For a function $f : T_q \to \mathbb{R}$, the nabla q-derivative of $f$ is given by

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in T_q - \{0\} \quad (1)$$

The nabla q-integral of $f$ is given by

$$\int_0^t f(s) \nabla_q s = (1-q)t \sum_{i=0}^{\infty} q^i f(tq^i) \quad (2)$$

and for $0 \leq a \in T_q$

$$\int_a^t f(s) \nabla_q s = \int_0^t f(s) \nabla_q s - \int_0^a f(s) \nabla_q s$$

On the other hand

$$\int_t^{\infty} f(s) \nabla_q s = (1-q)t \sum_{i=1}^{\infty} q^{-i} f(tq^{-i}) \quad (3)$$

and for $0 < b < \infty$ in $T_q$

$$\int_t^b f(s) \nabla_q s = \int_t^{\infty} f(s) \nabla_q s - \int_b^{\infty} f(s) \nabla_q s \quad (4)$$
By the fundamental theorem in q-calculus we have
\[ \nabla_q \int_0^t f(s) \nabla_q s = f(t) \] (5)
and if \( f \) is continuous at 0, then
\[ \int_0^t \nabla_q f(s) \nabla_q s = f(t) - f(0) \] (6)
Also the following identity will be helpful
\[ \nabla_q \int_a^t f(t, s) \nabla_q s = \int_a^t \nabla_q f(t, s) \nabla_q s + f(qt, t) \] (7)
Similarly the following identity will be useful as well
\[ \nabla_q \int_b^t f(t, s) \nabla_q s = \int_b^{qt} \nabla_q f(t, s) \nabla_q s - f(t, t) \] (8)
The q-derivative in (7) and (8) is applied with respect to \( t \).
From the theory of q-calculus and the theory of time scale more generally, the following product rule is valid
\[ \nabla_q (f(t)g(t)) = f(qt) \nabla_q g(t) + \nabla_q f(t)g(t) \] (9)
The q-factorial function for \( n \in \mathbb{N} \) is defined by
\[ (t - s)_q^n = \prod_{i=0}^{n-1} (t - q^i s) \] (10)
When \( \alpha \) is a non positive integer, the q-factorial function is defined by
\[ (t - s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{1 - \frac{s}{t} q^i}{1 - q^{i+\alpha}} \] (11)
We summarize some of the properties of q-factorial functions, which can be found mainly in [6], in the following lemma

**Lemma 1.** (i) \((t - s)_q^{\alpha + \gamma} = (t - s)_q^\alpha (t - q^\gamma s)_q^\gamma\)
(ii) \((at - as)_q^\beta = a^\beta (t - s)_q^\beta\)
(iii) The nabla q-derivative of the q-factorial function with respect to \( t \) is
\[ \nabla_q (t - s)_q^\alpha = \frac{1 - q^\alpha}{1 - q} (t - s)_q^{\alpha-1} \]
(iv) The nabla q-derivative of the q-factorial function with respect to \( s \) is
\[ \nabla_q (t - s)_q^\alpha = -\frac{1 - q^\alpha}{1 - q} (t - qs)_q^{\alpha-1} \]
where \( \alpha, \gamma, \beta \in \mathbb{R} \).
Definition 2. [1] Let $\alpha > 0$. If $\alpha \notin \mathbb{N}$, then the $\alpha$–order Caputo (left) q-fractional derivative of a function $f$ is defined by

$$qC^\alpha_a f(t) \triangleq qI^{(n-\alpha)}(t)\nabla^nf(s)\nabla_qs$$  \hspace{1cm} (12)

where $n = \lfloor \alpha \rfloor + 1$.

If $\alpha \in \mathbb{N}$, then $qC^\alpha_a f(t) \triangleq \nabla^n_q f(t)$

It is clear that $qC^\alpha_a$ maps functions defined on $T_q$ to functions defined on $T_q$, and that $qC^\alpha_q$ maps functions defined on $T_q^{1-\alpha}$ to functions defined on $T_q$.

The following identity which is useful to transform Caputo q-fractional difference equations into q-fractional integrals, will be our key in solving the q-fractional linear type equation by using successive approximation.

Proposition 3. [1] Assume $\alpha > 0$ and $f$ is defined in suitable domains. Then

$$qI^\alpha_a qC^\alpha_a f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma_q(k+1)} \nabla^k_q f(a)$$  \hspace{1cm} (13)

and if $0 < \alpha \leq 1$ then

$$qI^\alpha_a qC^\alpha_a f(t) = f(t) - f(a)$$  \hspace{1cm} (14)

The following identity [14] is essential to solve linear q-fractional equations

$$qI^\alpha_a (x-a)^\mu_q = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)} (x-a)^{\mu+\alpha} \quad (0 < a < x < b)$$  \hspace{1cm} (15)

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$. The q-analogue of Mittag-Leffler function with double index $(\alpha, \beta)$ is introduced in [1]. It was defined as follows:

Definition 4. [1] For $z, z_0 \in \mathbb{C}$ and $\Re(\alpha) > 0$, the q-Mittag-Leffler function is defined by

$$qE_{\alpha,\beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z-z_0)^{ak}}{\Gamma_q(ak+\beta)}.$$  \hspace{1cm} (16)

When $\beta = 1$ we simply use $qE_{\alpha}(\lambda, z - z_0) := qE_{\alpha,1}(\lambda, z - z_0)$.

2 Main Results

The following is to be the q-analogue of the generalized Mittag-Leffler function introduced by Kilbas and Saigo [18] (see also [17] page 48).
Definition 5. For $\alpha, l, \lambda \in \mathbb{C}$ are complex numbers and $m \in \mathbb{R}$ such that $\Re(\alpha) > 0$, $m > 0$, $a \geq 0$ and $\alpha(jm + l) \neq -1, -2, -3, \ldots$, the generalized q-Mittag-Leffler function (of order 0) is defined by

$$qE_{\alpha,m,l}(\lambda, x-a) = 1 + \sum_{k=1}^{\infty} \lambda^k q^{-(k-1)\alpha} c_k(x-a)^{\alpha km}$$

where

$$c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm + l) + 1]}{\Gamma_q[\alpha(jm + l + 1) + 1]}, \quad k = 1, 2, 3, \ldots$$

While the the generalized q-Mittag-Leffler function (of order r), $r = 0, 1, 2, 3, \ldots$, is defined by

$$qE^r_{\alpha,m,l}(\lambda, x-a) = 1 + \sum_{k=1}^{\infty} \lambda^k q^{-r\alpha} c_k(x-a)^{\alpha km}.$$  

Remark 6. In particular, if $m = 1$, then the generalized q-Mittag-Leffler function is reduced to the q-Mittag-Leffler function, apart from a constant factor $\Gamma_q(\alpha l + 1)$. Namely,

$$qE_{\alpha,1,l}(\lambda, x-a) = \Gamma_q(\alpha l + 1) qE_{\alpha,\alpha l+1}(\lambda, x-a)$$  

This turns to be the q-analogue of the identity $E_{\alpha,1,l}(z) = \Gamma(\alpha l + 1)E_{\alpha,\alpha l+1}(z)$ (see [17]) page 48).

Example 7. Consider the q-Caputo difference equation

$$qC^\alpha_a y(x) = \lambda(x-a)^\beta q(y(q^{-\beta} x)), \quad y(a) = b$$

where

$$0 < \alpha < 1, \quad \beta > -\alpha, \quad \lambda \in \mathbb{R}, \quad b \in \mathbb{R}.$$  

Applying Proposition 3 we have

$$y(x) = y(a) + \lambda_q I^\alpha_a [(x-a)\beta y(q^{-\beta} x)].$$

The method of successive applications implies that

$$y_m(x) = y(a) + \lambda_q I^\alpha_a [(x-a)\beta y_{m-1}(q^{-\beta} x)], \quad m = 1, 2, 3, \ldots,$$

where $y_0(x) = b$. Then by the help of (17) we have

$$y_1(x) = b + \lambda_q \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\beta + \alpha + 1)}(x-a)^{\beta + \alpha}.$$
and

\[ y_2(x) = b + b\lambda q \Gamma^\alpha_a[(x-a)^\beta] \{1 + \lambda \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\beta + \alpha + 1)}(q^{-\beta}x-a)^{\beta+\alpha}\} \]

Then by (i) and (ii) of Lemma 1

\[ y_2(x) = b + b\lambda q \Gamma^\alpha_a[(x-a)^\beta] + \lambda \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\beta + \alpha + 1)}q^{-\beta(\alpha+\beta)}(x-a)^{2\beta+\alpha} \]

Again by (15) we conclude

\[ y_2(x) = b + b\lambda q \Gamma^\alpha_a[(x-a)^\beta] + \lambda \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\beta + \alpha + 1)}q^{-\beta(\alpha+\beta)}(x-a)^{2\beta+\alpha} \]

Then (15) leads to

\[ y_2(x) = b[1 + \lambda \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\beta + \alpha + 1)}(x-a)^{\beta+\alpha} + \lambda^2 \frac{\Gamma_q(2\beta + \alpha + 1)}{\Gamma_q(2\beta + 2\alpha + 1)}q^{-\beta(\alpha+\beta)}(x-a)^{2\beta+2\alpha}] \]

Proceeding inductively, for each \( m = 1, 2, \ldots \) we obtain

\[ y_m(x) = b[1 + \sum_{k=1}^{m} \lambda^k q^{-\beta\frac{k(k-1)}{2}(\alpha+\beta)} c_k(x-a)^{k(\alpha+\beta)}] \quad (20) \]

where

\[ c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]} \quad m = 1 + \frac{\beta}{\alpha}, \quad l = \frac{\beta}{\alpha}, \quad k = 1, 2, 3, \ldots \]

If we let \( m \to \infty \), then we obtain the solution

\[ y(x) = b \left[1 + \sum_{k=1}^{\infty} \lambda^k q^{-\beta\frac{k(k-1)}{2}(\alpha+\beta)} c_k(x-a)^{k(\alpha+\beta)}\right] \]

which is exactly

\[ y(x) = b q E_{\alpha,1+\frac{\beta}{\alpha}+\frac{\beta}{\alpha}}(\lambda, x-a). \]

**Remark 8.**

1) If in (18) \( \beta = 0 \), then in accordance with (17) and Example 10 in \([1]\) we have

\[ qE_{\alpha,1,0}(\lambda, x-a) = qE_{\alpha,1}(\lambda, x-a) = qE_{\alpha}(\lambda, x-a) \]

2) The solution of the \( q \)-Cauchy problem

\[ (qC^1_a y)(x) = \lambda(x-a)^\beta y(q^{-\beta}x), \quad y(a) = b \quad (21) \]
where
\[ 0 < \alpha < 1, \quad \beta > -\frac{1}{2}, \quad \lambda \in \mathbb{R}, \quad b \in \mathbb{R} \]
is given by
\[ y(x) = b q \mathcal{E}_{\frac{1}{2}, 1 + 2\beta, 2\beta}^\alpha (\lambda, x - a). \]

3) By the help of (13) and Lemma [1] and by applying the successive approximation with
\[ y_0(x) = \sum_{k=0}^{n-1} \frac{(t-a)^k}{q(k+1)} \nabla_q^k f(a), \]
Example [1] can be generalized for arbitrary \( \alpha > 0 \). Namely, the solution of the q-initial value problem

\[ (qC^\alpha_q y)(x) = \lambda(x - a) q^{-\beta} y(q^{\beta} x), \quad y^{(k)}(a) = b_k \quad (b_k \in \mathbb{R}, \quad k = 0, 1, ..., n-1) \quad (22) \]
where
\[ n - 1 < \alpha < n, \quad \beta > -\alpha, \quad \lambda \in \mathbb{R}, \quad b \in \mathbb{R} \]
is given by
\[ y(x) = \sum_{r=0}^{n-1} \frac{b_r}{\Gamma_q(r+1)} (x - a)^r q \mathcal{E}_{\frac{1}{2}, 1 + \alpha, \alpha + \beta}^{\frac{\alpha}{2}, \frac{\alpha}{2}} (\lambda, x - a). \]

Note that when \( 0 < \alpha < 1 \), i.e., \( n = 1 \), the solution of Example [1] is recovered.

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