Banach Algebra of Bounded Complex-Valued Functionals

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Summary. In this article, we describe some basic properties of the Banach algebra which is constructed from all bounded complex-valued functionals.

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The notation and terminology used in this paper are introduced in the following articles: [2], [16], [9], [14], [7], [8], [3], [18], [17], [4], [19], [5], [15], [1], [20], [12], [11], [10], [21], [13], and [6].

Let $V$ be a complex algebra. A complex algebra is called a complex subalgebra of $V$ if it satisfies the conditions (Def. 1).

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of $V$,
(ii) the addition of it = (the addition of $V$) $\upharpoonright$ (the carrier of it),
(iii) the multiplication of it = (the multiplication of $V$) $\upharpoonright$ (the carrier of it),
(iv) the external multiplication of it = (the external multiplication of $V$)$\upharpoonright$(C $\times$ the carrier of it),
(v) $1_{it} = 1_V$, and
(vi) $0_{it} = 0_V$.

We now state the proposition

(1) Let $X$ be a non empty set, $V$ be a complex algebra, $V_1$ be a non empty subset of $V$, $d_1, d_2$ be elements of $X$, $A$ be a binary operation on $X$, $M$ be a function from $X \times X$ into $X$, and $M_1$ be a function from $C \times X$ into $X$. Suppose that $V_1 = X$ and $d_1 = 0_V$ and $d_2 = 1_V$ and $A = (the$ addition of $V) \upharpoonright (V_1)$ and $M = (the$ multiplication of $V) \upharpoonright (V_1)$ and $M_1 = (the$ external multiplication of $V)\upharpoonright(\mathbb{C} \times V_1)$ and $V_1$ has inverse. Then $\langle X, M, A, M_1, d_2, d_1 \rangle$ is a complex subalgebra of $V$. 

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Let $V$ be a complex algebra. One can check that there exists a complex subalgebra of $V$ which is strict.

Let $V$ be a complex algebra and let $V_1$ be a subset of $V$. We say that $V_1$ is $\mathbb{C}$-additively-linearly-closed if and only if:

(Def. 2) $V_1$ is add closed and has inverse and for every complex number $a$ and for every element $v$ of $V$ such that $v \in V_1$ holds $a \cdot v \in V_1$.

Let $V$ be a complex algebra and let $V_1$ be a subset of $V$. Let us assume that $V_1$ is $\mathbb{C}$-additively-linearly-closed and non empty. The functor $\text{Mult}(V_1, V)$ yielding a function from $\mathbb{C} \times V_1$ into $V_1$ is defined as follows:

(Def. 3) $\text{Mult}(V_1, V) = (\text{the external multiplication of } V)|_{(\mathbb{C} \times V_1)}$.

Let $X$ be a non empty set. The functor $\mathbb{C}$-BoundedFunctions $X$ yielding a non empty subset of $\text{CAlgebra}(X)$ is defined by:

(Def. 4) $\mathbb{C}$-BoundedFunctions $X = \{ f : X \to \mathbb{C} : f|_{X}$ is bounded $\}$.

Let $X$ be a non empty set. Note that $\text{CAlgebra}(X)$ is scalar unital.

Let $X$ be a non empty set. One can verify that $\mathbb{C}$-BoundedFunctions $X$ is $\mathbb{C}$-additively-linearly-closed and multiplicatively-closed.

Let $V$ be a complex algebra. Observe that there exists a non empty subset of $V$ which is $\mathbb{C}$-additively-linearly-closed and multiplicatively-closed.

Let $V$ be a non empty CLS structure. We say that $V$ is scalar-multiplication-cancelable if and only if:

(Def. 5) For every complex number $a$ and for every element $v$ of $V$ such that $a \cdot v = 0_V$ holds $a = 0$ or $v = 0_V$.

One can prove the following two propositions:

(2) Let $V$ be a complex algebra and $V_1$ be a $\mathbb{C}$-additively-linearly-closed multiplicatively-closed non empty subset of $V$.

Then $\langle V_1, \text{mult}(V_1, V), \text{Add}(V_1, V), \text{Mult}(V_1, V), \text{One}(V_1, V), \text{Zero}(V_1, V) \rangle$ is a complex subalgebra of $V$.

(3) Let $V$ be a complex algebra and $V_1$ be a complex subalgebra of $V$. Then

(i) for all elements $v_1, w_1$ of $V_1$ and for all elements $v, w$ of $V$ such that $v_1 = v$ and $w_1 = w$ holds $v_1 + w_1 = v + w$,

(ii) for all elements $v_1, w_1$ of $V_1$ and for all elements $v, w$ of $V$ such that $v_1 = v$ and $w_1 = w$ holds $v_1 \cdot w_1 = v \cdot w$,

(iii) for every element $v_1$ of $V_1$ and for every element $v$ of $V$ and for every complex number $a$ such that $v_1 = v$ holds $a \cdot v_1 = a \cdot v$,

(iv) $1_{(V_1)} = 1_V$, and

(v) $0_{(V_1)} = 0_V$.

Let $X$ be a non empty set. The $\mathbb{C}$-algebra of bounded functions of $X$ yielding a complex algebra is defined by:

(Def. 6) The $\mathbb{C}$-algebra of bounded functions of $X = \langle \mathbb{C}$-BoundedFunctions $X, \text{mult} (\mathbb{C}$-BoundedFunctions $X, \text{CAlgebra}(X))$,
Banach algebra of bounded complex-valued . . .

\text{Add}(\text{C-BoundedFunctions} \, X, \text{CAlgebra}(X)),
\text{Mult}(\text{C-BoundedFunctions} \, X, \text{CAlgebra}(X)),
\text{One}(\text{C-BoundedFunctions} \, X, \text{CAlgebra}(X)),
\text{Zero}(\text{C-BoundedFunctions} \, X, \text{CAlgebra}(X))).

One can prove the following proposition

(4) For every non empty set \(X\) holds the \(\mathbb{C}\)-algebra of bounded functions of \(X\) is a complex subalgebra of \(\text{CAlgebra}(X)\).

Let \(X\) be a non empty set. Note that the \(\mathbb{C}\)-algebra of bounded functions of \(X\) is vector distributive and scalar unital.

Next we state several propositions:

(5) Let \(X\) be a non empty set, \(F, G, H\) be vectors of the \(\mathbb{C}\)-algebra of bounded functions of \(X\), and \(f, g, h\) be functions from \(X\) into \(\mathbb{C}\). Suppose \(f = F\) and \(g = G\) and \(h = H\). Then \(H = F + G\) if and only if for every element \(x\) of \(X\) holds \(h(x) = f(x) + g(x)\).

(6) Let \(X\) be a non empty set, \(a\) be a complex number, \(F, G\) be vectors of the \(\mathbb{C}\)-algebra of bounded functions of \(X\), and \(f, g\) be functions from \(X\) into \(\mathbb{C}\). Suppose \(f = F\) and \(g = G\). Then \(G = a \cdot F\) if and only if for every element \(x\) of \(X\) holds \(g(x) = a \cdot f(x)\).

(7) Let \(X\) be a non empty set, \(F, G, H\) be vectors of the \(\mathbb{C}\)-algebra of bounded functions of \(X\), and \(f, g, h\) be functions from \(X\) into \(\mathbb{C}\). Suppose \(f = F\) and \(g = G\) and \(h = H\). Then \(H = F \cdot G\) if and only if for every element \(x\) of \(X\) holds \(h(x) = f(x) \cdot g(x)\).

(8) For every non empty set \(X\) holds \(0\) the \(\mathbb{C}\)-algebra of bounded functions of \(X\) is \(X \mapsto 0\).

(9) For every non empty set \(X\) holds \(1\) the \(\mathbb{C}\)-algebra of bounded functions of \(X\) is \(X \mapsto 1\).

Let \(X\) be a non empty set and let \(F\) be a set. Let us assume that \(F \in \text{C-BoundedFunctions} \, X\). The functor \(\text{modetrans}(F, X)\) yields a function from \(X\) into \(\mathbb{C}\) and is defined by:

(Def. 7) \(\text{modetrans}(F, X) = F\) and \(\text{modetrans}(F, X)|_X\) is bounded.

Let \(X\) be a non empty set and let \(f\) be a function from \(X\) into \(\mathbb{C}\). The functor \(\text{PreNorms}(f)\) yields a non empty subset of \(\mathbb{R}\) and is defined by:

(Def. 8) \(\text{PreNorms}(f) = \{ |f(x)| : x \text{ ranges over elements of } X \}\).

We now state two propositions:

(10) For every non empty set \(X\) and for every function \(f\) from \(X\) into \(\mathbb{C}\) such that \(f|_X\) is bounded holds \(\text{PreNorms}(f)\) is upper bounded.

(11) Let \(X\) be a non empty set and \(f\) be a function from \(X\) into \(\mathbb{C}\). Then \(f|_X\) is bounded if and only if \(\text{PreNorms}(f)\) is upper bounded.
Let $X$ be a non empty set. The functor $\mathbb{C}\text{-BoundedFunctionsNorm \, X}$ yields a function from $\mathbb{C}\text{-BoundedFunctions \, X}$ into $\mathbb{R}$ and is defined by:

(Def. 9) For every set $x$ such that $x \in \mathbb{C}\text{-BoundedFunctions \, X}$ holds $(\mathbb{C}\text{-BoundedFunctionsNorm \, X})(x) = \sup \text{PreNorms}(\text{modetrans}(x, X))$.

One can prove the following two propositions:

(13) For every non empty set $X$ and for every function $f$ from $X$ into $\mathbb{C}$ such that $f \mid X$ is bounded holds $\text{modetrans}(f, X) = f$.

(14) For every non empty set $X$ and for every function $f$ from $X$ into $\mathbb{C}$ such that $f \mid X$ is bounded holds $(\mathbb{C}\text{-BoundedFunctionsNorm \, X})(f) = \sup \text{PreNorms}(f)$.

Let $X$ be a non empty set. The $\mathbb{C}$-normed algebra of bounded functions of $X$ yielding a normed complex algebra structure is defined by:

(Def. 10) The $\mathbb{C}$-normed algebra of bounded functions of $X$ =

$$\langle \mathbb{C}\text{-BoundedFunctions \, X}, \text{mult}(\mathbb{C}\text{-BoundedFunctions \, X}, \text{CAlgebra}(X)), \text{Add}(\mathbb{C}\text{-BoundedFunctions \, X}, \text{CAlgebra}(X)), \text{Mult}(\mathbb{C}\text{-BoundedFunctions \, X}, \text{CAlgebra}(X)), \text{One}(\mathbb{C}\text{-BoundedFunctions \, X}, \text{CAlgebra}(X)), \text{Zero}(\mathbb{C}\text{-BoundedFunctions \, X}, \text{CAlgebra}(X)), \mathbb{C}\text{-BoundedFunctionsNorm \, X} \rangle.$$ 

Let $X$ be a non empty set. One can verify that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is non empty.

Let $X$ be a non empty set. One can check that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is unital.

We now state a number of propositions:

(15) Let $W$ be a normed complex algebra structure and $V$ be a complex algebra. Suppose $(\text{the carrier of } W, \text{the multiplication of } W, \text{the addition of } W, \text{the external multiplication of } W, \text{the one of } W, \text{the zero of } W) = V$. Then $W$ is a complex algebra.

(16) For every non empty set $X$ holds the $\mathbb{C}$-normed algebra of bounded functions of $X$ is a complex algebra.

(17) Let $X$ be a non empty set and $F$ be a point of the $\mathbb{C}$-normed algebra of bounded functions of $X$.

Then $(\text{Mult}(\mathbb{C}\text{-BoundedFunctions \, X}, \text{CAlgebra}(X)))(1_{\mathbb{C}}, F) = F$.

(18) For every non empty set $X$ holds the $\mathbb{C}$-normed algebra of bounded functions of $X$ is a complex linear space.

(19) For every non empty set $X$ holds $X \longrightarrow 0 = 0$, the $\mathbb{C}$-normed algebra of bounded functions of $X$.

(20) Let $X$ be a non empty set, $x$ be an element of $X$, $f$ be a function from $X$ into $\mathbb{C}$, and $F$ be a point of the $\mathbb{C}$-normed algebra of bounded functions of $X$. If $f = F$ and $f \mid X$ is bounded, then $|f(x)| \leq \|F\|$.

\footnote{The proposition (12) has been removed.}
(21) For every non empty set $X$ and for every point $F$ of the $\mathbb{C}$-normed algebra of bounded functions of $X$ holds $0 \leq \|F\|$.

(22) Let $X$ be a non empty set and $F$ be a point of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $F = 0$ the $\mathbb{C}$-normed algebra of bounded functions of $X$. Then $0 = \|F\|$.

(23) Let $X$ be a non empty set, $f, g, h$ be functions from $X$ into $\mathbb{C}$, and $F, G, H$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) + g(x)$.

(24) Let $X$ be a non empty set, $a$ be a complex number, $f, g$ be functions from $X$ into $\mathbb{C}$, and $F, G$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element $x$ of $X$ holds $g(x) = a \cdot f(x)$.

(25) Let $X$ be a non empty set, $f, g, h$ be functions from $X$ into $\mathbb{C}$, and $F, G, H$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) \cdot g(x)$.

(26) Let $X$ be a non empty set, $a$ be a complex number, and $F, G$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Then

(i) if $\|F\| = 0$, then $F = 0$ the $\mathbb{C}$-normed algebra of bounded functions of $X$.

(ii) if $F = 0$ the $\mathbb{C}$-normed algebra of bounded functions of $X$, then $\|F\| = 0$,

(iii) $\|a \cdot F\| = |a| \cdot \|F\|$, and

(iv) $\|F + G\| \leq \|F\| + \|G\|$.

Let $X$ be a non empty set. Note that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, scalar unital, discernible, reflexive, and complex normed space-like.

We now state two propositions:

(27) Let $X$ be a non empty set, $f, g, h$ be functions from $X$ into $\mathbb{C}$, and $F, G, H$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F - G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) - g(x)$.

(28) Let $X$ be a non empty set and $s_1$ be a sequence of the $\mathbb{C}$-normed algebra of bounded functions of $X$. If $s_1$ is C.Cauchy, then $s_1$ is convergent.

Let $X$ be a non empty set. Observe that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is complete.

Next we state the proposition

(29) For every non empty set $X$ holds the $\mathbb{C}$-normed algebra of bounded functions of $X$ is a complex Banach algebra.
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