Cosmological Dark sector from a Mimetic-Metric-Torsion perspective

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Abstract

We generalize the basic theory of mimetic gravity by extending its purview to the general metric-compatible geometries that admit torsion, in addition to curvature. This essentially implies reinstating the mimetic principle of isolating the conformal degree of freedom of gravity in presence of torsion, by parametrizing both the physical metric and torsion in terms of the scalar ‘mimetic’ field and the metric and torsion of a fiducial space. We assert the requisite torsion parametrization from an inspection of the fiducial space Cartan transformation which, together with the conformal transformation of the fiducial metric, preserve the physical metric and torsion. In formulating the scalar-tensor equivalent Lagrangian, we consider an explicit contact coupling of the mimetic field with torsion, so that the former can manifest itself geometrically as the source of a torsion mode, and most importantly, give rise to a viable ‘dark universe’ picture from a mimicry of an evolving dust-like cosmological fluid with a non-zero pressure. A further consideration of higher derivatives of the mimetic field in the Lagrangian leads to physical bounds on the mimetic-torsion coupling strength, which we determine explicitly.

1 Introduction

General Relativity (GR), in spite of its immense success, has been subject to quite a bit of skepticism on its adequacy and correctness in predictability at cosmological scales, when confronted with the challenges in dealing with the dark constituents of the universe, viz. dark energy (DE) and dark matter (DM) [1–5]. A considerable amount of interest has therefore been developed on the studies of the cosmological aspects of the extensions or alternatives of GR, which are collectively known as the theories of modified gravity (MG) [6–15]. Importance has primarily been given to the MG equivalent scalar-tensor formulations which, in a cosmological setting, give rise to scenarios of a scalar field induced DE, that interacts with matter fields (including the DM sources) under conformal transformations [16–20]. Consequently, one gets a platform for exploring a plausible ‘geometric unification’ of the cosmological dark sector via some mechanism of emulating the DM (or part thereof) using the same scalar field artifact(s) of geometry from which the DE emerges. Such explorations have had a fresh impetus in recent years with the emergence of the mimetic gravity theory [21], and a host of its extensions proposed on various grounds.

The remarkable aspect of mimetic gravity is that it provides an exact fluid description of an irrotational dust, supposedly the cold dark matter (CDM) in the Friedmann-Robertson-Walker (FRW) cosmological framework, while preserving the conformal symmetry of the physical space-time metric $g_{\mu\nu}$ [21–29]. This actually is a consequence of the diffeomorphism invariance of gravitational theories à la GR, which allows one to parametrize $g_{\mu\nu}$ by a fiducial metric $\tilde{g}_{\mu\nu}$ and a scalar field $\phi$, in a non-invertible disformal manner in general [30–33]. The mimetic parametrization though is a simplified form of a disformation, viz. $g_{\mu\nu} = -\kappa^2\tilde{g}_{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\tilde{g}_{\mu\nu}$, where $\kappa$ denotes the Planck length scale, and the scalar ‘mimetic field’ $\phi$ is assumed dimensionless [21]. Such a parametrization not only ensures the invariance of $g_{\mu\nu}$ under a conformal
transformation of $g_{\mu\nu}$, but also implicates that $\phi$ is left non-dynamical by the condition for the invertibility of $g_{\mu\nu}$, viz. $-\kappa^2 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 1$. This condition could be implemented in the theory as a constraint, using a Lagrange multiplier $\lambda$ in an equivalent mimetic action. A further equivalence of the latter with the singular Brans-Dicke (BD) action (in which the BD parameter $w = -3/2$) eventually shows that mimetic gravity is indeed a ratification of GR being raised to the status of a scalar-tensor equivalent MG theory \[21\].

Detailed studies have revealed many attractive features of the basic mimetic model of Chamseddine and Mukhanov (henceforth, the CM model [21]) and its various extensions, from the perspectives of both cosmology and astrophysics [34–60]. In particular, the CM Lagrangian extended by a potential $V(\phi)$ leads to the scenarios of a cosmologically evolving fluid — the so-called ‘mimetic fluid’ — which characterizes dust, albeit with a non-vanishing pressure [22]. As such, no propagating scalar perturbation mode is there to suppress the growth of structures at smaller (sub-galactic) scales [26,27]. Nor it is possible to define the quantum fluctuations to the (non-dynamical) field $\phi$, so that the latter can provide the seeds of the observed large scale structure of the universe [22,23]. A reasonable supposition has therefore been to extend the CM theory further by incorporating higher derivative (HD) term(s) for $\phi$, e.g. $(\Box \phi)^2$, which lead(s) to a non-vanishing sound speed $c_s$ of the mimetic matter perturbations, keeping the background solution unaffected qualitatively [22,23]. However, the HD terms in general make the theory susceptible to Ostrogradsky ghost or(and) gradient instabilities [26,27], possible wayouts of which point to more complicated extensions of the CM model. For instance, one may require to incorporate explicit HD couplings with the Ricci curvature scalar $R$ [66,69], or resort to degenerate higher-order scalar-tensor (DHOST) Lagrangians [70,71]. Noteworthy mimetic extensions from various other considerations include the mimetic $f(R)$ gravity and its unimodular, non-local and several more variants [15,36–42,72–74], mimetic Horndeski theory [76,77], mimetic Born-Infeld theory [78,79], mimetic brane-world gravity [80,82], mimetic massive gravity [83–86], and so on [87,91]. Interestingly, it has also been shown that mimetic gravity has a close correspondence with the scalar version of the Einstein-Æther theory, and hence appears in the infrared limit of the projectable Horava-Lifshitz gravity [94–99].

Nevertheless, some arbitrariness is often implicit in many of the extended mimetic formulations, for e.g. in the assertion of the potential $V(\phi)$, or in the choice of appropriate HD terms or(and) their couplings with curvature invariants, or in some other context. Scope therefore remains for further extensions or generalizations, particularly from the point of view of asking the following:

- **Can there be a proper geometrical significance of the mimetic field, notwithstanding its role in encoding the conformal degree of freedom of gravity?**

Or, more specifically,

- **Can the mimetic field manifest itself geometrically, say for e.g. acting as the source of a purely geometric entity?**

A positive answer to this may be reckoned within the option of considering the requisite entity to be torsion, which is often regarded as important a space-time characteristic as curvature. In fact, torsion is the only geometric entity, other than curvature, which can be extracted from a general metric-compatible affine connection [100,104]. Torsion’s physical significance could be drawn from its indispensability in forming a classical background geometry for quantized matter fields with arbitrary spin. As such, torsion could be a low energy manifestation of a fundamental theory of quantum gravity [107,110]. For e.g. an axial torsion may have its source in the string theoretic Kalb-Ramond field [113,114], many implications of which have been revealed from extensive studies [115–136]. Also appealing are the physical aspects of e.g. the propagating torsion theories [137,143], the parity violating metric-torsion theories [144,150], the extended gravity theories with torsion [151,153], the skewon theory of gravity with torsion [154,156], the scalar-torsion theory that

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1Such scenarios are actually reminiscent of the dusty dark energy (DDE) models [61,63], that preceded the original CM paper [21].

2In fact, the dusty fluid with pressure is rendered imperfect, with or without a violation of the shift symmetry ($\phi \rightarrow \phi + \text{constant}$) of the theory, in presence of the HD terms [23,27,28].
demonstrates Cartan gauge invariance [157], the square-torsion theory [158–161], the degenerate tetrad formalism [162–166], the non-minimal metric-scalar-torsion coupling formalism [167–169], and so on. Moreover, a lot of attention has been drawn in recent years by modified versions of the (curvature-free) teleparallel gravity theories [170–185], apart from the modern refinements of Poincaré gauge theory of gravity [186–196].

Now, given such a wide applicability of torsion, it seems really interesting to look for a generalized mimetic gravity formulation in presence of torsion, especially from the point of view of getting an observational support for the latter, amidst its miniscule experimental evidence till date [197–203]. In fact, there had been some expectation that the mimetic field could be identified with, or at least correlated to, a torsion degree of freedom [45]. However, only a few attempts on formulating a mimetic-torsion theory have so far been made, and that too in the specific contexts of e.g. Lyra geometry, or modified teleparallelism [87,204,205]. Whether such a formulation be substantiated within the purview of the rather general Riemann-Cartan geometry, that admits both curvature and torsion, is nonetheless an open issue worth a proper attention and study. It is our objective in this paper to do so, of course with an endeavour to see the role of torsion, and that of its coupling with the mimetic field, in leading to viable cosmological scenarios.

Let us outline the steps we take in our approach towards formulating a mimetic-metric-torsion (MMT) theory, and see its cosmological consequences:

The first step concerns the conformal properties of torsion. Note that in a given space-time with torsion, a conformal transformation of the metric is in general associated with the Cartan transformation, which although affects only the trace mode $T\mu$ of torsion, is quite non-trivial in its ‘strong’ form [107,157,209]. Nevertheless, a simplification could be reasoned from certain standpoints, which we discuss in section 2 of this paper, after briefly recapitulating the general aspects of the metric-torsion theories therein.

The second (and the most crucial) step is to ensure the conformal symmetry of the MMT theory at the field (metric and torsion) level. This requires us to resort to not only the parametrization of $g_{\mu\nu}$ in terms of $\hat{g}_{\mu\nu}$ and the mimetic field $\phi$, but also that of the physical torsion $T^\alpha_{\mu\nu}$ in terms of a fiducial torsion $\hat{T}^\alpha_{\mu\nu}$ and $\phi$. We look to assert such a torsional parametrization in the first part of section 3 taking account of the requisite criteria:

(i) In the first place, the metricity of $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$ must hold in the respective spaces with torsion fields $T^\alpha_{\mu\nu}$ and $\hat{T}^\alpha_{\mu\nu}$.

(ii) $T^\alpha_{\mu\nu}$ should remain invariant (just like $g_{\mu\nu}$) under the conformal and Cartan transformations of the fiducial fields, $\hat{g}_{\mu\nu}$ and $\hat{T}^\alpha_{\mu\nu}$.

The third (and perhaps the most rigorous) step is to formulate equivalent MMT Lagrangians. This is done in the second part of section 3 wherein the forms of such Lagrangians are motivated from a number of considerations, most importantly that of explicit contact couplings of the mimetic field $\phi$ with the individual torsion terms. The purpose for this is to get a constraint $T^\mu_\mu \propto \partial_\mu \phi$, which is essential not only for a geometric manifestation of $\phi$ (as the source of the torsion trace mode $T^\mu_\mu$), but also for retaining the dust-like characteristic of the mimetic fluid in space-times with torsion. We further consider certain mutual relationships between those couplings (which we call ‘MMT couplings’) such that the equivalent MMT Lagrangians look elegant and simplified, with just one independent MMT coupling function.

The fourth step is to derive the field equations corresponding to the equivalent MMT Lagrangian in the Einstein frame. This is done in the last part of section 3 wherein we ignore for brevity any external sources for the torsion irreducible modes. We show that the system of MMT equations of motion look exactly similar to that for an evolving dust-like mimetic fluid with a non-zero pressure (which of course results from the MMT coupling, rather than from a potential term in the Lagrangian).

3These are the terms that generally appear in the Lagrangians for the conventional metric-torsion theories, such as that formulated originally by Cartan and Einstein in early 1920s.
The fifth step is to work out the MMT cosmological equations in the FRW framework. This we do in section 4, and subsequently resort to a specific form of the MMT coupling function $\beta(\phi) \sim \phi^2$, which leads to a $\Lambda$CDM cosmological evolution, where $\Lambda$ is an effective cosmological constant. We illustrate the various phases of evolution of $\beta(\phi)$ and the norm of $T_\mu$ (which we consider as a torsion parameter), and estimate their values at the present epoch using the latest PLANCK results [206].

The sixth (and the final) step is to see, in section 5, the outcome of incorporating a $(\Box \phi)^2$ term in our MMT formalism. Such a term of course leads to a non-zero sound speed $c_s$ of the linear matter perturbations, without affecting the background (ACDM) solution qualitatively. As such, there is no change in the torsion parameter (norm of $T_\mu$) but the MMT coupling $\beta(\phi)$ effectively gets rescaled by a factor inversely proportional to $(1+3c_s^2)$. Consequently, the physical limits of $c_s^2$ determine the bounds on $\beta(\phi)$ evaluated at the present epoch.

We conclude with a summary and some open questions in section 6. In the Appendix, we justify the requirement $T_\mu \propto \partial_\mu \phi$ from a purely geometrical standpoint. We work out the mimetic fluid acceleration $\tilde{a}_\mu$ in a background space-time with torsion, and show that it vanishes (i.e. the fluid velocity becomes tangential to the time-like auto-parallel curves, as in the case of dust) only when $T_\mu \sim \partial_\mu \phi$.

Conventions and Notations: Throughout this paper we use (i) metric signature $(-, +, +, +)$, and (ii) natural units (the speed of light $c = 1$). We denote (i) the determinant of the metric tensor $g_{\mu\nu}$ by $g$, (ii) the Planck length parameter by $\kappa = \sqrt{8\pi G_N}$ (where $G_N$ is the Newton’s gravitational constant) and (iii) the values of parameters or functions at the present epoch by affixing a subscript ‘0’.

2 Metric-Torsion formalism and Conformal transformation

Let us begin with the formal definition of the torsion tensor, viz.

$$T^\alpha_{\mu\nu} := 2 \tilde{\Gamma}^\alpha_{[\mu\nu]} = \tilde{\Gamma}^\alpha_{\mu\nu} - \tilde{\Gamma}^\alpha_{\nu\mu}$$, \hspace{1cm} (2.1)

in the four dimensional Riemann-Cartan ($U_4$) space-time geometry, characterized by a general (asymmetric) affine connection

$$\tilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + K^\alpha_{\mu\nu}$$, \hspace{1cm} \left[ K^\alpha_{\mu\nu} = \frac{1}{2} \left( T^\alpha_{\mu\nu} - T^\alpha_{\mu\nu} - T^\alpha_{\nu\mu} \right) \right]$$, \hspace{1cm} (2.2)

where $\Gamma^\alpha_{\mu\nu}$ is the four dimensional Riemannian ($R_4$) Levi-Civita connection, and the tensor $K^\alpha_{\mu\nu}$ is known as ‘contorsion’. Eq. (2.2) is actually a re-statement of the condition of metric-compatibility $\tilde{\nabla}_\mu g_{\nu\rho} = 0 = \nabla_\mu g_{\nu\rho}$, where $\tilde{\nabla}_\alpha$ is the $U_4$ covariant derivative defined in terms of $\tilde{\Gamma}^\alpha_{\mu\nu}$ in the same way as the $R_4$ covariant derivative $\nabla_\alpha$ is defined in terms of $\Gamma^\alpha_{\mu\nu}$. Note also that $K_{\alpha\mu\nu} = K_{\mu\rho\nu\beta}$, unlike $T_{\alpha\mu\nu} = T_{\alpha[\mu\nu]}$.

The $U_4$ analogue of the Riemannian curvature tensor $R^\alpha_{\mu\nu\rho}$ is given by

$$\tilde{R}^\alpha_{\mu\nu\rho} := \partial_\rho \tilde{\Gamma}^\alpha_{\mu\nu} + \tilde{\Gamma}^\alpha_{\tau\rho} \tilde{\Gamma}^\tau_{\mu\nu} - (\rho \leftrightarrow \nu)$$, \hspace{1cm} (2.3)

which (unlike $R^\alpha_{\mu\nu\rho}$) neither have any cyclicity property nor exhibit any symmetry under the interchange of the first and last pairs of indices. Accordingly, the $U_4$ analogue of the Ricci tensor $R_{\mu\nu}$, viz. $\tilde{R}_{\mu\nu} := \tilde{R}^\alpha_{\mu\nu\alpha}$ is not symmetric in its indices.

Also well-known is the following decomposition of the torsion tensor

$$T^\alpha_{\mu\nu} = \frac{1}{3} (T^\alpha_{\mu\nu} - T^\alpha_{\nu\mu}) + \frac{1}{6} \epsilon^{\alpha\beta\gamma\rho} A_{\rho} + Q^\alpha_{\mu\nu}$$, \hspace{1cm} (2.4)

in its irreducible modes, viz. (i) the trace vector $T^\mu_{\mu} := T^\nu_{\nu\mu}$; (ii) the pseudo-trace vector $A^\mu := \epsilon^{\alpha\beta\gamma\mu} T_{\alpha\beta\gamma}$, and (iii) the (pseudo)-tracefree tensor $Q^\mu_{\nu\mu} := Q^\mu_{\nu[\mu]}$ and $Q^\mu_{\mu\alpha} = 0 = \epsilon^{\alpha\beta\gamma\mu} Q_{\alpha\beta\gamma}$. In terms of these modes,
the $U_4$ connection, and the $U_4$ analogue of the Riemannian curvature scalar $R$, are given respectively by

$$
\tilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \frac{1}{3} (T^\alpha g_{\mu\nu} - T_\mu^\alpha \delta_\nu^\sigma - \frac{1}{2} \xi^{\alpha}_{\mu\nu\rho} A^\rho + Q^\alpha_{\nu\mu} ) ,
$$

(2.5)

$$
\tilde{R} = g^{\mu\nu} \tilde{R}_{\mu\nu} = \tilde{R} - 2 \nabla_\mu T^\mu - \frac{2}{3} \tilde{T}^\mu T^\mu + \frac{1}{24} A_\mu A^\mu + \frac{1}{2} Q^{\alpha\mu} Q^\alpha_{\mu\nu} .
$$

(2.6)

Usually, $\tilde{R}$ is taken to replace the standard Einstein-Hilbert Lagrangian $R$ in the conventional metric-torsion theories, viz. the original Einstein-Cartan formulation and a host of its extensions [101–104, 107].

Now, in a metric-compatible space-time with torsion, a conformal transformation

$$
g_{\mu\nu} \rightarrow g'_{\mu\nu} = e^{2\sigma} g_{\mu\nu} ,
$$

(2.7)

is in general associated with the Cartan transformation equations [107, 157]:

$$
T^\alpha_{\mu\nu} \rightarrow T'^{\alpha}_{\mu\nu} = T^\alpha_{\mu\nu} + q (\delta^\alpha_{\mu} \partial_\nu \sigma - \delta^\alpha_{\nu} \partial_\mu \sigma ) ; \quad [q = \text{a constant}],
$$

(2.8)

$$
T_\mu \rightarrow T'_\mu = T_\mu - 3q \partial_\mu \sigma , \quad A_\mu \rightarrow A'_\mu = A_\mu , \quad Q^\alpha_{\mu\nu} \rightarrow Q'^{\alpha}_{\mu\nu} = Q^\alpha_{\mu\nu} ,
$$

(2.9)

for any given scalar function of coordinates $\sigma(t, \vec{x})$. Note that the transformed torsion tensor $T'^{\alpha}_{\mu\nu}$ has the same decomposition (2.4) in terms of the transformed modes $T'_{\mu\nu}$, $A'_\mu$, and $Q'^{\alpha}_{\mu\nu}$ (of course, with the indices being raised or lowered using the conformal metric $g'_{\mu\nu}$). In other words, Eqs. (2.8) and (2.9) are defined in a way that $T'_{\mu\nu}$, $A'_\mu$, and $Q'^{\alpha}_{\mu\nu}$ are truly the irreducible modes of $T^{\alpha}_{\mu\nu}$. [107, 157, 207, 213]. The constant $q$ is an arbitrary numerical parameter, that typifies the conformal symmetry of the $U_4$ theory (if any). More specifically, the $U_4$ theory is said to be conformally symmetric in the ‘weak’ or ‘strong’ form [107, 157], if the $U_4$ action is conformally invariant for $q = 0$ or $\neq 0$. In this paper however, we shall resort to the particular setting $q = 1$, which is of significance from the following points of view.\(^4\)

(a) In general, the metric and the connection are taken as independent variables in a geometric theory of gravity. For any general affine connection $\tilde{\Gamma}^\alpha_{\mu\nu}$, the corresponding curvature constructs $\tilde{R}^{\alpha}_{\mu\nu\rho}$, $\tilde{R}_{\mu\nu}$ and $\tilde{R}$, as well as the Einstein tensor analogue $\tilde{G}^{\alpha}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R}$, remain invariant under the so-called Einstein’s $\lambda$-transformation [207]:

$$
\tilde{\Gamma}^\alpha_{\mu\nu} \rightarrow \tilde{\Gamma}'^\alpha_{\mu\nu} = \tilde{\Gamma}^\alpha_{\mu\nu} + \delta^\alpha_\mu \partial_\nu \lambda ,
$$

(2.10)

where $\lambda = \lambda(t, \vec{x})$ is an arbitrary scalar function of coordinates. Now, given the specific form (2.5) of $\tilde{\Gamma}^\alpha_{\mu\nu}$, verify that the transformations (2.7)-(2.9) imply

$$
\tilde{\Gamma}^\alpha_{\mu\nu} \rightarrow \tilde{\Gamma}'^\alpha_{\mu\nu} = \tilde{\Gamma}^\alpha_{\mu\nu} + \delta^\alpha_{\mu} \partial_\nu \sigma + (q - 1) \left( g_{\mu\nu} g^{\alpha\beta} \partial_\beta \sigma - \delta^\alpha_\nu \partial_\mu \sigma \right) .
$$

(2.11)

For $q = 1$ and $\sigma = \lambda$, this corresponds to the $\lambda$-transformation (2.10), with one caveat though — invariance of $\tilde{R}^{\alpha}_{\mu\nu\rho}$, $\tilde{R}_{\mu\nu}$ and $\tilde{G}^{\alpha}_{\mu\nu}$, but $\tilde{R} \rightarrow e^{2\sigma} \tilde{R}$.

(b) Recall that in the Riemannian space-time ($R_4$), one can construct using the curvature tensor $R^{\alpha}_{\mu\nu\rho}$, and its contractions $R_{\mu\nu}$ and $R$, the Weyl tensor

$$
W^{\alpha}_{\mu\nu\rho} := R^{\alpha}_{\mu\nu\rho} - ( \Gamma^{\alpha}_{\mu\rho} g_{\nu\alpha} - \Gamma^{\alpha}_{\mu\nu} g_{\rho\alpha} ) + \frac{1}{6} ( g^{\alpha\beta} g_{\nu\alpha} - g^{\alpha\beta} g_{\rho\alpha} ) R .
$$

(2.12)

This has the same (anti)symmetry and cyclicity properties of $R^{\alpha}_{\mu\nu\rho}$, but is irreducible and conformally covariant, i.e. preserved in the mixed form $W^{\alpha}_{\mu\nu\rho}$ under the conformal transformation (2.7). In the $U_4$ space-time however, the conformal covariance of a tensor implies the invariance of the latter, in a certain

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4 For convenience of our discussions later on, we purposefully call Eq. (2.7) the ‘conformal transformation’ equation, and Eqs. (2.8) and (2.9) together as the ‘Cartan transformation’ equations. In the literature though, the ‘Cartan transformations’ generally refer to the full set (2.7)-(2.9) [107, 157].

5 The general case of an arbitrary $q$ is considered in our subsequent works [214, 215].
specified form, under the full set of transformations \([2.7] - [2.9]\), which involve the arbitrary parameter \(q\). Now, there is no straightforward way to find the \(U_4\) equivalent of the Weyl tensor, that is uniquely defined for all values of \(q\) and constructed out of the \(U_4\) curvature analogues \(\tilde{R}_{\alpha\mu\nu}\), \(\tilde{R}_{\mu\nu}\) and \(\tilde{R}\). Nevertheless, a close inspection of the Jacobi-Bianchi identities in presence of torsion has led to the suggestion that the following extension of \(\tilde{R}_{\alpha\mu\nu}\) may be treated as the effective \(U_4\) curvature tensor \([209]\):

\[
\tilde{R}_{\alpha\mu\nu}^\text{eff} := \tilde{R}_{\alpha\mu\nu} + \left(\frac{q-1}{3q}\right) (T_\alpha T_{\mu\nu} - T_{\mu} T_{\alpha\nu}) .
\]

(2.13)

Accordingly, the contractions \(\tilde{R}_{\alpha\mu\nu}^\text{eff} = g^{\alpha\rho} \tilde{R}_{\alpha\rho\mu\nu}\) and \(\tilde{R}_{\mu\nu}^\text{eff} = g^{\mu\rho} \tilde{R}_{\rho\mu\nu}\) may serve as the effective \(U_4\) Ricci tensor and curvature scalar respectively. Note that \(\tilde{R}_{\alpha\mu\nu}^\text{eff}\) bears the antisymmetry properties of \(\tilde{R}_{\alpha\mu\nu}\), and so does the irreducible tensor

\[
\tilde{W}_{\alpha\mu\nu} := \tilde{R}_{\alpha\mu\nu}^\text{eff} - \left(\tilde{R}_{\alpha[\rho}^\text{eff} g_{\nu]\mu} - \tilde{R}_{\mu[\rho}^\text{eff} g_{\nu]\alpha}\right) + \frac{1}{6} (g_{\alpha[\rho} g_{\nu]\mu] - g_{\mu[\rho} g_{\nu]\alpha}) \tilde{R}_{\alpha\mu\nu}^\text{eff} ,
\]

(2.14)

which can therefore be considered as the effective \(U_4\) Weyl tensor \([209]\), since it is preserved in the mixed form \(\tilde{W}_{\alpha\mu\nu}\) under the transformations \([2.7] - [2.9]\). For \(q = 0\) however, the above definition of \(\tilde{R}_{\alpha\mu\nu}^\text{eff}\) is invalid. So the case of the weak conformal symmetry cannot be addressed this way. On the other hand, the setting \(q = 1\) simply means \(\tilde{R}_{\alpha\mu\nu}^\text{eff} = \tilde{R}_{\alpha\mu\nu}\), i.e. we have a straightforward construction of the \(U_4\) Weyl tensor \(\tilde{W}_{\alpha\mu\nu}\) in exact analogy of Eq. (2.12).

3 Mimetic theory and its generalization in presence of Torsion

3.1 Conformal invariance of the physical fields

Refer to the basic CM formalism that exploits the diffeomorphism invariance of GR in parametrizing the physical metric \(g_{\mu\nu}\) by a fiducial metric \(\tilde{g}_{\mu\nu}\), and the dimensionless ‘mimetic’ scalar field \(\phi\), as \([21, 22, 23]\):

\[
g_{\mu\nu} = \hat{X} \tilde{g}_{\mu\nu} , \quad \text{where} \quad \hat{X} = -\kappa^2 \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi .
\]

(3.1)

This parametrization is non-invertable, as it represents a mapping of \(10 \to 11\) variables in \(R_4\). Observe that:

(i) The physical metric \(g_{\mu\nu}\) remains invariant under the conformal transformation

\[
\tilde{g}_{\mu\nu} \to e^{2\sigma} \tilde{g}_{\mu\nu} \quad \text{and} \quad \phi \to \phi ,
\]

(3.2)

where \(\sigma = \sigma(t, \vec{x})\) is a scalar function of coordinates\(^6\). (ii) The physical metric is non-singular (i.e. its inverse exists) only when

\[
X \equiv -\kappa^2 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 1 .
\]

(3.3)

(iii) The Christoffel symbols corresponding to \(g_{\mu\nu}\) and \(\tilde{g}_{\mu\nu}\) have the relationship

\[
\Gamma^\alpha_{\mu\nu} = \tilde{\Gamma}^\alpha_{\mu\nu} + \frac{1}{2X} \left( \delta^\alpha_\mu \partial_\nu \hat{X} + \delta^\alpha_\nu \partial_\mu \hat{X} - \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \partial_\beta \hat{X} \right) ,
\]

(3.4)

which implies that \(\nabla_\alpha g_{\mu\nu} = 0 = \hat{X} \hat{\nabla}_\alpha \tilde{g}_{\mu\nu}\), i.e. the spaces of \(g_{\mu\nu}\) and \(\tilde{g}_{\mu\nu}\) are indeed metric spaces (with the respective covariant derivatives \(\nabla_\alpha\) and \(\hat{\nabla}_\alpha\)).

\(^6\) The conformal transformation relations would have been \(g_{\mu\nu} \to e^{2\sigma} \tilde{g}_{\mu\nu}, \phi \to e^{-\sigma} \phi\), if \(\phi\) would have had a mass dimension = 1, instead of being dimensionless (see for e.g. Maldacena \([216]\) and references therein).
Now, in order to extend the mimetic formalism to the metric-torsion scenario, we first need to find an appropriate analogous parametrization of a (conformally covariant) physical torsion field $T_{\mu\nu}^\alpha$, in terms of the mimetic field $\phi$ and a fiducial torsion $\hat{T}_{\mu\nu}^\alpha$ having its irreducible modes defined in the fiducial space as follows:

$$\hat{T}_{\mu} = \hat{g}^{\alpha\nu} \hat{T}_{\alpha\mu\nu} , \quad \hat{A}_{\mu} = \hat{g}_{\mu\nu} \epsilon^{\alpha\beta\gamma\nu} \hat{T}_{\alpha\beta\gamma} ,$$

$$\hat{Q}_{\mu\nu}^\alpha = \hat{T}_{\mu\nu}^\alpha - \frac{2}{3} \hat{T}_{[\mu} \delta_{\nu]}^\alpha - \frac{1}{6} \hat{g}^{\alpha\beta} \hat{g}_{\beta\mu\rho} \epsilon_{\rho\nu\alpha} \hat{A}_{\lambda} . \quad (3.5)$$

A close inspection of (i) how the quantity $\hat{X} = -\kappa^2 \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ and its derivatives transform under the conformal transformation \cite{(3.2)}, and (ii) the form of the Cartan transformation of the fiducial torsion, viz.

$$\hat{T}_{\mu\nu}^\alpha \to \hat{T}_{\mu\nu}^\alpha + \delta_\mu^\alpha \partial_\nu \sigma - \delta_\nu^\alpha \partial_\mu \sigma , \quad (3.6)$$

indeed reveals that one may conveniently resort to the parametrization

$$T_{\mu\nu}^\alpha := \hat{T}_{\mu\nu}^\alpha + \delta_\mu^\alpha \partial_\nu (\ln \hat{X}) , \quad (3.7)$$

which ensures the conformal covariance of the physical torsion, i.e. invariance in the mixed form $T_{\mu\nu}^\alpha$ under the transformations \cite{(3.2)} and \cite{(3.6)}. The conformal covariance of the irreducible modes of $T_{\mu\nu}^\alpha$ can consequently be verified from their relationships with the irreducible modes of $\hat{T}_{\mu\nu}^\alpha$:

$$T_{\mu} = \hat{T}_{\mu} - \frac{3}{2} \partial_\mu (\ln \hat{X}) , \quad A_{\mu} = \hat{A}_{\mu} , \quad \text{and} \quad Q_{\mu\nu}^\alpha = \hat{Q}_{\mu\nu}^\alpha . \quad (3.8)$$

Moreover, using Eq. \cite{(3.7)} one gets $\nabla_{\alpha} \hat{g}_{\mu\nu} = 0 = \hat{X} \nabla_{\alpha} \hat{g}_{\mu\nu}$, which verifies the metricity of $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$ in the respective spaces with torsion fields $T_{\mu\nu}^\alpha$ and $\hat{T}_{\mu\nu}^\alpha$, and covariant derivatives denoted by $\nabla_{\alpha}$ and $\hat{\nabla}_{\alpha}$.

### 3.2 Equivalent Lagrangian formulation

Refer back again to the basic CM theory of mimetic gravity, the action for which is no different from the standard Einstein-Hilbert action in presence of matter fields

$$S = S^{(m)} + \frac{1}{2k^2} \int d^4 x \sqrt{-g} R(g_{\mu\nu}) ; \quad [S^{(m)} = \int d^4 x \sqrt{-g} L(m)] , \quad (3.9)$$

where $L(m)$ is the matter Lagrangian. A recourse to the parametrization $g_{\mu\nu} (\hat{g}_{\mu\nu}, \phi)$, given by Eq. \cite{(3.1)}, leads to the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 \left[ T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(d)} \right] , \quad (3.10)$$

$$\nabla_\mu \left[ g^{\mu\nu} T^{(d)} _{\nu} \partial_\nu \phi \right] \equiv 0 , \quad (3.11)$$

where $T_{\mu\nu}^{(m)}$ is the matter energy-momentum tensor, whose mimetic modification is

$$T_{\mu\nu}^{(d)} = \left[ R + \kappa^2 T_{\mu\nu}^{(m)} \right] \partial_\mu \phi \partial_\nu \phi , \quad (3.12)$$

$T^{(m)} = g^{\mu\nu} T_{\mu\nu}^{(m)}$ and $T^{(d)} = g^{\mu\nu} T_{\mu\nu}^{(d)} = - [ \kappa^{-2} R + T^{(m)} ]$ being the respective traces.

Eq. \cite{(3.12)} can be recast in the form of the energy-momentum tensor due to a pressureless fluid (dubbed the ‘mimetic fluid’), viz.

$$T_{\mu\nu}^{(d)} = \rho^{(d)} u_\mu u_\nu , \quad \text{[of energy density: } \rho^{(d)} = - T^{(d)} = \kappa^{-2} R + T^{(m)} \text{]} , \quad (3.13)$$

Note that the parametrization \cite{(3.1)} of the physical metric $g_{\mu\nu}$ implies that the physical space Levi-Civita tensor is parametrized as $\epsilon_{\alpha\beta\gamma\delta} = \hat{X}^7 \epsilon_{\alpha\beta\gamma\delta}$.
by identifying the fluid velocity as \( u_\mu \equiv \kappa \partial_\mu \phi \), in analogy with k-essence cosmologies [217][221], and in accord with the relation \( X = -\kappa^2 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 1 \) [cf. Eq. (3.31)]. Note that it is this relation which makes the velocity normalization condition \( g^{\mu\nu} u_\mu u_\nu = -1 \) hold. Moreover, this relation implies that the fluid acceleration

\[
a_\mu = g^{\alpha\nu} u_\alpha \nabla_\nu u_\mu = \kappa^2 \nabla_\nu \left( \frac{1}{2} g^{\alpha\nu} \partial_\alpha \phi \partial_\mu \phi \right) = 0 ,
\]

i.e. the mimetic fluid has the flow lines of its elements following the time-like geodesics [61], and hence behaves as the standard dust (supposedly the cold dark matter in the FRW cosmological framework [21][22][29]).

It is also easy to see that the above Eqs. (3.10) and (3.11) can be derived from the equivalent action

\[
S = S^{(m)} + \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ R(g_{\mu\nu}) + \lambda \cdot (X - 1) \right] ,
\]

where \( \lambda \) is a scalar Lagrange multiplier field (of mass dimension = 2) that enforces the constraint \( X = 1 \) [21][29]. A host of other equivalent mimetic actions have also been found in the literature [24].

Let us consider, for an illustration (and for our subsequent discussions), the following Brans-Dicke (BD) action in the fiducial metric space:

\[
\hat{S} = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-\hat{g}} \left[ \hat{X} R(\hat{g}_{\mu\nu}) - \frac{w}{X} \hat{g}^{\mu\nu} \partial_\mu \hat{X} \partial_\nu \hat{X} - \hat{\lambda} \cdot (\hat{X} + \kappa^2 \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi) \right] ,
\]

where \( w \) is the constant BD parameter. Note that \( \hat{X} \) is treated here just as a (dimensionless) BD scalar field, and not as a quantity pre-assigned to be equal to \( -\kappa^2 \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \) [what we had in Eq. (3.11)]. In fact, a scalar Lagrange multiplier field \( \hat{\lambda} \) (of mass dimension = 2) is used instead, to impose \( \hat{X} = -\kappa^2 \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \) as a constraint. Moreover, \( \hat{X} \) being dimensionless, the effective gravitational coupling is determined as \( \kappa^2_{\text{eff}} = \kappa^2 \hat{X}^{-1} \), for a suitable reference setting, for e.g. the stipulation that at the present epoch \( t = t_0 \), \( \hat{X}(t_0) = 1 \), so that \( \kappa^2_{\text{eff}}(t_0) = \kappa^2 = 8\pi G_N \).

Under the conformal transformation

\[
\hat{g}_{\mu\nu} \to g_{\mu\nu} = \hat{X} \hat{g}_{\mu\nu} , \quad \phi \to \phi , \quad \hat{\lambda} \to \lambda = \hat{X}^{-1} \hat{\lambda} ,
\]

the action (3.16) transforms to (3.15), albeit at the free theoretical level (i.e. without the matter action \( S^{(m)} \)), if one sets \( w = -\frac{3}{2} \) (specific to the singular BD theory) [24].

Now, a generalization of the CM action, or equivalents thereof, in presence of torsion, crucially requires us to ponder on the following:

1. Non-minimal metric-torsion couplings with scalar fields are seemingly more favourable than the usual minimal couplings which lead to a well-known uniqueness problem while defining equivalent Lagrangians by integrating out boundary terms [107][167][169]. Such a problem however arises only when the torsion has its trace \( T_\mu \neq 0 \) and the scalar fields are dynamical, unlike the mimetic field \( \phi \) in the basic CM theory. So, apparently there is no concern in generalizing this theory by coupling the non-dynamic \( \phi \) minimally with curvature and torsion. Nevertheless, provisions for a dynamical \( \phi \) is often desired in order to have extended mimetic formulations by incorporating higher derivative terms, e.g. \((\Box \phi)^2\), in any of the equivalent actions (3.9) and (3.15) describing a minimal gravitational coupling with \( \phi \) [22][29]. In such a situation, the generalizations of these actions in presence of a generic torsion (with \( T_\mu \neq 0 \)) are not straightforward, as the uniqueness problem becomes imminent. A better option is therefore to generalize the explicitly non-minimal (or, Jordan frame) action \( \hat{S} \) given by Eq. (3.16).

2. The Jordan frame action (3.16), being defined in the fiducial space with metric \( \hat{g}_{\mu\nu} \), its generalization amounts to that in presence of the fiducial torsion \( \hat{T}_\alpha^{\mu\nu} \). Appropriate conformal and Cartan transformations may then lead to the Einstein frame action in the physical space, i.e. the equivalent action generalizing (3.15), and hence (3.9). Note also that the action (3.16) does not include any matter Lagrangian \( L^{(m)} \). In fact, a fiducial space action is anyhow not ideal for describing matter fields, since the
latter usually couple to the physical metric $g_{\mu\nu}$ (and torsion $T^a_{\mu\nu}$, if existent). Therefore, while taking matter into consideration it is preferable to first generalize the Jordan frame action (3.10), then get to the equivalent action in the physical space, and finally incorporate the $\mathcal{L}^{(m)}$ term therein.

3. The matter Lagrangian $\mathcal{L}^{(m)}$ could be due to a host of external fields — scalars, pseudoscalars, and fields with different spin (e.g. fermions, vector bosons, etc.), some of which may induce the various torsion modes. However, the mimetic theory is not expected to get affected qualitatively if torsion exists only in presence of certain external matter fields. What would be really intriguing is an appropriate generalization of the CM action such that the mimetic field $\phi$ could manifest itself geometrically, by acting as a potential source of a torsion mode.

4. The torsional generalization would have its true significance only if it leads to a dust-like fluid component, mimicking cold dark matter, in the standard cosmological setup. By ‘dust-like’ we mean the fluid with not necessarily a vanishing pressure, but a vanishing acceleration $a_{\mu}$, so that the fluid velocity $u_{\mu}$ is tangential to the time-like geodesics [61]. Now, in referring to geodesics in space-times with torsion, one has to be careful since the so-called affine geodesics (or the auto-parallel curves), that transport their tangent vectors parallely along themselves, do not in general coincide with the (Riemannian) metric geodesics which extremize the space-time interval $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ along themselves [103,107,111,112,157]. Therefore, in presence of torsion it is necessary to see whether the mimetic fluid velocity $u_{\mu}$ is tangential to the time-like auto-parallels, and if so, then in what circumstances. In other words, under what condition (if any) the effective fluid acceleration $\ddot{a}_{\mu} = u^\alpha \tilde{\nabla}_\alpha u_{\mu}$ vanishes, thereby confirming the auto-parallel equation? As we show in the Appendix, one can indeed have $\ddot{a}_{\mu} = 0$, provided the torsion trace $T_{\mu} = \partial_{\mu} \phi$, where $\phi$ is the mimetic field. The latter therefore, must act as the source of the torsion mode $T_{\mu}$ in order that the dust-like character of the mimetic fluid is retained in space-times with torsion.

Taking all these into consideration, let us resort to the following action:

$$\tilde{S} = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-\tilde{g}} \left[ \tilde{X} \sum_{n=0}^{4} \beta_n(\phi) P_n(\tilde{g}_{\mu\nu}, \tilde{T}^a_{\mu\nu}) - \left\{ \frac{\nu(\phi)}{\tilde{X}} \tilde{g}^{\mu\nu} \partial_\mu \tilde{X} \partial_\nu \tilde{X} + \eta(\phi) \Box \tilde{X} \right\} \right],$$

(3.18)

where $\Box \equiv \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu$, and $P_n$'s denote the various terms in the $U_4$ curvature scalar analogue $R$ [cf. Eq. (2.6)] (in the fiducial space of course):

$$P_0 = R(\tilde{g}) , \quad P_1 = \tilde{\nabla}_\mu \left( \tilde{g}^{\mu\nu} \tilde{T}_\nu \right) , \quad P_2 = \tilde{g}^{\mu\nu} \tilde{T}_\mu \tilde{T}_\nu , \quad P_3 = \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu , \quad P_4 = \tilde{g}_{\mu\alpha} \tilde{g}^{\nu\beta} \tilde{g}^{\lambda\gamma} \tilde{Q}_{\nu\lambda}^{\mu} \tilde{Q}_{\alpha}^{\beta} \gamma .$$

(3.19)

This action (3.18) has a direct bearing on the actions in the literature that have long been suggested for non-minimal metric-torsion couplings to a dynamical scalar field $\Phi$ with a self-interacting potential $U(\Phi)$. Such actions are of the general form

$$S = \int d^4 x \sqrt{-g} \left[ \Phi^2 \sum_{n=0}^{4} \beta_n P_n - \frac{1}{2} \partial_\mu \Phi \partial^{\mu} \Phi - U(\Phi) \right],$$

(3.20)

with constant $\beta_n$'s, and $P_n$'s the same as in Eq. (3.19) except with the hats over the variables removed (see for e.g. Shapiro [107] and references therein). The above action (3.18) is actually a further generalization of (3.20) in the following respects:

8A specific combination of $\beta_n$'s, in presence of matter fields, can in fact give rise to cosmological scenarios that exhibit a weakly dynamic dark energy [107,109].
• There are two scalar fields involved (in the fiducial space though) — one is $\hat{X}$, which is a dynamical field (an analogue of $\Phi$), and the other one is $\phi$, which has no dynamical terms to begin with.  

• The couplings $\beta_n$ have in general been taken to be functions of $\phi$, so as to make it act as a source of the trace mode of torsion, via the corresponding equation of motion (see the next subsection).

• The self-terms for $\hat{X}$ have also been taken as general as possible, viz.

   (i) The term $\hat{X}^{-1}\hat{g}^{\mu\nu}\partial_\mu\hat{X}\partial_\nu\hat{X}$ is considered to have a variable coefficient $w(\phi)$, that replaces the constant BD parameter $w$ in Eq. (3.16).

   (ii) The term $\Box\hat{X}$ is considered to have a variable coefficient $\eta(\phi)$, which of course prevents it from being a surface term.

Note also that the non-minimal coupling of $\phi$ with $R(\hat{g})$ (or $P_0$) in the action (3.18) can always be removed by absorbing the coefficient $\beta_0(\phi)$ in a redefinition of the field $\hat{X}$. Appropriate redefinitions of the other coupling functions $\beta_n(\phi)$ (for $n = 1$ to $4$), as well as $w(\phi)$ and $\eta(\phi)$, can then leave Eq. (3.18) same as before, except with $\beta_0 = 1$. Finally, we can make out a ‘mimetic field’ interpretation of $\phi$ by including in (3.18) a Lagrange multiplier term that enforces $\hat{X}$ to be equal to $-\kappa^2\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, in the same way as in the old (torsion-free) action (3.16). The end result would be

$$\hat{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\hat{g}} \left[ R(\hat{g}_{\mu\nu}) + \sum_{n=1}^4 \beta_n(\phi) P_n(\hat{g}_{\mu\nu}, \hat{T}^{\alpha}_{\mu\nu}) \right] - \left\{ w(\phi) \hat{g}^{\mu\nu}\partial_\mu\hat{X}\partial_\nu\hat{X} + \eta(\phi) \Box\hat{X} \right\} - \lambda \left( \hat{X} + \kappa^2\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right).$$

(3.21)

This we propose here as the generalization of the action (3.16) in presence of torsion, i.e. our scalar-tensor equivalent mimetic-metric-torsion (MMT) action.

Under the set of conformal and Cartan transformations, viz.

$$\hat{g}_{\mu\nu} \rightarrow g_{\mu\nu} = \hat{X}\hat{g}_{\mu\nu}, \quad \phi \rightarrow \phi, \quad \hat{X} \rightarrow \lambda = \hat{X}^{-1}, \quad \hat{\lambda} \rightarrow \lambda = \hat{X}^{-1}, \quad [\text{cf. Eq. (3.17)}]$$

$$\hat{T}_\mu \rightarrow T_\mu = \hat{T}_\mu - \frac{3}{2}\partial_\mu(\ln \hat{X}), \quad \hat{A}_\mu \rightarrow A_\mu = \hat{A}_\mu, \quad \hat{Q}^{\alpha}_{\mu\nu} \rightarrow Q^{\alpha}_{\mu\nu} = \hat{Q}^{\alpha}_{\mu\nu}, \quad (3.22)$$

Eq. (3.21) transforms to (after eliminating surface terms):

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R(g_{\mu\nu}) + \lambda(X - 1) + \beta_1(\phi)\nabla_\mu T^\mu + \beta_2(\phi)\nabla_\mu \hat{T}^\mu + \beta_3(\phi)A_\mu A^\mu + \beta_4(\phi)Q_{\mu\nu\lambda}Q^{\mu\nu\lambda} \right.$$  
$$\left. + \left\{ \partial_\mu \left( \eta(\phi) - \frac{3}{2}\beta_1(\phi) \right) - \left[ \beta_1(\phi) - 3\beta_2(\phi) \right] \right\} \frac{\partial_\mu\hat{X}\hat{g}^{\mu\nu}\hat{X}}{X^2} \right],$$

(3.23)

where $X = -\kappa^2\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ as before [cf. Eq. (3.3)], and we have raised and lowered the tensor indices using the transformed metric $g_{\mu\nu}$.

Let us now make the following set of choices of the coupling functions:

$$\beta_1(\phi) = 3\beta_2(\phi) = -2\beta(\phi); \quad \eta(\phi) = 3\beta(\phi); \quad \omega(\phi) = -\frac{3}{2} [1 - \beta(\phi)];$$

$$\beta_3(\phi) = \frac{1}{24} \beta(\phi); \quad \beta_4(\phi) = \frac{1}{2} \beta(\phi).$$

(3.24)

(3.25)

Later on, of course, we shall consider a kinetic term of $\phi$, and constrain it such that it is interpreted as the mimetic field.
The first set (3.24) leaves no dynamical term for $\hat{X}$ in the above action (3.23). The second set (3.25), on the other hand, reduces (3.23) further to the particularly simplified form

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R(g_{\mu\nu}) + \lambda \cdot (X - 1) + \beta(\phi) \Theta(g_{\mu\nu}, T_\mu^\alpha) \right],$$

(3.26)

where we have recalled the expression (2.6) for the $U_4$ curvature scalar analogue $\tilde{R}$, and specifically used its torsion-dependent part:

$$\Theta(g_{\mu\nu}, T_\mu^\alpha) := \tilde{R}(g_{\mu\nu}, T_\mu^\alpha) - R(g_{\mu\nu})$$

$$= -2\nabla_{\mu} T^\mu - \frac{2}{3} T_\mu T^\mu + \frac{1}{24} A_\mu A^\mu + \frac{1}{2} Q_{\alpha\mu\nu} Q^{\alpha\mu\nu}.$$  

(3.27)

Eq. (3.26) is the final form of our proposed MMT action (without matter fields though) in which the mimetic field $\phi$ does not couple non-minimally with the Riemannian curvature scalar $R$, but does so with torsion. Note that in both the limits $\beta \to 0$ and $\beta \to 1$ the action (3.26) corresponds to Eq. (3.15), the torsion-free mimetic action (with the matter part $S^{(m)}$ ignored). In fact, such a correspondence is evident for any constant value of $\beta$, whence the last term in Eq. (3.26) becomes a surface term. The scalar-tensor action equivalent to (3.26), i.e. the version of Eq. (3.21) one gets after making the choices (3.24) and (3.25), is given by

$$\hat{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R(\hat{g}_{\mu\nu}) + \beta(\phi) \Theta(\hat{g}_{\mu\nu}, \hat{T}_\mu^\alpha) \right]$$

$$+ \frac{3[1 - \beta(\phi)]}{2X} \hat{g}^{\mu\nu} \partial_\mu \hat{X} \partial_\nu \hat{X} - 3\beta(\phi) \Box \hat{X}$$

$$- \lambda \cdot (\hat{X} + \kappa^2 \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi).$$

(3.28)

This of course corresponds to the torsion-free BD action (3.16), with the BD parameter $w = -\frac{3}{2}$, in the limit $\beta \to 0$. However, such a correspondence is not apparent in the limit $\beta \to 1$, whence $\hat{X}$ is rendered an auxiliary field, as its self terms in the second line of Eq. (3.28) get obliterated in entirety.$^{10}$ Using the auxiliary equation of motion, it can indeed be verified that the action (3.28), albeit on-shell, corresponds to (3.16) with $w = -\frac{3}{2}$, in the limit $\beta \to 1$. Such a correspondence, while remaining on-shell, happens for any constant value of $\beta$ as well. However, in that case $\hat{X}$ is not auxiliary, rather it is the torsion mode $\hat{T}_\mu$ (within $\Theta(\hat{g}_{\mu\nu}, \hat{T}_\mu^\alpha)$), which can be made to act as the requisite auxiliary field, by eliminating a surface term.

### 3.3 Field equations in presence of matter

In accord with our prior supposition that matter fields couple only to the physical metric and torsion, we shall consider the full MMT action to be that given by Eq. (3.26) augmented with the matter action $S^{(m)}$. However, we ignore for brevity any matter field sources for the torsion modes $A_\mu$ and $Q^{\alpha\mu\nu}$. Such modes would therefore vanish by virtue of the equations of motion $\delta S/\delta A_\mu = 0$ and $\delta S/\delta Q^{\alpha\mu\nu} = 0$. Consequently, the full MMT action is given, up to a surface term, as

$$S = S^{(m)} + \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R(g_{\mu\nu}) + \lambda \cdot (X - 1)$$

$$+ 2\beta(\phi) T_\mu \partial^\mu \phi - \frac{2}{3} \beta(\phi) T_\mu T^\mu \right];$$

(3.29)

where $\beta(\phi) \equiv d\beta/d\phi$. The torsion trace $T_\mu$, thus being ‘auxiliary’, is derived from $\phi$ via the field equation

$$\frac{\delta S}{\delta T_\mu} = 0 \quad \Rightarrow \quad T_\mu = \frac{3 \beta(\phi)}{2 \beta(\phi)} \partial_\mu \phi.$$  

(3.30)

$^{10}$Note that a constant times $\sqrt{-\hat{g}} \Box \hat{X}$ is merely a surface term.
Of course, the field $\phi$ has its dynamical degree of freedom taken care of by the equation of motion

$$\frac{\delta S}{\delta \lambda} = 0 \implies X \equiv -\kappa^2 \partial_\mu \phi \partial^\mu \phi = 1 : \text{the mimetic constraint.} \quad (3.31)$$

This leaves us with the following two field equations:

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \implies R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 \left[ T_{\mu\nu}^{(m)} - \left\{ W(\phi) - \lambda \right\} \partial_\mu \phi \partial_\nu \phi \right] - \frac{1}{2}g_{\mu\nu}W(\phi), \quad (3.32)$$

$$\frac{\delta S}{\delta \phi} = 0 \implies \nabla_\mu \left( \left\{ W(\phi) - \lambda \right\} \partial^\mu \phi \right) = -\frac{W_\phi(\phi)}{2\kappa^2}, \quad (3.33)$$

where we have defined

$$W(\phi) := -\frac{3\beta_3^2(\phi)}{2\beta(\phi)} \partial_\mu \phi \partial^\mu \phi = \frac{3\beta_3^2(\phi)}{2\kappa^2\beta(\phi)}; \quad [\text{by Eq. (3.31)}], \quad (3.34)$$

and $W_\phi \equiv dW/d\phi$. The trace of Eq. (3.32) determines

$$\lambda = -R + \kappa^2 T^{(m)} - W(\phi), \quad (3.35)$$

which when substituted back in Eqs. (3.32) and (3.33) yields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 \left[ T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(d)} \right], \quad (3.36)$$

$$\nabla_\mu \left( \left\{ \kappa^2 T^{(d)} + 2W(\phi) \right\} \partial^\mu \phi \right) = -\frac{W_\phi(\phi)}{2\kappa^2}, \quad (3.37)$$

where $T_{\mu\nu}^{(d)}$, the mimetic modification of the matter energy-momentum tensor $T_{\mu\nu}^{(m)}$, is now given by

$$T_{\mu\nu}^{(d)} = \left[ R + \kappa^2 T^{(m)} - 2W(\phi) \right] \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \frac{W(\phi)}{2\kappa^2}, \quad (3.38)$$

with trace $T^{(d)} = -[\kappa^{-2}R + T^{(m)}]$ remaining the same as that we had earlier in subsection 3.2.

It is easy to verify that the above field equations (3.36) and (3.37) would also follow from the action

$$S = S^{(m)} + \int d^4x \sqrt{-g} \left[ R(g_{\mu\nu}) + \lambda \cdot (X - 1) \right] - \frac{W(\phi)}{2}. \quad (3.39)$$

Therefore, the function $W(\phi)$ can naturally be interpreted as an effective mimetic potential, the same as that one considers in the torsion-free mimetic extensions [22][29]. Moreover, the expression (3.38) for $T_{\mu\nu}^{(d)}$ is of the form of a perfect fluid, viz.

$$T_{\mu\nu}^{(d)} = \left[ \rho^{(d)} + p^{(d)} \right] u_\mu u_\nu + g_{\mu\nu} p^{(d)}, \quad (3.40)$$

with $u_\mu \equiv \kappa \partial_\mu \phi$, and the energy density and pressure identified respectively as

$$\rho^{(d)} = \frac{R}{\kappa^2} + T^{(m)} - \frac{3W(\phi)}{2\kappa^2}, \quad p^{(d)} = -\frac{W(\phi)}{2\kappa^2}. \quad (3.41)$$

### 4 Cosmological aspects of the MMT formalism

Let us resort to the standard spatially flat FRW cosmological framework, viz. that of the four-dimensional space-time foliated to three-dimensional maximally symmetric hypersurfaces of constant cosmic time $t$. In order have such a foliation and hence the FRW metric structure preserved in presence of torsion, one requires the torsion modes to be constrained as follows [222]:

```
Only the component $T_0$ of $T_\mu$ can in general exist.

Only the component $A_0$ of $A_\mu$ can in general exist.

All the components of $Q^\alpha_{\mu\nu}$ have to vanish identically.

These constraints of course hold for our MMT formalism here, since we have $A_\mu = 0 = Q^\alpha_{\mu\nu}$, and Eq. (3.30) implies that among the components of $T_\mu$, only $T_0 \neq 0$ in general, once we have the mimetic field $\phi = \phi(t)$ in the standard cosmological setting. In fact, in the FRW space-time the mimetic constraint $X = 1$ [cf. Eq. (3.31)] implies $\dot{\phi} = d\phi/dt = 1/\kappa$. So, without loss of generality, we can identify $\kappa \phi \equiv t$, the cosmic time.

Now, since we are primarily interested in the late-time cosmological evolution, we take into consideration that leads to the $\Lambda$CDM solution expressed as $\Lambda(t)\equiv \dot{\phi}/\kappa$. Therefore, without loss of generality, we have the coupling that leads to the $\Lambda$CDM solution expressed as

$$
\beta(\phi) = \beta_0 \left( \frac{\phi}{\phi_0} \right)^2,
$$

where $\phi_0 = \frac{t_0}{\kappa}$, $\beta_0 = \beta|t_0 = \frac{\kappa^2 \Lambda t_0^2}{3}$. (4.8)
Such a $\phi^2$ coupling has actually been the trademark of metric-scalar-torsion theories in the literature [107, 167-169]. However, in our MMT formalism we crucially have the mimetic constraint (3.31), that takes care of the dynamics of $\phi$, and hence of the mode $T_\mu$, thus leaving us with a constant effective potential $W$.

Now, for $\Lambda$CDM the Friedmann equation (4.3) can be expressed in the form

$$H^2(a) = H_0^2 \frac{\Omega^{(A)}_0}{\Omega^{(A)}(a)},$$

(4.9)

where $H_0 = H|_{t_0}$ is the Hubble constant, and

$$\Omega^{(A)}(a) = \frac{\Lambda}{\rho(a)} = \frac{\Omega^{(A)}_0 a^3}{1 - \Omega^{(A)}_0 (1 - a^3)},$$

(4.10)

is the $\Lambda$-density parameter, with value $\Omega^{(A)}_0$ at the present epoch $t = t_0$ (or $a = 1$).

Solving the conservation equation (4.4) one then obtains

$$\phi(a) = \phi_0 \left[ \tanh^{-1} \sqrt{\Omega^{(A)}_0(a)} \right].$$

(4.11)

Conversely, since $\phi/\phi_0 = t/t_0$, we get the well-known $\Lambda$CDM cosmological solution

$$a(t) = \left[ \frac{1 - \Omega^{(A)}_0}{\Omega^{(A)}_0} \sinh^2 \left( \frac{t}{t_0} \tanh^{-1} \sqrt{\Omega^{(A)}_0} \right) \right]^{1/3}.$$  

(4.12)

It also follows from the above equations that

$$t_0 = \frac{2}{3H_0} \frac{\tanh^{-1} \sqrt{\Omega^{(A)}_0}}{\sqrt{\Omega^{(A)}_0}},$$

(4.13)

which is of course the standard expression for the present age of the $\Lambda$CDM universe.

### 4.2 Characteristics of the torsion parameter and the MMT coupling function

Let us have a closer look into the factors responsible for the $\Lambda$CDM evolution, viz.

(i) the trace mode $T_\mu$ of torsion, and

(ii) the mimetic coupling $\beta(\phi)$ which induces it.

In general, in metric-torsion theories it is often convenient to treat the norms of the torsion modes, viz.

$$T = \sqrt{-T_\mu T^\mu}, \quad A = \sqrt{-A_\mu A^\mu}, \quad \text{and} \quad Q = \sqrt{-Q_\mu\nu Q^{\mu\nu}},$$

(4.14)

as the torsion parameters that quantify the contribution of torsion in observable phenomena [107]. However, since in the present context $A_\mu = 0 = Q^\alpha_{\mu\nu}$, we have only one torsion parameter — the norm (or length) $T$ of the torsion trace vector $T_\mu$.

For a physical interpretation of $T$, let us refer back to the action (3.29) in subsection 3.3. If the coupling $\beta(\phi)$ had not been there (i.e. if we would have taken the limit $\beta(\phi) \to 1$), then the mode $T_\mu$ would not have been sourced by the mimetic field $\phi$, but may have been sourced by some matter field described by the matter action $S^{(m)}$. In such a case, we would have had the bare energy-momentum tensor due to torsion

$$T^{(T)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left( -\sqrt{-g} \frac{\delta}{3\kappa^2} T_\alpha T^\alpha \right) = \frac{2}{3\kappa^2} \left( T_\mu T_\nu - \frac{1}{2} g_{\mu\nu} T_\alpha T^\alpha \right),$$

(4.15)
and consequently in the FRW framework, the bare torsional energy density
\[ \rho^{(T)} = -g^{00} T^{(T)}_{00} = \frac{T^2}{3\kappa^2}. \] (4.16)

For a non-trivial coupling \( \beta(\phi) \) however, \( T_{\mu} \) is sourced by the mimetic field \( \phi \) via the equation of motion (3.30). The norm of \( T_{\mu} \) (i.e. the torsion parameter) is then
\[ T \equiv \sqrt{-g} T_{\mu} T^\mu = \frac{3|\beta(\phi)|}{2\kappa\beta(\phi)}. \] (4.17)

Working out again the energy-momentum tensor due to the terms involving torsion in the action (3.29), one finds that \( \rho^{(T)} \), which is of course no longer the bare torsion density, gets modulated by the factor \( \beta(\phi) \) to give the net contribution of torsion to the total energy density. In fact, as is evident from the field equations (3.32) and (3.33), the mimetic constraint (3.31) reduces this torsional contribution to \( \frac{1}{3} W(\phi) \), where \( W(\phi) \) is the effective potential defined by Eq. (3.34). Indeed, from the equations (3.34), (4.16) and (4.17) we get
\[ W(\phi) = \frac{2}{3} \beta(\phi) T^2 = 2\kappa^2 \beta(\phi) \rho^{(T)}, \] (4.18)

which explicitly shows that \( W(\phi) \) results from \( \rho^{(T)} \) modulated by \( \beta(\phi) \).

Now, the \( \Lambda \)CDM solution is a consequence of having \( W = 2\kappa^2 \Lambda \), a constant. It would therefore be interesting to see how this constant results, when both the constituting factors, viz. the torsion parameter \( T \) and the coupling \( \beta \), evolve with time (or the scale factor \( a \)). By Eq. (4.11), the given expressions (4.17) and (4.3), for \( T \) and \( \beta \) respectively, reduce to
\[ T(a) = T_o \left[ \frac{\tanh^{-1} \sqrt{\Omega_0^{(A)}}}{\tanh^{-1} \sqrt{\Omega_0^{(A)}}(a)} \right] ; \quad T_o = T|_{t_o} = \frac{3}{t_o}, \] (4.19)
\[ \beta(a) = \beta_o \left[ \frac{\tanh^{-1} \sqrt{\Omega_0^{(A)}}(a)}{\tanh^{-1} \sqrt{\Omega_0^{(A)}}} \right]^2 ; \quad \beta_o = \frac{\kappa^2 t_o}{3} = (H_o t_o)^2 \Omega_0^{(A)}. \] (4.20)

Using Eq. (4.13), and the best fit \( \Omega_0^{(A)} = 0.6889 \) obtained from the combined analysis of Planck TT,TE,EE+lowE, Lensing and Baryon Acoustic Oscillation (BAO) data for the base \( \Lambda \)CDM model [206], we estimate
\[ T_o = \frac{9 H_o \sqrt{\Omega_0^{(A)}}}{2 \tanh^{-1} \sqrt{\Omega_0^{(A)}}} = 3.1436 \text{ in units of } H_o, \] (4.21)
\[ \beta_o = \frac{4}{9} \left( \tanh^{-1} \sqrt{\Omega_0^{(A)}} \right)^2 = 0.6274. \] (4.22)

Moreover, using Eqs. (4.3), (4.7), (4.9) and (4.18) we have the \( \Lambda \)-density parameter
\[ \Omega_0^{(A)} = \frac{\Lambda}{\rho} = \left( \frac{T}{3H} \right)^2 \beta. \] (4.23)

Refer now to the left panel of Fig. 1, which shows how the rationalized parameters \( \frac{T_{\phi}}{T_o} \) and \( \frac{\beta_{\phi}}{\beta_o} \) evolve with time, or rather with the redshift \( z = a^{-1} - 1 \), from a certain epoch in a fairly distant past, say \( z = 4 \), to the extreme future limit \( z \to -1 \). As we see, with decreasing \( z \), \( \frac{T_{\phi}}{T_o} \) falls off steeply in the past, until beginning to slow down a bit near \( z \approx 1 \). It continues to decrease with \( z \) at slower and slower rates till the present epoch (indicated by the vertical line at \( z = 0 \)), when \( T = T_o \), and in the future (\( z < 0 \)), and finally begins to diminish quickly to zero asymptotically as \( z \to -1 \). On the other hand, as \( z \) decreases, \( \frac{\beta_{\phi}}{\beta_o} \) increases very slowly in the distant past and grows more and more rapidly afterwards from \( z \approx 1 \) to the present epoch \( z = 0 \), when \( \beta = \beta_o \), and in the future, till blowing up in the limit \( z \to -1 \).
Figure 1: [Left panel] Evolution of the rationalized parameters $\mathcal{T}/T_0$ and $\beta/\beta_0$, over a redshift range $z = 4$ (in the past) to $z = -1$ (extreme future). [Right panel] Evolution of $(\mathcal{T}/3H)^2$, $\beta$ and their resultant $\Omega^{(A)}$ over the same redshift range. All the plots are specific to $\Lambda$CDM, and the vertical line at $z = 0$ in either panel corresponds to the present epoch.

The right panel of Fig. 1 depicts the evolution of the factors in Eq. (4.23), viz. $(\mathcal{T}/3H)^2$ and $\beta$, and their resultant $\Omega^{(A)}$, with $z$ in the same range $(-1, 4]$. From the distant past to a fairly distant future, the evolution of $\Omega^{(A)}$ is almost similar to that of $\beta$, i.e. a slow increase with decreasing $z$ for $z \gtrsim 1$, followed by a progressively rapid growth as $z$ reduces further and further. Deep in the future ($z \lesssim -0.5$) however, the growth of $\Omega^{(A)}$ slows down considerably, unlike that of $\beta$, and in the asymptotic limit $\Omega^{(A)} \to 1$. This is of course what one expects in a $\Lambda$CDM universe, however the crucial point to note here is that the factors $\mathcal{T}$ and $\beta$, which are responsible for the $\Lambda$CDM solution, have their course of evolution changing drastically at late times. At early times ($z \gtrsim 4$), the torsion parameter $\mathcal{T}$ just scales as $4.5$ times the Hubble rate $H$, whereas both the mimetic-torsion coupling $\beta$ and the $\Lambda$-density parameter $\Omega^{(A)}$ are negligible. It therefore seems reasonable to limit one’s attention up to a redshift $z \simeq 4$ in the past, beyond which $\mathcal{T}$ and $\beta$ are not expected to have any further change in their course of evolution depicted in Fig. 1.

Nevertheless, for completeness, let us resort to Fig. 2 which shows the parametric variation with $\Omega^{(A)}$, over the entire physically acceptable domain $0 \leq \Omega^{(A)} \leq 1$ (not just over a limited range of redshift $z \in (-1, 4]$ as in Fig. 1). The left panel of Fig. 2 shows the plots for $\mathcal{T}T_0$ and $\beta/\beta_0$, whereas the right panel shows the plots for the factors $(\mathcal{T}/3H)^2$ and $\beta$, and their resultant $\Omega^{(A)}$. The vertical line in either panel corresponds to the present epoch at which $\Omega^{(A)} = \Omega^{(A)}_0 = 0.6889$ (Planck TT,TE,EE+lowE+Lensing+BAO best fit). As $\Omega^{(A)}$ increases from zero, $\mathcal{T}/T_0$ falls off sharply, but begins to slow down considerably near $\Omega^{(A)} \simeq 0.3$. It continues that way till at $\Omega^{(A)} \simeq 0.9$ its rate of decrease starts to become faster again. $\mathcal{T}/H$ however, decreases steadily with increasing $\Omega^{(A)}$ until diminishing to zero rapidly as $\Omega^{(A)} \to 1$. On the other hand, $\beta/\beta_0$ (and in fact $\beta$ as well) increases slowly as $\Omega^{(A)}$ increases from zero, until growing at progressively faster rates for $\Omega^{(A)} \gtrsim 0.3$. Thus, once again we see that $\mathcal{T}$ and $\beta$ have their course of evolution changing drastically in the range $0.3 \lesssim \Omega^{(A)} \lesssim 0.9$, which corresponds to the late-time evolution $(-0.4 \lesssim z \lesssim 0.7$, not stretching to very distant future though).
On the whole, the above Eqs. \((4.20) - (4.23)\), as well as the parametric evolution shown in Figs. \([1, 2]\) lead us to infer the following:

- Despite getting weakened further and further as time progresses, torsion has the strength of its coupling with the mimetic field increasing at such a rate that the evolution of the universe is effectively driven by a constant potential \(W \propto \beta(\phi)T^2\), in addition to the pressureless (baryonic plus mimetic) matter, which we have already had in the torsion-free scenario.

- In particular, a dominant dark sector at late times is perceivable mainly from the rapid growth of the coupling \(\beta(\phi)\) of the mimetic field \(\phi\) with torsion, although the latter decays to progressively smaller amounts with time (which could be a plausible reason behind the miniscule experimental evidence of direct effects of torsion searched extensively till date \([197-203]\)).

5 Higher derivative extension and bounds on MMT coupling strength

So far we have only had the cosmological evolution of a mimetic fluid with a non-zero pressure, resulting from an effective potential \(W(\phi)\). However, the presence of the potential makes with no characteristic difference of the fluid with pure dust, since the sound speed \(c_s\) of the mimetic matter perturbations remains vanishing. The reason is that the mimetic constraint \((3.31)\) prevents the scalar field \(\phi\) to exercise its usual dynamical degree of freedom to admit any wave-like or oscillatory solution that one would usually expect in the linear perturbation analysis. As such, there arises the well-known problem of defining quantum fluctuations to \(\phi\) from which the large scale structures of the universe can originate \([22, 23, 26, 27, 29]\). Also, for the universe constituted by the dust-like mimetic matter, it is not possible to suppress the over-abundance of small scale structures \([23, 27, 28]\). A possible way to ameliorate these issues is by the including higher derivative (HD) terms for \(\phi\) in the mimetic action. The most commonly used such term has been \((\Box \phi)^2\), which does not
qualitatively alter the cosmological solution one has in absence of it. This term could effectively be obtained from the action of another scalar field coupled to the mimetic field $\phi$ in a quite non-trivial way \cite{22}. Moreover, a deeper theoretical reasoning for the inclusion of such a term comes from an insight to the correspondence of the mimetic theory with the projectable Horava-Lifshitz gravity, the infrared limit of which has precisely the $(\Box \phi)^2$ term appearing in the corresponding action \cite{99}.

Let us see the cosmological consequences of incorporating the $(\Box \phi)^2$ term in our MMT action \cite{3.29}, or more conveniently, the equivalent action \cite{3.39}. The latter takes the form

$$S = S^{(m)} + \int d^4x \sqrt{-\gamma} \left[ R(g_{\mu\nu}) + \lambda \cdot (X - 1) - \frac{W(\phi)}{2} + \frac{\gamma}{2} (\Box \phi)^2 \right], \quad (5.1)$$

where $\gamma$ is a dimensionless coupling factor, which we presume here to be a constant\footnote{It is worth pointing out here that the $(\Box \phi)^2$ term may actually be an inherent part of the MMT formalism itself, since as shown explicitly in our subsequent papers \cite{214,215}, such a term could be induced by the axial mode $A_\mu$ of torsion, and couplings thereof with the mimetic field $\phi$.}. As is well-known, the $(\Box \phi)^2$ augmentation only leads to a rescaling of the effective potential by a dimensionless constant in the FRW space-time \cite{22}. So, the action $(5.1)$ could be recast in its earlier form $(3.39)$, after only a replacement:

$$W(\phi) \rightarrow \tilde{W}(\phi) = \left(1 - \frac{3\gamma}{2}\right)^{-1} W(\phi). \quad (5.2)$$

Consequently, the cosmological equations and the $\Lambda$CDM solution we have had earlier (in subsection 4.1) remain unaltered, except that the $\Lambda$-density parameter is now given by $\Omega^{(\Lambda)} = \tilde{\Lambda} / \rho_0$, where

$$\tilde{\Lambda} = \left(1 - \frac{3\gamma}{2}\right)^{-1} \Lambda, \quad (5.3)$$

is the new (rescaled) cosmological constant. From Eq. $(4.19)$ it is evident that the torsion parameter $T$ remains unchanged as well. However, the MMT coupling factor $\beta$, given by Eq. $(4.20)$, gets rescaled since its strength $\beta_0$ at the present epoch $t_0$ is determined by the old value of the cosmological constant, i.e. $\Lambda = \frac{W(\rho)}{2\kappa^2}$. In terms of the new value $\tilde{\Lambda}$,

$$\beta_0 = \left(1 - \frac{3\gamma}{2}\right) \frac{\kappa^2 \tilde{\Lambda} t_0^2}{3} = \left(1 - \frac{3\gamma}{2}\right) \left(\frac{H_0 t_0}{\kappa^2} \right)^2 \Omega^{(\Lambda)} = \left(1 - \frac{3\gamma}{2}\right) \left(\frac{H_0 t_0}{\kappa^2} \right)^2 \Omega^{(\Lambda)}, \quad (5.4)$$

where $H_0 t_0$ is as given by Eq. $(4.13)$, and we now have $\Omega^{(\Lambda)} = \Omega^{(\Lambda)}|_{t_0} = \frac{\tilde{\Lambda}}{\rho_0}$, with $\rho_0 = \rho|_{t_0} = \frac{3H_0^2}{\kappa^2}$ being the value of the critical density $\rho$ at the present epoch $t_o$.

Now, the striking outcome of the $(\Box \phi)^2$ augmentation is the non-zero sound speed of the mimetic matter perturbations \cite{22}

$$c_s = \sqrt{\frac{2\gamma}{2 - 3\gamma}}. \quad (5.5)$$

Therefore, using Eq. $(4.13)$ we get from Eq. $(5.4)$,

$$\beta_0 = \frac{4}{9(1 + 3c_s^2)} \left(\tanh^{-1} \sqrt{\Omega^{(\Lambda)}}\right)^2. \quad (5.6)$$

Demanding that the squared sound speed should be within its physical limits, viz. $0 \leq c_s^2 \leq 1$, we have $0 \leq \gamma \leq \frac{1}{2}$, and hence

$$\frac{1}{9} \left(\tanh^{-1} \sqrt{\Omega^{(\Lambda)}}\right)^2 \leq \beta_0 \leq \frac{4}{9} \left(\tanh^{-1} \sqrt{\Omega^{(\Lambda)}}\right)^2, \quad (5.7)$$

where the lower(upper) bounds are those that correspond to the upper(lower) bounds of $c_s^2$. Given the Planck TT,TE,EE+lowE+ Lensing+BAO best fit $\Omega^{(\Lambda)} = 0.6889$ \cite{206}, one has $\beta_0 \in [0.1568, 0.6274]$.

\footnote{Note that apart from the technical simplification, the constancy of the coefficient of the $(\Box \phi)^2$ term implies that latter does not disturb the scale invariance of the mimetic theory.}
6 Conclusions

We have thus generalized the basic formalism of the mimetic gravity theory [21] for the metric-compatible (U_4) geometries characterized by both curvature and torsion. Essentially, we have looked to extend the mimetic principle of isolating the conformal degree of freedom of gravity in the U_4 space-time, via the parametrization of not just the physical metric g_{μν} by a fiducial metric ̂g_{μν} and the mimetic scalar field φ, but the physical torsion T^α_{μν} by a fiducial torsion ̂T^α_{μν} and φ as well. Form a close inspection of (i) how the quantity ̂X = −κ^2g^{μν}∂_μφ∂_νφ and its derivatives transform under the conformal transformation of ̂g_{μν}, and (ii) the form of the associated Cartan transformation equation for ̂T^α_{μν}, we have been able to assert the latter and how it parametrizes T^α_{μν}. Of course, the foremost criterion for such an assertion has been to make sure that both the physical fields g_{μν} and T^α_{μν} are preserved under the conformal and Cartan transformations of ̂g_{μν} and ̂T^α_{μν} respectively. The Cartan transformation however, has been considered to be that of a specific class, which provides some technical simplifications discussed in section 2 of this paper.

While formulating the mimetic-metric-torsion (MMT) Lagrangian we have considered explicit contact couplings of the individual torsion terms with the mimetic field φ, following certain criteria which are enlisted in subsection 3.2. The outcome of such couplings is twofold: (i) they implicate T^α_{μν} ∝ ∂_μφ, meaning that the mimetic field φ manifests itself geometrically as the source of the torsion trace mode T_μ, and (ii) they give rise to an effective potential W(φ), so that the dust-like mimetic fluid can have a non-zero pressure. If, for a specific choice of W(φ), the pressure is negative and large enough, a dark universe picture could be perceived in the standard FRW cosmological framework, just as in the phenomenological extensions of the original CM model of mimetic gravity [22][29].

The dark universe interpretation however depends on whether in the first place the mimetic fluid retains its dust-like nature in space-times with torsion. So, there arises the question: can we still have zero mimetic fluid acceleration, i.e. the mimetic fluid velocity tangential to the time-like auto-parallel in presence of torsion? The answer is: yes, and as we have shown rigorously in the Appendix, the fluid acceleration indeed vanishes whenever the torsion trace T_μ ∝ ∂_μφ. One of the criteria mentioned above has therefore been to set up the effective Lagrangian in such a way that one gets T_μ ∝ ∂_μφ naturally as an equation of motion. This is actually what prompted us to consider the explicit MMT coupling terms in our proposed action (3.18) in subsection 3.2. Note also that such an action is a scalar-tensor action, involving non-minimal gravitational and MMT couplings with a Brans-Dicke scalar field ̂X, as well as the self terms for ̂X and a Lagrangian multiplier term that enforces the constraint ̂X = −κ^2g^{μν}∂_μφ∂_νφ. We have chosen this action following another major criterion which is motivated from the point of view of avoiding a well-known uniqueness problem with the usual minimal couplings of scalar fields with torsion. Thereafter, performing the conformal and Cartan transformations we have worked out the equivalent minimally coupled (Einstein frame) MMT action (3.23), and for a certain restrictive choice of the coupling functions, the rather simplified action (3.26). Augmenting the latter straightaway with the external matter action, we have then determined the corresponding field equations, which show that the mimetic fluid acquires an effective pressure p^{(d)} ∝ −W(φ). For simplicity however, we have ignored any matter field source for the other torsion modes, viz. the pseudo-trace vector A_μ and the (pseudo-)tracefree tensor Q^α_{μν}.

A detailed study of the cosmological aspects of our MMT formalism in section 4 has revealed a plausible ΛCDM evolution in the FRW framework, if the MMT coupling function is set to be β(φ) ∼ φ^2. In fact, for such a setting the potential W is a constant, which could be identified as an effective cosmological constant Λ, modulo a factor (2κ)^2. The mimetic field φ, on the other hand, has its usual identification with the cosmic time t in the FRW space-time. Using the solution for φ (or t) in terms of the scale factor a for ΛCDM, we have made a careful examination of how the induced torsion field, quantified by the norm T of the trace vector T^μ, and the quadratic coupling β(φ) ∼ φ^2, evolve with time (or, rather with the decreasing redshift z = a^{−1} − 1). It is found that the torsion parameter T is weakened more and more, whereas the coupling β gets enhanced increasingly rapidly as time progresses. The combined effect of the two is of course the constant potential W. Moreover, both T and β have their course of evolution changing drastically at late times. Therefore, despite a significant suppression in the amount of torsion, we have a late time dominance of the dark sector.
constituents, viz. the cosmological constant $\Lambda$ (or $W$) and the dust-like mimetic matter which acts as the cold dark matter. The values of $T$ and $\beta$ at the present epoch, viz. the quantities $T_0$ and $\beta_0$, have been estimated using the Planck 2018 results, combined with weak lensing and BAO data [206]. In particular, the Planck TT,TE,EE+lowE+Lensing+BAO best fit value of the $\Lambda$-density parameter at the present epoch, $\Omega_0^{(\Lambda)}$, has been used to get the plots showing the evolution of $T$ and $\beta$ in Figs. 1 and 2 in section 4. This is reasonable since the 1$\sigma$ errors on the estimated parameters are very low for such a combination of refined data.

The coupling $\beta(\phi) \sim \phi^2$, which leads to the $\Lambda$CDM solution, is reminiscent of that in the metric-scalar-torsion (MST) theories which involve dynamical scalar fields coupled non-minimally to curvature and torsion [107,167–169]. In the mimetic theory however, the scalar field $\phi$ cannot exercise its dynamical degree of freedom because of the mimetic constraint. Nevertheless, it is often desired to have an implicitly dynamical $\phi$ in an extended mimetic gravity formulation that incorporates higher derivative term(s), e.g. $(\Box\phi)^2$, in the Lagrangian [22]. In section 5 we have considered such a $(\Box\phi)^2$ term to go with the coupling $\beta(\phi) \sim \phi^2$, in our MMT formalism, which is therefore liable to correspond to the MST theories, if the mimetic constraint is enforced in the latter using a Lagrange multiplier.

Now, the significance of the $(\Box\phi)^2$ augmentation of the mimetic Lagrangian is in making the sound speed $c_s$ of the mimetic matter perturbations non-vanishing [22]. This is essential for defining the quantum fluctuations to the mimetic field $\phi$ in the usual way, so that in the standard cosmological setting $\phi$ can provide the seeds of the observed large scale structure of the universe [22,29]. The background (unperturbed) cosmological solutions are however not altered qualitatively by the $(\Box\phi)^2$ term, since the latter only leads to a rescaling of the effective potential of the mimetic field by a constant factor, that depends on $c_s^2$, in the FRW space-time. Accordingly, the $\Lambda$CDM solution, that has resulted from the $\beta(\phi) \sim \phi^2$ coupling in our MMT formalism, remains as the background solution, albeit with a $c_s^2$-dependent rescaling of the cosmological constant $\Lambda$. Taking the observed density parameter $\Omega_0^{(\Lambda)}$ as that due to the rescaled $\Lambda$, we find the torsion parameter $T$ unchanged, but the MMT coupling strength at the present epoch, $\beta_0$, rescaled by a factor $(1 + 3c_s^2)^{-1}$. The upper and lower bounds on $\beta_0$ are then determined from the demand that $c_s^2$ should be within its physical limits 0 and 1 respectively.

On the whole, no matter how self-sufficient our MMT formalism may seem, there remains some open questions, such as the following:

- What about the Ostrogradsky ghost and gradient instabilities that usually arise in the mimetic gravity theory whenever one incorporates the $(\Box\phi)^2$ and other higher derivative terms in order to have a propagating scalar perturbation mode [66–69]? Can we resolve the problem of having such instabilities via explicit non-minimal couplings of some primordial field with curvature, torsion and the mimetic field $\phi$, or by assuming one of the torsion modes to be propagating and coupled non-minimally to $\phi$?

- Would it anyhow worth extending our MMT formalism by considering the torsion modes $A_{\mu}$ or(and) $Q^{\alpha}_{\mu\nu}$ to be existent? Under what conditions such extensions would lead to a $\Lambda$CDM cosmological dark sector, or a small deviation from that?

- How would the torsional reparametrization and consequently the equivalent Lagrangian formulation be affected if we resort to the most general form of the Cartan transformation equations? What is actually the status of the Cartan gauge invariance when the MMT formalism is compared to the (dynamical) scalar-torsion theories in the literature [107,157]?

- What about the first order (Palatini) reformulation of mimetic theory, for a non-minimal coupling of the gauge field strength with the mimetic field that would inevitably involve torsion (or, rather the contorsion part of the spin connection)?

Some of these are being addressed in a few of our subsequent works [214,215], which we hope to report soon.
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Appendix: Mimetic fluid acceleration in space-times with torsion

Let us resort to the general action, used to describe a dusty dark energy (DDE) fluid, with non-vanishing pressure [61]:

\[ S = \int d^4x \sqrt{-g} \left[ R(g_{\mu\nu}) + \lambda \cdot (X - V(\phi)) \right] / (2\kappa^2) + P(\phi, X) \]  \hspace{1cm} (6.1)

where \( \phi \) is a dimensionless scalar field with a potential \( V(\phi) \), \( X = -\kappa^2 \partial_\mu \phi \partial^\mu \phi \), \( \lambda \) is a scalar Lagrange multiplier field (of mass dimension = 2) and \( P(\phi, X) \) is a non-canonical kinetic term for \( \phi \) (similar to what one finds in the k-essence theories [217–221]). For \( V(\phi) = 1 \) and \( P(\phi, X) = 0 \), one recovers the action (3.15) for mimetic gravity (in absence of external matter fields though).

While the Lagrange multiplier \( \lambda \) in the action (6.1) implements the constraint

\[ X \equiv -\kappa^2 \partial_\mu \phi \partial^\mu \phi = V(\phi) \] \hspace{1cm} (6.2)

the conserved total energy-momentum tensor, due to the fields \( \phi \) and \( \lambda \), is

\[ T_{\mu\nu} = \left( 2\kappa^2 \partial P / \partial X + \lambda \right) \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu}P \] \hspace{1cm} (6.3)

This is of the form of a perfect fluid, viz.

\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu + g_{\mu\nu}p \] \hspace{1cm} (6.4)

once we identify, in analogy with the k-essence models [217–221], the fluid velocity as

\[ u_\mu = \frac{\kappa \partial_\mu \phi}{\sqrt{X}} \] \hspace{1cm} [so that \( u_\mu u^\mu = -1 \)] \hspace{1cm} (6.5)

and the fluid energy density and pressure respectively as

\[ \rho = 2X \partial P / \partial X - P + \frac{\lambda X}{\kappa^2} \] \hspace{1cm} and \hspace{1cm} \( p = P \) \hspace{1cm} (6.6)

The fluid acceleration is then

\[ a_\mu := u^\nu \nabla_\nu u_\mu = \kappa^2 \left( \nabla_\mu \partial_\nu \phi - \frac{\partial_\mu \phi \partial_\nu X}{2X} \right) \frac{\partial^\nu \phi}{X} = 0 \] \hspace{1cm} (6.7)

by virtue of the constraint (6.2). So, the fluid velocity is tangential to the time-like geodesics in the Riemannian space-time, i.e. the fluid emulates the dust, even with a non-zero pressure. Hence the name dusty fluid.

Now, the geodesics of actual geometrical significance are the affine geodesics (or the auto-parallel curves), which parallel transport their tangent vectors along themselves. Strictly speaking therefore, the dusty fluid is the one for which the fluid velocity is tangential to the time-like auto-parallel. This is ensured by the vanishing of the fluid acceleration \( a_\mu \) [cf. Eq. (6.7)] in the Riemannian space-time, in which the auto-parallels coincide with the metric geodesics that extremize the space-time interval \( ds^2 = g_{\mu\nu}dx^\mu dx^\nu \) between two neighbouring points. In non-Riemannian space-times however, such a coincidence is not guaranteed [103, 107, 111, 112, 157, 222]. Hence, the auto-parallel equation that should correspond to the vanishing of the dusty fluid acceleration, is not ensured because the latter in general differs from its Riemannian expression (6.7). To be more specific, consider the situation in Riemann-Cartan \((U_4)\) space-time that involves a non-zero
torsion $T^\alpha_{\mu\nu}$. Even if we resort to the minimal coupling prescription, i.e. simply replace the Riemannian covariant derivatives $\nabla_\mu$ everywhere by the Riemann-Cartan ones, $\tilde{\nabla}_\mu$, the Riemannian expressions for the velocity and acceleration, $u_\mu$ and $a_\mu$, would in general get altered. In fact, the generalization of $u_\mu$ gives back its Riemannian expression (6.5):

$$u_\mu \rightarrow \tilde{u}_\mu = \frac{\kappa \tilde{\nabla}_\mu \phi}{\sqrt{X}} = \frac{\kappa \partial_\mu \phi}{\sqrt{X}} = u_\mu ,$$

but $a_\mu$ generalizes to a different form:

$$a_\mu \rightarrow \tilde{a}_\mu = \tilde{u}_\nu \tilde{\nabla}_\nu \tilde{u}_\mu = \kappa^2 \left( \tilde{\nabla}_\mu \partial_\nu \phi - \frac{\partial_\mu \phi \partial_\nu X}{2X} - \left[ \tilde{\nabla}_\mu, \tilde{\nabla}_\nu \right] \phi \right) \frac{\partial_\nu \phi}{X} .$$

Comparing this with Eq. (6.7) we see that $a_\mu = 0$ does not necessarily mean $\tilde{a}_\mu = 0$, because of the presence of certain extra terms. If these extra terms vanish as well, then only the dusty fluid will have its properties retained in the $U_4$ space-time. Let us figure out the criterion for the extra terms to vanish:

From the definition of $U_4$ covariant derivative of the gradient of a scalar

$$\tilde{\nabla}_\mu \partial_\nu \phi := \nabla_\mu \partial_\nu \phi - K^\alpha_{\nu\mu} \partial_\alpha \phi ,$$

where $K^\alpha_{\nu\mu} = \frac{1}{2} \left( T^\alpha_{\nu\mu} - T^\alpha_{\nu\mu} - T^\alpha_{\mu\nu} \right)$ is the contorsion tensor, and the well-known commutation relation

$$\left[ \tilde{\nabla}_\mu, \tilde{\nabla}_\nu \right] \phi = T^\alpha_{\mu\nu} \partial_\alpha \phi ,$$

it is easy to show that Eq. (6.9) reduces to

$$\tilde{a}_\mu = a_\mu - \frac{\kappa^2 T^\alpha_{\mu\nu} \partial_\alpha \phi \partial_\nu \phi}{X} ,$$

with $a_\mu$ having the expression (6.7). In terms of the irreducible torsion modes, given by Eq. (2.4), we have

$$\tilde{a}_\mu = a_\mu + \frac{1}{3} \left( T_\mu + \frac{\kappa^2 \partial_\mu \phi \partial_\nu \phi T_\nu}{X} \right) - Q^\alpha_{\mu\nu} \partial^\alpha \phi \partial^\nu \phi .$$

The last term vanishes identically by the properties $Q^\alpha_{\mu\nu} = 0 = \epsilon^{\rho\mu\nu\alpha} Q_{\rho\mu\nu}$. Moreover, since $X = -\kappa^2 \partial_\mu \phi \partial_\mu \phi$, we infer that the second term would vanish as well (leaving $\tilde{a}_\mu = a_\mu = 0$), if $T_\mu \propto \partial_\mu \phi$. Note also that the relationship (6.13) between $\tilde{a}_\mu$ and $a_\mu$ is irrespective of the enforcement of the constraint (6.2), which has of course been responsible for the original result $a_\mu = 0$ [cf. Eq. (6.7)].

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