THE PROPERTIES OF POSITIVE SOLUTIONS TO SEMILINEAR EQUATIONS INVOLVING THE FRACTIONAL LAPLACIAN

Rongrong Yang and Zhongxue Lü*
School of Mathematics and Statistics, Jiangsu Normal University
Xuzhou 221116, China

(Communicated by Wenxiong Chen)

Abstract. Let \( \Omega \) be either a unit ball or a half space. Consider the following Dirichlet problem involving the fractional Laplacian
\[
\begin{cases}
(\mathcal{L}_\alpha u)(x) = f(u), & \text{in } \Omega, \\
u = 0, & \text{in } \Omega^c,
\end{cases}
\]
where \( \alpha \) is any real number between 0 and 2. Under some conditions on \( f \), we study the equivalent integral equation
\[
u(x) = \int_\Omega G(x, y)f(u(y))dy,
\]
here \( G(x, y) \) is the Green's function associated with the fractional Laplacian in the domain \( \Omega \). We apply the method of moving planes in integral forms to investigate the radial symmetry, monotonicity and regularity for positive solutions in the unit ball. Liouville type theorems-non-existence of positive solutions in the half space are also deduced.

1. Introduction. In this paper, we analyze the behavior of solutions to the Dirichlet problem for semilinear equation (1) which involves the fractional Laplacian, here \( \Omega = B_1 \) or \( \mathbb{R}_+^n \). And \( B_1 = B_1(0) = \{ x \in \mathbb{R}^n : |x| < 1 \} \) is the unit ball in \( \mathbb{R}^n \), \( \mathbb{R}_+^n = \{ x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n > 0 \} \) is the upper half Euclidean space. \( B_1^c \) is the complement of \( B_1 \), \( \mathbb{R}_+^n = \{ x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n \leq 0 \} \) is the the complement of \( \mathbb{R}_+^n \).

For the various applications of the fractional Laplacian in anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics, and relativistic quantum mechanics of stars, probability and finance, we refer the readers to see [1, 2, 4, 6, 7, 13, 18, 20].

The fractional Laplacian in \( \mathbb{R}^n \) is a nonlocal differential operator, it has three equivalent definitions.

The first definition takes the form
\[
(\mathcal{L}_\alpha u)(x) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x-z|^{n+\alpha}} dz,
\]

2000 Mathematics Subject Classification. Primary: 35J60, 35B06, 35B09, 35B53, 45G10, 45M20.

Key words and phrases. The fractional Laplacian, semi-linear equations, symmetry, monotonicity, regularity, Liouville theorem.

The second author is supported by NSFC(No.11271166), NSF of Jiangsu Province(No. BK2010172), sponsored by Qing Lan Project.

* Corresponding author.
where $\alpha$ is any real number between 0 and 2 and $P.V.$ stands for the Cauchy principle value.

In addition, the definition of fractional Laplacian in a domain $\Omega \subset \mathbb{R}^n$ is to restrict the integration to the domain:

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} P.V. \int_{\Omega} \frac{u(x) - u(z)}{|x - z|^{n+\alpha}} dz,$$

known as the regional fractional Laplacian [15].

The second definition is using the extension method introduced by Caffarelli and Silvestre in [5]:

$$\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u = -C \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial U}{\partial y}, \\
\text{div} (y^{1-\alpha} \nabla U) = 0, \\
U(x, 0) = u(x).
\end{cases} \tag{4}$$

The third definition is in $\mathcal{S}$, the Schwartz space of rapidly decreasing $C^\infty$ functions in $\mathbb{R}^n$, by the Fourier transform

$$(\widehat{-\Delta}^\frac{\alpha}{2} u)(\xi) = |\xi|^{\alpha} \hat{u}(\xi), \tag{5}$$

where $\hat{u}$ is the Fourier transform of $u$. And it can be extended to a wider space of distributions

$$\mathcal{L}_2^\alpha = \{ u : \mathbb{R}^n \to \mathbb{R} | \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \}.$$ 

Then in this space, we define $(-\Delta)^{\frac{\alpha}{2}} u$ as a distribution by

$$< (-\Delta)^{\frac{\alpha}{2}} u(x), \phi > = \int_{\mathbb{R}^n} u(x) (-\Delta)^{\frac{\alpha}{2}} \phi(x) dx, \ \forall \ \phi \in C^\infty_0(\Omega).$$

For any $f \in L^1_{loc}(\Omega)$, we say that $u \in \mathcal{L}_2^\alpha$ is a solution of

$$(-\Delta)^{\frac{\alpha}{2}} u = f(x), \ x \in \Omega$$

if and only if

$$\int_{\mathbb{R}^n} u(-\Delta)^{\frac{\alpha}{2}} \phi dx = \int_{\mathbb{R}^n} f(x) \phi(x) dx, \ \forall \ \phi \in C^\infty_0(\Omega). \tag{6}$$

In this paper, we consider the distributional solutions in sense of (6).

For the properties of solutions for equations involving the fractional Laplacian in $\mathbb{R}^n$, we refer the readers to see [12, 14, 17, 21].

In [16], the solutions of (1) when $\Omega = B_1$ under the condition $f(u) \in L^q(B_1)$ can be expressed as

$$u(x) = \int_{B_1} G_1(x, y) f(u(y)) dy. \tag{7}$$

Here Green’s function $G_1(x, y)$ satisfies:

$$\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} G_1(x, y) = \delta(x - y), \quad \text{in } B_1, \\
G_1(x, y) = 0, \quad \text{in } B_1^c.
\end{cases} \tag{8}$$

And it is in the form:

$$G_1(x, y) = \frac{A_{n,\alpha}}{s^{\frac{n+\alpha}{2}}} \left[ 1 - \frac{B_{n,\alpha}}{(t + s)^{\frac{n+\alpha}{2}}} \int_0^t \left( \frac{s - tb)^{\frac{n+\alpha}{2}}}{b^\alpha (1 + b)} \right) db \right],$$

where $s = |x - y|^2$ and $t = (1 - |x|^2)(1 - |y|^2)$.
In [8], Chen, Fang and Yang considered the Dirichlet problem involving the fractional Laplacian (1), where \( \Omega \) is either a unit ball or a half space \( \mathbb{R}_+^n \). Instead of using the conventional extension method of Cafarelli and Silvestre [5], the authors have employed a new and direct approach by studying the equivalent integral equation
\[
  u(x) = \int_\Omega G(x, y) f(u(y)) dy,
\]
where \( G(x, y) \) is the Green’s function associated with the fractional Laplacian in the domain \( \Omega \). Applying the method of moving planes in integral forms, they established the symmetry, monotonicity of the positive solutions to the Dirichlet problem for semilinear equation
\[
\begin{cases}
  (-\Delta)^\frac{\alpha}{2} u = f(u), & \text{in } B_1, \\
  u = 0, & \text{in } B_1^c
\end{cases}
\]
under the conditions
\[(1) \quad f : [0, \infty) \to [0, \infty) \text{ is increasing, } f(0) = 0, \quad (10)\]
and
\[
(2) \quad f'(\cdot) \text{ is monotonic and } f'(u) \in L^\frac{\alpha}{\beta}(B_1) \text{ or } \quad (2') \quad |f'(u)| \leq C_1|u|^{\beta_1} + C_2|u|^{\beta_2} + C_3
\]
where \( C_1, C_2, C_3 \) and \( \beta_1 \) can be any nonnegative constants, while \( \beta_2 \) is some nonpositive constant. If \( C_1 > 0 \), we require \(|u|^{\beta_1} \in L^\frac{\alpha}{\beta}(B_1)\), and if \( C_2 > 0 \), we need \(|u|^{\beta_2} \in L^\frac{\alpha}{\beta}(B_1)\).

Under the conditions
\[
|\frac{f(u)}{u}| \leq C_1 + C_2|u|^{\beta},
\]
for some \( \beta > \frac{\alpha}{n-\alpha} \) and
\[
u(x) \in L^\frac{\alpha}{\beta}(B_1),
\]
they also proved the regularity of the positive solutions to the Dirichlet problem for semilinear equation (9) or integral equation (7).

In [9], Chen, Fang and Yang established the equivalence between the Dirichlet problem involving the fractional Laplacian (1) when \( \Omega \) is a half space \( \mathbb{R}_+^n \) and \( f(u) = u^p \),
\[
\begin{cases}
  (-\Delta)^\frac{\alpha}{2} u = u^p, & \text{in } \mathbb{R}_+^n, \\
  u = 0, & \text{in } \mathbb{R}_+^n
\end{cases}
\]
and integral equation
\[
  u(x) = \int_{\mathbb{R}_+^n} G_\infty(x, y) u^p(y) dy.
\]
Where
\[
  G_\infty(x, y) = \frac{A_{n, \alpha}}{s^{\frac{n-\alpha}{2}}} \left[ 1 - \frac{B_{n, \alpha}}{(t + s)^{\frac{n-\alpha}{2}}} \int_0^t \frac{(s - tb)^{\frac{n-2}{2}}}{b^2(1 + b)} db \right],
\]
here \( s = |x - y|^2 \) and \( t = 4x_n y_n \). And the non-existence of the solution of (14) has also been proved.

In this paper, for (0.1), when \( \Omega = B_1 \), we consider some other conditions different with (12) and (13): \( \frac{f(u)}{u} \) is decreasing and \( |\frac{f(u)}{u}| \leq C_1 + C_2|u|^{\beta}, \) and \(|u|^{\beta} \in L^\frac{\alpha}{\beta}(B_1)\), where \( C_1 \) and \( C_2 \) are nonnegative constants, \( \beta \) is any real number. When \( \Omega = \mathbb{R}_+^n \),
relative to (14) and (15), we consider \( f(u) \) for a more general case to get the Liouville theorem and the equivalence between

\[
\begin{cases}
(-\Delta)^\frac{\beta}{2} u(x) = f(u(x)), & \text{in } \mathbb{R}^n_+ \\
u(x) = 0, & \text{in } \mathbb{R}^n_-
\end{cases}
\] (16)

and

\[u(x) = \int_{\mathbb{R}^n_+} G_\infty(x,y)f(u(y))dy.\] (17)

Our main results as follows.

**Theorem 1.1.** Assume that \( f \) is increasing and \( f(0) = 0 \), \( \frac{f(u)}{u} \) is decreasing and \(|\frac{f(u)}{u}| \leq C_1 + C_2 u^\beta, \) \( C_1 \) and \( C_2 \) are nonnegative constants, \( \beta \) is any real number . If \(|u|^\beta \in L^{\frac{\alpha}{\beta}}(B_1)\), then every positive solution of integral equation (7) is radially symmetric about the origin and strictly decreasing in the radial direction.

**Corollary 1.** Under the conditions of Theorem 1, if \( u \) is a positive solution of (9) with \( f(u) \in L^q(B_1) \), then it is radially symmetric about the origin and strictly decreasing in the radial direction.

**Theorem 1.2.** Let \( u(x) \) be a positive solution of (7) or of (9). Assume that

\[|\frac{f(u)}{u}| \leq C_1 + C_2 u^\beta \text{ for } -1 < \beta < \frac{\alpha}{\alpha - n}(n \geq 4)\] (18)

and \( u^\beta \in L^{\frac{\alpha}{\beta}}(B_1), u \in L^{\frac{n\alpha}{n-\alpha}}(B_1) \). Then \( u \) is uniformly bounded in \( B_1 \).

**Remark 1.** When \( \beta > \frac{\alpha}{n - \alpha} \), we know that \( u \in L^{\frac{n\alpha}{n-\alpha}}(B_1) \) and \( u^\beta \in L^{\frac{\alpha}{\beta}}(B_1) \) are equivalent. In our paper, we consider some negative \( \beta \). \( u \in L^{\frac{n\alpha}{n-\alpha}}(B_1) \) and \( u^\beta \in L^{\frac{\alpha}{\beta}}(B_1) \) can not be equivalent as we wish. So we take them as above conditions.

Under the conditions \( u \in L^{\frac{n\alpha}{n-\alpha}}(B_1) \) and \( u^\beta \in L^{\frac{\alpha}{\beta}}(B_1) \), we find that the uniform boundedness of the solution of (7) or of (9) without regularity lifting, at the same time, we only need \(|\frac{f(u)}{u}| \leq C_1 + C_2 u^\beta, \) \( \beta < \frac{\alpha}{\alpha - n} \) and \( n \geq 3 \).

**Remark 2.** Under conditions (12) and (13), we can get the same result with another method by estimating \( \|f(u)\|_{L^{\frac{n\alpha}{\alpha - n}}(B_1)} \).

**Theorem 1.3.** Assume \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function. If \( u \) is a locally bounded positive solution of (16), and there exists a sequence \( \{x^k\} \in \mathbb{R}_+^n \) such that

\[\frac{u(x^k)}{(\alpha_n)^\frac{\beta}{2}} \to 0.\] (19)

Then \( u \) is also a solution of (17) and vice versa.

**Theorem 1.4.** Assume \( f \) is increasing, \( f(0) = 0 \) and \( f' \in L^\frac{\alpha}{\alpha - n}(\mathbb{R}_+^n) \),

\[f \in L^{\frac{n\alpha}{n + \alpha - n}}(\mathbb{R}_+^n), \frac{n}{n - \alpha} < q < \infty.\] (20)

If \( u \) is a nonnegative solution of (17). Then \( u \equiv 0 \).

2. Symmetry of solutions in the ball. Similar to [8], in this section, we obtain the radial symmetry and monotonicity of positive solutions for integral equation (7) by using the method of moving planes in integral forms.
2.1. Properties of the Green’s function and $f(u)/u$. Let $\lambda \in (-1, 0)$ be a real number and $T_\lambda = \{ x \in \mathbb{R}^n \mid x_1 = \lambda \}$.

$$\Sigma_\lambda = \{ x = (x_1, x_2, \ldots, x_n) \in B_1 \mid -1 < x_1 < \lambda \}$$

is the region in the ball $B_1$ between the plane $x_1 = -1$ and the plane $x_1 = \lambda$, $\Sigma_\lambda^c = B_1 \setminus \Sigma_\lambda$ is the complement of $\Sigma_\lambda$ in $B_1$.

Let $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$ be the reflection of the point $x = (x_1, x_2, \ldots, x_n)$ about the plane $T_\lambda$, $\Sigma_\lambda = \{ x^\lambda \mid x \in \Sigma_\lambda \}$ be the reflection of $\Sigma_\lambda$, and let $u_\lambda(x) = u(x^\lambda)$ and $\omega_\lambda = u(x^\lambda) - u(x)$.

To use the method of moving planes, we need the following lemmas from [8]. They are about the Green’s function $G_1(x, y)$ and $f(u)/u$.

**Lemma 2.1.** (i) For any $x, y \in \Sigma_\lambda$, $x \neq y$, we have

$$G_1(x^\lambda, y^\lambda) > \max\{G_1(x^\lambda, y), G_1(x, y^\lambda)\}$$

and

$$G_1(x^\lambda, y^\lambda) - G_1(x, y) > |G_1(x^\lambda, y) - G_1(x, y^\lambda)|.$$

(ii) For any $x \in \Sigma_\lambda$, $y \in \Sigma_\lambda^c$, it holds

$$G_1(x^\lambda, y) > G_1(x, y).$$

**Lemma 2.2** For any $x \in \Sigma_\lambda$, it holds

$$u(x) - u_\lambda(x) \leq \int_{\Sigma_\lambda} [G_1(x^\lambda, y^\lambda) - G_1(x, y^\lambda)][f(u(y)) - f(u_\lambda(y))]|dy.$$

The proofs of these two lemmas are standard and can be founded in [8].

**Lemma 2.2** (An equivalent form of the Hardy-Littlewood-Sobolev inequality). Assume $0 < \alpha < n$ and $\Omega \subset \mathbb{R}^n$. Let $g \in L^{\frac{n\alpha}{n-\alpha}}(\Omega)$ for $\frac{n}{n-\alpha} < p < \infty$. Define

$$Tg(x) := \int_{\Omega} \frac{1}{|x - y|^{n-\alpha}}g(y)dy.$$

Then

$$||Tg||_{L^p(\Omega)} \leq C(n, p, \alpha) \|g\|_{L^{\frac{n}{n-\alpha}}(\Omega)}.$$

The proof of this lemma can be seen in [10] or [11].

**Lemma 2.3.** Assume $u \geq 0, f(u)$ is increasing and $f(0) = 0$. If $f(u)/u$ is decreasing, then

$$\frac{f(u_1)}{u_1} > \frac{f(u_1) - f(u_2)}{u_1 - u_2}$$

and

$$\frac{f(u_2)}{u_2} > \frac{f(u_1) - f(u_2)}{u_1 - u_2}.$$

Proof. Since $f(u)/u$ is decreasing, without loss of generality, we assume $u_1 > u_2$, then $\frac{f(u_1)}{u_1} < \frac{f(u_2)}{u_2}$, and $f(u_1)(u_1 - u_2) > u_1(f(u_1) - f(u_2))$. So we have

$$\frac{f(u_1)}{u_1} > \frac{f(u_1) - f(u_2)}{u_1 - u_2}.$$

The same procedure may be easily adapted to obtain

$$\frac{f(u_2)}{u_2} > \frac{f(u_1) - f(u_2)}{u_1 - u_2}.$$

This completes the proof of the lemma. □
2.2. The proof of Theorem 1. The proof of Theorem 1 contains two steps.

Step 1: We first prove by contradiction

$$\omega(x) = u(x) - u(x) \geq 0, \text{ a.e. } x \in \Sigma_\lambda. \tag{22}$$

Define $$\Sigma^\lambda = \{x \in \Sigma_\lambda | u(x) > u_\lambda(x)\}$$. For any $$x \in \Sigma_\lambda$$, we show that $$\Sigma^\lambda$$ is almost empty.

By Lemma 2.2, we have the below result:

$$0 < u(x) - u_\lambda(x)$$

$$\leq \int_{\Sigma_\lambda} [G_1(x^\lambda, y^\lambda) - G_1(x, y^\lambda)] [f(u(y)) - f(u_\lambda(y))] dy$$

$$= \int_{\Sigma^\lambda} [G_1(x^\lambda, y^\lambda) - G_1(x, y^\lambda)] [f(u(y)) - f(u_\lambda(y))] dy \tag{23}$$

$$+ \int_{\Sigma_\lambda \setminus \Sigma^\lambda} [G_1(x^\lambda, y^\lambda) - G_1(x, y^\lambda)] [f(u(y)) - f(u_\lambda(y))] dy.$$  

Since $$f$$ is increasing and $$u(y) < u_\lambda(y)$$ in $$\Sigma_\lambda \setminus \Sigma^\lambda$$, then

$$f(u(y)) - f_\lambda(u(y)) < 0. \tag{24}$$

Applying (21) and (24) to (23), we arrive at: for any $$x \in \Sigma^\lambda$$,

$$0 < u(x) - u_\lambda(x)$$

$$\leq \int_{\Sigma^\lambda} [G_1(x^\lambda, y^\lambda) - G_1(x, y^\lambda)] [f(u(y)) - f(u_\lambda(y))] dy$$

$$\leq \int_{\Sigma^\lambda} G_1(x^\lambda, y^\lambda) [f(u(y)) - f(u_\lambda(y))] dy$$

$$= \int_{\Sigma^\lambda} G_1(x^\lambda, y^\lambda) \frac{f(u(y)) - f(u_\lambda(y))}{u(y) - u_\lambda(y)} |u(y) - u_\lambda(y)| dy \tag{25}$$

$$\leq \int_{\Sigma^\lambda} \frac{1}{|x - y|^{n-\alpha}} \frac{f(u(y)) - f(u_\lambda(y))}{u(y) - u_\lambda(y)} \omega_\lambda(y) dy$$

$$\leq \int_{\Sigma^\lambda} \frac{1}{|x - y|^{n-\alpha}} \frac{f(u(y))}{u(y)} \omega_\lambda(y) dy$$

$$\leq \int_{\Sigma^\lambda} \frac{1}{|x - y|^{n-\alpha}} |C_1 + C_2| |u|^\beta |\omega_\lambda(y)| dy.$$

In above inequalities we have used Lemma 2.4, the definition of $$\Sigma^\lambda$$, the monotonicity of $$f$$ and

$$|G_1| \leq \frac{C}{|x - y|^{n-\alpha}}, \tag{26}$$

here (26) is obviously derived by the expression of $$G_1$$.

Applying H-L-S inequality and Hölder inequality to (25), we arrive at, for any $$q > \frac{n}{n-\alpha},$$

$$\|\omega_\lambda\|_{L^q(\Sigma^\lambda)} \leq C \|(C_1 + C_2 |u|^\beta)\|_{L^{\frac{q}{n-\alpha}}(\Sigma^\lambda)}$$

$$\leq C \|(C_1 + C_2 |u|^\beta)\|_{L^{\frac{q}{n-\alpha}}(\Sigma^\lambda)} \|\omega_\lambda\|_{L^q(\Sigma^\lambda)}. \tag{27}$$
By assumption $|u|^\beta \in L^\frac{1}{\beta}(B_1)$, for $\lambda$ sufficiently close to -1, it holds $C\|(C_1 + C_2|u|^\beta)\|_{L^\frac{1}{\beta}(\Sigma_\eta)} \leq \frac{1}{2}$. Then $\|\omega_\lambda\|_{L^\lambda(\Sigma_\eta)} \leq \frac{1}{2}\|\omega_\lambda\|_{L^\lambda(\Sigma_\eta)}$. This implies that $\|\omega_\lambda\|_{L^\lambda(\Sigma_\eta)} = 0$, therefore $\Sigma_\eta$ must be measure zero.

**Step 2:** We now move the plane $T_\lambda$ continuously towards the right as long as (22) holds.

Define

$$\lambda_0 = \sup\{\lambda \in (-1, 0) \mid \omega_\eta(x) \geq 0, x \in \Sigma_\eta, \eta \leq \lambda\}. \quad (28)$$

We prove that $\lambda_0$ must be 0.

Otherwise, we suppose $\lambda_0 < 0$. First, we will show that $\omega_{\lambda_0} > 0$ in the interior of $\Sigma_{\lambda_0}$, by contradiction.

Due to the definition of $\Sigma_\lambda$, for any $x \in \Sigma_\lambda$, we have $G_1(x, y)f(u) \mid_{\Sigma_\lambda} = G_1(x, y^\lambda)f(u_\lambda) \mid_{\Sigma_\lambda}$ and

$$u(x) = \int_{\Sigma_\lambda} G_1(x, y)f(u(y))dy + \int_{\Sigma_\lambda} G_1(x, y)f(u(y))dy \quad + \int_{\Sigma_\lambda \setminus \Sigma_\lambda} G_1(x, y)f(u(y))dy \quad (29)$$

Similarly,

$$u(x^\lambda) = \int_{\Sigma_\lambda} G_1(x^\lambda, y)f(u(y))dy + \int_{\Sigma_\lambda} G_1(x^\lambda, y)f(u_\lambda(y))dy \quad + \int_{\Sigma_\lambda \setminus \Sigma_\lambda} G_1(x^\lambda, y)f(u(y))dy \quad (30)$$

Due to (29) and (30), it holds

$$u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} [G_1(x^\lambda, y) - G_1(x, y)]f(u(y))dy \quad + \int_{\Sigma_\lambda} [G_1(x^\lambda, y^\lambda) - G_1(x, y^\lambda)]f(u_\lambda(y))dy \quad + \int_{\Sigma_\lambda \setminus \Sigma_\lambda} [G_1(x^\lambda, y) - G_1(x, y)]f(u(y))dy \quad (31)$$

By Lemma 2.1(i), for any $x, y \in \Sigma_\lambda, x \neq y$, we have

$$G_1(x^\lambda, y^\lambda) - G_1(x, y) \geq G_1(x, y^\lambda) - G_1(x^\lambda, y).$$

That is

$$G_1(x^\lambda, y) - G_1(x, y) \geq G_1(x, y^\lambda) - G_1(x^\lambda, y^\lambda). \quad (32)$$
From Lemma 2.1(i), the monotonicity of \( f \) and (31),(32), we get
\[
\begin{align*}
    u_\lambda(x) - u(x) & \geq \int_{\Sigma_\lambda} \left[ G_1(x, y^\lambda) - G_1(x^\lambda, y^\lambda) \right] f(u(y)) dy \\
    & \quad + \int_{\Sigma_\lambda} \left[ G_1(x^\lambda, y^\lambda) - G_1(x, y^\lambda) \right] f(u_\lambda(y)) dy \\
    & \quad + \int_{\Sigma_\lambda} \left[ G_1(x^\lambda, y) - G_1(x, y) \right] f(u(y)) dy \\
    & = \int_{\Sigma_\lambda} \left[ G_1(x^\lambda, y) - G_1(x, y) \right] \left[ f(u_\lambda(y)) - f(u(y)) \right] dy \\
    & \quad + \int_{\Sigma_\lambda} \left[ G_1(x^\lambda, y) - G_1(x, y) \right] f(u(y)) dy \\
    & \geq \int_{\Sigma_\lambda} \left[ G_1(x^\lambda, y) - G_1(x, y) \right] f(u(y)) dy.
\end{align*}
\]

The above last inequality has used Lemma 2.1(ii), \( u < u_\lambda \) and the monotonicity of \( f \).

If \( \omega_{\lambda_0} > 0 \) is violated, then there exists some point \( x_0 \in \Sigma_{\lambda_0} \) such that \( u(x_0) = u_\lambda(x_0) \). So, the left side of (33) can be zero. Since \( G_1(x^\lambda, y) > G_1(x, y), \forall y \in \Sigma_\lambda \setminus \Sigma_0 \) and the monotonicity of \( f \), the right side of (33) \( \geq 0 \).

If (33) holds, we require \( f(u(y)) \equiv 0, \forall y \in \Sigma_{\lambda_0} \setminus \Sigma_0 \). Because \( f \) is increasing and \( f(0) = 0 \), we have \( u(y) \equiv 0, \forall y \in \Sigma_{\lambda_0} \setminus \Sigma_0 \). This is a contradiction with our assumption \( u > 0 \). Therefore, \( \omega_{\lambda_0} > 0 \).

According to the Lusin’s Theorem, for any \( \delta > 0 \), there exists a closed subset \( F_\delta \) of \( \Sigma_{\lambda_0} \), with \( \mu(\Sigma_{\lambda_0} \setminus F_\delta) < \delta \), such that \( \omega_{\lambda_0}|_{F_\delta} \) is continuous with respect to \( x \), hence \( \omega_{\lambda_0}|_{F_\delta} \) is continuous with respect to \( \lambda \) for \( \lambda \) close to \( \lambda_0 \). By \( \omega_{\lambda_0} > 0 \), there exists \( \epsilon > 0 \) such that for all \( \lambda \in [\lambda_0, \lambda_0 + \epsilon] \), it holds \( \omega_\lambda \geq 0, \forall x \in F_\delta \). It follows that, for such \( \lambda \),

\[
\mu(\Sigma^-_\lambda) \leq \mu(\Sigma_{\lambda_0} \setminus F_\delta) < \delta + \mu(\Sigma_\lambda \setminus \Sigma_{\lambda_0}) \leq \delta + 2\epsilon.
\]

Similar to step 1, we can choose \( \lambda \) and \( \epsilon \) sufficiently small such that
\[
C \int_{\Sigma^-_\lambda} |C_1 + C_2| |u|^2 dx \leq \frac{1}{2}.
\]

Then \( \|\omega_\lambda\|_{L^6(\Sigma^-_\lambda)} = 0 \). Hence \( \Sigma^-_\lambda \) is measure zero and
\[
\omega_\lambda(x) \geq 0 \quad \text{a.e } x \in \Sigma_\lambda, \lambda \in [\lambda_0, \lambda_0 + \epsilon).
\]

This is in contradiction with the definition of \( \lambda_0 \). So \( \lambda_0 = 0 \). This completes the proof of Theorem 1.

2.3. The proof of Corollary 1. We find that the equivalence between (7) and (9) when \( f(u) \in L^q(B_1) \) has been proved in [8]. So if \( u \) is a positive solution of (9) with \( f(u) \in L^q(B_1) \), then it is radially symmetric about the origin and strictly decreasing in the radial direction.

3. Regularity of solutions. In this section, we first prove the uniform boundedness of solutions of (7) or of (9) by Regularity Lifting Lemma which is introduced in [11]. We then get the same result with another method by estimating \( \|f(u)\|_{L^{\frac{nq}{n(q-1)}}(B_1)} \).
3.1. Regularity lifting. Let $V$ be a Hausdorff topological vector space. Suppose there are two extended norms defined on $V$,

$$\| \cdot \|_X, \| \cdot \|_Y : V \rightarrow [0, \infty].$$

Assume that the spaces

$$X := \{ v \in V : \|v\|_X < \infty \} \text{ and } Y := \{ v \in V : \|v\|_Y < \infty \}$$

are complete under corresponding norms, and the convergence in $X$ or in $Y$ implies the convergence in $V$.

**Lemma 3.1** (Regularity Lifting). Let $T$ be a contracting map from $X$ into itself and from $Y$ into itself. Assume that $f \in X$, and that there exists a function $g \in Z := X \cap Y$ such that $f = Tg + g$ in $X$. Then $f$ also belongs to $Z$.

3.2. The proof of Theorem 2. Proof: We only consider integral equation of (7). We will show that for any $p > \frac{n}{n-\alpha}$,

$$u \in L^p(B_1)$$

by using the Regularity Lifting Lemma.

For any real number $a > 0$, let $A = \{ x \in B_1 \mid \frac{1}{|u(x)|} \leq a \}$ and

$$u_a(x) = \begin{cases} u(x), & \text{if } x \in A, \\ 0, & \text{elsewhere.} \end{cases}$$

Set $u_b(x) = u - u_a(x)$. Then $f(u) = f(u_a)\chi_A + f(u_b)\chi_D$, here $\chi_A$ is the characteristic function on the set $A$ and $D = B_1 \setminus A$.

Define the linear operator

$$T_a\omega(x) = \int_{B_1} G_1 \frac{f(u_a(y))\chi_A(y)}{u_a(y)} \omega(y)dy$$

and

$$J(x) = \int_{B_1} G_1 f(u_b(y))\chi_D(y)dy. \quad (35)$$

Then

$$T_a u(x) = \int_{B_1} G_1 \frac{f(u_a(y))\chi_A(y)}{u_a(y)} (u_a(y) + u_b(y))dy$$

$$= \int_{B_1} G_1 f(u_a(y))\chi_A(y)dy. \quad (36)$$

Obviously, we have $u(x) = T_a u(x) + J(x), \forall x \in B_1$.

According to Regularity Lifting Lemma, we need $T_a$ is a contracting map from $L^p(B_1)$ to $L^p(B_1)$ and $J(x) \in L^p(B_1)$.

We first prove that for a sufficiently small $a$, $T_a$ is a contracting map from $L^p(B_1)$ to $L^p(B_1)$ for any $p > \frac{n}{n-\alpha}$.

$$|T_a \omega(x)| = \left| \int_{B_1} G_1 \frac{f(u_a(y))\chi_A(y)}{u_a(y)} \omega(y)dy \right|$$

$$\leq \int_{B_1} \frac{1}{|x - y|^{n-\alpha}} \frac{|f(u_a(y))|\chi_A(y)}{u_a(y)} |\omega(y)|dy. \quad (37)$$

By H-L-S inequality and Hölder inequality, we get

$$\|T_a \omega(x)\|_{L^p(B_1)} \leq C \|\frac{f(u_a)\chi_A}{u_a}\|_{L^{\frac{np}{n-\alpha}}(B_1)} \leq \|\frac{f(u)}{u}\|_{L^\infty(A)} \|\omega\|_{L^p(B_1)}. \quad (38)$$
Choose a sufficiently small $a$, so that the measure of $A$ is small, then $\| T_0 \omega(x) \|_{L^p(B_1)} \leq \frac{1}{2} \| \omega \|_{L^p(B_1)}$. Therefore $T_0$ is a contracting map from $L^p(B_1)$ to $L^p(B_1)$.

Next we estimate $J(x)$.

Similarly, due to H-L-S inequality, Hölder inequality and (35), it holds
\[
\| J(x) \|_{L^p(B_1)} \leq \| f(u_b) \chi_D \|_{L^{\frac{n\beta}{n\beta + \sigma}}(B_1)}
\leq C \| f(u) \|_{L^{\frac{np}{p + \sigma}}(D)}
\leq C \| C_1 u + C_2 |u|^{\beta + 1} \|_{L^{\frac{np}{p + \sigma}}(D)}
\leq C_a.
\]

Since $u(x) < \frac{1}{a}$ for $x \in D$ and $\beta + 1 > 0$, we get the last above inequality. Hence $J(x) \in L^p(B_1)$, for any $p > \frac{n}{n - \alpha}$.

Due to $u \in L^{\frac{np}{p + \sigma}}(B_1)$, $|\frac{np}{p + \sigma}| > \frac{n}{n - \alpha}$, and regularity lifting, we get (34).

In what follows, we prove that $u$ is uniformly bounded.

By (26), (18) and Hölder inequality, we have
\[
u(x) = \int_{B_1} G_1(x, y) f(u(y)) dy
\leq C \left( \int_{B_1} \frac{1}{|x - y|^{r(n - \alpha)}} dy \right)^{\frac{1}{r}} \left( \int_{B_1} (|C_1 u + C_2| u |^{\beta + 1}|) dy \right)^{\frac{1}{\beta}}
\leq C \left( \int_{B_1} \frac{1}{|x - y|^{r(n - \alpha)}} dy \right)^{\frac{1}{r}}.
\]

Here
\[
\frac{1}{r} + \frac{1}{s} = 1, \quad 1 < r < \frac{n}{n - \alpha}, \quad s > \frac{n}{\alpha}.
\]

Thus $0 < (\beta + 1)s < s$ and for $n \geq 4$,
\[
s > \frac{n}{\alpha} > \frac{n}{n - \alpha}.
\]

Combining (34) and (42), then the last inequality of (40) holds.

According to (41), we have $n - \alpha < r(n - \alpha) < n$, that is $1 - \alpha < r(n - \alpha) - (n - 1) < 1$. So we obtain
\[
u(x) \leq C \left( \int_{B_1} \frac{1}{|x - y|^{r(n - \alpha)}} dy \right)^{\frac{1}{r}}
\quad = C \left[ \int_{B_1} \frac{1}{|x - y|^{r(n - \alpha)}} dy + \int_{B_1 \setminus B_\varepsilon(x)} \frac{1}{|x - y|^{r(n - \alpha)}} dy \right]^{\frac{1}{r}},
\]

where $\varepsilon$ is a sufficiently small number between 0 and 1. Therefore $u$ is uniformly bounded.

3.3. The proof of Remark 2. By $|\frac{f(u)}{u}| \leq C_1 + C_2 |u|^\beta$ for some $\beta < \frac{\alpha}{\alpha - n}$ ($n \geq 3$), we have
\[
|f(u)| \leq |u|(C_1 + C_2 |u|^\beta).
\]

By Hölder inequality, and $|u|^\beta \in L^{\frac{n\beta}{\alpha}}(B_1)$, $u \in L^{\frac{np}{p + \sigma}}(B_1)$, we have
\[
\| f(u) \|_{L^{\frac{n\beta}{n\beta + \sigma}}(B_1)} \leq \| u \|_{L^{\frac{np}{p + \sigma}}(B_1)} \| C_1 + C_2 |u|^\beta \|_{L^{\frac{n\beta}{\alpha}}(B_1)},
\]

(44)
and hence

\[ f(u) \in L^{\frac{n|\beta|}{\alpha(|\beta|+1)}}(B_1), \]  

(45)

where \( \frac{n|\beta|}{\alpha(|\beta|+1)} > \frac{n}{\alpha} \).

Due to (26), (45) and Hölder inequality, we have

\[
 u(x) = \int_{B_1} G_1(x, y) f(u(y)) dy \\
 \leq \int_{B_1} \frac{1}{|x-y|^{n-\alpha}} f(u(y)) dy \\
 \leq C \left( \int_{B_1} \frac{1}{|x-y|^{r(n-\alpha)}} dy \right)^{\frac{1}{r}} \left( \int_{B_1} (f(u))^s dy \right)^{\frac{1}{s}} \\
 \leq C \left( \int_{B_1} \frac{1}{|x-y|^{r(n-\alpha)}} dy \right)^{\frac{1}{r}},
\]

(46)

here \( r, s \) satisfying

\[
 1 < \frac{n|\beta|}{(n-\alpha)|\beta| - \alpha} < r < \frac{n}{n-\alpha}, \quad \frac{n}{\alpha} < s < \frac{n|\beta|}{\alpha(|\beta|+1)}.
\]

(47)

From (47), we have

\[
 n - \alpha < r(n-\alpha) < n, \quad 1 - \alpha < r(n-\alpha) - (n-1) < 1.
\]

(48)

Then by (46) and (48), similar to the proof of Theorem 2, \( u \) is uniformly bounded.

4. Equivalence between the two equations on \( \mathbb{R}^n_+ \).

4.1. Some useful lemmas. In this section, we establish the equivalence between (16) and (17). We need some lemmas.

**Lemma 4.1** (Boundary Harnack, see [5] or [3]). Let \( f, g : \mathbb{R}^n_+ \to \mathbb{R} \) be two nonnegative functions such that \((-\Delta)^{\frac{\alpha}{2}} f = (-\Delta)^{\frac{\alpha}{2}} g = 0\) in a domain \( \Omega \). Suppose that \( x_0 \in \partial \Omega, f(x) = g(x) = 0 \) for any \( x \in B_1 \setminus \Omega \), and \( \partial \Omega \cap B_1 \) is a Lipschitz graph in the direction of \( x_1 \) with Lipschitz constant less than 1. Then there is a constant \( C \) depending only on dimension such that

\[
 \sup_{x \in \Omega \cap B_2} \frac{f(x)}{g(x)} \leq C \inf_{x \in \Omega \cap B_2} \frac{f(x)}{g(x)}.
\]

The following uniqueness of \( \alpha \)-harmonic functions on half spaces was derived based on the above Harnack inequality in [8].

**Lemma 4.2** (Uniqueness of \( \alpha \)-harmonic functions on half spaces). Assume that \( \omega \) is a nonnegative solution of

\[
 \begin{cases} 
 (-\Delta)^{\frac{\alpha}{2}} \omega = 0, & x \in \mathbb{R}^n_+, \\
 \omega = 0, & x \in \mathbb{R}^n_+. 
\end{cases}
\]

Then there is a constant \( c_0 > 0 \) such that for any two points \( x = (x_1, x_2, \cdots, x_n) \) and \( y = (y_1, y_2, \cdots, y_n) \) in \( \mathbb{R}^n_+ \), we have

\[
 \frac{\omega(y)}{(y_n)^{\frac{\alpha}{2}}} \geq c_0 \frac{\omega(x)}{(x_n)^{\frac{\alpha}{2}}}. 
\]
Consequently, we have either $\omega(x) \equiv 0, x \in \mathbb{R}^n$ or there exists a $a_0 > 0$, such that

$$\omega(x) \geq a_0(x_n)^{\frac{2}{n}}, \forall x \in \mathbb{R}^n.$$ 

The following lemma is a Maximum Principle from [19].

**Lemma 4.3** (Maximum Principle). Let $\Omega \in \mathbb{R}^n$ be a bounded open set, and let $f$ be a lower-semicontinuous function in $\overline{\Omega}$ such that $(-\Delta)f \geq 0$ in $\Omega$ and $f \geq 0$ in $\mathbb{R}^n \setminus \Omega$. Then $f \geq 0$ in $\mathbb{R}^n$.

4.2. **The proof of Theorem 3.** Proof. Set

$$v_R(x) = \int_{B_R(P_R)} G_R(x, y)f(u(y))dy.$$ 

From the continuous assumption of $f$, one can see that, for each $R > 0$, $v_R(x)$ is well-defined and is continuous, and

$$\begin{cases}
  (-\Delta)\tilde{\omega}_R = f(u(x)), & x \in B_R(P_R), \\
  \omega_R(x) = 0, & x \notin B_R(P_R).
\end{cases}$$

Let $\omega_R(x) = u(x) - v_R(x)$, then

$$\begin{cases}
  (-\Delta)\tilde{\omega}_R = f(u(x)), & x \in B_R(P_R), \\
  \omega_R(x) \geq 0, & x \notin B_R(P_R).
\end{cases}$$

By Maximum Principle, $\omega_R(x) \geq 0, \forall x \in B_R(P_R)$. Let $R \to \infty$, we have $u(x) \geq \int_{\mathbb{R}^n_+} G_\infty(x, y)f(u(y))dy$. Let $v(x) = \int_{\mathbb{R}^n_+} G_\infty(x, y)f(u(y))dy$. Then $(-\Delta)\tilde{\omega}(x) = f(u(x)), \forall x \in \mathbb{R}^n$. Set $\omega = u - v$, then

$$\begin{cases}
  (-\Delta)\tilde{\omega}(x) = 0, & x \in \mathbb{R}^n_+, \\
  \omega(x) \equiv 0, & x \in \mathbb{R}^n.
\end{cases}$$

We see from Lemma 4.2 that $\omega \equiv 0, \forall x \in \mathbb{R}^n$. Therefore

$$u(x) = v(x) = \int_{\mathbb{R}^n_+} G_\infty(x, y)f(u(y))dy.$$ 

On the other hand, suppose $u(x)$ is a solution of integral equation (17), applying $(-\Delta)\tilde{\omega}$ to both sides, then by exchanging $(-\Delta)\tilde{\omega}$ with the integral on the right hand side, we get (16). So we finish the proof.

5. **The Liouville type theorem in $\mathbb{R}^n_+$.**

5.1. **The properties of the Green’s function in this half space.** Let $\lambda$ be a positive real number and $T_\lambda = \{x \in \mathbb{R}^n_+ | x_n = \lambda\}$.

$$\Sigma_\lambda = \{x = (x_1, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n_+ | 0 < x_n < \lambda\}$$

is the region between the plane $x_n = 0$ and the plane $x_n = \lambda$, $\Sigma_\lambda^C = \mathbb{R}^n_+ \setminus \Sigma_\lambda$ is the complement of $\Sigma_\lambda$ in $\mathbb{R}^n_+$, $\Sigma_\lambda = \{x^\lambda | x \in \Sigma_\lambda\}$ is the reflection of $\Sigma_\lambda$ about the plane $T_\lambda$.

Let $x^\lambda = (x_1, \cdots, x_{n-1}, 2\lambda - x_n)$ be the reflection of the point $x = (x_1, \cdots, x_{n-1}, x_n)$ about the plane $T_\lambda$, and set $u_\lambda(x) = u(x^\lambda)$ and $\omega_\lambda(x) = u_\lambda(x) - u(x)$.

The following three lemmas will be used in the proof of Theorem 4.
Lemma 5.1. (i) For any \( x, y \in \Sigma \lambda, x \neq y \), we have

\[
G_\infty(x^\lambda, y^\lambda) > \max\{G_\infty(x^\lambda, y), G_\infty(x, y^\lambda)\}
\]

and

\[
G_\infty(x^\lambda, y^\lambda) - G_\infty(x, y) > |G_\infty(x^\lambda, y) - G_\infty(x, y^\lambda)|.
\]

(ii) For any \( x \in \Sigma \lambda, y \in \Sigma \lambda^C \), it holds

\[
G_\infty(x^\lambda, y) > G_\infty(x, y).
\]

Lemma 5.2. For any \( x \in \Sigma \lambda \), it holds

\[
u(x) - u_\lambda(x) \leq \int \Sigma \lambda [G_\infty(x^\lambda, y^\lambda) - G_\infty(x, y^\lambda)] [f(u(y)) - f(u_\lambda(y))] dy.
\]

Proof. Due to

\[
G_\infty(x, y)f(u)|_{\Sigma \lambda^C} = G_\infty(x, y^\lambda)f(u_\lambda)|_{\Sigma \lambda},
\]

we have

\[
u(x) = \int \Sigma \lambda G_\infty(x, y)f(u(y))dy + \int \Sigma \lambda G_\infty(x, y)f(u(y))dy
\]

\[
+ \int \Sigma \lambda^C \Sigma \lambda G_\infty(x, y)f(u(y))dy
\]

\[
= \int \Sigma \lambda G_\infty(x, y)f(u(y))dy + \int \Sigma \lambda G_\infty(x, y^\lambda)f(u_\lambda(y))dy
\]

\[
+ \int \Sigma \lambda^C \Sigma \lambda G_\infty(x, y)f(u(y))dy. \tag{49}
\]

Similarly,

\[
u(x^\lambda) = \int \Sigma \lambda G_\infty(x^\lambda, y)f(u(y))dy + \int \Sigma \lambda G_\infty(x^\lambda, y^\lambda)f(u_\lambda(y))dy
\]

\[
+ \int \Sigma \lambda^C \Sigma \lambda G_\infty(x^\lambda, y)f(u(y))dy. \tag{50}
\]
Due to (49), (50) and Lemma 5.1, we get
\[
u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} [G_\infty(x, y) - G_\infty(x, y^\lambda)] f(u(y)) dy \\
+ \int_{\Sigma_\lambda} [G_\infty(x, y^\lambda) - G_\infty(x, y^\lambda)] f(u(y)) dy \\
+ \int_{\Sigma_\lambda \setminus \Sigma_\lambda^0} [G_\infty(x, y) - G_\infty(x, y^\lambda)] f(u(y)) dy \\
\leq \int_{\Sigma_\lambda} [G_\infty(x, y) - G_\infty(x, y^\lambda)] f(u(y)) dy \\
- \int_{\Sigma_\lambda} [G_\infty(x, y^\lambda) - G_\infty(x, y^\lambda)] f(u(y)) dy \\
\leq \int_{\Sigma_\lambda} [G_\infty(x, y^\lambda) - G_\infty(x, y^\lambda)] f(u(y)) dy \\
- \int_{\Sigma_\lambda} [G_\infty(x, y^\lambda) - G_\infty(x, y^\lambda)] f(u(y)) dy \\
= \int_{\Sigma_\lambda} [G_\infty(x, y^\lambda) - G_\infty(x, y^\lambda)] f(u(y)) - f(u_\lambda(y)) dy.
\]
(51)

So we finish the proof of Lemma 5.2. \(\square\)

**Lemma 5.3.** Let \(\Omega \subset \mathbb{R}^n_+\) be an open set, and let \(u\) be a lower-semi-continuous function in \(\Omega\) such that
\[
(-\Delta)^2 u \geq 0 \text{ and } u \geq 0 \text{ in } \mathbb{R}^n_+ \setminus \Omega.
\]
Then \(u \geq 0\) in \(\mathbb{R}^n_+\). Moreover, if \(u(x) = 0\) for some point inside \(\Omega\), then \(u \equiv 0\) in all \(\mathbb{R}^n_+\).

**5.2. The proof of Theorem 4.**

**Proof.** From Lemma 5.3, we know that a nonnegative solution \(u\) of (17) is either strictly positive or identically zero in \(\mathbb{R}^n_+\). Without loss of generality, we assume that \(u > 0\) in \(\mathbb{R}^n_+\), then we will derive a contradiction.

The proof of Theorem 4 consists of two steps.

Step 1. Define \(\Sigma_\lambda^\pm = \{x \in \Sigma_\lambda \mid \omega_\lambda < 0\}\). That is \(u(x^\lambda) < u(x), \forall x \in \Sigma_\lambda^\pm\). We will prove that for sufficiently small \(\lambda\), \(\Sigma_\lambda^\pm\) must be measure zero.

According to Lemma 5.2, Lemma 5.1 and the definition of \(\Sigma_\lambda^\pm\), for \(\forall x \in \Sigma_\lambda^\pm\), it holds
\[
0 < u(x) - u_\lambda(x) \\
\leq \int_{\Sigma_\lambda} [G_\infty(x, y^\lambda) - G_\infty(x, y^\lambda)] [f(u(y)) - f(u_\lambda(y))] dy \\
= \int_{\Sigma_\lambda} [G_\infty(x, y^\lambda) - G_\infty(x, y^\lambda)] [f(u(y)) - f(u_\lambda(y))] dy \\
+ \int_{\Sigma_\lambda \setminus \Sigma_\lambda^0} [G_\infty(x, y^\lambda) - G_\infty(x, y^\lambda)] [f(u(y)) - f(u_\lambda(y))] dy \\
\leq \int_{\Sigma_\lambda} [G_\infty(x, y^\lambda) - G_\infty(x, y^\lambda)] [f(u(y)) - f(u_\lambda(y))] dy.
\]
\[
\leq \int_{\Sigma^-} G_\infty(x^\lambda, y^\lambda)[f(u(y)) - f(u_\lambda(y))] \, dy
= \int_{\Sigma^-} G_\infty(x^\lambda, y^\lambda) f'(\psi(y))(u(y) - u_\lambda(y)) dy.
\] (52)

Where \(\psi(y)\) is valued between \(u_\lambda(y)\) and \(u(y)\). Hence on \(\Sigma^-\), we have
\[
0 \leq u_\lambda(y) \leq \psi(y) \leq u(y).
\]

By the expression of \(G_\infty(x, y)\), we can see that \(G_\infty(x, y) \leq \frac{A_{n, \alpha}}{|x-y|^{n-\alpha}}\). So
\[
0 < u(x) - u_\lambda(x) \leq \int_{\Sigma^-} \frac{C}{|x-y|^{n-\alpha}} |f'(\psi(y))| |u(y) - u_\lambda(y)| dy.
\]

By the Hardy-Littlewood-Sobolev inequality and Hölder inequality, we obtain, for any \(\alpha > \frac{n}{n-\alpha}\),
\[
\|\omega\|_{L^q(\Sigma^-)} \leq C \|f'(\psi(y))\|_{L^\frac{\alpha n}{\alpha n - \alpha}}(\Sigma^-) \leq C \|f'(\psi(y))\|_{L^\frac{\alpha n}{\alpha n - \alpha}}(\Sigma^-) \|\omega\|_{L^q(\Sigma^-)}.
\] (53)

Since \(f'(\psi) \in L^\frac{\alpha n}{\alpha n - \alpha}(\mathbb{R}^n_+)\), choose a small positive constant \(\lambda\) such that
\[
C \|f'(\psi(y))\|_{L^\frac{\alpha n}{\alpha n - \alpha}(\Sigma^-)} = C \left(\int_{\Sigma^-} |f'(\psi(y))|^{\frac{\alpha n}{\alpha n - \alpha}} dy\right)^{\frac{\alpha n - \alpha}{\alpha n}} \leq \frac{1}{2}.
\]

Then
\[
\|\omega\|_{L^q(\Sigma^-)} \leq \frac{1}{2} \|\omega\|_{L^q(\Sigma^-)}.
\] (54)

(54) implies that \(\|\omega\|_{L^q(\Sigma^-)} = 0\). Thus, \(\Sigma^-\) is measure zero.

**Step 2.** Due to Step 1, we can start from the small \(\lambda\). So the plane \(T_\lambda\) can be moved as long as \(\omega_\lambda(x) = u_\lambda(x) - u(x) \geq 0\) a.e. \(x \in \Sigma_\lambda\) holds.

Define \(\lambda_0 = \sup \{\lambda : u_\lambda(x) - u(x) \geq 0, \text{ a.e. } x \in \Sigma_\lambda\}\). We want to prove the limit position \(T_{\lambda_0}\) of the plane \(T_\lambda\) locally at infinity. If \(\lambda_0 = +\infty\), then \(u\) is always increasing. That is
\[
\|u(x)\|_{L^q(\mathbb{R}^n_+)} = \left(\int_{\mathbb{R}^n_+} |u|^q dx\right)^{\frac{1}{q}} = +\infty.
\] (55)

But on the other hand, by (17) and (26), we have \(|u(x)| \leq C \int_{\mathbb{R}^n_+} \frac{1}{|x-y|^{n-\alpha}} |f(u)| dy\).

By H-L-S inequality and (20), it holds
\[
\|u(x)\|_{L^q(\mathbb{R}^n_+)} \leq C \|f(u)\|_{L^\frac{\alpha n}{\alpha n - \alpha}(\mathbb{R}^n_+)} \leq C_1,
\] (56)

where \(C\) and \(C_1\) are some positive constants. (56) contradicts with (55). Then our assumption \(u > 0\) is not right. According to Lemma 5.3, the positive solution of (17) does not exists.

So we first prove \(\lambda_0 = +\infty\). Otherwise, we suppose \(\lambda_0 < +\infty\). So \(u\) is symmetric about the plane \(T_{\lambda_0}\), i.e.
\[
\omega_{\lambda_0} \equiv 0, \text{ a.e. } x \in \Sigma_{\lambda_0}
\] (57)

under the assumption \(\lambda_0 < +\infty\). We postpone the proof of (57) for a while.

By (57), we can deduce that the boundary \(\partial \mathbb{R}^n_+\) and the plane \(x_n = 2\lambda_0\) are symmetric about the plane \(T_{\lambda_0}\). That is \(u(x_1, x_2, \cdots, 2\lambda_0) = u(x_1, x_2, \cdots, 0)\).

By
Thus, what remaining is to prove (60). The method is the same as [8].

In the same way, we suppose in the contrary that for such a \( \lambda_0 \),

\[
\omega_{\lambda_0} \geq 0, \text{ but } \omega_{\lambda_0} \neq 0 \text{ a.e. } x \in \Sigma_{\lambda_0}.
\]  

(58)

Under this assumption, we can show that there exists an \( \varepsilon > 0 \), such that for \( \forall \lambda \in [\lambda_0, \lambda_0 + \varepsilon] \)

\[
\omega_{\lambda} \geq 0, \text{ a.e. } x \in \Sigma_{\lambda},
\]  

(59)

this implies the plane can be moved further up. This contradicts the definition of \( \lambda_0 \). So the assumption (58) is not right, which means (57) holds.

Now we prove (59).

Similar to step 1, to prove (59) is to prove that the measure of \( \Sigma_{\lambda}^- \) is zero. So the key is

\[
C \left( \int_{\Sigma_{\lambda}^-} |f'(\psi(y))| \frac{\mu}{\pi} dy \right)^{\frac{n}{2}} \leq \frac{1}{2}.
\]  

(60)

Thus, what remaining is to prove (60). The method is the same as [8].

Since \( C \left( \int_{R^+ \setminus B_R} |f'(\psi(y))| \frac{\mu}{\pi} dy \right)^{\frac{n}{2}} < \eta \), then we only need to show that \( \Sigma_{\lambda}^- \cap B_R \) is sufficiently small for \( \lambda \) close to \( \lambda_0 \).

First we show that \( \omega_{\lambda_0} > 0, \forall x \in \Sigma_{\lambda_0} \). Due to Lemma 5.2, Lemma 5.1 and the definition of \( \lambda_0 \), we have

\[
\begin{aligned}
& u_{\lambda}(x) - u(x) \geq \int_{\Sigma_{\lambda}} [G_{\infty}(x^\lambda, y^\lambda) - G_{\infty}(x, y^\lambda)] [f(u^\lambda(y)) - f(u(y))] dy \\
& \quad + \int_{\Sigma_{\lambda}^C \setminus \Sigma_{\lambda}} [G_{\infty}(x^\lambda, y) - G_{\infty}(x, y)] f(u(y)) dy \\
& \quad \geq \int_{\Sigma_{\lambda}^C \setminus \Sigma_{\lambda}} [G_{\infty}(x^\lambda, y) - G_{\infty}(x, y)] f(u(y)) dy.
\end{aligned}
\]  

(61)

If \( \omega_{\lambda_0} > 0 \) is violated, then there exists some point \( x_0 \in \Sigma_{\lambda_0} \) such that \( u(x_0) = u_{\lambda_0}(x_0) \). Then \( f(u(y)) \equiv 0, \forall y \in \Sigma_{\lambda_0} \setminus \Sigma_{\lambda} \). That is \( u(y) \equiv 0, \forall y \in \Sigma_{\lambda_0} \setminus \Sigma_{\lambda} \). It’s a contradiction with \( u > 0 \).

For any \( \gamma > 0 \), let \( E_{\gamma} = \{ x \in \Sigma_{\lambda_0} \cap B_R \mid \omega_{\lambda_0}(x) > \gamma \} \), \( F_{\gamma} = (\Sigma_{\lambda_0} \cap B_R) \setminus E_{\gamma} \).

Then \( \lim_{\gamma \to 0} \mu(F_{\gamma}) = 0 \). For \( \lambda > \lambda_0, \) let \( D_{\lambda} = (\Sigma_{\lambda} \setminus \Sigma_{\lambda_0}) \cap B_R \). Hence, \( \lim_{\lambda \to \lambda_0} \mu(D_{\lambda}) = 0 \) and

\[
(\Sigma_{\lambda}^- \cap B_R) \subset (\Sigma_{\lambda}^- \cap E_{\gamma}) \cup F_{\gamma} \cup D_{\lambda}.
\]

For \( \forall x \in \Sigma_{\lambda}^- \cap E_{\gamma} \), we have

\[
\omega_{\lambda}(x) = u_{\lambda}(x) - u(x) = u_{\lambda_0}(x) - u_{\lambda_0}(x) + u_{\lambda_0}(x) - u(x) < 0.
\]

That is \( u_{\lambda}(x) - u_{\lambda_0}(x) > u_{\lambda_0}(x) - u(x) = \omega_{\lambda_0}(x) > \gamma \). Thus,

\[
(\Sigma_{\lambda}^- \cap E_{\gamma}) \subset G_{\gamma} \equiv \{ x \in B_R \mid u_{\lambda_0}(x) - u(x) > \gamma \}.
\]

By Chebyshev inequality, we have \( \mu(G_{\gamma}) \leq \frac{1}{p+1} \int_{G_{\gamma}} |u_{\lambda_0}(x) - u(x)|^{p+1} dx \). For each fixed \( \gamma \) as \( \lambda \) close to \( \lambda_0 \), the right hand side of the above inequality can be made as small as we wish.

This completes the proof of Theorem 4. \( \square \)
Acknowledgments. We would like to thank the referee for his or her helpful suggestions to improve the writing of the paper.

REFERENCES

[1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, 2nd ed, Cambridge Studies in Advanced Mathematics, 116, Cambridge University Press, Cambridge, 2009.

[2] J. Bertoin, *Lévy Processes*, Cambridge Tracts in Mathematics, 121 Cambridge University Press, Cambridge, 1996.

[3] K. Bogdan, The boundary Harnack principle for the fractional Laplacian, *Studia Math.*, 123 (1997), 43–80.

[4] J. P. Bouchard and A. Georges, Anomalous diffusion in disordered media, Statistical mechanics, models and physical applications, *Physics reports*, 195 (1990).

[5] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. in PDE*, 32 (2007), 1245–1260.

[6] L. Caffarelli and L. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, *Ann. of Math. (2)*, 171 (2010), 1903–1930.

[7] X. Cabré and J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, *Adv. in Math.*, 224 (2010), 2052–2093.

[8] W. Chen, Y. Fang and R. Yang, Semilinear equations involving the fractional Laplacian on domains, arXiv:1309.7499v1.

[9] W. Chen, Y. Fang and R. Yang, Liouville theorems involving the fractional Laplacian on a half space, *Adv. in Math.*, 274 (2015), 167–198.

[10] W. Chen and C. Li, Regularity of solutions for a system of integral equation, *Comm. Pure Appl. Anal.*, 4 (2005), 1–8.

[11] W. Chen and C. Li, *Methods on Nonlinear Elliptic Equations*, AIMS. Ser. Differ. Equ. Dyn. Syst. vol.4 2010.

[12] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, *Comm. Pure Appl. Math.*, 59 (2006), 330–343.

[13] P. Constantin, Euler equations, Navier-Stokes equations and turbulence, in *Mathematical Foundation of Turbulent Viscous Flows*, Vol. 1871 of Lecture Notes in Math. 1–43, Springer, Berlin, 2006.

[14] P. Felmer and Y. Wang, Symmetry and non-existence of solutions for a nonlinear system involving the fractional Laplacian, *Comm. Cont. Math.*, 16 (2014), 1350023.

[15] Q. Guan, Integration by parts formula for regional fractional Laplacian, *Comm. Math. Phys.*, 266 (2006), 289–329.

[16] T. Kulczycki, Properties of Green function of symmetric stable processes, *Probability and Mathematical Statistics*, 17 (1997), 339–364.

[17] Yan Li, A semilinear equation involving the fractional Laplacian in R^n, *J. Math. Anal. Appl.*, 7 (2015).

[18] E. Nezza, G. Palatucci and E. Valdinoci, Hitchhikers guide to the fractional Sobolev spaces, *Bull. Sci. math.*, 136 (2012), 521–573.

[19] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.*, 60 (2007), 67–112.

[20] V. Tarasov and G. Zaslavsky, Fractional dynamics of systems with long-range interaction, *Comm. Nonl. Sci. Numer. Simul.*, 11 (2006), 885–889.

[21] R. Zhuo, W. Chen, X. Cui and Z. Yuan, Radial symmetry of positive solutions to equations involving the fractional Laplacian, *Discrete Contin. Dyn. Syst.*, 36 (2016), 1125–1141.

Received December 2017; revised April 2018.

E-mail address: lvxl0@tom.com
E-mail address: 1119392960@qq.com