Integrability and non-integrability of periodic non-autonomous Lyness recurrences

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This paper studies non-autonomous Lyness-type recurrences of the form $x_{n+2} = (a_n + x_{n+1})/x_n$, where $\{a_n\}$ is a $k$-periodic sequence of positive numbers with primitive period $k$. We show that for the cases $k \in \{1, 2, 3, 6\}$, the behaviour of the sequence $\{x_n\}$ is simple (integrable), while for the remaining cases satisfying this behaviour can be much more complicated (chaotic). We also show that the cases where $k$ is a multiple of 5 present some different features.

Keywords: Integrability and non-integrability of discrete systems; numerical chaos; periodic difference equations; QRT maps; rational and meromorphic first integrals

1. Introduction and main results

The dynamical study of the Lyness difference equation [1–4] and its generalizations to higher order Lyness-type equations, [5–8] or to difference equations with periodic coefficients, [9–14] has been the focus of an active research activity in the last two decades. In more recent dates, Lyness-type equations have also been approached using different points of view: from algebraic geometry [15–17] to the theory of discrete integrable systems [14,18–21].

This paper deals with the problem of the integrability and non-integrability of non-autonomous planar Lyness difference equations of the form

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n},$$

where $\{a_n\}$ is a cycle of $k$ positive numbers, that is, $a_{n+k} = a_n$ for all $n \in \mathbb{N}$, $k$ being the primitive period and we consider positive initial conditions $x_1$ and $x_2$. As we will see, the behaviour of the sequences $\{x_n\}$ can be essentially different according to whether $k \in \{1, 2, 3, 6\}$, $k$ is a multiple of 5 or not.

In this section, we summarize our main results on Equation (1) in terms of $k$. We also give an account of the tools that we have developed for this study that we believe might be interesting by themselves. We start by introducing the notations and definitions used in the paper.

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1.1. Notations and definitions

Given a periodic sequence \( \{a_n\} \) of primitive period \( k \), we will say that its rank is \( m \) if

\[
\text{Card}\{a_1, a_2, \ldots, a_k\} = m \in \mathbb{N}.
\]

The values \( a_1, a_2, \ldots, a_k \) will be usually called parameters. In our context, the recurrence (1) is called persistent if for any sequence \( \{x_n\} \) there exist two real positive constants \( c \) and \( C \), which depend on the initial conditions, such that for all \( n, 0 < c < x_n < C < \infty \).

For each \( k \), the composition maps are

\[
F_{a_k \ldots a_2, a_1} := F_{a_k} \circ \cdots \circ F_{a_2} \circ F_{a_1},
\]

where each \( F_{a_i} \) is defined by

\[
F_{a_i}(x, y) = \left( y, \frac{a_i + y}{x} \right)
\]

and \( a_1, a_2, \ldots, a_k \) are the \( k \) elements of the cycle. When there is no confusion, for the sake of shortness, we will also use the notation \( F_{[k]} := F_{a_k \ldots a_2, a_1} \). Note that these maps are birational maps and are always well defined in the open invariant set \( \mathcal{Q}^+ = \{(x, y) : x > 0, y > 0\} \subset \mathbb{R}^2 \). Moreover,

\[
(x_1, x_2) \xrightarrow{F_{a_1}} (x_2, x_3) \xrightarrow{F_{a_2}} (x_3, x_4) \xrightarrow{F_{a_3}} (x_4, x_5) \xrightarrow{F_{a_4}} (x_5, x_6) \xrightarrow{F_{a_k}} \cdots
\]

and, in general,

\[
F_{[k]}(x_1, x_2) = (x_{k+1}, x_{k+2}).
\]

There are two concepts coexisting in this context, the non-autonomous invariants and the first integrals, that we will use in this paper. Given a difference equation of the form Equation (1), a non-autonomous invariant is a function \( V(x, y, n) \), such that

\[
V(x_{n+1}, x_{n+2}, n + 1) = V(x_n, x_{n+1}, n),
\]

for all initial conditions and all \( n \in \mathbb{N} \). On the other hand, when the difference equation has \( k \)-periodic coefficients, a first integral is a function \( H \), which is a first integral for the discrete dynamical system generated by \( F_{[k]} \), that is, \( H(F_{[k]}(x, y)) = H(x, y) \), for all points in an open set. In terms of the recurrence,

\[
H(x_{n+k}, x_{n+k+1}) = H(x_n, x_{n+1}),
\]

for all initial conditions \( (x_n, x_{n+1}) \). We will relate both concepts in Section 3.

Two analytic functions \( P, Q : U \subset \mathbb{C}^2 \to \mathbb{C} \) are said to be coprime if the points of the set \( \{(x, y) \in U : P(x, y) = Q(x, y) = 0\} \) are isolated. A function \( H = P/Q \), with \( P \) and \( Q \) coprime, will be called a meromorphic function. A meromorphic first integral of an analytic map \( F : U \to \mathbb{C}^2 \) is a meromorphic function \( H = P/Q \) such that

\[
P(F(x, y))Q(x, y) = P(x, y)Q(F(x, y)) \quad \text{for all} \quad (x, y) \in U.
\]
Observe that from this definition, $H(F(x, y)) = H(x, y)$ for all points of $\mathcal{U}$, for which both terms of this last equality are well defined. When $P$ and $Q$ are polynomials, then it is said that $H$ is a rational first integral. Similarly, we can talk about meromorphic or rational invariants, and in this sense we will talk about rational or meromorphic integrability.

Finally, we will say that a planar map $F$ has structurally stable numerical chaos (SSNC) when, studying numerically several of its orbits, we observe that it presents all the features of a non-integrable perturbed twist map, that is, many invariant curves and, between them, couples of orbits of $p$-periodic points (for several values of $p$), half of them of elliptic type and the other half of hyperbolic saddle type. Moreover, the separatrices of these hyperbolic saddles intersect transversally (see, for instance, [22, Chapter 6]).

1.2. Main results

This subsection collects the general outlines of all our results about the recurrence (1), in terms of $k$. Figure 1 shows some typical behaviours of the orbits of $F|_k$. In fact, we consider some maps $G|_k$, which are conjugate to $F|_k$, because the pictures are much more clear. See Lemma 7 for the definition of $G|_k$.

Cases $k \in \{1, 2, 3, 6\}$ and other concrete integrable cases. For $k \in \{1, 2, 3\}$, it is already known that the recurrences (1) are persistent. Moreover, either each sequence $\{x_n\}$ is periodic, with period a multiple of $k$ or it densely fills at most $k$ disjoint intervals of $\mathbb{R}^+$ (see [10]). A key point for the proof is the existence of a rational first integral for $F|_k$. When $k = 6$, we can also prove the existence of a similar first integral (see Corollary 4) and the persistence of recurrence (1). Moreover, we are confident that the same characterization of the sequences $\{x_n\}$ holds but we have only been able to prove the result when $F|_6$ has a unique fixed point in the first quadrant (see Lemma 11 and Proposition 16).

It is also satisfied that for any $k \neq 5$, there are values $a_1, \ldots, a_k$, with primitive period $k$ and high rank, satisfying the property that all the sequences $\{x_n\}$ given by Equation (1) are either periodic, with period a multiple of $k$ or they densely fill at most $k$ disjoint intervals (see Theorem 18). For $k = 5$, there are also some cases with this dynamics; consider, for instance, the trivial case $F_{a,a,a,a,a} = F_5^5$.

Cases $k$ being a multiple of 5. When $k$ is a multiple of 5 (from now on denoted by $k = 5$), apart from the behaviours described above, there appear others for an open set of values of $a_j$, $j = 1, \ldots, k$ and initial conditions. For instance, we can find sequences $\{x_n\}$ such that

$$\liminf_{n \to \infty} x_n = 0 \quad \text{and} \quad \limsup_{n \to \infty} x_n = +\infty,$$

Figure 1. Different possible behaviours of the orbits of $G|_k$, according to $k$. Other behaviours are possible for $k$ being a multiple of 5.
and others such that their adherence (i.e. the sequence itself plus its accumulation points) consists of \( k \) points (see Theorem 6). The existence for \( k = 5 \) of values of \( a_j, j = 1, \ldots, 5 \), for which the sequence \( \{x_n\} \) has the first behaviour, has already been established in previous works (see [6, Example 5.43.1] or [23]), but only for very concrete initial conditions and parameters \( a_1, \ldots, a_5 \).

Moreover, in this case, we can prove that for most values of the parameters the map \( F_{[k]} \) has no meromorphic first integral (see Theorem 19). Furthermore, the phase portrait of the map \( F_{[k]} \) does not always coincide with the ones found in all the rational integrable cases (see, for instance, the second picture in Figure 1). In this case, apart from the celebrated Lyness map \( F_1 \), which satisfies \( F_{1}^{5} = F_{1,1,1,1,1} = Id \), there are values of the parameters \( a_1, \ldots, a_5 \) such that \( F_{[k]}^{m} = Id \) (see Corollary 13) and others for which the number of fixed points of the maps is a one-dimensional manifold, or 2, 1 or 0 points (see Lemma 11). Finally, when \( k \geq 15 \), there are cases presenting SSNC (see Section 6).

**Cases \( k \notin \{1, 2, 3, 5, 6, 10\} \).** When \( k \in \{4, 7, 11, 15\} \), for some values of the parameters, \( a_1, \ldots, a_5 \), we have numerically found SSNC (see Section 6). In fact, we prove in Lemma 22 that based on these examples we can obtain values \( a_1, \ldots, a_5 \), all different, with a similar behaviour for all the remaining values of \( k \). So, for all these values of \( k \), there are situations for which the sequence \( \{x_n\} \) can have different behaviours to those given in the above situations. For instance, there appear sequences which fill more than \( k \) intervals. Some concrete examples for \( k = 4 \) are shown in Section 6.

Observe that as an application of our results we can show an interesting and curious phenomenon that can be understood as a kind of ‘chaos regularization’: consider a map \( G = F_{[2]} \), which is a rationally integrable map, and a map \( H = F_{[4]} \), which has has chaotic behaviour, then both maps \( G \circ H \) and \( H \circ G \) are rationally integrable because they are of type \( F_{[6]} \). Thus, \( G \) regularizes \( H \).

Finally, note that the above results show that the only cases for which recurrence (1) can have a rational invariant for all values of the parameters \( a_1, \ldots, a_5 \) are \( k \in \{1, 2, 3, 6\} \).

### 1.3. Main tools

In this subsection, we present several results that we have obtained, which we believe are interesting by themselves. Other technical results will be given in Section 2.

The first result is a necessary condition for the meromorphic integrability of planar maps near a fixed point. Our approach follows the guidelines of Poincaré when he studied the same problem for ordinary differential equations, [24] and the references therein for the approach to ordinary differential equations. In Section 5, we will apply the result below to study the case \( k = 5 \).

**Theorem 1:** Let \( F : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) be an analytic map defined in \( U \), an open neighbourhood of the origin, such that \( F(0, 0) = (0, 0) \) and \( DF(0, 0) \) is diagonalizable with eigenvalues \( \lambda \) and \( \mu \). Assume that \( F \) has a meromorphic first integral \( H \) in \( U \).

(i) If \( \lambda \mu \neq 0 \), then there exists \( (p, q) \in \mathbb{Z}^2 \), \( (p, q) \neq (0, 0) \), such that \( \lambda^p \mu^q = 1 \).
(ii) If \( \lambda \neq 0 \) and \( \mu = 0 \), then there exists \( n \in \mathbb{N}^+ \) such that \( \lambda^n = 1 \).

When the map \( F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is real valued and of class \( C^2(U) \), the proof of Theorem 1 can be adapted following the same steps. Taking into account that in this case, when \( \lambda \in \mathbb{C} \) is an eigenvalue of \( DF(0, 0) \), then \( \lambda \) is also, and we have to deal with the resonant condition \( \lambda^p \bar{\lambda}^q = 1 \), we obtain the following result.
Corollary 2: Let \( F : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2 \) be a \( C^2(\mathcal{U}) \) map such that \( F(0, 0) = (0, 0) \in \mathcal{U} \) and \( DF(0, 0) \) is diagonalizable, with eigenvalues \( \lambda \) and \( \mu \). Assume that \( F \) has a meromorphic first integral \( H \) in \( \mathcal{U} \).

(i) If \( \lambda, \mu \in \mathbb{R}, \lambda \mu \neq 0 \), then there exists \( (0, 0) \neq (p, q) \in \mathbb{Z}^2 \) such that \( \lambda^p \mu^q = 1 \).

(ii) If \( 0 \neq \lambda \in \mathbb{C} \setminus \mathbb{R} \) (hence \( \mu = \lambda \)), then either \( |\lambda| = 1 \) or \( |\lambda| = |\lambda| e^{i\theta} \) and there exists \( n \in \mathbb{N} \) such that \( (e^{i\theta})^n = 1 \).

(iii) If \( \lambda \neq 0 \) and \( \mu = 0 \), then there exists an \( n \in \mathbb{N}^+ \) such that \( \lambda^n = 1 \).

The above results will be applied to prove the meromorphic non-integrability of many cases when \( k = 5 \) (see Theorem 19). On the other hand, the next result will be the key point to prove the existence of rational integrable cases for all \( k \neq 5 \) (see Theorem 18).

For the recurrence (1) we look for non-autonomous invariants of the form

\[
V(x, y, n) = \frac{\Phi_n(x, y)}{xy}, \quad (3)
\]

where

\[
\Phi_n(x, y) = A_n + B_n x + C_n y + D_n x^2 + F_n y^2 + G_n x^3 + H_n x^2 y + I_n x y^2 \\
+ J_n x^3 y + K_n x^4 + L_n x^3 y + M_n x^2 y^2 + N_n x y^3 + O_n y^4,
\]

with all the sequences of positive numbers. This method is introduced in [11] and the special form of \( V \) is inspired by this paper and the known invariant of the Lyness recurrences (see [1,12,13]). We prove the following theorem.

Theorem 3: If the recurrence (1) has a non-autonomous invariant of the form (3), then \( a_{n+6} = a_n \) and

\[
\Phi_n(x, y) = a_n F_{n+1} + (F_{n+2} + a_{n+1} F_{n+1}) x + (F_{n+1} + a_n F_n) y + F_{n-3} x^2 \\
+ F_n y^2 + F_{n-2} x^2 y + F_{n-1} x y^2, \quad (4)
\]

where \( \{F_n\}_n \) satisfies that \( F_{n+6} = F_n \) and \( a_{n+1} F_{n+2} - a_n F_{n-3} = 0 \).

Corollary 4:

(i) The non-autonomous \( k \)-periodic recurrence (1) has invariants of the form (3) if and only if \( k \in \{1, 2, 3, 6\} \).

(ii) The first integrals of the maps \( F_{[k]} \), for \( k \in \{1, 2, 3, 6\} \), corresponding to the invariants given in Theorem 3 are

\[
V_a(x, y) = \frac{a + (a + 1) x + (a + 1) y + x^2 + y^2 + x y^2}{xy}, \\
V_{b,a}(x, y) = \frac{ab + (a + b^2) x + (b + a^2) y + bx^2 + ay^2 + ax^2 y + bxy^2}{xy}, \\
V_{c,b,a}(x, y) = \frac{ac + (a + bc) x + (c + ab) y + bx^2 + by^2 + cx^2 y + axy^2}{xy}, \\
V_{f,e,d,c,b,a}(x, y) = \frac{af + (a + bf) x + (f + ae) y + bx^2 + ey^2 + cx^2 y + dxy^2}{xy}.
\]
Remark 5: Our proof of Theorem 3 does not require the sequence of parameters \( \{a_n\} \) to be periodic.

Observe that

\[
V_a(x, y) + 2 + a = \frac{(x + 1)(y + 1)(a + x + y)}{xy}
\]

is the usual first integral (invariant) of the map \( F_a \) associated to the classical Lyness recurrence (see, for instance, [1]). It is already known [10,12,13] that in the two- and three-periodic cases, the functions \( V_{b,a} \) and \( V_{a,b,a} \) are first integrals of the maps \( F_{b,a} \) and \( F_{c,b,a} \), respectively. These first integrals play a crucial role for the understanding of the recurrence (1) when \( k = 2, 3 \) (see [9] and [10]). To the best of our knowledge, the existence of a first integral for the general non-autonomous six-periodic case was not known. In Section 4, we use it to describe the dynamics in this case.

Also observe that, due to the form of the invariants, all the maps \( F_k \), for \( k \in \{1, 2, 3, 6\} \), preserve a foliation of the plane given by biquadratic curves, which are elliptic except for a finite number of level sets. In fact, these maps are particular cases of the celebrated Quispel-Roberts-Thomson (QRT) family of planar maps. Perhaps a further algebraic-geometric approach, like the one presented in [16], by studying the maps induced by each \( F_k \) on the corresponding elliptic surface could give more information about the reason why the cases \( k \in \{1, 2, 3, 6\} \) are special.

It is also interesting to notice that other integrable QRT maps with periodic coefficients have been found recently [14] as well as another major family of maps, the Hirota–Kimura–Yahagi-type ones [25].

As we have already commented, it is known that for very concrete values of \( a_1, \ldots, a_5 \) and suitable initial conditions, the behaviour of \( \{x_n\} \) is different to the ones appearing when \( k \in \{1, 2, 3\} \) and, in particular, Equation (1) is non-persistent (see [6, Example 5.43.1] or [23]). This behaviour can also be seen considering

\[
F_{a,1,1,1,1}(x, y) = \left( x, \frac{(x + a)y}{1 + x} \right).
\]

Since \( F_{a,1,1,1,1}(1, y) = (1, (1 + a)y/2) \), it is clear that for \( a > 1 \) the orbits of the points of the form \( (1, y) \) with \( y \neq 0 \) are unbounded.

Our next result allows to establish, when \( k = 5 \), the non-persistence of the recurrence (1) for many values \( a_1, \ldots, a_k \).

**Theorem 6:** Consider recurrence (1) for \( k = 5 \). Set

\[
\phi_i := \prod_{n \equiv i \pmod{5}} a_n, \quad \text{for} \quad i = 1, 2, \ldots, 5.
\]

If for all \( i = 1, \ldots, 5 \), \( \phi_i \neq 1 \) and

\[
\min_{i=1,\ldots,5} \{\phi_i\} < 1 < \max_{i=1,\ldots,5} \{\phi_i\}, \tag{5}
\]

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then Equation (1) is non-persistent. In fact, for an open set of initial conditions, \( \lim \inf_{n \to \infty} (x_n) = 0 \) and \( \lim \sup_{n \to \infty} (x_n) = +\infty \).

In the above result, the open set of initial conditions for which the result holds is sometimes the whole first quadrant \( Q^+ \). For instance, this is the case when \( k = 5 \) and \( a_1 = a, a_2 = ac, a_3 = c, a_4 = 1/a \) and \( a_5 = 1/(ac) \), when \( a > 1 \) and \( ac > 1 \), because

\[
F_{\frac{1}{5}, \frac{1}{5}, c, ac, a}(x, y) = \left( \frac{x}{a}, \frac{y}{ac} \right)
\]  

is a linear map with a stable node at the origin.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminary results, while in Section 3, we prove the main tools described in Section 1.3. Section 4 is devoted to the cases for which we find rational integrability, while in Section 5 we prove the non-integrability results when \( k \) is a multiple of 5. Finally, in Section 6, we present some numerical evidence of chaos.

2. Preliminary results

This section contains some technical preliminary results and other known results that we will use in the proofs given in subsequent sections.

The next result will be useful for our numerical simulations. As we will see, some new variables allow to ‘observe’ much better the numerical non-integrability studied in Section 6. Its proof is straightforward.

**Lemma 7:** The sequence (1) in the variables \( z_n : \log(x_n) \) becomes

\[
z_{n+2} = -z_n + \log(a_n + \exp(z_{n+1}))
\]

and the corresponding maps \( F_{[k]} \) are conjugate to \( G_{[k]} \), where \( G_{[k]} = G_{a_2, a_3, \ldots, a_5, a_1} \),

\[
G_a(x, y) = (y, -x + \log(a + \exp(y))
\]

and each \( G_a \) is defined on the whole plane, \( \mathbb{R}^2 \).

Observe that the maps \( G_{[k]} \) are area preserving. In fact, it is easy to see that the maps \( F_{[k]} \) and \( G_{[k]} \) satisfy the following properties.

**Lemma 8:** For every choice of positive numbers \( a_1, \ldots, a_k \),

(i) The map \( F_{[k]} \) preserves the measure \( m(B) = \int_B \frac{1}{xy} \, dx \, dy \), or, in other words, it preserves the symplectic form \( \omega := \frac{1}{xy} \, dx \wedge dy \). In consequence, it holds that

\[
\mu(F_{[k]}(x, y)) = \det(DF_{[k]}(x, y)) \mu(x, y),
\]

where \( \mu(x, y) = xy \).

(ii) The map \( G_{[k]} \) preserves the Lebesgue measure \( n(B) = \int_B dx \, dy \), that is, it preserves the canonical symplectic form \( dx \wedge dy \). Hence, it holds that

\[
\det(DG_{[k]}(x, y)) = 1.
\]
A related issue in connection with the above lemma is the fact \[15, \text{Theorem 1}\] that the group of symplectic birational transformations of the plane (which is the group of birational transformations of \(C^2\), which preserve the differential form \(\omega\)) is generated by compositions of the Lyness map \(F_1\) (the 5-periodic case of \(F_{[1]}\), with \(a = 1\)), a scaling and a map of the form \((x, y) \rightarrow (x^ay^b, x^cy^d)\), where the matrix \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) \(\in SL(2, \mathbb{Z})\). This was conjectured by Usnich in \[17\] and recently proved by Blanc in the above mentioned reference.

The next result allows to know the dynamics of each \(F_{[k]}\) when the map has a smooth first integral and its applicability to characterize the dynamics of any integrable \(F_{[k]}\) is guaranteed by Lemma 8.

**Theorem 9 (\[26\]):** Let \(U \subset \mathbb{R}^2\) be an open set and let \(F : U \rightarrow U\) be a diffeomorphism such that it has a smooth regular first integral \(V : U \rightarrow \mathbb{R}\) and there exists a smooth function \(\mu : U \rightarrow \mathbb{R}^+\) such that for any \((x, y) \in U\), \(\mu(F(x, y)) = \det(DF(x, y))\). Then, the following holds.

(i) If a level set \(\Gamma_h := \{(x, y) \in U : V(x, y) = h\}\) is a simple closed curve invariant under \(F\), then the map \(F\) restricted to \(\Gamma_h\) is conjugate to a rotation.

(ii) If \(\Gamma_h\) is diffeomorphic to an open-interval curve and invariant under \(F\), then the map \(F\) restricted to \(\Gamma_h\) is conjugate to a translation.

The next result, which is a consequence of Lemma 8, proves that all the fixed points \(p\) of some map \(F_{[k]}\) in \(\mathbb{R}^2\) such that \(F_{a_j, a_{j-1}, \ldots, a_1}(p)\), for all \(j \leq k\), is well defined, are resonant. As we will see in Proposition 20, the hypothesis on the maps \(F_{a_j, a_{j-1}, \ldots, a_1}(p)\) is unavoidable, because for \(j = 5\) there appear some cancellations that make that the maps \(F_{[5]}\) have as fixed point \(p = (0, 0)\) and this point can be of saddle type with arbitrary eigenvalues. In fact, this property will be the key point for our proof of non-existence of meromorphic first integrals for most \(F_{[k]}\), \(k = 5\) (see Theorem 19).

**Corollary 10:** The following statements hold.

(i) Let \(p \in \mathbb{R}^2\) be a fixed point of a composition map \(F_{[k]}\) and such that \(F_{a_j, a_{j-1}, \ldots, a_1}(p)\) is well defined for all \(j \leq k\). Then,

\[
\det(DF_{[k]}(p)) = 1. \tag{7}
\]

(ii) Let \(p \in \mathbb{R}^2\) be an \(m\)-periodic point of a composition map \(F_{[k]}\) such that \(F_{a_j, a_{j-1}, \ldots, a_1}(p)\) is well defined for all \(j \leq km\). Then,

\[
\det(DF_{[k]}^m(p)) = 1.
\]

The following lemma studies the number of fixed points of \(F_{[k]}\) for \(k = 4, 5, 6\).

**Lemma 11:**

(i) There is a unique fixed point of \(F_{d, c, b, a}\) in \(Q^+\) and it satisfies

\[
\begin{align*}
  x &= y^2 + (a - c)y - d, \\
  y &= x^2 + (d - b)x - a.
\end{align*}
\]
(ii) There are either 0, 1, 2 or a continuum of fixed points of $F_{e, d, c, b, a}$ in $Q^+$ and they satisfy
\[
\begin{align*}
(b - 1)x + (1 - d)y + (a - e) &= 0, \\
(c - 1)(b - 1)x^2 + (2e - 1 + bd + ac - ec - eb - ad)x + (e - 1)(e - ad) &= 0.
\end{align*}
\]

(iii) Let $F_{f, e, d, c, b, a} \subset Q^+$ be the set of fixed points of map $F_{f, e, d, c, b, a}$ and let $S_{f, e, d, c, b, a} \subset Q^+$ be the set of singular points of its first integral $V_{f, e, d, c, b, a}$ given in Corollary 4. Then, $F_{f, e, d, c, b, a} = S_{f, e, d, c, b, a}$ and both sets coincide with the set of points of $Q^+$ satisfying
\[
\begin{align*}
y^2 &= \frac{(f + x)(a + bx)}{e + dx}, \\
x^2 &= \frac{(a + y)(f + ey)}{b + cy}.
\end{align*}
\]

Moreover, $\text{Card}(F_{f, e, d, c, b, a}) \geq 1$.

**Proof:**

(i) The set of fixed points of $F_{d, c, b, a}$ is exactly the set of points satisfying
\[
F_d(F_c(F_b(F_a(x, y)))) = (x, y),
\]
but it is not easy to handle these two equations. On the other hand, the equivalent condition,
\[
F_b(F_a(x, y)) = F_c^{-1}(F_d^{-1}(x, y)),
\]
leads us to the system of the statement. Clearly, both parabolas meet at a unique point in $Q^+$.

(ii) Studying the condition
\[
F_c(F_b(F_a(x, y))) = F_d^{-1}(F_e^{-1}(x, y)),
\]
we obtain the system of the statement. The $x$-coordinate of a fixed point has to satisfy the quadratic equation
\[
(c - 1)(b - 1)x^2 + (2e - 1 + bd + ac - ec - eb - ad)x + (e - 1)(e - ad) = 0.
\]
From this equation, we easily obtain the result. Notice that simple cases having infinitely many fixed points appear, for instance, when $b = d = 1$ and $e = a$.

(iii) The two conditions given by
\[
F_e(F_b(F_a(x, y))) = F_d^{-1}(F_c^{-1}(x, y))
\]
directly lead to the system of the statement. The set of singular points of $V_{f, e, d, c, b, a}$ is formed by the points satisfying
\[
\left\{(x, y) \in Q^+ : \frac{\partial}{\partial x}V(x, y) = \frac{\partial}{\partial y}V(x, y) = 0\right\}.
\]
The two equations describing the above set exactly coincide again with the two equations given in the statement. So, \( F_{f.e.d.c.b.a} = S_{f.e.d.c.b.a} \). The Card(\( F_{f.e.d.c.b.a} \)) \( \geq 1 \) can be seen studying the behaviour of the functions, \( \frac{(f+x)(a+by)}{e+dx} \) and \( \frac{(a+y)(f+xy)}{b+cy} \), near 0 and \( +\infty \). \( \square \)

Finally, we will also use the following results.

**Lemma 12:** The map \( F_{1/a, c, ac, a} \) is conjugate to the Lyness’ map \( F_{1/(ac^2)} \).

**Proof:** Observe that \( F_{1/a, c, ac, a}(x, y) = \left( \frac{1+cx}{y}, \frac{x}{a} \right) \) and \( F_{1/ac}({u, v}) = \left( v, \frac{1+v}{u} \right) \). If we consider the linear map

\[
\varphi(x, y) = \left( \frac{y}{c}, \frac{x}{ac} \right),
\]

it holds that \( F_{1/ac} = \varphi \circ F_{1/a, c, ac, a} \circ \varphi^{-1} \), as we wanted to prove. \( \square \)

A nice consequence of the five-global periodicity of the Lyness map \( F_1(x, y) = (y, (1 + y)/x) \) and the above lemma is the following result.

**Corollary 13:** Recurrence (1) with \( k = 4 \) and \([a_1, a_2, \ldots] = [1/e^2, 1/c, c, c^2, 1/e^2, 1/c, \ldots] \) is globally 20-periodic, that is, \( F_{e^2,c,1/e^2}(x, y) = (x, y) \) for all \((x, y) \in Q^+ \).

### 3. Proof of the main tools

This section is devoted to proving Theorems 1, 3 and 6.

**Proof of Theorem 1:** Write

\[
H(x, y) = \frac{P(x, y)}{Q(x, y)} = \frac{P_\bar{\eta}(x, y) + O(\bar{\eta} + 1)}{Q_\bar{m}(x, y) + O(\bar{m} + 1)},
\]

where \( P_\bar{\eta} \) and \( Q_\bar{m} \) are homogeneous polynomials with degrees \( \bar{\eta} \geq 0 \) and \( \bar{m} \geq 0 \), respectively, and \( O(k) \) denotes terms of order at least \( k \). Firstly, we prove that it is not restrictive to assume that \( \bar{\eta} \geq \bar{m} \) and that, if \( \bar{n} = \bar{m} \), then \( P_\bar{n}(x, y)/Q_\bar{m}(x, y) \) is not constant. Notice that if \( H \) is a first integral, then \( 1/H \) is also. Hence, we can assume that \( \bar{n} \geq \bar{m} \). If \( \bar{n} = \bar{m} \) and \( P_\bar{n} = \eta Q_\bar{n} \) for some \( 0 \neq \eta \in \mathbb{C} \), take \( \tilde{H} = H - \eta \). Clearly, \( \tilde{H} \) is a new first integral of the form \( \tilde{H}(x, y) = (O(\bar{n} + 1))/(Q_\bar{n}(x, y) + O(\bar{m} + 1)) \), as we wanted to see.

It is also clear that in a neighbourhood of the origin, we can assume that \( F(x, y) = (\lambda x + O(2), \mu y + O(2)) \).

(i) We start studying the case \( \lambda, \mu \neq 0 \). By imposing that \( H \) is a first integral of \( F \) in \( \mathcal{U} \), we have that

\[
P(F(x, y))Q(x, y) = P(x, y)Q(F(x, y)).
\]

By taking the lower order terms of the above equality, we obtain

\[
P_\bar{\eta}(\lambda x, \mu y)Q_\bar{m}(x, y) = P_\bar{\eta}(x, y)Q_\bar{m}(\lambda x, \mu y).
\]

Define

\[
P_\bar{\eta}(x, y) = \frac{P_\bar{\eta}(x, y)}{\gcd(P_\bar{\eta}(x, y), Q_\bar{m}(x, y))}, \quad Q_\bar{m}(x, y) = \frac{Q_\bar{m}(x, y)}{\gcd(P_\bar{\eta}(x, y), Q_\bar{m}(x, y))},
\]

where \( \gcd \) denotes the greatest common divisor and \( \mathcal{U} \) is defined as the open ball centered at the origin with radius 1.
where \( n \geq m \) are suitable non-negative integers. By using the homogeneity of \( P_n \) and \( Q_m \), equation (8) becomes

\[
\mu^n y^n P_n(\lambda x/(\mu y), 1) y^m Q_m(x/y, 1) = y^n P_n(x/y, 1) \mu^m y^m Q_m(\lambda x/(\mu y), 1),
\]

where, notice that we have cancelled the common factor of \( P_n \) and \( Q_m \). By introducing the polynomials in one variable \( p_n(w) = P_n(w, 1) \), \( q_m(w) = Q_m(w, 1) \), with respective maximum degrees \( n \) and \( m \), and \( \rho = \lambda/\mu \), \( w = x/y \), Equation (9) becomes

\[
\mu^{n-m} p_n(\rho w) q_m(w) = p_n(w) q_m(\rho w), \tag{10}
\]

where we know that \( p_n \) and \( q_m \) are not identically zero and have no common root.

Notice that equality (10) implies that if \( w = w^* \) is a root of \( p_n \), then \( \rho w^* \) is also, and hence \( \rho^j w^* \), for any \( j \in \mathbb{N} \), is a root of \( p_n \). Since \( p_n \) has, at most, \( n \) roots, if \( w^* \neq 0 \), we have that \( \rho^j = 1 \) for some \( k \leq n \), proving the theorem, because \( \lambda^j \mu^{-j} = 1 \). A similar reasoning can be done for \( q_m \). Hence, it only remains to study the cases

\[ p_n(w) = aw^\hat{n}, \quad 0 \leq \hat{n} \leq n \quad \text{and} \quad q_m(w) = bw^\hat{m}, \quad 0 \leq \hat{m} \leq m, \]

for some complex numbers \( a \) and \( b \), \( ab \neq 0 \). Remember that we know that both polynomials have no common roots. So, at least one of the two numbers \( \hat{n} \) or \( \hat{m} \) has to be 0. In any case, Equation (10) becomes

\[
\mu^{n-m} a \rho^\hat{n} w^\hat{n} b \rho^\hat{m} w^\hat{m} = a w^\hat{n} b \rho^\hat{m} w^\hat{m},
\]

giving \( \mu^{n-m}(\lambda/\mu)^{\hat{n} - \hat{m}} = \lambda^{\hat{n} - \hat{m}} \mu^{n+\hat{m} - m - \hat{n}} = 1 \), as we wanted to prove.

(ii) When \( \lambda \neq 0 \) and \( \mu = 0 \), Equation (8), after dropping the common factor of \( P_\hat{n} \) and \( Q_\hat{m} \), becomes

\[ P_n(\lambda x, 0) Q_m(x, y) = P_n(x, y) Q_m(\lambda x, 0). \]

Notice that \( P_n(0, 0) = ax^n, Q_m(0, 0) = bx^m \) and \( (a, b) \neq (0, 0) \), because, otherwise, \( P_n \) and \( Q_m \) would have \( y \) as a common factor. Hence,

\[ a \lambda^n x^n Q_m(x, y) = b \lambda^m x^m P_n(x, y), \]

and so \( ab \neq 0 \). Therefore, \( P_n(x, y) = a \lambda^{n-m} x^{n-m} Q_m(x, y)/b \). Since \( P_n \) and \( Q_m \) have no common factor, we get that \( m = 0 \) and so \( Q_m = b \). Hence, this last equality becomes \( P_n(x, y) = a \lambda^n x^n \). Then, \( ax^n = P(x, 0) = a \lambda^n x^n \), giving \( \lambda^n = 1 \), as we wanted to prove. \( \square \)

To illustrate the above result, in the next remark we present some examples of maps having (or not having) meromorphic first integrals.

**Remark 14:**

(i) The linear maps \( F(x, y) = (\lambda x, \mu y) \), with \( \lambda \) and \( \mu \) satisfying the resonant condition \( \lambda^p \mu^q = 1 \), are the simplest maps with meromorphic first integrals \( H(x, y) = x^p y^q \).

(ii) The map \( F(x, y) = (x + y(x - y), 0) \), with an eigenvalue 0, has the first integral \( H(x, y) = (x - y + 1)(y + 1) \).
(iii) For maps with identically zero linear part, we can have existence or not of meromorphic first integrals. For instance, the map \( F(x, y) = (x^2, xy) \) has the first integral \( H(x, y) = x/y \). On the other hand, by using the same tools as that in our proof of Theorem 1, we can prove that the map \( F(x, y) = (x^2, y^2) \) has no meromorphic first integral.

**Proof of Theorem 3:** The condition that a function \( V(x, y, n) \) of the form (3) is a non-autonomous invariant of the recurrence (1) becomes

\[
V \left( y, \frac{a_n + y}{x}, n + 1 \right) - V(x, y, n) = 0,
\]

for all \((x, y) \in Q^+ \) and all \( n \in \mathbb{N} \). Imposing that each one of the coefficients of the 31 monomials \( x^iy^j \) vanishes identically and playing a little bit with these conditions, we obtain that Equation (4) holds and, moreover, that

\[
a_{n+1}F_{n+2} - a_nF_{n-3} = 0, \tag{11}
\]

\[
F_{n+3} - F_{n-3} + a_{n+2}F_{n+2} - a_nF_{n-2} = 0.
\]

From the first equation, we get that

\[
a_n = \frac{F_{n+2}}{F_{n-3}} a_{n+1} = \frac{F_{n+3}}{F_{n-2}} \frac{F_{n+2}}{F_{n-3}} a_{n+2}.
\]

Plugging this equation in the second one, we obtain that

\[
\frac{F_{n+3} - F_{n-3}}{F_{n-3}} (F_{n-3} - a_{n+2}F_{n+2}) = 0.
\]

Clearly, the above equation holds if either \( \{F_n\}_n \) is a six-periodic sequence or \( F_{n-3} = a_n + 2F_{n+2} \).

In the first situation, let us prove that if \( \{F_n\}_n \) is a \( p \)-periodic sequence, \( p \in \{1, 2, 3, 6\} \), then \( a_n \) also has to be \( p \)-periodic. Assume, for instance, that \( p = 3 \); then, using Equation (11), we have

\[
a_{n+3} \quad \quad a_{n+2} \quad \quad a_{n+1} \quad \quad a_n \quad \quad \frac{F_{n-1}}{F_{n-3}} \frac{F_{n-2}}{F_{n+3}} \frac{F_{n-3}}{F_{n+2}} = 1,
\]

as we wanted to see. The other cases follow similarly.

In the second situation, we have that \( F_{n-3} = a_{n+2}F_{n+2} \). Using this equation and equality (11), we have that

\[
a_{n+2} = \frac{F_{n-3}}{F_{n+2}} = \frac{a_{n+1}}{a_n}.
\]

which is a well-known six-periodic recurrence, as we wanted to see. Finally, using Equation (11) six times, we get that \( F_{n+6} = F_n \). \( \square \)

Before proving Corollary 4, we explain here how non-autonomous invariants and first integrals are related. Consider a recurrence with \( k \)-periodic coefficients and having a
non-autonomous invariant \( V(x, y, n) \) that satisfies \( V(x, y, n) = V(x, y, n + k) \). Then, \( H(x, y) : V(x, y, 1) \) is a first integral of \( F_{[k]} \). Conversely, if \( H(x, y) \) is a first integral, then

\[
V(x, y, n) := H(F_{a_1, a_2, \ldots, a_k}(x, y)), \quad \text{where} \quad 1 \leq \ell \leq k, \quad n - \ell = k
\]
is a non-autonomous periodic invariant of the recurrence. These relations are used in the next corollary for constructing the first integrals of \( F_{[k]}, k = 1, 2, 3, 6 \) using the invariant found in the above theorem. Notice also that there is another way to relate both concepts. Indeed, it is possible to replace the non-autonomous \( k \)-periodic recurrence by an autonomous map on an enlarged phase space of dimension \( \mathbb{R}^2 \times \mathbb{R}^k = \mathbb{R}^{k+2} \), simply considering the map \((x_n, x_{n+1}, a_1, a_2, \ldots, a_k) \rightarrow (x_{n+1}, x_{n+2}, a_2, a_3, \ldots, a_k, a_1)\). In this case, any non-autonomous invariant is just an ordinary first integral for the above map. This approach is not used in this paper.

**Proof of Corollary 4:**

(i) This result is proved along the proof of Theorem 3, above.

(ii) We only give the details for \( k = 6 \). The other cases follow similarly. We introduce the following notations for the non-autonomous six-periodic recurrences.

\[
\{a_n\} = a_1, a_2, \ldots = a, b, c, d, e, f, a, b, c, d, e, f, a, \ldots \\
\{F_n\} = F_1, F_2, \ldots = 1, \ell, m, n, o, p, 1, \ell, m, n, o, p, 1, \ldots
\]

From the relations \( a_{n+1}F_{n+2} - a_nF_{n-3} = 0 \), we obtain that

\[
\ell = \frac{f}{e}, \quad m = \frac{a}{e}, \quad n = \frac{b}{e}, \quad o = \frac{c}{e} \quad \text{and} \quad p = \frac{d}{e}.
\]

Hence, with the notations of Theorem 3, we get that

\[
\Phi_1(x, y) = a_1F_2 + (F_3 + a_2F_2)x + (F_2 + a_1F_1)y + F_0x + F_1y + F_1x^2 + F_0x + F_0x^2 + F_0y^2 \\
= a\ell + (m + b\ell)x + (a - \ell)y + nx^2 + y^2 + ax^2 + px^2
\]

\[
= \frac{af + (a + bf)x + (f + ae)y + bx^2 + ey^2 + cx^2 + dx^2}{e}.
\]

Hence, \( V_{f, e, d, c, b, a}(x, y) = e\Phi_1(x, y)/(xy) \) is a first integral of \( F_{[6]} \), as we wanted to prove. □

We first prove Theorem 6 for \( k = 5 \).

**Proposition 15:** Consider recurrence (1) with \( k = 5 \) and \( a_i \neq 1 \), for \( i = 1, 2, \ldots, 5 \) and satisfying

\[
\min\{a_1, a_2, a_3, a_4, a_5\} < 1 < \max\{a_1, a_2, a_3, a_4, a_5\}.
\] (12)

Then, the recurrence (1) is non-persistent. Moreover, for an open set of initial conditions \( \lim\inf_{n\to\infty} x_n = 0 \) and \( \lim\sup_{n\to\infty} x_n = +\infty \).
Proof: A computation shows that $F_{[5]}(x, y) = (P_1(x, y), P_2(x, y))$ where
\[
P_1(x, y) = \frac{x (a_2 xy + a_4 y^2 + a_2 x + (a_1 a_4 + 1) y + a_1)}{(a_1 + y)(a_1 + a_2 x + y)},
\]
\[
P_2(x, y) = \frac{y N(x, y)}{(a_1 + a_2 x + y + a_3 xy)(a_1 + a_2 x + y)}
\]
and
\[
N(x, y) = a_3 x^2 y + a_4 xy^2 + a_2 x^2 + (a_1 a_4 + a_2 a_5 + 1) xy + a_5 y^2 + a_1(1 + a_2 a_5)x + 2 a_1 a_5 y + a_1^2 a_5.
\]

First, observe that, contrary to what happens for $F_{[k]}$, $k < 4$, $F_{[5]}$ can be extended to a neighbourhood of $Q^+$. Note also that $(0, 0)$ is a fixed point and
\[
DF_{[5]}(0, 0) = \begin{pmatrix} \frac{1}{a_2} & 0 \\ 0 & a_5 \end{pmatrix}.
\]

So, under our hypotheses, the origin is a hyperbolic fixed point of $F_{[5]}$. Arguing similarly with the shifted maps $F_{a_1, a_5, a_3, a_2}, F_{a_2, a_1, a_5, a_3}, F_{a_3, a_2, a_1, a_5, a_4}$ and $F_{a_4, a_3, a_2, a_1, a_5}$, we obtain that the origin is also a hyperbolic fixed point for these maps, with Jacobian matrices
\[
\begin{pmatrix} \frac{1}{a_2} & 0 \\ 0 & a_1 \end{pmatrix}, \begin{pmatrix} \frac{1}{a_3} & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} \frac{1}{a_4} & 0 \\ 0 & a_3 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{a_5} & 0 \\ 0 & a_4 \end{pmatrix},
\]
respectively.

Condition (12) implies that there exists at least a parameter with a value less than 1, and others greater than 1. Since there are no parameters with value equal to 1 and the sequence is cyclic, we can choose two contiguous parameters such that $a_i < 1$ and $a_{i+1} > 1$. This implies that the origin is an attractive fixed point for some of the five shifted maps. For example, suppose that $a_2 < 1$ and $a_3 > 1$, then the origin is a stable node for $F_{a_2, a_1, a_5, a_4, a_3}$.

Taking an initial condition $(x_0, y_0)$, with positive coordinates and in the basin of attraction of the origin for the corresponding shifted map, we obtain that $\liminf_{n \to \infty} x_n = 0$ for the solution of Equation (1), with initial condition $x_1 = x_0$ and $x_2 = y_0$.

Recall $F_{a_j}(x, y) = (y, (a_j + y) / x)$, $a_j \neq 0$. Thus, the fact that $\limsup_{n \to \infty} x_n = +\infty$ follows because if some $\{(x_{n_i}, y_{n_i})\}_{n_i}$ tends to $(0, 0)$, then the second component of $\{F_{a_j}(x_{n_i}, y_{n_i})\}_{n_i}$ tends to $+\infty$. \hfill \Box

Proof of Theorem 6: Set $F_{[k]}$ for $k = 5m$. We can write $F_{[k]} = F_{a_{5m}, \ldots, a_{5m-4}} \circ \ldots \circ F_{a_5, \ldots, a_1}$, so $F_{[k]}$ can be extended to a neighbourhood of $Q^+$. Furthermore, observe that
\[
DF_{[k]}(0, 0) = DF_{a_{5m}, \ldots, a_{5m-4}}(0, 0) \circ \ldots \circ DF_{a_5, \ldots, a_1}(0, 0) = \begin{pmatrix} \frac{1}{\phi_2} & 0 \\ 0 & \phi_1 \end{pmatrix}.
\]
Similarly, the Jacobian matrices of the shifted maps have the form
\[
\begin{pmatrix} \frac{1}{\phi_{i+1}} & 0 \\ 0 & \phi_i \end{pmatrix}.
\]
Arguing as in Proposition 15, the relation (5) implies that there exists at least a couple \((\phi_i, \phi_{i+1})\) such that one of the values is greater than 1 and the others less than 1. So, the origin is an attractive fixed point for some of the \(m\)-shifted maps. Now, the proof follows again as in Proposition 15.

4. Rational integrability and associated dynamics

As we have already explained in Section 1.2, the cases \(k = 1, 2, 3\) are very similar and totally understood. From Corollary 4, we can prove in the next proposition a similar result when \(k = 6\). Before stating the result, we introduce the following notation:

\[
P_{f,e,d,c,b,a} := \{(a, b, c, d, e, f) \in (\mathbb{R}^+)^6 : \text{system (13) has a unique solution in } Q^+\}
\]

\[
\begin{align*}
  y^2 &= \frac{(f + x)(a + bx)}{dx + e}, \\
  x^2 &= \frac{(a + y)(f + ey)}{b + cy}.
\end{align*}
\]

**Proposition 16:** For \(k = 6\), the recurrence (1) is persistent. Moreover, if \((f, e, d, c, b, a) \in P_{f,e,d,c,b,a}\), any sequence \(\{x_n\}\) generated by Equation (1) is either periodic, with period a multiple of 6 or it densely fills, at most, six disjoint intervals of \(\mathbb{R}^+\).

**Proof:** We follow the same steps as in the proof of [10, Theorem 1]. To prove the persistence of Equation (1), it suffices to show that each level curve \(\{(x, y) : V_{f,e,d,c,b,a}(x, y) = h\} \cap Q^+\) is bounded. Since

\[
\frac{af}{xy} + \frac{a + bf}{y} + \frac{f + ae}{x} + \frac{bx}{y} + \frac{ey}{x} + cx + dy = h,
\]

we know that

\[
\frac{f + ae}{h} \leq x \leq \frac{h}{c} \quad \text{and} \quad \frac{a + bf}{h} \leq y \leq \frac{h}{d}
\]

and the persistence follows.

By Lemma 11(iii), under our hypotheses, the set of fixed points of \(F_{[6]}\) and the set of singular points of \(V_{[6]}\) coincide and consist of a single point. Following again the same guidelines of the proof of [10, Theorem 1], which in turn is based on [27, Proposition 2.1], we prove that all the level curves of \(V_{[6]}\) in \(Q^+\), apart from the fixed point, are diffeomorphic to circles. Hence, by using Lemma 8 and Theorem 9(i), the proposition follows.

**Remark 17:** We believe that \(P_{f,e,d,c,b,a}\) is the whole of \((\mathbb{R}^+)^6\), but we have not been able to prove this equality. In any case, it is easy to find sufficient conditions to ensure that some \((a, b, c, d, e, f)\) belongs to \(P_{f,e,d,c,b,a}\). For instance, since

\[
\begin{align*}
  \frac{\partial}{\partial x} \left(\frac{(f + x)(a + bx)}{e + dx}\right) &= \frac{bdx^2 + 2bex + ae + bef - adf}{(e + dx)^2}, \\
  \frac{\partial}{\partial y} \left(\frac{(a + y)(f + ey)}{b + cy}\right) &= \frac{cey^2 + 2bey + bf + abe - acf}{(b + cy)^2},
\end{align*}
\]
when both numerators have no positive real roots, the point \((a, b, c, d, e, f)\) is in the set, because the functions that we have derived are both increasing and so the curves defined by system (13) cut at a single point.

Next result collects our integrability results for any \(k \neq 5\).

**Theorem 18:**

(i) For any \(k \geq 15\), there exist sequences \(\{a_n\}\) of prime period \(k\) and rank \(k\) such that \(F[k] = F_{a_k\ldots a_2 a_1}\) is rationally integrable and the corresponding recurrence (1) is persistent.

(ii) For any \(k < 15\), \(k \neq 5\), there exist sequences \(\{a_n\}\) of prime period \(k\) with the ranks as in Table 1, such that \(F[k]\) is rationally integrable and the corresponding recurrence (1) is persistent.

(iii) Moreover, it is possible to take in all the above cases parameters \(a_1, a_2, \ldots a_k\) such that each sequence \(\{x_n\}\) is either periodic, with period a multiple of \(k\) or it densely fills, at most, \(k\) disjoint intervals of \(\mathbb{R}^+\).

**Proof:** We start by introducing some notation. Given \(a_i\) and \(c_i\) are positive, we consider the sets \(S_i := \left\{ \frac{1}{a_i c_i}, \frac{1}{a_i}, c_i, a_i c_i, a_i \right\}\). Assume that \(a_i\) and \(c_i\) are such that \(\text{Card}(S_i) = 5\) and consider

\[
\Phi_i(x, y) = F_{\frac{1}{a_i c_i}, \frac{1}{a_i}, c_i, a_i c_i, a_i}(x, y) = \left( \frac{x}{a_i}, \frac{y}{a_i c_i} \right),
\]

where we have used expression (6). Notice that \(1 \notin S_i\). Given any natural number \(m \geq 1\), we also consider \(m\) sets \(S_1, S_2, \ldots, S_m\), and define \(\Phi^{[m]} = \Phi_m \circ \Phi_{m-1} \circ \cdots \circ \Phi_1\). Then,

\[
\Phi^{[m]}(x, y) = \left( \frac{x}{m \prod_{i=1}^{m} a_i}, \frac{y}{m \prod_{i=1}^{m} a_i c_i} \right).
\]

When \(m = 0\), we consider \(\Phi^{[0]}(x, y) = (x, y)\). Finally, choosing the values of the parameters such that \(\prod_{i=1}^{m} a_i = 1\) and \(\prod_{i=1}^{m} c_i = 1\), we obtain that for all \(m \geq 3\), \(\Phi^{[m]}(x, y) = (x, y)\). Moreover, for these values of \(m\), the parameters \(a_i\) and \(c_i\) can be chosen such that \(\text{Card}(\bigcup_{i=0}^{m} S_i) = 5m\). Observe also that when \(m = 1\), it is not possible to choose \(a_1 = 1\) and \(c_1 = 1\). When \(m = 2\), it is again possible, but with \(\text{Card}(\bigcup_{i=0}^{2} S_i) = 5\).

Now we can start the proof of the theorem. First, we study the cases \(k \leq 4\) and \(k \geq 15\). Consider \(k = 5m + \ell\) with \(\ell \in \{0, 1, 2, 3, 4\}\) and \(m = 0\) or \(m \geq 3\).
Now, taking \( \Psi_1(x, y) \):

\[
\begin{aligned}
\Psi(x, y) := \\
&\begin{cases} \\
(x, y) & \text{for } \ell = 0, \\
F_a(x, y) & \text{for } \ell = 1, \\
F_{b,a}(x, y) & \text{for } \ell = 2, \\
F_{c,b,a}(x, y) & \text{for } \ell = 3, \\
F_{d,c,a,c,a}(x, y) & \text{for } \ell = 4,
\end{cases}
\end{aligned}
\]

(14)

with suitable values of \( a, b \) and \( c \), we obtain that the orbits of

\[
F_{k}(x, y) := \Psi \circ \Phi^{[m]}(x, y) = \Psi(x, y)
\]

are like the ones of \( \Psi \) and the rank(\( \{ a_n \} \)) = \( k \). Then, by using the known results for \( k = 1, 2, 3 \) and Lemma 12, the result follows for \( k \geq 15 \) and \( k \leq 4 \).

When \( k = 6 \), the result is proved in Proposition 16. Finally, for \( 7 \leq k \leq 14 \), we consider

\[
k = 7, \ F_{b,a,1,1,1,1} \quad \text{with rank 3,}
\]

\[
k = 8, \ F_{c,b,a,1,1,1,1} \quad \text{with rank 4,}
\]

\[
k = 9, \ F_{1/c,c,a,c,a,1,1,1,1} \quad \text{with rank 5,}
\]

and \( \Psi \circ F_{\tilde{a},\tilde{c},1/\tilde{c},1/(\tilde{a}\tilde{c}),1/\tilde{a},1/(\tilde{a}\tilde{c}),1/\tilde{a},\tilde{c},\tilde{a},\tilde{c},\tilde{a}} = \Psi \) for \( 10 \leq k \leq 14 \), with \( \tilde{a} \) and \( \tilde{c} \) suitably chosen.

All these \( F_{[k]} \) have ranks \( k - 5 \), as we wanted to prove.

5. Meromorphic non-integrability for the case \( k = 5 \)

Our main result is the following theorem.

**Theorem 19:** For \( k = 5 \) and most values of \( \{ a_n \} \), the map \( F_{[k]} \) has no meromorphic first integral.

Its proof is a consequence of the following result.

**Proposition 20:** For \( k = 5 \), let \( \phi_i, i = 1, \ldots, 5 \) be as in Theorem 6,

\[
\phi_i = \prod_{n = i \text{ (mod 5)}}^{n = 1, \ldots, k} a_n.
\]

Then, if \( \{ \phi_2, \phi_3, \phi_4, \phi_5 \} \not\subset \{ \phi_1', r \in \mathbb{Q} \} \), the map \( F_{[k]} \) has no meromorphic first integral.

**Proof:** For simplicity, we prove the result for the case \( k = 5 \), being the proof in the general case similar. Note that the above condition reads as

\[
\{ b, c, d, e \} \not\subset \{ a', r \in \mathbb{Q} \}.
\]

If \( F_{5} = F_{c,d,e,b,a} \) has a meromorphic first integral, the same holds for all the other maps, \( F_{a,e,d,c,b}, F_{b,a,e,d,c}, F_{c,b,a,e,d} \) and \( F_{d,c,b,a,e} \). These five maps have the origin \((0, 0)\) as a fixed point and are analytic in its neighbourhood (see the proof of Proposition 15). Moreover, the corresponding couples of eigenvalues of their linear parts at zero are \( 1/a, \)}
e; 1/b, a; 1/c, b; 1/d, c and 1/e, d, respectively. Hence, applying Theorem 1, we obtain the following necessary conditions for the existence of a meromorphic first integral.

\[a^{n_1}e^{m_1} = a^{n_2}b^{m_2} = b^{n_3}c^{m_3} = c^{n_4}d^{m_4} = d^{n_5}e^{m_5} = 1,\]

for some \(n_i, m_i \in \mathbb{Z}, i = 1, \ldots, 5\). From these equalities, we get that \(\{b, c, d, e\} \subset \{a^r, r \in \mathbb{Q}\}\). So, the result follows.

A simple corollary of the above result is as follows.

**Corollary 21:** The map \(F_{e, d, c, b, 1}\) has a meromorphic first integral if and only if \(b = c = d = e = 1\).

Of course, all the known rationally integrable cases, like \(F_5 = F_{a, 5}, F_{10} = F_{b, 5}, F_{15} = F_{c, b, a}^3\) and \(F_{20} = F_{1/a, c, a, c, a}^5\), satisfy \(\{\phi_2, \phi_3, \phi_4, \phi_5\} \subset \{\phi_r^t, r \in \mathbb{Q}\}\).

6. **Numerical evidences of chaos**

Our simulations show that there are examples of maps \(F_k\) exhibiting SSNC when \(k \in \{4, 7, 8, 11, 15\}\). These behaviours can be seen by plotting some orbits of the corresponding conjugated maps \(G_k\) for the \(k\)-periodic sequences of parameters 2, 2, \ldots, 2, 3. More visual examples can be obtained by studying the maps \(G_{6, 7, 4, 2}, G_{1, 8, 1, 8, 7, 4, 2}, G_{8/2, 1, 6, 1, 8, 7, 4, 2}, G_{1, 8, 1, 8, 7, 4, 2}\) and \(G_{7, 4, 2, 6, 1, 8, 2, 8/2, 1, 8, 6, 8, 7, 4, 2}\) with \(\delta = 0.001\). See some pictures in Figures 1 and 2.

In fact, we prove the following.

**Lemma 22:** If there exist maps of the form \(F_4, F_7\) and \(F_{11}\) exhibiting SSNC, then, for any \(k \geq 7, k \notin \{10, 15\}\), there exist maps of the form \(F_k\) with \(\text{rank}([a_n]) = k\) also having SSNC.

**Proof:** For \(m \geq 0\), the maps,

\[F_{5m+11} := F_{1, 1, 1, 1, 1}^m \circ F_{11} = F_{11},\]

![Figure 2. Some orbits of \(G_{0.001, 7, 4, 2}\) and a zoom with much more orbits.](image-url)
will also have SSNC. Note that, since $20 \equiv 0$, $11 \equiv 1$, $7 \equiv 2$, $8 \equiv 3$ and $4 \equiv 4 \pmod{5}$, the above maps cover all the values of $k$ given in the statement. Since one of the features of these maps is the existence of transversal homoclinic points, which is a structurally stable property, we can perturb each of the corresponding $a_j$ by $a_j + \varepsilon_j$, with all the $\varepsilon_j$ sufficiently small, to obtain $k$-periodic sequences of parameters having SSNC and $\text{rank}((a_n)) = k$, as we wanted to prove. 

We have (only numerically) shown the existence of SSNC for $k \in \{4, 7, 11, 15\}$, but note that the above lemma allows to reduce all the other cases to these four ones.

Although the complicated behaviour of the maps, as in Figure 2, leads one to believe that even there may be no upper bound for the number of intervals given by the adherence of a sequence (formed by the sequence itself and its accumulation points), we only present here some simple examples. Concretely, for $k = 4$, we give a map $F_4$ and two sets of initial conditions such that the adherence of the sequences $\{x_n\}$ generated by Equation (1) consists of more than $k$ intervals.

We have that for $a = 2$, $b = 4$, $c = 7$ and $d = 0.001$,

- the sequence starting at 13.35, 7.27 is formed by 20 intervals;
- the sequence starting at 14.8, 8.25 is formed by 7 intervals.

For instance, this last assertion can be seen by making the phase portraits of the orbit of $G_{0.01, 7, 4, 2}$ starting at (14.8, 8.25), which is formed by 5 islands, together with their images through $G_2$, $G_4$, 2 and $G_{7, 4, 2}$ and their projections in the $x$-axis (see Figure 3). The property of the existence of sequences $\{x_n\}$ generated by Equation (1) such that their adherence consists of more that $k$ intervals should be true for all the values of $k$ given in Lemma 22. We also want to comment that, for these values of $k$, the initial conditions lying on the stable manifolds of the $q$-periodic saddle points of $G_k$ also have a curious behaviour; the adherence of $\{x_n\}$ is the sequence itself, together with $q$ more points, corresponding

![Figure 3. Case $k = 4$. An orbit of $G_4$, their images through $G_{a_i, d_i, \ldots, -a_1}$, $i = 1, 2, 3$, and the projection corresponding to $\{x_n\}$.](image-url)
to the saddle points. In general, $q$ also is greater than $k$. On the other hand, the most complicated orbits, that is, the ones between two big invariant curves, whose adherence seems to fill a region of positive measure give rise to a single interval when we consider their projections given by the sequence $\{x_n\}$.

A final remark
After the first version of this paper was finished, one of the authors (A. Cima) and S. Zafar obtained a proof of the non-rational integrability of generic maps $F_{[4]}$ and $F_{[5]}$ by computing its dynamical degree [28].

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