HAAR-OPEN SETS: A RIGHT WAY OF GENERALIZING THE STEINHAUS SUM THEOREM TO NON-LOCALLY COMPACT GROUPS

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Abstract. Let $X$ be the countable product of Abelian locally compact Polish groups and $A, B \subseteq X$ be two Borel sets, which are not Haar-null in $X$. We prove the then the sum-set $A + B := \{a + b : a \in A, \ b \in B\}$ is 
Haar-open in the sense that for any non-empty compact subset $K \subseteq X$ and point $p \in K$ there exists a point $x \in X$ such that the set $K \cap (A + B + x)$ is a neighborhood of $p$ in $K$. This is a generalization of the classical Steinhaus Theorem (1920) to non-locally compact groups. We do not know if this generalization holds for Banach spaces.

In [9] Steinhaus proved that for any Borel subsets $A, B$ of positive Haar measure in the real line the sum-set $A + B := \{a + b : a \in A, \ b \in B\}$ has non-empty interior and the difference set $A - A := \{a - b : a, b \in A\}$ is a neighborhood of zero in the real line. In [10] Weil generalized this theorem of Steinhaus to all locally compact topological groups. The Steinhaus-Weil Theorem has many important applications, for example, to automatic continuity [8] or to functional equations [6].

In [2] Christensen introduced the notion of Haar-null set in a topological group and generalized the difference part of the Steinhaus Theorem to non-locally compact Polish groups proving that for any Borel subset $A$ of a Polish Abelian group $X$ the difference $A - A$ is a neighborhood of zero in $X$ if the set $A$ is not Haar-null.

Following Christensen, we define a universally measurable subset $A$ of an Abelian topological group $X$ to be Haar-null if there exists a $\sigma$-additive Borel probability measure $\mu$ on $X$ such that $\mu(A + x) = 0$ for all $x \in X$. A subset $A$ of a topological space $X$ is called universally measurable is it is measurable with respect to any Radon-$\sigma$-additive Borel probability measure on $X$. A Borel probability measure $\mu$ on a topological space $X$ is called Radon if for any Borel subset $B \subseteq X$ and any $\varepsilon > 0$ there exists a compact set $K \subseteq B$ such that $\mu(B \setminus K) < \varepsilon$. Any Radon measure $\mu$ on a topological space is regular, which means that for any $\varepsilon > 0$, any closed set $F \subseteq X$ has an open neighborhood $U \subseteq X$ such that $\mu(U \setminus F) < \varepsilon$.

Let us observe that in contrast to the difference part of the Steinhaus Theorem, its sum-part does not generalize to non-locally compact groups.

Example 1. The subset $A = [0,\infty)^\omega$ is not Haar-null in the Polish group $\mathbb{R}^\omega$, yet $A = A + A$ has empty interior in $\mathbb{R}^\omega$.

Nonetheless, in this paper we generalize the sum-part of the Steinhaus Theorem using a proper interpretation of non-empty interior for $A + B$ in non-locally compact case.

Definition 2. A subset $A$ of a topological group $X$ is called Haar-open if for any compact subset $K \subseteq X$ and any point $x \in K$ there exists a point $y \in X$ such that the set $K \cap (A + y)$ is a neighborhood of $x$ in $K$.

Haar-open sets in Abelian complete metric groups admit the following characterization.

Theorem 3. A subset $A$ of an Abelian complete metric group $X$ is Haar-open if and only if for any compact set $K \subseteq X$ there exists a point $x \in X$ such that the set $K \cap (A + x)$ has non-empty interior in $K$.

Proof. The “only if” part is trivial. To prove the “if” part, take any non-empty compact space $K \subseteq X$ and a point $p \in K$. Let $\rho$ be an invariant complete metric generating the topology of the group $X$.

For every $n \in \omega$ consider the compact set $K_n := \{x \in K : \rho(x, p) \leq \frac{1}{2^n}\}$ neighborhood of the point $p$ in $K$.

Next, consider the compact space $\Pi := \prod_{n \in \omega} K_n$ and the map

$$f : \Pi \to X, \ f : (x_n)_{n \in \omega} \mapsto \sum_{n=0}^{\infty} x_n,$$

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which is well-defined and continuous because of the completeness of the metric ρ and the convergence of the series \( \sum_{n=0}^{\infty} \frac{x_n}{\lambda_n} \).

If for the compact set \( f(\Pi) \) in \( X \) there exists \( x \in X \) such that the intersection \( f(\Pi) \cap (A + x) \) contains some non-empty open subset \( U \) of \( f(\Pi) \), then we can choose any point \( (x_i)_{i \in \omega} \in A + x \) and find \( n \in \omega \) such that \( \{(x_i)_{i < n} \times \prod_{i \geq n} K_i \subset f^{-1}(U) \). Then for the point \( s = \sum_{i < n} x_i \in \sum_{i < n} K_i \subset X \), we get \( s + K_n \subset U \subset A + x \) and hence the intersection \( K \cap (A + x) \supset K_n \) is a neighborhood of \( p \) in \( K \).

It is clear that a subset of a locally compact topological group is Haar-open if and only if it has non-empty interior. That is why the following theorem can be considered as a generalization of the Steinhaus sum theorem.

**Main Theorem.** Let \( X := \prod_{n \in \omega} X_n \) be the countable product of locally compact Abelian topological groups. If universally measurable sets \( A, B \subset X \) are not Haar-null, then their sum \( A + B \) is Haar-open in \( X \).

This theorem follows from Lemmas 7 and 8 proved below. In the proofs we shall use special measures, introduced in Definitions 4 and 5. For a Borel measure \( \mu \) on a topological space \( X \) a Borel set \( B \subset X \) will be called \( \mu \)-positive if \( \mu(B) > 0 \).

**Definition 4.** A Borel probability measure \( \mu \) on an Abelian topological group \( X \) is called Steinhaus-like if

1. \( \mu(-B) = \mu(B) \) for any Borel set \( B \subset X \) and
2. for any \( \mu \)-positive Borel sets \( A, B \subset X \), the intersection \( (A + a) \cap (B + b) \) is \( \mu \)-positive for some points \( a, b \in X \).

**Definition 5.** Let \( K \) be a compact set in an Abelian topological group \( X \). A \( \sigma \)-additive Borel probability measure \( \mu \) on \( X \) is called locally \( K \)-invariant if there exists an increasing sequence \( (M_n)_{n \in \omega} \) of Borel sets in \( X \) such that

1. \( \lim_{n \to \infty} M_n = 1 \) and
2. for every \( n \in \mathbb{N} \) there exists a neighborhood \( U \subset X \) of \( \theta \) such that \( \mu(B + x) = \mu(B) \) for any Borel set \( B \subset M_n \) and any \( x \in U \cap K \).

**Lemma 6.** If a Radon Borel probability measure \( \mu \) on an Abelian topological group \( X \) is locally \( K \)-invariant for some compact set \( K \supset \theta \), then for any \( \mu \)-positive compact set \( C \) the set \( P := \{ x \in K : \mu(C \cap (C + x)) > 0 \} \) is a neighborhood of \( \theta \) in \( K \).

**Proof.** Let \( (M_n)_{n \in \omega} \) be an increasing sequence of Borel sets, witnessing that the measure \( \mu \) is locally \( K \)-invariant. Since \( \lim_{n \to \infty} \mu(M_n) = 1 \), there exists \( n \in \mathbb{N} \) such that \( \mu(C \cap M_n) > \frac{3}{4} \mu(C) \) and hence \( \mu(C') > \frac{3}{4} \mu(C) \) for some compact subset \( C' \subset C \cap M_n \). By Definition 5 for the number \( n \) there exists a neighborhood \( U \subset K \) of \( \theta \) such that \( \mu(B + x) = \mu(B) \) for any Borel set \( B \subset M_n \) and any \( x \in U \). By the regularity of the measure \( \mu \), we can assume the neighborhood \( U \) to be so small that \( \mu(C' + U) < \frac{5}{4} \mu(C') \). It remains to prove that \( K \cap U \subset P \).

Indeed, for any \( x \in K \cap U \), the set \( C' \cap M_n \) has measure \( \mu(C') = \mu(C' + x) \). Since \( C' + x \subset C' + U \), we conclude that

\[
\begin{align*}
\mu(C \cap (C + x)) &\geq \mu(C' \cap (C' + x)) = \mu(C') + \mu(C' + x) - \mu(C' \cup (C' + x)) \\
&\geq 2 \mu(C') - \mu(C' + U) > 2 \frac{3}{4} \mu(C) - \frac{5}{4} \mu(C) = \frac{1}{4} \mu(C) > 0,
\end{align*}
\]

which means that \( x \in P \) and \( K \cap U \) is a neighborhood of \( \theta \) in \( K \).
Lemma 8. Let $X = \prod_{n \in \omega} X_n$ be the Tychonoff product of locally compact Abelian topological groups. Then for every compact subset $K \subset X$ the group $X$ admits a Steinhaus-like locally $K$-invariant $\sigma$-additive Borel measure with compact support.

Proof. Fix a compact set $K \subset X$. Replacing $K$ by $K \cup \{\theta\} \cup (-K)$ we can assume that $\theta \in K = -K$.

For every $n \in \omega$ identify the group $X_n$ with the subgroup $\{(x_i)_{i \in \omega} : x_n = \theta_n\}$ in $X$. Here by $\theta_n$ we denote the neutral element of the group $X_n$. Also we identify the products $\prod_{i < n} X_i$ and $\prod_{i \geq n} X_i$ with the subgroups $X_{<n} := \{(x_i)_{i \in \omega} : \forall i < n \ x_i = \theta_i\}$ and $X_{\geq n} := \{(x_i)_{i \in \omega} : \forall i \geq n \ x_i = \theta_i\}$, respectively.

Let $\text{pr}_n : X \to X_n$ and $\text{pr}_{<n} : X \to X_{<n}$ be the natural coordinate projections. For every $n \in \omega$ let $K_n$ be the projection of the compact set $K$ onto the locally compact group $X_n$.

Fix a decreasing sequence $(O_{i,n})_{i \in \omega}$ of open neighborhoods of zero in the group $X_n$ such that $\overline{O_{0,n}}$ is compact and $O_{i+1,n} + O_{i+1,n} \subset O_{i,n} = -O_{i,n}$ for every $i \in \omega$. For every $m \in \omega$ consider the neighborhood $U_{m,n} := O_{1,n} + O_{2,n} + \cdots + O_{m,n}$ of zero in the group $X_n$. By induction it can be shown that $-U_{m,n} = U_{m,n} \subset O_{0,n}$ and $U_{m,n} \subset U_{m+1,n}$ for all $m \in \omega$. Let $U_{o,n} = \bigcup_{m \in \omega} U_{m,n}$.

Fix a Haar measure $\lambda_n$ in the locally compact group $X_n$. Since the locally compact Abelian group $X_n$ is amenable, by the Følner Theorem [7, 4.13], there exists a compact set $\Lambda_n = -\Lambda_n \subset X$ such that $\lambda_n((\Lambda_n + K_n + U_{o,n}) \setminus \Lambda_n) < \frac{1}{2^n} \lambda_n(\Lambda_n)$. Multiplying the Haar measure $\lambda_n$ by a suitable positive constant, we can assume that $\lambda_n(\Omega_n) = 1$ where $\Omega_n := \Lambda_n + K_n + U_{o,n}$. Now consider the probability measure $\mu_n$ on $X_n$ defined by $\mu_n(B) = \lambda_n(B \cap \Omega_n)$ for any Borel subset $B$ of $X$. If follows from $\Omega_n = -\Omega_n$ that $\mu_n = -\mu_n$.

Let $\mu := \otimes_{n \in \omega} \mu_n$ be the product measure of the probability measures $\mu_n$. It is clear that $\mu = -\mu$. We claim that the measure $\mu$ is locally $K$-invariant and Steinhaus-like.

Claim 9. The measure $\mu$ is locally $K$-invariant.

Proof. For every $k \in \omega$ consider the set $M_k := \prod_{n < k} (\Lambda_n + K_n + U_{k,n}) \times \prod_{n \geq k} \Lambda_n$. We claim that $\lim_{k \to \infty} M_k = 1$. Indeed, for every $\varepsilon > 0$ we can find $m \in \mathbb{N}$ such that $\prod_{n \geq m} (1 - \frac{\varepsilon}{2^n}) > 1 - \varepsilon$ and then for any $n < m$ by the $\sigma$-additivity of the Haar measure $\lambda_n$, find $i_n > m$ such that $\lambda_n(\Lambda_n + K_n + U_{i,n}) > (1 - \varepsilon)^{1/n}$. Then for any $k \geq \max i_n > m$, we obtain the lower bound

$$
\mu(M_k) = \prod_{n < k} \lambda_n(\Lambda_n + K_n + U_{k,n}) \cdot \prod_{n \geq k} \lambda_n(\Lambda_n) \geq \prod_{n < k} \lambda_n(\Lambda_n + K_n + U_{k,n}) \cdot \prod_{n \geq m} \lambda_n(\Lambda_n) > \prod_{n < m} (1 - \varepsilon)^{1/n} \cdot \prod_{n \geq m} (1 - \frac{\varepsilon}{2^n}) > (1 - \varepsilon)^2.
$$

Now we show that the sequence $(M_k)_{k \in \omega}$ satisfies the second condition of Definition [4]. Given any number $k \in \mathbb{N}$, consider the neighborhood $U_k := \prod_{n < k} O_{k+1,n} \times \prod_{n \geq k} X_n$ of $\theta$ in $X$, and observe that for any $x \in U \cap K$, we have $M_k + x \subset \prod_{n \in \omega} \Omega_n$. Consequently, for any Borel subset $B \subset M_k$ we get $\mu(B) = \mu(B + x)$ by the definition of the measure $\mu$.

In the following claim we prove that the measure $\mu = -\mu$ is Steinhaus-like.

Claim 10. For any $\mu$-positive Borel sets $A, B \subset X$ there are points $a, b \in X$ such that $\mu((A + a) \cap (B + b)) > 0$.

Proof. We lose no generality assuming that the sets $A, B$ are compact and are contained in $\prod_{n \in \omega} \Omega_n$. By the regularity of the measure $\mu$, there exists a neighborhood $U \subset X$ of $\theta$ such that $\mu(A + U) < \frac{\theta}{2} \mu(A)$ and $\mu(B + U) < \frac{\theta}{2} \mu(B)$. Replacing $U$ by a smaller neighborhood, we can assume that $U$ is of the basic form $U = V + X_{<n}$ for some $n \in \omega$ and some open set $V \subset X_{<n}$. Consider the tensor product $\lambda = \lambda' \otimes \mu'$ of the measures $\lambda = \otimes_{k < n} \lambda_k$ and $\mu' = \otimes_{k \geq n} \mu_k$ on the subgroups $X_{<n}$ and $X_{\geq n}$ of $X$, respectively. It follows from $A \cup B \subset \prod_{n \in \omega} \Omega_n$ that $\lambda(A) = \mu(A)$ and $\lambda(B) = \mu(B)$. Observe also the the measure $\lambda$ is $X_{<n}$-invariant in the sense that $\lambda(C) = \lambda(C + x)$ for any Borel set $C \subset X$ and any $x \in X_{<n}$.

Consider the projection $A'$ of $A$ onto $X_{<n}$ and for every $y \in X_{\geq n}$ let $A_y := X_{<n} \cap (A - y)$ be a shifted $y$-th section of the set $A$. Taking into account that $A_y \subset A'$, we conclude that

$$
\lambda'(A_y) \leq \lambda'(A') = \mu(A' + X_{\geq n}) \leq \mu(A + U) < \frac{\theta}{2} \mu(A).
$$

By the Fubini Theorem, $\mu(A) = \int_{A'} \lambda'(A_y)dy > \frac{\theta}{2} \mu(A' + X_{\geq n}) = \frac{\theta}{2} \lambda'(A')$. Consider the set

$$
L_A := \{y \in \prod_{k \geq n} \Omega_k : \lambda'(A_y) > \frac{1}{3} \lambda'(A')\}.
$$
and observe that
\[ \frac{5}{6} \lambda'(A') < \mu(A) \leq \mu'(L_A) \cdot \lambda'(A') + \frac{1}{3} \lambda'(A') \cdot (1 - \mu'(L_A)) \]
and hence \( \mu'(L_A) > \frac{1}{3} \).

By analogy, for the set \( B \) consider the projection \( B' \) of \( B \) onto \( X_{<n} \) and for every \( x \in B' \) put \( B_x := X_{>n} \cap (B - x) \). Repeating the above argument we can show that the set \( L_B := \{ y \in \prod_{k \geq n} \Omega_k : \mu'(B - x) > \frac{1}{3} \lambda'(B') \} \) has measure \( \mu'(L_B) > \frac{1}{3} \). Then \( \mu'(L_A \cap L_B) > \frac{1}{3} \).

Now we are ready to find \( s \in X_{<n} \) such that \( A \cap (B + s) \neq \emptyset \). Observe that a point \( z \in X \) belongs to \( A \cap (B + s) \) if and only if \( \chi_A(z) \cdot \chi_{B+s}(z) > 0 \), where \( \chi_A \) and \( \chi_{B+s} \) denote the characteristic functions of the sets \( A \) and \( B + s \) in \( X \).

So, it suffices to show that the function \( \chi_A \cdot \chi_{B+s} \) has a non-zero value. For this write \( z \) as a pair \((x, y)\) in \( X_{<n} \times X_{\geq n} \) and for every \( s \in X_{<n} \) consider the function
\[ g(s) := \int_X \chi_A(z) \cdot \chi_{B+s}(z) dz = \int_X \int_{x'} \chi_A(x, y) \cdot \chi_{B+s}(x, y) dy dx \]
and its integral, transformed with help of Fubini’s Theorem:
\[
\int_X g(s) ds = \int_X \int_{x'} \chi_A(x, y) \cdot \chi_{B+s}(x, y) dy ds dx = \int_X \int_{x'} \chi_A(x, y) \cdot \chi_{B}(x - s, y) ds dy dx = \\
\int_{x'} \int_X \chi_A(x, y) \cdot \chi_{B}(x - s, y) ds dy dx = \int_{x'} \int_X \chi_A(x, y) \lambda(B_y) dy dx = \\
\int_{x'} \lambda(B_y) \int_X \chi_A(x, y) dy dx = \int_{x'} \lambda(B_y) \cdot \lambda(A_y) dy > \frac{1}{9} \lambda(A') \cdot \lambda(B') \cdot \mu'(L_A \cap L_B) > 0.
\]

Now we see that \( g(s) > 0 \) for some \( s \in X_{<n} \) and hence \( A \cap (B + s) \neq \emptyset \).

Main Theorem raises many intriguing open problems.

Problem 11. Is the conclusion of the Main Theorem true for Banach spaces? Is it true for the classical Banach space \( c_0 \) or \( \ell_2 \)?

A category version of the Haar-null set was recently introduced by Darji [3], who defined a Borel subset \( B \) of a Polish Abelian group \( X \) to be \( \text{Haar-meager} \) if there exists a continuous map \( f : K \to X \) defined on a compact metrizable space such that \( f^{-1}(A + x) \) is meager in \( K \) for every \( x \in X \). More information on Haar-meager sets can be found in [11, 14, 15]. Observe that a closed subset of a Polish Abelian group is Haar-meager if and only if it is not Haar-open.

Problem 12. Let \( A, B \) two Borel subsets of an Abelian Polish group \( X \in \{ \mathbb{R}^\omega, c_0, \ell_2 \} \) such that \( A, B \) are not Haar-meager in \( X \). Is the sum-set \( A + B \) (or \( A + A \)) Haar-open?

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