A Nonlinear differential equation model of Asthma effect of environmental pollution using LHAM

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Abstract. In this paper, we investigated a nonlinear differential equation mathematical model to study the spread of asthma in the environmental pollutants from industry and mainly from tobacco smoke from smokers in different type of population. Smoking is the main cause to spread Asthma in the environment. Numerical simulation is also discussed. Finally by using Liao’s Homotopy analysis Method (LHAM), we found that the approximate analytical solution of Asthmatic disease in the environmental.

1. Introduction

Asthma is a chronic disease in which it will mainly spread airways and other reason for spreading this disease is smoking in the public environment [4 - 7]. The people who are affected by Asthma disease suffer lack of oxygen, breathlessness, chest tightness and chest pain and these are the symptoms of the disease [1]. If a person affected by disease is smoking in public environmental then it will circulated to other people in the environmental section. Smoking is affected our body in many ways but it mainly affected lungs and if once lungs is affected then it will lead to asthma disease [8].

Now smoking can be seen is the main impact exposure in the public environmental among the children. Recently survived 63% of children those who are smoking in home or public environmental are affected by Asthmatic disease [6, 7]. Nearly 15 million children having habit of secondhand smoke and chain smoke. People are also affected in various causes like power stations, cocktails, Cement Company, brick and lime work etc [2 - 3]. Recently Ghosh constructed an Asthmatic model to study the effect of Asthma spread. In that it is clearly mentioned that the main cause of the disease is continuous smoking in the public environmental section [4]. Due to continuous smoking the people who are susceptible by asthmatic disease they will affected by this disease [9]. It is a better solution of this disease is to avoid smoking or we need to isolate the smokers.

Liao’s Homotopy Analysis Method is one of the easy methods to solve non linear differential equations in any field [10 - 11]. Here we are going to find an approximate analytical solution of the non linear differential equation model of Asthmatic disease. This method is used in many Engineering and Science side to solve linear and non linear differential equation. From this method we can find the accurate solution of the model [12 - 13]. Appendix A, the more general detail method of LHAM is discussed. Numerical simulation is also presented to explain that the rate of change of infected people with respect to time [14]. The very important aspect is that giving separate place of smokers has been added in this model [15].
2. Liao’s Homotopy Analysis Method (LHAM) – Basic Methodology:

Now we can consider the equation

$$B[w(t)] = 0 \quad (1)$$

Here we consider $B$ as non linear operator and the $w(t)$ is unknown identity function with respect to time $t$. Initially in this method we ignore all boundary and initial conditions [16 - 17]. The most generalized constructed Liao’s Homotopy Analysis method of equation is

$$(1-p)N[\lambda(t, p) - w_0(t)] = pgG(t)N[\lambda(t; p)]$$

The embedding parameter of the value always lies between 0 and 1 [18]. We treat $G$ as auxiliary value of the system. At the end points the auxiliary value of the system is always holds.

$$\lambda(t : 0) = w_0(t) \text{ and } \lambda(t : 1) = w(t) \quad (2)$$

Thus, as $p$ increases from 0 to 1, the solution $\phi(t : p)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Expanding $\phi(t : p)$ in Taylor series with respect to $p$, we have [19 - 20]:

$$\lambda(t : p) = w_0(t) + \sum_{z=1}^{\infty} w_z(t) p^z \quad (3)$$

Then

$$w_z(t) = \frac{1}{z} \frac{\partial^z \lambda(t : p)}{\partial p^z} \bigg|_{p=0} \quad (4)$$

The equation (5) converges when we are using initial conditions it will become

$$w(t) = w_0(t) + \sum_{z=1}^{\infty} w_z(t) \quad (5)$$

Let us consider the vector equation of the form

$$w_n = \{w_0, w_1, w_2, \ldots, w_n\} \quad (6)$$

If we differentiate the system of equation with respect to the value of parameter embedding $p$, then we will get

$$N[w_z - \mu_z w_{z-1}] = gG(t)D_z(w_{z-1}) \quad (7)$$

Then we have

$$D_z(w_{z-1}) = \left[ \frac{1}{z-1} \frac{\partial^{z-1} B[\chi(t : p)]}{\partial p^{z-1}} \right]_{p=0} \quad (8)$$

where

$$\mu_z = 0, z \leq 1$$

$$\mu_z = 1, z > 1 \quad (9)$$

Taking $N^{-1}$ of the above equation

$$w_z(t) = \mu_z w_{z-1}(t) + gN^{-1}[G(t)D_z(w_{z-1})]$$

To get the higher order of this system of model is

$$w(t) = \sum_{z=1}^{\infty} w_z(t)$$

This method is used to find approximate solution of the fair non linear differential equations when the parameter value of $z$ tends to infinity. This method will give perfect solution when compare with all methods which involved in solving non linear ordinary differential equation.
3. Mathematical Modeling

We construct the model is

\[
\frac{dx}{dt} = \lambda - \mu x - \beta x z \quad (10)
\]

\[
\frac{dy}{dt} = \theta_1 z - \alpha y - \mu y \quad (11)
\]

\[
\frac{dz}{dt} = \theta - \theta_1 z - \theta k - \mu z \quad (12)
\]

These initial and boundary conditions are needed to find the accurate solution of the above system:

\[ x(0) = 0, y(0) = 0, z(0) = 0 \]

\[ x(0) = 5500, y(0) = 4000, z(0) = 200 \]

In this model we classified the amount of population in to three departments as the susceptible people, the number of infected people and the smoker free people. The first stage of this model is how many people are migrated from susceptible stage to infected stage. The main cause of this disease is smoking in public place environment. The approximate solution of the model using the Liao’s Homotopy Analysis Method is

\[
x(t) = \frac{\lambda}{\mu} \left(1 - e^{-\mu t}\right) + \left(5500 + \frac{1}{B^2} - \frac{A}{B} \left(\frac{1}{\mu} + \frac{1}{\mu - A}\right)\right) e^{-\mu t} + \frac{A e^{-At}}{B(\mu - A)} - \frac{A e^{-At}}{B} - \frac{e^{-(A + \mu)t}}{B^2}
\]

\[
y(t) = \frac{q \theta_1}{NM} \left(1 - e^{-N t}\right) + \frac{q \theta_1}{M(N - M)} \left(e^{-N t} - e^{-M t}\right) + 40000 e^{-N t} + \frac{200 \theta_1}{(N - M)} \left(e^{-M t} - e^{-N t}\right)
\]

\[
z(t) = \frac{q}{M} \left(1 - e^{-M t}\right) + 200 e^{-M t} \quad (13)
\]

Where

\[ N = \alpha + \mu \]

\[ M = \theta_1 + \theta + \mu \]

\[ A = h \beta \lambda q \]

\[ B = (\theta_1 + \theta + \mu) \mu \]

4. Numerical Simulation

To investigate the disease spread of Asthma we divided the population into three departments. We studied the model numerically to analyze the spread asthma effect. In figures (1) – (3) the rate of change of susceptible population with respect to time and different values of \(\beta\), \(\lambda\) and \(h\) are discussed in this model. Then in figures (4) – (6), the rate of change of infected population with respect to time and different values of \(\mu\), \(\alpha\) and \(\theta_1\) are also discussed here. Finally in figures (7) – (9), smoker class with respect to time and different values of \(\mu\), \(\theta_1\) and \(\theta\). The model description is given in Table 1. Hence it is clear that the rate of change of infected population with respect to time is increased.
Figure 1: Asthmatic susceptible with respect to time for $\beta = 0.0002, 0.0012, 0.0022$

Figure 2: Asthmatic susceptible with respect to time for $\lambda = 0.0001, 0.0011, 0.0021$
Figure 3: Asthmatic susceptible with respect to time for $h = 0.1, 0.15, 0.2$

Figure 4: Asthmatic infective with respect to time for $\mu = 0.014, 0.024, 0.034$
Figure 5: Asthmatic infective with respect to time for $\alpha = 0.018, 0.038, 0.058$

Figure 6: Asthmatic infective when $\theta_1 = 0.0002, 0.0052, 0.0102$
Figure 7: Smoker class with respect to time for $\mu = 0.014, 0.024, 0.034$

Figure 8: Smoker class with respect to time for $\theta_1 = 0.0002, 0.0012, 0.0022$
Figure 9: Smoker class with respect to time for $\theta = 0.002, 0.012, 0.022$

5. Conclusion

In this paper we analyzed a nonlinear differential equation mathematical model for the spread of asthma in different type of population by dividing in to three compartments with the effects of pollutants in the environment. Clearly we see that the people whose are susceptible become infected when they are regularly smoking in the environment. Numerical simulation are also expressed that the rate of change of infected pollutants is increased with respect to time when they are regularly smoking in the environment. In table 1 parameter values are given. Hence the spread of disease can be controlled if the smokers are restricted in the environment.

6. Appendix A

Solution of the system of equation (10) – (12) using LHAM Method

\[
\frac{dx}{dt} - \lambda + \mu x + \beta x z = 0 \quad (A1)
\]

\[
\frac{dy}{dt} - \theta_1 z + \alpha y + \mu y = 0 \quad (A2)
\]

\[
\frac{dz}{dt} - q + \theta_1 z + \theta \epsilon + \mu \epsilon = 0 \quad (A3)
\]

To find the solution of the system, the Homotopy is as:

\[
(1 - p) \left( \frac{dx}{dt} - \lambda + \mu x \right) = h p \left( \frac{dx}{dt} - \lambda + \mu x + \beta x z \right) \quad (A4)
\]

\[
(1 - p) \left( \frac{dy}{dt} - \theta_1 z + \alpha y + \mu y \right) = h p \left( \frac{dy}{dt} - \theta_1 z + \alpha y + \mu y \right) \quad (A5)
\]

\[
(1 - p) \left( \frac{dz}{dt} - q + \theta_1 z + \theta \epsilon + \mu \epsilon \right) = h p \left( \frac{dz}{dt} - q + \theta_1 z + \theta \epsilon + \mu \epsilon \right) \quad (A6)
\]

We can compare the coefficient of $p$ in the above system, we have

\[
p^0: \frac{dx_0}{dt} - \lambda + \mu x_0 = 0
\]
\[ p^0: \frac{dy_0}{dt} - \theta_1 z_0 + \alpha y_0 + \mu y_0 = 0 \quad (A7) \]
\[ p^0: \frac{dz_0}{dt} - q + (\theta_1 + \theta + \mu) z_0 = 0 \]
\[ p^1: \frac{dx_1}{dt} + \mu x_1 - \frac{dx_0}{dt} + \lambda - \mu x_0 - h \left( \frac{dx_0}{dt} - \lambda + \mu x_0 + \beta x_0 z_0 \right) = 0 \]
\[ p^1: \frac{dy_1}{dt} - \theta_1 z_1 + N y_1 - \frac{dy_0}{dt} + \theta_1 z_0 - N y_0 - h \left( \frac{dy_0}{dt} - \theta_1 z_0 + N y_0 \right) = 0 \]
\[ p^1: \frac{dz_1}{dt} + M z_1 - \frac{dz_0}{dt} + q - M z_0 - h \left( \frac{dz_0}{dt} - q + M z_0 \right) = 0 \]

Using initial conditions on the above system, we get:
\[
\begin{align*}
    x_0 &= \frac{\lambda}{\mu} \left(1 - e^{-\mu t}\right) \\
    y_0 &= \frac{q \theta_1}{NM} \left(1 - e^{-\lambda t}\right) + \frac{q \theta_1}{M (N - M)} \left(e^{-\lambda t} - e^{-\mu t}\right) \\
    z_0 &= \frac{q}{M} \left(1 - e^{-\mu t}\right)
\end{align*}
\]

The result using the initial conditions is as:
\[
\begin{align*}
    x_i &= \left[ 5500 + \frac{1}{B^2} - \frac{A}{B} \left( \frac{1}{\mu} + \frac{1}{\mu - A} \right) \right] e^{-\mu t} + \frac{A e^{-\lambda t}}{B (\mu - A)} - \frac{A \theta_1}{B} e^{-(\lambda + \mu) t} \\
    y_i &= 40000 e^{-\lambda t} + \frac{200 \theta_1}{(N - M)} \left(e^{-\mu t} - e^{-\lambda t}\right) \\
    z_i &= 2000 e^{-\mu t}
\end{align*}
\]

**Table 1: Parameter values of Asthma Model**

| Parameter description                                      | Symbol | Parameter Value |
|-----------------------------------------------------------|--------|-----------------|
| Interaction rate of susceptible with smoker                | \( \beta \) | 0.0002          |
| Natural death rate                                         | \( \mu \) | 0.014           |
| Disease induced death rate                                 | \( \alpha \) | 0.018           |
| Recruitment rate of smokers                               | \( q \) | 60              |
| Rate at which smokers quit smoking                         | \( \theta \) | 0.002           |
| Interaction of exposed persons with pollutants             | \( \lambda \) | 0.0001          |
| Rate at which smokers become infected                      | \( \theta_1 \) | 0.0002          |
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