CHARACTERIZATIONS OF OPERATOR MONOTONICITY VIA OPERATOR MEANS AND APPLICATIONS TO OPERATOR INEQUALITIES

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Abstract. We prove that a continuous function \( f : (0, \infty) \to (0, \infty) \) is operator monotone increasing if and only if \( f(A \mid_t B) \leq f(A) \mid_t f(B) \) for any positive operators \( A, B \) and scalar \( t \in [0, 1] \). Here, \( \mid_t \) denotes the \( t \)-weighted harmonic mean. As a counterpart, \( f \) is operator monotone decreasing if and only if the reverse of preceding inequality holds. Moreover, we obtain many characterizations of operator-monotone increasingness/decreasingness in terms of operator means. These characterizations lead to many operator inequalities involving means.

1. Introduction

Let \( B(\mathcal{H}) \) be the algebra of bounded linear operators on a complex Hilbert space \( \mathcal{H} \). The cone of positive operators on \( \mathcal{H} \) is written by \( B(\mathcal{H})^+ \). For selfadjoint operators \( A, B \in B(\mathcal{H}) \), the partial order \( A \leq B \) means that \( B - A \in B(\mathcal{H})^+ \), while the notation \( A < B \) indicates that \( B - A \) is an invertible positive operator.

A useful and important class of real-valued functions is the class of operator monotone functions, introduced by Löwner in a seminal paper [14]. Let \( I \subseteq \mathbb{R} \) be an interval. A continuous function \( f : I \to \mathbb{R} \) is said to be operator monotone (increasing) if

\[
A \leq B \implies f(A) \leq f(B)
\]  

(1.1)

for all operators \( A, B \in B(\mathcal{H}) \) whose spectra contained in \( I \) and for all Hilbert spaces \( \mathcal{H} \). If the reverse inequality in the right hand side of (1.1) holds, then we say that \( f \) is operator monotone decreasing. It is well known that (see e.g. [12, Example 2.5.9]) the function \( f(x) = x^\alpha \) is operator monotone on \([0, \infty)\) if and only if \( \alpha \in [0, 1] \), and it is operator monotone decreasing if and only if \( \alpha \in [-1, 0] \). The function \( x \mapsto \log(x + 1) \) is operator monotone on \((0, \infty)\). More concrete examples can be found in [10].

Operator monotony arises naturally in matrix/operator inequalities (e.g. [2, 6, 16]). It plays a major role in the so-called Kubo-Ando theory of operator means (e.g. [13]). It has applications in many areas, including electrical networks (see e.g. [1]), elementary particles ([15]) and entropy in physics ([8]).

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A closely related concept to operator monotonicity is the concept of operator concavity. A continuous function \( f : (0, \infty) \to \mathbb{R} \) is said to be operator concave if
\[
f((1-t)A + tB) \geq (1-t)f(A) + tf(B) \tag{1.2}
\]
for any \( A, B > 0 \) and \( t \in [0, 1] \). The continuity of \( f \) implies that \( f \) is operator concave if and only if \( f \) is operator midpoint-concave, in the sense that the condition (1.2) holds for \( t = 1/2 \).

Löwner \[14\] characterized operator monotonicity in terms of the positivity of matrix of divided differences and an important class of analytic functions, namely, Pick functions. Hansen and Pedersen \[11\] provided various characterizations of operator monotonicity in terms of operator inequalities, using 2-by-2 block matrix techniques. The following fact is well known:

**Theorem 1.1.** (see \[11\] or \[12, Corollary 2.5.4\]) A continuous function \( f : (0, \infty) \to (0, \infty) \) is operator monotone if and only if \( f \) is operator concave.

Kubo and Ando \[13\] characterized operator monotony in terms of operator connections (see details in the next section). Systematic explanations of operator monotonicity/concavity can be found in \[5, Chapter V\] and \[12, Section 2\].

In the present paper, we focus on the relationship between the operator monotonicity of functions and operator means. See some related discussions in \[3, 4\]. Note that the condition (1.2) can be restated in terms of weighted arithmetic means \( \nabla_t \) as follows
\[
f(A \nabla_t B) \geq f(A) \nabla_t f(B) \tag{1.3}
\]
for any \( A, B > 0 \) and \( t \in [0, 1] \). The reverse inequality of (1.3) is equivalent to the operator convexity of \( f \). Consider the \( t \)-weighted harmonic mean
\[
A!_t B = [(1-t)A^{-1} + tB^{-1}]^{-1}, \quad A, B > 0.
\]

We prove an interesting fact about operator monotone functions:
\[
f(A!_t B) \leq f(A)!_t f(B), \quad A, B > 0 \text{ and } t \in [0, 1]. \tag{1.4}
\]
Conversely, the above property characterizes the operator monotonicity of \( f \). Moreover, \( f \) is operator monotone decreasing if and only if the reverse inequality of (1.4) holds. Many characterizations of operator monotone increasing/decreasing functions in this type are established in Section 3. This gives a natural way to derive operator inequalities involving means in Section 4.

2. Preliminaries on operator means

In this section, we review Kubo-Ando theory of operator means (see e.g. \[6, Chapter 4\], \[12, Section 3\]). We begin with the axiomatic definition of an operator mean. Then we mention fundamental results and give practical examples of operator means which will be used in later discussions.

An **operator connection** is a binary operation \( \sigma \) on \( B(\mathcal{H})^+ \) such that for all positive operators \( A, B, C, D \):

(M1) monotonicity: \( A \leq C, B \leq D \implies A \sigma B \leq C \sigma D \)

(M2) transformer inequality: \( C(A \sigma B)C \leq (CAC) \sigma (CBC) \)
(M3) continuity from above: for $A_n, B_n \in B(\mathcal{H})^+$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \sigma B_n \downarrow A \sigma B$. Here, $X_n \downarrow A$ indicates that $(X_n)$ is a decreasing sequence converging strongly to $X$.

An operator mean is an operator connection with property that $A \sigma A = A$ for all $A \geq 0$.

Classical examples of operator means are the arithmetic mean (AM), the harmonic mean (HM), the geometric mean (GM) and their weighted versions. For each $t \in [0, 1]$, we define the $t$-weighted arithmetic mean, the $t$-weighted harmonic mean and the $t$-weighted geometric mean for invertible positive operators $A$ and $B$ as follows:

\[
A \nabla_t B = (1-t)A + tB \quad A \downarrow \quad A^\downarrow_t B = [(1-t)A^{-1} + tB^{-1}]^{-1} \quad A \#_t B = A^{\frac{t}{2}}(A^{-\frac{t}{2}}BA^{-\frac{t}{2}})^t A^{\frac{t}{2}}.
\]

These means can be extended to arbitrary $A, B \geq 0$ by continuity. For example,

\[
A \#_t B = \lim_{\epsilon \downarrow 0} (A + \epsilon I) \#_t (B + \epsilon I),
\]

here the limit is taken in the strong-operator topology. We abbreviate $\nabla = \nabla_{1/2}$, $! =!_{1/2}$ and $\# = \#_{1/2}$.

A famous theorem in this theory is the one-to-one correspondence between operator connections and operator monotone functions:

**Theorem 2.1.** ([13, Theorem 3.4]) There is a one-to-one correspondence between operator connections $\sigma$ and operator monotone functions $f : [0, \infty) \rightarrow [0, \infty)$ given by the relation

\[
f(A) = I \sigma A, \quad A \geq 0.
\]  

Moreover, $\sigma$ is an operator mean if and only if $f(1) = 1$. Every connection $\sigma$ is uniquely determined on the set of invertible positive operators. Indeed, for $A, B \geq 0$ we have $A + \epsilon I, B + \epsilon I > 0$ for any $\epsilon > 0$ and then by the monotonicity (M1) and the continuity from above (M3),

\[
A \sigma B = \lim_{\epsilon \downarrow 0} (A + \epsilon I) \sigma (B + \epsilon I).
\]

Using (M2), every operator connection $\sigma$ is congruent invariant in the sense that

\[
C(A \sigma B)C = CAC \sigma CBC
\]

for any $A \geq 0, B \geq 0$ and $C > 0$. Theorem 2.1 serves a simple proof of operator versions of the weighted AM-GM-HM inequalities:

**Proposition 2.2.** (see e.g. [12, Proposition 3.3.2]) For each $A, B \geq 0$ and $t \in [0, 1]$, we have

\[
A \nabla_t B \geq A \downarrow_t B \geq A \#_t B.
\]

An operator connection $\sigma$ is symmetric if $A \sigma B = B \sigma A$ for all $A, B \geq 0$, or equivalently, $f(x) = xf(1/x)$ for all $x > 0$. Let $\sigma$ be a nonzero operator
connection. Then $A \sigma B > 0$ for any $A, B > 0$ (see e.g. [7]). We define the adjoint of $\sigma$ to be the operator connection

$$\sigma^* : (A, B) \mapsto (A^{-1} \sigma B^{-1})^{-1}.$$  

If $\sigma$ has the representing function $f$, then the representing function of $\sigma^*$ is given by the following operator monotone function:

$$f^*(x) = \frac{1}{f(1/x)}, \quad x > 0.$$  

Next, we introduce two important classes of parametrized means. For each $p \in [-1, 1]$ and $\alpha \in [0, 1]$, consider the operator monotone function (see e.g. [5])

$$f_{p, \alpha}(x) = (1 - \alpha + \alpha x^p)^{1/p}, \quad x \geq 0.$$  

We define $f_{p, \alpha}(0) \equiv 0$ and $f_{p, \alpha}(1) \equiv 1$ by continuity. When $p = 0$, it is understood that we take limit at $p$ tends to 0 and, by L'Hôpital's rule,  

$$f_{0, \alpha}(x) = x^\alpha.$$  

Each function $f_{p, \alpha}$ gives rise to a unique operator mean, namely, the quasi-arithmetic power mean $\#_{p, \alpha}$ with exponent $p$ and weight $\alpha$. Note that

$$\#_{1, \alpha} = \nabla_\alpha, \quad \#_{0, \alpha} = \#_\alpha, \quad \#_{-1, \alpha} = !_\alpha.$$  

The second class is a class of symmetric means. Let $r \in [-1, 1]$ and consider the operator monotone function (see [9])

$$g_r(x) = \frac{3r - 1}{3r + 1} x^{\frac{3r+1}{2}} - 1, \quad x \geq 0.$$  

This function satisfies $g_r(1) = 1$ and $g_r(x) = xg_r(1/x)$. Thus it associates to a unique symmetric operator mean, denoted by $\Diamond_r$. In particular,

$$\Diamond_1 = \nabla, \quad \Diamond_0 = \#, \quad \Diamond_{-1} = !.$$  

The operator means $\Diamond_{1/3}$ and $\Diamond_{-1/3}$ are known as the logarithmic mean and its adjoint.

3. Characterizations of operator monotonicity via operator means

In this section, we characterize operator monotone increasing/decreasing functions in terms of operator means. Let us start with a simple observation about operator monotony:

**Lemma 3.1.** A function $f : (0, \infty) \to (0, \infty)$ is operator monotone (operator monotone decreasing) if and only if $f^*$ is operator monotone (operator monotone decreasing, resp.).

**Proof.** We consider only for the case of operator monotonicity since a proof for another case is similar to this one. It is easy to see that the continuity of $f$ and $f^*$ are equivalent. Suppose that $f$ is operator monotone and consider $A, B > 0$ such that $A \geq B$. Then $A^{-1} \leq B^{-1}$ by the operator decreasingness of the map.
$t \mapsto t^{-1}$ on $(0, \infty)$. Hence $f(A^{-1}) \leq f(B^{-1})$ by the operator monotonicity of $f$. Since $f^*(t) = f(t^{-1})^{-1}$ for all $t > 0$, it follows that
$$f^*(A) = f(A^{-1})^{-1} \geq f(B^{-1})^{-1} = f^*(B).$$
For the converse, use the fact that $(f^*)^* = f$.

The following result dualizes operator inequalities and will be used many times later.

**Proposition 3.2.** Let $\sigma$ and $\eta$ be binary operations for invertible positive operators. Let $f, g, h : (0, \infty) \to (0, \infty)$ be continuous functions. Then the following statements are equivalent:

1. $f(A \sigma B) \leq g(A) \eta h(B)$ for all $A, B > 0$;
2. $f^*(A \sigma^* B) \geq g^*(A) \eta^* h^*(B)$ for all $A, B > 0$.

**Proof.** Assume (1) and consider $A, B > 0$. By definition of $f^*$ and the operator-monotone decreasingness of the map $t \mapsto t^{-1}$, we have
$$f^*(A \sigma^* B) = f^* (\sigma^{-1} B^{-1} A)$$
$$= [f(A^{-1} \sigma B^{-1})]^{-1}$$
$$\geq [g(A^{-1}) \eta h(B^{-1})]^{-1}$$
$$= [g^*(A)^{-1} \eta^* h^*(B)^{-1}]^{-1} = g^*(A) \eta^* h^*(B).$$
To prove (2) $\Rightarrow$ (1), apply (1) $\Rightarrow$ (2) to continuous functions $f^*, g^*, h^*$ and binary operations $\sigma^*, \eta^*$.

Recall the following result.

**Proposition 3.3.** ([3, Theorem 2.3]) Let $f : (0, \infty) \to (0, \infty)$ be a continuous function. The following statements are equivalent:

(i) $f$ is operator monotone;
(ii) $f(A \triangledown B) \geq f(A) \# f(B)$ for all $A, B > 0$;
(iii) $f(A \triangledown B) \geq f(A) \sigma f(B)$ for all $A, B > 0$ and for all symmetric means $\sigma$;
(iv) $f(A \triangledown B) \geq f(A) \sigma f(B)$ for all $A, B > 0$ and for some symmetric mean $\sigma \neq !$.

The next theorem further characterizes operator monotonicity.

**Theorem 3.4.** Let $f : (0, \infty) \to (0, \infty)$ be a continuous function. Then the following statements are equivalent:

(i) $f$ is operator monotone (increasing);
(ii) $f(A \triangledown B) \geq f(A) \triangledown f(B)$ for all $A, B > 0$;
(iii) $f(A \triangledown t B) \geq f(A) \triangledown_t f(B)$ for all $A, B > 0$ and for all $t \in [0, 1]$;
(iv) $f(A ! B) \leq f(A) \# f(B)$ for all $A, B > 0$;
(v) $f(A ! B) \leq f(A) \#_t f(B)$ for all $A, B > 0$ and for all $t \in [0, 1]$;
(vi) $f(A ! B) \leq f(A) ! f(B)$ for all $A, B > 0$;
(vii) $f(A ! t B) \leq f(A) !_t f(B)$ for all $A, B > 0$ and for all $t \in [0, 1]$;
Proof. The following implications are clear: (I3) \(\Rightarrow\) (I2), (I5) \(\Rightarrow\) (I4) and (I7) \(\Rightarrow\) (I6). The equivalence (I1) \(\Leftrightarrow\) (I3) is a restatement of Theorem 1.1.

(I2) \(\Rightarrow\) (I1). Assume (I2), i.e. \(f\) is operator midpoint-concave. The continuity of \(f\) implies that \(f\) is operator concave. Hence \(f\) is operator monotone by Theorem 1.1.

(I1) \(\Leftrightarrow\) (I4). The operator monotonicity of \(f\) and \(f^*\) are equivalent by Lemma 3.1. By the equivalent (i) \(\Leftrightarrow\) (ii) in Proposition 3.3, the operator monotonicity of \(f^*\) reads
\[
f^*(A \ominus B) \geq f^*(A) \#^* f^*(B) \quad \text{for all } A, B > 0.
\]
Proposition 3.2 says that this condition is equivalent to \(f(A!B) \leq f(A) \# f(B)\) for all \(A, B > 0\) since \(\triangle = !\) and \(\#^* = \#\).

(I1) \(\Leftrightarrow\) (I6). Note that the operator monotonicity of \(f\) and \(f^*\) are equivalent by Lemma 3.1. Using the equivalence (I1) \(\Leftrightarrow\) (I2), the operator monotonicity of \(f^*\) can be expressed as
\[
f^*(A \ominus_t B) \geq f^*(A) \ominus_t f^*(B).
\]
Hence, by Proposition 3.2, we obtain (I5) since \(\triangle_t = !^t\) and \(#_t^* = \#_t\).

(I1) \(\Leftrightarrow\) (I7). The proof is similar to that of (I1) \(\Leftrightarrow\) (I6). In this case, we utilize the equivalence (I1) \(\Leftrightarrow\) (I3).

(I1) \(\Leftrightarrow\) (I8). By using (i) \(\Leftrightarrow\) (iii) in Proposition 3.3, the operator monotonicity of \(f\) (hence, of \(f^*\)) is equivalent to the condition that
\[
f^*(A \ominus B) \geq f^*(A) \sigma f^*(B)
\]
for \(A, B > 0\) and for all symmetric means \(\sigma\). By Proposition 3.2, this condition is then equivalent to the following:
\[
f(A!B) \leq f(A) \sigma^* f(B)
\]
for all symmetric means \(\sigma\). Since the map \(\sigma \mapsto \sigma^*\) is bijective on the set of symmetric means, we arrive at (I8).

(I1) \(\Leftrightarrow\) (I9). The proof is similar to that of (I1) \(\Leftrightarrow\) (I8). Here, we use (i) \(\Leftrightarrow\) (iv) in Proposition 3.3 and the fact that \(!^* = \triangle\).

Next, we turn to operator monotone decreasingness. Recall the following result:

**Proposition 3.5.** ([3, Theorems 2.1 and 3.1]) Let \(f : (0, \infty) \rightarrow (0, \infty)\) be a continuous function. The following statements are equivalent:

(i) \(f\) is operator monotone decreasing;
(ii) \( f(A \triangledown B) \leq f(A) \# f(B) \) for all \( A, B > 0 \);

(iii) \( f(A \triangledown B) \leq f(A) \sigma f(B) \) for all \( A, B > 0 \) and for all symmetric means \( \sigma \);

(iv) \( f(A \triangledown B) \leq f(A) \sigma f(B) \) for all \( A, B > 0 \) and for some symmetric mean \( \sigma \neq \triangledown \).

(v) \( f \) is operator convex and \( f \) is decreasing.

The next theorem is a counterpart of Theorem 3.4.

**Theorem 3.6.** Let \( f : (0, \infty) \to (0, \infty) \) be a continuous function. Then the following statements are equivalent:

1. \( f \) is operator monotone decreasing;
2. \( f(A \# B) \geq f(A) \# f(B) \) for all \( A, B > 0 \);
3. \( f(A \#_t B) \geq f(A) \#_t f(B) \) for all \( A, B > 0 \) and for all \( t \in [0, 1] \);
4. \( f(A \#_t B) \geq f(A) \#_t f(B) \) for all \( A, B > 0 \) and \( f \) is decreasing;
5. \( f(A \#_t B) \geq f(A \#_t) f(B) \) for all \( A, B > 0 \) and for all \( t \in [0, 1] \) and \( f \) is decreasing;
6. \( f(A \#_t B) \geq f(A) \sigma f(B) \) for all \( A, B > 0 \) and for all symmetric means \( \sigma \);
7. \( f(A \#_t B) \geq f(A) \sigma f(B) \) for all \( A, B > 0 \) and for some symmetric mean \( \sigma \neq \#_t \).

**Proof.** It is clear that (D3) \( \Rightarrow \) (D2), (D5) \( \Rightarrow \) (D4) and (D6) \( \Rightarrow \) (D7).

(D1) \( \Leftrightarrow \) (D2). The operator-monotone decreasingness of \( f \) and \( f^* \) are equivalent. According to the equivalent (i) \( \Leftrightarrow \) (ii) in Proposition 3.5, the fact that \( f^* \) is operator monotone decreasing is equivalent to

\[
f^*(A \triangledown B) \leq f^*(A) \# f^*(B)
\]

for all \( A, B > 0 \). Proposition 3.2 states that this condition is equivalent to \( f(A \#^t B) \geq f(A) \#_t f(B) \) for all \( A, B > 0 \).

(D1) \( \Rightarrow \) (D3). Assume that \( f \) is operator monotone decreasing. Then so is \( f^* \) by Lemma 3.1. Consider \( A, B > 0 \) and \( t \in [0, 1] \). Recall the weighted AM-GM inequality for operators (Proposition 2.2):

\[
A \triangledown_t B \geq A \#_t B.
\]

It follows that \( f^*(A \triangledown_t B) \leq f^*(A \#_t B) \). By Proposition 3.2, we have \( f(A \#_t B) \geq f(A \#_t B) \).

(D1) \( \Rightarrow \) (D4). Assume that \( f \) is operator monotone decreasing. By (D2) and the GM-HM inequality for operators (Proposition 2.2), we have

\[
f(A \#_t B) \geq f(A) \# f(B) \geq f(A) \#_t f(B)
\]

for all \( A, B > 0 \). It is trivial that \( f \) is decreasing in usual sense.

(D4) \( \Rightarrow \) (D1). Suppose that \( f(A \#_t B) \geq f(A) \#_t f(B) \) for all \( A, B > 0 \) and \( f \) is decreasing. Then \( f^* \) is also a decreasing function. By Proposition 3.2, we have \( f^*(A \triangledown B) \leq f^*(A) \triangledown f^*(B) \) for all \( A, B > 0 \), i.e. \( f^* \) is operator convex. The implication (v) \( \Rightarrow \) (i) in Proposition 3.5 tells us that \( f^* \) is operator monotone decreasing. Hence, so is \( f \) by Lemma 3.1.
(D1) ⇒ (D5). The proof is similar to that of (D1) ⇒ (D4). Here, we use the weighted GM-HM inequality (2.3):
\[ A \#_t B \geq A \!_t \! B \]
for any \( A, B > 0 \) and \( t \in [0, 1] \).

(D1) ⇒ (D6). Assume that \( f \) is operator monotone decreasing. Then so is \( f^* \) by Lemma 3.1. By applying the equivalence (i) ⇔ (iii) in Proposition 3.5 to \( f^* \), we obtain that
\[ f^*(A \vee B) \leq f^*(A) \sigma f^*(B) \]
for all \( A, B > 0 \) and for all symmetric means \( \sigma \). Now, use Proposition 3.2.

(D7) ⇒ (D1). This is a combination of Proposition 3.2, the equivalence (i) ⇔ (iv) in Proposition 3.5 and Lemma 3.1. □

4. Applications to Operator Inequalities Involving Means

Let us derive operator inequalities involving means by making use of Theorems 3.4 and 3.6. A simple way is to take a specific operator monotone increasing/decreasing function.

**Corollary 4.1.** For each \( A, B \geq 0 \) and \( \alpha, t \in [0, 1] \), we have
\[ (A \!_t B)^\alpha \leq A^\alpha \!_t B^\alpha. \]

*Proof.* By applying Theorem 3.4 (I1) ⇒ (I7) to the operator monotone function \( f(x) = x^\alpha \), we get
\[ (A \!_t B)^\alpha \leq A^\alpha \!_t B^\alpha \]
for any \( A, B > 0 \). For general \( A, B \geq 0 \), consider \( A + \epsilon I, B + \epsilon I > 0 \) for \( \epsilon > 0 \) and then use the continuity from above (M3).

**Corollary 4.2.** For each \( A, B > 0 \) and \( t \in [0, 1] \), we have
\[ \log ((A \!_t B) + I) \leq \log(A + I) \!_t \log(B + I). \]

*Proof.* Apply Theorem 3.4 (I1) ⇒ (I7) to the operator monotone function \( f(x) = \log(x + 1) \).

**Theorem 4.3.** Let \( \sigma \) be an operator connection. Then for any \( A, B, C \geq 0 \) and \( t \in [0, 1] \), we have
\[ A \sigma (B \!_t C) \leq (A \sigma B) \!_t (A \sigma C), \quad (4.1) \]
\[ A \sigma (B \triangledown_t C) \geq (A \sigma B) \triangledown_t (A \sigma C). \quad (4.2) \]

*Proof.* It is trivial when \( \sigma \) is the zero connection. Suppose that \( \sigma \) is nonzero. By Theorem 2.1, there is an operator monotone function \( f : [0, \infty) \to [0, \infty) \) such that \( f(A) = I \sigma A \) for any positive operator \( A \). Note that \( f(x) > 0 \) for any \( x > 0 \); otherwise \( \sigma \) is the zero connection. This implies that it suffices to consider \( f : (0, \infty) \to (0, \infty) \). Theorem 3.4 (I1) ⇒ (I7) implies that for each \( A, B > 0 \),
\[ I \sigma (A \!_t B) = f(A \!_t B) \leq f(A) \!_t f(B) = (I \sigma A) \!_t (I \sigma B). \]
It follows from the property (1.1) and the congruent invariance (2.2) that, for \( A, B, C > 0 \),
\[
A \sigma (B^t C) = A^{\frac{1}{2}} I A^{\frac{1}{2}} \sigma (A^{\frac{1}{2}} A^{-\frac{1}{2}} B A^{-\frac{1}{2}} A^t A^{\frac{1}{2}} A^{-\frac{1}{2}} C A^{-\frac{1}{2}} A^t)
\]
\[
= A^{\frac{1}{2}} \left[ I \sigma (A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{-\frac{1}{2}} \sigma (A^{\frac{1}{2}} A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \right] A^{\frac{1}{2}}
\]
\[
\leq A^{\frac{1}{2}} \left[ (I \sigma A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{-\frac{1}{2}} A^{\frac{1}{2}} (I \sigma A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \right] A^{\frac{1}{2}}
\]
\[
= (A \sigma B) A^t (A \sigma C).\]

Finally, for general \( A, B, C \geq 0 \), use the continuity from above. The proof of the inequality (4.2) is similar to (4.1). In this case, we use Theorem 3.4 (I1) \( \Rightarrow \) (I3).

Many interesting operator inequalities are obtained as special cases of Theorem 4.3. For example, for any \( p \in [-1, 1] \) and \( \alpha, t \in [0, 1] \), we have
\[
A \&_{p,\alpha}^t (B^t C) \leq (A \&_{p,\alpha} B) A^{\frac{1}{2}} (A \&_{p,\alpha} C),
\]
\[
A \&_{p,\alpha} (B \nabla C) \geq (A \&_{p,\alpha} B) \nabla (A \&_{p,\alpha} C)
\]
hold for all positive operators \( A, B, C \).

The next theorem is a symmetric counterpart of Theorem 4.3.

**Theorem 4.4.** Let \( \sigma \) be an operator connection and \( \eta \) a symmetric operator mean. For each \( A, B, C \geq 0 \), we have
\[
A \sigma (B^t C) \leq (A \sigma B) \eta (A \sigma C),
\]
\[
A \sigma (B \nabla C) \geq (A \sigma B) \eta (A \sigma C).
\]

**Proof.** The proof is similar to that of Theorem 4.3. In this case, we apply Theorem 3.4 (I1) \( \Rightarrow \) (I3).

Consider the symmetric mean \( \diamond_r \) for each \( r \in [-1, 1] \). The previous theorem implies that for any operator connection \( \sigma \) and positive operators \( A, B \)
\[
A \sigma (B^t C) \leq (A \sigma B) \diamond_r (A \sigma C),
\]
\[
A \sigma (B \nabla C) \geq (A \sigma B) \diamond_r (A \sigma C).
\]

**Corollary 4.5.** For each \( A, B \geq 0 \), \( r \in [-1, 0] \) and \( t \in [0, 1] \), we have
\[
(A^t B)^r \geq A^r \#_t B^r \geq A^r \#_t B^r.
\]

**Proof.** The first inequality is obtained when applying Theorem 3.6 (D1) \( \Rightarrow \) (D3) to the operator monotone decreasing function \( f(x) = x^r \) The second inequality comes from the weighted GM-HM inequality (2.3).

Our final result is a symmetric counterpart of Corollary 4.5.

**Corollary 4.6.** Let \( \sigma \) be a symmetric operator mean. For each \( A, B \geq 0 \) and \( r \in [-1, 0] \), we have
\[
(A^t B)^r \geq A^r \sigma B^r.
\]

**Proof.** Apply Theorem 3.6 (D1) \( \Rightarrow \) (D6) to the operator monotone decreasing function \( f(x) = x^r \).
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