The Foldy-Lax Approximation for the Full Electromagnetic Scattering by Small Conductive Bodies of Arbitrary Shapes

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Abstract

We deal with the electromagnetic waves propagation in the harmonic regime. We derive the Foldy-Lax approximation of the scattered fields generated by a cluster of small conductive inhomogeneities arbitrarily distributed in a bounded domain Ω of \( \mathbb{R}^3 \). This approximation is valid under a sufficient but general condition on the number of such inhomogeneities \( m \), their maximum radii \( \epsilon \) and the minimum distances between them \( \delta \), of the form

\[
(\ln m)^{\frac{1}{3}} \frac{\epsilon}{\delta} \leq C,
\]

where \( C \) is a constant depending only on the Lipschitz characters of the scaled inhomogeneities. In addition, we provide explicit error estimates of this approximation in terms of aforementioned parameters, \( m, \epsilon, \delta \) but also the used frequencies \( k \) under the Rayleigh regime. Both the far-fields and the near-fields (stated at a distance \( \delta \) to the cluster) are estimated. In particular, for a moderate number of small inhomogeneities \( m \), the derived expansions are valid in the mesoscale regime where \( \delta \sim \epsilon \).

At the mathematical analysis level and based on integral equation methods, we prove a priori estimates of the densities in the \( L^2_{\text{div}} \) spaces instead of the usual \( L^2 \) spaces (which are not enough). A key point in such a proof is a derivation of a particular Helmholtz type decomposition of the densities. Those estimates allow to obtain the needed qualitative as well as quantitative estimates while refining the approximation. Finally, to prove the invertibility of the Foldy-Lax linear algebraic system, we reduce the coercivity inequality to the one related to the scalar Helmholtz model. As this linear algebraic system comes from the boundary conditions, such a reduction is not straightforward.

Keywords: Electromagnetic scattering, Small bodies, Multiple scattering, Foldy-Lax approximation.

AMS subject classification: 35J08, 35Q61, 45Q05.

1 Introduction and main results

Let \((B_i)_{i=1}^m\) be \(m\) open, bounded and simply connected sets containing the origin, with Lipschitz boundaries. To these sets, we correspond the small bodies \((D_i)_{i=1}^m\) which are defined as the translations and contractions of the \(m\) bodies \((B_i)_{i=1}^m\), that is

\[
D_i = \epsilon B_i + z_i, i = 1, ..., m
\]

(1.1)

where \(z_i, i = 1, ..., m\) are given positions in \(\mathbb{R}^3\) and \(\epsilon\) a small parameter.

We consider the scattering of a time-harmonic electromagnetic plane wave by the perfectly conducting small bodies \((D_i)_{i=1}^m\) formulated as follows (see [10])

\[
\nabla \times E - ikH = 0 \text{ in } D^+ := \mathbb{R}^3 \setminus \bigcup_{i=1}^m D_i,
\]

\[
\nabla \times H + ikE = 0 \text{ in } D^+,
\]

(1.2)
with the boundary condition
\[ \nu \times E = 0 \quad \text{on } \partial D = \bigcup_{i=1}^{m} \partial D_i. \quad (1.3) \]

The total electromagnetic fields are expressed as \( E = E^{\text{inc}} + E^{\text{sca}}, \) \( H = H^{\text{inc}} + H^{\text{sca}} \) where the indices “inc” and “sca” indicate the incident wave and the the scattered wave, respectively. The condition (1.3) corresponds to the case of perfectly conducting obstacle and \( \nu \) expresses the unit outward normal vector to the boundary of \( D = \bigcup_{i=1}^{m} D_i. \) Here the wave number \( k \) is defined through the relation \( k^2 = (\xi \omega + i \sigma) \omega \) where \( \xi, \sigma, \mu \) are respectively the electric permittivity, electric conductivity and the magnetic permeability.

In the case where \( \sigma = 0, \) then \( k \) is real, the scattered wave \( (E^{\text{sca}}, H^{\text{sca}}) \) must satisfy the outgoing radiation condition
\[ \lim_{|x| \to \infty} (H^{\text{sca}}(x) \times x - |x|E^{\text{sca}}(x)) = 0. \quad (1.4) \]

Motivated by applications, we restrict ourselves to incident waves of the form \( E^{\text{inc}}(x) = Pe^{ikx}e^{i\theta}, \) where \( \theta \) is the incident direction, \( P \in \mathbb{R}^3 \) is the polarization that is orthogonal to \( \theta. \)

We introduce the diameters \( \epsilon_i = \max_{x,y \in D_i} d(x,y), \) \( i \in \{1, ..., m\}, \) and the distance between two bodies \( D_i, D_j, i \neq j, \) as \( \delta_{i,j} = \min_{x \in D_i, y \in D_j} d(x,y), \) for every \( i,j \in \{1, ..., m\}; i \neq j \) where \( d(\cdot, \cdot) \) stands for the Euclidean distance. We set
\[ \epsilon := \max_{i \in \{1, ..., m\}} \epsilon_i, \quad \delta := \min_{i \neq j \in \{1, ..., m\}} \delta_{i,j}. \quad (1.5) \]

We suppose in addition that \( \bigcup_{i=1}^{m} \overline{D_i} \subset \Omega, \) where \( \Omega \) is a bounded Lipschitz domain such that
\[ d(\partial \Omega, \bigcup_{i=1}^{m} \overline{D_i}) \geq \delta. \]

Let us recall that a bounded open connected domain \( B, \) is said to be a Lipschitz domain with character \( (l_{\partial B}, L_{\partial B}) \) if for each \( x \in \partial D \) there exist a coordinate system \( (y_i)_{i=1,2,3}, \) a truncated cylinder \( C \) centered at \( x \) whose axis is parallel to \( y_3 \) with length \( l \) satisfying \( l_{\partial B} \leq l \leq 2L_{\partial B}, \) and a Lipschitz function \( f \) that is \( |f(s_1) - f(s_2)| \leq L_{\partial B} |s_1 - s_2| \) for every \( s_1, s_2 \in \mathbb{R}^2, \) such that \( B \cap C = \{ (y_i)_{i=1,2,3} : y_3 > f(y_1, y_2) \} \) and \( \partial B \cap \partial C = \{ (y_i)_{i=1,2,3} : y_3 = f(y_1, y_2) \}. \)

In this work, we assume that the sequence of Lipschitz characters \( (l_{\partial D_i}, L_{\partial D_i})_{i=1}^{m} \) of the bodies \( B_i, i = 1, ..., m, \) is bounded from above and below.

The scattering problem (1.2) under the boundary condition (1.3) and the radiating condition (1.4) is well posed in appropriate spaces under appropriate conditions (see [10, 20]) which will described later. In addition, when \( \Im k \) is different from zero, the scattered electromagnetic fields are fastly decaying at infinity as we have attenuation. But when \( \Im k = 0, \) i.e. in the absence of attenuation, we have the following behavior (as spherical-waves) of the scattered electric fields far away from the sources \( D_i, \)'s
\[ E^{\text{sca}}(x) = \frac{e^{ik|x|}}{|x|} \{ E^{\infty}(\tau) + O(|x|^{-1}) \}, \quad |x| \to \infty, \quad (1.6) \]
and we have a similar behavior for the scattered magnetic field as well:
\[ H^{\text{sca}}(x) = \frac{e^{ik|x|}}{|x|} \{ H^{\infty}(\tau) + O(|x|^{-1}) \}, \quad |x| \to \infty. \quad (1.7) \]
where \( (E^{\infty}(\tau), H^{\infty}(\tau)) \) is the electromagnetic far field pattern in the direction of propagation \( \tau := \frac{x}{|x|}. \)

The goal of this work is to derive the Foldy-Lax approximation (also called the point interaction approximation) of the electromagnetic fields taking into account the whole parameters defining the model, i.e. the three parameters \( \epsilon, \delta, m \) defining the conductors but also the wave number \( k. \) In addition, the error of these approximations are uniform in terms of these parameters where the uniform bounds depend only on the a priori bounds of the Lipschitz character, of the set of conductors, described above. In particular, we deal with the mesoscale regime where \( \epsilon \sim \delta. \)
To describe the results precisely, let us recall some properties and notations. For \( i = 1, \ldots, m \), we recall the single layer operator \([S^k_{i,i,D}] : L^2(\partial D_i) \to H^1(\partial D_i)\), defined as
\[
[S^k_{i,i,D}](\psi)(x) = \int_{\partial D_i} \Phi_k(x,y)\psi(y) \, ds(y), \quad x \in \partial D_i,
\]
and the double layer operator \([K^k_{i,i,D}] : L^2(\partial D_i) \to L^2(\partial D_i)\),
\[
[K^k_{i,i,D}](\psi)(x) = \int_{\partial D_i} \frac{\partial \Phi_k}{\partial \nu_y}(x,y)\psi(y) \, ds(y), \quad x \in \partial D_i,
\]
with its adjoint
\[
[(K^k_{i,i,D})^*](\psi)(x) = \int_{\partial D_i} \frac{\partial \Phi_k}{\partial \nu_y}(x,y)\psi(y) \, ds(y), \quad x \in \partial D_i,
\]
where \( \Phi_k(x,y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|} \) is the Green function for the Helmholtz equation at the wave number \( k \).

The operator \([\lambda I + (K^0_{i,i,D})^*]\) is invertible from \( L^2(\partial D_i) \) onto itself for any complex number \( \lambda \) such that \(|\lambda| \geq \frac{1}{2}\), see [3][17] for instance. The following two quantities will play an important role in the sequel:
\[
[\mathcal{P}_{\partial D_i}] := \int_{\partial D_i} \left[ -\frac{1}{2} I + (K^0_{i,i,D})^* \right]^{-1}(\nu)(y)(y-z_i)^T \, ds(y),
\]
and
\[
[\mathcal{T}_{\partial D_i}] := \int_{\partial D_i} \left[ \frac{1}{2} I + (K^0_{i,i,D})^* \right]^{-1}(\nu)(y)y^T \, ds(y).
\]
The tensor \([\mathcal{P}_{\partial D_i}]\) is negative-definite symmetric matrix and \([\mathcal{T}_{\partial D_i}]\) is positive-definite symmetric matrix, (see Lemma 5 and Lemma 6 in [24] or Theorem 4.11 in [5]). Further, we have the following scales:
\[
[\mathcal{P}_{\partial D_i}] = e^{3}[\mathcal{P}_{\partial B_i}], \quad \text{and} \quad [\mathcal{T}_{\partial D_i}] = e^{3}[\mathcal{T}_{\partial B_i}].
\]
Indeed, \footnote{Recall that \( \int_{\partial D_i} \left[ -\frac{1}{2} I + (K^0_{i,D})^* \right]^{-1}(\nu)(y)(y-z_i)^T \, ds(y) = \int_{\partial D_i} \left[ -\frac{1}{2} I + (K^0_{i,D})^* \right]^{-1}(\nu)(y) \, ds(y) = 0 \).}
\[
[\mathcal{P}_{\partial D_i}] = \int_{\partial D_i} \left[ -\frac{1}{2} I + (K^0_{i,i,D})^* \right]^{-1}(\nu)(y) (y-z_i)^T \, ds(y),
\]
\[
= \int_{\partial B_i} \left[ -\frac{1}{2} I + (K^0_{i,i,D})^* \right]^{-1}(\nu)(s_i) (\epsilon s + z_i - z_i)^T \epsilon^2 \, ds(y),
\]
\[
= e^{3}[\mathcal{P}_{\partial B_i}],
\]
and we get it in the same way for the second identity, after noticing that \footnote{We have \( [\frac{1}{2} I + K^0_{i,i,D}] |(z_i) = z_i [\frac{1}{2} I + K^0_{i,i,D}] (1) = 0 \).}
\[
[\mathcal{T}_{\partial D_i}] := \int_{\partial D_i} \nu_y \left[ \frac{1}{2} I + K^0_{i,i,D} \right]^{-1}(x)^T ds(y) = \int_{\partial D_i} \nu_y \left[ \frac{1}{2} I + K^0_{i,i,D} \right]^{-1} (x-z_i)^T \, ds(y).
\]
For \( i \in \{1, \ldots, m\} \) let \((\mu_i^T)^+, (\mu_i^T)^-\) be the respective maximal eigenvalues of \([\mathcal{T}_{\partial B_i}], -[\mathcal{P}_{\partial B_i}]\), and let \((\mu_i^T)^-, (\mu_i^T)^+\) be their minimal ones. We define
\[
\mu^+ = \max_{i \in \{1, \ldots, m\}} ((\mu_i^T)^+, (\mu_i^T)^+), \quad \mu^- = \min_{i \in \{1, \ldots, m\}} ((\mu_i^T)^-, (\mu_i^T)^-).
\]
We introduce generic functions \( \epsilon \) and the dyadic Green’s function is given by

\[
\Pi(x, y) := k^2 \Phi_k(x, y) I + \nabla_x \nabla_x \Phi_k(x, y) = k^2 \Phi_k(x, y) I - \nabla_x \nabla_y \Phi_k(x, y).
\]

Theorem 1.1. Let \( S \) and \( \Omega \) be the solutions of the following linear system

\[
A_i = -[P_{\partial D_i}] \sum_{(j \neq i) \geq 1} (\Pi_k(z_i, z_j)A_j - k^2 \nabla \Phi_k(z_i, z_j) \times B_j) - [P_{\partial D_i}] \text{curl} E^{inc}(z_i),
\]

\[
B_i = [T_{\partial D_i}] \sum_{(j \neq i) \geq 1} (- \nabla_x \Phi_k(z_i, z_j) \times A_j + \Pi_k(z_i, z_j)B_j) - [T_{\partial D_i}] E^{inc}(z_i),
\]

which is invertible under the following sufficient condition

\[
C_{L_1} := 1 - C_{L_2} \frac{\mu^+ \epsilon^3}{\delta^3} > 0.
\]

There exists a constant \( C \) depending only on the Lipschitz character of the \( B_i \)’s such that if

\[
|k|^2 \epsilon + (1 + |k|^2) \mu^+ \epsilon^3 \frac{\delta}{3} + \left( \frac{\ln (\frac{1}{k})}{\delta^3} + \frac{2|k|^{\frac{4}{3}}}{\delta^2} + \frac{m^2}{2 \delta^2} \right) \epsilon^3 < C,
\]

then

1. the scattered electric field has the following expansion, for \( \forall k \geq 0 \), for \( x \in \mathbb{R}^3 \setminus \bigcup_{i=1}^{m} D_i \) such that \( \min_{1 \leq i \leq m} d(x, D_i) = \delta \),

\[
E^{sc}(x) = \sum_{i=1}^{m} \left( \nabla \Phi_k(x, z_i) \times \mathcal{A}_i + \text{curl} \Phi_k(x, z_i) \mathcal{B}_i \right)
\]

\[
+ \left( C_{L_2} \mu^- \right)^{-1} \times O^\epsilon \left( \frac{4}{\delta^4} \right) + \left( C_{L_2} \mu^- \right)^{-1} \times O^\epsilon \left( \frac{\epsilon^2}{\delta^3} \right),
\]

with

\[
C_{L_2} = 1 - 4 \mu^+ \left( \frac{\ln (\frac{1}{k})}{\delta^3} + \frac{2|k|^{\frac{4}{3}}}{\delta^2} + \frac{m^2}{2 \delta^2} \right) \epsilon^3.
\]
2. the far field pattern has the following expansion for $\Im k = 0$

$$E^\infty(r) = \frac{ik}{4\pi} \sum_{i=1}^{m} e^{-ik\tau_i \tau} \times (\hat{A}_i - ik\tau \times \hat{B}_i) + O\left((|k|^3 + |k|^2) \frac{m\epsilon}{C_{Lm}}\right)$$

and the errors in (1.19) and (1.20) correspond to

$$O^{k}\left(\frac{\epsilon^7}{\delta^7}\right) = \mathcal{O}\left(\frac{\epsilon^7}{\delta^7} + \epsilon(\delta^6, |k| + |k|^2)\epsilon^2 + \epsilon(\delta^5, |k|^2)\epsilon^2\right),$$

$$O^{k}\left(\frac{\epsilon^4}{\delta^4}\right) = \mathcal{O}\left(\frac{\epsilon^4}{\delta^4} + (1 + |k|)\epsilon_{k,\delta,m}\epsilon^4 + \max(1 + |k|, |k|^2)\epsilon\right).$$

The approximation in (1.19) and (1.20) are called the point-interaction approximations or the Foldy-Lax approximations as the dominant field is reminiscent to the field describing the interactions between the points $z_i$, $i = 1, \ldots, m$, with the scattering coefficients given by the polarization tensors $[P_{BD_i}]$ and $[T_{BD_i}]$, see [6, 12] for particular situations. Since the pioneering works of Rayleigh till Foldy, the first and original goal of such approximations was to reduce the computation of the fields generated by a cluster of small bodies to inverting an algebraic system (called the Foldy linear algebraic system), see [12] for more information. With our approximations above, and regarding the full Maxwell system, such a goal is reached with high generality as we take into account all the parameters, $m, \epsilon, \delta$ and $k$, modeling the scattering by the cluster of small conductors $D_i$'s.

In the recent twenty years or so, there were different and highly important fields where such kind of approximations are key tools. Let us mention few of them which are of particular interest to us:

1. Let us start with the mathematical imaging field where the small bodies can model impurities or small tumors, for instance, that one should localize and estimate the sizes from the measured fields (either near fields or far-fields), see [4]. In this case, based on our approximations, we can indeed localize the small bodies by reconstructing the points $z_i$, via MUSIC type algorithms [4, 6], and then estimate the polarization tensors. From these polarization tensors, one can derive lower and upper estimates of the small bodies’ sizes, see [2]. The small bodies can also model electromagnetic nanoparticles. Imaging using electromagnetic nanoparticles as contrast agent is a recent and highly attractive imaging modality that uses special properties of the nanoparticles to create high contrasts in the tissue and then enhance the resolution of permittivity reconstruction for instance. There are at least two types of such special properties: one is related to the plasmonic nanoparticles (which are nearly resonant nanoparticles but might create high dissipation) and the other one related to the all dielectric nanoparticles (which are characterized by their high refraction indices), see [11], for instance.

2. A second field where this kind of approximations are useful is the material sciences. Indeed, arranging appropriately the small bodies in a given bounded domain, the whole cluster will generate electromagnetic fields which are close to the fields generated by related indices of refraction (or permittivities and permeabilities). These indices of refraction are dependent on the properties of the small bodies, as the size, geometry and their own possible contrasts in addition to the used frequencies. This opens the door to the possibility of creating desired and new materials. Such ideas are already tested and justified to some extent mathematically in the framework of the homogenization theory. However, this theory is based on the periodicity (or randomness) in distributing the small bodies. As we can see it from the approximations we provide above, we can achieve similar goals but without assuming the periodicity. In addition, and as far the electromagnetic waves are concerned, we can handle in a unified way, the generation of volumetric metamaterials, Gradient metasurfaces and also metawires, [23, 25]. These properties will be quantified and justified in a future work were we plan to handle more general type of inhomogeneities than the conductive ones described in this work.
Our contribution in this work is to have succeeded in handling the full Maxwell model by taking into account (explicitly) all the parameters modeling the small conductors $m, \epsilon$ and $\delta$ but also the used frequency $k$. To our best knowledge, there is no result in the literature where such approximations are provided with such generality and precision. At the mathematical analysis level, and as we are using integral equations methods, we needed to derive an a priori estimate of the related densities. The first key observation here is to derive it in the $L^2_{\text{Div}}$ spaces instead of the usual $L^2$ spaces. As a second observation, to derive such estimates, we used a particular decomposition of the densities, see Proposition 2.1 or Theorem 2.3 which allows to obtain the needed qualitative as well as quantitative estimates while refining the approximation. Finally, to prove the invertibility of the algebraic system (1.16) under the general condition (1.16), the key point is to have reduced the coercivity inequality to the one related to scalar Helmholtz model. Let us emphasize here that as this linear algebraic system comes from the boundary conditions, such a reduction of Maxwell to Helmholtz is not a trivial one (even, a priori, not a natural one).

The only restriction we have in our condition (1.18) is the appearance of the factor $\ln(m)$. At the technical level, see the last part of the proof of Theorem 2.3 its appearance is due to the fact the singularity of the dyadic Green’s function (1.14) is of the order 3 (in contrast to the ones of the Green’s functions for the Laplace or Lamé related models). We believe that this factor can be removed using more pde tools to invert the Calderon operator, see (2.14), and hope to report on this in the near future.

The closest published works (i.e. deriving the Foldy-Lax type of approximations) are those by A. Ramm in one side and those by V. Maz’ya, A. Movchan and M. Nieves in another side. The several works by A. Ramm on Maxwell are derived more in a formal way, see [21, 22]. In addition, condition of the type $\frac{\epsilon}{\delta} << 1$ are used (meaning at least that the mesoscale regimes where $\epsilon \sim \delta$ are not handled) and without clarifying the rates. Finally, and unfortunately, to our opinion the form of the derived algebraic system is unclear and questionable. In a series of works dedicated to the Laplace and Lamé models (assuming $k = 0$), V. Maz’ya, A. Movchan and M. Nieves proposed a method which indeed takes into account the parameter and state the results in the mesoscale regimes too, see [13–15]. One possible limit to their approach is the need of the maximum principle in handling the link between the system on the boundary to fields outside. This might be a handicap for tackling the Maxwell system for which such maximum principles are not at our hands.

The remaining part of the paper is dedicated to the proof of Theorem 1.1 and it is arranged as follows. In section 2 we recall and discuss the well posedness of the scattering problem via integral equation method and then derive the key a priori estimates of the densities. In section 3 we derive the fields approximations with the corresponding non homogeneous linear system. In section 4 we justify and quantify the invertibility of the algebraic system and in section 5 we combine the estimates in the last two sections to complete the proof of Theorem 1.1.

2 Existence, unique solvability and an a priori estimation of the density

2.1 Preliminaries

Let us recall few properties of the surface divergence which will be important in our later analysis, see (Section 4 in [13] and Chapter 2 in [9]) for more details. First, we recall the surface gradient of a smooth function $\phi$ on $\partial D$, $\nabla_s$, as

$$\nabla_s \phi := \nabla \phi - (\nu \cdot \nabla \phi)\nu$$

where $\nu$ is the exterior unit normal to $\partial D$. Then the (weak) surface divergence for a tangential field $a$ is defined using the duality

$$\int_{\partial D} \phi \text{Div} a \; ds = - \int_{\partial D} \nabla_s \phi \cdot a \; ds,$$  \hspace{1cm} \text{(2.1)}$$

for every $\phi \in C^\infty(\partial D)$.

If $a$ is a tangential field for which $\text{Div} a$ exists in the sense above, and it is in $L^1(\partial D)$ for instance, then,
taking $\phi(x) = x_i$ in (2.1), we have
\[ \int_{\partial D} x \text{Div} a(x) \, ds(x) = -\int_{\partial D} a(x) \, ds(x) \] (2.2)
and taking $\phi(x) = 1$, we have
\[ \int_{\partial D} \text{Div} a(x) \, ds(x) = 0. \] (2.3)

When $a := \nu \times u$ for a certain sufficiently smooth vector field $u$, we get
\[ \text{Div} a = -\nu \cdot \text{curl} u. \] (2.4)

Further, for a scalar function $\psi$, being $\psi$ a tangential (i.e. $\psi \cdot \nu = \psi (a \cdot \nu) = 0$), the following identity holds,
\[ \text{Div} (\psi a) = \nabla \psi \cdot a + \psi \text{Div} a, \]
and let be $L^p(\partial D)$ and $L^p, \text{Div}(\partial D)$ denote respectively the space of all tangential fields of of $L^p(\partial D)$; and the subspace of $L^p(\partial D)$ that have an $L^p$ weak surface divergence, precisely
\[ L^p_\nu(\partial D) = \{ a \in L^p(\partial D); a \cdot \nu = 0 \}, \]
\[ L^p, \text{Div}_\nu(\partial D) = \{ a \in L^p(\partial D); \text{Div} a \in L^p_\nu(\partial D) \} \]
where $L^p_\nu(\partial D) := \{ u \in L^p(\partial D), \text{ such that } \int_{\partial D} u \, ds = 0 \}.$

### 2.2 Existence and uniqueness of the solution

The solution to the problem (1.2) under the boundary condition (1.3) and the radiating conditions (1.4) can be expressed in terms of boundary integral equation (see [10, 20]), under certain conditions in appropriate spaces that will be specified later, using either one of the representations
\[ E(x) = E^i(x) + \text{curl} \int_{\partial D^+} \Phi_k(x, y) a(y) \, ds(y), \quad x \in \mathbb{R}^3 \setminus (\bar{\bar{D}} := \cup_{i=1}^m \bar{D}_i) \] (2.6)
\[ E(x) = E^i(x) + \text{curl} \int_{\partial D^+} \Phi_k(x, y) a(y) \, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D} \] (2.7)
or a linear combination of the two, where $a$ is the unknown vector density to be found to solve the problem. Let us consider the representation (2.6), i.e. for a tangential field $a$ and $s \in D^+$
\[ E^\text{sca}(s) = \text{curl} \int_{\partial D^+} \Phi_k(s, y) a(y) \, ds(y), \] (2.8)
and let be $\{ \Gamma_+(x), \Gamma_-(x), \, x \in \cup_{i=1}^m \partial D_i \}$ a family of doubly truncated cones with a vertex at $x$ such that $\Gamma_\pm(x) \cap D^\pm = \emptyset$, we have for almost every $x \in \cup_{i=1}^m \partial D_i$ (see [18])
\[ \lim_{s \to x} E^\text{sca}(s) = \left( \mp \frac{1}{2} \nu \times a + \text{curl} \int_{\partial D} \Phi_k(x, y) a(y) \, ds(y) \right) \] (2.9)
where the integral is taken in the principal value of Cauchy sense, and the identity must be understood in the sense of trace operator, then using the condition (1.3), we get

\[ \nu \times \lim_{s \to \pm} E^{\text{sca}}(s) = \pm \frac{1}{2} I + M^k_{\partial D_i}(a)(x) := \pm \frac{1}{2} a + \nu \times \text{curl} \int_{\partial D} \Phi_k(x, y)a(y) \, ds(y) \quad (2.10) \]

where \( \pm \frac{1}{2} I + M^k_{\partial D_i} \) is called the magnetic dipole operator. Consequently, to solve the scattering problem we need to solve the integral equation

\[ \left[ \frac{1}{2} I + M^k_{\partial D_i} \right](a)(x) = -\nu \times E^{\text{inc}}, \quad \text{on } \cup_{i=1}^m \partial D_i \quad (2.11) \]

or,

\[ \left[ \frac{1}{2} I + M^k_{\partial D_i} \right](a)(x) + \left[ \sum_{(j \neq i) \geq 1} M^k_{i,j} \right](a) = -\nu \times E^{\text{inc}}(x_i), \quad x_i \in \partial D_i. \quad (2.12) \]

In this case the magnetic field is represented by

\[ H^{\text{sca}}(x) = -ik \text{curl} \int_{\partial D^+} \Phi_k(x, y)a(y) \, ds(y). \]

Further it satisfies

\[ \nu \times H^{\text{sca}}(s) = \left[ N^k_{\partial D} \right](a) = k^2 [\nu \times S_{\partial D}^k](a) + [\nu \times \nabla S_{\partial D}^k](\partial \text{div} a) \quad (2.13) \]

which means that \( \nu \times (H^{\text{sca}}) = (\nu \times H^{\text{sca}})^\ast \) i.e \( H \) is continuous across the boundary, the operator \( N_k \) is called electric dipole operator.

We can write the equations in (2.12) in a compact form as follows

\[ \left( \frac{1}{2} I + M_D + M_N \right) A = -\nu \times E^I, \quad (2.14) \]

where \( A = (a_1, \ldots, a_m)^T \) is a column matrix which vectorial components are \( a_i := a/\partial D_i \). Similarly, \( E^I = (E^i_{\text{inc}}, \ldots, E^i_{\text{inc}}) \) with \( E^i_{\text{inc}} = E^\text{inc}/\partial D_i \) and \( M_D \) is the diagonal matrix operator given by

\[ M_D := \begin{cases} M_{i,j}^k & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \]

and finally \( M_N \) is the matrix operator with null diagonal

\[ M_N := \begin{cases} 0 & \text{if } i = j \\ M_{i,j}^k & \text{if } i \neq j \end{cases} \]

where

\[ M_{i,j}^k a(x) := \nu \times \int_{\partial D_j} \nabla_x \Phi_k(x, y) \times a(y) \, ds(y), \text{ for every } x \in \partial D_i. \]

For \( k \in \mathbb{C} \setminus \{0\} \) such that \( \Im k \geq 0 \), the operators \( (\pm \frac{1}{2} I + M^k_{\partial D_i}) \) are Fredholm with index zero from \( L^2_{\text{div}}(\partial D_i) \) into itself, furthermore \( (\pm \frac{1}{2} I + M^k_{\partial D_i}) \) are Fredholm from \( L^2_{\text{div}}(\partial D_i)/L^2_0(\partial D_i) \) into itself, likewise for \( L^2(\partial D_i) \). Moreover if \( k \) is not a Maxwell eigenvalue for \( D_i \) then the operators are in fact isomorphisms, see Theorem 5.3 [18] and the remark after, and [19].

Let us notice that when \( \Im k > 0 \), then \( k \) is not a Maxwell eigenvalue. In addition, when \( \Im k = 0 \) and as the radius of \( D_i \) is small, by a scaling argument, this condition on \( k \) is obviously fulfilled. As \( \pm \frac{1}{2} I + M_D \) is an isomorphism and \( M_N \) is compact (since the kernel of each component is of class \( C^\infty \)), the operators

\[ \pm \frac{1}{2} I + M_D + M_N : \prod_{i=1}^m \mathcal{E}(\partial D_i) \longrightarrow \prod_{i=1}^m \mathcal{E}(\partial D_i), \quad (2.15) \]
where \( \mathcal{E} := L^2_t \), \( L^2_{t, \text{Div}} \), are Fredholm with zero index, so to show that the operator above is in fact an isomorphism it is enough to show that the homogeneous problem (i.e. \( E^\text{inc} = 0 \)), has the unique identically null solution density that is \( \mathcal{A} = 0 \). We derive from (2.11), for \( E^\text{inc} \equiv 0 \) and the uniqueness to the exterior boundary problem that

\[
E^\text{sca}(x) = \text{curl} \int_{\partial D^+} \Phi_k(x, y) a(y) \, ds(y) \equiv 0, \quad x \in \mathbb{R}^3 \setminus \overline{D},
\]

(2.16)

taking the rotational, we obtain

\[
H^\text{sca}(x) = -ik \text{curl} \text{curl} \int_{\partial D^+} \Phi_k(x, y) a(y) \, ds(y) \equiv 0, \quad x \in \mathbb{R}^3 \setminus \overline{D},
\]

(2.17)

then going to the boundary using (2.13) we get

\[
\nu \times H^\text{sca} = 0, \quad x \in \partial D = \bigcup_{i=1}^m \partial D_i.
\]

As the homogeneous interior boundary problem admits the unique identically null solution, we get \( H^\text{sca}(x) = 0, \quad x \in D \) and hence taking the rotational gives us \( E^\text{sca}(x) = 0, \quad x \in D \) and ultimately \( (\nu \times E^\text{sca}) = 0 \) on \( \partial D \). Finally we have

\[
[\pm \frac{1}{2} I + M^k_{i,i} + \sum_{(j \neq i) \geq 1} M^k_{j,i}] a = 0
\]

(2.18)

and taking the difference of the two identities we get \( a_i = 0 \) for every \( i \in \{1, \ldots, m\} \), and yield that \( a \equiv 0 \) on \( \partial D = \bigcup_{i=1}^m \partial D_i \). This shows that we have existence of the solution of our original scattering problem and it can be represented as (2.10) with a unique tangential density \( a \). The uniqueness of the solution for the original scattering problem is deduced in the same way as it is done in Theorem 6.10 in [10] for instance.

### 2.3 A priori estimates of the densities

In order to derive suitable estimates of the densities \( a_i, \quad i = 1, \ldots, m \), we need to use the Helmholtz decomposition based on the following operators, which are isomorphism (see Theorem 5.1 and Theorem 5.3 in [13]),

\[
\begin{aligned}
\nu \cdot \text{curl} S^0_{i,i} : L^2_{t, \text{Div}}(\partial D_i) &\longrightarrow L^2_{0}(\partial D_i), \\
\nu \times \nabla S^0_{i,i} : L^2_{0}(\partial D_i) &\longrightarrow L^2_{2}(\partial D_i), \\
\nu \times S^0_{i,i} : L^2_{t, \text{Div}}(\partial D_i) &\longrightarrow (L^2_{t, \text{Div}} \setminus L^2_{0})(\partial D_i) := L^2_{t, \text{Div}}(\partial D_i) \setminus L^2_{t, \text{Div}}(\partial D_i).
\end{aligned}
\]

(2.19)

The following decomposition holds,

**Proposition 2.1.** Each element \( V \) of \( L^p_{t, \text{Div}}(\partial D_i) \) can be decomposed as

\[
V = \Psi + \nu \times \nabla \nu
\]

(2.20)

where

\[
\begin{aligned}
\Psi &\in L^p_{t, \text{Div}}(\partial D_i) \setminus L^p_{t, 0}(\partial D_i), \\
\nu \times \nabla \nu &\in L^p_{t, 0}(\partial D_i), \quad \nu \in H^1(\partial D_i) \setminus C;
\end{aligned}
\]

(2.21)

which satisfy

\[
\begin{aligned}
\|\Psi\|_{L^2(\partial D_i)} &\leq C_1 \|V\|_{L^2_{t, \text{Div}}(\partial D_i)}, \\
\|\Psi\|_{L^2_{t, \text{Div}}(\partial D_i)} &\leq C_2 \|V\|_{L^2_{t, \text{Div}}(\partial D_i)}, \\
\|\nu \times \nabla \nu\|_{L^2_{t, 0}(\partial D_i)} &\leq C_3 \|V\|_{L^2_{t, \text{Div}}(\partial D_i)},
\end{aligned}
\]

(2.22)

(2.23)

(2.24)

and

\[
\|\nu\|_{L^2(\partial D_i)} \leq C_4 \|a\|_{L^2_{t, \text{Div}}(\partial D_i)}
\]

(2.25)

where \( (C_i)_{i=1,2,3,4} \) are constants which depend only on \( B'_i \)'s,
To prove Proposition 2.1 we need the following identities.

Lemma 2.2. For \( x_i \in \partial D_1, s_i \in \partial B_1 \), with \( x_i = \epsilon s_i + z_i \), and \( \hat{a}(s_i) = a(\epsilon s_i + z_i) \), we have the following scaling identities

\[
\| \epsilon \hat{a} \|_{L^2_0(\partial B_1)} = \| a \|_{L^2_0(\partial D_1)}, \quad \| \text{Div} \hat{a} \|_{L^2_0(\partial B_1)} = \| \text{Div} a \|_{L^2_0(\partial D_1)},
\]

and

\[
\| [\nu \times \nabla S^0_{\nu, i, D}]^{-1} \|_{L^2_0(\partial D_1), L^2_0(\partial D_1)} = \| [\nu \times \nabla S^0_{\nu, D}]^{-1} \|_{L^2_0(\partial B_1), L^2_0(\partial B_1)},
\]

\[
\| \nu \cdot \text{curl} S^0_{\nu, i, D} \|_{L^2_0(\partial D_1), L^2_0(\partial D_1)} = \| \nu \cdot \text{curl} S^0_{\nu, D} \|_{L^2_0(\partial B_1), L^2_0(\partial B_1)}.
\]

Proof. We derive the identities in (2.26) as follows:

\[
\int_{\partial D_i} \text{Div}_x a(x) \phi(x) \, ds(x) = \int_{\partial D_i} a(x) \cdot \nabla \phi(x) \, ds(x) = \int_{\partial D_i} \hat{a}(s) \cdot \frac{1}{\epsilon} \nabla_s \phi(s) \, ds(x),
\]

\[
= \int_{\partial D_i} \text{Div}_s \hat{a}(s) \frac{1}{\epsilon} \phi(s) \, ds(x) = \int_{\partial D_i} \frac{1}{\epsilon} \text{Div}_s \hat{a}(s) \phi(s) \, ds(x),
\]

then \( \text{Div}_x a(x) = \frac{1}{\epsilon} \text{Div}_s \hat{a}(s) \). It remains to take the norm to conclude.

Concerning (2.27) we have,

\[
[\nu \times \nabla S^0_{\nu, i, D}](u)(x) = \nu \times \nabla \left( \frac{1}{4\pi} \int_{\partial D_i} \frac{1}{|x - y|} u(y) \, ds(y) \right) = \nu \times \frac{1}{4\pi} \int_{\partial D_i} \frac{1}{|x - y|^3} (x - y) u(y) \, ds(y),
\]

hence, for \( x_i = \epsilon s_i + z_i \) and \( y_i = \epsilon t_i + z_i \), and \( \hat{u}(s_i) = u(\epsilon t_i + z_i) \)

\[
[\nu \times \nabla S^0_{\nu, i, D}](u)(x) = \nu_{x_i} \times \frac{1}{4\pi} \int_{\partial B_1} \frac{1}{|s_i - t_i|^3} (s_i + z_i - \epsilon t_i - z_i) u(\epsilon t_i + z_i) \, ds(t),
\]

\[
= \nu_{x_i} \times \frac{1}{4\pi} \int_{\partial B_1} \frac{1}{\epsilon^3 |s_i - t_i|^3} \epsilon (s_i - t_i) u(\epsilon t_i + z_i) \, ds(t),
\]

\[
= \nu_{x_i} \times \frac{1}{4\pi} \int_{\partial B_1} \frac{1}{s_i - t_i} (s_i - t_i) \hat{u}(t_i) \, ds(t) = [\nu \times \nabla S^0_{\nu, i, D}](\hat{u})(s_i).
\]

From the equality \( [\nu \times \nabla S^0_{\nu, D}][[\nu \times \nabla S^0_{\nu, D}]^{-1}(u)] = u \), we get replacing in the previous identity \( u \) by \( [\nu \times \nabla S^0_{\nu, D}]^{-1}(u) \), that \( [\nu \times \nabla S^0_{\nu, D}][[\nu \times \nabla S^0_{\nu, D}]^{-1}(u)] = \hat{u} \). Finally inverting the left-hand side operator, we have the scales

\[
[\nu \times \nabla S^0_{\nu, D}]^{-1}(u) = [\nu \times \nabla S^0_{\nu, D}]^{-1} \hat{u}.
\]

As

\[
\| [\nu \times \nabla S^0_{\nu, D}]^{-1} \|_{L^2_0(\partial D_1), L^2_0(\partial D_1)} = \sup_{\substack{u \in L^2_0(\partial D_1) \setminus \{0\}}} \frac{\| [\nu \times \nabla S^0_{\nu, D}]^{-1}(u) \|_{L^2(\partial D_1)}}{\| u \|_{L^2(\partial D_1)}}.
\]

3The gradient stands for the surface gradient.
Taking the surface divergence we have, \( \nu \) inequality (2.23) is an immediate consequence. Now, (2.30) becomes using (2.31) for the last coming inequality, we have

\[
\begin{align*}
\text{Proof.} \quad \text{(Of Proposition 2.1)} & \quad \text{It suffices to seek for the solution of the following equation} \\
\nu \times \nabla S^0_{\nu,D}(v) + \nu \times S^0_{\nu,D}(w) = V. 
\end{align*}
\]

Taking the surface divergence we have, \( \nu \cdot \text{curl} S^0_{\nu,D}(w) = \text{Div} V \) and then using (2.14), \( w = [\nu \cdot \text{curl} S^0_{\nu,D}]^{-1} \text{Div} V \). Using (2.27) we get the estimate

\[
\|w\|_{L^2_B(\partial D_i)} \leq \|\nu \cdot \text{curl} S^0_{\nu,D}\|_{L^2_B(\partial D_i)}\|\text{Div} V\|_{L^2_B(\partial D_i)}. 
\]

Put \( \mathcal{W} = \nu \times S^0_{\nu,D}(w) \),

\[
\|\mathcal{W}\|_{L^2(\partial D_i)} = \left( \int_{\partial D_i} |\nu \times S^0_{\nu,D}(w)|^2 \, ds \right)^{\frac{1}{2}} = \left( \int_{\partial D_i} \left( \nu \times \int_{\partial D_i} \frac{1}{\epsilon|s-t|}\hat{w}(t)\epsilon^2 s(t) \, ds \right)^2 \, ds \right)^{\frac{1}{2}},
\]

\[
= \epsilon^2 \left( \int_{\partial D_i} \left( \nu \times \int_{\partial D_i} \frac{1}{|s-t|}\hat{w}(t) \, ds(t) \right)^2 \, ds \right)^{\frac{1}{2}} = \epsilon^2 \|\nu \times S^0_{\nu,D}(\hat{w})\|_{L^2(\partial D_i)}. 
\]

Using (2.31) for the last coming inequality, we have

\[
|\mathcal{W}|_{L^2(\partial D_i)} = \epsilon^2 \|\nu \times S^0_{\nu,D}(\hat{w})\|_{L^2(\partial D_i)},
\]

\[
\leq \epsilon^2 \left( \|\nu \times S^0_{\nu,D}(\hat{w})\|_{L^2(\partial D_i)} + \|\text{Div} (\nu \times S^0_{\nu,D}(\hat{w}))\|_{L^2_B(\partial D_i)} \right),
\]

\[
\leq \|\nu \times S^0_{\nu,D}(\hat{w})\|_{L^2_B(\partial D_i)},
\]

\[
\leq \epsilon^2 \|\nu \times S^0_{\nu,D}\|_{L^2_B(\partial D_i)}\|\text{Div} V\|_{L^2_B(\partial D_i)}^2,
\]

\[
\leq C_1^\epsilon \epsilon^2 \|\text{Div} V\|_{L^2_B(\partial D_i)},
\]

\[
\text{here} \quad C_1^\epsilon := \|\nu \times S^0_{\nu,D}\|_{L^2_B(\partial D_i)}\|\text{Div} V\|_{L^2_B(\partial D_i)}, \quad \text{which gives (2.22). The inequality (2.22) is an immediate consequence. Now, (2.30) becomes} \\
\nu \times \nabla S^0_{\nu,D}(v) = \mathcal{W}, \quad \text{with} \ \text{Div} (V - \mathcal{W}) = 0, \quad \text{then we get successively}
\]

\[
\|v\|_{L^2_B(\partial D_i)} \leq \|\nu \times S^0_{\nu,D}\|_{L^2_B(\partial D_i)}|V - \mathcal{W}|_{L^2_B(\partial D_i)},
\]

\[
\leq \|\nu \times S^0_{\nu,D}\|_{L^2_B(\partial D_i)}\|\nu \times \nabla S^0_{\nu,D}(v)\|_{L^2_B(\partial D_i)} \leq C_3^\epsilon \|V\|_{L^2_B(\partial D_i)},
\]

\[
\text{where} \quad C_3^\epsilon := \|\nu \times \nabla S^0_{\nu,D}\|_{L^2_B(\partial D_i)}\|\nu \times \nabla S^0_{\nu,D}(v)\|_{L^2_B(\partial D_i)}\|\nu \times \nabla S^0_{\nu,D}(v)\|_{L^2_B(\partial D_i)} = \|V - \mathcal{W}\|_{L^2_B(\partial D_i)}^2. \quad \text{Then with (2.27) in mind, we derive the estimate (2.24)}
\]

\[
\|\nu \times \nabla v\|_{L^2_B(\partial D_i)} = \|V - \mathcal{W}\|_{L^2_B(\partial D_i)} \leq \|V\|_{L^2_B(\partial D_i)} + \|\mathcal{W}\|_{L^2_B(\partial D_i)},
\]
which satisfy
\[
\leq \|V\|_{L^2_t,\text{div}((\partial D_i))} + C_{B_1}^\epsilon \|V\|_{L^2_t,\text{div}((\partial D_i))},
\]
\[
\leq (C_{B_3}^\epsilon := (1 + C_{B_1}^\epsilon)) \|V\|_{L^2_t,\text{div}((\partial D_i))}.
\]
Concerning (2.25) we have \(\|v\|_{L^2(\partial D_i)} = \|S_{0,i}^0(v)\|_{L^2(\partial D_i)}\), hence
\[
\|v\|_{L^2(\partial D_i)} = \left( \int_{\partial B_i} \left( \int_{\partial B_i} \frac{1}{|s-t|} \hat{v}(t) e^2 ds(t) \right)^2 e^2 ds(s) \right)^{1/2},
\]
\[
= \epsilon^2 \|S_{0,i}^0(v)\|_{L^2(\partial B_i)} \leq \epsilon^2 \left( \|S_{0,i}^0(\hat{v})\|_{L^2(\partial B_i)} + \|\nabla v S_{0,i}^0(\hat{v})\|_{L^2(\partial B_i)} \right),
\]
\[
\leq \epsilon^2 \|S_{0,i}^0\|_{L(L^2(\partial B_i),H^1(\partial B_i))} \|\hat{v}\|_{L^2(\partial B_i)} = \epsilon \|S_{0,i}^0\|_{L(L^2(\partial B_i),H^1(\partial B_i))} \|v\|_{L^2(\partial D_i)},
\]
and with (2.33) we get,
\[
\|v\|_{L^2(\partial D_i)} \leq \epsilon (C_{4}^\epsilon := \|S_{0,i}^0\|_{L(L^2(\partial B_i),H^1(\partial B_i))} C_{3}^\epsilon) \|V\|_{L^2_t,\text{div}((\partial D_i))},
\]
to conclude we put for \(i = 1, 2, 3, 4\), \(C_i = \max i \in \{1, ..., m\} C_{i}^\epsilon\).

We have the following theorem

**Theorem 2.3.** There exist constants \(C_{B,2}, C_{B,1}\) and \(C_e\) which depend only on \(B_i\)'s and independent of their number such that if
\[
C_{B,1}|k|^2 \epsilon < 1, \quad C_{B,2} \left( \frac{\ln m^{3/2}}{\delta^3} + \frac{2km^{3/2}}{\delta^2} + \frac{m^{3/2} k^2}{2\delta} \right) \epsilon^3 < 1
\]
then
\[
\|a\|_{L^2_t,\text{div}} \leq C_e \epsilon
\]

Further, in view of Proposition 2.7 each \(a_i \in L^2_t,\text{div}((\partial D_i))\) can be decomposed as the sum of
\[
a_i^{[1]} \in L^p_t,\text{div}((\partial D_i)) \setminus L^0_t,\text{div}((\partial D_i)), \text{ and,}
\]
\[
a_i^{[2]} = \nu \times \nabla u_i \in L^0_t,\text{div}((\partial D_i)), \quad u \in H^1(\partial D_i) \setminus C;
\]
which satisfy
\[
\|a^{[1]}\|_{L^2(\partial D_i)} \leq C_1 \epsilon \|a\|_{L^2_t,\text{div}((\partial D_i))},
\]
\[
\|a^{[1]}\|_{L^2_t,\text{div}((\partial D_i))} \leq C_2 \|a\|_{L^2_t,\text{div}((\partial D_i))},
\]
\[
\|a^{[2]}\|_{L^2_t,\text{div}((\partial D_i))} \leq C_3 \|a\|_{L^2_t,\text{div}((\partial D_i))},
\]
\[
\|u_i\|_{L^2(\partial D_i)} \leq C_4 \epsilon \|a\|_{L^2_t,\text{div}((\partial D_i))},
\]
where \((C_i)_{i=1,2,3,4}\) are constants which depends only on the shape of the \(B_i\)'s (i.e their Lipschitz character) and not on their number \(m\).

From (2.36), we have successively
\[
\left( \frac{1}{2} I + M_D \right) \left( I + \frac{1}{2} I + M_D \right)^{-1} M_N A = -\nu \times E',
\]
\[
\left( I + \frac{1}{2} I + M_D \right)^{-1} M_N A = -\left( \frac{1}{2} I + M_D \right)^{-1} \nu \times E',
\]
and if
\[
\left( \frac{1}{2} I + M_D \right)^{-1} \|M_N\| < 1,
\]

(2.41)
For every \( 1 \leq n \leq m \) and set conclusions of Proposition 2.1 and do not rely on (2.34).

Proof. We prove that under the condition (2.34), we have (2.35). The properties (2.36)-(2.40) are immediate from here

Then, by inverting the operators in each side, under the condition that

we get

\[
\left\| \frac{1}{2} I + M_{i,D} \right\|_{L^2(\partial D_i)} = \left\| \frac{1}{2} I + M_{i,D} \right\|_{L^2(\partial D_i)} \frac{1}{2} I + M_{i,D} \left[ \frac{1}{2} I + M_{i,D} \right]^{-1} \left[ \frac{1}{2} I + M_{i,D} \right]^{-1},
\]

As

\[
\left( \frac{1}{2} I + M_{i,D} \right)^{-1} = \left( \frac{1}{2} I + M_{i,D} \right)^{-1} \left( \frac{1}{2} I + M_{i,D} \right)\left( \frac{1}{2} I + M_{i,D} \right)^{-1},
\]

we have finally

\[
\left\| \frac{1}{2} I + M_{i,D} \right\|_{L^2(\partial D_i)} \leq \left\| \frac{1}{2} I + M_{i,D} \right\|_{L^2(\partial D_i)} \left( \frac{1}{2} I + M_{i,D} \right)^{-1} \left( \frac{1}{2} I + M_{i,D} \right)^{-1}.
\]

From here \( L(E) := L(E, E) \) denotes the space of continuous linear operators which are defined from \( E \) to \( E \).

Hence the proof of (2.35), based on the condition (2.34), is reduced to the following two estimates:

\[
\left\| \frac{1}{2} I + M_{i,D} \right\|_{L^2(\partial D_i)} \leq \left\| \frac{1}{2} I + M_{i,D} \right\|_{L^2(\partial D_i)} \left( \frac{1}{2} I + M_{i,D} \right)^{-1} \left( \frac{1}{2} I + M_{i,D} \right)^{-1} \left( \frac{1}{2} I + M_{i,D} \right)^{-1}.
\]

and

\[
\left\| M_N \right\|_{L^2(\partial D_i)} \leq 2^6 C B \left( \frac{\ln (n + 1)}{\delta} + \frac{2 n k}{\delta} + \frac{n}{2} (n + 1) k^2 \right),
\]

where \( n = O(m^2) \), and \( C_B \) is a constant which depends exclusively on \( B_i \). In some places of the next computations, we use the notation

\[
C_0 := 2^6.
\]
To justify (2.44) and (2.45), we need the following lemma.

**Lemma 2.4.** For \( x_i \in \partial D_i, s_i \in \partial B_i \), with \( x_i = \varepsilon s_i + z \) the following scaling estimation

\[
\| \frac{1}{2} I + M_{ii,D}^0 \|_{L_2^2(\partial D_i)}^{-1} = \| \frac{1}{2} I + M_{ii,B}^0 \|_{L_2^2(\partial B_i)}, \tag{2.46}
\]

and

\[
\| \frac{1}{2} I - K_{ii,D}^0 \|_{L_2^2(\partial D_i)}^{-1} = \| \frac{1}{2} I - K_{ii,B}^0 \|_{L_2^2(\partial B_i)}, \tag{2.47}
\]

and

\[
\| [M_{ii,D}^k - M_{ii,B}^0] \|_{L_2^2(\partial D_i), L_2^2(\partial B_i)} \leq 2C_b \| \partial B \| k^2 \| \partial \|_{L_2^2(\partial D_i)}. \tag{2.48}
\]

**Proof.** For (2.46) and (2.47), we first have

\[
M_{ii,D}^0 (a)(x_i) = \nu \int \mathcal{A}(x_i, y)(x_i - y) \times a(y) \, ds(y),
\]

which leads to \( \frac{1}{2} I + M_{ii,D}^0 \)^{-1}(b) = \( \frac{1}{2} I + M_{ii,B}^0 \)^{-1}b. With this in mind, considering (2.20) we get

\[
\| \frac{1}{2} I + M_{ii,D}^0 \|_{L_2^2(\partial D_i)}^{-1} = \sup_{(b \neq 0) \in L_2^2(\partial D_i)} \frac{\| \frac{1}{2} I + M_{ii,D}^0 \|^{-1}(b) \|_{L_2^2(\partial D_i)}}{\| b \|_{L_2^2(\partial D_i)}} = \sup_{(b \neq 0) \in L_2^2(\partial B_i)} \frac{\| \frac{1}{2} I + M_{ii,B}^0 \|^{-1}(b) \|_{L_2^2(\partial B_i)}}{\| b \|_{L_2^2(\partial B_i)}}.
\]

We obtain (2.47) in the same way. For (2.45), by Mean-value-theorem, we have

\[
(\Phi_k(x, y) - \Phi_0(x, y)) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|} = \int_0^1 ik \frac{1}{4\pi} e^{ik|x-y|} \, ds.
\]

Taking the gradient gives

\[
\nabla_x (\Phi_k(x, y) - \Phi_0(x, y)) = \frac{(ik)^2}{4\pi} \int_0^1 \frac{e^{ik|x-y|}}{|x-y|} (x-y) \, ds,
\]

thus, being \( \Im k = 0 \), \( |\nabla_x (\Phi_k(x, y) - \Phi_0(x, y)) \times b(y)| \leq \frac{|k|^2}{4\pi} |b(y)| \leq \frac{|k|^2}{4\pi} |b(y)|, \)

and

\[
|\Phi_k(x, y) - \Phi_0(x, y)| b(y) \leq \frac{|k|^2}{4\pi} |b(y)|, \tag{2.51}
\]

then it comes

\[
|M_{ii,D} - M_{ii,B}^0| b(y) \leq \int_{\partial D_i} \nabla (\Phi_k - \Phi_0)(x_i, y) \times b(y) \, ds(y) \leq \int_{\partial D_i} \frac{|k|^2}{4\pi} |b(y)| \, ds(y),
\]

and

\[
\int_{\partial D_i} |b(y)|^2 \, ds(y) \frac{1}{4\pi} \epsilon \left( \int_{\partial D_i} |b(y)|^2 \, ds(y) \right)^{\frac{1}{2}}.
\]
Taking the norm in both sides,
\[
\| [M_{i,D}^k - M_{i,D}^0] (b) \|_{L^2(\partial D_i)}^2 \leq \left( \frac{|k|^2 (\| \partial B_j \| \| \partial B_j \|)^\frac{1}{2} e^2}{4\pi} \| b \|_{L^2(\partial D_i)} \right)^2. \tag{2.52}
\]

We have the identities, \( \text{Div} \ [M_{i,D}^k - M_{i,D}^0] (b) (x) = -[k^2 \nu \cdot S_{i,D}^k] (b) - [\frac{1}{2} I - (K_{i,D}^0)^\star] \text{Div} b \). To estimate \([K_{i,D}^k - K_{i,D}^0]^\star \) \( \text{Div} b \), we can reproduce the same steps as we did to obtain (2.52). We obtain
\[
\| [(K_{i,D}^k - K_{i,D}^0)^\star] \|_{L^2(\partial D_i)}^2 \leq \left( \frac{|k|^2 (\| \partial B_j \| \| \partial B_j \|)^\frac{1}{2} e^2}{4\pi} \| \text{Div} b \|_{L^2(\partial D_i)} \right)^2. \tag{2.53}
\]

Using \( (2.51) \), we deduce that
\[
\| [k^2 \nu \cdot S_{i,D}^k] (b) \| = \| [k^2 \nu \cdot (S_{i,D}^k - S_{i,D}^0)] (b) + [k^2 \nu \cdot S_{i,D}^0] (b) \|,
\]
\[
\leq \frac{|k|^3 |\partial B_i| e}{4\pi} \| b \|_{L^2(\partial D_i)} + \left\| [k^2 \nu \cdot S_{i,D}^0] (b) \right\| ,
\]
and taking the norm, we get \( 3 \)
\[
\| [k^2 \nu \cdot S_{i,D}^k] (b) \|_{L^2(\partial D_i)} \leq \frac{|k|^3 |\partial B_i| e}{4\pi} \| b \|_{L^2(\partial D_i)} + |k|^2 \| S_{i,D}^0 \| \| b \|_{L^2(\partial D_i)},
\]
\[
\leq C_b |\partial B| \| b \|_{L^2(\partial D_i)},
\]
where \( |\partial B| = \max_i |\partial B_i| \) and \( C_b \) is the maximum of the constants that appear in the inequalities \( (2.52) \) and \( (2.53) \). Hence
\[
\| [M_{i,D}^k - M_{i,D}^0] (b) \|_{L^2(\partial D_i)} \leq 2C_b |\partial B| \| b \|_{L^2(\partial D_i)}. \tag{2.55}
\]

To prove (2.44), let us recall that we have
\[
\| \frac{1}{2} I + M_{i,D}^0 \|_{L^2(\partial D_i)} = \sup_{b \neq 0} \left( \frac{\| \frac{1}{2} I + M_{i,D}^0 \|_{L^2(\partial D_i)} - \| \text{Div} \ b \|_{L^2(\partial D_i)}^2}{\| b \|_{L^2(\partial D_i)}^2} \right)^{\frac{1}{2}}. \tag{2.56}
\]
Considering the fact that \( \text{Div} \left( \frac{1}{2} I + M_{i,D}^0 \right)^{-1} = \left[ \frac{1}{2} I - (K_{i,D}^0)^\star \right]^{-1} \text{Div} \), then (2.56) gives
\[
\| \frac{1}{2} I + M_{i,D}^0 \|_{L^2(\partial D_i)} \leq \sup_{b \neq 0} \left( \frac{\| \frac{1}{2} I + M_{i,D}^0 \|_{L^2(\partial D_i)} - \| \text{Div} \ b \|_{L^2(\partial D_i)}^2}{\| b \|_{L^2(\partial D_i)}^2} \right)^{\frac{1}{2}},
\]
\[
\leq \sup_{b \neq 0} \left( \frac{\| \frac{1}{2} I + M_{i,D}^0 \|_{L^2(\partial D_i)} - \| \text{Div} \ b \|_{L^2(\partial D_i)}^2}{\| b \|_{L^2(\partial D_i)}^2} \right)^{\frac{1}{2}},
\]
\[
+ \sup_{b \neq 0} \left( \frac{\| \frac{1}{2} I - (K_{i,D}^0)^\star \|_{L^2(\partial D_i)}^2}{\| b \|_{L^2(\partial D_i)}^2} \right)^{\frac{1}{2}},
\]
\]

\[\text{Notice that } \epsilon |b|_{L^2(\partial B_i)} = |b|_{L^2(\partial D_i)}. \]
\[\text{It suffices to write, for } b, c \in L^2(\partial D_i) \text{ such that } b = \left[ \frac{1}{2} I + M_{i,D}^0 \right]^{-1} c, \text{ Div } c = \text{Div} \left[ \frac{1}{2} I + M_{i,D}^0 \right] b = \left[ \frac{1}{2} I - (K_{i,D}^0)^\star \right] \text{Div } b \text{ and inverting } \left[ \frac{1}{2} I - (K_{i,D}^0)^\star \right] \text{ to get } \text{Div } b = \left[ \frac{1}{2} I - (K_{i,D}^0)^\star \right]^{-1} \text{Div } c.\]
where the last inequality is due to the fact that $\alpha^2 + \beta^2 \leq \alpha^2 + \beta^2 + 2\alpha\beta$ for $\alpha, \beta \geq 0$.

Finally, as $\alpha^2 \leq \beta^2 + \alpha^2$, we get

$$\left\| \frac{1}{2} I + M^0_{i,j,D} \right\|^{-1} \leq \sup_{b \neq 0} \left( \frac{\left\| \frac{1}{2} I + M^0_{i,j,D} \right\|^{-1} b \right\|^2}{\| b \|^2} \right)^{\frac{1}{2}} + \sup_{b \neq 0} \left( \frac{\left\| \frac{1}{2} I - (K^0_{i,j,D})^\ast \right\|^{-1} \parallel \text{Div} b \parallel^2}{\| \text{Div} b \|^2} \right)^{\frac{1}{2}}.$$  

To prove (2.55), we will need the following lemma

**Lemma 2.5.** For every $i, j \in \{1, \ldots, m\}$, under the condition that $\epsilon \leq \delta < 1$ we have

$$\| M_{i,j,D}^k \|_{L^2(D;\mathbb{R},L^2(D))} \leq \frac{4(\| \partial B_i \| \| \partial B_j \|)}{\pi \delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right)^2 \epsilon^3. \quad (2.57)$$

**Proof.** For $i \neq j$, $x_i \in \partial D$, we have, recalling (2.5), that

$$\int_{\partial D} \Phi_k(x_i, y)b(y) \, ds(y) = \int_{\partial D_j} \left( \nabla_x (y - z_j) \right) \left( \Phi_k(x_i, y)b(y) \right) \, ds(y),$$

$$= \int_{\partial D_j} (y - z_j) \text{Div} \left( \Phi_k(x_i, y)b(y) \right) \, ds(y),$$

$$= -\int_{\partial D_j} (y - z_j) \Phi_k(x_i, y) \text{Div} b(y) \, ds(y)$$

$$- \int_{\partial D_j} (y - z_j) \nabla_y \Phi_k(x_i, y) \cdot b(y) \, ds(y). \quad (2.58)$$

Being $-M_{i,j,D}^k(b) = -\nu \times \nabla_x \int_{\partial D_j} \Phi_k(x_i, y) \times b(y) \, ds(y)$, it comes from (2.58)

$$-M_{i,j,D}^k(b) = -\nu \times \nabla_x \int_{\partial D_j} (y - z_j) \Phi_k(x_i, y) \text{Div} b(y) \, ds(y)$$

$$+ \nu \times \nabla_x \int_{\partial D_j} (y - z_j) \nabla_y \Phi_k(x_i, y) \cdot b(y) \, ds(y),$$

$$= \nu \times \int_{\partial D_j} (y - z_j) \times \nabla_x \Phi_k(x_i, y) \text{Div} b(y) \, ds(y)$$

$$+ \nu \times \int_{\partial D_j} (y - z_j) \times \nabla_x \nabla_y \Phi_k(x_i, y) \cdot b(y) \, ds(y). \quad (2.59)$$

As

$$-\nabla_x \nabla_y \Phi_k(x, y) = (4\pi)^4 \Phi_k(x, y) \Phi_0^2 \left( (\Phi_0 - ik)^2 + (\Phi_0 - i\kappa)\Phi_0 + \Phi_0^2 \right) (x, y) ((x - y)(x - y)^T)$$

$$+ (4\pi)^2 \Phi_k \Phi_0 (\Phi_0 - i\kappa)(x, y) I,$$

where $(x - y)^T$ stands for the transpose vector of $(x - y)$, we get

$$|\nabla_x \nabla_y \Phi_k(x, y)| \leq \frac{3}{4\pi} \frac{1}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right)^2, \quad (2.60)$$

and

$$|\nabla_y \Phi_k(x, y)| \leq \frac{1}{4\pi} \frac{1}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right). \quad (2.61)$$

---

6With $L^2$-norm.

7 Notice that $(x - y)(x - y)^T b = (b \cdot (x - y))(x - y) \times (x - y) = 0$.

8 As $e^{-\| k \delta_{i,j} \|} \leq 1$. 
Taking the surface divergence of from which it follows that and, we conclude by combining (2.62) and (2.64).

Using the Mean-value-theorem, we get

The sum of the three last inequalities, gives us the estimates

Taking the surface divergence of $M^k_{ij,D}(a)$ we have

and using (2.68)

Using the Mean-value-theorem, we get successively

and

The sum of the three last inequalities, gives us the estimates

We conclude by combining (2.62) and (2.64).

\[\int_{\partial D_i} \nabla \Phi_k(x_i, z_j) \operatorname{Div} b \, ds(y) = \nabla \Phi_k(x_i, z_j) \int_{\partial D_i} \operatorname{Div} b \, ds(y) = 0\]
Based on Lemma 2.5, let us show the proof of (2.65). Draw \( l \) spheres \( (S_\delta(z_i))_{i=1,2,...,n} \) centered at \( z_i \) with radius \( l \delta \), where \( n \) will be determined later, let \( R_l = S_{l+1} - S_1 \), and \( R_0 = S_1 \) the volume of each \( R_l \) is given by

\[
Vol(R_l) = \frac{4\pi ((l + 1)\delta)^3}{3} - \frac{4\pi (l\delta)^3}{3} = \frac{4\pi (3l^2 + 3l + 1)\delta^3}{3}.
\]

For \( j \neq i \), we consider the spheres \( S_\delta(z_j) \). We claim that for \( j \neq 1 \), \( int(S_\delta(z_j)) \cap int(S_\delta(z_1)) = \emptyset \), where \( int(S_\delta) \) stands for the interior of \( S_\delta \). Indeed, if \( t \) was in the intersection, we would have \( d_{z_1, z_2} = \min\{d_{x, y} \leq d(z_{j_1}, z_{j_2}) \leq d(z_{j_1}, t) + d(t, z_{j_2}) < \frac{\delta}{2} + \frac{\delta}{2} = \delta \}, \) which contradicts the fact that \( \delta \) is the minimum distance. Hence, each domain \( D_j \) located in \( z_{j_1} \) occupying the volume of \( S_\delta(z_{j_2}) \) which is not shared with another \( D_j \). Then the maximum number of \( D_j \)'s that could occupy each \( R_l \), for \( l = 1, ..., m \), corresponds to the maximum number of spheres \( S_\delta(z_j) \) that could fit in the closure of \( R_l \). Considering the case where \( z_j \) is on \( \partial R_l \), only the half of the ball is in \( R_l \), see the figure.

If \( m_l \) corresponds to the maximum amount of locations \( z_j \) that are in the closure of \( R_l \) then \( m_l \leq Vol(R_l)/\frac{Vol(S_\delta)}{2} = 2^4(3l^2 + 3l + 1) \leq 2^6l^2 \), whenever \( l \geq 4 \), then, being \( \sum_{l=1}^{n} m_l = m \leq \sum_{l=1}^{n} 2^4(3l^2 + 3l + 1) = 2^4n(n^2 + 3n + 3) \), we get \( n = O(m^4) \). Now, considering lemma 2.5, we have

\[
\| M_{k_{i,0}}^k \|_{L^2_{L^2_{\text{Div}}}(\partial B), L^2_{L^2_{\text{Div}}}(\partial B)} \leq \frac{4(|\partial B_1| |\partial B_j|)^{\frac{1}{2}}}{\pi \delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right)^2 \epsilon^3 \leq \frac{C_{i,j}}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right)^2 \epsilon^3,
\]

hence, for \( C_B = \max_{i,j \in \{1, ..., m\}} C_{i,j} \), we have

\[
\sum_{(j \neq i) \geq 1}^{m} \| M_{k_{i,0}}^k \|_{L^2_{L^2_{\text{Div}}}(\partial B), L^2_{L^2_{\text{Div}}}(\partial B)} \leq C_B \sum_{l=1}^{n} \sum_{z_j \in R_l} \frac{1}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right)^2 \epsilon^3.
\]

(2.65)

Figure 1: Possible configuration for the scatterers.
Since for every \(z_j \in \tilde{R}_i\), \(l\delta \leq \delta_{i,j}\) we get

\[
\sum_{j \neq i}^m \|M^k_{ij, B}\|_{L^2(\partial B), L^2(\partial B))} \leq C_B \sum_{l=1}^n \sum_{z_j \in \tilde{R}_i} \frac{1}{\delta^3} \left( \frac{1}{\delta} + k \right)^2 \varepsilon^3,
\]

\[
\leq C_B \sum_{l=1}^n 2^4(3l^2 + 3l + 1) \frac{1}{\delta} \left( \frac{1}{\delta} + k \right)^2 \varepsilon^3,
\]

\[
\leq 2^6 C_B \sum_{l=1}^n l^2 \frac{1}{\delta} \left( \frac{1}{\delta} + k \right)^2 \varepsilon^3,
\]

\[
\leq \frac{2^6 C_B}{\delta} \left( \frac{\ln (n + 1) + 2kn}{\delta} + \frac{n}{2} (n + 1) k^2 \right) \varepsilon^3.
\]

(2.66)

Considering (2.46) and (2.48), the condition (2.42) is acquired if

\[
\frac{|\partial B_i| e^{C_B}}{4\pi \text{diam}(B)} (k\varepsilon)^2 \|\left[ \frac{1}{2} I + M^0_{i, B} \right]^{-1}\|_{L^2(\partial B_i), L^2(\partial B_i))} < 1,
\]

(2.67)

If we set \(C_{B_i} = \frac{|\partial B_i| e^{C_B}}{4\pi \text{diam}(B)} \|\left[ \frac{1}{2} I + M^0_{i, B} \right]^{-1}\|_{L^2(\partial B_i), L^2(\partial B_i))}\) we get

\[
\|\left[ \frac{1}{2} I + M^k_{i, B} \right]^{-1}\| \leq C_{i, \varepsilon} := \frac{\|\left[ \frac{1}{2} I + M^0_{i, B} \right]^{-1}\|_{L^2(\partial B_i), L^2(\partial B_i))}}{1 - C_{B_i} (e^{k\varepsilon} - 1) k\varepsilon},
\]

(2.68)

so we find \(\|\left[ \frac{1}{2} I + M_{B} \right]^{-1}\| \leq C_{\varepsilon} := \max_{i \in \{1, \ldots, m\}} C_{i, \varepsilon}\), under the condition (2.67).

### 3 Fields approximation and the linear algebraic systems

Based on the representation (2.0), the expression of the far field pattern is given by

\[
E^\infty(\tau) = \frac{ik}{4\pi} \tau \times \int_{\partial D} a(y) e^{-i k \tau y} ds(y),
\]

where \(\tau = (x/|x|) \in S^2\). We put

\[
A_i := \int_{\partial D_i} a_i^{[1]} ds, \quad B_i := \int_{\partial D_i} \nu u_i ds,
\]

where \(a_i^{[1]}\) and \(u_i\) are defined in Theorem 2.3 with the notation (1.15) repeating the same calculations as in (2.66), we derive the estimate

\[
\sum_{j \geq 1} \frac{1}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right)^3 + \left( \frac{1}{\delta_{i,j}} + |k| \right)^2 + \left( \frac{1}{\delta_{i,j}} + |k| \right)
\]

\[
= O\left( \frac{1}{\delta^3} + \frac{|k| + 1}{\delta^3} \ln(m) + \frac{(|k| + 1)^2}{\delta^2} m \right) + \left( \frac{|k| + 1}{\delta} \right)^3 m \right) =: \varepsilon_{k, \delta, m}.
\]

(3.1)

**Proposition 3.1.** For \(3k = 0\), the far field pattern can be approximated by

\[
E^\infty(\tau) = \frac{ik}{4\pi} \sum_{i=1}^m e^{-i k \tau \cdot z_i} \times \{ A_i - i k \tau \times B_i \} + O\left( (|k|^3 + |k|^2) m \varepsilon^4 \right).
\]

(3.2)
For \( \exists k \geq 0 \), and for every \( x \in \mathbb{R}^3 \setminus \bigcup_{i=1}^{m} \overline{D_i} \), such that \( d_x := d(x, \bigcup_{i=1}^{m} \overline{D_i}) \geq \delta \), the scattered electric field \( E^\text{sc}(x) \) has the following expansion,

\[
E^\text{sc}(x) = \sum_{i=1}^{m} (\nabla_x \Phi_k(x, z_i) \times A_i + \nabla_y \times (\Phi_k(x, z_j) B_j)) + O\left(\frac{e^4}{\delta^4} + \epsilon_{k,\delta, m} e^4\right). \tag{3.3}
\]

The elements \((A_i)_{i=1}^{m}\) and \((B_i)_{i=1}^{m}\) are solutions of the following linear algebraic system

\[
A_i = -[P_{\partial D_i}] \sum_{(j \neq i) \geq 1} \left( \Pi_k(z_i, z_j) A_j - k^2 \nabla \Phi_k(z_i, z_j) \times B_j \right) - [P_{\partial D_i}] \text{curl} \ E^\text{inc}(z_i) + O\left(\frac{e^7}{\delta^4} + |k|\epsilon_{k,\delta, m} e^7 + |k|^2 e^4\right).
\]

\[
B_i = [T_{\partial D_i}] \sum_{(j \neq i) \geq 1} \left( - \nabla_x \Phi_k(z_i, z_j) \times A_j + \Pi_k(z_i, z_j) B_j \right) - [T_{\partial D_i}] E^\text{inc}(z_i) + O\left(\frac{e^7}{\delta^4} + \epsilon_{k,\delta, m} e^7 + (1 + |k|)e^4\right). \tag{3.4}
\]

### 3.1 Justification of \((3.2)\) and \((3.3)\)

**Lemma 3.2.** For \( \exists k = 0 \), the far field pattern can be approximated by

\[
E^\infty(\tau) = \frac{ik}{4\pi} \sum_{i=1}^{m} e^{-ik\tau z_i} \times \left\{ \int_{\partial D_i} a_i \ ds - \int_{\partial D_i} (ik\tau(y - z_i)) a_i(y) \ ds(y) \right\} \tag{3.6}
\]

with an error estimate given by \( O\left(\frac{|k|e^m}{m^2} |k|^3 e^4\right) \). Precisely, in view of the decomposition \((2.36)\) of Theorem 2.3, the far field admits the following expansion

\[
E^\infty(\tau) = \frac{ik}{4\pi} \sum_{i=1}^{m} e^{-ik\tau z_i} \times \{ A_i - ik \tau \times B_i \} + O\left(\frac{|k|^3 + |k|^2}{m e^4}\right). \tag{3.7}
\]

**Proof.** To prove \((3.6)\), we write

\[
E^\infty(\tau) = \frac{ik}{4\pi} \sum_{i=1}^{m} e^{-ik\tau z_i} \times \int_{\partial D_i} a_m(y) \ ds(y) + \frac{ik}{4\pi} \sum_{i=1}^{m} \tau \times \int_{\partial D_i} \left( e^{-ik\tau y} - e^{-ik\tau z_i} \right) a_m(y) \ ds(y)
\]

for every \( i \in \{1, \ldots, m\} \) and evaluate the term

\[
Q_{(\tau, z_m)} := \tau \times \int_{\partial D_i} \left( e^{-ik\tau y} - e^{-ik\tau z_m} \right) a_m(y) \ ds(y).
\]

Developing the exponential in Taylor series, we obtain

\[
Q_{(\tau, z_m)} = e^{-ik\tau z_m} \times \int_{\partial D_i} \left( e^{-ik\tau (y - z_m)} - 1 \right) a_m(y) \ ds(y),
\]

\[
e^{-ik\tau z_m} \times \int_{\partial D_i} \sum_{n \geq 1} \frac{(-ik\tau (y - z_m))^n}{n!} a_m(y) \ ds(y),
\]

\[
e^{-ik\tau z_m} \times \left( \int_{\partial D_i} \sum_{n \geq 2} \frac{(-ik\tau (y - z_m))^n}{n!} a_m(y) \ ds(y) + \int_{\partial D_i} (-ik\tau (y - z_m)) a_m(y) \ ds(y) \right).
\]
As $|y - z_i| \leq \epsilon$, the first term gives us\(^{10}\)

$$
|e^{-ik\tau \cdot z_i} \times \int_{\partial D_i} \sum_{n \geq 2} \frac{(-ik\tau \cdot (y - z_i))^n}{n!} a_i(y) \, ds(y) | \leq |\partial B_i|^{\frac{\delta}{2}} \sum_{n \geq 2} \frac{|k| \epsilon^n}{n!} \|a\|_{L^2(\partial D_i)},
$$

$$
\leq |\partial B_i|^{\frac{\delta}{2}} \sum_{n \geq 2} \frac{|k| \epsilon^{n-2}}{n!} |k|^2 \epsilon^4 \leq |\partial B_i|^{\frac{\delta}{2}} e^{-|k|^4} |k|^2 \epsilon^4.
$$

Taking the sum over $i$, we get

$$
E^\infty(\tau) = \frac{ik}{4\pi} \sum_{i=1}^m e^{-ik\tau \cdot z_i} \times \left( A_i - \int_{\partial D_i} (ik\tau \cdot (y - z_i)) a_i(y) \, ds(y) \right) + O(|k| \epsilon^3 |k|^3 m \epsilon^4), \quad (3.8)
$$

Now, considering the decomposition\(^{2,3,6}\) of theorem\(^{2,3}\) we have

$$
\int_{\partial D_i} (ik\tau \cdot (y - z_i)) a_i(y) \, ds(y) = \int_{\partial D_i} (ik\tau \cdot (y - z_i)) (a_i^{[1]} + a_i^{[2]}) \, ds(y),
$$

$$
= \int_{\partial D_i} (ik\tau \cdot (y - z_i)) a_i^{[2]} \, ds(y) + O(|k| \epsilon^4),
$$

where $O(|k| \epsilon^4)$ comes from

$$
\left| \int_{\partial D_i} ik\tau \cdot (y - z_i) a_i^{[1]} \, ds(y) \right| \leq |k| |\epsilon^2| \|\partial B_i\|_{L^2(\partial D_i)} \leq |k| |\epsilon^2| \partial B_i |C_1 \epsilon \|a\|_{L^2(\partial D_i)} \leq |k| |\partial B_i| |C_1 C \epsilon| \epsilon^4.
$$

Then $\int_{\partial D_i} ik\tau \cdot (y - z_i) \, ds(y) = \int_{\partial D_i} ik\tau \cdot (y - z_i) \, \nu \times \nabla u_i \, ds(y) + O(|k| \epsilon^4)$. Multiplying by $e^{-ik\tau \cdot z_i}$ and taking the sum over $i$, we obtain

$$
\sum_{i=1}^m e^{-ik\tau \cdot z_i} \int_{\partial D_i} ik\tau \cdot (y - z_i) \, a_i \, ds(y) = \sum_{i=1}^m e^{-ik\tau \cdot z_i} \left( \int_{\partial D_i} ik\tau \cdot (y - z_i) \, \nu \times \nabla u_i \, ds(y) + O(|k| \epsilon^4) \right),
$$

With this last approximation, (3.8) gives

$$
E^\infty(\tau) = \frac{ik}{4\pi} \sum_{i=1}^m e^{-ik\tau \cdot z_i} \times \left( A_i - \int_{\partial D_i} ik\tau \cdot (y - z_i) \, \nu \times \nabla u_i \, ds(y) \right) + O \left( (e^{-|k|^4} + |k|^2) m \epsilon^4 \right).
$$

Finally, integrating by part the second term of the second member, we obtain

$$
\int_{\partial D_i} ik\tau \cdot (y - z_i) \, \nu \times \nabla u_i \, ds(y) = -ik \int_{\partial D_i} (\nu \times \nabla y \tau \cdot (y - z_i)) \, u_i \, ds(y),
$$

$$
= +ik \int_{\partial D_i} (\tau \times \nu) \, u_i \, ds(y) = ik \times B_i.
$$

**Lemma 3.3.** The Electric field has the following asymptotic expansion

$$
E^{\text{scat}}(x) = \sum_{i=1}^m \left( \nabla_x \Phi_k(x, z_i) \times A_i + \nabla_y \times \nabla_x \times (\Phi_k(x, z_i) B_i) \right) + \frac{1}{\delta} \left( \frac{1}{\delta^3} + \frac{3k}{\delta^2} + \frac{5k^2}{\delta^2} \right) \epsilon^4 \left| k \right|^4,
$$

$$
+ O\left( \sum_{i \neq (i \neq 0)}^m \left( \frac{1}{\delta^3} \left| k \right| + \frac{3k}{\delta^2} \right) \epsilon^4 \right) \left| k \right|^4.
$$

\(^{10}\)We have $\sum_{n \geq 2} \frac{|k|^n}{n!} \leq \sum_{n \geq 2} \frac{|k|^n}{(n-2)!} = e^{-|k|^4}$.\]
Proof. For \( x \in \mathcal{R}^3 \setminus \bigcup_{i=1}^{m} \overline{D}_i \), using Taylor formula with integral reminder, we get from the representation \( \Phi \)

\[
E^{sca}(x) = \sum_{i=1}^{m} \int_{\partial D_i} \left( \nabla_x \Phi_k(x, z_i) + (\nabla_y \nabla_x \Phi_k(x, z_i) \cdot (y - z_i)) \right) \times a_i(y) \, ds(y)
\]

\[+ \sum_{i=1}^{m} \int_{\partial D_i} \int_{0}^{1} D^3 \Phi_k(x, ty + (1 - t)z_i) \circ (y - z_i)(y - z_i) \times a_i(y) \, ds(y),\]

which is\(^{11}\)

\[
E^{sca}(x) = \sum_{i=1}^{m} \left( \nabla_x \Phi_k(x, z_i) \times A_i + \int_{\partial D_i} \left( -\nabla_x \left( \nabla_x \Phi_k(x, z_i) \cdot (y - z_i) \right) \right) \times a_i(y) \, ds(y) \right) + \sum_{i=1}^{m} \int_{\partial D_i} \int_{0}^{1} D^3 \Phi_k(x, ty + (1 - t)z_i) \circ (y - z_i)(y - z_i) \times a_i(y) \, ds(y),
\]

(3.10)

As it was done in (3.35), with \( d_{x,i} := d(x, \partial D_i) \), we have

\[
\left| \int_{\partial D_i} \int_{0}^{1} D^3 \Phi_k(x, ty + (1 - t)z_i) \circ (y - z_i)(y - z_i) \times a_i(y) \, ds(y) \right| \leq e^{-3k\delta_{x,i}} d_{x,i} \left( \frac{1}{d_{x,i}} + |k| \right)^3 \epsilon^2 \int_{\partial D_i} |a_i| \, ds.
\]

For a fixed \( x \in \mathcal{R}^3 \setminus \Omega \), set \( d_x := \min_{i \in \{1, \ldots, m\}} d_{x,i} \), hence there exists some \( i_0 \) such that \( d_x = d(x, \partial D_{i_0}) \), further \( \delta_{i_0,i} = d(\partial D_{i_0}, \partial D_i) \leq d_{x,i} + d_{x,i_0} \) from which follows

\[
\frac{1}{d_{x,i}} \leq \frac{2}{\delta_{i_0,i}}.
\]

(3.11)

Summing over \( i \), the reminder, remain smaller then

\[
O\left( \sum_{(i \neq i_0) \geq 1} \frac{e^{-3k\delta_{i_0,i}/2}}{\delta_{i_0,i}} \left( \frac{1}{\delta_{i_0,i}} + |k| \right)^3 \epsilon^4 + \frac{e^{-3k\delta}}{\delta} \left( \frac{1}{\delta} + |k| \right)^3 \epsilon^4 \right).
\]

(3.12)

The second term under the sum of (3.10) is precisely

\[
\nabla_y \times \nabla_x \times (\Phi_k(x, z_i) B_i) + O \left( \frac{e^{-3k\delta_{i_0,i}/2}}{\delta_{i_0,i}} \left( \frac{1}{\delta_{i_0,i}} + |k| \right)^2 \epsilon^4 \right).
\]

(3.13)

Indeed,

\[
\int_{\partial D_i} \nabla_x (\nabla_y \Phi_k(x, z_i) \cdot (y - z_i)) \times a_i(y) \, ds(y) = \int_{\partial D_i} \nabla_x \times \left( [(\nabla_y \Phi_k(x, z_i) \cdot (y - z_i)) a_i(y) \right) \, ds(y).
\]

As we did for the far field approximation, we get in view of decomposition \( \Phi \)

\[
\int_{\partial D_i} (\nabla_x \Phi_k(x, z_i) \cdot (y - z_i)) a_i(y) \, ds(y) = \int_{\partial D_i} (\nabla_x \Phi_k(x, z_i) \cdot (y - z_i)) \left( a_i^{[1]} + a_i^{[2]} \right) (y) \, ds(y),
\]

(3.14)

and being

\[
\left| \int_{\partial D_i} \nabla_x (\nabla_y \Phi_k(x, z_i) \cdot (y - z_i)) \times a_i^{[1]}(y) \, ds(y) \right| \leq C e^{-3k d_{x,i}} \left( \frac{1}{d_{x,i}} + |k| \right)^2 \epsilon \int_{\partial D_i} |a_i^{[1]}| \, ds,
\]

\(^{11}\)Recall that \( \nabla_y \nabla_x \Phi_k(x, y) = -\nabla_x \nabla_x \Phi_k(x, y) \).
and differentiating, we get
\[
\int_{\partial D_i} \nabla (\nabla \Phi_k(x, z_i) \cdot (y - z_i)) \times a_i^{[1]}(y) \, ds(y) = 0 \left( \frac{e^{-\frac{1}{2} d_{x,i}}}{d_{x,i}} \left( \frac{1}{d_{x,i}} + |k| \right)^2 \epsilon^4 \right).
\]
(3.15)

Further, integrating by part, in the second step of the following identities
\[
\int_{\partial D_i} (\nabla \Phi_k(x, z_i) \cdot (y - z_i)) a_i^{[2]}(y) \, ds(y) = \int_{\partial D_i} (\nabla \Phi_k(x, z_i) \cdot (y - z_i)) \nu \times \nabla u_i(y) \, ds(y),
\]
\[
= - \int_{\partial D_i} \nu \times \nabla_g (\nabla \Phi_k(x, z_i) \cdot (y - z_i)) u_i(y) \, ds(y),
\]
\[
= \int_{\partial D_i} \nabla \Phi_k(x, z_i) \times \nu u_i(y) \, ds(y) = \nabla \Phi_k(x, z_i) \times B_i,
\]
and differentiating, we get
\[
- \nabla \times \int_{\partial D_i} (\nabla \Phi_k(x, z_i) \cdot (y - z_i)) a_i^{[2]}(y) \, ds(y) = \nabla_y \nabla (\Phi_k(x, z_i) B_i).
\]

Hence, considering (3.15), (3.13) follows from (3.14).

Replacing (3.13) and (3.12) in (3.10) gives
\[
E^\text{ sca}(x) = \sum_{i=1}^{m} \left( \nabla \Phi_k(x, z_i) \times A_i + \nabla_y \times \nabla_x \times (\Phi_k(x, z_i) B_i) + O \left( \frac{e^{-\frac{1}{2} d_{x,i}}}{d_{x,i}} \left( \frac{1}{d_{x,i}} + |k| \right)^2 m \epsilon^4 \right) \right)
\]
\[
+ O \left( \sum_{i=1}^{m} \frac{e^{-\frac{1}{2} d_{x,i}}}{d_{x,i}^{3/2}} \left( \frac{1}{d_{x,i}} + |k| \right)^3 \epsilon^4 \right).
\]

Repeating the calculations done to get (3.12), we obtain
\[
E^\text{ sca}(x) = \sum_{i=1}^{m} \left( \nabla \Phi_k(x, z_i) \times A_i + \nabla_y \times \nabla_x \times (\Phi_k(x, z_i) B_i) \right)
\]
\[
+ O \left( \frac{e^{-\frac{1}{2} d_{x,i}}}{d_{x,i}} \left[ \left( \frac{1}{d_{x,i}} + |k| \right)^2 + \left( \frac{1}{d_{x,i}} + |k| \right)^3 \right] \epsilon^4 \right)
\]
\[
+ O \left( \sum_{(i \neq j) \geq 1} \frac{e^{-\frac{1}{2} d_{x,i}}}{d_{x,i}^{3/2}} \left[ \left( \frac{1}{d_{x,i}} + |k| \right)^2 + \left( \frac{1}{d_{x,i}} + |k| \right)^3 \right] \epsilon^4 \right).
\]

The approximation (3.3) follows using (3.1).

\[\Box\]

### 3.2 Justification of (3.4) and (3.5)

We provide the justification of (3.5) and then the one of (3.4).

#### 3.2.1 Justification of (3.5)

Let \( \psi \) be any smooth enough vectorial function. Multiplying by (3.11) and integrating over \( \partial D_i \), we get
\[
\int_{\partial D_i} \psi \cdot \left[ \frac{1}{2} I + M_{i, j}^k \right] a_j \, ds + \sum_{(j \neq i) \geq 1} \int_{\partial D_i} \psi \cdot [M_{i, j}^k] (a_j) \, ds = - \int_{\partial D_i} \psi \cdot \nu_i \times E^\text{ inc} \, ds.
\]
(3.16)
Recalling the scaling (2.46) and the estimate (2.52)\(^\text{12}\), we have

\[
\left| \int_{\partial D_i} \psi \cdot \left[ \frac{1}{2} I + M_{ii,D}^k \right] a_i^{[1]} \right| ds \leq \| \psi \| \left( \frac{1}{2} I + M_{ii,D}^0 \right) a_i^{[1]} + \left\| \left[ M_{ii,D}^k - M_{ii,D}^0 \right] a_i^{[1]} \right\|, \\
\leq \| \psi \| L^2(\partial D_i) \left( C_{\partial D_i} + \frac{|k|^2 |\partial B| \epsilon^2}{4\pi} \right) \left\| a_i^{[1]} \right\| L^2(\partial D_i).
\]

In view of the decomposition (2.36), we obtain that

\[
\left\| \psi \right\| L^2(\partial D_i) = O(\| \psi \| L^2(\partial D_i) \epsilon^2),
\]

and then (3.10) gives

\[
O(\| \psi \| L^2(\partial D_i) \epsilon^2) + \int_{\partial D_i} \psi \cdot \left[ \frac{1}{2} I + M_{ii,D}^0 \right] a_i^{[2]} \ ds + \sum_{(j \neq i) \geq 1} \int_{\partial D_i} \psi \cdot [M_{ij,D}^k (a_j^{[1]} + a_j^{[2]} \ ds) ds = \int_{\partial D_i} \psi \cdot \nu_i \times E^{inc} ds.
\]

Using (2.52) and the estimates (2.39) of the decomposition (2.36), in the left hand side, we get

\[
O(\| \psi \| L^2(\partial D_i) \epsilon^2) + \int_{\partial D_i} \psi \cdot \left[ \frac{1}{2} I + M_{ii,D}^0 \right] a_i^{[2]} \ ds + O(\| \psi \| L^2(\partial D_i) \epsilon^2),
\]

and for the approximation

\[
\int_{\partial D_i} \psi_i \cdot \left[ \frac{1}{2} I + M_{ii,D}^0 \right] a_i^{[2]} \ ds = O(\epsilon^4 + |k|^2 \epsilon^5) + \int_{\partial D_i} \nu_i \times u_i \ ds.
\]

Now, we show how we choose appropriate candidates \(\psi\) to derive the estimates (3.3) and (3.5).

Lemma 3.4. There are functions \((\psi_i)_{i=1,2,3}\) such that \(\nu \times \psi_i \in L^2(\partial D_i)\) and satisfying, for constants \(C_{(M_{ii,B}^0)} C_{(M_{ii,B}^0 K_{ii,B}^0)}\), which depends only on \(|\partial B|\),

\[
\| \nu \times \psi_i \| L^2(\partial D_i) \leq C_{(M_{ii,B}^0)} |\partial B| \epsilon^2, \quad \| \nu \times \psi_i \| L^2(\partial D_i) \leq C_{(M_{ii,B}^0 K_{ii,B}^0)} |\partial B| \epsilon,
\]

and for which the approximation

\[
\int_{\partial D_i} \psi_i \cdot \left[ \frac{1}{2} I + M_{ii,D}^0 \right] a_i^{[2]} \ ds = O(\epsilon^4 + |k|^2 \epsilon^5) + \int_{\partial D_i} \nu_i \times u_i \ ds,
\]

hold, where \(\nu \times \nabla u_i = a_i^{[2]}\).

Proof. Let \((b_i)_{i=1,2,3}\) be the solution of the following equation

\[
\left[ -\frac{1}{2} I + M_{ii,D}^0 \right] (b_i) = -\nu \times V_i,
\]

where, \((e_i)_{i=1,2,3}\) being the canonical base of \(\mathbb{R}^3\): \(V_1 = (0,0,(x-z_i) \cdot e_2)\), \(V_2 = ((x-z_i) \cdot e_3,0,0)\) and \(V_3 = (0,(x-z_i) \cdot e_1,0)\). It is evident that \(\text{curl} V_i = e_i\), further \(b_i \in L^2(\partial D_i)\), and \(\nu \times (\nu \times b_i) = (\nu \cdot b_i) \nu - (\nu \cdot \nu) b_i = -b_i\).

We put \(\psi_i := -\nu \times b_i\), hence \(\nu \times \nu_i = b_i\) and \(\text{Div} (\nu \times \psi_i) = \text{Div} b_i\). Solving (3.20) is amount to solve the following problem (it suffices to take the surface divergence in the identity (3.21))

\[
\left[ -\frac{1}{2} I + (K_{ii,D}^0)^* \right] (\nu \cdot \text{curl} \psi_i) = -\nu_i.
\]

\(^{12}\)With \(L^2(\partial D_i)\) norms.

\(^{13}\)Having \(\nu \cdot b_i = 0\) and \(\nu \cdot \nu = 1\).
Further, as it was done in (2.46) and (2.44), the following estimates hold

\[
\| \nu \times \psi \|_{L^2(\partial D_i)} \leq \| \left[ -\frac{1}{2} I + M^{0}_{\nu,i,B} \right]^{-1} \|_{L^2(\partial B)} \| \nu \times V_i \|_{L^2(\partial D_i)} \leq C_{M^{0}_{\nu,i,B}} \| \partial B \| |e|^2,
\]

\[
\| \nu \times \psi \|_{L^2,div(\partial D_i)} \leq \| \left[ -\frac{1}{2} I + M^{0}_{\nu,i,B} \right]^{-1} \|_{L^2,div(\partial B)} \| V_i \|_{L^2,div(\partial D_i)} \leq C_{M^{0}_{\nu,i,B},K^{0}_{\nu,i,D}} \| \partial B \| |e|.
\]

We have the following relations (see Lemma 5.11 \[18\])

\[
\left[ \frac{1}{2} I + M^{0}_{\nu,i,D} \right] (a_i^{[2]}) = \left[ \frac{1}{2} I + M^{0}_{\nu,i,D} \right] (\nu \times \nabla u_i) = \nu \times \nabla \left[ \frac{1}{2} I + K^{0}_{\nu,i,D} \right] (u_i),
\]

(3.23)

and, for every scalar function \( w \), \[14\]

\[
\int_{\partial D_i} w \nu \cdot \text{curl} \psi \ ds = -\int_{\partial D_i} \psi \cdot (\nu \times \nabla w) \ ds.
\]

(3.24)

Hence, the term under the integral of the left hand side of (3.17), using (3.23) and (3.24), gives

\[
\int_{\partial D_i} \psi \cdot \left[ \frac{1}{2} I + M^{0}_{\nu,i,D} \right] (a_i^{[2]}) \ ds = \int_{\partial D_i} \psi \cdot \nu \times \nabla \left[ \frac{1}{2} I + K^{0}_{\nu,i,D} \right] (u_i) \ ds,
\]

(3.25)

\[
= -\int_{\partial D_i} (\nu \cdot \text{curl} \psi) \left( \frac{1}{2} I + K^{0}_{\nu,i,D} \right) (u_i) \ ds,
\]

\[
= -\int_{\partial D_i} \left[ \frac{1}{2} I + (K^{0}_{\nu,i,D})^{*} \right] (\nu \cdot \text{curl} \psi) u_i \ ds.
\]

Using (3.25) with \( \psi \) as in (3.20), we get

\[
O((\epsilon^2 + |k|\epsilon^2)\epsilon^2) + \int_{\partial D_i} \left[ \frac{1}{2} I + (K^{0}_{\nu,i,D})^{*} \right] (\nu \cdot \text{curl} \psi) u_i \ ds = O(\epsilon^4) + O(|k|^2 \epsilon^4) - \int_{\partial D_i} \nu_i^i u_i \ ds,
\]

to conclude that

\[
\int_{\partial D_i} \psi \cdot \left[ \frac{1}{2} I + M^{0}_{\nu,i,D} \right] (a_i^{[2]}) \ ds = O(\epsilon^4 + |k|^2 \epsilon^5) + \int_{\partial D_i} \nu_i^i u_i \ ds.
\]

(3.26)

We recall the notations, for \( l = 1, 2, 3 \),

\[
[T_{\partial D_i}]^l := \int_{\partial D_i} \nu_i^l \left[ \frac{1}{2} I + K^{0}_{\nu,i,D} \right]^{-1} (x - z_i) \ ds.
\]

Lemma 3.5. The second term of the left hand side of (3.17) admits the following approximations

\[
\int_{\partial D_i} \psi_i \cdot [M^{k}_{j,l,D} ] (a_i^{[2]}) \ ds = [T_{\partial D_i}]^l \cdot \nabla_x \Phi_k(z_i, z_j) \times A_j + O\left( \frac{1}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right)^{\epsilon^7} \right)
\]

(3.27)

and

\[
\int_{\partial D_i} \psi_i \cdot [M^{k}_{j,l,D} ] (a_i^{[2]}) \ ds = [T_{\partial D_i}]^l \cdot (-k^2 \Phi_k(z_i, z_j) I + \nabla_x \nabla_y \Phi_k(z_i, z_j)) B_j
\]

\[
+ O\left( \frac{1}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right)^3 + \left( \frac{1}{\delta_{i,j}} + |k| \right) \right)^{\epsilon^7}.
\]

(3.28)

In addition, we have the approximation

\[
\int_{\partial D_i} \psi_i \cdot \nu_i \times E^{inc} \ ds = \int_{\partial D_i} \nu_i^l \left[ \frac{1}{2} I + K^{0}_{\nu,i,D} \right]^{-1} (x - z_i) \ ds \cdot E^{inc}(z_i) + O(k\epsilon^4).
\]

(3.29)

\[14\] A direct application of (2.3) with \( a = w\psi \).
Proof. Adding and subtracting $\nabla_x \Psi_k(z_i, y)$, we write
\[
\int_{\partial D_i} \psi_i \cdot [M_{ij}^k] a_j^1 \, ds = \int_{\partial D_i} \psi_i \cdot (\nu_{x_i} \times \int_{\partial D_i} (\nabla_x \Psi_k(x_i, y) - \nabla_x \Psi_k(z_i, y)) \times a_j^1(y) \, ds(y)) \, ds(x)
+ \int_{\partial D_i} \psi_i \cdot (\nu_{x_i} \times \int_{\partial D_i} (\nabla_x \Psi_k(z_i, y)) \times a_j^1(y) \, ds(y)) \, ds(x). \tag{3.30}
\]
For the first integral of the right hand side, we get, using Holder’s inequality then the Mean-value-theorem, with $L^2(\partial D_i)$ norm,
\[
\left| \int_{\partial D_i} \psi_i \cdot (\nu_{x_i} \times \int_{\partial D_i} (\nabla_x \Psi_k(x_i, y) - \nabla_x \Psi_k(z_i, y)) \times a_j^1(y) \, ds(y)) \, ds(x) \right|
\leq \|\psi_i\| \| \int_{\partial D_i} (\nabla_x \Psi_k(x_i, y) - \nabla_x \Psi_k(z_i, y)) \times a_j^1(y) \, ds(y) \|
\]
As
\[
\left| (\int_{\partial D_i} (\nabla_x \Psi_k(x_i, y) - \nabla_x \Psi_k(z_i, y)) \times a_j^1(y)) \, ds(y) \right|
\leq \sup_{x \in \partial D_i} |(x - z_i)| \|a_j^1\| \sup_{x \in \partial D_i, y \in \partial D_i} |\nabla_x \nabla_x \Psi_k(x_i, y)| \epsilon |\partial B_i|^{1/2},
\]
with (2.60), we get
\[
\left| (\int_{\partial D_i} (\nabla_x \Psi_k(x_i, y) - \nabla_x \Psi_k(z_i, y)) \times a_j^1(y)) \, ds(y) \right| \, ds(x) \leq \frac{e^{-\Im k \delta_{i,j}}}{4\pi} \frac{1}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right) \epsilon^4.
\]
Finally
\[
\left| \int_{\partial D_i} \psi_i \cdot (\nu_{x_i} \times \int_{\partial D_i} (\nabla_x \Psi_k(x_i, y) - \nabla_x \Psi_k(z_i, y)) \times a_j^1(y) \, ds(y)) \, ds(x) \right| \leq \frac{e^{-\Im k \delta_{i,j}}}{4\pi\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right) \epsilon^7,
\]
which is
\[
\int_{\partial D_i} \psi_i \cdot (\nu_{x_i} \times \int_{\partial D_i} (\nabla_x \Psi_k(x_i, y) - \nabla_x \Psi_k(z_i, y)) \times a_j^1(y) \, ds(y)) \, ds(x) = O\left( \frac{1}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right) \epsilon^7. \right) \tag{3.31}
\]
For the second integral of the right hand side of (3.30), we get
\[
\int_{\partial D_i} \psi_i \cdot (\nu_{x_i} \times \int_{\partial D_i} \nabla_x \Psi_k(z_i, y) \times a_j^1(y) \, ds(y)) \, ds(x)
= \int_{\partial D_i} \psi_i \cdot (\nu_{x_i} \times \int_{\partial D_i} \nabla_x (\Psi_k(z_i, y) - \Phi_k(z_i, z_j)) \times a_j^1(y) \, ds(y)) \, ds(x)
+ \int_{\partial D_i} \psi_i \cdot (\nu_{x_i} \times \nabla_x \Phi_k(z_i, z_j) \times \int_{\partial D_i} a_j^1(y) \, ds(y)) \, ds(x).
\]
Using again the Mean-value-theorem as in (3.31) for the first integral of the second member, we get
\[
\int_{\partial D_i} \psi_i \cdot (\nu_{x_i} \times \int_{\partial D_i} \nabla_x \Phi_k(z_i, y) \times a_j^1(y) \, ds(y)) \, ds(x) = O\left( \frac{1}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right) \epsilon^7 \right)
+ \int_{\partial D_i} \psi_i \cdot (\nu_{x_i} \times \nabla_x \Phi_k(z_i, z_j) \times \int_{\partial D_i} a_j^1 \, ds(y)) \, ds(x). \tag{3.32}
\]
In addition, considering the fact that, for any vectors \( a, b, c \) of \( \mathbb{R}^3 \) we have \( a \cdot (b \times c) = -c \cdot (b \times a) \), we write
\[
\int_{\partial D_i} \psi_I \cdot (\nu_{z_i} \times \nabla_x \Phi_i(z_i, z_j) \times \int_{\partial D_j} a_i^{[1]} \) ds \) = \int_{\partial D_i} \psi_I \cdot \nu_{z_i} \times \nabla ((x - z_i) \cdot \nabla_x \Phi_i(z_i, z_j) \times A_j) \) ds \),
\[
= - \int_{\partial D_i} \nu_{z_i} \cdot \psi_I \cdot ((x - z_i) \cdot \nabla_x \Phi_i(z_i, z_j) \times A_j) \) ds \). 
\]
Integrating by parts, and considering \(3.32\), we have
\[
\int_{\partial D_i} \psi_I \cdot (\nu_{z_i} \times \nabla_x \Phi_i(z_i, z_j) \times A_j) \) ds \) \int_{\partial D_i} \text{Div} (\nu \times \psi_I) ((x - z_i) \cdot \nabla_x \Phi_i(z_i, z_j) \times A_j) \) ds \),
\[
= - \int_{\partial D_i} \nu \cdot \text{curl} \psi_I ((x - z_i) \cdot \nabla_x \Phi_i(z_i, z_j) \times A_j) \) ds \),
\[
= \left( \int_{\partial D_i} \frac{1}{2} \nu + (K^0_{ii,i}) \cdot \text{curl} \psi_I \cdot (x - z_i) \) ds \right) \cdot \nabla_x \Phi_i(z_i, z_j) \times A_j \). 
\]
Replacing in \(3.32\), summing over \( j \) gives the first approximation. For \(3.32\), being \( a_j^{[2]} = \nu \times \nabla u_j \) we have (see Lemma 5.11 \[18\])
\[
[M_{i,j}^k] \nu \times \nabla u_j = \nu \times \nabla [K_{i,j}^k] u_j - k^2 \nu \times [S_{i,j}^k] (\nu \cdot u_j),
\]
then
\[
\int_{\partial D_i} \psi_I \cdot [M_{i,j}^k] (a_j^{[2]}) \) ds \) = \psi_I \cdot (\nu \times \nabla [K_{i,j}^k] u_j - k^2 \nu \times [S_{i,j}^k] (\nu \cdot u_j)). \) (3.33)
\]
The first term of the right hand side gives \([16]\)
\[
\int_{\partial D_i} \psi_I \cdot \nu \times \nabla [K_{i,j}^k] u_j \) ds \) = - \int_{\partial D_i} \nu \cdot \text{curl} \psi_I (x) \int_{\partial D_j} \nu \cdot \nabla_y \Phi_i(x, y) u_j(y) \) ds(y) \) ds \),
\[
= - \int_{\partial D_i} \nu \cdot \text{curl} \psi_I (x) \int_{\partial D_j} \nu \cdot \nabla_y (\Phi_i(x, y) - \Phi_i(z_i, y)) u_j(y) \) ds(y) \) ds \),
\)
By Taylor formula at the first order, \([17]\)
\[
\nabla_y (\Phi_i(x, y) - \Phi_i(z_i, y)) = \nabla_x \nabla_y \Phi_i(z_i, y) (x - z_i) + O\left( \frac{1}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right)^2 \right), \) (3.35)
\]
\([3.34]\) gives
\[
\int_{\partial D_i} \psi_I \cdot \nu \times \nabla [K_{i,j}^k] u_j \) ds \) = \int_{\partial D_i} \nu \cdot \text{curl} \psi_I (x) \int_{\partial D_j} \nabla \nabla_y \Phi_i(z_i, y) (x - z_i) \cdot \nu \cdot u_j(y) \) ds(y) \),
\[
+ \int_{\partial D_i} \nu \cdot \text{curl} \psi_I (x) \int_{\partial D_j} \nu \cdot \nabla_y (\Phi_i(x, y) - \Phi_i(z_i, y)) u_j(y) \) ds(y) \) ds \),
\]
Adding and subtracting \(\nabla_x \nabla_y \Phi_i(z_i, z_j)\) under the integral, with the approximation \(3.35\), yields
\[
\int_{\partial D_i} \psi_I \cdot \nu \times \nabla [K_{i,j}^k] u_j \) ds \) = \int_{\partial D_i} \nu \cdot \text{curl} \psi_I (x) \int_{\partial D_j} \nabla \nabla_y \Phi_i(z_i, z_j) (x - z_i) \cdot \nu \cdot u_j(y) \) ds(y) \),
\]
\[15\] Recall the definition of \( A_j = \int_{\partial D_i} (a_i^{[1]} + a_i^{[2]}) \) ds = \int_{\partial D_i} a_i^{[1]} \) ds.
\[16\] Being \( \int_{\partial D_i} \nu \cdot \text{curl} \psi_i C = 0 \), for any constant vector \( C \).
\[17\] Actually \( \int_0^1 \Phi_k(tx + (1 - t)z_i, y) dt \cdot (x - z_i) \) \( (x - z_i) \) \( \leq \frac{C}{\delta_{i,j}} \left( \frac{1}{\delta_{i,j}} + |k| \right)^2 \).
In view of (2.40), we have

\[
\left| \int_{\partial D_i} \nu \cdot \nabla \psi \right| \left( \int_{\partial D_j} \nu_g \cdot O \left( \frac{1}{\delta_{i,j}} \right) u_j(y) \ ds(y) \right) \ ds(x)
\]

hence, being \((\nabla_x \nabla_y \Phi_k(z_i, z_j))^T\) standing for the transpose,

\[
\int_{\partial D_i} \psi_l \cdot \nu \cdot \nabla |K_{i,j}^k| u_j \ ds = - \int_{\partial D_i} \nu \cdot \nabla \psi \nu_g \cdot O \left( \frac{1}{\delta_{i,j}} \right) u_j(y) \ ds(x)
\]

\[
+ O \left( \frac{1}{\delta_{i,j}} \right) (1 + |k|^3 \epsilon^7).
\]

It follows, with \((\nabla_x \nabla_y \Phi_k(z_i, z_j))^T = \nabla_x \nabla_y \Phi_k(z_i, z_j)\), we get in view of (3.34) that

\[
\int_{\partial D_i} \psi_l \cdot \nu \cdot \nabla |K_{i,j}^k| u_j \ ds = O \left( \frac{1}{\delta_{i,j}} \right) (1 + |k|^3 \epsilon^7)
\]

\[
- \int_{\partial D_i} \frac{1}{2} f (K_{i,j}^o)^* (-v'_i)(x - z_i) \ ds \cdot (\nabla_x \nabla_y \Phi_k(z_i, z_j) B_j).
\]

Now, consider the second term of (3.33), \(^{18}\)

\[- k^2 \int_{\partial D_i} \psi_l \cdot (\nu \cdot [S^{k}_{i,j}] (\nu_g u_j(y))) \ ds = k^2 \int_{\partial D_i} \nu \cdot \psi \nu_g \cdot O \left( \frac{1}{\delta_{i,j}} \right) u_j(y) \ ds(x),
\]

we have

\[
\int_{\partial D_j} \Phi_k(x_i, y)(\nu_g u_j(y)) \ ds(y) = \int_{\partial D_j} \Phi_k(z_i, y) (\nu_g u_j(y)) \ ds(y)
\]

\[
+ \int_{\partial D_j} \int_0^1 (\nabla_x \Phi_k(tx + (1 - t)z_i, y) dt \cdot (x - z_i)) (\nu_g u_j(y)) \ ds(y),
\]

and repeating the same approximation in \(y\), with the following estimate

\[
\left| \int_{\partial D_j} \int_0^1 (\nabla_x \Phi_k(tx + (1 - t)z_i, y) dt \cdot (x - z_i)) (\nu_g u_j(y)) \ ds(y) \right| \leq C \frac{\epsilon^2}{\delta_{i,j}} (1 + |k|) \|u_j\|^2_{L^2(\partial D_j)},
\]

\[
\leq C \frac{1}{\delta_{i,j}} (1 + |k|) \epsilon^4,
\]

we get

\[
\int_{\partial D_j} \Phi_k(x_i, y)(\nu_g u_j(y)) \ ds(y) = \Phi_k(z_i, z_j) \int_{\partial D_j} \nu_g u_j \ ds + 2 O \left( \frac{1}{\delta_{i,j}} (1 + |k|) \epsilon^4 \right),
\]

\(^{18}\)With the following product rule \(\nu \cdot (v \times w) = -w \cdot (v \times u)\).
Replacing in (3.37), gives
\[-k^2 \int_{\partial D_i} \psi \cdot (\nu \times [S_{ij,o}^k](\nu_y u_j)) \, ds = k^2 \int_{\partial D_i} (\nu_x \times \psi_i(x)) \cdot (\Phi_k(z_i, z_j)B_j + O(\frac{1}{\delta_{i,j}} (\frac{1}{\delta_{i,j}} + |k|)^2)) \, ds(x)\]

Considering the estimate (3.18), we have
\[-k^2 \int_{\partial D_i} \psi \cdot (\nu \times [S_{ij,o}^k](\nu_y u_j)) \, ds = k^2 \int_{\partial D_i} (\nu_x \times \psi_i(x)) \cdot \Phi_k(z_i, z_j)B_j \, ds(x) + O(\frac{1}{\delta_{i,j}} (\frac{1}{\delta_{i,j}} + |k|)^2),\]
and then, with (2.22) for the second inequality, we derive
\[-k^2 \int_{\partial D_i} \psi \cdot (\nu \times [S_{ij,o}^k](\nu_y u_j)) \, ds = k^2 \int_{\partial D_i} (\nu_x \times \psi_i(x)) \cdot \Phi_k(z_i, z_j) \nabla ((x - z_i) \cdot B_j) \, ds(x) + O(\frac{1}{\delta_{i,j}} (\frac{1}{\delta_{i,j}} + |k|)^2),\]
\[-k^2 \int_{\partial D_i} \psi \cdot (\nu \times [S_{ij,o}^k](\nu_y u_j)) \, ds = -k^2 \int_{\partial D_i} \text{Div}(\nu_x \times \psi_i(x)) \Phi_k(z_i, z_j) ((x - z_i) \cdot B_j) \, ds(x) + O(\frac{1}{\delta_{i,j}} (\frac{1}{\delta_{i,j}} + |k|)^2).\]

As consequence, being \(\text{Div}(\nu \times \psi) = -\nu \cdot \text{curl} \psi\), we obtain
\[-k^2 \int_{\partial D_i} \psi \cdot (\nu \times [S_{ij,o}^k](\nu_y u_j)) \, ds = k^2 \int_{\partial D_i} \left[ \frac{1}{2} I + (R_{ii,o}^0)^* \right]^{-1} (-\nu_x)(x) ((x - z_i) \cdot \Phi_k(z_i, z_j)B_j) \, ds(x) + O(\frac{1}{\delta_{i,j}} (\frac{1}{\delta_{i,j}} + |k|)^2).\]

(3.38)

It remain to put together (3.36), (3.38) and to sum over \(j\) to get the conclusion.

Concerning (3.20), doing as in (3.37)
\[\int_{\partial D_i} \psi_i(x) \cdot \nu_x \times E^{inc}(x) \, ds(x) = -\int_{\partial D_i} \nu_x \times \psi_i \cdot E^{inc}(z_i) \, ds(x) - \int_{\partial D_i} \nu_x \times \psi_i \cdot (E^{inc}(x_i) - E^{inc}(z_i)) \, ds(x),\]
\[= -\int_{\partial D_i} \nu_x \times \psi_i \cdot E^{inc}(z_i) \, ds(x) + O(\|\nu_x \times \psi_i\|_{L^2(\partial D_i)} \|E^{inc}(x_i) - E^{inc}(z_i)\|_{L^2(\partial D_i)}).\]

With the Mean Value Theorem, we get
\[
\|E^{inc}(x_i) - E^{inc}(z_i)\|_{L^2(\partial D_i)} = \|\int_0^1 \nabla E^{inc}(tx_i + (1-t)z_i) dt \cdot (x - z_i)\|_{L^2(\partial D_i)} = O(ke^2),
\]
thus, considering (3.18), and (2.22) for the last identity, we end up with
\[\int_{\partial D_i} \psi_i(x) \nu_x \times E^{inc}(x) \, ds(x) = -\int_{\partial D_i} \nu_x \times \psi_i(x) \cdot \nabla ((x - z_i) \cdot E^{inc}(z_i)) \, ds(x) + O(ke^4),\]
\[= \int_{\partial D_i} -\nu_x \cdot \text{curl} \psi_i(x) \cdot ((x - z_i) \cdot E^{inc}(z_i)) \, ds(x) + O(|k|e^4).\]

Using (3.22) gives the conclusion.

Finally, the approximation for the \(B_i\)’s, with \([T_{\partial D_i}]\) as defined in (1.10),
\[B_i = [T_{\partial D_i}] \sum_{(j \neq i) \geq 1} (-\nabla_x \Phi_k(z_i, z_j) \times A_j + \Pi_k(z_i, z_j)B_j) - [T_{\partial D_i}] E^{inc}(z_i)\]
+ O\left(\frac{1}{\delta_{i,j}} \left(\frac{1}{\delta_{i,j}} + |k|^3 + \left(\frac{1}{\delta_{i,j}} + |k|\right)^2 + \left(\frac{1}{\delta_{i,j}} + |k|\right)^4\right) \epsilon^7\right) \\
+ O((1 + |k| + |k|^2 \epsilon) \epsilon^4),
\]
holds. It suffices, for \( l = 1, 2, 3 \), to replace the approximations of Lemma 3.4 and Lemma 3.5 in (3.17) to conclude. Developing the approximation error of the above equation, as pointed in (3.1), gives (3.5).

### 3.2.2 Justification of (3.4)

Let \( \phi \) be any smooth enough scalar function. Multiply each side of (2.11) by \( \nabla \phi \) and integrate over \( \partial D_i \) to get, using the relation (2.1),

\[
\int_{\partial D_i} \phi \text{Div} \left[ \frac{1}{2} I + M_{i,j}^k \right] a \, ds + \sum_{(j \neq i) \geq 1} \int_{\partial D_i} \phi \text{Div} \left[ M_{i,j}^k \right](a) \, ds = - \int_{\partial D_i} \phi \text{Div} \left( \nu_x \times E^{\text{inc}} \right) \, ds. \tag{3.39}
\]

As \( \text{curl}^2 = \text{curl} \text{curl} = -\Delta + \nabla \text{div} \), we have

\[
\int_{\partial D_i} \phi \left[ \frac{1}{2} I - (K_{\partial D_i}^k)^* \right] \text{Div} a - k^2 \nu_x \cdot [S_{i,j}^k] a \, ds \\
- \sum_{(j \neq i) \geq 1} \int_{\partial D_i} \phi \left[ (K_{i,j}^k)^* \right] \text{Div} a + k^2 \nu_x \cdot [S_{i,j}^k] a \, ds = - \int_{\partial D_i} \phi \nu_i \cdot \text{curl} E^{\text{inc}} \, ds. \tag{3.40}
\]

Let now \( \phi \) be the solution to the following integral equation

\[
\left[ -\frac{1}{2} I + K_{i,i}^0 \right] (\phi)(x) = (x - z_i), \tag{3.41}
\]

then, as result of (2.37), \( \phi \) satisfies the following estimate

\[
\|\phi\|_{L^2(\partial D_i)} \leq C_{K_{i,i}} \epsilon^2. \tag{3.42}
\]

The tensor \( [P_{\partial D_i}] \) is defined in (1.9). The justification of (3.4) is a direct consequence of the following expansions.

**Lemma 3.6.** With the previous notation we have the following three approximations

\[
\int_{\partial D_i} \phi \left[ \frac{1}{2} I - (K_{\partial D_i}^k)^* \right] \text{Div} a - k^2 \nu_x \cdot [S_{i,j}^k] a \, ds = A_i + O((\epsilon + 1)|k|^2 \epsilon^4). \tag{3.43}
\]

\[
\int_{\partial D_i} \phi \left[ (K_{i,j}^k)^* \right] \text{Div} a + k^2 \nu_x \cdot [S_{i,j}^k] a \, ds = [P_{\partial D_i}] \left( \Pi_k (z_i, z_j) A_j - k^2 \nabla \Phi_k (z_i, z_j) \times E_j \right) \\
+ O\left(\left(\frac{1}{\delta_{i,j}} + |k|^3 + \left(\frac{1}{\delta_{i,j}} + |k|\right)^2 + \left(\frac{1}{\delta_{i,j}} + |k|\right)^4\right) \epsilon^7\right),
\]

\[
\int_{\partial D_i} \phi \nu_i \cdot \text{curl} E^{\text{inc}} \, ds = [P_{\partial D_i}] \text{curl} E^{\text{inc}}(z_i) + O(|k|^2 \epsilon^4). \tag{3.44}
\]

**Proof.** We have for (3.13)

\[
\int_{\partial D_i} \phi \left[ \frac{1}{2} I - (K_{i,i}^0)^* \right] \text{Div} a + k^2 \nu_x \cdot [S_{i,i}^k] a \, ds \\
= \int_{\partial D_i} \phi \left[ \frac{1}{2} I - (K_{i,i}^0)^* \right] \text{Div} a + \left( (K_{i,i}^0)^* - (K_{i,i}^0)^* \right) \text{Div} a + k^2 \nu_x \cdot [S_{i,i}^k] a \, ds, \tag{3.46}
\]
Repeating the same computations for \( x = 20 \), with the notation (1.9), we get

Concerning the second term of the first member of (3.44), we have in view of (2.36) theorem 2.3, which gives as we did it in (3.35)

hence as

and then

Repeating the same computations for \( x = 20 \), we get, in consideration of (3.42)

Concerning (3.44), we obtain, after developing the first term of the first member in Taylor series,

which gives as we did in (3.36) [19]

Repeating the same computations for \( x = 20 \), we get, in consideration of (3.42)

With the notation (1.9), we get [20]

Concerning the second term of the first member of (3.44), we have in view of (2.36) theorem 2.3

and then

\[ \int_{\partial D_i} \phi \kappa^2 \nu_i \cdot [S^k_{i,j,d}] (a_j) \, ds = \int_{\partial D_i} \phi \kappa^2 \nu_i \cdot [S^k_{i,j,d}] (a_j^{[1]} + a_j^{[2]}) \, ds, \]

Note that \( \int_{\partial D_i} \nabla_x \Phi_k(x, z_j) \text{Div} a = 0 \).

Recall that \( \int_{\partial D_j} (y - z_j) \text{Div} a(y) \, ds(y) = - \int_{\partial D_j} a(y) \, ds(y) = - A_j \).
Using Mean-value-theorem, we get for the right-hand side of the above equation
\[
\int_{\partial D_i} \phi k^2 \nu_i \cdot \left( \int_{\partial D_j} O \left( \frac{1}{|\delta_{ij}|} + |k| \right) \epsilon a_j^{[1]}(y) \, ds(y) \right) \, ds(x) + k^2 \Phi_k(z_i, z_j) \int_{\partial D_j} \phi \nu_i \cdot \left( \int_{\partial D_j} a_j^{[1]}(y) \, ds(y) \right) \, ds(x),
\]
then considering the estimates (2.37) and (3.42), with H"older’s inequality give
\[
\int_{\partial D_i} \phi k^2 \nu_i \cdot [S_{ij,D}^k](a_j^{[1]}) \, ds = \left[ \mathcal{P}_{\partial D_j} \right] k^2 \Phi_k(z_i, z_j) A_j + O \left( \frac{1}{|\delta_{ij}|} + |k| \epsilon^7 \right). \tag{3.50}
\]
The second term of the second member of (3.49) gives, again considering (2.36),
\[
\int_{\partial D_i} \phi k^2 \nu_i \cdot [S_{ij,D}^k](a_j^{[2]}) \, ds = \int_{\partial D_i} \phi(x) k^2 \nu_i \cdot \int_{\partial D_j} \Phi_k(x,y) \nu \times \nabla u_j \, ds(y) \, ds(x),
\]
which, by integrating by parties, give
\[
\int_{\partial D_i} \phi k^2 \nu_i \cdot [S_{ij,D}^k](a_j^{[2]}) \, ds = \int_{\partial D_i} \phi(x) k^2 \nu_i \cdot \left( - \int_{\partial D_j} u_j(y) \nu y \times \nabla \Phi_k(x,y) \, ds(y) \right) \, ds(x).
\]
Now, doing a first order approximation, we have
\[
\int_{\partial D_i} \phi(x) k^2 \nu_i \cdot \left( - \int_{\partial D_j} u_j \nu_j \, ds \right) \times \nabla \Phi_k(x,z_j) \, ds(x)
+ \int_{\partial D_i} \phi(x) k^2 \nu_i \cdot \left( - \int_{\partial D_j} u_j \nu_j \, ds \right) \times \nabla \Phi_k(x,z_j) \, ds(x)
+ \int_{\partial D_i} \phi(x) k^2 \nu_i \cdot \left( - \int_{\partial D_j} u_j \nu_j \, ds \right) \times \nabla \Phi_k(x,z_j) \, ds(x),
\]
and similarly to (4.31) the right-hand side is equal to
\[
\int_{\partial D_i} \phi(x) k^2 \nu_i \cdot \left( - \int_{\partial D_j} u_j \nu_j \, ds \right) \times \nabla \Phi_k(x,z_j) \, ds(x)
+ \int_{\partial D_i} \phi(x) k^2 \nu_i \cdot \left( - \int_{\partial D_j} u_j \nu_j \, ds \right) \times \nabla \Phi_k(x,z_j) \, ds(x) + O \left( \frac{1}{|\delta_{ij}|^2} + |k|^2 \epsilon^7 \right).
\]
Repeating the same calculation, for \( x \in \partial D_i \), gives
\[
\int_{\partial D_i} \phi k^2 \nu_i \cdot \left( - \int_{\partial D_j} u_j \nu_j \, ds \right) \times \nabla \Phi_k(z_i, z_j) \, ds(x)
+ O \left( \frac{|k|^2}{|\delta_{ij}|} + |k|^2 \epsilon^7 \right).
\]
Hence, being \( V \times U = -U \times V \) for any vectors \( U, V \), and \( \nabla \Phi_k(x,y) = -\nabla_x \Phi_k(x,y) \) we get with the notations (1.3)
\[
\int_{\partial D_i} \phi k^2 \nu_i \cdot \left( - \int_{\partial D_j} u_j \nu_j \, ds \right) \times \nabla \Phi_k(z_i, z_j) \times B_j + O \left( \frac{|k|^2}{|\delta_{ij}|} + |k|^2 \epsilon^7 \right).
\]
The last approximation of Lemma (3.10) being obvious, we end the proof of (3.44) by taking the sum over \( j \) of the two first approximations and replacing in (3.44). \( \square \)
4 Invertibility of the linear system

For $\mu^+$ and $\mu^-$ defined as in (1.12) and $\mathcal{E} = (\mathcal{E})_{i=1}^m$ defined as

$$\mathcal{E}_i = \begin{cases} E^{inc}(z_i), & i \in \{1, \ldots, m\}, \\ \text{curl} E^{inc}(z_{i-m}), & i \in \{m+1, \ldots, 2m\}, \end{cases}$$

we have the following proposition.

**Proposition 4.1.** Under the condition

$$C_{L_i} := 1 - C_{L_s} \frac{\mu^+}{\delta^3} > 0,$$

for some constant $C_{L_s}$, the following linear system is invertible

$$\hat{\mathbf{E}}_i = [\mathcal{T}_{\partial D_i}] \sum_{(j \neq i) \geq 1}^{m} \left( - \nabla_x \Phi_k(z_i, z_j) \times \hat{A}_j + \Pi_k(z_i, z_j) \hat{B}_j \right) - [\mathcal{T}_{\partial D_i}] E^{inc}(z_i),$$

$$\hat{A}_i = -[\mathcal{P}_{\partial D_i}] \sum_{(j \neq i) \geq 1}^{m} \left( \Pi_k(z_i, z_j) \hat{A}_j - k^2 \nabla \Phi_k(z_i, z_j) \times \hat{B}_j \right) - [\mathcal{P}_{\partial D_i}] \text{curl} E^{inc}(z_i),$$

and the solution satisfies the following estimate

$$\left( \sum_{i=1}^{m} (|\hat{\mathbf{A}}_i|^2 + |\hat{\mathbf{B}}_i|^2) \right)^{\frac{1}{2}} := \left( \langle \hat{\mathbf{E}}, \hat{\mathbf{E}} \rangle_{\mathbb{C}^3 \times m} + \langle \hat{\mathbf{A}}, \hat{\mathbf{A}} \rangle_{\mathbb{C}^3 \times m} \right)^{\frac{1}{2}} \leq \frac{1}{C_{L_i} \mu^-} e^{3 \langle \mathcal{E}, \hat{\mathbf{E}} \rangle_{\mathbb{C}^3 \times 2m}}.$$  \hspace{1cm} (4.4)

Further, if the condition (2.34) is satisfied, then the system could be inverted using Neumann series with the following estimate

$$|\hat{\mathbf{A}}_i| \leq \frac{1}{C_{L_i} \mu^-} e^{3 |\mathcal{E}_i|}, \quad |\hat{\mathbf{B}}_i| \leq \frac{1}{C_{L_i} \mu^-} e^{3 |\mathcal{E}_{i+m}|}.$$  \hspace{1cm} (4.5)

To prove this result, we need to introduce some notations. Let $(\hat{\mathbf{C}}_i)_{i \in \{1, \ldots, 2m\}}$ be defined as

$$\hat{\mathbf{C}}_i = \begin{cases} [\mathcal{T}_{\partial D_i}]^{-1} \hat{\mathbf{E}}_i, & i \in \{1, \ldots, m\}, \\ -[\mathcal{P}_{\partial D_i-1}]^{-1} \hat{\mathbf{A}}_{i-1}, & i \in \{m+1, \ldots, 2m\}, \end{cases}$$

and let $Q$ be the following diagonal Bloc matrix

$$Q := \begin{cases} [\mathcal{T}_{\partial D_i}] & \text{for } i \in \{1, \ldots, m\}, \\ -[\mathcal{P}_{\partial D_i}] & \text{for } i \in \{m+1, \ldots, 2m\}, \end{cases}$$  \hspace{1cm} (4.6)

with $Q_{1,1} = \text{Diag}(Q_{i})_{i=1}^{m}$, and $Q_{2,2} = \text{Diag}(Q_{i+m})_{i=1}^{m}$.

Consider

$$\hat{\theta}_{ij} \hat{\mathbf{C}}_j := \nabla_x \Phi_k(z_i, z_j) \times \hat{\mathbf{C}}_j = \begin{bmatrix} 0 & - \partial_3 \Phi_k(z_i, z_j) & \partial_2 \Phi_k(z_i, z_j) \\ \partial_3 \Phi_k(z_i, z_j) & 0 & - \partial_1 \Phi_k(z_i, z_j) \\ - \partial_2 \Phi_k(z_i, z_j) & \partial_1 \Phi_k(z_i, z_j) & 0 \end{bmatrix} \hat{\mathbf{C}}_j,$$  \hspace{1cm} (4.7)

and define the following bloc matrix

$$\Sigma^k := \begin{bmatrix} \Sigma_{1,1}^k & 0_{\mathbb{C}^m \times \mathbb{C}^m} \\ 0_{\mathbb{C}^m \times \mathbb{C}^m} & \Sigma_{2,2}^k \end{bmatrix} = (\sigma_{ij}^k)_{i,j=1}^{2m}, \quad \Theta^k := \begin{bmatrix} 0_{\mathbb{C}^m \times \mathbb{C}^m} & \Theta_{1,2}^k \\ \Theta_{2,1}^k & 0_{\mathbb{C}^m \times \mathbb{C}^m} \end{bmatrix} = (\theta_{ij}^k)_{i,j=1}^{2m},$$

$^{21}$The constant $C_{L_s}$ is provided in (4.28).
where
\[
\sigma_{i,j}^k := \begin{cases} 
- \Pi_k(z_i, z_j), & i \neq j, i, j \in \{1, \ldots, m\}, \\
- \Pi_k(z_{i-m}, z_{j-m}), & i \neq j, i, j \in \{m+1, \ldots, 2m\}, \\
0, & \text{otherwise,}
\end{cases}
\]
and
\[
\theta_{i,j}^k := \begin{cases} 
\hat{\partial}_{i,j-m}, & i \in \{1, \ldots, m\}, j \in \{1+m, \ldots, 2m\}, \\
k^2 \hat{\partial}_{i-m,j}, & j \in \{1, \ldots, m\}, i \in \{1+m, \ldots, 2m\}, \\
0, & \text{otherwise.}
\end{cases}
\]

With these notations, solving the system \((\text{4.3})\) is equivalent to solve the equation
\[
\hat{\mathcal{C}} + \Sigma^k \mathcal{Q} \hat{\mathcal{C}} + \Theta^k \mathcal{Q} \hat{\mathcal{C}} = \mathcal{E}.
\]

If we multiply both sides of the last system by \(\mathcal{Q} \hat{\mathcal{C}}\) we get
\[
\langle \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}} + \langle \Sigma^k \mathcal{Q} \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}} + \langle \Theta^k \mathcal{Q} \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}} = \langle \mathcal{E}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}}
\]
where \(\langle \cdot, \cdot \rangle_{C^{3 \times 2m}}\) stands for the usual scalar product in \(C^{3 \times 2m}\).

Adding and subtracting \(\langle \Sigma^0 \mathcal{Q} \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}}\) gives
\[
\langle \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}} + \langle (\Sigma^k - \Sigma^0) \mathcal{Q} \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}} + \langle \Sigma^0 \mathcal{Q} \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}} + \langle \Theta^k \mathcal{Q} \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}} = \langle \mathcal{E}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}},
\]
where \(\langle \cdot, \cdot \rangle_{L^2(\partial \Omega)}\) denotes the characteristic function on \(\partial \hat{\Omega} = \bigcup_{m=1}^m \partial B_{3/4}(z_i)\) where \(B_r(z) := B(z, r)\) denotes a ball of center \(z\) and radius \(r\), and \(\langle \cdot, \cdot \rangle_{L^2(\partial \Omega)}\) denotes the usual scalar product of \(L^2(\partial \Omega)\).

**Lemma 4.2.** For \(U := \sum_{i=1}^m \nu_{z_i} \mathcal{Q} \hat{\mathcal{C}} \chi_{\partial B_{3/4}^i} \) and \(V := \sum_{i=m+1}^{2m} \nu_{z_{i-m}} \mathcal{Q} \hat{\mathcal{C}} \chi_{\partial B_{3/4}^{z_{i-m}}}\) with \(\nu_{z_i}\) being the outward unit normal vector to \(\partial B_{3/4}^{z_i}\), we have
\[
\Sigma^0_{1,1} \mathcal{Q} \hat{\mathcal{C}} \cdot \mathcal{Q} \hat{\mathcal{C}} = \frac{48^2}{\pi \delta^2} \left( \langle S^0_{\partial \hat{\Omega}} U, U \rangle_{L^2(\partial \hat{\Omega})} - \sum_{i=1}^m \langle S^0_{\partial B_{3/4}^{z_i}} U, U \rangle_{L^2(\partial B_{3/4}^{z_i})} \right),
\]
\[
\Sigma^0_{2,2} \mathcal{Q} \hat{\mathcal{C}} \cdot \mathcal{Q} \hat{\mathcal{C}} = \frac{48^2}{\pi \delta^2} \left( \langle S^0_{\partial \hat{\Omega}} V, V \rangle_{L^2(\partial \hat{\Omega})} - \sum_{i=m+1}^{2m} \langle S^0_{\partial B_{3/4}^{z_{i-m}}} V, V \rangle_{L^2(\partial B_{3/4}^{z_{i-m}})} \right),
\]
\[
|\langle (\Sigma^k - \Sigma^0) \mathcal{Q} \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}}| \leq \frac{63^2}{4 \pi \delta} m^3 \sum_{i=1}^m |\mathcal{Q} \hat{\mathcal{C}}|^2 = \frac{63^2}{4 \pi \delta^3} \frac{D(\Omega)}{\delta^3} \langle \mathcal{Q} \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}},
\]
and
\[
|\langle \Theta^k \mathcal{Q} \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}}| \leq \frac{1}{8 \pi} \frac{(1 + |k|^2) C_{k,D(\Omega)}}{\delta^3} \langle \mathcal{Q} \hat{\mathcal{C}}, \mathcal{Q} \hat{\mathcal{C}} \rangle_{C^{3 \times 2m}}.
\]

where \(C_{k,D(\Omega)} := \max(C_0 D(\Omega)^{\frac{1}{2}} \frac{1 + |k|^2}{\delta^3} + D(\Omega)^{\frac{1}{2}})\).

**Proof.** To prove \((\text{4.14})\), using Mean-value-theorem for harmonic function, for \(j \neq i\), we have
\[
\nabla_x \nabla_y \Phi_0(z_i, z_j) = \frac{3 \times 4^3}{4 \pi \delta^3} \int_{B(z_j, \frac{3}{4})} \nabla_x \nabla_y \Phi_0(z_i, y) dy = \frac{48}{\pi \delta^3} \int_{B(z_j, \frac{3}{4})} \nabla_y \Phi_0(z_i, y) dy
\]
then using Gauss divergence theorem,

$$\nabla_x \nabla_y \Phi_0(z_i, z_j) = \frac{48}{\pi^3} \nabla \int_{\partial B_{\delta/4}^i} \Phi_0(z_i, y) \nu_y \; ds(y).$$

Repeating the same for $z_i$ we get

$$\nabla_x \nabla_y \Phi_0(z_i, z_j) = \frac{48^2}{\pi^2 \delta^6} \int_{\partial B_{\delta/4}^i} \Phi_0(x, y) \nu_x \nu_y^T \; ds(y) \; ds(x), \quad (4.17)$$

and from $\langle \Sigma_{1,1}^0 \hat{C}_1, \hat{C}_1 \rangle_{C^{3 \times m}} = \sum_{i=1}^{m} \sum_{1 \leq j \neq i}^{m} (\nabla_x \nabla_y \Phi_0(z_i, z_j)Q_j\hat{C}_j) \cdot Q_i\hat{C}_i$, using (4.17), we get

$$\langle \Sigma_{1,1}^0 \hat{C}_1, \hat{C}_1 \rangle_{C^{3 \times m}} = \frac{48^2}{\pi^2 \delta^6} \sum_{i=1}^{m} \left( \sum_{j=1}^{m} \int_{\partial B_{\delta/4}^i} \Phi_0(x, y) \nu_x \nu_y^T \; ds(y) \; ds(x) \right) Q_j\hat{C}_j \cdot Q_i\hat{C}_i,$$

$$= \frac{48^2}{\pi^2 \delta^6} \sum_{i=1}^{m} \int_{\partial B_{\delta/4}^i} \Phi_0(x, y) \nu_x \nu_y \; ds(y) \; ds(x) \cdot Q_i\hat{C}_i,$$

Adding and subtracting $\sum_{i=1}^{m} \int_{\partial B_{\delta/4}^i} \Phi_0(x, y) \nu_y \; Q_j\hat{C}_i \; ds(y) \nu_x \cdot Q_i\hat{C}_i \; ds(x)$, gives

$$\langle \Sigma_{1,1}^0 \hat{C}_1, \hat{C}_1 \rangle_{C^{3 \times m}} = \frac{48^2}{\pi^2 \delta^6} \sum_{i=1}^{m} \int_{\partial B_{\delta/4}^i} \Phi_0(x, y) \nu_y Q_j\hat{C}_i \; ds(y) \nu_x \cdot Q_i\hat{C}_i \; ds(x),$$

$$- \frac{48^2}{\pi^2 \delta^6} \sum_{i=1}^{m} \int_{\partial B_{\delta/4}^i} \Phi_0(x, y) \nu_y \; Q_j\hat{C}_i \; ds(y) \nu_x \cdot Q_i\hat{C}_i \; ds(x), \quad (4.18)$$

and then (4.18) becomes,

$$\langle \Sigma_{1,1}^0 \hat{C}_1, \hat{C}_1 \rangle_{C^{3 \times m}} = \frac{48^2}{\pi^2 \delta^6} \int_{\partial B_{\delta/4}^i} \Phi_0(x, y) \nu_y \; ds(y) \; ds(x) \cdot Q_j\hat{C}_i,$$

$$- \frac{48^2}{\pi^2 \delta^6} \sum_{i=1}^{m} \int_{\partial B_{\delta/4}^i} \Phi_0(x, y) \nu_y \; Q_j\hat{C}_i \; ds(y) \nu_x \cdot Q_i\hat{C}_i \; ds(x).$$

The same arguments remain valid for (4.14).

Concerning (4.15), we have

$$\left| \left( -k^2 \Phi_k(z_i, z_j) + \nabla_x \nabla_y \Phi_k(z_i, z_j) - \nabla_x \nabla_y \Phi_0(z_i, z_j) \right) \right| \leq \frac{6|k|^2}{4\pi \delta_{i,j}} \quad \quad (4.20)$$

Indeed, recalling (2.49) and (2.50)

$$\nabla_y (\Phi_k(x, y) - \Phi_0(x, y)) = \frac{-(ik)^2}{4\pi} \int_0^1 \frac{te^{ik|x-y|}}{|x-y|} (x-y) \; dl$$

then we have also

$$\nabla_x \nabla_y (\Phi_k(x, y) - \Phi_0(x, y)) = \frac{-(ik)^2}{4\pi} \int_0^1 t^2 e^{ik|x-y|} (x-y)^T \frac{1}{|x-y|^2} \; dl$$

$$- \frac{-(ik)^2}{4\pi} \int_0^1 t e^{ik|x-y|} \left( \frac{I}{|x-y|} + \frac{(x-y)(x-y)^T}{|x-y|^3} \right).$$
Integrating by part the first term of the right-hand side, gives

$$\nabla_x \nabla_y (\Phi_k(x, y) - \Phi_0(x, y)) = (k^2 \Phi_k(x, y) - \Phi_0(x, y)) - \frac{k^2}{4\pi} \int_0^1 2i e^{ikl|x-y|} \frac{(x-y)(x-y)^T}{|x-y|^2} dl + \frac{-(ik)^2}{4\pi} \int_0^1 l e^{ikl|x-y|} dl \left( \frac{I}{|x-y|} + \frac{(x-y)(x-y)^T}{|x-y|^3} \right).$$  (4.21)

Hence, we get

$$|\Pi_k(z_i, z_j) - \Pi_0(z_i, z_j)| = |k^2 \Phi_k(z_i, z_j) I - \nabla_x \nabla_y \Phi_k(z_i, z_j) + \nabla_x \nabla_y \Phi_0(z_i, z_j)|,$$

$$\leq |k^2 \Phi_k(z_i, z_j)| + |\nabla_x \nabla_y \Phi_k(z_i, z_j) - \nabla_x \nabla_y \Phi_0(z_i, z_j)| \leq \frac{6|k|^2}{4\pi \delta_{i,j}}.$$

Now, as

$$\langle (\Sigma_{1,1}^k - \Sigma_{1,1}^0) \mathcal{Q}_{1,1} \mathcal{C}_1, Q_1 \mathcal{C}_1 \rangle_{C^3_{\times m}} = \sum_{i=1}^m \left( \sum_{1 \leq j \neq i}^m (\Pi_k(z_i, z_j) - \Pi_0(z_i, z_j)) Q_j \mathcal{C}_j \right) \cdot \overline{Q_i \mathcal{C}_i},$$  (4.22)

using Holder’s inequality, for the inner sum, we get

$$\left| \langle (\Sigma_{1,1}^k - \Sigma_{1,1}^0) \mathcal{Q}_{1,1} \mathcal{C}_1, Q_1 \mathcal{C}_1 \rangle \right| \leq \sum_{i=1}^m \left( \sum_{1 \leq j \neq i}^m |\Pi_k(z_i, z_j) - \Pi_0(z_i, z_j)| \right)^\frac{1}{2} \left( \sum_{j=1}^m |Q_j \mathcal{C}_j|^2 \right)^\frac{1}{2} |Q_i \mathcal{C}_i|.$$

which gives in view of (4.20)

$$\left| \langle (\Sigma_{1,1}^k - \Sigma_{1,1}^0) \mathcal{Q}_{1,1} \mathcal{C}_1, Q_1 \mathcal{C}_1 \rangle \right| \leq \sum_{i=1}^m \left( \sum_{1 \leq j \neq i}^m \frac{3|k|^2}{8\pi \delta_{i,j}} \right)^\frac{1}{2} \left( \sum_{j=1}^m |Q_j \mathcal{C}_j|^2 \right)^\frac{1}{2} |Q_i \mathcal{C}_i|.$$

Repeating Holder’s inequality for the outer sum, we obtain

$$\left| \langle (\Sigma_{1,1}^k - \Sigma_{1,1}^0) \mathcal{Q}_{1,1} \mathcal{C}_1, Q_1 \mathcal{C}_1 \rangle \right| \leq \sum_{i=1}^m \sum_{1 \leq j \neq i}^m \left( \frac{3|k|^2}{2\pi \delta_{i,j}} \right)^2 \sum_{j=1}^m \sum_{i=1}^m |Q_j \mathcal{C}_j|^2 \right)^\frac{1}{2} \sum_{j=1}^m |Q_j \mathcal{C}_j|^2 \right)^\frac{1}{2},$$  (4.23)

$$\leq \left( \sum_{i=1}^m \sum_{1 \leq j \neq i}^m \frac{6|k|^2}{4\pi \delta_{i,j}} \right)^\frac{1}{2} \left( \sum_{j=1}^m |Q_j \mathcal{C}_j|^2 \right) |Q_i \mathcal{C}_i|.$$

The inner sum gives, as we did it in (2.66),

$$\sum_{1 \leq j \neq i}^m \frac{6|k|^2}{4\pi \delta_{i,j}} \leq \sum_{l=2}^m \frac{7l^2}{2\pi} \frac{3^2|k|^4}{4\pi^2} = \frac{7m^4}{2\pi} \frac{3^2|k|^4}{4\pi^2 \delta^2},$$

and then

$$\sum_{l=2}^m \frac{7l^2}{2\pi} \frac{3^2|k|^4}{4\pi^2} \leq \frac{m^4}{2\pi} \frac{3^2|k|^4}{4\pi^2 \delta^2} \leq \frac{63^2|k|^2 m^4}{4\pi \delta}.$$

Repeating the same calculation for \( \langle (\Sigma_{2,2}^k - \Sigma_{2,2}^0) \mathcal{Q}_{2,2} \mathcal{C}_2, Q_{2,2} \mathcal{C}_2 \rangle_{C^3_{\times m}} \) leads to the conclusion. For the last assertion (4.10), we proceed as follows:
\begin{equation}
\left\langle \Theta^k \mathcal{Q} \mathcal{C}, \mathcal{Q} \mathcal{C} \right\rangle_{\mathbb{C}^{3 \times 2m}} = \sum_{i=1}^{m} \left( \sum_{j \neq i}^{m} \nabla \Phi_k(z_i, z_j) \times Q_{j+m} \mathcal{C}_{j+m} \right) \cdot \overline{Q_{i} \mathcal{C}_{i}}
+ \sum_{i=1}^{m} \left( \sum_{j \neq i}^{m} k^2 \nabla \Phi_k(z_i, z_j) Q_{j} \mathcal{C}_{j} \right) \cdot \overline{Q_{i+m} \mathcal{C}_{i+m}},
\end{equation}

(4.24)

The first term of the right hand side of (4.24) is smaller then

\[ \frac{1}{4\pi} \sum_{i=1}^{m} \sum_{j \neq i}^{m} \left| Q_{j+m} \mathcal{C}_{j+m} \right| \delta_{i,j} \left( \frac{1}{\delta_{i,j}} + |k| \right) |Q_{i} \mathcal{C}_{i}|, \]

which is,

\[ \frac{1}{4\pi} \sum_{i=1}^{m} \sum_{j \neq i}^{m} \left| Q_{j+m} \mathcal{C}_{j+m} \right| \frac{|Q_{i} \mathcal{C}_{i}|}{\delta_{i,j}} + \frac{1}{4\pi} \sum_{i=1}^{m} \sum_{j \neq i}^{m} \frac{|k|}{\delta_{i,j}} \frac{|Q_{j+m} \mathcal{C}_{j+m}|}{\delta_{i,j}} \frac{|Q_{i} \mathcal{C}_{i}|}{\delta_{i,j}}, \]

and do not exceed \(^{22}\)

\[ \frac{1}{8\pi} \sum_{i=1}^{m} \sum_{j \neq i}^{m} \left( \left| Q_{j+m} \mathcal{C}_{j+m} \right|^2 + \left| Q_{i} \mathcal{C}_{i} \right|^2 \right) \frac{1}{\delta_{i,j}} + \frac{1}{8\pi} \sum_{i=1}^{m} \sum_{j \neq i}^{m} \frac{|k|}{\delta_{i,j}} \frac{|Q_{j+m} \mathcal{C}_{j+m}|^2}{\delta_{i,j}} + \frac{|k|}{\delta_{i,j}} \frac{|Q_{i} \mathcal{C}_{i}|^2}{\delta_{i,j}}, \]

which in its turn, is not greater than \(^{23}\)

\[ \frac{1}{8\pi} \sum_{j=1}^{m} \sum_{i \neq j}^{m} \left| Q_{j+m} \mathcal{C}_{j+m} \right|^2 \frac{1}{\delta_{i,j}} + \frac{1}{8\pi} \sum_{i=1}^{m} \sum_{j \neq i}^{m} |Q_{i} \mathcal{C}_{i}|^2 \frac{1}{\delta_{i,j}} \]

\[ + \frac{|k|}{8\pi} \sum_{j=1}^{m} \sum_{i \neq j}^{m} \left| Q_{j+m} \mathcal{C}_{j+m} \right|^2 \frac{1}{\delta_{i,j}} + \frac{|k|}{8\pi} \sum_{i=1}^{m} \sum_{j \neq i}^{m} |Q_{i} \mathcal{C}_{i}|^2 \frac{1}{\delta_{i,j}}. \]

We have, as done in (2.60), for \( m \leq D(\Omega)/\delta^3 \),

\[ \sum_{j \neq i}^{m} \frac{1}{\delta_{i,j}^2} = \sum_{l=2}^{(m/2)^{\frac{1}{2}}} C_0 \frac{t^2}{l^2 \delta^2} = \frac{C_0 (\frac{\pi}{2})^{\frac{1}{2}}}{\delta^2} \leq \frac{C_0 (\delta (\Omega))^{\frac{1}{2}}}{\delta^3} = C_0 (\frac{\Omega}{D (\Omega)}). \]

We get, in a similar way,

\[ \sum_{j \neq i}^{m} \frac{1}{\delta_{i,j}^2} = \sum_{l=2}^{(m/2)^{\frac{1}{2}}} C_0 \frac{t^2}{l^2 \delta^2} = \frac{C_0 (\frac{\pi}{2})^{\frac{1}{2}} (\frac{\pi}{2})^{\frac{1}{2}} + 1}{2 \delta} \leq \frac{C_0 (\delta (\Omega))^{\frac{1}{2}} + D (\Omega)^{\frac{1}{2}}}{2 \delta^3} = \frac{C_0 (\frac{\Omega}{D (\Omega)})}{\delta^3}. \]

Then, for \( C_{k,D (\Omega)} := \max (C_0 (\frac{\Omega}{D (\Omega)}), |k| C_0 (\frac{\pi}{2})^{\frac{1}{2}}) \), we obtain

\[ \left| \sum_{i=1}^{m} \sum_{j \neq i}^{m} \nabla \Phi_k(z_i, z_j) \times Q_{i+m} \mathcal{C}_{i+m}, \overline{Q_{i} \mathcal{C}_{i}} \right| \leq \frac{1}{8\pi} \frac{C_{k,D (\Omega)}}{\delta^3} \sum_{i=1}^{2m} \left| Q_{i} \mathcal{C}_{i} \right|. \]

Repeating the same argument for the second term of the right-hand side of (4.24) gives the conclusion. \( \square \)

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\(^{22}\) Comes from \( 2ab \leq a^2 + b^2 \) for every real numbers \( a, b \).

\(^{23}\) being \( \sum_{i=1}^{m} \sum_{j \neq i} a_{i,j} = \sum_{j=1}^{m} \sum_{i \neq j} a_{i,j} \), for every real numbers \( a_{i,j} \).
Proof. (of Proposition 1.1) If we consider (1.12), in view of (1.13) and (1.14), we get
\[
(\tilde{C}, Q\tilde{C})_{C^{3,2m}} + \left(\Theta^k Q\tilde{C}, Q\tilde{C}\right)_{C^{3,2m}} + \left(\Sigma^k - \Sigma^0\right) Q\tilde{C}, Q\tilde{C}\right)_{C^{3,2m}} + \frac{48^2}{\pi^2 \delta^6} \left(\frac{1}{\delta} \sum_{i=1}^m \left(S_{\partial B^{i+4}}^0 U, U\right)_{L^2(\partial B^{i+4})} - \frac{2m}{\delta^4} \left(S_{\partial B^{i+4}_0}^0 \mathcal{V}, \mathcal{V}\right)_{L^2(\partial B^{i+4}_0)}\right) \leq (\mathcal{E}, Q\tilde{C})_{C^{3,2m}}.
\]

Using the fact that \(\left(\frac{\pi \delta}{2}\right)^2 \sum_{i=1}^m \left(S_{\partial B^{i+4}}^0 U, U\right)_{L^2(\partial B^{i+4})} + \sum_{i=1}^m \left(S_{\partial B^{i+4}_0}^0 \mathcal{V}, \mathcal{V}\right)_{L^2(\partial B^{i+4}_0)}\right) = (\mathcal{E}, Q\tilde{C})_{C^{3,2m}}.
\]

Using (1.13) and the inequalities (1.13), we have
\[
\left(\frac{\pi \delta}{2}\right)^2 \sum_{i=1}^m \left(S_{\partial B^{i+4}}^0 U, U\right)_{L^2(\partial B^{i+4})} + \sum_{i=1}^m \left(S_{\partial B^{i+4}_0}^0 \mathcal{V}, \mathcal{V}\right)_{L^2(\partial B^{i+4}_0)}\right) \leq (\mathcal{E}, Q\tilde{C})_{C^{3,2m}}.
\]

Further, the following scaling inequality holds
\[
\left(\frac{\pi \delta}{2}\right)^2 \sum_{i=1}^m \left(S_{\partial B^{i+4}}^0 U, U\right)_{L^2(\partial B^{i+4})} + \sum_{i=1}^m \left(S_{\partial B^{i+4}_0}^0 \mathcal{V}, \mathcal{V}\right)_{L^2(\partial B^{i+4}_0)}\right) \leq (\mathcal{E}, Q\tilde{C})_{C^{3,2m}}.
\]

As \(\left(\frac{\pi \delta}{2}\right)^2 \sum_{i=1}^m \left(S_{\partial B^{i+4}}^0 U, U\right)_{L^2(\partial B^{i+4})} + \sum_{i=1}^m \left(S_{\partial B^{i+4}_0}^0 \mathcal{V}, \mathcal{V}\right)_{L^2(\partial B^{i+4}_0)}\right) \leq (\mathcal{E}, Q\tilde{C})_{C^{3,2m}}
\]
we get, with (1.12)
\[
\left(1 - \left(\frac{(1 + |k|)^2}{8\pi} + \frac{63^2 |k|^2 D(\Omega)^{3/2}}{4\pi} \right)\frac{\mu^3}{\delta^3}\right) \left(\tilde{C}, Q\tilde{C}\right)_{C^{3,2m}} \leq (\mathcal{E}, Q\tilde{C})_{C^{3,2m}}
\]

which is precisely

\[
\left(1 - C_{L_s} \frac{\mu^+ \epsilon^3}{\delta^3}\right) \langle \tilde{\mathcal{C}}, \mathcal{Q} \tilde{\mathcal{C}} \rangle_{C^3 \times 2m} \leq \langle \mathcal{E}, \mathcal{Q} \mathcal{E} \rangle_{C^3 \times 2m} \tag{4.27}
\]

where we set

\[
C_{L_s} := \left[ \frac{(1 + |k|^2)C_0 \left(1 + \frac{|k|^2}{\delta^2}D(\Omega) + \frac{|k|^2}{\delta^2}D(\Omega)\right)}{8\pi} + \frac{12^2\|S_{\mu 1}\|}{\pi} + \frac{63^2|k|^2D(\Omega)}{4\pi} \right] \tag{4.28}
\]

recalling the constants \(C_{k,D(\Omega)}\).

Then a sufficient condition for the solvability of (4.3) is given by \(C_{L_s} \frac{\mu^+ \epsilon^3}{\delta^3} < 1\). Further, if the previous condition is satisfied, then from (4.27), considering (1.13), we get

\[
\left(1 - C_{L_s} \frac{\mu^+ \epsilon^3}{\delta^3}\right) \mu^- \epsilon^3 \langle \tilde{\mathcal{C}}, \tilde{\mathcal{C}} \rangle_{C^3 \times 2m} \leq \mu^+ \epsilon^3 \langle \mathcal{E}, \mathcal{E} \rangle_{C^3 \times 2m} \tag{4.29}
\]

and the definition of \(\mathcal{C}\), yields inverting (1.13)

\[
\langle \tilde{\mathcal{C}}, \tilde{\mathcal{C}} \rangle_{C^3 \times 2m} = \left(\langle \mathcal{Q}_{1,1}^{-1} \mathcal{B}, \mathcal{Q}_{1,1}^{-1} \mathcal{B} \rangle_{C^3 \times m} + \langle \mathcal{Q}_{2,2}^{-1} \tilde{\mathcal{A}}, \mathcal{Q}_{2,2}^{-1} \tilde{\mathcal{A}} \rangle_{C^3 \times m} \right)^{\frac{1}{2}},
\]

\[
\geq \mu^+ \epsilon^3 \left(\langle \mathcal{B}, \mathcal{B} \rangle_{C^3 \times m} + \langle \tilde{\mathcal{A}}, \tilde{\mathcal{A}} \rangle_{C^3 \times m} \right)^{\frac{1}{2}}.
\]

Replacing in (4.29), we obtain

\[
\left(1 - C_{L_s} \frac{\mu^+ \epsilon^3}{\delta^3}\right) \mu^- \mu^+ \left(\langle \tilde{\mathcal{B}}, \tilde{\mathcal{B}} \rangle_{C^3 \times m} + \langle \tilde{\mathcal{A}}, \tilde{\mathcal{A}} \rangle_{C^3 \times m} \right)^{\frac{1}{2}} \leq \mu^+ \epsilon^3 \langle \mathcal{E}, \mathcal{E} \rangle_{C^3 \times 2m}.
\]

Concerning (4.5), it suffice to observe that

\[
|\Sigma^k| \leq 2 \sum_{i=1}^{m} |\Pi_k(z_i, z_j)| \leq \frac{1}{2\pi} \sum_{i=1}^{m} \left(\frac{|k|^2}{\delta_{i,j}} + \frac{3}{\delta_{i,j}} \left(\frac{1}{\delta_{i,j}} + |k|\right)^2\right),
\]

\[
|\Theta^k| \leq 2 \sum_{i=1}^{m} |\nabla \Phi_k(z_i, z_j)| \leq \frac{1}{2\pi} \sum_{i=1}^{m} \frac{1}{\delta_{i,j}} \left(\frac{1}{\delta_{i,j}} + |k|\right)
\]

which, summing as in (2.36) gives,

\[
|\Sigma^k \mathcal{Q}| + |\Theta^k \mathcal{Q}| \leq |\Sigma^k| + |\Theta^k| \leq 4 \mu^+ \left(\frac{\ln m\epsilon^\frac{1}{2}}{\delta^3} + \frac{2km\epsilon^\frac{1}{2}}{\delta^2} + \frac{m\epsilon^\frac{3}{2}}{2\delta^2}k^2\right) \epsilon^3
\]

with this, it comes that, for

\[
C_{L_s}^2 = 1 - 4 \mu^+ \left(\frac{\ln m\epsilon^\frac{1}{2}}{\delta^3} + \frac{2km\epsilon^\frac{1}{2}}{\delta^2} + \frac{m\epsilon^\frac{3}{2}}{2\delta^2}k^2\right) \epsilon^3
\]

\[
|\mathcal{C}| \leq \frac{1}{C_{L_s}^2} \mathcal{E}_i. \tag{4.30}
\]
5 End of the proof of Theorem 1.1

With the notations of the previous section, the linear system \((5.5), (3.4)\) becomes

\[
C + \Sigma^k Q C + \Theta^k Q C = E + \epsilon(\epsilon, \delta, |k|, m)e^3, \tag{5.1}
\]

with \((C_i)_{i \in \{1, ..., 2m\}}\) defined as

\[
C_i := \begin{cases} 
    [T_{D_i}]^{-1} B_i, & i \in \{1, ..., m\}, \\
    - [P_{D_i,\downarrow}]^{-1} A_{i-m}, & i \in \{m+1, ..., 2m\},
\end{cases} \tag{5.2}
\]

and \(\epsilon(\epsilon, \delta, |k|, m) = (\epsilon_i(\epsilon, \delta, |k|, m))_{i=1}^{2m}\) with

\[
\epsilon_i(\epsilon, \delta, |k|, m) := \begin{cases} 
    \frac{e^4}{\delta^4} + \epsilon_k, \delta, m \epsilon^4 + (1 + |k|)\epsilon, & i \in \{1, ..., m\}, \\
    \frac{e^4}{\delta^4} + |k|\epsilon_k, \delta, m \epsilon^4 + |k|^2\epsilon, & i \in \{m+1, ..., 2m\}.
\end{cases} \tag{5.3}
\]

The difference between\( (5.1) \) and \((4.10)\) implies

\[
(C - \tilde{C}) + \Sigma^k Q(C - \tilde{C}) + \Theta^k Q(C - \tilde{C}) = \epsilon(\epsilon, \delta, |k|, m), \tag{5.4}
\]

which gives, with the estimates \((4.4)\)

\[
\sum_{i=1}^{m} (|\tilde{A}_i - A_i|^2 + |\tilde{B}_i - B_i|^2) \leq \frac{1}{CL^3 \mu^-} \left( \frac{e^4}{\delta^4} + (1 + |k|)\epsilon_k, \delta, m \epsilon^4 + \max(|k|^2, 1 + |k|)\epsilon \right)^2 2m \epsilon^6, \tag{5.5}
\]

and, with \((4.5)\)

\[
\begin{aligned}
    |\tilde{A}_i - A_i| &\leq \frac{1}{CL^3 \mu^-} \left( \frac{e^4}{\delta^4} + (1 + |k|)\epsilon_k, \delta, m \epsilon^4 + \max(|k|^2, 1 + |k|)\epsilon \right)^2 \epsilon^3, \\
    |\tilde{B}_i - B_i| &\leq \frac{1}{CL^3 \mu^-} \left( \frac{e^4}{\delta^4} + (1 + |k|)\epsilon_k, \delta, m \epsilon^4 + \max(|k|^2, 1 + |k|)\epsilon \right)^2 \epsilon^3.
\end{aligned} \tag{5.6}
\]

We set

\[
O\epsilon\left( \frac{e^4}{\delta^4} \right) := O\left( \frac{e^4}{\delta^4} + (1 + |k|)\epsilon_k, \delta, m \epsilon^4 + \max(|k|^2, 1 + |k|)\epsilon \right).
\]

Lemma 5.1. We have the following asymptotic approximation for the far field,

\[
E^{\infty}(\tau) = \frac{ik}{4\pi} \sum_{i=1}^{m} e^{-ik\tau \cdot z_i} \times (\tilde{A}_i - ik \times \tilde{B}_i) + O\left( \left( |k|^3 + |k|^2 \right) m e^4 \right)
\]

\[
+ \frac{|k|}{2\pi} \max(1, |k|) \frac{1}{CL^3 \mu^-} O\epsilon\left( \frac{e^4}{\delta^4} \right) m \epsilon^3
\]

and the following one for the scattered field

\[
E^{\text{scat}}(x) = \sum_{i=1}^{m} \left( \nabla \Phi_k(x, z_i) \times \tilde{A}_i + \text{curl} \\text{curl} (\Phi_k(x, z_i) \tilde{B}_i) \right) + \frac{(CL^3 \mu^-)^{-1}}{\mu^+} \times O\epsilon\left( \frac{e^4}{\delta^4} \right)
\]

\[
+ O\left( (CL^3 \mu^-)^{-1} \frac{e^7}{\delta^7} + \epsilon(\delta^6, |k| + |k|^2 + |k|^3) e^7 + \epsilon(\delta^5, |k|^2) e^7 \right).
\]
Proof. Recalling the approximation of the far field in Proposition 3.1, we have
\[
E^\infty(\tau) = \frac{ik}{4\pi} \sum_{i=1}^{m} e^{-ikr_i \tau} \times ((A_i - \hat{A}_i) - i k \tau \times (B_i - \hat{B}_i))
\]
\[+ \frac{ik}{4\pi} \sum_{i=1}^{m} e^{-ikr_i \tau} \times (\hat{A}_i - i k \tau \times \hat{B}_i) + O\left((|k|^3 + |k|^2) m \epsilon^4\right).
\]
(5.7)

For the first term of the right hand side, we have
\[
\left| \frac{ik}{4\pi} \sum_{i=1}^{m} e^{-ikr_i \tau} \times ((A_i - \hat{A}_i) - i k \tau \times (B_i - \hat{B}_i)) \right|
\]
\[\leq 2 \frac{|k|}{4\pi} \max(1, |k|) m^{\frac{1}{2}} \left(\sum_{i=1}^{m} (|A_i - \hat{A}_i|^2 + |B_i - \hat{B}_i|^2)\right)^{\frac{1}{2}},
\]
\[\leq \frac{|k| \max(1, |k|)}{2\pi C_{L,\mu^-}} \epsilon^4 \left(\frac{\epsilon^4}{\delta^4}\right) m \epsilon^3.
\]

With this estimate, (5.7) becomes
\[
E^\infty(\tau) = \frac{ik}{4\pi} \sum_{i=1}^{m} e^{-ikr_i \tau} \times (\hat{A}_i - i k \tau \times \hat{B}_i) + O\left((|k|^3 + |k|^2) m \epsilon^4\right)
\]
\[+ \frac{|k| \max(1, |k|)}{2\pi C_{L,\mu^-}} \epsilon^4 \left(\frac{\epsilon^4}{\delta^4}\right) m \epsilon^3.
\]
(5.8)

Again, in view of Proposition 3.1, we have
\[
E^{\text{ sca}}(x) = \sum_{i=1}^{m} \left( \nabla \Phi_k(x, z_i) \times (A_i - \hat{A}_i) + \text{curl} \text{curl}(\Phi_k(x, z_i)(B_i - \hat{B}_i)) \right)
\]
\[+ \sum_{i=1}^{m} \left( \nabla \Phi_k(x, z_i) \times \hat{A}_i + \text{curl} \text{curl}(\Phi_k(x, z_i)\hat{B}_i) \right)
\]
\[+ O\left(\frac{\epsilon^4}{\delta^4} + \epsilon_{k,\delta,m} \epsilon^4\right).
\]
(5.9)

Let \(i_0\) be as in (3.11), from the representation of the linear system we have \(^{24}\)
\[
\sum_{(i \neq i_0) \geq 1} \left( \nabla \Phi_k(z_{i_0}, z_i) \times (A_i - \hat{A}_i) + \text{curl} \text{curl}(\Phi_k(z_{i_0}, z_i)(B_i - \hat{B}_i)) \right)
\]
\[= [T_0 D_{i_0}]^{-1} (B_{i_0} - \hat{B}_{i_0}) + \epsilon_{i_0}(\epsilon, \delta, |k|, m),
\]
\(^{24}\)Notice that \(-H_k(x, y) = \nabla_y \times \nabla_x \times \Phi_k(x, y) I\).
hence, adding and subtracting the last identity to \(5.9\) gives

\[
E_{\text{sc}}(x) = \left( \nabla \Phi_k(x, z_{i0}) \times (A_{i0} - \hat{A}_{i0}) + \text{curl} \left( \Phi_k(x, z_{i0}) (B_{i0} - \hat{B}_{i0}) \right) \right)
\]

\[
+ \sum_{i \neq i_0} \left[ (\nabla \Phi_k(x, z_i) - \nabla \Phi_k(z_{i0}, z_i)) \times (A_{i0} - \hat{A}_{i0}) \right.
\]

\[
+ \text{curl} \left( \Phi_k(x, z_i) - \Phi_k(z_{i0}, z_i) \right) (B_i - \hat{B}_i) \left] \right. \right)
\]

\[+ \left[ T_{\partial D_{i0}} \right]^{-1} (B_{i0} - \hat{B}_{i0}) + \epsilon_{i0} (\epsilon, \delta, |k|, m)
\]

\[
+ \sum_{i=1}^{m} \left( \nabla \Phi_k(x, z_i) \times \hat{A}_i + \text{curl} (\Phi_k(x, z_i) \hat{B}_i) \right)
\]

\[+ O \left( \frac{\epsilon^4}{\delta^4} + \epsilon_{k, \delta, m} \epsilon^4 \right).
\]

For \(x \in \partial \Omega, \frac{1}{\delta x_{i0}} = \frac{1}{\delta} \) and then the first term of the right hand side of \(5.10\) is smaller then

\[
\left( \frac{1}{\delta} \left( \frac{1}{\delta} + |k| \right) |A_{i0} - \hat{A}_{i0}| + \left( \frac{|k|^2}{\delta} + \frac{1}{\delta} \left( \frac{1}{\delta} + |k| \right)^2 \right) |B_{i0} - \hat{B}_{i0}| \right)
\]

which gives, considering \(5.6\)

\[
(C_{L_2} \mu^\ast)^{-1} \left( \frac{1}{\delta^3} + \frac{1 + 2|k| + |k| + 2|k|^2}{\delta^2} \right) \left( \frac{\epsilon^4}{\delta^4} + (1 + |k|) \epsilon_{k, \delta, m} \epsilon^4 + \max(|k|^2, 1 + |k|) \epsilon^2 \right)
\]

which is

\[
O \left( (C_{L_2} \mu^\ast)^{-1} \frac{\epsilon^7}{\delta^7} + \epsilon (\delta^6, |k|) \epsilon^7 + \epsilon (\delta^5, |k|^2) \epsilon^7 \right).
\]

The second term of \(5.10\) is exactly

\[
\sum_{(i \neq i_0) \geq 1} \left( \int_{[0,1]} \nabla_x \nabla_x \Phi_k(t x + (1 - t) z_{i0}, z_i) dt \cdot (x - z_{i0})) \times (A_{i0} - \hat{A}_{i0}) \right.
\]

\[
+ \left( \int_{[0,1]} \nabla_x \Pi_k(t x + (1 - t) z_{i0}, z_i) dt \cdot (x - z_{i0})) (B_i - \hat{B}_i) \right],
\]

which turns out to be not greater then

\[
\sum_{(i \neq i_0) \geq 1} \left[ \frac{1}{\delta x_{i0,i}} \left( \frac{1}{\delta x_{i0,i}} + |k| \right)^2 \delta |A_{i0} - \hat{A}_{i0}| + \left( \frac{|k|^2}{\delta x_{i0,i}} + |k| \right) \right.
\]

\[
+ \left( \frac{1}{\delta x_{i0,i}} \left( \frac{1}{\delta x_{i0,i}} + |k| \right)^3 \right) \delta |B_i - \hat{B}_i| \right],
\]

and, due to \(5.6\), not exceed

\[
\sum_{(i \neq i_0) \geq 1} \left[ \frac{1}{\delta x_{i0,i}} \left( \frac{1}{\delta x_{i0,i}} + |k| \right)^2 \delta + \left( \frac{|k|^2}{\delta x_{i0,i}} + |k| \right) \right.
\]

\[
+ \left( \frac{1}{\delta x_{i0,i}} \left( \frac{1}{\delta x_{i0,i}} + |k| \right)^3 \right) \delta \right]
\]

\[
\times (C_{L_2} \mu^\ast)^{-1} \left( \frac{\epsilon^4}{\delta^4} + (1 + |k|) \epsilon_{k, \delta, m} \epsilon^4 + \max(|k|^2, 1 + |k|) \epsilon^2 \right) 2 \epsilon^3,
\]

hence summing as in \(2.66\) gives

\[
O \left( \frac{1}{\delta^3} + \frac{(1 + |k|) \ln(m^{1/3})}{\delta^3} + \frac{(|k| + |k|^2) m^{1/3}}{\delta^2} + \frac{(|k|^3 + |k|^2) m^{2/3}}{\delta^3} \right) \delta
\]
\( \times (C_{L^2}^i \mu^-)^{-1} \left( \frac{\epsilon_k^4}{\delta^4} + (1 + |k|) \epsilon_{k,\delta,m} \epsilon^4 + \max(|k|^2, 1 + |k|) \epsilon \right)^2 2\epsilon^3, \)

which is, for \( m = O(1/\delta^3) \), the analogue of (5.11) with the following additional term

\[ \epsilon (\delta^5, |k|^2 + |k|^3) \epsilon^7 + \epsilon (\delta^5, |k|^2) \epsilon^7. \] (5.12)

The third term of (5.10), due to (1.13) and (5.6), is bounded by

\[ \frac{(C_{L^2}^i \mu^-)^{-1}}{\mu^+} \left( \frac{\epsilon_k^4}{\delta^4} + (1 + |k|) \epsilon_{k,\delta,m} \epsilon^4 + \max(|k|^2, 1 + |k|) \epsilon \right)^2 = \frac{(C_{L^2}^i \mu^-)^{-1}}{\mu^+} \times O\left( \frac{\epsilon^4}{\delta^4} \right), \] (5.13)

compiling this last error with (5.11) and the additional term (5.12) gives us the result.

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