ON MODULAR INEQUALITIES IN VARIABLE $L^p$ SPACES

ANDREI K. LERNER

Abstract. We show that the Hardy-Littlewood maximal operator and a class of Calderón-Zygmund singular integrals satisfy the strong type modular inequality in variable $L^p$ spaces if and only if the variable exponent $p(x) \sim \text{const}$.

1. Introduction

Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the Banach space of measurable functions $f$ on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} \, dx < \infty,$$

with norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} \, dx \leq 1 \right\}.$$

The spaces $L^{p(\cdot)}(\mathbb{R}^n)$ are a special case of Musielak-Orlicz spaces (cf. [9]). The behavior of some classical operators in harmonic analysis on $L^{p(\cdot)}(\mathbb{R}^n)$ is intensively investigated during several last years. Among numerous papers appeared in this area, let us mention only those of specific interest to us, to be precise those where different aspects concerning the boundedness on $L^{p(\cdot)}(\mathbb{R}^n)$ of the Hardy-Littlewood maximal operator [1, 2, 3, 10, 11, 12] and the Calderón-Zygmund operators [4, 8] were studied.

We recall that the Hardy-Littlewood maximal operator is defined for any $f \in L^{1}_{\text{loc}}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B} |f(y)| \, dy,$$

where the supremum is taken over all balls $B$ containing $x$.

2000 Mathematics Subject Classification. Primary 42B20, 42B25.

Key words and phrases. Maximal function, singular integral, $A_\infty$ weight, variable $L^p$. 

1
Let \( p_- = \text{ess inf}_{x \in \mathbb{R}^n} p(x) > 1 \) and \( p_+ = \text{ess sup}_{x \in \mathbb{R}^n} p(x) < \infty \). It has been proved by Diening [2] that if \( p \) satisfies the following uniform continuity condition
\[
|p(x) - p(y)| \leq \frac{c}{-\log |x - y|}, \quad |x - y| < 1/2,
\]
and if \( p \) is a constant outside some large ball, then
\[
\|Mf\|_{L^p(\cdot)} \leq c\|f\|_{L^p(\cdot)}
\]
for all \( f \in L^p(\mathbb{R}^n) \). After that the second condition on \( p \) has been improved independently in several directions by Cruz-Uribe, Fiorenza, and Neugebauer [1] and Nekvinda [10]. For example, it is shown in [1] that if \( p \) satisfies (1.1) and
\[
|p(x) - p(y)| \leq \frac{c}{\log(e + |x|)}, \quad |y| \geq |x|,
\]
then (1.2) holds.

Diening and Růžička [4] (see also [8, Theorem 2.7] and [3, Section 8]) have proved that if \( p_- > 1 \) and \( p_+ < \infty \), then a large class of Calderón-Zygmund operators is bounded on \( L^p(\mathbb{R}^n) \) provided the Hardy-Littlewood maximal operator is bounded on \( L^p(\mathbb{R}^n) \) and on \( L^{(p)/(s)}(\cdot) \) for some \( 0 < s < 1 \), where \( p'(x) = p(x)/(p(x) - 1) \).

A natural question arises about conditions on \( p \) implying the strong type inequality
\[
\int_{\mathbb{R}^n} |Rf(x)|^{p(x)}dx \leq c \int_{\mathbb{R}^n} |f(x)|^{p(x)}dx
\]
(so-called modular inequality in terminology of Musielak [9]), where \( R \) is any of the above-mentioned classical operators. Note that in [1] the weak type modular inequality
\[
|\{x \in \mathbb{R}^n : Mf(x) > \alpha\}| \leq c \int_{\mathbb{R}^n} |f(x)/\alpha|^{p(x)}dx \quad (\alpha > 0)
\]
is proved under extremely weak assumptions on \( p \). It is easy to see that (1.3) yields the norm inequality
\[
\|Rf\|_{L^p(\cdot)} \leq c\|f\|_{L^p(\cdot)},
\]
and therefore one should expect that the class of functions \( p \), for which (1.3) holds, must be smaller than the corresponding class implying (1.2). Nevertheless, our main result is somewhat surprising, since it says that this class is trivial. More precisely, we have the following.
Theorem 1.1. Let $p_- > 1$ and $p_+ < \infty$. Then the inequality
\begin{equation}
\int_{\mathbb{R}^n} (Mf(x))^{p(x)} dx \leq c \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx
\end{equation}
holds for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $p(x) \sim \text{const}$.

It is noteworthy that analogous questions on singular integrals are very similar to those when the boundedness on weighted $L^p_\omega$ implies $\omega \in A_p$ (cf. [13, p. 210]). We shall deal with a singular integral operator $Tf = f \ast K$, with kernel $K$ satisfying the standard conditions
\[ \| \hat{K} \|_\infty \leq c, \quad |K(x)| \leq c/|x|^n, \]
\[ |K(x) - K(x - y)| \leq c|y|/|x|^n+1 \text{ for } |y| < |x|/2, \]
and an additional nondegeneracy condition
\[ |K(tu_0)| \geq c'|tu_0|^n \]
for some unit vector $u_0$ and any $t \in \mathbb{R}$. Observe that this class of operators contains, for instance, any one of the Riesz transforms.

Theorem 1.2. Let $p_- > 1$ and $p_+ < \infty$. Then the inequality
\begin{equation}
\int_{\mathbb{R}^n} |Tf(x)|^{p(x)} dx \leq c \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx
\end{equation}
holds for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $p(x) \sim \text{const}$.

2. Proofs

By a weight we mean any non-negative locally integrable function on $\mathbb{R}^n$. Given a ball $B$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, let $f_B = |B|^{-1} \int_B f$. For measurable $f$ and $g$ the notation $f \sim g$ means $f(x) = g(x)$ a.e.

We say that a weight $\omega$ satisfies $A_\infty$ Muckenhoupt’s condition if for any $\alpha$, $0 < \alpha < 1$, there exists a $\beta$, $0 < \beta < 1$, such that $|E| \geq \alpha |B|$ implies $\int_E \omega dx \geq \beta \int_B \omega dx$ for all balls $B$ and all subsets $E \subset B$. There are many equivalent characterizations of $A_\infty$ (see, e.g., [13, Ch. 5]). In particular, $\omega \in A_\infty$ if and only if (see [6, p. 405] or [7])
\begin{equation}
\left( \frac{1}{|B|} \int_B \omega dx \right) \exp \left( \frac{1}{|B|} \int_B \log(1/\omega) dx \right) \leq A.
\end{equation}

We say that a family of weights $\{\omega_\alpha\}_{\alpha \in A}$ satisfies $A_\infty$ condition uniformly in $\alpha$ if $\omega_\alpha \in A_\infty$ for any $\alpha \in A$ with corresponding $A_\infty$ constants independing of $\alpha$.

Lemma 2.1. Let $p$ be a non-negative measurable function on $\mathbb{R}^n$. The family of weights $\{t^{p(x)}\}_{t > 0}$ satisfies $A_\infty$ condition uniformly in $t$ if and only if $p(x) \sim \text{const}$.
Proof. When \( p(x) \sim \text{const} \) the statement of the lemma is trivial. Thus, we assume that \( t^{p(x)} \in A_\infty \) uniformly in \( t \). Applying (2.1) to \( \omega_t(x) = t^{p(x)} \) yields

(2.2) \[
\frac{1}{|B|} \int_B t^{p(x) - p_B} \, dx \leq A
\]

for any ball \( B \) and all \( t > 0 \). Now, if \(|\{x \in B : p(x) > p_B\}| > 0\), we get a contradiction by letting \( t \to \infty \) in (2.2). Analogously, if \(|\{x \in B : p(x) < p_B\}| > 0\), we get a contradiction by letting \( t \to 0 \) in (2.2). Therefore, \( p(x) = p_B \) for a.e. \( x \in B \) and for all balls \( B \). Hence, the limit \( p_\infty = \lim_{|B| \to \infty} p_B \), where it is taken over all balls \( B \) in \( \mathbb{R}^n \) as the measure \(|B|\) tends to infinity, exists, and \( p(x) = p_\infty \) for a.e. \( x \in \mathbb{R}^n \). \( \square \)

We are now in a position to prove Theorems 1.1 and 1.2. Since for \( p(x) \sim \text{const} \) both theorems represent known classical results, we need to prove only the converse directions.

Proof of Theorem 1.1. It follows from (1.4) that for any ball \( B \) and any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \),

(2.3) \[
\int_B (|f|_B)^{p(x)} \, dx \leq c \int_B |f(x)|^{p(x)} \, dx.
\]

Let \( E \subset B \) be an arbitrary measurable subset with \(|E| \geq \alpha|B|\), \( 0 < \alpha < 1 \). Taking in (2.3) \( f = t \chi_E, t > 0 \), we get

\[
\alpha^{p^+} \int_B t^{p(x)} \, dx \leq c \int_E t^{p(x)} \, dx.
\]

Therefore, the family of weights \( \{t^{p(x)}\}_{t>0} \) satisfies \( A_\infty \) condition uniformly in \( t \). Now we invoke Lemma 2.1 to complete the proof. \( \square \)

Proof of Theorem 1.2. We use the following property of singular integrals (see [13, Ch. 5, 4.6]): for any ball \( B \) there exists a ball \( B' \) such that \( f_B \leq c |(Tf)\chi_{B'}| \) for any non-negative \( f \in C^0_\infty \) supported in \( B \) and \( f_{B'} \leq c |(Tf)\chi_{B'}| \) for any non-negative \( f \in C^0_\infty \) supported in \( B' \). It follows from this and from (1.5) that for such \( f \),

(2.4) \[
\int_{B'} (f_{B'})^{p(x)} \, dx \leq c \int_{B'} (f(x))^{p(x)} \, dx
\]

and

(2.5) \[
\int_B (f_{B'})^{p(x)} \, dx \leq c \int_B (f(x))^{p(x)} \, dx.
\]
A simple limiting argument extends these estimates for any \( f \geq 0 \). Taking in (2.4) \( f = t\chi_E, t > 0 \), where \( E \subset B \) with \( |E| \geq \alpha|B| \), we get

\[
\alpha p^+ \int_{B'} t^{p(x)} dx \leq c' \int_{E} t^{p(x)} dx.
\]

However (2.5), with \( f = t\chi_{B'} \), yields

\[
\int_{B} t^{p(x)} dx \leq c' \int_{B'} t^{p(x)} dx,
\]

and therefore,

\[
\alpha p^+ \int_{B} t^{p(x)} dx \leq c'^2 \int_{E} t^{p(x)} dx.
\]

This gives the desired result exactly as in the previous proof. \( \Box \)

3. Concluding remarks

**Remark 3.1.** We recall that a weight \( \omega \) is doubling if there exists a constant \( c > 0 \) such that \( \int_{2B} \omega dx \leq c \int_{B} \omega dx \) for any ball \( B \subset \mathbb{R}^n \). It is well known that any \( A_\infty \) weight is doubling but the converse is not true. In Lemma 2.1 the \( A_\infty \) condition, in general, can not be replaced by a wider doubling condition. Indeed, one can construct on the real line disjoint sets \( E_1 \) and \( E_2 \) of positive measure whose union is \( \mathbb{R}^1 \), while \( \chi_{E_1} \) and \( \chi_{E_2} \) are doubling measures (see [13, Ch. 1, 8.8]). Let now \( p(x) = c_1 \chi_{E_1} \times \mathbb{R}^{n-1} + c_2 \chi_{E_2} \times \mathbb{R}^{n-1} \), where \( c_1 \neq c_2 \). Then \( p(x) \not\sim \text{const} \), while it is easy to check that \( \int_{2B} t^{p(x)} dx \leq c \int_{B} t^{p(x)} dx \) for any ball \( B \subset \mathbb{R}^n \) and all \( t > 0 \).

However, assuming additionally that \( p \) is continuous, one can show that the family \( \{t^{p(x)}\}_{t>0} \) is doubling uniformly in \( t \) if and only if \( p(x) = \text{const} \).

**Remark 3.2.** Musielak-Orlicz spaces (cf. [9]) consist of all measurable \( f \) such that for some \( \lambda > 0 \),

\[
\int_{\mathbb{R}^n} \varphi(x, |f(x)/\lambda|) dx < \infty,
\]

where \( \varphi: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+ \) satisfies specific conditions. These spaces contain, i.e., weighted Lebesgue spaces \( L^p_\omega \) (when \( \varphi(x, \xi) = \xi^p \omega(x) \)) and Orlicz spaces (when \( \varphi(x, \xi) \) is constant in the first variable).

Theorems 1.1 and 1.2 show that in the case \( \varphi(x, \xi) = \xi^p \omega(x) \) the corresponding modular inequality for \( M \) or \( T \) holds if and only if \( \varphi(x, \xi) \) is constant in the first variable. It is easy to see that in general an analogous result does not hold. For example, one can take \( \varphi(x, \xi) = \xi^p \omega(x) \) with \( \omega \) satisfying the \( A_p \) Muckenhoupt condition.
On the other hand, it is known (see, e.g., [5]) that in the context of Orlicz spaces the modular inequality for $M$ is equivalent to the norm inequality. Theorem 1.1 shows that this is not the case in the context of Musielak-Orlicz spaces.

Acknowledgement. The author thanks Alexei Karlovich for helpful discussions.

References

[1] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer, The maximal function on variable $L^p$ spaces, Ann. Acad. Sci. Fenn. Math., 28 (2003), 223–238, and 29 (2004), 247-249.

[2] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(x)}$, Fakultät für Mathematik und Physik, Albert-Ludwigs-Universität Freiburg, preprint no. 2, 2002. Math. Inequal. Appl., to appear.

[3] L. Diening, Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces, Fakultät für Mathematik und Physik, Albert-Ludwigs-Universität Freiburg, preprint no. 21, 2003.

[4] L. Diening and M. Růžička, Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(x)}$ and problems related to fluid dynamics, J. reine angew. Math., 563 (2003), 197–220.

[5] D. Gallardo, Orlicz spaces for which the Hardy-Littlewood maximal operator is bounded, Publ. Mat. 32 (1988), no. 2, 253–257.

[6] J. Garca-Cuerva and J. Rubio de Francia, Weighted norm inequalities and related topics, North-Holland Math. Studies 116, North-Holland, Amsterdam, (1985).

[7] S.V. Hruščev, A description of weights satisfying the $A_\infty$ condition of Muckenhoupt, Proc. Amer. Math. Soc. 90 (1984), 253–257.

[8] A.Yu. Karlovich and A.K. Lerner, Commutators of singular integrals on generalized $L^p$ spaces with variable exponent, submitted. Preprint is available from http://front.math.ucdavis.edu/math.CA/0401348

[9] J. Musielak, Orlicz Spaces and Modular Spaces, Lecture Notes in Mathematics, 1034, Springer-Verlag, Berlin, 1983.

[10] A. Nekvinda, Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$, Faculty of Civil Engineering, Czech Technical University in Prague, preprint no. 2, 2002. Math. Inequal. Appl., to appear.

[11] A. Nekvinda, A note on maximal operator on $\ell^{p(x)}$ and on $L^{p(x)}(\mathbb{R})$, Faculty of Civil Engineering, Czech Technical University in Prague, preprint no. 1, 2004.

[12] L. Pick and M. Růžička, An example of a space of $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded, Expo. Math. 19 (2001), no. 4, 369–371.

[13] E.M. Stein, Harmonic Analysis, Princeton Univ. Press, Princeton 1993.

Department of Mathematics, Bar-Ilan University, 52900 Ramat Gan, Israel

E-mail address: aklerner@netvision.net.il