On the Symmetry of Real-Space Renormalisation

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Abstract

A natural geometry, arising from the embedding into a Hilbert space of the parametrised probability measure for a given lattice model, is used to study the symmetry properties of real-space renormalisation group (RG) flow. In the projective state space this flow is shown to have two contributions: a gradient term, which generates a projective automorphism of the state space for each given length scale; and an explicit correction. We then argue that this structure implies the absence of any symmetry of a geodesic type for the RG flow when restricted to the parameter space submanifold of the state space. This is demonstrated explicitly via a study of the one dimensional Ising model in an external field. In this example we construct exact expressions for the beta functions associated with the flow induced by infinitesimal rescaling. These constitute a generating vector field for RG diffeomorphisms on the parameter space manifold, and we analyse the symmetry properties of this transformation. The results indicate an approximate conformal Killing symmetry near the critical point, but no generic symmetry of the RG flow globally on the parameter space.

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I. INTRODUCTION

The construction of effective theories for the low energy modes from their underlying microscopic dynamics or, equivalently, the study of the thermodynamic behaviour of spin and lattice field systems is a fundamental problem in modern physics. The standard theoretical framework for this procedure, the renormalisation group [1], provides a methodology for systematically integrating out high momentum or short distance modes leaving an effective dynamical system for the long wavelength physics. This effective theory then determines, in particular, the critical behaviour of field theories or alternatively the thermodynamic behaviour of lattice or statistical mechanical spin systems.

The fundamental obstacle in following this procedure through to completion is that coarse graining the system, regardless of the detailed approach employed, generically leads to a highly complex effective theory introducing an often arbitrarily high number of additional interactions. While some simplifications occur near the upper critical dimension for a given system, allowing systematic techniques such as the $\epsilon$–expansion to be employed, generically one requires drastic approximations in order to render the RG procedure practically implementable. In particular, we focus on real-space renormalisation [2] methods which implement an intuitive blocking picture in attempting to explain certain universal aspects of critical behaviour, often near the lower critical dimension. However, except in a few tractable cases, such calculational schemes involve significant truncations, and there is rarely a useful criterion for selecting a particular approximation. This is also a significant problem in more recent attempts to study field theories via Wilsonian RG techniques [3] as the dominant infrared degrees of freedom in such cases may differ greatly from the relevant microscopic fields.

With these problems in mind it is clearly of interest to study the structure of the RG in some detail. Generically the Hamiltonian for a spin system, or the Wilsonian action for a field theory, at a given scale may be represented in the form, $H = \sum_i \theta_i H_i$. This is generally an infinite series of interaction Hamiltonians $H_i$ and their associated scale dependent couplings $\theta_i(t)$. The RG acts on these couplings as one coarse grains the system, and one may gain insight into this process by studying the geometry of the parameter space, whose singularity structure and possible symmetries determine the properties of the RG flow, in particular the fixed points. However, the manifold $\mathcal{M}$ of the parameter set dealt with here is typically a subspace of an infinite dimensional Hilbert space $\mathcal{H}$ and this infinite dimensionality, which in effect is the cause of the singularities, gives rise to various technical problems that have to be treated with caution. Nevertheless, the ability to formulate certain RG related questions in a geometric formalism allows the utilisation of various powerful techniques in differential geometry.

The Riemannian structure of this parameter manifold $\mathcal{M}$, as we discuss in more detail below, is inherited from the structure of the probability measure of the system over its configuration space [4]. Various authors [4–8] have utilised this structure to study aspects of statistical mechanical systems and field theories, and it has been shown that [8] the RG acts via diffeomorphisms of this parameter space manifold generated by the $\beta$–function vector field. It is then a clear corollary that the geometry can place restrictions on the $\beta$–function of the theory if other symmetries of the parameter space exist, or indeed if the RG flow itself is a symmetry of $\mathcal{M}$.
In this paper, we investigate this last question, and consider whether the RG diffeomorphism actually corresponds to a symmetry of the parameter space. This is motivated by earlier work [5,9] which suggests that an approximate symmetry holds near the critical points, and also to some extent by recent work [10,11] on the geometric structure of the state space of statistical mechanical systems, and in particular the parametric evolution of the states. A consequence of this latter work is that for a system whose action is expressible in the form \( \sum \theta_i H_i \), the parametric evolution of the state corresponds to a projective automorphism of the projective Hilbert space, which is the most general transformation of a Riemannian manifold mapping geodesics into geodesics.

In generalising this result to systems where all the parameters depend on a single dimensionful scale, we show in Section 2 that the generating vector field of the flow, when restricted to a constant scale, still generates a projective automorphism of the state–space. However, in considering the full scale dependence of the states, it is shown that the integral curve corresponding to RG evolution only generates a projective automorphism of the state-space up to an anomalous correction. Since the parameter space can be viewed as a submanifold of the projective state-space, this suggests the absence of such a geodesic type symmetry of the RG flow on the parameter space itself. The generality of this construction allows us to conjecture that in general the RG does not generate any standard global symmetry of this type on the parameter space.

In order to test this conjecture we provide an explicit counter-example in which both the RG trajectory, and the underlying parameter space geometry, can be exactly determined. While the RG procedure generally requires various approximations, there are a number of exactly tractable models known in statistical mechanics [7,8], and in the present paper we study one such model exhaustively by use of real-space RG methods, in order to determine the properties of RG flow in the relevant parameter space.

The example we consider here is the one dimensional Ising model in an external magnetic field. As we shall discuss in more detail below, the parameter space in this case is a two dimensional manifold endowed with a Riemannian structure in terms of the Fisher-Rao metric. In Section 3 we consider the exact parameter space geometry of this system both for a finite \( N \)-spin chain and in the thermodynamic limit. In Section 4 we utilise a transfer matrix method to obtain an exact expression for the vector field associated with an infinitesimal change of the lattice spacing, the components being the differential RG \( \beta \)-functions. Using this result we study, in Section 5, the symmetry associated with the flow. We find, in particular, that the generating vector field corresponds to a conformal Killing vector field in the vicinity of the critical point. However, this vector field does not generate such a transformation, nor the most general additional class of mappings, a projective automorphism, globally on the manifold. Therefore we are led to conclude, via this explicit counter-example, that there is no generic global symmetry structure of a standard form for the RG flow. We finish in Section 6 with some concluding remarks.

II. STATE SPACE RENORMALISATION

We begin by considering the effect of renormalisation group flow in the actual state space of a given system, which can be viewed as the space of rays through the origin of a real Hilbert space \( \mathcal{H} \), that is, the real projective \( n \)-space \( \mathbb{R}P^n \) (possibly infinite dimensional).
Before studying the behaviour of RG flow in this space we review briefly the notion of the state space in statistical mechanics [11].

In a purely statistical mechanical context we are given the parametrised family of probability distributions, taking the form of the Gibbs measure,

\[ p(x, \theta) = q(x) \exp \left[ - \sum_j \theta^j H_j(x) - W_\theta \right], \]  

(1)

where the variable \( x \) ranges over the configuration space, \( H_j(x) \) represents the form of the energy, \( W_\theta \) is a normalisation factor, and \( q(x) \) determines the distribution at \( \theta^j = 0 \). In a field theoretic context the operators \( H_j \) effectively determine the form of the action, and the parameters \( \theta^j \) are viewed as the coupling constants. Now, by taking the square-root of the distribution function (1) we can map, for each given value of \( \theta^j \), the space of probabilities onto a manifold in a real Hilbert space \( \mathcal{H} \). Since the probability distribution is normalised, we find that for each fixed value of \( \theta^j \), the probability state corresponds to a point on the unit sphere \( S \subset \mathcal{H} \). Then, by choosing a suitable basis of vectors in \( \mathcal{H} \), we can consider the vector \( \psi^a(\theta) \) as representing the point characterised by \( \sqrt{p(x, \theta)} \). In other words, we regard \( \psi^a(\theta) \) as a state vector corresponding to the distribution (1).

First we would like to formulate a Hilbert space characterisation of this distribution. In fact, it can be shown [10,11] that the state vector \( \psi^a(\theta) \) in \( \mathcal{H} \) corresponding to the Gibbs distribution (1) satisfies the differential equation,

\[ \frac{\partial \psi^a}{\partial \theta^j} = -\frac{1}{2} \tilde{H}^a_{jb} \psi^b, \]  

(2)

where the operators \( H^a_{jb} \) in \( \mathcal{H} \) represent the energy \( H_j(x) \), \( \tilde{H}_{j,ab} = H_{j,ab} - g_{ab} E[H_j] \), with \( E[\cdot] \) denoting the expectation. The solution to this equation is given by an exponential family [4] of states

\[ \psi^a(\theta) = \exp \left[ -\frac{1}{2} \left( \sum_j \theta^j H^a_{jb} + \tilde{W}_\theta \delta^a_b \right) \right] q^b, \]  

(3)

where \( \tilde{W}_\theta = W_\theta - W_0 \) and \( q^a = \psi^a(0) \) is the prescribed state at \( \theta^j = 0 \).

The real Hilbert space \( \mathcal{H} \), however, is not what we view here as the true state space of the physical model under study since there is an extra degree of freedom, namely, the overall normalisation. That is to say, the expectation of any physical observable is independent of the value of the normalisation. Hence we can gauge this extra degree of freedom away by identifying all the points in \( \mathcal{H} \) along the given ray that passes through the origin of \( \mathcal{H} \), corresponding to different normalisation factors. The resulting space obtained is the real projective \( n \)-space \( \mathbb{RP}^n \) which we view as the actual state space, with the state vectors \( \psi^a \) in \( \mathcal{H} \) also representing the homogeneous coordinates for \( \mathbb{RP}^n \). Note that we encounter an analogous situation in quantum mechanics where the overall phase degree of freedom associated with a given quantum state can be eliminated [12] to recover the quantum phase space \( \mathbb{CP}^n \).

Now, given the exponential states (3), we may wish to ask if there is any symmetry associated with the flow induced by changing the canonical parameters \( \theta^j \) in \( \mathbb{RP}^n \). This
line of enquiry has been investigated recently \cite{11}, and it was shown that the induced flow gives rise to a projective automorphism on the state manifold—a general transformation of the kind that projects geodesics onto geodesics. Furthermore, it has also been shown that the induced flow is a Hamiltonian gradient flow with respect to the natural ‘spherical’ metric on the state space. In other words, the field generated by the tangent vectors of the homogeneous coordinates $\zeta^a_j = \partial_j \psi^a$ is given by the gradient

$$\zeta^a_j = -\frac{1}{4} \nabla^a H_j,$$  \hspace{1cm} (4)

where $H_j$ is a globally defined function on $RP^n$, given by the expectation value $H^a_{j b} \psi^b \psi^a$, of the Hamiltonian operator $H_{j a b}$ in the state $\psi^a$, and $\nabla^a = g^{a b} \nabla_b$ denotes the gradient operator with respect to the homogeneous coordinates. This implies that if we consider the coordinate transformation induced by changing each of the coupling constants $\theta^j$ according to the differential equation (2), then in local coordinates the vector field $\zeta^a = \partial_j \psi^a$ for each given $j$, in local coordinates, satisfies the general equation for projective transformations:

$$\zeta^c_{i a b} + R^c_{i a d} \zeta^d = \delta^c_{(a} \phi_{b)} ,$$  \hspace{1cm} (5)

where $\phi_a$ is a covector given by $\phi_a = (\zeta^b_j)_{;a}$, $R^c_{i a d}$ is the curvature tensor, and the expression $X_{;a}$ denotes the covariant derivative of $X$ with respect to the natural spherical metric.

Having in mind the fairly general results noted above on the symmetry associated with the parameter development of the equilibrium states $\psi^a$, we would now like to specialise further and consider the flow induced by real-space renormalisation transformations. In particular, we now view the coupling constants $\theta^j(t)$ themselves as dependent on a single scale parameter $t$. As a consequence, we may view the gradient flow in (4) as a one–parameter family of flows, corresponding to each given scale. In this case, it is not difficult to show that the induced flow due to a change of scale is not quite a Hamiltonian gradient flow in an ordinary sense, and we require some appropriate modifications.

To develop this further, we simply apply the chain rule in the equation (2) for the parameter development of the state vector. Hence, if we write $\psi^a$ for $\partial \psi^a / \partial t$, we have

$$\psi^a = -\frac{1}{2} \phi^j H^a_{j b} \psi^b.$$  \hspace{1cm} \text{Furthermore, by noticing that } \nabla^a H_j = 2 \delta^a_{j b} \psi^b,$$  \hspace{1cm} (6)

we can write the defining equation for the integral curve induced by renormalisation transformations in the form

$$\frac{\partial \psi^a}{\partial t} = -\frac{1}{4} \nabla^a \Phi + \frac{1}{4} H_i \nabla^a \beta^i,$$

where the potential function $\Phi$ is given by the contraction $\Phi = \beta^i H_i$ of the Hamiltonian function with the RG $\beta$-functions, given by $\beta^i = \partial \theta^i / \partial t$. One observes that this integral curve does not in general correspond to a gradient flow of the form (4). However, we note that the potential function $\Phi$ is scale dependent, i.e. $\Phi = \Phi(t)$, and is a global function on $RP^n \times R_+$. If we define the restriction of this function to the state space at a constant but arbitrary scale $t = t_0$ as $\Phi_{t_0}$, then the generating vector field for the flow, restricted to the given constant scale, takes the form

$$\left. \frac{\partial \psi^a}{\partial t} \right|_{t_0} = -\frac{1}{4} \nabla^a \Phi_{t_0}.$$  \hspace{1cm} (7)
From this expression, we can make the following observation, namely, the flow induced by the renormalisation transformations on the state space manifold $R^p$ induces, at each given scale, a projective vector field. However, the integral curves associated with such a flow cannot be obtained from any global projective vector field, since the flow changes with respect to the given scale characterised by the variable $t$.

The situation just described is quite analogous to the mechanics of a time dependent Hamiltonian. In this case, the Hamiltonian ‘gradient’ flow gives rise to, at each instant of time, a Killing vector field. However, the integral curve along any given time evolution does not correspond to a global isometry of the manifold, since the Killing field changes in time.

The general property of the induced flow in the state space that we have observed is worth bearing in mind when we specialise to study the flow restricted to the parameter space itself. In particular, since the parameter space manifold is a subspace of the state-space, it is natural to consider how the above structure might naturally project down onto the parameter space. In order to address this question we now study an explicit example, namely, the one-dimensional Ising model in an applied magnetic field. In this case the space of coupling constants is two-dimensional, a submanifold of the possibly infinite dimensional state space, endowed with the natural Fisher-Rao metric.

### III. PARAMETER SPACE GEOMETRY FOR THE ISING CHAIN

Consider the dimensionless Hamiltonian for the one-dimensional Ising model, given by

$$ H = -K \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{N} \sigma_i. \tag{8} $$

The probability distribution for the energy $H$ is given, as usual, by the Gibbs measure $p(H) = \exp(-H)/Z$, where $Z = \text{Tr}[\exp(-H)]$ is the partition function. Here, the symbol $\text{Tr}$ denotes summation over the energy levels corresponding to all possible configurations.

The totality of the space of such probability distributions $\{p(H)\}$ can be mapped into a real Hilbert space $H$, as discussed earlier, by taking the square root, whereupon we obtain a state vector for the system $\psi^a(H)$, which satisfies the normalisation condition $g_{ab} \psi^a \psi^b = 1$, determined by the Hilbert space metric. This implies that $\psi^a$ is an element on the unit sphere $S$ in $H$. The relevant parameter space submanifold $M$ on the sphere is determined by $\psi^a(\theta)$ where $\{\theta^i\} = (K, h)$ are local parameters.

A Riemannian metric on $M$, induced by the spherical geometry of $S$, is given by the Fisher-Rao metric which, in local coordinates, is expressed as

$$ G_{ij} = 4g_{ab} \partial_i \psi^a \partial_j \psi^b, \tag{9} $$

where $\partial_i = \partial/\partial \theta^i$ denotes differentiation with respect to the coordinates $(\theta^1, \theta^2) = (K, h)$. In particular, when the distribution is of Gibbs type, the Fisher-Rao metric reduces to the form $G_{ij} = -\partial_i \partial_j \ln Z$, which does not explicitly involve the trace operation.

In the case of an $N$-spin Ising chain, the components of the Fisher-Rao metric can be calculated explicitly and are given by
which is always positive. If we observe that when the size of the system $N$ is below certain critical values $N_c(K,h)$, the scalar curvature is negative. This is illustrated in Fig. 1 where the scalar curvature is plotted against the system size $N$ for the parameter values $K = h = 1$. As $N$ increases the curvature tends asymptotically to the thermodynamic value $R$, while for $N < N_c$ ($N_c = 7$ in this case) the curvature is negative, indicating a radically different parameter space geometry.
FIG. 1. A plot of the Ricci scalar $R(N)$ at the point $K = h = 1$. The value in the thermodynamic limit $R \approx 2.30$ is approached as $N$ becomes large.

Nevertheless, since we are only interested in the limit of large $N$, and as this limit appears to be well defined, we shall generally work henceforth with the Riemannian structure obtained after taking thermodynamic limit.

We conclude this section with a brief discussion of the physical significance of the Ricci curvature. This is a natural question to address since the curvature can be expressed [8] in terms of a combination of higher order central moments of the Hamiltonian $H$, while geometrically it is an invariant quantity with respect to reparametrisation. The expression based on the central moments of $H$ allows a study of the scaling behaviour of $R$ in the vicinity of the critical points for those models exhibiting phase transitions, from which Ruppeiner [8] conjectured that the curvature is a measure of the correlation volume of the system (see also [7]).

In the present case we can compare $R$ directly with the correlation length $\xi$ of the Ising chain. If we write $\cot(2\phi) = e^{2K} \sinh(h)$, then the correlation function $C(r) = \langle \sigma_i \sigma_{i+r} \rangle - \langle \sigma_i \rangle \langle \sigma_{i+r} \rangle$ can be expressed, after taking the limit $N \to \infty$, in the simple form

$$C(r) = \sin^2(2\phi) \left(\frac{\lambda_+}{\lambda_-}\right)^r,$$

(14)

where $\lambda_{\pm} = e^{K}(c \pm \eta)$ are the eigenvalues of the transfer matrix $V$ discussed below in Section 4. Therefore, for the correlation length $\xi$ we find that

$$\xi = \frac{1}{\ln(\lambda_+/\lambda_-)} = -\ln \left(1 - \frac{2}{R}\right),$$

(15)

where we have used (13) to determine the exact relation between $\xi$ and the Ricci scalar $R$. Using standard techniques for inverting a power series, this relation may be expressed in the form,

$$\xi = \sum_{k=0}^{\infty} c_k \left(\frac{2}{R}\right)^k = \frac{R}{2} \left(1 - \frac{1}{R} - \frac{1}{3R^2} - O\left(\frac{1}{R^3}\right)\right),$$

(16)
where the coefficients $c_k$ are given by $c_k = - \sum_{p=1}^{k} \frac{1}{p+1} c_{k-p}$ with the initial value $c_0 = R/2$. From this relation we observe that near the critical point $\xi \sim R/2$ and thus $R/2$ provides a good quantitative measure of the correlation length. Such a connection breaks down, however, in any regime where $R \sim 1$.

Having discussed the Riemannian geometry associated with the parameter space of the 1D Ising model, we shall, in the next section, determine the exact diffeomorphism of this manifold induced by an infinitesimal rescaling of the lattice spacing. Subsequently, we shall then consider the symmetry properties of this mapping.

**IV. CALCULATION OF THE RG $\beta$–FUNCTIONS**

The standard approach to real-space renormalisation is implemented by means of a discrete scaling transformation, known as block-spin decimation. In the case of the 1D Ising chain this transformation can be performed exactly. However, such transformations are not advantageous in seeking differentiable structures on the parameter space, such as we are investigating here. Therefore, in order to extract a generating vector field induced by these transformations, an appropriate form of analytic continuation is required. In this section we present a relatively simple technique based on the transfer matrix for achieving this aim.

We consider first a discrete decimation of the $N$-spin system via the blocking of lattice sites by an integer factor $l$ at each stage of the decimation process. Thus, if we denote by $a$ the lattice spacing between the spins, then at each iteration the lattice size increases by a factor of $l$, that is, $a \to la$. In addition, we assume a periodical boundary condition on the lattice. The renormalised parameters $K'$ and $h'$, after rescaling, are determined implicitly by the relation

$$Z_{N/l}(K', h') = f Z_N(K, h),$$  \hspace{1cm} (17)

where $f$, a function of $K$ and $h$, is an overall scaling factor. Representing the partition function by $Z_N = \text{Tr}(V^N)$ in terms of the transfer matrix $V$, given by

$$V = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix},$$  \hspace{1cm} (18)

we can reexpress the RG relation (17) in terms of individual configurations as

$$V(K', h') = f^{l/N} V^l(K, h).$$  \hspace{1cm} (19)

In particular, we see that the following relations

$$e^{2h'} = \frac{(V^l)_{11}}{(V^l)_{22}}, \quad e^{4K'} = \frac{(V^l)_{11}(V^l)_{22}}{(V^l)_{12}^2}$$  \hspace{1cm} (20)

hold. Therefore, in order to extract the recursion relations we require the transfer matrix raised to an arbitrary power $l$. This is achieved by the use of a similarity transformation $S$ to diagonalise the transfer matrix $V$. Since the eigenvalues of the matrix $V$ are given by

$$\lambda_{\pm} = e^K (\cosh h \pm \eta),$$  \hspace{1cm} (21)
if we denote the diagonalisation of \( V \) by \( \Lambda \), i.e. \( V = S\Lambda S^{-1} \), we find
\[
V^l = S \left( \begin{array}{cc} \lambda^l_+ & 0 \\ 0 & \lambda^l_- \end{array} \right) S^{-1}.
\] (22)

Recalling the definition \( \cot(2\phi) = e^{2K} \sinh h \), this similarity transformation can be expressed in a simple form:
\[
S = \left( \begin{array}{cc} 1 & -\tan \phi \\ \tan \phi & 1 \end{array} \right).
\] (23)

Thus, from the expressions in (20), the recursion relations for the rescaled coupling constants can be obtained as
\[
e^{2h'} = \frac{(c + \eta)^l + (c - \eta)^l \tan^2 \phi}{(c - \eta)^l + (c + \eta)^l \tan^2 \phi},
\] (24)
\[
e^{4K'} = 1 + \frac{4e^{4K} \eta^2 (1 - e^{-4K})^l}{[(c + \eta)^l - (c - \eta)^l]^2}.
\] (25)

Here, for clarity, we recall the notation \( s \equiv \sinh h \) and \( c \equiv \cosh h \). One readily verifies that these expressions reproduce the standard recursion relations for 2-spin blocking if we set \( l = 2 \), and also the recursion relation for arbitrary \( l \), when \( h = 0 \), as discussed in [14].

In the present context we assume that we can analytically continue these results to arbitrary \( l \in \mathbb{R}_+ \). It is convenient therefore to proceed by setting \( l = 1 + \epsilon \), where \( \epsilon \ll 1 \). Then in expressions (24) and (23) the parameters now have a full lattice spacing dependence of the form \( \theta = \theta(a/a_0) \) and \( \theta' = \theta'(a/a_0 + \epsilon a/a_0) \) where \( \theta = (K, h) \) and \( a_0 \) is the short distance cutoff scale. We then perform a Taylor series expansion of the left hand side of (24) and (23) to \( O(\epsilon) \). The right hand side can be expanded to the same order using the standard relation \( x^\epsilon \approx 1 + \epsilon \ln x + O(\epsilon^2) \). Equating terms of \( O(\epsilon) \) we obtain the \( \beta \)-functions for the parameters \( K \) and \( h \). In particular, it is convenient to write the lattice spacing as \( a/a_0 = e^\epsilon \), so that the expressions \( \beta^\epsilon \equiv a \partial \theta'/\partial a \) take the form,
\[
\beta^K \equiv \frac{\partial K}{\partial t} = \frac{e^{-2K} \sinh 2K \cosh h}{2\eta} \ln \left( \frac{\cosh h - \eta}{\cosh h + \eta} \right)
\] (26)
\[
\beta^h \equiv \frac{\partial h}{\partial t} = \frac{e^{-2K} \sinh 2K \sinh h}{\eta} \ln \left( \frac{\cosh h + \eta}{\cosh h - \eta} \right).
\] (27)

Note that \( \beta^K \) and \( \beta^h \) have a simple proportionality relation \( \beta^h = -2 \tanh h \beta^K \). From these expressions one may readily verify various limiting cases. In particular, for arbitrary \( h \) there is a line of trivial fixed points at \( K = 0 \). In the limit \( h \to 0 \), the first component \( \beta^K \) reduces to the familiar form \( \beta^K_{h=0} = \frac{1}{2} \sinh 2K \ln(\tanh K) \), while if we work to \( O(h) \) we also find that \( \beta^h = -2h \beta^K_{h=0} + O(h^3) \), as discussed in [2].
With these generating functions at hand we can view the infinitesimal RG transformations as inducing the following diffeomorphism on the parameter space manifold,

$$\theta^i \rightarrow \theta^i + \epsilon \xi^i,$$

(28)

where $\epsilon$ is an infinitesimal parameter and the generating vector field $vec\xi$ is given by the RG $\beta$-functions,

$$\vec{\xi} = \beta^i \frac{\partial}{\partial \theta^i}.$$

(29)

The behaviour of this vector field is displayed in Fig. 2, indicating the flow of the RG trajectories away from the critical point at $K = \infty$ towards the line of stable high temperature fixed points at $K = 0$.

In the next section we shall study the geometrical properties of this vector field in various limiting cases. Before proceeding to do so, we note that certain physical features of the model contained in the vector field $\xi^i$ are only made explicit when the underlying Riemannian structure is taken into account. To this end, we consider the conventional definition of the fixed points, as identified with the zeros of the $\beta$-functions. This definition is not entirely satisfactory, since there are cases where the $\beta$-function does not vanish at the critical point, such as the example considered here. While the $\beta$-functions vanish at one of the fixed points, $K = 0$, they tend to constant values for any given value of $h$ when $K \rightarrow \infty$.

Clearly, if the underlying geometry were Euclidean, then this implies that $(K = \infty, h = 0)$ cannot be regarded as a proper fixed point. However, the point here is that the parameter space is a curved manifold endowed with the Fisher-Rao metric. Therefore, any fixed point of a given vector field generating a diffeomorphism on the manifold, in general, can be (cf. \[15\]) identified with the zeros of the ‘velocity function’ defined by

$$v = \sqrt{G_{ij} \beta^i \beta^j}.$$

(30)

The behaviour of the velocity function in the case of the 1D Ising model is illustrated in Fig. 3. It is clear that the proper velocity of the flow vanishes at both fixed points. An
alternative interpretation of this point is available in two dimensions where the existence of the Zamolodchikov $C$–function allows the $\beta$ function to be constructed as $\beta^i = G^{ij} \partial_j C$ near the fixed points [10]. When the renormalization group generates a gradient flow of this form [7], it is then clear that the fundamental covector $\beta_j = \partial_j C$ also encodes the correct fixed point structure of the flow.

![Diagram](image_url)

FIG. 3. A plot of the velocity function $v(K, h)$ over the parameter space of the one dimensional Ising model. The velocity of the flow vanishes for an arbitrary value of $h$ at the high temperature fixed points at $K = 0$, and also at the critical point given by $K = \infty$ and $h = 0$.

V. THE SYMMETRY OF RG DIFFEOMORPHISMS

As we have discussed in Section 2, the RG generating vector field in the projective state space may be interpreted as generating a projective automorphism of the fixed scale state-space manifold. On the other hand, if we consider the global flow induced by the RG, then there seems to be no generic symmetry associated with the flow that can be expressed in terms of conventional diffeomorphisms in Riemannian geometry. This fact then leads us to conjecture that such a symmetry should also be absent for RG flow viewed in its normal setting of the parameter space submanifold. In this case the restriction of the natural state-space metric to the parameter space can be shown [11] to coincide with the Fisher-Rao metric which, as we have discussed in Section 3, can be explicitly evaluated in local coordinates which are parameters of the Hamiltonian. The components of the generating vector field for the full scale-dependent flow then reduce to the $\beta$–functions for these parameters. Thus we are led to enquire whether the vector field $\xi^i$ is a projective vector field on the parameter space manifold. Indeed, this is also a natural question on general grounds from the standard theory of Riemannian geometry, since projective automorphisms are the most general diffeomorphisms preserving geodesics on a manifold, and include affine and Killing transformations as special cases.

In fact, as we shall observe below, by explicit analysis of a counter example—the one dimensional Ising model—the flow induced by the RG in general does not possess any standard global symmetry. In particular, the exact generating vector field for RG flow in
the 1D Ising model, $\xi^i$, is not a projective vector field. This is of course consistent with the observations in state space where there is an explicit correction term.

This statement is verified in the case of the 1D Ising model in a straightforward manner since for $\xi^i$, evaluated in Section 4, to be a projective vector field it must satisfy the relation (3),

$$\xi^k_{ij} + R^k_{jil}\xi^l = \delta^k_{ij}\phi_i,$$

for some covariant vector $\phi_i$. In two dimensions, the components of this equation are explicitly given by

$$\begin{align*}
\xi^1_{2,2} + G^{11}R_{1221}\xi^1 &= 0 \\
\xi^2_{1,1} + G^{12}R_{2112}\xi^2 &= 2\phi_1 \\
\xi^1_{1,2} + G^{12}R_{2121}\xi^1 &= 2\phi_2 \\
\xi^1_{2,1} + G^{11}R_{1212}\xi^2 &= \phi_2
\end{align*}$$

(32)

and by direct substitution of the expressions for the metric, curvature, and the vector field $\xi^i$, we find that the equation is not satisfied globally on the parameter space for any choice of $\phi_i$ for the geometry at finite $N$ and also in the thermodynamic limit. Thus $\xi^i$ is not a projective vector field on the parameter space manifold. Since we have knowledge of the exact geometric quantities, and the RG $\beta$–functions, in this case the 1D Ising model in an external field serves as a counter-example to any conjecture of the general validity of such a symmetry, as stated above.

It is interesting to note that as we are dealing in this case with a two-dimensional parameter space, this allows additional control through the fact that one may coordinatise the surface with a complex Riemann coordinate and consider general conformal transformations. However, as we wish to consider this model purely as a tractable example of higher dimensional (generically infinite dimensional) cases we shall avoid for the moment any aspects which appear particular to two dimensions.

Nevertheless, we can consider various limiting cases by restricting attention to local regions of the parameter space. The most relevant region is of course the vicinity of the critical point. Therefore, it would be convenient to linearise the vector field around $K = \infty$ and $h = 0$. However, as we notice from the relations (26) and (27), the generic dependence on the temperature parameter $T = 1/K$ is of the form $\exp(\pm 2/T)$, implying that $T = 0$ is an essential singularity. Thus, following standard practice for the study of critical exponents, it is convenient to introduce a new variable $\tau = \exp(-2K)$, and consider a linearisation of the RG transformation in the variables $(\tau, h)$ about $(0, 0)$. Assuming that $\tau$ and $h$ are of $O(\epsilon)$, and expanding equations (26) and (27) up to $O(\epsilon^4)$, we obtain

$$\beta^K = -\frac{1}{2} + \left(\frac{\tau^2}{3} - \frac{h^2}{6}\right) + \left(\frac{2\tau^2h^2}{15} + \frac{\tau^4}{15} + \frac{h^4}{90}\right) + O(\epsilon^5),$$

(33)

$$\beta^h = h - \left(\frac{2\tau^2h}{3}\right) + O(\epsilon^5).$$

(34)

Note that various geometric quantities, e.g., the curvature, remain singular at the critical point in these variables. However, in terms of $\tau$ and $h$ we can construct a Laurent expansion and consider the structure of the lowest order terms.
Rather than considering general projective automorphisms in the neighbourhood of the critical point, it is now convenient to analyse particular transformations which have had some attention in the literature. In particular, we consider the degree to which RG flow violates the geodesic equation, and whether there are any special flows which are exact geodesics of the metric. We also investigate the extent to which $\xi^i$ may be regarded as a conformal Killing vector near the critical point. Conformal Killing transformations, preserving the angles between vectors, are not a subset of projective automorphisms but one readily verifies that, for the 1D Ising model, the differential RG does not generate such a symmetry globally.

We first analyse the relationship between RG diffeomorphisms and geodesic flow on the parameter manifold. In fact, using the standard discrete decimation procedure, Dolan [5] suggested that the induced flow might be a geodesic along the line $h = 0$. Although an explicit solution for the geodesic equations has not been obtained, we can nevertheless determine in which regimes the vector field $\xi^i$ induced by the infinitesimal scale change satisfies the geodesic equation $\mathbf{\xi}^j \nabla_j \xi^i - A \xi^i = 0$, where $\nabla_j$ denotes the covariant derivative compatible with the underlying Fisher-Rao metric $G_{ij}$. The function $A$ in this expression vanishes only if the parameter $t = \ln a/a_0$ is an affine parameter.

Numerical analysis indicates that along the lines $h = 0$ and $K = 0$ the flow is an exact geodesic flow, and that the deviation becomes linear in $h$, as $K$ increases. Hence our results confirm the expectation in [5]. In order to see this explicitly, we expand the geodesic equations in $\tau$ and $h$ to obtain

$$\xi^j \nabla_j \xi^1 - A \xi^1 = 0 + \frac{2A - 1}{4} + \frac{h}{2} + O(\epsilon^2) ,$$  \hspace{1cm} (35)$$

$$\xi^j \nabla_j \xi^2 - A \xi^2 = 0 + (1 - A)h + O(\epsilon^2) .$$  \hspace{1cm} (36)$$

The common factor of $A$ is given by $A = 1/2 + O(\epsilon)$, and the deviation from geodesic flow is clearly observed to be linear in $h$ with a positive coefficient of $1/2$ for each component.

We turn now to an investigation of whether $\xi^i$ corresponds to a conformal Killing vector field in particular regions of the parameter space. Justification for the conjecture that $\xi^i$ may have such a structure at least locally on the manifold in the neighbourhood of the critical point is provided by the analysis of Diósi et al. [5] who have shown that one recovers the standard critical exponents of a system such as the 1D Ising model under the assumption that $\xi^i$ linearised around the critical point is a conformal Killing vector field. In other words, the components of the metric tensor $G_{ij}$ should satisfy $\mathcal{L}_\xi G - dG = 0$ where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the vector field $\xi$, and $d$ is the spacetime dimension.

As we have noted earlier, explicit evaluation of this equation in the present example indicates that the symmetry does not hold globally on the parameter space. However, $\xi^i$ is indeed a conformal Killing field to a good approximation near the critical point $K = \infty$, $h = 0$. Indeed significant deviations are only seen as one moves close to the high temperature fixed points where $K = 0$.

The approximate symmetry near the critical point may be shown more quantitatively by expanding the conformal Killing equation in a Laurent expansion in the variables $\tau$ and $h$. For example, the $(1, 1)$ and $(2, 2)$ components take the form

$$(\mathcal{L}_\xi G)_{11} - dG_{11} = 0 + \left(\frac{8(1 - d)}{h} - 8(2 - d)\frac{8(5 - d)h}{3} + O(h^2)\right)\tau^2 + O(\epsilon^4) .$$  \hspace{1cm} (37)$$
\[(\mathcal{L}_{\xi}G)_{22} - dG_{22} = 0 + \left(\frac{(1 - d)}{h^3} - \frac{2}{3h} \frac{(11 + 3d)h}{45} + O(h^2)\right) \tau^2 + O(\epsilon^4), \tag{38}\]

and thus \(\xi^i\) in fact corresponds to a Killing vector field up to \(O(\tau)\), and to a conformal Killing vector field with \(d = 1\) up to \(O(\tau^2)\) for the \((1,1)\) component and up to \(O(\tau^2/h)\) for the \((2,2)\) component. We note that this expansion is not a strict linearisation of the RG equations about the critical point, due to their singular structure. Nevertheless these results verify, in this example, the claim of Diósi et al. explicitly.

VI. DISCUSSION

We have presented an analysis of the Riemannian parameter space geometry of the 1D Ising model, and the characteristics of renormalisation group trajectories on this manifold. While the generator of this transformation is, up to higher order corrections, a conformal Killing vector field near the critical point, there is apparently no such interpretation globally on the manifold. This is consistent with the general structure observed for RG flow in the state space of the system.

It should be pointed out, however, that in other models of interest such symmetries may exist due to particular properties of the theory itself or the individual parameters. The question of the existence of other independent parameter space symmetries is also of importance as these may provide additional, generically nonperturbative, constraints on the \(\beta\)-functions of the theory. Discrete parameter space symmetries are of this kind, while in the present case we observe that there is a class of diffeomorphisms of the parameter space manifold which preserve the RG flow in the sense that, for the generator \(X\) of such a transformation, \(\mathcal{L}_X \eta = 0\) where \(\eta = \eta_i d\theta^i\) is the 1-form dual to the vector field \(\xi^i\). One may verify that \(\eta\) has the form \(\eta = (\eta_1, 0)\) and thus any diffeomorphism generated by a vector field of the form \(X = (0, x_2)\) will preserve the RG flow in the sense described above. Such a structure is clearly quite model dependent. Nevertheless, the presence of a commuting flow may well have some more general validity.

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