LIE-ADMISSIBLE INVARIANT ORIGIN OF
IRREVERSIBILITY FOR MATTER AND ANTIMATTER
AT THE CLASSICAL AND OPERATOR LEVELS

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Abstract

It was generally believed throughout the 20-th century that irreversibility is a purely
classical event without operator counterpart. However, a classical irreversible system
cannot be consistently decomposed into a finite number of reversible quantum particles
(and, vice versa), thus establishing that the origin of irreversibility is basically unknown
at the dawn of the 21-th century. To resolve this problem, we adopt the historical an-
alytic representation of irreversibility by Lagrange and Hamilton with external terms
in their analytic equations; we show that, when properly written, the brackets of the
time evolution characterize covering Lie-admissible algebras; we show that the for-
malism has a fully consistent operator counterpart given by the Lie-admissible branch
of hadronic mechanics; we identify catastrophic mathematical and physical inconsis-
tencies when irreversible formulations are treated with the conventional mathematics
used for reversible systems; and show that, when the dynamical equations are treated
with a novel irreversible mathematics, Lie-admissible formulations are fully consistent
because invariant at both the classical and operator level. The case of closed-isolated
systems verifying conventional total conservation laws, yet possessing an irreversible
structure, is treated via the Lie-isotopic subclass of Lie-admissible formulations. The
analysis is conducted for both matter and antimatter at the classical and operator
levels to prevent insidious inconsistences occurring for the sole study of matter or,
separately, antimatter, as known known for the problem of grand unifications.

1. INTRODUCTION

1.1 The Scientific Imbalance Caused by Irreversibility. As it is well known, rather
vast studies have been conducted in the 20-th century in attempting a reconciliation of
the manifest irreversibility in time of our macroscopic environment with the reversibility of
quantum mechanics, resulting in the rather widespread belief that irreversibility solely exists
at the macroscopic level because it ”disappears” at quantum mechanical level (see, e.g., the
reprint volume by Shoeber and vast literature quoted therein).

The above belief created a large scientific imbalance because irreversible systems were
treated with the mathematical and physical formulations developed for reversible systems.
Since these formulations are themselves reversible, the attempt in salvaging quantum me-
chanics vis a vis the irreversibility of physical reality caused serious limitations in virtually
all quantitative sciences.
The imbalance originated from the fact that all used formulations were of *Hamiltonian type* (i.e., the formulations are entirely characterized by the sole knowledge of the Hamiltonian), under the awareness that all known Hamiltonians are reversible because all known potentials (such as the Coulomb potential $V(r) = \frac{q_1 q_2}{r}$) are reversible.

The academic belief of the purely classical character of irreversibility was disproved in 1985 by Santilli [2] via the following theorem whose proof is instructive for serious scholars in the field

**THEOREM 1.1:** A classical irreversible system cannot be consistently reduced to a finite number of quantum particles all in reversible conditions and, vice versa, a finite ensemble of quantum particles all in reversible conditions cannot consistently reproduce an irreversible macroscopic system under the correspondence or other principles.

The implications of the above theorem are rather deep because the theorem establishes that, rather than adapting to quantum mechanics all possible physical events in the universe, it is necessary to seek a *covering of quantum mechanics* permitting a consistent treatment of elementary particles in irreversible conditions, where the notion of covering is intended to be such that, when irreversibility is removed, quantum mechanics is recovered identically and uniquely.

In the final analysis, the orbit of an electron in an atomic structure is indeed reversible in time, with consequential validity of quantum mechanics. However, the idea that the same electron has an equally reversible orbit when moving in the core of a star is repugnant to reason as well as scientific vigour because, e.g., said idea would imply that the electron has to orbit in the core of a star with a locally conserved angular momentum (from the basic rotational symmetry of all Hamiltonians), with consequential direct belief of the existence of the perpetual motion within physical media.

In short, contrary to a popular belief of the 20-th century, Theorem 1.1 establishes that *irreversibility originates at the most primitive levels of nature, that of elementary particles, and then propagates all the way to our macroscopic environment.*

In this paper, we complete studies in the origin of irreversibility initiated by the author during his graduate studies in physics at the University of Torino, Italy, in the late 1960s [7-9], and continued at Harvard University under DOE support in the late 1970s [6,11,12], which studies achieved mathematical and physical consistency only recently. The decades required by the completion of the research is an indication of the complexity of the problem.

In fact, it was relativity easy to identify since the early efforts irreversible generalizations of Heisenberg’s equations [7-9]. However, these generalized equations subsequently resulted to lack *invariance*, here intended as the capability by quantum mechanics of predicting the same numerical values for the same conditions but under the time evolution of the theory.

In turn, the achievement of invariance predictably required the laborious prior effort of identifying a *new irreversible mathematics*, namely, a new mathematics specifically constructed for the invariant treatment of irreversible systems while being a covering of conventional mathematics in the above indicated sense. Following the achievement of the new mathematics, the resolution of the scientific imbalance of the 20-th century caused by irreversibility was direct and immediate.

By looking in retrospective, rather than being demeaning for quantum mechanics, the search for its invariant irreversible covering has brought into full light the majestic axiomatic
consistency of quantum mechanics, and how difficult resulted to be the preservation of the same axiomatic consistency at the covering irreversible level.

In other words, only art masterpieces, such as Michelangelo’s La Pietà, are eternal. By comparison, quantitative sciences will never admit final theories because, no matter how majestic a given theory may appear at a given time, its surpassing by a broader theory for more complex physical conditions is only a matter of time.

As we shall see, the studies presented in this paper required several preceding contributions over an extended period of time, although their coordination for the invariant treatment of irreversibility is presented in this paper for the first time. A comprehensive presentation of these studies will appear in monographs [18c] following the publication of this paper.

1.2 The Forgotten Legacy of Newton, Lagrange and Hamilton. The scientific imbalance on irreversibility was created in the early part of the 20-th century when, to achieve compatibility with quantum mechanics and special relativity, the entire universe was reduced to potential forces and the analytic equations were “truncated” with the removal of the external terms.

In reality, Newton [3] did not propose his celebrated equations to be restricted to reversible forces derivable from a potential $F = \partial V/\partial r$, but proposed them for the most general possible irreversible systems,

$$m_a \times \frac{dv_{ka}}{dt} = F_{ka}(t, r, v), \quad k = 1, 2, 3; \quad a = 1, 2, \ldots, N,$$

where the conventional associative product of numbers, matrices, operators, etc., is denoted hereon with the symbol $\times$ so as to distinguish it from numerous other products needed later on.

Similarly, to be compatible with Newton’s equations, Lagrange [4] and Hamilton [5] decomposed Newton’s force into a potential and a nonpotential component, represented all potential forces with functions today known as the Lagrangian and the Hamiltonian, and represented nonpotential forces with external terms. Therefore the true Lagrange and Hamilton equations are given, respectively, by

$$\frac{d}{dt} \frac{\partial L(t, r, v)}{\partial v^k_a} - \frac{\partial L(t, r, v)}{\partial r^k_a} = F_{ka}(t, r, v), \quad (1.2a)$$

$$\frac{dr^k_a}{dt} = \frac{\partial H(t, r, p)}{\partial p_{ak}}, \quad \frac{dp_{ak}}{dt} = -\frac{\partial H(t, r, p)}{\partial r^k_a} + F_{ak}(t, r, p), \quad (1.2b)$$

$$L = \Sigma_a \frac{1}{2} \times m_a \times v^2_a - V(t, r, v), \quad H = \Sigma_a \frac{p^2_a}{2 \times m_a} + V(t, r, p), \quad (4.1.2c)$$

$$V = U(t, r)_{ak} \times v^k_a + U_o(t, r), \quad F(t, r, v) = F(t, r, p/m). \quad (1.2d)$$

The analytic equations used throughout the 20-th century, those without external terms, shall be referred to as the truncated Lagrange and Hamilton equations.

More recently, Santilli [6a] conducted comprehensive studies on the integrability conditions for the existence of a potential, or a Lagrangian, or a Hamiltonian, called conditions of variational selfadjointness. These study permit the rigorous decomposition of Newtonian
forces into a component that is variationally selfadjoint (SA) and a component that is not (NSA),

\[ m_a \times \frac{dv_{ka}}{dt} = F_{ka}^{SA}(t, r, v) + F_{ka}^{NSA}(t, r, v). \]  

Consequently, the true Lagrange and Hamilton equations can be more technically written

\[ \left[ \frac{d}{dt} \frac{\partial L(t, r, v)}{\partial v_k^a} - \frac{\partial L(t, r, v)}{\partial r_k^a} \right]^{SA} = F_{ak}^{NSA}(t, r, v), \]  

\[ \left[ \frac{dr_k^a}{dt} - \frac{\partial H(t, r, p)}{\partial p_{ak}} \right]^{SA} = 0, \quad \left[ \frac{dp_{ak}}{dt} + \frac{\partial H(t, r, p)}{\partial r_k^a} \right]^{SA} = F_{ak}^{NSA}(t, r, p). \]

The forgotten legacy of Newton, Lagrange and Hamilton is that irreversibility originates precisely in the truncated external terms, because all known potential-SA forces are reversible. The scientific imbalance of Section 1.1 is then due to the fact that no serious scientific study on irreversibility can be conducted with the truncated analytic equations and their operator counterpart, since these equations can only represent reversible systems.

Being born and educated in Italy, during his graduate studies the author had the opportunity of reading in the late 1960s the original works by Lagrange that were written in Torino (where this paper has been written) and mostly in Italian.

In this way, the author had the opportunity of verifying Lagrange’s analytic vision of representing irreversibility precisely via the external terms, due to the impossibility of representing all possible physical events via the sole use of the Lagrangian, since the latter was conceived for the representation of reversible and potential events. As the reader can verify, Hamilton had, independently, the same vision.

Consequently, the truncation of the analytic equations caused the impossibility of a credible treatment of irreversibility at the purely classical level. The lack of a credible treatment of irreversibility then propagated at the subsequent quantum mechanical and quantum field theoretical levels.

It then follows that quantum mechanics cannot possibly be used for serious studies on irreversibility because the discipline was constructed for the description of reversible quantized atomic orbits and not for irreversible systems.

While the validity of quantum mechanics for the arena of its original conception and construction is beyond scientific doubt, the assumption of quantum mechanics as the final operator theory for all conditions existing in the universe, such as orbits of particles in the core of a star, is outside the boundaries of serious science.

This establishes the need for the construction of a covering of quantum mechanics specifically conceived and constructed for quantitative treatments of irreversible systems.

1.3 Early Representations of Irreversible Systems. As it is well known, the brackets of the time evolution of the truncated analytic equations, the familiar Poisson brackets, characterize a Lie algebra, a feature that persists at the quantum level, thus establishing Lie algebras as the ultimate foundations of the physics of the 20-th century.

By contrast, the brackets of the time evolution of an observable \( A(r, p) \) in phase space according to the analytic equations with external terms,

\[ \frac{dA}{dt} = (A, H, F) = \frac{\partial A}{\partial r_k^a} \times \frac{\partial H}{\partial p_{ka}} - \frac{\partial H}{\partial r_k^a} \times \frac{\partial A}{\partial p_{ka}} + \frac{\partial A}{\partial r_k^a} \times F_{ka}, \]  

(1.5)
cannot characterize any algebra as commonly understood in mathematics, because the brackets violate the right associative and scalar laws.

Therefore, the presence of external terms in the analytic equations causes not only the loss of all Lie algebras in the study of irreversibility, but actually the loss of all possible algebras.

To resolve this problem, the author initiated a long scientific journey following the reading of Lagrange’s papers.

The original argument [7-9], still valid today, is to select analytic equations characterizing brackets in the time evolution that verify the following conditions:

(1) Said brackets must verify the right and left associative and scalar laws to characterize an algebra;

(2) Said brackets must not be invariant under time reversal as a necessary condition to represent irreversibility ab initio; and

(3) The underlying algebras must be a covering of Lie algebras as a necessary condition to characterize a covering of the truncated analytic equations, namely, as a condition for the selected representation of irreversibility to admit reversibility as a particular case.

Condition (1) requires that said brackets should be bilinear, e.g., of the form \((A, B)\) with basic algebraic axioms

\[
(n \times A, B) = n \times (A, B), \quad (A, m \times B) = m \times (A, B); \quad n, m \in C, \quad (1.6a)
\]

\[
(A \times B, C) = A \times (B, C), \quad (A, B \times C) = (A, B) \times C. \quad (1.6b)
\]

Condition (2) can be first realized by requiring that brackets \((A, B)\) are not totally antisymmetric as the conventional Poisson brackets,

\[
(A, B) \neq -(B, A), \quad (1.7)
\]

because time reversal is realized via the use of Hermitean conjugation.

Condition (3) implies that brackets \((A, B)\) characterize Lie-admissible algebras in the sense of Albert [10], with operator form resulting to be also Jordan-admissible according to the following

**DEFINITION 1.1:** A generally nonassociative algebra \(U\) with elements \(a, b, c, \ldots\) and abstract product \(ab\) is said to be Lie-admissible when the attached algebra \(U^-\) characterized by the same vector space \(U\) equipped with the product \([a, b] = ab - ba\) verifies the Lie axioms

\[
[a, b] = -(b, a), \quad (1.8a)
\]

\[
[[a, b], c] + [[b, c], a] + [[c, b], a] = 0. \quad (1.8b)
\]

Said generally nonassociative algebra \(U\) is said to be Jordan-admissible when the attached algebra \(U^+\) characterized by the vector space \(U\) equipped with the product \(\{a, b\} = ab + ba\) verifies the Jordan axioms

\[
\{a, b\} = \{b, a\}, \quad (1.9a)
\]

\[
\{\{a, b\}, a^2\} = \{a, \{b, a^2\}\}. \quad (1.9b)
\]
In essence, the condition of Lie-admissibility requires that the brackets \((A, B)\) contain a totally antisymmetric component with a residual totally symmetric part, thus admitting the decomposition
\[
(A, B) = [A, B]^* + \{A, B\}^*.
\]
where the * denotes expected generalizations of conventional brackets.

The antisymmetric brackets \([A, B]^*\) generally result to be Lie at both classical and operator levels. However, as illustrated below, for reasons yet unknown the symmetric brackets \(\{A, B\}^*\) always verify Jordan’s first axiom (1.9a), but they generally violate the second axiom (1.9b) in their classical form and verify it only in their operator form.

After identifying the above lines, Santilli [9] proposed in 1967 the following generalized analytic equations
\[
\frac{dr_k}{dt} = \alpha \times \frac{\partial H(t, r, p)}{\partial r_k}, \quad \frac{dp_k}{dt} = -\beta \times \frac{\partial H(t, r, p)}{\partial p_k},
\]
(1.11)
where \(\alpha\) and \(\beta\) are real non-null parameters) that are manifestly irreversible. The time evolution is then given by
\[
\frac{dA}{dt} = (A, H) = \alpha \times \frac{\partial A}{\partial r_k} \times \frac{\partial H}{\partial p_k} - \beta \times \frac{\partial H}{\partial r_k} \times \frac{\partial A}{\partial p_k},
\]
(1.12)
whose brackets are manifestly Lie-admissible because the attached antisymmetric brackets are proportional to the conventional Poisson brackets,
\[
[A, B]^* = (A, B) - (B, A) = (\alpha + \beta) \times \left( \frac{\partial A}{\partial r_k} \times \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial r_k} \times \frac{\partial A}{\partial p_k} \right).
\]
(1.13)
However, the attached symmetric brackets
\[
[A, B]^* = (A, B) + (B, A) = (\alpha - \beta) \times \left( \frac{\partial A}{\partial r_k} \times \frac{\partial H}{\partial p_k} + \frac{\partial H}{\partial r_k} \times \frac{\partial A}{\partial p_k} \right),
\]
(1.14)
do not characterize a Jordan algebra because they verify first axiom (1.9a) but violate the second axiom (1.9b) as the reader is encouraged to verify.

The above analytic equations characterize the time-rate of variation of the energy
\[
\frac{dH}{dt} = (\alpha - \beta) \times \frac{\partial H}{\partial r_k} \times \frac{\partial H}{\partial p_k}.
\]
(1.15)

Also in 1967, Santilli [7,8] proposed an operator counterpart of the preceding classical setting consisting in the first known Lie-admissible parametric generalization of Heisenberg’s equation in the following infinitesimal and finite forms
\[
i \times \frac{dA}{dt} = (A, B) = p \times A \times H - q \times H \times A = m \times (A \times B - B \times A) + n \times (A \times B + B \times A),
\]
(1.16a)
\[ A(t) = W(t) \times A(0) \times W^\dagger(t) = e^{i \times H \times q \times t} \times A(0) \times e^{-i \times p \times H}, \] (1.16b)

\[ W \times W^\dagger \neq I. \] (1.16c)

where \( p, q, p \pm q \) are non-null parameters, and

\[ m = p + q, \quad n = q - p, \] (1.17)

Brackets \((A, B)\) are manifestly Lie-admissible with attached antisymmetric part

\[ [A, B]^* = (A, B) - (B, A) = (p - q) \times [A, B]. \] (1.18)

The same brackets are also Jordan-admissible as interested readers are encouraged to verify, thus illustrating the peculiar occurrence whereby symmetric brackets that are apparently similar are Jordan admissible only at the operator level.

Despite this limitation, the Jordan-admissibility of the operator brackets establishes Jordan’s dream on the physical applications of his algebras, although the latter occur for irreversible systems. As a matter of fact, the above Jordan-admissibility establishes the impossibility for Jordan algebras to have applications at the purely quantum level, due to its purely reversible, thus Lie character.

The time evolution of Eqs. (1.16) is manifestly irreversible (for \( p \neq q \)) with nonconservation of the energy

\[ i \times \frac{dH}{dt} = (H, H) = (p - q) \times H \times H \neq 0. \] (1.19)

Subsequent to papers [7-9], Santilli realized that the above formulations are not invariant under their own time evolution because Eq. (1.16b) is manifestly nonunitary.

The application of nonunitary transforms to time evolution (1.16) then led to the proposal in memoir [11,12] of 1978 of the following Lie-admissible operator generalization of Heisenberg equations in the following infinitesimal and finite forms

\[ i \times \frac{dA}{dt} = A < H - H > A = A \times R \times H - H \times S \times A = (A, H)^*, \] (1.20a)

\[ A(t) = W(t) \times A(0) \times W^\dagger(t) = (e^{i \times H \times S \times t}) \times A(0) \times (e^{-i \times R \times H}), \quad W \times W^\dagger \neq I, \] (1.20b)

\[ R = R(t, r, p, \psi, \partial \psi, ...) = S^\dagger. \] (1.20c)

where \( R, S \) and \( R \pm S \) are now nonsingular operators (or matrices), and Eq. (1.20c) is a basic consistency condition explained later on.

Eqs. (1.20) are the fundamental equations of hadronic mechanics 12,18]. Their basic brackets are manifestly Lie-admissible and Jordan admissible with structure

\[ (A, B)^* = A < B - B > A = A \times R \times B - B \times S \times A = \]

\[ = (A \times T \times B - B \times T \times A) + (A \times Q \times B + B \times Q \times A), \] (1.21a)

\[ T = R + S, \quad T = S - R. \] (1.21b)

The generalized classical equations proposed in Refs. [11,12] jointly with Eqs. (1.20] are given in unified phase space notation by

\[ \frac{db^\mu}{dt} = S^{\mu \nu}(t, b) \times \frac{\partial H(b)}{\partial b^\nu}, \] (1.22a)
where the tensor $S^{\mu\nu}$ is Lie-admissible (but not jointly Jordan admissible), namely, the attached antisymmetric tensor
\[ \Omega^{\mu\nu} = S^{\mu\nu} - S^{\nu\mu}, \]
characterizes Birkhoff’s equations (see monograph [6b] for comprehensive studies), with solution
\[ S_{\mu\nu} = \frac{\partial R_{\nu}(t, b)}{\partial b_{\mu}}, \quad S_{\mu\nu} = [(S_{\alpha\beta})^{-1}]^{\mu\nu}. \]
The canonical particular case is recovered for the value $R = (R_{\mu}) = (p_k, 0)$ as the reader is encouraged to verify [6b,11].

It is easy to see that the application of a nonunitary transform to the parametric equations (1.16) leads to the operator equations (1.20) and that the application of additional nonunitary transforms preserves their Lie-admissible and Jordan-admissible characters.

Consequently, fundamental equations (1.20) are “directly universal” in the sense of admitting as particular cases all possible brackets characterizing an algebra (universality) without the use of the transformation theory (direct universality).

1.4 Catastrophic Inconsistencies of Early Irreversible Formulations. Despite their direct universality, Eqs. (1.20) are not invariant under their own time evolution and, consequently, are afflicted by catastrophic inconsistencies [29-37].

To clarify this crucial point, let us recall that the capability by quantum mechanics to predict the same numerical values under the same conditions at different times is given by the unitary character of Heisenberg’s time evolution for Hermitean Hamiltonians,
\[ A(t) = U(t) \times A(0) \times U^\dagger = (e^{i\times H\times t}) \times A(0) \times (e^{-i\times t\times H}), \]
\[ U \times U^\dagger = U^\dagger \times U = I, \quad H = H^\dagger, \]
that characterizes the following type of invariance of the Lie brackets,
\[ U \times [A, B] \times U^\dagger = A' \times B' - B' \times A' = [A', B'], \]
\[ A' = U \times A \times U^\dagger, \quad B' = U \times B \times U^\dagger. \]

namely, an invariance specifically referred to the preservation of the conventional associative product, $\times' \equiv \times$.

For the case of the broader Lie-admissible equations (1.20) the situation is different because the Hamiltonian remains Hermitean, but the time evolution is nonunitary, in which case the application of the time evolution leads to the new brackets
\[ W \times (A, B)^* \times W^\dagger = A' <' B' - B' >' A' = A' \times R' \times B' - B' \times S' \times S' \times A' = A' \times (W^{-1} \times R \times W^{-1}) \times B' - B' \times (W^{-1} \times R \times W^{-1}) \times A' = (A', B')^{*'}, \]
\[ W \times W^\dagger \neq I, \quad H = H^\dagger, \]
namely, the transformed product remains Lie-admissible due to its direct universality, but the product itself is altered, $R \rightarrow R' \neq R, S \rightarrow S' \neq S$. 

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As we shall see in Section 4, different values of the $R$ and $S$ operators characterize different physical systems. Consequently, the above change of values of the $R$ and $S$ operators caused by the time evolution of the theory cannot preserve numerical predictions in time.

It is instructive to verify that essentially the same occurrences hold for the classical Lie-admissible equations [11] due to the noncanonical character of the time evolution.

Without proof we, therefore, quote the following:

**THEOREM 1.2 [29]:** Whether Lie or Lie-admissible, all classical theories possessing noncanonical time evolutions are afflicted by catastrophic mathematical and physical inconsistencies.

**THEOREM 1.3 [29]:** All operator theories possessing a nonunitary time evolution formulated on conventional Hilbert spaces $\mathcal{H}$ over conventional fields of complex numbers $\mathbb{C}$ are afflicted by catastrophic mathematical and physical inconsistencies. In particular, said nonunitary theories
1) do not possess invariant units of time, space, energy, etc., thus lacking physically meaningful application to measurements;
2) do not conserve Hermiticity in time, thus lacking physically meaningful observables;
3) do not possess unique and invariant numerical predictions;
4) generally violate probability and causality laws; and
5) violate the basic axioms of Galileo’s and Einstein’s relativities.

A comprehensive presentation of the above inconsistency theorems is available in Chapter 1 of monograph [18c]. For the limited scope of this paper it is sufficient to indicate that the mathematical inconsistencies are rather serious. All mathematical formulations used in physics are based on fields that, in turn, are centrally dependent on the basic unit. However, noncanonical or nonunitary transformations do not preserve the basic unit by central assumption.

Hence, a given field at the initial time is no longer applicable at a subsequent time for all noncanonical or nonunitary theories, due to the lack of time invariance of the basic unit. In turn, the loss under the time evolution of the base field causes the catastrophic collapse of all mathematical formulations.

For example, the formulation of a noncanonical classical theory on the Euclidean space $\mathcal{E}$ over the field of Real numbers $\mathbb{R}$, or of a nonunitary operator theory on a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathbb{C}$ has no mathematical consistency, again, because of the lack of invariance of the basic unit of the fields (and, hence, of the field themselves) under the time evolution.

Conventional mathematics can at best be used for the treatment of noncanonical or nonunitary theories for the representation of systems at a fixed value of time without any possible dynamics. In this case no irreversibility can be consistently treated due to the need of a time evolution for its very manifestation.

The physical inconsistencies are equally catastrophic, as illustrated by occurrences 1) to 5) of Theorem 1.3 (see Ref. [29] for details). It is sufficient here to recall that the basic unit of the three-dimensional Euclidean space $I = \text{Diag.}(1,1,1)$ represents in an abstract dimensionless form the assumed measurement units, e.g., $I = \text{Diag.}(1\text{cm},1\text{cm},\text{cm})$. The
loss of the measurement units under the time evolution then illustrates the catastrophic character of the inconsistencies due, e.g., to the consequential lack of invariant numerical predictions.

Similarly, it is easy to prove that the condition of Hermiticity at the initial time,

$$\langle \phi \times H^\dagger \times |\psi\rangle \equiv \langle \phi \times (H \times |\psi\rangle), \quad H = H^\dagger,$$

is violated at subsequent times for theories with nonunitary time evolution when formulated on $\mathcal{H}$ over $\mathbb{C}$. This additional catastrophic inconsistency (known as Lopez’s lemma [31-32]), can be expressed by

$$[\langle \psi \times W^\dagger \times (W \times W^\dagger)^{-1} \times W \times H \times W^\dagger \times W|\psi\rangle =$$

$$= \langle \psi \times W^\dagger \times [(W \times H \times W^\dagger) \times (W \times W^\dagger)^{-1} \times W \times |\psi\rangle =$$

$$= ([\hat{\psi} \times T \times H_r^\dagger]) \times |\psi\rangle = \hat{\psi} \times (\hat{H} \times T \times |\psi\rangle), \quad (1.29a)$$

$$|\hat{\psi}\rangle = W \times |\psi\rangle, \quad T = (W \times W^\dagger)^{-1} = T^\dagger, \quad (1.29b)$$

$$H_r^\dagger = T^{-1} \times \hat{H} \times T \neq H. \quad (1.29c)$$

As a result, nonunitary theories treated with conventional mathematics do not admit physically meaningful observables.

Perhaps more insidious is the catastrophic inconsistency caused by the general violation of causality by theories with nonunitary time evolutions since, the verification of causality laws by quantum mechanics is deeply linked to the unitarity of its time evolution, as well known.

By no means the above catastrophic inconsistencies solely apply to Santilli’s early formulations of irreversibility. In fact, due to the ”direct universality” of theories (1.20), the same inconsistencies apply to a rather vast number of theories, such:

1) Dissipative nuclear theories [25] represented via an imaginary potential in non-Hermitean Hamiltonians,

$$H = H_0 = iV \neq H^\dagger \quad (1.30)$$

lose all algebras in the brackets of their time evolution (requiring a bilinear product) in favor of the triple system,

$$i \times dA/dt = A \times H - H^\dagger \times A = [A, H, H^\dagger] \quad (1.31)$$

This causes the loss of nuclear notions such as “protons and neutrons” as conventionally understood, e.g., because the definition of their spin mandates the presence of a consistent algebra in the brackets of the time evolution.

2) Statistical theories with an external collision term $C$ (see Ref. [38] and literature quoted therein) and equation of the density

$$i \frac{d\rho}{dt} = \rho \odot H = [\rho, H] + C, \quad H = H^\dagger, \quad (1.32)$$

violate the conditions for the product $\rho \odot H$ to characterize any algebra, as well as the existence of exponentiation to a finite transform, let alone violating the conditions of unitarity.
3) The so-called “q-deformations” of the Lie product (see, e.g., [39-44] and very large literature quoted therein)
\[ A \times B - q \times B \times A, \]  
where \( q \) is a non-null scalar, that are a trivial particular case of Santilli’s \((p,q)\)-deformations
(1.11) introduced in 1967 [7].

4) The so-called “k-deformations” [45-48] that are a relativistic version of the \( q \)-deformations, thus also being a particular case of general structures (1.4.42).

5) The so-called “star deformations” [49] of the associative product
\[ A \star B = A \times T \times B, \]  
where \( T \) is fixed, and related generalized Lie product
\[ A \star B - B \star A, \]  
are manifestly nonunitary and coincide with Santilli’s Lie-isotopic algebras introduced in 1978 [11,12].

6) Deformed creation-annihilation operators theories [50,51].

7) Nonunitary statistical theories [52].

8) Irreversible black holes dynamics [53] with Santilli’s Lie-admissible structure (1.20).

9) Noncanonical time theories [54-56].

10) Supersymmetric theories [57] with product
\[ (A, B) = [A, B] + \{A, B\} = 
= (A \times B - B \times A) + (A \times B + B \times A), \]  
are an evident particular case of Santilli’s Lie-admissible product (1.4.46) with \( T = W = I \).

11) String theories (see ref. [37] and literature quoted therein) generally have a noncanonical structure due to the inclusion of gravitation with additional catastrophic inconsistencies when including supersymmetries.

12) The so-called squeezed states theories [58,59] due to their manifest nonunitary character.

13) All quantum groups (see, e.g., refs. [60-62]) with a nonunitary structure.

14) Kac-Moody superalgebras [63] are also nonunitary and a particular case of Santilli’s Lie-admissible algebra (1.20) with \( T = I \) and \( Q \) a phase factor.

Numerous additional theories are also afflicted by the catastrophic inconsistencies of Theorem 1.3, such as quantum groups, quantum gravity, and other theories with nonunitary time evolution formulated on conventional Hilbert spaces over conventional fields the reader can easily identify in the literature.

All the above theories have a nonunitary structure formulated via conventional mathematics and, therefore, are afflicted by the catastrophic mathematical and physical inconsistencies of Theorem 1.3.

Additional generalized theories were attempted via the relaxation of the linear character of quantum mechanics [35]. These theories are essentially based on eigenvalue equations with the structure
\[ H(t, r, p, |\psi\rangle) \times |\psi\rangle = E \times |\psi\rangle, \]  
(1.37)
(i.e., $H$ depends on the wavefunction).

Even though mathematically intriguing and possessing a seemingly unitary time evolution, these theories also possess rather serious physical drawbacks, such as: they violate the superposition principle necessary for composite systems such as a hadron; they violate the fundamental Mackay imprimitivity theorem necessary for the applicability of Galileo’s and Einstein’s relativities and possess other drawbacks \cite{18b} so serious to prevent consistent applications.

Yet another type of broader theory is Weinberg’s nonlinear theory \cite{64} with brackets of the type

$$
A \odot B - B \odot A = \frac{\partial A}{\partial \psi} \times \frac{\partial B}{\partial \psi^\dagger} - \frac{\partial B}{\partial \psi} \times \frac{\partial A}{\partial \psi^\dagger},
$$

(1.38)

where the product $A \odot B$ is nonassociative.

This theory violates Okubo’s No-Quantization Theorem \cite{30}, prohibiting the use of nonassociative envelopes because of catastrophic physical consequences, such as the loss of equivalence between the Schrödinger and Heisenberg representations (the former remains associative, while the latter becomes nonassociative, thus resulting in inequivalence).

Weinberg’s theory also suffers from the absence of any unit at all, with consequential inability to apply the theory to measurements, the loss of exponentiation to a finite transform (lack of Poincaré-Birkhoff-Witt theorem), and other inconsistencies studied in Ref. \cite{11}.

These inconsistencies are not resolved by the adaptation of Weinberg’s theory proposed by Jordan \cite{65} as readers seriously interested in avoiding the publication of theories known to be inconsistent \textit{ab initio} are encouraged to verify.

In conclusion, by the late 1970’s Santilli had identified classical and operator generalized theories \cite{11,12} that were proved to be directly universal and include as trivial particular cases a plethora of simpler versions by various other authors.

However, all these theories subsequently resulted in being catastrophically inconsistent on mathematical and physical grounds because they are noninvariant under their time evolution when elaborated with conventional mathematics, thus mandating the search for a new mathematics capable of restoring the invariance of quantum mechanics.

1.5 Guide to the Research. The first need for a serious study in the problem considered is the identification of a new mathematics specifically constructed for the invariant treatment of irreversibility at the classical and operator levels.

As an example, classical Lie-admissible equatiions (1.15) and (1.22) are manifestly not derivable from a potential (because variationally nonselfadjoint \cite{6}), and, consequently, they do not admit a unique and ambiguous map into an operator form.

As a matter of fact, the lack of a universal representation of nonconservative forced via a variational principle is a main historical reason for the inability of the physics of the 20-th century to conduct quantitative studies on the origin of irreversibility.

As a result, a primary objective of the needed new mathematics is that of achieving a directly universal representation of all (sufficiently smooth) nonconservative and irreversible systems via a variational principle. The needed new mathematics will be presented in the next section. Our classical invariant treatment of irreversibility is presented in Section 3 and its operator counterpart is presented in Section 4.
The second need for a serious study in the origin of irreversibility is the inclusion of antimatter ab initio.

Note that all the preceding theories were solely intended for the representation of matter and are inapplicable to antimatter for numerous reasons. For instance, classical equations (1.15) and (1.22) permit no differentiation at all between neutral matter and antimatter stars. Assuming that said equations admit a sort of operator map, the operator image of a classical representation of antimatter with the sole change of the sign of the charge (as solely done in the 20-th century) would be given by a "particle" with the wrong sign of the charge and definitely not by a charge conjugated "antiparticle."

In any case, one of the historical scientific imbalances of the 20-th century has been the treatment of matter at all possible levels, from Newtonian mechanics to quantum field theory, while antimatter was treated at the sole level of second quantization. In turn this imbalance has prohibited the initiation of the study whether a far away galaxy or quasar is made up of matter of antimatter (since such a study requires a consistent classical gravitational theory of antimatter with null total charge, thus without the use of the charge for the characterization of antimatter).

In view of these and other insufficiencies, the author had to conduct a separate laborious construction of a new mathematics specifically conceived for the treatment of antiparticle at the classical level in such a way to yield operator images equivalent to charge conjugated states.

This new mathematics, today known as Santilli’s isodual mathematics, is based on the application of the isodual map, here generically expressed by

\[ Q(t, r, \psi, \partial \psi, ...) \rightarrow Q^d(t^d, r^d, \psi^d, \partial^d \psi^d, ...) = -Q^\dagger(-t^\dagger, -r^\dagger, -\psi^\dagger, -\partial^\dagger(-\psi^\dagger), ...) \]  

applied to the totality of the mathematical and physical formulations used for matter, resulting in this way in the novel isodual numbers, isodual fields, isodual spaces, isodual differential calculus, isodual functional analysis, isodual geometries, Lie-Santilli isodual theory, isodual symmetries, etc.

The classical and operator isodual theory of antimatter cannot possibly be reviewed in this paper to avoid a prohibitive length. We must, therefore, refer the reader to the existing literature, such as monographs [18,22] (see monographs [66-72] for independent studies) and vast literature quoted therein.

The reader should be alerted that the restriction of the studies on the origin of irreversibility solely to matter, as done in the 20-th century, is insufficient and actually insidious because, as now known for grand unifications [22], certain basic insufficiencies emerge only when the study of antimatter is included ab initio.

The third need for a serious study of irreversibility is the joint consideration of open, nonconservative and irreversible as well as closed, conservative and irreversible systems.

As a matter of fact, some of the most relevant irreversible systems can be considered as isolated from the rest of the universe at the classical level (such as the study of the structure of Jupiter) as well as at the operator level (such as the study of the structure of a star), in which case these systems verify all ten conventional total conservation laws, while being intrinsically irreversible.

In view of such a need, Santilli proposed his Lie-admissible formulations as a covering, not of conventional Lie formulation, but as a covering of the broader irreversible Lie-isotopic
formulations [11,12]. The latter were originally based on the following operator equations

\[ i \times \frac{dA}{dt} = A \times H - H \times A = \]

\[ = A \times T(t, r, \psi, \partial \psi, ...) \times H - H \times T(t, r, \psi, \partial \psi, ...) A = [A, H]^*, \quad (1.40a) \]

\[ A(t) = W(t) \times A(0) \times W^\dagger(t) = (e^{i \times H \times T \times t}) \times A(0) \times (e^{-i \times H \times T \times H}), \quad (1.40b) \]

\[ W \times W^\dagger \neq I, \quad T = T^\dagger > 0, \quad H = H^\dagger, \quad (1.40c) \]

with classical counterpart given by Birkhoff’s equations [6b]

\[ \frac{dA}{dt} = \frac{\partial A}{\partial b^\mu} \times \Omega_{\mu\nu}(t, b) \times \frac{\partial H(b)}{\partial b^\nu} = [A, H]^*. \quad (1.41) \]

In both cases the brackets of the time evolution \([A, H]^*\) are totally antisymmetric, thus permitting the conservation law of the energy

\[ i \times \frac{dH}{dt} = [H, H]^* \equiv 0, \quad (1.42) \]

as well as all other conventional conservation laws [18], under a full representation of irreversibility given by

\[ T(t, r, \psi, \partial \psi, ...) \neq T(-t, r, \psi, \partial \psi, ...). \quad (1.43) \]

In particular, the generalized brackets \([A, B]^*\) verify the Lie axioms, although they have a nontrivial generalized structure (e.g., due to the general noncommutativity of \(H\) and \(T\)) and they characterize a formulation, today known as Lie-Santilli isotheory [6b,11,18,22-72] that includes isoenveloping algebras, Lie-Santilli isoalgebras, Lie-Santilli isogroups, isorepresentation theory, isotransformation theory, etc.

We cannot possibly review in this paper the latter formulations and are regrettably forced to refer the reader to the existing literature (see, e.g., monographs [18] and vast contributions quoted therein). For the limited scope of this paper we merely indicate that closed irreversible conditions are generally obtained by restricting the Lie-admissible brackets to be antisymmetric, yet generalized form. As an illustration, this condition is reached for the operator case by requiring that the generally different \(R\) and \(S\) operators coincide, are Hermitean and actually positive definite (for certain topological conditions required by the underlying isotopology), \(R = S = T = T^\dagger > 0\).

In summary, in this paper we shall consider: 1) Conventional, closed and reversible systems of particles represented with conventional symbols such as \(t, r, H\), etc., conventional associative product \(\times\) and conventional Lie theory; 2) Closed irreversible systems of particles represented with \(\hat{\ }\) over conventional symbols, and the Lie-Santilli isotheory; 3) Open irreversible systems of particles moving forward in time represented with the upper index \(>\), and related Santilli’s Lie-admissible theory; 4) Open irreversible systems of particles moving backward in time represented with the upper index \(<\), and related Santilli’s Lie-admissible theory; and 5) The antiparticle counterpart of all preceding systems 1-4 represented with the superscript \(d\) denoting isoduality.

Needless to say, the polyhedric character of problem as well as the complexity of each individual aspect, are such that, despite the decades of studies by the author, studies the
origin of irreversibility are at their infancy because so much remains to be done. It is hoped that this paper will stimulate young minds of any age to contribute to one of the ultimate frontiers of knowledge.

2. ELEMENTS OF SANTILLI GENOMATHEMATICS AND ITS ISODUAL

2.1 Genounits, Genoproducts and their Isoduals

In the author’s view, there cannot be a truly new physical theory without a new mathematics, and there cannot be a truly new mathematics without new numbers. Therefore, the resolution of the problem of invariance for nonunitary theories required laborious efforts in the search of new numbers capable of representing irreversibility via their own axiomatic structure.

After failed attempts and a futile search in the mathematical literature, Santilli proposed in Refs. [11,12] of 1978 the construction of a new mathematics specifically conceived for the indicated task, whose number theory eventually reached mathematical maturity only in paper [13] of 1993, mathematical maturity for the new differential calculus in memoir [14] of 1996 and, finally, an invariant formulation of Lie-admissible formulations only in paper [15] of 1997.

The new Lie-admissible mathematics is today known as Santilli genomathematics [66-76], where the prefix “geno” suggested in the original proposal [11,12] is used in the Greek meaning of “inducting” new Axioms (as compared to the prefix “iso” of the preceding chapter denoting the preservation of the axioms).

The basic idea is to lift the isounits of the Lie-isotopic theory [18] into a form that is still nowhere singular, but non-Hermitean, thus implying the existence of two different generalized units, today called Santilli genounits for the description of matter, that are generally written

\[
\hat{I}^* = 1/\hat{T}^*, \quad \langle \hat{I} \rangle = 1/\langle \hat{T} \rangle, \quad (2.1a)
\]

\[
\hat{I}^* \neq \hat{I}, \quad \hat{I}^* = (\langle \hat{I} \rangle)^*, \quad (2.1b)
\]

with two additional isodual genounits for the description of antimatter [14]

\[
(\hat{I}^*)^d = -\langle \hat{I} \rangle^\dagger = -\langle \hat{I} \rangle = -1/\langle \hat{T} \rangle, \quad (\langle \hat{I} \rangle)^d = -\hat{I}^* = -1/\hat{T}^*. \quad (2.2)
\]

Jointly, all conventional and/or isotopic products \(A \times B\) among generic quantities (numbers, vector fields, operators, etc.) are lifted in such a form to admit the genounits as the correct left and right units at all levels, i.e.,

\[
A > B = A \times \hat{T}^* \times B, \quad A > \hat{I}^* = \hat{I}^* > A = A, \quad (2.3a)
\]

\[
A < B = A \times \hat{T} \times B, \quad A < \hat{I} = \hat{I} < A = A, \quad (2.3b)
\]

\[
A >^d B = A \times \hat{T}^d \times B, \quad A >^d \hat{I}^d = \hat{I}^d >^d A = A, \quad (2.3c)
\]

\[
A <^d B = A \times \hat{T}^d \times B, \quad A <^d \hat{I}^d = \hat{I}^d <^d A = A, \quad (2.3d)
\]

for all elements \(A, B\) of the set considered.

As we shall see in Section 3, the above basic assumptions permit the representation of irreversibility with the most primitive possible quantities, the basic units and related products.
In particular, genounits permit an invariant representation of the external forces in Lagrange’s and Hamilton’s equations. As such, they are generally dependent on time, coordinates, momenta, wavefunctions and any other needed variable, e.g., $\hat{I}^> = \hat{I}^>(i^>, r^>, p^>, \psi^>, \ldots)$.

The assumption of all ordered product to the right $>$ permits the representation of matter systems moving forward in time, the assumption of all ordered products to the left $<$ can represent matter systems moving backward in time, with corresponding antimatter systems represented by the respective isodual ordered products $>^d = - >^\dagger$ and $<^d = - <^\dagger$. Irreversibility is represented ab initio by the inequality $A > B \neq A < B$ for matter and $>^d \neq <^d$ for antimatter.

Note that the simpler isotopic cases are given by $\hat{I}^> = < \hat{I}^> = \hat{I}^*>^0$ for matter and $\hat{I}^d >^d = < \hat{I}^d = \hat{I}^d^d < 0$ for antimatter.

In conclusion, the reader should be aware that genomathematics consists of four branches, the forward and backward genomathematics for matter and their isoduals for antimatter, each paid being interconnected by time reversal, the two pairs being interconnected by isodual map (1.39) that, as it is well known [22], is equivalent to charge conjugation.

2.2. Genonumbers, Genofunctional Analysis and their Isoduals Genomathematics began to reach maturity with the discovery made, apparently for the first time in paper [13] of 1993, that the axioms of a field still hold under the ordering of all products to the right or, independently, to the left.

This unexpected property permitted the formulation of new numbers, that can be best introduced as a generalization of the isonumbers [18], although they can also be independently presented as follows:

**DEFINITION 2.1 [13]:** Let $F = F(a, +, \times)$ be a field of characteristic zero as per Definitions 2.2.1 and 3.2.1. Santilli’s forward genofields are rings $\hat{F}^> = \hat{F}(\hat{a}^>, +^>, \times^>)$ with: elements

$$\hat{a}^> = a \times \hat{I}^>,$$

where $a \in F$, $\hat{I}^> = 1/\hat{I}^>$ is a non singular non-Hermitean quantity (number, matrix or operator) generally outside $F$ and $\times$ is the ordinary product of $F$; the genosum $+^>$ coincides with the ordinary sum $+$,

$$\hat{a}^>+\hat{b}^> \equiv \hat{a}^> + \hat{b}^>, \ \forall \hat{a}^>, \hat{b}^> \in \hat{F}^>,$$

consequently, the additive forward genounit $\hat{0}^> \in \hat{F}^>$ coincides with the ordinary $0 \in F$; and the forward genoproduct $>$ is such that $\hat{I}^> >^\times a^> = \hat{I}^> \equiv a^>$, $\forall a^> \in \hat{F}^>.$

Santilli’s forward genofields verify the following properties:

1) For each element $\hat{a}^> \in \hat{F}^>$ there is an element $\hat{a}^>-^1>^>$, called forward genoinverse, for which

$$\hat{a}^> > \hat{a}^>-^1>^> = \hat{I}^>, \ \forall a^> \in \hat{F}^>.$$

2) The genosum is commutative

$$\hat{a}^>+\hat{b}^> = \hat{b}^>+\hat{a}^>,$$
and associative
\[
(a^+ b^+ c^+) = (a^+ b^+) c^+, \quad \forall a, b, c \in \hat{F};
\]  
(2.9)

3) The forward genoproduct is associative
\[
\hat{a}^+ (\hat{b}^+ \hat{c}^+) = (\hat{a}^+ \hat{b}^+) \hat{c}^+, \quad \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}^>;
\]  
(2.10)

but not necessarily commutative
\[
\hat{a}^+ \hat{b}^+ \neq \hat{b}^+ \hat{a}^+;
\]  
(2.11)

4) The set \( \hat{F}^> \) is closed under the genosum,
\[
\hat{a}^+ \hat{b}^+ = \hat{c}^+ \in \hat{F}^>;
\]  
(2.12)

the forward genoproduct,
\[
\hat{a}^+ \hat{b}^+ = \hat{c}^+ \in \hat{F}^>;
\]  
(2.13)

and right and left genodistributive compositions,
\[
\hat{a}^+ (\hat{b}^+ \hat{c}^+) = \hat{a}^+ \hat{b}^+ \hat{c}^+, \quad \forall \hat{a}, \hat{b}, \hat{c}, \hat{d} \in \hat{F}^>;
\]  
(2.14a)

\[
(\hat{a}^+ \hat{b}^+ \hat{c}^+) = \hat{a}^+ \hat{b}^+ \hat{c}^+, \quad \forall \hat{a}, \hat{b}, \hat{c}, \hat{d} \in \hat{F}^>;
\]  
(2.14b)

5) The set \( \hat{F}^> \) verifies the right and left genodistributive law
\[
\hat{a}^+ (\hat{b}^+ \hat{c}^+) = (\hat{a}^+ \hat{b}^+) \hat{c}^+, \quad \forall \hat{a}, \hat{b}, \hat{c}, \hat{d} \in \hat{F}^>.
\]  
(2.15)

In this way we have the forward genoreal numbers \( \hat{R}^> \), the forward genocomplex numbers \( \hat{C}^> \) and the forward genoquaternionic numbers \( \hat{Q}^C \) while the forward genooctonions \( \hat{O}^> \) can indeed be formulated but they do not constitute genofields [14].

The backward genofields and the isodual forward and backward genofields are defined accordingly. Santilli’s genofields are called of the first (second) kind when the genounit is (is not) an element of \( F \).

The basic axiom-preserving character of genofields is illustrated by the following:

**LEMMA 2.1 [13]:** Genofields of first and second kind are fields (namely, they verify all axioms of a field).

Note that the conventional product “2 multiplied by 3” is not necessarily equal to 6 because, for isodual numbers with unit –1 it is given by –6 [13].

The same product “2 multiplied by 3” is not necessarily equal to +6 or –6 because, for the case of isonumbers, it can also be equal to an arbitrary number, or a matrix or an integrodifferential operator depending on the assumed isounit [13].

In this section we point out that “2 multiplied by 3” can be ordered to the right or to the left, and the result is not only arbitrary, but yielding different numerical results for different orderings, \( 2 \times 3 \neq 2 < 3 \), all this by continuing to verify the axioms of a field per each order [13].
Once the forward and backward genofields have been identified, the various branches of genomathematics can be constructed via simple compatibility arguments.

For specific applications to irreversible processes there is first the need to construct the genofunctional analysis, studied in Refs. [6,18] that we cannot review here for brevity. The reader is however warned that any elaboration of irreversible processes via Lie-admissible formulations based on conventional or isotopic functional analysis leads to catastrophic inconsistencies because it would be the same as elaborating quantum mechanical calculations with genomathematics.

As an illustration, Theorems 1.5.1 and 1.5.2 of catastrophic inconsistencies are activated unless one uses the ordinary differential calculus is lifted, for ordinary motion in time of matter, into the following forward genodifferentials and genoderivatives

\[
\hat{d}^\times x = \hat{T}_x^\times \times dx, \quad \frac{\hat{\partial}^\times}{\hat{\partial}^\times x} = \hat{I}_x^\times \times \frac{\partial}{\partial x}, \quad \text{etc.} \tag{2.16}
\]

with corresponding backward and isodual expressions here ignored.

Similarly, all conventional functions and isofunctions, such as isosinus, isocosinus, isolog, etc., have to be lifted in the genoform

\[
\hat{f}^\times (\hat{x}^\times) = f(\hat{x}^\times) \times \hat{I}^\times, \tag{2.17}
\]

where one should note the necessity of the multiplication by the genounit as a condition for the result to be in \(\hat{R}^\times\), \(\hat{C}^\times\), or \(\hat{O}^\times\).

### 2.3. Genogeometries and Their Isoduals

Particularly intriguing are the genogeometries [16] (see also monographs [18] for detailed treatments). They are best characterized by a simple genotopy of the isogeometries, although they can be independently defined.

As an illustration, the Minkowski-Santilli forward genospace \(\hat{M}^\times(\hat{x}^\times, \hat{\eta}^\times, \hat{R}^\times)\) over the genoreal \(\hat{R}^\times\) is characterized by the following spacetime, genocoordinates, genometric and genoinvariant

\[
\hat{x}^\times = x \hat{I}^\times = \{x^\mu\} \times \hat{I}^\times, \quad \hat{\eta}^\times = \hat{T}^\times \times \eta, \quad \eta = \text{Diag.}(1,1,1,-1), \tag{4.18a}
\]

\[
\hat{x}^{\times2\times} = \hat{x}^{\times\mu} \times \hat{\eta}_{\mu\nu} \times \hat{x}^{\times\nu} = (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I}^\times, \tag{2.18b}
\]

where the first expression of the genoinvariant is on genospaces while the second is its projection in our spacetime.

Note that the minkowski-Santilli genospace has, in general, an explicit dependence on spacetime coordinates. Consequently, it is equipped with the entire formalism of the conventional Riemannian spaces covariant derivative, Christoffel’s symbols, Bianchi identity, etc. only lifted from the isotopic form of the preceding chapter into the genotopic form.

A most important feature is that genospaces permit, apparently for the first time in scientific history, the representation of irreversibility directly via the basic genometric. This is due to the fact that genometrics are nonsymmetric by conception, e.g.,

\[
\hat{\eta}_{\mu\nu} \neq \hat{\eta}_{\nu\mu}. \tag{2.19}
\]

Consequently, genotopies permit the lifting of conventional symmetric metrics into nonsymmetric forms,

\[
\eta_{\text{Symm}} \rightarrow \hat{\eta}_{\text{NonSymm}} \tag{2.20}
\]
Remarkably, nonsymmetric metrics bare indeed permitted by the axioms of conventional spaces as illustrated by the invariance

\[
(x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \equiv [x^\mu \times (\hat{T}^\times \times \eta_{\mu\nu}) \times x^\nu] \times T^{>\times -1} \equiv (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{T},
\]

where \(\hat{T}\) is assumed in this simple illustration to be a complex number.

Interested readers can then work out backward genogeometries and the isodual forward and backward genogeometries with their underlying genofunctional analysis.

This basic geometric feature was not discovered until recently because hidden where nobody looked for, in the basic unit. However, this basic geometric advance in the representation of irreversibility required the prior discovery of basically new numbers, Santilli’s genonumbers with nonsymmetric unit and ordered multiplication [14].

2.4. Santilli Lie-Admissible theory and its Isodual Particularly important for irreversibility is the lifting of Lie’s theory and Lie-Santilli’s isotheories permitted by genomathematics, first identified by Ref. [11] of 1978 (and then studied in various works, e.g., [6,18-22]) via the following genotopies:

(1) The forward and backward universal enveloping genoassociative algebra \(\hat{\xi}^>\), \(\langle\hat{\xi}\rangle\), with infinite-dimensional basis characterizing the Poincaré-Birkhoff-Witt-Santilli genotheorem

\[
\hat{\xi}^>: \hat{I}^>, \hat{X}_i, \hat{X}_i > \hat{X}_j, \hat{X}_i > \hat{X}_j > \hat{X}_k, \ldots, i \leq j \leq k,
\]

(2.22a)

\[
\langle\hat{\xi}\rangle: \hat{I}, \langle\hat{X}_i\rangle, \hat{X}_i < \hat{X}_j, \hat{X}_i < \hat{X}_j < \hat{X}_k, \ldots, i \leq j \leq k;
\]

(2.22b)

where the “hat” on the generators denotes their formulation on genospaces over genofields and their Hermiticity implies that \(\hat{X}^> = \langle\hat{X}\rangle = \langle\hat{X}\rangle\)

(2) The Lie-Santilli genoalgebras characterized by the universal, jointly Lie- and Jordan-admissible brackets,

\[
\langle\hat{L}\rangle: (\hat{X}_i;\hat{X}_j) = \hat{X}_i < \hat{X}_j - \hat{X}_j > \hat{X}_i = C_{i\bar{j}}^k \times \hat{X}_k,
\]

(2.23)

here formulated formulated in an invariant form (see below);

(3) The Lie-Santilli genotransformation groups

\[
\langle\hat{G}\rangle: \hat{A}(\hat{w}) = (\hat{e}^{i\hat{X}\hat{x} \times \hat{w}}) > \hat{A}(0) \langle\hat{e}^{-i\hat{X}\hat{x} \times \hat{w}}\hat{X}\rangle = (\hat{e}^{i\hat{X} \times \hat{T}^\times \times \hat{w}}) \times A(0) \times (\hat{e}^{-i\hat{X} \times \hat{T}^\times \times \hat{w}}\hat{X}),
\]

(2.24)

where \(\hat{w}^> \in \hat{R}^>\) are the genoparameters; the genorepresentation theory, etc.

2.5. Genosymmetries and Nonconservation Laws The implications of the Santilli Lie-admissible theory are significant mathematically and physically. On mathematical grounds, the Lie-Santilli genoalgebras are “directly universal” and include as particular cases all known algebras, such as Lie, Jordan, Flexible algebras, power associative algebras, quantum, algebras, supersymmetric algebras, Kac-Moody algebras, etc. (Section 1.5).

Moreover, when computed on the genobimodule

\[
\langle\hat{B}\rangle = \langle\hat{\xi} \times \hat{\xi}\rangle,
\]

(2.25)
Lie-admissible algebras verify all Lie axioms, while deviations from Lie algebras emerge only in their projection on the conventional bimodule

\[ \langle B \rangle = \langle \xi \times \hat{\xi} \rangle, \]  

(2.26)
of Lie’s theory (see Ref. [17] for the initiation of the genorepresentation theory of Lie-admissible algebras on bimodules).

This is due to the fact that the computation of the left action \( A \times \langle \hat{T} \times \hat{B} \rangle \) on \( \langle \hat{\xi} \rangle \) (that is, with respect to the genounit \( \langle \hat{I} \rangle = 1 / \langle \hat{T} \rangle \)) yields the same value as the computation of the conventional product \( A \times B \) on \( \langle \xi \rangle \) (that is, with respect to the trivial unit \( I \)), and the same occurs for the value of \( A \times \rangle \) on \( \langle \hat{\xi} \rangle \).

The above occurrences explain the reason the structure constant and the product in the r.h.s. of Eq. (4.2.23) are those of a conventional Lie algebra.

In this way, thanks to genomathematics, Lie algebras acquire a towering significance in view of the possibility of reducing all possible irreversible systems to primitive Lie axioms.

The physical implications of the Lie-Santilli genotheory are equally far reaching. In fact, Noether’s theorem on the reduction of reversible conservation laws to primitive Lie symmetries can be lifted to the reduction, this time, of irreversible nonconservation laws to primitive Lie-Santilli genosymmetries.

As a matter of fact, this reduction was the very first motivation for the construction of the genotheory in memoir [12] (see also monographs [6,18,19,20]). The reader can then foresee similar liftings of all remaining physical aspects treated via Lie algebras.

The construction of the isodual Lie-Santilli genotheory is an instructive exercise for readers interested in learning the new methods.

3. LIE-ADMISSIBLE CLASSICAL MECHANICS FOR MATTER AND ITS ISODUAL FOR ANTIMATTER

3.1. Fundamental Ordering Assumption on Irreversibility

Another reason for the inability during the 20-th century for in depth studies of irreversibility is the general belief that motion in time has only two directions, forward and backward (Eddington historical time arrows). In reality, motion in time admits four different forms, all essential for serious studies in irreversibility, given by: 1) motion forward to future time characterized by the forward genotime \( \hat{\xi} \); 2) motion backward to past time characterized by the backward genotime \( \langle \hat{t} \rangle \); 3) motion backward from future time characterized by the isodual forward genotime \( \hat{t}^d \); and 4) motion forward from past time characterized by the isodual backward genotime \( \langle \hat{t}^d \rangle \).

It is at this point where the necessity of both time reversal and isoduality appears in its full light. In fact, time reversal is only applicable to matter and, being represented with Hermitean conjugation, permits the transition from motion forward to motion backward in time, \( \hat{\xi} \rightarrow \langle \hat{t} \rangle = (\hat{\xi})^\dagger \). If used alone, time reversal cannot identify all four directions of motions. The only additional conjugation known to this author that is applicable at all levels of study and is equivalent to charge conjugation, is isoduality [22].

The additional discovery of two complementary orderings of the product and related units, with corresponding isoduals versions, individually preserving the abstract axioms of a field has truly fundamental implications for irreversibility, since it permits the axiomatically consistent and invariant representation of irreversibility via the most ultimate and primitive axioms, those on the product and related unit. We, therefore, have the following:
FUNDAMENTAL ORDERING ASSUMPTION ON IRREVERSIBILITY [15]: Dynamical equations for motion forward in time of matter (antimatter) systems are characterized by genoproducts to the right and related genounits (their isoduals), while dynamical equations for the motion backward in time of matter (antimatter) are characterized by genoproducts to the left and related genounits (their isoduals) under the condition that said genoproducts and genounits are interconnected by time reversal expressible for generic quantities \( A, B \) with the relation,

\[
(A > B)\dagger = (A > \hat{T} > B\dagger = B\dagger \times (\hat{T} >)\dagger \times A\dagger,
\]

namely,

\[
\hat{T} > = (<\hat{T})\dagger
\]

thus recovering the fundamental complementary conditions (4.1.17) or (4.2.2).

Unless otherwise specified, from now on physical and chemical expression for irreversible processes will have no meaning without the selection of one of the indicated two possible orderings.

3.2. Geno-Newtonian Equations and Their Isoduals

Recall that, for the case of isotopies, the basic Newtonian systems are given by those admitting nonconservative internal forces restricted by certain constraints to verify total conservation laws called closed non-Hamiltonian systems [6b,18].

For the case of the genotopies under consideration here, the basic Newtonian systems are the conventional nonconservative systems without subsidiary constraints, known as open non-Hamiltonian systems, with generic expression (1.3), in which case irreversibility is entirely characterized by nonselfadjoint forces, since all conservative forces are reversible.

As it is well known, the above equations are not derivable from any variational principle in the fixed frame of the observer [6], and this is the reason all conventional attempts for consistently quantizing nonconservative forces have failed for about one century. In turn, the lack of achievement of a consistent operator counterpart of nonconservative forces lead to the belief that they are illusory because they disappear at the particle level.

The studies presented in this paper have achieved the first and only physically consistent operator formulation of nonconservative forces known to the author. This goal was achieved by rewriting Newton’s equations (1.3) into an identical form derivable from a variational principle. Still in turn, the latter objective was solely permitted by the novel genomathematics.

It is appropriate to recall that Newton was forced to discover new mathematics, the differential calculus, prior to being able to formulated his celebrated equations. Therefore, readers should not be surprised at the need for the new genodifferential calculus as a condition to represent all nonconservative Newton’s systems from a variational principle.

Recall also from Section 3.1 that, contrary to popular beliefs, there exist four inequivalent directions of time. Consequently, time reversal alone cannot represent all these possible motions, and isoduality results to be the only known additional conjugation that, when combined with time reversal, can represent all possible time evolutions of both matter and antimatter.

The above setting implies the existence of four different new mechanics first formulated by Santilli in memoir [14] of 1996, and today known as Newton-Santilli genomechanics, namely:
A) *Forward genomechanics* for the representation of forward motion of matter systems;

B) *Backward genomechanics* for the representation of the time reversal image of matter systems;

C) *Isodual backward genomechanics* for the representation of motion backward in time of antimatter systems, and

D) *Isodual forward genomechanics* for the representation of time reversal antimatter systems.

These new mechanics are characterized by:

1) Four different times, *forward and backward genotimes for matter systems and the backward and forward isodual genotimes for antimatter systems*

\[ \hat{t}^> = t \times \hat{t}^>, \quad -\hat{t}^>, \quad \hat{t}^{>d}, \quad -\hat{t}^{>d}, \quad (3.3) \]

with (nowhere singular and non-Hermitean) *forward and backward time genounits and their isoduals* (Note that, to verify the condition of non-Hermiticity, the time genounits can be complex valued),

\[ \hat{I}_t^> = 1/\hat{T}_t^>, \quad -\hat{I}_t^>, \quad \hat{I}_t^{>d}, \quad -\hat{I}_t^{>d}, \quad (3.4) \]

2) The *forward and backward genocoordinates and their isoduals*

\[ \hat{x}^> = x \times \hat{I}_x^>, \quad -\hat{x}^>, \quad \hat{x}^{>d}, \quad -\hat{x}^{>d}, \quad (3.5) \]

with (nowhere singular non-Hermitean) *coordinate genounit*

\[ \hat{I}_x^> = 1/\hat{T}_x^>, \quad -\hat{I}_x^>, \quad \hat{I}_x^{>d}, \quad -\hat{I}_x^{>d}, \quad (3.6) \]

with *forward and backward coordinate genospace and their isoduals* \( \hat{S}_x^> \), etc., and related *forward coordinate genofield and their isoduals* \( \hat{R}_x^> \), etc.;

3) The *forward and backward genospeeds and their isoduals*

\[ \hat{v}^> = \dot{\hat{x}}^> / \dot{\hat{t}}^>, \quad -\hat{v}^>, \quad \hat{v}^{>d}, \quad -\hat{v}^{>d}, \quad (3.7) \]

with (nowhere singular and non-Hermitean) *speed genounit*

\[ \hat{I}_v^> = 1/\hat{T}_v^>, \quad -\hat{I}_v^>, \quad \hat{I}_v^{>d}, \quad -\hat{I}_v^{>d}, \quad (3.8) \]

with related *forward speed backward genospaces and their isoduals* \( \hat{S}_v^> \), etc., over *forward and backward speed genofields* \( \hat{R}_v^> \), etc.;

The above formalism then leads to the *forward genospace for matter systems*

\[ \hat{S}_\text{tot}^> = \hat{S}_t^> \times \hat{S}_x^> \times \hat{S}_v^>, \quad (3.9) \]

defined over the *it forward genofield*

\[ \hat{R}_\text{tot}^> = \hat{R}_t^> \times \hat{R}_x^> \times \hat{R}_v^>, \quad (3.10) \]

with *total forward genounit*

\[ \hat{I}_\text{tot}^> = \hat{I}_t^> \times \hat{I}_x^> \times \hat{I}_v^>, \quad (3.11) \]
and corresponding expressions for the remaining three spaces obtained via time reversal and isoduality.

The basic equations are given by:

I) The forward Newton-Santilli genoequations for matter systems \[14\], formulated via the genodifferential calculus,

\[
\hat{m}_a \hat{d} \hat{v}_a = -\hat{\partial} \hat{V} \hat{\partial} \hat{x}_a; \quad (3.12)
\]

II) The backward genoequations for matter systems that are characterized by time reversal of the preceding ones;

III) the backward isodual genoequations for antimatter systems that are characterized by the isodual map of the backward genoequations,

\[
\hat{m}_a \hat{d} \hat{v}_a = -\hat{\partial} \hat{V} \hat{\partial} \hat{x}_a; \quad (3.13)
\]

IV) the forward isodual genoequations for antimatter systems characterized by time reversal of the preceding isodual equations.

Newton-Santilli genoequations (4.3.12) are “directly universal” for the representation of all possible (well behaved) Eqs. (1.3) in the frame of the observer because they admit a multiple infinity of solution for any given nonselfadjoint force.

A simple representation occurs under the conditions assumed for simplicity,

\[N = \hat{I}_t = \hat{I}_v = 1,\]

for which Eqs. (3.12) can be explicitly written

\[
\hat{m} \hat{d} \hat{v} = m \times \frac{d \hat{v}}{dt} = m \times \frac{d \hat{v}}{dt} = m \times \frac{d (x \times \hat{I}_x)}{dt} = m \times \frac{d v}{dt} \times \hat{I}_x + m \times x \times \frac{d \hat{I}_x}{dt} = \hat{I}_x \times \frac{\partial V}{\partial x}, \quad (3.15)
\]

from which we obtain the genorepresentation

\[F^{NSA} = -m \times x \times \frac{1}{\hat{I}_x} \times \frac{d \hat{I}_x}{dt}, \quad (3.16)\]

that always admit solutions here left to the interested reader since in the next section we shall show a much simpler, universal, algebraic solution.

As one can see, in Newton’s equations the nonpotential forces are part of the applied force, while in the Newton-Santilli genoequations nonpotential forces are represented by the genounits, or, equivalently, by the genodifferential calculus, in a way essentially similar to the case of isotopies.

The main difference between iso- and geno-equations is that isounits are Hermitean, thus implying the equivalence of forward and backward motions, while genounits are non-Hermitean, thus implying irreversibility.
Note also that the topology underlying Newton’s equations is the conventional, Euclidean, local-differential topology which, as such, can only represent point particles.

By contrast, the topology underlying the Newton-Santilli genoequations is given by a genotopy of the isotopology studied in the preceding chapter, thus permitting for the representation of extended, nonspherical and deformable particles via forward genounits, e.g., of the type

\[
\hat{I}^\gamma = \text{Diag.}(n_1^2, n_2^2, n_3^2, n_4^2) \times \Gamma^\gamma(t, r, v, \ldots),
\]

where \(n_k^2, k = 1, 2, 3\) represents the semi-axes of an ellipsoid, \(n_4^2\) represents the density of the medium in which motion occurs (with more general nondiagonal realizations here omitted for simplicity), and \(\Gamma^\gamma\) constitutes a nonsymmetric matrix representing nonselfadjoint forces, namely, the contact interactions among extended constituents occurring for the motion forward in time.

### 3.3. Lie-Admissible Classical Genomechanics and its Isodual

In this section we show that, once rewritten in their identical genoform (4.3.12), Newton’s equations for nonconservative systems are indeed derivable from a variational principle, with analytic equations possessing a Lie-admissible structure and Hamilton-Jacobi equations suitable for the first known consistent and unique operator map studied in the next section.

The most effective setting to introduce real-valued non-symmetric genounits is in the 6\(N\)-dimensional forward genospace (genocotangent bundle) with local genocoordinates and their conjugates

\[
\hat{a}^\gamma_\mu = a^\rho \times \hat{I}^\gamma_1, \quad (\hat{a}^\gamma_\mu) = \left(\hat{a}^\gamma_\alpha \atop \hat{p}^\gamma_{\rho \alpha}\right)
\]

and

\[
\hat{R}^\gamma_\mu = R_\rho \times \hat{I}^\gamma_2, \quad (\hat{R}^\gamma_\mu) = (\hat{p}^\gamma_{\rho \alpha}, \hat{0}),
\]

\[
\hat{I}^\gamma_1 = 1/\hat{T}^\gamma_1 = (\hat{I}^\gamma_2)^T = (1/\hat{T}^\gamma_2)^T,
\]

\(k = 1, 2, 3; \; \alpha = 1, 2, \ldots, N; \; \mu, \; \rho = 1, 2, \ldots, 6N,\)

where the superscript \(T\) stands for transposed, and nowhere singular, real-valued and non-symmetric genometric and related invariant

\[
\hat{\delta}^\gamma = \hat{T}^\gamma_{16N \times 6N} \delta_{6N \times 6N} \times \delta_{6N \times 6N},
\]

\[
\hat{a}^\gamma_\mu > \hat{R}^\gamma_\mu = \hat{a}^\gamma_\rho \times \hat{T}^\gamma_1 \times \hat{R}^\gamma_\beta = a^\rho \times \hat{I}^\gamma_2 \times R_\beta.
\]

In this case we have the following genoaction principle [14]

\[
\hat{\delta}^\gamma \hat{A}^\gamma = \hat{\delta}^\gamma \int\left[\hat{R}^\gamma_\mu \hat{a}^\gamma_\mu - \hat{H}^\gamma \hat{d}^\gamma \hat{t}^\gamma\right] = \delta \int[R_\mu \times \hat{T}^\gamma_1(t, x, p, \ldots) \times d(a^\beta \times \hat{I}^\gamma_2^\beta) - H \times dt] = 0,
\]

where the second expression is the projection on conventional spaces over conventional fields and we have assumed for simplicity that the time genounit is 1.
It is easy to prove that the above genoprinciple characterizes the following forward Hamilton-Santilli genoequations, (originally proposed in Ref. [11] of 1978 with conventional mathematics and in Ref. [14] of 1996 with genomathematics (see also Refs. [18,19,20])

\[
\dot{\omega}^\mu_\nu > \frac{\dot{\hat{A}}^\nu >}{\dot{\hat{a}}^\mu >} - \frac{\dot{\hat{H}}^\nu >}{\dot{\hat{a}}^\mu >} = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \times \left( \begin{array}{c} \frac{dr}{dt} \\ \frac{dp}{dt} \end{array} \right) - \left( \begin{array}{cc} 1 & K \\ 0 & 1 \end{array} \right) \times \left( \begin{array}{c} \frac{\partial H}{\partial r} \\ \frac{\partial H}{\partial p} \end{array} \right) = 0,
\]

(3.22a)

\[
\dot{\omega}^\gamma = \left( \frac{\partial^\gamma R^\gamma_\nu}{\partial^\gamma \hat{a}^\nu >} - \frac{\partial^\gamma \hat{R}^\gamma_\nu}{\partial^\gamma \hat{a}^\nu >} \right) \times \hat{\gamma} = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \times \hat{\gamma},
\]

(3.22b)

\[
K = F^{NSA} / (\partial H / \partial p),
\]

(4.3.22c)

where one should note the “direct universality” of the simple algebraic solution (3.22c).

The time evolution of a quantity \( \hat{A}^\gamma > (\hat{a}^\gamma >) \) on the forward geno-phase-space can be written in terms of the following brackets

\[
\frac{\dot{\hat{A}}^\gamma >}{\dot{\hat{a}}^\gamma >} = (\hat{A}^\gamma >, \hat{H}^\gamma >) = \frac{\partial \hat{A}^\gamma >}{\partial \hat{a}^\gamma >} \times \frac{\partial \hat{H}^\gamma >}{\partial \hat{a}^\gamma >} = \partial \hat{A}^\gamma > \times S^{\mu\nu} \times \frac{\partial \hat{H}^\gamma >}{\partial \hat{a}^\gamma >} = \left( \frac{\partial \hat{A}^\gamma >}{\partial \hat{a}^\gamma >} \right) \times \left( \frac{\partial \hat{H}^\gamma >}{\partial \hat{a}^\gamma >} \right) + \frac{\partial \hat{A}^\gamma >}{\partial \hat{p}^\gamma_{ka}} \times F^{NSA}_{ka},
\]

(3.23a)

\[
S^{\gamma\mu\nu} = \omega^{\mu\rho} \times \hat{I}^{\gamma}_{\rho}, \omega^{\mu\nu} = (|\omega_{\alpha\beta}|)^{-1} \mu\nu,
\]

(3.23b)

where \( \omega^{\mu\nu} \) is the conventional Lie tensor and, consequently, \( S^{\mu\nu} \) is Lie-admissible in the sense of Albert [7].

As one can see, the important consequence of genomathematics and its genodifferential calculus is that of turning the triple system \( (A, H, F^{NSA}) \) of Eqs. (1.5) in the bilinear form \( (A; B) \), thus characterizing a consistent algebra in the brackets of the time evolution.

This is the central purpose for which genomathematics was built (note that the multiplicative factors represented by \( K \) are fixed for each given system). The invariance of such a formulation will be proved shortly.

It is an instructive exercise for interested readers to prove that the brackets \( (A; B) \) are Lie-admissible, although not Jordan-admissible.

It is easy to verify that the above identical reformulation of Hamilton’s historical time evolution correctly recovers the time rate of variations of physical quantities in general, and that of the energy in particular,

\[
\frac{dA^\gamma >}{dt} = (A^\gamma >, H^\gamma >) = [A^\gamma >, H^\gamma >] + \frac{\partial A^\gamma >}{\partial p^\gamma_{ka}} \times F^{NSA}_{ka},
\]

(3.24a)

\[
\frac{dH}{dt} = [H^\gamma >, H^\gamma >] + \frac{\partial H^\gamma >}{\partial p^\gamma_{ka}} \times F^{NSA}_{ka} = v^k_\alpha \times F^{NSA}_{ka}.
\]

(3.24b)
It is easy to show that genoaction principle (4.3.21) characterizes the following Hamilton-Jacobi-Santilli genoequations [14]

\[
\frac{\hat{\mathcal{J}}\gamma}{\partial \hat{t}} + \hat{H} = 0, \tag{3.25a}
\]

\[
\left(\frac{\hat{\mathcal{J}}\gamma}{\partial \hat{a}_{\mu}}\right) = \left(\frac{\hat{\mathcal{J}}\gamma}{\partial x^k}, \frac{\hat{\mathcal{J}}\gamma}{\partial p_k\gamma}\right) = (\hat{R}_{\mu}^\gamma) = (\hat{p}_{k\gamma}, \hat{0}), \tag{3.25b}
\]

which confirm the property (crucial for genoquantization as shown below) that the genoaction is indeed independent of the linear momentum.

Note the direct universality of the Lie-admissible equations for the representation of all infinitely possible Newton equations (1.3) (universality) directly in the fixed frame of the experimenter (direct universality).

Note also that, at the abstract, realization-free level, Hamilton-Santilli genoequations coincide with Hamilton’s equations without external terms, yet represent those with external terms.

The latter are reformulated via genomathematics as the only known way to achieve invariance and derivability from a variational principle while admitting a consistent algebra in the brackets of the time evolution [38].

Therefore, Hamilton-Santilli genoequations (3.6.66) are indeed irreversible for all possible reversible Hamiltonians, as desired. The origin of irreversibility rests in the contact nonpotential forces \(F_{NSA}\) according to Lagrange’s and Hamilton’s teaching that is merely reformulated in an invariant way.

The above Lie-admissible mechanics requires, for completeness, three additional formulations, the backward genomechanics for the description of matter moving backward in time, and the isoduals of both the forward and backward mechanics for the description of antimatter.

The construction of these additional mechanics is left to the interested reader for brevity.

4. LIE-ADMISSIBLE OPERATOR MECHANICS FOR MATTER AND ITS ISODUAL FOR ANTIMATTER

4.1. Basic Dynamical Equations

A simple genotopy of the naive or symplectic quantization applied to Eqs. (3.24) yields the Lie-admissible branch of hadronic mechanics [18] comprising four different formulations, the forward and backward genomechanics for matter and their isoduals for antimatter. The forward genomechanics for matter is characterized by the following main topics:

1) The nowhere singular (thus everywhere invertible) non-Hermitean forward genounit for the representation of all effects causing irreversibility, such as contact nonpotential interactions among extended particles, etc. (see the subsequent chapters for various realizations)

\[
\hat{I} = 1/\hat{T} \neq (\hat{I})^\dagger, \tag{4.1}
\]

with corresponding ordered product and genoreal \(\hat{R}\) and genocomplex \(\hat{C}\) genofields;

2) The forward genotopic Hilbert space \(\hat{H}\) with forward genostates \(|\hat{\psi}\rangle\) and forward genoinner product

\[
<\hat{\psi}|\hat{\psi}\rangle \times \hat{I} = <\hat{\psi}|\hat{T} \times |\hat{\psi}\rangle \times \hat{I} \in \hat{C}, \tag{4.2}
\]
and fundamental property
\[ \hat{I} > |\hat{\psi} > = |\hat{\psi} >, \quad (4.3) \]
holding under the condition that \( \hat{I} > \) is indeed the correct unit for motion forward in time, and forward genounitary transforms
\[ \hat{U} > (\langle \hat{U} \rangle^\dagger > = (\langle \hat{U} \rangle^\dagger > = \hat{U} = \hat{I} >; \quad (4.4) \]

3) The fundamental Lie-admissible equations, first proposed in Ref. [12] of 1974 (p. 783, Eqs. (4.18.16)) as the foundations of hadronic mechanics, formulated on conventional fields, and first formulated in Refs. [14,18] of 1996 on genospaces and genodifferential calculus on genofields, today’s known as Heisenberg-Santilli genoequations, that can be written in the finite form
\[ \hat{A}(\hat{t}) = \hat{U} > \hat{A}(0) <\hat{U} = (e^{i\hat{H}\hat{t}}) > \hat{A}(0) <(e^{-i\hat{H}\hat{t}}) = \]
\[ = (e^{i\hat{H}\hat{t}\times T^\times \hat{t}}) \times A(0) \times \times e^{i\hat{t}\times T^\times \hat{H}}, \quad (4.5) \]
with corresponding infinitesimal version
\[ i\times \frac{d\hat{A}}{dt} = (\hat{A},\hat{H}) = \hat{A} < \hat{H} - \hat{H} > \hat{A} = \]
\[ = \hat{A} < T(\hat{\hat{t}},\hat{\hat{r}},\hat{\hat{p}},\hat{\hat{\psi}},\ldots) \times \hat{H} - \hat{H} \times T^\times \hat{t}(\hat{\hat{t}},\hat{\hat{r}},\hat{\hat{p}},\hat{\hat{\psi}},\ldots) \times \hat{A}, \quad (4.6) \]
where there is no time arrow, since Heisenberg’s equations are computed at a fixed time.

4) The equivalent Schrödinger-Santilli genoequations, first suggested in the original proposal [12] to build hadronic mechanics (see also Refs. [17,23,24]), formulated via conventional mathematics and in Refs. [14,18] via genomathematics, that can be written
\[ \hat{H} > |\hat{\psi} > = \hat{H} > |\hat{\psi} > = \]
\[ = \hat{H}(\hat{\hat{t}},\hat{\hat{\psi}}) \times T^\times \hat{t}(\hat{\hat{t}},\hat{\hat{r}},\hat{\hat{p}},\hat{\hat{\psi}}) \times |\hat{\psi} > = E > |\psi >, \quad (4.7) \]
where the time orderings in the second term are ignored for simplicity of notation.

5) The forward genomomentum that escaped identification for two decades and was finally identified thanks to the genodifferential calculus in Ref. [14] of 1996
\[ \hat{p}_k > |\psi > = -i\hat{\psi} > = -i\hat{\hat{\psi}} > \times \hat{I} > |\hat{\psi} >, \quad (4.8) \]

6) The fundamental genocommutation rules also first identified in Ref. [14],
\[ (\hat{r}^i;\hat{p}_j) = i \times \hat{\delta}_j^i \times \hat{I} >, \quad (\hat{r}^i;\hat{r}^j) = (\hat{p}_i;\hat{p}_j) = 0, \quad (4.9) \]

7) The genoexpectation values of an observable for the forward motion \( \hat{A} > [14,19]
\[ \langle \langle \hat{\psi} | \hat{A} > |\psi > \times \hat{I} > \in \hat{C} >, \quad (4.10) \]
under which the genoexpectation values of the genounit recovers the conventional Planck’s unit as in the isotopic case,

\[
\frac{\langle \hat{\psi} | \hat{I} > | \hat{\psi} \rangle}{\langle \hat{\psi} | | \hat{\psi} \rangle} = I.
\] (4.11)

The following comments are now in order. Note first in the genoaction principle the crucial independence of isoaction \( \hat{A} \) in form the linear momentum, as expressed by the Hamilton-Jacobi-Santilli genoequations (4.3.25). Such independence assures that genoquantization yields a genowavefunction solely dependent on time and coordinates, \( \hat{\psi} > = \hat{\psi} > (t, r) \).

Other geno-Hamiltonian mechanics studied previously [7] do not verify such a condition, thus implying genowavefunctions with an explicit dependence also on linear momenta, \( \hat{\psi} > = \hat{\psi} > (t, r, p) \) that violate the abstract identity of quantum and hadronic mechanics whose treatment in any case is beyond our operator knowledge at this writing.

Note that forward geno-Hermiticity coincides with conventional Hermiticity. As a result, all quantities that are observables for quantum mechanics remain observables for the above genomechanics.

However, unlike quantum mechanics, physical quantities are generally nonconserved, as it must be the case for the energy,

\[
\hat{i} > \frac{d > \hat{H} >}{d > \hat{t} >} = \hat{H} \times \langle \hat{T} - \hat{T} > \rangle \times \hat{H} \neq 0.
\] (4.12)

Therefore, the genotopic branch of hadronic mechanics is the only known operator formulation permitting nonconserved quantities to be Hermitian as a necessary condition to be observability.

Other formulation attempt to represent nonconservation, e.g., by adding an “imaginary potential” to the Hamiltonian, as it is often done in nuclear physics [25]. In this case the Hamiltonian is non-Hermitian and, consequently, the nonconservation of the energy cannot be an observable.

Besides, said “nonconservative models” with non-Hermitian Hamiltonians are nonunitary and are formulated on conventional spaces over conventional fields, thus suffering all the catastrophic inconsistencies of Theorem 1.3.

We should stress the representation of irreversibility and nonconservation beginning with the most primitive quantity, the unit and related product. Closed irreversible systems are characterized by the Lie-isotopic subcase in which

\[
\hat{i} \times \frac{d \hat{A}}{dt} = [\hat{A}; \hat{H}] = \hat{A} \times \hat{T}(t, \ldots) \times \hat{H} - \hat{H} \times \hat{T}(t, \ldots) \times \hat{A}, \quad 4.13a
\]

\[
\langle \hat{T}(t, \ldots) \rangle = \hat{T} > (t, \ldots) = \hat{T} (t, \ldots) = \hat{T} (t, \ldots) \neq \hat{T} (-t, \ldots), \quad (4.13b)
\]

for which the Hamiltonian is manifestly conserved. Nevertheless the system is manifestly irreversible. Note also the first and only known observability of the Hamiltonian (due to its iso-Hermiticity) under irreversibility.

As one can see, brackets \((A, B)\) of Eqs. (4.6) are jointly Lie- and Jordan-admissible.

Note also that finite genotransforms (4.4.5) verify the condition of genohermiticity, Eq. (4.4).
We should finally mention that, as it was the case for isotheories, genotheories are also admitted by the abstract axioms of quantum mechanics, thus providing a broader realization. This can be seen, e.g., from the invariance under a complex number $C$

$$<\psi|x|\psi> \times I = <\psi|xC^{-1} \times |\psi> \times (C \times I) = <\psi> \times |\psi> \times I^\dagger.$$  \hspace{1cm} (4.14)

Consequently, genomechanics provide another explicit and concrete realization of “hidden variables” [26], thus constituting another “completion” of quantum mechanics in the E-P-R sense [27]. For the studies of these aspects we refer the interested reader to Ref. [28].

The above formulation must be completed with three additional Lie-admissible formulations, the backward formulation for matter under time reversal and the two additional isodual formulations for antimatter. Their study is left to the interested reader for brevity.

### 4.2. Simple Construction of Lie-Admissible Theories

As it was the case for the isotopies, a simple method has been identified in Ref. [44] for the construction of Lie-admissible (geno-) theories from any given conventional, classical or quantum formulation. It consists in identifying the genounits as the product of two different non unitary transforms,

$$\hat{I}^\dagger = (<\hat{I})^\dagger = U \times W^\dagger, \quad <\hat{I} = W \times U^\dagger,$$

(4.15a)

$$U \times U^\dagger \neq 1, \quad W \times W^\dagger \neq 1, \quad U \times W^\dagger = \hat{I}^\dagger,$$

(4.15b)

and subjecting the totality of quantities and their operations of conventional models to said dual transforms,

$$I \rightarrow \hat{I}^\dagger = U \times I \times W^\dagger, \quad I \rightarrow <\hat{I} = W \times I \times U^\dagger;$$

(4.16a)

$$a \rightarrow \hat{a}^\dagger = U \times a \times W^\dagger = a \times \hat{I}^\dagger,$$

(4.16b)

$$a \rightarrow <\hat{a} = W \times a \times U^\dagger = <\hat{I} \times a,$$

(4.16c)

$$a \times b \rightarrow \hat{a}^\dagger \times \hat{b}^\dagger = U \times (a \times b) \times W^\dagger =$$

$$= (U \times a \times W^\dagger) \times (U \times W^\dagger)^{-1} \times (U \times b \times W^\dagger),$$

(4.16d)

$$\partial/\partial x \rightarrow \hat{\partial}/\hat{\partial}^\dagger \hat{x}^\dagger = U \times (\partial/\partial x) \times W^\dagger = \hat{I}^\dagger \times (\partial/\partial x),$$

(4.16e)

$$<\psi]|\psi> \rightarrow <\hat{\psi}|\hat{\psi}>= U \times (<\psi]|\psi>) \times W^\dagger,$$

(4.16f)

$$H \times |\psi> \rightarrow \hat{H}^\dagger > |\psi^\dagger> =$$

$$= (U \times H \times W^\dagger) \times (U \times W^\dagger)^{-1} \times (U \times \psi > W^\dagger), \text{ etc.}$$

(4.16g)

As a result, any given conventional, classical or quantum model can be easily lifted into the genotopic form.

Note that the above construction implies that all conventional physical quantities acquire a well defined direction of time. For instance, the correct genotopic formulation of energy, linear momentum, etc., is given by

$$\hat{H}^\dagger = U \times H \times W^\dagger, \quad \hat{p}^\dagger = U \times p \times W^\dagger, \text{ etc.}$$

(4.17)

In fact, under irreversibility, the value of a nonconserved energy at a given time $t$ for motion forward in time is generally different than the corresponding value of the energy for $-t$ for motion backward in past times.
This explains the reason for having represented in this section energy, momentum and other quantities with their arrow of time $>$. Such an arrow can indeed be omitted for notational simplicity, but only after the understanding of its existence.

Note finally that a conventional, one dimensional, unitary Lie transformation group with Hermitean generator $X$ and parameter $w$ can be transformed into a covering Lie-admissible group via the following nonunitary transform

$$Q(w) \times Q^\dagger(w) = Q^\dagger(w) \times Q(w) = I, \ w \in R,$$  \hspace{1cm} (4.18a)

$$U \times U^\dagger \neq I, \ W \times W^\dagger \neq 1,$$ \hspace{1cm} (4.18b)

$$A(w) = Q(w) \times A(0) \times Q^\dagger(w) = e^{X \times w \times i} \times A(0) \times e^{-i \times w \times X} \rightarrow$$

$$\rightarrow U \times (e^{X \times w \times i} \times A(0) \times e^{-i \times w \times X}) \times U^\dagger =$$

$$= [U \times (e^{X \times w \times i}) \times W^\dagger \times (U \times W^\dagger)^{-1} \times A \times A(0) \times$$

$$\times U^\dagger \times (W \times U^\dagger)^{-1} \times [W \times (e^{-i \times w \times X}) \times U^\dagger] =$$

$$= (e^{i \times X \times X}) > A(0) < (e^{-1 \times w \times X}) = \hat{U} > A(0) < \hat{U},$$ \hspace{1cm} (4.18c)

which confirm the property of Section 4.2, namely, that under the necessary mathematics the Lie-admissible theory is indeed admitted by the abstract Lie axioms, and it is a realization of the latter broader than the isotopic form.

4.3. Invariance of Lie-Admissible Theories Recall that a fundamental axiomatic feature of quantum mechanics is the invariance under time evolution of all numerical predictions and physical laws, which invariance is due to the unitary structure of the theory.

However, quantum mechanics is reversible and can only represent in a scientific way beyond academic beliefs reversible systems verifying total conservation laws due to the antisymmetric character of the brackets of the time evolution.

As indicated earlier, the representation of irreversibility and nonconservation requires theories with a nonunitary structure. However, the latter are afflicted by the catastrophic inconsistencies of Theorem 1.3.

The only resolution of such a basic impasse known to the author has been the achievement of invariance under nonunitarity and irreversibility via the use of genomathematics, provided that such genomathematics is applied to the totality of the formalism to avoid evident inconsistencies caused by mixing different mathematics for the selected physical problem.

Let us note that, due to decades of protracted use it is easy to predict that physicists and mathematicians may be tempted to treat the Lie-admissible branch of hadronic mechanics with conventional mathematics, whether in part or in full. Such a posture would be equivalent, for instance, to the elaboration of the spectral emission of the hydrogen atom with the genodifferential calculus, resulting in an evident nonscientific setting.

Such an invariance was first achieved by Santilli in Ref. [15] of 1997 and can be illustrated by reformulating any given nonunitary transform in the genounitary form

$$U = \hat{U} \times \hat{T}^{>1/2}, \ W = \hat{W} \times \hat{T}^{>1/2},$$ \hspace{1cm} (4.19a)

$$U \times W^\dagger = \hat{U} \times \hat{W}^\dagger = \hat{W}^\dagger \times \hat{U} = \hat{T}^{>1/2} = 1/\hat{T}^{>},$$ \hspace{1cm} (4.19b)
and then showing that genounits, genoproducts, genoexponentiation, etc., are indeed invariant under the above genounitary transform in exactly the same way as conventional units, products, exponentiations, etc. are invariant under unitary transforms,

\[
\hat{I}^> \rightarrow \hat{I}'^> = \hat{U} \hat{I}^> \hat{W}^\dagger = \hat{I}^>,
\]

(4.20a)
\[
\hat{A} \hat{B} \rightarrow \hat{U} (A \hat{B}) \hat{W}^\dagger = \hat{A}' \hat{B}',
\]

(4.20b)

from which all remaining invariances follow, thus resolving the catastrophic inconsistencies of Theorem 1.3.

Note the numerical invariances of the genounit \(\hat{I}^> \rightarrow \hat{I}'^> \equiv \hat{I}^>\), of the genotopic element \(\hat{T}^> \rightarrow \hat{T}'^> \equiv \hat{T}^>\), and of the genoproduct \(\hat{A} \hat{B} \rightarrow \hat{A}' \hat{B}' \equiv \hat{A} \hat{B}'\) that are necessary to have invariant numerical predictions.

5. SOCIETAL AND SCIENTIFIC IMPLICATIONS

The societal, let alone scientific implications of the proper treatment of irreversibility are rather serious. Our planet is afflicted by increasingly catastrophic climactic events mandating the search for basically new, environmentally acceptable energies and fuels.

All known energy sources, from the combustion of carbon dating to prehistoric times to the contemporary nuclear energy, are based on structurally irreversible processes. By comparison, all established doctrines of the 20-th century, such as quantum mechanics and special relativity, are structurally reversible, that is, reversible in their basic axioms, let alone their physical laws.

It is then easy to see that the serious search for basically new energies and fuels will require basically new theories that are as structurally irreversible as the process they are expected to describe. At any rate, all possible energies and fuels that could be predicted by quantum mechanics and special relativity were discovered by the middle of the 20-th century. Hence, the insistence in continuing to restrict irreversible processes to comply with preferred reversible doctrines may perhaps yield myopic short term benefits, but also cause a potentially historical condemnation by posterity.

An effective way to illustrate the need for new irreversible theories is given by nuclear fusions. All efforts to date in the field, whether for the ”cold fusion” or the ”hot fusion,” have been studiously restricted to verify special relativity and relativistic quantum mechanics. However, whether ”hot” or ”cold,” all fusion processes are strictly irreversible, while special relativity and relativistic quantum mechanics are strictly reversible.

It has been shown in Ref. [73] that some of the reasons for the failure to date by both the ”cold” and the ”hot” fusions to achieve industrial value is due precisely to the treatment of structurally irreversible nuclear fusions with structurally reversible mathematical and physical methods.

In the event of residual doubt due to protracted use of preferred theories, it is sufficient to compute the quantum mechanical probability for two nuclei to ”fuse” into a third one, and then compute its time reversal image. In this way the serious scholar will see that special
relativity and relativistic quantum mechanics predict a fully causal spontaneous disintegration of nuclei following their fusion, namely, a prediction outside the boundary of serious science.

The inclusion of irreversibility in quantitative studies of new energies then recommend the development, already partially achieved at the industrial level (see Chapter 8 of Ref. [19]), of a new, controlled "intermediate fusion" of light nuclei [73], that is, a fusion occurring at minimal energies firstly needed to expose nuclei as a pre-requisite for their fusion (a feature absent in the "cold fusion" due to insufficient energies), and secondly necessary to prevent instabilities at the very high energies of the "hot fusion" that have been uncontrollable precisely because due to irreversible processes described by reversible doctrines.

In view of these pressing societal needs, the construction of Lie-admissible and/or Lie-isotopic coverings of special relativity and relativistic quantum mechanics is advocated, followed by experimental verifications and specific applications to new clean energies so much needed by mankind.

Recall that Lorentz, Poincaré, Einstein, Minkowski and others insisted in the reversible character of special relativity because necessary for the description of the physical events for which the relativity was built for.

It is easy to see that, following the above historical teaching, a basically new relativity must be developed for irreversible processes. In fact, the original proposal of a Lie-admissible generalization of Lie's theory of 1978 [11] was intended, specifically, for the construction of a Lie-admissible covering of Galileo relativity indicated beginning with the title. The proposed relativity is expected to be a genotopic covering of the isorelativity already constructed and verified [18b,18c].

Another illustration of the implications of the studies herein considered is given by the fact that all inelastic scatterings in particle physics are irreversible, yet they have been elaborated in the 20-th century via the conventional (potential) scattering theory which is structurally reversible. Once possible contributions from irreversibility are duly taken into account, some of the "experimental results" in inelastic scatterings of the 20-th century may well turn out to be "experimental beliefs."

In order to implement a serious scientific process, rather than follow a scientific religion, it is necessary to construct Lie-admissible (Lie-isotopic) scattering theory for open (closed) inelastic reactions via the procedure given in Section 4.2, re-examine existing results on inelastic scatterings and establish whether or not irreversibility requires a revision of existing data. In particular, the proposed irreversible scattering theory is expected to be an extension of the isotopic scattering theory already embrionically constructed (see Ref. [18b] and contributions quoted therein).

Far from being inessential, the re-examination of irreversible particle processes via an axiomatically consistent and invariant irreversible mechanics appears to require a revision of hadron physics for which hadronic mechanics was built for [12,18]. In fact, various studies (see, e.g., Refs. [74,75] and literature quoted therein) have shown the need for a serious re-examination of quarks, neutrinos and other conjectures, again, to prevent that possible myopic short term gains in reality set the foundations for a potentially historical condemnation by posterity.

After all, quarks and neutrino cannot be detected directly; their existence is claimed only on grounds of conventional particles predicted from inelastic scattering elaborated via
a reversible scattering theory; and the resulting events admits numerous alternative interpretations, including those via photons emitted by antiparticles. In any case, quarks are known not to admit gravity (because they can only be defined in a mathematics internal space while gravity can only be defined in our spacetime), and admit a rather long list of additional basically unsolved insufficiencies [18c,73-75].

In closing, the main objectives of this paper are to establish beyond credible doubt that the problem of the origin of irreversibility is more open then ever; to illustrate the societal, let alone scientific need for its systematic study; to propose a quantitative, axiomatically consistent representation of irreversibility at the classical and operator levels as well as for matter and antimatter; and to solicit alternative formulations by interested colleagues, under the condition that they are as directly universal and invariant as the proposed Lie-admissible and Lie-isotopic formulations.

Finally, a few words on the limitations of our Lie-admissible and Lie-isotopic treatments of irreversibility are in order. We have stressed in Section 1 that physics will never admit final theories. That is the fate also for our formulations. In fact, their most visible limitations are due to the fact that the basic genounits, their genoproducts and other operations are “single-valued,” namely, the result of genooperations are given by one single quantity.

Such a feature is today known to be effective for irreversible classical and operator physical processes, but said feature is also known to be insufficient for the representation of biological structure, since the latter require not only clearly irreversible methods but also a multi-valued generalization of the (single-valued) Lie-admissible and Lie-isotopic formulations, whose construction has been already embrionically initiated via the hyperstructural branch of hadronic mechanics [18-22].

It is hoped that serious scholars will participate with independent studies on the above, as well as numerous other, basically open problems because, in the final analysis, lack of participation in basic advances is a gift of scientific priorities to others.

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