PROPERTIES OF MODIFIED RIEMANNIAN EXTENSIONS

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Abstract. Let $M$ be an $n-$dimensional differentiable manifold with a symmetric connection $\nabla$ and $T^*M$ be its cotangent bundle. In this paper, we study some properties of the modified Riemannian extension $\tilde{g}_{\nabla,c}$ on $T^*M$ defined by means of a symmetric $(0,2)$-tensor field $c$ on $M$. We get the conditions under which $T^*M$ endowed with the horizontal lift $HJ$ of an almost complex structure $J$ and with the metric $\tilde{g}_{\nabla,c}$ is a Kähler-Norden manifold. Also curvature properties of the Levi-Civita connection and another metric connection of the metric $\tilde{g}_{\nabla,c}$ are presented.

1. Introduction

Let $M$ be an $n-$dimensional differentiable manifold and $T^*M$ be its cotangent bundle. There is a well-known natural construction which yields, for any affine connection $\nabla$ on $M$, a pseudo-Riemannian metric $\tilde{g}_{\nabla}$ on $T^*M$. The metric $\tilde{g}_{\nabla}$ is called Riemannian extension of $\nabla$. Riemannian extensions were originally defined by Patterson and Walker [17] and further investigated in Afifi [2], thus relating pseudo-Riemannian properties of $T^*M$ with the affine structure of the base manifold $(M, \nabla)$. Moreover, Riemannian extensions were also considered in Garcia-Rio et al. [8] in relation to Osserman manifolds (see also Derdzinski [5]). Riemannian extensions provide a link between affine and pseudo-Riemannian geometries. Some properties of the affine connection $\nabla$ can be investigated by means of the corresponding properties of the Riemannian extension $\tilde{g}_{\nabla}$. For instance, $\nabla$ is projectively flat if and only if $\tilde{g}_{\nabla}$ is locally conformally flat [2]. For Riemannian extensions, also see [1, 7, 10, 12, 14, 18, 19, 21, 22]. In [3, 4], the authors introduced a modification of the usual Riemannian extensions which is called modified Riemannian extension.

Almost complex Norden manifolds are among the most important geometrical structures which can be considered on a manifold. Let $M_{2k}$ be a $2k$-dimensional differentiable manifold endowed with an almost complex structure $J$ and a pseudo-Riemannian metric $g$ of signature $(k,k)$ such that $g(JX,Y) = g(X,JY)$, i.e. $g$ is pure with respect to $J$ for arbitrary vector fields $X$ and $Y$ on $M_{2k}$. Then the metric $g$ is called Norden metric. Norden metrics are referred to as anti-Hermitian metrics or $B$-metrics. The study of such manifolds is interesting because there exists a difference between the geometry of a $2k-$dimensional almost complex manifold with Hermitian metric and the geometry of a $2k-$dimensional almost complex manifold with Norden metric. A notable difference between Norden metrics and Hermitian
metrics is that \( G(X, Y) = g(X, JY) \) is another Norden metric, rather than a differential 2-form. Some authors considered almost complex Norden structures on the cotangent bundle \( [6, 15, 16] \).

In this paper we will use a deformation of the Riemannian extension on the cotangent bundle \( T^*M \) over \((M, \nabla)\) by means of a symmetric tensor field \( c \) on \( M \), where \( \nabla \) is a symmetric affine connection on \( M \). The metric is so-called modified Riemannian extension. In section 3, in the particular case where \( \nabla \) is the Levi-Civita connection on a Riemannian manifold \((M, g)\), we get the conditions under which the almost complex manifold with Norden metric \((T^*M, HJ, \tilde{g}_{\nabla, c})\) is a Kähler-Norden manifold, where \( HJ \) is the horizontal lift of an almost complex structure \( J \) and \( \tilde{g}_{\nabla, c} \) is the modified Riemannian extension. In section 4 and 5, we show that the geometric properties of the Levi-Civita connection and another metric connection of the modified Riemannian extension \( \tilde{g}_{\nabla, c} \).

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class \( C^\infty \). Also, we denote by \( \mathcal{A}^p_q(M) \) the set of all tensor fields of type \((p, q)\) on \( M \), and by \( \mathcal{A}^p_q(T^*M) \) the corresponding set on the cotangent bundle \( T^*M \). The Einstein summation convention is used, the range of the indices \( i, j, s \) being always \( \{1, 2, \ldots, n\} \).

2. Preliminaries

2.1. The cotangent bundle. Let \( M \) be an \( n \)-dimensional smooth manifold and denote by \( \pi : T^*M \to M \) its cotangent bundle with fibres the cotangent spaces to \( M \). Then \( T^*M \) is a \( 2n \)-dimensional smooth manifold and some local charts induced naturally from local charts on \( M \), may be used. Namely, a system of local coordinates \((U, x^i)\), \( i = 1, \ldots, n \) in \( M \) induces on \( T^*M \) a system of local coordinates \((\pi^{-1}(U), x^i, x^i = p_i)\), \( i = n + i = n + 1, \ldots, 2n \), where \( x^i = p_i \) is the components of covectors \( p \) in each cotangent space \( T^*_xM \), \( x \in U \) with respect to the natural coframe \( \{dx^i\} \).

Let \( X = X^i \frac{\partial}{\partial x^i} \) and \( \omega = \omega_i dx^i \) be the local expressions in \( U \) of a vector field \( X \) and a covector (1-form) field \( \omega \) on \( M \), respectively. Then the vertical lift \( V\omega \) of \( \omega \), the horizontal lift \( HX \) and the complete lift \( CX \) of \( X \) are given, with respect to the induced coordinates, by

\[
V\omega = \omega_i \frac{\partial}{\partial x^i},
\]

\[
HX = X^i \partial_i + p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^i}
\]

and

\[
CX = X^i \partial_i - p_h \partial_i X^h \partial_i,
\]

where \( \partial_i = \frac{\partial}{\partial x^i} \), \( \partial_x = \frac{\partial}{\partial x} \) and \( \Gamma^h_{ij} \) are the coefficients of a symmetric (torsion-free) affine connection \( \nabla \) in \( M \).

The Lie bracket operation of vertical and horizontal vector fields on \( T^*M \) is given by the formulas

\[
\begin{aligned}
[H^X, H^Y] &= H^X Y + V (p \circ R(X, Y)) \\
[H^X, V^\omega] &= V (\nabla_X \omega) \\
[V \theta, V^\omega] &= 0
\end{aligned}
\]
for any $X, Y \in \mathfrak{X}_0(M)$ and $\theta, \omega \in \mathfrak{X}_0(M)$, where $R$ is the curvature tensor of the symmetric connection $\nabla$ defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ (for details, see [24]).

2.2. Expressions in the adapted frame. We insert the adapted frame which allows the tensor calculus to be efficiently done in $T^*M$. With the symmetric affine connection $\nabla$ in $M$, we can introduce adapted frames on each induced coordinate neighborhood $\pi^{-1}(U)$ of $T^*M$. In each local chart $U \subset M$, we write $X(j) = \frac{\partial}{\partial x^j}$, $\theta(j) = dx^j$, $j = 1, ..., n$. Then from (2.1) and (2.2), we see that these vector fields have, respectively, local expressions

$$H_{X(j)} = \partial_j + p_{\alpha a} \Gamma_{h j}^a \partial_h$$

$$V_{\theta(j)} = \partial_j$$

with respect to the natural frame $\{\partial_j, \partial_J\}$. These $2n$ vector fields are linearly independent and they generate the horizontal distribution of $\nabla$ and the vertical distribution of $T^*M$, respectively. The set $\{H_{X(j)}, V_{\theta(j)}\}$ is called the frame adapted to the connection $\nabla$ in $\pi^{-1}(U) \subset T^*M$. By denoting

$$E_j = H_{X(j)},$$

$$E_J = V_{\theta(j)},$$

we can write the adapted frame as $\{E_\alpha\} = \{E_j, E_J\}$. The indices $\alpha, \beta, \gamma, ... = 1, ..., 2n$ indicate the indices with respect to the adapted frame.

Using (2.1), (2.2) and (2.4), we have

$$V_\omega = \begin{pmatrix} 0 \\ \omega_j \end{pmatrix},$$

and

$$H_X = \begin{pmatrix} X^i \\ 0 \end{pmatrix}$$

with respect to the adapted frame $\{E_\alpha\}$ (for details, see [24]). By the straightforward calculations, we have the lemma below.

**Lemma 1.** The Lie brackets of the adapted frame of $T^*M$ satisfy the following identities:

$$[E_i, E_j] = p_{s} R_{sij} E_s,$$

$$[E_i, E_J] = -\Gamma_{ij}^s E_s,$$

$$[E_J, E_J] = 0,$$

where $R_{ijkl}$ denote the components of the curvature tensor of the symmetric connection $\nabla$ on $M$. 
An almost complex Norden manifold \((M, J, g)\) (an almost complex manifold with a Norden metric) is defined to be a real \(2n\)-dimensional differentiable manifold \(M\), i.e. \(J\) is an almost complex structure and \(g\) is a pseudo-Riemannian metric of neutral signature \((n, n)\) on \(M\) such that:

\[
J^2 X = -X, \quad g(JX, Y) = g(X, JY)
\]

for all \(X, Y \in \mathcal{X}(M)\). An Kähler-Norden (anti-Kähler) manifold can be defined as a triple \((M, J, g)\) which consists of a smooth manifold \(M\) endowed with an almost complex structure \(J\) and a Norden metric \(g\) such that \(\nabla J = 0\), where \(\nabla\) is the Levi-Civita connection of \(g\). It is well known that the condition \(\nabla J = 0\) is equivalent to C-holomorphy (analyticity) of the Norden metric \(g\) \([11]\), i.e. \(\Phi_J g = 0\), where \(\Phi_J\) is the Tachibana operator \([23,20]\): \((\Phi_J g)(X, Y, Z) = (JX)(g(Y, Z)) - X(g(JY, Z)) + g((L_Y J)X, Z) + g(Y, (L_Z J)X)\). Also note that \(G(Y, Z) = (g \circ J)(Y, Z) = g(JY, Z)\) is the twin Norden metric. Since in dimension 2 an Kähler-Norden manifold is flat, we assume in the sequel that \(\dim M \geq 4\).

Next, for a given symmetric connection \(\nabla\) on \(M\), the cotangent bundle \(T^*M\) may be equipped with a pseudo-Riemannian metric \(\tilde{g}_\nabla\) of signature \((n, n):\) the Riemannian extension of \(\nabla\) \([17]\), given by

\[
\tilde{g}_\nabla (C X, C Y) = -\gamma(\nabla_X Y + \nabla_Y X)
\]

where \(C X, C Y\) denote the complete lifts to \(T^*M\) of vector fields \(X, Y\) on \(M\). Moreover for any vector field \(Z\) on \(M\), \(Z = Z^i \partial_i, \gamma Z\) is the function on \(T^*M\) defined by \(\gamma Z = p_i Z^i\). The Riemannian extension is expressed by

\[
\tilde{g}_\nabla = \begin{pmatrix}
-2p_k \Gamma_{ij}^k & \delta_i^j \\
\delta_i^j & 0
\end{pmatrix}
\]

with respect to the natural frame.

Now we give a deformation of the Riemannian extension above by means of a symmetric \((0, 2)\)-tensor field \(c\) on \(M\), which the metric is called modified Riemannian extension. The modified Riemannian extension is expressed by

\[
\tilde{g}_\nabla, c = \tilde{g}_\nabla + \pi^* c = \begin{pmatrix}
-2p_k \Gamma_{ij}^k + c_{ij} & \delta_i^j \\
\delta_i^j & 0
\end{pmatrix}
\]

with respect to the natural frame. It follows that the signature of \(\tilde{g}_\nabla, c\) is \((n, n)\).

Denote by \(\nabla\) the Levi-Civita connection of a pseudo-Riemannian metric \(g\). In the section, we will consider \(T^*M\) equipped with the metric \(\tilde{g}_\nabla, c\) over a pseudo-Riemannian manifold \((M, g)\). Since the vector fields \(H X\) and \(V \omega\) span the module of vector fields on \(T^*M\), any tensor field is determined on \(T^*M\) by their actions on \(H X\) and \(V \omega\). The modified Riemannian extension \(\tilde{g}_\nabla, c\) has the following properties

\[
\tilde{g}_\nabla, c(H X, H Y) = c(X, Y)
\]

\[
\tilde{g}_\nabla, c(H X, V \omega) = g_{\nabla, c}(V \omega, H X) = \omega(X)
\]

\[
\tilde{g}_\nabla, c(V \omega, V \theta) = 0
\]

for all \(X, Y \in \mathcal{X}(M)\) and \(\omega, \theta \in \mathcal{Y}(M)\), which characterize \(\tilde{g}_\nabla, c\).
The horizontal lift of a \((1,1)\)-tensor field \(J\) to \(T^*M\) is defined by:

\[
\begin{align*}
H J(H X) & = H(JX) \\
H J(V \omega) & = V(\omega \circ J)
\end{align*}
\]

for any \(X \in \mathfrak{X}^1(M)\) and \(\omega \in \mathfrak{X}^0(M)\). Moreover, it is well known that if \(J\) is an almost complex structure on \((M, g)\), then its horizontal lift \(H J\) is an almost complex structure on \(T^*M\) \[24\].

Putting

\[
A\left(\tilde{X}, \tilde{Y}\right) = \tilde{g}_{\nabla, c}\left(H J\tilde{X}, \tilde{Y}\right) - \tilde{g}_{\nabla, c}\left(\tilde{X}, H J\tilde{Y}\right)
\]

for any \(\tilde{X}, \tilde{Y} \in \mathfrak{X}^1(T^*M)\). For all vector fields \(\tilde{X}\) and \(\tilde{Y}\) which are of the form \(V \omega, \theta\) or \(H X, H Y\), from \[3.2\] and \[3.3\], we have

\[
\begin{align*}
A(H X, H Y) & = \tilde{g}_{\nabla, c}\left(H J(H X), H Y\right) - \tilde{g}_{\nabla, c}\left(H X, H J(H Y)\right) \\
& = \tilde{g}_{\nabla, c}\left(H(J X), H Y\right) - \tilde{g}_{\nabla, c}\left(H X, H (J Y)\right) \\
& = c(JX,Y) - c(X, JY) \\
A(H X, V \theta) & = \tilde{g}_{\nabla, c}\left(H J(H X), V \theta\right) - \tilde{g}_{\nabla, c}\left(H X, H J(V \theta)\right) \\
& = \tilde{g}_{\nabla, c}\left(H(J X), V \theta\right) - \tilde{g}_{\nabla, c}\left(H X, V(\theta \circ J)\right) \\
& = \theta(JX) - (\theta \circ J)(X) \\
A(V \omega, H Y) & = \tilde{g}_{\nabla, c}\left(H J(V \omega), H Y\right) - \tilde{g}_{\nabla, c}\left(V \omega, H J(H X)\right) \\
& = (\omega \circ J)(Y) - \omega(JX) \\
A(V \omega, V \theta) & = \tilde{g}_{\nabla, c}\left(H J(V \omega), V \theta\right) - \tilde{g}_{\nabla, c}\left(V \omega, H J(V \theta)\right) \\
& = \tilde{g}_{\nabla, c}\left(H(J V \omega), V \theta\right) - \tilde{g}_{\nabla, c}\left(V \omega, H J(V \theta)\right) \\
& = \tilde{g}_{\nabla, c}\left(V(\omega \circ J), V \theta\right) - \tilde{g}_{\nabla, c}\left(V \omega, V(\theta \circ J)\right) \\
& = 0.
\]

If \(g\) and \(c\) is pure with respect to \(J\), we say that \(A\left(\tilde{X}, \tilde{Y}\right) = 0\), i.e. \(g_{\nabla, c}\) is pure with respect to \(H J\).

We now are interested in the holomorphy property of the metric \(g_{\nabla, c}\) with respect to \(H J\). We calculate

\[
(\Phi_{\nabla, c}\tilde{g}_{\nabla, c})(\tilde{X}, \tilde{Y}, \tilde{Z}) = (H J\tilde{X})(\tilde{g}_{\nabla, c}(\tilde{Y}, \tilde{Z})) - \tilde{X}(\tilde{g}_{\nabla, c}(H J\tilde{Y}, \tilde{Z})) + \tilde{g}_{\nabla, c}(L_{\tilde{Y}} H J\tilde{X}, \tilde{Z}) + \tilde{g}_{\nabla, c}(L_{\tilde{Z}} H J\tilde{X})
\]
for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathbb{S}_1(T^*M)$. Then we obtain the following equations

\[
\begin{align*}
(\Phi_{\nu} J_{\nu})^\nu_\theta^H Z &= 0, \\
(\Phi_{\nu} J_{\nu})^\nu_\theta^V \sigma &= 0, \\
(\Phi_{\nu} J_{\nu})^\nu_\theta^V \sigma &= 0, \\
(\Phi_{\nu} J_{\nu})^\nu_\theta^H Y^H Z &= (\omega \circ \nabla Y J)(Z) + (\omega \circ \nabla Z J)(Y), \\
(\Phi_{\nu} J_{\nu})^H X^\nu_\omega^H Z &= (\Phi_J g)(X, \bar{\omega}, Z) - g((\nabla \omega J)X, Z), \\
(\Phi_{\nu} J_{\nu})^H X^\nu_\omega^V \sigma &= 0, \\
(\Phi_{\nu} J_{\nu})^H X^H Y^H Z &= (\Phi_J c)(X, Y, Z) \\
&+ (p \circ R(Y, JX) - p \circ R(Y, X) J)(Z) \\
&+ (p \circ R(Z, JX) - p \circ R(Z, X) J)(Y), \\
(\Phi_{\nu} J_{\nu})^H X^H Y^V \sigma &= (\Phi_J g)(X, Y, \bar{\sigma}) - g(Y, \nabla \bar{\omega} J)X),
\end{align*}
\]

where $\bar{\omega} = g^{-1} \circ \omega = g^{ij} \omega_j$ is the associated vector field of $\omega$. On the other hand, we know that the Riemannian curvature $R$ of Kähler-Norden manifolds is totally pure. Also, the condition $\Phi_J g = 0$ is equivalent to $\nabla J = 0$, where $\nabla$ is the Levi-Civita connection of $g$. Hence, we say following result.

**Theorem 1.** Let $(M, J, g)$ is a Kähler-Norden manifold. Then $T^*M$ is a Kähler-Norden manifold equipped with the metric $\tilde{g}_\nabla$ and the almost complex structure $HJ$ if and only if the symmetric $(0, 2)$ tensor field $c$ on $M$ is a holomorphic tensor field with respect to the almost complex structure $J$.

### 4. Curvature properties of the Levi-Civita connection of the modified Riemannian extension $\tilde{g}_\nabla$

From now on we will consider $T^*M$ equipped with the modified Riemannian extension $\tilde{g}_\nabla$ for a given symmetric connection $\nabla$ on $M$. By virtue of (2.3) and (2.6), the modified Riemannian extension $\tilde{g}_\nabla$ and its inverse $\overline{g}_\nabla$ have the following components with respect to the adapted frame $\{E_\alpha\}$:

\[
(\tilde{g}_\nabla)_{\beta\gamma} = \begin{pmatrix} 0 & c_{ij} & \delta^i_j \\
\delta^i_j & 0 & 0 \\
c_{ij} & 0 & 0 \end{pmatrix},
\]

\[
(\overline{g}_\nabla)_{\beta\gamma} = \begin{pmatrix} 0 & \delta^i_j \\
\delta^i_j & -c_{ij} \end{pmatrix}.
\]

The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}_\nabla$ is characterized by the Koszul formula:

\[
2\tilde{g}_\nabla(\tilde{\nabla}_X \tilde{Y}, \tilde{Z}) = \tilde{X}(\tilde{g}_\nabla(\tilde{Y}, \tilde{Z})) + \tilde{Y}(\tilde{g}_\nabla(\tilde{Z}, \tilde{X})) - \tilde{Z}(\tilde{g}_\nabla(\tilde{X}, \tilde{Y}))
\]

\[
-\tilde{g}_\nabla(\tilde{X}, [\tilde{Y}, \tilde{Z}]) + \tilde{g}_\nabla(\tilde{Y}, [\tilde{Z}, \tilde{X}]) + \tilde{g}_\nabla(\tilde{Z}, [\tilde{X}, \tilde{Y}])
\]

for all vector fields $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$ on $T^*M$. One can verify the Koszul formula for pairs $\tilde{X} = E_1, E_2$ and $\tilde{Y} = E_j, E_j$ and $\tilde{Z} = E_k, E_\nu$. By using (4.1), Lemma 1, we obtain the following result.

**Proposition 1.** The Levi-Civita connection $\tilde{\nabla}$ of the modified Riemannian exten-
sion \( \tilde{\nabla} \) is given by

\[
\begin{align*}
\tilde{\nabla}_{E_i} E_j &= 0, \\
\tilde{\nabla}_{E_i} E_j &= -\Gamma_{ijh} E_h, \\
\tilde{\nabla}_{E_i} E_j &= \Gamma_{ijh} E_n + \left\{ p_s R_{hji}^s + \frac{1}{2} \left( \nabla_i c_{jh} + \nabla_j c_{ih} - \nabla_h c_{ij} \right) \right\} E_n
\end{align*}
\]

where \( R_{hji}^s \) are the local coordinate components of the curvature tensor field \( R \) of the symmetric connection \( \nabla \) on \( M \).

An important geometric problem is to find the geodesics on the smooth manifolds with respect to the Riemannian metrics. Let \( C \) be a curve in \( M \) expressed locally by \( x^h = x^h(t) \). We define a curve \( \tilde{C} \) in \( T^* M \) by

\[
\begin{align*}
\begin{cases}
x^h = x^h(t) \\
x^\pi \overset{def}{=} p_h = \omega_h(t)
\end{cases}
\end{align*}
\]

where \( \omega_h(t) \) is a covector field along \( C \). The geodesics of the connection \( \tilde{\nabla} \) is given by the differential equations

\[
\begin{align*}
\delta^2 x^A \frac{dt^2}{dt^2} &= \frac{d^2 x^A}{dt^2} + \tilde{\Gamma}^A_{CB} \frac{dx^C}{dt} \frac{dx^B}{dt} = 0
\end{align*}
\]

with respect to the induced coordinates \( (x^h, x^\pi) \), where \( t \) is the arc length of a curve in \( T^* M \).

We write down the form equivalent to (4.4), namely,

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\theta^a}{dt} \right) + \tilde{\Gamma}^a_{\gamma \beta} \frac{\theta^\gamma}{dt} \frac{\theta^\beta}{dt} = 0
\end{align*}
\]

with respect to adapted frame \( \{ E_a \} \), where

\[
\begin{align*}
\frac{\theta^h}{dt} &= \frac{dx^h}{dt}, \\
\frac{\theta^h}{dt} &= \frac{\delta p_h}{dt}
\end{align*}
\]

along a curve \( x^A = x^A(t) \) in \( T^* M \) [24]. Taking account of Proposition 1 then we have

\[
\begin{align*}
\begin{cases}
(a) & \frac{d^2 x^h}{dt^2} + \Gamma_{ijh} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0, \\
(b) & \frac{\delta^2 p_h}{dt^2} + \left\{ p_s R_{hji}^s + \frac{1}{2} \left( \nabla_i c_{jh} + \nabla_j c_{ih} - \nabla_h c_{ij} \right) \right\} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0
\end{cases}
\end{align*}
\]

where \( \frac{\delta^2 p_h}{dt^2} = \frac{d}{dt} \left( \frac{\delta p_h}{dt} \right) - \Gamma^a_{ih} \frac{\delta p_a}{dt} \frac{dx^i}{dt} \). Thus we have the following result.

**Theorem 2.** Let \( \tilde{C} \) be a curve on \( T^* M \) and locally expressed by \( x^h = x^h(t) \), \( x^h = p_h(t) \) with respect to the induced coordinates \( (x^h, x^\pi) \) in \( \pi^{-1}(U) \subset T^* M \). The curve \( \tilde{C} \) is a geodesic in \( M \) with the symmetric connection \( \nabla \) and \( p_h(t) \) satisfies the differential equation (b) in (4.5).
The Riemannian curvature tensor \( \tilde{R} \) of \( T^*M \) with the modified Riemannian extension \( \tilde{g}_{\nabla,c} \) is obtained from the well-known formula

\[
\tilde{R}
\left( \tilde{X}, \tilde{Y}, \tilde{Z} \right) = \nabla \tilde{X} \nabla \tilde{Y} \tilde{Z} - \tilde{Z} \nabla \tilde{X} \nabla \tilde{Y} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z}
\]

for all \( \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}_0(\*T^*M) \). Then from Lemma [1] and Proposition [1] we get the following proposition.

**Proposition 2.** The components of the curvature tensor \( \tilde{R} \) of the cotangent bundle \( T^*M \) with the modified Riemannian extension \( \tilde{g}_{\nabla,c} \) are given as follows:

\[
(\ref{eq:curvature_components}) \quad \tilde{R}(E_i, E_j)E_k = R_{ijk}^h E_h + \{p_s(\nabla_s R_{hkr}) - \nabla_j R_{hki}^s \}
+ \frac{1}{2} \{ \nabla_i (\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j (\nabla_k c_{ih} - \nabla_h c_{ik}) 
- R_{ijk}^m c_{mh} - R_{ijh}^m c_{km} \} \}
\]

\[
\tilde{R}(E_i, E_j) \sigma = R_{jih}^k E_k \sigma, \quad \tilde{R}(E_i, E_j) \epsilon = -R_{hki}^j E_j \epsilon, \quad \tilde{R}(E_i, E_j) \pi = R_{hkj}^i E_i \pi
\]

with respect to the adapted frame \( \{ E_\alpha \} \).

Proposition [2] leads to the following result.

**Theorem 3.** Let \( \nabla \) be a symmetric connection on \( M \) and \( T^*M \) be the cotangent bundle with the modified Riemannian extension \( \tilde{g}_{\nabla,c} \) over \( (M, \nabla) \). Then \( (T^*M, \tilde{g}_{\nabla,c}) \) is locally flat if and only if \( (M, \nabla) \) is locally flat and the components \( c_{ij} \) of \( c \) satisfy the condition

\[
(\ref{eq:flat_condition}) \quad \nabla_i (\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j (\nabla_k c_{ih} - \nabla_h c_{ik}) = 0.
\]

Let \( \tilde{X} \) and \( \tilde{Y} \) be vector fields of \( T^*M \). The curvature operator \( \tilde{R}(\tilde{X}, \tilde{Y}) \) is a differential operator on \( T^*M \). Similarly, for vector fields \( X \) and \( Y \) of \( M \), \( R(X, Y) \) is a differential operator on \( M \). Now, we operate the curvature operator to the curvature tensor. That is, for all \( \tilde{Z}, \tilde{W} \) and \( \tilde{U} \), we consider the condition \( \tilde{R}(\tilde{X}, \tilde{Y})(\tilde{Z}, \tilde{W})\tilde{U} = 0 \).

The tensor \( (\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U} \) has components

\[
(\ref{eq:curvature_tensor}) \quad ((\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U})_{\alpha\beta\gamma\delta}^\varepsilon
= \tilde{R}_{\alpha\beta\gamma\delta}^\varepsilon - \tilde{R}_{\alpha\delta\beta\gamma}^\varepsilon - \tilde{R}_{\alpha\beta\gamma\delta}^\varepsilon - \tilde{R}_{\alpha\delta\beta\gamma}^\varepsilon,
\]

with respect to the adapted frame \( \{ E_\alpha \} \). Similarly, for all \( X, Y, Z, W, U \) on \( M \),

\[
((R(X, Y)R)(Z, W)U)_{ijklm}^n = R_{ijkl}^n R_{kln}^m - R_{ijkp}^n R_{plm}^n - R_{ijlm}^n R_{kpn}^m - R_{ijlm}^n R_{kpl}^m
= 2\nabla_{[i} \nabla_j \nabla_{l} \nabla_k \nabla_m \nabla_n \]

where \( 2\nabla_{[i} \nabla_j \nabla_{l} \nabla_k \nabla_m \nabla_n \).
Case of $\alpha = i, \beta = j, \gamma = k, \theta = l, \sigma = m, \varepsilon = n$ in (4.8), by virtue of (4.6) the equation (4.8) reduces to

$$
(4.9) \quad (\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U})_{ijklm n} = 0.
$$

Case of $\alpha = i, \beta = j, \gamma = k, \theta = l, \sigma = m, \varepsilon = n$ in (4.8), by virtue of (4.6), we have

$$
(4.10) \quad (\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U})_{ijklm n} = 0.
$$

Case of $\alpha = \bar{I}, \beta = j, \gamma = k, \theta = l, \sigma = m, \varepsilon = n$ in (4.8), by virtue of (4.6), it follows that

$$
(4.11) \quad (\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U})_{ijklm n} = 0.
$$

Case of $\alpha = \bar{I}, \beta = j, \gamma = k, \theta = l, \sigma = m, \varepsilon = n$ in (4.8), we obtain

$$
(4.12) \quad (\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U})_{ijklm n} = 0.
$$

Case of $\alpha = \bar{I}, \beta = j, \gamma = k, \theta = l, \sigma = m, \varepsilon = n$ in (4.8), by virtue of (4.6), we get

$$
(4.13) \quad (\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U})_{ijklm n} = 0.
$$

The other coefficients of $(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U}$ reduce to one of (4.9), (4.10) or (4.11) by the property of the curvature tensor.

From (4.9)-(4.13), we have the following result.
Theorem 4. Let $\nabla$ be a symmetric connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\tilde{g}_{\nabla,c}$ over $(M, \nabla)$. Then $(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U} = 0$ if and only if the following conditions hold:

i) $(R(X, Y)R)(Z, W)U = 0$

ii) $R_{ijk}^{\sigma} = 0$, from which it follows that $\nabla_i R_{hkj}^s - \nabla_j R_{hki}^s = 0$ and $\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) - R_{ijk}^m c_{mh} - R_{ijk}^m c_{km} = 0$, where $R$ and $\tilde{R}$ is the curvature tensors of the symmetric connection $\nabla$ and the Levi-Civita connection $\nabla$ of the modified Riemannian extension $\tilde{g}_{\nabla,c}$ respectively.

Theorem 4 immediately give the following result.

Corollary 1. If the symmetric connection $\nabla$ on $M$ is locally symmetric, then $(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{R})(\tilde{Z}, \tilde{W})\tilde{U} = 0$ if and only if

\[
\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) - R_{ijk}^m c_{mh} - R_{ijk}^m c_{km} = 0.
\]

Now, we consider the components of $\tilde{\nabla}\tilde{R}$. Using Proposition 1 and (4.6), by direct computation, we obtain following relations

\[
\begin{align*}
\tilde{\nabla}_l \tilde{R}_{ijk}^h &= \nabla_l R_{ijk}^h, \\
\tilde{\nabla}_l \tilde{R}_{ijk}^\pi &= \sigma_l (\nabla_i \nabla_j R_{hk}^s - \nabla_j \nabla_i R_{hki}^s) + \frac{1}{2} \nabla_l \nabla_i (\nabla_k c_{jh} - \nabla_h c_{jk}) \\
&\quad - \nabla_i \nabla_j (\nabla_k c_{ih} - \nabla_h c_{ik}) - (\nabla_l \nabla_i R_{hj}^m) c_{km} - R_{ijk}^m (\nabla_l c_{km}) \\
\tilde{\nabla}_l \tilde{R}_{ijk}^k &= \nabla_l R_{jh}^k, \\
\tilde{\nabla}_l \tilde{R}_{ik}^j &= -\nabla_l R_{hki}^j, \\
\tilde{\nabla}_l \tilde{R}_{ij}^k &= \nabla_l R_{hk}^j, \\
\tilde{\nabla}_l \tilde{R}_{ijk}^l &= \nabla_l R_{hj}^l - \nabla_j R_{hki}^l,
\end{align*}
\]

all the others being zero, with respect to the adapted frame $\{E_\alpha\}$. Hence we have the following.

Theorem 5. Let $\nabla$ be a symmetric connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\tilde{g}_{\nabla,c}$ over $(M, \nabla)$. Then $(T^*M, \tilde{g}_{\nabla,c})$ is locally symmetric if and only if $(M, \nabla)$ is locally symmetric and the components $c_{ij}$ of $c$ satisfy the condition

\[
\begin{align*}
\nabla_i \nabla_j (\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j \nabla_i (\nabla_k c_{ih} - \nabla_h c_{ik}) \\
- R_{ijk}^m (\nabla_l c_{mh}) - R_{ijh}^m (\nabla_l c_{km}) &= 0.
\end{align*}
\]

We now turn our attention to the Ricci tensor and scalar curvature of the modified Riemannian extension $\tilde{g}_{\nabla,c}$. Let $\tilde{R}_{\alpha\beta} = \tilde{R}_{\sigma\alpha\beta}^\sigma$ and $\tilde{r} = (\tilde{g}_{\nabla,c})^{\alpha\beta} \tilde{R}_{\alpha\beta}$ denote the Ricci tensor and scalar curvature of the modified Riemannian extension $\tilde{g}_{\nabla,c}$, respectively. From (4.6), the components of the Ricci tensor $R_{\alpha\beta}$ are characterized
Theorem 6. Let $\nabla$ be a symmetric connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\tilde{\nabla}_{\nabla,c}$ over $(M, \nabla)$. Then $(T^*M, \tilde{\nabla}_{\nabla,c})$ is Ricci flat if and only if the Ricci tensor of $\nabla$ is skew symmetric (for Riemannian extension, see [17]).

Now, we operate the curvature operator to the Ricci tensor. The tensors $(\tilde{\nabla}(\tilde{\nabla} \nabla, \tilde{\nabla} \nabla)\nabla)\nabla = \tilde{\nabla}_{\nabla,c}$ have coefficients

$$(\tilde{\nabla}(\tilde{\nabla} \nabla, \tilde{\nabla} \nabla)\nabla)\nabla_{\alpha\beta\gamma\theta} = \tilde{\nabla}_{\alpha\beta\gamma\theta} \tilde{\nabla}_{\epsilon\theta} + \tilde{\nabla}_{\alpha\beta\gamma\epsilon} \tilde{\nabla}_{\epsilon\theta}$$

and

$$(\tilde{\nabla}(\tilde{\nabla} \nabla, \tilde{\nabla} \nabla)\nabla)\nabla_{ijkl} = \tilde{\nabla}_{ijkl} \tilde{\nabla}_{p\epsilon} + \tilde{\nabla}_{ijkl} \tilde{\nabla}_{p\epsilon}$$

respectively. By putting $\alpha = i, \beta = j, \gamma = k, \theta = l$, it follows that

$$(\tilde{\nabla}(\tilde{\nabla} \nabla, \tilde{\nabla} \nabla)\nabla)\nabla_{ijkl} = \tilde{\nabla}_{ijkl} \tilde{\nabla}_{p\epsilon} + \tilde{\nabla}_{ijkl} \tilde{\nabla}_{p\epsilon}$$

all the others being zero. Let the base manifold $M$ be a Riemannian manifold with the metric $g$ and $\nabla$ be the Levi-Civita connection of $g$. Then

$$(\tilde{\nabla}(\tilde{\nabla} \nabla, \tilde{\nabla} \nabla)\nabla)\nabla_{ijkl} = 2\tilde{\nabla}_{ijkl} \tilde{\nabla}_{p\epsilon} + 2\tilde{\nabla}_{ijkl} \tilde{\nabla}_{p\epsilon}$$

Therefore we get the following.

Theorem 7. Let $\nabla$ be the Levi-Civita connection on the Riemannian manifold $(M, g)$ and $T^*M$ be its cotangent bundle with the modified Riemannian extension $\tilde{\nabla}_{\nabla,c}$. Then $(\tilde{\nabla}(\tilde{\nabla} \nabla, \tilde{\nabla} \nabla)\nabla)\nabla_{ijkl} = 0$ if and only if $(\tilde{\nabla}(\tilde{\nabla} \nabla, \tilde{\nabla} \nabla)\nabla)\nabla_{ijkl} = 0$.

From (4.2) and (4.10), the scalar curvature of the modified Riemannian extension $\tilde{\nabla}_{\nabla,c}$ is given by

$$\tilde{\sigma} = 0.$$

Theorem 8. Let $\nabla$ be a symmetric connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\tilde{\nabla}_{\nabla,c}$ over $(M, \nabla)$. Then $(T^*M, \tilde{\nabla}_{\nabla,c})$ is a space of constant scalar curvature $0$.

In the following we give the conditions under which the cotangent bundle $(T^*M, \tilde{\nabla}_{\nabla,c})$ is locally conformally flat. The cotangent bundle $T^*M$ with the modified Riemannian extension $\tilde{\nabla}_{\nabla,c}$ is locally conformally flat if and only if its Weyl tensor $\tilde{\nabla}_{\nabla,c}$ vanishes, where the Weyl tensor is given by

$$\tilde{\nabla}_{\alpha\beta\gamma\sigma} = \tilde{\nabla}_{\alpha\beta\gamma\sigma} + \tilde{\nabla}_{\alpha\beta\gamma} \tilde{\nabla}_{\epsilon\sigma} \tilde{\nabla}_{\eta\theta} - (\tilde{\nabla}_{\alpha\beta\gamma} \tilde{\nabla}_{\epsilon\sigma} \tilde{\nabla}_{\eta\theta})$$

$$- \frac{1}{2(n-1)} \tilde{\nabla}_{\alpha\beta\gamma} \tilde{\nabla}_{\epsilon\sigma} \tilde{\nabla}_{\eta\theta} - (\tilde{\nabla}_{\alpha\beta\gamma} \tilde{\nabla}_{\epsilon\sigma} \tilde{\nabla}_{\eta\theta})$$

$$+ \frac{1}{2(n-1)} \tilde{\nabla}_{\alpha\beta\gamma} \tilde{\nabla}_{\epsilon\sigma} \tilde{\nabla}_{\eta\theta} - (\tilde{\nabla}_{\alpha\beta\gamma} \tilde{\nabla}_{\epsilon\sigma} \tilde{\nabla}_{\eta\theta}).$$
Theorem 9. Let $\nabla$ be a symmetric connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\tilde{g}_{\nabla,c}$ over $(M, \nabla)$. Then $(T^*M, \tilde{g}_{\nabla,c})$ is locally conformally flat if and only if $(M, \nabla)$ is projectively flat and the components $c_{ij}$ of $c$ satisfy the condition

$$\nabla_i(\nabla_k c_{jn} - \nabla_n c_{jk}) - \nabla_j(\nabla_k c_{in} - \nabla_n c_{ik}) - R_{ijk}^h c_{hn} - R_{ijh}^h c_{kh} = 0$$

(also see, [2]).

Finally, we finish the section by the projective curvature tensor of the modified Riemannian extension $\tilde{g}_{\nabla,c}$. A manifold is said to be projectively flat if the projective curvature tensor vanishes. The projective curvature tensor is defined by

$$\tilde{P}_{\alpha\beta\gamma\sigma} = \tilde{R}_{\alpha\beta\gamma\sigma} - \frac{1}{2(n-1)}((\tilde{g}_{\nabla,c})_{\alpha\sigma} \tilde{R}_{\beta\gamma} - (\tilde{g}_{\nabla,c})_{\beta\sigma} \tilde{R}_{\alpha\gamma}).$$

Case of $\alpha = i, \beta = j, \gamma = k, \sigma = n$, we have

$$\tilde{P}_{ijkn} = \tilde{R}_{ijkn} - \frac{1}{2(n-1)}((\tilde{g}_{\nabla,c})_{in} \tilde{R}_{jk} - (\tilde{g}_{\nabla,c})_{jn} \tilde{R}_{ik})$$

$$= R_{jin}.$$
Case of $\alpha = i, \beta = j, \gamma = k, \sigma = n$, we obtain
\[
\tilde{R}_{ijkn} = \tilde{R}_{ijkn} - \frac{1}{(2n-1)}(\tilde{g}_{\nabla, c})_{in}\tilde{R}_{jk} - (\tilde{g}_{\nabla, c})_{jn}\tilde{R}_{ik}
\]
\[
= R_{ij}^{\ h}c_{hn} + \rho_{s}(\nabla_{s}R_{nk})^{\ h} - \nabla_{j}R_{nk}^{\ h} + \frac{1}{2}(\nabla_{i}(\nabla_{k}c_{jn} - \nabla_{n}c_{jk}) - \nabla_{j}(\nabla_{k}c_{in} - \nabla_{n}c_{ik}) - R_{ij}^{\ h}c_{hn} - R_{ij}^{\ h}c_{kh})
\]
\[
- \frac{1}{2n-1}(c_{in}(R_{jk} + R_{kj}) - c_{jn}(R_{ik} + R_{ki})).
\]

The above equations give the following result.

**Theorem 10.** Let $\nabla$ be a symmetric connection on $M$ and $T^{*}M$ be the cotangent bundle with the modified Riemannian extension $\tilde{g}_{\nabla, c}$ over $(M, \nabla)$. Then $(T^{*}M, \tilde{g}_{\nabla, c})$ is projectively flat if and only if $(M, \nabla)$ is flat and the components $c_{ij}$ of $\nabla$ satisfy the condition
\[
(4.18) \quad \nabla_{i}(\nabla_{k}c_{jn} - \nabla_{n}c_{jk}) - \nabla_{j}(\nabla_{k}c_{in} - \nabla_{n}c_{ik}) = 0.
\]

**Remark 1.**

i) If $c_{ij} = 0$, then the conditions $(4.7)$, $(4.14)$, $(4.15)$, $(4.17)$ and $(4.18)$ are identically fulfilled.

ii) If $c_{ij}$ is parallel with respect to $\nabla$, then the conditions $(4.7)$, $(4.14)$, $(4.15)$, $(4.17)$ and $(4.18)$ are identically fulfilled.

iii) If $c_{ij}$ satisfies the relation $\nabla_{i}c_{jk} - \nabla_{j}c_{ik} = \nabla_{k}\omega_{ij}$, where the components $\omega_{ij}$ define a 2-form on $M$ and if $(M, \nabla)$ is flat then the condition $(4.7)$, $(4.14)$, $(4.15)$, $(4.17)$ and $(4.18)$ are identically verified.

5. Curvature properties of another metric connection of the modified Riemannian extension $\tilde{g}_{\nabla, c}$

Let $\nabla$ be a linear connection on an $n$-dimensional differentiable manifold $M$. The connection $\nabla$ is symmetric if its torsion tensor vanishes, otherwise it is non-symmetric. If there is a Riemannian metric $g$ on $M$ such that $\nabla g = 0$, then the connection $\nabla$ is a metric connection, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In section 4, we have considered the Levi-Civita connection $\tilde{g}_{\nabla, c}$ of the modified Riemannian extension $\tilde{g}_{\nabla, c}$ on the cotangent bundle $T^{*}M$ over $(M, \nabla)$. The connection is the unique connection which satisfies $\tilde{\nabla}_{\alpha}(\tilde{g}_{\nabla, c})_{\beta\gamma} = 0$ and has a zero torsion. Hayden [9] introduced a metric connection with a non-zero torsion on a Riemannian manifold. Now we are interested in a metric connection with a non-zero torsion $^{(M)}\tilde{g}_{\nabla, c}$ whose torsion tensor $^{(M)}\tilde{T}_{\alpha\beta} = ^{(M)}\tilde{R}_{\alpha\beta}$ is skew-symmetric in the indices $\gamma$ and $\beta$. We denote components of the connection $^{(M)}\tilde{\nabla}$ by $^{(M)}\tilde{\Gamma}$. The metric connection $^{(M)}\tilde{\nabla}$ satisfies
\[
^{(M)}\tilde{\nabla}_{\alpha}(\tilde{g}_{\nabla, c})_{\beta\gamma} = 0 \text{ and } ^{(M)}\tilde{\Gamma}_{\alpha\beta}^{\gamma} - ^{(M)}\tilde{\Gamma}_{\alpha\gamma}^{\beta} = ^{(M)}\tilde{T}_{\alpha\beta}^{\gamma}.
\]

On the equation $(5.1)$ is solved with respect to $^{(M)}\tilde{\Gamma}_{\alpha\beta}^{\gamma}$, one finds the following solution [9]
\[
^{(M)}\tilde{\Gamma}_{\alpha\beta}^{\gamma} = \tilde{\Gamma}_{\alpha\beta}^{\gamma} + U_{\alpha\beta}^{\gamma},
\]
where $\Gamma_{\alpha \beta}^\gamma$ is components of the Levi-Civita connection of the modified Riemannian extension $\tilde{g}_{\nabla, c}$.

\begin{equation}
\tilde{U}_{\alpha \beta \gamma} = \frac{1}{2} (^{(M)} \tilde{\nabla} T_{\alpha \beta} + ^{(M)} \tilde{\nabla} T_{\gamma \alpha \beta} + ^{(M)} \tilde{\nabla} T_{\gamma \beta \alpha})
\end{equation}

and

\begin{equation}
\tilde{U}_{\alpha \beta \gamma} = \tilde{U}_{\alpha \beta}(\tilde{g}_{\nabla, c})_{\epsilon \gamma}, ^{(M)} \tilde{\nabla} T_{\alpha \beta \gamma} = T_{\alpha \beta}(\tilde{g}_{\nabla, c})_{\epsilon \gamma}.
\end{equation}

If we put

\begin{equation}
^{(M)} \tilde{\nabla} T_{ij}^g = -p_s R_{ij}^s
\end{equation}

terms not related to $^{(M)} \tilde{\nabla} T_{ij}^g$ being assumed to be zero. We choose this $^{(M)} \tilde{\nabla} T_{ij}^g$ in $T^*M$ which is skew-symmetric in the indices $\gamma$ and $\beta$ as torsion tensor and determine a metric connection in $T^*M$ with respect to the modified Riemannian extension $\tilde{g}_{\nabla, c}$. By using (5.3) and (5.4), we get non-zero component of $\tilde{U}_{\alpha \beta}$ as follows:

\begin{equation}
\tilde{U}_{ij}^g = p_s R_{h j}^s
\end{equation}

with respect to the adapted frame. From (5.3) and Proposition 1 we have

**Proposition 3.** Let $\nabla$ be a symmetric connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\tilde{g}_{\nabla, c}$ over $(M, \nabla)$. The metric connection $(M) \tilde{\nabla}$ with respect to $\tilde{g}_{\nabla, c}$ is as follows:

\begin{align}
^{(M)} \tilde{\nabla} E_i E_j &= 0, ^{(M)} \tilde{\nabla} E_i E_j = 0, \\
^{(M)} \tilde{\nabla} E_i E_j &= -\Gamma_{ij}^h E_h, \\
^{(M)} \tilde{\nabla} E_i E_j &= \Gamma_{ij}^h E_h + \frac{1}{2} (\nabla_i c_{jh} + \nabla_j c_{ih} - \nabla_h c_{ij}) E_h
\end{align}

with respect to the adapted frame, where $R_{h j}^s$ are the local coordinate components of the curvature tensor field $R$ of the symmetric connection $\nabla$ on $M$.

**Remark 2.** If $\nabla_i c_{jh} + \nabla_j c_{ih} - \nabla_h c_{ij} = 0$, the metric connection $^{(M)} \tilde{\nabla}$ in $T^*M$ of the modified Riemannian extension $\tilde{g}_{\nabla, c}$ coincides with the metric connection $H \nabla$ of the Riemannian extension $\tilde{g}_{\nabla, c}$, where $H \nabla$ is the horizontal lift of the symmetric connection $\nabla$ on $M$.

For the curvature tensor $(M) \tilde{R}$ of the metric connection $(M) \tilde{\nabla}$, we state the following result.

**Proposition 4.** Let $\nabla$ be a symmetric connection on $M$ and $T^*M$ be the cotangent bundle with the modified Riemannian extension $\tilde{g}_{\nabla, c}$ over $(M, \nabla)$. The curvature tensor $(M) \tilde{R}$ of the metric connection $(M) \tilde{\nabla}$ satisfies the followings:

\begin{align}
^{(M)} \tilde{R}(E_i, E_j) E_k &= R_{ij}^k E_h \\
&+ \frac{1}{2} \{ \nabla_i (\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j (\nabla_k c_{ih} - \nabla_h c_{ik}) \\
&- R_{ijk}^m c_{mh} - R_{ijk}^m c_{km} \} E_h \\
^{(M)} \tilde{R}(E_i, E_j) E_k &= R_{ij}^k E_h \\
^{(M)} \tilde{R}(E_i, E_j) E_k &= 0, ^{(M)} \tilde{R}(E_i, E_j) E_k = 0, ^{(M)} \tilde{R}(E_i, E_j) E_k = 0
\end{align}
with respect to the adapted frame.

The non-zero component of the contracted curvature tensor field (Ricci tensor field) $(\bar{\mathcal{M}})\bar{R}_{\gamma\beta} = (\mathcal{M})\bar{\nabla} \bar{R}_{\alpha\beta\gamma}$ of the metric connection $(\mathcal{M})\bar{\nabla}$ is as follows:

$$(\mathcal{M})\bar{R}_{jk} = \bar{R}_{jk}.$$  

For the scalar curvature $(\mathcal{M})\bar{\nabla}$ of the metric connection $(\mathcal{M})\bar{\nabla}$ with respect to $\bar{g}_{\gamma,c}$, we obtain

$$(\mathcal{M})\bar{\nabla} = 0.$$  

Thus we have the following theorem.

**Theorem 11.** The cotangent bundle $T^*M$ with the metric connection $(\mathcal{M})\bar{\nabla}$ has a vanishing scalar curvature with respect to the modified Riemannian extension $\bar{g}_{\gamma,c}$.

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