Decay for Skyrme wave maps

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Abstract
We consider the decay problem for global solutions of the Skyrme and Adkins–Nappi equations. We prove that the energy associated to any bounded energy solution of the Skyrme (or Adkins–Nappi) equation decays to zero outside the light cone (in the radial coordinate). Furthermore, we prove that suitable polynomial weighted energies of any small solution decays to zero when these energies are bounded. The proof consists of finding three new virial type estimates, one for the exterior of the light cone, based on the energy of the solution, and a more subtle virial identity for the weighted energies, based on a modification of momentum-type quantities.

Keywords Skyrme · Adkins–Nappi · Decay · Virial

Mathematics Subject Classification 35B40 · 35J62 · 81T10

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1 Introduction

This work is concerned with decay properties of global solutions of two nonlinear quantum field models, also known in the literature as Skyrme and Adkins–Nappi equations. Physically these models intend to describe interactions between nucleons and π mesons. Classical nonlinear field theories played an important role in the description of particles as solitonic objects. A well-known example of these nonlinear theories is the $SU(2)$ sigma model \cite{11}, obtained as a formal critical point from the action

$$S(\psi) = \int_{\mathbb{R}^{1,3}} \eta^{\mu\nu} (\psi^* g)_{\mu\nu} = \int_{\mathbb{R}^{1,3}} \eta^{\mu\nu} \partial_\mu \psi^A \partial_\nu \psi^B g_{AB} \circ \psi.$$  \hspace{1cm} (1.1)

Here, $\psi$ is a map from $(\mathbb{R}^{1,3}, \eta)$ to $SU(2) \simeq S^3$. Unfortunately, the $SU(2)$ sigma model does not admit solitons and it develops singularities in finite time \cite{4, 10, 18}. To avoid these inconveniences and to prevent the possible breakdown of the system in finite time, Skyrme \cite{19} modified the associated Lagrangian to (1.1) by adding higher-order terms which break the scaling invariance of the initial model, which in spherical coordinates $(t, r, \theta, \varphi)$ on $\mathbb{R}^{1,3}$, and co-rotational maps $\psi(t, r, \theta, \varphi) = (u(t, r), \theta, \varphi)$, the Skyrme model leads to a scalar quasilinear wave equation satisfied by the angular variable $u$, as it will be shown in (1.2).

This equation has a unique static solution with boundary values $u(0) = 0$ and $\lim_{r \to \infty} u(r) = \pi$, and which is currently known as Skyrmion \cite{17}. This existence was proved in \cite{12} and \cite{17} by using variational methods and ODE techniques, respectively. As far as we know, the Skyrmion is not known in a closed form. Recently, Lawrie and Rodriguez \cite{14} established the existence, uniqueness, and asymptotic stability of topologically nontrivial stationary solutions for the Adkins–Nappi equation. Furthermore, they showed the stable soliton resolution for this equation, conditional on a certain non-conserved norm remaining bounded throughout the evolution.

For the Adkins–Nappi equation, Geba and Rajeev \cite{5} proved that solutions remain continuous at the origin and tend to zero as $(t, r)$ goes to zero. Furthermore, in \cite{6}, they proved that the energy associated to equivariant solutions does not concentrate. Finally, Lawrie \cite{13} studied the large data dynamics by proving that there is no type II blow-up in the class of maps with topological degree zero. In particular, any degree zero map whose critical norm stays bounded must be global-in-time and scatter to zero as $t$ tends to infinity.
For the Skyrme model, Geba and Da Silva [7] proved that the energy does not concentrate in the 2+1-dimensional equivariant model. Recently, the large data global regularity for the equivariant case was studied in [8], proving that this is valid for initial data in $H^s \times H^{s-1} (\mathbb{R}^3)$ with $s > 7/2$. Recently, for the (3+1)-dimensional case, Li [15] proved the unconditional global-well posedness in $H^4_{rad} \times H^3_{rad} (\mathbb{R}^5)$, introducing a new method to set global wellposedness for arbitrarily large initial data.

After that, Geba, Nakanishi, and Rajeev [9] proved global existence and scattering for small data in critical homogeneous Sobolev–Besov space (i.e., $\dot{B}^s_{p,q}$) for the Skyrme and Adkins–Nappi equations. In particular, considering the change of variable $u = rv$, they showed that the equation obtained to $v$ from Skyrme is globally well posed in $C(\mathbb{R}; \dot{B}_{2,1}^{3/2} \cap L^2 (\mathbb{R}^5)) \cap L^2 (\mathbb{R}; \dot{B}_{4,1}^{3/4} \cap \dot{B}_{4,2}^{-3/4} (\mathbb{R}^5))$. Analogously, they proved that the equation obtained to $v$ from Adkins–Nappi model is globally well posed in $C(\mathbb{R}; \dot{H}^{1/4} \cap L^2 (\mathbb{R}^5)) \cap L^2 (\mathbb{R}; \dot{B}_{4,2}^{1/4} \cap \dot{B}_{4,2}^{-3/4} (\mathbb{R}^5))$. See [9] for further details.

1.1 Main results

In this paper, we are interested in the long time asymptotics of two relevant mathematical physics models. Firstly, the Skyrme model is written as

$$\left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_{tt} - u_{rr}) - \frac{2}{r} u_r$$

$$+ \frac{\sin(2u)}{r^2} \left[1 + \alpha^2 \left(u_t^2 - u_r^2 + \frac{\sin^2(u)}{r^2}\right)\right] = 0. \quad (1.2)$$

where $\alpha$ is a positive constant having the dimension of length and which will not have any key role in our results. The second model is a short of generalization of supercritical wave maps as it was presented in [1]. This is a simplified version of the Skyrme model (1.2) and it is currently known as Adkins–Nappi model

$$u_{tt} - u_{rr} - \frac{2}{r} u_r + \frac{\sin(2u)}{r^2} + \frac{(u - \sin(u) \cos(u)) (1 - \cos(2u))}{r^4} = 0. \quad (1.3)$$

These two models have the following low-order conserved quantities (subindices “S” and “AN” for Skyrme and Adkins–Nappi models, respectively)

$$E_S[u, u_t](t) = \int_0^\infty r^2 \left[\left(1 + \frac{2\alpha^2 \sin^2(u)}{r^2}\right)(u_t^2 + u_r^2) + \frac{2\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4}\right],$$

$$E_{AN}[u, u_t](t) = \int_0^\infty r^2 \left[u_t^2 + u_r^2 + \frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u) \cos(u))^2}{r^4}\right]. \quad (1.4)$$

(Here, $\int_0^\infty$ means $\int_0^\infty dr$.) The energies $E_S$ and $E_{AN}$ are well-defined in the Sobolev spaces $H^{7/4} \cap \dot{H}^1 (\mathbb{R}^3)$ and $H^{5/3} \cap \dot{H}^1 (\mathbb{R}^3)$, respectively.
We denote by $E^X_n$ the space of all finite energy data of degree $n$, namely
\[ E^X_n = \left\{ (u_0, u_1) \left| E_X[u_0, u_1] < \infty, \quad u_0(0) = 0, \quad \lim_{r \to \infty} u_0(r) = n\pi \right. \right\}, \quad (1.6) \]
where hereafter $X = S$ refers to the Skyrme model or when $X = AN$ to the Adkins–Nappi model. In what follows, we consider $(u, u_t)$ as the solution of the model $X$ associated to $(u_0, u_1) \in E^X_0$ and such that is a global solution of (1.2) or (1.3), respectively.

The main goal of this work is to prove that small global solutions with enough regularity of Skyrme (1.2) and Adkins–Nappi (1.3) equations decay to zero in a certain region of the light cone. Furthermore, we also study the decay of an associated weighted energy for both equations, and for which we need to analyze their corresponding long time behavior.

More precisely, let $b > 0$ and consider the following time depending subset
\[ R(t; b) = \{ x \in \mathbb{R}^3 \mid |x| > (1 + b)t \} \subset \mathbb{R}^3, \quad (1.7) \]
which is the complement of the ball of radius $(1 + b)t$, for $b > 0$. We will show that any global solution $u$ to (1.2) or (1.3), which is sufficiently regular and without a previous smallness condition, must be concentrated inside the light cone. Namely

**Theorem 1.1** (Decay in exterior light cones for the Skyrme and Adkins–Nappi models)

Let $(u_0, u_1) \in E^X_0$, defined in (1.6), such that $u$ is a global solution, for (1.2) when $X = S$, or (1.3) when $X = AN$, respectively. Then, for any $b > 0$ there is strong decay to zero of the energy $E_X$ over the set $R(t; b)$ (defined in (1.7)), in particular:
\[ \lim_{t \to \infty} \|(u_t(t), u_r(t))\|_{L^2 \times L^2(\mathbb{R}^3 \cap R(t; b))} = 0. \quad (1.8) \]

Additionally, for any $c_0 > 0$, one has the mild rate of decay for $|\sigma| > 1$:
\[ \int_2^\infty \int_0^\infty e^{-c_0|\sigma'|} r^2 (u_t^2 + u_r^2) dr dt \lesssim c_0 1. \quad (1.9) \]

**Remark 1.1.1** The spaces $E^X_0$ are not empty. In fact, for the Skyrme and Adkins–Nappi equations, the corresponding energies are well-defined in the Sobolev spaces $\dot{H}^{7/4} \cap H^1(\mathbb{R}^3)$ and $\dot{H}^{5/3} \cap H^1(\mathbb{R}^3)$, respectively.

For the next results, we have to introduce a weighted version of the spaces (1.6). Let $E^X_n$ the space of all global solutions of $X$ of degree $n$ which has finite $\phi$-weighted energy
\[ E^X_n = \left\{ (u, u_t) \left| E_{X,\phi}[u, u_t](t) < \infty, \quad u(0, r) = u_0(r), \quad u_0(0) = 0, \quad \lim_{r \to \infty} u_0(r) = n\pi \right. \right\}, \]
where $E_{X,\phi}$ is written for the Skyrme model as
\[ E_{S,\phi}[u, u_t](t) = \int_0^\infty \phi(r) \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2). \]
+ 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right]. \quad (1.10)

and for the Adkins–Nappi model as

\[ E_{AN,\phi}[u, u_t](t) = \int_0^\infty \phi(r) \left[ u^2_t + u^2_r + 2 \frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u) \cos(u))^2}{r^4} \right]. \quad (1.11) \]

In fact, one can see that if \( E_X[u, u_t](t) = E_X[u, u_t](t) \), then \( E_n^{X,r^2} = E_n^X \), for \( X \in \{S, AN\} \).

Our second result shows that the energy \( E_X \) associated to any global solution \( (u, u_t) \in E_X^{r_n} \cap E_X^{r_{n-1}} \) of (1.2) or (1.3) decays to zero when \( t \) goes to infinity. This means that for any global solution \( u \) which is sufficiently regular and it satisfies a weighted integrability on \( r \), its energy \( E_X^{r_n} \) decays to zero when \( t \) goes to infinity for both \( X = S \) or \( X = AN \) cases.

**Theorem 1.2** (Decay of weighted energies) Assume that for \( \delta > 0 \) sufficiently small, the global solution \( (u, u_t) \) of (1.2) or (1.3), with \( (u, u_t) \in E_X^{r_n} \) satisfies

\[ \sup_{t \in \mathbb{R}} E_X[u, u_t](t) < \delta, \quad \text{for } X = AN, S. \quad (1.12) \]

Then, the modified energy \( E_{X,\phi}[u, u_t](t) \) with \( \phi(r) = r^n \) decays to zero, for \( n > 7 \) (\( X = S \) case) or for \( n \in \left[ \frac{3+\sqrt{41}}{2}, 10 \right] \) (\( X = AN \) case), respectively. In particular,

\[ \lim_{t \to \infty} \| r^{\frac{n-2}{2}} (u_t, u_r)(t) \|_{L^2 \times L^2(\mathbb{R}^3)} = \lim_{t \to \infty} E_{X,r^n}(t) = 0. \quad (1.13) \]

The next remark will be useful in the proof of Theorem 1.2.

**Remark 1.2.1** ([5, 13]) Note that finite energy smooth solutions of Skyrme (1.2) and Adkins–Nappi (1.3) equations are uniformly bounded as follows

\[ \| u \|_{L^\infty_t} \leq C(E_X[u, u_t](0)), \quad \text{where } X \in \{S, AN\}, \]

and \( C(s) \to 0 \) as \( s \to 0 \).

**1.2 Idea of the proof**

In order to prove Theorem 1.1, we follow some ideas appeared in [2, 3, 16], where decay for Camassa-Holm, Born-Infeld and Improved-Boussinesq models were considered. The main tool in these works was a suitable virial functional for which the dynamic of solutions is converging to zero when it is integrated in time.

In this paper, the new virial functionals give us relevant information about the dynamics of global solutions of Skyrme and Adkins–Nappi equations. Using a proper
virial estimate, we prove that the corresponding energies associated to Skyrme and Adkins–Nappi equations decay to zero in the subset $R(t; b)$ for any $b > 0$ (1.7).

Furthermore, to prove Theorem 1.2, we will study the growth rate of polynomial weight energies of the Skyrme and Adkins–Nappi equations. After that, assuming that their growth is bounded, we will prove that this growth decays to zero as $t$ tends to infinity. To prove this result, we introduce a functional associated with a sort of weighted momentum. It happens that the virial identity associated to this functional shows no evidence of good sign conditions, i.e., that the derivative of the functional is negative. Therefore, we have to introduce a new functional as a linear combination of these two virial identities and for which there is a good sign property. This ensures the integrability in time of polynomial weighted energies of degree $n$. Moreover, it also guarantees the decay of a polynomial weighted energy of degree $n+1$ over a subsequence of times. Combining these two facts, we conclude that the polynomial weighted energies, which are bounded, decay to zero as $t$ tends to infinity (over $\mathbb{R}^3$).

**Organization of this paper**

This chapter is organized as follows: Sect. 2 is splitted in two subsections where a series of virial identities are presented: in Sects. 2.1 and 2.2 we show the virial identities used for to prove the decay of the energy in the Skyrme equation (1.2) and (1.3), respectively. Section 3 deals with the proof of Theorem 1.1 for the Skyrme and Adkins–Nappi equations. Finally, Sect. 4 deals with the proof of Theorem 1.2 for both models.

**2 Virial identities**

In this section, three virial identities for the Skyrme and Adkins–Nappi models (1.2)–(1.3) are presented. One of the virial functionals is related with the exterior light cone behavior (Theorem 1.1), and the other ones are useful for understanding the decay of the weighted energy for Skyrme and Adkins–Nappi models (Theorem 1.2). Moreover, we remark here that the energies $E_S[u, u_t]$ and $E_{AN}[u, u_t]$, defined in (1.4) and (1.5) are bounded in spaces $\mathcal{E}_0^S$ and $\mathcal{E}_0^{AN}$, respectively. Furthermore, it is well-known that these energies are well defined in the Sobolev spaces $\dot{H}^{7/4} \cap \dot{H}^1(\mathbb{R}^3)$ and $\dot{H}^{5/3} \cap \dot{H}^1(\mathbb{R}^3)$ for the Skyrme and Adkins–Nappi equations, respectively.

**2.1 Virial identities for the Skyrme model**

Let $\varphi = \varphi(t, r)$ be a smooth, bounded weight function, to be chosen later. For each $t \in \mathbb{R}$, we consider the following functional

$$I_S(t) = \int_0^\infty \varphi r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) \left( u_t^2 + u_r^2 \right) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right],$$

(2.1)
which is a generalization of the energy introduced in (1.4), and well-defined for 
\((u, u_t) \in (\dot{H}^{7/4} \cap \dot{H}^1) \times L^2(\mathbb{R}^3)\). Moreover, if \(\varphi\) only depends on \(r\) and it can be 
written as \(\varphi(r) = \phi/r^2\), then we recover \(E_{S, \phi}\), which is the weighted energy defined 
in (1.10). The following identities will be useful for the proof of Theorems 1.1–1.2.

The following result shows the variation of the localized energy for the Skyrme 
equation:

**Lemma 2.1** (Energy local variations: Skyrme Model) For any \(t \in \mathbb{R}\), \(\varphi(t, r)\) a smooth 
function previously defined, and \(I_{S}(t)\) as in (2.1), we have that

\[
\frac{d}{dt} I_{S}(t) = \int_0^\infty \varphi_t r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right]
- \int_0^\infty \varphi_r r^2 \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) 2u_t u_r.
\]

(2.2)

**Proof of Lemma 2.1** Derivating (2.1) with respect to time, and using a basic trigono-
metric relation, we have

\[
\frac{d}{dt} I_{S}(t) = \int_0^\infty \varphi_t r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right]
+ 2 \int_0^\infty \varphi r^2 u_t \left[ \frac{\alpha^2 \sin(2u)}{r^2} (u_t^2 + u_r^2) + \frac{\sin(2u)}{r^2} + \frac{\alpha^2 \sin^2(u) \sin(2u)}{r^4} \right]
+ 2 \int_0^\infty \varphi r^2 u_t \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) u_{tt} + \int_0^\infty \varphi r^2 \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) u_r u_{rt}
:= I_1 + I_2 + I_3 + I_4.
\]

Now, using Eq. (1.2) in \(I_3\), we have

\[
I_3 = 2 \int_0^\infty \varphi r^2 u_t \left\{ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) u_{rr} + \frac{2}{r} u_r
- \frac{\sin(2u)}{r^2} \left[ 1 + \alpha^2 \left( u_t^2 - u_r^2 + \frac{\sin^2(u)}{r^2} \right) \right] \right\}.
\]

And integrating by parts in the last integral \(I_4\), we obtain

\[
\frac{1}{2} I_4 = - \int_0^\infty \varphi r^2 \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) u_r u_t
- \int_0^\infty \varphi r^2 u_t \left( \left( \frac{2}{r} u_r + \frac{2\alpha^2 \sin(2u)}{r^2} u_r^2 \right) - \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) u_{rr} \right).
\]
Finally, we have

\[ I_2 + I_3 + I_4 = -2 \int_0^\infty \varphi_r r^2 \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) u_t u_r, \]

and we get that

\[
\frac{d}{dt} I_S(t) = \int_0^\infty \varphi_r r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2) + \frac{2\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right] \\
- 2 \int_0^\infty \varphi r r^2 \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) u_t u_r.
\]

This concludes the proof.

**Remark 2.1.1** With the change of variables \( \varphi = \phi/r^2 \), we avoid the term \( r^2 \) in the weighted function (2.1), and which coming from the dimension of the problem. Then, \( E_{S,\phi} \) is recovered from \( I_S(t) \) and applying Lemma 2.1, we get

\[
\frac{d}{dt} E_{S,\phi} = \int_0^\infty \phi_t \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2) + \frac{2\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right] \\
- 2 \int_0^\infty \left( \phi' - 2\frac{\phi}{r} \right) \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) u_t u_r. \tag{2.3}
\]

This relation will be useful in the proof of Theorem 1.2.

Now, we define two functionals that we will use to prove the decay of the weighted energy \( E_{X,\phi} \). Firstly, denote by \( f \) the following function

\[
f(u) = 1 + \frac{2\alpha^2 \sin^2(u)}{r^2}. \tag{2.4}
\]

Now, considering \( \psi \) and \( \phi \) smooth weight functions of \( r \), which will be chosen later, we define the functional \( K_S(t) \) associated with a sort of momentum, given by

\[
K_S(t) = \int_0^\infty \psi f(u) u_t u_r, \tag{2.5}
\]

and the functional \( P_S(t) \), which corrects the bad sign of the variation in time on the functional \( K_S(t) \), and which is given by

\[
P_S(t) = \int_0^\infty \phi f(u) u_t u. \tag{2.6}
\]
Lemma 2.2 Let $t \in \mathbb{R}$, $\psi$ be a smooth weight function and $K_S(t)$ defined as in (2.5). If $u \in E_0^{S,\psi}$ and $p(r) = \left( \frac{\psi}{r^2} - 4 \frac{\psi}{r^3} \right)$, then we have

$$
\frac{d}{dt} K_S(t) = -\frac{1}{2} \int_0^\infty \frac{\psi'}{r^2} r^2 u_r^2 - \frac{1}{2} \int_0^\infty p(r) r^2 u_r^2 - \frac{\alpha^2}{2} \int_0^\infty p(r) \sin^2(u) (u_t^2 + u_r^2)
+ \int_0^\infty \left( \frac{\psi'}{r^2} - 2 \frac{\psi}{r^3} \right) \sin^2(u) + \frac{\alpha^2}{2} \int_0^\infty p(r) \frac{\sin^4(u)}{r^2}.
$$

(2.7)

Proof First of all, we notice that the time and radial derivatives of $f$ (2.4) are

$$(f(u))_t = 2\alpha^2 \frac{\sin(2u) u_t}{r^2}, \quad (f(u))_r = 2\alpha^2 \frac{\sin(2u) u_r}{r^2} - 4\alpha^2 \frac{\sin^2(u)}{r^3}. \quad (2.8)$$

Secondly, derivating the functional (2.5) with respect to time, we get

$$
\frac{d}{dt} K_S(t) = \int_0^\infty \psi (f(u))_t u_t u_r + \int_0^\infty \psi f(u) (u_t u_r + u_r u_{tr})
= \int_0^\infty \psi (f(u))_t u_t u_r + \int_0^\infty \psi f(u) u_{tr} u_r + \frac{1}{2} \int_0^\infty \psi f(u) (u_t^2)_r.
$$

Integrating by parts in the last term of the RHS, we obtain

$$
\frac{d}{dt} K_S(t) = \int_0^\infty \psi (f(u))_t u_t u_r + \int_0^\infty \psi f(u) u_{tr} u_r - \frac{1}{2} \int_0^\infty \psi f(u) u_t^2
- \frac{1}{2} \int_0^\infty \psi (f(u))_r u_r^2 := K_1 + K_2 + K_3 + K_4.
$$

(2.9)

For $K_2$, using (1.2), we obtain

$$
K_2 = \int_0^\infty \psi u_r \left\{ f(u) u_{rr} + \frac{2}{r} u_r - \frac{\sin(2u)}{r^2} \left[ 1 + \alpha^2 \left( u_t^2 - u_r^2 + \frac{\sin^2(u)}{r^2} \right) \right] \right\}
= -\int_0^\infty \left( \psi f(u) \right)_r \frac{u_r^2}{2} + 2 \int_0^\infty \psi \frac{u_r^2}{r}
- \int_0^\infty \psi \frac{u_r^2}{r^2} \sin(2u) u_r \left[ 1 + \alpha^2 \left( u_t^2 - u_r^2 + \frac{\sin^2(u)}{r^2} \right) \right]
= -\int_0^\infty \left[ \psi' f(u) + \psi (f(u))_r \right] \frac{u_r^2}{2} + 2 \int_0^\infty \psi \frac{u_r^2}{r}.
$$
\[ K_2 = - \int_0^\infty \psi f(u) \frac{u^2}{2} - \alpha^2 \int_0^\infty \psi \frac{\sin(2u)}{r^2} u^3 r + 2\alpha^2 \int_0^\infty \psi \frac{\sin^2(u)}{r^3} u^2 r^2 - 2 \int_0^\infty \psi \sin(2u) u_r \]
\[ \times \left[ 1 + \alpha^2 \left( u_1^2 - u_2^2 + \frac{\sin^2(u)}{r^2} \right) \right] \]
\[ (2.10) \]

Using (2.10) and (2.8) in (2.9), one can see
\[ \frac{d}{dt} K_S(t) = - \frac{1}{2} \int_0^\infty \psi' \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_1^2 + u_2^2) \]
\[ + 2\alpha^2 \int_0^\infty \frac{\sin^2(u)}{r^3} (u_1^2 + u_2^2) \]
\[ + 2 \int_0^\infty \frac{\psi}{r} u_r^2 - \int_0^\infty \frac{\psi}{r^2} \sin^2(u) \left( 1 + \frac{\alpha^2 \sin^2(u)}{r^2} \right) \]
\[ - 2 \int_0^\infty \frac{\psi}{r^3} \sin^2(u) \left( 1 + \frac{\alpha^2 \sin^2(u)}{r^2} \right) \]
\[ = - \frac{1}{2} \int_0^\infty \frac{\psi'}{r^2} r^2 u_1^2 - \frac{1}{2} \int_0^\infty p(r) r^2 u_2^2 - \frac{\alpha^2}{2} \int_0^\infty p(r) \sin^2(u) (u_1^2 + u_2^2) \]
\[ + \int_0^\infty \left( \frac{\psi'}{r^2} - 2 \frac{\psi}{r^3} \right) \sin^2(u) + \frac{\alpha^2}{2} \int_0^\infty p(r) \frac{\sin^4(u)}{r^2} . \]

Finally, integrating by parts and regrouping terms, we get
\[ \frac{d}{dt} K_S(t) = \ldots \]

and we conclude. \( \square \)

Similarly, we have the following result for the correction term \( P_S(t) \) (2.6).
Lemma 2.3 Let $t \in \mathbb{R}$, $\phi$ be a smooth weight function and $P_S(t)$ as in (2.6). Then, if $u \in \mathcal{E}_{0,r}^{s,\phi}$, we have

$$
\frac{d}{dt} P_S(t) = \alpha^2 \int_0^\infty \frac{\phi}{r^2} \left( \sin(2u)u + 2 \sin^2(u) \right) (u_t^2 - u_r^2) + \int_0^\infty \phi(u_t^2 - u_r^2) \\
- \int_0^\infty \left[ \left( \frac{\phi}{r} - \frac{1}{2} \frac{\phi'}{r^2} \right) u^2 - \int_0^\infty \frac{\phi}{r^2} \sin(2u) \right] u \\
+ \alpha^2 \int_0^\infty \left( r \phi'' - 4 \phi' + \frac{6 \phi}{r} \right) \frac{\sin^2(u)}{r^3} u^2 \\
- \alpha^2 \int_0^\infty \frac{\phi}{r} \sin(2u) \frac{\sin^2(u)}{r} u r u^2.
$$

(2.11)

**Proof** Derivating the functional (2.6) with respect to time, we have

$$
\frac{d}{dt} P_S(t) = \int_0^\infty \phi \left( f(u) u_r + \int_0^\infty \phi f(u) (u_{tt} u + u_t^2) \right) \\
= 2 \alpha^2 \int_0^\infty \frac{\phi}{r^2} \sin(2u) u_r^2 + \int_0^\infty \phi f(u) u_t^2 + \int_0^\infty \phi f(u) u_{tt} u \\
:= P_1 + P_2 + P_3.
$$

(2.12)

Using (1.2) in $P_3$, we get

$$
P_3 = \int_0^\infty \phi \left\{ f(u) u_{rr} + \frac{2}{r} u_r - \frac{\sin(2u)}{r^2} \left[ 1 + \alpha^2 \left( u_t^2 - u_r^2 + \frac{\sin^2(u)}{r^2} \right) \right] \right\}.
$$

Integrating by parts the first term on the RHS, we get

$$
\int_0^\infty \phi f(u) u_{rr} = - \int_0^\infty \phi' f(u) u_{rr} - \int_0^\infty \phi(f(u)) u_{rr} - \int_0^\infty \phi f(u) u_t^2 \\
= \frac{1}{2} \int_0^\infty (\phi'' f(u) + \phi'(f(u)) u^2) - \int_0^\infty \phi f(u) u_{rr} \\
- \int_0^\infty \phi f(u) u_r^2.
$$

Having in mind derivatives in (2.8), we get

$$
\int_0^\infty \phi f(u) u_{rr} = \frac{1}{2} \int_0^\infty \phi'' f(u) u^2 + \alpha^2 \int_0^\infty \phi' \left( \frac{\sin(2u) u_r}{r^2} - 2 \frac{\sin^2(u)}{r^3} \right) u^2 \\
- 2 \alpha^2 \int_0^\infty \phi \left( \frac{\sin(2u) u_r}{r^2} - 2 \frac{\sin^2(u)}{r^3} \right) u_r - \int_0^\infty \phi f(u) u_r^2.
$$
= \frac{1}{2} \int_0^\infty \phi'' f(u)u^2 + \alpha^2 \int_0^\infty \phi' \frac{\sin(2u)}{r^2} u^2 - 2\alpha^2 \int_0^\infty \phi' \frac{\sin^2(u)}{r^3} u^2

- 2\alpha^2 \int_0^\infty \phi \frac{\sin(2u)}{r^2} u u_r^2 + 4\alpha^2 \int_0^\infty \phi \frac{\sin^2(u)}{r^3} u u_r - \int_0^\infty \phi f(u) u_r^2. \quad (2.13)

Now, integrating by parts the second term in the RHS of the above line, we obtain

\begin{align*}
4\alpha^2 \int_0^\infty \phi \frac{\sin^2(u)}{r^3} u u_r &= -2\alpha^2 \int_0^\infty \left( \frac{\phi' \sin^2(u)}{r^3} \right) u^2 \\
&= -2\alpha^2 \int_0^\infty \phi' \frac{\sin^2(u)}{r^3} u^2 \\
&\quad -2\alpha^2 \int_0^\infty \phi \left( \frac{\sin(2u) u_r}{r^3} - 3 \frac{\sin^2(u)}{r^4} \right) u^2. \quad (2.14)
\end{align*}

Then, substituting into (2.13), we get

\begin{align*}
P_3 &= \frac{1}{2} \int_0^\infty \phi'' f(u)u^2 + \alpha^2 \int_0^\infty \left( -4\phi' + 6 \frac{\phi}{r} \right) \frac{\sin^2(u)}{r^3} u^2 \\
&\quad - \alpha^2 \int_0^\infty \phi \frac{u \sin(2u) \sin^2(u)}{r^3} \\
&\quad + \alpha^2 \int_0^\infty \left( \phi' - 2 \frac{\phi}{r} \right) \frac{\sin(2u)}{r^2} u u^2 - \int_0^\infty \phi \left( \alpha^2 \frac{\sin(2u)}{r^2} u + f(u) \right) u_r^2 \\
&\quad - \int_0^\infty \left( \frac{\phi}{r} \right) u^2 - \int_0^\infty \phi \frac{\sin(2u)}{r^2} u u_r^2 - \alpha^2 \int_0^\infty \phi \frac{\sin(2u)}{r^2} uu_r^2.
\end{align*}

Replacing (2.4) and regrouping again, we obtain

\begin{align*}
P_3 &= \alpha^2 \int_0^\infty \left( r \phi'' - 4 \phi' + 6 \frac{\phi}{r} \right) \frac{\sin^2(u)}{r^3} u^2 - \alpha^2 \int_0^\infty \phi \frac{u \sin(2u) \sin^2(u)}{r^3} \\
&\quad + \alpha^2 \int_0^\infty \left( \phi' - 2 \frac{\phi}{r} \right) \frac{\sin(2u)}{r^2} u u^2 - \alpha^2 \int_0^\infty \phi \left( \alpha^2 \frac{\sin(2u)}{r^2} u + f(u) \right) u_r^2 \\
&\quad + \int_0^\infty \left( \frac{1}{2} \phi'' - \left( \frac{\phi}{r} \right) \right) u^2 - \int_0^\infty \phi \frac{\sin(2u)}{r^2} u - \int_0^\infty \phi u_r^2 \\
&\quad - \alpha^2 \int_0^\infty \phi \frac{\sin(2u)}{r^2} uu_r^2. \quad (2.15)
\end{align*}

Collecting $P_3$ in (2.15), (2.12), and using (2.4), we get

\frac{d}{dt} P_3(t) = 2\alpha^2 \int_0^\infty \phi \frac{\sin(2u)}{r^2} u u_r^2

+ \int_0^\infty \phi f(u) u_r^2 + \alpha^2 \int_0^\infty \left( r \phi'' - 4 \phi' + 6 \frac{\phi}{r} \right) \frac{\sin^2(u)}{r^3} u^2
\[-\alpha^2 \int_0^\infty \frac{\phi}{r} \left( \frac{u \sin(2u) \sin^2(u)}{r^3} \right) + \alpha^2 \int_0^\infty \left( \frac{\phi'}{r^2} - \frac{2}{r} \phi \right) \frac{\sin(2u)}{r^2} u_r u^2 \]

\[-\alpha^2 \int_0^\infty \frac{\phi}{r^2} \left( \sin(2u) u + 2 \sin^2(u) \right) u_r^2 \]

\[-\int_0^\infty \left( \frac{\phi}{r} \right) u^2 + \frac{1}{2} \int_0^\infty \phi'' u^2 - \int_0^\infty \phi' \frac{\sin(2u)}{r^2} u - \int_0^\infty \phi u_r^2 \]

\[-\alpha^2 \int_0^\infty \phi \frac{\sin(2u)}{r^2} u u_r^2.\]

Finally, regrouping terms, we conclude

\[
\frac{d}{dt} \mathcal{P}_S(t) = \alpha^2 \int_0^\infty \frac{\phi}{r^2} \left( \sin(2u) u + 2 \sin^2(u) \right) (u_t^2 - u_r^2) + \int_0^\infty \phi (u_t^2 - u_r^2) \]

\[+ \alpha^2 \int_0^\infty \left( r \phi'' - 4 \phi' + 6 \frac{\phi}{r} \right) \frac{\sin^2(u)}{r^3} u^2 - \alpha^2 \int_0^\infty \frac{\phi}{r} u \sin(2u) \sin^2(u) \]

\[+ \alpha^2 \int_0^\infty \left( \phi' - 2 \frac{\phi}{r} \right) \sin(2u) \frac{r}{r^2} u_r u^2 - \int_0^\infty \left[ \left( \frac{\phi}{r} \right) - \frac{1}{2} \phi'' \right] u^2 \]

\[\quad - \int_0^\infty \phi \frac{\sin(2u)}{r^2} u. \tag{2.17}\]

\[\square \]

### 2.2 Virial identities for the Adkins–Nappi Model

Let \( \rho = \rho(t, r) \) a smooth, weight function, to be chosen later. Similarly to the previous section, for the Adkins–Nappi equation we introduce a suitable functional, as a weighted generalization of the energy (1.5), and given by

\[\mathcal{I}_{AN}(t) = \int_0^\infty \rho r^2 \left[ u_t^2 + u_r^2 + 2 \frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u) \cos(u))^2}{r^4} \right], \tag{2.16}\]

for each \( t \in \mathbb{R} \).

Recalling Remark 1.1.1, if \( \rho \) is a bounded function, the functional \( \mathcal{I}_{AN}(t) \) is well-defined for \( (u, u_t) \in (H^{5/3} \cap \dot{H}^1 \times L^2)(\mathbb{R}^3) \). The following result describes the time variation of (2.16).

**Lemma 2.4** (Energy local variations: Adkins–Nappi Model) For any \( t \in \mathbb{R} \), one has

\[
\frac{d}{dt} \mathcal{I}_{AN}(t) = \int_0^\infty \rho r^2 \left[ u_t^2 + u_r^2 + 2 \frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u) \cos(u))^2}{r^4} \right] - 2 \int_0^\infty \rho r^2 u_t u_r. \tag{2.17}\]
Proof Derivating the functional (2.16) with respect to time and using basic trigonometric identities, we obtain

\[
\frac{d}{dt} I_{AN}(t) = \int_0^\infty \rho r^2 \left[ u_t^2 + u_r^2 + 2 \frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u) \cos(u))^2}{r^4} \right]
\]

\[
+ 2 \int_0^\infty \rho r^2 u_t u_r
\]

\[
+ \int_0^\infty 2 \rho u_r^2 \left[ u_{rr} + \frac{\sin(2u)}{r^2} + \frac{(u - \sin(u) \cos(u)) (1 + \sin^2(u) - \cos^2(u))}{r^4} \right].
\]

(2.18)

Now, using Eq. (1.3) and integrating by parts in \( J_1 \), we have

\[
J_1 = 2 \int_0^\infty \rho u_r^2 \left[ u_{rr} + \frac{2}{r} u_r \right]
\]

\[
= 4 \int_0^\infty \rho r u_t u_r - 2 \int_0^\infty \left( \rho_r r^2 u_t + \rho u_t + \rho r^2 u_{rt} \right) u_r.
\]

Finally, substituting \( J_1 \) in (2.18), we get

\[
\frac{d}{dt} I_{AN}(t) = \int_0^\infty \rho r^2 \left[ u_t^2 + u_r^2 + 2 \frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u) \cos(u))^2}{r^4} \right]
\]

\[
- 2 \int_0^\infty \rho r^2 u_t u_r.
\]

(2.19)

This ends the proof of the lemma. \( \square \)

Remark 2.4.1 Similarly to Remark 2.1.1, using the change of variables \( \rho = \phi / r^2 \), the term \( r^2 \) in \( I_{AN}(t) \) is avoided, and therefore recovering the functional \( E_{AN,\phi}[u, u_t] \) (1.11). Furthermore, by Lemma 2.4 we have the following identity for the time variation of \( E_{AN,\phi} \):

\[
\frac{d}{dt} E_{AN,\phi}(t) = \int_0^\infty \phi_t \left[ u_t^2 + u_r^2 + 2 \frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u) \cos(u))^2}{r^4} \right]
\]

\[
- 2 \int_0^\infty \left( \phi_r - \frac{2 \phi}{r} \right) u_t u_r.
\]

(2.20)

This relation will be useful in the proof of Theorem 1.2.

Now, let \( \psi \) and \( \phi \) smooth weight functions of \( r \), which will be chosen later. We define the functional \( M_{AN}(t) \) associated with a sort of momentum, given by

\[
M_{AN}(t) = \int_0^\infty \psi u_t u_r.
\]

(2.21)
and the functional $\mathcal{R}_{AN}(t)$, which is the term that corrects the bad sign of the variation on the functional $\mathcal{M}_{AN}(t)$, given by

$$\mathcal{R}_{AN}(t) = \int_0^\infty \phi u_t u.$$  \hfill (2.22)

The following results show the time variation of these functionals, which will be used in the proof of Theorem 1.2.

**Lemma 2.5** Let $t \in \mathbb{R}$, $\psi$ be a smooth weight function and $\mathcal{M}_{AN}(t)$ as in (2.21). Then, if $u \in E_{0,\psi}^{AN}$, we have

$$\frac{d}{dt} \mathcal{M}_{AN}(t) = -\frac{1}{2} \int_0^\infty \psi' u_t^2 - \int_0^\infty \left(\frac{\psi'}{2} - \frac{2\psi}{r}\right) u_r^2$$

$$- \frac{1}{2} \int_0^\infty \left(\frac{2\psi}{r} - \psi'\right) \sin^2(u)$$

$$- \frac{1}{2} \int_0^\infty \left(\frac{4\psi}{r} - \psi'\right) \frac{(u - \sin(u) \cos(u))^2}{r^4}.$$ \hfill (2.23)

**Proof** Just derivating the functional (2.21) with respect to time, we obtain

$$\frac{d}{dt} \mathcal{M}_{AN}(t) = \int_0^\infty \psi (u_{tt} u_r + u_t u_{rt})$$

$$= -\frac{1}{2} \int_0^\infty \psi' u_t^2 + \int_0^\infty \psi u_{tt} u_r := M_1 + M_2.$$

For $M_2$, using (1.3) and integrating by parts, we have

$$M_2 = \int_0^\infty \psi u_r \left(u_{rr} + \frac{2}{r} u_r\right)$$

$$- \int_0^\infty \psi u_r \left(\sin(2u) + \frac{u - \sin(u) \cos(u)}{r^4} (1 - \cos(2u))\right)$$

$$= - \int_0^\infty \left(\frac{\psi'}{2} - \frac{2\psi}{r}\right) u_r^2$$

$$- \int_0^\infty \psi u_r \left(\sin(2u) + \frac{u - \sin(u) \cos(u)}{r^4} (1 - \cos(2u))\right)$$

$$= M_{21} + M_{22}.$$

With respect to $M_{22}$, note that rewriting it and integrating by parts, we obtain

$$M_{22} = -\frac{1}{2} \int_0^\infty \psi r^2 (\sin(u))_r - \frac{1}{2} \int_0^\infty \psi \frac{u}{r^4} (u - \sin(u) \cos(u))^2$$

$$= -\frac{1}{2} \int_0^\infty \left(\frac{2\psi}{r} - \psi'\right) \frac{\sin^2(u)}{r^2} - \frac{1}{2} \int_0^\infty \left(\frac{4\psi}{r} - \psi'\right) \frac{(u - \sin(u) \cos(u))^2}{r^4}.$$
Finally, collecting $M_1, M_{21},$ and $M_{22},$ we get

\[
\frac{d}{dt} M_{AN}(t) = -\frac{1}{2} \int_0^\infty \psi' u_i^2 - \int_0^\infty \left( \frac{\psi'}{2} - \frac{2\psi}{r} \right) u_r^2 \\
- \frac{1}{2} \int_0^\infty \left( \frac{2\psi}{r} - \psi' \right) \frac{\sin^2(u)}{r^2} \\
- \frac{1}{2} \int_0^\infty \left( \frac{4\psi}{r} - \psi' \right) \frac{(u - \sin(u) \cos(u))^2}{r^4}.
\]

This ends the proof of this lemma. \(\square\)

**Lemma 2.6** Let \(t \in \mathbb{R}, \phi\) be a smooth weight function and \(R_{AN}(t)\) as in (2.22). Then, if \(u \in \mathcal{E}_0^{AN,r^2}\phi\), we have

\[
\frac{d}{dt} R_{AN}(t) = \int_0^\infty \phi u_t^2 + \int_0^\infty \int_0^\infty \left[ \phi' r - \phi - \frac{r^2 \phi_{rr}}{2} \right] u^2 r^2 - \int_0^\infty \phi u_r^2 \\
- \int_0^\infty \left( \frac{\phi}{r^2} \right) u \sin(2u) \\
- \int_0^\infty \left( \frac{\phi}{r^4} \right) u (u - \sin(u) \cos(u)) (1 - \cos(2u)).
\] (2.24)

**Proof** Just derivating the functional (2.22) with respect to time, we obtain

\[
\frac{d}{dt} R_{AN}(t) = \int_0^\infty \phi u_t^2 + \int_0^\infty \phi u_{tt} u := R_1 + R_2. 
\] (2.25)

For \(R_2\), using (1.3) and integrating by parts, we get

\[
R_2 = \int_0^\infty \phi \frac{r^2}{2} u^2 - \int_0^\infty \phi u_r^2 - \int_0^\infty \left( \frac{\phi}{r} \right) u^2 r^2 \\
- \int_0^\infty \phi u \left( \frac{\sin(2u)}{r^2} + \frac{u - \sin(u) \cos(u)}{r^4} (1 - \cos(2u)) \right).
\]

Regrouping terms, we obtain

\[
R_2 = \int_0^\infty \left( \phi \frac{r^2}{2} - r \phi_r + \phi \right) \frac{u^2}{r^2} - \int_0^\infty \phi u_r^2 \\
- \int_0^\infty \left( \phi \frac{r^2}{2} u \sin(2u) + \phi \frac{u}{r^4} (u - \sin(u) \cos(u)) (1 - \cos(2u)) \right). 
\] (2.26)
Then, substituting (2.26) in (2.25), we obtain
\[
\frac{d}{dt} R_{AN}(t) = \int_{0}^{\infty} \phi(u^2_t - u^2_r) - \int_{0}^{\infty} \left[ \phi' r - \phi - r^2 \phi_{rr} \right] \frac{u^2_r}{r^2} - \int_{0}^{\infty} \frac{\phi}{r^2} u \sin(2u) - \int_{0}^{\infty} \frac{\phi}{r^4} u (u - \sin(u) \cos(u)) (1 - \cos(2u)).
\]

This concludes the proof of the lemma. \(\square\)

3 Decay in exterior light cones for the Skyrme and Adkins–Nappi models

This section deals with the proof of Theorem 1.1 for the Skyrme and Adkins–Nappi equations. In what follows, fix \(\sigma \in \mathbb{R}\) such that \(|\sigma| > 1\). Recalling the identity (2.2) and using the weight function \(\varphi = \varphi\left( \frac{r + \sigma t}{L} \right)\), we get
\[
\frac{d}{dt} I_S(t) = \frac{\sigma}{L} \int_{0}^{\infty} \varphi' r^2 \left[ (1 + 2\alpha^2 \sin^2(u)) (u^2_t + u^2_r) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right] - \frac{1}{L} \int_{0}^{\infty} \varphi' r^2 (1 + 2\alpha^2 \sin^2(u)) 2u_t u_r. \tag{3.1}
\]

Furthermore, from Lemma 2.4, we have:
\[
\frac{d}{dt} I_{AN}(t) = \frac{\sigma}{L} \int_{0}^{\infty} \varphi' r^2 \left[ u^2_t + u^2_r + 2 \frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u) \cos(u))^2}{r^4} \right] - \frac{1}{L} \int_{0}^{\infty} \varphi' r^2 u_t u_r. \tag{3.2}
\]

Now, we are ready to prove a first virial estimate.\(^1\)

**Lemma 3.1** Let \(L > 0\), \(\sigma = -(1 + b) < -1\), and \(\rho = \varphi = \tanh\left( \frac{r + \sigma t}{L} \right)\). Then
\[
(1) \quad \frac{d}{dt} I_S(t) \lesssim L, b - \int_{0}^{\infty} \varphi' r^2 \left[ (1 + 2\alpha^2 \sin^2(u)) (u^2_t + u^2_r) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right]. \tag{3.3}
\]

\(^1\) We denote by \(a \lesssim h b\) if there is a constant \(C\) depending on \(h\) such that \(a \leq C(h) b\).
\( \frac{d}{dt} \mathcal{I}_{AN}(t) \lesssim L, b - \int_0^\infty \rho' r^2 \left[ u_t^2 + u_r^2 + 2 \frac{\sin^2(u)}{r^2} + \frac{(u - \sin(u) \cos(u))^2}{r^4} \right] \).

(3.4)

Proof Firstly we prove (3.3). Focusing on the last term in the RHS of (3.1), note that, if \( \varphi' > 0 \), then using a Cauchy–Schwarz inequality, we have

\[
\left| \int_0^\infty \varphi' r^2 \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) 2u_t u_r \right| \leq \int_0^\infty \varphi' r^2 \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2).
\]

Therefore, if \( b > 0 \), \( \sigma = -(1 + b) < -1 \), and \( \varphi = \tanh \), we have from (3.1)

\[
\frac{d}{dt} \mathcal{I}_S(t) \leq \frac{\sigma}{L} \int_0^\infty \varphi' r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right]
+ \frac{1}{L} \int_0^\infty \varphi' r^2 \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2).
\]

Consequently, we obtain (3.3)

\[
\frac{d}{dt} \mathcal{I}_S(t) \lesssim L, b - \int_0^\infty |\varphi'| r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2)
+ 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right].
\]

The proof of (3.4) proceeds in a similar way. Only note that the last term in (3.2) verifies the following inequality

\[
\left| \int_0^\infty \varphi' r^2 2u_t u_r \right| \leq \int_0^\infty \varphi' r^2 (u_t^2 + u_r^2), \text{ for } \varphi' > 0,
\]

the rest of the proof follows the same lines as in the Skyrme case, and hence, for the sake of simplicity, we do not show it here.

Finally, we can observe that integrating in time on (3.3) and (3.4), we have proved (1.9) in Theorem 1.1.

3.1 Proof of Theorem 1.1: Skyrme and Adkins–Nappi equations

Firstly, we focus on the Skyrme case. It only remains to prove (1.8). We must to show decay in the right-hand side region, namely \((1 + b)t, +\infty\), \( b > 0 \). Now we choose \( \varphi(r) = \frac{1}{2} (1 + \tanh(r)), \sigma = -(1 + b), \) and \( \tilde{\sigma} = -(1 + b/2) \) with \( b > 0 \). Consider the modified energy functional, for \( t \in [2, t_0] \):

\[ \Box \]
\[ \mathcal{I}_{S, t_0}(t) := \frac{1}{2} \int_0^\infty \varphi \left( \frac{r + \sigma t_0 - \tilde{\sigma}(t_0 - t)}{L} \right) r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right]. \]

Note that \( \sigma < \tilde{\sigma} < 0 \). From Lemma 2.1 and proceeding exactly as in (3.3), we have

\[
\frac{d}{dt} \mathcal{I}_{S, t_0}(t) \leq b, L
\]

\[- \int_0^\infty \text{sech}^2 \left( \frac{r + \sigma t_0 - \tilde{\sigma}(t_0 - t)}{L} \right) r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right] \leq 0,
\]

which means that the new functional \( \mathcal{I}_{S, t_0} \) is decreasing in \([2, t_0]\). Therefore, we have

\[
\int_2^{t_0} \frac{d}{dt} \mathcal{I}_{S, t_0}(t) dt = \mathcal{I}_{S, t_0}(t_0) - \mathcal{I}_{S, t_0}(2) \leq 0 \implies \mathcal{I}_{S, t_0}(t_0) \leq \mathcal{I}_{S, t_0}(2).
\]

On the other hand, since \( \lim_{x \to -\infty} \varphi(x) = 0 \), we have

\[
\limsup_{t \to \infty} \int_0^\infty \varphi \left( \frac{r - \beta t - \gamma}{L} \right) r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right] (v, r) = 0,
\]

for \( \beta, \gamma, \nu > 0 \) fixed. This yields

\[
0 \leq \int_0^\infty \varphi \left( \frac{r - (1 + b)t_0}{L} \right) r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right] (t_0, r)
\]

\[
\leq \int_0^\infty \varphi \left( \frac{r - \frac{b}{2}t_0 - (2 + b)}{L} \right) r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right] (2, r),
\]

which implies

\[
\limsup_{t \to \infty} \int_0^\infty \varphi \left( \frac{r - (1 + b)t}{L} \right) r^2 \left[ \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2) + 2 \frac{\sin^2(u)}{r^2} + \frac{\alpha^2 \sin^4(u)}{r^4} \right] (t, r) dr = 0.
\]

This means that the energy over \( R(t; b) \), for any \( b > 0 \), (see (1.7)) converges to zero, implying (1.8) and then we conclude the Skyrme case.

For the Adkins–Nappi case, the proof is analogous but this time considering the modified energy functional

\[
\mathcal{I}_{AN, t_0}(t) := \int_0^\infty \rho \left( \frac{r + \sigma t_0 - \tilde{\sigma}(t_0 - t)}{L} \right) r^2 \left[ u_t^2 + u_r^2 + 2 \frac{\sin^2(u)}{r^2} \right]
\]
and repeating the same steps as in the Skyrme case. This concludes the proof of Theorem 1.1.

4 Decay of weighted energies

Firstly, we study the growth rate of the modified energies introduced in (1.10) and (1.11).

4.1 Growth rate for the modified energy in the Skyrme and Adkins–Nappi equations

In this section, we study the growth rate for the power-type weighted energy of the Skyrme and Adkins–Nappi equations

Proposition 4.1 Let $u$ a global solution of (1.2) (or (1.3)) such that $u \in \cap_{i=2}^{n} E_{X,r^{i}}$, for $X = S$ or $X = AN$. Then the corresponding weighted energy satisfies

$$E_{X,r^{n}}[u,u_{t}](t) = O(t^{n-2}),$$

where $E_{X,r^{n}}[u,u_{t}](t)$ is given in (1.10) and (1.11), respectively.

Proof Firstly, we consider $X = S$. We note that for $\psi = \phi/r^{2}$, we get

$$I_{S}(t) = E_{S,\psi}[u,u_{t}](t).$$

(4.1)

Then, using (2.20) with $\phi = r^{n}$, one can see

$$\frac{d}{dt} I_{S}(t) = -2K_{S}(t),$$

where $K_{S}(t)$ is given by (2.5) and $\psi = \phi' - 2\phi' = (n-2)r^{n-1}$. Now using (4.1), we get

$$\left| \frac{d}{dt} E_{S,r^{n}}[u,u_{t}](t) \right| \lesssim E_{S,r^{n-1}}[u,u_{t}](t),$$

and for $n = 3$, we obtain

$$\left| \frac{d}{dt} E_{S,r^{3}}[u,u_{t}](t) \right| \lesssim E_{S,r^{2}}[u,u_{t}](t) = E_{S}[u,u_{t}](0),$$

$$\left| E_{S,r^{3}}[u,u_{t}](t) \right| \lesssim E_{S}[u,u_{t}](0)t + |E_{S,r^{2}}[u,u_{t}](0)|.$$
Similarly, for \( n = 4 \) and using the last inequality, we get

\[
\left| \frac{d}{dt} E_{S,r^4}[u, u_t](t) \right| \lesssim E_{S,r^3}[u, u_t](t) \lesssim E_S[u, u_t](0) t + |E_{S,r^3}[u, u_t](0)|.
\]

Now, integrating with respect to time, we have

\[
\left| E_{S,r^4}[u, u_t](t) \right| \lesssim E_S[u, u_t](0) \frac{t^2}{2} + |E_{S,r^3}[u, u_t](0)| t + |E_{S,r^4}[u, u_t](0)|.
\]

Repeating this procedure, we conclude

\[
\left| E_{S,r^n}[u, u_t](t) \right| \lesssim E_S[u, u_t](0) t^{n-2} + \sum_{j=0}^{n-3} t^j |E_{S,r^{n-j}}[u, u_t](0)|.
\]

This ends the proof for the case \( X = S \). Analogously, following the same ideas, it can be proved for the case \( X = AN \) case. This completes the proof. \( \Box \)

### 4.2 Decay to zero for modified energies: Proof of Theorem 1.2

In the spirit of [2, 3, 16], we consider a suitable linear combination of virials \( K_S(t) \) and \( P_S(t) \) (see (2.5) and (2.6)), and \( M_{AN}(t) \) and \( R_{AN}(t) \) (see (2.21) and (2.22)), for the Skyrme and Adkins–Nappi models, respectively. Let

\[
\mathcal{H}_S(t) = K_S(t) + \gamma_S P_S(t),
\]

and

\[
\mathcal{H}_{AN}(t) = M_{AN}(t) + \gamma_{AN} R_{AN}(t),
\]

be new virials, where \( \gamma_S \) and \( \gamma_{AN} \) will be chosen later. These new virials introduce \( u^2 \) terms, which allow us to simplify the problem considering Taylor expansions for the involved trigonometric functions.

#### 4.2.1 Decay to zero for modified energy: Proof of Theorem 1.2 for the Skyrme model

**Lemma 4.2** Let \( u \) be a global solution of (1.2) such that \( \|u\|_{L^\infty} \leq \delta, u \in \mathcal{E}_0^{S,\psi} \), and \( \psi = r\phi \) (where \( \psi \) and \( \phi \) are the weight functions presented in (2.5) and (2.6)). Then, \( \mathcal{H}_S(t) \) in (4.2) satisfies the following identity
\[
\frac{d}{dt} \mathcal{H}_S(t) = -\frac{1}{2} \int_0^\infty \left( \psi' - 2\gamma_S \frac{\psi}{r} \right) u_r^2 - \frac{1}{2} \int_0^\infty \left( \psi' + (2\gamma_S - 4) \frac{\psi}{r} \right) u_r^2 \\
- \int_0^\infty \left( (2 - \gamma_S) \frac{\psi}{r} + (2\gamma_S - 1) \psi' - \frac{\gamma_S}{2} r \psi'' \right) \frac{u_r^2}{r^2} \\
- \alpha^2 \int_0^\infty \left( \psi' - (2 + 4\gamma_S) \frac{\psi}{r} \right) \frac{u_r^2}{r^2} (u_r^2 + u_t^2) \\
- \alpha^2 \int_0^\infty \left( -\frac{1}{2} (1 - 6\gamma_S) \psi' + (2 - 4\gamma_S) \frac{\psi}{r} - \frac{\gamma_S}{2} r \psi'' \right) \frac{u_r^4}{r^4} + H_e(t),
\]

where

\[
H_e(t) = \frac{1}{9} \alpha^2 \int_0^\infty \left( -\gamma_S r \psi'' + (-3 + 6\gamma_S) \psi' + (6\gamma_S + 12) \frac{\psi}{r} \right) \left( \frac{u_r^6}{r^4} + \frac{O(u^8)}{r^4} \right) \\
- \frac{1}{3} \int_0^\infty \left( \psi' - 2(1 + 2\gamma_S) \frac{\psi}{r} \right) \frac{u_r^4}{r^2} + \frac{2}{45} \int_0^\infty \left( \psi' - 2(1 + \gamma_S) \frac{\psi}{r} \right) \left( \frac{u_r^6}{r^4} + \frac{O(u^8)}{r^2} \right) \\
- \alpha^2 \int_0^\infty \left[ \frac{1}{3} \left( -\psi' + 2(1 + 6\gamma_S) \frac{\psi}{r} \right) u^2 \\
+ \frac{2}{45} \left( \psi' - 2(1 + 4\gamma_S) \frac{\psi}{r} \right) \left( u^4 + O(u^6) \right) \right] \frac{u_r^2}{r^2} (u_r^2 + u_t^2).
\]

**Proof** Collecting (2.11) and (2.7) and regrouping terms, we get

\[
\frac{d}{dt} \mathcal{H}_S(t) = -\frac{1}{2} \int_0^\infty \left( \psi' - 2\gamma_S \phi \right) u_r^2 - \frac{1}{2} \int_0^\infty \left( \psi' + 2\gamma_S \phi - 4 \frac{\psi}{r} \right) u_r^2 \\
+ \int_0^\infty \left( \frac{\phi'}{r^2} - \frac{2 \psi}{r^3} \right) \sin^2(u) - \gamma_S \int_0^\infty \phi \frac{u \sin(2u)}{r} - \gamma_S \int_0^\infty \left( \frac{\phi}{r} - \frac{1}{2} \phi'' \right) u^2 \\
+ \gamma_S \alpha^2 \int_0^\infty \left( \frac{\phi''}{r^2} - \frac{4 \phi'}{r^3} + \frac{6 \phi}{r^4} \right) \sin^2(u) u^2 - \gamma_S \alpha^2 \int_0^\infty \phi \frac{u \sin(2u) \sin^2(u)}{r^2} \\
+ \frac{\alpha^2}{2} \int_0^\infty \left( \frac{\phi'}{r^3} - \frac{2 \phi'}{r^2} \right) \sin^4(u) + \gamma_S \alpha^2 \int_0^\infty \left( \frac{\phi'}{r^3} - \frac{4 \phi'}{r^2} \right) \sin(2u) u^2 u'^2 \\
- \alpha^2 \int_0^\infty \left[ \frac{\psi'}{r^2} \sin^2(u) - \frac{2 \psi}{r^3} \sin^2(u) - \gamma_S \frac{\phi}{r} \left( \sin(2u) u + 2 \sin^2(u) \right) \right] (u_r^2 + u_t^2).
\]

Now, let \( \delta > 0 \) small enough such that \( \|u\|_{L^\infty} < \delta \) (by Remark 1.2.1), we note

\[
\sin^2(u) = u^2 - \frac{1}{3} u^4 + \frac{2}{45} u^6 + O(u^8), \\
u \sin(2u) = 2u^2 - \frac{4}{3} u^4 + \frac{4}{15} u^6 + O(u^8), \\
2 \sin^2(u) + u \sin(2u) = 4u^2 - 2u^4 + \frac{16}{45} u^6 + O(u^8), \\
u \sin^2(u) \sin(2u) = 2u^4 - 2u^6 + O(u^8), \\
\sin^4(u) = u^4 - \frac{2}{3} u^6 + O(u^8).
\]

Then, we obtain the following decomposition

\[
\frac{d}{dt} \mathcal{H}_S(t) = H_1 + H_2 + H_3 + H_4 + H_5,
\]
where

\[ H_1 = -\frac{1}{2} \int_0^\infty \left( \frac{\psi'}{r^2} - 2\gamma S \phi \right) u_r^2 - \frac{1}{2} \int_0^\infty \left( \frac{\psi'}{r} + 2\gamma S \phi - 4\frac{\psi}{r} \right) u_t^2, \]

\[ H_2 = \int_0^\infty \left( \frac{\psi'}{r^2} - 2\frac{\psi}{r^3} \right) \left[ u_r^2 - \frac{1}{3} u^4 + \frac{2}{45} u^6 + O(u^8) \right] \]
\[ - \gamma S \int_0^\infty \left( \left( \frac{\phi}{r} \right)_r - \frac{1}{2} \phi'' \right) u^2 \]
\[ - \gamma S \int_0^\infty \frac{\phi}{r^2} \left( 2u_r^2 - \frac{4}{3} u^4 + \frac{4}{15} u^6 + O(u^8) \right), \]

\[ H_3 = -\alpha^2 \int_0^\infty \left[ \left( \frac{\psi'}{r^2} - 2\frac{\psi}{r^3} \right) \left( u_r^2 - \frac{1}{3} u^4 + \frac{2}{45} u^6 + O(u^8) \right) \right. \]
\[ - \gamma S \frac{\phi}{r^2} \left( 4u_r^2 - 2u^4 + \frac{16}{45} u^6 + O(u^8) \right) \left. \right] \left( u_r^2 + u_t^2 \right) \]

\[ H_4 = \gamma S \alpha^2 \int_0^\infty \left[ \left( \frac{\phi''}{r^2} - 4\frac{\psi'}{r^3} + 6\frac{\phi}{r^4} \right) \left( u^4 - \frac{1}{3} u^6 + O(u^8) \right) \right. \]
\[ - 2\frac{\phi}{r^4} \left( u^4 - u^6 + O(u^8) \right) \]
\[ + \left. \frac{\alpha^2}{2} \int_0^\infty \left( \frac{\psi'}{r^4} - 4\frac{\psi}{r^5} \right) \left( u^4 - \frac{2}{3} u^6 + O(u^8) \right) \right] \]

and

\[ H_5 = \gamma S \alpha^2 \int_0^\infty \left( \frac{\phi'}{r^2} - 2\frac{\phi}{r^3} \right) \sin(2u)u_t u_r^2. \]  

(4.5)

Regrouping terms of the same order, we get

\[ H_2 = \int_0^\infty \left[ \left( \gamma S \left[ \frac{1}{2} \frac{\phi''}{r^2} - \frac{\phi}{r^3} \right] - \frac{2}{3} \left( \frac{\psi'}{r^2} - \psi \right) - 2\frac{\phi}{r^3} \right) u_r^2 \right. \]
\[ + \frac{2}{3} \left( \frac{\psi}{r^3} - \frac{2\psi}{r^2} + 2\gamma S \frac{\phi}{r^2} \right) u^4 \]
\[ + \frac{2}{45} \int_0^\infty \left( \frac{\psi'}{r^2} - 2\frac{\psi}{r^3} - \gamma S \frac{6\phi}{r^2} \right) \left( u^6 + O(u^8) \right) \left. \right] \left( u_r^2 + u_t^2 \right) \]

Similarly, for \( H_3 \) we get

\[ H_3 = -\alpha^2 \int_0^\infty \left( \frac{\psi'}{r^2} - 2\frac{\psi}{r^3} - 4\gamma S \frac{\phi}{r^2} \right) u^2 (u_t^2 + u_r^2) \]
\[ - \alpha^2 \int_0^\infty \left[ \frac{1}{3} \left( \frac{2\psi}{r^3} - \frac{\psi'}{r^2} + 6\gamma S \frac{\phi}{r^2} \right) u^4 \right. \]
\[ + \frac{2}{45} \left( \frac{\psi'}{r^2} - \frac{2\psi}{r^3} - 8\gamma S \frac{\phi}{r^2} \right) \left( u^6 + O(u^8) \right) \left. \right] (u_r^2 + u_t^2). \]
For $H_4$, we have

$$H_4 = \alpha^2 \int_0^\infty \left( \frac{1}{2} \psi' \frac{1}{r^4} - 2 \psi' \frac{1}{r^5} - 2 \gamma S \frac{\phi'}{r^4} + \gamma S \frac{\phi''}{r^5} - 4 \gamma S \frac{\phi'}{r^3} + 6 \gamma S \frac{\phi}{r^4} \right) u^4$$

$$+ \frac{1}{5} \alpha^2 \int_0^\infty \left( -\gamma S \frac{\phi''}{r^2} + 4 \gamma S \frac{\phi'}{r^3} - 6 \gamma S \frac{\phi}{r^4} - 4 \gamma S \frac{\phi}{r^3} + 4 \frac{\psi}{r^5} \right)$$

$$\left( u^6 + O(u^8) \right).$$

For $H_5$, first we note

$$\sin(2u) u_r u^2 = \frac{1}{4} \frac{d}{dr} \left( 2u \sin(2u) - 2u^2 \cos(2u) + \cos(2u) - 1 \right).$$

Now, replacing in $H_5$ and integrating by parts, we get

$$H_5 = -\frac{\gamma S}{4} \alpha^2 \int_0^\infty \left( \frac{\phi'}{r^2} - 2 \frac{\phi}{r^3} \right) \left( 2u \sin(2u) - 2u^2 \cos(2u) + \cos(2u) - 1 \right),$$

using its Taylor expansion and regrouping terms, we have

$$H_5 = -\frac{\gamma S}{4} \alpha^2 \int_0^\infty \left( \frac{\phi''}{r^2} - 4 \frac{\phi'}{r^3} + 6 \frac{\phi}{r^4} \right) \left( 2u^4 - \frac{8}{9} u^6 + O(u^8) \right)$$

$$= \alpha^2 \frac{\gamma S}{2} \left\{ \int_0^\infty \left( 4 \frac{\phi'}{r^3} - \frac{\phi''}{r^2} - 6 \frac{\phi}{r^4} \right) u^4 + \frac{2}{9} \int_0^\infty \left( \frac{\phi''}{r^2} - 4 \frac{\phi'}{r^3} + 6 \frac{\phi}{r^4} \right) \right. \right.$$

$$\left. \left( u^6 + O(u^8) \right) \right\}. \quad (4.6)$$

Collecting the last equation and $H_4$, we obtain

$$H_4 + H_5 = \alpha^2 \int_0^\infty \left( \frac{1}{2} \psi' - 2 \psi' \frac{1}{r^5} + \gamma S \frac{\phi'}{r^4} - 2 \gamma S \frac{\phi'}{r^3} + \frac{\gamma S}{2} \frac{\phi'}{r^2} \right. \right.$$

$$\left. \left( 1 \psi' - 2 \psi' \frac{1}{r^5} + \gamma S \frac{\phi'}{r^4} - 2 \gamma S \frac{\phi'}{r^3} + \frac{\gamma S}{2} \frac{\phi'}{r^2} \right) u^4$$

$$\left. + \alpha^2 \int_0^\infty \left( -\frac{1}{9} \gamma S \frac{\phi''}{r^2} + \frac{4}{9} \gamma S \frac{\phi'}{r^3} - \frac{1}{3} \gamma S \frac{\phi}{r^4} + \frac{4 \psi}{3} + \frac{1}{3} \gamma S \frac{\phi}{r^4} \right) \right.$$

$$\left. \left( u^6 + O(u^8) \right) \right\}. \quad (4.6)$$

Having in mind that $\psi = r \phi$, we have

$$\phi' = \frac{\psi'}{r} - \frac{\psi}{r^2} \quad \text{and} \quad \phi'' = \frac{\psi'}{r} - 2 \frac{\psi'}{r^2} + \frac{\psi}{r^3}. \quad (4.6)$$

Now, rewriting $H_i$, for $i = 1, \ldots, 5$, in terms of $\psi$ and its derivatives, we get

$$H_1 = -\frac{1}{2} \int_0^\infty \left( \psi' - 2 \gamma S \frac{\psi}{r} \right) u_t^2 - \frac{1}{2} \int_0^\infty \left( \psi' - 2 \gamma S - 4 \frac{\psi}{r} \right) u_r^2, \quad (4.7)$$
where we obtain

\[ H_2 = \int_0^\infty \left( (\gamma_s - 2) \frac{\psi}{r^3} + (1 - 2\gamma_s) \frac{\psi'}{r^2} + \frac{\gamma_s \psi''}{2} \right) u^2 \\
- \int_0^\infty \left( \frac{\psi'}{3r^2} - \frac{2}{3}(1 + 2\gamma_s) \frac{\psi}{r^3} \right) u^4 \\
+ \int_0^\infty \left( \frac{2}{45} \frac{\psi'}{r^2} - \frac{4}{45} \frac{1 + \gamma_s}{r^3} \right) \left( u^6 + O(u^8) \right). \] (4.8)

Finally, collecting (4.7), (4.8), (4.9), and regrouping terms of the same order, we obtain

\[ H_3 = \alpha^2 \int_0^\infty \left( (2 + 4\gamma_s) \frac{\psi}{r^3} - \frac{\psi'}{r^2} \right) u^2 (u_i^2 + u_r^2) \\
+ \alpha^2 \int_0^\infty \left[ \left( \frac{1}{3} \frac{\psi'}{r^2} - \frac{2}{3}(1 + 6\gamma_s) \frac{\psi}{r^3} \right) u^2 \\
- \left( \frac{2}{45} \frac{\psi'}{r^2} - \frac{4}{45} (1 + 4\gamma_s) \frac{\psi}{r^3} \right) \left( u^4 + O(u^6) \right) \right] u^2 (u_i^2 + u_r^2), \] (4.9)

and

\[ H_4 + H_5 = \alpha^2 \int_0^\infty \left( \frac{1}{2} (1 - 6\gamma_s) \frac{\psi'}{r^4} - (2 - 4\gamma_s) \frac{\psi}{r^5} + \frac{\gamma_s \psi''}{2} \right) u^4 \\
+ \alpha^2 \int_0^\infty \left[ \left( \frac{6}{9} \gamma_s + \frac{4}{3} \right) \frac{\psi}{r^5} + \left( \frac{6}{9} \gamma_s - \frac{1}{3} \right) \frac{\psi'}{r^4} - \frac{1}{9} \gamma_s \frac{\psi''}{r^3} \right] \times \left( u^6 + O(u^8) \right). \] (4.10)

Finally, collecting (4.7), (4.8), (4.9), and regrouping terms of the same order, we obtain

\[ \frac{d}{dt} H_S(t) = -\frac{1}{2} \int_0^\infty \left( \psi' - 2\gamma_s \frac{\psi}{r} \right) u_i^2 - \frac{1}{2} \int_0^\infty \left( \psi' + (2\gamma_s - 4) \frac{\psi}{r} \right) u_r^2 \\
- \int_0^\infty \left( 2 - \gamma_s \right) \frac{\psi}{r} + (2\gamma_s - 1) \psi' - \frac{\gamma_s \psi''}{2} \right) u_r^2 \\
- \alpha^2 \int_0^\infty \left( \psi' - (2 + 4\gamma_s) \frac{\psi}{r} \right) \frac{u^2}{r^2} (u_i^2 + u_r^2) \\
- \alpha^2 \int_0^\infty \left( -\frac{1}{2} (1 - 6\gamma_s) \psi' + (2 - 4\gamma_s) \frac{\psi}{r} - \frac{\gamma_s \psi''}{2} \right) \frac{u^4}{r^4} + H_e(t), \]

where

\[ H_e(t) = \frac{1}{9} \alpha^2 \int_0^\infty \left( (6\gamma_s + 12) \frac{\psi}{r} + (-3 + 6\gamma_s) \psi' - \gamma_s r \psi'' \right) \left( \frac{u^6}{r^4} + O(u^8) \right) \\
- \frac{1}{3} \int_0^\infty \left( \psi' - 2(1 + 2\gamma_s) \frac{\psi}{r} \right) \frac{u^4}{r^2} + \frac{2}{45} \int_0^\infty \left( \psi' - 2(1 + \gamma_s) \frac{\psi}{r} \right) \frac{u^6}{r^2} \right) \]
\[-\frac{\alpha^2}{45} \int_0^\infty \left[ 15 \left( 2(1 + 6\gamma_S) \frac{\psi}{r} - \psi' \right) 
+ 2 \left( \psi' - 2(1 + 4\gamma_S) \frac{\psi}{r} \right) \left( u^2 + O(u^4) \right) \right] \frac{u^4}{r^2} (u_t^2 + u_r^2). \]

This ends the proof. \(\square\)

Under the hypothesis of Lemma 4.2, the functionals \(K_S(t)\) and \(P_S(t)\) (see (2.5)–(2.6)) are well-defined. In fact, using the Cauchy–Schwarz inequality, we have

\[ K_S(t) \leq \int_0^\infty \psi \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2), \]

and

\[ P_S(t) \leq \int_0^\infty r\phi \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_t^2 + u_r^2). \]

Then, assuming \(\psi = r\phi\), and \(u \in \mathcal{E}_0^{S,\psi}\), we get

\[ |K_S(t)| + |P_S(t)| \leq E_{S,\psi}[u, u_t](t), \]

hence concluding that the functionals \(K_S(t)\) and \(P_S(t)\) in (2.5)–(2.6) are well-defined.

**Corollary 4.3** Let \(n, \gamma_S \in \mathbb{R}\) and \(\psi = r^n \chi\). Then, under the hypothesis of Lemma 4.2, the following identity holds:

\[
\frac{d}{dt} H_S(t) = -\frac{1}{2} \int_0^\infty \left( (n - 2\gamma_S) r^{n-1} \chi + r^n \chi' \right) u_t^2 - \frac{1}{2} \int_0^\infty \left( (n + 2\gamma_S - 4) r^{n-1} \chi + r^n \chi' \right) u_r^2, \\
- \int_0^\infty \left( (2 - \gamma_S + n(2\gamma_S - 1) - \frac{\gamma_S}{2} n(n-1)) r^{n-1} \chi + (\gamma_S (2 - n - 1) r^n \chi' - \frac{\gamma_S}{2} r^{n+1} \chi'') \right) \frac{u^2}{r^2}, \\
- \frac{\alpha^2}{2} \int_0^\infty \left( \frac{1}{2} (4 - n - 8\gamma_S - \gamma_S (-7 + n)n) r^{n-1} \chi + (\gamma_S (3 - n) - \frac{1}{2} r^n \chi' - \frac{\gamma_S}{2} r^{n+1} \chi'') \right) \frac{u^4}{r^4}, \\
- \alpha^2 \int_0^\infty \left( (n - 2 - 4\gamma_S) r^{n-1} \chi + r^n \chi' \right) \frac{u^2}{r^2} (u_t^2 + u_r^2) + H_e(t)
\]

and for \(H_e\) holds

\[ H_e(t) = -\frac{1}{3} \int_0^\infty \left( (n - 2(1 + 2\gamma_S)) r^{n-1} \chi + r^n \chi' \right) \frac{u^4}{r^2}. \]
+ \frac{2}{45} \int_0^\infty \left( (n - 2 (1 + \gamma_S)) r^{n-1} \chi + r^n \chi' \right) \left( \frac{u^6}{r^2} + \frac{O(u^8)}{r^2} \right) \right] \\
- \alpha^2 \int_0^\infty \left[ \frac{1}{3} \left( (-n + 2(1 + 6\gamma_S)) r^{n-1} \chi + r^n \chi' \right) u^2 \right] \frac{u^2}{r^2} (u_t^2 + u_r^2) \\
- \frac{2}{45} \alpha^2 \int_0^\infty \left[ (n - 2(1 + 4\gamma_S)) r^{n-1} \chi + r^n \chi' \right] \left( u^4 + O(u^6) \right) \frac{u^2}{r^2} (u_t^2 + u_r^2) \\
+ \frac{1}{9} \alpha^2 \int_0^\infty \left[ (\gamma_S n (5 - n) - 3n + 6\gamma_S + 12) r^{n-1} \chi' \right] \left( \frac{u^6}{r^4} + \frac{O(u^8)}{r^4} \right).

**Proof** The proof follows directly replacing \( \psi = r^n \chi \) in Lemma 4.2.

Now, if we set \( \chi = 1 \), we obtain the following result:

**Corollary 4.4** Let \( \delta > 0 \) small enough, \( \psi = r^n \) and \( u \) be a global solution of (1.2) such that \( u \in E_0^{S,r^n,n} \) and \( \|u\|_{L^\infty} < \delta \). Then, for \( \gamma_S = -1 \) and \( n \geq 2 \), the functional \( H_S(t) \) satisfies

\[
\frac{d}{dt} H_S(t) = -\frac{1}{2} \int_0^\infty r^{n-1} \left[ (n + 2)u_t^2 + (n - 6)u_r^2 + (n - 6)(n - 1) \frac{u^2}{r^2} \\
+ 2\alpha^2 (n + 2) \frac{u^2}{r^2} (u_t^2 + u_r^2) + \alpha^2 (n - 6)(n - 2) \frac{u^4}{r^4} \right] + H_e(t),
\]

with \( |H_e(t)| \leq \delta^2 E_{S,r^n,n-1}(t) \).

Assuming \( \delta > 0 \) small enough, \( n \geq 6 \) and applying Corollary 4.4, we obtain the following virial inequality

\[
- \frac{d}{dt} H_S(t) \geq \frac{1}{4} \int_0^\infty r^{n-1} \left[ (n + 2)u_t^2 + (n - 6)u_r^2 + (n - 6)(n - 1) \frac{u^2}{r^2} \\
+ 2\alpha^2 (n + 2) \frac{u^2}{r^2} (u_t^2 + u_r^2) + \alpha^2 (n - 6)(n - 2) \frac{u^4}{r^4} \right] \geq 0. \quad (4.11)
\]

In particular, as an application of (4.11), we obtain the following result for \( r^{6+\epsilon} \) and \( r^{7+\epsilon} \) weighted energies.

**Corollary 4.5** Let \( \epsilon > 0 \) and \( u \) be a small global solution of (1.2) in the class \( E_0^{S,r^{6+\epsilon}} \cap E_0^{S,r^{7+\epsilon}} \). Then,

1. **Integrability in time:**

\[
\int_0^\infty \int_0^\infty (r^{6+\epsilon} + r^{7+\epsilon}) \left[ u_t^2 + u_r^2 + \frac{u^2}{r^2} + 2\alpha^2 \frac{u^2}{r^2} (u_t^2 + u_r^2) + \alpha^2 u^4 \right] \ d \tau \ d t \leq \epsilon.
\]
(2) Sequential decay to zero: there exists $s_n$, $t_n \uparrow \infty$ such that

$$\lim_{n \to \infty} E_{S, r^6 + \epsilon}[u, u_t](t_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} E_{S, r^7 + \epsilon}[u, u_t](s_n) = 0.$$  \hfill (4.12)

**Proof** The proof of the first statement is obtained directly integrating (4.11).

The second statement follows from the approximation of the energy $E_{S, \phi}[u, u_t](t)$ in (1.4) for $\|u\|_{L^\infty}$ small enough. Then, using (4.4), we get

$$E_{S, \phi}[u, u_t](t) \leq \int_0^\infty \phi(r) \left[ u_t^2 + u_r^2 + 2\alpha^2 \frac{u_t^2}{r^2} (u_t^2 + u_r^2) + 2\frac{u_r^2}{r^2} + \alpha^2 \frac{u^4}{r^4} \right]$$

$$+ \|u\|^2_{L^\infty} \int_0^\infty \phi(r) \left[ 2\alpha^2 \frac{u_t^2}{r^2} (u_t^2 + u_r^2) + 2\frac{u_r^2}{r^2} + \alpha^2 \frac{u^2}{r^4} \right]$$

$$\lesssim \delta \int_0^\infty \phi(r) \left[ u_t^2 + u_r^2 + 2\alpha^2 \frac{u_t^2}{r^2} (u_t^2 + u_r^2) + 2\frac{u_r^2}{r^2} + \alpha^2 \frac{u^4}{r^4} \right].$$  \hfill (4.13)

We conclude replacing $\phi = r^6 + \epsilon$ and using the first statement. \hfill \Box

4.2.2 Decay to zero for modified energy: Proof of Theorem 1.2 for the Adkins–Nappi model

Similarly to the Skyrme equation, we will need the following technical lemmas.

**Lemma 4.6** Let $u$ be a global solution of (1.3) such that $u \in E_{0}^{AN, \psi}$, and $\psi = r \phi$ (where $\psi$ and $\phi$ are the weight functions presented in (2.21) and (2.22)). Then, the functional $\mathcal{H}_{AN}(t)$, defined in (4.3), satisfies the following identity

$$\frac{d}{dt} \mathcal{H}_{AN}(t)$$

$$= -\frac{1}{2} \int_0^\infty \left( \psi' - 2\gamma_{AN} \frac{\psi}{r} \right) u_t^2 - \frac{1}{2} \int_0^\infty \left( \psi' - 2(\gamma_{AN} - 2) \frac{\psi}{r} \right) u_r^2$$

$$- \frac{1}{2} \int_0^\infty \left[ 2(1 - \gamma_{AN}) \frac{\psi}{r} - \psi' + \gamma_{AN} r \psi'' \right] \frac{u_t^2}{r^2}$$

$$+ \frac{1}{3} \int_0^\infty \left[ (4\gamma_{AN} + 1) \frac{\psi}{r} - \frac{1}{2} \psi' \right] \frac{u_t^4}{r^2}$$

$$- \frac{1}{45} \int_0^\infty \left[ 2(1 + 6\gamma_{AN}) \frac{\psi}{r} - \psi' \right] \left( \frac{u_t^6}{r^2} + O(u^8) \right)$$

$$- \frac{1}{9} \int_0^\infty \left[ 4(2 + 3\gamma_{AN}) \frac{\psi}{r} - 2\psi' \right] \left( \frac{u_t^6}{r^4} + O(u^8) \right).$$  \hfill (4.14)
Proof First, we note that $\mathcal{M}_{AN}(t)$ and $\mathcal{R}_{AN}(t)$ are well defined in $\mathcal{E}_0^{AN, \psi}$. Collecting (2.23) and (2.24), we get that $\mathcal{H}_{AN}(t)$ is given by

$$\frac{d}{dt} \mathcal{H}_{AN}(t) = -\frac{1}{2} \int_0^\infty \left( \psi' - 2 \gamma_{AN} \phi \right) u_r^2 - \int_0^\infty \left( \frac{\psi' - 2 \psi}{r} \right) u_r^2 - \gamma_{AN} \int_0^\infty \phi u_r^2$$

$$- \frac{1}{2} \int_0^\infty \left( \frac{\psi' - \psi}{r} \right) \sin^2(u) - \frac{1}{2} \int_0^\infty \left( \frac{4 \psi - \psi'}{r} \right) \left( u - \sin(u) \cos(u) \right)^2$$

$$- \gamma_{AN} \int_0^\infty \left[ \phi' r - \phi - \frac{r^2 \phi r}{2} \right] u_r^2 - \gamma_{AN} \int_0^\infty \left( \frac{\phi}{r^4} \right) u \sin(2u)$$

$$- \gamma_{AN} \int_0^\infty \left( \frac{\phi}{r^4} \right) u \left( u - \sin(u) \cos(u) \right) \left( 1 - \cos(2u) \right).$$

Now, let $\delta > 0$ small enough and using the Taylor approximation for $\|u\|_{L^\infty} < \delta$, we have

$$u \sin(2u) = 2 u^2 - \frac{4}{3} u^4 + \frac{4}{15} u^6 + O(u^8),$$

$$\sin^2(u) = u^2 - \frac{1}{3} u^4 + \frac{2}{45} u^6 + O(u^8),$$

$$(u - \sin u \cos u)^2 = \frac{4}{9} u^6 - \frac{8}{45} u^8 + O(u^{10}),$$

$$u \left( u - \sin(u) \cos(u) \right) \left( 1 - \cos(2u) \right) = -\frac{4}{3} u^6 - \frac{32}{47} u^8 + O(u^{10}).$$

Replacing in $\frac{d}{dt} \mathcal{H}_{AN}(t)$ and regrouping terms of same order, we get

$$\frac{d}{dt} \mathcal{H}_{AN}(t)$$

$$= -\frac{1}{2} \int_0^\infty \left( \psi' - 2 \gamma_{AN} \phi \right) u_r^2 - \int_0^\infty \left( \frac{\psi' - 2 \psi}{r} + \gamma_{AN} \phi \right) u_r^2$$

$$- \int_0^\infty \left[ \frac{\psi}{r} - \frac{1}{2} \psi' + \gamma_{AN} \left( \phi' r + \phi - \frac{r^2 \phi r}{2} \right) \right] u^2 r^2$$

$$- \int_0^\infty \left[ \frac{1}{6} \psi' - \frac{1}{3} \psi + 4 \gamma_{AN} \phi \right] u^4 r^2$$

$$- \frac{1}{45} \int_0^\infty \left[ \frac{2 \psi}{r} - \psi' + 12 \gamma_{AN} \phi \right] \left( \frac{u^6}{r^2} + \frac{O(u^8)}{r^2} \right)$$

$$- \frac{2}{9} \int_0^\infty \left[ \frac{4 \psi}{r} - \psi' + 12 \gamma_{AN} \phi \right] \left( \frac{u^6}{r^4} + \frac{O(u^8)}{r^4} \right).$$

Since $\psi = r \phi$, we get

$$\phi' = \frac{\psi'}{r} - \frac{\psi}{r^2}, \quad \phi'' = \frac{\psi''}{r} - \frac{2 \psi'}{r^2} + \frac{2 \psi}{r^3}.$$
Then, rewriting $\mathcal{H}_{AN}(t)$ in terms of $\psi$, we get

\[
\frac{d}{dt} \mathcal{H}_{AN}(t) = -\frac{1}{2} \int_0^\infty \left( \psi' - 2\gamma_{AN} \frac{\psi}{r} \right) u_t^2 - \frac{1}{2} \int_0^\infty \left( \psi' - 2(\gamma_{AN} - 2) \frac{\psi}{r} \right) u_r^2
\]

\[
- \frac{1}{2} \int_0^\infty \left[ 2(1 - \gamma_{AN}) \frac{\psi}{r} - \psi' + \gamma_{AN} r^\psi \right] u_r^2
\]

\[
- \frac{1}{3} \int_0^\infty \left[ -(4\gamma_{AN} + 1) \frac{\psi}{r} + \frac{1}{2} \psi' \right] u_r^4
\]

\[
- \frac{1}{45} \int_0^\infty \left[ 2(1 + 6\gamma_{AN}) \frac{\psi}{r} - \psi' \right] \left( \frac{u^6}{r^2} + \frac{O(u^8)}{r^2} \right)
\]

\[
- \frac{1}{9} \int_0^\infty \left[ 4(2 + 3\gamma_{AN}) \frac{\psi}{r} - 2\psi' \right] \left( \frac{u^6}{r^4} + \frac{O(u^8)}{r^4} \right).
\]

This ends the proof. \(\square\)

Similarly to Skyrme equation, using $\psi = r\phi$ and the Cauchy–Schwarz inequality, we get

\[
|\mathcal{M}_{AN}(t)| + |\mathcal{R}_{AN}(t)| \leq E_{AN, \psi}[u, u_t](t).
\]

Then, the functionals $\mathcal{M}_{AN}(t)$ and $\mathcal{R}_{AN}(t)$ are well-defined if $u \in C^1_{0, \psi}$.

**Corollary 4.7** Under the hypothesis of Lemma 4.6 and assuming $n, \gamma_{AN} \in \mathbb{R}$ and $\psi = r^n \chi$, the following holds:

\[
\frac{d}{dt} \mathcal{H}_{AN}(t) = -\frac{1}{2} \int_0^\infty \left[ (n - 2\gamma_{AN}) r^{n-1} \chi + r^n \chi' \right] u_t^2
\]

\[
- \frac{1}{2} \int_0^\infty \left[ (n + 4 - 2\gamma_{AN}) r^{n-1} \chi + r^n \chi' \right] u_r^2
\]

\[
- \frac{1}{2} \int_0^\infty \left[ (n - 2)(1 - \gamma_{AN}n - \gamma_{AN}) r^{n-1} \chi + (\gamma_{AN} - 1)r^n \chi' + \gamma_{AN} r^{n+1} \chi'' \right] u_r^2
\]

\[
- \frac{1}{3} \int_0^\infty \left[ -(4\gamma_{AN} + 1) + \frac{1}{2} n \right] r^{n-1} \chi + r^n \chi' \right] u_r^4
\]

\[
- \frac{1}{45} \int_0^\infty \left[ (2(1 + 6\gamma_{AN}) - n) r^{n-1} \chi - r^n \chi' \right] \left( \frac{u^6}{r^2} + \frac{O(u^8)}{r^2} \right)
\]

\[
- \frac{1}{9} \int_0^\infty \left[ (4(2 + 3\gamma_{AN}) - 2n) r^{n-1} \chi - 2r^n \chi' \right] \left( \frac{u^6}{r^4} + \frac{O(u^8)}{r^4} \right).
\]

**Proof** The proof follows directly using (4.13) and replacing $\psi = r^n \chi$. \(\square\)
Now, considering that $\chi = 1$, we obtain the following result:

**Corollary 4.8** Let $\psi = r^n$, and $u$ be a global solution of (1.3) such that $u \in E_0^{AN,r^n}$. Then, for $\gamma_{AN} = (n - 2)/8$ and $n \geq 2$, the functional $H_{AN}$, defined in (4.3), satisfies the following identity

$$
\frac{d}{dt} H_{AN}(t) \leq -\frac{1}{2} \int_0^\infty r^{n-1} \left( \frac{3n+2}{4} u_t^2 + \frac{3(6+n)}{4} u_r^2 + \frac{(n-2)(n^2 - n - 10)}{8} \frac{u^2}{r^2} 
+ n - 2 \left( \frac{u^6}{r^2} + \frac{O(u^8)}{r^2} \right) + \left( \frac{10-n}{9} \frac{u^2}{r^4} + \frac{O(u^8)}{r^4} \right) \right).$$

For $n \in \left[ \frac{1+\sqrt{41}}{2}, 10 \right]$, by Corollary 4.8, we obtain the following inequality

$$
\frac{d}{dt} H_{AN}(t) \geq \frac{1}{4} \int_0^\infty r^{n-1} \left( \frac{3n+2}{4} u_t^2 + \frac{3(6+n)}{4} u_r^2 + \frac{(n-2)(n^2 - n - 10)}{8} \frac{u^2}{r^2} 
+ n - 2 \frac{u^6}{r^2} + \frac{(10-n)u^6}{9r^4} \right) \geq 0,
$$

which is essential to obtain the integrability property. In particular, we obtain the following result for the $r^{4+\epsilon}$ and $r^{5+\epsilon}$ weighted energies.

**Corollary 4.9** Let $u$ be a global solution of (1.3) in the class $E_0^{AN,r^{4+\epsilon}} \bigcap E_0^{AN,r^{5+\epsilon}}$ for $\epsilon \in [0, 4]$. Then,

1. **Integrability in time:**
   $$
   \int_2^\infty \int_0^\infty (r^{4+\epsilon} + r^{5+\epsilon}) \left( u_t^2 + u_r^2 \right) \left( \frac{u^2}{r^2} + \frac{u^6}{r^4} \right) dr dt \lesssim u_0^1.
   $$

2. **Sequential decay to zero:** there exists $s_n, t_n \uparrow \infty$ such that
   $$
   \lim_{n \to \infty} E_{AN,r^{5+\epsilon}}[u, u_t](t_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} E_{AN,r^{4+\epsilon}}[u, u_t](s_n) = 0.
   $$

The proof of above corollary follows directly from (4.14). With these results, we are ready to conclude the proof of Theorem 1.2 for the Skyrme and Adkins–Nappi equations.

### 4.3 End of the proof of Theorem 1.2

Consider $E_{S,\varphi}$ as in (1.10) with $\varphi = r^{7+\epsilon}$. From (2.3), we have

$$
\frac{d}{dt} E_{S,\varphi}[u, u_t](t) = -2 \int_0^\infty \left( \varphi' - 2\frac{\varphi}{r} \right) \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) u_t u_r.
$$
Therefore,

\[
\left| \frac{d}{dt} E_{S,\varphi}[u, u_t](t) \right| \lesssim \int_0^\infty \left| \varphi' - 2\frac{\varphi}{r} \right| \left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_r^2 + u_t^2) dt.
\]

Integrating in \([t, t_n] \), we have

\[
\left| E_{S,\varphi}[u, u_t](t) - E_{S,\varphi}[u, u_t](t_n) \right| \lesssim \int_t^{t_n} \int_0^\infty \left| \varphi' - 2\frac{\varphi}{r} \right| \\
\left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_r^2 + u_t^2) dr dt.
\]

Sending \(n \) to infinity, we have from (4.12) that \( E_{S,\varphi}[u](t_n) \rightarrow 0 \) and

\[
\left| E_{S,\varphi}[u, u_t](t) \right| \lesssim \int_t^\infty \int_0^\infty \left| \varphi' - 2\frac{\varphi}{r} \right| \\
\left( 1 + \frac{2\alpha^2 \sin^2(u)}{r^2} \right) (u_r^2 + u_t^2) dr dt.
\]

Finally, if \( t \rightarrow \infty \), we conclude. Since \( E_{S,\varphi}[u, u_t](t) \gtrsim \| (r^{\frac{5+\epsilon}{2}} u_t, r^{\frac{5+\epsilon}{2}} u_r)(t) \|_{L^2 \times L^2(\mathbb{R}^3)}^2 \), this proves Theorem 1.2 for the Skyrme equation.

The proof in the Adkins–Nappi case is analogous considering \( E_{AN,\varphi} \) in (1.11) with \( \varphi = r^{5+\epsilon} \).

This concludes the proof of Theorem 1.2.

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Declarations

Conflict of Interest Statement On behalf of all authors, the corresponding author states that there is no conflict of interest.

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