Almost Periodic Solutions to Impulsive Stochastic Delay Differential Equations Driven by Fractional Brownian Motion With $\frac{1}{2} < H < 1$

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In this article, we study the existence and uniqueness of square-mean piecewise almost periodic solutions to a class of impulsive stochastic functional differential equations driven by fractional Brownian motion. Moreover, the stability of the mild solution is obtained. To illustrate the results obtained in the paper, an impulsive stochastic functional differential equation driven by fractional Brownian motion is considered.

Keywords: fractional Brownian motion, square-mean piecewise almost periodic solution, impulsive systems, stochastic functional differential equation, stability

1 INTRODUCTION

Impulsive systems arise naturally in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time. Impulsive stochastic modeling has come to play an important role in many branches of science where more and more people have encountered impulsive stochastic differential equations. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [1] as a closed system with a simplified heart, one organ, or capillary bed, and recirculation of blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. Recently, there has been a significant development in impulsive stochastic differential equations (ISDEs). The existence and stability of ISDEs were investigated in [2–11] and the references therein.

On the other hand, in recent years, there has been considerable interest in studying fractional Brownian motions (fBms) due to their compact properties and applications in various scientific areas, including telecommunications [12, 13], turbulence [14], image processing [15], and finance [16]. Stochastic differential equations (SDEs) driven by fBms attract the interest of researchers [2, 3, 17–21]. Taking the time delay into account, the theory of stochastic differential equations has been generalized to stochastic functional differential equations; it makes the dynamics more complex and the system may lose stability and show almost periodicity. Arthi et al. [2] considered the existence and exponential stability for neutral stochastic integrodifferential equations with impulses driven by fractional Brownian motion (fBm), and Caraballo [3] studied the existence of mild solutions to stochastic delay evolution equations with fBm and impulses.

In this paper, we are concerned with the existence and stability of almost periodic mild solutions to the following impulsive stochastic functional differential system driven by fBm with Hurst index $H \in (1/2, 1)$:

\[
\begin{align*}
\frac{dx(t)}{dt} &= [Ax(t) + b(t, x_t)]dt + \sigma(t)dB^H(t), \\
\Delta x(t_i) &= x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i \in \mathbb{Z}, \\
x_{t_0} &= \xi = [\xi(t): -\theta \leq t \leq 0],
\end{align*}
\]

(1)
where $\mathbb{Z}$ is the set of integers, for any $i, k \in \mathbb{Z}$, and the sequence $\{t_i\}$ is such that the derived sequence $\{t_i^k = t_{i+k} - t_i\}$ is equipotentially almost periodic. Moreover, $A: D(A) \subset H \to H$ is a linear bounded operator, $\rho(A)$ is the resolvent set of $A$, and $\lambda \in \rho(A)$, $R(\lambda, A)$ is the resolvent of $A$. In addition, $b$, $\sigma$, and $I_t$ are appropriate functions, $x_t(\cdot) \colon [-\theta, 0] \to \mathbb{H}$ is given by $x_t(s) = x(t + s)$, for any $s \in [-\theta, 0]$, and $\xi \in C_0$ is an $\mathcal{F}_0$-measurable random variable such that $E\|\xi\|^2 < \infty$. Let $\theta > 0$ be a given constant and let $C_0 = \{\phi \colon [-\theta, 0] \times \Omega \to \mathbb{H}, \phi$ be continuous everywhere except for a finite number of points $s$ at which $\phi(s')$ and $\phi(s')$ exist and satisfy $\phi(s') = \phi(s)$, endowed with the norm $\|\phi\|_C = \sup_{0 \leq s \leq R} (E|\phi(s)|^2)^{1/2}$, such that $\phi(s, \cdot)$ is $\mathcal{F}_0$-measurable for each $s \in [-\theta, 0]$ and $\sup_{s \in [-\theta, 0]} E|\phi(s)|^2 < \infty$.

There are several difficulties with our problems. First, there is the delay for the impulsive stochastic differential equations. Second, about the stochastic differential equations driven by fractional Brownian motion, the classical stochastic integral failed for lack of the martingale property. Third, there is no known solution for stochastic partial delay differential equations driven by fractional Brownian motion. The lifting space method, mild solutions, fixed point theorem, and semigroup theory will be used to overcome these difficulties.

The paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. Section 3 is devoted to stating the existence and uniqueness of the mild square-mean piecewise almost periodic solution to (1). In Section 4, we show the stability of the mild square-mean piecewise almost periodic solution. An example is provided to illustrate the effectiveness of the results.

2 PRELIMINARIES

Let $(\mathbb{H}, \|\cdot\|_\mathbb{H}, (\cdot, \cdot)_\mathbb{H})$ and $(\mathbb{K}, \|\cdot\|_\mathbb{K}, (\cdot, \cdot)_\mathbb{K})$ denote two real separable Hilbert spaces. We denote by $L(\mathbb{H}, \mathbb{K})$ the set of all linear bounded operators from $\mathbb{H}$ into $\mathbb{K}$, equipped with the usual operator norm $\|\cdot\|$ and use $|$ to denote the Euclidean norm of a vector. In this article, we use the symbol $\|\cdot\|$ to denote the norms of operators regardless of the spaces involved when no confusion possibly arises. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual condition.

2.1 Fractional Brownian Motion

In this subsection, we briefly introduce some useful results about fBm and the corresponding stochastic integral taking values in a Hilbert space. For more details, refer to Hu [22], Mishura [23], Nualart [24], and references therein.

A real standard fractional Brownian motion $\{\beta^H(t), t \in \mathbb{R}\}$ with Hurst parameter $H \in (0, 1)$ is a Gaussian process with continuous sample paths such that $E[\beta^H(t)] = 0$ and $E[\beta^H(t)\beta^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$ for all $s, t \geq 0$. It is known that fBm $\{\beta^H(t), t \geq 0\}$ with $H > \frac{1}{2}$ admits the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s)dW(s),$$

where $W$ is a standard Brownian motion and the kernel $K_H(t, s)$ is given by

$$K_H(t, s) = c_H \int_s^t (u-s)^{H-\frac{1}{2}} \left(\frac{u}{s}\right)^{\frac{H}{2}} du, s < t,$$

where $c_H > 0$ is a constant satisfying $E[\beta^H(t)]^2 = 1$. For any function $\sigma \in L^2(0, T)$, the Wiener integral of $\sigma$ with respect to $\beta^H$ is defined by

$$\sigma^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n^H(t),$$

where $\lambda_n, n = 1, 2, \cdots$ are independent fBms with the same Hurst parameter $H \in (\frac{1}{2}, 1)$, $\{e_n, n \in N\}$, which is a complete orthonormal basis in $\mathbb{K}$, $\{\lambda_n, n \in N\}$ a bounded sequence of non-negative real numbers satisfying $Qe_n = \lambda_n e_n$, and $Q$ is non-negative self-adjoint trace class operator with $TrQ = \sum_{n=1}^{\infty} \lambda_n < +\infty$.

Let $L_2^2(\mathbb{H}, \mathbb{K})$ denote the space of all $\sigma \in L^2(\mathbb{H}, \mathbb{K})$ such that $Q^\sigma$ is a Hilbert–Schmidt operator. The norm is defined by $\|\sigma\|_2^2 = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} e_n^\sigma\|^2$. Generally, $\sigma$ is called a Q-Hilbert–Schmidt operator from $\mathbb{H}$ to $\mathbb{K}$.

Definition 2.1. Let $\sigma \colon [0, T] \to L_2^2(\mathbb{H}, \mathbb{K})$ such that

$$\sum_{n=1}^{\infty} \|K_H^s(\sigma e_n^\sigma)\|_2^2 < \infty,$$

then the stochastic integral of $\sigma$ with respect to fBm $\beta^H$ is defined by

$$\int_0^t \sigma(s)d\beta^H(s) := \sum_{n=1}^{\infty} \int_0^t \sigma(s)Qe_n^\sigma d\beta^H(s) = \sum_{n=1}^{\infty} \int_0^t (K_H^s(\sigma(s) e_n^\sigma)) (s)dW(s).$$

Remark. If $\{\lambda_n, n \in N\}$ is a bounded sequence of non-negative real numbers such that the nuclear operator $Q$ satisfies $Qe_n = \lambda_n e_n$, assuming that there exists a positive constant $K_\sigma$ such that $\|\sigma\|_2^2 \leq K_\sigma$ uniformly in $[0, T]$, then it is obvious that $\sum_{n=1}^{\infty} \|\sigma e_n^\sigma\|_2^2$ is uniformly convergent for $t \in [0, T]$.

2.2 Piecewise Almost Periodic Stochastic Processes

In this subsection, we recall some notations about the square-mean piecewise almost periodic stochastic process and introduce
some lemmas. For further details, we refer to Takens and Teissier [25] and Liu [26].

Recall that a stochastic process $X: \mathbb{R} \to L^2(\Omega; \mathbb{H})$ is said to be continuous if
\[
\lim_{t \to s} E\|X(t) - X(s)\|^2 = 0,
\]
for all $s \in \mathbb{R}$, and it is said to be bounded if there exists $N > 0$, such that $E\|X(t)\|^2 \leq N$ for all $t \in \mathbb{R}$. For convenience, we list the following concepts and notations:

- $L^2(\Omega; \mathbb{H})$ is Banach space when it is equipped with norm $\| \cdot \|_{L^2(\Omega; \mathbb{H})}$.
- Let $T$ be the set consisting of all real sequences $(t_i)_{i \in \mathbb{Z}}$ such that $\alpha = \inf_{i \in \mathbb{Z}} (t_{i+1} - t_i) > 0$, and $\lim t_i = \infty$, $\lim t_i = -\infty$, $x(t_i)$ and $x(t_n)$ represent the left and right limits of $x(t)$ at the point $t_i$.
- For any $\phi \in C_0$, $b(\cdot, \phi)$ is stochastically continuous at point $t$ for any $t \in [t_i, t_{i+1})$.
- For $k < t_i < t_{i+1}$, $t = t_{i+1}$, and $(i - k)\alpha$, if $(t_i)_{i \in \mathbb{Z}} \in T$.

Definition 2.2. ([28]). The family of the sequence $(t_{i+k})_{i \in \mathbb{Z}}$ holds for each $\tau \in \mathbb{Q}_\varepsilon$ and $i \in \mathbb{Z}$ such that the inequality
\[
|t_{i+k} - t_i - \tau| < \varepsilon,
\]
holds for each $i \in \mathbb{Q}_\varepsilon$.

Definition 2.3. A function $(b(t), t \geq 0)$ is said to be square-mean piecewise almost periodic if the following conditions are fulfilled:

a) For any $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points $t'$ and $t''$ belong to the same interval of continuity and $|t' - t''| < \delta$, then $E\|b(t') - b(t'')\|^2 < \varepsilon$.

b) For any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$, such that every interval of length $l(\varepsilon)$ contains a number $\tau$ with the property
\[
\sup_{t \in \mathbb{R}} E\|b(t + \tau - b(t))\|^2 < \varepsilon,
\]
which satisfies the condition $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$.

Let $AP_T(\mathbb{R}, L^2(\Omega; \mathbb{H}))$ denote the space of all square-mean piecewise almost periodic functions. Obviously $AP_T(\mathbb{R}, L^2(\Omega; \mathbb{H}))$ endowed with the supremum norm is a Banach space. Let $UPC(\mathbb{R}, L^2(\Omega; \mathbb{H}))$ be the space of all functions $b \in PC(\mathbb{R}, L^2(\Omega; \mathbb{H}))$ such that $b$ satisfies the condition (a) in Definition 2.3. It is easy to check that $UPC(\mathbb{R}, L^2(\Omega; \mathbb{H}))$ is a Banach space with the norm
\[
\|X\|_\infty = \sup_{t \in \mathbb{R}} E\|X(t)\|^2 < \varepsilon,
\]
for each $X \in UPC(\mathbb{R}; L^2(\Omega; \mathbb{H}))$.

Definition 2.4. (compare with [28]). A sequence $(x_t): \mathbb{Z} \to L^2(\Omega; \mathbb{H})$ is called square-mean almost periodic if for any $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for each $k \in \mathbb{Z}$, there is at least one integer $p \in (k, k + N)$, for which inequality
\[
E\|x_{t+p} - x_t\|^2 < \varepsilon,
\]
holds for all $i \in \mathbb{Z}$.

Definition 2.5. The function $b(t, \phi) \in PC(\mathbb{R} \times C_0, L^2(\Omega, \mathbb{H}))$ is said to be square-mean piecewise almost periodic in $t \in \mathbb{R}$ uniformly in $\phi$, where $\Delta \subseteq C_0$ is compact if for any $\varepsilon > 0$, there exists $(\varepsilon, \Lambda) > 0$ such that any interval of length $l(\varepsilon, \Lambda)$ contains at least a number $\tau$ for which
\[
\sup_{t \in \mathbb{R}} E\|b(t + \tau, x) - b(t, x)\|^2 < \varepsilon,
\]
for each $x \in \Lambda$, $t \in \mathbb{R}$, satisfying $|t - t_i| < \varepsilon$. The collection of all such processes is denoted by $AP_T(\mathbb{R} \times C_0, L^2(\Omega, \mathbb{H}))$.

Lemma 2.1. Let the function $f: \mathbb{R} \times C_0 \to L^2(\Omega; \mathbb{H})$ be square-mean piecewise almost periodic in $t \in \mathbb{R}$ uniformly for $y \in C_0$, where $\Lambda \subseteq C_0$ is compact. If $f$ is a Lipschitz function in the following sense,
\[
E\|f(t, y) - f(t, \tilde{y})\|^2 < M_2(\|y - \tilde{y}\|_{L^2(\Omega; \mathbb{H})}),
\]
for all $y, \tilde{y} \in C_0$, $t \in \mathbb{R}$, and a constant $M_2 > 0$, then for any $\phi(t) \in AP_T(\mathbb{R}, L^2(\Omega; \mathbb{H}))$, $f(\cdot, \phi(t)) \in AP_T(\mathbb{R}, L^2(\Omega; \mathbb{H}))$.

Proof. Noting that $\phi(t_i)$ is square-mean almost periodic, we can conclude that $\phi(t_i) - \theta \leq s \leq 0, \theta > 0)$ is square-mean almost periodic by Theorem 1.2.7 of [29]. Thus, for each $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that every interval with the length $l(\varepsilon)$ contains a number $\tau$ satisfying
\[
E\|f(t + \tau, \phi(t)) - f(t, \phi(t))\|^2 < \varepsilon,
\]
for any $x \in \Lambda(\subseteq C_0)$, $t \in \mathbb{R}$ with $|t - t_i| > \varepsilon$. Using the elementary inequality $|a + b|^2 \leq 2|a|^2 + |b|^2$ and condition (3), we have
\[
E\|f(t + \tau, \phi_{i+1}) - f(t, \phi_{i+1})\|^2 \leq 2E\|f(t + \tau, \phi_{i+1}) - f(t, \phi_{i+1})\|^2 + 2E\|f(t, \phi_{i+1}) - f(t, \phi_i)\|^2 \leq 2E\|f(t + \tau, \phi_{i+1}) - f(t, \phi_{i+1})\|^2 + 2M_2^2 E\|\phi_{i+1} - \phi_i\|_{L^2(\Omega; \mathbb{H})}^2,
\]
for all $t \in \mathbb{R}$. Combining (4) and (5), one can show that
which implies that \( f(t, \phi(t)) \) is square-mean piecewise almost periodic.

### 3 Existence of Square-Mean Piecewise Almost Periodic Solution

In this section, we study the existence of the square-mean piecewise almost periodic solution to (1). We first present some assumptions as follows:

(H1) Let the bounded linear operator \( A \) be an infinitesimal generator of an analytic semigroup \( \{S(t), t \geq 0\} \) such that

\[
\|S(t)\| \leq M e^{-\gamma t}, \quad t \geq 0
\]

for some \( \gamma > 0, M > 0 \). Moreover, \( R(\lambda, A) \) is almost periodic, where \( \lambda \in \rho(A) \).

(H2) Let \( b \in \mathcal{AP}_T(\mathbb{R} \times C_b, L^2(\Omega, \mathbb{H})) \). Moreover, there exists a positive constant \( M_b \) such that

\[
E\|b(t, x) - b(t, \bar{x})\|^2 \leq M_b \|x - \bar{x}\|^2_{C_b},
\]

for any \( x, \bar{x} \in C_b \).

(H3) Let \( \sigma \in \mathcal{AP}_T(\mathbb{R} L^3_b(\Omega, L^2(\Omega, \mathbb{H}))) \) and let \( \{I_i x(t_i), i \in \mathbb{Z}\} \) be a square-mean piecewise almost periodic sequence satisfying

\[
E\|I_i x(t_i) - I_i(y_i)\|^2 \leq M_I E\|x - y\|^2,
\]

for some positive constant \( M_I \).

Recall the notion of a mild solution for Eq. 1.

**Definition 3.1.** An \( \mathcal{F}_T \)-progressive process \( \{x(t)\}_{t \in \mathbb{R}} \) is called a mild solution of the system (1) on \( \mathbb{R} \) if it satisfies the corresponding stochastic integral equation

\[
x(t) = S(t) x_0 + \int_{t_0}^{t} S(t - s) b(s, x_s) ds + \int_{t_0}^{t} \sum_{t_0 < t_i < t} S(t - t_i) I_i(x(t_i)), \quad t \geq t_0, \quad t_0 \in \mathbb{R},
\]

where

\[
x(t) = \int_{-\infty}^{t} S(t - s) b(s, x_s) ds + \int_{-\infty}^{t} S(t - s) \sigma(s) dB_s(s) + \sum_{t_0 < t_i < t} S(t - t_i) I_i(x(t_i)),
\]

with \( t \geq t_0 \). It is easy to verify that the above equation is equivalent to (7). Define the operator \( \mathcal{L} \) on \( \mathcal{AP}_T(\mathbb{R}, L^2(\Omega, \mathbb{H})) \) by

\[
\mathcal{L} x(t) = \int_{-\infty}^{t} S(t - s) b(s, x_s) ds + \int_{-\infty}^{t} S(t - s) \sigma(s) dB_s(s) + \sum_{t_0 < t_i < t} S(t - t_i) I_i(x(t_i))
\]

for all \( t \in \mathbb{R} \). To prove the theorem, it is sufficient to show that the next statements hold:

I) \( \mathcal{L} x(t) \) is square-mean piecewise almost periodic.

II) \( \mathcal{L} \) admits a unique fixed point.

**Proof of Statement (I)** This will be done in two steps.

**Step 1.** We claim that \( \mathcal{L} x(t) \in \mathcal{UPC} \).

Let \( i \in \mathbb{Z} \). By the uniform continuity of \( S(t) \), we can see that, for any \( \epsilon > 0 \), there exists a number \( \delta > 0 \) between 0 and \( \epsilon \) such that

\[
\|S(t' - t'')\| = \min \left\{ \frac{\gamma^3 \epsilon}{\|b\|_{\infty}^2}, \frac{2H \gamma^2 \epsilon}{\sigma_{\infty}^2}, \frac{2H \gamma^2 \epsilon}{\sigma_{\infty}^2} \right\} \leq \delta
\]

for all \( t', t'' \in (t_0, t_{n+1}), t'' < t' \) as \( 0 < \epsilon < t'' < t' < \delta \), where

\[
\|b\|_{\infty} = 36 M^2 \|b\|_{\infty}^2, \quad \|\sigma\|_{\infty} = 36 H (2H - 1) M^2 \|\sigma\|_{\infty}, \quad \|\bar{I}\|_{\infty} = 9 M^2 \|\bar{I}\|_{\infty}.
\]

It follows from the inequality \( \|a + b + c\| \leq 3(a^2 + b^2 + c^2) \) that

\[
\|\mathcal{L} x(t) - \mathcal{L} x(t')\| \leq 3 \epsilon \|\mathcal{F}_1(t') - \mathcal{F}_1(t'')\|_{\|\cdot\|_{1}} + 3 \epsilon \|\mathcal{F}_2(t') - \mathcal{F}_2(t'')\|_{\|\cdot\|_{1}} + 3 \epsilon \|\mathcal{F}_3(t') - \mathcal{F}_3(t'')\|_{\|\cdot\|_{1}}
\]

for all \( t', t'' \in (t_0, t_{n+1}), t'' < t' \). By the assumptions (H1), (H2), and (H3), we have that

\[
E\|\mathcal{L} x(t) - \mathcal{L} x(t')\|^2 \leq 3 \epsilon \|\mathcal{F}_1(t') - \mathcal{F}_1(t'')\|^2 + 3 \epsilon \|\mathcal{F}_2(t') - \mathcal{F}_2(t'')\|^2 + 3 \epsilon \|\mathcal{F}_3(t') - \mathcal{F}_3(t'')\|^2,
\]

and

\[
E\left\| \int_{t_0}^{t} S(t' - s) \sigma(s) dB_s(s) ds \right\|^2 \leq M^2 \left(\int_{t_0}^{t} e^{-\gamma (t' - s)} ds \right) \leq M^2 \left(\int_{t_0}^{t} e^{-\gamma (t' - s)} ds \right)
\]

for all \( t' > t'' \). Moreover, we also have that

\[
\Theta := \frac{M^2 M_b}{\gamma^2} + \frac{M^2 M_I}{(1 - e^{-\gamma})^2} < 1.
\]
for all \( t', t'' \in (t_i, t_{i+1}) \), \( t' < t'' \) provided \( |t'' - t'| < \delta \). Thus, we have shown that the estimate

\[
E[\|L x(t) - \mathcal{L} x(t')\|^2] < \varepsilon,
\]
holds for all \( t', t'' \in (t_i, t_{i+1}) \), \( t' < t'' \) provided \( |t'' - t'| < \delta \), which means \( \mathcal{L} x(t) \in \mathcal{U}_{PC} \).

**Step 2.** We prove the almost periodicity of \( \mathcal{L} x(t) \).

For \( \Phi_1(t) \), let \( t_i < t < t_{i+1} \); by (H1), (H2), and Hölder’s inequality, we have that

\[
E[\|\Phi_1(t + \tau) - \Phi_1(t)\|^2] \\
= E\left[\int_{-\infty}^{t} S(t + \tau - s)b(s, x_s)ds - \int_{-\infty}^{t} S(t-s)b(s, x_s)ds\right]^2 \\
\leq E\left[\int_{-\infty}^{t} (S(t + \tau - s) - S(t-s))b(s, x_s)ds\right]^2 \\
\leq E\left[\sup_{r \in [\tau]}\|S(t + \tau - s) - S(t-s)\|_{\sigma}\right]E[\|b(s, x_s)\|^2]ds \\
\leq E\left[\sum_{j=0}^{i-1} \int_{t_{j+1} \eta}^{t_j \eta} e^{\gamma(t' - t_s)} E[\|b(s, x_s)\|^2]ds \right] \\
+ \sum_{j=0}^{i-1} \int_{t_{j+1} \eta}^{t_j \eta} e^{\gamma(t' - t_s)} E[\|b(s, x_s)\|^2]ds \\
+ \sum_{j=0}^{i-1} \int_{t_{j+1} \eta}^{t_j \eta} e^{\gamma(t' - t_s)} E[\|b(s, x_s)\|^2]ds \\
+ \int_{t_i \eta}^{t_{i+1} \eta} e^{\gamma(t' - t_s)} E[\|b(s, x_s)\|^2]ds,
\]
where \( \eta = \min(\varepsilon, 2^{-1}) \). By Lemma 2.1 and (H2), we find that for any \( \varepsilon > 0 \) and \( i \in \mathbb{Z} \), there exists a real number \( l(\varepsilon, \Lambda) > 0 \) such that every interval of length \( l(\varepsilon, \Lambda) \) contains at least a constant \( \tau \) and

\[
E[\|b(t + \tau, x_{s+}) - b(t, x_s)\|^2] < \varepsilon, \quad \forall \tau \in \mathbb{R},
\]
for each \( x \in \Lambda, \|t - t_i\| > \varepsilon \), since \( b \in \mathcal{A}_{\tau}(\mathbb{R}, L^2(\Omega, \mathbb{H})) \), where \( \Lambda \subset \mathcal{C}_b \) is compact.

For \( s \in [t_j + \eta, t_{j+1} - \eta], j \in \mathbb{Z}, j \leq i, t - s \geq t - t_i + t_{i+1} - (t_{j+1} - t_i) \geq t - t_i + \alpha(i + j - 1) + \eta \), we have

\[
\sum_{j=0}^{i-1} \int_{t_{j+1} \eta}^{t_j \eta} e^{\gamma(t' - t_s)} E[\|b(s, x_s)\|^2]ds \\
\leq \varepsilon \sum_{j=0}^{i-1} \int_{t_{j+1} \eta}^{t_j \eta} e^{\gamma(t' - t_s)} ds \leq \varepsilon \sum_{j=0}^{i-1} \int_{t_{j+1} \eta}^{t_j \eta} e^{\gamma(t' - t_s)} ds \\
\leq \frac{\varepsilon}{\gamma} \sum_{j=0}^{i-1} e^{-\gamma\eta} \leq \frac{\varepsilon}{\gamma(1 - e^{-\eta})}.
\]
For \( s \in [t_j, t_{j+1}) \), by the mean value theorem of integral, we get that
\[
\int_{t_j}^{t_{j+1}} e^{\gamma(t-s)} E\|b(s + \tau, x_{s+\tau}) - b(s, x_s)\|^2 ds \\
\leq 2 \sup_{s \in \mathbb{R}} E\|b(s, x_s)\| \int_{t_j}^{t_{j+1}} e^{\gamma(t-s)} ds \leq 2 \|b\|_{\text{ess}}^2 e^{\gamma t}.
\]
Similarly, we can show that
\[
\int_{t_k}^{t_{k+1}} e^{\gamma(t-s)} E\|b(s + \tau, x_{s+\tau}) - b(s, x_s)\|^2 ds \leq C_1 \epsilon,
\]
where \( C_1, C_2 \) are two positive constants. Thus, we have introduced the next estimate:
\[
E\|\Phi_1(t + \tau) - \Phi_1(t)\|^2 < N_1 \epsilon,
\]
where \( N_1 \) is a positive constant, which implies that \( \Phi_1(t) \) is square-mean piecewise almost periodic.

We now show that \( \Phi_2(t) \) is square-mean piecewise almost periodic. Recall that \( t \rightarrow \sigma(t) \) is piecewise almost periodic if for each \( \epsilon > 0 \) there exists a real number \( k(\epsilon) > 0 \) such that the estimate
\[
\|\sigma(s + \tau) - \sigma(s)\|^2 < \epsilon, \forall t \in \mathbb{R}, |t - t_k| > \epsilon, i \in \mathbb{Z}
\]
holds for every interval of length \( k(\epsilon) \) containing a number \( \tau \). By using (11) and the computation of \( \mathbb{E} \), we have
\[
E\|\Phi_1(t + \tau) - \Phi_2(t)\|^2 \\
= E\left[ \int_{t}^{t+\tau} S(t + s)\sigma(s)dB(s) - \int_{t}^{t+\tau} S(t - s)\sigma(s)dB(s) \right]^2 \\
\leq \left\| \int_{t}^{t+\tau} S(t + s)\sigma(s)dB(s) \right\|^2 + \left\| \int_{t}^{t+\tau} S(t - s)\sigma(s)dB(s) \right\|^2 \\
= H(2H - 1) \sum_{n=1}^{\infty} \int_{t-n\tau}^{t-n\tau+\tau} \left\| S(t - (u + \tau)) - S(u) \right\| Q(e(u)) \times \left\| \sigma(v + \tau) - \sigma(v) \right\| Q(e(u)) |u - v|^{2H-2} du dv \\
\leq H(2H - 1) M^2 \sum_{n=1}^{\infty} \int_{t-n\tau}^{t-n\tau+\tau} e^{\gamma(u-v)} \left\| \sigma(u + \tau) - \sigma(u) \right\| Q(e(u)) \times e^{\gamma(u-v)} \left\| \sigma(v + \tau) - \sigma(v) \right\| Q(e(u)) |u - v|^{2H-2} du dv \\
\leq H(2H - 1) M^2 \left( \int_{t-n\tau}^{t-n\tau+\tau} e^{\gamma(t-s)} \left\| \sigma(s + \tau) - \sigma(s) \right\|_2^2 ds \right)^{2H}.
\]
Furthermore, by Hölder’s inequality, we have
\[
E\|\Phi_2(t + \tau) - \Phi_2(t)\|^2 \\
\leq H(2H - 1) M^2 \left( \int_{t-n\tau}^{t-n\tau+\tau} e^{\gamma(t-s)} \left\| \sigma(s + \tau) - \sigma(s) \right\|_2^2 ds \right)^{2H}.
\]
where \( \eta = \min \{ \epsilon, \tilde{\epsilon} \} \). In the same way as that of handling \( \Phi_1(t) \), one can introduce the estimate
\[
E\|\Phi_2(t + \tau) - \Phi_2(t)\|^2 < N_2 \epsilon,
\]
where \( N_2 \) is a positive constant, and hence \( \Phi_2(t) \) is piecewise square-mean almost periodic.

For \( \Phi_3(t) = \sum_{i \in \mathbb{Z}} E\{ S(t - t_i)I_i(x(t_i)) \} \), let \( \beta_i = I_i(x(t_i)) \). For \( t_i < t \leq t_{i+1}, |t - t_i| > \epsilon, |t - t_{i+1}| > \epsilon, i \in \mathbb{Z} \), one has \( t_{i+n+1} > t + \tau > t_{i+n} \). From (H3), it follows that \( \beta_i \) is a square-mean almost periodic sequence, for any \( \epsilon > 0 \); there exists such a natural number \( N = N(\epsilon) \) that, for an arbitrary \( k \in \mathbb{Z} \), there is at least one integer \( p > 0 \) in the segment \( [k, k + N] \) such that the inequality
\[
E\|\beta_i - \beta_i\|^2 < \epsilon,
\]
holds for all \( i \in \mathbb{Z} \). We get
\[
E\|\Phi_3(t + \tau) - \Phi_3(t)\|^2 \\
= E\left[ \sum_{t_i < t < t_{i+1}} S(t - t_i)\beta_i - \sum_{t_i < t} S(t - t_i)\beta_i \right]^2 \\
\leq E\left[ \sum_{t_i < t} S(t - t_i)(\beta_i - \beta_i) \right]^2 \\
\leq M^2 \sum_{t_i < t} e^{\gamma(t-t_i)} \sum_{t_i < t} e^{\gamma(t-t_i)} E\|\beta_i - \beta_i\|^2 \\
\leq M^2 \epsilon \left( 1 - e^{-\gamma \epsilon} \right)^{2H}/(1 - e^{-\gamma \epsilon})^{2H},
\]
which implies that \( \Phi_3(t) \in \mathcal{AP}^T(\mathbb{R}, L^2(\Omega, \mathbb{H})) \). Thus, we have proved that \( L x(t) \in \mathcal{AP}^T(\mathbb{R}, L^2(\Omega, \mathbb{H})) \) and \( L x(t) \) is square-mean piecewise almost periodic.

**Proof of Statement (II).** Given \( B = \{ u \in \mathcal{AP}^T(\mathbb{R}, L^2(\Omega, \mathbb{H})) \} \) and assuming that \( x(t), y(t) \in B \) are both almost periodic solutions of (1) and \( x(t) \neq y(t) \), then we have
From (H1), (H2), (H3) and the Cauchy–Schwarz inequality, we have that

\[
A_1 \leq E \left( \int_{-\infty}^{t} M^2 e^{-2\gamma_1 s} \|b(s, x_s) - b(s, y_s)\| ds \right)^2 \\
\leq \left( \int_{-\infty}^{t} M^2 e^{-\gamma_1 s} ds \right) \left( \int_{-\infty}^{t} e^{-\gamma_1 s} E \|b(s, x_s) - b(s, y_s)\|^2 ds \right) \\
\leq \frac{M^2}{\gamma} \int_{-\infty}^{t} e^{-\gamma_1 s} M_1 E \|x_s - y_s\|^2_2 ds \\
\leq \frac{M^2}{\gamma} M_1 \sup_{r \in \mathbb{R}} E \|x(r) - y(r)\|^2 = \frac{M^2 M_1}{\gamma^2} \|x - y\|^2_\infty,
\]

and

\[
A_2 \leq E \left( \sum_{t \leq t_i} M^2 e^{-2\gamma_1 s} \|I_t(x(t)) - I_t(y(t))\|^2 \right) \\
\leq \left( \sum_{t \leq t_i} M^2 e^{-\gamma_1 s} \right) \left( \sum_{t \leq t_i} e^{-\gamma_1 s} E \|I_t(x(t)) - I_t(y(t))\|^2 \right) \\
\leq \frac{M^2}{1 - e^{-\gamma_1}} \left( \sum_{t \leq t_i} e^{-\gamma_1 s} M_1 E \|x(t) - y(t)\|^2 \right) \\
\leq \frac{M^2 M_1}{(1 - e^{-\gamma_1})^2} \sup_{r \in \mathbb{R}} E \|x(r) - y(r)\|^2 = \frac{M^2 M_1}{(1 - e^{-\gamma_1})^2} \|x - y\|^2_\infty.
\]

It follows that

\[
E \|Lx(t) - Ly(t)\|^2 \leq \Theta \|x - y\|^2_\infty,
\]

for each \( t \in \mathbb{R} \), which implies that

\[
\|Lx(t) - Ly(t)\|_\infty \leq \sqrt{\Theta} \|x - y\|_\infty.
\]

This means that \( L \) is a contraction when (8) holds and statement (II) follows.

## 4 ASYMPTOTIC STABILITY

In this section, we are interested in the asymptotical stability of the almost periodic mild solution to (1) with \( t_0 = 0 \). For convenience, we rewrite the equation as follows:

\[
dx(t) = \left[ A(t) + b(t, x(t)) dt + \sigma(t) dB^\Gamma(t) \right] dt + \sigma(t) dB^\Gamma(t), \quad t \in [t_0, \infty), \quad t \neq t_i, \quad i \in \mathbb{Z},
\]

\[
\Delta x(t_i) = x(t_i^-) - x(t_i) = I_t(x(t_i)), \quad i \in \mathbb{Z},
\]

\[
x_{t_0} = \xi = \xi(t) = [\xi(t) : -\theta \leq t \leq 0].
\]

(15)

**Lemma 4.1. ([30]).** Let a nonnegative piecewise continuous function \( t \mapsto v(t) \) satisfy the inequality

\[
v(t) \leq C + \int_{t_0}^{t} u(\sigma) v(\sigma) d\sigma + \sum_{\sigma_\ell < \sigma_i \leq t} \alpha_i v(\sigma_i),
\]

for \( t \geq t_0 \) where \( C \geq 0, u(\sigma) > 0, \alpha_i \geq 0, i \in \mathbb{Z}, \) and \( \sigma_\ell, \sigma_i \in \mathbb{Z} \) are the first kind discontinuity points of the function \( v \). Then, the following estimate holds:

\[
v(t) \leq C \prod_{\sigma_\ell < \sigma_i < t} (1 + \alpha_i) e^{\int_{t_0}^{\sigma_i} u(\sigma) d\sigma}.
\]

**Theorem 4.1.** Assume that (H1) – (H3) hold. The almost periodic solutions to (15) are asymptotically stable in the square-mean sense if

\[
\frac{1}{\alpha} \ln \left( 1 + \frac{3M^2 M_1}{1 - e^{-\gamma_1}} \right) - y + \frac{3M^2 M_1}{\gamma} < 0,
\]

(16)

**Proof.** Let \( x(t) \) and \( x'(t) \) be two square-mean piecewise almost periodic mild solutions of (15); then we have that

\[
E \|x(t) - x'(t)\|^2 = E \left\| S(t) [\xi - \xi'] + \int_{0}^{t} S(t-s) [b(s, x_s) - b(s, x'_s)] ds + \sum_{0 \leq s \leq t} S(t-s) [I_t(x(t)) - I_t(x'(t))] \right\|^2,
\]

for all \( t \geq 0 \). By using Cauchy–Schwartz’s inequality, Fubini’s theorem, and assumptions (H1) – (H3), we deduce that

\[
E \|x(t) - x'(t)\|^2 \leq 3E \|S(t) [\xi - \xi']\|^2 + 3E \left\| \int_{0}^{t} S(t-s) [b(s, x_s) - b(s, x'_s)] ds \right\|^2 + 3E \left\| \sum_{0 \leq s \leq t} S(t-s) [I_t(x(t)) - I_t(x'(t))] \right\|^2,
\]

\[
\leq 3M^4 e^{-\gamma_1 t} \|\xi - \xi'\|^2 + 3M^4 e^{-\gamma_1 t} \left( \sum_{0 \leq s \leq t} e^{\gamma_1 s} [b(s, x_s) - b(s, x'_s)] \right)^2 + 3M^4 e^{-\gamma_1 t} \left( \sum_{0 \leq s \leq t} e^{\gamma_1 s} [I_t(x(t)) - I_t(x'(t))] \right)^2,
\]

\[
\leq 3M^4 e^{-\gamma_1 t} \|\xi - \xi'\|^2 + 3M^4 M_1 \left( \sum_{0 \leq s \leq t} e^{\gamma_1 s} E \|x(t) - x'(t)\|^2 \right) + 3M^4 M_1 \left( \sum_{0 \leq s \leq t} e^{\gamma_1 s} E \|x(t) - x'(t)\|^2 \right) ds.
\]

(15)

for \( t \geq 0 \). Multiplying both sides of the above inequality by \( e^{\gamma t} \), we get

\[
e^{\gamma t} E \|x(t) - x'(t)\|^2 \leq 3M^2 e^{\gamma t} \|\xi - \xi'\|^2 + \frac{3M^2}{\gamma} E \left( \sum_{0 \leq s \leq t} e^{\gamma s} E \|x(t) - x'(t)\|^2 \right) ds + \frac{3M^2}{1 - e^{-\gamma_1}} M_1 \left( \sum_{0 \leq s \leq t} e^{\gamma s} E \|x(t) - x'(t)\|^2 \right).
\]
for \( t \geq 0 \), which implies that
\[
\sup_{0 \leq s \leq t} e^{\alpha s} \|x(s) - x^*(s)\|^2 \leq 3M^2 \|\xi - \xi^*\|^2 + \frac{3M^2}{\gamma} M_b \int_0^t \sup_{0 \leq s \leq r} e^{\alpha s} \|x(r) - x^*(r)\|^2 \ ds
\]
\[+ \frac{3M^2}{1 - e^{-\gamma t}} M_i \left( \sum_{i \leq c} \sup_{0 \leq s \leq r \leq t} e^{\alpha s} \|x(r) - x^*(r)\|^2 \right),
\]
for \( t \geq 0 \). Combining this with Lemma 4.1, we get that
\[
\sup_{0 \leq s \leq t} e^{\alpha s} \|x(s) - x^*(s)\|^2 \leq 3M^2 \|\xi - \xi^*\|^2 \left( 1 + \frac{3M^2}{1 - e^{-\gamma t}} M_i \right)_{\frac{t}{\alpha} e^{\gamma t} M_b t},
\]
for \( t \geq 0 \). So,
\[
e^{\alpha t} \|x(t) - x^*(t)\|^2 \leq 3M^2 \|\xi - \xi^*\|^2 \left( 1 + \frac{3M^2}{1 - e^{-\gamma t}} M_i \right)_{\frac{t}{\alpha} e^{\gamma t} M_b t},
\]
for \( t \geq 0 \). Thus, we get the desired estimate
\[
E \|x(t) - x^*(t)\|^2 \leq 3M^2 \|\xi - \xi^*\|^2 \left( 1 + \frac{3M^2}{1 - e^{-\gamma t}} M_i \right)_{\frac{t}{\alpha} e^{\gamma t} M_b t},
\]
and the square-mean piecewise almost periodic solution of (15) is asymptotically stable in the square-mean sense because of (16). This completes the proof.

5 AN EXAMPLE

Consider the semilinear impulsive stochastic partial functional differential equations of the following form:
\[
d\nu(t, x) = \left\{ \begin{array}{ll}
\frac{\partial}{\partial x^2} \nu(t, x) + 2a \sin (x(t - r)) \sin t \\
+ \cos t \db^{H} (t), \\
\Delta x(t) = x(t_i) - x(t_i^+) = 2a \cos (x(t_i)), \\
\nu(t, 0) = \nu(t, \pi) = 0, \\
x_{i} = \xi + [\xi(s) - \theta s \leq s \leq 0],
\end{array} \right.
\]
(17)
where \( r \) is a constant and \( \db^{H} (t) \) is a fractional Brownian motion. Denote \( X = L^2(\Omega, L^2([0, \pi])) \) and define \( A : D(A) \subseteq X \to X \) given by \( A = \frac{\partial}{\partial x^2} \) with the following domain:
\[
D(A) = \{ \nu(\cdot) \in X : \nu' \in X, \nu' \in X \}
\]
are absolutely continuous on \([0, \pi]\).

It is well known that a strongly continuous semigroup \( S(t) \) generated by the operator \( A \) satisfies \( \|S(t)\| \leq e^t \), for \( t \geq 0 \). Take
\[
b(t, x_i) = 2a \sin (x_i) \sin t,
\]
and
\[
I_i(x(t_i)) = 2a \cos (x(t_i)).
\]
Thus, one has
\[
E \|b(t, x) - b(t, y)\|^2 \leq 4a^2 \|x - y\|^2,
\]
and
\[
\|I_i(x) - I_i(y)\|^2 \leq 4a^2 \|x - y\|^2.
\]

6 CONCLUSION

In this article, we have investigated the existence and asymptotic stability of square-mean piecewise almost periodic mild solutions for a class of impulsive stochastic delay differential equations driven by fractional Brownian motion with the Hurst parameter \( H \in (\frac{1}{4}, 1) \) in a Hilbert space. An example is presented to illustrate our theoretical results. Fractional Brownian motion \( \db^{H} \) with \( H \in (0, \frac{1}{4}) \) admits different Wiener integral representation from fractional Brownian motion with \( H \in (\frac{1}{4}, 1) \). It is difficult to get the square-mean piecewise almost periodic mild solutions of ISDEs driven by fractional Brownian motion with \( H \in (0, \frac{1}{4}) \) in a Hilbert space properly due to estimation without moment.

DATA AVAILABILITY STATEMENT

The raw data supporting the conclusion of this article will be made available by the authors, without undue reservation.

AUTHOR CONTRIBUTIONS

LG and XS carried out the mathematical studies, participated in the sequence alignment, drafted the manuscript and participated in the design of the study and performed proof of results. All authors read and approved the submitted version.

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