Maximal entanglement, collective coordinates and tracking the King

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Received 30 September 2012, in final form 27 December 2012
Published 29 January 2013
Online at stacks.iop.org/JPhysA/46/075303

Abstract
Maximal entangled states provide a basis to two $d$-dimensional particles in Hilbert space, $d = \text{prime} \neq 2$. The maximally entangled states forming this basis are uniquely related to product states in the collective, center of mass and relative, coordinates. These states are associated (underpinned) with lines of finite geometry whose constituent points are associated with product states carrying mutual unbiased bases labels. This representation is shown to be convenient for the study of the mean King problem and a variant thereof, termed ‘tracking the King’, which proves to be a novel quantum communication channel. The main topics and notations used are reviewed in an attempt to keep the paper self contained.

PACS numbers: 03.65.Ta, 03.65.Wj, 02.10.Ox

1. Introduction
The mean King problem (MKP) introduced in [20] and studied extensively since [17], is a fundamental quantum mechanical problem [18]. In this problem Alice determines the outcome of the King’s measurement of one particle in an orthonormal basis $(b)$ of his choice subject to the following protocol: Alice prepares a state of her choosing that the King measures. Subsequently she is allowed one control measurement. It, like the state she prepares, may involve two particles, one inaccessible to the King. After she completes her control measurement the King informs her the basis $(b)$ he used and her task is to specify the outcome $(m)$ of his measurement.

In tracking the King, introduced here, her task is to determine, via her control measurement, the basis used by the King.

The analysis we adopt involves two recent studies, both relating to the notions of maximally (or completely) entangled states (MES) as analyzed, e.g. by [25], and to mutually unbiased bases (MUB), e.g. [1, 11, 17]. The first relates the MES to collective, viz center of ‘mass’ and relative coordinates of the constituent (two) particles. Thus the two particle system may always be accounted for by center of ‘mass’ and relative coordinates. We identify
product states in the collective coordinates that form MES [22] in the particles’ coordinates and show that these states provide a MES orthogonal basis spanning the two $d$-dimensional particles in Hilbert space. This observation simplifies the analysis considerably. For the sake of completeness and notational clarity we give in the next section, section 2, a brief review of both the single particle MUB [11, 17] and two particles mutual unbiased collective bases (MUCB) [22]. The second study involves the association of MES with finite geometry [23]. This allows an intuitive interpretation to the solution of the original MKP [17, 20, 21] and visualization of the herewith introduced tracking problem. In essence, the geometrical approach associates (underpins) states or operators in Hilbert space with lines and points of the geometry. E.g. the approach was applied to a one particle $d$-dimensional Hilbert space and provided a convenient formulation of finite dimensional Radon transformation [23]. Here we extend this single particle study to two particle systems. In so doing we identify what is termed [24] a balancing term, designated by $R$, with the aid of which Hilbert space operators/states that are interrelated via the geometry have their interrelation inverted. We summarize this topic in section 3. Thus the aim of the two succeeding sections, sections 2 and 3, is to provide the background and may be viewed as a reference for both the mathematics and the notations.

Section 4 contains the explicit expressions for the MES that play a dominant role in the analysis. We indicate how product states that relate to points in the geometry build up these MES that relate to (i.e. are ‘underpinned’ by) geometrical lines. In the last paragraphs of the section we use Fivel’s [25] results to relate these MES to our MUCB states, [22]. It will be seen that, in terms of MUCB, the proof that the $d^2$ ‘line’ underpinned MES form an orthonormal basis for the Hilbert space is trivial. It is the orthogonality and completeness of the MES that allows the definition of the measurement operator in both the standard MKP and the King’s tracking one.

The derivation of our central result is given in section 6. The first part contains the basic and intuitively obvious result that the two particle product state (underpinned by a geometrical point) has a non vanishing overlap with the MES (underpinned by a line) only when the point is on the line. We then discuss the overlap in the case of a single particle case which bears directly on the King measurement. The solution of the MKP is now outlined and the maximally entangled state prepared by Alice in this case is identified with the relevant balancing term ($R$) rather than a line underpinned MES. Next it is proved that with the line underpinned MES as the state prepared by Alice, the basis used by the King in his measurement is tracked: her control measurement allows her to infer the basis ($b$). Thus the procedure provides a novel quantum communication channel where the signal, sent by the King, is the identity of the basis, $b$, that he chose for his measurement.

Following Weyl [8] and Schwinger [6] we use unitary operators to represent physical quantities [17].

2. Brief review: mutually unbiased bases (MUB) and mutually unbiased collective bases (MUCB)

In a $d$-dimensional Hilbert space, two complete, orthonormal vectorial bases, $B_1$, $B_2$, are said to be MUB if and only if ($B_1 \neq B_2$)

$$\forall |u\rangle, |v\rangle \in B_1, B_2 \text{ resp., } |\langle u|v\rangle| = 1/\sqrt{d}. \tag{1}$$

The maximal number of MUB allowed in a $d$-dimensional Hilbert space is $d + 1$ [10, 12]. A variety of methods for construction of the $d + 1$ bases for $d = p^m$ are now available [2, 4, 11, 14]. Our present study is confined to $d = p \ (a \ prime) \neq 2$. 

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It is convenient \([2, 4, 11, 14]\) to list the \(d + 1\) MUB bases in terms of the so-called computational basis (CB). The CB states \(|n\rangle, \ n = 0, 1, \ldots d - 1, \ |n + d\rangle = |n\rangle\), are eigenfunctions of \(\hat{Z}\),

\[
\hat{Z}|n\rangle = \omega^n|n\rangle; \ \omega = e^{i2\pi/d}.
\]

We now give explicitly the MUB states in conjunction with the algebraically complete operators \([6, 9]\) set, \(\hat{Z}\), and the shift operator, \(\hat{X}|n\rangle = |n + 1\rangle\). In addition to the CB, the \(d\) other bases, each labeled by \(b\), are \([11]\)

\[
|m; b\rangle = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \alpha^{m(n-1)-nm}|n\rangle; \ b, m = 0, 1, \ldots d - 1,
\]

where \(m\) labels states within a basis. Each basis relates to a unitary operator \([11]\), \(\hat{X}^n|m; b\rangle = \omega^n|m; b\rangle\). For later reference we shall refer to the CB by \(b = 0\). Thus the \(d + 1\) bases are \(b = 0\) and \(b = 0, \ldots d - 1\). The total number of states is \(d(d + 1)\) and they are grouped in \(d\) sets each of \(d\) states. When no confusion may arise we abbreviate the states in the CB \(|m, 0\rangle\), i.e. the state \(m\) in the basis \(0\), by \(|\tilde{m}\rangle\), or simply \(|m\rangle\); we abbreviate \(|m, 0\rangle\), i.e. the \(m\) state in the basis \(b = 0\) by \(|m_0\rangle\).

We choose the phase of the CB nil, and note that the MUB set is closed under complex conjugation,

\[
\langle n|m, b\rangle^* = \langle n|\tilde{m}, \tilde{b}\rangle, \ \Rightarrow |\tilde{m}, \tilde{b}\rangle = |d - m, d - b\rangle, \ b \neq \tilde{b},
\]

\[
\langle n|m\rangle^* = \langle n|m\rangle, \ b = \tilde{b},
\]

as can be verified from equation (3).

Several studies \([13, 17, 21, 22, 25]\) consider the entanglement of two \(d\)-dimensional particles in Hilbert space via MUB state labeling. We shall now outline briefly the approach adopted by \([22]\) that will be used in later sections.

Guided by the continuum case, \(d \to \infty [22]\), where it is natural to consider collective coordinates (and operators) that refer to relative and center of mass coordinates, we consider the definitions for ‘relative’ and ‘center of mass’ for the finite dimensional Hilbert spaces. Thus the Hilbert space is spanned by the single particle computational bases, \(|n_1\rangle|n_2\rangle\) (the subscripts denote the particles). These are eigenfunctions of \(\hat{Z}_i, i = 1, 2\): \(\hat{Z}_i|n_i\rangle = \omega^n_i|n_i\rangle, \ \omega = e^{i\pi/2}\). Similarly \(\hat{X}_i|n_i\rangle = |n_i + 1\rangle\), \(i = 1, 2\). We now define our collective coordinates and collective operators (we remind the reader that the exponents are modular variables, e.g. \(1/2 \mod[d = 7] = (d + 1)/2 = 4\):

\[
\hat{Z}_x \equiv \hat{Z}_1^{1/2}\hat{Z}_2^{-1/2}; \ \hat{Z}_c \equiv \hat{Z}_1^{-1/2}\hat{Z}_2^{1/2} \leftrightarrow \hat{Z}_1 = \hat{Z}_x\hat{Z}_c; \ \hat{Z}_2 = \hat{Z}_c^{-1/2}\hat{Z}_x^{1/2},
\]

and, in a similar manner,

\[
\hat{X}_x \equiv \hat{X}_1\hat{X}_2^{-1}; \ \hat{X}_c \equiv \hat{X}_1^{-1/2}\hat{X}_2^{1/2} \leftrightarrow \hat{X}_1 = \hat{X}_x\hat{X}_c; \ \hat{X}_2 = \hat{X}_c^{-1/2}\hat{X}_x^{1/2}.
\]

We note that \(\hat{Z}_x^d = \hat{X}_x^d = 1\), and \(\hat{X}_x\hat{Z}_x = \omega\hat{Z}_x\hat{X}_x, \ s = r, c; \ \hat{X}_c\hat{Z}_c = \hat{Z}_x\hat{X}_c, \ s \neq c\).

The sets \(\hat{Z}_c, \hat{X}_c; \ i = 1, 2\) span the \(d^2\) dimensional Hilbert space. The sets \(\hat{Z}_x, \hat{X}_x; \ i = 1, 2\) are algebraically complete in this space \([6]\), i.e. every (non trivial) operator is a function of these operators. The eigenfunctions of \(\hat{Z}_q\) \(|n_q, n_q\rangle\) with \(\hat{Z}_q|n_q, n_q\rangle = \omega^n_q|n_q, n_q\rangle\). \(\hat{Z}_c|n_q, n_r\rangle = \omega^{n_q}|n_q, n_r\rangle\). We note, e.g. \([7]\), that \(|n_q, n_r\rangle\) is equivalent to \(|n_r, n_q\rangle\) when, as is the present case, the two sets, \(\hat{Z}_q, \hat{X}_q; q = c, r\) are compatible.

Clearly \(|n_1\rangle|n_2\rangle; \ n, n' = 0, 1, \ldots d - 1\), is a \(d^2\) orthonormal basis spanning the two \(d\)-dimensional particles in Hilbert space. We may consider their respective computational eigen-bases and with them the whole set of MUB bases \([22]\),

\[
\hat{Z}_a|n_s\rangle = \omega^{n_s}|n_s\rangle, \ \hat{X}_a\hat{Z}_b^0|m_s, b_s\rangle = \omega^{b_s}|m_s, b_s\rangle; \ \langle n_s|m_s, b_s\rangle = \omega^{b_s}n_s|n_s-1\rangle|m_s, b_s\rangle, s = r, c.
\]
States in the particle coordinates may clearly be expressed in terms of the product states of the collective coordinates, as both form a complete orthonormal basis that span the two particles in $d$-dimensional Hilbert space,

$$
|n_1⟩|n_2⟩ = \sum n_{e, n} |n_e, n⟩ |n_1⟩ |n_2⟩.
$$

(8)

The matrix element $⟨n_e, n | n_1⟩ |n_2⟩$ is readily evaluated [22],

$$
⟨n_1, n_2 | n_e, n⟩ = \delta_{n, (n_1 - n_2)/2} \delta_{n_1 + n_2}/2.
$$

(9)

We then have

$$
|n_e, n⟩ \sim |n_1, n_2⟩, \text{ for } n_e = (n_1 - n_2)/2,
$$

$$
n_e = (n_1 + n_2)/2 \Rightarrow n_1 = n_e + n, \text{ } n_2 = n - n_e.
$$

(10)

There are, of course, $d + 1$ MUB bases for each of the collective modes. Here too, we adopt the notational simplification $b_s \rightarrow 0_s$, $s = r, c$.

### 3. Finite geometry and Hilbert space operators: a brief outline

The association (underpinning) scheme of finite dimensional Hilbert space operators with finite plane geometry that we adopt is given in [24]. In [24], we studied the underpinning of Hilbert space single particle projectors onto a MUB state,

$$
\hat{A}_\alpha=⟨m, b| |m, b⟩
$$

with (the available) geometrical points, $S_\alpha$, $\alpha = 1, 2, \ldots, d(d + 1)$: $S_\alpha \rightarrow \hat{A}_\alpha$. In the present work we shall associate the two particle state with a geometrical point,

$$
|\tilde{m}, \tilde{b}⟩ \equiv |m, b⟩ |\tilde{m}, \tilde{b}⟩.
$$

(11)

$|\tilde{m}, \tilde{b}⟩$ is given in equation (4), the numerical subscripts refer to the particles. This new association requires that we reconsider the point–line interrelation that underpins the interrelation among the Hilbert space’s two particle state, that is underpinned by a point, and the states that correspond to lines. To this end, for the sake of clarity we briefly review the essential features of finite geometry [1, 3, 5, 15, 16, 23, 24].

A finite plane geometry is a system possessing a finite number of points ($\alpha$) and lines ($j$). A line, $L_j$ is formed by aggregate of points, $S_\alpha$; i.e. $\alpha \in j$ constitute the line $j$. There are two kinds of finite plane geometry, affine and projective. We confine ourselves to affine plane geometry (APG).

It can be shown [5, 16] that for $d = p^n$ (a power of prime) APG can be constructed (our study here is for $d = p \neq 2$. Furthermore the existence of APG implies the existence of its dual geometry DAPG wherein the points and lines are interchanged. Since we, in practice, utilize DAPG we list its properties [5, 16]. We shall refer to these by DAPG(\).

(a) The number of lines is $d^2$, $L_j$, $j = 1, 2, \ldots, d^2$. The number of points is $d(d + 1)$, $S_\alpha$, $\alpha = 1, 2, \ldots, d(d + 1)$.

(b) A pair of points on a line determine a line uniquely. Two (distinct) lines share one and only one point.

(c) Each point is common to $d$ lines. Each line contains $d + 1$ points: $S_\alpha = \bigcup_{j=1}^d L_j$; $L_j = \bigcup_{\alpha=1}^{d+1} S_\alpha$.

(d) The $d(d + 1)$ points may be grouped in sets of $d$ points; no two of a set share a line. Such a set is designated by $\alpha' \in \{\alpha \cup M_\alpha\}$, $\alpha' = 1, 2, \ldots, d$, ($M_\alpha$ contain all the points not connected to $\alpha$, i.e. they are not on a line that contain $\alpha$—they are not connected among
themselves), i.e. such a set contains \( d \) disjoint (among themselves) points. There are \( d + 1 \) such sets:

\[
\bigcup_{\alpha=1}^{d(d+1)} S_\alpha = \bigcup_{\alpha=1}^{d} R_\alpha; \quad R_\alpha = \bigcup_{\alpha' \in \cup M_\alpha} S_{\alpha'}; \quad R_\alpha \bigcap R_{\alpha'} = \emptyset, \quad \alpha \neq \alpha'. \quad (12)
\]

(e) Each point of a set of disjoint points is connected to every other point not in its set.

DAPG(c) allows the transcription, which we adopt, of \( S_\alpha \) in terms of addition of \( L_j \). This acquires a meaning upon viewing the points \( (S_\alpha) \) and the lines \( (L_j) \) as designating Hilbert space entities, e.g. projectors or states, which need to be specified. This point is further discussed at the end of this section:

\[
S_\alpha = \frac{1}{d} \sum_{j \in \alpha} L_j \Rightarrow \sum_{\alpha' \in \cup M_\alpha} S_{\alpha'} = \frac{1}{d} \sum_{j} L_j. \quad (13)
\]

DAPG(d) via equation (13) implies

\[
\sum_{\alpha' \in \cup M_\alpha} S_{\alpha'} = \frac{1}{d} \sum_{j} L_j = \frac{1}{d+1} \sum_{\alpha} L_\alpha \equiv \mathcal{R} \text{ independent of } \alpha. \quad (14)
\]

(Equation (14) reflects the relation among equivalent classes within the geometry [5].) Thus, consistency of the transcription requires that \( \mathcal{R} \) be ‘universal’ (i.e. independent of \( \alpha \)). Equation (14) will be referred to as the balance formula, with \( \mathcal{R} \) serving as a balancing term. Thus equations (13) and (14) imply, as can be verified by substitution,

\[
L_j = \sum_{\alpha \in J} S_\alpha - \sum_{\alpha' \in \cup M_\alpha} S_{\alpha'} = \sum_{\alpha \notin J} S_\alpha - \mathcal{R}. \quad (15)
\]

A particular arrangement of lines and points that satisfies DAPG, \( x = a, b, c, d, e \) is referred to as a realization of DAPG.

We now consider a generic realization of DAPG of dimensionality \( d = p \neq 2 \), which is the basis of our study [23, 24]. We arrange the \( d(d + 1) \) points, \( \alpha \), in a \( d \cdot (d + 1) \) matrix as a rectangular array of \( d \) rows and \( d + 1 \) columns. Each column is made up of a set of \( d \) points \( R_\alpha = \bigcup_{\alpha' \in \cup M_\alpha} S_{\alpha'} \); DAPG(d). We label the columns by \( b = 0, 1, 2, \ldots, d - 1 \) and the rows by \( m = 0, 1, 2, \ldots, d - 1 \). (Note that the first column label of \( b = 0 \) is for convenience with no numerical implication.) \( \alpha = m(b) \) designates a point by its row, \( m \), and its column, \( b \); when \( b \) is allowed to vary, it designates the point’s row position in every column. We label the leftmost column by \( b = 0 \) and with increasing values of \( b \), the basis label, we move to the right. Thus the rightmost column is \( b = d - 1 \).

Consider a realization of DAPG via Hilbert space operators or states, let \( \mathcal{A} \) stand for the Hilbert space entity underpinned with the coordinated point, \( (m,b) \). In this scheme the ‘universality’ of \( \mathcal{R} \) means that the sum along a fixed column, \( \sum_{\alpha \in \cup} \mathcal{A}_{\alpha \in \cup} \) is independent of the column, \( b \). In [24] \( \mathcal{A} \) stood for a projector, \( \mathcal{A}_{\alpha \in \cup} \rightarrow \mathcal{A}_\alpha = [m,b] \). In the present work \( \mathcal{A} \) signifies the two particles’ product state stipulated above, see equation (11). We now assert that the \( d + 1 \) points, \( m_j(b), b = 0, 1, 2, \ldots, d - 1 \), forming the line \( j \) contain the two (specified) points \( (m, 0) \) (denoted by \( \tilde{m} \)) and \( (m, 0) \) (denoted \( m_0 \)) is given by (we forfeit the subscript \( j \)—it is implicit in the two points, \( j = (\tilde{m}, m_0) \)),

\[
m(b) = m_0 + \frac{b}{2}(2\tilde{m} - 1), \quad b \neq 0,
\]

\[
m(\tilde{m}) = \tilde{m}, \quad b = 0. \quad (16)
\]
These lines obviously satisfy the geometrical requirements, e.g.: (i) since \( \tilde{m}, m_0 = 0, 1, 2, \ldots, d - 1 \), there are \( d \) lines, DAPG(\( a \)). (ii) Two points determine a line, DAPG(\( b \)). (iii) Every two lines has one common point and every line contains \( d + 1 \) points, DAPG(\( c \)).

For the underpinning considered in [23, 24], viz \( \mathcal{R} \rightarrow \hat{A}_\alpha \) the balancing term is \( \mathcal{R} = \sum_{m=0}^{d-1} |m, b \rangle \langle b, m| = 1 \), manifestly independent of \( \alpha \), thus consistent with the balancing requirement, equation (14). Corresponding to \( \mathcal{R} \rightarrow \hat{A}_\alpha \), we have via equation (15) \( L_j \rightarrow \hat{P}_j \), a ‘line’ operator: \( \hat{P}_j = \sum_{\alpha \in \mathcal{M}_0} \hat{A}_\alpha = 1 \). In [23, 24] this operator is shown to abide by

\[
|n|\hat{P}_j|n'\rangle = \delta_{n + w', 2m0}^{-\omega^{-\langle n - n'\rangle m_0}} \hat{P}_j^2 = 1; \forall j \; \text{tr} \hat{P}_j \hat{P}_f = d\delta_{j,f}.
\]  

(17)

The proof of the validity of the balance equation, equation (14), for the present case where the transcription is \( \hat{A}_{\alpha\rightarrow(m,b)} \rightarrow |\hat{A}_{\alpha\rightarrow(m,b)}\rangle \) is given in the next section.

It is possible to consider a different transcription corresponding to DAPG(\( c \)) [15], e.g. instead of equation (13) one may take

\[
L_j = \frac{1}{d+1} \sum_{\alpha \in \mathcal{M}_0} S_\alpha.
\]

This transcription, though consistent, leads to a more complicated formalism.

4. Geometric underpinning of two particle states

We now undertake the explicit DAPG underpinning for states in two \( d \)-dimensional particles in Hilbert space. The coordination scheme is as given above, the ‘point’ \( \alpha = (m, b) \); the ‘line’, \( j = (\tilde{m}, m_0) \). The line equation is given by equation (16). However the point \( \alpha \) underpins the state, \( |\hat{A}_{\alpha\rightarrow(m,b)}\rangle \),

\[
|\hat{A}_{\alpha\rightarrow(m,b)}\rangle = |m, b \rangle |\tilde{m}, \tilde{b}\rangle_2.
\]  

(18)

The numerical subscripts refer to the particles and \( |\tilde{m}, \tilde{b}\rangle \) is given by equation (4). Equations (13) and (15) now read

\[
|\hat{A}_{\alpha}\rangle = \frac{1}{d} \sum_{j \in \mathcal{M}_0} |P_j\rangle \quad \Rightarrow \quad |P_j\rangle = \sum_{\alpha \in \mathcal{M}_0} |\hat{A}_{\alpha}\rangle - |\mathcal{R}\rangle; \quad |\mathcal{R}\rangle = \sum_{\alpha' \in \mathcal{M}_0} |\hat{A}_{\alpha'}\rangle.
\]  

(19)

(Note that the ‘line’ state, \( |P_j\rangle \) is not normalized.) We now utilize our choice, equation (4), to show the universality of \( |\mathcal{R}\rangle \) i.e. in this case, it is independent of the basis, \( b \), since the sum over \( \alpha' = \alpha \cup M_{\alpha} \) is a sum for a fixed \( b \) [17, 25]:

\[
|\mathcal{R}\rangle = \sum_{\alpha' \in \mathcal{M}_0} |\hat{A}_{\alpha}\rangle = \sum_{m \in \mathcal{B}} |m, b \rangle |\tilde{m}, \tilde{b}\rangle = \sum_{m,n,n'} |n_1\rangle |n_2\rangle \langle m, b \rangle \langle n_1 | m, b \rangle \langle n_2 | \tilde{m}, \tilde{b}\rangle = \sum_{n} |n_1\rangle |n_2\rangle, \quad \text{indep. of } b.
\]  

(20)

The relation among the matrix elements of the projectors, \( \hat{A}_{(m,b)} = |m, b \rangle \langle b, m| \), residing on the line, equation (16) given in [23, 24], and the two particle states \( |\hat{A}_{(m,b)}\rangle = |m, b \rangle |\tilde{m}, \tilde{b}\rangle \), residing on the equivalent geometrical line, equation (16), are now used to obtain an explicit expression for the ‘line’ state, \(|P_j\rangle\)

\[
|P_j\rangle_{(m_0, m_0)} = \frac{1}{\sqrt{d}} \sum_{m(b) \in j} |m, b \rangle |\tilde{m}, \tilde{b}\rangle_2 - |\mathcal{R}\rangle
\]

\[
= \frac{1}{\sqrt{d}} \sum_{n,n'} |n_1\rangle |n_2\rangle \left[ |n\rangle \sum_{m(b) \in j} \hat{A}_{(m,b)} - |n'\rangle \right]
\]

\[
= \frac{1}{\sqrt{d}} \sum_{n,n'} |n_1\rangle |n_2\rangle \delta_{n + w', 2m0}^{-\omega^{-\langle n - n'\rangle m_0}}.
\]  

(21)
We used equations (4), (17) and (20). This formula will now be put in a more pliable form [25] and then expressed in terms of the collective coordinates

\[
|P_{j=(m_0, m)}\rangle = \frac{1}{\sqrt{d}} \sum_{n,m'} |n\rangle_1 |n'\rangle_2 \delta_{m+n', m_0} \langle n'\rangle_2 \mathrm{e}^{-i\tilde{\alpha}m_0}
= \frac{\omega^{2|m_0|}}{\sqrt{d}} \sum_n |n\rangle_1 |2\tilde{m} - n\rangle_2 \mathrm{e}^{-2im_0} = \frac{\omega^{2|m_0|}}{\sqrt{d}} \sum_n |n\rangle_1 \hat{X}^{2\tilde{m} \mod 2|m_0|} |n\rangle_2
= \frac{\omega^{2|m_0|}}{\sqrt{d}} \sum_m |m, b\rangle_1 \hat{X}^{2\tilde{m} \mod 2|m_0|} |\tilde{m}, \tilde{b}\rangle_2 = |\tilde{m}\rangle_c |2m_0\rangle_r.
\]  

(22)

The inversion operator \( I \) is defined via \( I|n\rangle = | - n\rangle \), and we used equation (4). The last equality in equation (22) follows, upon noting that

\[
|m\rangle_c |2m_0\rangle_r = \frac{1}{\sqrt{d}} \sum_m |m\rangle_1 \omega^{-2m_0} = \frac{1}{\sqrt{d}} \sum_n |\tilde{m} + n\rangle_1 |\tilde{m} - n\rangle_2 \omega^{-2m_0}.
\]

(23)

That the \( d^2 \) vectors \(|P_{j=(m_0, m)}\rangle\) form an orthonormal set is manifest in the collective coordinates formulation.

The central result of our geometrical approach is the following intuitively obvious overlap relation

\[
\langle A_{\alpha=(m,b)} | P_{j=(m_0,m)} \rangle \equiv \langle m, b \rangle_1 |\tilde{m}, \tilde{b}\rangle_2 P_{j=(m_0,m)} = \frac{1}{\sqrt{d}} \delta_{m,(m_0 + \frac{b}{2}(2\tilde{m} - 1))}, b \neq \tilde{b},
\]

\[
A_{\alpha=(m,b)} | P_{j=(m_0,m)} \rangle \equiv \langle n_1 | n_2 \rangle_2 \langle P_{j=(m_0,m)} | \rangle_2 = \frac{1}{\sqrt{d}} \delta_{n,\tilde{m}}, b = \tilde{b}.
\]

(24)

Thus the overlap of \(|A_{\alpha=(m,b)}\rangle\) with \(|P_j\rangle\) vanishes for \( \alpha \notin j \), i.e. for \( m \neq m_0 + b/2(2\tilde{m} - 1) \): only if the point \((m,b)\) is on the line \( j \) is the overlap not zero. This, as we shall see shortly, is the key to the solution of the MKP [21]. This is a remarkable attribute as it holds for the particle pair while each of its constituent particles by itself does not abide by it, see equation (26). We further note that the probability of finding the state \(|A_{\alpha}\rangle\), given that the state of the system is \(|P_j\rangle\) with \( \alpha \in j \), is \( 1/d \), while the number of points \((\alpha)\) on the line is \( d + 1 \), exposing these probabilities to not be mutually exclusive.

5. Tracking the mean King

The maximally entangled state, equation (22), was viewed geometrically as a 'line', \( j \), state, as given by the sum of product states each underpinned by the geometrical point, \( \alpha = (m,b) \), equation (18) that form the line \( j \) given by equation (16). The physical, i.e. Hilbert space, meaning of this is expressed in, cf equation (24),

\[
|\langle A_{\alpha} | P_j \rangle|^2 = |\langle \tilde{b}, \tilde{m} \rangle_2 |\langle b, m | \tilde{m} \rangle_1 |2m_0\rangle_r|^2 = \begin{cases} \frac{1}{d}, \alpha \in j, \\ 0, \alpha \notin j. \end{cases}
\]

(25)

This particle pair, viz the particle and its mate, the tilde particle, whose coordinates are \( \alpha = m, b \), do as a whole belong to the \( d \) lines that share this coordinate. However each of the constituent particles (either 1 or 2) is equally likely to be in any of the \( d^2 \) lines:

\[
\langle b, m | \tilde{m} \rangle_c |2m_0\rangle_r = \frac{1}{\sqrt{d}} |\tilde{m} + \Delta, \tilde{b}\rangle_2 \mathrm{e}^{-2i\tilde{m} \Delta}, \tilde{m} = m_0 + \frac{b}{2}(2\tilde{m} - 1); \Delta = \tilde{m} - m.
\]

(26)
It is this attribute that allows the tracking of the King’s measurement alignment.

We briefly outline and discuss the solution to the MKP as initiated by [20], and which was analyzed in several publications, e.g. as listed in [17]. The state prepared by Alice is \( |\mathcal{R}\rangle \), equation (20). The King measures in the alignment \( b \), of his choice, the operator

\[
\hat{K}_b = \sum_{m=0}^{d} |m, b\rangle K_m^{(b)}(b, m) \tag{28}
\]

and observes, say \( K_m^{(b)} \). The King’s measurement projects the state \( |\mathcal{R}\rangle \) to the state \( |A_{\alpha m, b}\rangle \), equation (18).

Now for her control measurement Alice measures the non degenerate operator

\[
\hat{\Gamma} = \sum_{m', m''} |\tilde{m}'\rangle_c |2m_0\rangle_r \Gamma_{m', m''} |2m''\rangle_r |\tilde{m}''\rangle_c , \tag{29}
\]

obtaining, say, \( \Gamma_{m', m''} \). Thence the quantity

\[
\langle A_{\alpha m, b} |P_{j=m', m''}\rangle \neq 0, \tag{30}
\]

implying, cf equation (24),

\[
m = m_0'' + (b/2)(2\tilde{m}'' - 1). \tag{31}
\]

Knowing \( \tilde{m}' \) and \( m_0'' \) through her control measurement, upon being informed of \( b \) she infers \( m \), the King’s measurement outcome, [20, 21]. This suggest the following geometrical view. The King’s measurement, equation (28), of the state prepared by Alice, \( |\mathcal{R}\rangle \), leaves the system at a geometrical point corresponding to equation (18). It, cf DAPG(c), is shared by \( d \) lines. Alice’s control measurement relate a line, \( |P_{j=m', m''}\rangle \), to this point, equation (24). The line’s equation gives \( m \), the vectorial coordinate, as a function of \( b \), the basis alignment. Thus upon being told the value of \( b \) Alice can deduce the outcome via \( m(b) \), equation (16).

The solution of the MKP wherein Alice can specify the outcome \( m \) of the King’s measurement upon being informed of the basis, \( b \), he used is seemingly paradoxical [18]. Thus she is seemingly able to specify for one state, the measurement outcome of each of the (possible) \( d + 1 \) incompatible bases. However this is illusionary [19]. The relation, equation (31), that gives \( m \) once \( b \) is known holds only for the basis, \( b \), that was actually used by the King through the presence of \( \tilde{m}' \) and \( m_0'' \) which relates to that basis. This relation is the central issue in tracking the King problem to which we turn now.

In our case of tracking the King—he does not inform Alice the basis he used—her control measurement is designed to track it. To this end the state that Alice prepares is one of the line vectors \( |P_{j=m', m_0}\rangle \), equation (22). Thus she knows \( \tilde{m} \) and \( m_0 \). The King’s measurement is as in the MKP case, equation (28), and, as specified above, he observed, \( K_m^{(b)} \). In this case the King’s measurement projects the state defined by Alice, \( |P_{j=m', m_0}\rangle \), to neglecting normalization to the much richer state,

\[
|m, b\rangle \langle b, m| |\tilde{m}\rangle_c |m_0\rangle_r . \tag{32}
\]

Now Alice, in her control measurement, measures, much like in the MKP, the operator \( \hat{\Gamma} \), equation (29) and obtains, say, like in the case above, \( \Gamma_{m', m''} \), implying that

\[
\langle 2m_0'' |, |\tilde{m}''\rangle_c |m, b\rangle \langle b, m| |\tilde{m}\rangle_c |m_0\rangle_r \neq 0. \tag{33}
\]
The lhs of this relation is via equation (26), given by, up to a phase
\[
\langle 2m_0''| \langle \bar{m}'| \frac{1}{d} \delta_{(m''-m_0),b(\bar{m}''-\bar{m})}, \quad b \neq \bar{0}. \tag{34}
\]
Thus,
\[
b = \frac{(m''_0 - m_0)}{(\bar{m}'' - \bar{m}')} \quad \bar{m} = \bar{m}' \rightarrow b = \bar{0}. \tag{35}
\]
Knowing the initial state, i.e. \( \bar{m}, m_0 \), and measuring (in the control measurement) \( \bar{m}' \) and \( m_0'' \), Alice tracks the King’s apparatus alignment, \( b \). The alignment \( b = \bar{0} \), the King’s use of the CB, gives \( \delta_{\bar{m},\nu} \). The case wherein both \( \bar{m}' = \bar{m} \) and \( m_0'' = m_0 \) is undetermined.

Note that Alice’s control measurement of \( \hat{\Gamma} \), equation (29), generates the state \( |P_{j=m_0''}| \), i.e. the system is reset for a succeeding message.

6. Concluding remarks

Two \( d \)-dimensional particles in Hilbert space are analyzed in terms of center of mass and relative coordinates. The product of the states, one in the center of mass and one in he relative coordinates that are maximally entangled (in their single particle coordinates) are used to span the Hilbert space. These states are associated with geometrical lines and form an orthonormal basis for the space. The formalism proves convenient in the study of the mean King problem (which is briefly summarized in the paper) and a variant thereof termed ‘tracking the King’.

This latter case may be viewed a novel quantum communication channel where the message sent by the King to Alice is given in terms of the base, \( b \), he used in his measurement (rather than its outcome). Alice’s (subsequent) measurement that retrieves the message sent by the King (i.e. obtains the identity of the basis he used) projects the state back to one of the above considered maximally entangled states and thus avails it for the King’s next message. Thus it is demonstrated that in quantum mechanics the measured entity (not its value) is itself an observable. The relative simplicity of the formalism and its intuition aiding geometrical facet lead us to expect (hope) that it could be of wide use in related problems.

Acknowledgments

The hospitality of the Perimeter Institute where this work was completed and discussions with Professor D Gottesman, L Hardy and M Mueller are gratefully acknowledged. I have benefited from comments by Professor W Unruh, W Debaere, A Mann and an essential correction by Dr A Kalev.

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