Complete Synchronization of Coupled Oscillators Based on Contraction Theory

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This paper studies contraction theory with the aim of exploring complete synchronization phenomenon in complex networks of coupled oscillators. We examine the conditions for complete synchronization in three network topologies: all-to-all, star, and ring. Specifically, we derive the conditions under which networks of linearly coupled (Duffing) van der Pol oscillators achieve complete synchronization. Additionally, by combining contraction theory with the trapping region method, we identify the conditions for complete synchronization of networks of linearly coupled Rayleigh van der Pol oscillators under specific initial conditions.

The study of coupled oscillators and complex systems spans various fields, including mathematics, engineering, robotics, and biology. Many of these complex systems can be described by networks, where nodes symbolize individual oscillators and links represent the couplings between them. The advancement of network science has significantly enhanced our understanding of complex systems. Synchronization is a key area of interest in this field, as it helps explain the underlying mechanisms of diverse collective behaviors in complex systems.

I. INTRODUCTION

Early studies observed synchronization phenomena in a variety of artificial devices. As research progressed, it became evident that synchronization is also prevalent in natural phenomena, such as Josephson junctions, nanomechanics, neurodynamics, the synchronous flickering of fireflies, the collective chirping of crickets, and recoil atomic lasers. Understanding these self-organized cooperative states is crucial not only for comprehending group dynamics in complex systems but also for conducting relevant experiments and exploring potential applications.

Complete synchronization, where both phase and amplitude are synchronized, is a significant aspect of this phenomenon. Contraction theory, rooted in fluid dynamics and differential geometry, has proven to be an effective tool for analyzing complete synchronization behaviors in nonlinear networks. Constructing a virtual system of the network and identifying its contraction region makes it possible to predict whether the network will achieve complete synchronization theoretically.

Van der Pol (vP) oscillators, initially developed to model electrical circuits employing vacuum tubes, are renowned for their ability to exhibit self-sustained oscillations. The dynamics of coupled vP oscillators have been extensively studied due to their nonlinear damping properties. Generalizations of vP oscillators, such as Duffing van der Pol (DvP) and Rayleigh van der Pol (RvP) oscillators, offer richer dynamics, including bifurcation and chaos, and are particularly useful in modeling the chaotic and complex coordinated behaviors in real-world systems and an electronic Central Pattern Generator that produces biped gait patterns for robotic systems. Using contraction theory, we analyze the conditions under which these coupled generalized vP oscillators achieve complete synchronization. We also verify our results using numerical simulations.

To address the issue of networks without virtual systems, we introduced the concept of a virtual network. This allows for constructing a higher-dimensional “virtual system”, enabling the identification of synchronization conditions for the original network. A critical challenge in applying contraction theory is ensuring that all oscillator trajectories in a network remain within the contraction region of the corresponding virtual system. For specific coupled-oscillator networks (e.g. coupled vP and DvP oscillators), adjusting coefficients can extend the contraction region to cover the entire phase plane, ensuring all trajectories remain within this region. However, for most complex networks (e.g. coupled RvP oscillators), the contraction regions are finite regardless of coefficients. Our approach involves finding a trapping region within the contraction region of the virtual system, ensuring that trajectories starting within the trapping region remain within the contraction region.

This paper is organized as follows: Section 2 introduces basic contraction theory. Section 3 discusses the application of contraction theory to complete synchronization in various network topologies, including all-to-all, star, and ring topologies. Section 4 examines the dynamics of linearly coupled vP, DvP, and RvP oscillators, providing the conditions for each to achieve complete synchronization.

In this paper, we will use the following notations: “$M \prec 0$” means that the matrix $M$ is negative (positive) definite; “$M \preceq 0$” means that $M$ is negative (positive) semi-definite.
II. CONTRACTION THEORY

Definition II.1. (Contracting Region) Given a system equation of the form

\[ \dot{x} = f(x(t)), \]

where \( x \in \mathbb{R}^n \) is a set of state variables and \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a smooth nonlinear vector function. A contraction region \( \mathcal{C} \) of this system is a region of the state space where the symmetric part of the Jacobian \( \frac{\partial f}{\partial x} \) is uniformly negative definite, i.e., \( \mathcal{C} = \{ x \in \mathbb{R}^n : \lambda_{\text{max}}(x) < 0 \} \), where \( \lambda_{\text{max}}(x) \) is the largest eigenvalue of \( \frac{1}{2} \left( \frac{\partial f}{\partial x} + \left( \frac{\partial f}{\partial x} \right)^T \right) \).

Theorem II.1. If any two trajectories, \( x_1(t) \) and \( x_2(t) \), of a system of the form (1), starting from different initial conditions, remain within \( \mathcal{C} \), then they converge exponentially to each other.

Proof. See Ref. [16].

In order to apply this theorem to the synchronization of coupled oscillators, the concept of the virtual system needs to be introduced. Consider a network of nonlinear systems

\[ \dot{x}_i = f_i(x_i) + u_i, i = 1, 2, \cdots, N, \]

where \( x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in \mathbb{R}^n \) is the state variables of node \( i \), \( f_i : \mathbb{R}^n \to \mathbb{R}^n \) are nonlinear smooth vector functions, \( u_i \) are coupling functions which depend on the difference between the state variables of \( i \)'th node and those of others that are linked with it.

Definition III.1. (Complete Synchronization) The network described by (2) is said to be completely synchronized if \( \forall j = 1, \cdots, n, \exists x_i(0) \in \mathbb{R}^n \) for \( i = 1, \cdots, N \) such that

\[ \lim_{t \to \infty} x_{i1}(t) = \lim_{t \to \infty} x_{i2}(t) = \cdots = \lim_{t \to \infty} x_{in}(t) \neq 0. \]

Theorem III.1. Consider a network of the form (2). Assume that there exists a virtual system of this network:

\[ \dot{y} = \Phi(y, x_1, x_2, \ldots, x_N), \]

where \( y \in \mathbb{R}^n \) is the virtual state variable, such that

\[ \Phi(x_1, x_1, x_2, \ldots, x_N) = f_1(x_1) + u_1, \]
\[ \Phi(x_2, x_1, x_2, \ldots, x_N) = f_2(x_2) + u_2, \]
\[ \vdots \]
\[ \Phi(x_N, x_1, x_2, \ldots, x_N) = f_N(x_N) + u_N. \]

Then if the trajectories of the oscillators, \( x_1(t), x_2(t), \ldots, x_N(t) \), remain within the contraction region of the virtual system (3) for some initial conditions, they are completely synchronized.

Proof. See Ref. [25].

In this paper, we only consider identical oscillators with linear couplings. Note that for a non-identical two-coupled-oscillator system with linear couplings, one may construct its virtual system as

\[ \dot{y} = (y - x_2) \frac{f_1(y)}{x_1 - x_2} + (y - x_1) \frac{f_2(y)}{x_2 - x_1} + A_1 x_2 + A_2 x_1 - (A_1 + A_2) y, \]

where \( A_1 \) and \( A_2 \) are oscillators 1 and 2 coupling matrices, respectively. However, this system forms singularities when the trajectories of the two oscillators coincide, and thus the right-hand-side function is not smooth. Therefore, the contraction theory is generally not applicable to the coupled non-identical oscillators. The coupled non-identical oscillators cannot generally be completely synchronized at an exponential rate.

III. COMPLETE SYNCHRONIZATION IN DIFFERENT NETWORK TOPOLOGIES

Consider a network of nonlinear systems

\[ \dot{x}_i = f_i(x_i) + u_i, i = 1, 2, \cdots, N, \]

where \( x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in \mathbb{R}^n \) is the state variables of node \( i \), \( f_i : \mathbb{R}^n \to \mathbb{R}^n \) are nonlinear smooth vector functions, \( u_i \) are coupling functions which depend on the difference between the state variables of \( i \)'th node and those of others that are linked with it.

A. All-to-all Topology

Proposition III.1. Consider \( N \) all-to-all coupled identical oscillators with symmetric linear couplings described by

\[ \dot{x}_i = f(x_i) + \sum_{j=1}^{N} A_j (x_j - x_i), \forall i = 1, \cdots, N, \]

Here, the "symmetric coupling" means that each oscillator has the same effect on all other oscillators. This network is completely synchronized if \( x_i(t) \) are always inside the region where \( J_{f_s} - \sum_{i=1}^{N} A_i J_s < 0 \) for all \( i \), where \( J_{f_s} \) refers to the symmetric part of the Jacobian of \( f \).

Proof. See Appendix [A1].

B. Star Topology

Proposition III.2. Consider \( N \) identical oscillators coupled by a star network with symmetric linear couplings described

\[ \dot{x}_i = f(x_i) + \sum_{j \neq i} A_{ij} (x_j - x_i), \forall i = 1, \cdots, N, \]

where \( A_{ij} \) are coupling matrices. This network is completely synchronized if \( x_i(t) \) are always inside the region where \( J_{f_s} - \sum_{i=1}^{N} A_i J_s < 0 \) for all \( i \), where \( J_{f_s} \) refers to the symmetric part of the Jacobian of \( f \).

Proof. See Appendix [A2].
by
\[
x_1 = f(x_1) + \sum_{j=2}^{N} A_j(x_j - x_1), \quad (5a)
\]
\[
x_i = f(x_i) + A_i(x_i - x_i), \quad \forall i = 2, \ldots, N, \quad (5b)
\]
This network is completely synchronized if \(x_1(t)\) and \(x_2(t)\) are always inside the region where \(f_{x_i} - \sum_{j=1}^{N} A_{ij} < 0\) and \(x_i(t)\) are always inside the region where \(f_{x_i} - A_{ii} < 0\) for all \(i\) from 2 to \(N\).

**Proof.** See Appendix [A.2]

The limitation of applying the above propositions is that when the symmetric part of the Jacobian of the virtual system of a network is not uniformly negative definite on the entire state space, it is difficult to determine whether the trajectories of the oscillators in this network are confined in the contraction region of the virtual system. Note that even if a trajectory starts in a contraction region, it may leave the region later. One way to determine whether a trajectory will remain within a contraction region based on the location of the trajectory’s starting point is to find a *trapping region* that is inside the contraction region. A trapping region \(N\) is a compact subset of the state space such that the flow of the system is inward everywhere on the boundary of \(N\), and therefore every trajectory that starts within \(N\) will remain there for all future time, details of which can be found in Section 4.

### C. Ring Topology

We cannot generally find virtual systems of bidirectional ring networks in which each oscillator affects its two neighboring oscillators and unidirectional ring networks in which each oscillator affects only its next oscillator. Instead, we introduce a multivariate “virtual network” as follows. For a network \(\Psi\), we attempt to construct

\[
\dot{y}_i = \Phi (y_1, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{N-1}, x_1, x_2, \ldots, x_N), i = 1, \ldots, N, \quad (6)
\]
for virtual variables \(y_1, \ldots, y_N\), which satisfies

\[
\Phi (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{N-1}, x_1, x_2, \ldots, x_N) = f(x_i) + u_i, \quad \forall i = 1, \ldots, N. \quad (7)
\]
where all subscripts are calculated modulo \(N\). We can turn this virtual network into a single \(nN\)-dimensional virtual system of \(Y\) by “concatenating” all original \(n\)-dimensional virtual state vectors \(y_i\) to form an \(nN\)-dimensional virtual state variable \(Y\), that is

\[
Y = (y_1, y_2 \cdots, y_N)^T \in \mathbb{R}^{nN}.
\]

Then we have

\[
\dot{Y} = \Psi (Y, x_1, x_2, \ldots, x_N) \equiv (\Phi (y_1, y_2, \ldots, y_N, x_1, x_2, \ldots, x_N), \Phi (y_2, \ldots, y_N, y_1, x_2, \ldots, x_N), \ldots, \Phi (y_N, y_1, \ldots, y_{N-1}, x_1, x_2, \ldots, x_N)) \quad (8)
\]

**Theorem III.2.** If network \(\Psi\) has a virtual system \(\Phi\) which is contracting with respect to \(Y\), then the network is completely synchronized regardless of the initial conditions.

**Proof.** Since the function \(\Phi\) satisfies Eq. (6), the trajectory \((x_1, x_2, \ldots, x_N)^T\) formed by concatenating the trajectories of all the oscillators in the original network \(\Psi\) and all its cyclic permutations are particular solutions of the system \((7)\). In particular, \((x_1, x_2, \ldots, x_N)^T\) and \((x_2, \ldots, x_N, x_1)^T\) are two solutions of \((7)\). Since the system is contracting, they converge exponentially to each other, which implies that \(x_1 \sim x_2, \ldots, x_{N-1} \sim x_N\) and \(x_N \sim x_1\), where \(a \sim b\) denotes that \(a\) and \(b\) are converged. Therefore, all oscillators converge into the same synchronous state vector.

We now consider ring networks consisting of \(N\) identical coupled oscillators whose links are either unidirectional with identical linear coupling or bidirectional with different linear couplings. Such networks can be expressed as the following two equations,

\[
\dot{x}_i = f(x_i) + A(x_{i-1} - x_i), \quad \forall i = 1, \ldots, N. \quad (8a)
\]

\[
\dot{x}_i = f(x_i) + A_{i+1,i}(x_{i+1} - x_i) + A_{i-1,i}(x_{i-1} - x_i), \quad \forall i = 1, \ldots, N, \quad (8b)
\]
where \(A_{i,j}\) denotes coupling from the \(i^{th}\) to \(j^{th}\) node, and the subscripts \(i, j\) are calculated modulo \(N\).

Wang and Slotine discussed the conditions for synchronization of an unidirectional ring network when \(A\) is a symmetric matrix and of a bidirectional ring network when the couplings are *interactional*, i.e., \(A_{i,j} = A_{j,i}^T\). In the following, we relax both of these conditions. We need the following lemma to prove Theorems III.3 and III.4.

**Lemma III.1.** Let \(M = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}\) be a positive semi-definite \(2 \times 2\) block matrix. Define \(M^{m,n}\) to be a block matrix of the form

\[
M^{m,n} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & (A)_{mn} & \cdots & (B)_{mn} \\ \cdots & \cdots & \cdots & \cdots \\ (B^T)_{mn} & \cdots & \cdots & (D)_{mn} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}
\]
where \((H)_{ij}\) denotes that the block \(H\) is in the \(i^{th}\) “row” and \(j^{th}\) “column” of the matrix \(M^{m,n}\). Here, all the blocks in \(M^{m,n}\) are zero matrices, except for the four blocks written out. Then for any unequal \(m\) and \(n\), \(M \succeq 0\) if \(M^{m,n} \succeq 0\).
One can be easily proved this lemma by noticing that \( x^T M x = y^T M^m n y \), where \( x \) is an arbitrary vector which has dimensions that are compatible with matrix \( M \), and \( x \) is exported to vector \( y \) by arbitrarily choosing the missing elements to have appropriate dimensions according to \( M^m n \).

**Theorem III.3.** The unidirectional ring network (8a) is completely synchronized if \( J_{f_s} + A_s \prec 0 \) on \( D \), for all \( i \), \( A_s \succ 0 \), and \( 4A_s - AA_s^{-1} A^T \preceq 0 \).

\[
\frac{\partial \Psi}{\partial y} = \begin{pmatrix} J_{f_1} & J_{f_2} & \cdots & J_{f_N} \\ J_{f_2} & J_{f_3} & \cdots & J_{f_1} \\ \vdots & \vdots & \ddots & \vdots \\ J_{f_N} & J_{f_1} & \cdots & J_{f_{N-1}} \end{pmatrix} - \begin{pmatrix} A_s - \frac{A^T}{2} & -\frac{\Psi}{2} & \cdots & -\frac{\Psi}{2} \\ -\frac{\Psi}{2} & A_s - \frac{A^T}{2} & \cdots & -\frac{\Psi}{2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\Psi}{2} & -\frac{\Psi}{2} & \cdots & A_s - \frac{A^T}{2} \end{pmatrix} = \begin{pmatrix} J_{f_1} + A_s & J_{f_2} + A_s & \cdots & J_{f_N} + A_s \\ J_{f_2} + A_s & J_{f_3} + A_s & \cdots & J_{f_1} + A_s \\ \vdots & \vdots & \ddots & \vdots \\ J_{f_N} + A_s & J_{f_1} + A_s & \cdots & J_{f_{N-1}} + A_s \end{pmatrix} - \begin{pmatrix} 2A_s - \frac{A^T}{2} & -\frac{\Psi}{2} & \cdots & -\frac{\Psi}{2} \\ -\frac{\Psi}{2} & 2A_s - \frac{A^T}{2} & \cdots & -\frac{\Psi}{2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\Psi}{2} & -\frac{\Psi}{2} & \cdots & 2A_s - \frac{A^T}{2} \end{pmatrix},
\]

where \( J_{fi} = \frac{\partial f(y_i)}{\partial y} \). By Theorem III.2, we need to prove that \( (\frac{\partial \Psi}{\partial y}) \) is uniformly negative definite under the given conditions. For an arbitrary \( \Psi \), since \( J_{fi} + A_s \prec 0 \) on \( D \), we have \( J_{fi} + A_s \prec 0 \) for all \( i \), and thus the first part in \( (\frac{\partial \Psi}{\partial y}) \) is uniformly negative definite.

Define block matrices:

\[
\tilde{A}_1 = \begin{pmatrix} A_s & -\frac{A^T}{2} \\ -\frac{\Psi}{2} & A_s \end{pmatrix},
\]

\[
\tilde{A}_2 = \begin{pmatrix} A_s & -\frac{A^T}{2} \\ -\frac{\Psi}{2} & A_s \end{pmatrix}.
\]

Observe that the second part in \( (\frac{\partial \Psi}{\partial y}) \), can be written as \( \sum_{i=1}^{N-1} \tilde{A}_i^{i+1} + \tilde{A}_N^{1,N} \). Since we have assumed that \( A_s \succ 0 \) and \( 4A_s - AA_s^{-1} A^T \preceq 0 \), Schur complement lemma implies that both \( \tilde{A}_1 \) and \( \tilde{A}_2 \) are positive semi-definite. Then, using Lemma III.1, we have \( \tilde{A}_i^{i+1} \succeq 0 \) for \( i \) from 1 to \( N - 1 \) and \( \tilde{A}_N^{1,N} \succeq 0 \), and thus the second part in \( (\frac{\partial \Psi}{\partial y}) \), is positive semi-definite. Therefore, \( (\frac{\partial \Psi}{\partial y}) \) is negative definite, and it follows that (8a) is completely synchronized. \( \square \)

**Theorem III.4.** The bidirectional ring network (8b) is completely synchronized if \( J_{f_i} \prec 0 \) and for each \( i \) from 1 to \( N \) at least one of the following two conditions is satisfied

1. \( A_{i+1,i} > 0 \) and \( A_{i+1,i} - \frac{1}{4}(A_{i,i+1} + A_{i+1,i}^T)A_{i,i+1}^{-1}(A_{i,i+1} + A_{i+1,i}^T) \succeq 0 \);
2. \( A_{i,i+1} > 0 \) and \( A_{i,i+1} - \frac{1}{4}(A_{i,i+1} + A_{i+1,i}^T)A_{i,i+1}^{-1}(A_{i,i+1} + A_{i+1,i}^T) \succeq 0 \).

**Proof.** The virtual network of (8b) can be constructed as

\[
\dot{y}_i = f(y_i) + A_{i+1,i}(y_{i+1} - y_i) + A_{i-1,i}(y_{i-1} - y_i), i = 1, \ldots, N.
\]

Define block matrices:

\[
\dot{\tilde{A}}_{j,i} = \begin{pmatrix} A_{i,j} & -\frac{A_{i,j} + A_{j,i}^T}{2} \\ -\frac{A_{i,j} + A_{j,i}^T}{2} & A_{i,j} \end{pmatrix}.
\]

Writing the virtual network (10) in the form of virtual system (7), the symmetric part of \( \frac{\partial \Psi}{\partial y} \) is \( D_{fi} = \sum_{i=1}^{N-1} \tilde{A}_i^{i+1} + \tilde{A}_N^{1,N} \), where \( D_{fi} \) is a block diagonal matrix whose main diagonal blocks are \( J_{fi} \). The fact that \( J_{fi} \) is negative definite guarantees that \( D_{fi} \prec 0 \). By Schur complement lemma, for each \( i \) from 1 to \( N \), when \( A_{i,i+1} \) is invertible, \( \tilde{A}_{i+1,i} \succeq 0 \) and \( \tilde{A}_{i,i+1} \succeq 0 \). Observe that the \( \tilde{A}_{i,i+1} \) is negative definite, and thus the second part in \( (\frac{\partial \Psi}{\partial y}) \), is positive semi-definite. Therefore, \( (\frac{\partial \Psi}{\partial y}) \) is negative definite, and it follows that (8b) is completely synchronized. \( \square \)

The following two corollaries can be drawn from Theorems III.3 and III.4.

**Corollary III.1.** The unidirectional ring network (8a) with identical symmetric coupling matrix \( A \) is completely synchronized if \( J_{fi} + A \prec 0 \) on \( D \) and \( A \succ 0 \);

**Corollary III.2.** The bidirectional ring network (8b) with interactional couplings, \( A_{i,j} = A_{j,i} \), is completely synchronized if \( J_{fi} \prec 0 \) and \( A_{i,i+1} \succeq 0 \) for all \( i \) from 1 to \( N \).

We’ve recovered the results obtained in Ref. [25].
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IV. NETWORKS OF THE (DUFFING/RAYLEIGH) VAN DER POL SYSTEMS AND NUMERICAL SIMULATIONS

A. Van der Pol Oscillators

We first consider a network of \( N \) all-to-all coupled identical vdP oscillators with symmetric linear couplings. In the absence of coupling, a single vdP oscillator has the following equation of motion

\[
\dot{x} + (\alpha x^2 - \gamma) \dot{x} + \omega^2 x = 0. \tag{11}
\]

It is usually written in two different two-dimensional forms, one by setting \( y = \dot{x} \) and the other by setting \( y = \frac{1}{\omega}(x + \frac{\omega}{\gamma}x^3 - \gamma x) \). We use the latter because, in this case, the Jacobian \( J_f \) has a globally negative semi-definite symmetric part if \( \alpha \geq 0 \) and \( \gamma \leq 0 \). Consider a network of \( N \) all-to-all coupled identical vdP oscillators with symmetric linear couplings described by Eq. (4). Write

\[
\sum_{j=1}^{N} A_{js} = \begin{pmatrix} A_{1s}^{11} & A_{1s}^{12} \\ A_{1s}^{21} & A_{1s}^{22} \end{pmatrix} \tag{12}
\]

with \( A_{is}^{12} = A_{is}^{21} \). We can then establish the following theorem.

**Theorem IV.1.** For a network of \( N \) all-to-all coupled identical vdP oscillators with symmetric linear couplings described by Eq. (4) in which \( f \) is defined by

\[
f(x, y) = \begin{pmatrix} \alpha y - \frac{\alpha}{\gamma} y^3 + \gamma y \\ -\omega x \end{pmatrix}, \tag{13}
\]

if \( \alpha \geq 0 \), \( \gamma < A_{is}^{11} - \frac{(A_{is}^{12})^2}{A_{is}^{22}} \), and \( A_{is}^{22} > 0 \), then the network is completely synchronized.

**Proof.** According to Proposition [III.1], the network is completely synchronized if

\[
(\gamma - \alpha x^2 - A_{is}^{11}) \preceq 0, \quad -A_{is}^{22} \leq A_{is}^{22} \leq 0,
\]

is uniformly negative definite on the entire state space \( \mathcal{D} \), or, equivalently, \( -(J_f - \sum_{j=1}^{N} A_{js}) \succ 0 \) on \( \mathcal{D} \). Since \( \gamma < A_{is}^{11} - \frac{(A_{is}^{12})^2}{A_{is}^{22}}, \alpha \geq 0 \) and \( A_{is}^{22} > 0 \), we have

\[
-(\gamma - \alpha x^2 - A_{is}^{11}) \geq -(\gamma - A_{is}^{11}) \geq -(\gamma - A_{is}^{11} + \frac{(A_{is}^{12})^2}{A_{is}^{22}}) > 0,
\]

and

\[
-(\gamma - A_{is}^{11})A_{is}^{22} - (A_{is}^{12})^2 > 0.
\]

Consequently,

\[
-(\gamma + \alpha x^2 + A_{is}^{11})^{-1} \geq -(\gamma + A_{is}^{11})^{-1},
\]

and

\[
0 < A_{is}^{22} - (\gamma + A_{is}^{11})^{-1}(A_{is}^{12})^2 \\
\leq A_{is}^{22} - (\gamma + \alpha x^2 + A_{is}^{11})^{-1}(A_{is}^{12})^2 \\
= A_{is}^{22} - A_{is}^{22}(\gamma + \alpha x^2 + A_{is}^{11})^{-1}A_{is}^{12}.
\]

Schr complement lemma then implies \( -(J_f - \sum_{j=1}^{N} A_{js}) \succ 0 \).

Next, we consider \( N \) identical vdP oscillators coupled by a star network with symmetric linear couplings described by Eq. (5), where the sum of coupling matrices is written as Eq. (12) and \( A_1 \), is given by

\[
A_1 = \begin{pmatrix} A_{1s}^{11} & A_{1s}^{12} \\ A_{1s}^{21} & A_{1s}^{22} \end{pmatrix}
\]

with \( A_{1s}^{12} = A_{1s}^{21} \). Then using Proposition [III.2] and following similar arguments as in the proof of Theorem [IV.1] we have

**Theorem IV.2.** For \( N \) identical vdP oscillators coupled by a star network with symmetric linear couplings described by Eq. (5) in which \( f \) is given by Eq. (7), if \( \alpha \geq 0, \gamma < \min\{A_{1s}^{11} - \frac{(A_{1s}^{12})^2}{A_{1s}^{22}}, A_{1s}^{11} - \frac{(A_{1s}^{12})^2}{A_{1s}^{22}}\} \), and \( \min\{A_{2s}^{22}, A_{2s}^{22}\} > 0 \), then the network is completely synchronized.

**Example.** Consider a star network of six coupled identical vdP oscillators described by Eqs. (9) and (7) with \( \alpha = 1, \omega = 1, \gamma = 5 \), where the coupling matrices are given by

\[
A_1 = \begin{pmatrix} 8 & 1 \\ 3 & 4 \end{pmatrix}, A_2 = \begin{pmatrix} 3 & 3 \\ 4 & -5 \end{pmatrix}, A_3 = \begin{pmatrix} 7 & -2 \\ -5 & -2 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}, A_5 = \begin{pmatrix} 8 & -3 \\ 1 & 10 \end{pmatrix}, A_6 = \begin{pmatrix} -3 & -4 \\ 2 & -6 \end{pmatrix}.
\]

In this case, \( \min\{A_{1s}^{22}, A_{1s}^{22}\} = \min\{4, 3\} = 3 > 0 \), and

\[
5 = \gamma < \min\{A_{1s}^{11} - \frac{(A_{1s}^{12})^2}{A_{1s}^{22}}, A_{1s}^{11} - \frac{(A_{1s}^{12})^2}{A_{1s}^{22}}\} = \min\{7, 21\} = 7.
\]

By Theorem [IV.2] the network is completely synchronized. Figure 7 shows the complete synchronization phenomenon of this network. It can be seen that all six oscillators converge into synchronous \( x \) and \( y \) states very quickly, as expected.

B. Duffing van der Pol Oscillators

We now introduce a Duffing term, \( x^3 \), into the vdP equation (1) so that the equation becomes
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Letting \( y = \dot{x} \), its two-dimensional form can be written as

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
\omega^2 + \frac{\alpha x^2}{3} - y \\
-\omega x - y
\end{pmatrix}. \tag{15}
\]

We assume all-to-all symmetric coupling as described in Eq. (4). We also assume full-state linear coupling, i.e., the coupling matrices \( A_i \) are diagonal for all \( i \). If \( x_i \) and \( y_i \) represent the “position” and “velocity” of \( i \)th node, respectively, then full-state coupling means that each node adjusts its position and velocity according to the difference between its position and velocity with other nodes, respectively. We can write

\[
J_{fs} - \sum_{j=1}^{N} A_{js} = \left( \begin{array}{ccc}
\frac{1}{N} \sum_{j=1}^{N} a_{ij} & 1 - 2ax - \omega^2 x \\
\omega & 1 - 2a \omega^2 - \gamma
\end{array} \right). \tag{17}
\]

\[ c \] is a constant coupling strength, and \( a_{ij} \) and \( b_{ij} \) are coefficients that represent the coupling effect of \( j \)th node on others. In this case, \( J_{fs} \) is neither negative definite nor negative semi-definite. Furthermore, the definiteness of \( J_{fs} - \sum_{j=1}^{N} A_{js} \) cannot be determined globally since it depends on the position \((x, y)\) in the virtual phase plane. We can nevertheless find the contraction region of the virtual system of this network:

**Lemma IV.1.** For a network of \( N \) all-to-all coupled identical RvdP oscillators with symmetric and full-state linear couplings as described in Eq. (6) in which \( f \) is given by Eq. (8), if \( \frac{1}{N} \sum_{j=1}^{N} a_{ij} > 0 \), then the network is completely synchronized if all trajectories, \((x_i(t), y_i(t))\), start within the region \( \mathcal{C}_{RvdP} := \left\{ (x,y)^T \in \mathbb{R}^2 : \frac{1}{N} \sum_{i=1}^{N} a_{ij} + \mu(x,y) \frac{1}{N} \sum_{j=1}^{N} v^2(x,y) > 0 \right\} \) and remain there for \( k = 1, \ldots, N \), where \( \mu(x,y) \equiv \alpha x^2 + 3 \beta y^2 - \gamma \) and \( v(x,y) \equiv 1 - 2ax - \omega^2 x \).

**Proof.** See Appendix B.  

To determine whether all trajectories will remain within \( \mathcal{C}_{RvdP} \), we need to find a trapping region of the virtual system that is contained in \( \mathcal{C}_{RvdP} \). For simplicity, we make a further assumption that the coupling matrices are identical and symmetric, i.e., \( a_{ij} = b_{ij} = a \) for all \( i \). The dynamics of this network can be expressed as

\[
\begin{aligned}
\dot{x}_i &= y_i + \frac{c}{N} a \sum_{j=1}^{N} (x_j - x_i) \\
\dot{y}_i &= - (\alpha x_i^2 + \beta y_i^2 - \gamma) y_i - \omega^2 x_i + \frac{c}{N} a \sum_{j=1}^{N} (y_j - y_i). \tag{19}
\end{aligned}
\]

Using the LaSalle’s invariance principle, we can obtain the following lemma.

**Lemma IV.2.** For a network of \( N \) all-to-all coupled identical RvdP oscillators with identical full-state coupling described by Eq. (79), the origin is globally asymptotically stable if \( \alpha \geq 0 \), \( \beta \geq 0 \), \( \gamma \geq 0 \), \( \omega \neq 0 \), and \( \gamma < 0 \).
Proof. See Appendix [B 2].

As a result, the solution of Eq. (19) with \( \alpha \geq 0, \beta \geq 0, ca \geq 0, \omega \neq 0, \) and \( \gamma < 0 \) will eventually converge to the origin regardless of the initial conditions. However, as stated in Definition [III.1], \( R_{vdP} = 0 \) is not treated as a synchronous state. We will show that all overdamped coupled RvdP oscillators can reach complete synchronization before they stop oscillating.

Observe that the virtual system of (19) can be constructed as follows:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
y + \frac{ca}{N} (\sum_{j=1}^{N} x_j - N x) \\
-(\alpha^2 + \beta y^2 - \gamma) y - \omega^2 x + \frac{ca}{N} (\sum_{j=1}^{N} y_j - N y)
\end{pmatrix}.
\]

(20)

Lemma IV.3. The disk of radius \( \mathcal{R}_{vdP \omega^2=1} := \{ (x, y) \in \mathbb{R}^2 : H(x, y) = x^2 + y^2 \leq \rho \} \)

is a trapping region of the virtual system (20) with \( \alpha \geq 0, \beta \geq 0, ac \geq 0, \omega^2 = 1, \) and \( \gamma < 0. \)

Proof. See Appendix [B 3].

By Lemma IV.1 when \( \omega^2 = 1, \) the contraction region \( \mathcal{C}_{vdP} \)

of the virtual system (20) becomes

\[
\mathcal{C}_{vdP \omega^2=1} = \{ (x, y) \in \mathbb{R}^2 : c^2 a^2 + ca \mu (x, y) - \nu^2 (x, y) > 0 \} = \{ (x, y) \in \mathbb{R}^2 : g(x, y) > 0 \},
\]

where \( g(x, y) \equiv c^2 a^2 - ca \gamma + ca \alpha^2 + 3 ca \beta y^2 - \alpha^2 y^2 \).

Lemma IV.4. Provided \( ca \geq \max \{ \gamma, 0 \}, \mathcal{R}_{vdP \omega^2=1} \) is contained in \( \mathcal{C}_{vdP \omega^2=1} \) in the following four cases

1. \( 3 \beta^2 > \gamma - ca \) and \( \alpha = 0; \)
2. \( r^2 \leq \frac{-ca - 3 ca \beta}{a^2} \) and \( 3 \beta^2 > \gamma - ca \) with \( 0 \neq \alpha > 3 \beta ; \)
3. \( r^2 \leq \frac{3 ca \beta - ca \alpha}{a^2} \) and \( \alpha^2 > \gamma - ca \) with \( 0 \neq \alpha < 3 \beta; \)
4. \( r^2 \geq \max \{ \frac{-ca - 3 ca \beta}{a^2}, \frac{3 ca \beta - ca \alpha}{a^2} \} \) and

\[
\frac{ca \alpha + 3 ca \beta - 2 \sqrt{3 c^2 a^2 \alpha^2 \beta + \alpha^2 ca(ca - \gamma)}}{a^2} < r^2 < \frac{ca \alpha + 3 ca \beta + 2 \sqrt{3 c^2 a^2 \alpha^2 \beta + \alpha^2 ca(ca - \gamma)}}{a^2}
\]

with \( \alpha \) and \( \beta \) satisfying \( 3 ca \alpha \beta + \alpha^2 (ca - \gamma) > 0. \)

Proof. See Appendix [B 4].

In these cases, any trajectory of the network (19) starting within \( \mathcal{R}_{vdP \omega^2=1} \) with appropriate coefficients will remain there due to Lemma IV.3 and hence in \( \mathcal{C}_{vdP \omega^2=1} \) due to Lemma IV.4. Consequently, Lemma IV.1 implies the following theorem.

Theorem IV.4. Consider a network of \( N \) all-to-all coupled identical RvdP oscillators with an identical full-state linear coupling described by Eq. (19) with \( \alpha \geq 0, \beta \geq 0, ca \geq 0, \gamma < 0, \) \( \omega^2 = 1. \) In the cases stated in Lemma IV.4, the network is completely synchronized if its starting points, \((x_0, y_0))\), are taken in the region \( \mathcal{R}_{vdP \omega^2=1} \) for all \( i \) from 1 to \( N. \)

Note that this theorem is based on the contraction theory, so the synchronization rate is exponential due to Theorem II.1. However, as can be seen in Lemma IV.6 below, the rate of convergence of the network to the origin is also exponential if the initial conditions are in a neighborhood of the origin. Therefore, if the initial conditions are in \( \mathcal{R}_{vdP \omega^2=1} \) and close to the origin, we usually cannot determine whether the network can reach complete synchronization before it decays to the equilibrium position without numerical methods. In general, for large coupling constants \( c \) and \( a \), the complete synchronization can be achieved first. For initial conditions in \( \mathcal{R}_{vdP \omega^2=1} \) and far from the origin, in most cases the network will get completely synchronized before the oscillations stop.

Example. Consider a network of six all-to-all coupled identical RvdP oscillators with an identical full-state linear coupling described by Eq. (19) with \( \alpha = 1, \beta = 1, \gamma = -0.1, \omega = 1, \) \( a = 1, \) \( c = 1, \) and \( N = 6. \)

Suppose the starting points of this system are

\[(x_1(0), y_1(0)) = (1.1, 5.1), (x_2(0), y_2(0)) = (0.3, -1.2), (x_3(0), y_3(0)) = (-0.5, -1.4), (x_4(0), y_4(0)) = (0.5, -0.5), (x_5(0), y_5(0)) = (1.7, 0.9), (x_6(0), y_6(0)) = (0.3, -1). \]

Choose the trapping region, \( \mathcal{R}_{vdP \omega^2=1}, \) to be a disk of radius \( r = 2. \) Then all the starting points are in \( \mathcal{R}_{vdP \omega^2=1}. \) By Lemma IV.4 (case 4 of the lemma), \( \mathcal{R}_{vdP \omega^2=1} \) is included in the contraction region, \( \mathcal{C}_{vdP \omega^2=1}, \) of the virtual system of this network since

\[
4 = r^2 \geq \max \{ \frac{ca \alpha - 3 ca \beta}{a^2}, \frac{3 ca \beta - ca \alpha}{a^2} \} = 2,
\]

\[
-0.05 \approx \frac{ca \alpha + 3 ca \beta - 2 \sqrt{3 c^2 a^2 \alpha^2 \beta + \alpha^2 ca(ca - \gamma)}}{a^2} < r^2 < \frac{ca \alpha + 3 ca \beta + 2 \sqrt{3 c^2 a^2 \alpha^2 \beta + \alpha^2 ca(ca - \gamma)}}{a^2}
\]

\[
ca \alpha + \alpha^2 (ca - \gamma) \approx 8.05,
\]

\[
3 ca \alpha \beta + \alpha^2 (ca - \gamma) \approx 4.1 > 0.
\]

By Theorem IV.4, the network is completely synchronized.

It can be seen from Fig. 2 that the trajectories, \((x_i(t), y_i(t)), \) of all oscillators in this network converge to the origin as predicted by Lemma IV.2, but all oscillators quickly reach complete synchronization before the amplitudes of the oscillations decrease to zero.

Next, we consider the case when \( \gamma > 0. \) In this case, the origin may not be an asymptotically stable equilibrium point, and the system may exhibit periodic behavior. We attempt to find the conditions that make the system (19) self-sustained so that a stable limit cycle exists in its state space.

The following lemma will be used in the proof of Lemma IV.6.

Lemma IV.5. For \( N \times N \) matrices \( W_N \) and \( S_N \) whose elements
are defined as

\[ W_{Nij} = \begin{cases} 
  a & \text{if } i = j \\
  b & \text{otherwise}
\end{cases} \]

and

\[ S_{Nij} = \begin{cases} 
  a & \text{if } i = j \\ 
  b & \text{otherwise}
\end{cases} \]

respectively, we have

\[ \det[W_N] = (a - b)^{N-1} [a + (N-1)b] \]
\[ \det[S_N] = b [a - b]^{N-1}. \] (21)

Proof. See Appendix B.5.

Lemma IV.6. Suppose \( ac > 0 \) and \( \omega \neq 0 \). When \( |\gamma| \geq 2|\omega| \) and \( \gamma < 0 \), the origin is a locally exponentially stable stationary point for the system (19) (overdamped oscillators for \( |\gamma| > 2|\omega| \)). When \( |\gamma| < 2|\omega| \) and \( \gamma < 2ca \), the system (19) experiences a Hopf bifurcation at the origin when \( \gamma \) crosses \( 0 \), and the origin is locally exponentially stable for \( \gamma < 0 \) (underdamped oscillators) and unstable for \( \gamma > 0 \), and the bifurcation is supercritical if \( \alpha \geq 0 \) and \( \beta \geq 0 \).

Proof. The Jacobian of the linearized system (19) about the origin is given by

\[ J = \begin{pmatrix} 
  -\frac{ca}{N} L_N - \lambda I_N & I_N \\
  -\omega^2 I_N & \gamma I_N - \frac{ca}{N} L_N
\end{pmatrix}, \]

where \( I_N \) is an \( N \times N \) identical matrix and \( L_N \) is an \( N \times N \) Laplacian matrix of complete graph whose elements are given by

\[ L_{Nij} = \begin{cases} 
  N-1 & \text{if } i = j \\
  -1 & \text{otherwise}
\end{cases} \]

Hence an eigenvalue \( \lambda \) of \( J \) satisfies

\[ \det \left( \begin{pmatrix} 
  -\frac{ca}{N} L_N - \lambda I_N & I_N \\
  -\omega^2 I_N & (\gamma - \lambda) I_N - \frac{ca}{N} L_N
\end{pmatrix} \right) = 0. \]

Since \( I_N \) is an identical matrix, the lower two blocks in matrix \( J \) commute. Using a property of block matrix determinant, we have

\[ 0 = \det \left( \begin{pmatrix} 
  -\frac{ca}{N} L_N - \lambda I_N & I_N \\
  -\omega^2 I_N & \gamma I_N - \frac{ca}{N} L_N
\end{pmatrix} \right) \]
\[ = \det((-\frac{ca}{N} L_N - \lambda I_N)((\gamma - \lambda) I_N - \frac{ca}{N} L_N) + \omega^2 I_N^2) \]
\[ = \det(\frac{c^2 a^2}{N^2} L_N^2 + \frac{ca}{N}(2\lambda - \gamma)L_N + [(\omega^2 - \lambda)(\gamma - \lambda)]I_N). \]

It can be calculated that

\[ L^2_{Nij} = \begin{cases} 
  N^2 - N & \text{if } i = j \\
  -N & \text{otherwise}
\end{cases} \]

We therefore have

\[ 0 = \det \left( \begin{pmatrix} 
  (p(\lambda)) q(\lambda) & \cdots & q(\lambda) \\
  q(\lambda) p(\lambda) & \cdots & q(\lambda) \\
  \vdots & \ddots & \vdots \\
  q(\lambda) & \cdots & p(\lambda)
\end{pmatrix}_{N \times N} \right), \]

where

\[ p(\lambda) = \lambda^2 + \frac{(2ca - 1)N}{N} - \gamma \lambda + c^2 a^2 - ca \gamma + \frac{ca \gamma - c^2 a^2}{N} + \omega^2, \]
\[ q(\lambda) = -\frac{2ca}{N} \lambda + \frac{ca \gamma - c^2 a^2}{N}. \]

Using Lemma IV.5 \( \lambda \) satisfies

\[ (p - q)^{N-1}[p + (N - 1)q](\lambda) = 0. \] (22)

One can find

\[ (p - q)(\lambda) = \lambda^2 + (2ca - \gamma) \lambda + c^2 a^2 - ca \gamma + \omega^2, \] (23)
\[ [p + (N - 1)q](\lambda) = \lambda^2 - \gamma \lambda + \omega^2. \] (24)

It follows that two roots of the function (23) are a pair of solutions of Eq. (22) with multiplicity \( N - 1 \) and two roots of the function (24) are a pair of solutions of Eq. (22) with multiplicity 1. One can find the roots of (23) and (24) are

\[ \lambda_{1\pm} = \frac{1}{2}(-2ac + \gamma \pm \sqrt{\gamma^2 - 4\omega^2}), \]
\[ \lambda_{2\pm} = \frac{1}{2}(\gamma \pm \sqrt{\gamma^2 - 4\omega^2}) \]
respectively. Assume that \( \alpha \geq 0 \) and \( \omega \neq 0 \). Then for \( \gamma < 0 \) and \( |\gamma| > 2 |\omega| \), \( \lambda_{1\pm} \) and \( \lambda_{2\pm} \) are all negative reals since

\[
\gamma + \sqrt{\gamma^2 - 4\omega^2} < 0.
\]

Therefore, in this case, \( J \) has \( 2N \) negative real eigenvalues, which implies the origin is locally exponentially stable. For \( |\gamma| > 2 |\omega| \), each pair of roots is distinct from each other, so the oscillations are overdamped. For \( |\gamma| < 2 |\omega| \), if \( \gamma < 2 \alpha \), \( J \) has the same \( N - 1 \) pairs of complex conjugate eigenvalues with negative real parts, \( \{\lambda_1, \lambda_{-1}\} \), and a pair of complex conjugate eigenvalues, \( \{\lambda_{2\pm}, \lambda_{-2\pm}\} \), which cross the imaginary axis due to a variation of \( \gamma \).

Note that

\[
\text{Re}(\lambda_{2\pm})(\gamma)|_{\gamma=0} = 0
\]

\[
\frac{d\text{Re}(\lambda_{2\pm})(\gamma)}{d\gamma}|_{\gamma=0} = -\frac{1}{2} > 0,
\]

where \( \text{Re}(\lambda_{2\pm}) \) denote the real parts of \( \lambda_{2\pm} \). According to the Hopf bifurcation theorem, a Hopf bifurcation arises from the origin at \( \gamma = 0 \), and the origin is locally exponentially stable for \( \gamma < 0 \) and unstable for \( \gamma > 0 \). For \( \gamma < 0 \), the oscillations are underdamped since \( J \) has \( 2N \) complex conjugate eigenvalues with negative real parts in this case. If we further have \( \alpha > 0 \) and \( \beta > 0 \), then the origin is globally asymptotically stable according to Lemma 4.5, which means that there is no limit cycle around the origin for \( \gamma < 0 \). Thus, the Hopf bifurcation must be supercritical.

By the Hopf bifurcation theorem for multi-dimensional systems, for sufficiently small values of \( \gamma \neq 0 \) a continuous family of stable limit cycles \( \{\xi(\gamma)\} \), with the parameter \( \gamma \), bifurcates from the origin into the region \( \gamma > 0 \) in a \( 2N \) dimensional state space when the supercritical Hopf bifurcation occurs in the coupled-oscillator system \( (19) \). By an empirical rule of thumb for supercritical Hopf bifurcations, the size of \( \xi(\gamma) \) grows continuously from zero and increases proportionally to \( \sqrt{\gamma} \). For fixed values of \( \alpha \) and \( \beta \) and a sufficiently small variable \( \gamma \), the “projection” of the limit cycle \( \xi(\gamma) \) onto each \( x_j y_j \)-plane is roughly a circle, \( \xi_j(\gamma) \), of radius, \( r_j(\gamma) \propto \sqrt{\gamma} \), around the origin, whose shape is distorted as \( \gamma \) gets larger. It is evident that if a network that starts near its stable limit cycle \( \xi(\gamma) \) is completely synchronized, then all projections \( \xi_j(\gamma) \) must coincide and have the same radius \( r(\gamma) \). For a sufficiently small value of \( \gamma \), the origin is, in fact, the only equilibrium point of the system \( (19) \) with \( \alpha > 0 \), \( \beta > 0 \), and \( \gamma > 0 \), and the bifurcated limit cycle \( \xi(\gamma) \) is also unique, which makes \( \xi(\gamma) \) a globally stable limit cycle so that every trajectory of \( (19) \) that does not start at the origin will spiral into \( \xi(\gamma) \) as time approaches infinity in this case.

When a globally stable limit cycle \( \xi(\gamma) \) that evolves around the origin exists in a coupled-oscillator system \( (19) \), it is plausible to conjecture that if the projected trajectories \( (x_j(t), y_j(t)) \) of \( (19) \) start in a disk of radius \( R \) that contains \( \xi_j(\gamma) \) for all \( j \) from 1 to \( N \), then we have

\[
\sum_{j=1}^{N} x_j^2(t) + y_j^2(t) \leq NR^2.
\]

Then we can see from the proof of Lemma IV.3 that at any point \( (x, y) \), on the boundary of the disk \( \mathcal{R}_{\text{RvdP}} \) of radius \( R \) we have

\[
\nabla H \cdot (\dot{x}(t), \dot{y}(t)) = 2x\ddot{x}(t) + 2y\ddot{y}(t) = -2(\alpha x^2 + \beta y^2 - \gamma) x^2
\]

\[
+ \frac{2ca}{N} \left( \sum_{j=1}^{N}(x_{j}(t) + y_{j}(t)) - N(x^2 + y^2) \right)
\]

\[
\leq 2\gamma(x^2 + y^2) + \frac{2ca}{N} \sum_{j=1}^{N}(x_{j}(t) + y_{j}(t)) - N(x^2 + y^2)
\]

\[
= \frac{2ca}{N} \left( \sum_{j=1}^{N}rr_{j}(t) - N(1 - \frac{\gamma}{ca})r^2 \right)
\]

\[
\leq 2\frac{ca}{N} \sum_{j=1}^{N}rr_{j}(t) - N(1 - \frac{\gamma}{ca})r^2.
\]

Suppose that \( x_{j0}^2 + y_{j0}^2 \leq (1 - \frac{\gamma}{ca})r^2 \) and that \( r_{j}(\gamma) \leq (1 - \frac{\gamma}{ca})r \) for all \( j \). By the Cauchy–Schwarz inequality and Eq. (25), we have

\[
\frac{1}{N} \sum_{j=1}^{N} r_{j}(t) \leq N(1 - \frac{\gamma}{ca})r^2.
\]

Hence,

\[
\sum_{j=1}^{N} r_{j}(t) \leq N(1 - \frac{\gamma}{ca})r^2.
\]

Therefore, \( \nabla H \cdot (\dot{x}(t), \dot{y}(t)) \leq 0 \) on the boundary of \( \mathcal{R}_{\text{RvdP}} \). It follows that the disk \( \mathcal{R}_{\text{RvdP}} \), having radius \( r > \max \{r_{j}(\gamma)\} \), is a “vague” trapping region of the virtual system \( (20) \) with \( \alpha \geq 0 \), \( \beta > 0 \), \( ca > 1 \), and with a sufficiently small value of \( \gamma \) satisfying \( 0 < \gamma < 2 \min \{ca, 1\} \) in the sense that every trajectory that starts within a smaller disk of radius \( (1 - \frac{\gamma}{ca})r \) which is included in \( \mathcal{R}_{\text{RvdP}} \) will remain in \( \mathcal{R}_{\text{RvdP}} \) as the system \( (20) \) evolves, and that the corresponding network \( (19) \) is completely synchronized in the cases given in Lemma IV.4 if \( x_{i0}^2 + y_{i0}^2 \leq (1 - \frac{\gamma}{ca})r^2 \) for all \( i \) from 1 to \( N \) and \( \sum_{i=1}^{N} x_{i0}^2 + y_{i0}^2 
eq 0 \).

We do not prove the inequality (25). However, this is not a crucial problem if we do not limit ourselves to exponential synchronization because any trajectory of \( (19) \) that does not start from the origin would eventually get infinitely close to \( \xi(\gamma) \) on which (25) holds for \( R \geq \max \{r_{j}(\gamma)\} \).

**Example.** Consider a network of six all-to-all coupled identical RvdP oscillators with an identical full-state linear cou-
pling described by Eq. (19) with \( \alpha = 1, \beta = 1, \gamma = 0.6, \omega = 1, a = 1, c = 1, \) and \( N = 6. \) Assuming \( \max\{r_{ij}(\gamma)\} \sim \sqrt{\gamma}, \) one may choose the “vague” trapping region, \( \mathcal{R}_{Rdp_{\omega^2=1}} \) to be a disk of radius \( r = 2 > \sqrt{\gamma(1 - \frac{Z}{ca})^{-1}} \approx 1.94. \) The starting points of this system are

\[
\begin{align*}
(x_{10}, y_{10}) &= (0.6, 0.5), (x_{20}, y_{20}) = (0.3, -0.2), \\
(x_{30}, y_{30}) &= (-0.7, -0.3), (x_{40}, y_{40}) = (0.5, -0.5), \\
(x_{50}, y_{50}) &= (0.1, 0.4), (x_{60}, y_{60}) = (0, 0).
\end{align*}
\]

We have \( x_{i0}^2 + y_{i0}^2 \leq (1 - \frac{Z}{ca})^2 \approx 0.64 \) for all \( i \) from 1 to 6, and it is obvious that \( \sum_{i=1}^{6} x_{i0}^2 + y_{i0}^2 \neq 0. \)

By Lemma IV.4 (case 4 of the lemma), \( \mathcal{R}_{Rdp_{\omega^2=1}} \) is included in \( \mathcal{C}_{Rdp_{\omega^2=1}} \) since

\[4 = r^2 \geq \max\left\{ \frac{c \alpha - 3 c \alpha \beta - c a \alpha}{\alpha^2}, \frac{3 c \alpha \beta - c a \alpha}{\alpha^2} \right\} = 2,
\]

\[4 - 2 \sqrt{3} \frac{1}{4} = \frac{c a a + 3 c a \beta - 2 \sqrt{3} c a a^2 \alpha^2 + a^2 c a (a - \gamma)}{\alpha^2} \leq r^2 < \frac{c a a + 3 c a \beta + 2 \sqrt{3} c a a^2 \alpha^2 + a^2 c a (a - \gamma)}{\alpha^2} = 4 + 2 \sqrt{3},
\]

\[3 c a a \beta + a^2 (a - \gamma) = 3.4 > 0.
\]

Then, by the above analysis, we can predict that this network is completely synchronized, which can be seen in Fig. 3.

(a) The time series of \( x_i \) for \( i \) from 1 to 6
(b) The time series of \( y_i \) for \( i \) from 1 to 6
(c) Phase portraits

FIG. 3: The time series and phase portraits of six self-sustained all-to-all coupled identical RvdP oscillators with an identical full-state linear coupling

Determining the direction of the Hopf bifurcation in the system (19) at \( \gamma = 0 \) is nontrivial when \( \alpha \) or \( \beta \) is negative. Moreover, the large number of oscillators in the network makes it even more difficult. For simplicity, we only consider the case of two coupled oscillators, i.e., \( N = 2. \)

For a system of the form (1) with the bifurcation parameter \( \nu, \) let \( J_A(\nu) \) be the Jacobian at the equilibrium point \( x_A(\nu), \) i.e., \( J_A(\nu) = \frac{\partial f(x_A(\nu))}{\partial x} \bigg\vert_{x=x_A(\nu)}. \) Suppose \( J_A(\nu) \) has \( m \) pairs of complex conjugate eigenvalues, \( \{\lambda_{+}, \lambda_{-}\}, \) where \( \lambda_{+} \) have positive imaginary parts and \( \lambda_{-} \) have negative ones, and \( n - 2m \) real eigenvalues, \( \lambda_{ij}. \) Let \( \text{Re}(\lambda_{i+}) \geq \text{Re}(\lambda_{i-}) \geq \cdots \geq \text{Re}(\lambda_{m+}) \) and let \( \lambda_{1+} \geq \lambda_{2+} \geq \cdots \geq \lambda_{m+} \geq 0. \) The system can be rewritten as

\[\dot{x} = f(x; \nu) = J_A(\nu)x + h(x), \quad \text{where } h(x) \text{ is the nonlinear term.}\]

The Hopf bifurcation theorem gives that the Hopf bifurcation occurs at \( \nu_c \) where \( \nu_c \) satisfies \( \text{Re}(\lambda_{i+}(\nu_c)) = 0, \) \( \frac{d\text{Re}(\lambda_{i+}(\nu_c))}{d\nu} \neq 0, \) \( \text{Im}(\lambda_{i+}(\nu_c)) \neq 0, \) \( \text{Re}(\lambda_{i+}(\nu_c)) < 0 \) for \( i \) from 2 to \( m, \) and \( \lambda_{i+}(\nu_c) < 0 \) for \( j \) from 1 to \( n - 2m. \)

Form a matrix \( P \equiv (\text{Re}(\nu_{i+}) - \text{Im}(\nu_{i+})) \cdots (\text{Re}(\nu_{m+}) - \text{Im}(\nu_{m+})) \nu_{1} \cdots \nu_{n-2m}, \) where \( \nu_{i+} \) and \( \nu_{j+} \) are (generalized) eigenvectors of \( J_A(\nu_c) \) corresponding to \( \lambda_{i+}(\nu_c) \) and \( \lambda_{j+}(\nu_c) \) respectively. We have the following lemma.

Lemma IV.7. \( P \) is invertible and \( P^{-1}J_A(\nu_c)P \) is in the Jordan canonical form, i.e.,

\[
P^{-1}J_A(\nu_c)P = \begin{pmatrix} 0 & -\text{Im}(\lambda_{i+}(\nu_c)) \\ \text{Re}(\lambda_{i+}(\nu_c)) & 0 \end{pmatrix}, \quad (26)
\]

where \( D \) is a block diagonal matrix of original dimension \( n - 2, \) whose diagonals are Jordan blocks \( D_{i} \) that are either of the form

\[
\begin{pmatrix}
B_i & I_{2 \times 2} \\
& \\ & \ddots \\
& & & I_{2 \times 2} \\
& & & & B_i
\end{pmatrix}
\]

for \( i \) from 2 to \( m \) with

\[
B_i = \begin{pmatrix} \text{Re}(\lambda_{i+}(\nu_c)) & -\text{Im}(\lambda_{i+}(\nu_c)) \\ \text{Im}(\lambda_{i+}(\nu_c)) & \text{Re}(\lambda_{i+}(\nu_c)) \end{pmatrix}
\]

for complex eigenvalues \( \lambda_{i+}(\nu_c), \) or of the form

\[
\begin{pmatrix}
\lambda_{rj}(\nu_c) & 1 \\
& \lambda_{rj}(\nu_c) \\
& & \ddots \\
& & & 1 \\
& & & & \lambda_{rj}(\nu_c)
\end{pmatrix}
\]

for real eigenvalue \( \lambda_{rj}(\nu_c) \) with \( j \) from 1 to \( n - 2m. \)

Perform coordinate transformations from \( x \) to \( y \) frames by \( x = Py + x_c(\nu_c). \) Then the system at \( \nu = \nu_c \) is converted into

\[
\dot{y} = P^{-1}x = P^{-1}f(Py + x_c(\nu_c)) = P^{-1}J_A(\nu_c)Py + P^{-1}J_A(\nu_c)x_c(\nu_c) + h(Py + x_c(\nu_c)) = F(y),
\]

where \( F = (F_1, \cdots, F_n) : \mathbb{R}^n \to \mathbb{R}^n \) is a vector-valued function. Then

\[
\frac{\partial F(y)}{\partial y} |_{y=0} = P^{-1}J_A(\nu_c)P, \quad \text{by Lemma IV.7.}
\]

the Ja-
The Hoft bifurcation is supercritical (subcritical) for

\[
g_{11} = \frac{1}{2} \left[ \frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} + \left( \frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} \right) \right],
\]

\[
g_{20} = \frac{1}{2} \left[ \frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} + 2 \left( \frac{\partial^2 F_2}{\partial y_1^2} \partial y_2^2 \right) i \left( \partial^2 F_2 / \partial y_1^2 \right) - \frac{\partial^2 F_2}{\partial y_1^2} \right],
\]

\[
g_{02} = \frac{1}{2} \left[ \frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} - 2 \left( \frac{\partial^2 F_2}{\partial y_1^2} \partial y_2^2 \right) i \left( \partial^2 F_2 / \partial y_1^2 \right) + \frac{\partial^2 F_2}{\partial y_1^2} \right],
\]

\[
G_{21} = \frac{1}{2} \left[ \frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} + \frac{\partial^2 F_2}{\partial y_1^2} \partial y_2^2 \right] + \left( \frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} \right),
\]

\[
g_{21} = G_{21} + \sum_{i=1}^{n-2} \left( 2G_{110}^{11} w_{11}^{i} + G_{101}^{11} w_{20}^{i} \right).
\]

where

\[
G_{j10} = \frac{1}{2} \left[ \frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} + \left( \frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} \right) \right],
\]

\[
G_{j10} = \frac{1}{2} \left[ \frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} + \left( \frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} \right) \right],
\]

for \( j = 3, \ldots, n \), and where \( w_{11} = \langle w_{11}^{i}, \ldots, w_{11}^{i-2} \rangle \) and \( w_{20} = \langle w_{20}^{i}, \ldots, w_{20}^{i-2} \rangle \) are the solutions of

\[
(Dw_{11} = -h_{11}, \quad (D - 2\text{Im}(\lambda_1 v_c)) w_{20} = -h_{20},
\]

respectively, with \( h_{11} = \langle h_{11}^{i}, \ldots, h_{11}^{i-2} \rangle \) and \( h_{20} = \langle h_{20}^{i}, \ldots, h_{20}^{i-2} \rangle \) whose elements are defined by

\[
h_{11}^{i-2} = \frac{1}{2} \left[ \frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} \right],
\]

\[
h_{20}^{i-2} = \frac{1}{2} \left[ \frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} - 2 \left( \frac{\partial^2 F_2}{\partial y_1^2} \right) \right],
\]

for \( j = 3, \ldots, n \), and with matrix \( D \) defined in Eq. (26). We then calculate

\[
\beta_2 = \text{Re} \left[ \frac{i}{\text{Im}(\lambda_1 v_c)} \left( g_{20} g_{11} - 2 |g_{11}|^2 - \frac{|g_0|^2}{3} \right) + g_{21} \right].
\]

The limit cycle is stable (unstable) if \( \beta_2 < 0 (\beta_2 > 0) \) on the side of \( v = v_c \) where the limit cycle appears, which indicates the Hopf bifurcation is supercritical (subcritical). We calculate the following quantities at \( y = (y_1, \cdots, y_n) = 0 \).

If \( \omega \neq 0 \) and \( \alpha + 3 \beta \omega^2 > 0 \) \((< 0)\), then \( \beta_2 < 0 \) \((> 0)\), and thus a supercritical (subcritical) Hopf bifurcation occurs. For systems with more oscillators, we can analyze the direction of the Hopf bifurcation in the same way, which will not be discussed in this paper. Combined with Lemma IV.6, we conclude

**Theorem IV.5.** For a two-oscillator system described by Eq. (19) with \( ca > 0 \) and \( N = 2 \), when \( |y|' < 2ca \) and \( \gamma < 2ca \), the origin is locally exponentially stable for \( \gamma < 0 \) and unstable for \( \gamma > 0 \), and a continuous one-parameter family of stable (unstable) limit cycles bifurcate from the origin into the region \( \gamma > 0 \) \((< 0)\) if \( \alpha + 3 \beta \omega^2 > 0 \) \((< 0)\).

In the case of a supercritical Hopf bifurcation, the conclusion about the conditions for synchronization of the network
\[ \alpha, \beta \geq 0 \text{ also applies here for the case } \alpha + 3 \beta \omega^2 > 0 \]

with a negative value of \( \alpha \) or \( \beta \). However, there are two issues to be noted, one is that the shape of the limit cycle becomes distorted more rapidly from a circle with increasing \( \gamma \) compared to the case for \( \alpha, \beta \geq 0 \), and the other is that the stability of the limit cycle becomes local since the origin may not be a unique equilibrium point even for a minimal value of \( \gamma \). Therefore, in the case where either \( \alpha < 0 \) or \( \beta < 0 \), the radius of a vague trapping region \( R_{\text{vdP}}^{(a_0=1)} \) of the virtual system (20) can be taken to be only slightly greater than \( \max\{r_j(\gamma)\}(1 - \frac{1}{c_0})^{-1} \).

Appendix A: Proofs of Propositions III.1 and III.2

1. Proof of Proposition III.1

**Proof.** Notice that the network (5) does not have a single virtual system, but it is easy to see that the virtual system of (5b) can be constructed as follows

\[ \dot{y} = f(y) + \sum_{j=1}^{N} A_j x_j - (\sum_{j=1}^{N} A_j)y. \]  \hspace{1cm} (A1)

We have

\[ \frac{\partial \Phi}{\partial y} = J_f - \sum_{j=1}^{N} A_j. \]

By Definition II.1, the contraction region of (A1) is the region in which \( \frac{\partial \Phi}{\partial y} \) is always in the region where \( \sum_{j=1}^{N} A_j < 0 \). Therefore, by Theorem III.1, if \( x_1(t) \) and \( x_2(t) \) remain within this region, (A1) is completely synchronized.

2. Proof of Proposition III.2

**Proof.** Notice that the network (5) does not have a single virtual system, but it is easy to see that the virtual system of (5b) is

\[ \dot{y} = f(y) + A_1(x_1 - y) \]  \hspace{1cm} (A2)

Since \( x_i(t) \) remain in the contraction region of (A2) where \( J_f - A_{i_j} < 0 \) at all times for all \( i \) from 2 to \( N \), by Theorem III.1, all the oscillators except the first one converge toward the same synchronous state exponentially. Hence Eq. (5b) will reduce to

\[ x_1 = f(x_1) + \sum_{j=2}^{N} A_j(x_2 - x_1), \]

which describes the state of the first oscillator in the network. Then, the virtual system of the subnetwork consisting of only the first two oscillators is

\[ \dot{y} = f(x) + A_1 x_1 + \left( \sum_{j=2}^{N} A_j \right)x_2 - (\sum_{j=1}^{N} A_j)y. \]  \hspace{1cm} (A3)

Therefore, the first two oscillators are completely synchronized as \( x_1(t) \) and \( x_2(t) \) are always inside the contraction region of (A3) where \( J_f - \sum_{j=1}^{N} A_{i_j} < 0 \), and thus the whole network is completely synchronized by transitivity.

Appendix B: Proofs of Lemmas IV.1-IV.5

1. Proof of Lemma IV.1

**Proof.** Since the coupling is all-to-all and symmetric, Proposition III.1 implies that the network is completely synchronized if \( (x_i(t), y_i(t))^T \) are always in the region where \( J_f - \sum_{j=1}^{N} A_{i_j} < 0 \) for all \( i \). Using Eq. (18) and Schur complement lemma, \( J_f - \sum_{j=1}^{N} A_{i_j} < 0 \) iff \( -J_f + \sum_{j=1}^{N} A_{i_j} > 0 \) and \( (\alpha y^2 + 3 \beta y^2) + \frac{\gamma}{\omega} \sum_{j=1}^{N} b_j - \frac{1}{2} \frac{\gamma}{\omega^2} (\sum_{j=1}^{N} a_j)^2 \). Therefore, the first two oscillators are completely synchronized.

2. Proof of Lemma IV.2

**Proof.** Note that the origin is an equilibrium point. Since \( \omega \neq 0 \), we can construct a continuously differentiable function \( V(x_i, y_i) = \sum_{i=1}^{N} \left( \frac{x_i^2}{2} + \frac{y_i^2}{2} \right) \). This is a Lyapunov function for the system (19). To see this, we take the derivative of \( V \) with respect to time along an arbitrary trajectory of Eq. (19).

\[ \dot{V}(x_i, y_i) = \sum_{i=1}^{N} \left( x_i \dot{x}_i + \frac{y_i \dot{y}_i}{\omega^2} \right) \]

\[ = \sum_{i=1}^{N} \left( x_i y_i - \frac{ca}{N} \sum_{j=1}^{N} x_i - x_j \right) \]

\[ - \frac{y_i}{\omega^2} \left( \alpha x_i^2 + \beta y_i^2 - \gamma \right) y_i - \frac{y_i}{\omega^2} \omega^2 x_i - \frac{ca}{N} \omega^2 y_i \sum_{j=1}^{N} \left( y_i - y_j \right) \]

\[ \leq \frac{\gamma}{\omega^2} \sum_{i=1}^{N} y_i^2 - \frac{ca}{N} \sum_{i,j=1}^{N} x_i (x_i - x_j) - \frac{ca}{N} \omega^2 \sum_{i,j=1}^{N} y_i (y_i - y_j) \]

\[ = \frac{\gamma}{\omega^2} \sum_{i=1}^{N} y_i^2 - \frac{ca}{2N} \sum_{i,j=1}^{N} \left( x_i^2 - 2 x_i x_j + x_j^2 \right) \]

\[ - \frac{ca}{2N} \omega^2 \sum_{i,j=1}^{N} \left( y_i^2 - 2 y_i y_j + y_j^2 \right) \]

Notice that \( \sum_{i,j=1}^{N} x_i^2 = \sum_{i,j=1}^{N} x_i^2 \) and that \( \sum_{i,j=1}^{N} y_i^2 = \]
\[ \sum_{j=1}^{N} y_j^2. \] We therefore have

\[ \dot{V}(x_i, y_i) \leq \frac{\gamma}{\omega^2} \sum_{i=1}^{N} y_i^2 - \frac{ca}{2N} \sum_{i,j=1}^{N} (x_i^2 - 2x_ix_j + x_j^2) \]

\[ \leq \frac{\gamma}{\omega^2} \sum_{i=1}^{N} y_i^2 - \frac{ca}{2N} \sum_{i,j=1}^{N} (y_i^2 - 2y_1y_j + y_j^2) \]

\[ \leq \frac{\gamma}{\omega^2} \sum_{i=1}^{N} y_i^2. \]

Since \( \gamma < 0 \), the equality holds if and only if \( x(t) = (x(t), y(t))^T = 0 \) for all \( t \geq 0 \). Note that

\[ V(0) = 0 \]
\[ V(x) > 0, \forall x \neq 0 \]
\[ V(x) \to \infty, \text{ as } ||x|| \to \infty \]

Thus, by LaSalle’s invariance principle, the origin is globally asymptotically stable. \( \square \)

3. Proof of Lemma [IV.3]

Proof. Recall that the gradient of \( H \) at a point on a level curve of \( H \) is perpendicular to the level curve and points toward the fastest increase. To verify that the vector field of (20) points inward on the boundary of the disk \( \mathcal{G}_{RvdP, \omega^2} = 1 \) at all times, it suffices to show that \( \nabla H \cdot (x, y) = 2x \dot{x} + 2y \dot{y} \leq 0 \) on the circle \( x^2 + y^2 = r^2 \) at all times. At any point \( (x, y) \) on \( x^2 + y^2 = r^2 \), we have

\[ \nabla H \cdot (\dot{x}(t), \dot{y}(t)) = 2x \dot{x}(t) + 2y \dot{y}(t) \]

\[ = 2xy + \frac{ca}{N} \left( \sum_{j=1}^{N} x_j(t) - Nx^2 \right) \]

\[ - 2 \left( \alpha x^2 + \beta y^2 - \gamma \right) \dot{x}^2 - 2xy + \frac{ca}{N} \left( \sum_{j=1}^{N} y_j(t) - Ny^2 \right) \]

\[ \leq 2 \frac{ca}{N} \left( \sum_{j=1}^{N} x_j(t) + y_j(t) - N(x^2 + y^2) \right) \]

\[ = 2 \frac{ca}{N} \left( \sum_{j=1}^{N} r_j(t) \cos \theta_j(t) - Nr^2 \right) \]

\[ \leq 2 \frac{ca}{N} \left( \sum_{j=1}^{N} r_j(t) - Nr^2 \right), \]

where \( r_j(t) \) represent the distances between the origin and the point \((x_j(t), y_j(t))\) at time \( t \) in the phase plane of (20) and \( \theta_j(t) \) represent the angles between the position vectors \((x, y)\) and \((x_j(t), y_j(t))\). By Lemma [IV.2], the origin is a globally asymptotically stable point of the original network, which means

\[ \sum_{j=1}^{N} r_j^2(t) \leq \sum_{j=1}^{N} r_j^2(t), \]

for all \( t > 0 \). Suppose that the starting points \((x_{j_0}, y_{j_0})\) of all oscillators lie in \( \mathcal{G}_{RvdP, \omega^2} = 1 \), or, in other words, \( r_j(0) \leq r \) for all \( j \) from 1 to \( N \). Then

\[ \sum_{j=1}^{N} r_j^2(t) \leq \sum_{j=1}^{N} x_{j_0}^2 + y_{j_0}^2 \leq N r^2. \]

Using the Cauchy–Schwarz inequality, we have

\[ \left( \sum_{j=1}^{N} r_j(t) \right)^2 \leq \sum_{j=1}^{N} r_j^2(t) \leq N r^2. \]

It follows that

\[ \sum_{j=1}^{N} r_j(t) \leq Nr, \]

or,

\[ \sum_{j=1}^{N} rr_j(t) \leq N r^2, \]

Therefore,

\[ \nabla H \cdot (\dot{x}(t), \dot{y}(t)) \leq 2 \frac{ca}{N} \left( \sum_{j=1}^{N} rr_j(t) - Nr^2 \right) \leq 0 \]

\( \square \)

4. Proof of Lemma [IV.4]

Proof. Since \( ca > 0 \) and \( ca > \gamma, c^2a^2 - ca\gamma = \gamma \gamma > 0 \),

Take an arbitrary point \((x^*, y^*)\) in \( \mathcal{G}_{RvdP, \omega^2} = 1 \). We want to show \( g(x^*, y^*) > 0 \) so that \((x^*, y^*) \in \mathcal{G}_{RvdP, \omega^2} = 1 \).

Case 1: \( 3\beta r^2 > \gamma - ca \) and \( \alpha = 0 \). In this case,

\[ g(x^*, y^*) = G(y^*) = ca(\gamma - \gamma) + 3ca\beta y^2 = ca(\gamma - \gamma + 3\beta y^2), \]

where \( y^* \in [-r, r] \). For \( \beta > 0 \), \( G(y) \) has a minimum at \( y = 0 \), and we have \( G(0) = ca(\gamma - \gamma) > 0 \). For \( \beta < 0 \) and \( y \in [-r, r] \), \( G(y) \) has minima at \( y = \pm r \). Since \( 3\beta r^2 > \gamma - ca \), \( G(\pm r) = ca(\gamma - \gamma + 3\beta r^2) > 0 \). Consequently, \( G(y) > 0 \) for all \( y \in [-r, r] \). Therefore, \( g(x^*, y^*) = G(y^*) > 0 \).

Case 2: \( r^2 \leq \frac{ca(\gamma - \gamma + 3\beta r^2)}{\alpha^2} \) and \( 3\beta r^2 > \gamma - ca \) with \( 0 < \alpha < 3\beta \).
The partial derivatives of \( g(x,y) \) are
\[
\begin{align*}
g_x(x,y) &= 2ca\alpha x - 2c^2 \alpha^2 y x \\
g_y(x,y) &= 6ca\beta y - 2c^2 \alpha^2 y.
\end{align*}
\]
Notice that \( g_x(x,y) = 0 \) when \( x = 0 \) or \( y^2 = \frac{ca}{\alpha} \) (if \( \alpha > 0 \)) and that \( g_y(x,y) = 0 \) when \( y = 0 \) or \( x^2 = \frac{3ca\beta}{\alpha^2} \) (if \( \alpha \neq 0 \) and \( \beta > 0 \)).

Thus, for \( \beta < 0, g(x,y) \) only has one critical point, \((0,0)\), at which \( g(0,0) = ca(\alpha - \gamma) > 0 \). For \( \beta \geq 0, g(x,y) \) has five critical points, \((0,0), (\pm \sqrt{\frac{3ca\beta}{\alpha^2}}, \pm \sqrt{\frac{ca}{\alpha}})\), if \( \alpha > 0 \). However, \[
r^2 \leq \frac{ca\alpha - 3ca\beta}{\alpha^2} \leq \frac{ca\alpha + 3ca\beta}{\alpha^2},
\]
which indicates that there is only one critical point, \((0,0)\), in the interior of \( \mathcal{R}_{\text{RedP}_{\alpha^2=1}} \), at which \( g(0,0) > 0 \).

On the boundary, \( x^2 + y^2 = r^2 \), of \( \mathcal{R}_{\text{RedP}_{\alpha^2=1}} \), we have
\[
g(x,y) = h(x) \equiv ca(\alpha - \gamma) + ca\alpha x^2 + 3ca\beta y^2 - \alpha^2 x^2(\gamma - y^2) = \alpha^2 x^4 + (ca\alpha - 3ca\beta - \alpha^2 r^2) x^2 + ca(\alpha - \gamma) + 3ca\beta r^2 > 0
\]

and
\[
h'(x) = 4\alpha^2 x^3 + 2(ca\alpha - 3ca\beta - \alpha^2 r^2) x.
\]

Since \( r^2 \leq \frac{ca\alpha - 3ca\beta}{\alpha^2} \), we have \( ca\alpha - 3ca\beta - \alpha^2 r^2 \geq 0 \). So \( h'(x) = 0 \) only at \( x = 0 \). Furthermore, \( h'(x) > 0 \) when \( x > 0 \) and \( h'(x) < 0 \) when \( x < 0 \). It follows that \( h(x) \) achieves a minimum at \( x = 0 \).

\[
h(0) = ca(\alpha - \gamma) + 3ca\beta r^2 = ca(\alpha - \gamma + 3\beta).
\]

Since \( 3\beta r^2 > \gamma - ca, h(0) > 0 \). Then \( h(x) > 0 \) for all \( x \). Thus, \( g(x', y') > 0 \).

**Case 2**: \( r^2 \leq \frac{3ca\beta - ca\alpha}{\alpha^2} \) and \( \alpha^2 r^2 > \gamma - ca \) with \( 0 \leq \alpha < 3\beta \).

For \( \alpha < 0 \), \( g(x,y) \) has only one critical point \((0,0)\) at which \( g(0,0) = ca(\alpha - \gamma) > 0 \). For \( \alpha > 0 \), \[
r^2 \leq \frac{3ca\beta - ca\alpha}{\alpha^2} \leq \frac{ca\alpha + 3ca\beta}{\alpha^2}.
\]

Only one critical point in \( \mathcal{R}_{\text{RedP}_{\alpha^2=1}} \) is still \((0,0)\), and \( g(0,0) > 0 \).

On the boundary of \( \mathcal{R}_{\text{RedP}_{\alpha^2=1}} \), \( g(x,y) = h(x) \) for \( x \in [-r, r] \). Since \( \alpha < 3\beta \), \( h'(x) = 0 \) at \( x = 0 \) and at \( x = x_{\pm} = \pm \sqrt{\frac{2(ca\alpha - 3ca\beta - \alpha^2 r^2)}{2\alpha}} \).

We have \( h''(x) = 12\alpha^2 x^2 + 2(ca\alpha - 3ca\beta - \alpha^2 r^2) \), so that
\[
h''(0) = 2(ca\alpha - 3ca\beta - \alpha^2 r^2) < 0, \quad h''(x_{\pm}) = -4(ca\alpha - 3ca\beta - \alpha^2 r^2) > 0.
\]

It follows that \( h(x) \) reaches its global minimum value at \( x = x_{+} \) or at \( x = x_{-} \). But since \( r^2 \leq \frac{3ca\beta - ca\alpha}{\alpha^2} \), \[
2\alpha^2 r^2 \leq -2(ca\alpha - 3ca\beta) \quad \quad 4\alpha^2 r^2 \leq -2(ca\alpha - 3ca\beta - \alpha^2 r^2) \quad \quad r^2 \leq \frac{-2(ca\alpha - 3ca\beta - \alpha^2 r^2)}{4\alpha^2} = x_{\pm}.
\]

Therefore, for \( x \in [-r, r], h(x) \) achieves its minimum at \( x = r \) or at \( x = -r \). Since \( \alpha^2 > \gamma - ca \),

\[
h(\pm r) = ca\alpha r^2 + ca(\alpha - \gamma) = ca(\alpha r^2 + ca - \gamma) > 0.
\]

So \( h(x) > 0 \) for all \( x \in [-r, r] \). Therefore, \( g(x', y') > 0 \).

**Case 4**: \( r^2 \geq \max\{\frac{ca\alpha - 3ca\beta}{\alpha^2}, \frac{3ca\beta - ca\alpha}{\alpha^2}\} \) and \( \frac{ca\alpha + 3ca\beta - 2\sqrt{3ca^2\alpha^2 + \alpha^2 ca(\alpha - \gamma)}}{\alpha^2} < r^2 < \frac{ca\alpha + 3ca\beta + 2\sqrt{3ca^2\alpha^2 + \alpha^2 ca(\alpha - \gamma)}}{\alpha^2} \) with \( 3ca\beta + \alpha^2 (ca - \gamma) > 0 \).

Notice from \( 3ca\alpha \beta + \alpha^2 (ca - \gamma) > 0 \) that \( \alpha \neq 0 \). In this case, \( g(x,y) \) has one critical point, \((0,0)\), if \( \alpha < 0 \) or \( \beta < 0 \), otherwise \( g(x,y) \) has five critical points, \((0,0)\) and \((\pm \sqrt{\frac{3ca\beta}{\alpha^2}}, \pm \sqrt{\frac{ca}{\alpha}})\), that could lie in the interior of \( \mathcal{R}_{\text{RedP}_{\alpha^2=1}} \),

\[
ca\alpha + 3ca\beta \leq ca\alpha + 3ca\beta + 2\sqrt{3ca^2\alpha^2 + \alpha^2 ca(\alpha - \gamma)} \leq \frac{ca\alpha + 3ca\beta + 2\sqrt{3ca^2\alpha^2 + \alpha^2 ca(\alpha - \gamma)}}{\alpha^2}
\]
for \( \alpha > 0 \) and \( \beta > 0 \). As before, \( g(0,0) > 0 \). At points \((\pm \sqrt{\frac{3ca\beta}{\alpha^2}}, \pm \sqrt{\frac{ca}{\alpha}})\),

\[
g(x,y) = ca(\alpha - \gamma) + \frac{3ca\beta}{\alpha^2} + \frac{ca\beta ca - \alpha^2}{\alpha^2} \leq ca(\alpha - \gamma) + 3\frac{ca\alpha^2 \beta}{\alpha} + \frac{3ca^2 \beta}{\alpha^2} - \frac{3ca^2 \beta}{\alpha^2} = ca(\alpha - \gamma) + \frac{3ca^2 \beta}{\alpha^2} = \frac{ca}{\alpha^2} [\alpha^2 (ca - \gamma) + 3ca\beta] > 0.
\]

On the boundary of \( \mathcal{R}_{\text{RedP}_{\alpha^2=1}} \), we have seen in Case 3 that \( h(x) \) reaches its global minimum value at \( x = x_+ \) or at
\( x = x_+ \) because \( r^2 \geq \frac{ca\alpha - 3ca\beta}{\alpha^2} \). In this case, \( x_+ \in [-r, r] \) since \( r^2 \geq \frac{3ca\beta - ca\alpha}{\alpha^2} \). Thus, for \( x \in [-r, r] \), \( h(x) \) has a minimum at \( x = x_+ \) or at \( x = -x_+ \). Note that

\[
h(x_+) = l(r^2) \equiv \frac{-(ca\alpha - 3ca\beta - \alpha^2 r^2)}{4\alpha^2} + ca(\alpha - \gamma) + 3ca\beta^2
\]

\[
= \frac{-1}{4} \alpha^2 + \frac{ca + 3ca\beta}{2} - \frac{(ca - 3ca\beta)^2}{4\alpha^2} + ca(\alpha - \gamma).
\]

Since \( 3ca\alpha + \alpha^2(\alpha - \gamma) > 0 \), \( 3ca^2\alpha \beta + \alpha^2 ca(\alpha - \gamma) > 0 \). Observe that \( r^2 = \frac{ca^3a\beta^2 + 2\sqrt{3ca^2\alpha^2\beta^2 + \alpha^2 ca(\alpha - \gamma)}}{\alpha^2} \), \( l(r^2) \) = 0. Hence \( l(r^2) > 0 \) for all \( x \in [-r, r] \), and therefore \( g(x^+, y^+) > 0 \). Therefore, \( \mathcal{R}_{rdP_{\alpha^2}=1} \subset \mathcal{C}_{rdP_{\alpha^2}=1} \) in all four cases. \( \square \)

5. Proof of Lemma IV.5

**Proof.** The case when \( N = 1 \) is trivial. We assume Eq. (21) is true for \( N \geq 1 \) and choose any row or column to calculate the determinants of \( W_{N+1} \) and \( S_{N+1} \). By switching rows, one can find

\[
det(W_{N+1}) = a(\det(W_N)) - Nb(\det(S_N))
\]

\[
= a(a-b)^N - Nb^2(a-b)^{N-1}
\]

\[
= (a-b)^{N-1}[a(\alpha + (N-1)b) - Nb^2(a-b)]
\]

\[
= (a-b)^{N-1}(a - (a-b)^N(a + Nb])
\]

\[
= (a-b)^N(a + Nb),
\]

and

\[
det(S_{N+1}) = b(\det(W_N)) - Nb(\det(S_N))
\]

\[
= b(a-b)^N - Nb^2(a-b)^{N-1}
\]

\[
= (a-b)^{N-1}[a(\alpha + (N-1)b) - Nb^2(a-b)]
\]

\[
= b(a-b)^N.
\]

By mathematical induction, this proof is completed. \( \square \)

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