Random-matrix model for thermalization

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Abstract
We show that for a system governed by a random-matrix Hamiltonian (a member of the time-reversal invariant Gaussian Orthogonal Ensemble (GOE) of random matrices of dimension \(N\)), all functions \(\text{Tr}(A\rho(t))\) in the ensemble thermalize: For \(N \to \infty\) every such function tends to the value \(\text{Tr}(A\rho_{\text{eq}}(\infty)) + \text{Tr}(A\rho(0))g^2(t)\). Here \(\rho(t)\) is the time-dependent density matrix of the system, \(A\) is a Hermitean operator standing for an observable, and \(\rho_{\text{eq}}(\infty)\) is the equilibrium density matrix at infinite temperature. The oscillatory function \(g(t)\) is the Fourier transform of the average GOE level density and falls off as \(1/|t|^{3/2}\) for large \(t\). With \(g(t) = g(-t)\), thermalization is symmetric in time. Analogous results, including the symmetry in time of thermalization, are derived for the time-reversal non-invariant Gaussian Unitary Ensemble of random matrices. Comparison with the ‘eigenstate thermalization hypothesis’ of (Srednicki 1999 J. Phys. A: Math. Gen. 32 1163) shows overall agreement but raises significant questions.

Keywords: random matrices, thermalization, eigenstate thermalization hypothesis

1. Introduction

Does an isolated quantum system thermalize? If so, under which conditions does that happen? Is thermalization linked to time-reversal invariance? These questions are important for the understanding of fundamental aspects of quantum statistical mechanics. Thermalization is seen as the quantum analogue of ergodicity in classical systems, see the reviews in [1–3]. Therefore, thermalization continues to be discussed intensely until today [4, 5]. Thermalization...
cannot occur in quantum systems that are integrable or show many-body localization, and the discussion is focused on chaotic quantum systems. Thermalization of a closed system is investigated experimentally by opening the system after some time and taking data. Repeating that process for different times yields the time dependence of thermalization. Attention is often focused on local observables where the emergence of local thermalized states causes thermalization [6]. That has been demonstrated experimentally, for instance, in [7].

In [1, 3], thermalization is defined in terms of the time-dependent function \( \text{Tr}(A\rho(t)) \). Here \( \rho(t) \) is the time-dependent density matrix of an isolated quantum system, and \( A \) is a Hermitian operator representing an observable used to test thermalization. The system is said to thermalize if for \( t \to \infty \), \( \text{Tr}(A\rho(t)) \) tends asymptotically to \( \text{Tr}(A\rho_{\text{eq}}) \) where \( \rho_{\text{eq}} \) is the time-independent density matrix of statistical equilibrium. Thus, thermalization provides a test of the general hypothesis that in the long-time limit, expectation values of operators tend to their equilibrium values.

A central role in the discussion is played by the ‘eigenstate thermalization hypothesis’ introduced by Srednicki [8]. In the basis of eigenstates \( \alpha, \beta, \ldots \) of the chaotic Hamiltonian with energy eigenvalues \( E_\alpha, E_\beta, \ldots \), the operator \( A \) is represented by the matrix \( A_{\alpha\beta} \). The hypothesis postulates that \( A_{\alpha\beta} \) has the form

\[
A_{\alpha\beta} = \mathcal{A}(E) \delta_{\alpha\beta} + \exp\{ -S(E)/2 \} F(E, \omega) R_{\alpha\beta},
\]

where \( \mathcal{A}(E) \) and \( F(E, \omega) \) are smooth functions of \( E = (1/2)(E_\alpha + E_\beta) \) and \( \omega = E_\alpha - E_\beta, S(E) \) is the thermodynamic entropy at energy \( E \), and \( R_{\alpha\beta} \) is a zero-centered Gaussian-distributed random variable with unit variance. (In [9], equation (1) has been generalized to non-Gaussian random variables). On the basis of equation (1) it is argued that in the semiclassical regime of high excitation energy, thermalization is generic for chaotic quantum systems [8].

That is an amazing assertion, for the following reason. The state of a quantum system is characterized by the Hermitian time-independent statistical operator \( \Pi \) that describes the distribution of the system over the states in Hilbert space. With \( H \) the Hamiltonian governing the system, the time-dependent density matrix \( \rho(t) \) is given by

\[
\rho(t) = \exp\{ -iHt/\hbar \} \Pi \exp\{ iHt/\hbar \}.
\]

In the eigenstate representation of \( H \), the diagonal matrix elements \( \rho_{\alpha\alpha} \) of \( \rho(t) \) are independent of time, and the non-diagonal elements \( \rho_{\alpha\beta} \) with \( \alpha \neq \beta \) carry the time-dependent phase factors \( \exp\{ -i(E_\alpha - E_\beta)t/\hbar \} \). These oscillate periodically in time. The operator \( A \) is independent of time. Therefore, the time-dependent function \( \text{Tr}(A\rho(t)) \) has a very simple structure, irrespective of whether the system is chaotic or not. How can that simplicity be reconciled with the statement that \( \text{Tr}(A\rho(t)) \) thermalizes, i.e. tends for \( t \to \infty \) to the limit \( \text{Tr}(A\rho_{\text{eq}}) \)? Why do chaotic systems thermalize while others do not? equation (1) indicates that the answer given in [8] is a probabilistic one. The random fluctuations of a chaotic quantum system encapsulated in the last term of equation (1) are used in [8] to show that with overwhelming probability, \( \text{Tr}(A\rho(t)) \) tends to \( \text{Tr}(A\rho_{\text{eq}}) \) in the long-time limit.

In the present paper we investigate thermalization and the eigenstate thermalization hypothesis (1) from a different point of view. The random fluctuations of eigenvalues and eigenfunctions of chaotic quantum systems agree locally in energy with random-matrix predictions [10]. That fact has motivated us to investigate the thermalization of a quantum system for which the Hamiltonian \( H \) is a random matrix. Throughout most of the paper, \( H \) is taken to be a member of the time-reversal invariant Gaussian Orthogonal Ensemble (GOE) of random matrices of dimension \( N \gg 1 \) while section 5 is devoted to the Gaussian Unitary Ensemble (GUE) that breaks time-reversal invariance. By definition, \( H \) is paradigmatic for the fluctuation properties that characterize chaotic quantum systems. Clearly, however, the structureless random-matrix
Hamiltonian $H$ which treats all states of the system on an equal footing, cannot represent a real chaotic quantum system with all its information content. Nevertheless, the following questions are relevant for an understanding of thermalization: Under which conditions does $\text{Tr}(\rho(t))$ thermalize for a GOE Hamiltonian? Is thermalization of $\text{Tr}(\rho(t))$ linked to time-reversal invariance, or does it occur in similar form also for a GUE Hamiltonian? How do our results compare with those of [8]? What causes the difference between the two approaches? These questions are fully answered in the course of the paper.

Each realization of the random-matrix Hamiltonian $H$ defined in section 2 produces another time-dependent test function $\text{Tr}(\rho(t))$. Jointly, these test functions form a 'stochastic process', i.e. a random ensemble of time-dependent functions. In sections 3 and 4 we address thermalization of that stochastic process. We determine analytically mean value (section 3) and variance (section 4) in the limit $N \gg 1$ of large matrix dimension. We calculate terms of order zero in $N$ as well as corrections of order $N^{-1}$. We thereby explore the conditions under which every member $\text{Tr}(\rho(t))$ of the stochastic process thermalizes. In section 5 we investigate thermalization for the GUE in the same manner. In section 6 we turn to the eigenstate thermalization hypothesis. We compare our approach and our results with those obtained in the framework of equation (1). Section 7 contains a discussion and the conclusions.

The present paper is the first in a sequence of two. In a forthcoming publication [11], we apply the concepts and the methods developed in the present paper to a chaotic many-body quantum system. We establish the conditions under which that system thermalizes. We identify the time scale for thermalization.

2. Definitions

All operators act in a Hilbert space of dimension $N \gg 1$, spanned by orthonormal basis states labeled $(\mu, \nu, \ldots)$. The function $\text{Tr}(\rho(t))$ depends on time $t$ via the density matrix $\rho(t)$ defined in equation (2). The Hermitian time-independent statistical operator $\Pi$ describes the distribution of the system over the states in Hilbert space, has eigenvalues $\pi_\kappa$, orthonormal eigenfunctions $|\kappa\rangle$ with $\kappa = 1, \ldots, N$, and is given by

$$\Pi_{\mu\nu} = \sum_\kappa \langle \mu | \kappa \rangle \pi_\kappa \langle \kappa | \nu \rangle. \quad (3)$$

The real eigenvalues $\pi_\kappa$ obey $0 \leq \pi_\kappa \leq 1$ for all $\kappa$. The matrix $\Pi$ has unit trace, $\sum_\kappa \pi_\kappa = 1$. It is capable of accounting for very different physical situations. If, for instance, $\pi_\kappa = 1/N$ for all $\kappa$, we have $\Pi_{\mu\nu} = (1/N) \delta_{\mu\nu} = (|\text{eq}\rangle \langle \text{eq}|)_{\mu\nu}$. That operator describes statistical equilibrium at infinite temperature where the Boltzmann factor is equal to unity and all states have equal occupation probability. If $\pi_\kappa = 2/N$ for $\kappa \leq N/2$ and $\pi_\kappa = 0$ for $\kappa > N/2$, the system is far from equilibrium. In both cases, however, $\text{Tr}(\Pi^2)$ is of order $1/N$ and, thus, small compared to $[\text{Tr}(\Pi)]^2 = 1$. A very different case is the one where $\pi_\kappa = 1$ for $\kappa = 1$ and $\pi_\kappa = 0$ otherwise. Then $\text{Tr}(\Pi^2) = 1$ and equal to $[\text{Tr}(\Pi)]^2 = 1$. Cases intermediate between these extremes also exist. If, for instance, $\pi_\kappa = N^{-\alpha}$ for $0 < \alpha < 1$ and for all $\kappa \leq N^\alpha$ and zero otherwise, we have $\text{Tr}(\Pi^2) = 1$ and $\text{Tr}(\Pi^2) = N^{-\alpha}$. These cases are distinguished by the rank $k$ of $\Pi$. In the first, second, third, and fourth example we have, respectively, $k = N$, $k = N/2$, $k = 1$, and $k = N^\alpha$, and $\text{Tr}(\Pi^2)$ is of order $1/k$. It is not always possible, however, to characterize $\text{Tr}(\Pi^2)$ in terms of the rank of $\Pi$. A counterexample is $\pi_\kappa = 1/(a \kappa)$ with $a = \sum_\kappa 1/\kappa$. Here $\text{Tr}(\Pi^2) = 1$ while $\text{Tr}(\Pi^2)$ is of order $1/\ln^2 N$. In discussing thermalization we keep in mind these different possibilities. They come into play in section 4.
The operator $\Pi$ is independent of time. The time evolution of the density matrix in equation (2) is entirely determined by the unitary operators $\exp\{\pm iHt/\hbar\}$. (We assume that $\Pi \neq \Pi_{eq}$ as otherwise $\rho(t) = \Pi_{eq}$ is independent of time, and the system remains in statistical equilibrium forever.) Except for section 5, the Hamiltonian $H$ is a member of the GOE. The matrix elements $H_{\mu\nu}$ are real zero-centered Gaussian random variables with second moments

$$\left\langle H_{\mu\nu}H_{\mu'\nu'} \right\rangle = \frac{\lambda^2}{N} (\delta_{\mu\mu'}\delta_{\nu\nu'} + \delta_{\mu\nu'}\delta_{\nu\mu'}) .$$  \hspace{1cm} (4)$$

Big angular brackets indicate the ensemble average, and $\lambda$ defines the width of the spectrum.

In terms of its eigenvalues $E_\alpha$ and eigenfunctions $O_{\mu\alpha}$ with $\alpha = 1, \ldots, N$, the Hamiltonian is given by

$$H_{\mu\nu} = \sum_\alpha O_{\mu\alpha}E_\alpha O_{\nu\alpha}.$$  \hspace{1cm} (5)$$

For $N \gg 1$ the matrix elements $O_{\mu\alpha}$ are real zero-centered Gaussian random variables \[12\] with second moments

$$\left\langle O_{\mu\alpha}O_{\mu'\alpha'} \right\rangle = \frac{1}{N} \delta_{\mu\mu'}\delta_{\alpha\alpha'} .$$  \hspace{1cm} (6)$$

The factor $(1/N)$ accounts for normalization. The eigenvalues $E_\alpha$ obey Wigner–Dyson statistics \[13\]. Eigenvectors and eigenvalues are statistically uncorrelated \[13\]. The unitary time-evolution operator $U = \exp\{-iHt/\hbar\}$ is given by

$$U_{\mu\nu}(t) = \sum_\alpha O_{\mu\alpha} \exp\{-iE_\alpha t/\hbar\} O_{\nu\alpha} .$$  \hspace{1cm} (7)$$

The operator $U$ carries the random variables $O_{\mu\alpha}$ and $E_\alpha$ and is, therefore, a matrix-valued stochastic process.

In analogy to the operator $\Pi$ in equation (3), the operator $A$ with eigenvalues $A_j$ and orthonormal eigenfunctions $\langle j \rangle$ is written as

$$A_{\mu\nu} = \sum_j \langle j|\mu\rangle A_j \langle j|\nu \rangle ,$$  \hspace{1cm} (8)$$

and we have

$$\text{Tr} (A\rho(t)) = \text{Tr} \left[ A U(t) \Pi U^\dagger(t) \right] .$$  \hspace{1cm} (9)$$

The operators $A$ and $\Pi$ are independent of time and not statistical. Both are Hermitean. Therefore, $\text{Tr}(A\rho(t))^\ast = \text{Tr}(A\rho(t))$ is real. The operator $U(t)$ defined in equation (7) is both time dependent and stochastic and so is, therefore, also $\text{Tr}(A\rho(t))$. We investigate thermalization by calculating the ensemble average and the variance of $\text{Tr}(A\rho(t))$. To that end we calculate in appendix A the low moments of $U(t)$ and in appendix B the time-dependent functions that control thermalization.

A caveat is called for regarding the use of the random-matrix ensemble defined in equations (5)–(7) for times near $t = 0$. These equations imply that random-matrix properties determine the time evolution (8) of the system for all times. That is unrealistic. Starting at time $t = 0$, a chaotic quantum system takes a finite time to build up the statistical correlations implied by equations (5)–(7). In using the random-matrix ensemble of equations (5)–(7) for all times, we disregard such transients. That causes an unrealistic feature of the ensemble average of $\text{Tr}(A\rho(t))$ at time $t = 0$ in section 3. We do not pay attention to that feature because we focus attention on the long-time properties of $\text{Tr}(A\rho(t))$. 

4
3. Ensemble average

We average separately over matrix elements $O_{\mu\alpha}$ and eigenvalues $E_{\alpha}$. We begin with the matrix elements. The average is indicated by an index $O$ on the big angular brackets. We use equation (9) and separate uncorrelated and correlated factors $U(t)$ to write

$$\langle \text{Tr}(A\rho(t)) \rangle_O = \text{Tr} \left[ A \langle U(t) \rangle_O \Pi \langle U^\dagger(t) \rangle_O \right] + \text{Tr} \left[ A \langle U(t) \Pi U^\dagger(t) \rangle_{O,\text{corr}} \right]. \quad (10)$$

We use the averages given in equations (37) and (38) of appendix A and $\text{Tr}(\Pi) = 1$ to obtain

$$\langle \text{Tr}(A\rho(t)) \rangle = \langle |f(t)|^2 \rangle \text{Tr}(\Pi) + \frac{1}{N} \text{Tr}(A) + \frac{1}{N} \text{Tr} \langle A^T\Pi \rangle. \quad (11)$$

Here

$$f(t) = \frac{1}{N} \sum_\alpha \exp \{-iE_\alpha t/\hbar\}. \quad (12)$$

The upper index $T$ in equation (11) denotes the transpose, the angular brackets on the right-hand side denote the remaining average over the eigenvalues. The function $f(t)$ is obviously of order zero in $N$. The third term on the right-hand side of equation (11) vanishes for $N \to \infty$ while the first and the second term do not. That can be verified using the estimates in section 4.

The second term on the right-hand side of equation (11) can be written as $\text{Tr}(A\rho_{eq}(\infty))$. Here $(\rho_{eq}(\infty))_{\alpha\beta} = (1/N)\delta_{\alpha\beta} = (\Pi_{eq})_{\alpha\beta}$ is the time-independent density matrix that describes statistical equilibrium at infinite temperature. If $\langle |f(t)|^2 \rangle$ tends to zero for large time, equation (11) implies thermalization to leading order in $N$ for the ensemble average of $\text{Tr}(A\rho(t))$. The rate of thermalization depends only on $\langle |f(t)|^2 \rangle$, i.e. on the eigenvalues $E_{\alpha}$ of $H$, and is independent of the test operator $A$ and of the statistical operator $\Pi$.

We note that at time $t = 0$, we have $U(0) = 1$ and, from equation (9), $\text{Tr}(A\rho(0)) = \text{Tr}(\Pi)$. That result (valid for $N \to \infty$) differs from the limit $t \to 0^+$ of expression (11). With $f(0) = 1$, the limit is given by $(1/N)\text{Tr}(A) + \text{Tr}(\Pi) + (1/N)\text{Tr}(A^T\Pi)$. Thus, $t = 0$ is a singular point in the time evolution of $\text{Tr}(A\rho(t))$. The reason is that the limits $t \to 0$ and $N \to \infty$ do not commute. The problem is resolved by requesting continuity in $t$ of $\langle \text{Tr}(A\rho(t)) \rangle$. Starting out at any value $t_0 \neq 0$, averaging, and using continuity, that yields the result (11) for all values of $t$. The singular point $t = 0$ is caused by our neglect of transients discussed at the end of section 2.

We write $\langle |f(t)|^2 \rangle$ as the sum of the uncorrelated and the correlated parts,

$$\langle |f(t)|^2 \rangle = \langle |f(t)|^2 \rangle_{\text{uncorr}} + \langle |f(t)|^2 \rangle_{\text{corr}}. \quad (13)$$

Both parts are calculated in appendix B. The resulting expressions involve the time scales

$$\tau_\lambda = \hbar/(2\lambda), \quad \tau_d = \hbar/d. \quad (14)$$

Here $2\lambda$ defines the width of the average GOE spectrum, and $d$ is the average GOE level spacing. At the center $E = 0$ of the spectrum, $d$ has the value $d = \pi \lambda/N$. Since $2\lambda$ is the largest and $d$ is the smallest natural GOE energy scale, $\tau_\lambda$ is the shortest and $\tau_d$ is the largest natural GOE time scale. The ratio is $\tau_\lambda/\tau_d = \pi/(2N) \ll 1$.

The uncorrelated part

$$\langle |f(t)|^2 \rangle = g(t/\tau_\lambda) = \frac{4}{\pi} \int_0^1 dx \sqrt{1-x^2} \cos (xt/\tau_\lambda) \quad (15)$$

is proportional to the Fourier transform of the average GOE level density in equation (43). The oscillatory function $g(t/\tau_\lambda)$ is given by

$$g(t/\tau) = 2J_1(t/\tau_\lambda) (t/\tau_\lambda). \quad (16)$$
Here \( J_\nu \) is the first Bessel function of index \( \nu \). The function \( g \) obeys \( g(0) = 1 \), is symmetric in time so that \( g(t/\tau_\lambda) = g(-t/\tau_\lambda) \), and for large time \(|t|\) is asymptotically proportional to \((|t|/\tau_\lambda)^{-3/2}\).

The correlated part

\[
\langle |f(t)|^2 \rangle_{\text{corr}} = \frac{4}{3\pi N} \int_{-\infty}^{+\infty} dy \, Y_2(y) \cos(yt/\tau_d) \tag{17}
\]

is proportional to the Fourier transform of the GOE two-level correlation function \( Y_2(y) \). With \( s(y) = \sin(\pi y)/(\pi y) \), the function \( Y_2 \) is given by

\[
Y_2(y) = s^2(y) + \left( \int_y^{+\infty} ds \, t(s) \left( \frac{d}{dy} s(y) \right) \right). \tag{18}
\]

The factor \( 1/N \) in equation (17) is due to the fact that \( Y_2(y) \) is essentially confined to values \( y \ll N \) of the argument.

In summary we have shown that

\[
\left\langle \text{Tr}(A \rho(t)) \right\rangle = \frac{1}{N} \text{Tr}(A) + \text{Tr}(\Pi) \left( g^2(t/\tau_\lambda) + \langle |f(t)|^2 \rangle_{\text{corr}} \right) + \frac{1}{N} \text{Tr}(A^T \Pi). \tag{19}
\]

The first term on the right-hand side represents the statistical equilibrium value \( \text{Tr}(\rho_{\text{eq}}(\infty)) \) of \( \text{Tr}(\rho(t)) \). The second term describes thermalization. Since both \( g(t/\tau_\lambda) \) and \( \langle |f(t)|^2 \rangle_{\text{corr}} \) are symmetric under the operation \( t \to -t \), thermalization is symmetric in time. The function \( g^2 \) falls off with time as \( \tau_\lambda^2/t^3 \) until it reaches the level of the statistical fluctuations given by \( \langle |f(t)|^2 \rangle_{\text{corr}} \) in equation (17). Since the fluctuations are of order \( 1/N \approx \tau_\lambda/\tau_d \), that happens when \( g^2 \approx 1/N \) or, with \( |g|^2 \approx \tau_\lambda^2/t^3 \), at time \( t \approx \tau_\lambda(\tau_d/\tau_\lambda)^{1/3} \), intermediate between the shortest time \( \tau_\lambda \) and the longest time \( \tau_d \). Beyond that time, the fluctuations take over. These are governed by \( \tau_d \) and, thus, are extremely slow. The last term in equation (19) is of order \((1/N)^{1/2}\) or smaller. It provides an offset for the fluctuations which prevents for finite \( N \) the test function \( \text{Tr}(A \rho(t)) \) from ever thermalizing completely. Additional terms of order \( 1/N \) arise from higher-order \( 1/N \) corrections to equation (6). The dependence on time both of \( g \) and of the correlated part in equation (17) is determined entirely by the statistical properties of the GOE and is independent of the test operator \( A \) and of the statistical operator \( \Pi \). These two operators determine the values of the last two traces in equation (19) and, thus, initial and final values of thermalization.

The phase factors \( \exp\{ \pm i E_n t/\hbar \} \) of \( U(t) \) and \( U^\dagger(t) \) in expression (10) represent solutions of the Schrödinger equation. That equation is invariant under a time shift \( t \to t + \tau_0 \). We are free to use that shift in expression (10) and, in consequence, in \( g(t/\tau_\lambda) \). The resulting function \( g((t + \tau_0)/\tau_\lambda) \) is a valid alternative to \( g(t/\tau_\lambda) \). The maximum is at \( t = -\tau_0 \). Depending on the sign of \( \tau_0 \), the maximum occurs at negative or positive values of \( t \). In the latter case thermalization has the unexpected feature that, starting at \( t = 0 \), \( g^2 \) first grows (with oscillations) with increasing \( t \), reaches the absolute maximum \( g = 1 \) at \( t = \tau_0 \), and decreases thereafter.

As mentioned in the Introduction, thermalization is understood [1] as a test of the hypothesis that in the long-time limit, expectation values of operators tend to their equilibrium values. Thermalization does not imply that the density matrix \( \rho(t) \) itself thermalizes, i.e. tends to the time-independent statistical equilibrium distribution. That is demonstrated explicitly by the freedom to shift time \( t \to t + \tau_0 \) which shows that the onset in time of thermalization can be chosen arbitrarily. Choosing \( \tau_0 < 0 \) shows that \( \rho(t) \) does not thermalize in the time interval \( \tau_0 \leq t \leq 0 \).
The results for $N \to \infty$ derived in the present and in the following section were obtained previously without the use of random-matrix theory in [14, 15] in the semiclassical limit, using an average over a large number of Hilbert-space states located in a narrow energy shell. It was left open whether such averaging is due to a randomness of the eigenbasis of $H$ or of that of $A$ and $\Pi$. It was argued that such averaging produces typical results. In the present framework of random-matrix theory, our results are exact in the limit of infinite matrix dimension $N$. That is the basis of the discussion in section 6. The corrections of order $1/N$ given above illustrate the approach to universality.

4. Variance

As a measure of the size of the fluctuations of individual members $\text{Tr}(A\rho(t))$ around the ensemble average in equation (19) we calculate the variance

$$\left\langle \text{Tr}(A\rho(t))\text{Tr}(A\rho(t)) \right\rangle - \left\langle \text{Tr}(A\rho(t)) \right\rangle \left\langle \text{Tr}(A\rho(t)) \right\rangle.$$

(20)

We first perform the average over the matrix elements $O_{\mu\nu}$. Following appendix A we distinguish contributions due to the first moment of $U$ and to the totally correlated parts of the second, third, and fourth moment of $U(t)$. Nonvanishing contributions to expression (20) are due to the following possibilities: (i) In each trace in the first term of expression (20), each of the two factors $U$ and $U^\dagger$ is averaged individually, or these two factors are replaced by the correlated part of $(UU^\dagger)$. In either case the only non-vanishing contribution to the variance (20) is due to the correlations between eigenvalues. (ii) In each trace one factor $U$ or $U^\dagger$ is part of the correlated part of the second moment, the other factor is replaced by its average (four cases); (iii) in each trace each factor $U, U^\dagger$ is paired with a factor $U$ or $U^\dagger$ in the other trace to form the correlated part of the second moment (two cases); (iv) the two factors $U$ and $U^\dagger$ in one trace combine with one factor $U$ or $U^\dagger$ in the second trace to form the correlated part of the third moment, the remaining factor $U^\dagger$ or $U$ is averaged separately (four cases); (v) the four factors $U$ and $U^\dagger$ form the correlated part of the fourth moment (one case).

For case (i) equation (37) shows that the contribution to the variance is given by

$$\left(\left\langle |f(t)|^2 |f(t)|^2 \right\rangle - \left\langle |f(t)|^2 \right\rangle \left\langle |f(t)|^2 \right\rangle \right) (\text{Tr}(A\Pi^2))^2.$$

(21)

Here $\text{Tr}(A\Pi^2)$ is of order zero in $N$. The variance of $|f(t)|^2$ involves the GOE two-point correlation function in equation (17) as well as the GOE three-point and four-point correlation functions. As pointed out below equation (17) and in appendix B, the factor $1/N$ in equation (17) is due to the fact that the integration variable $y$ is essentially confined to values of order unity. In the GOE three-point (four-point) functions, two (three) integration variables are similarly confined. That gives rise, respectively, to factors $1/N^2$ and $1/N^3$ multiplying these functions. In evaluating the variance in expression (21) we confine ourselves to contributions of order $1/N$. These are due to the two-point function and given by

$$\left(\left\langle f(t) \right\rangle \left\langle f(t) \right\rangle \left\langle f^\ast(t)f^\ast(t) \right\rangle_{\text{corr}} + c.c. \right) + \left(\left\langle f(t) \right\rangle \left\langle f^\ast(t) \right\rangle \left\langle f^\ast(t)f(t) \right\rangle_{\text{corr}} + c.c. \right).$$

(22)

Here $\left\langle f(t) \right\rangle$ is given in equation (15) and the two-point correlation function is given in equation (17). All terms are of order $1/N$ and are of second order in $g$ and, thus, disappear with $r^{-3}$ for large $|t|$. For case (ii) the results carry the factor $1/N$, factors $f(2t)\left\langle f^\ast(t) \right\rangle^2$ or $|f(t)|^2$, and traces of the form $\text{Tr}(A^2\Pi^2)$, $\text{Tr}(A\Pi^2\Pi^2)$, or $\text{Tr}(\Pi\Pi)^2$. To leading order in $N$, all time-dependent factors are proportional to $g^k$ with $k \geq 2$ and tend to zero for large $|t|$. The factor $\text{Tr}(A^2\Pi^2)$ is bounded in magnitude by $N^{3/2}$. That follows from writing that factor in the eigenbasis of $\Pi$, using the
Cauchy–Schwarz inequality, and the fact that \( \text{Tr}(\Pi^4) \leq 1 \). For the remaining two traces the same result follows under the assumption that

\[
(\text{Tr}(A))^2 \gg \text{Tr}(A^2),
\]

the inequality indicating a ratio of order 1/N. The postulate (23) resembles the eigenstate thermalization hypothesis of equation (1). It does not, however, invoke any statistical assumption on \( A \). If condition (23) is met, all terms under item (ii) are at most of order 1/N^1/2, and, thus, negligible.

For case (iii) the results carry a factor 1/N^2, a factor unity or \(|f(2t)|^2\), and factors \( \sum_{\mu\nu} A_{\mu\nu}^2 \Pi_{\mu\nu}, \text{Tr}(A^2)\text{Tr}(\Pi^2) \), or \( |\text{Tr}(\Pi)|^2 \). All these terms are small of order 1/N or smaller, without any constraints on \( A \). For case (iv) the results carry a factor \((1/N)^2|f(t)|^2\) and factors \( \text{Tr}(\Pi^2) \text{Tr}(A\Pi) \), or \( \text{Tr}(A\Pi)\text{Tr}(\Pi) \), or \( \text{Tr}(\Pi^2) \) [Tr(A)]. These are small of order 1/N.

We have, thus, shown that all terms in the variance (20) are of order 1/N^1/2 or smaller, provided the constraint (23) applies. Stronger bounds that are independent of that constraint hold for statistical operators of rank \( N \) for which the distribution of eigenvalues is sufficiently narrow and centered at 1/N.

Summarizing the results of the previous and of the present section, we have shown that for matrix dimension \( N \to \infty \), all members \( \text{Tr}(A\rho(t)) \) of the stochastic process defined by the GOE Hamiltonian \( H \) tend to a common limit,

\[
\text{Tr}(A\rho(t)) \to \frac{1}{N} \text{Tr}(A) + \text{Tr}(\Pi) g^2(t/\tau_\lambda). \tag{24}
\]

The first term on the right-hand side represents the statistical equilibrium value \( \text{Tr}(A\rho_{\text{eq}}(\infty)) \) of \( \text{Tr}(A\rho(t)) \) at infinite temperature where the Boltzmann factor equals unity. In our random-matrix ensemble, all states in Hilbert space are treated on the same footing. That is a consequence of the orthogonal invariance of the ensemble and is the cause for the occurrence of infinite temperature. The oscillatory function \( g^2 \) has its maximum \( g^2 = 1 \) at \( t = 0 \), is bounded for large \( |t| \) from above by \((2\tau_\lambda/|t|)^2\), and, thus, tends to zero for large time \( |t| \). Thus, each of the functions \( \text{Tr}(A\rho(t)) \) thermalizes. Since \( g(t/\tau_\lambda) = g(-t/\tau_\lambda) \), thermalization is symmetric in time. The result (24) holds in full generality if the rank of \( \Pi \) is of order \( N \) and for which the distribution of eigenvalues is sufficiently narrow and centered at 1/N.

5. Gaussian unitary ensemble

The symmetry in time of the thermalization process in equation (24) is evident already in equation (11). Since \( f(t)^* = f(-t) \), the function \( \langle |f(t)|^2 \rangle \) obeys \( \langle |f(t)|^2 \rangle = \langle |f(-t)|^2 \rangle \). To test whether that symmetry is caused by the time-reversal invariance of the GOE, we study thermalization for the Gaussian Unitary Ensemble (GUE) of random matrices. That ensemble is not invariant under time reversal. The GUE consists of Hermitian matrices \( H_U \) of dimension \( N \gg 1 \). The elements \( H_{U;\mu\nu} \) are complex zero-centered Gaussian random variables with second moments

\[
\langle H_{U;\mu\nu}H_{U;\nu'\mu}' \rangle = \frac{\lambda^2}{N} \delta_{\mu\nu}\delta_{\nu'\mu}'. \tag{25}
\]
Here $\lambda_U$ defines the width of the GUE spectrum. In terms of its eigenvalues $E_{U;\alpha}$ and eigenfunctions $U_{\mu\alpha}$ with $\alpha = 1, \ldots, N$, the Hamiltonian matrix $H_U$ is given by

$$H_{U;\mu\nu} = \sum_\alpha U_{\mu\alpha} E_{U;\alpha} U^{*}_{\nu\alpha}. \quad (26)$$

The elements $U_{\mu\alpha}$ of the unitary matrix $U$ are complex zero-centered random variables that carry random phases. Therefore, $\langle U_{\mu\alpha} U^{*}_{\nu\alpha'} \rangle = 0$ while

$$\langle U_{\mu\alpha} U^{*}_{\mu\alpha'} \rangle = \frac{1}{N} \delta_{\alpha\alpha'} \delta_{\mu\mu'}. \quad (27)$$

The eigenvalues $E_{U;\alpha}$ obey GUE Wigner–Dyson statistics. Eigenvectors and eigenvalues are statistically uncorrelated [13]. The unitary time-evolution operator $U_U(t) = \exp\{-iH_U t/\hbar\}$ is given by

$$U_{U;\mu\nu}(t) = \sum_\alpha U_{\mu\alpha} \exp\{-iE_{U;\alpha} t/\hbar\} U^{*}_{\nu\alpha}. \quad (28)$$

With the replacement $U(t) \rightarrow U_U(t)$, the stochastic process $\text{Tr}(\rho(t))$ is given by equation (9). We average $\text{Tr}(\rho(t))$ first over the eigenvectors $U_{\mu\alpha}$. That is done by contracting each of the two factors $U_{\mu\alpha}$ with one of the two factors $U^{*}_{\mu\alpha}$ and use of equation (27). There are two contraction patterns. The result is

$$\langle \text{Tr}(\rho(t)) \rangle = \frac{1}{N} \text{Tr}(A) + \langle |f_U(t)|^2 \rangle \text{Tr}(\Pi). \quad (29)$$

Here $f_U(t)$ is defined as in equation (12) except for the replacement $E_\alpha \rightarrow E_{U;\alpha}$. Proceeding as in appendix B we see that $\langle f_U(t) \rangle$ is proportional to the Fourier transform of the average GUE level density, and that the correlated part $\langle |f_U(t)|^2 \rangle_{\text{corr}}$ is proportional to the Fourier transform of the GUE level correlation function $Y_2(r) = \sin^2(\pi r) / (\pi r)^2$ [13]. The average level densities of the GOE and of the GUE coincide [13]. Thus, $\langle f_U(t) \rangle$ equals $g(t/\tau_\lambda)$ defined in equation (15), whereas $\langle |f_U(t)|^2 \rangle_{\text{corr}}$ is obtained from $\langle |f(t)|^2 \rangle_{\text{corr}}$ in equation (17) by replacing the GOE two-level correlation function by its GUE counterpart. The steps in section 4 can be followed to show that the GUE analogue to the variance in equation (20) vanishes for $N \rightarrow \infty$. Thus, thermalization is very similar for the GUE and for the GOE. Since $f_U^*(t) = f_U(-t)$ we have $\langle |f_U(t)|^2 \rangle = \langle |f_U(-t)|^2 \rangle$. The time dependence of thermalization for the GUE is symmetric about $t = 0$. We conclude that the symmetry of thermalization with respect to the replacement $t \rightarrow -t$ is not a consequence of time-reversal invariance. It follows from the fact that $\rho(t)$ is Hermitian. That causes $\rho(t)$ to have the forms $U(t)\Pi U^\dagger(t)$ for the GOE and $U_U(t)\Pi U^\dagger_U(t)$ for the GUE. These forms, in turn, cause the appearance of the time-symmetric factors $\langle |f(t)|^2 \rangle$ in equation (24) and $\langle |f_U(t)|^2 \rangle$ in equation (29).

6. The eigenstate thermalization hypothesis

We compare our procedure and results with those of [8]. We use the eigenbasis of the Hamiltonian $H$ with eigenvectors $|\alpha\rangle$ and eigenvalues $E_\alpha$ and write equation (8) as

$$\text{Tr}(A \rho(t)) = \sum_{\alpha} A_{\alpha\beta} \Pi_{\beta\alpha} \exp \{i (E_\alpha - E_\beta) t/\hbar\}. \quad (30)$$

Here $A_{\alpha\beta} = \langle \alpha | A | \beta \rangle$ and $\Pi_{\beta\alpha} = \langle \beta | \Pi | \alpha \rangle$. Equation (30) is completely general, i.e. holds for every Hamiltonian, and is not restricted to the model defined in section 2. In [8] it is assumed
that the system is in a single state $|\psi(t)\rangle = \sum_\alpha c_\alpha \exp\{-iE_\alpha t/\hbar\}|\alpha\rangle$. That is equivalent to assuming that $\Pi$ has the special form

$$\Pi_{\beta\alpha} = c_\beta c_\alpha^*,$$  \hspace{1cm} (31)

i.e. that $\Pi$ has rank one.

The basic assumptions used in [8] and in the present work differ. In [8] it is assumed that $A_{\alpha\beta}$ has the form of equation (1), with $R_{\alpha\beta}$ a zero-centered Gaussian random variable. The occupation amplitudes $c_\beta$ and eigenvalues $E_\alpha$ are taken to be fixed, i.e. non-statistical. Using a statistical assumption only on the operator $A$ is in line with the original approach of von Neumann [16]. In the random-matrix approach used in the present paper, the matrices $A_{\alpha\beta}$ and $\Pi_{\beta\alpha}$ are, respectively, given by $\sum_{\mu\nu} O_{\mu\alpha} A_{\nu\beta} O_{\beta\nu}^*$ and $\sum_{\mu\nu} O_{\mu\beta} \Pi_{\nu\alpha} O_{\alpha\nu}^*$. The matrix elements $O_{\mu\nu}$ are random variables. Therefore, both $A_{\alpha\beta}$ and $\Pi_{\beta\alpha}$ are stochastic, and so are the eigenvalues $E_\alpha$.

Our random-matrix approach is motivated by the fact that chaotic quantum systems are known to display stochastic features. These manifest themselves as fluctuation properties of eigenvalues and eigenfunctions of the Hamiltonian [10]. We interpret equation (1) and especially the occurrence of the random variable $R_{\alpha\beta}$ as due to that fact. Consistency then requires that both the eigenvalues $E_\alpha$ and the elements $\Pi_{\beta\alpha}$ are treated as stochastic variables, too. That is not done in [8]. We note that recent works (see, for instance, [17]), while close in spirit to [16], also abandon the solitary statistical treatment of $A$. The difference between the approach of [8] and the random-matrix model is significant and leads to different results. We demonstrate that fact below equation (35) for the function $C_{\text{th}}(t)$.

Aside from using the eigenstate thermalization hypothesis equation (1), the approach of [8] investigates thermalization of generic chaotic quantum systems with the help of long-time averages of powers of $\text{Tr}(A\rho(t))$ defined for $m = 1, 2, \ldots$ as

$$\overline{\text{Tr}(A\rho(t))^m} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle \text{Tr}(A\rho(t))^m \rangle.$$  \hspace{1cm} (32)

For $m = 1$ the long-time average gives the mean value of $\text{Tr}(A\rho(t))$, and powers of higher order are used to determine the fluctuations of $\text{Tr}(A\rho(t))$. We investigate that approach by applying it to $\text{Tr}(A\rho(t))$ defined in equation (9) in terms of the manifestly chaotic Hamiltonian (5), and by comparing the results with those of the random-matrix approach. We do so first for the mean values and then for the time dependence of thermalization.

In [8], the long-time average of $\text{Tr}(A\rho(t))$ is shown to be equal to the thermal average of $A$ at the appropriate temperature provided that the energy spread of the system is sufficiently small compared to the mean energy. That constrains the statistical operator $\Pi$. For the random Hamiltonian (5), calculation of the long-time average of $\text{Tr}(A\rho(t))$ in equation (9) gives

$$\sum_{\mu\nu\rho\sigma} A_{\mu\nu} O_{\nu\rho} \Pi_{\rho\sigma} O_{\sigma\mu}.$$  \hspace{1cm} (33)

Averaging over the random matrix elements $O_{\mu\nu}$ yields three terms, two of which are of order $1/N$. With $\text{Tr}(\Pi) = 1$ the leading-order term is $(1/N)\text{Tr}(A)$. That agrees with the first term on the right-hand side of equation (24) and, thus, with $\text{Tr}(A\rho_{\text{eq}}(\infty))$, in agreement with the result of [8]. (Actually, in the random-matrix approach there is no need for any constraint on $\Pi$ since equation (11) is an immediate consequence of orthogonal invariance.) We note that in both approaches, the average terms are actually present for all times $t > 0$. In our case that is seen directly from equation (11). For [8] it follows from the fact that fluctuations around the long-time average are calculated as fluctuations of $(\langle \text{Tr}(A\rho(t)) \rangle - \text{Tr}(A\rho_{\text{eq}}))$. 

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We turn to thermalization. In [8] the time dependence of thermalization is determined in two steps. In the first step, higher moments of \( \text{Tr}(A_{\rho}(t)) - \text{Tr}(A_{\rho_{eq}}) \) are estimated with the help of equation (1). It is shown that in the semiclassical regime and provided that the spread in energy of the system is sufficiently small, the time evolution of \( \text{Tr}(A_{\rho}(t)) - \text{Tr}(A_{\rho_{eq}}) \) is with overwhelming probability given by \( C(t)\text{Tr}(A_{\rho}(0)) \). Here \( C(t) \) is independent of the initial value \( \text{Tr}(A_{\rho}(0)) \) and is given by

\[
C(t) = \frac{\text{Tr}(A_{\rho}(t + t')) \text{Tr}(A_{\rho}(t'))}{\text{Tr}(A_{\rho}(t'))^2}
\]

(34)

where the long-time average is over \( t' \). In the second step, the numerator on the right-hand side of equation (34) is written in the eigenstate representation of the Hamiltonian as \( \sum_{\alpha\beta}(A_{\alpha\beta})^2(\Pi_{\beta\alpha})^2\exp\{i(E_{\alpha} - E_{\beta})t/\hbar\} \). Replacing \( A_{\alpha\beta} \) by the last term in equation (1) and averaging over the ensemble results in

\[
C_{\text{eth}}(t) \propto \int_{-\infty}^{+\infty} d\omega \, |F(E, \omega)|^2 \exp\{i\omega t/\hbar\}
\]

(35)

with \( F(E, \omega) \) introduced in equation (1). The index ‘eth’ stands for the eigenstate thermalization hypothesis. As mentioned earlier, in deriving equation (35) it is assumed that the eigenvalues \( E_{\alpha} \) are ordinary nonstochastic variables. If the \( E_{\alpha} \) were taken as random variables, one would have to average \( \sum_{\alpha\beta}(A_{\alpha\beta})^2(\Pi_{\beta\alpha})^2\exp\{i(E_{\alpha} - E_{\beta})t/\hbar\} \) over the joint probability distribution of \( E_{\alpha} \) and \( E_{\beta} \). Averaging would have to take into account that the random variables \( E_{\alpha} \) and \( E_{\beta} \) occur not only in the exponential but also as arguments of the function \( F(E, \omega) \) in equation (1). The result would differ from equation (35).

We compare the first step taken in [8] with the random-matrix model. There it is obvious almost from the outset (see equation (11)) that the time dependence of thermalization is independent of the initial value \( \text{Tr}(A_{\rho}(0)) \) and is given by \( \langle \langle f(t) \rangle \rangle \). To compare with that result, we calculate \( C(t) \) from equation (34) and use equation (9). We focus attention on the numerator on the right-hand side of equation (34). In averaging over the matrix elements \( O_{\mu\alpha} \) we take into account only contraction patterns that retain the dependence on \( t \). That is consistent with [8]. Using \( C(0) = 1 \) we find

\[
C_{\text{rm}}(t) = \langle \langle f(t) \rangle \rangle
\]

(36)

where the index ‘rm’ stands for the random-matrix model. Equation (36) is in agreement with equation (24), confirming the definition equation (34) of \( C(t) \) given in [8] and the claim that the time evolution of thermalization is given by \( C(t)\text{Tr}(A_{\rho}(0)) \).

The second step taken in [8] leads to the explicit expression for \( C_{\text{eth}}(t) \) in equation (35). That expression differs from \( C_{\text{rm}}(t) \) in equation (36). The function \( C_{\text{eth}}(t) \) in equation (35) is the Fourier transform of the non-negative function \( |F(E, \omega)|^2 \). Therefore, \( C_{\text{eth}}(t) \) oscillates in time about the value zero. Except for the assumption on smoothness, the dependence on \( E \) and \( \omega \) of the function \( F(E, \omega) \) in equation (1) is not defined nor is it obvious how information on that function might be obtained. In [8] the band width of the function \( F(E, \omega) \) with respect to \( \omega \) is used as a parameter and is estimated with the help of physical arguments. If the band width in \( \omega \) of \( |F(E, \omega)|^2 \) is finite, \( C_{\text{eth}}(t) \) is asymptotically \( (t \to \infty) \) inversely proportional to \( t \). In contrast, the function \( C_{\text{rm}}(t) \) in equation (36) is non-negative for all times \( t \). To leading order in \( 1/N \), \( \langle \langle f(t) \rangle \rangle \) is equal to the square of the Fourier transform \( g(t/\tau_{\text{ch}}) \) in equation (15) of the average level density and, for large \( |t| \), falls off as \( 1/|t|^3 \). We see that the time dependence of thermalization is qualitatively different from the one predicted in the framework of equation (1). Within the random-matrix model, it is fully determined analytically.
The fundamental difference between the random-matrix model and the eigenstate thermalization hypothesis lies in the identification of the source of randomness. In the random-matrix model, it is the operator $U(t)$ defined in equation (7) and appearing in equation (9) that carries the random variables and that causes the function $\text{Tr}(A \rho(t))$ to be a stochastic process. The operator $A$, in contradistinction, is written in the fixed basis of states $(\mu, \nu)$ as in equation (8) and, therefore, is not random. The central result, equation (11), is valid for any operator $A$. The approach of [8] takes the converse point of view. The elements $\rho_{\alpha\beta}(t)$ of the density matrix of the system are treated as ordinary (non-statistical) variables. The stochasticity of the chaotic system is contained entirely in the eigenstate representation $A_{\alpha\beta}$ of the operator $A$.

Equation (1) is tailored to yield thermalization. From the point of view of formal logic, equation (1) formulates a sufficient condition for thermalization. It is the great merit of [8] to have in that way given a first analytical approach to and understanding of thermalization. A number of questions remains. Why is randomness attached only to the elements $A_{\alpha\beta}$ of the operator $A$? Does equation (1) impose a constraint on the operator $A$? If so, which operators obey that constraint? Do all operators that obey the constraint possess the same function $F(E, \omega)$, or does that function depend on the operator $A$ under consideration? How does $F(E, \omega)$ (beyond being smooth) actually depend on the variables $E$ and $\omega$? The answers should link equation (1) with the Hamiltonian of the chaotic quantum system. Without that link, equation (1) remains a hypothesis.

Quantum chaos and stochasticity of the Hamiltonian are closely linked [10]. Therefore, it seems more natural to analyze thermalization in terms of a stochastic model for the Hamiltonian rather than with the help of equation (1). The random-matrix model defined in section 2 is too restricted for that purpose. The approach of [18] would seem a suitable candidate. Unfortunately, and in contrast to [8], it does not offer the chance of an analytical treatment. The development of an approach to thermalization of closed chaotic many-body quantum systems that identifies and uses statistical properties of the Hamiltonian is clearly called for.

7. Results, discussion, conclusions

The motivation for our use of a random-matrix model to investigate thermalization has been twofold. (i) The model is intended to illustrate generic features of thermalization, and to test aspects of the eigenstate thermalization hypothesis. (ii) We expect that the concepts and techniques developed for the model will form the basis of a general treatment of thermalization of a closed chaotic many-body quantum system.

As for point (i), we have shown that in the random-matrix model, every test function $\text{Tr}(A \rho(t))$ thermalizes, i.e. attains the limit (24) for $N \to \infty$. That holds in full generality if the rank of the statistical operator $\Pi$ is of order $N$ and for which the distribution of eigenvalues is sufficiently narrow and centered at $1/N$. It holds with the constraint (23) on the test operator $A$ in all other cases. Thermalization is due to the orthogonal invariance of the GOE: For $N \gg 1$, orthogonal invariance causes the matrix elements $O_{\mu\nu}$ in equation (5) to be zero-centered Gaussian random variables that obey equation (6). That is the only fact used in deriving equation (11) and, for matrix dimension $N \to \infty$, equation (24). In particular, orthogonal invariance implies that all states in Hilbert space are treated on the same footing. That fact determines the form of the equilibrium term $(1/N)\text{Tr}(A)$. Formally, that expression corresponds to the Boltzmann factor being equal to unity and, thus, to infinite temperature. The last term in equation (24) gives the time dependence of thermalization. One might have expected that the time dependence of thermalization is determined by destructive interference of the phase factors $\exp\{\pm iE_{i}/\hbar\}$ in equation (2) for the density matrix. That is not the case. The
leading-order contribution to thermalization is given by the function \( \langle f(t) \rangle \), i.e. by the Fourier transform \( g(t/\tau_\lambda) \) of the average GOE level density. That function does not contain interference terms. The time scale \( \tau_\lambda \) is set by the range of the average GOE level density. The GOE two-point level correlation function does contain interference terms but only yields a term of order \( 1/N \). Both \( \langle f(t) \rangle \) and the two-point function are determined by orthogonal invariance and are independent of the form of the test operator \( A \). Corresponding statements hold for the GUE where unitary invariance takes the role of orthogonal invariance.

Application of the random-matrix model to the eigenstate thermalization hypothesis requires a caveat. As stated earlier, the random-matrix Hamiltonians in equations (4) and (25) cannot be taken as realistic models of chaotic many-body systems. That is made evident by the special point \( t = 0 \) (see sections 2 and 3) which is caused by the fact that the random-matrix approach does not allow for transients. The statement also applies to the form (43) of the average level density, and to the form of the time dependence of \( g(t/\tau_\lambda) \) that follows from it. It applies, likewise, to the statement that thermalization occurs at infinite temperature. Nevertheless, a meaningful comparison with the eigenstate thermalization hypothesis equation (1) is possible as that hypothesis should apply in the present case as well. The comparison shows that long-time averages agree with random-matrix results. The predicted time dependence of thermalization is only qualitatively correct. The difference to the exact random-matrix results is caused by the different statistical assumptions used. In equation (1) the matrix elements \( A_{\alpha\beta} \) of the operator \( A \) are treated as stochastic variables while in [8] the matrix elements \( \Pi_{\beta\alpha} \) of the statistical operator \( \Pi \) are held fixed, i.e. are non-stochastic. In the random-matrix approach, on the other hand, the statistical properties of \( H \) imply that both \( A_{\alpha\beta} \) and \( \Pi_{\beta\alpha} \) are stochastic, and are correlated.

Concerning point (ii), the results of the present paper have strengthened our conviction that the statistical assumptions used in equation (1) reflect properties of the chaotic Hamiltonian, and that consistency requires corresponding assumptions to be used for \( \Pi \) as well. In [11] we implement these ideas, using the methods and results of the present paper for the statistical treatment of the Hamiltonian of a closed chaotic quantum system. In such a system, the basis states of Hilbert space are not completely mixed with each other. Mixing occurs only over a limited energy interval. The resulting distribution of the basis states in Hilbert space over the eigenstates of the Hamiltonian differs from equation (43) and is Gaussian [19, 20]. That leads to precise conditions for thermalization and to a prediction for the time scale of thermalization.

We conclude with a few comments on general aspects of thermalization. Thermalization as defined in section 1 is a property of the test function \( \text{Tr}(A\rho(t)) \). It characterizes the system but is not an obvious property of the density matrix \( \rho(t) \) itself. Thermalization is caused by the statistical properties of the time-evolution operator \( U(t) \) in equation (9). Thermalization is symmetric in time. That symmetry holds for both, the time-reversal invariant GOE and the time-reversal non-invariant GUE.

Thermalization is very different from equilibration in open quantum systems [21]. The time dependence of equilibration is described by a master equation that violates time-reversal invariance. The statistical operator of the system changes with time and tends toward the statistical equilibrium distribution. The process is exponential in time. In contrast, thermalization occurs in an isolated system. The statistical operator of the system does not change with time, the elements of the density matrix of the system keep oscillating in time forever. Only the test function \( \text{Tr}(A\rho(t)) \) thermalizes.

In [4], the eigenstate thermalization hypothesis is formulated without reference to equation (1) and in more general terms by stating that ‘every eigenstate in an energy shell represents thermal equilibrium’. We believe that our random-matrix models are paradigmatic for the eigenstate thermalization hypothesis in that more general form. The energy shell is the
entire Hilbert space. All eigenstates $|\alpha\rangle = \sum_{\mu} O_{\mu \alpha} |\mu\rangle$ are random superpositions of the basis states $|\mu\rangle$. The statistical weight of every component vector $|\mu\rangle$ is the same, independent of the eigenvalue $E_{\alpha}$ to which the state belongs.

Thermalization tests the hypothesis that in an isolated system and in the long-time limit, expectation values of operators tend to their equilibrium values. That hypothesis is seen as the quantum analogue of the ergodic hypothesis in classical statistical mechanics [1]. It is not surprising that in the random-matrix models, thermalization follows from orthogonal or unitary invariance. Such invariance implies that all states in $N$-dimensional matrix space are equivalent. That is similar in spirit to classical ergodicity.

Data availability statement

No new data were created or analyzed in this study.

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Appendix A. Low moments of the time-evolution operator $U$

The matrix $U(t)$, defined in equation (7), depends on the random variables $O_{\mu \alpha}$ and $E_{\alpha}$. The former are Gaussian and obey equation (6). The latter obey Wigner–Dyson statistics. We calculate the moments as averages over the matrix elements $O_{\mu \alpha}$. We do not, at this stage, average over the eigenvalues $E_{\alpha}$. Averaging over the Gaussian-distributed matrix elements $O_{\mu \alpha}$ is indicated by the index $O$ and is done by applying equation (6) to all pairs of matrix elements (‘contracting’ pairs of matrix elements). Our averaging procedure is justified in [22].

The ensemble average of $U(t)$ is

$$\langle U_{\mu \nu} (t) \rangle_O = \delta_{\mu \nu} f(t)$$

(37)

where $f(t)$ is defined in equation (12). The correlated part of the second moment of $U(t)$ is

$$\langle U_{\mu \nu} (t) U_{\mu' \nu'} (t) \rangle_{O \text{corr}} = \frac{1}{N} (\delta_{\mu \mu'} \delta_{\nu \nu'} + \delta_{\mu \nu'} \delta_{\mu \nu'}) f(2t),$$

$$\langle U_{\mu \nu} (t) U_{\mu' \nu'}^*(t) \rangle_{O \text{corr}} = \frac{1}{N} (\delta_{\mu \mu'} \delta_{\nu' \nu'} + \delta_{\mu \nu'} \delta_{\mu \nu'}) .$$

(38)

For the totally correlated part of the third moment of $U(t)$ we need only the contribution that contains two factors $U(t)$ and one factor $U^*(t)$. That term is given by

$$\langle U_{\mu_1 \nu_1} (t) U_{\mu_2 \nu_2} (t) U_{\mu_3 \nu_3}^* (t) \rangle_{O \text{corr}} = \frac{1}{N^2} f(t) (\delta_{\nu_1 \mu_2} \delta_{\nu_2 \mu_3} \delta_{\nu_3 \mu_1} + \ldots) .$$

(39)

The dots indicate the remaining seven possibilities to contract pairs of matrix elements so as to yield the totally correlated part. For the totally correlated part of the fourth moment of $U(t)$ we need only the part that is bilinear in both $U(t)$ and in $U^*(t)$. That part is given by

$$\langle U_{\mu_1 \nu_1} (t) U_{\mu_2 \nu_2} (t) U_{\mu_3 \nu_3} (t) U_{\mu_4 \nu_4}^* (t) \rangle_{O \text{corr}} = \frac{1}{N^3} (\delta_{\nu_1 \mu_2} \delta_{\nu_2 \mu_3} \delta_{\nu_3 \mu_4} \delta_{\nu_4 \mu_1} + \ldots) .$$

(40)

The dots indicate the remaining possibilities to contract pairs of matrix elements so as to yield the totally correlated part of the fourth moment.
Appendix B. Time dependence of thermalization

For the uncorrelated part in equation (13) we write the function \( f(t) \), defined in equation (12), as

\[
f(t) = \frac{1}{N} \int dE \, r(E) \exp \{-iEt/\hbar\}
\]

where

\[
r(E) = \sum_\alpha \delta(E-E_\alpha)
\]

is the non-averaged GOE level density normalized to the total number \( N \) of states. The ensemble average of \( r(E) \) is [13]

\[
\langle r(E) \rangle = \frac{N}{\pi \lambda} \sqrt{1 - \left(E/(2\lambda)^2\right)},
\]

where the range of \( E \) is \(-2\lambda \leq E \leq 2\lambda\). That gives

\[
\langle f(t) \rangle = \frac{1}{\pi \lambda} \int_{-2\lambda}^{2\lambda} dE \, \sqrt{1 - (E/(2\lambda)^2)} \exp \{-iEt/\hbar\}.
\]

Thus, \( \langle f(t) \rangle \) is the Fourier transform of the average GOE level density. We substitute \( x = E/(2\lambda) \), use equation (14), and obtain equation (15). The explicit expression for \( g(t/\tau) \) is given in equation (16). For \( |t| \gg \tau \lambda \) we have asymptotically

\[
g(t/\tau \lambda) \sim -2\sqrt{2/\pi} \cos(\pi|t/\tau \lambda|) \sim |t|^{-3/2}.
\]

The correlated part of \( f(t) \), written in full generality as a function of two time arguments \((t_1, t_2)\), is given by

\[
\langle f(t_1)f^*(t_2) \rangle_{\text{corr}} = \frac{1}{N^2} \int_{-2\lambda}^{+2\lambda} dE_1 \int_{-2\lambda}^{+2\lambda} dE_2 \langle r(E_1) r(E_2) \rangle_{\text{corr}} \exp \{-i (E_1 t_1 - E_2 t_2)/\hbar\}.
\]

For \( N \gg 1 \) the correlated part of the GOE level density is given by [13]

\[
\langle r(E_1) r(E_2) \rangle_{\text{corr}} = -\langle r(E_1) \rangle \langle r(E_2) \rangle Y_2(y).
\]

Here \( Y_2(y) \) with \( y = (E_1 - E_2)/d \) is the two-level correlation function of the GOE given in equation (18), and \( d \) is the mean GOE level spacing. We define new integration variables \( E = (1/2)(E_1 + E_2), \varepsilon = E_1 - E_2 \). Then \( y = \varepsilon/d \). The two-point function \( Y_2 \) effectively confines \( y \) to values \( y \ll N \) and, thus, \( \varepsilon \) to values of order \( d \ll 2\lambda \). With \( E_1 = E + (1/2)\varepsilon, E_2 = E - (1/2)\varepsilon, \) equation (43) shows that the dependence on \( \varepsilon \) of \( \langle r(E_1) \rangle \) and of \( \langle r(E_2) \rangle \) in equation (46) may be neglected. That gives

\[
\langle f(t_1)f^*(t_2) \rangle_{\text{corr}} = \frac{(-1)}{(\pi \lambda)^2} \int_{-\infty}^{+\infty} d\varepsilon \, \frac{Y_2(\varepsilon/d)}{\varepsilon} \exp \{-i/2\varepsilon(t_1 + t_2)/\hbar\} \times \int_{-2\lambda}^{+2\lambda} dE \, \left[1 - (E/(2\lambda))^2\right] \exp \{-i(E(t_1 - t_2)/\hbar)\}.
\]

We define \( x = E/(2\lambda), \pi \lambda = Nd \), use the definition (14), and carry out the integral in the last line. We obtain
\[ \langle f(t_1) f^*(t_2) \rangle_{\text{corr}} = \frac{8}{\pi} \left( \frac{\cos((t_1 - t_2)/\tau_\lambda)}{(t_1 - t_2)^2} - \frac{\sin((t_1 - t_2)/\tau_\lambda)}{(t_1 - t_2)^3} \right) \times \frac{1}{N} \int_{-\infty}^{+\infty} dy \ Y_2(y) \cos(y(t_1 + t_2)/(2\tau_d)). \] 

(49)

The factor $1/N$ is due to the substitution $\varepsilon/((\pi \lambda) \rightarrow y/N$. It accounts for the fact that the two-point correlation function of the GOE is essentially confined to distances in energy that are of order $d \ll 2\lambda$. The time-dependent functions in the second (third) line of equation (49) display oscillations on the time scale $\tau_\lambda$ ($\tau_d$, respectively). These are the smallest (largest) time scales of the GOE. For $N \gg 1$ the oscillations of the function in the third line are extremely slow compared to those of the function in the second line. Putting $t_1 = t = t_2$ we obtain equation (17).

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