Asymptotically Coupled Coincidence Points and Asymptotically Coupled Fixed Points in Fuzzy Semi-Metric Spaces

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Abstract: Asymptotically coupled coincidence points and asymptotically coupled fixed points in fuzzy semi-metric spaces are studied in this paper. The fuzzy semi-metric space is taken into account, which lacks symmetric conditions. In this case, the desired results are separately investigated based on four different types of triangle inequalities. The uniqueness of asymptotically coupled coincidence points cannot be guaranteed, and it can only be addressed in a weak sense of uniqueness. However, the uniqueness of asymptotically coupled fixed points can be guaranteed using different arguments.

Keywords: asymptotically coupled coincidence points; asymptotically coupled fixed points; fuzzy semi-metric space

MSC: 54E35; 54H25

1. Introduction

The so-called asymptotically coupled coincidence points and asymptotically coupled fixed points in fuzzy semi-metric space $(U, \delta)$ are studied in this paper. Let $\{F_n\}_{n=1}^\infty$ be a sequence of functions with $F_n : U \times U \to U$ for all $n$ and let $f : U \to U$ be another function. We say that an element $(u, v) \in U \times U$ is an asymptotically coupled coincidence point of functions $\{F_n\}_{n=1}^\infty$ and $f$ when we have the following convergence

$$F_n(u, v) \xrightarrow{\delta} f(u) \quad \text{and} \quad F_n(v, u) \xrightarrow{\delta} f(v) \quad \text{as} \quad n \to \infty.$$

On the other hand, we say that an element $(u, v) \in U \times U$ is an asymptotically coupled fixed point of functions $\{F_n\}_{n=1}^\infty$ and $f$ when we have the following convergence

$$F_n(u, v) \xrightarrow{\delta} f(u) = u \quad \text{and} \quad F_n(v, u) \xrightarrow{\delta} f(v) = v \quad \text{as} \quad n \to \infty.$$

The convergence $F_n(u, v) \xrightarrow{\delta} f(u)$ is based on the fuzzy semi-metric space $(U, \delta)$, which will be defined in Section 2. In particular, suppose that all the functions $F_n$ are identical to $f$ for all $n$.

- If $(u, v) \in U \times U$ is an asymptotically coupled coincidence point, then it is also a coupled coincidence point in the sense of

$$F(u, v) = f(u) \quad \text{and} \quad F(v, u) = f(v).$$

- If $(u, v) \in U \times U$ is an asymptotically coupled fixed point, then it is also a common coupled fixed point in the sense of

$$F(u, v) = f(u) = u \quad \text{and} \quad F(v, u) = f(v) = v.$$

More detailed arguments will be presented in the context of this paper.
The symmetric condition will not be assumed in fuzzy semi-metric space. In this case, we shall propose four different concepts of triangle inequalities. The asymptotically coupled coincidence points and asymptotically coupled fixed points will be separately studied using these four different triangle inequalities.

The idea of fuzzy metric space was inspired by the probabilistic metric space. For more details on the topic of probabilistic metric space, we may refer to Schweizer and Sklar [1–3], Hadžić and Pap [4] and Chang et al. [5]. A special kind of probabilistic metric space called the Menger space was adopted by Kramosil and Michalek [6] to define the so-called fuzzy metric space. In this paper, we are going to consider a weak sense of fuzzy metric space, which is called the fuzzy semi-metric space (ref. Wu [7,8]). The main purpose of this paper is to study the asymptotically coupled coincidence points and asymptotically coupled fixed points in fuzzy semi-metric space.

Shen and Chen [9], Singh and Chauhan [10], Vasuki [11] and Wang et al. [12] have proposed many different methodologies to study fixed points in fuzzy metric spaces. On the other hand, Wu [13–15] also studied many different kinds of fixed points in fuzzy semi-metric spaces. In this paper, the purpose is to study the asymptotically coupled coincidence points and asymptotically coupled fixed points in fuzzy semi-metric spaces. Because of the lack of symmetric conditions in fuzzy semi-metric space, the desired results will be separately studied using the four different types of triangle inequalities. The uniqueness of asymptotically coupled coincidence points cannot be guaranteed, and it can only be addressed in a weak sense of uniqueness. However, we can prove the uniqueness of asymptotically coupled fixed points using different arguments in this paper.

In Section 2, the concept of fuzzy semi-metric space, which is endowed with four different triangle inequalities is introduced. In Section 3, the Cauchy sequences in fuzzy semi-metric space are provided, which will be used to study the asymptotically coupled coincidence points and asymptotically coupled fixed points. In Section 4, we establish the theorems for asymptotically coupled coincidence points in fuzzy semi-metric spaces using the four different triangle inequalities. In Section 5, we also establish the asymptotically coupled fixed points in fuzzy semi-metric spaces.

### 2. Fuzzy Semi-Metric Spaces

We are going to introduce the concept of fuzzy semi-metric space based on t-norm. A function \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called a t-norm (triangular norm) when the following conditions are satisfied:

- \( \alpha * 1 = \alpha \) for any \( \alpha \in [0, 1] \);
- \( \alpha * \beta = \beta * \alpha \) for any \( \alpha, \beta \in [0, 1] \);
- \( \beta < \gamma \) implies \( \alpha * \beta \leq \alpha * \gamma \) for any \( \alpha, \beta, \gamma \in [0, 1] \);
- \( (\alpha * \beta) * \gamma = \alpha * (\beta * \gamma) \) for any \( \alpha, \beta, \gamma \in [0, 1] \).

We can also treat the t-norm \( \alpha * \beta \) as a function \( \eta(\alpha, \beta) = \alpha * \beta \). The commutativity of t-norm indicates that \( \eta(\alpha, \beta) = \eta(\beta, \alpha) \). Given any fixed \( b \), when \( \eta(. , b) \) is continuous on \([0, 1] \), we say that the function \( \eta \) is continuous with respect to the first argument. Similarly, given any fixed \( a \), when \( \eta(a, .) \) is continuous on \([0, 1] \), we also say that the function \( \eta \) is continuous with respect to the second argument. Since the function \( \eta \) is commutative, we have the following observations.

- If \( \eta \) is continuous with respect to the first argument, then it is also continuous with respect to the second argument.
- If \( \eta \) is continuous with respect to the second argument, then it is also continuous with respect to the first argument.

**Remark 1.** We want to claim that \( \alpha * \beta = 1 \) implies \( \alpha = 1 = \beta \). Since \( \alpha \leq 1 \), according to the increasing property of t-norm, it follows that \( 1 = \alpha * \beta \leq 1 * \beta = \beta \). Similarly, since \( \beta \leq 1 \), we also have \( 1 = \alpha * \beta \leq 1 * \alpha = \alpha \), which implies \( \alpha = 1 = \beta \).
Given a sequence \( \{a_n\}_{n=1}^{\infty} \) in \([0,1]\), we write \( a_n \to a^- \) as \( n \to \infty \) to mean that the sequence \( \{a_n\}_{n=1}^{\infty} \) converges to \( a \) from the left.

**Proposition 1** (Wu [7] (Proposition 2)). Given any fixed \( a, b \in [0,1] \), assume that the \( t \)-norm \( * \) is left-continuous at \( a \) and \( b \) with respect to the first or second component. If the sequences \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) in \([0,1]\) satisfy

\[
a_n \to a^- \quad \text{and} \quad b_n \to b^- \quad \text{as} \quad n \to \infty,
\]

then we have

\[
a_n \ast b_n \to a \ast b \quad \text{as} \quad n \to \infty.
\]

**Definition 1.** Let \( U \) be a universal set. We consider a function \( \mathcal{F} : U \times U \times [0,\infty) \to [0,1] \). The ordered pair \((U, \mathcal{F})\) is called a fuzzy semi-metric space when the following conditions are satisfied.

- For any fixed \( u, v \in U \), we have \( \mathcal{F}(u,v,t) = 1 \) for all \( t > 0 \) if and only if \( u = v \).
- For any fixed \( u, v \in U \) with \( u \neq v \), we have \( \mathcal{F}(u,v,0) = 0 \).

The symmetric condition is said to be satisfied when we have

\[
\mathcal{F}(u,v,t) = \mathcal{F}(v,u,t) \quad \text{for any} \quad u, v \in U \quad \text{and} \quad t \geq 0.
\]

Since the symmetric condition is not necessarily satisfied in the fuzzy semi-metric space \((U, \mathcal{F})\), four different kinds of triangle inequalities should be considered as follows.

**Definition 2.** Let \( U \) be a universal set and let \( * \) be a \( t \)-norm. We consider a function \( \mathcal{F} : U \times U \times [0,\infty) \to [0,1] \).

- The function \( \mathcal{F} \) is said to satisfy \( \triangleright \)-triangle inequality when

\[
\mathcal{F}(u,v,t) \ast \mathcal{F}(v,w,s) \leq \mathcal{F}(u,w,t+s) \quad \text{for all} \quad u, v, w \in U \quad \text{and} \quad s, t > 0.
\]

- The function \( \mathcal{F} \) is said to satisfy \( \triangleright \)-triangle inequality when

\[
\mathcal{F}(u,v,t) \ast \mathcal{F}(w,v,s) \leq \mathcal{F}(u,w,t+s) \quad \text{for all} \quad u, v, w \in U \quad \text{and} \quad s, t > 0.
\]

- The function \( \mathcal{F} \) is said to satisfy \( \prec \)-triangle inequality when

\[
\mathcal{F}(v,u,t) \ast \mathcal{F}(v,w,s) \leq \mathcal{F}(u,w,t+s) \quad \text{for all} \quad u, v, w \in U \quad \text{and} \quad s, t > 0.
\]

- The function \( \mathcal{F} \) is said to satisfy \( \prec \)-triangle inequality when

\[
\mathcal{F}(v,u,t) \ast \mathcal{F}(w,v,s) \leq \mathcal{F}(u,w,t+s) \quad \text{for all} \quad u, v, w \in U \quad \text{and} \quad s, t > 0.
\]

**Definition 3.** Let \((U, \mathcal{F})\) be a fuzzy semi-metric space.

- The semi-metric \( \mathcal{F} \) is said to be nondecreasing when, given any fixed \( u, v \in U \), the following inequality is satisfied

\[
\mathcal{F}(u,v,t_1) \geq \mathcal{F}(u,v,t_2) \quad \text{for} \quad t_1 > t_2.
\]

- The semi-metric \( \mathcal{F} \) is said to be symmetrically nondecreasing when, given any fixed \( u, v \in U \), the following inequality is satisfied

\[
\mathcal{F}(u,v,t_1) \geq \mathcal{F}(v,u,t_2) \quad \text{for} \quad t_1 > t_2.
\]

**Proposition 2** (Wu [7] (Proposition 4)). Let \((U, \mathcal{F})\) be a fuzzy semi-metric space.

(i) Suppose that the \( \triangleright \)-triangle inequality is satisfied. Then, the semi-metric \( \mathcal{F} \) is nondecreasing.

(ii) Suppose that the \( \triangleright \)-triangle inequality or the \( \prec \)-triangle inequality is satisfied. Then, the semi-metric \( \mathcal{F} \) is both nondecreasing and symmetrically nondecreasing.
(iii) Suppose that the \(\circ\)-triangle inequality is satisfied. Then, semi-metric \(\mathfrak{F}\) is symmetrically nondecreasing.

Let \(\{u_n\}_{n=1}^{\infty}\) be a sequence in a metric space \((U,d)\). We write \(u_n \xrightarrow{d} u\) as \(n \to \infty\) to mean

\[
\lim_{n \to \infty} d(u_n, u) = 0
\]

in which \(u\) is also called a \(d\)-limit of the sequence \(\{u_n\}_{n=1}^{\infty}\).

**Definition 4.** Let \(\{u_n\}_{n=1}^{\infty}\) be a sequence in a fuzzy semi-metric space \((U, \mathfrak{F})\).

- We write \(u_n \xrightarrow{\mathfrak{F}} u\) as \(n \to \infty\) to mean
  
  \[
  \lim_{n \to \infty} \mathfrak{F}(u_n, u, t) = 1 \text{ for all } t > 0.
  \]
  In this case, \(u\) is also called a \(\mathfrak{F}\)-limit of the sequence \(\{u_n\}_{n=1}^{\infty}\).

- We write \(u_n \xrightarrow{\mathfrak{F}^{-1}} u\) as \(n \to \infty\) to mean
  
  \[
  \lim_{n \to \infty} \mathfrak{F}(u_n, u, t) = 1 \text{ for all } t > 0.
  \]
  In this case, \(u\) is also called a \(\mathfrak{F}^{-1}\)-limit of the sequence \(\{u_n\}_{n=1}^{\infty}\).

- We write \(u_n \xrightarrow{\mathfrak{F}} u\) as \(n \to \infty\) to mean
  
  \[
  \lim_{n \to \infty} \mathfrak{F}(u_n, u, t) = \lim_{n \to \infty} \mathfrak{F}(u_n, u, t) = 1 \text{ for all } t > 0.
  \]
  In this case, \(u\) is also called a \(\mathfrak{F}\)-limit of the sequence \(\{u_n\}_{n=1}^{\infty}\).

The following interesting results will be used for the further study.

**Proposition 3 (Wu [8] (Proposition 7)).** Let \((U, \mathfrak{F})\) be a fuzzy semi-metric space along with a \(t\)-norm \(*\) and let \(\{u_n, v_n, t_n\}_{n=1}^{\infty}\) be a sequence in \(U \times U \times (0, \infty)\). Assume that the \(t\)-norm \(*\) is left-continuous with respect to the first or second component, and that the following inequality is satisfied

\[
\sup_n (\alpha_n \ast \beta_n) \geq \left( \sup_n \alpha_n \right) \ast \left( \sup_n \beta_n \right)
\]

for any two sequences \(\{\alpha_n\}_{n=1}^{\infty}\) and \(\{\beta_n\}_{n=1}^{\infty}\) in \([0,1]\). Suppose that \(\infty\)-triangle inequality is satisfied and that the following limits

\[
t_n \to t^*, \quad u_n \xrightarrow{\mathfrak{F}} u^* \text{ and } v_n \xrightarrow{\mathfrak{F}} v^* \text{ as } n \to \infty
\]

exist. Then, we have the following properties.

- If the function \(\mathfrak{F}(u^*, v^*, \cdot) : (0, \infty) \to [0, 1]\) is continuous at \(t^*\), then we have
  
  \[
  \lim_{n \to \infty} \mathfrak{F}(u_n, v_n, t_n) = \mathfrak{F}(u^*, v^*, t^*).
  \]

- If the function \(\mathfrak{F}(v^*, u^*, \cdot) : (0, \infty) \to [0, 1]\) is continuous at \(t^*\), then we have
  
  \[
  \lim_{n \to \infty} \mathfrak{F}(v_n, u_n, t_n) = \mathfrak{F}(v^*, u^*, t^*).
  \]

**Definition 5.** Let \(\{u_n\}_{n=1}^{\infty}\) be a sequence in a fuzzy semi-metric space \((U, \mathfrak{F})\).
• We say that \( \{u_n\}_{n=1}^{\infty} \) is a \( \times \)-Cauchy sequence when, for any pair \( (r, t) \) satisfying \( t > 0 \) and \( 0 < r < 1 \), there exists an integer \( n_{r,t} \) such that
\[
m > n \geq n_{r,t} \text{ implies } \mathfrak{S}(u_m, u_n, t) > 1 - r
\]
for any pairs \( (m, n) \) of integers \( m \) and \( n \).

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\]
for any pairs \( (m, n) \) of integers \( m \) and \( n \).

• We say that \( \{u_n\}_{n=1}^{\infty} \) is a Cauchy sequence when, for any pair \( (r, t) \) satisfying \( t > 0 \) and \( 0 < r < 1 \), there exists an integer \( n_{r,t} \) such that
\[
m, n \geq n_{r,t} \text{ with } m \neq n \text{ implies } \mathfrak{S}(u_m, u_n, t) > 1 - r \text{ and } \mathfrak{S}(u_m, u_n, t) > 1 - r
\]
for any pairs \( (m, n) \) of integers \( m \) and \( n \).

**Definition 6.** We consider the different completeness as follows.

• The fuzzy semi-metric space \((U, \mathfrak{S})\) is said to be \((\times, \to)\)-complete when each \( \times \)-Cauchy sequence is convergent with \( u_n \xrightarrow{\mathfrak{S}} u \).

• The fuzzy semi-metric space \((U, \mathfrak{S})\) is said to be \((\times, \leftarrow)\)-complete when each \( \times \)-Cauchy sequence is convergent with \( u_n \xrightarrow{\mathfrak{S}} u \).

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**Definition 7.** Let \((U, \mathfrak{S})\) be a fuzzy semi-metric space and let \( \{u_n\}_{n=1}^{\infty} \) be a sequence in \( U \). Consider a function \( f : U \to U \).

• The function \( f \) is said to be \((\to, \times)\)-continuous with respect to \( \mathfrak{S} \) when
\[
\text{for any } n \to \infty \text{ implies } f(u_n) \xrightarrow{\mathfrak{S}} f(u) \text{ as } n \to \infty.
\]

• The function \( f \) is said to be \((\leftarrow, \times)\)-continuous with respect to \( \mathfrak{S} \) when
\[
\text{for any } n \to \infty \text{ implies } f(u_n) \xrightarrow{\mathfrak{S}} f(u) \text{ as } n \to \infty.
\]

• The function \( f \) is said to be \((\times, \to)\)-continuous with respect to \( \mathfrak{S} \) when
\[
\text{for any } n \to \infty \text{ implies } f(u_n) \xrightarrow{\mathfrak{S}} f(u) \text{ as } n \to \infty.
\]

• The function \( f \) is said to be \((\times, \leftarrow)\)-continuous with respect to \( \mathfrak{S} \) when
\[
\text{for any } n \to \infty \text{ implies } f(u_n) \xrightarrow{\mathfrak{S}} f(u) \text{ as } n \to \infty.
\]

Under the above settings, we can study the common coupled coincidence points and common coupled fixed points of nonlinear contractive functions. We shall first present the properties of Cauchy sequences below.
3. Cauchy Sequences

Let $U$ be a universal set and let $\{F_n\}_{n=1}^{\infty}$ be a sequence of functions $F_n : U \times U \rightarrow U$ for all $n$. Consider a function $f : U \rightarrow U$ satisfying

$$F_n(U, U) \subseteq f(U) \text{ for all } n. \tag{1}$$

For any two initial elements $u_0, v_0 \in U$, according to the inclusion (1), there exist $u_1, v_1 \in U$ satisfying

$$f(u_1) = F_1(u_0, v_0) \text{ and } f(v_1) = F_1(v_0, u_0).$$

Similarly, there also exist $u_2, v_2 \in U$ satisfying

$$f(u_2) = F_2(u_1, v_1) \text{ and } f(v_2) = F_2(v_1, u_1).$$

Continuing this process, we can construct two sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ in $U$ satisfying

$$f(u_n) = F_n(u_{n-1}, v_{n-1}) \text{ and } f(v_n) = F_n(v_{n-1}, u_{n-1}) \tag{2}$$

for all $n$.

**Definition 8.** We say that the function $\lambda : [0, 1] \rightarrow [0, 1]$ is left-continuous at $a_0 \in [0, 1]$ in the left sense when, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < a_0 - \alpha < \delta \text{ implies } 0 < \lambda(a_0) - \lambda(\alpha) < \epsilon.$$

Suppose that $\lambda$ is left-continuous at $a_0$ in the left sense. Then, it is clear to see that

$$\alpha \rightarrow a_0 - \text{ implies } \lambda(\alpha) \rightarrow \lambda(a_0).$$

We first provide a useful lemma.

**Lemma 1.** Let $(U, \mathfrak{g})$ be a fuzzy semi-metric space. Suppose that the following conditions are satisfied:

- the $t$-norm $*$ is left-continuous at 1 with respect to the first or second component.
- the functions $F_n : U \times U \rightarrow U$ and $f : U \rightarrow U$ satisfy $F_n(U, U) \subseteq f(U)$ for all $n \in \mathbb{N}$;
- the function $\lambda : [0, 1] \rightarrow [0, 1]$ is left-continuous on $[0, 1]$ in the left sense and satisfies the following strict inequality

$$\lambda(\alpha) * \lambda(\alpha) > \alpha \text{ for any } \alpha \in [0, 1]; \tag{3}$$

- for any two functions $F_i$ and $F_j$, the following inequality is satisfied

$$\mathfrak{g}(F_i(u_1, v_1), F_j(u_2, v_2), t) \geq \lambda_{ij}(\mathfrak{g}(f(u_1), f(u_2), t) * \mathfrak{g}(f(v_1), f(v_2), t)), \tag{4}$$

where the functions $\lambda_{ij} : [0, 1] \rightarrow [0, 1]$ satisfy $\lambda_{ij}(\alpha) \geq \lambda(\alpha)$ for all $\alpha \in [0, 1]$ and all $i, j \in \mathbb{N}$.

Given any initial elements $u_0, v_0 \in U$, two sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are constructed according to Formula (2). We define two sequences of functions $\{\xi_n\}_{n=1}^{\infty}$ and $\{\eta_n\}_{n=1}^{\infty}$ on $(0, +\infty)$ by

$$\xi_n(t) = \mathfrak{g}(f(u_n), f(u_{n+1}), t) * \mathfrak{g}(f(v_n), f(v_{n+1}), t) \tag{5}$$

and

$$\eta_n(t) = \mathfrak{g}(f(u_{n+1}), f(u_n), t) * \mathfrak{g}(f(v_{n+1}), f(v_n), t). \tag{6}$$

Then, we have

$$\lambda(1) = 1. \tag{7}$$
For any \( t > 0 \), we also have
\[
\lim_{n \to \infty} \zeta_n(t) = \lim_{n \to \infty} [\mathcal{F}(f(u_n), f(u_{n+1}), t) + \mathcal{F}(f(v_n), f(v_{n+1}), t)] = 1
\] (8)
and
\[
\lim_{n \to \infty} \xi_n(t) = \lim_{n \to \infty} [\mathcal{F}(f(u_{n+1}), f(u_n), t) + \mathcal{F}(f(v_{n+1}), f(v_n), t)] = 1
\] (9)

**Proof.** Since the function \( \lambda \) is left-continuous at 1 in the left sense, we have that
\[
\alpha \to 1^- \implies \lambda(\alpha) \to \lambda(1)^-.
\]
Then, we obtain
\[
1 \geq \lambda(1) * \lambda(1) = \lim_{\alpha \to 1^-} [\lambda(\alpha) * \lambda(\alpha)] \quad \text{(using Proposition 1)}
\]
\[
\geq \lim_{\alpha \to 1^-} \alpha = 1 \quad \text{(using (3))},
\]
which indicates that
\[
1 = \lambda(1) * \lambda(1).
\] (10)
Since \( \lambda(1) \leq 1 \), using the increasing property of t-norm, we also have
\[
1 = \lambda(1) * \lambda(1) \leq 1 * \lambda(1) = \lambda(1),
\]
which implies (7).

Now, we have
\[
\mathcal{F}(f(u_n), f(u_{n+1}), t) = \mathcal{F}(f(u_{n-1}, v_{n-1}), f_{n+1}(u_n, v_n), t) \quad \text{(using (2))}
\]
\[
\geq \lambda_{n,n+1}(\mathcal{F}(f(u_{n-1}, f(u_n), t) * \mathcal{F}(f(v_{n-1}), f(v_n), t)) \quad \text{(using (4))}
\]
\[
\geq \lambda(\mathcal{F}(f(u_{n-1}), f(u_n), t) * \mathcal{F}(f(v_{n-1}), f(v_n), t))
\]
\[
= \lambda(\zeta_{n-1}(t))
\] (11)
and
\[
\mathcal{F}(f(v_n), f(v_{n+1}), t) = \mathcal{F}(f(v_{n-1}, u_{n-1}), f_{n+1}(v_n, u_n), t) \quad \text{(using (2))}
\]
\[
\geq \lambda_{n,n+1}(\mathcal{F}(f(v_{n-1}, f(v_n), t) * \mathcal{F}(f(u_{n-1}), f(u_n), t)) \quad \text{(using (4))}
\]
\[
\geq \lambda(\mathcal{F}(f(v_{n-1}), f(v_n), t) * \mathcal{F}(f(u_{n-1}), f(u_n), t))
\]
\[
= \lambda(\xi_{n-1}(t)) \quad \text{(using the commutativity of t-norm)}.
\] (12)

We similarly have
\[
\mathcal{F}(f(u_{n+1}), f(u_n), t) = \mathcal{F}(f_{n+1}(u_n, v_n), f_{n+1}(u_{n-1}, v_{n-1}), t) \quad \text{(using (2))}
\]
\[
\geq \lambda(\mathcal{F}(f(u_n), f(u_{n-1}), t) * \mathcal{F}(f(v_n), f(v_{n-1}, t))) \quad \text{(using (4))}
\]
\[
= \lambda(\zeta_{n-1}(t))
\] (13)
and
\[
\mathcal{F}(f(v_{n+1}), f(v_n), t) = \mathcal{F}(f_{n+1}(v_n, u_n), f_{n+1}(v_{n-1}, u_{n-1}), t)
\]
\[
\geq \lambda(\mathcal{F}(f(v_n), f(v_{n-1}), t) * \mathcal{F}(f(u_n), f(u_{n-1}, t))
\]
\[
= \lambda(\xi_{n-1}(t)) \quad \text{(using the commutativity of t-norm)}.
\] (14)

Using (5), (11), (12) and the increasing property of t-norm, we obtain
\[
\zeta_n(t) \geq \lambda(\zeta_{n-1}(t)) * \lambda(\xi_{n-1}(t)).
\] (15)
We consider the following two cases

- Suppose that $\zeta_{n-1}(t) \in [0, 1)$. Using (15) and (3), we have
  \[ \zeta_n(t) > \zeta_{n-1}(t). \]

- Suppose that $\zeta_{n-1}(t) = 1$. Using (15) and (10), we have
  \[ \zeta_n(t) \geq \lambda(1) \ast \lambda(1) = 1 = \zeta_{n-1}(t). \]

The above two cases indicate $\zeta_n(t) \geq \zeta_{n-1}(t)$. We can similarly obtain

\[ \zeta_n(t) \geq \lambda(\zeta_{n-1}(t)) \ast \lambda(\zeta_{n-1}(t)) \geq \zeta_{n-1}(t), \]

which indicate that $\{\zeta_n\}_{n=1}^{\infty}$ and $\{\bar{\zeta}_n\}_{n=1}^{\infty}$ are two increasing sequences of functions. Therefore, their limits exist. For any fixed $t > 0$, we define

\[ \zeta(t) = \sup_{n \in \mathbb{N}} \zeta_n(t) = \lim_{n \to \infty} \zeta_n(t) \quad \text{and} \quad \bar{\zeta}(t) = \sup_{n \in \mathbb{N}} \bar{\zeta}_n(t) = \lim_{n \to \infty} \bar{\zeta}_n(t). \tag{16} \]

Then, we see that $0 \leq \zeta(t) \leq 1$ for all $t > 0$. We are going to claim $\zeta(t) = 1$ for all $t > 0$. Suppose that there exists $t^0 > 0$ satisfying $\zeta(t^0) < 1$. Since $\zeta_n(t^0) \uparrow \zeta(t^0)$ from (16), i.e., $\zeta_n(t^0) \to \zeta(t^0)$, using the left-continuity of $\lambda$ in the left sense, we have

\[ \lambda(\zeta_n(t^0)) \to \lambda(\zeta(t^0)) \quad \text{as} \quad n \to \infty. \tag{17} \]

Applying (17) to Proposition 1, we have

\[ \lim_{n \to \infty} [\lambda(\zeta_n(t^0)) \ast \lambda(\zeta_n(t^0))] = \lambda(\zeta(t^0)) \ast \lambda(\zeta(t^0)). \tag{18} \]

Therefore, we obtain

\[
\begin{align*}
\zeta(t^0) &= \lim_{n \to \infty} \zeta_{n+1}(t^0) \quad \text{(using (16))} \\
&\geq \lim_{n \to \infty} [\lambda(\zeta_n(t^0)) \ast \lambda(\zeta_n(t^0))] \quad \text{(using (15))} \\
&= \lambda(\zeta(t^0)) \ast \lambda(\zeta(t^0)) \quad \text{(using (18))} \\
&> \zeta(t^0) \quad \text{(using (3)),}
\end{align*}
\]

which leads to a contradiction. This shows that $\zeta(t) = 1$ for all $t > 0$. In other words, from (5), we obtain

\[ \lim_{n \to \infty} \zeta_n(t) = \lim_{n \to \infty} [\mathcal{B}(f(u_n), f(u_{n+1}), t) \ast \mathcal{B}(f(v_n), f(v_{n+1}), t)] = 1 \]

for any $t > 0$. We can similarly prove $\bar{\zeta}(t) = 1$ for all $t > 0$ and the proof is complete. \( \Box \)

**Remark 2.** Suppose that the $t$-norm satisfies

\[ \alpha \ast \beta \geq \alpha \cdot \beta \quad \text{for all} \quad \alpha, \beta \in [0, 1], \]

and that the function $\lambda$ satisfies

\[ \lambda(\alpha) > \sqrt{\alpha} \quad \text{for all} \quad \alpha \in (0, 1). \]

Then, it is clear to see that the strict inequality (3) is satisfied.

**Proposition 4.** (Satisfying the $\triangleright$-Triangle Inequality). Let $(\mathcal{U}, \mathcal{F})$ be a fuzzy semi-metric space. Suppose that the following conditions are satisfied:

- the $\triangleright$-triangle inequality is satisfied;
• the t-norm * is left-continuous on [0, 1] with respect to the first or second component;
• for any fixed u, v ∈ U, the function \( \mathfrak{g}(u, v, \cdot) : (0, \infty) \to [0, 1] \) is left-continuous at each point \( t \in (0, \infty) \);
• the functions \( F_n : U \times U \to U \) and \( f : U \to U \) satisfy \( F_n(U, U) \subseteq f(U) \) for all \( n \in \mathbb{N} \);
• the function \( \lambda : [0, 1] \to [0, 1] \) is left-continuous on \([0, 1]\) in the left sense and satisfies the strict inequality
  \[ \lambda(\alpha) \ast \lambda(\alpha) > \alpha \text{ for any } \alpha \in [0, 1]; \]  
• for any two functions \( F_i \) and \( F_j \), the following inequality is satisfied
  \[ \mathfrak{g}(F_i(u_1, v_1), F_j(u_2, v_2), t) \geq \lambda_{ij}(\mathfrak{g}(f(u_1), f(u_2), t) \ast \mathfrak{g}(f(v_1), f(v_2), t)), \]  
where the functions \( \lambda_{ij} : [0, 1] \to [0, 1] \) satisfy \( \lambda_{ij}(\alpha) \geq \lambda(\alpha) \) for all \( \alpha \in [0, 1] \) and all \( i, j \in \mathbb{N} \).

Given any initial elements \( u_0, v_0 \in U \), two sequences \( \{u_n\}_{n=1}^{\infty} \) and \( \{v_n\}_{n=1}^{\infty} \) are constructed according to Formula (2). Then, \( \{f(u_n)\}_{n=1}^{\infty} \) and \( \{f(v_n)\}_{n=1}^{\infty} \) are both \( \ast \)-Cauchy and \( \ast \)-Cauchy sequences.

**Proof.** Suppose that \( \{f(u_n)\}_{n=1}^{\infty} \) is not a \( \ast \)-Cauchy sequence. We want to lead to a contradiction. By definition, there exists a pair \((r_0, t_0)\) with \( t_0 > 0 \) and \( 0 < r_0 < 1 \) such that, for each integer \( k \), there exists a pair \((i_k, j_k)\) of integers \( i_k \) and \( j_k \) (depend on \( k \)) satisfying \( i_k > j_k \geq k \) and
  \[ \mathfrak{g}(f(u_{i_k}), f(u_{j_k}), t_0) \leq 1 - r_0. \]  
(21)
Since the function \( \mathfrak{g}(u, v, \cdot) \) is nondecreasing by part (i) of Proposition 2, using (21), we also have
  \[ \mathfrak{g}(f(u_{i_k}), f(u_{j_k}), t^*) \leq 1 - r_0 \text{ for } t^* < t_0. \]  
(22)
Therefore, for \( t^* < t_0 \), we have
  \[ \mathfrak{g}(f(u_{i_k}), f(u_{j_k}), t^*) \ast \mathfrak{g}(f(v_{i_k}), f(v_{j_k}), t^*) \leq 1 - r_0. \]  
(23)
Given any fixed \( t^* < t_0 \), let \( i_k^* \) be the smallest integer such that (23) holds true, where \( i_k^* \) depends on \( t^* \). Since we are going to take \( k \to \infty \), we may assume that \( i_k^* \geq 3 \). Since \( i_k^* \) is the smallest integer, it means that the inequality (23) will be violated for \( i_k^* - 1 \) satisfying \( i_k^* - 1 > j_k^* \geq k \), which indicates that
  \[ \mathfrak{g}(f(u_{i_k^*-1}), f(u_{i_k^*}), t^*) \ast \mathfrak{g}(f(v_{i_k^*-1}), f(v_{i_k^*}), t^*) > 1 - r_0. \]  
(24)
Since the function \( \mathfrak{g}(u, v, \cdot) \) is nondecreasing by part (i) of Proposition 2 and is left-continuous at \( t^* \) by the assumption, we have
  \[ \mathfrak{g}(f(u_{i_k^*-1}), f(u_{i_k^*}), t) \uparrow \mathfrak{g}(f(u_{i_k^*-1}), f(u_{i_k^*}), t^*) \text{ as } t \uparrow t^* \]  
(25)
and
  \[ \mathfrak{g}(f(v_{i_k^*-1}), f(v_{i_k^*}), t) \uparrow \mathfrak{g}(f(v_{i_k^*-1}), f(v_{i_k^*}), t^*) \text{ as } t \uparrow t^* \].  
(26)
Using (24), (25) and Proposition 1, we obtain
  \[ \mathfrak{g}(f(u_{i_k^*-1}), f(u_{i_k^*}), t) \ast \mathfrak{g}(f(v_{i_k^*-1}), f(v_{i_k^*}), t) \]  
\[ \to \mathfrak{g}(f(u_{i_k^*-1}), f(u_{i_k^*}), t^*) \ast \mathfrak{g}(f(v_{i_k^*-1}), f(v_{i_k^*}), t^*) \text{ as } t \uparrow t^* \].  
(27)
Using (24), let
\[ \epsilon = \delta \left( f(u_{t_k}^-), f(u_j^k), t^* \right) * \delta \left( f(v_{t_k}^-), f(v_j^k), t^* \right) - (1 - r_0) > 0. \] (28)

Since the function \( \delta(u, v, \cdot) \) is left-continuous at each point \( t \in (0, \infty) \) by the assumption and the t-norm * is left-continuous at \( t^* \) with respect to each component, using (27) and (28), there exists \( \delta > 0 \) with 0 < \( t^* - \delta < t^* \) satisfying
\[
\left| \delta \left( f(u_{t_k}^-), f(u_j^k), t^* - \delta \right) * \delta \left( f(v_{t_k}^-), f(v_j^k), t^* - \delta \right) - \delta \left( f(u_{t_k}^-), f(u_j^k), t^* \right) * \delta \left( f(v_{t_k}^-), f(v_j^k), t^* \right) \right| < \epsilon,
\]
which implies
\[
\delta \left( f(u_{t_k}^-), f(u_j^k), t^* - \delta \right) * \delta \left( f(v_{t_k}^-), f(v_j^k), t^* - \delta \right) > 1 - r_0.
\] (29)

Then, we have
\[
1 - r_0 \geq \delta \left( f(u_{t_k}^-), f(u_j^k), t^* \right) * \delta \left( f(v_{t_k}^-), f(v_j^k), t^* \right) \quad \text{(using (23) since \( i_k^* > j_k^* \geq k \))} \quad (30)
\]
\[
\geq \left[ \delta \left( f(u_{t_k}^-), f(u_j^k), t^* - \delta \right) * \delta \left( f(v_{t_k}^-), f(u_j^k), t^* - \delta \right) \right]
\times \left[ \delta \left( f(u_{t_k}^-), f(v_j^k), t^* - \delta \right) * \delta \left( f(v_{t_k}^-), f(v_j^k), t^* - \delta \right) \right]
\]
(\text{using the } \infty \text{-triangle inequality and increasing property of t-norm})
\[
= \delta \left( f(u_{t_k}^-), f(u_j^k), t^* - \delta \right) * \delta \left( f(v_{t_k}^-), f(v_j^k), t^* - \delta \right) * \xi_{t_k}^{-1}(\delta)
\]
(\text{using (6), the commutativity and associativity of t-norm})
\[
\geq (1 - r_0) * \xi_{t_k}^{-1}(\delta) \quad \text{(using (29) and increasing property of t-norm).} \quad (31)
\]

Since \( i_k^* > k, \) if \( k \to \infty, \) i.e., \( i_k^* \to \infty, \) then \( \xi_{t_k}^{-1}(\delta) \to 1 \) using (9). Since \( \xi_{t_k}^{-1}(\delta) \leq 1 \) for all \( k, \) we can also say that the sequence \( \{\xi_{t_k}^{-1}(\delta)\}_{k \in \mathbb{N}} \) converges to 1 from the left, i.e., \( \xi_{t_k}^{-1}(\delta) \to 1 - \) as \( k \to \infty. \) Therefore, we obtain
\[
1 - r_0 \geq \limsup_{k \to \infty} \delta \left( f(u_{t_k}^-), f(u_j^k), t^* \right) * \delta \left( f(v_{t_k}^-), f(v_j^k), t^* \right) \quad \text{(using (30))}
\]
\[
\geq \limsup_{k \to \infty} (1 - r_0) * \xi_{t_k}^{-1}(\delta) \quad \text{(using (31))}
\]
\[
= \lim_{k \to \infty} (1 - r_0) * \xi_{t_k}^{-1}(\delta) = (1 - r_0) * 1 = 1 - r_0 \quad \text{(using Proposition 1).}
\]

Similarly, we can obtain
\[
1 - r_0 \geq \liminf_{k \to \infty} (1 - r_0) * \xi_{t_k}^{-1}(\delta) = \lim_{k \to \infty} (1 - r_0) * \xi_{t_k}^{-1}(\delta) = (1 - r_0) * 1 = 1 - r_0,
\]
which implies
\[
\lim_{k \to \infty} \left[ \delta \left( f(u_{t_k}^-), f(u_j^k), t^* \right) * \delta \left( f(v_{t_k}^-), f(v_j^k), t^* \right) \right] = 1 - r_0 \quad \text{for } t^* < t_0.
\]

Equivalently, using (23), we have
\[
\delta \left( f(u_{t_k}^-), f(u_j^k), t^* \right) * \delta \left( f(v_{t_k}^-), f(v_j^k), t^* \right) \to (1 - r_0) - \quad \text{as } k \to \infty \quad \text{for } t^* < t_0. \quad (32)
\]
where we need to emphasize that, for each integer \( k \), the integers \( i_k^* \) and \( j_k^* \) depend on \( t^* \). Since \( i_k^* > j_k^* \geq k \) and considering \( k \to \infty \), we must have \( i_k^* \to \infty \) and \( j_k^* \to \infty \). It follows that (32) is equivalent to the following statement

\[
\mathfrak{v} \left( f(u_{i_k}^*), f(u_{j_k}^*), \hat{t} \right) * \mathfrak{v} \left( f(v_{i_k}^*), f(v_{j_k}^*), \hat{t} \right) \to (1 - r_0) - \text{ as } k \to \infty \text{ for } \hat{t} < \tilde{t}_0, \tag{33}
\]

where \( i_k > j_k \geq k \), since we also have \( i_k \to \infty \) and \( j_k \to \infty \) as \( k \to \infty \).

Now, given any fixed \( \hat{t} < \tilde{t}_0 \), by referring to (23), we have

\[
\mathfrak{v} \left( f(u_{i_k}^*), f(u_{j_k}^*), \hat{t} \right) * \mathfrak{v} \left( f(v_{i_k}^*), f(v_{j_k}^*), \hat{t} \right) \leq 1 - r_0, \tag{34}
\]

where \( \hat{t}_k \) is the smallest integer such that \( \hat{t}_k > j_k \geq k \) and (34) is satisfied. For some \( \varepsilon > 0 \) with \( \hat{t} - 2\varepsilon > 0 \), we can obtain

\[
\mathfrak{v} \left( f(u_{i_k}^*), f(u_{j_k}^*), \hat{t} \right) * \mathfrak{v} \left( f(v_{i_k}^*), f(v_{j_k}^*), \hat{t} \right) \\
\geq \left[ \mathfrak{v} \left( f(u_{i_k}^*), f(u_{j_k}^*), f(v_{i_k}^*), f(v_{j_k}^*) \right) * \mathfrak{v} \left( \hat{t} - 2\varepsilon \right) * \mathfrak{v} \left( f(u_{j_k}^*), f(u_{i_k}^*), f(v_{j_k}^*), f(v_{i_k}^*) \right) \right] \\
\geq \mathfrak{v} \left( f(u_{i_k}^*), f(u_{j_k}^*), f(v_{i_k}^*), f(v_{j_k}^*) \right) * \mathfrak{v} \left( \hat{t} - 2\varepsilon \right) * \mathfrak{v} \left( f(u_{j_k}^*), f(u_{i_k}^*), f(v_{j_k}^*), f(v_{i_k}^*) \right) \tag{using (20)}
\]

We write \( \hat{t} - 2\varepsilon = \tilde{t} < \tilde{t}_0 \). Using (34) and (35), we have

\[
1 - r_0 \geq \lambda \left( \mathfrak{v} \left( f(u_{i_k}^*), f(u_{j_k}^*), \tilde{t} \right) * \mathfrak{v} \left( f(v_{i_k}^*), f(v_{j_k}^*), \tilde{t} \right) \right) \\
\geq \lambda \left( \mathfrak{v} \left( f(u_{i_k}^*), f(u_{j_k}^*), \tilde{t} \right) * \mathfrak{v} \left( f(v_{i_k}^*), f(v_{j_k}^*), \tilde{t} \right) \right) * \mathfrak{v} \left( \tilde{t} - 2\varepsilon \right) * \mathfrak{v} \left( f(v_{j_k}^*), f(v_{i_k}^*), \tilde{t} - 2\varepsilon \right) \tag{using (20)}
\]

According to (33), since \( \lambda \) is left-continuous on \([0, 1]\) in the left sense, it follows that

\[
\lim_{k \to \infty} \lambda \left( \mathfrak{v} \left( f(u_{i_k}^*), f(u_{j_k}^*), \tilde{t} \right) * \mathfrak{v} \left( f(v_{i_k}^*), f(v_{j_k}^*), \tilde{t} \right) \right) = \lambda(1 - r_0). \tag{37}
\]

According to (33), since \( \lambda \) is left-continuous on \([0, 1]\) in the left sense, it follows that

\[
\lim_{k \to \infty} \lambda \left( \mathfrak{v} \left( f(u_{i_k}^*), f(u_{j_k}^*), \tilde{t} \right) * \mathfrak{v} \left( f(v_{i_k}^*), f(v_{j_k}^*), \tilde{t} \right) \right) = \lambda(1 - r_0). \tag{37}
\]

Therefore, we obtain

\[
1 - r_0 \geq \lambda(1 - r_0) * \lambda(1 - r_0) * 1 * 1 \tag{using Proposition 1, (8), (9), (36) and (37)}
\]

which leads to a contradiction. Therefore, we conclude that \( \{f(u_n)\}_{n=1}^{\infty} \) is indeed a \( \prec \)-Cauchy sequence. We can similarly show that \( \{f(v_n)\}_{n=1}^{\infty} \) is a \( \prec \)-Cauchy sequence.
Now, suppose that \( \{f(u_n)\}_{n=1}^{\infty} \) is not a \( \kappa \)-Cauchy sequence. Then, by definition, there exists a pair \((r_0, l_0)\) with \( l_0 > 0 \) and \( 0 < r_0 < 1 \) such that, for each integer \( k \), there exists a pair \((i_k, j_k)\) of integers \( i_k \) and \( j_k \) (depend on \( k \)) satisfying \( i_k > j_k \geq k \) and

\[
\delta(f(u_{i_k}), f(u_{j_k}), l_0) \leq 1 - r_0.
\]

We can similarly show that \( \{f(u_n)\}_{n=1}^{\infty} \) and \( \{f(v_n)\}_{n=1}^{\infty} \) are \( \kappa \)-Cauchy sequences. This completes the proof. \( \square \)

**Remark 3.** When \( \delta \) satisfies the \( \succ \)-triangle inequality, or \( \prec \)-triangle inequality or \( \ll \)-triangle inequality, we can similarly obtain the desired results in Proposition 4. As a matter of fact, some trick modifications are needed and are left for the readers.

### 4. Asymptotically Coupled Coincidence Points

Let \( U \) be a universal set. We consider two functions \( F : U \times U \to U \) and \( f : U \to U \).

- The functions \( F \) and \( f \) are said to be commuted when
  
  \[ f(F(u, v)) = F(f(u), f(v)) \]

  for all \( u, v \in U \).

- We say that an element \((u, v) \in U \times U\) is a coupled coincidence point of functions \( F \) and \( f \) when
  
  \[ F(u, v) = f(u) \text{ and } F(v, u) = f(v). \]

- We say that an element \((u, v) \in U \times U\) is a common coupled fixed point of functions \( F \) and \( f \) when
  
  \[ u = f(u) = F(u, v) \text{ and } v = f(v) = F(v, u). \]

**Definition 9.** Let \( U \) be a universal set. Given a function \( f : U \to U \) and a sequence of functions \( \{F_n\}_{n=1}^{\infty} \) with \( F_n : U \times U \to U \), we say that an element \((u, v) \in U \times U\) is an asymptotically coupled coincidence point of functions \( \{F_n\}_{n=1}^{\infty} \) and \( f \) when

\[
F_n(u, v) \stackrel{\delta}{\to} f(u) \text{ and } F_n(v, u) \stackrel{\delta}{\to} f(v) \text{ as } n \to \infty.
\]

**Remark 4.** We remark that \( F_n(u, v) \stackrel{\delta}{\to} f(u) \) as \( n \to \infty \) if and only if

\[
\lim_{n \to \infty} \delta(f(u), F_n(u, v), t) = \lim_{n \to \infty} \delta(F_n(u, v), f(u), t) \text{ for all } t > 0;
\]

that is,

\[
F_n(u, v) \stackrel{\delta}{\to} f(u) \text{ and } F_n(u, v) \stackrel{\delta}{\to} f(u) \text{ as } n \to \infty.
\]

In particular, suppose that all the functions \( F_n \) are identical to \( F \) for all \( n \). If \((u, v) \in U \times U\) is an asymptotically coupled coincidence point, then we have

\[
\lim_{n \to \infty} \delta(f(u), F(u, v), t) = \lim_{n \to \infty} \delta(F(u, v), f(u), t) \text{ for all } t > 0
\]

and

\[
\lim_{n \to \infty} \delta(f(v), F(v, u), t) = \lim_{n \to \infty} \delta(F(v, u), f(v), t) \text{ for all } t > 0
\]

which implies

\[ F(u, v) = f(u) \text{ and } F(v, u) = f(v). \]

This indicates that \((u, v)\) is a coupled coincidence point.

**Theorem 1.** *(Satisfying the \( \ll \)-Triangle Inequality).* Let \((U, \delta)\) be a fuzzy semi-metric space. Suppose that the \( \ll \)-triangle inequality is satisfied and that the following conditions are satisfied:
• the $t$-norm $*$ is left-continuous with respect to the first or second component;

• for any fixed $u, v \in U$, the function $\mathfrak{F}(u, v, \cdot) : (0, \infty) \to [0, 1]$ is left-continuous at each point $t \in (0, \infty)$;

• the function $\lambda : [0, 1] \to [0, 1]$ is left-continuous on $[0, 1]$ in the left sense and satisfies the following strict inequality

\[
\lambda(\alpha) \cdot \lambda(\alpha) > \alpha \text{ for any } \alpha \in [0, 1);
\]  

(38)

• the functions $F_n : U \times U \to U$ and $f : U \to U$ satisfy $F_n(U, U) \subseteq f(U)$ for all $n \in \mathbb{N}$;

• the functions $f$ and $F_n$ commute all $n \in \mathbb{N}$;

• for any two functions $F_i$ and $F_j$, the following inequality is satisfied

\[
\mathfrak{F}(F_i(u_1, v_1), F_j(u_2, v_2), t) \geq \lambda_{ij} (\mathfrak{F}(f(u_1), f(u_2), t) \cdot \mathfrak{F}(f(v_1), f(v_2), t)),
\]  

(39)

where the functions $\lambda_{ij} : [0, 1] \to [0, 1]$ satisfy $\lambda_{ij}(\alpha) \geq \lambda(\alpha)$ for all $\alpha \in [0, 1]$ and all $i, j \in \mathbb{N}$;

• any one of the following conditions is satisfied:

  (a) function $f$ is $(+, +)$-continuous and $(+, -)$-continuous with respect to $\mathfrak{F}$ and the space $(U, \mathfrak{F})$ is $(+, +)$-complete or $(+, -)$-complete;

  (b) the function $f$ is $(+, +)$-continuous and $(+, -)$-continuous with respect to $\mathfrak{F}$ and the space $(U, \mathfrak{F})$ is $(+, +)$-complete or $(+, -)$-complete.

Then, we have the following properties.

(i) The function $f$ and the sequence of functions $\{F_n\}_{n=1}^\infty$ have an asymptotically coupled coincidence point $(u^*, v^*)$ in the sense of

\[
F_n(u^*, v^*) \overset{\mathfrak{F}}{\to} f(u^*) \text{ and } F_n(v^*, u^*) \overset{\mathfrak{F}}{\to} f(v^*) \text{ as } n \to \infty.
\]

(ii) Given any two functions $F_i$ and $F_j$, we further assume that the following inequality is satisfied

\[
\mathfrak{F}(F_i(u_1, v_1), F_j(u_2, v_2), t) \geq \lambda_{ij} (\mathfrak{F}(f(u_1), f(u_2), 3t) \cdot \mathfrak{F}(f(v_1), f(v_2), 3t)).
\]  

(40)

Suppose that $(\tilde{u}, \tilde{v})$ is another asymptotically coupled coincidence point of $f$ and $\{F_n\}_{n=1}^\infty$.

Then, we have

\[
f(u^*) = f(\tilde{u}) \text{ and } f(v^*) = f(\tilde{v}).
\]

(iii) Given any two functions $F_i$ and $F_j$, we further assume that the following inequality is satisfied

\[
\mathfrak{F}(F_i(u_1, v_1), F_j(u_2, v_2), t) \geq \lambda_{ij} (\mathfrak{F}(f(u_1), f(u_2), 3t) \cdot \mathfrak{F}(f(v_1), f(v_2), 3t)).
\]

Then, there exists $(u^*, v^*) \in U \times U$ such that $(f(u^*), f(v^*)) \in U \times U$ is the common coupled fixed point of the functions $\{F_n\}_{n=1}^\infty$ in the sense of

\[
f(u^*) = F_n(f(u^*), f(v^*)) \text{ and } f(v^*) = F_n(f(v^*), f(u^*)) \text{ for all } n.
\]

Moreover, the element $(u^*, v^*) \in U \times U$ can be obtained as follows.

• When condition (a) is satisfied, the element $(u^*, v^*) \in U \times U$ can be obtained from the following limits

\[
f(u_n) \overset{\mathfrak{F}}{\to} u^* \text{ and } f(v_n) \overset{\mathfrak{F}}{\to} v^* \text{ as } n \to \infty.
\]

• When condition (b) is satisfied, the element $(u^*, v^*) \in U \times U$ can be obtained from the following limits

\[
f(u_n) \overset{\mathfrak{F}}{\to} u^* \text{ and } f(v_n) \overset{\mathfrak{F}}{\to} v^* \text{ as } n \to \infty.
\]

The sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are constructed from any given initial element $(u_0, v_0) \in U \times U$ according to Formula (2).
We are going to prove

\[ \text{• Suppose that condition (a) is satisfied. Since } (U, \mathfrak{G}) \text{ is } (\kappa, \kappa)-\text{complete and } \mathfrak{G} \text{ is } (\kappa, \kappa)-\text{continuous, there exist } u^*, v^* \in U \text{ satisfying} \]

\[ f(u_n) \xrightarrow{\mathfrak{G}} u^* \text{ and } f(v_n) \xrightarrow{\mathfrak{G}} v^* \text{ as } n \to \infty. \]

Since \( f \) is \((\iota, \iota\iota)-\text{continuous} \) and \((\iota, \iota\iota)-\text{continuous} \) with respect to \( \mathfrak{G} \), we have

\[ f(f(u_n)) \xrightarrow{\mathfrak{G}} f(u^*) \text{ and } f(f(v_n)) \xrightarrow{\mathfrak{G}} f(v^*) \text{ as } n \to \infty \]

and

\[ f(f(u_n)) \xrightarrow{\mathfrak{G}} f(u^*) \text{ and } f(f(v_n)) \xrightarrow{\mathfrak{G}} f(v^*) \text{ as } n \to \infty, \]

which indicate

\[ \mathfrak{G}(f(f(u_n)), f(u^*), t) \to 1 - , \quad \mathfrak{G}(f(f(v_n)), f(v^*), t) \to 1 - \]

\[ \mathfrak{G}(f(u^*), f(f(u_n)), t) \to 1 - \text{ and } \mathfrak{G}(f(v^*), f(f(v_n)), t) \to 1 - \] (41)

for all \( t > 0 \).

\[ \text{• Suppose that condition (b) is satisfied. Since } (U, \mathfrak{G}) \text{ is } (\kappa, \kappa)-\text{complete and } \mathfrak{G} \text{ is } (\kappa, \kappa)-\text{continuous, there exist } u^*, v^* \in U \text{ satisfying} \]

\[ f(u_n) \xrightarrow{\mathfrak{G}} u^* \text{ and } f(v_n) \xrightarrow{\mathfrak{G}} v^* \text{ as } n \to \infty. \]

Since \( f \) is \((\iota, \iota\iota)-\text{continuous} \) and \((\iota, \iota\iota)-\text{continuous} \) with respect to \( \mathfrak{G} \), we can similarly obtain (41).

From (41), using Proposition 1, we obtain

\[ \mathfrak{G}(f(f(u_n)), f(u^*), t) \ast \mathfrak{G}(f(f(v_n)), f(v^*), t) \to 1 \ast 1 = 1 \text{ as } n \to \infty. \]

From (41), we also obtain ANS: Fixed

\[ \mathfrak{G}(f(u^*), f(f(u_n)), t) \ast \mathfrak{G}(f(v^*), f(f(v_n)), t) \to 1 \ast 1 = 1 \text{ as } n \to \infty. \]

Since \( \lambda \) is left-continuous at 1 in the left sense, using (7), we obtain

\[ \lambda(\mathfrak{G}(f(f(u_n)), f(u^*), t) \ast \mathfrak{G}(f(f(v_n)), f(v^*), t)) \to \lambda(1) = 1 \text{ as } n \to \infty. \] (42)

and

\[ \lambda(\mathfrak{G}(f(u^*), f(f(u_n)), t) \ast \mathfrak{G}(f(v^*), f(f(v_n)), t)) \to \lambda(1) = 1 \text{ as } n \to \infty. \] (43)

Using (2) and the commutativity of \( F_n \) and \( f \), we obtain

\[ f(f(u_{n+1})) = f(F_{n+1}(u_n, v_n)) = F_{n+1}(f(u_n), f(v_n))) \] (44)

and

\[ f(f(v_{n+1})) = f(F_{n+1}(v_n, u_n)) = F_{n+1}(f(v_n), f(u_n))). \]

We are going to prove

\[ f(u^*) = F_n(u^*, v^*) \text{ and } f(v^*) = F_n(v^*, u^*) \text{ for all } n \in \mathbb{N}. \]
Now, we have
\[
\delta(f(f(u_{n+1})), F_n(u^*, v^*), t) = \delta(F_{n+1}(f(u_n), f(v_n)), F_n(u^*, v^*), t) \text{ (using (44))}
\]
\[
\geq \lambda_{n+1} \delta(f(f(u_n), f(u^*), t) + \delta(f(f(v_n), f(v^*), t)) \text{ (using (39))}
\]
\[
\geq \lambda \delta(f(f(u_n)), f(u^*), t) \ast \delta(f(f(v_n)), f(v^*), t)),
\]
(45)
and
\[
\delta(F_n(u^*, v^*), f(f(u_{n+1})), t) = \delta(F_n(u^*, v^*), F_{n+1}(f(u_n), f(v_n)), t) \text{ (using (44))}
\]
\[
\geq \lambda_{n+1} \delta(f(u^*), f(f(u_n)), t) + \delta(f(v^*), f(f(v_n)), t)) \text{ (using (39))}
\]
\[
\geq \lambda \delta(f(u^*), f(f(u_n)), t) \ast \delta(f(v^*), f(f(v_n)), t)),
\]
(46)
which imply
\[
1 \geq \limsup_{n \to \infty} \delta(f(f(u_{n+1})), F_n(u^*, v^*), t) \geq \liminf_{n \to \infty} \delta(f(f(u_{n+1})), F_n(u^*, v^*), t)
\]
\[
\geq \liminf_{n \to \infty} \lambda \delta(f(f(u_n)), f(u^*), t) \ast \delta(f(f(v_n)), f(v^*), t)) \text{ (using (45))}
\]
\[
= 1 \text{ (using (42)).}
\]
and
\[
1 \geq \limsup_{n \to \infty} \delta(F_n(u^*, v^*), f(f(u_{n+1})), t) \geq \liminf_{n \to \infty} \delta(F_n(u^*, v^*), f(f(u_{n+1})), t)
\]
\[
\geq \liminf_{n \to \infty} \lambda \delta(f(u^*), f(f(u_n)), t) \ast \delta(f(v^*), f(f(v_n)), t)) \text{ (using (46))}
\]
\[
= 1 \text{ (using (43)).}
\]
Therefore, we obtain
\[
\delta(f(f(u_{n+1})), F_n(u^*, v^*), t) \to 1 - \text{ and } \delta(F_n(u^*, v^*), f(f(u_{n+1})), t) \to 1 - .
\]
(47)
Using the \(\delta\)-triangle inequality, we have
\[
\delta(f(u^*), F_n(u^*, v^*, 2t) \geq \delta(f(u^*), f(f(u_{n+1})), t) + \delta(f(f(u_{n+1})), F_n(u^*, v^*, t).
\]
Applying Proposition 1 to (41) and (47), we obtain
\[
\lim_{n \to \infty} \delta(f(u^*), F_n(u^*, v^*, 2t) = 1 \text{ for all } t > 0.
\]
(48)
Using the \(\delta\)-triangle inequality again, we also have
\[
\delta(F_n(u^*, v^*), f(u^*, 2t) \geq \delta(F_n(u^*, v^*), f(f(u_{n+1})), t) + \delta(f(f(u_{n+1})), f(u^*), t).
\]
Applying Proposition 1 to (41) and (47), we obtain
\[
\lim_{n \to \infty} \delta(F_n(u^*, v^*), f(u^*, 2t) = 1 \text{ for all } t > 0.
\]
This shows \(F_n(u^*, v^*) \xrightarrow{n \to \infty} f(u^*)\) as \(n \to \infty\). We can similarly obtain
\[
\lim_{n \to \infty} \delta(f(v^*), F_n(v^*, u^*, 2t) = 1 = \lim_{n \to \infty} \delta(F_n(v^*, u^*), f(v^*, 2t) \text{ for all } t > 0,
\]
which indicates \(F_n(v^*, u^*) \xrightarrow{n \to \infty} f(v^*)\) as \(n \to \infty\). This proves part (i).
To prove part (ii), since \((\bar{u}, \bar{v})\) is another asymptotically coupled coincidence point, we have
\[
\lim_{n \to \infty} \delta(f(\bar{u}), F_n(\bar{u}, \bar{v}), t) = 1 = \lim_{n \to \infty} \delta(F_n(\bar{u}, \bar{v}), f(\bar{u}), t) \text{ for all } t > 0.
\]
(49)
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and
\[
\lim_{n \to \infty} \mathfrak{F}(f(\bar{\sigma}), F_n(\bar{\sigma}, \bar{u}), t) = 1 = \lim_{n \to \infty} \mathfrak{F}(F_n(\bar{\sigma}, \bar{u}), f(\bar{\sigma}), t) \quad \text{for all } t > 0
\]

Using the \(\infty\)-triangle inequality, we have
\[
\mathfrak{F}(u^\ast, f(\bar{u}), 3t) \\
\geq \mathfrak{F}(f(u^\ast), F_n(u^\ast, v^\ast), t) * \mathfrak{F}(F_n(u^\ast, v^\ast), F_n(\bar{\sigma}, \bar{v}), t) * \mathfrak{F}(F_n(\bar{\sigma}, \bar{v}), f(\bar{u}), t).
\]

We also have
\[
\mathfrak{F}(F_n(u^\ast, v^\ast), F_n(\bar{\sigma}, \bar{v}), t) \geq \lambda_{n, t}(\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t)) \quad \text{(using (40))}
\]
\[
\geq \lambda(\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t))
\]

From (50) and (51), using the increasing property of t-norm, we obtain
\[
\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) \geq \lambda(\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t)) * 1
\]
\[
= \lambda(\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t)).
\]

Using the \(\infty\)-triangle inequality, we have
\[
\mathfrak{F}(f(v^\ast), f(v), 3t) \\
\geq \mathfrak{F}(f(v^\ast), F_n(v^\ast, u^\ast), t) * \mathfrak{F}(F_n(v^\ast, u^\ast), F_n(\bar{\sigma}, \bar{u}), t) * \mathfrak{F}(F_n(\bar{\sigma}, \bar{u}), f(\bar{v}), t).
\]

We can similarly obtain
\[
\mathfrak{F}(f(v^\ast), f(v), 3t) \geq \lambda(\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t)).
\]

From (54) and (56), using the increasing property of t-norm, we obtain
\[
\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t) \geq [\lambda(\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t))]
\]
\[
* [\lambda(\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t))].
\]

Suppose that
\[
\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t) < 1.
\]

Using (38), we have
\[
[\lambda(\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t))] * [\lambda(\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t))]
\]
\[
* \mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t),
\]

which contradicts (55). Therefore, we must have
\[
\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) * \mathfrak{F}(f(v^\ast), f(v), 3t) = 1 \quad \text{for all } t > 0.
\]

According to Remark 1, we obtain
\[
\mathfrak{F}(f(u^\ast), f(\bar{u}), 3t) = 1 = \mathfrak{F}(f(v^\ast), f(v), 3t) \quad \text{for all } t > 0,
\]

which also indicates \(f(u^\ast) = f(\bar{u})\) and \(f(v^\ast) = f(v)\). This proves part (ii).

To prove part (iii), using the commutativity of \(F_n\) and \(f\), we have
\[
f(F_n(u^\ast, v^\ast)) = F_n(f(u^\ast), f(v^\ast)) = F_n(F_n(u^\ast, v^\ast), F_n(v^\ast, u^\ast))
\]
and
\[
    f(F_n(v^*, u^*)) = F_n(f(v^*), f(u^*)) = F_n(f(F_n(v^*, u^*)), F_n(u^*, v^*)) .
\]
(57)

By regarding \( F_n(u^*, v^*) \) as \( \bar{a} \) and \( F_n(v^*, u^*) \) as \( \bar{v} \), the equalities (56) and (57) become
\[
    f(\bar{a}) = F_n(\bar{a}, \bar{v}) \quad \text{and} \quad f(\bar{v}) = F_n(\bar{v}, \bar{a}) \quad \text{for all } n .
\]

Then, we have
\[
    \lim_{n \to \infty} \mathfrak{S}(f(\bar{a}), F_n(\bar{a}, \bar{v}), t) = 1 = \lim_{n \to \infty} \mathfrak{S}(F_n(\bar{a}, \bar{v}), f(\bar{a}), t) \quad \text{for all } t > 0
\]

and
\[
    \lim_{n \to \infty} \mathfrak{S}(f(\bar{v}), F_n(\bar{v}, \bar{a}), t) = 1 = \lim_{n \to \infty} \mathfrak{S}(F_n(\bar{v}, \bar{a}), f(\bar{v}), t) \quad \text{for all } t > 0 .
\]

Therefore, using part (ii), we obtain
\[
    f(u^*) = f(\bar{a}) = f(F_n(u^*, v^*)) = F_n(f(u^*), f(v^*))
\]
and
\[
    f(v^*) = f(\bar{v}) = f(F_n(v^*, u^*)) = F_n(f(v^*), f(u^*)),
\]
which indicates that \((f(u^*), f(v^*)) \in U \times U\) is the common coupled fixed point of the functions \( \{F_n\}_{n=1}^{\infty} \). This completes the proof. □

**Remark 5.** We are going to claim that if the function \( \lambda_{ij} \) is nondecreasing then the inequality (40) implies the inequality (39). Using part (i) of Proposition 2, we have
\[
    \mathfrak{S}(f(u_1), f(u_2), 3t) \geq \mathfrak{S}(f(u_1), f(u_2), t) \quad \text{and} \quad \mathfrak{S}(f(v_1), f(v_2), 3t) \geq \mathfrak{S}(f(v_1), f(v_2), t) .
\]

Using the increasing property of \( t \)-norm, we also have
\[
    \mathfrak{S}(f(u_1), f(u_2), 3t) \ast \mathfrak{S}(f(v_1), f(v_2), 3t) \geq \mathfrak{S}(f(u_1), f(u_2), t) \ast \mathfrak{S}(f(v_1), f(v_2), t) .
\]

Since \( \lambda_{ij} \) is assumed to be nondecreasing, it follows that inequality (40) implies inequality (39).

**Theorem 2. (Satisfying the \( \triangleright \)-Triangle Inequality).** Let \((U, \mathfrak{S})\) be a fuzzy semi-metric space. Suppose that the \( v \)-triangle inequality is satisfied, and that the following conditions are satisfied:

- the first six conditions in Theorem 1 are satisfied;
- any one of the following conditions is satisfied:
  - (a) the function \( f \) is \((\triangleright, \triangleright)\)-continuous with respect to \( \mathfrak{S} \), and the space \((U, \mathfrak{S})\) is \((\triangleright, \triangleright)\)-complete or \((\triangleright, \triangleright)\)-complete;
  - (b) the function \( f \) is \((\triangleright, \triangleright)\)-continuous with respect to \( \mathfrak{S} \), and the space \((U, \mathfrak{S})\) is \((\triangleright, \triangleright)\)-complete or \((\triangleright, \triangleright)\)-complete.

Then, we have the following properties.

(i) The function \( f \) and the sequence of functions \( \{F_n\}_{n=1}^{\infty} \) have an asymptotically coupled coincidence point \((u^*, v^*)\) in the sense of
\[
    F_n(u^*, v^*) \xrightarrow{\mathfrak{S}} f(u^*) \quad \text{and} \quad F_n(v^*, u^*) \xrightarrow{\mathfrak{S}} f(v^*) \quad \text{as } n \to \infty .
\]

(ii) Given any two functions \( F_i \) and \( F_j \), we further assume that the following inequality is satisfied
\[
    \mathfrak{S}(F_i(u_1, v_1), F_j(u_2, v_2), t) \geq \lambda_{ij}(\mathfrak{S}(f(u_1), f(u_2), 3t) \ast \mathfrak{S}(f(v_1), f(v_2), 3t)) .
\]

Suppose that \((\bar{a}, \bar{v})\) is another asymptotically coupled coincidence point of \( \{F_n\}_{n=1}^{\infty} \) and \( f \). Then, we have
\[
    f(u^*) = f(\bar{a}) \quad \text{and} \quad f(v^*) = f(\bar{v}) .
\]
Given any two functions $F_i$ and $F_j$, we further assume that the following inequality is satisfied

$$
\mathfrak{G}(F_i(u_1, v_1), F_j(u_2, v_2), t) \geq \lambda_{ij}(\mathfrak{G}(f(u_1), f(u_2), 3t) \ast \mathfrak{G}(f(v_1), f(v_2), 3t)).
$$

Then, there exists $(u^*, v^*) \in U \times U$ such that $(f(u^*), f(v^*)) \in U \times U$ is the common coupled fixed point of the functions $\{F_n\}_{n=1}^{\infty}$ in the sense of

$$
f(u^*) = F_n(f(u^*), f(v^*)) \text{ and } f(v^*) = F_n(f(v^*), f(u^*)) \text{ for all } n.
$$

Moreover, the element $(u^*, v^*) \in U \times U$ can be obtained as follows.

- Suppose that condition (a) is satisfied. Then, the element $(u^*, v^*) \in U \times U$ can be obtained from the following limits

$$
f(u_n) \xrightarrow{\mathfrak{G}} u^* \text{ and } f(v_n) \xrightarrow{\mathfrak{G}} v^* \text{ as } n \to \infty.
$$

- Suppose that condition (b) is satisfied. Then, the element $(u^*, v^*) \in U \times U$ can be obtained from the following limits

$$
f(u_n) \xrightarrow{\mathfrak{G}} u^* \text{ and } f(v_n) \xrightarrow{\mathfrak{G}} v^* \text{ as } n \to \infty.
$$

The sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are constructed from any given initial element $(u_0, v_0) \in U \times U$ according to Formula (2).

**Proof.** Using Remark 3 and Proposition 4 by considering the $\triangleright$-triangle inequality, we see that $(f(u_n))_{n=1}^{\infty}$ and $(f(v_n))_{n=1}^{\infty}$ are both $\triangleright$-Cauchy and $\triangleright$-Cauchy sequences. We consider the following cases

- When condition (a) is satisfied, there exists $u^* \in U$ satisfying $f(u_n) \xrightarrow{\mathfrak{G}} u^*$ as $n \to \infty$. Since $f$ is $(., -)$-continuous with respect to $\mathfrak{G}$, it follows that (41) are satisfied.

- When condition (b) is satisfied, there exists $u^* \in U$ satisfying $f(u_n) \xrightarrow{\mathfrak{G}} u^*$ as $n \to \infty$. Since $f$ is $(., -)$-continuous with respect to $\mathfrak{G}$, it follows that (41) are satisfied.

From (41), using Proposition 1, we obtain

$$
\mathfrak{G}(f(u^*), f(f(u_n)), t) \ast \mathfrak{G}(f(v^*), f(f(v_n)), t) \to 1 \ast 1 = 1 \text{ as } n \to \infty.
$$

Since $\lambda$ is left-continuous at 1 in the left sense, using (7), we obtain

$$
\lambda(\mathfrak{G}(f(u^*), f(f(u_n)), t) \ast \mathfrak{G}(f(v^*), f(f(v_n)), t)) \to \lambda(1) = 1 \text{ as } n \to \infty.
$$

Now, we have

$$
\mathfrak{G}(F_n(u^*, v^*), f(f(u_{n+1})), t) = \mathfrak{G}(F_n(u^*, v^*), F_{n+1}(f(u_n), f(v_n)), t) \text{ (using (44))}
\geq \lambda_{n+1}(\mathfrak{G}(f(u^*), f(f(u_n)), t) \ast \mathfrak{G}(f(v^*), f(f(v_n)), t)) \text{ (using (39))}
\geq \lambda(\mathfrak{G}(f(u^*), f(f(u_n)), t) \ast \mathfrak{G}(f(v^*), f(f(v_n)), t)),
$$

which indicates, by using (58),

$$
1 \geq \limsup_{n \to \infty} \mathfrak{G}(F_n(u^*, v^*), f(f(u_{n+1})), t) \geq \liminf_{n \to \infty} \mathfrak{G}(F_n(u^*, v^*), f(f(u_{n+1})), t) \geq 1.
$$

Therefore, we obtain

$$
\mathfrak{G}(F_n(u^*, v^*), f(f(u_{n+1})), t) \to 1.
$$

Using the $\triangleright$-triangle inequality, we have

$$
\mathfrak{G}(f(u^*), F_n(u^*, v^*), 2t) \geq \mathfrak{G}(f(u^*), f(f(u_{n+1})), t) \ast \mathfrak{G}(F_n(u^*, v^*), f(f(u_{n+1})), t).
$$
Applying Proposition 1 to (41) and (59), we obtain
\[
\lim_{n \to \infty} \diamondsuit(f(u^*), F_n(u^*, v^*), 2t) = 1 \text{ for all } t > 0.
\]

Using the $\triangleright$-triangle inequality, we also have
\[
\diamondsuit(F_n(u^*, v^*), f(u^*), 2t) \geq \diamondsuit(F_n(u^*, v^*), f(u_{n+1}), t) \ast \diamondsuit(f(u^*), f(u_{n+1}), t).
\]

Applying Proposition 1 to (41) and (59), we obtain
\[
\lim_{n \to \infty} \diamondsuit(F_n(u^*, v^*), f(u^*), 2t) = 1 \text{ for all } t > 0.
\]

We can similarly obtain
\[
\lim_{n \to \infty} \diamondsuit(f(v^*), F_n(v^*, u^*), 2t) = 1 = \lim_{n \to \infty} \diamondsuit(F_n(v^*, u^*), f(v^*), 2t) \text{ for all } t > 0.
\]

This proves part (i).

To prove part (ii), using the $\triangleright$-triangle inequality, we have
\[
\diamondsuit(f(u^*), f(\bar{u}), 3t) \\
\geq \diamondsuit(f(u^*), F_n(u^*, v^*), t) \ast \diamondsuit(f(\bar{u}), F_n(u^*, v^*), 2t) \\
\geq \diamondsuit(f(u^*), F_n(u^*, v^*), t) \ast \diamondsuit(f(\bar{u}), F_n(\bar{u}, v), t) \ast \diamondsuit(F_n(u^*, v^*), F_n(\bar{u}, v), t).
\]

(60)

From (60) and (51), using the increasing property of \(t\)-norm, we obtain
\[
\diamondsuit(f(u^*), f(\bar{u}), 3t) \geq \diamondsuit(f(u^*), F_n(u^*, v^*), t) \\
\ast \diamondsuit(f(\bar{u}), F_n(\bar{u}, v), t) \ast \lambda(\diamondsuit(f(u^*), f(\bar{u}), 3t) \ast \diamondsuit(f(v^*), f(v), 3t)).
\]

(61)

From (61), applying Proposition 1 to (48) and (49), we also obtain
\[
\diamondsuit(f(u^*), f(\bar{u}), 3t) \geq \lambda(\diamondsuit(f(u^*), f(\bar{u}), 3t) \ast \diamondsuit(f(v^*), f(v), 3t)) \\
\geq \lambda(\diamondsuit(f(u^*), f(\bar{u}), 3t) \ast \diamondsuit(f(v^*), f(v), 3t)).
\]

Using the $\triangleright$-triangle inequality, we have
\[
\diamondsuit(f(v^*), f(\bar{v}), 3t) \\
\geq \diamondsuit(f(v^*), F_n(v^*, \bar{v}, 3t) \ast \diamondsuit(f(\bar{v}), F_n(\bar{v}, \bar{u}, t) \\
\geq \diamondsuit(f(v^*), F_n(v^*, u^*), t) \ast \diamondsuit(f(\bar{v}), F_n(\bar{v}, \bar{u}, t) \ast \diamondsuit(F_n(v^*, u^*), F_n(\bar{v}, \bar{u}, t).
\]

We can similarly obtain
\[
\diamondsuit(f(v^*), f(\bar{v}), 3t) \geq \lambda(\diamondsuit(f(u^*), f(\bar{u}), 3t) \ast \diamondsuit(f(v^*), f(\bar{v}), 3t).
\]

The remaining proof follows from the similar proof of part (ii) of Theorem 1.

Part (iii) can be obtained by the similar proof of part (iii) of Theorem 1, since the $\triangleright$-$\triangleright$-triangle inequality was not used in that proof. This completes the proof. □

**Theorem 3. (Satisfying the $\triangleright$-Triangle Inequality).** Let \((U, \diamondsuit)\) be a fuzzy semi-metric space. Suppose that the $\triangleright$-triangle inequality is satisfied and that the following conditions are satisfied:

- the first six conditions in Theorem 1 are satisfied;
- any one of the following conditions is satisfied:
  - (a) the function \(f\) is \((+, -)\)-continuous with respect to \(\diamondsuit\), and the space \((U, \diamondsuit)\) is \((\pi, +)\)-complete or \((\pi, -)\)-complete;
  - (b) the function \(f\) is \((-\pi, -)\)-continuous with respect to \(\diamondsuit\), and the space \((U, \diamondsuit)\) is \((\pi, -\pi)\)-complete or \((\pi, -\pi)\)-complete.
Then, we have the following properties.

(i) The function $f$ and the sequence of functions $\{F_n\}_{n=1}^{\infty}$ have an asymptotically coupled coincidence point $(u^*, v^*)$ in the sense of

$$F_n(u^*, v^*) \xrightarrow{\mathcal{F}} f(u^*) \text{ and } F_n(v^*, u^*) \xrightarrow{\mathcal{F}} f(v^*) \text{ as } n \to \infty.$$  

(ii) Given any two functions $F_i$ and $F_j$, we further assume that the following inequality is satisfied

$$\mathcal{F}(F_i(u_1, v_1), F_j(u_2, v_2), t) \geq \lambda_{ij}(\mathcal{F}(F_i(u_1), F_i(u_2), 3t) + \mathcal{F}(F_j(v_1), F_j(v_2), 3t)).$$

Then, we have

$$f(u^*) = F_n(f(u^*), f(v^*)) \text{ and } f(v^*) = F_n(f(v^*), f(u^*)) \text{ for all } n.$$  

Moreover, the element $(u^*, v^*) \in U \times U$ can be obtained as follows.

- Suppose that condition (a) is satisfied. Then, the element $(u^*, v^*) \in U \times U$ can be obtained from the following limits

$$f(u_n) \xrightarrow{\mathcal{F}} u^* \text{ and } f(v_n) \xrightarrow{\mathcal{F}} v^* \text{ as } n \to \infty.$$  

- Suppose that condition (b) is satisfied. Then, the element $(u^*, v^*) \in U \times U$ can be obtained from the following limits

$$f(u_n) \xrightarrow{\mathcal{F}} u^* \text{ and } f(v_n) \xrightarrow{\mathcal{F}} v^* \text{ as } n \to \infty.$$  

The sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are constructed from any given initial element $(u_0, v_0) \in U \times U$ according to Formula (2).

**Proof.** Using Remark 3 and Proposition 4 by considering the \(\circ\)-triangle inequality, we see that $\{f(u_n)\}_{n=1}^{\infty}$ and $\{f(v_n)\}_{n=1}^{\infty}$ are both \(\times\)-Cauchy and \(\times\)-Cauchy sequences. We consider the following cases

- When condition (a) is satisfied, there exists $u^* \in U$ satisfying $f(u_n) \xrightarrow{\mathcal{F}} u^*$ as $n \to \infty$. Since $f$ is $(\cdot, \cdot)$-continuous with respect to $\mathcal{F}$, it follows that (41) is satisfied.

- When condition (b) is satisfied, there exists $u^* \in U$ satisfying $f(u_n) \xrightarrow{\mathcal{F}} u^*$ as $n \to \infty$. Since $f$ is $(\cdot, \cdot)$-continuous with respect to $\mathcal{F}$, it follows that (41) is satisfied.

The remaining proof follows from the similar proofs of Theorems 1 and 2. □

**Theorem 4. (Satisfying the \(\circ\)-Triangle Inequality).** Let $(U, \mathcal{F})$ be a fuzzy semi-metric space. Suppose that the \(\circ\)-triangle inequality is satisfied and that the following conditions are satisfied:

- the first six conditions in Theorem 1 are satisfied;

- any one of the following conditions is satisfied:
  
  (a) the function $f$ is $(\cdot, \cdot)$-continuous and $(\cdot, \cdot)$-continuous with respect to $\mathcal{F}$ and the space $(U, \mathcal{F})$ is $(\times, \cdot)$-complete or $(\times, \cdot)$-complete;

  (b) the function $f$ is $(\cdot, \cdot)$-continuous and $(\cdot, \cdot)$-continuous with respect to $\mathcal{F}$ and the space $(U, \mathcal{F})$ is $(\times, \cdot)$-complete or $(\times, \cdot)$-complete.
Then, we have the following properties.

(i) Given any two functions $F_i$ and $F_j$, we further assume that the following inequality is satisfied

$$F(F_i(u_1), F_j(u_2), t) \geq \lambda_{ij}(F(u_1), F(u_2), 3t) \ast \mathcal{F}(F(U), F(V), 3t)).$$

Suppose that (a) is satisfied. Then, the element $(\bar{u}, \bar{v})$ is another asymptotically coupled coincidence point of $\{F_n\}_{n=1}^\infty$ and $f$. Then, we have

$$f(u^*) = f(\bar{u})$$

and

$$f(v^*) = f(\bar{v}).$$

(ii) Given any two functions $F_i$ and $F_j$, we further assume that the following inequality is satisfied

$$F(F_i(u_1), F_j(u_2), t) \geq \lambda_{ij}(F(u_1), F(u_2), 3t) \ast \mathcal{F}(F(U), F(V), 3t)).$$

Then, there exists $(u^*, v^*) \in U \times U$ such that $(f(u^*), f(v^*)) \in U \times U$ is the common coupled fixed point of the functions $\{F_n\}_{n=1}^\infty$ in the sense of

$$f(u^*) = F_n(f(u^*), f(v^*))$$

and

$$f(v^*) = F_n(f(v^*), f(u^*))$$

for all $n$.

Moreover, the element $(u^*, v^*) \in U \times U$ can be obtained as follows.

- Suppose that condition (a) is satisfied. Then, the element $(u^*, v^*) \in U \times U$ can be obtained from the following limits

$$f(u_n) \xrightarrow{\mathcal{F}} u^*$$

and

$$f(v_n) \xrightarrow{\mathcal{F}} v^*$$

as $n \to \infty$.

- Suppose that condition (b) is satisfied. Then, the element $(u^*, v^*) \in U \times U$ can be obtained from the following limits

$$f(u_n) \xrightarrow{\mathcal{F}_1} u^*$$

and

$$f(v_n) \xrightarrow{\mathcal{F}_1} v^*$$

as $n \to \infty$.

where the sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are constructed from any given initial element $(u_0, v_0) \in U \times U$ according to Formula (2).

Proof. Using Remark 3 and Proposition 4 by considering the $\circ$-triangle inequality, we see that $\{f(u_n)\}_{n=1}^\infty$ and $\{f(v_n)\}_{n=1}^\infty$ are both $\circ$-Cauchy and $\times$-Cauchy sequences. We consider the following cases

- When condition (a) is satisfied, there exists $u^* \in U$ such that $f(u_n) \xrightarrow{\mathcal{F}_1} u^*$ as $n \to \infty$.
- When condition (b) is satisfied, there exists $v^* \in U$ such that $f(u_n) \xrightarrow{\mathcal{F}_1} u^*$ as $n \to \infty$.

The remaining proof follows from the similar proofs of Theorems 1 and 2. □

5. Asymptotically Coupled Fixed Points

Let $U$ be a universal set. We say that an element $(u, v) \in U \times U$ is a common coupled fixed point of functions $f : U \to U$ and $F : U \times U \to U$ when

$$u = f(u) = F(u, v)$$

and

$$v = f(v) = F(v, u).$$
Definition 10. Let \( U \) be a universal set. Given a function \( f : U \to U \) and a sequence of functions \( \{F_n\}_{n=1}^{\infty} \) with \( F_n : U \times U \to U \), we say that an element \((u, v) \in U \times U\) is an asymptotically coupled fixed point of functions \( f \) and \( \{F_n\}_{n=1}^{\infty} \) when

\[
F_n(u, v) \xrightarrow{n \to \infty} f(u) = u \quad \text{and} \quad F_n(v, u) \xrightarrow{n \to \infty} f(v) = v \quad \text{as} \quad n \to \infty.
\]

Theorem 5. (Satisfying the \( \infty \)-Triangle Inequality). Let \((U, \delta)\) be a fuzzy semi-metric space. Suppose that the \( \infty \)-triangle inequality is satisfied and that the following conditions are satisfied:

- the following inequality is satisfied

\[
\sup_n (\alpha_n \ast \beta_n) \geq \left( \sup_n \alpha_n \right) \ast \left( \sup_n \beta_n \right)
\]

for any two sequences \( \{\alpha_n\}_{n=1}^{\infty} \) and \( \{\beta_n\}_{n=1}^{\infty} \) in \([0, 1]\);

- the \( t \)-norm \( \ast \) is left-continuous with respect to the first or second component;

- for any fixed \( u, v \in U \), the function \( \delta(u, v, \cdot) : (0, \infty) \to [0, 1] \) is left-continuous at each point \( t \in (0, \infty) \);

- the function \( \lambda : [0, 1] \to [0, 1] \) is left-continuous on \([0, 1]\) in the left sense and satisfies the following strict inequality

\[
\lambda(\alpha) \ast \lambda(\alpha) > \alpha \quad \text{for any} \quad \alpha \in [0, 1);
\]

- the functions \( F_n : U \times U \to U \) and \( f : U \to U \) satisfy \( F_n(U, U) \subseteq f(U) \) for all \( n \in \mathbb{N} \);

- the functions \( f \) and \( F_n \) commute all \( n \in \mathbb{N} \);

- given any two functions \( F_i \) and \( F_j \), the following inequalities are satisfied

\[
\delta(F_i(u_1, v_1), F_j(u_2, v_2), t) \geq \lambda_{ij}(\delta(f(u_1), f(u_2)), t) \ast \delta(f(v_1), f(v_2), t))
\]

and

\[
\delta(F_i(u_1, v_1), F_j(u_2, v_2), t) \geq \lambda_{ij}(\delta(f(u_1), f(u_2)), 2t) \ast \delta(f(v_1), f(v_2), 2t)),
\]

where the functions \( \lambda_{ij} : [0, 1] \to [0, 1] \) satisfy \( \lambda_{ij}(\alpha) \geq \lambda(\alpha) \) for all \( \alpha \in [0, 1] \) and all \( i, j \in \mathbb{N} \);

- given any fixed \( v^* \in U \) and \( t^* > 0 \), if \( u_n \xrightarrow{n \to \infty} u^* \) as \( n \to \infty \) for a given sequence \( \{u_n\}_{n=1}^{\infty} \), then

\[
\delta(u_n, v^*, t^*) \leq \delta(u^*, v^*, t^*) \quad \text{for all} \quad n;
\]

- any one of the following conditions is satisfied:
  
  (a) the function \( f \) is \((\l, \r)-\)continuous and \((\l, -\r)-\)continuous with respect to \( \delta \) and the space \((U, \delta)\) is \((\infty, \l)-\)complete and \((\infty, -\r)-\)complete;

  (b) the function \( f \) is \((-\l, \r)-\)continuous and \((-\l, -\r)-\)continuous with respect to \( \delta \) and the space \((U, \delta)\) is \((-\l, \r)-\)complete and \((-\l, -\r)-\)complete.

Then, the function \( f \) and the sequence \( \{F_n\}_{n=1}^{\infty} \) of functions have a unique asymptotically coupled fixed point \((u^*, v^*)\), which is obtained by taking the following limits

\[
f(u_n) \xrightarrow{\infty} u^* \quad \text{or} \quad f(v_n) \xrightarrow{\infty} v^*
\]

and

\[
f(v_n) \xrightarrow{\infty} v^* \quad \text{or} \quad f(v_n) \xrightarrow{\infty} v^*,
\]

where the sequences \( \{u_n\}_{n=1}^{\infty} \) and \( \{v_n\}_{n=1}^{\infty} \) are constructed from any given initial element \((u_0, v_0) \in U \times U\) according to Formula (2).
We consider the following cases.

Applying (67) to Proposition 3 by taking the constant sequence and for all

In other words, we have

Using the triangle inequality, we have

In other words, we have

Since the function is left-continuous on \([0,1]\) in the left sense, it follows that

Using the \(\infty\)-triangle inequality, we have

\[
\begin{align*}
\mathfrak{f}(f(u_n), f(u^*), t) & = \mathfrak{f}(F_n(u_n, v_n), f(u^*), t) \\
& \geq \mathfrak{f}(F_{n+1}(u_n, v_n), F_n(u^*, v^*), t) + \mathfrak{f}(F_n(u^*, v^*), f(u^*), t).
\end{align*}
\]
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We also have
\[ \mathfrak{A}(F_{n+1}(u_n, v_n), F_n(u^*, v^*), t) \]
\[ \geq \lambda_{n+1, n}(\mathfrak{A}(f(u_n), f(u^*), 2t) \ast \mathfrak{A}(f(v_n), f(v^*), 2t)) \text{ (using (64))} \]
\[ \geq \lambda(\mathfrak{A}(f(u_n), f(u^*), 2t) \ast \mathfrak{A}(f(v_n), f(v^*), 2t)). \] (70)

From (69) and (70), using the increasing property of t-norm, we obtain
\[ \mathfrak{A}(f(u_{n+1}), f(u^*), 2t) \]
\[ \geq \lambda(\mathfrak{A}(f(u_n), f(u^*), 2t) \ast \mathfrak{A}(f(v_n), f(v^*), 2t)) \ast \mathfrak{A}(f_n(u^*, v^*), f(u^*), t). \] (71)

We also obtain
\[ \mathfrak{A}(u^*, f(u^*), 2t) = \lim_{n \to \infty} \mathfrak{A}(f(u_{n+1}), f(u^*), 2t) \text{ (using (67) and Proposition 3)} \]
\[ \geq \lambda(\mathfrak{A}(u^*, f(u^*), 2t) \ast \mathfrak{A}(v^*, f(v^*), 2t)) \ast 1 \text{ (using (66), (68), (71) and Proposition 1)} \]
\[ \geq \lambda(\mathfrak{A}(u^*, f(u^*), 2t) \ast \mathfrak{A}(v^*, f(v^*), 2t)). \] (72)

We can similarly obtain
\[ \mathfrak{A}(v^*, f(v^*), 2t) \geq \lambda(\mathfrak{A}(u^*, f(u^*), 2t) \ast \mathfrak{A}(v^*, f(v^*), 2t)). \] (73)

By applying the increasing property of t-norm to the inequalities (72) and (73), it follows that
\[ \mathfrak{A}(u^*, f(u^*), 2t) \ast \mathfrak{A}(v^*, f(v^*), 2t) \]
\[ \geq [\lambda(\mathfrak{A}(u^*, f(u^*), 2t) \ast \mathfrak{A}(v^*, f(v^*), 2t))] \ast [\lambda(\mathfrak{A}(u^*, f(u^*), 2t) \ast \mathfrak{A}(v^*, f(v^*), 2t))]. \] (74)

Suppose that
\[ \mathfrak{A}(u^*, f(u^*), 2t) \ast \mathfrak{A}(v^*, f(v^*), 2t) < 1. \]

Using (62), we obtain
\[ [\lambda(\mathfrak{A}(u^*, f(u^*), 2t) \ast \mathfrak{A}(v^*, f(v^*), 2t))] \ast [\lambda(\mathfrak{A}(u^*, f(u^*), 2t) \ast \mathfrak{A}(v^*, f(v^*), 2t))] \]
\[ > \mathfrak{A}(u^*, f(u^*), 2t) \ast \mathfrak{A}(v^*, f(v^*), 2t), \]

which contradicts (74). Therefore we must have
\[ \mathfrak{A}(u^*, f(u^*), 2t) \ast \mathfrak{A}(v^*, f(v^*), 2t) = 1 \text{ for all } t > 0. \]

According to Remark 1, it follows that
\[ \mathfrak{A}(u^*, f(u^*), t) = 1 = \mathfrak{A}(v^*, f(v^*), t) \text{ for all } t > 0, \]

which indicates that \( u^* = f(u^*) \) and \( v^* = f(v^*) \). This shows that \((u^*, v^*)\) is an asymptotically coupled fixed point of \( \{F_n\}_{n=1}^\infty \) and \( f \).

To prove the uniqueness, since \((u, v)\) is another asymptotically coupled fixed point, we have
\[ F_n(u, v) \xrightarrow{\mathfrak{A}} f(u) = u \text{ and } F_n(v, u) \xrightarrow{\mathfrak{A}} f(v) = v \text{ as } n \to \infty. \] (75)

Applying (67) to Proposition 3 by taking the constant sequence \( t_n = t \) for all \( n \in \mathbb{N} \), we have
\[ \lim_{n \to \infty} \mathfrak{A}(f(u_n), \bar{u}, t) = \mathfrak{A}(u^*, \bar{u}, t) \text{ and } \lim_{n \to \infty} \mathfrak{A}(f(v_n), \bar{v}, t) = \mathfrak{A}(v^*, \bar{v}, t). \]

Since
\[ \mathfrak{A}(f(u_n), \bar{u}, t) \leq \mathfrak{A}(u^*, \bar{u}, t) \text{ and } \mathfrak{A}(f(v_n), \bar{v}, t) \leq \mathfrak{A}(v^*, \bar{v}, t) \]
We also have
\[ g(f(u_n), \bar{u}, t) \to g(u^*, \bar{u}, t) \quad \text{as } n \to \infty. \]
and
\[ g(f(v_n), \bar{v}, t) \to g(v^*, \bar{v}, t) \quad \text{as } n \to \infty. \]

Using the left-continuity of t-norm and Proposition 1, we obtain
\[ g(f(u_n), \bar{u}, t) \ast g(f(v_n), \bar{v}, t) \to (g(u^*, \bar{u}, t) \ast g(v^*, \bar{v}, t)) \quad \text{as } n \to \infty. \]

Since the function \( \lambda \) is left-continuous on \([0, 1]\) in the left sense, it follows that
\[ \lim_{n \to \infty} \lambda(g(f(u_n), \bar{u}, t) \ast g(f(v_n), \bar{v}, t)) = \lambda(g(u^*, \bar{u}, t) \ast g(v^*, \bar{v}, t)). \] (76)

Using the \( \infty \)-triangle inequality, we have
\[
\begin{align*}
\bar{g}
\leq & \bar{g}(f(u_{n+1}, \bar{u}, 2t) = \bar{g}(f(u_{n+1}, v_n), \bar{u}, t) \\
\geq & \bar{g}(f(u_{n+1}, v_n), F_n(u, \bar{v}), t) \ast \bar{g}(F_n(u, \bar{v}), \bar{u}, t). \tag{77}
\end{align*}
\]

We also have
\[
\begin{align*}
\lambda_n(u_{n+1}, \bar{u}, 2t) & = \lambda(g(f(u_n), f(\bar{u}), 2t) \ast g(f(v_n), f(\bar{v}), 2t)) \quad \text{using (64)} \\
\geq & \lambda(g(f(u_n), f(\bar{u}), 2t) \ast g(f(v_n), f(\bar{v}), 2t)) \\
= & \lambda(g(f(u_n), f(\bar{u}), 2t) \ast g(f(v_n), f(\bar{v}), 2t)). \tag{78}
\end{align*}
\]

From (77) and (78), using the increasing property of t-norm, we obtain
\[
\bar{g}(f(u_{n+1}, \bar{u}, 2t) \geq \lambda(g(f(u_n), \bar{u}, 2t) \ast g(f(v_n), \bar{v}, 2t)) \ast \bar{g}(F_n(\bar{u}, \bar{v}), \bar{u}, t). \tag{79}
\]

Using (67), (76), (75), (79) and Proposition 3, we also obtain
\[
\bar{g}(u^*, \bar{u}, 2t) \geq \lambda(g(u^*, \bar{u}, 2t) \ast g(v^*, \bar{v}, 2t)). \tag{80}
\]

We can similarly obtain
\[
\bar{g}(v^*, \bar{v}, 2t) \geq \lambda(g(u^*, \bar{u}, 2t) \ast g(v^*, \bar{v}, 2t)). \tag{81}
\]

By applying the increasing property of t-norm to the inequalities (80) and (81), it follows that
\[
\begin{align*}
\bar{g}(u^*, \bar{u}, 2t) & \ast \bar{g}(v^*, \bar{v}, 2t) \\
\geq & [\lambda(g(u^*, \bar{u}, 2t) \ast g(v^*, \bar{v}, 2t))] \ast [\lambda(g(u^*, \bar{u}, 2t) \ast g(v^*, \bar{v}, 2t))]. \tag{82}
\end{align*}
\]

Suppose that
\[
\bar{g}(u^*, \bar{u}, 2t) \ast \bar{g}(v^*, \bar{v}, 2t) < 1.
\]

Using (62), we have
\[
[\lambda(g(u^*, \bar{u}, 2t) \ast g(v^*, \bar{v}, 2t))] \ast [\lambda(g(u^*, \bar{u}, 2t) \ast g(v^*, \bar{v}, 2t))]
> \bar{g}(u^*, \bar{u}, 2t) \ast \bar{g}(v^*, \bar{v}, 2t),
\]

which contradicts (82). Therefore, we must have
\[
\bar{g}(u^*, \bar{u}, 2t) \ast \bar{g}(v^*, \bar{v}, 2t) = 1 \text{ for all } t > 0.
\]
According to Remark 1, it follows that
\[ \mathfrak{g}(u^*, \bar{u}, t) = 1 = \mathfrak{g}(v^*, \bar{v}, t) \text{ for all } t > 0, \]
which indicates \( u^* = \bar{u} \) and \( v^* = \bar{v} \). This completes the proof. \( \square \)

**Remark 6.** By referring to Remark 5, when the function \( \lambda_{ij} \) is nondecreasing, the inequality (64) implies the inequality (63).

**Theorem 6.** (Satisfying the \( \triangleright \)-Triangle Inequality). Let \((U, \mathfrak{f})\) be a fuzzy semi-metric space. Suppose that the \( \triangleright \)-triangle inequality is satisfied and that the following conditions are satisfied:

- the first eight conditions in Theorem 5 are satisfied;
- the function \( f \) is \((\triangleright, \triangleright)\)-continuous or \((\ll, \triangleright)\)-continuous with respect to \( \mathfrak{f} \);
- any one of the following conditions is satisfied:
  1. the space \((U, \mathfrak{f})\) is \((\triangleright, \triangleright)\)-complete and \((\ll, \triangleright)\)-complete;
  2. the space \((U, \mathfrak{f})\) is \((\triangleright, \triangleright)\)-complete and \((\ll, \ll)\)-complete.

Then, the function \( f \) and the sequence \( \{F_n\}_{n=1}^\infty \) of functions have a unique asymptotically coupled fixed point \((u^*, v^*)\), which is obtained by taking the following limits
\[ f(u_n) \xrightarrow{\mathfrak{f}} u^* \text{ or } f(u_n) \xrightarrow{\mathfrak{f}} u^* \]
and
\[ f(v_n) \xrightarrow{\mathfrak{f}} v^* \text{ or } f(v_n) \xrightarrow{\mathfrak{f}} v^*, \]
where the sequences \( \{u_n\}_{n=1}^\infty \) and \( \{v_n\}_{n=1}^\infty \) are constructed from any given initial element \((u_0, v_0) \in U \times U\) according to Formula (2).\]

**Proof.** Using Remark 3 and Proposition 4 by considering the \( \triangleright \)-triangle inequality, we see that \( \{f(u_n)\}_{n=1}^\infty \) and \( \{f(v_n)\}_{n=1}^\infty \) are both \( \triangleright \)-Cauchy and \( \triangleright \)-Cauchy sequences. Using a proof similar to that of Theorem 5, we can obtain the desired results. \( \square \)

**Theorem 7.** (Satisfying the \( \ll \)-Triangle Inequality). Let \((U, \mathfrak{f})\) be a fuzzy semi-metric space. Suppose that the \( \ll \)-triangle inequality is satisfied and that the following conditions are satisfied:

- the first eight conditions in Theorem 5 are satisfied;
- the function \( f \) is \((\ll, \ll)\)-continuous or \((\triangleright, \ll)\)-continuous with respect to \( \mathfrak{f} \);
- any one of the following conditions is satisfied:
  1. the space \((U, \mathfrak{f})\) is \((\triangleright, \triangleright)\)-complete and \((\ll, \ll)\)-complete;
  2. the space \((U, \mathfrak{f})\) is \((\triangleright, \triangleright)\)-complete and \((\ll, \ll)\)-complete.

Then, the function \( f \) and the sequence \( \{F_n\}_{n=1}^\infty \) of the functions have a unique asymptotically coupled fixed point \((u^*, v^*)\), which is obtained by taking the following limits
\[ f(u_n) \xrightarrow{\mathfrak{f}} u^* \text{ or } f(u_n) \xrightarrow{\mathfrak{f}} u^* \]
and
\[ f(v_n) \xrightarrow{\mathfrak{f}} v^* \text{ or } f(v_n) \xrightarrow{\mathfrak{f}} v^*, \]
where the sequences \( \{u_n\}_{n=1}^\infty \) and \( \{v_n\}_{n=1}^\infty \) are constructed from any given initial element \((u_0, v_0) \in U \times U\) according to Formula (2).\]

**Proof.** Using Remark 3 and Proposition 4 by considering the \( \ll \)-triangle inequality, we see that \( \{f(u_n)\}_{n=1}^\infty \) and \( \{f(v_n)\}_{n=1}^\infty \) are both \( \ll \)-Cauchy and \( \ll \)-Cauchy sequences. Using a proof similar to that of Theorem 5, we can obtain the desired results. \( \square \)
Theorem 8. (Satisfying the $\bowtie$-Triangle Inequality). Let $(U, \mathcal{F})$ be a fuzzy semi-metric space. Suppose that the $\bowtie$-triangle inequality is satisfied and that the following conditions are satisfied:

- the first eight conditions in Theorem 5 are satisfied;
- Suppose that any one of the following conditions is satisfied:
  - the function $f$ is $(\bowtie, \bowtie)$-continuous and $(\bowtie, \bowtie)$-continuous with respect to $\mathcal{F}$;
  - the function $f$ is $(\bowtie, \bowtie)$-continuous and $(\bowtie, \bowtie)$-continuous with respect to $\mathcal{F}$;
- any one of the following conditions is satisfied:
  - the space $(U, \mathcal{F})$ is $(\bowtie, \bowtie)$-complete and $(\bowtie, \bowtie)$-complete;
  - the space $(U, \mathcal{F})$ is $(\bowtie, \bowtie)$-complete and $(\bowtie, \bowtie)$-complete.

Then, the function $f$ and the sequence $\{F_n\}_{n=1}^{\infty}$ of functions have a unique asymptotically coupled fixed point $(u^*, v^*)$, which is obtained by taking the following limits

$$f(u_n) \xrightarrow{\mathcal{F}} u^* \text{ or } f(u_n) \xrightarrow{\mathcal{F}} u^*$$

and

$$f(v_n) \xrightarrow{\mathcal{F}} v^* \text{ or } f(v_n) \xrightarrow{\mathcal{F}} v^*,$$

where the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are constructed from any given initial element $(u_0, v_0) \in U \times U$ according to Formula (2).

Proof. Using Remark 3 and Proposition 4 by considering the $\bowtie$-triangle inequality, we see that $\{f(u_n)\}_{n=1}^{\infty}$ and $\{f(v_n)\}_{n=1}^{\infty}$ are both $\bowtie$-Cauchy and $\bowtie$-Cauchy sequences. Using a proof similar to that of Theorem 5, we can obtain the desired results. □

6. Conclusions

The fuzzy semi-metric space considered in this paper does not assume the symmetric condition. In this case, the triangle inequality should be carefully treated. Therefore, four different triangle inequalities called $\bowtie\bowtie$-triangle inequality, $\bowtie$-triangle inequality, $\bowtie\bowtie$-triangle inequality and $\bowtie$-triangle inequality are considered. Under these settings, the different concepts of Cauchy sequence and completeness are also introduced without considering the symmetric condition.

Under the fuzzy semi-metric space, the asymptotically coupled coincidence points and asymptotically coupled fixed points are studied. Given a sequence of functions $\{F_n\}_{n=1}^{\infty}$ with $F_n : U \times U \to U$ and a function $f : U \to U$, an element $(u, v) \in U \times U$ is called an asymptotically coupled coincidence point of functions $\{F_n\}_{n=1}^{\infty}$ and $f$ when

$$F_n(u, v) \xrightarrow{\mathcal{F}} f(u) \text{ and } F_n(v, u) \xrightarrow{\mathcal{F}} f(v) \text{ as } n \to \infty.$$

Theorems 1–4 present the desired results based on the different conditions in which the four different types of triangle inequalities are separately considered. The assumptions are different when the triangle inequalities are different. The uniqueness of asymptotically coupled coincidence points cannot be guaranteed, and it can only be addressed in a weak sense of uniqueness.

An element $(u, v) \in U \times U$ is called an asymptotically coupled fixed point of functions $\{F_n\}_{n=1}^{\infty}$ and $f$ when

$$F_n(u, v) \xrightarrow{\mathcal{F}} f(u) = u \text{ and } F_n(v, u) \xrightarrow{\mathcal{F}} f(v) = v \text{ as } n \to \infty.$$

Theorems 5–8 present the desired results based on the different conditions in which the four different types of triangle inequalities are separately considered. The assumptions are different when the triangle inequalities are different. Although the uniqueness of asymptotically coupled coincidence points cannot be guaranteed, the uniqueness of asymptotically coupled fixed points can be guaranteed using different arguments.
The fuzzy semi-metric space is a general space. Since there are a lot of interesting fixed-point-related theorems that have been established in fuzzy metric space, in the future research, we can extend our work to study the fixed-point-related theorems in fuzzy semi-metric space.

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