Abstract

By using the cohomology theory of quandles, quandle cocycle invariants and shadow quandle cocycle invariants are defined for oriented links and surface-links via broken surface diagrams. By using symmetric quandles, symmetric quandle cocycle invariants are also defined for unoriented links and surface-links via broken surface diagrams. A marked graph diagram is a link diagram possibly with 4-valent vertices equipped with markers. S. J. Lomonaco, Jr. and K. Yoshikawa introduced a method of describing surface-links by using marked graph diagrams. In this paper, we give interpretations of these quandle cocycle invariants in terms of marked graph diagrams, and introduce a method of computing them from marked graph diagrams.

1 Introduction

A surface-link is a closed 2-manifold smoothly (or piecewise linearly and locally flatly) embedded in the Euclidian 4-space $\mathbb{R}^4$. Two surface-links $\mathcal{L}$
and $L'$ are said to be equivalent if there exists an orientation preserving homeomorphism $h : \mathbb{R}^4 \to \mathbb{R}^4$ such that $h(L) = L'$. When $L$ and $L'$ are oriented, it is assumed that $h|_L : L \to L'$ is also an orientation preserving homeomorphism.

A broken surface diagram of a surface-link is a projection image in $\mathbb{R}^3$ with over/under sheet information at each double point curve. It is known that two broken surface diagrams present equivalent surface-links if and only if they are related by a finite sequence of Roseman moves (cf. [29]).

A marked graph diagram is a link diagram possibly with 4-valent vertices equipped with markers. S. J. Lomonaco, Jr. [27] and K. Yoshikawa [32] introduced a method of describing surface-links by using marked graph diagrams. Yoshikawa introduced local moves on marked graph diagrams, which are so-called Yoshikawa moves. Two marked graph diagrams present equivalent surface-links if and only if they are related by a finite sequence of Yoshikawa moves ([21, 23, 31]). So one can use marked graph diagrams for studying surface-links and their invariants (cf. [1, 13, 14, 22, 23, 24, 25, 26, 30]).

A quandle is a set $X$ with a binary operation $*: X \times X \to X$ satisfying certain conditions derived from Reidemeister moves for classical link diagrams ([15, 28]). By using the cohomology theory of quandles ([5, 7, 8, 9, 10, 12]), quandle cocycle invariants and shadow quandle cocycle invariants are defined for oriented links and surface-links via broken surface diagrams ([11, 5, 6]). On the other hand, by using symmetric quandles, symmetric quandle cocycle invariants are also defined for unoriented links and surface-links via broken surface diagram ([17, 18]). These invariants for surface-links are defined as state-sums over all quandle colorings of sheets and corresponding Boltzmann weights that are evaluations of a cocycle at triple points in broken surface diagrams.

The aim of this paper is to interpret of these quandle cocycle invariants in terms of marked graph diagrams, and introduce a method of computing the quandle cocycle invariants from marked graph diagrams.

This paper is organized as follows: In Section 2 we prepare some preliminaries about broken surface diagrams and marked graph diagrams. Section 3 contains a review of quandle cocycle invariants of oriented surface-links. In Section 4 we describe quandle cocycle invariants via marked graph diagrams and give a method of computing the quandle cocycle invariants from marked graph diagrams. Section 5 contains shadow colorings and shadow quandle cocycle invariants of oriented surface-links. In Section 6 we describe how to compute shadow quandle cocycle invariants from marked graph diagrams. In Section 7 we recall symmetric quandles and symmetric quandle cocycle invariants.
cle invariants of unoriented surface-links. Section 8 is devoted to giving a method of computing symmetric quandle cocycle invariants from marked graph diagrams.

2 Marked graph diagrams of surface-links

In this section, we recall broken surface diagrams and marked graph diagrams presenting surface-links.

Let $L$ be a surface-link. By deforming $L$ by an ambient isotopy of $\mathbb{R}^4$ if necessary, we may assume that the restriction map $q|_L : L \rightarrow \mathbb{R}^3$ is a general position map, where $q : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ denotes the projection $(x, y, z, t) \mapsto (x, y, t)$. Along the double point curves, one of the sheets (called the over-sheet) lies above the other (under-sheet) with respect to the $z$-coordinate. The under-sheets are coherently broken in the projection. The union $B$ of such broken surfaces is called a broken surface diagram of $L$. When $L$ is an oriented surface-link, we assume that the sheets of are co-oriented such that the pair (orientation, co-orientation) matches the given (right-handed) orientation of $\mathbb{R}^3$. In [29], D. Roseman introduced seven moves of broken surface diagrams, called Roseman moves. Two surface-links are equivalent if and only if their broken surface diagrams are related by a finite sequence of Roseman moves. For more details, see [2, 29].

A marked graph is a spatial graph $G$ in $\mathbb{R}^3$ which satisfies the following:

(1) $G$ is a finite regular graph with 4-valent vertices, say $v_1, v_2, \ldots, v_n$.

(2) Each $v_i$ is a rigid vertex; that is, we fix a rectangular neighborhood $N_i$ homeomorphic to $\{(x, y)| -1 \leq x, y \leq 1\}$, where $v_i$ corresponds to the origin and the edges incident to $v_i$ are represented by $x^2 = y^2$.

(3) Each $v_i$ has a marker, which is the interval on $N_i$ given by $\{(x, 0)| -1 \leq x \leq 1\}$.

Two marked graphs are said to be equivalent if they are ambient isotopic in $\mathbb{R}^3$ with keeping the rectangular neighborhoods and markers.

An orientation of a marked graph $G$ is a choice of an orientation for each edge of $G$ such that every vertex in $G$ looks like $\boxed{\text{left-right or right-left}}$. A marked graph $G$ is said to be orientable if it admits an orientation. Otherwise, it is said to be non-orientable. Figure [ ] shows an oriented marked graph and a
non-orientable marked graph. Marked graphs can be described by diagrams on $\mathbb{R}^2$ with some 4-valent vertices equipped with markers.

A surface-link $\mathcal{L}$ in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ can be described in terms of its cross-sections $\mathcal{L}_t = \mathcal{L} \cap \mathbb{R}^3 \times \{t\}$, $t \in \mathbb{R}$ (cf. [11]). Let $p : \mathbb{R}^4 \to \mathbb{R}$ be the projection given by $p(x_1, x_2, x_3, x_4) = x_4$, and we denote by $p_\mathcal{L} : \mathcal{L} \to \mathbb{R}$ the restriction to $\mathcal{L}$. It is known ([19, 20, 27]) that any surface-link $\mathcal{L}$ is equivalent to a surface-link $\mathcal{L}'$, called a hyperbolic splitting of $\mathcal{L}$, such that the projection $p_{\mathcal{L}'} : \mathcal{L}' \to \mathbb{R}$ satisfies that all critical points are non-degenerate, all the index 0 critical points (minimal points) are in $\mathbb{R}^3 \times \{-1\}$, all the index 1 critical points (saddle points) are in $\mathbb{R}^3 \times \{0\}$, and all the index 2 critical points (maximal points) are in $\mathbb{R}^3 \times \{1\}$.

Let $\mathcal{L}$ be a surface-link and let $\mathcal{L}'$ be a hyperbolic splitting of $\mathcal{L}$. The cross-section $\mathcal{L}'_0 = \mathcal{L}' \cap \mathbb{R}^3 \times \{0\}$ at $t = 0$ is a 4-valent graph in $\mathbb{R}^3 \times \{0\}$. We give a marker at each 4-valent vertex (saddle point) that indicates how the saddle point opens up above as illustrated in Figure 2. The resulting marked graph $G$ is called a marked graph presenting $\mathcal{L}$. As usual, $G$ is described by a diagram $\Gamma$ on $\mathbb{R}^2$ which is a generic projection on $\mathbb{R}^2$ with over/under crossing information for each double point such that the restriction to a rectangular neighborhood of each marked vertex is an embedding. Such a diagram is called a marked graph diagram or a ch-diagram (cf. [30]) presenting $\mathcal{L}$.

When $\mathcal{L}$ is an oriented surface-link, we assume that $\mathcal{L}'_0$ has the induced orientation as the boundary of the oriented surface $\mathcal{L}' \cap (\mathbb{R}^3 \times (-\infty, 0])$.

Let $\Gamma$ be a marked graph diagram and $\Gamma_0$ the singular link diagram obtained from $\Gamma$ by removing all markers. Let $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ be the set of all vertices of $\Gamma$. For each $i$ ($i = 1, \ldots, n$), consider four points $v_i^1, v_i^2, v_i^3, v_i^4$ on $\Gamma$ in a neighborhood of $v_i$ as in Figure 3. We define

$$\Gamma_+ = [\Gamma_0 \setminus \bigcup_{i=1}^n (\bigcup_{j=1}^4 |v_i, v_i^j|)] \cup \bigcup_{i=1}^n (|v_i^1, v_i^2| \cup |v_i^3, v_i^4|),$$

Figure 1: Marked graphs
which is called the positive resolution of $\Gamma$, and

$$\Gamma_- = \left[ \Gamma_0 \setminus \bigcup_{i=1}^n \left( \bigcup_{j=1}^4 |v_i, v^j_i| \right) \right] \cup \left[ \bigcup_{i=1}^n (|v^1_i, v^3_i| \cup |v^2_i, v^4_i|) \right],$$

the negative resolution of $\Gamma$, where $|v, w|$ is the line segment connecting $v$ and $w$. When both resolutions $\Gamma_-$ and $\Gamma_+$ are diagrams of trivial links, we call $\Gamma$ admissible.

When $\Gamma$ is admissible, we construct a surface-link as follows (cf. [17, 19, 20, 32]). Let $L_0$ denote a graph in $\mathbb{R}^3$ whose diagram is $\Gamma_0$. Let $w^j_i$ and $w_i$ be points on $L_0$ such that $\pi(w^j_i) = v^j_i$, $\pi(w_i) = v_i$, respectively, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection $(x, y, z) \mapsto (x, y)$. For each $t \in [0, 1]$, let $w^j_i(t)$ be the point $(1 - t)w_i + tw^j_i \in \mathbb{R}^3$.

For each $t \in [0, 1]$, let $L^+_t$ be a link defined by

$$L^+_t = \left[ L_0 \setminus \bigcup_{i=1}^n \left( \bigcup_{j=1}^4 |w_i, w^j_i(t)| \right) \right] \cup \left[ \bigcup_{i=1}^n (|w^1_i(t), w^2_i(t)| \cup |w^3_i(t), w^4_i(t)|) \right],$$

and for each $t \in [-1, 0]$, let $L_-$ be a link defined by

$$L^-_t = \left[ L_0 \setminus \bigcup_{i=1}^n \left( \bigcup_{j=1}^4 |w_i, w^j_i(-t)| \right) \right] \cup \left[ \bigcup_{i=1}^n (|w^1_i(-t), w^3_i(-t)| \cup |w^2_i(-t), w^4_i(-t)|) \right].$$
Put $L_+ = L_1^+$ and $L_- = L_1^-$. Then $L_+$ and $L_-$ have diagrams $\Gamma_+$ and $\Gamma_-$, respectively. Let $B_1^+, \ldots, B_\mu^+$ be mutually disjoint 2-disks in $\mathbb{R}^3$ with $\partial(B_1^+ \cup \cdots \cup B_\mu^+) = L_+$, and let $B_1^-, \ldots, B_\lambda^-$ be mutually disjoint 2-disks in $\mathbb{R}^3$ with $\partial(B_1^- \cup \cdots \cup B_\lambda^-) = L_-$. Let $B_1^+, \ldots, B_\mu^+$ be mutually disjoint 2-disks in $\mathbb{R}^3$ with $\partial(B_1^+ \cup \cdots \cup B_\mu^+) = L_+$, and let $B_1^-, \ldots, B_\lambda^-$ be mutually disjoint 2-disks in $\mathbb{R}^3$ with $\partial(B_1^- \cup \cdots \cup B_\lambda^-) = L_-$. Let $F(\Gamma)$ be a surface-link in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ defined by

$$
F(\Gamma) = (B_1^- \cup \cdots \cup B_\lambda^-) \times \{-2\} \cup L_- \times (-2, -1) \cup \left( \cup_{t \in [0, 1]} L_t^- \times \{t\} \right) \cup L_0 \times \{0\} \cup \left( \cup_{t \in [0, 1]} L_t^+ \times \{t\} \right) \cup L_+ \times (1, 2) \cup (B_1^+ \cup \cdots \cup B_\mu^+) \times \{2\}.
$$

We say that $F(\Gamma)$ is a surface-link associated to $\Gamma$. It is uniquely determined from $\Gamma$ up to equivalence (see [19]).

A surface-link $\mathcal{L}$ is said to be presented by a marked graph diagram $\Gamma$ if $\mathcal{L}$ is equivalent to the surface-link $F(\Gamma)$. Any surface-link can be presented by an admissible marked graph diagram. Two admissible marked graph diagrams present equivalent surface-links if and only if they are related by a finite sequence of Yoshikawa moves ([21, 23, 31]).

S. Ashihara introduced a method of constructing a broken surface diagram of a surface-link from its marked graph diagram [1]. For our later use, we describe here his construction. In what follows, by $D \rightarrow D'$ we mean that a link diagram $D'$ is obtained from a link diagram $D$ by a single Reidemeister move (Figure 4) or an ambient isotopy of $\mathbb{R}^2$.

![Figure 4: Reidemeister moves of type $R_1$, $R_2$ and $R_3$](image)

Let $\Gamma$ be an admissible marked graph diagram, and let $\Gamma_+$ and $\Gamma_-$ be the positive and the negative resolutions.

Since $\Gamma_+$ is a diagram of a trivial link, there is a sequence of link diagrams from $\Gamma_+$ to a trivial link diagram $O$ related by ambient isotopies of $\mathbb{R}^2$ and Reidemeister moves:

$$
\Gamma_+ = D_1 \rightarrow D_2 \rightarrow \cdots \rightarrow D_r = O.
$$

For each $i (i = 1, \ldots, r - 1)$, let $\{f_t^{(i)}\}_{t \in I}$ be a 1-parameter family of homeomorphisms from $\mathbb{R}^3$ to $\mathbb{R}^3$ which satisfies

$$
f_0^{(i)} = id, \quad f_1^{(i)}(L(D_i)) = L(D_{i+1}),
$$
where $L(D_i)$ denotes a link in $\mathbb{R}^3$ with diagram $D_i$ for $i = 1, \ldots, r$. Without loss of generality, we may assume that $L(D_1) = L_+$ and the following two conditions are satisfied.

- When the move $D_i \rightarrow D_{i+1}$ is an ambient isotopy of $\mathbb{R}^2$, let $\{h_t^{(i)}\}_{t \in I}$ be an ambient isotopy of $\mathbb{R}^2$ such that $h_t^{(i)}(D_i) = D_{i+1}$. Then $f_t^{(i)}$ satisfies $\pi(f_t^{(i)}(L(D_i))) = h_t^{(i)}(\pi(L(D_i)))$ for $t \in I$.

- When the move $D_i \rightarrow D_{i+1}$ is a Reidemeister move, let $B_{(i)}$ be a disk in $\mathbb{R}^2$ where the move is applied, and let $M_{(i)}$ be the subset of $B_{(i)} \times I \subset \mathbb{R}^3$ determined by $\pi(M_{(i)} \cap (B_{(i)} \times \{t\})) = \pi(f_t^{(i)}(L(D_i))) \cap B_{(i)}$ for $t \in I$. Then $M_{(i)}$ is as in Figure 5 or 6 or 7.

![Figure 5: $M_{(i)}$ for $R_1$](image)

![Figure 6: $M_{(i)}$ for $R_2$](image)

![Figure 7: $M_{(i)}$ for $R_3$](image)
Take real numbers $t_1, \ldots, t_r$ with $1 < t_1 < \cdots < t_r < 2$. For each $i$ ($i = 1, \ldots, r - 1$), we define a homeomorphism $F^{(i)} : \mathbb{R}^4 (= \mathbb{R}^3 \times \mathbb{R}) \to \mathbb{R}^4$ by

$$F^{(i)}(x, t) = \begin{cases} (x, t) & (t \leq t_i), \\ (f^{(i)}(x), t) & (t_i < t < t_{i+1}), \\ (g^{(i)}_1(x), t) & (t \geq t_{i+1}), \end{cases}$$

where $\phi(t) = (t - t_i)/(t_{i+1} - t_i)$.

Similarly, consider a sequence of link diagrams from $\Gamma$ to a trivial link diagram $O'$, related by ambient isotopies of $\mathbb{R}^2$ and Reidemeister moves:

$$\Gamma = D'_1 \to D'_2 \to \cdots \to D'_s = O'.$$

For each $j$ ($j = 1, \ldots, s - 1$), let $\{g^{(j)}_t\}_{t \in I}$ be a 1-parameter family of homeomorphisms from $\mathbb{R}^3$ to $\mathbb{R}^3$ which satisfies

$$g^{(j)}_0 = \text{id}, \quad g^{(j)}_1(L(D'_j)) = L(D'_{j+1}).$$

Without loss of generality, we may assume that $L(D'_1) = L$ and the following two conditions are satisfied.

- When the move $D'_j \to D'_{j+1}$ is an ambient isotopy of $\mathbb{R}^2$, let $\{h^{(j)}_t\}_{t \in I}$ be an ambient isotopy of $\mathbb{R}^2$ such that $h^{(j)}_1(D'_j) = D'_{j+1}$. Then $g^{(j)}_t$ satisfies $\pi(g^{(j)}_t(L(D'_j))) = h^{(j)}_t(\pi(L(D'_j)))$ for $t \in I$.

- When the move $D'_j \to D'_{j+1}$ is a Reidemeister move, let $B'_j$ be a disk in $\mathbb{R}^2$ where the move is applied, and let $M'_j$ be the subset of $B'_j \times I \subset \mathbb{R}^3$ determined by $\pi(M'_j \cap (B'_j \times \{t\})) = \pi(g^{(j)}_t(L(D'_j))) \cap B'_j$ for $t \in I$. Then $M'_j$ is as in Figure 5, 6, or 7.

Take real numbers $t'_1, \ldots, t'_s$ with $-1 > t'_1 > \cdots > t'_s > -2$. For each $j$ ($j = 1, \ldots, s - 1$), we define a homeomorphism $G^{(j)} : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$G^{(j)}(x, t) = \begin{cases} (x, t) & (t \geq t'_j), \\ (g^{(j)}_t(x), t) & (t'_j < t < t'_{j+1}), \\ (g^{(j)}_1(x), t) & (t \leq t'_{j+1}), \end{cases}$$

where $\psi(t) = (t'_j - t)/(t'_{j+1} - t'_j)$.

Let $F' = G^{(s-1)} \circ G^{(s-2)} \circ \cdots \circ G^{(2)} \circ F^{(r-1)} \circ F^{(r-2)} \circ \cdots \circ F^{(1)}(F(\Gamma))$. Then $F'$ is equivalent to $F(\Gamma)$.  

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Let $B_1, \ldots, B_\mu$ be mutually disjoint 2-disks in $\mathbb{R}^3$ such that $\partial(B_1 \cup \cdots \cup B_\mu) = L(O)$ and $\pi|_{B_1 \cup \cdots \cup B_\mu}$ is an embedding. Let $B'_1, \ldots, B'_\lambda$ be mutually disjoint 2-disks in $\mathbb{R}^3$ such that $\partial(B'_1 \cup \cdots \cup B'_\lambda) = L(O')$ and $\pi|_{B'_1 \cup \cdots \cup B'_\lambda}$ is an embedding. Finally we define $F$ to be the surface constructed as follows:

$$F = (B'_1 \cup \cdots \cup B'_\lambda) \times \{-2\} \cup (F' \cap (\mathbb{R}^3 \times (-2, 2))) \cup (B_1 \cup \cdots \cup B_\mu) \times \{2\}.$$ 

It is in general position with respect to the projection $q : \mathbb{R}^4 \to \mathbb{R}^3$, $(x, y, z, t) \mapsto (x, y, t)$. The broken surface diagram of $F$ obtained from $q(F)$ is called a broken surface diagram associated to $\Gamma$, and denoted by $B(\Gamma)$.

### 3 Quandle cocycle invariants of oriented surface-links

We recall quandle cocycle invariants of oriented surface-links from [5]. A quandle is a set $X$ with a binary operation $*: X \times X \to X$ satisfying that (i) for any $x \in X$, $x*x = x$, (ii) for any $x, y \in X$, there is a unique $u \in X$ such that $x = u*y$, and (iii) for any $x, y, z \in X$, $(x*y)*z = (x*z)*(y*z)$. In (ii), the unique element $u$ is denoted by $x*y$, and then $x = u*y = (x*y)*y$.

**Example 3.1.** (1) The dihedral quandle of order $n$ is the set $R_n = \{0, 1, \ldots, n-1\}$ with the binary operation $i*j = 2j - i \text{ (mod } n)$ for each $i, j \in R_n$.

(2) Let $S_4 = \{0, 1, 2, 3\}$. Define a binary operation $*: S_4 \times S_4 \to S_4$ by

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 3 | 1 |
| 1 | 3 | 1 | 0 | 2 |
| 2 | 1 | 3 | 2 | 0 |
| 3 | 2 | 0 | 1 | 3 |

Then $S_4$ is a quandle, which is called the tetrahedral quandle.

(3) Let $G$ be a group. The conjugation quandle, denoted by $\text{conj}(G)$, is $G$ with the operation $x*y = y^{-1}xy$.

Let $X$ be a quandle. For each positive integer $n$, let $C^R_n(X)$ be the free abelian group generated by $n$-tuples $(x_1, \ldots, x_n)$ of elements of $X$. We assume $C^R_n(X) = \{0\}$ for $n \leq 0$. Define a homomorphism $\partial : C^R_n(X) \to C^R_{n-1}(X)$ by

$$\partial_n(x_1, x_2, \ldots, x_n) = \sum_{i=2}^n (-1)^i [(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

$$- (x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)]$$

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for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$. Then $C^R_n(X) = \{C^R_n(X), \partial_n\}$ is a chain complex. Let $C^D_n(X)$ be the subset of $C^R_n(X)$ generated by $n$-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some $i \in 1, \ldots, n-1$ if $n \geq 2$; otherwise let $C^D_n(X) = 0$. Then $C^D_n(X) = \{C^D_n(X), \partial_n\}$ is a sub-complex of $C^R_n(X)$. Consider the quotient chain complex $C^Q_n(X) = \{C^Q_n(X), \partial_n\}$, where $C^Q_n(X) = C^R_n(X)/C^D_n(X)$. For an abelian group $A$, we define chain and cochain complexes by $C^Q_n(X; A) = C^Q_n(X) \otimes A$ and $C^Q_n(X; A) = \text{Hom}(C^Q_n(X), A)$. The homology and cohomology groups are denoted by $H^Q_n(X; A)$ and $H^Q_n(X; A)$, respectively. The cycle and boundary groups (or cocycle and coboundary groups, resp.) are denoted by $Z^Q_n(X; A)$ and $B^Q_n(X; A)$ (or $Z^Q_n(X; A)$ and $B^Q_n(X; A)$, resp.). We will omit the coefficient group $A$ as usual if $A = \mathbb{Z}$.

A homomorphism $\theta : C^R_3(X) \to A$ is regarded as a 3-cocycle of the cochain complex $C^Q_3(X; A)$, called a quandle 3-cocycle, if and only if $\theta$ satisfies the following two conditions (where $A$ is written multiplicative):

- $\theta(x, x, y) = 1$ and $\theta(x, y, y) = 1$ for all $x, y \in X$, where 1 is the identity element in $A$.
- $\theta(x, z, w)\theta(x, y, z)\theta(x \ast z, y \ast z, w) = \theta(x, y, w)\theta(x \ast y, z, w)\theta(x \ast w, y \ast w, z \ast w)$ for each $x, y, z, w \in X$.

Let $\mathcal{B}$ be a broken surface diagram of an oriented surface-link $\mathcal{L}$, and let $S(\mathcal{B})$ be the set of sheets of $\mathcal{B}$. Let $X$ be a quandle. A coloring of $\mathcal{B}$ by $X$ is a map $C : S(\mathcal{B}) \to X$ satisfying the condition that at each double point curve, if the co-orientation of the over-sheet $y$ is from the under-sheet $x$ to $z$, then $C(z) = C(x) \ast C(y)$. See the left of Figure 8. Let $\text{Col}_X(\mathcal{B})$ denote the set of all colorings of $\mathcal{B}$ by $X$.

![Figure 8: A double point curve and a triple point](image-url)

Let $\tau$ be a triple point of $\mathcal{B}$. The sign of $\tau$ is positive if the co-orientations
of the top, the middle and the bottom sheets at \( \tau \) in this order match the given (right-handed) orientation of \( \mathbb{R}^3 \). Otherwise, the sign is negative. There are eight complementary regions of \( \mathcal{B} \) around \( \tau \). (Some of them may be the same.) There is a unique region such that the co-orientations of the sheets facing the region point from the region to the opposite regions. We call this region the \textit{source region} of \( \tau \).

For a 3-cocycle \( \theta \in Z_3^0(X; \mathcal{A}) \), the quandle cocycle invariant \( \Phi_\theta(\mathcal{L}) \) of an oriented surface-link \( \mathcal{L} \) associated to \( \theta \) is defined as follows. Let \( \mathcal{B} \) be a broken surface diagram of \( \mathcal{L} \). Let \( C : S(\mathcal{B}) \to X \) be a coloring of \( \mathcal{B} \). Let \( \tau \) be a triple point of \( \mathcal{B} \) and let \( x_1, x_2, \) and \( x_3 \) be colors of the bottom, the middle, and the top sheets facing the source region of \( \tau \), respectively. Let \( \epsilon(\tau) \) denote the sign of \( \tau \). See Figure 8, where \( \epsilon(\tau) = 1 \). The (Boltzmann) weight \( B_\theta(\tau, C) \) at \( \tau \) with respect to \( C \) is defined to be

\[
B_\theta(\tau, C) = \theta(x_1, x_2, x_3) \epsilon(\tau).
\]

The \textit{partition function} or \textit{state-sum} of \( \mathcal{B} \) (associated to \( \theta \)) is

\[
\Phi_\theta(\mathcal{B}) = \sum_{C \in \text{Col}_X(\mathcal{B})} \prod_{\tau \in T(\mathcal{B})} B_\theta(\tau, C) \in \mathbb{Z}[\mathcal{A}],
\]

where \( T(\mathcal{B}) \) is the set of all triple points in \( \mathcal{B} \).

\textbf{Theorem 3.2 \cite{5}}. Let \( \mathcal{L} \) be an oriented surface-link and let \( \mathcal{B} \) be a broken surface diagram of \( \mathcal{L} \). The partition function \( \Phi_\theta(\mathcal{B}) \) does not depend on the choice of \( \mathcal{B} \). Thus it is an invariant of \( \mathcal{L} \).

We call \( \Phi_\theta(\mathcal{B}) \) the \textit{quandle cocycle invariant} of \( \mathcal{L} \) associated to \( \theta \), and denote it by \( \Phi_\theta(\mathcal{L}) \).

\section{How to compute quandle cocycle invariants from marked graph diagrams}

In this section we introduce a method of computing quandle cocycle invariants from marked graph diagrams.

Let \( \Gamma \) be an oriented marked graph diagram and let \( V(\Gamma) \) denote the set of all marked vertices of \( \Gamma \). By an \textit{arc} of \( \Gamma \) we mean a connected component of \( \Gamma \setminus V(\Gamma) \). (At a crossing of \( \Gamma \) the under-arcs are assumed to be cut.) Let \( A(\Gamma) \) denote the set of arcs of \( \Gamma \). Since \( \Gamma \) is oriented, we assume that it is co-oriented: The co-orientation of an arc of \( \Gamma \) satisfies that the pair (orientation, co-orientation) matches the (right-handed) orientation of the
plane. At a crossing, if the pair of the co-orientation of the over-arc and that of the under-arc matches the (right-handed) orientation of the plane, then the crossing is called positive; otherwise it is negative. The crossing in (a) of Figure 9 is positive and that in (b) is negative.

Definition 4.1. Let \( X \) be a quandle and let \( \Gamma \) be an oriented marked graph diagram. A coloring of \( \Gamma \) by \( X \) is a map \( C : A(\Gamma) \to X \) satisfying the following conditions (1) and (2):

(1) For each crossing \( c \), let \( s_2 \) be the over-arc and let \( s_1 \) and \( s_3 \) be the under-arcs as shown in (a) or (b) of Figure 9 such that the co-orientation of \( s_2 \) points from \( s_1 \) to \( s_3 \). Then \( C(s_3) = C(s_1) \ast C(s_2) \).

(In this case, \( s_1 \) is called the source arc and \( s_3 \) is called the target arc at \( c \). The quandle element \( C(s_i) \) is called a color of the arc \( s_i \).)

(2) For each marked vertex \( v \), let \( s_1, s_2, s_3 \) and \( s_4 \) be the arcs of \( \Gamma \) as shown in (c) or (d) of Figure 9. Then \( C(s_1) = C(s_2) = C(s_3) = C(s_4) \).

Figure 9: Labels at a crossing

We denote by \( \text{Col}_X(\Gamma) \) the set of colorings of \( \Gamma \) by \( X \).

Theorem 4.2. Let \( L \) be an oriented surface-link. Let \( \Gamma \) and \( B \) be a marked graph diagram and a broken surface diagram presenting \( L \), respectively. Then there is a bijection from \( \text{Col}_X(\Gamma) \) to \( \text{Col}_X(B) \).

Proof. The fundamental quandle \( Q(\Gamma) \) is defined by a quandle generated by \( A(\Gamma) \) and the defining relations \( s_3 = s_1 \ast s_2 \) for \( s_1, s_2, s_3 \) as in (a) or (b) in Figure 9 and \( s_1 = s_2 = s_3 = s_4 \) for \( s_1, \ldots, s_4 \) as in (c) or (d). Without loss of generality, we may assume that \( B \) is a broken surface diagram associated to \( \Gamma \). Then by the same argument with [1] we see that there is a natural isomorphism from \( Q(\Gamma) \) to the fundamental quandle \( Q(B) \) of \( B \). Since \( \text{Col}_X(\Gamma) \) is identified with \( \text{Hom}(Q(\Gamma), X) \) and \( \text{Col}_X(B) \) is identified with \( \text{Hom}(Q(B), X) \), we have a bijection from \( \text{Col}_X(\Gamma) \) to \( \text{Col}_X(B) \). 

\[ \square \]
Let $\Gamma$ be a marked graph diagram of an oriented surface-link $\mathcal{L}$ and $\Gamma_+$ the positive resolution of $\Gamma$. Let $\Gamma_+ = D_1 \rightarrow D_2 \rightarrow \cdots \rightarrow D_r = O$ be a sequence of link diagrams from $\Gamma_+$ to a trivial link diagram $O$ related by ambient isotopies of $\mathbb{R}^2$ and oriented Reidemeister moves. Let $I_+^3 = \{i \mid D_i \rightarrow D_{i+1} \text{ is a move of type } R_3\}$. For each $i \in I_+^3$, let $B_{(i)}$ be a disk in $\mathbb{R}^2$ where the move $D_i \rightarrow D_{i+1}$ is applied.

Similarly, let $\Gamma_-$ be the negative resolution of $\Gamma$ and $\Gamma_- = D'_1 \rightarrow D'_2 \rightarrow \cdots \rightarrow D'_s = O'$ be a sequence of link diagrams from $\Gamma_-$ to a trivial link diagram $O'$ related by ambient isotopies of $\mathbb{R}^2$ and Reidemeister moves. Let $I_-^3 = \{j \mid D'_j \rightarrow D'_{j+1} \text{ is a move of type } R_3\}$. For each $j \in I_-^3$, let $B'_{(j)}$ be a disk in $\mathbb{R}^2$ where the move $D'_j \rightarrow D'_{j+1}$ is applied.

We define two functions $\epsilon_{tm}$ and $\epsilon_b$ from the disjoint union $I_+^3 \sqcup I_-^3$ to $\{\pm 1\}$ as follows:

Let $i \in I_+^3$ (or $i \in I_-^3$, resp.) and let $c$ be the crossing between the top arc and the two middle arcs in $D_i \cap B_{(i)}$ (or $D'_i \cap B'_{(i)}$, resp.) and let $n_1$ be the co-orientation of the bottom arc. Define $\epsilon_{tm}(i)$ and $\epsilon_b(i)$ for $i \in I_+^3 \sqcup I_-^3$ by

$$
\epsilon_{tm}(i) = \text{sign}(c),
$$

$$
\epsilon_b(i) = \begin{cases} 
1 & \text{if } n_1 \text{ points from } c, \\
-1 & \text{otherwise}.
\end{cases}
$$

**Definition 4.3.** Let $\Gamma$ be a marked graph diagram of an oriented surface-link $\mathcal{L}$. Let $\mathcal{C} : A(\Gamma) \rightarrow X$ be a coloring of $\Gamma$ and let $\theta \in Z_3^0(X; A)$.

Let $i \in I_+^3 \sqcup I_-^3$. Let $R$ be the source region of the crossing $c$, i.e., the quadrant from which all co-orientations of the top arc and the middle arc point outwards. Let $R'$ be the opposite region of $R$ with respect to the top arc. The (Boltzman) weight $B_\theta(i, \mathcal{C})$ at $i$ with respect to $\mathcal{C}$ is defined by

$$
B_\theta(i, \mathcal{C}) = \theta(x_1, x_2, x_3)^{\epsilon_{tm}(i)\epsilon_b(i)},
$$

where $x_2$ and $x_3$ are the colors of the middle arc and the top semi-arc facing $R$, respectively, and $x_1$ is the color of the bottom semi-arc which is in $R$ or $x_1$ is the element with $x_1 = a \ast \frac{x_3}{x_2}$, where $a$ is the color of the bottom semi-arc which is in $R'$. See Figure [10]

**Definition 4.4.** Let $\Gamma$ be a marked graph diagram of an oriented surface-link $\mathcal{L}$. The partition function or state-sum of $\Gamma$ (associated to $\theta$) is

$$
\Phi_\theta(\Gamma) = \sum_{\mathcal{C} \in \text{Col}_X(\Gamma)} \left( \prod_{i \in I_+^3} B_\theta(i, \mathcal{C}) \prod_{j \in I_-^3} B_\theta(j, \mathcal{C})^{-1} \right).
$$

13
Theorem 4.5. Let $\mathcal{L}$ be an oriented surface-link and $\Gamma$ a marked graph diagram of $\mathcal{L}$. Then for any $\theta \in \mathbb{Z}^3(\mathbb{X}; A)$, $\Phi_\theta(\mathcal{L}) = \Phi_\theta(\Gamma)$.

Proof. Let $\mathcal{B} = \mathcal{B}(\Gamma)$ be a broken surface diagram associated to $\Gamma$. It is sufficient to prove that $\Phi_\theta(\Gamma) = \Phi_\theta(\mathcal{B})$.

Since there is a natural bijection between $\text{Col}_\mathcal{X}(\Gamma)$ and $\text{Col}_\mathcal{X}(\mathcal{B})$ (as in the proof of Theorem 4.2), it suffices to show the following claim.

Claim: For each coloring $C \in \text{Col}_\mathcal{X}(\Gamma)$,

$$\prod_{i \in I_+^3} B_\theta(i, C) \prod_{j \in I_-^3} B_\theta(j, C)^{-1} = \prod_{\tau \in T(\mathcal{B})} B_\theta(\tau, C),$$

where $C \in \text{Col}_\mathcal{X}(\mathcal{B})$ in the right hand side is the corresponding coloring.

Proof of Claim. Let $\mathcal{B}_i^j = \mathcal{B} \cap (\mathbb{R}^2 \times [t'_j, t_i])$ for $i = 1, \ldots, r$ and $j = 1, \ldots, s$. Let $\phi : (\mathbb{R}^2, \Gamma_0) \to (\mathbb{R}^2 \times [t'_1, t_1], \mathcal{B}_1^1)$ be the natural embedding at $t = 0$ as in Figure 11. The vertices of $\Gamma_0$ correspond to the saddle points in $\mathcal{B}_1^1$ and the crossings of $\Gamma_0$ correspond to the intersection of $\mathbb{R}^2 \times \{0\}$ and the double point curves in $\mathcal{B}_1^1$. There are no triple points in $\mathcal{B}_1^1$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10.png}
\caption{A (Boltzmann) weight at $i \in I_+^3 \cap I_-^3$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11.png}
\caption{$\phi : (\mathbb{R}^2, \Gamma_0) \to (\mathbb{R}^2 \times [t'_1, t_1], \mathcal{B}_1^1)$}
\end{figure}
Let $B_i = B \cap (\mathbb{R}^2 \times [t_i, t_{i+1}])$ for $i = 1, \ldots, r-1$ and $B_j' = B \cap (\mathbb{R}^2 \times [t'_{j+1}, t'_j])$ for $j = 1, \ldots, s-1$. Note that $T(B) = \left( \bigcup_{i=1}^{r-1} T(B_i) \right) \cup \left( \bigcup_{j=1}^{s-1} T(B_j') \right)$.

If the move $D_i \rightarrow D_{i+1}$ is an ambient isotopy of $\mathbb{R}^2$, then $D_i \times [t_i, t_{i+1}] \cong B_i$, and there are no triple points in $B_i$.

Suppose that the move $D_i \rightarrow D_{i+1}$ is a Reidemeister move. Since $D_i \setminus B(i)$ and $D_i+1 \setminus B(i)$ are identical, there are no triple points in $B_i \setminus M(i)$ and we have $T(B_i) = T(M(i))$, where $M(i)$ is a subset of $B(i) \times I$ determined by $\pi(M(i)) \cap (B(i) \times \{t\}) = \pi(f_1(t)(L(D_i))) \cap B(i)$ for $t \in I$ and a homeomorphism $f_1(t) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfying $f_1(t) = \text{id}$ and $f_1(t)(L(D_i)) = L(D_{i+1})$.

If the move $D_i \rightarrow D_{i+1}$ is of type $R_1$ or $R_2$, then there are no triple points in $M(i)$. See Figures 5 and 6.

If the move $D_i \rightarrow D_{i+1}$ is of type $R_3$, then there is a triple point $\tau_i$ in $M(i)$ as in Figure 12 and $T(B_i) = \{ \tau_i \}$. Then $\bigcup_{i=1}^{r-1} T(B_i) = \{ \tau_i \mid i \in I_+^3 \}$.

![Figure 12: A (Boltzman) weight](image-url)
Similarly, suppose that the move $D'_j \to D'_{j+1}$ is a Reidemeister move and $M'_{(j)}$ is a subset of $B'_{(j)} \times I$ determined by $\pi(M'_{(j)} \cap (B'_{(j)} \times \{t\})) = \pi(g_{(j)}(L(D'_j))) \cap B'_{(j)}$, where $g_{(j)} : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism satisfying $g_0^{(j)} = \text{id}$ and $g_1^{(j)}(L(D'_j)) = L(D'_{j+1})$ for $t \in I$. There is a triple point $\tau'_j \in M'_{(j)}$ for $j \in I^3$. We have that $\bigcup_{j=1}^{s-1} T(B'_j) = \{\tau'_j \mid j \in I^3\}$.

Now we have

$$T(B) = \{\tau_i \mid i \in I^3_+\} \cup \{\tau'_j \mid j \in I^3_\pm\}. \quad (4.3)$$

Let $i \in I^3_+$, i.e., $D_i \to D_{i+1}$ is a move of type $R_3$ and let $\tau_i$ be the triple point in $M_{(i)}$. Let $n_1$, $n_2$ and $n_3$ be the co-orientations of the bottom, the middle and the top arcs of $D_i$ in $B_{(i)}$, respectively. By an ambient isotopy, we deform $M_{(i)}$ in $B_{(i)} \times I$ to the standard form of the neighborhood of the triple point $\tau_i$ as in Figure 12. Let $\bar{n}_1$, $\bar{n}_2$, and $\bar{n}_3$ be the normal vectors corresponding to $n_1$, $n_2$, and $n_3$, respectively. Without loss of generality, we may assume $\bar{n}_3 = e_1$, $\bar{n}_2 = e_2$ and $\bar{n}_1 = \varepsilon \varepsilon' e_3$ for some $\varepsilon, \varepsilon' \in \{1, -1\}$. Here $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. See Figure 12.

Let $c$ be the crossing between the top and the middle arcs in $B_{(i)}$. It is clear from Figure 12 that $\varepsilon = \text{sign}(c)$. By (4.1), $\varepsilon = \text{sign}(c) = \varepsilon_{tm}(i)$. Hence $\varepsilon = \varepsilon_{tm}(i)$.

The sign $\varepsilon'$ depends on the co-orientation $n_1$ of the bottom arc. If $n_1$ points from $c$, then $\varepsilon' = 1$. If $n_1$ points toward $c$, then $\varepsilon' = -1$. So, by (4.2),

$$\varepsilon' = \varepsilon_b(i) \quad \text{and} \quad \varepsilon = \varepsilon_{tm}(i) \varepsilon_b(i). \quad (4.4)$$

On the other hand, by definition, the sign $\varepsilon(\tau_i)$ of the triple point $\tau_i$ is positive if the co-orientations of the top, the middle and the bottom sheets in this order match the given (right-handed) orientation of $\mathbb{R}^3$. Otherwise, the sign $\varepsilon(\tau_i)$ is negative. This gives

$$\varepsilon(\tau_i) = \begin{cases} 1 & \text{if } (\bar{n}_3, \bar{n}_2, \bar{n}_1) \in A, \\ -1 & \text{if } (\bar{n}_3, \bar{n}_2, \bar{n}_1) \in B, \end{cases}$$

where $A = \{(e_1, e_2, e_3), (e_1, -e_2, -e_3)\}$ and $B = \{(e_1, -e_2, e_3), (e_1, e_2, -e_3)\}$. Therefore for each $i \in I^3_+$,

$$\varepsilon(\tau_i) = \varepsilon_{tm}(i) \varepsilon_b(i). \quad (4.4)$$

Let $j \in I^3_-$, i.e., $D'_j \to D'_{j+1}$ is a move of type $R_3$. Let $\tau'_j$ be the triple point in $M'_{(j)}$. Let $n_1$, $n_2$ and $n_3$ be the co-orientations of the bottom, the middle and the top arcs of $D'_j$ in $B'_{(j)}$, respectively. By an ambient isotopy,
we deform $M'_j$ to the standard form of the neighborhood of the triple point $\tau'_j$. Let $\bar{n}_1$, $\bar{n}_2$, and $\bar{n}_3$ be the co-orientations corresponding to $n_1$, $n_2$, and $n_3$, respectively. Without loss of generality, we may assume $\bar{n}_3 = e_1$, $\bar{n}_2 = e_2$ and $\bar{n}_1 = e' e_3$ for some $\epsilon, \epsilon' \in \{1, -1\}$.

Let $c$ be the crossing between the top and the middle arcs in $B'_j$. It is easily seen that $\epsilon = \text{sign}(c)$ (cf. Figure 12). By (4.1), $\epsilon = \text{sign}(c) = \epsilon_{tm}(j)$. Hence $\bar{n}_2 = \epsilon_{tm}(j)e_2$.

The sign $\epsilon'$ depends on the co-orientation $n_1$ of the bottom arc. If $n_1$ points from $c$, then $\epsilon' = -1$. If $n_1$ points toward $c$, then $\epsilon' = 1$. So, by (4.2), $\epsilon' = -\epsilon_b(j)$ and hence $\bar{n}_1 = -\epsilon_b(j)e_3$.

On the other hand, by definition, $\epsilon(\tau'_j) = 1$ if $\bar{n}_3, \bar{n}_2, \bar{n}_1$ in this order match the given (right-handed) orientation of $\mathbb{R}^3$. Otherwise, $\epsilon(\tau'_j) = -1$. This gives

$$\epsilon(\tau'_j) = \begin{cases} 1 & \text{if } (\bar{n}_3, \bar{n}_2, \bar{n}_1) \in B, \\ -1 & \text{if } (\bar{n}_3, \bar{n}_2, \bar{n}_1) \in A. \end{cases}$$

Therefore for each $j \in I^3$,

$$\epsilon(\tau'_j) = -\epsilon_{tm}(j)\epsilon_b(j). \quad (4.5)$$

We show that for each $i \in I^3_+$, $B_\theta(i,C) = B_\theta(\tau_i,C)$ and that for each $j \in I^3_-$, $B_\theta(j,C) = B_\theta(\tau'_j, C)^{-1}$.

Let $i \in I^3_+$ (or $j \in I^3_-$). There are two cases: The bottom arc meets the source region of the crossing $c$ or not (see Figure 12). In this proof, we denote $M_i$ (or $M'_j$) by $M$ and $[t_i, t_{i+1}]$ (or $[t'_j, t'_{j+1}]$) by $I$.

**Case I**: Consider $i \in I^3_+$ (or $j \in I^3_-$) such that the bottom arc in $B_{(i)}$ (or $B'_{(j)}$) hits the source region $R$ of $c$.

The top (or the middle, resp.) sheet in $M$ corresponds to the top (or the middle, resp.) arc times $I$. As shown in Figure 12, $R \times I$ is divided into two (3-dimensional) regions by the bottom sheet whose color is $x_1$. One of them is the source region $R$ of the triple point $\tau_i$ (or $\tau'_j$). The colors of the top arc and the middle arc facing the source region $R$ of $c$ are the colors of the top and the middle sheets facing $R$. From the equalities (4.4) and (4.5), we see that $B_\theta(i,C) = \theta(x_1, x_2, x_3)^{\epsilon_{tm}(i)\epsilon_b(i)} = \theta(x_1, x_2, x_3)^{\epsilon(\tau_i)} = B_\theta(\tau_i, C)$ and $B_\theta(j,C) = \theta(x_1, x_2, x_3)^{\epsilon_{tm}(j)\epsilon_b(j)} = \theta(x_1, x_2, x_3)^{-\epsilon(\tau'_j)} = B_\theta(\tau'_j, C)^{-1}$.

**Case II**: Consider $i \in I^3_+$ (or $j \in I^3_-$) such that the bottom arc in $B_{(i)}$ (or $B'_{(j)}$) does not meet the source region $R$ of $c$.

Similar to the case I, the second and third coordinates of $B_\theta(i,C)$ (or $B_\theta(j,C)$) are the same as those of $B_\theta(\tau_i, C)$ (or $B_\theta(\tau'_j, C)$). As illustrated in
Figure 12 R' × I is divided into two (3-dimensional) regions by the bottom sheet whose color is a, where a is the color of the bottom arc in R'. Since the co-orientation of the top sheet is from R × I to R' × I, a = a × a. Thus x_1 = a × a. Therefore B_θ(i, C) = θ(x_1, x_2, x_3)^ε_m(i)ε_s(i) = θ(x_1, x_2, x_3)^ε(τ_i) = B_θ(τ_i, C) and B_θ(j, C) = B_θ(τ'_j, C)^−1.

This completes the proof of Claim and hence the proof of Theorem 4.5.

Example 4.6. We consider the oriented marked graph diagram 10_2 of the 2-twist spun trefoil L in Figure 15. Let

θ = χ_{0,1,0}−1χ_{0,2,0}−1χ_{0,0,2}−1χ_{1,0,1}χ_{0,2,2}χ_{2,0,2}χ_{2,1,2} ∈ Z_3^3(R_3; Z_3),

where χ_{x,y,z}(a,b,c) = u if (x,y,z) = (a,b,c), χ_{x,y,z}(a,b,c) = 1 otherwise, and Z_3 = ⟨u | u^3 = 1⟩ is the cyclic group of order 3. Consider sequences of link diagrams from the positive and negative resolutions to trivial link diagrams are as shown in Figures 13 and 14 respectively. Then I^3_+ = φ and I^3_− = {2, 3, 4, 5, 8, 10}. The (Boltzmann) weights are B_θ(2, C) = θ(y, y, x) = 1, B_θ(3, C) = θ(x, y, x)^−1 = 1, B_θ(4, C) = θ(x, x, y, x)^−1 = 1, B_θ(5, C) = θ(x, y, x)^−1 = 1, B_θ(8, C) = θ(y, x, y) and B_θ(10, C) = θ(x, y, x, x, y) for x, y ∈ R_3. Therefore

Φ_θ(L) = \sum_{(x,y) ∈ R_3 × R_3} θ(x, y, x, x, y, x, y) θ(y, x, y, x, y, x, y, x, y)^−1

= 3 + 6u.

This matches the computation in [3].

For a surface-link L, the ch-index χ(L) is defined by min_{Γ} χ(Γ), where Γ is a marked graph diagram presenting L and χ(Γ) is the sum of the number of crossings of Γ and that of vertices of Γ.

Example 4.7. Let L be an oriented surface-link with χ(L) ≤ 10 presented by a marked graph diagram in Figure 15 (see [32], for more details).

Let R_3 be the dihedral quandle of order 3 and θ the 3-cocycle in Example 4.6. Let S_4 be the tetrahedral quandle in Example 3.1 and let η =

χ_{0,1,0}χ_{0,2,1}χ_{0,0,2}χ_{1,0,1}χ_{0,3,2}χ_{1,0,3}χ_{1,1,0}χ_{2,0,3}χ_{1,2,0}χ_{0,1,3}χ_{1,2,3}χ_{2,1,3}χ_{2,3,2}

in Z_4^3(S_4; Z_2), where χ_{x,y,z}(a,b,c) = t if (x, y, z) = (a, b, c), χ_{x,y,z}(a,b,c) = 1 otherwise, and Z_2 = ⟨t | t^2 = 1⟩. Then Φ_θ(L) and Φ_η(L) are as in the table below.
Figure 13: A sequence of link diagrams for positive resolution of $10_2$

Figure 14: A sequence of link diagrams for negative resolution of $10_2$
Figure 15: Oriented marked graph diagrams $\Gamma$ with $\chi(\Gamma) \leq 10$

\begin{table}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$\mathcal{L}$ & $\Phi_\theta(\mathcal{L})$ & $\Phi_\eta(\mathcal{L})$ & $\mathcal{L}$ & $\Phi_\theta(\mathcal{L})$ & $\Phi_\eta(\mathcal{L})$ \\
\hline
$0_1$ & 3 & 4 & $10_2$ & 3 & 4+6u \\
$2^1_1$ & 3 & 4 & $10_3$ & 3 & 4+12t \\
$6^{0,1}_1$ & 3 & 4 & $10^{1,1}_1$ & 9 & 16 \\
$8_1$ & 9 & 16 & $10^{0,1,0,1}_1$ & 3 & 4 \\
$8^{1,1}_1$ & 3 & 4 & $10^{2,1}_1$ & 3 & 4 \\
$9_1$ & 9 & 16 & $10^{2,1}_1$ & 3 & 4 \\
$9^{0,1}_1$ & 3 & 4 & $10^{0,1,0,1}_1$ & 9 & 16 \\
$10_1$ & 3 & 4 & $10^{0,1,0,1}_1$ & 9 & 16 \\
\hline
\end{tabular}
\end{table}

Table: $\Phi_\theta(\mathcal{L})$ and $\Phi_\eta(\mathcal{L})$ with $\chi(\mathcal{L}) \leq 10$
In [32], K. Yoshikawa introduced the notion of a marked graph diagram of \textit{triangle type}. It is seen that the quandle cocycle invariant $\Phi_{\theta}(L)$ of an oriented surface-link $L$ presented by a marked graph diagram of triangle type is equal to $\#\text{Col}_X(L)$ for any finite quandle $X$ and any 3-cocycle $\theta \in Z^3_Q(X; A)$, where $\#\text{Col}_X(L)$ denotes the cardinality of the set $\text{Col}_X(L)$.

In [30], M. Soma gave an enumeration of surface-links presented by marked graph diagrams of \textit{square type}; $A_n, B_n, C_n, D_n, E_n, F_n, G_n, H_n$ and $I$ (See [22], [30, Theorems 1.1 and 1.2]). We remark that surface-links presented by marked graph diagrams $A_n$ and $B_n$ are orientable for all $n \geq 2$, and surface-links presented by marked graph diagrams $E_n, F_n, G_n, H_n$ and $I$ are also orientable for all odd integers $n \geq 3$. See Figure 16.

We observe that for any finite quandle $X$ and $\theta \in Z^3_Q(X; A)$, the quandle cocycle invariant $\Phi_{\theta}(L)$ of an oriented surface-link $L$ presented by a marked graph diagram of square type is equal to $\#\text{Col}_X(L)$ except for the surface-link presented by $G_n$.

![Figure 16: Marked graph diagrams of square type](image-url)
For an oriented surface-link \( L \), we denote the same surface-link as \( L \) but with the opposite orientations on all the components of \( L \) by \( -L \). An oriented surface-link \( L \) is said to be \textit{invertible} if it is equivalent to \( -L \); otherwise \textit{non-invertible}. The quandle cocycle invariant provides a diagrammatic method of detecting non-invertibility of surface-links (cf. [8 Section 3]).

**Theorem 4.8.** For every integer \( k \geq 0 \), the oriented surface-links presented by marked graph diagrams \( G_{18k+3} \) and \( G_{18k+15} \) in Figure 16 are non-invertible.

**Proof.** Let \( X \) be the dihedral quandle of order 3 and \( \theta \in \mathbb{Z}_4^3(X; A) \) the 3-cocycle in Example 4.6. Let \( L \) be the oriented surface-link presented by the oriented marked graph diagram \( G_n \) in Figure 16. Then \( \Phi_\theta(L) = 3 + 6u \) and \( \Phi_\theta(-L) = 3 + 6u^2 \) if \( n = 18k+3 \), and \( \Phi_\theta(L) = 3 + 6u^2 \) and \( \Phi_\theta(-L) = 3 + 6u \) if \( n = 18k + 15 \) for any integer \( k \geq 0 \). This shows that \( L \) and \( -L \) are not equivalent for any \( k \geq 0 \) and completes the proof. \( \square \)

On the other hand, it is shown that for every integer \( m \geq 1 \), the oriented surface-links presented by marked graph diagrams \( F_{2m+1} \) and \( H_{2m+1} \) in Figure 16 are all non-invertible [22 Theorem 7.4].

**5 Shadow quandle cocycle invariants of oriented surface-links**

In this section, we recall shadow quandle cocycle invariants of oriented surface-links (cf. [4]).

Let \( X \) be a quandle and let \( \mathcal{B} \) be a broken surface diagram of an oriented surface-link \( L \). Let \( S(\mathcal{B}) \) be the set of sheets of \( \mathcal{B} \) and \( R(\mathcal{B}) \) be the set of the complementary regions of \( \mathcal{B} \) in \( \mathbb{R}^3 \). Let \( C : S(\mathcal{B}) \to X \) be a coloring of \( \mathcal{B} \). A \textit{shadow coloring} of \( \mathcal{B} \) (extending a given coloring \( C \)) is a map \( \tilde{C} : S(\mathcal{B}) \cup R(\mathcal{B}) \to X \) satisfying the conditions:

- The restriction of \( \tilde{C} \) to \( S(\mathcal{B}) \) is a given coloring \( C \).
- If two adjacent regions \( f_1 \) and \( f_2 \) are separated by a sheet \( e \) and the co-orientation of \( e \) points from \( f_1 \) to \( f_2 \), then \( \tilde{C}(f_1) \circ \tilde{C}(e) = \tilde{C}(f_2) \).

Let \( \text{Col}_X^S(\mathcal{B}) \) be the set of all shadow colorings of \( \mathcal{B} \) by \( X \).

Let \( \tilde{C} \) be a shadow coloring of \( \mathcal{B} \). Let \( \tau \) be a triple point and let \( \mathcal{R} \) be the source region of \( \tau \). Let \( \theta \in \mathbb{Z}_4^3(X; A) \). Define the \textit{shadow (Boltzman) weight} at \( \tau \) by

\[
B_\theta^S(\tau, \tilde{C}) = \theta(y, x_1, x_2, x_3)^{\epsilon(\tau)},
\]
where $\epsilon(\tau)$ is the sign of $\tau$, $y$ is the color of $\mathcal{R}$ and $x_1, x_2$ and $x_3$ are the colors of the bottom, the middle and the top sheets facing $\mathcal{R}$, respectively. See Figure 17 for $\epsilon(\tau) = 1$. The shadow partition function of $\mathcal{B}$ (associated to $\theta$) is defined by

$$\Phi^s_\theta(\mathcal{B}) = \sum_{\tilde{C} \in \text{Col}^s_X(\mathcal{B})} \prod_{\tau \in \mathcal{T}(\mathcal{B})} B^s_\theta(\tau, \tilde{C}) \in \mathbb{Z}[A].$$

![Figure 17: Shadow (Boltzman) weight at $\tau$ with $\epsilon(\tau) = 1$](image)

**Theorem 5.1** ([4]). Let $\mathcal{B}$ be a broken surface diagram of an oriented surface-link $\mathcal{L}$. The shadow partition function $\Phi^s_\theta(\mathcal{B})$ does not depend on the choice of a broken surface diagram. Thus it is an invariant of $\mathcal{L}$.

We denote $\Phi^s_\theta(\mathcal{B})$ by $\Phi^s_\theta(\mathcal{L})$ and call it a shadow quandle cocycle invariant of $\mathcal{L}$ associated to $\theta \in Z^4_Q(X; A)$.

There is a generalized version of the shadow quandle cocycle invariant.

Let $X$ be a quandle. The associated group, $G_X$, of $X$ is $(x \in X; x * y = y^{-1}xy \ (x, y \in X))$. An $X$-set is a set $Y$ equipped with a right action of the associated group $G_X$. We denote by $y * g$ the image of an element $y \in Y$ by the action $g \in G_X$.

Let $X$ be a quandle and $Y$ an $X$-set. For each positive integer $n$, let $C^n_R(X)_Y$ be the free abelian group generated by the elements $(y, x_1, \ldots, x_n)$ where $y \in Y$ and $x_1, \ldots, x_n \in X$. Let $C_0(X)_Y = \mathbb{Z}(Y)$, the free abelian group on $Y$, and let $C^n_R(X)_Y$ be $\{0\}$ for $n < 0$. Define a homomorphism
Let $\partial_n : C^R_n(X)_Y \rightarrow C^R_{n-1}(X)_Y$ be

$$
\partial_n(y, x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i \left[ (y, x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - (y \ast x_i, x_1 \ast x_i, x_2 \ast x_i, \ldots, x_{i-1} \ast x_i, x_{i+1}, \ldots, x_n) \right]
$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$. Then $C^R_n(X)_Y = \{C^R_n(X)_Y, \partial_n\}$ is a chain complex. This chain complex is due to R. Fenn, C. Rourke and B. Sanderson (\cite{sands}). Let $D^Q_n(X)_Y$ be the subgroup of $C^R_n(X)_Y$ generated by $(y, x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \ldots, n-1\}$ if $n \geq 2$; otherwise let $D^Q_n(X)_Y = \{0\}$. Then $C^D_n(X)_Y = \{D^Q_n(X)_Y, \partial_n\}$ is a subcomplex of $C^R_n(X)_Y$. Put $C^Q_n(X)_Y = C^R_n(X)_Y / D^Q_n(X)_Y$, and consider the quotient chain complex $C^Q_n(X)_Y = \{C^Q_n(X)_Y, \partial_n\}$. For an abelian group $A$, we define chain and cochain complexes by $C^Q_n(X, A)_Y = C^Q_n(X)_Y \otimes A$ and $C^Q_*(X, A)_Y = \text{Hom}(C^Q_*(X)_Y, A)$. The homology and cohomology groups are denoted by $H^Q_*(X, A)_Y$ and $H^Q_*(X, A)_Y$, respectively. For more details, see \cite{[17]}.

Let $X$ be a quandle, $Y$ an $X$-set and let $B$ be a broken surface diagram of an oriented surface-link $L$. Let $S(B)$ be the set of sheets of $B$ and $R(B)$ the set of the complementary regions of $B$ in $\mathbb{R}^3$. Let $\mathcal{C} : S(B) \rightarrow X$ be a coloring of $B$. A shadow coloring of $B$ (extending a given coloring $\mathcal{C}$) by $(X, Y)$ is a map $\tilde{\mathcal{C}} : S(B) \cup R(B) \rightarrow X \cup Y$ satisfying the conditions:

- $\tilde{\mathcal{C}}(S(B)) \subset X$ and $\tilde{\mathcal{C}}(R(B)) \subset Y$.
- The restriction of $\tilde{\mathcal{C}}$ to $S(B)$ is a given coloring $\mathcal{C}$.
- If two adjacent regions $f_1$ and $f_2$ are separated by a sheet $e$ and the co-orientation of $e$ points from $f_1$ to $f_2$, then $\tilde{\mathcal{C}}(f_1) \ast \tilde{\mathcal{C}}(e) = \tilde{\mathcal{C}}(f_2)$.

We denote by $\text{Col}^S_{(X, Y)}(B)$ the set of all shadow colorings of $B$ by $(X, Y)$.

**Proposition 5.2** (cf. \cite{[4]}). If $B$ and $B'$ present equivalent oriented surface-links, then there is a bijection between $\text{Col}^S_X(B)$ and $\text{Col}^S_X(B')$, and there is a bijection between $\text{Col}^S_{(X, Y)}(B)$ and $\text{Col}^S_{(X, Y)}(B')$.

Let $\tilde{\mathcal{C}}$ be a shadow coloring of a broken surface diagram $B$ by $(X, Y)$. Let $\tau$ be a triple point and let $R$ be the source region of $\tau$. Let $\theta \in Z^3_0(X; A)_Y$. Define the shadow (Boltzman) weight at $\tau$ by

$$
B^S_\theta(\tau, \tilde{\mathcal{C}}) = \theta(y, x_1, x_2, x_3)^{\epsilon(\tau)},
$$

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where $\epsilon(\tau)$ is the sign of $\tau$, $y$ is the color of $R$ and $x_1, x_2$ and $x_3$ are the colors of the bottom, the middle and the top sheets facing $R$, respectively. See Figure 17. The shadow partition function of $B$ (associated to $\theta$) is defined by

$$
\Phi_\theta^S(B) = \sum_{\tilde{C} \in \text{Col}_{(X,Y)}^S(\Gamma) \cap \tau(B)} \prod_{\tau \in T(B)} B_\theta^S(\tau, \tilde{C}) \in \mathbb{Z}[A].
$$

**Theorem 5.3** (cf. [4]). Let $B$ be a broken surface diagram of an oriented surface-link $L$. The shadow partition function $\Phi_\theta^S(B)$ does not depend on the choice of a broken surface diagram. Thus it is an invariant of $L$.

We denote $\Phi_\theta^S(B)$ by $\Phi_\theta^S(L)$ and call it a shadow quandle cocycle invariant of $L$ associated to $\theta \in \mathbb{Z}_3^3(X; A)_Y$.

### 6 How to compute shadow quandle cocycle invariants from marked graph diagrams

In this section we give a method of computing shadow quandle cocycle invariants from marked graph diagrams.

Let $\Gamma$ be a marked graph diagram of an oriented surface-link $L$. Let $A(\Gamma)$ be the set of arcs of $\Gamma$ and $R(\Gamma)$ the set of complementary regions of $\Gamma$ in $\mathbb{R}^2$. Let $X$ be a quandle and let $Y$ be an $X$-set. Let $C: A(\Gamma) \to X$ be a coloring of $\Gamma$ by a quandle $X$. A shadow coloring of $\Gamma$ (extending a given coloring $C$) by $X$ (or by $(X,Y)$, resp.) is a map $\tilde{C}: A(\Gamma) \cup R(\Gamma) \to X$ (or a map $\tilde{C}: A(\Gamma) \cup R(\Gamma) \to X \cup Y$, resp.) satisfying the conditions (2) and (3) (or the conditions (1)–(3), resp.):

1. $\tilde{C}(A(\Gamma)) \subset X$ and $\tilde{C}(R(\Gamma)) \subset Y$.
2. The restriction of $\tilde{C}$ to $A(\Gamma)$ is a given coloring $C$.
3. If two adjacent regions $f_1$ and $f_2$ are separated by an arc $e \in A(\Gamma)$ and the co-orientation of $e$ points from $f_1$ to $f_2$, then $\tilde{C}(f_1) \ast \tilde{C}(e) = \tilde{C}(f_2)$.

Let $\text{Col}_{X}^S(\Gamma)$ (or $\text{Col}_{(X,Y)}^S(\Gamma)$, resp.) denote the set of all shadow colorings of $\Gamma$ by $X$ (or by $(X,Y)$, resp.).

**Theorem 6.1.** Let $\Gamma$ be a marked graph diagram of an oriented surface-link $L$ and $B = B(\Gamma)$ an associated broken surface diagram of $\Gamma$. There is a bijection from $\text{Col}_{X}^S(\Gamma)$ to $\text{Col}_{X}^S(B)$, and a bijection from $\text{Col}_{(X,Y)}^S(\Gamma)$ to $\text{Col}_{(X,Y)}^S(B)$. 

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Proof. Consider a shadow coloring of $\mathcal{B}$. The 0-level cross-section with the colors induced by the shadow coloring of $\mathcal{B}$ is a shadow coloring of $\Gamma$. By the same argument as in [1], we see that this gives a bijection from $\text{Col}^S_X(\Gamma)$ to $\text{Col}^S_X(\mathcal{B})$ and a bijection $\text{Col}^S_{(X,Y)}(\Gamma)$ to $\text{Col}^S_{(X,Y)}(\mathcal{B})$. □

Corollary 6.2. If $\Gamma$ and $\Gamma'$ present equivalent oriented surface-links, then there is a bijection from $\text{Col}^S_X(\Gamma)$ to $\text{Col}^S_X(\Gamma')$, and there is a bijection from $\text{Col}^S_{(X,Y)}(\Gamma)$ to $\text{Col}^S_{(X,Y)}(\Gamma')$.

Proof. Let $\mathcal{B}(\Gamma)$ and $\mathcal{B}(\Gamma')$ be broken surface diagrams associated to $\Gamma$ and $\Gamma'$, respectively. By Proposition 5.2 and Theorem 6.1, we see the result. □

Let $\Gamma$ be a marked graph diagram of an oriented surface-link $L$ and let $\Gamma_+ = D_1 \to D_2 \to \cdots \to D_r = O$, $\Gamma_- = D'_1 \to D'_2 \to \cdots \to D'_s = O'$, $\epsilon_{tm}$ and $\epsilon_b$ be as in Section 4. Let $\tilde{C} : A(\Gamma) \cup R(\Gamma) \to X$ or $\tilde{C} : A(\Gamma) \cup R(\Gamma) \to X \cup Y$ be a shadow coloring of $\Gamma$. Let $i \in I^+ \ (\text{or} \ j \in I^-)$. Let $R$ be the source region of the crossing $c$ between the top arc and the middle arc in $D_i \cap B(i)$ (or $D'_j \cap B'(j)$). Let $R'$ be the opposite region of $R$ with respect to the top arc.

Let $x_1$, $x_2$ and $x_3$ be as in Section 4. There are two cases, the bottom arc intersects with the source region $R$ or not. If not, we consider two cases, $\epsilon_b(i) = 1$ or $\epsilon_b(i) = -1$ (See Figure 18). In the case where the bottom arc hits the source region $R$, the region $R$ is divided by the bottom arc. Let $y$ be the color of the divided region of $R$ such that the co-orientation of the bottom arc points from that region. In the case where the bottom arc does not intersect with the source region $R$ and $\epsilon_b(i) = 1$, let $y$ be the element $s = \tilde{C}(R)$. In the case where the bottom arc does not meet the source region $R$ and $\epsilon_b(i) = -1$, let $y$ be the element $s \ast \overline{x_1}$, where $s = \tilde{C}(R)$. For $j \in I^- \cup I^+$, let $y$ be the element defined in the same way with $i \in I^+$.

Definition 6.3. Let $\theta \in Z^4_Q(X; A)$ be a 4-cocycle or let $\theta \in Z^3_Q(X; A)_Y$ be a 3-cocycle. The shadow (Boltzman) weight for $i \in I^+ \cup I^-$ is defined by

$$B^S_\theta(i; \tilde{C}) = \theta(y, x_1, x_2, x_3)^{\epsilon_{tm}(i)\epsilon_b(i)}.$$
Definition 6.4. Let $\Gamma$ be a marked graph diagram of an oriented surface-link $L$. The shadow partition function of $\Gamma$ (associated to $\theta$) is defined by

$$
\Phi^\theta_s(\Gamma) = \sum_{\tilde{C}} \left( \prod_{i \in I_+^3} B^S_\theta(i, \tilde{C}) \prod_{j \in I_-^3} B^S_\theta(j, \tilde{C})^{-1} \right),
$$

where $\tilde{C}$ runs all shadow colorings of $\Gamma$ by $X$ when $\theta \in Z_Q^4(X; A)$ or all shadow colorings of $\Gamma$ by $(X, Y)$ when $\theta \in Z_Q^3(X; A)_Y$.

Figure 18: The triple point $\tau_i$
**Theorem 6.5.** Let $\mathcal{L}$ be an oriented surface-link and let $\Gamma$ be a marked graph diagram presenting $\mathcal{L}$. For any $\theta \in Z^4_Q(X; A)$ or $\theta \in Z^3_Q(X; A)_Y$, $\Phi^\theta_\mathcal{L}(\mathcal{L}) = \Phi^\theta_\mathcal{L}(\Gamma)$.

*Proof.* Let $\mathcal{B} = \mathcal{B}(\Gamma)$ be a broken surface diagram associated to $\Gamma$. It suffices to show that $\Phi^\theta_\mathcal{B}(\mathcal{B}) = \Phi^\theta_\mathcal{B}(\Gamma)$.

We define $M^{(i)}$ and $M^{(j)}$ for any $i \in I^3_+$ and $j \in I^3_+$ as in the proof of Theorem 4.5. We have that $T(\mathcal{B}) = \{\tau_i \mid i \in I^3_+\} \cup \{\tau_j^i \mid j \in I^3_+\}$, where $\tau_i$ is the triple point in $M^{(i)}$ for $i \in I^3_+$ and $\tau_j^i$ is the triple point in $M^{(j)}$ for $j \in I^3_+$.

Let $\tilde{\mathcal{C}}$ be a shadow coloring of $\mathcal{B}$ by $X$ when $\theta \in Z^4_Q(X; A)$ or a shadow coloring of $\mathcal{B}$ by $(X, Y)$ when $\theta \in Z^3_Q(X; A)_Y$, and let $\mathcal{C} \in \text{Col}_X(\mathcal{B})$ be the restriction of $\tilde{\mathcal{C}}$ to the set $S(\mathcal{B})$.

We show that $B^S_\theta(i, \tilde{\mathcal{C}}) = B^S_\theta(\tau_i, \tilde{\mathcal{C}})$ for each $i \in I^3_+$.

The exponents appearing in $B^S_\theta(i, \tilde{\mathcal{C}})$ and $B^S_\theta(\tau_i, \tilde{\mathcal{C}})$ are identical. The second, the third and the fourth coordinates of $B^S_\theta(\tau, \tilde{\mathcal{C}})$ are the same as $B_\theta(\tau, \mathcal{C})$ for every triple point $\tau$. Also, the second, third and fourth coordinates of $B^S_\theta(i, \tilde{\mathcal{C}})$ are the same as $B_\theta(i, \mathcal{C})$ for each $i \in I^3_+$. Combining these facts, the second, the third and the fourth coordinates of $B^S_\theta(i, \tilde{\mathcal{C}})$ are the same as those of $B^S_\theta(\tau_i, \tilde{\mathcal{C}})$ for any $i \in I^3_+$.

It remains to show that the first coordinate of $B^S_\theta(\tau_i, \tilde{\mathcal{C}})$ is identical with that of $B^S_\theta(i, \tilde{\mathcal{C}})$ for any $i \in I^3_+$. The first coordinate of $B^S_\theta(\tau_i, \tilde{\mathcal{C}})$ is the color of the source region $\mathcal{R}$ of the triple point $\tau_i$.

**Case I :** The bottom arc intersects with the source region $R$ of the crossing between the top and middle arc in $B^{(i)}$.

The top (or the middle, resp.) sheet corresponds to the top (or the middle, resp.) arc times $[t_i, t_{i+1}]$. Also, the quadrant between the top and middle sheets with the co-orientations outward is divided into two (3-dimensional) regions by a bottom sheet whose color is the same as that of the bottom arc in $R$. Therefore, the first coordinate of $B^S_\theta(\tau_i, \tilde{\mathcal{C}})$ is $\tilde{\mathcal{C}}(R)$, where $R$ is the source region of the crossing $c$ between the top and middle arcs in $B^{(i)}$. Therefore $B^S_\theta(i, \tilde{\mathcal{C}}) = B^S_\theta(\tau_i, \tilde{\mathcal{C}})$ for all $i \in I^3_+$.

**Case II :** The bottom arc does not hit the source region $R$ of the crossing between the top and middle arc in $B^{(i)}$.

Let $\tilde{\mathcal{C}}(R) = s$. The quadrant corresponding to $R \times [t_i, t_{i+1}]$ is divided into two (3-dimensional) regions by a bottom sheet whose color is $x_1 = a \ast \overline{x_3}$. If $\epsilon_b(i) = 1$, then the co-orientation of the bottom sheet in that quadrant is from the region which has a color $s$ (see the case II-1 in Figure 18). Therefore, the color $y$ of the source region $\mathcal{R}$ of the triple point $\tau_i$ is $s$. Otherwise, the
co-orientation of the bottom sheet in that quadrant points to the region whose color is \( s \) (see the case II-2 in Figure 13). In addition, the color of the bottom sheet in that quadrant is \( x_1 \). Thus the color \( y \) of \( \mathcal{R} \) is \( s * x_1 \). Therefore \( B^S(i, \mathcal{C}) = B^S(\tau_i, \mathcal{C}) \) for all \( i \in I^3_+ \).

For \( j \in I^2 \), it is similarly seen that \( B^S(j, \mathcal{C}) = B^S(\tau_j, \mathcal{C})^{-1} \).

Hence we have \( \Phi^S_0(B) = \Phi^S_0(\Gamma) \) for all \( j \in I^2 \). \( \square \)

7 Symmetric quandle cocycle invariants of unoriented surface-links

This section is devoted to recalling symmetric quandle cocycle invariants of unoriented surface-links (cf. [17, 18]).

Let \( X \) be a quandle. A map \( \rho : X \to X \) is a good involution if it is an involution (i.e., \( \rho \circ \rho = \text{id} \)) such that \( \rho(x \ast y) = \rho(x) \ast y \) and \( x \ast \rho(y) = x \ast y \) for any \( x, y \in X \). Such a pair \((X, \rho)\) is called a quandle with a good involution or a symmetric quandle.

**Example 7.1.** ([17, 18]) Let \( G \) be a group. The inversion, \( \text{inv}(G) : G \to G ; g \mapsto g^{-1} \), is a good involution of \( \text{conj}(G) \). We call \((\text{conj}(G), \text{inv}(G))\) the conjugation symmetric quandle.

The associated group, \( G_{(X, \rho)} \), of a symmetric quandle \((X, \rho)\) is \( G_{(X, \rho)} = \langle x \in X ; x * y = y^{-1}xy \ (x, y \in X), \ \rho(x) = x^{-1} \ (x \in X) \rangle \). An \((X, \rho)\)-set is a set \( Y \) equipped with a right action of the associated group \( G_{(X, \rho)} \). We denote by \( y * g \) the image of an element \( y \in Y \) by the action \( g \in G_{(X, \rho)} \).

Let \((X, \rho)\) be a symmetric quandle and \( Y \) an \((X, \rho)\)-set. Let \( C^R_n(X)_Y = \{ C^R_n(X)_Y, \partial_h \} \) be the chain complex of \( X \) with \( Y \), and \( C^D_n(X)_Y = \{ D^D_n(X)_Y, \partial_h \} \) be the sub-complex of \( C^R_n(X)_Y \) as in Section 5.

Let \( D^D_n(X)_Y \) be the subgroup of \( C^R_n(X)_Y \) generated by

\[
(y, x_1, \ldots, x_n) + (y * x_j, x_1 * x_j, \ldots, x_{j-1} * x_j, \rho(x_j), x_{j+1}, \ldots, x_n)
\]

for \( j = 1, \ldots, n \) if \( n \geq 2 \); otherwise let \( D^D_0(X)_Y = \{ 0 \} \).

Define \( C^Q_n(X)_Y \) to be \( C^R_n(X)_Y / (D^D_n(X)_Y + D^D_0(X)_Y) \), and we have the quotient complex \( C^Q_n(X)_Y = \{ C^Q_n(X)_Y, \partial_h \} \). For an abelian group \( A \), we define chain and cochain complexes by \( C^Q_n(X, A)_Y = C^Q_n(X)_Y \otimes A \) and \( C^Q_n(X, A)_Y = \text{Hom}(C^Q_n(X)_Y, A) \), respectively. The homology and cohomology groups are denoted by \( H^Q_n(X, A)_Y \) and \( H^Q_n(X, A)_Y \), respectively. For details, see [17, 18].
Let $B$ be an unoriented broken surface diagram. When we divide oversheets at the double curves, we call the sheets of the result semi-sheets of $B$. Each semi-sheet is a compact orientable surface in $\mathbb{R}^3$ (cf. [16]).

Consider an assignment of normal orientation and an element of $X$ to each semi-sheet of $B$. A basic inversion is an operation which reverses the normal orientation of a semi-sheet and changes the element $x$ assigned to the semi-sheet by $\rho(x)$. See Figure 19.

![Figure 19: A basic inversion](image)

We would rather use the terminology ‘normal orientation’ than ‘co-orientation’ when $B$ is an unoriented broken surface diagram.

An $(X, \rho)$-coloring of $B$ is the equivalence class of an assignment of a normal orientation and an element of $X$ to each semi-sheet of $B$ satisfying the coloring condition below. Here the equivalence relation is generated by basic inversions.

- By basic inversions, assume the normal orientations of semi-sheets around a double point curve to be as in Figure 20. Then $x_1 \ast x_3 = x_2$ and $x_3 = x_4$.

![Figure 20: Coloring conditions](image)

Let $Y$ be an $(X, \rho)$-set. An $(X, \rho)_Y$-coloring of $B$ is an $(X, \rho)$-coloring of $B$ with an assignment of an element of $Y$ to each complementary region of $B$ satisfying the following condition.
Suppose that adjacent regions $f_1$ and $f_2$ separated by a semi-sheet $e$ are labeled by $y_1$ and $y_2$. If the semi-sheet $e$ is labeled by $x$ and the normal orientation of $e$ points from $f_1$ to $f_2$, then $y_1 \star x = y_2$.

Proposition 7.2 ([17][18]). Let $(X, \rho)$ be a symmetric quandle and $Y$ an $(X, \rho)$-set. If two broken surface diagrams present equivalent unoriented surface-links, then there is a bijection between the sets of $(X, \rho)$-colorings of the broken surface diagrams, and there is a bijection between the sets of $(X, \rho)_Y$-colorings of them.

Let $B$ be an unoriented broken surface diagram. Fix an $(X, \rho)_Y$-coloring of $B$, say $\tilde{C}$. For a triple point $\tau$ of $B$, there are eight complementary regions of $B$ around $\tau$ (Some of them may be the same). Choose one of them, say $f$, which we call a specified region for $\tau$, and let $y$ be the label of $f$.

Let $e_1$, $e_2$ and $e_3$ be the bottom semi-sheet, the middle semi-sheet and the top semi-sheet at $\tau$, respectively, which face the region $f$. By basic inversions, we assume that the normal orientations $n_1$, $n_2$ and $n_3$ of them point from $f$ to the opposite regions. Let $x_1$, $x_2$ and $x_3$ be the labels of them, respectively. The sign of $\tau$ with respect to the region $f$ is $+1$ (or $-1$) if the triple of normal orientations $(n_3, n_2, n_1)$ does (or does not) match the orientation of $\mathbb{R}^3$. Let $\theta \in Z^3_{Q_{\rho}}(X, A)_Y$. The symmetric (Boltzmann) weight $B^{\text{sym}}_{\theta}(\tau, \tilde{C})$ of $\tau$ is defined to be

$$B^{\text{sym}}_{\theta}(\tau, \tilde{C}) = \theta(y, x_1, x_2, x_3)^{\epsilon(\tau)},$$

where $\epsilon(\tau)$ is the sign of $\tau$. See Figure 21.

![Symmetric Boltzmann weights](image.png)

Figure 21: Symmetric Boltzmann weights

The symmetric partition function of $B$ (associated to $\theta$) is defined by

$$\Phi^{\text{sym}}_{\theta}(B) = \sum_{\tilde{C}} \prod_{\tau \in T(B)} B^{\text{sym}}_{\theta}(\tau, \tilde{C}) \in \mathbb{Z}[A].$$

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where the sum is taken over all possible \((X, \rho)\)-colorings \(\tilde{C}\) of \(B\). (The value of \(B^\text{Sym}_\theta(\tau, \tilde{C})\) is in the coefficient group \(A\) written multiplicatively).

**Theorem 7.3** ([17], [18]). Let \(B\) be a broken surface diagram of an unoriented surface-link \(L\). The symmetric partition function \(\Phi^\text{Sym}_\theta(B)\) is an invariant of the unoriented surface-link \(L\).

We denote \(\Phi^\text{Sym}_\theta(B)\) by \(\Phi^\text{Sym}_\theta(L)\) and call it the symmetric quandle cocycle invariant of \(L\) associated to \(\theta\).

### 8 How to compute symmetric quandle cocycle invariants from marked graph diagrams

Let \(\Gamma\) be a marked graph diagram of an unoriented surface-link \(L\) and let \((X, \rho)\) be a symmetric quandle.

A semi-arc of \(\Gamma\) is a connected component of \(\Gamma \setminus (C(\Gamma) \cup V(\Gamma))\), where \(C(\Gamma)\) is the set of crossings and \(V(\Gamma)\) is the set of marked vertices of \(\Gamma\).

A basic inversion is an operation which reverses the normal orientation of a semi-arc and changes the element \(x\) assigned to the semi-arc by \(\rho(x)\). See Figure 22.

\[
\begin{array}{c}
\xrightarrow{\Leftrightarrow} x \\
\end{array} = \begin{array}{c}
\xrightarrow{\rho(x)}
\end{array}
\]

**Figure 22:** A basic inversion

We say that an assignment of a normal orientation and an element of \(X\) to each semi-arc of \(\Gamma\) satisfies the coloring conditions if it satisfies the following conditions.

- For each marked vertex, using basic inversions, we assume that normal orientations of semi-arcs are as in Figure 23. Then \(x_1 = x_2\).

- For each crossing, using basic inversions, we assume that normal orientations of semi-arcs are as in Figure 23. Then \(x_1 * x_3 = x_2\) and \(x_3 = x_4\).

An \((X, \rho)\)-coloring of \(\Gamma\) is the equivalence class of an assignment of a normal orientation and an element of \(X\) to each semi-arc of \(\Gamma\) satisfying the coloring conditions. Here the equivalence relation is generated by basic inversions.
Let $Y$ be an $(X, \rho)$-set. An $(X, \rho)_Y$-coloring of $\Gamma$ is an $(X, \rho)$-coloring with an assignment of an element of $Y$ to each complementary region of $\Gamma$ satisfying the following condition.

- Suppose that two adjacent regions $f_1$ and $f_2$ separated by a semi-arc $e$ are labeled by $y_1$ and $y_2$. If the semi-arc $e$ is labeled by $x$ and the normal orientation of $e$ points from $f_1$ to $f_2$, then $y_1 \ast x = y_2$.

**Theorem 8.1.** Let $(X, \rho)$ be a symmetric quandle and let $Y$ be an $(X, \rho)$-set. Let $\Gamma$ be an admissible marked graph diagram, and let $B = B(\Gamma)$ be a broken surface diagram associated with $\Gamma$. There is a bijection from the set of $(X, \rho)_Y$-colorings of $B$ to that of $\Gamma$.

**Proof.** By the same argument as in the proof of Theorem 6.1, we see the result. \qed

Let $\Gamma$ be an admissible marked graph diagram. Fix an $(X, \rho)_Y$-coloring of $\Gamma$, say $\tilde{C}$. Then both resolutions $\Gamma_+$ and $\Gamma_-$ have induced colorings.

![Figure 24: Induced colorings on $\Gamma_+$ and $\Gamma_-$](image)

Let $\Gamma_+ = D_1 \to D_2 \to \cdots \to D_r = O$ and $\Gamma_- = D'_1 \to D'_2 \to \cdots \to D'_s = O'$ be sequences of link diagrams as before. Let $i \in I^3_+ \amalg I^3_-$ and $f$ the
complementary region of $D_i$ in $B_{(i)}$ or $B'_{(i)}$ such that $f$ does not intersect with the boundary $\partial B_{(i)}$ or $\partial B'_{(i)}$, respectively. Let $e_1$, $e_2$ and $e_3$ be the bottom, the middle and the top semi-arcs facing the region $f$, respectively. By basic inversions, we assume that the normal orientations $n_1$, $n_2$ and $n_3$ of them point outwards. Let $x_1$, $x_2$ and $x_3$ be the labels of them, respectively. Define $\epsilon_{tm}(i) = 1$ if $(n_3, n_2)$ matches with the given (right-handed) orientation of $\mathbb{R}^2$ and $-1$ otherwise. For a given 3-cocycle $\theta \in Z^3_{Q,\rho}(X,A)_Y$, we define the symmetric (Boltzman) weight at $i$ to be

$$B^{\text{Sym}}_{\theta}(i, \tilde{C}) = \theta(y, x_1, x_2, x_3)^{\epsilon_{tm}(i)}.$$  

Figure 25: The symmetric (Boltzman) weight at $i \in I^3_+ \cup I^3_-$.

For a marked graph diagram $\Gamma$ and an $(X, \rho)_Y$-coloring $\tilde{C}$, we define the symmetric partition function by

$$\Phi^{\text{Sym}}_{\theta}(\Gamma) = \sum_{\tilde{C}} \left( \prod_{i \in I^3_+} B^{\text{Sym}}_{\theta}(i, \tilde{C}) \prod_{j \in I^3_-} B^{\text{Sym}}_{\theta}(j, \tilde{C})^{-1} \right),$$

where $\tilde{C}$ runs over all $(X, \rho)_Y$-colorings of $\Gamma$.

**Theorem 8.2.** Let $L$ be an unoriented surface-link and let $\Gamma$ be a marked graph diagram presenting $L$. For any 3-cocycle $\theta \in Z^3_{Q,\rho}(X,A)_Y$, the symmetric partition functions $\Phi^{\text{Sym}}_{\theta}(\Gamma)$ is equal to $\Phi^{\text{Sym}}_{\theta}(\mathcal{L})$.

**Proof.** The proof of this theorem is similar to that of Theorem 6.5. Let $\mathcal{B} = \mathcal{B}(\Gamma)$ be the broken surface diagram associated to $\Gamma$, and let $\tilde{C} \in \text{Col}^{\text{Sym}}_{\theta}(\mathcal{B})$ be an $(X, \rho)_Y$-coloring of $\mathcal{B}$. We denote by the same symbol $\tilde{C}$ for the corresponding $(X, \rho)_Y$-coloring of $\Gamma$. As the oriented case, the set of triple points is $T(\mathcal{B}) = \{ \tau_i \mid i \in I^3_+ \} \cup \{ \tau'_j \mid j \in I^3_- \}$, where $\tau_i$ is the triple point in $M(i)$ for $i \in I^3_+$ and $\tau'_j$ is the triple point in $M'(j)$ for $j \in I^3_-$. Let $i \in I^3_+$. Since

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we choose the normal orientation of the bottom arc such that $\epsilon_b(i) = 1$, we have $\epsilon(\tau_i) = \epsilon_{lm}(i)$. Thus $B^\text{Sym}_\theta(i, \tilde{C}) = B^\text{Sym}_\theta(\tau_i, \tilde{C})$. Similarly, for $j \in I^3$, we have $B^\text{Sym}_\theta(j, \tilde{C}) = B^\text{Sym}_\theta(\tau_j', \tilde{C})^{-1}$. Hence we have $\Phi^\text{Sym}_\theta(\Gamma) = \Phi^\text{Sym}_\theta(B)$.

Example 8.3. Let $\Gamma$ be the unorientable marked graph diagram in Figure 27 representing two component $\mathbb{R}P^2$-link $L$ (\Gamma is a marked graph diagram $8_{1}^{-1,-1}$ in Yoshikawa table [32]).

Let $X$ be the dihedral quandle of order 4, in which we rename the elements 0, 1, 2, 3 by $e_1, e_2, e'_1, e'_2$, respectively. Let $\rho : X \to X$ be the antipodal
map, i.e., $\rho(e_i) = e'_i$ ($i = 1, 2$). Let $Y = \{e\}$, which is an $(X, \rho)$-set. Let

$$
\theta = \chi \in C(e, e_1, e_2, e_1 X e, e_1, e_2, e'_1 X e, e_1, e_2, e'_1)
\chi \in C(e, e_1, e_2, e_1 -1 X e, e_1, e_2, e'_1 -1 X e, e_1, e_2, e'_1 -1)
\chi \in C(e, e_1, e_2, e'_1, e_2 X e, e_1, e_2, e'_1, e_1)
\chi \in C(e, e_1, e_2, e'_1, e_2 X e, e_1, e_2, e'_1, e_1') \in Z^3_\mathbb{Q}(X; \mathbb{Z}),
$$

where $\chi_{xyzw}(a, b, c, d) = t$ if $(x, y, z, w) = (a, b, c, d)$, $\chi_{xyzw}(a, b, c, d) = 1$ otherwise, and $\mathbb{Z} = \langle t \rangle$ is the infinite cyclic group (cf. [18, Example 9.3]).

Consider sequences of link diagrams from the positive and negative resolutions to trivial link diagrams as in Figures 28 and 29, respectively. From those figures, we get $I^3_+ = \{2, 3, 4, 6\}$ and $I^3_- = \emptyset$. The symmetric (Boltzmann) weights are $B^{Sym}_\theta(2, C) = \theta(e, y, \rho(y), \rho(x)) = \theta(e, y, \rho(y), \rho(x))^{-1} = 1$, $B^{Sym}_\theta(3, C) = \theta(e, y, \rho(x), \rho(x)) = 1$, $B^{Sym}_\theta(4, C) = \theta(e, x, y, \rho(x))$, $B^{Sym}_\theta(6, C) = \theta(e, \rho(y), x, y)^{-1}$ for $(x, y) \in E$, where $E = \{(e_1, e_2), (e_1, e_2'), (e_1', e_2), (e_1, e_2', (e_2, e_1), (e_2', e_1), (e_2', e_1'))$. Therefore

$$
\Phi_\theta(\mathcal{L}) = \sum_{\mathcal{C}} \left( \prod_{i \in I^3_+} B^ {Sym}_\theta(i, \mathcal{C}) \prod_{j \in I^3_-} B^ {Sym}_\theta(j, \mathcal{C})^{-1} \right)
= \theta(e, x, y, \rho(x)) \theta(e, \rho(y), x, y)^{-1}
= 4 + 2t^2 + 2t^{-2}.
$$

**Acknowledgements.** The first author was supported by JSPS KAKENHI Grant Number 26287013. The third author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2013R1A1A2012446).
Figure 28: A sequence of link diagrams for $\Gamma_+$

Figure 29: A sequence of link diagrams for $\Gamma_-$
References

[1] S. Ashihara, Calculating the fundamental biquandles of surface-links from their ch-diagrams. *J. Knot Theory Ramifications* **21** (2012), no. 10, 1250102 (23 pages).

[2] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, American Mathematical Society, 1998.

[3] J. S. Carter, S. Kamada and M. Saito, Surfaces in 4-space, *Springer*, 2004.

[4] J. S. Carter, S. Kamada and M. Saito, Geometric interpretations of quandle homology, *J. Knot Theory Ramifications* **10** (2001), no. 3, 345–386.

[5] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito, Quandle cohomology and state-sum invariants of knotted curves and surfaces, *Trans. Amer. Math. Soc.* **355** (2003), 3947–3989.

[6] J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada and Masahico Saito, Computations of quandle cocycle invariants of knotted curves and surfaces, *Adv. in Math.* **157** (2001), no. 1, 36–94.

[7] J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada and Masahico Saito, Quandle homology groups, their Betti numbers, and virtual knots, *J. Pure Appl. Algebra*, **157** (2001), no. 2, 135–155.

[8] R. Fenn, C. Rourke and B. Sanderson, Trunks and classifying spaces, *Appl. Categ. Structures* **3** (1995), no. 4, 321–356.

[9] R. Fenn, C. Rourke and B. Sanderson, James bundles and applications, preprint (1996), [http://www.maths.warwick.ac.uk/cpr/ftp/james.ps](http://www.maths.warwick.ac.uk/cpr/ftp/james.ps)

[10] J. Flower, Cyclic Bordism and Rack Spaces, Ph.D. Dissertation, Warwick, 1995.

[11] R. H. Fox, A quick trip through knot theory, in *Topology of 3-manifolds and Related Topics*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962, 120–167.

[12] M. T. Greene, Some Results in Geometric Topology and Geometry, Ph.D. Dissertation, Warwick, 1997.
[13] Y. Joung, J. Kim and S. Y. Lee, Ideal coset invariants for surface-links in $\mathbb{R}^4$, *J. Knot Theory Ramifications* **22** (2013), no. 9, 1350052 (25 pages).

[14] Y. Joung, S. Kamada and S. Y. Lee, Applying Lipson’s state models to marked graph diagrams of surface-links, preprint (2014), arXiv:1411.5740 [math.GT]

[15] D. Joyce, A classifying invariant of knots, the knot quandle, *J. Pure Appl. Algebra* **23** (1982), 37–65.

[16] S. Kamada, Wirtinger presentations for higher dimensional manifold knots obtained from diagrams, *Fund. Math.* **168** (2001), 105–112.

[17] S. Kamada, Quandles with good involutions, their homologies and knot invariants, in: *Intelligence of Low Dimensional Topology 2006*, Eds. J. S. Carter et. al., pp. 101–108, World Scientific Publishing Co., 2007.

[18] S. Kamada and K. Oshiro, Homology groups of symmetric quandles and cocycle invariants of links and surface-links, *Trans. Amer. Math. Soc.* **362** (2010), no. 10, 5501–5527.

[19] A. Kawauchi, T. Shibuuya and S. Suzuki, Descriptions on surfaces in four-space, I Normal forms, *Math. Sem. Notes Kobe Univ.* **10** (1982), 75–125.

[20] A. Kawauchi, *A survey of knot theory*, Birkhäuser, 1996.

[21] C. Kearton and V. Kurlin, All 2-dimensional links in 4-space live inside a universal 3-dimensional polyhedron, *Algebr. Geom. Topol.* **8** (2008), 1223–1247.

[22] J. Kim, Y. Joung and S. Y. Lee, On the Alexander biquandles of oriented surface-links via marked graph diagrams, *J. Knot Theory Ramifications* **23** (2014), no. 7, 1460007 (26 pages).

[23] J. Kim, Y. Joung and S. Y. Lee, On generating sets of Yoshikawa moves for marked graph diagrams of surface-links (preprint), 2014, arXiv:1412.5288 [math.GT]

[24] S. Y. Lee, Invariants of surface links in $\mathbb{R}^4$ via classical link invariants, in *Intelligence of low dimensional topology 2006*, Series on Knots Everything, Vol. **40**, World Scientific Publishing, Hackensack, NJ, 2007, 189–196.
[25] S. Y. Lee, Invariants of surface links in $\mathbb{R}^4$ via skein relation, *J. Knot Theory Ramifications* **17** (2008), 439–469.

[26] S. Y. Lee, Towards invariants of surfaces in 4-space via classical link invariants, *Trans. Amer. Math. Soc.* **361** (2009), 237–265.

[27] S. J. Lomonaco, Jr., The homotopy groups of knots I. How to compute the algebraic 2-type, *Pacific J. Math.* **95** (1981), 349–390.

[28] S. V. Matveev, Distributive groupoids in knot theory. (Russian) Mat. Sb. (N.S.) **119** (161) (1982), no. 1, 78–88; Math. USSR-Sb. **47** (1984), 73–83 (in English).

[29] D. Roseman, Reidemeister-type moves for surfaces in four dimensional space, *Banach Center Publications* **42** (1998) Knot theory, 347–380.

[30] M. Soma, Surface-links with square-type ch-graphs, Proceedings of the First Joint Japan-Mexico Meeting in Topology (Morelia, 1999), *Topology Appl.* **121** (2002), 231–246.

[31] F. J. Swenton, On a calculus for 2-knots and surfaces in 4-space, *J. Knot Theory Ramifications* **10** (2001), 1133–1141.

[32] K. Yoshikawa, An enumeration of surfaces in four-space, *Osaka J. Math.* **31** (1994), 497–522.