Neutron star masses in $R^2$-gravity

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Abstract

We address the issue of the existence of inequivalent definitions of gravitational mass in $R^2$-gravity. We present several definitions of gravitational mass, and discuss the formal relations between them. We then consider the concrete case of a static and spherically symmetric neutron star, and solve numerically the equations of motion for several values of the free parameter of the model. We compare the features of the mass-radius relations obtained for each definition of gravitational mass, and we comment on their dependence on the free parameter. We then argue that $R^2$-gravity is a valuable proxy to discuss the existence of inequivalent definitions of gravitational mass in a generic modified gravity theory, and present some comments on the general case.

Keywords: modified gravity, f(R) gravity, neutron stars.

1 Introduction

Infrared modifications of gravity have become a very popular way of addressing the problem of the late-time acceleration of the Universe. The possibility of explaining the cosmic...
acceleration without introducing exotic and experimentally unobserved forms of energy, has generated a lot of interest in theories like \( f(R) \), braneworlds, massive gravity and generalisations thereof. Currently, they are undergoing an intense scrutiny as a result of the recent birth of the field of multimessenger astronomy.

A notoriously delicate aspect of modified theories of gravity (MTG) is that, since modifying General Relativity (GR) introduces additional degrees of freedom (DsOF) in the theory, often the late time acceleration is achieved at the expense of the correct behaviour of gravity at small scales. Indeed, an efficient screening of the additional DsOF is mandatory to ensure that a MTG passes the solar system tests. The physics of compact objects provides another important test bench, this time regarding the strong gravity regime. Changing the behaviour of gravity in fact generally impacts both on the macroscopic properties of massive bodies and on their evolution history.

A compelling case is that of neutron stars (NS). In the static and spherically symmetric case GR makes rather stark predictions, like the existence of a minimum and a maximum mass for NS. This implies that, provided one is able to account for the effects of rotation and to model reliably their internal structure, observations of NS can be used to test (and potentially falsify) GR. As a matter of fact, observations like PSR J1614–2230 [1] and PSR J0348–0432 [2] already represent a challenge for GR. However, a satisfactory understanding of the behaviour of nuclear matter at the extreme conditions realised in the interior of a NS is still lacking, so it is hard to say whether these observations point to a breakdown of GR or to a poor understanding of the equation of state (EoS). An important tool to break this degeneracy is the mass-radius (M–R) relation. The interest in this field is presently very high due to the recent detection of gravitational wave signals by the LIGO and VIRGO collaborations, exemplified by the signal GW 170817 emitted by a merging neutron stars binary system [3, 4, 9, 6, 5, 7, 8].

However, a subtle and perhaps underestimated point is that, when we speak of the mass of a star in MTG, we are speaking of a not well-defined concept. In GR we are used to consider different definitions of mass, depending on the specific problem under consideration, which are nonetheless equivalent. A crucial remark is that these definitions are in general not equivalent in MTG, when additional degrees of freedom enter into play, so possible confusion arises when we speak generically of “gravitational mass” in MTG without specifying to what definition exactly we are referring to. Clarifying this point is evidently very important, especially to be able to compare different types of observations, theoretical predictions with observations, and different theoretical studies (see on this respect the case of [10, 11] and [12], which use different definitions of mass and find different M–R relations).

Our aim here is to give a systematic discussion of this fact. Although the inequivalence is, as we propose, inherent to MTG, considering a general MTG from the outset would imply the risk of not being able to identify clearly the physical reason for the inequivalence. To avoid generality to hinder clarity of analysis, we prefer to consider a concrete model of MTG, to be used as a proxy which is meant to represent general features of MTG. For this reason we focus on the quadratic \( f(R) \) model \( f(R) = R + \alpha R^2 \) (to which we refer to as \( R^2 \)-gravity), also known as the Starobinsky model [13, 14] (which is one of the most successful models in inflationary cosmology [15]). Coherently with the spirit of our analysis of using this model as a proxy, we leave \( \alpha \) completely free, although it is known that its value is observationally severely constrained both from cosmological observations [16, 17] and from solar system tests [18, 19]. For the sake of concreteness, and to be able to grasp
the quantitative relevance of the inequivalence, we apply our discussion to the M–R relation of neutron stars in $R^2$-gravity, investigating how the features of the curves change when we change the definition of gravitational mass.

The paper is structured as follows: in Section 2 we review the equations of motion in $R^2$-gravity, specialising to a static and spherically symmetric system. In Section 3 we introduce several definitions of gravitational mass, discussing their inequivalence. In Section 4 we consider a neutron star, and study its M–R relation using the definitions of gravitational mass previously introduced. We discuss our results and present our conclusions in Section 5.

We adopt the “mostly plus” signature $(-, +, +, +)$ and, unless stated otherwise, we use units of measure where $c = 1$.

2 The equations of motion and the behaviour of weak gravity

The general $f(R)$ gravity action $[20, 21, 22, 23]$ is given by

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} f(R) + S_M,$$

where $g$ is the determinant of the metric $g_{\mu\nu}$, $R$ is the Ricci scalar and $S_M$ is the action of matter fields. The case of $R^2$-gravity corresponds to the choice

$$f(R) = R + \alpha R^2,$$

with $\alpha > 0$. In the metric formulation, which we adopt throughout the paper, the connection is taken to be the Levi-Civita one and the metric obeys the equation of motion

$$(1 + 2\alpha R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R + \alpha R^2) - 2\alpha (\nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \Box) R = 8\pi G_N T_{\mu\nu},$$

where the Ricci scalar is a functional of the metric $R = R[g_{\mu\nu}]$, $\Box = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu}$ is the (curved space) d’Alembert operator and the stress-energy tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}.$$

As is well-known, the theory can be recast in a scalar-tensor form. Introducing the scalar degree of freedom

$$\zeta = \alpha R,$$

the fourth-order equation (2.3) can be shown to be equivalent to the second-order system for the metric $g_{\mu\nu}$ and the scalar field $\zeta$ [24]:

$$(1 + 2\zeta) G_{\mu\nu} = 2 \left( g^{\nu\lambda} \nabla_{\mu} \partial_\lambda - \delta_{\mu}^{\nu} \Box \right) \zeta - 3m^2 \zeta^2 \delta_{\mu}^{\nu} + 8\pi G_N T_{\mu\nu},$$

$$\Box \zeta - m^2 \zeta = \frac{4\pi G_N}{3} T,$$

where $T$ is the trace of the stress-energy tensor and we defined the mass associated to $\zeta$ as

$$m = \frac{1}{\sqrt{6\alpha}}.$$
Although linked by the relation (2.5), the metric and the scalar field are independent degrees of freedom as far as the initial value problem of the system (2.6)–(2.7) is concerned. It is possible to perform a field redefinition to avoid having \((1+2\zeta)\) multiply \(G_{\mu\nu}\), at the expense of introducing a non-minimal coupling with matter (Einstein frame). See e.g. [25] for an analysis in that direction. In this work, we don’t follow that path and work only in the Jordan frame.

2.1 Non-rotating stars

To study static and spherically symmetric stars in \(R^2\)-gravity, we consider the line element

\[
ds^2 = -B(r) \, dt^2 + A(r) \, dr^2 + r^2 \left( d\theta^2 + \sin^2\theta \, d\phi^2 \right),
\]

and model the star as a perfect fluid, so the stress-energy tensor reads

\[
T_{\mu \nu} = \text{diag} \left( -\rho, p, p, p \right),
\]

where \(\rho(r)\) and \(p(r)\) are the energy density and the pressure, respectively. We postpone to Section 4 the detailed discussion of the equation of state we employ to model the interior of a neutron star.

The staticity condition implies that the covariant conservation of \(T_{\mu \nu}\) gives the equation of hydrostatic equilibrium

\[
p' = -\frac{B'}{2B} \left( \rho + p \right),
\]

while the equation (2.7) for the scalar DOF takes the form

\[
\zeta'' + \left( \frac{2}{r} + \frac{B'}{2B} - \frac{A'}{2A} \right) \zeta' = A \left[ m^2 \zeta + \frac{4\pi G_N}{3} (3p - \rho) \right],
\]

where we indicated \(\zeta' \equiv d\zeta/dr\). By taking suitable linear combinations of the components of (2.6), we obtain the equations for \(B\) and \(A\)

\[
\left( 1 + 2\zeta + r\zeta' \right) \frac{1}{A} \frac{B'}{B} = \frac{1 + 2\zeta}{r^2} \left( 1 - \frac{1}{A} \right) - 3m^2 \zeta^2 - \frac{4}{Ar} \zeta' + 8\pi G_N \rho,
\]

\[
\left( 1 + 2\zeta + r\zeta' \right) \frac{1}{A} \frac{A'}{A} = \frac{1 + 2\zeta}{r^2} \left( 1 - \frac{1}{A} \right) + 3m^2 \zeta^2 + \frac{2}{A} \left( \zeta'' + \frac{2}{r} \zeta' \right) + 8\pi G_N \rho.
\]

2.2 The behaviour of weak gravity

In light of the discussion to follow, it is worthwhile to recall the features of static and weak gravitational fields in \(R^2\)-gravity. Although non-linear effects do become important for neutron stars, the results of this subsection will be useful for the discussion of Section 3 on the concept of gravitational mass. To prevent any possible confusion we underline that, when studying neutron stars in Section 4, we consider the full non-linear equations of motion.

The behaviour of the gravitational field outside a static and spherically symmetric star in \(f(R)\) gravity has been extensively studied in the literature. We follow here the analysis
In particular, we consider the spacetime to be asymptotically flat, which is a consistent boundary condition when the function $f(R)$ is analytic in $R = 0$. The behaviour of the metric and of the scalar DOF is controlled by two characteristic radii $r_\zeta$ and $r_g$, which somehow play in $R^2$-gravity a role analogous to that of the Schwarzschild radius in GR. Indicating with $r_\star$ the radius of the star, and slightly changing the notation with respect to [26], the characteristic radii take the form

$$r_\zeta = 2G_N \left( \frac{\bar{M}_\rho}{3} - \bar{P} \right)$$

(2.14a)

$$r_g = 2G_N \left[ \left( \frac{\bar{M}_\rho}{3} - \bar{P} \right) e^{-mr_\star} \left( 1 + mr_\star \right) + \frac{2M_\rho}{3} + P + \Xi \right]$$

(2.14b)

where

$$M_\rho = 4\pi \int_0^{r_\star} \rho(r) r^2 dr$$

$$P = 4\pi \int_0^{r_\star} p(r) r^2 dr$$

(2.15a)

$$\bar{M}_\rho = 4\pi \int_0^{r_\star} \frac{\sinh mr}{mr} \rho(r) r^2 dr$$

$$\bar{P} = 4\pi \int_0^{r_\star} \frac{\sinh mr}{mr} p(r) r^2 dr$$

(2.15b)

$$\Xi = \frac{m^2}{G_N} \int_0^{r_\star} \zeta(r) r^2 dr$$

(2.15c)

Note that $\bar{M}_\rho \to M_\rho$, $\bar{P} \to P$ and $\Xi \to 0$ when $m \to 0$.

In the weak-field approximation, the solutions outside of the star explicitly read [26]

$$\zeta(r) = \frac{r_\zeta}{6r} e^{-mr}$$

(2.16a)

$$A(r) = 1 + \left( 1 + mr \right) \frac{r_\zeta}{3r} e^{-mr} - \frac{r_g}{r}$$

(2.16c)

It is apparent that, at distances $r \gg m^{-1}$ larger than the range of the scalar DOF, the Schwarzschild solution is recovered with $r_g$ as the effective (asymptotic) Schwarzschild radius. In this region, to which we refer to as the asymptotic region, GR is recovered. On the other hand, at distances $r \sim m^{-1}$ comparable to the range of the scalar DOF, the metric components do not even have a Newtonian behaviour, due to the presence of the exponential function. We refer to this region as the transition region. Finally, when the range of the scalar DOF is much larger than the radius of the star ($r_\star \ll m^{-1}$), there exist another interesting region $r_\star \leq r \ll m^{-1}$ outside the star, to which we refer to as the nearby region.

1Note however that, to facilitate the comparison with [27, 28], we changed notation calling now $B$ and $A$ respectively what we called $A$ and $B$ in [26].
In this region $\zeta$, $B$ and $A$ display a Newtonian behaviour

$$\zeta(r) \simeq \frac{\zeta}{6r} ,$$  \hspace{1cm} (2.17a)

$$B(r) \simeq 1 - \frac{\zeta + 3\rho}{3r} ,$$  \hspace{1cm} (2.17b)

$$\frac{1}{A(r)} \simeq 1 + \frac{\zeta - 3\rho}{3r} ,$$  \hspace{1cm} (2.17c)

but the gravitational potentials have different amplitude, since they are sourced in a different way by the scalar DOF. More precisely, the PPN parameter $\gamma$ in the nearby region reads [26]

$$\gamma_\star \simeq \frac{2M_\rho + 3P}{4M_\rho - 3P} ,$$  \hspace{1cm} (2.18)

where neglecting the pressure we recover the well-known result $\gamma_\star = 1/2$ [29]. Clearly GR is not recovered here, apart from the hypothetical case where exotic physics inside the star pushes the pressure to be of the same order of the mass, since solar system observations constrain $\gamma$ to be unity within few parts in $10^5$ [19]. Nevertheless, if we limit our attention to non-relativistic bodies, Newtonian gravity is recovered.

### 3 On the definition of gravitational mass in $R^2$-gravity

To put the discussion of the inequivalent definitions of gravitational mass in the proper context, and to spell out the subtleties which become relevant in the $R^2$ case, it is worthwhile to recall first how the concept of gravitational mass emerges in Newtonian gravity and in GR.

#### 3.1 The gravitational mass in Newton’s and Einstein’s theories

In Newtonian gravity, the concept of gravitational mass is borne out of the experimental observation that the external gravitational potential generated by a spherical body is proportional to $1/r$ independently of the composition and of the radius of the body itself. It follows that to characterise completely the external gravitational field only one number is needed, the proportionality constant, from which the Newton’s constant $G_N$ is factored out for dimensional reasons. This number, the “gravitational charge” of the body, is conventionally called the gravitational mass. This notion is extended to the non-spherically symmetric case by selecting the monopole term in the multipole expansion of the external gravitational field, or equivalently considering the asymptotic behaviour of the latter (since the monopole term is the one with the slowlest decay).

General Relativity, although being more complex and having more degrees of freedom, shares the above mentioned property. Choosing the gauge suitably, the metric outside of a spherical body can always be written in the form

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) ,$$  \hspace{1cm} (3.1)
so also in this case the external field is completely characterised by one number, the characteristic radius $r_g$. Mathematically, the role of the source term (i.e. $\rho$ and $p$, in our case) is to provide boundary conditions at the body’s surface for the external solution (3.1), conditions which are to be found by solving the equations of motion inside the body. The gravitational mass in GR is usually defined by resorting to the Newtonian limit of the theory [31, 30], that is setting

$$M \equiv \frac{r_g}{2G_N} = \lim_{r \to \infty} \frac{r}{2G_N} \left(1 - B(r)\right) = \lim_{r \to \infty} \frac{r}{2G_N} \left(1 - \frac{1}{A(r)}\right).$$  \hspace{1cm} (3.2)

This definition can be given a more elegant and coordinate-independent form, and generalised to stationary and asymptotically flat space-times which are not spherically symmetric, such as in the Komar expression for $M$.\(^2\) The rationale is that the gravitational mass is a measure of the external gravitational field produced by a body, it is a global quantity (in the sense that it is associated to the spacetime itself, not to the volume occupied by the star), and its value is linked to the effect the gravitational field has on test bodies in the Newtonian limit. The idea of identifying the physical parameters of isolated astrophysical bodies by studying the asymptotic behaviour of the gravitational field is indeed ubiquitous in gravitational physics.

One can however use the equations of motion to link the gravitational mass (defined as above) to the properties of the source. Integrating in the Schwarzschild gauge (2.9) the time-time component of the Einstein equations, it is easy to derive the relation between $M$ and the energy density of the source

$$M = 4\pi \int_0^{r_*} \rho(r) r^2 \, dr \hspace{1cm} (3.3)$$

Since the energy density is a scalar, the expression on the right hand side of (3.3) is not invariant with respect to radial re-parametrisations. Indeed, including the proper volume element $\sqrt{g^{(3)}} \, dx^3$ on the constant-time spatial hypersurfaces, one obtains the *proper mass* of the body [30]

$$M_p = \int_{V_*} \rho \sqrt{g^{(3)}} \, dx^3 = 4\pi \int_0^{r_*} \rho(r) \sqrt{A(r)} \, r^2 \, dr \hspace{1cm} (3.4)$$

which is invariant ($V_*$ indicates the volume occupied by the star). The fact that the equations of motion indicate (3.3) as the correct relation is interpreted, *a posteriori*, assuming that the expression on the right hand side of (3.3) evaluated in Schwarzschild coordinates takes into account also the gravitational binding energy, thereby enforcing the equivalence between energy and gravitational mass for extended and self-gravitating objects.

It is customary to rephrase the discussion of the mass of spherical stars in terms of the so-called *mass function*

$$\mathcal{M}(r) = \frac{r}{2G_N} \left(1 - \frac{1}{A(r)}\right).$$  \hspace{1cm} (3.5)

Indicating with $\mathcal{B}_r(0)$ the sphere of radius $r$ centered at the origin, in Schwarzschild coordinates we have

$$\int_{\mathcal{B}_r(0)} G_{t \ell} \, y^2 \, dy \, d\Omega = -8\pi G_N \mathcal{M}(r).$$  \hspace{1cm} (3.6)

\(^2\)We remind that the Komar and ADM mass coincide for stationary, asymptotically flat spacetimes if the initial data set of the ADM construction is chosen suitably [30].
The Einstein’s equations imply that $M(r)$ is constant outside the star, where it coincides with the effective Newtonian mass (3.2), and taking into account (2.10) we get

$$M(r) = 4\pi \int_0^r \rho(y) y^2 dy .$$

(3.7)

The continuity of $M$ across the star’s surface then implies (3.3). Although we cannot in general associate the concept of gravitational energy to a finite volume, since there is no unique way to introduce a (local) gravitational stress-energy tensor, for the case of spherical stars (and just for this case) it is meaningful to interpret $M(r)$ as the gravitational mass enclosed in the sphere of radius $r$ [31]. This is a consequence of the non-existence of spherically symmetric gravitational radiation in Einstein’s gravity. Considering non-static (but still spherical) configurations, it follows that the energy inside a sphere of radius $r$ can change only because of heat fluxes and radiation of particles across the sphere’s surface, or by work done on the surface by pressure forces. Therefore, the fact that we can associate a (localised) mass $M(r)$ to the sphere of radius $r$ is due to the circumstance that in the spherical case any transfer of energy is detectable by local measurements.

Clearly this situation breaks down as long as we depart from spherical symmetry. On the other hand there is no difficulty in defining the proper mass contained in any chosen spatial volume, independently of the configuration being spherically symmetric or not, since the proper mass does not include the gravitational binding energy.

3.2 The case of $R^2$-gravity

In $R^2$-gravity the situation is qualitatively different. As can be seen from the expressions (2.16b) and (2.16c) for the metric components in the weak-field approximation, the external gravitational field is determined by two numbers, $r_\zeta$ and $r_g$, or in other words by two gravitational charges. This is a consequence of the fact that the metric components, beside by $\rho$ and $p$, are sourced also by the scalar degree of freedom $\zeta$ which dynamically extends outside the body (being itself sourced by $\rho$ and $p$ and obeying a Klein-Gordon equation).

It follows that, for what concerns the metric, the energy density and the pressure do not generate simply a boundary condition for the external gravitational field. The impossibility of describing the external geometry with one gravitational charge, which is fundamentally due to the presence of the extra scalar DOF and therefore not peculiar to the weak-field approximation, has profound implications as we can see.

3.2.1 On the usual definitions of gravitational mass

It is quite easy to see that, in $R^2$-gravity, the relations (3.2) and (3.3) are not compatible. Let us define for ease of notation

$$M_g = \lim_{r \to \infty} \frac{r}{2G_N} \left( 1 - \frac{1}{A(r)} \right) , \quad M_\mu = 4\pi \int_0^r \rho(r) r^2 dr ,$$

(3.8)

and introduce the operator

$$\mathcal{D}_{\mu\nu} = 2 \left( g^{\mu\lambda} \nabla_\mu \partial_\lambda - \delta_{\mu\nu} \Box \right) .$$

(3.9)

Rewriting the equation (2.6) as follows

$$G_{\mu\nu} = \mathcal{D}_{\mu\nu} \zeta - 3m^2 \zeta^2 \delta_{\mu\nu} - 2\zeta G_{\mu\nu} + 8\pi G_N T_{\mu\nu} ,$$

(3.10)
the integration of the $tt$ component on the sphere of radius $r$ gives

$$M(r) = \frac{1}{8\pi G_N} \int_{\mathbb{R}^3} \left( - \partial_t^t \zeta + 3m^2 \zeta^2 + 2\zeta \partial_t^t \right) dV + 4\pi \int_0^r \rho(y) y^2 dy , \quad (3.11)$$

where $dV$ indicates the flat volume element. Sending $r \to \infty$ we get

$$M_g - M_\rho = \frac{1}{8\pi G_N} \int_{\mathbb{R}^3} \left( - \partial_t^t \zeta + 3m^2 \zeta^2 + 2\zeta \partial_t^t \right) dV . \quad (3.12)$$

Note that, since $\zeta$ has a Yukawa behaviour with range $m^{-1}$, only the sphere of approximate radius $m^{-1} \sim \sqrt{\alpha}$ contributes to the integral. The quantity on the right hand side of (3.12) does not vanish in general, as can be checked numerically (see Section 4), so the definitions (3.8) of gravitational mass are not equivalent in $R^2$-gravity.

It is important to understand which physical meaning can be given to $M_g$ and $M_\rho$ in this context, and whether one of the two definitions is eligible as “the” definition of gravitational mass. In the asymptotic region the weak-field approximation is always valid, and therefore (2.16b)-(2.16c) hold. Moreover, the terms containing $r_\zeta$ are exponentially suppressed. This means that $M_g$ is the Newtonian mass felt by test-bodies orbiting in the asymptotic region, so $M_g$ maintains the interpretation it has in GR although its spatial validity is limited (it is a “distant observers” mass). The story for $M_\rho$ is more complicated. It may be thought that the failure of $M_\rho$ to coincide with the asymptotic Newtonian mass $M_g$ is due to $M_\rho$ being linked to the properties of the spacetime near the star, while $M_g$ is linked to the properties of the spacetime far away. This idea is however not correct, as can be seen as follows. Considering a configuration where $mr_\ast \ll 1$ and the weak-field approximation holds, test-bodies orbiting close to the star (i.e. well inside the range of $\zeta$) feel an effective Newtonian mass equal to

$$M_n = \frac{r_\ast}{2G_N} \left( 1 - B(r_\ast) \right) = \frac{1}{2G_N} \frac{r_\zeta + 3r_\rho}{3} , \quad (3.13)$$

and taking the limit $m \to 0$ we get

$$M_n = \frac{4}{3} M_\rho - P . \quad (3.14)$$

Even neglecting the pressure we have $M_\rho \neq M_n$. Moreover, away from the limit $m \to 0$ the ratio between $M_\rho$ and $M_n$ does not remain constant (and equal to $4/3$ when $P = 0$), but displays a very complicated dependence on $m$, $r_\ast$, $\rho$ and $p$. Therefore $M_\rho$ is not simply related to the effective Newtonian mass in any region of spacetime. This situation may be regarded as inconvenient enough to abandon the idea of interpreting $M_\rho$ as a gravitational mass, as already pointed out in [27] where $M_\rho$ is regarded as merely a parameter (a “tag”) characterising families of solutions, without a specific physical interpretation.

### 3.2.2 On a unique definition of gravitational mass

One may however take the totally opposite point of view, turning the impossibility of describing the external metric with a unique gravitational charge into an indication that in $R^2$-gravity the concept of gravitational mass has to be disentangled from the behaviour of test bodies in the Newtonian limit. This argument may be used in favour of regarding $M_\rho$
as the definition of gravitational mass in $R^2$-gravity, owing to its interpretation as the total energy of matter.\footnote{We are here loosely using the word “matter” to mean “matter, radiation and every form of energy-momentum included in $T_{\mu\nu}$."
} Recall in fact that (in the Jordan frame) the stress-energy tensor couples only to the metric, in the sense that there is no direct coupling between $\zeta$ and $T$. Moreover, the coupling term is exactly that of GR. This means in particular that matter feels only the spacetime metric or, in other words, that regarding the dynamical behaviour of matter the role of the scalar $\zeta$ is just to indirectly influence the metric configuration $g_{\mu\nu}$ via the equations of motion. Since it is assumed that the integral which defines $M_\rho$ correctly takes into account the gravitational binding energy in GR, it is reasonable to expect this be true also for theories where $T_{\mu\nu}$ couples only to $g_{\mu\nu}$ via the same coupling term.\footnote{Of course, a configuration $(\rho(r), p(r))$ cannot be an equilibrium profile at the same time for GR and $R^2$-gravity. But the binding energy is not defined only for equilibrium configurations.}

Following this line of reasoning, the relation (3.12) would lend itself to the heuristic interpretation that the asymptotic mass $M_g$ is given by the sum of the energy of matter (including the binding energy), i.e. $M_\rho$, and the energy associated to the additional degree of freedom, i.e. the right hand side. This interpretation can be formalised by associating to the extra DOF $\zeta$ an effective stress-energy tensor $\mathcal{T}_{\mu\nu}$ defined as follows

$$8\pi G_N \mathcal{T}_{\mu\nu} = \mathcal{D}_{\mu\nu} - 3m^2 \zeta^2 - 2\zeta G_{\mu\nu},$$

(3.15)

which is covariantly conserved as a consequence of the equation of motion (3.10). Accordingly, we may indicate $\mathcal{R} = -\mathcal{T}^t_t$ and consider it as the effective energy density of the extra DOF

$$8\pi G_N \mathcal{R} = -\mathcal{D}^t_t + 3m^2 \zeta^2 + 2\zeta G^t_t.$$  

(3.16)

It is worthwhile to point out that in the literature it is customary to introduce the effective stress-energy tensor associated to the extra DOF in a different way (see for example [32, 33]), to wit

$$8\pi G_N \mathcal{J}_{\mu\nu} = \frac{1}{1 + 2\zeta} \left[ 2 \left( g^{\mu\lambda} \nabla_\lambda \partial_\mu - \delta_\mu^\lambda \Box \right) \zeta - 3m^2 \zeta^2 \delta_\mu^\nu \right],$$

(3.17)

so that the equation (2.6) becomes

$$G_{\mu\nu} = 8\pi G_N \left( \mathcal{J}_{\mu\nu} + \frac{1}{1 + 2\zeta} T_{\mu\nu} \right).$$  

(3.18)

One may then integrate the $tt$ component of (3.18) over the whole 3D space to derive a relation similar to (3.12), obtaining a different expression on the right hand side and, on the left hand side, the integral of $\rho/(1 + 2\zeta)$ instead of $M_\rho$. It is not clear however what the interpretation of this relation should be. For example, it is doubtful that the integral of $\rho/(1 + 2\zeta)$ should be interpreted as the total matter energy. Moreover, both $\mathcal{J}_{\mu\nu}$ and $T_{\mu\nu}/(1 + 2\zeta)$ are not (separately) covariantly conserved (unless $\zeta$ is constant). For this reason, we believe (3.12) and (3.15) to be physically more meaningful.

### 3.2.3 Gravisphere and surface redshift

According to the above interpretation of the relation (3.12), the difference between $M_g$ and $M_\rho$ is to be found in the contribution of the extra DOF, both inside and outside the star.
This may suggest that a useful characterisation of the properties of spacetime outside and near the star may be given by introducing a new definition of gravitational mass which, if possible, takes into account both the contribution of matter and of the extra DOF inside the star’s surface.

Such a definition has indeed been proposed in [28], by suitably using the function $\mathcal{M}(r)$ defined in (3.5) (see also [27]). More specifically, the attention is cast upon the quantity

$$M_s = \mathcal{M}(r_\star) = \frac{r_\star}{2G_N} \left(1 - \frac{1}{\mathcal{A}(r_\star)} \right), \quad (3.19)$$

which is referred to as the “stellar mass bounded by the star’s surface”, as opposed to the mass $M_g = \lim_{r \to \infty} \mathcal{M}(r)$ which is referred to as the “gravitational mass measured by a distant observer”. Evaluating the relation (3.11) at $r = r_\star$ we obtain

$$M_s = M_\rho + \int_{V_\star} \mathcal{R} \, dV, \quad (3.20)$$

which, again assuming that $\mathcal{T}_{\mu \nu}$ indeed is the legitimate stress-energy tensor for the extra DOF, indeed suggests that $M_s$ takes into account also the energy of the extra DOF inside the star. Here $dV$ indicates the flat space volume element. Indicating with $V_\star^C$ the (set) complement of $V_\star$, or in other words the region outside the star, the relation (3.12) can then be rewritten as

$$M_g = M_s + \int_{V_\star^C} \mathcal{R} \, dV, \quad (3.21)$$

where, as for (3.12), only the spherical shell of radii $r_\star < r \lesssim m^{-1}$ significantly contributes to the integral. We may then interpret (3.21) as if the difference between $M_g$ and $M_s$ were given by the energy associated to the extra DOF in the above mentioned spherical shell. The analysis of [28] is indeed centered on the idea that the curvature present inside a region called gravisphere, which surrounds the star and has external radius approximately equal to $\sqrt{\alpha}$, itself contributes to the the distant observer’s mass.

Clearly, assigning a mass to the volume enclosed by the star’s surface (and, complementary, to the gravisphere) raises immediate concerns, since (3.19) is meant to include the gravitational binding energy (it is not a proper mass in the sense of (3.4)), which in general cannot be localised. Remember in fact that in GR the identification of $\mathcal{M}(r)$ with the mass contained inside the sphere of radius $r$ is possible only thanks to the absence of spherically symmetric gravitational waves; however, spherically symmetric radiation do exists in $R^2$-gravity, being the (scalar) waves of $\zeta$. The validity of the interpretation of [28] therefore strongly relies on our ability of defining a local and covariantly conserved stress-energy tensor for $\zeta$, the obvious candidate being (3.15). If, in the spherically symmetric and time-dependent case, one trusts (3.15) to describe the flux of energy and momentum associated to $\zeta$ across the star’s surface, then also in this case $M_s$ can change only because of fluxes of energy and momentum detectable by local measurements (now including also scalar radiation).

The mass $M_s$ is without doubt a legitimate “tag” to characterise the spherically symmetric solutions. It is however important to point out that, contrary to the claim of [28], $M_s$ does not characterise to the surface gravitational redshift of the star [34]. The gravitational redshift $z_\star$ undergone by an electromagnetic wave emitted (with frequency $\omega_e$) on
the surface of the star and detected (with frequency $\omega_d$) in the asymptotic region $r \gg m^{-1}$ is given by

$$z_* = \frac{\omega_e - \omega_d}{\omega_d} = \frac{1}{\sqrt{|g_{tt}(r_\star)|}} - 1 = \frac{1}{\sqrt{B(r_\star)}} - 1 \quad ,$$

(3.22)

so it is controlled by the $tt$ component of the metric on the surface. On the other hand $M_s$ is determined by the $rr$ component of the metric on the star’s surface, and in $R^2$-gravity there is no simple and general relation between $|g_{tt}(r_\star)|$ and $g_{rr}(r_\star)$ (unlike in GR, where they are one the inverse of the other). We could define an effective “surface redshift” mass $M_{sr}$ by means of the relation

$$z_* = \frac{1}{\sqrt{1 - \frac{2G_NM_{sr}}{r_\star}}} - 1 \quad (3.23)$$

which holds in GR between $z_*$ and the mass, that is defining

$$M_{sr} = \frac{r_\star}{2G_N} \frac{z_* (2 + z_*)}{(1 + z_*)^2} \quad .$$

(3.24)

The effective mass $M_{sr}$ defined this way is different from $M_s$, and indeed is none else than the nearby mass $M_n$, introduced in (3.13). The latter seems therefore better suited than $M_s$ to describe the gravitational field in the proximity of the star.

### 4 Neutron stars and numerical mass-radius relations

To render our analysis more concrete we now turn to the case of neutron stars in $R^2$-gravity and their M–R relation. This gives us the possibility of assessing quantitatively the relevance of the difference between the several definitions of mass described above, in a context where these differences may be potentially relevant. See [35, 36, 37, 38] for related work.

To achieve this, we numerically solve the equations (2.12) (2.13a) and (2.13b) by means of a shooting method. For definiteness we model the interior of the neutron star with the Sly equation of state [39, 40] expressed by the following analytic formula [41]:

$$\log_{10} p = \frac{a_1 + a_2 \log_{10} \rho + a_3 (\log_{10} \rho)^3}{\exp[a_5 (\log_{10} \rho - a_6)] + 1} + \frac{a_7 + a_8 \log_{10} \rho}{1 + a_4 \log_{10} \rho} + \frac{a_9 (\log_{10} \rho)^3}{\exp[a_{10} (\log_{10} \rho)] + 1} + \frac{a_{11} + a_{12} \log_{10} \rho}{\exp[a_{13} (\log_{10} \rho)] + 1} \quad ,$$

(4.1)

where the pressure is given here in units of dyn/cm$^2$ and the density in units of g/cm$^3$, and the 18 coefficients $a_i$ come from a numerical fit and are tabulated in [41]. As done in [26], prior to the numerical integration we recast the system of equations in terms of dimensionless quantities. For example, we normalise the parameter $\alpha$ to the Sun’s half Schwarzschild radius $r_0 = G_M/c^2 \approx 1.5$ km, so we work with the adimensional parameter $\hat{\alpha} = \alpha/r_0^2$. Regarding the shooting, whose initial conditions are given at the centre of the star, the only free parameters are the central density $\rho_0$ and the central value $\zeta_0$ of the extra DOF ($\sim$ curvature), since the central pressure is determined by $\rho_0$ via the equations of state.
To individuate the solution we choose a priori a grid of values for the central density, and for each of them we determine $\zeta_0$ via the shooting method, selecting the solution for $\zeta$ which decays exponentially to zero far from the star (meaning at $r \gg \sqrt{2A} \sim m^{-1}$). It follows that our shooting solutions, and therefore our curves in the M–R plane, are parametrised by the value of the central density. The star’s radius is determined to be that for which $\rho$ vanishes (strictly speaking, for which it becomes negative). We refer to [26] for a more detailed discussion of our procedure.\(^5\)

For definiteness we consider the representative values $\hat{\alpha} = 1, 10$ and $100$, which permit to gain an understanding of how the properties of the star depend on the parameter $\alpha$.

### 4.1 Numerical results

#### 4.1.1 The asymptotic mass $M_g$

Let us begin our discussion by considering the results for the asymptotic mass $M_g$. Doing so allows to clearly highlight the main features of the curves and their dependence on $\hat{\alpha}$, and facilitates the subsequent comparison with the other definitions of mass. In Fig. 1 the M–R relation (left) and the dependence of $M_g$ with the central density (right) are shown for the three chosen values of $\hat{\alpha}$. These results are in qualitative agreement with the findings of [10, 42, 43, 11]. The central density increases monotonically along the M–R curves, the lowest values of $\rho_0$ corresponding to the bottom-right part of the curves and the highest values to top-left part. In particular, the mass increases monotonically with $\rho_0$. It is apparent that the maximum mass can get comfortably well above two solar masses, and that it increases with increasing $\alpha$.

An evident feature of the M–R curves is the presence of an intermediate mass range where the radius increases with increasing mass and central density. In GR this behaviour is usually associated with a thermodynamic instability, which sets in at the points where the derivative $dM/dr$ vanishes (turning point instability [44], for a review see [45]).\(^6\) This does not immediately imply an instability in $R^2$-gravity, since the $dM/dr > 0$ part of the

\(^5\)Again we remind that in [26] slightly different conventions are used, in that the role of $A$ and $B$ is interchanged.

\(^6\)At least for most equations of state, Sly included. There are indeed cases, such as self-bound strange stars and stars with condensates, where the radius can increase with the total mass even in GR. [46]
M–R curves is delimited by points where \( dM/dr \) diverges (or equivalently, where \( dr_*/dM \) vanishes). A hint to the presence of two regimes is visible also in the M-\( \rho_0 \) curves, where the value \( M_g \sim 1.5M_* \) separates two different behaviours. For smaller values of \( M_g \), the neutron star’s mass in \( R^2 \)-gravity is smaller (and even more so the higher \( \alpha \) becomes) than the analogous star in GR with the same central density. The opposite behaviour happens for values of \( M_g \lesssim 1.5M_* \).

### 4.1.2 The definitions of mass

Let us now compare the M–R curves relative to the definitions of mass discussed in Section 3. In Figure 2, the M–R curves relative to \( M_g \), \( M_\rho \), \( M_\nu \), \( M_n \) and \( M_s \) are displayed respectively for \( \hat{\alpha} = 1 \), \( \hat{\alpha} = 10 \) and \( \hat{\alpha} = 100 \), with the curve for GR (\( \hat{\alpha} = 0 \)) included in all the plots for the sake of comparison.

Several comments are in order. On general grounds note that, despite the difference in the curves, all of them share the feature mentioned above of having a region where \( dr_*/dM > 0 \), which is the more evident the bigger the value of \( \alpha \). Moreover, all the curves nearly coincide for small central density, while the difference is more pronounced when we approach the maximum mass. Note that there is a perceptible difference between the small mass limit of \( R^2 \)-gravity and that of GR: this is not surprising, since the weak field limit of \( R^2 \)-gravity is different from GR’s. This difference becomes less important when \( \alpha \) gets smaller, coherently with GR being reproduced in the \( \alpha \to 0 \) limit of \( R^2 \)-gravity.
Focusing on the region close to the maximum mass, it is apparent that the proper mass is significantly higher than the others. This is sensible since the gravitational binding energy is negative, and we are considering compact objects where gravity is not weak. It is also apparent that \( M_s \) is smaller than \( M_g \) and \( M_\rho \), and significantly so when \( \hat{\alpha} = 10 \) and \( \hat{\alpha} = 100 \). This can be understood, in terms of our analysis of Section 3, by referring to the behaviour of the extra DOF. We found that, generically, the function \( \zeta(r) \) is concave inside the star and convex outside. Note that the effective energy density explicitly reads

\[
8\pi G_N R = \frac{2}{A} \left[ \zeta'' + \left( \frac{2}{r} - \frac{A'}{2A} \right) \zeta' \right] + 3m^2 \zeta^2 + 2\zeta G_t^t ,
\]

so its sign is influenced by the second derivative of \( \zeta \). If \( \zeta'' \) dominates the other terms inside the star, then \( R \) is negative inside the star and positive outside, so the contribution of \( R \) is negative in Eq. (3.20) and positive in Eq. (3.21), explaining why \( M_s \) is both smaller than \( M_g \) and \( M_\rho \). Furthermore, since the Equation (3.12) can be rewritten as

\[
M_g - M_\rho = \int_{\mathbb{R}^3} R \, dV ,
\]

the net effect of the competition between the negative (inside) and the positive (outside) contributions of \( R \), with the preponderance of the former, is a relatively small difference between \( M_g \) and \( M_\rho \). This qualitative analysis is corroborated by the numerical results for \( \zeta \) and for the adimensional quantity \( \hat{R} = R G_N \rho_0^2 / c^4 \) as functions of the adimensional radius \( r/r_0 \), plotted in Figure 3 for the case \( \hat{\alpha} = 1 \), \( \rho_0 = 10^{15} \) g/cm\(^3\) e \( \zeta_0 = 0.0170299 \).

![Figure 3: The extra DOF \( \zeta \) (left) and the adimensional quantity \( \hat{R} \) (right), as functions of \( r/r_0 \). The continuous vertical line marks the star's surface while the dashed line marks the range of \( \zeta \).](image)

**4.1.3 Numerical values**

For completeness, we provide here a quantitative comparison between the M–R curves relative to different definitions of mass. To do this, we concentrate on the characteristic features of the curves mentioned above, to wit the maximum mass \( M^{\text{max}} \) and the mass range \( \Delta M \) where \( dr_\star/dM \) is positive. In Table 1 and 2 we list the values obtained numerically for
these quantities, for all the definitions of mass discussed above. Since our datapoints are
discrete, we evaluate $M_{\text{max}}$ by approximating the maximum point of the M–R curve with
the maximum-value datapoint. We give the numerical value with three significant digits,
without attempting to assess the uncertainty. Regarding $\Delta M$, we estimate the minimum
and maximum mass of the interval by using the mass of the datapoint where the radius
has an extremum (again, extremum with respect to the other datapoints). The error is
estimated to be the difference in mass with the datapoints adjacent to the extremum.

\[
\hat{\alpha} \quad M_{g}^{\text{max}} \quad M_{p}^{\text{max}} \quad M_{p}^{\text{max}} \quad M_{s}^{\text{max}} \quad M_{n}^{\text{max}}
\]

| $\hat{\alpha}$ | 2.05 | 2.05 | 2.84 | 2.05 | 2.05 |
|----------------|------|------|------|------|------|
| GR             | 2.06 | 2.08 | 2.85 | 1.98 | 1.85 |
| 1              | 2.14 | 2.22 | 2.96 | 1.95 | 2.09 |
| 10             | 2.21 | 2.35 | 3.09 | 1.96 | 2.20 |

Table 1: Maximum masses for GR and for $\hat{\alpha} = 1, 10, 100$. Solar mass units are employed.

| $\hat{\alpha}$ | $\Delta M_{g}$ | $\Delta M_{p}$ | $\Delta M_{s}$ | $\Delta M_{n}$ |
|----------------|----------------|----------------|----------------|----------------|
| 1              | 0.6 – 1.2      | 0.6 – 1.3      | 0.6 – 1.4      | 0.5 – 1.1      | 0.4 – 1.0 |
| 10             | 0.4 – 1.5      | 0.4 – 1.6      | 0.4 – 1.8      | 0.3 – 1.2      | 0.4 – 1.5 |
| 100            | 0.4 – 1.7      | 0.4 – 1.8      | 0.4 – 2.0      | 0.3 – 1.3      | 0.5 – 1.7 |

Table 2: Mass interval where $dr_*/dM > 0$. Solar mass units are employed, and the error is
±0.1.

It is apparent that, when $\hat{\alpha}$ lies in the range between 1 and 100, the maximum masses
$M_{g}^{\text{max}}, M_{p}^{\text{max}}$ and $M_{p}^{\text{max}}$ roughly display a logarithmic behaviour $M^\text{max}(\hat{\alpha}) = M^\text{max}(1) +
k \log_{10}(\hat{\alpha})$, with $k \simeq 0.07, \simeq 0.13$ and $\simeq 0.12$ respectively. We are not sure whether or
not this approximate behaviour continues to hold for larger values of $\hat{\alpha}$. The behaviour of
$M_{s}^{\text{max}}$ and $M_{n}^{\text{max}}$ are instead peculiar. In particular, the latter is significantly lower than the
other masses when $\hat{\alpha} = 1$ but becomes very close to $M_{g}^{\text{max}}$ when $\hat{\alpha} = 100$. It is worthwhile
to recall that $M_{n}^{\text{max}}$ can be interpreted as the mass felt by orbiting test bodies only for $\hat{\alpha}$
sufficiently big, i.e. when the nearby region exists. For example this does not happen when
$\hat{\alpha} = 1$, since in that case the range of the extra DOF is smaller than the radius of the star.

Regarding the mass range $\Delta M$, it is interesting to note that quite generically the interval
becomes wider as $\hat{\alpha}$ increases, but without shifting significantly its position. In particular,
the lower extreme of the interval decreases between $\alpha = 1$ and $\alpha = 10$ and then remains
approximately constant between $\alpha = 10$ and $\alpha = 100$. Again, the case of the nearby mass
is peculiar.
4.2 Equation of state and degeneracy

An important question is how (and how much) the results of the previous section depend on the equation of state. Specifically, we would like to understand whether there is degeneracy between the EoS and the profiles of the M–R curves relative to the various definitions of gravitational mass. While a thorough investigation is beyond the scope of the present work, it is worthwhile to perform here a preliminary investigation to shed light on this point. Therefore, we consider below the equations of state BSk19 and BSk20 [47], and perform again the analysis done above for the Sly EoS.

4.2.1 Qualitative comparison

The first comment is that the M–R curves and the M–\(\rho_0\) curves relative to the BSk equations of state share the same qualitative features with those relative to the Sly. In particular, for any definition of gravitational mass, the mass increases monotonically with \(\rho_0\) and there is an intermediate mass range where \(dr_\star/dM > 0\) (noteworthy, for the BSk20 EoS this range exists already in GR). Moreover, there exists a critical value \(\rho_0^c\) of the central density (which depends on the definition of mass under consideration, and weakly on \(\alpha\)) such that for \(\rho_0 < \rho_0^c\) the mass in \(R^{2}\)-gravity is lower than the mass in GR with the same central density, while the opposite happens for \(\rho_0 > \rho_0^c\). This is apparent from the Figure 4, relative to the BSk19 EoS, and from the Figure 5, relative to the BSk20 EoS.

![Figure 4: The M–R curves relative to the different definitions of mass for the BSk19 EoS, respectively for \(\hat{\alpha} = 1\), \(\hat{\alpha} = 10\) and \(\hat{\alpha} = 100\). The GR curve is present in all the plots. The bottom-right plot shows the M–\(\rho_0\) curves for the asymptotic mass \(M_g\).](image-url)
4.2.2 Quantitative comparison

A more effective comparison is provided by focusing on the features considered in Tables 1 and 2, that is the value of the maximum mass and the mass interval for which $dr_*/dM > 0$. The analogous results for the BSk equations of state are given respectively in Tables 3 and 5, for the BSk19 EoS, and in Tables 4 and 6, for the BSk20 EoS.

\[
\hat{\alpha} \quad M_{g}^{\text{max}} \quad M_{\rho}^{\text{max}} \quad M_{s}^{\text{max}} \quad M_{n}^{\text{max}}
\]

| $\hat{\alpha}$ | GR | 1   | 10  | 100 |
|-----------------|----|-----|-----|-----|
| $M_{g}^{\text{max}}$ | 1.86 | 1.88 | 1.95 | 2.01 |
| $M_{\rho}^{\text{max}}$ | 1.86 | 1.89 | 2.03 | 2.14 |
| $M_{s}^{\text{max}}$ | 2.58 | 2.59 | 2.70 | 2.81 |
| $M_{n}^{\text{max}}$ | 1.86 | 1.80 | 1.77 | 1.78 |

Table 3: Maximum masses for GR and for $\hat{\alpha} = 1, 10, 100$, using the BSk19 EoS. Solar mass units are employed.

Let us start commenting on the maximum gravitational mass. On a general ground, we can see that the maximum mass is higher with the BSk20 EoS than with the Sly EoS, and is lower with the BSk19 EoS. Apart from this, the behaviour of $M_{g}^{\text{max}}$ displays evident similarities for the three equations of state. At fixed $\hat{\alpha}$, the values of $M_{g}^{\text{max}}$ are ordered as...
Table 4: Same as in Table 3, but for the BSk20 EoS.

\[
M_p^{\text{max}} > M^\alpha_p > M_g^{\text{max}} > M_s^{\text{max}}, \quad (4.4)
\]

and for every value of \( \hat{\alpha} \) we have \( M_g^{\text{max}} > M_s^{\text{max}} \) and \( M_s^{\text{max}} < M_G^{\text{max}} \). The analysis of section 4.1.2 is compatible with these results, and provides a theoretical explanation about why these properties are shared by all the equations of state considered here. It may indeed suggest these properties to be quite general. The value of \( M_n^{\text{max}} \) on the other hand does not obey a clear ordering in relation to the other definitions of mass, although for all the equations of state considered it gets very close to \( M_g^{\text{max}} \) when \( \hat{\alpha} = 100 \).

Regarding the dependence of \( M_p^{\text{max}} \) on \( \hat{\alpha} \) when \( 1 \leq \hat{\alpha} \leq 100 \), note that for all the equations of state under consideration the values of \( M_g^{\text{max}} \), \( M_P^{\text{max}} \), \( M_P^{\text{max}} \), and \( M_n^{\text{max}} \) increase with increasing \( \hat{\alpha} \), while the value of \( M_s^{\text{max}} \) doesn’t. Moreover, focusing on \( M_g^{\text{max}} \), \( M_P^{\text{max}} \) and \( M_P^{\text{max}} \), the approximate logarithmic behaviour \( M^{\text{max}}(\hat{\alpha}) = M^{\text{max}}(1) + k \log_{10}(\hat{\alpha}) \) for \( 1 \leq \hat{\alpha} \leq 100 \) (discussed in section 4.1.3) still holds with the BSk equations of state (although the agreement is not so good for \( M_P^{\text{max}} \) with the BSk19). Intriguingly, as far as we can say from the study of the Sly, BSk19 and BSk20, the value of \( k \) seems to depend very little (if at all) on the EoS.

Table 5: Mass interval where \( dr_s/dM > 0 \), using the BSk19 EoS. Solar mass units are employed, and the error is \( \pm 0.1 \).

| \( \hat{\alpha} \) | \( \Delta M_g \) | \( \Delta M_P \) | \( \Delta M_P \) | \( \Delta M_s \) | \( \Delta M_n \) |
|-------------------|----------------|----------------|----------------|----------------|----------------|
| 1                 | 0.6 – 1.0      | 0.6 – 1.0      | 0.7 – 1.1      | 0.6 – 0.9      | 0.5 – 0.8      |
| 10                | 0.4 – 1.2      | 0.4 – 1.3      | 0.5 – 1.5      | 0.3 – 1.0      | 0.4 – 1.3      |
| 100               | 0.4 – 1.4      | 0.4 – 1.5      | 0.4 – 1.7      | 0.3 – 1.0      | 0.5 – 1.5      |

Let us now pass to the intermediate mass range where \( dr_s/dM > 0 \). Also in this case the behaviour of the interval \( \Delta M \) displays evident similarities for the three equations of state. It is in fact apparent that \( \Delta M \) becomes wider as \( \hat{\alpha} \) increases, without shifting significantly its position. To be more specific, with the only exception of \( \Delta M_n \), the lower extreme of the interval decreases for \( 1 \leq \hat{\alpha} \leq 10 \) and saturates for \( 10 \leq \hat{\alpha} \leq 100 \), and its “saturated” value seems not to depend on the EoS. Furthermore, focusing on \( \Delta M_g \), \( \Delta M_P \) and \( \Delta M_P \), the lower extreme of the interval saturates to the same value. The higher extreme of the interval, on the other hand, is monotonically increasing with increasing \( \hat{\alpha} \), with the only exception of
\[ \Delta M_g \]. Its value, differently from the lower extreme, depends both on the definition of mass and on the equation of state.

4.2.3 Comments on degeneracy

Overall, the answer to the question whether there is degeneracy between the EoS and the definitions of gravitational mass seems to depend at least in part on what we exactly mean by saying that there is degeneracy. For example, it is well known (and our analysis confirms it) that the maximum mass in general changes when the EoS changes. From our analysis it seems conceivable that the maximum mass correspondent to different definitions of mass can be made to have the same value by suitably changing the equation of state. Therefore, if our idea of degeneracy concerns only the maximum mass, and more specifically only two definitions of gravitational mass, then we can say that there is degeneracy.

However, we may not be concerned only with the maximum value of two definitions of gravitational mass, but we may be interested also with the relation between the various definitions of mass, and with several other features of the M–R curves. From this latter point of view, our analysis gives indications that there are similarities/regularities in the features of the families of M–R curves which cannot be averted by changing the equation of state. An example is the hierarchy (4.4) of the values of the maximum mass calculated with different definitions. This is especially true of the definitions \( M_g, M_{\rho} \) and \( M_p \), while \( M_s \) and \( M_n \) have at times peculiar behaviours, depending on the feature under consideration.

Of course our analysis, having considered only three equations of state, cannot give definite answers on this point but can merely give indications. To settle this point a thorough investigation, scanning a large number of equations of state, would be needed, but this may well be the subject of a separate publication.

5 Discussion and conclusions

In the previous sections we discussed several possible definitions of gravitational mass for a static and spherically symmetric star in \( R^2 \) gravity, and unambiguously proved, by numerical means, that these definitions are indeed quantitatively different. This confirms that, in modified theories of gravity, when speaking of the mass of a star it is not possible to avoid specifying which definition is used, and caution has to be taken when estimating the properties of a star from observational data, especially when information about a star is obtained combining different observational techniques.
As we manifestly declared, our numerical results were obtained for values of $\alpha$ which are not compatible with observations. Our declared aim, however, was to use $R^2$-gravity as a proxy to study the issue related to the definition of gravitational mass, due to its simplicity. It is fact known that modifying GR always introduces new degrees of freedom, and often introduces new characteristic scales. $R^2$-gravity is a very convenient proxy because it is simple yet nontrivial: it introduces only one extra DOF, whose potential is very simple (being just a mass term), and only one new scale (the mass $m$ or, in terms of length, the range $m^{-1}$). Choosing to work with values of $\alpha$ which span a wider domain than that allowed by observations has also advantages: it permits to appreciate more clearly the difference between the various definitions, and to perceive better how this difference depends on $\alpha$. For this reason, we believe our numerical results remain relevant, because document a phenomenon which is likely to be much more general than the particular model we considered.

5.1 General considerations

At first sight, the idea of having several inequivalent definitions of gravitational mass is surprising (disturbing, even). However, it is important to reflect on the fact that in Newtonian gravity and in GR we use the concept of mass to characterise several a priori different things, such as for example the motion of test bodies in different locations in space (e.g. close to the star/far from the star), the gravitational redshift of radiation, the energy content of a star. We are so used to GR and Newton’s theory that it is easy to take for granted that the effect of the gravitational field outside a static and spherically symmetric star can be described by a unique gravitational charge. The usefulness of using a specific model (here, $R^2$-gravity) to investigate a general problem shows up already here, since the study of static and spherically symmetric configurations in the weak field limit permits to show explicitly that the external gravitational field depends on two gravitational charges.

Indeed we would like to reverse the perspective, and propose that, if we admit that a theory of gravity may contain others degrees of freedom, apart from those of the massless and spin-2 graviton of GR, and new characteristic scales, it is natural that a unique gravitational charge fails to describe the external gravitational field. It is just that the behaviour of the gravitational field is more complex. As the example of $R^2$-gravity shows, the crux of the problem is that the presence of matter excites the gravitational potentials and at the same time the extra DOF, which in turn couples to the gravitational potentials. In the process, the characteristic scale associated to the extra DOF remains imprinted in the behaviour of the gravitational potentials, along with the characteristic of the source. From this point of view, it is GR and Newton’s theory that are special in their simplicity.

In our opinion, it is not that fruitful to try to establish which one is the definition of gravitational mass, and which are ancillary definitions. Choosing one definition over the others would in some sense be like forcing the richer phenomenology of modified gravity into the conceptual structure of GR. We feel that it is better to live with the fact that gravity is more complex when you modify GR, and accept that different definitions of mass just describe different aspects of the gravitational field. On the other hand we deem important that every definition of mass be tightly linked to specific observable phenomena, in continuity with the Newtonian’s idea behind the introduction of gravitational mass.
5.2 Comments on some specific definitions

This of course does not mean that all definitions are equally useful, in practice. Regarding the motion of test bodies in the star’s gravitational field, it is probably quite generic in MTG (at least in those with new characteristic scales) that the spacetime outside a star may be made up of different domains. In some of these domains Newtonian gravity is not reproduced, while in others it is, but the value of the effective Newtonian mass felt by test bodies (as well as the value of the local PPN parameters) vary from one domain to the other. In such a situation, to each domain where Newtonian gravity is reproduced we may assign an effective gravitational mass according to Kepler’s third law

\[ M_{\text{eff}} = \omega^2 a^3, \tag{5.1} \]

where \( \omega \) is the angular frequency (\( \omega = 2\pi/T \), where \( T \) is the period) and \( a \) is the semi-major axis of the elliptical orbit.

On the other hand, we don’t feel the definition (3.19) of mass \( M_s \) to be as useful as the others. First of all, its link to observable phenomena is weak. The idea of treating separately the extra DOF \( \zeta \) inside and outside the star therefore seems dictated mainly by the desire to keep the concept of gravitational mass inside GR’s conceptual framework. Secondly, it relies on the possibility of localising the energy of the extra DOF inside the star’s surface, which in turn relies on the possibility of associating to \( \zeta \) a (conserved) stress-energy tensor in a unique way. While a heuristic candidate for \( T_{\mu\nu} \) has been proposed, it is worthwhile to remind that, in the original fourth-order theory, the extra DOF is just a part of the gravitational field (it is essentially the curvature). To put the definition of \( M_s \) on a firm basis, it would be advisable to embed its definition in a general analysis of the concept of energy of the gravitational field in the context of the fourth order theory.

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Declarations of interest

None

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