SOME ESTIMATES FOR IMAGINARY POWERS OF THE LAPLACE OPERATOR IN VARIABLE LEBESGUE SPACES AND APPLICATIONS

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Abstract. In this paper we study some estimates of norms in variable exponent Lebesgue spaces for singular integral operators that are imaginary powers of the Laplace operator in $\mathbb{R}^n$. Using the Mellin transform argument, from these estimates we obtain the boundedness for a family of maximal operators in variable exponent Lebesgue spaces, which are closely related to the (weak) solution of the wave equation.

1. Introduction

In the recent paper [8] we studied the boundedness of Stein’s spherical maximal function $M$ in variable exponent Lebesgue spaces. The proof is based on the Rubio De Francia extrapolation method and one of the corresponding results in weighted Lebesgue spaces. The Stein’s spherical maximal functions are closely related to the solution of the wave equation in $\mathbb{R}^3$. In order to study the wave equation in $\mathbb{R}^n$, $n > 3$ we need to consider the more general spherical maximal function $M^\alpha$, $\alpha = \frac{3-n}{2}$ ([15]). To investigate such operators we use a new approach based on a Mellin transform argument used for the first time by Cowling and Mauceri in [1], which reduces the problem to that one to find sharp estimates for norms of imaginary power of the Laplace operator.

Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all measurable functions $p : \mathbb{R}^n \to [1, \infty]$, which will be called variable exponents. Set $p^- = \text{essinf}_{x \in \mathbb{R}^n} p(x)$ and $p^+ = \text{esssup}_{x \in \mathbb{R}^n} p(x)$. If $p^+ < \infty$, then the variable exponent $p$ is bounded. For $p \in \mathcal{P}(\mathbb{R}^n)$, let $p' \in \mathcal{P}(\mathbb{R}^n)$ be defined through $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, where we adopt the convention $\frac{1}{\infty} := 0$. The function $p'$ represents the dual variable exponent of $p$ (and the symbol should not be confused with the derivative).

For $p \in \mathcal{P}(\mathbb{R}^n)$, let $\mathbb{R}_\infty^n$ be the set where $p = +\infty$. The symbol $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of measurable functions real or complex valued $f$ on $\mathbb{R}^n$ such that for some $\lambda > 0$

$$\rho_{p(\cdot)} \left( \frac{f}{\lambda} \right) = \int_{\mathbb{R}^n \setminus \mathbb{R}_\infty^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx + \left\| \frac{f}{\lambda} \right\|_{L^{\infty}(\mathbb{R}_\infty^n)} < \infty.$$
This set becomes a Banach function space when equipped with the norm
\[ \|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot)} \left( \frac{f}{\lambda} \right) \leq 1 \right\}. \]

Let \( B(x,r) \) denote the open ball in \( \mathbb{R}^n \) of radius \( r \) and center \( x \). By \( |B(x,r)| \) we denote the \( n \)-dimensional Lebesgue measure of \( B(x,r) \). The Hardy-Littlewood maximal operator \( M \) is defined on locally integrable functions \( f \) on \( \mathbb{R}^n \) by the formula
\[ Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy. \]

Define the Spherical Maximal operator \( \mathcal{M} \) by
\[ \mathcal{M}f(x) := \sup_{t>0} |\mu_t \ast f(x)| = \sup_{t>0} \left| \int_{\{y \in \mathbb{R}^n : |y|=1\}} f(x-ty) \, d\mu_t(y) \right| \]
where \( \mu_t \) denotes the normalized surface measure on the sphere of center 0 and radius \( t \) in \( \mathbb{R}^n \). The Hardy-Littlewood maximal operator \( M \), which involves averaging over balls, is clearly related to the spherical maximal operator, which averages over spheres. Indeed, by using polar coordinates, one easily verifies the pointwise inequality \( Mf(x) \leq \mathcal{M}f(x) \) for any (continuous) function.

A function \( p : \mathbb{R}^n \to (0, \infty) \) is said to be locally log-Hölder continuous on \( \mathbb{R}^n \) if there exists \( c_1 > 0 \) such that
\[ |p(x) - p(y)| \leq c_1 \frac{1}{\log(e + |x - y|)} \]
for all \( x, y \in \mathbb{R}^n \), \( |x - y| < 1/2 \). Moreover, \( p(\cdot) \) satisfies the log-Hölder decay condition if there exist \( p_{\infty} \in (0, \infty) \) and a constant \( c_2 > 0 \) such that
\[ |p(x) - p_{\infty}| \leq c_2 \frac{1}{\log(e + |x|)} \]
for all \( x \in \mathbb{R}^n \). We say that \( p(\cdot) \) is globally log-Hölder continuous on \( \mathbb{R}^n \) if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. We write \( p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) if \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and \( p(\cdot) \) is globally log-Hölder continuous on \( \mathbb{R}^n \). If \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) with \( p_{+} < \infty \), then \( p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) if and only if \( p(\cdot) \) is globally log-Hölder continuous on \( \mathbb{R}^n \). If \( p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) and \( p_{-} > 1 \), then the classical boundedness theorem for the Hardy-Littlewood maximal operator can be extended to \( L^{p(\cdot)} \) (see \([1, 4, 2]\)). For more information on variable Lebesgue spaces, log-Hölder continuity conditions and their relationship with the boundedness of the Hardy-Littlewood maximal operator, see the monographs \([3, 7]\). If \( n \geq 3 \), \( p \in \mathcal{P}^{\log}(\mathbb{R}^n) \) and \( \frac{n}{n-2} < p_{-} \leq p_{+} < p_{-}^{n-1}, \) then the boundedness theorem for the spherical maximal function \( \mathcal{M} \) in \( L^{p(\cdot)} \) was proved in \([8]\).

We will denote by \( \mathcal{B}(\mathbb{R}^n) \) the class all measurable functions \( p : \mathbb{R}^n \to (0, \infty) \) for which the Hardy-Littlewood maximal operator is bounded on \( L^{p(\cdot)} \).

Throughout the paper, we denote by \( c, C, c_1, C_1, c_2, C_2, \) etc. positive constant which is independent of the main parameters but which may vary from line to line.

2. Imaginary Power of Laplace Operator in Variable Lebesgue Spaces

Let \( S(\mathbb{R}^n) \) denote the Schwartz space, consisting of sufficiently smooth functions that are rapidly decreasing at infinity. Let \( \Delta \) be the standard Laplace operator in
$\mathbb{R}^n$, given by

$$\Delta = \sum_{j=1}^{n} \partial_j^2.$$  

If $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(x) \, dx$, then

$$(-\Delta f)\wedge(\xi) = |2\pi \xi|^2 \hat{f}(\xi), \quad f \in S(\mathbb{R}^n).$$

Starting from this relation, it is natural to define $\Delta^{\beta/2}$ for any complex exponent $\beta$ by

$$((-\Delta)^{\beta/2} \hat{f})(\xi) = (2\pi|\xi|)^{\beta} \hat{f}(\xi), \quad f \in S(\mathbb{R}^n).$$

In particular, for each $0 < \alpha < n$, the operator

$$I_\alpha : f \mapsto (-\Delta)^{-\alpha/2} f$$

is known as the Riesz potential. Here $I_\alpha$ may be expressed as

$$I_\alpha f = K_\alpha * f$$

where $K_\alpha(x) = \pi^{-n/2} 2^{-\alpha} \Gamma \left( \frac{n-\alpha}{2} \right) / \Gamma \left( \frac{n}{2} \right) |x|^{-n+\alpha}$, the symbol $\Gamma$ denoting the standard gamma function, and therefore (see [15] p. 117; see also [6, 3]) $I_\alpha$ is an integral operator.

In this paper we shall consider the operator $I_{iu}$, $u \in \mathbb{R}\setminus\{0\}$, given by

$$I_{iu} f = K_{iu} * f, \quad f \in S(\mathbb{R}^n),$$

which makes sense via

$$(I_{iu} f)\wedge(\xi) = (2\pi|\xi|)^{-iu} \hat{f}(\xi), \quad f \in S(\mathbb{R}^n),$$

that is $I_{iu} = (-\Delta)^{-iu}$, an imaginary power of $-\Delta$. This operator was studied by Muckenhoupt [14] in 1960 and used by Cowling and Mauceri [1] in 1978 to prove E.M. Stein's theorem on the spherical maximal function [15].

Note that $|\hat{K}(\xi)| = |(2\pi|\xi|)^{-iu}| = 1$, so that by Plancherel’s theorem we have in $L^2(\mathbb{R}^n)$

$$\|I_{iu} f\|_2 = \|f\|_2. \tag{2.1}$$

By using further properties of the kernel $K_{iu}$, particularly the fact that it is locally integrable away from the origin and satisfies

$$|K_{iu}(x)| \leq C(1 + |u|)^{n/2} |x|^{-n}$$

and

$$|\nabla K_{iu}(x)| \leq C(1 + |u|)^{n/2 + 1} |x|^{-n-1}$$

for $x \neq 0$ (see [3] and [10]), one may observe that $I_{iu}$ also extends to a bounded operator on $L^p_w(\mathbb{R}^n)$. By $w$ we mean a weight, i.e. a non-negative, locally integrable function on $\mathbb{R}^n$. When $1 < p < \infty$, we say $w \in A_p$ if for every ball $B$

$$\frac{1}{|B|} \int_B w(x) \, dx \left( \frac{1}{|B|} \int_B w(x)^{1-p'} \, dx \right)^{p-1} \leq C < \infty.$$  

By $A_{p,w}$ we denote the infimum over the constants on the right-hand side of the last inequality.

Theorem 2.1 (III). Let be $1 < p < \infty$ and $w \in A_p$. For each $\delta \in (0, 1)$ and $u \in \mathbb{R}\setminus\{0\}$, the following weighted estimate holds whenever $w \in A_p$,

$$1 < p < \infty:$

$$\|I_{iu} f\|_{p,w} \leq C(1 + |u|)^{n/2 + \delta} \|f\|_{p,w}, \quad f \in L^p_w(\mathbb{R}^n). \tag{2.2}$$
Our extension of this theorem in the variable Lebesgue spaces setting is the following

**Theorem 2.2.** Let be $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then for all $\alpha \in (0, 1)$ there exists a constant $C$ such that, for all $u \in \mathbb{R}\setminus\{0\}$

\[
\|I_u f\|_{p(\cdot)} \leq C(1 + |u|)^{n/2 + \delta} \|f\|_{p(\cdot)}, \quad f \in L^{p(\cdot)}(\mathbb{R}^n).
\]

To prove this Theorem we need the extrapolation theorem for variable Lebesgue spaces. By $\mathcal{F}$ we will denote a family of ordered pairs of non-negative, measurable functions $(f, g)$. We say that an inequality

\[
\int_{\mathbb{R}^n} f(x)^{p_0} w(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w(x) dx, \quad (0 < p_0 < \infty)
\]

holds for any $(f, g) \in \mathcal{F}$ and $w \in A_q$ (for some $q, 1 < q < \infty$) if it holds for any pair in $\mathcal{F}$ such that the left-hand side is finite, and the constant $C$ depends only on $p_0$ and on the constant $A_{q, w}$.

**Theorem 2.3.** ([7] Theorem 7.2.1, page 214, see also [6] Theorem 4.25, page 87), [4] Chap. 5). Given a family $\mathcal{F}$, assume that $(2.3)$ holds for some $1 < p_0 < \infty$, for every weight $w \in A_{p_0}$ and for all $(f, g) \in \mathcal{F}$. Let exponent $p(\cdot)$ be such that there exists $1 < p_1 < p_\alpha$, with $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$. Then

\[
\|f\|_{p(\cdot)} \leq C\|g\|_{p(\cdot)}
\]

for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

**Proof.** Using Theorem 2.2 estimate (2.2) and the fact that if $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ then $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < p_1 < p_\alpha$, (see [7] Theorem 5.7.2, page 181) we obtain (2.3). □

**Corollary 2.4.** Let $\frac{1}{p(\cdot)} = \frac{1-\theta}{p_1} + \frac{\theta}{p_\alpha}$ for some $0 < \theta < 1$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then for all $0 < \delta < 1$ there exists a constant $C$ such that, for all $u \in \mathbb{R}\setminus\{0\}$

\[
\|I_u f\|_{p(\cdot)} \leq C(1 + |u|)^{\delta n/2 + \theta \delta} \|f\|_{p(\cdot)}, \quad f \in L^{p(\cdot)}(\mathbb{R}^n).
\]

**Proof.** By using the complex interpolation theorem for variable exponent Lebesgue spaces (see [7] Theorem 1.2, page 215), we have $L^{p(\cdot)}(\mathbb{R}^n) = [L^2(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n)]_{\theta}$.

Therefore, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and

\[
\|I_u\|_{L^{p(\cdot)} \rightarrow L^{p(\cdot)}} \leq \|I_u\|_{L^2 \rightarrow L^2}^{1-\theta} \|I_u\|_{L^{p(\cdot)} \rightarrow L^{p(\cdot)}} \leq C(1 + |u|)^{\theta n/2 + \theta \delta}.
\]

□

3. The spherical maximal function

For $\alpha > 0$, let $m_{\alpha}(x) = (1 - |x|^2)^{\alpha - 1}/\Gamma(\alpha)$, where $|x| < 1$, and $m_{\alpha}(x) = 0$ if $|x| \geq 1$. With $m_{\alpha,t}(x) = m_{\alpha}(x/t)t^{-n}$, $t > 0$, we define spherical means of (complex) order $Re \alpha > 0$, by

\[
\mathcal{M}_{t}^{\alpha} f(x) = (m_{\alpha,t} * f)(x).
\]

Note that the Fourier transform of $m_{\alpha}$ is given by

\[
\hat{m}_{\alpha}(\xi) = \pi^{-\alpha+1}|\xi|^{-n/2-\alpha+1}J_{\alpha/2+\alpha-1}(2\pi|\xi|).
\]

The definition of $\mathcal{M}_{t}^{\alpha}$ can be extended to the region $Re \alpha \leq 0$ by the analytic continuation. Indeed for complex $\alpha$ in general we can define the operator $\mathcal{M}_{t}^{\alpha}$ by

\[
(\mathcal{M}_{t}^{\alpha} f)^{\wedge}(\xi) = \hat{m}_{\alpha}(t\xi)\hat{f}(\xi), \quad f \in S(\mathbb{R}^n).
\]
Theorem 3.2. The spherical maximal operator of order $\alpha$ is defined by

\[ M^\alpha f(x) = \sup_{t > 0} |M_{t^\alpha} f(x)|. \]

Note that for $\alpha = 0$ we have $M^\alpha f(x) = cMf(x)$ for an appropriate constant $c$.

**Theorem 3.1** (E.M. Stein [15]). The inequality $\|M^\alpha f\|_p \leq A_{p,\alpha} \|f\|_p$ holds in the following circumstances:

(a) if $1 < p \leq 2$, when $\alpha > 1 - n + n/p$,

(b) if $2 \leq p \leq \infty$, when $\alpha > (1/p)(2 - n)$.

Note that for $\alpha$ in Steins’s theorem we have the restriction $1 - n/2 < \alpha < 1$ and, moreover, we have

a) if $\alpha = 0$, then $n \geq 3$, $p > \frac{n}{n+1}$,

b) if $0 < \alpha < 1$ then $n \geq 2$, $\frac{n}{n+1-\alpha} < p \leq \infty$,

c) If $1 - n/2 < \alpha < 0$ then $n \geq 3$, $\frac{n}{n+1-\alpha} < p < \frac{2n}{\alpha}$.

In this section we study boundedness properties of the Stein’s spherical maximal operator $M^\alpha$ on the variable exponent Lebesgue spaces. Our main result is following

**Theorem 3.2.** Let $1 - n/2 < \alpha < 1$ and let $\tilde{p}(\cdot) := \frac{2\tilde{p}(\cdot)\theta}{\pi(2(1-\theta)p(\cdot))} \in \mathcal{B}(\mathbb{R}^n)$ for some $0 < \theta < 1 - \frac{2}{n} + \frac{2}{\alpha} n$. Then the spherical maximal operator $M^\alpha$ is bounded on $L^{\tilde{p}(\cdot)}(\mathbb{R}^n)$.

*Proof.* Let $F_\alpha(\lambda) = \pi^{-\alpha+1}\lambda^{-n/2-\alpha+1}J_{n/2+\alpha-1}(2\pi\lambda)$, $\lambda > 0$, where $J_\nu$ denotes the Bessel function of order $\nu$.

Then we have

\[ (M^\alpha f)^\vee(\xi) = F_\alpha(t(|\xi|))\tilde{f}(\xi) \]

Let $F^*_\alpha(\lambda) = F_\alpha(\lambda) - F_\alpha(0)e^{-\lambda^2}$, so that $F^*_\alpha(0) = 0$. Using the Mellin transform we have

\[ F^*_\alpha(\lambda) = \int_\mathbb{R} A_\alpha(u)\lambda^{iu}du, \quad \lambda > 0, \]

that is, $F^*_\alpha$ is the Mellin transform of $A_\alpha(u)$ (for the Mellin transform see [9]). By the Fourier Inversion Theorem, this holds if and only if

\[ A_\alpha(u) = \frac{1}{2\pi} \int_0^\infty F^*_\alpha(\lambda)\lambda^{-1-iu}d\lambda, \quad u \in \mathbb{R}. \]

For $f \in \mathcal{S}(\mathbb{R}^n)$ we have

\[ m_\alpha * f(x) = (m_\alpha^\alpha(\xi) \cdot \tilde{f}(\xi))^\vee(x) = (F_\alpha(t(|\xi|)) \cdot \tilde{f}(\xi))^\vee(x) \]

\[ = \left( (F^*_\alpha(t(|\xi|)) + F_\alpha(0)e^{-|t||\xi|^2}) \cdot \tilde{f}(\xi) \right)^\vee(x). \]

and

\[ (F^*_\alpha(t(|\xi|)) \cdot \tilde{f}(\xi))^\vee(x) = \left( \int_R A_\alpha(u)(t(|\xi|)u\tilde{f}(\xi))^\vee(u) \right)(x) \]

\[ = \int_R A_\alpha(u)t^{iu} \left( |\xi|^{iu} \tilde{f}(\xi) \right)^\vee(x)du \]

\[ = \int_R A_\alpha(u)t^{iu}(2\pi)^{-iu} \left( (2\pi|\xi|)^{iu} \tilde{f}(\xi) \right)^\vee(x)du \]

\[ = \int_R A_\alpha(u)t^{iu}(2\pi)^{-iu}I_{su}f(x)du. \]
Since $|(2\pi)^{-iu}f(x)| = 1$, by using Minkovski’s inequality and Corollary 2.4 we obtain

$$
\| (F^\alpha_\alpha (t|\xi|) \cdot \hat{f}(\xi))^{\vee} (x) \|_{p(\cdot)} = \left\| \int_R A_\alpha(u)(2\pi)^{-iu} I_{tu} f(x) du \right\|_{p(\cdot)} \\
\leq \int_R \| A_\alpha(u)(2\pi)^{-2iu} I_{tu} f(x) \|_{p(\cdot)} du \\
\leq \int_R \| A_\alpha(u) \| I_{tu} f(\cdot) \|_{p(\cdot)} du \\
\leq C \int_R \| A_\alpha(u) \| (1 + |u|)^{\theta n/2 + \theta \delta} du \| f \|_{p(\cdot)}.
$$

Using the following expression for $A_\alpha(u)$ (see [11])

$$
A_\alpha(u) = \frac{\Gamma(\alpha + n/2 - 1/2)}{4\pi^{n/2}} \left[ \frac{2^{-iu}}{\Gamma(\alpha + n/2 + iu/2)} - \frac{1}{\Gamma(\alpha + n/2)} \right]
$$

we have $A_\alpha(u) = O \left( (1 + |u|)^{-\alpha - n/2} \right)$. Finally we get

$$
\| (F^\alpha_\alpha (t|\xi|) \cdot \hat{f}(\xi))^{\vee} (x) \|_{p(\cdot)} \leq C \int_R (1 + |u|)^{-\alpha - n/2} \| f \|_{p(\cdot)} du \\
\leq C \int_R (1 + |u|)^{-\alpha - n/2 + \theta n/2 + \theta \delta} du \| f \|_{p(\cdot)} < \infty
$$

and therefore

$$
(3.2) \quad \| (F^\alpha_\alpha (t|\xi|) \cdot \hat{f}(\xi))^{\vee} (x) \|_{p(\cdot)} \leq C \| f \|_{p(\cdot)}.
$$

It is known that (see [16], p. 61),

$$
\left| (F_\alpha(0)e^{-|\xi|^2} \cdot \hat{f}(\xi))^{\vee} (x) \right| \leq C \| P_t * |f| \| (x),
$$

where $P_t$ is the Poisson kernel. Since (see [16] Theorem 1, p.62)

$$
\sup_{t \geq 0} P_t * |f| (x) \leq C M f (x),
$$

where $M$ is the Hardy-Littlewood maximal function, using the assumption $\tilde{p} (\cdot) \in \mathcal{B}(\mathbb{R}^n)$, the identity $\frac{1}{p(\cdot)} = \frac{1 - \theta}{2} + \frac{\theta}{p(\cdot)}$ and the complex interpolation method

$$
L^p(\mathbb{R}^n) \subset [L^2(\mathbb{R}^n), L^p(\mathbb{R}^n)]_\theta
$$

, we obtain that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and therefore

$$
(3.3) \quad \left\| (F_\alpha(0)e^{-|\xi|^2} \cdot \hat{f}(\xi))^{\vee} (x) \right\|_{p(\cdot)} \leq C \| M f \|_{p(\cdot)} \leq C \| f \|_{p(\cdot)}.
$$

Using (3.2) and (3.3) we get

$$
\| \mathcal{M}^\alpha f \|_{p(\cdot)} \leq C \| f \|_{p(\cdot)}.
$$
Finally, it is not hard to prove that equality (3.4) holds. We need the following Lemma 3.3.

**Lemma 3.3.** Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$. If $1 - n/2 < \alpha < 1$ and $-\frac{n}{n-1+\alpha} < p_- \leq p_+ < \frac{n}{1-\alpha}$, then there exist $\theta$, $0 < \theta < 1 - \frac{2}{n} + \frac{2}{n} \alpha$ and variable exponent $\tilde{p}(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$, such that we have

$$
\frac{1}{\tilde{p}(\cdot)} = \frac{1 - \theta}{2} + \frac{\theta}{p(\cdot)}.
$$

**Proof.** We need to find $\theta$ such that $0 < \theta < 1 - \frac{2}{n} + \frac{2}{n} \alpha$ and exponent $\tilde{p}(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$ such that (3.4) holds.

We have

$$
\frac{1}{n} - \frac{\alpha}{n} < \inf_{x \in \mathbb{R}^n} \frac{1}{p(x)} \leq \sup_{x \in \mathbb{R}^n} \frac{1}{p(x)} < 1 - \frac{1}{n} + \frac{\alpha}{n}.
$$

Let $\frac{1}{\tilde{p}(x)} = \frac{1}{2} + r(x)$. It is easy to see that

$$
\frac{1}{n} - \frac{\alpha}{n} - \frac{1}{2} < \inf_{x \in \mathbb{R}^n} r(x) \leq \sup_{x \in \mathbb{R}^n} r(x) < 1 - \frac{1}{n} + \frac{\alpha}{n}.
$$

Equation (3.5) is equivalent to

$$
\frac{1}{2} + \frac{r(x)}{\theta} = \frac{1}{\tilde{p}(x)}.
$$

Using (3.5) we may take a small $\delta > 0$ such that

$$
\frac{1}{n} - \frac{\alpha}{n} - \frac{1}{2} + \delta < \inf_{x \in \mathbb{R}^n} r(x) \leq \sup_{x \in \mathbb{R}^n} r(x) < 1 - \frac{1}{n} + \frac{\alpha}{n}.
$$

Then for $\theta, 0 < \theta < 1 - \frac{2}{n} + \frac{2}{n} \alpha$, $\theta = 1 - \frac{2}{n} + \frac{2}{n} \alpha - \theta_0$, $\theta_0 > 0$ we have

$$
\frac{1}{n} - \frac{\alpha}{n} - \frac{1}{2} + \delta < \inf_{x \in \mathbb{R}^n} \frac{r(x)}{\theta} \leq \sup_{x \in \mathbb{R}^n} \frac{r(x)}{\theta} < 1 - \frac{1}{n} + \frac{\alpha}{n} - \theta_0.
$$

and taking $\theta_0 < 2\delta$ we get

$$
-\frac{1}{2} < \inf_{x \in \mathbb{R}^n} \frac{r(x)}{\theta} \leq \sup_{x \in \mathbb{R}^n} \frac{r(x)}{\theta} < \frac{1}{2}.
$$

From (3.6) and (3.7) it follows

$$
0 < \inf_{x \in \mathbb{R}^n} \frac{1}{\tilde{p}(x)} \leq \sup_{x \in \mathbb{R}^n} \frac{1}{\tilde{p}(x)} < 1.
$$

Finally, it is not hard to prove that $\tilde{p}(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$. Indeed we may use the simple equality

$$
\left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| = \theta \left| \frac{1}{\tilde{p}(x)} - \frac{1}{\tilde{p}(y)} \right|
$$

\[ \boxed{\blacksquare} \]

**Remark 3.4.** In fact in the Lemma 3.3 we use only the simple fact that if $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ then also $\tilde{p}(\cdot) \in \mathcal{P}^{log}(\mathbb{R}^n)$. For applications we need to consider such classes of exponents with this property, because they make the Hardy-Littlewood maximal function bounded on variable exponent Lebesgue spaces. In general the class of variable exponents $\mathcal{B}(\mathbb{R}^n)$, for which the Hardy-Littlewood maximal function are bounded on variable exponent Lebesgue spaces have not such property (see [13] and the digression in [3 Chap. 4]). The opposite statement by interpolation...
Theorem 3.6. Let \( \frac{n}{n-1+\alpha} < p_- \leq p_+ < \frac{n}{1-\alpha} \). Then spherical maximal functions \( M^\alpha \) are bounded on \( L^{p_+}(\mathbb{R}^n) \).

**Proof.** Using Lemma 3.3 and the complex interpolation method for variable exponent Lebesgue spaces, from Theorem 3.2 we deduce the desired result. \( \square \)

**Theorem 3.6.** Let \( n \geq 2 \) and \( p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) with \( \frac{n}{n-1+\alpha} < p_- \leq p_+ < \frac{n}{1-\alpha} \), \( 0 \leq \alpha < 1 \). Then the spherical maximal operator \( M^\alpha \) is bounded on \( L^{p_+}(\mathbb{R}^n) \).

**Proof.** Fix \( \gamma \) such that \( 1 < \gamma < \frac{p_-(n-1+\alpha)}{n} \) and \( p_+ \gamma < \frac{p_-(n-1+\alpha)}{1-\alpha} \). Define the variable exponent

\[
\overline{p}(x) := \frac{p(x)\gamma}{p_-(n-1+\alpha)}.
\]

It is clear that \( \overline{p} \in \mathcal{P}^{\log}(\mathbb{R}^n) \),

\[
\overline{p}_- = \frac{n\gamma}{n-1+\alpha} > \frac{n}{n-1+\alpha}
\]

and

\[
\overline{p}_+ = \frac{p_+ n\gamma}{p_-(n-1+\alpha)} < \frac{n}{n-\alpha}.
\]

By Lemma 3.3 there exists \( \theta \), \( 0 < \theta < 1 - \frac{2}{n} + \frac{2}{n} \alpha \) and a variable exponent \( \bar{p} \in \mathcal{P}^{\log}(\mathbb{R}^n) \), such that

\[
\frac{1}{\bar{p}(\cdot)} = \frac{1}{p(\cdot)} = \frac{1}{\overline{p}(\cdot)} + \frac{\theta}{\overline{p}(\cdot)}.
\]

Therefore, by the complex interpolation theorem for variable exponent Lebesgue spaces, we have

\[
L^{\bar{p}(\cdot)}(\mathbb{R}^n) = [L^2(\mathbb{R}^n), L^{\overline{p}(\cdot)}(\mathbb{R}^n)]_{\theta}.
\]

From Theorem 3.2 we deduce that the spherical maximal functions \( M^\alpha \) are bounded on \( L^{\bar{p}(\cdot)}(\mathbb{R}^n) \).

By the complex interpolation method for variable exponent Lebesgue spaces, we have

\[
[L^\infty(\mathbb{R}^n), L^{\bar{p}(\cdot)}(\mathbb{R}^n)]_{\theta} = L^{p(\cdot)}(\mathbb{R}^n)
\]

for \( \theta = \frac{n\gamma}{p_-(n-1+\alpha)} \in (0, 1) \). Finally, by using the fact that if \( 0 \leq \alpha < 1 \) the spherical maximal functions \( M^\alpha \) are bounded on \( L^\infty(\mathbb{R}^n) \), we obtain that the spherical maximal functions \( M^\alpha \) are bounded on \( L^{p(\cdot)}(\mathbb{R}^n) \). \( \square \)

Taking \( \alpha = 0 \) in Theorem 3.6 we get immediately the following

**Corollary 3.7.** Let \( n \geq 3 \), \( p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \) and \( \frac{n}{n-1} < p_- \leq p_+ < p_-(n-1) \). Then the spherical maximal function \( M \) is bounded on \( L^{\bar{p}(\cdot)}(\mathbb{R}^n) \).
4. An application

Spherical averages often make their appearance as solutions of certain partial differential equations. We will state two corollaries, whose proofs are immediate consequences of the boundedness of $M^\alpha$ in the variable exponent Lebesgue spaces and of (3.1).

a) Let $\alpha = \frac{3-n}{2}$. For an appropriate constant $c_n$, we have that $u(x,t) = c_n t M^\alpha_t(x)$, where $u$ is the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} (x,t) = \Delta_x (u)(x,t),$$
$$u(x,0) = 0,$$
$$\frac{\partial u}{\partial t} (x,t) = f(x),$$

(see [16], page 519).

**Corollary 4.1.** Let $n \geq 3$ and $p(\cdot) \in P^{\log}(\mathbb{R}^n)$. If $\frac{2n}{n+1} < p_- \leq p_+ < \frac{2n}{n-1}$, then for the weak solution $u = u(x,t)$ of the wave equation with the initial data in $f \in L^p(\mathbb{R}^n)$, we have the following a priori estimate:

$$\left\| \sup_{t>0} \frac{|u(x,t)|}{t} \right\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)},$$

where $C$ depends on $p(\cdot)$ only.

Also, $\lim_{t \to 0} \frac{|u(x,t)|}{t} = f(x)$ a.e. $x \in \mathbb{R}^n$.

The estimate (4.1) tells that the integrability property of the initial velocity of propagation of a wave is preserved in the solution, under an assumption of regularity of the exponent, expressed in terms of a logarithmic Dini-type condition. In some sense (which is made precise by such logarithmic Dini-type condition), in the places where the initial velocity does not blow up (in the sense of the boundedness of the norm in a Lebesgue space which may vary pointwise), also the wave does not. For a more detailed digression, see [8].

b) Let $\alpha = \frac{3-n}{2}$. For an appropriate constant $c_n$, the spherical average $u(x,t) = c_n M^\alpha_t(x)$ solves Darboux’s equation (see [12])

$$\frac{\partial^2 u}{\partial t^2} (x,t) + 2 \frac{\partial u}{\partial t} (x,t) = \Delta_x (u)(x,t),$$
$$u(x,0) = f(x),$$
$$\frac{\partial u}{\partial t} (x,t) = 0.$$

**Corollary 4.2.** Let $n \geq 3$ and $p(\cdot) \in P^{\log}(\mathbb{R}^n)$. If $\frac{2n}{n+1} < p_- \leq p_+ < \frac{2n}{n-1}$, then for the weak solution $u = u(x,t)$ of the Darboux’s equation with the initial data $f \in L^p(\mathbb{R}^n)$ we have the following a priori estimate:

$$\| \sup_{t>0} u(x,t) \|_{p(\cdot)} \leq C \|f\|_{p(\cdot)},$$

where $C$ depends on $p(\cdot)$ only.

Also, $\lim_{t \to 0} |u(x,t)|t = f(x)$ a.e. $x \in \mathbb{R}^n$. 
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