Scalarized charged black holes with scalar mass term

De-Cheng Zou\textsuperscript{a,\textcopyright} and Yun Soo Myung\textsuperscript{\dag}

\textsuperscript{a}Institute of Basic Sciences and Department of Computer Simulation, Inje University
Gimhae 50834, Korea
\textsuperscript{b}Center for Gravitation and Cosmology and College of Physical Science and Technology,
Yangzhou University, Yangzhou 225009, China

Abstract

We study the scalarized charged black holes in the Einstein-Maxwell-Scalar (EMS) theory with scalar mass term. In this work, the scalar mass term is chosen to be \( m_\phi^2 = \frac{\alpha}{\beta} \), where \( \alpha \) is a coupling parameter and \( \beta \) is a mass-like parameter. It turns out that any scalarized charged black holes are not allowed for the case of \( \beta \leq 4.4 \) with \( q = 0.7(M = 0.5, Q = 0.35) \) because this case implies the stable Reissner-Nordström (RN) black holes. In the massless limit of \( \beta \rightarrow \infty \), one recovers the case of the EMS theory. We note that the unstable RN black hole implies the appearance of scalarized charged black holes. The other unstable case of \( \beta > 4.4 \) with \( q = 0.7 \) allows us to obtain the \( n = 0, 1, 2, \cdots \) scalarized charged black holes for \( \alpha(\beta) \geq \alpha_{\text{th}}(\beta) \) where \( \alpha_{\text{th}}(\beta) \) represents the threshold of instability for the RN black hole. Furthermore, it is shown that the \( n = 0 \) black hole is stable against radial perturbations, while the \( n = 1 \) black hole is unstable. This stability result is independent of the mass parameter \( \beta \).

\textsuperscript{\dag}e-mail address: dczou@yzu.edu.cn
\textsuperscript{\dag}e-mail address: ysmyung@inje.ac.kr
1 Introduction

Recently, the scalarized black holes have been found from the Einstein-Gauss-Bonnet-Scalar (EGBS) theory which includes a scalar-Gauss-Bonnet coupling term \[ f(\phi)G \] \[ \text{[1, 2, 3]} \]. Here, \( f(\phi) = \alpha \phi^2 / 2 \) is chosen for quadratic coupling and \( f(\phi) = \alpha (1 - e^{-6\phi^2}) / 12 \) for exponential coupling, indicating different types from linear and dilatonic couplings. In these models, the appearance of scalar hairy black hole results from the unstable Schwarzschild black holes. The scalar-Gauss-Bonnet coupling term induces negative potential outside the horizon and the coupling constant \( \alpha \) plays the role of a spectral parameter in the linearized scalar equation around the Schwarzschild black hole. It is meaningful to say that the scalar hairy black holes appear through a spontaneous scalarization for a large coupling constant in the full EGBS system.

More recently, introducing a scalar mass term has an effect on the bifurcation points where the scalarized black holes branch out of the Schwarzschild black hole without scalar hair in the EGBS theory \[ \text{[4, 5, 6]} \]. This theory includes a quadratic scalar term with mass as well as the scalar-Gauss-Bonnet coupling term. In other words, the mass term changes the threshold for scalarization surely and it may give the black hole mass range over which scalarized black holes can exist. Moreover, it is suggested that a quartic scalar term is sufficient to make a stable \( n = 0 \) black hole against the radial perturbations without introducing an exponential coupling term. However, we note that this indicates a feature of the EGBS theory with quadratic coupling. In this direction, it is worth noting that the scalarized charged black holes were found from the Einstein-Maxwell-Scalar (EMS) theory \[ \text{[7, 8]} \]. We would like to mention that the \( n = 0 \) black hole are stable in the EMS theory with exponential and quadratic couplings \[ \text{[9]} \].

On the other hand, it is curious to know why a single branch of the non-Schwarzschild black hole with Ricci tensor hair exists in the Einstein-Weyl (EW) gravity whose Lagrangian takes the form of \( \mathcal{L}_{\text{EW}} = \sqrt{-g} [R - C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} / 2m_2^2] \) with coupling parameter \( m_2^2 \) \[ \text{[10]} \]. Actually, an apparent difference implies that many branches of \( \alpha_{n=0,1,2,\ldots} = \{8.019, 40.84, 99.89, \ldots\} \) for \( q = 0.7 \) exist in the EMS theory with exponential coupling term \( e^{\alpha \phi^2} F^2 \), while a single branch of \( m_2^2 = 0.767 \) exists for the EW gravity \[ \text{[11]} \]. This seems to appear because an asymptotic form of Zerilli potential \( V \to 0 \) in the EMS theory \[ \text{[12]} \]. This means that the scalar perturbation vanishes asymptotically \( (\phi_\infty \to 0) \) in the EMS theory, while the \( s(l = 0) \)-mode of Ricci
tensor perturbation takes a normalizable form \((\psi_\infty \rightarrow e^{-m_2 r})\) in the EW gravity. An asymptotic correspondence would be met naively when one proposes a mass term of \(V_\phi = 2 m^2 \phi^2\). In this case, a scalar potential takes the form of \(V_{\text{mass}}(r) = f(r)[2M/r^3 + m^2 - (m^2 + 2)Q^2/r^4]\) which shows a similar asymptote \((V_{\text{mass}} \rightarrow m^2, \text{as } r \rightarrow \infty)\) to \(V_Z(r)\). However, it turns out that for this mass term, all potentials are positive definite outside the horizon, providing the sufficient condition for stability. Therefore, this choice does not allow any scalarized charged black holes. Of course, an independent choice of mass parameter is available and it may lead to the scalarized charged black holes on the analogy of the EGBS theory with scalar mass term.

In this work, we wish to investigate how the number of bifurcation points can be changed when including a specific mass term of \(V_\phi = 2(\alpha/\beta)\phi^2\). Here, \(\alpha\) is a coupling parameter and \(\beta\) is a mass parameter in the EMS theory with scalar mass term. The original motivation is mainly to explain a difference between many branches in the EMS theory and a single branch in the EW theory. However, it turns out that for \(\beta > 4.4\) with \(q = 0.7\), the number of bifurcation points remains unchanged when including such a mass term. Instead, for \(\beta \leq 4.4\) with the same \(q\), there is no unstable RN black hole and thus, one could not find any scalarized charged black holes. In the massless limit of \(\beta \rightarrow \infty\), one recovers the case of the EMS theory. This implies that the role of scalar mass term provides either nothing or all bifurcation points, but it does not lead to a single branch of scalarized charged black holes. This indicates a difference between scalar and tensor hairs. Finally, we show that the \(n = 0\) black hole is stable against radial perturbations, while the \(n = 1\) black hole is unstable. This result is independent of the mass parameter \(\beta\).

## 2 EMS theory

The EMS theory with scalar mass term takes the form

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - 2\nabla_\mu \phi \nabla^\mu \phi - 2m_\phi^2 \phi^2 - e^{\alpha \phi^2} F^2 \right],
\]

where the mass squared is chosen to be \(m_\phi^2 = \alpha/\beta > 0\). Here \(\alpha(\beta)\) are coupling (mass) parameters and the type of scalar coupling to the Maxwell term is exponential. The other case of \(m_\phi^2 < 0(\beta < 0)\) corresponds to a genuinely tachyonic instability and, therefore, this
will be excluded from our consideration. First, we derive the Einstein equation

\[ G_{\mu\nu} = 2\nabla_\mu \phi \nabla_\nu \phi - \left[ (\nabla \phi)^2 + \frac{\alpha}{\beta} \phi^2 \right] g_{\mu\nu} + 2e^{\alpha \phi^2} T_{\mu\nu} \]  

(2)

with \( G_{\mu\nu} \) the Einstein tensor and \( T_{\mu\nu} = F_{\mu\rho} F_{\nu}^{\ \rho} - F^2 g_{\mu\nu}/4 \). The Maxwell equation is coupled to scalar as

\[ \nabla^\mu F_{\mu\nu} - 2\alpha \phi \nabla^\mu (\phi) F_{\mu\nu} = 0. \]  

(3)

We obtain the scalar field equation

\[ \nabla^2 \phi - \frac{\alpha}{\beta} \phi - \frac{\alpha}{2} e^{\alpha \phi^2} F^2 \phi = 0. \]  

(4)

Taking into account \( \bar{\phi} = 0 \) and electrically charged \( \bar{A}_t = Q/r \), the RN solution is found when solving (2) and (3)

\[ ds^2_{\text{RN}} = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \]  

(5)

with the metric function

\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \]  

(6)

The outer (inner) horizon is located at \( r = r_\pm = M(1 \pm \sqrt{1 - q^2}) \) with \( q = Q/M \). We stress that the RN solution (5) is a black hole solution to the EMS theory with scalar mass term, being independent of \( \alpha \) and \( \beta \). Hereafter, we will choose a particular case of \( q = 0.7(M = 0.5, Q = 0.35) \) as a representative of non-extremal RN black holes. In this case, solving \( f(r) = 0 \) determines the outer horizon at \( r = r_+ = 0.857 \) and the inner horizon \( r = r_- = 0.143 \).

Finally, we would like to note that the case of \( \bar{\phi} = \text{const} \) may provide a different solution because their equations are given by

\[ \bar{G}_{\mu\nu} = -\frac{\alpha}{\beta} \bar{\phi}^2 g_{\mu\nu} + 2e^{\alpha \bar{\phi}^2} \bar{T}_{\mu\nu}, \quad \nabla^\mu \bar{F}_{\mu\nu} = 0, \quad \frac{1}{\beta} = -\frac{1}{2} e^{\alpha \bar{\phi}^2} F^2. \]  

(7)

In this case, the last relation reduces to

\[ \frac{1}{\beta} = e^{\alpha \bar{\phi}^2} \frac{Q^2}{r^4} \]  

(8)

which means that \( \beta \) is not a proper coupling constant. So, we exclude the case of \( \bar{\phi} = \text{const} \) from our consideration.
3 Stability for RN black hole

The linearized theory around the RN black hole could be obtained to investigate the stability analysis of a RN black hole with $q = 0.7$. The perturbed fields are introduced by considering metric tensor ($g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$), vector ($A_{\mu} = \bar{A}_{\mu} + a_{\mu}$), and scalar ($\phi = \bar{\phi} + \varphi$) with $\bar{\phi} = 0$. We note that there are two ways to obtain the linearized theory. One way is first to bilinearize the action (1) and then, obtain its linearized equations by varying perturbed fields. The other is to linearize equations (2)-(4) directly. Adapting the latter leads to the linearized Einstein-Maxwell equations

$$\delta G_{\mu\nu}(h) = 2\delta T_{\mu\nu}, \quad \bar{\nabla}^{\mu}f_{\mu\nu} = 0$$

(9)

with a decoupled scalar equation

$$\left[\bar{\nabla}^2 - \frac{\alpha}{\beta} + \alpha Q^2 r^4\right]\varphi = 0.$$  

(10)

In the EMS theory, the last term in (10) develops a negative potential outside the horizon and thus, it may induce the instability. In the EMS theory with scalar mass term, however, there exists a competition between mass term and the last term to give a negative potential outside the horizon. Therefore, the instability is harder to realize for large scalar masses.

Concerning the stability analysis of the RN black hole, we consider the two linearized equations in (9) first because two of metric $h_{\mu\nu}$ and vector $a_{\mu}$ are coupled to each other as in the Einstein-Maxwell theory. It is worth noting that these are exactly the same linearized equations for the Einstein-Maxwell theory. We briefly review the stability of RN black hole in the Einstein-Maxwell theory. In this case, one obtained the Zerilli-Moncrief equation describing two physical degrees of freedom (DOF) for the odd-parity perturbations [13, 14], while the even-parity perturbations for two physical DOF were investigated in [15, 16]. It is known that the RN black hole is stable against the tensor-vector perturbations.

Hence, the instability of RN black holes in the EMS theory with scalar mass term will be determined entirely by the linearized scalar equation (10), indicating a feature of the EMS theory with scalar mass term. Now, let us introduce the separation of variables around a spherically symmetric RN background (5)

$$\varphi(t, r, \theta, \chi) = \frac{u(r)}{r} e^{-i\omega t} Y_{lm}(\theta, \chi).$$

(11)
Choosing a tortoise coordinate \( r_* \) defined by \( r_* = \int dr/f(r) \), a radial part of the scalar equation takes the form
\[
\frac{d^2 u}{dr_*^2} + \left[ \omega^2 - V(r) \right] u(r) = 0.
\]
(12)

Here the scalar potential \( V(r) \) is given by
\[
V(r) = f(r) \left( \frac{2M}{r^3} + \frac{l(l+1)}{r^2} + \frac{\alpha}{\beta} - \frac{2Q^2}{r^4} - \frac{\alpha Q^2}{r^4} \right),
\]
(13)
which seems to be a complicated form. The \( s(l = 0) \)-mode is an allowable mode for the scalar perturbation and thus, it could be used to test the instability of the RN black hole. Hereafter, we confine ourselves to the \( l = 0 \) mode. It is interesting to note that \( V(r) \rightarrow \alpha/\beta \) as \( r \rightarrow \infty \), compared to the massless potential of \( V_{\beta \rightarrow \infty}(r) \rightarrow 0 \) in the EMS theory. From the potential (13), the condition for positive definite potential which corresponds to sufficient condition for stability could be found as \[9\]
\[
V(r) \geq 0 \rightarrow \beta \leq G(r, \alpha) = \frac{\alpha r^4}{Q^2(\alpha + 2) - 2Mr}.
\]
(14)

We observe the behavior of \( G(r, \alpha) \) function with \( M = 0.5 \) and \( Q = 0.35 \) pictorially. Its minimum stays near \( r = r_* \) as \( \alpha \) increases for \( r \in [r_+ = 0.857, 2] \) and \( \alpha \in [0.01, 100] \). A minimum value of \( G(r, \alpha) \) locates at 5 around \( r = r_* \) for \( \alpha = 1000 \). We read off the stability bound from \( G(r, \alpha) \) as
\[
\beta \leq G(r_+, \alpha \rightarrow \infty) = \frac{r_{+}^4}{Q^2} = 4.4.
\]
(15)

However, it is not easy to obtain the instability condition from the potential (13) directly. In this direction, we need to look for the negative region of potential outside the horizon because it may indicate a signal of instability. Taking into account the stability condition (15), one expects that a negative region may allow for \( \beta > 4.4 \) and \( \alpha < \infty \). As an example, we wish to display the negative region of potential (13) as function of \( r \) and \( \alpha \) for \( \beta = 811 \) in Fig. 1(Left). We find from Fig. 1(Right) that the width and depth of negative region in \( V(r, \alpha) \) increase as \( \alpha \) increases. It is conjectured that if the potential \( V(r) \) is negative in some region, a growing perturbation may appear in the spectrum, indicating an instability of a RN black hole. However, this is not always true. A determining condition for whether a black hole is stable or not depends on whether the time-evolution of the scalar perturbation is decaying or not. The linearized scalar equation (12) around RN black hole may allow an unstable (growing) mode like \( e^{\Omega t} \) for a scalar perturbation and thus, it indicates the
Figure 1: (Left) The $\alpha$-dependent potential $V(r, \alpha, \beta = 811)$ as function of $r \in [r_+, 4.0]$ and $\alpha \in [0.01, 50]$ for $q = 0.7$. The shaded region along $\alpha$-axis represents negative region of the potential. (Right) Plots of three potentials $V(r, \alpha, \beta = 811)$ with three different $\alpha = \{8, \alpha_{\text{th}} = 8.82, 20\}$ from top to bottom near the $V$-axis.

sign for instability of the black hole. Importantly, it is stated that the instability of RN black holes implies the appearance of scalarized charged black holes. Therefore, we have to solve (12) after replacing $\omega = -i\Omega$ numerically by imposing boundary conditions: purely ingoing wave near the horizon and purely outgoing wave at infinity. From Fig. 2, we read off the threshold of instability $[\alpha_{\text{th}}(\beta)]$. Hence, the instability bound can be determined numerically by

$$\alpha(\beta) \geq \alpha_{\text{th}}(\beta)$$

with $\alpha_{\text{th}}(\beta) = \{9.345(356), 8.82(811), 8.60(1400), 8.019(\infty)\}$. On the other hand, one always finds stable RN black holes for $\alpha(\beta) < \alpha_{\text{th}}(\beta)$. From (Right) Fig. 1, for $\beta = 811$, one finds $\alpha < \alpha_{\text{th}} = 8.82$ for stable RN black hole and $\alpha \geq \alpha_{\text{th}}$ for unstable RN black holes.

4 Static scalar perturbation: bifurcation points

Now, let us check the instability bound (16) again because the precise value of $\alpha_{\text{th}}(\beta)$ determines the appearance of scalarized charged black holes. This can be confirmed by obtaining a static scalar solution [scalar cloud: $\varphi(r)$] to the linearized equation (12) with $u(r) = r\varphi(r)$ and $\omega = 0$ on the RN background. For a given $l = 0$ and $q = 0.7$, requiring an asymptotically normalizable solution ($\varphi_{\infty} \to e^{-\sqrt{\alpha/\beta}r/r}$) leads to the fact that the existence of a smooth scalar determines a discrete set for $\alpha_n(\beta)$ where $n = 0, 1, 2, \cdots$ denotes the
Figure 2: Four graphs of $\Omega$ in $e^{\Omega}$ as functions of $\alpha$ are used to determine the thresholds of instability [$\alpha_{th}(\beta)$] around RN black hole. These correspond to the crossing points at $\alpha$-axis appearing in the magnification of the enclosed region. We find $\alpha_{th}(\beta) = 9.345(356)$, $8.82(811)$, $8.60(1400)$, $8.019(\infty)$. The last one denotes the massless limit (the EMS theory) at $\beta = \infty$, whereas the other stable boundary at $\beta = 4.4$ could not represent here because it appears at $\alpha = \infty$.

Table 1: List for the first two bifurcation points depending on $\beta$: $\alpha_{n=0}(\beta)$ represents the fundamental branch and $\alpha_{n=1}(\beta)$ denotes the first excited branch. In the limit of $\beta \to 4.4$, one recovers stable RN black hole, while one recovers $\alpha_{n=0,1}$ for the EMS theory in the massless limit of $\beta \to \infty$. The underlined cases are used for stability analysis.

Figure 3: Plot of radial profiles $\varphi(z) = u(z)/z$ as function of $z = r/2M$ for the first three static perturbed scalar solutions with $\beta = 811$ and $q = 0.7$. The number $n$ of zero nodes describes the $n = 0, 1, 2$ scalarized charged black holes.
Figure 4: Plot of $\alpha_{n=0,1}(\beta)$ based on Table 1 shows effects of the mass term $\beta$ on the scalarization. Observing $\alpha_{n=0}(\beta \to 4.4) \to \infty$ implies that unstable RN black holes exit for $\beta > 4.4$. This implies that any scalarized charged black holes would be found for $\beta > 4.4$. Also, we note that $\alpha_{n=0,1}(\beta \to \infty)$ reduces to $\alpha_{n=0,1} = \{8.019, 40.84\}$ for the EMS theory.

The number of zero crossings for $\varphi(r)$ (or order number). See Fig. 3 for static scalar solutions $\varphi(z)$ for $z = r/2M$ with $\beta = 811$ and $q = 0.7$. The $n = 0$ scalar mode represents the fundamental branch of scalarized charged black holes, while the $n = 1, 2$ scalar modes denote $n = 1, 2$ higher branch of scalarized charged black holes. It is noted that this corresponds to finding the first three bifurcation points from the RN black hole.

Consequently, we confirm from Fig. 2 and Table 1 that for given $\beta$,

$$\alpha_{th}(\beta) = \alpha_{n=0}(\beta)$$

which states that the threshold of instability for RN black hole is precisely the appearance of $n = 0$ scalarized charged black holes. We find from Fig. 4 that $\alpha_{n=0,1}(\beta)$ increases as $\beta$ decreases. The instability is therefore harder to realize for larger scalar masses (as $\beta \to 4.4$).

This picture is similar to Fig. 1(Left) in Ref. [5], where no upper limit appears because they used an independent mass term. We find that in the massless limit of $\beta \to \infty$, $\alpha_{n=0,1}(\beta)$ approaches $\alpha_{n=0,1} = \{8.019, 40.84\}$ for the EMS theory. Also, we observe the other limit that $\alpha_{n=0}(\beta) \to \infty$, as $\beta \to 4.4$. In other words, we show that unstable RN black holes exist for $\beta > 4.4$ [see the opposite bound (15) for stable RN black holes]. This implies that the $n = 0$ scalarized charged black hole would be found for $\alpha \geq \alpha_{n=0}(\beta)$ with $8.019 \leq \alpha_{n=0}(\beta) < \infty$ for $\beta \in (4.4, \infty]$, showing a significant shift of the $n = 0$ scalarized charged black holes in compared to the massless case ($\alpha \geq \alpha_{n=0}(\beta \to \infty) = 8.019$) in the EMS theory. Particularly, an unallowable region for scalarization is given by $0 < \beta \leq 4.4$.
where the unstable RN black holes are never found for any $\alpha > 0$. Finally, we note that the case of $m_{\phi}^2 = 2\alpha$ with $\beta = 1$ corresponds to the stable RN black hole. Therefore, one could not find any scalarized black holes from this case.

5 Scalarized charged black holes

First of all, we would like to mention that the RN black hole is allowed for any value of $\alpha$, while a scalarized charged black hole solution may exist only for $\alpha(\beta) \geq \alpha_{th}(\beta)$ and $\beta > 4.4$. The threshold of instability for a RN black hole denotes an exact appearance of the $n = 0$ scalarized charged black hole. So, we derive the $n = 0$ scalarized RN black hole for $q = 0$.

Also, we consider the $U(1)$ potential $A_\mu = \{v(r), 0, 0, 0\}$ and scalar $\phi(r)$. Substituting these into Eqs. (2)-(4) leads to four equations for $\{A(r), B(r), v(r), \phi(r)\}$ as

\begin{align*}
\frac{1}{r^2} + \frac{1}{B}(-\frac{1}{r^2} + \frac{\alpha \phi^2}{\beta}) + \frac{A'}{rA} + e^{\alpha \phi^2} \frac{r v'^2}{B} - \phi'^2 &= 0, \\
-\frac{\alpha \phi^2}{\beta} + \frac{1 - B - r B'}{r^2} - e^{\alpha \phi^2} \frac{B v'^2}{A} - B \phi'^2 &= 0, \\
Q + e^{\alpha \phi^2} r^2 \sqrt{\frac{B}{A}} v' &= 0, \\
\phi'' + \left(\frac{2}{r} + \frac{A'}{2A} + \frac{B}{2B'}\right) \phi' + \left(-\frac{\alpha}{\beta B} + \frac{\alpha e^{\alpha \phi^2} v'^2}{A}\right) \phi &= 0.
\end{align*}

One finds an approximate solution to equations in the near horizon

\begin{align*}
A(r) &= A_1(r - r_+) + A_2(r - r_+)^2 + \ldots, \\
B(r) &= B_1(r - r_+) + B_2(r - r_+)^2 + \ldots, \\
\phi(r) &= \phi_0 + \phi_1(r - r_+) + \ldots, \\
v(r) &= v_1(r - r_+) + v_2(r - r_+)^2 + \ldots
\end{align*}
Figure 5: Plots of a scalarized charged black hole with $\alpha = 8.82$ in the $n = 0$ branch of $\alpha(\beta = 811) \geq 8.82$ and $q = 0.7$. (Left) Metric function $\delta(r) = \ln[B(r)/A(r)]/2$, $A(r)$, and $f(r)$ for the RN black hole. (Right) Scalar hair $\phi(r)$ and scalar hair $\phi_{ml}$ for the EMS theory with scalar charge $Q_s = 0.105$.

with the first-order three coefficients
\[
B_1 = \frac{1}{r_+} \left(1 - \frac{Q^2 e^{-\alpha\phi_0^2}}{r_+^2} - \frac{\alpha r_+^2 \phi_0^2}{\beta}\right), \quad \phi_1 = \frac{\alpha(Q^2 - r_+^2 e^{\alpha\phi_0^2})\phi_0}{Q^2 r_+\beta + r_+^3(-\beta + \alpha r_+^2 e^{\alpha\phi_0^2})}, \quad (27)
\]
\[
v_1 = -\frac{e^{-\alpha\phi_0^2}Q\sqrt{A_1}}{\sqrt{r_+ (r_+^2 e^{-\alpha\phi_0^2} Q^2 - \frac{\alpha r_+^2 \phi_0^2}{\beta})}}.
\]

Here $A_1$ is a free parameter. $\phi_0 = \phi(r_+)$ will be determined when matching (23)-(26) with the asymptotic solutions in the far region of $r \gg r_+$

\[
A(r \gg r_+) = 1 - \frac{2M}{r} + \ldots, \quad B(r \gg r_+) = 1 - \frac{2M}{r} + \ldots, \quad 
\phi(r \gg r_+) = \phi_{ml} e^{-\sqrt{\beta}r} + \ldots, \quad v(r \gg r_+) = \Phi + \frac{Q}{r} + \ldots, \quad (28)
\]

where $\phi_{ml} = Q_s/r$ denotes the scalar hair for the EMS theory and $\Phi = Q/r_+$ denotes the electrostatic potential. In addition, $M$, $Q_s$, and $Q$ denote the ADM mass, the scalar charge, and the electric charge, respectively. In the massless limit of $\beta \to \infty$, one recovers the asymptotic solution for the EMS theory.

Consequently, we obtain the $n = 0$ scalarized charged black hole solution shown in Fig. 5 for $\alpha = 8.82$ at $\beta = 811$. The metric function $A(r)$ has a different horizon at $\ln[r] = -0.303$ in comparison to the RN horizon at $\ln[r] = -0.154$ and it approaches the RN metric function $f(r)$ as $\ln[r]$ increases. Also, $\delta(r) = \ln[B(r)/A(r)]/2$ decreases as $\ln[r]$ increases, while $\delta_{RN}(r) = 0$ remains zero because of $B(r)/A(r) = 1$ for the RN case. From
we observe a difference between $\phi(r)$ and $\phi_{\text{ml}}$ for the EMS theory in the asymptotic region. The other scalarized charged black holes for $\beta = 258$, 356, 1400, 2000 are found similarly.

6 Stability of scalarized charged black holes

It turns out that the $n = 0 (\beta = \infty)$ black hole is stable, while the $n = 1, 2, \cdots (\beta \to \infty)$ black holes are unstable in the EMS theory with exponential and quadratic couplings [9]. Now, let us analyze the stability of $n = 0, 1$ black holes the EMS theory with scalar mass term. For this purpose, we choose three scalar masses of $\beta = 258$, 356, 811, 1400, 2000 whose $n = 0$ and $n = 1$ bifurcation points are given by $\alpha_{n=0} = \{9.619, 9.345, 8.82, 8.60, 8.493\}$ and $\alpha_{n=1} = \{56.73, 54.01, 49.15, 47.03, 45.96\}$, respectively. We focus on larger $\beta$ which provides smaller scalar mass $m^2_\phi$ for computation.

For simplicity, we perform radial (spherically symmetric) perturbations by choosing three perturbations of $H_0(t, r), H_1(t, r), \delta \phi(t, r)$ as

$$d{s^2_{\text{RP}}} = -A(r) (1 + \epsilon H_0) dt^2 + \frac{dr^2}{B(r) (1 + \epsilon H_1)} + r^2 (d\theta^2 + \sin^2 \theta d\chi^2),$$

$$\phi = \phi(r) + \epsilon \delta \phi,$$

(29)

where $A(r), B(r), \phi(r)$ denote a scalarized charged black hole and $\epsilon$ is a control parameter of perturbations. Considering the separation of variables

$$\delta \phi(t, r) = \phi_1(r)e^{\Omega t},$$

(30)

we obtain the Schrödinger-type equation for scalar perturbation

$$\frac{d^2 \phi_1(r)}{dr_*^2} - \left[ \Omega^2 + V_{\text{SBH}}(r) \right] \phi_1(r) = 0,$$

(31)

with $r_*$ is the tortoise coordinate defined by

$$\frac{dr_*}{dr} = \frac{1}{\sqrt{A(r)B(r)}},$$

(32)
Figure 6: Three scalar potentials $V_{SBH}$ with $l = 0$ scalar mode for $\beta = 811$. (Left) Around $n = 0$ black hole. Even though they contain small negative regions outside the horizon, these show stable black holes. (Right) Around $n = 1$ black hole. They indicate unstable black holes because their potentials include large negative regions outside the horizon.

and its potential reads as

$$V_{SBH}(r) = \frac{\alpha}{\beta} A \left( 1 + \alpha \phi^2 - 2 \alpha^2 \phi^4 + r\phi (4 + 5 \alpha \phi^2) \phi' \right)$$

$$-\frac{B'A}{2r} \left( -1 - 2\alpha + 4\alpha^2 \phi^2 - 10 r \alpha \phi' \phi + 3r^2 \phi'^2 \right)$$

$$-\frac{\alpha A}{r^2} (1 - B) (1 - 2 \alpha \phi^2 + 5 r \alpha \phi \phi') + \frac{AB'(r^2 \phi'^2 - 1)}{2r}$$

$$+ AB\phi'^2 (-2 - \alpha + 2 \alpha^2 \phi^2 - 5 r \alpha \phi \phi' + r^2 \phi'^2). \quad (33)$$

It is suggested from Fig. 6 that the potentials around the $n = 0$ black hole indicates small negative regions around the horizon, suggesting the instability. On the other hand, the potentials around the $n = 1$ black hole indicates large negative regions outside the horizon, showing the instability. However, the former case may be not true. The potential $V_{SBH}$ with $\alpha = 8.820 (\beta = 811)$ with small negative region does not imply the instability, but it might support the stability. The linearized scalar equation (31) around the $n = 0, 1$ scalarized charged black holes may allow either a stable (decaying) mode with $\Omega < 0$ or an unstable (growing) mode with $\Omega > 0$.

We solve (31) numerically with imposing a boundary condition that $\phi_1(r)$ vanishes at the horizon and at infinity. We find from Figs. 7 and 8 that the $n = 0$ black hole is stable against the $l = 0$ scalar mode, while the $n = 1$ black hole is unstable against the $l = 0$ scalar mode. Furthermore, we show that that the (in) stability of $n = 0 (n = 1)$ black holes is independent of the mass parameter $\beta$. 

13
Figure 7: The negative $\Omega$ is given as function of $\alpha$ for the $l = 0$ scalar mode around the $n = 0$ black hole, showing stability. Here we consider five different cases of $\beta = 258, 356, 811, 1400, $ and 2000. Five dotted curves start from $\alpha_{n=0} = 9.649, 9.345, 8.82, 8.60, 8.493$. Five red lines denote the RN black holes [See Fig. 4].

Figure 8: The positive $\Omega$ is given as function of $\alpha$ for the $s$-mode of scalar around the $n = 1$ black hole, indicating instability. Here we consider five different cases of $\beta = 258, 356, 811, 1400, $ and 2000. Five dotted curves start from $\alpha_{n=1} = 56.73, 54.01, 49.25, 47.03, 45.96$. Five red lines represent the RN black holes.

7 Discussions

One of original motivations to study this work is to understand the difference between infinite branches in the EMS theory and a single branch in the EW theory. The infinite branches of $n = 0, 1, 2, \cdots$ scalarized charged black holes in the EMS theory are not changed for $\beta > 4.4$ even for including a scalar mass term $m_\phi^2 = \alpha/\beta$, whereas these all disappear
for $0 < \beta \leq 4.4$. This is so because the bifurcation points is determined solely by the exponential coupling to the Maxwell term in the scalar equation (1). This implies that the role of scalar mass term provides either nothing or all bifurcation points, but it never lead to a single branch of scalarized charged black holes. On the other hand, the single branch is determined by the static Lichnerowicz-Ricci tensor equation $[(\triangle_L + m_2^2)\delta R_{\mu\nu} = 0]$ where a single bifurcation point is given by $m_2^2 = 0.7677$. This indicates a difference between scalar and Ricci-tensor hairs.

In this work, we have investigated the scalarized charged black holes in the EMS theory with a specific choice of scalar mass $m_2^2 = \alpha/\beta$. The computing process is as follows: detecting instability of RN black holes $\rightarrow$ prediction of scalarized charged black holes (bifurcation points) $\rightarrow$ obtaining the $n = 0, 1$ scalarized charged black holes $\rightarrow$ performing (in)stability analysis of $n = 0, 1$ scalarized charged black holes.

We find that the first two bifurcation points of $\alpha_{n=0,1}(\beta)$ increases as $\beta$ decreases. The RN instability is therefore harder to realize for larger scalar masses. We found two limits. In the massless limit of $\beta \rightarrow \infty$, $\alpha_{n=0,1}(\beta)$ approaches $\alpha_{n=0,1} = \{8.019, 40.84\}$ for the EMS theory. The other limit is given by $\alpha_{n=0}(\beta) \rightarrow \infty$, as $\beta \rightarrow 4.4$. In other words, we have stated that unstable RN black holes exist for $\beta > 4.4$ [see the opposite bound (15) for stable RN black holes]. This implies that the $n = 0$ scalarized charged black hole was found for $\alpha \geq \alpha_{n=0}(\beta)$ with $8.019 \leq \alpha_{n=0}(\beta) < \infty$ for $\beta \in (4.4, \infty]$, showing a shift from $\alpha_{n=0}(\beta \rightarrow \infty) = 8.019$. Also, the $n = 1$ scalarized charged black hole was found for $\alpha \geq \alpha_{n=1}(\beta)$ with $40.84 \leq \alpha_{n=1}(\beta) < \infty$ for $\beta \in (4.4, \infty]$, showing a shift from $\alpha_{n=1}(\beta \rightarrow \infty) = 40.84$. Interestingly, an unallowable region for scalarization is given by $0 < \beta \leq 4.4$ where the unstable RN black holes are never found for any $\alpha > 0$.

Finally, we have shown that the $n = 0$ black hole is stable against radial perturbations, while the $n = 1$ black hole is unstable. Further, it was shown that the stability result of $n = 0, 1$ black holes is independent of the mass parameter $\beta$, even though it changes the bifurcation points significantly.

Acknowledgments

This work was supported by the National Research Foundation of Korea (NRF) grant
funded by the Korea government (MOE) (No. NRF-2017R1A2B4002057).
References

[1] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. 120, no. 13, 131103 (2018) doi:10.1103/PhysRevLett.120.131103 arXiv:1711.01187 [gr-qc].

[2] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou and E. Berti, Phys. Rev. Lett. 120, no. 13, 131104 (2018) doi:10.1103/PhysRevLett.120.131104 arXiv:1711.02080 [gr-qc].

[3] G. Antoniou, A. Bakopoulos and P. Kanti, Phys. Rev. Lett. 120, no. 13, 131102 (2018) doi:10.1103/PhysRevLett.120.131102 arXiv:1711.03390 [hep-th].

[4] Y. Brihaye and L. Ducobu, Phys. Lett. B 795, 135 (2019) doi:10.1016/j.physletb.2019.06.006 arXiv:1812.07438 [gr-qc].

[5] C. F. B. Macedo, J. Sakstein, E. Berti, L. Gualtieri, H. O. Silva and T. P. Sotiriou, Phys. Rev. D 99, no. 10, 104041 (2019) doi:10.1103/PhysRevD.99.104041 arXiv:1903.06784 [gr-qc].

[6] D. D. Doneva, K. V. Staykov and S. S. Yazadjiev, Phys. Rev. D 99, no. 10, 104045 (2019) doi:10.1103/PhysRevD.99.104045 arXiv:1903.08119 [gr-qc].

[7] C. A. R. Herdeiro, E. Radu, N. Sanchis-Gual and J. A. Font, Phys. Rev. Lett. 121, no. 10, 101102 (2018) doi:10.1103/PhysRevLett.121.101102 arXiv:1806.05190 [gr-qc].

[8] P. G. S. Fernandes, C. A. R. Herdeiro, A. M. Pombo, E. Radu and N. Sanchis-Gual, Class. Quant. Grav. 36, no. 13, 134002 (2019) doi:10.1088/1361-6382/ab23a1 arXiv:1902.05079 [gr-qc].

[9] Y. S. Myung and D. C. Zou, Eur. Phys. J. C 79, no. 8, 641 (2019) doi:10.1140/epjc/s10052-019-7176-7 arXiv:1904.09864 [gr-qc].

[10] H. Lu, A. Perkins, C. N. Pope and K. S. Stelle, Phys. Rev. Lett. 114, no. 17, 171601 (2015) doi:10.1103/PhysRevLett.114.171601 arXiv:1502.01028 [hep-th].

[11] H. L, A. Perkins, C. N. Pope and K. S. Stelle, Phys. Rev. D 96, no. 4, 046006 (2017) doi:10.1103/PhysRevD.96.046006 arXiv:1704.05493 [hep-th].

[12] Y. S. Myung and D. C. Zou, Eur. Phys. J. C 79, no. 3, 273 (2019) doi:10.1140/epjc/s10052-019-6792-6 arXiv:1808.02609 [gr-qc].
[13] F. J. Zerilli, Phys. Rev. D 9, 860 (1974). doi:10.1103/PhysRevD.9.860

[14] V. Moncrief, Phys. Rev. D 9, 2707 (1974). doi:10.1103/PhysRevD.9.2707

[15] V. Moncrief, Phys. Rev. D 10, 1057 (1974). doi:10.1103/PhysRevD.10.1057

[16] V. Moncrief, Phys. Rev. D 12, 1526 (1975). doi:10.1103/PhysRevD.12.1526