Units in quasigroups with classical Bol-Moufang type identities

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Abstract

We prolong Kunen's research about existence of units (left, right, two-sided) in quasigroups with classical Bol-Moufang type identity. These Bol-Moufang type identities are listed in Fenvesh's article [9].

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1 Introduction

We research the existence of units in quasigroups with classical Bol-Moufang type identity.

Binary groupoid $(G, \ast)$ is a non-empty set $G$ together with a binary operation “$\ast$” which is defined on the set $G$.

Groupoid $(Q, \ast)$ is called a quasigroup, if the following conditions are true [2]:

$(\forall u, v \in Q)(\exists! x, y \in Q)(u \ast x = v \& y \ast u = v)$. This is a so-called existential definition [22].

The following definition is called an equational definition of quasigroup.

Definition 1. [3, 4, 8]. A groupoid $(Q, \cdot)$ is called a quasigroup if on the set $Q$ there exist operations “$\setminus$” and “$/\rangle$” such that in the algebra $(Q, \cdot, \setminus, /)$ identities

\[
\begin{align*}
x \cdot (x \setminus y) &= y, \\
(y / x) \cdot x &= y, \\
x \setminus (x \cdot y) &= y, \\
(y \cdot x) / x &= y,
\end{align*}
\]

(1) (2) (3) (4)
are fulfilled.
In any quasigroup \((Q, \cdot, \backslash, /)\) the following identities are true:

\[
x / (y \backslash x) = y, \quad (5)
\]
\[
(x / y) \backslash x = y. \quad (6)
\]

Standard introduction in quasigroup theory is given in \([2, 23, 27]\).
Let \((Q, \cdot)\) be a quasigroup.

**Definition 2.** An element \(i \in Q\) is an *idempotent* of \((Q, \cdot)\) if and only if \(i \cdot i = i\).

An element \(f \in Q\) is a *left unit* of \((Q, \cdot)\) if and only if \(f \cdot x = x\) for all \(x \in Q\).
An element \(e \in Q\) is a *right unit* of \((Q, \cdot)\) if and only if \(x \cdot e = x\) for all \(x \in Q\).
An element \(e \in Q\) is a *left and right unit* of \((Q, \cdot)\) if and only if it is both left and right unit.
An element \(m \in Q\) is a *middle unit* of \((Q, \cdot)\) if and only if \(x \cdot x = m\) for all \(x \in Q\).

**Definition 3.** A quasigroup is a left (right) loop if it has a left (right) unit.
A quasigroup is a loop if it has both left and right units.

An identity based on a single binary operation is of Bol-Moufang type if “both sides consist of the same three different letters taken in the same order but one of them occurs twice on each side” \([9]\). We use list and denotation of 60 Bol-Moufang type identities given in \([15]\).

**Remark 1.** There exist other (“more general”) definitions of Bol-Moufang type identities and, therefore, other lists and classifications of such identities \([1, 7]\).

Quasigroups and loops, in which a Bol-Moufang type identity is true, are central and classical objects of Quasigroup Theory. We recall, works of R. Moufang, G. Bol, R. H. Bruck, V. D. Belousov, K. Kunen, S. Gagola and of many other mathematicians are devoted to the study of quasigroups and loops with Bol-Moufang type identities \([22, 5, 2, 9, 13, 10, 11, 24, 7, 1]\).

Reformulating title of Gagola’s article “How and why Moufang loops behave like groups” \([11]\) we can say that quasigroups with Bol-Moufang identities “behave like groups”. This is one of reasons why we study these quasigroup classes. Notice, information on right and left unit elements in quasigroups with any Bol and any Moufang identity is known \([2, 12, 17, 14, 26]\).

In some cases we have used Prover9 \([21]\) and Mace4 \([20]\) for finding the proofs and constructing counterexamples. The nearest article to our paper is Kunen’s article \([18]\). Notice, Professor Kunen in his researches have used some versions of Prover and Mace.
2 Results

2.1 (12)-parastrophes of identities

We recall, (12)-parastroph of groupoid \((G, \cdot)\) is a groupoid \((G, \ast)\) in which operation \(\ast\) is obtained by the following rule:

\[
x \ast y = y \cdot x.
\]  

(7)

It is clear that for any groupoid \((G, \cdot)\) there exists its (12)-parastroph groupoid \((G, \ast)\).

Cayley table of groupoid \((G, \ast)\) is a mirror image of the Cayley table of groupoid \((G, \cdot)\) relative to main diagonal. Notice, for any binary quasigroup there exist five its parastrophes \([2, 23, 27]\) more.

Suppose that an identity \(F\) is true in groupoid \((G, \cdot)\). Then we can obtain (12)-parastrophic identity \(F^\ast\) of the identity \(F\) replacing the operation \(\cdot\) with the operation \(\ast\) and changing the order of variables using rule (7).

Remark 2. In quasigroup case, similarly to (12)-parastrophe identity other parastrophe identities can be defined. See \([25]\) for details.

It is clear that an identity \(F\) is true in groupoid \((G, \cdot)\) if and only if in groupoid \((Q, \ast)\) identity \(F^\ast\) is true. The following lemma is evident. See \([13]\).

Lemma 1. For classical Bol-Moufang type identities over groupoids the following equalities are true:

\[
(F_1)^\ast = F_3, \quad (F_2)^\ast = F_4, \quad (F_3)^\ast = F_{10}, \quad (F_6)^\ast = F_8, \quad (F_7)^\ast = F_9, \\
(F_{11})^\ast = F_{24}, \quad (F_{12})^\ast = F_{23}, \quad (F_{13})^\ast = F_{22}, \quad (F_{14})^\ast = F_{21}, \quad (F_{15})^\ast = F_{30}, \quad (F_{16})^\ast = F_{29}, \\
(F_{17})^\ast = F_{27}, \quad (F_{18})^\ast = F_{28}, \quad (F_{19})^\ast = F_{26}, \quad (F_{20})^\ast = F_{25}, \quad (F_{31})^\ast = F_{34}, \\
(F_{32})^\ast = F_{33}, \quad (F_{35})^\ast = F_{40}, \quad (F_{36})^\ast = F_{39}, \quad (F_{37})^\ast = F_{37}, \quad (F_{38})^\ast = F_{38}, \quad (F_{39})^\ast = F_{39}, \\
(F_{40})^\ast = F_{40}, \quad (F_{41})^\ast = F_{53}, \quad (F_{42})^\ast = F_{54}, \quad (F_{43})^\ast = F_{51}, (F_{44})^\ast = F_{52}, \quad (F_{45})^\ast = F_{60}, \quad (F_{46})^\ast = F_{56}, \\
(F_{47})^\ast = F_{58}, \quad (F_{48})^\ast = F_{57}, \quad (F_{49})^\ast = F_{59}, \quad (F_{50})^\ast = F_{55}.
\]

For quasigroups, analogue of Theorem [1] is given in [18]. It is easy to see that any group satisfies any of identities \(F_1 - F_{60}\). Therefore the cyclic group \(Z_3\) is a counter-example to proposition that “there exist a quasigroup with an identity from list of identities \(F_1 - F_{60}\) and which has middle unit”.

Lemma 2. If a quasigroup \((Q, \cdot)\) has a left (right) unit element, then (12)-parastrophe of quasigroup \((Q, \cdot)\) has right (left) unit element.

Proof. It is easy to see. □

Lemma 3. If in quasigroup \((Q, \cdot)\) the variables \(x\) and \(y\) run through the whole set \(Q\) (\(x \neq y\)), then the term \(x \cdot y\) runs through the whole set \(Q\), too.

Proof. If we fix variable \(x\), for example, \(x = a, a \in Q\), then even the term \(ay\) runs through the whole set \(Q\). See, also, [27]. □
Notice, situation is another when $x = y$. See, for example, identity $F_{12}$, Theorem 24.

**Lemma 4.** If in quasigroup $(Q, \cdot)$ identity of associativity is true, then this quasigroup is a group.

**Proof.** In order to show for readers of this paper methods of the proofs, which usually are used in this paper, we give the well known standard proof. See, for example, [19].

Here we use Lemma 3. In identity of associativity $x \cdot yz = xy \cdot z$ we put $x = f_y$ ($f_y y = y$). Then we have $f_y y \cdot yz = yz$. Therefore element $f_y$ is a left unit of quasigroup $(Q, \cdot)$.

In identity of associativity $x yz = xy \cdot z$ we put $z = e_y$ ($ye_y = y$). Then we have $xy = xy \cdot e_z$. Therefore element $e_z$ is a right unit of quasigroup $(Q, \cdot)$.

**Lemma 5.** If quasigroup $(Q, \cdot)$ is a group, then quasigroup $(Q, \cdot)^{(12)}$ is a group.

**Proof.** This is mathematical folklore.

**Theorem 1.** Quasigroup $(Q, \cdot)$ with identity $F_1$ is a group.

**Proof.** Using Lemma 3, we can rewrite identity $F_1 (xy \cdot zx = (xy \cdot z)x)$ in the form $t \cdot zx = tz \cdot x$, where $t = xy$. We obtain identity of associativity. By Lemma 4 quasigroup $(Q, \cdot)$ is a loop, moreover, it is a group. Then quasigroup $(Q, \cdot)$ is a group.

**Corollary 1.** Quasigroup $(Q, \cdot)$ with identity $F_3$ is a group.

**Proof.** By Theorem 1 $F_3 = (F_1)^*$.

**Theorem 2.** Quasigroup $(Q, \cdot)$ with identity $F_4$ and $F_2$ is a loop.

**Proof.** The fact that quasigroup $(Q, \cdot)$ with identity $F_4$ is a loop has been proved in [12, 26]. The rest follows from Lemma 3 since by Theorem 1 $(F_2)^* = F_4$.

It is well known that there exist non-associative Moufang loops [6].

**Theorem 3.** Quasigroup $(Q, \cdot)$ with identities $F_5$ and $F_{10}$ is a group.

**Proof.** It is well known that any quasigroup $(Q, \cdot)$ has left and right cancellative property [2, 23, 27]. Then from identity $(xy \cdot z)x = (x \cdot yz)x$ we have $xy \cdot z = x \cdot yz$. The last follows from Lemma 4.

**Theorem 4.** Quasigroup $(Q, \cdot)$ with identity $F_6$ $(xy \cdot z)x = x(y \cdot zx)$ (extra identity) is a loop.

**Proof.** See [18].

There exist non-associative extra loops [16].
Theorem 5. Quasigroup \((Q, \cdot)\) with identity \(F_7\) \((xy \cdot z)x = x(yz \cdot x)\) has left and has not right unit.

Proof. If we put \(x := f_y\) in identity \(F_7\), then we have \(yz \cdot f_y = f_y(yz \cdot f_y)\). It is clear that term \((yz \cdot f_y)\) runs all elements of the set \(Q\).

The following counterexample shows that quasigroup \((Q, \cdot)\) with identity \(F_7\) has not right unit.

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 1 & 2 & 0 \\
1 & 0 & 1 & 2 \\
2 & 2 & 0 & 1 \\
\end{array}
\]

\(\square\)

Corollary 2. Quasigroup \((Q, \cdot)\) with identity \(F_8\) \(((x \cdot yz)x = x(yz \cdot x))\) has right and has not left unit.

Proof. The proof follows from Theorem 5, Lemma 2 and the fact that \(F_8 = (F_7)^*\) (Theorem 1). \(\square\)

Theorem 6. Quasigroup \((Q, \cdot)\) with identity \(F_9\) \(((x \cdot yz)x = x(yz \cdot x)\) ) has not left and has not right unit.

Proof. The following quasigroup \((Q, \cdot)\) satisfies identity \(F_9\) and it has not left and right unit.

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 1 & 0 & 2 \\
1 & 0 & 2 & 1 \\
2 & 2 & 1 & 0 \\
\end{array}
\]

\(\square\)

Theorem 7. Quasigroup \((Q, \cdot)\) with identity \(F_{11}\) \(xy \cdot xz = (xy \cdot x)z\) is a group.

Proof. We can rewrite identity \(F_{11}\) \(xy \cdot xz = (xy \cdot x)z\) in the form \(t \cdot xz = tx \cdot z\), where \(t = xy\). The rest follows from Lemma 4. \(\square\)

Corollary 3. Quasigroup \((Q, \cdot)\) with identity \(F_{24}\) \(yx \cdot zx = y(x \cdot zx)\) is a group.

Proof. By Theorem 1 \(F_{24} = (F_{11})^*\). \(\square\)

Theorem 8. Quasigroup \((Q, \cdot)\) with identity \(F_{12}\) \((xy \cdot xz = (x \cdot yx)z)\) is a group.

Proof. If we put \(x := f_z\) in identity \(F_{12}\), then \(f_y \cdot f_z = (f_z \cdot yf_z)z\). After cancellation in the last equality we have \(y = yf_z\). The last equality means that quasigroup with identity \(F_{12}\) has right unit.
If we put \( x = y = e \) in identity \( F_{12} \), then we have \( ee \cdot ez = (e \cdot ee)z, e \cdot ez = ez. \) Therefore quasigroup \( (Q, \cdot) \) with identity \( F_{12} \) has left unit, and, finally, this quasigroup is a loop.

If we put \( z = 1 \) in identity \( F_{12} \), then we have

\[
xy \cdot x = x \cdot yx
\]  
(8)

If we apply equality (8) to identity \( F_{12} \), then we have

\[
xy \cdot xz = (xy \cdot x)z.
\]  
(9)

If we denote term \( xy \) by the letter \( t \), then we can re-write identity (9) in the form

\[
t \cdot xz = tx \cdot z.
\]  
(10)

**Corollary 4.** Quasigroup \( (Q, \cdot) \) with identity \( F_{23} \) \( yx \cdot zx = y(xz \cdot x) \) is a group.

**Proof.** By Theorem 1 \( F_{23} = (F_{12})^\ast \).

**Theorem 9.** Quasigroup \( (Q, \cdot) \) with identity \( F_{13} \) \( xy \cdot xz = x(yx \cdot z) \) is a loop.

**Proof.** See [16].

**Corollary 5.** Quasigroup \( (Q, \cdot) \) with identity \( F_{22} \) \( yx \cdot zx = (yx \cdot z)x \) is a loop.

**Proof.** By Theorem 1 \( F_{22} = (F_{13})^\ast \).

Examples of non-associative loops with identities \( F_{13} \) and \( F_{22} \) are given in [16].

**Theorem 10.** Quasigroup \( (Q, \cdot) \) with identity \( F_{14} \) \( xy \cdot xz = x(y \cdot xz) \) is a group.

**Proof.** We can put \( t = xz \). Further proof is similar to the proof of Theorem 7.

**Corollary 6.** Quasigroup \( (Q, \cdot) \) with identity \( F_{21} \) \( yx \cdot zx = (yx \cdot z)x \) is a group.

**Proof.** By Theorem 1 \( F_{21} = (F_{14})^\ast \).

**Theorem 11.** Quasigroup \( (Q, \cdot) \) with identity \( F_{15} \) \( (xy \cdot x)z = (x \cdot yx)z \) does not have left and right unit.

**Proof.** The following quasigroup \( (Q, \cdot) \) satisfies identity \( F_{15} \) and it has not left and right unit.

\[
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
\hline
0 & 1 & 0 & 2 \\
1 & 0 & 2 & 1 \\
2 & 2 & 1 & 0 \\
\end{array}
\]
Corollary 7. Quasigroup \((Q, \cdot)\) with identity \(F_{30} y(xz \cdot x) = y(x \cdot xz)\) does not have left and right unit element.

**Proof.** By Theorem \(F_{30} = (F_{15})^*\).

\[\square\]

Theorem 12. Quasigroup \((Q, \cdot)\) with identity \(F_{16} ((xy \cdot x)z = x(yz \cdot z))\) has left unit and it has not right unit.

**Proof.** If we put \(x := y, y := y \backslash x\) in identity \(F_{16}\), then we have \((y(y \backslash x) \cdot y)z = y((y \backslash x)y \cdot z)\) and after application of identity (1) we have

\[xy \cdot z = y((y \backslash x)y \cdot z)\]  \hspace{1cm} (11)

Using the operation \(\backslash\) in equality (11) we have

\[(y \backslash x)y \cdot z = y \backslash (xy \cdot z).\]  \hspace{1cm} (12)

If in equality (12) we put \(xy = t\), then \(x = t/y\),

\[(y \backslash (t/y))y \cdot z = y \backslash (t \cdot z).\]  \hspace{1cm} (13)

If in equality (13) we put \(y = t\) and apply identity (3), then we have

\[(y \backslash (y/y))y \cdot z = z.\]  \hspace{1cm} (14)

The last equality demonstrates that quasigroup \((Q, \cdot)\) has left identity element.

The following quasigroup satisfies identity \(F_{16}\) and it does not have right identity element.

| · | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | 2 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 2 | 0 | 1 |

\[\square\]

Corollary 8. Quasigroup \((Q, \cdot)\) with identity \(F_{29} (y \cdot xz)x = y(x \cdot xz)\) does not have left unit and it has right unit element.

**Proof.** By Theorem \(F_{29} = (F_{16})^*\).

\[\square\]

Theorem 13. Quasigroup \((Q, \cdot)\) with identity \(F_{17} (\text{left Moufang}) (xy \cdot x)z = x(yz \cdot x)\) is a loop.

**Proof.** See [14, 17, 26].

\[\square\]

Corollary 9. Quasigroup \((Q, \cdot)\) with identity \(F_{27} (\text{right Moufang}) (yx \cdot z)x = y(x \cdot xz)\) is a loop.

\[\square\]
Proof. By Theorem 1, \( F_{27} = (F_{17})^* \).

**Theorem 14.** Quasigroup \((Q, \cdot)\) with identity \(F_{18}\) (\((x \cdot y)x = x(y \cdot z)\)) is a group.

Proof. Denote the term \(yx\) by variable \(t\). Then identity \(F_{18}\) takes the form \(xt \cdot z = x \cdot tz\). The rest follows from Lemma 4.

**Corollary 10.** Quasigroup \((Q, \cdot)\) with identity \(F_{28}\) (\((y \cdot x)z = y(x \cdot z)\)) is a group.

Proof. By Theorem 1, \( F_{28} = (F_{18})^* \).

**Theorem 15.** Quasigroup \((Q, \cdot)\) with identity \(F_{19}\) (left Bol) (\((x \cdot y)x = x(y \cdot z)\)) does not have left unit and it has right unit element.

Proof. Quasigroup \((Q, \cdot)\) with identity \(F_{19}\) does not have left unit.

|   | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | 0 | 2 |
| 1 | 2 | 1 | 0 |
| 2 | 0 | 2 | 1 |

Quasigroup \((Q, \cdot)\) with identity \(F_{19}\) has right unit. If in identity \((x \cdot y)x = x(y \cdot z)\) we put \(z := e_x\), then we have \((x \cdot y)e_x = x \cdot yx\) for all \(x, y \in Q\).

**Corollary 11.** Quasigroup \((Q, \cdot)\) with identity \(F_{26}\) (right Bol) (\((yx \cdot z) \cdot x = y(xz \cdot x)\)) has left unit and does not have right unit.

Proof. By Theorem 1, \( F_{26} = (F_{19})^* \).

**Theorem 16.** Quasigroup \((Q, \cdot)\) with identity \(F_{20}\) (\((x \cdot y \cdot z) = x(y \cdot xz)\)) is a group.

Proof. If we perform cancellation from the left in identity \(x(y \cdot xz) = x(y \cdot xz)\), then we have \(y \cdot xz = y \cdot xz\). Further we can apply Lemma 4.

**Corollary 12.** Quasigroup \((Q, \cdot)\) with identity \(F_{25}\) (\((y \cdot z)x = (y \cdot xz)x\)) is a group.

Proof. By Theorem 1, \( F_{25} = (F_{20})^* \).

**Theorem 17.** Quasigroup \((Q, \cdot)\) with identity \(F_{31}\) (\((x \cdot tz = y(x \cdot xz)\)) is a group.

Proof. If we denote term \(yx\) as \(t\), then from identity \(F_{31}\) we obtain identity of associativity \(t \cdot xz = t \cdot xz\).

**Corollary 13.** Quasigroup \((Q, \cdot)\) with identity \(F_{34}\) (\((y \cdot xz) = y(x \cdot xz)\)) is a group.

Proof. By Theorem 1, \( F_{34} = (F_{31})^* \).

**Theorem 18.** Quasigroup \((Q, \cdot)\) with identity \(F_{32}\) (\((y \cdot xz) = (y \cdot xz)z\)) is a group.
Proof. We prove that quasigroup \((Q, \cdot)\) with identity \(F_{32}\) has left unit. In identity \(F_{32} \ yx \cdot xz = (y \cdot xx)z\) we change \(y \to x/zz\), \(x \to z\), \(z \to y\) and obtain
\[
((x/zz)zz)y = ((x/zz)z) \cdot zy,
\]
and after application of identity (15) we have
\[
xy = ((x/zz)z) \cdot zy. \tag{16}
\]
If we denote term \(zy\) by the letter \(t\), then \(y = z \backslash t\). And we can rewrite equality (16) in the form
\[
x(\backslash tz) = ((x/zz)z) \cdot t. \tag{17}
\]
If we put \(x = z\) in equality (17), then we have
\[
x(z \backslash t) \overset{11}{=} t = ((x/xx)x) \cdot t. \tag{18}
\]
We have proved that quasigroup \((Q, \cdot)\) with identity \(F_{32}\) has left unit \(f = ((x/xx)x)\).

If we substitute \(x = e\) in identity \(F_{32}\), then we have
\[
y \cdot ez = yz, ez = z, e = f. \tag{19}
\]
Since in quasigroup \((Q, \cdot)\) there exists left unit, then there exists right unit too.

If in identity \(F_{32}\) we put \(z = e\), then we have
\[
yx \cdot x = y \cdot xx. \tag{20}
\]
If we apply identity (20) to the right side of identity \(F_{32}\), then we have
\[
yx \cdot xz = (yx \cdot x)z. \tag{21}
\]
In order to prove that quasigroup with identity \(F_{32}\) is a group we can denote term \(yx\) in identity (21) by the letter \(t\) and use Lemmas 3 and 4.

Corollary 14. Quasigroup \((Q, \cdot)\) with identity \(F_{33} \ yx \cdot xz = y(xx \cdot z)\) is a group.

Proof. By Theorem \(F_{33} = (F_{32})^*\).

Theorem 19. Quasigroup \((Q, \cdot)\) with identity \(F_{35} \ (yx \cdot x)z = (y \cdot xx)z\) does not have left unit and it has right unit element.

Proof. Prove that quasigroup \((Q, \cdot)\) with identity \(F_{35}\) has right unit. In identity \(F_{35}\) we make cancellation from the right side and we have:
\[
yx \cdot x = y \cdot xx. \tag{22}
\]
If in identity (22) we put \(yx = t\), then \(t/x = y\). Using this substitution we can rewrite identity (22) as follows:
\[
tx = (t/x) \cdot xx. \tag{23}
\]
Suppose that \( x := y\backslash x \) in identity \( F_{35} \). Then we have:

\[
(y(y\backslash x) \cdot (y\backslash x))z = (x \cdot (y\backslash x))z = (y \cdot (y\backslash x)(y\backslash x))z.
\]

(24)

After cancellation in identity (24) from the right side we have:

\[
x \cdot (y\backslash x) = y \cdot (y\backslash x)(y\backslash x).
\]

(25)

From identity (25) using the operation of right division “\( \backslash \)” we have

\[
(y\backslash x)(y\backslash x) = y\backslash (x \cdot (y\backslash x)).
\]

(26)

If we put \( x = y \) in identity (26), then we have:

\[
(x\backslash x)(x\backslash x) = x\backslash (x(x\backslash x)) \equiv x\backslash x.
\]

(27)

In identity (23) we put \( x := z\backslash z \) for some \( z \in Q \). Then we have:

\[
t(z\backslash z) = (t/(z\backslash z)) \cdot (z\backslash z)(z\backslash z)
\]

(28)

The following example demonstrates that quasigroup \((Q, \cdot)\) with identity \( F_{35} \) does not have left unit.

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 2 | 3 | 4 | 5 |
| 1 | 0 | 1 | 3 | 2 | 5 | 4 |
| 2 | 3 | 2 | 4 | 5 | 0 | 1 |
| 3 | 2 | 3 | 5 | 4 | 1 | 0 |
| 4 | 5 | 4 | 0 | 1 | 2 | 3 |
| 5 | 4 | 5 | 1 | 0 | 3 | 2 |

Corollary 15. Quasigroup \((Q, \cdot)\) with identity \( F_{40} \) \( y(xx \cdot z) = y(x \cdot xz) \) has left unit and it does not have right unit element.

Proof. By Theorem 1 \( F_{35} = (F_{40})^* \). □

Theorem 20. Quasigroup \((Q, \cdot)\) with identity \( F_{36} \) (RC identity) \( (yx \cdot x)z = y(xx \cdot z) \) has left unit and it does not have right unit element.

Proof. Quasigroup \((Q, \cdot)\) with identity \( F_{36} \) has left unit. If we put \( y := f_x \) in identity \( (yx \cdot x)z = y(xx \cdot z) \), then we have

\[
(f_x(x \cdot x))z = f_x(xx \cdot z), xx \cdot z = f_x(xx \cdot z).
\]

(29)
From equalities (29) it follows that quasigroup \((Q, \cdot)\) with identity \(F_{36}\) has left unit.

The following example demonstrates that right identity element does not exist.

\[
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
\hline
0 & 1 & 2 & 0 \\
1 & 0 & 1 & 2 \\
2 & 2 & 0 & 1 \\
\end{array}
\]

\[\square\]

**Corollary 16.** Quasigroup \((Q, \cdot)\) with identity \(F_{39}\) (LC identity) \((y \cdot xx)z = y(x \cdot xz)\) does not have left unit and it has right unit element.

*Proof.* By Theorem \(\text{IV } F_{39} = (F_{38})^*\).

\[\square\]

**Theorem 21.** Quasigroup \((Q, \cdot)\) with identity \(F_{37}\) (C identity) \((yx \cdot x)z = y(x \cdot xz)\) does not have left unit and it does not have right unit.

*Proof.* We give necessary example.

\[
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
\hline
0 & 1 & 0 & 2 \\
1 & 0 & 2 & 1 \\
2 & 2 & 1 & 0 \\
\end{array}
\]

\[\square\]

**Theorem 22.** Quasigroup \((Q, \cdot)\) with identity \(F_{38}\) \((y \cdot xx)z = y(xx \cdot z)\) is a loop.

*Proof.* In identity \(F_{38}\) we put \(y := f_{xx}\). Then we have

\[(f_{xx} \cdot xx)z = f_{xx}(xx \cdot z), xx \cdot z = f_{xx}(xx \cdot z). \quad (30)\]

In identity \(F_{38}\) we put \(z := e_{xx}\). Then we have

\[(y \cdot xx)e_{xx} = y(xx \cdot e_{xx}), (y \cdot xx)e_{xx} = y \cdot xx. \quad (31)\]

Therefore in quasigroup \((Q, \cdot)\) with identity \(F_{38}\) there exists right unit.

\[\square\]

The following example demonstrates that quasigroup with identity \(F_{38}\) is not associative; loop with identity \(F_{38}\) has not Lagrange property.

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 3 & 4 \\
2 & 2 & 4 & 0 & 1 \\
3 & 3 & 2 & 4 & 0 \\
4 & 4 & 3 & 1 & 2 \\
\end{array}
\]
Theorem 23. Quasigroup \((Q, \cdot)\) with identity \(F_{41}\) (LC identity \(xx \cdot yz = (x \cdot xy)z\)) is a loop.

Proof. We prove that quasigroup \((Q, \cdot)\) with identity \(F_{41}\) has left identity element. In identity \(F_{41}\) we put \(y := e_x\). Then we have:

\[
xx \cdot e_x z = (x \cdot e_x x)z, xx \cdot e_x z = xx \cdot z, e_x z = z.
\] (32)

We prove that quasigroup \((Q, \cdot)\) with identity \(F_{41}\) has right identity element. In identity \(F_{41}\) we put \(y := f_z\). Then we have:

\[
xx \cdot f_z z = (x \cdot x f_z)z, xx \cdot z = (x \cdot x f_z)z, xx = x \cdot x f_z, x = x f_z.
\] (33)

The following example demonstrates that loop with identity \(F_{41}\) is not associative.

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 0 & 1 & 2 & 4 & 3 \\
2 & 4 & 0 & 1 & 3 & 2 \\
3 & 4 & 1 & 0 & 2 & 5 \\
4 & 2 & 5 & 0 & 4 & 1 \\
5 & 3 & 4 & 1 & 0 & 2 \\
\end{array}
\]

Corollary 17. Quasigroup \((Q, \cdot)\) with identity \(F_{53}\) (RC identity \(yz \cdot xx = y(zx \cdot x)\)) is a loop.

Proof. By Theorem 1 \(F_{53} = (F_{41})^*\).

\[
\]

Theorem 24. Quasigroup \((Q, \cdot)\) with identity \(F_{42}\) (xx \cdot yz = (xx \cdot y)z) is a left loop.

Proof. We prove that quasigroup \((Q, \cdot)\) with identity \(F_{42}\) has left identity element. In identity \(F_{42}\) we put \(y := e_{xx}\). Then we have:

\[
xx \cdot e_{xx} z = (xx \cdot e_{xx} x)z, xx \cdot e_{xx} z = xx \cdot z, e_{xx} z = z.
\] (34)

The following example demonstrates that left loop with identity \(F_{42}\) has no right identity element.

\[
\begin{array}{cc}
0 & 1 \\
0 & 1 \\
1 & 0 \\
2 & 1 \\
\end{array}
\]

\]

\]

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Corollary 18. Quasigroup \((Q, \cdot)\) with identity \(F_{54}\) \((yz \cdot xx = y(z \cdot xx))\) is a right loop.

Proof. By Theorem 1 \(F_{54} = (F_{42})^*\). \(\square\)

Theorem 25. Quasigroup \((Q, \cdot)\) with identity \(F_{43}\) \((xx \cdot yz = x(x \cdot yz))\) is a left loop.

Proof. We prove that quasigroup \((Q, \cdot)\) with identity \(F_{43}\) has left identity element. In identity \(F_{43}\) we put \(x = y = f_z\). Then we have:

\[
f_z f_z \cdot f_z z = f_z(f_z \cdot f_z z), \quad f_z f_z \cdot z = f_z z, \quad f_z f_z = f_z.
\] (35)

Further, in identity \(F_{43}\) we put \(x = f_x\). Then we have:

\[
f_x f_x \cdot yz = f_x(f_x \cdot yz), \quad \text{using equality (35),}\quad yz = f_x(f_x \cdot yz), \quad yz = f_x \cdot yz.
\] (36)

The following example demonstrates that left loop with identity \(F_{43}\) has no right identity element.

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 0 & 3 & 2 & 5 \\
1 & 0 & 1 & 2 & 3 & 4 \\
2 & 2 & 3 & 4 & 5 & 0 \\
3 & 3 & 2 & 5 & 4 & 1 \\
4 & 4 & 5 & 0 & 1 & 2 \\
5 & 5 & 4 & 1 & 0 & 3 \\
\end{array}
\]

\(\square\)

Corollary 19. Quasigroup \((Q, \cdot)\) with identity \(F_{51}\) \((yz \cdot xx = (yz \cdot x) x))\) is a right loop.

Proof. By Theorem 1 \(F_{51} = (F_{43})^*\). \(\square\)

Theorem 26. Quasigroup \((Q, \cdot)\) with identity \(F_{44}\) \((xx \cdot yz = x(xy \cdot z))\) is a left loop.

Proof. We prove that quasigroup \((Q, \cdot)\) with identity \(F_{44}\) has left identity element. In identity \(F_{44}\) we put \(x = f_y\). Then we have:

\[
f_y f_y \cdot yz = f_y(f_y y \cdot z), \quad f_y f_y \cdot yz = f_y \cdot yz, \quad f_y f_y = f_y.
\] (37)

Further we put \(x = f_x\) in identity \(F_{44}\). Then we have:

\[
f_x f_x \cdot yz = f_x(f_x y \cdot z), \quad \text{we use (37),}\quad f_x \cdot yz = f_x(f_x y \cdot z), \quad yz = f_x y \cdot z, \quad y = f_x y.
\] (38)
The following example demonstrates that left loop with identity $F_{44}$ has no right identity element.

| · | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 0 | 1 | 2 |
| 2 | 2 | 0 | 1 |

**Theorem 27.** Quasigroup $(Q, \cdot)$ with identity $F_{45}$ ($x \cdot xy = (xx \cdot y)z$) is a left loop.

*Proof.* We prove that quasigroup $(Q, \cdot)$ with identity $F_{45}$ has left identity element. In identity $F_{45}$ we make cancellation from the right side. Then we have:

\[ x \cdot xy = xx \cdot y. \] (39)

In identity (39) we put $x = f_y$ and obtain

\[ f_y \cdot f_y y = f_y f_y \cdot y, f_y = f_y f_y. \] (40)

Further we put $x = f_x$ in identity (39). We have:

\[ f_x \cdot f_x y = f_x f_x \cdot y, \text{ we use (40), } f_x \cdot f_x y = f_x y, f_x y = y. \] (41)

The following example demonstrates that left loop with identity $F_{45}$ has no right identity element.

| · | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 3 | 2 | 5 | 4 | 1 | 0 |
| 4 | 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 5 | 4 | 1 | 0 | 3 | 2 |

**Corollary 20.** Quasigroup $(Q, \cdot)$ with identity $F_{60}$ ($y(xz \cdot x) = y(z \cdot xx)$) is a right loop.

*Proof.* By Theorem 1 $F_{60} = (F_{45})^*$. □

**Theorem 28.** Quasigroup $(Q, \cdot)$ with identity $F_{46}$ (LC identity) $(x \cdot xy)z = x(x \cdot yz)$ does not have left and right unit.
Proof.

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 0 & 2 & \\
1 & 0 & 2 & 1 & \\
2 & 2 & 1 & 0 & \\
\end{array}
\]

Corollary 21. Quasigroup \((Q, \cdot)\) with identity \(F_{56}\) (RC identity) \((yz \cdot x)x = y(zx \cdot x)\) does not have left and right unit.

Proof. By Theorem 1 \(F_{56} = (F_{46})^*\).

Theorem 29. Quasigroup \((Q, \cdot)\) with identity \(F_{47}\) \(((x \cdot xy)z = x(xy \cdot z))\) is a group.

Proof. If in identity \(F_{47}\) we denote term \(xy\) by variable \(t\), then we have \(xt \cdot z = x \cdot tz\).

Corollary 22. Quasigroup \((Q, \cdot)\) with identity \(F_{58}\) \(((y \cdot zx)x = y(zx \cdot x))\) is a group.

Proof. By Theorem 1 \(F_{58} = (F_{48})^*\).

Theorem 30. Quasigroup \((Q, \cdot)\) with identity \(F_{48}\) \(((xx \cdot y)z = x(x \cdot yz))\) is a left loop.

Proof. We prove that quasigroup \((Q, \cdot)\) with identity \(F_{48}\) has left identity element. In identity \(F_{48}\) we put \(x = f_{yz}\). Then we have:

\[
(f_{yz}f_{yz} \cdot y)z = f_{yz}(f_{yz} \cdot yz), (f_{yz}f_{yz} \cdot y)z = yz, f_{yz}f_{yz} \cdot y = y. \quad (42)
\]

The following example demonstrates that left loop with identity \(F_{48}\) has no right identity element.

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 4 & 3 & 0 \\
1 & 3 & 0 & 4 & 2 \\
2 & 0 & 1 & 2 & 3 \\
3 & 2 & 3 & 1 & 4 \\
4 & 4 & 2 & 0 & 1 \\
\end{array}
\]

Corollary 23. Quasigroup \((Q, \cdot)\) with identity \(F_{57}\) \(((yz \cdot x)x = y(z \cdot xx))\) is a right loop.

Proof. By Theorem 1 \(F_{57} = (F_{48})^*\).
Theorem 31. Quasigroup \((Q, \cdot)\) with identity \(F_{49}\) \(((xx \cdot y)z = x(xy \cdot z))\) is a left loop.

Proof. We prove that quasigroup \((Q, \cdot)\) with identity \(F_{49}\) has left identity element. In identity \(F_{49}\) we change \(y \rightarrow (xx)\backslash y\). Then we have:

\[
(xx \cdot (xx)\backslash y)z = yz = x((xx)\backslash y) \cdot z).
\]

(43)

Further we use right division “\(\backslash\)” in equality (43).

\[
(x((xx)\backslash y))z = x\backslash(yz).
\]

(44)

If we put \(x = y\) in equality (44), then we have:

\[
(x((xx)\backslash x))z = x\backslash(xz) = z.
\]

(45)

The following example demonstrates that left loop with identity \(F_{49}\) has no right identity element.

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 1 & 2 & 0 \\
1 & 0 & 1 & 2 \\
2 & 2 & 0 & 1 \\
\end{array}
\]

Corollary 24. Quasigroup \((Q, \cdot)\) with identity \(F_{59}\) \(((y \cdot zx)x = y(z \cdot xx))\) is a right loop.

Proof. By Theorem \(F_{59} = (F_{49})^*\).

Theorem 32. Quasigroup \((Q, \cdot)\) with identity \(F_{44}\) \((xx \cdot yz = x(xy \cdot z))\) is a left loop.

Proof. We put \(x = f_y\) in identity \(F_{44}\). Then we have:

\[
f_yf_y \cdot yz = f_y(f_yy \cdot z), f_yf_y \cdot yz = f_y \cdot yz, f_yf_y = f_y.
\]

(46)

We put \(x = f_x\) in identity \(F_{44}\). Then we have:

\[
f_xf_x \cdot yz = f_x \cdot yz = f_x(f_xy \cdot z), yz = f_xy \cdot z, y = f_xy.
\]

(47)

From the last equality in (47) it follows that quasigroup \((Q, \cdot)\) with identity \(F_{44}\) has left unit.

The following example demonstrates that left loop with identity \(F_{44}\) has no right identity element.

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 1 & 2 & 0 \\
1 & 0 & 1 & 2 \\
2 & 2 & 0 & 1 \\
\end{array}
\]

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Corollary 25. Quasigroup \((Q, \cdot)\) with identity \(F_{32} (yz \cdot xx = (y \cdot zz)x)\) is a right loop.

Proof. By Theorem \(F_{32} = (F_{44})^*\).

3 Type of classical Bol-Moufang identity

We try to find some invariants of Bol-Moufang identities which demonstrate (without any additional researches) that a quasigroup with this identity (in what this identity is true) has left (right, middle) unit element, or it is a loop, a group.

We define type of classical Bol-Moufang identity.

Definition 4. The order of execution of operations in the left and right side of a quasigroup identity we name a type of the identity.

In Table 1 we also indicate the places of double variable \(x\). For example, in the identity \(F_1 xy \cdot zx = (xy \cdot z)x\) these places are \(\{1, 4\}\).

4 Table

| Name  | Abbrev. | Identity                  | f | e | Lo. | Gr. | Type                      |
|-------|---------|---------------------------|---|---|-----|-----|---------------------------|
| \(F_1\) |         | \(xy \cdot zx = (xy \cdot z)x\) | + | + | +   | +   | \(23) = \varepsilon, \{1, 4\}\ |
| \(F_3\) |         | \(xy \cdot zx = x(y \cdot zx)\) | + | + | +   | +   | \(23)=\{13\}, \{1, 4\}\ |
| \(F_5\) |         | \((xy \cdot z)x = (x \cdot yz)x\) | + | + | +   | +   | \(\varepsilon = (12), \{1, 4\}\ |
| \(F_{10}\) |        | \(x(y \cdot zx) = x(yz \cdot x)\) | + | + | +   | +   | \(13) = (132), \{1, 4\}\ |
| \(F_{11}\) |        | \(xy \cdot xz = (xy \cdot x)z\) | + | + | +   | +   | \(23) = \varepsilon, \{1, 3\}\ |
| \(F_{12}\) |        | \(xy \cdot xz = (x \cdot yx)z\) | + | + | +   | +   | \(23)=\{12\}, \{1, 3\}\ |
| \(F_{14}\) |        | \(xy \cdot xz = (y \cdot xz)\) | + | + | +   | +   | \(23)=\{13\}, \{1, 3\}\ |
| \(F_{18}\) |        | \((x \cdot yx)z = x(yx \cdot z)\) | + | + | +   | +   | \(12)=\{132\}, \{1, 3\}\ |
| $F_{20}$ | $x(yx \cdot z) = x(y \cdot xz)$ | + | + | + | $(132) = (13)$, \{1, 3\} |
| $F_{21}$ | $yx \cdot zx = (yx \cdot z)x$ | + | + | + | $(23) = \varepsilon$, \{2, 4\} |
| $F_{23}$ | $yx \cdot zx = y(xz \cdot x)$ | + | + | + | $(23) = (132)$, \{2, 4\} |
| $F_{24}$ | $yx \cdot zx = y(x \cdot zx)$ | + | + | + | $(23) = (132)$, \{2, 4\} |
| $F_{25}$ | $(yx \cdot z)x = (y \cdot xz)x$ | + | + | + | $\varepsilon = (12)$, \{2, 4\} |
| $F_{28}$ | $(y \cdot xz)x = y(xz \cdot x)$ | + | + | + | $(12) = (132)$, \{2, 4\} |
| $F_{31}$ | $yx \cdot xz = (yx \cdot x)z$ | + | + | + | $(23) = \varepsilon$, \{2, 3\} |
| $F_{32}$ | $yx \cdot xz = (y \cdot xx)z$ | + | + | + | $(23) = (12)$, \{2, 3\} |
| $F_{33}$ | $yx \cdot xz = y(xx \cdot z)$ | + | + | + | $(23) = (132)$, \{2, 3\} |
| $F_{34}$ | $yx \cdot xz = y(x \cdot xz)$ | + | + | + | $(23) = (132)$, \{2, 3\} |
| $F_{37}$ | $(x \cdot xy)z = x(xy \cdot z)$ | + | + | + | $(12) = (132)$, \{1, 2\} |
| $F_{50}$ | $x(x \cdot yz) = x(xy \cdot z)$ | + | + | + | $(13) = (132)$, \{1, 2\} |
| $F_{55}$ | $(yz \cdot x)x = (y \cdot zx)x$ | + | + | + | $\varepsilon = (12)$, \{3, 4\} |
| $F_{58}$ | $(y \cdot zx)x = y(zx \cdot x)$ | + | + | + | $(12) = (132)$, \{3, 4\} |
| $F_i$ | Type | Identity | Hinge | Schist | Meso | Set |
|------|------|----------|-------|--------|------|-----|
| $F_4$ | middle Moufang | $xy \cdot zx = x(yz \cdot x)$ | + | + | + | - | (23) = (132), $\{3, 4\}$ |
| $F_2$ | middle Moufang | $xy \cdot zx = (x \cdot yz)x$ | + | + | + | - | (23) = (12), $\{3, 4\}$ |
| $F_6$ | extra identity | $(xy \cdot z)x = x(y \cdot zx)$ | + | + | + | - | $\varepsilon = (13)$, $\{1, 4\}$ |
| $F_{13}$ | extra identity | $xy \cdot xz = x(yz \cdot x)$ | + | + | + | - | (23) = (12), $\{1, 3\}$ |
| $F_{17}$ | left Moufang | $(xy \cdot x)z = x(y \cdot xz)$ | + | + | + | - | $\varepsilon = (13)$, $\{1, 3\}$ |
| $F_{22}$ | extra identity | $yx \cdot zx = (y \cdot zx)x$ | + | + | + | - | (23) = (12), $\{2, 4\}$ |
| $F_{27}$ | right Moufang | $(yx \cdot z)x = y(x \cdot zx)$ | + | + | + | - | $\varepsilon = (13)$, $\{2, 4\}$ |
| $F_{38}$ | | $(y \cdot xx)z = y(xx \cdot z)$ | + | + | + | - | (23) = (12), $\{2, 3\}$ |
| $F_{41}$ | LC identity | $xx \cdot yz = (x \cdot xy)z$ | + | + | + | - | (23) = (12), $\{1, 2\}$ |
| $F_{33}$ | RC identity | $yz \cdot xx = y(zx \cdot x)$ | + | + | + | - | (23) = (12), $\{3, 4\}$ |
| $F_7$ | | $(xy \cdot z)x = x(yz \cdot x)$ | + | - | - | - | $\varepsilon = (132)$, $\{1, 4\}$ |
| $F_{16}$ | | $(xy \cdot x)z = x(yx \cdot z)$ | + | - | - | - | $\varepsilon = (132)$, $\{1, 3\}$ |
| $F_{26}$ | right Bol | $(yx \cdot z)x = y(xz \cdot x)$ | + | - | - | - | $\varepsilon = (132)$, $\{2, 4\}$ |
| $F_{36}$ | RC identity | $(yx \cdot x)z = y(xx \cdot z)$ | + | - | - | - | $\varepsilon = (132)$, $\{2, 3\}$ |
| $F_{40}$ | | $y(xx \cdot z) = y(x \cdot xz)$ | + | - | - | - | (13) = (13), $\{1, 2\}$ |
| $F_{42}$ | $xx \cdot yz = (xx \cdot y)z$ | + | - | - | (23) = $\varepsilon$, \{1, 2\} |
| $F_{43}$ | $xx \cdot yz = x(x \cdot yz)$ | + | - | - | (23) = (13), \{1, 2\} |
| $F_{44}$ | $xx \cdot yz = x(xy \cdot z)$ | + | - | - | (23) = (132), \{1, 2\} |
| $F_{45}$ | $(x \cdot xy)z = (xx \cdot y)z$ | + | - | - | (12) = $\varepsilon$, \{1, 2\} |
| $F_{48}$ | LC identity $(xx \cdot y)z = x(x \cdot yz)$ | + | - | - | $\varepsilon = (13)$, \{1, 2\} |
| $F_{49}$ | $(xx \cdot y)z = x(xy \cdot z)$ | + | - | - | $\varepsilon = (132)$, \{1, 2\} |
| $F_{48}$ | $(x \cdot yz)x = x(y \cdot zx)$ | - | + | - | (12) = (13), \{1, 4\} |
| $F_{49}$ | left Bol $(x \cdot yx)z = x(y \cdot zx)$ | - | + | - | (12) = (13), \{1, 3\} |
| $F_{51}$ | $(yx \cdot x)z = (y \cdot xx)z$ | - | + | - | $\varepsilon = (12)$, \{2, 3\} |
| $F_{52}$ | LC identity $(y \cdot xx)z = y(x \cdot xx)$ | - | + | - | (12) = (13), \{2, 3\} |
| $F_{53}$ | $yz \cdot xx = (yz \cdot x)x$ | - | + | - | (23) = $\varepsilon$, \{3, 4\} |
| $F_{54}$ | $yz \cdot xx = (y \cdot zx)xx$ | - | + | - | (23) = (12), \{3, 4\} |
| $F_{55}$ | $yz \cdot xx = y(z \cdot xx)$ | - | + | - | (23) = (13), \{3, 4\} |
| $F_{57}$ | RC identity $(yz \cdot x)x = y(z \cdot xx)$ | - | + | - | $\varepsilon = (13)$, \{3, 4\} |
| $F_{59}$ | $(y \cdot zx)x = y(z \cdot xx)$ | - | + | - | (12) = (13), {3, 4} |
| $F_{60}$ | $y(zx \cdot x) = y(z \cdot xx)$ | - | + | - | (132) = (13), {3, 4} |
| $F_9$ | $(x \cdot yz)x = x(yz \cdot x)$ | - | - | - | (12) = (132), {1, 4} |
| $F_{15}$ | $(xy \cdot x)z = (x \cdot yx)z$ | - | - | - | $\varepsilon$ = (12), {1, 3} |
| $F_{30}$ | $y(xz \cdot x) = y(x \cdot zx)$ | - | - | - | (132) = (13), {2, 4} |
| $F_{37}$ | C identity | $(yx \cdot x)z = y(x \cdot xz)$ | - | - | - | $\varepsilon$ = (13), {2, 3} |
| $F_{46}$ | LC identity | $(x \cdot xy)z = x(x \cdot yz)$ | - | - | - | (12) = (13), {1, 2} |
| $F_{56}$ | RC identity | $(yz \cdot x)x = y(zx \cdot x)$ | - | - | - | $\varepsilon$ = (132), {3, 4} |

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