\[ \Psi = W e^{\pm \Phi} \] quantum cosmological solutions for Class A Bianchi Models

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Abstract

We find solutions for quantum Class A Bianchi models of the form \( \Psi = W e^{\pm \Phi} \) generalizing the results obtained by Moncrief and Ryan in standard quantum cosmology. For the II and IX Bianchi models there are other solutions \( \tilde{\Phi}_2, \tilde{\Phi}_9 \) to the Hamilton-Jacobi equation for which \( \Psi \) is necessarily zero, in contrast with solutions found in supersymmetric quantum cosmology.

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1 Introduction

In recent years, progress has been made to find solutions\cite{1,2} to the canonical constraints of the full general relativity theory. However, the canonical quantization program is far from complete. It is then hoped that the study of some particular models could illustrate the behaviour of the general theory. The Bianchi cosmologies are the prime example. Even in these simplified cases little progress has been achieved. It was just recently that solutions were found for the more generic Bianchi Class A models, in particular the Bianchi IX model, resembling the situation that one faces in the full theory.

It was first remarked by Kodama\cite{3} that solutions to the Wheeler-DeWitt equation (WDW) in the formulation of Arnowitt, Deser and Misner (ADM) and Ashtekar (in the connection representation) are related by $\Psi = \Psi_A e^{\pm i\Phi_A}$, where $\Phi_A$ is the homogeneous specialization of the generating functional\cite{1} of the canonical transformation from the ADM variables to Ashtekar’s. This function was calculated explicitly for the diagonal Bianchi type IX model by Kodama\cite{3}, he also found $\Psi_A = \text{const.}$ as solution. Since $\Phi_A$ is pure imaginary, for a certain factor ordering, one expects a solution of the form $\Psi = W e^{\pm \Phi}$, $W=$const., $\Phi = i\Phi_A$. In fact these type of solutions have been found for the diagonal Bianchi type IX model\cite{4}. For the special case of the Taub model\cite{4,5} it is also possible to find a solution for which $W = \text{const.} e^{\alpha} e^{\beta+}$.

In superquantum cosmology the same kind of solutions have been found by means of two different approaches. Using supergravity N=1 it was shown\cite{3} for the Bianchi type I model that the general solution has the form $\Psi = C_1 h^{-\frac{1}{2}} e^{-\Phi} + C_2 h^{-\frac{3}{2}} \psi^{2n} e^{+\Phi}$, where $C_1 = C_2 = \text{const.}$, $h$ is the determinant of the three metric, and $\psi^{2n}$ express symbolically the expansion of the wave function $\Psi$ in even powers (this guarantees Lorentz invariance) of the gravitino field. The function $\Phi$ in this particular case is zero but it was suggested that for the Bianchi Class A models the solution has exactly this
form with their corresponding Φ function. This conjecture has been confirmed in a series of publications using the ADM [7, 8] and the Ashtekar formulation [9, 10]. A more general postulate for the Lorentz invariance seems to allow solutions also in the $\psi^2$ and $\psi^4$ terms [11]. Similar solutions exist for a WDW equation derived for the bosonic sector of the heterotic string [12].

A second approach [13] considers the WDW equation also in the ADM and Ashtekar formulation [14], for the Bianchi model of interest and proceeds by finding appropriate operators which are the “square root” of this equation. This procedure has the disadvantage that one has to introduce fermionic variables without a direct physical meaning. However, for the physical quantities of interest (like $\Psi^* \Psi$) one integrate over these variables [15], getting information about their influence on the unnormalized probability function.

The three previous procedures virtually result in the same kind of quantum state and are of interest because for some of these models (by ex. Bianchi IX) these are the only known solutions. It is remarkable that they appear in the three different approaches mentioned. However, these kind of solutions have been found in standard quantum cosmology, only for the Bianchi type IX, Taub and FRW models [4, 5]. The main point of this paper is to generalize the results of Moncrief and Ryan to the diagonal Bianchi Class A models. We will show that all solutions are of the form $\Psi = W e^{\pm \Phi}$, where $W$ is in general a function and can be reduced to a constant for the Bianchi models VIII and IX, depending on the factor ordering in the WDW equation. For the Bianchi II and IX models there exist [16] others real $\tilde{\Phi}_2$ and $\tilde{\Phi}_9$, however, it is surprising that contrarily to the results claimed in supersymmetric quantum cosmology [11], $\Psi = W e^{\pm \tilde{\Phi}_9}$ is not a solution of the WDW equation because for this $\tilde{\Phi}_9$, $W$ is necessarily zero. Also $W$ result to be zero for $\Psi = W e^{\pm \tilde{\Phi}_2}$. 
Let us recall here the canonical formulation in the ADM formalism of the diagonal Bianchi Class A models. The metrics have the form

\[ ds^2 = -dt^2 + e^{2\alpha(t)} (e^{2\beta(t)})_{ij} \omega^i \omega^j, \]  

where \( \alpha(t) \) is a scalar and \( \beta_{ij}(t) \) a 3x3 diagonal matrix, \( \beta_{ij} = \text{diag}(x+\sqrt{3}y, x-\sqrt{3}y, -2x) \), \( \omega^i \) are one-forms that characterize each cosmological Bianchi type model, and that obey

\[ d\omega^i = \frac{1}{2} C_{jk}^i \omega^j \wedge \omega^k, \]  

\( C_{jk}^i \) the structure constants of the corresponding invariance group.

The ADM action has the form

\[ I = \int (P_x dx + P_y dy + P_\alpha d\alpha - N \mathcal{H}_\perp) dt, \]  

where

\[ \mathcal{H}_\perp = e^{-3\alpha} \left( -P_\alpha^2 + P_x^2 + P_y^2 + e^{4\alpha} V(x, y) \right), \]  

and \( e^{4\alpha} V(x, y) = U(q^\mu) \) is the potential term of the cosmological model under consideration.

The WDW equation for these models is achieved by replacing \( P_{q^\mu} \) by \( -i \partial_{q^\mu} \) in (1.3), with \( q^\mu = (\alpha, x, y) \). The factor \( e^{-3\alpha} \) may be factor ordered with \( \hat{P}_\alpha \) in many ways. Hartle and Hawking\[17\] have suggested what might be called a semi-general factor ordering which in this case would order \( e^{-3\alpha} \hat{P}_\alpha^2 \) as

\[ -e^{-(3-Q)\alpha} \partial_\alpha e^{Q\alpha} \partial_\alpha = -e^{-3\alpha} \partial_\alpha^2 + Q e^{-3\alpha} \partial_\alpha, \]  

where \( Q \) is any real constant. With this factor ordering the Wheeler-DeWitt equation becomes

\[ \Box \Psi + Q \frac{\partial \Psi}{\partial \alpha} - U(q^\mu) \Psi = 0, \]  

where \( \Box \) is the three dimensional d’Lambertian in the \( q^\mu \) coordinates.

The paper is then organized as follows. In Sec. II, we introduce the Ansatz \( \Psi = W e^{-\Phi} \) in (1.5) and set the general equations for the Bianchi Class A models. In Sec.
III we present solutions for the cosmological Class A Bianchi models. Only for the cases of Bianchi VIII and IX, W can be directly reduced to a constant. For all other Bianchi models W is a function in contrast with the solutions found in superquantum cosmology. For the Bianchi type VI $h=-1$, $\Phi$ does not coincide with the general form that appears in superquantum cosmology. Sec. IV, is dedicated to final remarks.

2 Transformation of the Wheeler-DeWitt equation

Under the Ansatz for the wave function $\Psi(q^\mu) = W(\alpha, x, y)e^{-\Phi}$, (1.3) is transformed in

$$\Box W - W \Box \Phi - 2 \nabla W \cdot \nabla \Phi + Q \frac{\partial W}{\partial \alpha} - QW \frac{\partial \Phi}{\partial \alpha} + W[(\nabla \Phi)^2 - U] = 0,$$

(2.1)

where $\Box = G^\mu_\nu \frac{\partial^2}{\partial q^\mu \partial q^\nu}$, $\nabla W \cdot \nabla \Phi = G^\mu_\nu \frac{\partial W}{\partial q^\mu} \frac{\partial \Phi}{\partial q^\nu}$, $(\nabla)^2 = -\left(\frac{\partial}{\partial \alpha}\right)^2 + (\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2$, with $G^\mu_\nu = \text{diag}(-1, 1, 1)$, $U$ is the potential term of the cosmological model under consideration. If one can solve the non-linear equation

$$(\nabla \Phi)^2 - U = 0,$$

(2.2)

for $\Phi$, then one can obtain a master equation for the function $W$.

$$\Box W - W \Box \Phi - 2 \nabla W \cdot \nabla \Phi + Q \frac{\partial W}{\partial \alpha} - QW \frac{\partial \Phi}{\partial \alpha} = 0.$$

(2.3)

(2.2) is the classical Einstein-Hamilton-Jacobi equation which can be obtained by replacing the momentum $P_{q^\mu} \rightarrow \frac{\partial \Phi}{\partial q^\mu}$ in (1.3).

We were able to solve (2.2), for the Class A Bianchi models, by doing the following change of coordinates $\beta_1 = \alpha + x + \sqrt{3}y$, $\beta_2 = \alpha + x - \sqrt{3}y$, $\beta_3 = \alpha - 2x$, and rewrite (2.2) in these new coordinates. With this change, the function $\Phi$ is obtained and will be given in section III, in general form for the Class A Bianchi Models. In particular, Moncrief and Ryan$^4$, have found in the case of the Bianchi type IX model an exact
solution for (2.2), being

\[ \Phi = \frac{1}{6} e^{2\alpha} [e^{-4x} + 2e^{2x} \cosh(2\sqrt{3}y)], \quad (2.4) \]

and then the solution for the wave function, where \( W=\text{const.} \), implying \( Q= -6 \), and a solution for the Taub model where the value of the \( Q \) parameter is zero and \( W = \text{const.} \ e^{\alpha+x} \).

Let us make an assumption which will allow us to solve more easily (2.3), we demand that

\[ \Box W + Q \frac{\partial W}{\partial \alpha} = 0, \quad (2.5) \]

which should be consistent with

\[ W \Box \Phi + 2 \nabla W \cdot \nabla \Phi + QW \frac{\partial \Phi}{\partial \alpha} = 0, \quad (2.6) \]

(2.5) is easier to solve than the original (1.3), because it does not contain any potential.

In the rest of this work, we will study the different solutions to Class A Bianchi models, where the \( Q \) parameter corresponds to different factor orderings in the quantum Wheeler-DeWitt equation.

3 \( \Psi = W e^{\pm \Phi} \) solutions

In this section, we obtain the solutions to the equations that appear in the decomposition of the WDW equation, (2.2), (2.5) and (2.6) and give them for the Class A Cosmological Bianchi models.

Let us present, by means of a different procedure, the already known solution\[^4\] to (2.2) for the Bianchi type IX model, because this procedure is used for the other Bianchi Class A models.
Using the change of variables \((\alpha, x, y) \rightarrow (\beta_1, \beta_2, \beta_3)\), where the law of the transformation between both set of variables is
\[
\begin{align*}
\beta_1 &= \alpha + x + \sqrt{3}y, \\
\beta_2 &= \alpha + x - \sqrt{3}y, \\
\beta_3 &= \alpha - 2x,
\end{align*}
\] (3.1)
the equation \(\nabla^2 = -\left(\frac{\partial}{\partial \alpha}\right)^2 + \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\) can be written in the following way
\[
\begin{align*}
\left[\nabla\right]^2 &= 3\left(\frac{\partial}{\partial \beta_1} + \frac{\partial}{\partial \beta_2} + \frac{\partial}{\partial \beta_3}\right)^2 - 12\left[\frac{\partial}{\partial \beta_1} \frac{\partial}{\partial \beta_2} + \frac{\partial}{\partial \beta_1} \frac{\partial}{\partial \beta_3} + \frac{\partial}{\partial \beta_2} \frac{\partial}{\partial \beta_3}\right] \\
&= 3\left(\frac{\partial}{\partial \beta_1} + \frac{\partial}{\partial \beta_2} + \frac{\partial}{\partial \beta_3}\right)^2 - 12\left[\frac{\partial}{\partial \beta_1} \frac{\partial}{\partial \beta_2} + \frac{\partial}{\partial \beta_1} \frac{\partial}{\partial \beta_3} + \frac{\partial}{\partial \beta_2} \frac{\partial}{\partial \beta_3}\right]. \quad (3.2)
\end{align*}
\]

The potential term of the Bianchi type IX is transformed in the new variables as
\[
U = \frac{1}{3}\left[\left(e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3}\right)^2 - 4e^{2(\beta_1 + \beta_2)} - 4e^{2(\beta_1 + \beta_3)} - 4e^{2(\beta_2 + \beta_3)}\right]. \quad (3.3)
\]
Then (2.2) for this models is rewritten in the new variables as
\[
3\left(\frac{\partial^2 \Phi}{\partial \beta_1^2} + \frac{\partial^2 \Phi}{\partial \beta_2^2} + \frac{\partial^2 \Phi}{\partial \beta_3^2}\right) - 12\left[\frac{\partial^2 \Phi}{\partial \beta_1 \partial \beta_2} + \frac{\partial^2 \Phi}{\partial \beta_1 \partial \beta_3} + \frac{\partial^2 \Phi}{\partial \beta_2 \partial \beta_3}\right] - \frac{1}{3}\left[\left(e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3}\right)^2 - 4e^{2(\beta_1 + \beta_2)} - 4e^{2(\beta_1 + \beta_3)} - 4e^{2(\beta_2 + \beta_3)}\right] = 0. \quad (3.4)
\]
Now, we can use the separation of variables method to get solutions to the last equation for the \(\Phi\) function, obtaining for the Bianchi type IX model \([4]\)
\[
\Phi_9 = \pm \frac{1}{6}\left(e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3}\right), \quad (3.5)
\]
and \([16]\)
\[
\tilde{\Phi}_9 = \pm \frac{1}{6}\left(e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3} - 2e^{(\beta_1 + \beta_2)} - 2e^{(\beta_1 + \beta_3)} - 2e^{(\beta_2 + \beta_3)}\right). \quad (3.6)
\]
But surprisingly enough this \(\tilde{\Phi}_9\) does not produce any new wave function because necessarily \(W=0\). This means that the recent solutions that have been claimed in supersymmetric quantum cosmology \([11]\) are not solutions of the standard WDW equation.
This same procedure is used for getting the $\Phi$ function for the others Bianchi Class A models. Also for the Bianchi type II model there exist a second solution \( \Phi_2 = \pm \frac{1}{6} e^{2\beta_1} + F[(\beta_1 + \beta_2)] \), where $F$ is any function of the argument. But for $F \neq 0$ the wave function vanishes (for $F=0$, $\Phi_2 \equiv \Phi_2$). We show these results in the table 1.

With this result, the solution to (2.5) and (2.6), give for the $W$ function

$$W_9 = W_0 \exp[(3 + \frac{Q}{2})\alpha].$$

(3.7)

where $W_0 = \text{const}$, and $Q = \pm 6.$, then the wave function has the following form

$$\Psi_9 = W_0 \exp[(3 + \frac{Q}{2})\alpha] \exp[\pm \Phi_9].$$

(3.8)

In the case of the Taub model, one replace in all terms only $y = P_y = 0$

$$\Phi_{\text{Taub}} = \frac{1}{6} e^{2\alpha} [2 e^{2x} + e^{-4x}],$$

(3.9)

and the function $W$

$$W = W_0 \exp(\alpha + x).$$

(3.10)

In this last case, the only value of the $Q$ parameter is zero. These solutions were given by Moncrief and Ryan[4].

In the case of FRW model, the value of $Q=2$ and $W=\text{constant}$ are obtained by means of this method. The solution is well known \( \Phi_{\text{FRW}} = \frac{1}{2} e^{2\alpha} \), and $\Psi_{\text{FRW}} = W_0 \exp[\pm \Phi_{\text{FRW}}].$

The functions $W$ for the Bianchi Class A models are shown in Table 2.

If one look at the expressions for the functions $\Phi_i$, one notes that there exist a general form to write them using the $3\times3$ matrix $m^{ij}$ that appear in the classification scheme of Ellis and MacCallum[18, 19], the structure constants are written in the form

$$C^i_{jk} = \epsilon_{jks} m^{si} + \delta^{i}_{[k} a_{j]},$$

(3.11)
where \( a_i = 0 \) for the Class A models.

If we define \( g_i(q^\mu) = (e^{\beta_1}, e^{\beta_2}, e^{\beta_3}) \), with \( \beta_i \) given in (3.1), all solutions to (2.2) can be written as

\[
\Phi(q^\mu) = \pm \frac{1}{6} [g_i M^{ij}_{\beta} (g_j)^T],
\]

where \( M^{ij} = m^{ij} \) for the Class A Bianchi models, excepting the Bianchi type VI\( h = -1 \) for which we redefine the matrix to be consistent with (3.12)

\[
M^{ij} = \frac{6}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

For the rest of the models (3.12) can be reduced to the expression given previously in the literature, in connection with superquantum cosmology [8, 12]

\[
\Phi(q^\mu) = \pm \frac{1}{6} [m^{ij} g_{ij}],
\]

where \( g_{ij} \) is the 3-metric. Then, for the Class A Bianchi models the wave function \( \Psi \) can be written in the general form

\[
\Psi = W \exp \left[ \pm \frac{1}{6} [g_i M^{ij}_{\beta} (g_j)^T] \right],
\]

and for each cosmological model under consideration the wave function of interest can be read from tables 1 and 2.

4 Final remarks

Wave functions of the form \( \Psi = W e^{\pm \Phi} \) are the only kind of solutions already found in supersymmetric quantum cosmology and also for the WDW equation defined in the bosonic sector of the heterotic strings. Also for the Bianchi type IX model these are the only known solutions in standard quantum cosmology. We have found solutions
of the same kind to the Class A Bianchi models. It is to be noted that \( \Psi = W e^{\pm \tilde{\Phi}_9} \) (and \( \Psi = W e^{\pm \tilde{\Phi}_2} \) too) which seems to be a solution [11] in supersymmetric quantum cosmology is not an allowed wave function in standard quantum cosmology. For the Bianchi Vi_{h=-1} model, (3.12) should be used instead of (3.13) to get the right \( \Phi_6 \) and consequently the corresponding wave function.

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FIGURE CAPTION

Table 1. Potential U and $\Phi$ function for the Class A Bianchi Models.

Table 2. W function and constraints between the constants in the solutions for Class A Bianchi models.
| Bianchi type | Potential U | $\Phi$ |
|--------------|-------------|--------|
| I            | 0           | 0      |
| II           | $\frac{1}{6}e^{4\beta_1}$ | $\pm\frac{1}{6}e^{2\beta_1}$ |
| VI$_{h=-1}$  | $\frac{4}{3}e^{2(\beta_1+\beta_2)}$ | $\pm\frac{1}{6}[2(\beta_1 - \beta_2)e^{(\beta_1+\beta_2)}]$ |
| VII$_{h=0}$  | $\frac{1}{3}[e^{4\beta_1} + e^{4\beta_2} - 2e^{2(\beta_1+\beta_2)}]$ | $\pm\frac{1}{6}[e^{2\beta_1} + e^{2\beta_2}]$ |
| VIII         | $\frac{1}{3}[e^{4\beta_1} + e^{4\beta_2} + e^{4\beta_3} - 2e^{2(\beta_1+\beta_2)} + 2e^{2(\beta_1+\beta_3)} + 2e^{2(\beta_2+\beta_3)}]$ | $\pm\frac{1}{6}[e^{2\beta_1} + e^{2\beta_2} - e^{2\beta_3}]$ |
| IX           | $\frac{1}{3}[e^{4\beta_1} + e^{4\beta_2} + e^{4\beta_3} - 2e^{2(\beta_1+\beta_2)} - 2e^{2(\beta_1+\beta_3)} - 2e^{2(\beta_2+\beta_3)}]$ | $\pm\frac{1}{6}[e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3}]$ |
| Bianchi type | W | constraint |
|--------------|---|------------|
| I            | $\exp [\vec{x} \cdot \vec{k}]$ | $a^2 - aQ - (b^2 + c^2) = 0$ |
| II           | $\exp[(3 + \frac{Q}{2} - a)\alpha + (b - a)x - \frac{b}{\sqrt{3}}y]$ | $108 - 72a + 24ab - 16b^2 - 3Q^2 = 0$ |
| VI_{h=-1}   | $\exp[\frac{1}{4}(\alpha + x)]$ | $Q = 0$ |
| VII_{h=0}   | $\exp[(3 + \frac{Q}{2} - a)\alpha - ax]$ | $36 - 24a - Q^2 = 0$ |
| VIII         | $\exp[(3 + \frac{Q}{2})\alpha]$ | $Q = \pm 6$ |
| IX           | $\exp[(3 + \frac{Q}{2})\alpha]$ | $Q = \pm 6$ |