Any \(\ell\)-state solutions of the Feynman propagator for the q-deformed Woods-Saxon potential in arbitrary dimensions

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Abstract. Using the space-time transformations, approximate analytical solutions of the D-dimensional propagator in the presence of q-deformed Woods-Saxon potential are obtained. The analytical expression of the energy eigenvalues is given for various quantum numbers and the corresponding normalized eigenfunctions are obtained in terms of hypergeometric function. Our results are compared with those given by The Nikivorov-Uvarov method.

1. Introduction

It is well known that the analytical solutions of the Schrödinger equations with some exponential-type potentials are impossible for any arbitrary l-wave states, approximation schemes such as the Pekeris-type (or a new improved approximation scheme) need to be employed to deal with the centrifugal term [1, 2]. These exponential-type potentials include the Pöschl-Teller potential [3, 4], Manning-Rosen potential [5, 6], Rosen-Morse-type potentials [7].

The deformed Woods-Saxon (dWS) potential is a short range potential and is given by

\[ V_{dWS}(r) = -\frac{V_0}{q + e^{(r - R)/a}}. \]  

(1)

where \( R = R_0 A_0^{1/3} \), \( V_0 = (40.5 + 0.13A_0)\text{MeV}, r >> a, q > 0 \). where \( V_0 \) is the depth of potential, \( q \) is a real parameter which determines the shape (deformation) of the potential, \( a \) is the diffuseness of the nuclear surface, \( R \) is the width of the potential, \( A_0 \) is the atomic mass number of target nucleus and \( r_0 \) is radius parameter.

This potential has been used in nuclear, particle, atomic, condensed matter and chemical physics [8-14]. It is known that for the Woods-Saxon potential, the Schrödinger equation can be solved exactly when \( \ell = 0 \) [15]. Unfortunately, for arbitrary \( \ell \)-state, the radial Schrödinger equation does not admit exact solutions. In this case, some authors have used an approximation scheme to study analytically arbitrary \( \ell \)-wave bound states of the Schrödinger equation for the Woods-Saxon potential [16]and the deformed Woods-Saxon [17].

This paper is organized as follows. In Sec. 2 we derive arbitrary \( \ell \)-states solutions of the deformed Woods-Saxon potential within the Feynman path integrals formalism by approximate method. In Sec.3 the analytical expressions are given to compare with those obtained by other method. The concluding remarks are given in Sec. 4.

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2. Feynman Propagator

Since the potential (1) is the deformed Woods-Saxon potential, we have a spherically symmetric system which may be conveniently described in polar coordinates. The propagator is written as [18]:

\[ K(r'', t''; r', t') = \frac{1}{4\pi r'' r'} \sum_{l=0}^{\infty} (2l + 1) K_l(r'', t''; r', t') P_l(\cos \theta). \] (2)

Where \( P_l(\cos \theta) \) is the Legendre polynomial with \( \theta \equiv (r'', r') \) and

\[ K_l(r'', t''; r', t') = \lim_{N \to \infty} \int \prod_{j=1}^{N} \exp \left[ \frac{i}{\hbar} S_j \right] \prod_{j=1}^{N} \left[ \frac{m}{2\pi i \hbar \varepsilon} \right]^{1 \over 2} \int_{r_j}^{r''} dr_j. \] (3)

With the sliced action \( S_j = \frac{m}{2\varepsilon} (\Delta r_j)^2 - \varepsilon V_{\text{eff}}(r_j) \), where

\[ V_{\text{eff}}(r_j) = \frac{\hbar^2}{2m} \left( l + \frac{D}{2} - 1 \right)^2 - \frac{1}{4} \frac{V_0}{q + e^{(r_j - R)}}, \] (4)

with the notations \( \Delta r_j = r_j - r_{j-1}, \epsilon = t_j - t_{j-1}, t' = t_0, t'' = t_N, T = t'' - t' \) and for any variable \( r: r_j = r(t_j) \).

If we take the special case \( D = 3 \), we get

\[ V_{\text{eff}}(r_j) = \frac{\hbar^2 l(l+1)}{2m r_j r_{j-1}} - \frac{V_0}{q + e^{(r_j - R)}}, \] (5)

Since the Feynman propagator, with dWS effective potential (3), has no analytical solution for \( l \neq 0 \) states, an approximation to the centrifugal term has to be made. The good approximation for the centrifugal barrier is taken as [17]:

\[ \frac{l(l+1)}{r_j^2} \approx \frac{l(l+1)}{R^2} \left[ D_0 + \frac{D_1}{q + e^{\epsilon R}} + \frac{D_2}{(q + e^{\epsilon R})^2} \right], \] (6)

where

\[ D_0 = 1 - \left( 1 + e^{-aR} \right)^2 \left( \frac{4aR}{1 + e^{-aR}} - 3 - aR \right), \]

\[ D_1 = 2 \left( 1 + e^{aR} \right) \times \left[ 3 \left( \frac{1 + e^{-aR}}{aR} \right) - (3 + aR) \left( \frac{1 + e^{-aR}}{aR} \right) \right], \]

\[ D_2 = \left( 1 + e^{aR} \right)^2 \left( \frac{1 + e^{-aR}}{aR} \right)^2 \times \left( 3 + aR - \frac{2aR}{1 + e^{-aR}} \right). \]

By substituting the centrifugal term in (5) by its approximation (6), we get

\[ V_{\text{eff}}(r_j) = A \tanh(\alpha r_j) - \frac{B}{\cosh^2(\alpha r_j)} + C, \] (7)

In the following, let us apply the procedure of path reparametrization [19]. The path integral for the potential (7) is

\[ K_l(r'', r'; T) = \int \mathcal{D}r(t) \exp \left[ \int_{t'}^{t''} \left( \frac{m}{2} \dot{r}^2 - V_{\text{eff}}(r) \right) dt \right]. \] (8)
A convenient way to derive the corresponding transformation formula is to use the energy-dependent Green function $G$ of the propagator $K_l$. By achieving the point canonical transformation and the time transformation,

$$\begin{cases} r = f(q) \\ dt = [f'(q)]^2 ds, \end{cases}$$

we obtain the following Fourier transform formula

$$K_l(r''', r'; T) = \frac{1}{2\pi i} \int G(r''', r', E)e^{-iET/\hbar} dE,$$

where

$$G(r'', r', E) = \frac{i}{\hbar} [f'(q')f'(q'')]^{1/2} \int_0^\infty \tilde{K}_i(q'', q'; s'') ds''.$$  \hspace{1cm} (11)$$

Here

$$\tilde{K}_i(q'', q'; s'') = \int Dq(s) \exp \left[ \frac{i}{\hbar} \int_0^{s''} \left( \frac{m}{2} q'^2 - f'^2(q) [V(f(q)) - E] - \Delta V(q) \right) ds \right],$$

and

$$\Delta V(q) = \frac{\hbar^2}{8m} \left[ \frac{3 [f''(q)]^2}{[f'(q)]^2} - 2 \frac{f'''(q)}{f'(q)} \right].$$  \hspace{1cm} (13)$$

The functions $f'$, $f''$ and $f'''$ are the successive derivatives of the function $f$.

$$\tilde{K}_i(q'', q'; s'') = \exp \left[ \frac{is''(E + C - A)}{\hbar \alpha^2} \right] \times \int Dq(s) \exp \left[ \frac{i}{\hbar} \int_0^{s''} \frac{m}{2} q'^2 - \frac{\hbar^2}{2m} \left( \frac{\eta(\eta - 1)}{\sinh^2 q} - \frac{\nu(\nu - 1)}{\cosh^2 q} \right) ds \right] = \exp \left[ \frac{is''(E + C - A)}{\hbar \alpha^2} \right] K^{\text{MPT}}_i(q'', q'; s'')$$

with

$$\eta = \frac{1}{2} \pm \sqrt{\frac{-2m(E + C + A)}{\alpha^2 \hbar^2}},$$

$$\nu = \frac{1}{2} \pm \sqrt{1 + 8mB \alpha^2 \hbar^2}$$

and $K^{\text{MPT}}_i$ is the path integral of the modified Pöschl-Teller potential which is a known solved problem \cite{?} for which the bound states wave functions are explicitly given by:

$$\chi_{l,n}^{(k_1,k_2)}(q) = N_{n}^{(k_1,k_2)}(\sinh q)^2k_{2} - 1/2(\cosh q)^{-2k_{1} + 3/2} \times \ {}_2F_1(-k_1+k_2+k, -k_1+k_2-k+1; 2k_2; -\sinh^2 q),$$

\hspace{1cm} (15)
where \( k = k_1 - k_2 - n \), and

\[
N_n^{(k_1,k_2)} = \frac{1}{\Gamma(2k_2)} \left( \frac{(2k-1)\Gamma(k_1+k_2-k)\Gamma(k_1+k_2+k-1)}{\Gamma(k_1-k_2+k)\Gamma(k_1-k_2-k+1)} \right)^{1/2},
\]

(16)

The eigenvalues are expressed by

\[
E_{n}^{MPT} = -\frac{\hbar^2}{2m} [2(k_1 - k_2 - n) - 1]^2.
\]

(17)

Calculating \( K_l(q'', q'; s'') \) allows us to obtain the Green function. Since this is known, the whole energy spectrum is obtained from its poles, and from the residues at the poles we obtain the corresponding wave functions.

Substituting (15) in (14), we get

\[
\hat{K}_l(q'', q'; s'') = \sum_{n=0}^{N_m} \exp \left\{ \frac{i}{\hbar} s'' \left[ \frac{E + C - A}{\alpha^2} - E_n^{MPT} \right] \right\} \chi^{(k_1,k_2)}_{l,n}(q'') \chi^{* (k_1,k_2)}_{l,n}(q')
\]

\[
+ \int_0^\infty dK \exp \left\{ \frac{i}{\hbar} s'' \left[ \frac{E + C - A}{\alpha^2} - \frac{\hbar^2 K^2}{2m} \right] \right\} \chi^{(k_1,k_2)}_{K}(q'') \chi^{* (k_1,k_2)}_{K}(q').
\]

(18)

Then, by integrating this latter over the pseudo-time parameters \( s'' \), the Green function \( G \) is written as:

\[
G_l(r'', r'; E) = \sum_{n=0}^{N_m} \frac{dW_S(k_1,k_2)(r'') \, dW_S*(k_1,k_2)(r')}{E_{n,l}^{MPT} - E}
\]

\[
+ \int_0^\infty dK \frac{\chi^{dW_S}_{l,K}(r'') \chi^{dW_S*}_{l,K}(r')}{(\hbar^2 K^2 \alpha^2/2m) - A - C - E}.
\]

(19)

Following [20], the values of \( k_1, k_2 \), satisfying the appropriate boundary conditions for \( r \to 0 \) and \( r \to \infty \), are respectively:

\[
k_1 = \frac{1}{2} \left( 1 + \sqrt{1 + \frac{8mE}{\alpha^2 \hbar^2}} \right) \equiv \frac{1}{2} (1 + s),
\]

\[
k_2 = \frac{1}{2} \left( 1 + \frac{1}{2} (s - 2n - 1) - \frac{2mA}{\alpha^2 (s - 2n - 1)} \right).
\]

3. Energy spectrum and wave functions

Substituting the values of \( k_1, k_2 \) in (15) and (17), and by considering

\[
u = \frac{1}{2} [1 + \tanh \alpha r],
\]

we get

\[
\chi_{l,n}^{dW_S(k_1,k_2)}(r) = \sqrt{\alpha} \chi_{n}^{(k_1,k_2)} u^{k_2-(1/2)} (1 - u)^{(s/2)-k_2-n}
\]

\[
\times F_1(-n, s - n; 2k_2; u)
\]

\[
= \left[ \left( 1 - \frac{4mA}{\hbar \alpha^2 (s - 2n - 1)^2} \right) \frac{(s - 2k_2 - 2n) n! \Gamma(s - n)}{\Gamma(s + 1 - n - 2k_2) \Gamma(2k_2 + n)} \right]^{1/2}
\]

\[
\times 2^{n+(1-s)/2} (1 - \tanh \alpha r)^{s/2-k_2-n} \times (1 + \tanh \alpha r)^{k_2-1/2} P_n^{(s-2k_2-2n,2k_2-1)}(\tanh \alpha r),
\]

(20)

\[
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\]
where

$$E_{n,l}^{WS} = \alpha^2 E_{n}^{MPT} + A - C,$$

$$= -\frac{\hbar^2(s - 2n - 1)^2\alpha^2}{8m} - \frac{2mA^2}{\hbar^2\alpha^2(s - 2n - 1)^2} + C.$$  \tag{21}$$

This expression is perfectly identical to that given in refs.[17], obtained with the framework of the Schrödinger formalism.

4. Concluding remarks

We have obtained the bound state solutions of the Feynman Propagator for the deformed Woods-Saxon potential using space-time transformations. The eigenvalue have been obtained from the poles of the Green function for various $n$ and $\ell$ quantum numbers. We find that our results are in excellent agreement with the findings of other methods.

To conclude, our method is efficient in solving this type of potentials. We intend in a near future, to expand the path integral method to a more general form of potentials and relativistic cases.

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