Generation of coherent superposition of Fock states in a cavity. Entanglement of atomic coherent states.

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A scheme for preparation of coherent superposition of Fock states of electromagnetic field is constructed. The superposition state is created inside the cavity via a strong interaction of a four-level atom with quantum field of the cavity and classic laser fields. We demonstrate the possibility to create desired arbitrary superposition of the cavity Fock states just by changing the relative delay of the fields. Then, as another application of our model, we study a means of creating entanglement of neutral four-level atoms using different sequences of interactions with the cavity and laser fields.

I. INTRODUCTION

The past few years have seen an upsurge of interest in the creation of non-classical states of electromagnetic field, such as squeezed states, Fock states [1]. These states have no classical analogies and exhibit particular statistical properties. Relevant examples are superpositions of Fock states, which can have applications in quantum computing [2].

Of particular interest in the generation of quantum states of light fields is that associated with cavity QED, in which beams of atoms strongly interact with a single cavity electromagnetic mode. It is assumed that atom-cavity coupling strengths much larger than dissipative rates.

Yet another class of theory and experiments relevant to quantum communication and computation procedures is generation of entangled states [3]. Entangled states are interesting because they exhibit correlations that have no classical analog.

A variety of proposals have been put forward for the generation of Fock states in a cavity. The most simple system is a single two-level atom coupled with cavity-mode field. Theoretical studies indicate the possibility to create and arbitrary prescribed superposition of Fock states in a cavity via two-level atom interacting with quantum and classical fields [1]. This model requires precise control of time sequences of amplitudes and phases of atom-cavity and atom-laser coupling fields. In a similar vein, Domokos et al. [4] demonstrated an experimental scheme for the generation of Fock states in a cavity. Initially excited Rydberg two-level atoms pass through and interact with two quantized field modes of the microwave cavity. Specifically, the first mode is a photon reservoir prepared in a coherent state and the next one stores the Fock states.

Finally, a Fock states preparation scheme was suggested by Parkins et al. [6], in which a beam of Λ-system atoms interact with a single quantized field mode of cavity and then with classical laser field. The initial eigenstate of the atom-cavity system, which corresponds to the atom being in first ground state, adiabatically evolves into a final eigenstate, which corresponds to the atom in the ground state and the cavity in a Fock state. So, if initially there is no photon in the cavity, it is created one-photon Fock state. Because of the system evolution corresponds to the dark state, which has no contribution from excited atomic state, spontaneous emission does not figure in the dynamics of the system.

A scheme for preparation of pairs of massive atoms in an entangled state was predicted by Cirac and Zoller [8] and then observed by Hagley et al. [7]. The states were produced by interaction with a high-Q cavity. The technique of Gerry and Grobe [9] for generation of an entangled state assumes a large number of two-level atoms confined in a cavity.

In the present paper we describe in detail a method for the generation of coherent superposition of Fock states in a cavity and demonstrate its feasibility by analytic and numerical calculations. Our scheme requires the passage of a four-level atom through overlapping two classical laser fields and quantum cavity-mode field. The first laser field of frequency \(\omega_1\) couples the atomic state \(|\psi_1\rangle\) with the excited state \(|\psi_2\rangle\). The second laser field of frequency \(\omega_2\) couples the state \(|\psi_3\rangle\) with the excited state \(|\psi_2\rangle\). The quantum cavity-mode field couples the states \(|\psi_4\rangle\) and \(|\psi_2\rangle\). Figure 1 illustrates the connections. Unanyan et al. [10] have shown that using appropriate delay and sequence of interaction pulses one can create robust coherent superposition of atomic states. Our scheme is based on those results. We show that, if one of the pulses is the quantum field of the cavity, a coherent superposition of quantum states is generated in the cavity. In fact, our scheme requires the following sequence of interactions with the atom (see Fig. 2): the cavity-mode field is turned on before the first laser field arrives and is turned off after \(V_1\) ceases. Further, the second laser fields \(V_2\) is turned on right after laser pulse and turned off the last. Of most interest for us is the case when the cavity quantum field is initially in the vacuum state \(|n = 0\rangle\) and the passage of the atom through the interaction
Thus, in case of exact resonance we have the following equations for the coefficients of transformation:

\[ i \hbar \frac{d}{dt} |\Psi(t)\rangle = H(t) |\Psi(t)\rangle, \]  

(1)

with the Hamiltonian

\[ H(t) = H_{at} + \hbar \omega c^+ c + (V_1(t) |1\rangle \langle 2| + H.c.) + (V_2(t) |3\rangle \langle 2| + H.c.) + \hbar (c^+ \beta(t) |4\rangle \langle 2| + H.c.), \]  

(2)

where \( c \) is the annihilation operator for the cavity-mode, \( \beta(t) \) is the atom-cavity-mode coupling strength and \( V_{1,2}(t) = W_{1,2} e^{i\omega_{1,2} t} + W_{1,2}^* e^{-i\omega_{1,2} t} \). The time dependence of \( W_{1,2} \) and \( \beta(t) \) is provided by the motion of the atom through the interaction region.

We express the solution of Schrödinger equation (1) as a superposition of atomic basis states:

\[ |\Psi(t)\rangle = \sum_i a_i(t) |\psi_i\rangle, \]  

(3)

here \( |\psi_i\rangle \) are the eigenvectors of the free atomic Hamiltonian

\[ H_{at} |\psi_i\rangle = \varepsilon_i |\psi_i\rangle, \quad (i = 1, 2, 3, 4). \]  

(4)

Further, we transform the coefficients through the following expression:

\[ a_i(t) = b_i(t) e^{-i(\varepsilon_i + \omega c^+ c)t} \]  

(5)

and, expanding the amplitudes \( b_i(t) \) in photon number states of the cavity, we have

\[ b_i(t) = \sum_n b_{i,n}(t) |n\rangle. \]  

(6)

Thus, in case of exact resonance we have the following equations for the coefficients of transformation:

\[ i \hbar \frac{d}{dt} B_n(t) = W(t) B_n(t), \]  

(7)

where \( B_n(t) \) is a column of coefficients \( b_{in}(t) \), and

\[ W(t) = \begin{pmatrix} 0 & W_1(t) & 0 & 0 \\ W_1(t) & 0 & W_2(t) & \sqrt{n + 1}\beta(t) \\ 0 & W_2(t) & 0 & 0 \\ 0 & \sqrt{n + 1}\beta(t) & 0 & 0 \end{pmatrix}. \]  

(8)

Here the matrix elements \( W_1(t), W_2(t), \beta(t) \) are assumed to be real-valued, since any constant phases of laser fields can be absorbed in the definition of amplitudes \( b_{in}(t) \).
We are interested in solutions of the equation (3), (8) in adiabatic approximation. Two adiabatic energy eigenstates are degenerate, with null eigenvalue:

\[
\Phi_1(t) = \begin{bmatrix} \cos \vartheta(t) \\ 0 \\ -\sin \vartheta(t) \end{bmatrix}, \quad \Phi_2(t) = \begin{bmatrix} \sin \varphi(t) \sin \vartheta(t) \\ 0 \\ -\cos \varphi(t) \sin \vartheta(t) \end{bmatrix},
\]

and:

\[
\lambda_{1,2}(t) = 0
\]

The remaining eigenvectors and eigenvalues are:

\[
\Phi_3(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos \varphi(t) \sin \vartheta(t) \\ \sin \varphi(t) \\ \cos \varphi(t) \cos \vartheta(t) \end{bmatrix}, \quad \Phi_4(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos \varphi(t) \sin \vartheta(t) \\ -1/\sqrt{2} \sin \varphi(t) \\ \cos \varphi(t) \cos \vartheta(t) \end{bmatrix},
\]

with:

\[
\lambda_3(t) = +\Omega(t), \quad \lambda_4(t) = -\Omega(t)
\]

where \( \Omega(t) = \sqrt{W_1(t)^2 + W_2(t)^2 + (n + 1)\beta(t)^2} \). \( \vartheta(t) \) and \( \varphi(t) \) are defined by

\[
\tan \vartheta(t) = \frac{W_1(t)}{\sqrt{n + 1} \beta(t)}, \quad \tan \varphi(t) = \frac{W_2(t)}{\sqrt{W_1(t)^2 + (n + 1)\beta(t)^2}}.
\]

It is easily seen that the degenerate eigenvectors \( \Phi_{1,2}(t) \) receive no contribution from the excited atomic state \( |2\rangle \). These states are therefore termed trapped or dark states. The significant feature of our scheme is that, if the system remains in some superposition of the two degenerate dark states, atomic spontaneous emission plays no role in the system dynamics.

### III. GENERATION OF CA VITY SUPERPOSITION STATES

In order to place diagonally the matrix \( W(t) \) (8) we introduce a unitary matrix

\[
U(t) = \begin{pmatrix} \cos \vartheta(t) & \sin \vartheta(t) & \frac{1}{\sqrt{2}} \sin \vartheta(t) \cos \varphi(t) & \frac{1}{\sqrt{2}} \sin \vartheta(t) \cos \varphi(t) \\ 0 & 0 & \frac{1}{\sqrt{2}} \sin \varphi(t) & \frac{1}{\sqrt{2}} \sin \varphi(t) \\ -\sin \vartheta(t) & -\cos \vartheta(t) & \frac{1}{\sqrt{2}} \cos \vartheta(t) \cos \varphi(t) & \frac{1}{\sqrt{2}} \cos \vartheta(t) \cos \varphi(t) \\ -\sin \vartheta(t) & \cos \vartheta(t) & \frac{1}{\sqrt{2}} \cos \vartheta(t) \sin \varphi(t) & \frac{1}{\sqrt{2}} \cos \vartheta(t) \sin \varphi(t) \end{pmatrix}
\]

Taking into consideration the non-adiabatic interactions between adiabatic energy eigenstates (8), (11) we obtain:

\[
i \frac{dC_n(t)}{dt} = \widetilde{W}_n(t)C_n(t),
\]

where

\[
\widetilde{W}(t) = U^{-1}(t)W(t)U(t) + i \frac{dU^{-1}(t)}{dt} U(t),
\]

\[
B_n(t) = U(t)C_n(t).
\]

Obviously, through the conditions

\[
\frac{d\vartheta(t)}{dt} \ll \Omega(t),
\]

\[
\frac{d\varphi(t)}{dt} \ll \Omega(t),
\]

are satisfied.
which imply that the effective pulse area is very large

\[ S' = \int_{-\infty}^{+\infty} d\tau \Omega(\tau) \gg 1, \]  

(20)

the coupling of the dark states (10) to the bright states (11) can be disregarded. Hence, the matrix \( \tilde{W}_n(t) \) is given by

\[
\tilde{W}_n(t) \approx \begin{pmatrix}
0 & -i \vartheta \sin \varphi(t) & 0 & 0 \\
i \vartheta \sin \varphi(t) & 0 & 0 & 0 \\
0 & 0 & \Omega(t) & 0 \\
0 & 0 & 0 & -\Omega(t)
\end{pmatrix}
\]  

(21)

Note that because of the degeneracy the coupling between dark states (10) still exists. The solution of the equation (15) with the matrix (21) has the simple form

\[ C_n(t) = R(t, -\infty)C_n(-\infty) \]  

(22)

Here \( C_n(-\infty) \) is defined by initial conditions and the evolution operator is given by

\[
R(t, -\infty) = \begin{pmatrix}
\cos \gamma(t) & -\sin \gamma(t) & 0 & 0 \\
\sin \gamma(t) & \cos \gamma(t) & 0 & 0 \\
0 & 0 & e^{-iS(t)} & 0 \\
0 & 0 & 0 & e^{iS(t)}
\end{pmatrix},
\]  

(23)

with

\[
\gamma(t) = \int_{-\infty}^{t} \vartheta(\tau) \sin \varphi(\tau) d\tau
\]  

(24)

\[
S(t) = \int_{-\infty}^{t} \Omega(\tau) d\tau
\]  

(25)

Proceed from (17) and (22) we obtain the general solutions for coefficients

\[ B_n(t) = U(t)R(t, -\infty)U^{-1}(-\infty)B_n(-\infty) \]  

(26)

The atom is supposed to be initially in the state \( |1\rangle \)

\[ B_n(-\infty) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]  

(27)

Thus, from (26) we obtain the solutions for initial condition (27):\n
\[
B_n(t) = \begin{pmatrix}
\cos \vartheta(t) \cos \gamma(t) + \sin \vartheta(t) \sin \varphi(t) & 0 & 0 \\
-\cos \varphi(t) \sin \gamma(t) & 0 & 0 \\
-\sin \vartheta(t) \cos \gamma(t) + \cos \vartheta(t) \sin \varphi(t) \sin \gamma(t) & 0 & 0
\end{pmatrix}
\]  

(28)

Our scheme requires the passage of the atom through the cavity field and laser field \( V_1(t) \). After that, the atom interacts with the second laser field \( V_2(t) \). Then, we consider the case, when laser field \( V_2(t) \) turn off after the cavity quantum field and laser field \( V_1(t) \) cease. To achieve this pulse sequence we choose \( \vartheta(+\infty) = 0, \quad \varphi(+\infty) = \frac{\pi}{2} \)
The result of this sequence of interactions is the state
\[
B_n(+\infty) = \begin{pmatrix}
\cos \gamma \\
0 \\
\sin \gamma 
\end{pmatrix}
\]  
\hspace{1cm} (30)

As easily can be seen from the superposition expressions (3), (11) and (30), in case of formation of dark states, and small non-adiabatic interaction of degenerate dressed states, the asymptotic wave function for \( t \rightarrow +\infty \) can be written as
\[
|\Psi\rangle = e^{-i(\varepsilon_1 + \omega_1) t} \left[ \cos \gamma |n\rangle |\psi_1\rangle + e^{i\varphi_1} e^{-i\omega_1 t} \sin \gamma |n + 1\rangle |\psi_4\rangle \right]
\]  
\hspace{1cm} (31)

here \( \varphi_1 \) is the constant phase factor of the laser field \( V_1(t) \), and
\[
\gamma = \int_{-\infty}^{+\infty} \dot{\vartheta}(t) \sin \varphi(t) dt
\]  
\hspace{1cm} (32)

Especially, if the cavity-mode is initially in the vacuum state, i.e. 0-photon Fock state, the resulting wave function is given by a superposition of vacuum and 1-photon states:
\[
\Phi(t) = e^{-i\varepsilon_1 t} \left[ \cos \gamma |0\rangle |\psi_1\rangle + e^{i\varphi_1} \sin \gamma e^{-i\omega_1 t} |1\rangle |\psi_4\rangle \right]
\]  
\hspace{1cm} (33)

One can achieve the desired superposition just by changing the relative delay between pulses.

Fig. 7 shows an example of population histories for this sequence of interaction of three Gaussian pulses. Fig. 8 demonstrates the dependence of mixing angle \( \gamma \) on the delay between pulses calculated numerically.

We note that the only conditions for successful operation (18), (19) can be easily fulfilled. The scheme is robust against small variations of parameters.

**IV. ENTANGLEMENT OF TWO FOUR-LEVEL ATOMS**

In the following we modify the scheme presented above to prepare an entanglement of atomic coherent states. Let us consider now two four-level atoms and a configuration of interaction fields when the laser pulses \( V_1(t) \) and \( V_2(t) \) equal each other (Fig. 6). We assume, that initially \( (t \rightarrow -\infty) \) the atom named as first is in \( |\psi_1\rangle \) state, the second one is in \( |\psi_4\rangle \) state, and the cavity field is in the vacuum state \( |n = 0\rangle \):
\[
|\psi_1\rangle_1 \otimes |\psi_4\rangle_2 \otimes |n = 0\rangle
\]  
\hspace{1cm} (34)

At the first step the first atom interacts with quantum field and then with the laser field (see Fig. 6). The initial state asymptotically coincides with the dressed state \( \Phi_1(t) \) at very early times. At the end of the interaction, as can be easily seen from (3), (11) the following connections exist between adiabatic and bare states.
\[
\Phi_1 \rightarrow |\psi_4\rangle_1 \otimes |n = 1\rangle
\]  
\hspace{1cm} (35)

and
\[
\Phi_2 \rightarrow \frac{1}{\sqrt{2}} (|\psi_1\rangle_1 - |\psi_3\rangle_1) \otimes |n = 0\rangle
\]  
\hspace{1cm} (36)

Again, under the conditions (18), (19) in the adiabatic limit, we must take into account only transitions between the degenerate dressed states \( \Phi_1(t) \), \( \Phi_2(t) \).

At the next step, the reverse sequence of pulses (i.e. the equal laser fields then the cavity field) acts on the second atom, which is still in \( |\psi_4\rangle_2 \) state (see Fig. 8). It is clear, that the result of this interaction depends on the number of photons in the cavity. If the cavity is in the zero-photon state \( |n = 0\rangle \) as (36), the second atom remains in the \( |\psi_4\rangle_2 \) state. It is straightforward to verify that for one-photon cavity state \( |n = 1\rangle \) (as in (33)) the result of this sequence of interactions is:
\[
\frac{1}{\sqrt{2}} (|\psi_1\rangle_2 - |\psi_3\rangle_2) \otimes |n = 0\rangle
\]  
\hspace{1cm} (37)

So, the initial state (34) turns into the entanglement state
\[
|\psi_1\rangle_1 \otimes |\psi_4\rangle_2 \otimes |n = 0\rangle \rightarrow \left( \frac{1}{2} (|\psi_1\rangle_1 - |\psi_3\rangle_1) \otimes |\psi_4\rangle_2 - \frac{1}{2} |\psi_4\rangle_1 \otimes (|\psi_1\rangle_2 - |\psi_3\rangle_2) \right) \otimes |n = 0\rangle
\]  
\hspace{1cm} (38)

Thus, we obtain the entanglement state of two spatially separated atoms.
V. SUMMARY AND CONCLUSIONS

In the present paper we have suggested a method to create a quantum superposition of Fock states in cavity. We obtained the wave function (31), (33) in case of formation of dark states, and non-adiabatic interaction of two degenerate ”dressed” states. As can be seen, the final result does not depend on the effective area of pulses (20), and depends only on the angle $\gamma$ or on the ratio of delay time and durations of two laser fields. That makes this method of creation of coherent superposition effective and robust against small variations of parameters. Arbitrary coherent superposition of Fock states in the cavity can be obtained by changing the value of the angle $\gamma$, i.e. the delay between laser pulses. Especially, in the case of $\theta = \pi/4$, we obtain Fock states of the cavity field.

A variation of these basic ideas for interaction of cavity and laser fields with atomic systems is to employ the particular properties of the cavity field for the generation of entanglement of atoms. In this arrangement two atoms initially prepared in $|\psi_1\rangle_1$ and $|\psi_4\rangle_2$ states, adiabatically interact with direct and reverse sequences of the cavity field and two equal laser fields. After the interaction multiparticle entanglement of spatially separated neutral atoms is formed.

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FIG. 7. Generation of entanglement of atomic coherent states. Sequence of interaction of pulses for the first atom.

FIG. 8. Generation of entanglement of atomic coherent states. Sequence of interaction of pulses for the second atom.
atom

\[ v_1 \quad v_2 \]
First atom in the state $\Psi_1$

Second atom in the state $\Psi_4$

Interaction potential $V_{1,2}$
$V_{1,2}$ and $\beta$