Some Congruences from the Karlsson-Minton Summation Formula

Junhang Li, Yezhenyang Tang, and Chen Wang

Abstract. Let $p$ be an odd prime. In this paper, by using the well-known Karlsson-Minton summation formula, we mainly prove two supercongruences as variants of a supercongruence of Deines-Fuselier-Long-Swisher-Tu, which confirm some recent conjectures of V.J.W. Guo.

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1. Introduction

For any $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, let $(x)_n = x(x + 1) \cdots (x + n - 1)$ denote the Pochhammer symbol. For $n, r \in \mathbb{N}$ and $a_0, \ldots, a_r, b_1, \ldots, b_r, z \in \mathbb{C}$ with $(b_1)_n, \ldots, (b_r)_n$ being nonzero, the truncated hypergeometric series $r+1 F_r$ are defined as

$$r+1 F_r \left[ a_0, a_1, \ldots, a_r \mid b_1, \ldots, b_r \right]_n = \sum_{k=0}^{n} \frac{(a_0)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_r)_k} \cdot z^k.$$

Clearly, they are partial sums of the classical hypergeometric series. Let $p$ be an odd prime. For any integer $n \geq 1$, the $p$-adic Gamma function introduced by Morita (cf. [8, 12]) is defined as

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1 \leq k < n \\ p \nmid k}} k.$$
Moreover, set $\Gamma_p(0) = 1$, and for any $p$-adic integer $x$ set

$$\Gamma_p(x) = \lim_{n \to x} \Gamma_p(n),$$

where $n$ runs through any sequence of positive integers $p$-adically approaching $x$.

Rodriguez-Villegas [13] investigated hypergeometric families of Calabi-Yau manifolds, and discovered (numerically) a number of possible supercongruences. Some of them have been proved in [9,10] where Mortenson, with the help of the Gross-Koblitz formula, determined $\frac{\binom{\alpha}{1} - \binom{1}{1}}{p-1}$ modulo $p^2$ for $\alpha \in \{1/2, 1/3, 1/4, 1/6\}$. For instance, he showed that

$$\frac{\binom{1}{2} - \binom{1}{1}}{p-1} \equiv (-1)^{(p-1)/2} \pmod{p^2}$$

for any prime $p \geq 5$. Using the Legendre relation of $p$-adic Gamma function (cf. [12, p. 370]), we may replace the right-hand side of (1.1) with $-\Gamma_p(1/2)^2$. Later, Sun [14] extended Mortenson’s result to the general $p$-adic integer $\alpha$. Let $\mathbb{Z}_p$ denote the ring of all $p$-adic integers and $\mathbb{Z}_p^\times := \{x \in \mathbb{Z}_p : p \nmid x\}$. Z.-H. Sun proved that for each odd prime $p$ and $\alpha \in \mathbb{Z}_p^\times$,

$$\frac{\binom{\alpha}{1} - \binom{1}{1}}{p-1} \equiv (-1)^{(\langle \alpha \rangle_p)} \pmod{p^2},$$

for any prime $p \geq 5$. Using the Legendre relation of $p$-adic Gamma function (cf. [12, p. 370]), we may replace the right-hand side of (1.1) with $-\Gamma_p(1/2)^2$. Later, Sun [14] extended Mortenson’s result to the general $p$-adic integer $\alpha$. Let $\mathbb{Z}_p$ denote the ring of all $p$-adic integers and $\mathbb{Z}_p^\times := \{x \in \mathbb{Z}_p : p \nmid x\}$. Z.-H. Sun proved that for each odd prime $p$ and $\alpha \in \mathbb{Z}_p^\times$,

$$\frac{\binom{\alpha}{1} - \binom{1}{1}}{p-1} \equiv (-1)^{(\langle \alpha \rangle_p)} \pmod{p^2},$$

where $\langle x \rangle_p$ is the least nonnegative residue of $x$ modulo $p$, i.e., $0 \leq \langle x \rangle_p \leq p - 1$ and $x \equiv \langle x \rangle_p \pmod{p}$. On the other hand, Deines et al. [1] obtained the following generalization of (1.1): for any integer $d > 1$ and prime $p \equiv 1 \pmod{d}$,

$$\frac{\binom{1}{d} - \binom{1}{1}}{p-1} \equiv \Gamma_p \left( \frac{1}{d} \right)^d \pmod{p^2}.$$  

(1.3)

In fact, Deines et al. also conjectured that for any integer $d \geq 3$ and prime $p \equiv 1 \pmod{d}$, the congruence (1.3) holds modulo $p^3$, and this conjecture was later confirmed by the third author and Pan [15].

In the past decade, many mathematicians studied $q$-analogues of (1.1) and its generalizations; among these $q$-congruences, the first one was obtained by Guo and Zeng [4]. Recently, via the so-called ‘creative microscoping’ method introduced by Guo and Zudilin [5], Guo [3] established a $q$-analogue of (1.3). Meanwhile, Guo obtained a variant of (1.3) as follows: for any integer $d > 1$ and prime $p \equiv 1 \pmod{d}$,

$$\sum_{k=0}^{p-1} k \left( \frac{d-1}{d} \right)^d \frac{k!}{k!} \equiv \frac{(d-1)\Gamma_p \left( \frac{1}{d} \right)}{2d} \pmod{p^2}.$$

(1.3)

The main purpose of this paper is to prove the following variants of (1.3) which confirm two conjectures of Guo [3, (5.4) and (5.5)].
Theorem 1.1. (i) Let $d \geq 4$ be an even integer. Then, for any prime $p \equiv -1 \pmod{d}$ with $p \geq 2d - 1$,

$$dF_{d-1} \left[ \left\lfloor \frac{1}{d} - 1, 1 + \frac{1}{d}, 1 + \frac{1}{d}, \ldots, 1 + \frac{1}{d} \right\rfloor \right]_{p-1} \equiv \frac{d-1}{d^2} \Gamma_p \left( -\frac{1}{d} \right)^d \pmod{p^2}. \quad (1.4)$$

(ii) Let $d \geq 3$ be an odd integer. Then, for any prime $p \equiv -1 \pmod{d}$,

$$dF_{d-1} \left[ \left\lfloor \frac{1}{d}, 1 + \frac{1}{d}, 1 + \frac{1}{d}, \ldots, 1 + \frac{1}{d} \right\rfloor \right]_{p-1} \equiv -\frac{1}{d^2} \Gamma_p \left( -\frac{1}{d} \right)^d \pmod{p^2}. \quad (1.5)$$

The second goal is to show another conjectural congruence [3, Conjecture 1.3].

Theorem 1.2. Let $p \equiv 1 \pmod{4}$ be a prime and $r \geq 1$. Then

$$\sum_{k=0}^{p^r-1} \left( k - \frac{p^{2r} - 1}{4} \right) \frac{1}{k!^2} \equiv 0 \pmod{p^{2r+1}}. \quad (1.6)$$

The rest of this paper is organized as follows. In the next section, we list some necessary lemmas which play key roles in the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.1. In Sect. 4, we prove Theorem 1.2. In Sect. 5, we shall pose a conjecture for further research.

2. Some necessary lemmas

The first key ingredient of our proofs is the following Karlsson-Minton summation formula (cf. [2, p. 19]).

Lemma 2.1. Let $m_1, m_2, \ldots, m_n$ be nonnegative integers. Then

$$n+1F_n \left[ -\left( m_1 + \cdots + m_n \right), b_1 + m_1, \ldots, b_n + m_n, 1 \right] = (-1)^{m_1 + \cdots + m_n} \frac{(m_1 + \cdots + m_n)!}{(b_1)_{m_1} \cdots (b_n)_{m_n}}. \quad (2.1)$$

Our proofs also rely on some properties of the $p$-adic Gamma functions.

Lemma 2.2. [12, p. 369] Let $p$ be an odd prime and $x \in \mathbb{Z}_p$. Then

$$\frac{\Gamma_p(x + 1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } p \nmid x, \\ -1, & \text{if } p \mid x. \end{cases} \quad (2.2)$$

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\langle -x \rangle_p}p^{-1}. \quad (2.3)$$

Remark 2.1. (a) By (2.2), it is easy to see that for any positive integer $n \leq p$,

$$\Gamma_p(n) = (-1)^n \Gamma(n). \quad (2.4)$$
(b) The identity (2.3) is a $p$-adic analogue of the following Legendre relation of the classical Gamma function:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$  

The next lemma concerns a $p$-adic approximation to $\Gamma_p$-quotients.

**Lemma 2.3** [7, Theorem 14]. For any prime $p \geq 5$ and $x \in \mathbb{Z}_p$, there exists $G_1(x) \in \mathbb{Z}_p$ such that for any $t \in \mathbb{Z}_p$,

$$\Gamma_p(x + tp) \equiv \Gamma_p(x)(1 + G_1(x)tp) \pmod{p^2}. \quad (2.5)$$  

For the properties of $G_1(x)$, the reader may consult [11]. The following lemma lists two identities involving the derivatives of $(1 + \alpha + x)_k$, which can be verified directly.

**Lemma 2.4.** For any integer $k \geq 0$ and $\alpha, \beta \in \mathbb{R}$,

$$\frac{d}{dx} (1 + \alpha + x)_k = (1 + \alpha + x)_k \sum_{j=1}^{k} \frac{1}{j + \alpha + x},$$

$$\frac{d}{dx} \left( \frac{1}{(1 + \beta + x)_k} \right) = -\frac{1}{(1 + \beta + x)_k} \sum_{j=1}^{k} \frac{1}{j + \beta + x}.$$  

3. **Proof of Theorem 1.1**

Throughout this section, we set $m = (p + 1)/d$. We first prove (1.4). To show (1.4), we need the following preliminary result.

**Lemma 3.1.** Under the assumptions of Theorem 1.1 (i), modulo $p$, we have

$$\sum_{k=0}^{p-1} \frac{(m - 1)_k(m + 1)_k^{d-1}}{(1)_k^d} \left( \sum_{j=0}^{k-1} \frac{1}{m - 1 + j} - \sum_{j=0}^{k-1} \frac{1}{m + 1 + j} \right) \equiv \frac{(p - 1)!}{(1)_{m-2}(1)_{m-1}} \left( \frac{1}{m} - \frac{1}{m - 1} \right).$$  

**Proof.** For $x, y \in (-1, +\infty)$, set

$$\Psi(x, y) = d+1 F_d \left[ \begin{array}{c} 1 - p, m - 1 + x, m + 1 + y, m + 1, \ldots, m + 1 \\ 1 + x, 1 + y, 1, \ldots, 1 \end{array} \right]_{p-1}.$$

Clearly, $\Psi(x, y)$ is smooth on $(-1, +\infty) \times (-1, +\infty)$. Since $(1 - p)_k = 0$ for all $k \geq p$, we have

$$\Psi(x, y) = d+1 F_d \left[ \begin{array}{c} 1 - p, m - 1 + x, m + 1 + y, m + 1, \ldots, m + 1 \\ 1 + x, 1 + y, 1, \ldots, 1 \end{array} \right].$$
As
\[ m - 2 + (d - 1)m = m - 2 + p + 1 - m = p - 1, \]
by Lemma 2.1,
\[ \Psi(x, y) = \frac{(p - 1)!}{(1 + x)m - 2(1 + y)m(1)^{d-2}}. \tag{3.2} \]

Now we calculate \( \Psi_x(0, 0) - \Psi_y(0, 0) \) in two different ways, where \( \Psi_x(0, 0) \) and \( \Psi_y(0, 0) \) stand for the partial derivatives of \( \Psi \) at \( (0, 0) \) with respect to \( x \) and \( y \). By (3.1) and Lemma 2.4, we obtain
\[ \Psi_x(0, 0) = \sum_{k=0}^{p-1} \frac{(1 - p)_k(m - 1)_k(m + 1)^{d-1}_k}{(1)_k^{d+1}} \left( \frac{1}{\sum_{j=0}^{k-1} m - 1 + j} - \frac{1}{\sum_{j=0}^{k-1} m + 1 + j} \right), \]
\[ \Psi_y(0, 0) = \sum_{k=0}^{p-1} \frac{(1 - p)_k(m - 1)_k(m + 1)^{d-1}_k}{(1)_k^{d+1}} \left( \frac{1}{\sum_{j=0}^{k-1} m - 1 + j} - \frac{1}{\sum_{j=0}^{k-1} m + 1 + j} \right), \]
where \( H_k = \sum_{j=1}^{k} 1/j \) denotes the harmonic number. Therefore,
\[ \Psi_x(0, 0) - \Psi_y(0, 0) = \sum_{k=0}^{p-1} \frac{(1 - p)_k(m - 1)_k(m + 1)^{d-1}_k}{(1)_k^{d+1}} \left( \frac{1}{\sum_{j=0}^{k-1} m - 1 + j} - \frac{1}{\sum_{j=0}^{k-1} m + 1 + j} \right). \tag{3.3} \]

On the other hand, by (3.2) and Lemma 2.4,
\[ \Psi_x(0, 0) - \Psi_y(0, 0) = \frac{(p - 1)!}{(1)_m - 2(1)_m^{d-1}} \left( \frac{1}{m} + \frac{1}{m - 1} \right). \tag{3.4} \]

Note that for \( k \) among 0, 1, \ldots, \( p - 1 \), \( (1 - p)_k \equiv (1)_k \) (mod \( p \)) and
\[ \frac{(m - 1)_k(m + 1)^{d-1}_k}{(1)_k^{d+1}} \left( \sum_{j=0}^{k-1} m - 1 + j - \sum_{j=0}^{k-1} m + 1 + j \right) \in \mathbb{Z}_p. \]

This, together with (3.3) and (3.4), gives the desired result. \( \square \)

**Proof of (1.4).** For any \( x, y \in \mathbb{Z}_p \), let
\[ \Phi(x, y) = a F_{d-1} \left[ \begin{array}{c} m - 1 + x, m + 1 + y, \ldots, m + 1 + y \\ 1, \ldots, 1 \end{array} \right]_{p-1}. \]

Obviously, for any \( s, t \in \mathbb{Z}_p \),
\[ \Phi(sp, tp) \equiv \Phi(0, 0) + sp \Phi_x(0, 0) + tp \Phi_y(0, 0) \pmod{p^2}. \tag{3.5} \]
In particular,
\[ \Phi(-p, 0) \equiv \Phi(0, 0) - p \Phi_x(0, 0) \pmod{p^2}. \tag{3.6} \]
Substituting (3.6) into (3.5), we get

$$\Phi(sp,tp) \equiv \Phi(-p,0) + (s+1)p\Phi_x(0,0) + tp\Phi_y(0,0) \pmod{p^2}.$$  

Taking \(s = t = -1/d\), in view of Lemmas 2.4 and 3.1, \(\Phi(−p, 0)\) is

$$\Phi\left(-\frac{p}{d} - \frac{p}{d}\right) = \sum_{k=0}^{p-1} \frac{(m-1)k(m+1)d-1}{1} \times \frac{1}{(1)^k} \times \frac{1}{m+1+j} \times \frac{1}{m-1+j} \equiv \Phi(-p,0) \pmod{p^2}.$$  

Now we evaluate \(\Phi(-p,0)\) modulo \(p^2\). Since \(m(d-1) = p + 1 - m\), by Lemma 2.1,

$$\Phi(-p,0) = (-1)^m \frac{(p + 1 - m)!}{(1)^d m}.$$  

With the help of (2.2), we obtain

$$\Phi(-p,0) = (-1)^m \frac{\Gamma(p - m + 2)}{\Gamma(m + 1)d-1} = (-1)^{p+2-(m+1)(d-1)} \frac{\Gamma_p(p - m + 2)}{\Gamma_p(m + 1)d-1} \times \frac{\Gamma(p - m)}{\Gamma_p(m + 1)d-1} \times \frac{\Gamma_p(p - m)}{\Gamma_p(m + 1)d-1} \times \frac{\Gamma_p(p - m)}{\Gamma_p(m + 1)d-1},$$  

(3.9)
where in the last step we have used the fact that \( d \) is even. In light of (2.3) and Lemma 2.3,

\[
\frac{\Gamma_p(p - m)}{\Gamma_p(m + 1)^{d - 1}} = (-1)^{m-1}(d-1)\Gamma_p(p - m)\Gamma_p(-m)^{d-1} \\
\equiv (-1)^{m-1}\Gamma_p\left(-\frac{1}{d}\right)^d\left(1 + \left(1 - \frac{1}{d}\right)pG\left(-\frac{1}{d}\right)\right) \\
\left(1 - \frac{p}{d}G\left(-\frac{1}{d}\right)\right)^{d-1} \\
\equiv (-1)^{m-1}\Gamma_p\left(-\frac{1}{d}\right)^d \pmod{p^2}. \tag{3.10}
\]

Moreover,

\[
(p - m)(p - m + 1) = \left(-\frac{1}{d} + \left(1 - \frac{1}{d}\right)p\right)\left(1 - \frac{1}{d} + \left(1 - \frac{1}{d}\right)p\right) \\
\equiv \frac{1 - d}{d^2} + \frac{(d - 1)(d - 2)p}{d^2} \pmod{p^2}.
\]

Combining this with (3.9) and (3.10), we obtain

\[
\Phi(-p, 0) \equiv \left(d - \frac{1}{d^2} - \frac{(d - 1)(d - 2)p}{d^2}\right)\Gamma_p\left(-\frac{1}{d}\right)^d \pmod{p^2}. \tag{3.11}
\]

Similarly, it is routine to verify that

\[
\left(1 - \frac{1}{d}\right)p \cdot \frac{(p - 1)!}{(1)m-2(1)^{d-1}} \left(\frac{1}{m} + \frac{1}{m - 1}\right) \\
\equiv \frac{(d - 1)(d - 2)p}{d^2}\Gamma_p\left(-\frac{1}{d}\right)^d \pmod{p^2}. \tag{3.12}
\]

Substituting (3.11) and (3.12) into (3.7), we immediately get (1.4). \(\square\)

In order to show (1.5), we also need an auxiliary lemma.

**Lemma 3.2.** Under the assumptions of Theorem 1.1 (ii), we have

\[
\sum_{k=0}^{p-1} (m)_k^2 (m + 1)^{d-2} \left(\frac{1}{m} - \sum_{j=0}^{k-1} \frac{1}{m + j}\right) \\
\equiv \frac{(p - 1)!}{(1)(m-1)(1)^{d-1}} \pmod{p}.
\]

**Proof.** For \( x, y \in (-1, +\infty) \), set

\[
\Upsilon(x, y) = \left[\begin{array}{c}
1 & 1 & 1 & \ldots & 1 \\
1 + x & 1 & 1 + y & \ldots & 1
\end{array}\right]_{p-1}.
\]

Similarly as in the proof of Lemma 3.1, we are led to the desired result by considering \( \Upsilon_x(0, 0) - \Upsilon_y(0, 0) \). \(\square\)
Proof of (1.5). For any \( x, y \in \mathbb{Z}_p \), set
\[
\Omega(x, y) = dF_{d-1} \left[ \frac{m + x, m + y, m + 1 + y, \ldots, m + 1 + y}{1, 1, \ldots, 1} \right]_{p-1}.
\]
Similarly as before, by Lemmas 2.1, 2.4 and 3.2,
\[
\Omega \left( -\frac{p}{d}, -\frac{p}{d} \right) = dF_{d-1} \left[ \frac{1, 1, 1 + \frac{1}{d}, 1 + \frac{1}{d}, \ldots, 1 + \frac{1}{d}}{1, 1, \ldots, 1} \right]_{p-1}
\]
\[
\equiv \Omega(-p, 0) + (1 - \frac{1}{d}) p\Omega_x(0, 0) - \frac{p}{d} \Omega_y(0, 0)
\]
\[
\equiv \Omega(-p, 0) + \left( 1 - \frac{2}{d} \right) p \sum_{k=0}^{p-1} \frac{(m)_k^2 (m + 1)_k^{d-2}}{(1)_k^{d}} \left( \sum_{j=0}^{k-1} \frac{1}{m + j} - \sum_{j=0}^{k-1} \frac{1}{m + 1 + j} \right)
\]
\[
\equiv \Omega(-p, 0) + \left( 1 - \frac{2}{d} \right) p \cdot \frac{\Gamma_p \left( -\frac{1}{d} \right)^d}{\Gamma_p \left( -\frac{1}{d} \right)^{d-1}} (\mod p^2).
\]
Then, in view of Lemmas 2.1–2.3, we have
\[
\Omega(-p, 0) \equiv \left( -\frac{1}{d^2} + \frac{(d - 2)p}{d^2} \right) \Gamma_p \left( -\frac{1}{d} \right)^d (\mod p^2)
\]
and
\[
\left( 1 - \frac{2}{d} \right) p \cdot \frac{(p - 1)!}{(1)_{m-1}(1)_{d-1}^{d-1}} \equiv \frac{(2 - d)p}{d^2} \cdot \Gamma_p \left( -\frac{1}{d} \right)^d (\mod p^2).
\]
The proof of (1.5) follows by combining the above. \( \square \)

4. Proof of Theorem 1.2

We need the following identity which can be verified by induction on \( n \).

Lemma 4.1. For any positive integer \( n \), we have
\[
\sum_{k=0}^{n-1} \frac{(4k + 1)}{k!^2} \left( \frac{1}{2} \right)^2 = \frac{n^2}{4} \left( \frac{2n}{n} \right)^2.
\]

Assuming \( p \equiv 1 (\mod 4) \) and putting \( x = 1/2 \) in [6, Corollary 1.4], we have the following result.

Lemma 4.2. Let \( p \equiv 1 (\mod 4) \) be a prime and \( r \) a positive integer. Then
\[
\sum_{k=0}^{p^r-1} \frac{(1/2)_k^2}{k!^2} \equiv 1 (\mod p^2).
\]
Proof of Theorem 1.2. Equivalently, we only need to show
\[ \sum_{k=0}^{p^r-1} (4k + 1) \left( \frac{1}{2} \right)_k \equiv \sum_{k=0}^{p^r-1} \left( \frac{2^r}{k!^2} \right) \pmod{p^{2r+1}}. \quad (4.1) \]
Putting \( n = p^{2r} \) in Lemma 4.1, we obtain
\[ \sum_{k=0}^{p^r-1} (4k + 1) \left( \frac{1}{2} \right)_k = \frac{p^{2r}}{4^{2p-1}} \left( \frac{2^r}{p} \right)^2. \quad (4.2) \]
By Fermat’s little theorem,
\[ 4^{2p-1} = 4 \times 16^{p-1} = 4 \times (16^{p-1})^{\frac{p-1}{2}} \equiv 4 \pmod{p}. \quad (4.3) \]
Using Lucas’s theorem, we have
\[ \left( \frac{2^{p^r}}{p^r} \right) \equiv \left( \frac{2^{p-1}}{p-1} \right) \equiv \cdots \equiv \left( \frac{2}{1} \right) = 2 \pmod{p}. \quad (4.4) \]
Substituting (4.3) and (4.4) into (4.2), the left-hand side of (4.1) becomes \( p^{2r} \) modulo \( p^{2r+1} \). In view of Lemma 4.2, we arrive at Theorem 1.2 at once. □

5. Concluding Remarks

It is routine to check that
\[ dF_{d-1} \left[ \frac{1}{d} - 1, 1 + \frac{1}{d}, \ldots, 1 + \frac{1}{d} \left| \begin{array}{l} 1 \end{array} \right. \right]_{p-1} \]
\[ + (d-1)dF_{d-1} \left[ \frac{1}{d}, \frac{1}{d}, 1 + \frac{1}{d}, \ldots, 1 + \frac{1}{d} \left| \begin{array}{l} 1 \end{array} \right. \right]_{p-1} \]
\[ = d \cdot dF_{d-1} \left[ \frac{1}{d} - 1, \frac{1}{d}, 1 + \frac{1}{d}, 1 + \frac{1}{d}, \ldots, 1 + \frac{1}{d} \left| \begin{array}{l} 1 \end{array} \right. \right]_{p-1}. \quad (5.1) \]
Note that Guo [3, Corollaries 4.2 and 4.4] proved that (1.4) and (1.5) also hold for odd integers \( d \geq 3 \) and even integers \( d \geq 4 \), respectively. This, together with (1.4), (1.5) and (5.1), gives that
\[ dF_{d-1} \left[ \frac{1}{d} - 1, \frac{1}{d}, 1 + \frac{1}{d}, 1 + \frac{1}{d}, \ldots, 1 + \frac{1}{d} \left| \begin{array}{l} 1 \end{array} \right. \right]_{p-1} \equiv 0 \pmod{p^2} \quad (5.2) \]
for any integer \( d \geq 3 \) and prime \( p \equiv -1 \pmod{d} \) with \( p \neq d-1 \). In fact, (5.2) can also be proved independently by using the method we used to prove Theorem 1.1 and the following Karlsson-Minton summation formula:
\[ n+1F_n \left[ a, b_1 + m_1, \ldots, b_n + m_n \left| \begin{array}{l} b_1, \ldots, b_n \end{array} \right. \right] = 0 \]
provided that \( m_1, \ldots, m_n \) are nonnegative integers and \( \Re(-a) > m_1 + \cdots + m_n \).
Motivated by (5.2) and based on some numerical calculations, we made the following conjecture for further study.
**Conjecture 5.1.** Let $d \geq 2$ be an integer. Let $n$ be a positive integer with $n \equiv -1 \pmod{d}$ and $n > d - 1$. Then

$$\left(\frac{n - 1}{n^2}\right)! d^{dn-d} \cdot d^{F_{d-1}} \left[ \frac{1}{d-1}, 1, 1 + \frac{1}{d}, 1 + \frac{1}{d}, \ldots, 1 + \frac{1}{d} \right]_{n-1} \in \mathbb{Z}. \quad (5.3)$$

We think it is possible that the ‘creative microscoping’ could be used to prove Conjecture 5.1. We hope that an interested reader will make some progress on it.

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