New Quantitative Deformation Lemma and New Mountain Pass Theorem

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Abstract In this paper, we obtain a new quantitative deformation lemma so that we can obtain more fixed points, especially for sup inf critical value $c$, $x = \varphi^{-1}(c)$ is a new fixed point. By the new quantitative deformation lemma, a new mountain pass theorem is obtained. Moreover, comparing with the mountain pass theorem in \cite{6}, $\varphi(e) \leq \varphi(0) < c_2$, but in our new mountain pass theorem, $\varphi(e) = c_2$, so our new mountain pass theorem can not be obtained by the quantitative deformation lemma in \cite{6}. Besides, in our new mountain pass theorem, if $\varphi$ satisfies $(PS)_c$ condition, we can obtain two new critical points $x = 0$ (valley point) and $x = e$ (peak point) which have not been obtained before.

Key words Fixed Points; Critical Points; Quantitative Deformation Lemma; Mountain Pass Theorem

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1 Introduction

In 1973, Ambrosetti and Rabinowitz \cite{1} presented the famous Mountain Pass Theorem. Later, there were many variants and generalizations. Specially, Willem \cite{6} gave the Quantitative Deformation Lemma and the corresponding mountain pass theorem. It is well known that quantitative deformation lemma is a very powerful tool to obtain mountain pass theorem,
and the mountain pass theorem has proved to be a power tool in many areas of analysis. But to our best knowledge, very few works have been done for quantitative deformation lemma or mountain pass theorem in the past two decades.

In this paper, we extend the quantitative deformation lemma in [6] so that we obtain more fixed points, especially for sup inf critical value $c$, $x = \varphi^{-1}(c)$ is a new fixed point. Moreover, as an application of our deformation lemma, a new mountain pass theorem is given. Comparing with the mountain pass theorem in [6], $\varphi(e) \leq \varphi(0) < c_2$, but in our new mountain pass theorem, $\varphi(e) = c_2$, so our new mountain pass theorem can not be obtained by the quantitative deformation lemma in [6]. Besides, in our new mountain pass theorem, if $\varphi$ satisfies $(PS)_c$ condition, we can obtain two new critical points $x = 0$ (valley point) and $x = e$ (peak point), but the mountain pass theorem in [6] can only obtain one critical point.

The organization of this paper is as following. In section 2, the quantitative deformation lemma in [6] and the corresponding mountain pass theorem in [6] are given. In section 3, on the basis of the quantitative deformation lemma in [6], we prove the new quantitative deformation lemma. In section 4, as an application of our deformation lemma, our new mountain pass theorem is given.

2 Preliminaries

For convenience, we introduce the Quantitative Deformation Lemma (See [6]) and the Corresponding Mountain Pass Type Theorem (See [6]) as the following:

**Lemma 2.1. (Quantitative deformation lemma)** Let $X$ be a Hilbert space, $\varphi \in C^2(X, \mathbb{R})$, $c \in \mathbb{R}$, $\varepsilon > 0$. Assume that

\[
(\forall u \in \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon])) : \|\varphi'(u)\| \geq 2\varepsilon.
\]

Then there exists $\eta \in C(X, X)$, such that

(a) $\eta(u) = u$, $\forall u \notin \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]).$

(b) $\eta(\varphi^{c+\varepsilon}) \subset \varphi^{c-\varepsilon}$, where $\varphi^{c-\varepsilon} := (-\infty, c - \varepsilon]$.

**Theorem 2.1. (Mountain pass theorem)** Let $X$ be a Hilbert space, $\varphi \in C^2(X, \mathbb{R})$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and

\[
b := \inf_{\|u\|=r} \varphi(u) > \varphi(0) \geq \varphi(e). \tag{2.1}
\]

Then, for each $\varepsilon > 0$, there exists $u \in X$ such that

(i) $c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon$,

(ii) $\|\varphi'(u)\| < 2\varepsilon$,
where
\[ c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)) \]
and
\[ \Gamma := \{ \gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e \}. \]

**Definition 2.1.** ([9]) Let \( X \) be a Banach space, \( \varphi \in C^1(X, \mathbb{R}) \) and \( c \in \mathbb{R} \). The function \( \varphi \) satisfies the \((PS)_c\) condition if any sequence \( (u_n) \subset X \) such that
\[ \varphi(u_n) \to c, \varphi'(u_n) \to 0 \]
has a convergent subsequence.

### 3 New Quantitative Deformation Lemma

**Theorem 3.1.** Let \( X \) be a Hilbert space, \( \varphi \in C^2(X, \mathbb{R}) \), \( c \in \mathbb{R} \), \( \varepsilon > 0 \). Assume that
\[ (\forall u \in \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon])) : \|\varphi'(u)\| \geq 2\varepsilon. \]
Then there exists \( \eta \in C(X, X) \), such that

1. \( a' \) \( \eta(u) = u, \forall u \notin \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \setminus D, \) where \( D \subseteq \varphi^{-1}([c - 0.5\varepsilon, c + \varepsilon]) \).
2. \( b' \) \( \eta(\varphi^{-1}[c - \varepsilon, c - 0.6\varepsilon]) \subset \varphi_{c+\varepsilon} \), where \( \varphi_{c+\varepsilon} \) denotes \( \varphi^{-1}([c + \varepsilon, +\infty)) \).
3. \( c' \) \( \eta(\varphi^{-1}[c + 0.6\varepsilon, c + \varepsilon]) \subset \varphi_{c-\varepsilon} \), where \( \varphi_{c-\varepsilon} \) denotes \( \varphi^{-1}((-\infty, c - \varepsilon]) \).

**Proof.** Let us define

\[
A := \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \setminus D,
B := \varphi^{-1}([c - \varepsilon, c - 0.6\varepsilon]),
C := \varphi^{-1}([c + 0.6\varepsilon, c + \varepsilon]),
\psi(u) := \frac{[\text{dist}(u, C) - \text{dist}(u, B)]\text{dist}(u, X \setminus A)}{[\text{dist}(u, C) + \text{dist}(u, B)]\text{dist}(u, X \setminus A) + \text{dist}(u, B)\text{dist}(u, C)},
\]
so that \( \psi \) is locally Lipschitz continuous, \( \psi = 1 \) on \( B \), \( \psi = -1 \) on \( C \) and \( \psi = 0 \) on \( X \setminus A \).

Let us also define the locally Lipschitz continuous vector field
\[
f(u) := \psi(u)\|\nabla \varphi(u)\|^{-2}\nabla \varphi(u), \quad u \in A,
:= 0, \quad u \in X \setminus A.
\]
It is clear that \( \|f(u)\| \leq (2\varepsilon)^{-1} \) on \( X \). For each \( u \in X \), the Cauchy problem
\[
\frac{d}{dt} \sigma(t, u) = f(\sigma(t, u)),
\]
\( \sigma(0,u) = u, \)

has a unique solution \( \sigma(\cdot,u) \) defined on \( \mathbb{R} \). Moreover, \( \sigma \) is continuous on \( \mathbb{R} \times X \) (see e.g. [10]). The map \( \eta \) defined on \( X \) by \( \eta(u) := \sigma(2\varepsilon,u) \) satisfies (a'). Since

\[
\frac{d}{dt} \varphi(\sigma(t,u)) = \left( \nabla \varphi(\sigma(t,u)), \frac{d}{dt} \sigma(t,u) \right) \\
= \left( \nabla \varphi(\sigma(t,u)), f(\sigma(t,u)) \right) \\
= \psi(\sigma(t,u)),
\]

(3.1)

If

\[
\sigma(t,u) \in \varphi^{-1}([c-\varepsilon, c-0.6\varepsilon]) = B, \quad \forall t \in [0,2\varepsilon],
\]

then

\[
\psi(\sigma(t,u)) = 1.
\]

So, we obtain from (3.1),

\[
\varphi(\sigma(2\varepsilon,u)) = \varphi(u) + \int_0^{2\varepsilon} \frac{d}{dt} \varphi(\sigma(t,u))dt \\
= \varphi(u) + \int_0^{2\varepsilon} \psi(\sigma(t,u))dt \\
\geq c - \varepsilon + 2\varepsilon = c + \varepsilon,
\]

and (b') is also satisfied.

Finally, similar to prove (b'), we prove (c').

If

\[
\sigma(t,u) \in \varphi^{-1}([c + 0.6\varepsilon, c + \varepsilon]) = C, \quad \forall t \in [0,2\varepsilon],
\]

then

\[
\psi(\sigma(t,u)) = -1.
\]

So, we obtain from (3.1),

\[
\varphi(\sigma(2\varepsilon,u)) = \varphi(u) + \int_0^{2\varepsilon} \frac{d}{dt} \varphi(\sigma(t,u))dt \\
= \varphi(u) + \int_0^{2\varepsilon} \psi(\sigma(t,u))dt \\
\leq c + \varepsilon - 2\varepsilon = c - \varepsilon,
\]

and (c') is also satisfied.

**Remark 3.1.** By Theorem 3.1, we get more fixed points than the Quantitative Deformation Lemma in [6]. All the domain \( D \) in Theorem 3.1, especially for sup inf critical value \( c \), \( x = \varphi^{-1}(c) \), are all new fixed points.

**Remark 3.2.** In Lemma 2.1, there are two conclusions, but in Theorem 3.1, there are three conclusions.
4 An Example (New Mountain Pass Theorem)

Let $X$ be a Hilbert space, $\varphi \in C^2(X, \mathbb{R})$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and

$$\varphi(0) = c_1, \quad \varphi(e) = c_2, \quad c_1 \neq c_2,$$

and

$$c_1 := \sup_{\gamma \in \Gamma} \min_{t \in [0,1]} \varphi(\gamma(t)), \quad c_2 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)),$$

where

$$\Gamma := \{ \gamma \in C([0,1], X) : \gamma(\frac{1}{4}) = 0, \gamma(\frac{1}{2}) = e \}.$$

Then, for each $\varepsilon > 0$, there exists $u^* \in X$ and $u^\Delta \in X$ such that

(I) $c_1 - 2\varepsilon \leq \varphi(u^*) \leq c_2 + 2\varepsilon,$

(II) $\|\varphi'(u^*)\| < 2\varepsilon.$

(III) $c_2 - 2\varepsilon \leq \varphi(u^\Delta) \leq c_2 + 2\varepsilon,$

(IV) $\|\varphi'(u^\Delta)\| < 2\varepsilon.$

Proof. Obviously, for each $\varepsilon > 0$, (I) and (III) are easy to get. Next, we prove (II) and (IV). Suppose that at least one of (II) and (IV) is not true. Then we can get the contradiction:

Case 1. We assume that (II) is not true. It means that there exist $\varepsilon_1$ such that

$$\|\varphi'(u^*)\| \geq 2\varepsilon.$$

From

$$c_1 := \sup_{\gamma \in \Gamma} \min_{t \in [0,1]} \varphi(\gamma(t)), \quad c_2 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)),$$

we get

$$c_1 \leq c_2.$$

By $c_1 \neq c_2$, we get $c_1 < c_2$.

Let $\varepsilon_1 = \min\{\frac{c_2 - c_1}{8}, \varepsilon\}$. It is clear that

$$\|\varphi'(u^*)\| \geq 2\varepsilon_1.$$

and for $\varepsilon_1$, (I) is still easy to get.

From $\varepsilon_1 = \min\{\frac{c_2 - c_1}{8}, \varepsilon\}$, we obtain

$$c_1 + 2\varepsilon_1 \leq c_2 + 2 \times \frac{c_2 - c_1}{8} = c_2 + \frac{c_2 - c_1}{4} = \frac{c_2}{4} + \frac{3c_1}{4} < c_2.$$

It means that

$$c_2 > c_1 + 2\varepsilon_1.$$
In Theorem 3.1, we can take \( D = \{ u \in X \mid \varphi(u) = c_1 \} \). Consider \( \beta = \eta \circ \gamma \), where \( \eta \) is given by Theorem 3.1. Using (a'), we have,

\[
\begin{align*}
\beta(\frac{1}{4}) &= \eta(\gamma(\frac{1}{4})) = \eta(0) = 0, \\
\beta(\frac{1}{2}) &= \eta(\gamma(\frac{1}{2})) = \eta(e) = e,
\end{align*}
\]

so that \( \beta \in \Gamma \). From \( c_1 := \sup_{\gamma \in \Gamma} \min_{t \in [0,1]} \varphi(\gamma(t)) \),

there exist \( \gamma \in \Gamma \) and \( \varepsilon_1 > 0 \) such that

\[
c_1 - 0.6\varepsilon_1 \leq \min_{t \in [0,1]} \varphi(\gamma(t)) \leq c_1 - \varepsilon_1.
\]

Then, from (b'), we have

\[
\min_{t \in [0,1]} \varphi\left(\eta(\gamma(t))\right) \geq c_1 + \varepsilon_1.
\]

It means that

\[
\min_{t \in [0,1]} \varphi(\beta(t)) \geq c_1 + \varepsilon_1.
\]

So

\[
c_1 + \varepsilon_1 \leq \min_{t \in [0,1]} \varphi(\beta(t)) \leq c_1.
\]

This is a contradiction. Therefore, (II) is true.

Case 2. We assume that (IV) is not true. It means that there exists \( \varepsilon \) such that

\[
\|\varphi'(u^\Delta)\| \geq 2\varepsilon.
\]

From

\[
c_1 := \sup_{\gamma \in \Gamma} \min_{t \in [0,1]} \varphi(\gamma(t)), \quad c_2 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)),
\]

we get

\[
c_1 \leq c_2.
\]

By \( c_1 \neq c_2 \), we get \( c_1 < c_2 \).

Let \( \varepsilon_1 = \min\{\frac{c_2 - c_1}{8}, \varepsilon\} \). It is clear that

\[
\|\varphi'(u^\Delta)\| \geq 2\varepsilon_1.
\]

and for \( \varepsilon_1 \), (III) is still easy to get.

From \( \varepsilon_1 = \min\{\frac{c_2 - c_1}{8}, \varepsilon\} \), we obtain

\[
c_2 - 2\varepsilon_1 \geq c_2 - 2 \times \frac{c_2 - c_1}{8} = c_2 - \frac{c_2 - c_1}{4} = \frac{c_2 - c_1}{4} + \frac{3c_1}{4} > c_1.
\]
It means that \( c_1 < c_2 - 2\varepsilon_1 \).

In Theorem 3.1 we can take \( D = \{ u \in X \mid \varphi(u) = c_2 \} \). Consider \( \beta = \eta \circ \gamma \), where \( \eta \) is given by Theorem 3.1. Using (a'), we have,

\[
\begin{align*}
\beta(\frac{1}{4}) &= \eta(\gamma(\frac{1}{4})) = \eta(0) = 0, \\
\beta(\frac{1}{2}) &= \eta(\gamma(\frac{1}{2})) = \eta(e) = e,
\end{align*}
\]

so that \( \beta \in \Gamma \). From

\[
c_2 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)),
\]

there exist \( \gamma \in \Gamma \) and \( \varepsilon_2 > 0 \) such that

\[
c_2 + 0.6\varepsilon_2 \leq \max_{t \in [0,1]} \varphi(\gamma(t)) \leq c_2 + \varepsilon_2.
\]

Then, from (c_1'), we have

\[
\max_{t \in [0,1]} \varphi\left(\eta(\gamma(t))\right) \leq c_2 - \varepsilon_2.
\]

It means that

\[
\max_{t \in [0,1]} \varphi(\beta(t)) \leq c_2 - \varepsilon_2.
\]

So

\[
c_2 \leq \max_{t \in [0,1]} \varphi(\beta(t)) \leq c_2 - \varepsilon_2.
\]

This is a contradiction. Therefore, (IV) is true. From Case 1 and Case 2, our new mountain pass theorem is proved.

\[\blacksquare\]

**Remark 4.1.** In Theorem 2.1 (Mountain pass theorem), \( c \) is defined as

\[
c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t))
\]

where

\[
\Gamma := \{ \gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e \}.
\]

But in our new mountain pass theorem, \( c_1 \) and \( c_2 \) are defined as

\[
c_1 := \sup_{\gamma \in \Gamma} \min_{t \in [0,1]} \varphi(\gamma(t)), \quad c_2 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)),
\]

where

\[
\Gamma := \{ \gamma \in C([0,1], X) : \gamma(\frac{1}{4}) = 0, \gamma(\frac{1}{2}) = e \}.
\]
Remark 4.2. In fact, in Theorem 2.1 (Mountain pass theorem),
\[ c_2 > \varphi(0) \geq \varphi(e) \]
But in our new mountain pass theorem,
\[ \varphi(0) = c_1, \quad \varphi(e) = c_2, \quad c_1 \neq c_2. \]
and in the proof of our new mountain pass theorem, we take \( D = \{ u \in X \mid \varphi(u) = c_1 \} \) in Case 1, and take \( D = \{ u \in X \mid \varphi(u) = c_2 \} \) in Case 2.

Remark 4.3. In the example, if we do not use our Theorem 3.1 (New quantitative deformation lemma), we can not obtain
\[ \beta(\frac{1}{4}) = \eta(\frac{1}{4}) = \eta(0) = 0. \]
Moreover, we can not obtain \( \beta \in \Gamma \).

Remark 4.4. An interesting point in the example is that, if \( \varphi \) satisfies \((PS)_c\) condition, it is easy to obtain two new critical points \( x = 0 \) and \( x = e \) which have not been obtained before.

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