Davenport constant of the multiplicative semigroup of the quotient ring $\mathbb{F}_p[x]/\langle f(x) \rangle$

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Abstract

Let $S$ be a finite commutative semigroup. The Davenport constant of $S$, denoted $D(S)$, is defined to be the least positive integer $d$ such that every sequence $T$ of elements in $S$ of length at least $d$ contains a subsequence $T'$ with the sum of all terms from $T'$ equaling the sum of all terms from $T$. Let $\mathbb{F}_p[x]$ be a polynomial ring in one variable over the prime field $\mathbb{F}_p$, and let $f(x) \in \mathbb{F}_p[x]$. In this paper, we made a study of the Davenport constant of the multiplicative semigroup of the quotient ring $\mathbb{F}_p[x]/\langle f(x) \rangle$. Among other results, we mainly

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prove that, for any prime $p > 2$ and any polynomial $f(x) \in \mathbb{F}_p[x]$ which can be factorized into several pairwise non-associated irreducible polynomials in $\mathbb{F}_p[x]$, then

$$D(S_{f(x)}^p) = D(U(S_{f(x)}^p)),$$

where $S_{f(x)}^p$ denotes the multiplicative semigroup of the quotient ring $\mathbb{F}_p[x]_{f(x)}$ and $U(S_{f(x)}^p)$ denotes the group of units of the semigroup $S_{f(x)}^p$.

Key Words: Davenport constant; Zero-sum; Finite commutative semigroups; Polynomial rings

1 Introduction

Let $G$ be an additive finite abelian group. A sequence $T$ of elements in $G$ is called a zero-sum sequence if the sum of all terms of $T$ equals to zero, the identify element of $G$. The Davenport constant $D(G)$ of $G$ is defined to be the smallest integer $d \in \mathbb{N}$ such that, every sequence $T$ of $d$ elements in $G$ contains a nonempty subsequence $T'$ with the sum of all terms of $T'$ equaling zero. H. Davenport [2] proposed the study of this constant in 1965, which aroused a huge variety of further researches (see [8] for a survey). Recently, G.Q. Wang and W.D. Gao [5] generalized the Davenport constant to finite commutative semigroups.

Definition. [5] Let $S$ be a finite commutative semigroup. Let $T$ be a sequence of elements in $S$. We call $T$ a reducible sequence if $T$ contains a proper subsequence $T'$ ($T' \neq T$) such that the sum of all terms of $T'$ equals the sum of all terms of $T$. Define the Davenport constant of the semigroup $S$, denoted $D(S)$, the smallest integer $d$ such that every sequence $T$ of length at least $d$ of elements in $S$ is reducible.

Moreover, the above two authors together with S.D. Adhikari also made a study of some related additive problems in semigroups (see [1] and [6]). Motivated by their pioneering work on additive problems in semigroups, we study the Davenport constant for the multiplicative semigroup of the quotient ring of a polynomial ring in one variable over the prime field $\mathbb{F}_p$. Our main result is as follows.

Theorem 1.1. For any prime $p > 2$, let $f(x) \in \mathbb{F}_p[x]$ such that $f(x)$ can be factorized into several irreducible polynomials which are not associated each other. Then

$$D(S_{f(x)}^p) = D(U(S_{f(x)}^p)),$$

where $S_{f(x)}^p$ denotes the multiplicative semigroup of the quotient ring $\mathbb{F}_p[x]_{f(x)}$. 

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Moreover, in the final concluding section, we conjecture that

\[ D(S^p_{f(x)}) = D(U(S^p_{f(x)})) \]

holds for any prime \( p > 2 \) and any non-constant polynomial \( f(x) \in \mathbb{F}_p[x] \), and in particular, we verify it for the case of \( f(x) = (x + 1)^2 \).

## 2 The proof of Theorem 1.1

We begin this section by giving some preliminaries.

Let \( S \) be a finite commutative semigroup. The operation on \( S \) is denoted by \(+\). The identity element of \( S \), denoted \( 0_S \) (if exists), is the unique element \( e \) of \( S \) such that \( e + a = a \) for every \( a \in S \). If \( S \) has an identity element \( 0_S \), let

\[ U(S) = \{a \in S : a + a' = 0_S \text{ for some } a' \in S\} \]

be the group of units of \( S \). The zero element of \( S \), denoted \( \infty_S \) (if exists), is the unique element \( z \) of \( S \) such that \( z + a = z \) for every \( a \in S \). Let

\[ T = x_1x_2 \cdots x_n = \prod_{x \in S} x^{v_x(T)}, \]

is a sequence of elements in the semigroup \( S \), where \( v_x(T) \) denotes the multiplicity of \( x \) in the sequence \( T \). Let \( T_1, T_2 \in \mathcal{F}(S) \) be two sequences on \( S \). We call \( T_2 \) a subsequence of \( T_1 \) if

\[ v_x(T_2) \leq v_x(T_1) \]

for every element \( x \in S \), in particular, if \( T_2 \neq T_1 \), we call \( T_2 \) a proper subsequence of \( T_1 \), and write

\[ T_3 = T_1 \cdot T_2^{-1} \]

to mean the unique subsequence of \( T_1 \) with \( T_2 \cdot T_3 = T_1 \). By \( \lambda \) we denote the empty sequence. If \( S \) has an identity element \( 0_S \), we allow \( T = \lambda \) to be empty and adopt the convention that \( \sigma(\lambda) = 0_S \). We say that \( T \) is reducible if \( \sigma(T') = \sigma(T) \) for some proper subsequence \( T' \) of \( T \) (Note that, \( T' \) is probably the empty sequence \( \lambda \) if \( S \) has the identity element \( 0_S \) and \( \sigma(T) = 0_S \)). Otherwise, we call \( T \) irreducible. For more related terminology used in additive problems in semigroups, one is referred to [6].

**Lemma 2.1.** ([4], Lemma 6.1.3) Let \( G \) be a finite abelian group, and let \( H \) be a subgroup of \( G \). Then, \( D(G) \geq D(G/H) + D(H) - 1 \).
For any finite commutative semigroup \( S \) with identity \( 0 \), since \( U(S) \) is a nonempty subsemigroup of \( S \), we have the following.

**Lemma 2.2.** (see [5], Proposition 1.2) Let \( S \) be a finite commutative semigroup with identity. Then \( D(U(S)) \leq D(S) \).

**Lemma 2.3.** Let \( k \geq 1 \), and let \( n_1, n_2, \ldots, n_k \geq 2 \) be positive integers. Let \( S_i = C_{n_i} \cup \{ \infty_i \} \) be a semigroup obtained by a cyclic group of order \( n_i \) adjoined with a zero element \( \infty_i \) for each \( i \in [1, k] \). Let \( S = S_1 \times S_2 \times \cdots \times S_k \) be the product of \( S_1, S_2, \ldots, S_k \). Then \( D(S) = D(U(S)) \).

**Proof.** By Lemma 2.2, we need only to show that \( D(S) \leq D(U(S)) \).

Observe that
\[
U(S) = U(S_1) \times \cdots \times U(S_k) = C_{n_1} \times \cdots \times C_{n_k}.
\]

Let \( g_i \) be the generator of the cyclic \( C_{n_i} \). Let \( a = (a_1, a_2, \ldots, a_k) \) be an element of \( S \). We define
\[
\mathcal{J}(a) = \{ i \in [1, k] : a_i = \infty_i \}.
\]

We see that for each \( i \in [1, k] \), either \( a_i = \infty_i \) or \( a_i = m_i g_i \) where \( m_i \in [1, n_i] \), and moreover, \( a \in U(S) \) if and only if \( \mathcal{J}(a) = \emptyset \).

For any index set \( I \subseteq [1, k] \), let \( \psi_I \) be the canonical epimorphism of \( S \) onto the semigroup \( \prod_{i \in I} S_i \) given by
\[
\psi_I(x) = (x'_1, x'_2, \ldots, x'_k)
\]
with
\[
x'_i = 0 \quad \text{for} \quad i \in I
\]
and
\[
x'_i = x_i \quad \text{for} \quad i \in [1, k] \setminus I,
\]
where \( x = (x_1, x_2, \ldots, x_k) \) denotes an arbitrary element of \( S \). Note that \( \psi_I(S) \) is a subsemigroup of \( S \) which is isomorphic to the semigroup \( \prod_{i \in I} S_i \), the product of the semigroups \( S_i \) with \( i \in I \).

Now take an arbitrary sequence \( T \) of elements of \( S \) of length at least \( U(S) \). By applying
Lemma 2.1 recursively, we have that

\[ |T| \geq D(S) \]
\[ \geq D(U(S)) \]
\[ = D(\prod_{i \in [1, k]} C_{n_i}) \]
\[ \geq D(\prod_{i \in [1, k-1]} C_{n_i}) + D(C_{n_k}) - 1 \geq D(\prod_{i \in [1, k]} C_{n_i}) + 1 \]
\[ \geq D(\prod_{i \in [1, k]} C_{n_i}) + 2 \]
\[ \vdots \]
\[ \geq D(C_{n_1}) + (k - 1) \]
\[ \geq 2 + (k - 1) \]
\[ = k + 1. \]

It suffices to show that \( T \) contains a proper subsequence \( T' \) with \( \sigma(T') = \sigma(T) \).

Suppose first that all the terms of \( T \) are from \( U(S) \), i.e., \( \mathcal{J}(x) = \emptyset \) for each term \( x \) of \( T \). Since \( |T| \geq D(U(S)) \), it follows that \( T \) contains a nonempty subsequence \( V \) with \( \sigma(V) = 0_S \), i.e., the sum of all terms from \( V \) is the identity element of \( S \). This implies that \( \sigma(TV^{-1}) = \sigma(TV^{-1}) + 0_S = \sigma(TV^{-1}) + \sigma(V) = \sigma(T) \). Then \( T' = TV^{-1} \) shall be the required proper subsequence of \( T \), we are done. Hence, we assume that not all the terms of \( T \) are from \( U(S) \), that is, \( \mathcal{J}(\sigma(T)) \neq \emptyset \).

Note that for each \( i \in \mathcal{J}(\sigma(T)) \), there exists at least one term of \( T \), say \( a_i \), such that

\[ i \in \mathcal{J}(a_i). \]

It follows that there exists a nonempty subsequence \( V \) of \( T \) of length at most \( |\mathcal{J}(\sigma(T))| \) such that

\[ \mathcal{J}(\sigma(V)) = \mathcal{J}(\sigma(T)). \]

Let

\[ L = [1, k] \setminus \mathcal{J}(\sigma(T)). \]

Note that \( \psi_L(TV^{-1}) \) is a sequence of elements in \( U(\psi_L(S)) \equiv U(\prod_{i \in L} S_i) = \prod_{i \in L} C_{n_i} \). By \( \prod \), we have that

\[ |TV^{-1}| \geq 1. \]
By applying Lemma 2.1 recursively, we have that

\[
D(U(\psi_L(S))) = D(\prod_{i \in L} C_n) \\
\leq D(\prod_{i \in [1,k]} C_n) - (k - |L|) \\
= D(\prod_{i \in [1,k]} C_n) - |\mathcal{F}(\sigma(T))| \\
\leq D(\prod_{i \in [1,k]} C_n) - |V| \\
= D(U(S)) - |V| \\
= |TV^{-1}| \\
= |\psi_L(TV^{-1})|.
\]

It follows that \(TV^{-1}\) contains a nonempty subsequence \(W\) such that \(\sigma(\psi_L(W))\) is the identity element of the group \(U(\psi_L(S))\), i.e.,

\[
\sigma(\psi_L(W)) = 0_{U(\psi_L(S))} = 0_S.
\]

Since \(V \mid TV^{-1}\), it follows that \(\mathcal{F}(\sigma(TV^{-1})) = \mathcal{F}(\sigma(V)) = \mathcal{F}(\sigma(T))\), which implies that

\[
\sigma(TV^{-1}) + \sigma(\psi_L(W)) = \sigma(TW^{-1}) + \sigma(W).
\]

Then we have that

\[
\sigma(TW^{-1}) = \sigma(TV^{-1}) + 0_S = \sigma(TW^{-1}) + \sigma(\psi_L(W)) = \sigma(TW^{-1}) + \sigma(W) = \sigma(T),
\]

and thus, \(T' = TW^{-1}\) is the required proper subsequence of \(T\), we are done. \(\square\)

**Proof of Theorem 1.1**

Let

\[
f(x) = f_1(x)f_2(x) \cdots f_k(x)
\]

where \(f_1(x), f_2(x), \ldots, f_k(x) \in \mathbb{F}_p[x]\) are irreducible and do not associate each other. By the Chinese Remainder Theorem, we have

\[
\frac{\mathbb{F}_p[x]}{(f(x))} \cong \frac{\mathbb{F}_p[x]}{(f_1(x))} \times \frac{\mathbb{F}_p[x]}{(f_2(x))} \times \cdots \times \frac{\mathbb{F}_p[x]}{(f_k(x))}.
\]

It follows that the multiplicative semigroup \(S^p_{f(x)}\) of the ring \(\frac{\mathbb{F}_p[x]}{(f(x))}\), is isomorphic to the product of the multiplicative semigroups of \(\frac{\mathbb{F}_p[x]}{(f_1(x))}, \frac{\mathbb{F}_p[x]}{(f_2(x))}, \ldots, \frac{\mathbb{F}_p[x]}{(f_k(x))}\), i.e.,

\[
S^p_{f(x)} \cong S^p_{f_1(x)} \times S^p_{f_2(x)} \times \cdots \times S^p_{f_k(x)}.
\]

Since the polynomial \(f_i(x)\) is irreducible for each \(i \in [1,k]\), we have \(\frac{\mathbb{F}_p[x]}{(f_i(x))}\) is a finite field, and thus, the semigroup \(S^p_{f_i(x)}\) is a cyclic group adjoined with a zero element. Then the conclusion follows from Lemma 2.3 immediately. \(\square\)
3 Concluding remarks

In the final section, we propose the further research by suggesting the following conjecture.

Conjecture 3.1. For any prime \( p > 2 \), let \( f(x) \in \mathbb{F}_p[x] \) with \( \deg(f(x)) \geq 1 \). Then
\[
D(S^p_{f(x)}) = D(U(S^p_{f(x)})).
\]

From Theorem 1.1, we need only to verify the case that there exists some irreducible polynomial \( g(x) \in \mathbb{F}_p[x] \) with \( g(x)^2 \mid f(x) \). Therefore, we close this paper by making a preliminary verification for example when \( f(x) = (x + 1)^2 \).

Proposition 3.2. For a prime \( p > 2 \),
\[
D(S^p_{(x+1)^2}) = D(U(S^p_{(x+1)^2})).
\]

Proof. In view of Lemma 2.2 we need only to show that \( D(S^p_{(x+1)^2}) \leq D(U(S^p_{(x+1)^2})) \). Take an arbitrary sequence \( T \) of elements in the semigroup \( S^p_{(x+1)^2} \) with length
\[
|T| = D(U(S^p_{(x+1)^2})). \tag{2}
\]
It suffices to show that \( T \) is reducible. Note that
\[
U(S^p_{(x+1)^2}) = \{ax + b : a, b \in \mathbb{F}_p \text{ and } a \neq b\}. \tag{3}
\]
Let \( g \) be a primitive root of the prime \( p \). Take the sequence \( V = a_1a_2 \cdots a_{p-1} \) of elements in \( S^p_{(x+1)^2} \), where \( a_1 = x \) and \( a_2 = \cdots = a_{p-1} = g \). It is easy to check that \( V \) is irreducible, which implies
\[
D(U(S^p_{(x+1)^2})) \geq |T| + 1 = p. \tag{4}
\]
Suppose first that all the terms of \( T \) are from \( U(S^p_{(x+1)^2}) \). By (2), we have that \( T \) is reducible, we are done.

Suppose that \( T \) contains two terms, say \( a_1, a_2 \), which are not in \( U(S^p_{(x+1)^2}) \). By (3), we have that \( (x + 1)^2 \) divides the product of the two polynomials \( a_1 \) and \( a_2 \), that is, the sum of the two elements \( a_1 \) and \( a_2 \) is the zero element of the semigroup \( S^p_{(x+1)^2} \), i.e.,
\[
a_1 + a_2 = \infty_{S^p_{(x+1)^2}}.
\]
Then \( \sigma(T) = \sigma(T \cdot a_1^{-1}a_2^{-1}) + \sigma(a_1a_2) = \sigma(T \cdot a_1^{-1}a_2^{-1}) + \infty_{S^p_{(x+1)^2}} = \infty_{S^p_{(x+1)^2}} = \sigma(a_1a_2) \). Combined with (4), we have that \( a_1a_2 \) is the required proper subsequence of \( T \), we are done.
It remains to consider the case that $T$ contains exactly one element outside the group $U(S_{(x+1)^2})$, say

$$a_1 \in S_{(x+1)^2} \setminus U(S_{(x+1)^2}) \text{ and } a_i \in U(S_{(x+1)^2}) \text{ for } i = 2, 3, \ldots, |T|.$$ 

By (3), we have that

$$a_1 = m(x + 1),$$

where $m \in \mathbb{F}_p$. We may assume $Ta_1^{-1}$ is irreducible (otherwise, $T$ shall be reducible, and we are done). By the definition of irreducible sequences, we have that

$$0_{S_{(x+1)^2}} = 0_{U(S_{(x+1)^2})} \notin \sum(Ta_1^{-1}),$$

where the set

$$\sum(Ta_1^{-1}) = \{\sigma(T') : T' \text{ is a nonempty subsequence of } Ta_1^{-1}\}$$

consists of all the elements of the semigroup $S_{(x+1)^2}$ that can be represented by the sum of several distinct terms from the sequence $Ta_1^{-1}$. We see that the sequence $Ta_1^{-1}$ is a zero-sum free sequence of elements in the group $U(S_{(x+1)^2})$ of length $D(U(S_{(x+1)^2})) - 1$. It follows that

$$\sum(Ta_1^{-1}) = U(S_{(x+1)^2}) \setminus \{0_{U(S_{(x+1)^2})}\},$$

which implies that there exists a nonempty subsequence $W$ of $Ta_1^{-1}$ with $\sigma(W) = x + 2 \in \mathbb{F}_p[x]$. We see that

$$\sigma(W) + a_1 = (x + 2) * m(x + 1) = m(x + 1) = a_1.$$

Let $T' = TW^{-1}$. Then we have that

$$\sigma(T') = \sigma(T'a_1^{-1}) + a_1 = \sigma(T'a_1^{-1}) + (a_1 + \sigma(W)) = \sigma(T') + \sigma(W) = \sigma(T),$$

and thus, $T'$ is the required proper subsequence of $T$. This completes the proof. \qed

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