On the Connection Between Sequential Quadratic Programming
and Riemannian Gradient Methods

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Abstract

We prove that a simple Sequential Quadratic Programming (SQP) algorithm for equality
constrained optimization has local linear convergence with rate $1 - 1/\kappa_R$, where $\kappa_R$ is the
condition number of the Riemannian Hessian. Our analysis builds on insights from Riemannian
optimization and indicates that first-order Riemannian algorithms and “simple” SQP algorithms
have nearly identical local behavior. Unlike Riemannian algorithms, SQP avoids calculating the
projection or retraction of the points onto the manifold for each itertes. All the SQP iterates
will automatically be quadratically close the the manifold near the local minimizer.

1 Introduction

In this paper, we consider the equality-constrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

subject to $x \in \mathcal{M} = \{x : F(x) = 0\}$,

(1)

where we assume $f : \mathbb{R}^n \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}^m$ are $C^2$ smooth functions with $m \leq n$.

The focus of this paper is on the local convergence rate of “first-order” methods for (1): methods
that only query $\nabla f(x)$ at each iteration (but can do whatever they want with the constraint). An
example is projected gradient descent, in which each iterate moves in the negative gradient direction
followed by a projection back onto $\mathcal{M}$. As each iterate only receives first-order information about
$f$, one expects local linear convergence of such algorithms, whose rate indicates the difficulty of
the problem in the same way that the condition number of $\nabla^2 f(x^*)$ tells about the difficulty of
unconstrained optimization.

While numerous “first-order” methods can solve problem (1) [Ber99, NW06], we will restrict
attention to two types of methods: Riemannian first-order methods and Sequential Quadratic
Programming, which we now briefly review. When $\mathcal{M}$ has a manifold structure near $x^*$, one could
use Riemannian optimization algorithms [AMS09], whose iterates are maintained on the constraint
set $\mathcal{M}$. Usually, a first-order Riemannian algorithm computes the Riemannian gradient and then
takes a descent step on the manifold based on this gradient. Classical Riemannian type first
order algorithms date back to geodesic gradient projection [Lue72] and geodesic steepest descent
algorithm [Gab82a]. These algorithms proposed to perform descent steps along the geodesics of
the manifold. Beyond several simple cases, the geodesics along a manifold is in general hard to

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compute, making these algorithms hard to implement in practice. Later, Riemannian algorithms are simplified by making use of retraction, a mapping from the tangent space to the manifold that can replace the necessity of computing the exact geodesics while still maintaining the same convergence rate. Intuitively, first-order Riemannian methods can be viewed as variants of projected gradient descent that utilize the manifold structure more carefully. Analyses of many such Riemannian algorithms are given in [AMS09, Section 4].

An alternative approach for solving problem (1) is Sequential Quadratic Programming (SQP) [NW06, Section 18]. Each iteration of SQP solves a quadratic programming problem which minimizes a quadratic approximation of \( f \) on the linearized constraint set \( \{ x : F( x_k ) + \nabla F( x_k ) ( x - x_k ) = 0 \} \). This makes SQP an infeasible method: each iterate may be infeasible but eventually they converge to a feasible point. When the quadratic approximation uses the Hessian of the objective function, this is equivalent to Newton method solving nonlinear equations. When the full Hessian is intractable, one can either approximate the Hessian with BFGS-type updates, or just use some raw estimate such as a big PSD matrix [BT95].

It is known that the convergence rate of many first-order algorithms is determined by the condition number \( \kappa_R \) of the restricted Hessian of the Lagrangian (e.g. [LY08, Section 11.6]). Many algorithms are shown to achieve local linear convergence with rate \( ( 1 - 1/\kappa_R ) \), which is called the canonical rate for problem (1) [LY08]. Riemannian algorithms that achieve the canonical rate include geodesic gradient projection [Lue72, Theorem 2], geodesic steepest descent [Gab82a, Theorem 4.4], Riemannian gradient descent [AMS09, Theorem 4.5.6]. The canonical rate is also achieved by the modified Newton method on the quadratic penalty function [LY08, Section 15.7], which resembles an SQP method in spirit.

Though analyses are well-established for both the Riemannian and the SQP approach (even by the same author in [Gab82a] for the Riemannian approach and [Gab82b] for the SQP approach), the connection of these algorithms did not receive much attention until the past 10 years. The connection between SQP and Riemannian method is re-emphasized in [ATMA09]: the authors pointed out that the feasibly-projected sequential quadratic programming (FP-SQP) method in [WT04] gives the same update as the Riemannian Newton update. However, this connection is established for second-order methods between the Riemannian and the SQP approaches, and such connection for first-order methods are not—we believe—explicitly pointed out yet.

1.1 Outline and our contribution

In this paper, we prove that the local convergence rate of a simple version of SQP is the canonical rate \( 1 - 1/\kappa_R \) (Theorem 1) from a Riemannian optimization perspective. The algorithm we consider is the following: given initialization \( x_0 \) close to a local minimizer \( x_\star \), iterate

\[
x_{k+1} = \arg \min x_k + \langle \nabla f( x_k ), x - x_k \rangle + \frac{1}{2\eta} \| x - x_k \|^2
\]

subject to \( F( x_k ) + \nabla F( x_k ) ( x - x_k ) = 0 \),

where \( \eta > 0 \) is the stepsize. This is a standard SQP in which the quadratic term uses the matrix \( I/(2\eta) \), and in practice is slightly easier to implement than BFGS-type SQP.

Generic convergence theorems for SQP show that algorithm (2) has local linear convergence [BT95, Theorem 3.2]. Though the rate is not the focus there, one could extract from their proof a rate that depends on the condition number of the Hessian of the Lagrangian, which is slightly slower than the canonical rate. Meanwhile, modified Newton methods that resemble an SQP are shown to achieve the canonical rate [LY08, Section 15.7]. Though these are compelling reasons for one
to believe that the SQP (2) converges with the canonical rate, the authors were unable to find an explicitly written proof.

Unlike classical analyses of SQP which often construct a merit function based on function values, our analysis focuses on the distance to optimality in the tangent and normal directions separately. Crucially, by observing the fact that the iterates will remain quadratically close to the manifold, we choose our merit function as

$$
\|P_{x^*}(x_k - x^*)\|^2_2 + \sigma \|P_{x^*}^\perp(x_k - x^*)\|^2_2
$$

for some $\sigma > 0$, where $P_{x^*}$ and $P_{x^*}^\perp$ are projection matrices onto correspondingly the tangent space and normal space of the manifold $\mathcal{M}$ at $x^*$. To show that this quantity converges linearly with the canonical rate, we make use of a “Riemannian Taylor expansion” result (Lemma 3.1). This result extends existing Riemannian Taylor expansion onto points off the manifold and is stated in an Euclidean form.

Our proof uses insights from Riemannian optimization and makes precise the connection between first-order Riemannian methods and SQP methods with “cheap” quadratic terms: locally they are about the same. More precisely, the fact that $\|P_{x^*}^\perp(x_k - x^*)\|^2_2$ is comparable with $\|P_{x^*}(x_k - x^*)\|^2_2$ suggests that the iterates $x_k$ are “quadratically close” to the manifold. Hence, performing these SQP steps quickly becomes nearly identical to performing gradient steps on the manifold, i.e. Riemannian methods. We illustrate this geometric observation in Figure 1. We hope that this observation can shed light on the analyses of constrained optimization problems and be of further interest.

The rest of this paper is organized as the following. In Section 2 we state our assumptions and give preliminaries on the Riemannian geometry on and off the manifold $\mathcal{M}$. We present our main convergence result in Section 3.1 and state the “Riemannian Taylor expansion” Lemma in Section 3.2. We prove our main result in Section 4, give an example in Section 5, and prove the needed technical results in the Appendix.

1.2 Notation

For a matrix $A \in \mathbb{R}^{m \times n}$, we denote $A^\dagger \in \mathbb{R}^{n \times m}$ as its Moore-Penrose inverse, $A^T \in \mathbb{R}^{n \times m}$ as its transpose, and $A^\dagger_T$ as the transpose of its Moore-Penrose inverse. As $n \geq m$, we have $A^\dagger = A^T(AA^T)^{-1}$. For a $k$th order tensor $T \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_k}$, and $k$ vectors $u_1 \in \mathbb{R}^{n_1}, \ldots, u_k \in \mathbb{R}^{n_k}$, we denote $T[u_1, \ldots, u_k] = \sum_{i_1, \ldots, i_k} T_{i_1 \cdots i_k} u_{i_1} \cdots u_{i_k}$ as tensor-vectors multiplication. The operator norm of tensor $T$ is defined as $\|T\|_{\text{op}} = \sup_{\|u_1\|_2 = 1, \ldots, \|u_k\|_2 = 1} T[u_1, \ldots, u_k]$. 
For a scalar-valued function $f : \mathbb{R}^n \to \mathbb{R}$, we write its gradient at $x \in \mathbb{R}^n$ as a column vector $\nabla f(x) \in \mathbb{R}^n$. We write its Hessian at $x \in \mathbb{R}^n$ as a matrix $\nabla^2 f(x) \in \mathbb{R}^{n \times n}$, and its third order derivative at $x$ as a third order tensor $\nabla^3 f(x) \in \mathbb{R}^{n \times n \times n}$. For a vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^m$, its Jacobian matrix at $x \in \mathbb{R}^n$ is an $m \times n$ matrix $\nabla F(x) \in \mathbb{R}^{m \times n}$, and its Hessian matrix at $x$ as a third order tensor $\nabla^2 F(x) \in \mathbb{R}^{m \times n \times n}$.

## 2 Preliminaries

### 2.1 Assumptions

Let $x_*$ be a local minimizer of problem (1). Throughout the rest of this paper, we make the following assumptions on problem (1). In particular, all these assumptions are local, meaning that they only depend on the properties of $f$ and $F$ in $B(x_*, \delta)$ for some $\delta > 0$.

**Assumption 1** (Smoothness). Within $B(x_*, \delta)$, the functions $f$ and $F$ are $C^2$ with local Lipschitz constants $L_f, L_F$, Lipschitz gradients with constants $\beta_f, \beta_F$, and Lipschitz Hessians with constants $\rho_f, \rho_F$.

**Assumption 2** (Manifold structure and constraint qualification). The set $M$ is a $m$-dimensional smooth submanifold of $\mathbb{R}^n$. Further, $\inf_{x \in B(x_*, \delta)} \sigma_{\min}(\nabla F(x)) \geq \gamma_F$ for some constant $\gamma_F > 0$.

Smoothness and constraint qualification together implies that the constraints $F(x)$ are well-conditioned and problem (1) is $C^2$ near $x_*$. In particular, we can define a matrix $\text{Hess}f(x_*)$ via the formula

$$\text{Hess}f(x_*) = \text{P}_{x_*} \nabla^2 f(x_*) \text{P}_{x_*} - \sum_{i=1}^{m} [\nabla F(x_*)]^T \nabla^2 f(x_*) [\nabla F(x_*)]_{i} \cdot \text{P}_{x_*} \nabla^2 F_i(x_*) \text{P}_{x_*},$$

where

$$\text{P}_{x_*} = I_n - \nabla F(x_*)^T \nabla F(x_*).$$

We can see later that $\text{Hess}f(x_*)$ is the matrix representation of the Riemannian Hessian of function $f$ on $M$ at $x_*$. 

**Assumption 3** (Eigenvalues of the Riemannian Hessian). Define

$$\lambda_{\max} = \sup\{ \langle u, \text{Hess}f(x_*) u \rangle : \|u\|_2 = 1, \nabla F(x_*) u = 0 \},$$

$$\lambda_{\min} = \inf\{ \langle u, \text{Hess}f(x_*) u \rangle : \|u\|_2 = 1, \nabla F(x_*) u = 0 \}.$$

We assume $0 < \lambda_{\min} \leq \lambda_{\max} < \infty$. We call $\kappa_R = \lambda_{\max} / \lambda_{\min}$ the condition number of $\text{Hess}f(x_*)$.

### 2.2 Geometry on the manifold $M$

Since we assumed that the set $M$ is a smooth submanifold of $\mathbb{R}^n$, we endow $M$ with the Riemannian geometry induced by the Euclidean space $\mathbb{R}^n$. At any point $x \in M$, the tangent space (viewed as a subspace of $\mathbb{R}^n$) is obtained by taking the differential of the equality constraints

$$T_x M = \{ u \in \mathbb{R}^n : \nabla F(x) u = 0 \}.$$  

Let $\text{P}_x$ be the orthogonal projection operator from $\mathbb{R}^n$ onto $T_x M$. For any $u \in \mathbb{R}^n$, we have

$$\text{P}_x(u) = [I_n - \nabla F(x)^T (\nabla F(x) \nabla F(x)^T)^{-1} \nabla F(x)] u.$$
Let $P^\perp_x$ be the orthogonal projection operator from $\mathbb{R}^n$ onto the complement subspace of $T_x\mathcal{M}$. For any $u \in \mathbb{R}^n$, we have
\[ P^\perp_x(u) = \nabla F(x)^T(\nabla F(x)\nabla F(x)^T)^{-1}\nabla F(x)u. \] (7)

With a little abuse of notations, we will not distinguish $P_x$ and $P^\perp_x$ with their matrix representations. That is, we also think of $P_x, P^\perp_x \in \mathbb{R}^{n \times n}$ as two matrices.

We denote $\nabla f(x)$ and $\text{grad} f(x)$ respectively the Euclidean gradient and the Riemannian gradient of $f$ at $x \in \mathcal{M}$. The Riemannian gradient of $f$ is the projection of the Euclidean gradient onto the tangent space
\[ \text{grad} f(x) = P_x(\nabla f(x)) = [l_n - \nabla F(x)^T(\nabla F(x)\nabla F(x)^T)^{-1}\nabla F(x)]\nabla f(x). \] (8)

Since $x_*$ is a local minimizer of $f$ on the manifold $\mathcal{M}$, we have $\text{grad} f(x_*) = 0$.

At $x \in \mathcal{M}$, let $\nabla^2 f(x)$ and $\text{Hess} f(x)$ be respectively the Euclidean and the Riemannian Hessian of $f$. The Riemannian Hessian is a symmetric operator on the tangent space and is given by projecting the directional derivative of the gradient vector field. That is, for any $u, v \in T_x\mathcal{M}$, we have (we use $\mathcal{D}$ to denote the directional derivative)
\[ \text{Hess} f(x)[u, v] = \langle v, P_x(\mathcal{D}\text{grad} f(x)[u]) \rangle = \langle v, P_x \cdot \nabla^2 f(x)u - P_x \cdot (\mathcal{D}P^\perp_x[u]) \cdot \nabla f \rangle \]
\[ = v^T P_x \nabla^2 f(x) P_x u - \nabla^2 F(x)[\nabla F(x)^T \nabla f, u, v]. \] (9)

With a little abuse of notation, we will not distinguish the Hessian operator with its matrix representation. That is
\[ \text{Hess} f(x) = P_x \nabla^2 f(x) P_x - \sum_{i=1}^m [\nabla F(x)^T \nabla f(x)]_i \cdot P_x \nabla^2 F_i(x) P_x. \] (10)

2.3 Geometry off the manifold $\mathcal{M}$

We can extend the definition of the matrix representations of the above Riemannian quantities outside the manifold $\mathcal{M}$. For any $x \in \mathbb{R}$, we denote
\[ P_x = I_n - \nabla F(x)^T(\nabla F(x)\nabla F(x)^T)^{-1}\nabla F(x), \]
\[ P^\perp_x = \nabla F(x)^T(\nabla F(x)\nabla F(x)^T)^{-1}\nabla F(x), \]
\[ \text{grad} f(x) = [l_n - \nabla F(x)^T(\nabla F(x)\nabla F(x)^T)^{-1}\nabla F(x)]\nabla f(x). \] (11)

By the constraint qualification assumption (Assumption 2), $\nabla F(x)\nabla F(x)^T$ is invertible and the quantities above are well defined in $\mathcal{B}(x_*, \delta)$. We call $\text{grad} f(x)$ the pseudo-Riemannian gradient of $f$ at $x$, which not only depends on function $f$ on the manifold $\mathcal{M}$, but also depends on the function outside the manifold $\mathcal{M}$, and the formulation of the constraint function $F$.

2.4 Closed-form expression of the SQP iterate

The above definitions makes it possible to have a concise closed-form expression for the SQP iterate (2). Indeed, as each iterate solves a standard QP, the expression can be obtained explicitly by writing out the optimality condition. Letting $x_k = x$, the next iteration $x_{k+1} = x_+$ is given by
\[ x_+ = x - \eta[l_n - \nabla F(x)^T(\nabla F(x)\nabla F(x)^T)^{-1}\nabla F(x)]\nabla f(x) - \nabla F(x)^T(\nabla F(x)\nabla F(x)^T)^{-1}F(x) \]
\[ = x - \eta P_x \nabla f(x) - \nabla F(x)^T(\nabla F(x)\nabla F(x)^T)^{-1}F(x) \]
\[ = x - \eta \text{grad} f(x) - \nabla F(x)^T F(x). \] (12)

We will frequently refer to this expression in our proof.
3 Main result

3.1 Convergence theorem

Under Assumptions 1, 2 and 3, we show that the SQP algorithm converges locally linearly with the canonical rate.

**Theorem 1** (Local linear convergence of SQP with canonical rate). There exists \( \varepsilon > 0 \) and a constant \( \sigma > 0 \) such that the following holds. Let \( x_0 \in B(x_*, \varepsilon) \) and \( x_k \) be the iterates of Equation (2) with stepsize \( \eta = 1/\lambda_{\text{max}}(\text{Hess}(f(x_*))) \). Letting

\[
\begin{align*}
  a_k &= \|P_{x_k}(x_k - x_*)\|_2, \\
  b_k &= \|P_{x_*}^\perp(x_k - x_*)\|_2,
\end{align*}
\]

we have

\[
a_{k+1}^2 + \sigma b_{k+1} \leq \left(1 - \frac{1}{2\kappa_R}\right)^2(a_k^2 + \sigma b_k),
\]

where \( \kappa_R = \lambda_{\text{max}}(\text{Hess}(x_*)/\lambda_{\text{min}}(\text{Hess}(x_*))) \) is the condition number of the Riemannian Hessian of function \( f \) on the manifold \( M \) at \( x_* \). Consequently, the distance \( \|x_k - x_*\|_2^2 = a_k^2 + b_k^2 \) also converges linearly with rate \( 1 - 1/(2\kappa_R) \).

**Remark** Theorem 1 requires choosing the stepsize \( \eta \) according to the maximum eigenvalue of \( \text{Hess}(x_*) \), which might not be known in advance. In practice, one could implement a line-search (for example as in [AMS09] Section 4.2) for Riemannian gradient methods to achieve the same optimal rate. However, line-search methods for the SQP will call for a different convergence analysis.

3.2 Riemannian Taylor expansion

The proof of Theorem 1 relies on a particular expansion of the Riemannian gradient off the manifold, which we state as follows.

**Lemma 3.1** (First-order expansion of Riemannian gradient). There exist constants \( \varepsilon_0 > 0 \) and \( C_{r,1}, C_{r,2} > 0 \) such that for all \( x \in B(x_*, \varepsilon_0) \),

\[
\text{grad} f(x) = \text{Hess}(x_*)(x - x_*) + r(x), \quad \|r(x)\|_2 \leq C_{r,1}\|x - x_*\|_2^2 + C_{r,2}\|P_{x_*}^\perp(x - x_*)\|_2.
\]

Lemma 3.1 extends classical Riemannian Taylor expansion (e.g. [AMS09] Section 7.1) to points off the manifold, where the remainder term contains an additional first-order error. While the error term is linear in \( \|P_{x_*}^\perp(x - x_*)\|_2 \), this expansion is particularly suitable when this distance is on the order of \( \|P_{x_*}(x - x_*)\|_2 \), which results in a quadratic bound on \( \|r(x)\|_2 \).

In the proof of Theorem 1, \( \text{grad} f(x) \) mostly appears through its squared norm and inner product with \( x - x_* \). We summarize the expansion of these terms in the following corollary.

**Corollary 3.2.** There exist constants \( \varepsilon_0 > 0 \) and \( C_{r,1}, C_{r,2} > 0 \) such that the following hold. For any \( \varepsilon \leq \varepsilon_0 \), \( x \in B(x, \varepsilon) \),

\[
\begin{align*}
  \langle \text{grad} f(x), x - x_* \rangle &= \langle x - x_*, \text{Hess}(x_*)(x - x_*) \rangle + R_1, \\
  \|\text{grad} f(x)\|_2^2 &= \langle x - x_*, [\text{Hess}(x_*)]_2^2(x - x_*) \rangle + R_2,
\end{align*}
\]

where

\[
\max \{ |R_1|, |R_2| \} \leq \varepsilon(C_{r,1}\|P_{x_*}(x - x_*)\|_2^2 + C_{r,2}\|P_{x_*}^\perp(x - x_*)\|_2).
\]
4 Proof of Theorem [1]

The following perturbation bound (see, e.g. [Sun01]) for projections is useful in the proof.

Lemma 4.1. For \( x_1, x_2 \in B(x_*, \delta) \), we have

\[
\|P_{x_1} - P_{x_2}\|_{op} \leq \beta_P \|x_1 - x_2\|_2, \quad \|P_{x_1}P_{x_2}^\perp\|_{op} \leq \beta_P \|x_1 - x_2\|_2.
\]

where \( \beta_P = 2\beta_F/\gamma_F \).

We now prove the main theorem.

Step 1: A trivial bound on \( \|x_+ - x_*\|^2_2 \). Consider one iterate of the algorithm \( x \to x_+ \), whose closed-form expression is given in (12). We have \( x_+ = x + \Delta \), where

\[
\Delta = -\eta \cdot \nabla f(x) - \nabla F(x)^\dagger F(x).
\]

We would like to relate \( \|x_+ - x_*\|_2 \) with \( \|x - x_*\|_2 \). Observe that \( \nabla f(x_*) = 0 \), we have

\[
\|\nabla f(x)\|_2 = \|\nabla f(x) - \nabla f(x_*)\|_2 \leq \beta_E \|x - x_*\|_2.
\]

Observe that \( F(x_*) = 0 \), we have

\[
\|\nabla F(x)^\dagger F(x)\|_2 = \|\nabla F(x)^\dagger (F(x) - F(x_*))\|_2 \leq \|\nabla F(x)^\dagger\|_2 \|F(x) - F(x_*)\|_2 \leq (\beta_F/\gamma_F)\|x - x_*\|_2.
\]

Accordingly, we have

\[
\|x_+ - x_*\|_2 \leq \|x - x_*\|_2 + \eta \|\nabla f(x)\|_2 + \|\nabla F(x)^\dagger F(x)\|_2 \leq [1 + \eta \beta_E + (\beta_F/\gamma_F)]\|x - x_*\|_2.
\]

Hence, for any stepsize \( \eta \), letting \( C_d = [1 + \eta \beta_E + (\beta_F/\gamma_F)]^2 \), for \( x \in B(x_*, \delta) \), we have

\[
\|x_+ - x_*\|^2_2 \leq C_d \|x - x_*\|^2_2.
\]

Step 2: Analyze normal and tangent distances. This is the key step of the proof. We look into the normal direction and the tangent direction separately. The intuition for this process is that the normal part of \( x - x_* \) is a measure of feasibility, and as we will see, converges much more quickly.

Now we look at equation \( x_+ - x_* = x - x_* + \Delta \). Multiplying it by \( P_x \) and \( P_x^\perp \) gives

\[
P_x^\perp(x_+ - x_*) = P_x^\perp(x - x_*) + P_x^\perp \Delta = P_x^\perp(x - x_*) - \nabla F(x)^\dagger F(x),
\]

\[
P_x(x_+ - x_*) = P_x(x - x_*) + P_x \Delta = P_x(x - x_*) - \eta \cdot \nabla f(x).
\]

We now take squared norms on both equalities and bound the growth. Define the normal and tangent distances as

\[
a = \|P_{x_*}(x - x_*)\|_2, \quad b = \|P_{x_*}^\perp(x - x_*)\|_2,
\]

and \((a_+, b_+)\) similarly for \( x_+ \). Note that the definitions of \( a, b \) use the projection at \( x_* \), so they are slightly different from quantities \( \{16\} \) and \( \{17\} \).

From now on, we assume that \( \|x - x_*\|_2 \leq \varepsilon_0 \), and \( \varepsilon_0 \) is sufficiently small such that (15) holds. The requirements on \( \varepsilon_0 \) will later be tightened when necessary.
The normal direction. We have
\[
\|P_x^\perp(x_+ - x_*)\|^2_2 = \|P_x^\perp(x - x_*)\|^2_2 + 2\langle x - x_*, P_x^\perp \Delta \rangle + \|P_x \Delta\|^2_2
\]
\[
= \|P_x^\perp(x - x_*)\|^2_2 - 2\langle x - x_*, \nabla F(x)^\top F(x) \rangle + \langle F(x), (\nabla F(x)\nabla F(x)^\top)^{-1}F(x) \rangle
\]
\[
= \|P_x^\perp(x - x_*)\|^2_2 - 2\langle \nabla F(x)(x - x_*, (\nabla F(x)\nabla F(x)^\top)^{-1}(\nabla F(x)(x - x_*) + r(x)) \rangle
\]
\[
+ \langle \nabla F(x)(x - x_*) + r(x), (\nabla F(x)\nabla F(x)^\top)^{-1}(\nabla F(x)(x - x_*) + r(x)) \rangle
\]
\[
= (r(x), (\nabla F(x)\nabla F(x)^\top)^{-1}r(x)),
\]
where \(r(x) = F(x) - \nabla F(x)(x - x_*)\). By the smoothness of function \(F\), we have
\[
\|r(x)\|_2 \leq \beta_F/2 \cdot \|x - x_*\|^2_2.
\]
Accordingly, we get
\[
\|P_x^\perp(x_+ - x_*)\|^2_2 \leq \beta_F^2/(4\gamma_F^2) \cdot \|x - x_*\|^2_2 = \beta_F^2/(4\gamma_F^2) \cdot \||P_x^\perp(x - x_*)\|^2_2 + \|P_x(x - x_*)\|^2_2\]
\[
= \beta_F^2/(4\gamma_F^2) \cdot (a^2 + b^2)^2.
\]
Applying the perturbation bound on projections (Lemma 4.1), we get
\[
b_+ = \|P_x^\perp(x_+ - x_*)\|_2 \leq \|P_x^\perp(x_+ - x_*)\|_2 + \|P_x - P_x^\perp\|_2 \|x_+ - x_*\|_2
\]
\[
\leq \beta_F/(2\gamma_F) \cdot (a^2 + b^2) + \beta_P \|x_+ - x_*\|_2 \|x_+ - x_*\|_2
\]
\[
\leq \beta_F/(2\gamma_F) \cdot (a^2 + b^2) + \beta_P C_d \cdot \|x - x_*\|^2_2
\]
\[
= \beta_F/(2\gamma_F) + \beta_P C_d \cdot (a^2 + b^2) = C_b(a^2 + b^2).
\]

The tangent direction. We have
\[
\|P_x(x_+ - x_*)\|^2_2 = \|P_x(x - x_*)\|^2_2 + 2\langle x - x_*, P_x \Delta \rangle + \|P_x \Delta\|^2_2
\]
\[
= \|P_x(x - x_*)\|^2_2 - 2\eta \langle \nabla f(x), x - x_* \rangle + \eta^2 \cdot \|\nabla f(x)\|^2_2.
\]
Applying Lemma 4.1, we get that for any vector \(v\),
\[
\|P_x v\|^2_2 - \|P_x v\|^2_2 = \|v, (P_{x_*^\perp} - P_x) v\| \leq \beta_P \|x - x_*\|^2_2 \cdot \|v\|^2_2.
\]
Applying this to vectors \(x_+ - x_*\) and \(x - x_*\) gives
\[
a_+^2 \leq a^2 - 2\eta \langle \nabla f(x), x - x_\* \rangle + \eta^2 \|\nabla f(x)\|^2_2 + \beta_P \|x - x_*\|^2_2 \|x_+ - x_*\|^2_2
\]
\[
\leq a^2 - 2\eta \langle \nabla f(x), x - x_\* \rangle + \eta^2 \|\nabla f(x)\|^2_2 + \beta_P (1 + C_d) \|x - x_*\|^2_2.
\]
Applying Corollary 3.2 and note that \(Hess(f(x_*)) = P_{x_*^\perp}Hess(f(x_*))P_{x_*}\) by the property of the Riemannian Hessian, we get
\[
a_+^2 \leq a^2 - 2\eta \langle x - x_\*, Hess(f(x_*))(x - x_\*) \rangle + \eta^2 \langle x - x_\*, Hess(f(x_*))^2(x - x_\*) \rangle
\]
\[
+ (-2\eta R_1 + \eta^2 R_2) + \varepsilon_0 C(a^2 + b^2)
\]
\[
= \langle P_{x_*^\perp}(x - x_\*), (1 - \eta Hess(f(x_*)))^2 P_{x_*}(x - x_\*) \rangle + (-2\eta R_1 + \eta^2 R_2) + \varepsilon_0 C(a^2 + b^2),
\]
where the remainders are bounded as
\[
\max \{|R_1|, |R_2|\} \leq \varepsilon_0 (C_{r,1} \|P_{x_*}^\perp(x - x_\*)\|^2_2 + C_{r,2} \|P_{x_*^\perp}(x - x_\*)\|_2) = \varepsilon_0 (C_{r,1} a^2 + C_{r,2} b).
\]
Choosing the stepsize as \(\eta = 1/\lambda_{\text{max}}(Hess(f(x_*)))\), we have \(1 - \eta Hess(f(x_*)) \leq (1 - 1/\kappa R) I\). For this choice of \(\eta\), using the above bound for \(R_1, R_2\), we get that there exists some constant \(C_1, C_2\) such that
\[
a_+^2 \leq \left(1 - \frac{1}{\kappa R}\right)^2 a^2 + \varepsilon_0 (C_1 a^2 + C_2 b).
\]
Putting together. Let $\sigma > 0$ be a constant to be determined. Looking at the quantity $a_+^2 + \sigma b_+$, by the bounds (19) and (21) we have
\[
a_+^2 + \sigma b_+ \leq \left( 1 - \frac{1}{\kappa_R} \right)^2 a^2 + \varepsilon_0 (C_1 a^2 + C_2 b) + \sigma C b (a^2 + b^2)
\]
\[
\leq \left( 1 - \frac{1}{2 \kappa_R} \right)^2 (1 - \kappa_R) a^2 + \varepsilon_0 (C_1 + \sigma C)b.
\]
We can choose $\sigma$ sufficiently small, then $\varepsilon_0$ sufficiently small, such that
\[
a_+^2 + \sigma b_+ \leq \left( 1 - \frac{1}{2 \kappa_R} \right)^2 (a^2 + b).
\]

Step 3: Connect the entire iteration path. Consider iterates $x_k$ of the first-order algorithm (2). Initializing $x_0$ sufficiently close to $x_\star$, we can guarantee that the descent on $a_k^2 + \sigma b_k$ ensures $a_k^2 + b_k^2 \leq \varepsilon_0^2$. Thus we chain this analysis on $(x, x_\star) = (x_k, x_{k+1})$ and get that $a_k^2 + \sigma b_k$ converges linearly with rate $1 - 1/(2 \kappa_R)$.

5 Example

We provide the eigenvalue problem as a simple example, showing that SQP algorithm is as good as power iterations. We emphasize that this is a simple and well-studied problem; our goal here is only to illustrate the connection between SQP and the Riemannian algorithms, in particular the canonical rate.

Example 1 (SQP for eigenvalue problems): Consider the eigenvalue problem for a symmetric matrix $A \in \mathbb{R}^{n \times n}$:
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^\top A x \\
\text{subject to} & \quad \|x\|^2_2 - 1 = 0.
\end{align*}
\]
This is an instance of problem (1) with $f(x) = (1/2) x^\top A x$ and $F(x) = \|x\|^2_2 - 1$. Let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $A$, and $v_i(A)$ be the corresponding eigenvectors.

The SQP iterate for this problem is $x \mapsto x_\star = x + \Delta$, where $\Delta$ solves the subproblem
\[
\begin{align*}
\text{minimize} & \quad \langle Ax, \Delta \rangle + \frac{1}{2 \eta} \|\Delta\|_2^2 \\
\text{subject to} & \quad \|x\|^2_2 + 2 \langle x, \Delta \rangle - 1 = 0.
\end{align*}
\]
Applying (12), we obtain the explicit formula
\[
x_\star = \frac{\|x\|^2_2 + 1}{2 \|x\|^2_2} x - \eta \left( I_n - \frac{x x^\top}{\|x\|^2_2} \right) A x.
\]
(22)

By Theorem 1, the local convergence rate is $1 - 1/\kappa_R$, which we now examine. At $x_\star = v_1(A)$, we have $\text{Hess} f(x_\star) = A - \lambda_1 I_n$, and the tangent space $T_{x_\star} \mathcal{M} = \text{span}(v_2(A), \ldots, v_n(A))$. Therefore, the condition number of the Riemannian Hessian is
\[
\kappa_R = \frac{\sup_{v \in T_{x_\star} \mathcal{M}, \|v\|_2 = 1} \langle v, \text{Hess} f(x_\star) v \rangle}{\inf_{v \in T_{x_\star} \mathcal{M}, \|v\|_2 = 1} \langle v, \text{Hess} f(x_\star) v \rangle} = \frac{\lambda_n - \lambda_1}{\lambda_2 - \lambda_1}.
\]
Hence, choosing the right stepsize $\eta$, the convergence rate of the SQP method is
\[
1 - \frac{1}{\kappa_R} = \frac{\lambda_n - \lambda_2}{\lambda_n - \lambda_1},
\]
matching the rate of power iteration (Riemannian gradient descent). The keen reader might notice that the iterate 22 converges globally to $x_*$ as long as $\langle x_0, x_* \rangle \neq 0$, which is another feature shared with power iteration. Whether such global convergence holds more generally would be an interesting direction for future study. ♦

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A  Proof of technical results

A.1  Some tools

Lemma A.1 ([MZ10]). For $x_1, x_2 \in B(x_*, \delta)$, we have
\[ \|\nabla F(x_1) - \nabla F(x_2)\|_\infty \leq \beta_D \|x_1 - x_2\|_2. \]
where $\beta_D = 2\beta_F / \gamma_F^2$.

Lemma A.2. The pseudo-Riemannian gradient $\nabla f(x)$ is $\beta_E$-Lipschitz in $B(x_*, \delta)$, where $\beta_E = \beta_p L_f + \beta_f$.

Proof. We have
\[ \|\nabla f(x_1) - \nabla f(x_2)\|_2 = \|\text{P}_{x_1} \nabla f(x_1) - \text{P}_{x_2} \nabla f(x_2)\|_2 \]
\[ \leq \|\text{P}_{x_1} - \text{P}_{x_2}\|_2 \|\nabla f(x_1)\|_2 + \|\text{P}_{x_2} (\nabla f(x_1) - \nabla f(x_2))\|_2 \]
\[ \leq (\beta_p L_f + \beta_f) \|x_1 - x_2\|_2. \]

A.2  Proof of Lemma 3.1

For any $x \in B(x_*, \varepsilon_0)$, denote $x_p = \text{P}_{x_*}(x - x_*) + x_*$, We prove the following equation first
\[ \nabla f(x_p) = \text{Hess} f(x_*)(x_p - x_*) + r(x_p), \tag{23} \]
where $r(x_p) \leq C_{r,1} \|x_p - x_*\|_2^2$.

First, we show that $\nabla f(x_p)$ can be well approximated by $\text{P}_{x_*} \nabla f(x_p)$. Denoting $r_0 = \text{P}_{x_*} \nabla f(x_p) - \nabla f(x_p)$, by Lemma A.1 and A.2, we have
\[ \|r_0\|_2 = \|\text{P}_{x_*} \nabla f(x_p)\|_2 = \|\text{P}_{x_*} \text{P}_{x_p} \nabla f(x_p)\|_2 \]
\[ \leq \|\text{P}_{x_*} \text{P}_{x_p}\|_\infty \cdot \|\nabla f(x_p)\|_2 \leq \beta_p \|x_p - x_*\|_2 \|\nabla f(x_p)\|_2 \leq \beta_p \beta_E \|x_p - x_*\|_2^2. \]

Then, for any $u \in \mathbb{R}^n$ with $\|u\|_2 = 1$, we have
\[ \langle u, \text{P}_{x_*} \nabla f(x_p) \rangle = \langle u, \text{P}_{x_*} \text{P}_{x_p} (\nabla f(x_*) + \nabla^2 f(x_*) (x_p - x_*) + 1/2 \cdot \nabla^3 f(x_p^\dagger) (\cdot, (x_p - x_*)^\otimes 2)) \rangle \]
\[ = \langle u, \text{P}_{x_*} \nabla f(x_*) + \nabla^2 f(x_*) (x_p - x_*) \rangle + r_1, \]
where
\[ |r_1| = 1/2 \cdot |\langle u, \text{P}_{x_*} \text{P}_{x_p} \nabla^3 f(x_p^\dagger) (\cdot, (x_p - x_*)^\otimes 2) \rangle| \leq 1/2 \cdot \rho_f \|x_p - x_*\|_2^2. \]

Then we have
\[ \langle u, \text{P}_{x_*} \nabla f(x_p) \rangle = \langle u, \text{P}_{x_*} \text{P}_{x_p} (\nabla f(x_*) + \nabla^2 f(x_*) (x_p - x_*)) \rangle + r_1, \]
\[ = \langle u, \text{P}_{x_*} \nabla f(x_*) \rangle + \langle u, \text{P}_{x_*} \nabla^2 f(x_*) (x_p - x_*) \rangle + \langle u, \text{P}_{x_*} (\text{P}_{x_p} - \text{P}_{x_*}) \nabla^2 f(x_*) (x_p - x_*) \rangle + r_1 \]
\[ = \langle u, \text{P}_{x_*} \nabla f(x_*) \rangle + \langle u, \text{P}_{x_*} \nabla^2 f(x_*) (x_p - x_*) \rangle + r_1 + r_2, \]
\[ |r_2| = |\langle u, P_{x_\star}(P_{x_p} - P_{x_\star})\nabla^2 f(x_\star)(x_p - x_\star)\rangle| \leq \beta_p \beta_f \|x_p - x_\star\|^2_2. \]

Then we look at the term I. We have
\[
I = -\langle u, P_{x_\star} \nabla F(x_p)^T \nabla F(x_p)^T \nabla f(x_\star) \rangle \\
= -\langle u, P_{x_\star}[\nabla F(x_p) - \nabla F(x_\star)]^T \nabla F(x_p)^T \nabla f(x_\star) \rangle \\
= -\langle u, P_{x_\star} \{ \nabla^2 F(x_\star) [\cdot, \cdot, x_p - x_\star] + 1/2 \cdot \nabla^3 F(\tilde{x}_p) [\cdot, \cdot, (x_p - x_\star)^{\otimes 2}] \}^T \nabla F(x_p)^T \nabla f(x_\star) \rangle \\
= -\sum_{i=1}^{m} [\nabla F(x_p)^T \nabla f(x_\star)]_i \nabla^2 F_i(x_\star)[x_p - x_\star, P_{x_\star} u] + r_3,
\]
where
\[
|r_3| = 1/2 \cdot \sum_{i=1}^{m} |\nabla F(x_p)^T \nabla f(x_\star)]_i \nabla^2 F_i(x_\star)[(x_p - x_\star)^{\otimes 2}, P_{x_\star} u]| \\
\leq \rho_F L_f/(2\gamma_F) \cdot \|x_p - x_\star\|^2_2.
\]

We further have
\[
I = -\sum_{i=1}^{m} [\nabla F(x_p)^T \nabla f(x_\star)]_i \nabla^2 F_i(x_\star)[x_p - x_\star, P_{x_\star} u] + r_3 \\
= -\sum_{i=1}^{m} [\nabla F(x_\star)^T \nabla f(x_\star)]_i \nabla^2 F_i(x_\star)[x_p - x_\star, P_{x_\star} u] + r_3 + r_4,
\]
where
\[
|r_4| = \sum_{i=1}^{m} |[\nabla F(x_p) - \nabla F(x_\star)]^T \nabla f(x_\star)]_i \nabla^2 F_i(x_\star)[x_p - x_\star, P_{x_\star} u] \leq \beta_D L_f \beta_F \cdot \|x_p - x_\star\|^2_2.
\]

Above all, combining all the terms, we have
\[
|\langle u, \nabla f(x_p) - \nabla f(x_\star) \rangle| \leq \|r_0\|_2 + |r_1 + r_2 + r_3 + r_4| \\
\leq (\beta_p \beta_E + 1/2 \cdot \rho_f + \beta_p \beta_f + \rho_F L_f/(2\gamma_F) + \beta_D L_f \beta_F) \|x_p - x_\star\|^2_2.
\]

Taking \( C_{r,1} = \beta_p \beta_E + 1/2 \cdot \rho_f + \beta_p \beta_f + \rho_F L_f/(2\gamma_F) + \beta_D L_f \beta_F \) we get Eq. \( 23 \). Then by Lemma A.2, we have
\[
\|\nabla f(x) - \nabla f(x_p)\| \leq C_{r,2} \|x - x_p\|_2.
\]

This proves the lemma.

### A.3 Proof of Corollary 3.2

Let \( \varepsilon_0 \) be given by Lemma 3.1, \( \varepsilon \leq \varepsilon_0 \), and \( x \in B(x, \varepsilon) \). We have
\[
\nabla f(x) = \text{Hess} f(x_\star)(x - x_\star) + r(x), \quad \|r(x)\|_2 \leq C_{r,1} \|x - x_\star\|^2_2 + C_{r,2} \|P_{x_\star}^\perp (x - x_\star)\|_2.
\]
Therefore we obtain the expansion
\[
\langle \nabla f(x), x - x_\star \rangle = \langle x - x_\star, \text{Hess} f(x_\star)(x - x_\star) \rangle + R_1.
\]
where $R_1$ is bounded as
\[
|R_1| = |\langle r(x), x - x_* \rangle| \leq C_{r,1} \|x - x_*\|^2_2 + C_{r,2}\|x - x_*\|_2 \|P_{x_*}^\perp (x - x_*)\|_2 \\
\leq \varepsilon C_{r,1} \|x - x_*\|^2_2 + \varepsilon C_{r,2}\|P_{x_*}^\perp (x - x_*)\|_2.
\] (26)

Similarly, we have the expansion
\[
\|\text{grad} f(x_*)\|^2_2 = \langle x - x_*, (\text{Hess} f(x_*))^2(x - x_*) \rangle + R_2,
\]
where $R_2$ is bounded as (letting $H = \lambda_{\text{max}}(\text{Hess} f(x_*))$ for convenience)
\[
|R_2| \leq 2\|\text{Hess} f(x_*)(x - x_*, r(x))\| + \|r(x)\|^2_2 \\
\leq 2H (C_{r,1} \|x - x_*\|^3_2 + C_{r,2}\|x - x_*\|_2 \|P_{x_*}^\perp (x - x_*)\|_2) \\
+ (C_{r,1}^2 \|x - x_*\|^2_2 + 2C_{r,1}C_{r,2}\|x - x_*\|^2_2 \|P_{x_*}^\perp (x - x_*)\|_2 + C_{r,2}^2\|P_{x_*}^\perp (x - x_*)\|_2^2) \\
\leq (2HC_{r,1}\varepsilon + C_{r,1}^2\varepsilon^2)\|x - x_*\|^2_2 + (2HC_{r,2}\varepsilon + 2C_{r,1}C_{r,2}\varepsilon^2)\|P_{x_*}^\perp (x - x_*)\|_2 + C_{r,2}^2\varepsilon\|P_{x_*}^\perp (x - x_*)\|_2 \\
\leq \varepsilon (2HC_{r,1} + C_{r,1}^2\varepsilon_0)\|x - x_*\|^2_2 + \varepsilon (2HC_{r,2} + 2C_{r,1}C_{r,2}\varepsilon_0 + C_{r,2}^2\varepsilon_0)\|P_{x_*}^\perp (x - x_*)\|_2 \\
= \varepsilon (C_{r,1} \|x - x_*\|^2_2 + C_{r,2}\|P_{x_*}^\perp (x - x_*)\|_2) .
\] (27)

Overloading the constants, we get
\[
\max \{ |R_1|, |R_2| \} \leq \varepsilon (C_{r,1} \|x - x_*\|^2_2 + C_{r,2}\|P_{x_*}^\perp (x - x_*)\|_2) \\
= \varepsilon (C_{r,1}\|P_{x_*}(x - x_*)\|^2_2 + C_{r,1}\|P_{x_*}^\perp (x - x_*)\|^2_2 + C_{r,2}\|P_{x_*}^\perp (x - x_*)\|_2) \\
\leq \varepsilon (C_{r,1} \|x - x_*\|^2_2 + (C_{r,2} + C_{r,1}\varepsilon_0)\|P_{x_*}^\perp (x - x_*)\|_2).
\]