RIGIDITY OF STABLE CYLINDERS IN THREE-MANIFOLDS

JOSÉ M. ESPINAR

(Communicated by Michael Wolf)

Abstract. In this paper we show how the existence of a certain stable cylinder determines (locally) the ambient manifold where it is immersed. This cylinder has to verify a bifurcation phenomenon; we make this explicit in the introduction. In particular, the existence of such a stable cylinder implies that the ambient manifold has infinite volume.

1. Introduction

A stable compact domain Σ on a minimal surface in a Riemannian three-manifold \( M \) is one whose area cannot be decreased up to second order by a variation of the domain leaving the boundary fixed. Stable oriented domains Σ are characterized by the stability inequality for normal variations \( \psi N \)

\[
\int_{\Sigma} \psi^2 |A|^2 + \int_{\Sigma} \psi^2 \text{Ric}_M (N, N) \leq \int_{\Sigma} |\nabla \psi|^2
\]

for all compactly supported functions \( \psi \in H^{1,2}_0 (\Sigma) \). Here \(|A|^2\) denotes the square of the length of the second fundamental form of \( \Sigma \), \( \text{Ric}_M (N, N) \) is the Ricci curvature of \( M \) in the direction of the normal \( N \) to \( \Sigma \) and \( \nabla \) is the gradient w.r.t. the induced metric.

One writes the stability inequality in the form

\[
\frac{d^2}{dt^2} \left. \text{Area}(\Sigma(t)) \right|_{t=0} = -\int_{\Sigma} \psi L \psi \geq 0,
\]

where \( L \) is the linearized operator of the mean curvature

\[
L = \Delta + |A|^2 + \text{Ric}_M.
\]

In terms of \( L \), stability means that \(-L\) is nonnegative; i.e., all its eigenvalues are nonnegative. \( \Sigma \) is said to have finite index if \(-L\) has only finitely many negative eigenvalues.

From the Gauss equation, one can write the stability operator as \( L = \Delta - K + V \), where \( \Delta \) and \( K \) are the Laplacian and Gauss curvature associated to the metric \( g \) respectively and \( V := 1/2|A|^2 + S \), where \( S \) denotes the scalar curvature associated to the metric \( g \).

Received by the editors August 2, 2010 and, in revised form, January 14, 2011.

2010 Mathematics Subject Classification. Primary 53A10; Secondary 53C24, 49Q05.

Key words and phrases. Stable surfaces, bifurcation.

The author is partially supported by the Spanish MEC-FEDER Grant MTM2010-19821 and Regional J. Andalucia Grants P06-FQM-01642 and FQM325.
The index form of this kind of operators is

\[ I(f) = \int_{\Sigma} \left\{ \|\nabla f\|^2 - Vf^2 + Kf^2 \right\} \]

where \( \nabla \) and \( \|\cdot\| \) are the gradient and norm associated to the metric \( g \). Thus, if \( \Sigma \) is stable, we have

\[ \int_{\Sigma} fLf = -I(f) \leq 0, \]

or equivalently

\[ (1.1) \quad \int_{\Sigma} f^2(1/2|A|^2 + S) \leq \int_{\Sigma} \left\{ \|\nabla f\|^2 + Kf^2 \right\}. \]

In a seminar paper [5], D. Fischer-Colbrie and R. Schoen proved:

**Theorem A.** Let \( M \) be a complete oriented three-manifold of nonnegative scalar curvature. Let \( \Sigma \) be an oriented complete stable minimal surface in \( M \). If \( \Sigma \) is noncompact, conformally equivalent to the cylinder and the absolute total curvature of \( \Sigma \) is finite, then \( \Sigma \) is flat and totally geodesic.

They also state [5, Remark 2]: We feel that the assumption of finite total curvature should not be essential in proving that the cylinder is flat and totally geodesic.

Recently, this question was partially answered in [3] under the assumption that the positive part of the Gaussian curvature is integrable, i.e., \( K^+ := \max \{0, K\} \in L^1(\Sigma) \). It was totally answered by M. Reiris [10]; he proved:

**Theorem B.** Let \( M \) be a complete oriented three-manifold of nonnegative scalar curvature. Let \( \Sigma \) be an oriented complete stable minimal surface in \( M \) diffeomorphic to the cylinder. Then \( \Sigma \) is flat and totally geodesic.

Besides, Bray, Brendle and Neves [11] were able to determine the structure of a three-manifold \( M \) under the assumption of the existence of an area minimizing two-sphere. Specifically, they proved:

**Theorem C.** Let \( M \) be a compact three-manifold with \( \pi_2(M) \neq 0 \). Denote by \( F \) the set of all smooth maps \( f : \mathbb{S}^2 \to M \) which represent a nontrivial element of \( \pi_2(M) \). Set

\[ A(M) := \inf \{ \text{area}(f(\mathbb{S}^2)) : f \in F \}. \]

Then

\[ A(M)\text{inf}_M R \leq 8\pi, \]

where \( R \) denotes the scalar curvature of \( M \). Moreover, if the equality holds, then the universal cover of \( M \) is isometric to the standard cylinder \( \mathbb{S}^2 \times \mathbb{R} \) up to scaling.

In this paper, we will go further. We will see how the existence of a stable cylinder verifying a bifurcation phenomenon determines the ambient manifold \( M \). First, let us make clear what we mean by bifurcation phenomenon:

**Definition 1.1.** We say that a complete minimal surface \( \Sigma \subset M \) bifurcates if there exist \( \delta > 0 \) and a smooth map \( u : \Sigma \times (-\delta, \delta) \to \mathbb{R} \) so that:

- For each \( p \in \Sigma \), we have \( u(x, 0) = 0 \) and \( \frac{\partial}{\partial t}|_{t=0} u(p, t) = 1 \). Moreover, \( u(p, t) \geq 0 \) if \( t > 0 \) and \( u(p, t) \leq 0 \) if \( t < 0 \).
For each $t \in (-\delta, \delta)$, the surface

$$\Sigma_t := \{ \exp_p(u(p,t))N(p) : p \in \Sigma \}$$

is a complete minimal surface. Here, $\exp$ denotes the exponential map in $M$.

Now, we can state

**Theorem 1.1.** Let $M$ be a complete oriented Riemannian three-manifold with nonnegative scalar curvature. Assume there exists $\Sigma \subset M$ a complete stable minimal surface conformally equivalent to a cylinder that bifurcates. Then $\Sigma$ is flat, totally geodesic, and $S$ vanishes along $\Sigma$. Moreover, there exists an open set $U \subset M$ so that $U$ is locally isometric to $S^1 \times \mathbb{R}$ or $\mathbb{T}^2 \times \mathbb{R}$ (here $\mathbb{T}^2$ is the flat tori).

We should point out that the condition that $\Sigma$ bifurcates is necessary. In fact, one can construct the following example: Let $C(-l, l)$ be the right cylinder of height $2l$ and radius 1 endowed with the flat metric. Close it up with two spherical caps $S_i$, $i = 1, 2$ (one on the top and another on the bottom). Now, smooth the surface $M^2 = C(-l, l) \cup S_1 \cup S_2$ so that it is flat on $C(-l + \varepsilon, l - \varepsilon)$, for some $\varepsilon > 0$, and has nonnegative Gaussian curvature.

Consider the three-manifold $M^3 = M^2 \times \mathbb{R}$. One can see that, if we take a closed geodesic $\gamma(t) \subset C(-l + \varepsilon, l - \varepsilon)$, $t \in (-l + \varepsilon, l - \varepsilon)$, the surface $\Sigma(t) := \gamma(t) \times \mathbb{R}$ is a complete stable minimal cylinder in $M$ that bifurcates, but, when we reach $t = l - \varepsilon$, this property might disappear (it could bifurcate as constant mean curvature surfaces at one side, but not minimal).

One interesting consequence of Theorem 1.1 is the following:

**Corollary 1.1.** Let $M$ be a complete oriented Riemannian three-manifold with nonnegative scalar curvature. Assume there exists $\Sigma \subset M$ a complete stable minimal surface conformally equivalent to a cylinder that bifurcates. Then,

$$\text{Vol}(M) = +\infty.$$  

Actually, the above conclusion (that is, Corollary 1.1) is also valid when the cylinder bifurcates only at one side.

**2. Preliminaries**

We denote by $M$ a complete connected orientable Riemannian three-manifold, with Riemannian metric $g$. Moreover, throughout this work, we will assume that its scalar curvature is nonnegative, i.e., $S \geq 0$. Also, $\Sigma \subset M$ will be assumed to be connected and oriented.

We denote by $N$ the unit normal vector field along $\Sigma$. Let $p_0 \in \Sigma$ be a point of the surface and let $D(p_0, s)$, for $s > 0$, denote the geodesic disk centered at $p_0$ of radius $s$. We assume that $D(p_0, s) \cap \partial \Sigma = \emptyset$. Moreover, let $r$ be the radial distance of a point $p$ in $D(p_0, s)$ to $p_0$. We write $D(s) = D(p_0, s)$.  

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
We also denote
\[\begin{align*}
l(s) &= \text{Length}(\partial D(s)), \\
a(s) &= \text{Area}(D(s)), \\
K(s) &= \int_{D(s)} K, \\
\chi(s) &= \text{Euler characteristic of } D(s).
\end{align*}\]

Let \(\Sigma \subset \mathcal{M}\) be a stable minimal surface diffeomorphic to the cylinder. Then, from Theorem B [10], \(\Sigma\) is flat and totally geodesic. We will give a (more general) proof of this result in the abstract setting of Schrödinger-type operators:

**Lemma 2.1.** Let \(\Sigma\) be a complete Riemannian surface. Let \(L = \Delta + V - aK\) be a differential operator on \(\Sigma\) acting on compactly supported \(f \in H^1_0(\Sigma)\), where \(a > 1/4\) is constant, \(V \geq 0\), and \(\Delta\) and \(K\) are the Laplacian and Gauss curvature associated to the metric \(g\) respectively.

Assume that \(\Sigma\) is homeomorphic to the cylinder and \(-L\) is nonnegative. Then, \(V \equiv 0\) and \(K \equiv 0\). Therefore,
\[\text{Ker}L := \{1\};\]
i.e., its kernel is the constant functions. Here, \(L\) denotes the Jacobi operator.

**Proof.** Set \(b \geq 1\) and let us consider the radial function
\[f(r) := \begin{cases}
(1 - r/s)^b, & r \leq s, \\
0, & r > s,
\end{cases}\]
where \(r\) denotes the radial distance from a point \(p_0 \in \Sigma\). Then, from [3, Lemma 3.1] (see also [9]), we have
\[\int_{D(s)} (1 - r/s)^2b V \leq 2a\pi G(s) + \frac{b(b(1 - 4a) + 2a)}{s^2} \int_0^s (1 - r/s)^2b - 1 l(r),\]
where
\[G(s) := -\int_0^s (f(r)^2)'\chi(r).\]

Therefore, since \(a > 1/4\), we can find \(b \geq 1\) so that \(b(1 - 4a) + 2a \leq 0\). So
\[\int_{D(s)} (1 - r/s)^2b V \leq 2a\pi G(s).\]

**Step 1:** \(V\) vanishes identically on \(\Sigma\).

Suppose there exists a point \(p_0 \in \Sigma\) so that \(V(p_0) > 0\). From now on, we fix the point \(p_0\). Then, there exists \(\epsilon > 0\) so that \(V(q) \geq \delta\) for all \(q \in D(\epsilon) = D(p_0, \epsilon)\). Since \(\Sigma\) is topologically a cylinder, there exists \(s_0 > 0\) so that for all \(s > s_0\) we have \(\chi(s) \leq 0\) (see [24, Lemma 1.4]).

Now, from the above considerations, there exists \(\beta > 0\) so that
\[0 < \beta \leq 2a\pi G(s).\]
But, following [3], we can see that

\[ G(s) = -\int_0^s (f(r)^2)' \chi(r) = -\int_0^{s_0} (f(r)^2)' \chi(r) - \int_{s_0}^s (f(r)^2)' \chi(r) \leq -\int_0^{s_0} (f(s_0)^2)' - f(0)^2 = -f(s_0)^2 + 1 \]

since \(-f_{s_0} (f(r)^2)' \chi(r) \geq 0\). Therefore,

\[ G(s) \leq 1 - (1 - s_0/s)^{2b} \to 0, \quad \text{as} \quad s \to +\infty, \]

which is a contradiction. Thus, \(V\) vanishes identically along \(\Sigma\).

\[ \bullet \text{ Step 2:} \quad K \text{ vanishes identically on } \Sigma. \text{ In particular, } \Sigma \text{ is parabolic.} \]

First, note that \(L := \Delta - aK\). From [4], there is a smooth positive function \(u\) on \(\Sigma\) such that \(Lu = 0\). Set \(\alpha := 1/a\). Then, from [9] (following ideas of [4]), the conformal metric \(\tilde{ds}^2 := u^{2\alpha} ds^2\), where \(ds^2\) is the metric on \(\Sigma\), is complete and its Gaussian curvature \(\tilde{K}\) is nonnegative; i.e., \(\tilde{K} \geq 0\).

On the one hand, the respective Gaussian curvatures are related by

\[ \alpha \Delta \ln u = K - \tilde{K} u^{2\alpha}. \]

On the other hand, since \(\Sigma\) is topologically a cylinder, the Cohn-Vossen inequality says

\[ \int_{\Sigma} \tilde{K} \leq 0; \]

that is, \(\tilde{K}\) vanishes identically.

Thus, \(K = \alpha \Delta \ln u\). From this last equation, we get

\[ aK = \frac{1}{u} \Delta u - \frac{|\nabla u|^2}{u^2}; \]

that is,

\[ \frac{|\nabla u|^2}{u} = \Delta u - aK u = 0. \]

This last equation implies that \(u\) is constant, and since \(u\) satisfies \(Lu = 0\), we have that \(K\) vanishes identically on \(\Sigma\). In particular, \(\Sigma\) is parabolic (see [6, Lemma 5]).

This implies that the Jacobi operator becomes \(L := \Delta\), and so the constant functions are in the kernel. But, since \(\Sigma\) is parabolic, such a kernel has dimension one (see [3]); therefore

\[ \text{Ker} L := \{1\}. \]

Let \(C := S^1 \times \mathbb{R}\) be the flat cylinder. Then we can parametrize \(\Sigma\) as the isometric immersion \(\psi_0 : C \to \mathcal{M}\) where \(\Sigma := \psi_0(C)\). Also, let \(N_0 : \mathcal{C} \to N \Sigma\) be the unit normal vector field along \(\Sigma\).

Assume \(\Sigma\) bifurcates (see Definition 1.1). Then there exist \(\delta > 0\) and a smooth map \(u : C \times (-\delta, \delta) \to \mathbb{R}\) so that the surface \(\Sigma_t := \psi_t(C)\), \(\psi_t : C \to \mathcal{M}\) where

\[ \psi_t(p) := \exp_{\psi_0(p)}(u(p,t)N_0(p)), \quad p \in \mathcal{C}, \]

is a complete minimal surface.
For each \( t \in (-\delta, \delta) \), the lapse function \( \rho_t : \Sigma \to \mathbb{R} \) is defined by
\[
\rho_t(p) = g \left( N_t(p), \frac{\partial}{\partial t} \psi_t(p) \right).
\]

Clearly, \( \rho_0(p) = 1 \) for all \( p \in \mathcal{C} \). Also, the lapse function satisfies the Jacobi equation
\[
(2.1) \quad \Delta_t \rho_t + (\text{Ric}(N_t) + |A_t|^2) \rho_t = 0,
\]

since \( \psi_t(\mathcal{C}) \) is minimal for all \( |t| < \delta \).

**Lemma 2.2.** There exists \( 0 < \delta' < \delta \) such that \( \Sigma_t \) is a stable minimal surface for each \( t \in (-\delta, \delta) \). Thus, \( \Sigma_t \) is flat, totally geodesic, and \( S \) vanishes along \( \Sigma_t \) for each \( t \in (-\delta, \delta) \).

**Proof.** First, note that the lapse function is not negative for all \( |t| < \delta \) and therefore, by \((2.1)\) and the Maximum Principle, either \( \rho_t \) vanishes identically or \( \rho_t > 0 \) for each \( |t| < \delta \).

So, since
\[
\rho_t \to \rho_0 \equiv 1 \text{ as } t \to 0,
\]
we can find a uniform constant \( 0 < \delta' < \delta \) such that \( \rho_t > 0 \) for all \( |t| \leq \delta' \).

Therefore, \( \rho_t \), \( |t| \leq \delta' \), is a positive function solving the Jacobi equation. This implies that \( \Sigma_t \) is stable for all \( |t| \leq \delta' \) (see \([4]\)).

The last assertion follows from Lemma \([2.1]\) and \( \Sigma_t \) is stable. \( \square \)

### 3. Proof of Theorem \([11]\)

From Definition \([1.1]\) and Lemma \([2.2]\) there exists \( \delta > 0 \) so that \( \Sigma_t \) is a complete minimal stable surface, which is flat, totally geodesic and \( S = 0 \) along \( \Sigma_t \), for each \( |t| < \delta \).

Now, we follow ideas of \([1]\). Since \( \text{Ric}(N_t) + |A_t|^2 = 0 \) and \( H(t) = 0 \) for each \( |t| < \delta \), from \([2.1]\) and \( \Sigma_t \) being parabolic, we obtain that \( \rho_t \) is constant. Thus, since \( \Sigma_t \) is totally geodesic,
\[
Y : \mathcal{C} \times (-\delta, \delta) \to \mathcal{M}
\]

is parallel. Also, the flow of \( N_t \) is a unit speed geodesic flow (see \([7]\)). Moreover, the map
\[
\Phi : \Sigma \times (-\delta, \delta) \to \mathcal{M}
\]

is a local isometry onto \( \mathcal{U} = \bigcup_{|t| < \delta} \Sigma_t \). Therefore, \( \Phi \) is a diffeomorphism onto \( \mathcal{U} \), which implies that \( Y : \mathcal{C} \times (-\delta, \delta) \to \mathcal{U} \) is a globally defined unit Killing vector field. This implies that \( \mathcal{U} \) is locally isometric to \( \mathcal{C} \times (-\delta, \delta) \).

Now, assume that any stable minimal complete cylinder bifurcates for a uniform \( \delta > 0 \). Then, we can start with a complete stable minimal cylinder \( \Sigma_0 \) that bifurcates, and then by the above considerations, \( \Sigma_t \), for each \( |t| < \delta \), is complete, flat, totally geodesic and \( S \) vanishes along \( \Sigma_t \). Moreover, \( \Sigma_t \) is strongly stable for each \( |t| < \delta \). Note that \( \Sigma_\delta \) is a strongly stable minimal surfaces conformally equivalent to a cylinder. Since it is the limit of strongly stable minimal surfaces \( \Sigma_t \) which are flat and totally geodesic, then \( \Sigma_\delta \) is totally geodesic, flat and \( S = 0 \) along \( \Sigma_\delta \). Therefore, by Definition \([1.1]\) and Lemma \([2.2]\) there exists \( \delta > 0 \) so that \( \Sigma_t, -\delta < t < 2\delta \),
is flat, totally geodesic and $S$ vanishes along $\Sigma_t$. Continuing this argument, $\Sigma_t$ is flat, totally geodesic and $S$ vanishes along $\Sigma_t$ for each $t \in I$, where $I = \mathbb{R}$ or $I = S^1$.

As we did above, since $\text{Ric}(N_t) + |A_t|^2 \equiv 0$ and $H(t) = 0$ for each $t \in I$, from (2.1) and $\Sigma_t$ being parabolic, we obtain that $\rho_t$ is constant. Thus, since $\Sigma_t$ is totally geodesic,

$$Y : \mathcal{C} \times I \rightarrow \mathcal{M} \quad (p, t) \mapsto Y(p, t) := N_t(p)$$

is parallel, where $I = \mathbb{R}$ or $I = S^1$. Also, the flow of $N_t$ is a unit speed geodesic flow (see [7]). Moreover, the map

$$\Phi : \Sigma \times I \rightarrow \mathcal{M} \quad (p, t) \mapsto \Phi(p, t) := \exp_{\psi_0(p)}(t N(p))$$

is a local isometry, which implies that it is a covering map. Therefore, $\Phi$ is a diffeomorphism, which implies that $Y : \mathcal{C} \times I \rightarrow \mathcal{M}$ is a globally defined unit Killing vector field. This implies that $\mathcal{M}$ is locally isometric either to $S^1 \times \mathbb{R}^2$ or $T^2 \times \mathbb{R}$ (here $T^2$ denotes the flat tori).

References

[1] H. Bray, S. Brendle and A. Neves: Rigidity of area-minimizing two-spheres in three-manifolds. Comm. Anal. Geom. 18 (2010), no. 4, 821–830. MR2765731
[2] P. Castillon: An inverse spectral problem on surfaces, Comment. Math. Helv. 81 (2006), no. 2, 271–286. MR2225628 (2007b:58042)
[3] J.M. Espinar and H. Rosenberg: A Colding-Minicozzi stability inequality and its applications. Trans. A.M.S. 363 (2011), 2447–2465. MR2763722
[4] D. Fischer-Colbrie: On complete minimal surfaces with finite Morse index in three manifolds, Invent. Math. 82 (1985), 121–132. MR808112 (87b:53090)
[5] D. Fischer-Colbrie and R. Schoen: The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature, Comm. Pure Applied Math. 33 (1980), 199–211. MR562550 (81i:53044)
[6] T. Klotz and R. Osserman: Complete surfaces in $E^3$ with constant mean curvature, Comment. Math. Helv. 41 (1966-67), 313–318. MR0211332 (35:2213)
[7] S. Montiel: Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds, Indiana Univ. Math. J. 48 (1999), 711–748. MR1722814 (2001f:53131)
[8] J. M. Manzano, J. Pérez and M. M. Rodríguez: Parabolic stable surfaces with constant mean curvature. To appear in Calc. Var. Partial Differential Equations.
[9] W. Meeks, J. Pérez and A. Ros: Stable constant mean curvature surfaces, Handbook of Geometric Analysis, volume 1 (2008), pages 301–380. International Press, edited by Liizhen Ji, Peter Li, Richard Schoen and Leon Simon. MR2483369 (2009k:53016)
[10] M. Reiris: Geometric relations of stable minimal surfaces and applications. Preprint.
[11] R. Schoen and S.T. Yau: Harmonic maps and the topology of stable hypersurfaces and manifolds of nonnegative Ricci curvature, Comm. Math. Helv. 39 (1976), 333–341. MR0438388 (55:11302)

Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain
E-mail address: jespinar@ugr.es