Higher order first integrals of motion in a
gauge covariant Hamiltonian framework

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Abstract

The higher order symmetries are investigated in a covariant Hamiltonian formulation. The covariant phase-space approach is extended to include the presence of external gauge fields and scalar potentials. The special role of the Killing-Yano tensors is pointed out. Some non-trivial examples involving Runge-Lenz type conserved quantities are explicitly worked out.

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1 Introduction

In a recent paper van Holten [1] proposed a technique for deriving conserved quantities in a covariant formalism of the dynamics of particles in external gauge fields. Using a completely covariant phase-space formulation, he studied a set of generalized Killing equations in order to produce constants of motion in a covariant way. This procedure applies to conserved quantities

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which are higher order polynomials in the momenta, as well as the spinning particle models in curved space-time, involving Grassmann variables to take into account fermionic degrees of freedom [2]. Van Holten’s algorithm was used successively to construct conserved quantities in a non-Abelian monopole field [3] and on generalized Euclidean Taub-NUT space [4].

The aim of this paper is to extend the technique from [1] for the dynamics of particles in external gauge fields and scalar potentials. The inclusion of scalar potentials permits us to extend the applicability of the covariant approach to more complex cases. For example, the motion in a Kepler-Coulomb (KC) potential with Runge-Lenz (RL) type conserved quantities is affected in a non-trivial way by the external gauge fields. Moreover, some generalizations of the KC systems have interesting integrability properties connected with the existence of additional hidden integrals of motion which are polynomial functions in the momenta [5, 6, 7, 8]. It is interesting to investigate the superintegrability of the generalized KC systems on $N$-dimensional curved spaces in conjunction with external gauge fields.

In the absence of gauge fields, the system of generalized Killing equations separates into two groups: one group involves the terms of the integral of motion of odd degree in the momenta and the other involves only the terms of even degree in the momenta [5]. In the presence of gauge fields, such a separation is not possible and the analysis of the system of differential equations is more intricate. A few examples will explicitly illustrate the complexity of the systems of conditions for the integrals of motion.

In general, the explicit and hidden symmetries of a space-time are encoded in the multitude of Killing vectors and higher order Stäckel-Killing (SK) tensors respectively. Another natural generalization of the Killing vectors is represented by the Killing-Yano (KY) tensors [9]. A KY tensor generates additional supercharges in the dynamics of pseudo-classical spinning particles, realizing a natural connection with supersymmetries [2]. Passing to quantum Dirac equation it was discovered [14] that KY tensors generate non-standard Dirac operators which commute with the standard one. These two generalizations of the Killing vectors could be related. In some cases a SK tensor is associated with a KY tensor, namely a rank 2 SK tensor could be written as a product of KY tensors.

The role of KY tensors with the framework extended to include electromagnetic interactions was pointed out by Tanimoto [10]. It was found the necessary condition of the electromagnetic field $F_{\mu\nu}$ to maintain the supersymmetry generated by a KY tensor. In this paper we retrace the argument
in [10] and see the role of KY tensors in connection with the covariant Hamiltonian framework.

The plan of the paper is as follows. In Section 2 we establish the generalized Killing equations in a covariant framework including external gauge fields and scalar potentials. In Section 3 we discuss the role of KY tensors using the condition on the electromagnetic tensor $F_{\mu\nu}$ from [10]. In the next Section we produce some non-trivial examples for a KC system in the presence of external fields: constant electric field, spherically symmetric magnetic fields and a magnetic field along a fixed direction. Finally, Section 5 is devoted to conclusions.

2 Conditions for a conserved quantity

Let $(\mathcal{M}, g)$ denote a $N$-dimensional manifold with the metric tensor $g$. The manifold $\mathcal{M}$ is usually the space-time manifold or its Euclidean version, i.e. a Riemannian manifold of dimension 4. However in many modern physical applications, such as superspace, Kaluza-Klein models and string theories, the manifold $\mathcal{M}$ can be an arbitrary manifold.

The classical dynamics of a point charge $q$ of mass $M$ in the external Abelian gauge field $A_i$ and a scalar potential $V(x^i)$ with respect to a system of local position coordinates $x^i$ is described by the Hamiltonian:

$$H = \frac{1}{2M} g^{ij}(p_i - qA_i)(p_j - qA_j) + V. \quad (1)$$

Here $g^{ij}$ are the components of the contravariant metric on $\mathcal{M}$ and $p_i$ are the canonical momenta conjugate to the coordinates $x^i$. We also adopt the notation $;i$ for partial differentiation with respect to $q^i$ and $;i$ for the covariant derivative constructed with the Levi-Civita connection on $\mathcal{M}$ with respect to the metric $g$. We shall also employ the usual summation convention over the repeated upper and lower indices from 1 to $N$.

The equations of motion can be written with the use of Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i}. \quad (2)$$

The disadvantage of this approach is that the canonical momenta $p_i$ and implicitly the Hamilton equations of motion are not manifestly gauge covariant. Using van Holten’s receipt [11], this drawback can be removed introducing
the gauge invariant momenta

$$\Pi = p - qA = M \dot{x}.$$  \hfill (3)

The Hamiltonian becomes

$$H = \frac{1}{2M} g^{ij} \Pi_i \Pi_j + V,$$  \hfill (4)

and the equations of motion are derived using the covariant Poisson brackets

$$\{ f, g \} = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \Pi_i} - \frac{\partial f}{\partial \Pi_i} \frac{\partial g}{\partial x^i} + q F_{ij} \frac{\partial f}{\partial \Pi_i} \frac{\partial g}{\partial \Pi_j}. \hfill (5)$$

where \( F_{ij} = A_{j;i} - A_{i;j} \) is the field strength.

The fundamental Poisson brackets are

$$\{ x^i, x^j \} = 0, \quad \{ x^i, \Pi_j \} = \delta^i_j, \quad \{ \Pi_i, \Pi_j \} = q F_{ij},$$  \hfill (6)

showing that the momenta \( \Pi \) are not canonical. A direct computation of the Hamilton’s equations gives:

$$\dot{x}^i = \{ x^i, H \} = \frac{1}{M} g^{ij} \Pi_j,$$  \hfill (7a)

$$\dot{\Pi}_i = \{ \Pi_i, H \} = q F_{ij} \dot{x}^j - V_{,i}.$$  \hfill (7b)

In terms of phase-space variables \((x^i, \Pi_i)\) the conserved quantities of motion read:

$$K = K_0 + \sum_{n=1}^{p} \frac{1}{n!} K^{i_1 \cdots i_n} (x) \cdots \Pi_i \Pi_{i_n},$$  \hfill (8)

where \( K^{i_1 \cdots i_n} \), \( n = 1, \cdots p \) are contravariant tensors on \( \mathcal{M} \) taken to be completely symmetric. Its bracket with the Hamiltonian vanishes \( \{ K, H \} = 0 \) and this yields the series of constraints:

$$K_1^i V_i = 0,$$  \hfill (9a)

$$K_{0,i} + q F_{ii} K_1^j = MK_2^j V_j.$$  \hfill (9b)

$$K_n^{(i_1 \cdots i_n;i_{n+1})} + q F_j^{(i_{n+1}} K_{n+1}^{i_1 \cdots i_n)j} = \frac{M}{(n + 1)} K_n^{i_1 \cdots i_{n+1}+j} V_j,$$

for \( n = 1, \cdots (p - 2) \), \hfill (9c)

$$K_{p-1}^{(i_1 \cdots i_{p-1};i_p)} + q F_j^{(i_p} K_p^{i_1 \cdots i_{p-1})j} = 0,$$  \hfill (9d)

$$K_p^{(i_1 \cdots i_p;i_{p+1})} = 0.$$  \hfill (9e)
Here the parentheses denote full symmetrization over the indices enclosed.

Examining the above hierarchy of constraints (9) we remark that in the absence of the gauge field strength $F_{ij}$, equations (9) separate into two groups. One group involves the terms of $K$ of odd degree in the momenta and the other involves only the terms of $K$ of even degree in the momenta [5]. Here, the presence of the gauge field strength $F_{ij}$ mixes up the terms of $K$ of even and odd degrees in the momenta and consequently the system of coupled equations (9) is more intricate. Moreover, only the last equation (9e) for the leading order term $K^i_2 ... i_p$ defines the component of a SK tensor of rank $p$. Note that again, in the absence of the field strength $F_{ij}$, equation (9d) also defines a SK tensor $K^{i_1 ... i_{p-1}}_2$, but here this is not the case.

### 3 Role of KY tensors

The next most simple objects that can be studied in connection with the symmetries of a manifold $(\mathcal{M}, g)$ after SK tensors are KY tensors [9]. Their physical utility remained unclear until Floyd [12] and Penrose [13] showed that a SK tensor of rank 2 of the 4-dimensional Kerr-Newman space-time admits a certain square root which defines a KY tensor. On the other hand it was realized [2] that a KY tensor generates additional supercharges in the dynamics of pseudo-classical spinning particles. In the quantum framework it was shown [14] that KY tensors produce conserved non-standard Dirac type operators which commute with the standard one.

A differential $p$-form $f$ is called a KY tensor if its covariant derivative $f_{\mu_1 ... \mu_p; \lambda}$ is totally antisymmetric. Equivalently, a tensor is called a KY tensor of rank $p$ if it is totally antisymmetric and satisfies the equation

$$f_{\mu_1 ... (\mu_p; \lambda)} = 0.$$  \hspace{1cm} (10)

These two generalizations SK (9e) and KY (10) of the Killing vectors could be related. Let $f_{\mu_1 ... \mu_p}$ be a KY tensor, then the tensor field

$$K_{2\mu\nu} = f_{\mu_2 ... \mu_p} f^{\mu_2 ... \mu_p}_{\mu\nu},$$  \hspace{1cm} (11)

is a SK tensor and one sometimes refers to it as the associated tensor with $f$.

A typical example is represented by the Euclidean Taub-NUT space which admits a RL vector whose components are SK tensors. These SK tensors
can be written as symmetrized products of KY tensors \[15\]. An important physical consequence of this possibility to decompose a SK tensor in terms of antisymmetric KY tensors is the absence of gravitational anomalies \[16, 17\].

The role of KY tensors in the motion of pseudo-classical spinning point particles was extended by Tanimoto \[10\] to include electromagnetic interactions. He obtained the condition of the electromagnetic field \(F_{\mu\nu}\) to maintain the non generic supersymmetry associated with a KY tensor \(f\) of rank \(p\). This condition can be expressed as

\[
F_{\nu[\mu_{p}\cdots\mu_{p-1}]} = 0 ,
\]

where the indices in square brackets are to be antisymmetrized. In particular, for a Killing vector \(K_{\nu}^1\) this condition is

\[
F_{\mu\nu}K_{\nu}^1 = 0 .
\]

(13)

In what follows we shall investigate the consequences of this condition for the series of constraints (9). To be more definite, we shall limit ourselves to the first three constraints (9a) - (9c) for \(n = 1\), assuming that the SK tensor \(K_{2\mu\nu}\) is associated with a KY tensor \(f_{\mu\nu}\)

\[
K_{2\mu\nu} = f_{\mu\lambda}f_{\nu}^\lambda .
\]

(14)

In this case, condition (12) for the electromagnetic field \(F_{\mu\nu}\) reads

\[
F_{\lambda[\mu}f_{\nu]}^\lambda = 0 .
\]

(15)

Using the antisymmetric properties of the KY tensors and electromagnetic field \(F_{\mu\nu}\), for a SK tensor of the form (14) in conjunction with condition (15) we get that in the l. h. s. of equation (9c) the term \(qF_{j^2} K_{i^2}^i\) vanishes. Consequently we have a relation between the odd terms \(K_1, K_3\) as in the absence of the electromagnetic field. The same argument applies to equation (9b) where condition (13) implies a relation only between even terms \(K_0, K_2\).

In conclusion, condition (12) which plays an important role in the construction of superinvariants for the motion of pseudo-classical spinning charged point particles proves to produce significant simplifications in the series of constraints (9) for the higher order integrals of motion.
4 Explicit examples

Let us illustrate these general considerations by some non trivial examples. In what follows we consider $\mathcal{M}$ to be a 3-dimensional Euclidean space $\mathbb{E}^3$ and in these circumstances it is more convenient to get rid of a difference between covariant and contravariant indices.

We are looking for constants of motion of the form

$$K = \frac{1}{2} K_{2ij} \Pi_i \Pi_j + K_{1i} \Pi_i + K_0.$$  \hspace{1cm} (16)

Recently the three-dimensional integrable systems were investigated assuming that there are additional integrals with quadratic dependence of momenta \[18, 19\]. In some cases the potentials are compatible with complete integrability and then one gets separation of variables. In the Hamiltonian-like constants the role of the metric is played by the SK tensor $K_{2ij}$ and the corresponding potential $K_0$ is related to the original one $V$ as in (9b). The reciprocal relation between the standard metric $g_{ij}$ and SK tensor $K_{2ij}$ has a geometrical interpretation \[20\]: it implies that if $K_{2ij}$ are the contravariant components of a SK tensor with respect to the inverse metric $g^{ij}$, then $g_{ij}$ must represent a SK tensor with respect to the inverse metric defined by $K_{2ij}$.

In what follows we investigate the constant of motion in a KC potential adding different types of electric and magnetic fields. To put in a concrete form, we consider the motion of a point charge $q$ of mass $M$ in the Coulomb potential $Q/r$ produce by a charge $Q$ when some external electric or magnetic fields are also present.

4.1 Constant electric field

In a first example we consider the electric charge $q$ moving in the Coulomb potential with a constant electric field $E$ present. Therefore in the potential $V(x^i)$ we include the Coulomb potential and $E \cdot r$ for the external electric field. The corresponding Hamiltonian is:

$$H = \frac{1}{2M} \Pi^2 + q \frac{Q}{r} - qE \cdot r,$$  \hspace{1cm} (17)

with $\Pi = M \dot{r}$ in spherical coordinates of $\mathbb{E}^3$.

As it is known the non relativistic KC problem admits two vector constants of motion, namely the angular momentum

$$L = r \times \Pi,$$  \hspace{1cm} (18)

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and the RL vector

$$K = \mathbf{\Pi} \times \mathbf{L} + M q Q \frac{\mathbf{r}}{r}.$$  \hfill (19)

The components $K_{2ij}$ of the constant of motion (16) are SK tensors, satisfying equation (9c) for $p = 2$. For the KC problem it proves adequate to choose for the SK tensor of rank 2 the simple form [21]

$$K_{2ij} = 2 \delta_{ij} \mathbf{n} \cdot \mathbf{r} - (n_i r_j + n_j n_i), \hfill (20)$$

written in spherical coordinates with $\mathbf{n}$ an arbitrary constant vector.

In the presence of a constant electric field $\mathbf{E}$ it proves convenient to choose $\mathbf{n}$ along $\mathbf{E}$ and we start to solve the hierarchy of constraint (9) with a solution of equation (9e) of the form (20) with $\mathbf{n} = \mathbf{E}$. Using this form for $K_{2ij}$ and the derivative of the potential $V$ corresponding to the Hamiltonian (17)

$$V_i = -\frac{qQ}{r^3} r_i - q E_i, \hfill (21)$$

we get from (9b) after a straightforward calculation

$$K_0 = \frac{M q Q}{r} \mathbf{E} \cdot \mathbf{r} - \frac{M q}{2} \mathbf{E} \cdot [\mathbf{r} \times (\mathbf{r} \times \mathbf{E})]. \hfill (22)$$

Concerning equation (9a) with the derivative of the potential (21), it is automatically satisfied with a vector $\mathbf{K}_1$ of the form

$$\mathbf{K}_1 = \mathbf{r} \times \mathbf{E}, \hfill (23)$$

modulo an arbitrary constant factor. This vector $\mathbf{K}_1$ contribute to a conserved quantity with a term proportional to the angular momentum $\mathbf{L}$ along the direction of the electric field $\mathbf{E}$.

In conclusion, when a uniform constant electric field is present, the KC system admits two constants of motion $\mathbf{L} \cdot \mathbf{E}$ and $\mathbf{C} \cdot \mathbf{E}$ where $\mathbf{C}$ is a generalization of the RL vector (19) (see also [22]):

$$\mathbf{C} = \mathbf{K} - \frac{M q}{2} \mathbf{r} \times (\mathbf{r} \times \mathbf{E}). \hfill (24)$$

For this system with two additional constants of motion, the separation of variables is possible as it was demonstrated in [18, 19]. Here we confine ourselves to mention that the separation of variables for this definite problem can be found in many textbooks. For example, L. D. Landau and E. M. Lifshitz [23] have given an expression in parabolic coordinates for the constant of motions for the KC problem plus a constant electric field.
4.2 Spherically symmetric magnetic field

In what follows we consider an external spherically symmetric magnetic field

\[ B = f(r)r, \]  

(25)

and the Coulomb potential acting on a electric charge \( q \).

Spherically symmetric magnetic fields appear in many interesting physical problems, the most notable configuration being the Dirac charge-monopole system. Many different formulations of the charge-monopole system have been well discussed in the literature, see e.g. [24].

Here we are interested in higher order constants of motion for spherically symmetric magnetic configurations involving a SK tensor of rank 2. Again we truncate the system (9) taking \( K_{p}^{i_{1} \ldots i_{p}} = 0 \) for all \( p \geq 3 \). For the beginning the scalar function \( f(r) \) is not fixed, its form will be determined from the hierarchy of constraints (9). For \( K_{2ij} \) we use the form (20) typical for the SC system with spherical symmetry. Equation (9c) for \( n = 1 \) with

\[ F_{ij} = \epsilon_{ijk}B_k = \epsilon_{ijk}r_kf(r), \]  

(26)

is

\[ K_{1(i,j)} = -q f(r)[(n \times r)(ir_j)], \]  

(27)

with \( n \) an arbitrary unit constant vector. It is easy to get for the vector \( K_1 \) the solution

\[ K_{1i} = q \left[ \int rf(r)dr \right] (n \times r)_i, \]  

(28)

and equation (9a) is obviously satisfied.

For \( K_0 \), equation (9b) can be solely solved making choice of a definite form for the function \( f(r) \)

\[ f(r) = \frac{g}{r^{5/2}}, \]  

(29)

with \( g \) a constant connected with the strength of the magnetic field. For this function \( f(r) \) the energy of the magnetic field diverges at \( r = 0 \) and \( r \to \infty \). Of course such a special magnetic field (25) could be prepared only in a finite region of space and all present considerations are limited to this space domain.

With this special form of the function \( f(r) \) we get

\[ K_0 = \left[ \frac{MqQ}{r} - \frac{2g^2q^2}{r} \right] (n \cdot r), \]  

(30)
and
\[ K_{1i} = -\frac{2gq}{r^{1/2}}(r \times n)_i. \]  

(31)

Collecting the terms \(K_0, K_{1i}, K_{2ij}\) the constant of motion (16) becomes
\[ K = n \cdot \left( K + \frac{2gq}{r^{1/2}}L - 2g^2q^2\frac{r^3}{r} \right), \]

(32)

with \(n\) an arbitrary constant unit vector and \(K, L\) given by (19), (18) respectively. In contrast with the pure Coulomb potential, the presence of a spherically magnetic field prevents the separate conservation of the angular momentum \(L\).

4.3 Magnetic field along a fixed direction

Let us consider a magnetic field directed along a fixed unit vector \(n\)
\[ \mathbf{B} = B(\mathbf{r} \cdot \mathbf{n}) \mathbf{n}, \]

(33)

where, for the beginning, \(B(\mathbf{r} \cdot \mathbf{n})\) is an arbitrary function.

As in the previous example we truncate the hierarchy of constraints (9) for \(p \geq 3\). Using (33) to evaluate the electromagnetic field strength \(F_{ij}\) and choosing for \(K_{2ij}\) the form (20) with the arbitrary vector \(n\) along the magnetic field, equation (9e) for \(p = 1\) reads
\[ K_{1(i,j)} = -qB(\mathbf{r} \times \mathbf{n})_{(i,n_j)}. \]  

(34)

\(K_{1i}\), solution of this equation, must satisfy also equation (9a) and after some straightforward calculations we get
\[ K_{1i} = q \left[ \int r B(\mathbf{r} \cdot \mathbf{n})d(\mathbf{r} \cdot \mathbf{n}) \right] (\mathbf{r} \times \mathbf{n})_i. \]  

(35)

Equation (9b) for \(K_0\) proves to be solvable for a particular form of the magnetic field
\[ \mathbf{B} = \frac{\alpha}{\sqrt{\alpha} \mathbf{r} \cdot \mathbf{n} + \beta} \mathbf{n}, \]  

(36)

with \(\alpha\) and \(\beta\) two arbitrary constants.
Finally we get for $K_0$ and $K_{1i}$

$$K_0 = \frac{MqQ}{r} (r \cdot n) + \alpha q^2 (r \times n)^2,$$

$$K_{1i} = -2q \sqrt{\alpha r \cdot n + \beta} (r \times n)_i.$$  \hfill (37)

With these solutions, the constant of motion (16) for this configuration of the magnetic field superposed on the Coulomb potential becomes:

$$K = n \cdot \left[ K + 2q \sqrt{\alpha r \cdot n + \beta} \ L \right] + \alpha q^2 (r \times n)^2.$$  \hfill (39)

As in the previous example the angular momentum $L$ is no longer conserved, forming part of the constant of motion $K$ (39).

\section{5 Concluding remarks}

In this paper we presented the hierarchy of constraints for integrals of motion in a gauge covariant Hamiltonian framework in the presence of Abelian gauge fields and scalar potentials. The formalism is valuable for the construction of the higher order constants of motions and deserves further studies.

An obvious extension is represented by the non-Abelian dynamics using the appropriate Poisson brackets [11, 3]. On the other hand it was observed that the RL-type vector plays a role in the non-linear supersymmetry of various systems: fermion-monopole system [25, 26], generalization of the Landau problem for non-relativistic electron coupled to electric and magnetic fields that produce a 1D crystal [27], reflectionless Poschl-Teller system in the context of the AdS/CFT holography and Aharonov-Bohm effect [28].

In order to exemplify the general considerations we worked out some simple examples in an Euclidean 3-dimensional space and restricted to SK tensors of rank 2. In a forthcoming paper [29] we shall illustrate the covariant Hamiltonian dynamics with more elaborate examples working on a $N$-dimensional curved space and involving higher ranks of SK tensors [7, 8].

We concentrated on classical analysis and in principle there are no obstacles in passing to quantum mechanics. The results and their derivations are valid quantum mechanically, provided care is taken with the order of operators.
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