Manipulating the Speed of Sound in a Two-Component Bose-Einstein Condensate

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Abstract

We consider a two-component weakly interacting Bose-Einstein condensate in the presence of an external field which couples the two components. We express the Hamiltonian in terms of the energy eigenstates of the single-body part of the Hamiltonian. These eigenstates are the atomic dressed states of quantum optics. When the energy difference between the two dressed states is much larger than the mean-field interactions, two-body interactions in the dressed state basis that do not conserve the number of atoms in each of the two dressed states are highly suppressed. The two-body interactions then take on a simplified form in the dressed basis with effective coupling constants that depend on the intensity and frequency of the external field. This implies that the chemical potential as well as the quasiparticle spectrum may be controlled experimentally in a simple manner. We demonstrate this by showing that one may achieve significant variations in the speed of sound in the condensate, a quantity which has been measured experimentally.

I. INTRODUCTION

The recent experimental realization of trapped Bose-Einstein condensates (BEC) with internal degrees of freedom corresponding to different hyperfine states [1] [2] has sparked much theoretical and experimental study of the properties of multi-component condensates. Multi-component condensates exhibit a rich variety of new phenomena not present in condensates with a scalar order parameter. For condensates with a vectorial order parameter, such as the $F = 1$ hyperfine multiplet of $^{23}Na$, there have been predictions of spin waves, instability of vortices with more than one unit of circulation, coreless vortices, dynamic spin localization [3] [4] [5] and the observation of spin domains [6]. For two-component condensates, such as $^{87}Rb$, there have been extensive theoretical and experimental investigations of the dynamics of the condensates in the presence of external fields which couple
the components. Examples include Ramsey fringes [7], non-linear Josephson oscillations [8], instabilities in the quasiparticle spectrum [9], the elimination of the mean-field shift in the energy [10], control of the spatial dependence of the two components [11], and collapse and revival of Rabi oscillations [12].

In this paper, a two-component homogenous weakly interacting BEC at zero temperature in the presence of an external field which couples the two states is investigated. The physical system is similar to that discussed in [9]. In quantum optics, the dressed states, which are the energy eigenstates of a two level atom interacting with an external radiation field, form a convenient basis for many problems [13]. More specifically, if one has a two-level system with states $|a\rangle$ and $|b\rangle$ governed by the Hamiltonian,

$$H = \frac{\hbar}{2} \begin{pmatrix} \delta & 2\Omega_R \\ 2\Omega_R & -\delta \end{pmatrix},$$

then the dressed states, $|c\rangle$ and $|d\rangle$, are simply the energy eigenstates of $H$ with eigenvalues $\pm \frac{\hbar}{2} \omega_{cd}$. The dressed states are a superposition of the states $|a\rangle$ and $|b\rangle$ with amplitudes determined by $\delta/\Omega_R$. We explore the use of a dressed state basis for describing an interacting two-component condensate.

When the two-body interactions in the second-quantized Hamiltonian are rewritten in terms of a dressed state basis, one finds that there are terms that conserve the number of atoms in each of the two dressed states and terms that change the number of atoms in each of the dressed states. When the energy difference between the dressed states is much larger than the single particle kinetic energies and the mean-field energy, the Hamiltonian simplifies since the terms that change the number of atoms in each of the dressed states may be neglected. In this limit, the ground state of the condensate consists of atoms in one of the dressed states only, and the calculation of the excited states becomes trivial. In this case, there are two branches to the spectrum of elementary excitations. One branch has the standard Bogoliubov dispersion relation while the other branch corresponds to single particle excitations. In addition, the speed of sound is an explicit function of $\delta/\Omega_R$. This
result indicates that the spectrum of collective excitations of the condensate depend on the coherence between the internal states of the atoms in a manner which can be experimentally controlled.

Before proceeding, it is helpful to review the hydrodynamics of a single component BEC at zero temperature. At zero temperature, the thermal component of the Bose gas is absent and the quantum depletion is negligible if the bosons are weakly interacting. In this case, the wave function (or order parameter) for the condensate, \( \phi(r, t) = \sqrt{n(r, t)} e^{iS(r, t)} \), obeys the time-dependent Gross-Pitaevskii equation \([14]\) which may be written as two coupled equations for the density, \( n(r, t) \), and velocity, \( \mathbf{v}(r, t) = \frac{\hbar}{m} \nabla S(r, t) \) \([13]\),

\[
\frac{\partial}{\partial t} n + \nabla \cdot (\mathbf{v} n) = 0 \tag{1}
\]

and

\[
m \frac{\partial}{\partial t} \mathbf{v} + \nabla \left( V(r) + Un - \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} + \frac{m|\mathbf{v}|^2}{2} \right) = 0. \tag{2}
\]

Here \( U = \frac{4\pi \hbar^2 a}{m} \), \( a \) is the s-wave scattering length which characterizes the two-body interactions between atoms, \( m \) is the atomic mass, and \( V(r) \) is an external potential. In general, the quantum pressure, \( \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} \), may be neglected in comparison to the mean-field interaction. For a static condensate ground state, this corresponds to the Thomas-Fermi limit for the density,

\[
n_o(r) = U^{-1}(\mu - V(r)), \tag{3}
\]

where \( \mu \) is the chemical potential. By considering small fluctuations, \( \delta n(r, t) \), about the static ground state density \( n_o(r) \), one may derive a linear wave equation for \( \delta n(r, t) \) from Eqs. (1-2),

\[
\partial_t^2 \delta n(r, t) = \nabla \cdot \left[ u^2(r) \nabla \delta n(r, t) \right], \tag{4}
\]

where \( u^2(r) = \frac{\partial P}{\partial n} n_o(r) \) is the local speed of sound. The sound speed may also be obtained from the relation, \( u^2(r) = \frac{1}{m} \frac{\partial P}{\partial n} \), where \( P \) is the pressure of the ground state of the condensate. The derivation of Eq. (4) neglects the quantum pressure term in Eq. (2).
$V(r) \equiv 0$, the condensate ground state is spatially homogenous so that $n_o(r) = n_o$. In this case, the quantum pressure may be neglected when the variations in the density are small over distances on the order of the healing length, $\xi = 1/\sqrt{8\pi a n_o}$, or equivalently, the elementary excitations have momenta satisfying $\hbar k \ll \mu$ where $k$ is the wave vector for the excitation and $u = u(r)$.

For a homogenous BEC (i.e. $V(r) \equiv 0$), the energies of the elementary excitations form a continuous spectrum. The dispersion relation for the long wavelength collective excitations of the condensate have the form $\omega = u k \ll \mu/\hbar$. These excitations correspond to phonons. The speed of sound in a homogenous BEC (sometimes called Bogoliubov or zero sound) was first derived by Bogoliubov [16] using a microscopic theory of weakly interacting bosons,

$$u = \sqrt{\frac{4\pi \hbar^2 a n_o}{m^2}}.$$  \hspace{1cm}(5)

The Bogoliubov theory is used in this paper to derive the excitation spectra for a two component condensate.

Bogoliubov sound in a condensate of sodium atoms has been experimentally studied by M. R. Andrews and coworkers in a highly elongated cigar shaped trap [17]. Note that $V(r) \neq 0$ in this case but along the long axis of the trap, the condensate can be treated as being locally homogenous. In this experiment, the repulsive dipole force of a focused blue-detuned laser beam was used to create a localized density perturbation at the center of the trap. The double peaked perturbation was subsequently imaged as it propagated along the long axis of the trap. When the density of atoms in the trap was varied, the measured sound speeds showed good agreement with Eq. (5) where $n_o$ refers to the density at the center of the trap divided by 2. The extra factor of a 1/2 has a simple interpretation. Since the sound propagation is confined to the long axis of the trap, the condensate can be treated as an effective one-dimensional system. In this case the effective density is given by averaging the condensate density over the directions transverse to the long axis which in the Thomas-Fermi limit is just the central density divided by 2 [18].

This experiment has been analyzed theoretically by several authors [18] [19] in the
Thomas-Fermi limit. In a trapped condensate, sound waves can propagate provided that the excitations satisfy both $\hbar k \ll \mu(r)$ and $kL \gg 1$ where $L$ is the size of the condensate in the direction of propagation. The latter condition corresponds to wavelengths much less than the condensate size so that the variations in $n_o(r)$ over distances on the order of the wavelength are negligible. Consequently, the condensate can be treated as locally homogeneous. As such, the results derived for a homogenous BEC using Bogoliubov theory may be used for trapped condensates with the substitution $n_o \rightarrow \bar{n}_o(r)$ where the bar denotes the possible averaging over transverse dimensions for anisotropic condensates.

In the following section a second-quantized Hamiltonian for a two-component BEC is given in the "bare" atomic basis and the Hamiltonian in the dressed basis is derived. In section III, the condensate ground state and elementary excitations in the dressed basis are determined. Finally, in section IV, the results are discussed including justification of the physical approximations used.

II. PHYSICAL MODEL

We consider a collection of $N$ bosonic atoms that have internal states $|a\rangle$ and $|b\rangle$ with energies $\hbar \omega_o/2$ and $-\hbar \omega_o/2$, respectively. There is a spatially uniform radiation field with frequency $\omega_e$ which couples the two internal states with a Rabi frequency $\Omega_R$. The atom field detuning is denoted by $\delta = \omega_o - \omega_e$. The atoms in states $|a\rangle$ and $|b\rangle$ may also be subject to an external trapping potential $V_a(r)$ and $V_b(r)$, respectively. Furthermore, the atoms interact via elastic two-body collisions through the interaction potentials $V_{ij}(r - r')$ for $i,j \in \{a,b\}$. The many-body Hamiltonian operator describing the system is given by,

$$\hat{H} = \hat{H}_{\text{atom}} + \hat{H}_{\text{coll}}$$

$$\hat{H}_{\text{atom}} = \int d^3r \left\{ \hat{\Psi}_a^\dagger(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_a(r) + \frac{\hbar \delta}{2} \right] \hat{\Psi}_a(r) + \hat{\Psi}_b^\dagger(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_b(r) - \frac{\hbar \delta}{2} \right] \hat{\Psi}_b(r) \right\}$$

$$+ \hbar \Omega_R \left\{ \hat{\Psi}_a^\dagger(r) \hat{\Psi}_b(r) + \hat{\Psi}_b^\dagger(r) \hat{\Psi}_a(r) \right\}$$

$$\hat{H}_{\text{coll}} = \frac{1}{2} \int d^3rd^3r' \left\{ \hat{\Psi}_a^\dagger(r)\hat{\Psi}_a^\dagger(r')V_{aa}(r - r')\hat{\Psi}_a(r')\hat{\Psi}_a(r) + \hat{\Psi}_b^\dagger(r)\hat{\Psi}_b^\dagger(r')V_{bb}(r - r')\hat{\Psi}_b(r')\hat{\Psi}_b(r) \right\}$$
\[
+2\hat{\Psi}_a^\dagger(r)\hat{\Psi}_b^\dagger(r')V_{ab}(r-r')\hat{\Psi}_a(r)\hat{\Psi}_b(r') \}. \tag{6c}
\]

Here, \(\hat{H}_{\text{atom}}\) is the single particle Hamiltonian and \(\hat{H}_{\text{coll}}\) represents two-body collisions.

The operators \(\hat{\Psi}_i(r)\) and \(\hat{\Psi}_i^\dagger(r)\) are bosonic annihilation and creation operators for an atom in state \(i = \{a, b\}\) at position \(r\) which satisfy the commutation relations \([\hat{\Psi}_i(r), \hat{\Psi}_j^\dagger(r')] = \delta_{ij} \delta(r-r')\) and \([\hat{\Psi}_i(r), \hat{\Psi}_j(r')] = 0\). The operators, \(\hat{\Psi}_i(r)\), have been written in a field interaction representation which is rotating at the frequency of the external field, \(\omega_e\),

\[
\hat{\Psi}_a(r) = \hat{\Psi}_a^{(N)}(r)e^{i\omega_e t/2}; \tag{7a}
\]
\[
\hat{\Psi}_b(r) = \hat{\Psi}_b^{(N)}(r)e^{-i\omega_e t/2}; \tag{7b}
\]

where \(\hat{\Psi}_i^{(N)}(r)\) are the field operators in the normal representation. This explains the appearance of the detuning in Eq. (3b).

The two-body interaction between atoms, \(V_{ij}(r-r')\), depends on the internal states of the atoms. For a dilute gas such that \(\bar{n}|a_{ij}|^3 \ll 1\) where \(\bar{n}\) is the average density and \(a_{ij}\) is the s-wave scattering length between atoms in states \(i\) and \(j\), the interaction may be written as \(V_{ij}(r-r') = U_{ij}\delta(r-r')\) where \(U_{ij} = \frac{4\pi\hbar^2 a_{ij}}{m}\). It is assumed that \(a_{ij} > 0\) corresponding to repulsive interactions.

For the remainder of this section and sections III and IV, we consider only the case when \(V_i(r) = 0\). One can expand the field operators in a basis of single particle momentum eigenstates,

\[
\hat{\Psi}_a(r) = \frac{1}{\sqrt{V}} \sum_p \alpha_{ap} e^{ip\cdot r/\hbar}, \tag{8a}
\]
\[
\hat{\Psi}_b(r) = \frac{1}{\sqrt{V}} \sum_p \alpha_{bp} e^{ip\cdot r/\hbar}, \tag{8b}
\]

where \(V\) is the quantization volume and \([\alpha_{ip}, \alpha_{jp}^\dagger] = \delta_{ij} \delta_{p,p'}\) and \([\alpha_{ip}, \alpha_{jp}^\dagger] = 0\). In this basis, the Hamiltonian becomes

\[
\hat{H}_{\text{atom}} = \sum_p \left\{ \left( \frac{\hbar^2}{2m} + \frac{\hbar\delta}{2} \right) \alpha_{ap}^\dagger \alpha_{ap} + \left( \frac{\hbar^2}{2m} - \frac{\hbar\delta}{2} \right) \alpha_{bp}^\dagger \alpha_{bp} + \hbar\Omega_R \left( \alpha_{bp}^\dagger \alpha_{ap} + \alpha_{ap}^\dagger \alpha_{bp} \right) \right\}, \tag{9a}
\]
\[
\hat{H}_{\text{coll}} = \frac{1}{2V} \sum_{p_1+p_2=p_3+p_4} \left\{ U_{aa} \alpha_{ap_1}^\dagger \alpha_{ap_2}^\dagger \alpha_{ap_3} \alpha_{ap_4} + U_{bb} \alpha_{bp_1}^\dagger \alpha_{bp_2}^\dagger \alpha_{bp_3} \alpha_{bp_4} + 2U_{ab} \alpha_{ap_1}^\dagger \alpha_{bp_2}^\dagger \alpha_{ap_3} \alpha_{bp_4} \right\}, \tag{9b}
\]
At this point it is advantageous to introduce the grand canonical Hamiltonian, \( \hat{K} = \hat{H} - \mu \hat{N} \) where \( \mu \) is the chemical potential and \( \hat{N} = \sum_{p} (\alpha_{ap}^{\dagger} \alpha_{ap} + \alpha_{bp}^{\dagger} \alpha_{bp}) \) is the number operator. The motivation for using \( \hat{K} \) is that the Bogoliubov prescription which will be used later, results in a Hamiltonian that does not conserve the number of particles. The chemical potential serves as a Lagrange multiplier which allows one to impose the constraint \( \langle \hat{N} \rangle = N \) [21]. On the other hand, since \( \mu \) is the energy of an atom in the condensate ground state, \( \hat{K} \) corresponds to a representation in which all single particle energies are measured relative to the condensate.

Dressed operators \( c_{p} \) and \( d_{p} \), which correspond to the atomic dressed states \( |c\rangle = \cos \theta |a\rangle + \sin \theta |b\rangle \) and \( |d\rangle = -\sin \theta |a\rangle + \cos \theta |b\rangle \), may be introduced which diagonalize \( \hat{H}_{\text{atom}} - \mu \hat{N} \). They are related to the operators \( \alpha_{ap} \) and \( \alpha_{bp} \) by

\[
\begin{pmatrix}
    c_{p} \\
    d_{p}
\end{pmatrix}
= \begin{pmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
    \alpha_{ap} \\
    \alpha_{bp}
\end{pmatrix},
\]

where the dressed state angle \( \theta \) is given by \( \cos \theta = \frac{1}{\sqrt{2}} \left( 1 + \frac{\delta}{\sqrt{\delta^2 + 4\Omega_{R}^2}} \right)^{1/2} \) or \( \tan 2\theta = \frac{2\Omega_{R}}{\delta} \). Note that \( \theta \) is a measure of the relative mixing of the atomic states \( |a\rangle \) and \( |b\rangle \) in the dressed states so that, for example, \( \theta = \pi/4 \) (\( \Omega_{R} \gg |\delta| \)) corresponds to dressed states which are equal superpositions of the states \( |a\rangle \) and \( |b\rangle \). In contrast, for \( \theta = 0 \) (\( \delta > 0 \) and \( \Omega_{R} \ll |\delta| \)) one has \( |c\rangle = |a\rangle \) and for \( \theta = \pi/2 \) (\( \delta < 0 \) and \( \Omega_{R} \ll |\delta| \)) one has \( |c\rangle = |b\rangle \) so that in these limits, the dressed states may be identified with the atomic states. It is easy to show that \([c_{p}, c_{p}^{\dagger}] = [d_{p}, d_{p}^{\dagger}] = \delta_{pp'} \) and \([c_{p}, c_{p'}] = [d_{p}, d_{p'}] = [d_{p}, c_{p'}] = [d_{p}, c_{p'}] = 0 \).

In the dressed basis \( \hat{H}_{\text{atom}} - \mu \hat{N} \) has the form

\[
\hat{H}_{\text{atom}} - \mu \hat{N} = \sum_{p} \left\{ \left( \frac{p^2}{2m} + \frac{\hbar \omega_{cd}}{2} - \mu \right) c_{p}^{\dagger} c_{p} + \left( \frac{p^2}{2m} - \frac{\hbar \omega_{cd}}{2} - \mu \right) d_{p}^{\dagger} d_{p} \right\};
\]

where \( \hbar \sqrt{\delta^2 + 4\Omega_{R}^2} \) is the energy difference between the two dressed states. However, in the dressed basis \( \hat{H}_{\text{coll}} \) has a significantly more complicated form,

\[
\hat{H}_{\text{coll}} = \frac{1}{2V} \sum_{p_1+p_2+p_3+p_4} \left\{ U_1 c_{p_1}^{\dagger} c_{p_2}^\dagger c_{p_3} c_{p_4} + U_2 d_{p_1}^{\dagger} d_{p_2}^\dagger d_{p_3} d_{p_4} + U_3 c_{p_1}^{\dagger} d_{p_2}^\dagger c_{p_3} d_{p_4} \right\}.
\]
\[ + U_4 \left( c_{p_1}^\dagger c_{p_2}^\dagger d_{p_3} d_{p_4} + d_{p_1}^\dagger d_{p_2}^\dagger c_{p_3} c_{p_4} \right) + U_5 \left( c_{p_1}^\dagger c_{p_2}^\dagger c_{p_3} d_{p_4} + d_{p_1}^\dagger c_{p_2}^\dagger c_{p_3} c_{p_4} \right) + U_6 \left( c_{p_1}^\dagger d_{p_2}^\dagger d_{p_4} + d_{p_1}^\dagger d_{p_2}^\dagger d_{p_3} c_{p_4} \right) \].

The collisional couplings \( U_i \) are given by,

\[ U_1 = U_{aa} \cos^4 \theta + U_{bb} \sin^4 \theta + \frac{1}{2} U_{ab} \sin^2 2\theta; \quad (13a) \]
\[ U_2 = U_{aa} \sin^4 \theta + U_{bb} \cos^4 \theta + \frac{1}{2} U_{ab} \sin^2 2\theta; \quad (13b) \]
\[ U_3 = (U_{aa} + U_{bb}) \sin^2 2\theta + 2U_{ab} \cos^2 2\theta; \quad (13c) \]
\[ U_4 = \frac{1}{4} (U_{aa} + U_{bb} - 2U_{ab}) \sin^2 2\theta; \quad (13d) \]
\[ U_5 = \sin 2\theta (U_{bb} \sin^2 \theta - U_{aa} \cos^2 \theta + U_{ab} \cos 2\theta); \quad (13e) \]
\[ U_6 = \sin 2\theta (U_{bb} \cos^2 \theta - U_{aa} \sin^2 \theta - U_{ab} \cos 2\theta). \quad (13f) \]

Even though the atom-radiation field interaction term has been eliminated, it would seem that nothing is gained by using the dressed basis because of the increased complexity of \( \hat{H}_{\text{coll}} \). However, one may make a transformation to an interaction representation with respect to the internal dressed state energies by defining new slowly varying operators \( \bar{c}_p \) and \( \bar{d}_p \) which are related to the normal dressed operators by,

\[ c_p = \bar{c}_p e^{-i\omega_{cd} t/2}; \quad (14a) \]
\[ d_p = \bar{d}_p e^{+i\omega_{cd} t/2}. \quad (14b) \]

The time evolution of \( \bar{c}_p \) and \( \bar{d}_p \) is governed by the Hamiltonian, \( \hat{K}_s \), given by

\[
\hat{K}_s = \sum_{p} \left\{ \left( \frac{\hat{p}^2}{2m} - \mu \right) \bar{c}_p^\dagger \bar{c}_p + \left( \frac{\hat{p}^2}{2m} - \mu \right) \bar{d}_p^\dagger \bar{d}_p \right\} + \\
\frac{1}{2V} \sum_{p_1+p_2=p_3+p_4} \left\{ U_1 \bar{c}_{p_1}^\dagger \bar{c}_{p_2}^\dagger \bar{c}_{p_3} \bar{c}_{p_4} + U_2 \bar{d}_{p_1}^\dagger \bar{d}_{p_2}^\dagger \bar{d}_{p_3} \bar{d}_{p_4} + U_3 \bar{c}_{p_1}^\dagger \bar{d}_{p_2}^\dagger \bar{c}_{p_3} \bar{d}_{p_4} + U_4 \left( e^{2i\omega_{cd} t} \bar{c}_{p_1}^\dagger \bar{c}_{p_2}^\dagger \bar{d}_{p_3} \bar{d}_{p_4} + e^{-2i\omega_{cd} t} \bar{d}_{p_1}^\dagger \bar{d}_{p_2}^\dagger \bar{c}_{p_3} \bar{c}_{p_4} \right) + U_5 \left( e^{i\omega_{cd} t} \bar{c}_{p_1}^\dagger \bar{c}_{p_2}^\dagger \bar{c}_{p_3} \bar{d}_{p_4} + e^{-i\omega_{cd} t} \bar{d}_{p_1}^\dagger \bar{d}_{p_2}^\dagger \bar{c}_{p_3} \bar{c}_{p_4} \right) + U_6 \left( e^{i\omega_{cd} t} \bar{c}_{p_1}^\dagger \bar{d}_{p_2}^\dagger \bar{d}_{p_3} \bar{c}_{p_4} + e^{-i\omega_{cd} t} \bar{d}_{p_1}^\dagger \bar{c}_{p_2}^\dagger \bar{c}_{p_3} \bar{d}_{p_4} \right) \right\}. \quad (15) \]

The terms proportional to \( U_4, U_5, \) and \( U_6 \) oscillate with a frequency that is a multiple of \( \omega_{cd} \). The time scales over which the operators \( \bar{c}_p \) and \( \bar{d}_p \) evolve are governed by the single particle
kinetic energy, \( \frac{p^2}{2m} \), and the mean-field interaction energies, \( n_o U_i \), which are on the order of \( n_o U_{aa}, n_o U_{bb}, \) and \( n_o U_{ab} \) where \( n_o \) is the density of atoms in the condensate. Consequently, when the dressed state energy splitting is sufficiently large so that

\[
\hbar \omega_{cd} \gg \frac{p^2}{2m}, n_o U_{aa}, n_o U_{bb}, n_o U_{ab},
\]

the oscillatory terms in \( \hat{K}_s \) will undergo many oscillations before \( \bar{c}_p \) and \( \bar{d}_p \) will have changed significantly. One may therefore consider the Heisenberg equation of motion for \( \bar{c}_p \) which is averaged over a time interval \( T = 2\pi/\omega_{cd} \),

\[
\frac{1}{T} \int_t^{t+T} \frac{d\bar{c}_p}{dt} dt = \frac{1}{T} \int_t^{t+T} \frac{1}{i\hbar} [\bar{c}_p, \hat{K}_s] dt.
\]

This allows one to define a coarse-grained derivative,

\[
\frac{\delta \bar{c}_p}{\delta T} = \frac{1}{T} \int_t^{t+T} \frac{d\bar{c}_p}{dt} dt = \frac{\bar{c}_p(t + T) - \bar{c}_p(t)}{T},
\]

When condition (16) is satisfied, the operators in \( \hat{K}_s \) may be treated as constant over the period \( T \) so that

\[
\frac{\delta \bar{c}_p}{\delta T} = \frac{1}{i\hbar} [\bar{c}_p, \hat{K}_{sr}] \]

where \( \hat{K}_{sr} \) is the "resonant" Hamiltonian,

\[
\hat{K}_{sr} = \sum_p \left\{ \left( \frac{p^2}{2m} - \mu \right) \bar{c}_p \bar{c}_p + \left( \frac{p^2}{2m} - \mu \right) \bar{d}_p \bar{d}_p \right\} + \frac{1}{2V} \sum_{p_1 + p_2 = p_3 + p_4} \left\{ U_{1} \bar{c}_{p_1} \bar{c}_{p_2} \bar{c}_{p_3} \bar{c}_{p_4} + U_{2} \bar{d}_{p_1} \bar{d}_{p_2} \bar{d}_{p_3} \bar{d}_{p_4} + U_{3} \bar{c}_{p_1} \bar{d}_{p_2} \bar{c}_{p_3} \bar{d}_{p_4} \right\}.
\]

Since \( \bar{c}_p \) changes very little in a time \( T \) one may identify the coarse grained derivative with the actual derivative,

\[
\frac{\delta \bar{c}_p}{\delta T} \approx \frac{d\bar{c}_p}{dt}.
\]

The same results hold for \( \bar{d}_p \). As a result, one may consider the time evolution of \( \bar{c}_p \) and \( \bar{d}_p \) to be governed by \( \hat{K}_{sr} \).
Neglecting the oscillatory terms in $\hat{K}_s$ is analogous to the rotating wave approximation in quantum optics [22] [23] and has a simple interpretation in terms of energy conservation. As an example, consider the term $\bar{c}^\dagger_{p_1} \bar{c}^\dagger_{p_2} \bar{d}_{p_3} \bar{d}_{p_4}$ in $\hat{K}_s$. This term corresponds to the destruction of two atoms in the dressed state $|d\rangle$ and the creation of two atoms in state $|c\rangle$ which requires an amount of energy equal to $2\hbar\omega_{cd}$. However when condition (16) is satisfied, the kinetic energy and mean-field energy of the atoms is insufficient to overcome the energy difference of $2\hbar\omega_{cd}$. Consequently, the process described by $\bar{c}^\dagger_{p_1} \bar{c}^\dagger_{p_2} \bar{d}_{p_3} \bar{d}_{p_4}$ can only occur over time scales consistent with the uncertainty relation $\Delta t \Delta E \approx \hbar$, which in this case is $\Delta t \approx 1/2\omega_{cd}$.

However, by coarse graining the time evolution of the operators $\bar{c}_p$ and $\bar{d}_p$, one ignores processes that occur on time scales less than $T > \Delta t$.

The slowly varying operators were useful for showing that several of the terms $\hat{H}_{\text{coll}}$ could be neglected. However, for the remainder of the paper, all calculations will be carried out using the operators $c_p$ and $d_p$ in the normal Heisenberg representation with the "resonant" Hamiltonian $\hat{K}_R$ given by,

$$\hat{K}_R = \sum_p \left\{ \left( \frac{p^2}{2m} + \frac{\hbar \omega_{cd}}{2} - \mu \right) c^\dagger_p c_p + \left( \frac{p^2}{2m} - \frac{\hbar \omega_{cd}}{2} - \mu \right) d^\dagger_p d_p \right\} + \frac{1}{2V} \sum_{p_1 + p_2 = p_3 + p_4} \left\{ U_1 c^\dagger_{p_1} c^\dagger_{p_2} c_{p_3} c_{p_4} + U_2 d^\dagger_{p_1} d^\dagger_{p_2} d_{p_3} d_{p_4} + U_3 c^\dagger_{p_1} d^\dagger_{p_2} c_{p_3} d_{p_4} \right\}. \tag{22}$$

Notice that Eq. (22) resembles Eq. (9a-9b) except that the external field coupling term is absent. Equation (22) is the central result of this paper. The coupling constants $U_1$, $U_2$, and $U_3$ in Eq. (22) are explicit functions of the dressed state angle. Consequently, the collective properties of the condensate which depend on the two-body interactions will be a function of $\theta$. In the following section, the condensate ground state and elementary excitations above the ground states are calculated using Eq. (22).

### III. CONDENSATE AND EXCITED STATES

The c-number equations for the wave function of the condensate ground state may be derived by dividing the field operators $\hat{\Psi}_c(r)$ and $\hat{\Psi}_d(r)$, in the dressed basis, into a part with
a non-zero expectation value and a fluctuating part, \( \delta \hat{\Psi}_i(r) \), that has a vanishing expectation value with respect to the ground state,

\[
\hat{\Psi}_c(r) = \frac{c_0}{\sqrt{V}} + \delta \hat{\Psi}_c(r); \quad (23a)
\]

\[
\hat{\Psi}_d(r) = \frac{d_0}{\sqrt{V}} + \delta \hat{\Psi}_d(r). \quad (23b)
\]

The wave functions for the condensate ground state are \( \phi_c \equiv \langle \hat{\Psi}_c \rangle = \langle c_0 \rangle \sqrt{V} + \delta \hat{\Psi}_c \) and \( \phi_d \equiv \langle \hat{\Psi}_d \rangle = \langle d_0 \rangle \sqrt{V} \). A pair of coupled equations for \( \phi_c, \phi_d, \) and \( \mu \) may be derived from the expectation value of the Heisenberg equations of motion and using the fact that \( \dot{\phi}_c = \dot{\phi}_d = 0 \) with respect to \( \hat{K}_R [21] \),

\[
\mu \phi_c = \left( \frac{\hbar}{2} \omega_{cd} + U_1 |\phi_c|^2 + \frac{1}{2} U_3 |\phi_d|^2 \right) \phi_c; \quad (24a)
\]

\[
\mu \phi_d = \left( -\frac{\hbar}{2} \omega_{cd} + U_2 |\phi_d|^2 + \frac{1}{2} U_3 |\phi_c|^2 \right) \phi_d; \quad (24b)
\]

with the constraint \( n_o \equiv \frac{N}{V} = |\phi_c|^2 + |\phi_d|^2 \). Equations (24a-24b) may also be derived by requiring that the energy of the ground state be an extremum with respect to variations in \( \phi_c \) and \( \phi_d \).

Due to the non-zero mean-field interactions, there are three possible solutions to Eqs. (24a-24b):

(i) \( |\phi_c|^2 = n_o, \phi_d = 0 \), and \( \mu_c = \frac{\hbar}{2} \omega_{cd} + U_1 n_o \);

(ii) \( |\phi_d|^2 = n_o, \phi_c = 0 \), and \( \mu_d = -\frac{\hbar}{2} \omega_{cd} + U_2 n_o \);

(iii) \( |\phi_c|^2 = \frac{(U_2 - \frac{1}{2} U_3) n_o - \hbar \omega_{cd}}{U_1 + U_2 - U_3}, \) \( |\phi_d|^2 = \frac{(U_1 - \frac{1}{2} U_3) n_o + \hbar \omega_{cd}}{U_1 + U_2 - U_3}, \)

and \( \mu = \frac{1}{2} \left( 2U_1 U_2 - \frac{1}{2} U_3^2 \right) n_o + (U_2 - U_1) \hbar \omega_{cd} \).

The derivation of solution (iii) requires \( \phi_d \neq 0 \) and \( \phi_c \neq 0 \), but it is easy to show that in the limit that \( |\phi_d|^2 \to 0 \) or \( |\phi_c|^2 \to 0 \) one recovers solutions (i) or (ii), respectively. When (25) is satisfied, (iii) is an unphysical solution since either \( |\phi_c|^2 < 0 \) or \( |\phi_d|^2 < 0 \) and consequently, case (iii) may be ignored. The solutions (i) and (ii) correspond to atoms of the condensate being in one of the dressed states, \( |c \rangle \) or \( |d \rangle \). Case (ii) corresponds to the thermodynamic
ground state since it has the lowest ground state energy. However, in what follows the
excitation spectrum for both cases is calculated.

The Bogoliubov approximation \[16\] may applied to $\hat{K}_{R}$ for the two condensate solutions (i) and (ii) to obtain a Hamiltonian which is quadratic in the operators for the excited states (those states with $p \neq 0$). For case (i), one obtains the linearized Hamiltonian,

$$
\hat{K}_{Rc} = K_{oc} + \sum_{p \neq 0} \left\{ \left( \frac{p^2}{2m} + U_{1}n_{o} \right) c^{\dagger}_{p}c_{p} + \frac{1}{2} U_{1}n_{o} \left( c^{\dagger}_{p}c_{-p} + c_{p}c_{-p} \right) \right\} 
+ \sum_{p \neq 0} \left( \frac{p^2}{2m} - \hbar \omega_{ed} - (U_{1} - \frac{1}{2} U_{3})n_{o} \right) d^{\dagger}_{p}d_{p},
$$

(25)

by taking $c^{\dagger}_{0} \approx c_{0} = \sqrt{N_{o}}$ and $d^{\dagger}_{0} \approx d_{0} = 0$ and neglecting terms which are cubic and quartic in operators with $p \neq 0$. Here $K_{oc} = E_{oc} - \mu_{c}N_{o}$ and $E_{oc} = \frac{\hbar \omega_{ed}}{2} N_{o} + \frac{1}{2} U_{1} N_{o}^{2}$ is the total energy of the ground state.

One can see directly from Eq. (25) that the excitations which are in the atomic state that is orthogonal to the ground state, i.e. $|d\rangle$, are single particle excitations with an energy spectrum of

$$
\hbar \omega_{d}(p) = \frac{p^2}{2m} - \hbar \omega_{ed} - (U_{1} - \frac{1}{2} U_{3})n_{o}.
$$

(26)

This spectrum corresponds to the free atoms in state $|d\rangle$ with a mean field shift of $\frac{1}{2} U_{3}n_{o}$ due to the interactions with the condensate. The term $-(\hbar \omega_{ed} + U_{1}n_{o})$ in $\omega_{d}(p)$ results from the fact that the single particle energies are measured relative to $\mu_{c}$. The reason $\omega_{d}(p) < 0$ is due to the fact that ground state (ii) is the global minimum in the condensate energy.

For excitations in state $|c\rangle$, one can carry out a canonical transformation by defining quasiparticle operators, $C_{p}$, such that

$$
\sum_{p \neq 0} \left\{ \left( \frac{p^2}{2m} + U_{1}n_{o} \right) c^{\dagger}_{p}c_{p} + \frac{1}{2} U_{1}n_{o} \left( c^{\dagger}_{p}c_{-p} + c_{p}c_{-p} \right) \right\} = \sum_{p \neq 0} \hbar \omega_{c}(p) C^{\dagger}_{p}C_{p} + E_{vac,c};
$$

(27)

where $E_{vac,c}$ is the vacuum energy for the quasiparticle vacuum. The $C_{p}$ obey the bosonic commutation relations $[C_{p}, C^{\dagger}_{p'}] = \delta_{p,p'}$ and $[C_{p}, C_{p'}] = 0$ and can be expressed in terms $c_{p}$ and $c_{p}^{\dagger}$ as
\[ C_p = \cosh \varphi_p c_p + \sinh \varphi_p c_p^\dagger; \] (28)

The solutions for the quasiparticle energies and \( \varphi_p \) are easily found to be [21],

\[ \hbar \omega_c(p) = \sqrt{p^2/2m + 2U_1n_o}; \] (29a)

\[ \tanh 2\varphi_p = \frac{U_1n_o}{p^2/2m + U_1n_o}. \] (29b)

The long-wavelength quasiparticle excitations for which \( p^2/2m \ll 2U_1n_o \) correspond to phonons with the dispersion relation \( \hbar \omega_c(p) = u_c p \) and a speed of sound \( u_c \) given by

\[ u_c = \sqrt{\frac{U_1n_o}{m}} = \sqrt{\frac{n_o}{m} \left( U_{aa} \cos^4 \theta + U_{bb} \sin^4 \theta + \frac{1}{2} U_{ab} \sin^2 2\theta \right)}; \] (30)

The speed of sound given by Eq.(30) is the same as that calculated from the pressure, \( P \), of the condensate using \( P = -\frac{\partial E_{oc}}{\partial V} \) along with \( u_c^2 = \frac{1}{m} \frac{\partial P}{\partial n_o} \). It is easy to show that the excitations in state \( |c\rangle \) give rise to density perturbations, \( \delta \hat{n}(r,t) \). The number density operator is

\[ \hat{n}(r) = \Psi_c^\dagger(r) \Psi_c(r) + \Psi_d^\dagger(r) \Psi_d(r) = \frac{1}{V} \sum_{p \neq 0} \left( c_p^\dagger c_p + d_p^\dagger d_p \right) e^{i(p-p') \cdot r/\hbar} \] (31)

which may be linearized around the condensate ground state to give \( \hat{n}(r) \approx n_o + \delta \hat{n}(r,t) \). The density perturbations can be expressed in terms of the quasiparticles as

\[ \delta \hat{n}(r,t) = \left( \frac{n_o}{V} \right)^{1/2} \sum_{p \neq 0} \left( \cosh \varphi_p - \sinh \varphi_p \right) \left( C_p^\dagger(t) e^{-i p \cdot r/\hbar} + C_p(t) e^{i p \cdot r/\hbar} \right), \] (32)

which has the form for the phonon states, \( p \ll mu_c \), of

\[ \delta \hat{n}(r,t) = \sum_{p \ll mu_c} \left( \frac{n_o P}{2Vmu_c} \right)^{1/2} \left( C_p^\dagger(0) e^{-i(p \cdot r - u_c \cdot t)/\hbar} + C_p(0) e^{i(p \cdot r - u_c \cdot t)/\hbar} \right). \] (33)

Equation (33) is the operator for the density perturbations which obeys Eq.(1) for a homogenous fluid [21]. Consequently, the excitations in state \( |d\rangle \) do not contribute (at least to order \( \sqrt{N_o} \)) to sound propagation.

The calculation for case (ii) proceeds in an identical manner and, as such, we quote only the main results. The linearized Hamiltonian is,
\[ \hat{K}_{RD} = K_{od} + \sum_{p \neq 0} \left\{ \left( \frac{P^2}{2m} + U_2 n_o \right) d^+_p d_p + \frac{1}{2} U_2 n_o \left( d^+_p d^{\dagger}_p + d_p d_{-p} \right) \right\} \\
+ \sum_{p \neq 0} \left( \frac{P^2}{2m} + \hbar \omega_{cd} - (U_2 - \frac{1}{2} U_3)n_o \right) c^+_p c_p; \]  

(34)

with \( K_{od} = E_{od} - \mu N_o \) and \( E_{od} = -\frac{\hbar \omega_{od}}{2} N_o + \frac{1}{2} U_2 N_o^2 \) is the ground state energy. The excitations in state \( |c\rangle \) are single particle excitations with an energy spectrum of \( \hbar \omega_c(p) = \frac{p^2}{2m} + \hbar \omega_{cd} - (U_2 - \frac{1}{2} U_3)n_o \). The quasiparticle operators \( D_p = \cosh \varphi_p d_p + \sinh \varphi_p d^{\dagger}_p \) diagonalize the terms in \( \hat{K}_{RD} \) involving \( d_p \) and \( d^{\dagger}_p \),

\[ \sum_{p \neq 0} \left\{ \left( \frac{P^2}{2m} + U_2 n_o \right) d^+_p d_p + \frac{1}{2} U_2 n_o \left( d^+_p d^{\dagger}_p + d_p d_{-p} \right) \right\} = \sum_{p \neq 0} \hbar \omega_d(p) D^+_p D_p + E_{vac,d} \]  

(35)

with quasiparticle energies

\[ \hbar \omega_d(p) = \sqrt{\frac{P^2}{2m} \left( \frac{P^2}{2m} + 2U_2 n_o \right)}. \]  

(36)

The long wavelength phonon excitations have a sound velocity \( u_d \) given by

\[ u_d = \sqrt{\frac{U_2 n_o}{m}} = \sqrt{\frac{n_o}{m} \left( U_{aa} \sin^4 \theta + U_{bb} \cos^4 \theta + \frac{1}{2} U_{ab} \sin^2 2\theta \right)}. \]  

(37)

Again, the sound speed given by Eq. (37) is identical to that calculated using \( P = -\frac{\partial E}{\partial V} \) and \( u^2_d = \frac{1}{m^2 n_o} \). The linearized density perturbations are now given by,

\[ \delta \hat{n}(\mathbf{r}, t) = \left( \frac{n_o}{V} \right)^{1/2} \sum_{p \neq 0} (\cosh \varphi_p - \sinh \varphi_p) \left( D^+_p(t) e^{-i p \cdot r / \hbar} + D_p(t) e^{i p \cdot r / \hbar} \right); \]

\[ = \sum_{p \leq m u_d} \left( \frac{n_o p}{2 V m u_d} \right)^{1/2} \left( D^+_p(0) e^{-i (p \cdot r - u_d t) / \hbar} + D_p(0) e^{i (p \cdot r - u_d t) / \hbar} \right); \]  

(38)

Equations (33) and (38) for \( \delta \hat{n}(\mathbf{r}, t) \) and Eqs. (30) and (37) for the speed of sound are the main results of this section. The expressions for \( u_c \) and \( u_d \) given by Eqs. (30) and (37) indicate that the speed at which a density perturbation propagates in the condensate, depends on the particular dressed state that the condensate atoms are in and the value of the dressed state angle. In the limit that \( U_{aa} = U_{bb} = U_{ab} = \frac{4\pi \hbar^2}{m} \), one obtains \( u_c = u_d = u \) where \( u \) is given by Eq. (3). Consequently, the inequality of the scattering lengths \( a_{ij} \) is crucial for this effect to be observed. Figures 1(a-c) show plots of \( u_c \) and \( u_d \) for various ratios of \( U_{bb} / U_{aa} \) and \( U_{ab} / U_{aa} \).
The atoms in the condensate can be prepared in either states (i) or (ii) by adiabatically turning on the external field. Suppose that at \( t = -\infty \), \( \Omega_R = 0 \) and \( \delta \neq 0 \) so that the atoms are in either \( |a\rangle \) or \( |b\rangle \). If the atoms initially in state \( |b\rangle = |d\rangle \) (\( \delta > 0 \)), then the atoms will remain in \( |d\rangle \) provided \( \theta \) varies sufficiently slowly. For free atoms, the atoms will adiabatically remain in the initial dressed state provided \( \dot{\theta} |\omega_{cd}^{-1} | \ll 1 \). However, for interacting atoms in a condensate, \( \hbar |\dot{\theta}| \) should be much less than all of the energies in the Heisenberg equations of motion which govern the time evolution of \( c_p \) and \( d_p \). Consequently, one must satisfy the more stringent requirement \( \hbar \omega_{cd} \gg \frac{p^2}{2m}, n_o U_{aa}, n_o U_{bb}, n_o U_{ab} \gg \hbar |\dot{\theta}| \).

IV. DISCUSSION

Up to this point, the oscillatory terms in Eq. (15) have been neglected under the assumption that their effect is small compared to the terms in \( \hat{K}_{sr} \). However, it is important to obtain an estimate of the leading order correction due to these terms. The lowest order effect of the oscillatory terms is an energy shift in the chemical potential and quasiparticle spectrum which is analogous to the Bloch-Siegert frequency shift in the atomic resonance of a two-level atom due to the counter-rotating terms in the atom-field coupling \[23\]. The shift in the chemical potential (i.e. \( \mu \to \mu + \delta \mu \) where \( \delta \mu \) is the shift) for cases (i) and (ii) may be calculated by including the two-body interactions proportional to \( U_i \) for \( i = 4, 5, 6 \) in Eqs. (24a-24b). Doing so, one finds that for case (i) the shift is \( \delta \mu_c = \frac{3(U_5 n_o)^2}{4\hbar \omega_{cd}} \) and for case (ii) the shift is \( \delta \mu_d = \frac{3(U_6 n_o)^2}{4\hbar \omega_{cd}} \).

To obtain the shift in the quasiparticle spectrum, one may write Eq. (15) as \( \hat{K}_s = \hat{K}_{sr} + \hat{H}_{cs, nr}(t) \) where \( \hat{K}_{sr} \) is given by Eq. (20) and \( \hat{H}_{cs, nr}(t) \) consists of the terms which oscillate at \( \omega_{cd} \). To calculate the energy shift, one first linearizes \( \hat{H}_{cs, nr}(t) \) around one of the ground states given by cases (i) or (ii) so that \( \hat{H}_{cs, nr}(t) \) is quadratic in \( c_p \) and \( d_p \) for \( p \neq 0 \). Since \( e^{i\omega_{cd}t} \) varies rapidly compared to \( c_p \) and \( d_p \), the coupling between the operators \( c_p \) and \( d_p \) in \( \hat{H}_{cs, nr}(t) \) may be adiabatically eliminated by integrating the Heisenberg equations of motion for the quasiparticles and substituting the solut-
tion back into $\hat{H}_{cs, nr}(t)$. One then finds for case (i), that Eq. (23) now has the form $\hat{K}_{\text{loc}} = K_{\text{ata}} + E_{\text{vac}} + \hbar \sum_{p \neq 0} \left[ (\omega_c(p) + \delta \omega_c(p)) C_p^d C_p + (\omega_d(p) + \delta \omega_d(p)) d_p^d d_p \right]$ where the lowest order energy shifts of the excited states, including the shift in the chemical potential, are given by $\hbar \delta \omega_c(p) = \frac{(U_{n_o})^2}{4\hbar \omega_{cd}} \left( \frac{7 \pi^2}{\hbar \omega_c(p)} \right)$ and $\hbar \delta \omega_d(p) = \frac{(U_{n_o})^2 - 3(U_{n_o})^2/2}{2\hbar \omega_{cd}}$. Note that the factor $\frac{7 \pi^2}{\hbar \omega_c(p)}$ in $\hbar \delta \omega_c(p)$ comes from expressing the $\tilde{c}_p$ operators in the linearized $\hat{H}_{cs, nr}(t)$ in terms of $C_p$ using Eq. (28). For the energy shift to be negligible for the phonon states (i.e. $\omega_c(p) \gg \delta \omega_c(p)$), one must satisfy the condition $\frac{p^2}{2m} \gg \frac{(U_{n_o})^2}{8\hbar \omega_{cd}}$. In a similar manner one may calculate the energy shift in the excited states corresponding to ground state (ii). In this case one finds that the energy shifts in Eq. (34) are $\hbar \delta \omega_c(p) = -\frac{(U_{n_o})^2 - 3(U_{n_o})^2/2}{2\hbar \omega_{cd}}$ and $\hbar \delta \omega_d(p) = -\frac{(U_{n_o})^2}{4\hbar \omega_{cd}} \left( \frac{7 \pi^2 - U_{n_o}}{\hbar \omega_c(p)} \right)$. Again, one finds that for case (ii), the energy shift for the phonon excitations is negligible provided $\frac{p^2}{2m} \gg \frac{(U_{n_o})^2}{8\hbar \omega_{cd}}$. One can see that the energy shifts are smaller than the unperturbed energies by factors which goes like $\frac{U_{n_o}}{\hbar \omega_{cd}} \ll 1$ and that even for the low energy phonon states, the energy shift is negligible.

The validity of the results in the beginning of this section as well as in sections II and III all rely on condition (16) being satisfied. Since we are primarily interested in the low energy phonon states for which $\frac{p^2}{2m} \ll n_o U_{aa}, n_o U_{bb}, n_o U_{ab}$, condition (16) reduces to

$$\hbar \omega_{cd} \gg n_o U_{aa}, n_o U_{bb}, n_o U_{ab}. \quad (39)$$

To satisfy this condition as the external field turns on, it is necessary to have $\hbar |\delta| \gg n_o U_{aa}, n_o U_{bb}, n_o U_{ab}$. At later times, however, once $\Omega_R$ is established, the detuning can be varied adiabatically to adjust the dressed state angle. For these times, to satisfy (39) it is sufficient that

$$\hbar 2 \Omega_R \gg n_o U_{aa}, n_o U_{bb}, n_o U_{ab}. \quad (40)$$

To estimate the mean-field interactions, one can take $a_{ij} \sim a$ and use the values of $a = 2.75nm$ for $^{23}Na$ and $a = 5.77nm$ for $^{87}Rb$ and a condensate density of $n_o \sim 10^{14} cm^{-3}$ [13]. The mean-field energy, $\frac{\hbar 2 a_{m}}{m}$, divided by $\hbar$ for $^{23}Na$ is then $9.5 \times 10^3 s^{-1}$ and for $^{87}Rb$ it is
\[ 5.3 \times 10^3 \text{s}^{-1} \]. Notice that this is comparable to the two-photon Rabi frequencies currently used in experiments with \(^{87}\text{Rb}\) which are typically \( \sim 2\pi \times 600\text{s}^{-1} \) \([20]\).

In order to obtain some feeling of the necessary field strengths needed to satisfy \((13)\), one can, for the sake of definiteness, consider single photon rf transitions between Zeeman states in a spinor condensate so that \( \hbar \Omega_R \sim \mu_B B \) where \( \mu_B \) is the Bohr magneton and \( B \) the magnitude of the rf magnetic field. Thus a magnetic field of \( B \gg 10^{-3} \text{G} \) would be required to satisfy \((10)\). One could also hold \( \delta \) constant and vary the rf field strength \( B \). The splitting between Zeeman states in the presence of a static uniform magnetic field \( B_o \) will be on the order of \( \omega_o \sim \mu_B B_o / \hbar \approx 9 \times 10^{10} \text{s}^{-1} \) for \( B_o = 10^4 \text{G} \). One could therefore achieve detunings such that \( \delta \ll \omega_o \) so that the rotating wave approximation remains valid but still have \( \hbar |\delta| \approx n_o U_{aa}, n_o U_{bb}, n_o U_{ab} \). Consequently, condition \((39)\) should be experimentally achievable.

As mentioned before, in order for the speed of sound to show a dependence on the internal states of the atoms, the scattering lengths \( a_{ij} \) must be different. For magnetically trapped \(^{87}\text{Rb}\) with the two \( 5S_{1/2} \) hyperfine states \( |a> = |F = 1, m = -1 > \) and \( |b> = |F = 2, m = 1 > , \) the scattering lengths are nearly equal and are in the ratios \( a_{aa} : a_{ab} : a_{bb} :: 1.03 : 1 : 0.97 \). This would preclude the use of \(^{87}\text{Rb}\). However, the recent demonstrations of the manipulation of the scattering length in condensates of \(^{85}\text{Rb}\) and \(^{23}\text{Na}\) using a Feshbach resonance \([24]\) opens up the possibility of using similar techniques in multicomponent condensates. An alternative method for manipulating the effective strength of the two-body interactions is based on a two-mode model for the condensate in the presence of an external trapping potential \([10, \ 25]\). This is discussed in appendix A.

\section*{V. SUMMARY AND CONCLUSIONS}

We have shown that the atomic dressed states, which diagonalize the single body Hamiltonian, are a useful basis for two-component condensates when the dressed state energy splitting is much larger than the mean-field energies. In this limit, the two-body interac-
tions take on a simple form since those interactions, in the dressed basis, which change the total number of atoms in each of the two dressed states are highly suppressed. As such, the Hamiltonian in the dressed basis has the same form as the Hamiltonian in the original atomic basis with no external field coupling the two atomic states. However, in the dressed basis, the coupling constants for the two-body interactions are functions of the dressed state angle, a quantity which is experimentally controllable. Consequently, collective properties of the condensate which depend on the strength of the mean field interactions, such as the speed of sound, may be controlled by adiabatically varying the dressed state angle. The key requirement that $\hbar \omega_{cd} \gg n_o U_{aa}, n_o U_{bb}, n_o U_{ab}$ represents a purely technical challenge of achieving sufficiently high Rabi frequencies and detunings.

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VII. APPENDIX A- ANISOTROPIC TRAP.

Up to now we have limited ourselves to the case of no external trapping potential, $V_i(r) \equiv 0$. However, another possibility for manipulating the strength of the two-body interactions involves using condensates in highly elongated cigar shaped traps. Let $Z$ be the length of the condensate along the long axis, which we take to be in the $z$ direction, and $R$ the radius of the condensate. When the condition $kZ \gg 1$ and $kR \ll 1$ is satisfied, then the propagation of sound is along the long axis of the trap and one may ignore changes in the radial direction. If one ignores the trapping potential in the $z$-direction and considers a radial harmonic potential with trapping frequencies $\omega_a$ and $\omega_b$ for the two-components,

$$V_i(\rho) = \frac{1}{2}m\omega_i^2 \rho^2$$

(41)
where \( \rho = \sqrt{x^2 + y^2} \) is the radial distance from the trap axis, then one may express the field operators as

\[
\hat{\Psi}_a(r) = \frac{1}{\sqrt{L}} \sum_p \alpha_{ap} \psi_a(\rho) e^{ipz/\hbar}; \quad (42a)
\]

\[
\hat{\Psi}_b(r) = \frac{1}{\sqrt{L}} \sum_p \alpha_{bp} \psi_b(\rho) e^{ipz/\hbar}; \quad (42b)
\]

where \( \psi_i(\rho) \) are the ground state radial wave functions for the two states. Equations (42a-42b) represent a two mode approximation with respect to the transverse coordinates. Physically, this approximation consists of neglecting excitations above the ground state in the directions transverse to the axis of the condensate.

When \( \hbar \omega_i \gg n_o U_{kl} \), the \( \psi_i(\rho) \) are the harmonic oscillator ground states

\[
\psi_i(\rho) = \left( \frac{m \omega_i}{\pi \hbar} \right)^{1/2} e^{-m \omega_i \rho^2 / 2 \hbar}. \quad (43a)
\]

By substituting Eqs. (42a-42b) into Eqs. (6b-6c) and using Eq. (43a), one obtains, after integration over the spatial coordinates, an effective one-dimensional Hamiltonian for the excitations in the axial direction which is given by,

\[
\hat{H}_{atom} = \sum_p \left\{ \left( \frac{p^2}{2m} + \frac{\hbar \delta}{2} + \hbar \omega_a \right) \alpha^\dagger_{ap} \alpha_{ap} + \left( \frac{p^2}{2m} - \frac{\hbar \delta}{2} + \hbar \omega_b \right) \alpha^\dagger_{bp} \alpha_{bp} + \hbar \tilde{\Omega}_R \left( \alpha^\dagger_{bp} \alpha_{ap} + \alpha^\dagger_{ap} \alpha_{bp} \right) \right\}; \quad (44a)
\]

\[
\hat{H}_{coll} = \frac{1}{2L} \sum_{p_1 + p_2 = p_3 + p_4} \left\{ \tilde{U}_{aa} \alpha^\dagger_{ap_1} \alpha^\dagger_{ap_2} \alpha_{ap_3} \alpha_{ap_4} + \tilde{U}_{bb} \alpha^\dagger_{bp_1} \alpha^\dagger_{bp_2} \alpha_{bp_3} \alpha_{bp_4} + 2 \tilde{U}_{ab} \alpha^\dagger_{ap_1} \alpha^\dagger_{bp_2} \alpha_{ap_3} \alpha_{bp_4} \right\}. \quad (44b)
\]

One now has a rescaled Rabi frequency, \( \tilde{\Omega}_R \), and one dimensional mean-field interactions, \( \tilde{U}_{ij} \) which are given by

\[
\tilde{\Omega}_R = \Omega_R \int 2\pi \rho d\rho \rho \psi_a(\rho) \psi_b(\rho) = \Omega_R \frac{2 \sqrt{\omega_a \omega_b}}{\omega_a + \omega_b}; \quad (45a)
\]

\[
\tilde{U}_{aa} = U_{aa} \left( \int 2\pi \rho d\rho |\psi_a(\rho)|^4 \right) = U_{aa} \frac{m \omega_a}{2 \pi \hbar}; \quad (45b)
\]

\[
\tilde{U}_{bb} = U_{bb} \left( \int 2\pi \rho d\rho |\psi_b(\rho)|^4 \right) = U_{bb} \frac{m \omega_b}{2 \pi \hbar}; \quad (45c)
\]

\[
\tilde{U}_{ab} = U_{ab} \left( \int 2\pi \rho d\rho |\psi_a(\rho)|^2 |\psi_b(\rho)|^2 \right) = U_{ab} \frac{m}{\pi \hbar} \frac{\omega_a \omega_b}{\omega_a + \omega_b}. \quad (45d)
\]

One sees that the Hamiltonian has the same form as Eq. (9a-9b) but for a one-dimensional homogenous condensate. The harmonic oscillator energies appearing in \( \hat{H}_{atom} \) can be
eliminated by a redefinition of the internal energy levels for \( |a \rangle \) and \( |b \rangle \). However, the one dimensional mean-field interactions, \( \tilde{U}_{ij} \), can now be adjusted by varying the trap frequencies. For example, if \( \omega_b/\omega_a = 2 \) and \( U_{ij} = U \) then \( \tilde{U}_{bb}/\tilde{U}_{aa} = 2 \) and \( \tilde{U}_{ab}/\tilde{U}_{aa} = 4/3 \). This case is shown in Fig. 1(c).

The system described by Eqs. (44a-44b) is different from the Thomas-Fermi limit considered in [18] [19]. In the Thomas-Fermi limit, the radial wave function is a superposition of harmonic oscillator excited states, \( \psi_i^{(TF)}(\rho) = \sum_m c(\mu, U_{aa}, U_{bb}, U_{ab})_m \psi_{i,m}(\rho) \), where the \( \psi_{i,m}(\rho) \) is the \( m^{th} \) excited state of the radial harmonic oscillator and the \( c(\mu, U_{aa}, U_{bb}, U_{ab})_m \) are functions of the chemical potential and coupling constants(see Eq. (3)). Consequently, the resulting one dimensional form of \( \hat{H}_{atom} \) will not be independent of the total number of atoms (which is determined by \( \mu \)) or the strength of the mean-field interactions in the Thomas-Fermi limit.
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Fig 1. Plot of the speed of sound in units of $u = \sqrt{\frac{u_o U_{aa}}{m}}$ as a function of the dressed state angle $\theta$. The dotted line is $u_c$ and the solid line is $u_d$. The dressed state angle is restricted to the range $0 \leq \theta \leq \pi/2$. (a) $U_{bb}/U_{aa} = 3/2$ and $U_{ab}/U_{aa} = 2$ (b) $U_{bb}/U_{aa} = 1/2$ and $U_{ab}/U_{aa} = 2$ (c) $U_{bb}/U_{aa} = 2$ and $U_{ab}/U_{aa} = 4/3$. 