On Stability of Nonlinear Differential System Via Cone-Perturbing Lyapunov Function Method

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Abstract
Totally equistable, practically -equistable, practically – equistable, practically $\phi_o$ – equistable of system of differential equations are studied. Cone valued perturbing Lyapunov functions method and comparison methods are our technique. Some results of these properties are given.

Keywords: Totally equistable, practically -equistable, practically – equistable, practically $\phi_o$ – equistable - Cone valued perturbing Lyapunov functions method.

1. Introduction
Consider the non linear system of ordinary differential equations
\[ x' = f(t, x), \quad x(t_0) = x_0 \] (1.1)
and the perturbed system
\[ x' = f(t, x) + \epsilon R(t, x), \quad x(t_0) = x_0 \] (1.2)

Let $R^n$ be Euclidean n-dimensional real space with any convenient norm $\| \cdot \|$, and scalar product $(\cdot, \cdot)$. Let for some $\rho > 0$
\[ S_\rho = \{ x \in R^n, \| x \| < \rho \} \]
where \( f, R \in C[\mathbb{R} \times S_\rho, R^n], I = [0, \infty) \) and \( C[\mathbb{R} \times S_\rho, R^n] \)
denotes the space of continuous mappings \( \mathbb{R} \times S_\rho \) into \( R^n \).

Consider the scalar differential equations with an initial condition
\[ u' = g_1(t, u), \quad u(t_0) = u_0 \] (1.3)
\[ \omega' = g_2(t, \omega), \quad \omega(t_0) = \omega_0 \] (1.4)
and the perturbing equations
\[ u' = g_1(t, u) + \phi_1(t, u), \quad u(t_0) = u_0 \] (1.5)
\[ \omega' = g_2(t, \omega) + \phi_2(t, \omega), \quad \omega(t_0) = \omega_0 \] (1.6)
where \( g_1, g_2, \phi_1, \phi_2 \in C[\mathbb{R} \times \mathbb{R}, \mathbb{R}], \phi_1, \phi_2 \in C[\mathbb{R}, \mathbb{R}] \) respectively.

The following definitions [1] will be needed in the sequel.

Definition 1.1
A proper subset $K$ of $R^n$ is called a cone if
\[ \langle \lambda x, x \rangle \geq 0, \quad (\lambda x, x) \in K, (\lambda x, x) \in K^* x \in K \]
where $K$ and $K^*$ denotes the closure and interior of $K$ respectively and $\partial K$ denote the boundary of $K$.

Definition 1.2
The set $K = \{ \phi \in R^n, \langle \phi, x \rangle \geq 0, x \in K \}$ is called the adjoint cone if it satisfies the properties of definition 3.1.
\[ x \in \partial K \text{ if } \langle \phi, x \rangle = 0 \text{ for some } \phi \in K^*, K_0 = \frac{K}{\{0\}} \]

Definition 1.3
A function $g : D \rightarrow K, D \subset R^n$ is called quasimonotone relative to the cone $K$ if for every \( f, R \in D \times \mathbb{R}^n \), there exists \( \phi_0 \in K_0 \) such that
\[ \langle \langle \phi_0, g(y) - g(x) \rangle, y - x \rangle > 0 \]

Definition 1.4
A function $a(t)$ is said to belong to the class $\mathcal{K}$ if $a \in [R^2, R^+]$, $a(0) = 0$ and $a(r)$ is strictly monotone increasing in $r$.

2. Totally equistable
In this section we discuss the concept of totally equistable of the zero solution of (1.1) using perturbing Lyapuniv functions method and Comparison principle method.

We define for
\[ V \in C[\mathbb{R} \times S_\rho, R^n], \text{ the function } D^+V(t, x) \text{ by} \]
\[ \sup \left\{ \frac{h(V(t + h, x) + \langle f(t, x) + R(t, x), -V(t, x) \rangle)}{h} \right\} \]

The following definition [2-10] will be needed in the sequel.

Definition 2.1
The zero solution of the system (1.1) is said to be $T_1$ – totally equistable (stable with respect to permanent perturbations) , if for every $\epsilon > 0, t_0 \in J$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that for every solution of perturbed equation (1.2), the inequality
\[ \| x(t, x_0) \| < \epsilon \text{ for } t \geq t_0 \]
holds provided that $\| x_0 \| < \delta_1$ and $\| R(t, x) \| < \delta_2$.

Definition 2.2
The zero solution of the equation (1.3) is said to be $T_1$ – totally equistable (stable with respect to permanent perturbations), if for every $\epsilon > 0, t_0 \in J$, there exist two positive numbers
\[ \delta_1^* = \delta_1^*(\epsilon) > 0 \text{ and } \delta_2^* = \delta_2^*(\epsilon) > 0 \] 
such that for every solution of perturbed equation (1.5), the inequality
\[ u(t_0, u_0) < \epsilon, \quad t \geq t_0 \]
holds, provided that \( u_0 < \delta_1^* \) and \( \varphi_1(t) < \delta_2^* \).

**Theorem 2.1**

Suppose that there exist two functions \( \delta_1, \delta_2 \in \mathbb{C}[\mathbb{R}^n \times \mathbb{R}] \) with \( \delta_1(t, 0) = \delta_2(t, 0) = 0 \) and there exist two Liapunov functions
\[ V_1(t, x) \in \mathbb{C}[\mathbb{R}^n \times \mathbb{R}] \quad \text{and} \quad V_2(t, x) \in \mathbb{C}[\mathbb{R}^n \times \mathbb{R}] \]
with \( \delta_1(t, 0) = \delta_2(t, 0) = 0 \) where \( S_0 = \{ x \in \mathbb{R}^n, ||x|| < \eta \} \) for \( \eta > 0 \) and \( S_0^c \) denotes the complement of \( S_0 \) satisfying the following conditions:

**\( \mathcal{H}_1 \)** \( V_1(t, x) \) is locally Lipschitzian in \( x \).

**\( \mathcal{H}_2 \)** \( V_2(t, x) \) is locally Lipschitzian in \( x \).

\[ b(||x||) \leq \delta_2(t, x) \in \mathbb{R}_+ \quad \forall (t, x) \in \mathbb{R}^n \times \mathbb{R}^n \]

where \( \delta, b \in \mathbb{R}_+ \) are increasing functions.

**Proof**

Since the zero solution of the system (1.4) is totally equistable, given \( b(\epsilon) > 0 \) there exist two positive numbers \( \delta_1^* = \delta_1^*(\epsilon) > 0 \) and \( \delta_2^* = \delta_2^*(\epsilon) > 0 \) such that for every solution \( \omega(t, t_0, u_0) \) of perturbed equation (1.6), the inequality
\[ \omega(t, t_0, u_0) < \epsilon, \quad t \geq t_0 \] 
holds, provided that \( \omega_0 < \delta_1^* \) and \( \varphi_2(t) < \delta_2^* \).

Since the zero solution of (1.3) is equistable, given \( b(\epsilon) > 0 \) there exists \( \delta = \delta(t_0, \epsilon) > 0 \) such that
\[ u(t_0, u_0) < \frac{\delta_0(\epsilon)}{2}, \quad \frac{\delta_0(\epsilon)}{2} < \delta \]
holds, provided that \( u_0 < \delta \).

From the condition \( \mathcal{H}_2 \) we can find
\[ \delta_1^* = \delta_1^*(\epsilon) > 0 \] 
and
\[ a(\delta_1^*) + \frac{\delta_0(\epsilon)}{2} < \delta_2^* \] 
(2.3)

To show that the zero solution of (1.1) is totally equistable, it must show that for every \( \epsilon > 0 \), \( t_0 \in \mathbb{J} \) there exist two positive numbers \( \delta_1 = \delta_1(\epsilon) > 0 \) and \( \delta_2 = \delta_2(\epsilon) > 0 \) such that for every solution \( x(t, t_0, x_0) \) of perturbed equation (1.2), the inequality
\[ ||x(t, t_0, x_0)|| < \epsilon \quad \forall t \geq t_0 \]
holds, provided that \( ||x_0|| < \delta_1 \) and \( ||R(t, x)|| < \delta_2 \).

Suppose that this is false, then there exists a solution \( x(t, t_0, x_0) \) of (1.2) with \( t_1 > t_0 \) such that
\[ ||x(t_1, t_0, x_0)|| < \epsilon \quad (2.4) \]
and
\[ ||R(t, x)|| < \delta_2 \]

from the condition (2.4) setting \( \delta_1 = \eta \) where \( V_1(t, x) \) and \( V_2(t, x) \) are Lipschitzian in \( x \) for constants \( M_1 \) and \( M_2 \) respectively.

Then
\[ D^\alpha V_1(t, x) + D^\alpha V_2(t, x) \leq g_1(t, V_1(t, x)) + g_2(t, V_2(t, x)) \]
where \( M = M_1 + M_2 \). From the condition \( \mathcal{H}_3 \) we obtain the differential inequality
\[ D^\alpha V_1(t, x) + D^\alpha V_2(t, x) \leq g_1(t, V_1(t, x)) + g_2(t, V_2(t, x)) \leq M ||R(t, x)|| \]
for \( t \in [t_0, t_1] \). Then we have

\[ D^\alpha m(t, x) \leq g_2(t, m(t, x)) + M ||R(t, x)|| \]
where \( m_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_2(t_0, x_0) \)
and \( \omega_0 \leq m(t_0, x_0) \).

Choosing \( M_1 = V_1(t_0, x_0) \) from the condition \( \mathcal{H}_1 \) and applying the comparison Theorem of [7], it yields
\[ m(t, x) \leq r_2(t, t_0, \omega_0) \]
where \( r_2(t, t_0, \omega_0) \) is the maximal solution of the perturbed equation (1.6)
\[ \varphi_2(t) = M ||R(t, x)|| \]
and \( \omega_0 < \delta_1 \) and \( \varphi_2(\epsilon) < \delta_2 \).

Choose \( \omega_0 = \delta_1 \). From the condition \( \mathcal{H}_1 \) and applying the comparison Theorem of [7], it yields
\[ V_1(t, x) \leq r_1(t, t_0, u_0) \]
where \( r_1(t, t_0, u_0) \) is the maximal solution of (1.3).

From (2.2) at \( t = t_0 \)
\[ V_1(t_0, x_0) \leq r_1(t_0, t_0, u_0) \leq \frac{\delta_0(\epsilon)}{2} \]
and
\[ \frac{\delta_0(\epsilon)}{2} < \delta \]
(2.5)

From the condition \( \mathcal{H}_2 \) and (2.4), at \( t = t_0 \)
\[ V_2(t_0, x_0) \leq a(||x_0||) \leq a(\delta_1) \] 
(2.6)

From (2.3), we get
\[ \omega_0 = V_1(t_0, x_0) + V_2(t_0, x_0) \]
and
\[ \omega_0 < \frac{\delta_0(\epsilon)}{2} + a(\delta_1) < \delta_2 \]
(2.3)

From (2.1), we get
\[ m(t, x) \leq r_2(t, t_0, \omega_0) < b(\epsilon) \quad (2.7) \]

Then from the condition \([H_2] \), \((2.4)\) and \((2.7)\)
we get \( t = t_1 \)
\[ b(\epsilon) \leq b(t_1) \leq b(t_1, x(t_1)) \leq m(t_1, x(t_1)) \leq r_2(t_1, t_0, \omega_0) < b(\epsilon). \]

This is a contradiction, then it must be
\[ \| x(t, t_0, \omega_0) \| < \epsilon \quad \text{for} \quad t \geq t_0 \]
holds provided that \( \| x_0 \| < \delta_1 \) and
\[ \| R(t, x) \| < \delta_2 \]
Therefore the zero solution of \((1.1)\) is totally equistable.

3. Totally \(\Phi_0\)-equistable.

In this section we discuss the concept of
Totally \(\Phi_0\)-equistable of the zero solution of
\((1.1)\) using cone valued perturbing Liapunov functions method and Comparison principle method.

The following definition [3] will be needed in the sequel.

Definition 3.1

The zero solution of the system \((1.1)\) is said to be
totally \(\Phi_0\)-equistable \((\Phi_0\)-equistable with respect to permanent perturbations\) if for every \(\epsilon > 0\),
\[ t_0 \in J \quad \text{and} \quad \Phi_0 \in K^\rho_0 \]
there exist two positive numbers \(\delta_1 = \delta_1(\epsilon) > 0\) and \(\delta_2 = \delta_2(\epsilon) > 0\)
such that the inequality
\[ \left( \Phi_0(x(t, t_0, \omega_0)) \right) < \epsilon \quad \text{for} \quad t \geq t_0 \]
holds provided that \(\Phi_0(x_0) < \delta_1\) and
\[ \| R(t, x) \| < \delta_2 \]
where \(x(t, t_0, \omega_0)\) is the maximal solution of perturbed equation \((1.2)\).

Let for some \(\rho > 0\)
\[ S^\rho = \{ x \in R^n, (\Phi_0(x) < \rho, \Phi_0 \in K^\rho_0) \} \]

Theorem 3.1

Suppose that there exist two functions \(g_1, g_2 \in C[0, R]\) with \(g_1(t, 0) = g_2(t, 0) = 0\)
and let there exist two valued Liapunov functions
\[ V_1 \in C \left[ J \times S^\rho \right], \quad V_2 \in C \left[ J \times S^\rho \right], \]
\[ V_1(t_0, 0) = V_2(t_0, 0) = 0 \]
with where
\[ S^\eta = \{ x \in K_\eta(x_0, x), (\Phi_0(x) \in K^\eta_0) \quad \text{for} \quad \eta > 0 \}
\[ S^\rho \subseteq S^\eta \]
denotes the complement of \(S^\eta\) satisfying the following conditions:

\[ (h_1) \quad V_1(t, x) \text{ is locally Lipschitzian in } x \quad \text{and} \quad V_2(t, x) \text{ is locally Lipschitzian in } x \]
\[ (h_2) \quad \text{with} \]
\[ \begin{align*}
\delta_1 = \delta_1(\epsilon) > 0 \quad \text{and} \quad \delta_2 = \delta_2(\epsilon) > 0 \quad \text{such that} \neg \quad \text{the inequality} \neg \\
\left( \Phi_0, x(t, t_0, \omega_0) \right) < \epsilon \quad \text{for} \quad t \geq t_0 \\
\| R(t, x) \| < \delta_2 \] holds provided that \(\Phi_0(x_0) < \delta_1\) and
\[ a(\delta_1) + \frac{\delta_2}{2} < \delta_1 \]
To show that the zero solution of \((1.1)\) is \(t_1\)-totally \(\Phi_0\)-equistable, it must be prove that for every \(\epsilon > 0\), \(t_0 \in J \quad \text{and} \quad \Phi_0 \in K^\rho_0 \)
there exist two positive numbers \(\delta_1 = \delta_1(\epsilon) > 0\)
and \(\delta_2 = \delta_2(\epsilon) > 0\) such that the inequality
\[ \left( \Phi_0, x(t, t_0, \omega_0) \right) < \epsilon \quad \text{for} \quad t \geq t_0 \]
holds provided that \(\Phi_0(x_0) < \delta_1\) and
\[ a(\delta_1) + \frac{\delta_2}{2} < \delta_1 \]

Suppose that is false, then there exists a solution
\[ x(t, t_0, \omega_0) \]
\[ (\Phi_0(x(t, t_0, \omega_0)) \]
\[ \left( \Phi_0, x(t, t_0, \omega_0) \right) < \epsilon \quad \text{for} \quad t \geq t_0 \]
holds provided that \(\Phi_0(x_0) < \delta_1\) and
\[ a(\delta_1) + \frac{\delta_2}{2} < \delta_1 \]

Since \(x(t, t_0, \omega_0)\) is the maximal solution of perturbed equation \((1.2)\).

Suppose that is false, then there exists a solution
\[ x(t, t_0, \omega_0) \]
\[ (\Phi_0(x(t, t_0, \omega_0)) \]
\[ \left( \Phi_0, x(t, t_0, \omega_0) \right) < \epsilon \quad \text{for} \quad t \geq t_0 \]
holds provided that \(\Phi_0(x_0) < \delta_1\) and
\[ a(\delta_1) + \frac{\delta_2}{2} < \delta_1 \]

Then
\[ D^+(\Phi_0(x_1(t, x)), 2 + D^+(\Phi_0(x_2(t, x)), 2 \leq D^+(\Phi_0(x_1(t, x)), 2 + D^+(\Phi_0(x_2(t, x)), 2 \leq M\|x(t, x)\| \]

where \( M = M_1 + M_2 \) From the condition \((h_2)\) we obtain the differential inequality
\[
D^+(\phi_0, V_1(t,x)) + D^+(\phi_0, V_2(t,x)) \leq \delta_1 (T(t,x) + V_2(t,x)) + M \| R(t,x) \|
\]
for \( t \in [t_0, t_1] \). Then we have
\[
D^+(\phi_0, m(t,x)) \leq \delta_2 (T(t,x) + V_2(t,x)) + M \| R(t,x) \|
\]
Let \( \omega_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_2(t_0, x_0) \).

Applying the comparison Theorem of [7], it yields
\[
(\phi_0, m(t,x)) \leq \delta_2 (T(t,x) + V_2(t,x)) + M \| R(t,x) \|
\]
for \( t \in [t_0, t_1] \).

Define \( \phi_2(t) = M \| R(t,x) \| \)
To prove that
\[
[(\phi_0, r_2(t, t_0, \omega_0))] < b(\epsilon).
\]
It must be shown that
\[
[(\phi_0, \omega_0)] \leq \delta_1 \quad \text{and} \quad \phi_2(t) < \delta_2.
\]
Choose \( u_0 = V_1(t_0, x_0) \). From the condition \((h_4)\) and applying the comparison Theorem [7], it yields
\[
[(\phi_0, \omega_0)] \leq [(\phi_0, r_1(t, t_0, u_0))]
\]
From (3.2) at \( t = t_0 \)
\[
(\phi_0, \omega_0) \leq [(\phi_0, r_1(t_0, t_0, u_0))] + \frac{\delta_0(\epsilon)}{2}
\]
(3.5)
From the condition \((h_2)\) and (3.4), at \( t = t_0 \)
\[
(\phi_0, \omega_0) \leq a(\delta_1)
\]
(3.6)
From (3.3), we get
\[
[(\phi_0, \omega_0)] = (\phi_0, \omega_0) + \frac{\delta_0(\epsilon)}{2} + a(\delta_1) < \delta_1.
\]
Since \( \phi_2(t) = M \| R(t,x) \| \)
\[
\text{From (3.1), we get}
\]
\[
(\phi_0, m(t,x)) \leq (\phi_0, r_2(t, t_0, \omega_0)) < b(\epsilon)
\]
(3.7)
Then from the condition \((h_2)\), (3.4) and (3.7) we get
\[
\text{This is a contradiction, then}
\]
\[
(\phi_0, x(t_0, t_0, x_0)) \leq \delta_1 \quad \text{and} \quad \| R(t,x) \| < \delta_2 \quad \text{where} \quad x(t, t_0, x_0) \quad \text{is the maximal solution of perturbed equation (1.2)}.
\]
Therefore the zero solution of (1.1) is totally \( \phi_0 \) - equistable.

4. Practically equistable

In this section, we discuss the concept of practically equistable of the zero solution of (1.1) using perturbing Liapunov functions method and Comparison principle method.

The following definition [5] will be needed in the sequel.

Definition 4.1

Let \( 0 < \lambda < A \) be given. The system (1.1) is said to be practically equistable if for \( t_0 \in [0,1] \) such that the inequality
\[
\| x(t, t_0, x_0) \| < \lambda \quad \text{for} \quad t \geq t_0
\]
holds provided that \( \| x(0) \| < \lambda \) where \( x(t, t_0, x_0) \) is any solution of (1.1).

In case of uniformly practically equistable, the inequality (4.1) holds for any \( t_0 \).

We define
\[
S(A) = \{ x \in R^n : \| x_0 \| < A, A > 0 \}.
\]

Theorem 4.1

Suppose that there exist two functions \( g_1, g_2 \in C([t_0, t] \times R, R) \) with \( g_1(t, 0) = g_2(t, 0) = 0 \) and there exist two Liapunov functions \( V_1 \in C([t_0, t] \times S(A), R^+) \) and \( V_2 \in C([t_0, t] \times S(A) \cap S(B)^c, R^+) \) with
\[
V_1(t, 0) = V_2(t, 0) = 0
\]
where
\[
S(B) = \{ x \in R^n : \| x_0 \| < B, 0 < B < A \} \quad \text{and} \quad S(B)^c
\]
denotes the complement of \( S(B) \) satisfying the following conditions:

(I) \( V_1(t,x) \) is locally Lipschitzian in \( x \).

II) \( V_2(t,x) \) is locally Lipschitzian in \( x \).

\( b(\| x \|) \leq V_2(t,x) < a(\| x \|) \) \( \forall (t,x) \in J \times S(A) \).

(IV) If the zero solution of (1.3) is equistable, and the zero solution of (1.4) is uniformly practically equistable.

Then the zero solution of (1.1) is practically equistable.

Proof

Since the zero solution of (1.4) is uniformly practically equistable, given \( 0 < \lambda_0 < A \) such that for every solution \( x(t, t_0, x_0) \) of (1.4) the inequality
\[
\omega(t, t_0, x_0) \leq \lambda_0 \quad \text{holds provided} \quad x_0 \leq \lambda_0
\]
holds provided \( x_0 \leq \lambda_0 \).

Since the zero solution of the system (1.3) is equistable, given \( \lambda_0 \) and \( t_0 \in R \), there exist
\[
\delta = \delta(t_0, \epsilon) > 0 \quad \text{such that} \quad \text{for every solution}
\]
\[
u(t, t_0, u_0) \leq \lambda_0 \quad \text{holds provided} \quad u_0 \leq \delta.
\]

From the condition (I), we can find \( \lambda > 0 \) such that
\[
\lambda + \frac{\lambda_0}{2} \leq \lambda_0
\]
To show that The zero solution of (1.1) practically equistable , it must be exist $0 < \lambda < A$ such that for any solution $x(t_0, x_0)$ of (1.1) the inequality

$$\|x(t, t_0, x_0)\| < A \quad \text{for} \quad t \geq t_0$$

holds .provided that $\|x_0\| < \lambda$  .

Suppose that this is false, then there exists a solution $x(t_0, t, x_0)$ of (1.1) with $t_1 > t_0$ such that

$$\|x(t_0, t, x_0)\| = \lambda , \quad \|x(t_0, t, x_0)\| = A$$ (4.5)

$$\lambda < \|x(t_0, t, x_0)\| \leq A \quad \text{for} \quad t \in [t_0, t_1].$$

Let $\lambda = \bar{B}$ and setting

$$m(t, x) = V_1(t_0, x) + V_2(t_0, x)$$

From the condition (III) we obtain the differential inequality for $t \in [t_0, t_1]$

$$D^\alpha m(t, x) \leq g_2(t, m(t, x))$$

Let $\omega_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_2(t_0, x_0)$ Applying the comparison Theorem [7] , yields

$$m(t, x) \leq \omega_2(t_0, \omega_0) \quad \text{for} \quad t \geq t_0$$

where $\omega_2(t_0, \omega_0)$ is the maximal solution of (1.4).

To prove that

$$\omega_2(t_0, \omega_0) < b(A).$$

It must be show that $\omega_0 < \lambda_0.$

Choose $u_0 = V_1(t_0, x_0), f$ from the condition (II) and applying the comparison Theorem of [7],

$$V_1(t, x) \leq \omega_2(t, t_0, u_0)$$

where $\omega_2(t, t_0, u_0)$ is the maximal solution of (1.3).

From (4.3) at $t = t_0$

$$V_1(t, x) \leq \omega_2(t, t_0, u_0) < \frac{\lambda_0}{2}$$

From the condition (II) and (4.5) , at $t = t_0$

$$V_2(t_0, x_0) \leq a(\|x(t_0)\|) \leq a(\lambda)$$

From (4.4),(4.6) and(4.7) , we get

$$\omega_0 = V_1(t_0, x_0) + V_2(t_0, x_0) \leq \lambda_0$$

From (4.2) we get

$$m(t, x) \leq \omega_2(t, t_0, \omega_0) < b(A)$$

Then from the condition (II) , (4.5) and (4.8) , we get at $t = t_1$

$$b(A) = b(\|x(t_0)\|) \leq V_2(t_0, x_0) + m(t, x(t_0)) \leq \omega_2(t_0, \omega_0) < b(A).$$

This is a contradiction ,then

$$\|x(t, t_0, x_0)\| < A \quad \text{for} \quad t \geq t_0$$

provided that $\|x_0\| < \lambda_0$ .

Therefore the zero solution of (1.1) is practically equistable.

5. practically $\phi_0$ - equistable

In this section we discuss the concept of practically $\phi_0$ - equistable of the zero solution of (1.1) using cone valued perturbing Liapunov functions method and Comparison principle method.

The following definitions [6] will be needed in the sequel .

Definition 5.1

Let $0 < \lambda < \bar{A}$ be given . The system (1.1) is said to be practically $\phi_0$ - equistable, if for $t_0 \in I$ and $\phi_0 \in K_0$ such that the inequality

$$(\phi_0, x(t_0, x_0)) < \bar{A} \quad \text{for} \quad t \geq t_0$$

holds .provided that $\|x_0\| < \lambda_0.$

where

$x(t, t_0, x_0)$ is the maximal solution of (1.1).

In case of uniformly practically $\phi_0$ - equistable ,the inequality (5.1) holds for any $t_0$.

We define $S^+(A) = \{x \in K, (\phi_0, x) < \lambda, \phi_0 \in K_0\}$

Theorem 5.1

Suppose that there exist two functions $g_1, g_2 \in C[I \times R,R]$ with

$g_1(t, 0) = g_2(t, 0) = 0$ and let there exist two cone valued Liapunov functions $V_1 \in C(I \times S^+(A), K)$ and $V_2 \in C(I \times S^+(A) \cap S^+(B)^C, K)$ with

$V_1(t, 0) = V_2(t, 0) = 0$ where $S^+(B) = \{x \in K, (\phi_0, x) < 0, B < A, \phi_0 \in K_0\}$ and $S^+(B)^C$ denotes the complement of $S^+(B)$ satisfying the following conditions:

(i) $V_1(t, x)$ is locally Lipschitzian in $x$ relative to $K$.

$$D^+(\phi_0, V_1(t, x)) \leq g_1(t, V_1(t, x)) \quad \forall(t, x) \in I \times S^+(A).$$

(ii) $V_2(t, x)$ is locally Lipschitzian in $x$ relative to $K$.

$$b(\phi_0, x) \leq a(\phi_0, x) \quad \forall(t, x) \in I \times S^+(A) \cap S^+(B)^C.$$

where $a, b \in K$ are increasing functions.

(iii) (4.6)

$$D^+(\phi_0, V_1(t, x)) + D^+(\phi_0, V_2(t, x)) \leq g_2(t, V_1(t, x) + V_2(t, x))$$

$\forall(t, x) \in I \times S^+(A) \cap S^+(B)^C.$

(iv) If the zero solution of (1.3) is $\phi_0$ - equistable, and the zero solution of (1.4) is uniformly practically $\phi_0$ - equistable.

Then the zero solution of (1.1) is practically $\phi_0$ - equistable.

Proof

Since the zero solution of the system (1.4) is uniformly practically $\phi_0$ - equistable, given given $0 < \lambda_0 < a(B)$ for $a(B) > 0$ such that the inequality

$$(\phi_0, r_2(t, t_0, \omega_0)) < a(B)$$

holds .provided $\|P_0, \omega_0\| < \lambda_0,$ where

$r_2(t, t_0, \omega_0)$ is the maximal solution of (1.4).

Since the zero solution of the system (1.3) is $\phi_0$ - equistable , given $\lambda_0 > 0$ and

there exist $\delta = \delta(t, t_0, \omega_0)$ such that the inequality

$$(\phi_0, r_1(t, t_0, \omega_0)) < \frac{\lambda_0}{2}$$

holds .provided $\|P_0, \omega_0\| < \lambda_0,$ where

$r_1(t, t_0, \omega_0)$ is the maximal solution of (1.3).
From the condition (ii), assume that 
\[ a(B) \leq b(A) \] (5.4)
also we can choose \( \lambda_0 > 0 \) such that
\[ a(\lambda) + \frac{\lambda_0}{2} \leq \lambda_0 \] (5.5)
To show that the zero solution of (1.1) is practically \( \theta_0 \) -equistable. It must be show that for \( \lambda \in \lambda_0, t_0 \in I \) and \( \phi_0 \in K^0 \) such that the inequality
\[ (\phi_0, x(t, t_0, x_0)) < A \quad \text{for all } t \geq t_0 \]
holds provided that \( (\phi_0, x_0) < \lambda \) where \( x(t, t_0, x_0) \) is the maximal solution of (1.1).
Suppose that is false, then there exists a solution \( x(t, t_0, x_0) \) of (1.1) with \( t_2 > t_1 > t_0 \) such that for \( (\phi_0, x_0) < \lambda \) where
\[ \lambda = \max f_{n_0}^{-1}(\lambda_1), \quad (\phi_0, x(t_1, t_0, x_0)) = A \]
(5.6)
Let \( \lambda_1 = B \) and setting
\[ m(t, x) = V_1(t, x) + V_2(t, x) \]
From the condition (ii), we obtain the differential inequality
\[ D^+ (\phi_0, m(t, x)) \leq (\phi_0, r_2(t, t_0, \omega_0)) \quad \text{for all } \lambda \in [t_1, t_2] \]
(5.7)
Applying the comparison theorem of [7], yields
\[ (\phi_0, m(t, x)) \leq (\phi_0, r_2(t, t_0, \omega_0)) \]
To prove that \( (\phi_0, r_2(t, t_0, \omega_0)) \) is a solution of (5.8)
It must be show that
\[ (\phi_0, m(t, x)) \leq \lambda_0 \]
Choose \( \omega_0 = V_1(t_0, x_0) \) From the condition (i) and applying the comparison theorem [7] it yields
\[ (\phi_0, V_1(t, x)) \leq (\phi_0, r_1(t, t_0, u_0)) \]
From (5.3) at \( t = t_1 \)
\[ (\phi_0, V_1(t, x)) \leq (\phi_0, r_1(t, t_0, u_0)) \leq \frac{\lambda_0}{2} \] (5.8)
From the condition (ii) and (5.6), at \( t = t_1 \)
\[ (\phi_0, V_{2N}(t, x(t_1))) \leq (\phi_0, x(t_1)) \leq a(\lambda_0) \]
(5.9)
From (5.5), (5.8) and (5.9), we get
\[ (\phi_0, V_{2N}(t, x(t_1))) \leq \lambda_0 \]
From (5.2), we get
\[ \left\langle [\phi_0, m(t, x)] \right\rangle \leq (\phi_0, r_2(t, t_0, \omega_0)) < a(B) \] (5.10)
Then from the condition (ii), (5.4), (5.6) and (5.10), we get at \( t = t_2 \)
\[ b(A) = b(\phi_0, x(t_2)) \]
\[ \leq (\phi_0, r_2(t_2, t_0, \omega_0)) \]
\[ < a(B) \]
\[ \leq a(\lambda), \]
which leads to a contradiction, then it must be \( (\phi_0, x(t, x_0)) < A \) for all \( t \geq t_0 \)
holds provided that \( (\phi_0, x_0) < \lambda \) Therefore the zero solution of (1.1) is practically \( \theta_0 \) -equistable.

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