The Post Minkowskii Expansion of General Relativity

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A post-Minkowskii approximation of general relativity is described as a power series expansion in \( G \), Newton’s gravitational constant. Material sources are hidden behind boundaries, and only the vacuum Einstein equations are considered. An iterative procedure is outlined which, in one complete step, takes any approximate solution of the Einstein equations and produces a new approximation which has the error decreased by a factor of \( G \). Each step in the procedure consists of three parts: first the equations of motion are used to update the trajectories of the boundaries; then the field equations are solved using a retarded Green function for Minkowskii space; finally a gauge transformation is performed which makes the geometry well behaved at future null infinity. Differences between this approach to the Einstein equations and similar ones are that we use a general (non-harmonic) gauge and formulate the procedure in a constructive manner which emphasizes its suitability for implementation on a computer.

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I. INTRODUCTION

Close binary systems are a possible source for the LIGO or VIRGO gravitational wave detectors, but the non-linearities of the Einstein equations have made such systems difficult subjects of analyses. Although, most notably, the post-Newtonian expansion [1–5] has now been used to calculate the gravitational wave form resulting from the inspiralling evolution of the binary through terms of order \((v/c)^5[6]\).

In anticipation of the day when such analytical results can be pushed no further, we have developed a purely constructive, iterative approach that allows for straightforward numerical implementation. We present a variation of a post-Minkowskii expansion for relativistic systems. Formulated as an iterative procedure, it takes an approximate solution to the Einstein equations and produces a better one—the error decreases by a factor of \( G \) with every iteration. The procedure reproduces the standard lowest order post-Newtonian equations of motion and the traditional quadrupole formula for gravitational waves. But higher orders are generally too complicated for analytical work without the slow-motion assumption of the post-Newtonian approximation. However, on a computer the \( n \)th iteration of the process is no more difficult than the first, and it is there where we envisage putting this procedure to good use.

Two sources in a close binary system are likely to contain regions of extreme curvature. However, when the sources are far enough apart the multipole moments of the individual sources depend very little on the presence of a companion and the evolution, say, of two neutron stars differs only slightly from that of two black holes or any other small, massive objects. In our approach a boundary surrounds each source—and we focus on the vacuum Einstein equations in the region outside of these boundaries.

In many ways our approach is a combination of methods developed by others. It starts with a formal expansion of the Einstein equations in powers of Newton’s gravitational constant, \( G \), in a manner similar to Kerr [7]. The first order results are formulated using a multipole expansion in terms of symmetric trace free tensors described by Pirani [8], Thorne [9] and Blanchet and Damour [10]. Particularly at first order our results formally appear quite similar to Thorne’s [9] analysis of linearized gravity, but our multipole expansion is done about the moving boundaries which hide the sources. At higher orders, we mimic the approach of Blanchet and Damour [10,11] but allow for a general (not harmonic) gauge which is restricted only enough to be well behaved at future null infinity. Throughout our analysis, we rely heavily upon the rigorous mathematical results provided by Blanchet and Damour [10,12].

One original aspect of our approach is the freedom from gauge requirements—at least within the constraints of the metric being considered a tensor field on a flat background space-time. Also original is our analysis of the equations of motion which, at lowest order, is similar to that of Bel et al. [13] except that we enclose the sources within boundaries and avoid any formally divergent integrals. At each order of the approximation we find the equations of motion as consequences of the desire to have the world line of the center of the boundaries actually match the trajectory of the physical source which is contained inside.

Also, we often use flat-space outgoing-null spherical coordinates attached to the accelerating world lines. These provide globally well behaved coordinate systems and seem particularly well suited for problems involving black holes as long as the relevant accelerations are small, \( m\dot{v} \ll 1 \).
In §II a summary of our assumptions and approximations is presented and followed by descriptions of the notation, coordinate systems and formulation of the general Einstein equations which we use. §II gives a description of the iterative procedure along with specific details of how to start the process at the first order, and how to iterate the field equations at nth order. But, for the field equations to have a well behaved solution it is first necessary to iterate the equations of motion as described in §V. The behavior of the gravitational field at future-null infinity is discussed in §VI. Some details are relegated to the Appendix including a description of the retarded Poincare transformation—a convenient generalization of the Lorentz transformation which relates outgoing-null spherical coordinates centered on different world lines.

Our process is described in a manner that should make implementation on a computer straightforward, particularly for the analysis of the binary inspiral problem from the time of the post-Newtonian applicability perhaps down to the innermost stable circular orbit, where speeds could be \( \sim c/2 \). We expect this approach to fail when the evolutionary time scale is comparable to the dynamical time scale.

II. BASIC FORMULATION

A. Assumptions and approximations

We assume that Newton’s gravitational constant, \( G \), is small, but not necessarily very small, when the units are such that a characteristic mass and distance are of order 1 for a specific problem. This is essentially equivalent to assuming that the metric of space-time deviates only modestly from being flat. We also use units where the speed of light is unity—the characteristic time corresponds to the light travel time across the characteristic distance. We specifically do not assume that the speeds are small, but we do assume that accelerations are, \( \dot{v}^a = O(G) \). This is consistent with the assumptions of weak fields—when fast, not-strong-field sources interact their accelerations are small because the fields are weak.

We deal only with the vacuum Einstein equations and assume that any material sources are shrouded by boundaries at surfaces of constant \( r \) in outgoing-null spherical coordinates centered on each source. Some geometrical data can be given on the boundary in order to distinguish, say, a system involving black holes from one of neutron stars. This distinction may be difficult to implement on a computer, and numerical results may only be of general validity and unable to carefully examine features of the geometry which depend upon the detailed nature of the sources. But, at least in principle, if this process converges to an accurate solution of the Einstein equations, then the detailed nature of the true physical geometry at the boundary can be examined to see whether it is physically consistent with any particular source of interest.

We have only modest gauge restrictions which are imposed to keep the geometry well behaved at future null infinity. In particular, we do not require the harmonic gauge.

In §II we assume that the mass monopole moment, \( A \), is larger than all others by \( O(G) \). This is not a required assumption, but it is reasonable on physical grounds and allow us to describe the dynamical equations in familiar terms.

And in §III we assume that there exists an initial spacelike hypersurface in the past of which the geometry is an exact solution of the Einstein equations. The geometrical data on this hypersurface must satisfy minimal gauge requirements at large values of \( r \). And the first order approximation to the Einstein equations, \( h^{ab}_1 \), must match up smoothly with the data on this initial hypersurface.

B. Notation

This formalism considers a gravitational field as a tensor field on a flat Minkowskii background. The mathematical notation is that of flat-space tensor analysis with Minkowskii coordinates \( (t,x,y,z) \), along with the usual flat metric, \( \eta_{ab} \), and its inverse, \( \eta^{ab} \), which are both \((-1,1,1,1)\) down the diagonal and zero elsewhere. The operator \( \nabla_a \) is just the usual derivative operator of flat space, and \( \nabla^2 \equiv \nabla_a \nabla^a \). Only \( \eta_{ab} \) and \( \eta^{ab} \) are used to lower and raise tensor indices, and a tensor index always refers to a component in Minkowskii coordinates.

For a binary system the positions of the two sources are described by two different world lines of the form \( z^a(s) \) parameterized by the Minkowskii proper time, \( s \). In the discussion the focus is usually upon only one world line at a time with four-velocity, \( v^a \). It is convenient to use \( z^a_s \equiv z^a(s) \).

Occasionally we use outgoing-null spherical coordinates \((s,r,\theta,\phi)\) based on the world line \( z^a(s) \). The scalar field \( s \) is defined at an event, \( P \), of Minkowskii space by \( s(P) = s(Q) \), where \( Q \) is the vertex of the future null cone from \( z^a(s) \) which contains \( P \); it is convenient to use \( s_r \equiv s(x^a) \). In a similar manner \( v^a \), as well as any other tensor defined
along the world line, can be promoted to a tensor field over all space-time by parallel transport of $v^a(Q)$ up the future null cone of $Q$ to $P$.

Any field derived from a tensor defined along the world line, has a particularly simple expression for its derivative. For example

$$\nabla_b v^a = -k_b \dot{v}^a$$

(1)

where

$$k_a \equiv -\nabla_a s,$$

(2)

and a dot denotes a derivative with respect to the retarded time $s$, so that

$$\dot{v}^a = v^b \nabla_b v^a = \frac{dv^a}{ds}.$$  

(3)

The quantity $r$, in the outgoing-null coordinates, is the Minkowskii spatial distance between $Q$ and $P$ as measured in the instantaneous rest frame of the world line at $Q$,

$$r(x^a) = -v_a[x^a - z^a(Q)].$$

(4)

Pirani shows that $k^a$ is the null vector field

$$k^a = (x^a - z^a(Q))/r,$$

pointing from $Q$ to $P$. Also

$$k^a v_a = -1,$$

(6)

and

$$\nabla_b r = -v_b + k_b (1 + rk_a \dot{v}^a) = n_b + rk_b k_a \dot{v}^a$$

(7)

where $n_b \equiv k_b - v_b$ is an outward-pointing, spatial unit vector. It is useful to know that

$$r \nabla_a k_b = \eta_{ab} + v_a k_b + v_b k_a - k_a k_b (1 + rk_c \dot{v}^c).$$

(8)

The angles $\theta$ and $\phi$ at $P$ are defined in the usual way with the origin at $Q$ and with a set of orthonormal basis vectors which is Fermi-Walker transported along the world line and parallel transported up the null cone.

The projection operator onto the spatial three manifold instantly orthogonal to $v^a$ at $Q$ is $f_{ab} \equiv \eta^{ab} + v^a v^b$, and the alternating tensor orthogonal to $v^a$ is $\epsilon^{abc} \equiv \epsilon^{abcd} v_d$. But, note that when $f^{ab}$ is promoted to a tensor field via parallel transport up the future null cone it does not become the spatial three metric of a constant $t$ surface if the world line is accelerating.

The description of tensor multipole moments often requires a large set of, say, $l$ indices; we follow Blanchet and Damour and define a tensor multi-index, $L \equiv d_1...d_l$, to denote a succession of $l$ space-time indices. We use $N^L \equiv n^{d_1}...n^{d_l}$ for the tensor outer product of $l$ vectors, $n^d$, and $\nabla_L \equiv \nabla_{d_1}...\nabla_{d_l}$ for a succession of $l$ derivatives. Oftentimes we sum over $l$ from 0 to $\infty$, and this summation is assumed to converge without justification being given. Sometimes a set of tensor indices are symmetric, spatial with respect to $v^a(s)$ and completely trace free, these are referred to as being SSTF. If a tensor has all of its indices SSTF and is a function of only $s$, then it is denoted by a capital, script base letter. Damour and Iyer give a host of useful formulae for decomposing a tensor into SSTF parts. We follow their notation and equivalently denote the SSTF part of a tensor $A^L$ by $\hat{A}^L \equiv A^{<L>} \equiv A^{<a_1...a_l>}$. Also, $[l/2]$ is just the largest integer less than or equal to $l/2$.

C. The Einstein equations on a flat background manifold.

A metric, $g_{ab}$, on a four dimensional space-time may be considered as a two indexed, symmetric invertible tensor field on a flat, background Minkowskii space. It is convenient to define $h_{ab}$ by

$$\sqrt{-g} g^{ab} \equiv \eta^{ab} - h^{ab}$$

(9)
and an Einstein tensor density as a functional of \( h^{ab} \),

\[
E^{ab}(h) \equiv (-g)(2R^{ab} - g^{ab}R),
\]

so that the vacuum Einstein equation is

\[
E^{ab}(h) = 0.
\]

Landau and Lifshitz \cite{13} give an exact form for \( E^{ab} \); we write this as

\[
E^{ab}(h) = -\nabla^2 h^{ab} + \nabla^a \nabla_c h^{cb} + \nabla^b \nabla_c h^{ca} - \eta^{ab} \nabla_d h^{cd} - 16\pi \tau^{ab}(h)
\]

where

\[
16\pi \tau^{ab}(h) \equiv -2\nabla_c h^{ab} \nabla_d h^{dc} - h^{ab} \nabla_c \nabla_d h^{cd} + h^{ad} \nabla_d \nabla_c h^{eb} + \nabla_c \nabla_d h^{be} - h^{cd} \nabla_c \nabla_d h^{ab} + \nabla_c h^{ad} \nabla_d h^{bc} + 16\pi (-g) \tau_{ab}^{\text{LL}}(h).
\]

The quantity \( \tau_{ab}^{\text{LL}}(h) \) is the Landau-Lifshitz pseudotensor, Eq. (96.9) in \cite{13} or Eq. (20.22) in Misner et al. \cite{16}.

The Bianchi identity translated onto the flat background takes the form

\[
\nabla_a E^{ab} = (\eta_c^b \Gamma_{da}^{cd} - \Gamma_{ac}^{bd}) E^{ac},
\]

where \( \Gamma_{ac}^{bd} \) is the usual Christoffel symbol for the real space-time metric, \( g_{ab} \), and is \( O(G) \).

### III. Iterative Procedure

We formally expand the gravitational field in powers of \( G \): \( h^{ab}_0 \) is zero, and at first order \( h^{ab}_1 = O(G^1) \) and exactly matches the geometrical data on the initial hypersurface. We iteratively assume that \( E^{ab}(h_n-1) = O(G^n) \), with no gauge restrictions on \( h^{ab}_{n-1} \) (in particular it need not be in the harmonic gauge), and look for a correction, \( \delta h^{ab}_n = O(G^n) \), such that

\[
h^{ab}_n(x) \equiv h^{ab}_{n-1}(x) + \delta h^{ab}_n(x; G)
\]

and \( E^{ab}(h_n) = O(G^{n+1}) \). The dependence of \( \delta h^{ab}_n \) on \( G \) is allowed to be more complicated than just being proportional to \( G^n \), and in Eq. (15) that functional dependence is explicit—usually the dependence on \( G \) is just understood.

For a given \( h^{ab}_{n-1} \), the next order approximation follows from a solution of

\[
\nabla^2 \delta h^{ab}_n = E^{ab}(h_{n-1}) + O(G^{n+1}),
\]

for \( \delta h^{ab}_n = O(G^n) \), with the additional restriction that

\[
\nabla_a \delta h^{ab}_n = O(G^{n+1}).
\]

That \( h^{ab}_{n-1} + \delta h^{ab}_n \) is a more accurate solution to the Einstein equations is revealed by substitution into Eq. (12) resulting in

\[
E^{ab}(h_{n-1} + \delta h_n)
\]

\[
= [E^{ab}(h_{n-1}) + \delta h_n] + \nabla^2 \delta h^{ab}_n + O(G^{n+1})
\]

\[
= \nabla^a \nabla_c \delta h^{cb} + \nabla^b \nabla_c \delta h^{ca} - \eta^{ab} \nabla_d h^{cd} - 16\pi \tau^{ab}(h_{n-1}) + O(G^{n+1}).
\]

The first equality follows from Eq. (10), and the second from Eq. (12). With the restriction (17) each of the first three terms on the right hand side are \( O(G^{n+1}) \), and \( \tau^{ab}(h) \) is quadratic in \( h^{ab} \) and its derivatives, so the fourth term is also \( O(G^{n+1}) \). Thus

\[
E^{ab}(h_{n-1} + \delta h_n) = O(G^{n+1}),
\]
and one full step of the iteration consists of solving Eq. (16) with the restriction (17).

The restriction (17) should not be considered a gauge condition. After all, at the nth step \( h_n^{ab} \) need not satisfy any particular gauge choice. And the restriction does not involve the residual value of \( \nabla_a h_n^{ab} \) in any manner. Thus, at nth order there is no limitation upon \( \nabla_a h_n^{ab} \). Also, the divergence of Eq. (16) with the Bianchi identity (4) implies that \( \nabla^a \nabla_a \delta h_n^{ab} = O(G^{n+1}) \); thus, with the proper choices of initial data and of boundary conditions the restriction follows naturally from the wave equation (16). To emphasize finally the gauge freedom allowed, note that after each iteration is complete a \( \lambda_n^{\alpha} = O(G^n) \) gauge transformation can change the metric by \( h_n^{ab} \rightarrow h_n^{ab} + \partial \lambda_n^{ab} \) where

\[
\partial \lambda_n^{ab} = \nabla^a \lambda_n^b + \nabla^b \lambda_n^a - \eta^{ab} \nabla_c \lambda_n^c.
\]

This transformation preserves the accuracy of the approximation and changes \( E_n^{ab}(h_n) \) only at \( O(G^{n+1}) \) and only through the change in \( \gamma_n^{ab}(h_n) \). Also, at the nth order, the metric may be changed by the addition of a small, arbitrary symmetric tensor, \( \gamma_n^{ab} = O(G^n) \); \( h_n^{ab} \rightarrow h_n^{ab} + \gamma_n^{ab} \) only changes \( E_n^{ab}(h_n) \) at \( O(G^{n+1}) \). Such changes are used in \( \nabla \) to insure proper asymptotic behavior.

As an iterative procedure this is slightly more general than the post-Minkowski expansion of Blanchet and Damour [1]. If any initial approximation to the Einstein equations has \( E_n^{ab}(h_1) = O(\epsilon) \) for some small epilson, then after one step \( E_n^{ab}(h_2) = O(\epsilon G) \). And if \( h_n^{ab} \) represents an exact solution, independent of gauge, then the procedure terminates.

The remainder of this paper focuses on a specific, constructive method for performing one full step.

A. First order approximation

We formally require at first order that \( h_n^{ab} \) must match the geometrical data on the initial hypersurface. However this limitation is only used in [3], in addition, for applications on a computer we are unlikely to use exact initial data. For these reasons in this section we just look for an \( h_n^{ab} \) which resembles two moving sources and has \( E_n^{ab}(h_1) = O(G^2) \) but does not necessarily match onto good initial data.

A general multipole source, \( M_1^{abL}(s) \), confined to a world line, \( z^a(s) \), has an \( h_n^{ab} = \delta h_1^{ab} \) which both satisfies

\[
\nabla^2 h_1^{ab} = -4\pi \sum_{l=0}^{\infty} \int M_1^{abL}(s) \nabla L \eta^{l}(x - z_s) \, ds
\]

and also is of the general form (see Appendix A)

\[
h_n^{ab} = \sum_{l=0}^{\infty} \int M_1^{abL} \nabla L G(x - z_s) \, ds = \sum_{l=0}^{\infty} \nabla L [r^{-1} M_1^{abL}(s)].
\]

We write \( M_1^{abL} \) as a sum of terms involving completely SSTF tensors, which are \( O(G) \) and functions only of s, in a manner which parallels Thorne’s Eq. (8.4) [3] or Blanchet and Damour’s Eq. (2.25) [10]. The resulting most general multipole source has

\[
M_1^{abL} = v^a v^b A^L_1 + 2v^a B^{abL}_1 + 2v^a \bar{e}^b (q_d C^{L-1} q) + 2v^a (\bar{e}^b (d_1 D^{L-1}) + f^{ab} \xi^L + F^{abL}_1 + 2 \eta^a (d_1 J^{L-1} G^{L-1}) (b) + f^{aL} K^{L-2} J^{L-1}) (b).
\]

The reverse parentheses imply symmetrization on \( a \) and \( b \). Note that of these SSTF tensors, \( B^{abL}_1 \), \( C^{L}_1 \), \( H^{L}_1 \) and \( J^{L}_1 \) always have \( l \geq 1 \), while \( F^{abL}_1 \) and \( G^{L}_1 \) have \( l \geq 2 \).

But the \( h_1^{ab} \) from Eq. (22) can represent the first term in the expansion of the gravitational field of the multipolar source of Eq. (23) only if the restriction (17) is also satisfied. A lengthy analysis of the consequences of restriction (17) invokes a useful identity described in Appendix B, Eq. (B2), liberally uses \( f^{ac} = \epsilon^{ac} + \epsilon^{ac} v_c \) and \( \nabla^2 G = -4\pi \delta^4(x - z_s) \) and ultimately results in

\[
\nabla_a h_1^{ab}(x) = \sum_{l=0}^{\infty} \int \left[ v^b A^L_1 + v^b B^{abL}_1 + \epsilon^b q \delta_1 \xi^L + f^{bd} D^{L-1} + v^b B^{abL}_1 + f^{bd} \xi^L + F^{abL}_1 + \epsilon^b q \delta_1 \xi^L + f^{bdL} H^{L-1} + v^b D^{abL}_1 + f^{bdL} \xi^L + \epsilon^b q \delta_1 \xi^L + f^{bdL} H^{L-1} \right] \nabla L G(x - z_s) \, ds + O(G^2).
\]
The completeness of the decomposition of SSTF vectors and tensors allows the decomposed parts in Eq. (24) to be matched up according to the location of the index $b$—whether it sits on $v^b$, $f^{bd}$, $e^b_{\ q\ d}$ or on an SSTF object. Thus the restriction (17) requires that the multipole source, $M_i^{abL}$, satisfy

$$
\mathcal{A}_1^L + B_1^L + D_1^L = O(G^2) \ [v^b], \\
B_1^L + F_1^L + H_1^L = O(G^2) \ [\text{STTF}^b], \\
\mathcal{C}_1^L + G_1^L + J_1^L = O(G^2) \ [e^b_{\ q\ d}], \\
D_1^L + \mathcal{E}_1^L + K_1^L = O(G^2) \ [f^{bd}].
$$

These are closely related to Eqs. (8.5) of Thorne [9] and Eqs. (2.26)–(2.28) of Blanchet and Damour [10]. The multipole moments $A_i^L$ ... $G_i^L$ are closely related to similar quantities used by Thorne [9]. The differences are that our

$$
A_i^L, C_i^L, D_i^L
$$

are defined with respect to an arbitrary world line, his are with respect to the origin of the Cartesian coordinates.

In this section we discuss the procedure by which the next order approximation is found, but we avoid issues of the non-vanishing to give the source both mass and angular momentum but no additional structure.

### B. The $n$th order approximation

In this section we discuss the procedure by which the next order approximation is found, but we avoid issues of the behavior of the approximation at large $r$—that subject is analyzed in §13.

A given $(n-1)$th order approximation to the Einstein equations has $E_n^{ab} \equiv E^{ab}(h_{n-1}) = O(G^n)$. A step in the iterative procedure requires a solution of Eq. (16), with $\delta h^{ab} = O(G^n)$ which satisfies the restriction (13). Following Blanchet and Damour [10,11], we represent $\delta h^{ab}$ as the sum of a particular solution of the inhomogeneous wave equation, $p_n^{ab}$, and a general solution of the homogeneous wave equation, $q_n^{ab}$; thus,

$$
\delta h_n^{ab} = p_n^{ab} + q_n^{ab},
$$

where

$$
\nabla^2 p_n^{ab} = E_n^{ab}
$$

and

$$
\nabla^2 q_n^{ab} = 0, \quad \text{except on the world line.}
$$

And $q_n^{ab}$ is chosen so that

$$
\nabla_a p_n^{ab} + \nabla_a q_n^{ab} = O(G^{n+1}).
$$

6
1. The solution for $p_{n}^{ab}$

The quantity $p_{n}^{ab}$ is formally given, with a retarded Green function, by

$$p_{n}^{ab} \equiv -\frac{1}{4\pi} \int E_{n-1}^{ab}(x')G(x-x')\,d^{4}x'.$$  \hspace{1cm} (33)

In a numerical implementation this evaluation is the single computationally intensive element. It would most likely be performed through standard finite differencing of the inhomogeneous wave equation. The fields would propagate freely through the interiors of the inner boundaries where the source, $E_{n-1}^{ab}$, would be set to zero. Boundary conditions for $p_{n}^{ab}$ would only be imposed in the wave zone. It is convenient to interpret $p_{n}^{ab}$ as resulting from the nonlinearity of the Einstein equations with $h_{n-1}^{ab}$ creating an effective stress-energy outside of the boundaries. Similarly, $q_{n}^{ab}$ appears to come from an $n$th order correction to the multipole moments of the source hidden behind the boundary. However, with a nonlinear theory, the split of $\delta h_{n}^{ab}$ into two parts in a manner which depends upon the location of the boundary is at least modestly arbitrary, and these interpretations are only suggestive.

2. The solution for $q_{n}^{ab}$

As a solution of the homogeneous wave equation, except on the world line, $q_{n}^{ab}$ has a general representation similar to that of $h_{n}^{ab}$ given in \textsection II A.

$$q_{n}^{ab}(x) = \sum_{l=0}^{\infty} \nabla_{L}[r^{-1}M_{n}^{abL}(s_{x})].$$  \hspace{1cm} (34)

The majority of this section shows how the SSTF components of $M_{n}^{abL}$ are determined by the SSTF components of $E_{n-1}^{ab}$ on the inner boundaries; the results are exhibited in Eqs. (37)–(40).

A consequence of the formal solution for $p_{n}^{ab}$ in terms of the retarded Green function is that

$$\nabla_{a}p_{n}^{ab} = \frac{1}{4\pi} \int \nabla_{a}'E_{n-1}^{ab}(x')G(x-x')\,d^{4}x' - \frac{1}{4\pi} \int \nabla_{a}'E_{n-1}^{ab}(x-x')\,d^{4}x',$$  \hspace{1cm} (35)

where use is made of integration by parts and the symmetry of the retarded Green function. From the Bianchi identity [41] and $\Gamma_{bc}^{a} = O(G)$, the second integrand is $O(G^{n+1})$, and

$$\nabla_{a}p_{n}^{ab} = \frac{1}{4\pi} \int \nabla_{a}'E_{n-1}^{ab}(x')G(x-x')\,d^{4}x' + O(G^{n+1}).$$  \hspace{1cm} (36)

This four-volume integral reduces to boundary integrals about each of the two sources and a third at large $r'$, which gives a vanishing contribution, for $x$ fixed, as $r'$ goes to infinity—this can be verified by a lengthy analysis which starts with the multipolar expansion of the Green function provided by Blanchet and Damour [4].

From this last equation it is clear that outside of the inner boundaries the $O(G^{n})$ part of $\nabla_{a}p_{n}^{ab}$ is a homogeneous solution of the vector wave equation. And we must obtain its multipolar decomposition in order to find the corresponding $q_{n}^{ab}$, a homogeneous solution of the spin-two wave equation which also happens to satisfy Eq. (32). The needed SSTF decomposition of $\nabla_{a}p_{n}^{ab}$ involves surprising subtlety and concludes in Eqs. (39) and (43) below. The boundary integral resulting from Eq. (39) is evaluated with $(s', r', \theta', \phi')$ coordinates and an expansion in terms of SSTF tensors. For simplicity we choose the boundary surrounding each source to be a surface of constant $r' = r_{0}$; this boundary is spherical in a comoving frame of reference for non accelerating world lines and is appropriately Lorentz contracted in a frame wherein the world line is moving. Each boundary integral gives

$$\nabla_{a}p_{n}^{ab} = -\frac{1}{4\pi} \oint \nabla_{a}'r'E_{n-1}^{ab}(x-x')r_{0}^{-2}\sin\theta'd\theta'd\phi'ds' + O(G^{n+1}).$$  \hspace{1cm} (37)
A Taylor series expansion of the Green function about the world line leads to the multipolar decomposition of the integrand in Eq. (37) on a two-sphere with constant retarded time, \( s' \). But, the Green function is non-vanishing only on the past null cone from the field point \( x^a \); \( \nabla_a p^{ab}_n \) picks up a contribution only where the past null cone intersects the three dimensional boundary, and this occurs for differing values of \( s \). The point of intersection closest to the field point will have null-coordinate value \( s \), but the point of intersection on the far side of the boundary will have null-coordinate value approximately \( s - 2r_0 \). Thus, the best two-sphere to use for the multipolar decomposition of \( \nabla_a p^{ab}_n \) is the one at \( s' = s - r_0 \). And the Taylor series expansion for the Green function about the point \( z^a(s', r') \) on a hypersurface which is orthogonal to the world line has \( x^a = z^a + r' n^a + O(r'^2 u^a) \) with the right hand side evaluated at \( s' + r' \); thus,\[ G(x^a - x'^a) = \sum_{l=0}^{\infty} \frac{(-r')^l}{l!} N^{ll} \nabla_L G(x^a - z^a(s' + r')) + O(G) . \] (38)

Now, with \( \nabla_a r'|_{s = r_0} = \nabla_a r'|_{s} + O(G) \) on the boundary and \( s' \rightarrow s' - r_0 \),\[ \nabla_a p^{ab}_n = -\sum_{l=0}^{\infty} \int \frac{(-r_0)^{l+2}}{4\pi l!} \oint \nabla_a r' E_{n-1}^{ab}(s' - r_0) N^{ll} \times \nabla_L G(x - z(s')) \sin \theta' \, d\theta' \, d\phi' \, ds' + O(G^{n+1}) . \] (39)

Below it is necessary that this integrand be in terms of SSTF tensors. To this end, \( N^L \) is equal to a sum of terms each of which is a symmetrized outer product of projection operators \( f^{a_1 a_2} \) and of the SSTF combinations \( N^L \); thus,\[ N^L = \sum_{m=1}^{\lfloor l/2 \rfloor} b_{l,m} f^{a_1 a_2} \cdots f^{a_{2m-1} a_{2m}} n^{<a_{2m+1} \cdots n^{a_l}>} , \] (40)

for a set of coefficients \( b_{l,m} \) which are obtained in Appendix C and given in Eq. (33). Now the substitution \( f^{a_1 a_2} = v^{a_1 a_2} + v^{a_1} v^{a_2} \) and the use of Eq. (A3) allows part of the integrand of Eq. (39) to be rewritten for all \( x \neq z_s \) as\[ N^{ll} \nabla_L G(x - z_s) = \sum_{m=1}^{\lfloor l/2 \rfloor} b_{l,m} v^{a_1} \cdots v^{a_{2m}} n^{<a_{2m+1} \cdots n^{a_l}>} \times \nabla_L G(x - z_s) . \] (41)

Application of the useful identity (B2) further transforms the \( v^{a_1} \cdots v^{a_{2m}} \) inside the integral into proper-time derivatives, and rearrangement of the summation ultimately yields\[ \nabla_a p^{ab}_n = -\sum_{l,m=0}^{\infty} b_{l+2m,m} \int \frac{(-r_0)^{l+2m+2}}{4\pi(l + 2m)!} \times \frac{d^{2m}}{ds^{2m}} \left[ \oint \nabla_a r' E_{n-1}^{ab}(s' - r_0) N^{ll} \sin \theta' \, d\theta' \, d\phi' \right] \times \nabla_L G(x - z(s')) \, ds' + O(G^{n+1}) , \] (42)

which has the \( L \) indices SSTF. Finally, the desired multipolar decomposition is\[ \nabla_a p^{ab}_n = -\sum_{l=0}^{\infty} \int K^{bL}_n(s) \nabla_L G(x - z_s) \, ds + O(G^{n+1}) , \] (43)

where we define\[ K^{bL}_n(s) \equiv \sum_{m=0}^{\infty} b_{l+2m,m} \frac{(-r_0)^{l+2m+2}}{4\pi(l + 2m)!} \times \frac{d^{2m}}{ds^{2m}} \int_{s, r_0} \nabla_a r E_{n-1}^{ab}(s - r_0) N^{ll} \sin \theta \, d\theta \, d\phi . \] (44)
The $L$ indices are explicitly SSTF, and $K_n^{bL}$ has SSTF components $\mathcal{P}_n^L$, $Q_n^L$, $\mathcal{R}_n^L$ and $S_n^L$ defined from

$$
K_n^{bL}(s) = v^b \mathcal{P}_n^L(s) + \epsilon^b_q (d_l^b q_l^{L-1})^q(s) + \mathcal{R}_n^{bL}(s) + \delta^{bc} d_l^{L-1} S_n^{L-1c}(s).
$$

With this decomposition of $\nabla_s \mathcal{P}_n^{ab}$ in hand, we return to the search for $q_n^{ab}$ of the form given in Eq. (23) which satisfies Eq. (24). This analysis closely follows \textsection IIIA. Let

$$
q_n^{ab} = \sum_{l=0}^{\infty} \int M_n^{abL}(s) \nabla_L G(x - z_s) \, ds
$$

where the definition of $M_n^{abL}(s)$ in terms of $\mathcal{A}_n^L \ldots \mathcal{K}_n^L$ is similar to Eq. (23). Now, when $\nabla_s q_n^{ab}$, as in Eq. (24), is added to $\nabla_s \mathcal{P}_n^{ab}$, as in Eqs. (43) and (45), and like terms are matched up, the result of Eq. (32) is

\begin{align*}
\mathcal{A}_n^L + \mathcal{B}_n^L + \mathcal{D}_n^L = O(G^{n+1}),
\mathcal{B}_n^L + \mathcal{C}_n^L + \mathcal{R}_n^L = O(G^{n+1}),
\mathcal{C}_n^L + \mathcal{G}_n^L + \mathcal{J}_n^L = O(G^{n+1}),
\mathcal{D}_n^L + \mathcal{E}_n^L + \mathcal{H}_n^L + \mathcal{K}_n^L = O(G^{n+1}).
\end{align*}

These form a set of coupled, linear, ordinary, inhomogeneous differential equations for $\mathcal{A}_n^L \ldots \mathcal{K}_n^L$ with sources involving $\mathcal{P}_n^L \ldots S_n^L$. And any solution to these equations gives a corresponding $q_n^{ab}$ via Eq. (24) which, along with $p_n^{ab}$ determines $\delta h_n^{ab}$ and formally yields an improved, approximate solution to the Einstein equations.

A particular solution to most of these equations results from setting $\mathcal{A}_n^L$, $\mathcal{C}_n^L$, $\mathcal{D}_n^L$, $\mathcal{H}_n^L$, $\mathcal{J}_n^L$ and $\mathcal{K}_n^L$ to zero, then

$$
\mathcal{B}_n^L = \mathcal{P}_n^L, \quad \mathcal{F}_n^L = \mathcal{R}_n^L - \mathcal{B}_n^L, \quad \mathcal{G}_n^L = \mathcal{Q}_n^L, \quad \text{and} \quad \mathcal{E}_n^L = S_n^L.
$$

The general solution to Eqs. (47)–(50) is this particular solution plus any homogeneous solution for the $\mathcal{A}_n^L \ldots \mathcal{K}_n^L$. And a homogeneous solution added in at the $n$th iteration is no different from starting the entire iterative process with a slightly different choice for the first order $\mathcal{A}_n^L \ldots \mathcal{K}_n^L$.

The specific solution of Eqs. (47)–(50) to be used should be determined by the physics of the interior. For example, the particular solution in Eq. (51) is easy to implement and leaves unchanged all of the mass and current moments, $\mathcal{A}_n^L$ and $\mathcal{C}_n^L$ respectively. This may be loosely interpreted as the appropriate solution for a steady object and is not unreasonable as a choice for any astrophysically interesting source in a binary as long as tidal effects are unimportant. Also, this particular solution coupled with the simple choice for the first order moment of only the mass monopole, $\mathcal{A}_1$, and current dipole, $\mathcal{C}_1$, being non-zero ought to be able to reproduce numerically the results of the higher order post-Newtonian analyses which are currently published and also contain no tidal effects. But, to find the solution appropriate for a tidally distorted star is more difficult—$p_n^{ab}$ within the boundary creates tidal forces, distorts the star and changes all of the moments, $\mathcal{A}_n^L \ldots \mathcal{K}_n^L$, in a manner which would need to be determined. Or, for a black hole within the boundary a perturbative analysis might be used to determine the appropriate solution of Eq. (17)–(20). In any event, a variety of different possibilities could be implemented; the actual choice made should be specific to the physics of the interior sources.

But, the particular solution, above, fails for the low multipoles $\mathcal{B}_n$, $\mathcal{F}_n^b$ and $\mathcal{G}_n^b$ because these SSTF tensors don’t exist and can’t satisfy Eq. (51). Thus, we still must contend with

\begin{align*}
\dot{\mathcal{A}}_n + \ddot{\mathcal{A}}_n - \mathcal{P}_n = O(G^{n+1}),
\dot{\mathcal{A}}_n^b + \ddot{\mathcal{A}}_n^b - \mathcal{P}_n^b = O(G^{n+1}),
\dot{\mathcal{B}}_n^b + \ddot{\mathcal{B}}_n^b - \mathcal{P}_n^b = O(G^{n+1}),
\dot{\mathcal{C}}_n^b + \ddot{\mathcal{C}}_n^b - \mathcal{Q}_n^b = O(G^{n+1}).
\end{align*}

These remaining ten ordinary linear differential equations have simple interpretations. $\dot{\mathcal{A}}_n + \ddot{\mathcal{A}}_n$ gives the rate of change of the mass monopole moment, and $\mathcal{P}_n$ is analogous to the rate energy flows into the source through the boundary. $\dot{\mathcal{C}}_n^b + \ddot{\mathcal{C}}_n^b$ is similar to the rate of change of spin angular momentum, and $\mathcal{Q}_n^b$ is analogous to the torque. $\dot{\mathcal{B}}_n^b + \ddot{\mathcal{B}}_n^b$ gives the rate of change of momentum of the source with respect to the world line, and $\mathcal{R}_n^b$ is analogous to the force. $\dot{\mathcal{A}}_n^b + \ddot{\mathcal{A}}_n^b$ is closely related to the rate of change of the dipole moment caused by the momentum of the source with respect to the world line and by $\mathcal{P}_n^b$, which has no common Newtonian analog.
Eqs. (52) and (53) may be integrated as ordinary differential equations. Then, after a proper time \( s = O(G^{-1}) \), \( \mathcal{A}_n \), for example, will typically have grown large enough that the order of the approximation will have decreased by one. This is not particularly troublesome, and just implies that to obtain an approximate solution to the Einstein equations with \( E^{ab}(h_n) = O(G^N) \) after a time \( s = O(G^{N-n}) \) requires that \( n = N + m - 1 \). And for a binary system the approximation loses one order only when the fractional change in the mass of one of the components is \( O(G^n) \).

But the physical interpretations of Eqs. (53) and (54) give cause for concern. So far in this formalism the world line has been given ahead of time. And the changing dipole moment and relative momentum of Eqs. (53) and (54) just reflect the fact that the true, physical source is moving with respect to the predetermined world line. But all of the moments of the source are calculated about the world line which does not necessarily follow the center of mass of the source. And as the source drifts away from the world line a rapidly growing number of multipole moments need to be monitored to adequately describe the source. This would be a disaster for any implementation.

Thus at the \( n - 1 \) iteration, before getting to this stage, we should have made certain that the trajectory of the world line was chosen so that

\[
\mathcal{R}_n^b - \hat{\mathcal{P}}_n^b = O(G^{n+1}).
\]

Then with \( \mathcal{B}_n^b = \mathcal{P}_n^b + O(G^{n+1}) \), Eqs. (53) and (54) are solved with all of \( \mathcal{A}_n^b \), \( \mathcal{D}_n^b \) and \( \mathcal{H}_n^b \) being zero. And with no growing dipole moment the plethora of required moments is avoided.

In the next section we show that Eq. (56) is essentially the \( (n - 1) \)th order equation of motion of the source and generally necessitates an \( O(G^{n-1}) \) adjustment of the world line.

At this point (if not previously) one might wonder what the effects of making a different choice for the radius of the boundary, \( r_0 \), might be. For example, if \( r_0 \) are increased then Eq. (52) shows that \( p_n^{ab} \) would change by the addition of a homogeneous solution to the wave equation whose divergence would account for any consequent change in the \( \mathcal{P}_n^L \ldots \mathcal{S}_n^L \). Thus the change could be absorbed by \( q_n^{ab} \).

IV. EQUATIONS OF MOTION

After the \( n \)th iteration of the field equations as described in §II B both \( h_n^{ab}(s) \) and \( z_n^{a-1}(s) \) are known. Before iterating the field equations again, it is first necessary to adjust the world line, \( z_n^{a-1}(s) \rightarrow z_n^a(s) \), in order to enforce the \( n \)th order equation of motion, \( \mathcal{R}_n^b - \hat{\mathcal{P}}_n^b = O(G^{n+1}) \).

Care must be taken to insure that this adjustment is accomplished while maintaining the order of accuracy of the current approximation to the metric. Thus, we require a satisfactory method for pulling the self-field of a source along a new world line. An aid in this task is the invariance of the Einstein equations under a Lorentz transformation, as formulated in §II C. As we see below, the appropriate adjustment of a world line necessarily involves changing its acceleration but only by a small amount of \( O(G^N) \). And, pulling the self-field along a new world line is nearly, but not quite, accomplished by a Lorentz transformation. However, the retarded Poincare transformation, described in Appendix D, as a generalization of the Lorentz transformation, allows for a time-dependent boost and is still adequately behaved globally. The Einstein equations are not strictly invariant under a retarded Poincare transformation; but, as demonstrated for the scalar wave equation in Appendix E, they are approximately so. And the retarded Poincare transformation is sufficient for the task.

First in this section, we show how to implement the retarded Poincare transformation to pull the self-field of a source along a new world line in a manner that maintains the accuracy of the approximation to the Einstein equations. Then, we show just how the new world line is chosen to satisfy the \( n \)th order equation of motion.

A. Adjusting \( h_n^{ab} \)

A retarded Poincare transformation, described in Appendix E, adjusts a world line by defining a new coordinate system with

\[
y^{a'} = \Lambda_n^{ab}(s_x)x^b + \zeta_n^{a'}(s_x),
\]

where \( \Lambda_n^{ab}(s_x) \) is a matrix of the form of a Lorentz transformation and a function of the retarded time, \( s_x \), at \( x^b \); also \( \zeta_n^{a'}(s_x) \) satisfies Eq. (56). The world line in these new coordinates is

\[
z_n^{a'} = \Lambda_n^{a'b}(s_x)z_n^{b} + \xi_n^{a'}(s_x),
\]

\[10\]
with the consequence, noted in Eq. [D11], that

\[ v_n^a = \Lambda^a^b(s) v_{n-1}^b. \]  

(59)

We show below how to choose \( \Lambda^a^b(s) \) to determine a new world line along which the equation of motion is satisfied; this has \( \Lambda^a^b(s = 0) = \delta^a^b \), to match smoothly to the initial data, and \( \Lambda = O(G^n) \) to keep the adjustment small.

The field \( h_{n}^{ab} \) is separated into a self field, \( h_{A}^{ab} \), and a background field, \( h_{B}^{ab} \),

\[ h_{n}^{ab} = h_{A}^{ab} + h_{B}^{ab} \]  

(60)

where \( h_{A}^{ab} \) contains at least the \( O(G^1) \) part of the source whose world line is under consideration, and \( h_{B}^{ab} \) contains at least the \( O(G^1) \) part of all other sources. The distribution of the remaining \( O(G^n; n \geq 2) \) parts of \( h_{n}^{ab} \) between \( h_{A}^{ab} \) and \( h_{B}^{ab} \) is immaterial.

There is no unique way to pull the self-field \( h_{A}^{ab} \) along with the new world line. But if \( \Lambda^a^b \) were constant then the usual Lorentz boost would be required. Thus, a natural choice for the self-field associated with a world line adjusted via a retarded Poincare transformation is

\[ h_{A \text{new}}^{a'b'}(y) = \Lambda^a^c \Lambda^b^d h_{A}^{cd}(x), \]  

(61)

and this seems particularly reasonable when \( \Lambda^c^a = O(G^n) \) and is small. Thus the new field, \( h_{\text{new}}^{a'b'}(y) \), in the new coordinates is chosen to be

\[ h_{\text{new}}^{a'b'}(y) \equiv \Lambda^a^c \Lambda^b^d h_{A}^{cd}(x) + \delta^a^c \delta^b^d h_{B}^{cd}(y) \]

\[ = \delta^a^c \delta^b^d h_{n}^{cd}(y) + [\Lambda^a^c \Lambda^b^d h_{A}^{cd}(x) - \delta^a^c \delta^b^d h_{A}^{cd}(y)] \]  

(62)

where \( x \) is the function of \( y \) consistent with the inverse of Eq. \([57]\). The background field \( h_{B}^{ab} \) is the same function of \( y^{a'} \) as it was of \( x^{a} \), while the self field is simultaneously pulled along and boosted by the time dependent Lorentz transformation, \( \Lambda^a^c(s) \). This choice is consistent with the derivation of Eq. \([73]\) below.

But, we must show that the nonlinearity of \( E^{ab}(h) \) combines with \( \Lambda^a^b \) to change \( E^{ab} \) only at \( O(G^{n+1}) \). The part of \( h_{\text{new}}^{a'b'}(y) \) in square brackets in Eq. \([12]\) vanishes when \( s \) is zero (because \( \Lambda^a^b(s = 0) = \delta^a^b \)), is proportional to \( h_{A}^{ab} \) and is, therefore, small and \( \sim s h_{A} \) = \( O(s G^{n+1}) \). Now, \( E^{a'b'}(h_{\text{new}}^{a'b'}(y)) \) can be expanded about its value at \( \delta^a^c \delta^b^d h_{n}^{cd}(y) \) and broken up into the parts

\[ E^{a'b'}(h_{\text{new}}^{a'b'}(y)) = E^{a'b'}[\delta^a^c \delta^b^d h_{n}^{cd}(y)] \]

\[ + E_{\text{linear}}^{a'b'}[\Lambda^a^c \Lambda^b^d h_{A}^{cd}(x) - \delta^a^c \delta^b^d h_{A}^{cd}(y)] \]

\[ + E_{\tau}^{a'b'} + O(s^{2}G^{2n+2}), \]  

(63)

where \( E_{\text{linear}}^{a'b'} \) denotes the linear part of the operator \( E^{a'b'} \), from Eq. \([2]\); and \( E_{\tau}^{a'b'} \) is the part derived from \( \tau^{ab} \) which is still linear in \( \Lambda^a^c \Lambda^b^d h_{A}^{cd}(x) - \delta^a^c \delta^b^d h_{A}^{cd}(y) \) but also depends upon \( \delta^a^c \delta^b^d h_{n}^{cd}(y) \). The \( O(s^{2}G^{2n+2}) \) terms remaining are at least quadratic in \( \Lambda^a^c \Lambda^b^d h_{A}^{cd}(x) - \delta^a^c \delta^b^d h_{A}^{cd}(y) \). To observe how well \( h_{\text{new}}^{a'b'} \) satisfies the Einstein equations, we analyze Eq. \([3]\) term by term.

The first term is \( O(G^{n+1}) \) by assumption.

The functional argument of the second term consists of the difference of two parts, each of which is \( O(G) \), but whose difference is \( O(s G^{n+1}) \). The \( O(G) \) piece of each part is a solution to the linear Einstein equations. And the difference of the \( O(G^2) \) pieces of the two parts is only \( O(s G^{n+1}) \). Thus this second term is \( O(s G^{n+1}) \).

The third term consists of a sum of terms each of which is of the order of at least the product of \( \delta^a^c \delta^b^d h_{n}^{cd}(y) \) with \( \Lambda^a^c \Lambda^b^d h_{A}^{cd}(x) - \delta^a^c \delta^b^d h_{A}^{cd}(y) \); the former is \( O(G) \), the latter is \( O(s G^{n+1}) \). Thus the third term is \( O(s G^{n+2}) \).

All together then

\[ E^{a'b'}[h_{\text{new}}^{a'b'}] = O(G^{n+1}) + O(s G^{n+2}). \]  

(64)

And we see that while \( s = O(G^{-1}) \) the error is \( O(G^{n+1}) \), after that the order of the approximation decreases by one in a manner similar to \([11\beta]\). This is not a severe limitation on applications of this method to binary systems, where the radius of the boundary can be chosen to be of the same order of magnitude as the separation between the components, \( R \). For a total mass of the system, \( M \), and a typical speed, \( V \), \( GM/R \approx (V/c)^2 = O(G) \). The order of the approximation decreases by one only when \( sc/R = O(G^{-2}) \), and this occurs when \( sV \approx R(c/V)^3 \). Thus the binary must orbit on the order of \((c/V)^3\) times before the order of approximation decreases.
B. Adjusting the world line

At this stage of an iterative step, we know \( z_n^0(s), h_n^a(x) \) and \( h_n^b(x) \). This section shows how a change in the acceleration of the world line, effected by a retarded Poincaré transformation with non-vanishing \( \dot{A}^a_{b}(s) \), insures that the \( n \)th order equation of motion is satisfied.

For simplicity we assume that out of the first order moments, \( A_1^a \ldots K_1^a \), only \( A_1 \) is \( O(G) \), and all of the others are smaller and \( O(G^2) \). This is a reasonable physical assumption for astrophysical objects, and it should be clear how to generalize this analysis if one desires to examine, say, spin-orbit coupling by including \( C_4^b \), yields smaller and

\[
\frac{1}{2} \dot{g}^{ab}(k_c \dot{v}^c + r^{-1} k_v \dot{v}^c + (k_v \dot{v}^c)^2)].
\]

The evaluation of \( R_A^b \) and \( P_A^b \) from Eqs. (44) and (45) requires the two integrals

\[
\frac{1}{4\pi} \int_{s, r_0} \nabla_a r E_A^{ab} r_0^2 \sin \theta d\theta d\phi = -A_1(v^b + r_0 \dot{v}^b) + O(G^3) \tag{66}
\]

and

\[
-\frac{1}{4\pi} \int_{s, r_0} \nabla_a r E_A^{ab} n^d r^3 \sin \theta d\theta d\phi = \frac{1}{3} A_1 r_0^2 v^b \dot{v}^d + O(G^3), \tag{67}
\]

where use is made of the fact that \( v_0 \dot{v}^b = -\dot{v}_0 \dot{v}^b = O(G^2) \). Now Eq. (66), along with \( b_{2m, m} = 1/(2m + 1) \) from Eq. (34), yields

\[
R_A^b(s) = -A_1 \sum_{m=0}^{\infty} \left[ \frac{r_0^{2m}}{(2m + 1)!} \frac{d^{2m}}{ds^{2m}} (\dot{v}^b + r_0 \dot{v}^b) \right]_{s - r_0} + O(G^3). \tag{68}
\]

That the right hand side here is evaluated at \( s - r_0 \) is a complicating consequence of Eq. (14). Also, from Eq. (72), along with \( \dot{b}_{1 + 2m, m} = 3/(2m + 3) \) from Eq. (35), it follows that

\[
P_A^b(s) = A_1 \sum_{m=0}^{\infty} \left[ \frac{r_0^{2m+2}}{(2m + 1)(2m + 3)!} \frac{d^{2m+2}}{ds^{2m+2}} \dot{v}^b \right]_{s - r_0} + O(G^3). \tag{69}
\]

These two equations combine to yield

\[
(R_A^b - P_A^b)_s = -(A_1 \dot{v}^b)_{s - r_0} - A_1 \sum_{m=0}^{\infty} \left[ \frac{r_0^{2m+2}}{(2m + 1)!} \frac{d^{2m}}{ds^{2m}} \dot{v}^b \right. \\
\left. + \frac{r_0^{2m+1}}{(2m + 1)!} \frac{d^{2m+1}}{ds^{2m+1}} \dot{v}^b + \frac{r_0^{2m+2}}{(2m + 1)(2m + 3)!} \frac{d^{2m+2}}{ds^{2m+2}} \dot{v}^b \right]_{s - r_0} + O(G^3); \tag{70}
\]

the first two terms inside the summation come from the \( \dot{v}^b \) and \( \ddot{v}^b \) parts of \( R_A^b \), respectively; the third term comes from \( P_A^b \). The two \( \dot{v}^b \) terms add directly, and the entire expression simplifies remarkably to

\[
(R_A^b - P_A^b)_s = -A_1 \sum_{m=0}^{\infty} \left( \frac{r_0^m}{m!} \frac{d^{m}}{ds^{m}} \dot{v}^b \right)_{s - r_0} + O(G^3). \tag{71}
\]

Finally, the right hand side is a Taylor series expansion of \( -A_1 \dot{v}^b \) about \( s - r_0 \) but evaluated at \( s \) so that
\[ R^b_a - \dot{\mathcal{P}}^b_a = -A_1 \dot{v}^b + O(G^3), \]  
(72)

and both sides of this equation are now evaluated at the same \( s \). The surprising simplicity of this last result might imply that a substantially more straightforward derivation could be found.

A retarded Poincare transformation, with Eq. (59), effects the acceleration of the world line by

\[ \dot{v}^b_n = \Lambda^b_a v^a_{n-1} + \dot{\Lambda}^b_a v^a_{n-1}. \]  
(73)

Now, we assume that \( \Lambda^b_a(s) \) has been determined consistently with the equation of motion for all \( s \) up to some value \( s_0 \). Then both \( R^b_a \) and \( \dot{\mathcal{P}}^b_a \) at \( s_0 \) can be found from Eqs. (44) and (45) along with the temporary assumption that \( \dot{\Lambda}^b_a(s_0) = 0 \). And we define

\[ F^b = (R^b_a - \dot{\mathcal{P}}^b_a)_{s_0} \quad \text{with} \quad \dot{\Lambda}^b_a(s_0) = 0. \]  
(74)

But, a value for \( \dot{\Lambda}^b_a(s_0) \) of \( O(G^n) \) changes \( (R^b_a - \dot{\mathcal{P}}^b_a)_{s_0} \) by \( -A_1 \dot{\Lambda}^b_a v^a_{n-1} + O(G^{n+2}) \), from Eqs. (72) and (73). Thus if we choose at \( s_0 \) that

\[ A_1 \dot{\Lambda}^b_a(s_0) v^a_{n-1} = F^b, \]  
(75)

then the adjusted world line will satisfy the \( n \)th order equation of motion at \( s_0 \) as well.

This differential equation for \( \Lambda^b_a \) is consistent with \( \dot{\Lambda}^b_a = O(G^n) \), as promised in \[ \text{V}, \text{A} \], and gives three equations for \( \Lambda^b_a \) (the \( b \) index is orthogonal to \( v^b \)); the remainder of \( \Lambda^b_a \) is determined by the requirement of Fermi-Walker transport Eq. (D21).

With Eq. (74) the differential equation can be rewritten as

\[ A_1 \dot{v}^b_n = A_1 \Lambda^b_a v^a_{n-1} + F^b, \]  
(76)

which has the expected form for an iteration of the equation of motion with \( F^b \), a residual force remaining on world line \( z_{n-1}(s) \). It is not difficult to show that at the first order this equation of motion is equivalent to the usual post-Newtonian result as presented by Bel et al. [13].

V. BEHAVIOR AT FUTURE NULL INFINITY

Now we reconsider the iterative procedure outlined in \[ \text{II}, \text{B} \] with particular attention given to the limit of large \( r \) while \( s = t - r \) is held constant. Thus we consider the approach to future null infinity and show how to insure that the outgoing radiation propagates along flat space null cones which match up asymptotically with the null cones of the true, physical space-time. At every iteration an \( O(G^n) \) gauge transformation, \( \partial \chi^{ab}_{n+1} \), and a small contribution, \( \chi^{ab}_{n+1} = O(G^{n+2}) \), insure that at large \( r \), \( h^{ab}_{n+1} \) can be written as an expansion in inverse powers of \( r \), times functions of retarded time, \( s \), and angle, \( \mathbf{n} \),--in particular there are to be no \( \ln r \) terms in this expansion. We refer to such an expansion as a proper expansion in inverse powers of \( r \).

We continue to use outgoing-null spherical coordinates, \((s, r, \theta, \phi)\), but now they are tied to a non-accelerating world line near the center of the binary system.

At large \( r \), \( h^{ab}_{1} \) admits a general multipolar decomposition just like that presented in Eqs. (22) and (23) and satisfying Eqs. (24)–(25) with the \( O(G^2) \) terms also being \( O(r^{-2}) \). Thus,

\[ h^{ab}_{1} = r^{-1} \chi^{ab}_{1}(s, \mathbf{n}) + O(r^{-2}) \]  
(77)

defines \( \chi^{ab}_{1} \), the dominant part of \( h^{ab}_{2} \) at large \( r \). But at second order, a difficulty immediately arises in evaluating \( p^{ab}_{2} \). Namely, \( E^{ab}_{1} = O(r^{-2}) \) and Eq. (25) gives a \( r^{-1} \ln r \) term to \( p^{ab}_{2} \) (Theorem 7.2 of Blanchet and Damour [10]). Such logarithmic behavior is the signature of a mismatch between the null cones of the background Minkowski space and of space-time. Blanchet [12] shows that when the \( O(r^{-2}) \) part of \( E^{ab}_{2} \) is of a particular form, Eq. (55) below, then the logarithmic terms and the mismatch can be removed by a gauge transformation. Our analysis follows Blanchet [12] closely, except that we differ on a choice of gauge for \( h^{ab}_{2} \) and that his analysis involves a clean separation of the powers of \( G \), while our \( O(G^n) \) terms contain further functional dependence on \( G \).

Analysis of the definition of \( r^{ab}(h) \) reveals that if \( k_a \chi^{ab}_{1} \) were zero then the offending \( r^{-2} \) part of \( E^{ab}_{2} \) would be easy to evaluate. Generally, \( k_a \chi^{ab}_{1} \) is not zero. In fact, it is straightforward, but not simple, to see that the restrictions \[ \text{24} - \text{28} \] imply that
\[ k_a \chi^{ab} = -v^b (A_1 + \dot{D}_1) - (B^b_1 + \dot{r}^b), \]  
(78)

where \( A_1 \), \( B^b_1 \), \( D_1 \) and \( \mathcal{H}^b_1 \) now refer to the multipole moments of \( h^{ab}_1 \) as measured with respect to the non-accelerating center of the Minkowski background geometry. With foresight, a Lorentz transformation removes the three-momentum, \((B_1^b + \mathcal{H}^b_1)/4\); a gauge transformation with \( \nabla^2 \chi^a = 0 \) (this preserves the general form of \( h^{ab}_1 \) and is discussed by Thorne [9]) sets \( D_1 = 0 \); and a second, preemptive, gauge transformation with \( \lambda^a = (\frac{1}{2} A_1 \ln r, -\frac{1}{2} A_1 n^a) \) finally results in an \( h^{ab}_1 \) whose lowest non-radiative multipoles take the simple form at large \( r \)

\[ h^{ab}_1 \approx \frac{i}{2r} A_1 k^a b^b. \]  
(79)

(Interestingly, this is the exact, at all orders, solution for \( h^{ab}_1 \) for a Schwarzschild black hole in outgoing Eddington-Finkelstein coordinates [14.] With these gauge choices, \( h^{ab}_1 \) is now in a form such that

\[ k_a \chi^{ab} = 0. \]  
(80)

With this last result, the asymptotic behavior of \( E^{ab}_1 \) is dominated by \( r^{ab}(h_1) \) and is of the form

\[ E^{ab}_1 = -r^{-2} k^a b^b \Psi_2(s, n) + \tilde{E}^{ab}_1 \]  
(81)

where \( \tilde{E}^{ab}_1 = O(G^2 r^{-3}) \) and

\[ \Psi_2(s, n) = \frac{1}{2} \chi^{ab} \chi_{1ab} - \frac{1}{2} \chi_{ab} \chi_{1b} = O(G^2). \]  
(82)

We note that \( \Psi_2 \) may be interpreted as being proportional to the effective energy density of the outgoing gravitational waves.

Generally, this behavior for \( E^{ab}_n \) occurs at every order. Rather than continuing a discussion with a focus on the second iterative order equations, we switch to the consideration of the \( n \)th iterative order and continue following Blanchet’s [14] analysis closely.

We iteratively assume that \( h^{ab}_{n-1} \) has a proper expansion in inverse powers of \( r \) times functions of \( s \) and \( n \),

\[ h^{ab}_{n-1} = r^{-1} \delta h^{ab}_{n-1}(s, n) + O(G n^{-2}), \]  
(83)

with \( \delta \chi^{ab}_{n-1} \equiv \chi^{ab}_{n-1} - \chi^{ab}_{n-2} = O(G n^{-1}), \)

\[ k_a \chi^{ab}_{n-1} = 0, \]  
(84)

and that

\[ E^{ab}_{n-1} = -r^{-2} k^a b^b \Psi_n(s, n) + \tilde{E}^{ab}_{n-1} \]  
(85)

where \( \tilde{E}^{ab}_{n-1} = O(G^n r^{-3}) \) and \( \Psi_n = O(G^n) \). First we seek \( h^{ab}_{n} \) and \( \tilde{E}^{ab}_{n} \) such that \( E^{ab}_{n}(h_{n-1} + \delta h_{n}) = O(G^n r^{-3}) \) where this \( \delta h^{ab}_{n} \) differs from that of [14] by a gauge transformation and a small addition and, also, contains no \( \ln r \) term at any order. Then we must reconsider the analysis of Eq. (85) to account for the difficulties caused by the \( O(G^n r^{-2}) \) behavior of \( E^{ab}_{n-1} \).

The initial task is to guarantee that \( E^{ab}_{n}(\delta h_{n}) = -E^{ab}_{n-1} + O(G^{n+1} r^{-3}) \) so that after the \( n \)th step \( E^{ab}_{n} \) will satisfy the incremented version of Eq. (85). The substitution of the Bianchi identity (14) into Eq. (85) results in

\[ \nabla_v P^{ab}_{n} = \frac{1}{4\pi} \int \nabla_v [E^{ab}_{n-1} G(x - x')] d^4 x' \]
\[ - \frac{1}{4\pi} \int E^{ac}_{n-1} \left( \eta_c b^{d} \Gamma_{da} - \Gamma_{ac}^{b} \right) G(x - x') d^4 x'. \]  
(86)

The second integrand is \( O(G^{n+1} r^{-3}) \), thus the integral is \( O(G^{n+1}) \) and has a proper expansion in inverse powers of \( r \) the leading term of which matches an outgoing solution of the homogeneous vector wave equation. In §11, \( q^{ab}_{n} \) was chosen to cancel just the \( O(G^n) \) part of \( \nabla_v P^{ab}_{n} \); now, \( q^{ab}_{n} \) in Eq. (82) can be chosen at no additional expense to cancel this \( O(G^{n+1} r^{-1}) \) leading term from the second integral as well. This results in

\[ \nabla_v (\varphi^{ab}_{n} + q^{ab}_{n}) = r^{-2} \tilde{E}^{ab}_{n+1}(s, n) + O(G^{n+1} r^{-3}) \]  
(87)

for some vector \( \tilde{E}^{ab}_{n+1}(s, n) = O(G^{n+1}) \). Further, if we choose \( \gamma^{ab}_{n+1} = O(G^{n+1} r^{-2}) \) such that
Lemma 2.1 that if both Eqs. (90) and (92) are used below. And so that it follows that

\[ \nabla H_n^{ab} = O(G^{n+1}r^{-3}). \]  

(90)

Also

\[ \nabla^2 H_n^{ab} = E_{n-1}^{ab} + \nabla^2 \gamma_{n+1}^{ab} = E_{n-1}^{ab} + O(G^{n+1}r^{-3}). \]  

(92)

Both Eqs. (90) and (92) are used below. And \( E_{\text{linear}}^{ab}(H_n) = -E_{n-1}^{ab} + O(G^{n+1}r^{-3}) \) as required, so \( H_n^{ab} \) would be the choice for \( \delta h_n^{ab} \) except for the \( \ln r \) behavior of \( p_n^{ab} \).

An \( O(G^n) \) gauge transformation resolves this difficulty and leaves \( E_{\text{linear}}^{ab} \) unchanged. Blanchet \[12\] shows with his Lemma 2.1 that if

\[ \lambda_n = \frac{1}{4\pi} \int \frac{1}{2r^2} k^a \int_{-\infty}^{s'} \Psi_a(t,n')dt \ G(x-x') \ d^4x' \]  

(93)

then

\[ \partial \lambda_n^{ab} = -\frac{1}{4\pi} \int \frac{k^a k^b}{r^2} \Psi_a(s',n') G(x-x') \ d^4x' \]  

\[ + O(G^n r^{-1}) \]  

(94)

and

\[ \nabla a \partial \lambda_n^{ab} = -\frac{1}{2r^2} k^a \int_{-\infty}^{s} \Psi_a(s',n') \ ds' \]  

(95)

Thus, from Eqs. (33), (58) and (59) the combination \( p_n^{ab} + \partial \lambda_n^{ab} \) is a proper expansion in inverse powers of \( r \). And, with

\[ \delta h_n^{ab} \equiv p_n^{ab} + q_n^{ab} + \partial \lambda_n^{ab} + \gamma_{n+1}^{ab} \]  

(96)

and \( h_n^{ab} = h_{n-1}^{ab} + \delta h_n^{ab} \), it follows that \( \delta h_n^{ab} \) has a proper expansion in inverse powers of \( r \), and Eq. (33) holds with \( n-1 \to n \).

The iterated versions of Eq. (84) and (85) remain to be checked. From Eqs. (90) and (95) it follows that \( \nabla_a \delta h_n^{ab} = O(G^n r^{-2}) \). But with \( \delta h_n^{ab} \equiv r^{-1} \delta \lambda_n^{ab}(s,n) + O(G^n r^{-2}) \), it must also be that

\[ \nabla_a \delta h_n^{ab} = -r^{-1} k_a \delta \lambda_n^{ab} + O(G^n r^{-2}). \]  

(97)

Hence \( k_a \delta \lambda_n^{ab} = 0; \) and, for \( n > 1 \), \( \delta \lambda_n^{ab} \) is zero on the initial hypersurface, where \( h_n^{ab} \) matches smoothly onto the initial data, so \( k_a \delta \lambda_n^{ab} = 0 \) always, and Eq. (84) holds with \( n-1 \to n \).

Finally, the analysis of Eq. (48) is modified by the presence of \( \partial \lambda_n^{ab} \) and \( \gamma_{n+1}^{ab} \); but,

\[ E^{ab}(h_n) = [E^{ab}(h_n) - E^{ab}(h_{n-1})] + E^{ab}(h_{n-1}) \]

\[ = -\nabla^2 \delta h_n^{cb} + \nabla^a \nabla_c \delta h_n^{cb} + \nabla^b \nabla_c \delta h_n^{ca} - \eta^{ab} \nabla_c \nabla_d \delta h_n^{cd} - 16 \pi [\tau^{ab}(h_n) - \tau^{ab}(h_{n-1})] + E^{ab}(h_{n-1}) \]

\[ = \nabla^a \nabla_c H_n^{cb} + \nabla^b \nabla_c H_n^{ca} - \eta^{ab} \nabla_c \nabla_d H_n^{cd} - 16 \pi [\tau^{ab}(h_n) - \tau^{ab}(h_{n-1})] + O(G^n r^{-3}) \]  

(98)

where the second equality follows from the definition of \( E^{ab}(h) \), Eq. (12). And the third equality is a consequence both of \( O(G^n) \) gauge invariance of \( E_{\text{linear}}^{ab} \) and also of Eq. (92). Now, Eq. (80) and the application of the iterated versions of Eqs. (83) and (84) to the definition of \( \tau^{ab}(h_n) \) imply that \( E^{ab}(h_n) \) is of the form of Eq. (83) with
\[
\Psi_{n+1}(s, n) = \frac{1}{2}(\chi_n^{ab} \chi_{nab} - \chi_n^{ab} \chi_{n-1 ab})
- \frac{1}{4}(\chi_n^{ab} \chi_{nab} - \chi_n^{ab} \chi_{n-1 ab}).
\]

An iterative step is thus formulated in a manner which leaves \( h_n^{ab} \) expressible as a proper expansion in inverse powers of \( r \) with Eqs. (33)–(35) holding at every iteration. And the gravitational waves asymptotically expand out along constant \( s \) surfaces, so the outgoing null cones of the true space-time metric asymptotically match up with the flat space null cones.

VI. CONCLUSIONS

We have given a prescription for iteratively improving an approximate solution to the Einstein equations which could be carried out, by computer, to any order. The lowest order approximation is just the familiar linearized approximation of general relativity.

The description of the iterative process in the text was intended to follow a logical order to provide motivation for each part, in turn, of one full iterative step. However, in practice the chronological order is slightly different. Thus, we now summarize the entire process with a brief chronological description of the procedure.

For the initial step we choose \( h^0 \) to be the algebraic sum of two terms like \( A_1 v^a v^b / r \), one for each of the two sources. Let this \( h^0 \) be considered \( h_n^{ab} \) and let the world lines be \( z_n^{a-1} \).

In order to iterate the equations of motion, first find \( E_n^{ab} \), on each boundary and then find \( R_{n+1}^{ab} \) and \( P_{n+1}^{ab} \) from Eqs. (33) and (15). Now, Eqs. (77) and (D2) determine the \( \Lambda_n \) which adjusts the world line to \( z_n \) and modifies \( h_n^{ab} \) to \( h_{n+1}^{new} \) with Eq. (32) while preserving \( E^{(h_{new})} = O(G^{n+1}) \).

Next the field equations are iterated by using Eq. (33) to determine \( p_{n+1}^{ab} \) and Eq. (40) for \( q_{n+1}^{ab} \), with the \((n+1)\)th order moments satisfying both Eqs. (47)–(50) and also appropriate conditions determined by the physics of the sources within the boundaries. The determination of \( p_{n+1}^{ab} \) involves solving the inhomogeneous wave equation for all independent components of the symmetric tensor. This is the single, computationally intensive part of every iterative step.

At this point \( E^{ab}(h_n + p_{n+1} + q_{n+1}) = O(G^{n+2}) \). But to preserve proper behavior at future null infinity with the outgoing null cones of flat space-time matching up asymptotically with those of the true, physical space-time, a gauge transformation, \( \partial \chi_{n+1}^{ab} \), from Eq. (93) is needed.

In preparation for the next iteration \( q_{n+1}^{ab} \) should also cancel the \( O(G^{n+2} r^{-1}) \) contribution from the second integral in Eq. (86). And \( h_{n+1}^{ab} \) should be changed by a small correction, \( \gamma_{n+2}^{ab} = O(G^{n+2} r^{-2}) \), which satisfies Eq. (95). Now, \( h_{n+1}^{ab} = h_n^{ab} + p_{n+1}^{ab} + q_{n+1}^{ab} + \partial \chi_{n+1}^{ab} + \gamma_{n+2}^{ab} \) completes one full step of the iterative procedure.

The freedom of this iterative process from the restriction of the harmonic gauge may provide an important aid to its implementation. For example the first order \( h^1 \) might be chosen to be the sum of two terms like \( A_1 v^a v^b / r \). Then individually each term would be the exact Schwarzschild geometry, if the source were not accelerating. And \( E^{ab}(h_1) \) would consist only of linear terms dependent upon the acceleration and cross terms between the two sources. This choice for \( h^1 \) would already be an accurate approximation to two Schwarzschild black holes even near the past event horizon of one hole where \( E^{ab}(h_1) \sim M/R \), with \( M \) and \( R \) being the mass of and distance to the companion hole.

One weakness described in §II stems from the inability to treat the conditions at the inner boundaries in a straightforward manner. In problems whose focus is on the emission of gravitational radiation from binary systems, this is a difficulty only when the system is tight enough that tidal deformations are important. To include tidal effects of any sort it is necessary to solve the internal problem, as described for example by Damour [17], and thus to obtain the specific solution of Eqs. (17)–(20) which matches the physics of the problem.

Work in progress applies similar methods to a Schwarzschild background geometry. In this case it appears that perturbation analysis ought to yield boundary conditions which can be properly imposed at the event horizon. This extension will provide a better method for the analysis of black hole binary systems.

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APPENDIX A: THE RETARDED GREEN FUNCTION.

The retarded Green function,

\[ G(x - x') = 2\theta(x^0 - x'^0)\delta(\Omega), \]  

(A1)

where \( \Omega \) is the square of the flat-space interval between two points,

\[ \Omega(x, x') \equiv \eta_{ab}(x^a - x'^a)(x^b - x'^b), \]  

(A2)

is a solution of

\[ \nabla^a \nabla_a G(x - x') = -4\pi\delta^4(x - x'). \]  

(A3)

For a generic problem we wish to find a particular solution of

\[ \nabla^2 h_{ab} = -\rho_{ab} \]  

(A4)

where \( \rho_{ab} \) is a multipolar skeleton source on \( z^a(s) \). Thus, if

\[ \rho_{ab}(x) = 4\pi \int M^{abL}(s) \nabla_L \delta^4(x - z_s) \, ds, \]  

(A5)

then

\[ h_{ab}(x) = \int G(x - x') \nabla'_a \int M^{abL}(s) \delta^4(x' - z_s) \, ds \, d^3x', \]  

(A6)

with \( \nabla'_a \) being the derivative operator with respect to \( x'^a \).

After integrating by parts \( l \) times, changing the derivatives to be with respect to \( x^a \) and integrating over \( x'^a \), we have

\[ h_{ab}(x) = \int M^{abL}(s) \nabla_L G(x - z_s) \, ds, \]  

(A7)

or after withdrawing \( \nabla_L \) from the integral we have

\[ h_{ab}(x) = \nabla_L[r^{-1} M^{abL}(s_x)]. \]  

(A8)

While this result may appear quite familiar, it is important to remember that it holds for a source which is moving along some accelerating world line and has time-changing multipole moments all the while. The consequent radiation results from both the acceleration as well as the varying multipole moments.

APPENDIX B: A USEFUL IDENTITY

A useful identity \([\text{[7]}]\) is

\[ \int f(s) v^a \nabla_a F(x - z_s) \, ds = -\int f(s) \frac{d}{ds} F(x - z_s) \, ds \]

\[ = -[f(s)F(x - z_s)]_{\infty}^{-\infty} + \int \frac{df(s)}{ds} F(x - z_s) \, ds. \]  

(B1)

For our applications this integral is over all proper time, \( s \), and \( F(x - z_s) \) involves the retarded Green function and is zero except where the past null cone from \( x^a \) intersects the world line. With these conditions the contribution from the limits of integration is always zero, and

\[ \int f(s) v^a \nabla_a F(x - z_s) \, ds = \int \frac{df(s)}{ds} F(x - z_s) \, ds. \]  

(B2)

This identity is particularly useful in the reduction leading to Eq. (24).
APPENDIX C: SSTF DECOMPOSITION OF $N^L$

By repeated subtraction of the trace parts, $N^L$ is expressed as

$$N^L = \sum_{m=0}^{[l/2]} b_{l,m} f(a_1 a_2 \ldots f(a_{2m-1} a_{2m}^n)^{a_2 m+1} \ldots a_l),$$

(C1)

for some set of coefficients $b_{l,m}$. Contraction with $f a_{l-1} a_l$ and use of Eq. (C1) with $l \rightarrow l - 2$ yields

$$b_{l-2,m} = \frac{(2m + 2)(2l - 2m - 1)}{l(l - 1)} b_{l,m+1}.$$  

(C2)

It is clear that $b_{l,0} = 1$, and with elementary methods we find that

$$b_{l,m} = \frac{l!(2l - 4m + 1)!}{2^m m!(l - 2m)!(2l - 2m + 1)!}.$$  

(C3)

Some special values which are of use for determining $R^b$ and $P^b$ are

$$b_{2,m,m} = 1/(2m + 1)$$  

(C4)

and

$$b_{1+2m,m} = 3/(2m + 3).$$  

(C5)

APPENDIX D: RETARDED POINCARE TRANSFORMATIONS

The retarded Poincare transformation is a little known method for relating outgoing-null coordinates associated with different world lines. This transformation is a mapping from one flat space-time to another, which transforms one world line, $z^a(s)$, into a second while preserving the values of the scalar fields $s$ and $r$; also, the future null cone of each event on the first world line is mapped to the future null cone of the corresponding event on the second world line. Strictly speaking this transformation is a diffeomorphism from the causal future of $z^a(s)$ (that is from the set of all events which can be reached from $z^a(s)$ by a future directed, non-spacelike curve) onto the causal future of the second world line. This technicality is required to allow for the possibility that one world line has constant acceleration in the distant past, and its causal future is, thus, not the entire Minkowski spacetime.

We first describe the mathematical formalism of the retarded Poincare transformation and then give an application which is closely related to the analysis of §IV A.

1. Mathematical formalism

We start with a given world line $z^a(s)$ in Minkowskii space covered with the usual Minkowskii coordinates, $x^a$, and define a coordinate transformation by

$$y^{a'} = \Lambda^{a'}_{b'}(s_x) x^b + \xi^{a'}(s_x),$$

(D1)

where $\Lambda^{a'}_{b'}$ and $\xi^{a'}$ are functions of $s_x$, explicitly, and of $x^a$, implicitly, and $\xi^{a'}(s)$ satisfies Eq. (D6), below. The matrix $\Lambda^{a'}_{b'}$ is a time dependent Lorentz transformation, i.e. it is a matrix of the general form of a Lorentz boost and a rotation, as described by Misner et al. [16], but with the boost and rotation parameters being functions of $s$; also, $\Lambda_{b'}^{a'}$ is the matrix inverse of $\Lambda^{a'}_{b'}$. Thus transformation (D1) reduces to a Lorentz transformation if $\Lambda^{a'}_{b'}$ is constant and $\xi^{a'} = 0$; thus, this transformation is a time dependent generalization of the Lorentz transformation which is reasonably well behaved in a global sense.

In this section a prime on a base letter identifies a geometrical object which is most naturally discussed in the $y^a$ coordinate system; a prime on an index refers to the components of a geometrical object in the $y^{a'}$ coordinate system. We also define
\( \eta^{a'b'} \equiv \Lambda^a_c \Lambda^{b'}_d \eta^{cd}. \)  
(D2)

From the algebraic properties of Lorentz transformations we know that the \( y^{a'} \) components of \( \eta^{a'b'} \) are \((-1,1,1,1)\) on the diagonal and zero elsewhere. While \( \eta^{a'b'} \) is the usual flat Minkowskii metric of the \( y^{a'} \) coordinate system, it is not the tensor equivalent of \( \eta^{ab} \) with the coordinate transformation (D1), because

\[
\eta^{a'b'} = \frac{\partial y^{a'}}{\partial x^c} \frac{\partial y^{b'}}{\partial x^d} \eta^{cd} \neq \eta^{a'b'}.
\]  
(D3)

We further define

\[
\eta'_{a'b'} \equiv \Lambda^a_c \Lambda^{b'}_d \eta^{cd}.
\]  
(D4)

Then \( \eta'_{a'b'} \) is the matrix inverse of \( \eta^{b'c'} \) because of the usual algebraic properties of Lorentz transformations. And we raise and lower primed indices on primed tensors with \( \eta^{b'c'} \) and \( \eta'_{a'b'} \); however, with two different metrics at hand we rarely raise or lower indices implicitly. From these definitions it follows that

\[
\eta'_{a'b'} \Lambda^{b'}_c = \eta_{cb} \Lambda^a_a,
\]  
(D5)

along with some index variations of this equation.

For the given world line, \( z^a(s) \), we choose \( \xi^a(s) \) so that

\[
\dot{\xi}^a = -\Lambda^a_b \dot{z}^b(s);
\]  
(D6)

this uniquely determines \( \xi^a(s) \) up to the addition of a constant vector. This choice is motivated below, after Eq. (D11).

The transformation of the components of tensors is governed by

\[
\frac{\partial y^{a'}}{\partial x^b} = \Lambda^a_b - \Lambda^a_c x^c k_b - \dot{\xi}^a k_b = \Lambda^a_b - r \Lambda^a_c c^c k_b.
\]  
(D7)

The inverse transformation is

\[
\frac{\partial x^b}{\partial y^{a'}} = \Lambda^b_a - r \Lambda^b_c d^c k_a;
\]  
(D8)

the derivation of this inverse involves some of the results derived below.

We are free to consider the coordinate transformation (D1) as a diffeomorphism from one manifold to a second. Then \( \eta^{ab} \) is a flat metric on the \( z^a \) manifold, and \( \eta^{a'b'} \) is a flat metric on the \( y^a \) manifold. With this point of view, the world line \( z^a(s) \) is mapped to a world line on the \( y^a \) manifold by

\[
z^{a'}(s) = \Lambda^a_b z^b(s) + \xi^{a'}(s).
\]  
(D9)

Then

\[
\dot{z}^{a'}(s) = \Lambda^a_b \dot{z}^b(s) + \dot{\Lambda}^a_b z^b(s) + \dot{\xi}^{a'};
\]  
(D10)

and with Eq. (D6), the four-velocities are related by

\[
v^{a'} \equiv \dot{z}^{a'} = \Lambda^a_b v^b,
\]  
(D11)

and it follows easily that \( \eta_{a'b'} v^{a'} v^{b'} = -1 \) demonstrating that \( s \) is the proper time for the world line \( z^{a'} \) as well.

The future null cone structure of the world line is preserved under the transformation in the sense that the future null cone of the event \( z^a(s) \) is mapped onto the future null cone of the event \( z^{a'}(s) \). For a proof consider the square of the interval between a generic point on the \( y^a \) manifold and a point on the world line,

\[
\Omega'(y^{a'},z^{a'}) \equiv (y^{a'} - z^{a'}) (y^{b'} - z^{b'}) \eta_{a'b'}
\]

\[
= \Omega(x^a,z^a) + 2 \Lambda^a_b (x^b - z^b) \eta_{a'c'} [\xi^{a'}(s_x) - \xi^{a'}(s_z)]
\]

\[
+ [\xi^a(s_x) - \xi^a(s_z)] \eta_{a'c'} [\xi^{a'}(s_x) - \xi^{a'}(s_z)],
\]  
(D12)
where the second equality follows from Eqs. (D11) and (D14). Thus, if \( x^a \) is on the future null cone of \( z^a \) then \( s_x = s_z \) and \( \Omega(x^a, z^a) = 0 \) so that \( \Omega(y^{a'}, z^{a'}(s)) = 0 \); it follows, then, that \( y^{a'} \) is on the future null cone of \( z^{a'} \) as determined by the \( \eta'_a \eta'_b \) metric.

We define a useful scalar field, \( r' \), similar to \( r \) in Eq. (3), by

\[
r' \equiv -v'_a(y^{a'} - z^{a'}) = -\eta_{cd}v^c[x^d - z^d(s_x)] = r.
\]

(D13)

In other words, \( r'(y(x)) = r(x) \).

Finally, from

\[
k_a = -\nabla_a s = (\Lambda^{b'}_a - r\Lambda_i^{b'}_c k^c k_a)\nabla_b s,
\]

and

\[
k_a = (\Lambda^{b'}_a - r\Lambda_i^{b'}_c k^c k_a)k'_b,
\]

and contraction with \( k^a \) reveals that \( k^a\Lambda^{b'}_a k'_b = 0 \); with some effort it also follows that \( \Lambda^{b'}_a k'_b k^a = 0 \) and that both

\[
k_a = \Lambda^{b'}_a k'_b,
\]

and

\[
k^a = \Lambda^a_{b'} k^{b'}.
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With these results at hand, we simplify and summarize the notation. The \( x^a \) manifold and the \( y^a \) manifold have some similar structures which are distinguished by a prime for the structure on the \( y^a \) manifold. Examples are \( z^a \) and \( \eta_{a'b} \), which are similar to \( z^a \) and \( \eta_{ab} \). But if we consider the mapping from \( x^a \) to \( y^a \) to be a coordinate transformation then we can denote the components of \( \eta' \) in the \( x^a \) coordinate system as \( \eta'_{ab} \), and as discussed above \( \eta'_{a'b} \neq \eta_{a'b} \). But from Eq. (D14), it does follow that \( k_a = k'_a \), so that \( k_a \) and \( k'_a \) are the same vector in different coordinates; hence we leave the prime off the base letter \( k \); and an index on \( k \) can be raised or lowered by either \( \eta \) or \( \eta' \). Additionally, both \( s \) and \( r \) evaluate to the same scalar fields on the two different manifolds—and we leave the primes off these fields as well.

Thus we see that the retarded Poincare transformation described in Eq. (D11) is a diffeomorphism which maps one world line into another while preserving its future null cone and the values of the scalar fields \( s \) and \( r \). Associated with each world line is a distinct flat metric, \( \eta_{ab} \) or \( \eta'_{a'b} \), which is of the usual Minkowskii diagonal form in the appropriate (resp. \( x^a \) or \( y^a \)) coordinate system. We find it most convenient to be able to move easily between these two manifolds.

It is not difficult to show that the composition of two retarded Poincare transformations can be described as a single transformation, and also that any retarded Poincare transformation has a unique inverse.

For any world line, \( z^{a'}(s) \), there are many retarded Poincare transformations from \( z^{a}(s) = (s, 0, 0, 0) \) to \( z^{a'}(s) \) which also rotate the coordinate basis vectors, \( e^{a'}_i \). But, if the \( e^{a'}_i \) are Fermi-Walker transported along \( z^{a'}(s) \), then, as we now show, the transformation is unique up to an initial rotation of the basis vectors. Fermi-Walker transport requires that

\[
e^{a'}_i = -\Omega^{a'}_{b'} e^{b'}_i,
\]

(D17)

where

\[
\Omega^{a'}_{b'} \equiv \dot{v}^{a'}_i v'_i - \dot{v'}^{a'}_i v'_i.
\]

(D18)

In addition, for a retarded Poincare transformation the coordinate basis vectors must also obey

\[
e^{a'}_i = \Lambda^{a'}_{b'} e^{b'}_i,
\]

(D19)

The substitution of Eq. (D13) into Eq. (D17), along with the orthonormality of the basis vectors, results in

\[
\dot{\Lambda}^{a'}_{b'} = -\Omega^{a'}_{b'} \dot{\Lambda}^{c'}_{b'},
\]

(D20)

which has a unique solution, given suitable initial conditions. This last equation, along with Eqs. (D6) and (D9), determines the retarded Poincare transformation that maps \( z^a(s) = (s, 0, 0, 0) \) into \( z^{a'}(s) \) with Fermi-Walker transport of the basis vectors.

In the more general circumstance that \( z^a(s) \) as well as \( z^{a'}(s) \) are arbitrary world lines, Fermi-Walker transport requires that

\[
\dot{\Lambda}^{a'}_{b'} = -\Omega^{a'}_{b'} \lambda^{c'}_{b'} + \Lambda^{a'}_{c'} \Omega^{c'}_{b'},
\]

(D21)

where \( \Omega^{c'}_{b'} \) is defined as in Eq. (D18) but with the primes removed.
2. An application

An interesting application of the retarded Poincare transformation, related to the analysis of §IV A, involves a scalar field which satisfies the wave equation with a $2^l$-pole source moving along some given world line and with no incoming radiation at infinity.

For a time dependent source at rest in the $x$ manifold, a simple expression for the scalar field is

$$\psi(x) = \sum_{k=0}^{l} \frac{(-1)^{l+k}(l-k)!}{2^{k}k!(l-k)!} \frac{(l-k)\mathcal{M}^{L}(s)\hat{N}_{L}}{r^{k+1}},$$

where $\mathcal{M}^{L}(s)$ are the retarded-time dependent SSTF $2^l$-pole moments of the source, and the prefix superscript of $\mathcal{M}^{L}$ denotes differentiation with respect to $s$; this equation is given by Thorne [9] in Eq. (2.53a).

More generally, let a retarded Poincare transformation generate a different world line, $z'\omega'(s)$, for the source on the $y'\omega'$ manifold via Eqs. (D1) and (D9); then the solution of the scalar wave equation, retaining the same $2^l$-pole moments as measured by a nearby comoving observer, is easily written as

$$\psi'(y) = \nabla'_{L}(\mathcal{M}'^{L}(s)/r),$$

where $r$ is defined as in Eq. (3), and $\mathcal{M}'^{L} \equiv \Lambda'_{K}\mathcal{M}^{K}$. But for an accelerating world line the evaluation of this right hand side for $\psi'$ is much more complicated than the right hand side of Eq. (D22): it involves some terms with up to $l$ derivatives of the velocity and others containing $\dot{v}^l$; and, even though the source is a $2^l$-pole, $\psi'$ has moments for the radiation all the way from a monopole up to a $2^{2l}$-pole. Thus, although Eq. (D23) looks simple, it is actually quite difficult to evaluate for an accelerating world line.

Now, consider a new scalar field on the $y'\omega'$ manifold defined in terms of the scalar field of Eq. (D22) on the $x$ manifold by

$$\psi'_{\text{new}}(y) \equiv \psi(x);$$

this is not a solution of the scalar wave equation on the $y'\omega'$ manifold, but for small accelerations it is nearly one. In fact, inductive evaluation of Eq. (D23) along with Eqs. (3) and (8) and the invariance of $s$ and $r$ under the retarded Poincare transformation, shows that

$$\psi'_{\text{new}}(y) = \psi'(y)(1 + O(\dot{\Lambda})),$$

and this is uniformly valid even for large $r$.

A more general analysis reveals that in a similar manner a retarded Poincare transformation can take a scalar wave solution for a multipole source moving along one accelerating world line and generate an approximate solution for the same multipole source moving along a different world line, as long as the difference in the accelerations of the world lines is small. Note that this final limitation restricts neither the total acceleration of the trajectory nor the size of the boost allowed in changing the world line, as long as the proper time derivative of the boost is small. In this same manner Eq. (61) gives a new self field, $h'_{\text{new}}$, when the world line of the source is changed via a retarded Poincare transformation.

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