ON DECOMPOSITIONS INTO EXPANDERS

FEDERICO VIGOLO

Abstract. In this note we give an alternative proof of the fact that graphs having no linearly small Følner sets can be decomposed into a union of expanders. We use this fact to prove a decomposition result for graphs having linearly small maximal Følner sets and we deduce that a family of such graphs must contain a family of expanders. We also note that the absence of linearly small Følner sets is a quasi-isometry invariant.

1. Introduction

This paper revolves around the fact that “graphs where linearly-small sub-sets have large boundaries can be decomposed into unions of expanders”. To make the statement clear, we need to introduce some terminology: let $X$ be a finite graph with no multiple edges nor loops. Given a finite set of vertices $A \subseteq X$, the boundary of $A$ is the set of edges connecting $A$ to its complement:

$$\partial A = \{\{v, w\} \in E(X) \mid v \in A, w \in X \setminus A\}.$$ 

Given $\epsilon > 0$, a non-empty set of vertices $A \subset X$ is a $\epsilon$-Følner set if $|A| \leq \frac{1}{2}|X|$ and $|\partial A| \leq \epsilon|A|$ (here $|X|$ is the number of vertices of $X$). The graph $X$ is a $\epsilon$-expander if it contains no $\epsilon$-Følner sets. Let $\deg(X) := \max\{\deg(v) \mid v \in X\}$ be the degree of $X$ and $D \in \mathbb{N}$ some number. Then $X$ is an $(\epsilon, D)$-expander if it is an $\epsilon$-expander and $\deg(X) \leq D$.

If $X$ is a connected finite graph, it is trivially a $\left(\frac{2}{|X|}, |X|\right)$-expander. On the other hand, it is generally hard and very interesting to prove that a graph $X$ is an $(\epsilon, D)$-expander for some constants $\epsilon, D$ that are fixed a priori and do not depend on $|X|$. A family of expander graphs is a sequence of $(\epsilon, D)$-expanders $(X_n)_{n \in \mathbb{N}}$ such that $|X_n| \to \infty$. We refer to [4] for more background and motivation.

A subset of vertices $Y \subset X$ can be made into a subgraph of $X$ by keeping all the edges in $X$ having both endpoints in $Y$. In this paper we will say that $X = X_1 \sqcup \cdots \sqcup X_n$ is a decomposition of $X$ if the $X_i$ are subgraphs arising from a partition of the set of vertices of $X$. We are not requiring that every edge of $X$ as an edge of $X_i$ for some $i$ (i.e. there might be edges connecting the $X_i$’s). We will be particularly interested in decompositions where the $X_i$ are $\epsilon$-expanders. If $\deg(X) \leq D$, the $X_i$ will then automatically be $(\epsilon, D)$-expanders.

Finally, given a constant $\alpha \in (0, 1)$ we say that a subset $A \subseteq X$ is $\alpha$-big if $|A| \geq \alpha|X|$, and that it is $\alpha$-small if $|A| < \alpha|X|$. Given nested subsets $A \subset Y \subset X$, we will avoid confusion by specifying whether $A$ is $\alpha$-big in $Y$ or in $X$ (and similarly for $\alpha$-small).
In this paper we wish to advertise the fact graphs having “linearly-small-size set expansion” can be decomposed into linearly-large expanders:

**Theorem 1.1** (Oveis Gharam–Trevisan). Let $X$ be a finite graph. If $X$ has no $\alpha$-small $\epsilon$-Følner sets, then it can be decomposed as $X = X_1 \sqcup \cdots \sqcup X_k$ where $k \leq \left\lfloor \frac{1}{\alpha} \right\rfloor$, all the $X_i$ are $\alpha$-big and they are $\delta$-expanders for $\delta = (3/8)^{k-1}\epsilon$.

Theorem 1.1 is a special case of [1, Theorem 1.5] (see Remark 3.2 for a more detailed comparison). The main contribution of this note is to provide a short proof (Section 3) and illustrate a few geometric consequences of this fact (Section 2). In particular, we find that Theorem 2.2 and Corollary 2.3 are of independent interest.

**Acknowledgments.** I am grateful to Emmanuel Breuillard for directing me to the paper [1] and pointing out the argument explained in Remark 3.2. I am also thankful to Henry Bradford, Ana Khukhro, Kang Li and Jiawen Zhang for their helpful comments.

2. Related results

2.1. A general decomposition theorem. It is convenient to introduce one piece of notation: given subsets $A, B \subset X$, we let $\partial^BA$ be the set of edges in $X$ joining a vertex of $A$ with a vertex of $B \setminus A$ (this is the subset of $\partial A$ consisting of those edges that land into $B$). It is interesting to combine Theorem 1.1 with the “maximal Følner set trick”:

**Lemma 2.1.** Let $X$ be a finite graph and $\epsilon > 0$ a fixed constant. If there exists a $\epsilon$-Følner set $F$ that is maximal with respect to inclusion, consider the subgraph $Y := X \setminus F$. Then every subset $A \subset Y$ such that $|A| \leq \frac{1}{2}|X| - |F|$ satisfies $|\partial^Y A| > \epsilon |A|$.

This sort of maximality argument is also used fairly often in the theory of von Neumann algebras and it was also a key ingredient in [5]. The proof of Lemma 2.1 is completely elementary and can also be found in [5, Lemma 3.1]. Together with Theorem 1.1 the maximality trick implies the following structure theorem:

**Theorem 2.2.** Let $X$ be a finite graph. If $X$ has a maximal $\epsilon$-Følner set $F$ that is $\alpha$-small for some $\alpha < \frac{1}{2}$, then there exists $\delta = \delta(\epsilon, \alpha)$ such that $X$ can be decomposed as

$$X = F \sqcup Y_1 \sqcup \cdots \sqcup Y_k$$

where the graphs $Y_i$ are $\delta$-expanders and are $(\frac{1}{2} - \alpha)$-big in $X$.

**Proof.** Let $F$ be an $\alpha$-small maximal $\epsilon$-Følner set and let $Y := X \setminus F$. If $A \subset Y$ is a $\epsilon$-Følner sets of $Y$, then by Lemma 2.1 we must have:

$$|A| \geq \frac{1}{2}|X| - |F| = |Y| - \frac{1}{2}|X| > \left(1 - \frac{1}{2(1-\alpha)}\right)|Y| = \frac{1}{2}\left(1 - \frac{2\alpha}{1-\alpha}\right)|Y|.$$

That is, $Y$ has no $\frac{1}{2}\left(\frac{1-2\alpha}{1-\alpha}\right)$-small $\epsilon$-Følner sets. We can hence apply Theorem 1.1 to obtain a decomposition of $Y$ into $\delta$-expanders. \qed
Corollary 2.3. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of finite graphs with \(\deg(X_n) \leq D\) and \(|X_n| \to \infty\). Given a constant \(\epsilon > 0\), either there exist \(\epsilon\)-Følner sets \(F_n \subseteq X_n\) such that \(\limsup \frac{|\partial F_n|}{|X_n|} = \frac{1}{2}\) or there exists \(\alpha > 0\) and \(\alpha\)-big subgraphs \(Y_n \subseteq X_n\) that are \((\delta, D)\)-expanders for some \(\delta > 0\).

Note in particular that the graphs \(Y_n\) in Corollary 2.3 would be a family of expander graphs. A sample application of this result could be for proving that some metric space \(Y\) contains families expander: it may be possible to prove that \(Y\) contains some graphs \(X_n\) that do not have Følner sets of size \(\approx |X_n|/2\), and Corollary 2.3 would then immediately imply that \(Y\) contains some genuine expanders as well. This is relevant e.g. in the study of coarse embeddings into Hilbert spaces [3, 5, 6].

2.2. Quasi-isometry invariance. Given \(L, A > 0\), a \((L, A)\)-quasi-isometry is a function between metric spaces \(f: (X, d_X) \to (Y, d_Y)\) such that
\[
\frac{1}{L} d_X(x, x') - A \leq d_Y(f(x), f(x')) \leq L d_X(x, x') + A
\]
for every \(x, x' \in X\), and such that for every \(y \in Y\) there is an \(x \in X\) with \(d_Y(f(x), y) \leq A\). This notion is a cornerstone of geometric group theory [2].

Connected graphs can be seen as metric spaces where the distance between two vertices is the length of the shortest path connecting them. It is well-known that quasi-isometries preserve expansion. More precisely, one can prove the following lemma (see [7, Lemma 2.7.5] for a proof):

Lemma 2.4. For every \(\epsilon > 0\) there exists an \(\eta = \eta(\epsilon, D, L, A)\) such that if \(X\) and \(Y\) are connected graphs with degree bounded by \(D\) and \(f: X \to Y\) is an \((L, A)\)-quasi-isometry, then for every subset \(F \subseteq Y\) with \(|\partial F| \leq \eta|F|\) the preimage \(T = f^{-1}(F)\) satisfies \(|\partial T| \leq \epsilon|T|\).

If \(f: X \to Y\) is an \((L, A)\)-quasi-isometry between graphs, then any two vertices of \(X\) that are at distance \(L(A + 2)\) or more are sent to distinct points in \(Y\). It follows that if \(X\) has degree bounded by \(D\) then
\[
|f^{-1}(y)| \leq \text{(cardinality of a ball of radius } L(A+2) \text{)} \leq D^{L(A+2)+1}.
\]
On the other hand, since every point in \(Y\) is within distance \(A\) from \(f(X)\), it follows that if \(Y\) has degree bounded by \(D\) then
\[
|Y| \leq D^{A+1}|f(X)| \leq D^{A+1}|X|.
\]
Combining these inequalities proves the following:

Lemma 2.5. Let \(X\) and \(Y\) be connected graphs with degree bounded by \(D\) and \(f: X \to Y\) an \((L, A)\)-quasi-isometry. For any \(\alpha > 0\) let \(\beta := D^{-L(A+2)-\alpha^2}\). Then the preimage of a \(\beta\)-small subset of \(Y\) is \(\alpha\)-small in \(X\).

The following is now immediate:

Proposition 2.6. For every \(\epsilon, \alpha, D, L, A > 0\) there exist \(\eta, \beta > 0\) such that if \(X\) and \(Y\) are connected graphs with degree bounded by \(D\), \(X\) has no \(\alpha\)-small \(\epsilon\)-Følner set and \(f: X \to Y\) is an \((L, A)\)-quasi-isometry, then \(Y\) has no \(\beta\)-small \(\eta\)-Følner set.
3. Proof of Theorem 1.1

Recall that, given subsets $A, B \subset X$, we denote by $\partial^B A$ the set of edges in $X$ joining a vertex of $A$ with a vertex of $B \setminus A$. Define the Cheeger constant of a finite graph as

$$h(X) := \min \left\{ \frac{|\partial A|}{|A|} \mid A \subset X, \ 0 < |A| \leq \frac{1}{2}|X| \right\}.$$ 

Note that $X$ is an $\epsilon$-expander if and only if $h(X) > \epsilon$.

**Lemma 3.1.** Let $\lambda := 3/8$, $X$ be any graph and $Y \subset X$ be a set with $0 < |Y| \leq \frac{1}{2}|X|$ that realizes the Cheeger constant $\frac{|\partial Y|}{|Y|} = h := h(X)$. Then every $\lambda$-Følner set of $Y$ is a $\epsilon$-Følner set of $X$.

Furthermore, letting $Z := X \setminus Y$ we also have that every $\lambda$-Følner set of $Z$ is a $\epsilon$-Følner set of $X$.

**Proof.** Let $A \subset Y$ be $\lambda$-Følner set of $Y$. Note that $\partial^X A = \partial^Y A \cup \partial^Z A$ and that $\partial^Z A \subset \partial^X Y$. We have:

$$\partial^X (Y \setminus A) = (\partial^X Y \setminus \partial^Z A) \cup (\partial^A (Y \setminus A)),$$

and hence

$$|\partial^X (Y \setminus A)| = |\partial^X Y| - |\partial^Z A| + |\partial^A (Y \setminus A)| = |\partial^X Y| - |\partial^Z A| + |\partial^Y A|$$

because $\partial^Y A = \partial^A (Y \setminus A)$. By minimality, we thus obtain:

(1) \hspace{1cm} \frac{|\partial^X Y|}{|Y|} \leq \frac{|\partial^X (Y \setminus A)|}{|Y \setminus A|} = \frac{|\partial^X Y| - |\partial^Z A| + |\partial^Y A|}{|Y \setminus A|}.

For convenience, let $t := |A|/|Y|$ and let $r := |\partial^Y A|/|\partial^X A|$. To conclude the proof it will be enough to show that $r \geq \lambda$, because in this case we would have

$$\lambda |A| \geq |\partial^Y A| = r |\partial^X A| \geq \lambda |\partial^X A|$$

and hence $A$ is a $\epsilon$-Følner set.

With the newly introduced notation, (1) becomes:

$$h \leq \frac{|\partial^X Y| - (1 - r)|\partial^X A| + r |\partial^X A|}{(1 - t)|Y|} = \frac{h}{1 - t} + \frac{2r - 1}{(1 - t)/t} \frac{|\partial^X A|}{|A|}.$$ 

Rearranging the terms we obtain:

$$-t^2 h \leq (2r - 1) \frac{|\partial^X A|}{|A|}$$

and hence

$$r \geq \frac{1}{2} - \frac{t^2}{2} \frac{|A|}{|\partial^X A|} \geq \frac{1}{2} - \frac{1}{8} = \frac{1}{4},$$

where the last inequality follows from $h \leq \frac{|\partial^X A|}{|A|}$ and $t \leq \frac{1}{2}$.

The proof of the “furthermore” is similar. As above, let $A \subset Z$ be $\lambda$-Følner set of $Z$. Now there are two possibilities. If $|Z \setminus A| \leq \frac{1}{2}|X|$ then we have an analogue of (1):

$$h \leq \frac{|\partial^X (Z \setminus A)|}{|Z \setminus A|} = \frac{|\partial^X Z| - |\partial^Y A| + |\partial^Z A|}{|Z \setminus A|}.$$
and the same argument implies that $\partial^X A \leq \epsilon |A|$.

On the other hand, if $|Z \setminus A| > \frac{1}{2} |X|$ then $|Y \cup A| < \frac{1}{2} |X|$, and therefore we have

$$h \leq \frac{|\partial^X (Y \cup A)|}{|Y \cup A|} = \frac{|\partial^X Y| - |\partial^Y A| + |\partial^Z A|}{|Y| + |A|} \leq h + \frac{|\partial^Z A| - |\partial^Y A|}{|Y| + |A|},$$

from which it follows that $|\partial^Z A| \geq |\partial^Y A|$ and hence $|\partial^X A| \leq 2|\partial^Z A| \leq 2\lambda \epsilon |A| < \epsilon |A|.$

**Proof of Theorem 1.1.** Let $X$ be a finite graph with no $\alpha$-small $\epsilon$-Følner sets. We will show that it can be decomposed as $X = X_1 \sqcup \ldots \sqcup X_k$ where $k \leq \left\lfloor \frac{1}{\alpha} \right\rfloor$, all the $X_i$ are $\delta$-expanders for $\delta := \lambda^{k-1} \epsilon$, where $\lambda = 3/8$.

The idea is to apply Lemma 3.1 recursively: if $X$ is not a $\epsilon$-expander then $h(X) \leq \epsilon$ and there exists a $Y_0 \subset X$ that realizes the Cheeger constant. Since $X$ has no $\alpha$-small $\epsilon$-Følner sets, we deduce that $|Y_0| \geq \alpha |X|$. Letting $Y_1 := X \setminus Y_0$, we have a decomposition $X = Y_0 \cup Y_1$ where both $Y_i$ are $\alpha$-large. Importantly, it follows from Lemma 3.1 that $\lambda \epsilon$-Følner sets of $Y_i$ are also $\epsilon$-Følner sets of $X$.

Let us now focus on $Y_0$: if it is a $\lambda \epsilon$-expander there is nothing to do. Otherwise, we can choose $Y_{00} \subset Y_0$ realizing the Cheeger constant. Such $Y_{00}$ is a $\lambda \epsilon$-Følner set in $Y_0$ and hence a $\epsilon$-Følner set of $X$. It follows that $|Y_{00}| \geq \alpha |X|$. On the other hand, $Y_{01} := Y_0 \setminus Y_{00}$ is at least as large as $Y_{00}$ and hence $|Y_{01}| \geq \alpha |X|$. Using Lemma 3.1 we deduce that the decomposition $Y_0 = Y_{00} \sqcup Y_{01}$ is such that every $\lambda \epsilon^2 \alpha$-Følner set in $Y_{00}$ is a $\lambda \epsilon$-Følner set in $Y_0$ and hence an $\epsilon$-Følner set in $X$.

One can thus continue to decompose the sets $Y_{0i_1 \ldots i_k}$ that appear using this procedure. This process ends because $X$ is a finite graph and all the subsets $Y_{0i_1 \ldots i_k}$ obtained during this process are $\alpha$-big in $X$. In particular, when the process ends one has decomposed $X$ into at most $\left\lfloor \frac{1}{\alpha} \right\rfloor$ sets $X_1, \ldots, X_k$. Moreover, the worst possible expansion constant is what is obtained by the longest chain of consecutive applications of Lemma 3.1. This gives rise to the lower bound $\delta \geq \lambda^{k-1} \epsilon$.

**Remark 3.2.** As already remarked, Theorem 1.1 is only a special case of the result of Oveis Gharam–Trevisan. In fact, for every $m \geq 1$ one can define a higher order Cheeger constant $\rho_m(X)$ as

$$\rho_m(X) := \min \left\{ \max_{1 \leq i \leq m} \frac{|\partial A_i|}{|A_i|} \right\} \quad A_1, \ldots, A_m \subset X \text{ disjoint}.$$

Theorem 1.5 implies that when $\rho_m(X) > 0$ one can always find a partition $X = X_1 \sqcup \ldots \sqcup X_t$ for some $t \leq m - 1$ where the graphs $X_i$ are $\zeta$-expanders for some $\zeta = \zeta(m, \rho_m(X), \deg(X))$. To prove Theorem 1.4 it is then enough to note that if there are no $\alpha$-small $\epsilon$-Følner sets in $X$ and $m = \left\lfloor \frac{1}{\alpha} \right\rfloor + 1$, then any for any choice of $m$ disjoint sets $A_1, \ldots, A_m$ at least one of them will be smaller than $\alpha |X|$ and hence $\rho_m(X) > \epsilon$. It will hence be possible to partition $X$ into at most $\left\lfloor \frac{1}{\alpha} \right\rfloor$-many $\zeta$-expanders.

The proof of Oveis Gharam–Trevisan appears to be somewhat more involved than the proof we gave (it follows from [1] Theorem 1.7]), but it has a
few significant advantages: it gives a bound on the number of edges connecting the sets in the partition, it applies to weighted graphs and it produces asymptotically better constants.

With regard to constants: we wrote that the constant $\zeta$ of Oveis Gharam–Trevisan depends on the degree of $X$ because what they actually estimate is the conductance. In particular, this makes it hard to compare directly the constants that we obtain. It appears that our approach provides sharper estimates when $k$ is very small (i.e. for large $\alpha$). On the other hand, our estimate degrades exponentially fast with $k$, while that of Oveis Gharam–Trevisan degrades only quadratically.

One small advantage of our proof is that it is not immediately clear from the result of Oveis Gharam–Trevisan that all the sets $X_1, \ldots, X_l$ appearing in the partition are all $\alpha$-big.

References

[1] S. O. Gharan and L. Trevisan. Partitioning into expanders. In Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms, pages 1256–1266. Society for Industrial and Applied Mathematics, 2014.
[2] M. Gromov. Geometric Group Theory (Asymptotic invariants), volume 2 of London Mathematical Society Lecture Note Series, chapter Asymptotic invariants of infinite groups, pages 1–295. Cambridge University Press, 1993.
[3] M. Gromov. Random walks in random groups. Geometric and Functional Analysis, 13(1):73–146, 2003.
[4] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. Bulletin of the American Mathematical Society, 43(4):439–561, 2006.
[5] A. Khukhro, K. Li, F. Vigolo, and J. Zhang. On the structure of asymptotic expanders. arXiv preprint arXiv:1910.13320, 2019.
[6] R. Tessera. Coarse embeddings into a Hilbert space, Haagerup property and Poincaré inequalities. Journal of Topology and Analysis, 1(01):87–100, 2009.
[7] F. Vigolo. Geometry of actions, expanders and warped cones. PhD thesis, University of Oxford, Trinity 2018.

---

1 This is not an important difference: it is not hard to modify our proof to cover this case as well.
2 This includes some normalization terms that take into account the degrees of the vertices. It is natural to consider conductances when one is planning to use spectral characterization of expansion (as Oveis Gharam–Trevisan do). In this note we preferred the approach via Cheeger constants because it is marginally simpler to introduce.