SOLUTIONS AND CHARACTERIZATIONS UNDER MULTICRITERIA MANAGEMENT SYSTEMS

YU-HSIEN LIAO*

Department of Applied Mathematics
National Pingtung University
900 Pingtung, Taiwan

(Communicated by Gerhard-Wilhelm Weber)

Abstract. In real situations, agents might take different activity levels to participate; agents might represent administrative areas of different scales. On the other hand, agents always face an increasing need to focus on multiple aims efficiently in their operational processes. Thus, we introduce two solutions to investigate distribution mechanism by applying the maximal level-marginal contributions among activity level (decision) vectors under multicriteria management systems. Based on a specific reduced game and some reasonable properties, we offer some characterizations to analyze the rationality for these two solutions. In order to desire that any utility could be distributed among the players and their activity levels in proportion to related differences, two weighted extensions are also proposed by means of different weight functions.

1. Introduction. Reduced game property is an important property in the characterizations on traditional games. It asserts that the recommendation made for any problem should always agree with the recommendation made in the subproblem that appears when the payoffs of some agents are settled on. This property has been always analyzed in different contexts by adopting reduced games, such as bargaining problems, cost allocation problems and so on.

In a transferable-utility (TU) game, each agent is either fully involved or totally out of participation with some other agents. Based on the notion of the marginal contributions, the equal allocation of non-separable costs (EANSC, Ransmeier (1942)) and the normalized marginal index are defined on traditional TU games respectively. Moulin (1985) introduced the complement-reduced game to show that the EANSC provides a fair rule for distributing utilities. Liao et al. (2017) also adopted the complement-reduction to characterize the normalized marginal index.

In a multi-choice TU game, each agent is permitted to participate with finite various activity levels (or decisions). Thus, a multi-choice TU game could be regarded as a natural extension of a traditional TU game. Nouweland et al. (1995) referred to other applications of multi-choice TU games, such as a large building project with a deadline and a penalty for every day if this deadline is overtime. The date of completion depends on the effort of how all of the people plunged into the project: the greater they exert themselves, the sooner the project will be completed. This

2020 Mathematics Subject Classification. Primary: 91A06, 91A12, 91A40, 91A80, 90B50.
Key words and phrases. The maximal level-marginal contribution, multicriteria management system, weighted extension.

* Corresponding author: Yu-Hsien Liao.
situation gives rise to a multi-choice game. The worth of a coalition where each agent works at a certain activity level is defined as the minus of the penalty which needs to be paid for giving the date of completion of the project when every agent makes the relative effort. Another application appears in a large company with many divisions, where the profit-making depends on their performance. This gives rise to a multi-choice TU game in which the agents are the divisions and the worth of a coalition where each division functions at a certain level is the corresponding profit made by the company. Many multi-choice solutions have been applied wildly, e.g., Branzei (2008), Branzei et al. (2009), Branzei et al. (2009, 2014), Hwang and Liao (2013), Liao (2018), and so on. By considering overall allocations for a given member on multi-choice TU games, Hwang and Liao (2010), Liao (2008, 2012) and Nouweland et al. (1995) proposed several extended allocations and related results for the core, the EANSC and the Shapley value respectively.

In different fields, from sciences to industry, engineering and the social sciences, managers face an increasing need to focus on multiple aims efficiently in their operational processes. Related situations include analyzing distribution tradeoffs, selecting optimal decision or process designs, or any other condition where you need an efficient solution with tradeoffs between two or more aims. In many cases these real world efficient situations could be formulated as multicriteria mathematical optimization models. The solutions of such situations requires appropriate techniques to offer optimal results that - unlike traditional viewpoints or methods - take several properties of the aims into account. Several pre-existing results considered multicriteria situations. For example, Bednarczuk et al. (2018) transformed the multiple-choice knapsack problem into a bi-objective optimization problem whose solution set contains solutions of the original multiple-choice knapsack problem. Goli et al. (2019) addressed the optimization of the multi-objective product portfolio problem under return uncertainty is addressed here. The contribution is based on the application of a hybrid improved artificial intelligence and robust optimization and presenting a new method for calculating the risk of a product portfolio. A two-objective (minimizing risk and maximizing return) mathematical model is also proposed. In order to maximize the increase in cash flow, maximize the total created jobs in the supply chain, and maximize the reliability of consumed raw materials, Goli et al. (2020) addressed the multiobjective, multiproducts and multiperiod closed-loop supply chain network design with uncertain parameters, whose aim is to incorporate the financial flow as the cash flow and debts’ constraints and labor employment under fuzzy uncertainty. By considering multi-criteria analysis methods in some complex situations (e.g., with different aspects to be considered and multilevel actors involved), the objective of the results proposed by Guarini et al. (2018) is to outline a procedure with which to select the method best suited to the specific queries of evaluation, which commonly arise while addressing decision-making problems. A flexible combinatorial optimization modeling approach introduced by Mustakerov et al. (2018) is developed for multi-choice complying with different decision maker requirements. The described approach is based on formulation of multicriteria linear mixed integer optimization tasks. Tirkolaee et al. (2019) addressed the multi-objective multi-mode resource-constrained project scheduling problem with payment planning where the activities can be done through one of the possible modes and the objectives are to maximize the net present value and minimize the completion time concurrently.
In real situations, the agents and their activity levels are different. Agents might represent constituencies of different sizes; Agents might have different bargaining abilities. Also, lack of symmetry may arise when different bargaining abilities for different agents are modeled. In line with the above interpretations, we would now desire that any utility could be distributed among the agents and their activity levels in proportion to weights. Weights come up naturally in the framework of utilities allocation. For example, when we deal with a problem of utility allocation among investment projects, then the weights could be associated to the profitability of the different projects. In a question of allocating travel costs among various places visited, the weights could be the number of days spent at each one (cf. Shapley (1982)). It is reasonable that weights could be assigned to the “agents” and the “levels” to distinguish the difference among the agents and their activity levels respectively.

The above mentioned results raise one motivation:

- whether some solutions could be introduced by considering multi-choice behavior, multicriteria situation and weights simultaneously.

The paper is devoted to investigate the motivation. The main results of this paper are as follows.

- Two new solutions, the multicriteria max-index (MMI) and the multicriteria normalized index (MNI), are introduced in Section 2. These two solutions are multi-choice extensions of the EANSC and the normalized marginal index under multicriteria situation.

- In order to analyze the rationality for these two solutions, we propose an extended reduction to offer some characterizations. In Section 3, we show that the MMI is the only solution satisfying the properties of multicriteria standard for games and multicriteria reduced game property. Further, we also show that the MMI is the only solution satisfying the properties of multicriteria pareto optimality, multicriteria equal treatment for equal, multicriteria covariance and multicriteria reduced game property.

- Since the MNI violates multicriteria reduced game property, in Section 4, we define the revised reduced game property to show that the MNI is the only solution satisfying the properties of normalized-marginal-standard of games and revised reduced game property.

- In Section 5, we adopt the weight function for agents and the weighted function for levels to propose two weighted extensions of the MMI and related characterizations.

2. Preliminaries. Let $U$ be the universe of agents. For $i \in U$ and $b_i \in \mathbb{N}$, we set $B_i = \{0, \cdots, b_i\}$ to be the action (decision) space of agent $i$ and $B_i^+ = B_i \setminus \{0\}$, where 0 denotes no participation. Let $B^N = \prod_{i \in N} B_i$ be the product set of the action spaces for agents in $N$. For all $T \subseteq N$, a agent-coalition $T \subseteq N$ corresponds in a canonical way to the multi-choice coalition $e^T \in B^N$, which is the vector with $e^T_i = 1$ if $i \in T$, and $e^T_i = 0$ if $i \in N \setminus T$. Denote $0_N$ the zero vector in $\mathbb{R}^N$. For $m \in \mathbb{N}$, let $0_m$ be the zero vector in $\mathbb{R}^m$ and $\mathbb{N}_m = \{1, 2, \cdots, m\}$.

A multi-choice TU game is a triplet $(N, b, v)$, where $N$ is a non-empty and finite set of agents, $b = (b_i)_{i \in N} \in \prod_{i \in N} B_i^+$ is the vector that describes the highest levels of activity for each agent, and $v : B^N \to \mathbb{R}$ is a function with $v(0_N) = 0$ which assigns to each action vector $\alpha = (\alpha_i)_{i \in N} \in B^N$ the worth that the agents can obtain when each agent $i$ plays at level $\alpha_i$. A multicriteria multi-choice TU
game is a triple \((N, b, V^m)\), where \(m \in \mathbb{N}\), \(V^m = (v^t)_{t \in \mathbb{N}_m}\) and \((N, b, v^t)\) is a multi-choice TU game for all \(t \in \mathbb{N}_m\). Denote the class of all multicriteria multi-choice TU games by \(\Gamma\).

A solution is a map \(\sigma\) assigning to each \((N, b, V^m) \in \Gamma\) an element

\[
\sigma(N, b, V^m) = (\sigma^t(N, b, V^m))_{t \in \mathbb{N}_m},
\]

where \(\sigma^t(N, b, V^m) = (\sigma^t_i(N, b, V^m))_{i \in \mathbb{N}} \in \mathbb{R}^N\) and \(\sigma^t_i(N, b, V^m)\) is the payoff of the agent \(i\) when \(i\) participates in \((N, b, v^t)\). Let \((N, b, V^m) \in \Gamma\), \(T \subseteq \mathbb{N}\) and \(\alpha \in \mathbb{R}^N\), we denote \(L(\alpha) = \{i \in N|\alpha_i \neq 0\}\), and denote \(\alpha_T \in \mathbb{R}^T\) to be the restriction of \(\alpha\) to \(T\). Given \(i \in \mathbb{N}\), we introduce the substitution notation \(\alpha_{-i}\) to stand for \(\alpha_N \setminus \{i\}\) and let \(\gamma = (\alpha_{-i}, p) \in \mathbb{R}^N\) be defined by \(\gamma_{-i} = \alpha_{-i}\) and \(\gamma_i = p\).

In the following, we provide different generalizations of the EANSC and the normalized marginal index.

**Definition 2.1.**

1. The **multicriteria max-index (MMI)**, \(\eta_i\), is defined by

\[
\eta^t_i(N, b, V^m) = \eta^t_i(N, b, V^m) + \frac{1}{|N|} \cdot [v^t(b) - \sum_{k \in N} \eta^t_k(N, b, V^m)]
\]

for all \((N, b, V^m) \in \Gamma\), for all \(t \in \mathbb{N}_m\) and for all \(i \in \mathbb{N}\). The value \(\eta^t_i(N, b, V^m) = \max_{j \in B^t} \{v^t(b_{-i, j}) - v^t(b_{-i, j-1})\}\) is the **maximal level-marginal contribution** of the agent \(i\) from level \(j - 1\) to \(j\) in \((N, b, v^t)\). From now on we restrict our attention to bounded multi-choice TU games, defined as those games \((N, b, v^t)\) such that, there exists \(M_v \in \mathbb{R}\) such that \(v^t(\alpha) \leq M_v\) for all \(\alpha \in B^N\). We adopt it to guarantee that \(\eta_i(N, b, v^t)\) is well-defined. Under the solution \(\eta_i\), all agents first receive their maximal level-marginal contribution, and further allocate the remaining utility equally.

2. The **multicriteria normalized index (MNI)**, \(\overline{X}_i\), is defined by

\[
\overline{X}^t_i(N, b, V^m) = \frac{v^t(b)}{\sum_{k \in N} \eta^t_k(N, b, V^m)} \cdot \eta^t_i(N, b, V^m)
\]

for all \((N, b, V^m) \in \Gamma^*,\) for all \(t \in \mathbb{N}_m\) and for all \(i \in \mathbb{N}\), where \(\Gamma^* = \{(N, b, V^m) \in \Gamma | \sum_{i \in N} \eta^t_i(N, b, V^m) \neq 0\text{ for all } t \in \mathbb{N}_m\}\). By the definition of \(\overline{X}_i\), all agents allocate the full-level utility proportionally by applying maximal level-marginal contributions.

**Remark 1.** As we mention in Introduction, multicriteria analysis (also known multiattribute analysis, multi-objective analysis, and so on) is a notion of multiple criteria analysis that is concerned with conditions involving more than one aim to be optimized simultaneously. Multicriteria analysis has been applied in many areas, including engineering, politics, economics, logistics where efficient decisions need to be used in the presence of trade-offs among two or more aims. For example, minimizing cost while maximizing comfort by buying a central air conditioning system, and maximizing efficiency whilst minimizing energies consumption and emission of pollutants are examples of multicriteria efficient problems involving two and three aims respectively. In many situations, there can be more than three aims. On the other hand, each agent could be allowed to participate with infinite various activity decisions (levels, strategies) in real situations respectively. Therefore, we consider the framework of multicriteria multi-choice TU games in this paper.
In the following we provide a brief application of multicriteria multi-choice TU games in the setting of “management”. This kind of problem can be formulated as follows. Let \( N = \{1, 2, \ldots, n\} \) be a set of all agents of a grand management system \((N, b, V^m)\). The function \( v^t \) could be treated as a utility function which assigns to each level vector \( \alpha = (\alpha_i)_{i \in N} \in B^N \) the worth that the agents can obtain when each agent \( i \) participates at operation strategy \( \alpha_i \in B_i \) in the sub-management system \((N, b, v^t)\). Modeled in this way, the grand management system \((N, b, V^m)\) could be considered as a multicriteria multi-choice TU game, with \( v^t \) being each characteristic function and \( b_i \) being the set of all operation strategies of the agent \( i \). In the following sections, we would like to show that the MMI and the MNI could provide “optimal allocation mechanisms” among all agents under multi-choice behavior and multicriteria situation.

3. The MMI and its characterizations. In this section, we adopt some properties to characterize the MMI. A solution \( \psi \) satisfies **multicriteria pareto optimality (MPO)** if for all \((N, b, V^m) \in \Gamma \) and for all \( t \in \mathbb{N}_m \), \( \sum_{i \in N} \psi_t^i(N, b, V^m) = v^t(b) \). A solution \( \psi \) satisfies **multicriteria standard for games (MSTFG)** if \( \psi(N, b, V^m) = \eta(N, b, V^m) \) for all \((N, b, V^m) \in \Gamma \) with \(|N| \leq 2\). A solution \( \psi \) satisfies **multicriteria equal treatment for equal (METE)** if \( \psi_t(N, b, V^m) = \psi_k(N, b, V^m) \) for all \((N, b, V^m) \in \Gamma \) with \( \eta^t_i(N, b, V^m) = \eta^t_k(N, b, V^m) \) for some \( i, k \in N \) and for all \( t \in \mathbb{N}_m \). A solution \( \psi \) satisfies **multicriteria covariance (MCOV)** if \( \psi(N, b, V^m) = \psi(N, b, W^m) + (f^t)_{t \in \mathbb{N}_m} \) for all \((N, b, V^m), (N, b, W^m) \in \Gamma \) with \( v^t(\alpha) = \psi_t^i(\alpha) + \sum_{i \in L(\alpha)} f^t_i \) for some \( f^t \in \mathbb{R}^N \), for all \( t \in \mathbb{N}_m \) and for all \( \alpha \in B^N \).

MPO asserts that all utility should be allocated completely. MSTFG is an extension of the two-person standardness due to Hart and Mas-Colell (1989). METE asserts that the payoffs should be the same if the maximal level-marginal contributions are the same. MCOV can be interpreted as an extremely weak kind of additivity. By Definition 1, it is easy to see that the MMI satisfies MPO, MSTFG, METE and MCOV.

Moulin (1985) defined the reduced game as that in which each coalition in the subgroup could attain payoffs to its agents only if they are compatible with the initial payoffs to “all” the agents outside of the subgroup. An extended Moulin-reduction on \( \Gamma \) is defined as follows. Let \((N, b, V^m) \in \Gamma \), \( S \subseteq N \) and \( \psi \) be a solution. The **reduced game** \((S, b_S, V^m_{S, \psi})\) is defined by \( V^m_{S, \psi} = (v^t_{S, \psi})_{t \in \mathbb{N}_m} \) and

\[
v^t_{S, \psi}(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0_S, \\ v^t(\alpha, b_N \setminus S) - \sum_{i \in N \setminus S} \psi^i_t(N, b, V^m) & \text{otherwise.} \end{cases}
\]

A solution \( \psi \) satisfies **multicriteria reduced game property (MCRGP)** if \( \psi^t_i(S, b_S, V^m_{S, \psi}) = \psi^t_i(N, b, V^m) \) for all \((N, b, V^m) \in \Gamma \), for all \( t \in \mathbb{N}_m \), for all \( S \subseteq N \) with \(|S| = 2\) and for all \( i \in S \).

**Lemma 3.1.** The MMI \( \eta \) satisfies MCRGP.

**Proof.** Let \((N, b, V^m) \in \Gamma \), \( S \subseteq N \) and \( t \in \mathbb{N}_m \). Assume that \(|N| \geq 2 \) and \(|S| = 2\).

By the definition of \( \eta \),

\[
\eta^t_i(S, b_S, V^m_{S, \eta}) = \eta^t_i(S, b_S, V^m_{S, \eta}) + \frac{1}{|S|} \cdot \left[ v^t_{S, \eta}(b_S) - \sum_{k \in S} \psi^t_k(S, b_S, V^m_{S, \eta}) \right].
\]  

(1)
for all \( i \in S \) and for all \( t \in \mathbb{N}_m \). By definitions of \( \eta^t \) and \( v^t_{S,\pi} \):
\[
\eta^t_i(S, b_S, V^m_{S,\pi}) = \max_{j \in B^t_i} \{ v^t_{S,\pi}(b_{S\setminus\{i\}}, j) - v^t_{S,\pi}(b_{S\setminus\{i\}}, j - 1) \} = \max_{j \in B^t_i} \{ v^t(b_{-i}, j) - v^t(b_{-i}, j - 1) \} = \eta^t_i(N, b, V^m).
\]

By equations (1), (2) and definitions of \( v^t_{S,\pi} \) and \( \pi \),
\[
\eta^t_i(S, b_S, V^m_{S,\pi}) = \eta^t_i(N, b, V^m) + \frac{1}{|S|} \left[ v^t_{S,\pi}(b_S) - \sum_{k \in S} \eta^t_k(N, b, V^m) \right] = \eta^t_i(N, b, V^m) + \frac{1}{|S|} \left[ v^t(b) - \sum_{k \in S} \eta^t_k(N, b, V^m) \right] - \sum_{k \in S} \eta^t_k(N, b, V^m) = \eta^t_i(N, b, V^m) + \frac{1}{|S|} \left[ v^t(b) - \sum_{k \in S} \eta^t_k(N, b, V^m) \right] - \sum_{k \in S} \eta^t_k(N, b, V^m) = \eta^t_i(N, b, V^m) + \frac{1}{|S|} \left[ v^t(b) - \sum_{k \in S} \eta^t_k(N, b, V^m) \right] - \sum_{k \in S} \eta^t_k(N, b, V^m) = \eta^t_i(N, b, V^m)
\]
for all \( i \in S \) and for all \( t \in \mathbb{N}_m \). So, the MMI satisfies MCRGP.

Inspired by Hart and Mas-Colell (1989), Maschler and Owen (1989) and Moulin (1985), we adopt MCRGP to characterize the MMI.

**Theorem 3.2.** On \( \Gamma \), the MMI is the only solution satisfying \( \text{MSTFG} \) and \( \text{MCRGP} \).

**Proof.** By Lemma 3.1, \( \pi \) satisfies MCRGP. Clearly, \( \pi \) satisfies MSTFG.

To prove uniqueness, suppose \( \psi \) satisfies \( \text{MSTFG} \) and \( \text{MCRGP} \). By \( \text{MSTFG} \) and \( \text{MCRGP} \) of \( \psi \), it is easy to derive that \( \psi \) also satisfies \( \text{MPO} \), hence we omit it. Let \( (N, b, V^m) \in \Gamma \). By \( \text{MSTFG} \) of \( \psi \), \( \psi(N, b, V^m) = \pi(N, b, V^m) \) if \(|N| \leq 2 \). The case \(|N| > 2 \): Let \( i \in N \), \( t \in \mathbb{N}_m \) and \( S = \{i, k\} \) for some \( k \in N \setminus \{i\} \).

\[
\psi^t(N, b, V^m) - \psi^t_k(N, b, V^m) = \psi^t_i(S, b_S, V^m_{S,\psi}) - \psi^t_k(S, b_S, V^m_{S,\psi}) \quad (\text{by \text{MCRGP} of } \psi)
\]
\[
\eta^t_i(S, b_S, V^m_{S,\psi}) - \eta^t_k(S, b_S, V^m_{S,\psi}) = \eta^t_i(N, b, V^m) - \eta^t_k(N, b, V^m) \quad (\text{by \text{MSTFG} of } \psi)
\]
\[
\max_{j \in B^t_i} \{ v^t_{S,\psi}(b_{S\setminus\{i\}}, j) - v^t_{S,\psi}(b_{S\setminus\{i\}}, j - 1) \} - \max_{j \in B^t_k} \{ v^t(b_{-i}, j) - v^t(b_{-i}, j - 1) \} = \max_{j \in B^t_i} \{ v^t(b_{-i}, j) - v^t(b_{-i}, j - 1) \} - \max_{j \in B^t_k} \{ v^t(b_{-i}, j - 1) \} = \eta^t_i(N, b, V^m) - \eta^t_k(N, b, V^m).
\]

Thus, \( \psi^t_i(N, b, V^m) - \psi^t_k(N, b, V^m) = \eta^t_i(N, b, V^m) - \eta^t_k(N, b, V^m) \). By \( \text{MPO} \) of \( \psi \) and \( \pi \),
\[
|N| \cdot \psi^t_i(N, b, V^m) - \psi^t(b) = \sum_{k \in N} \left[ \psi^t_i(N, b, V^m) - \psi^t_k(N, b, V^m) \right] = \sum_{k \in N} \left[ \eta^t_i(N, b, V^m) - \eta^t_k(N, b, V^m) \right] = |N| \cdot \eta^t_i(N, b, V^m) - \psi^t(b).
\]
Hence, \( \psi^t_i(N, b, V^m) = \eta^t_i(N, b, V^m) \) for all \( i \in N \) and for all \( t \in \mathbb{N}_m \).
Next, we characterize the MMI by means of related properties of MPO, METE, MCOV and MCRGP.

**Lemma 3.3.** If a solution \( \psi \) satisfies MPO, METE and MCOV, then \( \psi \) satisfies MSTFG.

**Proof.** Assume that a solution \( \psi \) satisfies MPO, METE and MCOV. Let \((N, b, V^m)\) \(\in \Gamma.\) The proof is completed by MPO of \( \psi \) if \(|N| = 1.\) Let \((N, b, V^m) \in \Gamma \) with \(N = \{i, k\} \) for some \( i \neq k.\) We define a game \((N, b, V^m)\) to be that \(w^t(\alpha) = v^t(\alpha) - \sum_{i \in L(\alpha)} \eta^t_i(N, b, V^m)\) for all \(\alpha \in B^N\) and for all \(t \in \mathbb{N}_m.\) By definition of \(W^m,\)

\[
\eta^t_i(N, b, W^m) = \max_{j \in B^+_i} \{w^t(j, b_k) - w^t(j - 1, b_k)\}
\]

\[
= \max_{j \in B^+_i} \{v^t(j, b_k) - v^t(j - 1, b_k) - \eta^t_i(N, b, V^m)\}
\]

\[
= \max_{j \in B^+_i} \{v^t(j, b_k) - v^t(j - 1, b_k)\} - \eta^t_i(N, b, V^m)
\]

\[
= \eta^t_i(N, b, V^m) - \eta^t_i(N, b, V^m) = 0.
\]

Similarly, \(\psi^t_k(N, b, W^m) = 0.\) Therefore, \(\eta^t_i(N, b, W^m) = \eta^t_k(N, b, W^m).\) By METE of \( \psi,\) \(\psi^t_i(N, b, W^m) = \psi^t_i(N, b, W^m).\) By MPO of \( \psi,\) \(w^t(b) = \psi^t_i(N, b, W^m) + \psi^t_k(N, b, W^m) = 2 \cdot \psi^t_i(N, b, W^m)\) i.e., \(\psi^t_i(N, b, W^m) = \frac{w^t(b)}{2} = \frac{1}{2} \cdot [v^t(b) - \eta^t_i(N, b, V^m) - \eta^t_k(N, b, V^m)].\) By MCOV of \( \psi,\)

\[
\psi^t_i(N, b, V^m) = \frac{1}{2} \cdot [v^t(b) - \eta^t_i(N, b, V^m) - \eta^t_k(N, b, V^m)]
\]

\[
= \frac{1}{2} \cdot \psi^t_i(N, b, V^m).
\]

Similarly, \(\psi^t_k(N, b, V^m) = \frac{1}{2} \cdot \psi^t_k(N, b, V^m).\) Hence, \( \psi \) satisfies MSTFG. \(\square\)

**Theorem 3.4.** On \( \Gamma, \) the MMI is the only solution satisfying MPO, METE, MCOV and MCRGP.

**Proof.** By Definition 2.1, \( \bar{\eta} \) satisfies MPO, METE and MCRGP. The remaining proofs follow from Theorem 3.2 and Lemmas 3.1, 3.3. \(\square\)

The following examples are to show that each of the axioms used in Theorems 3.2 and 3.4 is logically independent of the remaining axioms.

**Example 1.** For all \((N, b, V^m) \in \Gamma,\) for all \(t \in \mathbb{N}_m\) and for all \(i \in N,\) we define the solution \( \psi \) to be

\[
\psi^t_i(N, b, V^m) = \begin{cases} 
\eta^t_i(N, b, V^m) & \text{if } |N| \leq 2, \\
0 & \text{otherwise}. 
\end{cases}
\]

Clearly, \( \psi \) satisfies MSTFG, but it violates MCRGP.

**Example 2.** For all \((N, b, V^m) \in \Gamma,\) for all \(t \in \mathbb{N}_m\) and for all \(i \in N,\) we define the solution \( \psi \) to be \(\psi^t_i(N, b, V^m) = \eta^t_i(N, b, V^m).\) Clearly, \( \psi \) satisfies METE, MCOV and MCRGP, but it violates MPO and MSTFG.

**Example 3.** We define the solution \( \psi \) to be \(\psi^t_i(N, b, V^m) = \frac{v^t(b)}{|N|}\) for all \((N, b, V^m) \in \Gamma,\) for all \(t \in \mathbb{N}_m\) and for all \(i \in N.\) Clearly, \( \psi \) satisfies MPO, METE and MCRGP, but it violates MCOV.
Example 4. For all \((N, b, V^m) \in \Gamma\), for all \(t \in \mathbb{N}_m\) and for all \(i \in N\), we define the solution \(\psi\) to be
\[
\psi^t_i(N, b, V^m) = \left[ v^t(b) - v^t(b_{-i}, 0) \right] + \frac{1}{|N|} \cdot \left[ v^t(b) - \sum_{k \in N} [v^t(b) - v^t(b_{-k}, 0)] \right].
\]
Clearly, \(\psi\) satisfies MPO, MCOV and MCRGP, but it violates METE.

Example 5. For all \((N, f, v) \in \Gamma\), for all \(t \in \mathbb{N}_m\) and for all \(i \in N\), we define the solution \(\psi\) to be
\[
\psi^t_i(N, b, V^m) = \eta^t_i(N, b, V^m) + \frac{d^t(i)}{\sum_{k \in N} d^t(k)} \cdot \left[ v^t(b) - \sum_{k \in N} \eta^t_k(N, b, V^m) \right],
\]
where for all \((N, b, V^m) \in \Gamma\), \(d^t : N \to \mathbb{R}^+\) is defined by \(d^t(i) = d^t(k) = \eta^t_i(N, b, V^m) = \eta^t_k(N, b, V^m)\). Define a solution \(\theta\) by for all \((N, b, V^m) \in \Gamma\), for all \(t \in \mathbb{N}_m\) and for all \(i \in N\),
\[
\theta^t_i(N, b, V^m) = \begin{cases} 
\eta^t_i(N, b, V^m) & \text{if } |N| \leq 2, \\
\psi^t_i(N, b, V^m) & \text{otherwise}.
\end{cases}
\]
Clearly, \(\theta\) satisfies MPO, METE and MCOV, but it violates MCRGP.

4. The MNI and its characterization. It is easy to show that the MNI satisfies MPO, METE and NMSG, but it violates MCOV. Similar to Theorem 1, we would like to characterize the MNI by means of reduced game property. Unfortunately, it is easy to see that \((S, b, V^m)\) does not exist if \(\sum_{i \in S} \eta^t_i(N, b, V^m) = 0\). Thus, we consider the revised reduced game property as follows. A solution \(\psi\) satisfies \underline{revised reduced game property (RERGP)} if \((S, b, V^m) \in \Gamma^*\) for some \((N, b, V^m) \in \Gamma\) and for some \(S \subseteq N\), it holds that \(\psi^t_i(S, b, V^m) = \psi^t_i(N, b, V^m)\) for all \(t \in \mathbb{N}_m\) and for all \(i \in S\). \(\psi\) satisfies \underline{normalized-marginal-standard for games (NMSG)} if \(\psi(N, b, V^m) = \lambda(N, b, V^m)\) for all \((N, b, V^m) \in \Gamma\), \(|N| \leq 2\).

Lemma 4.1. The MNI satisfies RERGP on \(\Gamma^*\).

Proof. Let \((N, b, V^m) \in \Gamma^*\). If \(|N| \leq 2\), then the proof is completed. Assume that \(|N| \geq 3\) and \(S \subseteq N\) with \(|S| = 2\). Similar to equation \((2)\),
\[
\eta^t_i(S, b, V^m) = \eta^t_i(N, b, V^m).
\]
for all \(i \in S\) and for all \(t \in \mathbb{N}_m\). Define that \(a_t = \frac{1}{\sum_{b \in \mathbb{N}} \eta^t_i(N, b, V^m)}\). For all \(i \in S\) and for all \(t \in \mathbb{N}_m\),
\[
\lambda_t^t(S, b, V^m) = \sum_{b \in \mathbb{N}_m} \eta^t_i(S, b, V^m) \cdot \eta^t_i(N, b, V^m)
\]
\[
= \frac{1}{\sum_{b \in \mathbb{N}_m} \eta^t_i(N, b, V^m)} \cdot \eta^t_i(N, b, V^m)
\]
\[
= a_t \cdot \eta^t_i(N, b, V^m) \quad \text{(by Definition 2.1)}
\]
By equations \((3)\) and \((4)\), the solution \(\lambda\) satisfies RERGP. \(\square\)
Theorem 4.2. On $\Gamma^*$, the solution $\overline{\lambda}$ is the only solution satisfying NMSG and RERGP.

Proof. By Lemma 4.1, $\overline{\lambda}$ satisfies RERGP. Clearly, $\overline{\lambda}$ satisfies NMSG.

To prove uniqueness, suppose $\psi$ satisfies RERGP and NMSG on $\Gamma^*$. By NMSG and RERGP of $\psi$, it is easy to derive that $\psi$ also satisfies MPO, hence we omit it. Let $(N, b, V^m) \in \Gamma^*$. We will complete the proof by induction on $|N|$. If $|N| \leq 2$, it is trivial that $\psi(N, b, V^m) = \overline{\lambda}(N, b, V^m)$ by NMSG. Assume that it holds if $|N| \leq p - 1$, $p \leq 3$. The case $|N| = p$: Let $i, j \in N$ with $i \neq j$ and $t \in \mathbb{N}_m$. By Definition 1, $\lambda_k(N, b, V^m) = \sum_{k \in N} \eta^t_k(N, b, V^m) \cdot \eta^t_k(N, b, V^m)$ for all $k \in N$. Assume that $\alpha^t_k = \sum_{k \in N} \eta^t_k(N, b, V^m)$ for all $k \in N$. Therefore,

$$\begin{align*}
\psi^t_i(N, b, V^m) &= \psi^t_i(N \setminus \{j\}, b_{N \setminus \{j\}}, V^m_{N \setminus \{j\}}, \psi) \quad \text{(by RERGP of } \psi) \\
&= \overline{\lambda}(N \setminus \{j\}, b_{N \setminus \{j\}}, V^m_{N \setminus \{j\}}, \psi) \quad \text{(by NMSG of } \psi) \\
&= \sum_{k \in N} \eta^t_k(N \setminus \{j\}, b_{N \setminus \{j\}}, V^m_{N \setminus \{j\}}, \psi) \cdot \eta^t_k(N, b, V^m) \\
&= \frac{v^t(b) - \psi^t_i(N, b, V^m)}{\eta^t(N, b, V^m)} \cdot \eta^t_i(N, b, V^m) \quad \text{(by equation (2))} \\
&= \frac{v^t(b) - \psi^t_i(N, b, V^m)}{\eta^t(N, b, V^m) + \sum_{k \in N} \eta^t_k(N, b, V^m)} \cdot \eta^t_i(N, b, V^m). \\
\end{align*}$$

By equation (5),

$$\begin{align*}
\psi^t_i(N, b, V^m) \cdot [1 - \alpha^t_j] &= [v^t(b) - \psi^t_i(N, b, V^m)] \cdot \alpha^t_j \\

\Rightarrow \sum_{i \in N} \psi^t_i(N, b, V^m) \cdot [1 - \alpha^t_j] &= [v^t(b) - \psi^t_i(N, b, V^m)] \cdot \sum_{i \in N} \alpha^t_j \\

\Rightarrow v^t(b) \cdot [1 - \alpha^t_j] &= [v^t(b) - \psi^t_i(N, b, V^m)] \cdot 1 \quad \text{(by MPO of } \psi) \\

\Rightarrow v^t(b) = v^t(b) \cdot \alpha^t_j = v^t(b) - \psi^t_i(N, b, V^m) \\

\Rightarrow \overline{\lambda}(N, b, V^m) = \psi^t_i(N, b, V^m). 
\end{align*}$$

The proof is completed. \qed

The following examples are to show that each of the axioms adopted in Theorem 4.2 is logically independent of the remaining axioms.

Example 6. For all $(N, b, V^m) \in \Gamma^*$, for all $t \in \mathbb{N}_m$ and for all $i \in N$, we define the solution $\psi$ to be $\psi^t_i(N, b, V^m) = 0$. Clearly, $\psi$ satisfies RERGP, but it violates NMSG.

Example 7. For all $(N, b, V^m) \in \Gamma^*$, for all $t \in \mathbb{N}_m$ and for all $i \in N$, we define the solution $\psi$ to be

$$\psi^t_i(N, b, V^m) = \begin{cases} 
\overline{\lambda}(N, b, V^m), & \text{if } |N| \leq 2, \\
0, & \text{otherwise}.
\end{cases}$$

Clearly, $\psi$ satisfies NMSG, but it violates RERGP.

5. Two weighted extensions. As mentioned in Introduction, weights come up naturally in the framework of utilities allocation. For example, we may be dealing with a problem of utility allocation among investment projects. Then the weights could be associated to the profitability of the different choices of all projects. Weights are also included in contracts signed by the owners of a condominium and
used to divide the cost of building or maintaining common facilities. Another example is data or patent pooling among firms where the size of the firms, measured for instance by their market shares, are natural weights. Thus, it is reasonable that weights could be assigned to the “agents” and the “levels” to distinguish the difference among the agents and their activity levels respectively.

If $d : U \rightarrow \mathbb{R}^+$ be a positive function, then $d$ is called a weight function for agents. If $w : B^U \rightarrow \mathbb{R}^+$ be a positive function, then $w$ is called a weight function for levels. These weights could be interpreted as a-priori measures of importance; they are taken to reflect considerations not captured by the characteristic function. By these two types of the weight function, two weighted revisions of the MMI is defined as follows.

Definition 5.1.

- The **P-weighted max-index (P-WMI)**, $\eta^d$, is defined by for all $(N, b, V^m) \in \Gamma$, for all weight function for agents $d$, for all $t \in \mathbb{N}_m$ and for all agent $i \in N$,

$$\eta^{d,t}_i(N, b, V^m) = \eta^d_i(N, b, V^m) + \frac{d(i)}{\sum_{k \in N} d(k)} \cdot \left[ v^t(b) - \sum_{k \in N} \eta^d_k(N, b, V^m) \right]. \quad (6)$$

Under the solution $\eta^d$, all agents first receive their maximal level-marginal contribution, and further allocate the remaining utilities proportionally by applying weights.

- The **L-weighted max-index (L-WMI)**, $\eta^w$, is defined by for all $(N, b, V^m) \in \Gamma$, for all weight function for agents $w$, for all $t \in \mathbb{N}_m$ and for all agent $i \in N$,

$$\eta^{w,t}_i(N, b, V^m) = \mu^{w,t}_i(N, b, V^m) + \frac{1}{|N|} \cdot \left[ v^t(b) - \sum_{k \in \mathbb{N}} \mu^{w,t}_k(N, b, V^m) \right], \quad (7)$$

where $\mu^{w,t}_i(N, b, V^m) = \max_{j \in B^+_i} \left\{ w(j) \cdot \left[ v^t(b_{-i}, j) - v^t(b_{-i}, j - i) \right] \right\}$. Under the solution $\eta^w$, all agents first receive their weighted maximal level-marginal contribution, and further allocate the remaining utility equally.

Remark 2. By Definition 5.1, it is easy to show that the P-WMI satisfies MPO and MCOV, but it violates METE. Similarly, the L-WMI satisfies MPO, but it violates METE and MCOV.

Similarly, a solution $\psi$ satisfies **P-weighted standard for games (PWSFG)** if $\psi(N, b, V^m) = \eta^d(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$ with $|N| \leq 2$ and for all weight function for agents $d$. A solution $\psi$ satisfies **L-weighted standard for games (LWSFG)** if $\psi(N, b, V^m) = \eta^w(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$ with $|N| \leq 2$ and for all weight function for levels $w$. Similar to the proofs of Lemma 3.1 and Theorem 3.2, we propose the analogies of Lemma 3.1 and Theorem 3.2.

- The P-WMI $\eta^d$ and the L-WMI $\eta^w$ satisfy MCRGP.
- On $\Gamma$, the P-WMI $\eta^d$ is the only solution satisfying PWSFG and MCRGP.
- On $\Gamma$, the L-WMI $\eta^w$ is the only solution satisfying LWSFG and MCRGP.

The following examples are to show that each of the axioms used in above characterizations is logically independent of the remaining axioms.

Example 8. For all $(N, b, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$, for all weight function $w$ and for all $i \in N$, we define the solution $\psi$ to be $\psi_i^t(N, b, V^m) = 0$. Clearly, $\psi$ satisfies MCRGP, but it violates PWSFG and LWSFG.
Example 9. For all \((N, b, V^m) \in \Gamma\), for all \(t \in \mathbb{N}_m\), for all weight function for agents \(d\) and for all \(i \in N\), we define the solution \(\psi_i\) to be
\[
\psi_i^t(N, b, V^m) = \begin{cases} \\
\eta_i^{d,t}(N, b, V^m) & \text{if } |N| \leq 2, \\
0 & \text{otherwise}. 
\end{cases}
\]
Clearly, \(\psi\) satisfies PWSFG, but it violates MCRGP.

Example 10. For all \((N, b, V^m) \in \Gamma\), for all \(t \in \mathbb{N}_m\), for all weight function for levels \(w\) and for all \(i \in N\), we define the solution \(\psi_i\) to be
\[
\psi_i^t(N, b, V^m) = \begin{cases} \\
\eta_i^{w,t}(N, b, V^m) & \text{if } |N| \leq 2, \\
0 & \text{otherwise}. 
\end{cases}
\]
Clearly, \(\psi\) satisfies LWSFG, but it violates MCRGP.

Remark 3. Here we provide an application with real data. Let \((N, b, V^m) \in \Gamma\) with \(N = \{i, j\}\), \(m = 2\) and \(b = (2, 1)\). Thus, \((N, b, V^m) = ((N, b, v^1), (N, b, v^2))\).

Let \(d(i) = 2\), \(d(j) = 3\), \(w(2) = 1\), \(w(1) = 5\) and \(w(1_i) = 2\), where \(k_p\) means level \(k\) of agent \(p\), \(p \in \{i, j\}\). Further, let \(v^1(2, 1) = 6\), \(v^1(2, 0) = 5\), \(v^1(1, 1) = 3\), \(v^1(0, 1) = 4\), \(v^1(1, 0) = 8\), \(v^1(0, 0) = 0\), \(v^2(2, 1) = 9\), \(v^2(2, 0) = 8\), \(v^2(1, 1) = 7\), \(v^2(0, 1) = 2\), \(v^2(1, 0) = 5\) and \(v^2(0, 0) = 0\). By Definitions 2.1 and 5.1,
\[
\eta_1^1(N, b, V^m) = 3, \quad \eta_2^1(N, b, V^m) = 1, \\
\eta_1^2(N, b, V^m) = 5, \quad \eta_2^2(N, b, V^m) = 1, \\
\eta_1^3(N, b, V^m) = 4, \quad \eta_2^3(N, b, V^m) = 2, \\
\eta_1^4(N, b, V^m) = \frac{13}{2}, \quad \eta_2^4(N, b, V^m) = \frac{5}{2}, \\
\eta_1^5(N, b, V^m) = \frac{3}{2}, \quad \eta_2^5(N, b, V^m) = \frac{3}{2}, \\
\eta_1^6(N, b, V^m) = \frac{15}{2}, \quad \eta_2^6(N, b, V^m) = \frac{3}{2}, \\
\eta_1^{d,1}(N, b, V^m) = \frac{19}{5}, \quad \eta_2^{d,1}(N, b, V^m) = \frac{11}{5}, \\
\eta_1^{d,2}(N, b, V^m) = \frac{31}{5}, \quad \eta_2^{d,2}(N, b, V^m) = \frac{14}{5}, \\
\mu_1^{w,1}(N, b, V^m) = 3, \quad \mu_2^{w,1}(N, b, V^m) = 2, \\
\mu_1^{w,2}(N, b, V^m) = 25, \quad \mu_2^{w,2}(N, b, V^m) = 2, \\
\eta_1^{w,1}(N, b, V^m) = \frac{7}{2}, \quad \eta_2^{w,1}(N, b, V^m) = \frac{5}{2}, \\
\eta_1^{w,2}(N, b, V^m) = 16, \quad \eta_2^{w,2}(N, b, V^m) = -7.
\]

6. Concluding remarks. Differing from pre-existing investigations on multichoice TU games, some results of this paper are provided as follows. Based on multicriteria situation and multi-choice behavior simultaneously, we consider the framework of multicriteria multi-choice TU games. By applying the maximal level-marginal contributions under multicriteria situation and multi-choice behavior simultaneously, we propose the MMI and the MNI. Further, we provide axiomatic results to present the rationality for the MMI and the MNI. In order to modify the difference among the agents and their activity levels respectively, we propose two weighted extensions of the MMI and related characterizations. we adopt different weight functions to introduce two weighted extensions of the MMI and related characterizations.

On the other hand, Davoodi et al. (2019) investigated an integrated model for relief operations in critical situations. The model is aimed at minimizing the late arrival of relief vehicles that cross points en route to disaster locations. Sangaiah et al. (2020) addressed a robust mixed-integer linear programming model for LNG sales planning over a given time horizon aiming to minimize the costs of the vendor. Since the parameter of the manufacturer supply has an uncertain nature in...
the real world, and this parameter is regarded to be interval-based uncertain. It is reasonable that the results due to Davoodi et al. (2019) and Sangaiah et al. (2020) could be extended to improve the method presentations of this paper by considering multicriteria situation and multi-choice behavior simultaneously. Based on the main results of this paper, it is also reasonable to consider that some traditional solutions could be extended by applying the maximal level-marginal contributions under multicriteria situation and multi-choice behavior simultaneously. This is left to the readers.

Acknowledgments. The author is grateful to the editor, the associate editor and the anonymous referees for very helpful suggestions and comments.

REFERENCES

[1] E. M. Bednarczuk, J. Miroforidis and P. Pyzel, A multi-criteria approach to approximate solution of multiple-choice knapsack problem, *Computational Optimization and Applications*, 70 (2018), 889–910.
[2] R. Branzei, On solution concepts for multi-choice cooperative games, *SEIO Bulletin*, 24 (2008), 13–19.
[3] R. Branzei, N. Llorca, J. Sanchez-Soriano and S. Tijis, Multi-choice clan games and their core, *TOP*, 17 (2009), 123–138.
[4] R. Branzei, N. Llorca, J. Sanchez-Soriano and S. Tijis, A constrained egalitarian solution for convex multi-choice games, *TOP*, 22 (2014), 860–874.
[5] R. Branzei, S. Tijis and J. M. Zarzuelo, Convex multi-choice cooperative games: Characterizations and monotonic allocation schemes, *European J. Oper. Res.*, 198 (2009), 571–575.
[6] S. M. R. Davoodi and A. Goli, An integrated disaster relief model based on covering tour using hybrid Benders decomposition and variable neighborhood search: Application in the Iranian context, *Computers and Industrial Engineering*, 130 (2019), 370–380.
[7] A. Goli, H. K. Zare, R. Tavakkoli-Moghaddam and A. Sadegheih, Hybrid artificial intelligence and robust optimization for a multi-objective product portfolio problem Case study: The dairy products industry, *Computers and Industrial Engineering*, 137 (2019), 106090.
[8] A. Goli, H. K. Zare, R. Tavakkoli-Moghaddam and A. Sadegheih, Multiobjective fuzzy mathematical model for a financially constrained closed-loop supply chain with labor employment, *Computational Intelligence*, 36 (2020), 4–34.
[9] M. R. Guarini, F. Battisti and A. Chiocchi, A methodology for the selection of multi-criteria decision analysis methods in real estate and land management processes, *Sustainability*, 10 (2018), 507–534.
[10] S. Hart and A. Mas-Colell, Potential, value and consistency, *Econometrica*, 57 (1989), 589–614.
[11] Y. A. Hwang and Y. H. Liao, The unit-level-core for multi-choice games: The replicated core for TU games, *Journal of Global Optimization*, 47 (2010), 161–171.
[12] Y. A. Hwang and Y. H. Liao, Reduction and dynamic approach for the multi-choice Shapley value, *Journal of Industrial and Management Optimization*, 9 (2013), 885–892.
[13] Y. H. Liao, The maximal equal allocation of nonseparable costs on multi-choice games, *Economics Bulletin*, 3 (2008), 1–8.
[14] Y. H. Liao, The duplicate extension for the equal allocation of nonseparable costs, *Operational Research: An International Journal*, 13 (2012), 385–397.
[15] Y. H. Liao, The precore: Converse consistent enlargements and alternative axiomatic results, *TOP*, 26 (2018), 146–163.
[16] Y. H. Liao, P. T. Liu and L. Y. Chung, The normalizations and related dynamic processes for two power indexes, *Journal of Control and Decision*, 4 (2017), 179–194.
[17] M. Maschler and G. Owen, The consistent Shapley value for hyperplane games, *International Journal of Game Theory*, 18 (1989), 389–407.
[18] H. Moulin, On additive methods to share joint costs, *The Japanese Economic Review*, 46 (1985), 303–332.
[19] I. Mustakerov, D. Borissova and E. Bantutov, Multiple-choice decision making by multicriteria combinatorial optimization, *Advanced Modeling and Optimization*, 14 (2012), 729–737.
[20] A. van den Nouweland, J. Potters, S. Tijs and J. M. Zarzuelo, Cores and related solution concepts for multi-choice games, ZOR-Mathematical Methods of Operations Research, 41 (1995), 289–311.

[21] J. S. Ransmeier, The Tennessee Valley Authority, Vanderbilt University Press, Nashville, TN, 1942.

[22] A. K. Sangaiah, E. B. Tirkolaee, A. Goli and S. Dehnavi–Arani, Robust optimization and mixed-integer linear programming model for LNG supply chain planning problem, Soft Computing, 24 (2020), 7885–7905.

[23] L. S. Shapley, Discussant’s Comment, Joint Cost Allocation, University of Oklahoma Press, Tulsa, 1982.

[24] E. B. Tirkolaee, A. Goli, M. Hematian, A. K. Sangaiah and T. Han, Multi-objective multi-mode resource constrained project scheduling problem using Pareto-based algorithms, Computing, 101 (2019), 547–570.

Received January 2020; 1st revision August 2020; 2nd revision October 2020.

E-mail address: twincos@ms25.hinet.net