Abstract

A Chen generating series, along a path and with respect to \( m \) differential forms, is a noncommutative series on \( m \) letters, with coefficients which are holomorphic functions over a simply connected manifold in other words a series with variable (holomorphic) coefficients. Such a series satisfies a first order noncommutative differential equation which is considered, by some authors, as the universal differential equation, \( i.e. \), in this case, universality can be seen by replacing each letter by constant matrices (resp. holomorphic vector fields) and then solving a system of linear (resp. nonlinear) differential equations.

Via rational series, on noncommutative indeterminates and with coefficients in rings, and their non-trivial combinatorial Hopf algebras (or pseudo-Hopf algebras), we propose a first step of a noncommutative Picard-Vessiot theory and we illustrate it with the case of linear differential equations with singular regular singularities thanks to the universal equation previously mentioned.
Combinatorial Picard-Vessiot (PV for short) theory of bilinear systems\(^1\) was realized by Fliess and Reutenauer\([28]\), as an application of differential algebra\([41,46]\).

This theory allows to employ, with success, linear algebraic groups in control theory (\textit{i.e.} as symmetry groups of linear differential equations), for which some questions were solved thanks to the theory of Hopf algebras\([11]\) and some combinatorial and effective aspects were set in\([45]\).

Let us, for instance, consider the following nonlinear dynamical system

\[
\dot{q}(z) = A_0(q)u_0(z) + \ldots + A_m(q)u_m(z), \quad q(z_0) = q_0, \quad y(z) = f(q(z)),
\]

where

\begin{enumerate}[(i)]
  
  \item \(y\) is the output,
  
  \item the vector state \(q = (q_1, \ldots, q_n)\) belongs to a complex holomorphic manifold \(\mathcal{M}\) of dimension \(n\),
  
  \item the observation \(f\) is defined within a fixed connected neighbourhood\(^2\) \(U\) of the initial state \(q_0\),
  
  \item the vector fields \((A_i)_{i=0,\ldots,m}\) are defined with respect to the coordinates as follows
\end{enumerate}

---

\(^{1}\) Namely - locally - linear of the states \(q_1, \ldots, q_N\) and linear of the inputs \(u_0, \ldots, u_m\)\([28]\).

\(^{2}\) In this introductory description the points are loosely identified with their coordinates through some chart \(\varphi_U : U \to \mathbb{C}^n\) likewise, in\([44]\), the space of holomorphic functions \(\mathcal{H}(U)\) is described by \(\mathcal{C}^\mathcal{O}[q_1, \ldots, q_n]\).
\[ A_i = \sum_{j=1}^{n} A_i^j(q) \frac{\partial}{\partial q_j}, \text{ with } A_i^j(q) \in \mathcal{H}(U), \]  

(2)

(v) the inputs \((u_i)_{i=0,...,m}\), as well as their inverses \((u_i^{-1})_{i=0,...,m}\), belong to a subring, \(\mathcal{O}_0\), of the ring of holomorphic functions \(\mathcal{H}(\Omega)\) with neutral element \(1_{\mathcal{H}(\Omega)}\) over the simply connected manifold \(\Omega\).

It is convenient (and possible) to separate the contribution of the vector fields \((A_i)_{i=0,...,m}\) and that of the differential forms \((\omega_i)_{i=0,...,m}\), defined by the inputs \(\omega_i(z) = u_i(z)dz\), through the encoding alphabet \(X = \{x_i\}_{i=0,...,m}\) which generates the monoid \(X^*\) with neutral element \(1_X^*\). Indeed, the output \(y\) can be computed by

\[ y(z_0, z) = \langle C_{z_0 \rightarrow z} \| \sigma f_{|q_0} \rangle = \sum_{w \in X^*} \alpha_{z_0}^z (w) \mathcal{Y}(w)[f]_{|q_0}, \]  

(3)

as the pairing (under suitable convergence conditions [30,33,35,44]) between the Chen series \(\mathcal{Y}\) of \((\omega_i)_{i=0,...,m}\) along the path \(z_0 \rightarrow z\) over \(\Omega\), \(C_{z_0 \rightarrow z} \in \mathcal{H}(\Omega)\langle X\rangle\) [10], and the generating series of (1), \(\sigma f_{|q_0} \in \mathcal{H}(U)\langle X\rangle\) [30], defined as follows

\[ C_{z_0 \rightarrow z} := \sum_{w \in X^*} \alpha_{z_0}^z (w) w \text{ and } \sigma f_{|q_0} := \sum_{w \in X^*} \mathcal{Y}(w)[f]_{|q_0} w, \]  

(4)

where, in (3)–(4), the iterated integral \(\alpha_{z_0}^z (w)\) and the differential operator \(\mathcal{Y}(w)\), are decoded, from the word \(w \in X^*\), recursively as follows

\[
\begin{cases}
\alpha_{z_0}^z (w) = 1_{\mathcal{H}(\Omega)} & \text{and } \mathcal{Y}(w) = \text{Id}, \quad \text{for } w = 1_{X^*}, \\
\alpha_{z_0}^z (w) = \int_{z_0}^z \omega_i(s) \alpha_{z_0}^s (v) & \quad \text{and } \mathcal{Y}(w) = A_i \circ \mathcal{Y}(v), \quad \text{for } w = x_i v, x_i \in X, v \in X^*.
\end{cases}
\]  

(5)

In this work, following this route, considering the differential ring \((\mathcal{H}(\Omega), \partial)\) and equipping \(\mathcal{H}(\Omega)\langle X\rangle\) with the derivation defined, for any \(S \in \mathcal{H}(\Omega)\langle X\rangle\), by

\[ dS = \sum_{w \in X^*} (\partial \langle S \| w \rangle) w, \]  

(6)

we can see that the Chen series satisfies the following noncommutative differential equation

\[ dS = MS \text{ with } M = u_0 x_0 + \ldots + u_m x_m, \]  

(7)

considered by many authors as the universal differential equation [10,14,17,18,38]. Universality can be seen by specialization, i.e. replacing the letters by constant matrices (resp. holomorphic vector fields) and therefore obtaining linear (resp. nonlinear) differential equations (see Remark 4.9 below) as well as their solutions.

---

3 This (usually one dimensional) manifold will be the support of the iterated integrals below.

4 By a Ree’s theorem [43], there is a primitive series \(L_{z_0 \rightarrow z} = \sum_{n \geq 1} L_n \in \mathcal{H}(\Omega)\langle X\rangle\) s.t. \(e^{L_{z_0 \rightarrow z}} = C_{z_0 \rightarrow z}\), meaning that \(C_{z_0 \rightarrow z}\) is group-like and \(L_n\) is (homogenous of degree \(n \geq 1\)) primitive series.
From equation (7), it follows (see, for example, [9]) that a PV theory of nonlinear systems (1) should be intimately connected with (7) (the reader may remark that, due to the connectedness of Ω, the constants of \((H(Ω)\langle\langle X\rangle\rangle, d)\) are

\[
\text{Const}(H(Ω)\langle\langle X\rangle\rangle) = \ker d = \mathbb{C} 1_{H(Ω)\langle\langle X\rangle\rangle}.
\] (8)

This culminates with the fact that the coefficients of any suitable\(^5\) solution is group-like, \(i.e.\) satisfies\(^6\), for any \(u, v \in X^*\) and \(x_i \in X,

\[
\partial \langle S \mid u \rangle = u_0 \langle S \mid v \rangle \quad \text{and} \quad \langle S \mid u \rangle = \langle S \mid u \rangle \langle S \mid v \rangle = \langle S \mid 1_{X^*} \rangle = 1_{H(Ω)}
\] (9)

Due to the fact that Ω is simply connected, the coordinate values of this series only depend on the endpoints and not on paths drawn on Ω. Denoting the subalgebra of \((H(Ω), \partial)\) generated by the family \((f_i)_{i \in I}\) and derivatives by \(C\{(f_i)_{i \in I}\}\) \(^{[48]}\) (\(i.e.\) the differential algebra generated by \((f_i)_{i \in I}\)), it follows that [28]

\[
\text{span}_{\mathbb{C}}\{d^I S \mid w\}_{w \in X^*, I \geq 0} \subset \text{span}_{\mathbb{C}}\{(u_i)_{i=0, \ldots, m}\} \{\langle S \mid w \rangle\}_{w \in X^*}
\] (10)

\[
\subset \text{span}_{\mathbb{C}}\{(u_i^{+1})_{i=0, \ldots, m}\} \{\langle S \mid w \rangle\}_{w \in X^*}
\] (11)

and then, in Section 4, the isomorphism between \(\text{span}_{\mathbb{C}}\{(u_i^{+1})_{i=0, \ldots, m}\} \{\alpha_{0}^2 (w)\}_{w \in X^*}\) and \(\mathbb{C}\{(u_i^{+1})_{i=0, \ldots, m}\} \otimes_{\mathbb{C}} \text{span}_{\mathbb{C}}\{\alpha_{0}^2 (w)\}_{w \in X^*}\) will be examined (Theorem 4.4) via the PV-extension related to (7) and, on the other hand, the output of (1) will be computed (Theorem 4.8) by pairing the series given in (4). As example, this calculation will be achieved according to the algebraic combinatorics of rational series, established beforehand in Sections 2 (Theorems 2.2, 2.5) and 3 (Theorems 3.2, 3.10).

## 2 Combinatorial framework

In this section, coefficients are taken in a commutative ring\(^7\) \(A\) and, unless explicitly stated, all tensor products will be considered over the ambient ring (or field).

### 2.1 Factorization in bialgebras

In section 1, the encoding alphabet \(X\) was already introduced. In particular, for \(m = 1 (i.e. X = \{x_0, x_1\})\), let us note that there are one-to-one correspondences

\[
(s_1, \ldots, s_r) \in \mathbb{N}_+^r \leftrightarrow x_0^{s_1-1} x_1 \ldots x_0^{s_r-1} x_1 \in X^* x_1 \overset{\pi_y}{\pi_x} y_s x_1 \ldots y_r \in Y^*,
\] (12)

where \(Y := \{y_k\}_{k \geq 1}\) and \(\pi_X\) is the \(\text{conc}\) morphism, from \(A\langle Y\rangle\) to \(A\langle X\rangle\), mapping \(y_k\) to \(x_0^{k-1} x_1\). This morphism \(\pi_X\) admits an adjoint \(\pi_Y\) for the two standard scalar

---

\(^5\) \text{\textit{i.e.} group-like at one - interior or frontier - point.}

\(^6\) In the first identity, also called Friedrichs criterion, is involved the shuffle product \((\cdot \cdot)\) \cite{[10,29,45]}.

\(^7\) although some of the properties already hold for a general commutative semiring \cite{[1]}.
products\(^8\) which has a simple combinatorial description: the restriction of \(\pi_Y\) to the subalgebra \((A_{1X} \oplus A(Y)x_1, \text{conc})\), is an isomorphism given by \(\pi_Y(x_k^{-1}x_1) = y_k\) (and the kernel of the non-restricted \(\pi_Y\) is \(A(X)x_0\)). For all matters concerning finite \((X\text{ and similar})\) or infinite \((Y\text{ and similar})\) alphabets, we will use a generic model noted \(\mathcal{X}\) in order to state their common combinatorial features. Let us recall also that the coproduct \(\Delta_{\text{conc}}\) is defined, for any \(w \in \mathcal{X}^*\), as follows
\[
\Delta_{\text{conc}}w = \sum_{u,v \in \mathcal{X}^*, uv = w} u \otimes v. \tag{13}
\]

As an algebra the \(A\)-module \(A(\mathcal{X})\) is equipped with the associative unital concatenation and the associative commutative and unital shuffle product. The latter being defined, for any \(x,y \in \mathcal{X}\) and \(u,v,w \in \mathcal{X}^*\), by the following recursion
\[
w \shuffle 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} \shuffle w = w \text{ and } xu \shuffle yv = x(u \shuffle yv) + y(xu \shuffle v) \tag{14}
\]
or, equivalently, by its dual comultiplication (which is a morphism for concatenations\(^9\)), defined, for each letter \(x \in \mathcal{X}\), by
\[
\Delta_x x = 1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*}. \tag{15}
\]

Once \(\mathcal{X}\) has been totally ordered\(^{10}\), the set of Lyndon words over \(\mathcal{X}\) will be denoted by \(\mathcal{L} \text{yn} \mathcal{X}\). A pair of Lyndon words \((l_1, l_2)\) is called the standard factorization of a Lyndon \(l\) (and will be noted \((l_1, l_2) = st(l)\)) if \(l = l_1l_2\) and \(l_2\) is the longest nontrivial proper right factor of \(l\) or, equivalently, its smallest such (for the lexicographic ordering, see \([42]\) for proofs and details). According to a theorem by Radford, the set of Lyndon words form a pure transcendence basis of the \(A\)-shuffle algebras \((A(\mathcal{X}), \shuffle, 1_{\mathcal{X}^*})\).

It is well known that the enveloping algebra \(\mathcal{U}(\mathcal{L} \text{ie}_A(\mathcal{X}))\) is isomorphic to the (connected, graded and co-commutative) bialgebra\(^{11}\) \(\mathcal{H}_{\text{cc}}(\mathcal{X}) = (A(\mathcal{X}), \text{conc}, 1_{\mathcal{X}^*}, \Delta_x, e)\) (the counit being here \(e(P) = \langle P \mid 1_{\mathcal{X}^*} \rangle\)) and, via the pairing
\[
A(\langle \mathcal{X} \rangle) \otimes A(\langle \mathcal{X} \rangle) \longrightarrow A, \tag{16}
\]
\[
T \otimes P \longrightarrow \langle T \mid P \rangle := \sum_{w \in \mathcal{X}^*} \langle T \mid w \rangle \langle P \mid w \rangle, \tag{17}
\]
we can, classically, endow \(A(\mathcal{X})\) with the graded\(^{12}\) linear basis \(\{P_w\}_{w \in \mathcal{X}^*}\) (expanded after any homogeneous basis \(\{P_i\}_{i \in \mathcal{L} \text{yn} \mathcal{X}}\) of \(\mathcal{L} \text{ie}_A(\mathcal{X})\)) and its graded

---

\(^8\) That is to say \(\langle 1_{\mathcal{X}} \rangle \otimes 1_{\mathcal{Y}} \in A(X)\) \(\forall \{p \in A(X), q \in A(Y)\}\) \((\langle 1_{\mathcal{X}} p \mid q \rangle_Y = \langle p \mid \pi X q \rangle_Y\).

\(^9\) On \(A(\mathcal{X})\) and \(A(\mathcal{X}) \otimes A(\mathcal{X})\), respectively.

\(^{10}\) For technical reasons, the orders \(x_0 < x_1\) (for \(X\)) and \(y_1 > y_2 > \ldots > y_{n+1} > \ldots\) (for \(Y\)) are usual.

\(^{11}\) In case \(A\) is a \(\mathbb{Q}\)-algebra, the isomorphism \(\mathcal{U}(\mathcal{L} \text{ie}_A(\mathcal{X})) \cong \mathcal{H}_{\text{cc}}(\mathcal{X})\) can also be seen as an easy application of the CQMM theorem.

\(^{12}\) For \(\mathcal{X} = X\) or \(= Y\) the corresponding monoids are equipped with length functions, for \(X\) we consider the length of words and for \(Y\) the length is given by the weight \(\ell(y_1, \ldots, y_n) = i_1 + \ldots + i_n\). This naturally induces a grading of \(A(\mathcal{X})\) and \(\mathcal{L} \text{ie}_A(\mathcal{X})\) in free modules of finite dimensions. For
dual basis \( \{ S_w \}_{w \in \mathcal{X}^*} \) (containing the pure transcendence basis \( \{ S_l \}_{l \in \mathcal{L}^{yn} \mathcal{X}} \) of the \( A \)-shuffle algebra). In the case when \( A \) is a \( \mathbb{Q} \)-algebra, we also have the following factorization \(^{13}\) of the diagonal series, i.e. \([45]\) (here all tensor products are over \( A \))

\[
\mathcal{D}_{\Delta} := \sum_{w \in \mathcal{X}^*} w \otimes w = \sum_{w \in \mathcal{X}^*} S_w \otimes P_w = \prod_{i \in \mathcal{L}^{yn} \mathcal{X}} e^{S_i \otimes P_i}
\]

and (still in case \( A \) is a \( \mathbb{Q} \)-algebra) dual bases of homogenous polynomials \( \{ P_w \}_{w \in \mathcal{X}^*} \) and \( \{ S_w \}_{w \in \mathcal{X}^*} \) can be constructed recursively as follows

\[
\begin{align*}
P_(x) &= x, & S_x &= x & \text{for } x \in \mathcal{X}, \\
P_l &= [P_{l_1}, P_{l_2}], & S_l &= y S_l', & \text{for } l = y l' \in \mathcal{L}^{yn} \mathcal{X} - \mathcal{X} \\
P_w &= \prod_{i_1 \cdots i_k} P_{i_1} \cdots P_{i_k}, & S_w &= \prod_{i_1 \cdots i_k} S_{i_1} \cdots S_{i_k}, & \text{for } w = i_1 \cdots i_k, \text{ with } l_1, \ldots, l_k \in \mathcal{L}^{yn} \mathcal{X}, l_1 > \ldots > l_k.
\end{align*}
\]  

The graded dual of \( \mathcal{H}_{\Delta}(\mathcal{X}) \) is \( \mathcal{H}^{\vee}_{\Delta}(\mathcal{X}) = (A\langle \mathcal{X} \rangle, \cup, 1_{\mathcal{X}}, \Delta_{\text{conc}}, e) \).

As an algebra, the module \( A(Y) \) is also equipped with the associative commutative and unital quasi-shuffle product defined, for \( u, v, w \in Y^* \) and \( y_i, y_j \in Y \), by

\[
w \ \cup \ \cup \ 1_Y = 1_Y \ \cup \ \cup \ w = w,
\]

\[
y_i u \ \cup \ \cup \ y_j v = y_i (u \ \cup \ \cup \ y_j v) + y_j (y_i u \ \cup \ \cup \ v) + y_{i+j} (u \ \cup \ \cup \ v).
\]

This product also can be dualized according to \( (y_k \in Y) \)

\[
\Delta_{\cup} y_k := y_k \otimes 1_Y + 1_Y \otimes y_k + \sum_{i+j=k} y_i \otimes y_j
\]

which is also a conc-morphism (see \([27]\)). We then get another (connected, graded and co-commutative) bialgebra which, in case \( A \) is a \( \mathbb{Q} \)-algebra, is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements,

\[
\mathcal{H}_{\cup}(Y) = (A\langle Y \rangle, \text{conc}, 1_{\mathcal{X}}, \Delta_{\cup}, e) \cong \mathcal{W}(\text{Prim}(\mathcal{H}_{\cup}(Y))),
\]

where \( \text{Prim}(\mathcal{H}_{\cup}(Y)) = \text{Im}(\pi_1) = \text{span}_A \{ \pi_1(w) | w \in Y^* \} \) and \( \pi_1 \) is the eulerian projector defined, for any \( w \in Y^* \), by \([36,37]\)

\[
\pi_1(w) = w + \sum_{k=2}^{\lfloor w \rfloor} \left( -\frac{1}{k} \right)^{k-1} \sum_{u_1, \ldots, u_k \in Y^+} \langle w | u_1 \wedge \ldots \wedge u_k \rangle u_1 \ldots u_k,
\]

and, for any \( w = y_{i_1} \cdots y_{i_k} \in Y^* \), \((w)\) denotes the number \( i_1 + \ldots + i_k \).

**Remark 2.1** By \((13)\) and \((15)\), any letter \( x \in \mathcal{X} \) is primitive, for \( \Delta_{\text{conc}} \) and \( \Delta_{\cup} \). By \((22)\), the polynomials \( \{ \pi_1(y_k) \}_{k \geq 2} \) and only the letter \( y_1 \) are primitive, for \( \Delta_{\cup} \).

\(^{13}\) Also called MSR factorization after the names of Mélançon, Schützenberger and Reutenauer.
Now, let \( \{ \Pi_w \}_{w \in Y^*} \) be the linear basis, expanded by decreasing Poincaré-Birkhoff-Witt (PBW for short) after any basis \( \{ \Pi_l \}_{l \in L^Y} \) of \( \mathcal{H}(Y) \) homogeneous in weight \(^{14}\), and let \( \{ \Sigma_w \}_{w \in Y^*} \) be its dual basis which contains the pure transcendence basis \( \{ \Sigma_l \}_{l \in L^Y} \) of the A-quasi-shuffle algebra. One also has the factorization of the diagonal series \( D_y \), on \( \mathcal{H}(Y) \), which reads \(^{15}\) [36,37,38]

\[
D_y := \sum_{w \in Y^*} w \otimes w = \sum_{w \in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l \in L^Y} e^{\Sigma_l \otimes \Pi_l}.
\] (25)

We are now in the position to state the following

**Theorem 2.2 ([37,38])** Let \( A \) be a \( Q \)-algebra, then the endomorphism of algebras \( \Phi_{\pi_1} : (A(Y), \text{conc}, 1_Y^*) \to (A(Y), \text{conc}, 1_Y^*) \) mapping \( y_k \) to \( \pi_1(y_k) \), is an automorphism of \( A(Y) \) realizing an isomorphism of bialgebras between \( \mathcal{H}(Y) \) and \( \mathcal{H}_{\pi_1}(Y) \cong \mathcal{U} \left( \text{Prim} \left( \mathcal{H}_{\pi_1}(Y) \right) \right) \).

In particular, it can be easily checked that the following diagram commutes

\[
\begin{diagram}
A(Y) & \xrightarrow{\Delta_{\pi_1}} & A(Y) \otimes A(Y) \\
\Phi_{\pi_1} \downarrow & & \downarrow \Phi_{\pi_1} \otimes \Phi_{\pi_1} \\
A(Y) & \xleftarrow{\Delta_{\pi_1}} & A(Y) \otimes A(Y)
\end{diagram}
\]

Hence, the bases \( \{ \Pi_w \}_{w \in Y^*} \) and \( \{ \Sigma_w \}_{w \in Y^*} \) of \( \mathcal{U} \left( \text{Prim} \left( \mathcal{H}_{\pi_1}(Y) \right) \right) \) are images by \( \Phi_{\pi_1} \) and by the adjoint mapping of its inverse, \( \Phi_{\pi_1}^{-1} \) of \( \{ P_w \}_{w \in Y^*} \) and \( \{ S_w \}_{w \in Y^*} \), respectively. Algorithmically, by Remark 2.1, the dual bases of homogeneous polynomials \( \{ \Pi_w \}_{w \in Y^*} \) and \( \{ \Sigma_w \}_{w \in Y^*} \) can be constructed directly and recursively by

\[
\begin{align*}
\Pi_y & = \pi_1(y), & \Sigma_y & = y, & \text{for } y \in Y, \\
\Pi_l & = [\Pi_{l_1}, \Pi_{l_2}], & \Sigma_l & = \sum_{(s)} \frac{y_{s_{l_1}} \ldots y_{s_{l_k}}}{i!} \sum_{l_1 \ldots l_k}, & \text{for } l \in L^Y - Y \text{ st } (l) = (l_1, l_2), \\
\Pi_w & = \Pi_{i_1} \ldots \Pi_{i_k}, & \Sigma_w & = \frac{\sum_{l_1} \ldots \sum_{l_k}}{i_1! \ldots i_k!}, & \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, \text{ with } l_1, \ldots, l_k \in L^Y, l_1 \geq \ldots \geq l_k.
\end{align*}
\] (26)

In (\(*\)), the sum is taken over all \( \{ k_1, \ldots, k_l \} \subset \{ 1, \ldots, k \} \) and \( l_1 \geq \ldots \geq l_n \) such that \( (y_{s_{l_1}}, \ldots, y_{s_{l_k}}) \notin (y_{s_{l_1}}, \ldots, y_{s_{l_k}}, l_1, \ldots, l_n) \), where \( \triangleq \) denotes the transitive closure of the relation on standard sequences, denoted by \( \triangleq [7,45] \).

\(^{14}\) Factorization (25) will be true in particular for the basis (26) explicitly constructed there.

\(^{15}\) Again all tensor products will be taken over \( A \). Note that this factorization holds for any enveloping algebra as announced in [45]. Of course, the diagonal series no longer exists and must be replaced by the identity \( Id_{\mathcal{U}} \) (see [25], coda for details).
To end this section, let us extend $\text{conc}$ and $\sqcup$, for any series $S, R \in A\langle\langle \mathcal{X} \rangle\rangle$, by

$$SR = \sum_{w \in \mathcal{X}^*} \left( \sum_{u,v \in \mathcal{X}^*} \langle S | u \rangle \langle R | v \rangle w \right) \text{ and } S \sqcup R = \sum_{u,v \in \mathcal{X}^*} \langle S | u \rangle \langle R | v \rangle u \sqcup v,$$

(27)

and $\sqcup$, for any series $S, R \in A\langle\langle Y \rangle\rangle$, by

$$S \sqcup R = \sum_{u,v \in Y^*} \langle S | u \rangle \langle R | v \rangle u \sqcup v.$$  

(28)

**Definition 2.3** Any series $S \in A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathcal{X} \rangle\rangle$) is said to be

(i) a $\sqcup$ (resp. $\sqcup$, conc)-character of $A\langle\langle X \rangle\rangle$, conc, 1$_{X^*}$) iff, for $u, v \in Y^*$ (resp. $\mathcal{X}^*$), one has $\langle S | 1_{Y^*} \rangle = 1_A$ (resp. $\langle S | 1_{\mathcal{X}^*} \rangle = 1_A$) and $\langle S | u \sqcup v \rangle = \langle S | u \rangle \langle S | v \rangle$ (resp. $\langle S | u \sqcup v \rangle = \langle S | u \rangle \langle S | v \rangle$, $\langle S | u v \rangle = \langle S | u \rangle \langle S | v \rangle$).

(ii) an infinitesimal $\sqcup$ (resp. $\sqcup$, conc)-character of $A\langle\langle X \rangle\rangle$, conc, 1$_{X^*}$) iff, for any $u, v \in Y^*$ (resp. $\mathcal{X}^*$), one has $\langle S | u \sqcup v \rangle = \langle S | u \rangle \langle v | 1_{Y^*} \rangle + \langle u | 1_{Y^*} \rangle \langle S | v \rangle$ (resp. $\langle S | u \sqcup v \rangle = \langle S | u \rangle \langle v | 1_{\mathcal{X}^*} \rangle + \langle u | 1_{\mathcal{X}^*} \rangle \langle S | v \rangle$, $\langle S | u v \rangle = \langle S | u \rangle \langle v | 1_{Y^*} \rangle$ + $\langle u | 1_{\mathcal{X}^*} \rangle \langle S | v \rangle$). Moreover, if $\langle S | 1_{Y^*} \rangle = 1_A$ (resp. $\langle S | 1_{\mathcal{X}^*} \rangle = 1_A$) then $\langle S | u \sqcup v \rangle = 0$ (resp. $\langle S | u \sqcup v \rangle = 0$, $\langle S | u v \rangle = 0$).

Let us also extend the coproduct $\Delta_{\sqcup}$ (resp. $\Delta_{\text{conc}}$ and $\Delta_{\sqcup}$) given in (22) (resp. (13) and (15)) over $A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathcal{X} \rangle\rangle$) by linearity as follows

$$\forall S \in A\langle\langle Y \rangle\rangle, \quad \Delta_{\sqcup} S = \sum_{w \in Y^*} \langle S | w \rangle \Delta_{\sqcup} w \in A\langle\langle Y^* \otimes Y^* \rangle\rangle,$$

(29)

$$\forall S \in A\langle\langle \mathcal{X} \rangle\rangle, \quad \Delta_{\sqcup} S = \sum_{w \in \mathcal{X}^*} \langle S | w \rangle \Delta_{\sqcup} w \in A\langle\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle\rangle,$$

(30)

$$\forall S \in A\langle\langle \mathcal{X} \rangle\rangle, \quad \Delta_{\text{conc}} S = \sum_{w \in \mathcal{X}^*} \langle S | w \rangle \Delta_{\text{conc}} w \in A\langle\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle\rangle.$$  

(31)

Now, let us see how these combinatorics will operate over rational series to describe, as illustration, solutions of linear differential equations in Section 4 below.

### 2.2 Representative series

Representative (or rational) series are the representative functions on the free monoid$^{16}$ [21] and their magic is that it rests on four (apparently distant) pillars:

- Separated coproduct (SC), uniquely for fields,
- Finite orbit by shifts (FS),
- Result of a rational expression (RE),
- Linear representation (LR).

$^{16}$ These functions were considered on groups in [11, 12].
Definition 2.4  Let $S \in A\langle \mathcal{X} \rangle$ (resp. $A\langle \mathcal{X}^* \rangle$) and $P \in A\langle \mathcal{X} \rangle$ (resp. $A\langle \mathcal{X}^* \rangle$).

(i) The left (resp. right) shift of $S$ by $P$, is $P \triangleright S$ (resp. $S \triangleleft P$) defined by

\[ \forall w \in \mathcal{X}^*, \langle P \triangleright S | w \rangle = \langle S | wP \rangle \quad (\text{resp.} \quad \langle S \triangleleft P | w \rangle = \langle S | Pw \rangle). \]

(ii) For any $S \in A\langle \mathcal{X} \rangle$ such that $\langle S | 1_{\mathcal{X}^*} \rangle = 0$, the Kleene star of $S$ is defined as $S^* = (1 - S)^{-1}$.

(iii) In case $A = K$ is a field, one can define also the Sweedler’s dual $\mathcal{H}_\triangleright^o (\mathcal{X})$ of $\mathcal{H}_\triangleright^o (\mathcal{X})$ by $S \in \mathcal{H}_\triangleright^o (\mathcal{X}) \iff \Delta_{\text{conc}} (S) = \sum_{i \in I} G_i \otimes D_i$ [45], for some $I$ finite, $\{G_i\}_{i \in I}$; $\{D_i\}_{i \in I}$ being series (as a matter of fact, it can be shown that they even can be chosen in $\mathcal{H}_\triangleright^o (\mathcal{X})$, see [38]).

Theorem 2.5 ([19,21,34,45]) For $S \in A\langle \mathcal{X} \rangle$, the following assertions are equivalent

(i) The shifts $\{S \triangleleft w\}_{w \in \mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w \in \mathcal{X}^*}$) lie in a finitely generated shift-invariant $A$-module.

(ii) The series $S$ belongs to the (algebraic) closure of $\hat{A}\mathcal{X}$ by the operations $\{\text{conc}, +, \ast\}$ (within $A\langle \mathcal{X} \rangle$).

(iii) There is a linear representation $(\nu, \mu, \eta)$, of rank $n$, for $S$ with $\nu \in M_{1,n}(A)$, $\eta \in M_{n,1}(A)$ and a morphism of monoids $\mu : \mathcal{X}^* \rightarrow M_{n,n}(A)$ such that

\[ S = \sum_{w \in \mathcal{X}^*} (\nu \mu(w) \eta)w. \]

A series satisfying one of the conditions of Theorem 2.5 is called rational. The set of these series, a $A$-module denoted by $A\text{rat}\langle \mathcal{X} \rangle$, is closed by $\{\text{conc}, +, \ast\}$.

We also have the following constructions of linear representations (only the last one is new and the first ones are already treated in [20,39]). In particular, the con-

---

17 Some schools (as Jacob one, see [39,31]) used to call this a residual. These actions are none other than the shifts of functions of harmonic analysis.

18 They are associative, commute with each other: $S \triangleleft (PR) = (S \triangleleft P) \triangleleft R$, $P \triangleright (R \triangleright S) = (P \triangleright R) \triangleright S$ and $(P \triangleleft S) \triangleright R = P \triangleleft (S \triangleright R)$ and, for $x, y \in \mathcal{X}, w \in \mathcal{X}^*, x \triangleright (yw) = (yw) \triangleleft x = \delta_{xw}$ (Kronecker delta).

19 Using one of the topologies of section 4.2 (adapted with $A$ replacing $\mathcal{H}(\Omega)$), we have $S^* = \sum_{n \geq 0} S^n$. We also get the fact that the space $\hat{A}\mathcal{X}$ (used below) of series of degree 1, i.e. the set $\{\sum_{x \in \mathcal{X}} \alpha(x) x\}_{\alpha \in A^\mathcal{X}}$ is the closure of the $A$-module $A\mathcal{X}$ generated by letters. In the case of a finite alphabet however (here $\mathcal{X} = X$) [21], $\hat{A}\mathcal{X} = A.\mathcal{X}$.

20 When $A$ is noetherian, first condition is equivalent to the fact that the module generated by $\{S \triangleleft w\}_{w \in \mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w \in \mathcal{X}^*}$) is finitely generated (and more precisely, in this case, by a finite number of those shifts). Unfortunately we are not in this case here, but our ring being without zero divisors (analytic functions), we can use the fraction field, here being realized by germs [15].

21 see [40].

22 In fact (we will see it) a unital $A$-algebra for conc and $\ast$. 
structions of $R_1 \sqcup R_2$ and $R_1 \sqcup R_2$ base on the tensor products (of representations) and use the expressions of the coproducts given in (15) and (22), respectively.

**Proposition 2.6** The module $A^{\text{rat}}(\langle \mathcal{X} \rangle)$ (resp. $A^{\text{rat}}(\langle \mathcal{Y} \rangle)$) is closed by $\sqcup$ (resp. $\sqcup$). Moreover, for $i = 1, 2$, let $R_i \in A^{\text{rat}}(\langle \mathcal{X} \rangle)$ and $(v_i, \mu_i, \eta_i)$ be its representation of dimension $n_i$. Then the linear representation of

$$R_i^*$$
is
given by

$$\begin{pmatrix} 0 & 1 \\ \mu_i + \eta_i v_i \mu_i & 0 \end{pmatrix}, \eta_i$$

that of $R_1 + R_2$ is

$$\begin{pmatrix} v_1 & v_2 \\ \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \eta_1$$

that of $R_1 R_2$ is

$$\begin{pmatrix} v_1 & 0 \\ \mu_1 \eta_1 v_2 \mu_2 & 0 \\ \mu_2 \end{pmatrix}, \eta_1 \eta_2$$

that of $R_1 \sqcup R_2$ is

$$\begin{pmatrix} v_1 \otimes v_2 \{ \mu_1 (x) \otimes I_{n_2} + I_{n_1} \otimes \mu_2 (x) \} \end{pmatrix}, \eta_1 \otimes \eta_2$$

that of $R_1 \sqcup R_2$ is

$$\begin{pmatrix} v_1 \otimes v_2 \{ \mu_1 (y) \otimes I_{n_2} + I_{n_1} \otimes \mu_2 (y) \} \end{pmatrix}_{k \geq 1}, \eta_1 \otimes \eta_2$$

**Example 2.7** [Identity $(-t^2 x_{0,1})^* \sqcup (t^2 x_{0,1})^* = (-4t^4 x_{0,1})^*$, [33,34]]

\[
\begin{array}{c}
\text{start} \rightarrow 1 \rightarrow 2 \rightarrow \text{start} \\
1 \xleftarrow{x_0, t} 2 \xleftarrow{x_1, t} 1 \xleftarrow{x_1, t} 2
\end{array}
\]

\[
(-t^2 x_{0,1})^* \leftrightarrow (v_2, \{ \mu_2 (x_0), \mu_2 (x_1) \}, \eta_2) (t^2 x_{0,1})^* \leftrightarrow (v_1, \{ \mu_1 (x_0), \mu_1 (x_1) \}, \eta_1).
\]

$v_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\mu_1 (x_0) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$, $\mu_1 (x_1) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$, $\eta_1 = \begin{pmatrix} 0 \end{pmatrix}$

$v_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\mu_2 (x_0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\mu_2 (x_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\eta_2 = \begin{pmatrix} 0 \end{pmatrix}$

\[
(-t^2 x_{0,1})^* \sqcup (t^2 x_{0,1})^* \leftrightarrow (v, \{ \mu (x_0), \mu (x_1) \}, \eta) \\
= (v_1 \otimes v_2, \{ \mu_1 (x_0) \otimes I_{n_2} + I_{n_1} \otimes \mu_2 (x_0), \\
\mu_1 (x_1) \otimes I_{n_2} + I_{n_1} \otimes \mu_2 (x_1), \eta_1 \otimes \eta_2 \}).
\]
Proposition 2.8 With the notations of Definition 2.4.(iii), if $A$ is a field $K$ then

(a) Assertions of Theorem 2.5 are equivalent to

(iv) There exists a finite double family of series $(G_i, D_i)_{i \in F}$ such that

\[
\Delta_{\text{conc}} S = \sum_{i \in F} G_i \otimes D_i
\]

(b) For $S \in \mathcal{H}^o_{\mathbb{L}}(\mathcal{X})$, since $A$ is a field then the previous identity is equivalent to

\[
\forall P, Q \in \mathcal{H}_{\mathbb{L}}(\mathcal{X}), \langle S \mid PQ \rangle = \sum_{i \in I} \langle G_i \mid P \rangle \langle D_i \mid Q \rangle.
\]

Therefore, $(K^{\text{rat}}(\langle X \rangle), \cup, 1_{X^*}, \Delta_{\text{conc}}, \epsilon)$ (resp. $(K^{\text{rat}}(\langle Y \rangle), \cup, 1_{Y^*}, \Delta_{\text{conc}}, \epsilon)$) is the Sweedler’s dual of $\mathcal{H}_{\mathbb{L}}(\mathcal{X})$ (resp. $\mathcal{H}_{\mathbb{U}}(\mathcal{Y})$).

\[23\] See [38] for a way to obtain this finite double family of series $(G_i, D_i)_{i \in F}$. 

11
3 Triangularity, solvability and rationality

3.1 Syntactically exchangeable rational series

Now, we have to study a special set of series in order to work with the rational series of this class: a series $S \in A(\langle x \rangle)$ is called syntactically exchangeable if and only if it is constant on multi-homogeneous classes, i.e.,

$$(\forall u, v \in \mathcal{X}^+)((\forall x \in \mathcal{X})(|u|_x = |v|_x)) \Rightarrow (S \mid u) = (S \mid v).$$

A series $S \in A(\langle x \rangle)$ is syntactically exchangeable iff it is of the following form

$$S = \sum_{\alpha \in \mathbb{N}^{N(\mathcal{X})}, \text{supp}(\alpha) = \{x_1, \ldots, x_k\}} s_{\alpha} x_1^{\alpha(x_1)} \ldots x_k^{\alpha(x_k)}.$$  (33)

The set of these series, a shuffle subalgebra of $A(\langle X \rangle)$, will be denoted $A_{\text{exc}}^\text{rat}(\langle X \rangle)$.

When $A$ is a field, the rational and exchangeable series are exactly those who admit a representation with commuting matrices (at least the minimal one is such, see Theorem 3.2 below). We will take this as a definition as, even for rings, this property implies syntactic exchangeability.

**Definition 3.1** A series $S \in A^\text{rat}(\langle \mathcal{X} \rangle)$ will be said rationally exchangeable if it admits a representation $(\nu, \mu, \eta)$ such that $\{\mu(x)\}_{x \in \mathcal{X}}$ is a set of commuting matrices, the set of these series, a shuffle subalgebra of $A(\langle X \rangle)$, will be denoted $A_{\text{exc}}^\text{rat}(\langle \mathcal{X} \rangle)$.

**Theorem 3.2 (See [23,38])** Let $A_{\text{exc}}^\text{synt}(\langle \mathcal{X} \rangle)$ denote the set of (syntactically) exchangeable series. Then

(i) In all cases, one has $A_{\text{exc}}^\text{rat}(\langle \mathcal{X} \rangle) \subset A_{\text{exc}}^\text{rat}(\langle \mathcal{X} \rangle) \cap A_{\text{exc}}^\text{synt}(\langle \mathcal{X} \rangle)$. The equality holds when $A$ is a field and

$$A_{\text{exc}}^\text{rat}(\langle X \rangle) = \bigcup_{x \in X} A_{\text{exc}}^\text{rat}(\langle x \rangle),$$

$$A_{\text{exc}}^\text{rat}(\langle Y \rangle) \cap A_{\text{fin}}^\text{rat}(\langle Y \rangle) = \bigcup_{k \geq 0} A_{\text{exc}}^\text{rat}(\langle y_1 \rangle) \cup \ldots \cup A_{\text{exc}}^\text{rat}(\langle y_k \rangle) \subseteq A_{\text{exc}}^\text{rat}(\langle Y \rangle),$$

where $A_{\text{fin}}^\text{rat}(\langle Y \rangle) = \bigcup_{F \subseteq_{\text{finite}} Y} A_{\text{exc}}^\text{rat}(\langle F \rangle)$, the algebra of series over finite subalphabets.

(ii) (Kronecker’s theorem [1,50]) One has $A_{\text{rat}}^\text{rat}(\langle x \rangle) = \{P(1 - xQ)^{-1}\}_{P,Q \in A[x]}$ (for $x \in \mathcal{X}$) and if $A = K$ is an algebraically closed field of characteristic zero one also has $K^\text{rat}(\langle x \rangle) = \text{span}_K \{((ax)^*) \cup K\langle x \rangle | a \in K\}$.  



---

The last inclusion is strict as shows the example of the following identity [6]

$$(ty_1 + t^2y_2 + \ldots)^* = \lim_{k \to +\infty} (ty_1 + \ldots + t^ky_k)^* = \lim_{k \to +\infty} (ty_1)^* \cup \ldots \cup (t^ky_k)^* = \bigcup_{k \geq 1} (t^ky_k)^*$$

which lives in $A_{\text{exc}}^\text{rat}(\langle Y \rangle)$ but not in $A_{\text{exc}}^\text{rat}(\langle Y \rangle) \cap A_{\text{fin}}^\text{rat}(\langle Y \rangle)$.  



---



---
(iii) The rational series \((\sum_{x \in \mathcal{A}} \alpha_x x)^*\) are conc-characters and any conc-character is of this form.

(iv) Let us suppose that \(A\) is without zero divisors and let \((\phi_i)_{i \in I}\) be a family within \(A\) which is \(\mathbb{Z}\)-linearly independent then, the family \(\mathcal{L}y((\mathcal{A}^*)) \cup \{\phi_i\}_{i \in I}\) is algebraically free over \(A\) within \((A)\).

(v) In particular, if \(A\) is a ring without zero divisors \(\{x^*\}_{x \in \mathcal{A}}\) (resp. \(\{y^*\}_{y \in Y}\)) are algebraically independent over \((A(\mathcal{A}^*), [\mathbb{Z}]_{\mathcal{A}^*})\) (resp. \((A(\mathcal{Y}^*), [\mathbb{Z}]_{\mathcal{Y}^*})\)) within \((A)\) (resp. \((A)\)).

Proof.

(i) The inclusion is obvious in view of (33). For the equality, it suffices to prove that, when \(A\) is a field, every rational and exchangeable series admits a representation with commuting matrices. This is true of any minimal representation as shows the computation of shifts (see [19,23,38]).

Now, if \(\mathcal{A}\) is finite, as all matrices commute, we have

\[
\sum_{w \in \mathcal{A}^*} \mu(w)w = \left( \sum_{x \in \mathcal{A}} \mu(x)x \right)^* = \bigcup_{x \in \mathcal{A}} (\mu(x)x)^*
\]

and the result comes from the fact that \(R\) is a linear combination of matrix elements. As regards the second equality, inclusion \(\supset\) is straightforward. We remark that the union \(\bigcup_{k \geq 1} A(\langle y_1 \rangle, [\mathbb{Z}]_{\mathcal{A}^*}) \cup \ldots \cup A(\langle y_k \rangle, [\mathbb{Z}]_{\mathcal{A}^*})\) is directed as these algebras are nested in one another. With this in view, the reverse inclusion comes from the fact that every \(S \in A(\mathcal{A}^*), \supset\) is a series over a finite alphabet and the result follows from the first equality.

(ii) Let \(\mathcal{A} = \{P(1-xQ)^{-1} P, \forall P \in A[x]\}. \) Since \(P(1-xQ)^{-1} = P(xQ)^*\) then it is obvious that \(\mathcal{A} \subset A(\langle x \rangle)\). Next, it is easy to check that \(\mathcal{A}\) contains \(A(\langle x \rangle)\) and it is closed by \(+, \mbox{conc}\), for instance,

\[
(1-xQ_1)(1-xQ_2) = (1-x(Q_1+Q_2-xQ_1Q_2)).
\]

We also have to prove that \(\mathcal{A}\) is closed for \(*\). For this to be applied to \(P(1-xQ)^{-1}\), we must suppose that \(P(0) = 0\) (as, indeed, \(\langle P(1-xQ)^{-1} \rangle = \langle P(0) \rangle\) and, in this case, \(P = xP_1\). Now

\[
\left( \frac{P}{1-xQ} \right)^* = \left( \frac{1-xQ}{1-xQ} \right)^{-1} = \frac{1-xQ}{1-x(Q+P_1)} \in \mathcal{A}.
\]

(iii) Let \(S = (\sum_{x \in \mathcal{A}} \alpha_x x)^*\) and since \(S = 1 + (\sum_{x \in \mathcal{A}} \alpha_x x)S\) then \(\langle S \mid 1, \mathcal{A}^* \rangle = 1_A\). If \(w = xu\) then we get \(\langle S \mid xu \rangle = \alpha_x \langle S \mid u \rangle\). Thus, by recurrence on the length, \(\langle S \mid x_1 \ldots x_k \rangle = \prod_{i=1}^k \alpha_{x_i}\) showing that \(S\) is a conc-character. Conversely, by Schützenberger’s reconstruction lemma saying that, for any series \(S\)
\[ S = \langle S \mid 1_{\mathcal{X}^*} \rangle \cdot 1_A + \sum_{x \in \mathcal{X}} x \cdot x^{-1}S \]

but, if \( S \) is a conc-character, \( \langle S \mid 1_{\mathcal{X}^*} \rangle = 1 \) and \( x^{-1}S = \langle S \mid x \rangle S \), then the previous expression reads \( S = 1_A + (\sum_{x \in \mathcal{X}} \langle S \mid x \rangle x)S \). The last equality is equivalent to \( S = (\sum_{x \in \mathcal{X}} \langle S \mid x \rangle x)^* \), proving the claim.

(iv) As \((A\langle \mathcal{X} \rangle, \shuffle, 1_{\mathcal{X}^*})\) and \((A\langle Y \rangle, \shuffle, 1_{Y^*})\) are enveloping algebras, this property is an application of the fact that, on an enveloping \( \mathcal{U} \), the characters are linearly independent w.r.t. to the convolution algebra \( \mathcal{U}^* \) (see the general construction and proof in [24] or [26]). Here, this convolution algebra \( (\mathcal{U}^*)^* \) contains the polynomials (is equal in case of finite \( \mathcal{X} \)). Now, consider a monomial

\[ (\varphi_{i_1})^{\shuffle} \cdots (\varphi_{i_n})^{\shuffle} = \left( \sum_{k=1}^{n} \alpha_{i_k} \varphi_{i_k} \right)^* \]

The \( \mathbb{Z} \)-linear independence of the monomials in \((\varphi_i)_{i \in I}\) implies that all these monomials are linearly independent over \( A \langle \mathcal{X} \rangle \) which proves algebraic independence of the family \((\varphi_i)_{i \in I}\).

To end with, the fact that \( \mathcal{L}yn(\mathcal{X}) \cup \{\varphi_i\}_{i \in I} \) is algebraically free comes from Radford theorem \((A\langle \mathcal{X} \rangle, \shuffle, 1_{\mathcal{X}^*}) \simeq A[\mathcal{L}yn(\mathcal{X})]\) and the transitivity of polynomial algebras (see [3] ch III.2 Proposition 8).

(v) Comes directly as an application of the preceding point.

\[ \square \]

**Remark 3.3** (Point (ii) of Theorem 3.2 above) Kronecker’s theorem which can be rephrased in terms of stars as \( A^{\text{rat}}(x) = \{P(xQ)^*\}_{P,Q \in A[x]} \) holds for every ring and is therefore characteristic free, unlike the shuffle version requiring algebraic closure and denominators.

Now, we are in situation to characterize conc-characters and infinitesimal conc-characters (see Definition 2.3) of \((A\langle \mathcal{X} \rangle, \text{conc}, 1_{\mathcal{X}^*})\).

**Corollary 3.4 (Kleene stars of the plane)** Let \( R,L \in A^{\text{rat}}(\langle \mathcal{X} \rangle), \langle R \mid 1_{\mathcal{X}^*} \rangle = 1_A, \langle L \mid 1_{\mathcal{X}^*} \rangle = 0 \), such that \( L^* = R \). The following assertions are equivalent

(i) \( R \) is a conc-character of \((A\langle \mathcal{X} \rangle, \text{conc}, 1_{\mathcal{X}^*})\).

(ii) There is a family of coefficients \((c_x)_{x \in \mathcal{X}}\) such that \( R = (\sum_{x \in \mathcal{X}} c_x x)^* \).

(iii) The series \( R \) admits a linear representation of dimension one\(^{25}\).

(iv) \( L \) belongs to the plane \( A.\mathcal{X} \).

(v) \( L \) is an infinitesimal conc-character of \((A\langle \mathcal{X} \rangle, \text{conc}, 1_{\mathcal{X}^*})\).

\(^{25}\) The dimension is here (as in [1]) the size of the matrices.
Moreover, \((\alpha_0x_0 + \alpha_1x_1)^* \sqcup (\beta_0x_0 + \beta_1x_1)^* = ((\alpha_0 + \beta_0)x_0 + (\alpha_1 + \beta_1)x_1)^*\) and\(^{26}\)

\[
\left( \sum_{s \geq 1} a_s y_s \right)^* \sqcup \left( \sum_{s \geq 1} b_s y_s \right)^* = \left( \sum_{s \geq 1} (a_s + b_s)y_s + \sum_{r,s \geq 1} a_s b_r y_{s+r} \right)^*,
\]

where, for any \(i = 0, 1 \) and \(s \geq 1\), \(\alpha_i, \beta_i, a_s, b_s \in \mathbb{C}\).

**Proof.** (i) \(\iff\) (ii): This corresponds to the point (iii) of Theorem 3.2 above.

(ii) \(\iff\) (iii) This is a direct consequence of Theorem 2.5.

(iii) \(\iff\) (iv): This is obvious, by construction (in which \(L\) is viewed as the \(\omega\)-logarithm of \(\mathcal{R}\), see Note 24). Indeed,

\[
R = \left( \sum_{x \in \mathcal{X}} c_x x \right)^* = \biguplus_{x \in \mathcal{X}} (c_x x)^* = \biguplus_{x \in \mathcal{X}} \exp_{\biguplus}(c_x x) = \exp_{\biguplus} \left( \sum_{x \in \mathcal{X}} c_x x \right).
\]

(iv) \(\iff\) (v): If \(L\) is an infinitesimal character then, by Definition 2.3, \(\langle L \mid uv \rangle = \langle L \mid u \rangle \langle v \mid 1_{\mathcal{Z}^\ast} \rangle + \langle u \mid 1_{\mathcal{Z}^\ast} \rangle \langle L \mid v \rangle\), for \(u, v \in \mathcal{Z}^\ast\). Hence, for \(w = uv \in \mathcal{Z}^2\) with \(u, v \in \mathcal{Z}^\ast\), one gets \(\langle L \mid w \rangle = \langle L \mid uv \rangle = 0\). In addition, for \(u = v = 1_{\mathcal{Z}^\ast}\), one also gets \(\langle L \mid 1_{\mathcal{Z}^\ast} \rangle = 0\) and it follows that \(L = \sum_{x \in \mathcal{X}} \langle L \mid x \rangle x\). Conversely, since for any \(u, v \in \mathcal{Z}^\ast\) and \(x \in \mathcal{X}^\ast\), one has \(\langle uv \mid x \rangle = \langle u \mid x \rangle \langle v \mid 1_{\mathcal{Z}^\ast} \rangle + \langle u \mid 1_{\mathcal{Z}^\ast} \rangle \langle v \mid x \rangle = 0\) then, by the pairing in (17), one deduces that

\[
\langle L \mid uv \rangle = \sum_{x \in \mathcal{X}} \langle L \mid x \rangle \langle uv \mid x \rangle = 0
\]

meaning that \(L\) is an infinitesimal conc-character.

To end, letting \(x_i \in X\) and using (15), one has

\[
\langle (\alpha_0x_0 + \alpha_1x_1)^* \sqcup (\beta_0x_0 + \beta_1x_1)^* \mid x_i \rangle = \langle (\alpha_0x_0 + \alpha_1x_1)^* \otimes (\beta_0x_0 + \beta_1x_1)^* \mid \Delta_{\omega} x_i \rangle = \langle (\alpha_0x_0 + \alpha_1x_1)^* \otimes (\beta_0x_0 + \beta_1x_1)^* \mid x_i \otimes 1_{\mathcal{X}^\ast} + 1_{\mathcal{X}^\ast} \otimes x_i \rangle = \alpha_i + \beta_i = \langle (\alpha_0 + \beta_0)x_0 + (\alpha_1 + \beta_1)x_1)^* \mid x_i \rangle.
\]

Similarly, letting \(y_i \in Y\) and using (22), one also has

\[
\langle \left( \sum_{s \geq 1} a_s y_s \right)^* \sqcup \left( \sum_{s \geq 1} b_s y_s \right)^* \mid y_i \rangle = \langle \left( \sum_{s \geq 1} a_s y_s \right)^* \otimes \left( \sum_{s \geq 1} b_s y_s \right)^* \mid \Delta_{\omega} y_i \rangle = \langle \left( \sum_{s \geq 1} a_s y_s \right)^* \otimes \left( \sum_{s \geq 1} b_s y_s \right)^* \mid y_i \otimes 1_{\mathcal{Y}^\ast} + 1_{\mathcal{Y}^\ast} \otimes y_i + \sum_{r,s \geq 1, r+s=t} y_s \otimes y_r \rangle = a_t + b_t + \sum_{r,s \geq 1, r+s=t} a_s b_r
\]

\(^{26}\)In particular, \(a_s y_s)^* \sqcup (a_r y_r)^* = (a_s y_s + a_r y_r + a_s a_r y_{s+r})^*\) and \((a_s y_s)^* \sqcup (-a_s y_s)^* = (-a_s^2 y_{2s})^*\).
\[ = \left( \sum_{s \geq 1} (a_s + b_s)y_s + \sum_{r,s \geq 1} a_s b_r y_{s+r} \right)^{*} | y_i \].

\[ \Box \]

**Remark 3.5** In Corollary 3.4, if \( A = K \) being a field, the points (i) and (v) can be rephrased as, respectively, “\( R \) is a group like element” and “\( L \) is a primitive element” of \( K^\text{rat}\langle \langle \mathcal{A} \rangle \rangle \), for \( \Delta_{\text{conc}} \). Indeed, in (29)–(31), if \( S \in K\langle \langle Y \rangle \rangle \) (resp. \( K\langle \langle \mathcal{A} \rangle \rangle \)) is a \( \varpi \) (resp. \( \varpi, \text{conc} \))-character of \( \langle K\langle X \rangle, \text{conc}, 1_X \rangle \) then

(i) Using the fact that \( S \otimes S = \sum_{u,v \in \mathcal{A}^*} \langle S | u \rangle \langle S | v \rangle u \otimes v \), one has \( \Delta_{\varpi} S = S \otimes S \) (resp. \( \Delta_{\varpi} S = S \otimes S, \Delta_{\text{conc}} S = S \otimes S \)) meaning that \( S \) is group like, for \( \Delta_{\varpi} \) (resp.\( \Delta_{\varpi}, \Delta_{\text{conc}} \)).

(ii) Using the fact that \( \Delta_{\varpi} \) (resp. \( \Delta_{\varpi}, \Delta_{\text{conc}} \)) and the maps \( T \mapsto T \otimes 1_{X^*}, T \mapsto 1_{Y^*} \otimes T \) (resp. \( T \mapsto T \otimes 1_{Y^*}, T \mapsto 1_{Y^*} \otimes T \)) are continuous homomorphisms, one has \( 27 \Delta_{\varpi} \log S = \log S \otimes 1_{Y^*} + 1_{Y^*} \otimes \log S \) (resp. \( \Delta_{\varpi} \log S = \log S \otimes 1_{Y^*} + 1_{Y^*} \otimes \log S, \Delta_{\text{conc}} \log S = \log S \otimes 1_{Y^*} + 1_{Y^*} \otimes \log S \)) meaning that \( \log S \) is primitive, for \( \Delta_{\varpi} \) (resp.\( \Delta_{\varpi}, \Delta_{\text{conc}} \)).

Hence, for \( \Delta_{\varpi}, \Delta_{\varpi}, \) and \( \Delta_{\text{conc}} \), \( S \) is group like iff \( \log S \) is primitive meaning that the equivalence, between (i) and (v), is an extension of the Ree’s theorem [43].

**Example 3.6** [Identity \((-t^2y_2)^* \varpi (t^2y_2)^* = (-4t^4y_4)^*, [33,34]\)]

\[ (-t^2y_2)^* \leftrightarrow (v_2, \mu_2(y_2), \eta_2) \quad (t^2y_2)^* \leftrightarrow (v_1, \mu_1(y_2), \eta_1) \quad (-t^4y_4)^* \leftrightarrow (v, \mu(y_4), \eta) \]

\[ = (1, -t^2, 1), \quad = (1, t^2, 1), \quad = (1, -t^4, 1). \]

### 3.2 Exchangeable rational series and their linear representations

As examples, one can consider the following forms \((F_0), (F_1)\) and \((F_2)\) of rational series in \( A^\text{rat}\langle \langle X \rangle \rangle \) [32,38]:

\[ (F_0) \ E_1 x_1 \ldots E_j x_j E_{j+1}, \text{ where } x_i, \ldots, x_j \in X, E_1, \ldots, E_j \in A^\text{rat}\langle \langle X_0 \rangle \rangle, \]

\[ (F_1) \ E_1 x_1 \ldots E_j x_j E_{j+1}, \text{ where } x_i, \ldots, x_j \in X, E_1, \ldots, E_j \in A^\text{rat}\langle \langle X \rangle \rangle, \]

\[ (F_2) \ E_1 x_1 \ldots E_j x_j E_{j+1}, \text{ where } x_i, \ldots, x_j \in X, E_1, \ldots, E_j \in A^\text{rat}_{\text{exc}}\langle \langle X \rangle \rangle. \]

\[ ^{27}\text{Here, } \log S \otimes 1_{Y^*} \text{ and } 1_{Y^*} \otimes \log S \text{ (resp. } \log S \otimes 1_{Y^*} \text{ and } 1_{Y^*} \otimes \log S \text{) commute.} \]
Using linear representations, we also have

**Theorem 3.7 (Triangular sub bialgebras of \( A^\text{rat}(\langle \mathcal{X} \rangle), \sqcup \sqcup, 1_{\mathcal{X}^*}, \Delta_{\text{conc}}, e \), [38])**

Let \( \rho = (\nu, \mu, \eta) \) a representation of \( R \in A^\text{rat}(\langle \mathcal{X} \rangle) \). Then

(i) If the matrices \( \{\mu(x)\}_{x \in \mathcal{X}} \) commute between themselves and if the alphabet is finite, every rational exchangeable series decomposes as

\[
R = \sum_{i=1}^{n} \sqcup \sqcup R_x^{(i)} \text{ with } R_x^{(i)} \in A^\text{rat}(\langle x \rangle).
\]

(ii) If \( \mathcal{L} \) consists of upper-triangular matrices then \( R \in A^\text{exc}(\langle \mathcal{X} \rangle) \sqcup A(\mathcal{X}) \).

(iii) For any \( x \in \mathcal{X} \), letting \( M(x) := \mu(x)x \) and then extending, in the obvious way, this representation to \( A(\langle \mathcal{X} \rangle) \) by \( M(S) = \sum w \in \mathcal{X}^* \langle S \mid w \rangle \mu(w)w \), we have

\[
R = \nu M(\mathcal{X}^* \eta).
\]

Moreover, we have

(a) If \( \{\mu(x)\}_{x \in \mathcal{X}} \) are upper-triangular then \( M(\mathcal{X}) = D(\mathcal{X}) + N(\mathcal{X}) \), where \( D(\mathcal{X}) \) and \( N(\mathcal{X}) \) are diagonal and strictly upper-triangular letter matrices, respectively, such that \(^{28}\)

\[
M(\mathcal{X}^*) = ((D(\mathcal{X}^*)N(\mathcal{X}^*))^*D(\mathcal{X}^*)).
\]

(b) We get \(^{29}\) (for \( \mathcal{X} = X \))

\[
M((x_0 + x_1)^*) = (M(x_1^*)M(x_0))^*M(x_1^*) = (M(x_0^*)M(x_1))^*M(x_0^*)
\]

and the modules generated by the families \( (F_0), (F_1) \) and \( (F_2) \) are closed by \( \text{conc}, \sqcup \sqcup \) (and coproducts if \( A = K \) is a field). From this, it follows that \( R \) is a linear combination of expressions in the form \( (F_0) \) (resp. \( (F_1) \)) if \( M(x_1^*)M(x_0) \) (resp. \( M(x_0^*)M(x_1) \)) is strictly upper-triangular.

(c) If \( A \) is a \( \mathbb{Q} \)-algebra then

\[
M(\mathcal{X}^*) = \prod_{l \in \text{Dyn} \mathcal{X}} e^{S_l \mu(P_l)}.
\]

**Remark 3.8** (i) The point (i) of Theorem 3.7 is no longer true for an infinite alphabet as shows the example of the series \( S = \sum_{k \geq 1} y_k \in A^\text{rat}(\langle Y \rangle) \).

(ii) On a general ring it can happen that \( R \) is exchangeable, \( \rho \) minimal and nevertheless \( \mathcal{L} \) is noncommutative, as shows the case of \( A = \mathbb{Q}[x,t]/t^3 \mathbb{Q}[x,t] \) and

\[
X = \{a,b\}, \mu(a) = t \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \mu(b) = t \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \nu = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

\(^{28}\) by Lazard factorization \([42,49]\).

\(^{29}\) *idem.*
With these data, \( R = 2 + (xt + 2t)(a + b) + (x^2t^2 + 2xt^2 + 2t^2)(ab + ba) \) which is an exchangeable polynomial but

\[
\mu(a)\mu(b) = \begin{pmatrix} t^2 & xt^2 \\ xt^2 & x^2t^2 + t^2 \end{pmatrix}, \quad \mu(b)\mu(a) = \begin{pmatrix} x^2t^2 + t^2 & xt^2 \\ xt^2 & t^2 \end{pmatrix}
\]

Now the representation is minimal because if it were of dimension 1, \( \frac{1}{2}R \) would be a conc-character, which is not the case. Otherwise, if it were of dimension 0, \( R \) would be zero.

In order to establish Theorem 3.10 below, we will use the following

**Lemma 3.9** Let \( (v, \tau, \eta) \) a representation of \( S \) of dimension \( r \) such that, for all \( x \in \mathcal{X} \), \( (\tau(x) - c(x)I_r) \) is strictly upper triangular, then \( S \in K_{\text{exc}}^{\text{rat}}(\langle \mathcal{X} \rangle) \cup K(\mathcal{X}) \).

**Proof.** Let \( (e_i)_{1 \leq i \leq r} \) be the canonical basis of \( K^{1 \times r} \). We construct the representations \( \rho_1 = (v, (x \mapsto \tau(x) - c(x)I_r), \eta), \rho_2 = (e_1, (x \mapsto c(x)I_r), e_1^\ast) \) of \( S_1 \) and \( S_2 \) and remark that \( S_1 \sqcup S_2 \) admits the representation

\[
\rho_3 = (v \otimes e_1, ((\tau(x) - c(x)I_r) \otimes I_r + I_r \otimes c(x)I_r)_{x \in \mathcal{X}}, \eta \otimes e_1^\ast)
\]

as \( I_r \otimes c(x)I_r = c(x)I_r \otimes I_r \), \( \rho_3 \) is, in fact, \( (v \otimes e_1, (\tau(x) \otimes I_r)_{x \in \mathcal{X}}, \eta \otimes e_1^\ast) \) which represents \( S \), the result now comes from the fact that \( S_1 \in K(\mathcal{X}) \) and \( S_2 = (\sum_{x \in \mathcal{X}} c(x)x)^\ast \in K_{\text{exc}}^{\text{rat}}(\langle \mathcal{X} \rangle) \).

We first begin by properties essentially true over algebraically closed fields.

**Theorem 3.10 (Triangular sub bialgebras of \( K^{\text{rat}}(\langle \mathcal{X} \rangle) \))** We suppose that \( K \) is an algebraically closed field and that \( \rho = (v, \mu, \eta) \) is a linear representation of \( R \in K_{\text{exc}}^{\text{rat}}(\langle \mathcal{X} \rangle) \) of minimal dimension \( n \), we note \( \mathcal{L} = \mathcal{L}(\mu) \subset K_{\text{exc}}^{n \times n} \) the Lie algebra generated by the matrices \( (\mu(x))_{x \in \mathcal{X}} \). Then

(i) \( \mathcal{L} \) is commutative iff \( R \in K_{\text{exc}}^{\text{rat}}(\langle \mathcal{X} \rangle) \),
(ii) \( \mathcal{L} \) is nilpotent iff \( R \in K_{\text{exc}}^{\text{rat}}(\langle \mathcal{X} \rangle) \cup K(\mathcal{X}) \),
(iii) \( \mathcal{L} \) is solvable iff \( R \) is a linear combination of expressions in the form \( (F_2) \).

Moreover, denoting \( K_{\text{nil}}^{\text{rat}}(\langle \mathcal{X} \rangle) \) (resp. \( K_{\text{sol}}^{\text{rat}}(\langle \mathcal{X} \rangle) \)), the set of rational series such that \( \mathcal{L}(\mu) \) is nilpotent (resp. solvable), we get a tower of sub Hopf algebras of the Sweedler’s dual, \( K_{\text{nil}}^{\text{rat}}(\langle \mathcal{X} \rangle) \subset K_{\text{sol}}^{\text{rat}}(\langle \mathcal{X} \rangle) \subset \mathcal{H}_{\text{exc}}^{\otimes}(\mathcal{X}) \).

**Proof.**

(i) Let us remark that, for \( x, y \in \mathcal{X}, p, s \in \mathcal{X}^\ast \), we have \( \langle R \mid pxy{s} \rangle = \langle R \mid py{x}s \rangle \) which is due to the commutation of matrices. Conversely, since \( \rho \) is minimal then there is \( P_i, Q_i \in K(\mathcal{X}), i = 1 \ldots n \) such that (see \([1,19,47]\)
\(\forall u \in \mathcal{D}^*, \mu(u) = (\langle P_i R Q_j | u \rangle)_{1 \leq i, j \leq n} = (\langle R | Q_j u P_i \rangle)_{1 \leq i, j \leq n}.\)

Now, for \(x, y \in \mathcal{D}^*\), we have

\[\mu(xy) = ((\langle R | Q_j x y P_i \rangle)_{1 \leq i, j \leq n})^* = (\langle R | Q_j y x P_i \rangle)_{1 \leq i, j \leq n} = \mu(yx)\]

equality \(\ast\) being due to exchangeability.

(ii) Let us consider \(K^n\) as the space of the representation of \(\mathcal{L}\) given by \(\mu\). Let \(K^n = \bigoplus_{j=1}^{m} V_j\) be a decomposition of \(K^n\) into indecomposable \(\mathcal{L}\)-modules (see [16], Theorem 1.3.19 where it is done for \(ch(K) = 0\), or [5] Chapter VII §1 Proposotion 9 for arbitrary characteristic), we know that each \(V_j\) is a \(\mathcal{L}\)-module and that the action of \(\mathcal{L}\) is triangularizable with constant diagonals inside each sector \(V_j\). Thus, it is an invertible matrix \(P \in \text{GL}(n, K)\) such that

\[\forall x \in \mathcal{D}^*, P \mu(x) P^{-1} = \text{blockdiag}(T_1, T_2, \ldots, T_k) = \begin{pmatrix}
T_1 & 0 & 0 & \cdots & 0 \\
0 & T_2 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}
\]

where the \(T_j\) are upper triangular matrices with scalar diagonal \(i.e.\) is of the form \(T_j(x) = \lambda_j(x)I + N(x)\) where \(N(x)\) is strictly upper-triangular\(^30\). Set \(d_j\) to be the dimension of \(T_j\) (so that \(n = \sum_{j=1}^{m} d_j\)), partitioning \(VP^{-1} = \psi'\) (resp. \(P \eta = \eta'\)) with these dimensions we get blocks so that each \((\psi'_j, T_j, \eta'_j)\) is the representation of a series \(R_j\) and \(R = \sum_{j=1}^{m} R_j\). It suffices then to prove that, for all \(j, R_j \in K_{\text{exc}}^{\text{rat}}(\langle \mathcal{D}^* \rangle) \equiv K(\mathcal{D}^*).\) This is a consequence of Lemma 3.9.

Conversely, if \(\rho_j = (v_i, \tau_i, \eta_i), i = 1, 2\), are two representations then \(\{\tau_1(x) \otimes I_r + I_r \otimes \tau_2(x), \tau_1(y) \otimes I_r + I_r \otimes \tau_2(y)\} = [\tau_1(x), \tau_1(y)] \otimes I_r + I_r \otimes [\tau_2(x), \tau_2(y)]\) and a similar formula holds for \(m\)-fold brackets (Dynkin combs), so that if \(\mathcal{L}(\tau_i)'s\) are nilpotent, the Lie algebra \(\mathcal{L}(\tau_1 \otimes I_r + I_r \otimes \tau_2)\) is also nilpotent. The point here comes from the fact that series in \(K_{\text{exc}}^{\text{rat}}(\langle \mathcal{D}^* \rangle)\) as well as in \(K(\mathcal{D}^*)\) admit nilpotent representations, so, let \((\alpha, \tau, \beta)\) such a representation and \((\alpha', \tau', \beta')\) its minimal quotient (obtained by minimization, see [1]), then \(\mathcal{L}(\tau')\) is nilpotent as a quotient of \(\mathcal{L}(\tau)\). Now two minimal representations being isomorphic, \(\mathcal{L}(\mu)\) is isomorphic to \(\mathcal{L}(\tau)\) and then it is nilpotent.

(iii) As \(\mathcal{L}\) is solvable and \(K\) algebraically closed, using Lie’s theorem, we can find a conjugate form of \(\rho = (\nu, \mu, \eta)\) such that the matrices \(\mu(x)\) are upper-triangular. Since this form also represents \(R\), letting \(D(\mathcal{D}^*)\) (resp. \(N(\mathcal{D}^*)\)) be

\(^{30}\) Even, as \(K\) is infinite, there is a global linear form on \(\mathcal{L}, \lambda_{\text{lin}}\) such that, for all \(g \in \mathcal{L}, PgP^{-1} - \lambda_{\text{lin}}(g)I\) is strictly upper-triangular.
the diagonal (rep. strictly upper-triangular) letter matrix such that \( M(\mathcal{X}) = D(\mathcal{X}) + N(\mathcal{X}) \) then
\[
R = vM(\mathcal{X}^*)\eta = v(D(\mathcal{X}^*)N(\mathcal{X}))^*D(\mathcal{X}^*)\eta.
\]
Since \( D(\mathcal{X}^*)N(\mathcal{X}) \) being nilpotent of order \( n \) then \( (D(\mathcal{X}^*)N(\mathcal{X}))^* = \sum_{i=0}^n (D(\mathcal{X}^*)N(\mathcal{X}))^i \). Hence, letting \( \mathcal{I} \) be the vector space generated by forms of type \( (F_2) \) which is closed by concatenation, we have \( D(\mathcal{X}^*)N(\mathcal{X}) \in \mathcal{I}^{n\times n} \) and then \( (D(\mathcal{X}^*)N(\mathcal{X}))^* \in \mathcal{I}^{n\times n} \). Finally, \( R = vM(\mathcal{X}^*)\eta \in \mathcal{I} \) which is the claim.

Conversely, as sums and quotients of solvable representations are solvable is suffices to show that a single form of type \( F_2 \) admits a solvable representation and end by quotient and isomorphism as in (ii). From Proposition (2.6), we get the fact that, if \( R_i \) admit solvable representations so does \( R_1R_2 \), then the claim follows from the fact that, firstly, single letters admit solvable (even nilpotent) representations and secondly series of \( \bigcup \{ K_{\text{rat}}(\langle x \rangle) \}_{x \in \mathcal{X}} \) admit solvable representations. Finally, we choose (or construct) a solvable representation of \( R \), call it \( (\alpha, \tau, \beta) \) and \( (\alpha', \tau', \beta') \) its minimal quotient, then \( \mathcal{L}(\tau') \) is solvable as a quotient of \( \mathcal{L}(\tau) \). Now two minimal representations being isomorphic, \( \mathcal{L}(\mu) \) is isomorphic to \( \mathcal{L}(\tau) \), hence solvable.

Moreover and ff.] Comes from the computation of the coproduct by insertion of identity \( \sum_{i=1}^n e_i^* e_i \).

\[ \square \]

**Remark 3.11** For an example of series \( S \) with solvable representation but such that \( S \not\in K_{\text{rat}}(\langle \mathcal{X} \rangle) \cup K(\langle \mathcal{X} \rangle) \). One can take \( \mathcal{X} = \{a, b\} \) and \( S = a^* b (-a)^* \).

To end this section (of combinatorial framework), for a need of the proof of Theorem 4.8 below, let us extend the pairing (17) as a partially defined map
\[
\text{Dom}(\langle \bullet \mid \bullet \rangle) \rightarrow A, \quad T \otimes S \rightarrow \langle T \mid S \rangle := \sum_{w \in \mathcal{X}^*} \langle T \mid w \rangle \langle S \mid w \rangle.
\]
\[
(34) \quad (35)
\]
where \( \text{Dom}(\langle \bullet \mid \bullet \rangle) \subset A(\langle \mathcal{X} \rangle) \otimes A(\langle \mathcal{X} \rangle) \).

Here, the family \( \sum_{w \in \mathcal{X}^*} \langle T \mid w \rangle \langle S \mid w \rangle \) is summable, for some topology on \( A \). Its sum is denoted by \( \langle T \mid S \rangle \) and the set of these series \( S \) is denoted by \( \text{Dom}_{\text{word}}(T) \).

This proof will also use the following lemma as a consequence of Theorem 2.5

**Lemma 3.12** For any ring \( A \) without zero divisors, let \( R \in A_{\text{rat}}(\langle \mathcal{X} \rangle) \) of linear representation \( (v, \mu, \eta) \) of dimension \( n \). Then any family \( \{R \preceq P_i | P_i \in A(\langle \mathcal{X} \rangle) \}_{i=1 \ldots m > n} \) is linearly dependent, i.e. there are \( \{\alpha_i\}_{i=1 \ldots m} \) in \( A \), not all zero, such that \( \sum_{i=1}^m \alpha_i (R \preceq P_i) = 0 \).
Towards a noncommutative Picard-Vessiot theory

Let \((\mathcal{A}, d)\) be a commutative associative differential ring (\(\ker(d) = k\) being a field), \(\mathcal{C}_0\) be a differential subring of \(\mathcal{A}\) (\(d(\mathcal{C}_0) \subset \mathcal{C}_0\)) which is an integral domain containing the field of constants and \(\mathbb{C}\{(g_i)_{i \in I}\}\) be the differential subalgebra of \(\mathcal{A}\) generated by \((g_i)_{i \in I}\), i.e. the \(k\)-algebra generated by \(g_i\)'s and their derivatives [48].

4.1 Noncommutative differential equations

Let us consider the following differential equation, with homogeneous series of degree 1 as multiplier (a polynomial in the case of finite alphabet).

\[
dS = MS; \langle S | 1 \rangle = 1, \text{where } M = \sum_{x \in \mathcal{X}} u_x x \in \mathcal{C}_0(\langle \mathcal{X} \rangle)
\]

**Example 4.1** [Drinfel’d equation] \(X = \{x_0, x_1\}\) and \(\Omega = \mathbb{C}\{(-∞, 0] \cup [1, +∞]\) \(\mathcal{W}_0 = \cdots \cup \mathcal{W}_m \in \mathcal{X}^r\).

\((KZ_2)\) \(dS = (x_0u_{x_0} + x_1u_{x_1})S\) with \(u_{x_0}(z) = z^{-1}, u_{x_1}(z) = (1 - z)^{-1}\).

This equation was introduced in [17,18] and a complete study was presented in [38] (solutions via polylogarithms and their special values, polyzetas).

**Example 4.2** \(Y = \{y_t\}_{t \geq 1}\) and \(\Omega = \{z \in \mathbb{C} | |z| < 1\}\).

\[
dS = (\sum_{i \geq 1} y_iu_{y_i})S \text{ with } u_{y_i}(z) = \partial\ell_i(z).
\]

where, denoting \(\gamma\) the Euler’s constant and \(\zeta\) the Riemann zeta function,

\[
\ell_1(z) := \gamma z - \sum_{k \geq 2} \zeta(k) \frac{(-z)^k}{k} \text{ and for } r \geq 2, \ell_r(z) := -\sum_{k \geq 1} \zeta(kr) \frac{(-z)^k}{k}.
\]

This equation was introduced in [9] to study the independence of a family of eulerian functions.

Let us also recall the following useful result for proving Theorem 4.8 bellow.

**Proposition 4.3** ([33,34,36]) Let \(S \in \mathcal{A}(\langle \mathcal{X} \rangle)\) be solution of (36). Then \(S\) satisfies the differential equations \(dS = Q_lS\), for \(l \geq 0\), where \(Q_l = \mathbb{C}\{(u_i)_{i \geq 0}\}\) satisfying the recursion \(Q_0 = 1\) and \(Q_l = Q_{l-1}M + dQ_{l-1}\).

More explicitly, \(Q_l\) can be computed as follows (summing over words \(w = x_{i_1} \cdots x_{i_l}\) and derivation multi-indices \(r = (r_1, \ldots, r_l)\) of degree \(\deg r = |w| = l\) and of weight \(\text{wt } r = l + r_1 + \ldots + r_l\))

\[
Q_l = \sum_{\text{wt } r = l} \prod_{i=1}^{\deg r} \left(\sum_{j=1}^{\deg i} \frac{r_j + j - 1}{r_j}\right) \tau_r(w) \text{ and } \tau_r(w) = \tau_{r_1}(x_{i_1}) \cdots \tau_{r_l}(x_{i_l}) = (\partial^r_{x_{i_1}} u_{x_{i_1}}) \cdot \cdots \cdot (\partial^r_{x_{i_l}} u_{x_{i_l}}) x_{i_l}.
\]
Theorem 4.4 Suppose that the \(\mathbb{C}\)-commutative ring \(\mathcal{A}\) is without zero divisors and equipped with a differential operator \(\partial\) such that \(\mathbb{C} = \ker \partial\).

Let \(S \in \mathcal{A}\langle\langle x\rangle\rangle\) be a grouplike solution of (36), in the following form

\[
S = 1 + \sum_{w \in x^*} \langle S \mid w \rangle w = 1 + \sum_{w \in x^*} \langle S \mid S_w \rangle P_w = \prod_{l \in \mathcal{L}} e^{[S(l)]P_l}.
\]

Then

(i) If \(H \in \mathcal{A}\langle\langle x\rangle\rangle\) is another grouplike solution of (36) then there exists \(C \in \mathcal{L}\)\(ie\)\(\mathcal{A}\langle\langle x\rangle\rangle\) such that \(S = He^C\) (and conversely).

(ii) The following assertions are equivalent

(a) \(\{\langle S \mid w \rangle\}_{w \in x^*}\) is \(\mathbb{C}_0\)-linearly independent,
(b) \(\{\langle S \mid l \rangle\}_{l \in \mathcal{L}' \setminus x^*}\) is \(\mathbb{C}_0\)-algebraically independent,
(c) \(\{\langle S \mid x \rangle\}_{x \in x}\) is \(\mathbb{C}_0\)-algebraically independent,
(d) \(\{\langle S \mid x \rangle\}_{x \in x^* \cup \{1 \}}\) is \(\mathbb{C}_0\)-linearly independent,
(e) The family \(\{u_x\}_{x \in x}\) is such that, for \(f \in \text{Frac}(\mathbb{C}_0)\) and \((c_x)_{x \in x} \in \mathbb{C}(x),\)

\[
\sum_{x \in x} c_x u_x = \partial f \implies (\forall x \in x)(c_x = 0).
\]

(f) The family \((u_x)_{x \in x}\) is free over \(\mathbb{C}\) and \(\partial\text{Frac}(\mathbb{C}_0) \cap \text{span}_\mathbb{C}\{u_x\}_{x \in x} = \{0\}.
\]

Proof. [Sketch] The first item has been treated in [34]. The second is a grouplike version of the abstract form of Theorem 1 of [15]. It goes as follows

- due to the fact that \(\mathcal{A}\) is without zero divisors, we have the following embeddings \(\mathbb{C}_0 \subset \text{Frac}(\mathbb{C}_0) \subset \text{Frac}(\mathbb{A}),\) \(\text{Frac}(\mathbb{A})\) is a differential field, and its derivation can still be denoted by \(\partial\) as it induces the previous one on \(\mathcal{A},\)
- the same holds for \(\mathcal{A}\langle\langle x\rangle\rangle \subset \text{Frac}(\mathbb{A})\langle\langle x\rangle\rangle\) and \(d\)
- therefore, equation (36) can be transported in \(\text{Frac}(\mathbb{A})\langle\langle x\rangle\rangle\) and \(M\) satisfies the same condition as previously.
- Equivalence between a-d comes from the fact that \(\mathbb{C}_0\) is without zero divisors and then, by denominator chasing, linear independences w.r.t \(\mathbb{C}_0\) and \(\text{Frac}(\mathbb{C}_0)\) are equivalent. In particular, supposing condition d, the family \(\{\langle S \mid x \rangle\}_{x \in x^* \cup \{1 \}}\) (basic triangle) is \(\text{Frac}(\mathbb{C}_0)\)-linearly independent which imply, by the Theorem 1 of [15], condition e,
- still by Theorem 1 of [15], e is equivalent to f, implying that \(\{\langle S \mid w \rangle\}_{w \in x^*}\) is \(\text{Frac}(\mathbb{C}_0)\)-linearly independent which induces \(\mathbb{C}_0\)-linear independence (i.e. a).

\[\square\]

Now, let us go back to notations of Section 1 and equip the differential rings of
(i) holomorphic functions over a simply connected domain \( \Omega \), \( (\mathcal{H}(\Omega), \partial) \), with the topology of compact convergence (CC),

(ii) formal series over \( \mathcal{X} \) and with coefficients in \( \mathcal{H}(\Omega), (\mathcal{H}(\Omega)/\langle \mathcal{X} \rangle, d) \), with the ultrametric distance defined by \( \delta(S, T) = 2^{-\sigma(S-T)} \).

Let us also consider again the Chen series of the differential forms \( (\omega_i)_{i\geq 1} \) defined by the inputs \( \omega_i = u_i dz \) along a path \( z_0 \rightsquigarrow z \) on \( \Omega \). By (18), it follows that

\[
C_{z_0 \rightsquigarrow z} = \sum_{w \in \mathcal{X}} \alpha_{z_0}^w (w) w = (\alpha_{z_0}^w \otimes \text{Id}) \mathcal{D}_{\mathcal{X}} = \prod_{l \in \mathcal{L}(\mathcal{X})} e^{\alpha_{z_0}^w (S_l) p_l}.
\]

This series satisfies (36) and is obtained as the limit, for the topology of (discrete) pointwise convergence over the words, of Picard iteration process initialized at \( (C_{z_0 \rightsquigarrow z} | \mathcal{X}^+) = 1_{\mathcal{H}(\Omega)} \).

Let us illustrate Theorem 4.4, with simple examples, for which \( \mathcal{C}_0 \) contains \( \mathbb{C}\{\{u_{+1}^\pm\}_{x \in \mathcal{X}}\} = \mathbb{C}[u_{x}^\pm, \partial^j u_x]_{j \geq 1, x \in \mathcal{X} \subset \mathcal{X} = (\mathcal{H}(\Omega), \partial) \}. \) In these examples, we use

**Proposition 4.5 ([32])** For \( \mathcal{X} = \{x\} \), since \( x^n = x^{1/n}/n! \) then

\[
\alpha_{z_0}^w (x^n) = \frac{(\alpha_{z_0}^w (x))^n}{n!}, C_{0 \rightsquigarrow z} = \sum_{n \geq 0} \alpha_{z_0}^w (x^n) x^n = e^{\alpha_{z_0}^w (x)^x}, \alpha_{0}^w (x^n) = e^{\alpha_{0}^w (x)}.
\]

**Example 4.6** Let us consider two positive cases over \( \mathcal{X} = \{x\} \).

(i) \( \Omega = \mathbb{C}, u_0(z) = 1_{\Omega}, \mathcal{C}_0 = \mathbb{C} \). Since \( \alpha_{0}^w (x^n) = z^n/n! \) then, by Proposition 4.5,

\[
C_{0 \rightsquigarrow z} = e^{x} \text{ and } dC_{0 \rightsquigarrow z} = zC_{0 \rightsquigarrow z}.
\]

Moreover, \( \alpha_{0}^w (x) = z \) which is transcendent over \( \mathcal{C}_0 \) and \( \{\alpha_{0}^w (x^n)\}_{n \geq 0} \in \mathcal{C}_0 \)-free. Now, let \( f \in \mathcal{C}_0 \) then \( \partial = 0 \). Hence, if \( \partial f = cu_x \) then \( c = 0 \).

(ii) \( \Omega = \mathbb{C}[z^{-1}] \to 0, u_0(z) = z^{-1}, \mathcal{C}_0 = \mathbb{C}[z^\pm 1] \subset \mathbb{C}(z) \). Since \( \alpha_{1}^w (x^n) = \log^n(z)/n! \) then, by Proposition 4.5,

\[
C_{1 \rightsquigarrow z} = z^z \text{ and } dC_{1 \rightsquigarrow z} = z^{-1} xC_{1 \rightsquigarrow z}.
\]

Moreover, \( \alpha_{1}^w (x) = \log(z) \) which is transcendent over \( \mathbb{C}(z) \) then over \( \mathcal{C}_0 \) and \( \{\alpha_{1}^w (x^n)\}_{n \geq 0} \in \mathcal{C}_0 \)-free. Now, let \( f \in \mathcal{C}_0 \) then \( \partial f \in \text{span}_\mathbb{C} \{z^{-n}\}_{n \in \mathbb{Z}, n \neq 1} \). Hence, if \( \partial f = cu_x \) then \( c = 0 \).

**Example 4.7** Let us consider two negative cases over \( \mathcal{X} = \{x\} \).

(i) \( \Omega = \mathbb{C}, u_0(z) = e^x, \mathcal{C}_0 = \mathbb{C}[e^\pm z] \). Since \( \alpha_{0}^w (x^n) = (e^z - 1)^n/n! \) then, by Proposition 4.5,
\[ C_{0 \rightarrow z} = e^{(e^z - 1)x} \text{ and } dC_{0 \rightarrow z} = e^z \cdot xC_{0 \rightarrow z}. \]

Moreover, \( a_0^z(x) = e^z - 1 \) which is not transcendent over \( \mathcal{C}_0 \) and \( \{ a_0^z(x^n) \}_{n \geq 0} \) is not \( \mathcal{C}_0 \)-free. If \( f(z) = ce^z \in \mathcal{C}_0 \) \((c \neq 0)\) then \( \partial f(z) = ce^z = cu_s(z). \)

(i) \( \Omega = \mathbb{C} \setminus ]-\infty, 0]\), \( u_s(z) = z^a, a \in \mathbb{C} \setminus \mathbb{Q}, \mathcal{C}_0 = \mathbb{C}\{z, z^\pm a\} = \text{span}_\mathbb{C}\{z^{ka+1}\}_{k, l \in \mathbb{Z}}. \)

Since \( a_0^z(x^n) = (a + 1)^{-n} z^{n(a+1)}/n! \) then, by Proposition 4.5,

\[ C_{0 \rightarrow z} = e^{(a+1)^{-1}z(a+1)x} \text{ and } dC_{0 \rightarrow z} = z^a \cdot xC_{0 \rightarrow z}. \]

Moreover, \( a_0^z(x) = z^{a+1}/(a+1) \) which is not transcendent over \( \mathcal{C}_0 \) and \( \{ a_0^z(x^n) \}_{n \geq 0} \) is not \( \mathcal{C}_0 \)-free. If \( f(z) = c z^{a+1}/(a + 1) \in \mathcal{C}_0 \) \((c \neq 0)\) then \( \partial f(z) = cz^a = cu_s(z). \)

4.2 First step of a noncommutative Picard-Vessiot theory

Let us recall that the vector space of solutions of (36) is a free \( (\mathbb{C}\langle\mathcal{A}\rangle)\)-right module of dimension one \(^{32}\) generated by \( C_{z_0 \rightarrow z} \) [34]. Hence, by Theorem 4.4, we have common traits with the ordinary case of first order differential equations,

(i) the differential Galois group of (36) + grouplike is the Hausdorff group

\[ \{ e^C \}_{C \in \mathcal{L} \text{ie} \mathcal{L}_1(\Omega) \langle \mathcal{A} \rangle} \] (group of characters of \( \mathcal{H} \langle \mathcal{L}_1(\Omega) \langle \mathcal{A} \rangle \)).

(ii) the PV extension related to (36) is \( \mathcal{C}\langle\mathcal{A}\rangle(C_{z_0 \rightarrow z}) \), where \( \mathcal{C} \subset \mathcal{A} = (\mathcal{H}(\Omega), \partial) \) such that \( \text{Const}(\mathcal{C}\langle\mathcal{A}\rangle) = \ker d = \mathbb{C}.1. \mathcal{H}(\Omega)\langle\mathcal{A}\rangle). \)

**Theorem 4.8 ([33,34,36])** Let \( R \in \mathbb{C}.1^{\text{fat}} \mathcal{H}(\Omega)\langle\mathcal{A}\rangle \). Then, for any path \( z_0 \leadsto z \) over \( \Omega \), we have \(^{33}\) \( R \in \text{Dom}_{\text{word}}(C_{z_0 \rightarrow z}) \) and the output of (1) can be computed by

\[ y(z_0, z) = \alpha^z_{z_0}(R) = \sum_{w \in \mathcal{A}^+} (v\mu(w)\eta) \alpha^z_{z_0}(w) = \langle C_{z_0 \rightarrow z} || R \rangle. \]

Now, let \( N \) be the least integer \( n \) such that \( y \) satisfies a (non-trivial) differential equation of order \( N \) (with coefficients in \( \mathcal{C} \)), the family \( \{ \partial y \}_{0 \leq k \leq N-1} \) is \( \mathcal{C} \)-linearly independent, i.e.

\[ (a_n \partial^N + \ldots + a_1 \partial + a_0)y = 0, \text{ with } a_N, \ldots, a_0 \in \mathcal{C}. \]

and, from what precedes, we have \( N \leq n = \text{rk}(R) \).

**Proof.** Due to this strong convergence condition, we have

(i) for any \( T \in \mathcal{H}(\Omega)\langle\mathcal{A}\rangle \) and \( P \in \mathcal{H}(\Omega)\langle\mathcal{A}\rangle, S \in \text{Dom}_{\text{word}}(T) \), we have

\[ S \in \text{Dom}_{\text{word}}(PT), S \cdot P \in \text{Dom}_{\text{word}}(T) \text{ and } \langle PT || S \rangle = \langle T || S \cdot P \rangle, \]

\(^{32}\) In fact, we will see that it is the \( \mathbb{C}\langle\mathcal{A}\rangle \)-right module \( C_{z_0 \rightarrow z} \cdot \mathbb{C}.1. \mathcal{H}(\Omega)\langle\mathcal{A}\rangle). \)

\(^{33}\) Once \( (z_0, z) \) is fixed on \( \Omega \), \( \text{Dom}_{\text{word}}(C_{z_0 \rightarrow z}) \) is the subset of \( \mathcal{A}\langle\mathcal{A}\rangle \) of series \( R \) such that \( \sum_{n \geq 0} a_n^z(R_n) \) is convergent for the standard topology, where \( R_n = \sum_{w=n}^{\infty} (R | w) w \) is a homogeneous component (we need to check that this series is convergent via majoration morphisms [33,34,36]).
(ii) from the continuity of \( \partial \), for any \( T \in \mathcal{H}(\Omega) \langle \langle 2 \rangle \rangle \) and \( S \in \text{Dom}_{\text{word}}(T) \), we have \( \partial(\langle T\|S \rangle) = \langle dT\|S \rangle + \langle T\|dS \rangle \).

Now, let \( (v, \mu, \eta) \) be a representation of \( R \in \mathbb{C}.1^{\text{rat}}_{\mathcal{H}(\Omega) \langle \langle 2 \rangle \rangle} \) of rank \( n \). Let us see that the family \( \{C_{z_0 \rightarrow z} \ | \ w\} \langle R \ | \ w \rangle \}_{w \in \mathcal{A}^*} \) is summable in \( \mathcal{H}(\Omega) \). Indeed, since the matrix norm is multiplicative then, for any \( w \in \mathcal{A}^* \) and \( B_1 > 0 \), we have

\[
\|\mu(w)\| \leq B_1^{\text{rat}} \quad \text{and} \quad v_\mu(w)\eta \leq k_1 \|v\|_r \|\mu(w)\| \|\eta\|_c.
\]

The Chen series \( C_{z_0 \rightarrow z} \) is exponentially bounded from above \({}^{35}\), i.e. for all compact \( \kappa \subset \Omega \), there is \( k_2, B_2 > 0 \) such that \({}^{36}\) \[33,34,36\]

\[\forall w \in \mathcal{A}^*, \|C_{z_0 \rightarrow z} \ | \ w\|_{\kappa} \leq k_2 B_2^{\text{rat}}/|w|!\].

Hence, choosing a compact \( \kappa \subset \Omega \), we obtain

\[
\sum_{w \in \mathcal{A}^*} \|C_{z_0 \rightarrow z} \ | \ w\|_{\kappa} \leq \sum_{w \in \mathcal{A}^*} \|C_{z_0 \rightarrow z} \ | \ w\|_{\kappa} \|\langle R \ | \ w \rangle \| \\
\leq \sum_{w \in \mathcal{A}^*} k_2 \frac{B_2^{\text{rat}}}{|w|!}(k_1 \|v\|_r B_1^{\text{rat}} \|\eta\|_c) < +\infty.
\]

Since \( y = y(z_0, z) = \alpha_{z_0}(R) \) and \( \partial \) is continuous for (CC) then, by Proposition 4.3, \( \partial^l y(z_0, z) = \langle d^l C_{z_0 \rightarrow z} || R \rangle \) and for \( l \leq n \), \( d^l C_{z_0 \rightarrow z} = Q_l(z) C_{z_0 \rightarrow z} \)

and then \[33,34,36\]

\[\partial^l y(z_0, z) = \langle Q_l(z) C_{z_0 \rightarrow z} || R \rangle = \langle C_{z_0 \rightarrow z} || R \triangleleft Q_l(z) \rangle \).

By Lemma 3.12, there is \( \{a_k\}_{k=0}^n \) in \( \mathbb{C} \), not all zero, such that \( \sum_{k=0}^n a_k(R \triangleleft Q_k) = 0 \) yielding the expected result. This linear independence holds in any module whatever the ring.

**Remark 4.9**

(i) The rational series in Theorem 4.8 is the generating series of the first order linear differential system, \( \partial q = (u_0 \mu(x_0) + \ldots + u_m \mu(x_m))q \), initialized at \( y(z_0) = \eta \). From [29], \( y(z) = \alpha_{z_0}(R) \). The \( N \)-th order differential equation in Theorem 4.8 is then the result, obtained by eliminating the states \( \{q_i\}_{i=0, \ldots, m} \) in this system.

(ii) The converse process is also possible thanks to the *compagnion form*.

(iii) Analogue results for nonlinear equations can be found in \[33,34,36\].

---

34 We choose a matrix norm *(i.e. multiplicative)* on \( \mathbb{C}^{n \times n} \), denoted \( \|M\| \), and two norms \( \|v\|_r, \|\eta\|_c \) on \( \mathbb{C}^{1 \times n}, \mathbb{C}^{n \times 1} \), respectively, and there is classically \( k_1 > 0 \) such that \( \|v M \eta\| \leq k_1 \|v\|_r \|M\| \|\eta\|_c \).

35 In the references the bounding is finer and adapted as well to infinite alphabet.

36 For any \( f \in \mathcal{H}(\Omega) \), we denote \( \|f\|_\kappa := \sup_{x \in \kappa} |f(x)| \).
5 Conclusion

In this work, we proposed a first step to construct a Picard-Vessiot theory for the class of noncommutative differential equations satisfied by the Chen series $C_{z_{0} \rightsquigarrow z}$ over the alphabet $\mathcal{X} = \{x_i\}_{i \geq 0}$ (along paths $z_0 \rightsquigarrow z$ belonging to a simply connected manifold $\Omega$ and with respect to the differential forms $(u_i dz)_{i \geq 0}$):

(i) The coefficients of these noncommutative generating series belong to the differential ring $C{\{(u_i)_{i \geq 0}]} \{C_{z_{0} \rightsquigarrow z} \mid w \}}_{w \in \mathcal{X}}$, which is closed by integration with respect to $(u_idz)_{i \geq 0}$.

(ii) The Picard-Vessiot extension of these noncommutative differential equations is defined as the module $C_{z_{0} \rightsquigarrow z} C1_{\mathcal{H}}(\Omega)$ and the Haussdorf group $\{e^C\}_{\mathcal{L}ieC(\mathcal{X})}$ plays the rôle of differential Galois group associated with this extension.

(iii) These differential equations were considered as universal differential equations by many authors [10,14,17,18,38]. Universality can be seen by replacing each letter by constant matrices (resp. holomorphic vector field, given in (2)) and then solving a system of linear (resp. nonlinear) differential equations, given in (1).

(iv) These solutions are obtained as a pairing between the series $C_{z_{0} \rightsquigarrow z}$ and the generating series of finite Hankel (resp. Lie-Hankel) rank [30,28,29,44], for linear (resp. nonlinear) differential equations explained by Remark 4.9.

(v) Via rational series (on noncommutative indeterminates and with coefficients in rings) [1,45] and their non-trivial combinatorial Hopf algebras (or pseudo Hopf algebras) (Theorems 2.2, 2.5, 3.2, 3.7 and 3.10), we illustrated this theory, still under construction, with the case of linear differential equations with singular regular singularities (Theorem 4.8) thanks to an equation satisfied by the Chen generating series.

This practical study allowed also to treat the noncommutative generating series of multiindexed polylogarithms and harmonic sums and as well as those of their special values (polyzetas). In particular, we proved the existence of well defined infinite sums of these polylogarithms and harmonic sums [9] in order to describe solutions of differential equations (Theorem 4.8).

References

[1] J. Berstel, C. Reutenauer. — *Rational series and their languages*, Spr.-Ver., 1988.

[2] N. Bourbaki. — *General Topology, Ch 1-3*, Hermann (1966)
[3] N. Bourbaki. — *Algebra I-III*, Springer-Verlag Berlin and Heidelberg GmbH & Co. K; (2nd printing 1989)

[4] N. Bourbaki. — *Commutative Algebra: Chapters 1-7*, Springer (1998)

[5] N. Bourbaki. — *Groupes et algèbres de Lie, Ch 7-8*, N. Bourbaki et Springer-Verlag Berlin Heidelberg 2006

[6] V.C. Bui, G.H.E. Duchamp, Hoang Ngoc Minh, L. Kane, C. Tollu. — *Dual bases for non commutative symmetric and quasi-symmetric functions via monoidal factorization*, Journal of Symbolic Computation, 1. 75, pp 56–73 (2016).

[7] V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo and C. Tollu. — *(Pure) Transcendence Bases in ϕ-Deformed Shuffle Bialgebras*, 74ème Sém. Lotharingien de Comb., Haus Schönenberg, Ellwangen (2018).

[8] V.C. Bui, G.H.E. Duchamp, Hoang Ngoc Minh. — *Structure of Polyzetas and Explicit Representation on Transcendence Bases of Shuffle and Stuffle Algebras*, J. of Sym. Comp., Volume 83, November–December 2017, Pages 93-111.

[9] V.C. Bui, Hoang Ngoc Minh, Q.H. Ngo. — *Families of eulerian functions involved in regularization of divergent polyzetas*, submitted.

[10] P. Cartier. — *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki, 687 (1987), 31–52.

[11] P. Cartier. — *A primer of Hopf algebras*, in: Cartier P., Moussa P., Julia B., Vanhove P. (eds) Frontiers in Number Theory, Physics, and Geometry II, (2007).

[12] R. Chari & A. Pressley. — *A guide to quantum group*, Cambridge (1994)

[13] C. Chevalley. — *Fundamental Concepts of Algebra*, Acad. Press, NY, 1956.

[14] P. Deligne. — *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math, 163, Springer-Verlag (1970).

[15] M. Deneufchâtel, G.H.E. Duchamp, Hoang Ngoc Minh, A.I. Solomon. — *Independence of hyperlogarithms over function fields via algebraic combinatorics*, in Lecture Notes in Computer Science (2011), Volume 6742/2011, 127-139.

[16] J. Dixmier. — *Enveloping algebras*, North-Holland Publishing Company (1977)

[17] V. Drinfel’d— *Quasi-Hopf Algebras*, Len. Math. J., 1, 1419-1457, 1990.

[18] V. Drinfel’d— On quasitriangular quasi-hopf algebra and a group closely connected with gal(¯q/q), Leningrad Math. J., 4, 829-860, 1991.
[19] G. Duchamp, C. Reutenauer. — *Un critère de rationalité provenant de la géométrie non-commutative*, Inventiones Mathematicae, 128, 613-622, (1997)

[20] G. Duchamp, M. Flouret, É. Laugerotte, J. G. Luque. — *Direct and dual laws for automata with multiplicities*, Theoretical Computer Science, 267, 105-120, (2001)

[21] G.H.E. Duchamp and C. Tollu. — *Sweedler’s duals and Schützenberger’s calculus*, In Combinatorics and Physics, p. 67-78, Amer. Math. Soc. (Contemporary Mathematics, vol. 539), 2011.

[22] G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo. — *Harmonic sums and polylogarithms at negative multi-indices*, J. of Sym. Comp., Volume 83, November–December 2017, Pages 166-186.

[23] G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo. — *Kleene stars of the plane, polylogarithms and symmetries*, Theoretical Computer Science, 800, p. 52-72, 2019.

[24] G. Duchamp, D. Grinberg, Hoang Ngoc Minh. — *Bialgebraic generalizations of linear independence of characters*, In preparation.

[25] GÉRARD H. E. DUCHAMP. — *MathOverflow question #203771: Local coordinates on (infinite dimensional) Lie groups, factorization of Riemann zeta functions*, https://mathoverflow.net/a/203771

[26] GÉRARD H. E. DUCHAMP. — *MathOverflow question #310354: Independence of characters with respect to polynomials*, https://mathoverflow.net/questions/310354

[27] J.Y. Enjalbert, G.H.E. Duchamp, V. Hoang Ngoc Minh, C. Tollu. — *The contrivances of shuffle products and their siblings*, Discrete Mathematics 340(9): 2286-2300 (2017).

[28] M. Fliess, C. Reutenauer. — *Théorie de Picard-Vessiot des systèmes réguliers (ou bilinéaires)*, dans “Outils et modèles mathématiques pour l’automatique, l’analyse de systèmes et le traitement du signal, CNRS-RCP 567 (1980).

[29] M. Fliess. — *Fonctionnelles causales non linéaires et indéterminées non commutatives*, Bull. Soc. Math. France, N°109, 1981, pp. 3-40.

[30] M. Fliess. — *Réalisation locale des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices*, Inv. Math., t 71, 1983, pp. 521-537.

[31] C. Hespel. — *Une étude des séries formelles noncommutatives pour l’Approximation et l’Identification des systèmes dynamiques*, thèse docteur d’état, Université Lille (1998).

[32] V. Hoang Ngoc Minh. — *Summations of Polylogarithms via Evaluation Transform*, in Math. & Computers in Simulations, 1336, pp 707-728, 1996.
[33] V. Hoang Ngoc Minh. — *Finite polyzêtas, Poly-Bernoulli numbers, identities of polyzêtas and noncommutative rational power series*, Proceedings of 4th International Conference on Words, pp. 232-250, 2003.

[34] V. Hoang Ngoc Minh. — *Differential Galois groups and noncommutative generating series of polylogarithms*, in “Automata, Combinatorics and Geometry”, 7th World Multi-conference on Systemics, Cybernetics and Informatics, Florida, 2003.

[35] V. Hoang Ngoc Minh. — *Algebraic combinatoric aspects of asymptotic analysis of nonlinear dynamical system with singular inputs*. Acta Academiae Aboensis, Ser. B 67(2), 117-126 (2007)

[36] V. Hoang Ngoc Minh. — *On a conjecture by Pierre Cartier about a group of associators*, Acta Math. Vietnamica (2013), 38, Issue 3, 339-398.

[37] V. Hoang Ngoc Minh. — *Structure of polyzetas and Lyndon words*, Vietnamese Math. J. (2013), 41, Issue 4, 409-450.

[38] V. Hoang Ngoc Minh. — *On the solutions of universal differential equation with three singularities*, in Confluentes Mathematici, Tome 11 (2019) no. 2, p. 25-64.

[39] G. Jacob. — *Réalisation des systèmes réguliers (ou bilinéaires) et séries génératrices non commutatives*, dans “Outils et modèles mathématiques pour l’automatique, l’analyse de systèmes et le traitement du signal, CNRS-RCP 567 (1980).

[40] G. Jacob. — Représentations et substitutions matricielles dans la théorie algébrique des transductions,thèse d’Etat, Univ. Paris 7, 1975.

[41] E.R. Kolchin. — *Differential Algebra and Algebraic Groups*, New York: Academic, 1973.

[42] M. Lothaire. — *Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, Addison-Wesley, 1983.

[43] R. Ree. — *Lie elements and an algebra associated with shuffles* Ann. Math 68 210–220, 1958.

[44] C. Reutenauer. — *The local realisation of generating series of finite Lie rank*. Algebraic and Geometric Methods In Nonlinear Control Theory, 33-43

[45] C. Reutenauer. — *Free Lie Algebras*, London Math. Soc. Monographs (1993).

[46] J.F. Ritt. — *Differential Algebra*, New York: AMS, 1950.

[47] M.-P., Schützenberger. — *On the definition of a family of automata*. Information and Control 4, 245–270, 1961
[48] M. Van der Put, M. F. Singer. — *Galois Theory of Linear Differential Equations*, Springer (2003)

[49] G. Viennot. — *Algèbres de Lie libres et monoïdes libres*, Lecture Notes in Mathematics, Springer-Verlag, 691, 1978.

[50] A. Zygmund. — *Trigonometric series*, Cambridge University Press 2002