$L^p$-Spaces as Quasi *-Algebras

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1. Introduction

Let $\mathcal{A}$ be a linear space and $\mathcal{A}'$ a $\ast$-algebra contained in $\mathcal{A}$. We say that $\mathcal{A}$ is a quasi $\ast$-algebra with distinguished $\ast$-algebra $\mathcal{A}'$ (or, simply, over $\mathcal{A}'$) if (i) the right and left multiplications of an element of $\mathcal{A}$ and an element of $\mathcal{A}'$ are always defined and linear; and (ii) an involution $\ast$ (which extends the involution of $\mathcal{A}'$) is defined in $\mathcal{A}$ with the property $(AB)\ast = B\ast A\ast$ whenever the multiplication is defined.

Quasi $\ast$-algebras \[1, 2\] arise in natural way as completions of locally convex $\ast$-algebras whose multiplication is not jointly continuous; in this case one has to deal with topological quasi $\ast$-algebras.

A quasi $\ast$-algebra $(\mathcal{A}, \mathcal{A}')$ is called topological if a locally convex topology $\tau$ on $\mathcal{A}$ is given such that:

(i) the involution $A \mapsto A\ast$ is continuous

(ii) the maps $A \mapsto AB$ and $A \mapsto BA$ are continuous for each $B \in \mathcal{A}$

(iii) $\mathcal{A}'$ is dense in $\mathcal{A}[\tau]$.

In a topological quasi $\ast$-algebra the associative law holds in the following two formulations

$$A(BC) = (AB)C; \quad B(AC) = (BA)C \quad \forall A \in \mathcal{A}, \forall B, C \in \mathcal{A}_0$$

Let $(X, \mu)$ be a measure space with $\mu$ a Borel measure on the locally compact Hausdorff space $X$. As usual, we denote by $L^p(X, d\mu)$ (or simply, $L^p(X)$ if no confusion is possible) the Banach space of all (equivalence classes of) measurable functions $f : X \rightarrow \mathbb{C}$ such that

$$\|f\|_p \equiv \left( \int_X |f|^p d\mu \right)^{1/p} < \infty.$$  

On $L^p(X)$ we consider the natural involution $f \in L^p(X) \mapsto f\ast \in L^p(X)$ with $f\ast(x) = f(x)$.

We denote with $C_0(X)$ the C*-algebra of continuous functions vanishing at infinity.

The pair $(L^p(X, \mu), C_0(X))$ provides the basic commutative example of topological quasi $\ast$-algebra.

From now on, we assume that $\mu$ is a positive measure.

In a previous paper \[3\] we introduced a particular class of topological quasi $\ast$-algebras, called CQ*-algebras. The definition we will give here is not the general one, but it is exactly what we need in the commutative case which we will consider in this paper.

A CQ*-algebra is a topological quasi $\ast$-algebra $(\mathcal{A}, \mathcal{A}')$ with the following properties
(i) $\mathcal{A}_0$ is a C*-algebra with respect to the norm $\| \cdot \|_0$ and the involution $^*$.

(ii) $\mathcal{A}$ is a Banach space with respect to the norm $\| \cdot \|$ and $\| A^* \| = \| A \|$ $\forall A \in \mathcal{A}$.

(iii) $\| B \|_0 = \max \left\{ \sup_{\| A \| \leq 1} \| AB \|, \sup_{\| A \| \leq 1} \| BA \| \right\}$ $\forall B \in \mathcal{A}_0$.

It is shown in [3] that the completion of any C*-algebra $(\mathcal{A}_0, \| \cdot \|_0)$ with respect to a weaker norm $\| \cdot \|_1$ satisfying

(i) $\| A^* \|_1 = \| A \|_1$ $\forall A \in \mathcal{A}_0$

(ii) $\| AB \|_1 \leq \| A \|_1 \| B \|_1$ $\forall A, B \in \mathcal{A}_0$

is a CQ*-algebra in the sense discussed above.

This is the reason why both $(L^p(X, \mu), C_0(X))$ and $(L^p(X, \mu), L^\infty(X, \mu))$ are CQ*-algebras.

$L^p$-spaces are examples of the $L_\rho$'s considered in [4]. Let $\mu$ be a measure in a non-empty point set $X$ and $M^+$ be the collection of all the positive $\mu$-measurable functions. Suppose that to each $f \in M^+$ it corresponds a number $\rho(f) \in [0, \infty]$ such that:

i) $\rho(f) = 0$ iff $f = 0$ a.e. in $X$;

ii) $\rho(f_1 + f_2) \leq \rho(f_1) + \rho(f_2)$;

iii) $\rho(af) = a\rho(f)$ $\forall a \in \mathbb{R}_+$;

iv) let $f_n \in M^+$ and $f_n \uparrow f$ a.e. in $X$. Then $\rho(f_n) \uparrow \rho(f)$.

Following [4], we call $\rho$ a function norm. Let us define $L_\rho$ as the set of all $\mu$-measurable functions such that $\rho(f) < \infty$. The space $L_\rho$ is a Banach space, that is it is complete, with respect to the norm $\| f \|_\rho = \rho(|f|)$. If the function norm $\rho$ satisfies the additional condition

$$\rho(|fg|) \leq \rho(|f|) \| g \|_\infty, \quad \forall f, g \in C_0(X),$$

then the completion of $C_0(X)$ with respect to this norm is an abelian CQ*-algebra over $C_0(X)$.

Of course, for $L^p$-spaces, $\| \cdot \| = \| \cdot \|_p$.

In this paper we will discuss some structure properties of $(L^p(X, \mu), C_0(X))$ as a CQ*-algebra. In Section 2, in particular, we will study a certain class of positive sesquilinear forms on $(L^p(X, \mu), C_0(X))$ which lead, in rather natural way, to a definition of *-semisimplicity. As is shown in [5], *-semisimple CQ*-algebras behave nicely and for them a refinement of the algebraic structure of quasi *-algebra to a partial *-algebra [6, 7] is possible. The abelian case is discussed in Section 3.
Finally, we characterize $\ast$-semisimple abelian CQ*-algebras as a CQ*-algebra of functions obtained by means of a family of $L^2$-spaces, generalizing in this way the concept of Gel'fand transform for C*-algebras.

2. Structure properties of $L^p$-spaces

**Lemma 2.1.** Let $g$ be a measurable function on $X$, with $\mu(X) < \infty$, and assume that $fg \in L^r(X)$ for all $f \in L^p(X)$ with $1 \leq r \leq p$. Then $g \in L^q(X)$ with $p^{-1} + q^{-1} = r^{-1}$.

*Proof.* Let us consider the linear operator $T_g$ defined in the following way:

$$T_g : f \in L^p(X) \longrightarrow fg \in L^r(X).$$

$T_g$ is closed. Indeed, let $f_n \overset{p}{\rightarrow} f$ and $T_g f_n \overset{r}{\rightarrow} h$; this implies the existence of a subsequence $f_{n_k}$ such that $f_{n_k} \overset{a.e.}{\rightarrow} f$ in $X$ and so $f_{n_k} g \overset{\text{in measure}}{\rightarrow} fg$ a.e. in $X$. Then, we have $f_{n_k} g \overset{\text{in measure}}{\rightarrow} fg$ in measure; on the other hand the fact that $T_g f_n \overset{r}{\rightarrow} h$ implies also that $f_n g$ converges to $h$ in measure. Thus, necessarily, $h(x) = f(x)g(x)$ a.e. in $X$.

Thus $T_g$ is closed and everywhere defined in $L^p(X)$; then, by the closed graph theorem, $T_g$ is bounded; i.e., there exists $C > 0$ (depending on $g$) such that

$$\| fg \|_r \leq C \| f \|_p.$$ 

If $r = 1$, this already implies that $g \in L^p(X)$ with $p^{-1} + p^{-1} = 1$.

Let now $r > 1$ and let $h \in L^{r'}(X)$ with $r^{-1} + r'^{-1} = 1$. If $f \in L^p(X)$ then, by Hölder inequality, $fh \in L^m(X)$ with $m^{-1} = p^{-1} + r'^{-1}$. Thus, the set

$$\mathcal{F} = \{ \{ g : \{ \in L^\bigvee(\mathcal{A}), \langle \in L^\bigvee'(\mathcal{A}) \} \}$$

is a subset of $L^m(X)$. Conversely, any function $\psi \in L^m(X)$ can be factorized as the product of a function in $L^p(X)$ and one in $L^{r'}(X)$. This is achieved by considering, for instance, the principal branches of the functions $\psi^{m/p}$ and $\psi^{m/r'}$.

We can now apply the first part of the proof to conclude that $g \in L^{m'}(X)$ with $m^{-1} + m'^{-1} = 1$. An easy computation shows now that $m' = q$.

$\square$

**Remark 2.2.** For $r = 1$ and $\mu(X)$ not necessarily finite, the above statement can also be found in [4, 8]. Using this fact, with the same technique as above, the statement of Lemma 2.1 can be extended to the case $\mu(X) = \infty$. Our proof involves functional aspects of $L^p$-spaces so we think is worth giving it.
Definition 2.3. Let \((A, A_0)\) be a CQ*-algebra. We denote as \(S(A)\) the set of sesquilinear forms \(\Omega\) on \(A \times A\) with the following properties:

(i) \(\Omega(A, A) \geq 0 \quad \forall A \in A\)
(ii) \(\Omega(AB, C) = \Omega(B, A^*C) \quad \forall A, B, C \in A_0\)
(iii) \(|\Omega(A, B)| \leq \|A\| \|B\| \quad \forall A, B \in A\)

The CQ*-algebra \((A, A_0)\) is called *-semisimple if \(\Omega(A, A) = 0\) \(\forall \Omega \in S(A)\) implies \(A = 0\).

From now on, we will only consider the case of compact \(X\) (in this case we denote as \(C(X)\) the C*-algebra of continuous functions on \(X\)) and we will focus our attention on the question whether \((L^p(X), C(X))\), is or is not *-semisimple. To this aim, we need first to describe the set \(S(L^p(X))\).

In what follows we will often use the following fact, which we state as a lemma for reader’s convenience.

Lemma 2.4. Let \(p, q, r \geq 1\) be such that \(p^{-1} + q^{-1} = r^{-1}\). Let \(w \in L^q(X)\). Then the linear operator \(T_w : f \in L^p(X) \mapsto fw \in L^r(X)\) is bounded and \(\|T_w\|_{p,r} = \|w\|q\).

Remark 2.5. Here \(\|T_w\|_{p,r}\) denotes the norm of \(T_w\) as bounded operator from \(L^p(X)\) into \(L^r(X)\).

For shortness, we put (if \(p = 2\), we set \(p \to (p-2) = \infty\)).

\[B^p_+ = \left\{ \chi \in L^p/(p-2), \chi \geq 0 \text{ and } \|\chi\|_{p/(p-2)} \leq 1 \right\}.\]

Proposition 2.6. 1. If \(p \geq 2\), then any \(\Omega \in S(L^p(X))\) can be represented as

\[\Omega(f, g) = \int_X f(x)\overline{g(x)}\psi(x) d\mu \quad (1)\]

for some \(\psi \in B^p_+\).

Conversely, if \(\psi \in B^p_+\), then the sesquilinear form \(\Omega\) defined by Eqn. (1) is in \(S(L^p(X))\).

2. If \(1 \leq p < 2\) then \(S(L^p(X)) = \{0\}\)

Proof. We notice, first, that any bounded sesquilinear form \(\Omega\) on \(L^p(X) \times L^p(X)\) can be represented as

\[\Omega(f, g) = \langle f, Tg \rangle = \int_X f(x)(Tg)(x) d\mu \quad (2)\]

where \(T\) is a bounded linear operator from \(L^p(X)\) into its dual space \(L^p'(X)\), \(p^{-1} + p'^{-1} = 1\) \([1, \S 40]\). From (ii) of Definition 2.3 and from Eqn. (2) it follows easily that

\[Tg = gTu \quad \forall g \in L^p(X)\]
where \( u(x) = 1 \ \forall x \in X \). Set \( Tu = \psi \); from (i) of Definition 2.3, we get \( \psi \geq 0 \).

(1) If \( p \geq 2 \) then by Lemma 2.1, we get \( \psi \in L^{p/(p-2)}(X) \). Making use of Lemma 2.4 it is also easy to check that \( \| \psi \|_{p/(p-2)} \leq 1 \).

(2) Let now \( 1 \leq p < 2 \) and \( \psi \neq 0 \). Since \( \psi \geq 0 \), we can choose \( \alpha > 0 \) in such a way that the set \( Y = \{ x \in X : \psi(x) > \alpha \} \) has positive measure.

Let \( f \in L^p(Y) \setminus L^2(Y) \) (such a function always exists because of the assumption on \( p \)). Now define

\[
\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in Y \\ 0 & \text{if } x \in X \setminus Y \end{cases}
\]

Clearly, \( \tilde{f} \in L^p(X) \).

Now,

\[
\Omega(f, f) = \int_X |\tilde{f}(x)|^2 \psi(x) d\mu(x) = \int_Y |f(x)|^2 \psi(x) d\mu(x) \geq \alpha \int_Y |f(x)|^2 d\mu(x) = \infty
\]

and this is a contradiction.

Remark 2.7. From the above representation theorem, it follows easily that if \( \Omega \in S(L^p(X)) \), then the sesquilinear form \( \Omega^* \) defined by \( \Omega^*(f, g) = \Omega(g^*, f^*) \) also belongs to \( S(L^p(X)) \).

Proposition 2.8. Let \( p \geq 2 \). \((L^p(X, d\mu), C(X))\) is \(*\)-semisimple.

Proof. We show first that \( \forall f \in L^p(X) \), there exists \( \Omega \in S(L^p(X)) \) such that \( \Omega(f, f) = \| f \|_{p}^2 \).

This is achieved by setting

\[
\tilde{\Omega}(g, h) = \| f \|_{p}^{-2} \int_X g \bar{h} |f|^{p-2} d\mu
\]

\( \tilde{\Omega} \in S(L^p(X)) \) since the function \( \psi = |f|^{p-2} \| f \|_{p}^{2-p} \) belongs to the set \( B_+^p \). A direct calculation shows that \( \tilde{\Omega}(f, f) = \| f \|_{p}^2 \).

Let us now suppose that \( \Omega(f, f) = 0 \ \forall \Omega \in S(L^p(X)) \). Then, in particular, \( \tilde{\Omega}(f, f) = \| f \|_{p}^2 = 0 \). Therefore \( f(x) = 0 \) and so \((L^p(X, \mu), C(X))\) is \(*\)-semisimple.

Positive sesquilinear forms which are normalized (in the sense that \( \Omega(\mathbb{I}, \mathbb{I}) = \| \mathbb{I} \|_{p}^2 \)) could be expected to play, in this framework, the same role as states on a \( C^* \)-algebra. Indeed, for such an \( \Omega \) we have

\[
\| \Omega \|_{\mathbb{F}} = \sup_{A \in A} \frac{\Omega(A, A)}{\| A \|_{2}^2} = 1
\]

The next Proposition shows, however, an essential difference between the two frameworks.
Proposition 2.9. In the CQ*-algebra \((L^p(X, \mu), C(X))\), \(p \geq 2\), there exists one and only one \(\Omega \in \mathcal{S}(L^p(X))\) such that \(\Omega(u, u) = \| u \|_p^2\), where \(u(x) = 1 \forall x \in X\).

Proof. The sesquilinear form

\[
\Omega_0(f, g) \equiv \mu(X)^{(2-p)/p} \int_X f(x)g(x) \, d\mu
\]

belongs to \(\mathcal{S}(L^p(X))\) and obviously satisfies the condition \(\Omega_0(u, u) = \| u \|_p^2\).

It remains to prove its uniqueness.

Let \(\Omega\) satisfy the assumptions of the Proposition.

By Proposition 2.6, there exists \(\psi \in \mathcal{B}_+^p\) such that

\[
\Omega(f, g) = \int_X f(x)g(x)\psi(x) \, d\mu.
\]

Therefore,

\[
\Omega(u, u) = \int_X \psi(x)dx = \| \psi \|_1
\]

and

\[
\| u \|_p^2 = \mu(X)^{2/p}
\]

and so we must have

\[
\| \psi \|_1 = \mu(X)^{2/p}.
\] (3)

On the other hand, since \(\psi \in \mathcal{B}_+^p\), using the inequality \(\| \psi \|_1 \leq \mu(X)^{2/p} \| \psi \|_{p/(p-2)}\), we conclude also that

\[
\| \psi \|_{p/(p-2)} = 1.
\] (4)

We will now prove that there exists only one \(\psi\) satisfying both (3) and (4). To show this, let us define on \(X\) a new measure \(\nu\) by \(\nu(Y) := \frac{\mu(Y)}{\mu(X)}\) for any \(\mu\)-measurable set \(Y \subseteq X\); then \(\nu(X) = 1\) and \(\| \psi \|_{1,\nu} \leq \mu(X)^{2/p} \| \psi \|_{p/(p-2),\nu}\) where \(\| \cdot \|_{r,\nu}\) denotes the \(L^r\)-norm with respect to the measure \(\nu\). From this it follows (3) that \(\psi\) is constant \(\nu\)-a.e. It is then easy to prove that \(\psi(x) = \mu(X)^{(2-p)/p} \mu\)-a.e. in \(X\). \(\square\)

We notice that a statement analogous to Proposition 2.9 does not hold for C*-algebras where the set of states is, in general, quite rich.

Remark 2.10. If a CQ*-algebra \((\mathcal{A}, \mathcal{A}_0)\) has only one normalized positive sesquilinear form \(\Omega_0\), then it is possible to identify easily \(\mathcal{A}/\text{Ker} \, \Omega_0\) with a linear subspace of the dual \(\mathcal{A}'\) of \(\mathcal{A}\) by

\[
[A] \in \mathcal{A}/\text{Ker} \, \Omega_0 \longrightarrow F_A \in \mathcal{A}'
\]

where \(F_A(B) \equiv \Omega_0(B, A) \forall B \in \mathcal{A}\). The known imbeddings of \(L^p\)-spaces on sets of finite measures, for \(p \geq 2\), provides examples of this situation (in this case \(\text{Ker} \, \Omega_0 = \{0\}\)).
In [3] we have introduced some norms on a semisimple CQ*-algebra which play an interesting role. Their definitions in the case of \((L^p(X, \mu), C(X))\) reads as follows

\[
\| f \|_\alpha = \sup \{ \Omega(f, f), \Omega \in \mathcal{S}(L^p(X)) \} \tag{5}
\]

and

\[
\| f \|_\beta = \sup \{ |\Omega(f\phi, \phi)|, \Omega \in \mathcal{S}(L^p(X)), \phi \in C(X), \| \phi \|_\infty \leq 1 \} \tag{6}
\]

**Proposition 2.11.** Let \(f \in L^p(X), p \geq 2\). Then

\[
\| f \|_\alpha = \| f \|_p \| f \|_\beta \leq \| f \|_{p/2} \tag{7}
\]

If \(f \geq 0\), then \(\| f \|_\beta = \| f \|_{p/2}\).

**Proof.** To prove the equality \(\| f \|_\alpha = \| f \|_p\), it is enough recalling that, as shown in the proof of Proposition 2.8, there exists a sesquilinear form \(\tilde{\Omega}\) such that \(\tilde{\Omega}(f, f) = \| f \|_p^2\). Therefore \(\| f \|_\alpha^2 \geq \tilde{\Omega}(f, f) = \| f \|_p^2\). The converse inequality follows from (iii) of Definition 2.3.

By Proposition 2.8 we get

\[
\| f \|_\beta = \sup \{ |\Omega(f\phi, \phi)|, \Omega \in \mathcal{S}(L^p(X)), \phi \in C(X), \| \phi \|_\infty \leq 1 \} = \sup \left\{ \int_X |f\phi|^2 \psi d\mu, \psi \in \mathcal{B}_+^2, \phi \in C(X), \| \phi \|_\infty \leq 1 \right\} \leq \| f \|_{p/2}
\]

On the other hand, for \(f \geq 0\) we get

\[
\| f \|_{p/2} = \sup \left\{ \int_X \overline{f} \psi d\mu, \| \psi \|_{p/(p-2)} \leq 1 \right\} = \sup \left\{ \int_X |f\phi|^2 \psi d\mu, \| \psi \|_{p/(p-2)} \leq 1, \phi \in C(X), \| \phi \|_\infty \leq 1 \right\}
\]

\[
= \sup \left\{ \int_X |f\phi|^2 \psi d\mu, \| \psi \|_{p/(p-2)} \leq 1, \phi \in C(X), \| \phi \|_\infty \leq 1 \right\}
\]

\[
= \sup \{ \Omega(f\phi, \phi), \Omega \in \mathcal{S}(L^p(X)), \phi \in C(X), \| \phi \|_\infty \leq 1 \} = \| f \|_\beta
\]

\(\square\)

Apart from \(A_0\), a \(^*-\)semisimple CQ*-algebra has another distinguished subset, denoted as \(A_\gamma\), which play an interesting role for what concerns the functional calculus and representations in Hilbert space. We give here the definition in the case \(A = L^p(X)\) with \(p \geq 2\). The general definition is an obvious extension of this one. We denote as \((L^p(X))_\gamma\) the set of all \(f \in L^p(X)\) such that

\[
\| f \|_{\gamma}^2 := \sup \left\{ \frac{\Omega(f\phi, f\phi)}{\Omega(\phi, \phi)}, \Omega \in \mathcal{S}(L^p(X)), \phi \in C(X), \Omega(\phi, \phi) \neq 0 \right\} < \infty \tag{8}
\]
Since \( \forall \phi(x) \in C(X) \) and \( \forall \Omega \in \mathcal{S}(L^p(X)) \) the sesquilinear form \( \Omega_{\phi} \), defined by \( \Omega_{\phi}(f,g) \equiv \frac{\Omega(f \phi, g \phi)}{\| \phi \|_2^\infty} \forall f, g \in L^p(X) \), also belongs to \( \mathcal{S}(L^p(X)) \), it turns out that
\[
\| f \|_2^\gamma = \sup \left\{ \frac{\Omega(f,f)}{\Omega(u,u)}, \Omega \in \mathcal{S}(L^p(X)) \right\},
\]
(9) where \( u(x) = 1 \forall x \in X \).

**Proposition 2.12.** \((L^p(X))_\gamma = L^\infty(X)\) and \( \| f \|_\gamma = \| f \|_\infty \).

**Proof.** From the previous discussion and from Proposition 2.3, it follows that \( f \in (L^p(X))_\gamma \) if, and only if,
\[
\| f \|_\gamma^2 = \sup_{\psi \in B_p^\gamma} \frac{\int_X |f|^2 \psi d\mu}{\| \psi \|_1} < \infty
\]
This means that there exists a constant \( C > 0 \) such that
\[
\omega_f(\psi) := \int_X |f|^2 \psi d\mu \leq C \| \psi \|_1 \quad \forall \psi \in B_p^\gamma
\]
Let now \( \psi \in L^{p/(p-2)}(X), \psi \geq 0 \). Then the function \( \psi_N \equiv \frac{\psi}{\| \psi \|_{p/(p-2)}} \in B_p^\gamma \). This implies that the above inequality also holds for all positive functions in \( L^{p/(p-2)}(X) \). Finally, if \( \psi \) is an arbitrary function in \( L^{p/(p-2)}(X) \), then
\[
|\omega_f(\psi)| = \left| \int_X |f|^2 \psi d\mu \right| \leq \int_X |f|^2 |\psi| d\mu \leq C \| \psi \|_1 = C \| \psi \|_1.
\]
Therefore \( \omega_f \) is a linear functional on \( L^{p/(p-2)}(X) \) and it is bounded with respect to \( \| \|_1 \). Since \( L^{p/(p-2)}(X) \) is dense in \( L^1(X) \), \( \omega_f \) has a unique continuous extension to \( L^1(X) \). Then \( |f|^2 \in L^\infty(X) \); this, in turn, implies that \( f \in L^\infty(X) \). Thus \((L^p(X))_\gamma \subseteq L^\infty(X) \). The converse inclusion is obvious. The norm of \( \omega_f \) as linear functional on the subspace \( L^{p/(p-2)}(X) \) of \( L^1(X) \) is \( \| \omega_f \| = \| |f|^2 \|_\infty = \| f \|_\infty^2 \) (the latter equality follows from the C*-nature of \( L^\infty(X) \)). Moreover, the Hahn- Banach theorem, used to extend \( \omega_f \) from \( L^{p/(p-2)}(X) \) to \( L^1(X) \), ensures that \( \| f \|_\gamma^2 = \| \omega_f \| \).

The role of \((L^p(X))_\gamma \) will be clearer if we consider the GNS-construction of an abstract CQ*-algebra \((\mathcal{A}, \mathcal{A}_0)\) obtained via a sesquilinear form \( \Omega \) in \( \mathcal{S}(\mathcal{A}) \). This problem was, from a general point of view, considered in [3]. We will give here a simplified version of the GNS-construction which is closer to that proved in [4] for general partial *-algebras.
Let \((A, A_0)\) be a CQ*-algebra and \(\Omega\) a positive sesquilinear form in \(S(A)\). Let \(K = \{ A \in A : \Omega(A, A) = 0 \}\). Let us consider the linear space \(A/K\); an element of this set will be denoted as \(\lambda_\Omega(A), A \in A\). Clearly, \(A/K = \lambda_\Omega(A)\) is a pre-Hilbert space with respect to the scalar product \((\lambda_\Omega(A), \lambda_\Omega(B)) = \Omega(A, B), A, B \in A\). We denote by \(H_\Omega\) the Hilbert space obtained by the \(\Lambda\alpha\) completion of \(\lambda_\Omega(A)\). Then \(\Omega\) is invariant in the sense of [7]. This means, in this case, that \(\Omega\) satisfies condition (ii) of Definition 2.3 and that \(\lambda_\Omega(A_0)\) is dense in \(H_\Omega\). Indeed, let \(\lambda_\Omega(A) \in \lambda_\Omega(A)\) and let \(\{ A_n \}\) be a sequence in \(A_0\) converging to \(A\) in the norm of \(A\). Then from \(\Omega(A - A_n, A - A_n) \leq \| A - A_n \|^2\) it follows that \(\lambda_\Omega(A_n) \rightarrow \lambda_\Omega(A)\) in \(H_\Omega\).

If we put \(\pi_\Omega(A) = \lambda_\Omega(AB) B \in A_0\), then \(\pi_\Omega(A)\) is a well-defined closable operator with domain \(\lambda_\Omega(A_0)\) in \(H_\Omega\). More precisely, it is an element of the partial O*-algebra \(L^+(\lambda_\Omega(A_0), H_\Omega)\) [6, 7]. The map \(A \mapsto \pi_\Omega(A)\) is a *-representation of partial *-algebras in the sense of [7].

We define now the following set:

\[
D_\Omega = \left\{ A \in A : \sup_{B \in A_0, \Omega(B, B) \neq 0} \frac{\Omega(AB, AB)}{\Omega(B, B)} < \infty \right\}
\]

then

(i) \(D_\Omega\) is a linear space;
(ii) \(D_\Omega \supset A_0\);
(iii) if \(A \in D_\Omega\) and \(B \in A_0\), then \(AB \in D_\Omega\)

If \(D_\Omega = A\) then \(\Omega\) is admissible in the sense of [4].

From the definition itself, it follows easily that \(\pi_\Omega(D_\Omega) \subseteq B(H_\Omega)\), i.e. each element of \(D_\Omega\) is represented by a bounded operator in Hilbert space.

As an example, let us consider the CQ*-algebra \((L^p(X, \mu), C(X))\) for \(p \geq 2\). Let us fix \(w \in B_+\) with \(w > 0\) and define

\[
\Omega(f, g) = \int_X f g w d\mu \quad f, g \in L^p(X).
\]

Thus \(\Omega \in S(L^p(X))\). It is clear that \(H_\Omega = L^2(X, \mu_w)\) where \(d\mu_w = wd\mu\). The representation \(\pi_\Omega\) is then defined by

\[
\pi_\Omega(f) \phi = f \phi \quad \phi \in C(X)
\]

for \(f \in L^p(X)\). A proof analogous to that of Proposition 2.12 shows that, in this case, \(D_\Omega = L^\infty(X, \mu_w) \cap L^p(X, \mu)\).
Remark 2.13. The set $A_\gamma$ is included in $D_\Omega \ \forall \Omega \in S(A)$. This makes clear the role of $A_\gamma$ as a relevant subset of $A$ represented by bounded operators $\forall \Omega \in S(A)$.

We conclude this Section by showing that each *-semisimple abelian CQ*-algebra, can be thought as CQ*-algebras of functions.

Let $X$ be a compact Hausdorff space and $\mathcal{M} = \{\mu_\alpha; \alpha \in \mathcal{I}\}$ be a family of Borel measures on $X$. Let us assume that there exists $C > 0$ such that $\mu_\alpha(X) \leq C \ \forall \alpha \in \mathcal{I}$. Let us denote by $\| \cdot \|_{p,\alpha}$ the norm in $L^p(X, \mu_\alpha)$ and define, for $\phi \in C(X)$

$$\| \phi \|_{p,\alpha} = \sup_{\alpha \in \mathcal{I}} \| \phi \|_{p,\alpha}.$$ 

Since $\| \phi \|_{p,\alpha} \leq C \| \phi \|_\infty \ \forall \phi \in C(X)$, then $\| \cdot \|_{p,\mathcal{I}}$ is finite on $C(X)$ and really defines a norm on $C(X)$ satisfying

(i) $\| \phi^* \|_{p,\mathcal{I}} = \| \phi \|_{p,\mathcal{I}} \ \forall \phi \in C(X)$
(ii) $\| \phi \psi \|_{p,\mathcal{I}} \leq \| \phi \|_{p,\mathcal{I}} \| \psi \|_\infty \ \forall \phi, \psi \in C(X)$.

Therefore the completion $L^p_I(X, \mathcal{M})$ of $C(X)$ with respect to $\| \cdot \|_{p,\mathcal{I}}$ is an abelian CQ*-algebra over $C(X)$. It is clear that $L^p_I(X, \mathcal{M})$ can be identified with a subspace of $L^p(X, \mu_\alpha) \ \forall \alpha \in \mathcal{I}$.

It is obvious that $L^p_I(X, \mathcal{M})$ contains also non-continuous functions.

For $p \geq 2$, the CQ*-algebra $(L^p_I(X, \mathcal{M}), C(X))$ is *-semisimple (this depends on the fact that each element of $S(L^p(X))$ gives rise, by restriction, to an element of $S(L^p_I(X, \mathcal{M}))$).

Proposition 2.14. Let $(A, A_0)$ be a *-semisimple abelian CQ*-algebra with unit $I$. Then there exists a family $\mathcal{M}$ of Borel measures on the compact space $X$ of characters of $A_0$ and a map $\Phi : A \in A \rightarrow \hat{\Phi}(A) \equiv \hat{A} \in L^2_I(X, \mathcal{M})$ with the properties

(i) $\Phi$ extends the Gel’fand transform of elements of $A_0$ and $\Phi(A) \supseteq C(X)$
(ii) $\Phi$ is linear and one-to-one
(iii) $\Phi(AB) = \Phi(A)\Phi(B) \ \forall A \in A, B \in A_0$.
(iv) $\Phi(A^*) = \Phi(A)^* \ \forall A \in A$.

Thus $A$ can be identified with a subspace of $L^2_I(X, \mathcal{M})$.

If $A$ is regular, i.e. if

$$\| A \|^2 = \sup_{\Omega \in S(A)} \Omega(A, A)$$

then $\Phi$ is an isometric *-isomorphism of $A$ onto $L^2_I(X, \mathcal{M})$.

Proof. Define first $\Phi$ on $A_0$ as the usual Gel’fand transform

$$\Phi : B \in A_0 \rightarrow \hat{B} \in C(X)$$
where $X$ is the space of characters of $A_0$.

As is known, the Gel'fand transform is an isometric *-isomorphism of $A_0$ onto $C(X)$. Let $\Omega \in \mathcal{S}(A)$ and define the linear functional $\omega$ on $C(X)$ by

\[ \omega(\hat{B}) = \Omega(B, \mathbb{I}). \]

It is easy to check that $\omega$ is bounded on $C(X)$; then by the Riesz representation theorem, there exists a unique positive Borel measure $\mu_\Omega$ on $X$ such that

\[ \omega(\hat{B}) = \Omega(B, \mathbb{I}) = \int_X \hat{B}(\eta) \mu_\Omega(\eta) \quad \forall \hat{B} \in A_\mu. \]

We have $\mu_\Omega(X) \leq \| I \|^p \quad \forall \Omega \in \mathcal{S}(A)$

Let $\mathcal{M} \equiv \{ \mu_\Omega : \Omega \in \mathcal{S}(A) \}$ and let $L^2_{\mathcal{S}(A)}(X, \mathcal{M})$ be the CQ*-algebra constructed as above. Now, if $A \in A$ there exists a sequence $\{ A_n \} \subset A_0$ converging to $A$ in the norm of $A$. We have then

\[ \| \hat{A}_n - \hat{A}_m \|_{2, \mathcal{S}(A)}^2 = \sup_{\Omega \in \mathcal{S}(A)} \Omega(A_n - A_m, A_n - A_m) \leq \| A_n - A_m \|_{2, \mathcal{S}(A)}^2 \rightarrow 0 \]

Let $\hat{A}$ be the $\| \|_{2, \mathcal{S}(A)}$-limit in $L^2_{\mathcal{S}(A)}(X, \mathcal{M})$ of $\{ \hat{A}_n \}$ and define

\[ \Phi(A) = \hat{A}. \]

Evidently, $\| \hat{A} \|_{2, \mathcal{S}(A)} = \sup_{\Omega \in \mathcal{S}(A)} \Omega(A, A)$. This implies that if $\hat{A} = 0$, then $\Omega(A, A) = 0 \quad \forall \Omega \in \mathcal{S}(A)$ and thus $A = 0$ for $A$ is *-semisimple. The proof of (ii), (iii) and (iv) is straightforward.

Now if $A$ is regular, from the above discussion it follows immediately that $\Phi$ is an isometry. We conclude by proving that in this case $\Phi$ is onto. Let $\hat{A}$ be an element of $L^2_{\mathcal{S}(A)}(X, \mathcal{M})$. Then there exists a sequence $\{ \hat{A}_n \} \subset C(X)$ converging to $\hat{A}$ with respect to $\| \|_{2, \mathcal{S}(A)}$. Then the corresponding sequence $\{ A_n \} \subset A_0$ converges to $A$ in the norm of $A$, since $\Phi$ is an isometry.

\[ \Box \]

3. The partial multiplication

In this final Section, we will discuss the possibility of refining the multiplication structure of $L^p$-spaces. Actually, we will show that $L^p(X)$ is really a partial *-algebra. For reader’s convenience we repeat here the definition.

A partial *-algebra is a vector space $A$ with involution $A \rightarrow A^*$ [i.e. $(A + \lambda B)^* = A^* + \overline{\lambda} B^* ; A = A^{**} ]$ and a subset $\Gamma \subset A \times A$ such that (i) $(A, B) \in \Gamma$ implies $(B^*, A^*) \in \Gamma$ ; (ii) $(A, B)$ and $(A, C) \in \Gamma$ imply $(A, B + \lambda C) \in \Gamma$ ; and (iii) if $(A, B) \in \Gamma$, then there exists an element
$AB \in \mathcal{A}$ and for this multiplication the distributive property holds in the following sense: if $(A, B) \in \Gamma$ and $(A, C) \in \Gamma$ then

$$AB + AC = A(B + C)$$

Furthermore $(AB)^* = B^*A^*$.

The product is not required to be associative.

The partial $*$-algebra $\mathcal{A}$ is said to have a unit if there exists an element $I$ (necessarily unique) such that $I^* = I$, $(I, \mathcal{A}) \in \Gamma$, $IA = AI$, $\forall A \in \mathcal{A}$.

If $(A, B) \in \Gamma$ then we say that $A$ is a left multiplier of $B$ [and write $A \in L(B)$] or $B$ is a right multiplier of $A$ [$B \in R(A)$].

There are at least two ways to introduce partial multiplications in $L^p(X)$. The first one is almost obvious, if the set of multicable elements is defined as follows:

$$\Gamma_1 = \{(f, g) \in L^p(X) \times L^p(X) : fg \in L^p(X)\}.$$

The second is obtained by defining the following set of real numbers

$$E(f) = \{q \in [1, \infty) : \|f\|_q < \infty\}.$$

The partial multiplication is then defined on the set

$$\Gamma_2 = \left\{(f, g) \in L^p(X) \times L^p(X) : \exists r \in E(f), s \in E(g); \frac{1}{r} + \frac{1}{s} = \frac{1}{p}\right\}.$$

Evidently, $\Gamma_2 \subseteq \Gamma_1$. If $\mu(X) < \infty$ then $(L^p(X), \Gamma_2)$ is a partial $*$-algebra. But for $\mu(X) = \infty$ the distributive property may fail (this is due to the fact that in the case $\mu(X) = \infty$, the family of $L^p$-spaces form a lattice which do not reduce to a chain). The set $\Gamma_2$ will be no longer considered here.

As shown before, for $p \geq 2$, $(L^p(X), C(X))$ is $*$-semisimple; then we can define in $L^p(X)$ a weak-multiplication in the following way. If $f, g \in L^p(X)$, we say that $f$ is a weak-multiplier of $g$, if there exists a unique element $h \in L^p(X)$ such that

$$\Omega(g\phi, f^*\psi) = \Omega(h\phi, \psi) \quad \forall \Omega \in \mathcal{S}(L^p(X)), \forall \phi, \psi \in C(X).$$

It is worth remarking that, in this case, the uniqueness of $h$ follows from Proposition 2.6 and so we do not need to require it.

Let $\Gamma_w$ denote the set of pairs $(f, g) \in L^p(X) \times L^p(X)$ such that $f$ is a weak-multiplier of $g$. It is very easy to see that $\Gamma_w = \Gamma_1$.

Another way to introduce a partial multiplication is to consider the so-called closable elements. In the following discussion we will not suppose that $X$ is compact (and so $\mu(X)$ is not necessarily finite).
Definition 3.1. Let $f \in L^p(X)$ we say that $f$ is closable if the linear map

$$T_f : \phi \in C_0(X) \mapsto f\phi \in L^p(X)$$

is closable as a densely defined linear map in $L^p(X)$. As is known [9, §36], $T_f$ is closable if, and only if, it has an adjoint $T'_f$ whose domain $D(T'_f)$ is weakly dense in $L^{p'}(X)$. It is easy to see that

$$D(T'_f) = \{ g \in L^{p'}(X) \mid f^*g \in L^p(X) \}.$$  \hfill (10)

For $p > 1$, this set is clearly weakly dense in $L^{p'}$, since it contains $C_0(X)$. Thus we have partially proved

**Proposition 3.2.** Let $1 < p < \infty$. Every $f \in L^p(X)$ is closable. If $\mu(X) < \infty$ then the statement holds also for $p = 1$.

**Proof.** It remains to prove the case $p = 1$, under the assumption $\mu(X) < \infty$. This can be shown by an argument similar to that used in the first part of the proof of Lemma 2.1. \hfill \Box

For each element $g$ in the domain $D(T_f)$ of the closure $\overline{T_f}$ we can then define as in [5] a strong multiplication \bullet setting

$$f \bullet g = T_f g.$$

But, taking into account that $\overline{T_f}$ coincides with the double-adjoint of $T_f$, in analogy to (10), one has, for $p > 1$

$$D(\overline{T_f}) = \{ g \in L^p(X) \mid fg \in L^p(X) \}.$$  \hfill (11)

If we denote with $\Gamma_s$ the set of pairs $(f,g) \in L^p(X) \times L^p(X)$ for which $f \bullet g$ is well-defined, from the above discussion it follows that, for $p \geq 2$, $\Gamma_s = \Gamma_w$. An interesting consequence of this fact is the following

**Corollary 3.3.** Let $f, g \in L^p(X)$, $1 < p < \infty$. The product $fg$ is in $L^p(X)$ if, and only if, there exists a sequence $g_n \in C_0(X)$ such that $g_n \overset{p}{\to} g$ and the sequence $fg_n$ converges in $L^p(X)$.

We end this paper by remarking that the problem of refining the multiplication in a quasi *-algebra arise naturally from very simple examples (think of two step functions in $(L^p(X, \mu), C_0(X))$). However, this problem has not always a positive answer. The relevant point here is that this is always possible, in non trivial way, for *-semisimple CQ*-algebras [5].
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