A BLOWUP FORMULA OF HIGH GENUS GROMOV-WITTEN INVARIANTS IN DIMENSION SIX

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ABSTRACT. Using the degeneration formula and localization technique, one studied the change of high genus Gromov-Witten invariants under the blowup for six dimensional symplectic manifolds and obtained a close blow-up formula for any genus Gromov-Witten invariants.

Key words: Gromov-Witten invariant, Blow-up, Degeneration formula, Localization

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1. INTRODUCTION

Gromov-Witten invariants count stable pseudo-holomorphic curves in a symplectic manifold. The Gromov-Witten invariants for semi-positive symplectic manifolds were first defined by Ruan [R1] and Ruan-Tian [RT1, RT2]. Gromov-Witten invariants can be applied to define a quantum product on the cohomology groups of a symplectic manifold in [RT1] and have many applications in symplectic geometry and symplectic topology, see [MS] and reference therein. Using the virtual moduli cycle technique, Li-Tian [LT1] defined the Gromov-Witten invariants purely algebraically for smooth projective varieties. During last two decades, there were a great deal of activities to remove the semi-positivity condition, see [B, FO, R2, S, LT2]. After its mathematical foundation was established, the study

Date: February 19, 2014.

Partially supported by NSFC Grant 11228101 and 11371381.
of Gromov-Witten theory focused on its computation and applications. We now know a lot about genus zero invariants of, say, toric manifolds, homogeneous spaces, etc. Some of the higher genus computations have also been done, but the understanding of higher genus Gromov-Witten invariants is still far from complete.

The computation of the Gromov-Witten invariants is known to be a difficult problem in geometry and physics. There are two major techniques: the degeneration formula and localization. Li-Ruan [LR] first obtained the degeneration formula, see [IP] for a different version and [Li] for an algebraic version. It used to be applied to the situations that a symplectic or Kahler manifold $X$ degenerates into a union of two pieces $X^\pm$ glued along a common divisor $Z$. The idea of degeneration formula is to express the Gromov-Witten invariants of $X$ in terms of relative Gromov-Witten invariants of the pairs $(X^\pm, Z)$. Localization played a very important role in the computation of Gromov-Witten invariants. Kontsevich [Ko2] first introduced this technique into this field, then Givental [Gi] and Lian-Liu-Yau [LLY] applied this technique to prove the mirror theorem in the genus zero case. So far the computation of high genus invariants is still a difficult task. The difficulty is that the localization technique often transfers the computation of high genus invariants into that of some Hodge integrals over $\overline{\mathcal{M}}_{g,n}$, which so far one does not have effective methods to compute. To obtain some general structures or close formulae of Gromov-Witten theory in many applications, we degenerate a symplectic or Kahler manifold into two toric relative pairs $(X^\pm, Z)$ and then use the localization technique to compute the associated relative invariants, see [HLR, MP]. The combination of the degeneration technique and localization technique has proven to be very powerful.

Ruan [R3] speculated that there should be a deep relation between quantum cohomology and birational geometry. The birational symplectic geometry program requires a thorough understanding of blow-up type formula of Gromov-Witten invariants and quantum cohomology because blow-up is the elementary birational surgery. Actually, it is rare to be able to obtain a general blow-up formula. For the last twenty years, only a few limited case were known, see [H1, H2, G]. Hu-Li-Ruan [HLR] studied the change of Gromov-Witten invariants under blow-up and obtained a blow-up correspondence of absolute/relative Gromov-Witten invariants. The named second author [H1, H2] obtained some blow-up formulae for genus zero Gromov-Witten invariants. In this paper, we try to apply the degeneration formula and localization technique to study the change of Gromov-Witten invariants under blowup and generalize the genus zero formula in [H1] to any genus case in six dimension.

Throughout this paper, let $X$ be a compact symplectic manifold of dimension six, $\tilde{X}$ be the blow-up of $X$ at a smooth point with exceptional divisor $E$. Let $e$ be the class of a line in $E$. Denote by $p : \tilde{X} \rightarrow X$ the natural projection. Let $A \in H_2(X)$, $\alpha_i \in H^*(X), (1 \leq i \leq m)$, and non-negative integers $d_i, 1 \leq i \leq m$. Denote by $\langle \tau_{d_1} \alpha_1, \cdots, \tau_{d_m} \alpha_m \rangle_{g,A}^X$ the descendant Gromov-Witten invariant of $X$ with the genus $g$ and the degree $A$. From the point of view of geometry, we could express the condition of counting curves with homology class $A$ passing through a generic point in $X$ in two ways: adding a point class, or blowing up a generic smooth
point and counting curves in \( \tilde{X} \) with \( p'(A) - e \). In genus zero case, the second author [H1] has proved that these two methods give the same GW invariant. The fourth author [Q] proved that the same result holds for any genus Gromov-Witten invariants in dimension 4. In this paper, we use degeneration formula to express the absolute invariants of \( X \) or \( \tilde{X} \) in terms of the relative invariants of \( (\tilde{X}, E) \). In these expressions, we found that the coefficients only depends on some special relative invariants of \( P^3 \) and \( \mathbb{P}^3 \). Then we use the localization technique to compute the associated invariants. Summarizing these result together, we get the following blow-up type formula:

**Theorem 1.1.** Suppose \( \alpha_i \in H^*(X, \mathbb{R}) \) (1 ≤ i ≤ m) with \( \deg \alpha_i > 0 \). Then for \( A \in H_2(X, \mathbb{Z}) \), \( g \geq 0 \) and non-negative integers \( d_i, 1 \leq i \leq m \), we have

\[
\langle \tau_{d_1} \alpha_1, \cdots, \tau_{d_m} \alpha_m, [pt] \rangle_{\tilde{X}}^{X} = \sum_{g_1 + g_2 = g} \frac{(-1)^{g_1} \cdot 2}{(2g_1 + 2)!} \langle \tau_{d_1} p^* \alpha_1, \cdots, \tau_{d_m} p^* \alpha_m \rangle_{\tilde{X}, p'(A) - e},
\]

where \( p : \tilde{X} \rightarrow X \) is the natural projection of the blow-up.

In particular, in the case of \( d_1 = \cdots = d_m = 0 \), if \( m + 2g \geq 3 \) or \( A \neq 0, m \geq 1 \), then the condition \( \deg \alpha_i > 0 \) can be removed.

It is illuminating to rephrase this using a genus \( g \) gravitational Gromov-Witten generating function. Suppose that \( T_0 = 1, T_1, \cdots, T_m \) is a basis for \( H^*(X, \mathbb{Q}) \). We introduce supercommuting variables \( t_d^i \) for \( d \geq 0 \) and \( 0 \leq j \leq m \) with \( \deg t_d^j = \deg T_j \). Set

\[
\gamma = \sum_{d=0}^{\infty} \sum_{j=1}^{m} t_d^j \tau_d T_j.
\]

Set

\[
F^X(u, t_d^i) = \sum_{g \geq 0} \sum_{n=0}^{\infty} \sum_{A \in H_2(X, \mathbb{Z})} \frac{u^{2g-2}}{n!} \langle \gamma^n, [pt] \rangle_{\tilde{X}}^{X} A
\]

and

\[
F^{\tilde{X}}(u, t_d^i) = \sum_{g \geq 0} \sum_{n=0}^{\infty} \sum_{A \in H_2(X, \mathbb{Z})} \frac{u^{2g-2}}{n!} \langle (p^*)^n \gamma^n \rangle_{\tilde{X}, p'(A) - e} A q^{p'(A) - e}.
\]

Then we have

\[
F^X(u, t_d^i) = \left( \frac{\sin \frac{u}{2}}{\frac{u}{2}} \right)^2 \cdot F^{\tilde{X}}(u, t_d^i),
\]

here we need change the variable \( q^A \) to \( q^{p'(A) - e} \).

### 2. Preliminaries

#### 2.1. Gromov-Witten invariant and its degeneration formula.

We use [LR] as our general reference on moduli spaces of stable maps, absolute/relative Gromov-Witten invariants and its degeneration formula.

Let \( X \) be a compact symplectic manifold and \( A \in H_2(X, \mathbb{Z}) \). Let \( \overline{M}_{g,k}(X, A) \) be the moduli space of \( n \)-pointed stable maps \( f : (\Sigma; x_1, \cdots, x_k) \rightarrow X \) from a nodal curve \( \Sigma \) with arithmetic genus \( g(\Sigma) = g \) and degree \( \{f(\Sigma)\} = A \). Let \( \mathcal{L} \) denote
the “cotangent line at the $i$th marked point $x_i$”, i.e., the line bundle over $\overline{\mathcal{M}}_{g,n}(X,A)$ whose fiber over the stable map $f : ([\Sigma; x_1, \ldots, x_n]) \to X$ is the cotangent space $T^*_x \Sigma$. Denote by $\psi_i$ the first Chern class $c_1(L_i)$ of $L_i$. Let $e_i : \overline{\mathcal{M}}_{g,k}(X,A) \to X$ be the evaluation maps $f \mapsto (f(x_i))$. The Gromov-Witten invariant for classes $\alpha_i \in H^*(X), 1 \leq i \leq k$ and non-negative integers $d_i, 1 \leq i \leq n$, is defined by

$$
\langle \tau_{d_1} \alpha_1, \ldots, \tau_{d_k} \alpha_k \rangle_{X,Z}^{\mathcal{G}} := \int_{\overline{\mathcal{M}}_{g,k}(X,A)} \psi_i^{d_i} e_i^* \alpha_1 \wedge \cdots \wedge e_i^* \alpha_k.
$$

The degeneration formula [LR, IP, Li] provides a rigorous formulation about the change of Gromov-Witten invariants under semi-stable degeneration, or symplectic cutting. The formula relates the absolute Gromov-Witten invariant of $X$ to the relative Gromov-Witten invariant of two smooth pairs.

Now we recall the relative invariants of a smooth relative pair $(X,Z)$ with $Z \subset X$ a smooth symplectic divisor. Let $A \in H_2(X,Z)$ and $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)}) \in \mathbb{N}^{\ell(\mu)}$ be a partition of $|\mu| := \sum_{i=1}^{\ell(\mu)} \mu_i = A \cdot Z$. We customarily used the relative graph to describe the topological type of relative stable map. A relative graph $\Gamma = (g,k,A,\mu)$ is defined to be a (connected) relative graph consisting of the following data:

1. a vertex decorated by $A \in H_2(X,Z)$ and genus $g$;
2. a tail for each absolute marked point;
3. a relative tail with the weight $\mu_i$ for each relative marked point.

Let $\overline{\mathcal{M}}_{\Gamma}(X,Z)$ be the moduli space of relative stable maps with the topological type $\Gamma$. Then it is well-known that this moduli space has virtual dimension $\text{vdim}_{\overline{\mathcal{M}}_{\Gamma}(X,Z)} = c_1(A) + k + \ell(\mu) - |\mu|$.

For $\alpha_i \in H^*(X), 1 \leq i \leq k$, $\delta_\mu \in H^*(Z)^{\otimes \ell(\mu)}$, and non-negative integers $d_i, 1 \leq i \leq k$, the relative invariant of stable maps with topological type $\Gamma$ (i.e. with contact order $\mu_i$ in $Z$ at the $i$-th relative point) is defined as

$$
\langle \tau_{d_1} \alpha_1, \ldots, \tau_{d_k} \alpha_k | \delta_\mu \rangle_{\overline{\mathcal{M}}_{\Gamma}(X,Z)} := \int_{\overline{\mathcal{M}}_{\Gamma}(X,Z)} \psi_i^{d_i} ev_i^* \alpha_1 \cup e_Z^* \delta_\mu
$$

where $ev_i : \overline{\mathcal{M}}_{\Gamma}(X,Z) \to X, e_Z : \overline{\mathcal{M}}_{\Gamma}(X,Z) \to Z^{\ell(\mu)}$ are evaluation maps on the $i$th absolute marked points and relative marked points respectively.

If $\Gamma = \bigsqcup x \Gamma^x$, the relative invariants (with disconnected domain curves)

$$
\langle \tau_{d_1} \alpha_1, \ldots, \tau_{d_k} \alpha_k | \delta_\mu \rangle_{\overline{\mathcal{M}}_{\Gamma}(X,Z)} := \prod_{x} \langle \tau_{d_1} \alpha_1, \ldots, \tau_{d_k} \alpha_k | \delta_\mu \rangle_{\overline{\mathcal{M}}_{\Gamma^x}(X,Z)}
$$

is defined to be the product of each connected component.

In the following, we shall discuss the degeneration formula which is one of the main tools employed in this paper.

Let $\pi : X \to D$ be a smooth symplectic manifold of dimension $2n + 2$ over a disk $D$ such that $\chi_t = \pi^{-1}(t) \cong X$ for $t \neq 0$ and $\chi_0$ is a union of two symplectic manifolds $X_1$ and $X_2$ intersecting transversely along a symplectic divisor $Z$. We write $\chi_0 = X_1 \cup_Z X_2$. Assume that $Z$ is simply connected.

Consider the natural maps

$$
i_t : X = \chi_t \to X, \quad i_0 : \chi_0 \to X,$$
and the gluing map

$$g = (j_1, j_2) : X_1 \bigsqcup X_2 \longrightarrow \chi_0.$$  

We have

$$H_2(X) \overset{i_*}{\longrightarrow} H_2(\chi) \overset{i_{0*}}{\leftarrow} H_2(\chi_0) \overset{g_*}{\leftarrow} H_2(X_1) \oplus H_2(X_2),$$

where $i_{0*}$ is an isomorphism since there exists a deformation retract from $\chi$ to $\chi_0$ (see [Li]) and $g_*$ is surjective from Mayer-Vietoris sequence. For $A \in H_2(X)$, there exist $A_1 \in H_2(X_1)$ and $A_2 \in H_2(X_2)$ such that

$$i_{0*}(A) = i_{0*}(j_1(A_1) + j_2(A_2)).$$

For simplicity, we write $A = A_1 + A_2$ instead.

Since the family $\chi \longrightarrow D$ comes from a trivial family, all cohomology classes $\alpha_i \in H^*(X)$ have global liftings and the restriction $\alpha_i(t)$ on $X_t$ is defined for all $t$.

For $\{d_i\}$ a basis of $H^*(D)$, with $\{d^i\}$ its dual basis and a partition $\mu$, denote $\delta_\mu = \delta_{d_1} \cdot \cdots \cdot \delta_{d_k}$ and its dual $\delta^\mu = \delta^{d_1} \cdot \cdots \cdot \delta^{d_k}$. The degeneration formula expresses the absolute invariants of $\chi$ in terms of the relative invariants of the two smooth pairs $(X_1, Z)$ and $(X_2, Z)$:

$$\langle \tau_{d_1} \alpha_1, \cdots, \tau_{d_k} \alpha_k \rangle^X_{\mu, \Gamma_i} = \sum_{\eta \in \Omega_\lambda} \sum_{\mu \in \Omega_\lambda} C_{\eta} \langle \tau_{d_1} j_1^* \alpha^+_1(0), \cdots, \tau_{d_k} j_1^* \alpha^+_k(0) | \delta^\mu \rangle^*_{\Gamma_i} 
\times \langle \tau_{d_1} j_2^* \alpha^+_1(0), \cdots, \tau_{d_k} j_2^* \alpha^+_k(0) | \delta^\mu \rangle^*_{\Gamma_2}.$$

Here $\eta = (\Gamma_1, \Gamma_2, I_{(\mu)})$, is an admissible triple which consists of (possibly disconnected) topological types

$$\Gamma_i = \bigsqcup_{\pi=1}^{\ell(\mu)} \Gamma^\pi_i$$

with the same contact order partition $\mu$ under the identification $I_\mu$ of relative marked points. The gluing $\Gamma_1 + I_{(\mu)} \Gamma_2$ has type $(g, n, A)$ and is connected. In particular, $\ell(\mu) = 0$ if and only if that one of the $\Gamma_i$ is empty. The total genus $g_\ell$, total number $n_\ell$ of absolute marked points and the total degree $A_\ell$ of $H_2(X_\ell)$ satisfy the splitting relations $g = g_1 + g_2 + \ell(\mu) + 1 - |\Gamma_1| - |\Gamma_2|$, $k_1 + k_2 = k$ and $A_1 + A_2 = A$.

The constants $C_{\eta} = m(\mu)|\text{Aut}_\eta|$, where $m(\mu) = \prod_{i} \mu_i$ and $\text{Aut}_\eta = \{\sigma \in S_{(\mu)} | \eta^\sigma = \eta\}$. We denote by $\Omega$ the equivalence class of all admissible triples, also by $\Omega_A$ and $\Omega_\mu$ the subset with fixed degree $A$ and fixed contact order $\mu$ respectively.

For the dimensions of the related moduli spaces in the degeneration formula, we have

**Lemma 2.1.** (Theorem 5.1 in [LR]) With the assumption as above,

$$\dim_\mathbb{C} \overline{\mathcal{M}}_{\Gamma_1} + \dim_\mathbb{C} \overline{\mathcal{M}}_{\Gamma_2} = \dim_\mathbb{C} \overline{\mathcal{M}}_{\Gamma} + 2\ell(\mu).$$

**Remark 2.2.** Symplectic cutting is a kind of surgery in symplectic geometry which is suitable for the above degeneration formula, see [LR]. Suppose that $X_0 \subset X$ is an open codimension zero submanifold with Hamiltonian $S^1$-action. Let $H : X_0 \longrightarrow \mathbb{R}$ be a Hamiltonian function with 0 as a regular value. If $H^{-1}(0)$ is a
separating hypersurface of $X_0$, then we obtain two connected manifolds $X_0^\pm$ with boundary $\partial X_0^\pm = H^{-1}(0)$, where the $+$ side corresponds to $H < 0$. Suppose further that $S^1$ acts freely on $H^{-1}(0)$. Then the symplectic reduction $Z = H^{-1}(0)/S^1$ is canonically a symplectic manifold. Collapsing the $S^1$-action on $\partial X^\pm = H^{-1}(0)$, we obtain two closed smooth manifolds $\bar{X}^\pm$ containing respectively real codimension 2 submanifolds $Z^\pm = Z$ with opposite normal bundles. Furthermore $\bar{X}^\pm$ admits a symplectic structure $\bar{\omega}^\pm$ which agrees with the restriction of $\omega$ away from $Z$, and whose restriction to $Z^\pm$ agrees with the canonical symplectic structure $\omega_Z$ on $Z$ from symplectic reduction. The pair of symplectic manifolds $(\bar{X}^\pm, \bar{\omega}^\pm)$ is called the symplectic cut of $X$ along $H^{-1}(0)$.

Suppose that $Y \subset X$ is a submanifold of $X$ of codimension $2k$. Denote by $N_Y$ the normal bundle. By the symplectic neighborhood theorem, and by possibly taking a smaller $\epsilon_0$, a tubular neighborhood $\mathscr{N}_0(Y)$ of $Y$ in $X$ is symplectomorphic to the disc bundle $N_Y(\epsilon_0)$ of $N_Y$. Denote by $\phi : \mathscr{N}_0(Y) \rightarrow N_Y(\epsilon_0)$ be such a symplectomorphism. Consider the Hamiltonian $S^1$-action on $X_0 = \mathscr{N}_0(Y)$ by complex multiplication. Fix $\epsilon$ with $0 < \epsilon < \epsilon_0$ and consider the moment map

$$H(u) = |\phi(u)|^2 - \epsilon, \quad u \in \mathcal{N}(\epsilon_0),$$

where $|\phi(u)|$ is the norm of $\phi(u)$ considered as a vector in a fiber of the Hermitian bundle $N_Y$. We cut $X$ along $H^{-1}(0)$ to obtain two closed symplectic manifolds $\bar{X}^\pm$. Notice that $\bar{X}^+ \cong \mathbb{P}_Y(N_Y \oplus \mathbb{C})$. $\bar{X}^-$ is called the blow-up of $X$ along $Y$, denoted by $\bar{X}$.

2.2. Virtual Localization. The classical localization theorem was formulated in the context of equivariant cohomology by Atiyah and Bott [AB]. Graber et al. generalized their result to the case of virtual fundamental class and applied it to compute (relative) GW invariants. Here we will recall notations and the (relative) virtual localization formula, and we refer readers to [GP] [GV] and the references therein for further details.

Let $X$ be a smooth projective variety with a $\mathbb{C}^*$-action. The moduli space $\bar{\mathcal{M}}_{g,k}(X, A)$ of stable maps admits a $\mathbb{C}^*$-action via acting on the target. There exists a $\mathbb{C}^*$-equivariant perfect obstruction theory on $\bar{\mathcal{M}}_{g,k}(X, A)$ that gives rise to a virtual fundamental class denoted by $[\bar{\mathcal{M}}_{g,k}(X, A)]^{vir}$. The fixed part of the perfect obstruction theory gives each $\mathbb{C}^*$-fixed locus $\mathcal{M}_i$ a virtual fundamental class and the non-fixed part of that gives a virtual normal bundle $\mathcal{N}_i$ to $\mathcal{M}_i$. The virtual localization formula ([GP]) is then:

$$[\bar{\mathcal{M}}_{g,k}(X, A)]^{vir} = t \sum_i \frac{[\mathcal{M}_i]^{vir}}{e^\mathbb{C}(\mathcal{N}^*_i)},$$

in $A^*_{\mathbb{C}}(\bar{\mathcal{M}}_{g,k}(X, A)) \otimes \mathbb{Q}[t, \frac{1}{t}]$ where $t$ is the generator of $H^*(BC^*)$.

Define $\mathcal{T}^1$ and $\mathcal{T}^2$ by the dual perfect obstruction theory on $\bar{\mathcal{M}}_{g,k}(X, A)$ (see [GP]). $\mathcal{T}^1$ is the tangent space of the moduli space and $\mathcal{T}^2$ is the obstruction
space. There is a tangent-obstruction exact sequence of sheaves:

\[
0 \rightarrow \text{Ext}^0(\Omega_C(D),\mathcal{O}_C) \rightarrow H^0(C,f^*TX) \rightarrow \mathcal{T}^1
\]

(4) \quad \text{Ext}^1(\Omega_C(D),\mathcal{O}_C) \rightarrow H^1(C,f^*TX) \rightarrow \mathcal{T}^2 \rightarrow 0,

where \(D\) represents the marked point divisor on \(C\). The four terms other than \(\mathcal{T}^1\) and \(\mathcal{T}^2\) form vector bundles as fibers. Restricting to the fixed locus \(\mathcal{M}_i\), each vector bundle decomposes as the direct sum of the fixed part and the moving part. This exact sequence implies:

\[
e^{C_i}(\mathcal{N}_i) = \frac{e(B_2^m)e(B_4^m)}{e(B_1^m)e(B_5^m)},
\]

where \(B_i^m\) denotes the moving part of the \(i\)-th term in (4).

In applications, we customarily use the dual graph \(\Gamma\) of the fixed stable maps to label the fixed locus \(\mathcal{M}_i\), denoted by \(\mathcal{M}_\Gamma\). If the \(C^i\)-action on \(X\) has isolated fixed points and isolated \(C^i\)-invariant curves, then \(\mathcal{M}_\Gamma\) may be expressed as a product of moduli spaces of curves. Therefore, when we integrate over both sides of (3), the RHS will reduce to Hodge integrals over the moduli spaces of curves. Graber-Vakil proved the localization formula in the case of relative Gromov-Witten invariants, see Theorem 3.6 of [GV].

2.3. Hodge integral. Let \(\mathcal{M}_{g,k}\) be the Deligne-Mumford moduli stack of genus \(g\), \(k\)-pointed stable curves, and denote an element of it by \([C; p_1, \ldots, p_k]\). Let \(\pi_k: \mathcal{M}_{g,k+1} \rightarrow \mathcal{M}_{g,k}\) be the universal family, which can be obtained by forgetting the \((k+1)\)-st marked point. Let \(\omega_C\) be the relative canonical sheaf. The Hodge bundle \(E \rightarrow \mathcal{M}_{g,k}\) is the rank \(g\) vector bundle \(E = \pi_*\omega_C\) with fiber \(H^0(C,\omega_C)\) over \([C; p_1, \ldots, p_k]\). Let \(\lambda_i = c_1(E) \in H^*(\mathcal{M}_{g,k},\mathbb{Q})\).

For each marked point, there is a natural section \(s_i: \mathcal{M}_{g,k} \rightarrow \mathcal{M}_{g,k+1}\), the image of a stable \(k\)-pointed curve under which is the stable \((k+1)\)-pointed curve obtained by attaching a 3-pointed rational curve at the \(i\)-th point and considering the remaining 2 points on that curve as the \(i\)th and \((k+1)\)st point. Let \(L_j = s_j^*\omega_C\) be the line bundle over \(\mathcal{M}_{g,k}\) with fiber \(T^{*p_j}C\) over the moduli point \([C; p_1, \ldots, p_k]\). Let \(\psi_j = c_1(L_j) \in H^*(\mathcal{M}_{g,k},\mathbb{Q})\).

A Hodge integral over \(\mathcal{M}_{g,k}\) is defined to be an integral of this form:

\[
\int_{\mathcal{M}_{g,k}} \psi_1^{\lambda_1} \ldots \psi_k^{\lambda_k} \lambda_1 \ldots \lambda_g.
\]

Hodge integrals naturally arise in the computation of Gromov-Witten invariants by virtual localization technique [GP, Ko2]. However, it is difficult to evaluate these integrals. In general, the calculation of Hodge integrals can be reduced to Hodge integrals involving only \(\psi\)-classes [F, FP], which can be computed recursively by the famous Witten-Kontsevich theorem [Ko1, W]. An important approach to obtain closed formulae for Hodge integrals is via virtual localization on suitable target varieties [FP].
In the proof of our technical lemmas, we need a result by G. Tian and J. Zhou about Hodge integrals [TZ], which was obtained by virtual localization on $\mathbb{P}^1$, as suggested in [FP].

**Lemma 2.3.** (Theorem 5.1 in [TZ]) Set $\Lambda^\vee_g(\alpha) := \sum_{j=0}^g (-1)^j \lambda_j \alpha^{g-j}$. Then

$$\sum_{g=0}^\infty \mu^{2g} \int_{\mathcal{M}_{g,1}} \frac{\Lambda^\vee_g(1) \Lambda^\vee_g(\alpha) \Lambda^\vee_g(\beta)}{1 - \psi_1} = \left( \frac{\sin \frac{\pi}{g}}{g} \right)^{\alpha+\beta}$$

**Remark 2.4.** In the above lemma, we have used the following convention to deal with the unstable case:

$$\int_{\mathcal{M}_{0,m}} \frac{1}{\alpha - \psi_1} = \alpha^{2-m}, m = 1, 2, 3, \ldots$$

**Remark 2.5.** Taking $\alpha = -k, \beta = -1$ in the lemma gives Theorem 2 in [FP].

### 3. A Comparison Theorem

Hu-Li-Ruan [HLR] obtained a blow-up correspondence between absolute and relative Gromov-Witten invariants. Their correspondence partially describe the change of Gromov-Witten invariant under blow-up. In this section, we first apply their method to obtain some comparison results between absolute and relative Gromov-Witten invariants, then use these comparison theorems and some results from localization to prove our main theorem.

**Proposition 3.1.** Let $X$ be a compact symplectic manifold of dimension 6. Suppose that $A \in H_2(X, \mathbb{Z})$, $\alpha_i \in H^*(X)$ ($1 \leq i \leq m$) with $\deg \alpha_i > 0$. Then for any genus $g \geq 0$ and non-negative integers $d_i$, $1 \leq i \leq m$, we have

$$\langle \tau_{d_1} \alpha_1, \ldots, \tau_{d_m} \alpha_m, [pt]\rangle^X_{g, A}$$

$$= \sum_{g^+ + g^- = g} \langle [pt] \rangle_{g^+ + g^-}^{\mathbb{P}^3, H} \langle \tau_{d_1} \alpha_1, \ldots, \tau_{d_m} \alpha_m | \mathbb{1} \rangle_{g^+ + g^-}^{X, E}$$

where $H \equiv \mathbb{P}^2$ is the hyperplane of $\mathbb{P}^3$, $L$ is the generator of $H_2(\mathbb{P}^3, \mathbb{Z})$ with $H \cdot L = 1$, and $\beta = 1^\vee \in H^4(\mathbb{P}^2; \mathbb{R})$ satisfying $\int_{\mathbb{P}^2} \beta = 1$.

**Proof:** We perform symplectic cutting along a point as in Remark 2.2. Without lose of generality, we may assume that the class $[pt]$ has support in $X^+$ and $\alpha_i$ has support in $X^-$. By the degeneration formula (1), we obtain the decomposition:

$$\langle \tau_{d_1} \alpha_1, \ldots, \tau_{d_m} \alpha_m, [pt]\rangle^X_{g, A} = \sum \Delta(\mathcal{B}) \langle [pt] \rangle_{\beta_{\mu_1}, \ldots, \beta_{\mu_{\ell(\mu)}}}^{\mathbb{P}^3, H}$$

$$\times \langle \tau_{d_1} \alpha_1, \ldots, \tau_{d_m} \alpha_m | \mathbb{1} \rangle_{g^+ + g^-}^{X, E}$$

where the summation runs over all the splitting of $(g, A)$, all partitions $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})$ of $A^+ \cdot Z^+ = A^- \cdot Z^-$ and all choices of the weighted classes $\beta_j$. Here $X^\vee \equiv \check{X}$, and $Z^\vee \equiv \mathbb{P}^2$ is the exceptional divisor $E$ in $\check{X}$. $\check{X}^\vee \equiv \mathbb{P}^3$, and $\check{Z}^\vee \equiv \mathbb{P}^2$ is the hyperplane of $\mathbb{P}^3$, and we will denote it by $H$. 


Let $\Gamma^+ = (g^+, 1, A^+, \mu)$ be a relative graph and $[u^+: \Sigma^+ \to \bar{X}^+ = \mathbb{P}^3]$ an element of $\mathcal{M}_{\Gamma^+}(\mathbb{P}^3, H)$. Then $u^+((\Sigma^+)) = A^+ \in H_2(\mathbb{P}^3; \mathbb{Z})$. Let $L$ be the generator of $H_2(\mathbb{P}^3; \mathbb{Z})$ with $L \cdot H = 1$. Let $A^+ = aL$, $a \in \mathbb{Z}$. We have $a = A^+ \cdot H = \Sigma \mu$, so $A^+ = \Sigma \mu L$. Hence we have the dimensions of these three moduli spaces corresponding to the three invariants in (8) as follows

$$
dim_{C, \mathcal{M}_{\Gamma^+}}(\mathbb{P}^3, H) = 4H \cdot \sum \mu_i L + 1 + \ell(\mu) - \sum \mu_i$$

$$= 3 \sum \mu_i + \ell(\mu) + 1,$$

$$
dim_{C, \mathcal{M}_{g,m+1}(X, A)} = c_1^X(A) + m + 1 = \frac{1}{2} \sum_1 \deg \alpha_i + \sum d_i + 3.
$$

By the formula (2),

$$
dim_{C, \mathcal{M}_{\Gamma^+}}(\bar{X}, E) = c_1^X(A) + 3(\ell(\mu) - \sum \mu_i) + m - 2\ell(\mu).
$$

The invariant $\langle \tau_d, \alpha_1, \cdots, \tau_m, \alpha_m | \beta_1, \cdots, \beta_m \rangle_{g, A^+, \mu}^{X, E}$ in (8) might be non-zero only if

$$
dim_{C, \mathcal{M}_{\Gamma^+}}(\bar{X}, E) - \frac{1}{2} \sum_1 \deg \alpha_i - \sum d_i = 3(\ell(\mu) - \sum \mu_i) + 2(1 - \ell(\mu)) \geq 0.
$$

Since $\deg \alpha_i > 0$ and by the connectedness of the stable maps to $X$, we see that $\mu \neq 0$. So the inequality holds only if the equality holds. This implies $\mu = (1)$. By the definition of relative Gromov-Witten invariant, we have $\beta^i = 1 \in H^0(\mathbb{P}^2; \mathbb{R})$. So $\Delta(\mathcal{F}_x) = 1$. Let $A^+ = p'(A)+e$. Since $A^+ \cdot E = \Sigma \mu_i = 1$ and $(p'(A)+e) \cdot E = -r$, we obtain that $r = -1$. Thus $A^+ = p'(A) - e$. This proves (7).

$\square$

**Proposition 3.2.** Let $X$ be a compact symplectic manifold of dimension 6, $\bar{X}$ be its blow-up at a smooth point. Suppose that $A \in H_2(X, \mathbb{Z})$, $\alpha_i \in H^*(X)$ ($1 \leq i \leq m$) with $\deg \alpha_i > 0$. Then for any genus $g \geq 0$ and non-negative integers $d_i$, $1 \leq i \leq m$, we have

$$
\langle \tau_d, p^* \alpha_1, \cdots, \tau_m p^* \alpha_m | \bar{X}, \mu \rangle_{g, p'(A)-e}^{\bar{X}} = \sum_{g^*+g^-=g} \langle p|_{g^*, F(1)}^{\mathbb{P}^3}, H \rangle_{g^*+g^-=g} \times \langle \tau_d, p^* \alpha_1, \cdots, \tau_m p^* \alpha_m | \bar{X}, \mu \rangle_{g^*+g^-=g, p'(A)-e(1)}^{\bar{X}, E},
$$

where $p : \bar{X} \to X$ is the natural projection of the blow-up, $H$ and $F$ are the hyperplane at infinity and the fiber class of $\mathbb{P}^3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}_E(-1))$, and $\beta = 1^\vee \in H^1(H; \mathbb{R})$ satisfying $\int_H \beta = 1$.

**Proof:** Performing the symplectic cutting along the exceptional divisor $E$ as in Remark 2.2, observe that $\bar{X}^- = \bar{X}$ and $\bar{X}^+ = \mathbb{P}(\mathcal{O} \oplus \mathcal{N}_{E|\bar{X}}) \cong \mathbb{P}^3$. It is easy to see that $Z^+ = H$ is the hyperplane at infinity of $\bar{X}^+ = \mathbb{P}^3$, $Z^- = E$ is the exceptional divisor in $\bar{X}^- \cong \bar{X}$. From the assumption of the theorem, we may assume that all cohomology classes $p^* \alpha_i$ ($1 \leq i \leq m$) support away from the exceptional divisor.
Therefore, by the degeneration formula (1), we obtain the decomposition:

\[
\langle \tau_{d_i} p^* \alpha_1 \cdots \tau_{d_m} p^* \alpha_m |_{\tilde{\mathcal{O}}_g^+, p^!(A) - e} \rangle = \sum \Delta(\mathcal{F}_i) \langle \beta_{i_1} \cdots \beta_{i_{\ell(i)}} |_{g^+, (p^!(A) - e)^+, \nu} \rangle \times \langle \tau_{d_i} p^* \alpha_1 \cdots \tau_{d_m} p^* \alpha_m \beta_1^\vee \cdots \beta_{i_{\ell(i)}}^\vee |_{g^+, (p^!(A) - e)^-, \nu} \rangle.
\]

(10)

Let \( \Gamma = \{ g^+, 0, (p^!(A) - e)^+, \nu \} \) and \( [h : \Sigma^+ \to \tilde{X}^+ = \mathbb{P}^3] \) be an element in \( \tilde{\mathcal{M}}_{\Gamma^+}(\mathbb{P}^3, H) \). Let \( (p^!(A) - e)^+ = aF + be \), where \( F \) is the class of a fiber of \( \mathcal{O}^g \oplus \mathcal{N}_{E|X} \to E \), which is the strict transform of a line in \( \mathbb{P}^3 \) passing through the blow-up point \( x \) with \( F \cdot E = 1 \) and \( F \cdot H = 1 \). Since \( e \cdot E = -1, e \cdot H = 0 \), we get

\[
\begin{align*}
\left\{ & a = (aF + be) \cdot H = (p^!(A) - e)^+ \cdot H = \sum \nu_i \\
& a - b = (aF + be) \cdot E = (p^!(A) - e)^+ \cdot E = 1.
\end{align*}
\]

Therefore we have \( (p^!(A) - e)^+ = \sum \nu_i F + (\sum \nu_i - 1)e \). Moreover,

\[
dim_{\mathbb{C}} \tilde{\mathcal{M}}_{\Gamma^+}(\mathbb{P}^3, H) = (4H - 2E) \cdot (\sum \nu_i F + (\sum \nu_i - 1)e) + \ell(\nu) - \sum \nu_i = 3 \sum \nu_i + \ell(\nu) - 2,
\]

\[
dim_{\mathbb{C}} \tilde{\mathcal{M}}_{\Gamma, m}(\tilde{X}, p^!(A) - e) = c_1^X(A) + m - 2 = \frac{1}{2} \sum_i \deg \alpha_i + \sum_i d_i.
\]

By the dimension relation (2), we have

\[
dim_{\mathbb{C}} \tilde{\mathcal{M}}_{\Gamma^-}(\tilde{X}^-, E) = \frac{1}{2} \sum_i \deg \alpha_i + \sum_i d_i + \ell(\nu) - 3 \sum \nu_i + 2,
\]

where \( \Gamma^- = \{ g^-, m, (p^!(A) - e)^-, \nu \} \). Since the image of a stable map representing homology class \( p^!(A) - e \) must intersect the exceptional divisor \( E \), and by the connectedness of stable maps to \( \tilde{X} \) and the assumption of the theorem, we have \( \nu \neq 0 \). By the definition, the relative invariant \( \langle \tau_{d_i} p^* \alpha_1 \cdots \tau_{d_m} p^* \alpha_m | \beta_1^\vee \cdots \beta_{i_{\ell(i)}}^\vee \rangle_{g^+, (p^!(A) - e)^-, \nu} \) in (10) is nonzero only if

\[
dim_{\mathbb{C}} \tilde{\mathcal{M}}_{\Gamma^-}(\tilde{X}^-, E) = \frac{1}{2} \sum_i \deg \alpha_i - \sum_i d_i = 2(1 - \ell(\nu)) + 3(\ell(\nu) - \sum \nu_i) \geq 0.
\]

This implies that \( \nu = (1) \), thus \( \beta \in H^4(\mathbb{P}^2; \mathbb{R}) \) is the volume form and \( \beta^\vee = 1 \). Furthermore, \( (p^!(A) - e)^+ = F \) and \( (p^!(A) - e)^- = p^!(A) - e \). This proves (9).

\[\square\]

**Remark 3.3.** In the genus zero case, using a geometric argument, the second named author [H1] proved that

\[
\langle [p^!]|\beta\rangle_{0,L_1}^{\mathbb{P}^3,H} = \langle \beta\rangle_{0,F,1}^{\mathbb{P}^3,H} = 1.
\]
**Theorem 3.4.** Under the same assumption and notations as Propositions 3.1 and 3.2. Denote $\langle \tau_{d_1} \alpha_1, \cdots, \tau_{d_m} \alpha_m \rangle_{g,A}^X$ and $\langle \tau_{d_1} \psi \alpha_1, \cdots, \tau_{d_m} \psi \alpha_m \rangle_{g,p'(A)-e}^X$ by $H_g$ and $P_g$ respectively. Then

$$H_g = \sum_{g_1 + g_2 = g} C_{g_1} P_{g_2},$$

where $C_g$’s can be determined by relative invariants $\langle [pt] \beta \rangle_{g,L(1)}^{2g_1} H$ and $\langle [pt] \beta \rangle_{g,F(1)}^{2g_1} H$.

**Proof.** Denote $\langle \tau_{d_1} \psi \alpha_1, \cdots, \tau_{d_m} \psi \alpha_m \rangle_{g,p'(A)-e(1)} E$, $\langle [pt] \beta \rangle_{g,L(1)}^{2g_1} H$ and $\langle [pt] \beta \rangle_{g,F(1)}^{2g_1} H$ by $K_g$, $I_g$ and $J_g$ respectively. Then for $g \geq 0$, we may write (7) and (9) as

$$H_g = I_g K_0 + I_{g-1} K_1 + \cdots + I_0 K_g,$$

$$P_g = J_g K_0 + J_{g-1} K_1 + \cdots + J_0 K_g,$$

or in matrix form

$$
\begin{bmatrix}
H_0 \\
H_1 \\
\vdots \\
H_g \\
P_0 \\
P_1 \\
\vdots \\
P_g
\end{bmatrix}
= 
\begin{bmatrix}
I_0 \\
I_1 \\
\vdots \\
I_g \\
J_0 \\
J_1 \\
\vdots \\
J_g
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
K_0 \\
K_1 \\
\vdots \\
K_g
\end{bmatrix},
$$

From Remark 3.3, we have that $I_0 = J_0 = 1$. So both matrices with entries $I_g$ and $J_g$ are invertible, denoted by $A_g$ and $B_g$ respectively. Then

$$H_g = \sum_{g_1 + g_2 = g} C_{g_1} P_{g_2},$$

where $C_g$ is the lower triangular matrix

$$A_g B_g^{-1} =
\begin{bmatrix}
C_0 & 0 \\
C_1 & C_0 \\
\vdots & \vdots \\
C_g & C_{g-1} & \cdots & C_0
\end{bmatrix}.$$

Now we will compute $C_g$’s which appear in Theorem 3.4 to get a blow-up formula. From the proof of Theorem 3.4, we see that $C_g$ is determined by relative invariants $\langle [pt] \beta \rangle_{g,L(1)}^{2g_1} H$ and $\langle [pt] \beta \rangle_{g,F(1)}^{2g_1} H$. The crucial observation here is that $C_g$ is
Lemma 3.5. If $X = \mathbb{P}^3$, $m=1$, $\alpha_1 = [pt], A = L$ and get:

\[
\langle [pt], [pt] \rangle_{g,L}^{\mathbb{P}^3} = \sum_{g_1+g_2=g} C_{g_1} \cdot \langle [pt] \rangle_{g_2, p'(L)-e}^{\mathbb{P}^3},
\]

where $p: \mathbb{P}^3 \to \mathbb{P}^3$ is the natural projection of the blow-up.

The only thing we need to do is to first compute the absolute Gromov-Witten invariants $\langle [pt], [pt] \rangle_{g,L}^{\mathbb{P}^3}$ and $\langle [pt] \rangle_{g_2, p'(L)-e}^{\mathbb{P}^3}$. Then we can get $C_g$’s by solving the equation (11). By the localization technique, we have

**Lemma 3.6.**

\[
\langle [pt] \rangle_{g,p'(L)-e}^{\mathbb{P}^3} = \delta_{g,0}.
\]

Before we give the proof of these two lemmas in next section, we first solve the equation (11) and get

\[
C_g = \frac{(-1)^g \cdot 2}{(2g + 2)!}.
\]

Plugging $C_g$ into Theorem 3.4, we get the blow-up formula.

**Theorem 3.7.** Under the same assumption as Theorem 3.4. Set $H_g := \langle \tau_1, \alpha_1, \cdots, \tau_d, \alpha_m, [pt] \rangle_{g,A}^X$, $P_g := \langle \tau_1, p^* \alpha_1, \cdots, \tau_d, p^* \alpha_m \rangle_{g, p'(A)-e}^X$. Then

\[
H_g = \sum_{g_1+g_2=g} \frac{(-1)^g \cdot 2}{(2g_1 + 2)!} \cdot P_{g_2}.
\]

**Remark 3.8.** In the case of $d_1 = \cdots = d_m = 0$, by the fundamental class axiom of Gromov-Witten invariants, both sides of (12) are zero if $m + 2g \geq 3$ or $A \neq 0$. Therefore, Theorem 1.1 holds.

It is illuminating to rephrase this using a genus $g$ gravitational Gromov-Witten generating function. Suppose that $T_0 = 1, T_1, \cdots, T_m$ is a basis for $H^*(X, \mathbb{Q})$. We introduce supercommuting variables $t_d^j$ for $d \geq 0$ and $0 \leq j \leq m$ with $\text{deg } t_d^j = \text{deg } T_j$. Set

\[
\gamma = \sum_{d=0}^{\infty} \sum_{j=1}^{m} t_d^j T_j.
\]

Define the genus $g$ gravitational Gromov-Witten generating function as

\[
F_{g}^X(t_d^j) = \sum_{n=0}^{\infty} \sum_{A \in H_2(X, \mathbb{Z})} \frac{1}{n!} \langle \gamma^n, [pt] \rangle_{g,A}^X q^A,
\]

\[
F_{g}^X(t_d^j) = \sum_{n=0}^{\infty} \sum_{A \in H_2(X, \mathbb{Z})} \frac{1}{n!} \langle (p^* \gamma)^n \rangle_{g, p'(A)-e}^X q^{p'(A)-e}.
\]
Set
\[ F^X(u, t_d^j) = \sum_{g \geq 0} u^{2g-2} F_g^X(t_d^j) \]
and
\[ F^\tilde{X}(u, t_d^j) = \sum_{g \geq 0} u^{2g-2} F_g^\tilde{X}(t_d^j). \]

Then from Theorem 3.7, we have

**Corollary 3.9.**
\[ F^X(u, \gamma) = \left( \frac{\sin \frac{\pi}{2}}{2} \right)^2 \cdot F^{\tilde{X}}(u, p^* \gamma), \]
where we need to change the variable \( q^A \) to \( q^{p(A) – e} \).

### 4. Proof of Technical Lemmas

In this section, we use virtual localization to prove Lemma 3.5 and 3.6. Let \( \mathbb{T} = (\mathbb{C}^*)^4 \) and we will use suitable \( \mathbb{T} \)-action on \( \mathbb{P}^3 \) and \( \mathbb{P}^3 \) to compute the relevant Gromov-Witten invariants. We refer readers to Chapter 27 in [MSym] for a friendly introduction to localization on moduli spaces of stable maps.

#### 4.1. Proof of Lemma 3.5.

Suppose that the \( \mathbb{T} \)-action on \( \mathbb{P}^3 \) is given by
\[ t \cdot [x_0 : x_1 : x_2 : x_3] = [t_0 x_0 : t_1 x_1 : t_2 x_2 : t_3 x_3], \]
where \( t = (t_0, t_1, t_2, t_3) \in \mathbb{T} \). Let \( H \) be the hyperplane class of \( \mathbb{P}^3 \). For \( (a_0, a_1, a_2, a_3) \in \mathbb{Z}^4 \), consider the \( \mathbb{T} \)-action on the line bundle \( \mathcal{O}_{\mathbb{P}^3}(1) \) given by
\[ t \cdot [x_0 : x_1 : x_2 : x_3 : x] = [t_0 x_0 : t_1 x_1 : t_2 x_2 : t_3 x_3 : t_0^{-a_0} t_1^{-a_1} t_2^{-a_2} t_3^{-a_3} x], \]
where \( t = (t_0, t_1, t_2, t_3) \in \mathbb{T} \). Then we can get the \( \mathbb{T} \)-equivariant lift of \( H \).

For \( (a_0, a_1, a_2, a_3) \) and \( i = 0, \cdots, 3 \) be the fixed points of the \( \mathbb{T} \)-action on \( \mathbb{P}^3 \). Then
\[ H_{(a_0, a_1, a_2, a_3)} = \sum_{k=0}^{3} a_k \alpha_k + \alpha_i, \quad i = 0, 1, 2, 3, \]
where \( \alpha_i, i = 0, \cdots, 3 \) are the weights of \( \mathbb{T} \)-actions.

Note that \([pt]\) = \( H^3 \) in \( \mathbb{P}^3 \), we may choose its two \( \mathbb{T} \)-equivariant lifts as follows
\[ [pt]_1 = H_{(-1,0,0,0)} \cdot H_{(0,0,-1,0)} \cdot H_{(0,0,0,-1)}, \]
\[ [pt]_2 = H_{(-1,0,0,0)} \cdot H_{(0,-1,0,0)} \cdot H_{(0,0,0,-1)}. \]

Thus we have
\[ \langle [pt], [pt] \rangle_{\mathbb{P}^3} = \int_{[\overline{M}_{0,2}(\mathbb{P}^3,L)]^{vir}} e^{v^*_1 [pt]_1} \wedge e^{v^*_2 [pt]_2}. \]
By the localization formula (5) we have

\[
\langle [pt], [pt] \rangle_{g,L}^{\mathbb{P}^3} = \sum_{\Gamma \in G_{g_1, g_2}^{p_1, p_2} (\mathbb{P}^3, L)} \frac{1}{|\mathcal{N}_1^\psi|} \int_{M_{\Gamma}} \prod_{k=1} (x_1 - x_k) \cdot \prod_{k=2} (x_2 - x_k) e_T(\mathcal{N}^uv_1),
\]

where \(G_{g_1, g_2}^{p_1, p_2} (\mathbb{P}^3, L)\) consists of dual graphs corresponding to \(T\)-fixed stable maps in \(\overline{M}_{g,2}(\mathbb{P}^3, L)\) such that the first marked point is mapped to \(p_1\) and the second marked point is mapped to \(p_2\).

Let \(l_{p_1, p_2}\) be the projective line in \(\mathbb{P}^3\) connecting \(p_1, p_2\). For each \(\Gamma \in G_{g_1, g_2}^{p_1, p_2} (\mathbb{P}^3, L),\) \(\Gamma\) has only one edge corresponding to the noncontracted component mapped to \(l_{p_1, p_2}\) with degree 1, and two vertices \(v_1\) and \(v_2\) corresponding to two contracted components mapped to \(p_1\) and \(p_2\) respectively. Let \(\Gamma_{g_1, g_2} \in G_{g_1, g_2}^{p_1, p_2} (\mathbb{P}^3, L)\) such that the contracted component corresponding to \(v_1, v_2\) have genera \(g_1, g_2\) respectively, i.e.,

\[
\{g_1\} \quad 1 \quad \{g_2\}
\]

**Figure 1.** Graph \(\Gamma_{g_1, g_2}\)

Then we have

\[
G_{g_1, g_2}^{p_1, p_2} (\mathbb{P}^3, L) = \{\Gamma_{g_1, g_2} : g_1 + g_2 = g\}.
\]

**Lemma 4.1.**

\[
\frac{1}{|\mathcal{N}_1^\psi|} \int_{M_{\Gamma_{g_1, g_2}}} \prod_{k=1} (x_1 - x_k) \cdot \prod_{k=2} (x_2 - x_k) e_T(\mathcal{N}^uv_1)
\]

\[
= \int_{M_{\Gamma_{g_1, g_2}}} \prod_{k=1} \Lambda^v_{g_1}(x_1 - x_k) \cdot \prod_{k=2} \Lambda^v_{g_2}(x_2 - x_k)
\]

\[
\frac{1}{|\mathcal{N}_1^\psi|} \int_{M_{\Gamma_{g_1, g_2}}} \prod_{k=1} (x_1 - x_k) \cdot \prod_{k=2} (x_2 - x_k) e_T(\mathcal{N}^uv_1)
\]

\[
= \int_{M_{\Gamma_{g_1, g_2}}} \prod_{k=1} \Lambda^v_{g_1}(x_1 - x_k) \cdot \prod_{k=2} \Lambda^v_{g_2}(x_2 - x_k)
\]

\[
1 - (x_1 - x_2 - \psi) \cdot \int_{M_{\Gamma_{g_1, g_2}}} \prod_{k=1} \Lambda^v_{g_1}(x_1 - x_k) \cdot \prod_{k=2} \Lambda^v_{g_2}(x_2 - x_k)
\]

**Proof.** This equality can be proved by a standard virtual localization calculation.

\(\square\)

From the above lemma, we have:

\[
\sum_{g=0}^{\infty} u^{2g} \cdot \langle [pt], [pt] \rangle_{g,L}^{\mathbb{P}^3} = \sum_{g_1 + g_2 = g} u^{2g_1} \int_{M_{\Gamma_{g_1, g_2}}} \prod_{k=1} \Lambda^v_{g_1}(x_1 - x_k) \cdot \prod_{k=2} \Lambda^v_{g_2}(x_2 - x_k)
\]

\[
= \left( \sum_{g_1=0}^{\infty} u^{2g_1} \int_{M_{\Gamma_{g_1, g_2}}} \prod_{k=1} \Lambda^v_{g_1}(x_1 - x_k) \cdot \prod_{k=2} \Lambda^v_{g_2}(x_2 - x_k) \right) \cdot \left( \sum_{g_2=0}^{\infty} u^{2g_2} \int_{M_{\Gamma_{g_1, g_2}}} \prod_{k=1} \Lambda^v_{g_1}(x_1 - x_k) \cdot \prod_{k=2} \Lambda^v_{g_2}(x_2 - x_k) \right)
\]

\[
= F_1(u) \cdot F_2(u).
\]
Considering dimensions, we have

\begin{equation}
F_1(u) = \sum_{g_1=0}^{\infty} u^{2g_1} \int_{M_{g_1,2}} \frac{\Lambda^\vee_{x_1}(1) \Lambda^\vee_{x_1} \alpha_{x_1-x_2} \Lambda^\vee_{x_1} \alpha_{x_1}}{1 - \psi_1}
= \sum_{g_1=0}^{\infty} u^{2g_1} \int_{M_{g_1,3}} \frac{\Lambda^\vee_{x_1}(1) \Lambda^\vee_{x_1} \alpha_{x_1-x_2} \Lambda^\vee_{x_1} \alpha_{x_1}}{1 - \psi_1}
= \left( \frac{\sin \frac{u}{2}}{2} \right)^{\alpha_{x_1-x_2} + \alpha_{x_1}} \frac{\alpha_{x_1-x_2} + \alpha_{x_1}}{\alpha_{x_1}} \cdot \sum_{g_1=0}^{\infty} u^{2g_1} \cdot \langle [pt], [pt] \rangle_{\overline{\mathbb{P}}^3} = F_1(u) \cdot F_2(u) = \left( \frac{\sin \frac{u}{2}}{2} \right)^2.
\end{equation}

Therefore,

\begin{equation}
\sum_{g=0}^{\infty} u^{2g} \cdot \langle [pt], [pt] \rangle_{\overline{\mathbb{P}}^3} = F_1(u) \cdot F_2(u) = \left( \frac{\sin \frac{u}{2}}{2} \right)^2.
\end{equation}

Expanding this generating function into Taylor series, we get Lemma 3.5.

4.2. **Proof of Lemma 3.6.** In this subsection, we shall use some elementary facts about toric geometry. We refer readers to Chapter 7 in [MSym] for a concise introduction to toric geometry.

Note that \( \mathbb{P}^3 \) is homogeneous, and we can let \( \overline{\mathbb{P}}^3 \to \mathbb{P}^3 \) be the blowup of \( \mathbb{P}^3 \) at \( p_3 \). Then \( \overline{\mathbb{P}}^3 \) is a toric variety given by

\[ \overline{\mathbb{P}}^3 = \mathbb{C}^3 \setminus \{0\} \times \mathbb{C}^2 \setminus \{0\} / (\mathbb{C}^*)^2, \]

where \( (\mathbb{C}^*)^2 \) acts on \( \mathbb{C}^3 \setminus \{0\} \times \mathbb{C}^2 \setminus \{0\} \) by

\[ u \cdot (x_0, x_1, x_2, x_3, x_4) = (u_1 u_2 x_0, u_1 u_2 x_1, u_1 u_2 x_2, u_1 x_3, u_2^{-1} x_4), \]

where \( u = (u_1, u_2) \in (\mathbb{C}^*)^2 \).

Let \( E \) be (the Poincaré dual of) the exceptional divisor, and \( \overline{H} \) be (the Poincaré dual of) the strict transform of a hyperplane in \( \mathbb{P}^3 \). Then we have

\begin{equation}
[pt] = E \overline{H}^2 \text{ in } H^6(\overline{\mathbb{P}}^3, \mathbb{R}).
\end{equation}

Let \( T \) act on \( \overline{\mathbb{P}}^3 \) by

\[ t \cdot [x_0, x_1, x_2, x_3, x_4] = [t_0 x_0 : t_1 x_1 : t_2 x_2 : t_3 x_3 : x_4], \]

where \( t = (t_0, t_1, t_2, t_3) \in T \). Then the \( T \)-fixed points are

\[ p_0 = [1 : 0 : 0 : 0 : 1], \quad q_0 = [1 : 0 : 0 : 1 : 0], \]
\[ p_1 = [0 : 1 : 0 : 0 : 1], \quad q_1 = [0 : 1 : 0 : 1 : 0], \]
\[ p_2 = [0 : 0 : 1 : 0 : 1], \quad q_2 = [0 : 0 : 1 : 1 : 0], \]
where \( p_0, p_1, p_2 \in \mathbb{P}^3 \setminus \{ p_3 \} \), and \( q_0, q_1, q_2 \) are in the exceptional divisor.

Let \( L_1 \) be the toric line bundle over \( \mathbb{P}^3 \) determined by the exceptional divisor. Then we have
\[
L_1 = \mathbb{C}^3 \setminus \{ 0 \} \times \mathbb{C}^2 \setminus \{ 0 \} \times \mathbb{C}/(\mathbb{C}^*)^2,
\]
where \( (\mathbb{C}^*)^2 \)-action on \( \mathbb{C}^3 \setminus \{ 0 \} \times \mathbb{C}^2 \setminus \{ 0 \} \times \mathbb{C} \) is given by
\[
u \cdot (x_0, x_1, x_2, x_3, x_4, x) = (u_1u_2x_0, u_1u_2x_1, u_1u_2x_2, u_1x_3, u_2^{-1}x_4, u_2^{-1}x),
\]
where \( u = (u_1, u_2) \in (\mathbb{C}^*)^2 \). For \( (a_0, a_1, a_2, a_3) \in \mathbb{Z}^4 \), consider the \( \mathbb{T} \)-action on \( L_1 \) given by
\[
\begin{align*}
t \cdot \{ x_0 : x_1 : x_2 : x_3 : x_4 : x \} &= \{ t_0x_0 : t_1x_1 : t_2x_2 : t_3x_3 : x_4 : t_0^{-a_0}t_1^{-a_1}t_2^{-a_2}t_3^{-a_3}x \},
\end{align*}
\]
where \( t = (t_0, t_1, t_2, t_3) \in \mathbb{T} \). Let
\[
E_{(a_0, a_1, a_2, a_3)} = e_{\mathbb{T}}(L_1)
\]
be the \( \mathbb{T} \)-equivariant lift of \( E \). Then for \( i = 0, 1, 2 \), we have
\[
\begin{align*}
E_{(a_0, a_1, a_2, a_3)}|_{p_i} &= \sum_{k=0}^3 a_k \alpha_k, \\
E_{(a_0, a_1, a_2, a_3)}|_{q_i} &= \sum_{k=0}^3 a_k \alpha_k - \alpha_i + \alpha_3,
\end{align*}
\]
where \( \alpha_i, 0 \leq i \leq 3 \), are the weights of \( \mathbb{T} \)-action on \( \mathbb{P}^3 \).

Let \( L_2 \) be the toric line bundle over \( \mathbb{P}^3 \) determined by a hyperplane in \( \mathbb{P}^3 \setminus E \). Then we have
\[
L_2 = \mathbb{C}^3 \setminus \{ 0 \} \times \mathbb{C}^2 \setminus \{ 0 \} \times \mathbb{C}/(\mathbb{C}^*)^2,
\]
where \( (\mathbb{C}^*)^2 \)-action on \( \mathbb{C}^3 \setminus \{ 0 \} \times \mathbb{C}^2 \setminus \{ 0 \} \times \mathbb{C} \) is given by
\[
u \cdot (x_0, x_1, x_2, x_3, x_4, x) = (u_1u_2x_0, u_1u_2x_1, u_1u_2x_2, u_1x_3, u_2^{-1}x_4, u_1u_2x),
\]
where \( u = (u_1, u_2) \in (\mathbb{C}^*)^2 \).

For \( (a_0, a_1, a_2, a_3) \in \mathbb{Z}^4 \), consider the \( \mathbb{T} \)-action on \( L_2 \) given by
\[
\begin{align*}
t \cdot \{ x_0 : x_1 : x_2 : x_3 : x_4 : x \} &= \{ t_0x_0 : t_1x_1 : t_2x_2 : t_3x_3 : x_4 : t_0^{-a_0}t_1^{-a_1}t_2^{-a_2}t_3^{-a_3}x \},
\end{align*}
\]
where \( t = (t_0, t_1, t_2, t_3) \in \mathbb{T} \). Let
\[
\overline{H}_{(a_0, a_1, a_2, a_3)} = e_{\mathbb{T}}(L_2)
\]
be the \( \mathbb{T} \)-equivariant lift of \( H \). Then for \( i = 0, 1, 2 \), we have
\[
\overline{H}_{(a_0, a_1, a_2, a_3)}|_{p_i} = \overline{H}_{(a_0, a_1, a_2, a_3)}|_{q_i} = \sum_{k=0}^3 a_k \alpha_k + \alpha_i.
\]
From (24), we have
\[
\langle [pt] \rangle_{g, p(L_2)\mathbb{P}^3} = \int_{[M_{BR}(\mathbb{P}^3, p^*(L_2)\mathbb{P}^3)]^{vir}} ev_1^*(E(0,0,0,0) \cdot \overline{H}(0,-1,0,0) \cdot \overline{H}(0,0,-1,0)).
\]
Now using similar calculation as in the last subsection, we have:

\[ \sum_{g=0}^{\infty} u^{2g} \cdot \langle [pt] \rangle_{g,L-e} = 1. \]

This implies Lemma 3.6.

Acknowledgements. The authors would like to thank Prof. Yongbin Ruan for his valuable discussions, and Pedro Acosta for his useful comments on earlier drafts. Huazhong would like to thank Prof. Jian Zhou for generously sharing his experience on localization calculation, and Xiaowen Hu and Hanxiong Zhang for helpful discussions. Weiqiang and Huazhong would like to thank Department of Mathematics of the University of Michigan for its hospitality during their visiting.

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