Testing $f(R)$ theories using the first time derivative of the orbital period of the binary pulsars

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ABSTRACT

In this paper, we use one of the post-Keplerian parameters to obtain constraints on $f(R)$ theories of gravity. Using the Minkowskian limit, we compute the prediction of $f(R)$ theories on the first time derivative of the orbital period of a sample of binary stars, and we use our theoretical results to perform a comparison with the observed one. Selecting a sample of relativistic binary systems, we estimate the parameters of analytic $f(R)$ gravity. We find that the theory is not ruled out if we consider only the double-neutron star systems, and in this case we can cover the existing gap between the general relativity prediction and the observed data.

Key words: gravitation.

1 INTRODUCTION

The gravitational waves (GWs) are one of the most promising tools to study astrophysical systems such as neutron stars (NS), coalescing binary systems, black holes and white dwarfs.

The pieces of observational indirect evidence of gravitational radiation were measured on the system B1913+16, known as the Hulse–Taylor binary pulsar, and confirmed in other relativistic binary systems. The prediction of general relativity (GR) on the first time derivative of the orbital period in binary pulsar systems was studied by Hulse & Taylor (1975) and Weisberg, Nice & Taylor (2010), for which the discrepancy on observed data with respect to the prediction is $\sim 1$ per cent. However, the observational results should be explained using a different formulation of gravity (Freire et al. 2012). As shown in De Laurentis & Capozziello (2011), these systems could represent a good test for extended theories of gravity (ETGs). Considering a class of analytic $f(R)$ theories, it is possible to evaluate the gravitational radiated power in a weak field limit. In this approximation, we find that the energy radiated depends on the third derivative of the quadrupole, as predicted by GR, and the fourth derivative representing the corrective contribution to the theory. This result can be used to set constraints on the theory, comparing the prediction on the first time derivatives of the orbital period with the observed one. The outline of the paper as follows: in Section 2, we briefly introduce the weak field limit approximation of $f(R)$ theories of gravity. In Section 3, we apply the theoretical results previously obtained to binary systems computing the energy lost through GW emission. In Section 4, we compute the first time derivative of the orbital period in $f(R)$ theories of gravity and compare the theoretical prediction with the observed data. Finally, in Section 5, we give our conclusions and remarks.

2 $f(R)$ GRAVITY BACKGROUND

The $f(R)$ theories are based on corrections and enlargements of the GR theory adding higher order curvature invariants and minimally or non-minimally coupled scalar fields into dynamics which come out from the effective action of quantum gravity (Capozziello & De Laurentis 2011).

Starting from the following field equations in $f(R)$ gravity (looking for major details at Capozziello & De Laurentis (2011), Nojiri & Odintsov (2007, 2011), Capozziello & Francaviglia (2008) and Capozziello, De Laurentis & Faraoni (2009)):\footnote{\text{\textsuperscript{1}}}

$$f(R)\mu\nu = \frac{f(R)}{2} g_{\mu\nu} - f'(R)_{\mu\nu} + g_{\mu\nu} \Box f'(R) = \frac{\chi}{2} T_{\mu\nu} , \quad (1)$$
$$3\Box f'(R) + f'(R) R - 2 f(R) = \frac{\chi}{2} T , \quad (2)$$

$$\text{\textsuperscript{1}} T_{\mu\nu} = \frac{-1}{2} \frac{\delta (\sqrt{-G} L)}{\delta g_{\mu\nu}}$$

is the energy momentum tensor of matter ($T$ is the trace), $\chi = \frac{\delta (\sqrt{-G} L)}{\delta g_{\mu\nu}}$ is the coupling. $f'(R) = \frac{\delta f(R)}{\delta R}$, $\Box = \partial^2_{\alpha\beta}$ and $\Box = \partial_{\alpha}\partial_{\beta}$.

We adopt a $(+, - , - , - )$ signature, and indicate with $\partial_i$ partial derivative and with $\Box$ covariant derivative with regard to $g_{\mu\nu}$; all Greek indices run from 0 to 3 and Latin indices run from 1 to 3; $g$ is the determinant.
the Minkowskian limit can be calculated for a class of analytic $f(R)$ Lagrangian (i.e. Taylor expandable in terms of the Ricci scalar$^2$):

$$f(R) = \sum_n \frac{f^n(R_0)}{n!} (R - R_0)^n \simeq f_0 + f'_0 R + \frac{f''_0}{2} R^2 + \cdots.$$  (3)

At the first order, in terms of the perturbations, the field equations become

$$f_0 \left[ R^{(1)}_{\mu\nu} - \frac{R^{(1)}}{2} \eta_{\mu\nu} \right] - f'_0 \left[ R^{(1)}_{\mu\nu} - \eta_{\mu\nu} \Box R^{(1)} \right] = \frac{\chi}{2} T^{(0)}_{\mu\nu},$$  (4)

where $f'_0 = \frac{df}{dR} |_{R=R_0}$, $f''_0 = \frac{d^2f}{dR^2} |_{R=R_0}$ and $\Box = \Box_{\mu\nu}$ is the $d'$Alembert operator. Here, the Ricci tensor and scalar read

$$R^{(1)}_{\mu\nu} = \hat{h}^\sigma_{\mu,\nu} - \frac{1}{2} \Box \hat{h}_{\mu\nu} - \frac{1}{2} \hat{h} \eta_{\mu\nu} + \Box \hat{h},$$

$$R^{(1)} = \hat{h} \Box_{\sigma\tau} - \Box \hat{h}.$$  (5)

Now, assuming that the source is localized in a finite region, as a consequence, outside this region $T_{\mu\nu} = 0$ and

$$R^{(1)}_{\mu\nu} = \Box \hat{h}_{\mu\nu} = 0.$$  (6)

From here it is possible to calculate the energy momentum tensor of gravitational field in $f(R)$ gravity, adopting the definition given in Landau & Lifshitz (1962) and De Laurentis & Capozziello (2011), so that it satisfies a conservation law as required by the Bianchi identities:

$$t^\mu_a = \frac{1}{\sqrt{-g}} \left[ \frac{\delta L}{\delta g^\rho_{\mu\lambda,\rho}} - \frac{\partial}{\partial g^\rho_{\mu\lambda,\rho}} \right] g_{\rho\lambda,a} + \frac{\delta L}{\delta g^\rho_{\mu\lambda,\rho \lambda}} \eta_{\rho\lambda,a} - \frac{1}{2} \delta^\mu_{\rho} \left( h^\rho_{\sigma,\rho} - \Box h \right),$$

$$t^\mu_a = \int 3 R_{\mu\nu} \xi^\nu d^3x / \Omega_1.$$  (7)

Starting from the above equation, De Laurentis & Capozziello (2011) have shown that the energy momentum tensor consists of a sum of a GR contribution plus a term coming from $f(R)$ gravity:

$$t^\mu_a = f_0 t^\mu_a^{GR} + f'_0 t^\mu_a^{(f(R))},$$  (8)

which in terms of the perturbation $h$ is

$$t^\mu_a \sim f_0 t^\mu_a^{GR} + f'_0 \left\{ \left( h^{\mu\nu}_{\sigma,\rho} - \Box h \right) \times \left[ h^\lambda_{\mu,\rho} - h^\lambda_{\rho,\mu} - \frac{1}{2} \delta^\lambda_{\rho} \left( h^{\mu\nu}_{\sigma,\rho} - \Box h \right) ight] ight. \
- \left. h^{\rho\nu}_{\sigma,\rho} h^\lambda_{\mu,\rho} + h^{\rho\nu}_{\sigma,\rho} h^\lambda_{\rho,\mu} \Box h \right\}.$$  (9)

To simplify the above equation, the weak field limit approximation is taken into account, i.e. the source $h_{\mu\nu}$ will be written as a function of a single scalar variable $\tau = t - r$, and as a consequence, it will be almost plane (Maggiore 2007; De Laurentis & Capozziello 2011).

Finally, the energy momentum tensor assumes the following form:

$$t^\mu_a = f_0 f^k_k k_a \left( h^{\mu\nu} h_{\rho\sigma} \right) \frac{1}{2} f'_0 f^j_k \left( k_a k_b k_c h^{\nu\sigma} \right)^2.$$  (10)

To be more precise, the first term, depending on the choice of the constant $f_0$, is the standard GR term, and the second is the $f(R)$ contribution. It is worth noticing that the order of derivative is increased by two degrees consistently to the fact that $f(R)$ gravity is of fourth order in the metric approach (De Laurentis & Capozziello 2011).

### 3 RADIATED ENERGY

In order to calculate the radiated energy of a GW source, De Laurentis & Capozziello (2011) suppose that $h_{\mu\nu}$ can be represented by a discrete spectral representation. The periodicity $T$ will be proportional to the inverse of the difference of the pair of frequency components in the wave, and then, the average of $\frac{dE}{dt}$ must be evaluated over an interval equal to or greater than $T$ (Landau & Lifshitz 1962; Maggiore 2007). The instantaneous flux of energy through a surface of area $r^2 d\Omega$ in the direction $\hat{x}$ is given by

$$\frac{dE}{dr} = r^2 d\Omega \hat{x}^{\nu} t^\nu_{\hat{x}},$$  (11)

and the average flux of energy can be written as

$$\left\langle \frac{dE}{dt} \right\rangle = r^2 d\Omega \langle \hat{x}^{\nu} t^\nu_{\hat{x}} \rangle.$$  (12)

Defining the following moments of the mass–energy distribution:

$$M(t) \simeq \int d^3 x T^{(0)}(x, t),$$  (13)

$$D^i(t) \simeq \int d^3 x x^i T^{(0)}(x, t),$$  (14)

$$Q^{ij}(t) \simeq \int d^3 x x^i x^j T^{(0)}(x, t),$$  (15)

and analysing the radiation in terms of multipoles, De Laurentis & Capozziello (2011) found

$$\left( \epsilon^j_t \right) = \left\langle \int f_0 k^j k_a k_0 \frac{4}{r^2} \left[ \left( \hat{x}^i \hat{x}^j \hat{Q}^{ij} \right)^2 - 2 \left( \hat{x}^i \hat{Q}^{ij} \right) \left( \hat{x}^j \hat{Q}^{ij} \right) + \left( \hat{Q}^{ij} \hat{Q}^{ij} \right) \right] ight\rangle$$

$$\left( \epsilon^j_t \right) = \left\langle \int f_0 f'_0 \delta^j_k \left( k_a k_0 \right)^2 \frac{2}{r^2} \left[ \left( \hat{x}^i \hat{x}^j \hat{Q}^{ij} \right)^2 - 2 \left( \hat{x}^i \hat{Q}^{ij} \right) \left( \hat{x}^j \hat{Q}^{ij} \right) + \left( \hat{Q}^{ij} \hat{Q}^{ij} \right) \right] \right\rangle.$$  (16)

Using the result in equation (12) and integrating over all directions, they computed the total average flux of energy due to the tensor wave,

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{G}{60} \left\langle \int f_0 \left( \hat{Q}^{ij} \hat{Q}^{ij} \right) - f'_0 \left( \hat{Q}^{ij} \hat{Q}^{ij} \right) \right\rangle.$$  (17)

Precisely, for $f''_0 \to 0$ and $f'_0 \to 0$, equation (17) becomes

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{G}{45} \left\langle \hat{Q}^{ij} \hat{Q}^{ij} \right\rangle,$$  (18)

which is the well-known result of GR (Landau & Lifshitz 1962; Weinberg 1972). An important remark is related to the absence of monopole and dipole terms in our considerations. In our case, all the calculations are performed in the Jordan frame so $f(R)$ gravity results as a mere extension of GR being $f(R) = R$, so any dipole term is null (as shown in Will 1993, table 10.2). In order to put in evidence such contributions, we have to pass in the Einstein frame where the additional degrees of freedom of the gravitational field can

\[\text{[Footnote: For convenience we will use } f \text{ instead of } f(R). \text{ All considerations are developed here in metric formalism. From now on, we assume physical units } G = c = 1.\]
be recast in terms of scalar fields. In this case, monopole and dipole terms are explicitly obtained (Will 1993; Damour & Esposito-Farese 1996; Naef & Jetzer 2011). The two approaches are conformally equivalent, but in the Einstein frame, monopole and dipole terms can be obtained (see e.g. Capozziello & De Laurentis 2011).

## 4 APPLICATION TO PULSAR BINARY SYSTEMS

Now, our goal is to use a sample of binary pulsar systems to fix bounds on \( f(R) \) parameters. To do this, we assume that the motion is Keplerian and the orbit is in the \((x, y)\) plane. We define \( m_p \) as the pulsar mass, \( m_c \) as the companion mass and \( \mu = \frac{m_cm_p}{m_c + m_p} \) as the reduced mass. In the \((x, y)\) plane, the quadrupole matrix is

\[
Q_{ij} = \mu r^2 \begin{pmatrix}
\cos^2 \psi & \sin \psi \cos \psi \\
\sin \psi \cos \psi & \sin^2 \psi
\end{pmatrix}_{ij},
\]

(19)

where \( i, j \) are the indices in the orbital plane, \( r \) is the equation of the elliptic Keplerian orbit and \( \psi \) is eccentric anomaly, and both of them are time dependent.

To compute the radiated power, we need the third and fourth derivatives of quadrupole, so we must compute the time derivatives using the following relation given in Maggiore (2007):

\[
\dot{\psi} = \left( \frac{Gm_c}{a^3} \right) \left( 1 - e^2 \right)^{-3/2} (1 + e \cos \psi)^2,
\]

(20)

where \( a \) is the semimajor axis, and \( e \) is the orbital eccentricity. We obtain the following relations for the time derivatives of the quadrupole:

\[
\ddot{Q}_{11} = \mathcal{H}_1 \sin 2\psi (e \cos \psi + 1)^2 (3e \cos \psi + 4),
\]

(21)

\[
\ddot{Q}_{22} = -\mathcal{H}_1 (8 \cos \psi + e (3 \cos 2\psi + 5)) \sin \psi (e \cos \psi + 1)^2,
\]

(22)

\[
\ddot{Q}_{12} = -\mathcal{H}_1 (e \cos \psi + 1)^2 \sin (5e \cos \psi + 3e \cos 3\psi + 8 \cos 2\psi),
\]

(23)

\[
\ddot{Q}_{11} = \mathcal{H}_2 \left[ 15e^2 \cos 4\psi + 50e \cos 3\psi \\
+ (12e^2 + 32) \cos 2\psi + 6e \cos \psi - 3e^2 \right],
\]

(24)

\[
\ddot{Q}_{22} = -\mathcal{H}_2 \left[ 15e^2 \cos 4\psi + 50e \cos 3\psi \\
+ (24e^2 + 32) \cos 2\psi + 14e \cos \psi - 7e^2 \right],
\]

(25)

\[
\ddot{Q}_{12} = 2\mathcal{H}_2 \sin \psi \left[ 15e^2 \cos 3\psi + 50e \cos 2\psi \\
+ (33e^2 + 32) \cos \psi + 30e \right],
\]

(26)

where

\[
\mathcal{H}_1 = \frac{(2\pi)^{5/3} G^{2/3} m_c m_p}{T^{5/3} \left( 1 - e^2 \right)^{5/2} \sqrt{m_c + m_p}},
\]

\[
\mathcal{H}_2 = \frac{2^{2/3} \pi^{8/3} G^{3/2} m_c m_p (e \cos \psi + 1)^3}{T^{8/3} \left( e^2 - 1 \right)^{3/2} \sqrt{m_c + m_p}}.
\]

### 4.1 Comparing theory prediction with data

It is well known that in the relativistic binary pulsar systems, there is a loss of energy due to GW emission. This energy loss, provided by GR, has been confirmed by the timing data analysis on the well-known binary pulsar B1913+16 (Hulse & Taylor 1975; Weisberg et al. 2010). We must also note that the systems like B1913+16 are optimal tools to constrain theories of gravitation (Damour & Esposito-Farese 1998) using the post-Keplerian parameters. For the sake of convenience, we choose the observed orbital period derivative \( \dot{T}_b \), because it is one of the best-observed post-Keplerian parameters. Moreover, we know, according to GR theory, that it is related to the foreseen orbital decay due to quadrupole gravitational radiation emitted by the binary systems.

As shown in Section 4, it is possible rewrite the first derivative of the orbital period in \( f(R) \) theories of gravity. In principle, if we know exactly which Lagrangian we have to use to describe those types of systems, then we can predict the energy loss through GWs radiation. Here, we want to make the inverse process, to get an estimation of the second derivative \( f'' \) imposing the strong hypothesis that the difference between the observed binary period variation \( \dot{T}_{b,\text{obs}} \pm \delta \) and the one obtained by the relativistic theory of gravitation, \( \Delta T_b = T_{b,\text{obs}} - T_{\text{GR}} \), is fully justified by imposing that

\[
\dot{T}_{b,\text{obs}} - \dot{T}_{\text{GR}} - f'' \dot{T}_{b, f(R)} = 0,
\]

(29)

\[
T_{b,\text{obs}} \pm \delta - T_{\text{GR}} - f''_{0,\text{obs}} \dot{T}_{b, f(R)} = 0,
\]

(30)

and propagating the experimental error, \( \pm \delta \), on the first derivative of the observed orbital period, \( T_{b,\text{obs}} \pm \delta \), into an uncertainty on second derivative of gravitational theory, \( f''_{0,\text{obs}} \). What we want to emphasize is that, where GR is not able to fully explain the loss of energy by emission of GW radiation then, the additive contribution of an ETGs can provide a way to fill the gap between theory and observations. We also have subtracted the external contributions to the orbital decay as galactic or Shklovskii acceleration when those values are available in the literature. Solving equations (29) and (30) for \( f'' \) and \( f''_{0,\text{obs}} \), we get an estimation of \( f'' \) and its upper and lower limits corresponding, respectively, to \( \mp \delta \). In this way, \( \Delta T_b \) is fully explained through the orbital period correction due to the
Table 1. Data for binary relativistic pulsars: in the order we reported the J-Name of the binary pulsar system, the orbital binary period $T_b$ in days, the orbital projected semimajor axis $a \sin(i)$ in light-second, the orbital eccentricity $\epsilon$, the observed orbital period variation $\dot{T}_{\text{Obs}}$, the predicted $\dot{T}_{\text{GR}}$ according to the GR theory, the experimental error $\pm \delta$ on $\dot{T}_{\text{Obs}}$ and the masses $m_p$ and $m_c$ of the binary system components in solar mass unit.

| Name         | $T_b$ (d) | $a$ (lsec) | $i$ (degrees) | $\epsilon$ | $\dot{T}_{\text{Obs}}$ ($10^{-12}$) | $\dot{T}_{\text{GR}}$ ($10^{-12}$) | $\pm \delta$ ($10^{-12}$) | $m_p$ (M$_\odot$) | $m_c$ (M$_\odot$) | References                        |
|--------------|-----------|------------|---------------|-------------|-----------------------------------|----------------------------------|----------------|----------------|----------------|-----------------------------------|
| J2129+1210C  | 0.335 282 049 | 2.518 45  | 60 0 0       | 0.681 395  | −3.96                                | −3.94                            | 0.05                          | 1.358          | 1.354          | Anderson et al. (1990); Jacoby et al. (2006) |
| J1915+1606   | 0.322 997 449 | 2.341 782  | 54 0 0       | 0.617 134  | −2.423                               | −2.403                           | 0.001                        | 1.4398         | 1.3886         | Hulse & Taylor (1975); Weisberg et al. (2010) |
| J0737−3039A  | 0.102 251 562 | 1.415 032  | 88 69 0      | 0.087 777  | −1.252                               | −1.248                           | 0.017                        | 1.3381         | 1.2489         | Burgay et al. (2003); Kramer et al. (2006) |
| J1141−6545   | 0.197 650 959 | 1.858 922  | 73 0 0       | 0.171 884  | −0.403                               | −0.387                           | 0.025                        | 1.27           | 1.02           | Kaspi et al. (2000); Bhat, Bailes & Verbist (2008) |
| J1537+1155   | 0.420 737 299 | 3.729 462  | 78 4 0       | 0.273 676  | −0.138                               | −0.192                           | 0.0001                       | 1.3332         | 1.3452         | Stairs et al. (2002); Konsac, Wolszczan & Stairs (2003) |
| J1738+0333   | 0.354 790 739 | 32 6 0     | 3 4E−7 0     | −0.017     | −0.0277                              | 0.0031                           | 1.46                         | 0.181          |                             | Freire et al. (2012) |
| J0751+1807   | 0.263 144 267 | 0.396 612  | 65 8 0       | 0.000 000 71 | −0.031                               | −0.017                           | 0.009                        | 1.7            | 0.67           | Lundgren, Zepka & Cordes (1995); Nice et al. (2008) |
| J0024−7204J  | 0.120 664 938 | 0.040 402  | 60 0 0       | −0.55       | −0.03                               | 0.13                             | 1.4                          | 0.024          |                             | Freire et al. (2003); Camilo et al. (2000) |
| J1701−3006B  | 0.144 545 417 | 0.252 756  | 8 7 0        | −5.12       | −0.09                                | 0.062                            | 1.4                          | 0.14           |                             | Possenti et al. (2003); Lynch et al. (2012) |
| J2051−0827   | 0.099 110 251 | 0.045 052  | 30 0 0       | −15.5       | −0.03                               | 0.8                              | 1.4                          | 0.027          |                             | Stappers et al. (1996); Doroshenko et al. (2001) |
| J1909−3744   | 1.533 449 475 | 1.897 991  | 86 4 0       | 1.302E−07   | −0.55                                | −0.003                           | 0.03                        | 1.57           | 0.212          | Jacoby et al. (2003); Verbist et al. (2009) |
| J1518+4904   | 8.634 005 096 | 20.044 002 | <47° 0       | 0.249 484 51 | 0.24                                 | −0.001                           | 0.22                        | 1.56           | 1.05           | Nice, Sayer & Taylor (1996); Janssen et al. (2008) |
| J1959+2048   | 0.381 966 607 | 0.089 225  | 65 0 0       | 14.7        | −0.003                               | 0.8                              | 1.4                          | 0.022          | Fruchter, Stinebring & Taylor (1988); Arzoumanian, Fruchter & Taylor (1994) |
| J2145−0750   | 6.838 93     | 10.164 108 | 0 0 0        | 0.000 019 18 | 0.159                                 | −0.0004                          | 0.283                       | 1.76           | 0.254          | Bailes et al. (1994); Verbist et al. (2009) |
| J0437+4715   | 5.741 046 46 | 3.366 697 0 | 137 58 0     | 0.000 019 18 | 0.159                                 | −0.0004                          | 0.283                       | 1.76           | 0.254          | Johnston et al. (1993); Verbist et al. (2008) |
| J0045−7319   | 51.169 451   | 174.2576   | 44° 0 0      | 0.807 949   | −3.03E+5                              | −0.0224                         | 9E+3                        | 8.8            |                             | McConnell et al. (1991); Kaspi et al. (1996) |
| J2019+2425   | 76.511 634 79 | 38.767 629 7 | 63 0       | 0.000 111 09 | −30.0                                | −0.0006                         | 60.0                        | 1.33           | 0.35           | Nice, Taylor & Fruchter (1993); Nice, Splaver & Stairs (2001) |
| J1623−2631   | 191.442 81   | 64.809 46  | 40 0 0       | 0.025 315 45 | 400.0                                | −0.0003                         | 600.0                       | 1.3            | 0.8            | Lyne et al. (1988); Thorsett et al. (1999) |
\[ \dot{f}_{0} = \frac{\Delta T_{0}}{T_{b_{0}}}, \tag{31} \]

\[ \dot{f}_{0,2} = \frac{\Delta T_{0}}{T_{b_{0,2}}}, \tag{32} \]

where \( \Delta T_{0} = T_{b_{0}} - \dot{T}_{b_{0}} - \delta - \dot{T}_{GR} \).

Thus, among the various binary star catalogues available in the literature, we choose a sample of observed relativistic binary pulsars such that the binary period \( T_{b_{0}} \), the observed orbital period variation \( T_{b_{0}} \), the computed orbital period variation from general relativistic theory \( T_{GR} \), the orbital eccentricity \( e \) and the masses of the components \( m_{1} \) and \( m_{2} \) are known with a fairly good precision.

For each system we have chosen, all previous parameters and their references are reported in Table 1, where we show the J-Name of the binary pulsar system, the observed orbital binary period \( T_{b_{0}} \), the observed time variation of the orbital period \( T_{b_{0}} \), the predicted one \( T_{GR} \) (according to the GR theory), the experimental error \( \delta \) on \( T_{b_{0}} \) and the masses \( m_{1} \) and \( m_{2} \) of the binary system components in solar mass unit.

Furthermore, in Table 2, we reported the J-Name of the systems, the difference \( \Delta T_{GR} \) between \( T_{b_{0}} \) and \( T_{GR} \) (equal to the correction \( f_{0} \)), the correction \( \dot{T}_{b_{0}} \), the corresponding \( f_{0} \) solution of (29) shown in (31), the interval centred on \( f_{0} \) and finally, the interval centred on \( f_{0,2} \) and computed from the difference \( f_{0,2} - f_{0,0} \), where \( f_{0,0} \) are the solutions of (29) shown in (31) taking into account the experimental errors \( \pm \delta \) on the observed orbital period variation \( T_{b_{0}} \).

Now, in Fig. 1, we report representative results of our numerical analysis on the sample of binary pulsars we choose. In both panels, we use the following notation: the black line shows the behaviour of the first derivative of the orbital binary period for the \( f(R) \) theories of gravity as computed in equation (28); the blue line represents the observed orbital period variation \( T_{b_{0}} \); the red lines give the error band determined by the experimental errors \( \pm \delta \); and finally, the green line is representative of the orbital period variation \( T_{GR} \) computed from the GR. For the binary pulsar system J2129+1210C, Fig. 1(a), the orbital period variation \( T_{GR} \) computed from the GR is included in the experimental error band \( \pm \delta \), as the observed orbital period variation \( T_{b_{0}} \). Moreover, it is possible to see from Fig. 1(a) that the GR value of \( T_{GR} \) is recovered for \( f_{0} = 0 \) (green square), whilst to justify the difference \( \Delta T_{GR} \) between \( T_{b_{0}} \) and \( T_{GR} \) we have, from the solution of (31), the values shown in Fig. 1 for \( f_{0} \) (red square) and for \( f_{0,2} \) (blue square). In panel (b) of Fig. 1, the same situation is reported for J0751+1807. In this case, \( T_{GR} \) is obtained from the error band determined by the experimental errors \( \pm \delta \). It is again possible to see for \( f_{0} = 0 \) that the GR value of \( T_{GR} \) is recovered, but in this case the \( f_{0} \) values are an order of magnitude greater than the one of the well-behaved case of J2129+1210C.

In Fig. 2, for the sake of convenience, the absolute values of \( f_{0} \) reported in Table 2 versus the ratio \( T_{b_{0}} / T_{GR} \) are shown in the logarithmic scale. We must note that for the first six binaries in tables, the ETGs are not ruled out 0.04 \( \leq f_{0} \leq 38 \). For those systems we get 0.5 \( \leq T_{b_{0}} / T_{GR} \leq 1.5 \); the difference between \( T_{GR} \) and \( T_{b_{0}} \) can be explained adding a new contribution from the theory of gravity. Instead, for most of binaries we have \( f_{0} \) values that can surely rule out the theory, since taking account of the weak field assumption we obtain 38 \( \leq f_{0} \leq 4 \times 10^{4} \). From these last values to the first ones, there is a jump of about four up to five orders of magnitude on \( f_{0} \). The origin of these strong discrepancies, perhaps, is due to the extreme assumption we made, to justify the difference between the observed \( T_{b_{0}} \) and the predicted \( T_{GR} \) using the ETGs.

### 5 Discussion and Remarks

We want point out in this preliminary work that, where the GR theory is not enough to explain the gap between the data and the theoretical estimation of the orbital decay, there is the possibility of extending the GR theory with a generic \( f(R) \) theory to cover the gap. Here, we simply verify that this possibility exists, but there is need to compute the post-Keplerian parameters in the \( f(R) \) theory to estimate correctly the masses of the binary systems to constrain correctly the analytic parameters of the ETGs. In the post-Minkowskian limit of analytic \( f(R) \) gravity models, the quadrupole radiation depends on the masses of the two bodies, on the orbital parameters and on the analytic parameters of the \( f(R) \) theory as the coefficients \( f_{0} \) and \( f_{0} \) of the Taylor expansion. The first result we present is the analytic solution of the quadrupole radiation rate in which it is possible to separate the GR contribution and the one due to the \( f(R) \) gravity. We should note that the correction depends on the eccentricity of the orbit and on the orbital period of the binary system, and specifically, the radiation rate is a function of \( f_{0} \) and \( f_{0} \). According to equation (28), we have selected a sample of relativistic binary systems for which the first derivative of the orbital period is observed, computed the theoretical quadrupole radiation rate, and finally, compared it to binary system observations. From Table 2, it is seen that the first five systems have masses determined in a manner quite reliable, while for the remaining sample, masses are estimated by requiring that the mass of the pulsar is 1.4 M\(_{\odot}\) and assuming for the orbital inclination one of the usual statistical values (\( i = 60^\circ \) or \( i = 90^\circ \)), and from here then comes the estimate of the mass of the
Figure 1. We report representative results of our numerical analyses on the sample of binary pulsars we have selected. In both figures, we use the following notation: the black line shows the behaviour of the first derivative of the orbital binary period for the $f(R)$ theory of gravitation as computed in equation (28); the blue line represents the observed orbital period variation $\dot{T}_{\text{Obs}}$; the red lines give the error band determined by the experimental errors $\pm \epsilon$ and finally, the green line is representative of the $\dot{T}_{\text{GR}}$ orbital period variation computed from the GR. In panel (a), for the system J2129 + 1210C, $\dot{T}_{\text{GR}}$ is included in the error band determined by the experimental errors $\pm \delta$, as $\dot{T}_{\text{Obs}}$. We point out that the GR value of $\dot{T}_{\text{GR}}$ is recovered for $f''(r) = 0$ (green square), while to justify the difference between $\dot{T}_{\text{Obs}}$ and $\dot{T}_{\text{GR}}$ we show the value of $f''(0)$ (blue square) and its error band $f''(0) \pm \delta$ (red square) as computed in equations (31) and (32). In the last panel (b), the same data are reported for J0751 + 1807, but in this case $\dot{T}_{\text{GR}}$ is OUT of the error band determined by the experimental errors $\pm \delta$. It is possible to see for $f''(0) = 0$ that the GR value of $\dot{T}_{\text{GR}}$ is recovered, but in this case the $f''(0)$ values are much greater than the previous ones.

Figure 1. We report representative results of our numerical analyses on the sample of binary pulsars we have selected. In both figures, we use the following notation: the black line shows the behaviour of the first derivative of the orbital binary period for the $f(R)$ theory of gravitation as computed in equation (28); the blue line represents the observed orbital period variation $\dot{T}_{\text{Obs}}$; the red lines give the error band determined by the experimental errors $\pm \epsilon$ and finally, the green line is representative of the $\dot{T}_{\text{GR}}$ orbital period variation computed from the GR. In panel (a), for the system J2129 + 1210C, $\dot{T}_{\text{GR}}$ is included in the error band determined by the experimental errors $\pm \delta$, as $\dot{T}_{\text{Obs}}$. We point out that the GR value of $\dot{T}_{\text{GR}}$ is recovered for $f''(r) = 0$ (green square), while to justify the difference between $\dot{T}_{\text{Obs}}$ and $\dot{T}_{\text{GR}}$ we show the value of $f''(0)$ (blue square) and its error band $f''(0) \pm \delta$ (red square) as computed in equations (31) and (32). In the last panel (b), the same data are reported for J0751 + 1807, but in this case $\dot{T}_{\text{GR}}$ is OUT of the error band determined by the experimental errors $\pm \delta$. It is possible to see for $f''(0) = 0$ that the GR value of $\dot{T}_{\text{GR}}$ is recovered, but in this case the $f''(0)$ values are much greater than the previous ones.

Companion star. So a primary cause of major discrepancies, not only for the ETGs, but also for the GR theory, between the variation of the observed orbital period and the predicted effect of emission of GWs could be a mistake in the estimation of the masses of the system. In addition, other causes may be attributable to the evolutionary state of the system, which, for instance, if it does not consist of two NSs may transfer mass from companion to the NS. In our sample, there are only five double NSs that can be used to test GR and ETGs. Taking into account of the strong hypothesis we made, the extended theory correction to $T_{\text{GR}}$ can also include the galactic acceleration term correction (Damour & Taylor 1991, 1992). Here, we give a preliminary result about the energy loss from binary systems, and we show that, when the nature of the binary systems can exclude energy losses due to trade or loss of matter, we can explain the gap between the first time derivative of the observed orbital period and the theoretical one predicted by GR, using an analytical $f(R)$ theory of gravity. In conclusion, to improve the estimation of the $f(R)$ coefficients, we need to consider the hydrodynamic effects due to the transfer of the matter in the binary system, in order to analyse different systems from double NS; and to improve the estimations.
Testing \( f(R) \) theories using binary pulsars

Figure 2. For the sake of convenience, the absolute values of \( f_0'' \) reported in Table 2 versus the ratio \( \frac{\dot{T}_{\text{obs}}}{\dot{T}_{\text{GR}}} \) are in logarithmic scale. We must note that for five binaries, the ETGs we are probing are not ruled out \( 0 < f_0'' \leq 38 \), for those systems the difference between \( \dot{T}_{\text{GR}} \) and \( \dot{T}_{\text{obs}} \) is tiny; indeed we get \( 0.5 \leq \frac{\dot{T}_{\text{obs}}}{\dot{T}_{\text{GR}}} \leq 1.5 \). Instead, for most of binaries we have \( f_0'' \) values that can surely rule out the theory, since taking account of the weak field assumption we obtain \( 38 \leq f_0'' \leq 4 \times 10^7 \). From these last values to the first ones, there is a jump of about four up to five orders of magnitude on \( f_0'' \).

of the mass of the stars in the binary systems without prior on pulsar mass and orbital inclination.

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