Highlights

1. A higher order polynomial equation in terms of amplitudes is developed to determine the jump points of the primary frequency response curve (FRC) in Duffing system by using a first-order Harmonic balance method (HBM), which are in good with the results of direct numerical simulation, and those given in other previous studies under the same system parameters.

\[ 27\zeta^2\gamma^3Z^{10} + 36\zeta^2\gamma^2Z^8 - 9\gamma^2a^2Z^6 + 24\gamma^2a^2Z^4 + 4a^2 = 0 \]  

(13)

2. Global stability criteria of the primary FRCs in both softening and hardening systems, which are analytical expressions composed of damping ratio, nonlinear coefficient and external excitation, are summarized in this paper.

\[ a_s = \left( \frac{256\zeta^3}{3\gamma} \left( \zeta \left( 4\zeta^2 + 1 \right) - \left( 4\zeta^2 + 1/3 \right) \left( \zeta^2 + 1/3 \right)^{1/2} \right) \right)^{1/2} \]  

(32)

\[ a_h = \left( \frac{256\zeta^3}{3\gamma} \left( \zeta \left( 4\zeta^2 + 1 \right) + \left( 4\zeta^2 + 1/3 \right) \left( \zeta^2 + 1/3 \right)^{1/2} \right) \right)^{1/2} \]  

(33)
On the primary resonance of Duffing oscillator: a novel approximate algebraic equation for solving jump points and a robust criterion for global stability

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Abstract Duffing oscillator is a classical nonlinear system widely used in various fields of science and engineering. In this paper, jump points and global stability of primary resonance of Duffing system, including softening and hardening, are further investigated. A novel higher order polynomial equation in terms of amplitudes is determined to calculate the jump points of primary frequency response curve (FRC) by using a first-order approximate HBM. The jump points determined by polynomial equation are in good with the results of direct numerical simulation, and those given in other previous studies under the same system parameters. Further, explicit analytical expressions of the boundary curves (BCs), that can be used to quickly determine the unstable response region of the primary FRC, is deduced by using Floquet theory. The BCs can also be considered as a set of jumping points of the FRCs under different excitations. According to a necessary condition of separation of FRC and BCs, the global stability criterion of FRC is transformed into the relationship between BCs and the extreme points of FRC. Noteworthily, the global stability criteria of FRC for both softening and hardening systems, that are analytical expressions composed of damping ratio, nonlinear coefficient and external excitation, can be determined. The criterion is provided a novel quantitative analysis method for the design of a Duffing nonlinear system to avoid or utilize the jump characteristic.

Keywords Duffing oscillator, Frequency response curve, Jump points, Boundary curves, Criteria of global stability

1. Introduction

Recent years, Duffing oscillator, an old-aged and classical nonlinear system, has been widely used to describe approximate models of the system in engineering, such as quasi-zero stiffness vibration isolation [1-4], vibration energy harvesting [5-7,22], and micro- and nano-electromechanical systems [8,9]. Duffing oscillator exhibits an enormous range of well-known behaviours in nonlinear dynamical systems, including homoclinic bifurcations, catastrophe, chaos, super- and sub-harmonic, etc [10]. Bifurcation of the frequency response curve (FRC) also is called the jump phenomenon, which is closely related to the stability of periodic response of the system. In this paper, a novel higher order polynomial equation in terms of amplitudes is determined the jump points of primary FRC by using a first-order approximate harmonic balance method (HBM). Furthermore, global stability criteria of the FRC for both softening and hardening systems, which are analytical expressions composed of damping ratio, nonlinear coefficient and external excitation,
can be determined by using Floquet theory.

A large number of approximate analytical methods, including classical perturbation methods (PMs) [11], multi-scale method (MSM) [12], HBM [13] and improved harmonic balance methods [14,15], homotopy method [16] and improved homotopy methods [17,18], as well as numerical methods, including time simulation, periodic extension and shooting method, have been applied to the solution of Duffing equation and analysis of its dynamic behaviours. Nayfeh and Mook [12] used MSM to determine the jump-down frequency and peak amplitude of the response. Kevorikian and Cole [19] determined the jump up and down frequencies and its corresponding amplitudes for a hardening system by the same method. Magnus [20] used HBM to determine the frequency of the jump-down point and the peak amplitude of a hardening system but did not consider the jump-up point. Brennan and Kovacic [21] determine the jump-down frequency and the corresponding amplitudes by HBM. They also summarised the approximate expressions of the jump-up point and jump thresholds given by PMs. Mallik [10] pointed out that FRC has vertical slope at the jump point, where \( \frac{d\omega}{dZ} = 0 \); here, \( \omega \) and \( Z \) represent the frequency and amplitude, respectively. Unfortunately, they did not elaborate the solution of the jump points. Recently, some literatures used these expressions [21] in the engineering application researches, such as nonlinear energy-harvesting devices [6, 22], Duffing-type suspension system [23], high-static-low-dynamic-stiffness isolator [24]. Ho and Lang [25] determined the output frequency response function to examine analytically the jump points of a Duffing oscillator. In the approximate analytical method [12,19-25], it is generally assumed that the damping of the system is small, so that the damping term above the second order in frequency response function (FRF) can be ignored to obtain analytical expressions. Therefore, the jump-down point is approximated to the maximum response of FRF, and the jump-up point given by PMs is considered to be a damping independent expression. However, with the increase of damping, we find that the higher-order damping term in FRF still affects the value of the jump point.

In numerical and hybrid analytical-numerical methods, Peleg [26], studying a hardening isolation system with friction and viscous damping, proposed an implicit amplitude-frequency equation. The jump frequencies and the corresponding amplitudes are determined by locating the vertical tangents to the FRC. Friswell and Penny [27] used a numerical approach based on Newton’s method, including terms up to the ninth harmonic, to compute the “exact” frequencies of the jump points. Worden [28] used a first-order HBM expansion and set the discriminant of a cubic polynomial in the square of amplitude equal to zero to obtain a higher-order polynomial equation in terms of jump frequency. Compared with the “exact” values of Friswell [27], the results show that the numerical solutions of the higher order polynomial proposed by Worden [28] is accurate enough. It is also noteworthy that Malatkar and Nayfeh [29] solve the jump points by constructing the Sylvester resultant of the first and second derivatives of FRF. A sixth order polynomial equation on jump frequencies determined by Sylvester resultant can provide accurate jump values, but it needs to be solved by a simple numerical method. They also given the expressions of jump frequency and jump threshold when the damping coefficient is equal to zero, which are approximately applied to the weak damping system. Numerical methods are accurate for solving values of jump points, but the disadvantage is that the solutions may converge slowly or to incorrect values.

Inspired by the Worden [28] and Malatkar [29], and using the property that FRC has a vertical slope at the jump points, a quintic polynomial equation in term of the square of jump amplitudes is derived in this paper, which is more concise than the equations in references [28,29]. The
corresponding jump frequencies can be obtained by substituting the jump amplitudes into FRC. In order to make the polynomial equation converge to an accurate value quickly, we use the approximate analytical formulas summarized in reference [21] to calculate the initial values of numerical simulation. The jump points determined by the polynomial equation given in this paper are in good agreement with the results of direct numerical simulation, and those given by Friswell [27] and Worden [28] under the same system parameters.

In addition, we use the Floquet theory to further study the global stability of the primary FRC of the system to determine the conditions for jump occurrence. Conversely, whether the FRC jumps at a certain point is determined by the stability of the response at that point. Lyapunov [30] and Floquet [31] theories are the commonly used methods to study the response stability of nonlinear systems, which mainly use the roots of characteristic equation to determine the stability of various systems [32-35].

In this study, the characteristic equation of the system is further expanded to obtain the explicit analytical expression of BCs, which can be used to quickly determine the unstable response region of FRC. The intersections of BCs and primary FRC are the jump points. More importantly, we further discuss how to determine the global stability of FRC, that is, the condition that there is no jump point in FRC. We find that the global stability of the system is dominated by the topological relationship between FRC and BCs, and can be transformed into the topological relationship between FRC and the extreme point of BCs. Finally, the global stability criteria in softening and hardening systems are determined, which are analytical expressions composed of damping ratio, nonlinear coefficient and external excitation. The global stability criterion is provided a fast and quantitative analysis method for the design of a Duffing nonlinear systems to avoid or utilize the jump characteristic.

2. Jump points of the primary FRC
2.1 Amplitude-frequency response

The differential equation of Duffing oscillator excited by a harmonic force \( A \cos(\omega t + \varphi) \) is given by

\[
m \dddot{z} + c \ddot{z} + k_1 z + k_3 z^3 = A \cos(\omega t + \varphi)
\]  
(1)

where \( m \) is the equivalent mass of the system, \( z \) is the displacement, \( c \) is the damping, and \( k_1 \) and \( k_3 \) are the linear and cubic stiffness terms, respectively.

The non-dimensional form of Eq. (1) can be written as

\[
z'' + 2 \zeta z' + z + \gamma z^3 = a \cos(\Omega \tau + \varphi)
\]  
(2)

where \( a = A/\sqrt{k_1}, \quad \zeta = \frac{c}{2m\omega_0}, \quad \gamma = \frac{k_3}{k_1}, \quad \omega_0' = \frac{k_1}{m}, \quad \Omega = \frac{\omega_0}{\omega_0'}, \quad \tau = \omega_0 t \) with \((\cdot)' = \left(\frac{d(\cdot)}{d\tau}\right)\). Parameter \( a \) denote the equivalent excitation of the system, and \( \gamma \) is called nonlinear coefficient that is the ratio of cubic stiffness to linear stiffness.

A complete FRC includes amplitude-frequency response (AFR) and phase-frequency response (PFR). The solution approach of jump points of AFR is explained here firstly. Applying HBM,
assuming a first-order approximate solution in the form of \( z = Z \cos(\Omega \tau) \), and neglecting the term of \( \cos 3 \Omega \tau \), the implicit AFR is given as

\[
\left( \frac{3}{4} y Z^2 + \left( 1 - \Omega^2 \right) Z \right)^2 + (2 \zeta \Omega Z)^2 = a^2
\]  

It can be expanded as a quadratic equation for \( \Omega^2 \):

\[
Z^2 \Omega^4 - 2Z^2 \left( \frac{3}{4} y Z^2 + 1 - 2\zeta^2 \right) \Omega^2 + Z^2 \left( \frac{3}{4} y Z^2 + 1 \right)^2 - a^2 = 0
\]  

Then, the explicit AFR derived from Eq. (4) are shown as

\[
\left( \frac{3}{4} y Z^2 + 1 - 2\zeta^2 \right) \Omega^2 + \frac{\left( a^2 - 4\zeta^2 (1 - \zeta^2) Z^2 - 3y \zeta^2 Z^4 \right)^{1/2}}{Z} = 0
\]

By setting Eq. (5a) and Eq. (5b) to be an equality, a quadratic equation in the form of \( Z^2 \) can be determined

\[
3y \zeta^2 Z^4 + 4\zeta^2 (1 - \zeta^2) Z^2 - a^2 = 0
\]

Solving Eq. (6) and deleting the negative root, the maximum amplitude of AFR is given as

\[
Z_m = \left( \frac{2}{3y} \left( 1 - \zeta^2 \right)^2 + \frac{3y a^2}{4\zeta^2} \right)^{1/2} - \frac{2}{3y} \left( 1 - \zeta^2 \right)
\]

Substituting Eq. (7) into Eq. (5a) or (5b) gives the corresponding frequency as

\[
\Omega_m = \left( \frac{1}{2} \left( 1 - \zeta^2 \right)^2 + \frac{3y a^2}{4\zeta^2} \right)^{1/2} + \frac{1}{2} - \frac{3}{2} \zeta^2
\]

When damping ratio \( \zeta \) is small enough, the term \( \zeta^2 \) can be neglected from Eqs. (7) and (8) to get the same results as those in reference [21]. Eqs. (7) and (8), which are valid for both hardening and softening systems, yield positive real values that satisfy the following inequalities:

\[
\frac{2}{3y} \left( 1 - \zeta^2 \right)^2 + \frac{3y a^2}{4\zeta^2} \left( 1 - \zeta^2 \right) > 0
\]  

and

\[
\frac{1}{2} \left( 1 - \zeta^2 \right)^2 + \frac{3y a^2}{4\zeta^2} + \frac{1}{2} - \frac{3}{2} \zeta^2 > 0
\]

and also

\[
1 - \zeta^2 + \frac{3y a^2}{4\zeta^2} > 0
\]

For hardening system with periodic response \((\gamma > 0, 0 \leq \zeta \leq 1)\), Eqs. (9a), (9b) and (9c) are always satisfied. However, it can be deduced from Eqs. (9a), (9b) and (9c) that the external excitation of softening system with periodic response \((\gamma < 0, 0 \leq \zeta \leq 1)\), should be less than a maximum value, which is given as
\[ a_{\text{max}} = 2\zeta \left( 1 - \zeta^2 \right) \left( -\frac{1}{3\gamma} \right)^{1/2} \]  \hspace{1cm} (10)

If \( a \) is greater than \( a_{\text{max}} \), \( \Omega_m \) and \( Z_m \) will become complex, which means that the potential energy cannot overcome the external excitation, so that the system no longer has periodic response. Substituting Eq. (6) into Eq. (5a) or Eq. (5b), the damping backbone curve which represents the damping free oscillation of system is given by

\[ \Omega^2 = \frac{3}{4} \gamma Z^2 + 1 - 2\zeta^2 \]  \hspace{1cm} (11)

Fig. 1 shows the AFR of Duffing system with damping ratio \( \zeta = 0.1 \) and excitation \( a = 1 \), as well as their damping backbone curves, maximum response and jump points. The AFR for hardening system bends to the right, while that for softening system bends to the left. The damping backbone curve intersects with AFR at the maximum response point.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{AFR_plot.png}
\caption{AFRs of Duffing oscillator with \( \zeta = 0.1 \) and \( a = 1 \). \( \square \) indicate unstable solutions, \( \triangle \) indicate jump-down points and \( \blacklozenge \) indicate jump-up points.}
\end{figure}

### 2.2 A higher-order polynomial equation for calculating jump points

AFR have vertical slopes at both jump down and up points, that is \( dZ/d\Omega = \infty \) or \( d\Omega/dZ = 0 \) \cite{[10]}. Therefore, applying \( d\Omega/dZ = 0 \) to the explicit AFR (Eqs. (5a, b)), an equation with amplitudes of the jump points can be written as

\[ \frac{3}{2} \gamma Z \pm \left( \frac{a^2}{Z^2} - 4\zeta^2 \left( 1 - \zeta^2 \right) - 3\gamma \zeta^2 Z^2 \right)^{1/2} \left( a^2 Z^3 + 3\gamma \zeta^2 Z \right) = 0 \]  \hspace{1cm} (12)

Expanding and simplifying Eq. (12), a quintic polynomial equation in terms of the square of jump amplitude is determined as

\[ 27\gamma^2 \zeta^2 Z^{10} + 36\zeta^2 \gamma^2 Z^8 - 9\gamma^2 a^2 Z^6 + 24\gamma \zeta^2 a^2 Z^4 + 4a^2 = 0 \]  \hspace{1cm} (13)

In references \cite{[28], [29]}, higher-order polynomial equations on jump frequencies have been given from different techniques, while in this paper, corresponding solutions have been given from the perspective of jump amplitude, and the polynomial equation presented in this paper is more concise. According to Galois theorem, there are no radical solutions for quintic equation and equations with more than five orders. In addition, it is complicated to use elliptic function to solve higher-order
equation, which is not involved in this paper. Therefore, Newton’s method was used to solve Eq. (13) in this paper. Firstly, Eq. (13) is rewritten as function $F(Z)$. The iterative equation is defined as

$$Z_{i+1} = Z_i - \frac{F(Z_i)}{F'(Z_i)}, \quad i = 0, 1, \ldots \quad (14)$$

where $i$ is the number of iterations, $Z_{i+1}$ is the amplitude after the $i$-th iteration. The iterative initial values are calculated by the formulas summarized in reference [21], which can make the system converge to accurate values rapidly. The iterative termination condition is set to $|Z_{i+1} - Z_i| < \varepsilon$, where $\varepsilon$ is the precision threshold. Then, substituting jump amplitudes into Eq. (5a) or Eq. (5b) yields the corresponding jump frequencies.

The evolution of jump points for hardening system with coefficient $\gamma$ in the range $[\gamma_h, 3\gamma_h]$ is shown in the Fig. 2. Parameter $\gamma_h$ called nonlinear threshold for hardening system will be explained in detail in section 3.2. Since different damping and excitation result in different nonlinear coefficient for the system jump, $\gamma$ is normalized to $[0, 1]$ represented by a symbol $\gamma_n$ for the convenience of comparison. Symbol $Z_d$ denote jump-down amplitude, $\Omega_d$ denote jump-down amplitude frequency, $Z_u$ denote jump-up amplitude and $\Omega_u$ denote jump-up frequency. Results in Fig. 2 show that the values of jump points given by Eq. (13) are in good agreement with the numerical sweep frequency simulation of equation Eq. (2). The jump-up amplitudes are decreasing with increase of $\gamma_n$, while the jump frequencies are increasing. This is due to the increase of the degree of nonlinearity, the AFR of hardening system bends more to the right. When $\gamma_n$ and $a$ are constant, jump amplitudes decrease and jump frequencies increase with the damping increases.

Similarly, for a softening system, the variation of jump points with $\gamma$ in the range $[1.5\gamma_s, \gamma_s]$ is shown in the Fig. 3. Parameter $\gamma_s$ called nonlinear threshold for softening system will be also defined.
in detail in section 3.2. Coefficient $\gamma$ is normalized to [-1,0] represented by the symbol $\gamma_n$. Fig. 3 show that, for the softening system, the jump points given by Eq. (13) also fit well with the direct numerical simulations of Eq. (2). Increasing value $\gamma_n$, the jump frequencies and jump-up amplitudes are increasing, while the jump-down amplitudes are decreasing. When $\gamma_n$ and $a$ are set to constant and the damping value increases, the jump frequencies and the corresponding amplitude will decrease.

Fig. 3. Jump points of softening system with $a=1$, where solid line --- represent solutions of Eq.(13) with $\zeta=0.05$, dotted line ---- represent solutions of Eq.(13) with $\zeta=0.1$, ○ and ◇ represent the numerical simulations of Eq.(2) at $\zeta=0.05$ and 0.1 respectively. (a) Jump-down amplitudes, (b) Jump-down frequencies, (c) Jump-up amplitudes, (d) Jump-up frequencies.

Furthermore, following the system parameters set in references [27,28], where the values $m=\gamma=1$, we further verify the accuracy of estimation formulas for the jump points proposed in this paper. The results in Table 1 and 2 show that the values in this paper are completely consistent with those in reference [28], and the percentage error with the “exact” value in reference [27] is basically less than 0.3%.

| Excitation ($a$) | 0.1 | 0.3 | 1.0 | 3.0 | 10.0 | 30.0 | 100.0 |
|-----------------|-----|-----|-----|-----|------|------|-------|
| 0.005 Damping ($\zeta$) | 3.0290 | 5.1464 | 9.3330 | 16.1340 | 29.4368 | 50.9762 | 93.0632 |
| 0.005 | -0.32% | -0.31% | -0.30% | -0.30% | -0.29% | -0.29% | -0.29% |
| 0.015 | 1.8520 | 3.0290 | 5.4196 | 9.3330 | 17.0052 | 29.4368 | 53.7332 |
| 0.015 | -0.31% | -0.32% | -0.31% | -0.30% | -0.29% | -0.29% | -0.29% |
| 0.05 | 1.2271 | 1.7742 | 3.0810 | 5.1465 | 9.3330 | 16.1341 | 29.4368 |
| 0.05 | -0.13% | -0.31% | -0.31% | -0.31% | -0.30% | -0.30% | -0.29% |
Table 2 Estimated jump-up frequencies estimated by Eq. (13) and percentage errors of “exact” values given in reference [27]

| Excitation (a) | 0.15 | 1.8571 | 3.0312 | 5.4207 | 9.3336 | 17.005 |
|---------------|------|---------|---------|---------|---------|---------|
| Damping (ζ)   | 0.005| -0.32%  | -0.33%  | -0.31%  | -0.30%  | -0.30%  |
|               | 0.015| -0.31%  | -0.31%  | -0.30%  | -0.30%  | -0.30%  |
|               | 0.05 | -0.30%  | -0.30%  | -0.29%  | -0.28%  | -0.28%  |
|               | 0.15 | -0.29%  | -0.30%  | -0.29%  | -0.29%  | -0.29%  |

2.3 Calculating for the jump points of PFR

The phase frequency response (PFR) [10] can be written as

$$\phi = \arctan \left( \frac{2\Omega}{\frac{3}{4} \gamma Z^2 + (1 - \Omega^2)} \right)$$

Substituting Eqs. (7) and (8) into Eq. (15) gives the phase \(\phi_m\) at the maximum response as

$$\phi_m = \arctan \left( \frac{\Omega_m}{\zeta} \right)$$

![Fig. 4 Phase frequency curves (PFCs) for a hardening system with a damping ratio \(\zeta=0.1\), \(a=1\) and various values of the nonlinear coefficient \(\gamma\).](image-url)
Substituting values of $\Omega_d$ and $Z_d$ into Eq. (15), the jump-down phase $\phi_d$ can be written as

$$\phi_d = \arctan \left( \frac{2\zeta \Omega_d}{\frac{3}{4} \gamma Z_d^2 + (1 - \Omega_d^2)} \right)$$  \hspace{1cm} (17)$$

Similarly, substituting values of $\Omega_u$ and $Z_u$ into Eq. (15), the jump-up phase $\phi_u$ can also be given as

$$\phi_u = \arctan \left( \frac{2\zeta \Omega_u}{\frac{3}{4} \gamma Z_u^2 + (1 - \Omega_u^2)} \right)$$  \hspace{1cm} (18)$$

PFRs of hardening systems with damping ratio $\zeta = 0.1$ and various values of the nonlinear coefficient $\gamma$ are shown in Fig. 4. It can be seen from the figure that all the PFRs bend to the right. The larger value $\gamma$ is, the more obvious the bending is. The phase $\phi_m$ is near $\pi/2$ and gradually approaches $\pi/2$ with the increase in $\gamma$. When $\gamma < \gamma_h$, there is no jump phenomenon in the PFR, as denoted by the green line. When $\gamma = \gamma_h$, there is only one jump point, that is, the jump-down point is equal to the jump-up point. When $\gamma > \gamma_h$, there are two jump points to form an unstable region, which becomes larger as the value of $\gamma$ increases.

3. Stability of the primary AFR

3.1 Boundary curve of the steady-state region

Stability of the periodic solutions of the system is calculated by applying Floquet theory as described in [33,34]. To this aim, a small disturbance $e(\tau)$ is introduced to the harmonic solutions $u(\tau)$ of the equation of motion given in Eq. (2), leading to

$$z = u(\tau) + e(\tau) = U \cos(\Omega \tau + \phi) + e(\tau)$$  \hspace{1cm} (19)$$

Applying Eq. (19) into the free oscillation corresponding to Eq. (2) that is, let $\cos(\omega t + \phi) = 0$, gives

$$u'' + 2\zeta u' + u + \gamma u^3 + e'' + 2\zeta e' + e + \gamma (3u^2 e + 3ue^2 + e^3) = 0$$  \hspace{1cm} (20)$$

Since $u(\tau)$ is a solution of Eq. (2), the subsequent expression can be equated to zero

$$u'' + 2\zeta u' + u + \gamma u^3 = 0$$  \hspace{1cm} (21)$$

As $e$ is a small parameter, neglecting the cubic and squared terms ($e^3$, $e^2$), the linear disturbance equation of the system is

$$e'' + 2\zeta e' + (1 + 3\gamma u^2) e = 0$$  \hspace{1cm} (22)$$

the solution of which is admitted to be a first-order Fourier series

$$e(\tau) = A \cos(\Omega \tau) + B \sin(\Omega \tau)$$  \hspace{1cm} (23)$$

Substituting Eq. (23) into Eq. (22) and equating the harmonic coefficients of the equation, Jacobian matrix of the system is yielded as
Generally, the stability of steady-state motion depends on the eigenvalues of the Jacobian matrix. The eigenvalues can be constructed to a criterion function of stability that is

\[ \Gamma = \left( 1 - \Omega^2 + \frac{9}{4} \gamma U^2 \right) \left( 1 - \Omega^2 + \frac{3}{4} \gamma U^2 \right) + 4 \zeta^2 \Omega^2 \]  

(25)

The function \( \Gamma > 0 \) means that the periodic solution is asymptotically stable, while \( \Gamma \leq 0 \) represents unstable periodic response. Consequently, \( \Gamma = 0 \) can determine the boundary between the steady and unsteady periodic responses of the system. A quadratic equation is obtained in terms of \( U^2 \) as a function of \( \Omega \)

\[ \frac{27}{16} \gamma^2 U^4 + 3 \gamma \left( 1 - \Omega^2 \right) U^2 + \Omega^4 + \left( 4 \zeta^2 - 2 \right) \Omega^2 + 1 = 0 \]  

(26)

in that two positive solutions are as follows:

\[ U_{1,2} = \left( -\frac{8}{9 \gamma} \left( 1 - \Omega^2 \right) + \frac{4}{9 \gamma} \left( \Omega^4 - \left( 12 \zeta^2 + 2 \right) \Omega^2 + 1 \right)^{1/2} \right) \]  

(27a, b)

BCs, given by Eqs. (27a) and (27b), plotted in the frequency-amplitude plane, provides the regions of instability (filled red areas) as shown in Fig. 5 and 6. The region is a function of the damping ratio \( \zeta \) and nonlinear coefficient \( \gamma \) but it does not depend on the displacement excitation \( a \). When the external excitation level \( a \) is low, the AFR curve is low and does not reach the unstable region, which indicates that the AFR is globally stable. While the external excitation \( a \) exceeds a certain critical value, part of AFR will cross the BCs to form unstable responses. These intersections between BCs and AFRs are the jump points.

Fig. 5. BCs for determining the stability of AFR in hardening system, where \( \zeta = 0.1 \) and \( \gamma = 0.05 \)
Fig. 6. BCs for determining the stability of AFR in softening system, where \( \zeta=0.1 \) and \( \gamma=-0.02 \)

The results in Fig. 7 show that the regions of instability formed by BCs varies with damping ratio \( \zeta \). When the nonlinear coefficient \( \gamma \) is constant, the unstable region of AFR decreases with the increase of \( \zeta \), whether it is hardening system or softening system. Those unstable regions with high damping ratio are surrounded by the unstable regions with low damping ratio. It should be noted that the jump points of AFRs just falls on its corresponding BCs. The jump-down point of undamped Duffing system is at infinity, so it cannot be plotted in Fig. 7.

Fig. 7. BCs of the system under different damping ratios \( \zeta \). (a) hardening system for \( \gamma=0.06 \), (b) softening system for \( \gamma=-0.02 \).

Fig. 8 show that the regions of instability formed by BCs varies with nonlinear coefficient \( \gamma \). When the damping ratio \( \zeta \) is constant, the unstable region of AFR also decreases with the increase of \( \gamma \) for both hardening system and softening system. Specifically, the amplitudes of the unstable response become smaller with the increase of \( \gamma \), while the corresponding frequency range has no obvious change.
Fig. 8. BCs of the system under different nonlinear coefficients $\gamma$. (a) hardening system for $\zeta=0.1$, (b) softening system for $\zeta=0.1$.

### 3.2 Global stability criteria of primary AFR

Extreme points of the BCs given by Eqs. (27) can be obtained by equal the discriminant of Eq. (26) to zero, that is

$$\Omega^4 - 2\left(6\zeta^2 + 1\right)\Omega^2 + 1 = 0$$  \hspace{1cm} (28)

Solving the quadratic Eq. (28) in term of $\Omega^2$, the frequencies of extreme points are given as

$$\Omega_{n,s} = \left(1 + 6\zeta^2 \pm 6\zeta \left(\zeta^2 + 1/3\right)^{1/2}\right)^{1/2}$$  \hspace{1cm} (29a, b)

The minimum frequency of BCs in hardening system is determined by Eq. (29a), while the maximum frequency of BCs in softening system is determined by Eq. (29b).

Substituting Eqs. (29a, b) into Eq. (27a) or Eq. (27b), the corresponding amplitudes of the extreme points are given as

$$U_{h,s} = \left\{\frac{16\zeta}{3\gamma} \left(\zeta \pm \left(\zeta^2 + 1/3\right)^{1/2}\right)\right\}^{1/2}$$  \hspace{1cm} (30a, b)

where $U_h$ given by Eq. (30a) is used for hardening system and $U_s$ given by Eq. (30b) is used for softening system. Considering the condition $\zeta>0$ in this paper, the values of $\Omega_s$, $\Omega_h$, $U_s$, and $U_h$ are always positive.

Substituting $U_s$ into Eq. (5a) to replace $Z$, a frequency of AFR for a softening system is determined as

$$\Omega_{s} = \left\{1 + 2\zeta^2 - 4\zeta \left(\zeta^2 + 1/3\right)^{1/2}\right\} - \left\{16\zeta^3 - \frac{9\gamma f^2}{16\zeta} \left(\zeta^2 + 1/3\right)^{1/2}\right\} - 4\zeta^2 - 28\zeta^4\right\}^{1/2}$$  \hspace{1cm} (31a)

Substituting $U_h$ into Eq. (5b) to replace $Z$, a frequency of AFR for a hardening system is determined as

$$\Omega_{h} = \left\{1 + 2\zeta^2 + 4\zeta \left(\zeta^2 + 1/3\right)^{1/2}\right\} + \left\{16\zeta^3 - \frac{9\gamma f^2}{16\zeta} \left(\zeta^2 + 1/3\right)^{1/2}\right\} - 4\zeta^2 - 28\zeta^4\right\}^{1/2}$$  \hspace{1cm} (31b)

Therefore, setting $\Omega_r=\Omega_{s}$ and simplifying, the excitation threshold of the global stability of a
softening system, which also determines whether the system jumps or not, is determined by the following equation

$$a_s = \left( \frac{256\zeta^3}{3\gamma} \left[ (4\zeta^2 + 1) - \left( 4\zeta^2 + 1/3 \right)(\zeta^2 + 1/3)^{1/2} \right] \right)^{1/2}$$  \hfill (32)$$

This means that the AFR of a softening system is globally stable when the condition \(a < a_s\) is satisfied. Contrarily, when the condition \(a \geq a_s\) is satisfied, the AFR of a softening system will form unstable regions, where the BCs passes through the jump points. Similarly, setting \(\Omega_h = \Omega_{fs}\), the excitation threshold of the global stability of a hardening system, is determined by

$$a_h = \left( \frac{256\zeta^3}{3\gamma} \left[ (4\zeta^2 + 1) + \left( 4\zeta^2 + 1/3 \right)(\zeta^2 + 1/3)^{1/2} \right] \right)^{1/2}$$  \hfill (33)$$

As shown in Fig. 9, when the nonlinear coefficient \(\gamma\) is constant, the excitation thresholds increase with the increase of damping ratio \(\zeta\) for both hardening and softening systems. At the same time, when the damping ratio \(\zeta\) is constant, with the increase of \(\gamma\), the excitation thresholds will also increase. A hardening system have similar criteria for determining the global stability of its AFR. Therefore, the global stability criteria (jump bifurcation conditions) of the AFRs for softening and hardening systems are summarized in Table 3.

| Category  | Stability of AFR | Jump points        | Criteria |
|-----------|------------------|---------------------|----------|
| Softening | Locally unstable | Jump down and up    | \(a_s < a \leq a_{max}\) |
|           | Unstable at a point | One point | \(a = a_s\) |
|           | Globally stable  | No                  | \(0 < a < a_s\) |
|           | Locally unstable | Jump down and up    | \(a > a_h\) |
| Hardening | Unstable at a point | One point | \(a = a_h\) |
|           | Globally stable  | No                  | \(0 < a < a_h\) |

Eq. (32) and Eq. (33) show that the excitation thresholds of global stability of AFR are determined by the damping and nonlinear coefficients of the system. The thresholds of other system parameters, derived from Eq. (32) or Eq. (33), can also be used to determine the global stability of
AFR. Therefore, nonlinear thresholds, represented by symbols $\gamma_s$ and $\gamma_h$ for the softening and hardening systems respectively, can be easily deduced as

$$\gamma_{s,h} = \frac{256 \zeta}{3a^2} \left( \zeta \left(4\zeta^2 + 1\right) + \frac{1}{3} \left(\zeta^2 + 1/3\right)^{1/2} \right)$$

Eq. (34a, b) can be expanded to a sixth order polynomial in term of damping ratio $\zeta$. Delete the damping terms higher than third order and setting $a=1$, the simplified thresholds of dimensionless nonlinearity are determined as

$$\gamma_{sd,hd} = \frac{\gamma_{s,h}}{a} = \frac{256}{3\sqrt{3}} \zeta^3$$

which are the same expressions as that in appendix A [21]. $\gamma_{sd}$ and $\gamma_{hd}$ are used for the softening and hardening systems respectively. Fig. 10 show that the absolute values of nonlinear thresholds will increase with the increase of the damping, which indicates that the larger the damping is, the stronger degree of nonlinearity required for the system to jump.

![Graph showing nonlinear thresholds vs. damping ratio](image)

**Fig. 10.** Comparison between the two expressions for solving the nonlinear thresholds under $0<\zeta<0.3$.

5. Conclusions

In this paper, a quintic polynomial equation in term of the square of the jump amplitudes is derived by HBM, which is more concise than the equation in references [28,29]. The corresponding jump frequencies can be obtained by substituting the jump amplitudes into FRC. In order to make the polynomial equation converge to an accurate value quickly, we use the approximate analytical formulas summarized in reference [21] to calculate the initial values of numerical simulation. The jump points determined by the polynomial equation given in this paper are in good agreement with the results of direct numerical simulation, and those given by Friswell [27] and Worden [28] under the same system parameters.

In addition, the characteristic equation of the system is obtained by using Floquet theory, and the explicit analytical expressions of BCs is derived, which can be used to quickly determine the unstable response region of FRC. Most importantly, we determine the global stability of FRC for softening and hardening systems, that are, the criteria of no jump phenomenon in FRC. These criteria are essentially determined by the topological relationship between FRC and BCs, and are analytical expressions composed of damping ratio, nonlinear coefficient and external excitation.
The criterion is provided a fast and quantitative analysis method for the design, analysis and application of a Duffing nonlinear system, which can be used to avoid or utilize the jump phenomenon.

**Appendix A. Some results from reference [21]**

Table A1 Normalised jump-up and -down frequencies and the normalised response amplitudes at these frequencies calculated using the harmonic balance method

|                          | Softening system                                                                 | Hardening system                                                                 |
|--------------------------|----------------------------------------------------------------------------------|----------------------------------------------------------------------------------|
| Jump-up frequency        | $\Omega_u = 1 - \frac{1}{2} \left( \frac{3}{2} \right)^{\gamma^2} | Y_u = \left( \frac{2}{3} \right)^{\gamma^2} \frac{1}{|y|^2}$ | $\Omega_u = 1 + \frac{1}{2} \left( \frac{3}{2} \right)^{\gamma^2} | Y_u = \left( \frac{2}{3} \right)^{\gamma^2} \frac{1}{|y|^2}$ (A1a, b) |
| Amplitude at the jump-up frequency | $Y_u = \left( \frac{2}{3} \right)^{\gamma^2} \frac{1}{|y|^2}$ | (A2)                                                                          |
| Jump-down frequency      | $\Omega_d = \frac{1}{2^{\gamma^2}} \left( 1 + \frac{3\gamma}{4\zeta^2} \right)^{1/2} + 1 \right)^{1/2}$ | (A3)                                                                          |
| Amplitude at the jump-down frequency | $Z_d = \left( \frac{2}{3\gamma} \left( \frac{1 + 3\gamma}{4\zeta^2} \right)^{1/2} - 1 \right)^{1/2}$ | (A4)                                                                          |
| (Maximum amplitude)      |                                                                                 |                                                                                 |
| Threshold value of $|y|$ for a jump to occur | $|y|^2 \geq \frac{256}{3^{5/3} \zeta^3}$ | (A5)                                                                          |

**Declarations**

**List of abbreviations**

FRC: Frequency response curve, BCs: Boundary curves, PMs: Perturbation methods, MSM: Multi-scale method, FRF: Frequency response function, AFR: Amplitude-frequency response, PFR: Phase-frequency response

**Ethics approval and consent to participate**

Not applicable.

**Consent for publication**

Not applicable.

**Availability of data and materials**

Not applicable. All the simulation data can be found in the manuscript.

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**Authors’ Contributions**

LC and BBS were in charge of the derivation of the formulas; LC wrote the manuscript; XGF, WJW and ZY were in charge of the simulation. All authors read and approved the final manuscript.
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