Exit asymptotics for small diffusion about an unstable equilibrium

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Abstract

A dynamical system perturbed by white noise in a neighborhood of an unstable fixed point is considered. We obtain the exit asymptotics in the limit of vanishing noise intensity. This is a refinement of a result by Kifer (1981).

1 Introduction

Random perturbations of dynamical systems have been studied intensively for several decades, see e.g. the classical book [FW98]. In particular, systems with unstable equilibrium points including Hamiltonian and related flows have been considered, see e.g. recent works [FW04], [Kor04]. See also [ASK03], [AB05], and [RHA06] for results on noisy heteroclinic networks and their applications.

The exit asymptotics for a neighborhood of an unstable fixed point was studied in [Kif81]. It was shown that as the intensity $\varepsilon$ of the white noise perturbation tends to 0, the exit distribution tends to concentrate around the invariant manifold associated to the highest Lyapunov exponent $\lambda > 0$, and that the exit time $\tau$ is asymptotically equivalent to $\lambda^{-1} \ln(\varepsilon^{-1})$ in probability.

In this paper we prove a refinement of this asymptotics for additive isotropic noise. In particular, we show that $\tau - \lambda^{-1} \ln(\varepsilon^{-1})$ converges almost surely to a random variable which we describe explicitly.

The approach we take also leads to a simpler proof of the main theorem of [Kif81] for this setting. Our main result will be useful in analysis of

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vanishing noise asymptotics for dynamics with heteroclinic networks, e.g. it provides a rigorous basis for some heuristic arguments from [SH90].

The paper is organized as follows. In Section 2 we describe the setting and state our main result. Its proof is given in Section 5 after a study of the linearized system in Section 3 and estimates on closeness of the linear approximation to the original nonlinear system in Section 4. Proofs of auxiliary lemmas are collected in Section 6.

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2 The setting and the main result

We suppose that there is a $C^2$-vector field $b : U \to \mathbb{R}^d$ defined on a bounded closed set $U \subset \mathbb{R}^d$ equal to the closure of its own interior. This vector field generates a uniquely defined flow $S^t$ associated with the ODE

$$\frac{d}{dt} S^t x = b(S^t x), \quad S^0 x = x.$$ 

This flow is well-defined for all $t \in [0, T_U(x)]$ where $T_U(x)$ is the first time the solution hits $\partial U$:

$$T_U(x) = \inf\{t \geq 0 : S^t x \in \partial U\}.$$ 

A white noise perturbation of $S^t$ is given by the following SDE:

$$dX_\varepsilon(t) = b(X_\varepsilon(t))dt + \varepsilon dW(t), \quad X_\varepsilon(0) = x.$$ 

Here $\varepsilon > 0$, and $W$ is a standard $d$-dimensional Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$. The SDE should be understood in the integral sense:

$$X_\varepsilon(t) = x + \int_0^t b(X_\varepsilon(s))ds + \varepsilon W(t),$$

and the (strong) solution can be obtained for $P$-almost every realization of $W$. We shall sometimes use the notation $S_{\varepsilon,W}^t x$ to denote the solution $X_\varepsilon(t)$ of (2) to stress the dependence on the initial condition and the realization of the noise. This solution is well-defined up to time $T_\varepsilon(x) = T_{\varepsilon,W}(x)$.
which is a (random) stopping time defined as the first time the solution hits $\partial U$:

$$T^U_\varepsilon(x) = T^U_{\varepsilon,W}(x) = \inf\{t \geq 0 : X_\varepsilon(t) \in \partial U\}.$$

Let $G \subset U$ be a closed set with piecewise smooth boundary. For each $x \in G$ we can consider equation (2) and define a stopping time

$$\tau_\varepsilon = \tau_\varepsilon(x) = T^G_{\varepsilon,W}(x) = \inf\{t \geq 0 : X_\varepsilon(t) \in \partial G\}$$

and the corresponding exit point

$$H_\varepsilon = H_\varepsilon(x) = X_\varepsilon(\tau_\varepsilon).$$

With probability 1 we have $(\tau_\varepsilon, H_\varepsilon) \in \partial G \times \mathbb{R}^+$, and we are going to study the asymptotics of this random vector in the limit as $\varepsilon \to 0$.

The limit behavior of $(\tau_\varepsilon, H_\varepsilon)$ depends very much on the vector field $b$ and point $x$. We proceed to describe a setting which is slightly more restrictive than that of [Ki81].

We shall assume that 0 belongs to the interior of $G$, $b(0) = 0$ and there are no other equilibrium points in $G$. We denote $A = J(0)$ where $J$ is the Jacobian matrix

$$J = (\partial_i b_j)_{i,j=1,\ldots,d}.$$

In this note we assume that $A$ has a simple positive eigenvalue $\lambda$ such that real parts of all other eigenvalues are less than $\lambda$. We denote one of the two unit eigenvectors associated to $\lambda$ by $v$. The vector subspace complement to $v$ and spanned by all the other Jordan basis vectors will be denoted by $L$. Projections on $\text{span}\{v\}$ along $L$ and on $L$ along $\text{span}\{v\}$ will be denoted $\Pi_v$ and $\Pi_L$ respectively.

The Hadamard–Perron Theorem (see [KH95, Theorem 6.2.8] and [Har64, Theorem 6.1]) implies that there is a locally $S^2$-invariant $C^1$-curve $\gamma$ containing 0 and tangent to $v$ at 0. This curve is contained in the unstable manifold of the origin, and if all the other Lyapunov exponents are negative, coincides with it. We shall assume that $\gamma \in C^2$ which is true in many important cases. In a small neighborhood of 0, the curve $\gamma$ can be represented as a graph of a map from $\text{span}\{v\}$ to $L$. For small $\delta$ we shall denote by $\gamma(\delta)$ the point $x$ on $\gamma$ such that $\Pi_v x = \delta v$. Notice that $|\Pi_L \gamma(\delta)| = O(\delta^2)$ as $\delta \to 0$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$.

We also assume that $\gamma$ intersects $\partial G$ transversally at two points $q_-$ and $q_+$ so that the part of $\gamma$ connecting $q_-$ and $q_+$ does not intersect $\partial G$ and contains points $\gamma(-\delta), 0, \gamma(\delta)$ (in this order) for some $\delta > 0$. 


We shall need the quantities $h_+$ and $h_-$ defined via:

$$
    h_{\pm} = \lim_{\delta \to 0} \left( \frac{\ln \delta}{\lambda} + t(\pm \delta, q_{\pm}) \right),
$$

(3)

where $t(\delta, q_+)$ and $t(-\delta, q_-)$ denote the times to get from $\gamma(\delta)$ to $q_+$ and from $\gamma(-\delta)$ to $q_-$ respectively:

$$
    t(\pm \delta, q_{\pm}) = T^G(\gamma(\pm \delta)),
$$

(4)

so that $S^{t(\pm \delta, q_{\pm})}(\gamma(\pm \delta)) = q_{\pm}$.

**Lemma 1** The numbers $h_{\pm}$ are well-defined by (3), i.e. finite limits in the r.h.s. exist.

A proof of this lemma is given in Section 6 and we proceed now to our main result.

**Theorem 1** Suppose $x$ belongs to the exponentially stable manifold of 0, i.e. there are positive constants $C$ and $\mu$ such that

$$
    |S^t x| \leq Ce^{-\mu t}, \quad t \geq 0.
$$

(5)

Then the following holds:

1. There is a positive number $\sigma = \sigma(x)$ and a standard Gaussian random variable $N$ defined on the probability space $(\Omega, \mathcal{F}, P)$ such that with probability 1

$$
    H_\varepsilon \to q_+ 1_{\{N > 0\}} + q_- 1_{\{N < 0\}},
$$

and

$$
    \tau_\varepsilon - \frac{\ln(1/\varepsilon)}{\lambda} \to h_+ 1_{\{N > 0\}} + h_- 1_{\{N < 0\}} - \frac{\ln(\sigma | N \rangle)}{\lambda}.
$$

2. The distribution of the random vector $(H_\varepsilon, \tau_\varepsilon - \frac{\ln(1/\varepsilon)}{\lambda})$ converges weakly to

$$
    \frac{1}{2} \delta_{q_+} \times \mu_{h_+, \sigma} + \frac{1}{2} \delta_{q_-} \times \mu_{h_-, \sigma},
$$

where $\mu_{h, \sigma}$ is the distribution of

$$
    h - \frac{\ln(\sigma | N \rangle)}{\lambda}.
$$

3. If $x = 0$, then $\sigma = (2\lambda)^{-1/2}$. 

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Remark 1 Notice that if $A$ has no negative eigenvalues, then the only point $x$ satisfying (5) for some $C, \mu > 0$ is the origin, i.e. the stable manifold is trivial, and our theorem is applicable only for the diffusion started at $x = 0$. In the opposite case, where there is at least one negative eigenvalue, the Hadamard–Perron theorem mentioned above guarantees the existence of a nontrivial stable manifold which plays the role of the unstable one for the system in the reverse time. Notice also that in the latter situation one can choose $\mu$ to be a constant independent of $x$ (namely, take any negative number that is closer to zero than any negative Lyapunov exponent), but $C$ depends on $x$ essentially.

Remark 2 It is an interesting phenomenon that in the situation where there is a variety of unstable directions, the system chooses the most unstable one so that the limiting dynamics is practically 1-dimensional if the leading eigenvalue of the linearization is real and simple. This was observed in [Kif81], where, in fact, a more general situation was considered as well. The leading eigenvalue $\lambda$ was not necessarily assumed real and simple. We can easily extend our approach to recover the main result of [Kif81]: the exit time grows as $\lambda^{-1}\ln(\varepsilon^{-1})$ and the exit measure tends to concentrate at the intersection of $\partial G$ and the invariant manifold corresponding to $\lambda$. However, without our assumptions on $\lambda$, the exit asymptotics analogous to Theorem 1 is more complicated and depends, in particular, on the shape of the set $G$.

Remark 3 The random variable $N$ is constructed explicitly in the proof of the theorem.

3 Linearization

We start our study of the SDE (1) with the analysis of its linearization:

$$\tilde{X}_\varepsilon(t) = S^t x + \varepsilon Y(t),$$

where $Y$ solves the equation in variations:

$$dY(t) = A(t) Y(t) dt + d W(t),$$

$Y(0) = 0$.

Here $A(t) = J(S^t x)$.

The main result of this section is the following lemma.
Lemma 2  There is a centered nondegenerate Gaussian random variable $N$, an a.s.-finite random variable $C_1$ and a number $\rho > 0$ such that with probability 1, for all $t$,

$$|e^{-\lambda t} Y(t) - Nv| \leq C_1 e^{-\rho t}.$$ 

Remark 4  The Gaussian random variable $N$ will be used to construct $\sigma$ and $N$ that appear in the statement of Theorem 1. Namely, $N = \sigma N$, see Section 5.

Proof. Let $Z$ be the solution of

$$dZ(t) = A(Z(t)) dt + dW(t),$$

$Z(0) = 0.$

Then

$$Z(t) = \int_0^t e^{A(t-s)} dW(s),$$

see [KS88, Section 5.6] for a treatment of stochastic linear equations. Let us denote $V(t) = e^{-\lambda t} Z(t)$. Since $e^{A(t-s)} v = e^{\lambda(t-s)} v$, and

$$|e^{A(t-s)} u| < C_2 e^{(\lambda - \Delta)(t-s)} |u|$$

for some positive constants $\Delta, C_2$ and all $u \in L$, we have

$$V(t) \to \Pi_v \int_0^\infty e^{-\lambda s} dW(s), \quad \text{as } t \to \infty,$$

and the convergence is exponentially fast.

Now let $D(t) = Y(t) - Z(t)$. Then

$$\frac{d}{dt} D(t) = AD(t) + (A(t) - A)Y(t),$$

so that

$$D(t) = \int_0^t e^{A(t-s)} (A(s) - A)Y(s) ds$$

$$= \int_0^t e^{\lambda(t-s)} \Pi_v (A(s) - A)Y(s) ds + \int_0^t e^{A(t-s)} \Pi_L (A(s) - A)Y(s) ds.$$

This implies

$$e^{-\lambda t} D(t) = \int_0^t e^{-\lambda s} \Pi_v (A(s) - A)Y(s) ds$$

$$+ e^{-\lambda t} \int_0^t e^{A(t-s)} \Pi_L (A(s) - A)Y(s) ds. \quad (8)$$
To estimate the r.h.s. we write
\[ Y(s) = \int_0^s \Phi_r(s)dW(r), \tag{9} \]
where \( \Phi_r(s) \) denotes the fundamental matrix solving the homogeneous system
\[ \frac{d}{ds}\Phi_r(s) = A(s)\Phi_r(s), \tag{10} \]
\[ \Phi_r(r) = I. \tag{11} \]

For a matrix \( B \), we denote \( |B| = \sup_{|x| \leq 1} |Bx| \).

**Lemma 3** For any \( \alpha > 0 \) there is a constant \( K_\alpha \) such that
\[ |\Phi_s(t)| \leq K_\alpha e^{(\lambda+\alpha)(t-s)} \]
for all \( t, s \) with \( t > s > 0 \).

We prove this lemma in Section 6. An almost immediate implication is the following statement:

**Lemma 4** For any \( \alpha > 0 \) there is an a.s.-finite random constant \( \tilde{K}_\alpha \) such that with probability 1,
\[ |Y(s)| \leq \tilde{K}_\alpha e^{(\lambda+\alpha)s}. \]
for all \( s \geq 0 \).

The proof of this lemma is also given in Section 6. It is important that the positive number \( \alpha \) can be chosen arbitrarily small. In fact, we shall use Lemmas 3 and 4 for \( \alpha < \mu \).

Since \( x \) belongs to the stable manifold of the origin, we have
\[ |A(s) - A| \leq C_3 e^{-\mu s} \]
for some \( C_3 \) and all \( s \geq 0 \), where \( \mu \) was introduced in (5). Therefore, Lemma 4 implies that as \( t \to \infty \), the first integral in (8) exponentially converges to
\[ \Pi_v \int_0^\infty e^{-\lambda s}(A(s) - A)Y(s)ds. \]
The same considerations and (7) imply that the second integral in (8) converges to 0 exponentially fast.
Therefore,
\[
\lim_{t \to \infty} e^{-\lambda t} Y(t) \overset{a.s.}{=} \Pi_v \left[ \int_0^\infty e^{-\lambda s} dW(s) + \int_0^\infty e^{-\lambda s} (A(s) - A) Y(s) \, ds \right].
\] (12)

The r.h.s. is a Gaussian random variable with distribution concentrated on \( \text{span}\{v\} \) since it is a finite linear functional of the Wiener process \( W \).

Our proof will be complete as soon as we show that this linear functional is non-degenerate. Using (9) we rewrite the r.h.s. of (12) as
\[
\Pi_v \left[ \int_0^\infty e^{-\lambda r} dW(r) + \int_0^\infty e^{-\lambda s} (A(s) - A) \Phi_r(s) dW(r) ds \right],
\] (13)
where \( I \) denotes the unit matrix.

Let us take a positive \( \alpha < \mu \). Lemma 3 implies that
\[
\left| \int_r^\infty e^{-\lambda s} (A(s) - A) \Phi_r(s) ds \right| \leq C_3 K \alpha \int_r^\infty e^{-\lambda s} e^{-(\lambda + \alpha)(s-r)} ds \leq \frac{C_3 K \alpha}{\mu - \alpha} e^{-(\lambda + \mu)r},
\]
and the expression in the stochastic integral in the r.h.s. of (13) cannot be identically equal to zero which completes the proof of Lemma 2.

**Lemma 5** If \( x = 0 \), then \( E N^2 = 1/(2\lambda) \), where \( N \) is the centered Gaussian random variable defined in Lemma 2.

**Proof.** If \( x = 0 \), then \( A(t) = A \) for all \( t \geq 0 \). Therefore, the second term in the r.h.s. of (12) vanishes, and the variance of the first term equals \( \int_0^\infty e^{-2\lambda s} ds = 1/(2\lambda) \) due to Itô’s isometry, see [Øks95, Lemma 3.5].

For every \( \delta > 0 \) we shall need a stopping time
\[
\tau(\tilde{X}_\varepsilon, \delta, v) = \inf\{t > 0 : |\Pi_v(\tilde{X}_\varepsilon(t) - S^tx)| \geq \delta\} = \inf\{t > 0 : \varepsilon |\Pi_v Y(t)| \geq \delta\},
\]
where \( \tilde{X}_\varepsilon \) is defined in (9).

**Lemma 6** For any \( \delta > 0 \),
\[
\lim_{\varepsilon \to 0} \left[ \tau(\tilde{X}_\varepsilon, \delta, v) - \frac{\ln \left( \frac{\delta}{\varepsilon |N|} \right)}{\lambda} \right] \overset{a.s.}{=} 0,
\]
where \( N \) is the centered Gaussian random variable defined in Lemma 2.
Proof. Obviously, \( \tau = \tau(\tilde{X}_\varepsilon, \delta, v) \xrightarrow{a.s.} \infty \) as \( \varepsilon \to 0 \), so that Lemma 2 implies
\[
\delta = \varepsilon|\Pi_v Y(\tau)| \sim \varepsilon e^{\lambda \tau}|N|,
\]
and the claim follows.

**Lemma 7** There is a positive number \( \beta \) such that for any \( \delta > 0 \) there is an a.s.-finite random variable \( C_4 = C_4(\delta) \) such that with probability 1
\[
\limsup_{\varepsilon \to 0} \frac{|\varepsilon \Pi_L Y(\tau)|}{\varepsilon^\beta} \leq C_4.
\]

Proof. Lemmas 2 and 6 imply that
\[
|\varepsilon \Pi_L Y(\tau)| \leq C_1 \varepsilon e^{\lambda \tau} e^{-\rho \tau} \sim C_1 \frac{\delta}{|N|} \left( \frac{\varepsilon |N|}{\delta} \right)^{\rho/\lambda},
\]
which proves our claim with \( \beta = \rho/\lambda \) and \( C_4(\delta) = C_1(\delta/|N|)^{1-\rho/\lambda} \).

4 The error of the linear approximation

In this section, we are going to compare the nonlinear diffusion process \( X_\varepsilon \) to its Gaussian linearization \( \tilde{X}_\varepsilon \) considered in Section 3.

**Lemma 8** There is a number \( \delta_0 > 0 \) and a random variable \( C_5 > 0 \) such that, with probability 1, if \( \delta \in (0, \delta_0) \), then
\[
\limsup_{\varepsilon \to 0} |X_\varepsilon(\tau(\tilde{X}_\varepsilon, \delta, v)) - \tilde{X}_\varepsilon(\tau(\tilde{X}_\varepsilon, \delta, v))| < C_5 \delta^2.
\]

Proof. In differential notation, the evolution of \( \tilde{X}_\varepsilon \) is given by
\[
d\tilde{X}_\varepsilon(t) = b(S^t x)dt + \varepsilon dY(t) = b(S^t x)dt + \varepsilon A(t)Y(t)dt + \varepsilon dW(t).
\]
Using \( Y(t) = (\tilde{X}_\varepsilon(t) - S^t x)/\varepsilon \), we obtain
\[
d\tilde{X}_\varepsilon(t) = b(S^t x)dt + A(t)(\tilde{X}_\varepsilon(t) - S^t x)dt + \varepsilon dW(t).
\]
Let us introduce \( U_\varepsilon(t) = X_\varepsilon(t) - \tilde{X}_\varepsilon(t) \), so that \( U_\varepsilon(0) = 0 \) and
\[
\frac{d}{dt} U_\varepsilon(t) = b(X_\varepsilon(t)) - b(S^t x) - A(t)(\tilde{X}_\varepsilon(t) - S^t x).
\]
Since $b \in C^2$, we have

$$b(X_\varepsilon(t)) = b(S^t x) + A(t)(X_\varepsilon(t) - S^t x) + Q(S^t x, X_\varepsilon(t) - S^t x)$$

where

$$|Q(y, z)| \leq C_6|z|^2$$

for a constant $C_6$ and all $y, z$, so that

$$\frac{d}{dt}U_\varepsilon(t) = A(t)U_\varepsilon(t) + Q(S^t x, X_\varepsilon(t) - S^t x).$$

Variation of constants yields:

$$V(t) = \int_0^t \Phi_s(t)Q(S^s x, X_\varepsilon(s) - S^s x)ds,$$

where $\Phi_s(t)$ is defined in (10)–(11). Since

$$|Q(S^s x, X_\varepsilon(s) - S^s x)| \leq C_6|U_\varepsilon(s)|^2 \leq 2C_6|U_\varepsilon(s)|^2 + 2C_6\varepsilon^2|Y(s)|^2,$$

Lemma 3 implies that for any $\alpha > 0$,

$$|U_\varepsilon(t)| \leq 2K_\alpha C_6 \int_0^t e^{(\lambda + \alpha)(t-s)}(|U_\varepsilon(s)|^2 + \varepsilon^2|Y(s)|^2)ds,$$

so that $|U_\varepsilon(t)| \leq u_\varepsilon(t)$, where $u_\varepsilon$ solves

$$\frac{d}{dt}u_\varepsilon(t) = (\lambda + \alpha)u_\varepsilon(t) + 2K_\alpha C_6 u_\varepsilon^2(t) + 2K_\alpha C_6\varepsilon^2|Y(t)|^2, \quad (14)$$

$$u_\varepsilon(0) = 0.$$ 

Obviously, $u_\varepsilon$ is a monotone nondecreasing function. Let us choose $\alpha < \lambda/2$ and denote

$$c = \frac{\frac{\lambda}{2} - \alpha}{2K_\alpha C_6}.$$ 

If $|u_\varepsilon(t)| \leq c$, then

$$(\lambda + \alpha)u_\varepsilon(t) + 2K_\alpha C_6 u_\varepsilon^2(t) \leq \frac{3}{2}\lambda u_\varepsilon(t).$$

Therefore,

$$1_{\{u_\varepsilon(t) \leq c\}} \frac{d}{dt}u_\varepsilon(t) \leq \frac{3}{2}\lambda u_\varepsilon(t) + 2K_\alpha C_6\varepsilon^2|Y(t)|^2,$$
so that
\[ u_\varepsilon(t)1_{\{u_\varepsilon(t) \leq c}\} \leq 2K_\alpha C_6 \varepsilon^2 \int_0^t e^{\frac{3}{2}\lambda(t-s)} |Y_\varepsilon(s)|^2 ds. \]

Since \( |Y_\varepsilon(t)| \sim e^{\lambda t} |N| \), the r.h.s. is asymptotically equivalent to
\[ 2K_\alpha C_6 N^2 \varepsilon^2 \int_0^t e^{\frac{3}{2}\lambda(t-s)} e^{2\lambda s} ds \sim 2K_\alpha C_6 N^2 \varepsilon^2 e^{2\lambda t}, \]
which implies that
\[ \limsup_{t \to \infty} \frac{u_\varepsilon(t)1_{\{u_\varepsilon(t) \leq c\}}}{2K_\alpha C_6 N^2 \varepsilon^2 e^{2\lambda t}} \leq 1 \quad (15) \]
uniformly in \( \varepsilon > 0 \). Next, let us consider \( \tau(u_\varepsilon, c) = \inf\{t \geq 0 : u_\varepsilon(t) \geq c\} \). Monotonicity of the r.h.s of (14) in \( \varepsilon \) implies \( \tau(u_\varepsilon, c) \to \infty \) as \( \varepsilon \to 0 \). Since \( |u_\varepsilon(s)| \leq c \) for all \( s \leq \tau(u_\varepsilon, c) \) and \( |u_\varepsilon(\tau(u_\varepsilon, c))| = c \), formula (15) implies
\[ \liminf_{\varepsilon \to 0} \frac{c}{2K_\alpha C_6 N^2 \varepsilon^2 e^{2\lambda \tau(u_\varepsilon, c)}} \leq 1, \]
i.e.
\[ \liminf_{\varepsilon \to 0} \left[ \tau(u_\varepsilon, c) - \frac{\ln(\frac{1}{|N|\varepsilon})}{\lambda} - \frac{\ln(\frac{c}{2K_\alpha C_6})}{2\lambda} \right] \geq 0. \]

The last relation and Lemma 6 imply that for sufficiently small \( \delta_0 \) and all \( \delta \in (0, \delta_0) \), there is an \( \varepsilon_0 = \varepsilon_0(\delta) \) such that if \( 0 < \varepsilon < \varepsilon_0 \), then
\[ \tau(\hat{X}_\varepsilon, \delta, v) < \tau(u_\varepsilon, c). \]

Now (15) implies that for these \( \delta \) and sufficiently small \( \varepsilon \)
\[ u_\varepsilon(\tau(\hat{X}_\varepsilon, \delta, v)) \leq 3K_\alpha C_6 N^2 \varepsilon^2 e^{2\lambda \tau(\hat{X}_\varepsilon, \delta, v)} \leq 4K_\alpha C_6 \delta^2, \]
where in the last inequality we used Lemma 6 again. The proof is complete.

5 Proof of the main result

We begin with auxiliary well-known statements. The first statement estimates closeness of perturbed trajectories to the orbits of the unperturbed system.

Lemma 9 Let \( W^*(t) = \sup_{s \in [0,t]} |W(s)| \). Then, for any \( y \in U \), any \( t < T^U(y) \), for a.e. Wiener trajectory \( W \) and \( \varepsilon < \varepsilon_0(W) \),
\[ |S_{\varepsilon,W}^t y - S^t y| \leq \varepsilon W^*(t)e^{Mt}, \]
where \( M \) is the Lipschitz constant of \( b \) on \( U \).
Proof of Lemma 9. Denote \( V_\varepsilon(t) = S^t_{\varepsilon,W}y - S^t x \). Then

\[
V(t) = \int_0^t (b(S^s_{\varepsilon,W}y) - b(S^s y))ds + \varepsilon W(s),
\]

so that

\[
|V_\varepsilon(t)| \leq \int_0^t M|V_\varepsilon(s)|ds + \varepsilon |W(s)|,
\]

and the lemma follows from Gronwall's inequality and a simple localization argument.

The next statement will estimate the closeness to the invariant curve \( \gamma \). We need more notation.

For \( K > 0 \) we introduce two sets

\[
D^{\pm}(\delta, K) = \{x \in \mathbb{R}^d : |x \mp \delta v| \leq K\delta^2\}.
\]

We shall need a closed set \( G' \subset U \) with the following properties: the boundary of \( G' \) is piecewise smooth, \( G \) is contained in the interior of \( G' \), and \( \gamma \) intersects \( \partial G' \) transversally.

**Lemma 10** For any \( K > 0 \) and sufficiently small \( \delta \), there are positive numbers \( T^+ = T^+(\delta) \) and \( T^- = T^-(\delta) \) such that

\[
S^{T^+}D^{\pm}(\delta, K) \subset U \setminus G'.
\]

For any \( K > 0 \)

\[
\lim_{\delta \to 0} \sup_{t \leq T'^+(\gamma(\pm \delta))} \sup_{y \in D^{\pm}(\delta, K)} |S^t y - S^t(\gamma(\pm \delta))| = 0.
\]

This lemma follows from the graph transform method of constructing the invariant manifold \( \gamma \) (see e.g. a version of Hadamard–Perron Theorem and its proof in [KH95, Section 6.2]).

Proof of Theorem 1. For any \( y \in G \) and any time \( \nu \geq 0 \) we define \( H^\nu_\varepsilon(y) \) and \( \tau^\nu_\varepsilon(y) \) analogously to \( H_\varepsilon(y) \) and \( \tau_\varepsilon(y) \), but using shifted trajectories \( W(\nu + \cdot) - W(\nu) \) instead of \( W(\cdot) \).

For sufficiently small \( \delta, \varepsilon > 0 \),

\[
H_\varepsilon(x) = H_\varepsilon^{\tau(\tilde{X}_\varepsilon, \delta, v)}(X_\varepsilon(\tau(\tilde{X}_\varepsilon, \delta, v))), \quad (16)
\]

\[
\tau_\varepsilon(x) = \tau(\tilde{X}_\varepsilon, \delta, v) + \tau_\varepsilon^{\tau(\tilde{X}_\varepsilon, \delta, v)}(X_\varepsilon(\tau(\tilde{X}_\varepsilon, \delta, v))). \quad (17)
\]
Let us now define the nondegenerate Gaussian random variable $N$ according to Lemma 2. Lemma 7 (estimating the closeness of the linearized process to $\text{sgn}(N)\delta v$ at the exit time $\tau(\hat{X}_\varepsilon, \delta, v)$) and Lemma 8 (estimating the closeness of the nonlinear process to the linearized process at the time $\tau(\hat{X}_\varepsilon, \delta, v)$) imply that there is a constant $\delta_0 > 0$ and a positive a.s.-finite random variable $C_7$ such that with probability 1 for $\delta \in (0, \delta_0)$ we have
\[
\limsup_{\varepsilon \to 0} \left| X_\varepsilon(\tau(\hat{X}_\varepsilon, \delta, v)) - \text{sgn}(N)\delta v \right| < C_7 \delta^2. \tag{18}
\]

To estimate the effect of the noise for the evolution along $\gamma$, we take a $K > 0$ and write
\[
\sup_{y \in D^\pm(\delta, K)} |S_t^\varepsilon, y - S_t^\varepsilon(\gamma(\pm\delta))| \leq \sup_{y \in D^\pm(\delta, K)} |S_t^\varepsilon, y - S_t^\varepsilon y| + \sup_{y \in D^\pm(\delta, K)} |S_t^\varepsilon y - S_t^\varepsilon(\gamma(\pm\delta))| \leq \varepsilon W^*(t)e^{Mt} + r(\pm\delta, K) \tag{19}
\]
if all the involved processes stay within $G'$ up to time $t$. Here we used Lemma 9 to bound the first term and denoted by $r(\pm\delta, K)$ the second term. Notice that for each $K > 0$ we have $r(\pm\delta, K) \to 0$ as $\delta \to 0$ due to Lemma 10.

So, we need an estimate on the exit time from $G'$. Let us fix $K > 0$ and small $\delta > 0$. Due to the continuous dependence of $S_t^\varepsilon, y$ on $y$ and $\varepsilon$, Lemma 10 allows us to choose constant times $\tilde{T}(\pm\delta, K) > 0$ and $\varepsilon_0 = \varepsilon_0(W) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, we have $(S_t^\varepsilon, y) \cap G' = \emptyset$. In particular, for fixed $\delta$ and $K$, and for almost every Wiener trajectory $W$, we have
\[
\limsup_{\varepsilon \to 0} \sup_{y \in D^\pm(\delta, K)} T_{\varepsilon, y}^G \leq \tilde{T}(\delta, K),
\]
and (19) implies that
\[
\limsup_{\varepsilon \to 0} \sup_{y \in D^\pm(\delta, K)} |S_t^\varepsilon, y - S_t^\varepsilon(\gamma(\pm\delta))| \leq r(\pm\delta, K) \tag{20}
\]
for all $t \leq \tilde{T}(\delta, K)$.

Since for any $y \in D^\pm(\delta, K)$, almost every $W$ and all sufficiently small $\varepsilon > 0$, we have $T_{\varepsilon, W}^G(\varepsilon) \leq \tilde{T}(\delta, K)$, we can combine (20) with the strong Markov property and estimate (18) to see that
\[
\limsup_{\varepsilon \to 0} \left| H_{\varepsilon}(\hat{X}_\varepsilon, \delta, v)(X_\varepsilon(\tau(\hat{X}_\varepsilon, \delta, v))) - H_0(\gamma(\text{sgn } N\delta))1_{\{C_7 < K\}} \right| \leq r_1(\pm\delta, K),
\]
\[13\]
and
\[
\limsup_{\varepsilon \to 0} |\tau(\tilde{X}, \delta, v)(X(\tau(\tilde{X}, \delta, v))) - T^G(\gamma(\text{sgn} N\delta))| 1_{\{C_7 < K\}} \leq r_1(\delta, K),
\]
for a deterministic function \(r_1(\delta, K) > 0\) such that \(\lim_{\delta \to 0} r_1(\delta, K) = 0\) for any fixed \(K > 0\).

Therefore, Lemma 6 and identities (16) and (17) imply that for fixed \(\delta\) and \(K\), on \(\{C_7 < K\}\) we have
\[
\limsup_{\varepsilon \to 0} \left| \tau(\varepsilon)(x) - \frac{\ln \left( \frac{\delta}{\varepsilon |N|} \right)}{\lambda} - T^G(\gamma(\text{sgn} N\delta)) \right| \leq r_2(\delta, K),
\]
and
\[
\limsup_{\varepsilon \to 0} |H(\varepsilon)(x) - H_0(\gamma(\text{sgn} N\delta))| \leq r_2(\delta, K)
\]
with \(\lim_{\delta \to 0} r_2(\delta, K) = 0\).

Since \(\{C_7 < K\} \uparrow \Omega\) as \(K \to \infty\), Part 1 of Theorem 1 follows with \(\sigma = \sqrt{EN^2} > 0\) and \(N = N/\sigma\).

Part 2 follows from Part 1 as soon as we notice that \(|N|\) and \(\text{sgn} N\) are independent, the latter taking values \(\pm 1\) with probability \(1/2\).

Part 3 of Theorem 1 follows from Lemma 5, and the proof is complete.

6 Auxiliary Lemmas

Proof of Lemma 1. Let us prove that \(h_+\) is well-defined by (3) (the same proof works for \(h_-\) as well). Let \(z(t) = |\Pi_\tau S^{-t}q_+|\). There is \(t_0 > 0\) such that on the semiline \((t_0, +\infty)\) the function \(z(t)\) is monotone decreasing and satisfies
\[
\dot{z}(t) = -\lambda z(t) + r(z(t)),
\]
where \(|r(z)| \leq K|z|^2\) for a constant \(K\) and all \(z\). Therefore,
\[
\frac{\ln \delta}{\lambda} + t(\delta, q_+) = \frac{\ln \delta}{\lambda} + t(\delta, S^{-t_0}q_+) + t_0
\]
\[
= -\int_{\delta}^{z(t_0)} \frac{dy}{\lambda y} + \int_{\delta}^{z(t_0)} \frac{dy}{\lambda y + r(y)} + \frac{\ln z(t_0)}{\lambda}
\]
\[
= -\int_{\delta}^{z(t_0)} \frac{r(y)dy}{\lambda y(\lambda y + r(y))} + \frac{\ln z(t_0)}{\lambda},
\]
where \(t(\delta, S^{-t_0}q_+)\) is defined analogously to (14). Our claim follows from the convergence of the integral in the r.h.s. as \(\delta \to 0\).
Proof of Lemma 3. Let us choose a new basis in \( \mathbb{R}^d \) so that in that basis the Euclidean norm of \( A \) (denoted by \( \| A \| \)) is bounded by \( \lambda + \alpha/2 \). This can be done as in [KH95, Section 1.2]. Now Lemma 4.1 from [Har64, Chapter IV] implies
\[
\| \Phi_s(t) \| \leq e^\int_s^t \| A(r) \| dr,
\]
and our claim follows from \( \lim_{r \to \infty} A(r) = A \).

Proof of Lemma 4. Itô’s isometry implies
\[
E|Y(t)|^2 = E \left( \int_0^t \Phi_r(t) dW(r) \right)^2 = \int_0^t \| \Phi_r(t) \|^2_2 dr,
\]
where \( \| B \|_2^2 = \sum_{i,j} B_{ij}^2 \) is the square of the quadratic norm of a matrix \( B \), so that due to Lemma 3 and the equivalence of any two norms in \( \mathbb{R}^d \), we have
\[
E|Y(t)|^2 \leq K'_{\alpha/2} \int_0^t e^{2(\lambda + \alpha/2)(t-r)} dr \leq K''_{\alpha/2} e^{2(\lambda + \alpha/2)t},
\]
for some constants \( K'_{\alpha/2}, K''_{\alpha/2} \), and all \( t \geq 0 \). Inequality (21) with the Borel–Cantelli Lemma implies the desired growth of \( Y(t) \) along integer values of \( t \). To interpolate between the integer times, we apply the standard Kolmogorov–Chentsov technique based on an estimate for increments of \( Y \).

For any \( z > 0 \), we have
\[
P\{|Y(t_2) - Y(t_1)| \geq z\} \leq \frac{4}{z^2} E \left( \int_{t_1}^{t_2} A(s)Y(s)ds \right)^2 + \frac{16}{z^4} E (W(t_2) - W(t_1))^4 \leq \hat{K}_{\alpha/2}(t_2 - t_1)^2 \left[ \frac{1}{z^2} e^{2(\lambda + \alpha/2)t_2} + \frac{1}{z^4} \right],
\]
where \( \hat{K}_{\alpha/2} \) is a positive constant.

For \( n, m \in \{0\} \cup \mathbb{N} \), we introduce \( D_{m,n} \) as the set of all the rationals of the form \( k/2^n \in [m, m+1] \) with integer \( k \). For each \( t \in D_{m,n} \) with \( n \in \mathbb{N} \), we define \( t_- = \sup\{s \in D_{m,n-1} : s \leq t\} \). Then \( |t - t_-| \leq 2^{-n} \). Pick any \( \rho \)
with $1 < \rho^4 < 2$. The continuity of the trajectories of $Y$ implies that

$$\mathbb{P}\left\{ \sup_{s \in [m,m+1]} |Y(s)| \geq e^{(\lambda+\alpha)m} \right\} \leq \mathbb{P}\left\{ |Y(m)| \geq e^{(\lambda+\alpha)m}\frac{\rho - 1}{\rho} \right\} + \sum_{n=1}^{\infty} \sum_{t \in D_{m,n}} \mathbb{P}\left\{ |Y(t) - Y(t_-)| \geq e^{(\lambda+\alpha)m}\frac{\rho - 1}{\rho^{n+1}} \right\}$$

$$\leq \frac{K''_{\alpha/2}}{e^{2(\lambda+\alpha/2)m}m^2(\rho - 1)^2} \sum_{n=1}^{\infty} 2^n \tilde{K}_{\alpha/2}(2^{-n})^2 \left[ \frac{e^{2(\lambda+\alpha/2)(m+1)}\rho^{n+2}}{e^{2(\lambda+\alpha)m}(\rho - 1)^2} + \frac{\rho^{4n+4}}{e^{4(\lambda+\alpha)m}(\rho - 1)^4} \right].$$

Due to our choice of $\rho$, the series in the r.h.s. converges exponentially, and the whole r.h.s. decays in $m$ as $e^{-\alpha m}$, so that we can finish the proof applying the Borel–Cantelli lemma.

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