Critical edge behavior in the singularly perturbed Pollaczek-Jacobi type unitary ensemble

Zhaoyu Wang, Engui Fan

Abstract: In this paper, we study the strong asymptotic for the orthogonal polynomials and universality associated with singularly perturbed Pollaczek-Jacobi type weight

\[ w_{p,2}(x,t) = e^{-\frac{t}{x(1-x)}}x^\alpha(1-x)^\beta, \]

where \( t \geq 0, \alpha > 0, \beta > 0 \) and \( x \in [0,1] \). Our main results obtained here include two aspects: I. Strong asymptotics: We obtain the strong asymptotic expansions for the monic Pollaczek-Jacobi type orthogonal polynomials in different interval \((0,1)\) and outside of interval \(\mathbb{C}\backslash(0,1)\), respectively; Due to the effect of \( \frac{t}{x(1-x)} \) for varying \( t \), different asymptotic behaviors at the hard edge 0 and 1 were found with different scaling schemes. Specifically, the uniform asymptotic behavior can be expressed as a Airy function in the neighborhood of point 1 as \( \zeta = 2nt \to \infty, n \to \infty \), while it is given by a Bessel function as \( \zeta \to 0, n \to \infty \). II. Universality: We respectively calculate the limit of the eigenvalue correlation kernel in the bulk of the spectrum and at the both side of hard edge, which will involve a \( \psi \)-functions associated with a particular Painlevé III equation near \( x = \pm 1 \). Further, we also prove the \( \psi \)-funcation can be approximated by a Bessel kernel as \( \zeta \to 0 \) compared with a Airy kernel as \( \zeta \to \infty \). Our analysis is based on the Deift-Zhou nonlinear steepest descent method for the Riemann-Hilbert problems.

Keywords: singularly perturbed Pollaczek-Jacobi type weight, strong asymptotic expansions, eigenvalue correlation kernel, universality, Riemann-Hilbert problem, Deift-Zhou nonlinear steepest descent method.

1 School of Mathematical Sciences, Fudan University, Shanghai 200433, P.R. China.
* Corresponding author and email address: faneg@fudan.edu.cn
1 Introduction

The orthogonal polynomials (OPS) associated with different weights have important applications in random matrix theory [1, 2], approximation theory [3] and the zeros distribution theory [4]. There are many ways to classify the weight functions. One way, just like the classification of random matrices ensembles, is to identify the weights into Hermite weight, Gaussian weight, Laguerre weight and Jacobi weight and so on [5]. Another way is to classify the weights into Szegö class or non-Szegö class according the following Szegö condition [3].

A weight $w(x)$ is said to be of the Szegö class if it has a definition in the domain $[-1, 1]$ and satisfies

$$\int_{-1}^{1} \frac{\ln w(x)}{\sqrt{1 - x^2}} dx > -\infty.$$ 

It is interest to study the weight in the Szegö class. It has been shown that a large number of weights belong to the Szegö class. Especially, the Jacobi weight is a typical example of the Szegö class, in details see [6].

The asymptotics of OPS with respect to the classical Jacobi weight has been studied in [3]. There are different generalizations to the classical Jacobi weight. For example, Kuijlaars et al investigated the OPS about the modified Jacobi weight

$$w(x) = (1 - x)\alpha(1 + x)\beta h(x), \quad x \in (-1, 1),$$

where $\alpha, \beta > -1$ and $h(x) > 0$ is a real analytic function for $x \in (-1, 1)$. They further obtained the asymptotic behavior for the Hankel determinants, the recurrence coefficients the leading coefficients of the OPS. Xu and Zhao have obtained the asymptotics of the polynomials orthogonal with respect to the perturbed Chebyshev weight in [7],

$$w(x) = (1 - x)^{-\frac{1}{2}}(t_n^2 - x^2)^\alpha, \quad x \in (-1, 1), t_n > 1.$$ 

At the same time, they give the critical edge behavior for this weight with different Painlevé equation according to the different property of $t_n$.

However, there are some weights that furnish a non-Szegö class such as the Pollaczek weight which will considered in this paper. The Pollaczek polynomials are orthogonal to the
following Pollaczek weight function

\[ w(x, a, b) = \exp(2\theta - \pi)h(\theta)[\cosh(\pi h(\theta))]^{-1}, \]

where \( \theta = \arccos\sqrt{x} \), \( h(\theta) = (a\cos\theta + b)/(2\sin\theta) \), and \( a \) and \( b \) are constants, and \( |b| < a \) [3, 8]. Compared by the Jacobi polynomials, the Pollaczek polynomials show a singular behavior in some items, such as the properties of the zero and the gap [3]. Ismail have studied the large-\( n \) asymptotic behavior of the Pollaczek polynomials using the confluent Horn function in [9]. Bo and Wong have investigated the asymptotics with the technique of the Airy function, see [10].

In recent years, there are some interests on the study on the associated weight functions with singular behavior in several different areas of mathematics and physics. For instance, Berry and Shukla studied the statistics for the zeros of the Riemann Zeta function in the Gaussian unitary ensemble of random matrices, with the singularly perturbed Gaussian weight [11]

\[ w(x; z, s) = \exp(-\frac{z^2}{2x^2} + \frac{s}{x} - \frac{x^2}{2}), z \in \mathbb{R} \setminus 0, s \in [0, \infty), x \in \mathbb{R}. \]

Then, Brightmore, Mezzadri and Mo studied the double scaling limit as \( N \to \infty \) and computed the asymptotics of the partition function when \( z \) and \( t \) are of order \( O(N^{-1/2}) \) which can be characterized as a particular Painlevé III equation [12]. Besides, Lukyanov calculated the finite temperature expectation values of the exponential fields to one-loop order of the semi-classical expansion under the singularly perturbed Laguerre weight

\[ w(x, t) = x^\alpha e^{-x-t/x}, \alpha > 0, t > 0, x \in (0, \infty), \]

which is the Laguerre weight \( x^\alpha e^{-x} \) perturbed by a multiplicative factor \( e^{s/x} \) inducing an infinitely strong zero at the origin [13]. In 2010, Chen and Its studied a certain linear statistics of the unitary Laguerre ensembles and the determination of the associated Hankel determinant and recurrence coefficients [14]. After that, Xu, Dai, and Zhao showed the eigenvalue correlation kernel associated with a particular Painlevé III equation. They also studied the transition of this limiting kernel to the Bessel and Airy kernels [15]. Further,
Chen and Dai introduced a sequence of polynomials orthogonal with respect to a one-parameter family of the Pollaczek-Jacobi type weight

\[ w(x, t) = x^\alpha (1 - x)^\beta e^{-t/x}, \quad t \geq 0, x \in [0, 1]. \quad (1.1) \]

If \( t = 0 \), this becomes to a shifted Jacobi weight. They expressed the recurrence coefficients in terms of a set of auxiliary quantities which equaled to a particular Painlevé V or allied functions using the ladder operator [16]. Chen and coauthors showed the uniform asymptotic expansions for the monic OPS and the recurrence coefficients and leading coefficient of the OPS were expressed in terms of a particular Painlevé III using the Riemann-Hilbert approach [17]. It is easy to check that this weight belongs to non-Szego since this weight is in some extent more “singular”.

The above weights considered admit the first order singularities. However, it is also interesting to study weights with higher order singularities. Atkin, Claey and Mezzadri obtained asymptotics for the partition functions associated to the Laguerre and Gaussian Unitary Ensembles perturbed with a pole of order \( k \) at the origin and their results were described in terms of a hierarchy of higher order analogs to the PIII equation [18]. Also, Dan, Xu and Zhang introduced the Fredholm determinant of an integrable operator whose kernel is constructed out of the \( \Psi \)-function associated with a hierarchy of higher order analogues to the Painlevé III equation and obtained the large \( s \) asymptotics of the Fredholm determinant using the RH method [19].

One of interesting phenomenon in random matrices is that as the order of the matrix goes to be large, the eigenvalues of different types of random matrix ensembles show the same statistical law. This phenomenon is known as universality, see [22, 23]. For example, for the Jacobi unitary ensemble, its limiting mean eigenvalue density admits the property

\[
\lim_{n \to +\infty} \frac{1}{n} K_n(x, x) = \frac{1}{\pi \sqrt{1 - x^2}}, \quad x \in (-1, 1).
\]

The limit of \( K_n \) is given by the sine kernels in the bulk of the spectrum [22, 24],

\[
S(x, y) = \frac{\sin \pi (x - y)}{x - y}, \quad (1.2)
\]
And the limit of $K_n$ shows the Bessel kernel at the hard edge of the spectrum:

$$
J_\alpha(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J'_\alpha(\sqrt{x})}{2(x-y)}.
$$

(1.3)

In this paper, we apply the Deift-Zhou nonlinear steepest descent method to study the OPS associated with the Pollaczek-Jacobi type weight involving two singularities at the edge

$$
w_{p,2}(x) = x^\alpha(1-x)^\beta e^{-t/x(1-x)},
$$

(1.4)

where $t \geq 0$, $\alpha > 0$, $\beta > 0$, $x \in [0,1]$. Compared to the weight (1.1), our weight (1.4) involves one more singular perturbation $1/(z-1)$. Our purpose is to see whether some new phenomena occur with the change of singularities. For our weight (1.4), we have to construct another model RH problem to match the original RH problem, rather than simply using the RH problem for the Bessel model in [17]. To build a suitable model, using the same method in [15, 21], we formulate a model RH problem, which is equivalent to a Cauchy-type singular integral equations, see [20]. Moreover, there is an interesting result that this model can transform from a modified Bessel model RH problem as $s \to 0$, $n \to \infty$ to another modified Airy function as $s \to \infty$, $n \to \infty$.

This paper is organized as follows. In Section 2, we state our main results which will be presented in this paper. In section 3, we propose a RH problem with respect to the weight (1.4) to characterize the Pollaczek-Jacobi type OPS. In Section 3, we apply the Deift-Zhou nonlinear steepest descent method to analyze the RH problem. In section 4, based on the asymptotic analysis, we provide detail proofs to all results stated in the section 2.

2 Statement of Main Results

Let introduce some notations used in this paper. Suppose that $\pi_n(x) = \pi_n(x; w_{p,2})$ denote the monic OPS of degree $n$ orthogonal with respect to the weight (1.4), then it holds that

$$
\int_0^1 \pi_n(x)\pi_m(x)w_{p,2}(x,t) \, dx = h_n(t)\delta_{nm}.
$$

(2.1)
Let $P_n(x) = P_n(x; w_{p,j}(x,t))$ denote normalized polynomials with respect to the weight $w_{p,j}(x,t)$, then we have

$$P_n(x) = \gamma_n \pi_n(x),$$

where $\gamma_n > 0$ is the leading coefficient of $P_n(x)$ and $h_n = \gamma_n^{-2}$.

According to basic property of orthogonal polynomials [1, 2, 5], then $\pi_n(x)$ meets the three-term recurrence relation

$$x \pi_n(x) = \pi_{n+1}(x) + \alpha_n(t) \pi_n(x) + \beta_n(t) \pi_{n-1}(x),$$

and the eigenvalue correlation kernel is expressed as

$$K_n(x, y; t) = \frac{\pi_n(x) \pi_{n-1}(y) - \pi_n(y) \pi_{n-1}(x)}{(x - y)}.$$  \hfill (2.4)

Using the Christoffel-Darboux formula [3], we can get

$$K_n(x, y; t) = \gamma_n^2 \left[ \frac{\pi_n(x) \pi_{n-1}(y) - \pi_n(y) \pi_{n-1}(x)}{(x - y)} \right] \left( 1 + O(\frac{1}{n}) \right).$$

### 2.1 Asymptotic for monic OPS

Using the Deift-Zhou steepest descent method, we get the uniform asymptotic expansion for the monic Pollaczek-Jacobi type OPS.

**Theorem 1.** For $z \in \mathbb{C} \setminus [0, 1]$, as $n \to \infty$, $\pi_n(z)$ has the following asymptotic approximation

$$\pi_n(z) = 2^{-2n(\alpha + \beta + 1)} z^{-\frac{2n+\alpha+\beta+1}{2}} (1 - z)^{-\frac{2n+\alpha+\beta+1}{4}} e^{\frac{z}{2(1-z)} (1 + O(\frac{1}{n}))},$$

where $t > 0$, $\phi(z) = 2z - 1 + 2\sqrt{z(z-1)}$ and the error terms have a uniform for $z$ on compact subsets of $\mathbb{C} \setminus [0, 1]$ as well as a full asymptotic expansion in terms of $1/n$.

We also calculate the strong asymptotic expansion in the bulk of the orthogonality interval $(0, 1)$.

**Theorem 2.** Let $K_1$ be a compact subset of $(0, 1)$. When $0 < x < 1$ and $n \to \infty$,

$$\pi_n(x) = \frac{2^{-(\alpha + \beta + 2n)}}{\sqrt{w_{p,j}(x,t)}} \left[ (1 + O(\frac{1}{n})) \cos((2n + \alpha + \beta + 1)\arccos x - \frac{\beta \pi}{2} + \frac{\pi}{4}) \right]$$

$$+ O\left( \frac{1}{n} \right) \cos((2n + \alpha + \beta + 1)\arccos x - \frac{\beta \pi}{2} - \frac{\pi}{4})$$

(2.6)
The above expansion is uniform for \( x \in \mathcal{K}_1 \).

As \( \zeta = 2n^2t \to 0 \), the asymptotic behavior of \( \pi_n \) near the endpoint 1 is described by the following theorem.

**Theorem 3.** There is a small and positive constant \( r \) which satisfies \( 0 < r < \frac{1}{2} \), such that \( x \in (1-r,1) \), as \( n \to \infty \), then

\[
\pi_n(x) = \frac{\sqrt{n}(2n \arccos \sqrt{x})^\frac{3}{2}}{2^{2n+\alpha+\beta} x^{\frac{3}{2}} (1-x)^{\frac{1}{2}} \sqrt{2w_n(x,t)}} \times [(1 + O(1/n) + O(\zeta))(\cos \theta_1 J_\beta(2n \arccos \sqrt{x}) + \sin \theta_1 J_\beta'(2n \arccos \sqrt{x}))
+ (O(1/n) + O(\zeta))(\cos \theta_2 J_\beta(2n \arccos \sqrt{x}) + \sin \theta_2 J_\beta'(2n \arccos \sqrt{x}))],
\]

(2.7)

where

\[ \theta_1 = (\alpha + \beta + 1) \arccos \sqrt{x}, \quad \text{and} \quad \theta_2 = (\alpha + \beta - 1) \arccos \sqrt{x}, \]

(2.8)

and \( J_\beta \) is the Bessel function of order \( \beta \). The error terms hold uniformly for \( x \) in compact subsets of \( (1-r,1) \cap \mathcal{D}_1 \), \( \mathcal{D}_1 = \{ x|2n \pi \arccos \sqrt{x} \in (0,\infty), n \to \infty \} \).

Near the endpoint 1, the strong asymptotic expansion can be calculated in terms of the Airy function as \( \zeta = 2n^2t \to \infty \).

**Theorem 4.** There is a small and fixed constant \( \eta \), \( 0 < \eta < \frac{1}{2} \), such that \( x \in (1-\eta,1) \) and \( \zeta = 2n^2t \to \infty \),

\[
\pi_n(x) = \frac{\sqrt{n}(2n \arccos \sqrt{x})^\frac{3}{2}}{2^{2n+\alpha+\beta} x^{\frac{3}{2}} (1-x)^{\frac{1}{2}} \sqrt{2w_n(x,t)}} \times \left\{ (1 + O(n^{-\frac{1}{2}}))[(\cos \theta_1 d_{11} + \sin \theta_1 (d_{21} - bd_{11}) (2n \arccos \sqrt{x})^{-1}) \text{Ai}(\xi)]
+ (\cos \theta_1 d_{12} - \sin \theta_1 (d_{22} + bd_{12}) (2n \arccos \sqrt{x})^{-1} \text{Ai}(\xi)] \right\}
+ \left\{ O(n^{-\frac{1}{2}})[(\cos \theta_2 d_{11} + \sin \theta_2 (d_{21} - bd_{11}) (2n \arccos \sqrt{x})^{-1}) \text{Ai}(\xi)]
+ (\cos \theta_2 d_{12} - \sin \theta_2 (d_{22} + bd_{12}) (2n \arccos \sqrt{x})^{-1} \text{Ai}(\xi)] \right\},
\]

(2.9)

where \( \theta_1, \theta_2 \) are in \( \left[ \frac{1}{2}, \frac{\pi}{2} \right] \), \( \xi = (\frac{3}{2})^\frac{3}{2} (1 - |\lambda|) \zeta^\frac{3}{2} |\lambda|^{-\frac{1}{2}}, \lambda = \zeta^{-\frac{1}{2}} n^2 f_1(z) \), \( f_1(z) = (\log(z))^2 \).

The error terms hold uniformly for \( x \) in compact subsets of \( (1-\eta,1) \cap \mathcal{D}_2 \), where

\[ \mathcal{D}_2 = \{ x|\left| \frac{3}{2} \right| (1 - |\lambda|) \zeta^\frac{3}{2} \lambda \in (-\infty,0), |\lambda| = \zeta^{-\frac{1}{2}} (2n \arccos \sqrt{x})^2, \zeta \to \infty, n \to \infty \}. \]
At the same time, \( b, d_{11}, d_{12}, d_{21}, \) and \( d_{22} \) are defined as follows:

\[
b = \frac{7}{72(|\lambda| - 1)} \tag{2.10}
\]

\[
d_{11} = \left(\frac{3}{2}\right)^{\frac{1}{2}} \zeta^{-\frac{1}{2}} |\lambda|^{-\frac{1}{2}} \cos \left(\frac{\beta}{2} \arccos \frac{1}{\sqrt{|\lambda|}}\right), \tag{2.11}
\]

\[
d_{22} = \left(\frac{3}{2}\right)^{-\frac{1}{2}} \zeta^{\frac{1}{2}} |\lambda|^{\frac{1}{2}} \cos \left(\frac{\beta}{2} \arccos \frac{1}{\sqrt{|\lambda|}}\right), \tag{2.12}
\]

\[
d_{12} = \left(\frac{3}{2}\right)^{-\frac{1}{2}} \zeta^{\frac{1}{2}} (|\lambda| - 1)^{-\frac{1}{2}} |\lambda|^{\frac{1}{2}} \sin \left(\frac{\beta}{2} \arccos \frac{1}{\sqrt{|\lambda|}}\right), \tag{2.13}
\]

\[
d_{21} = \left(\frac{3}{2}\right)^{-\frac{1}{2}} \zeta^{\frac{1}{2}} (|\lambda| - 1)^{\frac{1}{2}} |\lambda|^{-\frac{1}{2}} \sin \left(\frac{\beta}{2} \arccos \frac{1}{\sqrt{|\lambda|}}\right). \tag{2.14}
\]

The strong asymptotic expansion is expressed by the Bessel function of order \( \alpha \) as \( \zeta = 2n^2t \to 0 \) in the neighborhood of the point 0.

**Theorem 5.** There exists a small and positive constant \( \delta, 0 < \delta \ll \frac{1}{2} \) so that for \( x \in (0, \delta) \) and \( \zeta = 2n^2t \), when \( n \to \infty, \zeta \to 0 \),

\[
\pi_n(x) = \frac{\sqrt{\pi(n\pi - 2\arccos \sqrt{x})}^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}x^{\frac{1}{4}}(1 - x)^{\frac{1}{4}} \sqrt{2w_{p,2}(x,t)}} \left[ (n\pi - 2\arccos \sqrt{x}) \right]^{\frac{1}{2}}
\]

\[
	imes [ \left( 1 + \mathcal{O}(1/n) + \mathcal{O}(\zeta) \right) (\sin \theta_3 J_\alpha(n\pi - 2\arccos \sqrt{x})) + \cos \theta_3 J'_\alpha(n\pi - 2\arccos \sqrt{x}))]
\]

\[
+ (\mathcal{O}(1/n) + \mathcal{O}(\zeta)) (\sin \theta_4 J_\alpha(n\pi - 2\arccos \sqrt{x})) + \cos \theta_4 J'_\alpha(n\pi - 2\arccos \sqrt{x}))]),
\]

where

\[
\theta_3 = (\alpha + \beta + 1) \arccos \sqrt{x} - \frac{(\alpha + \beta)\pi}{2} \tag{2.15}
\]

\[
\theta_4 = (\alpha + \beta - 1) \arccos \sqrt{x} - \frac{(\alpha + \beta)\pi}{2} \tag{2.16}
\]

and \( J_\alpha \) is the Bessel function of order \( \alpha \). The error terms is uniformly for \( x \) in compact subsets of \( (0, \delta) \cap D_3 \), where \( D_3 = \{ x | n\pi - 2\arccos \sqrt{x} \in (0, \infty), n \to \infty \} \).

We can obtain similar asymptotic behavior in terms of Airy function of order \( \alpha \) near the endpoint 0 as \( \zeta = 2n^2t \to \infty \).
Theorem 6. There exists a small and fixed $\epsilon$, $0 < \epsilon \ll \frac{1}{2}$, so that $x \in (0, \epsilon)$, if $n \to \infty$, $\zeta = 2n^2t \to \infty$,

$$\pi_n(x) = \frac{\sqrt{\pi(n(\pi - 2\arccos \sqrt{x}))^{\frac{1}{2}}}}{2(1-x)^{\frac{1}{2}} \sqrt{2\omega_{\rho}(x,t)}} \times \{[1 + O(n^{-\frac{1}{2}})][\sin b_3d_{11} + \cos b_3(n(\pi - 2\arccos \sqrt{x}))^{-1}(bd_{11} + d_{21})]Ai(\hat{\xi})
+ (\sin b_3d_{11} + \cos b_3(n(\pi - 2\arccos \sqrt{x}))^{-1}(bd_{11} - d_{22}))Ai'(\hat{\xi})\}$$

(2.17)

where $Ai$ is the Airy function, $\hat{\xi} = (\frac{3}{2})(1 - |\lambda|)\xi\hat{\lambda}^{-\frac{3}{2}}$, $\hat{\lambda} = \zeta^{-\frac{3}{2}} n^2 f_0(z)$ with $f_0(z) = (\log^{\frac{3}{2}}(z))^{2}$. The error terms hold uniformly for $x \in K_2$, where $K_2$ is any compact subset of $(0, \epsilon) \cap D_4, t \in (0, c], c > 0$ and

$$D_4 = \{x|\frac{3}{2}(1 - |\lambda|)\hat{\xi}\hat{\lambda} \in (-\infty, 0), |\lambda| = \zeta^{-\frac{3}{2}} n^2 (\pi - 2\arccos \sqrt{x})^{\frac{3}{2}}, \zeta \to \infty, n \to \infty\},$$

(2.18)

where $b, \theta_3, \theta_4$ is given by (2.11), (2.15), (2.16) respectively and $d_{11}, d_{12}, d_{21}, d_{22}$ can be obtained by changing $\beta \to \alpha$ in (2.11)-(2.12).

2.2 Universality

Here we state asymptotic of the correlation kernel in the bulk of the spectrum and at both side of the hard-edge.

Theorem 7. The kernel $K_n(x, y)$ associated with weight (1.4) admits properties

(1) When $y \in (0, 1)$, it is easy to get

$$\frac{1}{n}K_n(y, y) = \frac{1}{\pi \sqrt{y(1-y)}} + O\left(\frac{1}{n}\right), \quad n \to \infty,$$

(2.19)

and the error term is uniformly in compact subsets of $(0, 1)$.

(2) For $a \in (0, 1), u, v \in \mathbb{R}$, as $n \to \infty$,

$$\frac{1}{n \rho(a)}K_n(a + \frac{u}{n \rho(a)}, a + \frac{v}{n \rho(a)}) = \frac{\sin(\pi(u - v))}{\pi(u - v)} + O\left(\frac{1}{n}\right),$$

(2.20)
where $\rho(y) = (\pi \sqrt{y(1-y)})^{-1}$, and the error term is uniformly for any $a$ and $u, v$ satisfying the above conditions.

(3) The large-$n$ behavior of the kernel near the right edge $x = 1$ is

$$
\frac{1}{4n^2} K_n(1 - \frac{u}{4n^2}, 1 - \frac{v}{4n^2}) = K_\Psi(u, v; 2nt) + O(\frac{1}{n^2}),
$$

(2.21)

as $n \to \infty$ with the uniform error term for $u, v \in (0, \infty)$ and $t \in (0, c]$ where $c > 0$, in which the $\Psi$-kernel is defined as

$$
K_\Psi(u, v; \zeta) = \frac{\psi_1(-v, \zeta)\psi_2(-u, \zeta) - \psi_1(-u, \zeta)\psi_2(-v, \zeta)}{2\pi i(u - v)},
$$

(2.22)

where the $\psi$-function is given by (5.33). Also it is noticeable that the $\Psi$-kernel is reduced to the Bessel kernel when $\zeta \to 0^+$ and then the Airy kernel when $\zeta \to +\infty$. Precisely, that is

$$
K_\Psi(u, v; \zeta) = J_\beta(u, v) + O(\zeta), \quad \zeta \to 0^+,
$$

(2.23)

where the error term holds uniformly for $u, v$ in compact subsets of $(0, \infty)$ and

$$
\mathbb{J}_\beta(x, y) = \frac{J_\beta(\sqrt{x})\sqrt{y}J'_\beta(\sqrt{y}) - J_\beta(\sqrt{y})\sqrt{x}J'_\beta(\sqrt{x})}{2(x - y)}.
$$

(2.24)

If the parameter $t \to 0^+$ and $n \to \infty$ so that,

$$
\lim_{n \to \infty} 2n^2 t = 0,
$$

then, we can obtain the Bessel kernel limit for $K_n$

$$
\lim_{n \to \infty} \frac{1}{4n^2} K_n(1 - \frac{u}{4n^2}, 1 - \frac{v}{4n^2}) = \mathbb{J}_\beta(u, v),
$$

(2.25)

which is uniform for $u, v$ in compact subsets of $(0, \infty)$.

However, as $\zeta \to +\infty$,

$$
\frac{1}{m^{4/9}} K_\Psi(\zeta^{2/3}(1 - \frac{u}{m\zeta^{2/3}}); \zeta^{2/3}(1 - \frac{v}{m\zeta^{2/3}}); \zeta) = \mathbb{A}(u, v) + O(\zeta^{-2/9}),
$$

(2.26)

in which $m = (\frac{1}{2})^{\frac{3}{4}}$ and the Airy kernel $\mathbb{A}$ is expressed by

$$
\mathbb{A}(x, y) = \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x - y},
$$

(2.27)
This formula is uniform for $u, v$ in compact subsets of $(-\infty, \infty)$. Following, if the parameter $t \in (0, d]$ and $n \to \infty$ such that

$$\lim_{n \to \infty} 2n^2 t = \infty,$$

the Airy kernel limit for $K_n$ is given by

$$\lim_{n \to \infty} \frac{s_n}{m \zeta^{2/9}} K_n(1 - s_n(1 - \frac{u}{m \zeta^{2/9}}), 1 - s_n(1 - \frac{v}{m \zeta^{2/9}}); \zeta) = \check{A}(u, v), \tag{2.28}$$

uniformly for $u, v$ in compact subsets of $(-\infty, \infty)$ with $\zeta = 2n^2 t$, $s_n = \zeta^{-\frac{2}{3}}/(4n^2)$.

(4) At left edge of the spectrum, namely near the endpoint 0, the limit kernel has the $\Psi$-kernel asymptotic behavior as $n \to \infty$,

$$\frac{1}{4n^2} K_n\left(\frac{u}{4n^2}, \frac{v}{4n^2}\right) = \check{K}_\Psi(u, v, 2n^2 t) + O\left(\frac{1}{n^2}\right) \tag{2.29}$$

uniformly for $u, v$ in compact subsets of $(0, \infty)$ and $t$ in $(0, c]$ with a constant $c > 0$. Here, the $\check{\Psi}$-kernel is defined as

$$\check{K}_\Psi(u, v; \zeta) = \frac{\check{\psi}_1(-v, \zeta)\check{\psi}_2(-u, \zeta) - \check{\psi}_1(-u, \zeta)\check{\psi}_2(-v, \zeta)}{2\pi i(u - v)} \tag{2.30}$$

where the scalar function $\check{\psi}_k(\xi, \zeta)$, $k = 1, 2$ are given in [21] (1.22)-(1.24). In addition, from [15] and [21], by the same way as we deal with $\Psi$-kernel, we can prove the limiting $\check{\Psi}$-kernel convert to the Bessel and Airy kernels when the parameter $\zeta$ changes from 0 to $\infty$.

**Remark 1.** It is also readily to show that $K_\Psi(u, v; s)$ and $\check{K}_\Psi(u, v; s)$ are the Painlevé type kernel associated with the solvable model RH problem, which admits Lax pair, see [15] and [21].

The above results can be established by applying Deift-Zhou nonlinear steepest descent approach to the asymptotic of Pollaczek-Jacobi type OPS, which should be first characterized with a appropriate RH problem.
3 RH Characterization for the OPS

Following the idea given by Fokas, Its and Kitaev in 1992 \[25]\), we construct a RH problem with respect to the weight function \( w_{p,2}(x, t) \) in (1.4) as follows.

**RH problem 1.** Find \(2 \times 2\) matrix-valued function \(Y(z)\) satisfying

(a) \(Y(z)\) is analytic for \(z \in \mathbb{C} \setminus [0, 1]\);

(b) \(Y(z)\) satisfies the jump condition

\[
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_{p,2}(x, t) \\ 0 & 1 \end{pmatrix}, \quad \text{for } x \in (0, 1);
\]

(c) The asymptotic behavior of \(Y(z)\) at infinity is

\[
Y(z) = \left( I + \mathcal{O}(z^{-1}) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty;
\]

(d) The asymptotic behavior of \(Y(z)\) at the origin is

\[
Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \quad z \to 0;
\]

(e) The asymptotic behavior of \(Y(z)\) at \(z = 1\) and \(\beta > 0\) is

\[
Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \quad z \to 1.
\]

In a similar way given in \([1, 25]\), we can show that the above RH problem for \(Y\) admits a unique solution

\[
y(z) = \begin{pmatrix} \pi_n(z) & \frac{1}{2\pi i} \int_0^1 \frac{\pi_n(x)w_{p,2}(x, t)}{x-z} \, dx \\ -2\pi i \gamma_{n-1}^2 \pi_{n-1}(z) & -\gamma_{n-1}^2 \int_0^1 \frac{\pi_{n-1}(x)w_{p,2}(x, t)}{x-z} \, dx \end{pmatrix}, \tag{3.1}
\]

where \(\pi_n(z)\) is the monic polynomial, and \(P_n(z) = \gamma_n \pi_n(z)\) is the orthonormal polynomial with respect to the weight \(w_{p,2}(x, t)\).
4 Asymptotic Analysis on the RH Problem

In this section, we will transform the original RH problem for $Y$ to a solvable RH problem via a series of invertible deformations $Y \to T \to S \to R$.

- The first deformation: $Y \to T$ means that we change the RH problem for $Y(z)$ to a normalized one, such that $T \sim I$, $z \to \infty$.
- The second deformation: $T \to S$ is that we obtain solvable RH problem for $S$ by removing the oscillation factors in RH problem for $T$.
- The third deformation: $S \to R$ is to makes a small norm RH problem for $R$ with the jumps close to the identity matrix by making of global parametrix $P^{(\infty)}(z)$ and local parametrixs $P^{(0)}(z)$ and $P^{(1)}(z)$ near the endpoints $z = 0$ and $z = 1$.

4.1 The first deformation $Y \to T$

In order to transform the matrix function $Y(z)$ at infinity into an identity matrix, we introduce the following transformation

$$T(z) = 4^n \sigma_3 Y(z)e^{-\frac{2n}{z(1-z)}\sigma_3 \varphi(z)^{-n} \sigma_3}, \quad z \in \mathbb{C} \setminus [0, 1], \quad (4.1)$$

where $\sigma_3$ is the Pauli matrix, $Y(z)$ is the unique solution of the above RH problem 1, and

$$\varphi(z) = 2z - 1 + 2\sqrt{z(z-1)}$$

is a conformal map from $\mathbb{C} \setminus [0, 1]$ onto the exterior of the unit circle. It can be shown that $\varphi(z)$ admits the property

$$\varphi(z) \sim 4z, \quad \text{for } z \to \infty; \quad \varphi_{+}(x)\varphi_{-}(x) = 1, \quad \text{for } x \in (0, 1).$$

Under the transformation (4.1), we can check that $T$ solves the following RH problem.

RH problem 2. Find $2 \times 2$ matrix-valued function $T(z)$ satisfying

(a) $T(z)$ is analytic for $z \in \mathbb{C} \setminus [0, 1]$.

(b) $T(z)$ satisfies the jump condition

$$T_{+}(x) = T_{-}(x) \begin{pmatrix} \varphi_{+}(x)^{-2n} & w(x) \\ 0 & \varphi_{-}(x)^{-2n} \end{pmatrix}, \quad \text{for } x \in (0, 1), \quad (4.2)$$
where \( w(x) = x^\alpha (1 - x)^\beta \), \( \alpha > 0, \beta > 0 \).

(c) The asymptotic behavior of \( T(z) \) at infinity is

\[
T(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty
\]  

(d) The asymptotic behavior of \( T(z) \) at the origin is

\[
T(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} e^{-\frac{2}{z} \sigma_3}, \quad z \to 0.
\]  

(e) The asymptotic behavior of \( T(z) \) at \( z = 1 \) is

\[
T(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} e^{-\frac{z}{2(1-z)} \sigma_3}, \quad z \to 1.
\]  

It is noteworthy that the transformation contains the factor \( e^{-\frac{z}{2(1-z)} \sigma_3} \) which will make \( T(z) \) has two singularities at the point of \( z = 1 \) and \( z = 1 \) respectively. We use the similar method in [27] to construct a new model RH problem for representing the property of singular point.

4.2 The second transformation \( T \to S \)

The jump matrix in (4.2) can be factorized in the form

\[
\begin{pmatrix} \varphi_+^{-2n} & w \\ 0 & \varphi_-^{-2n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -w^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varphi_+^{-2n} w^{-1} & 1 \end{pmatrix},
\]

which contains two oscillation factors \( \varphi_+^{-2n} \) and \( \varphi_-^{-2n} \). To remove these oscillation factors, according to above factorization, we define the transformation

\[
S(z) = \begin{dcases}
T(z), & \text{for } z \text{ outside the lens region;} \\
T(z) \begin{pmatrix} 1 & 0 \\ -w(z)^{-1} \varphi(z)^{-2n} & 1 \end{pmatrix}, & \text{for } z \text{ in the upper lens region;} \\
T(z) \begin{pmatrix} 1 & 0 \\ w(z)^{-1} \varphi(z)^{-2n} & 1 \end{pmatrix}, & \text{for } z \text{ in the lower lens region,}
\end{dcases}
\]
Figure 1: The contour for the RH problem for $\Sigma S(z)$

where $\text{arg } z \in (-\pi, \pi)$. Then $S$ solves the problem:

**RH problem 3.** Find $2 \times 2$ matrix-valued function $S(z)$ satisfying

(a) $S(z)$ is analytic for $z \in \mathbb{C} \setminus \bigcup_{j=1}^{3} \Sigma_j$, illustrated in Figure 1.

(b) $S_+(z) = S_-(z) Js(z)$ for $z \in \bigcup_{j=1}^{3} \Sigma_j$, and the jump $Js$ is given by

$$
\begin{align*}
Js(z) &= \begin{pmatrix} 1 & 0 \\
\frac{w(z)^{-1} \varphi(z)^{-2n}}{w(z)^{-1} \varphi(z)^{-2n}} & 1 
\end{pmatrix}, \quad \text{for } z \in \Sigma_1 \cup \Sigma_3, \\
Js(x) &= \begin{pmatrix} 0 & \frac{w(x)}{w(x)^{-1}} \\
\frac{w(x)^{-1}}{w(x)^{-1}} & 0 
\end{pmatrix}, \quad \text{for } x \in \Sigma_2.
\end{align*}
$$

(c) When $z \to \infty$, we get the asymptotic behavior of $S(z)$ is

$$
S(z) = I + O(z^{-1}), \quad z \to \infty
$$

(d) As $z \to 0$, the asymptotic behavior of $S(z)$ is sector wise as follows:

$$
S(z) = O(1)e^{-\frac{2\pi}{3} \sigma_3} \begin{cases}
I, & \text{outside the lens region;} \\
\begin{pmatrix} 1 & 0 \\
-w(z)^{-1} \varphi(z)^{-2n} & 1 
\end{pmatrix}, & \text{in the upper lens region;} \\
\begin{pmatrix} 1 & 0 \\
w(z)^{-1} \varphi(z)^{-2n} & 1 
\end{pmatrix}, & \text{in the lower lens region.}
\end{cases}
$$
(e) As $z \to 1$, the asymptotic behavior of $S(z)$ is sector wise as follows:

$$S(z) = O(1)e^{-\frac{1}{\pi(1-z)^{-1}}} \begin{cases} 
I, & \text{outside the lens region;} \\
1 & 0 \\
-w(z)^{-1} \varphi(z)^{-2n} & 1 \\
1 & 0 \\
w(z)^{-1} \varphi(z)^{-2n} & 1 
\end{cases}, \quad \text{in the upper lens region;} \quad (4.9)
$$

in the lower lens region.

4.3 Global parametrix

When $n \to \infty$, we see that the jump matrix $J_S$ on $\Sigma_1$ and $\Sigma_3$ converges to the identity matrix with an exponentially small error terms. Neglecting the exponentially small terms, $S(z)$ can be approximated by a solution of the following RH problem for $P^{(\infty)}(z)$:

**RH problem 4.** Find $2 \times 2$ matrix-valued function $P^{(\infty)}(z)$ satisfying

(a) $P^{(\infty)}(z)$ is analytic for $z \in \mathbb{C} \setminus [0, 1]$.

(b) $P^{(\infty)}(z)$ satisfies the jump condition

$$P_+^{(\infty)}(x) = P_-^{(\infty)}(x) \begin{pmatrix} 0 & w(x) \\
-w(x)^{-1} & 0 \end{pmatrix}, \quad \text{for } x \in (0, 1).$$

(c) The asymptotic behavior of $P^{(\infty)}(z)$ at infinity is

$$P^{(\infty)}(z) = (I + O(z^{-1})), \quad z \to \infty.$$

We can construct a solution to the above RH problem in the outside region applying the method in [28],

$$P^{(\infty)}(z) = D(\infty)^{-\sigma_3} M^{-1} a(z)^{-\sigma_3} M D(z)^{-\sigma_3}, \quad (4.10)$$

where $M = \frac{1 + i \sigma_3}{\sqrt{2}}$, $a(z) = (\frac{z-1}{z})^{1/4}$ with $\arg z \in (-\pi, \pi)$ and $\arg (z-1) \in (-\pi, \pi)$, and

$$D(z) = \varphi^{-\frac{\alpha+\beta}{2}} z^{-\frac{\alpha}{2}} (z-1)^{-\frac{\beta}{2}} \quad (4.11)$$
is the Szegö function associated with the weight \( w(z) \), which is a nonzero analytic function on \( \mathbb{C} \setminus [0,1] \). Besides it is easy to verify \( D_+(x)D_-(x) = w(x) \), for \( x \in (0,1) \). \( D(\infty) = \lim_{z \to +\infty} D(z) = 2^{\alpha+\beta} \).

After same computations,

\[
S(z)P^{(\infty)}(z)^{-1} = I + \mathcal{O}(e^{-cn}), \quad n \to \infty, \tag{4.12}
\]

where the error term is uniformly for \( z \) away from the end-points 0 and 1. However, the jump matrices of \( S(z)P^{(\infty)}(z)^{-1} \) are not uniformly close to the unit matrix because of the factor \( a(z) \) which has singularity. Thus, it is necessary to construct the local parametrix in the neighborhoods of the end-points.

### 4.4 Local parametrix at \( P^{(1)}(z) \) at \( z = 1 \)

Firstly, we construct the local parametrix \( P^{(1)}(z) \) in \( U(1,r) = \{ z \in \mathbb{C} \mid z - 1 \mid < r \} \) for a small and fixed \( r \), which solves the following RH problem:

**RH problem 5.** Find 2 × 2 matrix-valued function \( U(1,r) \) satisfying

(a) \( P^{(1)}(z) \) is analytic for \( z \in U(1,r) \setminus \bigcup_{j=1}^{3} \Sigma_j \), \( \Sigma_j \) for \( j = 1,2,3 \) is in Figure 1.

(b) \( P^{(1)}(z) \) satisfies the same jump condition with \( S(z) \) on \( U(1,r) \cap \Sigma_j, j = 1,2,3 \).

(c) \( P^{(1)}(z) \) fulfills the following matching condition on \( \partial U(1,r) \) and \( n \to \infty \)

\[
P^{(1)}(z)P^{(\infty)}(z)^{-1} = I + \mathcal{O}(n^{-2/3}). \tag{4.13}
\]

(d) The asymptotic behavior of \( P^{(1)}(z) \) at \( z = 1 \) is the same as \( S(z) \) in (4.9).

For converting all the jumps of the \( P^{(1)}(z) \) to constant jumps, we apply a transformation by defining

\[
P^{(1)}(z) = \hat{P}^{(1)}(z)W(z)^{-\sigma_3} \varphi(z)^{-n\sigma_3}, \tag{4.14}
\]

where the function

\[
W(z) = \begin{cases} 
  e^{\frac{\beta \pi i}{2}} w(z)^{\frac{1}{2}}, & \text{for } \text{Im} z > 0, \\
  e^{-\frac{\beta \pi i}{2}} w(z)^{\frac{1}{2}}, & \text{for } \text{Im} z < 0,
\end{cases} \tag{4.15}
\]

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where \( W_+(x)W_-(x) = w(x) \) for \( x \in (0, 1) \). Thus, it is readily seen that \( \hat{P}^{(1)}(z) \) satisfies the RH problem:

**RH problem 6.** Find \( 2 \times 2 \) matrix-valued function \( \hat{P}^{(1)}(z) \) satisfying

(a) \( \hat{P}^{(1)}(z) \) is analytic for \( z \in U(1, r) \setminus \{ \bigcup_{j=1}^{3} \Sigma_j \} \), see Figure 1.

(b) \( \hat{P}^{(1)}(z) \) satisfies the jump conditions:

\[
\hat{P}^{(1)}_+(z) = \hat{P}^{(1)}_-(z) + \begin{pmatrix} 1 & 0 \\ e^{\beta \pi i} & 1 \end{pmatrix}, \quad z \in U(1, r) \cap \Sigma_1, \\
\hat{P}^{(1)}_+(z) = \hat{P}^{(1)}_-(z) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in U(1, r) \cap (0, 1), \\
\hat{P}^{(1)}_+(z) = \hat{P}^{(1)}_-(z) + \begin{pmatrix} 1 & 0 \\ e^{-\beta \pi i} & 1 \end{pmatrix}, \quad z \in U(1, r) \cap \Sigma_3. 
\]

(c) The asymptotic behavior at the center \( z = 1 \) and \( \beta > 0 \),

\[
\hat{P}^{(1)}(z) = O(1)e^{-2\pi \Gamma_{\sigma_3} W(z)^{\sigma_3} \varphi(z)^{n_{\sigma_3}}}
\]

\[
\begin{cases}
I, & \text{outside the lens shaped region;} \\
\begin{pmatrix} 1 & 0 \\ -e^{\beta \pi i} & 1 \end{pmatrix}, & \text{in the upper lens region;} \\
\begin{pmatrix} 1 & 0 \\ e^{-\beta \pi i} & 1 \end{pmatrix}, & \text{in the lower lens region.}
\end{cases}
\]

In order to express the above RH problem to a RH problem whose analyticity and other properties have known, using the conformal mapping, we first build a crucial model RH problem for \( \Psi(\xi) = \Psi(\xi, s) \) to tackle the problem of OPS with singularities in the weight, which is similar to the model given by Xu, Dai, and Zhao \[15\].

**RH problem 7.** Find \( 2 \times 2 \) matrix-valued function \( \Psi(\xi) \) satisfying

(a) \( \Psi(\xi) \) is analytic for \( \xi \in \mathbb{C} \setminus \bigcup_{j=1}^{3} \gamma_j \), where \( \gamma_j \) are illustrated in Figure 2.
(b) $\Psi(\xi)$ satisfies the following jump conditions:

$$
\Psi_+(\xi) = \Psi_-(\xi) \begin{pmatrix} 1 & 0 \\ e^{\pm \beta \pi i} & 1 \end{pmatrix}, \quad \text{as } \xi \in \gamma_1 \text{ and } -\text{as } \xi \in \gamma_3,
$$

(4.18)

$$
\Psi_+(\xi) = \Psi_-(\xi) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } x \in \gamma_2,
$$

(4.19)

(c) As $\xi \to \infty$, the asymptotic behavior of $\Psi(\xi)$ is

$$
\Psi(\xi, s) = (I + \frac{C_1(s)}{\xi} + \mathcal{O}(\frac{1}{\xi^2}))\xi^{-\frac{i\alpha_3}{2}}\frac{I + i\sigma_1}{\sqrt{2}}e^{\sqrt{2}\sigma_3},
$$

(4.20)

where

$$
\arg \xi \in (-\pi, \pi), \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, C_1(s) = \begin{pmatrix} q(s) & -ir(s) \\ it(s) & -q(s) \end{pmatrix}
$$

and from Proposition 1 in [15], $q(s)$ and $t(s)$ are expressed as a combination of $r(s)$ and $r'(s)$, respectively.
(d) The asymptotic behavior of $\Psi(\xi)$ at $\xi = 0$ is

$$\Psi(\xi, s) = Q(s)(I + O(\xi))e^{\frac{\xi^2}{2} \frac{1}{\xi^2}}$$

$$\begin{cases}
  I, & \xi \in \Omega_1 \cup \Omega_4; \\
  \left( \begin{array}{cc}
    1 & 0 \\
    -e^{\beta \pi i} & 1
  \end{array} \right), & \xi \in \Omega_2; \\
  \left( \begin{array}{cc}
    1 & 0 \\
    e^{-\beta \pi i} & 1
  \end{array} \right), & \xi \in \Omega_3.
\end{cases} \tag{4.21}$$

where $\arg \xi \in (-\pi, \pi)$ and the explicit formula for $Q(s)$ can be given by (4.4) in [27].

The solvability of the model RH problem can be given by the similar method in [15]. What’s more, it has been proved by the Proposition 3 in [15] that for $\beta > 0$,

$$r(s) = \frac{1 - 4\beta^2}{8s} + \frac{s}{\beta} + O(s^2), \quad s \to 0,$$

$$r(s) = \frac{3}{2}s^{2/3} - \beta s^{2/3} + O(1), \quad s \to +\infty.$$

After comparing the RH problem for $\hat{P}^{(1)}$ with the above RH problem for $\Psi(\xi, s)$, we construct a solution $\hat{P}^{(1)}$ using the conformal mapping as follows,

$$\hat{P}^{(1)}(z) = E_1(z)\Psi(n^2 f_1(z), 2n^2 t) \tag{4.22}$$

and then, by the formula (4.14) we get

$$P^{(1)}(z) = E_1(z)\Psi(n^2 f_1(z), 2n^2 t)W(z)^{-\sigma_3} \phi(z)^{-n\sigma_3}, \tag{4.23}$$

where

$$E_1(z) = P^{(\infty)}(z)W(z)^{\sigma_3}M^{-1}(n^2 f_1(z))^{\frac{1}{2}\sigma_3}, \tag{4.24}$$

and $f_1(z) = (\log \phi(z))^2$.

Remark 2. It is readily to verify that $f_1(z)$ is analytic in $(0, 1)$, and $f_1(1) = 0$. $f_1(z)$ is a conformal mapping in the neighborhood of $z = 1$ and as $z \to 1$,

$$f_1(z) = 4(z - 1)(1 - \frac{1}{3}(z - 1) + O(z - 1)^2).$$
We transfer the RH problem for $P^{(1)}(z)$ in the $z$-plane to a model RH problem in the $\xi$-plane by the conformal mapping $\xi = n^2 f_1(z)$. Besides, we can prove that $E_1(z)$ is analytic for $z \in U(1, r)$ owning a weak singularity at $z = 1$, and by (4.25) and (4.10), we know
\[ P^{(1)}(z)P^{(\infty)}(z)^{-1} = I + O(n^{-1}). \] (4.25)

Meanwhile, the error terms is uniformly for $z \in \partial U(1, r) \setminus \Sigma_j, j = 1, 2, 3$.

In this situation, when $z \to 1$,
\[ \xi = n^2 f_1(z) \approx 4n^2 z, \text{ and } \zeta = 2n^2 t. \] (4.26)

So we can apply the behavior of $\Psi(\xi, \zeta)$ as $\zeta \to 0$ and $\zeta \to \infty$, which have been found in [17] and [15], to study the asymptotic behavior of $P^{(1)}(z)$, but we use the different parametrix $\beta$ to replace the parametrix $\alpha$ and different scaling process. For the convenience of the later part of the article, we prove the behavior briefly.

4.4.1 **The approximation of $\Psi(\xi, \zeta)$ for $\zeta \to 0$**

As we can see, the essential singularity at the point $z = 0$ vanishes when $\zeta = 0$. Thus, the singular factor $e^{\xi \sigma_3}$ can be ignored for small $\zeta$. In this case, we construct
\[
F(\xi, \zeta) = \pi^{\frac{1}{2}} \sigma_3 \begin{pmatrix}
\xi^{-\frac{\beta}{2}} I_\beta(\sqrt{\xi}) & \frac{\xi^2}{\beta} A_1(\sqrt{\xi}) \\
\frac{1}{\pi \xi^{\frac{1-\beta}{2}}} I'_\beta(\sqrt{\xi}) & -\xi^{\frac{\beta+1}{2}} A_2(\sqrt{\xi})
\end{pmatrix} \begin{pmatrix}
1 & f(\xi, \zeta) \\
0 & 1
\end{pmatrix} \xi^{\frac{\beta}{2}} \sigma_3 e^{\xi \sigma_3} J
\] (4.27)

where
\[
J = \begin{cases}
I, & \xi \in \Omega_1 \cup \Omega_4; \\
\begin{pmatrix}
1 & 0 \\
e^{-\beta \pi i} & 1
\end{pmatrix}, & \xi \in \Omega_2; \\
\begin{pmatrix}
1 & 0 \\
e^{-\beta \pi i} & 1
\end{pmatrix}, & \xi \in \Omega_3.
\end{cases}
\] (4.28)

and due to the formulas of (10.25.2), (10.27.4), and (10.31.1) in [29], it is readily find if $\beta > 0, \beta \notin \mathbb{N}$,
\[
A_1(z) = \frac{\pi}{2 \sin(\beta \pi)} I_{-\beta}(z), \quad A_2(z) = \frac{\pi}{2 \sin(\beta \pi)} I'_{-\beta}(z),
\]

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then if $\beta \in \mathbb{N}$,

$$A_1(z) = \frac{1}{2} \left(\frac{z}{2}\right)^{-\beta} \sum_{j=0}^{\beta-1} \frac{(-1)^j (\beta-j-1)! z^{2j}}{\Gamma(j+1) \Phi} + \frac{1}{2} \left(\frac{z}{2}\right)^{\beta} \sum_{j=0}^{\infty} \frac{(\Phi(j+1) + \Phi(j + \beta + 1)) z^{2j}}{\Gamma(j+1) \Gamma(j + \beta + 1) 4^j},$$

$$A_2(z) = A_1'(z) + \frac{(-1)^{\beta + 1}}{z} I_\beta(z),$$

with $\Phi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. At the same time,

$$f(\xi, \zeta) = -\frac{1}{4 \pi \sin(\beta \pi)} \int_{-\beta}^{\tau} \frac{e^{\frac{2\zeta}{\tau - \xi}}}{\tau - \xi} \, d\tau = \frac{\xi^\beta}{2 \sin(\beta \pi) i} (1 + O(\frac{\zeta}{\xi})), \quad \beta > 0, \beta \notin \mathbb{N};$$

$$f(\xi, \zeta) = \frac{(-1)^{\beta + 1}}{2 \pi^2} \int_{-\beta}^{\tau} \frac{e^{\frac{2\zeta}{\tau - \xi}} \ln\left(\frac{\sqrt{\tau}}{\tau - \xi}\right)}{\tau - \xi} \, d\tau = \frac{(-1)^{\beta} \xi^\beta \ln\left(\frac{\sqrt{\tau}}{\tau - \xi}\right)}{4 \pi} (1 + O(\frac{\zeta}{\xi}) + O(\frac{\xi^\beta + 1}{\xi} \ln \zeta)), \quad \beta \in \mathbb{N},$$

as $\Gamma$ has been shown in Figure 3.

After the discussion of Section 5.1 in [15], we know $\Psi(\xi, \zeta)$ is approximated by $F(\xi, \zeta)$. More precisely,

$$R_0(\zeta) = \begin{cases} 
\Psi(\xi, \zeta) F(\xi, 0)^{-1}, & \text{for } |\xi| > \epsilon_0; \\
\Psi(\xi, \zeta) F(\xi, \zeta)^{-1}, & \text{for } |\xi| < \epsilon_0; 
\end{cases}$$

(4.29)
and the jump on the circle $|\xi| = \epsilon_0$ is

$$
J_{R_0}(\zeta) = \begin{cases} 
I + O(\zeta) + O(\zeta^{\beta + 1}), & \text{for } \beta > 0, \beta \notin \mathbb{N}; \\
I + O(\zeta) + O(\zeta^{\beta + 1} \ln \zeta), & \text{for } \beta \in \mathbb{N};
\end{cases} \tag{4.30}
$$

By the norm estimation of Cauchy operator, we obtain

$$
R_0(\zeta) = \begin{cases} 
I + O(\zeta) + O(\zeta^{\beta + 1}), & \text{for } \beta > 0, \beta \notin \mathbb{N}; \\
I + O(\zeta) + O(\zeta^{\beta + 1} \ln \zeta), & \text{for } \beta \in \mathbb{N};
\end{cases} \tag{4.31}
$$

where the error terms hold uniformly for $|\xi| = \epsilon_0$ as $\zeta \to 0$.

4.4.2 The approximation of $\Psi(\xi, \zeta)$ for $\zeta \to \infty$

In order to removal of singularity of $\Psi(\xi, \zeta)$ as $\zeta \to \infty$, we rescale $\xi$ as $\zeta^{\frac{2}{3}} \lambda$, then we have

$$
H(\lambda) = \zeta^{\frac{1}{2} \sigma_3} \Psi(\zeta^{\frac{2}{3}} \lambda, \zeta)e^{-\zeta^{\frac{1}{2} \theta(\lambda)} \sigma_3}, \quad \theta(\lambda) = \frac{(\lambda + 1)^{\frac{2}{3}}}{\lambda}. \tag{4.32}
$$

which solves the following RH problem:

**RH problem 8.** Find $2 \times 2$ matrix-valued function $H(\lambda)$ satisfying
(a) $H(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus \bigcup_{j=1}^{3} \gamma_{j}$, see Figure [1].

(b) $H(\lambda)$ fulfills the jump conditions:

$$H_{+}(\lambda) = H_{-}(\lambda) \begin{cases} 
    \begin{pmatrix} 1 & 0 \\ e^{\beta \pi i} e^{-2\zeta \frac{1}{2} \theta(\lambda)} & 1 \end{pmatrix}, & \lambda \in \gamma_{1}, \\
    \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \lambda \in (-\infty, -1), \\
    \begin{pmatrix} 0 & e^{2\zeta \frac{1}{2} \theta(\lambda)} \\ -e^{-2\zeta \frac{1}{2} \theta(\lambda)} & 0 \end{pmatrix}, & \lambda \in (-1, 0), \\
    \begin{pmatrix} 1 & 0 \\ e^{-\beta \pi i} e^{-2\zeta \frac{1}{2} \theta(\lambda)} & 1 \end{pmatrix}, & \lambda \in \gamma_{3}; 
\end{cases} \quad (4.33)$$

(c) The asymptotic behavior of $H(\lambda)$ when $\lambda \to \infty$ is

$$H(\lambda) = (I + \mathcal{O}(\frac{1}{\lambda}))\lambda^{-\frac{1}{2} \pi \sigma_{3}} M; \quad (4.34)$$

(d) The asymptotic behavior of $H(\lambda)$ when $\lambda \to 0$ is

$$H(\lambda) = \mathcal{O}(1)\lambda^{\frac{\beta}{2} \pi \sigma_{3}} \begin{cases} 
    I, & \lambda \in \Pi_{1} \cup \Pi_{4}; \\
    \begin{pmatrix} 1 & 0 \\ -e^{\beta \pi i} e^{-2\zeta \frac{1}{2} \theta(\lambda)} & 1 \end{pmatrix}, & \lambda \in \Pi_{2}; \\
    \begin{pmatrix} 1 & 0 \\ e^{-\beta \pi i} e^{-2\zeta \frac{1}{2} \theta(\lambda)} & 1 \end{pmatrix}, & \lambda \in \Pi_{3}. 
\end{cases} \quad (4.35)$$

Then, to move the jumps from $\gamma_{1}$ and $\gamma_{3}$ to $\hat{\gamma}_{1}$ and $\hat{\gamma}_{3}$, we introduce the following transfor-
\[ N(\lambda) = \begin{cases} 
H(\lambda) \begin{pmatrix} 1 & 0 \\
e^{\beta \pi i e^{-2\zeta \theta(\lambda)}} & 1 \end{pmatrix}, & \lambda \in \Pi_5, \\
H(\lambda) \begin{pmatrix} 1 & 0 \\
e^{-\beta \pi i e^{-2\zeta \theta(\lambda)}} & 1 \end{pmatrix}, & \lambda \in \Pi_6, \\
H(\lambda), & \text{otherwise.} 
\end{cases} \quad (4.36) \]

So, the RH problem for \( N(\lambda) \) is constructed as follows:

**RH problem 9.** Find \( 2 \times 2 \) matrix-valued function \( N(\lambda) \) satisfying

(a) \( N(\lambda) \) is analytic for \( \lambda \in \mathbb{C} \setminus \hat{\gamma}_1 \cup \hat{\gamma}_3 \cup (-\infty, -1) \cup (-1, 0) \), see Figure 4.

(b) \( N(\lambda) \) fulfills the jump conditions:

\[
N_+(\lambda) = N_-(\lambda) \begin{cases} 
\begin{pmatrix} 1 & 0 \\
e^{\beta \pi i e^{-2\zeta \theta(\lambda)}} & 1 \end{pmatrix}, & \lambda \in \hat{\gamma}_1, \\
\begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}, & \lambda \in (-\infty, -1), \\
\begin{pmatrix} 0 & e^{2\zeta \theta(\lambda)} \\
-e^{-2\zeta \theta(\lambda)} & 0 \end{pmatrix}, & \lambda \in (-1, 0), \\
\begin{pmatrix} 1 & 0 \\
e^{-\beta \pi i e^{-2\zeta \theta(\lambda)}} & 1 \end{pmatrix}, & \lambda \in \hat{\gamma}_3; 
\end{cases} \quad (4.37) 
\]

(c) The asymptotic behavior of \( N(\lambda) \) when \( \lambda \to \infty \) is

\[
N(\lambda) = (I + O(\frac{1}{\lambda}))\lambda^{-\frac{2}{3} \sigma_3} M; \quad (4.38) 
\]

(d) The asymptotic behavior of \( N(\lambda) \) when \( \lambda \to 0 \) is

\[
N(\lambda) = O(1)\lambda^{\frac{2}{3} \sigma_3}. \quad (4.39) 
\]
Next, the global parametrix $U(\lambda)$ can be constructed by ignoring the exponentially small entries in the above jumps, which satisfies the following RH problem:

**RH problem 10.** Find $2 \times 2$ matrix-valued function $U(\lambda)$ satisfying

(a) $U(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus (-\infty, 0)$.

(b) $U(\lambda)$ fulfills the jump conditions:

\[
U_+ (\lambda) = U_- (\lambda) \begin{cases} 
 e^{\beta \pi i \sigma_3}, & \lambda \in (-1, 0), \\
 0 & 1 \
-1 & 0 
\end{cases}, \quad \lambda \in (-\infty, -1);
\]

(c) The asymptotic behavior of $U(\lambda)$ at infinity is the same as $N(\lambda)$.

(d) The asymptotic behavior of $U(\lambda)$ at the origin is also the same as $N(\lambda)$.

By the similar technique in [30], we find

\[
U(\lambda) = (\lambda + 1)^{-\frac{3}{2} \sigma_3} M \left( \frac{\sqrt{\lambda + 1} + 1}{\sqrt{\lambda + 1} - 1} \right)^{-\frac{3}{2} \sigma_3},
\]

with $\arg (\lambda + 1) \in (-\pi, \pi)$ and $\arg \lambda \in (-\pi, \pi)$. It is obvious that $N(\lambda)$ can be approximated by $U(\lambda)$ except for the neighborhood of the singularity at $\lambda = -1$, namely $U(-1, r)$, so we need to construct an extra model for the local parametrix, which is a Airy model RH problem. That means that the local parametrix can be expressed as

\[
U_1(\lambda) = E_{01}(\lambda)\Psi_A(\zeta \frac{3}{2} f_{01}(\lambda)) e^{-\zeta \frac{3}{2} \theta(\lambda) \sigma_3} e^{\pm \frac{3}{2} \beta \pi i}, \quad \text{for } \pm \text{Im} \lambda > 0, \lambda \in U(-1, r),
\]

where $\Psi_A$ is the Airy model RH problem in section 7 of [31] and

\[
E_{01}(\lambda) = U(\lambda) e^{\frac{3}{2} \zeta \sigma_3} M^{-1}(\zeta \frac{3}{2} f_{01}(\lambda))^{\frac{3}{2} \sigma_3}, \quad \text{for } \pm \text{Im} \lambda > 0, \lambda \in U(-1, r),
\]

\[
f_{01}(\lambda) = \left( -\frac{3}{2} \theta(\lambda) \right)^{\frac{3}{2}}.
\]

Then, we introduce the final transformation

\[
R_{01} = \begin{cases} 
 N(\lambda) U(\lambda)^{-1}, & |\lambda + 1| > r, \\
 N(\lambda) U_1(\lambda)^{-1}, & |\lambda + 1| < r.
\end{cases}
\]
and it is also easy to check that, for $\zeta \to \infty$

$$J_{R_{01}} = \begin{cases} 
    I + \mathcal{O}(e^{-\zeta^{2}}), & \Sigma_{R_{01}} \setminus \partial U(-1, r), \\
    I + \mathcal{O}(\zeta^{2}), & |\lambda + 1| = r.
\end{cases} \quad (4.46)$$

Applying the norm estimation of Cauchy operator, it follows that

$$R_{01}(\lambda) = I + \mathcal{O}(\zeta^{-\frac{1}{2}}), \quad \zeta \to \infty. \quad (4.47)$$

Then, using the technique showed in section 8 of [6] or the result in section 2.6.2 of [17], we have

$$R_{01}(\lambda) = I + \left(\frac{M_{1}}{\lambda + 1} + \frac{M_{2}}{(\lambda + 1)^{2}}\right)\zeta^{-\frac{1}{2}} + \mathcal{O}(\zeta^{-\frac{3}{2}}), \quad \zeta \to \infty. \quad (4.48)$$

where $M_{1} = \frac{7}{72} \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}$ and $M_{2} = \frac{5}{72} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$.

### 4.5 Local parametrix at $P^{(0)}(z)$ at $z = 0$

Next, the local parametrix $P^{(0)}(z)$ in the neighborhood of $U(0, r) = \{z \in \mathbb{C} : |z| < r\}$ with a small and fixed $r$ is constructed, which satisfies the following RH problem:

**RH problem 11.** Find $2 \times 2$ matrix-valued function $P^{(0)}(z)$ satisfying

(a) $P^{(0)}(z)$ is analytic for $U(0, r) \setminus \bigcup_{j=1}^{3} \Sigma_{j}$, where $\Sigma_{j}$ is in Figure II

(b) The jump condition of $P^{(0)}(z)$ is same as $S(z)$ on $U(0, r) \cap \Sigma_{j}, j = 1, 2, 3$.

(c) The matching condition on the boundary $\partial U(0, r)$ is when $n \to \infty$

$$P^{(0)}(z)P^{(\infty)}(z)^{-1} = I + \mathcal{O}(n^{-\frac{1}{2}}),$$

and the error terms is uniformly for $z \in \partial U(0, r) \setminus \Sigma_{j}, j = 1, 2, 3$.

(d) The asymptotic behavior of $P^{(0)}(z)$ at $z = 0$ is also the same as $S(z)$ in (4.8).

To transform the jumps of $P^{(0)}(z)$ into constant jumps, we introduce the transformation as follow:

$$P^{(0)}(z) = \hat{P}^{(0)}(z)V(z)^{-\sigma_{3}}\varphi(z)^{-n\sigma_{3}} \quad (4.50)$$
in which the function $V(z)$ is described by

$$V(z) = \begin{cases} e^{-\frac{\alpha}{2} \pi i} w(z)^{\frac{1}{2}}, & \text{for } \text{Im} z > 0, \\ e^{\frac{\alpha}{2} \pi i} w(z)^{\frac{1}{2}}, & \text{for } \text{Im} z < 0, \end{cases} \quad (4.51)$$

and $V_+(x)V_-(x) = w(x)$ for $x \in (0, 1)$. Also, when $z \to 0$, $V(z)$ can be approximated by $(-z)^{\frac{1}{2}}$.

Thus, we can directly verify that $\hat{P}^{(0)}(z)$ subjects to the RH problem as follow:

**RH problem 12.** Find $2 \times 2$ matrix-valued function $\hat{P}^{(0)}(z)$ satisfying

(a) $\hat{P}^{(0)}(z)$ is analytic for $z \in \mathbb{C} \setminus \bigcup_{j=1}^{3} \Sigma_j$, see Figure [I]

(b) $\hat{P}^{(0)}(z)$ satisfies the jump conditions:

$$\hat{P}^{(0)}_+(z) = \hat{P}^{(0)}_-(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{-\alpha \pi i} & 1 \end{pmatrix}, & z \in U(0, r) \cap \Sigma_1, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in U(0, r) \cap (0, 1), \\ \begin{pmatrix} 1 & 0 \\ e^{\alpha \pi i} & 1 \end{pmatrix}, & z \in U(1, r) \cap \Sigma_3. \end{cases} \quad (4.52)$$

(c) The asymptotic behavior at the center $z = 0,$

$$\hat{P}^{(0)}(z) = \mathcal{O}(1) e^{-\frac{\alpha}{2} \pi i} V(z)^{\sigma_3} \phi(z)^{\sigma_3}$$

$$\begin{cases} I, & \text{outside the lens region;} \\ \begin{pmatrix} 1 & 0 \\ -e^{-\alpha \pi i} & 1 \end{pmatrix}, & \text{in the upper lens region;} \\ \begin{pmatrix} 1 & 0 \\ e^{\alpha \pi i} & 1 \end{pmatrix}, & \text{in the lower lens region.} \end{cases} \quad (4.53)$$
After given the RH problem of \( \hat{P}(0)(z) \), we think about a model RH problem which is introduced by Xu, Dai and Zhao \[15\] and then developed by Chen and other experts \[17\] to deal with the local parametrix of \( \hat{P}(0)(z) \). To prevent confusion, we record this model as \( \bar{\Psi}(\bar{\xi}) = \bar{\Psi}(\bar{\xi}, \bar{s}) \).

In order to find out the conformal mapping, we first define auxiliary function \( h(z) \):

\[
    h(z) = \log \varphi(z) = \log (2z - 1 + 2\sqrt{z(1 - z)}) \tag{4.54}
\]

and \( h(z) \) is analytic in \( C \setminus (-\infty, 1] \) apparently. As \( \varphi_+(x)\varphi_-(x) = 1 \), we get \( h_+(x) = -h_-(x) \).

Then we define

\[
    f_0(z) = (h(z) - h(0))^2 = (\log \frac{\varphi(z)}{\varphi(0)})^2 \tag{4.55}
\]

and when \( z \to 0 \), the asymptotic behavior of \( f_0(z) \) is \( f_0(z) = -4z(1 + \frac{1}{3}z + \mathcal{O}(z^2)) \) and \( f_0'(0) = -4 \). Therefore \( f_0(z) \) is a conformal mapping in \( U(0, r) \) where \( r > 0 \) is small and fixed.

Compared the model RH problem of \( \bar{\Psi}(\bar{\xi}, \bar{s}) \) with the local parametrix \( \hat{P}(0)(z) \), the consequence is that \( \hat{P}(0)(z) \) can be expressed by

\[
    \hat{P}(0)(z) = E_0(z)\bar{\Psi}(n^2f_0(z), 2n^2t)e^{-\frac{2\pi i}{3}\sigma_3} \tag{4.56}
\]

This means \( \bar{\xi} = n^2f_0(z) \) and \( \bar{\zeta} = 2n^2t \) which equals to \( \zeta \). Therefore, in order to avoid repetition, we will use the same symbol \( \zeta \) in a later article. In the meantime, by the formula \( (4.56) \),

\[
    P(0)(z) = E_0(z)\Psi(n^2f_0(z), 2n^2t)e^{-\frac{2\pi i}{3}\sigma_3}V(z)^{-\sigma_3\varphi(z)^{-n\sigma_3}} \tag{4.57}
\]

where

\[
    E_0(z) = P(\infty)(z)^{n\sigma_3}V(z)^{\sigma_3}e^{\frac{2\pi i}{3}\sigma_3}\left(\frac{I + i\sigma_1}{\sqrt{2}}\right)^{-1}(n^2f_0(z))^{\frac{1}{2}\sigma_3} \tag{4.58}
\]

Just as we thought about \( E_1(z) \), \( E_0(z) \) is analytic for \( z \in U(0, r) \) with a weak singularity at the point \( z = 0 \), and

\[
    P(0)(z)P(\infty)(z)^{-1} = I + \mathcal{O}\left(\frac{1}{n}\right) \tag{4.59}
\]

can be achieved by the formula \( (4.10) \) and \( (4.57) \), where the error terms hold uniformly for \( z \in \partial U(0, r) \setminus \Sigma_j, j = 1, 2, 3 \).
In a similar way to the asymptotic properties of $P^{(1)}(z)$, we also can study the behavior of $P^{(0)}(z)$ by using the asymptotic behavior of $\tilde{\Psi}(\tilde{\xi}, \tilde{s})$ which have been researched by Chen and other scholars in [17]. Here we refer directly to the results in section 2.6 in the reference [17].

4.6 The final transformation $S \rightarrow R$

In the final transformation, we define $R(z)$ as follow, with the results of the global parametrix $P^{(\infty)}(z)$ and local parametrixs $P^{(0)}(z)$ and $P^{(1)}(z)$,

$$
R(z) = \begin{cases} 
S(z)P^{(\infty)}(z)^{-1}, & z \in \mathbb{C} \setminus U(1, r) \cap U(0, r) \cap \Sigma_S; \\
S(z)P^{(1)}(z)^{-1}, & z \in U(1, r) \setminus \Sigma_S; \\
S(z)P^{(0)}(z)^{-1}, & z \in U(0, r) \setminus \Sigma_S.
\end{cases} \quad (4.60)
$$

Thus, $R(z)$ is said to satisfy the following RH problem:

**RH problem 13.** Find $2 \times 2$ matrix-valued function $R(z)$ satisfying

(a) $R(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma_R$, as in Figure 5

(b) The jump condition of $R(z)$ is

$$
R_+(z) = R_-(z)J_R(z), \quad z \in \Sigma_R, \quad (4.61)
$$
where
\[
J_R(z) = \begin{cases} 
P^{(\infty)}(z)J_S(z)P^{(\infty)}(z)^{-1}, & z \in \Sigma_1 \cup \Sigma_3, \\
P^{(0)}(z)P^{(\infty)}(z)^{-1}, & z \in \partial U(0, r), \\
P^{(1)}(z)P^{(\infty)}(z)^{-1}, & z \in \partial U(1, r).
\end{cases}
\] (4.62)

and \(J_S(z)\) represents the jump matrices of \(S(z)\) in (4.7).

(c) The asymptotic behavior of \(R(z)\) at infinity:
\[
R(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty
\]

At the same time, using the relations in (4.12), (4.13), (4.49) and (4.62), we find,
\[
J_R(z) = \begin{cases} 
I + \mathcal{O}(e^{-cn}), & z \in \Sigma_1 \cup \Sigma_3, \\
I + \mathcal{O}(n^{-\frac{2}{3}}), & z \in \partial U(1, r) \cup \partial U(0, r).
\end{cases}
\] (4.63)

where \(c\) is a positive constant, and the error terms is uniformly for \(z\) on the corresponding contours. Hence, we have
\[
||J_R(z) - I||_{L^2 \cup L^\infty(\Sigma_R)} = \mathcal{O}(n^{-\frac{2}{3}}) \quad (4.64)
\]

Hence, applying the Plemelj formula in [1] and [31],
\[
R(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{R_{-}(\tau)(J_R(\tau) - I)}{\tau - z} d\tau, \quad z \not\in \Sigma_R, \quad (4.65)
\]

Finally, combined (4.64) with (4.63), we have
\[
R(z) = I + \mathcal{O}(n^{-\frac{2}{3}}), \quad (4.66)
\]
uniformly for \(z\) in the whole complex plane.

So far, we have completed the Deift-Zhou steepest descent analysis.

5 The Proofs of Main Results

We first start with the proof of Theorem 1 about the asymptotics for monic OPS with the two singularities Pollaczek-Jacobi type weight.
Proof of Theorem 1. For $z \in \mathbb{C} \setminus [0,1]$, using the formula (4.1), (4.6), (4.10), and (4.60),

$$Y(z) = 4^{-n\sigma_3}T(z)e^{\frac{t}{2}\sigma_3}\varphi(z)^{n\sigma_3}$$

$$= 4^{-n\sigma_3}S(z)e^{\frac{t}{2}\sigma_3}\varphi(z)^{n\sigma_3}$$

$$= 4^{-n\sigma_3}R(z)P(\infty)(z)e^{\frac{t}{2}\sigma_3}\varphi(z)^{n\sigma_3}$$

$$= 4^{-n\sigma_3}R(z)D(\infty)^{-\sigma_3}M^{-1}a(z)^{-\sigma_3}MD(z)^{-\sigma_3}(z)e^{\frac{t}{2}\sigma_3}\varphi(z)^{n\sigma_3}$$

due to

$$\begin{pmatrix}
Y_{11}(z) \\
Y_{22}(z)
\end{pmatrix} = Y(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then, we get

$$\begin{pmatrix}
Y_{11}(z) \\
Y_{22}(z)
\end{pmatrix} = 4^{-n\sigma_3}R(z)\begin{pmatrix} 2^{-(\alpha+\beta)}a(z)^{-1+\alpha}D(z)e^{\frac{t}{2}\sigma_3}\varphi(z)^{\alpha} \\ 2^{-(\alpha+\beta)}a(z)^{-1+\beta}D(z)e^{\frac{t}{2}\sigma_3}\varphi(z)^{\beta} \end{pmatrix}.$$ 

Plus, $Y_{11}(z) = \pi_n(z)$, $a(z) + a(z)^{-1} = \varphi(z)^{\frac{1}{4}}(z - 1)^{-\frac{1}{4}}z^{-\frac{1}{4}}$, $D(z) = \varphi(z)^{\frac{\alpha+\beta}{2}}(z - 1)^{-\frac{\beta}{2}}z^{-\frac{\alpha}{2}}$. 

Finally, after some calculation, the result of Theorem 1 has been obtained. In order to reduce complex calculations, our next proofs start from the first column of matrix $Y(z)$, which is enough for us to calculate $\pi_n(z)$.

Secondly, the asymptotic behavior of $\pi_n(z)$ in the domain $(0,1)$ has been proved as follow.

Proof of Theorem 2. $\pi_n(z)$ is a polynomial function, which is analytical and continuous in the complex plane. So it is certainly continuous on the whole axis, especially for $z \in (0,1)$. According to the definition of the continuity of complex functions,

$$\pi_n(z) = \lim_{z' \to z} \pi_n(z')$$

where $z' \in \mathbb{C}$, and the limit that tends to $z$ is from any orientation in the field of $z$ in the complex plane. Therefore, we consider the condition that $z'$ tends to $z$ on upper plane. Besides, by tracing back the steps $Y \rightarrow T \rightarrow S \rightarrow R$, that is (4.1), (4.6), (4.10), and (4.60),

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for $z \in (0,1)$, we obtain that

$$
\begin{pmatrix}
Y_{11}(z) \\
Y_{22}(z)
\end{pmatrix}
= 4^{-n\sigma_3} R(z) 2^{-n(\alpha+\beta)\sigma_3}
\begin{pmatrix}
\frac{a_+(z)^{-1} + a_+(z)}{2} & i \frac{a_+(z)^{-1} - a_+(z)}{2} \\
\frac{a_+(z)-a_+(z)^{-1}}{2} & \frac{a_+(z)^{-1} + a_+(z)}{2}
\end{pmatrix}
\begin{pmatrix}
D_+(z) \varphi_+(z)^n e^{zn(1-z)} \\
D_+(z)^{-1} z^{-\alpha_1} (1-z)^{-\beta_1} \varphi_+(z)^{-n} e^{zn(1-z)}
\end{pmatrix}
$$

By the formulas (4.10) and (4.11), it is easily to check that

$$
D_+(z) = e^{i((\alpha+\beta) \arccos \sqrt{z} - \beta u/2) z^{-\beta_1} (1-z)^{-\alpha_1}}
$$

$$
\varphi_+(z)^n = e^{i2n\arccos \sqrt{z}}
$$

$$
a_+(z)^{-1} + a_+(z) = \varphi_+(z)^{\frac{1}{2}} (z - 1)^{-\frac{i}{4}} z^{-\frac{i}{4}}
$$

$$
i(a_+(z)^{-1} - a_+(z)) = e^{i(-\arccos \sqrt{z} + \frac{\pi}{4}) (1-z)^{-\frac{i}{4}} z^{-\frac{i}{4}}}
$$

then the above matrix turns into

$$
\begin{pmatrix}
Y_{11}(z) \\
Y_{22}(z)
\end{pmatrix}
= (w_{p,r}(z,t))^{-\frac{i}{4}} z^{-\frac{i}{4}} (1-z)^{-\frac{i}{4}} 4^{-n\sigma_3} R(z) 2^{-n(\alpha+\beta)\sigma_3}
\begin{pmatrix}
\cos((2n+\alpha+\beta+1) \arccos \sqrt{z} - \frac{\beta u}{2} - \frac{\pi}{4}) \\
-i \cos((2n+\alpha+\beta-1) \arccos \sqrt{z} - \frac{\beta u}{2} - \frac{\pi}{4})
\end{pmatrix}.
$$

Apparently, the fact is $R(z) = I + O(n^{-1})$. The error terms hold uniformly for $x$ in compact subsets of $(0,1)$. Then, Theorem 2 is given by some calculations.

As we have known in Section 2, $\Psi(\xi,\zeta)$ can be approximated by a Bessel model RH problem as $\zeta \to 0^+$, while it convert to a Airy function as $\zeta \to \infty$. For expressing the strong asymptotic expansion near the endpoint 1, we use the fact that the local parametrix can be approximated by the Bessel function as $\zeta = 2n^2 t \to 0^+$.

**Proof of Theorem 3.** After taking a series of inverse transformation of $Y(z) \to T(z) \to S(z) \to R(z)$ with the formulas (4.1), (4.6), (4.10),and (4.60) as well as (4.23), we find,
\[
\begin{pmatrix}
Y_{11}(z) \\
Y_{22}(z)
\end{pmatrix} = 4^{-n\sigma_3} R(z) P_{-1}(z) \begin{pmatrix}
1 & 0 \\
-w(z)^{-1} \varphi_-(z)^{-2n} & 1
\end{pmatrix} \varphi_-(z)^{n\sigma_3} e^{\frac{\pi i}{2} (1 - z) \sigma_3} \begin{pmatrix}
1 \\
0
\end{pmatrix}
\]
\[
= 4^{-n\sigma_3} R(z) E_1(z) \Psi_-(\xi, \zeta) W_-(z)^{-\sigma_3} \varphi_-(z)^{-n\sigma_3} \\
\begin{pmatrix}
1 & 0 \\
-w(z)^{-1} \varphi_-(z)^{-2n} & 1
\end{pmatrix} \varphi_-(z)^{n\sigma_3} e^{\frac{\pi i}{2} (1 - z) \sigma_3} \begin{pmatrix}
1 \\
0
\end{pmatrix}
\]
\[
= 4^{-n\sigma_3} R(z) E_1(z) \Psi_-(\xi, \zeta) \left( e^{\frac{\beta \pi i}{\sigma_3}} - e^{-\frac{\beta \pi i}{\sigma_3}} \right) (w_{p,2}(z, t))^{-\frac{1}{2}}
\]

where \( \xi = n^2 f_1(z) \) and \( \zeta = 2n^2 t \). Then using the facts that
\[
\varphi_-(z) = e^{-2i\text{arccos} \sqrt{z}} \text{ and } \sqrt{n^2 f_1(z)} = -2n\text{arccos} \sqrt{z},
\]
and the conclusion that \( J_\beta(z e^{\frac{\pi i}{\sigma_3}}) = J_\beta(z e^{\frac{\pi i}{\sigma_3}}) \) with \( \arg z \in (-\pi, \frac{\pi}{2}) \), we get
\[
\begin{pmatrix}
Y_{11}(z) \\
Y_{22}(z)
\end{pmatrix} = 4^{-n\sigma_3} R(z) E_1(z) R_0(\zeta) F_1(z, \zeta) (w_{p,2}(z, t))^{-\frac{1}{2}} \begin{pmatrix}
e^{\frac{\beta \pi i}{\sigma_3}} \\
n e^{-\frac{\beta \pi i}{\sigma_3}}
\end{pmatrix}
\]
\[
= 4^{-n\sigma_3} R(z) E_1(z) R_0(\zeta) \sqrt{\pi e^{\frac{\pi i}{\sigma_3}}} (w_{p,2}(z, t))^{-\frac{1}{2}} \begin{pmatrix}e^{\frac{\beta \pi i}{\sigma_3}} I_\beta(\sqrt{\xi}) \\
\frac{\beta \pi i}{\sigma_3} \sqrt{\zeta - 1} I_\beta'(\sqrt{\xi})
\end{pmatrix}
\]
\[
= 4^{-n\sigma_3} R(z) E_1(z) R_0(\zeta) \begin{pmatrix}
J_\beta(2n\text{arccos} \sqrt{z}) \\
i2n\text{arccos} \sqrt{z} J'_\beta(2n\text{arccos} \sqrt{z})
\end{pmatrix} \sqrt{\pi e^{\frac{\pi i}{\sigma_3}}} (w_{p,2}(z, t))^{-\frac{1}{2}}
\]
where \( J_\beta \) is the Bessel function of order \( \beta \). To study the asymptotic behavior of \( E_1(z) \) in (4.10), we compute the asymptotic behavior of \( P(\infty) \) from the upper plane. From the expression of \( P(\infty) \) in (4.10), we find
\[
P_{\infty}(z) = \frac{e^{-\frac{\pi i}{\sigma_3}}}{2x^+(1 - x)^{\frac{1}{2}}} 2^{-(\alpha + \beta)\sigma_3} \begin{pmatrix}e^{i\theta_1} \\
i e^{-i\theta_1}
\end{pmatrix} e^{-\frac{\beta \pi i}{\sigma_3} \sigma_3 x - \frac{\beta \pi i}{\sigma_3} (1 - x) - \frac{\pi i}{\sigma_3} \sigma_3}
\]
where \( \theta_1 = (\alpha + \beta + 1)\text{arccos} \sqrt{z} \) and \( \theta_2 = (\alpha + \beta - 1)\text{arccos} \sqrt{z} \). It is readily found that
\[
(n^2 f_1(z))_{\infty}^{\frac{1}{2}} = e^{\frac{\pi i}{\sigma_3} (2n\text{arccos} \sqrt{z})^{\frac{1}{2} \sigma_3}},
\]

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Combining $E_1(z)$ in (4.24) with $W(z)$ in (4.15), we obtain

$$E_1(z) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2x^+(1-x)^+}} 2^{-(\alpha+\beta)\sigma_3} \begin{pmatrix} \cos \theta_1 & \sin \theta_1 \\ -i \cos \theta_2 & -i \sin \theta_2 \end{pmatrix} e^{\pm \sigma_3 (2n\arccos \sqrt{x}) \frac{i}{2}\sigma_3}. \quad (5.10)$$

By substituting the formulas (5.10) into (5.7) and considering the result $Y_{11}(z) = \pi_n(z)$, $R(z) = I + O(n^{-1})$, and $R_0(\zeta) = I + O(\zeta)$, we prove the conclusion of Theorem 3, where the error terms is uniformly for $z \in D_1$, as $\zeta = 2n^2 t \to 0$, and $D_1 = \{x|2n\arccos \sqrt{x} \in (0,\infty), n \to \infty\}$.

In order to obtain the strong asymptotic expansion near the endpoint 1, we take advantage of the fact $\Psi(\xi,\zeta)$ can be approximated by the Airy function when $\zeta = 2n^2 t \to \infty$.

**Proof of Theorem 4.** It is convenient to start with (5.5), and we list it here

$$\begin{pmatrix} Y_{11}(z) \\ Y_{22}(z) \end{pmatrix} = 4^{-n\sigma_3} R(z) E_1(z) \Psi_{-}(\xi,\zeta) \begin{pmatrix} e^{\frac{\alpha \pi i}{2}} \\ e^{-\frac{\alpha \pi i}{2}} \end{pmatrix} (w_{pJ_2}(z,t))^{-\frac{1}{2}},$$

where $E_1(z)$ is in (4.24), $\xi = n^2 f_1(z)$, and $\zeta = 2n^2 t$.

Combining (4.32), (4.36), (4.42) and (4.45), we get

$$\Psi(\zeta^{\frac{3}{2}} \lambda, \zeta) = \zeta^{-\frac{3}{2}\sigma_3} R_{01}(\lambda) E_{01}(\lambda) \Psi_A(\zeta^{\frac{3}{2}} f_{01}(\lambda)) e^{\pm \frac{3\pi i}{2}\sigma_3} \quad (5.11)$$

where $\pm \text{Im} \lambda > 0$, $\lambda \in U(-1, r)$, $\Psi_A$ is the Airy kernel. By integrating equations (4.41), (4.47) and (4.43), we acquire

$$E_{01}(\lambda) = (\lambda + 1)^{-\frac{3}{2}\sigma_3} \frac{1 + i\sigma_1}{\sqrt{2}} \frac{\sqrt{\lambda + 1} + 1}{\sqrt{\lambda + 1} - 1} e^{\mp \frac{3\pi i}{2}\sigma_3} \frac{1 + i\sigma_1}{\sqrt{2}} (\lambda^{\frac{3}{2}} f_{01}(\lambda)) \frac{1}{\lambda^{\frac{3}{2}}} \quad (5.12)$$

for $\pm \text{Im} \lambda > 0$, $f_{01}(\lambda) = (-\frac{3}{2} \theta(\lambda))^\frac{3}{2}$, $\theta(\lambda) = \frac{(\lambda + 1)^{\frac{3}{2}}}{\lambda^{\frac{3}{2}}}$. 

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Then, after rescaling, we obtain,
\[
\begin{pmatrix}
Y_{11}(z) \\
Y_{22}(z)
\end{pmatrix} = 4^{-n\sigma_3} R(z) E_1(z) \xi^{-\frac{1}{2n^2}} R_0(\xi) E_01(\lambda)
\]
\[
\Psi_{\lambda_+}(\xi f_{01}(\lambda)) \begin{pmatrix} 1 \\ -1 \end{pmatrix} (w_p(z, t))^{-\frac{1}{\lambda}}
\]
\[
= 4^{-n\sigma_3} R(z) E_1(z) \xi^{-\frac{1}{2n^2}} R_0(\xi) E_01(\lambda)
\]
\[
\sqrt{2\pi} \begin{pmatrix} Ai - (\xi f_{01}(\lambda)) \\ -iAi' - (\xi f_{01}(\lambda)) \end{pmatrix} (w_p(z, t))^{-\frac{1}{\lambda}}
\]

By some simple calculations, it is easily to verify that
\[
(\lambda + 1)^{-\frac{1}{2n^2}} = (|\lambda| - 1)^{-\frac{1}{2n^2}} e^{\frac{\pi i}{4}},
\]
\[
\left(\frac{\sqrt{\lambda + 1 + 1}}{\sqrt{\lambda + 1 - 1}}\right)^{\frac{1}{2n^2}} = \exp(-i \arccos \frac{1}{\sqrt{|\lambda|}} + i\pi),
\]
\[
(f_{01}(\lambda))^{-\frac{1}{2n^2}} = \frac{2}{3} (|\lambda| - 1)^{-\frac{1}{2n^2}} |\lambda| e^{\frac{3\pi i}{2}}.
\]

Substituting (5.14), (5.15) and (5.16), we derive
\[
\xi^{-\frac{1}{2n^2}} E_01(\lambda) = \begin{pmatrix} d_{11} & id_{12} \\ id_{21} & d_{22} \end{pmatrix}
\]
where \(d_{11}, d_{12}, d_{21}, \) and \(d_{22}\) are defined as (2.11), (2.13), (2.14) and (2.12) respectively.

With the help of the formula (4.48), it is readily to show, as \(\xi \to \infty\)
\[
\xi^{-\frac{1}{2n^2}} R_0(\xi) E_01(\lambda) = (I + \mathcal{O}(\xi^{-\frac{1}{2n^2}})) \begin{pmatrix} 1 & 0 \\ \frac{7}{2(\lambda + 1)^{-\frac{1}{4}}} & 1 \end{pmatrix}.
\]

Inserting the above results into (5.13) as well as the facts that \(R_0(\xi) = I + \mathcal{O}(\xi^{-\frac{1}{2n^2}})\), which is uniformly for \(z \in \mathbb{D}_2\), as \(\xi = 2n^2t \to \infty\) and \(R(z) = I + \mathcal{O}(n^{-1})\) which are uniformly for \(z \in (1 - r, 1)\), we follow Theorem 4.

To consider the strong asymptotic expansions of the monic polynomial, we recall some results in [17] that \(\tilde{\Psi}(\xi, \tilde{\xi})\) can be approximated by a modified Bessel model RH problem as
\( \zeta \to 0 \) and then prove the strong asymptotic expansion of monic OPS for \( z \in (0, \delta) \), where \( \delta \) is a small and positive constant.

**Proof of Theorem 5.** As we say above, \( \pi_n(z) \) is a polynomial function and we just consider the limit from the upper plane. Then, using the equations (4.1), (4.6), (4.10), and (4.60), it is readily to show

\[
\begin{pmatrix}
Y_{11}(z) \\
Y_{22}(z)
\end{pmatrix} = Y_+(z)
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

\[
= 4^{-n\sigma_3} R(z) P_+(0)(z)
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

\[
= 4^{-n\sigma_3} R(z) E_0(z) \psi_-(\xi, \bar{\zeta}) e^{(\alpha-1)n\sqrt{\pi} (w(p_j, z, t))^{-\frac{1}{2}}} e^{-\frac{1}{2}J_0(\sqrt{|\xi|})}
\]

where \( w(z) = z^n(1-z)^{\beta}, \xi = n^2 f_0(z), \bar{\zeta} = 2n^2 t, \) and \( E_0 \) is in (5.19). Besides, the boundary value on the positive side \( P_+(0) \) corresponds to the value \( \bar{\psi}_- \) on the negative side, as can be seen from the correspondence \( \bar{\xi} \approx -4n^2 z \) when \( z \to 0 \). According to the results of Chen has proven, see (86), (87) and (89) in [17], then

\[
\begin{pmatrix}
Y_{11}(z) \\
Y_{22}(z)
\end{pmatrix} = 4^{-n\sigma_3} R(z) E_0(z) \psi_-(\xi, \bar{\zeta}) e^{(\alpha-1)n\sqrt{\pi} (w(p_j, z, t))^{-\frac{1}{2}}} e^{-\frac{1}{2}J_0(\sqrt{|\xi|})}
\]

accompanied with the facts that

\[
\bar{\xi} = n^2 f_0(-z) = e^{-\pi i n^2(\pi - 2arccos\sqrt{z})^2}
\]

and

\[
\sqrt{|\xi|} = n(\pi - 2arccos\sqrt{z}).
\]

Also, we give a more precise expression for \( E_0(z) \) by the formula (4.58), (4.10), (4.51) and (5.8)

\[
E_0(z) = \frac{(-1)^n e^{-\frac{\pi i}{2}}}{\sqrt{2x^*(1-x)^*} \pi^{2-(\alpha+\beta)\sigma_3}} \begin{pmatrix}
-sin\theta_3 & cos\theta_3 \\
isin\theta_4 & -icos\theta_4
\end{pmatrix}
\]

\[
e^{-\frac{\pi i}{2} \sigma_3 (n(\pi - 2arccos\sqrt{z}))^{\frac{1}{2}}}
\]

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Bringing the results of (5.8) and (5.23) back to (5.20), we obtain the result of Theorem 5. What’s more, 
\[ R(z) = \mathcal{I} + \mathcal{O}(n^{-1}) \] 
uniformly for \( x \in (0, \delta) \) with \( 0 < \delta \ll \frac{1}{n} \) and 
\[ R_{0S}(\bar{\zeta}) = I + \mathcal{O}(\bar{\zeta}) \] 
uniformly for \( z \in \mathbb{D}_3 \) as \( \bar{\zeta} = 2n^2t \to 0 \), where \( \mathbb{D}_3 = \{ x|n(\pi - 2\arccos\sqrt{x}) \in (0, \infty), n \to \infty \} \).

We continue to consider the strong asymptotic expansion for \( z \in (0, \varepsilon) \), by using the theorem in [17] that the model RH problem for \( \Psi(\bar{\xi}, \bar{\eta}) \) can be approximated by the Airy model when \( \bar{\zeta} = 2n^2t \to \infty \).

**Proof of Theorem 6.** For your convenience, this application start with (5.19), that is
\[
\begin{pmatrix}
Y_{11}(z) \\
Y_{22}(z)
\end{pmatrix}
= 4^{-n\sigma_3} R(z) E_0(z) \Psi_-(\bar{\xi}, \bar{\eta}) e^{\frac{(\sigma_1-1)n\sigma_3}{2}} \left( \begin{array}{cc}
1 & 0 \\
1 & 1
\end{array} \right) \left( \begin{array}{c}
w_{p, 2}(z, t) \quad 1 \\
1 \quad 0
\end{array} \right),
\]
(5.24)
where \( E_0(z) \) is in (4.58), \( \bar{\xi} = n^2 f_0(z) \) and \( \bar{\zeta} = 2n^2t \).

Then, we use the same way as we deal with the situation for \( z \in (1-r, 1) \). Thus, we rescale \( \bar{\xi} \) by \( \bar{\xi} \bar{\lambda} \) with the help of the equation (92), (93), (95) and (96) in [15](To avoid the symbol confusion, we add a superscript in these equation), and we obtain
\[
\begin{pmatrix}
Y_{11}(z) \\
Y_{22}(z)
\end{pmatrix}
= 4^{-n\sigma_3} R(z) E_0(z) \bar{\zeta}^{-\frac{i}{2}} R_{01} E_{01}(\bar{\lambda})
\]
\[
\Psi_{A-}(\bar{\zeta}^2 f_{01}(\bar{\lambda}))(1)^{-n\sigma_3} \left( \begin{array}{c}
-1 \\
i
\end{array} \right) \left( \begin{array}{c}
w_{p, 2}(z, t) \quad 1 \\
1 \quad 0
\end{array} \right)^{-\frac{1}{2}}
\]
(5.25)
Finally, we can easily find the similar properties in (5.17) and (5.18) for \( E_{01} \) and \( R_{01} \). By substituting these formulas into (5.25), we have reached the desired conclusion. Besides, 
\( R(z) = I + \mathcal{O}(n^{-1}) \) holds uniformly for \( z \in (0, \varepsilon) \) while \( R_{01} = I + \mathcal{O}(\bar{\zeta}^{-\frac{1}{2}}) \) is uniformly for \( z \in \mathbb{D}_4 \), see (2.18), as \( \bar{\zeta} = 2n^2t \to \infty \).

The next step of this paper is to proof the limit behavior of the kernel \( K_n(x, y) \) in the bulk of the spectrum.
Proof of Theorem 7. For convenience, we start with (2.4) and rewrite it here:

$$K_n(x, y; t) = \gamma_n^2 \sqrt{w_{p2}(x, t)} \sqrt{w_{p2}(y, t)} \pi_n(x) \pi_{n-1}(y) - \pi_n(y) \pi_{n-1}(x)$$ \quad (x - y) \quad (5.26)$$

Then, by the facts that $Y_{11}(z) = \pi_n(z)$ and $\pi_n(z)$ is analytic in the whole complex plane, one finds

$$K_n(x, y; t) = \sqrt{w_{p2}(x, t)} \sqrt{w_{p2}(y, t)} Y^{-1}_+(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2n-\beta} \end{pmatrix} \pi_n(x) \pi_{n-1}(y) - \pi_n(y) \pi_{n-1}(x) \quad (x - y) \quad (5.27)$$

In order to express $Y(z)$, we take the inverse procedure of $Y \rightarrow T \rightarrow S \rightarrow R$ in (4.1), (4.6) and (4.60) and get

$$Y_+(x) = 4^{-n \sigma_3} R(x) P_+^{(\infty)}(x) \left( \begin{array}{c} 1 \\ w(x)^{-1} \varphi_+(x)^{-2n} \\ 1 \end{array} \right) e^{(n-\sigma_3) \pi n \varphi_+(x)^{n \sigma_3}}$$

$$= 4^{-n \sigma_3} R(x) 2^{-(\alpha+\beta) \sigma_3} M^{-1} a(x)^{-\sigma_3} M \varphi_+(x)^{n \sigma_3}$$

$$= \left( \begin{array}{c} 1 \\ w(x)^{-1} \varphi_+(x)^{-2n} \\ 1 \end{array} \right) e^{(n-\sigma_3) \pi n \varphi_+(x)^{n \sigma_3}}$$

where $M = \frac{I + i \sigma_1 \sqrt{2}}{2}$, $a(z) = (z^{-1})^{\frac{1}{2}}$, $D_+(x) = \varphi_+(x)^{\frac{\alpha+\beta}{2}} x^{-\frac{\beta}{2}} (x-1)^{-\frac{\alpha}{2}}$ and $R(z) = I + O(n^{-\frac{2}{3}})$. By the facts that

$$\varphi_+(x) = e^{2i \arccos \sqrt{x}}$$

we find,

$$Y_+(x)(w_{p2}(x, t))^{\frac{1}{2}} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = 4^{-n \sigma_3} R(x) 2^{-(\alpha+\beta) \sigma_3} M^{-1} a(x)^{-\sigma_3} M$$

$$\left( \begin{array}{c} \exp(-\frac{\beta \pi i}{2} + i(\alpha + \beta + 2n) \arccos \sqrt{x}) \\ \exp(-\frac{\alpha \pi i}{2} + i(\alpha + \beta + 2n) \arccos \sqrt{x}) \end{array} \right)$$

Moreover, it can be easily proved that

$$M^{-1} a(y)^{\sigma_3} a(x)^{-\sigma_3} M = I + O(x - y),$$

(5.30)
where the error terms holds uniformly for any \( x, y \) in the compact subsets of \((0, 1)\). Inserting (5.29) and (5.30) into (5.28), one finds

\[
K_n(x, y; t) = \frac{\sin[(2n + \alpha + \beta)(\arccos \sqrt{y} - \arccos \sqrt{x})]}{\pi(x - y)} + O(1). \tag{5.31}
\]

If \( x \to y \), then

\[
1/n K_n(y, y) = \frac{1}{\pi \sqrt{y(1 - y)}} + O(\frac{1}{n}) \quad n \to \infty,
\]

for any \( y \in (0, 1) \) with the uniform error terms.

Let \( a \in (0, 1) \), \( u, v \in \mathbb{R} \), and \( \rho = (\pi(\sqrt{x(1 - x)})^{-1} \). Then, \( x \) is rescaled as \( a + \frac{u}{n\rho(a)} \) while \( y \) is rescaled as \( a + \frac{v}{n\rho(a)} \) in (5.31). After some calculations, we obtain

\[
(2n + \alpha + \beta)(\arccos \sqrt{a + \frac{v}{n\rho(a)}} - \arccos \sqrt{a + \frac{u}{n\rho(a)}}) = \pi(u - v) + O(\frac{1}{n}), \tag{5.32}
\]

uniformly for \( a \) in any compact subsets of \((0, 1)\). Thus, combining (5.31) with (5.32), the result can be proved.

For the sake of expressing the limiting kernel at the right edge 1, we define the \( \psi \)-functions

\[
\begin{pmatrix}
\psi_1(\xi, \zeta) \\
\psi_2(\xi, \zeta)
\end{pmatrix} = \Psi_+(\xi, \zeta) \begin{pmatrix}
e^{-\frac{\beta \pi i}{2}} \\
e^{\frac{\beta \pi i}{2}}
\end{pmatrix}, \quad \xi < 0. \tag{5.33}
\]

The functions are determined via a model RH problem \( \Psi(\xi, \zeta) \) related to a special solution of a third-order nonlinear differential equation which can be reduced to a certain Painlevé III equation, see [15]. We also give the expression of the \( \Psi \)-kernel as

\[
K_\Psi(u, v, \zeta) = \frac{\psi_1(-u, \zeta)\psi_2(-v, \zeta) - \psi_1(-v, \zeta)\psi_2(-u, \zeta)}{2\pi i(u - v)} \tag{5.34}
\]

for \( u, v, \zeta \in (0, \infty) \). It is convenient to begin with (5.27).

\[
K_n(x, y; t) = \left( \sqrt{w_{p,2}(x; t)} \right)_{y=\sqrt{w_{p,2}(y; t)}} \frac{(0 \ 1) Y_+^{-1}(y) Y_+(x)}{2\pi i(x - y)} \tag{5.35}
\]

Tracing back the transformation \( R \to S \to T \to Y \) with the formulas (4.1), (4.6), and (4.10), and
\[ Y_+(z) = 4^{-n\sigma_3} R(z) P_+^{(1)}(z) \left( \begin{array}{cc} 1 & 0 \\ w(z)^{-1} \varphi_+(z)^{-2n} & 1 \end{array} \right) e^{\frac{2\pi i}{3} \sigma_3} \varphi_+(z)^{n\sigma_3} \]

\[ = 4^{-n\sigma_3} R(z) E_1(z) \Psi_+ (\xi, \zeta) e^{-\frac{2\pi i}{3} \sigma_3} \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) w_{p,2}(z,t)^{-\frac{3}{2}} \]  

(5.36)

Accordingly, we obtain

\[ Y_+(x) w_{p,2}(x,t)^{\frac{1}{2}} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = 4^{-n\sigma_3} R(x) E_1(x) \Psi_+ (\xi, \zeta) e^{-\frac{2\pi i}{3} \sigma_3} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \]

\[ = 4^{-n\sigma_3} R(x) E_1(x) \Psi_+ (\xi, \zeta) \left( \begin{array}{c} e^{-\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} \end{array} \right). \]  

(5.37)

Inserting (5.37) into (5.35), it is easy to achieve

\[ K_n(x,y) = \frac{(-\psi_2(g_n(y)), \psi_1(g_n(y))) E_1(y)^{-1} R(y)^{-1} R(x) E_1(x) (\psi_1(g_n(x)), \psi_2(g_n(x)))^T}{2\pi i(x-y)} \]  

(5.38)

where \( g_n(z) = n^2 f_1(z). \)

Let \( x = 1 - \frac{u}{n^2} \) and \( y = 1 - \frac{v}{n^2} \) with \( u, v = \mathcal{O}(1). \) Besides, due to

\[ g_n(z) = n^2 f_1(z) = n^2 [4(z-1) + \mathcal{O}(z-1)^2], \quad z \to 1, \]

we have

\[ g_n(x) = -u(I + \mathcal{O}(\frac{1}{n^2})), \quad g_n(y) = -v(I + \mathcal{O}(\frac{1}{n^2})) \]  

(5.39)

Since \( E_1(z) \) is a matrix function analytic in \( U(1, r) \), it is directly to show

\[ E_1(y)^{-1} E_1(x) = I + E_1(x)^{-1}(E_1(x) - E_1(y)) \]

\[ = I + \mathcal{O}(x-y) \]  

\[ = I + (u-v) \mathcal{O}(n^{-2}) \]  

(5.40)

for bounded \( u, v. \) Similarly, the analyticity of \( R(z) \) for \( z \in U(1, r) \) implies that

\[ R^{-1}(y) R(x) = I + \mathcal{O}(x-y) = I + (u-v) \mathcal{O}(n^{-2}), \]  

(5.41)
here again with uniform error terms. Therefore, by the above formulas, we obtain

$$K_n(x, y) = \frac{(-\psi_2(g_n(y)), \psi_1(g_n(y)))(I + O(x - y))(\psi_1(g_n(x)), \psi_2(g_n(x)))^T}{2\pi i(x - y)}$$

(5.42)

for $x, y \in (1 - r, 1)$ where $r > 0$. Meanwhile, the equation (5.39) implies that

$$\psi_k(g_n(x), \zeta) = \psi_k(-u, \zeta) + O(n^{-2})$$

and

$$\psi_k(g_n(y), \zeta) = \psi_k(-v, \zeta) + O(n^{-2})$$

(5.43)

for $k = 1, 2$, where the uniform error terms hold uniformly for $u$ and $v$ lying in compact subsets of $(0, \infty)$.

Hence, combining the above formula, we have

$$\frac{1}{4n^2} K_n(1 - \frac{u}{4n^2}, 1 - \frac{v}{4n^2}) = \frac{\psi_1(-u, \zeta)\psi_2(-u, \zeta) - \psi_1(-v, \zeta)\psi_2(-v, \zeta)}{2\pi i(u - v)} + O\left(\frac{1}{n^2}\right),$$

(5.44)

as $n \to \infty$.

It is also direct to proof the $\Psi$-kernel is reduced to the Bessel kernel when $\zeta \to 0^+$. After we substitute (4.27) and (4.29) into (5.33), we find

$$\begin{pmatrix}
\psi_1(\xi, \zeta) \\
\psi_2(\xi, \zeta)
\end{pmatrix} = R_0(\xi) e^{\frac{\pi i}{2} \sigma_3} \begin{pmatrix}
I_\beta(\sqrt{\xi} e^{\pi i}) \\
i\pi \sqrt{\xi} J_\beta(\sqrt{\xi} e^{\pi i})
\end{pmatrix}
\begin{pmatrix}
I_\beta(\sqrt{\xi} e^{\pi i}) \\
i\pi \sqrt{\xi} J_\beta(\sqrt{\xi} e^{\pi i})
\end{pmatrix},$$

(5.45)

During this process, we have use the equation $e^{-\frac{1}{2} \beta \pi i} I_\beta(z) = J_\beta(ze^{-\frac{1}{2} \pi i})$, arg$z \in (0, \frac{\pi}{2})$ (see (10.27.6) in [28]), and $X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as det$X = 1$, wher $X^T$ means the transpose of a matrix $X$. It follows that, as $\zeta \to 0^+$,

$$\frac{1}{4n^2} K_n(1 - \frac{u}{4n^2}, 1 - \frac{v}{4n^2}) = \frac{\psi_1(-u, \zeta)\psi_2(-v, \zeta) - \psi_1(-v, \zeta)\psi_2(-u, \zeta)}{2\pi i(u - v)} + O\left(\frac{1}{n^2}\right) + O(1/n^2),$$

(5.46)

where $J_\beta$ is the Bessel kernel, and the error terms are uniform in compact subsets of $u, v \in (0, \infty)$. Now we consider the asymptotic behaviors for $\Psi$-kernel as $\zeta \to \infty$, which transfers to
the Airy kernel. Taking a series of transformations \( \Psi(\zeta, \xi) \rightarrow H(\lambda, \xi) \rightarrow N(\lambda, \zeta) \rightarrow R_{01}(\lambda) \) and \( R_{01}(\lambda) = I + \mathcal{O}(\zeta^{-1/\lambda - 1}) \), \( \zeta \rightarrow \infty \), \( \lambda \rightarrow \infty \), we obtain, as \( \zeta \rightarrow \infty \), \( \lambda \rightarrow \infty \)

\[
\Psi(\zeta^{\frac{1}{2}}(\lambda, \xi) = \zeta^{-\frac{1}{2}\sigma_3}(I + \mathcal{O}(\zeta^{-1/3}\lambda^{-1}))U(\lambda)e^{\frac{1}{2}\theta(\lambda)\sigma_3}. \tag{5.47}
\]

With the help of the approximation of \( U(\lambda) \) as \( \lambda \rightarrow \infty \), it gives that, as \( \zeta \rightarrow \infty \), \( \lambda \rightarrow \infty \)

\[
\Psi(\zeta^{\frac{1}{2}}(\lambda, \xi)e^{-\frac{\lambda}{2\sqrt{\lambda}}\sigma_3} = (\zeta^{\frac{1}{2}}(\lambda) - \frac{1}{2}\sigma_3[I + \mathcal{O}(\zeta^{\frac{1}{2}}(\lambda)\lambda^{\frac{1}{2}})]M \tag{5.48}
\]

For \(|\lambda + 1| < r\), we begin with the formula (5.11) and substitute this formula into (5.33),

\[
\begin{align*}
\begin{bmatrix}
\psi_1(\xi, \zeta) \\
\psi_2(\xi, \zeta)
\end{bmatrix} &= \zeta^{-\frac{1}{2}\sigma_3}R_{01}(\lambda)E_{01}(\lambda)\Psi_A(\zeta^{\frac{2}{3}}f_{01}(\lambda))e^{\frac{2\theta(\lambda)}{2\lambda}} \left( e^{\frac{\alpha \pi i}{2}} \right)
\end{align*}
\tag{5.49}
\]

Let \( \lambda = -1 + \frac{\lambda}{\sqrt{\lambda}} \), then it is easy to verify that \( \zeta^{2/9}f_{01}(\lambda) = u[1 + \mathcal{O}(u\zeta^{-\frac{1}{2}})] \). Thus, the approximation of \( K \Psi \) can be given by (2.10), and \( K_n \) followed by (2.28) as \( \zeta \rightarrow \infty \). By now, we have finished the proof of case 4 in Theorem 7.

Then we focus on the large-\( n \) behavior of the kernel near the edge \( x = 0 \). We also begin with the equation (5.35). In this case, \( Y(z) \) has been found in (5.19), and after a compilation of equations, we show, for \( x \in (0, d) \)

\[
Y_+(x)(w_{p, 2}(x, t)) \frac{1}{2} \begin{pmatrix}
1 \\
0
\end{pmatrix} = 4^{-n\sigma_3}R(x)E_0(x)\Psi_-(f_n(x), \xi)e^{\frac{(\alpha-1)\pi i}{2}\sigma_3} \begin{pmatrix}
1 \\
1
\end{pmatrix}
\tag{5.50}
\]

where \( f_n(x) = n^2f_0(x) \) and \( \xi = 2n^2t \).

We rewrite the fact (4.4) in (21), for \( \xi \in (-\infty, 0) \),

\[
\begin{align*}
\begin{bmatrix}
\bar{\psi}_1(\xi) \\
\bar{\psi}_2(\xi)
\end{bmatrix} &= \begin{bmatrix}
\bar{\psi}_1(\xi, \zeta) \\
\bar{\psi}_2(\xi, \zeta)
\end{bmatrix} = \bar{\Psi}_-(\xi, \zeta) \begin{pmatrix}
e^{\frac{(\alpha-1)\pi i}{2}} \\
e^{-\frac{(\alpha-1)\pi i}{2}}
\end{pmatrix} \tag{5.51}
\end{align*}
\]
Substituting (5.50) and (5.51) into (5.35), we get
\[ K_n(x, y) = \frac{(-\bar{\psi}_2(f_n(y)), \bar{\psi}_1(f_n(y)))E_0(y)^{-1}R(y)^{-1}R(x)E_0(x)(\bar{\psi}_1(f_n(x)), \bar{\psi}_2(f_n(x)))^T}{2\pi i(x - y)} \]  
(5.52)

Now specifying
\[ x = \frac{u}{4n^2}, \quad y = \frac{v}{4n^2}, \]  
(5.53)
where \( u, v \in (0, \infty) \) and of the size \( O(1) \). Also, we know
\[ f_n(z) = n^2f_0(z) = -4n^2z(1 + O(z)), \text{ for } z \to 0 \]

Then it follows that
\[ f_n(x) = -u(I + O(1/n^2)), \quad f_n(y) = -v(I + O(1/n^2)) \]  
(5.54)

The matrix function \( E_0(z) \) is analytic in \( U(0, d) \), so
\[ E_0^{-1}(y)E_0(x) = I + E_0^{-1}(y)(E_0(x) - E_0(y)) \]
\[ = I + O(x - y) \]
\[ = I + (u - v)O(n^{-2}) \]  
(5.55)
for the above \( u, v \). Similarly, the analyticity of \( R(z) \) in \( U(0, d) \) tells us
\[ R^{-1}(y)R(x) = I + (u - v)O(n^{-2}) \]  
(5.56)
and also the error term holds uniformly for \( u, v \in (0, \infty) \).

Hence, substituting (5.55) and (5.56) into (5.52), we obtain
\[ K_n(x, y) = \frac{(-\bar{\psi}_2(f_n(y)), \bar{\psi}_1(f_n(y)))(I + O(x - y))(\bar{\psi}_1(f_n(x)), \bar{\psi}_2(f_n(x)))^T}{2\pi i(x - y)} \]  
(5.57)
the error term is uniformly for \( x, y \in (0, d) \), with \( d > 0 \) being constant. Also the formula in (5.53) implies that
\[ \bar{\psi}_k(f_n(x), \zeta) = \bar{\psi}_k(-u, \zeta) + O(n^{-2}) \]  
(5.58)
for \( k = 1, 2 \), again with uniform error terms.

Therefore, we have
\[ \frac{1}{4n^2}K_n(\frac{u}{4n^2}, \frac{v}{4n^2}) = \bar{K}_\Phi(u, v; \zeta) + O(\frac{1}{n^2}) \]  
(5.59)
for $n \to \infty$, where
\[
\tilde{K}_\phi(u, v; \zeta) = \frac{\bar{\psi}_1(-u, s)\bar{\psi}_2(-v, s) - \bar{\psi}_1(-v, s)\bar{\psi}_2(-u, s)}{2\pi i(u-v)}
\] (5.60)
is the Painlevé type kernel. The error term $O(n^{-2})$ in uniform for $u, v$ in any compact subsets of $(0, \infty)$ and $\zeta \in (0, \infty)$, thus finishing the proof of Theorem 7.

**Remark 3.** In the study of [13] and [21], we obtain the properties of $\bar{\Psi}$-kernel. To be specific, $\bar{\Psi}$-kernel is approximated by the Bessel kernel as $\zeta \to 0^+$
\[
\tilde{K}_\phi(u, v, \zeta) = J_\alpha(u, v) + O(\zeta),
\]
uniformly for $u, v$ in compact subsets of $(0, \infty)$. $J_\alpha$ is the Bessel kernel. Using this property, the Bessel kernel limit of $K_n$ has been given under some conditions. Similarly, the $\bar{\Psi}$-kernel can be approximated by the Airy kernel as $\zeta \to +\infty$ and the Airy kernel limit has been expressed with some assumptions using the same method as we deal with the $\Psi$-kernel.

Thus, we can get the similar results for the $\bar{\Psi}$-kernel and $K_n$ with different parameters.

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