An algorithm for full parametric solution of problems on the statics of orthotropic plates by the method of boundary states with perturbations

To cite this article: V B Penkov et al 2018 J. Phys.: Conf. Ser. 973 012015

View the article online for updates and enhancements.

Related content

- Optimal location of piezoelectric patches for active vibration control
  Mohammad F. Labanie, J S Mohamed Ali and M.S.I Shaik Dawood

- The comparative study for the isotropic and orthotropic circular plates
  C Popa and G Tomescu

- Application of generalized equations of the method of finite differences to the calculation of continuous orthotropic plates
  Natalia Uvarova and Maxim Aleksandrovskiy
An algorithm for full parametric solution of problems on the statics of orthotropic plates by the method of boundary states with perturbations

V B Penkov¹, D A Ivanychev¹, O S Novikova¹ and L V Levin¹

¹ Department of General mechanics, Lipetsk State Technical University, Moskovskaya 30, Russia
² Department of Applied Mathematics, Lipetsk State Technical University, Moskovskaya 30, Russia

E-mail: vbpenkov@mail.ru

Abstract. The article substantiates the possibility of building full parametric analytical solutions of mathematical physics problems in arbitrary regions by means of computer systems. The suggested effective means for such solutions is the method of boundary states with perturbations, which aptly incorporates all parameters of an orthotropic medium in a general solution. We performed check calculations of elastic fields of an anisotropic rectangular region (test and calculation problems) for a generalized plane stress state.

1. Introduction

Historically, there are three phases in the methodology for solving problems that aim to determine the strength of linear-elastic bodies.

Classical (pre-computer) method. It is characterized by an analytical approach to the formulation and solution of problems. The results are presented in a strict (closed or quadrature) or approximate (series-based) form and contain all the parameters of the problem (FPS: analytical full parametric solution). The obvious features and advantages of this approach are: wide use and concomitant development of the mathematical apparatus (special functions expanded into series, functions of a complex variable, integral transforms, ...); simplicity of computational procedures used in engineering and design work (including for optimization of solutions as regards geometric and physical parameters). The main disadvantages are: a relative narrowness of the classes available for the analysis of bodies in relation to their "geometry" (unlimited 3D or 2D space, half-space, layers, cylindrical bodies, cones (wedges), etc.); limited options for types of boundary conditions (BC; first main, second main, main combined, contact problems); and rareness of problems focusing on the interaction of homogeneous elastic bodies of different materials.

Computational (computer) method. The widespread use of computer technology has practically removed restrictions with regard to classes of geometric configurations of bodies considered. It became possible to find precise solutions for problems with numerous boundary conditions and problems involving piecewise-homogeneous and inhomogeneous bodies. However, the advantages inherent in the classical approach turned into disadvantages: a change of any parameter (outside the P–theorem [1]) requires time-consuming and resource-intensive re-computation; this spurred the need to compare the results of the application of various computational techniques; the energy intensity of
optimization problems dramatically increased. Nevertheless, sophisticated numerical computing systems, especially their finite-element and boundary-element options, are now the ultimate analysis tool used by structural engineers.

Computer algebraic (numerical analytical) method. The development of modern computing systems supporting "computer algebra" essentially allows combining the advantages of the classical and computational approaches. The (quite solvable) bottleneck here is the plurality of geometric parameters of bodies and physical constants of media. The methodology of FPS for isotropic media has been by and large carefully thought out by now [2].

The purpose of this paper is to develop a method for structuring FPSs for homogeneous anisotropic bodies. It is to be achieved through a system of interconnected procedures: correct problem formulation, non-dimensionalization (P-theorem), choice of solution methods, and incorporation of parameters of boundary conditions and physical parameters of the medium in the explicit solution.

Some of the above tasks require clarification. An effective means of building FPSs is the modern power-operated method of boundary states (MBS) [3], initially created to address computerized algebra issues. Its further enhancement to incorporate the perturbation method (MBSP) [3] allows to effectively address particular physical properties of media.

Below we provide a methodology for structuring solutions taking into account medium parameters, using a two-dimensional problem seeking to determine the anisotropic elasticity value as an example.

The most complete presentation of materials on the state of two-dimensional bodies (isotropic, orthotropic) is contained in the fundamental work of S. Timoshenko and S. Woinowsky-Krieger [4] and S. Lehnitsky [5].

2. Reduction of the model of weakly orthotropic environment by the perturbation method
The state of the environment is governed by the generalized Hooke's law (the coordinate axes coinciding with the axes of orthotropy), which expresses the stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ with their respective deformations $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$ ($\theta = \varepsilon_{xx} + \varepsilon_{yy}$):

$$\begin{align*}
\sigma_{xx} &= \lambda_{1} \theta + 2 \mu_{1} \varepsilon_{xx}; \\
\sigma_{yy} &= \lambda_{1} \theta + 2 \mu_{2} \varepsilon_{yy}; \\
\sigma_{xy} &= 2 \mu_{0} \varepsilon_{xy},
\end{align*}$$

where $\lambda_{1}, \mu_{1}, \mu_{2}, \mu_{0}$ are the modulus of volume expansion and the shift moduli, respectively. In the case of a generalized plane stress state it is convenient to use the following isotropic medium parameters: $\mu = (E_{1} + E_{2})/(4(1 + \nu)), \quad \lambda = (E_{1} + E_{2})\nu/(2(1 + \nu)(1 - \nu))$, where $E_{1}, E_{2}, \nu$ are the moduli of elasticity and Poisson's ratio for an orthotropic medium, respectively.

Such an assignment allows us to describe real elastic constants using small parameters $\alpha, \beta, \delta$, with the parameter $\alpha$ used twice: $\mu_{1} = \mu(1 + \beta), \quad \mu_{2} = \mu(1 + \alpha), \quad \lambda_{1} = \lambda(1 + \alpha), \quad \mu_{0} = \mu(1 + \delta)$. Now Hooke's law (2.1) is rewritten in a convenient form:

$$\begin{align*}
\sigma_{xx} &= \lambda(1 + \alpha) \theta + 2 \mu(1 + \beta) \varepsilon_{xx}; \\
\sigma_{yy} &= \lambda(1 + \alpha) \theta + 2 \mu(1 + \alpha) \varepsilon_{yy}; \\
\sigma_{xy} &= 2 \mu(1 + \delta) \varepsilon_{xy},
\end{align*}$$

Zero values of the parameters correspond to isotropy.

Detailed summary of features on the asymptotic perturbation method is given in the monograph of Nayfeh A. [6] and Minaeva N. [7].

For brevity we introduce the designation $\sum_{m,n,k=0}^{\infty} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty}$. ...
Below in asymptotic series the upper indices, which correspond to the powers of the small parameters, identify particular elements of the expansion:

\[
\{\sigma_{ij}, e_{ij}, u_i, \theta\} = \sum_{m n k} \alpha^m \beta^n \delta^k \{\sigma^{m n k}_{ij}, e^{m n k}_{ij}, u^{m n k}_i, \theta^{m n k}\}, \quad \theta^{m n k} = e^{m n k}_{x x} + e^{m n k}_{y y}.
\]

Expansion of the values incorporated into Hooke's law (2.2) after replacement of the summation and postulation variables with zero values for any formally non-existent element of expansion having at least one index with a negative value \((m^{m n k} = 0, m < 0 \lor n < 0 \lor k < 0)\) leads to the following consequence:

\[
\begin{align*}
\sigma^{m n k}_{x x} &= \lambda \theta^{m n k} + 2 \mu e^{m n k}_{x x} + \bar{\sigma}^{m n k}_{x x}, \quad \bar{\sigma}^{m n k}_{x x} = \lambda \theta^{m-1 n k} + 2 \mu e^{m n-1 k}_{x x}, \\
\sigma^{m n k}_{y y} &= \lambda \theta^{m n k} + 2 \mu e^{m n k}_{y y} + \bar{\sigma}^{m n k}_{y y}, \quad \bar{\sigma}^{m n k}_{y y} = \lambda \theta^{m-1 n k} + 2 \mu e^{m n-1 k}_{y y}, \\
\sigma^{m n k}_{x y} &= 2 \mu e^{m n k}_{x y} + \bar{\sigma}^{m n k}_{x y}, \quad \bar{\sigma}^{m n k}_{x y} = 2 \mu e^{m n k-1}.
\end{align*}
\]

After a change of notation (tensordial and indicial notation combined) [8]

\[
s^{m n k}_{ij} = \sigma^{m n k}_{ij} - \bar{\sigma}^{m n k}_{ij}
\]

we present the elements of the expansion in the form conventional for the generalized Hooke's law for isotropic bodies:

\[
s^{m n k}_{ij} = \lambda \theta^{m n k} + 2 \mu e^{m n k}_{ij}
\]

The Cauchy formula is put in the same notation:

\[
e^{m n k}_{ij} = \frac{1}{2} \left( u^{m n k}_{i,j} + u^{m n k}_{j,i} \right)
\]

Using \(X^{0}_{i}\) to denote volume forces and assuming the expansion

\[
X^{0}_{i} = \sum_{m n k} \alpha^m \beta^n \delta^k X^{0 m n k}_{i}
\]

as known, we rewrite the equilibrium equations in the following form:

\[
s^{m n k}_{i,j} + X^{m n k}_{i} = 0, \quad X^{m n k}_{i} = X^{0 m n k}_{i} + \bar{e}^{m n k}_{i,j}.
\]

Equations (2.4) – (2.6) formally describe the plane deformation state of isotropic elastic bodies (or generalized plane stress, in which case, though, the modulus of volumetric strain will have a different value [9]). Assuming the elastic field formed based on the fictitious body forces \(X^{m n k}_{i}\), we define the state \(\{u_i, e^{m n k}_{ij}, s^{m n k}_{ij}\}\) through the biharmonic equation for the Airy stress function [9], for which the Kolosov–Muskhelishvili general representation formula is obtained. This allows to obtain a basis of space of internal states by the MBS method [3]. Since the original basis is independent from the parameters \(\alpha, \beta, \delta\), an orthonormal basis ("body in the sense of MBS") is obtained only once with high precision and then used in each iteration.

3. The basics of the method of boundary states with perturbations

The MBS is based on the isomorphism of spaces \(\Xi\) and \(\Gamma\), \(\Xi\) being the space of internal states and \(\Gamma\) being the space of boundary states.
The main provisions of the perturbation method as applied to the medium state concept are as follows. Operator $A$ governs the constitutive relations of a medium. By making operator $A$ act on the internal state $\xi$, we obtain the following right-hand members of constitutive relations $f$:

$$A(\beta)\xi = f(\beta), \quad x \in V,$$

where $\beta$ is a small parameter. Nonlinear operator $L$ generates a set of boundary conditions $\varphi$ from attributes of the boundary state $\gamma$:

$$L(\beta)\gamma = \varphi(\beta), \quad x \in \partial V.$$ (3.2)

We will analyze a class of operators $A$ representable by series in terms of powers $\beta$ involving linear operators $A_k$:

$$A = \sum_{k=0}^{\infty} \beta^k A_k.$$ Let the right-hand member (2.6) be presented by the following series:

$$f(\beta) = \sum_{s=0}^{\infty} \beta^s f_s,$$

then the operator equation (3.1) takes the following form:

$$\sum_{k=0}^{\infty} \beta^k A_k \xi = \sum_{s=0}^{\infty} \beta^s f_s.$$ (3.3)

The internal state is treated as a series in terms of the small parameter $\beta$:

$$\xi = \sum_{j=0}^{\infty} \beta^j \xi^j,$$ (3.4)

where $\xi^j$ is a state of $j$-order in the method of perturbations. Ratios (3.3) adjusted for equation (3.4) deliver the following:

$$\sum_{k=0}^{\infty} \beta^k A_k \sum_{j=0}^{\infty} \beta^j \xi^j = \sum_{s=0}^{\infty} \beta^s f_s,$$

which is later converted as follows:

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \beta^{k+j} A_k \xi^j = \sum_{s=0}^{\infty} \beta^s f_s.$$ (3.5)

After bringing (3.5) to a common index $s$ we obtain the following:

$$\sum_{s=0}^{\infty} \sum_{k=0}^{s} \beta^s A_k \xi^{s-k} = \sum_{s=0}^{\infty} \beta^s f_s.$$

For approximation $s$ we have:

$$\sum_{k=0}^{s} A_k \xi^{s-k} = f_s,$$

which brings us to the operator equation:
\[ A_0 \xi^s = f_s - \sum_{k=1} A_k \xi^{s-k} . \]  

These calculations are also valid for operator \( L \), which is applied to boundary state \( \gamma \). The resultant formula of the boundary state is:

\[ L_0 \gamma^s = \varphi^s - \sum_{k=1} L_k \gamma^{s-k} . \]  

The result of the decomposition of the original nonlinear boundary value problem (3.1), (3.2) is reduced to a sequence of linear problems (3.6), (3.7).

4. The method of boundary states with perturbations for a multiple-parameter physical medium

Any internal state of a linear isotropic elastostatic medium is a set of displacements, strains, and stresses that correlate with each other through constitutive relations: \( \xi = \{u_i, \varepsilon_{ij}, \sigma_{ij}\} \in \Xi \). Their trace at the boundary \( \partial V \) of the region \( V \) with an outer unit normal \( n_j \) contains information about the displacements and forces at work along the boundary \( \gamma = \{u_i, p_j\} \in \Gamma \), \( p_j = \sigma_{ij} n_j \) and corresponds to the boundary state. The spaces of possible internal and boundary states are isomorphic: \( \Xi \leftrightarrow \Gamma \); moreover, this is a Hilbert isomorphism [3]:

\[ a_1 \xi^{(1)} + a_2 \xi^{(2)} \leftrightarrow a_1 \gamma^{(1)} + a_2 \gamma^{(2)} , \quad (\xi^{(1)}, \xi^{(2)})_\Xi = (\gamma^{(1)}, \gamma^{(2)})_\Gamma . \]

Any well-defined problem is reduced to an infinite system of linear algebraic equations (ISE) [3]

\[ Qc = q \]  

relative to the vector of Fourier coefficients \( c \), the unknown state being expanded into a series with an orthonormal basis

\[ \xi = \sum_i c_i \xi^{(i)} , \]  

where all the coefficients \( c_i \) are related to any component of the element basis. The matrix \( Q \) ("the skeleton of the problem") is determined structurally only by the type of boundary conditions and numerically through an orthonormal basis (with identity matrices used in basic problems). The vector of the right-hand members includes the information about the specific boundary conditions.

When using the MBSP, it is convenient to organize the enumeration of exponents of small parameters by assigning the consolidated index \( m + n + k = p \) in ascending order:

\[ p = 0, 1, 2, 3, \ldots ; \quad m = 0, \ldots, p ; \quad n = 0, \ldots, p - m ; \quad k = 0, \ldots, p - m - n \]

and solving specific problems for each combination of \( m, n, k \).

An ISE is formulated at each step (4.1) in accordance with the boundary condition of the iteration. In practice, it suffices to take the real boundary conditions into account only when \( p = 0 \), solve only the basic problem with \( Q = E \) in subsequent iterations, and take account of the corrections brought about by the new fictitious volume forces (which are not actually potential forces, but are polynomial) in the right-hand members. A general method for searching for internal states for a class of such forces has been recently described [10].

There are certain steps to be taken before the iterations: the bases of the spaces \( \Xi \) and \( \Gamma \) are formed using the general solution of Lamé’s equations and the basis of functions harmonic in
V ∪ ∂V; isomorphic orthonormal bases are made; and members \( X_i^{0 \, mnk} \) are set for expansion series for \( X_i^{0} \).

At step \( p = 0 \) (\( m = n = k = 0 \)): state \( \xi^{000} \) caused by volume forces \( X_i^{0 \, 000} \) is searched for; a correction corresponding to this state is made in the real boundary conditions; an ISE (4.1) \( Qc^{000} = q \) is formed; its solution and a linear combination (4.2) lead to the internal state value \( \xi^{000} \); this value added to that of the state caused by volume forces prepares an initial approximation for \( \xi \):

\[
\xi = \xi^{000} + \xi^{000}.
\]

Based on the previous (2.3) formulas, the tensor \( \sigma_{ij}^{000} \) is set.

At step \( m + n + k = p > 0 \): the tensor \( s_{ij}^{mnk} = \sigma_{ij}^{mnk} - \bar{\sigma}_{ij}^{mnk} \) and vector \( X_i^{mnk} \) are drawn; the state \( \xi^{mnk} \) caused by volume forces \( X_i^{mnk} \) is searched for in accordance with formula (2.6); the boundary conditions are amended accordingly; the first basic problem for a system of Lamé’s equations (2.4) – (2.6) is solved; the resultant state value is added to that of the state caused by fictitious volume forces; the stress field is adjusted in accordance with (2.3) \( \sigma_{ij}^{mnk} = s_{ij}^{mnk} + \bar{\sigma}_{ij}^{mnk} \); and the new value is added to the resultant state being accumulated with an index of \( \alpha^m \beta^n \delta^k \).

After performing a sufficient number of approximations it is necessary to make a final substitution \( \beta = \mu / \mu - 1, \alpha = \lambda / \lambda - 1, \delta = \mu_0 / \mu - 1 \) and turn to dimensional values. The result is an FPS.

5. Full parametric solution of problems with rectangular plates

The FPS methodology described above is tested on a fairly simple first basic problem featuring an aluminium plate occupying the area \( D = \{(x, y) | x| \leq 2R, |y| \leq R \} \). Following a non-dimensionalization (scales: on stresses – \( \eta^* \), linear – \( R \), we formulated and solved a test problem and a calculation problem (figure 1).

![Figure 1. Orthotropic plate: a) test; b) calculation.](image)

**Test problem.** The lateral faces of the plate are subjected to forces (figure 1,a):

\[
\{ p_x, p_y \} = \begin{cases} 
{-1, 1}, (x, y) \in S_1 \cup S_4 \\
{1, -1}, (x, y) \in S_2 \cup S_3 
\end{cases}.
\]

There are no volume forces at work: \( X_i^{0} = 0 \).
The solution of the homogeneous problem is plain: \( \sigma_{xx} = 1 \), \( \sigma_{yy} = 1 \), \( \tau_{xy} = -1 \). For aluminium \((\eta = 10^6 \text{ Pa})\), dimensionless elasticity parameters are taken as \( \lambda = 19.96 \), \( \mu = 24.46 \), \( \mu_s = 31.61 \), \( \nu = 0.29 \) the deformation values are: \( \varepsilon_{xx} = 0.0118 \), \( \varepsilon_{xy} = 0.00917 \), \( \varepsilon_{yy} = -0.01724 \).

The MBSP applied allows to analyze an isotropic medium with a dimensionless Young’s modulus of \( E = 70.75 \) and a Poisson's ratio of \( \nu = 0.24 \) (hence \( \mu = 28.53 \), \( \lambda = 18.018 \)), which correspond to the small parameters of \( \alpha = 0.108 \), \( \beta = -0.142 \), and \( \delta = 0.0165 \), at each iteration. The third-order solution delivers the following components of the displacement vector \( \mathbf{u} = \{u_x, u_y \} \) (with the highest degrees of the product of small parameters greater than the third order being dropped):

\[
\begin{align*}
    u_x & \approx (10.742x - 17.526y - 2.079x\alpha + 0.4x\alpha^3 - 8.663x\beta + 1.676x\alpha^2\beta + 1.676x\alpha\beta^2 + 6.986x\beta^3 - 17.526y\delta - 17.526y\delta^3) \cdot 10^{-3}; \\
    u_y & \approx (-17.526x + 10.742y - 12.821y\alpha + 13.223y\alpha^3 + 2.079y\beta - 0.402y\alpha^2\beta - 1.676y\beta^3 + 17.526x\delta - 17.526x\delta^3) \cdot 10^{-3}.
\end{align*}
\]

Using the above real values of \( \alpha \), \( \beta \), \( \delta \), we obtain: \( \sigma_{xx} = 1 \), \( \sigma_{yy} = 1 \), \( \tau_{xy} = -1 \), \( \varepsilon_{xx} = 0.01173 \), \( \varepsilon_{xy} = 0.00917 \), \( \varepsilon_{yy} = -0.01723 \), \( u_x = 0.01173x - 0.01723y \), \( u_y = -0.01723x + 0.00917y \). The relative asymptotic errors for \( \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy} \) are 0.59, 0.79, and 0.23\%, respectively.

**Calculation problem.** As an example of how the proposed method for taking account of physical parameters for media is to be refined, we are going to solve the first basic problem for an orthotropic plate (fig.1,b) in its heterogeneous state, it being understood that its orthotropic parameters are known. The non-dimensionalized load parameters are matched by the following equilibrium:

\[
\{ p_x, p_y \} = \begin{cases} 
-1 + y^2, & (x, y) \in S_1, \\
0, & (x, y) \in S_2 \cup S_3, \\
2/3, & (x, y) \in S_4.
\end{cases}
\]

We propose a second-order solution of the problem. To get around the awkwardness of the formulas involved, we will provide the components of the displacement vector in a truncated form. Their structures are:

\[
\begin{align*}
    u_x & \approx (9.568x - 0.238x^2 + 0.197x^3 - 0.094x^4 + K + 0.157y^2 - 0.68xy^2 + 0.98x^2y^2 + K + 0.634xy^3 + K + 73.635xy^4\alpha\beta^2 - K - 5.983xy^2\beta\delta^2 - K - 2.29lx^2y^6\alpha\delta^2 + K) \cdot 10^{-3}; \\
    u_y & \approx (-2.45y + 0.576xy - 0.937x^2y + 0.207x^3y + K + 0.038x^2y^5\alpha - K - 17.465x^6y\alpha\delta^2 - K - 1.31x^2y^7\alpha\delta^2 - K - 4.07lx^8y^5\delta^3 + K) \cdot 10^{-3}.
\end{align*}
\]

Reverse parameterization delivers the following:

\[
\begin{align*}
    u_x & \approx R(9.568x - 0.238x^2 + 0.197x^3 - 0.094x^4 + K + 0.157y^2 - 0.68xy^2 + 0.98x^2y^2 + K + 0.634xy^3 + K + 73.635xy^4\alpha\beta^2 - K - 5.983xy^2\beta\delta^2 - K - 2.29lx^2y^6\alpha\delta^2 + K) \cdot 10^{-3}; \\
    u_y & \approx R(-2.45y + 0.576xy - 0.937x^2y + 0.207x^3y + K + 0.038x^2y^5\alpha - K - 17.465x^6y\alpha\delta^2 - K - 1.31x^2y^7\alpha\delta^2 - K - 4.07lx^8y^5\delta^2 + K) \cdot 10^{-3}.
\end{align*}
\]

\[7\]
We note here that \( x, y \) are to be understood as their actual values assigned to \( R \).

Displacements, corresponding basis element of the space of internal states have the form (given 10 elements, table 1):

**Table 1.** The basic elements of displacements.

| \( n \) | \( u_x \) | \( u_y \) |
|---|---|---|
| 1 | 0.021x | 0.021y |
| 2 | 0.004x^2 - 0.074y^2 | 0.078xy |
| 3 | -0.035x | 0.035y |
| 4 | -0.078xy | 0.074x^2 - 0.004y^2 |
| 5 | 0.035y | 0.035x |
| 6 | -0.014x^3 - 0.17xy^2 | 0.17x^2y + 0.014y^3 |
| 7 | -0.053x^2 + 0.053y^2 | 0.11lx |
| 8 | -0.064x^2y + 0.092y^3 | -0.064xy^2 + 0.092x^3 |
| 9 | 0.1lx | 0.053x^2 - 0.053y^2 |
| 10 | -0.031x^4 - 0.23x^2y^2 + 0.11y^4 | 0.3x^3y - 0.016x^3y^3 |

The accuracy of the approximate parametric solution of the problem mainly depends on the accuracy of the solution in the zero approximation in which given boundary conditions are set on the boundary of the body. We present several explicit and implicit characteristics of the convergence of the problem in the zero approximation (table 2).

**Table 2.** Nonzero Fourier coefficients.

| \( k \) | \( c_k \) | \( k \) | \( c_k \) | \( k \) | \( c_k \) | \( k \) | \( c_k \) | \( k \) | \( c_k \) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.138 | 6 | 0.012 | 11 | -0.014 | 18 | -0.012 | 23 | 0.005 |
| 2 | -0.006 | 7 | 0.005 | 14 | 0.015 | 19 | -0.017 | 26 | -0.004 |
| 3 | -0.177 | 10 | -0.015 | 15 | 0.021 | 22 | 0.007 | 27 | 0.003 |
| 4 | 0.002 | 30 | 0.002 | 35 | 0.003 |
| 5 | 0.001 | 46 | 0.001 |

Figure 2 is a graph illustrating the saturation of the Bessel sum (left part of the inequality Bessel). It does not prove anything, but convinces us of the correctness of the received solution.

![Figure 2. Saturation of the Bessel sum.](image)
Figures 3, 4 and 5 show comparison of boundary conditions (bar) with the same value from the boundary state, obtained by solving problem (solid line). In the drawings there has been a convergence with the given calculated values with a sufficient degree of accuracy.

Figure 3. Verification of the boundary conditions with calculated values on the boundary $S_1$.

Figure 4. Verification of the boundary conditions with calculated values on the boundary $S_2$.

Figure 5. Verification of the boundary conditions with calculated values on the boundary $S_3$.

Figure 6 shows isolines of the components of the stress tensor $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ and shear stress $T_{max}$, drawn based on the actual parameters of the material.
As we can see from the drawings, the greatest tensile stress $\sigma_{xx}$ along the axis of symmetry is focused along the left edge in the center of the plate, whereas compressive stress is localized at its edges. Stress $\sigma_{yy}$ has approximately the same character. The nonzero shear stress $\sigma_{xy}$ is concentrated near the left side and each side approximately the same distance (20% of the width of the plate). The character of the maximum shear stress $T_{\text{max}}$ differs little from the character of the axial tensile stresses, except that its highest concentration is not adjacent to the left side.

The foregoing analysis leads to the conclusion that MBSP proves to be an effective means of explicit full parametric solution of mechanics problems containing a finite number of physical parameters of the medium.

The research was supported by RFFR with grant No. 16-41-480729 “r_a”

References
[1] Sedov L I 1977 Similarity methods and dimensional analysis in mechanics (Moscow: Nauka) p 440
[2] Levina L V, Novikova O S and Penkov V B 2016 Full parametric solution of problems of the theory of elasticity of simply connected finite bodies Bulletin of Lipetsk State Technical University 2 (28) pp 16-24
[3] Penkov V B and Satalinka L V 2012 Method of boundary states with the perturbation: the inhomogeneous and nonlinear problems in the theory of elasticity and thermoelasticity (Germany: LAP LAMBERT Academic Publishing GmbH & Co.) p 108
[4] Timoshenko S and Woinowsky-Krieger S 1959 Theory of plates and shells (USA: Mcgraw-Hill Book Company, Inc) p 580
[5] Lehnitsky S G 1957 Anisotropic plate Editor Feldman G I (Moscow: State publishing house of technical-theoretical literature) p 463
[6] Nayfeh A H 1993 Introduction to perturbation techniques (USA: A Wiley-interscience publication. John Wiley & Sons, Inc ) p 519
[7] Minaeva N I 2002 The perturbation method in mechanics of deformable bodies (Moscow: Scientific book) p 156
[8] Mase G E 1970 Theory and problems of continuum mechanics (New York, St. Louis, San Francisco, London, Sydney, Toronto, Mexico and Panama: Mcgraw-Hill Book Company, Inc) p 221
[9] Muskhelishvili N I 1966 Some basic problems of the mathematical theory of elasticity (Moscow: Nauka) p 70
[10] Penkov V B, Levina L V, Levin M Y and Kuzmenko N V 2016 A new method for analyzing the effect of body forces induced by nanodispersed magnetic fluids on states of elastic solids Science in the Central Russia (journal on science and production technology) 2(20) pp 12-16