Quantum carpets: a tool to observe decoherence

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Abstract. Quantum carpets—the spatio-temporal de Broglie density profiles—woven by an atom or an electron in the near-field region of a diffraction grating bring to light, in real time, the decoherence of each individual component of the interference term of the Wigner function characteristic of superposition states. The proposed experiments are feasible with present-day technology.

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1. Introduction

Decoherence [1] is an essential ingredient of quantum mechanics. Very often it is used to explain the emergence of the classical world [2–5] from a microscopic level. However, even macroscopic molecules [6–8] such as C_{60}, C_{70}, C_{60}F_{48} or C_{44}H_{30}N_{4} and interacting atomic Bose–Einstein condensates [9] display interference properties. Although in experiments involving a cavity field [10–12], superconducting quantum circuits [13], an ion stored in a Paul trap [14], C_{70} fullerenes [15] and a neutron in an interferometer [16] one has observed certain features of the decoherence of Fock states or of the superposition of two coherent states [17], the quest [18] to record the decay of the total Wigner function [19] of such non-classical states remains an important issue in studies in quantum mechanics. In the present work, we show that the space–time structures in the position probability distribution of a de Broglie wave after a diffraction grating, i.e. quantum carpets [20], allow us to observe the decay of each individual component of the Wigner function in real time.

Under appropriate conditions the quantum mechanical probability density of a non-relativistic particle displays [20], when represented in space–time, characteristic valleys and ridges as shown in figure 1. These striking patterns have led to the name quantum carpet. Its design is most pronounced when the energy spectrum, determining the propagation of the particle, produces a strong degeneracy of internode traces [21], as is the case for the particle caught in a box [22] or diffracted from an infinite periodic grating. The latter situation is closely related to the Talbot effect [20, 22]. Moreover, some features, such as the full and fractional revivals, also persist in the presence of small relativistic corrections [23] or even in a fully relativistic case [24], described by the Dirac equation. The ridges and valleys do not follow classical trajectories but the world lines created by interference. Consequently, the design of the carpet is purely an interference effect and is, therefore, sensitive to decoherence.

We show that a quantum carpet in the presence of decoherence allows us to observe the decay of each momentum component of the interference Wigner contribution. Our proposal relies on three guiding principles: (i) a quantum carpet, which is the position probability density \( P(x, t) \) to find the particle at the transverse position \( x \) at a distance \( z = vt \) after a diffraction
Figure 1. Quantum carpets in the absence of decoherence represented by the position probability density $P(x,t)$ of a beam of atoms or electrons after it has been diffracted from a grating aligned along the $x$-axis. To enhance the pattern visibility, the plot is in the logarithmic scale. Bright or dark colours represent high or low probability densities as indicated by the thermometer located on the right-hand side of the figure. Time translates into distance $z = v_z t$ from the grating by the macroscopic velocity $v_z$ of the beam orthogonal to the $x$-axis. Here we have chosen an initial Gaussian wave packet centred in $\bar{x} = L/2$, having an average momentum $\bar{p} = 10\hbar/L$ and width $\Delta x = 0.03L$.

We consider two types of carpets corresponding to two distinctly different mechanisms of decoherence: a carpet woven by (i) a two-level atom that can spontaneously go to its ground state and thereby receives a single random momentum kick, and (ii) an electron that passes close to a metal plate and thereby gets damped by its image charge and undergoes Brownian motion. In both cases the design of the carpet gets washed out as the distance to the grating increases.

Present-day technology even allows us to implement our proposal. Indeed, quantum carpets for several types of de Broglie waves [26–31] and of light [32] have been observed. Moreover, the influence of a single spontaneously emitted photon on the far-field interference pattern of an atomic wave has been measured [33]. Likewise, the field of electron optics [34, 35] has made remarkable progress and a double-slit experiment in the presence of a metal plate has already been carried out [36] at the University of Tübingen. Hence, the proposed observation of the space–time patterns in the probability density can serve as a flexible tool to monitor decoherence in a variety of experimental settings and conditions. A deeper understanding of decoherence not only is of interest to investigate the quantum to classical transition, but also has practical implications for the development of reliable quantum technologies.

This paper is organized as follows. In section 2 we connect the design of a quantum carpet to the individual components of the initial Wigner function. We then, in section 3, consider two different mechanisms of decoherence and study their effect on the space–time structures of the
probability density. Finally, we summarize and discuss our results in section 4. For technical
details of the derivation of the Wigner function produced by a periodic array of wave functions,
see the appendix.

2. Quantum carpets woven by Wigner functions

We start our discussion by connecting the design of a carpet to the individual components
of the initial Wigner function. For this purpose, we study the diffraction of a non-relativistic
quantum mechanical particle from a grating and emphasize that many such experiments have
been performed with atoms [26–28], molecules [8], electrons [29–31, 36], neutrons [16] or
light [32].

2.1. Carpet represented as a superposition of Wigner function components

Here we express the probability density in space–time forming the quantum carpet as an infinite
superposition of Wigner functions corresponding to an array of Schrödinger cat states. This state
corresponds to the initial state. The time dependence enters as a displacement of the original
Wigner function along straight lines in space–time.

A mechanical or optical grating of period $2L$ produces a periodic array

$$\psi(x; t = 0) \equiv \sum_{l=-\infty}^{\infty} \phi(x - 2lL)$$ (1)

of wave functions. Here $x$ denotes the coordinate along the grating and $\phi = \phi(x)$ is the wave
packet created by a single slit.

When we substitute this wave function into the definition [19]

$$W(x, p; t = 0) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \psi \left( x + \frac{1}{2}\xi, t \right) \psi^* \left( x - \frac{1}{2}\xi, t \right) e^{-ip\xi/\hbar}$$ (2)

of the Wigner function, we find in view of the results obtained in the appendix that the Wigner
function

$$W_{\phi}(x, p; t = 0) = \frac{\pi\hbar}{2L} \sum_{n,l} (-1)^n \delta(p - p_n) W_{\phi}(x - lL, p_n)$$ (3)

of such an array with period $2L$ consists of a periodic array of Wigner functions

$$W_{\phi}(x, p) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \phi \left( x + \frac{1}{2}\xi \right) \phi^* \left( x - \frac{1}{2}\xi \right) e^{-ip\xi/\hbar}$$ (4)

separated by $L$.

Due to the spatial periodicity, the momentum

$$p_n \equiv \frac{n\pi\hbar}{2L}$$ (5)

is discrete. We also note, in (3), the presence of the alternating weight factor $(-1)^{n+l}$ for the
individual contributions of the Wigner function.
After the grating the particle evolves freely as expressed by the Liouville equation [19]

\[
\hat{L} W_\psi(x, p; t) = 0,
\]

(6)

where

\[
\hat{L} \equiv \frac{\partial}{\partial t} + \frac{p}{M} \frac{\partial}{\partial x}
\]

(7)
denotes the Liouville operator of free motion for a particle of mass \(M\). It is easy to verify that the expression

\[
W_\psi(x, p; t) = W_\psi\left(x - \frac{p}{M} t, p; t = 0\right)
\]

(8)
in terms of the initial Wigner function is the solution of the Liouville equation (6).

With the help of expression (3), we find that

\[
W_\psi(x, p; t) = \pi \bar{h}^2 L \sum_{n,l} (-1)^n l \delta(p - p_n) W_\phi[\chi_{n,l}(x, t), p_n]
\]

(9)

with the straight space–time trajectories

\[
\chi_{n,l}(x, t) \equiv x - \frac{p_n}{M} t - lL.
\]

(10)
The position distribution \(P(x, t)\) at a distance \(z \equiv v_z t\) from the grating forms the carpet. Here \(v_z\) is the velocity component orthogonal to the grating. Since definition (2) of the Wigner function implies the identity

\[
P(x, t) = \int_{-\infty}^{\infty} dp \, W_\psi(x, p; t),
\]

(11)

we find \(P(x, t)\) from (9) by integration over \(p\), which yields

\[
P(x, t) = \frac{\pi \bar{h}}{2L} \sum_{n,l} (-1)^n W_\phi[\chi_{n,l}(x, t), p_n].
\]

(12)

Hence, the cut of the Wigner distribution along the \(x\)-axis for a fixed momentum \(p_n\) moves along the straight space–time trajectory \(\chi_{n,l}\) with a fixed speed determined by \(p_n\). Thus as time increases, the individual cuts of the Wigner distribution at neighbouring momenta separate. We can distinguish them when the size of the Wigner structure is smaller than the separation between two of them. The fastest separation occurs for almost vertical space–time trajectories, that is, for small momenta contained in the Wigner function of the original state. Small momenta are usually a characteristic feature of superposition states.

A consequence of momentum quantization is the periodic revival of the probability density \(P(x, t)\). Indeed, there exists a revival time \(T_1\) such that \(P(x, t + T_1) = P(x, t)\). This rephasing takes place when all components of the Wigner functions \(W_\phi[\chi_{n,l}(x, t), p_n]\) have travelled a transverse distance, which is an integer multiple of the period \(2L\) of the grating. Therefore, the revival time is set by the smallest momentum component \(p_1 \equiv \pi \bar{h}/(2L)\), that is, by

\[
T_1 \equiv \frac{2L}{p_1/M} = \frac{4L^2 M}{\pi \bar{h}}.
\]

(13)
2.2. Schrödinger cats

Next we recall [17] the Wigner function corresponding to the anti-symmetric superposition
\[ \phi(x) \equiv \varphi(x) - \varphi(-x), \]  
(14)
built from the wave packet \( \varphi = \varphi(x) \). Indeed, when we substitute this superposition into definition (2) of the Wigner function we obtain, owing to the bilinearity of \( W \) in \( \varphi \), the expression
\[ W_\varphi(x, p) = W_\varphi(x, p) + W_\varphi(-x, -p) + W_{\text{int}}(x, p), \]  
(15)
where
\[ W_\varphi(x, p) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi \left( x + \frac{1}{2} \xi \right) \varphi^* \left( x - \frac{1}{2} \xi \right) e^{-i p \xi / \hbar} \]  
(16)
represents the Wigner function of the basis packet, and the interference term \( W_{\text{int}} \) is given by
\[ W_{\text{int}}(x, p) \equiv -\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \varphi \left( x + \frac{1}{2} \xi \right) \varphi^* \left( x - \frac{1}{2} \xi \right) \]  
(17)
\[ + \varphi \left( x + \frac{1}{2} \xi \right) \varphi^* \left( -\left( x - \frac{1}{2} \xi \right) \right) \]  
\[ e^{-i p \xi / \hbar}. \]
Hence, the Wigner function of the superposition state (14) is the sum not only of the Wigner functions of the basis packet and its phase space inversion, that is, of \( W_\varphi(x, p) \) and \( W_\varphi(-x, -p) \), but also contains the term \( W_{\text{int}} \). It is this interference term which is extremely sensitive to decoherence.

The elementary example of a Gaussian wave function
\[ \varphi(x) \equiv \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{\Delta x}} \exp \left[ -\frac{1}{2} \left( \frac{x - \bar{x}}{\Delta x} \right)^2 \right] \exp \left[ \frac{i \bar{p} (x - \bar{x})}{\hbar} \right] \]  
(18)
yields the Gaussian Wigner distribution
\[ W_\varphi(x, p) \equiv \frac{1}{\pi \hbar} \exp \left[ -\left( \frac{x - \bar{x}}{\Delta x} \right)^2 \right] \exp \left[ -\left( \frac{p - \bar{p}}{\Delta p} \right)^2 \right], \]  
(19)
centred around \( \bar{x} \) and momentum \( \bar{p} \). Here the widths \( \Delta x \) and \( \Delta p \equiv \hbar / \Delta x \) are measures of the extension of the quantum state in phase space indicating when the Gaussian has decayed to \( 1/e \).

In contrast, the interference term
\[ W_{\text{int}}(x, p) = -\frac{2}{\pi \hbar} \cos \left[ \frac{2}{\hbar} (\bar{p} x - p \bar{x}) \right] e^{-\left( x / \Delta x \right)^2} e^{-\left( p / \Delta p \right)^2} \]  
(20)
is always located at the origin of phase space independent of the initial position \( \bar{x} \) and momentum \( \bar{p} \). Moreover, it assumes negative values whereas \( W_\varphi \) is everywhere positive.

2.3. Quantum carpet for a Schrödinger cat

Indeed, for the example of the superposition state \( \phi \) with a Gaussian wave packet \( \varphi \) of a macroscopic average momentum \( \bar{p} \gg p_1 \), we find a clear separation into momenta around \( \bar{p} \) giving rise to rather flat trajectories in space–time and small momenta creating steep world lines.
The latter result from the interference term $W_{\text{int}}$. Hence, $W_{\text{int}}$ translates itself into the design of the carpet.

But how to create a superposition state such as $\phi$ from a grating? Many possibilities offer themselves. In the case of atoms [19], we can use an optical grating which automatically creates the desired superposition. In the case of a mechanical grating [21], we can send two de Broglie waves under an appropriate angle onto the grating. In figure 1 we show a quantum carpet woven by a particle prepared in the superposition state defined by (14) and (18). Due to the parity of the initial wave function, it suffices to study the probability density in the space region from 0 to $L$ and up to the revival time $T_{\text{rev}} = T_1/2$.

3. Quantum carpets in the presence of decoherence

We now derive expressions for quantum carpets in the presence of decoherence. Here we consider two quantum systems with different mechanisms of decoherence: (i) a two-level atom undergoing spontaneous emission and (ii) an electron moving over a metal surface creating image charges.

3.1. Decoherence due to spontaneous emission

Our discussion starts with a two-level atom moving under the influence of spontaneous emission. The Wigner functions $W_e$ and $W_g$ corresponding, respectively, to atoms in the excited and ground states satisfy the generalized optical Bloch equations, which describe the coupled dynamics of internal and external atomic degrees of freedom [37]

\[ \hat{L} W_e(x, p; t) = -\gamma W_e(x, p; t) \]  
\[ \hat{L} W_g(x, p; t) = \gamma \int_{-1}^{1} d\epsilon f_s(\epsilon) W_e(x, p + \epsilon \hbar k; t), \]

where $\gamma$ is the rate of spontaneous emission and $\hat{L}$ denotes the Liouville operator of free motion defined by (7). The term in (21) proportional to $\gamma$ describes the relaxation due to spontaneous emission. Moreover, the term at the right-hand side of (22) describes the transfer of population from the excited to the ground state by spontaneous emission. The distribution $f_s = f_s(\epsilon)$ with

\[ \int_{-1}^{1} d\epsilon f_s(\epsilon) = 1 \]

reflects the momentum component along the $x$-axis.

When the atoms exit the grating at time $t = 0$ in the excited state, the solutions of these equations read

\[ W_e(x, p; t) = e^{-\gamma t} W_\psi \left( x - \frac{p}{M} t, p; t = 0 \right) \]  
and

\[ W_g(x, p; t) = \int_{0}^{\gamma t} d\tau e^{-\gamma t} \int_{-1}^{1} d\epsilon f_s(\epsilon) W_\psi \left( x - \frac{p}{M} t - \epsilon \delta x_\tau, p + \epsilon \hbar k; t = 0 \right) \]
with the displacement
\[ \delta x_s \equiv \frac{\hbar k}{M} \gamma \]  
(26)
in position during the decay time \( 1/\gamma \) due to the recoil of the atom giving it a velocity \( \hbar k/M \).

A comparison between expressions (24) and (8) for the Wigner functions \( W_e(x, p; t) \) and \( W(x, p; t) \) in the excited state and the standard one shows that the influence of decoherence is just a multiplication of \( W \) by the decay factor \( \exp(-\gamma t) \). As a result, we find the quantum carpet

\[ P_e^{(s)}(x, t) = e^{-\gamma t} P(x, t) \]  
(27)
formed by the atoms in the excited state to be the original quantum carpet, (12), in the absence of decoherence. However, its intensity decays as a function of time, that is, separation from the grating.

Next we turn to the carpet woven by the ground state atoms. For this purpose, we substitute the initial Wigner function (3) into (25) and obtain the formula

\[ W_g(x, p; t) = \frac{\pi \hbar}{2L} \sum_{n,l} (-1)^n l \int_0^{\gamma t} \int_{-1}^{1} \mathrm{d}\tau \mathrm{d}\varepsilon f_s(\varepsilon) \delta(p + \varepsilon \hbar k - p_n) \times W_\phi \left(x - \frac{p}{M} \tau - lL - \varepsilon \delta x_s \tau, p + \varepsilon \hbar k\right), \]  
(28)
which after integration over momentum with the help of the delta function yields the expression

\[ P_g^{(s)}(x, t) = \frac{\pi \hbar}{2L} \sum_{n,l} (-1)^n l \overline{W}_\phi^{(s)} \left[x_{n,l}(x, t), p_n; t\right]. \]  
(29)
Hence, while the intensity of the carpet formed by the excited atoms fades away, the carpet formed by the atoms in the ground state builds up in intensity. However, we now have to average the original Wigner function \( W_\phi \) over the displacement \( \delta x_s \) due to the spontaneously emitted photon, that is,

\[ \overline{W}_\phi^{(s)}(x, p; t) \equiv e^{-\gamma t} \int_0^{\gamma t} \mathrm{d}\tau \mathrm{d}\varepsilon f_s(\varepsilon)W_\phi(x + \varepsilon \delta x_s \tau, p). \]  
(30)
The averaged Wigner function \( \overline{W}_\phi^{(s)} \) depends explicitly on time. In particular, after a specific time the characteristic structures of the Wigner function are averaged out.

In figure 2 we illustrate these features by showing the quantum carpets for a beam of two-level atoms, initially prepared in the excited state. The left column displays the same ideal quantum carpet in the absence of decoherence (\( \gamma = 0 \)), which we use as a benchmark and a guide to the eye. In the other columns, we gradually increase the rate of spontaneous emission from \( \gamma = 1 \) (centre) to \( \gamma = 5 \) (right). In all our numerical calculations, we set \( \hbar = 1, L = 1 \) and \( M = 1 \). Moreover, we choose the photon momentum \( \hbar k = 1 \) and use a constant distribution \( f_s = 1/2 \).

In the top and bottom rows, we show the carpets corresponding to the excited and ground states, respectively. We clearly observe the exponential damping of the probability density \( P_e^{(s)} \) to find the atom in the excited state (top sequence) indicated by (27). Indeed, the design of the quantum carpet remains identical to the case without decoherence (left), but the contrast in the space–time structures becomes fainter as the atom moves away from the grating.
Figure 2. Decoherence of an atomic quantum carpet. Position probability density $P(x, t)$ of a beam of atoms after it has been diffracted from a grating aligned along the $x$-axis. To enhance the pattern visibility, the plots are in the logarithmic scale. Bright or dark colours represent high or low probability densities as indicated by the thermometer located on the right-hand side of the left figure. Time translates into distance $z = v_z t$ from the grating by the macroscopic velocity $v_z$ of the beam orthogonal to the $x$-axis. The left column displays the ideal quantum carpet in the absence of decoherence ($\gamma = 0$). This plot provides us with a benchmark against which we compare the quantum carpets for increasing values of the damping rate: $\gamma = 1$ (centre) and $\gamma = 5$ (right). Top: probability density $P_e(x, t)$ to find the atom in the excited state given by (27). Bottom: probability density $P_g(x, t)$ to find the atom in the ground state determined by (29) and (30). Here we have chosen an initial Gaussian wave packet centred in $\bar{x} = L/2$, having an average momentum $\bar{p} = 10\hbar/L$ and a width $\Delta x = 0.03L$.

At the same time, the probability $P_g(x)$ of finding the atom in the ground state given by (29) and (30) (bottom centre and right) builds up in intensity. However, the random recoil due to spontaneous emission produces a less distinct pattern in the corresponding quantum carpets. In this case, one can only appreciate the revival of the Gaussian wave packet at $T_{\text{rev}}/4$, $T_{\text{rev}}/2$, $3T_{\text{rev}}/4$ and $T_{\text{rev}}$. The pattern of the quantum carpet appears blurred already after $T_{\text{rev}}/4$.

In order to obtain the carpet for the excited state, we have directly used the expression for $P_e(x)$, (27), and computed the series using a convergence tolerance of $10^{-4}$. We found that this tolerance was sufficient to bring out the structures in the quantum carpet.

To plot $P_g(x)$, one could, in principle, use the analytical expression given by (29). However, in practice, we found the convergence of the series to be too slow. Thus, we numerically evaluate
the expression
\[ P_g(s, t) = \frac{\pi \hbar}{2L} \int_0^{\gamma_l} dt \int_{-1}^1 d\varepsilon e^{-\varepsilon} f_s(\varepsilon) \sum_{n,l} (-1)^n W_0 \left( x - \frac{p_n - \varepsilon \hbar k}{M} t - lL - \varepsilon \delta x_s \tau, p_n \right); \]  
that is, we interchange summations and integrations. We now first perform the summation over \( n \) and \( l \) and then we average over \( \varepsilon \) and \( \tau \).

Here we use an adaptive Simpson quadrature rule to calculate the integral and an error tolerance of \( 10^{-4} \) for the convergence of the series in the integrand. We found that this procedure is sufficient to illustrate the effect of decoherence in the carpet structures.

### 3.2. Decoherence due to image charges

We now turn to the second model of decoherence: an electron propagates above a metal surface as it creates the quantum carpet. The image charge in the metal [38] leads to a damped Brownian motion of the electron described by the Langevin equation, i.e. a Heisenberg equation of motion for the position operator \( \hat{x}(t) \) of the particle in equilibrium with a linear passive heat bath. For the case of a free particle interacting with an Ohmic bath, this equation takes on the well-known form [39]
\[ \frac{d^2 \hat{x}}{dt^2} + \Gamma \frac{d\hat{x}}{dt} = \frac{\hat{F}(t)}{M}. \]  
Here \( \Gamma \) and \( \hat{F} \) denote the damping constant and the Gaussian quantum force, respectively.

The explicit solution
\[ \hat{x}(t) = \hat{x}(0) + \frac{\hat{p}(0)}{M\Gamma} \left( 1 - e^{-\Gamma t} \right) + \hat{f}(t) \]  
of the Langevin equation, (32), with the force operator
\[ \hat{f}(t) \equiv \frac{1}{M} \int_0^t dt' \int_0^{t'} dt'' e^{-\Gamma(t' - t'')} \hat{F}(t'') \]  
allows us to analyse the corresponding quantum carpet
\[ P^{(B)}(x, t) \equiv \langle \delta[\hat{x}(t) - x] \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk |e^{ik\hat{x}(t)}| e^{-ikx} \]  
in the presence of Brownian motion.

In order to evaluate the expectation value \( \langle \exp[ik\hat{x}(t)] \rangle \) with the time-dependent position operator \( \hat{x}(t) \), we recall that the noise operator \( \hat{f}(t) \) at time \( t \) commutes with the operators \( \hat{x}(0) \) and \( \hat{p}(0) \) of the electron at the initial time \( t = 0 \), when it starts interacting with its image charge. Hence, the average \( \exp[ik\hat{x}(t)] \) factors into one over the electron and the other over the reservoir, that is,
\[ \langle e^{ik\hat{x}(t)} \rangle = \left\langle \exp \left\{ ik \left[ \hat{x}(0) + \left( 1 - e^{-\Gamma t} \right) \frac{\hat{p}(0)}{M\Gamma} \right] \right\} \right\rangle \langle e^{ik\hat{f}(0)} \rangle. \]
We can perform the average of the symmetrically ordered operator pertaining to the electron using the Wigner function, (3), and arrive at

\[ \langle e^{ik\hat{\mathbf{r}}(t)} \rangle = \frac{\pi \hbar}{2L} \sum_{n,l} (-1)^{n+l} \int dx(0) \int dp(0) \exp \left\{ ik \left[ x(0) + \left( 1 - e^{-\Gamma t} \right) \frac{p(0)}{\hbar M \Gamma} \right] \right\} \]

\[ \times \delta \left( p(0) - p_n \right) W_\phi (x(0) - IL, p_n) \left( e^{ik\hat{f}(t)} \right), \]  

which after integration over \( p(0) \) using the delta function and the new integration variable \( y' \equiv x(0) + (1 - e^{-\Gamma t}) p(0)/(M \Gamma) \) reduces to

\[ \langle e^{ik\hat{\mathbf{r}}(t)} \rangle = \frac{\pi \hbar}{2L} \sum_{n,l} (-1)^{n+l} \int dy' e^{iky'} W_\phi \left( y' - (1 - e^{-\Gamma t}) \frac{p_n}{\hbar M \Gamma} - IL, p_n \right) \left( e^{ik\hat{f}(t)} \right). \]  

In the final step, we substitute this expression into the space–time distribution (35) for the carpet and find that

\[ P^{(B)}(x, t) = \frac{\pi \hbar}{2L} \sum_{n,l} (-1)^{n+l} \int dy W_\phi \left( x^{(B)}_{n,l} (x, t) + y, p_n \right) \frac{1}{2\pi} \int dk e^{iky} \left( e^{ik\hat{f}(t)} \right) \]

where we have introduced \( y \equiv y' - x \) and the space–time trajectory

\[ x^{(B)}_{n,l} (x, t) \equiv x - \frac{p_n}{\hbar M \Gamma} \left( 1 - e^{-\Gamma t} \right) - IL, \]  

which is curved due to the damping.

It is convenient to arrange the terms in the form

\[ P^{(B)}(x, t) = \frac{\pi \hbar}{2L} \sum_{n,l} (-1)^{n+l} \overline{W}_\phi^{(B)} \left[ x^{(B)}_{n,l} (x, t), p_n; t \right] \]  

with the averaged Wigner function

\[ \overline{W}_\phi^{(B)} (x, p; t) \equiv \int_{-\infty}^{\infty} dy W_\phi (x + y, p) \mathcal{K}(y; t) \]

and the time-dependent kernel

\[ \mathcal{K}(y; t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{iky} \left( e^{ik\hat{f}(t)} \right) \]

of Brownian motion.

Hence, the Brownian motion averages the initial Wigner function over a time-dependent kernel \( \mathcal{K} \) and the characteristic interference features of the Wigner function disappear.

The similarities and differences between the carpets of spontaneous emission and Brownian motion stand out most clearly in the classical limit where \( \mathcal{K} \) reduces to a Gaussian [5, 39]

\[ \mathcal{K}(y; t) \equiv \frac{1}{\sqrt{\pi} \delta x_B(t)} \exp \left[ - \left( \frac{y}{\delta x_B(t)} \right)^2 \right], \]

with a time-dependent width

\[ \left[ \delta x_B(t) \right]^2 \equiv D \left[ 2\Gamma t - 2 (1 - e^{-\Gamma t}) - (1 - e^{-\Gamma t})^2 \right] \]

and \( D \equiv k_B T/(M \Gamma) \). Here, \( k_B \) and \( T \) denote Boltzmann’s constant and the temperature of the metal surface, respectively.

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Figure 3. Decoherence of an electronic quantum carpet. Position probability density $P^{(B)}(x, t)$ of a beam of electrons propagating on top of a metal surface after it has been diffracted from a grating aligned along the $x$-axis. To enhance the pattern visibility, the plots are in the logarithmic scale. Bright or dark colours represent high or low probability densities, respectively, as indicated by the thermometer located on the right-hand side of the left figure. Time translates into distance $z = v_z t$ from the grating by the macroscopic velocity $v_z$ of the beam orthogonal to the $x$-axis. Moving from left to right, we increase the damping rate from $\Gamma = 0$ to $\Gamma = 0.1$ and 0.5. Here we have chosen an initial Gaussian wave packet centred at $\bar{x} = L/2$, having an average momentum $\bar{p} = 10\hbar/L$ and a width $\Delta x = 0.03L$. The other parameters are $k_b = 1$ and $T = 4$.

At first glance, it might appear that this result is different from the corresponding results in [4]. However, the work by Ford et al [4] is concerned with ‘entanglement at all times’, in which case no divergences appear, even at zero temperature.

On the other hand, the work by Ford and O’Connell [5] is concerned with the solution of the exact master equation (which was shown to be equivalent to the solution of the Langevin equation for the initial value problem) and, in this case, it was shown that serious divergences arise for low bath temperatures. However, as stressed in the abstract of this paper, worthwhile results may be obtained for high temperatures but one must distinguish between two cases: (a) a particle at zero (or low) temperature which is suddenly coupled to a bath at high temperature and (b) a particle whose initial temperature is the same as the bath temperature (the case where the initial temperature and the bath temperature are both high but different was not analysed in detail but it follows closely the analysis for the previous case).

It is scenario (a) that is relevant in the present context, where we envisage an electron at a relatively low temperature being brought into the confines of a metal surface at a much higher temperature.

When we substitute the Gaussian kernel (44) into (42), we find the averaged Wigner function

$$\overline{W}_\phi^{(B)}(x, p; t) = \int_{-\infty}^{\infty} d\varepsilon f_B(\varepsilon) W_\phi(x + \varepsilon \delta x_B(t), p)$$

(46)

due to the Brownian motion, with the Gaussian weight function

$$f_B(\varepsilon) \equiv \frac{1}{\sqrt{\pi}} \exp \left( - \varepsilon^2 \right).$$

In figure 3 we show the probability density $P^{(B)}(x, t)$ to find the electron at a transverse position $x$ and a distance $v_z t$ from the grating given by (41) and (46). The left column displays again the
ideal quantum carpet in the absence of decoherence. We then increase the value of the damping rate from $\Gamma = 0.1$ (middle) to $\Gamma = 0.5$ (right). As a result the Brownian motion quickly destroys the space–time structures in $P^{(B)}(x, t)$ and after $T_{\text{rev}}/2$ the design of the quantum carpet gets completely washed out.

3.3. Discussion

We conclude by comparing the quantum carpets of an atom undergoing spontaneous emission and an electron moving over a metal surface. According to (46) the averaged Wigner function $\overline{W}^{(B)}_\phi$ due to Brownian motion takes a form similar to the distribution $\overline{W}^{(s)}_\phi$ defined in (30) and originating from spontaneous emission. However, there are three major differences: (i) whereas $\overline{W}^{(s)}_\phi$ contains two averages—one over the decay time and one over the momentum change—$\overline{W}^{(B)}_\phi$ involves only the average over the displacement in position. (ii) The weight function $f_B$ of the Brownian motion is always a Gaussian, whereas the corresponding distribution $f_s$ of spontaneous emission depends on the type of the transition and ranges from being constant to a trigonometric function. (iii) The displacement $\delta x_s$ of spontaneous emission in (30) is linear in time for all times. In contrast, the displacement squared $[\delta x_B]^2$ of Brownian motion, (45), grows for $\Gamma t \ll 1$ as $t^3$ but only turns into a linear dependence for $\Gamma t \to \infty$.

4. Summary and outlook

In conclusion, we have analysed the influence of decoherence on quantum carpets woven by de Broglie waves. Spontaneous emission or Brownian motion, the sources of decoherence, manifest themselves directly in the destruction of the design of the carpet. In this way, we can observe in real time the decay of the interference terms of the Wigner function.

We emphasize that the proposed technique to bring out the influence of decoherence is fundamentally different from the one used in the landmark experiments using photons in a cavity [10–12], an ion stored in a trap [14] or the circuit QED analogues [13]. Whereas they require the reconstruction [40, 41] of the Wigner function from measurements, such as the occupation probabilities of atomic, phononic or photonic states, the present method of using a quantum carpet measures directly the parts of the Wigner function most susceptible to decoherence. Indeed, here we take advantage of three facts: (i) we can prepare a de Broglie wave in an array of superposition states corresponding to a Schrödinger cat by diffracting it from a grating, (ii) the free propagation of the matter wave after the grating forming the carpet is classical when described by the Wigner function of the array of superposition states and (iii) the design of the carpet is intimately connected to the shape of the interference terms in the Wigner function.

Indeed, the structures of the individual world lines with their valleys and crests criss-crossing the carpet are the Wigner function components of the interference term caused by the superposition state at a given low momentum. Since we consider an infinitely long periodic grating, these momenta are discrete and structure develops in the carpet only at discrete inclinations with respect to the axes defining space–time, that is, in the near-field diffraction pattern.

Decoherence attacks predominantly the interference terms of the Wigner function and thereby fills the valleys. As a result, the design of the carpet gets washed out. However, it is
only due to the discreteness of the momenta and the time evolution of a free particle after the
grating that these components separate in the form of a fan and we can observe the filling-up of
each component.

Our proposal to use the design of a quantum carpet as a tool to observe decoherence is
particularly suited for diffraction experiments such as the pioneering ones based on atoms [42],
large molecules [6–8, 15, 25], light [32] or electrons [36]. Although most of these articles have
analysed the apparent decoherence based on specific models, they have not made use of the
direct connection between the near-field diffraction pattern defining the carpet and the Wigner
function. However, it is in this way that we gain deeper insight into the nature of decoherence
and, in particular, in crossing the border between the microscopic and the macroscopic world.

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Appendix. The Wigner function corresponding to a periodic array of wave functions

In this appendix, we derive an expression for the Wigner function corresponding to an array

\[ \psi(x) = \sum_{l=-\infty}^{\infty} \phi(x - 2lL) \]  

(A.1)
of wave functions with period 2L. Here we make no assumptions on \( \phi \) except that it is
integrable. Moreover, for simplicity in notation throughout this appendix the summation indices
run over all integers and the integrals have the limits \(-\infty\) and \(+\infty\).

When we substitute the wave function \( \psi \) given by (A.1) into definition (2) of the Wigner
function, we find that

\[ W_\psi(x, p) = \frac{1}{2\pi \hbar} \sum_{k,l} \int d\xi \left( x - 2kL + \frac{1}{2}\xi \right) \phi^* \left( x - 2lL - \frac{1}{2}\xi \right) e^{-ip\xi/\hbar}. \]  

(A.2)

We first change the summation indices \( k \) and \( l \) to \( r \) and \( s \), where \( k - l \equiv 2r \) and \( k + l \equiv 2s \) when
\( k - l \) and \( k + l \) are both even, and \( k - l \equiv 2r + 1 \) and \( k + l \equiv 2s + 1 \) when \( k - l \) and \( k + l \) are both
odd. After these transformations, (A.2) takes the form

\[ W_\psi(x, p) = \frac{1}{2\pi \hbar} (I_e + I_o), \]  

(A.3)

where

\[ I_e \equiv \sum_{r,s} \int d\xi \left( x - 2(s + r)L + \frac{1}{2}\xi \right) \phi^* \left( x - 2(s - r)L - \frac{1}{2}\xi \right) e^{-ip\xi/\hbar} \]  

(A.4)
contains the even and
\[ I_0 = \sum_{r,s} \int dx \, \phi \left( x - 2(s + r + 1) + \frac{1}{2} \xi \right) \phi^* \left( x - 2(s - r) - \frac{1}{2} \xi \right) e^{-ip\xi/h} \quad (A.5) \]
the odd terms.

Next we apply the Poisson summation formula \[ g_m = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\mu \, g(\mu) \, e^{2\pi i n\mu}, \quad (A.6) \]
where \( g(\mu) \) is an extension of the function \( g_m \) to the whole real axis, to the summation over \( r \) and obtain the expressions
\[ I_e = \sum_{n,s} \int dx \, \int d\xi \, \phi \left( x - 2(s + \mu) + \frac{1}{2} \xi \right) \phi^* \left( x - 2(s - \mu) - \frac{1}{2} \xi \right) e^{-ip\xi/h} e^{2\pi i n\mu} \quad (A.7) \]
and
\[ I_o = \sum_{n,s} \int dx \, \int d\xi \, \phi \left( x - 2(s + \mu + 1) + \frac{1}{2} \xi \right) \phi^* \left( x - 2(s - \mu) - \frac{1}{2} \xi \right) e^{-ip\xi/h} e^{2\pi i n\mu} \quad (A.8) \]

It is convenient to first slightly rewrite the arguments of \( \phi \) and cast the integrals into the form
\[ I_e = \sum_{n,s} \int dx \, \int d\xi \, \phi \left( x - 2sL + \frac{1}{2}(\xi - 4\mu \mu) \right) \phi^* \left( x - 2sL - \frac{1}{2}(\xi - 4\mu \mu) \right) e^{-ip\xi/h} e^{2\pi i n\mu} \quad (A.9) \]
and
\[ I_o = \sum_{n,s} \int dx \, \int d\xi \, \phi \left( x - (2s + 1) + \frac{1}{2}(\xi - (4\mu + 2)L) \right) \]
\[ \times \phi^* \left( x - (2s + 1) - \frac{1}{2}(\xi - (4\mu + 2)L) \right) e^{-ip\xi/h} e^{2\pi i n\mu} \quad (A.10) \]
and then introduce in \( I_e \) and \( I_o \) the new integration variables \( y = 4\mu L \) and \( (4\mu + 2)L \), respectively, which yields
\[ I_e = \frac{1}{4L} \sum_{n,s} \int dy \, \int dx \, \phi \left( x - 2sL + \frac{1}{2}(\xi - y) \right) \]
\[ \times \phi^* \left( x - 2sL - \frac{1}{2}(\xi - y) \right) e^{-ip\xi/h} \exp \left( i\pi n \frac{y}{2L} \right) \quad (A.11) \]
and
\[ I_o = \frac{1}{4L} \sum_{n,s} (-1)^n \int dy \, \int dx \, \phi \left( x - (2s + 1) + \frac{1}{2}(\xi - y) \right) \]
\[ \times \phi^* \left( x - (2s + 1) - \frac{1}{2}(\xi - y) \right) e^{-ip\xi/h} \exp \left( i\pi n \frac{y}{2L} \right) \quad (A.12) \]

Here we have made use of the identity \( e^{i\pi n} = (-1)^n \).
When we substitute (A.11) and (A.12) back into (A.3), we can combine both terms by noting that $(-1)^{n^2} = 1$ and $(-1)^{n(2s+1)} = (-1)^n$, which yields

$$W_\psi(x, p) = \frac{1}{4L} \frac{1}{2\pi\hbar} \sum_{l,n} (-1)^n \int dy \int d\xi \phi \left(x - lL + \frac{1}{2}\xi\right) \times \phi^*(x - lL - \frac{1}{2}\xi) e^{-i\nu\xi/\hbar} \exp \left[-i \left(p - n\frac{\pi\hbar}{2L}\right) \frac{y}{\hbar}\right], \quad (A.13)$$

where we have introduced the new integration variable $\xi' \equiv \xi - y$.

Next we interchange the two integrations, recall definition (2) of the Wigner function and arrive at

$$W_\psi(x, p) = \frac{1}{4L} \sum_{l,n} (-1)^n W_\phi(x - lL, p) \int dy e^{-i(p-p_n)y/\hbar}, \quad (A.14)$$

where we have defined the Wigner function

$$W_\phi(x, p) \equiv \frac{1}{2\pi\hbar} \int d\xi \phi \left(x - \frac{1}{2}\xi\right) \phi^*(x + \frac{1}{2}\xi) e^{-i\nu\xi/\hbar} \quad (A.15)$$

of the wave function $\phi$ and the discrete momentum

$$p_n = \frac{n\pi\hbar}{2L} \quad (A.16)$$

introduced by the periodicity of the array.

With the help of the Fourier representation

$$\frac{1}{2\pi\hbar} \int dy e^{-ipy/\hbar} = \delta(p) \quad (A.17)$$

of the delta function, we obtain the result

$$W_\psi(x, p) = \frac{\pi\hbar}{2L} \sum_{l,n} (-1)^n W_\phi(x - lL, p) \delta(p - p_n). \quad (A.18)$$

References

[1] Zurek W H 1991 Phys. Today 44 36
[2] Giulini D J W, Joos E, Zeh H D, Kiefer C, Kupsch J and Stamatescu I-O 1996 Decoherence and the Appearance of a Classical World in Quantum Theory (New York: Springer)
[3] Schlosshauer M 2005 Rev. Mod. Phys. 76 1267
[4] Schlosshauer M 2007 Decoherence and the Quantum-to-Classical Transition (Berlin: Springer)
[5] Ford G W and O’Connell R F 2001 Phys. Rev. A 64 032101
[6] Ford G W and O’Connell R F 2001 Phys. Rev. D 64 105020
[7] Arndt M, Nairz O, Vos-Andreae J, Keller C, van der Zouw G and Zeilinger A 1999 Nature 401 680
[8] Hackermüller L, Utenthaler S, Hornberger K, Reiger E, Brezger B, Zeilinger A and Arndt M 2003 Phys. Rev. Lett. 91 090408

New Journal of Physics 15 (2013) 013052 (http://www.njp.org/)
[38] Anglin J R and Zurek W H 1996 *Dark Matter in Cosmology, Quantum Measurements and Experimental Gravitation* ed R Ansari et al (Gif-sur-Yvette: Editions Frontières) p 263
   See also Anglin J R, Paz J P and Zurek W H 1997 *Phys. Rev. A* 55 4041
[39] Ford G W, Lewis J T and O’Connell R F 1988 *Phys. Rev. A* 37 4419
[40] Freyberger M, Bardroff P, Leichtle C, Schrade G and Schleich W P 1997 *Phys. World* 10 41
[41] Leibfried D, Pfau T and Monroe C 1998 *Phys. Today* 51 22
[42] Kurtsiefer Ch, Pfau T and Mlynek J 1997 *Nature* 386 150
[43] See, for example, Courant R and Hilbert D 1937 *Methods of Mathematical Physics* (Berlin: Springer)