Abstract

Plasma instabilities are parametrically the dominant nonequilibrium dynamics of a weakly coupled quark-gluon plasma. In recent years the time evolution of the corresponding collective colour fields has been studied in stationary anisotropic situations. Here I report on recent numerical results on the time evolution of the most unstable modes in a longitudinally expanding plasma as they grow from small rapidity fluctuations to amplitudes where non-Abelian self-interactions become important.

The experimental findings at the Relativistic Heavy Ion Collider RHIC [1] suggest a much faster thermalization than can be accounted for by the (weak-coupling) bottom-up thermalization scenario of Ref. [2]. However, as first pointed out by Arnold et al. [3], this scenario did not include the inevitable presence of plasma instabilities in an anisotropic quark-gluon plasma [4,5]. In fact, these are even the most important collective effects to leading order. Much effort has by now been invested recently in studying the specific non-Abelian dynamics of plasma instabilities, in particular in the crucial regime where the associated colour fields are still too weak to affect the distribution of hard particles but already strong enough for essential non-Abelian self-interactions.

Initial numerical studies that concentrated on the most unstable modes, namely long-wavelength coherent colour fields that are constant in transverse directions, suggested continued exponential growth also in this nonlinear regime [6]. Subsequent studies [7,8] with fully 3+1-dimensional colour fluctuations found however a late-time transition from exponential to (rather weak) linear growth. On the other hand, the more recent simulations of Ref. [9] found a continued exponential growth of initially small perturbations in the case of very strong momentum anisotropy, at least for small initial field configurations, which is similar to the behaviour of the 1+1-dimensional case.

All the above-mentioned investigations were performed within the hard-loop effective theory of a collisionless stationary anisotropic plasma [10]. Clearly, in view of the crucial dependence of the fate of non-Abelian plasma instabilities on the amount of momentum...
anisotropy, which rapidly increases during the free-streaming stage, a generalization to Bjorken expansion is desirable\footnote{In the color-glass framework, the issue of instabilities in an expanding system was studied in Ref. \cite{11}}.

In the stationary anisotropic situation, the coloured fluctuations of a colour-neutral space-time-independent background of hard particles can be factored according to

\[ \delta f^\alpha(x; p) = -g W^\mu_\alpha(t, x; v) \partial_\mu f_0(p) \]  

(1)

where the auxiliary fields \( W \) depend only on velocities \( v^\mu = p^\mu / |p| = (1, v) \) and are governed by a Vlasov equation

\[ [v \cdot D(A)] W_\mu(x; v) = F_{\mu\nu}(A) v^\nu \]  

(2)

in which the hard momentum scale \( p^0 \) does not appear. The latter can be “integrated out”, determining just the mass parameter in the induced current \( j^\mu[\delta f] = \frac{m_B^2}{2} \int W(x; v) \), where \( W \) is one particular linear combination of the components \( W^\nu \). The collective colour fields associated with plasma instabilities can then be determined by solving the equation for \( W \) together with the non-Abelian Maxwell equations \( D^\mu(A) F^\mu_\nu = j^\nu(x) \).

Remarkably, this formalism can largely be taken over to the nonstationary case of a plasma undergoing Bjorken expansion\footnote{In the color-glass framework, the issue of instabilities in an expanding system was studied in Ref. \cite{11}}.

In comoving (rapidity) variables \( x^\alpha = (\tau = \sqrt{t^2 - z^2}, x^i, \eta = \text{atanh}(z/t)) \) and \( p^\alpha = (p_\perp, (\cosh(y - \eta), \cos \phi, \sin \phi, \tau^{-1} \sinh(y - \eta)) \) with \( y = \text{atanh}(p^0/p^z) \), a free-streaming background distribution is given by

\[ f_0(p, x) = f_{\text{iso}} \left( \sqrt{p_\perp^2 + p_\eta^2/\tau_{\text{iso}}^2} \right) = f_{\text{iso}} \left( \sqrt{p_\perp^2 + (p^z / \tau_{\text{iso}})^2} \right) \]  

(3)

where \( p'_\perp \) is the boosted longitudinal momentum.

One can again introduce auxiliary fields \( W \)

\[ \delta f^\alpha(x; p) = -g W^\mu_\alpha(\tau, x^i, \eta; \phi, y) \partial_\mu f_0(p_\perp, p_\eta) \]  

(4)

that obey \( v \cdot D W_\alpha(\tau, x^i, \eta; \phi, y) |_{\phi, y} = v^\beta F_{\alpha\beta} \), but now

\[ v^\alpha \equiv p^\alpha / |p_\perp| = (\cosh(y - \eta), \cos \phi, \sin \phi, \tau^{-1} \sinh(y - \eta)). \]  

(5)

The induced current in the non-Abelian Maxwell equations is given by

\[ \frac{1}{\tau} D_\alpha(\tau F^{\alpha\beta}) = j^\beta(\tau, x^i, \eta) = -\frac{m_B^2}{2} (\tau = \tau_{\text{iso}}) \int \frac{d\phi}{2\pi} \int dy W(\tau, x^i, \eta; \phi, y) \]  

(6)

with \( W = v^i W_i - \frac{\tau_{\text{iso}}}{\tau_{\text{iso}}} \sinh(y - \eta) W_\eta \).

In numerical simulations one now ends up discretizing velocity/rapidity space that has the form of a cylinder parametrized by \( \phi, y \) instead of a sphere \( v^2 = 1 \). These numerical simulations have to start at a finite proper time \( \tau = \tau_0 \), both for practical reasons and for the physical reason that a plasma description makes sense at the earliest around \( \tau \sim Q_s^{-1} \), where \( Q_s \) is the saturation scale of perturbative QCD. Having chosen a background distribution function of the form \( \delta f^\alpha(x; p) \), the only free parameters are \( \tau_{\text{iso}} \), the (possibly
Fig. 1. Results [19] from a 1D+3V real-time lattice simulation of non-Abelian plasma instabilities in Bjorken expansion, seeded by small initial rapidity fluctuations with a spectrum modelled after Ref. [13].

We show the proper-time dependence of the total chromo-field energy density and its individual components $\mathcal{E} = \mathcal{E}_{BT} + \mathcal{E}_{ET} + \mathcal{E}_{BL} + \mathcal{E}_{EL} = \mathcal{E}_T + \mathcal{E}_L$ as well as the chromo-field energy gain rate $R$ defined by $R = d\mathcal{E}/d\tau + 2\mathcal{E}_T/\tau$.

In the results displayed in Fig. 1, $\tau_{iso} = 0.1\tau$, i.e. the distribution starts out already in oblate form and becomes increasingly oblate as time goes on. The mass parameter is matched to results from the saturation (colour-glass) scenario as explained in the appendix. In this numerical calculation, only the (most unstable) modes which are transversely constant and thus effectively 1+1-dimensional have been considered. These instabilities have been seeded by small rapidity fluctuations with a spectrum modelled after Ref. [13]. Given the findings in stationary anisotropic plasmas, the growth of non-Abelian plasma instabilities found here probably gives an upper bound on more generic cases. Considering that for $Q_s \simeq 1$ GeV for RHIC and $\simeq 3$ GeV for the LHC and that thus the maximal time in Fig. 1 corresponds to some 20 fm/c for RHIC and 7 fm/c for the LHC, one finds an uncomfortably long delay for the onset of plasma instabilities at least for RHIC. Further studies of more generic initial conditions (including then also strong initial fields) is work in progress.

Appendix A. Parameters of the numerical simulation

For fixing the dimensionful parameters of the numerical simulation in a way that makes contact with heavy-ion physics, the so-called Colour-Glass-Condensate framework [14,15] is invoked, and as starting time for the plasma phase $\tau_0 \simeq Q_s^{-1}$ is chosen, where $Q_s$ is the so-called saturation scale.

The only other dimensionful parameter, the Debye mass $m_D$ at the (fictitious because pre-plasma) time $\tau_{iso}$, is fixed by assuming a squashed Bose-Einstein distribution function for the hard particle distribution function [3] with $f_{iso}(p) = N/(2N_g)/(e^{p/T} - 1)$ where $N_g = N_g^2 - 1$ is the number of gluons and $N$ a normalization that is adjusted such that at $\tau = \tau_0$ the hard-gluon density of CGC estimates is matched. Since the expansion is by
assumption purely longitudinal, $T$ is a constant transverse temperature, and it has indeed been found in CGC calculations that the gluon distribution is approximately thermal in the transverse directions, with $T = Q_s/d$ and $d^{-1} \simeq 0.47$ according to Ref. [15]. The normalization $N$ can then be fixed by following Ref. [16], who write the initial hard-gluon density as

$$n(\tau_0) = \frac{c}{4\pi^2 N_c \alpha_s(Q_s \tau_0)} N_g Q_s^3,$$  \hspace{2cm} (A.1)

where $c$ is the gluon liberation factor, for which different estimates can be extracted from the literature.

We adopted the value $c = 2 \ln 2 \approx 1.386$ from an analytical estimate in Ref. [17] which turned out to be fairly close to the most recent numerical result $c \approx 1.1$ in Ref. [18]. With $\tau_{iso}$ remaining a free parameter which determines how anisotropic the gluon distribution is at $\tau_0$, the normalization $N$ is now fixed by

$$n(\tau_0) \frac{\tau_0}{\tau_{iso}} = n(\tau_{iso}) = \frac{2\zeta(3)}{\pi^2} N_g T^3.$$  \hspace{2cm} (A.2)

when $c = 2 \ln 2$ and $N_c = 3$. We adopt this value for our simulations where $N_c = 2$, since in previous studies of the stationary anisotropic situation little difference was found between the SU(2) and the SU(3) case provided $m_D$ was the same [8]. With our choice of an initial anisotropy given by $\tau_0/\tau_{iso} = 10$, equating $\tau_0 = Q_s^{-1}$ and using units where $\tau_0 = 1$, the above result corresponds to the value $m_D = 3.585$ employed in Fig. 1. This corresponds to an energy density of hard particles $\mathcal{E}(\tau_{iso} = 0.1\tau_0) = N 8\pi^2 Q_s^4 d^{-4}/15$ and (using Eq. (15) of [19]) $\mathcal{E}(\tau_0) \approx 0.0789 \mathcal{E}(\tau_{iso}) \approx 0.0938 Q_s^4(Q_s \tau_0)^{-1} \alpha_s^{-1}$.  

References

[1] M. J. Tannenbaum, Rept. Prog. Phys. 69 (2006) 2005–2060.
[2] R. Baier, A. H. Mueller, D. Schiff, D. T. Son, Phys. Lett. B502 (2001) 51–58.
[3] P. Arnold, J. Lenaghan, G. D. Moore, JHEP 08 (2003) 002.
[4] S. Mrówczyński, Phys. Lett. B214 (1988) 587.
[5] P. Romatschke, M. Strickland, Phys. Rev. D68 (2003) 036004.
[6] A. Rebhan, P. Romatschke, M. Strickland, Phys. Rev. Lett. 94 (2005) 102303.
[7] P. Arnold, G. D. Moore, L. G. Yaffe, Phys. Rev. D72 (2005) 054003.
[8] A. Rebhan, P. Romatschke, M. Strickland, JHEP 0509 (2005) 041.
[9] D. Bödeker, K. Rummukainen, JHEP 07 (2007) 022.
[10] S. Mrówczyński, A. Rebhan, M. Strickland, Phys. Rev. D70 (2004) 025004.
[11] P. Romatschke and R. Venugopalan, Rev. D74 (2006) 045011.
[12] P. Romatschke, A. Rebhan, Phys. Rev. Lett. 97 (2006) 252301.
[13] K. Fukushima, F. Gelis, L. McLerran, Nucl. Phys. A786 (2007) 107–130.
[14] L. D. McLerran, R. Venugopalan, Phys. Rev. D49 (1994) 2233–2241.
[15] E. Iancu, R. Venugopalan, The color glass condensate and high energy scattering in QCD, in: R. C. Hwa, X.-N. Wang (Eds.), Quark-gluon plasma 3, World Sci., Singapore, 2003, pp. 249–336.
[16] R. Baier, A. H. Mueller, D. Schiff, D. T. Son, Phys. Lett. B539 (2002) 46–52.
[17] Y. V. Kovchegov, Nucl. Phys. A692 (2001) 557–582.
[18] T. Lappi, Eur. Phys. J. C55 (2008) 285–292.
[19] A. Rebhan, M. Strickland, M. Attems, Phys. Rev. D78 (2008) 045023.