Compressing Branch-and-Bound Trees

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Branch and Bound Trees

Branch and bound (BB) tree:
- Each node $v$ corresponds to $Q(v)$
- Each non-leaf node $v$ has children:
  \[ Q(v) \cap \{ x : \pi^T x \leq \pi_0 \} \text{ and } \]
  \[ Q(v) \cap \{ x : \pi^T x \geq \pi_0 + 1 \} \]
  where $\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$.

Tree dual bound:
\[
d(T, c) = \min_{v \in L(T)} \min \{ c^T x : x \in Q(v) \} \]
Previous Research

• **Variable branching rules:**
  - Pseudocost branching
  - Strong branching
  - Reliability branching

  Benichou et al (1971)
  Applegate, Bixby, Chvátal & Cook (1995)
  Achterberg, Koch & Martin (2005)

• **Branching on general directions:**
  - Owen & Mehrotra (2001)
  - Mahajan & Ralphs (2009)
  - Cornuejols, Liberti, Nannicini (2011)
  - Gamrath & al (2015)

• **Bounds on tree size:**
  - Exponential size with var. disjunctions
  - Exponential size with general disjunctions
  - Size under limited support size
  - Full strong branching tree size

  Jeroslow (1974), Chvatal (1980)
  Dadush et al. (2020), Dey et al. (2022)
  Basu, Conforti, Di Summa & Jiang (2021)
  Dey, Dubey & Molinaro (2022)
Work Overview

- **Research question:** Can we make a BB tree smaller without deteriorating dual bound?
- **Motivation:**
  - Small dual certificates
  - Strong disjunctions for instance families
  - Training data for ML branching methods
- **Talk outline:**
  1. Tree Compression Problem (TCP)
  2. Complexity & lower bound results
  3. Exact & heuristic algorithms
  4. MIPLIB 3 & 2017 computational experiments
Tree Operations

\[
\text{drop}(T, \nu)
\]

\[
\text{replace}(T, \nu, \pi, \pi_0)
\]

\[
\pi \leq \pi_0
\]

\[
\pi x \geq \pi_0 + 1
\]
The Tree Compression Problem

**Compression**: $T_k$ is a compression of $T_1$ if $\exists T_1, T_2, \ldots, T_k$ such that:

1. $T_i = \text{drop}(T_{i-1}, \nu)$ or $T_i = \text{replace}(T_{i-1}, \nu, \pi, \pi_0)$; and
2. $|T_i| < |T_{i-1}|$; and
3. $d(T_i, c) \geq d(T_{i-1}, c)$

**Tree Compression Problem (TCP)**: Given a branch and bound tree $T$, an objective vector $c$, and a set of branching directions $\mathcal{D}$, is there a compression $T'$ of $T$?
NP-Completeness I

Disjunctive Infeasibility (DI):

- Let $S = \{x \in \mathbb{R}^n : Ax \leq b\}$ where $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$.
- Does there exist $\pi \in \mathbb{Z}^n \setminus \{0\}$, $\pi_0 \in \mathbb{Z}$ such that:

$$S \subseteq \{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}$$

Mahajan & Ralphs (2009): (DI) is NP-complete.

Theorem 1. (TCP) is NP-complete when $D = \mathbb{Z}^n$ and $c = 0$. 
NP-Completeness II

Proof Sketch:

1. Let $S$ be an instance of (DI)
2. Let $x^* \in S \setminus \mathbb{Z}^n$. WLOG $x_1 \notin \mathbb{Z}$.
3. Let

$$P = \text{conv}\left(\left\{\left(\frac{x^*}{0}, \frac{x^*}{1}\right)\right\} \cup \left\{\left(\frac{x}{\frac{1}{2}}\right) : x \in S\right\}\right)$$

4. Build the tree on the right.
5. If (DI) has a YES answer $(\pi, \pi_0)$ then replace($T, r, (\pi, 0), \pi_0$) is a compression.
6. If tree is compressible, it must be with replace($T, r, (\pi, \pi_{n+1}), \pi_0$), where $\pi_{n+1} = 0$ and $r$ is the root.
Theorem 2. There exists a tree $T$ with root polyhedron $P \subseteq \mathbb{R}^{n+1}$ such that:

1. $|T| \geq 2^{n+1}$ and $d(T, 0) = \infty$
2. Best compression of $T$ has at least $\frac{2^n-1}{n}$ nodes
3. There exists $T'$ with root $P$ s.t. $|T'| = 7$ and $d(T', 0) = \infty$.

Proposition: Suppose $T$ is generated with full strong branching and best bound on directions $D \subseteq \mathbb{Z}^n$. Let $T'$ be a compression of $T$ using the same directions. Then:

1. Dual bound does not improve: $d(T, c) = d(T', c)$
2. Drop operation is sufficient
Exact Algorithm

Observation: \( \text{replace}(T, \nu, \pi, \pi_0) \) is a compression of \( T \) if and only if:

\[
\min\{c^T x : x \in Q(\nu), \pi^T x \leq \pi_0\} \geq d(T, c) \quad \text{and} \\
\min\{c^T x : x \in Q(\nu), \pi^T x \geq \pi_0 + 1\} \geq d(T, c)
\]

MILP Formulation [Mahajan & Ralphs (2009)]:

\[
\begin{align*}
\max_{\delta, p, q, \pi, \\ \pi_0, s_L, s_R} & \quad A^T p - s_L c - \pi = 0, \quad p^T b - d(T, c) s_L - \pi_0 \geq \delta \\
\text{s.t.} & \quad A^T q - s_R c + \pi = 0, \quad q^T b - d(T, c) s_R - \pi_0 \geq \delta - 1 \\
& \quad p, q \geq 0, \quad s_L, s_R \geq 0, \quad \pi \in \mathbb{Z}^n, \quad \pi_0 \in \mathbb{Z}
\end{align*}
\]

Exact Algorithm: Solve MILP for every node.
Heuristic Algorithm

Heuristic for General Branching Directions:

- Owen & Mehrotra (2001)
- Cornuejols, Liberti, Nannicini (2011)
- Karamanov & Cornuejols (2011)
- Mahmoud & Chinneck (2013)
- Gamrath et al. (2015)

Owen & Mehrotra’s Heuristic:

- Find best single variable direction \((\pi, \pi_0)\)
- For each fractional \(x_i^*\) consider \(\pi + e_i\) and \(\pi - e_i\).
- Repeat until no further improvement
MIPLIB 3 Experiments: Setup

Questions

1. How compressible are realistic BB trees?
2. How much compression is achievable in short running times?

Branching rules considered:

1. Full strong branching (FSB)
2. Reliability branching (RB) with plunging

Implementation & Environment:

1. Julia, JuMP, Gurobi 9.5
2. MIPLearn: Custom B&B Implementation
3. Tree generation: 10k node limit, no time limit
4. Tree compression: 24-hour limit for exact, 15-minute for heuristic
5. AMD Ryzen 9 7950x (5.7GHz, 16C, 32T, 128 GB RAM)
MIPLIB 3 Experiments: FSB/Heuristic
MIPLIB 3 Experiments: RB/Exact
MIPLIB 3 Experiments: RB/Heuristic

![Graph showing performance of different models](image-url)
MIPLIB 2017: Setup

**Challenge:** Node subproblems become too expensive

**Node orderings:**
1. Random
2. DFS
3. NodeId
4. SubtreeSize
5. Gap
6. Expert

**Implementation:**
- Precomputed compressibility info
- Reliability branching without plunging
## MIPLIB 2017: Results

| Node Ordering | AUC (%) | Compression Ratio (%) |
|---------------|---------|-----------------------|
|               | 1-hour  | 15-min | 1-hour | 4-hour |
| Expert        | 65.4    | 30.8   | 34.4   | 35.1   |
| Gap           | 76.4    | 18.2   | 25.7   | 30.5   |
| NodeId        | 79.6    | 15.5   | 21.6   | 27.9   |
| SubtreeSize   | 79.6    | 15.6   | 21.7   | 28.1   |
| Random        | 80.7    | 13.3   | 21.2   | 28.7   |
| DFS           | 83.3    | 12.9   | 17.1   | 24.0   |
Conclusion & Future Work

In this talk:
- Tree compression problem
- NP-completeness and bound results
- Algorithms and MIPLIB experiments

Future work:
- Provably compressible trees
- Better heuristics
- Use directions found to accelerate MIP

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