EFFECTIVE COUNTING ON TRANSLATION SURFACES

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Abstract. We prove an effective version of a celebrated result of Eskin and Masur: for any $\text{SL}_2(\mathbb{R})$-invariant locus $\mathcal{L}$ of translation surfaces, there exists $\kappa > 0$, such that for almost every translation surface in $\mathcal{L}$, the number of saddle connections with holonomy vector of length at most $T$, grows like $cT^2 + O(T^2 - \kappa)$. We also provide effective versions of counting in sectors and in ellipses.

1. Introduction

The main goal of this paper is the effectivization of a celebrated result of Eskin and Masur [EM01] which we recall. A translation surface $x$ is a compact oriented surface equipped with an atlas of planar charts, whose transition maps are translations, where the charts are defined at every point of the surface except finitely many singular points at which the planar structure completes to form a cone point of angle an integer multiple of $2\pi$. Such structures arise in many contexts in geometry, complex analysis and dynamics, and have various equivalent definitions, see the surveys [MT02], [Zor06] for more details. The collection of all translation surfaces of a fixed genus, fixed number of singular points, and fixed cone angle at each singular point is called a stratum, and has a natural structure of a linear orbifold. Furthermore each connected component of the subset of area one surfaces in a stratum is the support of a natural smooth probability measure which we will call flat measure.

A saddle connection on a translation surface $x$ is a segment connecting two singular points which is linear in each planar chart and contains no singular points in its interior. The holonomy vector of a saddle connection is the vector in the plane obtained by integrating the pullback of the planar form $(dx, dy)$, along the saddle connection. We denote the collection of all holonomy vectors for $x$ by $V(x)$. The large scale geometry of $V(x)$ has been intensively studied, and one of the main results of [EM01] is that there is $c > 0$ such that for a.e. $x$ (with respect to the flat measure), the number $N(T, x) = |V(x) \cap B(0, T)|$ satisfies

$$N(T, x) = cT^2 + o(T^2).$$

When this holds we will say that $x$ satisfies quadratic growth.

The main purpose of this paper is to estimate the error term in the above result, that is to establish that

$$N(T, x) = cT^2 + O(T^{2(1-\kappa)})$$

for some $\kappa > 0$. In order to state our result in its full generality we need to introduce more precise terminology.

In this paper, the notations $g = O(f)$ and $g = O_A(f)$ mean respectively that $f, g$ are functions of a variable $x$, $A$ is a parameter, and there is a constant $C$ (depending on $A$) such that for all $x$, $g(x) \leq Cf(x)$. We will use $f \ll g$ and $f \ll_A g$ synonymously with $f = O(g)$ and $f = O_A(g)$. Let $H$ be a stratum of translation surfaces, let $G = \text{SL}_2(\mathbb{R})$ and let $\mathcal{L} \subset H$ be the closure of a $G$-orbit in $H$. By recent breakthrough results of Eskin, Mirzakhani and Mohammadi [EM13, EMM15], $\mathcal{L}$ is the intersection of $H$ with a linear suborbifold, and is the support of a smooth ergodic probability measure $\mu$, which we will call the flat measure of $\mathcal{L}$. We will refer to $(\mathcal{L}, \mu)$ as a locus (the terminology ‘affine invariant manifold’ is also in common use).

A cylinder on a translation surface is an isometrically embedded image of the annulus $[a_1, a_2] \times \mathbb{R}/c\mathbb{Z}$, for some $a_1 < a_2$ and $c > 0$. The image of a curve $\{b\} \times \mathbb{R}/c\mathbb{Z}$ for $a_1 < b < a_2$
is called a waist curve of the cylinder and the integral along a waist curve of the pullback of \((dx, dy)\) is called the holonomy vector of the cylinder. One can also study the asymptotic growth of \(V^{cyl}(x) \cap B(0, T)\), where \(V^{cyl}(x)\) is the collection of holonomy vectors of cylinders on \(x\). Furthermore, in [EMZ03 §3], Eskin, Masur and Zorich defined configurations which are a common generalization of saddle connections and cylinders. We will not need to repeat the definition of a configuration in this paper; in order to give the idea, we note three other examples of configurations: (i) \(C\) consists of a saddle connection joining some fixed singularity to itself, (ii) \(C\) is a saddle connection joining distinct fixed singularities, (iii) \(C\) consists of two homologous saddle connections joining two distinct fixed singularities, and forming a slit which disconnects the surface into components with a fixed topology. For each configuration \(C\) one can then define a collection of holonomy vectors \(V^C(x)\) of the saddle connections or cylinders comprising the configuration, and study the asymptotic growth of \(N^C(T, x) = |V^C(x) \cap B(0, T)|\). A remarkable feature of [EM01, EMZ03] is the authors’ foresight: they proved their results in an abstract framework which later (in [EM18]) was proved to be sufficient to cover all \(G\)-invariant ergodic measures and all configurations. Namely, they proved that for any locus \((L, \mu)\) and any configuration \(C\) there is \(c = c(L, C)\) such that for \(\mu\)-a.e. \(x \in L\) one has \(N^C(T, x) = cT^2 + o(T^2)\). Furthermore, [EMZ03] also discussed counting with multiplicities (that is, vectors in \(\mathbb{R}^2\) are counted according to the number of saddle connections which have them as holonomy vectors). In the notation of this paper \(N^C(T, x)\) may refer to counting either with or without multiplicity, i.e. the count in question is assumed to be a part of the data associated with \(C\). Finally, for the case \(L = H\), an algorithm for computing the constants \(c\) in the above asymptotic was described, in terms of so-called Siegel-Veech constants introduced by Veech in [VE98].

An additional improvement, due to Vorobets [Vor05 Thm. 1.9], concerns counting in sectors. Let \(\varphi_1 < \varphi_2\) with \(\varphi_2 - \varphi_1 \leq 2\pi\) and let \(N(T, x, \varphi_1, \varphi_2)\) denote the cardinality of the intersection of \(V(x)\) with the sector

\[
S_{T, \varphi_1, \varphi_2} = \{r(\cos \varphi, \sin \varphi) : 0 \leq r \leq T, \ \varphi_1 \leq \varphi \leq \varphi_2\} \subset \mathbb{R}^2.
\]

Vorobets showed that there is \(c > 0\) such that for a.e. \(x \in H\) (with respect to the flat measure on \(H\)), \(N(T, x, \varphi_1, \varphi_2) = c(\varphi_2 - \varphi_1)T^2 + o(T^2)\). Our main result is an effective version of the above-mentioned results. Setting \(N^C(T, x, \varphi_1, \varphi_2)\) for the number of holonomy vectors corresponding to the configuration \(C\) on \(x\) with holonomy vector in \(S_{T, \varphi_1, \varphi_2}\), we have:

**Theorem 1.1.** For any locus \((L, \mu)\) there is a constant \(\kappa > 0\) such that for any configuration \(C\) there is a constant \(c > 0\) such that for any \(\varphi_1 < \varphi_2\) with \(\varphi_2 - \varphi_1 \leq 2\pi\), for \(\mu\)-a.e. \(x\) we have

\[
N^C(T, x, \varphi_1, \varphi_2) = \frac{c}{2}(\varphi_2 - \varphi_1)T^2 + O_{x, \varphi_2 - \varphi_1}(T^{2(1-\kappa)}).
\]

Here, in the basic case that \(L = H\) is a stratum and \(C\) is one saddle connection (i.e. \(V^C(x) = V(x)\)), the constant \(c\) is the Siegel-Veech constant of [Vee98] (this is the reason for the denominator 2 appearing in \((1.2)\)). As we shall see below, \(\kappa\) can be estimated explicitly in terms of the size of the spectral gap in the unitary representation of \(G\) in \(L^2(L)\).

We have chosen to normalize our power saving exponent \(\kappa\) so that the error is written in the form \(2(1 - \kappa)\) rather than \(2 - \kappa\), that is to estimate the error as a power of the area growth, in order to permit easier comparisons with other bounds appearing in the literature on related problems. Note that in \((1.2)\), the dependence of the implicit constant in the \(O\)-notation on \(x\) is unavoidable given the existence of surfaces with different quadratic growth coefficients.

For a recent application of Theorem 1.1 see [CK18].

The proof of Theorem 1.1 does not give any insight into the set of full measure of \(x\) which satisfy \((1.2)\). In fact it is expected that every translation surface \(x\) satisfies quadratic growth (see [EM11] for a remarkable result in this direction). Thus it is of interest to exhibit explicit surfaces which satisfy quadratic growth with an effective error estimate (in particular, where \(\kappa\) is known). It is also of interest to count points in the intersection of \(V(x)\) with more general subsets of \(\mathbb{R}^2\). These questions are discussed in [BNRW19].
The expectation that any translation surface satisfies quadratic growth, and that the constant \( c \) appearing in (1.1) depends only on the orbit closure \( G^x \), leads to the expectation that the set of surfaces satisfying (1.1) is \( G \)-invariant. Since the assignment \( x \mapsto V(x) \) satisfies \( V(gx) = gV(x) \), this can be equivalently stated as a problem on counting in ellipses: in the definition of \( N(T, x) \), one should be able to replace Euclidean balls of radius \( T \), with dilates of any fixed ellipse centered at the origin, and the same should be true for \( N(T, x, \varphi_1, \varphi_2) \). The issue of existence of a full measure \( G \)-invariant set of surfaces with quadratic growth was not discussed in [EM01], but could probably be derived from the arguments in [EM01, Vor05]. Moreover, in the case that \( L \) is a stratum, it can be derived from a recent result of Athreya, Cheung and Masur [ACM19], in combination with an argument of Veech [Vee98, Thm. 14.11]. Using our technique we obtain the following effective strengthening.

**Theorem 1.2.** For any locus \((L, \mu)\), there is \( \kappa > 0 \) such that for every configuration \( C \) there is \( c > 0 \) such that for \( \mu \)-a.e. \( x \), for every \( \varphi_1 < \varphi_2 \) with \( \varphi_2 - \varphi_1 \leq 2\pi \), and for every \( g \in G \),

\[
N^C(T, gx, \varphi_1, \varphi_2) = \frac{c}{2}(\varphi_2 - \varphi_1)T^2 + O_{\varphi_2 - \varphi_1; \delta} \left( T^{2(1-\kappa)} \right).
\]

Note that Theorem 1.2 implies Theorem 1.1, but we present the proof of Theorem 1.1 separately. This is because the proof of Theorem 1.2 presents additional technicalities which may obscure the main ideas, and also because our proof of Theorem 1.2 gives slightly weaker estimates on \( \kappa \).

### 1.1. Ingredients of the proofs

**Theorem 1.1** follows the strategy of [EM01] (which in turn was inspired by [EMM98, Vee98]) of reducing the counting problem to an ergodic theoretic problem regarding the convergence of the translated circle averages \( \pi_L(\Sigma_t)f(x) = \int_K f(a_kx)\,dm_K \) (the notation is introduced in §2), as \( t \to \infty \). In the treatment of [EM01], \( f \) is the Siegel-Veech transform of an indicator of a rectangle in \( \mathbb{R}^2 \), and the required convergence of \( \pi_L(\Sigma_t)f(x) \) was proved by replacing \( f \) with a smoothed version of \( f \), developing various estimates to bound the amount of time the translated circle average spends outside large compact subsets of \( H \), and appealing to a pointwise ergodic theorem of the first-named author (see [Nev17]).

Our proof of Theorem 1.1 uses all of the above ingredients and more. The essential new ingredient is the fact that any \((L, \mu)\) possesses a spectral gap (see §3 for the definition). This was proved by Avila, Gouëzel and Yoccoz [AGY06] for the case of strata, and by Avila and Gouëzel [AG13] for general loci (again, in an abstract framework, as [AG13] also preceded [EM18]). Using the spectral gap it is possible to obtain an effective estimate of the difference \( |\pi_L(\Sigma_t)f(x) - \int f\,d\mu| \), in case \( f \) is a \( K \)-smooth function and \( t \) is large enough (depending on \( x \) and \( f \)). See [§3] for the definition of \( K \)-smooth functions. The estimate is valid for \( x \) in a set of large measure depending on \( f \) and \( t \). Using a Borel-Cantelli argument (see Theorem 4.3) we upgrade this to a set of full measure and a countable collection of \( K \)-smooth functions, which we then use in order to estimate effectively the integrals appearing in the counting problem, and thus the numbers \( N^C(T_n, x, \varphi_1, \varphi_2) \) for a countable collections of radii \((T_n)\). In order to pass from a countable collection of functions to the results, it is advantageous to replace the rectangle used in [EM01], or the trapezoid used in [Esk06], with a triangle with an apex at the origin.

Theorem 1.2 improves Theorem 1.1 in two ways: uniform counting with an error term in all sectors and in all ellipses. These improvements require two additional ingredients. First we note that the same Borel-Cantelli argument, and further approximation arguments, make it possible to use countably many functions in order to approximate all sectors and all ellipses simultaneously. That is, instead of working only with a countable set of radii, we work with a countable set of radii, a countable set of ellipses, and a countable collections of sectors. Furthermore, for uniform counting in ellipses, we replace the circle averages with ellipse averages \( \pi_L(\Sigma_t^{(g)})f(x) = \int_K f(a_kgx)\,dm_K \), and obtain an estimate on the rate at which \( \pi_L(\Sigma_t^{(g)}) (x) \to \int_L f\,d\mu \), which is uniform as \( g \) ranges over compact subsets of \( G \).
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2. Preliminaries

In this section we will collect results which we will need concerning the moduli space of translation surfaces.

2.1. The Siegel-Veech formula and the function $\ell$. We recall the Siegel-Veech summation formula:

**Theorem 2.1.** [Vee98, Thm. 0.5] For any locus $\mathcal{L}$ with flat measure $\mu$, and any configuration $\mathcal{C}$, there exists $c = c(\mathcal{L}, \mathcal{C}) > 0$ (called a Siegel-Veech constant) such that for any $\psi \geq 0$ Borel measurable on $\mathbb{R}^2$, if we let $\hat{\psi}(x) = \sum_{v \in V_c(x)} \psi(v)$, then

$$\int_{\mathcal{L}} \hat{\psi}(x) d\mu(x) = c \int_{\mathbb{R}^2} \psi(x) dx.$$

We stress that the definition of $\hat{\psi}$ depends on a choice of configuration $\mathcal{C}$, but this choice will not play an important role in what follows, and will be suppressed from the notation.

Let $\ell(x)$ be the Euclidean length of a shortest saddle connection in $x$. Building on earlier work in [Mas90] and [EMM98], a fundamental bound on the number of saddle connections in a compact set was established by Eskin and Masur as follows.

**Theorem 2.2.** [EM01, Theorem 5.1] For any stratum $\mathcal{H}$, any configuration $\mathcal{C}$, any compact set $B \subset \mathbb{R}^2$, any $x \in \mathcal{H}$, and any $\alpha_1 > 1$,

$$|V^c(x) \cap B| \ll_{\mathcal{H}, B, \alpha_1} \ell(x)^{-\alpha_1}.$$

Note that in [EM01], the bound was only stated for the set $V(x)$ of all saddle connection holonomies, that is the case in which the configuration $\mathcal{C}$ consists of any saddle connection; however since any cylinder contains saddle connections along its boundary, the bound for $V(x)$ implies the same bound for $V^c(x)$ for any configuration $\mathcal{C}$.

2.2. Translated circle averages. Consider the elements

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad k_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

and let $K = \{k_\theta : \theta \in [0, 2\pi)\} \subset G$. When $G$ acts ergodically by measure preserving transformations on a standard Borel probability space $(X, \mu)$, we will say that $(X, \mu)$ is an ergodic p.m.p. $G$-space. We let $\pi_X$ denote the unitary representation of $G$ in $L^2(X)$, given by $\pi_X(g)f(x) = f(g^{-1}x)$. We extend $\pi_X$ to a representation of the convolution algebra $M(G)$ of bounded complex Borel measures on $G$. Each $\sigma \in M(G)$ acts as an operator on $L^2(X)$ via the formula

$$\pi_X(\sigma)f(x) = \int_{G} f(g^{-1}x)d\sigma(g), \text{ for } f \in L^2(X).$$

For any two measures $\sigma_1, \sigma_2 \in M(G)$, we have $\pi_X(\sigma_1 * \sigma_2) = \pi_X(\sigma_1) \circ \pi_X(\sigma_2)$.

Let $m_K$ denote the probability Haar measure on the circle $K$ given in coordinates by $\frac{1}{2\pi}d\theta$, and denote the probability measure $m_K * \delta_{a_{-t}}$, by $\Sigma_t$. Thus for $f : \mathcal{L} \to \mathbb{R}$

$$\pi_X(\Sigma_t)f(x) = \int_{K} f(a_t k x) dm_K(k).$$

An important property of integrability of the function $\ell$, and a bound on its translated circle averages, were established by Eskin and Masur:

**Theorem 2.3** (See [EM01, Thm. 5.2, Lem. 5.5] and [Vee98, Cor 2.8].) For any $x \in \mathcal{L}$, and for any $1 \leq \alpha_2 < 2$,

$$\sup_{t > 0} \pi_X(\Sigma_t) \left( \ell(x)^{-t} \right) < \infty.$$
The bound can be taken to be uniform as \( x \) ranges over compact sets in \( \mathcal{L} \). Furthermore, for any locus \((\mathcal{L}, \mu)\), we have \( \ell(\cdot)^{-2\mathcal{L}_3} \in L^1(\mathcal{L}, \mu) \).

To account for sectors, we will use the family of measures on the circle

\[
\pi_X(\Sigma_{\nu,t}) f(x) = \int_{\mathcal{K}} f(a,kx) \nu(k) dm_{\mathcal{K}}(k)
\]

where \( \nu \) is a bounded density on \( \mathcal{K} \). In fact, in this paper \( \nu \) will be a characteristic function of an angular sector \( I = I_{\varphi_1, \varphi_2} = [\varphi_1, \varphi_2] \), so that \( d\nu = \chi_I dm_{\mathcal{K}} \). We will also consider below densities \( \nu_t \) corresponding to intervals which constitute a slight contraction or a slight expansion of \( I \). It is clear that (2.2) also holds for such \( \Sigma_{\nu,t} \), uniformly for all \( \nu \leq 1 \).

3. Spectral gap and pointwise ergodic theorem

3.1. Spectral gap and matrix coefficients estimate. Let \((X, \mu)\) be an ergodic p.m.p. \( G \)-space, and denote by \( L^2_0(X, \mu) \) the zero mean functions in \( L^2(X, \mu) \). By ergodicity, there are no nonzero invariant vectors in \( L^2_0(X, \mu) \). The action is said to have a spectral gap if the associated unitary representation of \( G \) is isolated from the trivial representation; equivalently, there does not exist a sequence of unit vectors \( (u_j)_{j \in \mathbb{N}} \) in \( L^2_0(X, \mu) \) which is asymptotically invariant under the representation, namely such that \( \lim_{j \to 0} \|\pi_X(g)u_j - u_j\| = 0 \) for every \( g \) in \( G \). Note that if \( (u_j)_{j \in \mathbb{N}} \) is an asymptotically invariant sequence, and \( K \) is a compact subgroup of \( G \), then \( v_j = \frac{\pi_X(m_K)u_j}{\|\pi_X(m_K)u_j\|} \) is well-defined for all but finitely many indices, and \( (v_j)_{j \in \mathbb{N}} \) is an asymptotically invariant sequence consisting of \( \pi_X(K) \)-invariant unit vectors.

Our results are based on the following important result:

**Theorem 3.1.** [AGY06] \( AG13 \) The representation of \( G \) on \( L^2_0(\mathcal{L}, \mu) \) possesses a spectral gap.

The functions \( g \mapsto \langle \pi_X(g)f_1, f_2 \rangle \), for \( f_1 \in L^2(X, \mu) \), are known as matrix coefficients for the action on \((X, \mu)\). \( f \) is called a \( K \)-eigenvector if there exists a character \( \chi \) of \( K \) such that \( \pi_X(k)f = \chi(k)f \) for all \( k \) in \( K \). If \( f \) is a finite linear combination of \( K \)-eigenvectors, it is called \( K \)-finite. Fix \( \omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) as a generator of the Lie algebra of \( K \). A function \( f \in L^2(X) \) is called \( K \)-smooth of degree one if

\[
\pi_X(\omega)f := \lim_{\phi \to 0} \frac{1}{\phi} \langle \pi_X(\exp(\phi\omega)f - f \rangle
\]

exists, where the convergence is with respect to the \( L^2(X) \)-norm (one may also consider the obvious extension to smoothness of degree \( d \) for \( \omega \), but we will not need this). Define the (degree one) Sobolev norm by

\[
S_K(f)^2 = \|f\|^2 + \|\pi_X(\omega)f\|^2.
\]

We denote the space of \( K \)-Sobolev functions with finite \( S_K(f) \)-norm by \( S_K(X) \), and set

\[
S_{K,0}(X) = S_K(X, \mu) \cap L^2_0(X, \mu).
\]

In the special case \( G = \text{SL}_2(\mathbb{R}) \) the spectral gap condition implies the following explicit quantitative estimate.

**Theorem 3.2.** Let \( G = \text{SL}_2(\mathbb{R}) \) and let \((X, \mu)\) be an ergodic p.m.p. \( G \)-space with a spectral gap. Then there are positive \( C, \lambda \) such that for any \( f_1, f_2 \in L^2_0(X) \) which are \( K \)-eigenvectors, and for any \( g \) in \( G \), written in Cartan polar coordinates as \( g = k_1 a_1 k_2 \), we have

\[
|\langle \pi_X(g)f_1, f_2 \rangle| \leq C e^{-|t|\lambda} \|f_1\|_2 \|f_2\|_2
\]

for any \( |t| \geq 1 \). Furthermore, for any matrix norm on \( \text{Mat}_2(\mathbb{R}) \), and \( \lambda \) for which (3.2) holds, and any \( K \)-Sobolev functions \( f_1, f_2 \in S_{K,0}(X) \),

\[
|\langle \pi_X(g)f_1, f_2 \rangle| \ll \|g\|^{-\lambda} S_K(f_1) S_K(f_2).
\]
The supremum of \( \lambda > 0 \) for which one can find \( C \) such that \( \|f_1 - f_2\| \leq C \|\tau_1 - \tau_2\| \) is satisfied for \( K \)-eigenvectors \( f_1, f_2 \), will be denoted by \( \lambda_X \) and will be called the size of the spectral gap. Note that the results of [AGY06, AG13] do not give explicit bounds on the size of the spectral gap.

Theorem 3.2 is well-known to experts as part of the general theory of unitary representations of simple Lie groups, but a convenient reference for the case at hand is hard to come by. We give a proof below. The proof we give below establishes 3.2 for \( K \)-eigenvectors and 3.3 for \( K \)-Sobolev functions in any unitary representation of \( G \) with a spectral gap, not only the representations arising from p.m.p. actions on probability spaces.

**Proof.** We use the concise exposition of the unitary representation theory of \( \text{SL}_2(\mathbb{R}) \) in [HT92], where a full parameterization of the unitary dual \( \hat{G} \) is given in [HT92, Ch. III, §3.1, Thm. 1.3.1], and an explicit construction of the corresponding irreducible unitary representations is given in [HT92, Ch. V, §3.1]. The unitary dual can be divided to four parts: the principal series (spherical and non-spherical), the complementary series, the countable set of discrete series representations and the two representations referred to as ‘limits of discrete series’. Let \( g = k_1 a_1 k_2 \) and consider the matrix coefficient \( \langle \tau(g)v_1, v_2 \rangle \), with \( \tau \) an irreducible non-trivial unitary representation and \( v_1, v_2 \) being \( K \)-eigenvectors of unit norm, including the case where \( v_1, v_2 \) are \( K \)-invariant. We shall always assume that \( |t| \geq 1 \).

For \( \tau \) in the principal series the matrix coefficients are bounded by \( C |t| \exp(-|t|) \), by [HT92, Ch. V, §3.1, eqs. (3.1.2), (3.1.4)], noting that the \( K \)-eigenvectors are just the characters of the circle group and hence are uniformly bounded functions, and that \( C \) is uniform in this case. For \( \tau \) a discrete series representation, the same uniform bound holds by [HT92, Ch. V, §3.2, Thm. 3.2.1]. This follows since a discrete series representation is a subrepresentation of the regular representation, using also the bound of the Harish-Chandra \( \Xi \)-function provided in the first estimate of [HT92, Ch. V, §3.1, Prop. 3.1.5].

The complementary series representations are parameterized by \( \tau_s \), \( 0 < s < 1 \), and the matrix coefficients of \( K \)-eigenvectors of unit norms are bounded by \( C_{\tau_s}|t|\exp(-(1-s)|t|) \), using the second estimate in [HT92, Ch. V, §3.1, Prop. 3.1.5] (with \( C_{\tau_s} \) possibly depending on \( s \) according to this estimate). In particular, for each such \( s \) there exists an integer \( n(s) \) such that the matrix coefficients, raised to the \( n(s) \)-power, are in \( L^2(G) \). It follows that the tensor power representation \( \tau_s^{n(s)} \) embeds as a subrepresentation of the regular representation by [HT92, Ch. V, §1.2, Cor. 1.2.4]. But then \( \langle \tau(g)v_1, v_2 \rangle^n \) satisfies the bound that a matrix coefficient associated with two \( K \)-eigenvectors in the regular representation satisfies, which is given in [HT92, Ch. V, §3.2, Thm. 3.2.1]. It follows that

\[
|\langle \tau_s(g)v_1, v_2 \rangle| \leq Cte^{-\frac{|t|}{8n(s)}},
\]

with \( C \) uniform over \( 0 < s < 1 \).

Finally, as to the two ‘limits of discrete series’ representations, by [HT92, Ch. V, Exer. 8, p. 242] the relevant matrix coefficients are all in \( L^{2+\varepsilon} \) for \( \varepsilon > 0 \) and so also in \( L^3 \), and hence the argument of the previous paragraph applies to give 3.4, with 3 replacing \( n(s) \), and \( C \) uniform.

We shall now use arguments appearing in [Rat87]. An arbitrary (separable strongly continuous) unitary representation \( \pi \) of \( G \) can decomposed as a direct integral of non-trivial irreducible representations (see [Rat87, p. 272ff]): Let \((Y, \zeta)\) be a standard Borel space and suppose there are non-trivial irreducible representations \( \tau_y \) of \( G \) for all \( y \in Y \) defined on separable Hilbert space \( H^y \) and some choice of orthonormal basis \( \{\phi^y_n\}_{n \in \mathbb{Z}} \) for each \( H^y \). We call a function (or more precisely, a section) \( f \) on \( Y \) with \( f^y \in L^p(Y^\circ) \) measurable w.r.t this choice of bases, if the inner products \( \langle f^y, \phi^y_n \rangle \) are measurable for any \( n \). The collection \( H \) of functions for which \( \int_Y \|f^y\|^2 d\zeta(y) \) is finite constitutes a separable Hilbert space, with inner product \( \langle f, h \rangle = \int_Y \langle f^y, h^y \rangle d\zeta(y) \) and the \( G \)-representation \( \int_Y \tau_y d\zeta(y) = \pi \) defined by \( \langle \tau_y(g)f^y \rangle = \tau_y(g)f^y \), for \( \zeta \)-a.e. \( y \in Y \). Conversely, for any representation \( \pi \) there exists such \((Y, \zeta)\) so that \( \pi \) is unitarily equivalent to \( \int_Y \tau_y d\zeta(y) \), and so we assume that the representation \( \pi_X \) is disintegrated in such a manner. We may further decompose
f^y = \sum f^y_g \text{ into isotypic components with respect to } K \text{ by assuming the basis of } H^y \text{ consists of } K\text{-eigenvectors (see } \text{ [RatS] Lem. 1.1}). \text{ If } f \text{ and } h \text{ are } K\text{-eigenvectors of } \pi_X, \text{ then their components } f^y \text{ and } h^y \text{ in the representations } \tau_y \text{ are } K\text{-eigenvectors of } \tau_y \text{ affording the corresponding characters (for } \zeta\text{-almost every } y \in Y').

Let us now note that a sequence } u_j \text{ of unit vectors is asymptotically invariant if and only if } |\langle \pi_X(g)u_j, u_j \rangle| \to 1 \text{ for every } y \in \hat{G} \text{ (or equivalently uniformly over compact subsets of } G). \text{ If } u_j \text{ are } K\text{-invariant, so are their direct components } u^y_j, \text{ for } \zeta\text{-a.e. } y \in \hat{G}, \text{ and then } |\langle \pi_X(g)u_j, u_j \rangle| \text{ obeys the bounds } (3.4).

Suppose now that the spectral measure } \zeta \text{ assigns zero measure to the set of complementary series representations given by } \{\tau_s; s > s_X\}, \text{ for some } 0 < s_X < 1. \text{ The bounds } (3.4) \text{ then immediately imply that there exist positive } C \text{ and } \lambda \text{ such for every } K\text{-eigenvectors } f, h \in L^2_G(X)

\begin{align*}
|\langle \pi_X(g)f, h \rangle| &= \left| \int_Y \langle \tau_y(g)f^y, h^y \rangle d\zeta(y) \right| \\
&\leq Ce^{-\lambda|t|} \int_Y \|f^y\|_y \|h^y\|_y d\zeta(y) \\
&\leq Ce^{-\lambda|t|}\|f\|_2 \|h\|_2,
\end{align*}

using the Cauchy-Schwarz inequality for the last inequality. In particular, it follows that the representation does not admit an asymptotically invariant sequence of } K\text{-invariant unit vectors in that case.

Conversely, if } \zeta \left( \left\{ \tau_s; s > 1 - \frac{1}{f} \right\} \right) > 0 \text{ for every } j \geq 1 (\tau_s, 0 < s < 1 \text{ being the complementary series), then } \pi_X \text{ does admit an asymptotically invariant sequence } u_j \text{ of } K\text{-invariant unit vectors. Indeed, each } \tau_s \text{ admits a unique } K\text{-invariant unit vector } v^s, \text{ up to a multiplication by a complex number of absolute value } 1, \text{ and independently of the choice of this scalar we have } \langle \tau_s(g)v^s, v^s \rangle = \Phi_s(g). \text{ Here } \Phi_s \text{ is the standard positive-definite and positive spherical function associated with the spherical representation } \tau_s. \text{ It is well known that for any fixed } g = k_1a_k2, \text{ we have } \lim_{s \to 1} \Phi_s(g) = 1. \text{ This follows immediately from the integral representation for the positive spherical function } \Phi_s, \text{ normalized so that } \Phi_s(e) = 1. \text{ Indeed, in the present case, for } f = h = 1 \text{ (corresponding to the trivial character of } K), \text{ the inequality } [\text{[HT92]} \text{ eq. 3.1.2, p. 215}] \text{ is in fact an identity because of the positivity of the integrand, and so}

\Phi_s(a_{k1}) = \frac{1}{2\pi} \int_0^{2\pi} \left( e^{-2|t| \cos^2 \phi} + e^{2|t| \sin^2 \phi} \right)^{-\frac{1}{2}(1-s)} d\phi.

Now let } u_j \text{ be any } K\text{-invariant unit vector } u_j \text{ in the subrepresentation of } \pi_X \text{ given by } \pi_j = \int_{\{s > 1 - \frac{1}{j}\}} \tau_s d\zeta(\tau_s). \text{ Such vectors do exist since the direct integral of irreducible representations each containing a } K\text{-invariant unit vector has the same property, and the sequence } u_j \text{ satisfies, for each fixed } g:

\begin{align*}
\langle \pi_j(g)u_j, u_j \rangle &= \int_{\{s > 1 - \frac{1}{j}\}} \langle \tau_s(g)v^s, v^s \rangle \|u^s_j\|^2 d\zeta(\tau_s) \\
&= \int_{\{s > 1 - \frac{1}{j}\}} \Phi_s(g) \|u^s_j\|^2 d\zeta(\tau_s) \to \int_{\{s > 1 - \frac{1}{j}\}} \|u^s_j\|^2 d\zeta(\tau_s) = 1, \text{ as } j \to \infty.
\end{align*}

Denoting by } s_X \text{ the infimum over } 0 < s < 1 \text{ for which } \zeta \left( \left\{ \tau'_s; s' > s \right\} \right) = 0, \text{ the previous argument shows that } \pi_X \text{ has a spectral gap if and only if } 0 \leq s_X < 1. \text{ We define } \lambda_X = 1/n(s_X), \text{ and then we can choose any } 0 < \lambda < \lambda_X \text{ and inequality } (3.5) \text{ is satisfied. Note that the } |t|\text{-factor appearing in the bound of matrix coefficients of the complementary series makes the constant } C \text{ used in } (3.5) \text{ depend on the choice of } \lambda. \text{ We will therefore write the bound as } \epsilon_X, \text{ from now on.}

Moving on to } K\text{-smooth functions, we note that } \omega \text{ defines an operator } \pi_X(\omega) \text{ acting on } K\text{-smooth vectors in } L^2(X), \text{ and the action of this operator is equivariant with respect to the decomposition, namely } \langle \pi_X(\omega)f, h \rangle = \int_Y \langle \tau_y(\omega)f^y, h^y \rangle d\zeta(y). \text{ Equivalently, for } \zeta\text{-almost
every \( y, f^y \) is \( K \)-smooth and \( (\pi_X(\omega)f)^y = \tau_y(\omega)f^y \) (\cite{Rat87} Lem. 1.2). Now note that if \( v_n \) is a \( K \)-eigenvector with character \( e^{in0} \) for a representation \( \tau \), then

\[
\tau(\omega)v_n = \frac{d}{d\phi} \bigg|_{\phi=0} \tau(\exp(\phi \omega))v_n = inv_n,
\]

and hence \( \|v_n\| = \frac{1}{n} \|\tau(\omega)v_n\| \). Furthermore, if \( v = \sum_{n \in \mathbb{Z}} v_n \) is the decomposition of \( v \) to isotypic components (whose components are mutually orthogonal), then using the previous identity and the Cauchy-Schwarz inequality

\[
\sum_{n \in \mathbb{Z}} \|v_n\| = \|v_0\| + \sum_{n \neq 0} \frac{1}{n} \|\tau(\omega)v_n\| \leq \|v_0\| + \left( \frac{\pi^2}{6} \right)^{1/2} \left( \sum_{n \neq 0} \|\tau(\omega)v_n\|^2 \right)^{1/2} \leq 2(\|v_0\| + \|\tau(\omega)v\|).
\]

Let \( f \) and \( h \) be two \( K \)-smooth vectors in \( L_0^2(X) \), and decompose their direct integral constituents \( f^y \) and \( h^y \) into their isotypic components: \( f^y = \sum f_n^y \), and \( h^y = \sum h_n^y \). We have

\[
|\langle \tau^y(g)f^y, h^y \rangle| = \sum_{n,m} |\langle \tau^y(g)f_n^y, h_m^y \rangle| \leq \lambda e^{-|\lambda|t} \sum_{n,m} \|f_n^y\|_y \|h_m^y\|_y
\]

\[
\leq \lambda e^{-|\lambda|t} \left( \int_Y \left( \|f^y\|_y + \|\tau_y(\omega)f^y\|_y \right)^2 d\zeta(y) \right)^{1/2} \left( \int_Y \left( \|h^y\|_y + \|\tau_y(\omega)h^y\|_y \right)^2 d\zeta(y) \right)^{1/2}
\]

\[
= \lambda e^{-|\lambda|t} \left( \|f\|_2^2 + \|\tau_X(\omega)f\|_2^2 \right)^{1/2} \left( \|h\|_2^2 + \|\tau_X(\omega)h\|_2^2 \right)^{1/2} = \lambda e^{-|\lambda|t} S_K(f) S_K(h).
\]

Finally, we may choose to replace the quantity \( e^{-|\lambda|t} \) with any matrix norm, and conclude that for any two \( K \)-eigenfunctions in \( L_0^2(X) \), or any two \( K \)-smooth functions:

\[
|\langle \pi_X(\omega)g, h \rangle| \leq \lambda \|g\|_{-\lambda} \|f\|_2 \|h\|_2.
\]

This follows from the fact that the Euclidean (sum-of-squares) norm on Mat_2(\( \mathbb{R} \)) satisfies \( \|k_1 a k_2\|_2 = \sqrt{2} \cosh 2\theta \ll \sqrt{2} e^{|\theta|} \), and any two linear norms on Mat_2(\( \mathbb{R} \)) are equivalent. \( \square \)

### 3.2. Effective pointwise ergodic theorem.

As shown by Eskin, Margulis and Mozes, from estimates such as those in Theorem \ref{lem:effective-pointwise-ergodic}, one can derive an estimate for the norm of the operator \( \pi_X(\Sigma_t) \), \( \nu \leq 1 \), viewed as an operator from the \( K \)-Sobolev space \( S_{K,0}(X) \) to \( L^2(X) \).

**Theorem 3.3** (See \cite{EMM98} (3.32) and \cite{Vee98}, §14). Let \( G = SL_2(\mathbb{R}) \) and let \( (X, \mu) \) be a p.m.p. \( G \)-space with a spectral gap of size \( \lambda_X \). Then for any \( \lambda < \lambda_X \), there exists \( C_\lambda > 0 \) such that for any interval \( I \subset \mathbb{R} \) of length \( |I| \neq 0 \), any \( f \in S_{K,0}(X) \), and all \( t > \frac{1}{2} \log \frac{1}{|I|} \),

\[
\|\pi_X(\Sigma_t) f\|_2 \leq C_\lambda e^{-2\lambda \eta |I|} S_K(f)^2 |I|^{2-\lambda \eta},
\]

where \( \eta = \frac{1}{1+\lambda} \) and \( \nu \) is the indicator function of the interval \( I \).

We note that if one normalizes \( \nu \) to be the density of a probability measure then the quality of the rate in (3.7) diminishes as the length of the interval decreases. We will not normalize \( \nu \) in this way because it will turn out to be less natural for some geometric considerations involved in the counting problem.
For completeness, and in order to have precise control of constants, we repeat the argument found in [EMM98, Vec98].

Proof. Let \( \nu^*(y) = \nu(g^{-1}) \). Then \( \nu^* \nu(k) = \int_K \nu(k^t) \nu(k) dm_K(k^t) = D_\nu(k) \), by \( G \)-invariance of the measure \( \mu \) one has

\[
\| \pi_X(\Sigma_{\nu,t}) f \|_1^2 = \langle \pi_X(\delta_{a_t} \ast \nu^* \ast \delta_{a_t}) f, f \rangle = \int_X \int_K f(a_t \delta_{a_t} x) \bar{f}(x) D_\nu(k) dm_K(k) dm(x).
\]

Replacing \( I \) if necessary by a disjoint union of at most 8 subinterval of length bounded by \( \pi/4 \), without loss of generality we can assume that \( |I| \leq \pi/4 \). Then using a rotation we can assume that \( I = [0, \phi_I] \subset [0, \pi/4] \). We identify \( K \) with \( \mathbb{S}^1 \) using (3.1) so that \( \nu(K) = |I|/2\pi \leq |I| \). For a parameter \( 0 < \lambda' < 1 \) to be fixed below, we set \( J = \{ \phi \in I : \sin \phi \exp(2t) < e^{2\lambda' t} \} \). To put ourselves in the case that \( J \) is a proper subset of \( I \), we assume that \( \sin \phi_I > e^{2t(\lambda' - 1)} \), which implies that \( 1 > \frac{\lambda'}{2} \geq \phi_I = |I| > \sin \phi_I > e^{2t(\lambda' - 1)} \), and thus \( 0 < \lambda' < \frac{\log |I|}{2t} + 1 < 1 \).

Write \( k = k_0 \) so that, using the supremum norm on \( \text{Mat}_2(\mathbb{R}) \), we have \( \|a_t k_{a_t} \| \geq e^{2\lambda' t} \) for \( k \in I \setminus J \). Since \( I \subset [0, \frac{\pi}{2}] \), \( D_\nu \) can be computed using convolution on \( \mathbb{R} \), and since \( \phi \leq 2 \sin \phi \) in the interval \( I \), and \( \|D_\nu\|_\infty \leq |I| \), we conclude that \( \int_J D_\nu(k) dm_K < 2 |I| e^{2t(\lambda' - 1)} \). Furthermore, clearly \( \int_{I \setminus J} D_\nu(k) dm_K \leq |I|^2 \). By Fubini, the matrix coefficient (3.8) is equal to

\[
\int_I \pi_X(a_t \delta_{a_t} x) f(x) D_\nu(k) dm_K(k) + \int_{I \setminus J} \pi_X(a_t \delta_{a_t} x) f(x) D_\nu(k) dm_K(k).
\]

We apply the previous estimate to the integral over \( J \), and apply \( \mathbf{[6.8]} \) to the integral over \( I \setminus J \), to arrive at

\[
\| \pi_X(\Sigma_{\nu,t}) f \|_1^2 \lesssim |I| e^{2t(\lambda' - 1)} \| f \|_1^2 \lesssim |I| e^{-2\lambda' t} S_K(f)^2 \lesssim \left( |I| e^{2t(\lambda' - 1)} + |I|^2 e^{-2\lambda' t} \right) S_K(f)^2.
\]

The best choice is to take \( \lambda' = \frac{1}{\lambda + 1} \left( \frac{1}{2t} \log |I| + 1 \right) \).

With this choice, using \( |I| < 1 \) we find \( 0 < \lambda' < \frac{\log |I|}{2t} + 1 < 1 \) provided \( t > \frac{1}{2} \log \left( \frac{1}{|I|} \right) \), and (3.7) holds.

The next result follows from the bound (3.8) combined with the Borel-Cantelli Lemma and the Markov inequality.

**Theorem 3.4.** Let \((X, \mu)\) be a p.m.p. \( G \)-space with a spectral gap of size \( \lambda_X \). Let \( \lambda < \lambda_X \), let \( t_n \in \mathbb{R}_+ \), let \( \eta = \frac{1}{\lambda t} \) and let \( \eta_1 \) be such that

\[
\sum_{n \in \mathbb{N}} e^{-\lambda \eta_1 t_n} < \infty.
\]

Let \( 0 \leq \nu_n \leq 1 \) be a sequence of functions on \( K \) as in Theorem [5.0] satisfying \( \nu_n(K) = \int \nu_n dm_K > e^{-2t_n} \). Let \( (f_n)_{n \in \mathbb{N}} \) be a collection of functions in \( S_{K,0}(X) \). Then for almost all \( x \in X \) there exists \( n_0 = n_0(x) \) such that if \( n \geq n_0 \) then

\[
|\pi_X(\Sigma_{\nu_n,t_n}) f_n(x)| \leq e^{-(\eta - \frac{\eta_1}{2}) \lambda t_n} S_K(f_n) \nu_n(K) 1^{-\frac{\eta_1}{2}}.
\]

Here \( \eta \) is as in (3.8).

Note that we will only be interested in the nontrivial case where the right hand side of (3.11) decays with \( t \), i.e. when \( \eta_1 \) satisfies \( 0 < \eta_1 < \eta \).

**Proof.** Using (3.8), there is \( C > 0 \) such that for \( f_n \in S_{K,0}(X) \) we have

\[
|\pi_X(\Sigma_{\nu_n,t_n}) f_n|_1^2 \leq C e^{-2\lambda \eta_1 t_n} C_n, \quad C_n = S_K(f_n)^2 \nu_n(K)^{2-\lambda_0}.
\]

Consider for each \( n \) the set of ‘bad points’

\[
U_n = \left\{ x : e^{-\lambda \eta_1 t_n/2} |\pi_X(\Sigma_{\nu_n,t_n}) f_n(x)| \geq e^{-\lambda \eta_1 t_n/2} \right\}.
\]
By Markov’s inequality and \( (3.12) \),
\[
\mu(U_n) \leq e^{-\lambda \eta t_n} \frac{\| \pi_X(\Sigma_{\nu_{n},t}) f_n \|_2^2}{e^{-2\lambda \eta t_n} C_n} \leq C e^{-\lambda \eta t_n}.
\]

By \( (3.10) \) \( \sum_{n \in \mathbb{N}} \mu(U_n) < \infty \), so by the Borel-Cantelli lemma, almost every \( x \in X \) belongs to at most finitely many of the sets \( U_n \). We conclude that for almost every \( x \in X \), there exists \( n_0 \) such that for all \( n \geq n_0 \) we have \( x \notin U_n \). \( \square \)

Now let
\[
\pi_X(\Sigma_{\nu_{t},\ell}) f(x) = \pi_X(\Sigma_{\nu_{t},\ell}) f(gx) = \int_K f(a_t k g x) \nu(k) dm_K(k)
\]
denote the ‘dilated ellipse average’ associated with \( g \in G \). For the proof of Theorem 1.2 we will need the following uniform versions of Theorems 3.3 and 3.4.

**Theorem 3.5.** With the notations of Theorems 3.3 and 3.4, for every \( \lambda < \lambda_X \) there exists \( C > 0 \) such that for all \( t > 1 \), any interval \( I \subset S^1 \) with \( |I| > e^{-2t} \), any \( f \in S_{K,0}(X) \), and any \( g \in G \), we have
\[
(3.13) \quad \|\pi_X(\Sigma_{\nu_{t},\ell}) f\|_2 \leq C e^{-\lambda \eta t} S_K(f)^2 |I|^{2-\omega}.
\]
Furthermore, if \( (t_n) \subset \mathbb{R}_+ \), \( \eta > 0 \) satisfy \( (3.11) \), \( 0 \leq \nu_n \leq 1 \) is a sequence of characteristic functions on \( K \) satisfying \( \nu_n(K) > e^{-2t_n} \), \( (f_n) \) is a sequence of functions in \( S_{K,0} \), and \( (g_n) \) is a countable subset of \( G \), then for almost all \( x \in X \) there is \( n_0 \) such that for all \( n \geq n_0 \) we have
\[
\pi_X(\Sigma_{t_n,\nu_n} g_n) f(x) \leq e^{-(\eta - \frac{1}{2\omega}) t_n} S_K(f_n) \nu_n(K)^{1-\frac{1}{2\omega}}.
\]

**Proof.** A change of variables \( y = g x \) shows that \( \|\pi_X(\Sigma(\nu_{t},\ell)) f\|_2 = \|\pi_X(\Sigma_{t,\ell}) f\|_2 \) and thus \( (3.13) \) follows from the same argument as for \( (3.1) \).

The proof of the second assertion for any fixed choice of sequence \( (g_n) \) is similar to the proof of Theorem 3.4 using \( (3.13) \) instead of \( (3.4) \). \( \square \)

4. Control over the cusp

The results of the previous section apply to every action with a spectral gap, and we will want to apply them to the action on the moduli space of flat surfaces, taking the functions \( f_n \) to be Siegel-Veech transforms of compactly supported functions on \( \mathbb{R}^2 \). However in this setting, the Sobolev norms \( S_K(f_n) \) might not be bounded, owing to a large contribution coming from surfaces in the thin part, i.e. surfaces \( \mathbf{x} \) with \( \ell(\mathbf{x}) \) small. When dealing with this issue it is helpful to note that the Sobolev norm we have used above involves only differentiation in the \( K \)-direction, and as we shall now see, this fact will allow us to use a simple argument for “cutting off the cusp”. We let \( M_\varepsilon = \ell^{-1}([\varepsilon, \infty)) \). By a well-known compactness criterion (see [AGY10], p. 152)) the sets \( M_\varepsilon \) are an exhaustion of \( \mathcal{H} \) by compact sets. Theorems 2.2 and 2.3 give bounds on the measure of the complement \( M_\varepsilon^c = \mathcal{H} \setminus M_\varepsilon \), and on the time a translated circle spends in \( M_\varepsilon^c \). We will use these to cut off any function at the cusp without affecting its asymptotic behavior.

Before proceeding with this argument, note that since we used the Euclidean metric in the definition of the function \( \ell \), the set \( M_\varepsilon^c \) is \( K \)-invariant, and hence its characteristic function is \( K \)-smooth. Below we let \( \partial_\theta \) denote the partial derivative in the spherical direction in polar coordinates. In terms of the action of \( K \) on the plane, it is defined as \( \pi_{\mathbb{R}^2}(\omega) \) in the notation \( (3.1) \). Equivalently, at a point \( \mathbf{y} \in \mathbb{R}^2 \),
\[
\partial_\theta \varphi(\mathbf{y}) = \left. \frac{d}{d\phi} \right|_{\phi=0} \varphi(\exp(\phi \omega)\mathbf{y}).
\]

**Lemma 4.1.** Suppose \( R > 0 \) and \( \psi : \mathbb{R}^2 \to \mathbb{R} \) is a non-negative bounded function which is supported in the ball \( B(0,R) \), such that \( \partial_\theta \psi \) is also bounded, and denote by \( f = \hat{\psi} \) its
Siegel-Veech transform as in Theorem 2.2 with respect to some configuration $C$. Let $\chi_\varepsilon$ denote the characteristic function of the cusp $M_\varepsilon$. Then the decomposition

$$f = f_{\text{main}} + f_\varepsilon,$$

where $f_{\text{main}} = f(1 - \chi_\varepsilon)$ and $f_\varepsilon = f\chi_\varepsilon$

satisfies for any $1 < [\varepsilon_1] < [\varepsilon_2] < 2$,

$$S_K(f_{\text{main}})^2 \ll_{R, \alpha_1} \max_{r \leq R} \left( \int_K \left( \psi^2 + |\partial \psi|^2 \right) (rke_1)dm_K(k) \right) \varepsilon^{-2\alpha_1},$$

(4.1)

$$\int_L f_\varepsilon d\mu \ll_{R, \alpha_1, \alpha_2} \|\psi\|_\infty^{L_2^{\varepsilon(x)}}$$

and

(4.2)

$$\pi_L(\Sigma_t)f_\varepsilon(x) \ll_{x, R, \alpha_1, \alpha_2} \|\psi\|_\infty^{L_2^{\varepsilon(x)}}$$

(4.3)

Moreover the implicit constant in (4.3) can be taken to be uniform as $x$ ranges over compact subsets of $L$.

Proof. We first bound the $L^2$-norm of $f_{\text{main}}$. Since the measure $\mu$ and the set $M_\varepsilon$ are $K$-invariant, and $y \mapsto V(y)$ is $K$-equivariant, we have

$$\|f_{\text{main}}\|^2_2 = \int_L |f(1 - \chi_\varepsilon)|^2 d\mu = \int_{(y) \geq \varepsilon} \left| \sum_{v \in V(y)} \psi(v) \right|^2 d\mu(y)$$

$$= \int_{(y) \geq \varepsilon} \int_K \left| \sum_{v \in V(ky)} \psi(v) \right|^2 dm_K(k) d\mu(y)$$

$$\leq \int_{(y) \geq \varepsilon} |V(y) \cap B(0, R)| \sum_{v \in V(y) \cap B(0, R)} \int_K |\psi(kv)|^2 dm_K(k) d\mu(y)$$

$$\leq \int_{(y) \geq \varepsilon} |V(y) \cap B(0, R)|^2 \left( \max_{r \leq R} \int_K |\psi(rke_1)|^2 dm_K(k) \right) d\mu(y).$$

In the first inequality above we have used Cauchy-Schwarz to get an estimate $|\sum_{v \in V(ky)} \psi(v)|^2 \leq \left( \sum_{v \in V(ky)} |\psi(v)|^2 \right) |V(y) \cap B(0, R)|$, and then exchanged summation and integration. For the second inequality, note that each $v \in V(y) \cap B(0, R)$ we can write $v = rke_1$ for some rotation $k = k(v) \in K$ and some positive scalar $r = r(v) \leq R$, and the estimate follows. Using Theorem 2.2 we conclude that

$$\|f_{\text{main}}\|^2_2 \ll_{R, \alpha_1} \varepsilon^{-2\alpha_1} \max_{r \leq R} \int_K |\psi(rke_1)|^2 dm_K(k).$$

We repeat this calculation for the angular derivative of $\psi$. Here we also use the fact that since the set of saddle connections satisfies $V(gy) = gV(y)$ for any $y \in L$ and $g \in G$, which implies that taking derivatives in the $K$ direction commutes with the Siegel-Veech transform. Namely, for any compactly supported $\psi : \mathbb{R}^2 \to \mathbb{R}$ for which $\partial \psi$ exists everywhere,

$$\pi_L(\omega) \hat{\psi} = \lim_{\phi \to 0} \frac{1}{\phi} \left( \pi_L(\exp(\phi_0)\psi - \hat{\psi}) = \lim_{\phi \to 0} \frac{1}{\phi} \left( \sum_{u \in V(\exp(\phi_0)y)} \psi(u) - \sum_{v \in V(y)} \psi(v) \right) \right)$$

$$= \lim_{\phi \to 0} \frac{1}{\phi} \left( \sum_{v \in V(y)} \left( \psi(\exp(\phi_0)v) - \psi(v) \right) \right) = \sum_{v \in V(y)} \partial \psi(v)$$

(where we have used the fact that $\psi$ is compactly supported to ensure that the sum is finite and hence we can switch the order of summation and differentiation). Thus

$$\pi_L(\omega) \hat{\psi} = \partial \psi.$$
and consequently, applying the argument used to prove inequality \( (4.4) \) to \( \chi_k \) and Lemma 4.1 to estimate the resulting orbit integrals. Since it relies on the Borel-Cantelli lemma, Theorem 3.4 only gives information about \( N \) outlined in the survey [Esk06], but replacing a trapezoid used in [Esk06] with a triangle. In Siegel-Veech transform of the indicator of a triangle. We will follow the simplified approach of values of \( T \) countably many values, which get denser and denser on a logarithmic scale, to all \( W \) and let \( \pi \).

The term \( \ell(x)^{-\beta} d\sigma \) is bounded by \( \|\ell(\cdot)^{-\alpha_2}\|_{L^1(\sigma)} \) for \( \alpha_2 = \alpha_1 + \beta \), which in turn is bounded by Theorem 2.3 for any \( \alpha_2 < 2 \) (where for \( \sigma = \pi(C) \) the bound depends on \( x \) uniformly on compact subsets of \( C \), and is independent of \( t \)).

5. Effective counting of saddle connections

We now give the proof of Theorem 4.1 dividing the argument into three steps. In the first we use a geometric counting method introduced in [EMM08] Lem. 3.6] and [EM01] Lem. 3.4] to estimate the quantity \( N(e^t, x, \varphi_1, \varphi_2) \) by orbit integrals \( \pi(C) f(x) \), where \( f \) is a Siegel-Veech transform of the indicator of a triangle. We will follow the simplified approach outlined in the survey [EM00], but replacing a trapezoid used in [EM00] with a triangle. In the second step we will replace \( f \) with certain smooth approximations and use Theorem 2.3 and Lemma 4.1 to estimate the resulting orbit integrals. Since it relies on the Borel-Cantelli lemma, Theorem 5.1 only gives information about \( N(T, x, \varphi_1, \varphi_2) \) for a countable number of values of \( T \). In the third and final step we use an interpolation argument to pass from countably many values, which get denser and denser on a logarithmic scale, to all \( T \).

**Step 1. Triangles, and reduction of counting to orbit integrals.** We fix a configuration \( C \) and use it to define a Siegel-Veech transform as in Theorem 2.1. For \( t \in (0, 1) \) we define two triangles \( W_1 = W_1(\theta) \) and \( W_2 = W_2(\theta) \) in the plane as follows. Let \( e_2 = (0, 1) \) and let \( W_1 \) have vertices \((0, 0), r\theta e_2, r_{-\theta} e_2 \) and \( W_2 \) have vertices \((0, 0), \frac{1}{\cos^2 \theta} e_2, \frac{1}{\cos \theta} e_2 \). That is, \( W_1 \) and \( W_2 \) are similar isosceles triangles with apex at the origin, apex angle \( 2\theta \), symmetric around the positive \( y \)-axis, and with height \( \cos \theta \) and 1 respectively. In particular \( W_1 \subset W_2 \). See Figure 1.

Now let \( t > 1 \) be a parameter. Applying the diagonal flow \( a_{-t} \) transforms \( W_1, W_2 \) into triangles with a narrow apex angle and large height, specifically the apex angle \( 2\theta_t \) of both \( a_{-t} W_1 \) and \( a_{-t} W_2 \) satisfies

\[
\tan \theta_t = e^{-2t} \tan \theta.
\] [(5.1)
We will obtain lower and upper bounds for $N^C(e^t, x, \varphi_1, \varphi_2)$ using radial averages over shrinking versions of these triangles.

Let $\varphi_1 < \varphi_2$ be as in Theorem 3.1. By a rotation, assume with no loss of generality that $\varphi_2 = \varphi > 0$ and $\varphi_1 = -\varphi$ so that $I = [-\varphi, \varphi]$ is symmetric around 0 and $\varphi_2 - \varphi_1 = 2\varphi$.

Recall the notation $r_n$ for an element of $K$ (see (2.1)). We will identify angles in $\mathbb{R}$ with their image modulo $2\pi\mathbb{Z}$ and functions on $K$ with functions on $\mathbb{R}/2\pi\mathbb{Z}$ without further mention.

Define
\begin{align}
I^-_t &= [-(\varphi - \theta_t), \varphi - \theta_t], \quad I^+_t = [-(\varphi + \theta_t), \varphi + \theta_t],
\end{align}
so that $I^-_t \subset I \subset I^+_t$, and let $\nu^-_t, \nu, \nu^+_t$ denote respectively the measures whose densities are the indicator functions of $I^-_t, I, I^+_t$ (note that the dependence of these indicators on $\varphi_2 - \varphi_1$ is suppressed from the notation). Also let $\mathbb{I}_{W_1}, \mathbb{I}_{W_2}$ denote the indicators of $W_1$ and $W_2$.

We claim that for any $x$,
\begin{align}
\pi_L(\Sigma_{\nu_t}, x) \leq \frac{\theta_t}{\pi} N^C(e^t, x, \varphi_1, \varphi_2) \leq \pi_L(\Sigma_{\nu_t}, x) \mathbb{I}_{W_1}(x).
\end{align}

To see the left hand inequality, recall that by the definition of the Siegel-Veech transform and the operator $\Sigma_{\nu_t}$ we have
\begin{align}
\pi_L(\Sigma_{\nu_t}, x) \mathbb{I}_{W_1}(x) = \sum_{v \in V^C(x)} \int_K \mathbb{I}_{W_1}(v(a_t kv)\nu_t^{-}(k)) dm_K(k).
\end{align}

We will estimate the contribution of each individual $v \in V^C(x)$ to the sum (5.4). For any $v \in \mathbb{R}^2$,
\begin{align}
\int_K \mathbb{I}_{W_1}(v(a_t kv)\nu_t^{-}(k)) dm_K(k) = \frac{1}{2\pi} \int_{-\varphi + \theta_t}^{\varphi - \theta_t} \mathbb{I}_{a_- W_1}(k\theta v) d\phi
\end{align}
is at most $\frac{a_t}{\pi}$, since the apex angle of $a_- W_1$ is $2\theta_t$. The quantity (5.5) vanishes if $|v| \geq e^t$ or $\angle(v, e_2) \notin I$, since in these cases the arc $K(I^-_t)v = \{k\beta v : \beta \in I^-_t\}$ never enters the triangle $a_- W_1$ (see Figure 2). Furthermore, if $|v| \leq e^t$ and $\angle(v, e_2) \in I$ then the arc $K(I^+_t)v$ intersects $a_- W_2$ along its entire apex angle, and so $\int_K \mathbb{I}_{W_2}(a_t kv)\nu_t^{+}(k)) dm_K(k) = \frac{\theta_t}{\pi}$, and this implies the right hand inequality.

**Step 2. Smooth approximations, ergodic theorem, and cutting off the cusp.**

Our goal will be to estimate the left and right hand sides of (5.3). To this end we will replace $\mathbb{I}_{W_1}, \mathbb{I}_{W_2}$ with smooth approximations and apply Theorem 3.4 and Lemma 4.1 to the approximating functions, where the $n$-th function will give a bound on (5.3) for a certain time $T_n = e^{tn}$. Our approximation depends on functions and parameters which we now describe (omitting their dependence on $n$). The first parameter $\theta$ controls the apex
angle of the triangles \( W_i \), as above. We will bound \( \mathbb{1}_{W_i} \) from above and below by \( K \)-smooth functions \( \psi(\cdot, \cdot) \) on the plane, supported respectively in a slightly contracted (resp. expanded) copy of \( W_1 \) (resp. \( W_2 \)), where the dilation is controlled by a smoothing parameter \( \delta \). The corresponding Siegel-Veech transforms \( \psi(\cdot, \cdot) \) will be denoted by \( f(\cdot, \cdot) \). They will be truncated using Lemma 5.1 along with a cutoff parameter \( \varepsilon \), for an appropriate choice of parameters \( \alpha_1, \alpha_2 \). Finally Theorem 3.4 will be applied to the main term \( (f(\cdot, \cdot))_{\text{main}} \), and all resulting errors will be collected and bounded.

We now make this discussion more precise and record some estimates for the errors incurred at the various stages. After collecting these bounds we will choose our parameters and optimize the error terms in the next step. Our optimization gives \( \theta = \delta^{1/2} \), and so in order to reduce the number of parameters we will use this dependence of \( \theta \) and \( \delta \) throughout.

For \( \theta < 1 \) we approximate both triangles \( W_1(\theta), W_2(\theta) \) by sectors around the positive vertical axis, that is by sets of the form

\[
S_{r_0, \varphi_0} = \{ r(\cos \beta, \sin \beta) : 0 \leq r \leq r_0, |\beta - \pi/2| \leq \varphi_0 \}.
\]

The sector \( S_1(\theta) = S_{\cos \theta, \theta} \) is contained in \( W_1(\theta) \) and the sector \( S_2(\theta) = S_{(\cos \theta) \cdot \theta, \theta} \) contains \( W_2(\theta) \) (see Figure 1). Let \( \delta = \theta^2 \) and let \( H : \mathbb{R} \to \mathbb{R} \) be a continuously differentiable function which vanishes outside \([-\theta, \theta] \), is equal to 1 on \([-\theta + \delta, \theta - \delta] \), and such that \( \|H\|_{\infty} \leq 2\delta^{-1} \), and define

\[
(5.6) \quad \psi(-, \cdot)(r \cos \beta, r \sin \beta) = \begin{cases} 
H(\beta - \pi/2) & r \leq \cos \theta \\
0 & r > \cos \theta
\end{cases}
\]

We have pointwise inequalities

\[
\psi(-, \cdot) \leq \mathbb{1}_{S_1(\theta)} \leq \mathbb{1}_{W_1(\theta)},
\]

and an estimate

\[
\|\partial_\beta \psi(-, \cdot)\|_{\infty} \ll \delta^{-1}.
\]

Similarly, we define functions \( \psi(+, \cdot) \) which satisfy a pointwise inequality \( \mathbb{1}_{W_2(\theta)} \leq \psi(+, \cdot) \) and also satisfy \( \|\partial_\beta \psi(+, \cdot)\|_{\infty} \ll \delta^{-1} \). Since \( \partial_\beta \psi(+, \cdot)(rke_1) \) is only supported on an arc of angular width \( 2\delta \), this implies that

\[
(5.7) \quad \max_r \left( \int_K |\partial_\beta \psi(+, \cdot)(rke_1)|^2 dm_K(k) \right) \ll \delta^{-1}.
\]

Since we also have pointwise bounds \( \mathbb{1}_{S_1(\theta - \delta)} \leq \psi(-, \cdot) \leq \psi(+, \cdot) \leq \mathbb{1}_{S_2(\theta + \delta)} \), we obtain a bound

\[
(5.8) \quad \int_{\mathbb{R}^2} (\psi(+, \cdot) - \psi(-, \cdot)) \, dx \leq \text{Area}(S_2(\theta + \delta) \setminus S_1(\theta - \delta)) \ll \delta + \theta^3 \ll \delta.
\]

Similarly, we obtain the bounds

\[
(5.9) \quad \int_{\mathbb{R}^2} (\mathbb{1}_{W_1(\theta)} - \psi(-, \cdot)) \, dx \ll \delta \quad \text{and} \quad \int_{\mathbb{R}^2} (\psi(+, \cdot) - \mathbb{1}_{W_2(\theta)}) \, dx \ll \delta.
\]

Since

\[
\text{Area}(W_1(\theta)) = \cos \theta \sin \theta \quad \text{and} \quad \text{Area}(W_2(\theta)) = \tan \theta,
\]

(5.9) also implies that

\[
(5.10) \quad \left| \int_{\mathbb{R}^2} \psi(-, \cdot) \, dx - \cos \theta \sin \theta \right| \ll \delta, \quad \left| \int_{\mathbb{R}^2} \psi(+, \cdot) \, dx - \tan \theta \right| \ll \delta.
\]

We introduce the notation

\[
\tilde{\theta} \overset{\text{Def}}{=} e^{2\theta} \theta = \theta + O(\theta^3).
\]

Using (5.11) and expanding the Taylor series for \( \sin, \cos, \arctan \) we find

\[
\frac{\sin \theta \cos \theta}{\theta} = \frac{\theta + O(\theta^3)}{\theta + O(\theta^3)} = 1 + O(\theta^2), \quad \text{and also} \quad \frac{\tan \theta}{\theta} = 1 + O(\theta^2),
\]

so that

\[
(5.11) \quad \tilde{\theta} \overset{\text{Def}}{=} \frac{1}{\theta} \int_{\mathbb{R}^2} \psi(\cdot, \cdot) \, dx = 1 + \frac{1}{\theta} O(\theta^2 + \delta) = 1 + O(\delta^{3/2}).
\]
Note that the appearance of $\theta^2 + \delta$ in this bound explains our choice $\theta^2 = \delta$. Note also that both $\hat{\theta}$ and $\hat{1}$ depend on $t$, but this dependence is suppressed from the notation.

Provided $\theta \leq \pi/8$, and recalling from (5.2) that $\nu_t$ is supported on $I_t^\pm$, we have

$$\left| \int_{K} \nu_t^\pm dm_K \right| \leq \frac{\varphi_2 - \varphi_1}{2\pi} \leq \frac{\theta_t}{\pi} \leq \frac{\tan \theta_t}{\pi} \leq e^{-2t} \tan \theta \leq e^{-2t}.$$ 

We will make our choices so that

(5.12) $e^{-2t} = o(\delta^{1/2})$,

so that (5.11) implies

(5.13) $|I| = \frac{\varphi_2 - \varphi_1}{2\pi} + O(\delta^{1/2})$, where $I := \int_{K} \nu_t^\pm dm_K$.

Let $f(\pm, \delta) = \psi(\pm, \delta)$ be the Siegel-Veech transform of the functions defined in (5.4). The transform preserves pointwise inequalities of functions, and so (5.3) implies

(5.14) $\pi \pi C (\sum \nu_t^\pm, I) f(-\delta, \delta)(x) \leq N^C (e^t, x, \varphi_1, \varphi_2) \leq \pi \pi C (\sum \nu_t^\pm, I) f(+\delta, \delta)(x)$.

We apply Lemma 4.1 with parameters $\alpha_1, \alpha_2, \varepsilon$. Decompose $f(\pm, \delta)$ as the sum $(f(\pm, \delta))_{\text{main}} + (f(\pm, \delta))_{\varepsilon}$, where $(f(\pm, \delta))_{\text{main}} = f(\pm, \delta)(1 - \chi_\varepsilon)$ and $(f(\pm, \delta))_{\varepsilon} = f(\pm, \delta) \chi_\varepsilon$, and let

$$f(\pm, \delta, \varepsilon) := f(\pm, \delta)(1 - \chi_\varepsilon) - I_{\delta, \varepsilon}^\pm,$$

i.e. $f(\pm, \delta, \varepsilon)$ is the projection of $(f(\pm, \delta))_{\text{main}}$ to the space of zero integral functions. We note that

(5.15) $S_K (f(\pm, \delta, \varepsilon)) \leq S_K ((f(\pm, \delta))_{\text{main}}) + S_K (I_{\delta, \varepsilon}^\pm) \ll \varepsilon^{-\alpha_1} \delta^{-1/2} + I_{(\delta, \varepsilon)}^\pm$.

Indeed, the first inequality follows from the triangle inequality, and in the second inequality we used (4.1) and (5.7) for the first summand and that fact that $I_{(\delta, \varepsilon)}^\pm$ is a constant. By Siegel’s formula, the term $I_{(\delta, \varepsilon)}^\pm$ is uniformly bounded (independently of $\varepsilon, \delta$).

Having recorded these bounds, we turn to the application of Theorem 4.1. We will choose a sequence $t_n \to \infty$, choose parameters $\lambda \in (0, \lambda_c)$, set $\eta = 1/\lambda$, and for each $n$, define parameters $\delta_n, \varepsilon_n$, thus giving functions

$$f_n^\pm = f(\pm, \delta_n, \varepsilon_n).$$

The theorem will be applied twice, to each of the two sequences $f_n^+, f_n^-$. We will choose $0 < \eta_n < 2\eta$ so that (3.10) is satisfied. Then, since the $\nu_n(K)$ are bounded below by $|I|/4\pi$ for $t_n \geq t_m$ (see (5.2)), using (5.15) in Theorem 4.1 we obtain the bound

(5.16) $\pi C (\sum \nu_{t_n}^\pm, t_n) f_n^\pm(x) \ll e^{-(\eta - \eta_n)M_n} D_n,$

where

(5.17) $D_n \equiv S_K (f_n^\pm) = \varepsilon_n^{-\alpha_1} \delta_n^{-1/2}.$

In what follows we continue with the set of full measure of $x$ for which (5.16) holds, and thus the implicit constants in the $\ll$ and $O(\cdot)$ notations may depend on $x$. Let $I_n = I_{(\delta_n, \varepsilon_n)}^\pm$. Since $f(\pm, \delta_n)(1 - \chi_\varepsilon) = f_n^\pm + I_n^\pm$, (4.2) and (5.16) imply

(5.18) $\pi C (\sum \nu_{t_n}^\pm, t_n) (f(\pm, \delta_n)(1 - \chi_\varepsilon))(x) = \int_{K} I_n^\pm \nu_{t_n}^\pm dm_K + O(e^{-(\eta - \eta_n)M_n} D_n)

(\int f_{(\pm, \delta_n)} d\mu) \left( \int \nu_{t_n}^\pm dm_K \right) + O(\varepsilon_n^\beta + e^{-(\eta - \eta_n)M_n} D_n),$

where

$$\beta = (\alpha_2 - \alpha_1).$$

Moreover (4.3) implies

(5.19) $\pi C (\sum \nu_{t_n}^\pm, t_n) (f(\pm, \delta_n) \chi_\varepsilon)(x) \leq \pi C (\sum \nu_{t_n}^\pm, t_n) (f(\pm, \delta_n) \chi_\varepsilon)(x) \ll \varepsilon_n^{\alpha_1} \varepsilon_n^\beta \varepsilon_n^\beta.$
By Theorem 5.1 we have \( \int f_{\pm, \delta_n} dm = c \int_{\mathbb{R}^2} \psi_{\pm, \delta_n} dx \), where \( c = c(L, C) \) is the Siegel-Veech constant. Combining this with (5.18) and (5.19) we obtain
\[
\pi \kappa (\Sigma_{\nu_n}, t_n) f_{\pm, \delta_n}(x) - c \left( \int_{\mathbb{R}^2} \psi_{\pm, \delta_n} dx \right) \left( \int_{K} \nu_{n} \, dm_{K} \right) = \ll_{x} \varepsilon_{n}^{\beta} + e^{-(\eta - \frac{\delta}{\sqrt{n}}) \lambda t_{n} D_{n}}.
\]
Divide by \( \kappa \theta_{t} = \theta + O(\theta^3) = \delta^{1/2} + O(\delta^{3/2}) \) to find
\[
\kappa \theta_{t}^{-1} \pi \kappa (\Sigma_{\nu_n}, t_n) f_{\pm, \delta_n}(x) = \ll_{x} \varepsilon_{n}^{\beta} + e^{-(\eta - \frac{\delta}{\sqrt{n}}) \lambda t_{n} D_{n}}.
\]
Combining (5.19) and (5.20) and using (5.21), we get
\[
\kappa \theta_{t}^{-1} \pi \kappa (\Sigma_{\nu_n}, t_n) f_{\pm, \delta_n}(x) \ll_{x} \varepsilon_{n}^{\beta} + e^{-(\eta - \frac{\delta}{\sqrt{n}}) \lambda t_{n} D_{n}}.
\]
Plugging this estimate into (5.14) and using (5.17), we find that for any \( n \),
\[
N^{C}(\epsilon t_{n}, x, \varphi_{1}, \varphi_{2}) - \epsilon_{n}^{\beta} \left( \varphi_{2} - \varphi_{1} \right) \ll_{x} \varepsilon_{n}^{\beta} + e^{-(\eta - \frac{\delta}{\sqrt{n}}) \lambda t_{n} D_{n}}.
\]

\[\textbf{Step 3. Choosing parameters and deriving bounds for any } T.\] Let \( \lambda \) be the size of the spectral gap for \( L, \mu \) as in [3] and let \( \lambda < \lambda \). We will show that \( \kappa = \frac{\lambda}{8(1+\lambda)} \) satisfies the conclusion of the theorem (see the discussion following the statement of Theorem 1.1). Let \( t_{n} \) be the sequence defined by the equation
\[
e_{n} = n^{\frac{\lambda}{2}}
\]
where \( \sigma > 1 \) will be chosen sufficiently large later (depending on \( \lambda \)). Fix \( 1 < \alpha_{1} < \alpha_{2} < 2 \) and \( \beta = \alpha_{2} - \alpha_{1} \) as in Lemma 4.1 and set
\[
\eta_{1} = \frac{\alpha_{1}}{\sigma}
\]
(note that with this choice, \( 0 < \eta_{1} < 2 \eta \)). This choice is necessary if we want to satisfy (3.10), since \( \alpha_{1} > 1 \) implies
\[
\sum_{n \in \mathbb{N}} e^{-\lambda \eta_{1} t_{n}} = \sum_{n \in \mathbb{N}} n^{-\sigma \eta_{1}} = \sum_{n \in \mathbb{N}} n^{-\alpha_{1}} \ll \infty.
\]
We bound the right hand side of (5.22). First, we choose \( \varepsilon_{n} = \delta_{n} \), then the first two terms are \( O(\delta_{n}^{-1/2}) \) and the last term is \( O(n^{-\sigma \eta - \sqrt{\lambda t_{n} D_{n}}}) \). To optimize the asymptotics of these terms, we equalize \( \delta_{n}^{-1/2} = n^{-A \delta_{n}^{-1/2}} \), or equivalently \( \delta_{n} = n^{-\frac{1}{\eta}} \) for \( A = \frac{\sigma \eta}{1+\lambda} \).

\[
\alpha = \beta - \frac{1}{2} + (1 + \alpha_{1}) = \frac{1}{2} + \alpha_{2}.
\]
The right hand side of (5.22) becomes (up to constants)
\[
n^{-\beta - \frac{1}{\eta} \frac{1}{\sqrt{n}}}. \]
We let \( \alpha_{1} \to 1 \) and \( \alpha_{2} \to 2 \), so that \( (\beta - \frac{1}{2} \frac{1}{\sqrt{n}}) \) is arbitrarily close to
\[
B \overset{\text{Def}}{=} \left( 1 - \frac{1}{2} \right) \left( \frac{\sigma \eta - \frac{1}{2}}{1/2 + 2} = \frac{1}{5} (\sigma \eta - \frac{1}{2}) \right).
\]
Thus for time squares \( T_{n}^{2} = e^{2 t_{n}} = n^{2 \frac{\lambda}{2}} \), (5.22) can be written as
\[
N^{C}(T_{n}, x, \varphi_{1}, \varphi_{2}) - \frac{c}{2} (\varphi_{2} - \varphi_{1}) T_{n}^{2} \ll T_{n}^{2(1-\kappa)}
\]
where
\[
\kappa < \kappa(\sigma) \overset{\text{Def}}{=} \frac{\lambda}{1-2\sigma B}.
\]
Now for arbitrary \( T \), let \( n \) satisfy \( T_{n} < T \leq T_{n+1} \). By monotonicity,
\[
\frac{\epsilon (\varphi_{2} - \varphi_{1})}{2} T_{n}^{2} \left( 1 - O(T_{n}^{-2 \kappa}) \right) \ll N^{C}(T_{n}, x, \varphi_{1}, \varphi_{2}) \leq \frac{\epsilon (\varphi_{2} - \varphi_{1})}{2} T_{n+1}^{2} \left( 1 + O(T_{n+1}^{-2 \kappa}) \right).
\]
Since \( e^{t_n} = n^{\frac{2}{3}} \), we have
\[
\max \left( \frac{T_2^2}{T_1^2}, \frac{T_2^{n+1}}{T_2^n} \right) \leq \frac{T_2^{n+1}}{T_2^n} = \left( 1 + \frac{1}{n^2} \right) \frac{2^T}{nT_2^n} = 1 + O \left( \frac{1}{n^2} \right).
\]
So both sides of (5.28) are \( O \left( \frac{1}{n^2} \right) \), i.e., by (5.27), \( B = 1 \) which leads via (5.26) and (5.27) to
\[\sigma = \frac{2}{\pi}.\]
We may choose \( \sigma \) large and increase our original choice of \( \lambda < \lambda_{\mathcal{C}} \), to see that we can take \( \kappa = \frac{2}{\pi \lambda (\lambda + 1)} \). We leave it to the reader to verify that with any choice of \( \sigma \), (5.12) is satisfied.

6. Effective counting in all sectors and all dilates of an ellipse

In order to change the order of quantifiers and obtain an estimate simultaneously true for all \( \varphi_1, \varphi_2 \) and all \( \{g \xi \colon g \in G\} \), we will use two distinct techniques. Firstly we will use Theorem 3.5 instead of 3.4, as this will allow us to control countably many ellipses. Secondly we will give an additional approximation argument which shows how to use countably many functions, approximating a countable dense set of sectors, along with a countable dense set of ellipses, of countably many radii, to simultaneously control all ellipses and all sectors. We proceed to the details.

Proof of Theorem 1.2. We will use the notations and estimates as in the proof of Theorem 1.1 and follow the same steps. We fix a configuration \( \mathcal{C} \) and use it throughout, and let \( c \) be the corresponding Siegel-Veech constant. In analogy with (5.2), for fixed \( \varphi_1, \varphi_2 \) we set
\[
I = [\varphi_1, \varphi_2], \quad I_{-\varphi_1, \varphi_2} = [\varphi_1 + \theta, \varphi_2 - \theta], \quad I_{+\varphi_1, \varphi_2} = [\varphi_1 - \theta, \varphi_2 + \theta],
\]
and let \( \nu, \nu_{-\varphi_1, \varphi_2}^+, \nu_{+\varphi_1, \varphi_2}^+ \) denote respectively the indicator functions of these intervals (where we have selected a different notation to reflect the dependence on \( \varphi_1, \varphi_2 \)). Using (5.3) and using that \( \pi_L \left( \sum_{\nu_{+\varphi_1, \varphi_2}^+} \right) (g \xi) = \pi_L \left( \sum_{\nu_{-\varphi_1, \varphi_2}^+} \right) (\xi) \), we find that for every \( x \in \mathcal{H} \) and every \( g \in G \),
\[
\pi_L \left( \sum_{\nu_{-\varphi_1, \varphi_2}^+} \right) \mathbb{I}_{W_1}(\theta)(\xi) \leq \frac{\theta}{\pi} N^c(e^t, g \xi, \varphi_1, \varphi_2) \leq \pi_L \left( \sum_{\nu_{+\varphi_1, \varphi_2}^+} \right) \mathbb{I}_{W_2}(\theta)(\xi).
\]
This estimate generalizes (5.3) and constitutes the first step of the proof.

In the second step of the proof we again need to record certain bounds, but this time we will record their dependence on three additional parameters. Namely, as before, we will have parameters \( 1 < \alpha_1 < \alpha_2 < 2, \beta = \alpha_2 - \alpha_1, \eta_1 > 0 \), as well as sequences of times \( t_n \rightarrow \infty \), smoothing parameters \( \delta_n \) and cutoff parameters \( \varepsilon_n \). In addition we will have sequences of ‘ellipse parameters’ \( \{g_n\} \subset G \) and ‘angular sector parameters’ \( \varphi_1^{(n)} < \varphi_2^{(n)} \) with \( \varphi_2^{(n)} - \varphi_1^{(n)} \leq 2\pi \).

Using a smoothing parameter \( \delta = \delta_n \), defining the functions \( f_{(-\delta)} \) (Siegel-Veech transforms of smooth approximations of \( \mathbb{I}_{W_1}, \mathbb{I}_{W_2} \)) as before, and in analogy with (5.14), we obtain
\[
\pi \theta_t \pi \left( \sum_{\nu_{-\varphi_1, \varphi_2}^+} \right) f_{(-\delta)}(\xi) \leq N^c(e^t, g \xi, \varphi_1, \varphi_2) \leq \pi \theta_t \pi \left( \sum_{\nu_{+\varphi_1, \varphi_2}^+} \right) f_{(+\delta)}(\xi).
\]
Note that these upper and lower bounds are valid for any \( g \in G \) and any \( \varphi_1, \varphi_2 \).

In the proof of (5.22), there are two sources for the dependence of the estimate on \( x \). The first arises in deriving (5.19) by way of (1.3), and gives rise to an estimate which is uniform as \( x \) ranges over a compact subset of \( \mathcal{H} \), and the second arises from Theorem 1.4 and gives rise to a condition \( n \geq n_0(x) \). Thus the same argument (with Theorem 3.5 instead
of Theorem 3.4 gives the generalization of (5.22),

\[ \left| N^{\mathcal{C}}(e^{\lambda n}, g_n x, \varphi_1^{(n)}, \varphi_2^{(n)}) \right| \leq \frac{C}{\lambda} \left( \varphi_2^{(n)} - \varphi_1^{(n)} \right) \]

(6.1)

\[ \leq \delta_n/2 + \frac{1}{2} \epsilon_n^{\beta} + \left( \varphi_2^{(n)} - \varphi_1^{(n)} \right)^{1-\eta/2} e^{-\eta/\lambda} \epsilon_n\delta_n^{1/2}, \]

as long as \( n \geq n_0(x) \) and where the implicit constant depends on \( g_n \) and \( x \) and can be taken to be uniform in compact subsets of \( G \) and \( \mathcal{L} \). This completes the second step of the proof.

We now choose \( \lambda, \alpha_1, \alpha_2 \) satisfying \( \lambda < \lambda_G \), \( 1 < \alpha_1 < \alpha_2 < 2 \). For each \( n \in \mathbb{N} \), we define an auxiliary variable

\[ m = m_n = \lfloor n^{1/7} \rfloor, \]

which we will refer to as the scale of \( n \). For fixed \( m \), let

\[ \mathcal{N}_m = \{ n : m_n = m \} \]

denote the indices of scale \( m \). Note that as \( n \to \infty \), the scales \( m_n \) also tend to infinity at a slower rate, and the cardinality of \( \mathcal{N}_m \) is approximately \( m^6 \). Now choose \( \varphi_1^{(n)}, \varphi_2^{(n)}, g_n \) so that for all large enough \( m \), the collection of triples

\[ \{ (\varphi_1^{(n)}, \varphi_2^{(n)}, g_n) : n \in \mathcal{N}_m \} \]

is \( m/(\log m) \)-dense in

(6.2) \[ \{ (\varphi_1, \varphi_2, g) : \varphi_1 \in [0, 2\pi], \varphi_2 - \varphi_1 \in [0, 2\pi], g \in G, \max(\|g\|, \|g^{-1}\|) < \log m \} \]

(with respect to the sup-norm in the first two coordinates and the operator norm in the third coordinate), and so that \( \varphi_2^{(n)} - \varphi_1^{(n)} \geq \frac{1}{m \log m} \). This is possible since (6.2) defines a 5-dimensional manifold of diameter \( O(\log m) \).

Now following (5.23) we choose \( t_n \) so that \( e^{\lambda t_n} = m_n^2 \), where \( \sigma \) is a parameter we will optimize. The optimal value will turn out to be

(6.3) \[ \sigma = 8.5(\lambda + 1), \]

assume for now it is large. Let

(6.4) \[ \eta_1 = \frac{7 \alpha_1}{\sigma} > \frac{7}{\sigma}, \]

so that

\[ \sum_{n \in \mathbb{N}} e^{-\lambda \eta_1 t_n} = \sum_{m \in \mathbb{N}} \sum_{n \in \mathcal{N}_m} m^{-\sigma \eta_1} \ll \sum_{m \in \mathbb{N}} m^{6-\sigma \eta_1} < \infty, \]

so that (6.10) holds. Also note that the lengths of intervals at scale \( m_n \) is bounded below by \( e^{-\frac{\lambda}{\sigma} t_n} / t_n \) and in particular, since \( \frac{\lambda}{\sigma} < 2 \), satisfies the lower bound \( |I| > e^{-2t_n} \) for all large enough \( n \). Thus we can apply Theorem 3.3 and deduce (6.1).

As before, we optimize the right hand side of (6.1) by setting all three summands equal to each other, and we obtain that it is bounded by a constant (depending on \( \|g_n\| \)) multiplied by the expression (5.25). Letting \( \alpha \to 2, \beta \to 1 \) in (5.25) and using (6.3) instead of (5.24), we find that the right hand side of (6.1) is on the order of \( n^{-\frac{1}{2}(\sigma \eta - \frac{1}{2})} = (e^{2t_n})^{-\kappa} \) where

\[ \kappa = \kappa(\sigma) = \frac{\lambda}{10} \left( \eta - \frac{7}{2\sigma} \right). \]

Denote

\[ S(r_0, \varphi_1, \varphi_2) = \{ r(\cos \beta, \sin \beta) : \beta \in [\varphi_1, \varphi_2], 0 \leq r \leq r_0 \}. \]

Let \( g \in G \) and \( \varphi_1 \in [0, 2\pi] \) and \( \varphi_2 \in \mathbb{R} \) with \( \varphi_2 - \varphi_1 \leq 2\pi \). When \( \|g_j - g\| < \frac{1}{m(\log m)} \), and \( g, g_j \) are as in (6.2), then \( \max(\|\text{Id} - g_j g^{-1}\|, \|\text{Id} - gg_j^{-1}\|) < \frac{1}{m} \). Thus there is a constant \( c_1 \) such that for all large enough \( m \) there are \( k, \ell \in \mathcal{N}_m \) such that

\[ \varphi_1^{(k)} < \varphi_1 < \varphi_1^{(k)} < \varphi_2^{(k)} < \varphi_2 < \varphi_2^{(\ell)}. \]
and for any $r_0$ we have the inclusions
\[ g^{-1}_k S \left( r_0 \left( 1 - \frac{c_1}{m} \right), \varphi_i^{(k)}, \varphi_k^{(k)} \right) \subset g^{-1}_k S (r_0, \varphi_1, \varphi_2) \subset g^{-1}_k S \left( r_0 \left( 1 + \frac{c_1}{m} \right), \varphi_i^{(k)}, \varphi_k^{(k)} \right). \]
Hence for all $T$,
\[ N^c \left( T \left( 1 - \frac{c_1}{m} \right), g^k x, \varphi_1^{(k)}, \varphi_2^{(k)} \right) \leq N^c \left( T, g^k x, \varphi_1, \varphi_2 \right) \leq N^c \left( T \left( 1 + \frac{c_1}{m} \right), g^k x, \varphi_i^{(k)}, \varphi_k^{(k)} \right). \]
Choosing $n$ so that $e^{tn} < T < e^{t_{n+1}}$, and assuming $T$ and hence $m$ are large enough so that the preceding estimates are all satisfied, arguing as in the preceding proof, we obtain the following analogue of (5.28):
\[ \left( \varphi_2^{(k)} - \varphi_1^{(k)} \right) \frac{c}{2} e^{2tn} \left( 1 - O((e^{2tn})^{-\kappa}) \right) \left( 1 - O \left( \frac{1}{m} \right) \right) \]
(6.5)
\[ \leq N^c (T, g^k x, \varphi_1, \varphi_2) \]
\[ \leq \left( \varphi_2^{(k)} - \varphi_1^{(k)} \right) \frac{c}{2} e^{2tn+1} \left( 1 + O((e^{2tn+1})^{-\kappa}) \right) \left( 1 + O \left( \frac{1}{m} \right) \right) \]
(with implicit constants depending on $\|g\|$). As before
\[ \frac{e^{2tn+1}}{e^{2tn}} = 1 + O \left( T^{2(-\frac{\lambda}{2\sigma})} \right), \]
and since
\[ \max \left[ \frac{\varphi_2^{(k)} - \varphi_1^{(k)}}{\varphi_2^{(k)} - \varphi_1^{(k)}}, \frac{\varphi_2^{(k)} - \varphi_1^{(k)}}{\varphi_2^{(k)} - \varphi_1^{(k)}} \right] = 1 + O \left( \frac{1}{m} \right) = 1 + O \left( (e^{2tn})^{\frac{\lambda}{2\sigma}} \right), \]
both sides of (6.5) are $(\varphi_2 - \varphi_1) \frac{c}{2} T^2 \left( 1 + O \left( T^{-\kappa'} \right) \right)$, where
\[ \kappa' = \min \left\{ \kappa(\sigma), \frac{\lambda}{2\sigma} \right\}. \]
Setting both of these terms equal to each other and computing $\sigma$ gives (6.3). When we plug this in we get the required estimate, with
\[ \kappa = \frac{\lambda}{17(\lambda + 1)}. \]
completing the proof. \hfill $\square$

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