New stability conditions for class of nonlinear discrete-time systems with time-varying delay

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Abstract

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Index Terms

Nonlinear discrete time systems, time varying delay, delay dependent stability, M-matrix, Lure Postnikov systems.

I. INTRODUCTION

Stability of delay systems has been examined intensively by the academics from the control community [1]-[13], because several physical systems, like networked control systems, biological systems and chemical systems, are generally associated with time delays, [14]-[19]. Indeed, time delay can vary over time. For example, in real time communication and control systems, the signals are transmitted through the networks and subject to variable time delays because of the network traffic changes/uncertainties. Thus, stability analysis and control of such dynamical systems with time varying delays is essential. To attain stability conditions, two main strategies can be followed due to the time varying nature of the delay. Independent of delay (i.o.d ) results applicable when the size of the delay is arbitrary or if there is no information about the delay. This deficiency leads to conservative criteria, particularly if the delay is relatively small. When information about the size of delay can be included, less conservative delay-dependent ( d.d ) conditions can be provided [20].

The majority of the literature on stability of delay systems gave stability conditions in terms of linear matrix inequalities (LMIs) [21]. This remains true till now in a huge volume of the publications on the topic. And the size of LMIs increases with order/complexity of the systems. Obviously, it is desirable to have a very few number of stability conditions, regardless of order/complexity of a delay system; that is why we try in this study, to determine easy to test stability conditions for nonlinear discrete time systems with time varying delay. New delay dependent stability conditions are obtained by transforming the studied system under an arrow form state space representation [9], [10] and [11], using the Koteleyanski lemma [22] and by applying Lyapunov functional technique and M-matrix properties. The main obtained result is simple, and in fact it consists of verifying a scalar condition, without the need of solving any LMIs. It allows a great freedom by a judicious choice of some scalar parameters. The obtained results can be applied to large class of systems. As an example of these systems, we may mention the famous Lure Postnikov system, see [9] and the references therein. Moreover, we show how to use our method to design a state feedback controller that stabilizes a discrete time Lure system with time varying delay and sector bounded nonlinearity [23]-[28]. Note that this system is one of the most important classes of nonlinear control systems and remains one of the main problems in control theory which is intensively examined due to its various practical applications [29]-[35].

This paper is organized as follows: the utilized notations, the definition of M-matrices as well as some preliminary results are described in Section 2. The main results of this paper are represented in Section 3. Subsequently, the utility of these results if applied to the well known Lure systems is shown in Section 4. Finally, in Section 5 and Section 6, we provide some illustrative examples and a brief conclusion, respectively.

II. PRELIMINARIES

We present, in this part, some preliminary results including some definitions and lemmas used in the proof of the main results. Let us first fix the notation used throughout this paper. The set of real number is denoted by \( R \), \( N \) designates the set of non-negative integers, and \( R^n \) denotes an \( n \)−dimensional linear vector space over the reals with the norm \( ||.|| \). The notations \( ||.|| \) refers to the Euclidean vector norm or the induced matrix norm, as appropriate. Let \( I_n \) denotes the \( n \times n \) identity matrix and \( M^T \) denotes the transpose of matrix \( M \). Matrices, if their dimensions are not explicitly stated, are assumed to have

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considered as an M-matrix. Any matrix having the form presented in (1) is (-M)-matrix when the following conditions are satisfied:

**Lemma 1. Kotelyanski lemma [22]**
The real parts of the eigenvalues of a matrix $M$ are inside the open disk of radius $\mu$ if and only if all those of the matrix $M = \mu I_n - M$, are positive.

**Remark 1.** It is obvious, for $\mu = 1$, that if the matrix $(I_n - M)$ checks the Kotelyanski conditions, matrix $(I_n - M)$ is considered as an M-matrix.

Consider the following arrow form matrix $\Lambda$, which will be used in the next section

\[
\Lambda = \begin{pmatrix}
\lambda_{1,1} - 1 & 0 & \cdots & 0 & \lambda_{1,n} \\
0 & \lambda_{2,2} - 1 & \cdots & \vdots & \lambda_{2,n} \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \lambda_{n-1,n-1} - 1 & \lambda_{n-1,n} \\
0 & \cdots & \lambda_{n,1} & \cdots & \lambda_{n,n} - 1
\end{pmatrix},
\]

where $\lambda_{i,n}, \lambda_{n,i} > 0$, $i = 1, \ldots, n - 1$; $\lambda_{i,i} < 1$, $i = 1, \ldots, n$.

**Lemma 2.** Any matrix having the form presented in (1) is (-M)-matrix when the following conditions are satisfied:

1) $\lambda_{i,i} < 1$

2) $\lambda_{n,n} - 1 - \sum_{i=1}^{n-1} \frac{\lambda_{n,i} \lambda_{i,n}}{\lambda_{i,i} - 1} < 0$.

**Proof.** In case the matrix $\Lambda$ is (-M)-matrix, $-\Lambda$ is an M-matrix. Based on Kotelyanski lemma and Remark 1, successive principal minors of $-\Lambda$ with positive signs yields to $\lambda_{i,i} < 1$, $\forall i = 1, \ldots, n - 1$. It comes the first condition of lemma. For $i = n$,

\[
sign \det(\Lambda) = sign \left( \prod_{j=1}^{n-1} (-\lambda_{i,i} + 1) \left( -\lambda_{n,n} + 1 - \sum_{i=1}^{n-1} \frac{\lambda_{n,i} \lambda_{i,n}}{\lambda_{i,i} - 1} + 1 \right) \right)
\]

\[
= sign \left( -\lambda_{n,n} + 1 - \sum_{i=1}^{n-1} \frac{\lambda_{n,i} \lambda_{i,n}}{\lambda_{i,i} - 1} \right),
\]

it comes,

\[
\lambda_{n,n} - 1 - \sum_{i=1}^{n-1} \frac{\lambda_{n,i} \lambda_{i,n}}{\lambda_{i,i} - 1} < 0
\]

which completes the proof.

### III. MAIN RESULTS

The class of nonlinear delay systems studied in this manuscript are governed by the following difference equation:

\[
S_1:\begin{cases}
y(k + n) + \sum_{i=0}^{n-1} f_i(y(k + i)) + \sum_{i=0}^{n-1} g_i(y(k + i - h(k))) = 0, \\
y(k + i) = \phi_i(k), \forall k = -h_m, \ldots, 1 \text{ and } \forall i = 0, \ldots, n - 1,
\end{cases}
\]

where $y$ is the system output, and $h(k) : \mathbb{N} \to \mathbb{N}$ denotes a time varying delay. In practice, the time delay may be unknown and can vary over time in a certain interval. It is thus assumed that $h(k)$ has an upper limit $h_m$ so that $h(k) \leq h_m$, $h_m \in \mathbb{N}$.
\[ f_i(\cdot), g_i(\cdot): \mathcal{D} \times \Omega \times \Omega \to \mathbb{R}, i = 0, \ldots, n - 1, \] are the nonlinear functions of the time \( k \), \( y(k) \), \( y(k + 1) \), \ldots \( y(k + n - 1) \) and \( y(k - h(k)) \), \( y(k + 1 - h(k)) \), \ldots \( y(k + n - 1 - h(k)) \) where \( \mathcal{D} = [-h_m, \infty] \), and \( \Omega \) is a connected domain of \( \mathbb{R}^n \). For ease of exposition, let \( \sup_{\cdot} |f(\cdot)| \) be the supremum of \( f(\cdot) \) calculated over \( \mathcal{D} \times \Omega \times \Omega \), where \( f(\cdot) \) can be any of \( f_i \) and \( g_i \) and their algebraic combination.

Define the state variables:
\[
x_i(k) = y(k + i - 1), \quad i = 1, \ldots, n,
\]
which leads to
\[
x_i(k + 1) = x_{i+1}(k), \quad i = 1, \ldots, n - 1.
\]

System (3) is reformulated as follows
\[
\begin{align*}
x_i(k + 1) &= x_{i+1}(k), \quad i = 1, \ldots, n - 1, \\
x_n(k + 1) &= -\sum_{i=1}^{n} f_i(\cdot) x_i(k) - \sum_{i=1}^{n} g_i(\cdot) x_i(k - h(k)).
\end{align*}
\]

Let \( x(k) = (x_1(k), \ldots, x_n(k))^T \in \mathbb{R}^n \). The system (3) can be re-written as
\[
\begin{align*}
\begin{cases}
x(k + 1) = F(\cdot) x(k) + G(\cdot) x(k - h(k)) \\
x(k) = \phi(\cdot), \forall k = -h_m, \ldots, 1
\end{cases}
\end{align*}
\]
where
\[
F(\cdot) = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}, \quad G(\cdot) = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-g_0(\cdot) & -g_1(\cdot) & \cdots & -g_n(\cdot)
\end{pmatrix}.
\]

Apply the state transformation,
\[
x = PX,
\]
where
\[
P = \begin{pmatrix}
1 & 1 & \cdots & 1 & 0 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 0 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 1
\end{pmatrix},
\]
with \( \alpha_i \neq \alpha_k, \forall i, k = 1, \ldots, n - 1 \).

The system (3) becomes
\[
X(k + 1) = A_0(\cdot) X(k) + A_1(\cdot) X(k - h(k))
\]
where
\[
A_0(\cdot) = \begin{pmatrix}
\alpha_1 & 0 & \cdots & 0 & \beta_1 \\
0 & \alpha_2 & \cdots & \beta_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \beta_{n-1} \\
\gamma_1(\cdot) & \cdots & \gamma_{n-1}(\cdot) & \gamma_n(\cdot)
\end{pmatrix}, \quad A_1(\cdot) = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\delta_1(\cdot) & \delta_2(\cdot) & \cdots & \delta_n(\cdot)
\end{pmatrix}.
\]
for any $i = 1, \ldots, n - 1$,

$$
\beta_i = \frac{\alpha_i - s}{\prod_{k=1}^{n-1} (s - \alpha_k)}
$$

where

$$
\gamma_i(.) = -(\alpha_i^2 + \sum_{j=0}^{n-1} f_j(.)\alpha_j^2),
$$

$$
= -p_{A_0(.)}(\alpha_i),
$$

$$
\delta_i(.) = -\sum_{j=0}^{n-1} g_j(.)\alpha_j^2
$$

$$
= -p_{A_1(.)}(\alpha_i),
$$

and

$$
\gamma_n(.) = -f_{n-1(.)} - \sum_{j=1}^{n-1} \alpha_j,
$$

$$
\delta_n(.) = -g_{n-1(.)}.
$$

We treat in the rest of this part the two cases of constant delay and variable delay.

A. Constant delay case

Before stating the main result, let us define the following matrix:

$$
M_1(S_1) = \begin{pmatrix}
|\alpha_1| - 1 & |\alpha_2| - 1 & |\beta_1| & 0 \\
|\alpha_2| - 1 & |\alpha_3| - 1 & |\beta_2| & 0 \\
\vdots & \vdots & \ddots & \vdots \\
m_1(.) & m_2(.) & \cdots & m_{n-1}(.) - 1 & m_n(.) - 1
\end{pmatrix}
$$

where

$$
m_i(.) = |\gamma_i(.)| + \sup_{(.)} |\delta_i(.)|,
$$

Theorem 1. The time delay system (3) with constant delay, $h(k) = h$, is asymptotically stable if there exist distinct real numbers, $|\alpha_i| < 1$, $i = 1, \ldots, n - 1$, such that the following inequality holds true,

$$
|\gamma_n(.)| + \sup_{(.)} |\delta_n(.)| + \sum_{i=1}^{n-1} \frac{|\gamma_i(.)| + \sup_{(.)} |\delta_i(.)|}{1 - |\alpha_i|} |\beta_i| < 1.
$$

Proof: Choosing a radially unbounded, positive definite Lyapunov function candidate such that

$$
V(k) = p(X(k))^T \rho = \sum_{i=1}^{n} \rho_i p_i(X(k)),
$$

where

$$
p(X(k)) = ( p_1(X(k)) \ p_2(X(k)) \ \cdots \ p_n(X(k)) )^T,
$$

$$
\rho = ( \rho_1 \ \rho_2 \ \cdots \ \rho_n )^T > 0,
$$

with

$$
p_i(X(k)) = |X_i(k)|, i = 1, \ldots, n - 1,
$$

$$
p_n(X(k)) = |X_n(k)| + \sum_{i=1}^{n} \sup_{(.)} |\delta_i| \sum_{j=k-h(k)}^{k-1} |X_i(j)|.
$$

Because $\rho > 0$, so that $V(k) > 0$. We obtain the difference $V(k+1) - V(k)$ under the solution of (8) as follows:

$$
V(k+1) - V(k) = \sum_{i=1}^{n} \rho_i (p_i(X(k+1)) - p_i(X(k))).
$$
We notice that
\[
p_i(X(k+1)) = |\alpha_i X_i(k) + \beta_i X_n(k)|
\]
\[
\leq |\alpha_i||X_i(k)| + |\beta_i||X_n(k)|, \quad i = 1, 2, \ldots, n - 1,
\]
therefore
\[
p_i(X(k+1)) - p_i(X(k)) \leq (|\alpha_i| - 1)|X_i(k)| + |\beta_i||X_n(k)|, \quad i = 1, 2, \ldots, n - 1.
\]
and
\[
p_n(X(k+1)) - p_n(X(k)) = |X_n(k+1)| + \sum_{i=1}^{n} \sup_{(\cdot)} |\delta_i(\cdot)| \sum_{j=k+1-h}^{k} |X_i(j)|
\]
\[
- |X_n(k)| - \sum_{i=1}^{n} \sup_{(\cdot)} |\delta_i(\cdot)| \sum_{j=k-h}^{k-1} |X_i(j)|.
\]
Knowing that
\[
\sum_{i=1}^{n} \sup_{(\cdot)} |\delta_i(\cdot)| \sum_{j=k+1-h}^{k} |X_i(j)| - \sum_{i=1}^{n} \sup_{(\cdot)} |\delta_i(\cdot)| \sum_{j=k-h}^{k-1} |X_i(j)| = \sum_{i=1}^{n} \sup_{(\cdot)} |\delta_i(\cdot)| |X_i(k)|
\]
\[
- \sum_{i=1}^{n} \sup_{(\cdot)} |\delta_i(\cdot)| |X_i(k-h)|,
\]
and
\[
|X_n(k+1)| \leq \sum_{i=1}^{n} |\gamma_i(\cdot)||X_i(k)| + \sum_{i=1}^{n} \sup_{(\cdot)} |\delta_i(\cdot)| |X_i(k-h)|,
\]
the substitution of (22) in (20) gives
\[
p_n(k+1) - p_n(k) = \left( |\gamma_i(\cdot)| + \sup_{(\cdot)} |\delta_i(\cdot)| - 1 \right) |X_n(k)| + \sum_{i=1}^{n-1} \left( |\gamma_i(\cdot)| + \sup_{(\cdot)} |\delta_i(\cdot)| \right) |X_i(k)|.
\]
it comes from (18), (22) and (20),
\[
V(k+1) - V(k) < |X(k)|^T M_1^T(S_1) \rho.
\]
As nonlinear elements of $M_1^T(S_1)$, in the last column, are isolated, we obtain constant eigenvector $v(\cdot)$ relative to the eigenvalue $\lambda_m$, where $\lambda_m$ is such that $\text{Re}(\lambda_m) = \max\{\text{Re}(\lambda), \lambda \in \lambda(M_1(S_1))\}$ [9]. Then $\Delta V(k) < 0$ if $M_1^T(S_1)$ is the opposite of an $M$-matrix.

In fact, by Definition 3, $\forall \eta > 0$, the equation, $(-M_1^T(S_1))y = \eta$, has a solution. Let $y = \rho > 0$ this solution, it comes from $(-M_1^T(S_1))^{-1} \eta = \rho$. Then from (23), we obtain:$V(k+1) - V(k) < |X(k)|^T M_1^T(S_1) \rho = |X(k)|^T M_1^T(S_1)(-M_1^T(S_1))^{-1})\eta = |X(k)|^T(-\eta) = \sum_{i=1}^{n} X_i(k)\eta_i < 0$. Moreover, $\alpha_i, i = 1, \ldots, n - 1$, are arbitrary, we choose $|\alpha_i| < 1$ with $\alpha_i \neq \alpha_k, \forall i, k = 1, \ldots, n - 1$. It is noted, from Lemmas 1 and 2, that when
\[
m_n(\cdot) - 1 - \sum_{i=1}^{n-1} \frac{m_i(\cdot)|\beta_i|}{|\alpha_i| - 1} < 0,
\]
$-M_1(S_1)$ is an $M$-matrix. Thus, the proof is finished.

**B. Time varying delay case**

We take into account, in this sub-section, system $S_1$ with time varying delay which satisfies the below condition:
\[
h_1 \leq h(k) \leq h_2,
\]
where $h_1 > 0, h_2 > 0$ and $h_2 > h_1, \varphi(i), i = h_2 - h_2 + 1, \ldots, 0$, are the initial conditions. In this case, some modifications are carried on the matrix $M_1(\cdot)$ to obtain the matrix $M_2(\cdot)$ relative to $S_1$ for stability condition.

\[
M_2(S_1) = \begin{pmatrix}
|\alpha_1| - 1 & |\beta_1| \\
|\alpha_2| - 1 & |\beta_2| \\
\vdots & \vdots \\
m_1(h_1, \cdot) & m_2(h_1, \cdot) & \cdots & m_n(h_1, \cdot) - 1
\end{pmatrix},
\]
The overvaluation of

where

\[ m_i(h, \ldots) = |\gamma_i(.)| + (\Delta h + 1) \sup_{(.)} |\delta_i(.)|, \quad i = 1, \ldots, n, \]

with \( \Delta h = h_2 - h_1 \).

**Theorem 2.** The time varying delayed system (3) is delay dependent asymptotically stable, if there exist distinct real numbers, \( |\alpha_i| < 1, \quad i = 1, \ldots, n - 1 \), such that the following inequality holds true

\[ |\gamma_n(.)| + (\Delta h + 1) \sup_{(.)} |\delta_n(.)| + \sum_{i=1}^{n-1} \left( |\gamma_i(.)| + (\Delta h + 1) \sup_{(.)} |\delta_i(.)| \right) |\beta_i| < 1. \]  

**Proof:** Since \( \alpha_i, \quad i = 1, \ldots, n - 1 \), are arbitrary, we choose \( |\alpha_i| < 1 \) with \( \alpha_i \neq \alpha_k, \quad \forall i, k = 1, \ldots, n - 1 \), so that \( M_2(S_1) \) is an arrow form matrix \( \Lambda \)-matrix. Thus, it follows from Lemmas 1 and 2 that if

\[ m_i(h, \ldots) - 1 - \sum_{i=1}^{n-1} \frac{m_i(h, \ldots)|\beta_i|}{|\alpha_i| - 1} < 0. \]  

Let \( \bar{\rho} > 0 \) be a constant vector so that \( M_2(S_1) \bar{\rho} < \bar{\eta} \) remains true for \( \bar{\eta} < 0 \). Therefore, we choose the radially unbounded, positive definite Lyapunov function candidate given below

\[ \bar{V}(k) = \bar{p}(X(k))^T \bar{p} = \sum_{i=1}^{n} \bar{p}_i(X(k)), \]

where

\[ \bar{p}(X(k)) = (\bar{p}_1(X(k)), \bar{p}_2(X(k)), \ldots, \bar{p}_n(X(k)))^T, \]

\[ \bar{p} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)^T > 0, \]

with

\[ \bar{p}_i(X(k)) = |X_i(k)|, \quad i = 1, \ldots, n - 1, \]

\[ \bar{p}_n(X(k)) = |X_n(k)| + \sum_{j=1}^{n} \sup_{(.)} |\delta_j(.)| |I_j(k)|, \]

\[ I_j(k) = \sum_{\ell=k-h(k)}^{k-1} |X_j(\ell)| + \sum_{\ell=-h_2+2}^{-h_1+1} \sum_{m=k+\ell-1}^{k-1} |X_j(m)|. \]

Because \( \bar{\rho} > 0 \), so that \( \bar{V}(k) > 0 \). The difference \( \bar{V}(k+1) - \bar{V}(k) \) under the solution of (8) is given by

\[ \bar{V}(k+1) - \bar{V}(k) = \sum_{i=1}^{n} \bar{p}_i \left( \bar{p}_i(X(k+1)) - \bar{p}_i(X(k)) \right). \]

It is seen that

\[ p_i(k+1) - p_i(k) = |X_i(k+1)| - |X_i(k)| \]

\[ \leq (|\alpha_i| - 1)|X_i(k)| + |\beta_i||X_n(k)|. \]

The overvaluation of \( p_n(k+1) - p_n(k) \) necessitates overvaluing of \( I_j(k+1) - I_j(k) \). The difference below is first computed:

\[ I_j(k+1) - I_j(k) = \sum_{\ell=k+1}^{k-h(k+1)} |X_j(\ell)| + \sum_{\ell=-h_2+2}^{-h_1+1} \sum_{m=k+\ell}^{k-1} |X_j(m)| \]

\[ - \sum_{\ell=k-h(k)}^{k-1} |X_j(\ell)| - \sum_{\ell=-h_2+2}^{-h_1+1} \sum_{m=k+\ell-1}^{k-1} |X_j(m)| \]

\[ = \sum_{\ell=k-h(k)}^{k-h(k+1)} |X_j(\ell)| - \sum_{\ell=k-h(k)}^{k-1} |X_j(\ell)| \]

\[ + \sum_{\ell=-h_2+2}^{-h_1+1} \left( \sum_{m=k+\ell}^{k-1} |X_j(m)| - \sum_{m=k+\ell-1}^{k-1} |X_j(m)| \right), \]
because
\[
\sum_{\ell=-h_2+1}^{-h_1+1} \left( \sum_{m=k+\ell}^{k} |X_j(m)| - \sum_{m=k+\ell-1}^{k-1} |X_j(m)| \right) = (h_2 - h_1)|X_j(k)| - \sum_{\ell=-h_2+1}^{-h_1+1} |X_j(k + \ell - 1)|
\]
\[
= (h_2 - h_1)|X_j(k)| - \sum_{\ell=k+1-h_2}^{k-h_1} |X_j(\ell)|,
\]
and since \( h_1 \leq h(k) \leq h_2 \), we have
\[
\sum_{\ell=k+1-h_1}^{k-1} |X_i(\ell)| - \sum_{\ell=k+1-h(k)}^{k-1} |X_i(\ell)| \leq 0
\]
and
\[
\sum_{\ell=k+1-h(k+1)}^{k-h_1} |X_i(\ell)| - \sum_{\ell=k+1-h_2}^{k-h_1} |X_i(\ell)| \leq 0.
\]
It then follows from (39)
\[
I_j(k + 1) - I_j(k) \leq (\Delta h + 1)|X_j(k)| - |X_j(k - h(k))|,
\]
(40)
which yields
\[
p_n(k + 1) - p_n(k) \leq |X_n(k + 1)| - |X_n(k)|
\]
\[
+ (\Delta h + 1) \sup_{j=1}^{n} \| \delta_j(.) \| |X_j(k)|
\]
\[
- \sum_{j=1}^{n} \sup_{j=1}^{n} |\delta_j(.)| |X_j(k - h(k))|,
\]
Knowing that
\[
|X_n(k + 1)| \leq \sum_{j=1}^{n} |\gamma_j(.)| |X_j(k)| + \sum_{j=1}^{n} |\delta_j(.)| |X_j(k - h(k))|
\]
\[
\leq \sum_{j=1}^{n} |\gamma_j(.)| |X_j(k)| + \sum_{j=1}^{n} \sup_{j=1}^{n} |\delta_j(.)| |X_j(k - h(k))|,
\]
this allows us to obtain
\[
p_i(k + 1) - p_i(k) \leq \left( |\gamma_i(.)| - 1 + (\Delta + 1) \sup_{j=1}^{n} |\delta_i(.)| \right) |X_i(k)|
\]
\[
+ \sum_{j=1}^{n-1} \left( \sup_{j=1}^{n} |\gamma_j(.)| + (\Delta + 1) \sup_{j=1}^{n} |\delta_j(.)| \right) |X_j(k)|,
\]
which yields
\[
\tilde{V}(k + 1) - \tilde{V}(k) < |X(k)|^T M_2^T(S_1) \tilde{p} - |X(k)|^T \tilde{\eta} = - \sum_{i=1}^{n} \tilde{\eta}_i |X_i(k)| < 0,
\]
since \( \tilde{\eta} > 0 \) and the proof is completed.

IV. APPLICATION TO DELAYED LURE SYSTEMS

Consider the Lure type discrete time system presented in Figure 1. The model consists of a static nonlinearity in cascade with a dynamic linear time delay system. The structure of this system where only the variable \( \varepsilon_n \) is nonlinearly modulated, allows us to investigate the Lure type discrete-time system by the following nonlinear regression equation:
\[
S_2 : \varepsilon_{k+n} + \sum_{i=1}^{n} \bar{a}_i \varepsilon_{k+n-i} + \sum_{i=1}^{n} g_i(\varepsilon_{k-h+n-i}) = 0.
\]
(41)
Setting the following variable:
\[
g_i(\varepsilon_{k-h+n-i}) = g_i^* (\varepsilon_{k-h+n-i}) \varepsilon_{k-h+n-i}, \quad g_i^* \in E[k_1^i k_2^i] \quad \forall i = 0, \ldots, n,
\]
(42)
where:

\[ D(z,.) = D(z) = z^n + \sum_{i=0}^{n-1} a_{n-i} z^i, \]

\[ \gamma_i = -D(\alpha_i), \]

\[ \gamma_n = -\alpha_1 - \sum_{i=1}^{n-1} \alpha_i, \]

\[ N(z,.) = N(z, \varepsilon_k, \ldots, \varepsilon_{k-h}, \ldots, \varepsilon_{k-h+n-1}) = \sum_{i=0}^{n} g_i^*(\varepsilon_{k-h+i}) z^{n-i}, \]

\[ \delta_i(.) = -N(\alpha_i, \varepsilon_k, \ldots, \varepsilon_{k-h}, \ldots, \varepsilon_{k-h+n-1}), \]

\[ \delta_n = -g_0^*(\varepsilon_{k-h+n}). \]

Applying Theorem 1 leads to sufficient condition of the same form (12), but depending on \( \varepsilon_{k-h}, \ldots, \varepsilon_{k-h+n-1}. \) The obtained results are often difficult to implement, furthermore its interpretations with respect to the linear and nonlinear characteristics of the studied processes are generally limited. These considerations are due to the fact that the matrix description, from which the study is conducted, comes with a base change. The choice of a prior representation of Frobenius type allows to set similar to the previous stability conditions in which the coefficients depend only \( \varepsilon_k. \) By introducing the following variable changes,

\[ x_{k+1} = -g_n^*(\varepsilon_{k-h}) \varepsilon_{k-h} + \bar{a}_n \varepsilon_k, \]

\[ x_{k+1} = -g_q^*(\varepsilon_{k-h}) \varepsilon_{k-h} + \bar{a}_q \varepsilon_k + x_{k+1}^q, \quad q = n - 1 \ldots 2, \]

\[ \varepsilon_{k+1} = -g_q^*(\varepsilon_{k-h}) \varepsilon_{k-h} + \bar{a}_1 \varepsilon_k + x_{k+1}^q, \]

and by choosing the state vector \( x_k = (x_k^1 \ldots x_k^2 \varepsilon_k)^T \) the corresponding expression in terms of state space representation (41) becomes:

\[ x_{k+1} = \hat{F} x_k + \hat{G}(\varepsilon_{k-h}) x_{k-h}, \]

where:

\[ \hat{F} = \begin{pmatrix} 0 & \cdots & 0 & 0 & -\bar{a}_{n-1} \\ 1 & \cdots & 0 & 0 & -\bar{a}_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & -\bar{a}_1 \\ 0 & \cdots & 0 & 1 & -\bar{a}_0 \end{pmatrix}, \]

\[ \hat{G}(\varepsilon_{k-h}) = \begin{pmatrix} 0 & \cdots & 0 & -g_{n-1}^*(\varepsilon_{k-h}) \\ 0 & \cdots & 0 & -g_{n-2}^*(\varepsilon_{k-h}) \\ 0 & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & -g_0^*(\varepsilon_{k-h}) \end{pmatrix}, \]

(43)

This system is particular case of (3) where \( D(z,.) = \bar{D}(z) = z^n + \sum_{i=0}^{n-1} \bar{a}_{n-1-i} z^i, \) and \( N(z,.) = g^*(.) \bar{N}(z) = \sum_{i=0}^{n-1} g^*(.) \bar{b}_i z^i \) where \( \frac{\bar{N}(z)}{D(z)} = Z \left( \frac{B_0(s)}{D(s)} \right), g^*(.) = \frac{g(.)}{\bar{b}_0}, \) where \( g(.) \) is a function satisfying the sector bound condition, \( Z \) is
the Z transform, $B_0(s) = \frac{1-e^{-Ts}}{s}$ is a zero order holder, $T_z$ the sampling time and $h = \frac{T_z}{T_s}$ the time delay.

A. Sufficient stability conditions: autonomous case

Let us first consider the autonomous case ($r = 0$). The obtained system is a special case of (3) where $\hat{f}_i(.) = \bar{a}_{n-1-i}$, $\hat{g}_i(.) = g^*(.)\bar{b}_{n-1-i}$ $\forall$ $i = 1, ..., n-1$, $\gamma_n(.) = \gamma_n = -a_0 - \sum_{i=1}^{n-1} \alpha_i$ and $\delta_n(.) = -g^*(.)\bar{b}_n$. The following change of coordinates is employed:

$$y_k = P_1x_k,$$

where

$$P_1 =\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & \alpha_{n-1} & \cdots & \alpha_{n-1}^{n-2} & \alpha_{n-1}^{n-1} \\
1 & \alpha_{n-2} & \cdots & \alpha_{n-2}^{n-2} & \alpha_{n-2}^{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \alpha_1 & \cdots & \alpha_1^{n-2} & \alpha_1^{n-1}
\end{pmatrix}. \tag{48}$$

The transformation results in the following system

$$y_{k+1} = Fy_k + G(.)y_{k-h}, \tag{49}$$

where

$$F = P_1 \tilde{F} P_1^{-1} = \begin{pmatrix}
\gamma_n & \beta_1 & \cdots & \beta_{n-1} \\
\gamma_1 & \alpha_1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{n-1} & \alpha_{n-1} & \cdots & \alpha_{n-1}^{n-1}
\end{pmatrix}, \tag{50}$$

and

$$G(.) = P_1 \tilde{G}(. )P_1^{-1} = \begin{pmatrix}
g_n(.) & \cdots & 0 & 0 \\
g_1(.) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
g_{n-1}(.) & \cdots & 0 & 0
\end{pmatrix}. \tag{51}$$

In which case we obtain

$$M_1(S_2) = \begin{pmatrix}
m_n(.) - 1 & |\beta_1| & \cdots & |\beta_{n-1}| \\
m_1(.) & |\alpha_1| - 1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
m_{n-1}(.) & \cdots & |\alpha_{n-1}| - 1
\end{pmatrix}. \tag{52}$$

A sufficient stability condition for this system is given in the following theorem.

**Theorem 3.** The Lure type discrete-time system presented in figure 1 is (i.o.d) asymptotically stable, if there exist distinct real numbers, $|\alpha_i| < 1$, $i = 1, \ldots, n-1$, such that the following inequality holds true

$$|\gamma_n| + \sup_{(.)} (|g^*(.)|)|\bar{b}_{n-1}| + \sum_{i=1}^{n-1} \frac{|\bar{D}(\alpha_i)| + \sup_{(.)} (|g^*(.)|)|\bar{N}(\alpha_i)| |\beta_i|}{(1 - |\alpha_i|)} < 1. \tag{53}$$

**Remark 3.** L. Hou et al. in [34] established ultimate boundedness results for PWM feedback systems which can be considered a particular case of Theorem 1 when $g^*$ is considered as sign(.) function. They show that the solutions are ultimately bounded only when system is Hurwitz stable. Our result stated in Theorem 3, is obviously more general because it remains true when the system contains one unstable root and with delay. If $\bar{N}(z)$ has all its roots $z_i$, $i = 1, \ldots, n-1$ such that $|z_i| < 1$, and $-\bar{D}(z_i)\beta_i > 0$, then condition of Theorem 3 simplifies considerably. The following corollary gives this simplified condition.

**Corollary 2.** The Lure type discrete-time system presented in Fig. 1 is (i.o.d) asymptotically stable, if there exist distinct real numbers, $|\alpha_i| < 1$, $i = 1, \ldots, n-1$, such that the following inequality holds true $\gamma_n > 0$, $-\bar{D}(z_i)\beta_i > 0$ and

$$|g^*(.)| < \frac{\bar{D}(1)}{\bar{b}_{n-1}Q(1)}. \tag{53}$$

**Proof:** It is sufficient to take $\alpha_i = z_i$ in the condition of Theorem 3, in this case $\bar{N}(\alpha_i) = \bar{N}(z_i) = 0$.

Another important condition can be obtained when $\bar{D}(z)$ admits $n-1$ distinct roots with module inside the unit circle and the $n^{th}$ root can be outside the unit circle. This condition is given by the following corollary.
Corollary 3. The Lure type discrete-time system presented in figure 1 is (i.o.d.) asymptotically stable, if there exist distinct real numbers, $|\alpha_i| < 1$, $i = 1, \ldots, n - 1$, such that the following inequality holds true

$$|g^*(\cdot)| < \frac{1 - |z_n|}{|\bar{b}_{n-1}| + \sum_{i=1}^{n-1} |\bar{N}(z_i)||\beta_i|}.$$  

Proof: It is sufficient to take $\alpha_i$ equal to the roots of $\bar{D}$ that are inside the unit circle. In this case terms in condition of Theorem 3 becomes $D(\alpha_i) = 0$ and $\gamma_n = -\bar{a}_{n-1} - \sum_{i=1}^{n-1} z_i = -z_n$.

Remark 4. The last condition can also be simplified. In fact, if the roots of $\bar{D}$ verify $\bar{N}(z_i)\beta_i > 0$, $i = 1, \ldots, n - 1$ and $\bar{b}_{n-1} > 0$, we obtain a new simple condition given by the following corollary.

Corollary 4. If the conditions of corollary 3 and remark 4 are satisfied the system is stable if the following condition is satisfied:

$$|g^*(\cdot)| < \frac{Q(1)(1 - |z_n|)}{N(1)}. \quad (54)$$

Proof: Assuming that $N(z_i)\beta_i > 0$, $i = 1, \ldots, n - 1$ and $\bar{b}_{n-1} > 0$ are satisfied then one can remark that $|\bar{b}_{n-1}| + \sum_{i=1}^{n-1} |N(z_i)| |\beta_i| = \bar{b}_{n-1} + \sum_{i=1}^{n-1} \frac{N(z_i)\beta_i}{1 - z_i}$, and knowing that $\bar{b}_{n-1} + \sum_{i=1}^{n-1} \frac{N(z_i)\beta_i}{1 - z_i} = \frac{Q(1)}{Q(1)}$ then the result of corollary is obtained.

B. Feedback stabilization

In this case, take $r(k) = -Kx(k)$ with $K = (k_0, k_1, \ldots, k_{n-1})$, then the obtained system has the same form as (3), with

$$\bar{g}_i(\cdot) = g^*_K(\cdot)(b_i + k_i) = g^*(-(B + K)Cx(k - h))(b_i + k_i).$$

The stabilizing values of $K$ can be obtained by making the following changes: $\gamma_n = -\bar{a}_{n-1} - \sum_{i=1}^{n-1} \alpha_i$, $\delta_n(\cdot) = \delta(\cdot, k_{n-1}) = -g^*_K(\cdot)(\bar{b}_{n-1} + k_{n-1})$ and $\bar{N}(\alpha_i, k_{i-1}) = \frac{1}{\sum_{i=0}^{n-1} (b_i + k_i)\alpha_i}$. Then a sufficient stability condition for this system is given in the following theorem.

Theorem 4. The Lure type discrete time system presented in figure 1 is stabilizable via feedback control gain $K$, if there exist distinct real numbers, $|\alpha_i| < 1$, $i = 1, \ldots, n - 1$, such that the vector gain satisfies the following inequality

$$|\gamma_n| + \sup_{(\cdot)} |\delta_n(\cdot, k_{n-1})| + \sum_{i=1}^{n-1} \frac{|\bar{D}(\alpha_i)| + |\bar{g}^*_K(\cdot)| |\bar{N}(\alpha_i, k_{i-1})|}{|1 - |\alpha_i||} |\beta_i| < 1. \quad (55)$$

Remark 5. The above result of Theorem 4 gives an explicit way how to calculate the stabilizing values of the feedback gain vector $K$.

V. EXAMPLES

Example 1. In order to compare the different obtained results, we consider the linearized Clark equation with variable delay $h_1 \leq h(k) < h_2$ (the usual situation in its applications to population dynamics), that is,

$$x(k + 2) = \alpha x(k + 1) - \beta(\cdot) x(k + 1 - h(k)), k \geq 0,$$

where $\alpha \in (0, 1)$. Using Theorem 1, and choosing $\alpha_1 = \alpha$ yields $\gamma_2 = 0$ and $\gamma_1 = -D(\alpha) = -(\alpha^2 - \alpha^2) = 0$ and the stability condition is:

$$(h_2 - h_1 + 1) \left( \sup_{(\cdot)} |\beta(\cdot)| + \frac{\sup_{(\cdot)} |\beta(\cdot)|}{1 - \alpha} \right) < 1.$$

or

$$\sup_{(\cdot)} |\beta(\cdot)| < \frac{1 - \alpha}{(h_2 - h_1 + 1)(2 - \alpha)}. \quad (56)$$

if the delay is constant $h(k) = h$ the last condition becomes:

$$\sup_{(\cdot)} |\beta(\cdot)| < \frac{1 - \alpha}{(h + 1)(2 - \alpha)}. \quad (57)$$
Fig. 2. Dynamics evolution of $x(k)$ for initial condition $x(k) = 100|\sin(k)|$, $k = -20, \ldots, 0$.

Fig. 3. Dynamics evolution of $x(k)$ for initial condition $x(k) = 70$, $k = -20, \ldots, 0$.

Fig. 4. Dynamics evolution of $x(k)$ for initial condition $x(k) = 100|\sin(k)|$, $k = -20, \ldots, 0$.

For $\alpha = 0.9$ and $h = 20\text{sec}$ from (57) we have $\beta(.) < 0.0043$. Taking for example $\beta(.) = 0.002 \sin(0.2x(k) + 0.8x(k - h))$, the dynamic evolution of $x(k)$ for an initial condition $x(k) = 100|\sin(k)|$ is given in figure 4 and $x(k) = 70$ is given in figure 3.

**Example 2.**

Consider the example in [26]

\begin{align*}
    x(k+3) + a_2 x(k+2) + a_1 x(k+1) + a_0 x(k) &= b_0 F_k(x(k), x(k-h)), \\
    y(k) &= c_1 x(k) + c_2 x(k-h),
\end{align*}

(58)

where $a_i$, $i = 1, \ldots, 3$ and $c_j$, $j = 1, 2$ are constants and $F_k$ satisfies the following condition

\begin{equation}
    |F_k(u, v)| \leq \tilde{q}_1 |u| + \tilde{\delta}_1 |v|,
\end{equation}

(59)

with $\tilde{q}_1$ and $\tilde{\delta}_1$ are nonnegative constants. In this example we have

\begin{align*}
    D(z,.) &= z^3 + a_2 z^2 + a_1 z + (a_0 - b_0 F^* c_1) = D(z) - F^* c_1 N(z), \\
    N(z,.) &= -F^* c_2 b_0 = -F^* c_2 N(z),
\end{align*}

(60)
where

\[ D(z) = z^3 + a_2z^2 + a_1z + a_0, \]
\[ N(z) = b_0. \] (61)

In our case, \( D(z) \) has real roots \( z_j, j = 1, 2, 3 \). We can consider \( 0 \leq z_i < 1 \). Hence, choosing \( \alpha_1 = z_1 \) and \( \alpha_2 = z_2 \), we get \( \gamma_1 = -D(z_1) - F^*c_1b_0 = D(z_1) - F^*c_1b_0 = 0 - F^*c_1b_0 = -F^*c_1b_0 \), \( \delta_i = F^*c_2b_0 \), \( |\beta_i| = |\frac{1}{\alpha_1 - \alpha_2}| \) and the stability condition for this system is given by:

\[ |\gamma_3| + \frac{|(F^*(c_1 + c_2)b_0)| |\beta_1|}{1 - \alpha_1} + \frac{|(F^*(c_1 + c_2)b_0)| |\beta_2|}{1 - \alpha_2} < |\gamma_3| + \frac{|(\tilde{q}_1 + \tilde{\delta}_1)b_0| |\beta_1|}{1 - \alpha_1} + \frac{|(\tilde{q}_1 + \tilde{\delta}_1)b_0| |\beta_2|}{1 - \alpha_2} < 1. \]

We can obtain:

\[ |\gamma_3| + \frac{|(\tilde{q}_1 + \tilde{\delta}_1)b_0| |\beta_1|}{1 - \alpha_1} + \frac{|(\tilde{q}_1 + \tilde{\delta}_1)b_0| |\beta_2|}{1 - \alpha_2} < 1. \]

this gives

\[ \frac{|(\tilde{q}_1 + \tilde{\delta}_1)b_0| |\beta_1|}{1 - \alpha_1} + \frac{|(\tilde{q}_1 + \tilde{\delta}_1)b_0| |\beta_2|}{1 - \alpha_2} < 1 - z_3. \]

Knowing that \( (1 - \alpha_2)(1 - \alpha_1) > 0 \), the above yields

\[ \left( |(\tilde{q}_1 + \tilde{\delta}_1)b_0| \right) \left( |\beta_1|(1 - \alpha_2) + |\beta_2|(1 - \alpha_1) \right) < (1 - z_3)(1 - \alpha_2)(1 - \alpha_1) = D(1), \]

which can be re-written as the following form

\[ \tilde{q}_1 + \tilde{\delta}_1 < \frac{D(1)}{b_0|\beta_1|(2 - (\alpha_2 + \alpha_1))}. \]

**Example. 3**

Consider the study of a DC motor controlled by pulse width modulation from a tachometer.

![Block representation of the studied system](image)

The control pulses are rectangular, with a constant amplitude equal to \( \mathcal{M} \) and the sign of the error signal is defined at the sampling instants. Let \( T_s \) be the sampling time, \( R_k = \theta|\varepsilon_k| \) be the duration of the impulse in unsaturated regime and \( \tau_i \), \( i = 1, 2 \) be the time constants of the DC motor. The output of the modulator is a sequence of pulses of height \( \mathcal{M} \) and the width of the control pulses is related to the error function at the sampling instants by a relationship of the form:

\[ R_k = \begin{cases} \theta|\varepsilon_k| & \text{if } |\varepsilon_k| \leq \frac{T_s}{\theta}, \\ \tau_i & \text{if } |\varepsilon_k| \geq \frac{T_s}{\theta}, \end{cases} \] (62)

or simply under the following relation:

\[ R_k = T_s \text{sat} \left( \frac{\theta}{T_s} |\varepsilon_k| \right), \] (63)
where
\[
\frac{N(s)}{D(s)} = \frac{\lambda_1 s + 1}{(1 + \tau_1 s)(1 + \tau_2 s)}.
\]
From which, we can have
\[
\tilde{N}(z) = z(1 - \xi_1 - \xi_2 - \lambda) + \xi_1 \xi_2 + \lambda,
\]
\[
\tilde{D}(z) = (z - \xi_1)(z - \xi_2) = z^2 - (\xi_1 + \xi_2)z + \xi_1 \xi_2,
\]
where \(\xi_i = e^{-\frac{\tau_i s}{T}}\), \(i = 1, 2\) and \(\lambda = \frac{\xi_2 \tau_1 - \xi_1 \tau_2}{\tau_2 - \tau_1}\). The choice of \(\alpha = \xi_1\) yields \(\tilde{D}(\xi_1) = \tilde{D}(\alpha) = 0\). By Theorem 3, the stability condition in this particular case is given by
\[
|\gamma_2| + |(1 - \xi_1 - \xi_2 - \lambda)| \sup |g^*| + \frac{|\tilde{N}(\alpha)| \sup |g^*|}{1 - |\xi_1|} < 1,
\]
where \(\gamma_2 = \xi_2\).

A simple calculation leads to the following equalities:
\[
\frac{|N(\alpha)|}{1 - |\xi_1|} = |\xi_1 + \lambda| = \frac{\tau_1}{\tau_2 - \tau_1}(\xi_2 - \xi_1) \quad \text{and} \quad |(1 - \xi_1 - \xi_2 - \lambda)| = \frac{2\tau_2 - \tau_1}{\tau_2 - \tau_1} \xi_2 - \frac{2\tau_2 - \tau_1}{\tau_2 - \tau_1} \xi_1 + 1|.
\]
Now, let \(c = \frac{\tau_1}{\tau_2} - 1\), it comes
\[
\sup |g^*| < \frac{c(1 - \xi_1)}{\xi_2 - \xi_1 + |(1 - c)\xi_2 - (1 + c)\xi_1 + c|}.
\]
Because \(g^*(.) = \frac{\lambda T}{\|x\|}\) and taking into account of (62) we obtain \(\|e_k\| = \frac{T}{\tau}\), which gives
\[
\hat{M}T \times < \frac{cT^2(1 - \xi_1)}{\xi_2 - \xi_1 + |(1 - c)\xi_2 - (1 + c)\xi_1 + c|}.
\]
The above-mentioned results are shown in figure 6.

![Fig. 6. Stability boundaries for sampling PWM control system with a DC-motor represented by delayed second-order plant.](image-url)

VI. CONCLUSION

In this work, we have presented the new stability conditions for delayed nonlinear discrete time systems. The conditions are explicit, scalar and easy to check. Indeed, the application of the proposed method to delayed Lure Postnikov system shows simplicity and effectiveness. Moreover, our approach is self-contained, and systematic, and it does not go through the Linear Matrix Inequalities (LMI). Our theorems can deal with time delays, non-linearity and discrete time systems, and thus have more general applicability than those in the related literature.
