Reformulation of Yang-Mills Theories with Space-time Tensor Fields

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ABSTRACT: We provide the reformulation of Yang-Mills theories in terms of gauge invariant metric-like variables in three and four dimensions. The reformulations are used to analyze the dimension two gluon condensate and give gauge invariant descriptions of gluon polarization. In three dimensions, we obtain a non-zero dimension two gluon condensate, whose value is similar to the square of photon mass in the Schwinger model. In four dimensions, we obtain a Lagrangian with the dual property, which shares the similar but different property with the dual superconductor scenario.

KEYWORDS: Metric-Like Variables, Dimension Two Condensate, Dual Superconductor

1 Introduction

Regarding quantum Yang-Mills theories as highly nonlinear theories, it is difficult to achieve an understanding of its infrared region by the perturbative method. There are several approaches which suggest that the infrared region could be described through reformulating the Yang-Mills fields in terms of new variables. In [1], it was proposed that the infrared limit of the $SU(2)$ Yang-Mills theory in 4 dimensions could be given by a nonlinear sigma model by using partially dual variables, which are the decomposition of $SU(2)$ gauge field in terms of the variables of its $U(1)$ subgroup [6-8]. In [9], it was further proposed that a complete off-shell decomposition of $SU(2)$ field can be implemented through the view of spin-charge separation inspired by the strong correlated electron system.

There are also proposals to reformulate Yang-Mills theories by making use of field strength variables or gauge invariant metric-like variables [10]. Similar to the $U(1)$ Maxwell theory, which can be expressed by the field strength variables, the $SU(2)$ Yang-Mills theory can also be expressed by the field strength variables albeit in terms of an infinite series [11]. In [12], the frame-like fields are used as the pre-potential, and the $SU(2)$ Yang-Mills theory is recast into a $R^2$ gravity theory in 3 dimensions. In [13], the authors employed the metric-like fields and proposed that Yang-Mills theories could be regarded as the diffeomorphism invariant gravity theory broken by the background dependent ether term [14].

From another different angle, the reformulation or decomposition of gauge field is useful to address physical issues which are closely related to the gauge invariance. In [15], the transverse part of gauge field is used to analyze the gauge invariant dimension two condensate [15–18], which is also been discussed by using the “remaining” part after subtracting an “Abelian” part from the original gauge field [19]. In [20], it was proposed that the gauge invariant contribution of gluon polarization to the nucleon spin can be described by decomposing the gauge field into its physical part and its pure gauge part. In [21], many properties of quantum chromodynamics are discussed by dressed gluon fields, which are gluon fields with its pure gauge parts subtracted as background fields.

Inspired by the above investigations, we attempt to provide analysis on the infrared region of Yang-Mills theory through a novel reformulation and decomposition of the gauge field. At first, we

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1 For its application to the confinement problem, see [2–4] and the review article [5].
decompose the gauge field into two parts as \( A_{\mu}^a = B_{\mu}^a + \frac{1}{\ell} e_{\mu}^a \). Here \( e_{\mu}^a \) can be regarded as the frame-like fields in gravity theory, which has the same transformation properties as the gauge field strength. \( B_{\mu}^a \) can be regarded as the gauge connection in gravity theory, which can be solved in terms of \( e_{\mu}^a \) through imposing compatibility conditions, then a reformulation of \( A_{\mu}^a \) in terms of \( e_{\mu}^a \) is obtained. \( \frac{1}{\ell} \) is a quantity of mass dimension, which provides a natural cutoff for the theory. We shall show that in the low momentum limit \( p \ll \frac{1}{\ell} \) or the long wavelength limit \( \frac{1}{\ell} \gg \ell \), the reformulated Yang-Mills theories are easily tractable to analyze the dimension two condensate and provide gauge invariant descriptions of gluon polarization.

This paper is organized as follows. In section 2, we discuss a simpler case, the \( SU(2) \) Yang-Mills theory in three dimensions (3D), where a dimension two condensate with the value \( \frac{9}{2 \kappa^2} \frac{s^4}{\pi^2} \) is derived by considering the one loop quantum correction\(^2\). In section 3, we discuss the \( SU(2) \) Yang-Mills theory in four dimensions (4D), where a Lagrangian of the dual property is derived. By means of this Lagrangian, we propose that a nontrivial condensate of gauge invariant order parameter can yield a nonzero dimension two condensate. Alternatively, in the dual view, a nonzero dimension two condensate yields a Lagrangian which could support classical solutions of finite energy. This phenomenon is similar to but different from the analysis in [1, 9]. We provide conclusions in 4. Three appendices A, B and C are used to provide more details of the paper.

2 \( SU(2) \) Gauge Theory in Three Dimensions

2.1 Formulation with space-time tensor fields

In 3 dimensional space-time, the Lagrangian of \( SU(2) \) Yang-Mills theory is

\[
\mathcal{L} = -\frac{1}{4k}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}^a F_{\mu\nu}^a, \tag{2.1}
\]

\[
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc}A_\mu^b A_\nu^c,
\]

where \( \eta^{\mu\nu} = \text{diag}(1, -1, -1) \) is the Lorentz metric in 3D. We first decompose the gauge field as \(^3\)

\[
A_\mu^a = B_\mu^a + \frac{1}{\ell} e_\mu^a. \tag{2.2}
\]

Here \( \ell \) is a parameter with length dimension, so \( e_\mu^a \) is dimensionless. The decomposition (2.2) is familiar with the setting in the 3 dimensional gravity, where this decomposition is used to recast the Chern-Simons theory into a geometrical formulation [23, 24]. Under the gauge transformation, \( e_\mu^a = e_\mu^a \tau^a \) and \( B_\mu^a = B_\mu^a \tau^a \), where \( \tau^a \) takes values in the \( SU(2) \) Lie algebra, can have the transformation

\[
e_\mu^a \rightarrow U e_\mu^a U^{-1}, \tag{2.3}
\]

\[
B_\mu^a \rightarrow U B_\mu^a U^{-1} - i \partial_\mu U U^{-1}. \tag{2.4}
\]

These are similar to that we decompose the gauge field into its background part and its quantum part, then the quantum part transforms as Eq. (2.3) and the background part transforms as Eq. (2.4). Under this decomposition, the field strength has the decomposition

\[
F_{\mu\nu}^a = B_{\mu\nu}^a + \frac{1}{\ell} T_{\mu\nu}^a + \frac{1}{\ell^2} \epsilon^{abc} e_\mu^b e_\nu^c, \tag{2.5}
\]

\[
B_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + \epsilon^{abc} B_\mu^b B_\nu^c, \tag{2.6}
\]

\[
T_{\mu\nu}^a = (\partial_\mu e_\nu^a + \epsilon^{abc} B_\mu^b e_\nu^c) - (\partial_\nu e_\mu^a + \epsilon^{abc} B_\nu^b e_\mu^c). \tag{2.7}
\]

Here \( B_{\mu\nu}^a \) is the curvature of \( B_\mu^a \), and \( T_{\mu\nu}^a \) has the similar structure to the torsion in gravity theory. We further require that \( B_{\mu\nu}^a \) and \( e_{\mu}^a \) satisfy the equation

\[
\partial_\mu e_\mu^a + \epsilon^{abc} B_\mu^b e_\nu^c = \Gamma_\mu^\nu e_\nu^a. \tag{2.8}
\]

\(^2\)\( \kappa^2 \) is the coupling constant in 3D, which has the mass dimension.

\(^3\)For a different approach based on matrix parametrization of \( SU(N) \) Yang-Mills theory in 3D, see [22].
This equation is similar to the compatibility equation in gravity theory. From Eq. (2.8), we can solve $B^a_\mu$ as

$$B^a_\mu = -\frac{1}{2} \varepsilon^{abc} E^\rho_b (\partial_\mu E^\rho_c - \Gamma^\sigma_{\mu\rho} E^\rho_\sigma).$$  \hspace{1cm} (2.9)$$

$E^\rho_a$ is the inverse of $e^a_\rho$, that is, $E^a_\rho e^\rho_b = \delta^a_b$ and $E^a_\rho E^\rho_\beta = \delta^a_\beta$. As in gravity theory, from Eq. (2.8), we can obtain the metric compatibility condition

$$\partial_\mu g_{\alpha\beta} = \Gamma^\rho_{\mu\alpha} g_{\rho\beta} + \Gamma^\rho_{\mu\beta} g_{\rho\alpha}.$$  \hspace{1cm} (2.10)$$

Here $g_{\alpha\beta} = e^a_\alpha e^a_\beta$ is the metric tensor, and $g^{\alpha\beta} = E^a_\alpha E^\beta_\alpha$ is its inverse. A solution of Eq. (2.10) is

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}).$$  \hspace{1cm} (2.11)$$

In the above we have used the torsion free condition, so the connection is the Levi-Civita connection. From Eq. (2.8) and using the connections (2.9) and (2.11), we can obtain

$$B^a_\mu = \frac{1}{2} R^{s}_{\rho\mu\nu} e^c_s E^\rho_b e^{abc}, \quad T^a_{\mu\nu} = 0,$$  \hspace{1cm} (2.12)$$

where

$$R^{s}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} + \Gamma^\tau_{\mu\nu} \Gamma^\sigma_{\rho\tau} - \Gamma^\tau_{\rho\mu} \Gamma^\sigma_{\nu\tau}.$$  \hspace{1cm} (2.13)$$

is the Riemann curvature. Using these results, the Lagrangian (2.1) can be rewritten as

$$-4 \kappa^2 \mathcal{L} = \mathcal{L}^{(0)} + \frac{1}{\ell^2} \mathcal{L}^{(2)} + \frac{1}{\ell^4} \mathcal{L}^{(4)},$$  \hspace{1cm} (2.14)$$

$$\mathcal{L}^{(0)} = \frac{1}{4} \eta^{\mu\alpha} \eta^{\nu\beta} (g^{\rho\theta} g_{\sigma\tau} - \delta^{\rho\theta} \delta^\tau_\sigma) R^{s}_{\rho\mu\nu} R^{s}_{\theta\alpha\beta},$$  \hspace{1cm} (2.15)$$

$$\mathcal{L}^{(2)} = \eta^{\mu\alpha} \eta^{\nu\beta} (\delta^{\rho\theta} g_{\rho\sigma} - \delta^{\rho\theta} g_{\rho\alpha}) R^{s}_{\rho\mu\nu},$$  \hspace{1cm} (2.16)$$

$$\mathcal{L}^{(4)} = \eta^{\mu\alpha} \eta^{\nu\beta} (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}).$$  \hspace{1cm} (2.17)$$

The Lagrangian $\mathcal{L}$ is divided into three parts in the above. The $\mathcal{L}^{(0)}$ part is not new, as it has been given in [12] in a Hamiltonian analysis of the $SU(2)$ Yang-Mills theory and recently in [25] through the similar approach as we just presented in the above. Because we have decomposed $A^a_\mu$ into two parts in Eq. (2.2), we obtain two additional parts $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$. In the limit $\ell \rightarrow \infty$, $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$ are negligible, and only $\mathcal{L}^{(0)}$ term is left. The Lagrangian $\mathcal{L}$ is also similar to the higher derivative gravity of Stelle [26]. In [21], it has been suggested that the infrared region of quantum chromodynamics could be described by the Stelle action, in which the higher derivative terms are responsible for the confinement.

\section*{2.2 Dimension Two Condensate}

The nonzero value\(^4\) of dimension two gluon condensate has been suggested [15–18] through several different approaches. From our results in the last subsection, we learned that the metric tensor $g_{\mu\nu}$ provides a natural candidate for the dimension two gluon condensate. In this subsection, we provide analysis to show that a nonzero vacuum expectation of $g_{\mu\nu}$ can be generated by quantum corrections.

In Eq. (2.14), we have obtain a pure space-time tensor formulation of the Yang-Mills Lagrangian. The dynamical variable of this Lagrangian is a spin-2 tensor $g_{\mu\nu}$, which is gauge invariant by our construction. $\mathcal{L}^{(0)}$ is quadratic about the Riemann curvature, which includes higher derivative terms. $\mathcal{L}^{(2)}$ is similar to the Ricci scalar, and $\mathcal{L}^{(4)}$ is a Fierz-Pauli type massive term. The coupling $\frac{1}{\ell}$ provides a natural cutoff for the $\mathcal{L}$. If the characteristic momentum $p$ we considered is much smaller than $\frac{1}{\ell}$,\(^5\)

\(^4\)Here we have two metrics $\eta_{\mu\nu}$ and $g_{\mu\nu}$. The conventions of usage about these metrics are given in appendix A.

\(^5\)However, for a null result from the operator product expansion analysis of the lattice result, see [27].
that is, \( p \ll \frac{1}{\ell} \), or the characteristic wavelength \( \frac{1}{\ell} \) is much longer than \( \ell \), that is, \( \frac{1}{p} \gg \ell \), then the higher derivative Lagrangian \( \mathcal{L}^{(0)} \) is negligible. Therefore, in the limit of \( p \ll \frac{1}{\ell} \), the Lagrangian \( \mathcal{L} \) can be approximated as

\[
-4\kappa^2 \mathcal{L} \approx \frac{1}{\ell^2} \mathcal{L}^{(2)} + \frac{1}{\ell^4} \mathcal{L}^{(4)} ,
\]

\[
\mathcal{L}^{(2)} = \eta^{\mu\alpha} \eta^{\nu\beta} (\delta^\rho_\alpha g_\sigma\beta - \delta^\rho_\beta g_\sigma\alpha) R^\sigma_{\mu\rho\nu} ,
\]

\[
\mathcal{L}^{(4)} = \eta^{\mu\alpha} \eta^{\nu\beta} (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) .
\]

This approximate Lagrangian only includes second order derivatives of \( g_{\mu\nu} \). \( g_{\mu\nu} \) can be decomposed as

\[
g_{\mu\nu} = \varphi^2(x) \eta_{\mu\nu} + h_{\mu\nu} .
\]

Here \( \varphi^2(x) = \frac{1}{3} \eta^{\mu\nu} g_{\mu\nu} \) is the trace part of \( g_{\mu\nu} \), and \( h_{\mu\nu} \) is its symmetrical traceless part. So Eq. (2.18) describes the propagation of a scalar and a spin-2 tensor. We further assume that \( h_{\mu\nu} \) is small, then the Lagrangian (2.18) can be reduced to

\[
\mathcal{L} \approx \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\kappa^2}{24} \phi^4 ,
\]

where we have defined \( \phi = \frac{\sqrt{\kappa}}{\pi^2} \varphi \), and a total divergence term

\[
\mathcal{L}_{\text{bound}} = -\frac{1}{3} \partial_\alpha (\phi \partial^\alpha \phi)
\]

has been subtracted. This is the conventional \( \phi^4 \) theory. On the classical level, the minimal value of Eq. (2.22) is at \( \phi = 0 \). If we consider the quantum corrections, we can obtain the following effective potential

\[
V_{\text{eff}}(\phi) = V(\phi) + \frac{1}{2} \int \frac{d^3 k_E}{(2\pi)^3} \log \left[ \frac{k^2 + V''(\phi)}{k^2_E} \right] + m_0^2 \phi^2
\]

\[
\approx V(\phi) + \frac{1}{8\pi^2} \left( \Lambda \kappa^2 \phi^2 - \sqrt{2\pi} \frac{\kappa^3 \phi^3}{6} \right) + m_0^2 \phi^2 ,
\]

where \( k_E \) is the Euclidean momentum and

\[
V(\phi) = \frac{\kappa^2}{24} \phi^4 , \quad V''(\phi) = \frac{d^2 V}{d\phi^2} .
\]

In the above, \( \Lambda \) is the cutoff to regularize the effective potential. In 3D, only the massive term is cutoff dependent, so only the bare mass \( m_0 \) is needed to renormalize the effective potential. We can use the renormalization condition \( \frac{d^2 V_{\text{eff}}}{d\phi^2} |_{\phi=0} = 0 \). Then we obtain

\[
V_{\text{eff}}(\phi) = \frac{\kappa^2}{24} \phi^4 - \frac{\sqrt{2}}{48\pi} \kappa^3 \phi^3 .
\]

This effective potential has the minimum at \( \phi = \frac{3\sqrt{6}}{8\pi} \kappa \). The dimension two condensate is related to this minimum by

\[
\frac{1}{2} \langle \eta^{\alpha\beta} e^\alpha_\alpha e^\beta_\beta \rangle = \frac{1}{2} \langle \eta^{\alpha\beta} g_{\alpha\beta} \rangle = \frac{3}{4} \langle \varphi^2(x) \rangle = \frac{9}{64\pi^2} \kappa^4 .
\]

In two dimensional space-time, a nonzero photon mass can be generated in the Schwinger model [28]. We see that the dimension two condensate (2.27) have the same structure with the square of the photon mass.
The above computations are based on two assumptions: One assumption is that \( p \ll 1 \). So the higher directive Lagrangian \( \mathcal{L}^{(0)} \) is negligible. The other assumption is that the traceless part \( h_{\mu\nu} \) is negligible. So the theory is reduced to a scalar theory, which is tractable to derive a condensate through quantum corrections. However, there are concerns how can we consider the contribution from \( h_{\mu\nu} \) and the higher directive Lagrangian \( \mathcal{L}^{(0)} \). Because the Lagrangian (2.14) includes the inverse metric \( g^{\mu\nu} \), the Lagrangian (2.14) is a infinite series under the expansion of small \( h_{\mu\nu} \). As the gravitational theory, it is not renormalizable generally. Therefore, it is difficult to perform predictable quantum computations.

One possible solution of this issue is that the Lagrangian (2.14) with the higher derivative term (2.15) is actually renormalizable, just as the higher derivative gravity of Stelle [26], where the higher derivative terms are the main cause for the renormalizability. Another possible solution of this issue is to use the functional renormalization group proposed in [29, 30]. Using this functional renormalization group, the Einstein gravity has been suggest to be asymptotic safe [31–33]. It is possible that the Lagrangian (2.14) is also asymptotic safe. One argument to support this possibility is that the Lagrangian (2.14) is basically a Yang-Mills Lagrangian in disguise. So the functional renormalization group could be a possible approach to consider the contributions from \( h_{\mu\nu} \) and the higher directive terms consistently.

2.3 Gauge Invariant Gluon Polarization

The gauge invariant description [34] of gluon angular momentum is an important ingredient to address the nucleon spin problem. For two recent reviews on this problem, see [35, 36]. In [20], a gauge invariant expression is provided through decomposing the gauge field into its physical parts and its pure gauge part

\[
A^{\mu}_{a} = A^{\mu}_{phys} + A^{\mu}_{pure},
\]

in which \( A^{\mu}_{phys} \) transforms as Eq. (2.3), and \( A^{\mu}_{pure} \) transforms as Eq. (2.4). The descriptions of gluon polarization and orbital angular momentum are given by [37–39]

\[
k^{2}M^{\mu\nu}_{gs} = -F^{\mu\alpha}A^{\alpha}_{phys} + F^{\mu\alpha\beta}A^{\alpha\beta}_{phys},
\]

\[
k^{2}M^{\mu\nu}_{go} = (x^{\beta}\mathcal{T}^{\mu\alpha} - x^{\alpha}\mathcal{T}^{\mu\beta})
\]

\[
+ x^{\beta}\eta_{\sigma\alpha}F^{\mu\mu\rho}D^{\sigma}A^{\alpha\rho}_{phys} - x^{\alpha}\eta_{\sigma\alpha}F^{\mu\mu\rho}D^{\sigma}A^{\alpha\rho}_{phys},
\]

where

\[
D^{\sigma}A_{phys}^{\alpha\beta} = \partial^{\sigma}A_{phys}^{\alpha\beta} + \epsilon^{abc}A_{phys}^{\beta\sigma}A_{phys}^{\alpha\alpha},
\]

is the gauge covariant derivative, and \( \mathcal{T}^{\mu\nu} \) is proportional to the energy-momentum tensor

\[
\mathcal{T}^{\mu\nu} = \eta_{\rho\sigma}F_{a}^{\mu\nu}F^{\alpha\sigma} + k^{2}\eta_{\mu\nu}\mathcal{L}.
\]

From Eq. (2.2), we know that \( A_{\mu}^{a} \) has been decomposed into two parts. Now it is appropriate to make the designation

\[
A^{\mu}_{phys} = \frac{1}{\ell} \eta^{\mu\sigma}e_{\sigma}^{a},
\]

as they have the same transformation. Through this designation, \( M^{\mu\alpha\beta}_{gs} \) can be expressed completely with the tensor fields as

\[
k^{2}M^{\mu\nu}_{gs} = -\frac{1}{\ell}\eta^{\mu\sigma}(\eta^{\alpha\theta}e_{\gamma}^{\beta} - \eta^{\beta\gamma}e_{\gamma}^{\alpha})\sqrt{|g|}\left(\frac{1}{2}R_{\rho\sigma\theta\gamma}g^{\rho\lambda}\epsilon_{\lambda\sigma} + \frac{1}{\ell^{2}}\epsilon_{\gamma\tau\rho}e_{\alpha}^{\tau}\right),
\]

where \( |g| \) is the absolute value of the determinant of \( g_{\mu\nu} \). In the \( p \ll 1/\ell \) limit, \( M^{\mu\alpha\beta}_{gs} \) shall be dominated by

\[
k^{2}M^{\mu\alpha\beta}_{gs} \approx -\frac{2}{\ell^{2}}\sqrt{|g|}e^{\mu\alpha\beta}.
\]
Similarly, \( M_{\mu\nu}^{\alpha\beta} \) can also be expressed by space-time tensor fields and has the decomposition

\[
\kappa^2 M_{\mu\nu}^{\alpha\beta} = M_{gs(0)}^{\alpha\beta} + \frac{1}{\ell} M_{\mu\nu(1)}^{\alpha\beta} + \frac{1}{\ell^2} M_{\mu\nu(2)}^{\alpha\beta} + \frac{1}{\ell^3} M_{\mu\nu(3)}^{\alpha\beta} + \frac{1}{\ell^4} M_{\mu\nu(4)}^{\alpha\beta} \tag{2.36}
\]

Here \( M_{\mu\nu(0)}^{\alpha\beta}, M_{\mu\nu(2)}^{\alpha\beta} \) and \( M_{\mu\nu(4)}^{\alpha\beta} \) are

\[
M_{\mu\nu(0)}^{\alpha\beta} = x^\beta \mathcal{F}^{\alpha\beta} - x^\alpha \mathcal{F}_{\mu\nu}(0),
\]

\[
M_{\mu\nu(2)}^{\alpha\beta} = x^\beta \mathcal{F}^{\alpha\beta} - x^\alpha \mathcal{F}_{\mu\nu}(2)
\]

\[
+ \frac{1}{2} x^\beta \eta^\mu \eta^\nu \eta^\tau \eta^\sigma R^\lambda_{\mu\nu\rho\sigma} (\delta^\rho \gamma g_{\lambda\gamma} - \delta^\rho g_{\lambda\sigma}) - (\alpha \leftrightarrow \beta),
\]

\[
M_{\mu\nu(4)}^{\alpha\beta} = x^\beta \mathcal{F}^{\alpha\beta} - x^\alpha \mathcal{F}_{\mu\nu}(4)
\]

\[
+ x^\beta \eta^\mu \eta^\nu \eta^\tau \eta^\sigma (g_{\rho\sigma} g_{\gamma\tau} - g_{\rho\sigma} g_{\tau\gamma}) - (\alpha \leftrightarrow \beta).
\]

In the above, \( \mathcal{F}_{\mu\nu}^{(0)} \), \( \mathcal{F}_{\mu\nu}^{(2)} \) and \( \mathcal{F}_{\mu\nu}^{(4)} \) are the components of \( \mathcal{F}_{\mu\nu} \)

\[
\mathcal{F}_{\mu\nu} = \mathcal{F}_{\mu\nu}^{(0)} + \frac{1}{\ell^2} \mathcal{F}_{\mu\nu}^{(2)} + \frac{1}{\ell^4} \mathcal{F}_{\mu\nu}^{(4)},
\]

and they have the expression

\[
\mathcal{F}_{\mu\nu}^{(0)} = \frac{1}{4} \eta^\alpha \eta^\beta \eta^\gamma \eta^\lambda (g^{\alpha\lambda} g_{\beta\gamma} - \delta^\alpha \gamma \delta^\beta \lambda) R^\sigma_{\rho\sigma\tau} R_{\lambda\beta\theta} - \frac{1}{4} \eta_{\mu\nu} \mathcal{L}^{(0)},
\]

\[
\mathcal{F}_{\mu\nu}^{(2)} = \frac{1}{4} \eta^\alpha \eta^\beta \eta^\gamma \eta^\lambda (\delta^\alpha \gamma g_{\beta\lambda} - \delta^\beta \lambda g_{\gamma\alpha}) R^\sigma_{\rho\sigma\tau} - \frac{1}{4} \eta_{\mu\nu} \mathcal{L}^{(2)},
\]

\[
\mathcal{F}_{\mu\nu}^{(4)} = \frac{1}{4} \eta^\alpha \eta^\beta \eta^\gamma \eta^\lambda (g_{\rho\sigma} g_{\beta\gamma} - g_{\rho\sigma} g_{\gamma\beta}) - \frac{1}{4} \eta_{\mu\nu} \mathcal{L}^{(4)}.
\]

\( M^{\alpha\beta}_{gs(0)}, M^{\alpha\beta}_{gs(2)} \) and \( M^{\alpha\beta}_{gs(4)} \) are

\[
M_{gs(0)}^{\alpha\beta} = \frac{1}{2} x^\beta \eta^\mu \eta^\nu \eta^\gamma \eta^\lambda \sqrt{|g|} R^\sigma_{\rho\sigma\tau} \eta^\rho \gamma R^\delta_{\tau\lambda\epsilon} \delta_{\mu\nu} - (\alpha \leftrightarrow \beta),
\]

\[
M_{gs(2)}^{\alpha\beta} = \sqrt{|g|} x^\beta \eta^\mu \eta^\nu \eta^\gamma \eta^\lambda \Gamma_{\delta\lambda\epsilon} \eta_{\mu\nu} - (\alpha \leftrightarrow \beta).
\]

In the \( p \ll \frac{1}{\ell} \) limit, \( M_{gs}^{\mu\nu} \) shall be mainly dominated by \( M_{gs(4)}^{\mu\nu} \).

In the above, the gauge invariant descriptions of gluon angular momentum have been expressed by the geometrical objects. \( M_{gs}^{\mu\nu} \) in Eq. (2.34) and \( M_{gs(4)}^{\mu\nu} \) in Eq. (2.38) both include the contribution from the Riemann curvature, though they have the different tensor structure. \( M_{gs(0)}^{\mu\nu}, M_{gs(2)}^{\mu\nu} \) and \( M_{gs(4)}^{\mu\nu} \) in Eqs. (2.44) and (2.45) also includes the contribution from the Levi-Civita connection. According to Eq. (2.21), \( g_{\mu\nu} \) is decomposed into its trace part \( \varphi^2 \) and its traceless part \( h_{\mu\nu} \). The above gluon angular momentum can be further decomposed into contributions from \( \varphi^2 \) and \( h_{\mu\nu} \). We shall give more discussions on the gluon angular momentum in subsection 3.2.

3 \( SU(2) \) Gauge Theory in Four Dimensions

3.1 Formulation with space-time tensor fields

After the discussion on 3 dimensions, we begin to discuss the \( SU(2) \) Yang-Mills theory in 4 dimensions in this section. Although the \( SU(2) \) Yang-Mills theories in 3D and 4D have the same Lie Algebra, the 4D case is more complicated than the 3D case because of the number of space-time dimension. In 4 dimensional space-time, the Lagrangian of \( SU(2) \) Yang-Mills theory is

\[
\mathcal{L} = -\frac{1}{4\kappa^2} \eta^{\alpha\mu} \eta^{\beta\nu} F_{\alpha\beta} F_{\mu\nu},
\]

\[
F_{\mu\nu}^\alpha = \partial_{\mu} A_{\nu}^\alpha - \partial_{\nu} A_{\mu}^\alpha + e_{abc} A_{\mu}^b A_{\nu}^c,
\]

\[
\Rightarrow -6 -
\]
where \( \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the Lorentz metric in 4D. As in section 2, we decompose the gauge field as

\[
A^\mu = B^\mu + \frac{1}{E} e^\mu, \tag{3.2}
\]

Their transformations are similar to that in Eqs. (2.3) and (2.4). Accordingly the field strength has the decomposition

\[
F^a_{\mu\nu} = B^a_{\mu\nu} + \frac{1}{E} T^a_{\mu\nu} + \frac{1}{E^2} \varepsilon^{abc} B^b_{\mu\nu} e^c, \tag{3.3}
\]

\[
B^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + \varepsilon^{abc} \alpha_\mu^b \beta_\nu^c, \tag{3.4}
\]

\[
T^a_{\mu\nu} = (\partial_\mu e^\nu_a - \epsilon^{abc} B^b_{\mu\nu} e^c_a) - (\partial_\nu e^\mu_a + \epsilon^{abc} B^b_{\nu\mu} e^c_a). \tag{3.5}
\]

We also require that \( B^a_\mu \) and \( e^a_\mu \) satisfy

\[
\partial_\mu e^\mu_a + \epsilon^{abc} B^b_{\mu\nu} e^c = \Gamma^a_{\mu\nu} e^\rho_a. \tag{3.6}
\]

Before solving the above equation, we provide some additional information about \( e^a_\alpha \). For \( SU(2) \) gauge theory in 4D, \( e^a_\alpha \) can be regarded as a 3 \times 4 matrix, which does not an inverse in the conventional meaning. However, we can define a right-inverse matrix of \( e^a_\alpha \) as

\[
e^a_\alpha E^a_b = \delta^a_b. \tag{3.7}
\]

\( E^a_b \) can be regarded as a 4 \times 3 matrix, and Eq. (3.7) imposes 3 \times 3 constraints, so \( E^a_b \) exists but are not determined uniquely. Notice that \( E^a_b \) is only a right-inverse matrix of \( e^a_\alpha \). The quantity

\[
n^\beta_\alpha = e^a_\alpha E^\beta_a \tag{3.8}
\]

is not equal to the identity matrix but is an independent tensor. Now we define the metric-like fields

\[
g_{\alpha\beta} = e^a_\alpha e^a_\beta, \quad g^{\alpha\beta} = E^a_a E^a_b. \tag{3.9}
\]

They are the gauge invariant space-time tensors. From Eq. (3.9), we know that \( g_{\alpha\beta} \) is the product of two 4 \times 3 matrices. So \( g_{\alpha\beta} \) is not an invertible matrix, and \( g^{\alpha\beta} \) is also not invertible. From Eq. (3.7), we obtain the relation between \( g_{\alpha\beta} \) and \( g^{\alpha\beta} \)

\[
g_{\alpha\rho} g^{\rho\sigma} g_{\sigma\beta} = g_{\alpha\beta}, \quad g^{\alpha\rho} g_{\rho\sigma} g^{\sigma\beta} = g^{\alpha\beta}, \quad g_{\alpha\rho} g^{\rho\beta} = n^\beta_\alpha. \tag{3.10}
\]

The relation (3.10) can be identified with the definition of the Moore-Penrose generalized inverse matrix. So \( g^{\alpha\beta} \) can be regarded as a generalized inverse matrix of \( g_{\alpha\beta} \). From Eq. (3.7), we also have the identities

\[
g_{\alpha\rho} n^\beta_\rho = g_{\alpha\beta}, \quad g^{\alpha\rho} n^\beta_\rho = g^{\alpha\beta}, \quad n^\rho_\alpha n^\beta_\rho = n^\beta_\alpha. \tag{3.11}
\]

So \( n^\beta_\alpha \) behaves like a projection tensor. From \( e^a_\alpha \) and \( E^a_b \), we can define the gauge invariant variables

\[
\mathcal{E}_{\alpha\beta\gamma} = \epsilon_{abc} e^a_\alpha e^b_\beta e^c_\gamma, \quad \mathcal{E}^{\alpha\beta\gamma} = \epsilon^{abc} E^a_a E^b_b E^c_c. \tag{3.12}
\]

They are totally antisymmetrical space-time tensors. From Eq. (3.7), we have the projection relations similar to Eq. (3.11)

\[
\mathcal{E}_{\alpha\beta\rho} n^\rho_\gamma = \mathcal{E}_{\alpha\beta\gamma}, \quad \mathcal{E}^{\alpha\beta\rho} n^\rho_\gamma = \mathcal{E}^{\alpha\beta\gamma}. \tag{3.13}
\]

\( \mathcal{E}_{\alpha\beta\gamma} \) and \( \mathcal{E}^{\alpha\beta\gamma} \) are totally antisymmetrical tensors in 4D, so they have 4 independent components respectively. We can define

\[
V_\mu = \frac{1}{6} \mathcal{E}_{\mu\alpha\beta\gamma} \mathcal{E}^{\alpha\beta\gamma}, \quad U^\mu = \frac{1}{6} \mathcal{E}^{\mu\alpha\beta\gamma} \mathcal{E}_{\alpha\beta\gamma}. \tag{3.14}
\]
or equivalently

\[ \mathcal{E}_{\alpha\beta\gamma} = -U^\mu \epsilon_{\mu\alpha\beta\gamma}, \quad \epsilon^{\alpha\beta\gamma} = -V_\mu \epsilon_{\mu\alpha\beta\gamma}, \]

(3.15)

where \( \epsilon^{\mu\alpha\beta\gamma} \) is totally antisymmetric with the convention \( \epsilon^{0123} = 1 \), and \( \epsilon_{\mu\alpha\beta\gamma} = \eta_{\mu\nu} \eta_{\alpha\rho} \eta_{\beta\sigma} \eta_{\gamma\tau} \epsilon^{\nu\rho\sigma\tau} \).

We also have the identities

\[ g_{\alpha\beta} U^\beta = 0, \quad g^{\alpha\beta} V_\beta = 0. \]

(3.16)

Eq. (3.16) shows \( U^\mu \) is an eigenvector of \( g_{\alpha\beta} \) with the eigenvalue 0, which is an equal statement that \( g_{\alpha\beta} \) is not invertible. From Eq. (3.7), we can obtain the identities

\[ n_\rho^\alpha = 3, \quad U^\rho V_\rho = -1. \]

(3.17)

From Eq. (3.6), \( B_\mu^a \) is solved as

\[ B_\mu^a = -\frac{1}{2} \epsilon^{abc} E_b^\rho (\partial_\mu e_c^\rho - \Gamma_{\rho\mu}^\alpha e_\alpha), \]

(3.18)

and Eq. (3.6) also yields the condition

\[ \partial_\mu g_{\alpha\beta} = \Gamma_{\mu\alpha}^\rho g_{\rho\beta} + \Gamma_{\mu\beta}^\rho g_{\rho\alpha}. \]

(3.19)

Substituting Eq. (3.18) into Eq. (3.6), we obtain

\[ \partial_\mu e_\nu^a + \epsilon^{abc} B_\mu^b e_\nu^c = \frac{1}{2} \epsilon^a_\tau (\partial_\mu n_\tau^\nu + \Gamma_\mu^\tau n_\nu^\alpha) + \frac{1}{2} E_\alpha^\rho (\partial_\mu g_{\tau\nu} - \Gamma_{\mu\tau}^\rho g_{\rho\nu}). \]

(3.20)

If \( n_\nu^\tau \) is the identity matrix, then Eq. (3.19) would imply that Eq. (3.20) is consistent with Eq. (3.6) automatically. However, here \( n_\nu^\tau \) is an independent tensor, so the consistency among Eqs. (3.6), (3.19) and (3.20) require the condition

\[ \partial_\mu n_\nu^\tau + \Gamma_\mu^\tau n_\nu^\rho - \Gamma_{\mu\nu}^\rho n_\tau^\rho = 0, \]

(3.21)

which mean that \( \Gamma_{\mu\rho}^\delta \) should also compatible with \( n_\nu^\tau \). Eqs. (3.20) and (3.21) together means that \( \Gamma_{\mu\rho}^\delta \) need to be compatible with \( g^{\alpha\beta} \)

\[ \partial_\mu g^{\alpha\beta} = -\Gamma_{\mu\rho}^\delta g^{\alpha\beta} - \Gamma_{\mu\beta}^\delta g^{\alpha\rho}. \]

(3.22)

In another way, Eqs. (3.20) and (3.22) also means that Eq. (3.21) is satisfied. If \( g_{\alpha\beta} \) is invertible, then we can derive Eq. (3.22) from Eq. (3.19). However, \( g_{\alpha\beta} \) is not invertible in the present situation, so Eq. (3.22) is an independent equation to be satisfied. A connection which satisfies Eqs. (3.19) and (3.22) is given as

\[ \Gamma_{\alpha\beta}^\delta = g^{\alpha\sigma} \Gamma_{\sigma\alpha\beta} - (n_\alpha^\tau \partial_\beta n_\sigma^\rho + n_\beta^\tau \partial_\alpha n_\sigma^\rho + \frac{1}{2} n_\alpha^\tau n_\beta^\rho (\partial_\tau n_\rho^\sigma + \partial_\rho n_\tau^\sigma)) + \frac{1}{2} n_\alpha^\tau n_\beta^\rho (\partial_\tau n_\rho^\sigma - \partial_\rho n_\tau^\sigma) + \Sigma_{\alpha\beta}^\rho, \]

(3.23)

in which

\[ \Sigma_{\alpha\beta}^\rho = \frac{1}{2} (\delta_\beta^\delta - n_\beta^\delta) g^{\rho\sigma} \Gamma_{\tau,\alpha\sigma} - \frac{1}{2} (\delta_\alpha^\delta - n_\alpha^\delta) g^{\rho\sigma} \Gamma_{\tau,\beta\sigma} - \frac{1}{2} n_\alpha^\tau n_\beta^\rho (\partial_\tau n_\rho^\sigma - \partial_\rho n_\tau^\sigma), \]

(3.24)

and

\[ \Gamma_{\sigma,\alpha\beta} = \frac{1}{2} (\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}) \]

(3.25)

is the Christoffel symbols of the first kind. The detail to derive the above solution is given in appendix C. The above connection (3.23) is much more complicated than the Levi-Civita connection, which is mainly
because $g_{\alpha\beta}$ is not invertible. If we suppose $g_{\alpha\beta}$ is invertible, then $n_{\alpha}^\beta = \delta_{\alpha}^\beta$ is the identity matrix. So the corrections from $n_{\alpha}^\beta$ vanish, and the connection (3.23) is reduced to the Levi-Civita connection.

Using the above results, we can obtain

$$R^a_{\mu
u} = \frac{1}{2} R^\sigma_{\rho\mu\nu} e^a_\sigma E^0_{\rho} e^{a\ell}, \quad T^a_{\mu
u} = 2 \Sigma^a_{\mu\nu} e^a_\rho, \quad (3.26)$$

where

$$R^\sigma_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\tau_{\nu\rho} \Gamma^\sigma_{\mu\tau} - \Gamma^\tau_{\mu\rho} \Gamma^\sigma_{\nu\tau} \quad (3.27)$$

is the Riemann curvature with the connection defined by (3.23). The Lagrangian (2.1) has a natural decomposition according to the power counting of

$$L = L^{(0)} + \frac{1}{\ell} L^{(1)} + \frac{1}{\ell^2} L^{(2)} + \frac{1}{\ell^3} L^{(3)} + \frac{1}{\ell^4} L^{(4)}. \quad (3.28)$$

Here $L^{(0)}$, $L^{(2)}$ and $L^{(4)}$ are

$$L^{(0)} = \frac{1}{4} \eta^{\mu\alpha} \eta^{\nu\beta} (g^{\rho\delta} g_{\sigma\tau} - n_{\rho}^\delta n_{\sigma\tau}^\theta) R^\rho_{\mu\nu} R^\tau_{\theta\alpha\beta}, \quad (3.29)$$

$$L^{(2)} = \eta^{\mu\alpha} \eta^{\nu\beta} (n_{\rho}^\delta g_{\sigma\tau} - n_{\rho}^\beta g_{\sigma\alpha}) R^\rho_{\mu\nu} + 4 \eta^{\mu\alpha} \eta^{\nu\beta} g_{\rho\sigma} \Sigma_{\alpha\beta} \Sigma_{\tau\sigma}, \quad (3.30)$$

$$L^{(4)} = \eta^{\mu\alpha} \eta^{\nu\beta} (g_{\rho\alpha} g_{\sigma\beta} - g_{\rho\beta} g_{\sigma\alpha}). \quad (3.31)$$

They are similar to the situation in 3D, but with the identity matrix replaced by $n_{\rho}^\delta$. In 4D, the torsion tensor is not zero any more, and we also have contribution from the torsion in Eq. (3.30). $L^{(1)}$ and $L^{(3)}$ are relevant to the torsion and the totally antisymmetric tensor

$$L^{(1)} = 2 \eta^{\mu\alpha} \eta^{\nu\beta} R^\rho_{\mu\nu} g^{\rho\delta} \Sigma_{\alpha\beta} \Sigma_{\delta\tau}, \quad (3.32)$$

$$L^{(3)} = 4 \eta^{\mu\alpha} \eta^{\nu\beta} \Sigma_{\mu\alpha} \Sigma_{\tau\beta}. \quad (3.33)$$

### 3.2 Dimension Two Condensate

Similar to that in the subsection 2.2, $g_{\mu\nu}$ in this subsection is gauge invariant and has the natural interpretation as the dimension two condensate. However, an important point different from section 2 is that $g_{\mu\nu}$ is not invertible as we discussed in the subsection 3.1. Because $g_{\mu\nu}$ is not invertible, we propose a decomposition of $g_{\mu\nu}$ as

$$g_{\mu\nu} = \gamma_{\mu\nu} + V_{\mu} V_{\nu} / \gamma^{\rho\sigma} V_{\rho} V_{\sigma}, \quad (3.34)$$

where $V_{\mu}$ is defined in Eq. (3.14), and $\gamma_{\mu\nu}$ is invertible and its inverse is $\gamma^{\mu\nu}$. From Eqs. (3.14) and (3.17), we also have

$$\gamma_{\mu\tau} U^\tau + V_{\mu} / \gamma^{\rho\sigma} V_{\rho} V_{\sigma} = 0, \quad (3.35)$$

which means that

$$U^\mu = - \gamma^{\mu\tau} V_{\tau} / \gamma^{\rho\sigma} V_{\rho} V_{\sigma}, \quad (3.36)$$

so $U^\mu$ is completely determined by $V_{\mu}$. $\gamma_{\mu\nu}$ can be further decomposed as

$$\gamma_{\mu\nu} = \varphi^2(x) \eta_{\mu\nu} + h_{\mu\nu}, \quad (3.37)$$

where $\varphi^2$ is the trace part of $\gamma_{\mu\nu}$. If we suppose that $h_{\mu\nu}$ is small, then $g_{\mu\nu}$ has the approximate expression

$$g_{\mu\nu} \approx \varphi^2(x) \left( \eta_{\mu\nu} - \frac{V_{\mu} V_{\nu}}{\eta^{\rho\sigma} V_{\rho} V_{\sigma}} \right) + h_{\mu\nu}. \quad (3.38)$$
We can further define\(^6\)

\[ a_\mu = \frac{V_\mu}{\sqrt{\eta^{\alpha\beta} V_\alpha V_\beta}}. \]  

(3.39)

So that

\[ \eta^{\alpha\beta} a_\alpha a_\beta = 1. \]  

(3.40)

Now \( g_{\mu\nu} \) has the reformulation

\[ g_{\mu\nu} \approx \varphi^2(x) \left( \eta_{\mu\nu} - a_\mu a_\nu \right), \]  

(3.41)

and \( g^{\mu\nu} \) and \( n^\beta_\alpha \) are approximated by

\[ g^{\mu\nu} \approx \varphi^2(x) \left( \eta^{\mu\nu} - a^{\mu} a^{\nu} \right), \]

\[ n^\beta_\alpha \approx \delta^\beta_\alpha - a_\alpha a^\beta, \]  

(3.42)

where \( a^\mu = \eta^{\mu\sigma} a_\sigma \).

In the \( p \ll \frac{1}{\ell} \) limit, \( \mathcal{L}^{(0)} \) and \( \mathcal{L}^{(1)} \) are much smaller than \( \mathcal{L}^{(2)} \), \( \mathcal{L}^{(3)} \) and \( \mathcal{L}^{(4)} \). The Lagrangian \( \mathcal{L} \) is dominated by

\[ -4\kappa^2 \mathcal{L} \approx \frac{1}{\ell^2} \mathcal{L}^{(2)} + \frac{1}{\ell^3} \mathcal{L}^{(3)} + \frac{1}{\ell^4} \mathcal{L}^{(4)}. \]  

(3.43)

For \( g_{\mu\nu} \), we use the approximation (3.41), then Eq. (3.43) has the expression

\[ \mathcal{L} \approx \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - \frac{\kappa^2}{24} \phi^4 \]  

(3.44)

\[ - \frac{3}{4} a^\alpha a^\beta \partial_\alpha \phi \partial_\beta \phi - \frac{1}{12} (2a^\beta \partial_\alpha a^\alpha + a^\alpha \partial_\alpha a^\beta) \phi \partial_\beta \phi \]

\[ - \frac{1}{24} \phi^2 (3\partial_\alpha a^\beta \partial_\beta a^\alpha - \partial_\alpha a^\beta \partial^\alpha a_\beta + a^\alpha a^\beta \partial_\alpha a^\sigma \partial_\beta a_\sigma), \]

where \( \phi = \frac{\sqrt{E}}{\kappa \ell} \), and a total divergence term

\[ \mathcal{L}_{\text{bound}} = \partial_\alpha \left( \frac{1}{12} \phi^2 (a^{\alpha} a^{\beta} a^{\alpha} - a^{\beta} \partial^\alpha a) + \frac{1}{3} \phi (a^{\alpha} a^{\beta} \partial_\beta \phi - \partial^\alpha \phi) \right) \]  

(3.45)

has been subtracted. Eq. (3.44) is different from Eq. (2.22), which is a pure scalar theory. In Eq. (3.44), a nonzero value of the quantity

\[ (3\partial_\alpha a^\beta \partial_\beta a^\alpha - \partial_\alpha a^\beta \partial^\alpha a_\beta + a^\alpha a^\beta \partial_\alpha a^\sigma \partial_\beta a_\sigma) = -m^2 \]

(3.46)

shall yield a nonzero vacuum expectation value of \( \phi \). Inversely, when \( \phi \) takes a nonzero constant value, that is

\[ \langle \phi \rangle = v, \]

(3.47)

the torsion (3.24) and the connection (3.23) have the concise expression

\[ \Sigma^\rho_{\alpha\beta} = \frac{1}{2} a^\rho (\partial_\alpha a_\beta - \partial_\beta a_\alpha) + \frac{1}{4} (a_\alpha \partial^\rho a_\beta - a_\beta \partial^\rho a_\alpha) \]

(3.48)

\[ + \frac{1}{4} (a_\alpha \partial^\rho a_\beta - a_\beta \partial^\rho a_\alpha) - \frac{3}{4} a^\rho a^\sigma (a_\alpha \partial_\sigma a_\beta - a_\beta \partial_\sigma a_\alpha), \]

\[ \Gamma^\rho_{\alpha\beta} = \frac{1}{2} (a_\alpha \partial_\rho a_\beta - a_\beta \partial_\rho a_\alpha) + (a^\rho \partial_\alpha a_\beta - a_\beta \partial_\alpha a^\rho) \]

(3.49)

\[ + \frac{1}{2} a_\alpha a^\rho (a_\beta \partial_\rho a^\alpha - a^\alpha \partial_\rho a_\beta) + a_\beta a^\rho a^\sigma \partial_\sigma a_\alpha. \]

\[ ^6\text{Here we have supposed that } \eta^{\alpha\beta} V_\alpha V_\beta > 0. \text{ If } \eta^{\alpha\beta} V_\alpha V_\beta < 0, \text{ then we can define } a_\mu = \frac{V_\mu}{\sqrt{-\eta^{\alpha\beta} V_\alpha V_\beta}}. \]
In this case, Eq. (3.28) is the Lagrangian of the unit vector $a_\mu$, it is possible that the Lagrangian (3.28) supports nontrivial classical solution of finite energy.

Now we give comparisons of the above result to the dual superconductor scenario [40–42]. In [1], the $SU(2)$ gauge field is decomposed in terms of a unit 3-vector $n$, an Abelian field $C_\mu$ and two scalar fields $\rho(x)$ and $\sigma(x)$. The condensate of the order parameter $n$ yields the Abelian-Higgs model, which supports vortex solutions [43]. The vortex solutions are responsible for the string-like force in the large distance. In the dual view, the condensate of $\rho(x)$ and $\sigma(x)$ yields a Lagrangian of the order parameter $n$. This Lagrangian has the structure of non-linear $\sigma$ model, which supports knot-like solutions of finite energy [9]. The picture of our above result of this subsection has the similar interpretation to this dual scenario. Here $\phi^2(x)$ as the trace part of $\gamma_{\mu\nu}$, has the natural interpretation as the candidate of the dimension two condensate. While the unit vector $a_\mu$ serves as the order parameter. In the approximation of small $h_{\mu\nu}$, Eq. (3.28) is reduced to the Lagrangian about $\phi$ and $a_\mu$. From Eq. (3.46), we knew that the nonzero condensate of $a_\mu$ yields a nonzero condensate of $\phi$. If $\phi$ obtains a nonzero condensate as Eq. (3.47), the Lagrangian (3.28) is a Lagrangian about $a_\mu$. We conjecture that it could support classical solutions of finite energy.

3.3 Gauge Invariant Gluon Polarization

In 4D, the gluon angular momentum operator are similar to that in Eqs. (2.29) and (2.30) in 3D. Using the variables in the subsection 3.1, we have

$$\kappa^2 M_{g\mu}^{\nu\beta} = -\frac{1}{\ell^2} \eta^{\mu\nu}(\eta_{\beta\gamma} - \eta_{\gamma\beta}) \left(\frac{1}{2} P_{\rho\tau\theta} g^{\rho\lambda} \gamma_{\lambda\sigma} + 2 \Sigma_{\tau\theta} g_{\rho\sigma} + \frac{1}{\ell^2} \epsilon_{\tau\theta}\right).$$  \hfill (3.50)

In the $p \ll \frac{1}{\ell}$ limit, $M_{g\mu}^{\nu\beta}$ shall be dominated by

$$\kappa^2 M_{g\mu}^{\nu\beta} \approx -\frac{2}{\ell^3} \epsilon_{\tau\theta}\epsilon_{\rho\sigma}.$$

The orbital angular momentum $M_{g\mu}^{\nu\beta}$ can also be expressed by space-time tensor fields. Many results are similar to the expression in subsection 2.3. Because the torsion (2.35) is not zero, $M_{g\mu}^{\nu\beta}$ also includes the contribution from the torsion.

For the Maxwell field, it has been shown that the Laguerre-Gaussian laser modes have a well-defined angular momentum [44, 45]. The Laguerre-Gaussian laser modes are described by classical solutions of vacuum Maxwell equations. In the above, we have given the gauge invariant expressions of the Lagrangian (3.28) which have well-defined angular momentum. It is interesting to consider whether there exist classical solutions of the Lagrangian (3.28) which have well-defined angular momentum. The above expressions of $M_{g\mu}^{\nu\beta}$ and $M_{g\mu}^{\nu\beta}$ are appropriate variables to perform the computations.

4 Conclusions

We employed new variables in terms of space-time tensor fields to recast $SU(2)$ Yang-Mills theories into gravity-like formulations, which are proposed as effective descriptions of the infrared region. The gravity-like formulations include the square of Riemann curvature, which are higher derivative theories. However, there is also a natural cutoff scale $\frac{1}{\ell}$ in these formulations. In the low energy or the long wavelength limit, the higher derivative terms are negligible, and we obtain Lagrangian with second order derivative, which are tractable to analyze the dimension two condensate. In 3 dimensions, a dimension two condensate is obtained. Compared to the case in 3 dimensions, there is an additional unit 4-vector in 4 dimensions. The effective Lagrangian we obtained has the property of duality between the dimension two condensate and the unit 4-vector, which provides a novel dual property of the $SU(2)$ Yang-Mills theory in 4 dimensions. Because the induced theories are non-polynomial and are with higher derivatives, we also propose the functional renormalization group as a possible approach to define consistent quantum theories. An interesting question is whether the $SU(3)$ Yang-Mills theories can also be reformulated into gravity-like formulations. In this regard, the methods used in [47–50] could be very relevant.

\footnote{For another different expression, see [46].}
A Conventions

Because we have two metrics $\eta_{\mu\nu}$ and $g_{\mu\nu}$, the convention is that we raise up or down the tensor indices by $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. However, we claim several exceptions in this paper. $g_{\mu\nu}$ is an exception. $g^{\mu\nu}$ in section 2 is its inverse or is its generalized inverse in section 3, but is not obtained by raising up indices by $\eta^{\mu\nu}$ from $g_{\mu\nu}$. $\gamma^{\mu\nu}$ and $\varepsilon^{\alpha\beta\gamma}$ in section 3 are also exceptions. In most of cases, we give the summation manifestly to avoid confusions.

B Property of Projection Tensor

As we have mentioned in section 3, the tensors $(\delta^\alpha_\beta - n^\alpha_\beta)$ and $n^\alpha_\beta$ behave like projection tensors. Using the definition (3.8), we can prove that they have the properties

$$n^\alpha_\alpha n^\beta_\beta = n^\beta_\beta, \quad (\delta^\alpha_\beta - n^\alpha_\beta)(\delta^\beta_\beta - n^\beta_\beta) = (\delta^\beta_\beta - n^\beta_\beta), \quad n^\alpha_\beta(\delta^\beta_\beta - n^\beta_\beta) = 0. \quad (B.1)$$

Acting on the metric $g_{\alpha\beta}$, they yield

$$n^\alpha_\sigma g_{\sigma\beta} = g_{\alpha\beta}, \quad (\delta^\alpha_\sigma - n^\alpha_\sigma)g_{\sigma\beta} = 0. \quad (B.2)$$

Similarly, acting on the generalized inverse metric $g^{\alpha\beta}$, they yield

$$n^\alpha_\sigma g^{\sigma\beta} = g^{\alpha\beta}, \quad (\delta^\sigma_\beta - n^\sigma_\beta)g^{\sigma\beta} = 0. \quad (B.3)$$

$(\delta^\alpha_\beta - n^\alpha_\beta)$ and $n^\alpha_\beta$ can also be used to decompose tensors. For example, for the tensor $\Sigma^\rho_{\alpha\beta}$, we have the identity

$$\Sigma^\rho_{\alpha\beta} = n^\rho_\alpha n^\rho_\beta n^\sigma_\gamma \Sigma^\theta_{\sigma\tau} + n^\rho_\beta n^\sigma_\alpha \Sigma^\theta_{\sigma\tau} + n^\rho_\alpha \delta^\beta_\gamma n^\rho_\beta \Sigma^\theta_{\sigma\tau} + n^\rho_\alpha (\delta^\beta_\gamma - n^\beta_\gamma) n^\rho_\beta \Sigma^\theta_{\sigma\tau}$$

$$+ (\delta^\alpha_\sigma - n^\alpha_\sigma)(\delta^\sigma_\beta - n^\sigma_\beta) n^\rho_\sigma \Sigma^\theta_{\sigma\tau} + (\delta^\alpha_\sigma - n^\alpha_\sigma)(\delta^\beta_\gamma - n^\beta_\gamma) n^\rho_\sigma \Sigma^\theta_{\sigma\tau}, \quad (B.4)$$

that is, $\Sigma^\rho_{\alpha\beta}$ has 8 different parts under the projection decomposition. The above discussions are useful to find solutions of connection with metric compatibility.

C Solution of Connection in 4 Dimensions

In this appendix, we give the procedure to solve Eq. (3.19) and (3.22). We attempt to find a connection which satisfies

$$\partial_\mu g_{\alpha\beta} = \Gamma^\rho_{\mu\alpha} g_{\rho\beta} + \Gamma^\sigma_{\mu\beta} g_{\rho\sigma} \quad (C.1)$$

and

$$\partial_\mu g^{\alpha\beta} = -\Gamma^\beta_{\mu\alpha} g^{\rho\alpha} - \Gamma^\alpha_{\mu\beta} g^{\rho\beta}. \quad (C.2)$$

If $g_{\alpha\beta}$ is invertible, then we can derive Eq. (C.2) from Eq. (C.1). However, $g_{\alpha\beta}$ is not invertible here. We need to consider these two equations at the same time.

At the first step, we begin to solve Eq. (C.1). From Eq. (C.1), we have

$$\Gamma^\rho_{\sigma\alpha} = \Gamma^\rho_{(\alpha\beta)} g_{\rho\sigma} + \Sigma^\rho_{\alpha\sigma} g_{\rho\beta} + \Sigma^\rho_{\beta\sigma} g_{\rho\alpha}, \quad (C.3)$$
where \( \Gamma_{(\alpha \beta)}^\rho = \frac{1}{2}(\Gamma_{\alpha \beta}^\rho + \Gamma_{\beta \alpha}^\rho) \) is the symmetric part of the connection and \( \Sigma_{\alpha \beta}^\rho = \frac{1}{2}(\Gamma_{\alpha \beta}^\rho - \Gamma_{\beta \alpha}^\rho) \) is its antisymmetric part. \( \Gamma_{\sigma, \alpha \beta} \) has been defined in Eq. (3.25). From Eq. (C.3), we have

\[
\Gamma_{(\alpha \beta)}^\rho n_\rho^\sigma = g^{\sigma \rho} \Gamma_{\rho, \alpha \beta} - g^{\sigma \tau} (\Sigma_{\alpha \tau}^\rho g_{\rho \beta} + \Sigma_{\beta \tau}^\rho g_{\rho \alpha}).
\]  
(C.4)

Because we have the decomposition

\[
\begin{align*}
\Gamma_{\alpha \beta}^\rho &= \Gamma_{(\alpha \beta)}^\rho + \Sigma_{\alpha \beta}^\rho = \Gamma_{(\alpha \beta)}^\rho n_\rho^\sigma + \Gamma_{(\alpha \beta)}^\rho (\delta_\rho^\sigma - n_\rho^\sigma) + \Sigma_{\alpha \beta}^\rho \\
&= g^{\rho \tau} \Gamma_{\rho, \alpha \beta} - g^{\sigma \tau} (\Sigma_{\alpha \tau}^\rho g_{\rho \beta} + \Sigma_{\beta \tau}^\rho g_{\rho \alpha}) + \Gamma_{(\alpha \beta)}^\rho (\delta_\rho^\sigma - n_\rho^\sigma) + \Sigma_{\alpha \beta}^\rho.
\end{align*}
\]  
(C.5)

We have used Eq. (C.4) to replace the part \( \Gamma_{(\alpha \beta)}^\rho n_\rho^\sigma \) of \( \Gamma_{\alpha \beta}^\rho \). Substituting Eq. (C.5) into Eq. (C.1), we find that Eq. (C.1) is satisfied if

\[
(\delta_\rho^\sigma - n_\rho^\sigma) \Gamma_{\rho, \alpha \beta} = (\delta_\rho^\sigma - n_\rho^\sigma) (\Sigma_{\alpha \tau}^\rho g_{\rho \beta} + \Sigma_{\beta \tau}^\rho g_{\rho \alpha}).
\]  
(C.6)

Multiplying the two sides of Eq. (C.6) by the projection tensor \( (\delta_\rho^\sigma - n_\rho^\sigma) \), we have

\[
(\delta_\rho^\sigma - n_\rho^\sigma)(\delta_\rho^\sigma - n_\rho^\sigma) \Gamma_{\rho, \alpha \beta} = (\delta_\rho^\sigma - n_\rho^\sigma)(\delta_\rho^\sigma - n_\rho^\sigma) (\Sigma_{\alpha \tau}^\rho g_{\rho \beta} + \Sigma_{\beta \tau}^\rho g_{\rho \alpha}),
\]  
(C.7)

which means

\[
(\delta_\rho^\sigma - n_\rho^\sigma)(\delta_\rho^\sigma - n_\rho^\sigma) \Sigma_{\alpha \beta}^\rho h_\rho^\lambda = (\delta_\rho^\sigma - n_\rho^\sigma)(\delta_\rho^\sigma - n_\rho^\sigma) g^{\lambda \beta} \Gamma_{\rho, \alpha \beta}.
\]  
(C.8)

From Eq. (C.6), we also have

\[
(\delta_\rho^\sigma - n_\rho^\sigma)n_\alpha^\rho n_\beta^\sigma \Gamma_{\rho, \theta \tau} = (\delta_\rho^\sigma - n_\rho^\sigma)(n_\alpha^\rho \Sigma_{\theta \tau}^\rho g_{\rho \beta} + n_\beta^\sigma \Sigma_{\theta \tau}^\rho g_{\rho \alpha}).
\]  
(C.9)

We have the decomposition

\[
(\delta_\rho^\sigma - n_\rho^\sigma)n_\alpha^\rho n_\beta^\sigma = \frac{1}{2}(\delta_\rho^\sigma - n_\rho^\sigma)(n_\alpha^\rho \Sigma_{\theta \tau}^\rho g_{\rho \beta} + n_\beta^\sigma \Sigma_{\theta \tau}^\rho g_{\rho \alpha})
\]  
\[+ \frac{1}{2}(\delta_\rho^\sigma - n_\rho^\sigma)(n_\alpha^\rho \Sigma_{\theta \tau}^\rho g_{\rho \beta} - n_\beta^\sigma \Sigma_{\theta \tau}^\rho g_{\rho \alpha}).
\]  
(C.10)

Using Eq. (C.9) to replace the symmetrical part of Eq. (C.10), then from Eq. (C.10), we can obtain

\[
(\delta_\rho^\sigma - n_\rho^\sigma)n_\alpha^\rho n_\beta^\sigma \Gamma_{\rho, \theta \tau} = \frac{1}{2}(\delta_\rho^\sigma - n_\rho^\sigma)n_\alpha^\rho g^{\theta \tau} \Gamma_{\rho, \theta \tau} + \frac{1}{2}(\delta_\rho^\sigma - n_\rho^\sigma)(n_\alpha^\rho \Sigma_{\theta \tau}^\rho g_{\rho \beta} - g^{\beta \rho} \Sigma_{\theta \tau}^\rho g_{\rho \alpha}).
\]  
(C.11)

Eqs. (C.8) and (C.11) together yield

\[
(\delta_\rho^\sigma - n_\rho^\sigma) \Sigma_{\alpha \beta}^\rho n_\rho^\sigma = \frac{1}{2}(\delta_\rho^\sigma - n_\rho^\sigma)n_\alpha^\rho g^{\theta \tau} \Gamma_{\rho, \theta \tau} + \frac{1}{2}(\delta_\rho^\sigma - n_\rho^\sigma)(n_\alpha^\rho \Sigma_{\theta \tau}^\rho g_{\rho \beta} - g^{\beta \rho} \Sigma_{\theta \tau}^\rho g_{\rho \alpha}),
\]  
(C.12)

which further yields

\[
\Sigma_{\alpha \beta}^\rho = (\delta_\rho^\sigma - n_\rho^\sigma) \Sigma_{\alpha \beta}^\rho + (\delta_\rho^\sigma - n_\rho^\sigma) \Sigma_{\alpha \beta}^\rho + (\delta_\rho^\sigma - n_\rho^\sigma) n_\alpha^\rho \Sigma_{\theta \tau}^\rho g_{\rho \beta} + n_\beta^\sigma \Sigma_{\theta \tau}^\rho g_{\rho \alpha}
\]  
\[= (\delta_\rho^\sigma - n_\rho^\sigma)(\delta_\rho^\sigma - n_\rho^\sigma)g^{\theta \tau} \Gamma_{\rho, \theta \tau} + \frac{1}{2}(\delta_\rho^\sigma - n_\rho^\sigma)n_\alpha^\rho g^{\theta \tau} \Gamma_{\rho, \theta \tau}
\]  
\[+ \frac{1}{2}(\delta_\rho^\sigma - n_\rho^\sigma)(n_\alpha^\rho \Sigma_{\theta \tau}^\rho g_{\rho \beta} - g^{\beta \rho} \Sigma_{\theta \tau}^\rho g_{\rho \alpha}),
\]  
(C.13)

Using the expression (C.13), and the identity

\[
(\delta_\rho^\sigma - n_\rho^\sigma)(\delta_\rho^\sigma - n_\rho^\sigma)(n_\beta^\sigma - n_\beta^\rho) \Gamma_{\rho, \theta \tau} = 0,
\]  
(C.14)
we can prove that Eq. (C.6) is satisfied. $\Sigma^\sigma_{\alpha\mu}$ is antisymmetric about $\alpha$ and $\mu$, but the expression of Eq. (C.13) is not. However, multiplying the two sides of Eq. (C.13) by $(\delta^\sigma_\mu - n^\sigma_\mu)$ and $n^\theta_\alpha$, we have

\[
(\delta^\sigma_\mu - n^\sigma_\mu)n^\theta_\alpha \Sigma^\sigma_{\alpha\mu} = \frac{1}{2}(\delta^\rho_\sigma - n^\rho_\sigma)n^\theta_\alpha g^{\rho\tau}T_{\tau,\rho\theta} + (\delta^\sigma_\mu - n^\sigma_\mu)n^\theta_\alpha \Sigma^\sigma_{\alpha\mu}(\delta^\rho_\rho - n^\rho_\rho) + \frac{1}{2}(\delta^\sigma_\mu - n^\sigma_\mu)(n^\theta_\alpha \Sigma^\rho_{\theta\rho} + g^{\rho\rho}g_{\rho\rho}).
\] (C.15)

Similarly, we have

\[
(\delta^\sigma_\mu - n^\sigma_\mu)(\delta^\rho_\alpha - n^\rho_\alpha)\Sigma^\sigma_{\alpha\mu} = (\delta^\rho_\alpha - n^\rho_\alpha)(\delta^\rho_\beta - n^\rho_\beta)g^{\rho\tau}T_{\tau,\rho\theta} + (\delta^\sigma_\mu - n^\sigma_\mu)(\delta^\rho_\alpha - n^\rho_\alpha)\Sigma^\sigma_{\alpha\mu}(\delta^\rho_\rho - n^\rho_\rho). (C.17)
\]

We have decomposition

\[
\Sigma^\sigma_{\alpha\mu} = n^\theta_\alpha (\delta^\rho_\mu - n^\rho_\mu)\Sigma^\sigma_{\sigma\mu} + n^\theta_\sigma (\delta^\rho_\mu - n^\rho_\mu)\Sigma^\sigma_{\alpha\sigma} + n^\theta_\alpha n^\rho_\mu \Sigma^\sigma_{\alpha\rho} + (\delta^\rho_\alpha - n^\rho_\alpha) n^\rho_\mu \Sigma^\sigma_{\alpha\mu},
\] (C.18)

whose right side is transparently antisymmetric about $\alpha$ and $\mu$. Substituting Eqs. (C.15) and (C.17) into the above decomposition, we have

\[
\Sigma^\sigma_{\alpha\mu} = \frac{1}{2}(\delta^\sigma_\mu - n^\sigma_\mu)g^{\sigma\rho}T_{\tau,\rho\alpha} + \frac{1}{2}(\delta^\sigma_\alpha - n^\sigma_\alpha)g^{\sigma\rho}T_{\tau,\rho\mu} + \frac{1}{2}(\delta^\sigma_\mu - n^\sigma_\mu)(n^\theta_\alpha \Sigma^\rho_{\theta\rho} + g^{\rho\rho}g_{\rho\rho}).
\] (C.19)

The right side of this expression is antisymmetric about $\alpha$ and $\mu$, and we can check that $\Sigma^\sigma_{\alpha\mu}$ given by Eq. (C.19) satisfies Eq. (C.6).

The connection (C.5) with the torsion (C.19) satisfies the metric compatibility condition (C.1). At the second step, we further require that Eq. (C.2) should be satisfied. Using the projection tensor, from Eq. (C.2), we have

\[
(\delta^\rho_\sigma - n^\rho_\sigma)\Gamma^\rho_{\mu\beta}g^{\rho\beta} = -(\delta^\rho_\sigma - n^\rho_\sigma)\partial_\mu g^{\rho\beta},
\] (C.20)

which further yields

\[
(\delta^\rho_\sigma - n^\rho_\sigma)\Gamma^\rho_{\mu\beta}n^\beta_\beta = -n^\beta_\beta \partial_\mu n^\sigma_\sigma,
\] (C.21)

which is equivalent to

\[
(\delta^\rho_\sigma - n^\rho_\sigma)\Gamma^\rho_{(\mu\beta)}n^\beta_\beta = -(\delta^\rho_\sigma - n^\rho_\sigma)\Sigma^\rho_{\mu\beta}n^\beta_\beta - n^\rho_\beta \partial_\mu n^\sigma_\sigma.
\] (C.22)

we can derive Eq. (C.2) from Eqs. (C.1) and (C.21). So Eq. (C.2) provides a new constraint (C.21) for the connection (C.5). From Eqs. (C.21) and (C.22), we can obtain

\[
n^\rho_\sigma n^\beta_\beta \Gamma^\rho_{\tau\sigma}(\delta^\rho_\rho - n^\rho_\rho) = -\frac{1}{2} n^\rho_\sigma n^\beta_\beta (\partial_\tau n^\rho_\rho + \partial_\rho n^\beta_\sigma),
\] (C.23)

\[
n^\rho_\sigma n^\beta_\beta \Sigma^\rho_{\tau\sigma}(\delta^\rho_\rho - n^\rho_\rho) = -\frac{1}{2} n^\rho_\sigma n^\beta_\beta (\partial_\tau n^\rho_\rho - \partial_\rho n^\beta_\sigma),
\] (C.24)

\[
(\delta^\rho_\rho - n^\rho_\rho)\Gamma^\rho_{(\tau\theta)}(\delta^\rho_\rho - n^\rho_\rho) = -\frac{1}{2} n^\rho_\rho n^\beta_\beta \Gamma^\rho_{\tau\theta}(\delta^\rho_\rho - n^\rho_\rho) = -(\delta^\rho_\rho - n^\rho_\rho)n^\beta_\beta \Sigma^\rho_{\tau\theta}(\delta^\rho_\rho - n^\rho_\rho) = -n^\beta_\beta \partial_\tau n^\rho_\rho.
\] (C.25)

Inversely, Eqs. (C.23), (C.24) and (C.25) make Eq. (C.2) satisfied. We have

\[
\Gamma^\rho_{\alpha\mu}(\delta^\rho_\rho - n^\rho_\rho) = n^\mu_\alpha \Gamma^\rho_{\alpha\tau}(\delta^\rho_\rho - n^\rho_\rho) + n^\rho_\alpha \delta^\rho_\mu - n^\rho_\mu \delta^\rho_\rho + n^\rho_\alpha \delta^\rho_\rho = -n^\rho_\alpha \partial_\rho n^\beta_\beta - n^\rho_\alpha \partial_\rho n^\beta_\rho + n^\rho_\alpha \delta^\rho_\rho + (\delta^\rho_\rho - n^\rho_\rho)n^\beta_\beta \Gamma^\rho_{\tau\theta}(\delta^\rho_\rho - n^\rho_\rho) + (\delta^\rho_\rho - n^\rho_\rho)n^\beta_\beta \Sigma^\rho_{\tau\theta}(\delta^\rho_\rho - n^\rho_\rho) + (\delta^\rho_\rho - n^\rho_\rho)\Gamma^\rho_{\tau\theta}(\delta^\rho_\rho - n^\rho_\rho),
\] (26)
where we have used Eqs. (C.22) and (C.25). Using Eq. (C.24), \( n^\alpha_\mu n^\beta_\alpha g^\rho_\gamma (\delta^\rho_\gamma - n^\rho_\gamma) \) and \( n^\alpha_\mu n^\beta_\alpha n^\gamma_\rho \) can be decomposed as
\[
n^\alpha_\mu n^\beta_\alpha g^\rho_\gamma (\delta^\rho_\gamma - n^\rho_\gamma) = n^\alpha_\mu n^\beta_\alpha g^\rho_\gamma (\delta^\rho_\gamma - n^\rho_\gamma) + (\delta^\alpha_\mu - n^\alpha_\mu) n^\beta_\alpha g^\rho_\gamma (\delta^\rho_\gamma - n^\rho_\gamma),
\]
\[
= -\frac{1}{2} n^\alpha_\mu n^\beta_\alpha (\partial_\gamma n^\rho_\gamma - \partial_\gamma n^\rho_\gamma) + (\delta^\alpha_\mu - n^\alpha_\mu) n^\beta_\alpha g^\rho_\gamma (\delta^\rho_\gamma - n^\rho_\gamma),
\]
(C.27)
\[
n^\alpha_\mu n^\beta_\alpha g^\rho_\gamma (\delta^\rho_\gamma - n^\rho_\gamma) = n^\alpha_\mu n^\beta_\alpha g^\rho_\gamma (\delta^\rho_\gamma - n^\rho_\gamma) + n^\alpha_\mu n^\beta_\alpha g^\rho_\gamma n^\rho_\gamma
\]
\[
= -\frac{1}{2} n^\alpha_\mu n^\beta_\alpha (\partial_\gamma n^\rho_\gamma - \partial_\gamma n^\rho_\gamma) + n^\alpha_\mu n^\beta_\alpha g^\rho_\gamma n^\rho_\gamma.
\]
(C.28)

Substituting Eq. (C.27) into (C.26), we have
\[
\Gamma^\gamma_\mu_\nu (\delta^\rho_\gamma - n^\rho_\gamma) = - (n^\rho_\mu \partial_\alpha n^\rho_\alpha + n^\rho_\mu \partial_\mu n^\rho_\alpha) + \frac{1}{2} n^\alpha_\mu n^\beta_\alpha (\partial_\gamma n^\rho_\gamma - \partial_\gamma n^\rho_\gamma)
\]
\[
- (\delta^\gamma_\mu - n^\gamma_\mu) n^\beta_\alpha g^\rho_\gamma (\delta^\rho_\gamma - n^\rho_\gamma) - (\delta^\gamma_\mu - n^\gamma_\mu) n^\beta_\alpha g^\rho_\gamma (\delta^\rho_\gamma - n^\rho_\gamma)
\]
\[
+ (\delta^\gamma_\mu - n^\gamma_\mu) (\delta^\gamma_\mu - n^\gamma_\mu) \Gamma^\rho_\alpha_\beta (\delta^\rho_\alpha - n^\rho_\alpha).
\]
(C.29)

Substituting Eq. (C.28) into (C.19), we have
\[
\Sigma^\sigma_\alpha_\mu = \frac{1}{2} (\delta^\sigma_\gamma - n^\sigma_\gamma) g^\rho_\sigma \Gamma_\tau_\rho_\sigma - \frac{1}{2} (\delta^\sigma_\gamma - n^\sigma_\gamma) g^\rho_\sigma \Gamma_\sigma_\nu_\rho - \frac{1}{2} n^\sigma_\nu n^\rho_\gamma (\partial_\gamma n^\rho_\gamma - \partial_\gamma n^\rho_\gamma)
\]
\[
+ \frac{1}{2} (\delta^\sigma_\gamma - n^\sigma_\gamma) n^\gamma_\sigma \delta^\rho_\gamma - g^\sigma_\rho \delta^\rho_\gamma, \gamma_\rho_\nu - \frac{1}{2} (\delta^\sigma_\gamma - n^\sigma_\gamma) n^\gamma_\sigma \delta^\rho_\gamma, \gamma_\rho_\nu
\]
\[
+ (\delta^\gamma_\mu - n^\gamma_\mu) n^\beta_\sigma, \gamma_\rho_\delta_\sigma - \delta^\gamma_\mu (\delta^\rho_\delta - n^\rho_\delta)
\]
\[
+ (\delta^\gamma_\mu - n^\gamma_\mu) (\delta^\rho_\delta - n^\rho_\delta) \Sigma^\sigma_\alpha_\beta (\delta^\rho_\alpha - n^\rho_\alpha).
\]
(C.30)

Substituting (C.29) and (C.30) into Eq. (C.5), we then obtain the connection \( \Gamma^\alpha_\mu_\nu \) which satisfies both Eqs. (C.1) and (C.2). The connection \( \Gamma^\alpha_\mu_\nu \) with the expressions (C.29) and (C.30) is the most general solution of Eqs. (C.1) and (C.2). When \( n^\rho_\alpha = \delta^\rho_\alpha \), this solution reduces to the conventional metric-compatible connection with torsion.

We give some comments on our above discussions. The basic method which we used to solve Eqs. (C.1) and (C.2) is: At first we use the projection tensor \( n^\alpha_\mu \) and \( (\delta^\rho_\delta - n^\rho_\delta) \) to decompose the connection and the torsion, such as, the torsion is decomposed into 8 different parts in Eq. (B.4): Then some parts can be determined from Eqs. (C.1) and (C.2), and the unconstrained parts are left free. Using this method, we were able to find the most general solution of Eqs. (C.1) and (C.2). If we set the unconstrained part to be zero, then we obtain the special solutions in Eqs. (3.24) and (3.25) in subsection 3.1.

There were relevant discussions to find the metric-compatible connection for a degenerate metric. For a different definition of the metric-compatibility condition and the solution of connection, see [51]. For a review about the degenerate metric from the mathematical aspect, see [52]. For a recent discussion of the degenerate metric related to the Galilean symmetry, see [53].

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