Analysis of Hilfer Fractional Integro-Differential Equations with Almost Sectorial Operators

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Abstract: We investigate a class of nonlocal integro-differential equations involving Hilfer fractional derivatives and almost sectorial operators. We prove existence results by applying Schauder’s fixed point technique. Moreover, we show fundamental properties of the solution representation by discussing two cases related to the associated semigroup. For that, we consider compactness and noncompactness properties, respectively. Furthermore, an example is given to illustrate the obtained theory.

Keywords: Hilfer fractional derivatives; mild solutions; almost sectorial operators; measure of non-compactness.

MSC: 26A33; 47B12

1. Introduction

We consider nonlocal integro-differential equations involving Hilfer fractional derivatives and almost sectorial operators:

\[ D^{\alpha,\gamma}_{0+} u(t) + A u(t) = g(t, u(t), \int_0^t k(t, s)f(s, u(s))ds), \quad t \in (0, T] = J, \]  \(1\)

\[ I^{(1-\alpha)(1-\gamma)}_{0+} [u(t)]_{t=0} + h(u(t)) = u_0, \]  \(2\)

where \(D^{\alpha,\gamma}_{0+}\) is the Hilfer fractional derivative of order \(\alpha \in (0, 1)\) and type \(\gamma \in [0, 1]\). We assume that \(A\) is an almost sectorial operator on a Banach space \(\mathcal{Y}\) with norm \(\| \cdot \|\). Let \(f : J \times \mathcal{Y} \to \mathcal{Y}, g : J \times \mathcal{Y}^2 \to \mathcal{Y}\) and \(h : C(J, \mathcal{Y}) \to \mathcal{Y}\) be given abstract functions, to be specified later. For brevity, we take

\[ Bu(t) = \int_0^t k(t, s)f(s, u(s))ds. \]

During the last decades, mathematical modeling has been supported by the field of fractional calculus, with several successful results and fractional operators showing to be an excellent tool to describe the hereditary properties of various materials and processes. Recently, this combination has gained a lot of importance, mainly because fractional differential equations have become powerful tools.
in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering, see, for instance, the basic text books [1–4] and recent researches [5–7]. In fact, abrupt changes, such as shocks, harvesting, or natural disasters, may happen in the dynamics of evolving processes. These short-term perturbations are often treated in the form of impulses. Recently, in so many published works, Hilfer fractional differential equations have received attention [8–17].

In [15], Jaiswal and Bahuguna study equations of Hilfer fractional derivatives with almost sectorial operators in the abstract sense:

\[ D_0^{\lambda,\nu}u(t) + Au(t) = g(t, u(t)), \quad t \in (0, T], \]

\[ T_0^{(1-\lambda)(1-\nu)}u(0) = u_0. \]

We also refer to the work in [8], where Ahmed et al. study the question of existence for nonlinear Hilfer fractional differential equations with controls. Sufficient conditions are also established, where the time fractional derivative is the Hilfer derivative. In [18], Zhang and Zhou study fractional Cauchy problems with almost sectorial operators of the form

\[ (L D_0^{q}x)(t) = Ax(t) + f(t, x(t)), \quad \text{for almost all } t \in [0, a], \]

\[ (I_0^{1-q}x)(0) = x_0, \]

where \( L D_0^{q} \) is the Riemann–Liouville derivative of order \( q \), \( I_0^{1-q} \) is the Riemann–Liouville integral of order \( 1 - q \), \( 0 < q < 1 \), \( A \) is an almost sectorial operator on a complex Banach space, and \( f \) is a given function. Motivated by these results, here we extend the previous available results of the literature to a class of Hilfer fractional integro-differential equations in which the closed operator is almost sectorial. Moreover, we also consider both compactness and noncompactness cases of the semigroup operator.

The paper is structured as follows. In Section 2, we present necessary information about the Hilfer derivative, almost sectorial operators, measure of non-compactness, mild solutions of equations (1)–(2), along with some useful definitions, results, and lemmas. We discuss fundamental results for mild solutions for the equations (1)–(2) in Section 3. In Section 4, we prove the solvability question in two cases, when associated semigroup is compact and noncompact, respectively. An example is then given in Section 5, to illustrate our main results. We end with Section 6 of conclusions.

2. Preliminaries

In this section we recall some necessary theory that will be used throughout the work in order to get the new results.

2.1. Fractional derivatives

We start by a short introduction of the main definitions in fractional calculus [13,19].

**Definition 1.** The left-sided Riemann–Liouville fractional integral of order \( \alpha > 0 \) with lower limit \( a \) for a function \( h : [a, +\infty) \to \mathbb{R} \) is defined as

\[ I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds, \]

provided the right hand side is defined a.e. on \([a, +\infty)\).
Remark 1. If $a = 0$, then we write $L_D^0 f(t) = (g_0 * f)(t)$, where

$$g_0(t) := \begin{cases} \frac{1}{\Gamma(\alpha) t_0^{\alpha-1}}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and, as usual, $*$ denotes the convolution of functions. Note that $\lim_{\alpha \to 0^+} g_0(t) = \delta(t)$ with $\delta$ the delta Dirac function.

Definition 2. The left-sided Riemann–Liouville fractional derivative of order $\alpha > 0$, $n - 1 \leq \alpha < n$, $n \in \mathbb{N}$, for a function $h : [a, +\infty) \to \mathbb{R}$, is defined by

$$L_D^a, h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{h(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > a,$$

provided the right hand side is defined a.e. on $[a, +\infty)$.

Definition 3. The left-sided Caputo’s fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$, for a function $h : [a, +\infty) \to \mathbb{R}$, is defined by

$$C_D^a, h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{h^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I_{a^+}^{n-\alpha} h^{(n)}(t), \quad t > a,$$

provided the right hand side is defined a.e. on $[a, +\infty)$.

Definition 4. The left-sided Hilfer fractional derivative of order $0 < \alpha < 1$ and type $\gamma \in [0, 1]$, of a function $h : [a, +\infty) \to \mathbb{R}$, is defined as

$$D_a^{\alpha, \gamma} h(t) = \left[ I_{a^+}^{1-\alpha)\gamma} D \left( I_{a^+}^{1-\alpha(1-\gamma)} h \right) \right](t).$$

Remark 2. (i) If $\gamma = 0$, $0 < \alpha < 1$, and $a = 0$, then the Hilfer fractional derivative corresponds to the classical Riemann–Liouville fractional derivative:

$$D_0^a, h(t) = \frac{d}{dt} I_{0^+}^{1-\alpha} h(t) = L_D^a, h(t).$$

(ii) If $\gamma = 1$, $0 < \alpha < 1$, and $a = 0$, then the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative:

$$D_0^a, h(t) = I_{0^+}^{1-\alpha} \frac{d}{dt} h(t) = C_D^a, h(t).$$

2.2. Measure of non-compactness

The motivation to consider our problem can be found in [18,19]. Here we generalize the results in [18,19]. Let $\mathcal{L} \subset \mathcal{Y}$ be bounded. The Hausdorff measure of non-compactness is considered as

$$\Theta(\mathcal{L}) = \inf \left\{ \theta > 0 \text{ such that } \mathcal{L} \subset \bigcup_{j=1}^m B_\theta(x_j), \quad \text{where} \quad x_j \in \mathcal{Y}, m \in \mathbb{N} \right\}$$
while the Kurtawoski measure of noncompactness $\Phi$ on a bounded set $B \subset Y$ is given by

$$
\Phi(L) = \inf \left\{ e > 0 \text{ such that } L \subset \bigcup_{j=1}^{m} M_j \text{ and } \text{diam}(M_j) \leq e \right\}
$$

with the following properties:

1. $L_1 \subset L_2$ gives $\Theta(L_1) \leq \Theta(L_2)$, where $L_1$, $L_2$ are bounded subsets of $Y$;
2. $\Theta(L) = 0$ if and only if $L$ is relatively compact in $Y$;
3. $\Theta(z \cup L) = \Theta(L)$ for all $z \in Y$ $L \subseteq Y$;
4. $\Theta(L_1 \cup L_2) \leq \max\{\Theta(L_1), \Theta(L_2)\}$;
5. $\Theta(L_1 + L_2) \leq \Theta(L_1) + \Theta(L_2)$;
6. $\Theta(rL) \leq |r|\Theta(L)$ for $r \in \mathbb{R}$.

Let $M \subset C(I, Y)$ and $M(r) = \{v(r) \in Y | v \in M\}$. One defines

$$
\int_0^t M(r)dr := \left\{ \int_0^t v(r)dr | v \in M \right\}, \quad t \in \mathcal{J}.
$$

**Proposition 1.** If $M \subset C(\mathcal{J}, Y)$ is equicontinuous and bounded, then $t \rightarrow \Theta(M(t))$ is continuous on $I$. Also,

$$
\Theta(M) = \max \left\{ \Theta(M(t)), \Theta \left( \int_0^t v(r)dr \right) \right\} \leq \int_0^t \Theta(v(r))dr, \quad \text{for } t \in I.
$$

**Proposition 2.** Let $\{v_n : \mathcal{J} \rightarrow Y, n \in \mathbb{N}\}$ be Bochner integrable functions. This implies that $\|v_n\| \leq m(t)$ a.e. for $n \in \mathbb{N}$ and $m \in L^1(I, R^+)$. Then, $\xi(t) = \Theta \left( \{v_n(t)\}_{n=1}^{\infty} \right) \in L^1(I, R^+)$ and satisfies

$$
\Theta \left( \left\{ \int_0^t v_n(r)dr : n \in \mathbb{N} \right\} \right) \leq 2 \int_0^t \xi(r)dr.
$$

**Proposition 3.** Let $M$ be a bounded set. Then, for any $\theta > 0$, there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset M$ such that

$$
\Theta(M) \leq 2\Theta \{v_n\}_{n=1}^{\infty} + \theta.
$$

### 2.3. Almost sectorial operators

Let $0 < \beta < \pi$ and $-1 < \beta < 0$. We define $S_{\beta}^0 := \{v \in C \setminus \{0\} \text{ such that } |\arg v| < \beta\}$ and its closure by $S_\beta$, such that $S_\beta = \{v \in C \setminus \{0\} \text{ with } |\arg v| \leq \beta\} \cup \{0\}$.

**Definition 5 (See [20]).** For $-1 < \beta < 0$ and $0 < \omega < \frac{\pi}{2}$, we define $\Theta_\omega^\beta$ as the family of all closed and linear operators $A : D(A) \subset Y \rightarrow Y$ such that

1. $\sigma(A)$ is contained in $S_{\omega}$;
2. for all $\beta \in (\omega, \pi)$ there exists $M_\beta$ such that

$$
\|R(z, A)\|_{L(Y)} \leq M_\beta |z|^\beta,
$$

where $R(z, A) = (zI - A)^{-1}$ is the resolvent operator and $A \in \Theta_\omega^\beta$ is said to be an almost sectorial operator on $Y$.

**Proposition 4 (See [20]).** Let $A \in \Theta_\omega^\beta$ for $-1 < \beta < 0$ and $0 < \omega < \frac{\pi}{2}$. Then the following properties hold:
1. $Q(t)$ is analytic and $\frac{d^n}{dt^n}Q(t) = (-A^nQ(t))(t \in S^0_\omega)$;
2. $Q(t+s) = Q(t)Q(s) \forall t,s \in S^0_\omega$;
3. $\|Q(t)\| \leq C_0t^{-\beta-1}(t > 0)$, where $C_0 = C_0(\beta) > 0$ is a constant;
4. If $\sum Q = \{x \in \mathcal{Y} : \lim_{t \to 0^+} Q(t) = x\}$, then $D(A^\theta) \subset \sum Q$ if $\theta > 1 + \beta$;
5. $R(r,A) = \int_0^\infty e^{-rt}Q(s)ds$, $r \in \mathbb{C}$ with $\text{Re}(r) > 0$.

We use the following Wright-type function [19]:

$$M_\alpha(\theta) = \sum_{n \in \mathbb{N}} \frac{(-\theta)^{n-1}}{\Gamma(1-\alpha n)(n-1)!}, \quad \theta \in \mathbb{C}.$$ 

For $-1 < \sigma < \infty$, $r > 0$, the following properties hold:

(A1) $M_\alpha(\theta) \geq 0$, $t > 0$;
(A2) $\int_0^\infty \theta^r M_\alpha d\theta = \frac{\Gamma(1+\sigma)}{\Gamma(1+\alpha r)}$;
(A3) $\int_0^\infty \frac{\alpha t^{\sigma}}{\theta^r} e^{-\theta t} M_\alpha(\frac{\alpha t}{\theta})d\theta = e^{-rt}$.

The characteristic operators $\{S_\alpha(t)\}_{t \in S^0_\omega}$ and $\{T_\alpha(t)\}_{t \in S^0_\omega}$ are defined by

$$S_\alpha(t) := \int_0^\infty M_\alpha(\theta)Q(t^\alpha \theta)d\theta$$

and

$$T_\alpha(t) := \int_0^\infty \alpha \theta M_\alpha(\theta)Q(t^\alpha \theta)d\theta.$$

**Theorem 1** (See Theorem 4.6.1 of [19]). For each fixed $t \in S^0_\omega$, $S_\alpha(t)$ and $T_\alpha(t)$ are bounded linear operators on $\mathcal{Y}$. Moreover,

$$\|S_\alpha(t)\| \leq C_1 t^{-\alpha(1+\beta)} , \quad \|T_\alpha(t)\| \leq C_2 t^{-\alpha(1+\beta)}, \quad t > 0,$$

where $C_1$ and $C_2$ are constants dependent on $\alpha$ and $\beta$.

**Theorem 2** (See [19]). The operators $S_\alpha(t)$ and $T_\alpha(t)$ are continuous in the uniform operator topology, for $t > 0$. For $s > 0$, the continuity is uniform on $[s, \infty]$.

Define $\Omega_r(\mathcal{J}) := \{y \in C(\mathcal{J}, \mathcal{Y}) | \|y\| \leq r\}$. Our main results are proved under the following hypotheses:

(H1) For $t \in \mathcal{J}$, $g(t, \cdot) : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ and $f(t, \cdot) : \mathcal{Y} \to \mathcal{Y}$ are continuous functions and, for each $u \in C(\mathcal{J}, \mathcal{Y}), g(\cdot, u, Bu) : \mathcal{J} \to \mathcal{Y}$ and $f(\cdot, u) : \mathcal{J} \to \mathcal{Y}$ are strongly measurable.

(H2) There exist functions $k_1, k_2 \in L^1(\mathcal{J}, \mathbb{R}^+)$ satisfying $\|g(t, \cdot)\| \leq k_1(t) + k_2(t)e^{-\delta t}$ for all $u \in \Omega_r(\mathcal{J})$ and almost all $t$ on $\mathcal{J}$ and

$$\int_0^\infty [k_1(t) + k_2(t)e^{-\delta t}] \in C(\mathcal{J}, \mathbb{R}), \quad \lim_{t \to 0^+} t^{1+\alpha \beta}(1-\gamma) \int_0^\infty [k_1(t) + k_2(t)e^{-\delta t}] = 0.$$

(H3) Function $h : C(\mathcal{J}, \mathcal{Y}) \to \mathcal{Y}$ is completely continuous and there exists a positive constant $k$ such that $\|h(u)\| \leq k$. 

(H4) We assume that
\[
\sup_{t \in [0,T]} (t^{1+\alpha\beta}(1-\gamma)\|S_{a,\gamma}(t)[u_0 + k]\| + t^{1+\alpha\beta}(1-\gamma) \int_0^t (t-r)^{-\alpha\beta-1}[k_1(r) + k_2(r)e^{-\delta\gamma}]dr) \leq r,
\]
for \( r > 0, u_0 \in D(\mathcal{A}^\theta), \) and \( \theta > 1 + \beta, \) where \( S_{a,\gamma}(t) = \int_0^t t^{a-1}T_a(t). \)

For the next two lemmas, we refer to \([9,12]\).

**Lemma 1** (See \([9,12]\)). The fractional Cauchy problem (1)–(2) is equivalent to the integral equation
\[
u(t) = \frac{[u_0 - h(u(t))]}{\Gamma(\gamma(1-\alpha) + \alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^1 (t-r)^{a-1}[-\mathcal{A}u(r) + g(r,u(r),(Bu)r)]dr, \quad t \in \mathcal{J}. \tag{3}
\]

**Lemma 2** (See \([9,12]\)). If \( u \) is a solution to the integral equation (3), then it satisfies
\[
u(t) = S_{a,\gamma}(t)[u_0 - h(u(t))] + \int_0^t R_a(t-r)g(r,u(r),(Bu)r)dr,
\]
where \( S_{a,\gamma}(t) = \int_0^t t^{a-1}R_a(t) \) with \( R_a = t^a T_a(t). \)

**Definition 6.** By a mild solution of the Cauchy problem (1)–(2), we mean a function \( u \in C(\mathcal{J},\mathcal{Y}) \) that satisfies
\[
u(t) = S_{a,\gamma}(t)[u_0 - h(u(t))] + \int_0^t R_a(t-r)g(r,u(r),(Bu)r)dr, \quad t \in \mathcal{J}.
\]

We define operator \( \mathcal{P} : \Omega_{c}(\mathcal{J}) \rightarrow \Omega_{c}(\mathcal{J}) \) as
\[
(\mathcal{P}u)(t) := S_{a,\gamma}(t)[u_0 - h(u(t))] + \int_0^t (t-r)^{a-1}T_a(t-r)g(r,u(r),(Bu)r)dr.
\]

**Lemma 3** (See \([15]\)). The operators \( R_a(t) \) and \( S_{a,\gamma}(t) \) are bounded linear operators on \( \mathcal{Y} \) for every fixed \( t \in S_{\frac{\pi}{2}}^0. \)

Also, for \( t > 0, \) we have
\[
\|R_a(t)x\| \leq C_2 t^{1-\alpha\beta}\|x\|, \quad \|S_{a,\gamma}(t)x\| \leq \frac{\Gamma\left(-\alpha\beta\right)}{\Gamma(\gamma(1-\alpha) - \alpha\beta)} C_2 t^{\gamma(1-\alpha)-\alpha\beta-1}\|x\|.
\]

**Proposition 5** (See \([15]\)). The operators \( R_a(t) \) and \( S_{a,\gamma}(t) \) are strongly continuous for \( t > 0. \)

3. Auxiliary results

Now, we are in position to start our original contributions.

**Theorem 3.** Let \( A \in \Theta_{\omega}^\theta \) for \(-1 < \beta < 0 \) and \( 0 < \omega < \frac{\pi}{2}. \) Assuming that (H1)–(H4) are satisfied, then the operator \( \{\mathcal{P}y : y \in \Omega_{c}(\mathcal{J})\} \) is equicontinuous, provided \( u_0 \in D(\mathcal{A}^\theta) \) with \( \theta > 1 + \beta. \)

**Proof.** For \( y \in \Omega_{c}(\mathcal{J}) \) and \( t_1 = 0 < t_2 \leq T, \) we have
\[
\|\mathcal{P}y(t_2) - \mathcal{P}y(0)\| = \left\| \left[ t_2^{1+\alpha\beta}(1-\gamma)\left(S_{a,\gamma}(t_2)[u_0 - h(u(t))] + \int_0^{t_2} (t_2-r)^{a-1}T_a(t_2-r)g(r,u(r),(Bu)r)dr \right) \right] \right\|
\]

Then, by using \( t_1, t_2 \rightarrow 0 \), as \( t_2 \rightarrow 0 \),

\[
\left\| \left\| t_2^{(1+\alpha\beta)(1-\gamma)} S_{\alpha,\gamma}(t_2) \right\| (u_0 + k) \right. \\
+ \left. t_2^{(1+\alpha\beta)(1-\gamma)} \int_0^{t_2} (t_2 - r)^{\alpha-1} T_a(t_2 - r) g(r, u(r), (Bu)r) dr \right\| \rightarrow 0,
\]

Now, let \( 0 < t_1 < t_2 \leq T \). One has

\[
\left\| \mathcal{P} y(t_2) - \mathcal{P} y(t_1) \right\| \leq \left\| t_2^{(1+\alpha\beta)(1-\gamma)} S_{\alpha,\gamma}(t_2)[u_0 - h(u(t))] - t_1^{(1+\alpha\beta)(1-\gamma)} S_{\alpha,\gamma}(t_1)[u_0 - h(u(t))] \right\| \\
+ \left\| t_2^{(1+\alpha\beta)(1-\gamma)} \int_0^{t_2} (t_2 - r)^{\alpha-1} T_a(t_2 - r) g(r, u(r), (Bu)r) dr \right\| \\
- \left\| t_1^{(1+\alpha\beta)(1-\gamma)} \int_0^{t_1} (t_1 - r)^{\alpha-1} T_a(t_1 - r) g(r, u(r), (Bu)r) dr \right\|.
\]

Using the triangle inequality, we get

\[
\left\| \mathcal{P} y(t_2) - \mathcal{P} y(t_1) \right\| \leq \left\| t_2^{(1+\alpha\beta)(1-\gamma)} S_{\alpha,\gamma}(t_2)[u_0 - h(u(t))] - t_1^{(1+\alpha\beta)(1-\gamma)} S_{\alpha,\gamma}(t_1)[u_0 - h(u(t))] \right\| \\
+ \left\| t_2^{(1+\alpha\beta)(1-\gamma)} \int_0^{t_2} (t_2 - r)^{\alpha-1} T_a(t_2 - r) g(r, u(r), (Bu)r) dr \right\| \\
- \left\| t_1^{(1+\alpha\beta)(1-\gamma)} \int_0^{t_1} (t_1 - r)^{\alpha-1} T_a(t_1 - r) g(r, u(r), (Bu)r) dr \right\| \\
+ \left\| t_1^{(1+\alpha\beta)(1-\gamma)} \int_0^{t_1} (t_1 - r)^{\alpha-1} T_a(t_1 - r) g(r, u(r), (Bu)r) dr \right\| =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4.
\]

From the strong continuity of \( S_{\alpha,\gamma}(t) \), we have \( \mathcal{J}_1 \rightarrow 0 \) as \( t_2 \rightarrow t_1 \). Also,

\[
\mathcal{J}_2 \leq C_2 \int_0^{t_1} (t_2 - r)^{-\alpha - \beta - 1} [k_1(r) + k_2(r)e^{-\delta r}] dr \\
\leq C_2 \int_0^{t_1} (t_2 - r)^{-\alpha - \beta - 1} [k_1(r) + k_2(r)e^{-\delta r}] dr \\
- \int_0^{t_1} (t_1 - r)^{-\alpha - \beta - 1} [k_1(r) + k_2(r)e^{-\delta r}] dr \\
\leq C_2 \int_0^{t_1} \left[ t_1^{(1+\alpha\beta)(1-\gamma)} (t_1 - r)^{-\alpha - \beta - 1} - t_2^{(1+\alpha\beta)(1-\gamma)} (t_2 - r)^{-\alpha - \beta - 1} \right] [k_1(r) + k_2(r)e^{-\delta r}] dr.
\]

Then, by using \((H2)\) and the dominated convergence theorem, \( \mathcal{J}_3 \rightarrow 0 \) as \( t_2 \rightarrow t_1 \). Since

\[
\mathcal{J}_3 \leq C_2 \int_0^{t_1} (t_2 - r)^{-\alpha - \beta} [t_2^{(1+\alpha\beta)(1-\gamma)} (t_2 - r)^{\alpha-1} - t_1^{(1+\alpha\beta)(1-\gamma)} (t_1 - r)^{\alpha-1}] [k_1(r) + k_2(r)e^{-\delta r}] dr,
\]

\[
\leq C_2 \int_0^{t_1} (t_2 - r)^{\alpha-1} [k_1(r) + k_2(r)e^{-\delta r}] + t_1^{(1+\alpha\beta)(1-\gamma)} (t_1 - r)^{\alpha-1} [k_1(r) + k_2(r)e^{-\delta r}] \\
\leq 2 t_1^{(1+\alpha\beta)(1-\gamma)} (t_1 - r)^{\alpha-1} [k_1(r) + k_2(r)e^{-\delta r}]
\]

\[
\leq 2 t_1^{(1+\alpha\beta)(1-\gamma)} (t_1 - r)^{\alpha-1} \left( [k_1(r) + k_2(r)e^{-\delta r}] + [k_1(r) + k_2(r)e^{-\delta r}] \right)
\]
and \( \int_0^{t_1} 2 t_1^{1+\alpha \beta}(1-\gamma)(t_1 - r)^{a-1}[k_1(r) + k_2(r)e^{-\delta r}]dr \) exists, we obtain \( I_3 \to 0 \) as \( t_2 \to t_1 \). For \( \epsilon > 0 \), we have

\[
\mathcal{J}_4 = \left\| \int_0^{t_1} t_1^{1+\alpha \beta}(1-\gamma) [T_a(t_2 - r) - T_a(t_1 - r)](t_1 - r)^{a-1} g(r, u(r), (Bu)(r))dr \right\| \\
\leq \int_0^{t_1} t_1^{1+\alpha \beta}(1-\gamma) \left\| T_a(t_2 - r) - T_a(t_1 - r) \right\|_{L(\mathcal{Y})} (t_1 - r)^{a-1}[k_1(r) + k_2(r)e^{-\delta r}] \\
+ \int_0^{t_1} t_1^{1+\alpha \beta}(1-\gamma) \left\| T_a(t_2 - r) - T_a(t_1 - r) \right\|_{L(\mathcal{Y})} (t_1 - r)^{a-1}[k_1(r) + k_2(r)e^{-\delta r}] \\
\leq t_1^{1+\alpha \beta}(1-\gamma) \int_0^{t_1} (t_1 - r)^{a-1}[k_1(r) + k_2(r)e^{-\delta r}]dr \sup_{s \in [0,t_1-\epsilon]} \left\| T_a(t_2 - s) - T_a(t_1 - s) \right\|_{L(\mathcal{Y})} \\
+ C_2 \int_0^{t_1} t_1^{1+\alpha \beta}(1-\gamma) \left\{ (t_2 - r)^{-a-a\beta} + (t_1 - r)^{-a-a\beta} \right\} (t_1 - r)^{a-1}[k_1(r) + k_2(r)e^{-\delta r}]dr \\
\leq t_1^{1+\alpha \beta}(1-\gamma) \int_0^{t_1} (t_1 - r)^{-a\beta-1}[k_1(r) + k_2(r)e^{-\delta r}]dr \sup_{s \in [0,t_1-\epsilon]} \left\| T_a(t_2 - s) - T_a(t_1 - s) \right\|_{L(\mathcal{Y})} \\
+ 2C_2 \int_0^{t_1} t_1^{1+\alpha \beta}(1-\gamma) (t_1 - r)^{-a\beta-1}[k_1(r) + k_2(r)e^{-\delta r}]dr.
\]

Since \( T_a(t) \) is uniformly continuous and \( \lim_{t_2 \to t_1} \mathcal{J}_2 = 0 \), then \( \mathcal{J}_4 \to 0 \) as \( t_2 \to t_1 \), independent of \( y \in \Omega_r(\mathcal{J}) \). Hence, \( \left\| \mathcal{P} y(t_2) - \mathcal{P} y(t_1) \right\| \to 0 \), independently of \( y \in \Omega_r(\mathcal{J}) \) as \( t_2 \to t_1 \). Therefore, \( \{ \mathcal{P} y : y \in \Omega_r(\mathcal{J}) \} \) is equicontinuous.

**Theorem 4.** Let \( -1 < \beta < 0, 0 < \omega < \frac{\pi}{2} \), and \( A \in \Theta^\beta_\omega \). Then, under hypotheses \((H1)\)–\((H4)\), the operator \( \{ \mathcal{P} y : y \in \Omega_r(\mathcal{J}) \} \) is continuous and bounded, provided \( u_0 \in \mathcal{D}(A^\theta) \) with \( \theta > 1 + \beta \).

**Proof.** We verify that \( \mathcal{P} \) maps \( \Omega_r(\mathcal{J}) \) into itself. Taking \( y \in \Omega_r(\mathcal{J}) \) and defining

\[ u(t) := t^{-(1+\alpha \beta)(1-\gamma)}y(t), \]

we have \( u \in \Omega_r(\mathcal{J}) \). Let \( t \in [0,T], \)

\[ \left\| \mathcal{P} \right\| \leq \left\| t^{1+\alpha \beta}(1-\gamma) S_{a,\gamma}(t)[u_0 - h(u(t))] \right\| + t^{1+\alpha \beta}(1-\gamma) \int_0^t (t-r)^{a-1} T_a(t-r) g(r, u(r), (Bu)(r))dr. \]

From \((H2)\)–\((H4)\), we get

\[ \left\| \mathcal{P} y(t) \right\| \leq t^{1+\alpha \beta}(1-\gamma) \left\| S_{a,\gamma}(t)[u_0 - h(u(t))] \right\| + t^{1+\alpha \beta}(1-\gamma) \int_0^t (t-r)^{-a\beta-1}[k_1(r) + k_2(r)e^{-\delta r}]dr \\
\leq \sup_{[0,T]} \left( t^{1+\alpha \beta}(1-\gamma) \left\| S_{a,\gamma}(t)[\|u_0\| + k] \right\| + t^{1+\alpha \beta}(1-\gamma) \int_0^t (t-r)^{-a\beta-1}[k_1(r) + k_2(r)e^{-\delta r}]dr \right) \\
\leq r. \]

Hence, \( \left\| \mathcal{P} y \right\| \leq r \) for any \( y \in \Omega_r(\mathcal{J}) \). Now, to verify \( \mathcal{P} \) is continuous in \( \Omega_r(\mathcal{J}) \), let \( y_n, y \in \Omega_r(\mathcal{J}) \), \( n = 1, 2, \ldots \) with \( \lim_{n \to \infty} y_n = y \), that is, \( \lim_{n \to \infty} y_n(t) = y(t) \); \( \lim_{n \to \infty} t^{-(1+\alpha \beta)(1-\gamma)}y_n(t) = t^{-(1+\alpha \beta)(1-\gamma)}y(t) \), and \( \lim_{n \to \infty} t^{-(1+\alpha \beta)(1-\gamma)}y_n(t) = t^{-(1+\alpha \beta)(1-\gamma)}y(t) \) on \( \mathcal{J} \). Then, \((H1)\) implies that

\[ g\left(t, u_n(t), B(u_n(t))\right) = g\left(t, t^{-(1+\alpha \beta)(1-\gamma)}y_n(t), t^{-(1+\alpha \beta)(1-\gamma)}B(y_n(t))\right) \]
as \( n \to \infty \). From (H2), we obtain the inequality
\[
(t - r)^{-a^{\beta - 1}}|g(r, u_n(r), B(u_n(r)))| \leq 2(t - r)^{-(a^{\beta})(1 - \gamma)}[k_1(r) + k_2(r)e^{-\delta r}],
\]
that is,
\[
\int_0^t (t - r)^{-a^{\beta - 1}}\|g(r, u_n(r), B(u_n(r))) - g(r, u(r), B(u(r)))\|dr \to 0 \quad \text{as} \quad n \to \infty.
\]
Let \( t \in [0, T] \). Now,
\[
\|Py_n(t) - Py(t)\| \leq t^{(1 + a^{\beta})(1 - \gamma)} \int_0^t (t - r)^{a^{\beta} - 1}T_a(t - r)\|g(r, u_n(r), B(u_n(r))) - g(r, u(r), B(u(r)))\|dr.
\]
Applying Theorem 1, we have
\[
\|Py_n(t) - Py(t)\| \leq C_2 t^{(1 + a^{\beta})(1 - \gamma)} \int_0^t (t - r)^{-a^{\beta} - 1}\|g(r, u_n(r), B(u_n(r))) - g(r, u(r), B(u(r)))\|dr,
\]
which tends to 0 as \( n \to \infty \), i.e., \( Py_n \to Py \) pointwise on \( J \). Moreover, Theorem 3 implies that \( Py_n \to Py \) uniformly on \( J \) as \( n \to \infty \), that is, \( P \) is continuous. \( \square \)

4. Main results
We prove existence of a mild solution to problem (1)–(2) when the associated semigroup is compact (Theorem 5) and noncompact (Theorem 6).

4.1. Compactness of the semigroup
Here we assume \( Q(t) \) to be compact.

**Theorem 5.** Let \(-1 < \beta < 0, 0 < \omega < \frac{1}{2}\) and \( A \in \Theta^\delta \). If \( Q(t)(t > 0) \) is compact and (H1)–(H4) hold, then there exists a mild solution of (1)–(2) in \( \Omega_r(J) \) for every \( u_0 \in D(A^\theta) \) with \( \theta > 1 + \beta \).

**Proof.** Because we assume \( Q(t) \) to be compact, then the equicontinuity of \( Q(t)(t > 0) \) is assured. Moreover, by Theorems 3 and 4, \( P : \Omega_r(J) \to \Omega_r(J) \) is continuous and bounded and \( \epsilon : \Omega_r(J) \to \Omega_r(J) \) is bounded, continuous, and \( \{\epsilon y : y \in \Omega_r(J)\} \) is equicontinuous. We can write \( e : \Omega_r(J) \to \Omega_r(J) \) by \( (ey)(t) = (e^1 y)(t) + (e^2 y)(t), \) where

\[
(e^1 y)(t) = t^{(1 + a^{\beta})(1 - \gamma)}S_{h, \gamma}(t)(u_0 - h(u)) = t^{1 - a^{\beta}(1 - \gamma)}\Gamma(\gamma(1 - \alpha))\int_0^t (t - r)^{\gamma(1 - a) - 1}r^{a - 1}\int_0^\infty a^{\theta}M_a(\theta)Q(r^\theta(\theta)(u_0 - h(u)))d\theta dr
\]

\[
\quad = \frac{t^{(1 + a^{\beta})(1 - \gamma)}}{\Gamma(\gamma(1 - \alpha))}\int_0^t (t - r)^{\gamma(1 - a) - 1}r^{a - 1}\int_0^\infty (t - r)^{-1}\theta M_a(\theta)Q(r^\theta(\theta)(u_0 - h(u)))d\theta dr
\]

and

\[
(e^2 y)(t) = t^{(1 + a^{\beta})(1 - \gamma)}\int_0^t (t - r)^{a - 1}T_a(t - r)g(r, u(r)Bv(r))dr.
\]
For $\sigma > 0$ and $\theta \in (0,t)$, we define an operator $\varepsilon^1_{\theta,\sigma}$ on $\Omega_r(J)$ by

\[
(\varepsilon^1_{\theta,\sigma}y)(t) = t^{(1+\alpha\beta)(1-\gamma)}\int_0^t \int_0^\theta \Gamma(1-\alpha) (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) Q(r^\theta u_0 - h(u)) dr d\theta dr
\]

\[
= \frac{\alpha t^{(1+\alpha\beta)(1-\gamma)}}{\Gamma(1-\alpha)} \int_0^t \int_0^\theta \Xi(\theta^2 \sigma) (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) Q(r^\theta \theta - \theta^2 \sigma)(u_0 - h(u)) dr d\theta dr.
\]

Since $\Xi(\theta^2 \sigma)$ is compact, $\mathcal{V}^1_{\theta,\sigma}(t) = \{\varepsilon^1_{\theta,\sigma}y(t), y \in \Omega_r(J)\}$ is precompact in $\mathcal{Y}$ for all $\theta \in (0,t)$ and $\delta > 0$. Moreover, for any $y \in \Omega_r(J)$, one has

\[
|| (\varepsilon^1_{\theta,\sigma}y)(t) - (\varepsilon^1_{\theta,\sigma}y)(t) ||
\]

\[
\leq K(\alpha,\gamma) t^{(1+\alpha\beta)(1-\gamma)} \int_0^t \int_0^\sigma (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) Q(r^\theta u_0 - h(u)) dr d\theta dr
\]

\[
+ K(\alpha,\gamma) \left( t^{(1+\alpha\beta)(1-\gamma)} \int_0^t \int_0^\sigma (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) Q(r^\theta u_0 - h(u)) dr d\theta dr \right)
\]

\[
\leq K(\alpha,\gamma) t^{(1+\alpha\beta)(1-\gamma)} \int_0^t \int_0^\sigma (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) Q(r^\theta u_0 - h(u)) dr d\theta dr
\]

\[
+ K(\alpha,\gamma) \left( t^{(1+\alpha\beta)(1-\gamma)} \int_0^t \int_0^\sigma (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) Q(r^\theta u_0 - h(u)) dr d\theta dr \right)
\]

\[
= K(\alpha,\gamma) t^{(1+\alpha\beta)(1-\gamma)} \int_0^t \int_0^\sigma (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) Q(r^\theta u_0 - h(u)) dr d\theta dr
\]

\[
+ K(\alpha,\gamma) \left( t^{(1+\alpha\beta)(1-\gamma)} \int_0^t \int_0^\sigma (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) Q(r^\theta u_0 - h(u)) dr d\theta dr \right)
\]

\[
\leq K t^{-\sigma\gamma(1+\beta)} ||(u_0 - h(u))|| d\theta dr
\]

\[
\rightarrow 0 \text{ as } \theta \rightarrow 0 \text{ and } \sigma \rightarrow 0,
\]

where $K(\alpha,\gamma) = \frac{\theta}{\Gamma(1-\alpha)}$. Therefore, $\mathcal{V}^2_{\theta,\sigma}(t) = \{\varepsilon^2_{\theta,\sigma}y(t), y \in \Omega_r(J)\}$ are arbitrarily close to $\mathcal{V}^1(t) = \{\varepsilon^1y(t), y \in \Omega_r(J)\}$ for $t > 0$. Hence, $\mathcal{V}^1(t)$, for $t > 0$, is precompact in $\mathcal{Y}$. For $\theta \in (0,t)$ and $\sigma > 0$, we can present an operator $\varepsilon^2_{\theta,\sigma}$ on $\Omega_r(J)$ by

\[
(\varepsilon^2_{\theta,\sigma}y)(t) = \alpha t^{(1+\alpha\beta)(1-\gamma)} \int_0^t \int_0^\theta \Gamma(1-\alpha) (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) (t-r)^a Q((t-r)^a \theta) g(r, u(r), (Bu)r) dr d\theta dr
\]

\[
+ \alpha t^{(1+\alpha\beta)(1-\gamma)} \Xi(\theta^2 \sigma) \int_0^t \int_0^\theta \Gamma(1-\alpha) (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) (t-r)^a Q((t-r)^a \theta - \theta^2 \sigma) g(r, u(r), (Bu)r) dr d\theta dr.
\]

Hence, due to the compactness of $Q(\theta^2 \sigma)$, $\mathcal{V}^2_{\theta,\sigma}(t) = \{\varepsilon^2_{\theta,\sigma}y(t), y \in \Omega_r(J)\}$ is precompact in $\mathcal{Y}$ for all $\theta \in (0,t)$ and $\sigma > 0$. For every $y \in \Omega_r(J)$, we get

\[
|| (\varepsilon^2 y)(t) - (\varepsilon^2_{\theta,\sigma}y)(t) ||
\]

\[
\leq \left|\left| \alpha t^{(1+\alpha\beta)(1-\gamma)} \left( \int_0^t \int_0^\sigma \theta M_a(\theta) (t-r)^a Q((t-r)^a \theta) g(r, u(r), (Bu)r) dr d\theta \right) \right| \right|
\]

\[
+ \left|\left| \alpha t^{(1+\alpha\beta)(1-\gamma)} \left( \int_0^t \int_0^\theta \Xi(\theta^2 \sigma) (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) Q((t-r)^a \theta) g(r, u(r), (Bu)r) dr d\theta \right) \right| \right|
\]

\[
+ \left|\left| \alpha t^{(1+\alpha\beta)(1-\gamma)} \left( \int_0^t \int_0^\theta \Xi(\theta^2 \sigma) (t-r)^{1-\alpha} r^{-\alpha-1} \theta M_a(\theta) Q((t-r)^a \theta - \theta^2 \sigma) g(r, u(r), (Bu)r) dr d\theta \right) \right| \right|
\]

\[
\rightarrow 0 \text{ as } \theta \rightarrow 0 \text{ and } \sigma \rightarrow 0.
\]
This implies the relative compactness of \( \Omega \). From Theorems 6, the fixed point theorem, existence of a fixed point \( y \in \Omega \) is compact for \( \varepsilon > 0 \). Now, we assume that \( Q(t) \) is arbitrary, then \( \Omega \) is equicontinuous. Also, we prove that there is a subset of \( \Omega \) for every \( u \). Let \( u^*(t) := t^{(1+\alpha\beta)(\gamma-1)}y^*(t) \). Then, \( u^* \) is a mild solution of (1)–(2).

4.2. Non-compactness of the semigroup

Now, we assume that \( Q(t) \) is noncompact. We need the following supplementary condition:

\((H5)\) There exists a constant \( k > 0 \) satisfying

\[
\Theta(g(t, \mathcal{E}_1, \mathcal{E}_2)) \leq k \Theta(\mathcal{E}_1, \mathcal{E}_2) \text{ for a.a. } t \in [0, T]
\]

and for every bounded subsets \( \mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{Y} \).

Theorem 6. Let \(-1 < \beta < 0, 0 < \omega < \frac{1}{\beta}, \) and \( A \in \Theta^\beta_\omega \). Suppose \((H1)–(H5)\) hold. Then, (1)–(2) has a mild solution in \( \Omega_r(\mathcal{J}) \) for every \( u_0 \in D(\phi^\beta) \) with \( \theta > 1 + \beta \).

Proof. From Theorems 3 and 4, we get that \( \varepsilon : \Omega_r(\mathcal{J}) \to \Omega_r(\mathcal{J}) \) is continuous, bounded, and \( \{ey : y \in \Omega_r(\mathcal{J})\} \) is equicontinuous. Also, we prove that there is a subset of \( \Omega_r(\mathcal{J}) \) such that \( \varepsilon \) is compact in it. For any bounded set \( \mathbb{P}_0 \subset \Omega_r(\mathcal{J}) \), set

\[
e^{(1)}(\mathbb{P}_0) = \{e^{(n)}(\mathbb{P}_0) = \varepsilon(\mathcal{O}(e^{(n-1)}(\mathbb{P}_0))), n = 2, 3, \ldots\}
\]

For any \( \varepsilon > 0 \), we can obtain from Propositions 1–3 a subsequence \( \{y^{(1)}_n\}_{n=1}^\infty \subset \mathbb{P}_0 \) satisfying

\[
\Theta(e^{(1)}(\mathbb{P}_0(t))) \leq 2\Theta\left(t^{(1+\alpha\beta)(1-\gamma)}\int_0^t (t-r)^{a-1}T_a(t-r)g(r, r^{(1+\alpha\beta)(1-\gamma)}y_n^{(1)}(r), By_n^{(1)}(r))\right)\]

\[
\leq 4C_p\theta t^{(1+\alpha\beta)(1-\gamma)}\left(\int_0^t (t-r)^{a-1}T_a(t-r)g(r, r^{(1+\alpha\beta)(1-\gamma)}y_n^{(1)}(r), By_n^{(1)}(r))\right)\]

\[
\leq 4C_p\theta t^{(1+\alpha\beta)(1-\gamma)}\Theta(\mathbb{P}_0)\left(\int_0^t (t-r)^{a-1}T_a(t-r)g(r, r^{(1+\alpha\beta)(1-\gamma)}y_n^{(1)}(r), By_n^{(1)}(r))\right)\]

\[
= 4C_p\theta t^{(1+\alpha\beta)(1-\gamma)}\Theta(\mathbb{P}_0)\frac{\Gamma(-a\beta)\Gamma(-a\beta + \gamma(1 + a\beta))}{\Gamma(-2a\beta + \gamma(1 + a\beta))}.
\]

Since \( \varepsilon \) is arbitrary, then

\[
\Theta(e^{(1)}(\mathbb{P}_0(t))) \leq 4C_p\theta t^{(1+\alpha\beta)(1-\gamma)}\Theta(\mathbb{P}_0)\frac{\Gamma(-a\beta)\Gamma(-a\beta + \gamma(1 + a\beta))}{\Gamma(-2a\beta + \gamma(1 + a\beta))}.
\]
Again, for any \( \epsilon > 0 \), we can get from Propositions 1–3 a subsequence \( \{y_n^{(2)}, B y_n^{(2)}\}_{n=1}^{\infty} \subset \mathcal{C}(\epsilon^{(1)}(\mathbb{P}_0)) \), which implies that

\[
\Theta(\epsilon^{(2)}(\mathbb{P}_0(t))) = \Theta(\epsilon(\mathcal{C}(\epsilon^{(1)}(\mathbb{P}_0(t))))).
\]

\[
\leq 2\Theta\left(t^{(1+a\beta)(1-\gamma)} \int_0^t (t-r)^{a-1} \mathcal{Q}_n(t-r) g(r, \{r^{-(1+a\beta)(1-\gamma)}(y_n^{(2)}(r), B y_n^{(2)}(r))\}_{n=1}^{\infty}) dr \right)
\]

\[
\leq 4C_p t^{(1+a\beta)(1-\gamma)} \left( \int_0^t (t-r)^{-a\beta-1} \Theta(g(r, \{r^{-(1+a\beta)(1-\gamma)}(y_n^{(2)}(r), B y_n^{(2)}(r))\}_{n=1}^{\infty}) dr \right)
\]

\[
\leq 4C_p t^{(1+a\beta)(1-\gamma)} \left( \int_0^t (t-r)^{-a\beta-1} \Theta(r^{-(1+a\beta)(1-\gamma)}(y_n^{(2)}(r), B y_n^{(2)}(r))_{n=1}^{\infty}) dr \right)
\]

\[
\leq 4C_p t^{(1+a\beta)(1-\gamma)} \left( \int_0^t (t-r)^{-a\beta-1} r^{-(1+a\beta)(1-\gamma)} \Theta(y_n^{(2)}(r), B y_n^{(2)}(r))_{n=1}^{\infty}) dr \right)
\]

\[
\leq \frac{(4C_p k^2 t^{(1+a\beta)(1-\gamma)} \Gamma(-a\beta) \Gamma(-a\beta + \gamma(1 + a\beta))}{\Gamma(-2a\beta + \gamma(1 + a\beta))} \Theta(\mathbb{P}_0).
\]

Now,

\[
\Theta(\epsilon^{(n)}(\mathbb{P}_0(t))) \leq \frac{(4C_p k^n t^{-(a\beta)} \Gamma^n(-a\beta) \Gamma(-a\beta + \gamma(1 + a\beta))}{\Gamma(-(n+1)a\beta + \gamma(1 + a\beta))} \Theta(\mathbb{P}_0), \quad n \in \mathbb{N}.
\]

Let \( M = 4C_p k^2 T^{-a\beta} \Gamma(-a\beta) \). We can find \( m, k \in \mathbb{N} \) big enough such that \( \frac{1}{k} < \alpha \beta < \frac{1}{k-1} \) and \( \frac{n+1}{k} > 2 \) for \( n > m \Gamma(-(n+1)a\beta + \gamma(1 + a\beta)) > \Gamma\left(\frac{n+1}{k}\right) \), that is,

\[
\frac{(4C_p k^n T^{-a\beta} \Gamma^n(-a\beta) \Gamma(-a\beta + \gamma(1 + a\beta))}{\Gamma(-(n+1)a\beta + \gamma(1 + a\beta))} < \frac{(4C_p k^n T^{-a\beta} \Gamma^n(-a\beta) \Gamma(-a\beta + \gamma(1 + a\beta))}{\Gamma\left(\frac{n+1}{k}\right)}.
\]

Replacing \( (n+1) \) by \( (j+1)k \), then the right-hand side of the inequality given above becomes

\[
\frac{M^{j+1}}{\Gamma(j+1)} \Gamma(-a\beta + \gamma(1 + a\beta)) = \frac{M^j}{\Gamma(j)} \Gamma(-a\beta + \gamma(1 + a\beta)) \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty.
\]

Therefore, there exists a constant \( n_0 \in \mathbb{N} \) such that

\[
\frac{(4C_p k^{n_0} T^{-a\beta} \Gamma^{n_0}(-a\beta) \Gamma(-a\beta + \gamma(1 + a\beta))}{\Gamma(-(n_0+1)a\beta + \gamma(1 + a\beta))} \leq \frac{(4C_p k^{n_0} T^{-a\beta} \Gamma^{n_0}(-a\beta) \Gamma(-a\beta + \gamma(1 + a\beta))}{\Gamma(-(n_0+1)a\beta + \gamma(1 + a\beta))} = p < 1.
\]

Now, \( \Theta(\epsilon^{(n_0)}(\mathbb{P}_0(t))) \leq p \Theta(\mathbb{P}_0) \). Since \( \epsilon^{(n_0)}(\mathbb{P}_0(t)) \) is bounded and equicontinuous, it follows from Proposition 1 that

\[
\Theta(\epsilon^{(n_0)}(\mathbb{P}_0)) = \max_{t \in [0, T]} \Theta(\epsilon^{(n_0)}(\mathbb{P}_0(t))).
\]
Hence, \( \Theta(\varepsilon^0(\mathbb{P}_0)) \leq p \Theta(\mathbb{P}_0) \), where \( p < 1 \). Using a similar technique as in Theorem 5, we obtain \( C \) in \( \Omega_{r}(\mathcal{J}) \) with \( \varepsilon(C) \subset C \) and \( \varepsilon(C) \) compact. By applying Schauder’s fixed point theorem, we obtain a fixed point \( y^* \) in \( \Omega_{r}(\mathcal{J}) \) of \( \varepsilon \). Let \( u^*(t) = t^{(1+\alpha\beta)/(\gamma-1)}y^*(t) \). Then, \( u^*(t) \) is a mild solution of (1)–(2).

5. Example

As an illustrative example, let us consider the following Hilfer fractional partial differential equation with a nonlocal condition:

\[
D_{0+}^{\frac{3}{4}+\frac{1}{2}} z(t,y) + \partial_y^2 z(t,y) = G(t,z(t,y),Bz(t,y)), \quad t \in (0,1], \quad y \in [0,\pi],
\]

\[
z(t,0) = z(t,\pi) = 0, \quad t \in (0,1],
\]

\[
I_{0+}^{\frac{1}{4}} [z(t,y)]_{t=0} + \sum_{i=1}^{m} c_i z(t_i, y) = z_0(y), \quad y \in [0,\pi],
\]

where \( \alpha = \frac{3}{4}, \gamma = \frac{1}{2} \), \( 0 < t_1 < \cdots < t_m < 1 \), and \( c_i, i = 1,\ldots,m \), are given constants. Let us take the nonlinear function \( G(t,z), Bz \) and the nonlocal function \( h(z(t), \cdot) = \sum_{i=1}^{m} c_i z(t_k, \cdot) \). Assume that \( \mathcal{Y} = L^2[0,\pi] \) and define \( \mathcal{A} : D(A) \subset \mathcal{Y} \rightarrow \mathcal{Y} \) by \( \mathcal{A}z = z_{yy} \) with domain

\[
D(A) = \{ z \in \mathcal{Y} : z_y, z_{yy} \in \mathcal{Y}, z(t,0) = z(t,\pi) = 0 \}.
\]

It follows from the work in [20] that there exists constants \( \delta, \epsilon > 0 \) such that \( \mathcal{A} + \delta \in \mathcal{O}_{\frac{2}{\pi} - \epsilon}(\mathcal{Y}) \). It is known that \( \mathcal{A} \) is the infinitesimal generator of a differentiable semigroup \( Q(t)(t > 0) \) in \( \mathcal{Y} \) given by

\[
(Q(t)x)(y) = \begin{cases} 
\int_0^t \psi(t,s)x(s)ds, & t > 0, \\
x(y), & t = 0,
\end{cases}
\]

where

\[
\psi(t,y) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{y^2}{4t}}, \quad t > 0, \quad 0 < y < \pi,
\]

and \( x(t)(y) = z(t,y) \). This implies \( \|Q(t)\| \leq 1 \) and leads to its compactness property. We can check that all hypotheses \( (H1)–(H4) \) are fulfilled. Hence, our Theorem 5 can be applied ensuring that problem (4) admits a mild solution.

6. Conclusion

In this paper, we applied Schauder’s fixed point theorem to investigate the solvability of a class of Hilfer fractional integro-differential equations involving almost sectorial operators. We discussed both cases of compactness and noncompactness, related to associated semigroup operators. The obtained existence results were subject to an appropriate set of sufficient conditions. As a future direction of research, it is desirable to consider the study of \( \psi \)-Hilfer fractional nonlocal nonlinear stochastic systems involving almost sectorial operators and impulsive effects, generalizing the current work. Another open line of research consists to develop numerical methods to approximate the mild solutions predicted by our Theorems 5 and 6.

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