Efficient resummation of high post-Newtonian contributions to the binding energy

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I. INTRODUCTION

The determination of the binding energy of spin-less compact binaries having been completed up to 4th order in the post-Newtonian (PN) approximation to General Relativity (GR) by calculations performed within at least three different theoretical approaches \textsuperscript{1–13}, with matching results, the attention can now be focused on the 5PN level, and possibly beyond.

Such high precision, ambitious target is believed \textsuperscript{14,15} to be central for maximising the physics output of the inspirals and coalescences that will be detected by the advanced versions of ground based interferometers \textsuperscript{16,17}, as well as by third generation detectors such as ET \textsuperscript{18} and by the space detector LISA \textsuperscript{19}.

The advent of new techniques imported from particle physics has given further impulse to this quest and the full determination of the 5PN sector is now a realistic goal for the near future. More in detail, the first post-Minkowskian part (i.e. the effective potential at first order in Newton constant $G$ and at all orders in the velocity, or 1PM for short) is known since quite some time \textsuperscript{20}, recently it has been re-derived \textsuperscript{21,22}, and extended to 2PM \textsuperscript{23}. Modern scattering amplitude techniques have been employed to determine the effective potential up to 3PM and at any order in the velocity expansion \textsuperscript{24,25} (not without some controversy in the interpretation of the results, see \textsuperscript{26–29}, confirming the necessity of multiple independent approaches to such difficult problems).

At the highest level of interaction at 5PN (that is $G^5$), the potential in the harmonic gauge has been determined in close succession by two groups employing effective field theory (EFT)-based techniques, \textsuperscript{30} and \textsuperscript{31}, the latter with a brute force calculation, the former exploiting a factorisation property that applies to all static diagrams at any odd-PN level; the extension of this factorisation property to non-static diagrams is the main focus of this work.

More recently, a collaboration including the same authors as \textsuperscript{31} has pushed the method up to computing the 3PM potential in the harmonic gauge, up to 6PN \textsuperscript{32}, while the contribution of hereditary effects to the conservative dynamics have been determined at 5PN in \textsuperscript{33} along the same path established for the 4PN case within the EFT approach \textsuperscript{34,35}.

In addition to the above mentioned efforts using effective field theory techniques, the combination of several techniques, spanning from PN and PM expansions, to self-force approach and effective-one-body re-summation has led to a systematisation of the center-of-mass Hamiltonian up to 6PN, in which the still unknown sectors have been condensed into few unknown coefficients, respectively at 5PN (in the $G^5$ and $G^6$ sector) \textsuperscript{36} and at 6PN (in the $G^5, G^6$ and $G^7$ sectors) \textsuperscript{37–39}. This is a remarkable result, not quite for the small number of unknown coefficients, but rather for the insight gained by combining information coming from the computation of different observables in different limits. A similar route has been followed in \textsuperscript{40,41}, where a formalism has been developed to map observables relative to bound states and unbound scatterings.

In particular, from the above analyses it has emerged that the scattering angle seems to store information about the two body dynamics in a more efficient way than other observables. This can be seen in some remarkable properties of the scattering angle, like its simple dependence on the symmetric mass ratio \textsuperscript{42}, or the fact that it can be determined, at a given PM order, by a drastically lower number of diagrams with respect to the binding energy.

This means that, conversely, part of the information contained in the binding energy expression at a given PM or PN order is somehow redundant, as it should be determined by the knowledge of lower order contributions. This

\[ V_{\text{eff}}(\xi) = \frac{G}{\sqrt{\xi}} \left[ A_0 + A_1 \xi + A_2 \xi^2 + \cdots \right] \]
is exactly what has been enlightened in [30], where it is shown that static sectors at odd-PN orders are actually “redundant”, in the sense that they are entirely composed by Feynman diagrams which are products of diagrams belonging to lower PN orders, hence we call such diagrams factorisable.

In the following section II we extend this procedure to the general case (i.e. to non-static diagrams) and we provide a simple way of expressing factorisable diagrams in terms of lower order ones also in presence of time derivatives. We then apply this method to the 5PM sector at 5PN, in section III, and we determine its “redundant” contribution to the effective potential (that is, roughly two thirds of the diagrams involved in this sector) with a simple code which takes less than a minute on a normal laptop computer \(^1\), as it involves only elementary algebraic manipulation. Finally, the implications of our work will be discussed in the conclusive section IV.

## II. EFT AND FACTORISATION OF NON STATIC DIAGRAMS

The details of the procedure for computing the near-zone contribution to the effective potential have been outlined and discussed in several works [9,12,30,44,45]; we recall here just the points needed in this paper, starting from the fundamental Path-Integral equation

\[
\exp[iS_{\text{eff}}(x_1, x_2)] = \int Dg_{\mu\nu} \exp[iS_{\text{bulk}}(g_{\mu\nu}) + iS_{\text{pp}}(g_{\mu\nu}, x_1, x_2)],
\]

which relates the effective action to the fundamental actions describing gravity coupled to point particles.

Working in the post-Newtonian framework, the above expression can be perturbatively evaluated in terms of Feynman diagrams, along the well-established formalism historically introduced in the quantum field theory context and applied to the compact binary problem in [46]. We adopt a Kaluza-Klein (KK) decomposition of the metric into one scalar, one spatial vector, and one symmetric tensorial field, respectively called \(\phi\), \(\vec{A}\) and \(\sigma_{ij}\); because of the purely algebraic method implemented in this work, the exact expression of this parametrisation is not needed here, nor are the explicit forms of the fundamental actions \(S_{\text{bulk}}(g_{\mu\nu}), S_{\text{pp}}(g_{\mu\nu}, x_1, x_2)\).

The only relevant detail is the expression of the particle-gravity vertex interactions with different combinations of the KK fields, expanded up to the needed order in terms of the particle velocity \(v\):

\[
\begin{align*}
\begin{array}{c}
\text{black} \\
\text{dashed blue} \\
\text{dotted red} \\
\text{green}
\end{array}
\& \quad \begin{array}{c}
\text{propagators} \\
\phi \\
\vec{A} \\
\sigma_{ij}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{Here} & \quad \text{n} & \quad \text{is} & \quad \text{the} & \quad \text{order} & \quad \text{of} & \quad \text{the} & \quad \text{vertex} & \quad \text{interaction} & \quad \text{in} & \quad \text{terms} & \quad \text{of} & \quad \text{the} & \quad \text{particle} & \quad \text{velocity} & \quad v:
\end{align*}
\]

\[
\begin{align*}
\text{n} & \quad \text{is} & \quad \text{the} & \quad \text{order} & \quad \text{of} & \quad \text{the} & \quad \text{vertex} & \quad \text{interaction} & \quad \text{in} & \quad \text{terms} & \quad \text{of} & \quad \text{the} & \quad \text{particle} & \quad \text{velocity} & \quad v:
\end{align*}
\]

\[
\begin{align*}
\text{n} & \quad \text{is} & \quad \text{the} & \quad \text{order} & \quad \text{of} & \quad \text{the} & \quad \text{vertex} & \quad \text{interaction} & \quad \text{in} & \quad \text{terms} & \quad \text{of} & \quad \text{the} & \quad \text{particle} & \quad \text{velocity} & \quad v:
\end{align*}
\]

where black, dashed blue, dotted red, and green lines stand respectively for the particle world-line and the \(\phi\), \(\vec{A}\) and \(\sigma_{ij}\) propagators, while \(\Lambda^{-2} \equiv 32\pi G L^d - 3\) is the d-dimensional gravitational coupling (we work in dimensional regularisation, with the arbitrary length-scale \(L\) eventually dropping out from observables) and \(m\) is the mass of the point particle.

In [30] it has been observed that the value of diagrams like the one at the left of figure \([1]\) can be easily computed

\[\text{Figure 1: Left: a factorisable diagram, non vanishing in the static limit. Right: a non-factorisable whose leading order involves time derivatives.}\]

\[\text{[1]: Considering that factorisable diagrams represent typically at least half of the total number, there is a sizable gain obtainable by a full-scale use of our method compared to the few hours of CPU time mentioned in [13] or the several weeks displayed in Table 1 of [32].}\]
in terms of the values of the two sub-diagrams in which it is naturally subdivided, according to a simple formula

\[ V_{\text{factorisable}} = (V_1 \times V_2) \times K \times C, \quad (3) \]

where: i) \( V \) and \( V_{1,2} \) are, respectively, the values of the factorisable diagram and of the two sub-diagrams, ii) \( K \) accounts for the new matter interaction vertex of \( V \) (emerging from the sewing) out of matter interaction vertices of the two sub-diagrams and can be determined using equation (2), and iii) \( C = C_{\text{factorisable}}/(C_1 \times C_2) \) where the \( C \)'s are the combinatoric factors associated with each graph.

In [30] this result was used to obtain all the contributions to the static part of the 5PN effective action, and the extension to the non-static case requires only some minor adjustments, the main difference being that non-static diagrams can contain time derivatives which can propagate across factors. This can be accounted for by keeping track of these derivatives, a feature which can be naturally implemented in the codes usually employed to write the amplitude corresponding to a given graph.

For instance, the diagram in the right part of figure 1 contains two times derivatives in the bulk \( \phi^5 \) vertex and has the following value:

\[
\frac{G^4m_1^2m_2}{r^4} \left[ \frac{8}{3} v_1.v_2 - \frac{4}{9} v_1^2 + \frac{8}{3} v_2^2 - \frac{16}{9} v_1 v_2^* + \frac{2}{3} (4v_2^* - 3v_1^*) (D_{1a} + D_{1b} + D_{1e} + D_{1d}) - \frac{8}{3} v_1^* D_2 \\
+ \frac{2}{3} (2D_{1a} + D_{1b} + D_{1c} + D_{1d}) (D_{1a} + D_{1b} + D_{1c} + D_{1d}) - \frac{2}{3} (D_{1a}^2 + D_{1b}^2 + D_{1c}^2 + D_{1d}^2) \right], \quad (4)
\]

the symbols \( D_I \)'s representing time derivative operators which act on whatever sub-graph is eventually attached to the vertex \( i \). By setting the \( D_I \)'s to zero one obtains the value of the diagram which contributes at the 4PN effective action as computed in [11].

In this way, equation (3) holds also in the non-static case, provided the \( \times \) symbol between \( V_1 \) and \( V_2 \) is interpreted in an extensive way to include also the action of the derivative operators \( D_I \)'s. A detailed example is discussed in the Appendix.

\section*{III. PROOF OF CONCEPT}

In this section we apply the method described above to the determination of factorisable diagrams up to order \( v^2 \) and up to 5PN (the procedure is generalisable to higher powers of \( v \) and we leave this to future work). As a first step, we have modified our codes to include derivative operators as in eq.(4), and we have recomputed all the non-factorisable Feynman diagrams containing up to two derivatives, and up to 4PN, in order to take the \( D_I \)'s into account. Second, we have used these prime diagrams as building blocks to generate all the factorisable ones. We have verified that we have been able in this way to reproduce the value of all the factorisable diagrams up to 4PN, recovering the values already computed in [11 14 50].

Next, we moved on to 5PN, and we provide here a schematic summary of the result, where the 1220 \( G^5 v^2 \) factorisable diagrams (roughly two thirds of the total) have been divided into subcategories. The diagram with a single \( A \) propagator can be combined with four Newtonian diagrams, as schematically represented below

\[
\begin{array}{c}
\includegraphics{fig1a} \\
\times \\
\includegraphics{fig1b} \\
\end{array}
\]

(5)

to give 16 \( G^5 v^2 \) factorisable diagrams, whose total contribution to the potential is

\[
V_{N^4G^2v^2} = \frac{1}{6} G^5m_1^2m_2}{r^5} v_1.v_2 + \frac{40}{3} G^5m_1^2m_2}{r^5} v_1.v_2 + \frac{45}{2} G^5m_1^2m_2}{r^5} v_1.v_2 + (1 \leftrightarrow 2). \quad (6)
\]

The combination of three Newtonian diagrams with the 7 \( G^2 v^2 \) prime diagrams

\[
\begin{array}{c}
\includegraphics{fig2a} \\
\times \\
\includegraphics{fig2b} \\
\end{array}
\]

(7)

gives 135 \( G^5 v^2 \) factorisable diagrams, whose total contribution to the potential is

\[
V_{N^3G^2v^2} = \frac{G^5m_1^2m_2}{r^5} \left[ \frac{7}{12} v^2 + \frac{5}{3} v_1^2 + \frac{1}{12} v_1 v^2 - \frac{7}{3} \frac{v_1^* v_2}{v_2^*} \right]. \quad (8)
\]
\[ G^5m_1^3m_2^2 \frac{28v}{r^2} \left[ \frac{1}{3} \left( 13e^2 - 154v_1v_2 - 17v_2^2 + 335e^{-2} - 478e_1^2v_2^2 + 277e_2^2 \right) \right] \\
+ \frac{G^5m_1^3m_2^3}{r^5} \left[ \frac{26v_1v}{\epsilon} + \frac{1}{2} \left( 49e_1^2 - 147v_1v_2 + 615e_1^{-2} - 569e_1^2v_2^2 \right) \right] + (1 \leftrightarrow 2), \]

with \( \frac{1}{\epsilon} = \frac{1}{(a-3)} - 5 \log \left( \frac{r\sqrt{r/2}}{2} \right), v_{1,2} = \frac{r_1}{r_2}, \) and \( v = v_1 - v_2 \)

The same 7 \( G^5v^2 \) prime diagrams can also be combined with the 3 static prime \( G^3 \) diagrams

\[
\begin{array}{ccccccccc}
\text{factorisable diagrams} & & & & & & & & \\
\text{factorisable diagrams} & & & & & & & & \\
\end{array}
\]

(9)

to give 85 \( G^5v^2 \) factorisable diagrams (17 of which trivially vanishing because the last but one diagram in eq. (9) is zero at leading order), whose total contribution to the potential is

\[ V_{G^3v^0 G^2v^2} = \frac{G^5m_1^3m_2}{r^5} \left[ \frac{7}{6} v^2 - \frac{2}{3} v_2^2 + \frac{1}{6} v^2 - \frac{2}{3} v_2^2 \right] \]  
\[ + \frac{2G^5m_1^3m_2}{3r^5} \left[ 19v_1^2 - 70v_1v_2 + 7v_2^2 + 65v_1^2 - 82v_1^2 - 37v_2^2 \right] \]  
\[ + \frac{G^5m_1^3m_2}{r^5} \left[ -3v_1^2 - 63v_1v_2 + 215v_1^2 - 153v_1v_2 \right] + (1 \leftrightarrow 2). \]  

The combination of two Newtonian diagrams with the 28 \( G^5v^2 \) prime diagrams

\[ \text{(28 diagrams)} \times \left( \begin{array}{c}
\text{factorisable diagrams}
\end{array} \right)^2 \]

(11)
gives 267 \( G^5v^2 \) factorisable diagrams (11 of which trivially vanishing), whose total contribution to the potential is

\[ V_{N2G^3v^2} = \frac{G^5m_1^3m_2}{r^5} \left[ \frac{1}{e} \left( \frac{41}{20} v_1^2 - 2v_1v_2 - \frac{203}{20} v_1^2 + 10v_1^2 \right) + \left( \frac{3647}{600} v_1^2 - \frac{19}{6} v_1v_2 - \frac{27971}{600} v_1^2 + \frac{287}{6} v_1^2 \right) \right] \]
\[ + \frac{G^5m_1^3m_2}{r^5} \left[ \frac{1}{e} \left( \frac{287}{10} v_1^2 - 2v_1v_2 - \frac{1191}{10} v_1^2 + 97v_1^2 + 20v_1^2 \right) + \left( \frac{37829}{300} - \frac{3}{2} v_1^2 \right) v_1^2 + \left( \frac{99}{32} - \frac{178}{3} \right) v_1v_2 \right] \]
\[ + \left( \frac{41}{2} - \frac{3}{2} \pi^2 \right) v_2^2 + \left( \frac{9}{4} - \frac{242297}{300} \right) v_2^2 + \left( \frac{2792}{3} - \frac{129}{32} \pi^2 \right) v_1^2v_2 + \left( \frac{3}{2} - 167 \right) v_2^2 \]
\[ + \frac{G^5m_1^3m_2}{r^5} \left[ \frac{1}{e} \left( \frac{369}{20} v_1^2 - 18v_1v_2 - \frac{927}{20} v_1^2 + 45v_1^2 \right) + \left( \frac{43741}{200} - 6\pi^2 \right) v_1^2 \right] \]
\[ + \left( \frac{99}{16} - \frac{117}{2} \right) v_1v_2 + \left( \frac{33}{4} - \frac{227913}{200} \right) v_1^2 + \left( \frac{2057}{2} - \frac{141}{16} \pi^2 \right) v_1v_2 \] + (1 \leftrightarrow 2). \]

The product of the Newtonian diagram with the 171 \( G^4v^2 \) prime diagrams

\[
\begin{array}{ccccccccc}
\text{factorisable diagrams} & & & & & & & & \\
\text{factorisable diagrams} & & & & & & & & \\
\end{array}
\]

(13)
gives 665 \( G^5v^2 \) factorisable diagrams (only 4 of which are trivially vanishing), whose total contribution to the potential is

\[ V_{NG^2v^2} = \frac{G^5m_1^3m_2}{r^5} \left[ \frac{4}{e} \left( v_1v - 5e_1v_2 \right) - \frac{2}{3} \left( v_1^2 + 5v_1v_2 - 11v_1^2 + 17e_1v_2 \right) \right] \]
\[ + \frac{G^5m_1^3m_2}{r^5} \left[ \frac{4}{e} \left( -\frac{16}{15} v_1^2 - \frac{2}{3} v_1v_2 + 2v_2^2 + \frac{47}{5} v_1^2 - \frac{8}{3} v_1^2 - 7v_2^2 \right) - \left( \frac{29}{4} - \frac{538}{75} \right) v_1^2 + \left( \frac{7}{2} \pi^2 - \frac{184}{3} \right) v_1v_2 \right] \]
\[ + \left( \frac{11}{4} \pi^2 - \frac{134}{3} \right) v_2^2 + \left( \frac{781}{12} - \frac{7216}{25} \right) v_1^2 - \left( \frac{451}{6} \pi^2 + 308 \right) v_1v_2 + \left( \frac{133}{12} \pi^2 + \frac{212}{3} \right) v_2^2 \]
\[ G^5 m_1^3 m_2^3 \frac{4}{\epsilon} \left( -\frac{34}{15} v_1^2 + 5v_1 v_2 + \frac{419}{15} v_1^2 - \frac{92}{3} v_1 v_2 \right) - \left( \frac{7}{12} \pi^2 + \frac{13012}{75} \right) v_1^2 + \left( \frac{21}{4} \pi^2 - 72 \right) v_1 v_2 + \left( \frac{841}{12} \pi^2 + \frac{24444}{25} \right) v_1^2 - \left( \frac{299}{4} \pi^2 + \frac{2312}{3} \right) v_1^2 v_2^* \left( 1 \leftrightarrow 2 \right). \]

The product of the Newtonian diagram, of the diagram with a single $A$ propagator, and of the three static prime $G^3$ diagrams

\[(\text{\includegraphics[width=0.3\textwidth]{diagram1}}) \times (\text{\includegraphics[width=0.3\textwidth]{diagram2}}) \times (\text{\includegraphics[width=0.3\textwidth]{diagram3}}) \tag{15}\]

gives 27 factorisable diagrams (5 of which are trivially vanishing), whose total contribution to the potential is

\[ V_{NAG^3v^0} = \frac{4}{3} G^5 m_1^3 m_2^3 v_1 v_2 + \frac{152}{3} G^5 m_1^3 m_2^3 v_1 v_2 + \frac{76}{3} G^5 m_1^3 m_2^3 v_1 v_2 + (1 \leftrightarrow 2). \tag{16} \]

Finally, there are 25 diagrams which allow a non-vanishing static limit, thus contributing also at the 4PN dynamics (as computed in [10]). The next-to-leading order evaluation of such diagrams needed at 5PN can also be carried along the factorisation technique used so far. So, the 19 4PN diagrams resulting from the product of the three $G^3 v^0$ with two Newtonian

\[(\text{\includegraphics[width=0.3\textwidth]{diagram1}})^2 \times (\text{\includegraphics[width=0.3\textwidth]{diagram4}}) \tag{17}\]

give, at next-to-leading order

\[ V_{N^2AG^3v^0}^{\text{NLO}} = \frac{G^5 m_1^3 m_2^3}{r^5} \left[ \frac{1}{6} \left( -\frac{13}{10} v_1^2 + v_1 v_2 + \frac{59}{10} v_1^2 - 5 v_1 v_2 \right) - \frac{3691}{1800} v_1^2 + \frac{17}{36} v_1 v_2 - \frac{9}{4} v_2^2 + \frac{6113}{1800} v_1^2 - \frac{115}{36} v_1 v_2 \right] + \frac{G^5 m_1^3 m_2^3}{r^5} \left[ \frac{1}{3} \left( -\frac{91}{10} v_1^2 + 7 v_1 v_2 + \frac{323}{10} v_1^2 - 26 v_1 v_2 \right) + \left( \frac{7}{32} \pi^2 - \frac{12329}{300} \right) v_1^2 + \left( -\frac{17}{32} \pi^2 + \frac{97}{18} \right) v_1 v_2 \right] + \frac{G^5 m_1^3 m_2^3}{r^5} \left[ \frac{1}{6} \left( -\frac{39}{20} v_1^2 + \frac{3}{2} v_1 v_2 + \frac{191}{60} v_1^2 - \frac{11}{6} v_1 v_2 \right) + \left( \frac{7}{8} \pi^2 - \frac{25669}{1800} \right) v_1^2 + \left( -\frac{17}{16} \pi^2 + \frac{65}{36} \right) v_1 v_2 \right] + \left( -\frac{53}{16} \pi^2 + \frac{1713}{200} \right) v_1^2 + \left( \frac{31}{8} \pi^2 + \frac{29}{4} \right) v_1 v_1^2 \right] + (1 \leftrightarrow 2). \tag{18} \]

To conclude, the 6 4PN diagrams resulting from the combination of six Newtonian

\[(\text{\includegraphics[width=0.3\textwidth]{diagram5}})^5 \tag{19}\]

give, at next-to-leading order

\[ V_{N^6}^{\text{NLO}} = \frac{G^5 m_1^3 m_2^3}{r^5} \left[ \frac{1}{16} v_1^2 - \frac{1}{48} v_1 v_2 - \frac{81}{80} v_2^2 - \frac{1}{12} v_1^2 - \frac{5}{48} v_1 v_2 \right] + \frac{G^5 m_1^3 m_2^3}{r^5} \left[ -\frac{5}{3} v_1 v_2 + 2 v_2^2 - 3 v_1^2 + \frac{13}{3} v_1 v_2^* + \frac{1}{3} v_1 v_2 \right] + \frac{G^5 m_1^3 m_2^3}{r^5} \left[ -\frac{9}{8} v_1^2 - \frac{45}{16} v_1 v_2 - \frac{11}{4} v_1 v_2 - \frac{89}{16} v_1 v_2 \right] + (1 \leftrightarrow 2). \tag{20} \]

Summing all the above contributions, we obtain the total contributions of factorisable diagrams to the $G^5 v^2$ sector of the 5PN potential

\[ V_{v^2,\text{fact}}^{5\text{PN}} = \frac{G^5 m_1^3 m_2^3}{r^5} \left[ \frac{35}{6} \left( v_1 v - 5 v_1 v^* \right) + \frac{727}{144} v_1^2 - \frac{1159}{144} v_1 v_2 - \frac{41}{8} v_2^2 - \frac{643}{18} v_1^2 - \frac{4739}{144} v_1 v_2^* - \frac{11}{4} v_2^2 \right] \tag{21} \]
Two main kinds of obstacle stand on the way of scaling precision computations of the two-body dynamics to higher order: intrinsically difficult master integrals (either in coordinate or in momentum space), and proliferation of terms. The former affect in equal measure all different methodologies employed so far, and the next hard step will be met in the high $G$ orders sectors at 6PN. On the contrary, the latter impacts unequally different observables and approaches; for instance the EFT determination of Lagrangian effective potential involves a very large number of diagrams compared to other techniques. Several independent factors concur to this state of things: gauge freedom, while providing several useful consistency checks, disperses information among several non-essential contributions. Furthermore, the explicit velocity counting usually employed in the post-Newtonian version of the EFT approach involves a major proliferation of diagrams, only partially tampered by an efficient metric parametrisation, with respect to a full post-Minkowskian calculation, because several PN diagrams correspond to the same PM ones (also called topologies in our previous works), as depicted in figure 2. This is the price to pay for dealing with simpler integrals with respect to full post-Minkowskian calculations, a reasonable price so far, considered the success of velocity-truncation approaches up to 4PN.

The third reason is rooted in the observable itself, the binding energy which, as mentioned in the introduction, appears to be less efficient than the scattering angle in storing information. As a result several contributions at a given PN or PM order are actually simple algebraic functions of lower order ones, meaning that a good share of the diagram (or topology) proliferation could actually be overcome by more efficient bookkeeping algorithms. We have explicitly exposed this mechanism by reducing the computation of all factorisable diagrams as algebraic functions of lower order ones, first in the static limit [30] and then paving the way to do this in full generality in the present work.

IV. CONCLUSION

The ratio of factorisable over non-factorisable diagrams or topologies is a generally growing function of the PN or PM order. The factorisable vs. non-factorisable diagram number ratio goes from 0.3 to 0.6 passing from 2PN to 4PN; considering diagrams with two powers of velocity, like the one computed in this paper, the ratio grows from 0.6 to 1.6 from 3PN to 5PN, while in the static case the progression is from 2.75 to 6.5, thus suggesting a growing trend of this ratio, implying a higher and higher relative weight of factorisable diagrams with respect to non-factorisable

Figure 2: On the right, one of the hundreds of possible diagrams deriving from the 3PM topology on the left.

\[ \left[ \frac{1}{\epsilon} \left( \frac{107}{5} v_1^2 - \frac{85}{3} v_1 v_2 + 8v_2 + \frac{941}{15} v_1^2 + \frac{185}{3} v_1 v_2^2 + \left( \frac{273}{32} \pi^2 + \frac{7112}{75} \right) v_1^2 \right) + \left( \frac{97}{16} \pi^2 - \frac{2717}{18} \right) v_1 v_2 + \left( \frac{47}{32} \pi^2 - \frac{122}{9} \right) v_2^2 + \left( \frac{6347}{96} \pi^2 - \frac{75856}{225} \right) v_1^2 \right] \]

\[ + \left( \frac{3707}{48} \pi^2 + \frac{7025}{18} \right) v_1 v_2^2 + \left( \frac{1163}{96} \pi^2 + \frac{71}{3} \right) v_2^2 \]

\[ + \frac{G^5 m_1^4 m_2^2}{r^5} \left[ \frac{1}{2\epsilon} \left( \frac{223}{15} v_1^2 + 7v_1 v_2 + \frac{2837}{15} \right) v_2^2 - 211 v_1 v_2^2 \right) + \left( \frac{92387}{1800} \pi^2 - \frac{137}{24} \pi^2 \right) v_2^2 + \left( \frac{83}{8} \pi^2 - \frac{24409}{144} \right) v_1 v_2 \]

\[ + \left( \frac{1275}{16} \pi^2 + \frac{8009}{48} \pi^2 \right) v_2^2 - \pi^2 \right) + (1 \leftrightarrow 2) \].

\[ \left[ 1 \right] \]

2 The comparison must be made between PN or PM orders with the same parity.
ones, at higher powers of $G$ (for the PM-oriented reader, there are 118 factorisable topologies contributing at 5PN, vs. 62 non factorisable ones). These numbers indicate that any strategy or approach to scale perturbative calculations at higher orders will benefit from optimisation procedures like the one put forward in this work.

The work [51] appeared online on the arXiv on the same day of this paper, containing the full near zone computation of the fifth post-Newtonian order two body dynamics. The results of the present paper are in agreement with theirs.

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**Appendix A: Detailed computation of one diagram**

We show here how the procedure described in section [11] works in detail on the following diagram

![Diagram](image_url)

that is the product of an H-shaped diagram with two Newtonian ones (one attached at the upper left vertex, another attached at the down left one). This belong to the most difficult category, because all factors have to be evaluated at next-to-leading order in time derivatives, meaning that derivative operators act in every direction (from the H-shaped to the Newtonians, and viceversa). The majority of the other diagrams have derivative operators, when present at all, acting only in one direction, from the higher-loop factor towards the Newtonians.

We can write the each of the Newtonian building blocks as

\[
N = N_0 + N_{\nu^2},
\]

where

\[
N_0 = -\frac{Gm_1m_2}{r}\left[1 - \epsilon \left(\log r - \frac{1}{2}\right)\right],
\]

\[
N_{\nu^2} = -\frac{Gm_1m_2}{2r}\left[3\nu_1^2 + 3\nu_2^2 + \nu_1\nu_2 - \nu_1\nu_2' - rD_2\nu_1' + rD_1\nu_2 - r^2D_1D_2\right],
\]

(we are setting here $L = \sqrt{4\pi\gamma E}$ in order to simplify notations) and we keep also note of the derivative of the static term (the derivative of $N_{\nu^2}$ would obviously be $o(\nu^2)$ so it is not needed here):

\[
\dot{N}_0 = \frac{Gm_1m_2}{r^2}\nu^r \left[1 - \epsilon \left(\log r - \frac{3}{2}\right)\right].
\]
As to the H-shaped diagram, it reads

\[ F_3 = \frac{G^3 m_1^2 m_2^2}{r^3} \left[ f_0 + f_{i \bar{v}^2} + r f_i D_i + r^2 f_{i j} D_i D_j \right], \tag{A5} \]

with \( i, j = 1a, 2a, 1b, 2b \) and

\[
\begin{align*}
  f_0 &= -1 \\
  f_{i \bar{v}^2} &= -\frac{161}{36} \left( v_1^2 + v_2^2 \right) + \frac{25}{36} v_1 v_2 + \frac{29}{12} \left( v_1^2 + v_2^2 \right) - \frac{49}{12} v_1 v_2 + \frac{\pi^2}{16} \left( v^2 - 3v'^2 \right) \\
  f_{1a} &= -\left( \frac{4}{9} v_1^2 + \frac{11}{36} v_1^2 - \frac{\pi^2}{16} v_1^2 \right) \\
  f_{2a} &= -\left( \frac{4}{9} v_2^2 + \frac{11}{36} v_1^2 - \frac{\pi^2}{16} v_1^2 \right) \\
  f_{1a,2a} &= \left( -\frac{1}{3(d-3)} + \log r - \frac{29}{18} + \frac{\pi^2}{8} \right) 
\end{align*}
\]

while we can ignore the other \( f_i \)’s because the corresponding operators do not act on anything. For the same reason we can also set \( D_1 \) to zero in the Newtonian attached to the upper vertex, and vice versa \( D_2 = 0 \) for the other one. Also, we observe that derivative operators of each factor acts just on the static part of the other factors (again, because we are working at \( O(\nu^2) \)) so, for the sake of computing the generalized product \( N \times N \times F_3 \), we can make the following replacements:

\[
\begin{align*}
  D_1 \text{[lower Newtonian factor]} &\rightarrow -4 \frac{v_r}{r} \\
  D_2 \text{[upper Newtonian factor]} &\rightarrow -4 \frac{v_r}{r}
\end{align*}
\]

and

\[
D_{1a}, D_{2a} \text{[H diagram]} \rightarrow \frac{\dot N_0}{N_0}. 
\]

This brings to

\[
N \times N \times F_3 \\
= \frac{G^3 m_1^2 m_2^2}{r^3} \left( N_0 + N_{i \bar{v}^2} |_{D_1=-4 \frac{\nu_r}{r}, D_2=0} \right) \left( N_0 + N_{i \bar{v}^2} |_{D_2=-4 \frac{\nu_r}{r}, D_1=0} \right) \left[ -1 + f_{i \bar{v}^2} + r f_{1a} \frac{N_0}{N_0} + r f_{2a} \frac{\dot N_0}{N_0} + r^2 f_{1a,2a} \left( \frac{N_0}{N_0} \right)^2 \right] \\
\simeq \frac{G^3 m_1^2 m_2^2}{r^3} \left[ N_0^2 \left( -1 + f_{i \bar{v}^2} + N_0 \dot N_0 (r f_{1a} + r f_{2a}) - N_0 \left( N_{i \bar{v}^2} |_{D_1=-4 \frac{\nu_r}{r}, D_2=0} + N_{i \bar{v}^2} |_{D_2=-4 \frac{\nu_r}{r}, D_1=0} \right) + r^2 f_{1a,2a} \left( \frac{\dot N_0}{N_0} \right)^2 \right] \right] . 
\]

Since the only pole is in \( f_{1a,2a} \) we can trade \( N_0 \) and its derivative for their three dimensional value everywhere except in the last term, where we have to keep the full \( O(d-3) \) expressions:

\[
N \ast N \ast F_3 \simeq \frac{G^5 m_1^4 m_2^4}{r^5} \left\{ -1 + f_{i \bar{v}^2} - \frac{\nu_r}{r} \left( r f_{1a} + r f_{2a} \right) - \left( 3v_1^2 + 3v_2^2 + v_1 v_2 - v_1 v_2 + 2v_1 v_r - 2v_2 v_r \right) \right\} \\
+ r^2 f_{1a,2a} \left( \frac{\nu_r}{r} \right)^2 \left[ 1 - 2 \nu_r \left( \log r - \frac{3}{2} \right) \right] \right\} \\
\simeq \frac{G^5 m_1^4 m_2^4}{r^5} \left\{ -1 + f_{i \bar{v}^2} - \left( \frac{5}{36} + \frac{\pi^2}{16} \right) \nu_r^2 - \left( 3v_1^2 + 3v_2^2 + v_1 v_2 + 2v_1^2 + 2v_2^2 - 5v_1 v_2 \right) \right\} \\
+ \nu_r^2 \left[ -\frac{1}{3(d-3)} + \log r - \frac{29}{18} + \frac{\pi^2}{8} + \frac{3}{2} \left( \log r - \frac{3}{2} \right) + O(\epsilon) \right] \right\} \\
\simeq \frac{G^5 m_1^4 m_2^4}{r^5} \left\{ -\frac{\nu_r^2}{3 \epsilon} - 1 - \frac{269}{36} \left( v_1^2 + v_2^2 \right) - \frac{11}{36} v_1 v_2 - \frac{7}{3} \left( v_1^2 + v_2^2 \right) + \frac{77}{12} v_1^2 v_2 + \frac{\pi^2}{16} \left( v_1^2 - 2v_1 v_2 \right) \right\} . \tag{A7} 
\]
Next, we have to multiply by the symmetry factor and by the vertex corrections:

\[
S_{\text{sym}} \times \left( \frac{V_{\phi\phi}}{V_{\phi}} \right)_1 \times \left( \frac{V_{\phi\phi}}{V_{\phi}} \right)_2 = 8 \frac{1 - \frac{d^2}{2(d-2)} v_1^2}{2m_1 \left( 1 + \frac{d}{2(d-2)} v_1^2 \right)^2} \frac{1 - \frac{d^2}{2(d-2)} v_2^2}{2m_2 \left( 1 + \frac{d}{2(d-2)} v_2^2 \right)^2}
\]

\[
\simeq \frac{2}{m_1 m_2} \left[ 1 - \frac{3d(2d-4)}{2(d-2)^2} \left( v_1^2 + v_2^2 \right) \right] = \frac{2}{m_1 m_2} \left[ 1 - \frac{15}{2} \left( v_1^2 + v_2^2 \right) + O(d-3) \right],
\]

(A8)

and we can neglect the \(O(d-3)\) part because the pole in the previous expression is already \(O(v^2)\).

The final result gives

\[
\frac{2G^2 m_1^3 m_2^3}{r^8} \left[ -v^2 \left( 1 + \frac{1}{36} \left( v_1^2 + v_2^2 \right) - \frac{11}{36} v_1 v_2 - \frac{7}{3} \left( v_1^2 + v_2^2 \right) \right) + \frac{77}{12} v_1 r \frac{v}{3} + \frac{\pi^2}{16} \left( v^2 - 2v^2 \right) \right].
\]

(A9)

All the sequences described in this appendix can be easily automatized and adapted to all the 1220 factorizable diagrams (we remind that the case displayed here is the most involved one), thus reducing the evaluation of all of them to a few minutes run.

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