MORPHISMS BETWEEN TWO CONSTRUCTIONS OF WITT VECTORS OF NON-COMMUTATIVE RINGS

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Abstract. Let $A$ be any unital associative, possibly non-commutative ring and let $p$ be a prime number. Let $E(A)$ be the ring of $p$-typical Witt vectors as constructed by Cuntz and Deninger in $[1]$ and $W(A)$ be the abelian group constructed by Hesselholt in $[2]$ and $[3]$. In $[4]$ it was proved that if $p = 2$ and $A$ is non-commutative unital torsion free ring then there is no surjective continuous group homomorphism from $W(A) \to HH_0(E(A)) := E(A)/[E(A), E(A)]$ which commutes with the Verschiebung operator and the Teichmüller map. In this paper we generalise this result to all primes $p$ and simplify the arguments used for $p = 2$. We also prove that if $A$ is a non-commutative unital ring then there is no continuous map of sets $HH_0(E(A)) \to W(A)$ which commutes with the ghost maps.

1. Introduction

Let $p$ be a prime number. Let $A$ be any unital associative, non-commutative ring. In $[4]$ we compared two constructions, one of a ring $E(A)$ by Cuntz and Deninger, given in $[1]$ and the other of an abelian group $W(A)$ given by Hesselholt in $[2]$ (see also $[3]$). Both $E(A)$ and $W(A)$ are topological groups and are equipped with the Verschiebung operator $V$ and the Teichmüller map $\langle \cdot \rangle$. Moreover, $W(A)$ and $E(A)$ are isomorphic to the classical construction of ring of $p$-typical Witt vectors when $A$ is commutative. It is natural to see how these constructions are related when $A$ is non-commutative.

L. Hesselholt asked whether for an associative ring $A$, $W(A)$ is isomorphic to $HH_0(E(A))$? Although this question is still open, it was proved in $[4$, Theorem 1.2$]$ that for $p = 2$ and $A = \mathbb{Z}\{X,Y\}$ there is no continuous surjective group homomorphism from $W(A) \to HH_0(E(A))$ which commutes with $V$ and $\langle \cdot \rangle$. One of the main results of this paper generalises this result to any prime number $p$.

Theorem 1.1. Let $A := \mathbb{Z}\{X,Y\}$ and $p$ be any prime number. Then there is no continuous surjective group homomorphism from $W(A) \to HH_0(E(A))$ which is compatible with $V$ and $\langle \cdot \rangle$.

It is also natural to see whether there is a map in the opposite direction giving relation between $HH_0(E(A))$ and $W(A)$. The next result of this paper will show that even this is not possible under some additional hypothesis in the case when $p$ is any prime number and $A = \mathbb{Z}\{X,Y\}$.

Theorem 1.2. Let $p$ be any prime number. Let $A = \mathbb{Z}\{X,Y\}$. Then there is no map of sets from $HH_0(E(A)) \to W(A)$ which commutes with the ghost maps $\eta : HH_0(E(A)) \xrightarrow{\eta} \left(\frac{A}{[A,A]}\right)^{\mathbb{N}_0}$ and $W(A) \xrightarrow{\omega} \left(\frac{A}{[A,A]}\right)^{\mathbb{N}_0}$.

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2. Preliminaries

In this section we will briefly recall the constructions $W(A)$ from $[2]$, $[3]$ and of $E(A)$ from $[1]$.

(1) Hesselholt’s construction of $W(A)$:
We will stick to the hypothesis on $A$ as in [3]. Suppose $A$ is any unital associative (need not be commutative) ring $A$. Let $p$ be a prime number and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Consider the map (called as ghost map)
\[
\omega : A^{N_0} \rightarrow \left( \frac{A}{[A,A]} \right)^{N_0}
\]
\[
\omega(a_0, a_1, a_2, \ldots) := \left( \omega_0(a_0), \omega_1(a_0, a_1), \omega_2(a_0, a_1, a_2), \ldots \right)
\]
where $\omega_i$'s are ghost polynomials defined by
\[
\omega_i(a_0, \ldots, a_i) := a_0 p^i + pa_1 p^{i-1} + p^2 a_2 p^{i-2} + \cdots + p^i a_i.
\]

$\omega$ is merely a map of sets and not a homomorphism of groups. For every integer $n \in \mathbb{N}_0$, we also have truncated versions of the above map (denoted again by $\omega$).

Hesselholt then inductively defines groups $W_n(A)$ (see [3]) such that the map $\omega$ factor through
\[
A^n \xrightarrow{q_n} W_n(A) \xrightarrow{\varpi} \left( \frac{A}{[A,A]} \right)^n
\]
and the following are satisfied

1. $W_1(A) = \frac{A}{[A,A]}$.
2. $q_n$ is surjective map of sets.
3. $\varpi$ is an additive homomorphism and is injective if $\frac{A}{[A,A]}$ is $p$-torsion free.

Define $W(A) := \lim_{\longleftarrow n} W_n(A)$ and the topology on $W(A)$ is the inverse limit topology. Clearly one also has a factorization of $A^{N_0} \xrightarrow{\omega} \left( \frac{A}{[A,A]} \right)^{N_0}$ as
\[
A^{N_0} \xrightarrow{q} W(A) \xrightarrow{\varpi} \left( \frac{A}{[A,A]} \right)^{N_0}
\]
where $q$ is always surjective and where $\varpi$ is injective if $\frac{A}{[A,A]}$ has no $p$-torsion.

We have the Verschiebung operator
\[
V : W(A) \rightarrow W(A)
\]
and the Teichmüller map
\[
\langle \rangle : A \rightarrow W(A)
\]
which satisfy
\[
V(a_0, a_1, \cdots) = (0, a_0, a_1, \cdots)
\]
and
\[
\langle a \rangle = (a, 0, 0, \ldots)
\]
One can show that $V$ and $\langle \rangle$ are well defined and that $V$ is a additive group homomorphism. Similarly for $n \in \mathbb{N}_0$, we have truncated versions (denoted by the same notation).

**Ghost map:** The group homomorphism $\varpi : W(A) \rightarrow \left( \frac{A}{[A,A]} \right)^{N_0}$ given by
\[
\varpi(a_0, a_1, a_2, \ldots) := \left( \omega_0(a_0), \omega_1(a_0, a_1), \omega_2(a_0, a_1, a_2), \ldots \right)
\]
will also be called as the ghost map and $\varpi$ is injective if $\frac{A}{[A,A]}$ is $p$-torsion free (See [3], page 56).

(2) Cuntz and Deninger’s construction of the ring $E(A)$:
Consider the ring $A^{\mathbb{N}_0}$ with the product topology where $A$ has the discrete topology.

(i) Let $V : A^{\mathbb{N}_0} \to A^{\mathbb{N}_0}$ be the map defined by $V(a_0, a_1, ...) := p(0, a_0, a_1, ...)$. 

(ii) For an element $a \in A$, define $\langle a \rangle \in A^{\mathbb{N}_0}$ by $\langle a \rangle := (a, ap^2, ap^3, ...)$. 

(iii) Let $X(A) \subset A^{\mathbb{N}_0}$ be the closed subgroup generated by 
$$\left\{ V^m(\langle a_1 \rangle \cdots \langle a_r \rangle) \mid m \in \mathbb{N}_0, r \in \mathbb{N}, a_i \in A \forall i \right\}.$$ 

Similarly, if $I \subset A$ is an ideal, we let $X(I)$ denote the closed subgroup generated by 
$$\left\{ V^m(\langle a_1 \rangle \cdots \langle a_r \rangle) \mid m \in \mathbb{N}_0, r \in \mathbb{N}, a_i \in I \forall i \right\}.$$ 

For $n \in \mathbb{N}_0$, we also have the truncated version $X_n(A), n \in \mathbb{N}$ (See Preliminaries in [1]). In fact $X(A) = \varprojlim_{n} X_n(A)$ as topological rings.

Let $ZA$ be the monoid algebra of the multiplicative monoid underlying $A$. Thus the elements of $ZA$ are formal sums of the form $\sum_{r \geq 0} n_r [r]$ with almost all $n_r = 0$. We have a natural epimorphism of rings from $ZA \to A$ and we let $I$ denote its kernel. One now defines

$$E(A) := \frac{X(ZA)}{X(I)} \quad \text{and} \quad E_n(A) := \frac{X_n(ZA)}{X_n(I)}.$$

Note that $E(A)$ is a Hausdorff topological ring equipped with the (multiplicative) Teichmüller map $\langle \rangle$ and the continuous additive operator $V$ give by

$$\langle \rangle : A \to E(A) \quad \langle a \rangle := ([a], [a]^p, [a]^{p^2}, ...) \mod X(I).$$

$$V : E(A) \to E(A) \quad V(a_0, a_1, ..., a_n) := p(0, a_0, a_1, ..., a_{n-1})$$

The above construction gives a functor $E$ from the category of associative rings to the category of associative rings which is compatible with the map $\langle \rangle$ and the additive homomorphism $V$.

**Ghost maps**: Let $X(A) \xrightarrow{\gamma} \left( \frac{A}{[A,A]} \right)^{\mathbb{N}_0}$ be the group homomorphism which is the composition

$$X(A) \hookrightarrow A^{\mathbb{N}_0} \to \left( \frac{A}{[A,A]} \right)^{\mathbb{N}_0}.$$ 

Let $E(A) \xrightarrow{\eta} \left( \frac{A}{[A,A]} \right)^{\mathbb{N}_0}$ denote the composition

$$E(A) \xrightarrow{\pi} X(A) \xrightarrow{\gamma} \left( \frac{A}{[A,A]} \right)^{\mathbb{N}_0}.$$ 

Let $HH_0(E(A)) := E(A)/[E(A), E(A)]$ where $[E(A), E(A)]$ is the closure of the commutator subgroup $[E(A), E(A)]$. The subgroup $[E(A), E(A)]$ is not an ideal of $E(A)$. We then have the following induced maps.

1. the Teichmüller map $\langle \rangle : A \to HH_0(E(A))$. 
2. Additive group homomorphism $V : HH_0(E(A)) \to HH_0(E(A))$ 
3. The group homomorphisms which are analogous to the ghost homomorphism $W(A) \xrightarrow{\pi} \left( \frac{A}{[A,A]} \right)^{\mathbb{N}_0}$.

(a) $HH_0(X(A)) \xrightarrow{\pi} \left( \frac{A}{[A,A]} \right)^{\mathbb{N}_0}$ 
(b) $HH_0(E(A)) \xrightarrow{\pi} \left( \frac{A}{[A,A]} \right)^{\mathbb{N}_0}$. 

Let $A$ be any associative, possibly non-unital ring $A$, $p$ be a prime number and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We will refer to [1, Preliminaries and Page 20].
Remark 2.1. Note that if $A$ is commutative $p$-torsion free ring then the ghost map $\overline{\eta}$ is a injective group homomorphism (See [1, Corollary 2.10]). If $A$ is not commutative then $\overline{\eta}$ and $\overline{\eta}$ need not be injective even if $\frac{A}{[A,A]}$ is $p$-torsion free (See [4, Theorem 4.2]).

Remark 2.2 (Commutative case). Let $A$ be a unital commutative ring. In this case the two constructions above are identified with the classical construction of the ring of $p$-typical Witt vectors. The discussion after [1, Corollary 2.10] establishes an isomorphism $W(A) \to E(A)$, natural in $A$, given by $\psi(r) = \sum_n V_n(r)$. This isomorphism preserves $V$ and $\langle \rangle$.

The following is an alternative formulation of Theorem 1.1 as suggested by the referee. We have two functors $A \mapsto W(A)$ and $A \mapsto HH_0(E(A))$ on the category of unital associative rings. Restricted to the subcategory of commutative rings, these functors are naturally isomorphic (see Remark 2.2).

**Theorem 2.3.** There is no natural map from $W(A) \to HH_0(E(A))$ which is a continuous surjective group homomorphism and which induces the natural isomorphism in the commutative case.

**Proof.** Let $\phi_A : W(A) \to HH_0(E(A))$ be any natural map which induces the given natural isomorphism in the commutative case. By (1.1), it is is enough to show that $\phi_A$ preserves $V$ and $\langle \rangle$. For an element $a \in A$ consider the map from $\mathbb{Z}[T] \xrightarrow{\bar{f}} A$ which sends $T$ to $a$. This gives us the following commutative diagram

$$
\begin{array}{ccc}
W(\mathbb{Z}[T]) & \xrightarrow{W(f)} & W(A) \\
\downarrow{\phi_{\mathbb{Z}[T]}} & & \downarrow{\phi_A} \\
E(\mathbb{Z}[T]) & = & HH_0(E(\mathbb{Z}[T])) \\
\end{array}
$$

That $\phi_A(\langle a \rangle) = \langle a \rangle$ follows from the fact that the other three maps in the diagram are compatible with $\langle \rangle$. To check compatibility of $\phi_A$ with $V$ it is enough to check that for all elements $a \in A$ and $n \in \mathbb{N}_0$

$$
\phi_A(V^n(\langle a \rangle)) = V^n(\langle a \rangle).
$$

This also follows from the above diagram. $\square$

It is not clear to us if a similar reformulation for Theorem 1.2, analogous to (2.3) can be proved.

3. **Proof of the Theorem 1.1 and 1.2**

To prove the Theorem 1.1 we will observe that if there exists a continuous map $W(A) \to HH_0(E(A))$ which commutes with $V$ and $\langle \rangle$ then it has to commute with the ghost maps $\overline{\eta}$ and $\overline{\omega}$. The argument given here is implicit in the proof of the Theorem 1.3 [4]. For the convenience it is given below.

**Lemma 3.1.** If there exists a continuous map $\Psi : W(A) \to HH_0(E(A))$ which commutes with $V$ and $\langle \rangle$ then the following diagram must commute

$$
\begin{array}{ccc}
HH_0(E(A)) & \xrightarrow{\overline{\eta}} & (\frac{A}{[A,A]})^{N_0} \\
\downarrow{\Psi} & & \downarrow{\Psi} \\
W(A) & \xrightarrow{\overline{\omega}} & (\frac{A}{[A,A]})^{N_0} \\
\end{array}
$$

**Proof.** Suppose there exists a continuous group homomorphism $W(A) \to HH_0(E(A))$ satisfying the above mentioned properties. Composing with the natural homomorphism $HH_0(E(A)) \to HH_0(X(A))$ we get a map

$$
\Phi : W(A) \to HH_0(X(A)).
$$
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To prove the result of the lemma it is thus enough to show that if $\Phi$ exists then it has to commute with the ghost map $\bar{\omega}$ and $\bar{\eta}$.

By following Hesselholt’s construction in [3], we know that the map $f : A^N_0 \to A^N_0$ given by

$$a := (a_0, a_1, \cdots) \mapsto (w_0(a), w_1(a), \cdots)$$

factors thorough $W(A)$ and we get the following, where $q$ is a surjective map.

$$A^N_0 \xrightarrow{q} W(A) \xrightarrow{\bar{\omega}} (A_{[A/A]}^N)^{N_0}$$

Consider the set map $\Omega : A^N_0 \to A^N_0$ defined by

$$\Omega(a) = (\omega_0(a), \omega_1(a), \cdots)$$

where $\omega_n(a) = a_0^n + p \cdot a_1^{p^{n-1}} + \cdots + p^n a_n$ are the Witt polynomials. The Lemma 4.1 in [4] proves that the image of $\Omega$ is contained in $X(A)$. The fact that both $\Omega$ and $\bar{\omega} \circ q$ are given by the same Witt polynomials, we have the following commutative diagram.

$$X(A) \xrightarrow{\pi} HH_0(X(A)) \xrightarrow{\bar{\omega}} (A_{[A/A]}^N)^{N_0}$$

Suppose there exists a continuous map $\Phi : W(A) \to HH_0(X(A))$. By Lemma 4.2 in [4], we know that following diagram is commutative.

$$X(A) \xrightarrow{\pi} HH_0(X(A))$$

As $q$ is a surjective map and $\bar{\eta} \circ \pi \circ \Omega = \bar{\omega} \circ q$ from the first diagram, this commutative square can be extended to the following commutative diagram.

$$X(A) \xrightarrow{\pi} HH_0(X(A)) \xrightarrow{\bar{\omega}} (A_{[A/A]}^N)^{N_0}$$

This proves that a continuous map $\Phi : W(A) \to HH_0(X(A))$ which commutes with $V$ and $\langle \cdot \rangle$ has to commute with the ghost map $\bar{\omega}$ and $\bar{\eta}$. Thus it will commute with the ghost maps $\bar{\omega}$ and $\bar{\eta}$. □

Proof of Theorem 1.1. We will show that there does not exist a continuous surjective map $\Phi : W(A) \to HH_0(E(A))$ which commutes with $V$ and $\langle \cdot \rangle$. As explained in Lemma 3.1, it is enough to show that there does not exist a continuos surjective map $W(A) \to HH_0(X(A))$ which commutes with the ghost maps.

Let $p$ be any prime number and $A = \mathbb{Z}\{X,Y\}$. Let $\langle X \rangle \langle Y \rangle \in HH_0(X(A))$ and let $\alpha := (\alpha_1, \alpha_2, \cdots) \in W(A)$ such that $\Phi(\alpha) = \langle X \rangle \langle Y \rangle$. 
\[ \overline{\gamma}(\langle X \rangle \langle Y \rangle) = \overline{\varphi}(\alpha) \quad \cdots \cdots \quad \text{(By Lemma 3.1)} \]

\[ = (\overline{\alpha_1}, \overline{\alpha_1}^p + \overline{p\alpha_2}, \overline{\alpha_1}^{p^2} + \overline{p^2\alpha_2} + \overline{p^3\alpha_3}, \cdots) \]

\[ = (\overline{\alpha_1}, \overline{\alpha_1}^p, \cdots) \pmod {pA} \]

This gives us,

\[ \overline{XY} = \overline{\alpha_1} \pmod {pA}, \quad \overline{XP} = \overline{\alpha_1}^p = \overline{XY}^p \pmod {pA}, \cdots \]

In the next Lemma, we will show that the equality \( \overline{XY}^p = \overline{XY}^p \pmod {pA} \) is not possible. \( \square \)

**Lemma 3.2.** Suppose \( p \) is any prime number and \( A = \mathbb{Z}\{X,Y\} \). Let \( \overline{A} := \mathbb{Z}/p\mathbb{Z}\{X,Y\} \). Then \( X^pY^p \neq (XY)^p \pmod {\overline{A}, \overline{A}} \)

**Proof.** It is enough to find a homomorphism \( f \) from \( \overline{A} \) to another ring \( B \) such that \( f(X)^pf(Y)^p \neq f(XY)^p \pmod {\overline{B}, \overline{B}} \).

Let \( B := M_2(\mathbb{F}_p) \). Consider a homomorphism

\[ f : \overline{A} \to M_2(\mathbb{F}_p) \]

\[ f(X) = R := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

\[ f(Y) = S := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

If \( X^pY^p - (XY)^p \in [\overline{A}, \overline{A}] \) then \( R^pS^p - (RS)^p \in [M_2(\mathbb{F}_p), M_2(\mathbb{F}_p)] \). This will imply that

\[ Tr(R^pS^p - (RS)^p) = 0. \]

Now \( R^p = S^p = 0 \) and \( Tr(R^pS^p - (RS)^p) = -1 \). This implies that \( X^pY^p - (XY)^p \notin [\overline{A}, \overline{A}] \).

\( \square \)

**Remark 3.3.** The above proof of the claim that \( X^pY^p - (XY)^p \notin [\overline{A}, \overline{A}] \) simplifies the arguments of the main Theorem 2.1 of [4] for \( p = 2 \) and generalises it to any prime number \( p \).

We will prove below the Theorem 1.2 by using the Lemma 3.2.

**Proof of Theorem 1.2.** Suppose \( p \) is any prime number, \( A = \mathbb{Z}\{X,Y\} \) and \( \overline{A} := \mathbb{Z}/p\mathbb{Z}\{X,Y\} \). Suppose there exists a map \( \rho : HH_0(E(A)) \to W(A) \) which commutes with the ghost maps i.e. \( \overline{\varphi} \circ \rho = \overline{\eta} \).

Consider the element \( \rho(\langle X \rangle \langle Y \rangle) = (\alpha_1, \alpha_2, \cdots) \in W(A) \). Thus we have the equality,

\[ \overline{\varphi} \circ \rho(\langle X \rangle \langle Y \rangle) = \overline{\eta}(\langle X \rangle \langle Y \rangle) \]

\[ \overline{\varphi}(\rho(\langle X \rangle \langle Y \rangle)) = (\alpha_1, \alpha_1^p + p\alpha_2, \alpha_1^{p^2} + p^2\alpha_2 + p^3\alpha_3, \cdots) \pmod {\overline{A}, \overline{A}} \]

\[ = (\alpha_1, \alpha_1^p, \alpha_1^{p^2}, \cdots) \pmod {\overline{A}, \overline{A}} \]

We also have,

\[ \eta(\langle X \rangle \langle Y \rangle) = (XY, X^pY^p, X^{p^2}Y^{p^2}, \cdots) \pmod {\overline{A}, \overline{A}} \]

\[ = (XY, X^pY^p, X^{p^2}Y^{p^2}, \cdots) \pmod {\overline{A}, \overline{A}} \]
Thus, $\alpha p^1 = (XY)^p = X^p Y^p$. This is not possible by the Lemma 3.2. Thus, there does not exist any map $\psi : HH_0(E(A)) \to W(A)$ which commutes with the ghost maps.

\[\square\]

References

[1] J. Cuntz and C. Deninger; Witt vector rings and the relative de Rham Witt complex. With an appendix by Umberto Zannier. J. Algebra 440 (2015), 545–593. 1, 3, 4

[2] Hesselholt, Lars; Witt vectors of non-commutative rings and topological cyclic homology. Acta Math. 178 (1997), no. 1, 109–141. 1

[3] Hesselholt, Lars; Correction to: “Witt vectors of non-commutative rings and topological cyclic homology”. Acta Math. 195 (2005), 55–60. 1, 2, 5

[4] Hogadi, Amit; Pisolkar, Supriya; On the comparison of two constructions of Witt vectors of non-commutative rings. J. Algebra 506 (2018), 379–396. 1, 4, 5, 6

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