Research Article

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A multiplicity result for asymptotically linear Kirchhoff equations

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Abstract: In this paper, we study the following Kirchhoff type equation:

\[-\left(1 + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \Delta u + u = a(x)f(u) \quad \text{in} \quad \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),\]

where $N \geq 3$, $b > 0$ and $f(s)$ is asymptotically linear at infinity, that is, $f(s) \sim O(s)$ as $s \to +\infty$. By using variational methods, we obtain the existence of a mountain pass type solution and a ground state solution under appropriate assumptions on $a(x)$.

Keywords: Kirchhoff type equations, asymptotically linear, ground state solution, variational methods.

MSC 2010: 35J60, 35J25, 35J20

1 Introduction and main result

In this paper, we study the following Kirchhoff type equations in $\mathbb{R}^N$ ($N \geq 3$):

\[-\left(1 + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \Delta u + u = a(x)f(u) \quad \text{in} \quad \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),\]

where $b > 0$, $a(x)$ and $f$ satisfy the following assumptions:

(A1) $f \in C(\mathbb{R}, \mathbb{R}^+)$, $f(s) \equiv 0$ if $s < 0$ and $f(s) = o(s)$ as $s \to 0^+$.

(A2) There exists $l \in (0, +\infty)$ such that $\frac{f(s)}{s} \to l$ as $s \to +\infty$.

(A3) $a(x) > 0$ is a continuous function and there exists $R_0 > 0$ such that

\[\sup\left\{ \frac{f(s)}{s} : s > 0 \right\} < \inf\left\{ \frac{1}{a(x)} : |x| \geq R_0 \right\}.\]

Throughout this paper, we denote by $H := H^1(\mathbb{R}^N)$ the usual Sobolev space equipped with the following inner product and norm:

\[(u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv) \, dx, \quad \|u\| = (u, u)^{\frac{1}{2}}.\]

Define the energy functional $I_b : H \to \mathbb{R}$ by

\[I_b(u) = \frac{\|u\|^2}{2} + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2 - \int_{\mathbb{R}^N} a(x)f(u) \, dx,\]

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where $F(s) = \int_0^s f(t) \, dt$. The functional $I_b$ is well defined for each $u \in H$ and belongs to $C^1(H, \mathbb{R})$. Moreover, for any $u, \varphi \in H$, we have
\[
\langle I_b(u), \varphi \rangle = \int_{\mathbb{R}^N} (\nabla u \varphi + u \varphi) \, dx + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \int_{\mathbb{R}^N} \nabla u \varphi \, dx - \int_{\mathbb{R}^N} a(x)f(u)\varphi \, dx.
\]

Clearly, the critical points of $I_b$ are the weak solutions for problem (1.1).

In recent years, much attention has been paid to Kirchhoff type equations. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. The following Kirchhoff problem with zero boundary data:
\[
-\left(a + b \int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(x, u), \quad x \in \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega, \quad (1.2)
\]
which is related to the stationary analogue of the equation
\[
\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 \, dx\right) \frac{\partial^2 u}{\partial x^2} = 0,
\]
was proposed by Kirchhoff [10] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In (1.2), $u$ denotes the displacement, $f(x, u)$ the external force and $b$ the initial tension, while $a$ is related to the intrinsic properties of the string, such as Young's modulus. We would like to point out that such nonlocal problems also appear in other fields such as biological systems, where $u$ describes a process which depends on the average of itself, for example, population density. It is worth mentioning that Fiscella and Valdinoci [5] proposed a stationary fractional Kirchhoff model, in bounded regular domains of $\mathbb{R}^N$, which takes into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. For some recent results about stationary Kirchhoff problems involving the fractional Laplacian, we refer to [18–20, 22, 23, 25, 30, 31, 34] and the references therein. For more mathematical and physical background for problem (1.2), we refer the readers to [1, 2, 5, 24] and the references therein.

Recently, problems like type (1.2) in bounded domains have been investigated by many authors, see, for instance, [6, 7, 16, 17, 21, 26, 32, 33]. In [16], Ma and Muñoz Rivera obtained positive solutions via variational methods. In [21], Perera and Zhang obtained a nontrivial solution via the Yang index and the critical group. Zhang and Perera [33], and Mao and Zhang [17] obtained multiple and sign-changing solutions via the invariant sets of descent flow. Shuai [26] obtained one least energy sign-changing solution via a constraint variational method and the quantitative deformation lemma. He and Zou [6, 7] obtained infinitely many solutions via the local minimum method and the fountain theorems.

Equations of type (1.1) in the whole space $\mathbb{R}^N (N \geq 3)$, but with the nonlinear term $a(x)f(u)$ being replaced by a more general nonlinear term $f(x, u)$, have also been studied extensively, see, for example, [8, 9, 12, 13, 15, 28, 35] and the references therein. More recently, the researchers paid their attention on asymptotically linear Kirchhoff equations. In [11], for a special type of a Kirchhoff equation with asymptotically linear term, in which the nonlocal term is like $1 + b \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) \, dx$ (this makes the functional contain a term like $\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) \, dx + \frac{b}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) \, dx)$, Li and Sun only needed to verify the (PS) condition in order to apply the mountain pass theorem to obtain the existence and multiplicity of solutions. Moreover, we noticed that the potential $V(x)$ in [11] was assumed to be radially symmetric, in order to consider the problem in a radial function Sobolev space in which the compact Sobolev embedding holds, and thus being easy to verify the (PS) condition. In some cases, this symmetric condition may be replaced by other compact conditions which make the compact embedding holds; here we just quote [3]. In [29], Wu and Liu studied the existence and multiplicity of nontrivial solutions for a Kirchhoff equation in $\mathbb{R}^3$ with asymptotically linear term via Morse theory and local linking. To overcome the loss of compactness, the usual strategy is to restrict the functional to a subspace of $H^1(\mathbb{R}^3)$, which embeds compactly into $L^2(\mathbb{R}^3)$ with certain assumptions on radially symmetric functions. In this paper, we will study directly problem (1.1) in $H$, not in any subspaces.
To overcome the loss of compactness, motivated by [14] in which Liu, Wang and Zhou studied an asymptotically linear Schrödinger equation, we will make a careful prior estimate for the Cerami sequence (defined later). Unlike the problem in [11], we not only show the convergence of the Cerami sequence, but also show that $\int_{\mathbb{R}^N}|\nabla u_n|^2 \, dx \to \int_{\mathbb{R}^N}|\nabla u|^2 \, dx$ as $n \to \infty$, where $\{u_n\}$ is a Cerami sequence. This makes the study of our problem more difficult. Through a careful observation, we show that this limit holds only by assuming that (A1), (A2) and (A3) hold, without any more assumptions.

Now let us state the main result of this paper.

**Theorem 1.1.** Assume that (A1), (A2) and (A3) hold, If $l > \mu$ with
\[
\mu = \inf_{u \in H} \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx : u \in H \text{ with } \int_{\mathbb{R}^N} a(x) u^2 \, dx = 1 \right\},
\]
then there exists $\tilde{b} > 0$ such that for any $b \in (0, \tilde{b})$, problem (1.1) admits at least two nontrivial nonnegative solutions, in which one is a mountain pass type solution and the other is a ground state solution.

We use the following notation:
- $C, C_1, C_2$, etc. will denote positive constants whose exact values are not essential.
- $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^{-1}$ and $H$, where $H^{-1}$ denotes the dual space of $H$.
- $\| \cdot \|_p$ is the norm of the space $L^p(\mathbb{R}^N)$.
- $2^* = \frac{2N}{N-2}$ if $N \geq 3$.
- $D^{1,2}(\mathbb{R}^N) := \{ u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N) \}$.
- $B_R(x)$ denotes the open ball centered at $x$ having radius $R$.

## 2 Preliminary lemmas

To prove Theorem 1.1, we use a variant version of the mountain pass theorem, which allows us to find a so-called Cerami type (PS) sequence. The properties of this kind of (PS) sequence are very helpful in showing the boundedness of the sequence in the asymptotically linear case.

**Theorem 2.1** ([4]). Let $E$ be a real Banach space with its dual space $E^*$, and suppose that $I \in C^1(E, \mathbb{R})$ satisfies
\[
\max\{I(0), I(e)\} \leq \mu < \eta \leq \inf_{\|u\| = \rho} I(u)
\]
for some $\mu < \eta, \rho > 0$ and $e \in E$ with $\|e\| = \rho$. Let $c \geq \eta$ be characterized by
\[
c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)),
\]
where $\Gamma = \{ \gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e \}$ is the set of continuous paths joining 0 and $e$. Then there exists a sequence $\{u_n\} \subset E$ such that
\[
I(u_n) \to c \geq \eta \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty.
\]
This kind of sequence is usually called a Cerami sequence.

**Lemma 2.1.** Assume that (A1), (A2) and (A3) hold. Then there exist $\rho > 0, \eta > 0$ such that
\[
\inf_{\|u\| = \rho} I(u) > \eta.
\]

**Proof.** By (A1) and (A2), for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that
\[
f(s) \leq \epsilon|s| + C_\epsilon|s|^{2^*-1} \quad \text{for all } x \in \mathbb{R},
\]
and then
\[
F(s) \leq \frac{\epsilon}{2}|s|^2 + \frac{C_\epsilon}{2^*}|s|^{2^*} \quad \text{for all } x \in \mathbb{R}.
\]
Moreover, by (A1), (A2) and (A3), there exists $C_1 > 0$ such that

$$a(x) \leq C_1 \quad \text{for all } x \in \mathbb{R}^N. \quad (2.3)$$

So, from (2.2), (2.3) and the Sobolev inequality, for any $u \in H$, we have

$$\left| \int \frac{e^{C_1}}{2} \int \|u\|^2 \, dx + \frac{C_1 C_2}{|x|^2} \int |u|^{2^*} \, dx \leq \frac{e^{C_1}}{2} \|u\|^2 + \frac{C_1 C_2}{|x|^2} \|u\|^{2^*} \right|.$$ 

Thus, one has

$$I_b(u) = \frac{\|u\|^2}{2} + \frac{b}{4} \left( \int |\nabla u|^2 \, dx \right)^2 - \int a(x)F(u) \, dx \geq \frac{1 - C_1}{2} \|u\|^2 - \frac{C_1 C_2}{|x|^2} \|u\|^{2^*},$$

thanks to the fact that $\frac{b}{4}(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx)$ is nonnegative. Fixing $\epsilon \in (0, C_1^{-1})$ and letting $\|u\| = \rho > 0$ be small enough, there exists $\eta > 0$ such that the desired conclusion holds.

**Lemma 2.2.** Assume that (A1), (A2) and (A3) hold. If $l > \mu$, then there is $b > 0$ such that for any $b \in (0, \hat{b})$, there exists $e \in H$ with $\|e\| > \rho$ such that $I_b(e) < 0$.

**Proof.** According to the definition of $\mu$ and $l > \mu$, there exists $\varphi \in H$, with $\varphi \geq 0$, such that

$$\int a(x)\varphi^2 \, dx = 1 \quad \text{and} \quad \mu \leq \int (|\nabla \varphi|^2 + \varphi^2) \, dx < l.$$ 

By (A2) and Fatou’s Lemma, we have

\[
\lim_{t \to +\infty} I_b(t\varphi) = \frac{\|\varphi\|^2}{2} - \lim_{t \to +\infty} \int a(x)\frac{F(t\varphi)}{t}\varphi^2 \, dx \leq \frac{1}{2} (\|\varphi\|^2 - l) < 0,
\]

by taking $e = t_0\varphi$ with $t_0$ large enough so that $I_0(e) = I_0(t_0\varphi) < 0$ and $\|e\| = t_0\|\varphi\| > \rho$. Since

\[
I_b(e) = I_0(e) + \frac{b}{4} \left( \int |e|^2 \, dx \right)^2
\]

is continuous and increasing in $b \geq 0$ and $I_0(e) < 0$, there exists $\hat{b} > 0$ sufficiently small such that $I_b(e) < 0$ for all $b \in (0, \hat{b})$.

By means of Lemmas 2.1–2.2 and Theorem 2.1, there exists a sequence $\{u_n\} \subset H$ such that

$$I_b(u_n) \to c \geq \eta \quad \text{and} \quad \left(1 + \|u_n\|\right)\|I_b'(u_n)\|_{H^{-1}} \to 0 \quad \text{as } n \to \infty. \quad (2.4)$$

In order to get the existence of a nontrivial nonnegative solution, we first show that this sequence is bounded.

**Lemma 2.3.** Let (A1), (A2) and (A3) hold, and let $l > \mu$. Then, for any $b \in (0, \hat{b})$, where $\hat{b}$ given by Lemma 2.2, the sequence $\{u_n\}$ defined in (2.4) is bounded in $H$.

**Proof.** Assume on the contrary that $\|u_n\| \to +\infty$ as $n \to +\infty$. Define $\omega_n = \frac{u_n}{\|u_n\|}$. Clearly, $\{\omega_n\}$ is bounded in $H$ and there exists $\omega \in H$ such that, going if necessary to a subsequence,

$$\omega_n \to \omega \quad \text{in } H, \quad \omega_n \to \omega \quad \text{in } L^2_{loc}(\mathbb{R}^N), \quad \omega_n \rightharpoonup \omega \quad \text{in } \mathbb{R}^N \text{ as } n \to +\infty. \quad (2.5)$$

We claim that $\omega \not\equiv 0$. Assume on the contrary that $\omega \equiv 0$. By (A3), there exists a constant $\theta \in (0, 1)$ such that

$$\sup \left\{ \frac{f(s)}{s} : s > 0 \right\} < \theta \inf \left\{ \frac{1}{a(x)} : |x| \geq R_0 \right\}. $$
For any $n \in \mathbb{N}$, this yields

$$
\int_{|x| \geq R_0} a(x) \frac{f(u_n)}{u_n} \omega_n^2 dx \leq \theta \int_{|x| \geq R_0} \omega_n^2 dx \leq \theta \|\omega_n\|^2 = \theta < 1.
$$

(2.6)

Since the embedding $H^1(B_{R_0}(0)) \hookrightarrow L^2(B_{R_0}(0))$ is compact, we have $\omega_n \rightharpoonup \omega$ in $L^2(B_{R_0}(0))$. According to [27, Lemma A.1], going if necessary to a subsequence, there exists $g \in L^2(B_{R_0}(0))$ such that

$$
|\omega_n| \leq g(x) \quad \text{a.e. in } B_{R_0}(0).
$$

(2.7)

By (A1) and (A2), there exists $C > 0$ such that

$$
\frac{f(t)}{t} \leq C \quad \text{for all } t \in \mathbb{R}.
$$

(2.8)

By (2.3), (2.7) and (2.8), for any $n \in \mathbb{N}$, we have

$$
0 \leq a(x) \frac{f(u_n)}{u_n} \omega_n^2 \leq Ca(x) \omega_n^2 \leq CC_1 \omega_n^2 \leq CC_1 g^2 \quad \text{a.e. in } B_{R_0}(0).
$$

(2.9)

Noting that $\omega_n \to \omega \equiv 0$ a.e. in $\mathbb{R}^N$, we obtain

$$
a(x) \frac{f(u_n)}{u_n} \omega_n^2 \to 0 \quad \text{a.e. in } \mathbb{R}^N.
$$

(2.10)

It follows, from (2.9), (2.10) and the dominated convergence theorem, that

$$
\lim_{n \to \infty} \int_{|x| < R_0} a(x) \frac{f(u_n)}{u_n} \omega_n^2 dx = 0.
$$

(2.11)

Thus, by (2.6) and (2.11), we get

$$
\limsup_{n \to \infty} \int_{\mathbb{R}^N} a(x) \frac{f(u_n)}{u_n} \omega_n^2 dx < 1.
$$

(2.12)

Since $\|u_n\| \to +\infty$ as $n \to \infty$, it follows from (2.4) that

$$
o(1) = \frac{\langle I'_b(u_n), u_n \rangle}{\|u_n\|^2} = 1 + \frac{b \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^2}{\|u_n\|^2} - \int_{\mathbb{R}^N} a(x) \frac{f(u_n)}{u_n} \omega_n^2 dx \geq 1 - \int_{\mathbb{R}^N} a(x) \frac{f(u_n)}{u_n} \omega_n^2 dx.
$$

Therefore,

$$
\int_{\mathbb{R}^N} a(x) \frac{f(u_n)}{u_n} \omega_n^2 dx + o(1) \geq 1,
$$

which contradicts (2.12), so $\omega \not\equiv 0$.

On the other hand, since $\|u_n\| \to +\infty$ as $n \to \infty$, it follows form (2.4) that

$$
\frac{\langle I'_b(u_n), u_n \rangle}{\|u_n\|^2} = o(1),
$$

that is,

$$
o(1) = \frac{1}{\|u_n\|^2} + b \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^2 - \frac{\int_{\mathbb{R}^N} a(x) \frac{f(u_n)}{u_n} \omega_n^2 dx}{\|u_n\|^2}.
$$

(2.13)

From (2.3) and (2.8), one has

$$
\frac{\int_{\mathbb{R}^N} a(x) \frac{f(u_n)}{u_n} \omega_n^2 dx}{\|u_n\|^2} = o(1).
$$

(2.14)

From (2.13) and (2.14), it is clear that

$$
b \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^2 = o(1).
$$
Fatou’s Lemma yields
\[
0 = \liminf_{n \to \infty} b \left( \int_{\mathbb{R}^n} |\nabla u_n|^2 \, dx \right)^2 \geq b \left( \int_{\mathbb{R}^n} |\nabla \omega|^2 \, dx \right)^2 \geq 0,
\]
that is,
\[
\left( \int_{\mathbb{R}^n} |\nabla \omega|^2 \, dx \right)^2 = 0. \tag{2.15}
\]

Since the embedding \( H \hookrightarrow D^{1,2}(\mathbb{R}^N) \) is continuous, \( \omega \) also belongs to \( D^{1,2}(\mathbb{R}^N) \). According to the definition of the norm of \( D^{1,2}(\mathbb{R}^N) \) and (2.15), \( \omega \equiv 0 \). This is a contradiction. So, the sequence \( \{u_n\} \) is bounded in \( H \). \( \square \)

To prove that the Cerami sequence \( \{u_n\} \) in (2.4) converges to a nonzero critical point of \( I_b \), we need the following compactness lemma.

**Lemma 2.4.** Assume that (A1), (A2) and (A3) hold. Then, for any \( \epsilon > 0 \), there exists \( R(\epsilon) > R_0 \) and \( n(\epsilon) \) such that
\[
\int_{|x| \geq R} (|\nabla u_n|^2 + u_n^2) \, dx \leq \epsilon
\]
for all \( R \geq R(\epsilon) \) and \( n \geq n(\epsilon) \).

**Proof.** Let \( \xi_R : \mathbb{R}^N \to [0, 1] \) be a smooth function such that
\[
\xi_R(x) = \begin{cases} 0, & 0 \leq |x| \leq R, \\ 1, & |x| \geq 2R, \end{cases} \tag{2.16}
\]
and, for some constant \( C_0 > 0 \) (independent of \( R \)),
\[
|\nabla \xi_R(x)| \leq \frac{C_0}{R} \quad \text{for all } x \in \mathbb{R}^N.
\]

Then, for all \( n \in \mathbb{N} \) and \( R \geq R_0 \), we have
\[
\int_{\mathbb{R}^N} |\nabla (u_n \xi_R)|^2 \, dx \leq 2 \int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R^2 \, dx + \int_{\mathbb{R}^N} |u_n|^2 |\nabla \xi_R|^2 \, dx
\]
\[
\leq 2 \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \frac{2C_0^2}{R^2} \int_{\mathbb{R}^N} |u_n|^2 \, dx
\]
\[
\leq 2 \left( 1 + \frac{C_0^2}{R^2} \right) \|u_n\|^2
\]
\[
\leq 2 \left( 1 + \frac{C_0^2}{R^2} \right) \|u_n\|^2.
\]

This implies that
\[
\|u_n \xi_R\| \leq \sqrt{2} \left( 2 + \frac{C_0^2}{R^2} \right)^{\frac{1}{2}} \|u_n\| \tag{2.17}
\]
for all \( n \in \mathbb{N} \) and \( R \geq R_0 \). By (2.4), \( \|I'(u_n)\|_{H^{-1}} \|u_n\| \to 0 \) as \( n \to \infty \). So, for any \( \epsilon > 0 \), there exists \( n(\epsilon) > 0 \) such that
\[
\|I'(u_n)\|_{H^{-1}} \|u_n\| \leq \frac{\epsilon}{\sqrt{2} \left( 2 + \frac{C_0^2}{R^2} \right)^{\frac{1}{2}}} \tag{2.18}
\]
for all \( n \geq n(\epsilon) \). Hence, it follows from (2.17) and (2.18) that
\[
|\langle I'(u_n), u_n \xi_R \rangle| \leq \|I'(u_n)\|_{H^{-1}} \|u_n \xi_R\| \leq \epsilon \tag{2.19}
\]
for all \( R \geq R_0 \) and \( n \geq n(\epsilon) \). Note that
\[
\langle I'(u_n), u_n \xi_R \rangle = \int_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R^2 \, dx + \int_{\mathbb{R}^N} u_n^2 \xi_R^2 \, dx + b \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \int_{\mathbb{R}^N} |\nabla \xi_R|^2 \, dx + \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \xi_R \, dx
\]
\[
+ b \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \xi_R \, dx - \int_{\mathbb{R}^N} a(x)f(u_n)u_n \xi_R \, dx. \tag{2.20}
\]
For any $\epsilon > 0$, there exists $R(\epsilon) > R_0$ such that
\[
\frac{1}{R^2} \leq \frac{4\epsilon^2}{C_0^2} \quad \text{for all } R \geq R(\epsilon).
\] (2.21)

By (2.21) and Young's inequality, for all $n \in \mathbb{N}$ and $R \geq R(\epsilon)$, we get
\[
\int_{\mathbb{R}^n} |u_n \nabla u_n \nabla \xi_R| \, dx \leq \epsilon \int_{\mathbb{R}^n} |\nabla u_n|^2 \, dx + \frac{1}{4\epsilon} \int_{|x| \leq 2R} |u_n|^2 \frac{C_0^2}{R^2} \, dx
\leq \epsilon \int_{\mathbb{R}^n} |\nabla u_n|^2 \, dx + \epsilon \int_{|x| \leq 2R} |u_n|^2 \, dx
\leq \epsilon \|u_n\|^2.
\] (2.22)

By (A1), (A2), (A3) and (2.16), there exists $\eta_1 \in (0, 1)$ such that for all $n \in \mathbb{N}$ and $R \geq R_0$,
\[
\int_{\mathbb{R}^n} |a(x)f(u_n)u_n \xi_R| \, dx \leq \eta_1 \int_{\mathbb{R}^n} u_n^2 \xi_R \, dx.
\] (2.23)

In virtue of the fact that $b \int_{\mathbb{R}^n}|\nabla u_n|^2 \, dx \int_{\mathbb{R}^n}|\nabla u_n|^2 \xi_R \, dx$ is nonnegative, together with (2.20), (2.22) and (2.23), for all $n \in \mathbb{N}$ and $R \geq R(\epsilon) \geq R_0$, we have
\[
\langle I'(u_n), u_n \xi_R \rangle \geq \int_{\mathbb{R}^n} |\nabla u_n|^2 \xi_R \, dx + (1 - \eta_1) \int_{\mathbb{R}^n} u_n^2 \xi_R \, dx - \epsilon \|u_n\|^2 - \epsilon b \|u_n\|^4.
\] (2.24)

Since the sequence $\{u_n\}$ is bounded in $H$, it follows from (2.19) and (2.24) that there exists $C_3 > 0$ such that for all $R \geq R(\epsilon)$ and $n \geq n(\epsilon)$,
\[
\int_{\mathbb{R}^n} |\nabla u_n|^2 \xi_R \, dx + (1 - \eta_1) \int_{\mathbb{R}^n} u_n^2 \xi_R \, dx \leq C_3 \epsilon.
\]

From $\eta_1 \in (0, 1)$ and (2.16), the desired conclusion easily follows. \hfill \Box

## 3 Proof of Theorem 1.1

Now we are in a position to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 2.3, the sequence $\{u_n\}$ defined in (2.4) is bounded in $H$. Since $H$ is a reflexive space, going if necessary to a subsequence, $u_n \rightharpoonup u$ in $H$ for some $u \in H$. In order to prove the theorem, we need to show that $\int_{\mathbb{R}^n}|\nabla u_n|^2 \, dx \rightarrow \int_{\mathbb{R}^n}|\nabla u|^2 \, dx$ and that the sequence $\{u_n\}$ has a strong convergence subsequence in $H$, that is, $\|u_n\| \rightarrow \|u\|$ as $n \rightarrow \infty$. Note that, by (2.4),
\[
\langle I'_b(u_n), u_n \rangle = \int_{\mathbb{R}^n} (|\nabla u_n|^2 + u_n^2) \, dx + b \int_{\mathbb{R}^n} |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^n} a(x)f(u_n)u_n \, dx = o(1)
\] (3.1)

and
\[
\langle I'_b(u_n), u \rangle = \int_{\mathbb{R}^n} (|\nabla u_n \nabla u + u_n u| \, dx + b \int_{\mathbb{R}^n} |\nabla u_n|^2 \, dx \int_{\mathbb{R}^n} \nabla u_n \nabla u \, dx - \int_{\mathbb{R}^n} a(x)f(u_n)u \, dx = o(1).
\] (3.2)

Since $u_n \rightharpoonup u$ in $H$, we have
\[
\int_{\mathbb{R}^n} (|\nabla u_n \nabla u + u_n u| \, dx = \int_{\mathbb{R}^n} (|\nabla u|^2 + u^2) \, dx + o(1).
\] (3.3)
Since the embedding $H \hookrightarrow D^{1,2}(\mathbb{R}^N)$ is continuous, $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$, and thus
\begin{equation}
\int_{\mathbb{R}^N} \nabla u_n \nabla u \, dx = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + o(1). \tag{3.4}
\end{equation}

To show that
\begin{equation*}
\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \to \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \quad \text{and} \quad \|u_n\| \to \|u\| \quad \text{as} \quad n \to \infty,
\end{equation*}
we first prove that
\begin{equation}
\int_{|x| \geq R(\epsilon)} a(x)f(u_n)u_n \, dx \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}
\end{equation}

For any $\epsilon > 0$, by Lemma 2.4, (A3) and Hölder’s inequality, for $n$ large enough, we have
\begin{align*}
\int_{|x| \geq R(\epsilon)} a(x)f(u_n)u_n \, dx & \leq \int_{|x| \geq R(\epsilon)} (a^{1/2}|u_n - u|)(a^{1/2}|f(u_n)|) \, dx \\
& \leq \left( \int_{|x| \geq R(\epsilon)} a|u_n - u|^2 \, dx \right)^{1/2} \left( \int_{|x| \geq R(\epsilon)} |f(u_n)|^2 \, dx \right)^{1/2} \\
& \leq \left( \int_{|x| \geq R(\epsilon)} a|u_n - u|^2 \, dx \right)^{1/2} \left( \int_{|x| \geq R(\epsilon)} |u|^2 \, dx \right)^{1/2} \\
& \leq C_4 \epsilon.
\end{align*}

This and the compactness of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^N)$ imply (3.5). Since $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$, we assume that $\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \to \lambda \geq 0$. If $\lambda = 0$, by Fatou’s Lemma,
\begin{equation}
\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \to \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = 0 = \lambda \quad \text{as} \quad n \to \infty, \tag{3.6}
\end{equation}
so $u \equiv 0$ in $H$. From (3.1)–(3.6), it is easy to see that
\begin{equation}
\|u_n\|^2 \to \|u\|^2 = 0 \quad \text{as} \quad n \to \infty. \tag{3.7}
\end{equation}

But (3.6) and (3.7) invoke a contradiction with (2.4) in which $c > 0$. So $\lambda > 0$, and by Fatou’s Lemma we have
\[ \lambda \geq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx. \]

If $\lambda = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$, from (3.1)–(3.5), it is also easy to see that $\|u_n\| \to \|u\|$ in $H$ as $n \to \infty$, hence the proof is completed.

If $\lambda > \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$, from (3.1)–(3.5), we obtain
\begin{align*}
o(1) &= \int_{\mathbb{R}^N} (|\nabla u_n|^2 + u_n^2) \, dx - \int_{\mathbb{R}^N} (\nabla u_n \nabla u + u_n u) \, dx + b\left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \right)^2 \\
& \quad - b \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \int_{\mathbb{R}^N} \nabla u_n \nabla u \, dx + \int_{\mathbb{R}^N} a(x)f(u_n)u_n \, dx - \int_{\mathbb{R}^N} a(x)f(u_n)u_n \, dx \\
& = \int_{\mathbb{R}^N} (|\nabla u_n|^2 + u_n^2) \, dx - \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx + b\lambda \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) + o(1).
\end{align*}

By Fatou’s Lemma, it is easy to see from our assumption that
\[ 0 \geq b\lambda \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) = C_5 > 0, \]
which is impossible. So,  
\[ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \to \int_{\mathbb{R}^N} |\nabla u|^2 \, dx > 0 \quad \text{as} \quad n \to \infty. \]
Moreover, we get \( \|u_n\| \to \|u\| \) as \( n \to \infty \). Now we show that the solution \( u \) is nonnegative. Multiplying equation (1.1) by \( u^- \) and integrating over \( \mathbb{R}^N \), where \( u^- = \min\{u(x), 0\} \), we find
\[ \|u^-\|^2 + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \int |\nabla u^-|^2 \, dx = 0. \]
Hence, \( u^- = 0 \) and \( u \) is a nonnegative solution of problem (1.1).

To get a ground state solution, we denote by \( K \) the nontrivial critical set of \( I_b \). Set
\[ m := \inf\{I_b(u) : u \in K\}. \]
It is easy to see that \( K \) is nonempty. For any \( u \in K \), we have
\[ 0 = \langle I'_b(u), u \rangle = \|u\|^2 + b \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2 - \int_{\mathbb{R}^N} a(x)f(u)u \, dx \geq \|u\|^2 - \int_{\mathbb{R}^N} a(x)f(u)u \, dx. \]
Now we choose \( \epsilon \in (0, C^{-1}_1) \) as in the proof of Lemma 2.1 and use (2.1), (2.3) and the Sobolev embedding theorem to get
\[ \left| \int_{\mathbb{R}^N} a(x)f(u)u \, dx \right| \leq \left( \epsilon C_1 u^2 + C_1 C_\epsilon |u|^{2^*} \right)^{\frac{2}{2^*}} \leq \epsilon C_1 \|u\|^2 + C_2 C_\epsilon \|u|^{2^*}. \]
Therefore, for any \( u \in K \), we have
\[ 0 \geq \|u\|^2 - \epsilon C_1 \|u\|^2 - C_2 C_\epsilon \|u|^{2^*}. \quad (3.8) \]
We recall that \( u \neq 0 \) whenever \( u \in K \), and (3.8) implies
\[ \|u\| \geq \left( \frac{1 - \epsilon C_1}{C_2 C_\epsilon} \right)^{\frac{1}{2^*}} > 0 \quad \text{for all} \quad u \in K. \quad (3.9) \]
Hence, any limit point of a sequence in \( K \) is different from zero.

We claim that \( I_b \) is bounded from below on \( K \), i.e., there exists \( M > 0 \) such that \( I_b(u) \geq -M \) for all \( u \in K \). Otherwise, there exists \( \{u_n\} \subset K \) such that
\[ I_b(u_n) < -n \quad \text{for all} \quad n \in \mathbb{N}. \quad (3.10) \]
It follows from (2.3) that
\[ I_b(u_n) \geq \frac{1}{4} \|u_n\|^2 - CC_\epsilon \|u_n|^{2^*}. \]
This and (3.10) imply that \( \|u_n\| \to +\infty \) as \( n \to \infty \). Let \( \omega_n = \frac{u_n}{\|u_n\|} \). There exists \( \omega \in H \) such that (2.5) holds. Note that \( I'_b(\omega_n) = 0 \) for \( u_n \in K \). As in the proof of Lemma 2.3, we obtain that \( \|u_n\| \to +\infty \) is impossible. Then \( I_b \) is bounded from below on \( K \). So \( M \geq -M \). Let \( \{u_n\} \subset K \) be such that \( I_b(u_n) \to m \) as \( n \to \infty \). Then (2.4) holds for the sequence \( \{\tilde{u}_n\} \) and \( m \). Following almost the same procedures as in the proofs of Lemmas 2.3 and 2.4, and using the above arguments, we can show that \( \{\tilde{u}_n\} \) is bounded in \( H \) and, going if necessary to a subsequence, \( \|u_n\| \to \|\tilde{u}\| \), where \( \tilde{u} \in H \setminus \{0\} \) and \( \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \to \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx \) as \( n \to \infty \). There is only one difference in showing that (3.6) does not hold. Based on the aforementioned discussions, we know that the possible critical value \( c > 0 \). Here we do not know if \( m > 0 \), but (3.9) holds, and so \( \lambda > 0 \). Moreover, \( I_b(\tilde{u}) = m \) and \( I'_b(\tilde{u}) = 0 \). Therefore, \( \tilde{u} \in H \setminus \{0\} \) is a ground state solution of problem (1.1). Finally, we can also show that the ground state solution \( \tilde{u} \) is nonnegative. \( \square \)

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