Product-type operators between minimal Möbius invariant spaces and Zygmund type spaces

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Abstract. In this paper, we consider product-type operators $T_{u,v,\phi}^m$ from minimal Möbius invariant spaces into Zygmund type spaces. So some characterizations for boundedness and essential norm of these operators are obtained. As a result some conditions for the compactness will be given.

1. Introduction

By $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ is denoted the space of all analytic functions on $\mathbb{D}$. The classic Zygmund space $Z$ consists of all functions $f \in H(\mathbb{D})$ which are continuous on the closed unit ball $\overline{\mathbb{D}}$ and

$$\sup_{h > 0} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $\theta \in \mathbb{R}$ and $h > 0$. By [3, Theorem 5.3], an analytic function $f$ belongs to $Z$ if and only if $\sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)| < \infty$. Motivated by this, for each $\alpha > 0$, the Zygmund type space $Z_\alpha$ is defined to be the space of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{Z_\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f''(z)| < \infty.$$

The space $Z_\alpha$ is a Banach space equipped with the norm

$$\|f\|_{Z_\alpha} = |f(0)| + |f'(0)| + \|f\|_{sZ_\alpha},$$

for each $f \in Z_\alpha$.

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Let $\text{Aut}(\mathbb{D})$ be the group of all conformal automorphisms of $\mathbb{D}$ which is also called Möbius group. It is well-known that each element of $\text{Aut}(\mathbb{D})$ is of the form

$$e^{i\theta} \sigma_a(z) = e^{i\theta} \frac{a-z}{1-\overline{a}z}, \quad a, z \in \mathbb{D}, \quad \theta \in \mathbb{R}.$$ 

Let $X$ be a linear space of analytic functions on $\mathbb{D}$, which is complete. $X$ is called Möbius invariant if for each function $f$ in $X$ and each element $\psi$ in $\text{Aut}(\mathbb{D})$, the composition function $f \circ \psi$ also lies in $X$ and satisfies that $\|f \circ \psi\|_X = \|f\|_X$. For example, the space $H^\infty$ of all bounded analytic functions is Möbius invariant. Also the Besov space $B_p(1 < p < \infty)$, is Möbius invariant which is the space of all $f \in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2}dA(z) < \infty. \quad (1)$$

If $p = 2$, we have the well-known Dirichlet space. For $p = \infty$, $B_\infty = \mathcal{B}$ the classic Bloch space. The space $B_1$ which is called the minimal Möbius invariant is defined separately. The function $f \in H(\mathbb{D})$ belongs to $B_1$ if and only if it has representation as

$$f(z) = \sum_{k=1}^{\infty} c_k \sigma_{a_k} \quad \text{where} \quad a_k \in \mathbb{D} \quad \text{and} \quad \sum_{k=1}^{\infty} |c_k| < \infty.$$ 

The norm on $B_1$ defines as infimum of $\sum_{k=1}^{\infty} |c_k|$ for which the above statement holds. $B_1$ is contained in any Möbius invariant space and it has been proved that is the set of all analytic functions $f$ on $\mathbb{D}$ such that $f''$ lies in $L^1(\mathbb{D}, dA)$. Also there exist positive constants $C_1$ and $C_2$ such that

$$C_1 \|f\|_{B_1} \leq |f(0)| + |f'(0)| + \int_{\mathbb{D}} |f''(z)|dA(z) \leq C_2 \|f\|_{B_1}. \quad (2)$$

Let $u, v, \varphi \in H(\mathbb{D})$ and $\varphi : \mathbb{D} \rightarrow \mathbb{D}$. The Stević-Sharma type operator is defined as follows

$$T_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$ 

Indeed $T_{u,v,\varphi} = uC_\varphi + vC_\varphi D$ where $D$ is the differentiation operator and $C_\varphi$ is composition operator. More information about this operator can be found in [4, 5, 6].

The generalized Stević-Sharma type operator $T_{u,v,\varphi}^m$ is defined by the second author of this paper and et al. in [1] as follows

$$T_{u,v,\varphi}^m f(z) = (uC_\varphi f)(z) + (D_{\varphi,v}^m f)(z) = u(z)f(\varphi(z)) + v(z)f^{(m)}(\varphi(z)),$$

where $m \in \mathbb{N}$ and $D_{\varphi,v}^m$ is the generalized weighted composition operator. When $v = 0$, then $T_{u,0,\varphi}^m = uC_\varphi$ is the well-known weighted composition operator. If $u = 0$, then $T_{0,v,\varphi}^m = D_{\varphi,v}^m$ and for $m = 1$, $T_{u,v,\varphi} = T_{u,v,\varphi}$ is Stević-Sharma type operator. $T_{u,v,\varphi}^m$ also include other operators as well as product type operators which are studied
in several papers in recent years. The results of the papers can be stated to many
operators and obtained the results of the papers published before.

For Banach spaces $X$ and $Y$ and a continuous linear operator $T : X \to Y$, the
essential norm is the distance of $T$ from the space of all compact operators, that is

$$
\|T\|_e = \inf \{\|T - K\| : K : X \to Y \text{ is compact} \}.
$$

$T$ is compact if and only if $\|T\|_e = 0$.

In this paper, we study the operator-theoretic properties in minimal M"obius
invariant space. In section 2, we first bring some lemmas on the space $B_1$ and then
obtain some characterizations for boundedness of operator $T_{u,v,\varphi}^m : B_1 \to Z_\alpha$. In
section 3, some estimations for the essential norm of these operators are given. As
a result, some new criteria for the compactness of $T_{u,v,\varphi}^m$ are presented.

By $A \succeq B$ we mean there exists a constant $C$ such that $A \geq CB$ and $A \approx B$
means that $A \succeq B \succeq A$.

2. Boundedness

In this section, we give some necessary and sufficient conditions for the general-
ized Stević-Sharma type operators to be bounded. Firstly, we state some lemmas
which are needed for proving the main results.

According to the definition of the norm in minimal M"obius invariant space, for
each $f \in B_1$, $\|f\|_\infty \leq \|f\|_{B_1}$. Thus, from [7, Proposition 5.1.2] and [8, Proposition
8] we have the following lemma.

**Lemma 2.1.** Let $n \in \mathbb{N}$. Then there exists a positive constant $C$ such that for
each $f \in B_1$

$$
(1 - |z|^2)^n |f^{(n)}(z)| \leq C \|f\|_{B_1}.
$$

As a similar proof in Lemma 2.5 of [2] we get the following lemma.

**Lemma 2.2.** Let

$$
f_{j,a}(z) = \left( \frac{1 - |a|^2}{1 - \bar{a}z} \right)^j, \quad j \in \mathbb{N}, \ a \in \mathbb{D}.
$$

Then

$$
f_{j,a} \in B_1, \quad \sup_{a \in \mathbb{D}} \|f_{j,a}\|_{B_1} < \infty.
$$

Moreover, $\{f_{j,a}\} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$.

The proof of the following lemmas are similar to the proof of Lemmas 2.6 and
2.5 [1], so they are omitted.
Lemma 2.3. For any \( m \in \mathbb{N} - \{1, 2\}, \ 0 \neq a \in \mathbb{D} \) and \( i, k \in \{0, 1, 2, m, m + 1, m + 2\} \), there exists a function \( g_{i,a} \in B_1 \) such that
\[
g^{(k)}_{i,a}(a) = \frac{\delta_{ik} a^k}{(1 - |a|^2)^k},
\]
where \( \delta_{ik} \) is Kronecker delta. For each \( i \in \{0, 1, 2\} \) and \( i \in \{m, m + 1, m + 2\} \) respectively
\[
g_{i,a}(z) = \sum_{j=1}^{3} c^i_j f_{j,a}(z), \quad g_{i,a}(z) = \sum_{j=m+1}^{m+3} c^i_j f_{j,a}(z)
\]
where \( c^i_j \) is independent of \( a \).

Lemma 2.4. Let \( m = 1 \) or \( 2, \ 0 \neq a \in \mathbb{D} \) and \( i, k \in \{0, 1, \cdots, m + 2\} \), there exists a function \( g_{i,a} \in B_1 \) such that
\[
g^{(k)}_{i,a}(a) = \frac{\delta_{ik} a^k}{(1 - |a|^2)^k}.
\]

Let \( f \in B_1 \). Then
\[
||T^m_{u,v,\varphi} f||_{Z_a} = |T^m_{u,v,\varphi} f(0)| + |(T^m_{u,v,\varphi} f)'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha ||T^m_{u,v,\varphi} f''(z)||.
\]

We compute the above sentences separately. We have
\[
(T^m_{u,v,\varphi} f)'(0) = u'(0)f(\varphi(0)) + u(0)\varphi'(0)f'(\varphi(0)) + v'(0)f^{(m)}(\varphi(0)) + v(0)\varphi'(0)f^{(m+1)}(\varphi(0)).
\]

And
\[
(T^m_{u,v,\varphi} f)''(z) = \sum_{i=0}^{2} (I_i(z)f^{(i)}(\varphi(z)) + I_{i+m}(z)f^{(i+m)}(\varphi(z))),
\]
where,
\[
I_0 = u'' \quad I_1 = 2u'\varphi' + u\varphi'' \quad I_2 = u\varphi^2 \\
I_m = v'' \quad I_{m+1} = 2v'\varphi' + v\varphi'' \quad I_{m+2} = v\varphi^2
\]

Theorem 2.5. Let \( \alpha > 0, \ u, v, \varphi \in H(\mathbb{D}), \ \varphi : \mathbb{D} \to \mathbb{D} \) and \( m > 2 \) be an integer. Then the following conditions are equivalent:

(i) The operator \( T^m_{u,v,\varphi} : B_1 \to Z_a \) is bounded.
(ii) For each \( j \in \{0, 1, 2, m, m + 1, m + 2\} = \Omega \)
\[
\max\{\sup_{a \in \mathbb{D}} ||T^m_{u,v,\varphi} f_{j,a}||_{Z_a}, \ \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |I_j(z)|\} < \infty.
\]
(iii)
\[
\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\alpha |I_k(z)|}{(1 - |\varphi(z)|^2)^k} < \infty, \quad k \in \Omega.
\]
PROOF. (iii) ⇒ (i) Suppose that \( f \in B_1 \). By using Lemma 2.1

\[
(1 - |z|^2)^\alpha |(T_{u,v,\phi}^m f)'(z)| = (1 - |z|^2)^\alpha \sum_{k \in \Omega} |I_k(z)| f^{(k)}(\phi(z)) \\
\leq \sum_{k \in \Omega} |I_k(z)|(1 - |z|^2)^\alpha |f^{(k)}(\phi(z))| \\
\leq C \sum_{k \in \Omega} (1 - |z|^2)^\alpha |I_k(z)| (1 - |\phi(z)|^2)^k \|f\|_{B_1}.
\]

Also using the fact that \( \|f\|_\infty \leq \|f\|_B \) and Lemma 2.1, we have

\[
|T_{u,v,\phi}^m f(0)| \leq |u(0) f(\phi(0))| + |v(0) f^{(m)}(\phi(0))| \\
\leq |u(0)||f|_{B_1} + C \frac{|v(0)|}{(1 - |\phi(0)|^2)^m} \|f\|_{B_1}
\]

and

\[
|(T_{u,v,\phi}^m f)'(0)| \leq C \left( |u'(0)| + \frac{|u(0) \phi'(0)|}{1 - |\phi(0)|^2} + \frac{|v'(0)|}{(1 - |\phi(0)|^2)^m} + \frac{|v(0) \phi'(0)|}{(1 - |\phi(0)|^2)^{m+1}} \right) \|f\|_{B_1}.
\]

Therefore \( T_{u,v,\phi}^m \colon B_1 \to \mathcal{L}_\alpha \) is bounded.

(i) ⇒ (ii) Suppose that \( T_{u,v,\phi}^m \colon B_1 \to \mathcal{L}_\alpha \) be a bounded operator. Lemma 2.2 implies that \( \|f_{j,a}\|_{B_1} < \infty \). So

\[
\|T_{u,v,\phi}^m f_{j,a}\|_{\mathcal{L}_\alpha} \leq \|T_{u,v,\phi}^m\| \|f_{j,a}\|_{B_1} < \infty.
\]

Then

\[
\sup_{a \in \mathbb{D}} \|T_{u,v,\phi}^m f_{j,a}\|_{\mathcal{L}_\alpha} \leq \|T_{u,v,\phi}^m\| \sup_{a \in \mathbb{D}} \sup_{j \in \Omega} \|f_{j,a}\|_{B_1} < \infty.
\]

Define \( f_0(z) = 1 \in B_1 \). Boundedness of the operator implies that

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |I_0(z)| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |u'(z)| \leq \|T_{u,v,\phi}^m f_0\|_{\mathcal{L}_\alpha} \leq \|T_{u,v,\phi}^m\| \|f_0\|_{B_1} < \infty.
\]

Take \( f_1(z) = z \in B_1 \). Then we have

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |u''(z)\phi(z) + 2u'(z)\phi'(z) + u(z)\phi''(z)| \leq \|T_{u,v,\phi}^m f_1\|_{\mathcal{L}_\alpha} \leq \|T_{u,v,\phi}^m\| \|f_1\|_{B_1} < \infty.
\]

Using the previous equations, we can get that

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |I_1(z)| < \infty.
\]

Similarly by employing the functions \( f_2(z) = z^2, f_m(z) = z^m, f_{m+1}(z) = z^{m+1} \) and \( f_{m+2}(z) = z^{m+2} \) for operator \( T_{u,v,\phi}^m \) we get the other part of (ii).
\[(ii) \Rightarrow (iii)\] For any \(i \in \Omega\) and \(a \in \mathbb{D}\), by applying Lemma 2.3 we have
\[
\frac{(1 - |z|^2) \alpha |I_i(a)||\varphi(a)|^i}{(1 - |\varphi(a)|^2)^i} \leq \frac{(1 - |z|^2) \alpha |(T_m^m u,v,\varphi g_i,\varphi(a))^\prime\prime(a)|}{(1 - |\varphi(a)|^2)^i} \\
\leq \sup_{a \in \mathbb{D}} \|T_m^m u,v,\varphi g_i,\varphi(a)\|_{\mathbb{D}} \\
\leq \sup_{a \in \mathbb{D}} \|T_m^m u,v,\varphi f_{j,a}\|_{\mathbb{D}}, \quad \sum_{j=m+1}^{m+3} c_j \sup_{a \in \mathbb{D}} \|T_m^m u,v,\varphi f_{j,a}\|_{\mathbb{D}} \\
< \infty.
\]
So, for any \(i \in \Omega\)
\[
\sup_{|\varphi(a)| > 1/3} \frac{(1 - |z|^2) \alpha |I_i(a)|}{(1 - |\varphi(a)|^2)^i} < \infty. \tag{4}
\]
On the other hand
\[
\sup_{|\varphi(a)| \leq 1/3} \frac{(1 - |z|^2) \alpha |I_i(a)|}{(1 - |\varphi(a)|^2)^i} \leq C \sup_{a \in \mathbb{D}} (1 - |z|^2) \alpha |I_i(a)| < \infty. \tag{5}
\]
From the last inequalities, we get the desired result. \(\square\)

In the special case \(m \leq 2\), by using Lemma 2.4, we have the following theorems which is stated without proof.

**Theorem 2.6.** Let \(\alpha > 0\), \(u,v,\varphi \in H(\mathbb{D})\) and \(\varphi : \mathbb{D} \to \mathbb{D}\). Then the following conditions are equivalent:

(i) The operator \(T_{u,v,\varphi}^2 : B_1 \to \mathbb{Z}_\alpha\) is bounded.

(ii) For \(j \in \{1, \ldots, 5\}\), \(\sup_{a \in \mathbb{D}} \|T^m_{u,v,\varphi} f_{j,a}\|_{\mathbb{D}} < \infty\) and
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2) \alpha |u''(z)| + |(2u' \varphi' + u \varphi'')(z)| + |(u \varphi'^2 + v \varphi'')(z)| < \infty,
\]
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2) \alpha |(2v' \varphi' + v \varphi'')(z)| + |(v \varphi'^2)(z)| < \infty.
\]

(iii)
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2) \alpha \left(\frac{|u''(z)|}{(1 - |\varphi(z)|^2)} + \frac{|(2u' \varphi' + u \varphi'')(z)|}{(1 - |\varphi(z)|^2)^2}\right) < \infty,
\]
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2) \alpha \left(\frac{|(2v' \varphi' + v \varphi'')(z)|}{(1 - |\varphi(z)|^2)^3} + \frac{|(v \varphi'^2)(z)|}{(1 - |\varphi(z)|^2)^4}\right) < \infty.
\]

**Theorem 2.7.** Let \(\alpha > 0\), \(u,v,\varphi \in H(\mathbb{D})\) and \(\varphi : \mathbb{D} \to \mathbb{D}\). Then the following conditions are equivalent:

(i) The operator \(T_{u,v,\varphi} : B_1 \to \mathbb{Z}_\alpha\) is bounded.
(ii) For \( j \in \{1, \ldots, 4\} \), \( \sup_{a \in \mathbb{D}} \|T_{u,v,\varphi} f_{j,a} \|_{Z_\alpha} < \infty \) and
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} |u''(z)| + |(2u'\varphi' + w\varphi'' + v''')(z)| < \infty,
\]
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} |(u\varphi'^2 + 2v'\varphi' + v\varphi''')(z)| + |(v\varphi'^2)(z)| < \infty.
\]

(iii)
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left( |u''(z)| + \frac{|(2u'\varphi' + w\varphi'' + v''')(z)|}{(1 - |\varphi(z)|^2)} \right) < \infty,
\]
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left( \frac{|(u\varphi'^2 + 2v'\varphi' + v\varphi''')(z)| + |(v\varphi'^2)(z)|}{(1 - |\varphi(z)|^2)^2} \right) < \infty.
\]

3. Essential Norm

In this section, some estimations for the essential norm of the operator \( T_{u,v,\varphi}^m \) from minimal M"obius invariant spaces into Zygmund type spaces are given.

**Theorem 3.1.** Let \( u, v, \varphi \in H(\mathbb{D}) \), \( \varphi : \mathbb{D} \to \mathbb{D} \) and \( 2 < m \in \mathbb{N} \). Let the operator \( T_{u,v,\varphi}^m : B_1 \to Z_\alpha \) be bounded then
\[
\|T_{u,v,\varphi}^m\|_e \approx \max\{E_i\}_{i=1}^6 \approx \max\{F_k\}_{k \in \{0,1,2,m,m+1,m+2\}}
\]
where,
\[
E_i = \limsup_{|a| \to 1} \|T_{u,v,\varphi} f_{i,a}\|_{Z_\alpha} \quad \text{and} \quad F_k = \limsup_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^{\alpha}|I_k(z)|}{(1 - |\varphi(z)|^2)^k}.
\]

**Proof.** First we prove the lower estimates. Suppose that \( K : B_1 \to Z_\alpha \) be an arbitrary compact operator. Since \( \{f_{i,a}\} \) is a bounded sequence in \( B_1 \) and converges to 0 uniformly on compact subsets of \( \mathbb{D} \) as \( |a| \to 1 \), we have \( \limsup_{|a| \to 1} \|K f_{i,a}\|_{Z_\alpha} = 0 \). So
\[
\|T_{u,v,\varphi}^m - K\|_{B_1 \to Z_\alpha} \geq \limsup_{|a| \to 1} \|(T_{u,v,\varphi}^m - K) f_{i,a}\|_{Z_\alpha} = E_i.
\]

Then
\[
\|T_{u,v,\varphi}^m\|_e = \inf_K \|T_{u,v,\varphi}^m - K\|_{B_1 \to Z_\alpha} \geq \max\{E_i\}_{i=1}^6.
\]

For the other part, let \( \{z_j\}_{j \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) such that \( |\varphi(z_j)| \to 1 \) as \( j \to \infty \). Since \( T_{u,v,\varphi}^m : B_1 \to Z_\alpha \) is bounded, using Lemmas 2.2 and 2.3 for any compact operator \( K : B_1 \to Z_\alpha \) and \( i \in \{0,1,2,m,m+1,m+2\} \), we obtain
\[
\|T_{u,v,\varphi}^m - K\|_{B_1 \to Z_\alpha} \geq \limsup_{j \to \infty} \|T_{u,v,\varphi}^m (g_j, \varphi(z_j))\|_{Z_\alpha} \geq \limsup_{j \to \infty} \|K(g_j, \varphi(z_j))\|_{Z_\alpha} \geq \limsup_{j \to \infty} \frac{(1 - |z_j|^2)^{\alpha}|\varphi(z_j)|^i I_i(z_j)}{(1 - |\varphi(z_j)|^2)^i} = F_i.
\]
Therefore,
\[ \|T_{u,v,\varphi}^m\|_{e} = \inf_{K} \|T_{u,v,\varphi}^m - K\|_{B_{1} \to Z_{\alpha}} \geq \max\{F_{i}\}. \]

Now we prove the upper estimates. Consider the operators \( K_{r} \) on \( B_{1} \), \( K_{r}f(z) = f rz \), where \( 0 < r < 1 \). \( K_{r} \) is a compact operator and \( \|K_{r}\| \leq 1 \). Let \( \{r_{j}\} \subseteq (0,1) \) be a sequence such that \( r_{j} \to 1 \) as \( j \to \infty \). For any positive integer \( j \), the operator \( T_{u,v,\varphi}^m K_{r_{j}} : B_{1} \to Z_{\alpha} \) is compact. Thus
\[ \|T_{u,v,\varphi}^m\|_{e} \leq \limsup_{j \to \infty} \|T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_{j}}\|. \]

So it will be sufficient to prove that
\[ \limsup_{j \to \infty} \|T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_{j}}\| \leq \min\{\max\{E_{i}\}, \max\{F_{i}\}\}. \]

For any \( f \in B_{1} \) such that \( \|f\|_{B_{1}} \leq 1 \),
\[ \|\left(T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_{j}}\right)f\|_{Z_{\alpha}} = A_{1} + A_{2} + A_{3} \]
where
\[ A_{1} := \|T_{u,v,\varphi}^m f(0) - T_{u,v,\varphi}^m f_{r_{j}}(0)\| = |u(0)(f - f_{r_{j}})(\varphi(0)) + v(0)(f - f_{r_{j}})^{(m)}(\varphi(0))| \]
\[ A_{2} := \|(T_{u,v,\varphi}^m f - T_{u,v,\varphi}^m f_{r_{j}})'(0)\| = |u'(0)(f - f_{r_{j}})(\varphi(0)) + u(0)\varphi'(0)(f - f_{r_{j}})'(\varphi(0)) + v'(0)(f - f_{r_{j}})^{(m)}(\varphi(0)) + v(0)\varphi'(0)(f - f_{r_{j}})^{(m+1)}(\varphi(0))| \]
\[ A_{3} := \sup_{z \in D}(1 - \|z\|^{2})^{\alpha}\left|\left(T_{u,v,\varphi}^m f - T_{u,v,\varphi}^m f_{r_{j}}\right)^{''}(z)\right| \]
\[ = \sup_{z \in D}(1 - \|z\|^{2})^{\alpha}\left|\sum_{k \in \{0,1,2,m,m+1,m+2\}} I_{k}(z)(f - f_{r_{j}})^{(k)}(\varphi(z))\right| \]
\[ \leq \sup_{|\varphi(z)| \leq r_{N}} (1 - \|z\|^{2})^{\alpha}\sum_{k \in \{0,1,2,m,m+1,m+2\}} I_{k}(z)(f - f_{r_{j}})^{(k)}(\varphi(z)) \]
\[ + \sup_{|\varphi(z)| > r_{N}} (1 - \|z\|^{2})^{\alpha}\sum_{k \in \{0,1,2,m,m+1,m+2\}} I_{k}(z)(f - f_{r_{j}})^{(k)}(\varphi(z)) \]
\[ =: A_{4} + A_{5} \]

Since \( (f - f_{r_{j}})^{(i)} \to 0 \) uniformly on compact subsets of \( D \) as \( j \to \infty \), for any nonnegative integer \( i \), then using Theorem 2.5, we get
\[ \limsup_{j \to \infty} A_{1} = \limsup_{j \to \infty} A_{2} = \limsup_{j \to \infty} A_{4} = 0. \]
About $A_5$, we get
\[
A_5 \leq \sum_{k \in \{0,1,2,m,m+1,m+2\}} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)\alpha |I_k(z)| |f^{(k)}(\varphi(z))| \\
+ \sum_{k \in \{0,1,2,m,m+1,m+2\}} \sup_{|\varphi(z)| > r_N} (1 - |z|^2)\alpha |I_k(z)| |r_j f^{(k)}(r_j \varphi(z))| \\
= \sum_{k \in \{0,1,2,m,m+1,m+2\}} A_{k,6} + \sum_{k \in \{0,1,2,m,m+1,m+2\}} A_{k,7}.
\]

For $A_{k,6}$, using Lemmas 2.1, 2.2 and 2.3, we obtain
\[
A_{k,6} = \sup_{|\varphi(z)| > r_N} \frac{(1 - |\varphi(z)|^2)^k f^{(k)}(\varphi(z))}{|\varphi(z)|^k} \frac{(1 - |z|^2)^\alpha |I_k(z)| |\varphi(z)|^k}{1 - |\varphi(z)|^2)^k} \\
\leq ||f||_{B_1} \sup_{|\varphi(z)| > r_N} \|(T_{u,v,\varphi}^{m} g_k \varphi(z))\| \|z\alpha \\
\leq \sum_{j=1}^{6} c_j^k \sup_{|a| > r_N} \|(T_{u,v,\varphi}^{m} f_{j,a})\| \|z\alpha
\]
where $k \in \{0,1,2,m,m+1,m+2\}$. As $N \to \infty$,
\[
\lim_{j \to \infty} \sup A_{k,6} \leq \sum_{j=1}^{6} \lim_{|a| \to 1} \sup \|(T_{u,v,\varphi}^{m} f_{j,a})\| \|z\alpha \leq \max\{E_j\}_{j=1}^{6}.
\]

Also for $A_{k,6}$ we can write
\[
A_{k,6} = \sup_{|\varphi(z)| > r_N} \frac{(1 - |\varphi(z)|^2)^k f^{(k)}(\varphi(z))}{|\varphi(z)|^k} \frac{(1 - |z|^2)^\alpha |I_k(z)|}{1 - |\varphi(z)|^2)^k} \\
\leq ||f||_{B_1} \sup_{|\varphi(z)| > r_N} \left( \frac{(1 - |\varphi(z)|^2)^\alpha |I_k(z)|}{(1 - |\varphi(z)|^2)^k} \right)
\]
which can be deduced that
\[
\lim_{j \to \infty} \sup A_{k,6} \leq \lim_{|\varphi(z)| \to 1} \sup \left( \frac{(1 - |\varphi(z)|^2)^\alpha |I_k(z)|}{(1 - |\varphi(z)|^2)^k} \right) \leq \max\{F_k\}_{1}^{6}.
\]

A similar argument can be done for $A_{k,7}$. Thus we prove that
\[
\sup_{||f||_{B_1} \leq 1} \|(T_{u,v,\varphi}^{m} - T_{u,v,\varphi}^{m} K_{r_j}) f\| \|z\alpha \leq \max\{E_j\}_{1}^{6}
\]
and
\[
\sup_{||f||_{B_1} \leq 1} \|(T_{u,v,\varphi}^{m} - T_{u,v,\varphi}^{m} K_{r_j}) f\| \|z\alpha \leq \max\{F_k\}_{1}^{6}.
\]

Finally we have
\[
\lim_{j \to \infty} \sup ||T_{u,v,\varphi}^{m} - T_{u,v,\varphi}^{m} K_{r_j}|| \|z\alpha \leq \min\{\max\{E_j\}_{1}^{6}, \max\{F_k\}_{1}^{6}\}.
\]
\[
\square
\]
In the case $m \leq 2$, the similar result can be stated using Theorems 2.6 and 2.7.

**Theorem 3.2.** Let $\alpha > 0$, $u, v, \varphi \in H(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ and the operator $T_{u,v,\varphi}^2 : B_1 \to Z_{\alpha}$ be bounded. Then

\[
\|T_{u,v,\varphi}^2\|_c \approx \max \{\limsup_{|\alpha| \to 1} \|T_{u,v,\varphi}^2 f_{j,a}\|_{Z_{\alpha}}\}^5 \approx
\]

\[
\limsup_{|\varphi(z)| \to 1} (1 - |z|^2)^{\alpha} \left( |u''(z)| + \frac{|2u' \varphi' + u \varphi''(z)|}{(1 - |\varphi(z)|^2)} + \frac{|(u \varphi'^2 + v')(z)|}{(1 - |\varphi(z)|^2)^2} \right) +
\]

\[
\limsup_{|\varphi(z)| \to 1} (1 - |z|^2)^{\alpha} \left( \frac{|(2v' \varphi' + v \varphi''(z))|}{(1 - |\varphi(z)|^2)^3} + \frac{|(v \varphi'^2)(z)|}{(1 - |\varphi(z)|^2)^4} \right).
\]

**Theorem 3.3.** Let $\alpha > 0$, $u, v, \varphi \in H(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ and the operator $T_{u,v,\varphi} : B_1 \to Z_{\alpha}$ be bounded. Then

\[
\|T_{u,v,\varphi}\|_c \approx \max \{\limsup_{|\alpha| \to 1} \|T_{u,v,\varphi} f_{j,a}\|_{Z_{\alpha}}\}^4 \approx
\]

\[
\limsup_{|\varphi(z)| \to 1} \left( |u''(z)| + \frac{|2u' \varphi' + u \varphi''(z)|}{(1 - |\varphi(z)|^2)} \right) +
\]

\[
\limsup_{|\varphi(z)| \to 1} (1 - |z|^2)^{\alpha} \left( \frac{|(u \varphi'^2 + 2v' \varphi' + v \varphi''(z))|}{(1 - |\varphi(z)|^2)^2} + \frac{|(v \varphi'^2)(z)|}{(1 - |\varphi(z)|^2)^3} \right).
\]

By using Theorem 3.1, we have the following Corollary.

**Corollary 3.4.** Let $u, v, \varphi \in H(\mathbb{D})$, $\varphi : \mathbb{D} \to \mathbb{D}$ and $2 < m \in \mathbb{N}$. Let operator $T_{u,v,\varphi}^m : B_1 \to Z_{\alpha}$ be bounded. Then the following conditions are equivalent:

(i) The operator $T_{u,v,\varphi}^m : B_1 \to Z_{\alpha}$ is compact.

(ii) $\limsup_{|\alpha| \to 1} \|T_{u,v,\varphi}^m f_{j,a}\|_{Z_{\alpha}} = 0, \quad i = 1, 2, \ldots, 6$

(iii) $\limsup_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^{\alpha} |I_k(z)|}{(1 - |\varphi(z)|^2)^k} = 0, \quad k \in \{0, 1, 2, m, m + 1, m + 2\}$.

From Theorems 3.2 and 3.3, we obtain similar results for compactness of operator $T_{u,v,\varphi}^2 : B_1 \to Z_{\alpha}$ and $T_{u,v,\varphi} : B_1 \to Z_{\alpha}$ respectively.

**Remark 3.5.** By taking $u = 0 (v = 0)$, we can get the results of the paper for generalized (weighted) composition operators.

**References**

[1] Abbasi E, Liu Y and Hassanlou M. Generalized Stević-Sharma type operators from Hardy spaces into nth weighted type spaces, (2021) 45: 1543 – 1554. doi:10.3906/mat-2011-67
[2] Zhu X. Abbasi E and Ebrahimi A. A class of operator-related composition operators from the Besov spaces into the Bloch space, Bull. Iran. Math. Soc. (2021) 47, 171–184. https://doi.org/10.1007/s41980-020-00374-w

[3] Duren PL. Theory of $H^p$ Spaces. Academic Press, New York and London, 1970.

[4] Liu Y, Yu Y. On Stević-Sharma type operators from the Besov spaces into the weighted-type space $H^\infty_\mu$. Math. Inequal. Appl. 2019; 22(3): 1037–1053. doi: 10.7153/mia-2019-22-71

[5] Stević S, Sharma A, Bhat A. Products of multiplication composition and differentiation operators on weighted Bergman spaces. Appl. Math. Comput. 2011; 217: 8115–8125. doi: 10.1016/j.amc.2011.03.014

[6] Stević S, Sharma A, Bhat A. Essential norm of products of multiplications composition and differentiation operators on weighted Bergman spaces. Appl. Math. Comput. 2011; 218: 2386–2397. doi: 10.1016/j.amc.2011.06.055

[7] Zhu K. Operator Theory in Function Spaces. Pure and Applied Mathematics, Marcel Dekker, Inc., New York and Basel, 1990.

[8] Zhu K. Bloch type spaces of analytic functions. Rocky Mountain J. Math 1993; 23: 1143–1177. doi:10.1216/rmjm/1181072549

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