Post-Newtonian Theory and Dimensional Regularization

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Abstract. Inspiralling compact binaries are ideally suited for application of a high-order post-Newtonian (PN) gravitational wave generation formalism. To be observed by the LIGO and VIRGO detectors, these very relativistic systems (with orbital velocities \( v \sim 0.5c \) in the last rotations) require high-accuracy templates predicted by general relativity theory. Recent calculations of the motion and gravitational radiation of compact binaries at the 3PN approximation using the Hadamard self-field regularization have left undetermined a few dimensionless coefficients called ambiguity parameters. In this article we review the application of dimensional self-field regularization, within Einstein’s classical general relativity formulated in \( D \) space-time dimensions, which finally succeeded in clearing up the problem, by uniquely fixing the values of all the ambiguity parameters.

1. INTRODUCTION

The problem of the motion and gravitational radiation of compact objects in post-Newtonian (PN) approximations of general relativity is of crucial importance, for at least three reasons. First, the motion of \( N \) objects at the 1PN level, according to the Einstein–Infeld–Hoffmann equations [1], is routinely taken into account to describe the Solar System dynamics. Second, the gravitational radiation-reaction force, in reaction to the emission of gravitational radiation, which appears in the equations of motion at the 2.5PN order \( 1\sim c^5 \), has been experimentally verified, by the observation of the secular acceleration of the orbital motion of the Hulse–Taylor binary pulsar PSR 1913+16 [2, 3, 4, 5].

Last but not least, the forthcoming detection and analysis of the gravitational waves emitted by inspiralling compact binaries — two neutron stars or black holes driven into coalescence by emission of gravitational radiation — will necessitate the prior knowledge of the equations of motion and radiation field up to high post-Newtonian order. Inspiralling compact binaries are extremely promising sources of gravitational waves for the detectors LIGO, VIRGO, GEO and TAMA. The two compact objects steadily lose by gravitational radiation their orbital binding energy; as a result, the orbital separation between them decreases, and the orbital frequency increases. The frequency of the gravitational-wave signal, which equals twice the orbital frequency for the dominant harmonics, “chirps” in time (i.e. the signal becomes higher and higher pitched) until the two objects collide and merge.

Strategies to detect and analyze the very weak signals from compact binary inspiral involve matched filtering of a set of accurate theoretical template waveforms against the output of the detectors. Several analyses [6, 7, 8, 9, 10, 11] have shown that, in order to get sufficiently accurate theoretical templates, one must include post-Newtonian effects up to the 3PN level at least. To date, the templates have been completed through 3.5PN order for the phase evolution [12, 13, 14], and 2.5PN order for the amplitude corrections [15, 16]. Spin effects are known for the dominant relativistic spin-orbit coupling term at 1.5PN order and the spin-spin coupling term at 2PN order [17, 18, 19, 20, 21, 22], and also for the next-to-leading spin-orbit coupling at 2.5PN order [23, 24, 25, 26].

The main point about modelling the inspiralling compact binary is that a model made of two structureless point particles, characterized solely by two mass parameters \( m_1 \) and \( m_2 \) (and possibly two spins), is sufficient. Indeed, most of the non-gravitational effects usually plaguing the dynamics of binary star systems, such as the effects of a magnetic field, of an interstellar medium, and so on, are dominated by gravitational effects. However, the real justification for a model of point particles is that the effects due to the finite size of the compact bodies are small. Consider for instance the influence of the Newtonian quadrupole moments \( Q_1 \) and \( Q_2 \) induced by tidal interaction between two neutron stars.

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1 As usual \( n\)PN refers to the terms of order \( (v/c)^n \), where \( v \) is the typical orbital velocity of the objects and \( c \) is the speed of light. We shall generally abbreviate \( (v/c)^n \) as \( 1\sim c^n \),.
We suppose that the neutron stars have no intrinsic spins. Let \( a_1 \) and \( a_2 \) be the radius of the stars, and \( L \) the distance between the two centers of mass. We have, for tidal moments,

\[
Q_1 = k_1 m_2 \frac{a_1^5}{L^3}; \quad Q_2 = k_2 m_1 \frac{a_2^5}{L^3};
\]

(1)

where \( k_1 \) and \( k_2 \) are the star’s dimensionless (second) Love numbers, which depend on their internal structure, and are, typically, of the order unity. On the other hand, for compact objects, we can introduce their “compactness”, defined by the dimensionless ratios

\[
K_1 = \frac{G m_1}{a_1 c^2}; \quad K_2 = \frac{G m_2}{a_2 c^2};
\]

(2)

which equal 0.2 for neutron stars (depending on their equation of state). The quadrupoles \( Q_1 \) and \( Q_2 \) will affect both the Newtonian binding energy \( E \) of the two bodies, and the emitted total gravitational wave flux \( F \) as computed, say, using the standard Einstein quadrupole formula. It is known that for inspiralling compact binaries the neutron stars are not co-rotating because the tidal synchronization time is much larger than the time left till the coalescence. The best models for the fluid motion inside the two neutron stars are the so-called Roche–Riemann ellipsoids [27], which have tidally locked figures (the quadrupole moments face each other at any instant during the inspiral), but for which the fluid motion has zero circulation in the inertial frame. In the Newtonian approximation we find that within such a

\[
\phi = \phi_0 + \frac{1}{8 \pi c^2} \int_0^t \left( \frac{x}{K} \right)^5 dt;
\]

(3)

where \( x = \sqrt{G m \omega c^3} \) is a standard dimensionless PN parameter \( 1 \approx c^2 \), and where \( k \) is the Love number and \( K \) is the compactness of the neutron star. The first term in the right-hand-side (RHS) of (3) corresponds to the gravitational-wave damping of two point masses; the second term describes the finite-size effect, which appears as a relative correction, proportional to \( (c=K)^5 \), to the latter radiation damping effect. Because the finite-size effect is purely Newtonian, its relative correction \( (c=K)^5 \) should not depend on \( c \); and indeed the factors \( 1 \approx c^2 \) cancel out in the ratio \( x = K \). However, the compactness \( K \) of compact objects is by Eq. (2) of the order unity (or, say, a few tenths), therefore the \( 1 \approx c^2 \) it contains should not be taken into account in order to find the magnitude of the effect in this case, and so the real order of magnitude of the relative contribution of the finite-size effect in Eq. (3) should be given by \( x^5 \) alone. This means that for non-spinning compact objects the finite-size effect should be comparable, numerically, to a post-Newtonian correction of magnitude \( x^5 \approx 1 \approx c^{10} \) namely 5PN order. This is a much higher post-Newtonian order than the one at which we shall investigate the gravitational effects on the phasing formula. Using \( k^0 \approx \text{const} \) \( 1 \) and \( K \approx 0.2 \) for neutron stars (and the bandwidth of a VIRGO detector between 10 Hz and 1000 Hz), we find that the cumulative phase error due to the finite-size effect amounts to less that one orbital rotation over a total of 16000 produced by the gravitational-wave damping of point masses. The conclusion is that the finite-size effect can in general be neglected in comparison with purely gravitational-wave damping effects. Thus the appropriate theoretical description of inspiralling compact binaries is by two point masses within the post-Newtonian approximation.

Our strategy to obtain the motion and radiation of a system of two point-like particles at the 3PN order is to start with a general form of the 3PN metric, that is valid for a general continuous (smooth) matter distribution. Applying such metric to a system of point particles, we find that most of the integrals become divergent at the location of the particles, \( i.e. \) when \( x \rightarrow y_1 \) \( t \) or \( y_2 \) \( t \), where \( y_1 \) \( t \) and \( y_2 \) \( t \) denote the two trajectories. Consequently, we must supplement the calculation by a prescription for how to remove the “infinite part” of these integrals. At this stage different choices for a “self-field” regularization, which will take care of the infinite self-field of point particles, are possible. Among them:

1. Hadamard’s self-field regularization, which has proved to be very convenient for doing practical computations (in particular, by computer), but suffers from the important drawback of yielding some ambiguity parameters, which cannot be determined within this regularization, at the 3PN order;

\[2 \text{ This result can be derived in the context of relativistic equations of motion, and yields a proof of the so-called “effacement” principle in general relativity, according to which the internal structure of the compact bodies does not show up in their motion and emitted radiation which depend only on the masses.}\]

\[3 \text{ But note that for non-compact or moderately compact objects (such as white dwarfs) the Newtonian tidal interaction dominates over the radiation damping.}\]
2. Dimensional self-field regularization, an extremely powerful regularization, that is free of any ambiguities (at least up to the 3PN level), and as we shall see permits to uniquely fix the values of the ambiguity parameters coming from Hadamard’s regularization. However, dimensional regularization will be implemented in the present problem not in the general case of an arbitrary space-time dimension $D$ but only in the limit where $D! = 4$. Dimensional regularization was invented by ‘t Hooft and Veltman \[33, 30, 31, 32\] as a mean to preserve the gauge symmetry of perturbative quantum field theories. Our basic problem here is to respect the gauge symmetry associated with the diffeomorphism invariance of the classical general relativistic description of interacting point masses. Hence, we use dimensional regularization not merely as a trick to compute some particular integrals which would otherwise be divergent, but as a powerful tool for solving in a consistent way the Einstein field equations with singular point-mass sources, while preserving its crucial symmetries. In doing this, we implicitly assume that the correct theory is the Einstein general relativity in $D$ space-time dimensions (Section 2).

Earlier work on the equations of motion of point masses at the 2PN approximation level \[28\] was based on the Riesz analytical continuation method \[33\], which consists of replacing the delta-function stress-energy tensor of point particles by an auxiliary, smoother source defined from the Riesz kernel, depending on a complex number $\alpha$. The generalization of the Riesz continuation method to higher post-Newtonian orders is not straightforward because of the appearance of poles, proportional to $\alpha^{-1}$. Further in the same context, we quote Ref. \[41\] for an alternative approach, based on a Feynman diagram expansion, showing how to renormalize using dimensional regularization.

In the meantime all calculations were performed using the more rudimentary Hadamard regularization \[34, 35, 36, 37\], yielding almost complete results at the 3PN order, i.e. complete but for a few ambiguity parameters, which turn out to be in fact associated with the latter poles at the 3PN order. Then it was shown \[38\] how to use dimensional regularization within the ADM canonical formalism of general relativity at the 3PN order for the problem of the out to be in fact associated with the latter poles at the 3PN order. Then it was shown \[38\] how to use dimensional regularization not merely as a trick to compute some particular integrals which would otherwise be divergent, but as a powerful tool for solving in a consistent way the Einstein field equations with singular point-mass sources, while preserving its crucial symmetries. In doing this, we implicitly assume that the correct theory is the Einstein general relativity in $D$ space-time dimensions (Section 2).

2. EINSTEIN’S FIELD EQUATIONS IN $D$ DIMENSIONS

The field equations of general relativity (in $D$-dimensional space-time with signature $+$) form a system of ten second-order partial differential equations obeyed by the space-time metric $g_{\alpha \beta}$,

\[
E^{\alpha \beta} [\partial g_{\alpha \beta} \partial^2 g] = \frac{8 \pi G}{c^4} T^{\alpha \beta} [g];
\]

where the Einstein tensor $E^{\alpha \beta}$ is generated, through the gravitational coupling constant $\kappa = 8 \pi G = c^4$, by the matter stress-energy tensor $T^{\alpha \beta}$. The gravitational constant $G$ is related to the usual three-dimensional Newton’s constant $G_N$ by

\[
G = G_N \gamma_0^3;
\]

where $\gamma_0$ denotes an arbitrary length scale. Among the ten Einstein equations, four govern, via the contracted Bianchi identity, the evolution of the matter system,

\[
\nabla_\mu E^{\alpha \mu} = 0 \quad \Rightarrow \quad \nabla_\mu T^{\alpha \mu} = 0;
\]

The space-time geometry is constrained by the six remaining equations, which place six independent constraints on the ten components of the metric $g_{\alpha \beta}$, leaving four of them to be fixed by a choice of a coordinate system.

In this paper we adopt the conditions of harmonic, or de Donder, coordinates. We define, as a basic variable, the gravitational-field amplitude

\[
h^{\alpha \beta} = \frac{\partial}{\partial g} g^{\alpha \beta} \eta^{\alpha \beta};
\]

\[\text{Here } g^{\alpha \beta} \text{ denotes the contravariant metric (satisfying } g^{\alpha \mu} g_{\alpha \beta} = \delta^{\alpha \beta}, \text{ where } g \text{ is the determinant of the covariant metric, } g = \det [g_{\alpha \beta}], \text{ and where } \eta^{\alpha \beta} \text{ represents an auxiliary Minkowskian metric.}\]
The harmonic-coordinate condition, which accounts exactly for the four equations (6), corresponding to the conservation of the matter tensor, reads
\[ \partial_\mu h^{\alpha\mu} = 0 \quad \text{(8)} \]

The equations (7,8) introduce into the definition of our coordinate system a preferred Minkowskian structure, with Minkowski metric \( \eta_{\alpha\beta} \). Of course, this is not contrary to the spirit of general relativity, where there is only one physical metric \( g_{\alpha\beta} \) without any flat prior geometry, because the coordinates are not governed by geometry (so to speak), but rather are chosen by researchers when studying physical phenomena and doing experiments. Actually, the coordinate condition (8) is especially useful when we view the gravitational waves as perturbations of space-time propagating on the fixed Minkowskian manifold with the background metric \( \eta_{\alpha\beta} \). This view is perfectly legitimate and represents a fruitful and rigorous way to think of the problem when using approximation methods. Indeed, the metric \( \eta_{\alpha\beta} \), originally introduced in the coordinate condition (8), does exist at any finite order of approximation (neglecting higher-order terms), and plays in a sense the role of some “prior” flat geometry.

The harmonic-coordinate condition, which accounts exactly for the four equations (6), is made of terms at least quadratic in the gravitational-field strength \( \alpha\beta \). In this form the only explicit dependence on the dimension \( d \) is the cubic and quartic pieces of \( \Lambda_{\alpha\beta} \), including all non-linearities, reads
\[ \Lambda_{\alpha\beta} = h^{\mu\nu} \partial_\mu h^{\alpha\nu} + \partial_\mu h^{\alpha\nu} \partial_\nu h^{\beta\mu} + \frac{1}{2} \frac{c^4}{16\pi G} \Lambda^{\alpha\beta} + \frac{1}{2} \frac{c^4}{16\pi G} \Lambda^{\alpha\beta} \quad \text{(11)} \]

In this form the only explicit dependence on the dimension \( D \) is in the last term of (11). As is clear from this expression, \( \Lambda_{\alpha\beta} \) is made of terms at least quadratic in the gravitational-field strength \( h \) and its first and second space-time derivatives. In the following, for the highest post-Newtonian order that we consider (3PN), we need the quadratic, cubic and quartic pieces of \( \Lambda_{\alpha\beta} \). With obvious notation, we can write them as
\[ \Lambda_{\alpha\beta} = N_{\alpha\beta} \parallel h \parallel + M_{\alpha\beta} \parallel h \parallel h \parallel + L_{\alpha\beta} \parallel h \parallel h \parallel h \parallel + \cdots \quad \text{(12)} \]

These various terms can be straightforwardly computed from Eq. (11); see Eqs. (3.8) in Ref. [35] for explicit expressions.

As said above, the condition (8) is equivalent to the matter equations of motion, in the sense of the conservation of the total pseudo-tensor \( \tau_{\alpha\beta} \),
\[ \partial_\mu \tau^{\alpha\mu} = 0 \quad \text{and} \quad \nabla_\mu \tau^{\alpha\mu} = 0 \quad \text{(13)} \]

When developing post-Newtonian approximations, we look for the solutions of the field equations (9, 10, 11, 13) under the following four hypotheses:

1. The matter stress-energy tensor \( T^{\alpha\beta} \) is of spatially compact support, i.e. can be enclosed into some time-like world tube, say \( r > a \), where \( r = \parallel x \parallel \) is the harmonic-coordinate radial distance. Outside the domain of the source, when \( r > a \), the gravitational source term, according to Eq. (13), is divergence-free,
\[ \partial_\mu \Lambda^{\alpha\mu} = 0 \quad \text{when} \ r > a \quad \text{(14)} \]

2. The matter distribution inside the source is smooth: \( T^{\alpha\beta} \in C^\infty (\mathbb{R}^d) \) where \( d = D - 1 \) is the space dimension. We have in mind a smooth hydrodynamical “fluid” system, without any singularities nor shocks (a priori), that is described by some Eulerian equations including high relativistic corrections.
3. The source is post-Newtonian in the sense of the existence of the small post-Newtonian parameter \( v=c = O\ (l=c)\). For such a source we assume the legitimacy of the method of matched asymptotic expansions for matching the inner post-Newtonian field, which is valid only in the source’s near zone, and the outer multipolar decomposition in the source’s exterior near zone.

4. The gravitational field has been independent of time (stationary) in some remote past, \textit{i.e.} before some finite instant \( T \) in the past, in the sense that

\[
\frac{\partial}{\partial t} h^{\alpha\beta}(x,t) = 0 \quad \text{when } t < T:
\]

The latter condition is a mean to impose (somewhat by brute force), the famous \textit{no-incoming} radiation condition, ensuring that the matter source is isolated from the rest of the Universe and does not receive any radiation from infinity. Ideally, the no-incoming radiation condition should be imposed at past null infinity. One can argue \cite{42} that the condition of stationarity in the past \cite{15}, although much weaker than the real no-incoming radiation condition, does not entail any physical restriction on the validity of the formulas we derive.

Subject to the condition \cite{15}, the Einstein differential field equations \cite{9} can be written equivalently into the form of the integro-differential equations

\[
h^{\alpha\beta}(x,t) = \frac{16\pi G}{c^4} \int d^4x' dt' G_{\text{ret}}(x,x') (\frac{x-x'}{c t})^{\alpha\beta} (x',t') \theta(t-t') (\frac{x-x'}{c t})^{\alpha\beta} (x',t'):
\]

where \( G_{\text{ret}}(x,t) \) is the scalar retarded Green function in \( D = d + 1 \) dimensions. The Green function for general \( d \) has no simple expression in \( \theta(x) \) space. However, starting from its well-known Fourier-space expression, one can write the following simple integral expression (see e.g. \cite{43}),

\[
G_{\text{ret}}(x,t) = \frac{\theta(t)}{2\pi} \int_0^\infty dk \frac{k^d}{r} \sin(kr) J_{\alpha-1}^d (kr):
\]

Notice that this is in fact a function of \( t \) and \( r = \frac{x}{c} \) only. Here \( \theta(t) \) is the Heaviside step function, and \( J_{\alpha-1}^d (kr) \) the usual Bessel function [see Eq. \cite{73} in the Appendix].

3. DIMENSIONAL REGULARIZATION OF THE EQUATIONS OF MOTION

As said before, work at the 3PN order using Hadamard’s self-field regularization showed the appearance of ambiguity parameters, due to an incompleteness of the Hadamard regularization employed for curing the infinite self-field of point particles. By ambiguity parameter, we mean a dimensionless coefficient which cannot be computed within the Hadamard regularization scheme. Nevertheless, the majority of terms could be computed unambiguously with Hadamard’s regularization \cite{34, 35}. We summarize here the determination using dimensional regularization of the ambiguity parameter \( \lambda \) which appeared in the 3PN equations of motion; note that \( \lambda \) is equivalent to the static ambiguity parameter \( \lambda_{\text{static}} \) originally introduced in Refs. \cite{44, 45}.

The post-Newtonian iteration of the Einstein field equations with point-like matter source (Dirac delta-functions with spatial supports \( y_1 \) and \( y_2 \)) yields a generic form for the functions representing the metric components in successive post-Newtonian approximations. The generic functions we have to deal with in 3 dimensions, say \( F(x) \), are smooth on \( \mathbb{R}^3 \) except at \( y_1 \) and \( y_2 \), around which they admit singular Laurent-type expansions in powers and inverse powers of \( r_1 = \frac{x-y_1}{l_1} \) and \( r_2 = \frac{x-y_2}{l_2} \) given by (say, for any \( N \))

\[
F(x) = \sum_{p=0}^\infty \sum_{N} r_1^p f_p N + o(\frac{1}{r_1}) + o(\frac{1}{r_2}) \]

and similarly for the other point 2. Here \( r_1 = \frac{x-y_1}{l_1} \geq 0 \), and the coefficients \( f_p \) of the various powers of \( r_1 \) depend on the unit direction \( n_1 = \frac{x-y_1}{r_1} \) of approach to the singular point.\(^5\) The powers \( p \) of \( r_1 \) are relative integers, and

\(^5\) The function \( F(x) \) depends also on time \( t \), through for instance its dependence on the velocities \( v_1 \) and \( v_2 \), but the (coordinate) \( t \) time is purely “spectator” in the regularization process, and thus will not be indicated.

\(^6\) The \( o \) Landau symbol for remainders takes its standard meaning.
are bounded from below \((p_0 \quad p)\). The coefficients \(f_{\rho p}\) (and \(2f_{\rho p}\)) for which \(p < 0\) can be referred to as the \textit{singular} coefficients of \(F\).

A function \(F\) defined in 3 dimensions being given, and admitting the singular expansion \((18)\), we define the Hadamard \textit{partie finie} of \(F\) at the location of the particle 1, where it is singular, as

\[
\langle F \rangle_1 = \frac{\partial}{\pi} f_0 (\mathbf{n}_1); \tag{19}
\]

where \(d\Omega_1\) denotes the solid angle element centered on \(\mathbf{y}_1\) and of direction \(\mathbf{n}_1\). An important feature we have to notice is that because of the angular integration in Eq. \((19)\), the Hadamard partie finie is “non-distributive” in the sense that \(\langle FG \rangle_1 \not= \langle F \rangle_1 \langle G \rangle_1\) in general. The non-distributivity of Hadamard’s partie finie will be the main source of the appearance of ambiguity parameters at the 3PN order; remarkably it does not affect any calculation before that order. The second notion of Hadamard partie finie (in short \(\text{Pf}\)) concerns that of the integral \(-d^3x\, F\), which is generically divergent at the location of the two singular points \(\mathbf{y}_1\) and \(\mathbf{y}_2\) (we assume for the moment that the integral converges at infinity). It is defined by

\[
\text{Pf}_{\mathbf{y}_1 \mathbf{y}_2} Z \, d^3x F = \lim_{s \to 0} \frac{Z}{\partial} d^3x F + 4\pi \sum_{a+3b=0} d^{a+3} \frac{F}{r_1^a} + 4\pi \ln \frac{s}{s_1} r_1^3 F \, 1 + 1 \, 2; \tag{20}
\]

The first term integrates over a domain \(S \, (s)\) defined as \(\mathbb{R}^3\) from which the two spherical balls \(r_1 \quad s\) and \(r_2 \quad s\) of radius \(s\) and centered on the two singularities are excised. The other terms, where the value of a function at point 1 takes the meaning \((19)\), are such that they cancel out the divergent part of the first term in the limit where \(s \to 0\) (the symbol \(1 \, 2\) means the same terms but corresponding to the other point 2). The Hadamard partie-finie integral depends on two strictly positive constants \(s_1\) and \(s_2\), associated with the logarithms present in Eq. \((20)\).

The post-Newtonian approximation consists of breaking the hyperbolic d’Alembertian operator into the elliptic Laplacian operator \(\Delta\) plus the famous retardation term \(c^{-2}q^2\) which is to be considered small in the post-Newtonian sense, and put in the RHS of the equation where it is iterated. As a consequence, we essentially have to deal with the regularization of Poisson integrals, or iterated Poisson integrals, of the generic function \(F\). In the case of a Poisson integral potential in 3 dimensions, say

\[
P(\mathbf{x}) = \frac{1}{4\pi} \frac{Z}{\partial} d^3x F(\mathbf{x}); \tag{21}
\]

the Hadamard partie finie integral must be defined in a more precise way. Indeed, the definition \((19)\) \textit{stricto sensu} is applicable when the expansion of the function \(F\), when \(r_1 \to 0\), does not involve logarithms of \(r_1\); see Eq. \((18)\). However, the Poisson integral \(P(\mathbf{x})\) of \(F(\mathbf{x})\) will typically involve such logarithms (these will appear precisely at the 3PN order), namely some \(\ln r_1^0\) where \(r_1^0 \quad \mathbf{x} \quad 0 \quad \mathbf{y}_1\) tends to zero (hence \(\ln r_1^0\) is formally infinite). The proper way to define the Hadamard partie finie in this case is to include the \(\ln r_1^0\) into its definition, and we arrive at \((46)\)

\[
\langle P \rangle_1 = \frac{1}{4\pi} \text{Pf}_{\mathbf{r}_1^0 \mathbf{r}_2^0} \frac{Z}{\partial} d^3x F(\mathbf{x}) \quad \psi_1^2 \langle F \rangle_1; \tag{22}
\]

The first term follows from Hadamard’s partie finie integral \((20)\); the second one is given by Eq. \((19)\). Notice that in this result the constant \(s_1\) entering the partie finie integral \((20)\) has been “replaced” by \(r_1^0\), which plays the role of a new regularization constant (together with \(r_2^0\) for the other particle), and which ultimately parametrizes the final Hadamard regularized 3PN equations of motion. It was shown that \(r_1^0\) and \(r_2^0\) are unphysical, in the sense that they can be removed by a coordinate transformation \((34, 35)\). On the other hand, the constant \(s_2\) remaining in the result \((22)\) is the source for the appearance of the physical ambiguity parameter, called \(\lambda\), as it will be related to it by Eq. \((28)\) below.

In \(d\) spatial dimensions, there is an analogue of the function \(F\), which results from the same post-Newtonian iteration process but performed in \(d\) spatial dimensions. Let us call this function \(F^{(d)}(\mathbf{x})\), where \(\mathbf{x} \in \mathbb{R}^d\). When \(r_1 \to 0\) the function \(F^{(d)}\) admits a singular expansion which is richer than in 3 dimensions. Posing \(\epsilon = d - 3\) we have

\[
F^{(d)}(\mathbf{x}) = \sum_{p_0 \quad p} \frac{\partial}{\partial} p_0^{p_0} q_0^{q_0} \frac{\partial}{\partial} q_0^{q_0} \sum_{N} r_1^{p} q_1^{q} f_{\mu_1}^{(d)} (\mathbf{n}_1) + o \psi_1^2; \tag{23}
\]
The coefficients $f_1^{\mu} (n_1)$ depend on the dimension; the powers of $r_1$ involve the relative integers $p$ and $q$ whose values are limited by some $p_0, q_0$ and $q_1$ as indicated. The Poisson integral of $F^{(d)}$, in $d$ dimensions, is given by the Green’s function for the Laplace operator,

$$P^{(d)} (x^0) = \frac{\tilde{k}}{4\pi} \frac{Z^2}{\kappa} \frac{d^d x}{x_0^d} F^{(d)} (x);$$

(24)

where $\tilde{k}$ is a constant related to the usual Eulerian $\Gamma$-function by

$$\tilde{k} = \frac{\Gamma \frac{d - 2}{2}}{\pi \frac{d - 2}{2}}.$$  

(25)

We have $\lim_{d \to 3} \tilde{k} = 1$. Notice also that $\tilde{k}$ is closely linked to the volume $\Omega_{d - 1}$ of the sphere with $d - 1$ dimensions; see Eq. (25) in the Appendix.

We need to evaluate the Poisson integral at the point $x^0 = y_1$ where it is singular; in contrast with the case of Hadamard regularization where the result is given by (22), this is quite easy in dimensional regularization, because the nice properties of analytic continuation allow simply to get $P^{(d)} (x^0) \big|_{x^0 = y_1}$ by replacing $x^0$ by $y_1$ into the explicit integral form (24). So we simply have

$$P^{(d)} (y_1) = \frac{\tilde{k}}{4\pi} \frac{Z^2}{\kappa} \frac{d^d x}{r_1^d} F^{(d)} (x);$$

(26)

The main technical step of our strategy to compute the ambiguity parameter $\lambda$ will consist of computing, in the limit $\epsilon \to 0$, the difference between the $d$-dimensional Poisson potential (26), and its 3-dimensional counterpart which is defined from Hadamard’s self-field regularization and given by (22). Denoting the difference between the dimensional and Hadamard regularizations by means of the script letter $\mathcal{D}$, we pose (for the result concerning the point 1)

$$\mathcal{D} P_1 \cdot P^{(d)} (y_1) \cdot (P)_1;$$

(27)

That is, $\mathcal{D} P_1$ is what we shall have to add to the Hadamard-regularization result in order to get the $d$-dimensional result. However, we shall only compute the first two terms of the Laurent expansion of $\mathcal{D} P_1$ when $\epsilon = d - 3 \to 0$, say $a_1 \epsilon^{-1} + a_0 + o (\epsilon)$. This is the information we need to determine the value of the ambiguity parameter. Notice that the difference $\mathcal{D} P_1$ comes exclusively from the contribution of terms developing some poles $\sim 1 = \epsilon$ in the $d$-dimensional calculation.

Let us next outline the way we obtain, starting from the computation of the “difference”, the 3PN equations of motion in dimensional regularization, and show how the ambiguity parameter $\lambda$ is fixed by the process. By contrast to $r_1$ and $r_2$ which are pure gauge, $\lambda$ is a genuine physical ambiguity, introduced in Refs. [46, 35] as the single unknown numerical constant parametrizing the ratio between $s_2$ and $r_2^0$ [where $s_2$ is the constant left in Eq. (22)] as

$$\ln \frac{r_2^0}{s_2} = \frac{159}{308} + \frac{\lambda}{m_1 + m_2} \ln \frac{m_1 + m_2}{m_2} \quad (\text{and } 1 \lessapprox 2);$$

(28)

where $m_1$ and $m_2$ are the two masses. The terms corresponding to the $\lambda$-ambiguity in the acceleration $a_1 = dv_1/dt$ of particle 1 read simply

$$\Delta a_1 [\lambda] = \frac{44 \lambda}{3} \frac{G \ni_1 m_1 m_2 (n_1 + m_2)}{r_{12} \epsilon^6 n_{12} i};$$

(29)

where the relative distance between particles is denoted $v_1 \times v_2 \cdot r_{12} n_{12}$ (with $n_{12}$ the unit vector pointing from particle 2 to particle 1). We start from the end result of Ref. [35] for the 3PN harmonic coordinates acceleration $a_1$ in Hadamard’s regularization, abbreviated as HR. Since the result was in fact obtained by means of a specific variant of HR, called the extended Hadamard’s regularization (in short EHR), we write it as

$$a_1^{\text{HR}} = a_1^{\text{EHR}} + \Delta a_1 [\lambda];$$

(30)

where $a_1^{\text{EHR}}$ is a fully determined functional of the masses $m_1$ and $m_2$, the relative distance $r_{12} n_{12}$, the coordinate velocities $v_1$ and $v_2$, and also the gauge constants $r_1^0$ and $r_2^0$. The only ambiguous term is the second one and is given by Eq. (29).
Our method is to express both the dimensional and Hadamard regularizations in terms of their common “core” part, obtained by applying the so-called “pure-Hadamard-Schwartz” (pHS) regularization. Following the definition of Ref. 39, the pHS regularization is a specific, minimal Hadamard-type regularization of integrals, based on the partie finie integral 20, together with a minimal treatment of “contact” terms, in which the definition 20 is applied separately to each of the elementary potentials, denoted $V, V_i, \tilde{W}_{ij}$, that enter the post-Newtonian metric. The pHS regularization also assumes the use of standard Schwartz distributional derivatives 47. The interest of the pHS regularization is that the dimensional regularization is equal to it plus the “difference”; see Eq. (33) below.

To obtain the pHS-regularized acceleration we need to subtract from the EHR result a series of contributions, which are specific consequences of the use of EHR 46. For instance, one of these contributions corresponds to the fact that in the EHR the distributional derivative differs from the usual Schwartz distributional derivative. Hence we define

$$a_1^{(\text{pHS})} = a_1^{(\text{EHR})} - \sum \delta_\lambda a_1;$$

where the $\delta_\lambda a_1$’s denote the extra terms following from the EHR prescriptions. The pHS-regularized acceleration (31) constitutes essentially the result of the first stage of the calculation of $a_1$, composed of plenty of terms and which are all perfectly well-defined.

The next step consists of evaluating the Laurent expansion, in powers of $\varepsilon = d - 3$, of the difference between the dimensional regularization and the pHS (3-dimensional) computation. As we said above this difference makes a contribution only when a term generates a pole $1 = \varepsilon$, in which case the dimensional regularization adds an extra contribution, made of the pole and the finite part associated with the pole [we consistently neglect all terms $\propto \varepsilon$]. One must then be especially wary of combinations of terms whose pole parts finally cancel (“cancelled poles”) but whose dimensionally regularized finite parts generally do not, and must be evaluated with care. We denote the above defined difference by

$$\Delta a_1 = \sum \Delta P_i;$$

It is made of the sum of all the individual differences of Poisson or Poisson-like integrals as computed in Eq. (27). The total difference (32) depends on the Hadamard regularization scales $r_1$ and $s_2$ (or equivalently on $\lambda$ and $r_1^2, r_2^2$), and on the parameters associated with dimensional regularization, namely $\varepsilon$ and the characteristic length scale $\gamma_0$ introduced in Eq. (5). Finally, our main result is the explicit computation of the dimensional regularization (DR) acceleration as

$$a_1^{(\text{DR})} = a_1^{(\text{pHS})} + \Delta a_1;$$

With this result we can prove two theorems 39.

**Theorem 1** The pole part $\propto 1 = \varepsilon$ of the DR acceleration (33) can be re-absorbed (i.e., renormalized) into some shifts of the two “bare” world-lines: $y_1, y_1 + \xi_1$ and $y_2, y_2 + \xi_2$, with, say, $\xi_1, \xi_2 \propto 1 = \varepsilon$, so that the result, expressed in terms of the “dressed” quantities, is finite when $\varepsilon = 0$.

The precise shifts $\xi_1$ and $\xi_2$ involve not only a pole contribution $\propto 1 = \varepsilon$ [which would define a renormalization by minimal subtraction (MS)], but also a finite contribution when $\varepsilon = 0$. Their explicit expressions read:

$$\xi_1 = \frac{11}{3} \frac{G_N}{c^6} m_1^2 \frac{1}{\varepsilon} 2 \ln \frac{r_1^{C-2}}{\gamma_0} \left( \frac{327}{1540} a_{N1} \right);$$

where $G_N$ is Newton’s constant, $\gamma_0$ is the characteristic length scale of dimensional regularization [cf. Eq. (5)], $a_{N1}$ is the Newtonian acceleration of the particle 1 in $d$ dimensions, and $\gamma = 4\pi e^C$ depends on Euler’s constant $C = 0.577$.

Note that when working at the level of the equations of motion (not considering the metric outside the world-lines), the effect of shifts can be seen as being induced by a coordinate transformation of the bulk metric as in Ref. 35.

**Theorem 2** The renormalized (finite) DR acceleration is physically equivalent to the Hadamard-regularized (HR) acceleration (end result of Ref. 35), in the sense that

$$a_1^{(\text{HR})} = \lim_{\varepsilon \to 0} a_1^{(\text{DR})} + \delta_\xi a_1;$$

where $\delta_\xi a_1$ denotes the effect of the shifts on the acceleration, if and only if the HR ambiguity parameter $\lambda$ entering the harmonic-coordinates equations of motion takes the unique value

$$\lambda = \frac{1987}{3080};$$

$$\xi = \frac{11}{3} \frac{G_N}{c^6} m_1^2 \frac{1}{\varepsilon} 2 \ln \frac{r_1^{C-2}}{\gamma_0} \left( \frac{327}{1540} a_{N1} \right);$$
Our notation is the following: the Hadamard self-field regularization \[36, 37\]. To apply dimensional regularization, we must use as we did for the

An alternative work, by Itoh and Futamase \[49, 50\], following previous investigations in Refs. \[51, 52\], has derived the 3PN equations of motion in harmonic coordinates by means of a variant of the famous “surface-integral” method introduced by Einstein, Infeld and Hoffmann \[1\]. The aim is to describe extended relativistic compact binary systems in the 3PN equations of motion \[38\], and the harmonic-coordinates equations of motion \[39\].

We now address the similar problem concerning the binary’s gravitational radiation field (3PN beyond the Einstein quadrupole formalism), for which three ambiguity parameters, denoted \(\xi\), \(\kappa\), and \(\zeta\), have been shown to appear due to the Hadamard self-field regularization \[36, 37\]. To apply dimensional regularization, we must use as we did for the equations of motion in Section 3 the \(d\)-dimensional post-Newtonian iteration; and, crucially, we have to generalize to \(d\) dimensions some key results from the gravitational wave generation formalism. The specific wave generation formalism we employ is based on a post-Newtonian expansion for the metric field in the near zone of the source, and on the so-called multipolar-post-Minkowskian expansion for the field in the exterior of the source, including the regions at infinity from the source where the detector is located \[54\]. The expression of the multipole moments describing the physical (post-Newtonian) source are then obtained by a technique of asymptotic matching performed in the overlapping region of common validity between the two types of expansion, namely the exterior part of the near zone \[53, 50\] (see \[42\] for a review).

Let us first recall the expressions of the source multipole moments \(I_L\) (mass-type moment) and \(J_L\) (current-type) of an isolated post-Newtonian source in ordinary 3-dimensional space. They are given, for multipolarities \(\ell \geq 2\), by \[50\]:

\[
I_L(\ell) = J_P^\ell \int \frac{d^3 x}{(2\pi)^3} \frac{1}{d^\ell} \int d\xi \frac{d_\delta(\ell) }{c^2} \frac{4 \left( \ell^\ell + 1 \right)}{(\ell^\ell + 1)(\ell^\ell + 3)} \delta_{\ell - 1}(\xi) \delta_{\ell L} \Sigma_i \\
&+ \frac{2 \left( \ell^\ell + 1 \right)}{c^4 (\ell + 1)(\ell + 2)(\ell + 3)} \delta_{\ell + 2}(\xi) \delta_{\ell L} \Sigma_{ij} \left( \Sigma_{iP} + z \Sigma_{Pc} \right); \\
J_L(\ell) &- J_P^\ell \int \frac{d^3 x}{(2\pi)^3} \frac{1}{d^\ell} \int d\xi \frac{d_\delta(\ell) }{c^2} \frac{2 \left( \ell^\ell + 1 \right)}{(\ell + 2)(\ell + 3)} \delta_{\ell + 1}(\xi) \delta_{\ell L} \Sigma_{1+0} \Sigma_{bc} \left( \Sigma_{iP} + z \Sigma_{Pc} \right); \\
&= \frac{2 \left( \ell^\ell + 1 \right)}{c^4 (\ell + 2)(\ell + 3)} \delta_{\ell + 1}(\xi) \delta_{\ell L} \Sigma_{1+0} \Sigma_{bc} \left( \Sigma_{iP} + z \Sigma_{Pc} \right); \\
\]

(37a)

(37b)

where the source densities \(\Sigma_{\mu\nu}\)'s are evaluated at the position \(x\) and at time \(t + z x P c\), and are defined by

\[
\Sigma = \frac{\tau^{00} + \tau^{0i}}{c^2}; \quad \Sigma_i = \frac{\tau^{0i}}{c}; \quad \Sigma_{ij} = \tau^{ij};
\]

(38)

Here \(\tau^{\alpha\beta}\) is the pseudo stress-energy tensor \(10\) in 3 dimensions and the overbar refers to its formal post-Newtonian expansion, \(\bar{\tau}^{\alpha\beta}\) \(\text{PN}\ [\alpha\beta]\). Let us note that the expressions \(37\) are “exact”, in the sense that they are formally valid.

---

\(^7\) Our notation is the following: \(L = i_1 i_2 :: i_l\) denotes a multi-index, made of \(l\) spatial indices; similarly we write for instance \(a L = ai_1 :: i_l\) or \(L - 1 = i_1 :: i_l - 1\). The symmetric-trace-free (STF) projection is denoted with a hat, so that \(\hat{L} = s_{iL}\) is the STF projection of the product of \(l\) spatial vectors, denoted \(s_{iL} = s_{i1} \cdots s_{iL}\). Sometimes we also indicate the STF projection by brackets surrounding indices \(\hat{L} = s_{iL}\). Note that an expansion into STF tensors \(\hat{L} = \hat{s}_{iL} \hat{m}\) (which are functions of the spherical angles \(\theta\) and \(\phi\)) is equivalent to the usual expansion in spherical harmonics \(Y_{LM}(\theta, \phi)\). The dots indicate successive time-derivations.
up to any PN order. Equations (37) involve an integration over the variable \(z\), with associated function \(\delta(\varphi)\) given by

\[
\delta(\varphi) = \frac{Q^* + 1}{2^* + 1 + \eta^*} (1 - e^{-z})^{1/2}; \quad \lim_{\eta^* \to +\infty} \delta(\varphi) = \delta(\varphi);
\]

where \(\delta(\varphi)\) is the usual Dirac’s one-dimensional delta-function. In practice, the post-Newtonian moments (37) are to be computed by means of the infinite post-Newtonian series

\[
\sum_{k=0}^{+\infty} \frac{Q^* + 1}{(2k+1)(2k+2k+1)} \frac{x^j}{c} \frac{\partial}{\partial t} \frac{2k}{\Sigma(\mathbf{x}_t)} = \sum_{k=0}^{+\infty} \frac{Q^* + 1}{(2k+1)(2k+2k+1)} \frac{x^j}{c} \frac{\partial}{\partial t} \frac{2k}{\Sigma(\mathbf{x}_t)} \Omega(\mathbf{x}_t);
\]

In Eqs. (37) there is a special process of taking the “finite part”, indicated by the symbol \(\mathcal{F} \mathcal{P} \), which is necessary in order to deal with the bound of the integral at infinity, corresponding to infra-red divergencies, in the limit \(x^j \to \infty\). Indeed, notice that the pseudo stress-energy tensor \(\tau^{\alpha\beta}\) includes the crucial contribution of the gravitational field, denoted \(\Lambda^{\alpha\beta}\) in Eq. (10), which has a spatially non-compact support. This fact, together with the presence of the multipolar factor \(\hat{\gamma}_L\) in the integrand, prevents one to (naively) write the standard expressions for the multipole moments valid for compact-supported sources; such expressions have no meaning in non-linear general relativity. The solution to this dilemma has been to introduce [53, 56] the specific finite part \(\mathcal{F} \mathcal{P} \), and to show how these specific multipole moments so defined are related to the physical wave form at infinity.

To proceed with dimensional regularization, we need the \(d\)-dimensional analogues of the multipole moments of the source, say \(I^{(d)}_L\) and \(J^{(d)}_L\), consequences of the D-dimensional Einstein field equations for isolated post-Newtonian sources. In the case of the mass-type moments we find [40]

\[
I^{(d)}_L = i = \int \frac{d}{d^2} \mathcal{F} \mathcal{P} \int^2 \frac{d^d x}{c^2} \hat{\gamma}_L \frac{\partial}{\partial t} \frac{2}{4 d^2 l^2} \left( \frac{2}{d} \frac{x^j}{c^2} + \Gamma \frac{2}{d} \frac{x^j}{c^2} \right) \left( \frac{2}{d} \frac{x^j}{c^2} \right) \frac{\partial}{\partial t} \frac{2}{4 d^2 l^2} \left( \frac{2}{d} \frac{x^j}{c^2} \right) \frac{\partial}{\partial t} \frac{2}{4 d^2 l^2} \left( \frac{2}{d} \frac{x^j}{c^2} \right)
\]

where we denote [generalizing Eq. (38)]

\[
\Sigma = \frac{2}{d} \frac{x^j}{c^2} + \Gamma \frac{2}{d} \frac{x^j}{c^2} \frac{\partial}{\partial t} \frac{2}{4 d^2 l^2} \left( \frac{2}{d} \frac{x^j}{c^2} \right) \frac{\partial}{\partial t} \frac{2}{4 d^2 l^2} \left( \frac{2}{d} \frac{x^j}{c^2} \right)
\]

where now \(\tau^{\alpha\beta}\) is the post-Newtonian expansion of the pseudo stress-energy tensor in \(D\) dimensions. For any of the latter source densities the underscript \(\langle \rangle\) means the infinite series

\[
\Sigma(\mathbf{x}_t) = \sum_{k=0}^{+\infty} \frac{1}{2^{2k+1}} \Gamma \frac{2}{d^2} \frac{x^j}{c^2} \frac{\partial}{\partial t} \frac{2}{4 d^2 l^2} \left( \frac{2}{d} \frac{x^j}{c^2} \right) \frac{\partial}{\partial t} \frac{2}{4 d^2 l^2} \left( \frac{2}{d} \frac{x^j}{c^2} \right)
\]

which constitutes the \(d\)-dimensional version of Eq. (40). At Newtonian order Eq. (41) reduces to the standard result

\[
I^{(d)}_L = \int^2 \frac{d^d x}{c^2} \mathcal{F} \mathcal{P} \hat{\gamma}_L \frac{\partial}{\partial t} \frac{2}{4 d^2 l^2} \left( \frac{2}{d} \frac{x^j}{c^2} \right) \frac{\partial}{\partial t} \frac{2}{4 d^2 l^2} \left( \frac{2}{d} \frac{x^j}{c^2} \right)
\]

like for the case of the equations of motion, the ambiguity parameters \(\xi, \kappa\) and \(\zeta\) come from a deficiency of the Hadamard regularization coming up at 3PN order and mainly due to its “non-distributivity”. The terms corresponding to these ambiguities are contained in the 3PN mass quadrupole moment \(I_{1j}\) and are found to be

\[
\Delta I_{1j} = \frac{44}{3} \frac{G m^3}{c^6} \xi + \kappa \frac{m_1 + m_2}{m_1} \left( \frac{y}{m_2} a_{ij} + \zeta v_i v_j + 1 \right) 2 i + 1
\]

where \(y_1, v_1\) and \(a_1\) denote the first particle’s position, velocity and acceleration (the brackets \(\hat{\mathcal{H}}\) surrounding indices refer to the STF projection). As in Section 5 we express the Hadamard and dimensional results in terms of the more basic pure-Hadamard-Schwartz (pHS) regularization. The first step of the calculation (37) is therefore to relate the Hadamard-regularized quadrupole moment \(I_{1j}^{(HR)}\), for general orbits, to its pHS part; we find:

\[
I_{1j}^{(HR)} = I_{1j}^{(pHS)} + \frac{1}{22} \kappa \xi + \frac{9}{110} ;
\]

(45)
In the RHS we see both the pHS part, and the effect of adding the ambiguities, with some numerical shifts of the ambiguity parameters coming from the difference between the specific Hadamard-type regularization scheme used in Ref. [36] and the pHS one. The pHS part is free of ambiguities but depends on the gauge constants \( r_1^2 \) and \( r_2^2 \) introduced in the harmonic-coordinates equations of motion [34, 35].

We next use the \( d \)-dimensional moment \((41)\) to compute the difference between the dimensional regularization (DR) result and the pHS one \([14, 40]\). As in the work on equations of motion, we find that the ambiguities arise solely from the terms in the integration regions near the particles \((i.e., r_1 = \mathbf{x}_1, r_2 = \mathbf{x}_2 \neq 0)\) that give rise to poles \( \propto 1 = \epsilon \), corresponding to logarithmic ultra-violet divergences in 3 dimensions. The infra-red region at infinity \((i.e., \mathbf{x} \neq 0)\) does not contribute to the difference DR \( \pmb{pHS} \). The compact-support terms in the integrand of \((41)\), proportional to the components of the matter stress-energy tensor \( T^{\alpha\beta} \), are also found not to contribute to the difference. We are therefore left with evaluating the difference linked with the computation of the non-compact terms in the expansion of the integrand in \((41)\) near the singularities that produce poles in \( d \) dimensions.

Let \( F^{(\nu)}(\kappa) \) be the non-compact part of the integrand of the quadrupole moment \((41)\) with indices \( L = ij \), where \( F^{(\nu)} \) includes the appropriate multipolar factors such as \( \hat{s}_{ij} \), so that

\[
I_{ij}^{(\nu)} = \int d^d x F^{(\nu)}(\kappa) ;
\]

We do not indicate that we are considering here only the non-compact part of the moments. Near the singularities the function \( F^{(\nu)}(\kappa) \) admits a singular expansion of the type \((22)\). In practice, the coefficients \( f_{ij}^{(\nu)} \) are computed by performing explicitly the post-Newtonian iteration. On the other hand, the analogue of Eq. \((46)\) in 3 dimensions is

\[
I_{ij} = Pf \int d^3 x F(\kappa) ;
\]

where \( Pf \) refers to the Hadamard partie finie defined by Eq. \((20)\). The difference \( \mathbb{D} I \) between the DR evaluation of the \( d \)-dimensional integral \((46)\), and its corresponding three-dimensional evaluation, \( i.e. \) the partie finie \((47)\), reads then

\[
\mathbb{D} I_{ij} = I_{ij}^{(\nu)} - I_{ij} ;
\]

Such difference depends only on the ultra-violet behavior of the integrands, and can therefore be computed “locally”, \( i.e. \) in the vicinity of the particles, when \( r_1 \neq 0 \) and \( r_2 \neq 0 \). We find that Eq. \((48)\) depends on two constant scales \( s_1 \) and \( s_2 \) coming from Hadamard’s partie finie \((20)\), and on the constants belonging to dimensional regularization, which are \( \epsilon = d - 3 \) and the length scale \( \ell_0 \) defined by Eq. \((5)\). The dimensional regularization of the 3PN quadrupole moment is then obtained as the sum of the pHS part, and of the difference computed according to Eq. \((48)\), namely

\[
I_{ij}^{(\nu)\text{DR}} = I_{ij}^{(\nu)\text{PHS}} + \mathbb{D} I_{ij} ;
\]

An important fact, hidden in our too-compact notation \((29)\), is that the RHS of \((49)\) does not depend on the Hadamard regularization scales \( s_1 \) and \( s_2 \), which cancel out from the two terms in the RHS. Therefore it is possible to re-express these two terms (separately) by means of the constants \( r_1^2 \) and \( r_2^2 \) instead of \( s_1 \) and \( s_2 \), where \( r_1^2, r_2^2 \) are the two fiducial scales entering the Hadamard-regularization result \((45)\). This replacement being made the pHS term in Eq. \((49)\) is exactly the same as the one in Eq. \((45)\). At this stage all elements are in place to prove the following theorem \([14, 40]\).}

**Theorem 3** The DR quadrupole moment \((49)\) is physically equivalent to the Hadamard-regularized one (end result of Refs. [36, 37]), in the sense that

\[
I_{ij}^{(\nu)\text{HR}} = \lim_{\epsilon \to 0} I_{ij}^{(\nu)\text{DR}} + \delta_\xi I_{ij}^\xi ;
\]

where \( \delta_\xi I_{ij} \) denotes the effect of the same shifts as determined in Theorems \([7, 2]\) if and only if the HR ambiguity parameters \( \xi, \kappa \) and \( \zeta \) take the unique values

\[
\xi = \frac{9871}{9240}, \quad \kappa = 0, \quad \zeta = \frac{7}{33} ;
\]

Moreover, the poles \( \epsilon \neq 0 \) separately present in the two terms in the brackets of \((50)\) cancel out, so that the physical (renormalized or “dressed”) DR quadrupole moment is finite and given by the limit when \( \epsilon \to 0 \) as shown in Eq. \((50)\).
This theorem finally provides an unambiguous determination of the 3PN radiation field by dimensional regularization.

It should be emphasized that though the values \( \kappa \) and \( \xi \), computed in Ref. [51], represent the end result of dimensional regularization, several alternative calculations have provided a check, independent of dimensional regularization, for all the parameters \( \xi \) and \( \kappa \). In Ref. [57] we computed the 3PN binary’s mass dipole moment \( I_1 \) using Hadamard’s regularization, and identified \( I_1 \) with the 3PN center of mass vector position \( q_i \), already known as a conserved integral associated with the Poincaré invariance of the 3PN equations of motion in harmonic coordinates \( \{0, \xi, \eta, \omega \} \). This yields \( \xi + \kappa = 9871 = 9240 \) in agreement with Eq. (51). Next, we considered \[58\] the limiting physical situation where the mass of one of the particles is exactly zero (say, \( m_2 = 0 \)), and the other particle moves with uniform velocity. Technically, the 3PN quadrupole moment of a boosted Schwarzschild black hole is computed and compared with the result for \( I_{1j} \) in the limit \( m_2 = 0 \). The result is \( \xi = 7833 \), and represents a direct verification of the global Poincaré invariance of the wave generation formalism. Finally, \( \kappa = 0 \) is proven \[40\] by showing that there are no dangerously divergent “diagrams” corresponding to non-zero \( \kappa \)-values, where a diagram is meant here in the sense of Ref. \[59\]. All these verifications confirm the validity of dimensional regularization for describing the dynamics of systems of compact bodies.

5. CONCLUSION

The determination of the values \( \xi \) and \( \kappa \) completes the problem of the general relativistic prediction for the templates of inspiralling compact binaries up to 3PN order, and actually up to 3.5PN order as the corresponding “tail terms” composing this order have already been determined \[60\]. The relevant combination of the parameters \( \xi \) and \( \kappa \), entering the 3PN energy flux in the case of circular orbits, namely \[\theta \kappa = \frac{11831}{9240} \], is now fixed to

\[\theta = \frac{\xi + 2\kappa + \zeta}{9240} = \frac{11831}{9240} \]

The orbital phase of compact binaries, in the adiabatic inspiral regime (i.e., evolving by radiation reaction), involves at 3PN order a linear combination of \( \theta \) and of the equation-of-motion related parameter \( \lambda \) [13], which is determined as

\[\hat{\theta} = \frac{7}{3}\lambda = \frac{1039}{4620} \]

The parameter \( \lambda \) appears here because the orbital phase follows from energy balance between the total radiated energy flux and the decrease of orbital center-of-mass energy which is computed from the equations of motion.

The practical implementation of the theoretical templates in the data analysis of detectors follows the standard matched filtering technique. The raw output of the detector \( o(\theta) \) consists of the superposition of the real gravitational wave signal \( h_{\text{real}}(\theta) \) and of noise \( n(\theta) \). The noise is assumed to be a stationary Gaussian random variable, with zero expectation value, and with (supposedly known) frequency-dependent power spectral density \( S_n(\omega) \). The experimenters construct the correlation between \( o(\theta) \) and a filter \( q(\theta) \), i.e.,

\[c(\theta) = \int_{-\infty}^{\infty} dt o(\theta) q(\theta + t) \]

and divide \( c(\theta) \) by the square root of its variance, or correlation noise. The expectation value of this ratio defines the filtered signal-to-noise ratio (SNR). Looking for the useful signal \( h_{\text{real}}(\theta) \) in the detector’s output \( o(\theta) \), the experimenters adopt the following formula for the filter

\[\tilde{q}(\omega) = \frac{\tilde{h}(\omega)}{S_n(\omega)} \]

where \( \tilde{q}(\omega) \) and \( \tilde{h}(\omega) \) are the Fourier transforms of \( q(\theta) \) and of the theoretically computed template \( h(\theta) \). By the matched filtering theorem, the filter \( \tilde{q}(\omega) \) maximizes the SNR if \( h(\theta) = h_{\text{real}}(\theta) \). The maximum SNR is then the best achievable with a linear filter. In practice, because of systematic errors in the theoretical modelling, the template \( h(\theta) \) will not exactly match the real signal \( h_{\text{real}}(\theta) \), but if the template is to constitute a realistic representation of nature the errors will be small. This is of course the motivation for computing high order post-Newtonian templates, in order to reduce as much as possible the systematic errors due to the unknown post-Newtonian remainder. The fact that the numerical value of the parameter \( \xi \) is quite small, \( \hat{\theta} \equiv 0.22489 \), indicates, following measurement-accuracy analyses \[10 \& 11\], that the 3PN (or better 3.5PN) templates should constitute an excellent approximation for the analysis of gravitational wave signals from inspiralling compact binaries.
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A. USEFUL FORMULAS IN d SPATIAL DIMENSIONS

This Appendix is intended to provide a compendium of (mostly well-known) formulas for working in a space with \( d \) dimensions. As usual, though we shall motivate some formulas below by writing some intermediate expressions which make complete sense only when \( d \) is a strictly positive integer, our final formulas are to be interpreted, by complex analytic continuation, for a general complex dimension, \( d \in \mathbb{C} \). Actually one of the main sources of the power of dimensional regularization [29, 30, 31, 32] is its ability to prove many results by invoking complex analytic continuation in \( d \).

We discuss first the volume of the sphere having \( d = 1 \) dimensions, \( i.e. \) embedded into Euclidean \( d \)-dimensional space. We separate out the infinitesimal volume element in \( d \) dimensions into radial and angular parts,

\[
d^d \mathbf{x} = r^d \, dr \, d\Omega_{d-1} ;
\]

where \( r = \mathbf{x} \) denotes the radial variable (\( i.e. \), the Euclidean norm of \( \mathbf{x} \in \mathbb{R}^d \)) and \( d\Omega_{d-1} \) is the infinitesimal solid angle sustained by the unit sphere with \( d = 1 \) dimensional surface. To compute the volume of the sphere, \( \Omega_d = \int \sigma \, d\Omega_d \), one notices that the following \( d \)-dimensional integral can be computed both in Cartesian coordinates, where it reduces simply to a Gaussian integral, and also, using (56), in spherical coordinates:

\[
d^d x \, r^2 = \int_0^\infty \, dr \, r^d \, e^{-r^2} = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^{\infty} \, dr \, r^d \, e^{-r^2} = \frac{1}{2} \Gamma \left( d \right) \Omega_d \frac{d}{2} ;
\]

where \( \Gamma \) in the last equation denotes the Eulerian function. This leads to the well known result

\[
\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} ;
\]

For instance one recovers the standard results \( \Omega_2 = 4\pi \) and \( \Omega_1 = 2\pi \), but also \( \Omega_0 = 2 \), which can be interpreted by remarking that the sphere with 0 dimension is actually made of two points. If we parametrize the sphere \( \Omega_d \) in \( d = 1 \) dimensions by means of \( d \) spherical coordinates \( \theta_1, \theta_2, \ldots, \), which are such that the sphere \( \Omega_d \) in \( d = 2 \) dimensions is then parametrized by \( \theta_2, \theta_3, \ldots, \), and so on for the lower-dimensional spheres, then we find that the differential volume elements on each of the successive spheres obey the recursive relation

\[
d\Omega_{d-1} = \sin \theta_d \, d\Omega_d \frac{d}{2} ;
\]

Note that this implies

\[
\frac{\Omega_d}{\Omega_{d-2}} = \frac{\pi}{\Gamma(1/2)} \frac{d^d}{d\theta_1 \, \sin \theta_1 \, d\theta_1 \, \, d\Omega_{d-2}} = \frac{\pi}{\Gamma(1/2)} \frac{d^d}{dx \, \sin \theta_1 \, d\theta_1 \, \, d\Omega_{d-2}} = \frac{\pi}{\Gamma(1/2)} \frac{d^d}{x \, 2 \, \Gamma(3/2)} ;
\]

which can also be checked directly by using the explicit expression (58).

Next we consider the Dirac delta-function \( \delta^{(d)}(\mathbf{x}) \) in \( d \) dimensions, which is formally defined, as in ordinary distribution theory [47], by the following linear form acting on the set \( S \) of smooth functions \( C^\infty(\mathbb{R}^d) \) with compact support: \( \mathbf{\delta} \in S^* \),

\[
\langle \delta^{(d)} ; \mathbf{\phi} \rangle = \int S \, \mathbf{\delta}^{(d)}(\mathbf{x}) \mathbf{\phi}(\mathbf{x}) = \mathbf{\phi}(\mathbf{0}) ;
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where the brackets refer to the action of a distribution on \( \mathbf{\phi} \in S \). Let us now check that the function defined by

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is a strictly positive integer, our final formulas are to be interpreted, by complex analytic continuation in \( d \).

We discuss first the volume of the sphere having \( d = 1 \) dimensions, \( i.e. \) embedded into Euclidean \( d \)-dimensional space. We separate out the infinitesimal volume element in \( d \) dimensions into radial and angular parts,

\[
d^d \mathbf{x} = r^d \, dr \, d\Omega_{d-1} ;
\]

where \( r = \mathbf{x} \) denotes the radial variable (\( i.e. \), the Euclidean norm of \( \mathbf{x} \in \mathbb{R}^d \)) and \( d\Omega_{d-1} \) is the infinitesimal solid angle sustained by the unit sphere with \( d = 1 \) dimensional surface. To compute the volume of the sphere, \( \Omega_d = \int \sigma \, d\Omega_d \), one notices that the following \( d \)-dimensional integral can be computed both in Cartesian coordinates, where it reduces simply to a Gaussian integral, and also, using (56), in spherical coordinates:

\[
d^d x \, r^2 = \int_0^\infty \, dr \, r^d \, e^{-r^2} = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^{\infty} \, dr \, r^d \, e^{-r^2} = \frac{1}{2} \Gamma \left( d \right) \Omega_d \frac{d}{2} ;
\]

where \( \Gamma \) in the last equation denotes the Eulerian function. This leads to the well known result

\[
\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} ;
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For instance one recovers the standard results \( \Omega_2 = 4\pi \) and \( \Omega_1 = 2\pi \), but also \( \Omega_0 = 2 \), which can be interpreted by remarking that the sphere with 0 dimension is actually made of two points. If we parametrize the sphere \( \Omega_d \) in \( d = 1 \) dimensions by means of \( d \) spherical coordinates \( \theta_1, \theta_2, \ldots, \), which are such that the sphere \( \Omega_d \) in \( d = 2 \) dimensions is then parametrized by \( \theta_2, \theta_3, \ldots, \), and so on for the lower-dimensional spheres, then we find that the differential volume elements on each of the successive spheres obey the recursive relation

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\]

where the brackets refer to the action of a distribution on \( \mathbf{\phi} \in S \). Let us now check that the function defined by

\[
\frac{d^d}{\pi^{d/2}} ;
\]
This leads to the famous Riesz kernels, here denoted by the integral expression:

\[ \Delta u = 4\pi \delta^{(d)}(\mathbf{x}) : \]  

(63)

For any \( \alpha \in \mathbb{C} \) we have \( \Delta r^{\alpha} = \alpha (d + 2) r^{\alpha - 2} \), thus we see that \( \Delta u = 0 \) in the sense of functions. Let us formally compute its value in the sense of distributions in \( \mathbb{R}^d \). We apply the distribution \( \Delta u \) on some test function \( \varphi \in \mathcal{D} \), use the definition of the distributional derivative to shift the Laplace operator from \( u \) to \( \varphi \), compute the value of the \( d \)-dimensional integral by removing a ball of small radius \( s \) surrounding the origin, say \( B(s) \), and finally compute that integral by inserting the Taylor expansion of \( \varphi \) around the origin. The proof of Eq. (63) is thus summarized in the following steps:

\[
< \Delta u; \varphi > = \lim_{s! 0} \int_{\mathbb{R}^d \setminus B(s)} d^d x u \Delta \varphi
\]

\[
= \lim_{s! 0} \int_{\mathbb{R}^d \setminus B(s)} d^d x \left[ \partial_i u \partial_i \varphi + \partial_i u \varphi \right]
\]

\[
= \lim_{s! 0} s^d \int d\Omega_d - 1 (n_i) \left[ \partial_i \varphi \partial_i u \varphi \right]
\]

\[
= \lim_{s! 0} s^d \int d\Omega_d - 1 (n_i) \left[ \vec{k} \cdot \nabla \varphi \partial_i u \right]
\]

\[
= \frac{\Omega_d}{\vec{k}} \left[ \varphi \right] \cdot (d^n \varphi) \cdot 0 :
\]

(64)

In the last step we used the relation between \( \vec{k} \) and the volume of the sphere, which is

\[
\vec{k} \Omega_d - 1 = \frac{4\pi}{d} :
\]

(65)

From \( u = \vec{k} r^{d - 2} \) one can next find the solution \( v \) satisfying the equation \( \Delta v = u \) (in a distributional sense), namely

\[
v = \frac{\vec{k} r^{d - 2}}{2 \left[ d \left( d - 2 \right) \right]} :
\]

(66)

From (66) we can then define a whole “hierarchy” of higher-order functions \( w \), satisfying the Poisson equations \( \Delta w = v \), in a distributional sense.

However, the latter hierarchy of functions \( u, v, w \) is better displayed using some different, more systematic notation. This leads to the famous Riesz kernels, here denoted \( \delta^{(d)}_{\alpha} \), in \( d \)-dimensional Euclidean space \( \mathbb{R}^d \). These kernels depend on a complex parameter \( \alpha \in \mathbb{C} \), and are defined by

\[
\delta^{(d)}_{\alpha}(x) = K_{\alpha} r^{\alpha - d} ;
\]

(67a)

\[
K_{\alpha} = \frac{\Gamma \left( \frac{d - \alpha}{2} \right)}{2\pi \Gamma \left( \frac{d - \alpha}{2} \right)} ;
\]

(67b)

For any \( \alpha \in \mathbb{C} \), and also for any \( d \in \mathbb{C} \), the Riesz kernels satisfy the recursive relations

\[
\Delta \delta^{(d)}_{\alpha + 2} = \delta^{(d)}_{\alpha} ;
\]

(68)

\[ \text{The usual verification of (63) is done in Fourier space.} \]

\[ \text{Besides the Euclidean kernels \( \delta^{(d)}_{\alpha} \), we also have the Minkowski kernels (denoted \( Z^{(d)}_{\alpha} \)), which are at the basis of the Riesz analytic continuation method.} \]
Furthermore, they obey also an interesting convolution relation, which reads simply, with the chosen normalization of the coefficients $K_\alpha$, as
\[ \delta_\alpha^{(d)} \delta_\beta^{(d)} = \delta_\alpha^{(d)} \delta_\beta^{(d)}. \]  
(69)

When $\alpha = 0$ we recover the Dirac distribution in $d$ dimensions, $\delta_0^{(d)} = K_0 r^d = \delta^{(d)}$ (the coefficient vanishes in this case, $K_0 = 0$), and we have $\alpha = 4\pi \delta_\alpha^{(d)}, \nu = 4\pi \delta_\nu^{(d)}$.

The convolution relation (69) is nothing but an elegant formulation of the Riesz formula in $d$ dimensions. To check it let us consider the Fourier transform of $r^d$ in $d$ dimensions,
\[ f_\alpha(k) = \frac{2}{\pi^d} \int_0^\infty dr r^{\alpha+d} e^{ikr}. \]  
(70)

Using (56) we can rewrite it as
\[ f_\alpha(k) = \frac{2}{\pi^d} \int_0^\infty dr r^{\alpha+d} \frac{2}{\pi} e^{ikr} \Omega_{d-2} \int_0^{\pi} d\theta_d \sin(\theta_d) \rho_1 r^d e^{ikr \cos(\theta_d)} = \Omega_d (2\pi)^\frac{d}{2} (kr)^\frac{d}{2} J_{\frac{d}{2}}(kr); \]  
(71)

where $k \in \mathbb{R}$ and where we adopt for the Bessel function the defining expression
\[ J_\nu(x) = \frac{1}{\Gamma(\frac{\nu+1}{2})} \int_0^\infty \frac{t^\nu}{2} e^{-tx} \cos(\nu \cos^{-1}(x)) \]  
(72)

The radial integration in Eq. (71) is then readily done from using the previous expression, and we obtain
\[ f_\alpha(k) = \frac{2}{\pi^d} \frac{\Gamma(\frac{\alpha+d}{2})}{\Gamma(\frac{\alpha}{2})} k^{\alpha+d}; \]  
(73)

where the factor in front of the power $k^{\alpha+d}$, say $A_\alpha$, is checked from the Parseval theorem for the inverse Fourier transform, which necessitates that $A_\alpha A_\nu = (2\pi)^d$. To obtain Eq. (74) we employ the integration formula
\[ \int_0^\infty dz z^\mu J_\nu(z) = \frac{\Gamma(\frac{1+\mu+\nu}{2})}{\Gamma(\frac{1+\mu}{2})}; \]  
(75)

Finally we can check the Riesz formula by going to the Fourier domain, using the previous relations. The result,
\[ \int_0^\infty dz z^\mu J_\nu(z) = \frac{\Gamma(\frac{1+\mu+\nu}{2})}{\Gamma(\frac{1+\mu}{2})}; \]  
(76)

is equivalent to Eq. (69).

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