Semi-Complex Analysis and Mathematical Physics

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March 24, 2022
Author’s Note

The aim of this exposition is two-fold. First, we wish to acquaint the reader with the semi-complex algebra formalism. This is dealt with in Parts One and Two. In Part Three, we discuss physics related topics (in particular, space-time and gravity) in the context of this formalism.

Presently, we know that quantum mechanics finds its most natural expression in the language of complex numbers. In particular, many fruitful concepts arise by adopting these numbers as the basic building blocks of the theory. In Einstein’s formulation of General Relativity, the essential building blocks are the real numbers, which are used in the construction of real manifolds (with curvature).

It seems, then, that the starting point of each theory differs in a fundamental way, since one is deeply rooted in the theory of complex numbers and unitary transformations, while the other makes no explicit mention of these concepts. One approach towards reconciling these two theories involves re-expressing the relevant parameters in general relativity in terms of complex valued quantities (see, for example, [9]).

Rather than making use of complex numbers directly, the approach adopted in this exposition involves constructing a formalism which looks like the usual complex algebra formalism. More precisely, we perform a commutative extension of the real numbers, but instead of arriving at the division algebra of complex numbers, we arrive at a non-division algebra, which naturally admits ‘conjugation’ of elements.

Adopting these numbers as our building blocks produces a theoretical framework which is intrinsically non-euclidean in character. Connections can then be made between the symmetries of space-time, and the complex unitary groups that arise in particle gauge theories.

Acknowledgements

This work was entirely supported by the Commonwealth Scholarship and Fellowship Plan (The British Council, U.K.).

1 or hyperbolic quasi-numbers; see [1]
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Part I

Semi-Complex Analysis in One Variable
Chapter 1

The Semi-Complex Number System

1.1 Definition and Terminology

An informal yet instructive way of introducing the complex number system $\mathbb{C}$ to a newcomer is to postulate the existence of ‘numbers’ having the form

$$z = x + iy$$  \hspace{1cm} (1.1)

where $x$ and $y$ are real numbers and $i$ is a commuting variable satisfying the relation

$$i^2 = -1.$$ \hspace{1cm} (1.2)

If we write $i^2 = +1$ instead of $i^2 = -1$, what kind of number system do we end up with? To explore this possibility we will consider ‘numbers’ of the form

$$w = t + jx$$ \hspace{1cm} (1.3)

where $t$ and $x$ are real and $j$ is a commuting variable satisfying the relation

$$j^2 = +1.$$ \hspace{1cm} (1.4)

The algebra defined by (1.3) and (1.4) will be denoted by the symbol $\mathbb{D}$, and we will often reserve the letter $w$ for elements in $\mathbb{D}$. This number system turns out to be a peculiar creature, which I have ventured to call the *semi-complex number system*. The precise reason for this will become clear as this exposition unfolds.

We can add and multiply any two semi-complex numbers by assuming (tentatively) that the usual rules of arithmetic apply, as well as stipulating that
CHAPTER 1. THE SEMI-COMPLEX NUMBER SYSTEM

\[ j^2 = +1. \] Doing so enables us to write down immediately the rules for addition and multiplication:

**Addition:**
\[
(t_1 + jx_1) + (t_2 + jx_2) = (t_1 + t_2) + j(x_1 + x_2) \quad (1.5)
\]

**Multiplication:**
\[
(t_1 + jx_1) \cdot (t_2 + jx_2) = (t_1t_2 + x_1x_2) + j(t_1x_2 + x_1t_2) \quad (1.6)
\]

We are now in a position to give a formal definition of the semi-complex algebra \( D \); namely, \( D \) is the set of ordered pairs of real numbers \((t, x)\), equipped with the following operations of addition and multiplication:

1. \((t_1, x_1) + (t_2, x_2) = (t_1 + t_2, x_1 + x_2)\)
2. \((t_1, x_1) \cdot (t_2, x_2) = (t_1t_2 + x_1x_2, t_1x_2 + x_1t_2)\)

The reader may like to check that the above rules do indeed yield an associative, distributive and commutative algebra.

With these definitions, it is easy to see that an element of the form \((t, 0)\) can be identified with the real number \(t\), and \((0, 1)\) can be identified with \(j\), since
\[
(0, 1) \cdot (0, 1) = (1, 0).
\]

Also,
\[
(0, 1) \cdot (x, 0) = (0, x),
\]
so we may legitimately identify the ordered pair \((t, x)\) with our old friend \(t + jx\). At any rate, we unabashedly write
\[
w = t + jx
\]
for any semi-complex number \(w \in D\), and it is this notation that we will adopt throughout. Incidentally, we choose to call \(t\) and \(x\) respectively the real and imaginary parts of \(w = t + jx\), where the overlap of terminology with the complex numbers is intentional.

### 1.2 Conjugation and Inversion

Of course, now that we have taken the liberty to view \(D\) as a ‘number system’, it remains to show that we can do more than just add and multiply different elements. In particular, we would like to know which elements in \(D\) have inverses.

First, we introduce the notion of conjugation; given any semi-complex number \(w = t + jx\), the conjugate of \(w\), written \(\overline{w}\), is defined to be
\[
\overline{w} = t - jx.
\]
Two simple consequences can be immediately deduced; for any \( w_1, w_2 \in D \), we have
\[
\begin{align*}
\overline{w_1 + w_2} &= \overline{w_1} + \overline{w_2} \quad \text{and} \\
\overline{w_1 - w_2} &= \overline{w_1} \cdot \overline{w_2}.
\end{align*}
\] (1.7) (1.8)

We also have the important identity
\[
\|w\|^2 = t^2 - x^2.
\] (1.9)

Hence \( \|w\|^2 \) is real for any semi-complex number \( w \), although unlike the complex case, it may take on negative values. In order to strengthen the analogy between the semi-complex and complex numbers, we often write
\[
|w|^2 = \overline{w} \cdot w
\]
where \( |w|^2 \) is referred to as the ‘modulus squared’ of \( w \). A nice consequence of these definitions can now be stated:

**Proposition 1.1** For any semi-complex numbers \( w_1, w_2 \in D \),
\[
|w_1 \cdot w_2|^2 = |w_1|^2 \cdot |w_2|^2.
\]

**Proof:** Exercise! □

Now observe that if \( |w|^2 \) does not vanish, the quantity
\[
w^{-1} = \frac{\overline{w}}{|w|^2}
\] (1.10)

is a well defined inverse for \( w \). So \( w \) fails to have an inverse if (and only if)
\[
|w|^2 = t^2 - x^2 = 0,
\] (1.11)

or simply when \( x = \pm t \). If we view \( t \) as time and let \( x \) be a space coordinate, relation (1.11) defines the light cone in a 1 + 1 spacetime (\( c = \text{velocity of light} = 1 \)). Using this jargon, the number \( w = t + jx \) fails to have an inverse if the spacetime point \( (t, x) \) lies on the light cone.

![Light cone diagram](image-url)
An important distinction between the complex and semi-complex numbers can now be discerned: $D$ fails to be a division algebra. On first impression, this seems to cast a dark shadow upon the respectability of $D$ as a potentially useful number system. However, we have just seen that those elements in $D$ that are not invertible can be assigned special physical significance. This suggests that focusing exclusively on number systems that are division algebras may be unnecessarily restrictive, especially if we are seeking to construct physical theories out of them.

### 1.3 The Order Properties of $D$ (Optional)

In this section (which may be omitted on a first reading) we discover that the semi-complex algebra possesses certain order properties analogous to those of the real numbers, and, moreover, that these are related to the causal structure of $D$ viewed as 1 + 1 space-time.

For any semi-complex number $w = t + jx$, we introduce two associated real quantities, $w_+$ and $w_-$, by writing
\[ w_+ = t + x \text{ and } w_- = t - x. \]  
(1.12)

A straightforward calculation yields the identities
\[ (w_1 \cdot w_2)_\pm = (w_1)_\pm \cdot (w_2)_\pm \]  
(1.13)

The order relation $\leq$ on $D$ is defined by writing
\[ w_1 \leq w_2 \text{ (or equivalently, } w_2 \geq w_1) \]
whenever
\[ (w_1)_+ \leq (w_2)_+ \text{ and } (w_1)_- \leq (w_2)_-. \]

Of course, if we replace every occurrence of the symbol $\leq$ in the above definition with the strict inequality $<$ (or $>$), we have the corresponding definition for the meaning of the relation $w_1 < w_2$ (or $w_2 > w_1$). Notice that these definitions give the usual meanings for $\leq$ and $<$ when restricted to the real numbers.

Three distinctive properties of $\leq$ can be immediately deduced:

1. **Reflexivity**: $w \leq w$ for any $w \in D$;
2. **Anti-Symmetry**: If $w_1 \leq w_2$ and $w_2 \leq w_1$, then $w_1 = w_2$;
3. **Transitivity**: If $w_1 \leq w_2$ and $w_2 \leq w_3$, then $w_1 \leq w_3$;

These properties imply that the semi-complex algebra (equipped with the relation $\leq$) forms a partially ordered set (or *poset*). However, we have additional properties, which we list in the next proposition:
Proposition 1.2 Let $a, b, c, d$ be elements in $D$.

1. If $a \leq b$, then $a + c \leq b + c$;
2. If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$;
3. If $a \leq b$ and $c \geq 0$, then $a \cdot c \leq b \cdot c$;
4. If $a \leq b$ and $c \leq 0$, then $b \cdot c \leq a \cdot c$;
5. If $a > 0$, then $1/a > 0$;
6. If $a < 0$, then $1/a < 0$.

Proof: Use the identities (1.13).

For the real numbers, we may define the subset of strictly positive numbers $P = \{a \in \mathbb{R}|a > 0\}$. Likewise, we may define the set $C_+$ of strictly positive semi-complex numbers by setting

$$C_+ = \{w \in D|w > 0\}. \quad (1.14)$$

As is the case for the subset $P$ of strictly positive reals, the subset $C_+$ is closed under the operations of addition, multiplication, and inversion.

The subset $C_+$ has a nice physical interpretation; one can see from the definitions that it consists of those points in space-time lying inside the future light cone emanating from the origin.

Note that if $|w|^2 \neq 0$, then precisely one of the following numbers,

$$w, -w, jw, -jw,$$

lies in $C_+$. We may now define a function which sends any $w \in D$ to a corresponding element in the set of ‘non-negative’ numbers $\{w|w \geq 0\}$ by setting

$$|w|_* = \begin{cases} 
  w & \text{if } w \geq 0 \\
  -w & \text{if } -w \geq 0 \\
  jw & \text{if } jw \geq 0 \\
  -jw & \text{if } -jw \geq 0 
\end{cases} \quad (1.15)$$

The reader may like to check that this is a well defined function on $D$, and moreover, gives rise to the following identities

$$|w_1 + w_2|_* \leq |w_1|_* + |w_2|_* \quad (1.16)$$
$$|w_1 \cdot w_2|_* = |w_1|_* \cdot |w_2|_* \quad (1.17)$$

Of course, restricting $| \cdot |_*$ to the reals yields the usual ‘absolute value’ function on $\mathbb{R}$. 

\[\square\]
1.4 The Semi-Complex Norm (Optional)

In this section (which can be omitted on a first reading) we show that one can define a norm on the semi-complex algebra which is analogous to the complex modulus.

To begin, consider the two functions $| \cdot |_+ : D \to [0, \infty)$ defined by

$$
|t + jx|_+ = |t + x| \\
|t + jx|_- = |t - x|
$$

Note that $w = t + jx$ is invertible if and only if both $|w|_+$ and $|w|_-$ are non-zero. It is now a straightforward exercise to check that, for any $w, w_1, w_2 \in D$,

1. $|w|_+ \geq 0$
2. $|w + w_1|_+ \leq |w_1|_+ + |w_2|_+$
3. $|w \cdot w_2|_+ = |w_1|_+ \cdot |w_2|_+$
4. $|w|_+ = 0 \iff w = 0$

where the notation $|w|_+ = 0$ means $|w|_+ = 0$ and $|w|_- = 0$. So the two functions $| \cdot |_+$ and $| \cdot |_-$ are semi-norms on $D$. Having defined these semi-norms, we are now in a position to define the semi-complex norm $\| \cdot \| : D \to [0, \infty)$; namely, for any $w = t + jx$, we write

$$
\|w\| = \sqrt{|w|_+^2 + |w|_-^2} = \sqrt{2(t^2 + x^2)}
$$

The properties of $| \cdot |_+$ and $| \cdot |_-$ listed earlier can be used to prove the following; for any $w, w_1, w_2 \in D$,

1. $\|w\| \geq |w|_+ \geq 0$
2. $|w^n| = 0 \iff w = 0$ for any positive integer $n$;
3. $\|w_1 + w_2\| \leq \|w_1\| + \|w_2\|$;
4. $\|w_1 \cdot w_2\| \leq \|w_1\| \cdot \|w_2\|$.

**Proof:** Easy exercise! ☐

In order to define limits in $D$, we proceed as in the case for complex numbers, except we replace the complex modulus (or norm) by the semi-complex norm defined above. The induced topology is just the familiar topology for $\mathbb{R}^2$. 
Chapter 2

The Calculus of Semi-Complex Functions

2.1 Some Elementary Functions

We define the exponential function $e^w$ by invoking a familiar series expansion:

$$e^w \overset{\text{def}}{=} 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots,$$

(2.1)

which converges for all $w$ in the semi-complex (i.e. $t$-$x$) plane. Since the elements in $D$ commute, we have, for any $w_1, w_2 \in D$, the identity

$$e^{w_1} \cdot e^{w_2} = e^{w_1+w_2}.$$  

(2.2)

For real $\theta$, we have

$$e^{j\theta} = \cosh \theta + j \sinh \theta,$$

(2.3)

which can be derived by substituting $w = j\theta$ into the expansion (2.1). So, in terms of the real and imaginary parts of $w$, the exponential $e^w$ takes the form

$$e^w = e^t (\cosh x + j \sinh x).$$

(2.4)

Of course, this last identity may be taken as our defining expression for the exponential function.

Notice that for real values of $\theta$,

$$|e^{j\theta}|^2 = \cosh^2 \theta - \sinh^2 \theta = 1.$$  

In the light of proposition [1.3] we may view $e^{j\theta}$ as a ‘phase factor’ which leaves the modulus squared of a semi-complex number unchanged after multiplication.
This is of course analogous to the complex phase factor $e^{i\theta}$ encountered in complex number theory.

Our next task is to define the logarithmic function $\log w$. Let $w = t + jx > 0$ be a given strictly positive semi-complex number (see Section 1.3 for a discussion on the use of inequalities in $D$). Then the expression

$$\log w = \frac{1}{2} \ln (t^2 - x^2) + j \tanh^{-1} \left( \frac{x}{t} \right),$$

(2.5)

ensures that the identity

$$e^{\log w} = w$$

(2.6)

holds, and, moreover, it is the unique such expression. Using this fact, and identity (2.3), we arrive at a familiar looking result:

$$\log w_1 w_2 = \log w_1 + \log w_2 \quad \text{for any } w_1, w_2 > 0. \quad (2.7)$$

The reader is reminded that the condition $w = t + jx > 0$ implies that the space-time point $(t, x)$ lies inside the future light cone with vertex at the origin.

### 2.2 Differentiation and Holomorphic Functions

#### 2.2.1 The Derivative

For an arbitrary $D$-valued function $f$ defined on some open region $U$ of the semi-complex plane, we may write

$$f(t + jx) = u(t, x) + jv(t, x), \quad t + jx \in U,$$

where $u$ and $v$ are real valued functions. Unless otherwise stated, we will always assume $u$ and $v$ to be smooth (or $C^\infty$) functions.

Defining the derivative of a function at some point usually involves investigating the behaviour of the quotient

$$\frac{f(w + \Delta w) - f(w)}{\Delta w}$$

(2.8)

as $\Delta w$ ‘tends to zero’. An alternative formulation of the derivative which will be convenient for our purposes is based on the ‘linear approximation’

$$\Delta f = f(w + \Delta w) - f(w) \approx f'(w) \cdot \Delta w,$$

which becomes ‘exact’ as $\Delta w$ becomes vanishingly small. Adopting this last point of view enables us to define the derivative of a semi-complex function without introducing the need to divide by the semi-complex parameter $\Delta w$, which arises in the quotient (2.8).
Definition 2.1 Let \( f : U \to \mathbb{D} \) be a semi-complex function defined on some open region \( U \) of the semi-complex plane. Let \( w_0 \in U \). Then \( f'(w_0) \in \mathbb{D} \) is said to be the derivative of \( f \) at \( w_0 \) if the quotient
\[
\frac{f(w_0 + \Delta w) - f(w_0) - f'(w_0) \cdot \Delta w}{||\Delta w||}
\]
tends to zero as \( ||\Delta w|| \to 0 \).

If \( f \) has a derivative at each point in its domain \( U \), it is said to be holomorphic on \( U \).

As in the theory of complex functions, the real and imaginary parts of a holomorphic semi-complex function \( u + jv \) satisfy certain relations. In the complex case, these are known as the Cauchy-Riemann Equations. To discover what the corresponding relations are in the semi-complex case, we observe that the existence of a derivative for \( f \) at \( w \in U \) implies
\[
f'(w) = \lim_{h \to 0} \frac{f(w + h) - f(w)}{h} = \lim_{h \to 0} \frac{f(w + jh) - f(w)}{jh}, \quad (2.9)
\]
where \( h \) is a real parameter. Hence, using the notation \( f = u + jv \), we deduce that
\[
f' = \frac{\partial u}{\partial t} + j\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x} + j\frac{\partial u}{\partial x}, \quad (2.10)
\]
Equating real and imaginary parts above, we arrive at the necessary condition for holomorphicity:
\[
\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial t}, \quad (2.11)
\]
which are analogs of the Cauchy-Riemann equations arising in complex analysis. An easy calculation shows that these relations force \( u \) and \( v \) (and thus \( f \)) to satisfy the one dimensional wave equation:
\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = 0.
\]
These observations suggest that holomorphic semi-complex functions might play a useful role in special relativity – a speculation that we will pursue in greater detail later on.

We now wish to prove that a semi-complex function with real and imaginary parts satisfying relations \((2.11)\) in some open region \( U \) is holomorphic. The argument follows very closely the one given in [2] for the analogous situation in complex function theory. From the calculus, we can write
\[
\begin{align*}
\frac{\partial u}{\partial t} & \Delta t + \frac{\partial u}{\partial x} \Delta x + \epsilon_1, \\
\frac{\partial v}{\partial t} & \Delta t + \frac{\partial v}{\partial x} \Delta x + \epsilon_2.
\end{align*}
\]
where the remainders \( \epsilon_1, \epsilon_2 \) tend to zero more rapidly than \( \Delta w = \Delta t + j\Delta x \), in the sense that \( \epsilon_1/||\Delta w|| \) and \( \epsilon_2/||\Delta w|| \) tend to zero as \( ||\Delta w|| \to 0 \).

With the notation \( f(w) = u(t, x) + jv(t, x) \), we obtain by virtue of the relations (2.11)

\[
f(w + \Delta w) - f(w) = \left( \frac{\partial u}{\partial t} + j\frac{\partial v}{\partial t} \right) \cdot \Delta w + \epsilon_1 + j\epsilon_2,
\]

(2.12)

and therefore the quotient

\[
\frac{f(w + \Delta w) - f(w) - (\frac{\partial u}{\partial t} + j\frac{\partial v}{\partial t}) \cdot \Delta w}{||\Delta w||} = \frac{\epsilon_1 + j\epsilon_2}{||\Delta w||}
\]

tends to zero as \( ||\Delta w|| \to 0 \). We conclude that \( f(w) \) is holomorphic (on \( U \)).

### 2.2.2 Semi-Complex Holomorphic Functions

Having shown that relations (2.11) are the necessary and sufficient conditions for holomorphicity, we may approach the whole subject of semi-complex holomorphic functions without any explicit reference to limiting procedures at all.

Let us first introduce the operators

\[
\frac{\partial}{\partial w} = \frac{1}{2} \left\{ \frac{\partial}{\partial t} + j\frac{\partial}{\partial x} \right\}, \quad \frac{\partial}{\partial \overline{w}} = \frac{1}{2} \left\{ \frac{\partial}{\partial t} - j\frac{\partial}{\partial x} \right\}.
\]

(2.13)

A straightforward calculation shows that

\[
df = \frac{\partial f}{\partial w} dw + \frac{\partial f}{\partial \overline{w}} d\overline{w},
\]

(2.14)

for any semi-complex valued function \( f \). This result is particularly useful, since it is tantamount to saying that the variables \( w \) and \( \overline{w} \) may be treated independently, a fact which will be well appreciated in later sections of this article.

Now consider the constraint equation

\[
\frac{\partial f}{\partial \overline{w}} = 0.
\]

(2.15)

Separating real and imaginary parts, condition (2.13) takes the form

\[
\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial t},
\]

(2.16)

which are precisely the relations (2.11) satisfied by a semi-complex holomorphic function. This enables us to define holomorphic functions as precisely those for which the constraint equation (2.13) holds. (Compare this with the definition of a complex holomorphic function).
CHAPTER 2. THE CALCULUS OF SEMI-COMPLEX FUNCTIONS

By virtue of the holomorphicity condition \((2.15)\), the identity \((2.14)\) simplifies to
\[
df = \frac{\partial f}{\partial w} dw,
\]
which shows that the derivative \(f'\) of a holomorphic function \(f\) (sometimes written \(df/dw\)) may be defined as
\[
f' = \frac{\partial f}{\partial w}.
\]

A simple calculation demonstrates that \(f'\) above has the explicit form
\[
f' = \frac{\partial u}{\partial t} + j \frac{\partial v}{\partial t} = \frac{\partial v}{\partial x} + j \frac{\partial u}{\partial x},
\]
which is of course consistent with our earlier observations.

### 2.2.3 Derivatives of Elementary Functions

Fortunately, the results just obtained yield familiar looking results. For example, the reader is invited to verify the following fundamental facts;
\[
\frac{d}{dw} w^n = n w^{n-1}, \quad \frac{d}{dw} e^w = e^w, \quad \frac{d}{dw} \log |w|_* = \frac{1}{w},
\]
where the first two functions above are defined everywhere, and the logarithm is defined for all \(w\) satisfying \(|w|^2 \neq 0\).

We remark that the holomorphic map \(f : (t, x) \rightarrow (u, v)\) induces a conformal transformation on the semi-complex plane in the following sense:
\[
du^2 - dv^2 = |f'|^2 (dt^2 - dx^2),
\]
which is a direct consequence of \((2.17)\) (take the modulus of both sides). The conformal factor is thus \(|f'|^2\). The coordinates \((u, v)\) define a coordinate system which possesses the natural property that the coordinate curves \((u = \text{const}, v = \text{const})\) intersect orthogonally with respect to the Lorentz inner product with signature \((+, -)\). This is of course perfectly analogous to the ‘isothermal coordinates’ one encounters in complex analysis.

Integration of semi-complex valued functions offers additional insights; we pursue this topic next.

### 2.3 Semi-Complex Integration
2.3.1 Definition

Integrating a semi-complex function \( f = u + jv \) along a piecewise smooth curve \( \gamma \) is defined in the obvious way; namely, we set

\[
\int_\gamma f \, dw = \int_\gamma (u + jv)(dt + jdx)
\]

\[
= \int_\gamma u \, dt + v \, dx + j \int_\gamma v \, dt + u \, dx,
\]

where the last line above involves the evaluation of two line integrals in the semi-complex plane. We will always assume that the path of integration is piecewise smooth.

2.3.2 Cauchy Type Theorems

If the path of integration is a simple closed curve, then under certain favourable conditions, we may invoke Green’s Theorem in the plane to convert the line integral into a surface integral. This is the content of the next Proposition;

**Proposition 2.1** Let \( f \) be a semi-complex function defined on and within a smooth, simple-closed curve \( C \). Let \( R \) denote the interior of \( C \). Then

\[
\oint_C f \, dw = \int\int_R \frac{\partial f}{\partial w} \, d\bar{w} \wedge dw.
\]

**Proof**: Write \( f = u + jv, \, dw = dt + jdx \), and use Green’s Theorem in the plane. \( \square \)

Of course, the last Proposition still holds for regions \( R \) that are not simply connected. The details are left to the reader. An immediate corollary of Proposition 2.1 turns out to be formally identical to Cauchy’s Theorem in complex analysis;

**Corollary 2.1** Let \( f \) be a semi-complex function which is holomorphic on and within a simple closed curve \( C \). Then

\[
\oint_C f(w) \, dw = 0.
\]  \hspace{1cm} (2.22)

**Proof**: Recall that \( f \) is holomorphic \( \iff \frac{\partial f}{\partial w} = 0 \). Then use Proposition 2.1. \( \square \)

So integrals of holomorphic functions along paths lying in the domain of holomorphicity depend only on the endpoints of those paths; consequently, the
integrals are path independent. In fact, if $F(w)$ is the anti-derivative\footnote{i.e. $F'(w) = f(w)$; all holomorphic functions possess an anti-derivative which is unique up to a constant.} of a holomorphic function $f(w)$, we may evaluate the integral

$$\int_{\gamma} f(w)dw$$

by calculating the values for $F$ on the endpoints of the path $\gamma$. For example, suppose $\gamma$ is any path from $w_1$ to $w_2$. Then

$$\int_{\gamma} w^2 dw = \left[ \frac{w^3}{3} \right]^{w_2}_{w_1} = \frac{w_2^3}{3} - \frac{w_1^3}{3}.$$ 

We now seek a result which is analogous to Cauchy’s Integral Formula. This last classical result implies that if the values of a complex holomorphic function are known on a circle, the values of the function inside the circle are uniquely determined. Now a circle in the complex plane (centered at the origin, say) is the locus of points $z \in \mathbb{C}$ satisfying $|z|^2 = a^2$ (where $a$ is a real constant). For the semi-complex case, the analogous object is the locus of points $w \in D$ satisfying $|w|^2 = a^2$, which is a hyperbola in the semi-complex plane.

Thus we anticipate that a semi-complex holomorphic function is uniquely determined everywhere\footnote{almost; if $a = 0$, then $f$ is uniquely defined for all points in $D$; otherwise it is uniquely defined for all points $|w|^2 \neq 0$.} by its values on the hyperbola $|w|^2 = t^2 - x^2 = a^2$. This is indeed the case, which can be deduced from the following Proposition:

**Proposition 2.2** Suppose $f$ is semi-complex holomorphic on an open set $U \subseteq D$. Let $\alpha = (1 + j)/2$. Then for any $p, q \in \mathbb{R}, w \in U$ for which $w + p\alpha$ and $w + q\alpha$ lie in $U$, we have

$$f(w) = \alpha f(w + p\alpha) + \overline{\alpha} f(w + q\alpha). \quad (2.23)$$

**Proof:** Let $w \in U$ be fixed. Set $F(p, q) = \alpha f(w + p\alpha) + \overline{\alpha} f(w + q\alpha)$. Using the fact $\overline{\alpha} \alpha = 0$, and the holomorphicity of $f$, we deduce

$$\frac{\partial F}{\partial p} = \frac{\partial F}{\partial q} = 0.$$ 

So $F(p, q)$ is constant. But $F(0, 0) = f(w)$, and so the proof is complete. \qed
CHAPTER 2. THE CALCULUS OF SEMI-COMPLEX FUNCTIONS

\(|w|^2 = 0\). If \(f\) is known to be holomorphic everywhere except on the light cone, then it is uniquely determined by its values on the hyperbola \(|w|^2 = a^2, a \neq 0\). The verification of these facts is a simple ‘proof by picture’ argument. The reader may wish to fill in the details!

2.3.3 A Counter Example

The reader may recall that Cauchy’s Integral Formula plays a crucial role in the proof of the analyticity property of complex holomorphic functions. In this section we show by way of a counter example that semi-complex holomorphic functions are not necessarily analytic. Thus the holomorphicity condition imposed on semi-complex valued functions turns out to be a much less restrictive condition than for the complex case.

Fortunately, we don’t have to look too hard to discover an example\(^3\) of a non-analytic function which is holomorphic in the entire \(D\) plane. First, we introduce the function

\[
f(w) = e^{-\frac{1}{w^2}},
\]

which is defined and holomorphic for all \(w\) satisfying \(|w|^2 \neq 0\). What happens as \(|w|^2\) approaches zero (i.e. as \(w\) approaches the null lines \(t = \pm x\)) ? A straightforward calculation shows that

\[
e^{-\frac{1}{w^2}} = \frac{1}{2}(1 + j)e^{-\frac{1}{(t+x)^2}} + \frac{1}{2}(1 - j)e^{-\frac{1}{(t-x)^2}}.
\]

Evidently, as \(t \to x\), the second term in (2.23) vanishes, while the first is finite. Similarly, the function \(e^{-1/w^2}\) is well behaved as \(t \to -x\). The whole expression vanishes as \(w\) approaches zero. These observations permit us to extend the function \(e^{-1/w^2}\) to the entire \(D\) plane in the following way: Let

\[
g(w) = \begin{cases} 
  e^{-\frac{1}{w^2}} & \text{if } |w|^2 \neq 0 \\
  \frac{1}{2}(1 \pm j) & \text{if } t = \pm x, t \neq 0 \\
  0 & \text{if } w = 0
\end{cases}
\]

It is now a straightforward (though somewhat tedious) exercise to show that \(g: D \to D\) is holomorphic everywhere. Note that \(g\) is non-invertible on the null lines.

\(^3\)noticed by Steve Lack (Cambridge) and the author.
Chapter 3

The Physics of Semi-Complex Numbers

3.1 The Lorentz Transformation

It turns out that Lorentz’s Transformation on 1 + 1 space-time involving a velocity boost can be elegantly expressed in terms of semi-complex numbers. The Lorentz Transformation between two coordinate systems \((t', x')\) and \((t, x)\) is given by the equations

\[
ct' = \frac{ct - \frac{v}{c}x}{\sqrt{1 - \frac{v^2}{c^2}}} \quad x' = \frac{x - \frac{v}{c}ct}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{3.1}
\]

If we define \(\theta\) by setting

\[
\tanh \theta = \frac{v}{c}, \tag{3.2}
\]

then the identity \(\cosh^2 \theta - \sinh^2 \theta = 1\) enables us to write

\[
\cosh \theta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \sinh \theta = \frac{v/c}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{3.3}
\]

Consequently, \((3.1)\) is equivalent to the expression

\[
ct' + jx' = (\cosh \theta - j \sinh \theta)(ct + jx), \tag{3.4}
\]

or simply

\[
w' = e^{-j\theta} \cdot w \tag{3.5}
\]

\(^1\)where the appearance of the constant \(c\) is now made explicit.
where \( w \) and \( w' \) are semi-complex numbers whose real and imaginary parts are the time and space variables (respectively) of the corresponding coordinate system. So if a space-time point \((ct, x)\) is represented by the semi-complex number \( w = ct + jx \), the Lorentz Transformation in (3.4) simply involves multiplying \( w \) by a phase factor \( e^{-j\theta} \). A physical theory which is required to be Lorentz invariant must therefore be invariant under the global phase transformation

\[
w \rightarrow w' = e^{-j\theta} \cdot w \quad (\theta = \text{constant}) \quad (3.6)
\]

The simplest physical theory which is invariant under this transformation will be studied next.

We should remark that our reformulation of Lorentz’s Transformation in terms of semi-complex numbers involves only one spatial coordinate, which obviously falls short of the three spatial dimensions that are known to exist. In order to handle transformations involving multiple space dimensions, we need to introduce higher dimensional structures that will appear formally identical to the concept of vector spaces and curved manifolds in ordinary analysis. We pursue these topics in Part II.

### 3.2 Classical Point Particles

Let us consider a classical point particle with rest mass \( m \) moving in \( 1 + 1 \) space-time. A suitable Lagrangian (see [3]) for describing such a particle is given by

\[
L = m \left[ \left( \frac{dt}{d\tau} \right)^2 - \left( \frac{dx}{d\tau} \right)^2 \right],
\]

or simply

\[
L = m \frac{dw}{d\tau} \frac{dw}{d\tau},
\]

where \( w = ct + jx \), and \( \tau \) is the proper time\(^2\), which is an invariant parameter of the trajectory.

Note that the Lagrangian is always real valued, and invariant under the global phase transformation (3.6). By carrying out the variation on the independent variables \( w \) and \( \bar{w} \), the Euler-Lagrange equations to be solved are:

\[
\frac{d}{d\tau} \frac{\partial L}{\partial \frac{dw}{d\tau}} - \frac{\partial L}{\partial w} = 0, \quad \frac{d}{d\tau} \frac{\partial L}{\partial \frac{d\bar{w}}{d\tau}} - \frac{\partial L}{\partial \bar{w}} = 0
\]

which yield the following equation of motion:

\[
m \frac{d^2w}{d\tau^2} = 0.
\]

\(^2\)In fact \( d\tau = \sqrt{dw \cdot d\bar{w}} = \sqrt{c^2 dt^2 - dx^2} \), and has the dimensions of length.
Solving this equation yields solutions of the form
\[ w = \alpha \tau + \beta, \] (3.10)
where \( \alpha \) and \( \beta \) are semi-complex constants. These solutions are, of course, the straight lines in Minkowski space (i.e., particles moving with constant velocity).

### 3.3 Local Gauge Invariance

The dynamics generated by the Lagrangian in (3.8) is simple enough — free particles in Minkowski space move with constant velocity. Moreover, the Lorentz invariance of the theory simply refers to its invariance under the global phase transformation (3.6).

More interesting dynamics occurs if we make the added assumption that our theory should remain invariant under a local phase transformation:
\[ w(\tau) \rightarrow w'(\tau) = e^{-j\theta(\tau)} \cdot w(\tau) \] (local phase invariance) (3.11)
where the phase angle \( \theta = \theta(\tau) \) is now allowed to vary with \( \tau \). The invariance demanded by (3.11) is just a generalization of the global phase invariance encountered in (3.6).

Leaping from a global phase invariance to a local one is a very popular theme in particle physics, and is usually referred to as the ‘gauge principle’ [5].

To begin our analysis, we need to find a suitable Lagrangian that is preserved by local phase transformations. The Lagrangian (3.3) as it stands, is not invariant under the local phase transformation (3.11), and so needs to be modified. This can be done as follows; replace the derivative \( \frac{d}{d\tau} \) by the derivative operator
\[ D = \frac{d}{d\tau} - \frac{j}{c^2}g. \] (3.12)
The gauge field \( g \) (which we assume to be real valued) has the dimensions of acceleration, and is defined to transform in the following way:
\[ g \rightarrow g' = g - c \frac{d\theta}{d\tau}. \] (3.13)
Thus the derivative operator transforms as follows:
\[ D \rightarrow D' = \frac{d}{d\tau} + \frac{j}{c^2}g' \]
\[ = \frac{d}{d\tau} + \frac{j}{c^2}g - j \frac{d\theta}{d\tau}. \]
These definitions, combined with the local phase transformation (3.11), lead us to the identity
\[ D'w' = e^{-j\theta(\tau)} \cdot Dw \] (3.14)
If we redefine our Lagrangian to be

$$L = \overline{Dw} \cdot Dw$$  \hspace{1cm} (3.15)

then the gauge transformation $w \rightarrow w'$, $g \rightarrow g'$ leaves $L$ above invariant. We should emphasise that the gauge field $g$, together with its transformation law, have to be introduced in order to ensure local phase invariance.

To see that the gauge field $g$ is a suitable candidate for some kind of force field, we invoke relation (3.2) to deduce the explicit form of the transformation law for $g$:

$$g \rightarrow g' = g - c^2 \frac{d}{d\tau} \tanh^{-1} \frac{v}{c}$$

$$= g - \frac{1}{(1 - v^2/c^2)^{3/2}} \cdot \frac{dv}{dt}.$$  

Now $-(dv/dt)/(1 - v^2/c^2)^{3/2}$ corresponds to the relativistic force acting on a unit mass (at rest in the fixed frame) as seen by an observer moving with velocity $v$ with respect to the fixed frame. This is precisely the way in which a force field would transform in a relativistic setting.

These observations suggest that the gauge group $\{e^{i\theta} | \theta \in \mathbb{R}\}$ might give rise to physical fields. Our analysis thus far does not offer any clues about the field equations governing $g$, but we will soon remedy this after we introduce the calculus of several semi-complex variables.
Part II

Several Semi-Complex Variables
Chapter 4

Semi-Complex Inner Product Spaces

4.1 Motivation

Chapter 3 highlighted some interesting connections that can arise between semi-complex analysis and special relativity in one space dimension. However, we live in a space which has manifestly three spatial dimensions, so we might ask whether our analysis can be extended in a natural way to encompass this higher dimensional case.

Mathematically speaking, the most natural higher dimensional objects that can be built from semi-complex numbers are perhaps $D$-modules\(^1\) and semi-complex manifolds. The latter will allow us to discuss the notion of ‘curvature’ and will be dealt with in Chapter 6.

There is of course the question of the appropriateness of this theory in modelling physical reality. The reader is invited to form his or her own opinion on this one, but it is perhaps safest to view this subject as a kind of mathematical curiosity, which admits now and again the possibility of physical application. In any case, speculations on the relevance of the theory to mathematical physics will be given free reign in Part II.

The vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ should be familiar to the reader; we now wish to explore the module space $D^n$. This space admits a very natural inner product that is non-euclidean in character. This is the content of the next section.

\(^1\) i.e. modules over the ring of semi-complex numbers
4.2 The Standard Inner Product on \( D^n \)

The rather elegant properties exhibited by the semi-complex numbers suggest that they might be useful in constructing higher dimensional objects. One such object is the module space \( D^n \). Elements of \( D^n \) are the \( n \times 1 \) matrices

\[
 w = \begin{pmatrix} w^1 \\ \vdots \\ w^n \end{pmatrix}, \quad w^i \in D. \tag{4.1}
\]

For each element \( w \in D^n \), we define \( w^* \) to be the row matrix

\[
 w^* = ( \overline{w^1} \cdots \overline{w^n} ) \tag{4.2}
\]

i.e. \( w^* \) is just the conjugate transpose of \( w \).

Now consider the map \( <,> : D^n \times D^n \to D \) given by the rule

\[
 <w,\omega> = w^* \cdot \omega \tag{4.3}
\]

In components, definition (4.3) takes the form

\[
 < \begin{pmatrix} w^1 \\ \vdots \\ w^n \end{pmatrix}, \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix} > = \begin{pmatrix} \overline{w^1} & \cdots & \overline{w^n} \end{pmatrix} \cdot \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix} \tag{4.4}
\]

\[
 = \overline{w^1} \omega^1 + \cdots + \overline{w^n} \omega^n. \tag{4.5}
\]

In particular, we have

\[
 <w,w> = |w^1|^2 + \cdots + |w^n|^2. \tag{4.6}
\]

If we write \( w^i = t^i + jx^i \), the last expression becomes

\[
 <w,w> = (t^1)^2 + \cdots + (t^n)^2 - (x^1)^2 - \cdots - (x^n)^2. \tag{4.7}
\]

So the most ‘natural’ inner product on \( D^n \) induces a non-euclidean metric on \( \mathbb{R}^{2n} \). This inner product will be called the standard inner product on \( D^n \).

We remark that the standard inner product on \( D^n \) is non-degenerate. That is, if \( <w,\omega> = 0 \) for all \( \omega \), then \( w \) must vanish. We will be restricting our investigations to such non-degenerate cases.

At this stage, it is interesting to note that if we introduce a new variable \( t \) by writing

\[
 t^2 = (t^1)^2 + \cdots + (t^n)^2, \tag{4.8}
\]

then expression (4.5) takes the form

\[
 <w,w> = t^2 - (x^1)^2 - \cdots - (x^n)^2, \tag{4.9}
\]

which, on its own, induces a Lorentzian metric on the real space of \( (n+1) \)-tuples \( \{(t,x^1,\ldots,x^n)\} \).
4.3 Unitary Symmetry on $D^n$

Consider the case where $D^n$ is equipped with the standard inner product defined by (4.3). Let $U$ be an $n \times n$ matrix with semi-complex entries, and let $U^*$ denote, as usual, the conjugate transpose of $U$. A simple calculation shows that

$$<Uw, U\omega> = <w, \omega>$$

(4.10)

for all $w, \omega \in D^n$ if and only if

$$U^*U = 1.$$  

(4.11)

Consequently, the set of all (semi-complex) linear transformations $U : D^n \to D^n$ which preserve the inner product forms a (non-compact) Lie group. We will denote this group by $U(n, D)$, which is to be distinguished from the corresponding complex unitary group $U(n, \mathbb{C})$.

4.4 General Inner Product Spaces

We are now in a position to list the main properties of the standard inner product: For all $w_1, w_2, w_3 \in D^n$, $\lambda \in D$, we have

1. $<w_1, w_2> \in D$,
2. $<w_1, w_2> = <w_2, w_1>$,
3. $<w_1, \lambda w_2> = \lambda <w_1, w_2>$,
4. $<w_1 + w_2, w_3> = <w_1, w_3> + <w_2, w_3>$, and
5. $<,>$ is non-degenerate.

If $M$ is any given module over $D$ which admits a map $<,>: M \times M \to D$ satisfying the properties listed above, it will be called a semi-complex inner product space. Obviously, $D^n$, together with the standard inner product, is an example of semi-complex inner product space. A non-trivial example is given below.

**Example 4.1** Suppose $H$ is an $n \times n$ hermitian matrix over the semi-complex numbers. Define a map $<\cdot|H|\cdot>: D^n \times D^n \to D$ by writing

$$<w|H|\omega> = w^*H\omega \quad \forall w, \omega \in D^n.$$  

(4.12)

Then $<\cdot|H|\cdot>$ is a semi-complex inner product whenever the hermitian matrix $H$ is non-singular. We leave the details to the reader.

---

3 i.e. $H^* = H$, where $H^*$ denotes the conjugate transpose of $H$. 
Chapter 5

The Semi-Complex Unitary Groups

5.1 The Hyperbola Group U(1, D)

If the modulus squared of a semi-complex number \( w \) is unity, \( w \) must have the form \( e^{i\theta} \) or \(-e^{i\theta}\) for some real parameter \( \theta \). Consequently,

\[
U(1, D) = \{ \pm e^{i\theta} \mid \theta \in \mathbb{R} \}. \tag{5.1}
\]

Geometrically, \( U(1, D) \) is the hyperbola \( t^2 - x^2 = 1 \). Thus it is a non-compact Lie group with two simply connected components, which are the branches of the hyperbola. The branch \( U_+(1, D) = \{ e^{i\theta} \mid \theta \in \mathbb{R} \} \) is the component of \( U(1, D) \) containing the identity, and in the light of expression (3.5), is identified as the restricted group of Lorentz transformations in 1 + 1 space-time. From these definitions, we deduce that

\[
U(1, D)/U_+(1, D) \cong \mathbb{Z}_2. \tag{5.2}
\]

Generalisations of the above identity occur for the higher dimensional groups, which we begin to investigate next.

5.2 The Groups U(2, D) and SU(2, D)

If \( U \) is a \( 2 \times 2 \) matrix over \( D \) with the property

\[
U^*U = 1, \tag{5.3}
\]

then

\[
\det(U^*U) = \overline{\det U} \cdot \det U = |\det U|^2 = 1. \tag{5.4}
\]
From the preceding discussion, we conclude that for such a matrix $U$, we have, for some real $\theta$, either
\[ \det U = +e^{j\theta} \quad \text{or} \quad \det U = -e^{j\theta}. \] (5.5)

The set of all elements in $U(2, D)$ for which the first relation above holds forms a proper subgroup which we denote by $U_+(2, D)$. In fact, any element $U \in U_+(2, D)$ can be written in the form
\[ U = e^{jH} \] (5.6)

where $H$ is a $2 \times 2$ hermitian matrix over $D$ (i.e. $H^* = H$).

Since $U_+(2, D)$ is the connected component of $U(2, D)$ containing the identity, it is a normal subgroup, and elements $\sigma$ in the quotient $U(2, D)/U_+(2, D)$ (5.7)

have the property
\[ \sigma^2 = 1. \] (5.8)

The special unitary group $SU(2, D)$ is defined to be the set of all elements $U \in U(2, D)$ with the property
\[ \det U = +1. \] (5.9)

Consider the following traceless, hermitian matrices
\[ \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (5.10)

Then any $2 \times 2$ hermitian matrix $H$ may be written in the form
\[ H = \theta.1 + \sum_{k=1}^{3} \phi_k \tau_k \] (5.11)
\[ = \theta.1 + \vec{\phi} \cdot \tau, \] (5.12)

where the parameters $\theta, \phi_k$ are real, and $1$ is the $2 \times 2$ identity matrix. Utilising the identity
\[ \det e^A = e^{\text{tr} A}, \] (5.13)

we deduce that any element of the special unitary group $SU(2, D)$ may be written in the form
\[ U = \exp \left( j\vec{\phi} \cdot \tau / 2 \right), \quad \phi_k \in \mathbb{R} \] (5.14)

and that any member of $U_+(2, D)$ is a product of an element in $SU(2, D)$ and a phase $e^{j\theta}$.
The anti-hermitian matrices\(^1\) \(E_k = \frac{1}{2}j\tau_k\) form a basis for the real Lie algebra \(su(2, D)\), and satisfy the following commutation relations:

\[
\begin{align*}
[E_1, E_2] &= E_3 \\
[E_2, E_3] &= E_1 \\
[E_3, E_1] &= -E_2
\end{align*}
\]

(5.15)

(5.16)

(5.17)

The structure of this Lie algebra becomes clear as soon as we introduce the new variables \(S_+, S_-, S_3\) defined in terms of the \(E_k\):

\[
\begin{align*}
S_+ &= j(E_1 - jE_2) \\
S_- &= j(E_1 + jE_2) \\
S_3 &= jE_3
\end{align*}
\]

(5.18)

(5.19)

(5.20)

From the commutation relations (5.15)–(5.17), we deduce the following:

\[
\begin{align*}
[S_3, S_\pm] &= \pm S_\pm \\
[S_+, S_-] &= 2S_3
\end{align*}
\]

(5.21)

(5.22)

These relations are precisely those given for the real Lie algebra \(sl(2, \mathbb{R})\) (see, for example, \([6]\)), and play a fundamental role in the quantum mechanics of spin and angular momentum \([7]\). The different representations (over \(\mathbb{R}\)) for this algebra yield the quantised values for spin, characterised by the Casimir operator \(S^2 = \frac{1}{2}(S_+S_- + S_-S_+) + S_3^2\). It can be shown directly from the commutation relations that the Casimir operator commutes with each of the basis elements \(S_+, S_-\) and \(S_3\).

We have so far demonstrated that the real Lie algebras \(sl(2, \mathbb{R})\) and \(su(2, D)\) are (semi-complex) isomorphic, but \(sl(2, \mathbb{R})\) is isomorphic\(^2\) with \(su(2, \mathbb{C})\) as well. The interesting observation here is that the quantised values of spin (or the eigenvalues of the Casimir operator) predicted by an \(SU(2, \mathbb{C})\) theory of spin are identical to an \(SU(2, D)\) theory (where \(SU(2, D)\) acts on \(D^2\)).

Perhaps this is not too surprising considering the similarities of the conjugation operation for complex and semi-complex numbers. It should come as no surprise, then, that the Lie algebras for \(SU(3, \mathbb{C})\) and \(SU(3, D)\) are intimately related.

We devote the next section to a (brief) study of \(U(3, D)\) and \(SU(3, D)\).

### 5.3 The Groups \(U(3, D)\) and \(SU(3, D)\)

If \(U\) is a \(3 \times 3\) matrix over \(D\) with the property

\[
U^*U = 1,
\]

(5.23)

\(^1\)A matrix \(A\) is anti-hermitian if \(A^* = -A\)

\(^2\)Strictly speaking, complex isomorphic; see [\([\ldots]\)]
then we conclude, as before, that for some real \( \theta \),

\[
\det U = +e^{j\theta} \quad \text{or} \quad \det U = -e^{j\theta}.
\]  

(5.24)

The subgroup \( U_+ (3, D) \) is defined to be the set of all elements in \( U(3, D) \) with the first property above, and represents the connected component of \( U(3, D) \) containing the identity. Equivalently, we may write

\[
U_+ (3, D) = \{ e^{jH} \mid H \text{ is } 3 \times 3 \text{ hermitian} \}.
\]  

(5.25)

The special unitary group \( SU(3, D) \) consists of those elements in \( U(3, D) \) with determinant equal to one.

An element \( \sigma \) in the quotient group \( U(3, D)/U_+ (3, D) \) satisfies \( \sigma^2 = 1 \).

In order to determine the Lie algebra of the special unitary group \( SU(3, D) \) we use the fact that any \( 3 \times 3 \) hermitian matrix \( H \) may be written in the form

\[
H = \theta \cdot 1 + \sum_{i=1}^{8} \alpha_i \mu_i
\]  

(5.26)

\[
= \theta \cdot 1 + \vec{\alpha} \cdot \mu,
\]  

(5.27)

where the parameters \( \theta, \alpha_i \) are real, \( 1 \) is the \( 3 \times 3 \) identity matrix, and \( \mu_i \) are the traceless hermitian matrices listed below:

\[
\mu_1 = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \quad \mu_2 = \begin{pmatrix}
0 & -j & 0 \\
j & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \quad \mu_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\mu_4 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} \quad \mu_5 = \begin{pmatrix}
0 & 0 & -j \\
0 & 0 & 0 \\
j & 0 & 0
\end{pmatrix} \quad \mu_6 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
\mu_7 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -j \\
j & 0 & 0
\end{pmatrix} \quad \mu_8 = \begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

With the help of identity (5.13), one can see that any element \( U \) in \( SU(3, D) \) may be written in the form

\[
U = \exp (j\vec{\alpha} \cdot \mu/2)
\]  

(5.28)

where the eight parameters \( \alpha_i \) are real. The anti-hermitian matrices \( F_k = \frac{1}{2}j\mu_k \) are readily seen to form a basis for the real Lie algebra \( su(3, D) \), and satisfy commutation relations

\[
[F_p, F_q] = f_{pqr} F_r.
\]  

(5.29)

Explicit values for the (real) structure constants \( f_{pqr} \) can easily be obtained, but in order to illuminate the structure of this Lie algebra, we introduce new
variables $I_3, Y, I_\pm, U_\pm, V_\pm$, defined in terms of the $F_k$:

\[
\begin{align*}
I_3 &= jF_3 \\
Y &= 2j\left(\frac{1}{3}F_3 - \frac{2}{3}F_8\right) \\
I_\pm &= j(F_1 \mp jF_2) \\
U_\pm &= j(F_6 \mp jF_7) \\
V_\pm &= j(F_4 \mp jF_5).
\end{align*}
\]

From these definitions, and the commutation relations (5.29), we deduce the following:

\[
\begin{align*}
[I_3, I_\pm] &= \pm I_\pm & [Y, I_3] &= 0 \\
[I_3, U_\pm] &= \mp \frac{1}{2}U_\pm & [Y, U_\pm] &= \pm U_\pm \\
[I_3, V_\pm] &= \pm \frac{1}{2}V_\pm & [Y, V_\pm] &= \pm V_\pm
\end{align*}
\]

Incidentally, these commutation relations played a significant role in a formal quark theory of matter ([8], [6]) dealing with the up, down and strange quarks\(^3\). In this ‘flavour’ theory of quarks, $I_3$ and $Y$ are isospin and hypercharge operators, while $U_\pm$, $V_\pm$, and $I_\pm$, are raising and lowering operators.

Our observations show that the predicted values for isospin and hypercharge in the old flavour theory of quarks (where the symmetry group SU(3, $C$) acts on $C^3$) can equally be obtained by an SU(3, $D$) theory, where elements in SU(3, $D$) act on $D^3$.

In quantum mechanics, the complex unitary groups arise from considering the ‘rotational’ symmetries of a vector wave function $\Psi$, which has complex components. At this stage, it is tempting to ask if a vector valued function with semi-complex valued components has any potential application to physics. In other words, can an element in $D^3$, say, represent some physical state of a system, or of space-time?

\(^3\)now known to be related to quark ‘flavours’
Chapter 6

Semi-Complex Manifolds and Curvature

Fortunately, the subject of semi-complex manifolds can be handled in precisely the same way as for complex manifolds. Our initial discussion follows very closely the one given in [5]. Other possible sources that may be helpful to the reader include [10] and [11].

6.1 Semi-Complex Manifolds

To begin, we define a holomorphic map on $\mathbb{D}^m$. A semi-complex valued function $f : \mathbb{D}^m \to \mathbb{D}$ is holomorphic if

$$\frac{\partial f}{\partial w^\mu} = 0 \quad (6.1)$$

for $\mu = 1, \ldots, m$ ($w^\mu = t^\mu + jx^\mu$). A map $(f^1, \ldots, f^n) : \mathbb{D}^m \to \mathbb{D}^n$ is called holomorphic if each function $f^\lambda (1 \leq \lambda \leq n)$ is holomorphic.

To define a semi-complex manifold, we invoke the usual definition given for a complex manifold, except we assume the transition functions are semi-complex holomorphic instead of complex holomorphic.

In fact, for most of this chapter, the reader may safely arrive at the correct formal definitions (which we will often omit for brevity) by referring to the corresponding complex case, and replacing the phrase ‘complex holomorphic’ with the alternative phrase ‘semi-complex holomorphic’.

**Definition 6.1** $M$ is a semi-complex manifold if

1. $M$ is a topological space;

2. $M$ is provided with a family of pairs $\{(U_i, \phi_i)\}$ such that $\{U_i\}$ is an open cover of $M$, and each $\phi_i$ is a homeomorphism from $U_i$ to an open subset of $\mathbb{D}^m$;
3. Given \( U_i \) and \( U_j \) with non-empty intersection, the transition function 
\[ \psi_{ij} = \phi_j \circ \phi_i^{-1} \]
from \( \phi_i(U_i \cap U_j) \) to \( \phi_j(U_i \cap U_j) \) is semi-complex holomorphic.

In words, a semi-complex manifold is a geometrical object which \textit{locally} looks like \( \mathbb{D}^m \). The number \( m \) is called the semi-complex dimension of \( M \) and we often write \( \dim_{\mathbb{D}} M = m \).

### 6.2 Calculus on Semi-Complex Manifolds

The reader is reminded that our discussion is intended to be brief, since the terminology we use is identical in form to the one used in the treatment of complex manifolds.

#### 6.2.1 The Holomorphic Tangent Module

Let \( M \) be a semi-complex manifold with \( \dim_{\mathbb{D}} M = m \). Take a point \( p \) in a chart \((U, \phi)\) of \( M \). The \( m \) semi-complex linear operators
\[
\left\{ \frac{\partial}{\partial w^1}, \ldots, \frac{\partial}{\partial w^m} \right\}
\]

at the point \( p \) generate a module space over \( \mathbb{D} \). We call it the holomorphic tangent module at \( p \), and denote it by \( T_p M^+ \). By convention, any element \( W \in T_p M^+ \) will be called a holomorphic vector.

The anti-holomorphic tangent module \( T_p M^- \) is the module generated by the anti-linear operators
\[
\left\{ \frac{\partial}{\partial \bar{w}^1}, \ldots, \frac{\partial}{\partial \bar{w}^m} \right\},
\]

and any element in \( T_p M^- \) will be called an anti-holomorphic vector.

#### 6.2.2 Hermitian Manifolds

The form \( dw^\mu \) will be viewed as a semi-complex linear map
\[
dw^\mu : T_p M^+ \to \mathbb{D}
\]

with the property
\[
dw^\mu \left( \frac{\partial}{\partial w^\nu} \right) = \delta^\mu_\nu.
\]

We may now introduce the anti-linear map
\[
d\bar{w}^\mu : T_p M^+ \to \mathbb{D}
\]
by writing
\[ dw^\mu(W) = \overline{dw^\mu(W)} \]
for any given holomorphic vector \( W \).

The tangent module \( T_pM^+ \) may be viewed as a semi-complex inner product space if there exists a map
\[ g : T_pM^+ \times T_pM^+ \rightarrow \mathbb{D} \]
satisfying, for any \( W, X, Y \in T_pM^+ \), \( \lambda \in \mathbb{D} \), the following properties:
1. \( g(X, Y) = g(Y, X) \);
2. \( g(X, \lambda Y) = \lambda g(X, Y) \);
3. \( g(W, X + Y) = g(W, X) + g(W, Y) \);
4. \( g \) is non-degenerate (see below).

The first three properties imply that \( g \) has the form
\[ g = g_{\mu\nu} dw^\mu \otimes dw^\nu, \quad (6.2) \]
where the semi-complex components \( g_{\mu\nu} \) transform tensorially. The last property of non-degeneracy means the matrix \( \{g_{\mu\nu}\} \) is invertible.

If we now write \( < X, Y > = g(X, Y) \), then the map \( <,> \) is obviously a semi-complex inner product on the tangent module \( T_pM^+ \) (refer to Section 4.4).

If at each point on a manifold there exists such a map, the manifold is said to be Hermitian, and the map \( g \) is called a Hermitian metric. Justifying this terminology is simple enough; the property \( g(X, Y) = g(Y, X) \) implies
\[ \overline{g_{\mu\nu}} = g_{\nu\mu}, \quad (6.3) \]
and so the matrix \( \{g_{\mu\nu}\} \) must be Hermitian.

### 6.2.3 Covariant Derivatives

Let \( M \) be a semi-complex manifold, and suppose \( X \) is a holomorphic vector field on \( M \) (i.e. \( X(p) \in T_pM^+ \) for each \( p \in M \)). If we assume that any holomorphic vector \( V \in T_pM^+ \) parallelly transported (by some connection map \( \Gamma \)) to another point \( q \) is again a holomorphic vector \( \tilde{V} \in T_qM^+ \), then the covariant derivative of \( X \) takes the form
\[ \nabla_k X^i = \frac{\partial X^i}{\partial w^k} + \Gamma^i_{jk} X^j, \quad (6.4) \]
where the field of numbers \( \Gamma^i_{jk} \) are known as connection coefficients.
6.2.4 Metric Compatibility

If we now endow the manifold $M$ with a Hermitian metric $g$, there turns out to be a natural way in which to uniquely specify the value of these coefficients. We simply impose the condition that the connection preserves the inner product induced by the metric $g$. In particular, if two holomorphic vectors $V$ and $W$ at the point $p$ are ‘parallel transported’ into the vectors $\tilde{V}$ and $\tilde{W}$ at $q$ (respectively), then we always have the identity

$$< V, W >_p = < \tilde{V}, \tilde{W} >_q,$$

where we have used the notation $< X, Y >= g(X, Y)$.

One can show that this metric compatibility condition takes the equivalent form

$$\nabla_k g_{ij} = 0. \quad (6.5)$$

Evaluating the covariant derivative of the metric $g$ in the last expression yields the condition

$$\frac{\partial g_{ij}}{\partial w^k} - g_{in} \Gamma^i_{jk} = 0. \quad (6.6)$$

The connection coefficients are thus uniquely determined, since we can express them in terms of the components of the metric:

$$\Gamma^i_{jk} = g^{im} \frac{\partial g_{mj}}{\partial w^k}, \quad (6.7)$$

where $\{g^{nm}\}$ is the inverse matrix of $\{g_{ij}\}$. Such a connection is called a Hermitian connection.

6.3 The Curvature

Our aim in this section is to list the main results concerning the curvature of a semi-complex manifold. A thorough treatment can be found (at least in form) in [5] and [11].

6.3.1 Torsion

The torsion tensor $T$ is defined by writing

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}. \quad (6.8)$$

For a Hermitian connection, it takes the form

$$T^i_{jk} = g^{im} \left\{ \frac{\partial g_{mj}}{\partial w^k} - \frac{\partial g_{mk}}{\partial w^j} \right\}. \quad (6.9)$$
So if the torsion vanishes, the metric components satisfy the following relations:

\[
\frac{\partial g_{ij}}{\partial w^k} = \frac{\partial g_{ik}}{\partial w^j}.
\] (6.10)

A Hermitian metric \(g\) satisfying these equations is called a Kähler metric.

### 6.3.2 Geodesics

A geodesic is a curve \(\gamma\) whose tangent vector is everywhere non-zero and proportional to a parallely propagated vector. Intuitively, it is the ‘straightest’ possible curve.

In a particular coordinate system, the geodesic \(\gamma\) has a parametrization \(w(\tau) = (w^1(\tau), \ldots, w^n(\tau))\) which can be shown to satisfy (see [11]) the geodesic equation

\[
d^2 w^i/d\tau^2 + \Gamma^i_{jk} dw^j d\tau dw^k d\tau = 0,
\] (6.11)

for some affine parameter \(\tau\).

As in Riemannian geometry, the geodesic equation may be derived from an action principle whenever the metric is torsion free (or Kähler). In fact, if we choose as our Lagrangian the expression

\[
L = g_{\mu \nu} dw^\mu / d\tau \cdot dw^\nu / d\tau,
\] (6.12)

then the Euler-Lagrange equations can be shown to give the geodesic equation (6.11) whenever \(g\) is Kähler.

### 6.3.3 The Riemann Tensor

We will now state the main results concerning the form of the curvature tensor (or Riemann Tensor \(R\)) for a semi-complex manifold. Again, for a complete explanation of the concepts involved, the reader should refer to [1], [11] or [12].

Suppose \(M\) is a semi-complex manifold with a connection \(\Gamma\). Then the components of the Riemann tensor satisfy the following relations:

\[
R^j_{irm} = \partial_r \Gamma^j_{im} - \partial_m \Gamma^j_{ir} + \Gamma^m_{im} \Gamma^j_{nr} - \Gamma^m_{ir} \Gamma^j_{nm};
\] (6.13)

\[
R^j_{\bar{r}m} = \partial_{\bar{r}} \Gamma^j_{im};
\] (6.14)

\[
R^j_{ir\bar{m}} = -\partial_m \Gamma^j_{ir}.
\] (6.15)

In the above expressions, we used the symbol \(\partial_r\) to denote the operator \(\partial / \partial w^r\), and likewise, \(\partial_{\bar{r}}\) is shorthand for writing \(\partial / \partial \bar{w}^r\).

From expressions (6.14) and (6.15), we deduce the identity

\[
R^j_{ir\bar{m}} = -R^j_{\bar{m}ir}.
\] (6.16)
For a Hermitian connection on $M$ (see Section 6.2.4), a nice cancellation occurs; namely, we deduce the following:

$$R^i_{jrm} = 0.$$  \hspace{1cm} (6.17)

(Hint: substitute (6.7) into (6.13)). So the only non-vanishing components of the Riemann tensor are the relatively simple expressions $R^i_{jrm}$ and $R^j_{irm}$.

### 6.3.4 The Ricci Form

New quantities of interest may be obtained by contracting indices of the Riemann tensor. To begin, let us write

$$R_{rm} = R^i_{jrm} = -\partial_m \Gamma^i_{jr}.$$ \hspace{1cm} (6.18)

For a Hermitian connection, a convenient simplification occurs\footnote{namely, we use the result $\partial_r \ln det g = g^{rm} \partial_r \ln g$.}, and we may write

$$R_{rm} = -\partial_r \partial_m \ln det g,$$ \hspace{1cm} (6.19)

where $g = \{g_{\mu\nu}\}$. See [5] for more details.

Since the determinant of a Hermitian matrix is real, $\ln det g$ is a real quantity. Consequently,

$$R_{\mu\nu} = -\partial_\mu \partial_\nu \ln det g = -\partial_\nu \partial_\mu \ln det g = R_{\nu\mu},$$ \hspace{1cm} (6.20)

and so the matrix of elements $\{R_{\mu\nu}\}$ forms a Hermitian matrix.

To define the Ricci form, we write

$$R = dR_{\mu\nu} dw^\mu \wedge d\overline{w}^\nu,$$ \hspace{1cm} (6.21)

which is a real form, since $\overline{R} = R$.

### 6.3.5 The Ricci Tensor

The Ricci tensor $Ric$ is also obtained by a contraction of indices:

$$Ric_{rm} = R^i_{rjm} = -\partial_m \Gamma^i_{j};$$ \hspace{1cm} (6.22)

For a Hermitian connection with vanishing torsion (i.e. for a Kähler metric),

$$Ric_{rm} = -\partial_m \Gamma^i_{j} = -\partial_m \Gamma^i_{jr} = R_{rm},$$ \hspace{1cm} (6.23)

and so the components of the Ricci form agree with $Ric_{rm}$. So for a Kähler metric, we have

$$Ric_{rm} = -\partial_m \partial_j \ln det g.$$ \hspace{1cm} (6.24)
If $\text{Ric} = \mathcal{R} = 0$, the Kähler metric is said to be *Ricci-flat*. In this case, finding solutions to the equation

$$\text{Ric}_{\mu\nu} = 0$$

(6.25)

amounts to solving

$$\partial_{\mu}\partial_{\nu} \ln \det g = 0$$

(6.26)

Further insights are gained by solving this equation on a local co-ordinate chart. Firstly, by virtue of the vanishing torsion condition expressed by equation (6.10), we may write locally the expression

$$g_{\mu\nu} = \partial_{\mu}\partial_{\nu} \phi,$$

(6.27)

where $\phi$ is some scalar valued function. Since $g$ is hermitian, $\phi$ must be real valued. Now equation (6.26) implies that

$$\ln \det g = F(w^1, \ldots, w^n) + F(\overline{w}^1, \ldots, \overline{w}^n),$$

(6.28)

where $F$ is an arbitrary semi-complex holomorphic function in the variables $w^1, \ldots, w^n$. Here, $n$ is the (semi-complex) dimension of the manifold.

Performing a local co-ordinate transformation, we can arrange for $F$ to have the constant value $j$, which means equation (6.28) becomes

$$\det g = 1.$$  

(6.29)

Making use of the expression $g_{\mu\nu} = \partial_{\mu}\partial_{\nu} \phi$, we have

$$\det \{\partial_{\mu}\partial_{\nu} \phi\} = 1.$$  

(6.30)

So, locally, solving for a Ricci flat metric $g$ involves choosing a real scalar $\phi$ which satisfies this last condition.

Whether there exist topological obstructions preventing the existence of a globally defined metric of this kind is an open question.

In the complex case, the necessary and sufficient condition for the existence of a unique Ricci flat metric is known; namely, the first Chern class, $c_1$, must vanish. See [5].
Part III

Living in a Semi-Complex World
Chapter 7

The Arrow of Time

7.1 A Semi-Complex View of Space-Time

Our analysis of the semi-complex number system in Part I yielded various results that seemed to be closely related with the theory of special relativity in one space dimension. For example, if we replace the space-time point \((t, x)\) by the single semi-complex variable \(w = t + jx\) (we have chosen units so that \(c = 1\)), we can effect a Lorentz transformation by multiplying \(w\) by a phase factor \(e^{-j\theta}\).

At first glance, it is not at all clear how one can apply our semi-complex formalism to the case of four dimensional space-time. In particular, is there a natural representation of the real 4-tuple,

\[
(t, x^1, x^2, x^3), \quad (7.1)
\]

\((t\ is\ time;\ the\ x^i's\ are\ spatial\ coordinates)\) in terms of semi-complex quantities?

Given any space-time point \((t, x^1, x^2, x^3)\), we can define an associated element \(w \in \mathbb{D}^3\) by writing

\[
w = \begin{pmatrix} t + jx^1 \\ jx^2 \\ jx^3 \end{pmatrix}. \quad (7.2)
\]

With the standard inner product \(<, >\) on \(\mathbb{D}^3\), we have

\[
< w, w > = w^* \cdot w \quad (7.3)
\]

\[
= |t + jx^1|^2 + |jx^2|^2 + |jx^3|^2 \quad (7.4)
\]

\[
= t^2 - (x^1)^2 - (x^2)^2 - (x^3)^2. \quad (7.5)
\]

Preserving this last quantity under coordinate transformations is then related to the problem of finding transformations on \(\mathbb{D}^3\) which preserve the standard inner product.
One such transformation which preserves the form of (7.2) is the matrix

\[ U = \begin{pmatrix} e^{-j\theta} & 1 \\ 1 & 1 \end{pmatrix}, \quad \theta \in \mathbb{R}. \tag{7.6} \]

Note that \( U^* U = 1 \), and \( \det U = e^{-j\theta} \), which implies that \( U \in \mathbb{U}_+(3, \mathbb{D}) \). This corresponds to a velocity boost in the direction of the \( x^1 \) axis.

The reader may have noticed that the representation of a general space-time point (7.1) by the column vector (7.2) is slightly lopsided, since the time variable \( t \) appears in the first row, and not in the second or third. Of course, we would still end up with the result (7.5) if the time variable appeared in the second or third row. In fact, a completely symmetrical representation can be achieved by 'distributing' the time variable amongst the three rows.

More precisely, suppose \( t^1, t^2 \) and \( t^3 \) are any real numbers satisfying

\[ (t)^2 = (t^1)^2 + (t^2)^2 + (t^3)^2 \tag{7.7} \]

and let us define

\[ w = \begin{pmatrix} t^1 + jx^1 \\ t^2 + jx^2 \\ t^3 + jx^3 \end{pmatrix}. \tag{7.8} \]

Then

\[
\langle w, w \rangle = |t^1 + jx^1|^2 + |t^2 + jx^2|^2 + |t^3 + jx^3|^2 = (t^1)^2 + (t^2)^2 + (t^3)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{7.9}
\]

\[
= t^2 - (x^1)^2 - (x^2)^2 - (x^3)^2. \tag{7.10}
\]

Again, preserving this last quantity is related to finding transformations on \( \mathbb{D}^3 \) which preserve the standard inner product \( \langle , \rangle \).

Now any element of \( \mathbb{U}_+(3, \mathbb{D}) \) acting on (7.8) will preserve the inner product \( \langle , \rangle \), and so it induces (by virtue of the relation (7.7)) a transformation on Minkowski space \( \{(t, x^1, x^2, x^3)\} \) which preserves the Lorentz metric on this space.

Incidentally, this induced transformation on Minkowski space is non-linear.

We now consider infinitesimal displacements of space and time. In special relativity, the space-time metric is flat and has the form

\[ ds^2 = dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{7.12} \]

If we write

\[ dw = \begin{pmatrix} dt + jdx^1 \\ jdx^2 \\ jdx^3 \end{pmatrix}, \tag{7.13} \]
then we can recover the space-time metric as follows:

\[<dw, dw> = dw^r \cdot dw = dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.\]  

A more symmetrical formulation can be achieved by introducing the quantities \(dt^1, dt^2, dt^3\) which are assumed to satisfy the identity

\[(dt)^2 = (dt^1)^2 + (dt^2)^2 + (dt^3)^2.\]

Then, setting

\[dw = \begin{pmatrix} dt^1 + jdx^1 \\ dt^2 + jdx^2 \\ dt^3 + jdx^3 \end{pmatrix},\]

we can recover the (flat) space-time metric in the usual way:

\[<dw, dw> = (dt^1)^2 + (dt^2)^2 + (dt^3)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.\]

### 7.2 Time’s Arrow

Notice that we can write \((7.8)\) in the form

\[w = \vec{t} + j\vec{x},\]  

where

\[\vec{t} = \begin{pmatrix} t^1 \\ t^2 \\ t^3 \end{pmatrix} \in \mathbb{R}^3\]

is called the time vector, and

\[\vec{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathbb{R}^3\]

is the usual position vector.

The time \(t\) (or physical time) is simply the length of the time vector:

\[t = |\vec{t}|.\]

Similarly, we may write \((7.17)\) in the form

\[dw = \vec{dt} + j\vec{dx},\]
where $\tilde{d}t$ is the infinitesimal time displacement vector, and $\tilde{d}x$ is the infinitesimal position displacement vector.

The physical time displacement $d\bar{t}$ is given by the length of the infinitesimal $\tilde{d}t$:

$$d\bar{t} = |\tilde{d}t|.$$  \hfill (7.23)

So far we have attempted to symmetrise the representation of a space-time point (or infinitesimal space-time displacement) in terms of our semi-complex formalism by introducing a vector quantity whose length specifies the usual time variable $t$ (or displacement $d\bar{t}$). However, this prescription does not tell us what particular element of the form (7.8) corresponds to the space-time point $(t, x_1, x_2, x_3)$, since the identity (7.7) does not uniquely define the variables $t_1, t_2, t_3$.

On the other hand, we can assign to each element in $D^3$ a unique space-time point by invoking this same identity. Formally speaking, we have an onto map from $D^3$ to Minkowski space:

$$
\begin{pmatrix}
    t^1 + j x^1 \\
    t^2 + j x^2 \\
    t^3 + j x^3
\end{pmatrix}
\rightarrow
(t, x^1, x^2, x^3),$$
\hfill (7.24)

where $t$ is obtained by invoking identity (7.8). In this sense, $D^3$ may be viewed as an ‘augmentation’ of Minkowski space; or, equivalently, Minkowski space may be viewed as a ‘derived’ space. In any case, we choose to work in the space $D^3$ since the mathematics appears to be formally more elegant. This is in large part due to the extra symmetry afforded by introducing three time variables which we can match with the three well known space variables, allowing us to work in three semi-complex dimensions, rather than four real dimensions.

For now, we should look upon the introduction of a ‘time vector’ as a pure mathematical convenience. Speculations concerning the physical significance of our formalism will be discussed later.

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1strictly speaking, there is a sign ambiguity here.
Chapter 8

General Relativity on Semi-Complex Manifolds

In this chapter we briefly examine Einstein’s General Theory of Relativity in the context of curved semi-complex manifolds. In particular, we investigate equation (6.25) for Ricci-flat (semi-complex) space-times.

As a preliminary, we introduce the wave equation for the semi-complex space-time $D^3$.

8.1 The Wave Equation on $D^3$

Let $\Phi : D^3 \to D$ be a semi-complex valued scalar field on $D^3$, and set

$$\eta = \begin{pmatrix} \frac{\partial \Phi}{\partial w^1} \\ \frac{\partial \Phi}{\partial w^2} \\ \frac{\partial \Phi}{\partial w^3} \end{pmatrix}.$$  \hspace{1cm} (8.1)

If we define the Lagrangian density $L$ by writing

$$L = \eta^* \cdot \eta = \frac{\partial \Phi}{\partial w^1} \cdot \frac{\partial \Phi}{\partial w^1} + \frac{\partial \Phi}{\partial w^2} \cdot \frac{\partial \Phi}{\partial w^2} + \frac{\partial \Phi}{\partial w^3} \cdot \frac{\partial \Phi}{\partial w^3},$$  \hspace{1cm} (8.2)

then the equation of motion for $\Phi$ is

$$\left( \frac{\partial^2}{\partial w^1 \partial w^1} + \frac{\partial^2}{\partial w^2 \partial w^2} + \frac{\partial^2}{\partial w^3 \partial w^3} \right) \Phi = 0,$$  \hspace{1cm} (8.4)

1i.e. the Euler-Lagrange equations
where
\[
\frac{\partial^2}{\partial w^\mu \partial w^\mu} = \frac{\partial}{\partial w^\mu} \cdot \frac{\partial}{\partial w^\mu} = \frac{1}{4} \left\{ \frac{\partial^2}{\partial (w^\mu)^2} - \frac{\partial^2}{\partial (x^\nu)^2} \right\}, \quad \mu = 1, 2, 3. \tag{8.5}
\]

As a matter of convenience, we choose to define
\[
\Box_\mu = \frac{\partial^2}{\partial w^\mu \partial w^\mu}. \tag{8.6}
\]

Adopting this notation enables us to rewrite (8.4) as
\[
(\Box_1 + \Box_2 + \Box_3)\Phi = 0. \tag{8.7}
\]

Of course, we can generalise our definition of the wave equation for the space $D^n$. Namely, the $n$-dimensional semi-complex wave equation is defined by
\[
\left( \sum_{\mu=1}^n \Box_\mu \right) \Phi = 0. \tag{8.8}
\]

We will be primarily concerned with the three dimensional case.

By virtue of relations (8.5), the wave equation (8.4) can be put into the more explicit form
\[
\frac{1}{4} \left( \frac{\partial^2}{\partial (t^1)^2} + \frac{\partial^2}{\partial (t^2)^2} + \frac{\partial^2}{\partial (t^3)^2} - \frac{\partial^2}{\partial (x^1)^2} - \frac{\partial^2}{\partial (x^2)^2} - \frac{\partial^2}{\partial (x^3)^2} \right) \Phi = 0. \tag{8.9}
\]

Recalling that the (physical) time variable $t$ is determined by the length of the time vector $\mathbf{t}$, we may write
\[
\begin{align*}
t^1 &= t \cos \theta \sin \phi \tag{8.10} \\
t^2 &= t \sin \theta \sin \phi \tag{8.11} \\
t^3 &= t \cos \phi \tag{8.12}
\end{align*}
\]
where we have chosen the usual spherical coordinates $\theta$ and $\phi$ to specify the direction of the time vector. The variable $t$ is then the radial coordinate determined by the distance of the point $(t^1, t^2, t^3) \in \mathbb{R}^3$ from the origin.

In terms of these spherical coordinates, the operator
\[
\frac{\partial^2}{\partial (t^1)^2} + \frac{\partial^2}{\partial (t^2)^2} + \frac{\partial^2}{\partial (t^3)^2} \tag{8.13}
\]
becomes
\[
\frac{1}{t^2} \frac{\partial}{\partial t} \left( t^2 \frac{\partial}{\partial t} \right) + \frac{1}{t^2 \sin^2 \phi} \frac{\partial^2}{\partial \theta^2} + \frac{1}{t^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial}{\partial \phi} \right). \tag{8.14}
\]
To say the function $\Phi$ is dependent only on the physical time $t$ and the spatial coordinates $x^1, x^2, x^3$ implies the condition

$$\frac{\partial \Phi}{\partial \theta} - \frac{\partial \Phi}{\partial \phi} = 0.\quad (8.15)$$

This last condition is tantamount to saying that $\Phi$ is spherically symmetric with respect to the time co-ordinates $t^\mu$, and enables us to view $\Phi$ as a well defined function on Minkowski space.

For this spherically symmetric case, the wave equation on $D^3$ takes the form

$$\frac{1}{t^2} \frac{\partial}{\partial t} \left( t^2 \frac{\partial \Phi}{\partial t} \right) - \frac{\partial^2 \Phi}{\partial (x^1)^2} - \frac{\partial^2 \Phi}{\partial (x^2)^2} - \frac{\partial^2 \Phi}{\partial (x^3)^2} = 0.\quad (8.16)$$

If we now make the substitution

$$\Phi = \frac{\Psi}{t}, \quad t \neq 0,\quad (8.17)$$

equation (8.16) becomes (for $t \neq 0$)

$$\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial (x^1)^2} - \frac{\partial^2 \Psi}{\partial (x^2)^2} - \frac{\partial^2 \Psi}{\partial (x^3)^2} = 0,\quad (8.18)$$

which is the standard wave equation on Minkowski space.

### 8.2 Ricci-Flat Space-Times

Ricci-flat space-times in Einstein’s General Theory of relativity are obtained by solving the equation

$$R_{ij} = 0,\quad (8.19)$$

where $R_{ij}$ is the usual Ricci tensor defined for a real four dimensional manifold.

For semi-complex manifolds, the analogous equation to be solved is

$$\text{Ric}_{\mu \nu} = 0,\quad (8.20)$$

where the Ricci tensor $\text{Ric}$ for semi-complex manifolds was introduced in section 6.3.4. We also showed that in the case of a Hermitian connection with vanishing torsion, equation (8.20) assumes the relatively simple form

$$\partial_\mu \partial_\nu \ln G = 0,\quad (8.21)$$

where $G$ denotes the determinant of the Hermitian matrix $\{g_{\mu \nu}\}$.

In one semi-complex dimension, this equation is simply

$$\frac{\partial}{\partial w^1} \left( \frac{\partial}{\partial w^1} \ln g_{T1} \right) = 1\frac{1}{4} \left( \frac{\partial^2}{\partial (t^1)^2} - \frac{\partial^2}{\partial (x^1)^2} \right) \ln g_{T1} = 0,\quad (8.22)$$
and so $\ln g_{\Gamma 1}$ satisfies the one dimensional wave equation. Therefore, the general solution for equation (8.21) is simply

$$g_{\Gamma 1} = e^{2\sigma},$$  (8.23)

where $\sigma$ is a solution of the one dimensional wave equation.

Incidentally, this is equivalent to the analytical solution one would obtain from Einstein’s equation (8.20), for a real two dimensional manifold. Of course, all real two dimensional manifolds are conformally flat, so our result does not say anything profound in a geometrical sense, but it does indicate that our formalism yields a correct analytical description of Einsteinian gravity for the low dimensional case, and so there is a suggestion that for higher dimensions, we may also obtain a correct description of gravity.

For semi-complex manifolds of higher dimension, however, equation (8.21) is non-linear. In fact, we saw in Section 6.3.5 that, locally, a Ricci flat Kähler metric is given by

$$g_{\mu\nu} = \partial_\mu \partial_\nu \phi,$$  (8.24)

where $\phi$ is a real valued scalar function satisfying the equation

$$\det \{ \partial_\mu \partial_\nu \phi \} = 1.$$  (8.25)

It is thus a challenge to show that in three semi-complex dimensions, we can find a solution $\phi$ that gives rise to a geometry corresponding to the Schwarzschild solution of general relativity (after suitably projecting the dynamics from the semi-complex space to a real four-manifold). Presently, the author knows of no such solution.

In any case, we can gain considerable insight by investigating approximate solutions for the case of ‘weak curvature’. We take up this topic next.

### 8.2.1 The Weak Curvature Approximation

Our aim in this section is essentially to linearise equation (8.21). We also choose to work in three semi-complex dimensions, although our method of approach generalises easily to other dimensions.

For the case of weak curvature, we expect the matrix of metric coefficients $g = \{g_{\mu\nu}\}$ to deviate only slightly from the identity matrix. So if we write

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu},$$  (8.26)

then the Hermitian matrix $h = \{h_{\mu\nu}\}$ measures just this deviation, and the entries $h_{\mu\nu}$ may be assumed to be small order quantities (i.e. $h_{\mu\nu} \ll 1$).

Retaining only quantities which are first order in $h_{\mu\nu}$, we have

$$G \equiv \det(g_{\mu\nu}) \approx 1 + h_{11} + h_{22} + h_{33},$$  (8.27)
and so

\[ \ln G \approx \ln(1 + h_{\mu_1} + h_{\mu_2} + h_{\mu_3}) \]  
\[ \approx h_{\mu_1} + h_{\mu_2} + h_{\mu_3} \]  
\[ = \text{trace}(h). \]  

Substituting this last result into equation (8.21) yields the following equations:

\[ \square_{\mu} \text{trace}(h) = 0 \] \[ \mu = 1, 2, 3. \]  
\[ (\square_1 + \square_2 + \square_3)h_{\mu\nu} = 0 \] \[ \mu \neq \nu. \]  

Equation (8.32) is obtained by making repeated use of the identities

\[ \frac{\partial h_{\mu\nu}}{\partial w^\lambda} = \frac{\partial h_{\mu\lambda}}{\partial w^\nu}, \quad h_{\mu\nu} = h_{\nu\mu}, \]  

which follow from the fact that \( g \) is a Kähler metric (refer to Section 6.3.1).

Equation (8.32) is immediately identified as the three dimensional semi-complex wave equation, which we saw was closely linked to the standard wave equation on four dimensional space-time. In particular, for the case where these fields are spherically symmetric with respect to time (and so well defined on Minkowski space), we conclude from (8.17) and (8.18) that the weak curvature solutions represent waves which propagate at constant speed, and are modulated by the amplitude \( 1/t \).

### 8.2.2 Newtonian Gravity

We now wish to consider the case where the metric components are time independent. In particular, we analyse the geodesics for a stationary, Ricci-flat (semi-complex) space-time in the weak curvature approximation.

As in Section 8.2.1, the matrix \( h = \{ h_{\mu\nu} \} \) will measure the (small) deviation away from flat space-time. The condition of time independence for the metric components \( g_{\mu\nu} \) implies (by virtue of relation (8.26))

\[ \frac{\partial h_{\mu\nu}}{\partial t^\lambda} = 0, \quad \lambda = 1, 2, 3. \]  

Hence, we may write

\[ \frac{\partial h_{\mu\nu}}{\partial w^\lambda} = \frac{1}{2} \frac{\partial h_{\mu\nu}}{\partial x^\lambda}. \]  

The first identity in (8.33) now takes the form

\[ \frac{\partial h_{\mu\nu}}{\partial x^\lambda} = \frac{\partial h_{\mu\nu}}{\partial x^\nu}. \]  

\(^{2}\text{we allow for the possibility that the Ricci tensor may not be defined at some spatial point} \)
Combining this last result with the Hermitian condition for $h$ (second identity in (8.33)) enables one to prove in three dimensions that the matrix $h = \{h_{\mu\nu}\}$ is real symmetric. The proof is slightly involved, but the interested reader is strongly encouraged to go through the details! Since the components of the matrix $h$ are now real, we will write $h = \{h_{\mu\nu}\}$. So the real-symmetric property of $h$ translates into the identity

$$h_{\mu\nu} = h_{\nu\mu}. \quad (8.37)$$

Since the components $h_{\mu\nu}$ are time independent, equations (8.31) and (8.32) reduce to

$$\frac{\partial^2}{\partial (x^\mu)^2} \text{trace}(h) = 0 \quad \mu = 1, 2, 3; \quad (8.38)$$

$$\nabla^2 h_{\mu\nu} = 0 \quad \mu \neq \nu, \quad (8.39)$$

where $\nabla^2$ is the usual (three dimensional) Laplacian. Also, to first order in small quantities, the connection coefficients $\Gamma^\mu_{\lambda\nu}$ defined by equation (6.7) are given by

$$\Gamma^\mu_{\lambda\nu} \approx \frac{1}{2} \frac{\partial h_{\mu\lambda}}{\partial x^\nu} = \frac{1}{2} \frac{\partial h_{\nu\mu}}{\partial x^\lambda}, \quad (8.40)$$

where in the last step we used the relations (8.36) and (8.37).

For this weak curvature approximation, the geodesic equation (6.11) becomes

$$\frac{d^2w^\mu}{dt^2} + \frac{1}{2} \frac{\partial h_{\lambda\nu}}{\partial x^\mu} \frac{dw^\lambda}{dt} \frac{dw^\nu}{dt} = 0. \quad (8.41)$$

The invariant parameter $\tau$ is the ‘proper time’, and is connected with the physical time $t$ through the relation

$$d\tau^2 = g_{\mu\nu} \frac{dw^\mu}{dt} \frac{dw^\nu}{dt} dt^2. \quad (8.42)$$

We will now assume that the velocity $(dx^1/dt, dx^2/dt, dx^3/dt)$ which can be calculated from the parametrization $w(\tau) = (w^1(\tau), w^2(\tau), w^3(\tau))$ using identity (7.11), has magnitude much less than the speed of light (which in our present units is one). In General Relativity, this is known as taking the non-relativistic or ‘Newtonian’ limit. With this assumption, we may replace $\tau$ by $t$ in the geodesic equation (8.41) to obtain

$$\frac{d^2 (t^\mu + j x^\mu)}{dt^2} + \frac{1}{2} \frac{\partial h_{\lambda\nu}}{\partial x^\mu} \frac{dt^\lambda}{dt} \frac{dt^\nu}{dt} = 0, \quad \mu = 1, 2, 3. \quad (8.43)$$

Equating real and imaginary parts, we get

$$\frac{d^2 t^\mu}{dt^2} = 0; \quad (8.44)$$

$$\frac{d^2 x^\mu}{dt^2} = -\frac{1}{2} \frac{\partial h_{\lambda\nu}}{\partial x^\mu} \frac{dt^\lambda}{dt} \frac{dt^\nu}{dt} \quad \mu = 1, 2, 3. \quad (8.45)$$
The reader may notice that equation (8.45) gives the actual acceleration of a point particle defined by the geodesic.

Let us define the variables \( k^1, k^2, k^3 \) by writing

\[
k^i = \frac{dt^i}{dt}, \quad i = 1, 2, 3.
\]  

(8.46)

Notice from equation (8.44) that the \( k^i \)'s are constant, and moreover, satisfy the identity 
\[
(k^1)^2 + (k^2)^2 + (k^3)^2 = 1.
\]

If we now introduce the potential function

\[
\phi = \frac{1}{2} h_{\mu\nu} k^\mu k^\nu,
\]  

(8.47)

we may rewrite (8.45) as the familiar expression

\[
\vec{a} = -\nabla \phi,
\]  

(8.48)

where \( \vec{a} \) denotes the acceleration vector.

From the definition of the potential \( \phi \), and the time independent equations (8.38) and (8.39), we deduce the identity

\[
\nabla^2 \phi = \frac{1}{2} \left[ (k^1)^2 \nabla^2 h_{11} + (k^2)^2 \nabla^2 h_{22} + (k^3)^2 \nabla^2 h_{33} \right].
\]  

(8.49)

From the Newtonian viewpoint, the right hand side of this equation represents a mass source term, which in this case is induced by the diagonal elements of the matrix \( h \), and the time component derivatives \( k^i \). Our theory therefore provides a mechanism for obtaining sources of mass from a massless field. Notice, however, that for the symmetrical case \( k^1 = k^2 = k^3 \), this source term vanishes (by an application of equation (8.38)). In this case, the potential satisfies

\[
\nabla^2 \phi = 0.
\]  

(8.50)

Combining this result with relation (8.48), we conclude that for the radially symmetric case (allowing for a singularity at the origin), the acceleration is governed by an inverse square law. Of course, this is the same conclusion obtained by (a rather involved) analysis of Einstein’s equations.
Chapter 9

U+(1, D) Gauge Theory

9.1 Introductory Remarks

Two very fundamental forces in nature that have been known for a long time are the electromagnetic force (giving rise to the attraction or repulsion between two charged particles) and the force of gravity. The first can be described by a very elegant mathematical theory known as gauge field theory. In particular, the underlying gauge group for electromagnetic interactions is the circle group

\[ U(1, \mathbb{C}) = \{ e^{i\theta} | \theta \in \mathbb{R} \} \]  

(9.1)

The electro-weak force, and the proposed interactions between quarks, are also modelled using gauge theory (see [13], [14]). For an excellent introduction into this subject, the reader is referred to [1].

In all of these cases, the underlying gauge group is relatively simple in form. Electro-weak theory is based on the complex gauge group \( SU(2) \times U(1) \), while interactions between quarks seem to satisfy the \( SU(3) \) gauge. It is a curious fact that all of these gauge groups (including the one for the electromagnetic interaction) are (or are composed of) the complex unitary groups.

Perhaps this is not too surprising if one recalls that the language of quantum mechanics is deeply rooted in the Hilbert space formalism, in which unitary operators enjoy the property of preserving the inner product.

We saw in Section 3.3 that the semi-complex group

\[ U_+(1, \mathbb{D}) = \{ e^{i\theta} | \theta \in \mathbb{R} \} \]  

(9.2)

may give rise to physical fields consistent with special relativity. We now take up this topic by first discussing \( U_+(1, \mathbb{D}) \) gauge invariance for a massless scalar field.
9.2 The Massless Scalar Field

The Lagrangian density
\[ \mathcal{L} = \frac{\partial \Phi}{\partial w^1} \cdot \frac{\partial \Phi}{\partial w^1} + \frac{\partial \Phi}{\partial w^2} \cdot \frac{\partial \Phi}{\partial w^2} + \frac{\partial \Phi}{\partial w^3} \cdot \frac{\partial \Phi}{\partial w^3} \]  
(9.3)

first discussed in Section 8.1 is invariant under the global phase transformation
\[ \Phi \rightarrow \Phi' = e^{j\theta} \cdot \Phi, \quad \theta = \text{constant}, \]  
(9.4)

and gives rise to the wave equation on \( D^3 \). We now invoke the ‘gauge principle’ by imposing the condition of local phase invariance—or invariance under the transformation
\[ \Phi \rightarrow \Phi' = e^{j\theta(w)} \cdot \Phi, \]  
(9.5)

where \( \theta = \theta(w) \) is now allowed to vary for different space-time points \( w \in D^3 \).

We first need to modify our Lagrangian density \( \mathcal{L} \) in order to satisfy the invariance demanded by (9.5). We start by introducing the derivative operators
\[ D_\mu = \frac{\partial}{\partial w^\mu} + j \Omega_\mu, \quad \mu = 1, 2, 3, \]  
(9.6)

where the gauge field \( \Omega = \{ \Omega_\mu \} \) obeys the following transformation law:
\[ \Omega_\mu \rightarrow \Omega'_\mu = \Omega_\mu - \frac{\partial \theta}{\partial w^\mu}. \]  
(9.7)

The operator now transforms as
\[ D_\mu \rightarrow D'_\mu = \frac{\partial}{\partial w^\mu} + j \Omega'_\mu. \]  
(9.8)

With this terminology, it is a straightforward exercise to show that
\[ D'_\mu \Phi' = e^{j\theta(w)} \cdot D_\mu \Phi. \]  
(9.9)

We may now redefine our Lagrangian density to be
\[ \mathcal{L} = D_1 \Phi \cdot D_1 \Phi + D_2 \Phi \cdot D_2 \Phi + D_3 \Phi \cdot D_3 \Phi, \]  
(9.10)

which is invariant under the gauge transformations \( \Phi \rightarrow \Phi' \) and \( \Omega_\mu \rightarrow \Omega'_\mu \).

The equation of motion determined by this Lagrangian density is:
\[ (\Box_s + |\Omega|^2) \Phi = -V \Phi, \]  
(9.11)

where \( \Box_s = \Box_1 + \Box_2 + \Box_3 \), \( |\Omega|^2 = |\Omega_1|^2 + |\Omega_2|^2 + |\Omega_3|^2 \), and
\[ V = j \left( \Omega_\mu \bar{\partial}_\mu + \bar{\Omega}_\mu \partial_\mu + \bar{\Omega}_\mu \Omega_\mu \right). \]  
(9.12)

We have also used the notation \( \bar{\partial}_\mu \equiv \partial_\mu \), and summed over repeated indices.

So far, we have said nothing about the likely equations of motion governing the field \( \Omega \). We address this topic next.
9.3 The $U_+(1, D)$ Field Tensor

The gauge group $U_+(1, D) = \{ e^{i\theta} | \theta \in \mathbb{R} \}$ and transformation law (9.7) for the gauge field $\Omega$ is analogous to the gauge theory formalism employed in a treatment of the electromagnetic field. We now strengthen the analogy by defining the quantity

$$ F_{\mu \nu} = j \left( \frac{\partial \Omega_{\mu}}{\partial w^\nu} - \frac{\partial \Omega_{\nu}}{\partial w^\mu} \right), \quad (9.13) $$

which is analogous to the electromagnetic field tensor. We will call $F$ the $U_+(1, D)$ field tensor.

The main reason for defining $F$ in this way is that it is invariant under the gauge transformation $\Omega_{\mu} \rightarrow \Omega'_{\mu}$ defined by equation (9.7) (easy exercise!). Notice that

$$ F_{\mu \nu} = F_{\nu \mu}, \quad (9.14) $$

which means the matrix $\{ F_{\mu \nu} \}$ is Hermitian, and so

$$ jF_{\mu \nu} dw^\mu \wedge dw'^\nu \quad (9.15) $$

is a real form.

In order to obtain the Lagrangian density for the gravitational field, we contract indices by writing

$$ \mathcal{L} = F_{\mu \nu} F^{\mu \nu}, \quad (9.16) $$

where indices are raised using the Kronecker delta $\delta^{\nu \mu}$, and lowered using $\delta_{\mu \nu}$.

The Euler-Lagrange equations for this Lagrangian yield the following field equations:

$$ \partial_{\nu} F_{\nu \mu} \equiv 0, \quad \nu = 1, 2, 3, \quad (9.17) $$

where $\partial_{\mu} \equiv \frac{\partial}{\partial w^\mu}$. In terms of the fields $\Omega_{\mu}$, this last equation becomes:

$$ \Box \Omega_{\mu} - \partial_{\mu}(\partial_{\nu} \Omega_{\nu}) = 0, \quad \mu = 1, 2, 3. \quad (9.18) $$

Since $\Omega$ is a semi-complex valued vector field, we may write

$$ \Omega_{\mu} = G_{\mu} + jH_{\mu}, \quad \mu = 1, 2, 3, \quad (9.19) $$

where $G = \{ G_\mu \}$ and $H = \{ H_\mu \}$ are real fields.

For the case where the fields are independent of time, equations (9.18) reduce to

$$ \nabla^2 G + \nabla(\nabla \cdot G) = 0 \quad (9.20) $$

$$ \nabla^2 H - \nabla(\nabla \cdot H) = 0 \quad (9.21) $$
CHAPTER 9. $U_+(1, \mathbb{D})$ GAUGE THEORY

Invoking a well known identity\footnote{If $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$}, equation (9.21) is equivalent to

$$\nabla \times (\nabla \times H) = 0,$$

(9.22)

which is analogous to Maxwell’s field equation for the vector potential in the static case. In fact, making use of the gauge freedom, we can choose a gauge in which $\nabla \cdot H$ vanishes, and so we end up with Laplace’s equation for $H$. We cannot do the same for the $G$ field, since the gauge transformation for $G$ involves only time coordinates.

Let us rewrite (9.20) in the form

$$2\nabla(\nabla \cdot G) - \nabla \times (\nabla \times G) = 0.$$  

(9.23)

Thus, the only spherically symmetric, radially directed (i.e. curless) solution to the above equation is

$$G = -\Lambda_0 \vec{r} - \frac{M_0 \vec{r}}{r^3},$$

(9.24)

where $\vec{r} = (x^1, x^2, x^3)$ (with length $r$), and $\Lambda_0, M_0$ are constants. If we view $G$ as some kind of force field, then the above solution corresponds to Hooke’s Law and Newton’s inverse square Law.

It is interesting to note that only these force laws give rise to stable, closed orbits of particles. See [3] for more details!

9.4 The Road Ahead

Perhaps the most compelling reason for wishing to study space-time and gravity within a semi-complex framework is the striking similarity between the semi-complex unitary groups (arising from a natural investigation of the symmetries of $\mathbb{D}^n$) and the complex unitary groups that underly the modern description of particle interactions.

In particular, the complex gauge group SU(3) is believed to model quark interactions, while the semi-complex gauge group SU(3, $\mathbb{D}$) preserves the inner product on the semi-complex space-time $\mathbb{D}^3$. Are these two facts related?

If any significant progress is to be made, we should at least start by investigating the gauge fields arising from the group SU(3, $\mathbb{D}$), and applying some kind of quantisation procedure. Presumably, we should expect a new kind of physics, but hopefully, not too unlike the world inhabited by quarks and gluons!
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