Hybrid Architecture for Engineering Magnonic Quantum Networks

C. C. Rusconi,1,2 M. J. A. Schuetz,3 J. Gieseler,3 M. D. Lukin,3 and O. Romero-Isart1,2

1Institute for Quantum Optics and Quantum Information of the Austrian Academy of Sciences, A-6020 Innsbruck, Austria.  
2Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria.  
3Physics Department, Harvard University, Cambridge, MA 02318, USA.

We theoretically show that a network of superconducting loops and magnetic particles can be used to implement magnonic crystals with tunable magnonic band structures. In our approach, the loops mediate interactions between the particles and allow magnetic excitations to tunnel over long distances. As a result, different arrangements of loops and particles allow one to engineer the band structure for the magnonic excitations. We demonstrate that magnons can serve as a quantum bus for long-distance magnetic coupling of spin qubits. The qubits are coupled efficiently to the magnets in the network by their local magnetic-dipole interaction and provide an integrated way to measure the state of the magnonic quantum network.

Magnons, collective excitations of magnetization in magnetically ordered materials, have recently attracted significant attention in the context of quantum information science. Strong quantum coherent coupling of magnons to a microwave resonator [1–7], optical photons [8–10], and superconducting qubits [11, 12] have been recently reported. Magnonic systems [13–15] with tailored magnonic propagation properties are also investigated as a magnon quantum bus to couple quantum emitters over long distances [16–18].

In this Letter, we propose to use a network of superconducting loops [19] to couple magnetic particles over distances larger than what can be achieved with magnetic dipole-dipole interactions in free space. A magnetic excitation in a particle can tunnel to other particles provided there is a superconducting loop between them. Accordingly, a periodic arrangement of magnetic particles and superconducting loops, as exemplified in Fig. 1.a, allows one to engineer artificial magnonic crystals. Furthermore, spin qubits can be interfaced with such magnonic crystals via dipolar coupling to the magnetic particles. For a sufficiently coherent system this configuration could be used for long-range magnetic coupling of spin qubits and to engineer effective magnon-magnon interactions. Therefore, our proposal offers an all-magnetic solid-state alternative to optical quantum emitters coupled to photonic crystals [20], with the potential to access artificial spin systems on extended lengthscales in novel parameter regimes (for example, because of the tunable effective magnon speed). By tuning the magnonic bandgap, the resulting dynamics of the qubit-qubit interactions can be designed to be predominantly conservative or dissipative [20], with the possibility to make this choice on-demand in real time, thereby giving direct access to various, very different many-body problems.

Let us first consider the elementary building block of the superconducting-loop network of magnetic particles (SLNM): two magnetic particles linked by a superconducting loop (see Fig. 1.b). A circular superconducting loop of radius \( l \) is placed at the origin of the \( zy \)-plane. The loop is modeled as an \( LC \)-oscillator, whose self-inductance \( L \) and capacitance \( C \) are of geometrical origin [21]. We consider two magnetic spherical particles of radius \( R \), with center-of-mass coordinates \( \mathbf{r}_1 = (h, 0, -l - d) \) and \( \mathbf{r}_2 = (h, 0, l + d) \), respectively. A constant magnetic field \( \mathbf{B}_0 \equiv -B_0 \mathbf{e}_z \) is applied to polarize the magnets. The magnetic field generated by such polarized magnets is given by the field generated by a point dipole located at the center of the sphere and with magnetic dipole moment \( \hat{\mu}_j \) \( (j = 1, 2) \). The magnetic particles are modeled within the macrospin approximation [22] as magnetic point dipoles of constant magnitude. The coherent dynamics of the system is modeled by the following quantum mechanical Hamiltonian (see FIG. 1: a) Schematic illustration of a general superconducting-loop network of magnetic particles. b) Scheme of the elementary cell of a SLNM: two magnetic particles positioned at a distance \( d \) and height \( h \) from two opposite points of a superconducting ring of radius \( l \).
supplemental material [21] for its derivation)

\[
\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2L} \left[ \hat{\Phi} - \sum_j \Phi_j(\hat{\mu}_j) \right]^2 - \sum_j \mathbf{B}_0 \cdot \hat{\mu}_j + \frac{1}{2L} \sum_{j=1}^{N} \Phi_j(\hat{\mu}_j)^2 + U_d(\{\hat{\mu}_j\}) + \sum_j V_j^{\prime}(\hat{\mu}_j).
\]  

(1)

Here, \( \hat{Q} \) (\( \hat{\Phi} \)) is the charge (flux) operator of the loop [23], and \( \Phi_j(\hat{\mu}_j) \) is the external flux induced in the coil by fluctuations of the magnetic dipole moment \( \Delta \hat{\mu}_j = \hat{\mu}_j - \langle \hat{\mu}_j \rangle_0 \) from its equilibrium value \( \langle \hat{\mu}_j \rangle_0 \). The first two terms in Eq. (1) represent the energy of the loop in the presence of the magnets. The third term refers to the Larmor precession of the magnetic moment of the particle. The second term in Eq. (2) describes magnon tunneling to nearest-neighbors in a magnetic array. The fourth term \( \sum_j \Phi_j(\hat{\mu}_j)^2/2L \) represents the total loop-mediated magnetic interaction between the magnets [24]. \( U_d(\{\hat{\mu}_j\}) \) represents the free space magnetic dipole-dipole interaction of the two magnets. The last term in Eq. (1) is the magnetic anisotropy energy of each particle which represents the energy cost of magnetizing the particle along a certain direction due to the interaction between its magnetic moment and its crystal structure [26].

For a sufficiently large \( B_0 \), \( \Delta \hat{\mu}_j \) can be expressed within the Holstein-Primakoff approximation as \( \Delta \hat{\mu}_j = h \gamma_0 \hat{f}_j \hat{\sigma}_j, \Delta \hat{\sigma}_j = h \gamma_0 \sqrt{2F}(\hat{f}_j^\dagger + \hat{f}_j)/2, \) and \( \Delta \hat{\sigma}_j^2 = h \gamma_0 \sqrt{2F}(\hat{f}_j^\dagger - \hat{f}_j)/(2L) \). Here, \( \gamma_0 \) is the gyromagnetic ratio of the magnet and \( F \) is the total spin defined by \( \langle \hat{\mu}_j \rangle_0 = -h \gamma_0 F \mathbf{e}_z \). Hereafter, we assume the magnets to be identical, namely with same radius \( R \), gyromagnetic ratio \( \gamma_0 \), and same factor \( F \). The operator \( \hat{f}_j(\hat{\sigma}_j) \) creates (annihilates) an excitation in the uniformly precessing magnonic mode of the \( j \)-th magnet, and satisfies \( [\hat{f}_j, \hat{\sigma}_j] = 0 \). Within the Holstein-Primakoff approximation, \( \{\hat{f}_j, \hat{\sigma}_j\} \ll 2F \), and the assumption that the LC-circuit is far detuned from the magnonic modes (such that the degrees of freedom of the circuit can be traced out), the coherent dynamics of the magnets reduce to [21]

\[
\hat{H}_M = \hbar \sum_j \omega_j \hat{f}_j^\dagger \hat{f}_j + \hbar \sum_{j \neq i} (\mathcal{J}_{ij} + \mathcal{J}_{ij}^*) \hat{f}_i^\dagger \hat{f}_j.
\]  

(2)

Counter-rotating terms (of the form \( \hat{f}_i^\dagger \hat{f}_j + \hat{f}_i \hat{f}_j^\dagger \)) have been neglected within the rotating-wave approximation. Here, \( \omega_j \) is the sum of the frequency associated to the magnetic anisotropy and the Larmor precession frequency caused by the external field, the field created by other magnets and the field created by the superconducting loops. The first term in Eq. (2) describes magnon tunneling between magnets. The total tunneling rate has two contributions. The contribution from the free space magnetic dipole-dipole interaction is given by \( \mathcal{J}_{ij}^d = -\hbar \gamma_0 \mu_0 F(3 \sin^2 \theta_{ij} - 2)/(8 \pi \gamma_0^3) \) where \( r_{ij} = |r_i - r_j| \) and \( \theta_{ij} \) is the angle between \( r_i - r_j \) and \( \mathbf{e}_z \). In the case shown in Fig. 1.b, one has \( r_{12} = 2(l + d) \) and \( \theta_{12} = 0 \). The contribution from the loop-mediated magnetic interaction is given by [21]

\[
\mathcal{J}_{ij} = \left( \frac{\hbar \gamma_0 \mu_0}{4 \pi d} \right)^2 \frac{I_{ij}}{2\hbar LF}.
\]  

(3)

Here, \( I_{ij} \) is a dimensionless factor which depends on the loop geometry, and the aspect ratios \( h/d \) and \( l/l \) [21]. The geometrical inductance of a circular coil is approximated as \( L \approx \mu_0 l \ln(8l/\tau) \) for \( l < \tau \), where \( \tau \) is the wire thickness. In the case of Fig. 1.b and for \( l \gg \tau \), one finds that \( \mathcal{J}_{12}/\mathcal{J}_{12}^d \approx (l/d)^2 21I_{12}^2/[\pi \ln(8l/\tau)] \gg 1 \) since \( I_{12} \) can be of the order of one by a careful positioning of the magnets, see [21]. We stress that \( \mathcal{J}_{ij} \) scales as \( 1/(d^2l) \), whereas the factor \( 1/d^2 \) arises from the \( 1/d \)-dependence of \( \Phi_j(\hat{\mu}_j) \) and the factor \( 1/l \) arises from the linear dependence of \( L \) on the loop radius. For fixed \( d \ll l \), the loop-mediated interaction thus leads to a magnon tunneling rate which scales as \( \sim 1/r_{12} \). As an example consider the configuration shown in Fig. 1.b with \( R = 1 \mu m, d = 1.5 \mu m, l = 30 \mu m, h = 0, \) and \( \tau = 50 \mu m, \) which leads to \( I_{12} \approx 1.9 \) [27]. The tunneling rate due to the inductive magnetic interaction is then \( \mathcal{J}_{12}/2\pi \approx 5.85 \) MHz whereas the one due to the magnetic dipole interaction is \( \mathcal{J}_{12}^{d}/2\pi \approx 0.09 \) MHz. We remark that larger tunneling rates could be obtained by inscribing the magnets in the contour defined by the loop (see [21] for an example). However, this configuration will not be considered further since it is not well suited for building large networks.

Let us now focus on how to build networks by periodic arrangements of the simple configuration shown in Fig. 1.b. In the following, we neglect magnetic dipole-dipole coupling \( (\mathcal{J}_{ij}^d = 0) \) and the flux generated in a coil by next-to-nearest neighbor magnets and neighboring superconducting coils. Furthermore, within the assumption of identical loops, magnetic particles, and relative positioning of particles and loops, the magnon frequency (tunneling rate) is site-independent, namely \( \omega_j \equiv \omega \forall j \) (\( \mathcal{J}_{ij} \equiv \mathcal{J} \forall i, j \)). Let us show three different examples of SLNMs. (i) A one dimensional SLNM, shown in Fig. 2.a, can be described by \( \hat{H}_M^d = \hbar \omega_0 \sum_j \hat{f}_j^\dagger \hat{f}_j + \hbar \mathcal{J} \sum_j (\hat{f}_j^\dagger \hat{f}_{j+1} + \hat{f}_{j+1}^\dagger \hat{f}_j) \). This textbook Hamiltonian describes magnon tunneling to nearest-neighbors in a one dimensional crystal with \( N \) lattice sites separated by a distance \( a = 2(d + l) \). Assuming periodic boundary conditions, \( \hat{H}_M^d \) can be diagonalized in the reciprocal space leading to a magnon dispersion relation \( \omega(k) = \omega_0 + 2\mathcal{J} \cos(ka) \), where \( k = 2\pi n/(Na) \) \((n \in [N/2, N/2 - 1])\). In the continuum limit \( (N \gg 1) \), the magnon propagation is thus restricted to the frequency band \( \omega \in [\omega_0 - 2\mathcal{J}, \omega_0 + 2\mathcal{J}] \). Note that a feature of the proposed SLNMs in this Letter is that their magnonic dispersion relation can be tuned in real time by simply modifying the external magnetic field \( B_0 \). (ii) A SLNM where
N magnets couple to each other with the same strength can be realized as shown in Fig. 2.b. The Hamiltonian is given by \( H^{\text{2D}}_{\text{M}} = \hbar \omega_0 \sum_{i=1}^{N} \hat{f}^\dagger_i \hat{f}_i + \hbar J \sum_{N \neq i, j = 1}^N \hat{f}^\dagger_i \hat{f}_j \), with an all-to-all interaction \( \sim J \) [cf. Eq. (3)]; here, the geometrical factor \( I_{ij} \) is different from before due to the fact that the magnets are now polarized perpendicularly to the superconducting coils. In principle, this Hamiltonian could be used to generate magnonic superradiance by enhancing dissipation in the coil and allowing the system to evolve beyond the quadratic approximation [28].

A two-dimensional SLNM can be realized as in Fig. 2.c. Owing to the checkerboard arrangements of superconducting loops, we distinguish two magnonic sublattices: magnons in the \( D \) (\( A \)) sublattice preferably tunnel along the direction of the main diagonal (anti-diagonal) in the \( yz \)-plane. This SLNM can thus be described as a 2D Bravais lattice with a basis, where each elementary cell contains the two types of sites \( D \) and \( A \) (see Fig. 2.c).

The operators \( \hat{f}^A_j, \hat{f}^A_j \dagger, \hat{f}^D_j, \hat{f}^D_j \dagger \) respectively create and annihilate a magnon in the sublattice \( A \) (\( D \)) within the cell at position \( j = j_y v_1 + j_z v_2 \) \( (j_y, j_z) \), \( (j_y, j_z) \in \mathbb{Z} \), where \( v_1 = (2a, 0) \) and \( v_2 = (a, a) \) are Bravais vectors. The Hamiltonian of this 2D SLNM is given by [21]

\[
\hat{H}^{\text{2D}}_{\text{M}} = \hbar J \left[ \sum_{j, \beta} \hat{f}^D_{j+\beta} \hat{f}^A_{j+\beta} + \sum_{j, \alpha} \hat{f}^A_{j+\alpha} \hat{f}^A_{j+\alpha} \right] + \sum_{j, \delta} \hat{f}^D_{j+\delta} \hat{f}^D_{j+\delta} + \text{H.c.}.
\]

Here, \( \beta \in \{ (\pm 1, 0), (0, \pm 1) \} \), \( \alpha \in \{ (1, -1), (1, 1) \} \), and \( \delta \in \{ (1, 1), (1, -1) \} \), with \( \alpha \) (\( \delta \)) and \( \beta \) connecting the nearest neighbors of a point along the main anti-diagonal (diagonal) and along the \( z, y \) direction (Fig. 2.c).

The magnon dispersion relation of Eq. (4) leads to two bands given by \( \omega_{\pm}(k) = 2J/4 \cos(k_x a) \cos(k_y a) \pm \sqrt{\Delta} \), where \( \Delta \equiv 4 + 4 \cos(k_x a) \cos(k_y a) - \cos(2k_x a) - \cos(2k_y a) + 2 \cos(2k_x a) \cos(2k_y a) \), with \( a = \sqrt{2(l + d)} \). As shown in Fig. 2.d, the upper band \( \omega_{+}(k) \) features saddle points at \( k = (\pm \pi/2a, \pm \pi/2a) \) where the density of state diverges [29]. As recently shown in [30], this type of exceptional points can give rise to very exotic features in the quantum dynamics of emitters coupled to a two dimensional crystal.

After having presented SLNMs as artificial long-range magnonic crystals, let us now address the possibility to magnetically couple the magnons in a given SLNM to spin qubits. Spin qubits could be used, for instance, to produce or read out excitations in a SLNM. This coupling can be achieved by local magnetic dipole-dipole interaction between a magnet and a spin qubit. In particular, consider the \( j \)-th magnet and an NV-center spin qubit, obtained from the \{0, −1\} subspace of the NV ground state triple, placed at a position \( r_a \) with respect to the center of the magnet. The Hamiltonian of this system is given by \( \hat{H}^{(\text{NV})}_{\text{M}} = \hbar \omega_0 \hat{\sigma}^z_j /2 - Jg_a \hat{\sigma}^z_j \cdot \mathbf{B}(r_a, \hat{\mu}_0) /2 \), where \( \omega_0 \equiv \Delta_{\text{NV}} - \gamma_a B_0 \), \( \gamma_a \) is the qubit gyromagnetic ratio (generally different from \( \gamma_0 \)), \( \Delta_{\text{NV}} \) the NV-center zero field splitting, and \( \mathbf{B}(r_a, \hat{\mu}_0) \) the magnetic field field generated by the magnet at the position of the qubit.

Within the Holstein-Primakoff approximation, the rotating wave approximation, and assuming the qubit to be positioned along the x-axis of a reference frame centered in the magnet and oriented as in Fig. 1.b (see [21] for the generalization to any other position), the qubit-magnon dynamics is described by the Jaynes-Cumming Hamiltonian \( \hat{H}^{(\text{NV})}_{\text{M}} \approx \hbar \omega_0 \hat{\sigma}^x_j /2 - \hbar g_a \hat{\sigma}^z_j \hat{f}^\dagger_j \hat{f}_j + \text{H.c.} \), where \( g_a \equiv 3\hbar \gamma_a \mu_0 \sqrt{2F}/(8\pi a^3) \). Here, the qubit frequency \( \omega_0 \) contains \( \omega_q = \omega_0 + \hbar \gamma_a \mu_0 A F/(4\pi a^3) \) already contains the shift introduced by the dipole-interaction. The dynamics of a general 2D SLNM with magnetically coupled spin qubits at each lattice site is described by the Jaynes-Cumming-Hubbard Hamiltonian \( \hat{H}_T = \hat{H}_{\text{M}} + \sum_
u \hat{H}^{(\text{NV})}_{\nu,\text{MQ}} \) namely in k-space

\[
\hat{H}_T = \hbar \sum_{\nu, k} \omega_\nu(k) \hat{f}^\dagger_{\nu k} \hat{f}_{\nu k} + \hbar \sum_j \frac{\omega_\nu}{2} \hat{\sigma}^z_j - \hbar \sum_{\nu, k} \left( g_{\nu k} \hat{f}^\dagger_{\nu k} \hat{\sigma}^x_j + \text{H.c.} \right). \tag{5}
\]

Here, we introduced the k-space magnonic operator \( \hat{f}_{\nu k} = (1/N) \sum_{j} \hat{f}^\dagger_{j \nu} \exp(-i\nu j \cdot k) \) which creates a magnon of momentum \( k \) in the \( \nu \)-magnonic band propagating in a \( N \times N \) 2D lattice characterized by the dispersion relation \( \omega_\nu(k) \), and the coupling rate \( g_{\nu k} \equiv (g_{\nu}/N) \exp(-i\nu j \cdot k) \), where \( g_{\nu} \) is the local coupling to a magnon in the \( \nu \)-band, \( a \) is the SLNM lattice constant and \( j \) labels the sites in a 2D SLNM [21]. In Eq. (5), we neglected the small interaction between the qubit and the loop as well as counter-rotating terms of the form \( \hat{\sigma}^x_j \hat{f}^\dagger_{\nu k} + \hat{\sigma}^x_j \hat{f}_{\nu k} \).
within the rotating wave approximation, valid provided $g, |\omega_\nu(k) - \omega_\sigma| \ll \omega_\nu(k)$.

The Hamiltonian Eq. (5) can lead to strongly correlated, coherent magnon physics [31, 32], provided that the relevant decoherence rates are sufficiently small compared to the coherent coupling rates in the system. While the coherent magnon tunneling $J$ can reach several MHz (as discussed above), the coherent magnon-qubit coupling can be quantified as $g/(2\pi) = 5.2 \times 10^5 (R [\text{nm}]^{1/2}/r_q [\text{nm}])^{3} \text{MHz}$, as a function of both the magnet size $R$ and magnet-qubit distance $r_q > R$; see caption of Fig. 3 for the remaining parameters. The main sources of decoherence arise from qubit dephasing and magnon decay, as any potential damping in the superconducting loop is suppressed by its large detuning. For a NV-center spin qubit, characteristic dephasing times $T_2^* \approx 200 \mu s$ have been reported [33], which can further be increased by dynamical decoupling schemes up to $T_2 \approx 0.5 \text{s}$ [34]. In the low-temperature regime $~\sim 1 \text{K}$ the magnon linewidth [35] for a pure millimeter-size single-crystal YIG sphere has been measured as $\kappa/2\pi \approx 0.5 \text{ MHz}$ [2], at a relatively high magnon frequency of $~10 \text{GHz}$; this number could potentially be further reduced by working at lower frequencies according to the linear frequency dependence of the Gilbert damping rate in YIG [36]. Accordingly, the regime $J > \pi/T_2^*, \kappa$ is within reach for particles of size $R \approx 1 \mu m$ (see above) with current experimental capabilities, while the regime $g > \kappa$ is found to be challenging with the current reported values of the magnon linewidth. However, the detrimental effects due to magnon decay can be reduced efficiently by operating in the dispersive regime, as detailed next.

To this end, let us consider two identical spin qubits coupled to the elementary configuration described in Fig. 3.a, and thus separated by a distance $2(d + l)$.

The system is described by the Hamiltonian $H_T = \sum_{j=1}^{2} (J_1 \hat{\sigma}_j^+ \hat{\sigma}_j^-) + J \sum_{j \neq j} \hat{\sigma}_j^+ \hat{\sigma}_j^-$ in the dispersive regime, when the qubits are detuned from the magnonic eigenmodes of the system, it is possible to adiabatically eliminate the magnonic degrees of freedom. The qubits dynamics are thus described by the effective spin-spin interaction Hamiltonian $H_{QQ} = \hbar \left[ \omega_\sigma - g^2/(2\Delta) - g^2/(\Delta - 2J) \right] \left( \hat{\sigma}_1^+ \hat{\sigma}_2^- + \hat{\sigma}_2^+ \hat{\sigma}_1^- \right)$, where $\Delta \equiv \omega_0 + J - \omega_\sigma$, and the effective spin-spin coupling strength reads [21] $g_{\text{eff}} = g^2[1/\Delta - 1/(\Delta - 2J)]$. The level structure and typical values of frequencies and couplings are shown in Fig. 3.b-c. $H_{QQ}$ can be used to swap excitations between the two qubits at a rate $\pi/g_{\text{eff}}$ whenever $g_{\text{eff}} \gg \gamma, \kappa_{\text{eff}}$, where $\gamma \equiv \pi/T^*_2$ and $\kappa_{\text{eff}} = \kappa g^2[1/\Delta^2 + 1/(\Delta - 2J)^2]$ is the qubit damping induced by the lossy magnonic bus [21]. In this strong coupling regime, the error $\varepsilon$ on the state transfer fidelity for optimized values of the detuning $\Delta$ and magnon-tunneling $J$ is given by $\varepsilon \approx \sqrt{\alpha_\sigma \alpha_\nu / (2C)}$ with cooperativity $C \equiv g^2/(\gamma \kappa)$ where we numerically estimate $\alpha_\nu \approx 0.779$ and $\alpha_\sigma \approx 0.006$ as detailed in [21]. In Fig. 3.d, values of $C$ are shown as function of magnon damping $\kappa$ and qubit dephasing times $T_2$ for a fixed values of the remaining parameters of the set-up. As qubit dephasing time $T_2 \approx 0.5 \text{s}$ are achievable by adopting dynamical decoupling schemes [34], the main limitation in state of the art experiments would be set by the magnon damping rate currently achievable [2].

In conclusion, we have shown that superconducting-loop networks of magnets can be used to implement artificial magnonic crystals with engineered band structures. This could be relevant for the field of artificial spin systems [37, 38], as an alternative platform to study magnetic crystallization and dynamics of a low density ensemble of nanomagnets embedded in a non-magnetic matrix. Along this line it would be interesting to use superconducting wire network [39, 40], instead than a lattice of loops, to study how the interplay between connectivity and superconductivity affect the dynamics of magnetic particles in the network. Furthermore, magnetic coupling of spin qubits to the magnets in the network allows to do magnetometry of an SLNM, namely to probe the state of the network as well as to use magnons as a quantum bus to magnetically couple spin qubits over long distances [18], analogously to what is done with quantum emitters coupled to photonic crystals [20], albeit in a different parameter regime. The potential of our proposal depends very much on the linewidth of magnons in a magnetic sphere. While the microscopic origin of such damping is still not completely understood, interesting strategies to possibly reduce the damping can be envi-
sioned. Smaller magnetic particles might show a lower damping at $T \gtrsim 1K$ due to the discretization of phononic modes in the sample. A levitated version of our proposal [41, 42] might allow to study the impact of the conservation of total angular momentum on the (dissipative) dynamics of the magnetization. Finally, we remark that the present discussion could be generalized beyond the macrospin approximation to include other magnonic modes inside the magnetic particles which might result in an improvement on the magnon linewidth [43].

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CCR and MJAS contributed equally to this work.

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Hybrid Architecture for Engineering Magnonic Quantum Networks
Supplemental Material

C. C. Rusconi,1,2 M. J. A. Schuetz,3 J. Gieseler,3 M. D. Lukin,3 and O. Romero-Isart1,2
1Institute for Quantum Optics and Quantum Information of the Austrian Academy of Sciences, A-6020 Innsbruck, Austria.
2Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria.
3Physics Department, Harvard University, Cambridge, MA 02318, USA.

I. DETAILS ON THE SUPERCONDUCTING LOOP

In the following, we present some details regarding the description of the superconducting loop.

A. Magnetic flux through a coil

We consider the inductive coupling between a magnet with magnetic moment \( \hat{\mu} = h\gamma_0 \hat{F} \) and a coil of arbitrary shape. We assume the magnet to be placed at a distance \( h \) above the coil and at an horizontal distance \( d \) from the coil’s closest wire (see Fig. 1.b in the main text). The flux induced in the coil by the magnet reads

\[
\Phi(\hat{F}) = \oint \mathbf{dl} \cdot \mathbf{A}(\mathbf{r}, \hat{F})
\]

(S1)

where \( \mathbf{A}(\mathbf{r}, \hat{F}) \) is the magnetic vector potential generated by the magnet. Eq. (S1) can be written as

\[
\Phi(\hat{F}) = h\gamma_0 \mu_0 \int_{\Delta r} \frac{A_{\nu}}{(4\pi d)} d\mathbf{r}
\]

(S2)

is a dimensionless factor which depends only on the shape of the coil and on the mutual position of the magnet and the coil. Here \( \Delta r \) is the distance between the magnet and a point in the coil. For instance, for a circular ring of radius \( l \) centered at \((0,0,l+d)\) and for a nanomagnet at a position \((h,0)\), the factors \( I_\nu \) in Eq. (S2) read

\[
I_z = \int_{-l/d}^{l/d} d\lambda F\left(\lambda, l/d, h/d\right),
\]

(S3)

and \( I_y = 0 \), where

\[
F(\lambda, x, y) = \frac{y\lambda/\sqrt{x^2 - \lambda^2}}{[y^2 + x^2 + (x + 1) + 2\lambda(x + 1)]^{3/2}},
\]

G(\(\lambda, x, y\)) \(=\frac{\sqrt{x^2 - \lambda^2} + \frac{x\lambda}{\lambda + 1}}{[y^2 + x^2 + (x + 1) + 2\lambda(x + 1)]^{3/2}}\).

(S4)

As shown in Fig. S1.a, the integrals in Eq. (S3) have an optimal value around unity as function of \( h/d \) in the limit of a large loop radius, \( l/d \gg 1 \).

B. Superconducting Loop as LC oscillator

Superconducting rings on top of a dielectric substrate (see Fig. S1.b) have been shown to behave as microwave multimode resonators [S1, S2] characterized by a large quality factor \( Q \approx 10^6 \) at GHz frequencies [S3, S4]. The spectrum of the resonator is double degenerate, each frequency corresponding to both a clockwise and counter-clockwise traveling wave. Within a transmission line model the mode frequencies can be approximated by \( \omega_n / 2\pi \equiv n / (2\pi \sqrt{L / C}) \) for \( n \in \mathbb{N} \), where \( L \) (\( C \)) is the inductance (capacitance) per unit length of the loop and \( l \) the loop radius.

Adjusting the external magnetic field \( B_0 \) such as to tune the Larmor precession frequency of the magnetic particle’s macroparticle close to fundamental resonance of the ring resonator, it is possible to neglect the coupling between \( \hat{F} \) and the higher resonant modes. Moreover the degeneracy of the fundamental mode can be broken by introducing small asymmetries or imperfections as done for instance in [S3, S4]. The ring thus behaves as a single mode LC-resonator of frequency \( \omega_c \equiv 1 / \sqrt{LC} \), where \( L \) (\( C \)) is the total inductance (capacitance) of the ring. \( C \) is the capacitance between the loop and the ground plate at the opposite end of the dielectric substrate, and can be arbitrarily reduced by careful design. \( L \) amounts to the geometrical self-inductance of the loop, which depends on the particular shape of the loop and on the thickness \( \tau \) of the wires as detailed in [S5]. For the case of a circular loop of radius \( l \) and wires of circular section, the self-inductance reads

\[
L = \mu_0 l \left[ \ln \left( \frac{8l}{\tau} \right) - \frac{7}{4} + O \left( \frac{\tau^2}{l^2} \right) \right].
\]

Here we are assuming for simplicity the electric permittivity (magnetic permeability) of the substrate supporting the loop, Fig. S1.b, to be \( \varepsilon_r \approx 1 \) (\( \mu_r \approx 1 \)).

C. Magnetic Field intensity at the wires of the Loop

The distance \( d \) at which a magnetic particle can be placed from a superconducting loop is limited by the requirement that the magnetic field produced by the particle at the position of the loop’s wire must be smaller than the critical field (first critical field) \( B_c \) of the type I (type II) superconductor constituting the loop. Consider the situation illustrated in Fig. S1.b, the distance at which
the center of the magnetic particle should be placed such that the $e_x$-component of the magnetic field at the closest point of the loop equals $B_c$ reads
\[
d_c = \frac{\tau}{2} + \left(\frac{2\mu_0 M_s}{3B_c}\right)^{1/3} R, \tag{S6}
\]
where $M_s$ is the saturation magnetization of the magnetic particle, $R$ the particle radius, and $\tau$ the wire thickness. In Fig. S1.c, $(d-\tau/2)/R$ is plotted as function of the field $B_c$ at the wire position. For the values used in Fig. 3.c, the field produced by the magnetic particle at the position of the loop wire is $\approx 110\text{mT}$, which is below the first critical field of many type II superconductor such as Nb [S6].

II. DERIVATION OF THE SYSTEM HAMILTONIAN

In the following, we consider $N$ magnetic particles with a magnetic moment $\mathbf{\mu}_j$ ($j = 1, \ldots, N$) of constant magnitude $\mu \equiv |\mathbf{\mu}_j|$ fixed in space at positions $\mathbf{r}_j$ in the vicinity of a $LC$-ring resonator. An external bias field $\mathbf{B}_0 = -B_0 \mathbf{e}_z$ is applied parallel to the plane containing the $LC$ circuit (see Fig. 1.b in the main text). Here, we derive the quantum mechanical Hamiltonian Eq. (1) describing the dynamics of the coupled system composed by the circuit and the magnetic moments.

A. Classical Equations of Motion of the System

Within the single mode approximation, a superconducting $LC$-ring resonator can be modeled as an $LC$-circuit (see Sec. 1.B). The equations of motion for the $LC$-circuit can be derived from Kirchhoff’s current and voltage laws, together with the constitutive relations which relate current and voltage at each element of the circuit. Defining $V_C = \partial_t \Phi_C$ ($V_L \equiv \partial_t \Phi_L$) the flux at capacitor (inductor) of the circuit, we write the constitutive relations for the capacitor as $\Phi_C = I_C/C$ and for the inductor as
\[
\Phi_L = LI_L + \sum_{j=1}^N \Phi_j(\mathbf{\mu}_j). \tag{S7}
\]
Here, $C$ ($L$) are the circuit capacitance (inductance), and $\Phi_j(\mathbf{\mu}_j)$ is the flux induced in the ring by the $j$-th magnetic [see Eq. (S1)]. The equation of motion for the circuit can be derived from Kirchhoff’s law as
\[
C\ddot{\Phi} + \frac{\Phi}{L} = \sum_{j=1}^N \frac{\Phi_j(\mathbf{\mu}_j)}{L}, \tag{S8}
\]
where $\Phi \equiv \Phi_L = -\Phi_C$ [S7].

The coherent dynamics of the magnetic moment $\mathbf{\mu}_j \equiv \mu(\cos \varphi_j \sin \theta_j, \sin \varphi_j \sin \theta_j, \cos \theta_j)$, for $\theta_j \in [0, \pi]$ and $\varphi_j \in [0, 2\pi]$, is described by the Landau-Lifshitz equation
\[
\dot{\mathbf{\mu}}_j = -\gamma_0 \mathbf{\mu}_j \times \mathbf{B}(\mathbf{r}_j), \quad \text{where} \quad \mathbf{B}(\mathbf{r}_j) \quad \text{is the total magnetic field acting on the} \ j\ \text{-th magnetic moment}. \quad \text{In term of the polar} \ \varphi_j \ \text{and azimuthal} \ \theta_j \ \text{angles the Landau-Lifshitz equations read} \ [S8]
\]
\[
\begin{align*}
\dot{\varphi}_j &= -\frac{\gamma_0}{\mu \sin \theta_j} \partial_{\varphi_j} U, \\
\dot{\theta}_j &= \frac{\gamma_0}{\mu \sin \theta_j} \partial_{\theta_j} U,
\end{align*} \tag{S9}
\]
where $U \equiv \sum_{j=1}^N V_d(\mathbf{\mu}_j) + U_0 + U_{\text{dip}} + U_{\text{ind}}$ is the magnetic interaction energy of the dipoles. $V_d(\mathbf{\mu}_j)$ represents the magnetic anisotropy energy of the $j$-th magnetic particle. $U_0 = -\sum_{j=1}^N \mathbf{\mu}_j \cdot \mathbf{B}_0$ represents the interaction energy of the dipoles in the external bias field. $U_{\text{dip}}$ represents the free-space dipole-dipole interaction energy between the magnetic moments
\[
U_{\text{dip}}(\{\mathbf{\mu}_j\}) = -\frac{1}{2} \sum_{j=1}^N \sum_{j\neq 1}^N \mathbf{\mu}_j \cdot \mathbf{B}_{\text{dip}}^j(\mathbf{r}_j). \tag{S10}
\]
Here, the dipole field created by the dipole moment $\mathbf{\mu}_i$ at position $\mathbf{r}$ reads
\[
\mathbf{B}_{\text{dip}}^i(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3\mathbf{\Delta} \cdot \mathbf{\Delta}}{|\mathbf{\Delta}|^5} = \frac{\mu_0}{4\pi} \frac{3\mathbf{\Delta} \cdot \mathbf{\Delta}}{|\mathbf{\Delta}|^5} - \frac{\mu_i}{|\mathbf{\Delta}|^3}, \tag{S11}
\]
with \( \Delta \mathbf{r}_i \equiv \mathbf{r}_i - \mathbf{r}_i \). \( U_{\text{ind}} = I_L \sum_j \Phi_j(\mathbf{\mu}_j) \), where \( I_L \) is given by Eq. (S7), represents the interaction between the nanomagnets and the field produced by the current flowing in the ring in terms of the induced magnetic flux [S9].

**B. Hamiltonian description and quantization**

The equations of motion Eq. (S8) and Eq. (S9) can be derived from the Lagrangian

\[
\mathcal{L} = \frac{C}{2} \mathbf{\dot{q}}^2 - \frac{\mu}{\gamma_0} \sum_{j=1}^{N} \dot{\varphi}_j \cos \theta_j - \frac{1}{2L} \left[ \Phi - \sum_{j=1}^{N} \Phi_j(\varphi_j, \theta_j) \right]^2 - \frac{1}{2L} \sum_{j=1}^{N} \mu B_0 \cos \theta_j - U_{\text{dip}} \\
- \sum_{j} V_d(\varphi_j, \theta_j).
\]  
(S12)

From Eq. (S12), the classical Hamiltonian of the system is obtained introducing the momentum \( Q \equiv C \mathbf{\dot{q}} (p_j \equiv \mu \cos \theta_j/\gamma_0) \) conjugated to \( \Phi (\varphi_j) \). Following the usual canonical quantization procedure one can then write the quantum mechanical Hamiltonian of the system as

\[
\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2L} \left[ \hat{\Phi} - \sum_{j=1}^{N} \Phi_j(\hat{\mu}_j) \right]^2 + \sum_{j=1}^{N} B_0 \hat{\mu}_j^z \\
+ \sum_{a} \hat{V}_a(\hat{\mu}_j) + \frac{1}{2L} \left[ \sum_{j=1}^{N} \Phi_j(\hat{\mu}_j) \right]^2 + U_{\text{dip}}(\{\hat{\mu}_j\}).
\]  
(S13)

Here, the charge \( \hat{Q} \) and flux \( \hat{\Phi} \) operators of the circuit satisfy canonical commutation relations \( [\hat{\Phi}, \hat{Q}] = i\hbar \). The components of the magnetic moment \( \mathbf{\mu}_j \equiv \mu(\sin \theta_j \cos \varphi_j, \sin \theta_j \sin \varphi_j, \cos \theta_j)^T \), commute as \([\hat{\mu}_j^x, \hat{\mu}_j^y] = i\mu_0 \epsilon_{\nu \eta \xi} \hat{\mu}_j^\xi \), for \( \nu, \eta, \xi = x, y, z \), according to the canonical quantization of the classical Poisson bracket

\[
\{f, g\} = -\sum_{j=1}^{N} \frac{1}{\mu \sin \theta_j} \left( \frac{\partial f}{\partial \varphi_j} \frac{\partial g}{\partial \theta_j} - \frac{\partial f}{\partial \theta_j} \frac{\partial g}{\partial \varphi_j} \right),
\]  
(S14)

for any \( f, g \) function of \( \theta_j, \varphi_j \).

The last two terms in Eq. (S13) represent the total magnetic interaction between the magnetic dipoles, which is decomposed into the circuit mediated interaction \( \sum_{j=1}^{N} \Phi_j(\hat{\mu}_j)^2/(2L) \) and the free space dipole-dipole interaction \( U_a(\{\hat{\mu}_j\}) \). An additional direct dipole-dipole interaction is present in the second term of Eq. (S13). This is a fictitious term which appears in the Hamiltonian as a consequence of the formulation of the coupling \( U_{\text{ind}} \) in terms of the flux variables \( \Phi \) and \( \Phi_j(\hat{\mu}_j) \) [S10]. In the regime where the circuit degrees of freedom can be adiabatically eliminated, this fictitious term is exactly canceled by the circuit-dipole interaction contained in the second terms of Eq. (S13) (see below). A different description in terms of the coupling between the dipoles and the magnetic field produced by the resonator would have contained only the interaction between the dipoles and the resonator mode and the free space magnetic dipole coupling \( U_a(\{\hat{\mu}_j\}) \) [S11, S12]. The adiabatic elimination of the LC-resonator, in the limit when it is far detuned from the dipole precession frequency, would then yield a circuit-mediated dipole-dipole interaction equivalent to the coupling \( \sum_{j=1}^{N} \Phi_j(\hat{\mu}_j)^2/(2L) \).

**III. MAGNON DYNAMICS IN A SLNM: HAMILTONIAN DERIVATION AND EXAMPLES**

Here, we start from Eq. (S13) to derive the quadratic Hamiltonian in Eq. (2) which describes the propagation of magnonic excitation in a SLNM. Starting from Eq. (S13), we define

\[
\hat{\Phi}_j(\hat{\mu}_j) \equiv \Phi_j^{\text{bias}} + \Phi_c \sum_{\nu=x,y,z} I_{\nu}^z \Delta \hat{F}_{\nu}^z.
\]  
(S15)

The first term \( \Phi_j^{\text{bias}} \) represents the flux induced in the loop by the equilibrium state \( \langle \hat{\mu}_j \rangle_0 \) of the \( j \)-th magnet. The second term on the right hand side of Eq. (S15) represents the flux in the circuit caused by a perturbation of the magnetic moment around this equilibrium state, \( \Delta \hat{\mu}_j \equiv \hat{\mu}_j - \langle \hat{\mu}_j \rangle_0 \). Here, \( \Phi_c \equiv \hbar \gamma_0 \mu_0/4\pi d \), \( d \) is the distance between the magnet and the closest loop wire in the plane containing the loop, and we expressed the magnetic moment in terms of the system macrospin according to the gyromagnetic relation \( \hat{\mu}_j = \hbar \gamma_0 \hat{F}_j \). \( I_{\nu}^z (\nu = x, y, z) \) is the dimensionless coefficient defined in Eq. (S2). Expressing the circuit operators in terms of creation and annihilation operators \( (\hat{\Phi} - \Phi_j^{\text{bias}}) = \Phi_c (\hat{a}^\dagger + \hat{a}) \) and \( Q \equiv (\hat{a}^\dagger - \hat{a})/(2\Phi_c) \), where \( \Phi_c \equiv \sqrt{\hbar/(2C\omega_c)} \), Eq. (S13) can be written as

\[
\hat{H} = \hbar \omega_c \hat{a}^\dagger \hat{a} - \frac{\Phi_c}{L} (\hat{a}^\dagger + \hat{a}) \hat{\Phi}_{\text{ext}} + \hbar \omega_L \hat{F}_{\nu}^z \\
+ \frac{(\hat{\Phi}_{\text{ext}})^2}{L} + \Phi^{\text{bias}} \frac{\Phi_{\text{ext}}}{L} + \hat{U}_a(\{\hat{\mu}_j\}) + \hat{V}_a(\hat{\mu}_j),
\]  
(S16)

where \( \omega_c \equiv 1/\sqrt{LC} \), \( \Phi^{\text{bias}} \equiv \sum_j \Phi_j^{\text{bias}} \), and \( \hat{\Phi}_{\text{ext}} \equiv \Phi_{\text{ext}} \sum_{j,\nu} I_{\nu}^z \Delta \hat{F}_{\nu}^z \).

We consider the applied field \( B_0 \) to be sufficiently large as to initially polarize the macrospin at the two nodes along \( -\mathbf{e}_z \), such that \( (\hat{F}_j)_{0} = -\mathbf{F}_j \mathbf{e}_z \). The fluctuations of \( \hat{F}_j \) around the equilibrium state can be described by a bosonic mode \( \hat{F}_j \), \( \hat{F}_j^\dagger \) (magnon) according to the Holstein-Primakoff approximation \( \hat{F}_j^z = -F + \hat{F}_j^\dagger \hat{F}_j, \) and \( \hat{F}_j^\dagger \approx \sqrt{2F} \hat{F}_j^\dagger \). In the limit of small fluctuations \( \langle \hat{F}_j^\dagger \hat{F}_j \rangle \ll 2F, \)
\( \hat{H} \) can be approximated by a quadratic Hamiltonian in the bosonic operators \( \hat{a}, \hat{a}^\dagger, \hat{f}_j, \) and \( \hat{f}_j^\dagger \) as

\[
\hat{H} \approx \hbar \omega_c \hat{a}^\dagger \hat{a} - \hbar (\hat{a}^\dagger + \hat{a}) \sum_{j=1}^{2} \left( \chi_j \hat{f}_j + \text{H.c.} \right) + \hbar \sum_{j=1}^{N} \left( \omega_j \hat{f}_j^\dagger \hat{f}_j + 2 \sum_i \left( \Lambda_{ij} \hat{f}_i \hat{f}_j^\dagger + \text{H.c.} \right) + \sum_{i,j} \{ (2 - \delta_{ij}) \mathcal{J}_{ij} + (1 - \delta_{ij}) \mathcal{J}_{ij}^d \} \hat{f}_i^\dagger \hat{f}_j \right) + \hbar \sum_{j} \left( \eta_j \hat{f}_j + \eta_j^* \hat{f}_j^\dagger \right).
\]

Here, we defined

\[
\omega_j = \frac{\gamma_0 B_0}{2} + 2 \frac{\gamma_0 k_a}{M_s} + \mathcal{J}_{ij} + \mathcal{J}_{ij}^d
\]

\[
\Lambda_{ij} = -3(1 - \delta_{ij}) \frac{\gamma_0^2 \mu_0}{16 \pi r_{ij}^3} F \sin^2 \theta_{ij} e^{i2\varphi_{ij}} + \frac{\Phi_i^2}{2hL} I_j, \quad \mathcal{J}_{ij}
\]

\[
\mathcal{J}_{ij} = \left( \frac{\gamma_0^2 \mu_0}{4 \pi d} \right)^2 I_{ij} \frac{2}{2hL} F, \quad \mathcal{J}_{ij}^d = -\frac{\gamma_0^2 \mu_0}{8 \pi r_{ij}^3} (3 \sin^2 \theta_{ij} - 2) F, \quad \chi_j = \frac{\Phi_i \Phi_c}{2hL} I_j \sqrt{2F}, \quad \eta_j = \sum_i \left( 3 \frac{\gamma_0^2 \mu_0^2}{8 \pi r_{ij}^3} \sqrt{2F}e^{i2\varphi_{ij}} \cos \theta_{ij} \sin \theta_{ij} + \frac{\Phi_i \Phi_{bias}}{2hL} \sqrt{2FI_j} \right)
\]

where \( I_{ij} = I_{ij}^{\pm} = I_{ij}^x + iI_{ij}^y \), \( k_a \) is the magnetic anisotropy energy density, \( M_s \) the saturation magnetization, and \( \theta_{ij} \) and \( \varphi_{ij} \) are the azimuthal and polar angle of the vector \( \mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j \) connecting the \( i \)-th and \( j \)-th magnet. We additionally assumed a easy magnetization axis of the magnetic anisotropy potential of the material, \( i.e. \) one of the directions along which the anisotropy energy is minimized, to be aligned along the direction of the applied magnetic field. In this case, the anisotropy energy contributes only as a shift to the magnon oscillation frequency within the quadratic approximation.

The linear term in Eq. (S17) shifts the equilibrium orientation of the magnetic moments and the equilibrium value of the flux in the loop. It can be formally eliminated from Eq. (S17) displacing the bosonic operators \( \hat{a}, \hat{a}^\dagger, \hat{f}_j, \) and \( \hat{f}_j^\dagger \) to represent the fluctuation around the new equilibrium values. The displacement in the bosonic operators in higher order terms (not shown in Eq. (S17) can induce a shift in the frequencies and coupling rates of the final quadratic Hamiltonian. The linear term in Eq. (S17) can however be neglected when the magnetic particle are placed on the plane of the LC-resonator (\( h = 0 \)), as in Fig. S1.b, and the distance between the magnet is such that the free-space dipole-dipole interaction is negligible [S13]. We thus neglect hereafter the last term in Eq. (S17) assuming the shift in the relevant couplings and frequencies to be negligible. We remark that all the quantitative predictions made in the main text are calculated for \( h = 0 \) and negligible dipole-dipole interaction which thus justifies the approximation made here.

Due to the large detuning between \( \omega_c \) and Eq. (S18-S22), we adiabatically eliminate the LC-resonator degrees of freedom assumed to be in the vacuum state. Within the rotating-wave approximation and taking into account the circuit-induced frequency and coupling shift, the effective Hamiltonian describing the magnon dynamics reads

\[
\hat{H}_M = \hbar \sum_{j=1}^{N} \omega_j \hat{f}_j^\dagger \hat{f}_j + \hbar \sum_{i \neq j=1}^{N} (\mathcal{J}_{ij} + \mathcal{J}_{ij}^d) \hat{f}_i^\dagger \hat{f}_j.
\]

Here, \( \mathcal{J}_{ij} \) and \( \mathcal{J}_{ij}^d \) represents the contributions to the magnon tunneling rate of the direct inductive magnetic interaction, and the magnetic dipole-dipole interaction respectively. For \( \mathcal{J}_{ij} \) decreases slowly with the magnets separation as compared to \( \mathcal{J}_{ij}^d \), we expect this latter to be negligible for large separation between the magnets. As shown in Fig. S2.a for the case of the simple configuration in Fig. 1.b, the magnetic dipole-dipole contribution to the magnon tunneling rate can be neglected for a sufficiently large separation \( a \equiv 2(l + d) \).

Let us now consider some particular SLNM and derive their corresponding Hamiltonian. We study first the case of a 2D SLNM as discussed in the main text. Later, we discuss the possibility of replacing the simple circular coil with a bone shape coil leading to an enhancement of the magnon tunneling rate.

### A. 2D SLNM

We derive the band structure of the 2D SLNM illustrated in Fig. 2.c. The elementary cell of such a configuration is shown in Fig. S2.b. For large loop size \( l \), \( \mathcal{J}_{ij}^d \ll \mathcal{J}_{ij} \), and thus the magnon tunneling rate is given by \( \mathcal{J}_{ij} = \mathcal{J}_e^{i\phi_{ij}} \), for

\[
\mathcal{J} = \frac{\phi_i^2 F}{2hL} (I_x^2 + I_y^2),
\]

where we defined \( I_x \equiv I_x^1 = I_x^2 = I_x^3 = I_x^4 \) and \( I_y \equiv I_y^1 = I_y^2 = -I_y^3 = -I_y^4 \), and \( \phi_{ij} \) is a function of \( I_x, I_y \). The magnon dynamics in the elementary cell is thus described by

\[
\hat{H}_M^{\text{el}} = \hbar \omega_0 \sum_{j=1}^{4} \hat{f}_j^\dagger \hat{f}_j + \hbar \mathcal{J} \sum_{i \neq j=1}^{4} \hat{f}_i^\dagger \hat{f}_j.
\]
where we redefined some of the magnonic operators to absorb the phase factor appearing in the tunneling rate $J_{12}$.

The extended 2D SLNM shown in Fig. 2.c is built by repetition of this elementary cell. As discussed in the main text, the magnon dynamics of such a 2D SLNM can be described by a two interacting sublattice model, labelled by $A$ and $D$ according to the Hamiltonian

$$
\hat{H}_M^{2D} = \hbar J \left[ \sum_{j,\beta} \hat{f}_j^{D\dagger} \hat{\Psi}_j^{\beta} + \sum_{j,\alpha} \hat{f}_j^{A\dagger} \hat{\Psi}_j^{\alpha} \right] + \sum_{j,\delta} \hat{f}_j^{D\dagger} \hat{\Psi}_j^{\delta} + \text{H.c.}
$$

(S27)

Here, the operators $\hat{f}_j^A, \hat{f}_j^{A\dagger}$ ($\hat{f}_j^D, \hat{f}_j^{D\dagger}$) respectively create and annihilate a magnon in the sublattice $A$ ($D$) within the cell at position $j = (j_x, j_y)$ and the vectors $\alpha$ and $\beta$ ($\delta$ and $\beta$), for $\beta \in \{(\pm 1, 0), (0, \pm 1)\}$, $\alpha \in \{(-1, 1), (1, -1)\}$, and $\delta \in \{(1, 1), (-1, -1)\}$, connect the nearest neighbors of a point in the sublattice $A$ ($D$).

In terms of the operators

$$
\hat{f}_j^D = \frac{1}{N} \sum_j e^{-ik_j} \hat{f}_j^D,
$$

(S28)

$$
\hat{f}_k^{D\dagger} = \frac{1}{N} \sum_j e^{ik_j} \hat{f}_j^{D\dagger},
$$

(S29)

$$
\hat{f}_j^A = \frac{1}{N} \sum_j e^{-ik_j} \hat{f}_j^A,
$$

(S30)

$$
\hat{f}_k^{A\dagger} = \frac{1}{N} \sum_j e^{-ik_j} \hat{f}_j^{A\dagger},
$$

(S31)

which create/annihilate a magnon of momentum $k = (k_x, k_y)$ in the sublattice $D$ or $A$, the Hamiltonian in Eq. (S27) can be written as $\hat{H}_M^{2D} = 2J \hat{\Psi}^\dagger M \hat{\Psi}$ where

$$
\hat{\Psi} \equiv (\hat{f}_k^D, \hat{f}_k^A)^T \quad \text{and}
$$

$$
M \equiv \begin{pmatrix}
2\cos((k_y + k_z)a) & \cos(k_ya) + \cos(k_za) \\
\cos(k_ya) + \cos(k_za) & 2\cos((k_za - k_ya)
\end{pmatrix},
$$

(S32)

The eigenvalues of $\hat{H}_M^{2D}$ read

$$
\omega_{\pm}(k) = 2J [4 \cos(k_z a) \cos(k_y a) \pm \sqrt{\lambda}],
$$

(S33)

where $\lambda \equiv 4 + 4 \cos(k_ya) \cos(k_za) - \cos(2ka) - \cos(k_za) + 2 \cos(2k_ya) \cos(ka), \text{ with } a = \sqrt{2(l + d)}$ the lattice constant. Eq. (S33) correspond to the magnon bands illustrated in Fig. 2.d.

B. Bone shape coil

Different coil geometries can be designed with the aim of enhancing the magnon tunneling rate for large separation between the magnets. An example is shown in Fig. S2.c, where two magnets are coupled through a bone-shape loop. Here, $d$ is the radius of the circular end-rings, $w$ the separation of the middle parallels wires, and $l$ their length. For $w \ll d/l$, the middle region connecting the two circular ends of the loop has a negligible contribution to the self inductance $L$ of the loop. Moreover for $R < d$, the magnetic flux produced by the magnetic particle is obtained as the flux generated by a magnetic moment $\mu$ placed at the origin of a circular coil and of intensity $\mu = M_s \pi R^2/3$. For $l \gg d$, the flux produced by a magnet in the loop at the opposite end of the coil can be neglected. The fluctuating magnetic moment $\hat{\mu}_j$ produces a fluctuating flux $\hat{\Phi}_j = h\gamma_0 \mu_0 \Delta \hat{F}_x/(4l)$, where $\Delta \hat{F}_x$ and $\Delta \hat{F}_z$ contribute only at higher order. In this configuration, the direct inductive magnetic coupling contribution to the magnon-tunneling rate thus reads

$$
J_{12}^{\text{bone}} = \frac{\gamma_0 \mu_0^2 h F}{8d^2 L},
$$

(S34)
Here, $L$ represents the inductance of a circular coil of radius $d$ [cf. Eq. (S5)].

In Fig. S2.a, $J_{12}^{\text{bone}}$ is plotted as a function of the magnet separation $a = 2(1 + d)$, keeping $d$ fixed. Realistically, at larger values of $l$ the contribution of the middle region to the total inductance will affect the scaling of $J_{12}^{\text{bone}}$. However, for a sufficiently small separation $w$ between the two parallel wires the tunneling rate is expected to vary only slightly with an increase of $l$. The bone-shape configuration (Fig. S2.b) thus allows to enhance the magnon tunneling rate for a given separation $a$ as compared to the simple configuration in Fig. 1.b.

### IV. SPIN QUBITS COUPLED TO A SLNM

Here, we derive the Hamiltonian describing the magnetic coupling of spin qubits to a general superconducting loop-network of magnetic particles. In particular, we consider to locally couple NV-center qubits to the magnetic particles at the nodes of the SLNM by magnetic dipole-dipole interactions. The total Hamiltonian of the system thus reads

$$\hat{H}_T = \hat{H}_M + \sum_{j=1}^{N} \hat{H}_{\text{MQ}}^j,$$

where $\hat{H}_M$ is given by Eq. (S24), and $\hat{H}_{\text{MQ}}^j$ represents the local interaction between the spin qubit and the magnet at site $j$. Let us now first derive the interaction Hamiltonian between a magnet and an NV-center at a single site. We will then later generalize this result to quantum emitters coupled to a general SLNM.

#### A. NV-Magnetic Particle Hamiltonian

The interaction Hamiltonian between a magnet and an NV-center located at $\mathbf{r}_q \equiv r_q (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ around the magnet (see inset Fig. S3) reads,

$$\hat{H}_{\text{MN}} = \hbar \Delta_{\text{NV}} \hat{S}_z^2 + \gamma_q B_0 \hat{S}_z - \hbar \gamma_q \hat{\mathbf{S}} \cdot \mathbf{B}^{\text{dip}}(\mathbf{r}_q),$$

where $\hat{\mathbf{S}}$ is the NV-center spin-1 operator, $\Delta_{\text{NV}}$ its zero field splitting, $\gamma_q$ its gyromagnetic ratio, and $\mathbf{B}^{\text{dip}}(\mathbf{r}_q)$ is the dipole field produced by the magnet at the NV position (see Eq. (S11)). As the derivation is the same at each node, we dropped the site index $j$.

Expressing the NV spin operators in terms of the eigenvalues of $\hat{S}_z$, namely $\hat{S}_z = |1\rangle\langle 1 | - |-1\rangle\langle -1 |$ and $\hat{S}_+ = (\hat{S}_z)^{1/2} = \sqrt{2} |0\rangle\langle -1 | + |1\rangle\langle 0 |$, the Hamiltonian Eq. (S36) can be rewritten as $\hat{H}_{\text{MN}} = \hat{H}_1 + \hat{H}_{-1}$, where $\hat{H}_1$ acts only on the states $|0 \rangle, |1 \rangle$ ($k = \pm 1$) of the NV center. In the following, we assume the frequency of the fluctuations of the magnetic moment of the magnets close to the NV center transition frequency between $|0 \rangle$ and $|-1 \rangle$. This can be achieved by appropriate values of the applied field $B_0$, magnet size $R$, and relative distance $r_q$ between the NV and the magnet. Within this assumption, the coupling between the magnet and the higher level $|1 \rangle$ is negligible, and the NV-magnet coupling is thus well approximated by $\hat{H}_{-1}$.

For small fluctuation of the macrospin $\hat{\mathbf{F}}$ around the direction of the applied field $B_0$, we can approximate the interaction by the spin qubit-magnon Hamiltonian

$$\hat{H}_{\text{MQ}} = \hbar \omega_\varphi (\theta) \hat{\sigma}_z^2 + \hbar \xi(\theta) (e^{i\varphi \hat{\sigma}_-} + \text{H.c.})$$

$$- \hbar \left\{ \left[ g(\theta)e^{-i2\varphi} \hat{f}^\dagger + \Omega(\theta) \hat{f} \right] \hat{\sigma}^- + \text{H.c.} \right\}$$

(S37)

where $\hat{\sigma}_z^2 \equiv | -1 \rangle\langle -1 | - |0\rangle\langle 0 |$, $\hat{\sigma}_+ \equiv | -1 \rangle\langle 0 |$, $\hat{\sigma}_- \equiv |0\rangle\langle -1 |$, and we defined the frequencies

$$\omega_\varphi (\theta) \equiv \Delta_{\text{NV}} - \gamma_q B_0 - \frac{\hbar \gamma_q \mu_0}{4 \pi r_q^3} F (3 \cos^2 \theta - 1),$$

(S38)

$$\xi(\theta) \equiv \frac{3 \hbar \gamma_q \mu_0}{4 \pi r_q^3} \sqrt{2} F \sin \theta \cos \theta,$$

(S39)

$$\Omega(\theta) \equiv \frac{\hbar \gamma_q \mu_0}{8 \pi r_q^3} \sqrt{F (3 \sin^2 \theta - 2)},$$

(S40)

$$g(\theta) \equiv \frac{3 \hbar \gamma_q \mu_0}{8 \pi r_q^3} \sqrt{F} \sin \theta,$$

(S41)

The spin qubit in Eq. (S37) can be diagonalized in terms of dressed states $|\pm \rangle$. These are obtained from the uncoupled state $|0 \rangle, |1 \rangle$ by the unitary transformation matrix

$$\hat{U} = \begin{pmatrix} e^{i\varphi/2} \cos(\Theta/2) & -e^{-i\varphi/2} \sin(\Theta/2) \\ e^{i\varphi/2} \sin(\Theta/2) & e^{i\varphi/2} \cos(\Theta/2) \end{pmatrix},$$

(S42)

where the angle $\Theta \in [0, \pi]$ is defined as

$$\Theta = \begin{cases} \pi - \arctan \left( \frac{\sqrt{F(\xi(\theta))}}{\omega_\varphi(\theta)} \right) & , \xi(\theta)/\omega_\varphi(\theta) < 0 \\ \arctan \left( \frac{\sqrt{F(\xi(\theta))}}{\omega_\varphi(\theta)} \right) & , \xi(\theta)/\omega_\varphi(\theta) > 0 \end{cases}.$$ (S43)

In the dressed state basis, Eq. (S37) reads

$$\hat{H}_{\text{MQ}} = \hbar \omega_\varphi \hat{\sigma}_z^2 + \hbar \xi(\varphi \hat{f}^\dagger + \text{H.c.})$$

$$- \hbar g(\hat{f} + \text{H.c.})$$

(S44)

The Pauli operator in Eq. (S44) refer now to the dressed states, namely $\hat{\sigma}_z^2 \equiv | + \rangle\langle + | - | - \rangle\langle - |$. The dressed state frequency is

$$\omega_\varphi \equiv \sqrt{\omega_\varphi(\theta)^2 + F \xi(\theta)^2},$$

(S45)

and the spin qubit-magnon couplings are

$$\xi \equiv \xi(\theta)e^{i\varphi \cos \Theta} + \left( \Omega(\theta)e^{-i\varphi} + g(\theta)e^{i2\varphi} \right) \sin \Theta,$$

(S46)

$$g \equiv \frac{1}{4} \xi(\varphi \cos \Theta + \Omega(\theta) \sin \Theta \cos \Theta - \Omega(\theta) \sin^2 \Theta / 2).$$

(S47)
where \( \xi, g, \) and \( \Omega \) vs the position angle \( \theta \). The remaining parameter have the following values: \( R = 350 \text{nm}, r_0 = R + 20 \text{nm}, B_0 = 70 \text{mT}, \) and \( \gamma_0, \gamma_6 \) as in the caption of Fig. 3. Inset: general configuration of the nanomagnet-qubit system.

Fig. S3 shows the dependence of the coupling \( \xi, g, \) and \( \Omega \) in Eq. (S46), Eq. (S47), and Eq. (S48) as function of \( \Omega \).

Fig. 3.a). In the following, we describe the effective dy-

V. EFFECTIVE SPIN-SPIN INTERACTION THROUGH A MAGNETIC QUANTUM BUS

We consider two spin qubits locally coupled by mag-
netic dipole-dipole interactions to two magnetic par-
ticles coupled by a superconducting loop resonator (see Fig. 3.a). In the following, we describe the effective dy-
namics of the system when the spin qubits are detuned from the magnon propagation frequencies and provide a figure of merit to estimate the efficacy of a SWAP gate operation performed by such a quantum bus.

A. Effective Master Equation for the Qubit Dynamics

The dynamics of the quantum state of the total system is described by the following master equation

\[
\dot{\rho} = -\frac{i}{\hbar}[\hat{H}_T, \rho] + \kappa \sum_{j=1}^2 \left( \hat{f}_j \rho \hat{f}_j^\dagger - \frac{1}{2} (\hat{f}_j^\dagger \hat{f}_j \rho - \rho \hat{f}_j^\dagger \hat{f}_j) \right) + \gamma \sum_{j=1}^2 (\hat{\sigma}_z^j \rho \hat{\sigma}_z^j - \rho),
\]

(S50)

where \( \hat{\rho} \) represents the quantum state of the two qubits and the magnons at site \( j = 1, 2, \kappa \) is the magnon damping rate, \( \gamma = \pi/T^2_\gamma \) the qubit dephasing rate, and \( \hat{H}_T \) is defined in Eq. (S49) for the simple case where \( i, j = 1, 2 \).

In terms of the modes \( \hat{f}_\pm \equiv (\hat{f}_1 \pm \hat{f}_2)/\sqrt{2} \) the Hamiltonian \( \hat{H}_T \) reads

\[
\dot{\hat{H}}_T = \hbar \omega_+ \hat{f}_\uparrow \hat{f}_\downarrow + \hbar \omega_- \hat{f}_\downarrow \hat{f}_\uparrow + \hbar \omega_\sigma \left( \frac{\hat{\sigma}_z^1}{2} + \frac{\hat{\sigma}_z^2}{2} \right)
- \frac{\Omega}{\sqrt{2}} \left[ \hat{\sigma}_z^1 \hat{\sigma}_z^2 \right] + \hat{\sigma}_y(\hat{\sigma}_z^1 + \hat{\sigma}_z^2) \]

\[
+ \text{H.c.},
\]

(S51)

where \( \omega_\pm \equiv \omega_0 \pm J \). The dissipative term in Eq. (S50) maintains the same structure where the magnonic operators \( \hat{f}_j, \hat{f}_j^\dagger \) (\( j = 1, 2 \)) are replaced by the normal modes \( \hat{f}_\pm, \hat{f}_\mp \).

In the limit of a large detuning between the spin qubits and the magnons, it is possible to adiabatically eliminate the magnonic degrees of freedom and obtain an effective master equation describing the effective dynamics of the spin qubits. Transforming the master equation describing the total system via the unitary operator

\[
\hat{U} \equiv \exp \left\{ -\frac{g}{\sqrt{2} \Delta} \left[ \hat{f}_\downarrow (\hat{\sigma}_1^- + \hat{\sigma}_2^-) - \text{H.c.} \right] - \frac{g}{\sqrt{2} (\Delta - 2J)} \left[ \hat{f}_\uparrow (\hat{\sigma}_1^+ + \hat{\sigma}_2^+) - \text{H.c.} \right] \right\},
\]

keeping terms up to second order in \( g/\Delta, g/(\Delta - 2J) \ll 1 \) and projecting the result on the vacuum subspace of the magnons Hilbert space, one obtains

\[
\dot{\rho}_{\text{eff}} = -\frac{i}{\hbar} [\hat{H}_{\text{QQ}}, \rho_{\text{eff}}] + \kappa_{\text{eff}} \sum_{j=1}^2 \mathcal{D}^{ij}_{\sigma} [\rho_{\text{eff}}]
+ \Omega_{\text{eff}} \sum_{i,j=1} \mathcal{D}^{ij}_{\sigma} [\rho_{\text{eff}}]
+ \gamma \sum_{j=1} (\hat{\sigma}_z^j \rho \hat{\sigma}_z^j - \rho),
\]

(S53)
where
\[
\hat{H}_{\text{QQ}} = \frac{\hbar}{2} \tilde{\omega}_{\sigma} (\hat{\sigma}^+_1 \hat{\sigma}^+_2 + \hat{\sigma}^-_1 \hat{\sigma}^-_2) - h g_{\text{eff}} (\hat{\sigma}^+_1 \hat{\sigma}^-_2 + \hat{\sigma}^-_1 \hat{\sigma}^+_2),
\]
(S54)
and
\[
D^{ij}_{\sigma} [\hat{\rho}_{\text{eff}}] \equiv \hat{\sigma}^+_i \hat{\rho}_{\text{eff}} \hat{\sigma}^+_j - \frac{1}{2} [\hat{\sigma}^+_j \hat{\sigma}^-_i, \hat{\rho}_{\text{eff}}].
\]
(S55)
Here, \(\hat{\rho}_{\text{eff}}\) represents the effective state of the two qubits and we defined the effective frequencies and decay rates
\[
\tilde{\omega}_{\sigma} \equiv \omega_{\sigma} - g^2 \left( \frac{1}{\Delta - 2J} + \frac{1}{\Delta} \right),
\]
(S56)
\[
g_{\text{eff}} \equiv g^2 \left( \frac{1}{\Delta} - \frac{1}{\Delta - 2J} \right),
\]
(S57)
\[
\kappa_{\text{eff}} \equiv \kappa g^2 \frac{\Delta^2 + (\Delta - 2J)^2}{\Delta^2 (\Delta - 2J)^2},
\]
(S58)
\[
\Omega_{\text{eff}} \equiv \kappa g^2 \frac{\Delta^2 - (\Delta - 2J)^2}{\Delta^2 (\Delta - 2J)^2}.
\]
(S59)
The Hamiltonian Eq. (S54) can be used to implement a long-range qubit-qubit interaction through the magnonic quantum bus provided by a SLNM [S14].

The intrinsic qubit dephasing \(\gamma\) as well as the bus-induced effective qubit damping \(\kappa_{\text{eff}}\) described in Eq. (S53), affect the performance of coherent exchange of excitation between the qubits. In the following, we describe the impact of these noise sources and derive a figure of merit for the performance of the coherent qubit coupling.

B. Figure of Merit for the Effective Qubit-Qubit Interaction

Let us consider a SWAP gate which transfers excitation from the first to the second qubit through the interaction described by Eq. (S54). The performance of the gate can be estimated in terms of the quantum state fidelity [S15] \(F(t) = (\text{Tr} \sqrt{\sqrt{\rho(t)} \hat{\rho}(t) \sqrt{\rho(t)}})\) between the state of the system \(\hat{\rho}(t)\) after the evolution governed by Eq. (S50) and the target state \(|\psi_t\rangle \equiv |01\rangle \otimes |\text{vac}\rangle\), where \(|\text{vac}\rangle\) is the vacuum of the magnon bus and \(|01\rangle\) is the two qubits state where only the second (taget) qubit is excited. We assume to prepare the system in the initial pure state \(\rho(0) = |\psi\rangle \langle \psi|\) where \(|\psi\rangle \equiv |10\rangle \otimes |\text{vac}\rangle\), where only the first qubit is excited. The performance of the SWAP gate can then be estimated by maximizing \(F(t)\) over the total evolution time \(t\), and calculating the fidelity error \(\varepsilon \equiv 1 - \text{max}_r [F(t)]\) in the presence of noise. A numerical optimization of \(\varepsilon\) over all the relevant parameters of the system \(g, J, \Delta, \kappa, \text{and } \gamma\) yields a figure of merit for the performance of the gate.

An analytical expression for the scaling of the optimal error can be obtained in the dispersive regime \(g/\Delta, g/(\Delta - 2J) \ll 1\). In the strong coupling limit \(g_{\text{eff}} \gg \kappa_{\text{eff}}, \gamma\), \(\varepsilon\) scales approximately linearly with the decoherence rates as \(\varepsilon \approx \alpha_\gamma \gamma/g_{\text{eff}} + \alpha_\kappa \kappa_{\text{eff}}/g_{\text{eff}}\) [S16], where the coefficients \(\alpha_\gamma\) and \(\alpha_\kappa\) are assumed to be approximately independent on the detuning \(\Delta\). This assumption can be numerically checked simulating the error scaling for different values of \(\Delta\) for the same values of \(\kappa_{\text{eff}}/g_{\text{eff}}, \gamma/g_{\text{eff}} \ll 1\). Substituting Eq. (S57) in the linear expansion for \(\varepsilon\) one obtains
\[
\varepsilon = -\alpha_\gamma \frac{\gamma (\Delta - 2J)}{2g^2 J} - \alpha_\kappa \frac{\kappa (\Delta^2 + (\Delta - 2J)^2)}{2 J (\Delta - 2J)}.
\]
(S60)
For the optimal values \(\Delta^* = J^*/(\pi \alpha_\gamma)\) [S17], Eq. (S60) reads
\[
\varepsilon = \sqrt{\frac{\alpha_\kappa \alpha_\gamma}{2 C}}.
\]
(S61)
where the cooperativity \(C\) is defined as
\[
C = \frac{g^2}{\gamma \kappa}.
\]
(S62)
In Fig. S4, the error \(\varepsilon\) optimized for \(\Delta = J\) is plotted as function of the normalized decoherence rates \(\kappa_{\text{eff}}/g_{\text{eff}}\) and \(\gamma/g_{\text{eff}}\). In the strong coupling regime \(\kappa_{\text{eff}}, \gamma \ll g_{\text{eff}}\), \(\varepsilon\) scales linearly with the decoherence rates and the coefficients \(\alpha_\kappa\) and \(\alpha_\gamma\) can be estimated through linear interpolations of the simulated results.

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FIG. S4: Distribution of the fidelity error $\varepsilon$ for the optimized detuning $\Delta^* = J$ for $\kappa = 0$ as function of $T_2^*/(\gamma g_{\text{eff}})$ (top panel, empty dots) and for $\pi/T_2^* = 0$ as function of $\kappa/g_{\text{eff}}$ (bottom panel, full dots). In each panel the linear regression curve (dashed line) is shown together with the interpolated values of $\alpha_\kappa$ and $\alpha_\gamma$ in each panel represent the linear regression curve. Other parameters as in the caption of Fig. 3.

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