Superstrings, Gauge Fields and Black Holes.

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Abstract

There has been spectacular progress in the development of string and superstring theories since its inception thirty years ago. Development in this area has never been impeded by the lack of experimental confirmation. Indeed, numerous bold and imaginative strides have been taken and the sheer elegance and logical consistency of the arguments have served as a primary motivation for string theorists to push their formulations ahead. In fact the development in this area has been so rapid that new ideas quickly become obsolete. On the other hand, this rapid development has proved to be the greatest hindrance for novices interested in this area. These notes serve as a gentle introduction to this topic. In these elementary notes, we briefly review the RNS formulation of superstring theory, GSO projection, $D$-branes, bosonic strings, dualities, dynamics of $D$-branes and the microscopic description of Bekenstein entropy of a black hole.

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1 Introduction

Superstring theories arose as an attempt to unify all the forces in Nature: the gravitational, the weak, the electromagnetic and the strong forces. An early attempt in this direction started with the Veneziano model in which one endeavored to reconcile the duality between s and t channels with the regularity in the spin-mass squared plot, namely the Regge behavior of S-matrix theory. The spectrum, which emerged from this early attempt, exhibited degeneracy at each mass level increasing exponentially with mass. Furthermore, this spectrum contained positive and negative norm states. However, the undesirable negative norm states (ghosts) could be removed if one restricted the dimension of space-time, \( d \), to less than or equal to 26. In particular, the choice of a critical dimension of \( d = 26 \) made the model unitary\(^2\). Veneziano model was formulated with bosons. In fact, the critical dimension for a fermionic version, called the dual pion model \(^1\) turned out to be 10.

It was subsequently found that the infinite particle spectrum in the Veneziano model could be derived in a more consistent manner in a quantized string theory. Originally proposed by Nambu, Goto, Nielson and Susskind, this formulation of a reparametrization-invariant string action showed that excitations of the one-dimensional (bosonic) string could be identified with the infinite particle spectrum of Veneziano model. The fermionic string theory was subsequently formulated by Ramond \(^2\), Neveu and Schwarz \(^3\). An ad-hoc procedure to incorporate both theories was later proposed by Gervais and Sakita \(^4\). An interesting feature of these string theories is that consistent quantization requires the fixation of space-time dimension to \( d = 26 \) for bosonic string and \( d = 10 \) for fermionic string, just like the Veneziano model.

Despite these developments, string theory was not readily accepted in these early attempts primarily due to the emergence of quantum chromodynamics and the failure of the dual models to describe experimental results at high energy scattering \(^1\). Moreover, the theory predicted a massless sector with a spin 2 particle and all experimental attempts to identify this particle failed. Fortunately, Scherk and Schwarz \(^5\) proposed a quick fix to the problem. They suggested that the spin 2 particle is really the graviton. Indeed, there were three subsequent discoveries \(^6\) that clearly pointed to the possibility of superstring theory as a strong candidate for the unifying theory of standard model with gravity. The

\(^2\)However, tachyons still exists in the bosonic model.
first observation was the discovery of the miraculous anomaly cancellation in superstring gauge theory. Furthermore, consistency at quantum level requires the gauge group to be $SO(32)$ or $E_8 \times E_8$. The second discovery was the development of heterotic string theories. It turned out that there were two new consistent heterotic superstring theories based on closed oriented strings with gauge groups $SO(32)$ or $E_8 \times E_8$. The third discovery was the realization that the heterotic string with $E_8 \times E_8$ gauge group admits solutions which results in 4d effective theory at low energies on Calabi-Yau compactification.

At the end of 1985, there were five totally self-consistent superstring theories in ten dimensions: Type I, non-chiral Type IIA, chiral Type IIB, $E_8 \times E_8$ heterotic string and $SO(32)$ heterotic string theory. One major obstacle remained. To make the necessary connections to our four dimensional world, these ten dimensional string theories have to be compactified. It turns out that there are thousands of ways of performing this compactification.

To reconcile the five superstring theories, it turned out that it was necessary to invoke the notion of duality symmetries. Existence of duality symmetries began as a conjecture and has remained a conjecture \[7\]. One such duality symmetry is T duality. This duality occurs when closed bosonic string compactified on a circle of radius $R_1$ has the same mass spectrum of physical states and scattering amplitudes as another string theory compactified on a radius of $R_2$ with $R_1 R_2 = \alpha'$, where $\alpha'$ is the universal Regge slope parameter. Using duality argument, one then shows that the Type IIA and IIB theories are T dual. So are the two heterotic string theories. Indeed, T duality is really a generalized Fourier transformation. Besides T duality, there were other dualities, namely S duality and U duality. S duality is essentially non-perturbative and exchanges weak and strong couplings while U duality describes a larger group of duality symmetries which encompasses both T and S dualities.

In 1995, Witten \[8\] invoked the power of duality symmetries to map all the strong coupling regimes of some string theory to the weak coupling regimes of another. Thus all the five superstring theories can be related to one another through duality symmetries. An important formulation using these duality symmetries is M theory. M theory is an 11 dimensional theory which reduces to 11 dimensional supergravity in its low energy limit and reproduces Type IIA string theory when compactified on a circle of vanishingly small radius. Moreover, it is also related to Type IIB string theory. To see this relation, one observes that Type IIA and IIB are T dual. Combining these facts, one sees that M theory compactified on
A torus is dual to Type IIB superstring theory compactified on a circle.

Type II superstring theories has a number of $(p + 1)$-dimensional membrane solutions that preserve half the supersymmetries and are called $p$-branes. A large class of these $p$-brane excitations are the Dirichlet or $D$-branes. Indeed, $D$-branes are classical solutions and can also be regarded as topological obstructions in superstring theories. Moreover, it is found that the longitudinal components can be gauged away leaving the transverse components as excitations. These in turn can be described by an effective theory given by supersymmetric Yang-Mill fields. In addition, $N$ $D$-branes can be superposed to give rise to an $U(N)$ nonabelain gauge theory for the lowest energy excitations.

This review arose from a series of lectures on superstring, $D$-branes and black holes by the first author in 1997. Since then, there have been many important discoveries. Indeed an entire arsenal of insightful tools have been invented and recent investigations into Maldacena conjectures concerning the AdS/CFT duality have shown that it is possible to study large $N$ gauge theory using strings and $D$-branes dynamics and vice versa [9,10]. M theory was first formulated by Witten to describe the 11-dimensional quantum theory whose effective description at low energy is the 11-dimensional supergravity. Recently, matrix theory has been formulated to provide a non-perturbative description of M theory. Not all the complexities regarding matrix quantum mechanics have been resolved, especially in the large $N$ limit.

Another recent development is the formulation of U theory [7] as the underlying fundamental theory whose limits result in the various string theories and their compactifications. Using duality symmetries, attempts have been made to understand this theory perturbatively through empirical conjectures regarding their effective actions. Indeed, there are numerous excellent reviews regarding these recent developments and their results [11].

Indeed duality arguments feature strongly in many superstring theories. Besides the tools in geometric engineering, a generalized Fourier transformation, there has been a tremendous explosion of activities around Maldacena conjecture which states that the quantum string theory on backgrounds of the form $AdS_d \times M_{D-d}$, where $AdS_d$ is an anti de-Sitter space of space time dimension $d$ and $M$ is a compactification space of dimension $D - d$, is dual to the conformally invariant quantum field theory on the boundary of the anti de Sitter space. In particular, it has been shown [13] that $\mathcal{N} = 4$ super Yang Mills theory on $M^4$ with gauge group $SU(N)$ is dual to Type IIB superstring theory on $AdS_5 \times M^5$. 
Starting with basic quantum field theory, we look at superstring theory. We consider the RNS formulation of superstring theory and look at how GSO projections can remove the existence of tachyons and impose space-time supersymmetry. We then briefly review and describe $D$-branes and bosonic strings. Duality is a fundamental concept in string theory. We look into the power of duality symmetries and review the relations between the various strings theories, namely the Type I, Type IIA and Type IIB. Dynamics of $D$-branes has provided much insight into M and U theories and we take a brief look at these dynamics. Finally, we review the microscopic description of Bekenstein entropy and Hawking radiation of a black hole and briefly describe the AdS/CFT correspondence.

2 RNS Formulation of Superstring Theory

When a particle traverses spacetime, it sweeps out a worldline. A similar traversal by a string naturally generate a worldsheet. Let $X^\mu(\sigma, \tau)$ be the position vector of a string in $d-1$ space-like dimensions. The simplest bosonic action can then be written in conformal gauge [14–16] as ($\alpha' = 1$)

$$S_B = \frac{1}{2\pi} \int d\sigma d\tau \partial_\alpha X^\mu(\sigma, \tau) \partial^\alpha X_\mu(\sigma, \tau).$$

Indeed we expect the string coordinates $X^\mu(\sigma, \tau)$ to be invariant under a reparametrization which constrains the state space. In conformal gauge, this can be realized by a super Virasoro algebra. The action (1) acts in $d$-Minkowski spacetime. Consistency of special relativity and quantum mechanics requires $d = 26$. This consistency requirement arises from the need to ensure that the Lorentz generators constructed in the theory can close properly.

Bosonic string has the following problems

- The particle spectrum contains a tachyon with $m^2 < 0$ but whose norm is positive definite.

- It is inconsistent at one loop calculation due to the tachyon.

- There are no spacetime fermions and so it cannot be used to describe Nature realistically.

The introduction of fermionic string with supersymmetry [17–20] solves these problems. There are two ways of introducing superstrings:
Green-Schwarz string. This string has genuine ten-dimensional spinor fields but the theory is not manifestly covariant.

- Ramond-Neveu-Schwarz (RNS) string. This formulation is manifestly covariant but the ten-dimensional spacetime supersymmetry is not obvious.

We follow the RNS-formulation. To attain supersymmetry, we introduce a fermionic field $\psi^\mu$ to the bosonic string action. This fermionic field $\psi^\mu$ is essentially a worldsheet spinor and can be expressed as a doublet

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}$$

but acts as a vector under $SO(1, D - 1)$ Lorentz group. The open superstring action is now written as

$$S = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \int_0^\pi d\sigma \{ \partial_\alpha X^\mu(\sigma, \tau) \partial^\alpha X_\mu(\sigma, \tau) - i \bar{\psi}^\rho \rho^\alpha \partial_\alpha \psi_\rho \}$$

where $\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$ are in the Majorana representation $\bar{\psi}^\mu = \psi^T \rho^0$.

This action possesses a supersymmetric invariance

$$\delta X^\mu = \epsilon \psi^\mu$$
$$\delta \psi^\mu = -i \rho^\alpha \partial_\alpha X^\mu \epsilon.$$ (4a)

(4b)

We introduce worldsheet light cone coordinates, $\partial_\pm = \partial_T \pm \partial_\sigma$. In this formalism, the fermionic part of the action reads

$$S_F = \frac{1}{\pi} \int d\tau \int d\sigma \{ \psi^-_\mu \partial_+ \psi^-_\mu + \psi^+_\mu \partial_- \psi^+\mu \}$$

so that

$$\delta S_F = \frac{1}{\pi} \int d\tau \int d\sigma \{ \delta \psi^-_\mu \partial_+ \psi^-_\mu + \delta \psi^+_\mu \partial_- \psi^+\mu \} + \frac{1}{2\pi} \int d\tau [\psi^-_\mu \delta \psi^-_\mu - \psi^+_\mu \delta \psi^+\mu].$$ (5)

Field equations of the fermionic action requires $\partial_\pm \psi^- = 0 = \partial_- \psi^+$ and we impose of the boundary conditions at $\sigma = 0$ and $\sigma = \pi$ so as to kill the surface terms. We can always choose $\psi^+(\pi, \tau) = \psi^-(\pi, \tau)$. However, at $\sigma = 0$, we have the following two distinct boundary conditions:

1. $\psi^+(0, \tau) = \psi^-(0, \tau)$, Ramond sector;
2. $\psi^+(0, \tau) = -\psi^-(0, \tau)$, Neveu-Schwarz sector.
For closed superstrings, the boundary conditions are given by

1. Periodic
\[ \psi_{\pm}(0, \tau) = \psi_{\pm}(2\pi, \tau) \], Ramond sector;

2. Anti-periodic
\[ \psi_{\pm}(0, \tau) = -\psi_{\pm}(2\pi, \tau) \], Neveu-Schwarz sector.

For the spectrum of states, we recall that superstrings are invariant under super-diffeomorphisms. Since we have chosen to work in the conformal gauge, we need to impose constraints on our state space, and these constraints are provided by the super Virasoro algebra. The Fourier expansion of the fermion strings in the Ramond and Neveu-Schwarz sectors are given by

\begin{align*}
\psi_{\mu}^{-} &= \sum_{n \in \mathbb{Z}} d_{n}^{\mu} \exp(-2in(\tau - \sigma)) \quad (7a) \\
\psi_{\mu}^{+} &= \sum_{n \in \mathbb{Z}} d_{n}^{\mu} \exp(-2in(\tau + \sigma)) \quad (7b)
\end{align*}

and

\begin{align*}
\psi_{\mu}^{-} &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_{r}^{\mu} \exp(-2ir(\tau - \sigma)) \quad (8a) \\
\psi_{\mu}^{+} &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} \tilde{b}_{r}^{\mu} \exp(-2ir(\tau + \sigma)) \quad (8b)
\end{align*}

respectively with the commutation relations

\begin{align*}
\{d_{n}^{\mu}, d_{m}^{\nu}\} &= \eta^{\mu\nu}\delta_{n+m} \text{, etc} \quad (9a) \\
\{b_{r}^{\mu}, b_{s}^{\nu}\} &= \eta^{\mu\nu}\delta_{r+s} \text{, etc} \quad (9b)
\end{align*}

Let \( \mathcal{F}^B \) and \( \tilde{\mathcal{F}}^B \) be the left and right movers for the bosons. Then for open strings, one sees that the state space is given by

\[ \mathcal{F}_{\text{open}} = \mathcal{F}^B \otimes (\mathcal{F}^R \oplus \mathcal{F}^NS). \quad (10) \]

In the case of closed strings, the space is given by

\[ \mathcal{F}_{\text{closed}} = \mathcal{F}_{\text{open}} \otimes \tilde{\mathcal{F}}_{\text{open}}. \quad (11) \]

Thus, in the Neveu-Schwarz sector, one demands that

\[ b_{r}|0, NS >= 0 \quad (12) \]

for \( r = \frac{1}{2}, \frac{3}{2}, \ldots \) and create higher mass states through \( b_{-r_{1}} b_{-r_{2}} \cdots b_{-r_{N}}|0, NS > \).

It will turn out that all excited states in the Neveu-Schwarz sector are spacetime
bosons. A similar analysis for the Ramond sector shows that the vacua transform
as a spinor of Spin(1, d − 1).

The energy-momentum tensor in the theory forms a closed algebra. From the
Fourier modes of the energy-momentum tensor, one can construct the world sheet
\( \mathcal{N} = 1 \) super Virasoro generators \((L_m, G_r)\) of the left moving sector for instance, satisfying

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{8}(m^3 - m)\delta_{m+n,0} \quad (13a)
\]

\[
[L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r} \quad (13b)
\]

\[
\{G_r, G_s\} = 2G_{r+s} + \frac{c}{2}(r^2 - \epsilon_R)\delta_{r+s,0} \quad (13c)
\]

where both \(L_n, G_r\) belong to either the \(R\) or \(NS\) sector and

\[
\epsilon_R = 0 \quad (14a)
\]

\[
\epsilon_{NS} = \frac{1}{4} \quad (14b)
\]

There is a similar algebra \(\tilde{L}_m, \tilde{G}_r\) for the right moving sector yielding a world
sheet \(\mathcal{N} = 2\) super Virasoro algebra. Superconformal invariance is achieved
through the introduction of ghost fields and obtaining a central extension \(c_{total} = 0\). It will be shown that by imposing the Gliozzi-Scherk-Olive condition, \(\mathcal{N} = 2\) world sheet supersymmetry yields a \(\mathcal{N} = 2\), \(d = 10\) spacetime supersymmetry
with \(2 \times 16 = 32\) supercharges.

3 Spectrum of Physical States and GSO Projection

In the previous lecture, we briefly touched on how to maintain super-conformal
invariance through the application of the super-Virasoro algebra by combining
the NS and R sector. In this lecture, we shall first consider the spectrum of the
physical states at low mass and show the existence of a state with mass\(^2 < 0\),
called a tachyon. To remove this unphysical state, we shall consider a technique
proposed by Gliozzi, Scherk and Olive in the seventies, commonly known as the
GSO projection.

Consider the Ramond sector for the open string. We shall consider left movers.
The ground state \(|0, \alpha, R\rangle\) satisfies the relation, for bosonic annihilators, \(\alpha_n\)
\( (n > 0), \)
\[
\alpha_n^\mu |0, \alpha, R > = 0 = d_n^\mu |0, \alpha, R > .
\]
(15)

From the commutation relations of the operators \( G_r \) and \( L_n \) in the previous lecture, one can easily show that \( G_0^2 = L_0 - \frac{c}{24} \). Since the ground state satisfies \( G_0 |0, \alpha, R > = 0 \), it follows that (for \( c = 0 \)) \( L_0 |0, \alpha, R > = 0 \). Further, by defining \( d_0^\mu \equiv \Gamma^\mu \), we have
\[
0 = G_0 |0, k, \alpha, R > ,
\]
(16a)
\[
= \alpha_0^\mu d_0^\mu |0, k, \alpha, R > ,
\]
(16b)
\[
= k^\mu \Gamma^\mu |0, k, \alpha, R > ,
\]
(16c)
giving us the massless Dirac equation. To obtain further excited massive states, one needs to consider the anti-symmetric polarization tensor.

In the Neveu-Schwarz sector, the ground state obeys
\[
b_r^{\mu}|0, k, NS > = 0, \quad r > 0,
\]
(17a)
\[
G_r |0, k, NS > = 0 \quad r \in \mathbb{Z} + \frac{1}{2}.
\]
(17b)
The physical state is subject to the mass shell condition
\[
(L_0 - \frac{1}{2})|0, k, NS > = 0
\]
(18a)
\[
\Rightarrow \left( \frac{1}{2} \alpha_0^2 - \frac{1}{2} \right)|0, k, NS > = 0
\]
(18b)
\[
\Rightarrow \left( \frac{1}{2} k^2 - \frac{1}{2} \right)|0, k, NS > = 0
\]
(18c)
\[
\Rightarrow k^2 - 1 = -m^2 - 1 = 0,
\]
(18d)
so that this state has mass \( m^2 < 0 \) and is indeed a tachyon.

### 3.1 Excited states

The first excited state \( |\zeta, k, NS > \) is obtained through the relation
\[
|\zeta, k, NS > = \zeta_\mu b_{-1/2}^\mu|0, k, NS >
\]
and satisfies
\[
0 = (L_0 - \frac{1}{2})|\zeta, k, NS > = (L_0 - \frac{1}{2})\zeta_\mu b_{-1/2}^\mu|0, k, NS >
\]
\[
\Rightarrow k^2 = 0,
\]
(20a)
\[
0 = L_1 |\zeta, k, NS >
\]
\[
\Rightarrow \zeta \cdot k = 0
\]
(20b)

\(^3\)Historically, Ramond considered the whole idea in reverse; beginning with the massless Dirac equation and applying it to the superstring theory.
Eq(20a) and Eq(20b) give rise to a massless vector state and a transversality condition. All other physical states can be obtained through the direct application of the operators $a^i_-$, $b^i_-$ and $d^i_-$ where $i = 1, \cdots, 8$.

For closed strings, we need to consider left and right movers in order to construct the complete spectrum. The fact that the origin for $\sigma$ is arbitrary for closed strings yields

$$p_L = p_R; \quad \alpha^\mu_0 = \tilde{\alpha}^\mu_0$$

(21)

The construction of the state space is easily done. Here, we shall summarize the result.

- **NS-NS bosons**
  - Ground state is a tachyon.
  - The massless states are graviton, $g_{\mu\nu}$, anti-symmetric tensor, $b_{\mu\nu}$ and the dilaton $\phi$.

- **NS-R (or R-NS) fermions**
  - No tachyon, since the Ramond sector only allows $\text{mass}^2 \geq 0$.
  - The massless ground state is reducible to the gravitino, and the dilatino.

- **R-R bosons**
  - No tachyon.
  - The massless states $|0, k, a, R \rangle_L \otimes |0, k, b, R \rangle_R$ decompose as a sum of the tensor representation of the group $SO(1, 9)$.

By combining the NS and the R sectors, we have constructed a string theory with bosons (e.g. the massless graviton) and fermions (e.g. gravitino). Despite the removal of negative norm states through the mass shell condition of the super Virasoro algebra, we still have states like tachyons. As it stands, this tachyon has no fermionic partner and the theory is essentially not space-time supersymmetric. To produce a spectrum with space-time supersymmetry, we need to consider the GSO projection.
3.2 GSO Projection

We define the GSO projection \[21\] in terms of the worldsheet fermion operator \( F \) in the form of \((-1)^F\). For the NS sector, we may represent \((-1)^F\), for the right movers, by

\[
(-1)^F = -(-1)^{h_{NS}},
\]

(22)

where \( h_{NS} = \sum_{r \in \mathbb{1}/2+Z^+} b_{-r} \cdot b_r \). For the Ramond sector,

\[
(-1)^F = \Gamma_{11}(-1)^{h_R}
\]

(23)

where \( h_R = \sum_{n>0} d_{-n} \cdot d_n \). The GSO condition is that, from eq(11),

\[
(-)^F \mathcal{F}_{\text{open}} = \mathcal{F}_{\text{open}}.
\]

(24)

There is a similar representation for left movers.

To obtain space-time supersymmetry, we note that in the NS sector, we have 8 physical degrees of freedom for \( A_\mu \); since in general a spinor in 10 dimensions has \( 2^{d/2} = 32 \) complex components or 64 real components, we need to impose both the Majorana and Weyl conditions. This reduces the dimensionality to 16 and the imposition of Dirac equation eliminates half the modes giving 8 physical modes. These 8 fermionic states are the superpartners of the 8 bosonic degrees of freedom carried by \( A_\mu \).

For the Ramond sector, we note that in the Clifford algebra defined by \( \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \), one finds, in the Majorana representation, that the real chirality matrix \( \Gamma_{11} \) is defined by

\[
\Gamma_{11} \equiv \Gamma^0\Gamma^1 \cdots \Gamma^9, \quad \Gamma_{11}^2 = 1
\]

(25)

and anti-commutes with all the generators in the Dirac spinor representation. For \( d = 2 \, (\text{mod} \, 8) \), a spinor can be reduced to a Weyl-Majorana spinor. This Weyl-Majorana condition effectively reduces the dimensionality of the ground state in the Ramond sector to an 8-component spinor.

Having analyzed the GSO projection in a single sector of the RNS string theory, we can next consider closed strings. Although a naive listing gives four possibilities, a more detailed analysis using parity operator shows that there are essentially two inequivalent closed string theories defined by the Ramond ground states:
• Type IIA, a non-chiral string theory in which the left-right Ramond vacua have opposite chirality and the theory is parity invariant under the exchange of left-right movers.

• Type IIB, a chiral string theory in which both the left-right Ramond vacua have the same chirality.

The state space of the two closed superstring theories yields an irrep of the $d = 10, \mathcal{N} = 2$ spacetime supersymmetry. Conjugation is defined for a Majorana spinor $Q$ by
\[ Q^\dagger = Q^T \Gamma_0 \equiv Q^T C. \] (26)

In the Majorana representation, $Q_\alpha$ is a real 32-component $d = 10$ (Majorana) spinor. A chiral spinor satisfies
\[ \Gamma_{11} Q^\pm = \pm Q^\pm. \] (27)

For Type IIA string theory, since the left and the right movers have supercharges with opposite chirality, they can be combined into a single Majorana spinor $Q_\alpha$ with $d = 10, \mathcal{N} = 2$ supersymmetry algebra ($\alpha = 1, 2, \cdots, 32$)
\[ \{Q_\alpha, Q_\beta\} = (C \Gamma^\mu)_{\alpha,\beta} P_\mu \quad \text{IIA} \] (28)

where $P_\mu$ is the $d = 10$ translation operator.

For Type IIB string theory, the left and the right supercharges have the same chirality and yield the chiral $d = 10, \mathcal{N} = 2$ superalgebra ($I = 1, 2, \alpha = 1, 2, \cdots, 32$)
\[ \{Q^+_\alpha, Q^+_\beta\} = \delta^{IJ} (C \Gamma^\mu \Gamma_+)_{\alpha,\beta} P_\mu \quad \text{IIB} \] (29)

with $\Gamma_+ = \frac{1}{\sqrt{2}} (1 + \Gamma_{11})$ and $Q^+_\alpha$ are Majorana-Weyl spinors. One of the many miracles of string theory is that the GSO projection eliminates the tachyon state and at the same time yields spacetime supersymmetry for closed strings.

4 Coupling of RR Fields to D-Branes

We recall that the imposition of Majorana-Weyl condition on the Ramond vacua yields an arbitrary vacuum state of the form
\[ F_{\alpha,\beta}|0, k, \alpha, R >_L |0, k, \beta, R >_R \] (30)
where $\alpha, \beta = 1, 2, \cdots 16$. The bispinor $F_{\alpha,\beta}$ is a $d = 10$ classical background field and specifies the vacua. It can be thought of as the vacuum condensate of the massless states of the RR-sector. Moreover, it can be decomposed in a complete basis of all antisymmetric gamma-matrix products as

$$F_{\alpha,\beta} = \delta_{\alpha,\beta} + \sum_{k=1}^{10} \frac{\gamma_k}{k!} F_{\mu_1 \cdots \mu_k} (\Gamma^{\mu_1 \cdots \mu_k})_{\alpha,\beta} \quad (31)$$

where $\Gamma^{\mu_1 \cdots \mu_k} = \Gamma^{[\mu_1 \cdots \mu_k]}$, and $F_{\mu_1 \cdots \mu_k}$ is an antisymmetric Lorentz tensor. We also recall that the RR-vacua has definite chirality and this implies:

$$\Gamma_{11} F = - F \Gamma_{11} = F \quad \text{Type IIA} \quad (32a)$$
$$\Gamma_{11} F = + F \Gamma_{11} = F \quad \text{Type IIB} \quad (32b)$$

Hence the antisymmetric tensors $F_{\mu_1 \cdots \mu_k}$ are not independent. To write the constraints on $F_{\mu_1 \cdots \mu_k}$, we note

$$\Gamma_{11} \Gamma^{\mu_1 \cdots \mu_k} = \frac{(-1)^{[k/2]} \epsilon^{\mu_1 \cdots \mu_{10}} \Gamma_{\mu_{k+1} \cdots \mu_{10}}}{(10 - k)!} \quad (33a)$$
$$\Gamma^{\mu_1 \cdots \mu_k} \Gamma_{11} = \frac{(-1)^{[(k+1)/2]} \epsilon^{\mu_1 \cdots \mu_{10}} \Gamma_{\mu_{k+1} \cdots \mu_{10}}}{(10 - k)!} \quad (33b)$$

For Type IIA, to satisfy eq(32a), we need $k$ to be even, and similarly for Type IIB, to satisfy eq(32b), $k$ has to be odd. The antisymmetric field tensors satisfy

$$F \Gamma_{11} = - F \quad \text{Type IIA} \quad (34a)$$
$$\text{or,} \quad F_k = - * F_{(10-k)} \quad (34b)$$

with $k$ even and which yields $F^{\mu_1 \cdots \mu_k} = \frac{(-1)^{[(k+1)/2]} \epsilon^{\mu_1 \cdots \mu_{10}} F_{\mu_{k+1} \cdots \mu_{10}}}{(10 - k)!}$. For Type IIB, the analogous equations are

$$F \Gamma_{11} = + F \quad \text{Type IIB} \quad (35a)$$
$$\text{or,} \quad F_k = + * F_{(10-k)} \quad (35b)$$

with $k$ odd.

We next recall that $F_{\alpha,\beta}$ has $16 \times 16$ components. We check that it has the correct components for Type II A and Type II B strings.
4.1 Superconformal Invariance

The super-Virasoro condition $G_0|\text{phys} >= 0$ yields

$$k_{\mu} \Gamma^\mu F = F k_{\mu} \Gamma^\mu = 0$$

(36)

which ultimately leads to two field equations

$$dF = 0 = d* F.$$  

(37)

The classical field equations in eq(37) arise from the condition of superconformal invariance of the RR vacuum. We note that for RR background fields, the free massless equation $dF = 0$ and the Bianchi identity $d* F = 0$ appear on the same footing showing explicit duality.

From Bianchi identity,

$$F_{\mu_1 \cdots \mu_k} = \frac{1}{(k-1)!} \partial_{[\mu_1} A_{\mu_2 \cdots \mu_k]}$$

where $A_{\mu_1 \cdots \mu_k}$ is the completely antisymmetric $U(1)$ gauge field so that $F_{(k)} = dA_{(k-1)}$. Hence we have the following RR-fields

| String Type | Independent Tensors | Number of components |
|-------------|----------------------|----------------------|
| Type II A   | $F_0$, $F_2$, $F_4$  | $1 + 10 \times 9$   |
|             |                      | $2! \times 9 \times 8 \times 7$ |
|             |                      | $4!$ |
|             |                      | Total: 256 |
| Type II B   | $F_1$, $F_3$, $F_5$  | $10 + 10 \times 9 \times 8$ |
|             |                      | $3! \times 9 \times 8 \times 7 \times 6$ |
|             |                      | $2 \times 5!$ |
|             |                      | Total: 256 |

| String Type | RR Background Fields | Number of components |
|-------------|----------------------|----------------------|
| Type II A   | $f^{(0)}$, $A^{(1)}_{\mu}$, $A^{(3)}_{\mu\nu\lambda}$ | 0 |
|             |                      | $8 \times 7 \times 6$ |
|             |                      | $3!$ |
|             |                      | Total: 64 |
| Type II B   | $A^{(0)}$, $A^{(2)}_{\mu\nu}$, $A^{(4)}_{\mu\nu\lambda\delta}$ self-dual | 1 |
|             |                      | $8 \times 7$ |
|             |                      | $2!$ |
|             |                      | $8 \times 7 \times 6 \times 5$ |
|             |                      | $2 \times 4!$ |
|             |                      | Total: 64 |
The sources for the $d = 10$ RR-fields are the $D$-branes. We list the $D$-brane couplings of Type-II strings. Note that in general for tensor field $A^{(p+1)}$ with electric coupling to $D$-$p$ brane, we also have the coupling of the dual $\tilde{A}^{q+1}$ to $D$-$q$ brane via magnetic couplings. Indeed, we have

$$A^{(p+1)} \rightarrow dA^{(p+2)} \rightarrow d\tilde{A}^{(8-p)} \rightarrow \tilde{A}^{(7-p)}$$

so that the tensor field $A^{(p+1)}$ couples to $D$-$p$ brane while its dual $\tilde{A}^{7-p}$ couples to a $D -(6 - p)$ brane. We have the following $D$-brane coupling in $D = 10$ dimensions:

| String Type | Couplings |
|-------------|-----------|
| Type II A   | $F_{(0)} \rightarrow \tilde{F}_{(10)} \rightarrow \tilde{A}^{(9)}$ couples to 8-brane |
|             | $A^{(1)} \mu$ couples to 0-brane |
|             | $\tilde{A}^{(7)}$ couples to 6-brane |
|             | $A^{(3)}$ couples to 2-brane |
|             | $\tilde{A}^{(5)}$ couples to 4-brane |
| Type II B   | $A^{(0)}$ couples to -1-brane |
|             | $\tilde{A}^{(8)}$ couples to 7-brane |
|             | $A^{(2)}$ couples to 1-brane |
|             | $\tilde{A}^{(6)}$ couples to 5-brane |
|             | $A^{(4)}$ couples to 3-brane (self-dual) |

Finally, we summarize everything in a ‘$D$-brane-scan’ [22]:

|  | $0_D$ | $2_D$ | $4_D$ | $6_D$ | $8_D$ |
|---|-----|-----|-----|-----|-----|
| Type II A | $0_D$ | $2_D$ | $4_D$ | $6_D$ | $8_D$ |
| Type II B | $-1_D$ | $1_D$ | $3_D^+$ | $5_D$ | $7_D$ |
| Type I | $1_D$ | $5_D$ |

### 4.2 Massive Spectra of Type II A and II B

Type II A and Type II B strings differ only for the massless states; the massive states are identical. While the massless states can be chiral, the massive spinor states cannot be chiral. For example, at the first excited level in the Ramond sector, we have states of opposite chirality, namely

$$\alpha^i_{-1}|0, k, \alpha, R>, \quad \tilde{d}^{i}_{-1}|0, k, \alpha, R>$$

which combine into a massive representation of a Majorana fermion. This means that the excited massive states are insensitive to the choice of the massless ground states as they should be since massive states have both chiralities.
If we compactify one space dimension into $S^1$ with radius $R_A$ for Type II A and with radius $R_B$ for Type II B, then we can show that the theories possess identical spectrums (including the massless sector) provided

$$R_A R_B = \alpha'.$$

(39)

This symmetry is called $T$-duality.

4.3 Coupling to $D$-branes

All antisymmetric tensor fields $A^{(k)}$ are $U(1)$ gauge fields and hence their charges can be defined by Gauss’ law. The field tensor $A^{(k)}$ couples to a $D - (k - 1)$ brane which has coordinates $X^\mu(\sigma_1, \cdots, \sigma_{k-1}, \tau)$. Let $J^{(k)}$ be the $k$-form tangent to the brane. This acts as the source for the RR-field given by

$$dF^{(k+1)} = \mu_{k-1} * J^{(k)}$$

(40)

where $\mu_{k-1}$ is the charge of the $(k - 1)$-brane. We can define ‘electric’ RR-charge by

$$e_k = \int_{S^{d-k-1}} *F^{(k+1)}$$

(41)

and a dual ‘magnetic’ RR-charge by its coupling to a $D - k - 3$-brane by

$$g_{d-k-2} = \int_{S^{k+1}} F_{k+1}.$$  

(42)

The equivalent Dirac quantization condition is

$$e_k, g_{d-k-2} = 2\pi n$$

(43)

where $n$ is an integer. It turns out that in string theory $n = 1$ for $D$-branes and the charges $e_k$ and $g_p$ are dimensionless only for $d = 2(k + 1)$. Thus

\begin{align*}
d & = 4 \quad k = 1 \quad \text{0-brane (particle)} \\
d & = 6 \quad k = 2 \quad \text{1-brane (string)} \\
d & = 10 \quad k = 4 \quad \text{3-brane}
\end{align*}

In the presence of a background gauge field living on the $k$-brane with field tensor $F = F_{\mu\nu} dx^\mu dx^\nu$, the coupling is given by

$$dF^{(k)} = \mu_{k-1} J^{(k)} * \text{tr} e^{F/2\pi}$$

(44)

which comes from a term in the $D$-brane action of the form

$$\mu_{k-1} \int_{\Sigma_k} c \wedge \text{tr} e^{F/2\pi}$$

(45)
where \( c \) is the differential form given by the sum of the RR-fields. For Type II A theory, for instance,

\[
c = A^{(1)}_\mu dx^\mu + A^{(3)}_{\mu\nu\lambda} dx^\mu dx^\nu dx^\lambda \equiv A^{(1)} + A^{(3)}.
\]

For example, using eq(45), we have \( \text{tr} \int \Sigma^1 A^{(1)} \wedge F \wedge F \) on a 4-brane so that the \( F \wedge F \) term couples to the field \( A^{(1)} \) giving a 0-brane charge equal to \( \left( \int \Sigma^4 F \wedge F \right) \int \Sigma^1 A^{(1)} \).

In other words, by having a non-trivial background gauge field in the 4-brane, we have effectively created a 0-brane embedded within the 4-brane carrying the right charge to couple to the RR field \( A^{(1)}_\mu \).

There is a symmetry in Type II strings which is a \( U(1) \) gauge symmetry \( \delta F = d\Lambda \). To preserve this symmetry we need to modify \( F \) to \( F + B \), where \( B_{\mu\nu} \) is the NS-NS field. The complete action for the single \((k+1)\)-form gauge field \( A^{(k+1)} \) is given by

\[
S = \frac{1}{2} \int \ast F^{(k+2)} \wedge F^{(k+2)} d^{10}x + \mu_k \int \Sigma c \wedge \text{tr} e^{\frac{F+B}{2\pi}}
+ i\mu_k \int \Sigma A^{(k+1)}_{\mu_1\cdots\mu_{k+1}} \partial_1 X^{[\mu_1} \cdots \partial_{k+1} X^{\mu_{k+1}]} d^{k+1} \sigma
\]

where \( \Sigma \) is the \( k \)-brane world volume. The charges satisfy the Dirac quantization condition

\[
\mu_{6-k} \mu_k = 2\pi.
\]

## 5 Bosonic String

Let \( X^\mu(\sigma, \tau) \) be the coordinates of the string in \( d \)-dimension with world volume \( M \). In the conformal gauge, we have \( (\alpha') \)

\[
S = \frac{1}{2\pi} \int_M d^2\sigma \partial^a X^\mu \partial_a X_\mu.
\]

Reparametrization invariance is obtained by imposing the Virasoro conditions on the state space. To obtain the field equations we have

\[
\delta S = -\frac{1}{\pi} \int_M d^2\sigma \delta X^\mu \partial^2 X_\mu + \frac{1}{2\pi} \int_{\partial M} d\tau \delta X^\mu \partial_n X_\mu
\]

where \( \partial_n \) is the derivative normal to \( \partial M \). We choose the boundary conditions so that the boundary term in \( \delta S \) reduces to zero. Consider Cartesian coordinates:
The boundary term with $\partial_n = \frac{\partial}{\partial \sigma} = ' is given by

$$\frac{1}{2\pi} \int \left[ \delta X^\mu X'^\mu(\pi, \tau) - \delta X^\mu X'^\mu(0, \tau) \right] d\tau.$$ 

This gives rise to several cases [24, 25]:

- **Closed String**
  $X_\mu(0, \tau) = X_\mu(\pi, \tau)$, periodic.
  Usually, in this case, one extends $\sigma$ to $0 \leq \sigma \leq 2\pi$.

- **Open String**
  There are two possible boundary conditions:
    - Neumann (NN) b.c.: $X'_0,\pi = 0$
    - Dirichlet (DD) b.c.: $\delta X|_{0,\pi} = 0$

  This gives a total of four possible boundary conditions on the two ends of the open strings, namely NN, ND, DN, DD.

For a $D$-$k$ brane in 10 dimensions,

- (NN) $X'_m(0) = 0 = X'_m(\pi), \ m = 0, 1, \cdots k$
- (DD) $X_i(0) = \text{constant} \quad X_i(\pi) = \text{constant}' \quad i = k + 1, \cdots 9$
The ends of the open strings are free to move in the \((k+1)\)-dimensional world-volume of the \(k\)-brane. The \(D\)-brane is thus a rigid submanifold of spacetime.

For the usual open string which is part of the type I superstring, we have \(k = 9\), i.e. all the components satisfy NN boundary conditions and the ends are free to move throughout spacetime.

What is the consequence of the NN and DD boundary conditions on the fermions? As before, for the open strings, the overall sign is fixed by \(\psi^\mu_+ (\pi, \tau) = \psi^\mu (\pi, \tau)\). To preserve worldsheet supersymmetry, we have

\[
\text{NN: } \psi^m_+(0, \tau) = \pm \psi^m_- (0, \tau) \quad m = 0, 1, \ldots, k
\]

\[
\text{DD: } \psi^i_+(0, \tau) = \mp \psi^i_- (0, \tau) \quad i = k + 1, \ldots, 9
\]

\(D\)-branes are BPS solitonic states and this is a consequence of the b.c’s of the worldsheet fields. Of the two supercharges of Type II theory \(Q^L_\alpha, Q^R_\alpha\), only the linear combination \(Q^L_\alpha + e^{i\phi_k}Q^R_\alpha\) is conserved in the presence of \(D\)-branes, with \(\phi_k\) being the phase coming from parity transformations for the transverse directions \(X^i, i = k, k+1, \ldots, 9\). Since the \(D\)-branes have half the supersymmetry of Type II, they form BPS states.

### 5.1 Spectrum of states

For the open string with NN b.c.’s, we solve \(\partial^2 X^\mu = 0\) to obtain

\[
X^\mu (0, \tau) = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{e^{in\tau}}{n} \alpha_n^\mu \cos(n\sigma)
\]

with \([X^\mu, p^\nu] = i\eta^{\mu\nu}, [\alpha_m^\mu, \alpha_n^\nu] = m \delta_{m+n} \eta^{\mu\nu}\).
The Virasoro operator \((\alpha_0^\mu = p^\mu)\),

\[
L_0 = \frac{1}{2} \alpha^2_0 + \sum_{m=1}^{\infty} \alpha_{-m} \alpha_m
\]

imposes constraint on physical states

\[(L_0 - 1)|\text{phy} >= 0\]

which in turn yields the mass spectrum as

\[M^2 = -p^\mu p_\mu = \sum_{m=1}^{\infty} \alpha_{-m} \alpha_m - 2\]

Define \(|0, k >_B\) by \(\alpha_m|0, k >_B = 0\) for all \(m > 0\). We then have the tachyon vacua and the massless state given respectively by

\[
\begin{align*}
tachyon: & \quad |0, k >_B, \quad M^2 = -1 \\
\text{photon: } & \quad \alpha^-_1|0, k >_B, \quad M^2 = 0
\end{align*}
\]

5.2 Chan-Paton factors

We can attach non-dynamical degrees of freedom to the ends of the open string.

The \(N \times N\) matrices \(\lambda_{ij}^a\) form a basis to decompose the string wave function into an irrep. These matrices are called the Chan-Paton factors [26]. We have the irreducible string state given by

\[
|k, i,j> = \sum_{ij} \lambda_{ij}^a |k, ij >
\]

Unitary demands \(\lambda^\dagger = \lambda\), so that \(\lambda\) is Hermitian. The scattering of two oriented strings appears as:

\[
\text{String state is } |k, ij >
\]
This diagram has a factor $\text{tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4)$. For an oriented string, one can think of the two ends carrying the fundamental $N$ and $\bar{N}$ as:

In order that the massless excitations of the open string can be precisely those of Yang-Mills gauge fields $N \otimes \bar{N} \sim \lambda^a$ since the fields have to be expressed in the adjoint representation of the Lie group, $G$. There are only three solutions for consistent open strings given by:

$$G = U(N) \quad \bar{N} = N^* \quad \text{orientable strings}$$

$$G = SO(N) \quad \bar{N} = N \quad \text{non-orientable strings}$$

$$G = Sp(N) \quad \bar{N} = N \quad \text{non-orientable strings}$$

We will later show that in $d = 10$ only the $SO(32)$ case is consistent. Hence we have to analyze the non-orientable strings.

The $D$-brane interpretation is that there are exactly 32 degenerate 9-branes if the open superstring is to avoid tadpole singularities. The subscripts $i, j$ in $\lambda_{ij}$ indicates on which of the degenerate 9-branes the two endpoints are on.
5.3 Non-oriented strings

For the case above in which the indices $i$ and $j$ are equivalent, we can associate a vertex operator $V_\Lambda(k, \tau)$ for every physical state $\Lambda$ with momentum $k^\mu$ by

$$V_\Lambda(k, \tau) = e^{i\tau L_0} V_\Lambda(k, 0) e^{-i\tau L_0}.$$  

This vertex operator describes the emission of the state $\Lambda$ from the $\sigma = 0$ end of the open string. For example, for the string propagator $\Delta = (L_0 - 1)^{-1}$, the amplitude $< 1 | V_2(k_2) \Delta V_3(k_3) \Delta \cdots V_{M-1}(k_M) | M >$ is represented by:

For non-orientable open string, emission from $\sigma = 0$ edge should be symmetric with emissions from $\sigma = \pi$ edge. This entails the requirement that the vertex operators be built from $X(\sigma = \pi, \tau)$ instead of $X(\sigma = 0, \tau)$. To achieve this, we introduce the ‘twist operator’ $\Omega$ which maps $\sigma$ to $\pi - \sigma$, so that

$$V(\sigma = \pi) = \Omega V(\sigma = 0) \Omega^{-1}.$$  

Clearly, $\Omega^2 = 1$. Replacing $\sigma = 0$ by $\sigma = \pi$ for the open string can be achieved by a change of variables $\sigma \rightarrow \pi - \sigma = \sigma'$; hence

$$X(\sigma', \tau) = x^\mu + p^\mu \tau + i \sum_n \frac{e^{-in\tau}}{2n} \alpha_n^{\mu} [e^{-in\pi} e^{i\sigma} + e^{+in\pi} e^{-i\sigma}].$$
\[
\begin{align*}
\Omega X(\sigma, \tau) \\
\Rightarrow \Omega \alpha_n &= (-1)^n \alpha_n, \quad \Omega x^\mu = x^\mu \\
\Omega p^\mu &= p^\mu, \quad \Omega(\sigma) = \pi - \sigma
\end{align*}
\]

Hence \( \Omega = (-1)^N B \), where \( N_B \) is a bosonic number operator with \([N_B, \alpha_n] = n\alpha_n\).

The vacuum has \( \Omega|0, k >_B = +|0, k >_B \). In terms of these operators, for the photon, we have
\[
\Omega \alpha^\mu_{-1}|0, k > = -\alpha^\mu_{-1}|0, k >
\]

The world sheet parity operator acts non-trivially on the Chan-Paton factors.
\[
\Omega \lambda_{ij}|k, ij > = \lambda'_{ij}|k, ij >
\]

where \( \lambda' \equiv M\lambda^T M^{-1} \). Since \( \Omega^2 = 1 \), we have in general
\[
\lambda^a = MM^{-T}\lambda^aT M^T M^{-1}.
\]

Since \( \lambda^a \) must form a complete set by CPT, we have by Schur’s lemma \( M^T M^{-1} = \pm 1_N \). There are two possibilities:

1. \( M^T = +M = I_N \)
   - tachyon: \( \Omega \lambda_{ij}|0, k, ij > = \lambda^T_{ij}|0, k, ij > \)
   - photon: \( \Omega \lambda_{ij}\alpha^\mu_{-1}|0, k, ij > = -\lambda^T_{ij}\alpha^\mu_{-1}|0, k, ij > \)

2. \( M^T = -M = i \begin{pmatrix} 0 & I_{N/2} \\ I_{N/2} & 0 \end{pmatrix} \equiv J \)
   - tachyon: \( \Omega \lambda_{ij}|0, k, ij > = (J\lambda^T J)_{ij}|0, k, ij > \)
   - photon: \( \Omega \lambda_{ij}\alpha^\mu_{-1}|0, k, ij > = -(J\lambda^T J)_{ij}\alpha^\mu_{-1}|0, k, ij > \)

For non-orientable strings, invariance under \( \Omega \) is a symmetry of the action and all amplitudes. To obtain a Hilbert space of states invariant under \( \Omega \), we take the oriented string spectrum and project out states with the projector \( P = \frac{1}{2}(1 + \Omega) \), with \( P^2 = P \).

For the open string with Chan-Paton charges, the operator \( P \) will project out the tachyon and retain the photon if we demand for:

Case 1 \( \lambda^T = -\lambda \) i.e. \( SO(N) \) gauge group
Case 2 \( J\lambda^T J = -\lambda \) i.e. \( Sp(N) \) gauge group

Nonorientable worldsheets arise in the following way. For every oriented string amplitude with vertex operators, the one loop diagram gives a sum over a complete set of intermediate states I.
But when the projector $\frac{1}{2}(1+\Omega)$ is inserted into the trace it switches the emission operators from $\sigma = 0$ to $\sigma = \pi$.

\[\sum_{\text{States I}}\]

5.4 Type I Superstring

Type I is constructed from open strings (both oriented and non-oriented) as well as the closed string sector. The Hilbert space $\mathcal{F}_{\text{open}}$ is $\Omega$-symmetric, i.e.

$$\Omega \mathcal{F}_{\text{open}} = \mathcal{F}_{\text{open}}$$

In GSO projected RNS superstring, the Type I superstring sector yields an $\mathcal{N} = 1$ supersymmetric Yang-Mills multiplets in $d = 10$ with $SO(32)$ being the only consistent gauge group. Thus

\[
\begin{align*}
\text{NS} : & \quad 8 \times 496 \quad A_\mu \\
\text{R} : & \quad 8 \times 496 \quad \psi^a
\end{align*}
\]

\text{space – time susy}
For the closed string sector, the action of $\Omega$ is to switch the left and right movers. To be symmetric under $\Omega$, the left and right movers must have the same parity. Hence, we must start with IIB and retain only states invariant under $\Omega$, i.e.

$$F_{\text{open, closed sector}} = F_{\text{IIB}} / \Omega.$$ 

On making this projection, we obtain the Type I closed string sector with space-time supersymmetric spectrum given by

| Type          | Number | Description     | Contributions |
|---------------|--------|-----------------|---------------|
| NS-NS bosons  | 35     | $G_{\mu\nu}$   | 1             |
| R-R bosons    | 28     | $A^{(2)}_{\mu\nu}$ |              |
| NS-R fermions | 56     | $\chi^\alpha_\mu$ |              |
| R-NS fermions | 8      | $\lambda_\alpha$  |               |
| Total         | 64     |                 |               |

From the RR-sector, we see that the field $A^{(2)}_{\mu\nu}$ in Type I couples to a D1-brane, which is dual to a D5-brane. It will turn out that it also couples to a D9-brane.

6 Background Fields: Supergravity

Recall that the Hilber spaces of Type I and the Heterotic strings yield an irrep of $d = 10, \mathcal{N} = 1$ spacetime supersymmetry, whereas Type II strings yield an irrep of $d = 10, \mathcal{N} = 2$ supersymmetry. We now study what are the allowed background fields for the various superstring theories. We will see that in the limit $\alpha' \to \infty$, spacetime supersymmetry will completely determine the allowed string vacua corresponding to different background field configurations. All the background fields are the result of the condensation of the massless excitations of the superstrings, since these condensates can be added to the action without

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violating superconformal invariance. Condensates of massive excitations have a mass scale and are therefore disallowed as they violate superconformal invariance.

Let us start with the bosonic string. Instead of expressing the world sheet action for the string in the conformal gauge as in eq(1), we write the action, without any particular choice of gauge, in the Polyakov formulation. In addition to dynamical quantum fields $X_{\mu}(\sigma, \tau)$, we have a dynamical world sheet metric $\gamma_{\alpha,\beta}$.

Let $g_{\mu\nu}(X)$ be the condensate of spin 2 excitations of the string, $B_{\mu\nu}(X)$, an anti-symmetric background field and $\phi(X)$, the dilaton. Taken together, $g_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\phi(X)$ are the $d=10$ Neveu-Schwarz (NS) background fields.

The fundamental open string $X_{\mu}$ couples to these background fields and the coupling is given by (restoring $\alpha'$)

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma}\left\{ \gamma^{ij} g_{\mu\nu}(X) \partial_i X^\mu \partial_j X^\nu + \epsilon^{ij} B_{\mu\nu}(X) \partial_i X^\mu \partial_j X^\nu + \alpha' R^{(2)}(\gamma) \phi(X) \right\}$$

(46)

where $R^{(2)}(\gamma)$ is the world sheet Ricci scalar curvature.

The quantum field theory is defined by

$$Z = \int D\gamma DX e^{S[X,\gamma]}$$

(47)

On performing the path integration, one finds that if one demands that the theory have superconformal invariance, this implies that all the beta functions must be zero. That is,

$$\beta_g = 0 = \beta_B = \beta_\phi$$

(48)

To lowest order (one-loop) in $\alpha'$, one finds that the background fields $g_{\mu\nu}$, $B_{\mu\nu}$ and $\phi$, together with some additional fields, must satisfy the classical field equations of $d=10$, $\mathcal{N}=1$ supergravity! In other words, as seen earlier for the case of RR-fields, superconformal invariance demands that the background fields must satisfy certain classical field equations which in turn can be obtained from an effective lagrangian field theory.

For concreteness, we analyze Type IIA in some detail. The irrep of the supersymmetry algebra for IIA given in eq(28) yields the following bosonic fields

$$g_{\mu\nu}, \phi, B_{\mu\nu} \quad \text{NS-NS}$$

IIA

$$A_{\mu}^{(1)}, A_{\mu\nu}^{(3)} \quad \text{RR}$$
The fermionic superpartners come from the $NS - R$ sector. The bosonic component of the non-chiral $\mathcal{N} = 2$ supergravity lagrangian, in the string frame, is given by

$$S_{IIA} = \int d^{10}X \sqrt{-g} \left\{ e^{-2\phi} \left[ R + 4|d\phi|^2 - \frac{1}{3}|H|^2 \right] - |K|^2 - \frac{1}{12}|G|^2 \right\} + \frac{1}{144} \int d^{10}X G \wedge G \wedge B \quad (49)$$

where $H = dB$, $K = dA^{(1)}$ and $G = dA^{(3)} + 12B \wedge K$. Note the significant fact that the $RR$-fields had no direct coupling to the fundamental string $X_\mu$, and they were ‘dragged into’ the action purely by demanding supersymmetry. These $RR$-fields, of course, exist in closed superstring theory, and are excited by the presence of $D$-branes in the ground state of the theory.

Actions similar to $S_{IIA}$ exist for the other consistent string theories. For Type IIB, we have the background fields given by

$$g_{\mu\nu}, \phi, B_{\mu\nu} \quad \text{NS-NS (as in Type IIA)}$$

IIB

$$A^{(0)}, A^{(2)}, A^{(4)+}_{\mu\nu\lambda\delta} \quad \text{RR}$$

and the $\mathcal{N} = 2, d = 10$ effective chiral supergravity action can be expressed as

$$S_{IIB} = \int d^{10}X \sqrt{-g} \left\{ e^{-2\phi} \left[ R + 4|d\phi|^2 - \frac{1}{3}|H|^2 \right] - 2|dA^{(0)}|^2 - \frac{1}{3}|H' - A^{(0)}H|^2 - \frac{1}{60}|M^+|^2 \right\} - \frac{1}{48} \int d^{10}XC^+ \wedge H \wedge H' \quad (50)$$

where $H = dB$, $H' = dA^{(2)}$ and the full non-linear Bianchi identity satisfied by the self-dual 5-form $M^+$ is now $dM^+ = H \wedge H'$ with $M^+ = dA^{(4)+}$. For Type I action, the background fields are

$$g_{\mu\nu}, \phi \quad \text{NS-NS}$$

Type I

$$A^{(1)}_\mu \quad \text{YM 1-form}$$

$$A^{(2)}_{\mu\nu} \quad \text{RR}$$

so that the bosonic sector of the Type I effective action is

$$S_I = \int d^{10}X \sqrt{-g} \left\{ e^{-2\phi} \left[ R + 4|d\phi|^2 \right] - e^{-\phi} \text{tr}[\mathcal{F}^2] - \frac{1}{3}|H'|^2 \right\} \quad (51)$$

where $\mathcal{F}$ is the YM 2-form and $H' = dA^{(2)}$ as before.

There are two important applications of the effective field theory for the background fields. One direction is compactification based on the classical solutions of eq(49)- eq(51) for which all the background fields are zero except for a constant dilaton and the metric which has the form $M_4 \times K_6$. Note that $M_4$ is 4-dimensional Minkowski space and $K_6$ is some 6-dimensional (compact) space.
with Euclidean signature. If one demands \( d = 4, \mathcal{N} = 1 \) supersymmetry for the state space \( \mathbb{R}^4 \) on \( M_4 \), it can be shown that the manifold \( K_6 \) must be Calabi-Yau space.

The other direction is to look for solitonic solutions of eq(49)–eq(51). In fact, it is known that the \( Dp \)-branes are solitonic solutions of the supergravity lagrangian. The solution for an arbitrary \( Dp \)-brane with the brane volume in the \((1, 2, \cdots, p)\) directions in \( M \) is given by

\[
ds^2 = f^{-1/2}(-dt^2 + dx_1^2 + \cdots + dx_p^2) + f^{1/2}(dx_{p+1}^2 + \cdots + dx_q^2) \tag{52}\]

with

\[
e^{-2\phi} = f^{p-3/2} \tag{53a}
\]
\[
A_{(p)}^{(p)} = -\frac{1}{2}(f^{-1} - 1) \tag{53b}
\]
\[
f = 1 + \frac{N_c}{r^{7-p}} \tag{53c}
\]
\[
r^2 = x_{p+1}^2 + \cdots + x_q^2 \tag{53d}
\]
\[
c = \frac{(2\pi \sqrt{\alpha'})^{7-p}}{(7-p)\Omega_8^{-p}}g_s \tag{53e}
\]
\[
\Omega_q = \frac{2\pi^{(q+1)/2}}{\Gamma[(q+1)/2]} \tag{53f}
\]

Note that for \( p = 3 \), the dilaton decouples from the D3-brane. The RR-field \( A^{(p)} \) is excited since the D \( p \)-brane carries RR-charge equal to \( N \). This solitonic solution is a BPS–state with mass/volume = \( N \). A black \( p \)-brane is a non-extremal black hole with two horizons. The D \( p \)-brane solution given above in eq(52) is a BPS-state since it is a charged object having the structure of an extremal black hole in that both the horizons have coalesced. On can also solve for a D3-brane on the space \( AdS_5 \times S^5 \).

7 T-Duality

For the closed string, we have

\[
\partial^2 X^\mu = 0
\]
with the boundary condition \( X^\mu(\sigma + 2\pi) = X^\mu(\sigma, \tau) \) and yields normal mode expansion.

\[\text{More details regarding this aspect can be found in the discussion on F-theory}\]
\[ X^\mu(\sigma, \tau) = x^\mu + p^\mu \tau + \frac{i}{2} \sum_n \frac{1}{n}(\alpha_n e^{-in(\tau-\sigma)} + \tilde{\alpha}_n e^{-in(\tau+\sigma)}) \] (54)

\[ = x^\mu_L + x^\mu_R + \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu(\tau - \sigma) + \sqrt{\frac{\alpha'}{2}} \tilde{\alpha}_0^\mu(\tau + \sigma) + \text{oscillators} \] (55)

\[ = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma) \] (56)

where

\[ p^\mu = \sqrt{\frac{1}{2\alpha'}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)} \]

Under \( \sigma \rightarrow \sigma + 2\pi \), \( X^\mu(\sigma, \tau) \) changes by \( \sqrt{\frac{\alpha'}{2}}(\tilde{\alpha}_0^\mu - \alpha_0^\mu)2\pi \). For non-compact spatial dimension \( X^\mu \) is single-valued and hence

\[ \tilde{\alpha}_0^\mu = \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu \]

Suppose \( X^{25} \) in a circle with radius \( R \). Then, under a shift of \( \sigma \rightarrow \sigma + 2\pi \), we can have

\[ X^{25}(\sigma + 2\pi, \tau) = X^{25}(\sigma, \tau) + 2\pi Rw \]

where \( w \) is the winding number \( \in \mathbb{Z} \).

Hence

\[ \alpha_0^{25} + \tilde{\alpha}_0^{25} = \frac{2n}{R} \sqrt{\frac{\alpha'}{2}} \] (57)

\[ \alpha_0^{25} - \tilde{\alpha}_0^{25} = wR \sqrt{\frac{2}{\alpha'}} \] (58)

where the Kaluza-Klein (KK) momentum is \( p^{25} = \frac{n}{R} \). Furthermore, we have

\[ \alpha_0^{25} = \left( \frac{n}{R} + \frac{wR}{\alpha'} \right) \sqrt{\frac{\alpha'}{2}} \] (59)

\[ \tilde{\alpha}_0^{25} = \left( \frac{n}{R} - \frac{wR}{\alpha'} \right) \sqrt{\frac{\alpha'}{2}} \] (60)

The mass spectrum is, putting back \( \alpha' \),

\[ M^2 = -p^\mu p_\mu = \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(L_0 - 1) \] (61)

\[ = \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(\tilde{L}_0 - 1) \] (62)

Note

\[ M^2(n, w, R) = M^2(w, n, R' = \frac{\alpha'}{R}) \]
i.e. identical mass spectrum for $n \leftrightarrow w, R \leftrightarrow \frac{\alpha'}{R}$. Under this transformation

$$\begin{align*}
\alpha_{0}^{25} & \rightarrow \alpha_{0}^{25} \\
\tilde{\alpha}_{0}^{25} & \rightarrow -\tilde{\alpha}_{0}^{25}
\end{align*}$$

(63) (64)

In other words, the $T$-dual theory is described by a parity transformation on left-movers, i.e.

$$T_{25}[X^{25}] = X'^{25}(\sigma, \tau) = X^{25}_{R}(\tau - \sigma) - X^{25}_{L}(\tau + \sigma)$$

$T$-duality is an exact symmetry of string theory. What this means is that the Hilbert spaces of the two $T$-dual theories are identical. In general, for $k$-compact directions we have $T_{\mu_{1}, \cdots, \mu_{k}}$ for the $T$-duality transformation \([27, 28]\).

The effective coupling in the $25^{th}$-dimension is $e^{\phi}R^{-1/2}$ ($\phi$: dilaton). Hence in the $T$-dual theory,

$$e^{\phi'}R'^{-1/2} = e^{\phi}R^{-1/2}$$

or,

$$e^{\phi'} = \sqrt{\alpha'}e^{\phi}R^{-1}$$

(65) (66)

### 7.1 Open Strings and $D$-branes

Recall with NN-conditions, we have for the open string,

$$X^{\mu}(\sigma, \tau) = x^{\mu} + p^{\mu}\tau(2\alpha') + \frac{i}{2} \sum \frac{\alpha^{\mu}}{n} (e^{-in(\tau-\sigma)} + e^{-in(\tau+\sigma)})$$

$$= X^{\mu}_{R}(\tau - \sigma) + X^{\mu}_{L}(\tau + \sigma).$$

(67) (68)

For the open string, left and right movers are reflected at the ends and form standing waves. We have

$$X^{\mu}_{R}(\tau - \sigma) = \frac{x^{\mu}}{2} + c + \alpha'(\tau + \sigma)p^{\mu} + \frac{i}{2} \sum \frac{\alpha^{-in(\tau-\sigma)}}{n} \alpha^{\mu}_{n}$$

$$X^{\mu}_{L}(\tau - \sigma) = \frac{x^{\mu}}{2} - c + \alpha'(\tau - \sigma)p^{\mu} + \frac{i}{2} \sum \frac{\alpha^{-in(\tau+\sigma)}}{n} \alpha^{\mu}_{n}$$

(69) (70)

Suppose $X^{25}$ is compact with radius $R$; this implies $p^{25} = \frac{n}{R}$. For closed strings, $T$-duality was a symmetry because we could interchange windings of the closed string in $X^{25}$ namely $w$ with the $KK$-modes due to the compactness of $X^{25}$. For the open string there is no winding number. So how do we obtain $T$-duality?

As in the case of closed strings, we define the $T$-duality for open string by

$$T_{25}[X^{25}] = X'^{25}(\sigma, \tau) \equiv X^{25}_{R}(\tau - \sigma) - X^{25}_{L}(\tau - \sigma)$$
or, \( X^{25}(\sigma, \tau) = 2c + 2\alpha' p^{25} \sigma - \sum_n e^{-in\tau} \sin(n\sigma) \alpha_n^{25} \). Note at the boundaries \( \sigma = 0, \pi \), the oscillators terms disappear and since there is no other dependence on \( \tau \), the boundaries do not move! By \( T \)-duality we have switched the \( NN \)-condition \( \frac{\partial X^{25}}{\partial \sigma}|_{0,\pi} = 0 \) to \( DD \)-condition \( \frac{\partial X^{25}}{\partial \tau}|_{0,\pi} = 0 \)!

Thus

\[
\begin{align*}
X^{25}(0, \tau) &= 2c \\
X^{25}(\pi, \tau) &= 2c + 2\pi \alpha' p^{25} \tag{72}
\end{align*}
\]

(73)

From eq(73), we see that in the dual theory with \( R' = \alpha'/R \), the \( X^{25} \) ends are fixed, i.e. \( X^{25}(0, \tau) = X^{25}(\pi, \tau) \) mod \( 2\pi R' \).

Since the \( X^{25} \) boundaries cannot move, it is meaningful if the \( X^{25} \) coordinates \textbf{winds} about the 25\textsuperscript{th} direction since the \( DD \)-conditions prevent it from un-winding.

We see that open strings have \( T \)-duality symmetry in a manner very different from closed strings. Instead of interchanging winding number with \( KK \)-momenta as is the case for closed strings, for open strings the \( DD \)-boundary conditions effectively switch the \( KK \)-momenta into a winding number in the dual theory!

### 7.2 \( R \to 0 \) Limit

For closed strings, as \( R \to 0 \) only the \( n = 0 \) \( KK \) mode survives and we have

\[
\lim_{R \to 0} M^2 = \frac{2}{\alpha'} \left( \frac{n}{R} + \frac{w R}{\alpha'} \right)^2 + \frac{4}{\alpha'} (L_0 - 1) \tag{74}
\]
\[
\rightarrow \frac{2}{\alpha'^3}(wR)^2 + \frac{4}{\alpha'}(L_0 - 1) \tag{75}
\]

\[
= \frac{2}{\alpha'^3}W^2 + \frac{4}{\alpha'}(L_0 - 1) \tag{76}
\]

where \( W = \omega R \in [-\infty, +\infty] \) is the effect on the mass spectrum due to the compact direction. This effect is a purely string effect since as \( R \to 0 \) the closed string finds it energetically more and more favorable to wind around a small \( R \) and winding number \( w \) becomes arbitrarily large yielding real continuous variable \( W = wR \): a new continuum of quantum states labeled by \( W \).

For open strings, as \( R \to 0 \), the KK momenta is \( n = 0 \) and hence in the dual picture winding number is also zero. Unlike the closed string, there is no new continuum of states. Recall

\[
X'^{25}(\pi) - X'^{25}(0) = 2\pi nR' \to 0
\]

That is, the open string is only free to move in \((d-1)\)-dimensions as \( R \to 0 \) unlike closed strings which continue to vibrate in \( d \)-dimensions even as \( R \to 0 \).

### 7.3 Wilson Lines, Open Strings and T-Duality

Recall for open strings we have non Abelian gauge fields \( A^a_\mu(x) \) in the expansion for the open string.

\[
A^a_\mu(x) = a_{ij}^a \alpha^\mu_{-1 | k, ij >}
\]

For compact direction \( X'^{25} \) we can consider the gauge field component \( A^{25}_{ij} = A^{a25}_2 \lambda^a_{ij} \) to be a background gauge field. Consider a \( U(N) \) oriented string with end points carrying \( N \) and \( \bar{N} \) irreps. One can consider Wilson line by

\[
P e^{i \int_{X'^{25}(0)}^X A^{25}_{ij} dx^{25}}
\]
with the background field in a fundamental irrep

\[
A^{25} = \frac{1}{2\pi R} \begin{pmatrix}
\theta_1 \\
\vdots \\
\theta_N
\end{pmatrix} = -i\Lambda^{-1} \frac{\partial \Lambda}{\partial x^{25}}
\]

which breaks the gauge symmetry from \( U(N) \) to \( U(1)^N \).

We have in the fundamental representation,

\[
\Lambda = \text{diag}(e^{i \frac{X^{25}_1 \theta_1}{2\pi R}}, \cdots, e^{i \frac{X^{25}_N \theta_N}{2\pi R}})
\]

We can set \( A^{25} \) to zero by a gauge transformation \( \Lambda \). We also have to simultaneously gauge transform the state by

\[
|i j \rangle \rightarrow \Lambda^{-1}(X^{25}(0))\Lambda_{ij}(X^{25}(\pi))|i j \rangle
\]

For the string state

\[
|\text{string} \rangle = e^{ip^{25}X^{25}}|i j \rangle
\]

and under \( X^{25} \to X^{25} + 2\pi R \), \( |i j \rangle \to e^{i(\theta_j - \theta_i)}|i j \rangle \). Hence \( p^{25} = \frac{1}{2\pi R}(\theta_j - \theta_i + 2n\pi) \). In other words, a Wilson line formed by breaking \( A^{25} \to < A^{25} > = -i\Lambda^{-1}\partial_{25}\Lambda \) is effectively imparting fractional KK-momenta to the open string given by \( p^{25} \).

Recall under \( T \) duality

\[
X'^{25}(\pi) - X'^{25}(0) = 2\pi \alpha' p^{25}
\]

\[
= \frac{\alpha'}{R}(2\pi n + \theta_j - \theta_i)
\]

The state of the string at \( \sigma = 0 \) should depend only on \( i \) and at \( \sigma = \pi \) on only \( j \) (otherwise we would violate locality). Therefore,

\[
X'^{25}(\pi) = 2\pi n R' + \theta_j R'
\]

\[
X'^{25}(0) = \theta_i R'
\]

In other words, for \( \theta_i \neq \theta_j \), in the dual spacetime, the open string endpoint for state \( i \) is located on a \( D \)-brane placed at \( \theta_i R' \) along \( X'^{25} \) and the other endpoint is located on a \( D \)-brane at \( \theta_j R' \) (both ends located modulo \( 2\pi R' \)).
A Wilson line introduces, in the dual theory, $D$-branes located at points inside the compact dimension $X'^{25}$.

8 D-Brane Dynamics and Gauge Fields

In this section, we look into the fluctuations of D-p branes [29–31]. We recall that for the case of open strings carrying fractional momenta $p^{25}$, the $T$-dual of the compact coordinate yields

$$M^2 = (p^{25})^2 + \frac{1}{\alpha'}(L_0 - 1)$$

$$= \left\{ \frac{R'}{2\pi\alpha'} [2\pi n + (\theta_i - \theta_j)] \right\}^2 + \frac{1}{\alpha'}(L_0 - 1)$$

Massless states can only arise for $n = 0$, i.e. for non-winding strings whose ends are on the same hyperplane ($\theta_i = \theta_j$) shown below. Massive string states are given by open strings stretching between $D$-branes with mass of the state given by the product of the string tension and the length of the string.
Continuing in the $T$-dual picture, when none of the $N$ D-branes coincide, there is just one massless $U(1)$ vector field in each D-brane with $U(1)^N$ being the unbroken group. If $m < n$ of the D-branes coincide, i.e. $\theta_1 = \theta_2 = \cdots = \theta_m$, there are $m^2$ massless vectors with $m^2 - m$ new massless states since open strings stretched between these branes have zero length. These $m^2$ massless vectors form the adjoint representation of a $U(m)$ gauge theory. This is a reflection of $U(m)$ subgroup being unbroken by the Wilson line.

It has been shown by Witten that $N$ parallel coinciding $D-p$ branes have low energy excitations described by a $U(N)$ Susy YM-theory dimensionally reduced from $d = 10$ to $p + 1$ dimensions.

### 8.1 Super Yang-Mills from $D$-Branes

Let $\psi$ be Majorana-Weyl spinor of $SO(1,9)$ acted on by $32 \times 32$ gamma matrices; let $A_\mu$ be the $U(N)$ gauge field given by an $N \times N$ Hermitian matrix. We then have the YM Lagrangian

$$S = \int d^{10}\xi ( - \frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\psi} \Gamma^\mu D_\mu \psi)$$

This action is invariant under supersymmetric transformation

$$\delta A_\mu = \frac{i}{2} \epsilon \Gamma_\mu \psi$$

$$\delta \psi = - \frac{1}{4} F_{\mu\nu} \Gamma^{[\mu} \Gamma^{\nu]} \epsilon$$ (85)

Moreover, 10-$D$ SYM has fields:
\( A_\mu \): 8 bosonic degrees of freedom.
\( \psi \): 8 fermionic degrees of freedom.
\( Q_\alpha \): 16 supercharges.

Consider \( N \) parallel \( D-p \) branes. To describe the \( D-p \) branes low energy dynamics, we make all the fields independent of the \( 9-p \) transverse (to the \( D-p \) brane) directions. The transverse oscillations of the \( N \) parallel \( D-p \) branes is described by the \( A_\alpha \) field.

Next, we notice that
\[
A_\alpha(\xi), \quad \alpha = 1, 2, \cdots, p+1: \text{propagate only inside the } D-p \text{ brane and}
\]
\[
A_i \equiv X_i(\xi), \quad i = p+2, \cdots, 10: \text{transverse position of the } N \text{ parallel } D-p \text{ brane move in the } 9-p \text{ transverse dimensions.}
\]

Ignoring fermions, we have
\[
S = \frac{1}{4g^2} \int d^{p+1}\xi \{-F_{\alpha\beta}F^{\alpha\beta} - (D_\alpha(A)X^i)^2 + [X^i, X^j]^2\}
\]

### 8.2 Classical Vacua

- Fermions vanish.
- \( X^i \)'s are constant and
- \( [X^i, X^j] = 0 \)

Thus we can simultaneously diagonalize all the \( X^i \)'s,
\[
X^i = \text{diag}(x_1^i, x_2^i, \cdots, x_N^i), \quad i = 1, 2, \cdots, k.
\]
or

where $\vec{x}_k$ is the (transverse) position of the $k$-th parallel $D-p$ brane. Configuration space is $(\mathbb{R}^{9-p})^N/S_N$ due to permutation symmetry (Branes are identical-bosons).

If $m$ of the branes are coincident $\vec{x} = \vec{x}_1 = \vec{x}_2 = \cdots = \vec{x}_m$ and

$$\vec{X} = \text{diag} (\vec{x}, \cdots, \vec{x}, \vec{x}_{m+1}, \cdots, \vec{x}_N)$$

It is instructive to look into an example. Consider $N$ 0-branes. In gauge $A_0 = 0$, we have $A^a \rightarrow X^a$, $a = 1, 2, \cdots, 9$. Hence,

$$L = \frac{1}{2\varrho^2} \{ \dot{X}^a \dot{X}_a + \sum_{a<b} \text{tr}[X^a, X^b]^2 + 2\theta^T(\dot{\theta} + \Gamma_a[X^a, \theta]) \}$$

where $\theta$ is 16-component real spinor.

The classical vacua is given by $[X^a, X^b] = 0$ where

$$\vec{X} = \text{diag} \left( \vec{X}_1, \vec{X}_2, \cdots, \vec{X}_N \right) : \text{positions of the } N0\text{-branes}$$
Configuration space = \((\mathbb{R}^9)^N/S_N\).

Off-diagonal \(X^a\) give a realization of non-commutative geometry.

### 8.3 T-Duality and Super Yang-Mills Theory

Consider strings on a circle of radius \(R\) giving an \(S^1 \times \mathbb{R}^9\) spacetime.

\(T\)-duality maps Neumann ⇔ Dirichlet.

Consider a \(D-p\) brane. Under \(T\)-duality we have

\[
\begin{array}{c}
p+1 \text{ brane if } S \notin \text{p-brane} \\
p \text{-} 1 \text{ brane if } S \in \text{p-brane}
\end{array}
\]

How do we realize \(T\)-duality for Super-Yang-Mills? Consider for simplicity \(N\) 0-branes. What is the realization for \(X^a, a = 1, 2, \ldots, 9\) for space \(S^1 \times \mathbb{R}^9\)? \((A_0 = 0\) gauge\) We construct \(S^1 = \mathbb{R}/2\pi R\mathbb{Z} = \mathbb{R}/\Gamma\).

The \(N \times N\) matrices \(X^a\) are now realized as infinite dimensional matrices which are infinitely many copies of \(X^a\) in \(\mathbb{R}^{10}\); we will quotient these matrices by \(\Gamma\) to obtain \(S^1 \times \mathbb{R}^9 = \mathbb{R}^{10}/\Gamma\).

Organize the \(\infty \times \infty\) matrices \(X^a\) into an infinite collection of \(N \times N\) matrices each labeled by \(m, n \in \mathbb{Z}\), i.e.

\[
X^a \rightarrow \{X^a_{mn}\}
\]
namely,
\[
X^a = \begin{pmatrix}
\vdots & \vdots \\
\cdots & X^a_{01} & X^a_{02} & \cdots \\
\cdots & X^a_{11} & X^a_{12} & \cdots \\
\vdots & \vdots 
\end{pmatrix}
\]

Suppose \( X^1 \) corresponds to \( S^1 \). Then quotient by \( \Gamma \) leads to the following symmetry
\[
X^i_{mn} = X^i_{(m-1)(n-1)}, \quad i > 1 \tag{87}
\]
\[
X^1_{mn} = X^1_{(m-1)(n-1)}, \quad m \neq n \tag{88}
\]
\[
X^1_{nn} = 2\pi RI + X^1_{(n-1)(n-1)} \tag{89}
\]

From above it follows that all information is contained in the \( N \times N \) matrices \( X^a_{0n} \equiv X_n^a \) with \( (X^a_n)^\dagger = X^a_n \). Using notation \( X_k = X^1_{0k} \) we have
\[
X^1 \sim M = \begin{pmatrix}
\cdots & X_1 & X_2 & X_3 & \cdots \\
X_{-1} & X_0 - 2\pi R & X_1 & X_2 & X_3 & \cdots \\
X_{-2} & X_{-1} & X_0 & X_1 & X_2 & \cdots \\
\cdots & X_{-2} & X_{-1} & X_0 + 2\pi R & X_1 & \cdots 
\end{pmatrix}
\]

\( T \)-duality is now defined by the transformation
\[
X^1 \rightarrow M = \sum_{n \in \mathbb{Z}} e^{inx/R} X^1_n \tag{90}
\]
\[
X^i \rightarrow X^i = \sum_{n \in \mathbb{Z}} e^{inx/R'} X^i_n, \quad \text{for } i > 1. \tag{91}
\]
and \( M = i\partial^1 + A^1 \) since by applying Fourier transformation to \( i\partial^1 \) we have
\[
i\partial^1 = \text{diag}(\cdots,-4\pi R,-2\pi R,0,2\pi R,4\pi R,\cdots).
\]

This is equivalent to the following for \( T \)-duality.
\[
X^\alpha \leftrightarrow (2\pi\alpha')(i\partial^\alpha + A^\alpha), \quad \alpha: \text{compact directions} \tag{92}
\]
\[
X^\beta \leftrightarrow X^\beta, \quad \beta: \text{noncompact directions} \tag{93}
\]

Consider the following example.
\[
A^1 = \frac{1}{2\pi R} \text{diag} (\theta_1, \cdots, \theta_N)
\]

Under \( T \)-duality
\[
2\pi\alpha' \left( i\partial^1 + A^1 \right) \rightarrow \frac{2\pi\alpha'}{2\pi R} \text{diag} (\theta_1, \cdots, \theta_N) = X^1
\]
That is, $\partial^1 \equiv 0$ as this has been dualized and lies inside the 1-brane and

$$X^1 = N \text{ positions of the } D1 - \text{branes} = \text{diag } (\theta_1 R', \ldots, \theta_N R')$$

On taking the $T$-dual of the 0-brane we have obtained a 1-brane since the open string has $DD$-b.c.'s on dual $S^1$. Note

$$[X^1, X^a] \rightarrow \int [i\partial^1 + A^1, X^a] dx^1$$

$$= i \oint (\partial^1 X^a - i[A^1, X^a])$$

$$= i \oint (D^1 X^a)^2 dx^1$$

$$(D^0 X^1)^2 = \{\partial^0 X^1 - i[A^0, X^1]\}^2$$

$$= \oint \{\partial^0 (i\partial^1 + A^1) - i[A^0, i\partial^1 + A^1]\}^2$$

$$= \oint \{\partial^0 A^1 - \partial^1 A^0 - i[A^0, A^1]\}^2$$

$$= \oint (F^0_{01})^2 dx^1$$

Hence by putting the prefactors in $(R' = \alpha/R)$

$$L = \frac{\text{tr}}{2g^2}(\hat{X}^1 \hat{X}_1 + ([X^1, X^a])^2 + \cdots)$$

$$\rightarrow \frac{1}{R' \frac{1}{2g^2}} \int dx^1 \{\text{tr} F^2_{01} - (D^1 X^a)^2 + \cdots\}$$

Hence the $T$-dual to 0-branes yields a 2D SYM with 16 supercharges. This result can be generalized to show that a $D - p$ brane on $T^{[p-q]}$ is $T$-dual to a D-q brane. We can also generalize the result to many compact dimensions. For directions $X^a$ and $X^b$ with radii $R_a$ and $R_b$ we have

$$([X^a, X^b])^2 \rightarrow -\frac{1}{R'_a R'_b} \oint dx^a dx^b (F^{ab})^2$$

### 9 M- and F-Theory

#### 9.1 M-Theory

The highest dimension with a unique and consistent classical theory of $\mathcal{N} = 1$ supergravity is $d = 11$ dimensional spacetime. The bosonic fields are

- $G_{MN}$: metric
- $C_{MNP}$: antisymmetric tensor potential
with \( M, N, P = 0, 1, 2, \ldots, 10 \). Note \( C_{MNP} \) couples to an \( M2 \)-brane and its dual couples to an \( M5 \)-brane.

On dimensionally reducing this theory from 11 to 10 dimensions, we obtain the \( \mathcal{N} = 2, d = 10 \) non-chiral supergravity Lagrangian of Type IIA strings given in eq\((49)\). In particular, the field reduction yields, for \( \mu, \nu, \rho = 0, 1, 2, \ldots, 9 \),

\[
G_{MN} \rightarrow \begin{cases} 
G_{10,10} = e^{\phi} \\
G_{10,\mu} = A_{\mu}^{(1)} \\
G_{\mu\nu} = g_{\mu\nu}
\end{cases}
\]

and

\[
C_{MNP} \rightarrow \begin{cases} 
C_{\mu\nu\rho} = A_{\mu\nu\rho}^{(3)} \\
C_{\mu(10)} = B_{\mu}
\end{cases}
\]

We see that the field content of Type IIA string theory emerges naturally from \( d = 11 \) supergravity fields.

Consider a \( d = 11 \) manifold \( M_{10} \times S^1 \), with radius \( R \) for \( S^1 \). The conjecture of M-Theory is that \( R = \sqrt{\alpha'} e^{\phi} \); and that for \( R \to 0 \), M-Theory reduces to Type IIA string theory. For \( R \to \infty \), M-Theory is a Lorentz invariant quantum theory in \( d = 11 \) such that its low energy infra-red limit is given by \( \mathcal{N} = 1, d = 11 \) supergravity.

Since string coupling constant \( g_{IIA} = e^{\phi} \), we see that Type IIA is the weak coupling (small radius) limit of M-Theory. Indeed, M-Theory has been subjected to several successful tests so far. For example, all the D-branes in Type IIA are seen to emerge from dimensional reduction and wrappings of the \( M2 \)- and \( M5 \)-branes of M-Theory. The compact direction \( S^1 \) has KK-momentum modes for a given point particle given by

\[
p_{10} = n \frac{2\pi}{R} = \frac{2\pi n}{e^{\phi} \sqrt{\alpha'}} , \quad n \in \mathbb{Z}
\]

What are the states in Type IIA which correspond to this? Recall the mass of a BPS D0-brane in Type IIA is given by

\[
m_0 = \frac{2\pi}{g_{IIA} \sqrt{\alpha'}} .
\]

Consider a collection of \( n \) D0-branes. Since this is a BPS state, their bound state has energy which is additive and yields total energy

\[
nm_0 = \frac{2\pi n}{g_{IIA} \sqrt{\alpha'}} = p_{10}
\]

Hence, although Type IIA does not explicitly have an eleventh dimension, imprints of this dimension are seen in the spectrum of states.
We have seen that Type IIA results from a $d=11$ higher dimensional theory. What about Type IIB? Is this the result of compactifying some higher dimensional theory? Consider Type IIB compactified on $M_9 \times S^1$ with radius $R' = \frac{\alpha'}{R}$ is dual to Type IIA on $M_9 \times S^1$ with radius $R$. It can be shown that M-Theory compactified on $M_9 \times T^2$ is equivalent to Type IIB with $g_{IIB} = \frac{R_{10}}{R_9}$. A simultaneous limit $R_9, R_{10} \to 0$ yields an arbitrary $g_{IIB}$. We see that the exact $SL(2, \mathbb{Z})$ symmetry of Type IIB results from the symmetry of exchanging $R_9, R_{10}$ within the compactification. M-Theory thus “geometrizes” the $SL(2, \mathbb{Z})$ symmetry of Type IIB.

### 9.2 F-Theory

So far, all compactifications of string theory from $M_{10} \to M_{10-n} \times K_n$ have been based on making all the background fields except the metric independent of the compact manifold $K_n$.

F-Theory stands for a more general compactification of Type IIB, where the background fields have non-trivial dependence on $K_n$. For usual compactifications, one solves for Einstein equation for $K_n$, which gives the Ricci flat condition,

$$R_{ij}^{(K)} = 0.$$  \hspace{1cm} (106)

For even $n$, unbroken $\mathcal{N} = 1$ supersymmetry for $M_{10-n}$ requires that $K_n$ be a Calabi-Yau manifold with $SU(n/2)$ holonomy.

Suppose in addition to the 10-dimensional metric $g_{\mu\nu}$, other background fields also depend on $K_n$. The simplest case is to allow this field be the complex scalar field $\lambda = A^{(0)} + i e^{-\phi}$, where $A^{(0)}$ is the RR-scalar of Type IIB theory. The Lagrangian for these classical background fields is given by the Type IIB classical supergravity action given in eq(50). We have from eq(50)

$$L = \int d^{10}X \sqrt{g} \left( R - \frac{1}{2} g^{\mu\nu} \partial_\mu \lambda \partial_\nu \bar{\lambda} \right)$$ \hspace{1cm} (107)

Suppose $g_{\mu\nu}$ is the metric for $M_8 \times S^2$. Clearly, $S^2$ is not a Calabi-Yau manifold since it is not Ricci flat. Hence, Einstein equations for this compactification must give a non-trivial dependence for the other background fields on $S^2$. We assume that $\lambda$ is a function only of $S^2$ The field equation for the $\lambda$-field, from eq(107), is given by the Einstein field equation

$$R_{ij}^{(2)} - \frac{1}{2} g_{ij}(2) R^{(2)} = T_{ij}^{\lambda},$$ \hspace{1cm} (108)
$(i,j = 1,2)$ where $T^\lambda_{ij}$ is the energy-momentum stress tensor of $\lambda$.

Let $z = X_8 + iX_9$, then eq.\((108)\) yields

$$\frac{\partial \lambda(z, \bar{z})}{\partial \bar{z}} = 0 \quad (109)$$

and hence $\lambda$ is holomorphic. However, since Type IIB has $SL(2, \mathbb{Z})$ symmetry, $\lambda(z)$ is not an arbitrary holomorphic function but rather must lie in the fundamental domain of the torus. Typically, we obtain, near $z = 0$,

$$\lambda = \frac{1}{2\pi i} \ln z \quad (110)$$

The monodromy of $\lambda$ about $z$ is given by $z \to e^{2\pi i}z$, $\lambda \to \lambda + 1$. In other words, at $z = 0$, there is a single RR-charge. Since $\lambda$ couples magnetically to a $D7$-brane, this implies that a single $D7$-brane is located at the “point” $z = 0$, and fills up the entire space transverse to $(X_8, X_9)$, i.e. fills up $(X_1, X_2, \cdots, X_7)$.

It is known that the $D7$-brane couples to the metric on $(X_8, X_9)$ and induces a deficit angle of $\pi/6$ at $z = 0$. Since the total solid angle of a sphere is $4\pi$, we need $4\pi \div \frac{\pi}{6} = 24$ $D7$-branes placed on the plane $(X_8, X_9)$ to make it ‘curl up’ into $S^2$. 

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In other words, near $z = z_i$,
\[ \lambda \approx \frac{1}{2\pi i} \ln(z - z_i), \quad i = 1, 2, \ldots, 24 \quad (111) \]
In summary, only by placing 24 $D7$-branes in the manifold $M_{10}$ transverse to $(X_8, X_9)$ can Type IIB theory be compactified on $M_8 \times S^2$. These $D7$-branes appear as points on $S^2$.

The variation of $\lambda(z)$ over $S^2 = \mathbb{CP}^1$ is known exactly, and indeed it is an elliptic fibration of $S^2$ with the total bundle space being equal to $K3$, the Calabi-Yau manifold with 2 complex dimensions.

In analogy with M-Theory where the dilaton $e^\phi$ of Type IIA was seen as the moduli of M-Theory defined on $M_{10} \times S^1$, one can view the complex dilaton $\lambda$ of Type IIB theory as the moduli of F-Theory defined on $M_8 \times K3$, a 12-dimensional manifold.

F-Theory refers to the as yet unknown theory in 12 dimensions which when compactified on $M_8 \times K$, where $K$ is an elliptic fibration by a torus of some manifold $B$, yields Type IIB string theory on $M_8 \times B$ with $\lambda$ identified as the complex structure of the torus. F-Theory is not as well understood as M-Theory since it is not clear whether the low energy supergravity Lagrangian for it even exists.

10 $D$-Branes from Gauge Fields

In this section, we briefly review the topological classification of gauge field configurations. Consider an $n$-dimensional base space $M$. Let the fiber $F$ be $k$-dimensional $\mathbb{R}^k$ and the bundle space $E$ locally is $M \times \mathbb{R}^k$. $E$ has dim = $n + k$.

Consider three patches of $E$. The transition function between patches $U$ and $V$ is given by $g_{uv}$ such that for $(x, f) \in U$ and $(x', f') \in V$
\[ f' = g_{uv} f \]
Bundle $E$ is called a vector bundle if $\{g\} \in GL(k, \mathbb{R})$. $E$ is a principal bundle if the fiber $F = G$ and transition functions $\{g\} = G$, where $G$ is a compact Lie group.

Transition function on a triple overlap satisfy the cocycle consistency condition

$$g_{uv}g_{vw}g_{wu} = I$$

This yields the Dirac quantization condition for monopoles.

A YM-connection $A = A^a_{\mu} T^a dx^\mu$ defines parallel transport for the principal bundle. On two overlapping patches

$$A' = gA g^{-1} - idg g^{-1}$$

A cross section or simply a section of a fiber bundle $E$ assigns a specific point $f(x) \in F$ for each point $x \in M$.

Matter fields are sections of associated bundles.

Suppose the base manifold $M$ is a compact space, say $T^{2n}$. How do we classify all the non-trivial principal bundles that can be defined on $M$? This is done by determining all the topological invariants of the bundle space $E$. In particular the Chern classes classify all principal bundles. Let $\Omega = F^a_{\mu\nu} T^a dx^\mu \wedge dx^\nu$; then the total Chern form is given by

$$c(\Omega) = \det (1 + \frac{i}{2\pi} \Omega)$$

$$= 1 + c_1(\Omega) + c_2(\Omega) + \cdots + c_N(\Omega).$$

The individual Chern classes are given by

$$c_1 = \frac{i}{2\pi} \text{tr} F$$

$$c_2 = \frac{1}{8\pi^2} \text{tr} F \wedge F - (\text{tr} F) \wedge (\text{tr} F)$$

$$c_3 = \cdots$$
Note the $c_i$’s are elements of integer-valued cohomology classes.

Recall that for a D-p brane, we have an interaction term

$$S_{cs} \sim \int \sum_{p+1} c_1 \wedge \text{tr } e^{\Omega}$$

where $c = \sum_{k=1}^{N} A^{(k)}_{\mu_1, \ldots, \mu_k} dx^{\mu_1} \cdots dx^{\mu_k}$ with $A^{(k)}_{\mu_1, \ldots, \mu_k}$ being the $R - R$ fields.

Consider the field tensors $F$ to be a background field on a $D-p$ brane. Then

$$S_{cs} \sim \int \sum_{p+1} c_1 \text{tr } F \sim \int \sum_{p=1} \int_{T^2} \text{tr } F \sim c_1 \int \sum_{p+1} A^{(p-1)}$$

In other words $c_1$ acts as a charge for a D-$(p - 2)$ brane which couples to the appropriate $RR$ field $A^{(p-2)}$. We consequently have that

- $\int F$ carries charge of a $(p - 2)$ brane
- $\int F \wedge F$ carries charge of a $(p - 4)$ brane
- $\int F \wedge F \wedge F$ carries charge of $(p - 6)$ brane, etc.

Hence, by having background gauge fields with non-trivial topology we in effect are creating D-branes embedded in the world volume of a $p$-brane. For example, $\int F \neq 0$ is equivalent to have a $(p - 2)$-brane embedded in a $p$-brane etc.

### 10.1 Gauge Fields on $T^d$ (’t Hooft)

In addition to the instanton number ($= c_2 =$ Second Chern class), for gauge fields in the absence of fermions in the fundamental representation, the $U(N)$ theory is invariant under $\mathbb{Z}_N$, where

$$Z_N = \{ e^{2\pi i n/N}, n = 0, 1, \ldots, N - 1 \}.$$

Since $Z_N^{-1} A_\mu Z_N = A_\mu$, $Z_N$ is a special gauge transformation and the theory is invariant under $U(N)/\mathbb{Z}_N$.

A consequence of $Z_N$ symmetry is the existence of another class of topological quantum numbers in addition to $c_2$. Let $\Omega A_\mu \equiv \Omega A_\mu \Omega^{-1} - i \Omega \partial_\mu \Omega^{-1}$. On a torus, periodicity yields the following.
Then
\[ A_{\mu}(L_1, X_2) = \Omega_1(0) A_{\mu}(L_1, 0) \] (117)
\[ A_{\mu}(X_1, L_2) = \Omega_2(0) A_{\mu}(X_1, 0) \] (118)

Consider two paths $I$ and $II$ from $(0, 0)$ to $(L_1, L_2) \equiv 0$.

\[ I : A_{\mu}(L_1, L_2) = \Omega_2(L_1) A_{\mu}(L_1, 0) \] (119)
\[ = \Omega_2(L_1) \Omega_1(0) A_{\mu}(0, 0) \] (120)

\[ II : A_{\mu}(L_1, L_2) = \Omega_1(L_2) A_{\mu}(0, L_2) \] (121)
\[ = \Omega_1(L_2) \Omega_2(0) A_{\mu}(0, 0) \] (122)

Since $(0, 0) \equiv (L_1, L_2)$, we have $A_{\mu}(L_1, L_2) = \Omega A_{\mu}(L_1, L_2) = z A_{\mu}(L_1, L_2)$ and hence, for non-trivial topology, consistency requires

\[ \Omega_1(L_2) \Omega_2(0) = \Omega_2(L_1) \Omega_1(0) z, \quad \text{for } z \in \mathbb{Z}_N \] (123)

where $\Omega_1, \Omega_2$ reflects the (non-trivial) topology of $A_{\mu}$.

For an arbitrary gauge transformation
\[ A_{\mu}(X_1, X_2) \rightarrow \Omega(X_1, X_2) A_{\mu}(X_1, X_2) \]
where we can have $\Omega_2(X_1)$ and $\Omega_1(X_2)$ arbitrary but constrained by \text{eq}(123)$. That is, we cannot take both $\Omega_1(X_2) \to I$ and $\Omega_2(X_1) \to I$ (trivial topology) since this violates \text{eq}(123).

Recall in \text{eq}(123) we choose plane $(1, 2)$ and hence $z = z(1, 2)$ by continuity. Since there are $\frac{1}{2}d(d - 1)$ independent planes in $T^d$, and $[z] = N$, the number of topological classes of gauge field configuration is $N^\frac{d(d-1)}{2}$. Introducing fermionic fields in the fundamental representation destroys the $\mathbb{Z}_N$ symmetry and consequently wipes out the $N^\frac{d(d-1)}{2}$ topological charges.

Let us consider an example. Consider a 2-brane wrapped $N$-times on $T^2$ giving a $U(N)$ bundle on $T^2$. These bundles are classified by $c_1 = \frac{\text{tr}}{2\pi} \int F$. Since $U(N) = \frac{U(1) \times SU(N)}{\mathbb{Z}_N}$, consider the $U(1)$ component. Suppose $F_{12} = \text{constant}$; then we can take $A_1 = 0, A_2 = XF_{12}$. On $T^2$, we have

\begin{align*}
X & \to X + 2\pi R_1 \\
Y & \to Y + 2\pi R_2 \\
A_2 & \to A_2 + 2\pi R_1 F_{12}
\end{align*}

Hence

\begin{align*}
\text{Wilson loop} & = e^{i \oint dy A_2} \\
& = e^{2\pi i R_2 XF_{12}} \\
& \to e^{2\pi i R_2 F_{12}(X+2\pi R_1)}.
\end{align*}

Hence, since Wilson loop must be single-valued, we have

\begin{align*}
R_1 R_2 F_{12} 2\pi & = k \in \mathbb{Z} \\
\text{or} \quad F_{12} & = \frac{k}{2\pi R_1 R_2} : \text{Quantization}
\end{align*}

which yields

\begin{equation}
c_1 = \frac{F_{12}}{2\pi} (2\pi)^2 R_1 R_2 = k
\end{equation}
For $c_1 = k$, the $U(1)$ flux is $F = \frac{k}{N} I$ and the $U(N)$ bundle has twist $z_k = e^{2\pi ik/N}$; in $d = 2$, there are $N^{2(d-1)} = N$ topological charges for $U(N)$ given by $z_k$. Let us choose $k = 1$; then the $U(N)$ principal bundle on $T^2$ is specified by

$$\frac{\text{tr}}{2\pi} \int F = 1$$

Choose the boundary conditions

$$\Omega_1(x_2) = Ve^{2\pi i x_2/L_2}T$$
$$\Omega_2(x_1) = I$$

where

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ 1 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$
$$T = \text{diag} \left( 0, 0, \cdots, \frac{1}{N} \right)$$

For this choice of $\Omega_1$ and $\Omega_2$, $F = \text{constant and}$

$$A_1 = 0$$
$$A_2 = x_1 F + \frac{2\pi}{L_2} \text{diag} \left( 0, \frac{1}{N}, \cdots, \frac{N-1}{N} \right)$$

with

$$F = \frac{2\pi}{NL_1L_2} I_{N \times N}$$

$$c_1 = \frac{1}{2\pi} \int \text{tr} F = \frac{2\pi N}{2\pi NL_1L_2} \cdot L_1L_2 = 1$$

Hence $c_1 = 1$ is the charge for a $(2-2) = 0$ brane; that is, due to the flux in the D2-brane, there is a 0-brane embedded in $T^2$ together with $N$ 2-branes. $c_1$. 

-1

-2

50
Let us $T$-dualize the 2-direction inside the 2-brane on $T^2$. We expect the following in the $X^2$-direction:

\[
\begin{array}{c|c|c}
N & 2\text{-branes} & N & 1\text{-brane} \\
1 & 0\text{-brane} & 1 & 1\text{-brane}
\end{array}
\]

$T$-duality yields

\[
X^2 = (2\pi \alpha') (i \partial^2 + A^2)
\]
\[
= (2\pi \alpha') A^2
\]

where $\partial^2 = 0$, being orthogonal to brane. Hence,

\[
X^2 = \frac{4\pi^2 \alpha'}{L_2} \frac{1}{N} \text{diag} \left( \frac{x_1}{L_1}, \frac{x_1}{L_1} + 1, \cdots, \frac{x_1}{L_1} + N - 1 \right)
\]
\[
= \text{coordinates of the brane in } \hat{2} \text{- direction.}
\]

Since we are on $T^2$ we have from eq(141)

The 1-brane coming from the $T$-dual of the 0-brane is in the $X_2$-direction and winds only once since $c_1 = 1$ around the $X_2$-direction.

The $N$-winding is due to the original brane configuration. The 1-brane $T$-dual to the 2-brane winds $N$-times since the 2-brane was wrapped $N$-times on $T^2$ to start with.

What happens when $c_1 = \frac{1}{2\pi} \int \text{tr } F = k$? We first of all expect $k$ 0-branes to be embedded in $T^2$. On $T$-duality thus we expect each 0-brane to go to its $T$-dual 1-brane. But since all these $k$ 0-branes come from the same non-trivial
background gauge field, we expect these $k$ 1-branes to wind in the $X^2$-direction $k$-times.

$S = -\tau_p \int d^{p+1}x \sqrt{-\det (\eta_{\mu\nu} + F_{\mu\nu} + \text{fermions})}$

where $G_{\mu\nu}, B_{\mu\nu}$ are the NS-fields and $F_{\mu\nu}$ is constructed from the ten-dimensional RR-fields. $\tau_p$ is the ‘tension’ of a p-brane.

In the static gauge, flat background and $U(1)$ gauge fields we have to dimensionally reduce

$S = -\int d^{10}x \sqrt{-\det (\eta_{\mu\nu} + F_{\mu\nu} + \text{fermions})}$

After dimensional reduction

$S = -\tau_p \int d^{p+1}x \sqrt{-\det (\eta_{\mu\nu} + \partial_\alpha X^a \partial_\beta X^a + 2\pi\alpha' F_{\alpha\beta}) + \text{fermions}}$

This simplified action can describe simple aspects of D-brane dynamics such as energy, fluctuations and scattering amplitudes.

Consider the following special D $p$-brane configurations

11 D-Brane Dynamics

The complete description of D-brane physics is given by the Born-Infeld lagrangian

$S = -\tau_p \int d^{p+1}x \sqrt{\frac{1}{2} (G + B + 2\pi\alpha' F) + \text{fermions}}$

Example: $(N,K) = (3,2)$
\[ X^a = 0, \text{ for } a = p + 1, \cdots, 9 \]

\[ F_{\alpha\beta} = \text{constant} \]

\[ [F_{\alpha\beta}, F_{\gamma\delta}] = 0 \]

This yields

\[ S = -\tau_p \int d^{p+1}\xi \, \text{tr} \sqrt{- \det (\eta_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta})} \]

The energy is given by

\[ E = \tau_p \int d^p\xi \, \text{tr} \sqrt{\det (\delta_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta})} \]

where the trace refers to non-abelian index and determinant is on Lorentz indices.

Note for a p-brane

\[ \tau_p = \frac{\text{Energy}}{\text{Volume of p-brane}} \]

From duality, for \( S^1 \) with radius \( R \), we have on taking T-dual, \( p \to p \pm 1 \) and

\[ \tau_p = \tau_{p+1}' (2\pi R'), \quad \tau_p (2\pi R) = \tau_{p-1}' \] (143)

\[ R = \alpha'/R' \] (144)

Examples

(a) \( N \) 2-branes with \( c_1 = q \) units of flux.

\[ \det (\delta_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta}) = \left( 1 + (2\pi \alpha' F_{12})^2 \right) \] (145)

Hence,

\[ E = \sqrt{1 + (2\pi \alpha' F_{12})^2} \tau_2 \int d^2\xi \, \text{Tr} (I) \]

\[ = \tau_2 NL_1 L_2 \sqrt{1 + 4\pi^2 \alpha'^2 F_{12}^2} \] (146)

Since \( F_{12} = \frac{q}{NL_1 L_2} \), we have

\[ E = \sqrt{(N\tau_2 L_1 L_2)^2 + (q\tau_0)^2} \]

where \( \tau_0 = 2\pi \alpha' \tau_2 \) = mass of 0-brane. Note for a system of \( N \) 2-branes and \( q \) 0-branes that are well separated we expect

\[ E_\infty = N\tau_2 L_1 L_2 + q\tau_0 \]

Since \( E < E_\infty \), the system forms a bound state.
On $T$-dualizing the $\hat{2}$-direction

\[
\tau_2 L_2 = \tau'_1
\]
\[
\tau_0 = \tau'_1 L'_2
\]  

(148)

(149)

Since $E$ is unchanged we have

\[
E = \sqrt{(\tau'_1 N L_1)^2 + (\tau'_1 L'_2 q)^2}
\]
\[
= \tau'_1 \sqrt{(N L_1)^2 + (q L'_2)^2}
\]

(150)

(151)

where from the diagram, we see that $E$ is the energy for $(N, q)$ windings on $T^2$.

(b) Consider 2 4-branes wrapped on $T^4$ with instanton number $k = 2$ and volume $V = L_1 L_2 L_3 L_4$. Choose linear connection

\[
A_1 = A_3 = 0
\]
\[
A_2 = \frac{2\pi X_1}{L_1 L_2} \tau_3, \quad A_4 = \frac{2\pi X_3}{L_3 L_4} \tau_3
\]

(152)

(153)

Then

\[
F_{12} = \frac{2\pi}{L_1 L_2} \tau_3, \quad F_{34} = \frac{2\pi}{L_3 L_4} \tau_3
\]
\[
E = \tau_4 V \text{Tr} \sqrt{\det (\delta_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta})}
\]

(154)

(155)

Then

\[
\det \begin{pmatrix}
I & 2\pi \alpha' F_{12} \tau_3 & 0 & 0 \\
-F_{12} \tau_3 & I & 0 & 0 \\
0 & 0 & I & F_{34} \tau_3 \\
0 & 0 & -F_{34} \tau_3 & I
\end{pmatrix} = I (1 + (2\pi \alpha' F_{12})^2) (1 + (2\pi \alpha' F_{34})^2)
\]

\[
\Rightarrow E = 2\tau_4 V \sqrt{(1 + (2\pi \alpha' F_{12})^2)(1 + (2\pi \alpha' F_{34})^2)}
\]

(156)

(157)
If $L_1L_2 = L_3L_4, F_{12} = F_{34}$ : self-dual, and hence

$$E = 2\tau_4V + 2\tau_0$$
$$= \text{energy of } 2\ 4-\text{branes} + \text{energy of } 2\ 0-\text{branes}$$

(158)
(159)

since $F$ is self-dual, the system satisfies the BPS condition and energy being additive is a reflection of the fact that $\frac{1}{2}$ susy is preserved.

## 12 Black Hole Entropy and D-Branes

General Relativity predicts gravitational collapse with the formation of a space-time singularity covered by an event horizon with Schwarschild radius $R_s$. The area of the event horizon $A_H$ always increases, i.e. $\Delta A_H \geq 0$.

Quantum fields in a black hole classical background exhibit Hawking thermal radiation emanating from the black hole of mass $M$ with temperature given by

$$T_H = \text{Hawking Temperature}$$
$$\sim \frac{\hbar}{R_S} \sim \frac{\hbar}{GM} \text{ in } d = 4$$

(160)
(161)

$$S_{BH} = \text{entropy of black hole}$$
$$= \frac{A_H}{4\hbar}$$

(162)
(163)

We also have $dM = T_HdS$ (First Law of Thermodynamics) where

$$S = \text{total entropy}$$
$$= S_{BH} + S_{\text{Radiation}}$$

(164)
(165)

$$\Delta S \geq 0$$

(166)

### 12.1 Charged Black Holes

For a black hole with mass $M$ and charge $Q$, the cosmic censorship hypothesis states that naked singularities do not form from generic smooth initial configurations. This hypothesis implies that

$$M \geq Q$$

---

5Restoring the $\hbar$
For the extremal case $M = Q$, we have the BPS condition that electric and gravitational forces exactly cancel giving no interaction energy.

Starting from $M > Q$, the black hole radiates off energy until it reaches equilibrium with $M = Q$. This violates the third Law of Thermodynamics which says that $S \to 0$ as $T \to 0$. For $M > 0$, the black hole emits mostly electrons since in appropriate units $m_e << e$.

The extremal black hole, with $M = Q$, has a finite horizon as well as non-zero entropy. This is a particularly useful case since there is no time dependence in the problem and its entropy can directly be calculated from its density of states.

12.2 Black Holes and Strings

One might at first sight think that black holes being macroscopic objects have nothing to do with strings. More precisely, since the Schwarschild radius is $R_S \sim GM_{BH}$ (which for $M_{BH} = M_\odot$ gives $R_S \sim 1.5$ km ) and strings are typically of Planck length $\ell_p \sim \sqrt{G} \sim 10^{-35}$ m, the scales do not match. Also, for an excited string state at level $N$, its mass is $M_{str} = \sqrt{N}/\ell_{str}$, where $\ell_{str}$ is the average length of a string, the density of states $\rho_{str}(M_{str}) \sim e^{M_{str}}$ whereas since $S_{BH} = \frac{A_H}{4G} \sim \frac{R_S^2}{G} \sim GM_{BH}^2$, we have $\rho_{BH}(M_{BH}) \sim e^{M_{BH}^2}$. Hence, naively, $\rho_{BH}$ for a black hole of mass $M$ does not match the $\rho_{str}$ for a string of the same mass.

So how can we resolve this contradiction? To start with, note that $G = g^2\ell_{str}^2$, where $g$ is the dimensionless effective string coupling constant. We can study the string in two limits, namely

- the string limit where $\ell_{str}$ is kept fixed at $\ell_p$ as $g$ varies and
• the Planck limit where $G$ is held fixed as $g$ varies.

Both approaches give the same answer, but we consider the Planck limit for clarity and treat $G = $ as a constant.

Consider

$$\frac{M_{BH}}{M_{str}} = \frac{R_S \ell_{str}}{G \sqrt{N}} = \frac{R_S}{g^2 \ell_{str} \sqrt{N}}$$

Clearly, for an excited state $N$, the mass of the black hole and string in general are not equal. Where should we match them? Clearly, we should set them equal where $R_S \sim \ell_{str}$, i.e.

$$1 = \frac{M_{BH}}{M_{str}} = \frac{1}{g^2 \sqrt{N}}$$

That is, for a given $N$, consider coupling $g$ such that $g^2 \sqrt{N} = 1$. For this coupling $M_{BH} = M_{str}$. Note that we have not assumed that $\ell_{str} \sim \ell_p$. On the contrary, since,

$$\ell_{str} = \frac{\sqrt{G}}{g} = N^{1/4} \sqrt{G}$$

we can consider highly excited states such that $\ell_{str} \sim R_S \sim 1$ km. For such large $N$, the description of the black hole as an excited string state becomes appropriate. The reason being that general relativity is valid only if the curvature of space is much less than $\frac{1}{\ell_{str}}$. When $R_S \sim \ell_{str}$, this description breaks down and black holes are more appropriately described by string theory.

We are interested in extremal black holes with $M = Q$. We need to consider supersymmetric black holes since BPS-states are then independent of coupling $g$. For such black holes, $M$ does not change with $g^2$ and hence $M_{BH} = M_{str} = M$ for all couplings. We immediately run into a problem. Most supersymmetric black holes have zero horizon. To obtain supersymmetric black holes with a finite horizon, we need several charges, namely three different charges in five dimension and four charges in four dimensions. These charges are carried by fundamental strings and $D$-branes which couple to ten dimensional $NS$-fields and $RR$-fields. Hence it was only with the introduction of open strings and D-branes into the theory that black holes could be addressed. We are interested in a perturbative description of string theory as $g \rightarrow 0$. Note that

1. p-branes carrying electric NS charge have mass $\sim 1$

2. p-branes carrying magnetic NS charge have mass $\sim \frac{1}{g^2}$

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6Supersymmetric black holes exist only for four and five dimensions
3. D-branes carrying RR charge have mass $\sim \frac{1}{g}$

Since $GM$ determines the gravitational field of mass $M$, and $G = g^2 \ell_5^2$, we see that as $g \to 0$ only (1) and (3) have $GM \to 0$.

In five spacetime dimensions, we need a three form field coupling to one of the required charges. A magnetic NS charge carrying p-brane has no weak coupling description. Fortunately, in Type IIB, we have a two form gauge field $A^{(2)}$ which couples to a D-1 brane. This two form gauge field is dual to $\tilde{A}^{(6)}$ which couples to a D5-brane. Since these D-branes have an exact description as $g \to 0$, we will be able to count all the microstates of these D-brane configurations.

### 13 Five Dimensional Charged Black Holes

We begin with Type IIB string theory in $d = 10$. The low energy effective background theory is 10-dimensional supergravity. One can find classical solutions for this theory in which the $D$-branes appear as solitons carrying RR-charge. One first obtains in $d = 10$ classical solutions with non-zero: $Q_5$, D5 brane charge; $Q_1$, D1-brane charge; $N$, momentum of D1-brane.
In Type IIB, consider compactification [32,33] on $M^5 \times T^5$ with $T^5 = T^4 \times S^1$. On compactifying from $d = 10$ to $d = 5$, one obtains a 5-dimensional black hole. Taking the extremal limit yields $T_H \to 0$ and entropy as

$$S_{BH} = 2\pi \sqrt{NQ_1Q_5}$$

Let the volume of $T^4$ be $(2\pi)^4V$, $R$ be the radius of the circle $S^1$ and $d\Omega_3$ be the line element on a unit three-sphere. The canonical black hole metric in the non-compact dimensions is given by

$$ds^2 = -\frac{1}{\lambda^{3/2}}dt^2 + \lambda^{1/3}(dr^2 + r^2d\Omega_3^2)$$  \hspace{1cm} (167)

where $\lambda = (1 + \frac{r_1^2}{r_2^2})(1 + \frac{r_5^2}{r_2^2})(1 + \frac{r_N^2}{r_2^2})$ and

$$r_1^2 = \frac{(RV)^{2/3}}{V\sqrt{g}}Q_1$$  \hspace{1cm} (168)

$$r_5^2 = (RV)^{2/3}\sqrt{g}Q_5$$  \hspace{1cm} (169)

$$r_N^2 = \frac{(RV)^{2/3}}{R^2V}N$$  \hspace{1cm} (170)

Why do we need $N$, $Q_1$ and $Q_5$? Essentially it is to obtain a finite area for the horizon of the black hole. As one approaches the D5-brane horizon, the volume parallel to the brane shrinks due to brane tension and the volume perpendicular to the brane expands. By superposing a D1-brane along one of the $T^5$ directions, the remaining directions are stabilized. The volume along the D1-brane due to tension also tends to shrink to zero and to balance this tension momentum $N$ is given to the D1-brane. Note that all the branes appear as point-like particles in $M^5$, and it is really the microstates of this ‘point’ particle that we are computing.

### 13.1 D-Brane Description of supersymmetric Black Holes

We wrap a D5-brane $Q_5$ times on $T^5$ and a D1-brane $Q_1$ times on $S^1$. Since the theory is boost invariant, the momentum $N$ can only come from the massless excitations of this system. Recall since D5- and D1-brane form a BPS system of bound states, we have

$$M = \frac{RV}{g}Q_5 + \frac{R}{g}Q_1 + \frac{N}{R}$$

where $g$ is the string coupling.

The massless modes must move along $S^1$ in one direction to maintain the BPS condition. A massive excitation would violate the BPS condition. Excitations of
the branes are described by open strings and the ones which contribute are those that go from 1-brane to 1-brane, namely (1,1), as well as (1,5), (5,1) and (5,5). One can do an explicit calculation to compute the open string contributions.

To compute the entropy of the charged black hole, we need to count all the microstates which can yield momentum $N$ for a $Q_5, Q_1$ BPS bound state of a D5 with a D1-brane. Since only massless excitations can contribute to the microstates, we can use the Yang-Mills description of D-branes to do this counting.

Start with $U(Q_5) N = 1$ super Yang-Mills in $d = 10$ and dimensionally reduce to $d = 5 + 1$

\[ A_\mu \rightarrow \begin{cases} A_\alpha, & \alpha = 0, 5, 6, 7, 8, 9 \\ A_I & I = 1, 2, 3, 4 \end{cases} \]

\[ S_{YM} = \frac{1}{g_{YM}^2} \int d^{5+1} \xi (\text{tr} \ F_{\alpha \beta} F^{\alpha \beta} + \frac{1}{2} \text{tr} \ D_\alpha X_I D^\alpha X^I + \frac{1}{4} \text{tr} \ [X_I, X_J]^2 + \text{fermions}) \]

where $g_{YM}^2 = g$ is the closed string coupling.

To introduce D1-branes we consider topologically non-trivial gauge field configurations for $A_\alpha$. Recall the interaction term for brane coupling is given by

\[ \int_{\Sigma_{5+1}} c \wedge \text{tr} F \]

where $c$ is the sum of the the $RR$-fields. Consider the term $\int_{\Sigma_{5+1}} A^{(2)} \text{tr} F \wedge F$ where $A^{(2)}$ is the $RR$ 2-form gauge field, with an instanton configuration in the compact directions 5,6,7,8, i.e.

\[ A_9 = 0, \ A_\alpha(x^5, \ldots, x^8) = \text{instanton} \]

having instanton number $Q_1$. We then obtain a D1-brane (since instanton is a soliton which is independent of $x^9$) coupled to the $RR$-field, namely,

\[ \int_{\Sigma_{5+1}} c \wedge \text{tr} e^F = Q_1 \int_{\Sigma_2} d^{1+1} \xi A^{(2)} \]
In effect, we have created \( Q_1 \) number of D1-branes fused together with the D5-brane.

To give momentum \( N \) to the \( Q_1 \) D1-brane along \( \hat{9} \), note that the instanton configuration depends on the moduli \( \zeta^a \), namely,

\[
A_\alpha(x^5, \cdots, x^8, \zeta^a)
\]

For \( Q_1 \) instantons in \( U(Q_5) \) gauge theory, the number of moduli parameters is \( 4Q_1Q_5 \), and moduli space is \( \mathcal{M} = \frac{(T^4)^{Q_1Q_5}}{S(Q_1Q_5)} \). The massless excitations of the D1-branes can be realized as small oscillations of the moduli, i.e. \( \zeta^a = \zeta^a(t + x^9) \), the moduli are all left movers so as to maintain the BPS condition. Hence

\[
S_{YM} = \frac{1}{g_Y^2} \int d^{1+1}\xi \int d^4\xi \text{tr} \ F^2
= \frac{Q_1}{g_Y^2} \int d^{1+1}\xi G_{ab}(\zeta) \partial^a \zeta^a \partial^a \zeta^a
\]

This action is an \( \mathcal{N} = 4 \) nonlinear sigma model, and together with its fermionic partner forms a (4,4) superconformal field theory on a space of length \( 2\pi R \). This theory is described by \( 4Q_1Q_5 \) free bosons and free fermions, i.e. \( B_B = N_F = 4Q_1Q_5 \).

The condition of only left movers \( \zeta^a(t + x^9) \) yields \( L_0 = N \) and \( \bar{L}_0 = 0 \). From conformal field theory,

\[
d(N) = \text{number of microstates for} \ N
= e^{2\pi \sqrt{cL_0}} = e^{S_{BH}}
\]

Since

\[
c = N_B + \frac{1}{2}N_F
= 6Q_1Q_5
\]

we finally obtain the result of Strominger and Vafa \[34\], namely

\[
S_{BH} = 2\pi \sqrt{Q_1Q_5N}
\]

Limitations of the calculation is that \( gN, gQ_1, gQ_5 \gg 1 \). For physically interesting cases, we need to understand non-susy black holes in four dimension. In these cases \( N \sim 1 \) and the formalism has to be extended.

In conclusion, a massive BPS D-brane state is identical to an extremal black hole. This seems paradoxical since string states are defined on a flat background.
spacetime. The magic of BPS is the answer. We start with weak coupling $g$ where we can count the number of microstates of the D-brane system. As we increase $g$, the counting is unchanged as the number of BPS states do not change. For large $g$, the $D$-brane system undergoes gravitational collapse and becomes a black hole with a horizon. Thus, a quantum black hole is a strongly coupled highly charged bound state of $D$-branes.

14 AdS/CFT Correspondence

In section 6, we saw that a system of $N$ coincident D $p$-branes is a classical solution of the low energy effective string action in which only the metric, the dilaton and the RR $(p+1)$-form potential are non-vanishing. The metric is given by eq(52). For large values of $r$, the metric becomes flat. Since the curvature is small, the classical supergravity theory provides a good description of D-brane [35, 36].

We specialize eq(52) to a D3-brane. For a D3-brane, the metric is given by

$$ds^2 = f^{-1/2}(-dt^2 + dx_\rho^2) + f^{1/2}(r^2 + r^2d\sigma_5^2)$$

(177)

where $x_\rho = (x_0, x_1, \cdots x_p)$ are the coordinates along the D3-brane worldvolume and

$$f = 1 + \frac{4\pi g_s N\alpha'^2}{r^4}$$

(178)

with $d\sigma_5^2$ as the metric on $S^5$. We can rewrite eq(177) as

$$ds^2 = \left\{f^{-1/2}(-dt^2 + dx_\rho^2) + f^{1/2}r^2\right\} + f^{1/2}r^2d\sigma_5^2.$$  

(179)

Note that the coordinate $r$ is the distance transverse to the D3-world brane. The terms within the bracket will ultimately yield in an appropriate limit the metric for $AdS_5$. Consider

$$r \to 0; \quad f^{1/2}r^2 \to \alpha'\sqrt{4\pi g_s N}, \text{constant.}$$

(180)

This means that $S^5$ which would become infinitesimally small to form the transverse space, due to the $r^2$ prefactor in the transverse direction, now forms a neck due to eq(179) with $S^5$ held fixed at some constant size as $r \to 0$. Also, note that the $AdS_5$ space is formed by combining the D3-brane world volume with an extra dimension from the transverse direction. The D3-brane world volume resides at “the end of the neck” (see figure below). In the limit $r \to 0$, $S^5$ carries a non-trivial RR-charge and provides the source for the $A^{(3)}$ RR-field emanating from the D3-brane.
Thus if we consider the near-horizon Maldacena limit in which
\[ r \to 0, \quad \alpha' \to 0 \] (181)
with \( U \equiv R / \alpha' \), held fixed, the metric in eq(52) becomes
\[
\frac{ds^2}{\alpha'} \to \frac{U^2}{\sqrt{4\pi N g_s}} (-dt^2 + dx_1^2 + dx_9^2) + \frac{\sqrt{4\pi N g_s}}{U^2} dU^2 + \sqrt{4\pi N g_s} d\Omega_5^2 \] (182)
Eq(182) is the metric of the manifold \( AdS_5 \times S^5 \) in which the two radii of \( AdS_5 \) and \( S^5 \) are equal and given by
\[
R^2_{AdS_5} = R^2_{S^5} \equiv b^2 = \alpha' \sqrt{4\pi N g_s}. \] (183)
It is also interesting to note that in this limit, the Yang-Mills coupling constant,
\[
g^2_{YM} = 2g_s (2\pi)^{p-2}(\alpha')^{(p-3)/2} \to 4\pi g_s \] (184)
becomes dimensionless, so that
\[
\lambda \equiv \frac{b^2}{\alpha'} = \sqrt{Ng^2_{YM}} \gg 1 \] (185)
This also implies that the four dimensional world volume is conformally invariant.

We have seen that a system of \( N \) coincident D 3-branes possesses \( \mathcal{N} = 4 \) super Yang-Mills gauge theory in 3+1 dimensions with \( U(N) \) gauge group. We see that the classical solution in eq(182) is a good approximation in the large \( N \) limit in which the radii of \( AdS_5 \) and \( S^5 \) are huge. Thus, Maldacena conjectures that the strongly interacting \( \mathcal{N} = 4 \) super Yang-Mills with gauge group \( U(N) \) is equivalent in the large \( N \) limit to the ten dimensional Type IIB superstring theory compactified on \( AdS_5 \times S^5 \). Indeed, since supergravity is not a consistent quantum theory, one can extend the conjecture to any value of \( \lambda \) and say that

\footnote{Note that the Maldacena limit is a weak string coupling limit in contrast with the strong \('t Hooft coupling.}
\( \mathcal{N} = 4 \) super Yang-Mills is equivalent to type IIB string theory compactified on
the special background of \( AdS_5 \times S^5 \).

We can also compare the global symmetries. Type IIB string theory on \( AdS_5 \times S^5 \) has
isometry group \( SO(4,2) \times SO(6) \) since \( S^5 \) has \( SO(6) \) symmetry. It turns out that
these symmetries are also the relevant symmetries for \( \mathcal{N} = 4 \) super Yang-Mills with
gauge group \( U(N) \) in 3+1 dimensions. Indeed the \( SO(4,2) \) or \( SU(2,2) \) is
realized as a conformal invariance in super Yang-Mills theory. Furthermore, in
10 dimensions, \( N = 1 \) pure super Yang-Mills contains gauge field potentials \( A_\mu \),
\( \mu = 0, 1, \ldots, 9 \) giving \( 10 - 2 = 8 \) degrees of freedom in the adjoint representation
of \( U(N) \) \[35\]. Together with the 8-dimensional Majorana-Weyl “gluinos” \( \lambda_\alpha \),
\( \alpha = 1, 2, \ldots, 8 \), the theory has 16 Majorana supercharges \( Q_\alpha \), \( \alpha = 1, 2, \ldots, 16 \).
Under dimensional reduction, the 16 supercharges becomes 4 sets of complex
Majoranas \( Q^A_\alpha, \bar{Q}^A_\dot{\alpha} \), \( \alpha = 1, 2, A = 1, \ldots, 4 \) which transform as the \( \{ 4 \} \) and \( \{ \bar{4} \} \) rep
of the \( R \)-symmetry group of \( SU(4) \) and the scalar fields \( \phi_i \) transform as \( \{ 6 \} \) under
\( SU(4) \). so that the theory is invariant under \( SU(4) \). However, Type IIB theory
has 32 supercharges and super Yang-Mills has only 16. From the perspectives of
the \( N \) coincident BPS D3-branes, half the Type IIB supersymmetries are broken.
One also notes that the remaining 16 fermionic generators can arise from the
extension of the conformal group under supersymmetries.

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