FEYNMAN PATH FORMULA FOR THE TIME FRACTIONAL SCHRÖDINGER EQUATION

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Abstract. In this paper, we define \( E_{\alpha}(t^\alpha A) \), where \( A \) is the generator of an uniformly bounded \((C_0)\) semigroup and \( E_{\alpha}(z) \) the Mittag-Leffler function. Since the mapping \( t \mapsto E_{\alpha}(t^\alpha A) \) has not the semigroup property, we cannot use the Trotter formula for representing the Feynman operator calculus. Thus for the Hamiltonian \( H_{\alpha} = -\hbar^2 \frac{\partial^2}{\partial x^2} + V(x) \), we express \( E_{\alpha}(t^\alpha H_{\alpha}) \) by subordination principle of the Feynman path integral and we retrieve the corresponding Green function.

1. Introduction. In the standard quantum mechanics, the photon energy is proportional to the angular frequency and the constant of proportionality is called the Planck constant \( \hbar (E = \hbar \omega) \), but in the new materials such as polymers, biopolymers, liquid crystals there exists a constant \( \alpha, 0 < \alpha < 1 \) such that \( E = \hbar_{\alpha} \omega^\alpha \) where \( \hbar_{\alpha} \) is the so called scaled Planck constant.

In the case \( \alpha = 1 \), we replace \( \hbar_{\alpha} \) by \( \hbar \). Thus the wave function \( \psi = \psi(x, t) \) is the solution of the standard Schrödinger equation

\[
\frac{i\hbar}{\partial t} \partial \psi = H \psi, \quad \psi(x, 0) = \psi_0(x).
\]

(1.1)

Here \( H = -\hbar^2 2m \Delta + V(x) \) and \( |\psi(x, t)|^2 \) is the position probability density, i.e. \( \int_{\mathbb{R}^d} |\psi(x, t)|^2 dx \) is the probability that the wave function’s position is in the Borel set \( \sigma \) at time \( t \). In the same manner if

\[
\hat{\psi}(\xi, t) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \psi(x, t) dx,
\]

is the Fourier transform of \( \psi \), then \( |\hat{\psi}(x, t)|^2 \) is the momentum probability density.

In (1.1), if \( \psi_0 \) is a measurable function belonging to the complex Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^d) \) and \( \int_{\mathbb{R}^d} |\psi_0|^2(x) dx = 1 \), then \( \psi(x, t) \) the solution of (1.1) satisfies also \( \int_{\mathbb{R}^d} |\psi(x, t)|^2 dx = 1 \) and in nonrelativistic quantum mechanic represent the density of particles traveling in \( \mathbb{R}^d \) under the influence of a potential \( V \).

Historically, there are two approaches to study the Feynman path integral, either we take the definition of Feynman path integral as it is presented in [8] and deduce the Schrödinger picture from that, or starting from Schrödinger equation, we try to end up at Feynman path integral (see [9, 10, 11, 12]). In [1], we can find the first approach and see how one can derive the Schrödinger equation from the Feynman path integral. For the reverse approach, let us denote by \( \Omega_x \) the set of all paths
(continuous function) \( \omega : \mathbb{R}_+ \to \mathbb{R}^d \) such that \( \omega(0) = x \). An outline of the Feynman path integral method is defining the wave function \( \psi(x,t) \) by the probability amplitude
\[
L \int_{\Omega_x} e^{iS(\omega,t)} \psi_0(\omega) d\omega,
\]
where \( S(\omega,t) \) is the action of the Lagrangian \( L(\omega, \dot{\omega}) = \frac{m}{2} |\dot{\omega}|^2 - V(\omega) \), that is
\[
S(\omega,t) = \int_0^t \left[ \frac{m}{2} |\dot{\omega}(s)|^2 - V(\omega(s)) \right] ds.
\]

Despite the fact that the physicists are very at ease with this formula, there are many difficulties to interpreting (1.2) rigorously. First the function \( \omega \) can be non-differentiable which makes no sense for the Lagrangian. Next \( \delta \omega = \lim_{n \to \infty} \Pi_{j=1}^n ds_j \), which has no meaning and one needs to introduce the Wiener measure as it is done in [12]. Finally as we will see the constant \( A \) is not finite. However the Trotter product formula can be very helpful for interpreting this formula. In fact if rewrite the equation (1.1) as
\[
\frac{\partial u}{\partial t} = (A + B)u, \quad u(0,x) = f(x),
\]
where \( A := \frac{\text{i} \hbar}{2m} \Delta \) and \( B \) is the multiplication operator \( B f(x) := -\frac{\text{i}}{\hbar} V(x) f(x) \), each of them is a skew-adjoint operator hence they generate the unitary groups
\[
S(t) f(x) = \left( \frac{m}{2\pi \text{i} \hbar t} \right)^{d/2} \int_{\mathbb{R}^d} \exp \left( \frac{-\text{i} m |x-y|^2}{2 \hbar t} \right) f(y) dy
\]
and
\[
V(t) f(x) = \exp(-\frac{\text{i} t}{\hbar} V(x)) f(x).
\]
One can impose some appropriate conditions on the potential \( V \) in order that the closure \( C = A + B \) becomes also skew-adjoint and the Trotter product formula
\[
\lim_{n \to \infty} \left[ S \left( \frac{t}{n} \right) V \left( \frac{t}{n} \right) \right]^n f = e^{tC} f
\]
can be applied. Here \( e^{tC} \) designates the unitary group generated by \( C \) and the convergence is uniform in
\( f \in L^2(\mathbb{R}^d) \) for \( t \) in any compact subset of \( \mathbb{R}^+ \).

In order to give a representation of
\[
U_n(t) := \left[ S \left( \frac{t}{n} \right) V \left( \frac{t}{n} \right) \right]^n,
\]
let us introduce the operator
\[
[K_j f](x_{j-1}) = \left( \frac{nm}{2 \pi \text{i} \hbar t} \right)^{d/2} \int_{\mathbb{R}^d} \exp \left( \frac{\text{i}}{\hbar} \left( \frac{m}{2} \frac{|x_j - x_{j-1}|^2}{t/n} - V(x_j) \right) \right) \left( \frac{t}{n} \right) f(x_j) dx_j.
\]
Thus \( U_n(t) f \) can be expressed as
\[
U_n(t) f = \left[ \prod_{j=1}^n K_j \right] (x).
\]
In other words, setting \( x_0 = x \in \mathbb{R}^d \) and
\[
S(x_0, \ldots, x_n; t) = \sum_{j=1}^n \left[ \frac{m}{2} \frac{|x_j - x_{j-1}|^2}{t/n} - V(x_j) \right] \left( \frac{t}{n} \right),
\]
then

\[ U_n(t)f(x) = \left( \frac{nm}{2\pi i H} \right)^{nd/2} \int_{\mathbb{R}^d} \exp\left\{ \frac{i}{\hbar} S(x_0, \ldots, x_n; t) \right\} f(x_n)dx_1 \cdots dx_n. \]  

(1.10)

Putting \( x_j = \omega(t_j/n) \), we see that \( S(x_0, \ldots, x_n; t) \) defined in (1.9) is a Riemann sum for the action integral (1.3) and formally as \( n \to \infty \), \( U_n \) defined in (1.10) goes to (1.2) in which \( L = \lim_{n \to \infty} \left( \frac{nm}{2\pi i H} \right)^{nd/2} = \infty \) and \( \delta \omega = \lim_{n \to \infty} dx_1 \cdots dx_n \) which is not a measure. In [12, p. 102] one can find many comments on these mathematical difficulties and gives many resolutions to avoid them. However, in the above procedure the fact that \( U_n(t)f \) converges to a solution of (1.1) is perfectly correct and we want to obtain a similar result for the solution of (1.15).

In the next section we introduce the Mittag-Leffler function \( E_\alpha(z) \). In fractional differential problems, Mittag-Leffler functions play the role of exponential functions in differential equations. One the most significant application of the Mittag-Leffler function in the fractional calculus is representation of the solution of fractional differential equation

\[
\begin{cases}
D_\alpha^n u = \lambda u \\
u(0) = f \in C,
\end{cases}
\]  

(1.11)

which depends on \( \lambda \) and \( f \) by

\[ u_{\lambda,f}(t) = E_\alpha(t^n \lambda)f. \]  

(1.12)

Now, consider a closed linear operator \( A \) densely defined in a Banach space \( X \) and consider the following Cauchy problem with time fractional derivative

\[
\begin{cases}
D_\alpha^n u = Au \\
u(0) = f \in X.
\end{cases}
\]  

(1.13)

The first conjecture that comes to mind is according (1.12) to write the solution of (1.13) as \( u(t) = u_{A,f}(t) := E_\alpha(t^n A)f \). But according to our knowledge \( E_\alpha(t^n A) \) is never defined when \( A \) is an unbounded operator. The section 3 is devoted to the \( \mathcal{M} \)-functional calculus for the generator of a semi-contractive semigroup. This material was introduced by the authors in [4] which permits to define \( E_\alpha(t^n A) \) as the family of solution operators. We justify our calculation by using the Wright function and subordination principle given by E. Bajlekova (see [2]). In Section 4 we show that the solution of fractional Schrödinger equation (1.15) can be expressed by \( E_\alpha(-it^n H_\alpha) \) which can be obtained by using the subordination principal in the fractional path integral.

Unfortunately a Trotter product type formula for fractional Schrödinger equation cannot be constructed, in fact according to an idea of P. Chernoff which is exposed in [3], if \( \{F(t) : t \in \mathbb{R}_+\} \) is a family of the bounded operators on the Banach space \( X \), such that \( \lim_{n \to \infty} F(t/n)^n = R(t) \) exists in the strong operator topology, then for any \( t, s \in [0, \infty) \),

\[ R(s + t) = R(s)R(t). \]  

(1.14)

This proves that if there exists an operator valued function \( F(t) \) such that \( s - \lim_{n \to \infty} F(t/n)^n = E_\alpha(t^n A) \) then we have to obtain \( E_\alpha(s^n A)E_\alpha(t^n A) = E_\alpha((s + t)^n A) \) which is not true, as it is shown in the counterexample of [15] ( for \( a = 1 \) and \( t = s = 1, E_\frac{1}{2}(s^{\frac{1}{2}}a)E_\frac{1}{2}(t^{\frac{1}{2}}a) \neq E_\frac{1}{2}((s + t)^{\frac{1}{2}}a) \)).
In [14], M. Naber proposed two options for considering the time fractional Schrödinger equation (TFSE), taking either
\[
i h_\alpha \mathbf{D}_t^\alpha \psi_\alpha(x,t) = H_\alpha \psi_\alpha(x,t), \quad \psi_\alpha(x,0) = \psi_0(x). \tag{1.15}
\]
or
\[
(i h_\alpha)^n \mathbf{D}_t^\alpha \psi_\alpha(x,t) = H_\alpha \psi_\alpha(x,t), \quad \psi_\alpha(x,0) = \psi_0(x). \tag{1.16}
\]
where \( \mathbf{D}_t^\alpha \) is the fractional derivative in the Caputo’s sense, which is defined in Definition 2.3 and \( H_\alpha \) is the Hamiltonian \( H_\alpha = -\frac{\hbar^2}{2m} \Delta + V(x) \) where \( m \) is the mass of particle.

In this paper we treat the problem (1.15). By denoting \( S(t) \) the operator solution of the problem
\[
\frac{\partial \psi}{\partial t}(x,t) = A_\alpha \psi(x,t), \quad \psi(x,0) = \psi_0(x). \tag{1.17}
\]
where \( A_\alpha = (i h_\alpha)^{-1} H_\alpha \) we prove that under some assumption on the potential \( V \), \( S(t) \) is a \( (C_0) \)-group. Despite non-constructibility of the product formula we can express \( E_\alpha(-it^\alpha H_\alpha) \) by subordination principle of the standard Feynman path integral, where we replace \( h \) with \( h_\alpha \). Finally in Section 5 we present the Green function for the family of solution operator \( \{ E_\alpha(-it^\alpha H_\alpha) \}_{t \geq 0} \) in the one dimensional case.

2. Preliminaries. In this paper we take the standard notations
\[
\|u\|_p = \begin{cases} \left( \int_{\mathbb{R}^d} |u(x)|^p \, dx \right)^{1/p}, & \text{for } 1 \leq p < \infty \\ \text{ess sup}_{x \in \mathbb{R}^d} |u(x)|, & \text{for } p = \infty. \end{cases}
\]

**Definition 2.1.** Let \( X \) be a Banach space and \( \tau \in \mathbb{R} \). Let \( g \) be a function of \( L^1_{loc}(0, \infty) \) and \( f \) be an element of \( L^1_{loc}((\tau, \infty), X) \). Then the convolution of \( g \) and \( f \) is the function of \( L^1_{loc}((\tau, \infty), X) \) defined by
\[
g \ast_\tau f(t) = \int_{\tau}^{t} g(t-y) f(y) \, dy, \quad \text{a.e. } t \in [\tau, \infty).
\]

**Definition 2.2.** For \( \beta \in (0, \infty) \), let us denote by \( g_\beta \) the function of \( L^1_{loc}(0, \infty) \) defined for a.e. \( t > 0 \) by
\[
g_\beta(t) = \frac{1}{\Gamma(\beta)} t^{\beta-1}. \tag{2.1}
\]

It is not hard to verify that for each \( \alpha, \beta \in (0, \infty) \), the following identity holds.
\[
g_\alpha \ast g_\beta = g_{\alpha + \beta},
\]
where \( \ast \) stands for \( \ast_0 \).

With this notation we can define the Riemann-Liouville integral
\[
\mathbf{I}_t^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_{\tau}^{t} (t-s)^{\alpha-1} f(s) \, ds = g_\alpha \ast_\tau f(t). \tag{2.2}
\]

Let us notice that, for any \( \alpha, \beta \in \mathbb{R} \), we have
\[
\mathbf{I}_t^\alpha \mathbf{I}_t^\beta = \mathbf{I}_t^{\alpha + \beta}. \tag{2.3}
\]

**Definition 2.3.** Let \( \alpha \in (0,1), \tau \in \mathbb{R} \) and \( f \in C([\tau, \infty), X) \). Let also \( I \) be any sub-interval of \([\tau, \infty)\). We say that \( f \) admits a fractional derivative of order \( \alpha \) in the sense of Caputo in \( C(I, X) \) if
\[
g_{1-\alpha} \ast_\tau (f - f(\tau)) \in C^1(I, X).
\]
and we denote
\[
D_{\alpha,t}^\alpha f := \frac{d}{dt}\{g_{1-\alpha}(f - f(\tau))\}, \quad (2.4)
\]
\[
D_\alpha^\alpha f := D_{0,t}^\alpha f. \quad (2.5)
\]

For \( \alpha = 1 \), \( D_t^\alpha = D_{0,t}^\alpha \) coincides with the standard derivative and we denote \( D_t := D_1^1 \).

If the function \( f \) is differentiable, the we can use the Riemann-Liouville integral and represent the fractional derivative of order \( \alpha \in (0,1] \) in the sense of Caputo by
\[
D_{\tau,t}^\alpha f := \frac{1}{\Gamma(1-\alpha)} \int^t_\tau \frac{D_s f(s)}{(t-s)^\alpha} ds = \Gamma^{1-\alpha}_{\tau} D_t f(t). \quad (2.6)
\]

**Definition 2.4.** For \( \alpha > 0 \) and \( \beta \in \mathbb{R} \), we define the generalised Mittag-Leffler function, \( E_{\alpha,\beta} \) by
\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad z \in \mathbb{C}.
\]

If \( \beta = 1 \) then we put \( E_\alpha := E_{\alpha,1} \) and \( E_\alpha \) is called the Mittag-Leffler function of order \( \alpha \).

Following [16], we introduce the definition of strong solution of (1.13).

**Definition 2.5.** Let \( \alpha \in (0,1] \) and \( f \) be in \( D(A) \). We say that a function \( u \) is a strong solution of (1.13) on \([0,\infty)\) if
(i) \( u \) belongs to \( C([0,\infty), D(A)) \) and \( u(0) = f \);
(ii) \( u \) admits a derivative of order \( \alpha \) in \( C([0,\infty), X) \);
(iii) \( D_t^\alpha u = Au \) in \( C([0,\infty), X) \).

**Definition 2.6.** The mappings \( \{S_\alpha(t) : \in \mathcal{L}(X) : \forall \alpha \in (0,1]\} \) are called solution operators for (1.13) if
(a) \( S_\alpha(0) = I \) for all \( \alpha > 0 \);
(b) \( S_\alpha(t) \) is strongly continuous for \( t > 0 \);
(c) \( S_\alpha(t) D(A) \subset D(A) \) and \( A S_\alpha(t) f = S_\alpha(t) Af \) for all \( f \in D(A), t \geq 0 \);
(d) For any \( f \in D(A), u = S_\alpha(t) f \) is a strong solution of (1.13).

According to the definition of Caputo’s fractional derivative it is not hard to verify that the solution operator of (1.13) for \( \alpha \in (0,1] \) satisfies in a unique manner the following Volterra integral equation (see [16]),
\[
S_\alpha(t) f = f + \int^t_0 g_\alpha(t - s) A u(s) ds.
\]

**Definition 2.7.** The solution operator \( S_\alpha(t) \) is called exponentially bounded if there are two constants \( M \geq 1 \) and \( \omega \geq 0 \) such that
\[
\|S_\alpha(t)\| \leq Me^{\omega t}, \quad t \geq 0. \quad (2.7)
\]

An operator \( A \) is said to belong to class \( C^\alpha(M,\omega) \), if the problem (1.13) admits a solution operator \( S_\alpha(t) \) satisfying (2.7).

**Remark 2.1.** The fact that \( S_\alpha(t) \) hasn’t the semigroup property for \( 0 < \alpha < 1 \) is related to the choice of lower bound \( \tau < \infty \) in the definition of fractional derivative in the sense of Caputo. By taking the lower bound \( \tau = -\infty \), we can obtain as in in [7] some standard properties of Schrödinger equation for its time fractional version.
If $\alpha = 1$ and the operator $A$ generates a $(C_0)$ semigroup of operators $\{e^{tA}\}_{t \geq 0}$ these operators are nothing but $\{S_1(t)\}_{t \geq 0}$ and in this case we denote $A \in C(M, 0)$. For two different reals $0 < \alpha < \beta \leq 2$ there is a relationship between $S_\alpha(t)$ and $S_\beta(t)$ which is called the subordination principle and is proved in [2, Theorem 3.1]. For announcing this theorem we need to define the Wright function.

**Definition 2.8.** For $0 < \alpha < 1$ the following function is called the *Wright function*,

$$\Phi_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-k\alpha + 1 - \alpha)} \text{ for all } z \in \mathbb{C}.$$  

The following relationship between the Wright function and Mittag-Leffler function is of interest,

$$E_\alpha(z) = \int_0^{\infty} \Phi_\alpha(s)e^{zs}ds, \quad z \in \mathbb{C}. \quad (2.8)$$

Furthermore $\Phi_\alpha$ can be considered as a probability density function, in fact

$$\Phi_\alpha(t) \geq 0, \quad t > 0 \quad \text{and} \quad \int_0^{\infty} \Phi_\alpha(s)ds = 1. \quad (2.9)$$

The following Theorem is slight modification of [2, Theorem 3.1] in which the operator $A$ is replaced by $A_\alpha$ as in (1.17)

**Theorem 2.1.** Let $A \in C(M, 0)$ and $0 < \alpha < \beta \leq 2$, $\gamma = \alpha/\beta$, $\omega \leq 0$. If $A_\beta = c_\beta A \in C^\beta(M, \omega)$, then there exists a constant $C > 0$ such that $A_\alpha = c_\alpha A \in C^\alpha(MC, \omega^{1/\gamma})$ and the following formula holds,

$$S_\alpha(t)f = \int_0^{\infty} t^{-\gamma}\Phi_\gamma(st^{-\gamma})S_\beta(s)f ds, \quad f \in X, \quad t > 0. \quad (2.10)$$

Whenever $S_\alpha(t)$ is defined by the above formula we say that $S_\alpha(t)$ is a subordinate solution operator from $S_\beta(t)$. This formula for $\beta = 1$ implies that if $A$ is a generator of a $(C_0)$ semigroup of operators $\{e^{tA}\}_{t \geq 0}$, then

$$S_\alpha(t)f = \int_0^{\infty} t^{-\alpha}\Phi_\alpha(st^{-\alpha})e^{sA}f ds, \quad f \in X, \quad t > 0. \quad (2.11)$$

By introducing the $\mathcal{M}$-functional calculus we give a short proof of (2.10) in which we represent $S_\alpha(t)$ by using the Mittag-Leffler function.

3. $\mathcal{M}$-functional calculus. In [4] the authors have introduced the $\mathcal{M}$-functional calculus as follows. Let

$$\mathcal{M}_+ := \{f \in C(\mathbb{R}) : \mathcal{F}(f) \in L^1(\mathbb{R}), \quad \text{supp } \mathcal{F}(f) \subset [0, \infty]\}$$

where $\mathcal{F}(f)$ is the one dimensional Fourier transform of $f$, i.e.

$$\mathcal{F}(f)(s) = \int_{-\infty}^{\infty} e^{-ixs}f(x)dx, \quad s \in \mathbb{R}.$$  

If $A$ is the generator of an uniformly bounded $(C_0)$ semigroup of operators $\{e^{tA}\}_{t \geq 0}$, satisfying

$$\sup_{t \geq 0} \|e^{tA}\| \leq M,$$

then for any $f \in \mathcal{M}_+$ we can define

$$f(-iA) = \frac{1}{2\pi} \int_0^{\infty} \mathcal{F}(f)(s)e^{sA}ds. \quad (3.1)$$
This defines a bounded linear operator \( f(-iA) \) satisfying
\[
\|f(-iA)\| \leq \frac{M}{2\pi} \|\mathcal{F}f\|_1.
\]

**Theorem 3.1.** Let \( 0 < \alpha < 1 \). If \( A \) is the generator of an uniformly bounded \((C_0)\) semigroup of operators \( \{e^{tA}\}_{t \geq 0} \), then there exists a constant \( C > 0 \) such that \( A \in C^\alpha(MC,0) \) and the following formula holds,
\[
E_\alpha(t^\alpha A)f := S_\alpha(t)f = \int_0^\infty t^{-\alpha}\Phi_\alpha(st^{-\alpha})e^{tA}fds, \quad f \in X, \ t > 0.
\]

**Proof.** Let us define the function \( f(x) = \mathcal{F}^{-1}H(s)(t^{-\alpha}\Phi_\alpha(st^{-\alpha})) \), where \( H(s) \) is the Heaviside function. This function belongs to \( \mathcal{M}_+ \). In fact the definition of the Heaviside function implies that \( \text{supp} \mathcal{F}f \subset [0,\infty) \). Furthermore since \( \mathcal{F}f = H(s)(t^{-\alpha}\Phi_\alpha(st^{-\alpha})) \), (2.9) implies that this function belongs to \( L^1(\mathbb{R}) \). According to (2.8)
\[
f(x) = \frac{1}{2\pi} \int_0^\infty t^{-\alpha}\Phi_\alpha(st^{-\alpha})e^{ixs}ds = \frac{1}{2\pi} \int_0^\infty \Phi_\alpha(\tau)e^{ix\tau}d\tau = \frac{1}{2\pi} E_\alpha(itx^\alpha).
\]
Hence by replacing \( f(-iA) \) by \( \frac{1}{2\pi} E_\alpha(t^\alpha A) \) in (3.1) we get (3.2).
As a consequence we can replace in the subordination relationship (2.10), \( S_\alpha(t) \) and \( S_\beta(s) \) by \( E_\alpha(t^\alpha A) \) and \( E_\alpha(s^\alpha A) \) which gives

**Corollary 3.1.** Under the assumptions of Theorem 2.1 we have
\[
E_\alpha(t^\alpha A)f = \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})E_\beta(s^\beta A)fds, \quad f \in X, \ t > 0.
\]

4. Fractional Feynman path integral. In all that follows we replace \( \hbar \) by \( h_\alpha \). Theorem 3.1 allows us to go back to Feynman path integral and taking the subordinate solution operator of \( U_n^\alpha(t) \) defined in (1.8) in the Hilbert space \( \mathcal{H} \) and by passing to the limit obtain \( E_\alpha(-it^\alpha H_\alpha) \) which is the solution of (1.15).

**Theorem 4.1.** The solution of (1.15) can be written by \( \psi_\alpha(t,x) = E_\alpha(-it^\alpha H_\alpha)\psi_0 \) and can be expressed as
\[
E_\alpha(-it^\alpha H_\alpha)\psi_0 = \lim_{n \to \infty} \int_0^\infty t^{-\alpha}\Phi_\alpha(st^{-\alpha})U_n^\alpha(s)\psi_0ds, \quad \psi_0 \in \mathcal{H}, \ t > 0.
\]

**Proof.** Let us leave aside the index \( \alpha \). Since \( \{S(t)\}_{t \in \mathbb{R}}, \ \{V(t)\}_{t \in \mathbb{R}} \) and \( \{e^{-it\mathcal{H}}\}_{t \in \mathbb{R}} \) are all the unitary groups in \( \mathcal{H} \),
\[
\left\|S\left(\frac{t}{n}\right)V\left(\frac{t}{n}\right)^n - e^{-it\mathcal{H}}\right\| \leq 2
\]
and thanks to the Trotter product formula (1.7),
\[
\lim_{n \to \infty} \left[S\left(\frac{t}{n}\right)V\left(\frac{t}{n}\right)^n \right] f = e^{-it\mathcal{H}}f
\]
uniformly in any compact \([0,T]\), hence for any \( \varepsilon > 0 \) there exists an \( n \) enough large such that
\[
\left\|S\left(\frac{t}{n}\right)V\left(\frac{t}{n}\right)^n f - e^{-it\mathcal{H}}f\right\| < \frac{\varepsilon}{2}\|f\|, \quad \text{for any } T > 0 \quad \text{and any } t \in [0,T].
\]
consequently, since \( \int_0^\infty t^{-\alpha} \Phi_\alpha(st^{-\alpha}) ds = 1 \), one can take \( T \) enough large such that \( \int_T^\infty t^{-\alpha} \Phi_\alpha(st^{-\alpha}) ds \leq \frac{\varepsilon}{T} \) and

\[
\left\| \int_0^\infty t^{-\alpha} \Phi_\alpha(st^{-\alpha}) [U_n(s)f - e^{-isH}f] ds \right\|
\leq \int_0^T t^{-\alpha} \Phi_\alpha(st^{-\alpha}) \|U_n(s)f - e^{-isH}f\| ds + \int_T^\infty t^{-\alpha} \Phi_\alpha(st^{-\alpha}) \|U_n(s)f - e^{-isH}f\| ds
\leq \left( \frac{\varepsilon}{\sqrt{2}} + \frac{2\varepsilon}{4} \right) \|f\| = \varepsilon \|f\|.
\]

By noting that \( E_\alpha(-it^\alpha H_\alpha)\psi_0 = \int_0^\infty t^{-\alpha} \Phi_\alpha(st^{-\alpha})e^{-isH_\alpha}\psi_0 ds \), we deduce (4.1). \( \square \)

5. The Green function for fractional Schrödinger equation. As we have proved in the Introduction that to get a Lie-Trotter type formula is unworkable for fractional Schrödinger equation, but despite that we can obtain a new version of the Green function for the fractional Schrödinger equation. For the standard version Schrödinger equation \((\alpha = 1)\) the Green function is

\[
G(x,t) = \left( \frac{m}{2\pi i\hbar t} \right)^{d/2} \exp \left( \frac{im|\xi|^2}{2\hbar t} \right),
\]

(5.1)
since the solution is \( u(x,t) = \int_{\mathbb{R}^d} G(x,y)f(y)dy \) (see (1.5)). For the fractional version of (5.1), let us take \( 0 < \alpha < 1 \). Then the Fourier transform of

\[
\mathcal{D}_t^\alpha u = \frac{i\hbar_\alpha}{2m} \Delta u, \quad u(x,0) = f(x)
\]

(5.2)
gives

\[
\mathcal{D}_t^\alpha \tilde{u}(\xi,t) = -\frac{i\hbar_\alpha}{2m}|\xi|^2 \tilde{u}, \quad \tilde{u}(\xi,0) = \tilde{f}(\xi)
\]

which according to (1.11) its solution introduces the Mittag-Leffler function: \( \tilde{u}(\xi,t) = E_\alpha(-\frac{i\hbar_\alpha}{2m}|\xi|^2 t^\alpha)\tilde{f}(\xi) \). Hence

\[
u(x,t) = \mathcal{F}^{-1}(E_\alpha(-\frac{i\hbar_\alpha}{2m}|\xi|^2 t^\alpha)) \ast f(x)
\]

(5.3)
and the Green function for the fractional Schrödinger equation is

\[
G_\alpha(x,t) = \mathcal{F}^{-1}(E_\alpha(-\frac{i\hbar_\alpha}{2m}|\xi|^2 t^\alpha)).
\]

(5.4)
According to [5, 6], taking \( P(\xi) = -\frac{i\hbar_\alpha}{2m}|\xi|^2 \), since \( |\arg(P(\xi))| = \frac{\pi}{2} \) there exists a constant \( C > 0 \) such that \( |E_\alpha(-\frac{i\hbar_\alpha}{2m}|\xi|^2 t^\alpha)| \leq C(1+t^\alpha|\xi|^2)^{-1} \), thus \( E_\alpha(-\frac{i\hbar_\alpha}{2m}|\cdot|^2 t^\alpha) \in L^1(\mathbb{R}^d) \) and (5.4) is well-defined.

In order to calculate \( G_\alpha(x,t) \) we use (2.8) and write (5.4) as

\[
G_\alpha(x,t) = \mathcal{F}^{-1} \left( \int_0^\infty \Phi_\alpha(s) \exp(-\frac{i\hbar_\alpha}{2m}|\xi|^2 s) ds \right)
\]

\[
= \int_0^\infty t^{-\alpha} \Phi_\alpha(st^{-\alpha}) \mathcal{F}^{-1} \left( \exp(-\frac{i\hbar_\alpha}{2m}|\xi|^2 s) \right) ds
\]
\[ \int_0^\infty t^{-\alpha} \Phi_{\alpha} (st^{-\alpha}) \left( \frac{m}{2\pi i \hbar_a s} \right)^{1/2} \exp \left( \frac{-im|x|^2}{2\hbar_a s} \right) ds \]

\[ = \int_0^\infty t^{-\alpha} \Phi_{\alpha} (st^{-\alpha}) G(x, s) ds. \]

This is exactly the subordinate formula for Green function.

In one dimensional case where \( d = 1 \) we can go one step further and we take the Laplace transform \( [\mathcal{L} f](\lambda) = \int_0^\infty e^{-\lambda s} f(x) dx \) of (5.4). Knowing that (see [13]) the Laplace transform and Fourier transform pairs for the Mittag-Leffler function is given by

\[ E_\alpha (zt^\alpha) \xleftrightarrow{\mathcal{L}} \frac{\lambda^{\alpha-1}}{\lambda^\alpha - z} \]

for \( Re(\lambda) > |z|^{1/\alpha} \) with \( z \in \mathbb{C} \). We get

\[ G_\alpha (x, t) = \mathcal{L}^{-1} \left( \mathcal{F}^{-1} \frac{\lambda^{\alpha-1}}{\lambda^\alpha + \frac{i\hbar}{2m} |\xi|^2} \right). \]

Since for any \( z \in \mathbb{C} \),

\[ \frac{\lambda^{\alpha-1}}{\lambda^\alpha + z|\xi|^2} \xleftrightarrow{\mathcal{F}} \frac{1}{2\sqrt{2}} \lambda^{\alpha/2-1} \exp \left( -\frac{|x|\lambda^{\alpha/2}}{\sqrt{2}} \right) \]

thus

\[ \mathcal{F}^{-1} \left( \frac{\lambda^{\alpha-1}}{\lambda^\alpha + \frac{i\hbar}{2m} |\xi|^2} \right) = \sqrt{\frac{m}{2i\hbar}} \lambda^{\alpha/2-1} \exp \left( -\sqrt{\frac{2m}{i\hbar}} \lambda^{\alpha/2} |x| \right) \]

and the Laplace transform pair for the Wright function is

\[ \frac{1}{t^\beta} \Phi_{\beta} \left( \frac{z}{t^\beta} \right) \xleftrightarrow{\mathcal{L}} \lambda^{\beta-1} \exp \left( -z \lambda^\beta \right), \]

thus we obtain the following Theorem.

**Theorem 5.1.** The Green function of the fractional free Schrödinger equation (5.2) is given by

\[ G_\alpha (x, t) = \left( \frac{m}{2i\hbar_a t^\alpha} \right)^{1/2} \Phi_{\alpha/2} \left( \frac{\sqrt{2m}|x|}{\sqrt{i\hbar_a} t^{\alpha/2}} \right). \] (5.5)

This expression is announced in [1] and the justification is given by taking the limit \( \alpha \to 1 \), since \( \Phi_{1/2} \) is nothing but the Gaussian

\[ \Phi_{1/2} (|x|) = \frac{1}{\sqrt{\pi}} e^{-|x|^2/4} \]

(see [13]), hence by replacing \( \alpha = 1 \) in (5.5), we obtain (5.1).

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