Second-order corrections to noncommutative spacetime inflation

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Abstract

We investigate how the uncertainty of noncommutative spacetime affects on inflation. For this purpose, the noncommutative parameter $\mu_0$ is taken to be a zeroth order slow-roll parameter. We calculate the noncommutative power spectrum up to second order using the slow-roll expansion. We find corrections arisen from a change of the pivot scale and the presence of a variable noncommutative parameter, when comparing with the commutative power spectrum. The power-law inflation is chosen to obtain explicit forms for the power spectrum, spectral index, and running spectral index. In cases of the power spectrum and spectral index, the noncommutative effect of higher-order corrections compensates for a loss of higher-order corrections in the commutative case. However, for the running spectral index, all higher-order corrections to the commutative case always provide negative spectral indexes, which could explain the recent WMAP data.

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I. INTRODUCTION

String theory as a candidate for the theory of everything can say something about cosmology [1]. Focusing on a universal property of string theory, it is very interesting to study its connection to cosmology. The universal property which we wish to choose here is a new uncertainty relation of $\Delta t_p \Delta x_p \geq l_s^2$ where $l_s$ is the string length scale [2]. This implies that spacetime is noncommutative. It is compared to a stringy uncertainty relation of $\Delta x_p \Delta p \geq 1 + l_s^2 \Delta p^2$. The former is considered as a universal property for strings as well as D-branes, whereas the latter is suitable only for strings. Spacetime noncommutativity does not affect the evolution of the homogeneous background. However, this leads to a coupling between the fluctuations generated in inflation and the flat background of Friedmann-Robertson-Walker (FRW) spacetime [3]. Usually the coupling appears to be nonlocal in time.

On the other hand, it is generally accepted that curvature perturbations produced during inflation are considered to be the origin of inhomogeneities necessary for explaining galaxy formation and other large-scale structure. The first year results of WMAP put forward more constraints on cosmological models and confirm the emerging standard model of cosmology, a flat $\Lambda$-dominated universe seeded by scale-invariant adiabatic gaussian fluctuations [4]. In other words, these results coincide with predictions of the inflationary scenario with an inflaton. Also WMAP brings about some new intriguing results: a running spectral index of scalar metric perturbations and an anomalously low quadrupole of the CMB power spectrum [5]. If inflation is affected by physics at a short distant close to string scale, one expects that the spacetime uncertainty must be encoded in the CMB power spectrum [6]. For example, the noncommutative power-law inflation may produce a large running spectral index to fit the data of WMAP [7–9].

Recently the noncommutative power spectrum, spectral index, and running spectral index of the curvature perturbations produced during inflation have been calculated with the slow-roll parameters $\epsilon_1$ and $\delta_n$ and noncommutative parameter $\mu_0$ [10]. Authors in [11] have examined whether or not the noncommutative parameter is considered as a slow-roll parameter. It turned out that the noncommutative parameter $\mu_0$ is considered as a zeroth order slow-roll parameter. In this work, we will make a further progress in this direction. We calculate the noncommutative power spectrum up to second order in the slow-roll expansion and up to first order in the noncommutative parameter. Also the spectral index and running spectral index are obtained for the noncommutative spacetime inflation. By choosing the power-law inflation, we find corrections to the commutative inflation. These result from a change of the pivot scale and the presence of $\mu_0 \neq \text{constant}$. In order to understand the role of $\mu_0$ further, we also calculate the power spectrum using $\mu_0 = \text{constant}$, and the spectral index and running spectral index using $\frac{d\mu_0}{d \ln k} \simeq -4\mu_0 \epsilon_1$.

The organization of this work is as follows. In Sec. II we review the slow-roll approximation in the noncommutative spacetime inflation. We calculate the power spectrum up to second order using the slow-roll expansion with $\mu_0 \neq \text{constant}$ in Sec. III. Sec. IV is devoted to obtaining the power spectrum up to second-order corrections by making use of the slow-roll expansion with $\mu_0 = \text{constant}$. This gives us an intermediate result between the slow-roll approximation and slow-roll expansion. Finally we discuss our results in Sec. V.
II. SLOW-ROLL APPROXIMATION

Our starting point is the effective action during inflation,

\[ S = \int \left[ -\frac{M_p^2}{2} R + \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) \right] \sqrt{-g} \, d^4x, \tag{1} \]

where \( M_p^2 \) is the reduced Planck mass defined by \( M_p = (8\pi G)^{1/2} \). For simplicity we choose \( M_p^2 = 1 \). The scalar metric perturbation to the homogeneous, isotropic background is expressed in the longitudinal gauge as \([12]\)

\[ ds_{\text{conf}}^2 = a^2(\eta) \left\{ (1 + 2A) d\eta^2 - (1 + 2\psi) \text{d}x \cdot \text{d}x \right\}, \tag{2} \]

where the conformal time \( \eta \) is given by \( d\eta = dt/a \). We get a relation of \( \psi = A \) because the stress-energy tensor does not have any off-diagonal component. It is convenient to express the density perturbation in terms of the curvature perturbation \( R_c \) of comoving hypersurfaces given by \([13]\)

\[ R_c = \psi - \frac{H}{\dot{\phi}} \delta \phi \tag{3} \]

during inflation, where \( \delta \phi \) is the perturbation in inflaton: \( \phi(x, \eta) = \phi(\eta) + \delta \phi(x, \eta) \). The overdot is derivative with respect to a comoving time \( t \) defined in the flat FRW line element: \( ds_{\text{FRW}}^2 = dt^2 - a(t)^2 \text{d}x \cdot \text{d}x \). Introducing

\[ z \equiv \frac{a\dot{\phi}}{H} \quad \text{and} \quad \varphi \equiv a \left( \delta \phi - \frac{\dot{\phi}}{H} \psi \right) = -z R_c, \tag{4} \]

the bilinear action for curvature perturbation is \([14]\)

\[ S = \int \frac{1}{2} \left[ \left( \frac{\partial \varphi}{\partial \eta} \right)^2 - (\nabla \varphi)^2 + \left( \frac{1}{z} \frac{d^2 \varphi}{d\eta^2} \right)^2 \right] d\eta \, d^3x. \tag{5} \]

At this stage we wish to note that \( z \) encodes information about inflation. Because the background is spatially flat, we can expand all perturbed fields in terms of Fourier modes as \( \varphi(x, \eta) = \int \frac{d^3k}{(2\pi)^3} \varphi_k(\eta) e^{i\mathbf{k} \cdot \mathbf{x}} \). In second quantization, these modes are given by \( \varphi_k(\eta) = b(\mathbf{k}) \varphi_k(\eta) + b^\dagger(-\mathbf{k}) \varphi^*_k(\eta) \). This quantum-to-classical behavior is a great success for the theory \([15]\). If it had failed, prediction for the power spectrum would have had nothing to do with reality.

For convenience we introduce another time coordinate \( \tau \) to incorporate the noncommutative spacetime. Then the perturbed metric in Eq.(2) can be rewritten as

\[ ds_{\text{non-p}}^2 = a^{-2}(\tau)(1 + 2A) d\tau^2 - a^2(\tau)(1 + 2\psi) \text{d}x \cdot \text{d}x. \tag{6} \]

The spacetime uncertainty relation of \( \Delta t_p \Delta x_p \geq l_s^2 \) becomes

\[ \Delta \tau \Delta x \geq l_s^2 \tag{7} \]

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for a cosmological purpose. This stringy spacetime uncertainty relation is compatible with a FRW background spacetime. However a nonlocal coupling between the fluctuation mode and the background arises because of the \(*\)-product. We propose the transition to noncommutative spacetime obeying Eq.(7) by taking the operator appearing in the bilinear action in Eq.(5) and replacing all multiplications by \(*\)-products [3]. Using the Fourier transform, the uncertainty relation leads to the modified action for \(\tilde{\phi}_k\):

\[
\tilde{S} = \frac{V_T}{2} \int \tilde{\eta} d^3k \int d^3\eta \left[ \frac{d\tilde{R}_{-k}}{d\eta} \frac{d\tilde{R}_{k}}{d\eta} - k^2 \tilde{R}_{-k} \tilde{R}_{k} \right],
\]

(8)

where \(V_T\) is the total spatial volume, and \(\tilde{\eta}\) is a new conformal time which is suitable for a noncommutative spacetime inflation. Here \(z_k\) is some smeared version of \(z\) or \(a\) over a range of time of characteristic scale \(\Delta \tau = l_s^2k\) defined by

\[
z^2_k(\tilde{\eta}) = z^2y^2_k(\tilde{\eta}), \quad y^2_k = \sqrt{\beta_k^+ \beta_k^-}, \quad \frac{d\tilde{\eta}}{d\tau} = \frac{\sqrt{\beta_k^+}}{\sqrt{\beta_k^-}},
\]

(9)

where

\[
\beta_k^\pm = \frac{1}{2} \left[ a^\pm(\tau + l_s^2k) + a^\pm(\tau - l_s^2k) \right].
\]

(10)

This representation has the advantage of preserving both spatial translational and rotational symmetry of the flat FRW metric, in compared to constructions based on the conventional noncommutative relations: \([x^\mu, x^\nu] = i\theta^{\mu\nu}\) [16,17]. Actually a spacetime noncommutativity does not affect the evolution of the homogeneous background. However, this leads to a coupling between the fluctuations generated in inflation and the flat background of FRW space. The coupling appears to be nonlocal in time as is shown in Eq.(10). Actually \(\Delta \tau = l_s^2k\) in Eq.(7) induces the uncertainty of time in defining \(a\). If one does not require the uncertainty relation, one finds easily commutative relations that \(y_k \rightarrow 1\), \(z_k \rightarrow z\), \(\tilde{\eta} \rightarrow \eta\), \(\tilde{\varphi}_k(\tilde{\eta}) \rightarrow \varphi_k(\eta)\).

From Eq.(8), we find the Mukhanov-type equation

\[
\frac{d^2\tilde{\varphi}_k}{d\tilde{\eta}^2} + \left( k^2 - \frac{1}{z_k} \frac{d^2z_k}{d\tilde{\eta}^2} \right) \tilde{\varphi}_k = 0.
\]

(11)

Our task is to solve Eq. (11). For this purpose we introduce slow-roll parameters \(\mu_0\), \(\epsilon_1\), \(\delta_n\) defined as

\[
\mu_0 = \left( \frac{kH}{aM_s^2} \right)^2, \quad \epsilon_1 = -\frac{\dot{H}}{H^2} = \frac{1}{2} \left( \frac{\dot{\varphi}}{H} \right)^2, \quad \delta_n \equiv \frac{1}{H^n\dot{\varphi}} \frac{d^{n+1}\varphi}{dt^{n+1}}.
\]

(12)

Here \(M_s = 1/l_s\) and the subscript denotes the order in the slow-roll expansion. We use the slow-roll approximation which means that \(\epsilon_1\), \(\delta_1\), \(\mu_0\) are taken to be approximately constant in calculation of the noncommutative power spectrum. That is, \(\epsilon_1\), \(\delta_1\), \(\mu_0\) =constant. For this purpose, we obtain relations up to first order [10]

\[
\frac{1}{z_k} \frac{d^2z_k}{d\tilde{\eta}^2} \sim 2(aH)^2 \left( 1 + \epsilon_1 + \frac{3}{2} \delta_1 - 2\mu_0 \right).
\]

(13)
and
\[ aH \simeq -\frac{1}{\eta}(1 + \epsilon_1 + \mu_0). \quad (14) \]

Then Eq.(11) takes the form
\[ \frac{d^2\tilde{\varphi}_k}{d\tilde{\eta}^2} + \left( k^2 - \frac{(\nu^2 - \frac{1}{4})}{\tilde{\eta}^2} \right) \tilde{\varphi}_k = 0 \quad (15) \]

with \( \nu = \frac{3}{2} + 2\epsilon_1 + \delta_1 \). We note here that this equation takes the same form as in the commutative case [18]. The asymptotic solution to Eq.(11) in the limit of \(-k\tilde{\eta} \to \infty\) takes the form
\[ \tilde{\varphi}_k = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}. \quad (16) \]

In the limit of \(-k\tilde{\eta} \to 0\), one finds asymptotic form of the Hankel function \( H_{\nu}^{(1)}(-k\tilde{\eta}) \)
\[ \tilde{\varphi}_k \simeq e^{i(\nu - \frac{1}{2})} 2^{\nu - \frac{1}{2}} \Gamma(\nu) \frac{1}{\Gamma(\frac{3}{2})} \sqrt{2k} (-k\tilde{\eta})^{\frac{1}{2} - \nu}. \quad (17) \]

Furthermore, from Eqs.(9) and (10), we have an expression up to first order
\[ y_k \simeq 1 + \mu_0. \quad (18) \]

The Fourier transform of curvature perturbation is given by \( \tilde{R}_k = -\tilde{\varphi}_k(\tilde{\eta})/z_k \). Then the noncommutative power spectrum is defined by
\[ \tilde{P}_{Rc}(k) = \left( \frac{k^3}{2\pi^2} \right) \lim_{-k\tilde{\eta} \to 0} \left| \frac{\tilde{\varphi}_k}{z_k} \right|^2. \quad (19) \]

Substituting equations (14), (17) and (18) into Eq.(19), one finds
\[ \tilde{P}_{Rc}(k) = \frac{H^4}{(2\pi)^2 \dot{\varphi}^2} \left[ 2^{\nu - \frac{3}{2}} \Gamma(\nu) \Gamma(\frac{3}{2}) \right]^2 \left( \frac{k}{aH} \right)^{-2(2\epsilon_1 + \delta_1)} 1 \left( 1 + \epsilon_1 + \mu_0 \right)^{2(1+2\epsilon_1 + \delta_1)} \left( 1 + \mu_0 \right)^{2}. \quad (20) \]

Making use of the Taylor expansions up to first order as
\[ 2^{\nu - \frac{3}{2}} \frac{\Gamma(\nu - \frac{3}{2} + \frac{3}{2})}{\Gamma(\frac{3}{2})} \simeq 1 + \alpha(2\epsilon_1 + \delta_1), \quad e^{-2(2\epsilon_1 + \delta_1)\ln \left( \frac{k}{aH} \right)} \simeq 1 - 2(2\epsilon_1 + \delta_1) \ln \left( \frac{k}{aH} \right), \quad (21) \]

we have [11]
\[ \tilde{P}_{Rc}^{1st}(k) = \frac{H^4}{(2\pi)^2 \dot{\varphi}^2} \left\{ 1 - 2\epsilon_1 - 4\mu_0 + 2 \left( \alpha - \ln \left( \frac{k}{aH} \right) \right) (2\epsilon_1 + \delta_1) \right\}. \quad (22) \]

In the limit of \( \mu_0 \to 0 \), \( \tilde{P}_{Rc}^{1st}(k) \) reduces to the commutative power spectrum [19,20]. In the noncommutative spacetime approach the horizon crossing occurs at \( k^2 = \frac{1}{z_k aH^2} \) [3].
Hence, from Eq.(13) we use the other pivot scale $k_* = \sqrt{2}aH$. As a result, we obtain the noncommutative power spectrum up to first order as

$$\hat{P}_{R_c}^{1st}(k) = \frac{1}{2\epsilon_1 (2\pi)^2} \frac{H^2}{k^2} \{1 - 2\epsilon_1 - 4\mu_0 + 2\alpha_*(2\epsilon_1 + \delta_1)\} \bigg|_{k = \sqrt{2}aH}$$

(23)

with $\alpha_* = \alpha - \frac{\ln 2}{2}$. Here we note that the right hand side is evaluated at $k = \sqrt{2}aH$.

Let us compare Eq.(23) with the commutative power spectrum. An apparent change of the pivot scale from $k_c = aH$ to $k_* = \sqrt{2}aH$ amounts to replacing $\alpha = 0.7296$ by $\alpha_* = 0.3831$ in the first-order calculation [20]. Further we note that $k_c = a(\vec{\eta}_c)H(\vec{\eta}_c)$ and $k_* = \sqrt{2}a(\vec{\eta}_*)H(\vec{\eta}_*)$. This means that the noncommutative horizon crossing time $\vec{\eta} = \vec{\eta}_*$ is different from $\vec{\eta} = \vec{\eta}_c$. In order to make its correction, we have to establish a connection between $H(\vec{\eta}_c)$ and $H(\vec{\eta}_*)$, $\epsilon_1(\vec{\eta}_c)$ and $\epsilon_1(\vec{\eta}_*)$, and so on [21]. At the level of slow-roll approximation, we consider slow-roll parameters to be constant everywhere, except in the factor $1/\epsilon_1$. Performing a series expansion around $\vec{\eta} = \vec{\eta}_c$ leads to

$$\frac{1}{\epsilon_1(\vec{\eta}_c)} = \frac{1}{\epsilon_1(\vec{\eta}_c)}[1 - 2(\epsilon_1(\vec{\eta}_c) + \delta_1(\vec{\eta}_c))\Delta\vec{\eta} + \cdots], \quad H^2(\vec{\eta}_c) = H^2(\vec{\eta}_c)[1 - 2\epsilon_1(\vec{\eta}_c)\Delta\vec{\eta} + \cdots]$$

(24)

and

$$\ln \left[ \frac{k}{a(\vec{\eta}_c)H(\vec{\eta}_c)} \right] = \ln \left[ \frac{k}{a(\vec{\eta}_c)H(\vec{\eta}_c)} \right] - \Delta\vec{\eta} + \cdots,$$

(25)

where the dots represent first-order terms and $\Delta\vec{\eta} = \vec{\eta}_* - \vec{\eta}_c$. Substituting everything into Eq.(22), one finds that all $\Delta\vec{\eta}$ cancel. When $\mu_0 = 0$, the result is the same form as in the commutative case except replacing $\alpha$ by $\alpha_*$

$$\hat{P}_{R_c}^{1st}(k) = \frac{1}{2\epsilon_1 (2\pi)^2} \frac{H^2}{k^2} \{1 - 2\epsilon_1 - 4\mu_0 + 2\alpha_*(2\epsilon_1 + \delta_1)\} \bigg|_{k = aH}.$$  

(26)

Hence we could use the slow-roll expansion at $k = k_c$ to obtain the noncommutative power spectrum at $k = k_*$. As is shown in Eq.(26), the noncommutative effect is to reduce the power spectrum by making use of $\alpha \to \alpha_*$ and $\mu_0 \neq 0$.

### III. SLOW-ROLL EXPANSION WITH $\mu_0 \neq$ CONSTANT

The slow-roll approximation could not be considered as a correct approach to calculate the power spectrum even for up to first order. In order to calculate the power spectrum up to second order correctly, one should use the slow-roll expansion based on Green’s function technique. In the case of $\mu_0=0$, two of the slow-roll approximation and slow-roll expansion give the same power spectrum up to first order. However, in the case of $\mu_0 \neq 0$, two provide different results. The assumption that slow-roll parameters are taken to be constant in the slow-roll approximation is not generally true.

The key step is to introduce a variable nature of slow-roll parameters during inflation:

$$\dot{\mu}_0 = -4H\mu_0\epsilon_1, \quad \dot{\epsilon}_1 = 2H(\epsilon_1^2 + \epsilon_1\delta_1), \quad \dot{\delta}_1 = H(\epsilon_1\delta_1 - \delta_1^2 + \delta_2)$$

(27)
which means that the derivative of slow-roll parameters with respect to time increases their order by one in the slow-roll expansion. However this expansion is useful for deriving the power spectrum at \( k = aH \) but not \( k = \sqrt{2}aH \) and thus it works well for commutative case and higher order case. As far as we know, there is no way to calculate the power spectrum up to second order at an arbitrary pivot scale using the slow-roll expansion. Hence, first we use the slow-roll expansion at \( k = aH \) to calculate the noncommutative power spectrum up to second order. Then, assuming the rule that the change of pivot scale from \( k = aH \) to \( k = \sqrt{2}aH \) amounts to replacing \( \alpha \) by \( \alpha_* \), our calculation may provide the result up to second order that will be derived from the slow-roll expansion at \( k = \sqrt{2}aH \).

Using notations of \( y = \sqrt{2k} \tilde{\varphi}_k \) and \( \tilde{x} = -k\tilde{\eta} \), we can reexpress Eq.(11) as

\[
\frac{d^2 y}{d\tilde{x}^2} + \left( 1 - \frac{1}{z_k} \frac{d^2 z_k}{d\tilde{x}^2} \right) y = 0. 
\tag{28}
\]

In general its asymptotic solutions are given by

\[
y \to \begin{cases} 
e^{i\tilde{x}} & \text{as } \tilde{x} \to \infty \\ \sqrt{2k} \tilde{A}_k z_k & \text{as } \tilde{x} \to 0. \end{cases} \tag{29}
\]

We solve Eq. (28) with the boundary condition Eq.(29) to eventually calculate \( \tilde{A}_k \). Now we can choose the ansatz that \( z_k \) takes the form

\[
z_k = \frac{1}{\tilde{x}} f(\ln \tilde{x}). \tag{30}
\]

Then we have

\[
\frac{1}{z_k} \frac{d^2 z_k}{d\tilde{x}^2} = \frac{2}{\tilde{x}^2} + \frac{1}{\tilde{x}^2} \tilde{g}(\ln \tilde{x}), \tag{31}
\]

where

\[
\tilde{g} = -3\tilde{f}' + \tilde{f}'' \tag{32}
\]

and the equation of motion is

\[
\frac{d^2 y}{d\tilde{x}^2} + \left( 1 - \frac{2}{\tilde{x}^2} \right) y = \frac{1}{\tilde{x}^2} \tilde{g}(\ln \tilde{x}) y. \tag{33}
\]

The homogeneous solution with correct asymptotic behavior at \( \tilde{x} \to \infty \) is

\[
y_0(\tilde{x}) = \left( 1 + \frac{i}{\tilde{x}} \right) e^{i\tilde{x}}. \tag{34}
\]

Using Green’s function technique, Eq.(33) with the boundary condition Eq.(29) can be written as the integral equation

\[
y(\tilde{x}) = y_0(\tilde{x}) + \frac{i}{2} \int_{\tilde{x}}^{\infty} du \frac{1}{u^2} \tilde{g}(\ln u) y(u) \left[ y_0^*(u)y_0(\tilde{x}) - y_0^*(\tilde{x})y_0(u) \right]. \tag{35}
\]
We are now in a position to solve Eq. (35) perturbatively using the slow-roll expansion. Introducing

\[ \ddot{x}z_k = \ddot{f}(\ln \bar{x}) = \sum_{n=0}^{\infty} \frac{\dot{f}_n}{n!}(\ln \bar{x})^n, \tag{36} \]

\( \dot{f}_n/\dot{f}_0 \) is of order \( n \) in the slow-roll expansion. This expansion is useful for \( \exp(-1/\xi) \ll \bar{x} \ll \exp(1/\xi) \) and for extracting information at \( \bar{x} = 1 \).

Considering a relation up to second order in the slow-roll expansion and up to first order in \( \mu_0 \) as

\[ \bar{x} = -k \eta = -k \int d\tau \left( \frac{\beta_0}{\beta_k^2} \right)^{1/2} \simeq \frac{k}{aH} \left\{ 1 + \epsilon_1 + 3\epsilon_1^2 + 2\delta_1 \epsilon_1 + \mu_0(1 - 2\epsilon_1) \right\}, \tag{37} \]

we can express the expansion coefficients \( \dot{f}_n \) in terms of \( \epsilon_1, \delta_n, \) and \( \mu_0 \) evaluated at \( k = aH \). In deriving the above expression, we use Eq.(27). From Eq.(36) we obtain

\[ \dot{f}_2 \simeq \frac{k \dot{\phi}}{H^2} \left\{ 8\epsilon_1^2 + 9\epsilon_1 \delta_1 + \delta_2 + \mu_0(2\delta_2 - 14\epsilon_1 \delta_1) \right\} \bigg|_{k=aH}, \tag{38} \]

\[ \dot{f}_1 \simeq -\frac{k \dot{\phi}}{H^2} \left\{ 2\epsilon_1 + \delta_1 + 6\epsilon_1^2 + 4\epsilon_1 \delta_1 + \mu_0(4\epsilon_1 - 2\delta_1 + 14\epsilon_1^2 + \epsilon_1 \delta_1) \right\} \bigg|_{k=aH}, \tag{39} \]

\[ \dot{f}_0 \simeq \frac{k \dot{\phi}}{H^2} \left\{ 1 + \epsilon_1 + 5\epsilon_1^2 + 3\epsilon_1 \delta_1 + \mu_0(2 + \epsilon_1 + \delta_1 + \epsilon_1^2 + 6\epsilon_1 \delta_1) \right\} \bigg|_{k=aH}, \tag{40} \]

\[ \frac{1}{f_0} \simeq \frac{H^2}{k \phi} \left\{ 1 - \epsilon_1 - 4\epsilon_1^2 - 3\epsilon_1 \delta_1 - \mu_0(2 - 3\epsilon_1 + \delta_1 - 15\epsilon_1^2 - 8\epsilon_1 \delta_1) \right\} \bigg|_{k=aH}, \tag{41} \]

\[ \frac{\dot{f}_1}{f_0} \simeq \left\{ -2\epsilon_1 - \delta_1 - 4\epsilon_1^2 - 3\epsilon_1 \delta_1 + \mu_0(8\epsilon_1 + 16\epsilon_1^2 + 10\epsilon_1 \delta_1 + \delta_1^2) \right\} \bigg|_{k=aH}, \tag{42} \]

\[ \frac{\dot{f}_2}{f_0} \simeq \left\{ 8\epsilon_1^2 + 9\epsilon_1 \delta_1 + \delta_2 - 16\mu_0 \epsilon_1(\epsilon_1 + 2\delta_1) \right\} \bigg|_{k=aH}. \tag{43} \]

Further Eqs.(32) and (36) give

\[ \bar{g}(\ln \bar{x}) = \sum_{n=0}^{\infty} \frac{\bar{g}_{n+1}}{n!}(\ln \bar{x})^n, \tag{44} \]

where \( \bar{g}_n \) is of order \( n \) in the slow-roll expansion and, up to second order

\[ \bar{g}_1 \simeq -3\frac{\dot{f}_1}{f_0} + \frac{\dot{f}_2}{f_0}, \quad \bar{g}_2 \simeq -3 \left( \frac{\dot{f}_1}{f_0} \right)^2 - 3\frac{\dot{f}_2}{f_0}. \tag{45} \]

Expanding \( y \) as

\[ y(\bar{x}) = \sum_{n=0}^{\infty} y_n(\bar{x}), \tag{46} \]
where \( y_0(\tilde{x}) \) is the homogeneous solution in Eq. (34), and \( y_n(\tilde{x}) \) is of order \( n \) in the slow-roll expansion. Following the procedure in commutative case \([22]\), we solve Eq. (35) perturbatively by substituting Eqs. (44) and (46) and equating terms of the same order. We obtain the asymptotic form for \( y \) up to first-order corrections

\[
y(\tilde{x}) \rightarrow i \left\{ \frac{1}{\tilde{x}} \left( \alpha + \frac{i\pi}{2} \right) + \frac{\tilde{\alpha}}{18} \left[ \alpha^2 - \frac{2}{3}x\alpha - 4 + \frac{x^2}{4} \right] + i\pi \left( \alpha - \frac{1}{3} \right) \right\} + \frac{2\tilde{\alpha}}{\tilde{x}} \left[ \alpha^2 - \frac{2}{3}x\alpha - \frac{x^2}{12} + i\pi \left( \alpha + \frac{1}{3} \right) \right] \]

Comparing this with Eq. (47), the coefficient of \( \tilde{x}^{-1} \) is the desired result because it will give \( \tilde{A}_k \) up to second-order corrections. The coefficient of \( \ln \tilde{x}/\tilde{x} \) simply gives the consistent asymptotic behavior, that is, proportional to \( \tilde{s}_k \). Substituting Eq. (45) into Eq. (47), matching the coefficient of \( \tilde{x}^{-1} \) with that in Eq. (48), the noncommutative power spectrum up to second order is

\[
\tilde{P}_{Re}^{2nd}(k) = \frac{k^3}{2\pi^2} |\tilde{A}_k|^2 = \frac{k^2}{(2\pi)^2} \tilde{f}_0 \frac{1}{f_0} \left[ 1 - 2\alpha \tilde{f}_1 + \left( 3\alpha^2 - 4 + \frac{5\pi^2}{12} \right) \left( \tilde{f}_1 / f_0 \right)^2 + \left( -\alpha^2 + \frac{\pi^2}{12} \right) \tilde{f}_2 \right].
\]

Substituting Eqs. (41), (42) and (43) into Eq. (49) leads to

\[
\tilde{P}_{Re}^{2nd}(k) = P_{Re}^{2nd}(k) = \frac{\mu_0}{2\epsilon_1 (2\pi)^2} \left\{ \frac{4 + (32\alpha - 10)\epsilon_1 + (8\alpha + 2)\delta_1}{4\epsilon_1 (2\pi)^2} \right\} + \frac{H^2}{2\epsilon_1 (2\pi)^2} \left\{ \frac{96\alpha^2 - 8\alpha - 232 + 24\pi^2}{4\epsilon_1 (2\pi)^2} \epsilon_1^2 + \frac{(28\alpha^2 + 32\alpha - 158 + 19\pi^2)\epsilon_1 \delta_1}{4\epsilon_1 (2\pi)^2} + \frac{12\alpha^2 + 6\alpha - 16 + \frac{5\pi^2}{3}}{4\epsilon_1 (2\pi)^2} \delta_1 - \frac{(4\alpha^2 - \frac{\pi^2}{3})\delta_2}{4\epsilon_1 (2\pi)^2} \right\}
\]

where the commutative contribution is given by

\[
P_{Re}^{2nd}(k) = \frac{1}{2\epsilon_1 (2\pi)^2} \left\{ \frac{H^2}{2\epsilon_1 (2\pi)^2} \left[ 1 - 2\epsilon_1 + 2(2\epsilon_1 + \delta_1) + \left( 4\alpha^2 - 23 + \frac{7\pi^2}{3} \right) \epsilon_1^2 + \left( 3\alpha^2 + \alpha - 22 + \frac{29\pi^2}{12} \right) \epsilon_1 \delta_1 + \left( 3\alpha^2 - 4 + \frac{5\pi^2}{12} \right) \delta_1^2 + \left( -\alpha^2 + \frac{\pi^2}{12} \right) \delta_2 \right\}
\]

and the right hand side should be evaluated at \( k = a\tilde{H} \). Comparing with the result Eq. (23) from the slow-roll approximation, we find additional terms depending \( \mu_0 \) even for up to
first-order corrections. We identify these with an effect of choosing the first in Eq.(27). Making use of Eq.(27), we obtain the following relations\(^1\) for calculating the spectral index and running spectral index:

\[
\frac{d\mu_0}{d\ln k} = \frac{1}{(1 - \epsilon_1 + 4\mu_0\epsilon_1)H} \frac{\partial \mu_0}{\partial t} + \frac{\partial \mu_0}{\partial \ln k} \simeq -4\mu_0\epsilon_1, \quad (52)
\]

\[
\frac{d\epsilon_1}{d\ln k} = \frac{1}{(1 - \epsilon_1 + 4\mu_0\epsilon_1)H} \frac{\partial \epsilon_1}{\partial t} \simeq 2(\epsilon_1^2 + \epsilon_1\delta_1), \quad (53)
\]

\[
\frac{d\delta_1}{d\ln k} = \frac{1}{(1 - \epsilon_1 + 4\mu_0\epsilon_1)H} \frac{\partial \delta_1}{\partial t} \simeq \epsilon_1\delta_1 - \delta_1^2 + \delta_2, \quad (54)
\]

\[
\frac{d\delta_2}{d\ln k} = \frac{1}{(1 - \epsilon_1 + 4\mu_0\epsilon_1)H} \frac{\partial \delta_2}{\partial t} \simeq 2\epsilon_1\delta_2 - \delta_1\delta_2 + \delta_3, \quad (55)
\]

\[
\frac{d\delta_3}{d\ln k} = \frac{1}{(1 - \epsilon_1 + 4\mu_0\epsilon_1)H} \frac{\partial \delta_3}{\partial t} \simeq 3\epsilon_1\delta_3 - \delta_1\delta_3 + \delta_4.
\]

Then the spectral index defined by

\[
\tilde{n}_s(k) = 1 + \frac{d\ln \tilde{P}_{R^2}}{d\ln k}
\]

can be easily calculated up to third order

\[
\tilde{n}_s(k) = n_s(k) + \mu_0 \left\{ 16\epsilon_1 + (32\alpha + 12)\epsilon_1^2 - (32\alpha - 10)\epsilon_1\delta_1 + 2\delta_1^2 - 2\delta_2 \\
+ (32\alpha + 12)\epsilon_1^3 + \left( -80\epsilon^2 - 140\alpha + 520 - \frac{148\pi^2}{3} \right) \epsilon_1^2 \delta_1 \\
+ (16\alpha^2 - 16\alpha + 64 - \frac{28\pi^2}{3}) \epsilon_1^2 \delta_1^2 \\
+ (16\alpha^2 - 20\alpha + 64 - \frac{28\pi^2}{3}) \epsilon_1 \delta_2 + 4\alpha \delta_3 - 4\alpha \delta_1 \delta_2 \right\}.
\]

where the right hand side should be evaluated at \( k = aH \). The commutative contribution up to third order is given by

\[
\tilde{n}_s(k) = 1 - 4\epsilon_1 - 2\delta_1 + (8\alpha - 8)\epsilon_1^2 + (10\alpha - 6)\epsilon_1\delta_1 - 2\alpha \delta_1^2 + 2\alpha \delta_2
\]

\[
+ \left( -16\alpha^2 + 40\alpha - 108 + \frac{28\pi^2}{3} \right) \epsilon_1^2 + \left( -31\alpha^2 + 60\alpha - 172 + \frac{199\pi^2}{12} \right) \epsilon_1^2 \delta_1
\]

\[
+ \left( -3\alpha^2 + 4\alpha - 30 + \frac{13\pi^2}{4} \right) \epsilon_1 \delta_1^2 + \left( -7\alpha^2 + 8\alpha - 22 + \frac{31\pi^2}{12} \right) \epsilon_1 \delta_2
\]

\[
+ \left( -2\alpha^2 + 8 - \frac{5\pi^2}{6} \right) \delta_1^3 + \left( 3\alpha^2 - 8 + \frac{3\pi^2}{4} \right) \delta_1 \delta_2 + \left( -\alpha^2 + \frac{\pi^2}{12} \right) \delta_3.
\]

Finally the running spectral index up to third order is given by

\[
\frac{\delta \tilde{P}_{R^2}}{d\ln k} = -2H\mu_0(1 + \epsilon_1) \quad \text{and} \quad \frac{\delta \mu_0}{d\ln k} = 2\mu_0 [11]. \quad \text{Actually} \quad \dot{\mu}_0 = -4H\mu_0\epsilon_1 \quad \text{in Eq.(27)} \quad \text{comes from taking into account these terms.}
\]
Thus slow-roll parameters are determined by

\[
\frac{d\tilde{n}_s}{d\ln k} = \frac{dn_s}{d\ln k} + \mu_0 \begin{cases}
-32\epsilon_1^2 + 32\epsilon_1\delta_1 - 32\epsilon_1^3 + (160\alpha + 70)\epsilon_1^2\delta_1 + (32\alpha + 6)\epsilon_1\delta_1^2 \\
+(-32\alpha + 14)\epsilon_1\delta_2 - 46\epsilon_1^3 + 64\delta_2 - 2\delta_3 \\
+(64\alpha - 8)\epsilon_1^4 - \left(80\alpha^2 - 212\alpha - 662 + \frac{148\pi^2}{3}\right)\epsilon_1^3\delta_1 \\
-(240\alpha^2 + 452\alpha - 1566 + 148\pi^2)\epsilon_1^2\delta_2 \\
-\left(512\alpha^2 + 172\alpha - 534 + \frac{148\pi^2}{3}\right)\epsilon_1^2\delta_2 \\
-\left(4\alpha + 4\right)\epsilon_1\delta_3 + (48\alpha^2 - 48\alpha + 206 - 28\pi^2)\epsilon_1\delta_1\delta_2 \\
+\left(16\alpha^2 - 20\alpha + 62 - \frac{28\pi^2}{3}\right)\epsilon_1\delta_3 \\
-12\alpha\delta_1^4 + 20\alpha\delta_1^2\delta_2 - 4\alpha\delta_2^2 - 4\alpha\delta_1\delta_3
\end{cases}.
\]

(57)

Also the commutative contributions up to third order take the form

\[
\frac{d}{d\ln k} n_s = -8\epsilon_1^2 - 10\epsilon_1\delta_1 + 2\delta_1^2 - 2\delta_2 + (32\alpha - 40)\epsilon_1^3 + (62\alpha - 60)\epsilon_1^2\delta_1 \\
+ (6\alpha - 4)\epsilon_1\delta_2^2 + (14\alpha - 8)\epsilon_1\delta_2 + 4\alpha\delta_3^2 - 6\alpha\delta_1\delta_2 + 2\delta_3 \\
+ \left(-96\alpha^2 + 272\alpha - 688 + 56\pi^2\right)\epsilon_1^4 + \left(-251\alpha^2 + 602\alpha - 1568 + \frac{1667\pi^2}{12}\right)\epsilon_1^3\delta_1 \\
+ \left(-105\alpha^2 + 202\alpha - 640 + \frac{251\pi^2}{4}\right)\epsilon_1^2\delta_2^2 + \left(-59\alpha^2 + 106\alpha - 268 + \frac{3323\pi^2}{12}\right)\epsilon_1^2\delta_2 \\
+ \left(-6\alpha^2 + 4\alpha + 24 - \frac{5\pi^2}{2}\right)\epsilon_1\delta_3^2 + \left(-4\alpha^2 + 10\alpha - 106 + \frac{34\pi^2}{3}\right)\epsilon_1\delta_1\delta_2 \\
+ \left(-10\alpha^2 + 10\alpha - 22 + \frac{17\pi^2}{6}\right)\epsilon_1\delta_3 + \left(6\alpha^2 - 24 + \frac{5\pi^2}{2}\right)\epsilon_1\delta_3 \\
+ \left(-12\alpha^2 + 40 - 4\pi^2\right)\epsilon_1\delta_2^2 + \left(4\alpha^2 - 8 + \frac{2\pi^2}{3}\right)\delta_1\delta_3 \\
+ \left(3\alpha^2 - 8 + \frac{3\pi^2}{4}\right)\epsilon_1\delta_2^2 + \left(-\alpha^2 + \frac{\pi^2}{12}\right)\delta_4.
\]

(58)

Up to now our calculation was done at the pivot scale of \(k = aH\). In order to obtain the correct expressions for noncommutative spacetime inflation, we have to change the pivot scale: \(k = aH \rightarrow k = \sqrt{2}aH\). Up to second-order corrections, we assume that \(k = aH \rightarrow k = \sqrt{2}aH\) corresponds to \(\alpha = 0.7296 \rightarrow \alpha_* = 0.3831\) in the above expressions [20]. Further, assuming that \(\Delta \tilde{n} = \tilde{n}_s - \tilde{n}_k\) is very small, we neglect higher order corrections like \((\Delta \tilde{n})^n\) for \(n \geq 2\) in obtaining the noncommutative power spectrum at \(\tilde{n} = \tilde{n}_s\) from the noncommutative power spectrum at \(\tilde{n} = \tilde{n}_k\).

As an example, we choose the power-law inflation like \(a(t) \sim t^p\) whose potential is given by

\[
V(\phi) = V_0 \exp \left( -\sqrt{\frac{2}{p}} \phi \right).
\]

(59)

Thus slow-roll parameters are determined by

\[
\epsilon_1 = \frac{1}{p}, \quad \delta_1 = -\frac{1}{p}, \quad \delta_2 = 2\delta_1^2 = \frac{2}{p^2}, \quad \delta_3 = 6\delta_1^3 = -\frac{6}{p^3}, \quad \delta_4 = 24\delta_1^4 = \frac{24}{p^4}.
\]

(60)
Then the noncommutative power spectrum takes the form

$$\tilde{P}_{Rc}^{P1,2nd}(k) = P_{Rc}^{P1,2nd}(k) + \mu_0 \frac{H^4}{(2\pi)^2 \dot{\phi}^2} \left\{ -4 + \frac{12(1 - 2\alpha_*)}{p} + \frac{1}{p^2} \left( 90 + 34\alpha_* - 72\alpha_*^2 - \frac{22\pi^2}{3} \right) \right\},$$

(61)

where the numerical values of the coefficients in this equation are $-4, 2.8056, 20.0812$, respectively.

Here $\mu_0$-independent terms are given by

$$P_{Rc}^{P1,2nd}(k) = \frac{H^4}{(2\pi)^2 \dot{\phi}^2} \left\{ 1 + \frac{2(\alpha_* - 1)}{p} + \frac{1}{p^2} \left( 2\alpha_*^2 - 2\alpha_* - 5 + \frac{\pi^2}{2} \right) \right\},$$

(62)

where the numerical values of the coefficients are $1, -1.2338, -0.537867$, respectively. The noncommutative spectral index can be easily calculated up to third order

$$\tilde{n}_s^{P1}(k) = n_s^{P1}(k) + \mu_0 \left\{ \frac{16}{p} + \frac{64\alpha_*}{p^2} + \frac{1}{p^3} \left( 128\alpha_*^2 + 120\alpha_* - 312 + 64\pi^2 \right) \right\},$$

(63)

where the numerical values of the coefficients in this expression are $16, 24.5184, 384.413$, respectively. Here

$$n_s^{P1}(k) = 1 - \frac{2}{p} - \frac{2}{p^2} - \frac{2}{p^3}.\,$$

(64)

Finally the running spectral index is found to be

$$\frac{d\tilde{n}_s^{P1}}{d\ln k} = \frac{dn_s^{P1}}{d\ln k} - \mu_0 \left\{ \frac{64}{p^2} + \frac{8(32\alpha_* + 8)}{p^3} + \frac{1}{p^4} \left( 512\alpha_*^2 + 736\alpha_* - 1184 + 256\pi^2 \right) \right\},$$

(65)

where the numerical values of the coefficients in this equation are $64, 162.074, 15.3118$, respectively. Here the commutative contribution is zero up to fourth order,

$$\frac{dn_s^{P1}}{d\ln k} = 0.$$

(66)

Comparing the above expressions with those of $\mu_0=$constant in Ref. [10], we find additional corrections in equations (61), (63), and (65). These are replacement of $\alpha \to \alpha_*$ from the change of pivot scale and additional contributions from a noncommutative parameter with $\mu_0 \neq$ constant. Actually the results from the slow-roll approximation with $\mu_0=$constant are those in the first terms $-4\mu_0, \frac{16\mu_0}{p}, \frac{-64\mu_0}{p^2}$ in Eqs. (61), (63), and (65), respectively. On the other hand, the slow-roll expansion with $\mu_0 \neq$ constant gives us $\tilde{P}_{Rc} \propto \mu_0 \left( -4 + \frac{2.8056}{p} + \frac{20.0812}{p^2} \right), \tilde{n}_s \propto \mu_0 \left( \frac{16}{p} + \frac{24.5184}{p^2} + \frac{384.413}{p^3} \right), \frac{dn_s}{d\ln k} \propto -\mu_0 \left( \frac{64}{p^2} + \frac{162.074}{p^3} + \frac{15.3118}{p^4} \right)$. Hence the slow-roll expansion decreases less the power spectrum than that of slow-roll approximation. On the other hand, the slow-roll expansion increases more the spectral index than that of slow-roll approximation. Finally, the slow-roll expansion decreases more the running spectral index than that of slow-roll approximation.
IV. SLOW-ROLL EXPANSION WITH $\mu_0=$CONSTANT

In the previous section we consider $\mu_0(t, k)$ to be a function of time $t$ and the comoving Fourier modes $k$ in the beginning. It is noted that near the horizon crossing time ($k \propto aH$), one has $\mu_0 \propto H^4$. This implies that $\mu_0$ may be considered to be nearly constant. In this section we use the case of $\mu_0=$constant to calculate the power spectrum and $\frac{\mu_0}{d\ln k} \simeq -4\mu_0\epsilon_1$ to compute the spectral index and running spectral index. We call it the slow-roll expansion with $\mu_0=$constant. This case provides us a new result between the slow-roll approximation in Sec. II and the slow-roll expansion in Sec. III.

Following the previous section, the noncommutative power spectrum is calculated as

$$\tilde{P}^{2nd}_{Re}(k) = P^{2nd}_{Re}(k)$$  \hspace{0.5cm} (67)

$$-\frac{\mu_0}{2\epsilon_1}(2\pi)^2 \bigg\{ \frac{4}{\epsilon_0^2} + (16\alpha - 2)\epsilon_1 + (8\alpha + 2)\delta_1 + (16\alpha^2 + 28\alpha - 84 + \frac{28\alpha^2}{3})\epsilon_1^2 + (12\alpha^2 + 32\alpha - 78 + \frac{29\alpha^2}{3})\epsilon_1^2 \bigg\}. \hspace{0.5cm} (68)$$

The spectral index takes the form

$$\tilde{n}_s(k) = n_s(k) + \mu_0 \left\{ -\frac{16\epsilon_1 + 28\epsilon_1^2 - 6\epsilon_1\delta_1 + 2\epsilon_1^2 - 2\epsilon_1^2}{32\epsilon_1^2 + 32\epsilon_1\delta_1 - 32\epsilon_1^3 + 150\epsilon_1^2\delta_1 - 10\epsilon_1\delta_1^2 - 2\epsilon_1\delta_2} \right\}. \hspace{0.5cm} (69)$$

Finally the running spectral index is

$$\frac{d\tilde{n}_s}{d\ln k} = \frac{dn_s}{d\ln k} + \mu_0 \left\{ -4 + \frac{4(1 - 2\alpha_*)}{p} + \frac{1}{p^2}(22 - 8\alpha_*^2 - 2\alpha_* - 2\pi^2) \right\}. \hspace{0.5cm} (70)$$

Choosing the power-law inflation, one has explicit forms for the power spectrum, spectral index, and running spectral index. The power spectrum is

$$\tilde{P}^{\text{PL,2nd}}_{Re}(k) = P^{\text{PL,2nd}}_{Re}(k) + \frac{\mu_0H^4}{(2\pi)^2\delta^2} \left\{ -4 + \frac{4(1 - 2\alpha_*)}{p} + \frac{1}{p^2}(22 - 8\alpha_*^2 - 2\alpha_* - 2\pi^2) \right\}. \hspace{0.5cm} (71)$$

The spectral index is given by

$$\tilde{n}_{s}^{\text{PL}}(k) = n_{s}^{\text{PL}}(k) + \mu_0 \left\{ \frac{16}{p} + \frac{32}{p^3} + \frac{1}{p^4}(8\alpha_* + 56) \right\}. \hspace{0.5cm} (72)$$

The running spectral index takes the form as

$$\frac{d\tilde{n}_s^{\text{PL}}}{d\ln k} = \frac{dn_s^{\text{PL}}}{d\ln k} - \mu_0 \left\{ \frac{64}{p^2} + \frac{192}{p^3} + \frac{1}{p^4}(32\alpha_* + 416) \right\}. \hspace{0.5cm} (73)$$

Finally, the slow-roll expansion with $\mu_0 =$ constant gives us $\tilde{P}_{Re} \propto \mu_0 \left( -4 + \frac{0.9352}{p} + \frac{0.3205}{p^2} \right)$, $\tilde{n}_s \propto \mu_0 \left( \frac{16}{p} + \frac{32}{p^2} + \frac{20.0648}{p^3} \right)$, $\frac{d\tilde{n}_s}{d\ln k} \propto -\mu_0 \left( \frac{64}{p^2} + \frac{192}{p^3} + \frac{428.257}{p^4} \right)$. Hence the slow-roll expansion with $\mu_0 =$constant decreases more the power spectrum than that of slow-roll expansion with $\mu_0 \neq$constant. On the other hand, the slow-roll expansion with $\mu_0 =$constant increases more the spectral index up to second order than that of slow-roll expansion with $\mu_0 \neq$constant. Finally, the slow-roll expansion with $\mu_0 =$constant decreases more the running spectral index up to third order than that of slow-roll expansion with $\mu_0 \neq$constant.
V. DISCUSSION

We investigate an effect of the uncertainty of noncommutative spacetime on the period of slow-roll inflation. For our purpose, the noncommutative parameter $\mu_0(t, k) = (k H / a M_s^2)^2$ arisen from the time uncertainty of $\triangle \tau = k / M_s^2$ is taken to be a zeroth order of slow-roll parameter. This is a key step in calculation of the noncommutative power spectrum. Actually $\mu_0$ is very small and thus one recovers the commutative result when $M_s \to \infty$.

When comparing with other slow-roll parameters $\epsilon_1(t)$ and $\delta_n(t)$, a difference is that $\mu_0$ is a function of time $t$ as well as comoving Fourier mode $k$. We calculate the noncommutative power spectrum up to second order in the slow-roll expansion. In deriving it, we have to remind the reader two important facts. First the slow-roll expansion is originally designed to calculate the power spectrum of curvature perturbation at the horizon crossing time when $k_c = a(\tilde{\eta}_c) H(\tilde{\eta}_c)$. However, in the noncommutative inflation, the horizon crossing time is delayed as $k_* = \sqrt{2} a(\tilde{\eta}_*) H(\tilde{\eta}_*)$. In the first-order corrections, this change of the pivot scale amount to replacing $\alpha = 2 - \ln 2 - \gamma = 0.7296$ by $\alpha_* = \alpha - \ln 2/2 = 0.3831$. Although we don’t know whether or not this substitution rule is valid up to second-order corrections, we assume this rule to be valid for calculating the noncommutative power spectrum up to second order.

Second we assume that $\triangle \tilde{\eta} = \tilde{\eta}_* - \tilde{\eta}_c$ is very small, thus we neglect higher order corrections like $(\triangle \tilde{\eta})^n$ for $n \geq 2$ in calculating the noncommutative power spectrum at $\tilde{\eta} = \tilde{\eta}_*$ from the noncommutative power spectrum at $\tilde{\eta} = \tilde{\eta}_c$. This implies that the form of the power spectrum at $\tilde{\eta} = \tilde{\eta}_c$ persists for the time at $\tilde{\eta} = \tilde{\eta}_*$.

Further we use the case of $\mu_0 =$ constant to calculate the power spectrum and $\frac{d \mu_0}{d \ln k} \simeq -4 \mu_0 \epsilon_1$ to compute the spectral index and running spectral index. This case provides us an intermediate result between the slow-roll approximation with $\mu_0 =$ constant and the slow-roll expansion with $\mu_0 \neq $ constant.

In conclusion, we observe important facts from our calculation using the slow-roll expansion. We find corrections arisen from a change of the pivot scale and the presence of a variable noncommutative parameter, when comparing with the commutative power spectrum. The power-law inflation is chosen to obtain explicit forms for the power spectrum, spectral index, and running spectral index. We find from Eqs.(61) and (62) that the noncommutative power spectrum always appears to be smaller than that of the commutative case. The noncommutative effect of higher-order corrections compensates for a loss of higher-order corrections in the commutative form. This suppression propagates the spectral index in a way that it is larger than the one in the commutative case. As is shown in Eqs.(63) and (64), the noncommutative effect of higher-order corrections again compensates for a loss of higher-order corrections in the commutative case. Furthermore we note that the spectral index for commutative form is invariant under transform of the pivot scale. According to the WMAP data [4], one finds $\frac{d \alpha_*}{d \ln k} = -0.031^{+0.0016}_{-0.0017}$ at $k=0.05$Mpc$^{-1}$. From Eq.(66), it is obvious that there is no contribution to the running spectral index from the commutative power-law inflation. It seems that a way to explain the above WMAP date is to consider the spacetime uncertainty during inflation. As is shown in Eq.(65), all higher-order corrections to the commutative case always provide us negative spectral indexes. Finally, we find the same observations even for the slow-roll expansion with $\mu_0 =$ constant.
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