ON THE CAUCHY PROBLEM
FOR A DYNAMICAL EULER’S ELASTICA

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Abstract

The dynamics for a thin, closed loop inextensible Euler’s elastica moving in three dimensions are considered. The equations of motion for the elastica include a wave equation involving fourth order spatial derivatives and a second order elliptic equation for its tension. A Hasimoto transformation is used to rewrite the equations in convenient coordinates for the space and time derivatives of the tangent vector. A feature of this elastica is that it exhibits time-dependent monodromy. A frame frame parallel-transported along the elastica is in general only quasi-periodic, resulting in time-dependent boundary conditions for the coordinates. This complication is addressed by a gauge transformation, after which a contraction mapping argument can be applied. Local existence and uniqueness of elastica solutions are established for initial data in suitable Sobolev spaces.
1 Introduction

1.1 Statement of the result

Let \( x(s, t) \) be a smooth closed curve in \( \mathbb{R}^3 \), parametrized by its arc length \( 0 \leq s \leq 2\pi \) and by time \( t \). Physically, the curve can be thought of as a loop of very thin inextensible wire which can move in space. We assume that the curve is flexible, with potential energy at time \( t \) determined by its curvature \( \kappa(s, t) = |\partial_s^2 x(s, t)| \),

\[
\mathcal{V} = \frac{1}{2} \int_0^{2\pi} \kappa^2(s, t) \frac{ds}{2\pi} . 
\]

(1.1)

Its kinetic energy at time \( t \) is given by

\[
\mathcal{T} = \frac{1}{2} \int_0^{2\pi} |\partial_t x(s, t)|^2 \frac{ds}{2\pi} . 
\]

(1.2)

We will refer to a dynamical curve \( x(s, t) \) with this potential and kinetic energy as a dynamical elastica. This choice of kinetic and potential energy gives rise to the variational problem of finding extreme solution curves for the Lagrangian \( \mathcal{L} = \mathcal{T} - \mathcal{V} \) subject to the constraint that \( s \) be in fact arc length. The constraint can be implemented by adding a suitable Lagrange multiplier term to the Lagrangian, enforcing the condition that the tangent vector \( u(s, t) = \partial_s x(s, t) \) be a vector of fixed modulus one. The variational problem we consider is then

\[
\delta \int_0^t \left\{ \mathcal{L}(x(s, t), \partial_t x(s, t)) - \frac{1}{2} \int_0^{2\pi} \lambda(s, t) \left(|\partial_s x(s, t)|^2 - 1\right) \frac{ds}{2\pi} \right\} \, dt = 0 .
\]

(1.3)

The variational derivative with respect to \( \lambda \) provides the arclength constraint. The resulting Euler-Lagrange equations are

\[
\partial_t^2 x = -\partial_s^4 x + \partial_s (\lambda \partial_s x) , \quad |\partial_s x|^2 = 1 , 
\]

(1.4)

where, we emphasize, the Lagrange multiplier \( \lambda(s, t) \) is a function of both \( s \) and \( t \). From this equation, it is apparent that \( \lambda(s, t) \) has the physical significance of being the tension of the curve at \( s \), at time \( t \). Note that \( \lambda \) can be negative, signifying compression. Eq. (1.4) is nonlinear in \( x \) due to the fact that at any time \( t \), the tension is itself the solution to an inhomogeneous elliptic equation involving the tangent vector \( u(s, t) = \partial_s x(s, t) \) and its derivatives (see Eq. (1.21) below). We note that omitting or relaxing the arclength constraint would require computing the curvature as \( \kappa = \partial_s x \times \partial_s^2 x / |\partial_s x|^3 \), leading to a rather forbidding quasilinear Euler-Lagrange equation.

Differentiating Eq. (1.4) with respect to \( s \), one obtains for the tangent vector the equation

\[
\partial_t^2 u = \partial_s^2 \left(-\partial_s^2 + \lambda\right) u , \quad u \in \mathbb{S}^2 .
\]

(1.5)
We study here the Cauchy problem for the wave equation Eq. (1.5), assuming periodic boundary conditions
\[ u(s+2\pi,t) = u(s,t) \tag{1.6} \]
and initial conditions
\[ u(s,0) = u_0(s) , \quad \partial_t u(s,0) = u_1(s) \tag{1.7} \]
where \( u_0 \) and \( u_1 \) are given \( 2\pi \)-periodic functions with
\[ |u_0(s)| = 1 , \quad u_0(s) \cdot u_1(s) = 0 \tag{1.8} \]
for all \( s \), that is, \( u_0(s) \) lies in \( \mathbb{S}^2 \), and \( u_1(s) \) is a tangent vector to the sphere at the point \( u_0(s) \). We are interested in weak solutions of Eq. (1.5), defined by the property that
\[ \frac{d}{dt} \int_0^{2\pi} \phi(s) \cdot \partial_t u(s,t) \frac{ds}{2\pi} = \int_0^{2\pi} \partial_s^2 \phi(s) \cdot (-\partial_s^2 u(s,t) + \lambda(s,t) u(s,t)) \frac{ds}{2\pi} \tag{1.9} \]
for any smooth \( 2\pi \)-periodic function \( \phi \). We require the map \( t \mapsto (u, \partial_t u) \) to be strongly continuous in a suitable Sobolev space, and interpret the time derivative on the left hand side in the sense of distributions (see Section 3 for a definition of the Sobolev spaces). The following is our main result.

**Theorem 1.1** Let \( (u_0, u_1) \) be a pair of \( 2\pi \)-periodic functions in a Sobolev space \( H^{r+2} \times H^r \) with values in \( \mathbb{R}^3 \) which satisfy the constraint in Eq. (1.8). For \( r \geq 1/2 \), there exists a time \( T > 0 \), which depends on \( \|u_0\|_{H^{r+2}} \) and on \( \|u_1\|_{H^r} \), such that the initial-value problem given by Eqs. (1.5)-(1.7) has a strongly continuous solution \( u \) on \([0,T]\) with \( u(\cdot,t) \in H^{r+2} \), \( \partial_t u(\cdot,t) \in H^r \), and \( \lambda(\cdot,t) \in H^{r+1} \). The solution is unique and depends continuously on the initial data. The conclusions hold for all \( r \geq 0 \) if the initial values \( u_0 \) and \( u_1 \) lie in a common plane through the origin. In this case, the solution is planar.

Theorem 1.1 implies that Eq. (1.4) with initial values
\[ x(s,0) = x_0(s) , \quad \partial_t x(s,0) = x_1(s) \tag{1.10} \]
satisfying the compatibility condition
\[ \partial_s x_0 \cdot \partial_s x_1 \equiv 0 \tag{1.11} \]
is well-posed in \( H^{7/2} \times H^{3/2} \) (in \( H^3 \times H^1 \), if the motion is planar). Periodicity of \( (x_0, x_1) \) implies that \( x(s+2\pi,t) - x(s,t) \) and its time derivative are zero at \( t = 0 \), whereas
\[ \partial_t^2 \left(x(s+2\pi,t) - x(s,t)\right) = \int_0^{2\pi} \partial_t^2 u(s',t) ds' = 0 \tag{1.12} \]
by Eq. (1.5), so that the curve $x(\cdot, t)$ remains a closed loop.

Theorem 1.1 applies also to infinite (expanding or contracting) helical curves $x(s, t)$ for which $u(\cdot, t)$ is still periodic. Eq. (1.12) implies that unless $u_1 = \partial_t u(\cdot, 0)$ averages to zero, $x(s + 2\pi, t) - x(s, t)$ grows linearly in time. Since it cannot grow beyond $2\pi$, the curve must disintegrate in a finite time under these circumstances.

Since the total energy $T + V$ is equivalent to the natural norm of $(x, \partial_t x)$ in $H^2 \times L^2$, it seems reasonable to consider the Eq. (1.4) in $H^2 \times L^2$. By conservation of energy, a small-time existence result for solutions of Eq. (1.4) in $H^1 \times L^2$ would imply that solutions exist globally in time. This would amount to proving Theorem 1.1 with $r = -1$ for initial values $(u_0, u_1)$ where $u_1$ averages to zero. It is an open question under what conditions on the initial values the solutions to Eq. (1.4) exist globally in time. Blowup, if there is any, must involve a transfer of energy to the high-frequency (large $n$) modes of the Fourier transform of the solution.

Our theorem should be compared with results of Caflisch and Maddocks [1], who have given a proof of global existence for a planar dynamical elastica. Their equations of motion include an additional rotational inertia term proportional to $\frac{1}{2} \int |\partial_s u|^2 \, ds$ in the kinetic energy $T$ (see their Eq. (2.16)). They assume that the initial values $u_0$ and $u_1$ are piecewise $C^2$ and piecewise $C^1$, respectively, a stronger assumption than we require in the planar case (although it appears that their smoothness assumptions can be relaxed somewhat). With this additional kinetic energy term, the equation of motion (for the tangent vector angle) can be written as a single second order semilinear wave equation with a non-local nonlinear term. Conservation of energy ultimately provides the a priori bounds needed to prove their global existence result.

### 1.2 Related geometrical and physical problems

The variational problem stated in Eq. (1.3) is closely related with wave maps. For a wave map problem, a typical choice for the kinetic energy would be

$$T_1 = \frac{1}{2} \int_0^{2\pi} |\partial_s u(s, t)|^2 \, ds.$$  

rather than that of Eq. (1.2) the potential energy is given by Eq. (1.1) with $\kappa = |\partial_s u|$, and $u$ is constrained to lie in a Riemannian manifold (see [2]). The resulting Euler-Lagrange equations for $u$ are just second order in space and time, and the Lagrange multipliers can be expressed in terms of the second fundamental form of the target manifold applied to first derivatives of $u$. Both local and global existence results have been obtained for various wave map problems [3, 4, 5, 6, 7, 8].

We note that choices for the potential energy of a geometric nature other than that of Eq. (1.3) are possible. Let $n(\cdot, t)$ and $b(\cdot, t)$ be the standard normal and the binormal to the curve $x(\cdot, t)$, as defined by the Serret-Frenet formulas, and let $A = A(s, t)$ be the $3 \times 3$-unitary matrix whose columns are $u(s, t)$, $n(s, t)$ and $b(s, t)$ (see Eq. (2.20) below). An alternate choice for the potential energy related to the Dirichlet form energies
for harmonic maps \([9, 10, 11, 12, 13]\)

\[ V_1 = \frac{1}{4} \int_0^{2\pi} tr \left( \partial_s A^+(s, t) \partial_s A(s, t) \right) \frac{ds}{2\pi} \]

\[ = \frac{1}{2} \int_0^{2\pi} (\kappa^2(s, t) + \theta^2(s, t)) \frac{ds}{2\pi} \]  

(1.14)

where \(\kappa\) is the curvature, and \(\theta\) is the torsion of the curve. Combining this with the kinetic energy in Eq. (1.2) results in adding a term of the form

\[ \partial^2_s \left( \frac{\theta}{\kappa^2} \partial_s u \times \partial^2_s u \right) + \partial^3_s \left( \frac{\partial_s \theta}{\kappa^2} \partial_s u \times \partial^2_s u \right) \]  

(1.15)

to the right hand side of Eq. (1.5), which is 6th order in the spatial derivatives of \(u\).

For a physical closed loop of wire with a small circular cross section of radius \(\rho > 0\), a more realistic expression for the elastic potential energy is given by

\[ V_2 = \frac{\pi}{2} \rho^4 \left\{ \left( \frac{\lambda}{4} + \frac{\mu}{2} \right) \int_0^{2\pi} \kappa(s, t)^2 \frac{ds}{2\pi} + \frac{\mu}{2} \left[ \int_0^{2\pi} \theta(s, t) \frac{ds}{2\pi} \right]^2 \right\} + o(\rho^4) \],  

(1.16)

where \(\lambda\) and \(\mu\) are the Lamé constants for a homogeneous isotropic hyperelastic material (see [14]). (For a wire made of material of a fixed density, the kinetic energy is of order \(\rho^2\). The kinetic energy due to twisting is of order \(\omega^2 \rho^4\), where \(\omega\) is the angular velocity.) The torsion term accounts for the expense of twisting the material frame. The equation is derived under the “quasi-static” assumption that the material arranges itself instantaneously about the central curve such that the contribution of the local twist to the elastic energy is as small as possible. The minimizing configuration of the material for a given curve is achieved by a local twist which is constant along the wire. For a dynamical elastica, that constant will in general change over time. The corresponding potential energy for a very thin, narrow ribbon would contain a torsion term proportional to the last term in Eq. (1.14). Maddocks and Dichmann [15], Coleman et al. [16] and others consider director theories, originated by Kirchhoff and Clebsch, in which there are further stress-strain relations between the tangent and two independent normal vectors (see [17]). The above choices for the potential and kinetic energy will not be pursued here.

Our equation is related to the Localized Induction Equation (LIE) first discussed by Da Rios [18] and rediscovered by Arms and Hama [19]. This is a second order equation for the approximate time evolution of a thin vortex filament \(x(s, t)\) (e.g. a smoke-ring) moving in a fluid, and is given by

\[ \partial_t x(s, t) = \partial_s x \times \partial^2_s x(s, t), \]  

(1.17)

again, \(s\) being arclength. (The equation ignores long-distance effects of the vortex acting upon itself.) Differentiating this equation with respect to \(s\), one obtains an equation for the tangent vector \(u\) to the vortex,

\[ \partial_t u(s, t) = u \times \partial^2_s u(s, t), \]  

(1.18)
known as the Landau-Lifshitz equation for the continuum Heisenberg ferromagnet [20]. We will elaborate on the connections between our equations and particularly the Landau-Lifshitz equation in the next subsection. A convenient reference for the LIE equation can be found in Newton [21].

The equations of motion for the elastica exhibit a rich variety of solutions. Langer and Singer found a countably infinite number of equilibrium configurations which are contained in tori of revolution and represent all but a finite number of torus knots, and showed that, up to the symmetries of the equations, every non-planar equilibrium configuration appears in their list [22]. They also relate these solutions to those of LIE [23].

At least in modern times, consideration of small amplitude vibrations of a rotating (in general extensible) ring seem to go back to Carrier’s 1945 paper [24]. He moreover considered the case in which the ring was constrained, or supported at points around the ring. Simmonds [25] also considered small planar vibrational modes for a nearly circular, extensible ring, in particular flexing modes in which extension is essentially negligible. His analysis provides a systematic treatment of the small amplitude approximations, and he shows for example that the vibrational frequencies decrease with amplitude. In a different direction, Coleman and Dill [26] examined the infinite length planar elastica, and showed that the solitary waves are of the form of a single loop and that traveling waves are a succession of periodically spaced loops, all of which satisfy differential equations similar to Euler’s equations for the static case. Note that their equations can include rotational inertia, cf. their equations (44a,b). They also find wave solutions that are periodic in time. Following this work, Coleman and Xu [27] numerically investigated solitary waves for an elastica of large length, and showed that the scattering was more than a simple phase shift, thereby providing compelling evidence that the elastica is not completely integrable. In [16], Coleman et al. considered the elastica moving in \( \mathbb{R}^3 \), showing existence of traveling and solitary waves exhibiting torsion so that the resulting curves are corkscrew-like. They include some discussion of the (finite) closed curve case. In their work on global existence of solutions for the planar elastica, Caflisch and Maddocks [1] also showed Liapunov stability for solutions near isolated relative minima of the potential \( V \).

### 1.3 Description of the proof

A first step towards solving Eq. (1.5) is to obtain an equation for the Lagrange multiplier \( \lambda \) in terms of the solution \( u \). In the related case of a wave map, an explicit expression for the Lagrange multiplier in terms of first derivatives of the solution is obtained by projecting the equation onto the normal of the target sphere [2]. In our problem, taking the inner product of Eq. (1.5) with \( u(s, t) \), writing \( |\partial_s u(s, t)| = \kappa(s, t) \), and using that \( u \cdot \partial_s u = u \cdot \partial_t u = 0 \) due to the constraint yields an elliptic boundary value problem for the tension \( \lambda(\cdot, t) \) at time \( t \),

\[
(-\partial_s^2 + \kappa^2) \lambda = |\partial_t u|^2 - u \cdot \partial_s^4 u .
\]  
(1.19)
A slightly less obnoxious form of this equation (involving lower order derivatives) results if one uses the identity

\[ 3|\partial_s^2 u|^2 + 4\partial_s u \cdot \partial_s^3 u + u \cdot \partial_s^4 u = 0, \tag{1.20} \]

which follows by differentiating the constraint \(|u|^2 = 1\) four times with respect to \(s\). With this identity the tension equation can be written as

\[ (-\partial_s^2 + \kappa^2) (\lambda + 2\kappa^2) = |\partial_t u|^2 + 2\kappa^4 - |\partial_s^2 u|^2. \tag{1.21} \]

We do not know how to make sense of Eq. (1.21) without requiring at least \(u(\cdot, t) \in H^2, \partial_t u(\cdot, t) \in L^2\).

Rewriting Eq. (1.5) as a system, we want to solve

\[
\begin{cases}
\partial_t u = v \\
\partial_t v = -\partial_s^4 u + \partial_s^2 (\lambda u),
\end{cases}
\tag{1.22}
\]

where \(\lambda\) is determined by Eq. (1.21). The linear part of Eq. (1.22) generates a strongly continuous semigroup on \(H^{r+2} \times H^r\) for any \(r \in \mathbb{R}\). Standard semilinear theory accommodates a nonlinearity that defines a locally Lipschitz continuous map from this space to itself as a perturbation. Since \(u \in H^{r+2}\), it would suffice to show that \(\lambda \in H^{r+2}\) to apply this technique. However, we only have \(\kappa \in H^{r+1}\), hence \(\kappa^2 \in H^{r+1}\) (provided \(r > -1/2\) ensuring that \(H^{r+1}\) is an algebra) while \(\lambda + 2\kappa^2\) is more regular by Lemma 3.5. Thus, we can only expect \(\lambda \in H^{r+1}\) and so Eq. (1.5) cannot be solved directly just by converting it to a Duhamel integral representation. A similar picture emerges if \(\lambda\) is inserted into Eq. (1.4).

Instead, we proceed as follows. Since Eq. (1.21) is the projection of Eq. (1.5) in the direction of \(u\), we combine it with the complementary projection

\[-u \times (u \times \partial_s^2 u) = u \times (u \times \partial_s^4 u) + 2(\partial_s \lambda) \partial_s u - \lambda u \times (u \times \partial_s^2 u). \tag{1.23}\]

We have used that \(u \times u = 0\) and \(u \cdot \partial_s u = 0\). The last two terms on the right hand side are locally Lipschitz from \(H^{r+2} \times H^r\) to \(H^r\). Now the problem is that the projection of the fourth derivative onto the orthogonal complement of \(u\) is a complicated nonlinear operation. Eqs. (1.23) is related to the Landau-Lifshitz equation, Eq. (1.18). This becomes apparent if we use Eq. (1.18) to formally write a differential equation for the second time derivative of \(u\). The resulting equation differs from our Eq. (1.23) in the last two terms: for us the tension is given by the solution to an elliptic problem, hence these terms are not local, while for the Landau-Lifshitz case the tension is given by a local expression of \(u\) and its derivatives. The similarity between the equations suggests how to proceed.

The Landau-Lifshitz equation is equivalent to a nonlinear Schrödinger equation via the so-called Hasimoto transformation (cf. [28]), which is is a nonlinear, solution-dependent change of variables where all partial derivatives of a curve are expressed with respect to a local coordinate frame transported along the curve. In Section 2, we will use the Hasimoto transformation to transform the projection of the second derivative operator
on the left hand side and the fourth derivative operator on the right hand side of Eq. (1.23) into linear differential operators plus a perturbation, while leaving the other terms essentially invariant (see Eq. (2.7)). Unless the motion described by Eq. (1.5) is planar, the Hasimoto transformation introduces a **monodromy** into the problem. In other words, the frame transported along the elastica is not periodic but rather quasiperiodic, with a generally **time-dependent** rotation of phase $2\pi \beta(t)$ about the tangent equal to the integral of the torsion over the curve (see Eq. (2.25)). We correct for the monodromy by performing an additional gauge transformation. Interestingly, the Hasimoto coordinate frame with the monodromy correction coincides in our problem with the natural material frame [23]. However, monodromy does not arise in the Kirchhoff-Clebsch director theories [15, 16].

Our Hasimoto transformation allows us to rewrite Eq. (1.5) in the form

$$\frac{d}{dt} Y(t) = G_{\beta(t)} Y(t) + F_{\beta(t)}(Y(t)),$$

(1.24)

where the linear part $G_{\beta(t)}$ is a differential operator which generates a strongly continuous evolution operator on a Sobolev space $\mathcal{Y}^r$ of periodic functions, and $F_{\beta(t)}$ is a nonlinear perturbation. The operator $G_{\beta(t)}$ depends on the on the monodromy, which is in turn determined by an auxiliary equation

$$\frac{d}{dt} \beta(t) = B(Y(t)),$$

(1.25)

see Eqs. (2.11) and (4.1).

In Section 3, we provide the basic estimates for the linear evolution operator $V_{\beta}$ generated by $G_{\beta(t)}$ and for the nonlinearity $F_{\beta(t)}$. In Lemma 3.3, we prove a positive lower bound on the spectrum of the Schrödinger operator $L_\kappa = -\partial_s^2 + \kappa^2$ appearing on the left hand side of Eq. (1.21) which may be of independent interest. We conjecture that the lowest eigenvalue of $L_\kappa$ is minimal when the elastica is a circle (see [30] for a related result on $-\partial_s^2 - \kappa^2$).

In Section 4, we set up a contraction mapping argument for Eqs. (1.24)-(1.25). A technical complication is that the dependence of the linear evolution operator $V_{\beta}$ generated by $G_{\beta(t)}$ is only strongly continuous, not norm-continuous in $\mathcal{Y}^r \times \mathbb{R}$. We overcome this difficulty by setting up the contraction mapping argument using a weaker norm.

Our local existence proof does not incorporate the more modern space-time methods such as Strichartz inequalities. Versions of these of these inequalities adapted to problems with periodic boundary conditions [31] have been used to obtain global existence of solutions for other related wave equations, for example the Boussinesq equation on a circle, which is also fourth order [32]. The methods do not immediately apply here however. The (spatial) derivative in the nonlinear term of the Boussinesq equation is of lower order than in our wave equation Eq. (1.5). Thus the Duhamel integral form of the equations of motion analogous to our Eqs. (1.22) make sense for example in $H^{r+1} \times H^r$ for suitable $r$, unlike the situation here which necessitates the Hasimoto change of variables. But again this change of variables leads to the monodromy issue and an associated linear problem with a **time-dependent** generator for which we do not have suitable space-time estimates.
Finding such estimates could be an avenue to relaxing our regularity assumptions for the initial data.

For the dynamical elastica moving in three dimensions, the question of global existence remains open. The picture which does emerge from our approach is that we can always integrate forward for an open interval of time, i.e. we have existence (and even uniqueness for planar motion), until \( \| (u'(\cdot, t), \partial_s u'(\cdot, t)) \|_{H^2 \times L^2} \) becomes unbounded. But at this moment, the tension \( \lambda(s) \) given by Eqs. (1.21) becomes infinite at some point \( s \) (or at least the individual terms on the right hand side of Eq. (1.21) are not integrable so that it is by no means clear that at this moment \( \lambda \) exists even as a distribution). Seemingly the elastica would break apart. It would be of interest to know whether indeed infinite tension can develop in a finite time.

We conclude with a couple of remarks about the infinite length elastica. A lower bound on the spectrum of \( L_\kappa \) is by no means apparent in this case, for example \( L_\kappa \) acting in \( L^2(\mathbb{R}) \) is typically not invertible, i.e. there is an infrared divergence and the estimates in Lemma 3.3 would fail. Physically, this divergence corresponds to elastica configurations with infinite tension. A local existence proof for the infinite elastica would thus presumably involve more subtly defined function spaces for which the tension is finite.

The second question concerns the role of the monodromy introduced by the Hasimoto transformation. Even for finite-energy solutions, it is not obvious how to remove the total twist by a gauge transformation.

2 A Change of Variables

2.1 Hasimoto transformation

Let \( u(s, t) \) be a smooth solution of Eq. (1.5). We will express the partial derivatives of \( u \) in terms of a positively oriented orthonormal frame which consists of \( u \) and two other unit vectors which we combine to a single complex vector \( \tilde{v} \). The vector \( \tilde{v}(s, t) \) is chosen so that for any fixed time \( t \),

\[
\partial_s \tilde{v} = -(\tilde{v} \cdot \partial_s u) u, \tag{2.1}
\]

that is, the real and imaginary parts of \( \tilde{v}(\cdot, t) \) are moved along the curve \( u(\cdot, t) \) by parallel transport on \( \mathbb{S}^2 \). Then

\[
\begin{align*}
\partial_s u &= \text{Re} \left[ \bar{q} \tilde{v} \right] \\
\partial_s \tilde{v} &= -qu \\
\partial_t u &= \text{Re} \left[ \bar{p} \tilde{v} \right] \\
\partial_t \tilde{v} &= -pu + i\tilde{\alpha} \tilde{v} \tag{2.2}
\end{align*}
\]

where \( p(s, t) \) and \( q(s, t) \) are complex-valued, and \( \tilde{\alpha}(s, t) \) is real-valued. Since \( \partial_s \partial_t u = \partial_t \partial_s u \) and \( \partial_s \partial_t \tilde{v} = \partial_t \partial_s \tilde{v} \), we must have

\[
(\partial_t - i\tilde{\alpha})q = \partial_s p, \quad \partial_s \tilde{\alpha} = \text{Im} \left[ \bar{q} p \right]. \tag{2.3}
\]
In these coordinates, we compute for the projection of Eq. (1.5) in the direction of \( u \),

\[-|p|^2 = \partial_s^2 (2|q|^2 + \lambda) - |q|^2 (|q|^2 + \lambda) - |\partial_s q|^2,\]

see Eq. (1.21). For the complementary projection, we find

\[(\partial_t - i\bar{\alpha})p = \partial_s^2 (-\partial_s q) + 2\partial_s ((|q|^2 + \lambda) q) - (|q|^2 + \lambda) \partial_s q + \text{Re} [\bar{q} \partial_s q],\]

see Eq. (1.23). If \( u \) is a smooth solution of Eq. (1.5), and \( \bar{v}, \bar{\alpha}, p, q \), are defined by Eqs. (2.2), and if we set

\[\mu = \lambda + 2|q|^2,\]

where \( \kappa = |q| \) is the curvature, then we arrive at the system

\[
\begin{cases}
(\partial_t - i\bar{\alpha})p &= -\partial_s^2 q - 4\text{Re} [\bar{q} \partial_s q] - i\text{Im} [\bar{q} \partial_s q] q + 2(\partial_s \mu) q + \mu \partial_s q \\
(\partial_t - i\bar{\alpha})q &= \partial_s p \\
\partial_s \bar{\alpha} &= \text{Im} [\bar{q} p] \\
(-\partial_s^2 + |q|^2)\mu &= |p|^2 + |q|^4 - |\partial_s q|^2.
\end{cases}
\]

Here, the first equation follows from Eq. (2.5), the second and third are the consistency relations in Eq. (2.3), and the last follows from Eqs. (2.4) and (2.6).

**Remark 2.1** There are many other ways to complete \( u \) to an orthonormal frame, which are all related by gauge transformations

\[\bar{v} \mapsto e^{i\gamma(s,t)} \bar{v}, \quad p \mapsto e^{i\gamma(s,t)} p, \quad q \mapsto e^{i\gamma(s,t)} q, \quad \bar{\alpha} \mapsto \partial_t \gamma(s,t) + \bar{\alpha}, \quad \mu \mapsto \mu\]

with some real-valued function \( \gamma(s,t) \). The choice in Eq. (2.1) has the special property that no second derivatives appear on the right hand sides of Eq. (2.7). As discussed in connection with Eq. (1.23), this is a key step towards solving Eqs. (1.5)-(1.7), because expressions containing first derivatives of \( q \) can, but expressions containing second derivatives of \( q \) cannot be treated as perturbations of the third derivative operator in the first line of Eq. (2.7).

### 2.2 The monodromy correction

Even when \( x(\cdot, t) \) is \( 2\pi \)-periodic, the frame \((u, \bar{v})\) defined by Eq. (2.1) is in general quasiperiodic,

\[\bar{v}(s+2\pi, t) = e^{2\pi i\beta(t)} \bar{v}(s, t),\]

where \( \beta \) is a real-valued function of time. The value of \( \beta(t) \) is determined modulo an integer by the monodromy of the parallel transported frame, which is related to the curve
torsion, see Eq. (2.25). It follows that $p$, $q$, and $\alpha$ satisfy time-dependent quasiperiodic boundary conditions.

Periodic boundary conditions are recovered by the gauge transformation

\[
\begin{align*}
\var{v}(s, t) &= e^{i(\bar{\beta}(t)-s\beta(t))}\tilde{\var{v}}(s, t) \\
\var{P}(s, t) &= e^{i(\bar{\beta}(t)-s\beta(t))}\var{p}(s, t) \\
\var{Q}(s, t) &= e^{i(\bar{\beta}(t)-s\beta(t))}\var{q}(s, t) \\
\var{\alpha}(s, t) &= \bar{\alpha}(s, t) + \partial_t(\bar{\beta}(t) - s\beta(t)) ,
\end{align*}
\]

where $\beta(t)$ and $\bar{\beta}(t)$ are chosen so that $\var{v}$ is periodic, and $\alpha$ averages to zero over a period of $2\pi$. We arrive at the system of equations

\[
\begin{align*}
(\partial_t - i\alpha)\var{P} &= - (\partial_s + i\beta)^2\var{P} - 4\text{Re} [\bar{\var{Q}} \partial_s \var{Q}] \var{Q} - i\text{Im} [\bar{\var{Q}} \partial_s \var{Q}] \var{Q} - i\beta|\var{Q}|^2\var{Q} + 2(\partial_s\mu)\var{Q} + \mu(\partial_s + i\beta)\var{Q} \\
(\partial_t - i\alpha)\var{Q} &= (\partial_s + i\beta)\var{P} \\
\partial_s\alpha + \partial_t\beta &= \text{Im} [\bar{\var{Q}} \var{P}] \\
(-\partial_s^2 + |\var{Q}|^2)\mu &= |\var{P}|^2 + |\var{Q}|^4 - |(\partial_s + i\beta)\var{Q}|^2 .
\end{align*}
\]

Here, $\var{P}(s, t), \var{Q}(s, t) \in \mathbb{C}$, and $\alpha(s, t), \beta(t), \mu(s, t) \in \mathbb{R}$ satisfy $2\pi$-periodic boundary conditions, with the constraint

\[
\int_0^{2\pi} \alpha(s, t) \frac{ds}{2\pi} = 0 .
\]

Initial conditions are given by

\[
\var{P}(s, 0) = P_0(s) , \quad \var{Q}(s, 0) = Q_0(s) , \quad \beta(0) = \beta_0 ,
\]

where $P_0$ and $Q_0$ are periodic complex-valued functions, and $\beta_0$ is the monodromy at time $t = 0$. We emphasize here that it is really this system, Eqs. (2.11)-(2.13) that we analyze in this paper. We will see in Section 3 that Eq. (2.11) can be written in the form of Eqs. (1.24)-(1.25) with $Y = (P, Q)$.

**Remark 2.2** Eqs. (2.11)-(2.12) and the periodic boundary conditions are invariant under the gauge transformations

\[
\begin{align*}
\var{P} &\mapsto e^{i(s_0 + ks)}\var{P} , \quad \var{Q} &\mapsto e^{i(s_0 + ks)}\var{Q} , \\
\alpha &\mapsto \alpha , \quad \beta &\mapsto \beta - k , \quad \mu &\mapsto \mu ,
\end{align*}
\]

when $k$ is an integer and $s_0 \in \mathbb{R}$.

We turn to the relation of the initial-value problem in Eqs. (2.11)-(2.13) with the initial-value problem in Eqs. (1.5)-(1.7) posed in the introduction. We say the initial conditions $(P_0, Q_0, \beta_0)$ for Eq. (2.11) are compatible with Eq. (1.5), if the linear system

\[
\begin{align*}
\partial_s \var{u} &= \text{Re} [\bar{Q}_0 \var{v}] \\
\partial_s \var{v} &= -Q_0 \var{u} - i\beta_0 \var{v} ,
\end{align*}
\]

has a $2\pi$-periodic solution forming an orthonormal frame.
**Lemma 2.3** For $r \geq 0$, the following statements are equivalent:

1. For each pair of initial values $(u_0, u_1) \in H^{r+2} \times H^r$ satisfying the condition in Eq. (1.8), the initial value problem in Eqs. (1.5)-(1.7) has a solution $(u, \partial_t u) \in H^{r+2} \times H^r$, defined on some short time interval, which is strongly continuous in $t$ and assumes the initial values at $t = 0$. The solution is unique and depends continuously on the initial values. If the initial values are smooth then the solution is smooth in both variables.

2. For each triple of initial values $(P_0, Q_0, \beta_0) \in H^r \times H^{r+1} \times \mathbb{R}$ which is compatible with Eq. (1.5), the initial value problem in Eqs. (2.11)-(2.13) has a solution $(P, Q, \beta) \in H^r \times H^{r+1} \times \mathbb{R}$, defined on a short time interval, which is strongly continuous in $t$ and assumes the initial values at $t = 0$. The solution is unique and depends continuously on the initial values. If the initial values are smooth then the solution is smooth in both variables.

**Remark 2.4** The regularity assumption $r \geq 0$ is needed only to ensure that the right hand side of the equation for $\mu$ in the fourth line of Eq. (2.11) makes sense as an $L^1$-function. In the proof of the lemma, we will construct a transformation $(u, \partial_t u) \mapsto (P, Q, \beta)$ between $H^{r+2} \times H^r$ and $H^r \times H^{r+1} \times \mathbb{R}$ which is continuous and has a continuous inverse for all $r \geq -1$.

**Proof of Lemma 2.3:** Given initial values $(u_0, u_1)$ for Eq. (1.5) satisfying the condition in Eq. (1.8), we determine initial values $(P_0, Q_0, \beta_0)$ for Eq. (2.11) by choosing a complex vector $\tilde{v}_0(0)$ whose real and imaginary parts complement $u_0(0)$ to a positively oriented orthonormal basis of $\mathbb{R}^3$, and then solving Eq. (2.1) with $u(s, 0) = u_0(s)$ to obtain $\tilde{v}_0(s)$. The initial monodromy $\beta_0$ is determined up to an additive integer by

$$\tilde{v}_0(2\pi) = e^{2\pi i \beta_0} \tilde{v}_0(0),$$

(2.16)

and $P_0$ and $Q_0$ are given by

$$P_0(s) = e^{-is \beta_0} \tilde{v}_0(s) \cdot u_1(s), \quad Q_0(s) = e^{-is \beta_0} \tilde{v}_0(s) \cdot \partial_s u_0(s).$$

(2.17)

This defines a continuous map from $H^{r+2} \times H^r$ to $H^r \times H^{r+1} \times \mathbb{R}$ for any $r \geq 0$. We have seen above that this transformation maps smooth solutions $u$ of Eq. (1.5)-(1.6) to smooth solutions $(P, Q, \beta)$ of Eq. (2.11)-(2.12). Since smooth functions are dense in $H^{r+2} \times H^r$, the transformation can be extended continuously to all of $H^{r+2} \times H^r$.

Conversely, given $(P_0, Q_0, \beta_0)$, let $(u_0, \nu_0)$ be a solution of the linear differential equation Eq. (2.15) which defines a periodic orthonormal frame, and set

$$u_1 = \text{Re} \left[ \tilde{P}_0 \nu_0 \right].$$

(2.18)
By construction, \((u_0, u_1)\) satisfy Eq. (1.8). This defines a continuous transformation from \(H^r \times H^{r+1} \times \mathbb{R}\) to \(H^{r+2} \times H^r\). Given a smooth solution of Eqs. (2.11)-(2.12), we can define a frame \((u, v)\) by solving

\[
\begin{align*}
\partial_s u &= \text{Re} \left[ \bar{Q} v \right], \\
\partial_s v &= -Q u - i\beta v, \\
\partial_t u &= \text{Re} \left[ \bar{P} v \right], \\
\partial_t v &= -P u - i\alpha v
\end{align*}
\]

(2.19)

with initial conditions \(u(s, 0) = u_0(s), v(s, 0) = v_0(s)\). Note that the third equation in Eq. (2.11) ensures that the two systems in Eq. (2.19) can be solved simultaneously. Since \(u_0\) and \(v_0\) are periodic, \(u(\cdot, t)\) and \(v(\cdot, t)\) are periodic for \(t > 0\) because the pair of equations on the right hand side of Eq. (2.19) preserves periodicity by the periodicity of \(P\) and \(\alpha\). The function \(u\) obtained in this way is smooth and solves Eqs. (1.5)-(1.7).

As above, the transformation can be extended continuously from the subset of smooth functions to all of \(H^r \times H^{r+1} \times \mathbb{R}\).

2.3 The standard normal frame

It is instructive to express \(v, \alpha, P,\) and \(Q\) in terms of standard normal coordinates, which are defined for smooth curves at any point where the curvature does not vanish. Assume \(u(s, t)\) describes the unit tangent vector of such a curve \(x(s, t)\), let \(n(s, t)\) be the unit normal to the curve in the direction of \(\partial_s u(s, t)\), and set \(b = u \times n\). The vector \(n\) is called the principal normal, and \(b\) the binormal of the curve at \(s\), at time \(t\). The standard normal frame \((u, n, b)\) is characterized by the Serret-Frenet differential equations

\[
\begin{align*}
\partial_s u &= \kappa n, \\
\partial_s (n + ib) &= -\kappa u - i\theta (n + ib)
\end{align*}
\]

(2.20)

where \(\kappa(s, t)\) is the curvature, and \(\theta(s, t)\) is the torsion of the curve. By definition, \(\kappa\) is nonnegative, and \(\theta\) is real-valued. The curvature can be expressed in the various frames as

\[
\kappa = |\partial_s u| = n \cdot \partial_s u = |q| = |Q|,
\]

(2.21)

and the torsion as

\[
\theta = \frac{u \cdot (\partial_s u \times \partial^2_s u)}{|\partial_s u|^2} = b \cdot \partial_s n = \text{Im} \left[ q^{-1} \partial_s q \right] = \text{Im} \left[ Q^{-1} \partial_s Q \right] + \beta.
\]

(2.22)

If we set

\[
\gamma(s, t) = \int_0^s \theta(s', t) \, ds'
\]

(2.23)

and define the functions \(\tilde{v}(s, t), \tilde{\alpha}(s, t), p(s, t),\) and \(q(s, t)\) by

\[
\begin{align*}
\tilde{v} &= e^{i\gamma} (n + ib), \\
\tilde{\alpha} &= \partial_t \gamma + n \cdot \partial_t b, \\
p &= e^{i\gamma} \partial_t u \cdot (n + ib), \\
q &= e^{i\gamma} \kappa,
\end{align*}
\]

(2.24)
then the differential equations in Eq. (2.2) and the consistency conditions in Eq. (2.3) are satisfied. The monodromy $\beta$ can be defined by

$$
\beta(t) = \int_0^{2\pi} \frac{\theta(s,t)}{2\pi} ds;
$$

(2.25)
in particular, the monodromy is an intrinsic quantity associated with the curve.

For planar curves, the Hasimoto transformation and the analysis of the transformed initial value problem simplify considerably. For a curve lying in the $x_1$-$x_2$-coordinate plane, we choose $\text{Re} [v]$ to be the unit vector obtained by rotating $u$ counterclockwise through an angle of $\pi/2$, and $\text{Im} [v]$ to be the unit vector in the $x_3$-direction, and set $\tilde{v} = v$. The partial derivatives of $u$ and $v$ satisfy Eq. (2.2) with $\tilde{\alpha} = 0$ and $p, q$ real-valued. The equations on the left hand side of Eq. (2.2) agree with the planar Serret-Frenet equations, and $q(s,t)$ is the signed curvature of the curve $x(s,t)$. In this case, no monodromy correction is required and $(P, Q) = (P, Q)$ satisfy Eqs. (2.11) with $\alpha = \beta = 0$. The resulting system can be written as a semilinear equation

$$
\frac{d}{dt} Y(t) = G Y(t) + F(Y(t))
$$

(2.26)
in a suitable Sobolev space of periodic functions, which can be solved by a standard fixed point argument (see Corollary 4.2).

### 3 Estimates

In the previous section, we have changed variables in the equations for or the dynamical Euler’s elastica, and transformed Eqs. (1.5)-(1.7) into the equivalent initial value problem given by Eqs. (2.11)-(2.13). To see that the resulting equations have the form of Eq. (1.24)-(1.25), set $Y = (P, Q)$, and define the linear operator $G_{\beta(t)}$ by

$$
G_{\beta(t)} \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} 0 & -(\partial_s + i\beta(t))^3 \\ (\partial_s + i\beta(t)) & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix},
$$

(3.1)
where $\beta$ is a real-valued function of time. Suppressing the time-dependence in the notation, we write the nonlinearity in Eq. (1.24) as a sum of three terms:

$$
F_{\beta}(Y) = i\alpha \begin{pmatrix} P \\ Q \end{pmatrix} + \left( 2(\partial_s \mu)Q + \mu(\partial_s + i\beta)Q \right) + \left( -4\text{Re} [\bar{Q} \partial_s Q] Q - i\text{Im} [\bar{Q} \partial_s Q] Q - i\beta |Q|^2 Q \right) =: F^{(1)}_{\beta}(Y) + F^{(2)}_{\beta}(Y) + F^{(3)}_{\beta}(Y),
$$

(3.2)
where $\alpha$ is determined by

$$
\partial_s \alpha = \text{Im} [P \bar{Q}] - \int_0^{2\pi} \text{Im} [P(s) \bar{Q}(s)] \frac{ds}{2\pi}, \quad \int_0^{2\pi} \alpha(s) \frac{ds}{2\pi} = 0,
$$

(3.3)
and μ solves the elliptic boundary value problem

$$(|Q|^2 - \partial_s^2)\mu = |P|^2 + |Q|^4 - |(\partial_s + i\beta)Q|^2$$

with periodic boundary conditions. The nonlinearity in Eq. (1.25) is given by

$$B(Y) = \int_0^{2\pi} \text{Im} [\overline{Q}(s)P(s)] \frac{ds}{2\pi}.$$  (3.5)

### 3.1 The linear part

We begin the analysis of the system in Eqs. (2.11)-(2.12) by solving the linear equation

$$\frac{d}{dt} \begin{pmatrix} P \\ Q \end{pmatrix} = G_{\beta(t)} \begin{pmatrix} P \\ Q \end{pmatrix}$$

with a given Lipschitz continuous function β defined on \([0, T]\). The fundamental solution of Eq. (3.6) will be denoted by \(V_\beta(t, t_0)\).

It is natural to consider Eq. (3.6) in the Fourier series representation

$$\frac{d}{dt} \begin{pmatrix} \hat{P}(n, t) \\ \hat{Q}(n, t) \end{pmatrix} = \begin{pmatrix} 0 & i(n + \beta(t))^3 \\ i(n + \beta(t)) & 0 \end{pmatrix} \begin{pmatrix} \hat{P}(n, t) \\ \hat{Q}(n, t) \end{pmatrix},$$

where it decouples into a sequence of linear ordinary differential equations on \(C^2\). Let \(\hat{G}_\beta(n, t)\) be the \(2\times2\) matrix appearing on the right hand side of Eq. (3.7). The fundamental solution \(\hat{V}_\beta(n, t, t_0)\) of Eq. (3.7) is given by the time-ordered exponential of \(\hat{G}_\beta(n, t)\).

Denote by \(H^r\) the Sobolev space of \(2\pi\)-periodic complex-valued functions (or distributions, when \(r < 0\)) having \(r\) fractional derivatives, with norm

$$\|f\|_{H^r}^2 = \sum_{n=-\infty}^{\infty} (1 + n^2)^r |\hat{f}(n)|^2.$$  (3.8)

Let \(Y^r\) be the space of \(2\pi\)-periodic functions \(Y = (P, Q)\) in \(H^r \times H^{r+1}\), with norm

$$\|Y\|_{Y^r}^2 = \|P\|_{H^r}^2 + \|Q\|_{H^{r+1}}^2.$$  (3.9)

For a vector \((a, b)\) in \(C^2\), we define its \(n\)-norm by

$$\left| \begin{pmatrix} a \\ b \end{pmatrix} \right|_n^2 = |a|^2 + w(n)^2|b|^2,$$

where \(w(n) = \sqrt{1 + n^2}\). With this notation, Eq. (3.9) becomes

$$\|Y\|_{Y^r}^2 = \sum_{n=-\infty}^{\infty} w(n)^{2r} \|\hat{Y}(n)\|_n^2,$$  (3.11)

where \(\hat{Y}(n) = (\hat{P}(n), \hat{Q}(n)) \in C^2\) is the Fourier transform of \(Y\).
Lemma 3.1 Suppose that $\beta$ is a real-valued Lipschitz-continuous function on $[0, T]$ with Lipschitz constant $\eta$, and that $|\beta(0)| \leq 1$. There exists an increasing continuous function $C$ of two variables with $\sup_{\eta} C(\eta, 0) < \infty$ such that for $0 \leq t_0 \leq t \leq T$ and any value of $r \in \mathbb{R}$,

$$\|V_\beta(t, t_0)\|_Y^r \leq C(\eta, T).$$ \hspace{1cm} (3.12)

Moreover, $V_\beta(t, t_0)$ is strongly continuous in $\mathbb{Y}^r$ with respect to $t$ and $t_0$.

PROOF: We first show that

$$\|\hat{V}_\beta(n, t, t_0)\|_n \leq C(\eta, T)$$ \hspace{1cm} (3.13)

uniformly in $n$, with $C$ as in the statement of the lemma. The idea is to rewrite Eq. (3.7) as

$$\frac{d}{dt} \left( \frac{\hat{P}(n, t)}{(n + \beta)\hat{Q}(n, t)} \right) = i(n + \beta)^2 \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \frac{\hat{P}(n, t)}{(n + \beta)\hat{Q}(n, t)} \right) + \frac{d\beta}{dt} \left( \begin{array}{c} 0 \\ \hat{Q}(n, t) \end{array} \right).$$ \hspace{1cm} (3.14)

Let $\hat{U}_\beta(n, t, t_0)$ be the unitary $2 \times 2$ matrix defined by

$$\hat{U}_n(\beta, t, t_0) = \exp \left\{ i \int_{t_0}^t (n + \beta(t'))^2 dt' \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \right\}.$$ \hspace{1cm} (3.15)

By the Duhamel integral formula, Eq. (3.14) is equivalent to

$$\left( \frac{\hat{P}(n, t)}{(n + \beta)\hat{Q}(n, t)} \right) = \int_{t_0}^t \hat{U}_\beta(n, t, t') \frac{d\beta}{dt} \left( \begin{array}{c} 0 \\ (n + \beta)\hat{Q}(n, t') \end{array} \right) dt' + \hat{U}_\beta(n, t, t_0) \left( \begin{array}{c} \hat{P}(n, t_0) \\ (n + \beta)\hat{Q}(n, t_0) \end{array} \right)$$ \hspace{1cm} (3.16)

provided that $|n|$ is sufficiently large so that $n + \beta(t)$ does not vanish anywhere on $[0, T]$. Since $U_\beta(n, t, t_0)$ is unitary, and $|\beta(t)| \leq 1 + \eta T$ for $0 \leq t \leq T$ by assumption, we can estimate for $|n| \geq 2(1 + \eta T)$,

$$\left| \left( \frac{\hat{P}(n, t)}{(n + \beta)\hat{Q}(n, t)} \right) \right| \leq \frac{2\eta}{|n|} \int_{t_0}^t \left( \left| \left( \begin{array}{c} 0 \\ (n + \beta)\hat{Q}(n, t') \end{array} \right) \right| \right) dt' + \left| \left( \begin{array}{c} \hat{P}(n, t_0) \\ (n + \beta)\hat{Q}(n, t_0) \end{array} \right) \right|.$$ \hspace{1cm} (3.17)

Applying Gronwall’s inequality, and using the fact that the left hand side is equivalent to the $n$-norm, we arrive at

$$\|V_\beta(n, t, t_0)\|_n \leq 6e^{2\eta T/|n|}, \quad (|n| \geq 1 + \eta T).$$ \hspace{1cm} (3.18)
For any value of $n$, we can bound the $n$-norm of $\hat{G}_\beta(n, t)$ of Eq. (3.7) on $C^2$ by
\[
\|\hat{G}_\beta(n, t)\|_n = \sup_{a^2 + w(n)^2b^2 = 1} \left| \begin{pmatrix} 0 \\ i(n + \beta(t))^3 \end{pmatrix} \right|_n.
\]
\[
\leq \max \left\{ \|n\| + 1 + \eta T w(n), \frac{(|n| + 1 + \eta T)^3}{w(n)} \right\}.
\]  
(3.19)

Estimating the right hand side and applying Gronwall’s inequality gives
\[
\|\hat{V}_\beta(n, t, t_0)\|_n \leq e^{2(|n| + 1 + \eta T)^2(1 + \eta T)} T.
\]  
(3.20)

Combining Eqs. (3.18) and (3.20) implies Eq. (claim:Vbetabound-n) with the value of the constant given by $C(\eta, T) = 6e^{8(1 + \eta T)^3} T$. This proves the claim in Eq. (3.12).

Clearly, each Fourier coefficient $\hat{V}_\beta(n, t, t_0)\hat{Y}(n)$ depends continuously on $t$ and $t_0$. Since Eq. (3.13) implies a uniform tail estimate on $\|\hat{V}_\beta(n, t, t_0)\hat{Y}(n)\|$, it follows that $V_\beta(t, t_0)$ is strongly continuous in both time variables.

We also need to bound the dependence of $V_\beta$ on $\beta$.

**Lemma 3.2** Assume that for some $T > 0$, the functions $\beta_1$ and $\beta_2$ are Lipschitz continuous on $[0, T]$ with Lipschitz constant $\eta$, and that $|\beta_1(0)|, |\beta_2(0)| \leq 1$. There exists an increasing continuous function $C$ of two variables such that
\[
\|V_{\beta_2}(t, t_0) - V_{\beta_1}(t, t_0)\|_{\mathcal{Y}^r} \leq C(\eta, T) \int_{t_0}^{t} |\beta_2(t') - \beta_1(t')| \, dt'.
\]  
(3.21)

Moreover, $V_\beta$ is strongly continuous in $\beta$ with respect to the $\mathcal{Y}^r$-topology in the sense that for every $Y \in \mathcal{Y}^r$ and any given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \eta, T, Y)$ such that
\[
\sup_{0 \leq t \leq T} |\beta_2(t) - \beta_1(t)| \leq \delta
\]  
(3.22)

implies
\[
\sup_{0 \leq t_0 \leq t \leq T} \|V_{\beta_2}(t, t_0) - V_{\beta_1}(t, t_0)\|_{\mathcal{Y}^r} \leq \varepsilon.
\]  
(3.23)

**PROOF:** We first show that for $0 \leq t_0 \leq t \leq T$,
\[
\|\hat{V}_{\beta_2}(n, t, t_0) - \hat{V}_{\beta_1}(n, t, t_0)\|_n \leq C(\eta, T) w(n) \int_{t_0}^{t} |\beta_2(t') - \beta_1(t')| \, dt'.
\]  
(3.24)

uniformly in $n$, which clearly implies Eq. (3.21). To see Eq. (3.24), we write
\[
\hat{V}_{\beta_2}(n, t, t_0) - \hat{V}_{\beta_1}(n, t, t_0) = \int_{t_0}^{t} \hat{V}_{\beta_2}(n, t, t') \Delta \hat{G}(n, t') \hat{V}_{\beta_1}(n, t', t_0) \, dt',
\]  
(3.25)
where
\[
\Delta \hat{G}(n, t) = \begin{pmatrix}
0 & i(n+\beta_2(t))^3 - i(n+\beta_1(t))^3 \\
i(\beta_2(t) - \beta_1(t)) & 0
\end{pmatrix}.
\] (3.26)

Using that \(|\beta_i(t)| \leq 1 + \eta T\) by assumption, we estimate
\[
\|\Delta \hat{G}(n, t)\|_n \leq |\beta_2(t) - \beta_1(t)| \left\| \begin{pmatrix}
0 & 3(|n| + 1 + \eta T^2) \\
1 & 0
\end{pmatrix} \right\|_n
\leq \max \left\{ \frac{3(|n| + 1 + \eta T^2)}{w(n)}, \frac{|\beta_2(t) - \beta_1(t)|}{w(n)} \right\}
\leq 3(1 + \eta T^2)\max \{w(n)\} |\beta_2(t) - \beta_1(t)|.
\] (3.27)

Inserting Eq. (3.27) and the bound in Eq. (3.13) of Lemma 3.1 into Eq. (3.25) yields Eq. (3.24), with
\[
C(\eta, T) = c(\eta, T)^3(1 + \eta T)^2,
\]
where \(c(\eta, T)\) is the constant from Lemma 3.1. This proves the claim in Eq. (3.21).

The strong continuity follows as in the proof of Lemma 3.1 by combining Eq. (3.24) with a uniform tail estimate obtained from Eq. (3.12).

3.2 The resolvent for \(-\partial_s^2 + \kappa^2\)

We provide a lower bound for the spectrum of the Schrödinger operator \(L_\kappa \equiv -\partial_s^2 + \kappa^2\) acting in \(L^2[0, 2\pi]\), with periodic boundary conditions.

Lemma 3.3 The operator \(L_\kappa\) has least eigenvalue \(e_0(\kappa)\) satisfying,
\[
e_0(\kappa) \geq 1/4.
\] (3.28)

In particular, \(L_\kappa\) is invertible in \(L^2\), and \(\|L_\kappa^{-1}\|_{L^2} \leq 4\).

Remark 3.4 It is natural to conjecture that \(\inf e_0(\kappa)\) is actually attained for \(x(s)\) a circle where \(\kappa(s) = 1\) and \(e_0(\kappa) = 1\). An analogous result due to Harrell and Loss says that the second-lowest eigenvalue of \(-\partial_s^2 - \kappa^2\) is maximal for a circle \([30]\).

Proof of Lemma 3.3: We have that
\[
\inf_{\{\psi : \|\psi\|_2 = 1\}} (\psi, L_\kappa \psi) = \inf_{\{\psi : \|\psi\|_2 = 1\}} \left( \|\partial_s \psi\|_2^2 + \|\psi \partial_s u\|_2^2 \right)
= \inf_{\{\psi : \|\psi\|_2 = 1\}} \left( \|\partial_s (\psi u)\|_2^2 \right)
\geq \inf_{\{\Psi\}} \|\partial_s \Psi\|_2^2
\] (3.29)
where the vector function $\Psi = \psi u$, and in the last line, the infimum is to be taken over all normalized $\Psi$ such that

$$\int u(s) \, ds = \int \frac{\Psi(s)}{|\Psi(s)|} \, ds = 0. \quad (3.30)$$

If these constraint integrals are zero, then each component $\psi_i$ of $\Psi$ must vanish at some point $s_i$, $i = 1, 2, 3$, i.e., each component must satisfy a Dirichlet condition and so $\|\partial_s \psi_i\|_2^2 \geq \frac{1}{4} \|\psi_i\|_2^2$, for each $i$ so that the infimum in the last line of Eq. (3.29) is bounded below by

$$\inf_{\sum \|\psi_i\|_2^2 = 1} \sum \|\partial_s \psi_i\|_2^2 \geq \inf_{\sum \|\psi_i\|_2^2 = 1} \frac{1}{4} \sum \|\psi_i\|_2^2 = \frac{1}{4}. \quad (3.31)$$

This completes the proof of the eigenvalue estimate.

Next, we provide bounds on the resolvent for $L_\kappa$, considered as mapping $H^r$ to $H^{r+2}$, assuming the spectral bound in Lemma 3.3 and additional bounds on the norm of $\kappa$. The inequalities given are by no means optimal, but they are adequate for our purposes; the estimates are in the spirit of Bessel kernel estimates (see [33]).

**Lemma 3.5** If $\kappa \in H^{r+1}$ for some $r \geq 0$, then $L_\kappa^{-1}$ defines a bounded linear operator from $H^{r-1}$ to $H^{r+1}$. More precisely, there exists a constant $C_1 = C_1(r)$ such that if $\mu$ solves

$$(-\partial_s^2 + \kappa^2)\mu = f \quad (3.32)$$

with $f \in H^{r-1}$ for some $\kappa$ with $\|\kappa\|_{H^{r+1}} \leq R$, then $\mu$ satisfies

$$\|\mu\|_{H^{r+1}} \leq C_1 \left(1 + R^2\right)^\nu \cdot \|f\|_{H^{r-1}}, \quad (3.33)$$

where $\nu = \nu(r)$ is the smallest the smallest integer at least as large as $(r+1)/2$ for $r \geq 1$, and $\nu(r) = 2$ for $0 \leq r < 1$. The Fourier coefficients of $\mu$ are bounded by

$$|\hat{\mu}(n)| \leq C_2 w(n)^{-2} \left(|\hat{f}(n)| + w(n)^{-1-r}(1 + R^2)^\nu \cdot \|f\|_{H^{r-1}}\right). \quad (3.34)$$

**Proof:** We compare $L_\kappa$ with the operator $-\partial_s^2 + 1$, using two forms of the resolvent identity:

$$L_\kappa^{-1} = (-\partial_s^2 + 1)^{-1} - (-\partial_s^2 + 1)^{-1}(\kappa^2 - 1)L_\kappa^{-1} \quad (3.35)$$

$$= (-\partial_s^2 + 1)^{-1} - L_\kappa^{-1}(\kappa^2 - 1)(-\partial_s^2 + 1)^{-1}. \quad (3.36)$$

Clearly,

$$\|(-\partial_s^2 + 1)^{-1}f\|_{H^{r'+1}} = \|f\|_{H^{r'-1}} \quad (3.37)$$
for any $r' \in \mathbb{R}$. Let $f \in L^2$. The first line of the resolvent identity shows that
\[
\| L_{\kappa}^{-1} f \|_{H^2} \leq \| f \|_{L^2} + \| (\kappa^2 - 1) L_{\kappa}^{-1} f \|_{L^2} 
\leq c_1 (1 + R^2) \| f \|_{L^2}
\] (3.38)
for some constant $c_1$. We have used Eq. (3.37) in the first line. In the second line, we have used Lemma 3.8, which is proved below, and the spectral bound of Lemma 3.3. This proves the claim in the case $r = 1$. More generally, the first line of the resolvent identity shows that for $r' \leq r$,
\[
\| L_{\kappa}^{-1} \|_{H^{r' + 3} \to H^{r + 3}} \leq c_2 (1 + R^2) \| L_{\kappa}^{-1} \|_{H^{r' - 1} \to H^{r + 1}}
\] (3.39)
for some constant $c_2 = c_2(r, r')$. Iterating this estimate we obtain the claim for $r \geq 1$.

The case $0 \leq r \leq 1$ follows from the second line of the resolvent identity, which implies that for $r' \leq r$
\[
\| L_{\kappa}^{-1} \|_{H^{r' - 3} \to H^{r - 3}} \leq c_3 (1 + R^2) \| L_{\kappa}^{-1} \|_{H^{r' - 1} \to H^{r + 1}} .
\] (3.40)

We will also need estimates on how $\mu$ varies as the curvature $\kappa$ varies.

**Lemma 3.6** Suppose that $\mu_1$ and $\mu_2$ solve the equations
\[
\begin{align*}
L_{\kappa_1} \mu_1 &= (-\partial^2_s + \kappa_1^2) \mu_1 = f \\
L_{\kappa_2} \mu_2 &= (-\partial^2_s + \kappa_2^2) \mu_2 = f ,
\end{align*}
\] (3.41)
where $\kappa_1, \kappa_2 \in H^{r+1}$. Suppose that $R \geq \max \{ \| \kappa_1 \|_{H^{r+1}} , \| \kappa_2 \|_{H^{r+1}} \}$. For $r \geq 0$ there exists a constant $C = C(r)$ such that
\[
\| \mu_2 - \mu_1 \|_{H^{r+1}} \leq CR (1 + R^2)^{2\nu} \| \kappa_2 - \kappa_1 \|_{H^r} \| f \|_{H^{r-1}} .
\] (3.42)
Here, $\nu = \nu(r)$ is the exponent from Lemma 3.5.

From the resolvent identity
\[
\mu_2 - \mu_1 = - L_{\kappa_2}^{-1} (\kappa_2^2 - \kappa_1^2) L_{\kappa_1}^{-1} f ,
\] (3.43)
we obtain for $r \geq 0$ with the help of Lemma 3.8, which is proved below,
\[
\| \mu_2 - \mu_1 \|_{r+1} \leq 2R \| L_{\kappa_2}^{-1} \|_{H^r \to H^{r+1}} \| \kappa_2 - \kappa_1 \|_{H^r} \| L_{\kappa_1}^{-1} \|_{H^{r-1} \to H^r} \| f \|_{H^{r-1}} .
\] (3.44)
The claim now follows from Lemma 3.5.
3.3 The nonlinearity

**Lemma 3.7** For \( Y = (P, Q) \in \mathcal{Y} \) and \( \beta \in \mathbb{R} \), let \( F_\beta \) be defined by Eq. (3.2), and let \( B \) be defined by Eq. (3.3). There exist increasing continuous functions \( C_1, C_2, \) and \( C_3 \) of two variables, which also depend on \( r \), such that the following estimates hold:

1. If \( r \geq 0 \), then for any \( Y \in \mathcal{Y} \) with \( \|Y\|_{\mathcal{Y}^r} \leq R \), and any \( \beta \in \mathbb{R} \) with \( |\beta| \leq b \),

\[
\begin{align*}
\|F_\beta(Y)\|_{\mathcal{Y}^r} & \leq C_1(b, R) \\
|B(Y)| & \leq R^2. 
\end{align*} \tag{3.45}
\]

2. If \( r \geq 0 \), then for any pair \( Y_1, Y_2 \in \mathcal{Y} \) with \( \|Y_i\|_{\mathcal{Y}^r} \leq R \) and any \( \beta_1, \beta_2 \in \mathbb{R} \) with \( |\beta_i| \leq b \),

\[
\begin{align*}
\|F_{\beta_2}(Y_2) - F_{\beta_1}(Y_1)\|_{\mathcal{Y}^r} & \leq C_2(b, R) \left( \|Y_2 - Y_1\|_{\mathcal{Y}^r} + |\beta_2 - \beta_1| \right) \\
|B(Y_2) - B(Y_1)| & \leq R\|Y_2 - Y_1\|_{\mathcal{Y}^r}. 
\end{align*} \tag{3.46}
\]

3. If \( r \geq 1/2 \), then for any pair \( Y_1, Y_2 \in \mathcal{Y} \) with \( \|Y_i\|_{\mathcal{Y}^r} \leq R \) and any \( \beta_1, \beta_2 \in \mathbb{R} \) with \( |\beta_i| \leq b \),

\[
\begin{align*}
\|F_{\beta_2}(Y_2) - F_{\beta_1}(Y_1)\|_{\mathcal{Y}^{r-1}} & \leq C_3(b, R) \left( \|Y_2 - Y_1\|_{\mathcal{Y}^{r-1}} + |\beta_2 - \beta_1| \right) \\
|B(Y_2) - B(Y_1)| & \leq R\|Y_2 - Y_1\|_{\mathcal{Y}^{r-1}}. 
\end{align*} \tag{3.47}
\]

The following lemma will be useful in estimating the various terms in the nonlinearity.

**Lemma 3.8** (Leibnitz rule for \( H^r \)-norms) There exist constants \( C_1 - C_7 \) which depend only on \( r \), so that each of the following inequalities holds whenever the right hand side is finite:

\[
\begin{align*}
\|fg\|_{H^r} & \leq C_1 \|f\|_{H^{r'}} \|g\|_{H^r} \quad (r \geq 1) \\
\|fg\|_{H^{r'}} & \leq C_2 \|f\|_{H^{r}} \|g\|_{H^{r'+1}} \quad (r \geq 0) \\
\|fg\|_{H^{r'+1}} & \leq C_3 \|f\|_{H^{r}} \|g\|_{H^{r'}} \quad (r \geq 0) \\
\|fg\|_{H^{r-1}} & \leq C_4 \|f\|_{H^{r-1}} \|g\|_{H^{r'+1}} \quad (r \geq 1/2) \\
\|fg\|_{H^{r-2}} & \leq C_5 \|f\|_{H^{r-1}} \|g\|_{H^{r'}} \quad (r \geq 1/2). 
\end{align*} \tag{3.48}
\]

**PROOF:** If \( r \) is an integer, then the first two inequalities follow immediately from the Leibnitz rule and the fact that \( H^1 \subset L^\infty \) in one space dimension. In general, we use that, for \( r \geq 0 \), \( w(n)^r \leq c_1 (w(k)^r + w(n-k)^r) \) with some constant \( c_1 = c_1(r) \) to get

\[
\begin{align*}
|fg(n)| & \leq w(n)^r \sum_k |\hat{f}(k)| \cdot |\hat{g}(n-k)| \\
& \leq w(n)^r \sum_k w(k)^r |\hat{f}(k)| \cdot |\hat{g}(n-k)| + \sum_k |\hat{f}(k)| \cdot w(n-k)^r |\hat{g}(n-k)| \quad (r \geq 1/2). 
\end{align*} \tag{3.49}
\]
Lemma 3.8, we have for
\[ |fg(n)| \leq c_2 w(n)^{-1} \sum_k w(k)^r |\hat{f}(k)| \cdot w(n-k)^r |\hat{g}(n-k)| , \tag{3.50} \]
where the constant \( c_2 \) depends only on \( r \). The sum is the Fourier transform of the product of two \( L^2 \)-functions, and hence in \( \ell^\infty \). Since \( w^{-1} \in \ell^2 \), it follows that \( w(n)^{-1} \hat{fg} \in \ell^2 \), which implies the third inequality.

For \( r \geq 1 \), the last two inequalities follow from the second and third with \( r \) replaced by \( r-1 \). For \( 1/2 \leq r < 1 \), we use that \( \sqrt{2}w(n) \geq w(n)w(n-k) \) and proceed as in the proof of the third inequality.

\[ \boxed{\text{Proof of Lemma 3.7}} \]

The estimates for \( F^{(1)} \), \( F^{(3)} \), and \( B \) follow by repeated applications of Lemma 3.8. We focus on the terms involving \( \mu \).

Consider the first claim, Eq. (3.45) for \( r \geq 0 \). We want to apply Lemma 3.5 with \( \kappa = |Q| \in H^{r+1} \) and \( f = |P|^2 + |Q|^4 - |(\partial_s + i\beta)Q|^2 \). By the third inequality of Lemma 3.8, we have for \( r \geq 0 \),
\[ \|f\|_{H^{r-1}} \leq c_1(1 + |\beta|^2)R^2. \tag{3.51} \]

By Lemma 3.5,
\[ \|\mu\|_{H^{r+1}} \leq c_2(1 + |\beta|^2)R^2(1 + R^2)^\nu, \tag{3.52} \]
and so by the second inequality of Lemma 3.8,
\[ \|F^{(2)}_\beta(Y)\|_{Y^r} = \|2(\partial_s\mu)Q + \mu(\partial_s + i\beta)Q\|_{H^r} \leq c_3(1 + |\beta|) \|\mu\|_{H^{r+1}} \|Q\|_{H^{r+1}} \leq c_4(1 + |\beta|^3)R^3(1 + R^2)^\nu, \tag{3.53} \]
where \( \nu = \nu(r) \) is the exponent from Lemma 3.3. This shows the bound in Eq. (3.45). The proof of the second claim, Eq. (3.46), is almost the same.

To see the third claim, Eq. (3.47), let \( Y_i = (P_i, Q_i) \), \( \kappa_i = |Q_i| \), and denote by \( f_i \) the right hand side of Eq. (3.44) corresponding to \( Y_i \) \( (i = 1, 2) \). By Lemma 3.8, we have for \( r \geq 1/2 \),
\[ \|f_2 - f_1\|_{H^{r-2}} \leq c_1(1 + b^2)R \|Y_1 - Y_2\|_{Y^{r-1}} + c_2 bR^2 |\beta_2 - \beta_1| \leq c_3(1 + b^2)(1 + R^2) \left( \|Y_2 - Y_1\|_{Y^{r-1}} + |\beta_2 - \beta_1| \right) \tag{3.54} \]
with suitable constants $c_1 - c_3$. By Lemmas 3.5 and 3.6, this implies
\[
\|\mu_2 - \mu_1\|_{H^r} \leq \|L^{-1}_{\kappa_2} (f_2 - f_1)\|_{H^r} + \|(L^{-1}_{\kappa_2} - L^{-1}_{\kappa_1}) f_1\|_{H^r}
\leq c_4 (1 + b^2) (1 + R^2)^\nu \|f_2 - f_1\|_{H^{r-2}}
+ c_5 R (1 + R^2)^{2\nu} \|\kappa_2 - \kappa_1\|_{H^r} \|f\|_{H^{r-1}}
\leq c_6 (1 + b^2) (1 + R^2)^{2\nu + 3/2} \left(\|Y_2 - Y_1\|_{\mathcal{Y}^{r-1}} + |\beta_2 - \beta_1|\right)
\] (3.55)

with suitable constants $c_4 - c_6$. Inserting this and Eq. (3.54) into $F^{(2)}_{\beta}$ and using the fourth inequality of Lemma 3.8, we see that $F^{(2)}_{\beta}$ satisfies the bound in Eq. (3.47).

4 Proof of Theorem 1.1

4.1 Existence of solutions to Eqs. (2.11)-(2.13)

In the notation of Eqs. (3.1)-(3.6), the Duhamel formula for Eqs. (2.11)-(2.13) is given by
\[
\begin{align*}
Y(t) &= V_\beta(t, 0) Y_0 + \int_0^t V_\beta(t, t') F_\beta(t') (Y(t')) \, dt' =: F_\beta(Y)(t) \\
\beta(t) &= \beta_0 + \int_0^t B(Y(t')) \, dt' =: B(Y)(t),
\end{align*}
\] (4.1)

where $Y_0 = (P_0, Q_0) \in \mathcal{Y}^r$ and $\beta_0 \in \mathbb{R}$ are given initial values. We begin by solving the first equation in Eq. (4.1) for a fixed function $\beta$.

Lemma 4.1 Let $\beta$ be a Lipschitz continuous real-valued function on $\mathbb{R}^+$ with Lipschitz constant $\eta$ and $|\beta(0)| \leq 1$, and let $Y_0 \in \mathcal{Y}^r$ for some $r \geq 0$. There exists a number $R < \infty$, which depends on $\|Y_0\|_{\mathcal{Y}^r}$, and a time $T > 0$ which depends on $\eta$ and $R$ such that the fixed point equation
\[
Y = F_\beta(Y)
\] (4.2)

has a unique solution $Y^*_\beta$ on $[0, T]$ in $\mathcal{Y}^r$. The solution satisfies the uniform bound
\[
\sup_{0 \leq t \leq T} \|Y^*_\beta(t)\|_{\mathcal{Y}^r} \leq R,
\] (4.3)

and depends continuously on the initial value $Y_0$ with respect to the $\mathcal{Y}^r$-norm. It is also strongly continuous in $\beta$ with respect to the $\mathcal{Y}^r$-norm, uniformly on $[0, T]$, in the sense that for any $\varepsilon$ there exists $\delta = \delta(\varepsilon, \beta, T, Y_0)$ such that
\[
\sup_{0 \leq t \leq T} |\beta'(t) - \beta(t)| \leq \delta
\] (4.4)

implies that the corresponding solutions $Y^*_\beta$ and $Y'^*_\beta$ satisfy
\[
\sup_{0 \leq t \leq T} \|Y^*_\beta(t) - Y'^*_\beta(t)\|_{\mathcal{Y}^r} \leq \varepsilon.
\] (4.5)
**PROOF:** We consider $\mathcal{F}_\beta$ as a map from the space

$$\mathcal{D}_R = \left\{ Y : [0, T] \mapsto \mathcal{Y} \mid Y \text{ continuous}, \sup_{0 \leq t \leq T} \|Y(t)\|_{\mathcal{Y}_r} \leq R \right\}$$  \quad (4.6)$$

with norm

$$|||Y|||_r = \sup_{0 \leq t \leq T} \|Y(t)\|_{\mathcal{Y}_r}$$  \quad (4.7)$$

into the space of real-valued continuous functions on $[0, T]$ with values in $\mathcal{Y}_r$. The values of $T$ and $R$ will be chosen below.

By Lemma 3.1, there exists an increasing continuous function of two variables $C_1$ with $\sup_\eta C_1(\eta, 0) < \infty$ such that for any $Z \in \mathcal{Y}_r$, we have

$$\sup_{0 \leq t_0 \leq t \leq T} \|V_\beta(t, t_0)Z\|_{\mathcal{Y}_r} \leq C_1(\eta, T) \|Z\|_{\mathcal{Y}_r}.$$  \quad (4.8)$$

By Eq. (3.46) of Lemma 3.7, there exists an increasing continuous function $C_2$ of two variables such that for any two continuous functions $Y_1, Y_2$ on $[0, T]$ with values in $\mathcal{Y}_r$ which are bounded uniformly by $R$, we have

$$\sup_{0 \leq t \leq T} \|F_\beta(Y_2(t)) - F_\beta(Y_1(t))\|_{\mathcal{Y}_r} \leq C_2(|\beta(t)|, R) \|Y_2(t) - Y_1(t)\|_{\mathcal{Y}_r}.$$  \quad (4.9)$$

for all $0 \leq t \leq T$. Combining the above two estimates and using that $|\beta(t)| \leq 1 + \eta T$, we see that

$$\sup_{0 \leq t \leq T} \|F_\beta(Y_2(t)) - F_\beta(Y_1(t))\|_{\mathcal{Y}_r} \leq TC_1(\eta, T)C_2(1 + \eta T, R) \sup_{0 \leq t \leq T} \|Y_2(t) - Y_1(t)\|_{\mathcal{Y}_r}.$$  \quad (4.10)$$

Note that by Lemma 3.1 and Eq. (3.46) of Lemma 3.7, the function $F_\beta(Y)$ is again a continuous function of $t$.

Fix

$$R \geq 4 \sup_\eta C_1(\eta, 0) \|Y_0\|_{\mathcal{Y}_r},$$  \quad (4.11)$$

and choose $T$ small enough so that

$$C_1(\eta, T) \leq 2C_1(\eta, 0), \quad TC_1(\eta, T)C_2(1 + \eta T, R) \leq \frac{1}{2}.$$  \quad (4.12)$$

Then $F_\beta$ is a contraction with Lipschitz constant $1/2$ on $\mathcal{D}_R$. Since, for $Y \in \mathcal{D}_R$,

$$\sup_{0 \leq t \leq T} \|F_\beta(Y)(t)\|_{\mathcal{Y}_r} \leq \sup_{0 \leq t \leq T} \|V_\beta(t, 0)Y_0\|_{\mathcal{Y}_r} + \sup_{0 \leq t \leq T} \|F_\beta(Y) - F_\beta(0)\|_{\mathcal{Y}_r} \leq R,$$  \quad (4.13)$$

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we see that $F_\beta$ maps $D_R$ into itself. By the contraction mapping principle, $F_\beta$ has a unique fixed point in $D_R$, which we denote by $Y^*_\beta$.

The function $F_\beta$ is clearly continuous in the initial value $Y_0$ with respect to the norm on $D_R$. By Lemma 3.2 and Eq. (3.46) of Lemma 3.7, it is also strongly continuous in $\beta$, in the sense that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \eta, T, Y)$ such that

$$\sup_{0 \leq t \leq T} |\beta'(t) - \beta(t)| \leq \delta$$

implies

$$\sup_{0 \leq t \leq T} \|F_\beta'(Y) - F_\beta(Y)\|_{Y^r} \leq \varepsilon,$$

where $Y$ is any continuous function on $[0, T]$ with values in $Y^r$, and $\beta$ is Lipschitz continuous with $|\beta(0)| \leq 1$ and Lipschitz constant $\eta$. By the uniform contraction principle (see [54], Theorem 2.2), the fixed point $Y^*_\beta$ inherits the claimed continuity properties from $F_\beta$. The modulus of continuity depends on $\beta$ and on $Y_0$ through the dependence of $Y^*_\beta$ on these parameters.

\[ \square \]

**Corollary 4.2** Let $Y_0 = (P_0, Q_0) \in Y^r$ for some $r \geq 0$, where $P$ and $Q$ are real-valued, and let $\beta_0 = 0$. There exists a time $T > 0$, which depends on $\|Y_0\|_{Y^r}$, such that the fixed point equations in Eq. (4.1) have a unique solution of the form $(Y^*, 0)$ on $[0, T]$, where $Y^* = (P^*, Q^*) \in Y^r$ are real-valued. The solution depends continuously on the initial values in $Y^r$.

**PROOF:** The claim follows immediately from Lemma 4.1 with $\beta \equiv 0$.

**Theorem 4.3** Let $r \geq 0$, and let initial values $Y_0 \in Y^r$ and $\beta_0 \in \mathbb{R}$ be given. There exists a time $T > 0$, which depends on $\|Y_0\|_{Y^r}$ and $|\beta_0|$, such that the pair of fixed point equations in Eq. (4.1) has a solution in $Y^r \times \mathbb{R}$ on $[0, T]$ which is strongly continuous in $t$ and assumes the initial values at $t = 0$.

**PROOF:** We may assume by the gauge invariance given in Eq. (2.14) that $|\beta_0| < 1$. Let $R = R(\|Y_0\|_{Y^r})$ be the constant appearing in the statement of Lemma (4.1), and consider the space

$$C_R = \{ \beta : [0, T] \mapsto \mathbb{R} \mid |\beta(0)| \leq 1, \text{ Lip}(\beta) \leq R^2 \}$$

(4.16)

with the norm of uniform convergence. Then $C_R$ is a compact convex subset of the space of continuous real-valued functions on $[0, T]$.

By Lemma 4.1, there exists a time $T = T(R) > 0$ such that for $\beta \in C_R$, the fixed point equation in Eqs. (4.2) has a unique solution $Y^*_\beta$ which is continuous in $t$ and satisfies

$$\sup_{0 \leq t \leq T} \|Y^*_\beta(t)\|_{Y^r} \leq R.$$
By Eq. (3.45) of Lemma (3.7), we have that
\[
|B(Y^*_{\beta})(t) - B(Y^*_{\beta})(t_0)| \leq \int_{t_0}^{t} B(Y^*_{\beta})(t') \, dt' \leq R^2 |t - t_0|, \tag{4.18}
\]
that is, \(B(Y^*_{\beta})\) is again Lipschitz continuous with Lipschitz constant \(R^2\), and hence in \(C_R\). Since the map \(\beta \mapsto B(Y^*_{\beta})\) is continuous on \(D_R\) by Lemma 4.1 and Eq. (3.46) of Lemma 3.7, Schauder’s theorem implies that it has a fixed point in \(C_R\), which we denote by \(\beta^*\). By construction, the pair \((Y^*_{\beta^*}, \beta^*)\) solves the fixed point equations in Eq. (4.1).

Remark 4.4 The dependence of \(R\) and \(T\) on the initial value \(\beta_0\) is due to the fact that the gauge transformation in Eq. (2.14) may change the norm of the initial values \(Y_0\). Let \(k\) be the integer closest to \(\beta_0\), and perform the gauge transformation in Eq. (2.14) with \(s_0 = 0\). Then \(|\beta_0 - k| \leq 1/2\), and for \(r \geq 0\),
\[
\|e^{iks}Y_0\|_{\mathcal{Y}^r} \leq (1 + k^2)^{(r+1)/2} \|Y_0\|_{\mathcal{Y}^r}. \tag{4.19}
\]

4.2 Well-posedness

In order to exploit the equivalence of Eqs. (2.11)-(2.13) with the original initial value problem in Eqs. (1.5)-(1.7) that was established in Lemma 2.3, we need to prove that the solutions of Eqs. (2.11)-(2.13) are unique, and depend continuously on the initial values in some reasonable norm. The difficulty is that the linear operator \(V_\beta\) is only strongly continuous, not norm-continuous, with respect to the parameter \(\beta\) (see Lemma 3.2). We deal with this problem by using a weaker norm to estimate \(F_\beta\) and \(B\).

Theorem 4.5 Let \(r \geq 1/2\). For given initial values \(Y_0 \in \mathcal{Y}^r\) and \(\beta_0 \in \mathbb{R}\), there exists a time \(T > 0\) which depends on \(\|Y_0\|_{\mathcal{Y}^r}\) and on \(|\beta_0|\) such that the pair fixed point equations in Eq. (4.1) has a unique solution in \(\mathcal{Y}^r \times \mathbb{R}\) on \([0, T]\) which is strongly continuous in \(t\). The solution depends continuously on the initial values with respect to the norm on \(\mathcal{Y}^r \times \mathbb{R}\).

PROOF: By the gauge invariance in Eq. (2.14), we may assume that \(|\beta_0| < 1\). For \(R\) to be chosen below, we consider the right hand side of Eq. (4.1) as a map on the space
\[
\mathcal{D}_R \times C_R = \left\{ (Y, \beta) : [0, T] \to \mathcal{Y}^r \middle| \begin{array}{l}
Y \text{ continuous,} \\
\sup_{0 \leq t \leq T} \|Y(t)\|_{\mathcal{Y}^r} \leq R, \\
\text{Lip}(\beta) \leq R^2, \ |\beta(0)| \leq 1
\end{array} \right\}, \tag{4.20}
\]
with the norm
\[
\||(Y, \beta)||| = \sup_{0 \leq t \leq T} \|Y(t)\|_{\mathcal{Y}^{r-1}} + \sup_{0 \leq t \leq T} |\beta(t)|. \tag{4.21}
\]
Note that $\mathcal{D}_R \times \mathcal{C}_R$ is complete with respect to the $||| \cdot |||$-norm by the convexity of the $\mathcal{Y}^r$-norm and the fact that the uniform limit of Lipschitz continuous functions with a given constant is again Lipschitz continuous with that constant.

We first bound the $\mathcal{Y}^r$-norm of $\mathcal{F}_\beta(Y)$ and the Lipschitz constant of $\mathcal{B}(Y)$. By Lemma 3.1, there exists an increasing continuous function $C_1$ of two variables with $\sup_\eta C_1(\eta, 0) < \infty$ such that for any Lipschitz continuous function $\beta$ and any $Z \in \mathcal{Y}^r$,

$$
\sup_{0 \leq t_0 \leq t \leq T} \| V_{\beta}(t_0, t) Z \|_{\mathcal{Y}^r} \leq C_1(\text{Lip}(\beta), T) \| Z \|_{\mathcal{Y}^r} \tag{4.22}
$$

By Eq. (3.45) of Lemma 3.7, there exists an increasing continuous function $C_2$ of two variables such that for any $(Y, \beta) \in \mathcal{D}_R \times \mathcal{C}_R$,

$$
\sup_{0 \leq t \leq T} \| F_{\beta(t)}(Y(t)) \|_{\mathcal{Y}^r} \leq C_2(\| \beta(t) \|, R) \tag{4.23}
$$

Combining Eqs. (4.22)-(4.23), we see that for $(Y, \beta) \in \mathcal{D}_R \times \mathcal{C}_R$,

$$
\sup_{0 \leq t \leq T} \| \mathcal{F}_\beta(Y, \beta)(t) \|_{\mathcal{Y}^r} \leq C_1(R^2, T) \| Y_0 \|_{\mathcal{Y}^r} + TC_1(R^2, T)C_2(1 + R^2T, R) \tag{4.24}
$$

By Eq. (3.45) of Lemma 3.7, we also have

$$
\text{Lip} \left( \mathcal{B}(Y) \right) = \sup_{0 \leq t_0 < t \leq T} \frac{| \mathcal{B}(Y)(t) - \mathcal{B}(Y)(t_0) |}{| t - t_0 |} \leq R^2 \tag{4.25}
$$

Next we bound the Lipschitz constant of $(\mathcal{F}_\beta, \mathcal{B})$ on \mathcal{X}. Let $(Y_1, \beta_1)$ and $(Y_2, \beta_2)$ be in $\mathcal{D}_R \times \mathcal{C}_R$. By Lemma 3.2, there exists an increasing continuous function $C_3$ of two variables such that for any two real-valued functions $\beta_1, \beta_2$ which are Lipschitz continuous with Lipschitz constant $R^2$,

$$
\| V_{\beta_2}(t_0, t) Z - V_{\beta_1}(t_0, t) Z \|_{\mathcal{Y}^{r-1}} \leq TC_3(R^2, T) \sup_{0 \leq t' \leq T} | \beta_2(t') - \beta_1(t') | \| Z \|_{\mathcal{Y}^r} \tag{4.26}
$$

for all $0 \leq t_0 \leq t \leq T$. By Eq. (3.47) of Lemma 3.7, there exists an increasing continuous function $C_4$ of two variables such that

$$
\| F_{\beta_2(t)}(Y_2(t)) - F_{\beta_1(t)}(Y_1(t)) \|_{\mathcal{Y}^{r-1}} \leq C_4(\| \beta \|, R) \left( \| Y_2(t) - Y_1(t) \|_{\mathcal{Y}^{r-1}} + | \beta_2(t) - \beta_1(t) | \right) \tag{4.27}
$$

for all $0 \leq t \leq T$. Combining Eqs. (4.22)-(4.23) with Eqs. (4.26)-(4.27), and using that $| \beta(t) | \leq 1 + R^2T$, we see that for $(Y, \beta) \in \mathcal{D}_R \times \mathcal{C}_R$,

$$
\sup_{0 \leq t \leq T} \| \mathcal{F}_{\beta_2}(Y_2, \beta_2)(t) - \mathcal{F}_{\beta_1}(Y_1, \beta_1)(t) \|_{\mathcal{Y}^{r-1}} \leq TC_5(R, T, \| Y_0 \|_{\mathcal{Y}^r}) \| (Y_2, \beta_2) - (Y_1, \beta_1) \|_{\mathcal{Y}^r} \tag{4.28}
$$
with some increasing continuous function \( C_5 \) of two variables. By Eq. (3.47) of Lemma 3.7,
\[
\sup_{0 \leq t \leq T} |B(Y_2)(t) - B(Y_1)(t)| \leq TR \|Y_2 - Y_1\|_{\mathcal{Y}_r^{-1}} .
\] (4.29)

In summary, Eqs. (4.28)-(4.29) show that
\[
\|\|(F_{\beta_2}(Y_2), B(Y_2)) - (F_{\beta_1}(Y_1), B(Y_1))\|\| \leq TC_6(R, T, \|Y_0\|_{\mathcal{Y}_r}) \|\|(Y_2, \beta_2) - (Y_1, \beta_1)\|\|
\] (4.30)
with a suitable increasing function \( C_6 \).

Choose
\[
R \geq 4 \sup_{\eta} C_1(\eta, 0) \|Y_0\|_{\mathcal{Y}_r} ,
\] (4.31)
and \( T \) small enough such that
\[
C_1(R^2, T) \leq 2C_1(R^2, 0) ,
\]
\[
TC_1(R^2, T)C_2(1 + R^2T, R) \leq R ,
\]
\[
TC_6(R, T) \leq \frac{1}{2} .
\] (4.32)

Then Eqs. (4.24)-(4.25) show that \((F_{\beta}, B)\) maps \( \mathcal{D}_R \times \mathcal{C}_R \) into itself, and Eq. (4.30) shows that \((F_{\beta}, B)\) is a contraction with Lipschitz constant \(1/2\). By the contraction mapping theorem, Eq. (4.1) has a unique solution in \( \mathcal{D}_R \times \mathcal{C}_R \). By the uniform contraction principle and the continuity properties of \( V_{\beta}(t, t') \) and \( F_{\beta} \), this solution is continuous with respect to the initial data \((Y_0, \beta_0)\) in the sense that for every \(\varepsilon > 0\) there exists \(\delta = \delta(\varepsilon, T, \beta_0, Y_0)\) such that
\[
|\beta'_0 - \beta_0| + \|Y'_0 - Y_0\|_{\mathcal{Y}_r^{-1}} \leq \delta
\] (4.33)
implies that the corresponding solutions \((Y, \beta)\) and \((Y', \beta')\) of Eq. (4.1) satisfy
\[
\|\|Y' - Y\|\| \leq \varepsilon .
\] (4.34)

It remains to prove the strong continuity of the solution with respect to \(Y_0\) and \(\beta_0\) in the natural norm on \( \mathcal{Y}_r \times \mathbb{R} \). Note that Eq. (4.34) implies in particular that
\[
\sup_{0 \leq t \leq T} |\beta'(t) - \beta(t)| \leq \varepsilon .
\] (4.35)

Fix initial values \((Y_0, \beta_0)\) in \( \mathcal{Y}_r \times \mathbb{R} \) with \(|\beta_0| < 1\), let \(\varepsilon > 0\) be given, and suppose that
\[
|\beta'_0 - \beta_0| + \|Y'_0 - Y_0\|_{\mathcal{Y}_r} \leq \delta
\] (4.36)
Let \((Y, \beta) \in \mathcal{D}_R \times \mathcal{C}_R\) be the solution of Eq. (4.1) with these initial values, defined on some interval strictly containing \([0, T]\), and let \((Y', \beta')\) be the solution to Eq. (4.1) with initial values \((Y'_0, \beta'_0)\), where \(Y_0 \in \mathcal{Y}_r\), and \(|\beta_0| \leq 1\). If \(Y'_0\) is sufficiently close to \(Y_0\), then
we may assume that \((Y'\beta')\) is defined on \([0, T]\). By Lemma 4.1, \(Y\) is the unique solution of the first fixed point equation in Eq. (4.1) with \(\beta\) fixed and the given initial value \(Y_0\). Let \(Z\) be the unique solution of the fixed point equation

\[
Z(t) = V_\beta(t, 0)Y_0' + \int_0^t V_\beta(t, t')F_{\beta(t')}(Z(t'))\, dt'
\]  

(4.37)

with the same function \(\beta\). Clearly,

\[
\sup_{0 \leq t \leq T} \|Y'(t) - Y(t)\|_{Y'} \leq \sup_{0 \leq t \leq T} \left(\|Y'(t) - Z(t)\|_{Y'} + \|Z(t) - Y(t)\|_{Y'}\right).
\]  

(4.38)

By the continuity statement in Eq. (4.35), we may assume that \(\sup_{0 \leq t \leq T} |\beta'(t) - \beta(t)|\) is as small as we please. Since \(Y'\) and \(Z\) solve the fixed point equation in the first line of Eq. (4.1) with the same initial value \(Y_0'\) but different functions \(\beta'\) and \(\beta\), and since \(Z\) and \(Y\) solve the equation with the same function \(\beta\) but different initial values \(Y_0'\) and \(Y_0\), the continuity statements of Lemma 4.1 imply the claim. The modulus of continuity depends on \(\beta_0\) and \(Y_0\) through the dependence of \((Y, \beta)\) on these parameters.

**Proof of Theorem 1.1** By Lemma 2.3, the well-posedness of Eqs. (1.5)-(1.7) asserted in Theorem 4.5 for the three-dimensional case and in Corollary 4.2 for the planar case implies the claims of Theorem 1.1.

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