SOME GENERAL STATISTICAL APPROXIMATION RESULTS FOR 
\( \lambda \)-BERNSTEIN OPERATORS

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ABSTRACT. In this article, we achieve some general statistical approximation results for \( \lambda \)-Bernstein operators in addition to some other approximation properties. We prove a statistical Voronovskaja-type approximation theorem. We also construct bivariate \( \lambda \)-Bernstein operators and study their approximation properties.

Keywords: Rate of weighted \( A \) statistical convergence, \( \lambda \)-Bernstein operators, bivariate \( \lambda \)-Bernstein operators, statistical approximation properties, GrüssVoronovskaja-type theorem, weighted \( A \)-statistical Voronovskaja-type theorem, weighted space

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1. Introduction

Bernstein used famous polynomials nowadays called Bernstein polynomials, in 1912, to obtain an alternative proof of Weierstrass’s fundamental theorem [2]. Approximation properties of Bernstein operators and their applications in Computer Aided Geometric Design and Computer Graphics have been extensively studied in many articles.

Bernstein basis of degree \( n \) on \( x \in [0, 1] \) is defined by

\[
b_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, \ldots, n,
\]

and \( n \)th order Bernstein polynomial is given by

\[
B_n(f; x) = \sum_{i=0}^{n} f \left( \frac{i}{n} \right) b_{n,i}(x)
\]

for any continuous function \( f(x) \) defined on \([0, 1]\).

In 2018, Cai et al. have introduced a new type \( \lambda \) Bernstein operators [4]

\[
B_{n,\lambda}(f; x) = \sum_{i=0}^{n} f \left( \frac{i}{n} \right) \tilde{b}_{n,i}(\lambda; x)
\]

with Bézier bases \( \tilde{b}_{n,i}(\lambda; x) \) [15]:

\[
\begin{align*}
\tilde{b}_{n,0}(\lambda; x) &= b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\
\tilde{b}_{n,i}(\lambda; x) &= b_{n,i}(x) + \lambda \left( \frac{n - 2i + 1}{n^2 - 1} b_{n+1,i}(x) - \frac{n - 2i - 1}{n^2 - 1} b_{n+1,i+1}(x) \right), \quad i = 1, 2, \ldots, n - 1, \\
\tilde{b}_{n,n}(\lambda; x) &= b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x),
\end{align*}
\]

where shape parameters \( \lambda \in [-1, 1] \).
2. Preliminary Results

In this part, we obtain global approximation formula in terms of Ditzian-Totik uniform modulus of smoothness of first and second order and give a local direct estimate of the rate of convergence by Lipschitz-type function involving two parameters for \( \lambda \) Bernstein operators. We also give a Grüss-Voronovskaja function and a quantitative Voronovskaja-type theorem.

Results in the following lemma were obtained for \( \lambda \) Bernstein operators in \([4, \text{Lemma 2.1}].\)

**Lemma 2.1.** We have following equalities for \( \lambda \) Bernstein operators:

\[
B_{n,\lambda}(1; x) = 1;
\]

\[
B_{n,\lambda}(t; x) = x + \frac{1 - 2x + x^{n+1} - (1 - x)^{n+1}}{n(n-1)} \lambda;
\]

\[
B_{n,\lambda}(t^2; x) = x^2 + \frac{2x - 4x^2 + 2x^{n+1}}{n(n-1)} + \frac{x^{n+1} + (1 - x)^{n+1} - 1}{n^2(n-1)} \lambda;
\]

\[
B_{n,\lambda}(t^3; x) = x^3 + \frac{3x^2(1 - x)}{n} + \frac{2x^3 - 3x^2 + x}{n^2} + \frac{6x^{n+1} - 6x^3 + 3x^2 - 3x^{n+1}}{n(n-1)} \\
+ \frac{9x^{n+1} - 9x^2}{n^2(n-1)} + \frac{4x^{n+1} - 4x}{n^3(n-1)} + \frac{1 - x^{n+1} + (1 - x)^{n+1}}{n^3(n-1)} \lambda;
\]

\[
B_{n,\lambda}(t^4; x) = x^4 + \frac{6x^3(1 - x)}{n} + \frac{7x^2 - 18x^3 + 11x^4}{n^2} + \frac{x - 7x^2 + 12x^3 - 6x^4}{n^3} \\
+ \frac{6x^2 - 2x^3 - 8x^4 + 4x^{n+1}}{n^2} + \frac{17x^{n+1} + 16x^4 - 32x^3 - x^2}{n^3} + \frac{x - x^{n+1}}{n^4} \\
+ \frac{7x^2 - 7x^{n+1}}{n^2(n-1)} + \frac{x - 23x^2 + 22x^{n+1}}{n^3(n-1)} + \frac{(1 - x)^{n+1} + x - 1}{n^4} \lambda.
\]

2.1. Global and local approximations. First we obtain global approximation formula in terms of Ditzian-Totik uniform modulus of smoothness of first and second order defined by

\[
\omega_1(f, \delta) := \sup_{0 < |h| \leq \delta} \sup_{x, x + h \xi(x) \in [0,1]} \{|f(x + h \xi(x)) - f(x)|\}
\]

and

\[
\omega_2(f, \delta) := \sup_{0 < |h| \leq \delta} \sup_{x, x + h \phi(x) \in [0,1]} \{|f(x + h \phi(x)) - 2f(x) + f(x - h \phi(x))|\},
\]

respectively, where \( \phi \) is an admissible step-weight function on \([a, b] \), i.e. \( \phi(x) = [(x - a)(b - x)]^{1/2} \) if \( x \in [a, b] \). \([5]\). Corresponding \( K \)-functional is

\[
K_{2,\phi(x)}(f, \delta) = \inf_{g \in W^2(\phi)} \{||f - g||_{C[0,1]} + \delta||\phi^2 g''||_{C[0,1]} : g \in C^2[0,1]\},
\]

where \( \delta > 0, W^2(\phi) = \{g \in C[0,1] : g' \in AC[0,1], \phi^2 g'' \in C[0,1]\} \) and \( C^2[0,1] = \{g \in C[0,1] : g', g'' \in C[0,1]\} \). Here, \( g' \in AC[0,1] \) means that \( g' \) is absolutely continuous on \([0,1]\). It is known by \([7]\) that there exists an absolute constant \( C > 0 \), such that

\[
C^{-1} \omega_2^2(f, \sqrt{\delta}) \leq K_{2,\phi(x)}(f, \delta) \leq C \omega_2^2(f, \sqrt{\delta}). \tag{2.2}
\]

**Theorem 2.2.** Let \( \lambda \in [-1, 1], f \in C[0,1] \) and \( \phi (\phi \neq 0) \) be an admissible step-weight function of Ditzian-Totik modulus of smoothness such that \( \phi^2 \) is concave. Then we have

\[
|B_{n,\lambda}(f; x) - f(x)| \leq C \omega_2^2 \left(f, \frac{\delta_n(x)}{2\phi(x)}\right) + \omega_1 \left(f, \frac{\beta_n(x)}{\xi(x)}\right)
\]

for \( x \in [0,1] \) and \( C > 0 \).
Theorem 2.4. and a quantitative Voronovskaja-type theorem for $B_{\phi}$ where
\[ B_{\phi}(f; x) = \lim_{n \to \infty} \sum_{k=0}^{n-1} \left( f(x + k) - f(x) \right) \]
for $k_1 \geq 0, k_2 > 0$, where $\eta \in (0, 1]$ and $M$ is a positive constant (see [11]).

Theorem 2.3. If $f \in Lip_M^{(k_1, k_2)}(\eta)$, then we have
\[ |B_{n, \lambda}(f; x) - f(x)| \leq M\alpha_n^2(x)(k_1x^2 + k_2x)^{-\eta} \]
for all $\lambda \in [-1, 1], x \in (0, 1]$ and $\eta \in (0, 1]$.

Theorem 2.4. The following inequality holds:
\[ |B_{n, \lambda}(f; x) - f(x)| \leq |\beta_n(x)| |f'(x)| + 2\sqrt{\alpha_n(x)}w(f', \sqrt{\alpha_n(x)}) \]
for $f \in C^1[0, 1]$ and $x \in [0, 1]$.

2.2. Voronovskaja-type theorems. In this part, we give a Grüss-Voronovskaja-type theorem and a quantitative Voronovskaja-type theorem for $B_{n, \lambda}(f; x)$.

We first obtain a quantitative Voronovskaja-type theorem for $B_{n, \lambda}(f; x)$ using Ditzian-Totik modulus of smoothness defined as
\[ \omega_{\phi}(f, \delta) := \sup_{0 < h \leq \delta} \left\{ \left| f(x + \frac{h\phi(x)}{2}) - f(x) - \frac{h\phi(x)}{2} \right|, x + \frac{h\phi(x)}{2} \in [0, 1] \right\}, \]
where $\phi(x) = (x(1-x))^{1/2}$ and $f \in C[0, 1]$, and corresponding Peetre’s $K$-functional is defined by
\[ K_{\phi}(f, \delta) = \inf_{g \in W_{\phi}[0, 1]} \{ ||f - g|| + \delta||\phi'g|| : g \in C^1[0, 1], \delta > 0 \}, \]
where $W_{\phi}[0, 1] = \{ g : g \in AC_{loc}[0, 1], ||\phi'g|| < \infty \}$ and $AC_{loc}[0, 1]$ is the class of absolutely continuous functions defined on $[a, b] \subset [0, 1]$. There exists a constant $C > 0$ such that
\[ K_{\phi}(f, \delta) \leq C \omega_{\phi}(f, \delta). \]

Theorem 2.5. Assume that $f \in C[0, 1]$ such that $f', f'' \in C[0, 1]$. Then, we have
\[ \left| B_{n, \lambda}(f; x) - f(x) - \beta_n f'(x) - \frac{\alpha_n + 1}{2} f''(x) \right| \leq C \frac{\rho^2(x)}{n} \omega_{\phi}(f'', n^{-1/2}) \]
for every $x \in [0, 1]$ and sufficiently large $n$, where $C$ is a positive constant, $\alpha_n$ and $\beta_n$ are defined in Theorem 2.2.

3. Statistical approximation properties by weighted mean matrix method

In this part, we study on statistical approximation properties and estimate rate of weighted $A$-statistical convergence. We also use statistical convergence to prove a Voronovskaja-type approximation theorem.

Theorem 3.1. Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix for $n, k \in \mathbb{N}$ and $q = (q_n)$ be a sequence of non-negative numbers such that $q_0 > 0$ and $Q_n = \sum_{k=0}^{n} q_k \to \infty$ as $n \to \infty$. For any $f \in C[0, 1]$, we have
\[ S_n^A - \lim_{n \to \infty} \|B_{n, \lambda}(f; x) - f(x)\|_{C[0, 1]} = 0. \]
3.1. A Voronovskaja-type approximation theorem. We prove a Voronovskaja-type approximation theorem by $\tilde{B}_{n,\lambda}(f; x)$ family of linear operators.

**Theorem 3.2.** Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix and let $(x_n)$ be a sequence of real numbers such that $\sum_{k=1}^{\infty} - \lim x_n = 0$. Also let $\tilde{B}_{n,\lambda}(f; x)$ be a sequence of positive linear operators acting from $C_B[0,1]$ into $C[0,1]$ defined by

$$ \tilde{B}_{n,\lambda}(f; x) = (1 + x_n)B_{n,\lambda}(f; x). $$

Then for every $f \in C_B[0,1]$, and $f', f'' \in C_B[0,1]$ we have

$$ \sum_{k=1}^{\infty} - \lim_{n \to \infty} n\{ \tilde{B}_{n,\lambda}(f; x) - f(x) \} = \frac{f''(x)}{2}x(1 - x). $$

4. Approximation properties for bivariate case

In this part, we construct bivariate $\lambda$ Bernstein operators and study their approximation properties.

Let $I = I_1 \times I_2 = [0,1] \times [0,1]$ and $(x, y) \in I$, then we construct bivariate $\lambda$ Bernstein operators as

$$ B_{n,m}(f; x, y; \lambda) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{m} f \left( \frac{k_1}{n}, \frac{k_2}{m} \right) \tilde{b}_{n,k_1}(\lambda; x) \tilde{b}_{m,k_2}(\lambda; y) $$

for $f \in C(I)$, where Bézier bases $\tilde{b}_{n,k_1}(\lambda; x)$, $\tilde{b}_{m,k_2}(\lambda; x)$ ($k_1 = 0, 1, \ldots, n$; $k_2 = 0, 1, \ldots, m$) are defined in [133].

**Lemma 4.1.** For any natural number $n$ ($n \geq 2$) the following equalities hold:

1. $\tilde{B}_{n,m}(1; x, y; \lambda) = 1$;

2. $\tilde{B}_{n,m}(s; x, y; \lambda) = x + \frac{1 - 2x + x^{n+1} - (1 - x)^{n+1}}{n(n - 1)} \lambda$;

3. $\tilde{B}_{n,m}(i; x, y; \lambda) = y + \frac{1 - 2y + y^{m+1} - (1 - y)^{m+1}}{m(m - 1)} \lambda$;

4. $\tilde{B}_{n,m}(s^2; x, y; \lambda) = x^2 + \frac{x(1 - x)}{n} + \frac{2x - 4x^2 + 2x^{n+1}}{n(n - 1)} \lambda$;

5. $\tilde{B}_{n,m}(i^2; x, y; \lambda) = y^2 + \frac{y(1 - y)}{m} + \frac{2y - 4y^2 + 2y^{m+1}}{m(m - 1)} \lambda$.

**Theorem 4.2.** The sequence $\tilde{B}_{n,m}(f; x, y; \lambda)$ of operators converges uniformly to $f(x, y)$ on $I$ for each $f \in C(I)$.

**Proof.** It is enough to prove the following condition

$$ \lim_{m,n \to \infty} \tilde{B}_{n,m}(e_{ij}(x, y); x, y; \lambda) = x^iy^j, \quad (i, j) \in \{(0, 0), (1, 0), (0, 1)\} $$

converges uniformly on $I$. We clearly have

$$ \lim_{m,n \to \infty} \tilde{B}_{n,m}(e_{00}(x, y); x, y; \lambda) = 1. $$

We have

$$ \lim_{m,n \to \infty} \tilde{B}_{n,m}(e_{10}(x, y); x, y; \lambda) = \lim_{n \to \infty} \left[ x + \frac{1 - 2x + x^{n+1} - (1 - x)^{n+1}}{n(n - 1)} \lambda \right] = e_{10}(x, y), $$

$$ \lim_{m,n \to \infty} \tilde{B}_{n,m}(e_{01}(x, y); x, y; \lambda) = \lim_{m \to \infty} \left[ y + \frac{1 - 2y + y^{m+1} - (1 - y)^{m+1}}{m(m - 1)} \lambda \right] = e_{01}(x, y). $$
Theorem 4.4. Peetre’s \( \delta > \) for all \( x \), and for every \( x \) converges uniformly. Bearing in mind the above conditions and Korovkin type theorem established by Volkov [14]

Let \( \lambda \) be the space of all functions defined on the real axis provided for every \((x, y) \in I_{ab}\) is defined as follows:

\[
\lim_{n,m \to \infty} B_{n,m} (f_{ij}(x, y); x, y; \lambda) = x^i y^j
\]

Peetre’s \( K \)-functional is given by

\[
K(f, \delta) = \inf_{g \in C^2(I_{ab})} \{\|f - g\|_{C(I_{ab})} + \delta\|g\|_{C^2(I_{ab})}\}
\]

for \( \delta > 0 \), where \( C^2(I_{ab}) \) is the space of functions of \( f \) such that \( f, \frac{\partial^2 f}{\partial x^2} \) and \( \frac{\partial^2 f}{\partial y^2} \) \((j = 1, 2)\) in \( C(I_{ab}) \) [12]. We now give an estimate of the rates of convergence of operators \( \bar{B}_{n,m}(f; x, y; \lambda) \).

Theorem 4.3. Let \( f \in C(I) \), then we have

\[
\lim_{n,m \to \infty} B_{n,m}(f; x, y; \lambda) - f(x, y) = 0
\]

for all \( x \in I \).

Now we investigate convergence of the sequence of linear positive operators \( \bar{B}_{n,m}(f; x, y; \lambda) \) to a function of two variables which defined on weighted space.

Let \( \rho(x, y) = x^2 + y^2 + 1 \) and \( B_{\rho} \) be the space of all functions defined on the real axis provided with \( |f(x, y)| \leq M f \rho(x, y) \), where \( M_f \) is a positive constant depending only on \( f \).

Theorem 4.4. For each \( f \in C^0_{\rho} \) and for all \((x, y) \in I \), we have

\[
\lim_{n \to \infty} \| B_{n,m}(f; x, y; \lambda) - f(x, y) \|_{\rho} = 0.
\]

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