STABLE TENSORS AND MODULI SPACE OF ORTHOGONAL SHEAVES

TOMÁS L. GÓMEZ AND IGNACIO SOLS

ABSTRACT. Let $X$ be a smooth projective variety over $\mathbb{C}$. We find the natural notion of semistable orthogonal bundle and construct the moduli space, which we compactify by considering also orthogonal sheaves, i.e. pairs $(E, \varphi)$, where $E$ is a torsion free sheaf on $X$ and $\varphi$ is a symmetric nondegenerate (in the open set where $E$ is locally free) bilinear form on $E$. We also consider special orthogonal sheaves, by adding a trivialization $\psi$ of the determinant of $E$ such that $\text{det}(\varphi) = \psi^2$; and symplectic sheaves, by considering a form which is skewsymmetric. More generally, we consider semistable tensors, i.e. multilinear forms on a torsion free sheaf, and construct their projective moduli space using GIT.

Let $X$ be a smooth projective variety of dimension $n$ over $\mathbb{C}$. If $X$ is a curve, the moduli space of vector bundles was constructed by Mumford, Narasimhan and Seshadri. If $\dim(X) > 1$, to obtain a projective moduli space we have to consider also torsion free sheaves, and this was done by Gieseker, Maruyama and Simpson.

An orthogonal bundle is a pair $(E, \varphi)$, where $E$ is a vector bundle and

$$\varphi : E \otimes E \rightarrow \mathcal{O}_X$$

is a bilinear symmetric nowhere degenerate form. The nondegeneracy means that it induces an isomorphism $E \rightarrow E^\vee$, hence $(\text{det} E)^2 \cong \mathcal{O}_X$.

A special orthogonal bundle is a triple $(E, \varphi, \psi)$ where $E$ and $\varphi$ are as before, and

$$\psi : \text{det} E \rightarrow \mathcal{O}_X$$

is an isomorphism such that $\text{det}(\varphi) = \psi^2$ (this equation means that for all points $x \in X$, if we choose a basis for the fiber $E_x$, the determinant of the matrix associated to $\varphi$ at $x$ is equal to the square of the scalar associated to $\psi$ at $x$).

A symplectic bundle is a pair $(E, \varphi)$, where $E$ is a vector bundle and $\varphi$ is a bilinear skewsymmetric nowhere degenerate form.

Note that giving an orthogonal (or special orthogonal, or symplectic) bundle is equivalent to giving a principal bundle with group structure group $O(r, \mathbb{C})$ (or $SO(r, \mathbb{C})$, or $Sp(r, \mathbb{C})$). To obtain a projective moduli space we have to consider also orthogonal, or special orthogonal or symplectic sheaves, i.e. allowing $E$ to be a torsion free sheaf, and then requiring $\varphi$ to be nondegenerate only on the open subset of $X$ where $E$ is locally free. We say that a subsheaf $F$ of $E$ is isotropic if $\varphi|_{F \otimes F} = 0$.

An orthogonal, or special orthogonal or symplectic sheaf is called stable (respectively semistable) if for all proper isotropic subsheaves $F$ of $E$

$$P_F + P_{F^\perp} \prec P_E$$

(respectively $\preceq$),

where $P_F$ is the Hilbert polynomial of $F$, $F^\perp$ is the sheaf perpendicular to $F$ with respect to the form $\varphi$, and, as usual, the inequality between polynomials $P_1 \prec P_2$.

Date: 21 January 2003.
Mathematical Subject Classification: Primary 14D22, Secondary 14D20.
(respectively $\leq$) means that $P_1(m) < P_2(m)$ (respectively $\leq$) for $m \gg 0$ (see sections 5 and 6 for precise definitions).

A similar problem was considered by Sorger [So]. He works on a curve $C$ (not necessarily smooth) on a smooth surface $S$, and constructs the moduli space of torsion free sheaves on $C$ together with a symmetric form taking values on the dualizing sheaf $\omega_C$. Faltings [Fa] has considered principal bundles on semistable curves. For $G$ orthogonal or symplectic he considers a torsion free sheaf with a quadratic form, and he also defines a notion of stability. For general reductive group $G$ he uses the approach of loop groups.

More generally, we can consider triples $(E, \varphi, u)$ where $E$ is a torsion free sheaf on $X$ and $\varphi$ is a non-zero homomorphism

$$\varphi : (E^\otimes s)^{\oplus c} \longrightarrow (\det E)^{\otimes b} \otimes D_u,$$

where $D_u$ is a locally free sheaf belonging to a fixed family $\{D_u\}_{u \in R}$ parametrized by a scheme $R$ (for instance, $R$ could be $\text{Pic}^a(X)$, and then $D_u$ is any line bundle with fixed degree $a$, or we can take $R$ to be a point, and then $D_u$ is a fixed locally free sheaf). We call these triples tensors. See section 1 for the precise definition. Tensors generalize several objects that have already appeared in the literature. If $s = 1$, $b = 0$, $c = 1$, and $R$ is a point, these are the framed modules of Huybrechts and Lehn. They gave two constructions of their moduli space: In [H-L1] for $\dim(X) \leq 2$, based in the ideas of Gieseker [Gi], and in [H-L2] for arbitrary dimension, following the ideas of Simpson [Si]. If $\dim(X) = 1$, $s = 2$, $b = 0$, $c = 1$, $R$ is a point and $D_u$ is a line bundle, these are the conic bundles of [G-S]. If $\dim(X) = 1$ and $D$ is a family of line bundles, these are the decorated vector bundles whose moduli space was constructed by Schmitt [Sch].

Using geometric invariant theory (GIT) as in [Si] and [H-L2], we construct the moduli space of semistable tensors (sections 1 to 4). This is used in sections 5 and 6 to construct the projective moduli space of classical sheaves.

Finally, in section 7, as a further application we obtain moduli spaces for $\text{GL}(r, \mathbb{C})$-representational pairs, i.e. pairs $(P, \sigma)$ consisting of a principal $\text{GL}(r, \mathbb{C})$-bundle $P$ and a section $\sigma$ of the vector bundle associated to a fixed representation of $\text{GL}(r, \mathbb{C})$. We can also consider a quasi-projective scheme $Y$ with an action of $\text{GL}(r, \mathbb{C})$, and then we can take $\sigma$ to be a section of the associated fiber bundle with fiber $Y$.

Banfield [Ba] and Mundet [MR] have given Hitchin-Kobayashi correspondences for these objects.

In a future paper we will consider principal $G$-bundles for more general groups.

**Notation.** If $f : Y \rightarrow Y'$ is a morphism, we denote $\bar{f} = \text{id}_X \times f$. If $E_S$ is a coherent sheaf on $X \times S$, we denote $E_S(m) := E_S \otimes p_X^* O_X(m)$. To simplify the notation, we will denote the complex groups $\text{GL}(r, \mathbb{C})$, $\text{O}(r, \mathbb{C})$, $\text{Sp}(r, \mathbb{C})$, ... by $\text{GL}(r)$, $\text{O}(r)$, $\text{Sp}(r)$. If $X$, $Y$, $Z$ are schemes, then $\pi_X$, $\pi_{Y \times Z}$, etc... denote the corresponding projections from $X \times Y \times Z$.

If $P_1$ and $P_2$ are two polynomials, we write $P_1 \prec P_2$ if $P_1(m) < P_2(m)$ for $m \gg 0$, and analogously for “$\leq$” and “$\preceq$”. We use the convention that whenever “(semi)stable” and “($\leq$)” appear in a sentence, two statements should be read: one with “semistable” and “($\leq$)” and another with “stable” and “$<$”.

**Acknowledgments.** We would like to thank U. Bhosle, N. Fakhruddin, M.S. Narasimhan, S. Ramanan, C.S. Seshadri and Ch. Sorger for discussions on this subject. The authors are members of VBAC (Vector Bundles on Algebraic Curves),
which is partially supported by EAGER (EC FP5 Contract no. HPRN-CT-2000-00099) and by EDGE (EC FP5 Contract no. HPRN-CT-2000-00101). T.G. was supported by a postdoctoral fellowship of Ministerio de Educación y Cultura (Spain).

1. Stability of tensors

Let \( X \) be a smooth projective variety over \( \mathbb{C} \). Fix an ample line bundle \( \mathcal{O}_X(1) \) on \( X \). Fix a polynomial \( P \) of degree \( n = \dim(X) \), and integers \( s, b, c \). We will denote by \( r \) and \( d \) the rank and degree of a sheaf with Hilbert polynomial \( P \). Fix a family \( \{ D_u \}_{u \in R} \) of locally free sheaves \( X \) parametrized by a scheme \( R \), i.e. we fix a locally free sheaf \( D \) on \( X \times R \), and given a point \( u \in R \), we denote by \( D_u \) the restriction to the slice \( X \times u \).

**Definition 1.1 (Tensor).** A tensor is a triple \((E, \varphi, u)\), where \( E \) is a coherent sheaf on \( X \) with Hilbert polynomial \( P_E = P \), \( u \) is a point in \( R \), and \( \varphi \) is a homomorphism

\[
\varphi : (E^{\otimes s})^{\oplus c} \rightarrow (\det E)^{\otimes b} \otimes D_u,
\]

that is not identically zero. Let \((E, \varphi, u)\) and \((F, \psi, v)\) be two tensors with \( P_E = P_F \), \( \det E \cong \det F \), and \( u = v \). A homomorphism between \((E, \varphi, u)\) and \((F, \psi, v)\) is a pair \((f, \alpha)\) where \( f : E \rightarrow F \) is a homomorphism of sheaves, \( \alpha \in \mathbb{C} \), and the following diagram commutes

\[
\begin{array}{ccc}
(E^{\otimes s})^{\oplus c} & \xrightarrow{(f^{\otimes s})^{\oplus c}} & (F^{\otimes s})^{\oplus c} \\
\downarrow{\varphi} & & \downarrow{\psi} \\
(\det E)^{\otimes b} \otimes D_u & \xrightarrow{\hat{f} \otimes \alpha} & (\det F)^{\otimes b} \otimes D_v
\end{array}
\]

where \( \hat{f} : \det E \rightarrow \det F \) is the homomorphism induced by \( f \). In particular, \((E, \varphi, u)\) and \((E, \lambda \varphi, u)\) are isomorphic for \( \lambda \in \mathbb{C}^* \).

**Remark 1.2.** We could have defined a more restrictive notion of isomorphism, considering only isomorphisms for which \( \alpha = 1 \). If we do this, we obtain a different category: for instance, if \( E \) is simple, the set of automorphisms of \((E, \varphi, u)\) is \( \mathbb{C}^* \), but if we require \( \alpha = 1 \), then the set of automorphisms is \( \mathbb{Z}/(rb - s)\mathbb{Z} \) (assuming \( rb - s \neq 0 \)). If \( rb - s \neq 0 \), even if the categories are not equivalent, the set of isomorphism classes will be the same (because \( \alpha \) can be absorbed in \( f \) by changing \( f \) into \( \alpha^{1/(rb-s)}f \)), and then the moduli spaces will be the same. But if \( rb - s = 0 \), then \( \alpha \) cannot be absorbed in \( f \), and the set of isomorphism classes is not the same.

Let \( \delta \) be a polynomial with \( \deg(\delta) < n = \dim(X) \)

\[
\delta = \delta_1 t^{n-1} + \delta_2 t^{n-2} + \cdots + \delta_n \in \mathbb{Q}[t],
\]

and \( \delta(m) > 0 \) for \( m \gg 0 \). We denote \( \tau = \delta_1 (n-1)! \). We will define a notion of stability for these tensors, depending on the polarization and \( \delta \), and we will construct, using geometric invariant theory (GIT), a moduli space for semistable tensors.

A weighted filtration \((E_\bullet, m_\bullet)\) of a sheaf \( E \) is a filtration of sheaves

\[
0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_t \subseteq E_{t+1} = E,
\]

and positive numbers \( m_1, m_2, \ldots, m_t > 0 \). Let \( r_i = \rk(E_i) \). If \( t = 1 \) (one step filtration), then we will take \( m_1 = 1 \). We will denote \( E^i = E/E_i \) and \( r^i = \rk(E^i) \).
The filtration is called \textit{saturated} if all sheaves $E_i$ are saturated in $E$, i.e. if $E^i$ is torsion free.

Consider the vector of $\mathbb{C}^r$ defined as $\gamma = \sum_{i=1}^t m_i \gamma_i$, where
\begin{equation}
(1.4) \quad \gamma^{(k)} = \left( k - r, \ldots, k - r, k, \ldots, k \right) \quad (1 \leq k \leq r - 1).
\end{equation}
Now let $\mathcal{I} = \{1, \ldots, t + 1\}^s$ be the set of all multi-indexes $I = (i_1, \ldots, i_s)$. Let $\gamma_j$ be the $j$-th component of the vector $\gamma$, and define
\begin{equation}
(1.5) \quad \mu(\varphi, E_\bullet, m_\bullet) = \min_{I \in \mathcal{I}} \{ \gamma_{r_{i_1}} + \cdots + \gamma_{r_{i_s}} : \varphi|_{(E_{i_1} \otimes \cdots \otimes E_{i_s})^{\otimes c} \neq 0} \}
\end{equation}

\textbf{Definition 1.3 (Stability).} Let $\delta$ be a polynomial as in (1.2). We say that $(E, \varphi, u)$ is $\delta$-(semi)stable if for all weighted filtrations it is
\begin{equation}
(1.6) \quad \left( \sum_{i=1}^t m_i (r \deg E_i - r_i \deg E) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta \leq 0
\end{equation}

Recall that we assume that $\varphi$ is not identically zero. It is easy to check that if $(E, \varphi, u)$ is semistable, then $E$ is torsion free. In this definition, it suffices to consider saturated filtrations, and with $\text{rk}(E_i) < \text{rk}(E_{i+1})$ for all $i$.

\textbf{Lemma 1.4.} There is an integer $A_1$ (depending only on $P$, $s$, $b$, $c$ and $D$) such that it is enough to check the stability condition (1.6) for weighted filtrations with $m_i \leq A_1$ for all $i$.

\textbf{Proof.} Again, let $\mathcal{I} = \{1, \ldots, t + 1\}^s$. Multi-indexes will be denoted $I = (i_1, \ldots, i_s)$. Note that (1.5) is a piece-wise linear function of $\gamma \in \mathcal{C}$, where $\mathcal{C} \subset \mathbb{Z}^r$ is the cone defined by $\gamma_1 \leq \ldots \leq \gamma_r$. This is because it is defined as the minimum among a finite set of linear functions, namely the functions $\gamma_{r_{i_1}} + \cdots + \gamma_{r_{i_s}}$ for $I \in \mathcal{I}$. There is a decomposition of $\mathcal{C} = \bigcup_{I \in \mathcal{I}} \mathcal{C}_I$ into a finite number of subcones
\begin{align*}
\mathcal{C}_I := \{ \gamma \in \mathcal{C} : \gamma_{r_{i_1}} + \cdots + \gamma_{r_{i_s}} \leq \gamma_{r_{i_1}' + \cdots + \gamma_{r_{i_s}'}} \text{ for all } I' \in \mathcal{I} \}
\end{align*}
and (1.5) is linear on each cone $\mathcal{C}_I$. Choose one vector $\gamma \in \mathbb{Z}^r$ in each edge of each cone $\mathcal{C}_I$. Multiply all these vectors by $r$, so that all their coordinates are divisible by $r$, and call this set of vectors $S$. All vectors in $S$ come from a collection of weights $m_i > 0$, $i = 1, \ldots, t + 1$, given by the formula $\gamma = \sum_{i=1}^t m_i \gamma_i$. Hence to obtain the finite set $S$ of vectors it is enough to consider a finite set of values for $m_i$, and hence there is a maximum value $A_1$.

Finally, it is easy to see that it is enough to check (1.6) for the weights associated to the vectors in $S$. Indeed, first note that since the first term in (1.6) is linear on $\mathcal{C}$, then it is also linear on each $\mathcal{C}_I$. Then the left hand side of (1.6) is linear on each $\mathcal{C}_I$, and hence it is enough to check it on all the edges of all the cones $\mathcal{C}_I$.

\textbf{Definition 1.5 (Slope stability).} We say that $(E, \varphi, u)$ is slope-$\tau$-(semi)stable if $E$ is torsion free, and for all weighted filtrations we have
\begin{equation}
(1.7) \quad \left( \sum_{i=1}^t m_i (r \deg E_i - r_i \deg E) \right) + \mu(\varphi, E_\bullet, m_\bullet) \tau \leq 0
\end{equation}

Recall. $\tau = \delta_1(n - 1)!$. As usual, we have the following implications
\begin{align*}
\text{slope-$\tau$-stable} \implies \delta\text{-stable} \implies 
\end{align*}
\[ \Rightarrow \text{δ-semistable} \Rightarrow \text{slope-τ-semistable} \]

The reason why we have to consider filtrations instead of just subsheaves is that (1.5) is not linear as a function of \( \{ m_i \} \). But we have the following result that will be used in the proof of theorem 2.5.

**Lemma 1.6.** Let \( (E_\bullet, m_\bullet) \) be a filtration as above, and let \( T' \) be a subset of \( T = \{ 1, \ldots , t \} \). Let \( (E'_\bullet, m'_\bullet) \) be the subfiltration obtained by considering only those terms \( E_i \) for which \( i \in T' \). Then

\[ \mu(\varphi, E_\bullet, m_\bullet) \leq \mu(\varphi, E'_\bullet, m'_\bullet) + \sum_{i \in T - T'} m_i s r_i. \]

**Proof.** Let \( I = \{ 1, \ldots , t+1 \}^\times s \) be the set of all multi-indexes \( I = (i_1, \ldots , i_s) \). Given a multi-index \( I \in I \), we have

\[ \gamma_{r_i} + \cdots + \gamma_{r_i} = \sum_{i=1}^{t} m_i (s r_i - \nu_i(I) r), \]

where \( \nu_i(I) \) is the number of elements \( k \) of the multi-index \( I = (i_1, \ldots , i_s) \) such that \( r_k \leq r_i \). If \( I \) is the multi-index giving minimum in (1.5), we will denote \( \varepsilon_i(\varphi, E_\bullet, m_\bullet) := \nu_i(I) \) (or just \( \varepsilon_i(E_\bullet) \) if the rest of the data is clear from the context). Then

\[ \mu(\varphi, E_\bullet, m_\bullet) = \sum_{i=1}^{t} m_i (s r_i - \varepsilon_i(E_\bullet) r). \]

We index the filtration \((E'_\bullet, m'_\bullet)\) with \( T' \). Let \( I' = (i'_1, \ldots , i'_s) \in \{ T' \cup \{ t+1 \} \}^\times s \) be the multi-index giving minimum for the filtration \((E'_\bullet, m'_\bullet)\). In particular, we have \( \varphi_{(E'_\bullet \otimes \cdots \otimes E'_i)^\oplus c} \neq 0 \). Then

\[ \mu(\varphi, E'_\bullet, m'_\bullet) = \min_{I \in I} \{ \gamma_{r_i} + \cdots + \gamma_{r_i} : \varphi_{(E'_i \otimes \cdots \otimes E'_i)^\oplus c} \neq 0 \} \]

\[ \leq \gamma_{r_i} + \cdots + \gamma_{r_i} \]

\[ = \sum_{i=1}^{t} m_i (s r_i - \nu_i(I') r) \]

\[ = \sum_{i=1}^{t} m_i (s r_i - \varepsilon_i(E'_\bullet) r) \]

\[ = \sum_{i \in T'} m_i (s r_i - \varepsilon_i(E'_\bullet) r) + \sum_{i \in T - T'} m_i (s r_i - \varepsilon_i(E'_\bullet) r) \]

\[ \leq \mu(\varphi, E'_\bullet, m'_\bullet) + \sum_{i \in T - T'} m_i s r_i. \]

\[ \square \]

A family of δ-(semi)stable tensors parametrized by a scheme \( T \) is a tuple \((E_T, \varphi_T, u_T, N)\), consisting of a torsion free sheaf \( E_T \) on \( X \times T \), flat over \( T \), that restricts to a torsion free sheaf with Hilbert polynomial \( P \) on every slice \( X \times t \), a morphism \( u_T : T \to R \), a line bundle \( N \) on \( T \) and a homomorphism \( \varphi_T \)

\[ \varphi_T : (E_T^{\otimes s})^{\otimes c} \to (\det E_T)^{\otimes b} \otimes u_T^* D \otimes \pi_T^* N, \]
(recall $u_T = \text{id}_Y \times u_T$) such that if we consider the restriction of this homomorphism on every slice $X \times t$
\[ \varphi_t : (E_t \otimes s)^{\boxtimes c} \longrightarrow (\text{det } E_t)^{\otimes b} \otimes D_{u(t)}, \]
the triple $(E_t, \varphi_t, u(t))$ is a $\delta$-(semi)stable tensor for every $t$ (in particular, $\varphi_t$ is not identically zero).

Two families $(E_T, \varphi_T, u_T, N)$ and $(E'_T, \varphi'_T, u'_T, N')$ parametrized by $T$ are isomorphic if $u_T = u'_T$ and there are isomorphisms $f : E_T \rightarrow E'_T$, $\alpha : N \rightarrow N'$ such that the induced diagram
\[
\begin{array}{ccc}
(E_T \otimes s)^{\boxtimes c} & \xrightarrow{(f \otimes s)^{\boxtimes c}} & (E'_T \otimes s)^{\boxtimes c} \\
\downarrow \varphi_T & & \downarrow \varphi'_T \\
(\text{det } E_T)^{\otimes b} \otimes u_T^*D \otimes \pi_T^*N & \xrightarrow{f \otimes \pi_T^*\alpha} & (\text{det } E'_T)^{\otimes b} \otimes u_T'^*D \otimes \pi_{T}'^*N'
\end{array}
\]

commutes.

Let $\mathcal{M}_\delta$ (respectively $\mathcal{M}_\delta^s$) be the contravariant functor from the category of complex schemes, locally of finite type, $(\text{Sch}/\mathbb{C})$ to the category of sets (Sets) which sends a scheme $T$ to the set of isomorphism classes of families of $\delta$-semistable (respectively stable) tensors parametrized by $T$, and sends a morphism $T' \rightarrow T$ to the map defined by pullback (as usual).

**Definition 1.7.** A scheme $Y$ corepresents a functor $F : (\text{Sch}/\mathbb{C}) \rightarrow (\text{Sets})$ if

1. There exists a natural transformation $f : F \rightarrow \underline{Y}$ (where $\underline{Y}$ is the functor of points represented by $Y$).
2. For every scheme $Y'$ and natural transformation $f' : F \rightarrow \underline{Y'}$, there exists a unique $g : \underline{Y} \rightarrow \underline{Y'}$ such that $f'$ factors through $f$.

If $Y$ exists, then it is unique up to unique isomorphism. If furthermore $f(\text{Spec } \mathbb{C}) : F(\text{Spec } \mathbb{C}) \rightarrow Y$ is bijective, we say that $Y$ is a coarse moduli space.

We will construct schemes $\mathcal{M}_\delta$, $\mathcal{M}_\delta^s$ corepresenting the functors $\mathcal{M}_\delta$ and $\mathcal{M}_\delta^s$. In general $\mathcal{M}_\delta$ will not be a coarse moduli space, because nonisomorphic tensors could correspond to the same point in $\mathcal{M}_\delta$. As usual, we declare two such tensors $\text{S-equivalent}$, and then $\mathcal{M}_\delta$ becomes a coarse moduli space for the functor of $\text{S-equivalence}$ classes of tensors.

**Theorem 1.8.** Fix $P$, $s$, $b$, $c$ and a family $D$ of locally free sheaves on $X$ parametrized by a scheme $R$. Let $d$ be the degree of a coherent sheaf whose Hilbert polynomial is $P$. Let $\delta$ be a polynomial as in (1.2).

There exists a coarse moduli space $\mathcal{M}_\delta$, projective over $\text{Pic}^d(X) \times R$, of $\text{S-equivalence}$ classes of $\delta$-semistable tensors. The closed points of $\mathcal{M}_\delta$ correspond to $\text{S-equivalence}$ classes of $\delta$-semistable tensors. There is an open set $\mathcal{M}_\delta^s$ corresponding to $\delta$-stable tensors. Points in this open set correspond to isomorphism classes of $\delta$-stable tensors.

In proposition 4.1 we give a criterion to decide when two tensors are $\text{S-equivalent}$. Theorem 1.8 will be proved in section 4.

**Remark 1.9.** Note that to define the functors we have used $\text{isomorphism}$ classes of families, but usually one uses $\text{equivalence}$ classes, declaring two families equivalent if they differ by the pullback of a line bundle $M$ on $T$. As a result, in general the
functors that we have defined will not be sheaves. The sheafified functors will be the same (this follows from the fact that if we shrink $T$ then $M$ will be trivial), and hence the corresponding moduli spaces are the same, because a scheme corepresents a functor if and only if it corepresents its sheafification (see [Si, p. 60]).

2. Boundedness

The objective of this section is theorem 2.5, where we reformulate the stability condition for tensors using some boundedness results. We start with some well known results. See [Si, cor 1.7] (also [H-L2, lemma 2.2]), [Gr, lemma 2.5] and [Ma].

Lemma 2.1 (Simpson). Let $r > 0$ be an integer. Then there exist a constant $B$ with the following property: for every torsion free sheaf $E$ with $0 < \text{rk}(E) \leq r$, we have

$$h^0(E) \leq \frac{1}{g^n-1!}\left( (\text{rk}(E)-1)(\mu_{\text{max}}(E) + B)_+^n + (\mu_{\text{min}}(E) + B)_+^n \right),$$

where $g = \text{deg} \mathcal{O}_X(1)$, $[x]_+ = \max\{0, x\}$, and $\mu_{\text{max}}(E)$ (respectively $\mu_{\text{min}}(E)$) is the maximum (respectively minimum) slope of the Mumford-semistable factors of the Harder-Narasimhan filtration of $E$.

Lemma 2.2 (Grothendieck). Let $T$ be a bounded set of sheaves $E$. The set of torsion free quotients $E''$ of the sheaves $E$ in $T$ with $|\text{deg}(E'')| \leq C''$ for some fixed constant $C''$, is bounded.

Theorem 2.3 (Maruyama). The family of sheaves $E$ with fixed Hilbert polynomial $P$ and such that $\mu_{\text{max}}(E) \leq C$ for a fixed constant $C$, is bounded.

Corollary 2.4. The set of $\delta$-semistable tensors $(E, \varphi, u)$ with fixed Hilbert polynomial is bounded.

Proof. Follows from theorem 2.3 and an easy calculation. \hfill $\Box$

The main theorem of this section is

Theorem 2.5. There is an integer $N_0$ such that if $m \geq N_0$, the following properties of tensors $(E, \varphi, u)$ with $E$ torsion free and $P_E = P$, are equivalent.

1. $(E, \varphi, u)$ is (semi)stable.
2. $P(m) \leq h^0(E(m))$ and for every weighted filtration $(E_\bullet, m_\bullet)$ as in (1.3)

$$\left( \sum_{i=1}^{t} m_i(\rho h^0(E_i(m)) - \tau_i P(m)) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta(m) \leq 0$$

3. For every weighted filtration $(E_\bullet, m_\bullet)$ as in (1.3)

$$\left( \sum_{i=1}^{t} m_i(\rho \tau P(m) - \rho h^0(E^i(m))) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta(m) \leq 0$$

Furthermore, for any tensor $(E, \varphi, u)$ satisfying these conditions, $E$ is $m$-regular.

Recall that a sheaf $E$ is called $m$-regular if $h^i(E(m-i)) = 0$ for $i > 0$. If $E$ is $m$-regular, then $E(m)$ is generated by global sections, and it is $m'$-regular for any $m' > m$. The set of tensors $(E, \varphi, u)$, with $E$ torsion free and $P_E = P$, satisfying the weak version of conditions 1-3 will be called $S^s$, $S'_m$, and $S''_m$. 
Lemma 2.6. There are integers $N_1, C$ such that if $(E, \varphi, u)$ belongs to $S = S^s \cup \bigcup_{m \geq N_1} S''_m$, then for all saturated weighted filtrations the following holds for all $i$:
\begin{equation}
\text{deg}(E_i) - r_i \mu_s \leq C,
\end{equation}

(where $\mu_s = (d - s \tau)/r$) and either $-C \leq \text{deg}(E_i) - r_i \mu_s$ or
\begin{enumerate}
\item $r h^0(E_i(m)) < r_i (P(m) - s \delta(m))$, if $(E, \varphi, u) \in S^s$ and $m \geq N_1$.
\item $r^i (P - s \delta) < r_i (P_{E^i} - s \delta)$, if $(E, \varphi, u) \in \bigcup_{m \geq N_1} S''_m$.
\end{enumerate}

Proof. Let $B$ be as in lemma 2.1. Choose $C$ large enough so that $C > s \tau$ and the leading coefficient of $G - (P - s \delta)/r$ is negative, where
\begin{equation}
G(m) = \frac{1}{g^n-1 n!} \left( (1 - \frac{1}{r}) (\mu_s + s \tau + mg + B)^n + \frac{1}{r} (\mu_s - \frac{1}{r} C + mg + B)^n \right)
\end{equation}

Choose $N_1$ large enough so that for $m \geq N_1$,
\begin{enumerate}
\item $\delta(m) \geq 0$,
\item $\mu_s - \frac{C}{r} + mg + B > 0$,
\item $G(m) - (P(m) - s \delta(m))/r < 0$.
\end{enumerate}

Since the filtration is assumed to be saturated, and since $E$ is torsion free, we have $0 < r_i < r$.

Case 1. Suppose $(E, \varphi, u) \in S^s$. For each $i$, consider the one step filtration $E_i \subseteq E$. The leading coefficient of the semistability condition applied to this filtration, together with $C > s \tau$, implies (2.1).

Let $E_{i,\text{max}} \subset E_i$ be the term in the Harder-Narasimhan filtration with maximal slope. Then the same argument applied to $E_{i,\text{max}}$ gives
\begin{equation}
\mu_{\text{max}}(E_i) = \mu(E_{i,\text{max}}) < \mu_s + s \tau.
\end{equation}

Now assume that the first alternative does not hold, i.e.
\[-C > \text{deg}(E_i) - r_i \mu_s.\]

This gives
\begin{equation}
\mu_{\text{min}}(E_i) \leq \mu(E_i) < \mu_s - \frac{C}{r}.
\end{equation}

Combining lemma 2.1 with (2.4), (2.6), (2.7) and (2.5), we have
\[r h^0(E_i(m)) < r_i (P(m) - s \delta(m)).\]

Case 2. Suppose $(E, \varphi, u) \in S''_m$ for some $m \geq N_1$. For each $i$, consider the quotient $E^i = E/E_i$. Let $E_{i,\text{min}}^i$ be the last factor of the Harder-Narasimhan filtration of $E^i$ (i.e. $\mu(E_{i,\text{min}}^i) = \mu_{\text{min}}(E^i)$). Let $E'$ be the kernel
\[0 \to E' \to E \to E_{i,\text{min}}^i \to 0,
\]

and consider the one step filtration $E' \subseteq E$. Equations (2.2) and (2.4) imply that $0 < G(m)$. Then a short calculation using (2.5), the fact that $(E, \varphi, u) \in S''_m$, (2.3) and lemma 2.1 shows
\[G(m) < \frac{h^0(E_{i,\text{min}}^i(m))}{\text{rk}(E_{i,\text{min}}^i)} \leq \frac{1}{g^n-1 n!} (\mu_{\text{min}}(E^i) + mg + B)^n.\]
It can be seen that if this inequality of polynomials holds for some \( m \geq N_1 \), then it holds for all larger values of \( m \), hence choosing \( m \) large enough and looking at the coefficients, we have

\[
\mu_{\min}(E^i) \geq \mu_s + (1 - \frac{1}{r}) s \tau - \frac{C}{r^2}.
\]

A short calculation using this, \( \mu_{\min}(E^i) \leq \mu(E^i) \) and \( 0 < \text{rk}(E^i) < \text{rk}(E) \) (hence \( \text{rk}(E) > 1 \)), yields (2.1).

Now assume that the first alternative does not hold, i.e.

\[-C > \text{deg}(E_i) - r_i \mu_s.\]

It follows that \( r^i \mu_s < \text{deg}(E^i) - s \tau \), and hence

\[r^i (P - s \delta) \prec r(P_{E^i} - s \delta).\]

\[\square\]

**Lemma 2.7.** The set \( S = S^s \cup \bigcup_{m \geq N_1} S_m'' \) is bounded.

**Proof.** Let \((E, \varphi, u) \in S\). Let \( E' \) be a subsheaf of \( E \), and \( \overline{E}' \) the saturated subsheaf of \( E \) generated by \( E' \). Using lemma 2.6

\[
\frac{\text{deg}(E')}{{\text{rk}(E')}} \leq \frac{\text{deg}(E)}{{\text{rk}(E)}} \leq \mu_s + \frac{C}{{\text{rk}(E)}} \leq \mu_s + C.
\]

Then by Maruyama’s theorem 2.3, the set \( S \) is bounded. \(\square\)

**Lemma 2.8.** Let \( S_0 \) be the set of sheaves \( E' \) such that \( E' \) is a saturated subsheaf of \( E \) for some \((E, \varphi, u) \in S\), and furthermore

\[|\text{deg}(E') - r' \mu_s| \leq C.\]

Then \( S_0 \) is bounded.

**Proof.** Let \( E' \in S_0 \). The sheaf \( E'' = E/E' \) is torsion free, and \( |\text{deg}(E'')| \) is bounded because the set \( S \) is bounded by

\[|\text{deg}(E'')| \leq |\text{deg}(E)| + |\text{deg}(E')| \leq \max_{E \in S} |\text{deg}(E)| + C + r|\mu_s|.\]

Then by Grothendieck’s lemma 2.2, the set of sheaves \( E'' \) obtained in this way is bounded, and hence also \( S_0 \). \(\square\)

**Lemma 2.9.** There is an integer \( N_2 \) such that for every weighted filtration \((E_\bullet, m_\bullet)\) as in (1.3) with \( E_i \in S_0 \), the inequality of polynomials

\[
\left( \sum_{i=1}^t m_i (r_{P_{E^i}} - r_i P) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta \leq 0
\]

holds if and only if it holds for a particular value of \( m \geq N_2 \).

**Proof.** Since \( S_0 \) is bounded, the set that consists of the polynomials \( \delta, P_0, r' P_0 \) and \( P_{E'} \) for \( E' \in S_0 \) is finite. On the other hand, lemma 1.4 implies that we only need to consider a finite number of values for \( m_i \), hence the result follows. \(\square\)
Proof of theorem 2.5. Let $N_0 > \max \{N_1, N_2 \}$ and such that all sheaves in $\mathcal{S}$ and $\mathcal{S}_0$ are $N_0$-regular, and $E_1 \otimes \cdots \otimes E_s$ is $sN_0$-regular for all $E_1, \ldots, E_s$ in $\mathcal{S}_0$.

2. $\Rightarrow$ 3. Let $(E, \varphi, u) \in \mathcal{S}_m'$. Consider a weighted filtration $(E_\bullet, m_\bullet)$. Then

$$
\left( \sum_{i=1}^{t} m_i \left( r^i P(m) - rh^0(E_i(m)) \right) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta(m) \leq \left( \sum_{i=1}^{t} \left( rh^0(E_i(m)) - r_i P(m) \right) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta(m) \leq 0
$$

1. $\Rightarrow$ 2. Let $(E, \varphi, u) \in \mathcal{S}_s^*$ and consider a saturated weighted filtration as in (1.3). Since $E$ is $N_0$-regular, $P(m) = h^0(E(m))$. If $E_i \in \mathcal{S}_0$, then $P_{E_i}(m) = h^0(E_i(m))$. If $E_i \notin \mathcal{S}_0$, then the second alternative of lemma 2.6 holds, and then

$$
r h^0(E_i(m)) < r_i (P(m) - s \delta(m)) .
$$

Let $\mathcal{T}' \subset \mathcal{T} = \{1, \ldots, t\}$ be the subset of those $i$ for which $E_i \in \mathcal{S}_0$. Let $(E_\bullet', m_\bullet')$ be the corresponding subfiltration. Lemma 1.6 and a short calculation shows that

$$
\left( \sum_{i=1}^{t} m_i \left( r h^0(E_i(m)) - r_i P(m) \right) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta(m) \leq \left( \sum_{i \in \mathcal{T}'} m_i \left( r P_{E_i}(m) - r_i P(m) \right) \right) + \mu(\varphi, E_\bullet', m_\bullet') \delta(m) + \left( \sum_{i \in \mathcal{T}' \setminus \mathcal{T}} m_i \left( r h^0(E_i(m)) - r_i P(m) - sr_i \delta(m) \right) \right) \leq \left( \sum_{i \in \mathcal{T}'} m_i \left( r P_{E_i}(m) - r_i P(m) \right) \right) + \mu(\varphi, E_\bullet', m_\bullet') \delta(m) \leq 0
$$

The condition that $E_i$ is saturated can be dropped, since $h^0(E_i(m)) \leq h^0(E_i(m))$ and $\mu(\varphi, E_\bullet, m_\bullet) = \mu(\varphi, \overline{E_\bullet}, m_\bullet)$, where $\overline{E_i}$ is the saturated subsheaf generated by $E_i$ in $E$.

3. $\Rightarrow$ 1. Let $(E, \varphi, u) \in \mathcal{S}_m''$ and consider a saturated weighted filtration $(E_\bullet, m_\bullet)$. Since $E$ is $N_0$-regular, $P(m) = h^0(E(m))$. If $E_i \in \mathcal{S}_0$, then $P_{E_i}(m) = h^0(E_i(m))$. Hence hypothesis 3 applied to the subfiltration $(E_\bullet', m_\bullet')$ obtained by those terms such that $E_i \in \mathcal{S}_0$ implies

$$
\left( \sum_{E_i \in \mathcal{S}_0} m_i \left( r^i P(m) - r P_{E_i}(m) \right) \right) + \mu(\varphi, E_\bullet', m_\bullet') \delta(m) \leq 0.
$$

This is equivalent to

$$
\left( \sum_{E_i \in \mathcal{S}_0} m_i \left( r P_{E_i}(m) - r_i P(m) \right) \right) + \mu(\varphi, E_\bullet', m_\bullet') \delta(m) \leq 0,
$$

and by lemma 2.9, this is in turn equivalent to

$$
\left( \sum_{E_i \in \mathcal{S}_0} m_i \left( r P_{E_i} - r_i P \right) \right) + \mu(\varphi, E_\bullet', m_\bullet') \delta (\leq 0).
$$

If $E_i \notin \mathcal{S}_0$, then the second alternative of lemma 2.6 holds, and then

$$
r P_{E_i} - r_i P + sr_i \delta < 0.
$$
Using lemma 1.6, (2.9) and (2.10)
\[
\left( \sum_{i=1}^{t} m_i (r P_{E_i} - r_i P) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta (\leq) 0.
\]
Again, we can drop the condition that the filtration is saturated, and this finishes the proof of theorem 2.5
\[\square\]

**Corollary 2.10.** Let \((E, \varphi, u)\) be \(\delta\)-semistable, \(m \geq N_0\), and assume that there is a weighted filtration \((E_\bullet, m_\bullet)\) with
\[
\left( \sum_{i=1}^{t} m_i (r h^0(E_i(m)) - r_i P(m)) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta(m) = 0.
\]
Then \(E_i \in S_0\) and \(h^0(E_i(m)) = P_{E_i}(m)\) for all \(i\).

**Proof.** By the proof of the part \((1 \Rightarrow 2)\) of theorem 2.5, if we have this equality then all inequalities in (2.8) are equalities, hence \(T' = T\), \(E_i \in S_0\) for all \(i\), and the result follows.
\[\square\]

Note that in theorem 2.5 we are assuming that \(E\) is torsion free. To handle the general case, we will use the following lemma

**Lemma 2.11.** Fix \(u \in R\). Let \((E, \varphi, u)\) be a tensor. Assume that there is a family \((E_t, \varphi_t, u_t)\) parametrized by a smooth curve \(C\) such that \((E_0, \varphi_0, u) = (E, \varphi, u)\) and \(E_t\) is torsion free for \(t \neq 0\). Then there exists a tensor \((F, \psi, u)\), a homomorphism
\[
(E, \varphi, u) \longrightarrow (F, \psi, u)
\]
such that \(F\) is torsion free with \(P_E = P_F\), and an exact sequence
\[
0 \longrightarrow T(E) \longrightarrow E \overset{\beta}{\longrightarrow} F,
\]
where \(T(E)\) is the torsion subsheaf of \(E\).

**Proof.** The family is given by a tuple \((E_C, \varphi_C, u_C, N)\) as in (1.9), where \(u_C\) is the constant map from \(C\) to \(R\) with constant value \(u\). Shrinking \(C\), we can assume that \(N\) is trivial. Let \(U = (X \times C) - \text{Supp}(T(E_0))\). Let \(F_C = j_*(EC|_U)\). Since it has no \(C\)-torsion, \(F_C\) is flat over \(C\). The natural map \(\beta : E_C \rightarrow F_C\) is an isomorphism on \(U\), hence we have a homomorphism \(\psi_U := \varphi_C|_U\) on \(U\), and this extends to a homomorphism \(\psi_C\) on \(X \times C\) because \(\pi_C^*D\) is locally free. Finally define \((F, \psi) = (F_0, \psi_0)\), and let \(\beta\) be the homomorphism induced by \(\beta\).
\[\square\]

3. GIT Construction

Let \(N \geq N_0\) be large enough so that for all \(i > 0\), all line bundles \(L\) of degree \(d\), all locally free sheaves \(D_u\) in the family parametrized by \(R\), and all \(m > N\), we have \(h^i(L^{\otimes b} \otimes D_u(sm)) = 0\) and \(L^{\otimes b} \otimes D_u(sm)\) is generated by global sections.

Fix \(m \geq N\) and let \(V\) be a vector space of dimension \(p = P(m)\). The choice of \(m\) implies that if \((E, \varphi, u)\) is \(\delta\)-semistable, then \(E(m)\) is generated by global sections and \(h^i(E(m)) = 0\) for \(i > 0\). Let \((g, E, \varphi, u)\) be a tuple where \((E, \varphi, u)\) is a \(\delta\)-semistable tensor and \(g\) is an isomorphism \(g : V \rightarrow H^0(E(m))\). This induces a quotient
\[
q : V \otimes O_X(-m) \rightarrow E.
\]
Let $\mathcal{H}$ be the Hilbert scheme of quotients of $V \otimes \mathcal{O}_X(-m)$ with Hilbert polynomial $P$. Let $l > m$ be an integer, and $W = H^0(\mathcal{O}_X(l - m))$. The quotient $q$ induces homomorphisms

\[
q : V \otimes \mathcal{O}_X(l - m) \rightarrow E(l) \\
q' : V \otimes W \rightarrow H^0(E(l)) \\
q'' : \wedge^{P(l)}(V \otimes W) \rightarrow \wedge^{P(l)}H^0(E(l)) \cong \mathbb{C}
\]

If $l$ is large enough, these homomorphisms are surjective, and give Grothendieck’s embedding

\[
\mathcal{H} \rightarrow \mathbb{P}(\wedge^{P(l)}(V^\vee \otimes W^\vee)),
\]

and hence a very ample line bundle $\mathcal{O}_\mathcal{H}(1)$ on $\mathcal{H}$ (depending on $m$ and $l$).

The tuple $(g, E, \varphi, u)$ induces a linear map

\[
(3.2) \quad \Phi : (V^\otimes)^{\otimes c} \rightarrow H^0((E(m)^\otimes)^{\otimes c}) \rightarrow H^0((\det E)^{\otimes b} \otimes D_u(sm)).
\]

Fix a Poincare bundle $\mathcal{P}$ on $J \times X$, where $J = \text{Pic}^d(X)$. Fix an isomorphism

\[
\beta : \det(E) \rightarrow \mathcal{P}|_{\det(E) \times X}.
\]

Then $\Phi$ induces a quotient

\[
(V^\otimes)^{\otimes c} \otimes H^0(\mathcal{P}|_{\det(E) \times X}^\otimes D_u(sm))^\vee \rightarrow \mathbb{C}.
\]

Choosing a different isomorphism $\beta$ will only change this quotient by a scalar, so we get a well defined point $[\Phi]$ in $P$, where $P$ is the projective bundle over $J \times R$ defined as

\[
P = \mathbb{P}\left( ((V^\otimes)^{\otimes c})^\vee \otimes \pi_{J \times R*}(\pi_{X \times J}^*\mathcal{P}^\otimes \otimes \pi_{X \times R}^*\mathcal{D}(sm)) \right) \rightarrow J \times R,
\]

where $\pi_{X \times J}$ (respectively $\pi_{J \times R, \ldots}$) denotes the natural projection from $X \times J \times R$ to $X \times J$ (respectively $J \times R, \ldots$). Recall that $\mathcal{D}(m) := \mathcal{D} \otimes \pi_X^*\mathcal{O}_X(m)$. Note that $\pi_{J \times R*}(\pi_{X \times J}^*\mathcal{P}^\otimes \otimes \pi_{X \times R}^*\mathcal{D}(sm))$ is locally free because of the choice of $m$. Replacing $\mathcal{P}$ with another Poincare bundle defined by tensoring with the pullback of a sufficiently positive line bundle on $J$, we can assume that $\mathcal{O}_P(1)$ is very ample (this line bundle depends on $m$).

A point $(q, [\Phi]) \in \mathcal{H} \times P$ associated to a tuple $(g, E, \varphi, u)$ has the property that the homomorphism $\Phi$ in (3.2) composed with evaluation factors as

\[
(3.3) \quad (V^\otimes)^{\otimes c} \otimes \mathcal{O}_X \xrightarrow{\Phi} (E(m)^\otimes)^{\otimes c} \xrightarrow{\varphi} H^0((\det E)^{\otimes b} \otimes D_u(sm)) \otimes \mathcal{O}_X \xrightarrow{\text{ev}} (\det E)^{\otimes b} \otimes D_u(sm). \]

Consider the relative version of the homomorphisms in (3.3), i.e. the commutative diagram on $X \times \mathcal{H} \times P$

\[
(3.4) \quad 0 \rightarrow \mathcal{K} \xrightarrow{f} (V^\otimes)^{\otimes c} \otimes \mathcal{O}_{X \times \mathcal{H} \times P} \xrightarrow{(p_{X \times \mathcal{H}}\mathcal{O}_\mathcal{H})^\otimes \otimes \mathcal{O}_X} (p_{X \times \mathcal{H}}\mathcal{E}_\mathcal{H}(m)^\otimes)^{\otimes c} \xrightarrow{\Phi_{X \times \mathcal{H}}} 0
\]

where

\[
A := p_{X \times J}^*\mathcal{P}^{\otimes b} \otimes p_{X \times R}^*\mathcal{D} \otimes p_X^*\mathcal{O}_X(sm).
\]
where \( p_{X \times F} \) (respectively \( p_X \), ...) denotes the natural projection from \( X \times H \times P \) to \( X \times F \) (respectively \( X \), ...), \( E_H \) is the tautological sheaf on \( X \times H \), and \( \Phi_{H \times P} \) is the relative version of the composition \( \Phi \circ \Phi \) in diagram (3.3).

The points \((q, \Phi)\) where the restriction \( \Phi_{H \times P | X \times (q, \Phi)} \) factors through \((E(m) \otimes \Theta) @ c\) (as in 3.3) are the points where \( f |_{X \times (q, \Phi)} \) is identically zero. We will need the following

**Lemma 3.1.** Let \( Y \) be a scheme, and let \( f : G \to F \) be a homomorphism of coherent sheaves on \( X \times Y \). Assume that \( F \) is flat over \( Y \). Then there is a unique closed subscheme \( Z \) satisfying the following universal property: given a Cartesian diagram

\[
\begin{array}{ccc}
X \times S & \xrightarrow{\pi} & X \times Y \\
| & & | \\
S & \xrightarrow{h} & Y \\
\end{array}
\]

\( h^* f = 0 \) if and only if \( h \) factors through \( Z \).

**Proof.** Uniqueness is clear. To show existence, assume that \( O_X(1) \) is very ample (taking a multiple if necessary) and let \( p : X \times Y \to Y \) be the projection to the second factor. Since \( F \) is \( Y \)-flat, taking \( m' \) large enough, \( p_sF(m') \) is locally free (recall \( F(m') = F \otimes p^*_X O_X(m') \)). The question is local on \( Y \), so we can assume, shrinking \( Y \) if necessary, that \( Y = \text{Spec} A \) and \( p_sF(m') \) is given by a free \( A \)-module. Now, since \( Y \) is affine, the homomorphism

\[
p_sF(m') : p_sG(m') \to p_sF(m')
\]

of sheaves on \( Y \) is equivalent to a homomorphism of \( A \)-modules

\[
M \xrightarrow{(f_1, \ldots, f_n)} A \oplus \cdots \oplus A
\]

The zero locus of \( f_i \) is defined by the ideal \( I_i \subset A \) image of \( f_i \), thus the zero scheme of \((f_1, \ldots, f_n)\) is given by the ideal \( I = \sum I_i \), hence \( Z_{m'} \) is a closed subscheme.

Since \( O_X(1) \) is very ample, if \( m'' > m' \) we have an injection \( p_sF(m') \hookrightarrow p_sF(m'') \) (and analogously for \( G \)), hence \( Z_{m''} \subset Z_{m'} \), and since \( Y \) is noetherian, there exists \( N' \) such that, if \( m' > N' \), we get a scheme \( Z \) independent of \( m' \).

To check the universal property first we will show that if \( h^* f = 0 \) then \( h \) factors through \( Z \). Since the question is local on \( S \), we can take \( S = \text{Spec} B \), \( Y = \text{Spec} A \), and the morphism \( h \) is locally given by a ring homomorphism \( A \to B \). Since \( F \) is flat over \( Y \), for \( m' \) large enough the natural homomorphism \( \alpha : h^*p_sF(m') \to p_s^*h^* F(m') \) (defined as in [Ha, Th. III 9.3.1]) is an isomorphism. Indeed, for \( m' \) sufficiently large, \( H^i(X, F(m')) = 0 \) and \( H^i(X, h^* F(m')) \) is 0 for all points \( y \in Y, s \in S \), and \( i > 0 \), and since \( F \) is flat, this implies that \( h^*p_sF(m') \) and \( p_s^*h^* F(m') \) are locally free. Then, to prove that the homomorphism \( \alpha \) is an isomorphism, it is enough to check at the fiber of every \( s \in S \), but this follows from [Ha, Th. III 12.11] or [Mu, II §5 Cor. 3].

Hence the commutativity of the diagram

\[
\begin{array}{ccc}
\text{ps}h^* G(m') & \xrightarrow{h^* p_s f(m') = 0} & \text{ps}h^* F(m') \\
\uparrow \text{ps}h^* \downarrow & & \uparrow \cong \\
\text{h}^* p_s G(m') & \xrightarrow{\text{h}^* p_s f(m')} & \text{h}^* p_s F(m')
\end{array}
\]
implies that $h^*p_f(m') = 0$. This means that for all $i$, in the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f_i} & A \\
\downarrow & & \downarrow \\
M \otimes_A B & \xrightarrow{f_i \otimes B} & A/I_i \\
\end{array} \quad \xrightarrow{0}
$$

it is $f_i \otimes B = 0$. Hence the image $I_i$ of $f_i$ is in the kernel $J$ of $A \rightarrow B$. Therefore $I \subset J$, hence $A \rightarrow B$ factors through $A \rightarrow A/I$, which means that $h : S \rightarrow Y$ factors through $Z$.

Now we show that if we take $S = Z$ and $h : Z \hookrightarrow Y$ the inclusion, then $\overline{h'}f = 0$. By definition of $Z$ we have $h^*p_f(m') = 0$ for any $m'$ with $m' > N'$. Showing that $\overline{h'}f = 0$ is equivalent to showing that

$$
\overline{h'}f(m') : \overline{h'}G(m') \rightarrow \overline{h'}F(m')
$$

is zero for some $m'$. Take $m'$ large enough so that $ev : p^*p_*G(m') \rightarrow G(m')$ is surjective. By the right exactness of $\overline{h'}$ the homomorphism $\overline{h'}ev$ is still surjective. The commutative diagram

$$
\begin{array}{ccc}
\overline{h'}G(m') & \xrightarrow{\overline{h'}f(m')} & \overline{h'}F(m') \\
\overline{h'}ev & & \downarrow \\
\overline{h'}p^*p_*G(m') & \xrightarrow{\overline{h'}p^*p_*f(m')} & \overline{h'}p^*p_*F(m') \\
\end{array}
$$

implies $\overline{h'}f(m') = 0$, hence $\overline{h'}f = 0$.

Let $Z'$ be the scheme given by this lemma for $Y = H \times P$ and the homomorphism $f : \mathcal{K} \rightarrow A$. Then $\overline{f} = 0$, and there is a commutative diagram on $X \times Z'$

$$
(3.5) \quad \overline{\mathcal{K}} \xrightarrow{\overline{f}} (V \otimes s)^{\otimes c} \otimes \mathcal{O}_{X \times Z'} \xrightarrow{(\overline{\varphi})^*p^*E_H(m)^{\otimes c}} 0
$$

and hence there is a universal family of based tensors parametrized by $Z'$

$$
(3.6) \quad \varphi_{Z'} : E_{Z'}^{\otimes s} \rightarrow (\det E_{Z'})^{\otimes b} \otimes p_{Z'}^*N
$$

Given a point $(q, [\Phi])$ in $Z'$, using the tautological family (3.6) we can recover the tuple $(q, E, \varphi, u)$ up to isomorphism, and if $H^0(q(m)) : V \rightarrow H^0(E(m))$ is an isomorphism, then we recover the tuple $(g = H^0(q(m)), E, \varphi, u)$ up to isomorphism, i.e. if $(g', E', \varphi', u')$ is another tuple corresponding to the same point, then there exists an isomorphism $(f, \alpha)$ between $(E, \varphi, u)$ and $(E', \varphi', u')$ as in (1.1), and $H^0(f(m)) \circ q = q'$.

Let $Z \subset Z'$ be the closure of the points associated to $\delta$-semistable tensors. Let $p_H$ and $p_P$ be the projections of $Z$ to $\mathcal{H}$ and $P$, and define a polarization on $Z$ by

$$
(3.7) \quad \mathcal{O}_Z(n_1, n_2) := p_H^*\mathcal{O}_H(n_1) \otimes p_P^*\mathcal{O}_P(n_2),
$$
where $n_1$ and $n_2$ are integers with

$$\frac{n_2}{n_1} = \frac{P(l)\delta(m) - \delta(l)P(m)}{P(m) - s\delta(m)}$$

The projective scheme $Z$ is preserved by the natural $SL(V)$ action, and this action has a natural linearization on $O_Z(n_1, n_2)$, using the natural linearizations on $O_H(1)$ and $O_P(1)$.

We have seen that the points of $Z$ for which $H^0(q(m))$ is an isomorphism correspond (up to isomorphism) to tuples $(g, E, \varphi, u)$, where $g$ is an isomorphism between $V$ and $H^0(E(m))$. To get rid of the choice of $g$, we have to take the quotient by $GL(V)$, but if $\lambda \in \mathbb{C}^*$, $(g, E, \varphi, u)$ and $(\lambda g, E, \varphi, u)$ correspond to the same point, and hence it is enough to divide by the action of $SL(V)$. In fact, the moduli space will be the GIT quotient of $Z$ by $SL(V)$.

In proposition 3.4, we will identify the GIT-(semi)stable points in $Z$ using the Hilbert-Mumford criterion. In theorem 3.6 we relate filtrations of sheaves with filtrations of the vector space $V$ to prove that GIT-(semi)stable points of $Z$ coincide with the points associated to $\delta$-(semi)stable points.

A nonconstant group homomorphism $\lambda : \mathbb{C}^* \rightarrow SL(V)$ is called a one-parameter subgroup of $SL(V)$. If $SL(V)$ acts on a projective scheme $Y$ with a given linearization, we denote by $\mu(y, \lambda)$ the minimum weight of the action of $\lambda$ on $y \in Y$.

A weighted filtration $(V_\bullet, m_\bullet)$ of the vector space $V$ is a filtration of vector spaces

$$0 \subseteq V_1 \subset V_2 \subset \cdots \subset V_t \subseteq V_{t+1} = V,$$

and positive numbers $m_1, m_2, \ldots, m_t > 0$. If $t = 1$ (one step filtration), then we will take $m_1 = 1$. Consider the vector of $\mathbb{C}^p$ defined as $\Gamma = \sum_{i=1}^t m_i \Gamma^{(\dim(V_i))}$, where

$$\Gamma^{(k)} = \left( \begin{array}{cc} p-k & \vdots & k-p \\ \vdots & \ddots & \vdots \\ k-p & \cdots & p-k \end{array} \right)$$

Now let $I = \{1, \ldots, t+1\}^{\times s}$ be the set of all multiindexes $I = (i_1, \ldots, i_s)$, and define

$$\mu(\Phi, V_\bullet, m_\bullet) = \min_{I \in \mathcal{I}} \{ \Gamma_{\dim(V_{i_1})} + \cdots + \Gamma_{\dim(V_{i_s})} : \Phi|_{(V_{i_1} \otimes \cdots \otimes V_{i_s})^{\oplus c}} \neq 0 \}$$

As we did in the proof of lemma 1.6, if $I$ is the multi-index giving minimum in (3.10), we will denote by $\epsilon_i(\Phi, V_\bullet, m_\bullet)$ (or just $\epsilon_i(V_\bullet)$ if the rest of the data is clear from the context) the number of elements $k$ of the multi-index $I$ such that $\dim V_k \leq \dim V_i$. Then we have, as in (1.8)

$$\mu(\Phi, V_\bullet, m_\bullet) = \sum_{i=1}^t m_i \left( s \dim V_i - \epsilon_i(V_\bullet) \dim V \right).$$

Given a subspace $V' \subset V$ and a quotient $q : V \otimes O_X(-m) \to E$, we define the subsheaf $E_{V'}$ of $E$ as the image of the restriction of $q$ to $V'$

$$V \otimes O_X(-m) \rightarrow E \quad \rightarrow \quad V' \otimes O_X(-m) \rightarrow E_{V'}$$

In particular, $E_{V'}(m)$ is generated by global sections.
On the other hand, if the quotient \( q : V \otimes O_X(-m) \to E \) induces an injection \( V \hookrightarrow H^0(E(m)) \) (we will later show that all quotients coming from GIT-semistable points of \( Z \) satisfy this property), and if \( E' \subset E \) is a subsheaf, we define

\[
V_{E'} = V \cap H^0(E'(m)).
\]

The following two lemmas are easy to check

**Lemma 3.2.** Given a point \((q, [\Phi]) \in Z\) such that \( q \) induces an injection \( V \hookrightarrow H^0(E(m)) \), and a weighted filtration \((E_\bullet, m_\bullet)\) of \( E \), we have:

1. \( E_{V_{E'}} \subset E_i \)
2. If \( \varphi(V_{E_{i+1}} \otimes \cdots \otimes V_{E_t})^c = 0 \) then \( \Phi(V_{E_{i+1}} \otimes \cdots \otimes V_{E_t})^c = 0 \)
3. \( \sum_i -m_i \epsilon_i(\varphi, E_\bullet, m_\bullet) \leq \sum_i -m_i \epsilon_i(\Phi, E_\bullet, m_\bullet) \)

Furthermore, if \( q \) induces an isomorphism \( V \cong H^0(E(m)) \), all \( E_i \) are \( m \)-regular and all \( E_{i_1} \otimes \cdots \otimes E_{i_s} \) are \( sm \)-regular, then (1) becomes an equality, (2) becomes “if and only if” and (3) an equality.

**Lemma 3.3.** Given a point \((q, [\Phi]) \in Z\) such that \( q \) induces an injection \( V \hookrightarrow H^0(E(m)) \), and a weighted filtration \((V_\bullet, m_\bullet)\) of \( V \), we have:

1. \( V_i \subset V_{E_{V_i}} \)
2. \( \varphi(V_{E_{i+1}} \otimes \cdots \otimes V_{E_t})^c = 0 \) if and only if \( \Phi(V_{E_{i+1}} \otimes \cdots \otimes V_{E_t})^c = 0 \)
3. \( \sum_i -m_i \epsilon_i(\varphi, V_\bullet, m_\bullet) = \sum_i -m_i \epsilon_i(\Phi, V_\bullet, m_\bullet) \)

**Proposition 3.4.** For sufficiently large \( l \), the point \((q, [\Phi])\) in \( Z \) is GIT-(semi)stable with respect to \( O_Z(n_1, n_2) \) if and only if for every weighted filtration \((V_\bullet, m_\bullet)\) of \( V \)

\[
(3.12) \quad n_1 \left( \sum_{i=1}^l m_i (\dim V_i P(l) - \dim V P_{E_{V_i}}(l)) \right) + n_2 \mu(\Phi, V_\bullet, m_\bullet) (\leq 0).
\]

Furthermore, there is an integer \( A_2 \) (depending only on \( m, P, s, b, c \) and \( \mathcal{D} \)) such that it is enough to consider weighted filtrations with \( m_i \leq A_2 \).

**Proof.** Given \( m \), the sheaves \( E_{V'} \) for \( V' \subset V \) form a bounded family, so if \( l \) is large enough, we will have

\[
\dim q'(V \otimes W) = h^0(E_{V'}(l)) = P_{E_{V'}}(l)
\]

for all subspaces \( V' \subset V \). By the Hilbert-Mumford criterion, a point is GIT-(semi)stable if and only if for all one-parameter subgroups \( \lambda \) of \( SL(V) \),

\[
\mu((q, [\Phi]), \lambda) = n_1 \mu(q, \lambda) + n_2 \mu([\Phi], \lambda) (\leq 0).
\]

A one-parameter subgroup of \( SL(V) \) is equivalent to a basis \( \{e_1, \ldots, e_p\} \) of \( V \) and a vector \( \Gamma = (\Gamma_1, \ldots, \Gamma_p) \in \mathbb{C}^p \) with \( \Gamma_1 \leq \ldots \leq \Gamma_p \). This defines a weighted filtration \((V_\bullet, m_\bullet)\) of \( V \) as follows: let \( \lambda_1 < \ldots < \lambda_{t+1} \) be the different values of \( \Gamma_k \), let \( V_i \) be the vector space generated by all \( e_k \) such that \( \Gamma_k \leq \lambda_i \), and let \( m_i = (\lambda_{i+1} - \lambda_i)/p \). Denote \( \mathcal{T}' = \{1, \ldots, t + 1\}^{P(l)} \). We have \(|S| \) or \([H-L2])

\[
\mu(q, \lambda) = \min_{I \in \mathcal{T}'} \{ \Gamma_{\dim V_i} + \cdots + \Gamma_{\dim V_{P(l)}} : q''(V_{i_1} \otimes \cdots \otimes V_{P(l)} \otimes W) \neq 0 \}
\]

\[
= \sum_{i=1}^t m_i (\dim V_i P(l) - \dim V P_{E_{V_i}}(l))
\]
and
\[
\mu([\Phi], \lambda) = \min_{l \in \mathbb{Z}} \{ \Gamma_{\dim V_{\lambda_1}} + \cdots + \Gamma_{\dim V_{\lambda_s}} : \Phi|_{(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s})^n} \neq 0 \} \\
= \mu(\Phi, V_*, m_*)
\]

The last statement follows from an argument similar to the proof of lemma 1.4, with \(Z\) replaced by \(Z^p\).

\(\square\)

**Proposition 3.5.** The point \((q, [\Phi])\) is GIT-(semi)stable if and only if for all weighted filtrations \((E_*, m_*)\) of \(E\)

\[
\sum_{i=1}^{t} m_i \left( \left( \dim V_i - \epsilon_i(E_*) \delta(m) \right) (P - s\delta) \right. \\
\left. - \left( P_{E_i} - \epsilon_i(E_*) \delta \right) \left( \dim V - s\delta(m) \right) \right) \leq 0
\]

(3.13)

Furthermore, if \((q, [\Phi])\) is GIT-semistable, then the induced map \(f_q : V \to H^0(E(m))\) is injective.

**Proof.** First we prove that if \((q, [\Phi])\) is GIT-semistable, then the induced linear map \(f_q\) is injective. Let \(V'\) be its kernel and consider the filtration \(V' \subset V\). We have \(E_{V'} = 0\) and \(\mu(\Phi, V' \subset V) = s \dim V'\). Applying proposition 3.4 we have

\[n_1 \dim V' P(l) + n_2 s \dim V' \leq 0,
\]

and hence \(V' = 0\).

Using (3.8) and (3.11), the inequality of proposition 3.4 becomes

\[
\sum_{i=1}^{t} m_i \left( \left( \dim V_i - \epsilon_i(V_*) \delta(m) \right) (P(l) - s\delta(l)) \right. \\
\left. - \left( P_{E_i}(l) - \epsilon_i(V_*) \delta(l) \right) \left( \dim V - s\delta(m) \right) \right) \leq 0
\]

(3.14)

An argument similar to lemma 2.9 (using \(A_2\) instead of \(A_1\)) shows that we can take \(l\) large enough (depending only on \(m, s, b, c, P, D\) and \(\delta\)), so that this inequality holds for \(l\) if and only if it holds as an inequality of polynomials.

Now assume that \((q, [\Phi])\) is GIT-(semi)stable. Take a weighted filtration \((E_*, m_*)\) of \(E\). Then lemma 3.2 and (3.14) applied to the associated weighted filtration \((V_{E_*, m_*})\) of \(V\) give (3.13).

On the other hand, assume that (3.13) holds. Take a weighted filtration \((V_*, m_*)\) of \(V\). Then lemma 3.3 and (3.13) applied to the associated weighted filtration \((E_{V_*, m_*})\) of \(E\) give (3.14), and it follows that \((q, [\Phi])\) is GIT-(semi)stable.

\(\square\)

**Theorem 3.6.** Assume \(m > N\). For \(l\) sufficiently large, a point \((q, [\Phi])\) in \(Z\) is GIT-(semi)stable if and only if the corresponding tensor \((E, \varphi, u)\) is \(\delta\)-(semi)stable and the linear map \(f_q : V \to H^0(E(m))\) induced by \(q\) is an isomorphism.

**Proof.** We prove this in two steps:

**Step 1.** \((q, [\Phi])\) GIT-semistable \(\implies (E, \varphi, u)\) \(\delta\)-semistable and \(q\) induces an isomorphism \(V \cong H^0(E(m))\).

The leading coefficient of (3.13) gives

\[
\sum_{i=1}^{t} m_i \left( \left( \dim V_{E_i} - \epsilon_i(E_* \delta(m)) \right) r_i \left( \dim V - s\delta(m) \right) \right) \leq 0.
\]
Note that even if \((q, [\Phi])\) is GIT-stable, here we only get weak inequality. This implies
\begin{equation}
\left( \sum_{i=1}^{t} m_i \left( r^i P(m) - rh^0(E^i(m)) \right) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta(m) \leq 0.
\end{equation}

To be able to apply theorem 2.5, we still need to show that \(E\) is torsion free. By lemma 2.11, there exists a tensor \((F, \psi, u)\) with \(F\) torsion free such that \(P_E = P_F\) and an exact sequence
\[
0 \rightarrow T(E) \rightarrow E \xrightarrow{\beta} F.
\]
Consider a weighted filtration \((F_\bullet, m_\bullet)\) of \(F\). Let \(F^i = F/F_i\), and let \(E^i\) be the image of \(E\) in \(F^i\). Let \(E_i\) be the kernel of \(E \rightarrow E^i\). Then \(\text{rk}(F_i) = \text{rk}(E_i) = r_i\), \(h^0(F^i(m)) \geq h^0(E^i(m))\), and \(\mu(\psi, F_\bullet, m_\bullet) = \mu(\varphi, E_\bullet, m_\bullet)\). Using this and applying (3.15) to \(E_i\) we get
\[
\left( \sum_{i=1}^{t} m_i \left( r^i P(m) - rh^0(F^i(m)) \right) \right) + \mu(\varphi, F_\bullet, m_\bullet) \delta(m) \leq 0,
\]
and hence theorem 2.5 implies that \((F, \psi, u)\) is \(\delta\)-semistable.

Next we will show that \(T(E) = 0\), and hence, since \(P_E = P_F\), we will conclude that \((E, \varphi, u)\) is isomorphic to \((F, \psi, u)\). Define \(E''\) to be the image of \(E\) in \(F\). Then
\[
P(m) - s\delta(m) = h^0(F(m)) - s\delta(m) \geq h^0(E''(m)) - s\delta(m) \geq P(m) - s\delta(m),
\]
where the last inequality follows from (3.15) applied to the one step filtration \(T(E) \subset E\). Hence equality holds at all places and \(h^0(F(m)) = h^0(E''(m))\). Since \(F\) is globally generated, \(F = E''\), and hence \(T(E) = 0\).

Finally, we have seen that \(f_q\) is injective, and since \((E, \varphi)\) is \(\delta\)-semistable, \(\dim V = h^0(E(m))\), hence \(f_q\) is an isomorphism.

**Step 2.** \((E, \varphi, u)\) \(\delta\)-stable (respectively strictly \(\delta\)-semistable) and \(q\) induces an isomorphism \(f_q : V \cong H^0(E(m)) \implies (q, [\Phi])\) GIT-stable (respectively strictly semistable).

Since \(f_q\) is an isomorphism, we have \(V_{E'} = H^0(E'(m))\) for any subsheaf \(E' \subset E\). Then theorem 2.5 implies that for all weighted filtrations
\begin{equation}
\left( \sum_{i=1}^{t} m_i \left( r \dim V_{E_i} - r_i P(m) \right) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta(m) (\leq 0)
\end{equation}
If the inequality is strict, then
\[
\sum_{i=1}^{t} m_i \left( \left( \dim V_{E_i} - \epsilon_i(E_\bullet) \delta(m) \right) (P - s\delta) - \left( P_{E_i} - \epsilon_i(E_\bullet) \delta \right) \left( \dim V - s\delta(m) \right) \right) < 0.
\]
If \((E, \varphi, u)\) is strictly \(\delta\)-semistable, by theorem 2.5 there is a filtration giving equality in (3.16), then corollary 2.10 implies that \(h^0(E_i(m)) = P_{E_i}(m)\), and by lemma 2.9
\[
\left( \sum_{i=1}^{t} m_i \left( r P_{E_i} - r_i P \right) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta = 0,
\]
and a short calculation using this and (3.16) gives
\[
\sum_{i=1}^{t} m_i \left( (\dim V_{E_i} - \epsilon_i(E_{\bullet})\delta(m)) (P - s\delta) - (P_{E_i} - \epsilon_i(E_{\bullet})\delta) (\dim V - s\delta(m)) \right) = 0.
\]
So we finish by using proposition 3.5.

\[\square\]

Given a one-parameter subgroup of \(\text{SL}(V)\), choose a basis \(\{e_j\}\) of \(V\) where it has a diagonal form
\[
\text{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_{t+1}, \ldots, \lambda_{t+1})
\]
This gives a weighted filtration \((V_{\bullet}, m_{\bullet})\) of \(V = H^0(E(m))\) (where \(V_i\) is the linear span of \(\{e_j\}\) with \(j \leq a_i\), and \(m_i = (\lambda_{i+1} - \lambda_i)/p\)) and a splitting of \(V = \oplus V^i\) of this filtration (with \(V^i\) the linear span of \(\{e_j\}\) with \(a_{i-1} < j \leq a_i\)). Defining \(E_{V_i} = q(V_i \otimes \mathcal{O}_X(-m))\) we obtain a weighted filtration \((E_{\bullet}, m_{\bullet})\) of \(E\).

Now let \((E_{\bullet}, m_{\bullet})\) be a weighted filtration of \(E\) and \(V = \oplus V^i\) a splitting of the filtration \(V_i = H^0(E(m))\). This gives a one-parameter subgroup \(\lambda\) of \(\text{SL}(V)\), defined as \(v^i \mapsto t^{\lambda_i} v^i\) for \(v^i \in V^i\), with \(\lambda_i\) such that \(m_i = (\lambda_{i+1} - \lambda_i)/p\) and \(\sum \lambda_i \dim V^i = 0\).

The following proposition will be used to prove the criterion for \(S\)-equivalence.

**Proposition 3.7.** Assume \(m > N\). Let \((E, \varphi, u)\) be a \(\delta\)-semistable tensor, \(f : V \cong H^0(E(m))\) an isomorphism, and let \((q, [\Phi]) \in Z\) be the corresponding GIT-semistable point. The above construction gives a bijection between one-parameter subgroups of \(\text{SL}(V)\) with \(\mu((q, [\Phi]), \lambda) = 0\) on the one hand, and weighted filtrations \((E_{\bullet}, m_{\bullet})\) of \(E\) with
\[
\left( \sum_{i=1}^{t} m_i (r P_{E_i} - r_i P) \right) + \mu(\varphi, E_{\bullet}, m_{\bullet}) \delta = 0
\]
together with a splitting of the filtration \(H^0(E_{\bullet}(m))\) of \(V = H^0(E(m))\) on the other hand.

**Proof.** Let \(\lambda\) be a one-parameter subgroup of \(\text{SL}(V)\) with \(\mu((q, [\Phi]), \lambda) = 0\). The proof of proposition 3.4 then gives equality in (3.12). Using (3.8) (relationship between \(n_2/n_1\) and \(\delta\), (3.11) (relationship between \(\epsilon(V_{\bullet})\) and \(\mu(V_{\bullet})\)) and lemma 3.3 (relationship between \(\epsilon(V_{\bullet})\) and \(\epsilon(E_{V_{\bullet}})\)), this equality becomes
\[
\sum_{i=1}^{t} m_i \left( (\dim V_i - \epsilon_i(E_{V_{\bullet}})\delta(m)) (P(l) - s\delta(l)) - (P_{E_{V_i}}(l) - \epsilon_i(E_{V_{\bullet}})\delta(l)) (\dim V - s\delta(m)) \right) = 0
\]
We have chosen \(l\) so large that this holds if and only if it holds as a polynomial in \(l\), hence taking the leading coefficient in \(l\) we obtain
\[
\sum_{i=1}^{t} m_i \left( (\dim V_i - \epsilon_i(E_{V_{\bullet}})\delta(m)) r - r_i (\dim V - s\delta(m)) \right) = 0.
\]
where \(r_i = \text{rk} E_{V_i}\) and \(r = \text{rk} E\). Using (1.8), this is
\[
\left( \sum_{i=1}^{t} m_i (r \dim V_i - r_i P(m)) \right) + \mu(\varphi, E_{\bullet}, m_{\bullet}) \delta(m) = 0
\]
By lemma 3.3, $V_i \subset V_{E_{V_i}} = H^0(E_{V_i}(m))$, hence

$$\left( \sum_{i=1}^{t} m_i (r h^0(E_{V_i}(m)) - r_i P(m)) \right) + \mu(\varphi, E_{V_i}, m_*) \delta(m) \geq 0$$

but by theorem 2.5 this must be nonpositive, hence $V_i = H^0(E_{V_i}(m)) = V_{E_{V_i}}$, and the last inequality is an equality. By corollary 2.10, $E_i \in S_0$, and hence $h^0(E_{V_i}(m)) = P(E_{V_i}(m))$ for all $i$, and then lemma 2.9 gives

$$\left( \sum_{i=1}^{t} m_i (r P_{E_{V_i}} - r_i P) \right) + \mu(\varphi, E_{V_i}, m_*) \delta = 0$$

Conversely, let $(E_*, m_*)$ be a filtration with

(3.17) $$\left( \sum_{i=1}^{t} m_i (r P_{E_i} - r_i P) \right) + \mu(\varphi, E_*, m_*) \delta = 0$$

together with a splitting of the filtration $H^0(E_i(m))$ of $V \cong H^0(E(m))$, and let $\lambda$ be the associated one-parameter subgroup of $\text{SL}(V)$. Equation (3.17) gives in particular

$$\left( \sum_{i=1}^{t} m_i (r P_{E_i}(m) - r_i P(m)) \right) + \mu(\varphi, E_*, m_*) \delta(m) = 0$$

By the proof of implication 3. \(\Rightarrow\) 1. in theorem 2.5, since we get an equality, it is $E_i \in S_0$ for all $i$, hence $P_{E_i(m)} = h^0(E_i(m)) = \dim V_{E_{V_i}}$ for all $i$, and the previous equality becomes

(3.18) $$\left( \sum_{i=1}^{t} m_i (r \dim V_{E_i} - r_i \dim V) \right) + \mu(\varphi, E_*, m_*) \delta(m) = 0$$

Furthermore, the strong version of lemma 3.2 gives $E_i = E_{V_{E_i}}$. Using (3.18) and (3.17), together with (1.8) and the strong form of lemma 3.2, we obtain

$$\sum_{i=1}^{t} m_i \left( (\dim V_{E_i} - \epsilon_i(V_{E_i}) \delta(m)) (P - s \delta) - (P_{E_i} - \epsilon_i(V_{E_i}) \delta) \left( \dim V - s \delta(m) \right) \right) = 0$$

Hence, we also get 0 after evaluating this polynomial in $l$, but by the proofs of propositions 3.4 and 3.5, this is equal to $\mu((q_1[\Phi]), \lambda)$.

We have seen that $V_i = V_{E_{V_i}}$ and $E_i = E_{V_{E_i}}$, and it is easy to check that this gives a bijection.

\[\square\]

4. Proof of Theorem 1.8

Proof of theorem 1.8. The main ingredient of the proof is theorem 3.6, showing that GIT-(semi)stable points correspond to $\delta$-(semi)stable tensors.

Using the notation of section 3, let $\mathcal{M}_\delta$ (respectively $\mathcal{M}_\delta^s$) be the GIT quotient of $Z$ (respectively $Z^s$) by $\text{SL}(V)$. Since $Z$ is projective, $\mathcal{M}_\delta$ is also projective. GIT gives that $\mathcal{M}_\delta^s$ is an open subset of the projective scheme $\mathcal{M}_\delta$. The restriction $Z^s \to \mathcal{M}_\delta^s$ to the stable part is a geometric quotient, i.e. the fibers are $\text{SL}(V)$-orbits, and hence the points of $\mathcal{M}_\delta^s$ correspond to isomorphism classes of $\delta$-stable tensors.
It only remains to show that \( \mathcal{M}_\delta \) corepresents the functor \( \mathcal{M}_\delta \). We will follow closely [H-L2, Proof of Main Theorem 0.1, p. 315]

Let \((E_T, \varphi_T, u_T, N)\) be a family of \( \delta \)-semistable tensors (cf. (1.9)) parametrized by a scheme \( T \). Then \( V := \pi_{T*}(E_T \otimes \pi_X^* \mathcal{O}_X(m)) \) is locally free on \( T \). The family \( E_T \) gives a map \( \Delta : T \to \text{Pic}^{d}(X) \), sending \( t \in T \) to \( \det E_T \). Cover \( T \) with small open sets \( T_i \). For each \( i \) we can find an isomorphism
\[
\beta_{T_i} : \det E_{T_i} \to \sum_{i} \mathcal{P}
\]
(where \( \mathcal{P} \) is the Poincare bundle in the definition of \( P \) at the beginning of section 3), and a trivialization
\[
g_{T_i} : V \otimes \mathcal{O}_{T_i} \to V|_{T_i}.
\]
Using this trivialization we obtain a family of quotients parametrized by \( T_i \)
\[
q_{T_i} : V \otimes \pi_X^* \mathcal{O}_X(-m) \to E_{T_i},
\]
giving a map \( T_i \to \mathcal{H} \). And using the quotient \( q_{T_i} \) and isomorphism \( \beta_{T_i} \) we have another family of quotients parametrized by \( T_i \)
\[
(V_{\otimes s})^{\otimes c} \otimes \left( \pi_{T_i*}(\sum_{i} \mathcal{P}^{\otimes b} \otimes \widetilde{\pi}_{T_i*} \mathcal{D} \otimes \pi_X^* \mathcal{O}_X(sm)) \right)^{V} \to N
\]
Then, using the representability properties of \( \mathcal{H} \) and \( P \), we obtain a morphism to \( \mathcal{H} \times P \), and by lemma 3.1 this morphism factors through \( Z' \) and since a \( \delta \)-semistable tensor gives a GIT-semistable point (theorem 3.6), the image is in \( Z^{ss} \). Composing with the geometric quotient to \( \mathcal{M}_\delta \) we obtain maps
\[
\hat{f}_i : T_i \xrightarrow{f_i} Z^{ss} \to \mathcal{M}_\delta
\]
The morphism \( f_i \) is independent of the choice of isomorphism \( \beta_{T_i} \). A different choice of isomorphism \( g_{T_i} \) will change \( f_i \) to \( h_i \cdot f_i \), where \( h_i : T_i \to \text{GL}(V) \), so \( \hat{f}_i \) is independent of the choice of \( g_{T_i} \). Then the morphisms \( \hat{f}_i \) glue to give a morphism
\[
\hat{f} : T \to \mathcal{M}_\delta,
\]
and hence we have a natural transformation
\[
\mathcal{M}_\delta \to \mathcal{M}_\delta.
\]
Recall there is a tautological family (3.6) of tensors parametrized by \( Z' \). By restriction to \( Z^{ss} \), we obtain a tautological family of \( \delta \)-semistable tensors parametrized by \( Z^{ss} \). If \( \mathcal{M}_\delta \to Y \) is another natural transformation, this tautological family defines a map \( Z^{ss} \to Y \), this factors through the quotient \( \mathcal{M}_\delta \), and it is easy to see that this proves that \( \mathcal{M}_\delta \) corepresents the functor \( \mathcal{M}_\delta \).

Note that in [H-L2], the moduli space of stable framed modules is a fine moduli space. In our situation this is not true in general, because the analog of the uniqueness result of [H-L2, lemma 1.6] does not hold in general for tensors. \( \square \)

Now we will give a criterion for \( S \)-equivalence. This is very similar to the criterion given in [G-S] for conic bundles. If \((E, \varphi, u)\) and \((F, \psi, u)\) are two \( \delta \)-stable tensors then we have seen that they correspond to the same point in the moduli space if and only if they are isomorphic. But if they are strictly \( \delta \)-semistable, it could happen that they are \( S \)-equivalent (i.e. they correspond to the same point in the moduli space), even if they are not isomorphic. Given a tensor \((E, \varphi, u)\), we will construct a canonical representative of its equivalence class \((E^S, \varphi^S, u)\), hence \((E, \varphi, u)\) will be \( S \)-equivalent to \((F, \psi, u)\) if and only if \((E^S, \varphi^S, u)\) is isomorphic to \((F^S, \psi^S, u)\).
Let \((E, \varphi, u)\) be strictly \(\delta\)-semistable, and let \((E_\bullet, m_\bullet)\) be an admissible weighted filtration, i.e. such that
\[
\left( \sum_{i=1}^{t} m_i (rP_{E_i} - r_i P) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta = 0
\]
Let \(I_0\) be the set of pairs \((k, I)\) where \(1 \leq k \leq c\) is an integer, and \(I = (i_1, \ldots, i_s)\) is a multi-index with \(1 \leq i_j \leq t + 1\), such that the restriction of \(\varphi\)
\[
\varphi_{k, I} : 0 \oplus \cdots \oplus 0 \oplus (E_{i_1} \oplus \cdots \oplus E_{i_s}) \oplus 0 \oplus \cdots \oplus 0 \rightarrow (\det E)^{\otimes b} \otimes D_u
\]
is nonzero, and
\[
\gamma_{r_{i_1}} + \cdots + \gamma_{r_{i_s}} = \mu(\varphi, E_\bullet, m_\bullet).
\]
If \((k, I) \in I_0\) and \(I' = (i'_1, \ldots, i'_s)\) is a multi-index with \(I' \neq I\) and \(i'_j \leq i_j\) for all \(j\), then
(4.1)
\[
\varphi_{k, I'} = 0,
\]
by definition of \(\mu(\varphi, E_\bullet, m_\bullet)\). Hence, if \((k, I) \in I_0\), the restriction \(\varphi_{k, I}\) defines a homomorphism in the quotient
\[
\varphi'_{k, I} : 0 \oplus \cdots \oplus 0 \oplus (E'_{i_1} \oplus \cdots \oplus E'_{i_s}) \oplus 0 \oplus \cdots \oplus 0 \rightarrow (\det E)^{\otimes b} \otimes D_u,
\]
where \(E'_i = E_i / E_{i+1}\). If \((k, I) \neq I_0\), then define \(\varphi'_{k, I} = 0\). Finally, we define
\[
E' = E'_1 \oplus \cdots \oplus E'_{t+1}, \quad \varphi' = \bigoplus_{(k, I)} \varphi'_{k, I}.
\]
In the definition of \(\varphi'\) we are using the fact that \(\det E \cong \det E'\), hence \((E', \varphi', u)\) is well-defined up to isomorphism, and it is called the admissible deformation associated to the admissible filtration \((E_\bullet, m_\bullet)\) of \(E\). Note that it depends on the admissible weighted filtration chosen.

**Proposition 4.1.** The tensor \((E', \varphi', u)\) is strictly \(\delta\)-semistable, and it is \(S\)-equivalent to \((E, \varphi, u)\). If we repeat this process, after a finite number of iterations the process will stop, i.e. we will obtain tensors isomorphic to each other. We call this tensor \((E^S, \varphi^S, u)\)

(1) The isomorphism class of \((E^S, \varphi^S, u)\) is independent of the choices made, i.e. the weighted filtrations chosen.

(2) Two tensors \((E, \varphi, u)\) and \((F, \psi, u)\) are \(S\)-equivalent if and only if \((E^S, \varphi^S, u)\) is isomorphic to \((F^S, \psi^S, u)\).

**Proof.** We start with a general observation about GIT quotients. Let \(Z\) be a projective variety with an action of a group \(G\) linearized on an ample line bundle \(O_Z(1)\). Two points in the open subset \(Z^{ss}\) of semistable points are GIT-equivalent (they are mapped to the same point in the moduli space) if there is a common closed orbit in the closures (in \(Z^{ss}\)) of their orbits. Let \(z \in Z^{ss}\). Let \(B(z)\) be the unique closed orbit in the closure \(\overline{G \cdot z}\) in \(Z^{ss}\) of its orbit \(G \cdot z\). Assume that \(z\) is not in \(B(z)\). There exists a one-parameter subgroup \(\lambda\) such that the limit \(z_0 = \lim_{t \to 0} \lambda(t) \cdot z\) is in \(\overline{G \cdot z} \setminus G \cdot z\) (for instance, we can take the one-parameter subgroup given by \([Si, Lemma 1.25]\)). Note that we must have \(\mu(z, \lambda) = 0\) (otherwise \(z_0\) would be unstable). Conversely, if \(\lambda\) is a one-parameter subgroup with \(\mu(z, \lambda) = 0\), then the limit is GIT-semistable (\([G-S, Prop. 2.14]\)). Note that \(G \cdot z_0 \subset \overline{G \cdot z} \setminus G \cdot z\), and
then $\dim G \cdot z_0 < \dim G \cdot z$. Repeating this process with $z_0$ we then get a sequence of points that eventually stops and gives $\tilde{z} \in B(z)$. Two points $z_1$ and $z_2$ will then be GIT-equivalent if and only if $B(z_1) = B(z_2)$.

Let $(E, \varphi, u)$ be a $\delta$-semistable tensor with an isomorphism $f : V \cong H^0(E(m))$, and let $z = (q, [\Phi]) \in Z$ be the corresponding GIT-semistable point. Recall from proposition 3.7 that there is a bijection between one-parameter subgroups of $SL(V)$ with $\mu(z, \lambda) = 0$ on the one hand, and weighted filtrations $(E_\bullet, m_\bullet)$ of $E$ with

$$
\left( \sum_{i=1}^{t} m_i (r P_{E_i} - r_i P) \right) + \mu(\varphi, E_\bullet, m_\bullet) \delta = 0
$$

together with a splitting of the filtration $H^0(E_\bullet(m))$ of $V = H^0(E(m))$ on the other hand.

The action of $\lambda$ on the point $z$ defines a morphism $\mathbb{C}^* \to R_3$ that extends to

$$
h : T = \mathbb{C} \to Z,
$$

with $h(t) = \lambda(t) \cdot z$ for $t \neq 0$ and $h(0) = \lim_{t \to 0} \lambda(t) \cdot z = z_0$.

Pulling back the universal family parametrized by $Z$ by $h$ we obtain the family $(q_T, E_T, \varphi_T, u)$

$$
E_T = \bigoplus_n E_n \otimes t^n \subset E \otimes \mathbb{C} t^{-N} \mathbb{C}[t] \subset E \otimes \mathbb{C} [t, t^{-1}]
$$

$$
q_T : V \otimes \mathcal{O}_X(-m) \otimes \mathbb{C}[t] \overset{\gamma}{\longrightarrow} \oplus_n V_n \otimes \mathcal{O}_X(-m) \otimes t^n \longrightarrow E_T
$$

$$
\varphi_T : (E_T \otimes \mathbb{C}) \otimes \mathbb{C} \longrightarrow (\det E_T)^{\otimes b} \otimes \mathcal{D} \otimes \pi_T^* N
$$

Then, as in [H-L2, §4.4], $(q_t, E_t, \varphi_t, u)$ corresponds to $h(t)$ (in particular, if $t \neq 0$, then $(E_t, \varphi_t, u)$ is canonically isomorphic to $(E, \varphi, u)$), and $(E_0, \varphi_0, u)$ is the admissible deformation associated to $(E_\bullet, m_\bullet)$.

\[\Box\]

5. Orthogonal and symplectic sheaves

In this section we apply the general theory of tensors to construct the moduli space of semistable orthogonal and symplectic sheaves. The only difference between these is whether the bilinear form is symmetric or skew-symmetric, hence we will first consider the orthogonal case, and at the end of the section we will add some comments about the symplectic case. We fix $D_u$ to be $\mathcal{O}_X$ (i.e. $R$ is one point and $D$ is $\mathcal{O}_{X \times U}$, and hence we can drop $u$ from the notation of tensors).

**Definition 5.1.** An orthogonal sheaf is a tensor

$$
(E, \varphi), \quad \varphi : E \otimes E \to \mathcal{O}_X
$$

such that

- (OS1) $(\det E)^{\otimes 2} \cong \mathcal{O}_X$
- (OS2) $\varphi$ is symmetric
- (OS3) $E$ is torsion free
- (OS4) $\varphi$ induces an isomorphism $E|_U \to E|_U^\vee$ on the open subset $U$ where $E$ is locally free.
An isomorphism of orthogonal sheaves is an isomorphism as tensors.

It is easy to see that, assuming (OS1) and (OS3), the last condition is equivalent to

- (OS4′) The induced homomorphism \( \text{det} \ E \rightarrow \text{det} \ E^\vee \) is nonzero (hence an isomorphism).

The following lemma justifies this definition for orthogonal sheaves.

**Lemma 5.2.** There is a bijection between the set of isomorphism classes of orthogonal sheaves with \( E \) locally free and the set of isomorphism classes of principal \( O(r) \)-bundles.

**Proof.** The category of principal \( O(r) \)-bundles is equivalent to the category whose objects are pairs \( (P, \sigma) \) (where \( \pi : P \rightarrow X \) is a principal GL(\( r \))-bundle, \( \sigma \) is a section of the associated fiber bundle \( P \times_{GL(r)} GL(r)/O(r) \)) and whose isomorphisms are isomorphisms \( f : P \rightarrow P' \) of principal bundles respecting \( \sigma \) (i.e. \( \pi' \circ f = \pi \) and \( (f \times \text{id}_{O(r)}) \circ \sigma = \sigma' \)). Note that this notion of isomorphism is not the same as isomorphism of reductions.

The category of principal GL(\( r \))-bundles is equivalent to the category of vector bundles of rank \( r \). The quotient \( GL(r)/O(r) \) is the set of invertible symmetric matrices (send \( A \in GL(r) \) to \( (T^A)^{-1} A^{-1} \)). Hence, a section \( \sigma \) is the same thing as a homomorphism \( \varphi \) as in (OS2). Now it is easy to check that there is a bijection between these sets of isomorphism classes.

\( \square \)

**Remark 5.3.** Note that the categories are not equivalent: for example, let \( P \) be a simple principal \( G \)-bundle, i.e. the set of automorphisms of \( P \) is the center of \( G \) (a finite group), but the set of automorphisms of the corresponding \( G \)-sheaf is \( \mathbb{C}^* \). We will have an equivalence of categories if we consider only isomorphisms \( (f, \alpha) \) with \( \alpha = 1 \), as in remark 1.2. This would be important if we wanted to construct the moduli stack, but since we are interested in the moduli space this is irrelevant, because the moduli space does not detect the group of automorphisms.

Let \( (E, \varphi) \) be an orthogonal (or symplectic) sheaf. A subsheaf of \( F \) of \( E \) is called isotropic if \( \varphi|_{F \otimes F} = 0 \). Given a subsheaf \( i : F \hookrightarrow E \), using the bilinear form \( \varphi \) we can associate the perpendicular subsheaf \( F^\perp = \ker(E \xrightarrow{\varphi} E^\vee \xrightarrow{\varphi^*} F^\vee) \),

where \( \varphi : E \rightarrow E^\vee \) is the homomorphism induced by \( \varphi \).

**Definition 5.4** (Stability). An orthogonal sheaf \( (E, \varphi) \) is (semi)stable if for all isotropic subsheaves \( F \subset E \),

\[
P_F + P_{F^\perp} (\preceq) P.
\]

An orthogonal sheaf \( (E, \varphi) \) is slope-(semi)stable if for all isotropic subsheaves \( F \subset E \),

\[
\deg(F) (\leq) 0.
\]

As usual, we can assume that \( F \) is saturated. A family of semistable orthogonal sheaves parametrized by \( T \) is a family of tensors

\[
(E_T, \varphi_T, N), \quad \varphi_T : E_T \otimes E_T \rightarrow \pi_T^* N,
\]

such that \( (\det E_T)^{\otimes 2} \) is isomorphic to the pullback of some line bundle on \( T \), \( \varphi_T \) is symmetric, and \( \varphi_T \) induces an isomorphism \( E_T|_U \rightarrow E_T^\vee \otimes \pi_T^* N|_U \) on the open
set \( U \) where \( E_T \) is locally free, and such that the restriction to \( X \times t \) for all closed points \( t \) is a semistable orthogonal sheaf.

Using this notion of family, we define the functor \( \mathcal{M}_{O(v)} \) of semistable orthogonal sheaves. We will construct a moduli space corepresenting this functor (theorem 5.9).

In proposition 5.7 we show that an orthogonal (or symplectic) sheaf \((E, \varphi)\) is (semi)stable in this sense if and only if it is \( \delta \)-(semi)stable as a tensor (definition 1.3), provided that \( \delta_1 > 0 \). Hence, the moduli space of semistable orthogonal (or symplectic) sheaves is a subscheme of the moduli space of \( \delta \)-semistable tensors. In theorem 5.9 we show that it is in fact projective. We can also ask about slope-semistability, and in proposition 5.8 we show that slope-(semi)stability in this sense and slope-\( \tau \)-(semi)stability as a tensor coincide if \( \tau > 0 \). If \( \delta_1 = 0 \), then the notion of \( \delta \)-semistability as a tensor is not equivalent to semistability as an orthogonal sheaf.

At the end of the section we give an example of this.

We start with some preliminaries. The intersection \( F \cap F^\perp \) is an isotropic subsheaf of \( F \). The following lemma gives exact sequences relating these subsheaves.

**Lemma 5.5.** With the previous notation:

1. Let \( U \) be the open set where \( F, E \) and \( E/F \) are locally free. There is an exact sequence on \( U \)

   \[
   0 \rightarrow F^\perp|_U \rightarrow E|_U \rightarrow F^\vee|_U \rightarrow 0,
   \]

   and hence \( \text{rk}(F^\perp) = \text{rk}(E) - \text{rk}(F) \). If furthermore \( F \) is saturated (i.e. \( E/F \) is torsion free), then \( \text{codim}(X - U) \geq 2 \) and hence \( \deg(F^\perp) = \deg(F) \).

2. If \( F \) is saturated, then \( F \cap F^\perp \) is also saturated.

3. There is an exact sequence

   \[
   0 \rightarrow F \cap F^\perp \rightarrow F \oplus F^\perp \rightarrow F + F^\perp \rightarrow 0
   \]

4. \( F + F^\perp \subseteq (F \cap F^\perp)^\perp \), \( \text{rk}(F + F^\perp) = \text{rk}((F \cap F^\perp)^\perp) \), and hence \( \deg(F + F^\perp) \leq \deg((F \cap F^\perp)^\perp) \).

5. Let \( F \) be a saturated subsheaf. If \( F \cap F^\perp \neq 0 \), then

   \[
   \deg(F) \leq \deg(F \cap F^\perp),
   \]

   and if \( F \cap F^\perp = 0 \), then

   \[
   \deg(F) \leq 0.
   \]

**Proof.** Since \( E/F|_U \) is locally free, the last term in the following exact sequence is zero

\[
0 \rightarrow (E/F)^\vee|_U \rightarrow E^\vee|_U \xrightarrow{i^\vee|_U} F^\vee|_U \rightarrow \text{Ext}^1((E/F)|_U, O_U) = 0,
\]

and hence \( i^\vee|_U \) is surjective. Combining this with (OS4) we get the exact sequence

\[
0 \rightarrow F^\perp|_U \rightarrow E|_U \cong E^\vee|_U \xrightarrow{i^\vee|_U} F^\vee|_U \rightarrow 0.
\]

If \( E/F \) is torsion free, then \( \text{codim}(X - U) \geq 2 \) and we can use this sequence to obtain \( \deg(F^\perp) = \deg(F) \).

To prove item 2, first we show that \( F^\perp \) is saturated. The composition \( E \rightarrow E^\vee \rightarrow F^\vee \) factors as

\[
E \rightarrow E/F^\perp \rightarrow F^\vee.
\]

The sheaf \( F^\vee \) is torsion free, and hence also \( E/F^\perp \) is torsion free.

We conclude by showing that the stalk \((E/(F \cap F^\perp))_x = E_x/(F_x \cap F^\perp_x)\) is torsion free for all points \( x \in X \). Let \( v \in E_x \) and let \( 0 \neq f \in m_x \) be a nonzero element in
the maximal ideal of the local ring of \( x \), such that \( f v \in F_x \cap F_x^\perp \). Since \( f v \in F_x \), and \( F_x \) is saturated, then \( v \in F_x \). The same argument applies to \( F_x^\perp \), and hence \( v \in F_x \cap F_x^\perp \).

Items 3 and 4 are easy to check. To show item 5, if \( F \cap F^\perp \neq 0 \), use the exact sequence (5.3), together with items 1, 2 and 4. If \( F \cap F^\perp = 0 \), then \( F \otimes F^\perp = F + F^\perp \) is a subsheaf of \( E \) of rank \( r \), then \( \deg(F) + \deg(F^\perp) \leq 0 \), and hence \( \deg(F) \leq 0 \).

The fact that on a generic fiber the quadratic form is nondegenerate has the following useful consequence:

**Lemma 5.6.** If \((E, \varphi)\) is an orthogonal or symplectic sheaf, then for all weighted filtrations

\[
(5.6) \quad \mu(\varphi, E_\bullet, m_\bullet) \leq 0.
\]

**Proof.** First we will show that if \( Q : W \otimes W \to \mathbb{C} \) is a bilinear nondegenerate form on a vector space \( W \), then \( Q \in \mathbb{P}(W^\vee \otimes W^\vee) \) is GIT-semistable under the natural action of \( \text{SL}(W) \) (with the natural linearization induced on \( O(1) \)). The point \( Q \) is unstable if and only if there is a one-parameter subgroup \( \lambda \) of \( \text{SL}(W) \) such that \( \lim_{t \to 0} \lambda(t) \cdot Q = 0 \). But this is impossible because \( \det(\lambda(t)) = 1 \), and then

\[
\det(\lambda(t) \cdot Q) = \det(\lambda(t) Q^T \lambda(t)) = \det(Q) \neq 0,
\]

hence \( Q \) is semistable. Then, using this and condition (OS4), it follows that

\[
\mu(\varphi, E_\bullet, m_\bullet) \leq 0
\]

for all weighted filtrations. \( \square \)

**Proposition 5.7.** Assume \( \delta_1 > 0 \). An orthogonal sheaf \((E, \varphi)\) is (semi)stable if and only if it is \( \delta \)-semistable as a tensor.

**Proof.** To see that \( \delta \)-semistable as a tensor implies (semi)stable as an orthogonal sheaf, we apply the stability condition to the weighted filtration \( F \subset F^\perp \subset E \) with weights \( m_1 = m_2 = 1 \). By lemma 5.5(1), \( r = \text{rk}(F) + \text{rk}(F^\perp) \). Since \( F \) is isotropic, \( \mu(\varphi, E_\bullet, m_\bullet) = 0 \), hence the stability condition (1.6) gives the result:

\[
r (P_F + P_{F^\perp} - P) = (r P_F - \text{rk}(F) P) + (r P_{F^\perp} - \text{rk}(F^\perp) P) \leq 0.
\]

Now we will show that if \((E, \varphi)\) is (semi)stable as an orthogonal sheaf, then it is \( \delta \)-semistable as a tensor. We start with a vector space \( W \) and a nondegenerate bilinear form \( Q : W \otimes W \to \mathbb{C} \). Let \((W_\bullet, m_\bullet)\) be a weighted filtration with

\[
(5.7) \quad \mu(Q, W_\bullet, m_\bullet) = 0.
\]

Denote \( r_i = \dim W_i \). Take a basis of \( W \) adapted to the filtration, and let \( \lambda \) be the one-parameter subgroup of \( \text{SL}(W) \) associated to this basis and weights \( m_\bullet \). Let \( \gamma = \sum_{i=1}^t m_i \gamma^{(r_i)} \) as in (1.4). Since \( \mu(Q, W_\bullet, m_\bullet) = 0 \), the limit \( Q' = \lim_{t \to 0} \lambda(t) \cdot Q \) exists, and \( \det Q' = \det Q \). Furthermore, we also have

\[
(5.8) \quad \mu(Q', W_\bullet, m_\bullet) = 0.
\]

Write \( Q \) and \( Q' \) as block matrices

\[
Q = \begin{pmatrix}
Q_{1,1} & Q_{1,2} & \cdots & Q_{1,t+1} \\
Q_{2,1} & Q_{2,2} & \cdots & Q_{2,t+1} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{t+1,1} & Q_{t+1,2} & \cdots & Q_{t+1,t+1}
\end{pmatrix}
\]

\[
Q' = \begin{pmatrix}
Q'_{1,1} & Q'_{1,2} & \cdots & Q'_{1,t+1} \\
Q'_{2,1} & Q'_{2,2} & \cdots & Q'_{2,t+1} \\
\vdots & \vdots & \ddots & \vdots \\
Q'_{t+1,1} & Q'_{t+1,2} & \cdots & Q'_{t+1,t+1}
\end{pmatrix}
\]

\[
\mu(Q, W_\bullet, m_\bullet) = 0.
\]
Note that if \( \gamma_{r_i} + \gamma_{r_j} < 0 \), then \( Q_{i,j} = 0 \) because of (5.7). We have

\[
Q'_{i,j} = \begin{cases} 
0, & \gamma_{r_i} + \gamma_{r_j} < 0 \\
Q_{i,j}, & \gamma_{r_i} + \gamma_{r_j} = 0 \\
0, & \gamma_{r_i} + \gamma_{r_j} > 0 
\end{cases}
\]

The weights \( \gamma_{r_i} + \gamma_{r_j} \) strictly increase with both \( i \) and \( j \). Assume \( Q'_{i,j} \neq 0 \). Then, if \( (a,b) \neq (i, j) \), and either \( a \leq i, \ b \leq j \), or \( a \geq i, \ b \geq j \), we have \( Q'_{a,b} = 0 \). In matrix form:

\[
Q' = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & Q'_{1,t+1} \\
0 & 0 & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & Q'_{t,2} & \cdots & 0 & 0 \\
Q'_{t+1,1} & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

Since \( \det Q' = \det Q \neq 0 \), in each row of \( Q' \) there must be at least one nonzero block (and the same for columns). This, together with (5.10) implies

\[
Q' = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & Q'_{1,t+1} \\
0 & 0 & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & Q'_{t,2} & \cdots & 0 & 0 \\
Q'_{t+1,1} & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

with nonzero blocks in the second diagonal, and zero everywhere else. Since \( Q' \) is nondegenerate, these blocks give isomorphisms for all \( 1 \leq i \leq t + 1 \)

\[
Q'_{i,t+2-i} : W_{t+2-i}/W_{t+1-i} \cong W_{i}/W_{i-1},
\]

and a short calculation then gives \( r_i = r - r_{t+1-i} \). This, together with (5.11), implies that

\[
W_i^\perp = W_{t+1-i}.
\]

Finally (5.9) and (5.11) imply that \( \gamma_{r_i} + \gamma_{r_{t+2-i}} = 0 \) for all \( 1 \leq i \leq t + 1 \). Then, using this and the definition of \( \gamma \),

\[
0 = (\gamma_{r_{t+1}} + \gamma_{r_{t+1-i}}) - (\gamma_{r_i} + \gamma_{r_{t+2-i}}) = r(m_i - m_{t+1-i}).
\]

Let \( (E_\bullet, m_\bullet) \) be a weighted filtration with \( \mu(\varphi, E_\bullet, m_\bullet) = 0 \). We can assume that all subsheaves \( E_i \) are saturated. Apply the previous argument to \( W = E|_x \), the fiber over a point where \( E \) is locally free, and \( Q \) the bilinear form induced by \( \varphi \) on the fiber. We have (5.12), hence it follows that \( E_i^\perp \supset E_{t+1-i} \). Furthermore, as we have just seen \( m_i = m_{t+1-i} \) and \( r_i = r - r_{t+1-i} \) for all \( i \). Hence we can write

\[
\sum_{i=1}^{t} m_i (r P_{E_i} - r_i P) = \sum_{i=1}^{[(t+1)/2]} m_i r (P_{E_i} + P_{E_{t+1-i}} - P) \leq \sum_{i=1}^{[(t+1)/2]} m_i r (P_{E_i} + P_{E_i^\perp} - P) \leq 0,
\]

where the last inequality is given by (5.1).
Let \((E_\bullet, m_\bullet)\) be a weighted filtration with \(\mu(\varphi, E_\bullet, m_\bullet) \neq 0\). By lemma 5.6, it is strictly negative.

We claim that \(\deg(E_i) \leq 0\) for all \(i\). Assume that this is not true. Then there is a saturated subsheaf \(F \subset E\) with \(\deg(F) > 0\). By lemma 5.5(5), \(N = F \cap F^\perp \neq 0\), and \(0 < \deg(F) \leq \deg(N)\). By lemma 5.5(2), \(N\) is saturated, and by lemma 5.5(1), \(\deg(N) = \deg(N^\perp)\). Consider the weighted filtration \(N \subset N^\perp \subset E\) with weights \(m_1 = m_2 = 1\). Since \(N\) is isotropic, \(\mu(\varphi, N \subset N^\perp) = 0\), and since \(\deg(N) = \deg(N^\perp) > 0\), this weighted filtration contradicts (5.1).

Hence, using \(\deg(E_i) \leq 0\) together with \(\delta_1 > 0\),
\[
\left(\sum_{i=1}^t m_i (r P_{E_i} - r_i P)\right) + \mu(\varphi, E_\bullet, m_\bullet) \delta =
\left(\sum_{i=1}^t \frac{r \deg(E_i)}{(n-1)!}\right) + \mu(\varphi, E_\bullet, m_\bullet) \delta_1 t^{n-1} + O(t^{n-2}) < 0
\]
□

**Proposition 5.8.** Assume \(\tau > 0\). An orthogonal sheaf \((E, \varphi)\) is slope-(semi)stable if and only if it is slope-\(\tau\)-(semi)stable as a tensor.

**Proof.** The proof of proposition 5.7, replacing the Hilbert polynomials \(P_F, P_{E_i}, P,\ldots\) by the degrees \(\deg(F), \deg(E_i), d,\ldots\), proves that \((E, \varphi)\) is slope-\(\tau\)-(semi)stable if and only if for all isotropic subsheaves \(F \subset E\),
\[
\deg(F) + \deg(F^\perp) \leq \deg(E).
\]
We can assume that \(F\) is saturated, hence \(\deg(F) = \deg(F^\perp)\) by lemma 5.5(1), and since \(\deg(E) = 0\), the result follows. □

Fix a polynomial \(P\). Recall that \(\mathcal{M}_O(r)\) is the functor of families of semistable orthogonal sheaves. Define \(\mathcal{M}_O(r)\) to be the subscheme of the moduli space of \(\delta\)-semistable tensors corresponding to orthogonal sheaves with Hilbert polynomial \(P\). The notion of S-equivalence for orthogonal sheaves is the same that was described in proposition 4.1.

**Theorem 5.9.** The scheme \(\mathcal{M}_{O(r)}\) is a coarse moduli space of S-equivalence classes of semistable orthogonal sheaves. There is an open subscheme \(\mathcal{M}_{O(r)}^0\) corresponding to semistable orthogonal bundles.

**Proof.** The proof that \(\mathcal{M}_{O(r)}\) corepresents the functor \(\mathcal{M}_{O(r)}\) is completely analogous to the proof of theorem 1.8 (see section 4), so we will not repeat it. The subscheme \(\mathcal{M}_{O(r)}^0\) is open because being locally free is an open condition.

Now we will prove that this moduli space is projective. Conditions (OS1) and (OS2) are closed conditions, so they define a projective subscheme \(\mathcal{M}_{1,2}\) of the moduli space of \(\delta\)-semistable tensors. The lemma will be proved by showing that \(\mathcal{M}_{O(r)} = \mathcal{M}_{1,2}\). If \((E, \varphi)\) is \(\delta\)-semistable then \(E\) is torsion free, so it only remains to check that if condition (OS4) does not hold, then \((E, \varphi)\) is \(\delta\)-unstable.

Assume that the homomorphism \(\det E \rightarrow \det E^\vee\) induced by \(\varphi\) is zero. Then the sheaf \(E^\perp\) defined as
\[
0 \rightarrow E^\perp \rightarrow E \xrightarrow{\varphi} E^\vee
\]
is nonzero. Let $C$ be the cokernel of $\nabla$

$$E \xrightarrow{\nabla} E^\vee \rightarrow C \rightarrow 0.$$ 

Taking the dual of this sequence and restricting to the open subset $U$ of $X$ where $E$ is locally free, we get

$$0 \rightarrow C^\vee|_U \rightarrow E^\vee|_U = E|_U \xrightarrow{\nabla|_U} E^\vee|_U.$$ 

By (OS2) we have $\nabla|_U = \nabla^\vee|_U$, hence $\ker(\nabla|_U) \cong \ker(\nabla^\vee|_U)$, and then $C^\vee|_U \cong E^\perp|_U$, and since $\text{codim}(X - U) \geq 2$, $\deg(C) = -\deg(E^\perp)$. The exact sequence on $U$

$$0 \rightarrow E^\perp|_U \rightarrow E|_U \rightarrow E^\vee|_U \rightarrow C|_U \rightarrow 0$$

implies that $\deg(E^\perp) = 0$. Consider the weighted filtration $0 \subset E^\perp \subset E$, $m_1 = 1$. We have

$$\mu(\varphi, E_\bullet, m_\bullet) > 0.$$ 

Recall that $\tau = \delta_1 (n - 1)!$. Then

$$r \deg(E^\perp) - \text{rk}(E^\perp) \deg(E) + \mu(\varphi, E_\bullet, m_\bullet)\tau = \mu(\varphi, E_\bullet, m_\bullet)\tau > 0,$$

and hence $(E, \varphi)$ is slope-$\tau$-unstable (definition 1.5), and in particular, $\delta$-unstable.

\[\square\]

**Remark 5.10.** The same proof gives that if $(E, \varphi)$ is a slope-$\tau$-semistable tensor with $\tau > 0$, satisfying conditions (OS1), (OS2) and (OS3), then condition (OS4) holds.

**Example.** We will give an example showing that, if we do not require $\delta_1$ to be positive, the notion of $\delta$-stability as a tensor (definition 1.3) is different from the notion of stability as an orthogonal sheaf (definition 5.4). We will check this by showing an example of an orthogonal sheaf whose $\delta$-stability really depends on $\delta$.

Let $X = \mathbb{P}^2$, let $p_1, p_2, p_3$ be three different points in $\mathbb{P}^2$, and consider the ideal sheaves $I_1 = I_{p_1}$ and $I_2 = I_{p_2 \cup p_3}$. Let

$$(E, \varphi) = \left( I_2 \oplus I_1 \oplus \mathcal{O}_X, \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \right)$$

In particular, the first summand $I_2$ of $E$ is isotropic, and $I_2^\perp = I_2 \oplus I_1$. Let $\delta = \delta_1 t + \delta_2 \in \mathbb{Q}[t]$ be a polynomial as in (1.2).

**Lemma 5.11.** If $\delta_1 = 0$ and $0 < \delta_2 < 3/2$, then $(E, \varphi)$ is $\delta$-unstable as a tensor. If $\delta_1 > 0$, then $(E, \varphi)$ is $\delta$-stable as a tensor.

**Proof.** The first claim is proved by considering the filtration $\mathcal{O}_X \subset I_2 \oplus I_1 \oplus \mathcal{O}_X$. If $\delta_1 = 0$, then this filtration does not satisfy (1.6), hence contradicts semistability.

Now we will prove that if $\delta_1 > 0$, then $(E, \varphi)$ is $\delta$-stable. Using proposition 5.7, we only have to study filtrations of the form

$$E_1 \subset E_2 \subset E^\perp \subset E,$$

with $\text{rk}(E_1) = 1$, $\text{rk}(E_2) = 2$ and $E_1$ isotropic and saturated. Using the Riemann-Roch formula we have

$$P_E_1 + P_E_2 - P = -2c_2(E_1) - 2c_2(E_2) + 2c_2(E),$$

so we need to estimate the second Chern classes of $E_1$ and $E_2$. 
The sheaf $E_2 = E_1^\perp$ is saturated (see the proof of lemma 5.5(2)). Define the torsion free rank one subsheaf $J$

$$0 \rightarrow E_2 \rightarrow I_2 \oplus I_1 \oplus O_X \stackrel{(a,b,c)}{\rightarrow} J \rightarrow 0,$$

where $a$, $b$ and $c$ are respectively elements of $\text{Hom}(I_2, J)$, $\text{Hom}(I_1, J)$ and $\text{Hom}(O_X, J)$. We have $\deg(J) = 0$, so $J$ is the ideal sheaf of a zero-dimensional subscheme of $\mathbb{P}^2$.

We distinguish several cases:

- If $c \neq 0$, then $J = O_X$, and $c_2(E) = 3$.
- If $a \neq 0$, $b \neq 0$, then again $J = O_X$, and $c_2(E) = 3$.
- If $a = 0$, $b \neq 0$ and $c = 0$, then $E_2$ does not contain a subsheaf $E_1$ with $E_2 \subseteq E_1^\perp$, hence this cannot happen.
- If $a \neq 0$, $b = 0$ and $c = 0$, then again $E_2$ does not contain a subsheaf $E_1$ with $E_2 \subseteq E_1^\perp$, hence this cannot happen.

So we conclude that $c_2(E_2) = 3$. The sheaf $E_1$ is a rank one subsheaf of $I_2 \oplus I_1 \oplus O_X$, hence $c_2(E_1) > 0$ unless $E_1$ is the third summand $O_X$, but this is not possible because the third summand is not isotropic. Putting everything together,

$$P_{E_1} + P_{E_2} - P = -2c_2(E_1) - 2c_2(E_2) + 2c_2(E) < 0,$$

hence $(E, \varphi)$ is $\delta$-stable by proposition 5.7. \qed

**Remark 5.12.** Note that $(E, \varphi)$ is stable as an orthogonal sheaf, but $E$ is Gieseker-unstable as a sheaf.

On the other hand, an orthogonal sheaf $(E, \varphi)$ is slope-semistable if and only if $E$ is slope-semistable as a sheaf. Indeed, if $F$ is a saturated subsheaf of $E$ with $\deg(F) > 0$, then lemma 5.5(5) shows that the isotropic subsheaf $F \cap F^\perp$ is nonzero and has positive degree, hence $(E, \varphi)$ is slope-unstable.

To obtain symplectic sheaves instead of orthogonal sheaves, we only need to take $\varphi$ skewsymmetric instead of symmetric. It follows that $\det E = O_X$ (recall that for orthogonal sheaves we only had $(\det E)^{\otimes 2} = O_X$).

There is a bijection between the set of isomorphism classes of symplectic bundles and principal $\text{Sp}(r)$-bundles. The proof is the same as with orthogonal bundles, noting that the quotient $\text{GL}(r)/\text{Sp}(r)$ is the set of invertible antisymmetric matrices (send $A \in \text{GL}(r)$ to $(TA^{-1}JA^{-1})$, where $J$ is the matrix representing the standard symplectic structure of $\mathbb{C}^r$).

All the results for orthogonal sheaves hold for symplectic sheaves, and in particular there is a coarse moduli space of $S$-equivalence classes of semistable symplectic sheaves.

### 6. Special orthogonal bundles

**Definition 6.1** (Special orthogonal sheaf). A special orthogonal sheaf is a triple

$$(E, \varphi : E \otimes E \rightarrow O_X, \psi : \det E \rightarrow O_X)$$

such that

- (SOS1) $\psi$ is an isomorphism.
- (SOS2) $\varphi$ is symmetric.
- (SOS3) $E$ is torsion free.
- (SOS4) $\varphi$ induces an isomorphism $E|_U \rightarrow E|_U^\vee$ on the open subset $U$ where $E$ is locally free.
• (SOS5) \( \det(\varphi) = \psi^2 \). More precisely, let \( \varphi' : E \to E^\vee \) and \( \psi : O_X \to \det E^\vee \) be the associated homomorphisms. Then we require \( \det(\varphi') = \psi \otimes \psi' \).

An isomorphism of special orthogonal sheaves is a pair \( (f, \lambda) \) such that \( f : E \to E' \) is an isomorphism, \( \lambda \in \mathbb{C}^* \) and the following diagrams commute

\[
\begin{array}{ccc}
E \otimes E & \xrightarrow{f \otimes 2} & E' \otimes E' \\
\varphi & \downarrow & \varphi' \\
O_X & \xrightarrow{\lambda^2} & O_X \\
\end{array}
\quad \quad
\begin{array}{ccc}
det E & \xrightarrow{\det f} & \det E' \\
\psi & \downarrow & \psi' \\
O_X & \xrightarrow{\lambda^r} & O_X \\
\end{array}
\]

It is easy to see that, assuming (SOS1) and (SOS3), condition (SOS4) is equivalent to

• (SOS4′) The induced homomorphism \( \det E \to \det E^\vee \) is nonzero (hence an isomorphism).

Condition (SOS5) is equivalent to

• (SOS5′). Let \( U \) be the open subset where \( E \) is torsion free. For all \( x \in U \), fix a basis of the fiber \( E_x \) of \( E \) on \( x \), Using this basis (and the canonical identification \( O_x = \mathbb{C} \)), \( \varphi \) restricted to \( x \) gives a symmetric matrix \( \varphi(x) \), and \( \psi \) restricted to \( x \) gives a complex number \( \psi(x) \). Then we require \( \det(\varphi(x)) = \psi(x)^2 \).

This definition of special orthogonal sheaf is justified by the following lemma.

**Lemma 6.2.** There is a bijection between the set of isomorphism classes of special orthogonal sheaves with \( E \) locally free and the set of isomorphism classes of principal \( \text{SO}(r) \)-bundles.

**Proof.** Let \( \text{SO}(r) \) act by multiplication on the right on \( \text{GL}(r) \), and consider the quotient \( \text{GL}(r)/\text{SO}(r) \). Let \( A \in \text{GL}(r) \), and let \([A]\) be the class in \( \text{GL}(r)/\text{SO}(r) \). To this class we associate the pair \((T^*A^{-1}, \det(A^{-1}))\). This gives a bijection between the set \( \text{GL}(r)/\text{SO}(r) \) and the set of pairs \((B, \beta)\), where \( B \) is a symmetric invertible matrix and \( \beta \) is a nonzero complex number such that

\[
\det B = \beta^2.
\]

Given a principal \( \text{GL}(r) \)-bundle \( P \) (or equivalently a vector bundle \( E \)), a reduction of structure group to \( \text{SO}(r) \) is a section \( \sigma \) of the associated bundle \( P \times_{\text{GL}(r)} \text{GL}(r)/\text{SO}(r) \), and then this is equivalent to a pair of homomorphisms \((\varphi, \psi)\) as in definition 6.1.

The rest of the proof is analogous to the proof of lemma 5.2.

**Definition 6.3** (Stability). A special orthogonal sheaf \((E, \varphi, \psi)\) is (semi)stable if the associated orthogonal sheaf \((E, \varphi)\) is (semi)stable.

A family of semistable special orthogonal sheaves parametrized by \( T \) is a tuple \((E_T, \varphi_T, \psi_T, N)\) such that \((E_T, \varphi_T, N)\) is a family of semistable orthogonal sheaves (cf. (5.2)), and \( \psi_T : \det E_T \to \pi_T^*L \) is an isomorphism, where \( L \) is a line bundle on \( T \). Two families are isomorphic if there is a pair \((f, \lambda : M \to M')\) where \( f : E_T \to E'_T \) is an isomorphism, \( M^{\otimes 2} \cong N \), \( M'^{\otimes 2} \cong N' \), \( \lambda \) is an isomorphism, and the relative versions of the diagrams (6.1) commute. In this section (theorem 6.7) we will construct the moduli space of semistable special orthogonal sheaves (with fixed Hilbert polynomial).
There is a map between isomorphism classes
\[
\begin{align*}
\{ \text{Special orthogonal sheaves} \} & \quad \xrightarrow{f} \quad \{ \text{Orthogonal sheaves such that } \det E \cong \mathcal{O}_X \} \\
(E, \varphi, \psi) & \quad \mapsto \quad (E, \varphi)
\end{align*}
\]
This map will induce a morphism between the corresponding moduli spaces.

**Lemma 6.4.** Let \((E, \varphi)\) be an orthogonal sheaf such that \(\det E \cong \mathcal{O}_X\).

If \(E\) has an automorphism \(f\) such that \(f \otimes f = \text{id}_{E \otimes E}\) and \(f = - \text{id}_{\det E}\), then the preimage of \((E, \varphi)\) under the map \(f\) consists of exactly one isomorphism class.

If \(E\) does not have such an automorphism, then the preimage consists of exactly two distinct isomorphism classes, represented by two special orthogonal sheaves \((E, \varphi, \psi)\) and \((E, \varphi, -\psi)\), differing in the sign of the isomorphism \(\psi\).

**Proof.** Property (SOS5) implies that to obtain the isomorphism \(\psi\) we have to extract a square root, so we obtain two special orthogonal sheaves \(P = (E, \varphi, \psi)\) and \(P' = (E, \varphi, -\psi)\) mapping to the given orthogonal sheaf. It only remains to check if these two objects are isomorphic or not.

If there is an automorphism \(f : E \to E\) with the above properties, then \((f, 1)\) is an isomorphism between \(P\) and \(P'\).

Conversely, assume that there is an isomorphism \((f, \lambda)\) between \(P\) and \(P'\). Then \(f' = (1/\lambda)f\) is an automorphism of \(E\) with \(f' \otimes f' = \text{id}\) and \(\det f' = - \text{id}\). \(\square\)

**Corollary 6.5.** If \(r\) is odd, there is a bijection between the set of isomorphism classes of special orthogonal sheaves and the set of isomorphism classes of orthogonal sheaves with \(\det E \cong \mathcal{O}_X\).

**Proof.** Apply lemma 6.4 to \(f = - \text{id}_E\) (multiplication by \(-1\)). \(\square\)

In particular, for \(r\) odd, the moduli space of (semi)stable special orthogonal sheaves consists of the components of the moduli space of (semi)stable orthogonal sheaves with trivial determinant. On the other hand, if \(r\) is even and \(E\) is simple, then for each orthogonal sheaf with trivial determinant, we have two nonisomorphic special orthogonal sheaves. From now on we will assume that \(r\) is even.

Fix a Hilbert polynomial \(P\). Let \(m\) be a large integer number as in section 3. Let \(V\) be a vector space of dimension \(P(m)\). Let \((g, E, \varphi, \psi)\) be a tuple where \((E, \varphi, \psi)\) is a semistable special orthogonal sheaf and \(g\) is an isomorphism between \(H^0(E(m))\) and \(V\). As in section 3, the homomorphism \(\varphi\) gives a vector
\[
\Phi \in (V^\otimes 2)^\vee \otimes H^0(\mathcal{O}_X(2m))
\]
We denote \(\Phi_s = \Phi^\otimes r/2\) the associated vector
\[
\Phi_s \in \text{Sym}^{r/2} \left( (V^\otimes 2)^\vee \otimes H^0(\mathcal{O}_X(2m)) \right).
\]
The homomorphism \(\psi\) induces a linear map
\[
\Psi : \bigwedge^r V \longrightarrow H^0(\det(E)(rm)) \longrightarrow H^0(\mathcal{O}_X(rm)),
\]
and hence a vector (that we denote with the same letter)
\[
\Psi \in \left( \bigwedge^r V \right)^\vee \otimes H^0(\mathcal{O}_X(rm)).
\]
These two quotients give a point \([\Phi_s, \Psi]\) in the projective space \(\tilde{P}\) defined as
\[
\mathbb{P} \left( \text{Sym}^{r/2} \left( (V^\otimes 2)^\vee \otimes H^0(\mathcal{O}_X(2m)) \right) \oplus \left( \bigwedge^r V \right)^\vee \otimes H^0(\mathcal{O}_X(rm)) \right).
\]
It is easy to check that the point only depends on the isomorphism class of the tuple. Here it is crucial that we took the $r/2$-symmetric power in (6.2): take the isomorphism $\lambda \text{id} : E \to E$ (multiplication by $\lambda \in \mathbb{C}^*$). It sends $\Phi$ to $\lambda^2 \Phi$, and $\Psi$ to $\lambda^r \Psi$, hence it sends $[\Phi_s, \Psi]$ to $[\lambda^r \Phi_s, \lambda^r \Psi]$, and this is the same point in the projective space.

Let $H$ be the Hilbert scheme of quotients as in section 3, and then given a tuple $(g, E, \varphi, \psi)$ we associate a point $(q, [\Phi_s, \Psi])$ in $H \times \tilde{P}$. The points obtained in this way have the following properties: the vector $\Phi_s$ is of the form $\Phi \otimes r/2$, $\Phi$ factors as $V \otimes 2 \otimes O_X(-2m)$ $\Phi_H 0(O_X(2m)) \otimes O_X(-2m)$ $E \otimes 2 \psi O_X$, the homomorphism $\Psi$ factors as $\wedge^r V \otimes O_X(-rm)$ $\Psi H 0(O_X(rm)) \otimes O_X(-rm)$ $\det E \psi O_X$, and $\det(\varphi) = \psi^2$ as in (SOS5).

Let $Z'$ be the closed subset of $H \times \tilde{P}$ defined by these properties. Given a point $z \in Z'$ we can recover the tuple up to isomorphism. Define the parameter space $\tilde{Z}$ as the closure in $Z'$ of those points obtained from semistable special orthogonal sheaves.

Let $\delta$ be a polynomial as in (1.2) and with $\delta_1 > 0$. Define a polarization on $\tilde{Z}$ by

$$O_{\tilde{Z}}(n_1, n_2) := p_H^*O_H(n_1) \otimes p_P^*O_P(\frac{2n_2}{r})$$

where $n_2$ is a multiple of $r/2$, $n_1$ is an integer, and

$$\frac{n_2}{n_1} = \frac{P(l)\delta(m) - \delta(l)P(m)}{P(m) - 2\delta(m)}$$

The projective scheme $\tilde{Z}$ is preserved by the natural SL($V$) action, and this action has a natural linearization on $O_{\tilde{Z}}(n_1, n_2)$.

**Proposition 6.6.** A point $(g, E, \varphi, \psi)$ is GIT-(semi)stable if and only if the special orthogonal sheaf $(E, \varphi, \psi)$ is (semi)stable (definition 6.3).

**Proof.** The parameter space $Z$ for orthogonal sheaves is a subscheme of $H \times P$, where

$$P = \mathbb{P}(V \otimes H^0(O_X(-2m)))$$

(this is a particular case of the parameter space defined in section 3). Let $O_{\tilde{Z}}(n_1, n_2)$ be the polarization defined in (3.7), and consider the natural linearization of the action of SL($V$) on this polarization. There is a morphism

$$\tilde{Z} \xrightarrow{f} Z$$

$$(g, E, \varphi, \psi) \mapsto (g, E, \varphi)$$

with $f^*O_{\tilde{Z}}(n_1, n_2) = O_{\tilde{Z}}(n_1, n_2)$. This morphism is equivariant with respect to SL($V$), and the linearizations are compatible. Property (SOS5') implies that $f$ is
finite étale (because $\tilde{Z}$ is given locally by the equation $\det(\varphi(x)) = \psi(x)^2$), and then it follows that a point in $\tilde{Z}$ is GIT-(semi)stable if and only if its image in $Z$ is GIT-(semi)stable. The result follows from theorem 3.6, proposition 5.7, and definition 6.3. □

Let $\mathcal{M}_{SO(r)}$ be the functor of families of semistable special orthogonal sheaves. Let $\mathcal{M}_{SO(r)}$ be the GIT quotient of $\tilde{Z}$ by $\text{SL}(V)$. Let $(E, \varphi, \psi)$ be a semistable special orthogonal sheaf. Let $E^S$ and $\varphi^S$ be defined as in section 4. There is a natural isomorphism between $\det E^S$ and $\det E$, then composing with $\psi$ we obtain an isomorphism $\psi^S: \det E^S \rightarrow \mathcal{O}_X$.

Let $(E, \varphi, \psi)$ and $(E', \varphi', \psi')$ be two semistable special orthogonal sheaves. They are $S$-equivalent if and only if $(E^S, \varphi^S, \psi^S)$ is isomorphic to $(E'^S, \varphi'^S, \psi'^S)$.

**Theorem 6.7.** The projective scheme $\mathcal{M}_{SO(r)}$ is the coarse moduli space of $S$-equivalence classes of semistable special orthogonal sheaves. There is an open subset $\mathcal{M}_{SO(r)}^0$ corresponding to semistable special orthogonal bundles.

The proof is completely analogous to the proof of theorem 1.8 (section 4).

### 7. GL($r$)-Representational Pairs

Once we have constructed the moduli space of tensors, it is easy to obtain moduli spaces for GL($r$)-representational pairs. In the case of $\dim(X) = 1$, this is done in [Sch], but since it does not depend on the dimension of the base $X$, the same arguments apply here. In [Ba], Banfield considered pairs $(P, \sigma)$, where $P$ is a principal $G$-bundle ($G$ any reductive group), and $\sigma$ is a section associated to $P$ by a fixed representation $\rho$. He defined stability, and proved a Hitchin-Kobayashi correspondence. Now we will construct the moduli space, when $G = \text{GL}(r)$.

Fix a polynomial $\delta$ as in (1.2). Let $\rho: \text{GL}(r) \rightarrow \text{GL}(n)$ be a representation sending the center of $\text{GL}(r)$ to the center of $\text{GL}(n)$. Consider a triple

$$(7.1) \quad (E, \psi: E_\rho \rightarrow D_u, u),$$

where $E$ is a vector bundle of rank $r$ on $X$, and $E_\rho$ is the vector bundle of rank $n$ associated to $E$ and $\rho$. Using [F-H, prop. 15.47], it can be shown that there exist integers $s > 0, b, c > 0$, and a vector bundle $F$ such that

$$(E^\otimes s)^c \otimes (\det E)^{-c_b} \cong E_\rho \oplus F$$

(see [Sch, cor 1.1.2] for details). Then a triple (7.1) is equivalent to a tensor $(E, \varphi, u)$ such that

$$(7.2) \quad \varphi|_F = 0,$$

and we say that the triple is $\delta$-(semi)stable if the corresponding tensor is. Since the condition (7.2) is closed, the moduli space of $\delta$-semistable triples is a closed subscheme of $\mathcal{M}_{\delta}^0$, the open subscheme corresponding to tensors with $E$ locally free. It is easy to check that the definition of stability in [Ba] coincides with our slope-$\tau$-stability.

In [MR], Mundet generalized Banfield’s work. He fixes a Kaehler manifold $Y$ and an action $\rho$ of a reductive group $G$ on $Y$, and considers pairs $(P, \sigma)$, where $\sigma$ is a section of the associated fiber bundle $P \times_G Y$. He defined stability and proved a Hitchin-Kobayashi correspondence. Now we will construct the moduli space, for the
case when \( G = \text{GL}(r) \), and \( Y \) is a projective (or more generally, quasi-projective) scheme.

Consider an action of \( \text{GL}(r) \) on a projective scheme \( Y \), together with a linearization of the action on an ample line bundle \( L \) on \( Y \). Assume that the center of \( \text{GL}(r) \) acts trivially on \( Y \). Consider a pair

\[
( P, \sigma : X \to P \times_{\text{GL}(r)} Y ),
\]

where \( P \) is a principal \( \text{GL}(r) \) bundle on \( X \), and \( \sigma \) is a section of the fiber bundle associated to \( P \) with fiber \( Y \). We fix the topology type of \( P \) and the homology class \([\sigma(X)]\) of the image. Fix \( k \) large enough so that we have a natural embedding \( F \hookrightarrow \mathbb{P}(H^0(F, L \otimes k)) \). Since the action of the center of \( \text{GL}(r) \) is trivial on \( Y \), the induced representation

\[
\rho : \text{GL}(r) \to \text{GL}(H^0(F, L \otimes k))
\]

sends the center of \( \text{GL}(r) \) to the center of \( \text{GL}(H^0(F, L \otimes k)) \). Let \( E \) be the rank \( r \) vector bundle corresponding to \( P \). Since we have fixed the topology type of \( P \), the Hilbert polynomial \( P_E \) of \( E \) is also fixed. The section \( \sigma \) gives a homomorphism

\[
\psi : E_\rho \to D_a,
\]

for some line bundle \( D_a \), whose degree \( a \) depends on the homology class \([\sigma(X)]\) of the image. Take \( R = \text{Pic}^a(X) \), and let \( D \) be a Poincare bundle. We obtain that a pair (7.3) is equivalent to a triple (7.1) with the property that the section \( \psi' : X \to \mathbb{P}(E_\rho) \) factors through \( P \times_{\text{GL}(r)} Y \). We define a pair (7.3) to be \( \delta \)-semistable if the corresponding triple is, and hence the moduli space of \( \delta \)-semistable pairs (7.3) is a closed subscheme of \( \mathcal{M}_0^\delta \). We can also take \( Y \) to be quasi-projective, and the moduli space will also be a subscheme (not necessarily closed) of \( \mathcal{M}_0^\delta \).

References

[Ba] D. Banfield, *Stable pairs and principal bundles*, Quarterly J. Math., 51 (2000), 417–436.

[Fa] G. Faltings, *Moduli-stacks for bundles on semistable curves*, Math. Ann. 304 (1996), 489–515.

[F-H] W. Fulton and J. Harris, Representation theory: a first course, Grad. Texts in Math. 129 Springer Verlag, 1991.

[Gi] D. Gieseker, *On the moduli of vector bundles on an algebraic surface*, Ann. Math., 106 (1977), 45–60.

[G-S] T. Gómez and I. Sols, *Stability of conic bundles*, Internat. J. Math., 11 (2000), 1027–1055.

[Gr] A. Grothendieck, *Techniques de construction et théorèmes d’existence en géométrie algébrique IV: Les schémas de Hilbert*, Séminaire Bourbaki, 1960/1961 221.

[Ha] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. 52, Springer Verlag, 1977.

[H-L1] D. Huybrechts and M. Lehn, *Stable pairs on curves and surfaces*, J. Algebraic Geom., 4 (1995), 67–104.

[H-L2] D. Huybrechts and M. Lehn, *Framed modules and their moduli*, Internat. J. Math., 6 (1995), 297–324.

[Ma] M. Maruyama, *On boundedness of families of torsion free sheaves*, J. Math. Kyoto Univ., 21 (1981), 673–701.

[MR] I. Mundet i Riera, *A Hitchin-Kobayashi correspondence for Kaehler fibrations*, J. Reine Angew. Math., 528 (2000), 41–80.

[Ma] D. Mumford, *Abelian varieties*. Oxford University Press, Bombay, 1970.

[Ra] A. Ramanathan, *Moduli for principal bundles over algebraic curves: I and II*, Proc. Indian Acad. Sci. (Math. Sci.), 106 (1996), 301–328, and 421–449.

[Sch] A. Schmitt, *A universal construction for moduli spaces of decorated vector bundles over curves*, preprint 2000, math.AG/0006029.
[Si] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I*, Publ. Math. I.H.E.S. **79** (1994), 47–129.

[So] C. Sorger, *Thêta-charactéristiques des courbes tracées sur une surface lisse*, J. reine angew. Math., **435** (1993), 83–118.

T. Gómez, School of Mathematics, Tata Institute of Fundamental Research, Mumbai 400 005 (India) (current address: Universidad Complutense de Madrid)

*E-mail address: tomas@math.tifr.res.in, tgomez@alg.mat.ucm.es*

I. Sols, Departamento de Algebra, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid (Spain)

*E-mail address: sols@mat.ucm.es*