Absorption and scattering by a self-dual black hole

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Abstract
In this paper we investigate the process of massless scalar wave scattering due to a self-dual black hole through the partial wave method. We calculate the phase shift analytically at the low energy limit and we show that the dominant term of the differential scattering cross section at the small angle limit is modified by the presence of parameters related to the polymeric function and minimum area of a self-dual black hole. We also find that the result for the absorption cross section is given by the event horizon area of the self-dual black hole at the low frequency limit. We also show that, contrarily to the case of a Schwarzschild black hole, the differential scattering/absorption cross section of a self-dual black hole is nonzero at the zero mass limit. In addition, we verify these results by numerically solving the radial equation for arbitrary frequencies.

Keywords Self-dual black holes · Massless scalar wave scattering · Partial wave method · Absorption and scattering cross sections

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1 Introduction

The so-called self-dual black hole corresponds to a simplified model that is obtained through a semiclassical analysis, consisting in symmetry reduced models corresponding to homogeneous spacetimes, of loop quantum gravity [1–5]. Loop quantum gravity is a quantum geometric theory constructed for the purpose of reconciling general relativity and quantum mechanics on the Planck scale. This theory is derived from the canonical quantization procedure of the Einstein equations obtained in terms of the Ashtekar variables [6]. The metric of the self-dual black hole with quantum gravity corrections has been found in [7,8]. This metric is characterized by its dependence on parameters $P$ and $a_0$, where $P$ is referred to as the polymeric parameter and $a_0$ as a minimum area in loop quantum gravity. It also features an event horizon and a Cauchy horizon. In addition, the condition of self-duality has the property of removing the singularity and replace it with another asymptotically flat region at distance $a_0/r$ as the radial coordinate $r \approx 0$ [8]. The Schwarzschild black hole solution is recovered in the limit as $P$ and $a_0$ go to zero. An analysis on the evaporation of self-dual black holes has been performed in [9–12]. By employing the tunneling formalism via the Hamilton-Jacobi method the thermodynamics of a self-dual black hole has been analyzed in [13]. In addition, the authors in [14], by applying this same method investigated the effect of the generalized uncertainty principle on the thermodynamic properties of the self-dual black hole. Studies on quasinormal modes of self-dual black holes using the WKB approximation have been performed in [15,16]. In [17] the authors have discussed the gravitational lensing effect of a self-dual black hole.

In the present work we have the main purpose of exploring the effect of quantum gravity corrections that contribute to the process of massless scalar wave scattering by a self-dual black hole. In this analysis we will make use of the partial wave method to calculate the absorption and differential scattering cross section. For this we will apply the technique implemented in [19] in order to determine the phase shift analytically at the low frequency limit ($m \omega \ll 1$). This technique has also been considered in [20] to examine the problem of scalar wave scattering by a noncommutative black hole. The partial wave approach has been widely applied in the black hole scattering process by considering various types of metrics [21–48]. This approach has also been applied to exploit scalar wave scattering by acoustic black holes [49–53,55–58] and also in [59] the differential scattering cross section of a noncommutative BTZ black hole has been obtained. In our analysis we have initially calculated the phase shift analytically and then computed the differential scattering cross section and the absorption. We find that the results obtained for the absorption/differential scattering cross section have their values increased when we vary the values of the polymeric parameter $P$ at the low frequency limit. In addition, we show that at the mass limit going to zero the absorption and differential scattering cross section tends to nonzero values that are proportional to
the minimum area $a_0$ of the self-dual black hole. Furthermore, we also have extended our computations for high energy regime by numerically solving the radial equation for arbitrary frequencies. Moreover, we also analytically explore the high frequency limit using the null geodesics method in order to check the validity of the numerical calculation. We show that the numerical results have a good agreement with the results obtained analytically in the low frequency limit. These investigations are particularly interesting as we combine these studies with the rate emission of particles of energy $v$, $R_{BH}(v, m_0)$, by ultra-light self-dual black holes with mass $m_0 \approx 10^{24}$eV today. The total absorption cross section that enters as a grey body factor that modifies the Planck law, i.e.,

$$R_{BH}(v, m_0) \sim \frac{8\pi v^2}{e^{v/T_H(m_0)} - 1} \times \sigma_{abs}(v, a_0),$$

may affect the estimated bounds for the parameters, such as the equilibrium temperature $T_{eq}$, for the ultra high energy cosmic rays produced by usual rate emission of particles by self-dual black holes [8].

The paper is organized as follows. In Sect. 2 we derive the phase shift and calculate the absorption/differential scattering cross section for a self-dual black hole by considering analytical and numerical analysis. In Sect. 3 we investigate the null geodesics to analyze the high frequency limit. In Sect. 4 we make our final considerations. We shall adopt the natural units $\hbar = c = k_B = G = 1$.

## 2 Self-dual black hole

In this section we introduce the self-dual black hole for the purpose of determining the differential scattering cross section and absorption for this model. Thus, we adopt the partial wave method to calculate the phase shift at the low energy limit. The spherically symmetrical self-dual black hole is described by the following line element

$$ds^2 = F(r)dt^2 - \frac{dr^2}{N(r)} - \rho^2(r) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).$$

The functions $F(r)$, $N(r)$ and $\rho(r)$ are given respectively by

$$F(r) = \frac{(r - r_+)(r - r_-)(r + r_+)^2}{r^4 + a_0^2},$$

$$N(r) = \frac{(r - r_+)(r - r_-)r^4}{(r + r_+)^2(r^4 + a_0^2)},$$

$$\rho(r) = r \sqrt{1 + \frac{a_0^2}{r^4}},$$
where $\rho$ denotes an effective radial coordinate that stands for the radius of the 2-sphere [8]. Here the event horizon, $r_+$, and the Cauchy horizon, $r_-$, read

$$r_+ = 2m, \quad r_- = 2m P^2.$$  (6)

We also define

$$r_* \equiv \sqrt{r_+ r_-} = 2m P,$$  (7)

where $P$ is the polymeric dimensionless function given by

$$P = \frac{\sqrt{1 + \epsilon^2} - 1}{\sqrt{1 + \epsilon^2} + 1},$$  (8)

with $\epsilon = \gamma \delta$, where $\gamma$ is the Barbero-Immirzi parameter and $\delta$ is the polymeric parameter. In addition, we have

$$a_0 = A_{\text{min}}/8\pi,$$  (9)

which is the area gap and $A_{\text{min}}$ is a minimum area in loop quantum gravity.

Finally, the value of $\rho$ that is related to the event horizon $r_+$ is given by the following relation:

$$\rho_h = \rho(r_+) = r_+ \sqrt{1 + \frac{a_0^2}{r_+^4}} = 2m \sqrt{1 + \frac{a_0^2}{(2m)^4}}.$$  (10)

### 2.1 Absorption and differential scattering cross section

One can interpret the black hole solution (2) as a solution coming from an effective matter fluid that simulates loop quantum gravity. The effective theory for gravity coupled to matter is then described by the Einstein equations

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu},$$  (11)

where $T_{\mu\nu} = (\rho, -P_r, -P_\theta, -P_\phi)$. At the semiclassical limit restricted to zeroth order in $\delta_b$ and $a_0$ one recovers the Schwarzschild solution that satisfies $G_{\mu\nu} = 8\pi T_{\mu\nu} = 0$. At higher order one captures the quantum corrections and $T_{\mu\nu} \neq 0$. For further discussions and explicit components of the corrected energy momentum-tensor see [8]—and references therein. Then, we can consider several types of matter fields that can be consistent with the aforementioned effective theory. Indeed scalar [15] and tensorial [16] perturbations have been already addressed recently in the context of quasinormal modes in this context.
We now consider the case of the massless scalar field equation to describe the scattered wave in the background (2), given by

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \Psi \right) = 0. \tag{12}
\]

We shall now apply a separation of variables into the equation above

\[
\Psi_{olm}(r, t) = \frac{\mathcal{R}_{ol}(r)}{\rho(r)} Y_{lm}(\theta, \phi) e^{-i\omega t}, \tag{13}
\]

where \(Y_{lm}(\theta, \phi)\) are the spherical harmonics and \(\omega\) is the frequency.

Thus, we obtain the following radial equation for \(\mathcal{R}_{ol}(r)\)

\[
\Lambda(r) \frac{d}{dr} \left( \Lambda(r) \frac{d\mathcal{R}_{ol}(r)}{dr} \right) + \left[ \omega^2 - V_{\text{eff}} \right] \mathcal{R}_{ol}(r) = 0, \tag{14}
\]

for \(\Lambda(r) = \sqrt{F(r)N(r)}\) and \(V_{\text{eff}}\) being the effective potential, given by

\[
V_{\text{eff}} = \frac{\Lambda(r)}{\rho(r)} \left[ \rho'(r) \frac{d\Lambda(r)}{dr} + \Lambda(r) \rho''(r) \right] + \frac{F(r)l(l+1)}{\rho^2(r)}, \tag{15}
\]

with

\[
\rho'(r) = \frac{\rho(r)}{r} \left( 1 - \frac{2a_0^2}{\rho^2(r)r^2} \right), \quad \rho''(r) = \frac{6a_0^2}{\rho(r)r^4} \left( 1 - \frac{2a_0^2}{3\rho^2(r)r^2} \right). \tag{16}
\]

Next, to write Eq. (14) in the form of a Schrödinger-type equation, we introduce the change of variables, \(\chi(r) = \Lambda^{1/2}(r) \mathcal{R}(r)\), so we have

\[
\frac{d^2 \chi(r)}{dr^2} + V(r) \chi(r) = 0, \tag{17}
\]

where

\[
V(r) = \frac{[\Lambda'(r)]^2}{4\Lambda^2(r)} - \frac{\Lambda''(r)}{2\Lambda(r)} + \frac{\omega^2}{\Lambda^2(r)} - \frac{V_{\text{eff}}}{\Lambda^2(r)} \tag{18}
\]

Now, by applying a power series expansion in \(1/r\) (for large \(r\), in comparison with \(\sqrt{a_0}\)) for the potential \(V(r)\), Eq. (17) becomes

\[
\frac{d^2 \chi(r)}{dr^2} + \left[ \omega^2 + \mathcal{U}_{\text{eff}}(r) \right] \chi(r) = 0, \tag{19}
\]
for the following effective potential
\[
U_{\text{eff}}(r) = \frac{4m(1 + P^2)\omega^2}{r} + \frac{12\ell^2}{r^2} + \frac{4m^2P^2\omega^2}{r^2} \left[ 1 + 3P^2 - \frac{3a_0^2(1 + P^2)}{16m^4P^2} \right] + \ldots,
\]  
(20)

with \(\ell^2\) defined as
\[
\ell^2 = -\frac{(l^2 + l)}{12} + \frac{\rho_0^2\omega^2}{4} \left( 1 + P^2 \right),
\]  
(21)
\[
= -\frac{(l^2 + l)}{12} + m^2\omega^2 \left( 1 + \frac{a_0^2}{16m^4} \right) \left( 1 + P^2 \right),
\]  
(22)
stands for the modification of the \(1/r^2\) term in the effective potential (20) [19,20]. We can observe that the suitable asymptotic behavior is satisfied, that is for \(r \to \infty\) we have \(U_{\text{eff}}(r) \to 0\).

Next combining Eq. (21) with \(l\) modes is reasonable to apply the following Ansatz to compute the phase shift [19,20]
\[
\delta_l \approx 2(l - \ell) = 2 \left( l - \sqrt{-\frac{(l^2 + l)}{12} + \frac{\rho_0^2\omega^2}{4} \left( 1 + P^2 \right)} \right).
\]  
(23)
And so taking the limit \(l \to 0\), we get the phase shift \(\delta_l\) given by
\[
\delta_l = \delta_0 + \delta_{l \geq 1}, \quad \delta_0 = -\omega \rho_0 \left( 1 + P^2 \right)^{1/2} = -2m\omega \left( 1 + \frac{a_0^2}{16m^4} \right)^{1/2} \left( 1 + P^2 \right)^{1/2},
\]  
(24)
Thereupon, knowing \(\delta_l\), we can now determine the differential scattering cross section and absorption. The differential scattering cross section can be determined by the formula
\[
\frac{d\sigma}{d\theta} = \left| f(\theta) \right|^2 = \left| \frac{1}{2i\omega} \sum_{l=0}^{\infty} (2l + 1) \left( e^{2i\delta_l} - 1 \right) P_l(\cos \theta) \right|^2,
\]  
(25)
where \(P_l(\cos \theta)\) are the Legendre polynomials. In order to obtain the differential scattering cross section we will consider the following expression [19,60,61]
\[
\frac{d\sigma}{d\theta} = \left| \frac{1}{2i\omega} \sum_{l=0}^{\infty} a_l P_l(\cos \theta) \right|^2,
\]  
(26)
where

\[
a_l^1 = a_l^0 - \frac{l + 1}{2l + 3} a_{l+1}^0 - \frac{l}{2l - 1} a_{l-1}^0,
\]

(27)

and

\[
a_l^0 = (2l + 1) \left( e^{2i\delta_l} - 1 \right).
\]

(28)

By applying the result of (24) the last two terms in (27) are null and so we can rewrite Eq. (26) as follows

\[
\frac{d\sigma}{d\theta} = \left| \frac{1}{2i\omega} \sum_{l=0}^{\infty} a_l^0 \frac{P_l(\cos \theta)}{1 - \cos \theta} \right|^2 = \left| \frac{1}{2i\omega} \sum_{l=0}^{\infty} (2l + 1) \left( e^{2i\delta_l} - 1 \right) \frac{P_l(\cos \theta)}{1 - \cos \theta} \right|^2.
\]

(29)

Now considering the small angle limit the above equation is rewritten as

\[
\frac{d\sigma}{d\theta} = \frac{4}{\omega^2 \theta^4} \left| \sum_{l=0}^{\infty} (2l + 1) e^{i\delta_l} \sin(\delta_l) P_l(\cos \theta) \right|^2,
\]

(30)

\[
= \frac{4}{\omega^2 \theta^4} \left| e^{i\delta_0} \sin(\delta_0) P_0(\cos \theta) + \sum_{l=1}^{\infty} (2l + 1) e^{i\delta_{l\geq1}} \sin(\delta_{l\geq1}) P_l(\cos \theta) \right|^2.
\]

(31)

Therefore, by applying the result of (24), Eq. (31) at the low frequency limit becomes

\[
\left. \frac{d\sigma}{d\theta} \right|_{\omega \to 0} = \frac{4}{\omega^2 \theta^4} b_0^2 = \frac{4}{\theta^4} \theta_0^2 \left( 1 + P^2 \right) = \frac{16m^2}{\theta^4} \left( 1 + \frac{a_0^2}{16m^2} \right) \left( 1 + P^2 \right) + \cdots.
\]

(32)

For \( P = 0 \) and \( a_0 = 0 \) we obtain the result for the Schwarzschild black hole case. Thus, we find that the differential scattering cross section of the self-dual black hole increases when the parameters \( P \) and/or \( a_0 \) increase. By comparing with the result of the Schwarzschild black hole the dominant term is modified by the polymeric parameter \( P \) and parameter \( a_0 \). At the limit of \( m \to 0 \) the dominant term of Eq. (32) becomes nonzero and is given by

\[
\left. \frac{d\sigma}{d\theta} \right|_{m \to 0} \approx \frac{a_0^2 \left( 1 + P^2 \right)}{\theta^4 m^2} = \frac{A_{\text{min}}^2 \left( 1 + P^2 \right)}{4\pi \theta^4 A_{\text{schwbh}}},
\]

(33)

where \( A_{\text{schwbh}} = 4\pi r_+^2 = 16\pi m^2 \) is the area of the Schwarzschild black hole and \( A_{\text{min}} \) is the minimal value of area in loop quantum gravity. Therefore, at this limit the
differential cross section is directly proportional to the minimum area \( A_{\text{min}} = 8\pi a_0 \) and inversely proportional to the area \( A_{\text{schwbh}} \) of the Schwarzschild black hole.

Now we will determine the absorption cross section for a self-dual black hole at the low frequency limit. Hence the total absorption cross section can be found through the following formula:

\[
\sigma_{\text{abs}} = \frac{\pi}{\omega^2} \sum_{l=0}^{\infty} (2l + 1) \left( 1 - |e^{2i\delta_l}|^2 \right),
\]

and so from quantum mechanics we have

\[
\sigma_{\text{abs}} = \frac{4\pi}{\omega^2} \sum_{l=0}^{\infty} (2l + 1) \sin^2(\delta_l),
\]

\[
= \frac{4\pi}{\omega^2} \left[ \sin^2(\delta_0) + \sum_{l=1}^{\infty} (2l + 1) \sin^2(\delta_{l\geq 1}) \right].
\]

Next we apply the low energy limit by taking \( \omega \to 0 \). In this case with \( \delta_l \) given by (24) the absorption reads

\[
\sigma_{\text{abs}}^{\text{lf}} = \frac{4\pi \delta_0^2}{\omega^2} = 4\pi \rho_h^2 \left( 1 + P^2 \right) = 16\pi m^2 \left( 1 + \frac{a_0^2}{16m^4} \right) \left( 1 + P^2 \right),
\]

\[
= A_{\text{schwbh}} \left( 1 + \frac{16\pi^2 a_0^2}{A_{\text{schwbh}}^2} \right) \left( 1 + P^2 \right).
\]

By making \( P = a_0 = 0 \) the result for the absorption of the Schwarzschild black hole is recovered. Note that when the parameters \( P \) and/or \( a_0 \) increase, the absorption cross section value increases. Thus, the presence of \( P^2 \) indicates that the absorption amplitude for the mode \( l = 0 \) will be increased as can be verified by the numerical calculation shown in Fig. 1. Furthermore, we can observe that at the zero mass limit the absorption is nonzero due to the contribution of the minimum area \( a_0 \). So for \( m \to 0 \) Eq. (37) becomes

\[
\sigma_{\text{abs}}^{\text{lf}} \approx \frac{\pi a_0^2}{m^2} \left( 1 + P^2 \right) = \frac{16\pi^2 a_0^2}{A_{\text{schwbh}}^2} \left( 1 + P^2 \right) = \frac{A_{\text{min}}^2}{4A_{\text{schwbh}}^2} \left( 1 + P^2 \right).
\]

It is interesting to note that, contrarily to the usual case, i.e., the Schwarzschild black hole, the absorption/differential scattering cross section of a self-dual black hole is different from zero as the mass goes to zero. Thus, at this limit the result for absorption cross section increases with the minimum area \( A_{\text{min}} \) and decreases with the area of the Schwarzschild black hole \( A_{\text{schwbh}} \). It is worth mentioning that in [58] we have considered an extended Abelian Higgs model with higher order derivatives terms from which an acoustic metric in 2+1 dimensions has been obtained. The absorption in such a context follows an analogous behavior, i.e.,
Fig. 1  Partial absorption cross section for $l = 0$, with $a_0 = \sqrt{3}/2$, $m = 1$ and $P = 0.1, 0.2, 0.3, 0.4$

\[
\sigma_{abs}^{lf} \approx 2\pi \lambda^2 C^2 \frac{D}{1 + 2\lambda^2}, \quad (40)
\]

in the limit of $D \to 0$. Here $\lambda$ is the parameter related to the strength of higher order derivatives. $C$ and $D$ are the circulation and draining parameters. This seems to reveal an interesting correspondence between the aforementioned acoustic model in $2 + 1$ dimensions and the present gravitational model in $3 + 1$ dimensions.

Equation (37) can also be rewritten in terms of the event horizon area of the self-dual black hole as follows

\[
\sigma_{abs}^{lf} = 4\pi \rho_h^2 \left(1 + P^2\right) \approx 4\pi \rho_h^2 = A_{sdbh}, \quad (41)
\]

where

\[
\rho_h = \rho(r_+) = \sqrt{4m^2 + \frac{a_0^2}{4m^2}} = 2m \left(1 + \frac{a_0^2}{16m^4}\right)^{1/2}, \quad (42)
\]

is the event horizon and $A_{sdbh} = 4\pi \rho_h^2$ is the area of the event horizon of the self-dual black hole.
2.2 Numerical analyses

Here we present the numerical results that were obtained by numerically solving the radial equation (14). For this purpose we have adopted the numerical procedure performed in [55]. The Table 1 shows the results for some values of \( m \), setting the value of \( a_0 = \sqrt{3}/2 \) and \( P = 0.1 \). All the results are divided by \( \pi \).

Note that the results of the Eqs. (37) and (39) are the same for small \( m \) as it should be, since Eq. (39) is valid only in the limit of very small values of \( m \). As can be seen in the Table 1, we get a good agreement between them for \( m = 0.01 \).

We can summarize the results as follows. In Fig. 1, we have plotted the partial absorption for mode \( l = 0 \) with \( m = 1 \) and \( a_0 = \sqrt{3}/2 \) and by adopting the following values for the polymeric parameter: \( P = 0.1, 0.2, 0.3, 0.4 \). Analyzing the curves, we find that the absorption amplitude of the self-dual black hole is increased as we vary the \( P \) polymeric parameter. And also a comparison between the absorption of the Schwarzschild black hole with that of the self-dual black hole is shown in Fig. 1. The graph for the absorption of the former corresponds to the case where \( a_0 = 0 \) and \( P = 0 \). Note that by setting \( a_0 = \sqrt{3}/2 \) and varying \( P \), the partial absorption for the \( l = 0 \) mode has its amplitude increased compared to that of the Schwarzschild black hole. In Fig. 2 we plot the partial absorption for \( l = 0 \) mode by setting the values of \( a_0 = \sqrt{3}/2 \) and \( P = 0.1 \). We can observe that when we decrease the mass value the absorption amplitude does not tend to zero, as seen earlier from the analytical result shown in Eq. (39). In Fig. 3 we plot the partial absorption for \( l = 0 \) mode by setting the values of \( a_0 = \sqrt{3}/2 \) and small mass \( m = 0.4 \). We observe that by varying the \( P \) parameter the absorption amplitude still increases. The partial absorption cross section graphs for \( l = 0, 1, 2, 3 \) modes are shown in Fig. 4. Thus, we have assigned values for the \( P \) polymeric parameter but keeping the values of \( a_0 = \sqrt{3}/2 \) and \( m = 1 \). Then, as \( P \) increases, the partial absorption increases in amplitude for the mode \( l = 0 \) and reduces in amplitude for the modes \( l = 1, 2, 3 \). For the modes \( l = 1, l = 2 \) and \( l = 3 \), note that the curves start from zero, increase in amplitude reaching a maximum value and then decrease in amplitude with the increasing of the frequency \( \omega \). It is also observed that the maximum partial absorption decreases as the \( l \) mode increases.

In Fig. 5 we plot the contributions of total absorption. Thus, we see an increase in amplitude when we increased \( P \) at the low frequency limit and a reduced amplitude

| Table 1 Analytical and numerical results for \( a_0 = \sqrt{3}/2 \) and \( P = 0.1 \) |
|---|---|---|
| \( m \) | Equation (37) | Equation (39) | Numerical result |
| 1 | 16.9175 | 0.75750 | 16.7515 |
| 0.5 | 7.07000 | 3.03000 | 7.00055 |
| 0.3 | 9.87107 | 8.41667 | 9.77417 |
| 0.2 | 19.5839 | 18.9375 | 19.3917 |
| 0.1 | 75.9116 | 75.7500 | 75.1662 |
| 0.05 | 303.040 | 303.000 | 299.964 |
| 0.01 | 7575.00 | 7575.00 | 7500.67 |
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Fig. 2 Partial absorption cross section for $l = 0$, with $a_0 = \sqrt{3}/2$, $P = 0.1$ and $m = 1, 0.5, 0.3, 0.2, 0.1$. at the high frequency limit. In Fig. 6 and Fig. 7 we plot the contributions of the differential scattering cross section.

3 Null geodesics

In this section to analyze the high frequency limit we investigate the null geodesics of the line element (2). Thus, at this limit, we can obtain an approximation for absorption in order to check the validity of our numerical results. We start by considering a Lagrangian density as follows

$$\mathcal{L} \equiv \frac{1}{2} g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu, \quad (43)$$

where $\dot{x}^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$ and the dot denotes the derivative with respect to an affine parameter. Applying the line element (2), we have

$$2\mathcal{L} = F(r) \dot{t}^2 - \frac{\dot{r}^2}{N(r)} - \rho^2 \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right). \quad (44)$$

Next, without loss of generality, we will consider the motion on the equatorial plane, $\theta = \pi/2$, of the spherically symmetric spacetime. Now using the Hamilton-Jacobi equation we can find two equations of motion,

$$E = F(r) \dot{t}, \quad L = \rho^2(r) \dot{\phi}, \quad (45)$$
Fig. 3 Partial absorption cross section for $l = 0$, with $a_0 = \sqrt{3}/2$, $m = 0.4$ and $P = 0.1, 0.2, 0.3, 0.4$

Fig. 4 Partial absorption cross section for modes $l = 0, 1, 2, 3$, with $a_0 = \sqrt{3}/2$, $m = 1$ and $P = 0.1, 0.2, 0.3, 0.4$
Fig. 5  Total absorption cross section with $a_0 = \sqrt{3}/2$, $m = 1$ and $P = 0.1, 0.2, 0.3, 0.4$. For $a_0 = 0$ and $P = 0$ we have the total absorption cross section of the Schwarzschild black hole. The lines represent the values for high frequency absorption obtained by the null geodesics analysis and the values obtained were 19.11, 20.26, 21.97, 24.33 and 27 for the Schwarzschild case.

Fig. 6  Differential scattering cross section with $a_0 = \sqrt{3}/2$, $m = 1$ and $P = 0.1, 0.2, 0.3, 0.4$. For $a_0 = 0$ and $P = 0$ we have the differential scattering cross section of the Schwarzschild black hole.
where $E$ and $L$ correspond to the conserved quantities for energy and angular momentum respectively. For the study of null geodesics we have, $g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu = 0$ and so we obtain

$$F(r)\dot{r}^2 - \frac{\dot{r}^2}{N(r)} - \rho^2(r)\dot{\phi}^2 = 0. \quad (46)$$

Substituting equations (45) in the result above and solving for $\dot{r}$, we find

$$\dot{r} = \frac{r^2}{(r + r_\ast)^2} \sqrt{E^2 - \frac{F(r)}{\rho^2(r)} L^2}, \quad (47)$$

and performing the second derivative we have

$$\ddot{r} = \frac{1}{2} \left[ \left( \frac{N'(r) - \frac{N(r) F'(r)}{F(r)}}{F(r)} \right) \frac{E^2}{F(r)} - \left( \frac{N'(r) - \frac{2N(r) \rho'(r)}{\rho(r)}}{\rho(r)} \right) \frac{L^2}{\rho^2(r)} \right]. \quad (48)$$

To obtain the critical impact parameter $b_c$ and the critical radius $r_c$ we apply the conditions $\dot{r} = 0$ and $\ddot{r} = 0$. So we find $b_c$ and $r_c$ respectively

$$b_c = \frac{L_c}{E_c} = \frac{\rho(r_c)}{F(r_c)}, \quad (49)$$

and

$$2\rho'(r_c)F(r_c) - \rho(r_c)F'(r) = 0, \quad (50)$$

![Fig. 7 Differential scattering cross section with $a_0 = \sqrt{3}/2$, $m = 1$ and $P = 0.1, 0.2, 0.3, 0.4$. For $a_0 = 0$ and $P = 0$ we have the differential scattering cross section of the Schwarzschild black hole](image-url)
with $E_c$ and $L_c$ being characteristic of the null circular geodesic. The absorption cross section at the high frequency limit can be determined by using the following relationship:

$$\sigma_{abs}^{hf} = \pi b_c^2 = \pi \frac{\rho^2(r_c)}{F(r_c)}. \quad (51)$$

Now to find the absorption cross section for high frequency, we just calculate the critical radius $r_c$ by solving equation (50) and use the result into equation (49) to obtain the critical impact parameter. In Fig. 5, we see that the lines obtained by the geodesic procedure fit perfectly with the total absorption curves obtained numerically.

### 4 Conclusions

In the present study, we investigated the process of massless scalar wave scattering due to a self-dual black hole through the partial wave method. We have computed the phase shift analytically at the low energy limit, and then have shown that the dominant contribution at the small angle limit of the differential scattering cross section is modified due to the parameters $a_0$ (minimum area) and $P$ (polymeric function). We have also found that the result for the absorption cross section is given by the event horizon area of the self-dual black hole at the low frequency limit. And mainly, contrarily to the Schwarzschild black hole, the differential scattering/absorption cross section of a self-dual black hole is nonzero at the zero mass limit. Thus, at the limit of $m \to 0$ the absorption cross section presents a dominant contribution that is inversely proportional to the mass squared, i.e., $\sigma_{abs}^{hf} \approx \pi a_0^2 (1 + P^2) / m^2$. In addition, we have verified these results by numerically solving the radial equation for arbitrary frequencies. Finally, we have obtained that the partial absorption amplitude of the self-dual black hole increases its value as we increase the values of the $P$ parameter for the mode $l = 0$ and reduces in amplitude for the modes $l = 1, 2, 3$ in several scenarios. However, we have shown an increase in amplitude when we increased $P$ at the low frequency limit and a reduced amplitude at the high frequency limit. Further investigations, such as exploring correspondence with analog models, may reveal new physics and should be addressed elsewhere.

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**References**

1. Ashtekar, A., Lewandowski, J.: Class. Quant. Grav. **21**, R53 (2004). https://doi.org/10.1088/0264-9381/21/15/R01. [gr-qc/0404018]
2. Han, M., Huang, W., Ma, Y.: Int. J. Mod. Phys. D **16**, 1397 (2007). https://doi.org/10.1142/S0218271807010894. [gr-qc/0509064]
3. Thiemann, T., Kastrup, H.A.: Nucl. Phys. B **399**, 211 (1993). https://doi.org/10.1016/0550-3213(93)90623-W. [gr-qc/9310012]
4. Ashtekar, A.: Gen. Relativ. Gravit. 41, 707 (2009). https://doi.org/10.1007/s10714-009-0763-4. arXiv:0812.0177 [gr-qc]
5. Bojowald, M., Morales-Tecotl, H.A.: Lect. Notes Phys. 646, 421 (2004). [gr-qc/0306008]
6. Ashtekar, A.: Phys. Rev. Lett. 57, 2244 (1986). https://doi.org/10.1103/PhysRevLett.57.2244
7. Modesto, L.: Int. J. Theor. Phys. 49, 1649 (2010). https://doi.org/10.1007/s10773-010-0346-x. arXiv:0811.2196 [gr-qc]
8. Modesto, L., Premont-Schwarz, I.: Phys. Rev. D 80, 064041 (2009). https://doi.org/10.1103/PhysRevD.80.064041. arXiv:0905.3170 [hep-th]
9. Hossenfelder, S., Modesto, L., Premont-Schwarz, I.: Phys. Rev. D 81, 044036 (2010). https://doi.org/10.1103/PhysRevD.81.044036. arXiv:0912.1823 [gr-qc]
10. Hossenfelder, S., Modesto, L., Premont-Schwarz, I.: Emission spectra of self-dual black holes. arXiv:1202.0412 [gr-qc]
11. Alesci, E., Modesto, L.: Gen. Relativ. Gravit. 46, 1656 (2014). https://doi.org/10.1007/s10714-013-1656-0. arXiv:1101.5792 [gr-qc]
12. Moulin, F., Martineau, K., Grain, J., Barrau, A.: Class. Quant. Grav. 36(12), 125003 (2019). https://doi.org/10.1088/1361-6382/ab207c. arXiv:1808.00207 [gr-qc]
13. Silva, C.A.S., Brito, F.A.: Phys. Lett. B 725(45), 456 (2013). https://doi.org/10.1016/j.physletb.2013.07.033. arXiv:1210.4472 [physics.gen-ph]
14. Anacleto, M.A., Brito, F.A., Passos, E.: Phys. Lett. B 749, 181 (2015). https://doi.org/10.1016/j.physletb.2015.07.072. arXiv:1504.06295 [hep-th]
15. Santos, V., Maluf, R.V., Almeida, C.A.S.: Phys. Rev. D 93(8), 084047 (2016). https://doi.org/10.1103/PhysRevD.93.084047. arXiv:1509.04306 [gr-qc]
16. Cruz, M.B., Silva, C.A.S., Brito, F.A.: Eur. Phys. J. C 79(2), 157 (2019). https://doi.org/10.1140/epjc/s10052-019-6565-2. arXiv:1511.08263 [gr-qc]
17. Sahu, S., Lochan, K., Narasimha, D.: Phys. Rev. D 91, 063001 (2015). https://doi.org/10.1103/PhysRevD.91.063001. arXiv:1502.05619 [gr-qc]
18. Regge, T., Wheeler, J.A.: Phys. Rev. 108, 1063 (1957)
19. Anacleto, M.A., Brito, F.A., Ferreira, S.J.S., Passos, E.: Phys. Lett. B 788, 231 (2019). https://doi.org/10.1016/j.physletb.2018.11.020. arXiv:1701.08147 [hep-th]
20. Anacleto, M.A., Brito, F.A., Campos, J.A.V., Passos, E.: Phys. Lett. B 803, 135334 (2020). https://doi.org/10.1016/j.physletb.2020.135334. arXiv:1907.13107 [hep-th]
21. Futterman, J.A., Handler, F.A., Matzner, R.A.: Scattering from Black Holes. Cambridge University Press, Cambridge (1988)
22. Matzner, R.A., Ryan, M.P.: Phys. Rev. D 16, 1636 (1977)
23. Westervelt, P.J.: Phys. Rev. D 3, 2319 (1971)
24. Peters, P.C.: Phys. Rev. D 13, 775 (1976)
25. Sánchez, N.G.: J. Math. Phys. 17, 688 (1976)
26. Sánchez, N.G.: Phys. Rev. D 16, 937 (1977)
27. Sánchez, N.G.: Phys. Rev. D 18, 1030 (1978)
28. Sánchez, N.G.: Phys. Rev. D 18, 1798 (1978)
29. De Logi, W.K., Kovács, S.J.: Phys. Rev. D 16, 237 (1977)
30. Doran, C.J.L., Lasenby, A.N.: Phys. Rev. D 66, 024006 (2002)
31. Dolan, S.R.: Phys. Rev. D 77, 044004 (2008). https://doi.org/10.1103/PhysRevD.77.044004. arXiv:0710.4252 [gr-qc]
32. Crispino, L.C.B., Dolan, S.R., Oliveira, E.S.: Phys. Rev. D 79, 064022 (2009). https://doi.org/10.1103/PhysRevD.79.064022. arXiv:0904.0999 [gr-qc]
33. Starobinsky, A.A., Churilov, S.M.: Sov. Phys. JETP 38, 1 (1974)
34. Gibbons, G.W.: Commun. Math. Phys. 44, 245 (1975)
35. Page, D.N.: Phys. Rev. D 13, 198 (1976)
36. Starobinskii, A.A., Churilov, S.M., Eksp, Zh: Teor. Fiz. 65, 3 (1973)
37. Moura, F.: JHEP 1309, 038 (2013). https://doi.org/10.1007/JHEP09(2013)038. arXiv:1105.5074 [hep-th]
38. Jung, E., Park, D.: Class. Quantum Grav. 21, 3717 (2004). arXiv:hep-th/0403251 [hep-th]
39. Jung, E., Kim, S., Park, D.: Phys. Lett. B 602, 105 (2004). arXiv:hep-th/0409145 [hep-th]
40. Doran, C., Lasenby, A., Dolan, S., Hinder, I.: Phys. Rev. D 71, 124020 (2005). arXiv:gr-qc/0503019 [gr-qc]
41. Dolan, S., Doran, C., Lasenby, A.: Phys. Rev. D 74, 064005 (2006). arXiv:gr-qc/0605031 [gr-qc]
42. Castineiras, J., Crispino, L.C.B., Meira Filho, D.P.: Phys. Rev. D 75, 024012 (2007)
43. Benone, C.L., de Oliveira, E.S., Dolan, S.R., Crispino, L.C.B.: Phys. Rev. D 89(10), 104053 (2014). https://doi.org/10.1103/PhysRevD.89.104053. arXiv:1404.0687 [gr-qc]
44. Marinho, C.I.S., de Oliveira, E.S.: arXiv:1612.05604 [gr-qc]
45. Das, S.R., Gibbons, G.W., Mathur, S.D.: Phys. Rev. Lett. 78, 417 (1997). https://doi.org/10.1103/PhysRevLett.78.417. [hep-th/9609052]
46. Macedo, C.F.B., Crispino, L.C.B., de Oliveira, E.S.: Int. J. Mod. Phys. D 25(9), 1641008 (2016). https://doi.org/10.1142/S021827181641008X. arXiv:1605.00123 [gr-qc]
47. de Oliveira, E.S.: Eur. Phys. J. C 78(11), 876 (2018). https://doi.org/10.1140/epjc/s10052-018-6316-9. arXiv:1805.04987 [gr-qc]
48. Hai, H., Yong-Jiu, W., Ju-Hua, C.: Chin. Phys. B 22(7), 070401 (2013)
49. Crispino, L.C.B., Oliveira, E.S., Matsas, G.E.A.: Phys. Rev. D 76, 107502 (2007)
50. Dolan, S.R., Oliveira, E.S., Crispino, L.C.B.: Phys. Rev. D 79, 064014 (2009)
51. Oliveira, E.S., Dolan, S.R., Crispino, L.C.B.: Phys. Rev. D 81, 124013 (2010)
52. Dolan, S.R., Oliveira, E.S., Crispino, L.C.B.: Phys. Lett. B 701, 485 (2011). https://doi.org/10.1016/j.physletb.2011.06.013
53. Anacleto, M.A., Brito, F.A., Passos, E.: Phys. Rev. D 86, 125015 (2012). arXiv:1208.2615 [hep-th]
54. Anacleto, M.A., Brito, F.A., Passos, E.: Phys. Rev. D 87, 125015 (2013). arXiv:1210.7739 [hep-th]
55. Dolan, S.R., Oliveira, E.S.: Phys. Rev. D 87(12), 124038 (2013). https://doi.org/10.1103/PhysRevD.87.124038. arXiv:1211.3751 [gr-qc]
56. Anacleto, M.A., Salako, I.G., Brito, F.A., Passos, E.: Phys. Rev. D 92(12), 125010 (2015). https://doi.org/10.1103/PhysRevD.92.125010. arXiv:1506.03440 [hep-th]
57. Anacleto, M.A., Brito, F.A., Mohammadi, A., Passos, E.: arXiv:1606.09231 [hep-th]
58. Anacleto, M.A., Brito, F.A., Campos, J.A.V., Passos, E.: arXiv:1810.13356 [hep-th]
59. Anacleto, M.A., Brito, F.A., Passos, E.: Phys. Lett. B 743, 184 (2015). arXiv:1408.4481 [hep-th]
60. Yennie, D.R., Ravenhall, D.G., Wilson, R.N.: Phys. Rev. 95, 500 (1954)
61. Cotaescu, I.I., Crucean, C., Sporea, C.A.: Eur. Phys. J. C 76(3), 102 (2016). https://doi.org/10.1140/epjc/s10052-016-3936-9. arXiv:1409.7201 [gr-qc]

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