CONVEX FUNCTIONS ON SUB-RIEMANNIAN MANIFOLDS. I

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ABSTRACT. We find a different approach to define convex functions in the sub-Riemannian setting. A function on a sub-Riemannian manifold is nonholonomically geodesic convex if its restriction to any nonholonomic (straightest) geodesic is convex. In the case of Carnot groups, this definition coincides with that by Danniell-Garofalo-Nieuhn (equivalent to that by Lu-Manfredi-Stroffolini). Nonholonomic geodesics are defined using the horizontal connection. A new distance corresponding to the horizontal connection has been introduced and near regular points proven to be equivalent to the Carnot-Carathéodory distance. Some basic properties of convex functions are studied. In particular we prove that any nonholonomically geodesic convex function locally bounded from above is locally Lipschitzian with respect to the Carnot-Carathéodory distance.

Keywords: sub-Riemannian manifolds, horizontal connection, nonholonomic geodesic, horizontal Hessian, nonholonomically geodesic convex functions, Lipschitz regularity.

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1. INTRODUCTION

Motivated by the role played by the theory of convex functions in the theory of fully nonlinear partial differential equations-the Monge-Ampère equations for instance-in the Euclidean or Riemannian case, mathematicians working in the field of subelliptic partial differential equations have proposed several notions of convexity of sets and functions in the setting of Carnot groups. The notion of horizontal convexity (h-convexity for short) originally formulated by Caffarelli, was rediscovered by Danielli, Garofalo and Nhieu in [6]. Roughly speaking, a subset Ω of a Carnot group G is said to be h-convex if the following condition holds: if two points on an integral curve of some left invariant, horizontal vector field on G belong to Ω, then the whole segment of the integral curve between these two points is also contained in Ω. A function f : Ω → R is h-convex if it is convex along the integral curves of the left invariant, horizontal vector fields on G. The notion of horizontal convexity in the sense of viscosity (v-convexity for short) was proposed and studied by Lu, Manfredi and Stroffolini, see [14] for the Heisenberg group case and [12] for general Carnot groups. Loosely speaking, an upper semicontinuous function f : Ω → R defined on open subset of a Carnot group G is v-convex if the horizontal Hessian of test functions touching f from above is positive semidefinite. It turns out that these two notions are equivalent, see [14] [12] [11] [15] [29] [19] [20]. So far many fundamental properties on horizontal convex functions have been obtained, see [6] [14] [7] [10] [11] [12] [1] [15] [29] [19] [20]. We in particular
remark that Rickly in [19] proved that measurable h-convex functions in Carnot groups are locally Lipschitzian with respect to the Carnot-Carathéodory distance.

In this paper we will, from a more geometric viewpoint, define a notion of convexity of functions on general sub-Riemannian manifolds. This notion is based on the concept of nonholonomic geodesics which researchers working in the field of nonholonomic mechanics studied (see e.g. [27, 28, 13] and references therein). Roughly speaking, nonholonomic geodesics in a sub-Riemannian manifold are the straightest horizontal curves which satisfy the equations of motion for the mechanical problem with a quadratic Lagrangian and nonholonomic linear constraints which only act by means of the reaction to them, i.e. in essence kinematically. These curves can be characterized using a nonholonomic connection (called horizontal connection in this paper). To be more precise, let $(M, \Sigma, g_c)$ be a sub-Riemannian manifold where $\Sigma$ is a subbundle of the tangent bundle $TM$ and $g_c$ a smooth inner product on $\Sigma$. Without generality we assume that $g_c$ is the restriction on $\Sigma$ of a Riemannian metric $g$ on $M$. For $X, Y \in \Gamma(\Sigma)$, define $D_X Y := \mathcal{P}(\nabla_X Y)$ where $\nabla$ is the Levi-Civita connection of $g$ and $\mathcal{P}$ denotes the projection onto $\Sigma$ with respect to the metric $g$. Given a complement of $\Sigma$, i.e., a decomposition $TM = \Sigma \oplus \Sigma'$, $D$ depends only on $g_c$ (see [24]). Nonholonomic geodesics are those curves satisfying

$$D_\dot{\gamma} \dot{\gamma} = 0, \quad \dot{\gamma} \in \Sigma_\gamma.$$ 

Nonholonomic geodesics are far different from sub-Riemannian geodesics which are the shortest horizontal curves realizing the Carnot-Carathéodory distance. In the literature the geometry studying the truncated connection $D$ (resp. sub-Riemannian geodesics) is called nonholonomic geometry, see e.g. [21, 25, 26] (resp. sub-Riemannian geometry, see e.g. [17]). However the connection $D$ plays an important role in the study of sub-Riemannian geometry, see e.g. [23, 24] where $D$ is used to define the horizontal mean curvature of hypersurfaces of sub-Riemannian manifolds, and [22] where we use $D$ to define the notion of sublaplacian on sub-Riemannian manifolds. In this paper we will use nonholonomic geodesics to give a notion of convex functions in the setting of sub-Riemannian geometry. In Section 3 we will use nonholonomic geodesics to define a new distance $d_H$. If a horizontal curve $\gamma$ consists of smooth segments and moreover each segment is a nonholonomic geodesic, we call $\gamma$ a broken geodesic. For $p, q \in M$, let $d_H(p, q)$ be the least length of all broken geodesics connecting $p$ and $q$. Assume $\Sigma$ be bracket generating and $M$ connected. In a neighborhood of each point we will construct a horizontal frame such that the integral curves of vector fields in this frame are nonholonomic geodesics. Thus we can use the Chow connectivity theorem to prove for $p, q \in M$ there exists at least a broken geodesic connecting $p$ and $q$, and $d_H$ is a distance. By definition $d_H \geq d_c$ where $d_c$ is the Carnot-Carathéodory distance. Indeed by the ball-box theorem $d_c \leq d_H \leq Cd_c$ locally holds for a local constant $C$. For the first Heisenberg group $\mathbb{H}^n$ we can prove $d_H = d_c$. These results tell us that the horizontal connection and its geodesics can provide us another method to study sub-Riemannian geometry.

To state our definition for convex functions in the setting of sub-Riemannian geometry, we first recall that in the Riemannian case a function on a Riemannian manifold $M$ is convex if so is its restriction to every Riemannian geodesic. Now let $f$ be a function on the sub-Riemannian manifold $(M, \Sigma, g_c)$. We define that $f$ is nonholonomic geodesically convex (n-convex for short) if its restriction to every nonholonomic geodesic is convex.
Thus if $\Sigma = TM$ this notion is just the Riemannian one. It is remarkable that when $M$ is a Carnot group, this notion is just the one by Danielli, Garofalo and Nhieu since now nonholonomic geodesics are accidentally the integral curves of the left invariant, horizontal vector fields. Monti and Rickly in [18] has proved that in the Heisenberg group every function whose restriction to every sub-Riemannian geodesic is convex must be constant. Thus in our definition nonholonomic geodesics can not be replaced by sub-Riemannian geodesics. We will also define a natural notion of horizontal Hessian for smooth functions on sub-Riemannian manifolds and then prove that a smooth function is $n$-convex if and only if its Hessian is positive semidefinite. It is natural to consider the regularity problem for $n$-convex functions in general sub-Riemannian manifolds. For Carnot groups, it is well known that each horizontal convex function locally bounded from above is locally Lipschitzian. The proof (see [19]) of this fact depends only on a metric property for Carnot groups $G$: there exist a constant $C$ and an integer $N$ such that every two points $p, q \in G$ can be connected by a broken geodesic, which is composed of $N$ nonholonomic geodesics and each of them has length less than $Cd_c(p, q)$. Fortunately this property locally holds for sub-Riemannian manifolds $(M, \Sigma, g_c)$ where $\Sigma$ is regular and $M$ is connected. Thus together with the local equivalence of $d_c$ and $d_H$ we prove local Lipschitz regularity for $n$-convex functions with local bound from above.

One motivation to study convex functions on such general sub-Riemannian manifolds is the role played by the theory of convex functions in the study of structures of Riemannian manifolds. It is well known that the existence of a convex function on a Riemannian manifold imposes strict limitations on its structure. We expect that the theory of convex functions developed in this paper could be used to study topology properties and structures of contact manifolds, Riemannian submersions, and so on. This program will be addressed in a forthcoming paper.

To end this introduction we give the structure of this paper. Some basic facts about sub-Riemannian manifolds will be given in the next section. We will adopt the viewpoint of Riemannian submersions to study Carnot groups. The definition of horizontal Hessian on general sub-Riemannian manifolds is new. Section 3 is devoted to nonholonomic geodesics and the distance $d_H$. In Section 4 we study the $n$-convex notion and prove some basic properties of $n$-convex functions.

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2. Horizontal Connection and Some Associated Differential Operators

Let $M$ be a smooth manifold of dimension $m$ endowed with a smooth distribution (horizontal bundle) $\Sigma$ of dimension $k$ with $k < m$. If we a priori equip $\Sigma$ with an inner product $g_c$ (sub-Riemannian metric), we call $(M, \Sigma, g_c)$ a sub-Riemannian manifold with the sub-Riemannian structure $(\Sigma, g_c)$. If $\Sigma$ is integrable, it is just the Riemannian geometry. We will assume $\Sigma$ is not integrable. A piecewise smooth curve $\gamma(t), t \in [a, b]$ in $M$ is horizontal if $\dot{\gamma}(t) \in \Sigma_{\nu(t)}$ a.e. $t \in [a, b]$. The length $\ell(\gamma)$ of the horizontal curve
\[ \gamma(t), t \in [a, b] \text{ is the integral } \int_a^b g_c(\dot{\gamma}(t), \dot{\gamma}(t))dt. \]

Denote by \( \Sigma_i \) the set of all vector fields spanned by all commutators of order \( \leq i \) of vector fields in \( \Sigma \) and let \( \Sigma_i(p) \) be the subspace of evaluations at \( p \) of all vector fields in \( \Sigma_i \). We call \( \Sigma \) satisfies the Chow(Hörmander) condition or say \( \Sigma \) is bracket generating if for any \( p \in M \), there exists an integer \( l(p) \) such that \( \Sigma_{l(p)}(p) = T_pM \) (the least such \( l \) is called the degree of \( \Sigma \) at \( p \)). If moreover \( \Sigma_i \) is of constant dimension (in a neighborhood of \( p \)) for all \( i \leq l \), \( \Sigma \) and also \( (M, \Sigma, g_c) \) are called regular (at \( p \)). If \( M \) is connected and \( \Sigma \) satisfies the Hörmander condition, the Chow connectivity theorem asserts that there exists at least one piecewise smooth horizontal curve connecting two given points (see [5, 3, 17]), and thus \( (\Sigma, g_c) \) yields a metric (called Carnot-Carathéodory distance) \( d_c \) by letting \( d_c(p, q) \) as the infimum among the lengths of all horizontal curves joining \( p \) to \( q \). Those horizontal curves realizing \( d_c \) are sub-Riemannian geodesics. \( (M, d_c) \) induces the same topology of \( M \). If \( (M, d_c) \) is complete, then for any two points there exists a shortest sub-Riemannian geodesic connecting them. Sub-Riemannian geometry is anyway not a trivial generalization of Riemannian geometry. Some phenomena in sub-Riemannian geometry never appear in Riemannian geometry. For example, in any neighborhood of a point \( p \) there exists a point \( q \) such that there are infinite many sub-Riemannian geodesics connecting them, and for some sub-Riemannian manifolds there are sub-Riemannian geodesics (called singular geodesics) not satisfying any ordinary differential equation. We refer to [3, 17] for more about sub-Riemannian geometry.

Sub-Riemannian manifolds are the common setting for control theory, nonholonomic mechanics and many geometric structures such as CR structures or contact metric structures, principal bundles with connections, and Riemannian submersions. In this section we mainly consider the horizontal connection which was originally introduced to characterize equations of motion in nonholonomic Lagrangian mechanics ([25]). We use the Einstein summation convention for expressions with indices: if in any term the same index name appears twice, as both an upper and a lower index, that term is assumed to be summed over all possible values of that index (for \( i, j, r \) from 1 to \( k \) and \( a, b, c, d \) from 1 to \( m \)).

**Example 2.1** (Riemannian submersions). Let a smooth map \( \pi \) between two Riemannian manifolds \( \pi: (M, g) \rightarrow (B, g') \) be a submersion, that is, \( \pi_*p \) has maximal rank at any point \( p \) of \( M \). Putting \( V_p = \text{ker}(\pi_p) \) for any \( p \in M \), we obtain an integrable distribution \( V \) which corresponds to the foliation of \( M \) determined by the fibres of \( \pi \), since each \( V_p \) coincides with the tangent space of \( \pi^{-1}(x) \) at \( p, \pi(p) = x \). \( V \) is called the vertical distribution whose section are the so-called vertical vector fields. Let \( \Sigma \) be the complementary distribution of \( V \) determined by the Riemannian metric \( g \). Thus \( (M, \Sigma, g|_\Sigma) \) is a sub-Riemannian manifold. Given \( X' \in \Gamma(TB) \), the horizontal vector field \( X \in \Gamma(\Sigma) \) satisfying \( \pi_*X = X' \) is called the horizontal lift of \( X' \). Note that horizontal lifts of all vector fields of \( B \) locally span \( \Sigma \). We further assume that \( \pi \) is a Riemannian submersion, that is, moreover at each point of \( p \in M \), \( \pi_p \) preserves the length of the horizontal vectors. Since \( p \) is a submersion, \( \pi_*p \) is a linear isomorphism between \( \Sigma_p \) and \( T_{\pi(p)}B \) and \( \pi_*p \) acts on \( \Sigma_p \) as a linear isometry. For fundamental properties of Riemannian submersions we refer to e.g. the book [8].

**Example 2.2** (Carnot groups). The most interesting models of sub-Riemannian manifolds are Carnot groups (called also stratified groups). The role played by Carnot groups
in sub-Riemannian geometry is the same as that by Euclidean spaces in Riemannian geometry (see e.g. [13]). A Carnot group \( G \) is a connected, simply connected Lie group whose Lie algebra \( \mathcal{G} \) admits the grading \( \mathcal{G} = V_1 \oplus \cdots \oplus V_l \), with \( [V_i, V_j] = V_{i+j} \), for any \( 1 \leq i \leq l - 1 \) and \( [V_1, V_l] = 0 \) (the integer \( l \) is called the step of \( G \)). Let \( \{ e_1, \cdots, e_m \} \) be a basis of \( \mathcal{G} \) with \( m = \sum_{i=1}^l \text{dim}(V_i) \). Let \( X_i(g) = (L_g)_* e_i \) for \( i = 1, \cdots, m \) where \( (L_g)_* \) is the differential of the left translation \( L_g(g') = gg' \). We call the system of left-invariant vector fields \( \Sigma := V_1 \oplus \cdots \oplus V_k \) the horizontal bundle of \( G \). If we equip \( G \) with an inner product \( g \) such that \( \{ X_1, \cdots, X_m \} \) is an orthonormal basis of \( T\mathcal{G} \), \( (G, \Sigma, g_c = g|_\Sigma) \) is a sub-Riemannian manifold. In \( (G, \Sigma, g_c) \) there exists a natural dilation homomorphism \( \delta_\lambda : \delta_\lambda p = \exp(\sum_{i=1}^l \lambda^i \xi_i) \) for \( p = \exp(\sum_{i=1}^l \xi_i) \), \( \xi_i \in V_i \). The most simplest Carnot group is the Heisenberg group \( \mathbb{H}^n \) which is, by definition, simply \( \mathbb{R}^{2n+1} \), with the noncommutative group law

\[
p p' = (x, y, t)(x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(\langle x', y \rangle - \langle x, y' \rangle))
\]

where we have let \( x, x', y, y' \in \mathbb{R}^n, t, t' \in \mathbb{R} \). A simple computation shows the left-invariant vector fields \( X_j(p) = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t} \), \( X_{n+j}(p) = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, j = 1, \cdots, n \), and \( T = \frac{\partial}{\partial t} \) span the Lie algebra \( \mathbb{R}^{2n+1} \) of \( \mathbb{H}^n \). Moreover \( [X_j, X_{n+k}] = -T \delta_{jk}, j, k = 1, \cdots, n \), and all other commutators are trivial. Note that the horizontal bundle \( \Delta = \text{span}\{X_1, \cdots, X_{2n}\} \) is the kernel of the 1-form \( \eta = \frac{1}{2} dt + \frac{1}{2} \sum_{i=1}^n (x_i dy_i - y_i dx_i) \) and the curvature form \( \omega = d\eta = \sum_{i=1}^n dx_i \wedge dy_i \) is the standard symplectic form in \( \mathbb{R}^{2n} \). Thus a piecewise smooth curve \( \gamma(s) = (x(s), y(s), \ell(s)) : [a, b] \to \mathbb{H}^n \) is horizontal if and only if

\[
\ell(s) = \sum_{i=1}^n y_i(s) \dot{x}_i(s) - x_i(s) \dot{y}_i(s), \quad \text{when } \gamma \text{ is smooth at } s \in [a, b]. \tag{2.1}
\]

**Lemma 2.3.** Any Carnot group can be regarded as the source manifold of a Riemannian submersion onto a standard Euclidean space whose dimension equals to the dimension of the first layer of its Lie algebra.

**Proof.** Let \( G \) be a Carnot group as above. Since the exponential map is globally diffeomorphic, we usually identify \( G \) with \( \mathbb{R}^n \) by the exponential map. It is easy to prove that

\[
X_j(x) = \frac{\partial}{\partial x_j} + \sum_{a=k+1}^m c_j^a(x) \frac{\partial}{\partial x_a}, \quad X_j(0) = e_j = \frac{\partial}{\partial x_j}, j = 1, \cdots, k \tag{2.2}
\]

where \( c_j^a(x) = c_j^a(x_1, \cdots, x_k) \) are polynomials such that \( c_j^a(\delta x) = \lambda^{w_a-1} c_j^a(x) \) where \( w_a \) is the weight of \( x_a \) \( \{2, 9\} \). For \( p = (x_1, \cdots, x_k) \in G \), we define \( \pi(p) = (x_1, \cdots, x_k) \in \mathbb{R}^k \). Then by direct computation it is easy to see that \( \pi \) is a Riemannian submersion from \( (G, g) \) onto the Euclidean space \( \mathbb{R}^k \) and the horizontal lift of \( \frac{\partial}{\partial x_j} \) is \( X_j, j = 1, \cdots, k \).

Although Lemma 2.3 is simple, via the theory of Riemannian submersions it may simplify our discussions on Carnot groups. For more information on Carnot groups we refer to [9].

Note that in a sub-Riemannian manifolds \( (M, g_c) \) we always can extend \( g_c \) to a Riemannian metric \( g \) in \( M \) such that \( TM \) can be \( g \)-orthogonally decomposed as \( TM = \)
where $\Sigma'$ is the distribution complementary to $\Sigma$. We call such $g$ an orthogonal extension of $g_c$. Obviously the orthogonal extension of $g_c$ is not unique. We will use $\Gamma(\Sigma)$ to denote the set of all smooth sections of $\Sigma$.

**Definition 2.4** (horizontal connection). Let $g$ be any orthogonal extension $g_c$ and let $\nabla$ be the Levi-Civita connection with respect to $g$. We define the horizontal connection $D$ on $\Sigma$ as

$$D_XY = P(\nabla_XY) := \sum_{i=1}^{k} g(\nabla_XY, X_i)X_i \text{ for any } X, Y \in \Gamma(\Sigma)$$

where $\{X_1, \cdots, X_k\}$ is an orthonormal basis of $\Sigma$.

It is obvious that $D$ is a linear connection on $\Sigma$. Since $\Sigma$ is in general not integrable, such $D$ is called nonholonomic connection in the literature.

**Proposition 2.5** (23). Given a decomposition of $TM$, $TM = \Sigma \oplus \Sigma^\perp$, $D$ is independent of the choice of orthogonal extensions of $g_c$ such that $\Sigma$ is orthogonal to $\Sigma^\perp$. Moreover $D$ is the unique nonholonomic connection on $\Sigma$ satisfying

1. $Z_{g_c}(X,Y) = g_c(DZ_XY, X) + g_c(X, DZ_Y)$;
2. $D_XY - D_YX = [X, Y]^H$

for any $X, Y, Z \in \Gamma(\Sigma)$, where $[X, Y]^H := P([X,Y])$ and $P$ denotes the (algebraic) projection onto $\Sigma$ with respect to the given decomposition.

Thus $D$ depends only on the sub-Riemannian structure $(\Sigma, g_c)$ and the splitting of $TM$ and thus is “almost” a sub-Riemannian object. For many sub-Riemannian structures such as contact metric structures, principal bundles with connections, and Riemannian submersions the horizontal bundle $\Sigma$ has a canonical complement. For these cases $D$ is canonical. Note that the sub-Riemannian geometry of $(M, \Sigma, g_c)$, i.e., the geometry of $(M, d_c)$, depends only on the sub-Riemannian structure $(\Sigma, g_c)$, not on complements of $\Sigma$ or extensions of $g_c$. What should a canonical or good complement of $\Sigma$, and then a canonical orthogonal extension of $g_c$ be? This may depend on the context. In a try to find the sublaplacian for sub-Riemannian manifolds the author in [22] proved the following statement.

**Theorem 2.6** (22). Let $(M, \Sigma, g_c)$ be a sub-Riemannian manifold. Then there exists a complement $\Sigma'$ of $\Sigma$, $TM = \Sigma \oplus \Sigma'$, such that for this decomposition there is an orthogonal extension $g$ of $g_c$ and an orthonormal basis $\{T'_1, \cdots, T'_{m-k}\}$ of $\Sigma'$ satisfying

$$P(\nabla_{T'_\beta} T'_\beta) = 0, \quad \beta = 1, \cdots, m - k$$

where $\nabla$ is the Riemannian connection of $g$. If $\Sigma$ is strong-bracket generating (that is, for each $p \in M$ and each nonzero horizontal vector $v \in \Sigma_p$ we have

$$\Sigma_p + [V, \Sigma]_p = T_p M$$

where $V$ is any horizontal extension of $v$), then such complement is unique.

We will not use the last result in this paper. We cite it here just to show that the selectivity of the splitting of the tangent bundle and orthogonal extensions of $g_c$ is helpful
in using $D$ to study sub-Riemannian geometry. To simplify the discussion, unless otherwise stated, we assume in the sequel that $g_{c}$ is the restriction to $\Sigma$ of a given Riemannian metric $g$ on $M$ and $TM$ admits the orthogonal decomposition, $TM = \Sigma \oplus \Sigma^\perp$.

The horizontal connection $D$ induces the directional derivative of a horizontal vector along a horizontal curve. Let $\{X_i\}_{i=1}^k$ be an orthonormal local basis of $\Gamma(\Sigma)$ and $Y$ be a horizontal vector field along a horizontal curve $\gamma$. Then $Y$ can be locally written as $Y(t) = Y^j(t)X_j$ and the directional derivative of $Y$ along $\gamma$ can be defined as

$$\frac{D}{dt}Y(t) = \left(\dot{Y}^r(t) + Y^j(t)\dot{\gamma}^i(t)\Gamma^r_{ij}\right)X_r$$

(2.3)

where $\Gamma^r_{ij}$ is Christoffel symbols such that $D_{X_i}X_j = \Gamma^r_{ij}X_r$. Note that the horizontal connection is natural in the sense that any isometry $\varphi$ between two sub-Riemannian manifolds $\varphi : (M^1, \Sigma^1, g^1_{c}) \to (M^2, \Sigma^2, g^2_{c})$ (i.e. $\varphi$ is a diffeomorphism such that $\varphi_*(\Sigma^1) = \Sigma^2$ and $\varphi^*g^2_{c} = g^1_{c}$) takes the horizontal connection $D^1$ of $(M^1, \Sigma^1, g^1_{c})$ to the horizontal connection $D^2$ of $(M^2, \Sigma^2, g^2_{c})$:

$$\varphi_* (D^1_{X} Y) = D^2_{\varphi_*(X)} (\varphi_* Y)$$

for any $X, Y \in \Gamma(\Sigma)$.

**Lemma 2.7.** In a Carnot group $G$, the horizontal connection has the simple form

$$DuV = U(V^i)X_i$$

for any $U, V = V^iX_i \in \Gamma(\Sigma)$

where $\{X_i\}_{i=1}^k$ is an orthonormal basis of the system of left invariant, horizontal vector fields.

**Proof.** Let $\{X_i\}_{i=1}^k$ be an orthonormal basis of left invariant vector fields with respect to $g$. Then by the grading condition of $G$ we have

$$\Gamma^r_{ij} = g_{c}(D_{X_i}X_j, X_r) = g(\nabla_{X_i}X_j, X_r)$$

$$= \frac{1}{2}\{g(X_i, [X_r, X_j]) - g(X_j, [X_i, X_r]) - g(X_r, [X_j, X_i])\}$$

$$= 0$$

Since $DuV = U(V^i)X_i + U^jV^i\Gamma^r_{ij}X_r$, the statement follows. □

**Definition 2.8** (horizontal gradient). For a smooth function $f : M \to R$ we define its gradient as the horizontal vector field $\nabla^H f$ such that $g_{c}(X, \nabla^H f) = Xf$ holds for any horizontal vector field $X \in \Gamma(\Sigma)$.

For every point $x \in M$ where $\nabla^H f(x) = 0$, by the definition $\nabla^H f(x)(\nabla^H f(x))$ is the horizontal direction along which $f$ increases (decreases) with the fastest velocity.

**Definition 2.9** (horizontal divergence). For $X \in \Gamma(\Sigma)$, its horizontal divergence is defined as $\text{div}^H X = \sum_{i=1}^k g_{c}(D_{X_i}X, X_i)$ where $\{X_i\}_{i=1}^k$ is an orthonormal local basis of $\Gamma(\Sigma)$. It’s clear that $\text{div}^H X$ is independent of the choice of orthonormal bases.

From Lemma [2.7] it’s easy to see that in a Carnot group $G$, $\text{div}X = \text{div}^H X$ for any horizontal vector field $X$. Here $\text{div}$ denotes the Riemannian divergence with respect to $g$. We can give a notion of sublaplacians on sub-Riemannian manifolds using the horizontal
gradient and divergence operators. We have given a detailed discussion about this topic in \cite{22}.

**Definition 2.10** (sublaplacian). The sublaplacian is defined as

$$\Delta^H := \text{div}^H \circ \nabla^H.$$ 

Next we define the horizontal Hessian of smooth functions on sub-Riemannian manifolds.

**Definition 2.11** (horizontal Hessian). The horizontal Hessian of a smooth function $f$ on $M$ is defined as

$$\text{Hess}^H f(X, Y) := \frac{1}{2} \{ g_c(D_X(\nabla^H f), Y) + g_c(D_Y(\nabla^H f), X) \}$$

where $X, Y \in \Gamma(\Sigma)$.

**Lemma 2.12.** Let $\text{Hess} f$ be the Riemannian Hessian of $f$. Then

$$\text{Hess}^H f(X, Y) = \text{Hess} f(X, Y) - \frac{1}{2} \{ B(X, Y) + B(Y, X) \}$$

for any $X, Y \in \Gamma(\Sigma)$, where $B(X, Y) := \nabla_X Y - D_X Y$.

It is obvious that $\text{Hess}^H f$ is a tensor on $\Gamma(\Sigma)$. Note that the trace of the horizontal Hessian is just the sublaplacian. The proof of the following statement is trivial.

**Proposition 2.13.** Let $f$ be a smooth function on a Carnot group. Then

$$\text{Hess}^H f(X_i, X_j) = \frac{1}{2}(X_i X_j f + X_j X_i f),$$

where $\{X_i\}_{i=1}^k$ as in Lemma 2.7.

From Proposition 2.13 we see that for Carnot groups the horizontal Hessian is just the ordinary symmetrized horizontal Hessian (e.g. \cite{6, 14}). In particular, in Carnot groups the sublaplacian is $\Delta^H = \sum_{i=1}^k X_i^2$ where $\{X_i\}_{i=1}^k$ as in Lemma 2.7.

### 3. Nonholonomic Geodesics

In nonholonomic Lagrangian mechanics, there are two well known approaches for the study of the constrained mechanics: d’Alembertian nonholonomic mechanics and the variational nonholonomic mechanics. The variational nonholonomic mechanics, also called vakonomic mechanics by the Russian school, is to solve a constrained variational problem: to find a curve $\gamma$ such that $\gamma$ minimizes the Lagrangian functional $\int L(\beta(t), \dot{\beta}(t))$ among all curves $\beta$ satisfying the (nonholonomic) constraints $\dot{\beta}(t) \in \Sigma_{\beta(t)}$, where $\Sigma$ is a (nonintegrable) distribution. When the Lagrangian $L$ is regular, and quadratic with respect to the velocity component, the required curve $\gamma$ is just a sub-Riemannian geodesic, see e.g. \cite{17} for details. While the dynamics studied by d’Alembertian nonholonomic mechanics is governed by the Lagrange-d’Alembert principle. The principle states that the equations of motion of a curve $q(t)$ in a configuration space are obtained by setting to zero
the variations in the integral of the Lagrangian subject to variations lying in the constraint
distribution and that the velocity of the curve \( q(t) \) itself satisfies the constraints:
\[
\dot{q}(t) \in \Sigma_{q(t)} \quad \text{and} \quad \delta L := \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = 0 \tag{3.1}
\]
for all variations \( \delta q \) such that \( \delta q \in \Sigma_q \). There is a huge literature on nonholonomic
mechanics. We refer to [28, 13, 4] and references therein.

The following theorem is well known, for its proof see e.g. [27, 28].

**Theorem 3.1.** Assume \((M, \Sigma, g_c)\) be a sub-Riemannian manifold where \( g_c \) is the restric-
tion to \( \Sigma \) of a given Riemannian metric \( g \). If the Lagrangian \( L \) is the kinetic energy
\( L(q, \dot{q}) = g(\dot{q}, \dot{q}) \), then by using the horizontal connection \( D \), the equations of motion
(3.1) can be rewritten as
\[
D\dot{q} = 0. \tag{3.2}
\]

**Definition 3.2** (nonholonomic geodesic). On a sub-Riemannian manifold \((M, \Sigma, g_c)\), any
horizontal curve \( \gamma \) satisfying \( D\dot{\gamma} = 0 \) is called a nonholonomic geodesic.

The equation (3.2) of nonholonomic geodesics is a system of second order differential
equations formulated in local coordinates as
\[
\frac{d^2 q^c}{dt^2} + (\Gamma^c)_{ab} \frac{dq^a}{dt} \frac{dq^b}{dt} = 0 \tag{3.3}
\]
where \((\Gamma^c)_{ab} = \Gamma^c_{ab} + (\mu_1)_{a:b}(\mu_1)^c, \Gamma^c_{ab}\) is the christoffel symbol, \((\mu_1)_{a:b} = \frac{\partial (\mu_1)_{a:b}}{\partial q^b} - \Gamma^d_{ab}(\mu_1)^d\) and \(\mu_1\) are functions such that \( \Sigma \) is locally described by \( \phi_i(q^a, \dot{q}^a) = (\mu_1)_i(q) \dot{q}^a = 0 \), see [21, 13].

It is well known that Riemannian geodesics are the projections on \( M \) of the integral
curves of a spray. For nonholonomic geodesics, the following theorem gives similar char-
acterization.

**Theorem 3.3** (e.g. [28, 13]). Nonholonomic geodesics are the projections on \( M \) of the integral
curves of a vector field \( \xi \) on \( \Sigma, \xi \in \Gamma(T\Sigma) \). Moreover there is an almost product
structure \((P, Q)\) on \( TM \) such that the vector field \( \xi \) can be explicitly given by projecting
the Riemannian spray \( \xi' \) to \( T\Sigma \), that is, \( \xi(p) = P(\xi'(p)) \) for \( p \in \Sigma \).

We call the vector field \( \xi \) in Theorem 3.3 a partial spray. Theorem 3.3 in particu-
lar implies that for any point \( x \) in \( M \), and any vector \( v \in \Sigma_x \), there exists a unique
smooth nonholonomic geodesic \( \gamma_v(t) \) such that \( \gamma_v(0) = x \) and \( \gamma_v(0) = v \). Note that any
nonholonomic geodesic \( \gamma \) has constant speed because
\[
\frac{d}{dt}(g_c(\gamma, \gamma)) = 2g_c(D\gamma, \gamma) = 0.
\]

From 3.3 we have

**Lemma 3.4.** Given a point \( x \in M \). For any \( v \in T_xM \) and \( c, t \in \mathbb{R} \),
\[
\gamma_{cv}(t) = \gamma_v(ct)
\]
whenever either side is defined.
So for \( x \in M \) there exists a neighborhood \((0 \in) \mathcal{E}_x \subset \Sigma_x\) such that for any \( v \in \mathcal{E}_x, \gamma_v(t) \) is defined on \([-2, 2]\). We define the horizontal (or partial) exponential

\[
\exp^H_x : \mathcal{E}_x \rightarrow M
\]

by \( \exp^H_x(v) = \gamma_v(1) \). Just as in the Riemannian case, it is direct to prove that \( \exp^H_x \) is a diffeomorphism from \( \mathcal{E}_x \) onto a \( k \)-dimensional submanifold containing \( x \). Sometimes for \( V = (x, v) \in \Sigma \), we write \( \exp^H_x(V) \) (or \( \gamma_V \)) for \( \exp^H_x(v) \) (or \( \gamma_v \)) when it is defined. Denote by \( \mathcal{E} \subset \Sigma \) the set on which a nonholonomic geodesic \( \gamma(t) \) is defined in \([-2, 2]\). Then \( \exp^H_x \) is smooth on \( \mathcal{E} \). The following theorem plays an important role in this paper.

**Theorem 3.5.** Given \( y \in (M, \Sigma, g_c) \). There exists a neighborhood \( \mathcal{O} \ni y \) and a horizontal frame \( \{E_1, \cdots, E_k\} \) on \( \mathcal{O} \) with \( \|E_i\|^2 := g_c(E_i, E_i) = 1 \) such that any integral curve of \( E_i \) is a nonholonomic geodesic in \( \mathcal{O} \), \( i = 1, \cdots, k \).

**Proof.** Let \( \{X_1, \cdots, X_k\} \) be an orthonormal basis of \( \Sigma \) in a neighborhood \( U \) of \( y \) in \( M \). For \( X_i \) we choose a hypersurface \( S_i \) containing \( y \) such that \( X_i(x) \) is not in \( T_xS_i \) for all \( x \) in \( S_i \cap U \). Now we can use the horizontal exponential map \( \exp^H \) to obtain a “local coordinate system” near \( S_i \) using nonholonomic geodesics with initial velocities given by \( X_i \). More explicitly, consider the smooth map \( \phi : (x, t) \mapsto \exp^H(tX_i(x)), \) with \( x \) in \( S_i \) and \( t \) in an interval \( (-\epsilon, \epsilon) \). By taking \( S_i \) and \( \epsilon \) sufficiently small and using the inverse function theorem one can prove that \( \phi \) is a smooth diffeomorphism onto an open neighborhood \( \mathcal{O}_i \subset U \) of \( y \) in \( M \). Now we can extend the vector field \( X_i|_{S_i} \) to the open set \( \mathcal{O}_i \) as follows: given \( p \) in \( \mathcal{O}_i \), we pick \( (x, t) \) with \( \phi_i(x, t) = p \) and define \( E_i(p) = \frac{\partial}{\partial t} \exp^H(tX_i(x)) \) in \( T_pM \). Just by definition it’s clear that the integral curves of \( E_i \) are nonholonomic geodesics. Because \( X_i \) has length 1, so is \( E_i \). Set \( \mathcal{O} = \cap_{i=1}^k \mathcal{O}_i \). Taking \( \mathcal{O} \) smaller if necessary, we get the desired frame \( \{E_1, \cdots, E_k\} \) in \( \mathcal{O} \). \( \square \)

If a continuous horizontal curve \( \gamma \) consists of segments each of them a nonholonomic geodesic, we call \( \gamma \) a broken geodesic. Thus broken geodesics are piecewise smooth. There is no Hopf-Rinow type theorem for nonholonomic geodesics. Most couples of points can not be connected by any nonholonomic geodesic. Through the horizontal exponential it is easy to see that the set accessible from a given point through nonholonomic geodesics is a smooth submanifold of dimension \( k \). But for broken geodesics we have

**Theorem 3.6.** Let \((M, \Sigma, g_c)\) be a sub-Riemannian manifold. If \( \Sigma \) satisfies the Chow condition and \( M \) is connected, for any two points \( p, q \in M \) there exists at least a broken geodesic joining \( p \) to \( q \).

**Proof.** For any \( p \in M \), let \( \mathcal{A}_p \) be the set of points which are reachable via broken geodesics starting from \( p \). To prove the statement it is sufficient to prove the openness of \( \mathcal{A}_p \) since \( M \) is connected and \( \mathcal{A}_p \) is trivially not empty. For \( y \in \mathcal{A}_p \), by Theorem 3.5 in a neighborhood \( \mathcal{O} \) of \( y \) there exists a frame \( \{E_1, \cdots, E_k\} \) such that integral curves of \( E_i \) are nonholonomic geodesics. Since \( \Sigma|_{\mathcal{O}} = \text{span}\{E_1, \cdots, E_k\} \) is bracket generating, by Chow connectivity theorem there exists a neighborhood of \( y \) such that any point in this neighborhood can be connected by a piecewise smooth horizontal curve starting from \( y \) and each piece is an integral curve of \( E_i \). \( \square \)

Now we define a new distance through broken geodesics.
**Definition 3.7.** Let \((M, \Sigma, g_c)\) be a sub-Riemannian manifolds with \(\Sigma\) bracket generating and \(M\) connected. For \(p, q \in M\), the distance \(d_H(p, q)\) is defined as the least length among all broken geodesics connecting \(p\) and \(q\).

From Theorem 3.6, we know \(d_H\) is well defined and by definition \(d_c \leq d_H\). To get more about the distance \(d_H\) we introduce some notations. Let \(Y_1\) and \(Y_2\) be smooth vector fields, with local flows \(\Psi_i(t) = \exp(tY_i)\). Then for small \(t\),

\[
\Psi_1(-t) \circ \Psi_2(-t) \circ \Psi_1(t) \circ \Psi_2(t)(p) = p + t^2[E_1, E_2](p) + O(t^2).
\]

Write \([Y_1(t), Y_2(t)]\) for \(\Psi_1(-t) \circ \Psi_2(-t) \circ \Psi_1(t) \circ \Psi_2(t)\). For \(y \in M\), let \(\{E_1, \cdots, E_k\}\) in \(O(\exists y)\) be the frame of norm 1 in Theorem 3.5. For multi-indices \(I = (i_1, \cdots, i_r)\), \(1 \leq i_j \leq k\), define vector fields \(E_I\) inductively by \(E_I = [E_{i_1}, E_{i_2}], \cdots, [E_{i_{r-1}}, E_{i_r}]\), where \(J = (j_2, \cdots, j_r)\) is the standard multi-index \(|J| = r - 1\). Similarly define \(\Psi_I(t) = \Psi_{i_1}(t) \circ \cdots \circ \Psi_{i_r}(t)\) as above for \(Y_1, Y_2\). Note that \(\Psi_I(t) = 1 + tE_I + O(t^{r+1})\) and \(\Psi_I(t)y\) is a concatenation of \((3 \cdot 2^{-r-1})\) nonholonomic geodesics, each one of length \(\epsilon\) if \(|t| \leq \epsilon\). If \(\Sigma\) is bracket generating, we can select a frame for the entire tangent bundle \((O)\) among the \(E_I\). We choose such a frame and relabel it \(\{E_1, \cdots, E_{n_1}, E_{n_1+1}, \cdots, E_{n_2}, E_{n_2+1}, \cdots, E_m\}\) where \(n_1 = k\), \(\{E_1, \cdots, E_n\}\) spans \(\Sigma_i := \Sigma_i + [\Sigma_{i-1}, \Sigma]\), \(\Sigma_i = \Sigma, i = 1, \cdots, l\) and \(n_l = m\). \((k, n_1, \cdots, n_l)\) is the growth vector of \(\Sigma\) at \(y\). For each chosen \(E_i\) of the form \(E_I\), let \(w_i\) be the length \(|I|\). We also relabel flows \(\Psi_I\) as \(\Psi_i, i = 1, \cdots, m\). Coordinates \(x_1, \cdots, x_m\) are said to be linearly adapted to \(\Sigma\) in \(O\) if \(\Sigma_i\) is annihilated by the differentials \(dx_{n_1+1}, \cdots, dx_m\) in \(O\). The weighted box of size \(\epsilon\) is the set

\[
\text{Box}^w(\epsilon) = \{x \in \mathbb{R}^m : |x_i| \leq \epsilon^{w_i}, i = 1, \cdots, m\}.
\]

Define the map \(F^y(t_1, \cdots, t_m) = \Psi_m(t_m) \circ \cdots \circ \Psi_1(t_1)(y) : \mathbb{R}^m \to O\).

In sub-Riemannian geometry, the ball-box theorem is well-known. The following theorem is essentially proven by [17, p.27-34], see also [2].

**Theorem 3.8.** Let \((M, \Sigma, g_c)\) be a sub-Riemannian manifold (with \(\Sigma\) bracket generating and \(M\) connected). Then for a regular point \(y_0 \in M\) there exist a neighborhood \(O\) and linearly adapted coordinates \(x_1, \cdots, x_m\) such that for any \(y \in O\) there exist positive continuous functions \(c(y) < C(y)\) and an integer \(N = \sum_{r=1}^l(n_r - n_{r-1})(3 \cdot 2^{-r-1})\) such that for all \(\epsilon < \epsilon_0\),

\[
B(y, \frac{c}{C} \epsilon) \subset F^y(\text{Box}^w(N-\epsilon)) \subset B(y, \epsilon) \quad (3.4)
\]

where \(\text{Box}^w(\epsilon) = \{(t_1, \cdots, t_m) \in \mathbb{R}^m : |t_i| \leq \epsilon\}\) is the standard \(\epsilon\)-cube and \(B(y, \epsilon) := \{p \in O : d_c(y, p) \leq \epsilon\}\) denotes the Carnot-Carathéodory ball centered at \(y\).

**Proof.** Note that because \(\Sigma\) is regular at \(y_0\), \(N\) is a constant in a neighborhood \(O\) of \(y_0\). From the proof of the ball-box theorem given in [17, p.27-34], we know that near a regular point there exist linearly adapted coordinates \(x_1, \cdots, x_m\) and positive constants \(c < C, \epsilon_0\) and an integer \(N = \sum_{r=1}^l(n_r - n_{r-1})(3 \cdot 2^{-r-1})\) such that for all \(\epsilon < \epsilon_0\),

\[
\text{Box}^w(\epsilon c) \subset F^y(\text{Box}^w(N-\epsilon)) \subset B(y, \epsilon) \subset \text{Box}^w(C \epsilon) \quad (3.5)
\]

and \(c, C\) and \(\epsilon_0\) continuously depend on \(y \in O\). We first use (3.5) replacing \(\epsilon\) by \(\frac{c}{C} \epsilon\) to get \(B(y, \frac{c}{C} \epsilon) \subset \text{Box}^w(\epsilon c)\). Using (3.5) again we obtain (3.4). \(\Box\)
Remark 3.9. If $y_0$ is not a regular point, the constants $c, C$ and $\epsilon_0$ are in general not continuous functions of $y$.

The ball-box theorem implies the Chow connectivity theorem and that the topology of $(M, d_c)$ is the same as the original one.

Corollary 3.10. Let $(M, \Sigma, g_c)$ be a sub-Riemannian manifold (with $\Sigma$ bracket generating and $M$ connected). Then for any regular point $y_0 \in M$ there exists a sub-Riemannian ball $B(y_0, \bar{\epsilon}_0)$ centered at $y_0$ with radius $\bar{\epsilon}_0$, and two constants $C_1, C_2$ such that for any two points $p, q \in B(y_0, \bar{\epsilon}_0)$

1. there exists a broken geodesic connecting $p$ and $q$, which consists of $N$ nonholonomic geodesics, each one of length less than $C_1 d_c(p, q)$;
2. $d_c(p, q) \leq d_{\mathcal{H}}(p, q) \leq C_2 d_c(p, q)$.

Proof. Let $O'$ be a neighborhood of $y_0$ such that $\bar{O}' \subset O$ where $O$ is the neighborhood in Theorem 3.8. In $O'$ set $c_1$ be the maximum of $C(y)$, $c_2$ the minimum of $c(y)$ and $\epsilon'_0$ the minimum of $\epsilon_0(y)$. Take a sub-Riemannian ball $B(y_0, \bar{\epsilon}_0) \subset O'$ with $\bar{\epsilon}_0 < \frac{c_2}{4c_1} \epsilon'_0$. Then by Theorem 3.8 for any $y \in B(y_0, \bar{\epsilon}_0)$ and $\epsilon < \epsilon'_0$ we have

$$B(y, \frac{c_2 \epsilon}{2c_1}) \subset B(y, \frac{c_2 \bar{\epsilon}}{c_1}) \subset F^p(\Box^s(N^{-1} \epsilon)). \quad (3.6)$$

For any $p, q \in B(y_0, \bar{\epsilon}_0)$, because $d_c(p, q) = \frac{c_2 \bar{\epsilon}}{c_1}$ for $\bar{\epsilon} = \frac{2c_1 d_c(p, q)}{c_2} < \frac{4c_1 \epsilon_0}{c_2} < \epsilon'_0$, by (3.6) we obtain $q \in F^p(\Box^s(N^{-1} \epsilon))$, that is, $q$ is the endpoint of a broken geodesic $\gamma$ starting from $p$ and consisting of $N$ nonholonomic geodesics, each one of length less than $N^{-1} \bar{\epsilon} = C_1 d_c(p, q)$ with $C_1 = \frac{2c_1}{Nc_2}$ (Take $\bar{\epsilon}_0$ smaller if necessary such that for any $p \in B(y_0, \epsilon_0)$, $F^p$ is well defined in $O'$). We also get $d_{\mathcal{H}}(p, q) \leq C_2 d_c(p, q)$ with $C_2 = \frac{2c_1}{c_2}$. \hfill $\Box$

Remark 3.11.

1. The condition “with $\Sigma$ bracket generating and $M$ connected” in Theorem 3.8 and Corollary 3.10 is not necessary, since we assume $y_0$ is regular, $d_c$ and $d_{\mathcal{H}}$ can be defined in $O$.
2. From the proof of Corollary 3.10 we see that moreover $M$ is compact then the constants $C_1, C_2$ and $\bar{\epsilon}_0$ can be taken universal constants.
3. If $\Sigma$ is regular and $M$ connected, the topology of $(M, d_{\mathcal{H}})$ is the same as that of $(M, d_c)$.
4. As pointed out in Section 2, the horizontal connection depends on the choice of the complement of $\Sigma$. So different choices of splitting lead to different sets of nonholonomic geodesics. However Corollary 3.10 holds for all choices with $C_1$ and $C_2$ possibly changed up to the choice.

In the remainder of this section we will consider nonholonomic geodesics and $d_{\mathcal{H}}$ for some special sub-Riemannian manifolds, in particular for Carnot groups.

Theorem 3.12. Let $(M, \Sigma, g_c)$ be a sub-Riemannian manifold where $g_c = g|_{\Sigma}$ for a Riemannian metric $g$ on $M$. If for any horizontal vector field $X \in \Gamma(\Sigma)$ and any vertical vector field in the orthogonal complement, $Y \in \Gamma(\Sigma^\perp)$ one has $[X, Y] \in \Gamma(\Sigma^\perp)$, then a horizontal curve is a nonholonomic geodesic of $(M, \Sigma, g_c)$ if and only if it is a Riemannian geodesic of $(M, g)$.
Proof. Let $\gamma$ be a nonholonomic geodesic (assume not a constant). Since $\gamma$ has constant speed it is regular and one can extend $\dot{\gamma}$ to a smooth horizontal vector field $Y$ in a small neighborhood $U$ of $\gamma(t_0)$ such that $\|Y\|^2 := g_c(\dot{\gamma}, \dot{\gamma}) \geq \frac{1}{2}c$ in $U$, where $c := g_c(\dot{\gamma}, \dot{\gamma})$.

Then $X := c\frac{Y}{\|Y\|}$ is also a smooth extension of $\dot{\gamma}$ with constant norm $c$. We claim that the projection on $\Sigma^\perp$ of $\nabla_X X$ vanishes, i.e., $(\nabla_X X)^\perp = 0$ in $U$. In fact, for any vertical vector field $W \in \Gamma(\Sigma^\perp)$, we have

$$g((\nabla_X X)^\perp, W) = -g(\nabla_X W, X)$$

$$= -g(\nabla_W X, X) + g([W, X], X)$$

$$= -W g(X, X)$$

$$= 0$$

where we used the condition that $[W, X] \in \Gamma(\Sigma^\perp)$ and $g(X, X) \equiv c$. Thus $(\nabla_{\dot{\gamma}}(t))^\perp = 0$ in a small neighborhood of $t_0$. Thus $\dot{\gamma}$ is a Riemannian geodesic. The inverse follows from a similar argument. □

Because for a Riemannian submersion the Lie bracket of a horizontal (projectable) vector field with a vertical vector field is vertical (see e.g. [6]), we have the following statement.

**Corollary 3.13.** Let $p$ be a Riemannian submersion $\pi : (M, g) \to (B, g')$ and let $\Sigma$ be the horizontal bundle of $(M, g)$. Then a horizontal curve is a nonholonomic geodesic of $(M, \Sigma, g_c = g|\Sigma)$ if and only if it is a Riemannian geodesic of $(M, g)$. Thus the set of nonholonomic geodesics of $(M, \Sigma, g_c)$ consists of horizontal lifts of all Riemannian geodesics of $(B, g')$ because the horizontal lift of every Riemannian geodesic of $(B, g')$ is a Riemannian geodesic of $(M, g)$.

**Corollary 3.14.** A horizontal curve in a Carnot group is a nonholonomic geodesic if and only if it is an integral curve of a left invariant horizontal vector field.

Proof. Let $G$ be a Carnot group. The submersion $\pi$ given in the proof of Lemma 2.3 satisfies $\pi(p, p') = \pi(p) + \pi(p')$ for any $p, p'$. Since geodesics in $\mathbb{R}^k$ are lines (or their intervals), the statement follows from Corollary 3.13 and a direct computation. □

Corollary 3.14 can also be verified by a direct computation using Lemma 2.7 and (2.3).

**Remark 3.15.** For Carnot groups $G$, due to a homogeneous structure in $G$ Folland and Stein ([9, Lemma 1.40]) proved the first statement of Corollary 3.10 holds globally with a universal constant $C_1$.

In any sub-Riemannian manifold most sub-Riemannian geodesics are not nonholonomic geodesics and broken geodesics. It is natural to ask whether or not a sub-Riemannian manifold $(M, \Sigma, g_c)$ admits a complement such that its corresponding $d_H$ satisfies $d_c = d_H$. So far we can give a positive answer only for the Heisenberg group $\mathbb{H}^n$. First we have

**Lemma 3.16.** In a Carnot group $G$ any nonholonomic geodesic $\gamma$ is a shortest (sub-)Riemannian geodesic. So if $p, q \in \gamma$, $d_c(p, q) = d_H(p, q)$.
Proof. By Corollary 3.13 and 3.14, $\gamma$ is a Riemannian geodesic in $G$. Since the interval or the line $\pi(\gamma)$ is obviously shortest in Euclidean space $\mathbb{R}^k$, the horizontal lift $\gamma$ must be a shortest Riemannian geodesic in $G$. Now the statement follows from $d_r \leq d_c \leq d_H$ where $d_c$ is the Riemannian distance. 

Proposition 3.17. In the Heisenberg group $\mathbb{H}^n$ we have $d_c = d_H$. 

Proof. It is enough to prove that for any $\lambda > 0$ and any $p^1, p^2 \in \mathbb{H}^n$ and any shortest sub-Riemannian geodesic $\gamma$ connecting $p^1, p^2$ there exists a broken geodesic $\beta$ connecting $p^1, p^2$ such that the difference between their lengths is less than $\lambda$. 

For $p^i = (x^i, y^i, t^i)$, let $(x^i, y^i) = \bar{p}^i = \pi(p^i) \in \mathbb{R}^{2n}, i = 1, 2$ and $\bar{\gamma} = \pi(\gamma)$ where $\pi$ as in the proof of Lemma 2.3. If $\bar{\gamma}$ is an interval of a line, then $\gamma$ is a nonholonomic geodesic. By Lemma 3.10, $d_c(p^1, p^2) = d_H(p^1, p^2)$.

If $\bar{\gamma}$ is an “arc” of a “circle”, let $A(\bar{\gamma}) = \int_{\bar{\gamma}} (ydx - xdy)$ be the symplectic area enclosing by $\bar{\gamma}$. Given $\epsilon > 0$ we always can find in $\mathbb{R}^{2n}$ a broken geodesic $\bar{\alpha}$ connecting $\bar{p}^1$ and $\bar{p}^2$ such that $|A(\bar{\gamma}) - A(\bar{\alpha})| < C\epsilon$ and $|\ell(\bar{\alpha}) - \ell(\bar{\gamma})| < \epsilon$ where $C$ is a constant depending only on $p^1$ and $p^2$. Let $\alpha$ be the unique horizontal lift of $\bar{\alpha}$ through $p^1$. Then $\alpha$ is a broken geodesic in $\mathbb{H}^n$. Denote by $p' = (x^2, y^2, t')$ the other endpoint of $\alpha$. Then from (2.1) we get $|t' - t^2| = |A(p') - A(p^2)| < C\epsilon$. So $d_c(p', p^2) = 2\sqrt{2\pi} \sqrt{|t' - t^2|} < C\epsilon$. Now we use Corollary 3.10 or [9, Lemma 1.40] to get a broken geodesic $\delta$ connecting $p'$ and $p^2$ with $\ell(\delta) < C^\lambda \sqrt{\epsilon}$ for a constant $C^\lambda$ depending only on $p^1$ and $p^2$. Concatenate $\alpha$ and $\delta$ to get a broken geodesic $\beta$ connecting $p^1$ and $p^2$. The difference between the lengths of $\gamma$ and $\beta$ satisfies 

$$|\ell(\beta) - \ell(\gamma)| = |\ell(\alpha) + \ell(\delta) - \ell(\gamma)|$$

$$= |\ell(\alpha) - \ell(\gamma)| + |\ell(\delta)|$$

$$\leq \epsilon + C^\lambda \sqrt{\epsilon}.$$ 

$\beta$ is the desired broken geodesic. \hfill \Box

4. GEODESICALLY CONVEX FUNCTIONS ON SUB-RIEMANNIAN MANIFOLDS

The theory of convex functions on Riemannian manifolds plays a very important role in the study of analysis and geometry on manifolds. We expect a similar theory in the setting of sub-Riemannian setting and wish that it could help the study of foliation theory, contact geometry, Riemannian submersions and so on. A function on a Riemannian manifold is convex if so is its restriction to any Riemannian geodesic. In a similar way we define convex functions on sub-Riemannian manifolds.

Definition 4.1 (n-convex sets). A subset $\Omega$ of a sub-Riemannian manifold $(M, \Sigma, g_c)$ is called nonholonomically geodesic convex (n-convex for short) if for any two points $p, q \in \Omega$ and if there exists a nonholonomic geodesic connecting them, then the segment between them is also in $\Omega$.

Definition 4.2 (n-convex functions). Let $\Omega$ be a n-convex subset of a sub-Riemannian manifold $(M, \Sigma, g_c)$. A function defined on $\Omega$ is called n-convex if its restriction to any nonholonomic geodesic contained in $\Omega$ is convex.
From Corollary 3.14 we have

**Proposition 4.3.** In Carnot groups n-convex functions are the same as h-convex functions defined by Danniell-Garofalo-Nieuhn [6].

By Theorem 3.12 every Riemannian convex function on a sub-Riemannian manifold satisfying the condition of Theorem 3.12 is n-convex.

**Theorem 4.4.** A smooth function $f$ on a sub-Riemannian manifold $(M, \Sigma, g_c)$ is n-convex if and only if its horizontal Hessian is positive semidefinite.

**Proof.** Since $\text{Hess}^H f$ is a tensor, the nonnegativity of $\text{Hess}^H f$ is equivalent to that of $\text{Hess}^H f(\dot{\gamma}, \dot{\gamma})$ for any nonholonomic geodesic $\gamma$. Let $\gamma$ be a nonholonomic geodesic. Because

$$\frac{d^2}{dt^2} (f \circ \gamma) = \dot{\gamma}(\dot{\gamma}(f))(\gamma)$$

$$= (\nabla_{\dot{\gamma}} (df)) (\dot{\gamma})$$

$$= \text{Hess} f(\dot{\gamma}, \dot{\gamma}) - (\nabla_{\dot{\gamma}} \dot{\gamma}) f$$

and by (2.4)

$$\text{Hess}^H f(\dot{\gamma}, \dot{\gamma}) = \text{Hess} f(\dot{\gamma}, \dot{\gamma}) - B(\dot{\gamma}, \dot{\gamma}) f$$

the statement follows from $D_{\dot{\gamma}} \dot{\gamma} = 0$ and the fact that $\frac{d^2}{dt^2} (f \circ \gamma)(t) = 0$ if and only $f \circ \gamma$ is convex. $\square$

**Definition 4.5.** A sub-Riemannian manifold $(M, \Sigma, g_c)$ is nonholonomically complete if every nonholonomic geodesic can be defined on the whole real line.

Given a Riemannian submersion $\pi : (M, g) \to (B, g')$, if $(B, g')$ is complete, then $(M, \Sigma, g|_\Sigma)$ is nonholonomically complete.

**Proposition 4.6.** Let $(M, \Sigma, g_c)$ be a nonholonomically complete sub-Riemannian manifold with $\Sigma$ regular and $M$ connected. If a n-convex function $f : M \to \mathbb{R}$ is upper bounded, then $f$ must be a constant.

**Proof.** Let $f : M \to \mathbb{R}$ be a n-convex function. Since $M$ is nonholonomically complete, every nonholonomic geodesic $\gamma(t)$ can be defined in $(-\infty, \infty)$. Let $p, q$ be in a nonholonomic geodesic $\beta(t) : [0, 1] \to M$ such that $\beta(0) = p$ and $\beta(1) = q$. Extend $\beta$ to $[0, t]$ for $t > 1$. By setting $u = st, s \in [0, 1], \beta$ is reparameterized to $\tilde{\beta}(s) := \beta(st)$. The convexity of $f$ on $\tilde{\beta}$ implies

$$f(\tilde{\beta}(s)) \leq (1 - s)f(\tilde{\beta}(0)) + sf(\tilde{\beta}(1))$$

$$= (1 - s)f(p) + sf(\beta(t))$$

for any $s \in [0, 1]$. In particular for $s = \frac{1}{t}$ we get

$$f(q) = f(\beta(1)) = f(\tilde{\beta}(\frac{1}{t}))$$

$$\leq (1 - \frac{1}{t})f(p) + \frac{1}{t}f(\beta(t)).$$
Since \( f \) is upper bounded, letting \( t \) go to \( \infty \) we obtain \( f(p) \leq f(q) \). Similarly we have \( f(p) \geq f(q) \). Thus \( f \) is constant on any nonholonomic geodesic. Now the statement follows from Theorem 3.6.

We turn to the regularity problem of \( n \)-convex functions on regular, connected sub-Riemannian manifolds. Balogh and Rickly (11) proved that for the Heisenberg group \( \mathbb{H}^n \) \( h \)-convex functions are locally Lipschitz with respect to the Carnot-Carathéodory distance. The arguments and regularity result in [11] were later extended to Carnot groups of step 2 by [19] and [20] independently. For general Carnot groups, [15] proved that any \( h \)-convex function with local upper bound is locally Lipschitz. For more properties, in particular the equivalence of several notions of \( h \)-convex functions, and their applications in nonlinear subelliptic PDEs, we refer to [6, 14, 7, 10, 11, 12, 11, 15, 29, 19] and references therein.

In the following we assume \( \Omega \) be a \( n \)-convex open subset in a sub-Riemannian manifold \((M, \Sigma, g_c)\) with \( \Sigma \) regular and \( M \) connected.

**Proposition 4.7.** Any \( n \)-convex function \( f : \Omega \to \mathbb{R} \) locally bounded above is also locally bounded below. More explicitly, if \( f \leq C \) in a neighborhood of \( y_0 \), then there exists a constant \( \bar{c}_0 \) such that \( f(p) \geq C(y_0) - (2^N - 1)C \) for any \( p \in B(y_0, \bar{c}_0) \), where \( N \) is the integer in Corollary 5.10.

**Proof.** For \( y_0 \in \Omega \) let \( \mathcal{O} \) be the neighborhood of \( y_0 \) as in Theorem 3.5 where there exists a horizontal frame \( \{E_1, \cdots, E_k\} \) of norm 1 such that integral curves of \( E_i \) are nonholonomic curves. Let \( \mathcal{O}' \) be a neighborhood of \( y_0 \) such that \( \mathcal{O}' \subset \mathcal{O} \) and there exists \( \bar{c}_0 > 0 \) such that for any \( p \in \mathcal{O}' \) and \( i = 1, \cdots, k \), the integral curve \( \Psi_i^p(t) \) of \( E_i \) with \( \Psi_i^p(0) = p \) is defined in the interval \([0, \bar{c}_0] \). Take \( \bar{c}_0 \) smaller if necessary such that \( B(y_0, \bar{c}_0) \subset \mathcal{O}' \). Let \( B(y_0, \bar{c}_0) \subset \mathcal{O}' \) be the ball as in Corollary 5.10. Again take \( \bar{c}_0 \) smaller if necessary such that \( NC_1 \bar{c}_0 < \bar{c}_0 \).

By Corollary 5.10 for any \( p \in B(y_0, \bar{c}_0) \), there exists a broken geodesic \( \gamma \) connecting \( y_0 \) and \( p \), which consists of \( N \) nonholonomic geodesics, each one of length less than \( C_1 d_c(y_0, p) \). Let \( q_i(i = 1, \cdots, N - 1) \) be the \( N - 1 \) breaks and \( \gamma_0 \) the nonholonomic geodesic from \( y_0 \) to \( q_1 \), \( \gamma_i \) the one from \( q_{i-1} \) to \( q_i \) and finally \( \gamma_{N-1} \) from \( q_{N-1} \) to \( p \). From our choice of \( \bar{c}_0 \) and \( \bar{c}_0 \), we have \( \gamma_i \subset \mathcal{O}' \) and \( \ell(\gamma_i) < \bar{c}_0 \), \( i = 1, \cdots, N - 1 \). Because \( E_i \)'s are of norm 1, \( \gamma_i \)'s are parameterized by length and thus \( \gamma_i \) can be extended to \( \tilde{\gamma}_i \) such that \( q_i \) is the middle point of \( \tilde{\gamma}_i \), \( i = 0, \cdots, N - 1 \) with \( q_0 = y_0 \). Also \( \tilde{\gamma}_i \subset \mathcal{O}' \).

Assume the \( n \)-convex function \( f \) in \( \mathcal{O}' \) is bounded by \( C \) from above. We claim \( f \) is bounded below in \( B(y_0, \bar{c}_0) \). In fact, because by definition \( f \) is convex on each \( \gamma_i \) and \( q_i \) is the middle point of \( \tilde{\gamma}_i \) \( i = 0, \cdots, N - 1 \), we have for \( p \in B(y_0, \bar{c}_0) \)

\[
f(p) \geq 2f(q_{N-1}) - C \\
\geq 2(2f(q_{N-2}) - C) - C = 2^2 f(q_{N-2}) - (1 + 2)C \\
\cdots \\
\geq 2^{N-1} f(q_1) - (1 + 2 + 2^2 + \cdots + 2^{N-2})C \\
\geq 2^{N-1}(2f(q_0) - C) - (1 + 2 + 2^2 + \cdots + 2^{N-2})C \\
= 2^N f(y_0) - (1 + 2 + 2^2 + \cdots + 2^{N-1})C \\
= 2^N f(y_0) - (2^N - 1)C
\]
Now we prove the locally Lipschitz continuity of n-convex functions locally bounded above. We follow closely the arguments in [19] for the case of Carnot groups.

**Theorem 4.8.** If a n-convex function \( f : \Omega \to \mathbb{R} \) is locally bounded above, then \( f \) is locally Lipschitz continuous with respect to \( d_c \) and \( d_H \).

**Proof.** Let \( y_0, \mathcal{O}, E_i, \mathcal{O}', \tilde{\varepsilon}_0, \tilde{\varepsilon}_0 \) be as in the proof of Proposition 4.7 such that \((2NC_1+1)\tilde{\varepsilon}_0 < \frac{1}{2}\tilde{\varepsilon}_0\). Let \( f \) be bounded in \( B(y_0, \tilde{\varepsilon}_0) \), |\( f \) | \( \leq \) \( C \). We first prove that \( f \) is Lipschitz on any integral curve \( \gamma \) of \( E_i \)'s which is contained in \( B(y_0, \frac{1}{4}\tilde{\varepsilon}_0) \). To this aim we assume \( \gamma(0) \in B(y_0, \frac{1}{7}\tilde{\varepsilon}_0) \). Since \( \gamma \) is parameterized by length and is contained in \( B(y_0, \frac{1}{4}\tilde{\varepsilon}_0) \), \( \ell(\gamma) \leq \frac{1}{2}\tilde{\varepsilon}_0 \). According to the choice of \( \tilde{\varepsilon}_0 \), \( \gamma \) can be extended. Still denote by \( \gamma(t) \) the extended \( \gamma \). Set \( t_- := \max\{t < 0|\gamma(t) \in \partial B(y_0, \frac{1}{2}\tilde{\varepsilon}_0)\} > -\tilde{\varepsilon}_0 \) and \( t_+ := \min\{t > 0|\gamma(t) \in \partial B(y_0, \frac{1}{2}\tilde{\varepsilon}_0)\} < \tilde{\varepsilon}_0 \). Then \( t_+ - t_- = [\frac{1}{3}\tilde{\varepsilon}_0, \tilde{\varepsilon}_0] \), and if \( t \in [t_-, t_+] \) and \( \gamma(t) \in B(y_0, \frac{1}{4}\tilde{\varepsilon}_0) \), \( t - t_- \geq \frac{1}{3}\tilde{\varepsilon}_0 \) and \( t_+ - t \geq \frac{1}{4}\tilde{\varepsilon}_0 \). Thus we can pick \( \lambda \in [\frac{1}{2}, \frac{3}{4}] \) such that \( t = (1-\lambda)t_- + \lambda t_+ \). Let \( t_1, t_2 \in [t_-, t_+] \) such that \( t_1 < t_2 \) and \( \gamma(t_1), \gamma(t_2) \in B(y_0, \frac{1}{4}\tilde{\varepsilon}_0) \). Then \( t_i = (1-\lambda_i)t_- + \lambda_i t_+ \) where \( \lambda_i \in [\frac{1}{4}, \frac{3}{4}] \), \( i = 1, 2 \) and \( \lambda_1 < \lambda_2 \). So

\[
\begin{align*}
t_1 &= \frac{\lambda_2 - \lambda_1}{\lambda_2}t_- + \frac{\lambda_1}{\lambda_2}t_2 \quad \text{and} \quad t_2 = \frac{1 - \lambda_2}{1 - \lambda_1}t_1 + \frac{\lambda_2 - \lambda_1}{1 - \lambda_1}t_+.
\end{align*}
\]

Because as an integral curve of some \( E_i \), \( \gamma \) is a nonholonomic geodesic, from the convexity of \( f \) we obtain

\[
\begin{align*}
f(\gamma(t_1)) - f(\gamma(t_2)) &\leq \frac{\lambda_2 - \lambda_1}{\lambda_2}f(\gamma(t_-)) + \frac{\lambda_1}{\lambda_2}f(\gamma(t_2)) \\
&= \frac{t_2 - t_1}{\lambda_2(t_+ - t_-)}\{f(\gamma(t_-)) - f(\gamma(t_2))\} \\
&= \frac{d_H(\gamma(t_1), \gamma(t_2))}{\lambda_2(t_+ - t_-)}\{f(\gamma(t_-)) - f(\gamma(t_2))\} \\
&\leq \frac{8C}{\tilde{\varepsilon}_0}d_c(\gamma(t_1), \gamma(t_2)),
\end{align*}
\]

where we use \( d_H \leq d_c \) and the fact that when \( \tilde{\varepsilon}_0 \) is small enough, \( \gamma \) is the unique nonholonomic geodesic passing through \( \gamma(0) \). Similarly we have

\[
f(\gamma(t_2)) - f(\gamma(t_1)) \leq \frac{8C}{\tilde{\varepsilon}_0}d_c(\gamma(t_1), \gamma(t_2)).
\]

We have proved that

\[
|f(p_1) - f(p_2)| \leq \frac{8C}{\tilde{\varepsilon}_0}d_c(p_1, p_2)
\]

for any \( p_1, p_2 \in \gamma \subset B(y_0, \frac{1}{4}\tilde{\varepsilon}_0) \) where \( \gamma \) is a segment of an integral curve of some \( E_i \).

Now for any \( p, q \in B(y_0, \varepsilon_0) \), by Corollary 5.10 there exists a broken geodesic \( \beta \) connecting \( p \) to \( q \), and \( \beta \) consists of \( N \) nonholonomic geodesics, as an integral curve of some
each one of length less than \( C_1 d_c(p, q) \). Since
\[
d_c(y_0, \beta) \leq d_c(\beta, p) + d_c(y_0, p) \\
\leq NC_1 d_c(p, q) + d_c(y_0, p) \leq (2NC_1 + 1)\tilde{\epsilon}_0 \\
\leq \frac{1}{2}\tilde{\epsilon}_0,
\]
each nonholonomic geodesic segment of \( \beta \) is contained in \( B(y_0, \tilde{\epsilon}_0) \). Thus to each segment we can use the above estimate to get
\[
|f(p) - f(q)| \leq \frac{8NC_1}{\tilde{\epsilon}_0} d_c(p, q) \leq \frac{8NC_1}{\tilde{\epsilon}_0} d_H(p, q).
\]
□

From Theorem 4.8 and Proposition 4.6 or 4.7 we have

**Corollary 4.9.** Let \((M, \Sigma, g_c)\) be a compact sub-Riemannian manifold with \( \Sigma \) regular and connected. Then on \( M \) any \( n \)-convex function locally bounded above is constant.

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