LOWER BOUNDS FOR SUP + INF AND SUP * INF AND AN EXTENSION OF
CHEN-LIN RESULT IN DIMENSION 3.

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Abstract. We give two results about Harnack type inequalities. First, on compact smooth
Riemannian surface without boundary, we have an estimate of the type \( \sup + \inf \). The second
result concerns the solutions of prescribed scalar curvature equation on the unit ball of \( \mathbb{R}^n \) with
Dirichlet condition.

Next, we give an inequality of the type \( (\sup_K u)^{2s-1} \times \inf_{\Omega} u \leq c \) for positive solutions of
\( \Delta u = Vu^s \) on \( \Omega \subset \mathbb{R}^3 \), where \( K \) is a compact set of \( \Omega \) and \( V \) is \( s \)-holderian, \( s \in ] -1/2, 1 ] \).
For the case \( s = 1/2 \), we prove that if \( \min_{\Omega} u > m > 0 \) and the holderian constant \( A \) of \( V \)
is small enough ( in certain meaning), we have the uniform boundedness of the supremum of the
solutions of the previous equation on any compact set of \( \Omega \).

1. INTRODUCTION AND RESULTS.

We denote \( \Delta = -\nabla^j (\nabla_j) \), the geometric Laplacian.

On compact smooth Riemann surface without boundary \( (M, g) \) we consider the following
equation :

\[
\Delta u + k = Ve^u, \quad (E_1)
\]

with, \( k \in \mathbb{R}_{+} \) and \( 0 \leq V \leq b \) ( \( V \not\equiv 0 \) ).

We suppose \( V \) smooth. The previous equation is of type prescribed scalar curvature. We
search to know if it’s possible to have a priori estimate of the type \( \sup + \inf \).

Note that in dimension 2, on \( \mathbb{R}^2 \), we have different results about \( \sup + \inf \) inequalities for the
following equation:

\[
\Delta u = Ve^u, \quad (E_2)
\]

see [B-L-S], [B-M], [C-L 2], [L 2] and [S].

In [S], Shafrir proved an inequality of the type \( \sup u + C \inf u < C' \) with minimal conditions
on the prescribed scalar curvature. In [B-L-S], Brezis-Li-Shafrir have proved a \( \sup u + \inf u \)
inequality with lipschitzian assumption on prescribed curvature. Finally, [C-L 2] have proved the
same result with holderian assumption on \( V \) in the equation \( (E_2) \).

Here, we are interested by the minoration of this sum. We can suppose that \( Volume(M) = 1 \).
We obtain,

Theorem 1. For all \( k, b > 0 \), there exists a constant \( c = c(k, b, M, g) \) such that, for all solution
of \( (E_1) \):

\[
\frac{k - 4\pi}{4\pi} \sup_M u + \inf_M u \geq c.
\]

We can remark that for \( k = 8\pi \), we have the same result than in [B 1]. Here there is no
restriction on \( k \).

Now we work in dimension \( n \geq 3 \), we set \( B = B_1(0) \) the unit ball of \( \mathbb{R}^n \). We try to study
some properties of the solutions of the following equation:

\[
\Delta u = Vu^{N-1-\epsilon}, \, u > 0 \text{ in } B, \, u = 0 \text{ on } \partial B \quad (E_3)
\]
with \(0 \leq V(x) \leq b < +\infty, 0 \leq \epsilon < 2/(n-2)\) and \(N = \frac{2n}{n-2}\) the critical Sobolev exponent.

Equation \((E_3)\) is the prescribed scalar curvature equation, it was studied a lot. We know, after using Pohozaev identity that, there is no solution for this equation if we assume \(\epsilon = 0\) and \(V \equiv 1\), see [P].

**Theorem 2.** For all compact \(K\) of \(B\), there exists one positive constant \(c = c(n, b, K)\) such that for all solution of \((E_3)\):

\[
(s_{\sup B} u)^7 \times \inf_{K} u \geq c.
\]

Recall that estimates like in the last theorem exist, see for example [B 1] et [B 2].

Now we work on \(\Omega \subset \mathbb{R}^3\) and we consider the following equation:

\[
\Delta u = Vu^5, \; u > 0, \quad (E_4)
\]

with,

\[
0 < a \leq V(x) \leq b \quad \text{and} \quad |V(x) - V(y)| \leq A|x - y|^s, \; s \in \left[\frac{1}{2}, 1\right], x, y \in \Omega. \quad (C)
\]

Without loss of generality, we suppose \(\Omega = B\) the unit ball of \(\mathbb{R}^3\).

The equation \((E_4)\) is the scalar curvature equation in three dimensions. It was studied a lot, see for example [B 3], [C-L 1], [L 1]. In [C-L 1], Chen and Lin have proved that if \(s > \frac{1}{2}\) then each sequence \((u_k)_k\) which are solutions of \((E_4)\) (with fixed \(V\)) are in \(L^\infty_{loc}\) if we suppose \(\min_B u_k > m > 0\). When \(s = 1\) they prove that the \(\sup \times \inf\) inequality holds. To prove those results, they use the moving-plane method.

In [L 1], Li proved (in particular) that the product \(\sup \times \inf\) is bounded if we replace \(\Omega\) by the three sphere \(S_3\). He used the notion of isolated and isolated simple blow-up points.

We can see in [B 3] another proof of the boundedness of \(\sup^{1/3} \times \inf\), also with the moving-plane method.

Note that, if we suppose \(\Omega\) a Riemannian manifold of dimension 3 (not necessarily compact), Li and Zhang (see [L-Z]) have proved that the \(\sup \times \inf\) holds when the prescribed scalar curvature is a constant.

Note that, in our work, we have no assumption on energy. There are many results, if we suppose the energy bounded.

Here, we use the moving-plane method to have \(\sup \times \inf\) inequalities. This method was developed by Gidas-Ni-Nirenberg, used by Chen-Lin and Li-Zhang, see [G-N-N], [C-L 1] and [L-Z]. In our work we follow and use the technique of Li and Zhang, see [L-Z].

**Theorem 3.** If \(s \in \left[\frac{1}{2}, 1\right]\), then, for all positive numbers \(a, b, A\) and all compact \(K\) of \(B\), there exists a positive constant \(c = c(a, b, A, s, K)\) such that:

\[
(s_{\sup K} u)^{2s-1} \times \inf_{B} u \leq c,
\]

where \(u\) is solution of \((E_4)\) with \(V\) satisfying \((C)\).

For \(s = \frac{1}{2}\) and \(a, b, m > 0\), there exists \(\delta = \delta(a, b, m) > 0\) such that for \(u\) solution of \((E_4)\) with \(A \in (0, \delta)\) for \(V\) in \((C)\) and \(u \geq m\), we have:

\[
\sup_{K} u \leq c = c(a, b, m, K),
\]

where \(K \subset \subset B_1\).
Note that in [B 3], for the dimension 4, we have a result like in the second part of the theorem 3.

About usual Harnack inequalities, we can find in [G-T] lots of those estimates. For harmonic functions ($\Delta u = - \sum_{i=1}^{n} \partial_{ii} u = 0$ on open set of $\mathbb{R}^n$), we have an estimate of the type:

$$\frac{\sup_{B_R} u}{\inf_{B_R} u} \leq 3^n,$$

on small ball $B_R$ of radius $R$ (see chapter 1 in [G-T]).

We have other results if we consider a general elliptic operator ($L = \partial_i (a^{ij} \partial_j) + \sum_{j=1}^{n} b^j \partial_j + c$ on open set of $\mathbb{R}^n$), we obtain:

$$\frac{\sup_{B_R} u}{\inf_{B_R} u} \leq C[n, R, c, (b^j), (a^{ij})],$$

for a non negative function $u$ such that $Lu = 0$ ($B_R$ is a ball of radius $R$). See for example theorem 8.20 in [G-T].

For subharmonic and superharmonic functions there is another type of Harnack inequalities linking their norm $L^p$ to their infimum or supremum. (See chapter 8 in [G-T]).

Here we follow the same idea and we try to compare the $\sup$ and the $\inf$ in a certain meaning.

2. PROOFS OF THE THEOREMS.

**Proof of Theorem 1:**

Consider the equation:

$$\Delta u_i + k = V_i e^{\alpha_i},$$

**Case 1: $\sup_M u_i \leq c < +\infty$.**

We set $x_i$ the point where $u_i$ is maximum, $u_i(x_i) = \sup_M u_i$, then:

$$0 \leq \Delta u_i(x) = V_i(x_i e^{u_i(x_i)} - k \leq be^{u_i(x_i)} - k,$$

thus,

$$\log \left( \frac{k}{b} \right) \leq u_i(x_i) \leq c'.$$

We denote $G$ the Green function of laplacian,

$$\Delta_{d, \text{distribution}} G(x, .) = 1 - \delta_x \text{ and } G(x, y) \geq 0, \int_M G(x, y) = C.$$ we can write,

$$\log \left( \frac{k}{b} \right) \leq u_i(x_i) = \int_M u_i dV_g - \int_M G(x_i, y) [V_i(y) e^{u_i(y)} - k] dV_g \leq \int_M u_i + C(be^c - k).$$

We deduce:

$$-\infty < c_2 \leq \int_M u_i \leq c_1 < +\infty, \forall i.$$ Now, we write,

$$\min_M u_i = u_i(y_i) = \int_M u_i + \int_M G(y_i, y) [V_i(y) e^{u_i(y)} - k] dV_g \geq c_2 - kC > -\infty.$$ Thus,
\[ ||u_i||_{L^\infty} \leq \epsilon' < +\infty, \forall i. \]

**Case 2:** \( \sup_i u_i \rightarrow +\infty. \)

According to T. Aubin (see [A]), we have,

\[ G(x, y) = -\frac{1}{2\pi} \log d(x, y) + g(x, y), \]

where \( g \) is a regular part of \( G \), it is a continuous function on \( M \times M \).

Let us note \( x_i \) the point where \( u_i \) is maximum, \( u_i(x_i) = \max_M u_i \). We can suppose that \( x_i \rightarrow x_0 \) and in the conformal isothermal coordinates around \( x_0 \) we set \( v_i(x) = u_i(x_i + xe^{-u_i(x_i)/2}) - u_i(x_i) \), then,

\[ \Delta v_i + h_i = \tilde{V}_i e^{v_i}, \quad h_i \rightarrow 0 \]

\( v_i(0) = 0, v_i(x) \leq 0, 0 \leq \tilde{V}_i(x) \leq b. \) We can use theorem 3 of [B-M] and we deduce after passing to the subsequence that:

\[ v_i(x) \geq C > -\infty, \quad \text{for} \ |x| \leq r. \]

Thus,

\[ u_i(y) \geq u_i(x_i) + C, \quad \text{if} \ d(y, x_i) \leq re^{-u_i(x_i)/2}, \]

Now, we work on \( M - B(x_i, re^{-u_i(x_i)/2}) \),

\[ G(x, y) \leq \frac{1}{4\pi} u_i(x_i) + C_1, \quad \text{on} \ \partial B(x_i, re^{-u_i(x_i)/2}) \]

\[ \Delta[u_i - kG(x_i, \cdot)] \geq 0, \quad \text{on} \ M - B(x_i, re^{-u_i(x_i)/2}) \]

\[ u_i(y) - kG(x_i, y) - \frac{4\pi - k}{4\pi} u_i(x_i) + kC_1 - C \geq 0, \quad \text{on} \ \partial B(x_i, re^{-u_i(x_i)/2}). \]

By maximum principle, we obtain:

\[ u_i \geq kG(x_i, \cdot) + \frac{4\pi - k}{4\pi} u_i(x_i) - kC_1 + C, \quad \text{on} \ M - B(x_i, re^{-u_i(x_i)/2}), \]

We use the fact, \( \int_M G(x, y) = \text{constant} \), and by integration of the last inequality we have,

\[ \inf_M u_i + \frac{k - 4\pi}{4\pi} \sup_M u_i \geq c > -\infty. \]

**Example with** \( V_i \rightarrow 0 \): we can take, \( u_i \equiv \log k + \log i \) and \( V_i \equiv 1/i. \)

**Remark:** If we suppose \( V_i \geq a > 0 \) uniformly, then, when \( k < 4\pi \) we can not have \( \sup_M u_i \rightarrow +\infty \). To see this, it is sufficient to integrate the equation.

**Proof of Theorem 2:**

We are going to prove that each sequence has a subsequence who has the searched inequality.

Next, we use the fact that, if we have possibility to extract a subsequence we do it and we denote \((u_i)_i\) the subsequence.

We have,

\[ \Delta u_i = V_i u_i^{N-1-\epsilon_i}, \quad u_i > 0 \quad \text{on} \ B, \quad (\tilde{E}) \]

with \( 0 \leq V_i(x) \leq b \ (V_i \neq 0). \)

Let us note \( G \) the Green function of the laplacian on unit ball with Dirichlet condition. \( G \) is of the form:

\[ u_i(x) = G(x, y) + \tilde{V}_i(x) e^{v_i}, \quad \text{on} \ M. \]
\[ G(x, y) = \frac{1}{n(n-2)\omega_n|x-y|^{n-2}} - \frac{1}{n(n-2)\omega_n(|x|^2|y|^2 + 1 - 2x.y)^{(n-2)/2}}. \]

Denote \( x_i \) the point where \( u_i \) is maximum. We write:

\[ u_i(x_i) = \int_B G(x_i, y)V_i(x)|u_i(y)|^{N-1-\epsilon_i}dy \leq b|u_i(x_i)|^{N-1-\epsilon_i} \int_B G(x_i, y)dy. \]

Consider the function \( h(x) = |x|^2 - 1 \), we have:

\[ \int_B G(x_i, y)dy = \frac{1 - |x_i|^2}{2n} \leq d(x_i, \partial B)/n. \]

We deduce:

\[ 0 < \frac{n}{b} \leq |u_i(x_i)|^{4/(n-2) - \epsilon_i} d(x_i, \partial B). \]

**Case 1:** \( \max_B u_i \leq c \)

Then, \( d(x_i, \partial B) \geq \epsilon' > 0 \). By elliptic estimates, \( u_i \rightharpoonup u \), with \( u > 0 \). Then, \( \inf_K u_i \geq \hat{c} > 0 \) with \( K \subset \subset B \).

To see this, we can write \((\mathcal{E})\) as:

\[ \Delta u_i = f_i \]

with, \( f_i \) uniformly in \( L^p \) for \( p > n \). We can use the elliptic estimates to have \( u_i \) uniformly in \( W^{2,p}(B) \) and by the Sobolev embedding, we have \( u_i \) uniformly in \( C^{1,\theta}(\overline{B}) \), for some \( \theta \in]0,1[ \).

Now, we can see that:

\[ \int_B \nabla u_i \cdot \nabla \varphi = \left( \int_B V_i u_i^{N-1-\epsilon_i} \varphi \right) \geq 0, \quad \text{for all } \varphi \in C_0^\infty(B), \varphi \geq 0 \text{ (distribution)}. \]

We can passe to the limit \( u_i \rightharpoonup u \geq 0 \) (subsequence) and \( u \in C^1(\overline{B}) \). Then, we have:

\[ \int_B \nabla u \cdot \nabla \varphi \geq 0, \quad \varphi \in C_0^\infty(B), \varphi \geq 0 \text{ (distribution)}. \]

We can use the strong maximum principle for weak solutions, see for example, Gilbarg-Trudinger, theorem 8.19 (applied to \( -u \leq 0 \)):

If, there is a point \( t \) in \( B \) such that, \( u(t) = 0 \) then, \( u \equiv 0 \). But we can see that \( u_i(x_i) \geq \hat{c}' > 0 \) with \( \hat{c}' \) do not depends on \( i \) and \( x_i \neq \partial B \) (subsequence).

Finally, \( u > 0 \) on \( B \).

**Remark 1.** Why do we do this? in fact, we have neither \( u_i \in C^2(\overline{B}) \) nor \( u \rightarrow u \) in \( C^2 \) norm because we don’t have more regularity on \( V_i \) and finally we don’t have \( \Delta u \geq 0 \) in the strong sense. We have weakly \( \Delta u \geq 0 \) with a good regularity on \( u \). Here, it is sufficient to have: \( C^1 \) regularity on \( u \) and an uniform boundedness for \( u_i \) in \( C^{1,\theta}(0 < \theta < 1) \), to obtain a good convergence for \( u_i \). After we can use a strong maximum principle for weak solutions.

**Remark 2.** If we take a sequence of functions \( V_i \) which converge uniformly to 0 (for example), the previous case 1 is not possible.

**Case 2:** \( \max_B u_i \rightarrow + \infty \)

1) \( x_i \rightarrow x_0 \in \partial B \):

To simplify our computations, we assume \( n/b > 1/2 \). Then, \( B(x_i, r_i) \subset B \), with \( r_i = 1/2[|u_i(x_i)|^{4/(n-2) - \epsilon_i}] \). We consider the following functions:

\[ u_i(x) = \frac{u_i[x_i + x/[u_i(x_i)]^{2/(n-2) - \epsilon_i}/2]}{u_i(x_i)}, \]

\[ \tilde{u}_i(x) = \frac{u_i[x_i + x/[u_i(x_i)]^{2/(n-2) - \epsilon_i}/2]}{u_i(x_i)} \]

\[ G_1(x, y) = \frac{1}{n(n-2)\omega_n|x-y|^{n-2}} - \frac{1}{n(n-2)\omega_n(|x|^2|y|^2 + 1 - 2x.y)^{(n-2)/2}}. \]
Those functions $v_i$, exist on $\Omega_0 = B(0, 5t_i)$, $t_i = 1/10|u_i(x_i)|^{2/(n-2) - \epsilon_i}/2$. We have:

$$\Delta v_i = \bar{V}_i v_i^{N-1-\epsilon_i}, \quad 0 < v_i(x) \leq v_i(0) = 1, \quad 0 \leq \bar{V}_i(x) \leq b.$$ 

with, $\bar{V}_i(x) = V_i[x_i + x/[u_i(x_i)]^{2/(n-2) - \epsilon_i}/2].$

We use Harnack inequality for $v_i$ (see Theorem 8.20 of [G-T]), we obtain:

$$\max_{B(0,t_i)} v_i \leq C \inf_{B(0,t_i)} v_i,$$

where $C = [C_0(n)]^{1+b}$ (see [G-T] and $t_i \leq 1$).

In 0, we obtain: $u_i(x) \geq C(n, b)u_i(x_i)$ for $|x| \leq s_i = 1/10|u_i(x_i)|^{4/(n-2) - \epsilon_i}$. Let us note that, $C(n, b) = C = [C_0(n)]^{1+b}$.

If we consider $B - B(x_i, s_i)$, then,

$$G(x_i, y) \leq c(n)|u_i(x_i)|^{4/(n-2)\epsilon_i}, \quad \text{for} \quad d(x_i, y) = s_i,$$

$$\Delta G(x_i, \cdot) = 0, \quad G(x_i, \cdot)_{|\partial B} = 0,$$

with, $c(n) = \frac{10^{n-2}}{n(n-2)\omega_n}$.

Thus,

$$u_i(y) - \frac{C(n, b)G(x_i, y)}{c(n)|u_i(x_i)|^{3/(n-2)\epsilon_i}} \geq 0, \quad \text{for} \quad d(y, x_i) = s_i, \quad \text{or, on} \quad \partial B.$$

By maximum principle, we have:

$$u_i(y) - \frac{C(n, b)G(x_i, y)}{c(n)|u_i(x_i)|^{3/(n-2)\epsilon_i}} \geq 0, \quad \text{on} \quad B - B(x_i, s_i).$$

In other terms,

$$u_i(y) \geq \frac{C(n, b)}{c(n)}G(x_i, y)|u_i(x_i)|^{-3/(n-2)\epsilon_i}, \quad \text{on} \quad B - B(x_i, s_i).$$

Now, we know that,

$$G(x_i, y) \geq c'(n)(1 - |y|)^{n-2} \times (1 - |x_i|)^{n-2}.$$

where $c'(n) = \frac{1}{2(n-2)2^{2(n-2)/2^2(n-2)}}\omega_n$.

We denote, $c'(n, b) = \frac{n}{b}$. Using the fact, $1 - |x_i| = d(x_i, \partial B) \geq c'(n, b)|u_i(x_i)|^{-4/(n-2) + \epsilon_i}$, we obtain,

$$u_i(y) \geq \frac{C(n, b)c'(n, b)}{c'(n)}(1 - |y|)^{n-2}|u_i(x_i)|^{-7+2(n-2)\epsilon_i}, \quad \text{on} \quad B - B(x_i, s_i).$$

On $B(0, k)$ with $k < 1$, by maximum principle we have: $\inf_{B(0, k)} u_i = \inf_{\partial B(0, k)} u_i$.

Then,

$$u_i(y) \geq C(n, b)(1 - k)^{n-2}|u_i(x_i)|^{-7+2(n-2)\epsilon_i}, \quad \text{on} \quad B(0, k) - B(x_i, s_i),$$

but, $x_i \rightarrow x_0 \in \partial B$, and for $i$ large we can conclude that $B(x_i, s_i) \cap B(0, k) = \emptyset$ and thus,

$$\inf_{B(0,k)} u_i \times |u_i(x_i)| \geq C(n, b, k).$$

We can remark that:
\[ C(n, b, k) = \frac{C(n, b)c'(n)c(n, b)}{c(n)(1 - k)^{n-2}}. \]

with, \( C(n, b) = C_0(n)^{1+b} \), \( c(n) = \frac{10^{n-2}}{n(n-2)\omega_n} \), \( c'(n) = \frac{1}{2(2n-2)^{2(n-2)}\omega_n} \) and \( c'(n, b) = \frac{n^b}{b} \).

Then,

\[ C(n, b, k) = \frac{C_0(n)^{1+b}2n^2(n - 2)2^{2(n-2)}\omega_n}{bn(n - 2)\omega_n}(1 - k)^{n-2} = \frac{[C_0(n)]^{1+b}2n^2(2n-2)}{b}(1 - k)^{n-2}. \]

**II.** \( x_i \to x_0 \in B : \)

Our computations are the same as in the previous case I), there are some modifications.

We take, \( t_i = 1 \) and \( s_i = [u_i(x_i)]^{2/(n-2) - \epsilon_i}/2. \) We have:

\[ G(x_i, y) \leq C(n)[u_i(x_i)]^{2-(n-2)\epsilon_i}/2. \]

After,

\[ u_i(y) \geq C'(n, b)G(x_i, y)[u_i(x_i)]^{-1+(n-2)\epsilon_i}/2, \]

we use the fact, \( x_i \to x_0 \in B, G(x_i, y) \geq C''(n, b, x_0)(1 - k)^{n-2}, \)

then,

\[ \inf_{B(0, R)} u_i \times [u_i(x_i)]^{-1-(n-2)\epsilon_i}/2 \geq c(n, b, k, x_0) > 0. \]

**Proof of the Theorem 3**

**Step 1: blow-up technique**

We are going to prove the following assertion:

\[ \exists c, R > 0 \text{ such that, } R \left[ \sup_{B(0, R)} u \right]^{2s-1} \times \inf_{B} u \leq c \text{ if } \frac{1}{2} < s \leq 1, \]

and,

\[ \exists c, R > 0 \text{ such that, } R \sup_{B(0, R)} u \leq c \text{ if } s = \frac{1}{2}. \]

We argue by contradiction (and after passing to a subsequence) and we suppose that for \( R_k \to 0 \) we have:

\[ R_k \left[ \sup_{B(0, R_k)} u_k \right]^{2s-1} \times \inf_{B} u_k \to +\infty, \text{ for } s \in [1/2, 1]. \]

\[ R_k \sup_{B(0, R_k)} u_k \to +\infty, \text{ for } s = 1/2. \]

Let \( x_k \) be the point such that \( u_k(x_k) = \sup_{B(0, R_k)} u_k \) and consider the following function:

\[ s_k(x) = \sqrt{(R_k - |x - x_k|)u_k(x)}. \]

Let \( a_k \) be the point such that: \( s_k(a_k) = \sup_{B(x_k, R_k)} s_k. \) We set \( M_k = u_k(a_k) \) and \( l_k = R_k - |a_k - x_k|. \) We have:

\[ M_k^{-1}u_k(x) \leq \sqrt{2}, \text{ for } |x - a_k| \leq \frac{l_k}{2} M_k^2. \]

We have:
\[
\frac{l_k}{2} M_k \to +\infty, \quad v_k(y) = M_k^{-1} u_k(a_k + M_k^{-2} y) \quad \text{for } |y| \leq \frac{l_k}{2} M_k^2,
\]
\[
\Delta v_k = V_k v_k^5, \quad v_k(0) = 1, \quad 0 < v_k \leq \sqrt{2}.
\]

We know, after passing to a subsequence, that:
\[
v_k \to U, \quad \text{with } \Delta U = V(0) U^5, \quad U > 0, \quad \text{on } \mathbb{R}^3.
\]

It easy to see that we can suppose \( V(0) = 1 \). The result of Caffarelli-Gidas-Spruck (see [C-G-S]) assures that \( U \) has an explicit form and is radially symmetric about some point.

**Step 2: The moving plane method**

Now, we use the Kelvin transform and we set for \( \lambda > 0 \):
\[
v_k^\lambda(y) = \frac{\lambda}{|y|} v_k(y^\lambda) \quad \text{with } y^\lambda = \frac{\lambda^2 y}{|y|^2}.
\]

We denote \( \Sigma_\lambda \) by:
\[
\Sigma_\lambda = B(0, R_k M_k^2) - \bar{B}(0, \lambda).
\]

We have the following boundary condition:
\[
\lim_{k \to +\infty} \min_{|y| = R_k M_k^2} (v_k(y)|y|) \to +\infty.
\]

We have:
\[
\Delta v_k^\lambda = V_k^\lambda (v_k^\lambda)^5.
\]

We set:
\[
w_\lambda = v_k - v_k^\lambda.
\]

Then,
\[
\Delta w_\lambda + \frac{n + 2}{n - 2} s^4 V_k w_\lambda = E_\lambda,
\]

with \( E_\lambda = (V_k - V_k^\lambda) (v_k^\lambda)^5 \).

Clearly, we have the following lemma.

**Lemma 1:**

We have:
\[
|E_\lambda| \leq A_k \times C(\lambda_1) M_k^{-2s} \lambda_1^s |y|^{s-5} \leq C(\lambda_1) \lambda_1^s \times A_k M_k^{-2s} \lambda_1^{s-5} |y|^{s-5}.
\]

Let
\[
h_\lambda = -C(s, \lambda_1) A_k M_k^{-2s} \left[ 1 - \left( \frac{\lambda}{|y|} \right)^{4-s} \right].
\]

**Lemma 2:**

\[\exists \lambda_0^k > 0 \text{ such that } w_\lambda + h_\lambda > 0 \text{ in } \Sigma_\lambda \forall 0 < \lambda \leq \lambda_0^k.\]

The proof of the lemma 2 is like the proof of the step 1 of the lemma 2 in [L-Z], we omit it here.

We set:
\[\lambda^k = \sup \{ \lambda \leq \lambda_1, \text{ such that } w_\mu + h_\mu > 0 \text{ in } \Sigma_\mu \text{ for all } 0 < \mu \leq \lambda \}.
\]

We have:
If \( s \in ]1/2, 1] \) then \( |h_{\lambda k}| R_k M_k^{2s} \leq C(s, \lambda_1) \sup_k A_k \), and thus,

\[ w_{\lambda k} + h_{\lambda k} > 0 \quad \forall \quad |y| = R_k M_k^{2s}. \]

If \( s = \frac{1}{2}, \min_M u_k \geq m > 0 \) and \( A_k \to 0 \), we obtain,

\[ \min_{|y|=(2 \lambda_1 M_k)/m} [v_k(y)|y|] \geq 2 \lambda_1 > 0, \]

thus, for \(|y| = \frac{2 \lambda_1 M_k}{m}\) and \( k \) large we have:

\[ w_{\lambda k} + h_{\lambda k} \geq \frac{(-\lambda^k v_k(y^\lambda) + 2 \lambda_1 - C(\lambda_1, s) A_k)}{(2 \lambda_1 M_k)/m} \geq \frac{m}{2 \lambda_1 M_k} [- (1 + \epsilon) \lambda_1 - \epsilon \lambda_1 + 2 \lambda_1] > 0, \]

where \( \epsilon > 0 \) is very small and \( v_k(y^\lambda) \to U(0) = 1 \).

For the case \( s = \frac{1}{2} \), we work in \( \Sigma_\lambda = B \left( 0, \frac{2 \lambda_1 M_k}{m} \right) - \bar{B}(0, \lambda) \). It is easy to see that, \( \frac{2 \lambda_1 M_k}{m} << R_k M_k^2 \). We define \( \lambda^k \) as in the case \( 1/2 < s \leq 1 \).

If we use the Hopf maximum principle, we prove that \( \lambda^k = \lambda_1 \) like in [L-Z]. We have the same contradiction as in [L-Z].

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**References:**

[A] T.Aubin, Some nonlinear problems in Riemannian Geometry. Springer-Verlag, 1998.

[B 1] S.S. Bahoura. Inégalités de Harnack pour les solutions d’équation du type courbure scalaire prescrite. C.R. Math. Acad. Sci. Paris. 341 (2005), no.1, 25-28.

[B 2] S.S. Bahoura. Estimations du type \( \sup u \times \inf u \) sur une variété compacte. Bull. Sci. Math. 1-13 (2006).

[B 3] S.S Bahoura. Majorations du type \( \sup u \times \inf u \leq c \) pour l’équation de la courbure scalaire sur un ouvert de \( \mathbb{R}^n \), \( n \geq 3 \). J. Math. Pures. Appl. (9) 83 2004 no, 9, 1109-1150.

[B-L-S], H. Brezis, YY. Li and I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.

[B M] H. Brezis, F. Merle, Uniform estimates and Blow-up Behavior for Solutions of \( -\Delta u = V(x)e^u \) in two dimension. Commun. in Partial Differential Equations, 16 (8 and 9), 1223-1253(1991).

[C-G-S] L. Caffarelli, B. Gidas, J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. 37 (1984) 369-402.

[C-L 1] C-C.Chen, C-S. Lin. Estimates of the conformal scalar curvature equation via the method of moving planes. Comm. Pure Appl. Math. L(1997) 0971-1017.

[C-L 2] C-C.Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in \( \mathbb{R}^2 \). Commun. Anal. Geom. 6, No.1, 1-19 (1998).

[G-T] D. Gilbarg, N.S. Trudinger. Elliptic Partial Differential Equations of Second order, Berlin Springer-Verlag, Second edition, Grundlehern Math. Wiss.,224, 1983.

[L 1] YY. Li. Prescribing scalar curvature on \( S_n \) and related Problems. C.R. Acad. Sci. Paris 317 (1993) 159-164. Part I: J. Differ. Equations 120 (1995) 319-410. Part II: Existence and compactness. Comm. Pure Appl.Math.49 (1996) 541-597.
[L 2] YY. Li, Harnack type Inequality, the Method of Moving Planes. Commun. Math. Phys. 200 421-444 (1999).

[L-Z] YY. Li, L. Zhang. A Harnack type inequality for the Yamabe equation in low dimensions. Calc. Var. Partial Differential Equations 20 (2004), no. 2, 133–151.

[P] S. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. Soviet. Math. Dokl., vol. 6 (1965), 1408-1411.

[S] I. Shafrir. A sup-inf inequality for the equation $-\Delta u = Ve^u$. C. R. Acad. Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.

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