Abstract

Decentralized optimization is a promising parallel computation paradigm for large-scale data analytics and machine learning problems defined over a network of nodes. This paper is concerned with decentralized non-convex composite problems with population or empirical risk. In particular, the networked nodes are tasked to find an approximate stationary point of the average of local, smooth, possibly non-convex risk functions plus a possibly non-differentiable extended valued convex regularizer. Under this general formulation, we propose the first provably efficient, stochastic proximal gradient framework, called ProxGT. Specifically, we construct and analyze several instances of ProxGT that are tailored respectively for different problem classes of interest. Remarkably, we show that the sample complexities of these instances are network topology-independent and achieve linear speedups compared to that of the corresponding centralized optimal methods implemented on a single node.

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1 Introduction

Decentralized optimization [34], also known as distributed optimization over graphs, is a general parallel computation model for minimizing a sum of cost functions distributed over a network of nodes without a central coordinator. This cooperative minimization paradigm, built upon local communication and computation, has numerous applications in estimation, control, adaptation, and learning problems that frequently arise in multi-agent systems [8, 17, 31, 57]. In particular, the sparse and localized peer-to-peer information exchange pattern in decentralized networks substantially reduces the communication overhead on the parameter server in the centralized networks, thus making decentralized optimization algorithms especially appealing in large-scale data analytics and machine learning tasks [4, 26, 70].

1.1 Background

In this paper, we study the following decentralized non-convex composite optimization problem defined over a network of $n$ nodes:

$$\min_{x \in \mathbb{R}^p} \Psi(x) := F(x) + h(x) \quad \text{such that} \quad F(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x).$$  \hfill (1)
Here, each $f_i : \mathbb{R}^p \to \mathbb{R}$ is $L$-smooth, possibly non-convex, and is only locally accessible by node $i$, while $h : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ is convex, possibly non-differentiable, and is commonly known by all nodes. Each $f_i$ is a cost function associated with local data at node $i$ while $h$ serves as a regularization term that is often used to impose additional problem structures such as convex constraints\footnote{The domain of $h$ acts effectively as the constraint set of Problem (1).} and/or sparsity; simple examples of $h$ include the $\ell_1$-norm or the indicator function of a convex set. The communication over the networked nodes is abstracted as a directed graph $G := (V, E)$, where $V := \{1, \cdots, n\}$ denotes the set of node indices and $E \subseteq V \times V$ collects ordered pairs $(i, r)$, $i, r \in V$, such that node $r$ sends information to node $i$. We adopt the convention that $(i, i) \in E, \forall i \in V$. Our focus in this paper is on the following formulations of the local costs $\{f_i\}_{i=1}^n$ that commonly appear in the context of statistical learning [46]:

- **Population risk:** In this case, each $f_i$ in Problem (1) is defined as
  \[
  f_i(x) := E_{\xi_i \sim D_i}[G_i(x, \xi_i)],
  \]
  where $\xi_i$ is a random data vector supported on $\Xi_i \subseteq \mathbb{R}^q$ with some unknown probability distribution $D_i$ and $G_i : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is a Borel function. The stochastic formulation (2) often corresponds to online scenarios such that samples are generated from the underlying data stream in real time at each node $i$, in order to construct stochastic approximation of $\nabla f_i$ for the subsequent optimization procedure [43].

- **Empirical risk:** As a special case of (2), i.e., when $\xi_i$ has a finite support set $\Xi_i := \{\xi_{i,1}, \cdots, \xi_{i,m}\}$ for some $m \geq 1$, each $f_i$ takes the deterministic form of
  \[
  f_i(x) := \frac{1}{m} \sum_{s=1}^m G_i(x, \xi_{i,s}).
  \]

As an alternate viewpoint, the formulation (3) may be considered as the sample average approximation of (2), where $\{\xi_{i,1}, \cdots, \xi_{i,m}\}$ take the role of offline samples generated from the distribution $D_i$ [46].

We are interested in modern-day big-data scenarios, where $m$ is very large and thus stochastic gradient methods are often preferable over exact gradient ones that use the entire local data per update.

The above formulations are quite general and has found applications in, e.g., sparse non-convex linear models [2], principle component analysis [45], and matrix factorization [9]. Our goal in this paper is thus on the design and analysis of efficient decentralized stochastic gradient algorithms to find an $\epsilon$-stationary point of the global composite function $\Psi$ in Problem (1) under both population risk (2) and empirical risk (3).

### 1.2 Literature Review

The last decade has witnessed a growing research interest and literature in the area of decentralized optimization. For convex (composite) problems, we refer the readers to, e.g., [1,18,20,21,23,24,32,40,49,56–59,64–66] and the references therein, for unifying frameworks and connections between the existing methods. On the other hand, the work on decentralized methods for non-convex composite problems is fairly limited. In the following, we review the existing results that are closely related to Problem (1) under either (2) or (3).

**Decentralized stochastic smooth non-convex optimization.** The special case $h(x) = 0$ of Problem (1) has been relatively well-studied. For this smooth formulation, variants of decentralized stochastic gradient descent (DSGD), e.g., [4,26,52,70], admit simple implementations yet provide competitive practical performance against centralized methods in homogeneous environments like data centers. When the data distributions across the network become heterogeneous, the performance of DSGD in both practice and theory degrades significantly [15,39,57,59,68]. To address this issue, stochastic methods that are robust to heterogeneous data have been proposed, e.g., D2 [51] that is derived from primal-dual formulations [22,25,47,69] and GT-DSGD [29,63] that is based on gradient tracking [10,33,38,41,67]. Reference [30] provides a lower bound for a class of population risk minimization problems and shows that combining GT-DSGD with multi-round accelerated consensus attains this lower bound. Built on top of D2 and GT-DSGD, recent work, e.g., [36,48,60–62] further leverages variance reduction techniques [11,27,54] to achieve accelerated convergence. These results are certainly promising, but they are not applicable to the general composite case where $h(x) \neq 0$. 
Decentralized non-convex composite optimization. The general case $h(x) \neq 0$ in Problem (1) is significantly less explored in the existing literature. The first algorithmic framework for decentralized non-convex composite optimization is due to [10], where $h$ is handled in a successive convex approximation scheme. Reference [71] presents decentralized proximal gradient descent which tackles $h$ via proximal mapping. These works [10, 71], however, require the gradient of $F$ and the subdifferential of $h$ to be uniformly bounded. This bounded (sub)gradient assumption is later removed in [7, 45], where compression and directed graphs are also considered respectively. A decentralized Frank-Wolfe method is proposed in [53] to handle the case where $h$ is an indicator function of a convex compact set. We note that the aforementioned results [7, 10, 45, 53, 71] are exact gradient methods, which are in general not applicable to the population risk (2) and also may not be efficient in the empirical risk setting (3) when the local data size $m$ is relatively large. Towards stochastic gradient methods, [6] analyzes a projected DSGD type method for problems with compact constraint set. Reference [50] establishes the asymptotic convergence of DSGD for a family of non-convex non-smooth functions that satisfy certain coercive property. A recent work [55] presents SPPDM, a decentralized stochastic proximal primal-dual method, and provides related convergence guarantees under the assumption that the epigraph of $h$ is a polyhedral set.

To the best of knowledge, in the literature of decentralized optimization, there is no non-asymptotic sample and communication complexity results of stochastic gradient methods for the non-convex composite problem with a general convex non-differentiable regularizer $h$. We address this gap in this paper.

1.3 Our Contributions

We develop a unified stochastic proximal gradient tracking framework, called ProxGT, for designing and analyzing decentralized methods for the general non-convex composite problem. ProxGT allows flexible construction of local gradient estimators, where a suitable one may be chosen in light of the underlying problem specifications and practical applications. We highlight several important aspects of ProxGT in the following.

- **Algorithm construction:** For definiteness, we present three instantiations of ProxGT. For the general population risk, we develop ProxGT-SA by using the minibatch stochastic approximation technique [12]. Leveraging SARAH type variance reduction schemes [19, 37, 54], we next provide two alternate algorithms, named ProxGT-SR-O and ProxGT-SR-E, for the population and empirical risk respectively that outperform ProxGT-SA when a mean-squared smoothness property holds [3].

- **Complexity results:** We establish the sample and communication complexities of the proposed ProxGT-SA, ProxGT-SR-O, and ProxGT-SR-E algorithms to find an $\epsilon$-stationary solution; see Table 1 for a summary. Remarkably, we show that their sample complexities at each node are network topology-independent and are $n$ times smaller than that of the centralized optimal algorithms implemented on a single node for the corresponding problem classes. In other words, ProxGT-SA, ProxGT-SR-O, and ProxGT-SR-E achieve a topology-independent linear speedup compared to their respective optimal centralized counterparts.

- **Analysis techniques:** Our convergence analysis is developed in a unified manner and can be used to analyze other instances of the ProxGT framework. In particular, we establish a novel stochastic descent inequality for the non-convex composite objective $\Psi$ and a new consensus error bound by carefully handling the proximal mapping. These intermediate technical results are of independent interest and may be used in analyzing other decentralized stochastic proximal first-order methods for non-convex composite problems.

- **Special cases:** For the special case $h = 0$, ProxGT-SR-E and ProxGT-SR-O also constitute improvements over the state-of-the-art decentralized variance-reduced methods GT-SARAH [60] and GT-HSGD [62] in the following sense. For the empirical risk, GT-SARAH attains the optimal centralized sample complexity when the local sample size $m$ is large enough. Similarly, for the population risk, GT-HSGD is optimal when the required accuracy is small enough. ProxGT-SR-O and ProxGT-SR-E improve these regime restrictions and attain the corresponding optimal centralized complexities by performing multiple rounds of (accelerated) consensus updates per iteration.
Table 1: A summary of the sample and communication complexities of the instances of ProxGT studied in this paper for finding an $\epsilon$-stationary point of the global composite function $\Psi$. In the table, $n$ is the number of the nodes, $(1 - \lambda) \in (0, 1]$ is the spectral gap of the weight matrix associated with the network, $L$ is the smoothness parameter for the risk functions, $\Delta$ is the function value gap, $\nu^2$ is the stochastic gradient variance under the expected risk, $\m$ is the local sample size under the empirical risk. The MSS column indicates whether the algorithm in question requires the mean-squared smoothness assumption.

| Algorithm     | Sample Complexity at Each Node                                                                 | Communication Complexity | MSS | Remarks                  |
|---------------|------------------------------------------------------------------------------------------------|--------------------------|-----|--------------------------|
| ProxGT-SA     | $\mathcal{O}\left(\frac{L\Delta}{c^2}\right)$                                               | $\mathcal{O}\left(\frac{L\Delta}{c^2} \log n \right)$ | x   | Population Risk (2)     |
| ProxGT-SS-O   | $\mathcal{O}\left(\frac{L\Delta}{c^2} + \frac{\nu^2}{n\epsilon^2}\right)$               | $\mathcal{O}\left(\frac{L\Delta}{c^2} + \frac{\nu^2}{n\epsilon^2} \log n \right)$ | ✓   | Population Risk (2)     |
| ProxGT-SS-E   | $\mathcal{O}\left(\frac{L\Delta}{c^2} \max \left\{\sqrt{\frac{m}{n}}, 1\right\} + \max \{m, \sqrt{mn}\} \right)$ | $\mathcal{O}\left(\frac{L\Delta}{c^2} \max \left\{\sqrt{\frac{m}{n}}, 1\right\} + \max \{m, \sqrt{mn}\} \log n \right)$ | ✓   | Empirical Risk (3)      |

1.4 Notation

The set of positive real numbers is denoted by $\mathbb{R}^+$. For an integer $z \geq 1$, we denote $[z] := \{1, \cdots, z\}$. The floor and ceiling function are written as $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ respectively. We use lowercase bold letters to denote $d$-dimensional column vectors of all ones and zeros are represented by $\mathbf{1}_d$ and $\mathbf{0}_d$ respectively. For a matrix $\mathbf{X} \in \mathbb{R}^{d \times d}$, its $(i, r)$-th entry is denoted by $[\mathbf{X}]_{i,r}$. We use $\otimes$ to denote the Kronecker product of two matrices $\mathbf{X}$ and $\mathbf{Y}$. The Euclidean norm of a vector or the spectral norm of a matrix is denoted by $\| \cdot \|$.

For an extended valued function $h : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$, we denote $\text{dom}(h) := \{x : h(x) < +\infty\}$, and $h$ is said to be proper if $\text{dom}(h)$ is nonempty. For $x \in \text{dom}(h)$, we denote the subdifferential of $h$ at $x$ by $\partial h(x)$, i.e., $\partial h(x) := \{u : h(y) \geq h(x) + \langle u, y - x \rangle, \forall y \in \text{dom}(h)\}$. The proximal mapping of $h$ is defined as

$$\text{prox}_h(x) := \arg\min_{u \in \mathbb{R}^p} \left\{ \frac{1}{2}\|u - x\|^2 + h(u) \right\}. \quad (4)$$

We work with a rich enough probability triple $(\Theta, \mathcal{F}, \mathbb{P})$, where all random objects are defined properly. Given a sub-$\sigma$-algebra $\mathcal{H} \subseteq \mathcal{F}$ and a random vector $x$, we write $x \in \mathcal{H}$ if $x$ is $\mathcal{H}$-measurable. We use $\sigma(\cdot)$ to denote the $\sigma$-algebra generated by the argument events and/or random vectors.

1.5 Roadmap

The remainder of the paper is organized as follows. Section 2 formulates the problems. Section 3 develops the proposed algorithmic framework and its instances of interest in this paper. Section 4 presents the main convergence results of the proposed algorithms and discuss their implications. Section 5 concludes the paper. Detailed proofs are provided in the appendix.

2 Problem Formulation

2.1 The Non-Convex Composite Model

We make the following assumption on the objective functions.

**Assumption 1 (Functions).** In Problem (1), the following statements hold:

(a) $h : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ is proper, closed, and convex;

(b) Each $f_i : \mathbb{R}^p \to \mathbb{R}$ is $L$-smooth, i.e., $\|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|$, $\forall x, y \in \mathbb{R}^p$, for some $L \in \mathbb{R}^+$;

(c) $\Psi$ is bounded below, i.e., $\Psi := \inf_{x \in \mathbb{R}^p} \Psi(x) > -\infty$.  

5
Assumption 1 describes the standard non-convex composite model [5, Section 10.1]. A simple example of the extended real-valued function $h$ that satisfies Assumption 1(a) is the indicator of a nonempty, closed, and convex set in $\mathbb{R}^p$. Under Assumption 1, we say that a point $\hat{x} \in \text{dom}(h)$ is stationary for Problem (1) if

$$-\nabla F(\hat{x}) \in \partial h(\hat{x}).$$

(5)

**Remark 1.** It is shown in [5, Theorem 3.7.2] that the stationary condition (5) is a necessary condition for a point $\hat{x}$ to be a local optimal solution of Problem (1).

Based on the definition of the proximal mapping (4), it can be shown that this stationarity condition is equivalent to a fixed point equation, i.e., $\hat{x} \in \mathbb{R}^p$ is stationary for Problem (1) if and only if

$$\hat{x} = \text{prox}_{\alpha h}(\hat{x} - \alpha \nabla F(\hat{x})), \quad \forall \alpha > 0.$$  

(6)

In view of (6), we define the gradient mapping for Problem (1):

$$s(x) := \frac{1}{\alpha}(x - \text{prox}_{\alpha h}(x - \alpha \nabla F(x))), \quad \forall x \in \text{dom}(h),$$

(7)

where $\alpha > 0$. We note that the gradient mapping $s(x)$ can be viewed as a generalized gradient of $\Psi$ at $x$ in the sense that $s(x) = \nabla F(x)$ if $h = 0$. The size of $s(\cdot)$ thus serves as a natural measure for the approximate stationarity of a solution [5, 19].

**Definition 1 (e-stationarity).** Let Assumption 1 hold. Then a random vector $x \in \text{dom}(h)$ is said to be an e-stationary solution for Problem (1) if $\mathbb{E}[\|s(x)\|^2] \leq \epsilon^2$, where $s(\cdot)$ is defined in (7).

2.2 The Network Model

We make the following assumption on the directed graph $G = (V, E)$ which characterizes the decentralized communication between the networked nodes.

**Assumption 2 (Network).** The directed network $G = (V, E)$ is strongly connected, i.e., there is a directed path from each node to every other node. Moreover, there exists a doubly stochastic weight matrix $W_\ast \in \mathbb{R}^{n \times n}$ associated with $G$, i.e., $W_\ast$ satisfies the following conditions:

(a) $[W_\ast]_{i,r} > 0$ if $(i, r) \in E$ and $[W_\ast]_{i,r} = 0$ if $(i, r) \notin E$.

(b) $W_\ast 1_n = W_\ast^\top 1_n = 1_n$.

Assumption 2 describes the standard consensus weight matrix $W_\ast$ [32]. Under this assumption, i.e., $W_\ast$ is primitive and doubly stochastic, it is well-known that

$$\lambda_\ast := \|W_\ast - \frac{1}{n} 1_n 1_n^\top\| \in [0, 1),$$

(8)

where $\lambda_\ast$ is the second largest singular value of $W_\ast$ and characterizes the connectivity of the network $G$ [14, 32]: if $G$ is fully connected then $\lambda_\ast = 0$; as the connectivity of $G$ becomes weaker, $\lambda_\ast$ approaches to 1. Therefore, we refer $(1 - \lambda_\ast) \in (0, 1]$ as the spectral gap of the network $G$. When $G$ is undirected, $W_\ast$ in Assumption 2 always exists and can be constructed efficiently via exchange of local degree information between the node [32]. Otherwise, $W_\ast$ may be found by decentralized recursive algorithms provided that it exists [13].

2.3 Stochastic Gradient Models

We make a blanket assumption that each node $i$ at every iteration $t$ is able to obtain i.i.d. minibatch samples \{\xi_{i,s} : s \in [b_t]\} for the local random data vector $\xi_i$. The induced natural filtration is given by

$$\mathcal{F}_t := \sigma(\xi_{i,r}^t : \forall i \in V, s \in [b_r], 1 \leq r \leq t - 1), \quad \forall t \geq 2,$$

(9)

$$\mathcal{F}_1 := \{\Theta, \phi\}.$$  

Intuitively, the filtration $\mathcal{F}_t$ represents the historical information of an algorithm that samples $\xi_i$ up to iteration $t$. We require that the stochastic gradient $\nabla G(\cdot; \xi_{i,s}^t)$ is conditionally unbiased with respect to $\mathcal{F}_t$. 


Assumption 3 (Unbiasedness). \( E[\nabla G_i(x, \xi_{i,s}) | \mathcal{F}_t] = \nabla f_i(x) \), \( \forall i \in \mathcal{V} \), \( \forall t \geq 1 \), \( \forall s \in [b_i] \), \( \forall x \in \mathcal{F}_t \).

Remark 2. Under the empirical risk (3), Assumption 3 amounts to uniform sampling at random from \([m]\).

We consider a standard empirical risk assumption [19] for \( \nabla G_i(\cdot; \xi_{i,s}) \).

Assumption 4 (Bounded Variance). Let \( \nu_i \in \mathbb{R}^+ \), \( \forall i \in \mathcal{V} \). We have \( E[\|\nabla G_i(x, \xi_{i,s}) - \nabla f_i(x)\|^2 | \mathcal{F}_t] \leq \nu_i^2 \), \( \forall t \geq 1 \), \( \forall s \in [b_i] \), \( \forall x \in \mathcal{F}_t \), \( \forall i \in \mathcal{V} \); \( \nu_i^2 := \frac{1}{b_i} \sum_{s=1}^{b_i} \nu_i^2 \).

We are also interested in the case when the stochastic gradients further satisfy the mean-squared smoothness property which is often satisfied in machine learning models [3, 19].

Assumption 5 (Mean-Squared Smoothness). Let \( L \in \mathbb{R}^+ \). In the case of population risk (2), we have

\[
E[\|\nabla G_i(x, \xi_i) - \nabla G_i(y, \xi_i)\|^2] \leq L^2 E[\|x - y\|^2],
\]

for all \( i \in \mathcal{V} \) and \( x, y \in \mathbb{R}^p \). In the case of empirical risk (3), the above statement reduces to

\[
\frac{1}{m} \sum_{s=1}^{m} \|\nabla G_i(x, \xi_{i,s}) - \nabla G_i(x, \xi_{i,s})\|^2 \leq L^2 \|x - y\|^2,
\]

for all \( i \in \mathcal{V} \) and \( x, y \in \mathbb{R}^p \).

Remark 3. Assumption 5 implies that each \( f_i \) is \( L \)-smooth by Jensen’s inequality.

3 Algorithm Development

In the centralized scenarios, a popular fixed-point method to solve (6) is the proximal gradient descent [5], which takes the following form:

\[
x_{t+1} = \text{prox}_{\alpha h}(x_t - \alpha \nabla F(x_t)), \quad \forall t \geq 1,
\]

where \( \alpha > 0 \). However, the recursion (10) cannot be directly implemented in a decentralized manner. The main challenge lies in the fact that the global gradient \( \nabla F \) is not locally available at any node and cannot be computed via one-shot aggregation of local gradient information in decentralized networks. Moreover, the local gradients \( \{\nabla f_i\}_{i=1}^n \) are often significantly different due to the heterogeneous data across the nodes, making the classical gradient consensus approaches [32] less effective especially in the non-convex settings [60]. One popular technique to overcome these issues is gradient tracking [10, 67], which has been adopted in various decentralized stochastic gradient methods for smooth non-convex problems, e.g., [48, 62, 63]. Inspired by these works, we propose a general proximal stochastic gradient tracking framework, termed as \text{ProxGT}, to tackle the non-convex non-smooth composite Problem (1).

3.1 A Generic Algorithmic Procedure

We now describe the proposed \text{ProxGT} framework. At every iteration \( t \), each node \( i \) in the network retains three local variables \( x_i \), \( v_i \), and \( y_i \), all in \( \mathbb{R}^p \), where \( x_i \) approximates an stationary point of Problem (1), \( v_i \) estimates the local exact gradient \( \nabla f_i(x_i) \) from the samples generated for \( \xi_i \), and \( y_i \) tracks the global gradient \( \nabla F(x_i) \) via a stochastic gradient tracking type update [38] from the local gradient estimates \( \{v_i\}_{i=1}^n \).

With the global gradient tracker \( y_i \) at hand, each node \( i \) performs a local inexact fixed point update for (6):

\[
z_{i+1} := \text{prox}_{\alpha h}(x_i - \alpha y_i), \quad \forall t \geq 1,
\]

where \( \alpha > 0 \) is the step-size. The local solution \( x_{i+1} \) at the next iteration is then updated by performing consensus on the intermediate variables \( \{z_{i+1}\}_{i=1}^n \) over the network. For the ease of presentation, we define

\[
W := W_s \otimes I_p.
\]
and the global variables $x_t$, $v_t$, $y_t$, $z_t$ which concatenate their corresponding local variables, i.e.,

$$
x_t = \begin{bmatrix} x_t^1 \\
\vdots \\
x_t^n \end{bmatrix}, \quad v_t = \begin{bmatrix} v_t^1 \\
\vdots \\
v_t^n \end{bmatrix}, \quad y_t = \begin{bmatrix} y_t^1 \\
\vdots \\
y_t^n \end{bmatrix}, \quad z_t = \begin{bmatrix} z_t^1 \\
\vdots \\
z_t^n \end{bmatrix},
$$

all in $\mathbb{R}^{np}$. With the help of these notations, we formally present ProxGT in Algorithm 1 from a global view.

**Remark 4.** In Algorithm 1, the decentralized propagation and averaging of local variables over the network appear as matrix-vector products, while the node-wise implementation of ProxGT can be obtained accordingly.

**Remark 5.** We note that $(W_\ast \otimes I_p)^K$ leads to $K$ decentralized averaging step(s) over $K$ rounds of communication in the corresponding update. This multi-consensus update with an appropriately chosen $K$ is often helpful to achieve faster convergence in the corresponding algorithms [16, 21, 49].

**Remark 6.** The $x$- and $y$-updates in Algorithm 1 take the adapt-then-combine form [8] which often leads improved practical performance.

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**Algorithm 1** ProxGT for Problem (1)

**Require:** $x_1 = 1_n \otimes x_1$; $K$; $\alpha$; $y_1 = 0_{np}$; $v_0 = 0_{np}$.

1: for $t = 1, \cdots, T$ do
2:  Generate an estimator $v_t^i$ of $\nabla f_i(x_t)$, $\forall i$.
3:  Tracking: $y_{t+1} = W^K(y_t + v_t - v_{t-1})$.
4:  Prox-Descent: $z_{t+1}^i = \text{prox}_{\alpha h}(x_t^i - \alpha y_{t+1}^i)$, $\forall i$.
5:  Consensus: $x_{t+1} = W^Kz_{t+1}$.
6: end for

Before we proceed, it is helpful to further clarify the role of $y_t^i$ and the choice of $v_t^i$ in Algorithm 1.

- With the help of the doubly stochasticity of the network weight matrix $W_\ast$, it is straightforward to show by induction that the $y$-update in Algorithm 1 satisfies an important dynamic tracking property:

$$
\frac{1}{n} \sum_{i=1}^{n} y_{t+1}^i = \frac{1}{n} \sum_{i=1}^{n} v_t^i, \quad \forall t \geq 1.
$$

(11)

In view of (11) and the recursion of Algorithm 1, it is expected that each $y_t^i$ approaches $\frac{1}{n} \sum_{i=1}^{n} v_t^i$ and thus asymptotically tracks the global gradient $\nabla F(x_t)$.

- Clearly, different choices of the gradient estimator $v_t^i$ lead to different instances of the ProxGT framework. Many local gradient estimation schemes are applicable here, such as the minibatch stochastic approximation [12, 43] and various variance reduction schemes, e.g., [11, 19, 27, 42, 54]. As we explicitly show, a suitable choice can be made in light of the underlying problem class and practical application.

### 3.2 Instances of Interest

In this section, we present several instances of ProxGT that are of particular interest for the population and empirical risk formulations considered in this paper.

#### 3.2.1 Population Risk Minimization

A natural choice of the gradient estimator $v_t^i$ in ProxGT is the minibatch stochastic approximation [12]. The resulting instance, called ProxGT-SA, is presented in Algorithm 2.

An alternate approach is to construct the gradient estimator $v_t^i$ in ProxGT via an online SARAH type recursive variance reduction scheme that effectively leverages the historical information to achieve faster convergence. When Assumption 5 holds, the resulting algorithm ProxGT-SR-O (given in Algorithm 3) shows superior performance over Algorithm 2.
Require: \( b \).
1: Obtain i.i.d samples \( \{ \xi_{i,s}^t : s \in [b] \} \) for \( \xi_i \).
2: Set \( v_i^t := \frac{1}{b} \sum_{s=1}^b \nabla G_i(x_i^t, \xi_{i,s}^t), \forall i \).

Algorithm 3 ProxGT-SR-O for Problem (1) with (2)

Ensure: Replace Line 2 in Algorithm 1 by the following for all \( i \).
Require: \( B, b, q \).
1: if \( t \mod q = 1 \) then
2: Obtain i.i.d samples \( \{ \xi_{i,s}^t : s \in [B] \} \) for \( \xi_i \).
3: Set \( v_i^t := \frac{1}{B} \sum_{s=1}^B \nabla G_i(x_i^t, \xi_{i,s}^t), \forall i \).
4: else
5: Obtain i.i.d samples \( \{ \xi_{i,s}^t : s \in [b] \} \) for \( \xi_i \).
6: Set \( v_i^t := \frac{1}{b} \sum_{s=1}^b \left( \nabla G_i(x_i^t, \xi_{i,s}^t) - \nabla G_i(x_i^{t-1}, \xi_{i,s}^t) \right) + v_i^{t-1}, \forall i \).
7: end if

3.2.2 Empirical Risk Minimization

We now consider Problem (1) under the empirical risk (3), where the support of each local random data \( \xi_i \) is a finite set \( \Xi_i = \{ \xi_{i,1}, \ldots, \xi_{i,l(m)} \} \). We recall that (3), in its essence, is a special case of the population risk (2) and therefore ProxGT-SA and ProxGT-SR-O developed in Section 3.2 remain applicable. However, the finite-sum structure of each \( f_i \) under (3) lends itself to faster stochastic variance reduction procedures [11,37,54]. In particular, we replace the periodic minibatch stochastic approximation step in ProxGT-SR-O by exact gradient computation. This corresponding implementation, named ProxGT-SR-E, is presented in Algorithm 4.

Algorithm 4 ProxGT-SR-E for Problem (1) with (3)

Ensure: Replace Line 2 in Algorithm 1 by the following for all \( i \).
Require: \( b, q \).
1: if \( t \mod q = 1 \) then
2: Set \( v_i^t := \nabla f_i(x_i^t), \forall i \).
3: else
4: Sample \( \{ \xi_{i,s}^t : s \in [b] \} \) uniformly at random from \( \Xi_i \), \( \forall i \).
5: Set \( v_i^t := \frac{1}{b} \sum_{s=1}^b \left( \nabla G_i(x_i^t, \xi_{i,s}^t) - \nabla G_i(x_i^{t-1}, \xi_{i,s}^t) \right) + v_i^{t-1}, \forall i \).
6: end if

4 Main Results

In this section, we present the main convergence results of the proposed algorithms and discuss their intrinsic features. Throughout the rest of this paper, we let Assumption 1, 2, and 3 hold without explicit statements. The iteration complexity of ProxGT and its instances is quantified in the following sense, while the sample and communication complexities can be obtained accordingly.

Definition 2 (Iteration Complexity). Consider the random vectors \( \{x_i^t\} \) generated by ProxGT. We say that ProxGT finds an \( \epsilon \)-stationary point of Problem (1) in \( T \) iterations if

\[
\frac{1}{T} \sum_{t=1}^T \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \|s(x_i^t)\|^2 + L^2 \|x_i^t - x_i \|^2 \right] \leq \epsilon^2,
\]

where \( x_i := \frac{1}{n} \sum_{i=1}^n x_i^t \) and the gradient mapping \( s(\cdot) \) is defined in (7).
In view of Definition 1, this metric concerns the stationary gap and the consensus error over the network. In particular, if (12) holds and we select the output, say $\hat{x}$, of ProxGT uniformly at random from $\{x_i^t : t \in [T], i \in V\}$, then $E[\|s(\hat{x})\|^2] \leq \epsilon^2$, i.e., $\hat{x}$ is an $\epsilon$-stationary solution for Problem (1).

### 4.1 Sample and Communication Complexity Results

For ease of presentation, we define

$$\Delta := \Psi(\bar{x}_1) - \Psi$$

and

$$\zeta^2 := \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(\bar{x}_1)\|^2.$$  \hspace{1cm} (13)

where recall that $\Psi$ is the global composite objective function. We note that $O(\cdot)$ in this section only hides universal constants that are not related to the problem parameters.

**Theorem 1** (Convergence of ProxGT-SA). Consider Problem (1) with the population risk (2). Let Assumption 4 hold and $\epsilon$ be the error tolerance. Set $K \asymp \frac{\log(n\zeta)}{\lambda_*}$, $\alpha \asymp \frac{1}{L}$, $b \asymp \frac{\nu^2}{\zeta}$ in Algorithm 2. Then Algorithm 2 finds an $\epsilon$-stationary solution in $O\left(\frac{L\Delta}{\epsilon^2} + \frac{\nu}{\zeta} \log(n\zeta) \frac{1}{1 - \lambda_*}\right)$ iterations, leading to $O\left(\frac{L\Delta \nu^2}{ne^4}\right)$ stochastic gradient samples at each node and $O\left(\frac{L\Delta}{\epsilon^2} \cdot \frac{\log(n\zeta)}{1 - \lambda_*}\right)$ rounds of communication over the network.

In view of Theorem 1, ProxGT-SA achieves an optimal and topology-independent sample complexity at each node that exhibits linear speedup against the centralized minibatch stochastic proximal gradient method $[3,12,30]$ executed on a single node. To the best of our knowledge, this is the first time such sample complexity result is established, in the literature of decentralized stochastic gradient methods for the general non-convex composite population risk minimization problem.

**Theorem 2** (Convergence of ProxGT-SR-O). Consider Problem (1) with the population risk (2). Let Assumption 4 and 5 hold and let $\epsilon$ be the error tolerance. Set $K \asymp \frac{\log(n\zeta)}{\lambda_*}$, $\alpha \asymp \frac{1}{L}$, $q \asymp \frac{\nu}{\zeta}$, $b \asymp \frac{\nu^2}{\zeta}$, $B \asymp \frac{\nu^2}{\zeta}$ in Algorithm 3. Then Algorithm 3 finds an $\epsilon$-stationary solution in $O\left(\frac{L\Delta}{\epsilon^2} + \frac{\nu}{\zeta}\right)$ iterations, leading to $O\left(\frac{L\Delta \nu}{ne^3} + \frac{\nu^2}{ne^2}\right)$ stochastic gradient samples at each node and $O\left(\left(\frac{L\Delta}{\epsilon^2} + \frac{\nu}{\epsilon}\right) \frac{\log(n\zeta)}{1 - \lambda_*}\right)$ rounds of communication over the network.

Two remarks are in order.

- **Theorem 2** shows that ProxGT-SR-O attains an optimal and topology-independent sample complexity at each node that exhibits linear speedup compared to the centralized optimal proximal online variance reduction methods $[3,19,27,37,54]$ implemented on a single node. To the best of our knowledge, this appears to be the first such sample complexity result in the literature of the decentralized non-convex composite population risk minimization problem with mean-squared smoothness.

- For the special case $h = 0$, ProxGT-SR-O also improves the state-of-the-art sample complexity result given by GT-HSGD $[62]$ in the following sense. GT-HSGD achieves the optimal sample complexity in the regime where the error tolerance $\epsilon$ of the problem is small enough, i.e., $\epsilon \lesssim (1 - \lambda_*)^{-3} n^{-1}$. ProxGT-SR-O removes this regime restriction by performing $K \asymp \frac{\log(n\zeta)}{1 - \lambda_*}$ rounds of consensus update per iteration.
Theorem 3 (Convergence of ProxGT-SR-E). Consider Problem (1) with the empirical risk (3). Let Assumption 4 and 5 hold and let $\epsilon$ be the error tolerance. Set $K \asymp \frac{\log \zeta}{\epsilon^2}$, $\alpha \asymp \frac{1}{L}$, $q \asymp \sqrt{nm}$, $b \asymp \max \left\{ \sqrt{m}, 1 \right\}$ in Algorithm 4. Then Algorithm 4 finds an $\epsilon$-stationary solution in $O\left( \frac{L \Delta}{\epsilon^2} \max \left\{ \sqrt{\frac{m}{n}}, 1 \right\} + \max \left\{ m, \sqrt{nm} \right\} \right)$ iterations, leading to $O\left( \frac{L \Delta}{\epsilon^2} \max \left\{ \sqrt{\frac{m}{n}}, 1 \right\} + \max \left\{ m, \sqrt{nm} \right\} \right)$ stochastic gradient samples at each node and $O\left( \left( \frac{L \Delta}{\epsilon^2} + \sqrt{nm} \right) \frac{\log \zeta}{1 - \lambda_*} \right)$ rounds of communication over the network.

Two remarks are in place:

- We conclude from Theorem 3 that under a moderate big-data condition $m \gtrsim n$, ProxGT-SR-E achieves a topology-independent sample complexity of $O\left( \frac{L \Delta}{\epsilon^2} \sqrt{\frac{m}{n}} + m \right)$ at each node, leading to a linear speedup compared to the centralized optimal proximal finite-sum variance reduction methods [19,37,54] implemented on a single node. To the best of our knowledge, this is the first such sample complexity result for the decentralized non-convex composite empirical risk minimization problem.

- For the special case $h = 0$, ProxGT-SR-E also improves the state-of-the-art sample complexity result achieved by GT-SARAH [60] in the following sense. GT-SARAH matches the centralized optimal methods in the regime that the local sample size is large enough, i.e., $m \gtrsim n(1 - \lambda_*)^{-6}$. ProxGT-SR-E improves this regime to $m \gtrsim n$ by performing $K \asymp \frac{\log \zeta}{\epsilon^2}$ rounds of consensus updates per iteration.

4.2 Improving Communication Complexity via Accelerated Consensus

As a standard practice, it is possible to employ accelerated consensus algorithms, e.g., [28,35,44], to implement the multiple consensus step $W^K$ in ProxGT to achieve improved communication complexities. The basic intuition is that the standard consensus algorithm $x_{t+1} = W x_t$ returns an $\delta$-accurate average of the initial states $\frac{1}{n} \sum_{i=1}^{n} x_i^t$ in $O\left( \frac{1}{\delta} \log \frac{1}{\delta} \right)$ rounds of communication, while the accelerated algorithms [28,35,44], only take $O\left( \frac{\log \log \frac{1}{\delta}}{\delta} \right)$ rounds of communication.

In particular, we can replace $W^K$ by a Chebyshev type polynomial of $W$; see, for instance, [44, Section 4.2], [23, Section 3.2], and [66, Section V-C] for detailed constructions. In this case, the communication complexity of ProxGT-SA stated in Theorem 1 improves to

$$O\left( \frac{L \Delta}{\epsilon^2} \cdot \frac{\log (n \zeta)}{\sqrt{1 - \lambda_*}} \right),$$

while the communication complexity of ProxGT-SR-0 stated in Theorem 2 improves to

$$O\left( \left( \frac{L \Delta}{\epsilon^2} + \frac{\nu}{\epsilon} \right) \frac{\log (n \zeta)}{\sqrt{1 - \lambda_*}} \right),$$

and the communication complexity of ProxGT-SR-E stated in Theorem 3 improves to

$$O\left( \left( \frac{L \Delta}{\epsilon^2} + \sqrt{nm} \right) \frac{\log \zeta}{\sqrt{1 - \lambda_*}} \right).$$

We omit the detailed calculations here for conciseness.

5 Conclusions

In this paper, we focus on decentralized non-convex composite optimization problems over networked nodes, where the network cost is the average of local, smooth, possibly non-convex risk functions plus an extended
valued, convex, possibly non-differentiable regularizer. To address this general formulation, we introduce a unified framework, called ProxGT, that is built upon local stochastic gradient estimators and a global gradient tracking technique. We construct several different instantiations of this framework by choosing appropriate local estimators for the corresponding problem classes. In particular, we develop ProxGT-SA and ProxGT-SR-O for the population risk, and ProxGT-SR-E for the empirical risk. Remarkably, we show that each algorithm achieves a network topology-independent sample complexity at each node, leading to a linear speedup compared to its centralized optimal counterpart.

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A Proofs of the Main Results

In this section, we describe a unified analysis for the proposed Prox-GT framework. Throughout the rest of the paper, we let Assumption 1, 2, and 3 hold without explicit statements.

A.1 Preliminaries

We start by introducing some additional notations for Algorithm 1, 2, 3, and 4. We find it convenient to abstract the local proximal descent step by a stochastic gradient mapping: \( \forall t \geq 1 \) and \( i \in V \),

\[
g_t^i := \frac{1}{\alpha} (x_t^i - z_{t+1}^i). \tag{14}
\]

For all \( t \geq 1 \), we let

\[
g_t := \begin{bmatrix} g_t^1 \\ \cdots \\ g_t^n \end{bmatrix}, \quad \nabla f(x_t) := \begin{bmatrix} \nabla f_1(x_t^1) \\ \cdots \\ \nabla f_n(x_t^n) \end{bmatrix},
\]

and define the following network mean states:

\[
x_t := \frac{1}{n} \sum_{i=1}^{n} x_t^i, \quad y_t := \frac{1}{n} \sum_{i=1}^{n} y_t^i, \quad z_t := \frac{1}{n} \sum_{i=1}^{n} z_t^i,
\]

\[
v_t := \frac{1}{n} \sum_{i=1}^{n} v_t^i, \quad g_t := \frac{1}{n} \sum_{i=1}^{n} g_t^i, \quad \nabla f(x_t) := \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x_t^i).
\]

In addition, we define the exact averaging matrix

\[
J := \left( \frac{1}{n} 1_n 1_n^\top \right) \otimes I_p.
\]

Averaging (14) over \( i \) from 1 to \( n \) gives: \( \forall t \geq 1 \),

\[
z_{t+1} = x_t - \alpha g_t.
\]

We multiply \( \frac{1}{n} (1_n^\top \otimes I_p) \) to the \( x \)-update of Algorithm 1 to obtain: \( \forall t \geq 1 \),

\[
\overline{x}_{t+1} = \overline{x}_{t+1}.
\]

Combining (15) and (16) yields: \( \forall t \geq 1 \),

\[
x_{t+1} = x_t - \alpha g_t.
\]

Throughout the analysis, we fix arbitrary \( K \geq 1 \) and denote

\[
\lambda := \lambda^K.
\]

A.2 Basic Facts

This section presents several basic facts that are used frequently in our analysis. We make use of a well-known non-expansiveness result for proximal mappings.

Lemma 1 ([5, Theorem 6.4.2]). Let \( h : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\} \) be a proper, closed, and convex function. Then we have the following:

\[
\| \text{prox}_h(x) - \text{prox}_h(y) \| \leq \| x - y \|, \quad \forall x, y \in \mathbb{R}^p.
\]

For ease of reference, we give a trivial accumulation formula for scalar sequences with contraction.
Lemma 2. Let \( \{a_t\} \) and \( \{b_t\} \) be scalar sequences and \( 0 < q < 1 \), such that
\[
a_{t+1} \leq qa_t + b_t, \quad \forall t \geq 1.
\]
Then for all \( T \geq 2 \), we have
\[
\sum_{t=1}^{T} a_t \leq \frac{1}{1-q} - q + \frac{1}{1-q} \sum_{t=1}^{T-1} b_t
\]
and
\[
\sum_{t=2}^{T+1} a_t \leq \frac{1}{1-q} - q + \frac{1}{1-q} \sum_{t=2}^{T} b_t.
\]
Proof. The proof follows from standard arguments of convolution sums and we omit the details.

Lemma 3. The following statements hold for all \( K \geq 1 \).
(a) \( W^K J = JW^K = J \).
(b) \( \|W^K - J\| = \lambda^K \).
(c) \( \|W^K x - Jx\| \leq \lambda^K \|x - Jx\|, \forall x \in \mathbb{R}^{np} \).
Proof. Since \( W \) is doubly stochastic, we have
\[
WJ = JW = J, \tag{19}
\]
which leads to part (a) by induction. Part (b) follows from
\[
\|W^K - J\| = \|W - J\|^K = \lambda^K,
\]
where the first equality uses (19) and the second equality uses the definition of the spectral norm of a matrix. Finally, part (c) is due to
\[
\|W^K x - Jx\| = \|(W^K - J)(x - Jx)\| \leq \|W^K - J\|\|x - Jx\| = \lambda^K \|x - Jx\|,
\]
where the first equality uses part (a) and \( J^2 = J \), and the last equality uses part (b).

Finally, we present a simple yet useful decomposition inequality.

Lemma 4. Consider the iterates generated by Algorithm 1. Then we have: \( \forall T \geq 2 \),
\[
\sum_{t=2}^{T} \|x_t - x_{t-1}\|^2 \leq 6 \sum_{t=1}^{T} \|x_t - Jx_t\|^2 + 3n\alpha^2 \sum_{t=1}^{T-1} \|\bar{e}_t\|^2,
\]
Proof. We note that \( \forall t \geq 2 \),
\[
\|x_t - x_{t-1}\|^2 = \|x_t - Jx_t + Jx_t - Jx_{t-1} + Jx_{t-1} - x_{t-1}\|^2
\leq 3\|x_t - Jx_t\|^2 + 3n\|\bar{e}_t - \bar{e}_{t-1}\|^2 + 3\|x_{t-1} - Jx_{t-1}\|^2,
\leq 3\|x_t - Jx_t\|^2 + 3n\alpha^2\|\bar{e}_{t-1}\|^2 + 3\|x_{t-1} - Jx_{t-1}\|^2, \tag{20}
\]
where the second line uses (17). Summing up (20) gives
\[
\sum_{t=2}^{T} \|x_t - x_{t-1}\|^2 \leq 3 \sum_{t=2}^{T} \left( \|x_t - Jx_t\|^2 + \|x_{t-1} - Jx_{t-1}\|^2 \right) + 3n\alpha^2 \sum_{t=2}^{T} \|\bar{e}_{t-1}\|^2
\leq 6 \sum_{t=1}^{T} \|x_t - Jx_t\|^2 + 3n\alpha^2 \sum_{t=2}^{T} \|\bar{e}_{t-1}\|^2
\]
which finishes the proof.
A.3 Descent Inequality and Error Bounds

We first establish a key descent inequality in terms of the value of the global composite objective function $\Psi$. This result plays a central role in our analysis.

**Lemma 5 (Descent).** Consider the iterates generated by Algorithm 1. If $0 < \alpha \leq \frac{1}{8q}$, then we have: $\forall t \geq 1$,

$$
\frac{1}{n} \sum_{t=1}^{T} \left( \sum_{i=1}^{n} \| s(x_i^t) \|^2 + L^2 \| x_t - Jx_t \|^2 \right) \leq \frac{8\Delta}{\alpha} - \frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \| g_i^t \|^2 + 76 \sum_{t=1}^{T} \| \nabla_t - \nabla f(x_t) \|^2 \\
+ \frac{6}{\alpha^2} \sum_{t=1}^{T} \| x_t - Jx_t \|^2 + \frac{10}{n} \sum_{t=2}^{T+1} \| y_t - Jy_t \|^2.
$$

**Proof.** See Appendix B. \qed

In light of Lemma 5, our analysis approach is to show that the accumulated descent effect of the stochastic gradient mappings $\sum_{i=1}^{n} \| g_i \|$ dominates the accumulated consensus, variance, and gradient tracking errors up to constant factors. To this aim, we establish useful error bounds for different algorithms. The following one is a consequence of the non-expansiveness of the proximal operator.

**Lemma 6 (Consensus).** Consider the iterates generated by Algorithm 1. We have: $\forall t \geq 1$,

$$
\sum_{t=1}^{T} \| x_t - Jx_t \|^2 \leq \frac{4\lambda^2\alpha^2}{(1-\lambda^2)^2} \sum_{t=2}^{T} \| y_t - Jy_t \|^2.
$$

**Proof.** See Appendix C. \qed

**Remark 7.** It is worth noting that Lemma 5 and 6 do not use any properties of the gradient estimator $v_t$. Therefore they may be of independent interest and used in other decentralized stochastic proximal gradient type methods for non-convex composite problems.

The next lemma establishes variance bounds for different algorithms.

**Lemma 7 (Variance).** The following statements hold.

(a) Let Assumption 4 hold and consider the iterates generated by Algorithm 2. Then we have: $\forall t \geq 1$,

$$
E \left[ \| \nabla_t - \nabla f(x_t) \|^2 \right] \leq \frac{\nu^2}{nb}.
$$

(b) Let Assumption 4 and 5 hold. Consider the iterates generated by Algorithm 3. Suppose that $T = Rq$ for some $R \in \mathbb{Z}^+$. Then we have: $\forall T \geq q$,

$$
\sum_{t=1}^{T} E \left[ \| \nabla_t - \nabla f(x_t) \|^2 \right] \leq \frac{6L^2q}{n^2b} \sum_{t=1}^{T} E \left[ \| x_t - Jx_t \|^2 \right] + \frac{qL^2\alpha^2}{nb} \sum_{t=1}^{T-1} E \left[ \| \xi_t \|^2 \right] + \frac{T\nu^2}{nb}.
$$

(c) Let Assumption 5 hold. Consider the iterates generated by Algorithm 4. Suppose that $T = Rq$ for some $R \in \mathbb{Z}^+$. Then we have: $\forall T \geq q$,

$$
\sum_{t=1}^{T} E \left[ \| \nabla_t - \nabla f(x_t) \|^2 \right] \leq \frac{6L^2q}{n^2b} \sum_{t=1}^{T} E \left[ \| x_t - Jx_t \|^2 \right] + \frac{qL^2\alpha^2}{nb} \sum_{t=1}^{T-1} E \left[ \| \xi_t \|^2 \right].
$$

**Proof.** See Appendix D. \qed
Finally, we give tracking error bounds for different algorithms in the following lemma.

**Lemma 8 (Tracking).** The following statements hold.

(a) Let Assumption 4 hold and consider the iterates generated by Algorithm 2. Then we have: $\forall T \geq 2,$
\[
\sum_{t=2}^{T+1} E[\|y_t - Jy_t\|^2] \leq \frac{2\lambda^2 n\varsigma^2}{1 - \lambda^2} + \frac{12\lambda^2 n\alpha^2 L^2}{(1 - \lambda^2)^2} \sum_{t=1}^{T-1} E[\|\varepsilon_t\|^2] + \frac{24\lambda^2 L^2}{(1 - \lambda^2)^2} \sum_{t=1}^{T} E[\|x_t - Jx_t\|^2] + 4T(2\lambda^2 n + 1)\nu^2 \frac{b}{b(1 - \lambda^2)}.
\]

(b) Let Assumption 4 and 5 hold. Consider the iterates generated by Algorithm 3. Let $T = Rq$ for some $R \in \mathbb{Z}^+$ and $R \geq 2$. Then we have:
\[
\sum_{t=2}^{T+1} E[\|y_t - Jy_t\|^2] \leq \frac{2\lambda^2 n\varsigma^2}{1 - \lambda^2} + \frac{96\lambda^2 L^2}{(1 - \lambda^2)^2} \sum_{t=1}^{T} E[\|x_t - Jx_t\|^2] + \frac{48\lambda^2 n\alpha^2 L^2}{(1 - \lambda^2)^2} \sum_{t=1}^{T-1} E[\|\varepsilon_t\|^2] + 14\lambda^2 Tn\nu^2 \frac{b}{(1 - \lambda^2)Bq}.
\]

(c) Let Assumption 5 hold. Consider the iterates generated by Algorithm 4. Let $T = Rq$ for some $R \in \mathbb{Z}^+$ and $R \geq 2$. Then we have:
\[
\sum_{t=2}^{T+1} E[\|y_t - Jy_t\|^2] \leq \frac{2\lambda^2 n\varsigma^2}{1 - \lambda^2} + \frac{96\lambda^2 L^2}{(1 - \lambda^2)^2} \sum_{t=1}^{T} E[\|x_t - Jx_t\|^2] + \frac{48\lambda^2 n\alpha^2 L^2}{(1 - \lambda^2)^2} \sum_{t=1}^{T-1} E[\|\varepsilon_t\|^2] + 14\lambda^2 Tn\nu^2 \frac{b}{(1 - \lambda^2)Bq}.
\]

**Proof.** See Appendix E.

### A.4 Proofs of the Main Theorems

We first use the consensus error bound in Lemma 6 to refine the descent inequality in Lemma 5.

**Proposition 1.** Consider the iterates generated by Algorithm 1. If $0 < \alpha \leq \frac{1}{8\lambda^4},$ then we have: $\forall t \geq 1,$
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \sum_{i=1}^{n} s(x^i_t) \right)^2 + L^2 \|x_t - Jx_t\|^2 \leq \frac{8\lambda^4 n\varsigma^2}{1 - \lambda^2} - \sum_{i=1}^{T} E[\|\varepsilon_t\|^2] + 76 \sum_{i=1}^{T} \|\vartheta_t - \vartheta T(x_t)\|^2 + 34 \sum_{i=1}^{T+1} \frac{(1 - \lambda^2)^2}{(1 - \lambda^2)^2} \sum_{i=1}^{T+1} \|y_t - Jy_t\|^2.
\]

**Proof.** This result follows by applying Lemma 6 to Lemma 5 and $\|\vartheta_t\|^2 \leq \frac{1}{n} \sum_{i=1}^{n} \|\vartheta_t\|^2.$

### A.4.1 Proof of Theorem 1

We apply Lemma 6 to Lemma 8(a) to obtain: $\forall T \geq 2,$
\[
\left(1 - \frac{96\lambda^2 n\alpha^2 L^2}{(1 - \lambda^2)^2}\right) \sum_{i=1}^{T+1} E[\|y_t - Jy_t\|^2] \leq \frac{2\lambda^2 n\varsigma^2}{1 - \lambda^2} + \frac{12\lambda^2 n\alpha^2 L^2}{(1 - \lambda^2)^2} \sum_{i=1}^{T-1} E[\|\varepsilon_t\|^2] + \frac{4T(2\lambda^2 n + 1)\nu^2}{b(1 - \lambda^2)}.
\] (21)

If $0 < \alpha \leq \frac{(1 - \lambda^2)^2}{14\lambda^4 L^2},$ then $1 - \frac{96\lambda^2 n\alpha^2 L^2}{(1 - \lambda^2)^2} \geq \frac{1}{2}$ and hence (21) implies that $\forall T \geq 2,$
\[
\sum_{i=1}^{T+1} E[\|y_t - Jy_t\|^2] \leq \frac{4\lambda^2 n\varsigma^2}{1 - \lambda^2} + \frac{24\lambda^2 n\alpha^2 L^2}{(1 - \lambda^2)^2} \sum_{i=1}^{T-1} E[\|\varepsilon_t\|^2] + \frac{8T(2\lambda^2 n + 1)\nu^2}{b(1 - \lambda^2)}.
\] (22)
Plugging Lemma 7(a) and (22) into Proposition 1 gives: if $0 < \alpha \leq \min \left\{ \frac{(1-\lambda^2)^2}{4\lambda^2}, \frac{1}{4} \right\}$, then

$$\frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| s(x_i^t) \right\|^2 + L^2 \left\| x_i^t - x_t \right\|^2 \right] \leq \frac{8\Delta}{\alpha} - \sum_{t=1}^{T} \mathbb{E} \left[ \left\| \overline{g}_t \right\|^2 \right] + \frac{76T\nu^2}{nb} + \frac{272T(2\lambda^2 + 1)\nu^2}{nb(1 - \lambda^2)^3} + \frac{136\lambda^2\zeta^2}{(1 - \lambda^2)^3} + \frac{816\lambda^2\alpha^2L^2}{(1 - \lambda^2)^4} \sum_{t=1}^{T-1} \mathbb{E} \left[ \left\| g_t \right\|^2 \right] \leq \frac{8\Delta}{\alpha} - \left( \frac{1 - 816\lambda^2\alpha^2L^2}{(1 - \lambda^2)^4} \right) \sum_{t=1}^{T} \mathbb{E} \left[ \left\| g_t \right\|^2 \right] + \frac{136\lambda^2\zeta^2}{(1 - \lambda^2)^3} + \frac{348T\nu^2}{nb(1 - \lambda^2)^3} + \frac{544\lambda^2\nu^2}{b(1 - \lambda^2)^3}. \quad (23)$$

From (23), we have: if $0 < \alpha \leq \min \left\{ \frac{(1-\lambda^2)^2}{4\lambda^2}, \frac{1}{4} \right\}$, then $\forall T \geq 2$,

$$\frac{1}{nT} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| s(x_i^t) \right\|^2 + L^2 \left\| x_i^t - x_t \right\|^2 \right] \leq \frac{8\Delta}{\alpha T} + \frac{136\lambda^2\zeta^2}{(1 - \lambda^2)^3 T} + \frac{348\nu^2}{nb(1 - \lambda^2)^3} + \frac{544\lambda^2\nu^2}{b(1 - \lambda^2)^3}. \quad (24)$$

Recall from (18) that $\lambda := \lambda^K_*$ and we set

$$K \asymp \frac{\log(n\zeta)}{1 - \lambda_*},$$

so that $\frac{1}{\lambda_*} = O(1)$, $\lambda_* = O(1)$, $\lambda n = O(1)$. As a consequence, from (24) we have: if $0 < \alpha \lesssim \frac{1}{T}$, then

$$\frac{1}{nT} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| s(x_i^t) \right\|^2 + L^2 \left\| x_i^t - x_t \right\|^2 \right] \lesssim \frac{\Delta}{\alpha T} + \frac{\nu^2}{nb}. \quad (25)$$

Finally, we observe that choosing

$$\alpha \asymp \frac{1}{L}, \quad b \asymp \frac{\nu^2}{nc^2}, \quad T \asymp \frac{L\Delta}{\epsilon^2}$$

in (25) gives $\frac{1}{nT} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| s(x_i^t) \right\|^2 + L^2 \left\| x_i^t - x_t \right\|^2 \right] \lesssim \epsilon^2$. The ensuing complexity results follow from the fact that each iteration of Algorithm 2 incurs $O(\log(n))$ stochastic gradient samples and $K$ rounds of communication.

### A.4.2 Proof of Theorem 2

Consider $T = Rq$ for some $R \in \mathbb{Z}^+$ and $R \geq 2$. Plugging Lemma 6 to Lemma 7(b) gives:

$$\sum_{t=1}^{T} \mathbb{E} \left[ \left\| \nabla_t - \nabla F(x_t) \right\|^2 \right] \leq \frac{24\lambda^2\alpha^2 L^2 q}{(1 - \lambda^2)^2 n b^2} \sum_{t=2}^{T} \mathbb{E} \left[ \left\| y_t - Jy_t \right\|^2 \right] + \frac{qL^2\alpha^2}{nb} \sum_{t=1}^{T-1} \mathbb{E} \left[ \left\| g_t \right\|^2 \right] + \frac{T\nu^2}{nb}. \quad (26)$$

In particular, if $0 < \alpha \leq \sqrt{\frac{nb}{24q}} \frac{1}{L}$, we have:

$$\sum_{t=1}^{T} \mathbb{E} \left[ \left\| \nabla_t - \nabla F(x_t) \right\|^2 \right] \leq \frac{\lambda^2}{(1 - \lambda^2)^2 n} \sum_{t=2}^{T} \mathbb{E} \left[ \left\| y_t - Jy_t \right\|^2 \right] + \frac{qL^2\alpha^2}{nb} \sum_{t=1}^{T-1} \mathbb{E} \left[ \left\| g_t \right\|^2 \right] + \frac{T\nu^2}{nb}. \quad (26)$$

Applying (26) to Proposition 1 yields: if $0 < \alpha \leq \min \left\{ \frac{1}{L}, \sqrt{\frac{nb}{24q}} \right\}$, then

$$\frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| s(x_i^t) \right\|^2 + L^2 \left\| x_i^t - x_t \right\|^2 \right] \leq \frac{8\Delta}{\alpha} - \left( \frac{1 - 76qL^2\alpha^2}{nb} \right) \sum_{t=1}^{T} \mathbb{E} \left[ \left\| g_t \right\|^2 \right] + \frac{110}{(1 - \lambda^2)^2 n} \sum_{t=1}^{T+1} \mathbb{E} \left[ \left\| y_t - Jy_t \right\|^2 \right] + \frac{76T\nu^2}{nb}.$$

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In particular, if $0 < \alpha \leq \min \left\{ \frac{1}{8}, \sqrt{\frac{nb}{152q}} \right\} \frac{1}{L}$, we have:

\[
\frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \|s(x_i^t)\|^2 + L^2 \|x_i^t - \mathbf{x}\|^2 \right] \leq \frac{8\Delta}{\alpha} + \frac{1}{2} \sum_{t=1}^{T} \mathbb{E} \left[ \|\mathbf{y}_t \|^2 \right] + \frac{110}{(1 - \lambda^2)^2 n} \sum_{t=2}^{T+1} \mathbb{E} \left[ \|y_t - J\mathbf{y}_t\|^2 \right] + \frac{76Tn^2}{nB}.
\]  

(27)

To proceed, we apply Lemma 6 to Lemma 8(b) to obtain:

\[
\left( 1 - \frac{384\lambda^2 \alpha^2 L^2}{(1 - \lambda^2)^4} \right) \sum_{t=2}^{T+1} \mathbb{E} \left[ \|y_t - J\mathbf{y}_t\|^2 \right] \leq \frac{2\lambda^2 n^2 \zeta^2}{1 - \lambda^2} + \frac{48\lambda^2 n^2 \alpha^2 L^2}{1 - \lambda^2} \sum_{t=1}^{T-1} \mathbb{E} \left[ \|\mathbf{y}_t \|^2 \right] + \frac{14\lambda^2 Tn^2}{(1 - \lambda^2)^2 Bq}.
\]

(28)

If $0 < \alpha \leq \frac{(1 - \lambda^2)^2}{28\lambda^2 L}$, (28) implies that

\[
\sum_{t=2}^{T+1} \mathbb{E} \left[ \|y_t - J\mathbf{y}_t\|^2 \right] \leq \frac{4\lambda^2 n^2 \zeta^2}{1 - \lambda^2} + \frac{96\lambda^2 n^2 \alpha^2 L^2}{1 - \lambda^2} \sum_{t=1}^{T-1} \mathbb{E} \left[ \|\mathbf{y}_t \|^2 \right] + \frac{28\lambda^2 Tn^2}{(1 - \lambda^2)^2 Bq}.
\]

(29)

Finally, plugging (29) to (27), we obtain: if $0 < \alpha \leq \min \left\{ \frac{1}{8}, \sqrt{\frac{nb}{152q}}, \frac{1 - \lambda^2)^2}{28\lambda^2 L} \right\} \frac{1}{L}$, then

\[
\frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \|s(x_i^t)\|^2 + L^2 \|x_i^t - \mathbf{x}\|^2 \right] \leq \frac{8\Delta}{\alpha T} + \frac{76Tn^2}{nB} + \frac{440\lambda^2 \zeta^2}{1 - \lambda^2} + \frac{3080\lambda^2 Tn^2}{(1 - \lambda^2)^2 Bq}
\]

\[
\frac{1}{2} \left( 1 - \frac{21120\lambda^2 \alpha^2 L^2}{(1 - \lambda^2)^4} \right) \sum_{t=1}^{T} \mathbb{E} \left[ \|\mathbf{y}_t \|^2 \right].
\]

Hence, if $0 < \alpha \leq \min \left\{ \frac{1}{8}, \sqrt{\frac{nb}{152q}}, \frac{1 - \lambda^2)^2}{140\lambda^2 L} \right\} \frac{1}{L}$, then

\[
\frac{1}{nT} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \|s(x_i^t)\|^2 + L^2 \|x_i^t - \mathbf{x}\|^2 \right] \leq \frac{8\Delta}{\alpha T} + \frac{76Tn^2}{nB} + \frac{440\lambda^2 \zeta^2}{1 - \lambda^2} + \frac{3080\lambda^2 Tn^2}{(1 - \lambda^2)^2 Bq}.
\]

(30)

Let $\epsilon > 0$ be given. Recall from (18) that $\lambda := \lambda_1^K$ and we set

\[ K \approx \log(n\zeta) \frac{1 - \lambda}{1 - \lambda_1}, \]

so that $\frac{1}{nT} = O(1), \lambda\zeta = O(1), \lambda n = O(1)$; moreover, we let

\[ q = nb \quad \text{and} \quad \alpha \approx \frac{1}{L}. \]

As a consequence, we have from (30) that

\[
\frac{1}{nT} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \|s(x_i^t)\|^2 + L^2 \|x_i^t - \mathbf{x}\|^2 \right] \lesssim \frac{L\Delta}{\epsilon^2} + \frac{\nu^2}{nB}.
\]

(31)

In view of (31), we further choose

\[ T \approx \frac{L\Delta}{\epsilon^2} + q \quad \text{and} \quad B \approx \frac{\nu^2}{ne^2}, \]

(32)

which lead to $\frac{1}{nT} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \|s(x_i^t)\|^2 + L^2 \|x_i^t - \mathbf{x}\|^2 \right] \lesssim \epsilon^2$. Since ProxGT-SR-0 requires $B$ samples every $q$ iterations and $b$ samples at each iteration, its total sample complexity is bounded by

\[ O \left( T \left( b + \frac{B}{q} \right) \right). \]

(33)
Setting $b = B/q$, together with $q = nb$ stated above, gives

$$b \asymp \sqrt{\frac{B}{n}} = \frac{\nu}{\sqrt{\epsilon}}$$ and $$q \asymp \frac{\nu}{\epsilon}.$$  \hfill (34)

Applying (32) and (34) to (33) concludes the ensuing sample complexity and the corresponding communication complexity is given by $TK$.

**A.4.3 Proof of Theorem 3**

Consider $T = Rq$ for some $R \in \mathbb{Z}^+$ and $R \geq 2$. Plugging Lemma 6 to Lemma 7(c) gives:

$$\sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla_t - \nabla f(x_t) \|^2 \right] \leq \frac{24\lambda^2 \alpha^2 L^2 q}{(1-\lambda^2)^2 b} \sum_{t=2}^{T} \mathbb{E} \left[ \| y_t - Jy_t \|^2 \right] + \frac{qL^2\alpha^2}{nb} \sum_{t=1}^{T-1} \mathbb{E} \left[ \| \xi_t \|^2 \right].$$

In particular, if $0 < \alpha \leq \sqrt{\frac{nb}{2\sqrt{q}}} \frac{1}{T}$, we have:

$$\sum_{t=1}^{T} \mathbb{E} \left[ \| \nabla_t - \nabla f(x_t) \|^2 \right] \leq \frac{\lambda^2}{(1-\lambda^2)^2 b} \sum_{t=2}^{T} \mathbb{E} \left[ \| y_t - Jy_t \|^2 \right] + \frac{qL^2\alpha^2}{nb} \sum_{t=1}^{T-1} \mathbb{E} \left[ \| \xi_t \|^2 \right]. \hfill (35)$$

Applying (35) to Proposition 1 yields: if $0 < \alpha \leq \min \left\{ \frac{1}{\sqrt{T}}, \sqrt{\frac{nb}{2\sqrt{q}}} \right\} \frac{1}{T}$, then

$$\frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \| s(x_t^i) \|^2 + L^2 \| x_t^i - x_t \|^2 \right] \leq \frac{8\Delta}{\alpha} - \frac{76qL^2\alpha^2}{nb} \sum_{t=1}^{T} \mathbb{E} \left[ \| \xi_t \|^2 \right] + \frac{110}{(1-\lambda^2)^2 b} \sum_{t=2}^{T} \mathbb{E} \left[ \| y_t - Jy_t \|^2 \right]. \hfill (36)$$

In particular, if $0 < \alpha \leq \min \left\{ \frac{1}{\sqrt{T}}, \sqrt{\frac{nb}{2\sqrt{q}}} \right\} \frac{1}{T}$, we have:

$$\frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \| s(x_t^i) \|^2 + L^2 \| x_t^i - x_t \|^2 \right] \leq \frac{8\Delta}{\alpha} - \frac{1}{2} \sum_{t=1}^{T} \mathbb{E} \left[ \| \xi_t \|^2 \right] + \frac{110}{(1-\lambda^2)^2 b} \sum_{t=2}^{T} \mathbb{E} \left[ \| y_t - Jy_t \|^2 \right]. \hfill (36)$$

To proceed, we apply Lemma 6 to Lemma 8(b) to obtain:

$$\left( 1 - \frac{384\lambda^4 \alpha^4 L^2}{(1-\lambda^2)^4} \right) \sum_{t=2}^{T+1} \mathbb{E} \left[ \| y_t - Jy_t \|^2 \right] \leq \frac{2\lambda^2 n \zeta^2}{1-\lambda^2} + \frac{48\lambda^2 \alpha^2 n^2 L^2}{(1-\lambda^2)^2} \sum_{t=1}^{T-1} \mathbb{E} \left[ \| \xi_t \|^2 \right]. \hfill (37)$$

If $0 < \alpha \leq \frac{(1-\lambda^2)^2}{288\lambda^2}$, (37) implies that

$$\sum_{t=2}^{T+1} \mathbb{E} \left[ \| y_t - Jy_t \|^2 \right] \leq \frac{4\lambda^2 n \zeta^2}{1-\lambda^2} + \frac{96\lambda^2 \alpha^2 n^2 L^2}{(1-\lambda^2)^2} \sum_{t=1}^{T-1} \mathbb{E} \left[ \| \xi_t \|^2 \right]. \hfill (38)$$

Finally, plugging (38) to (36), we obtain: if $0 < \alpha \leq \min \left\{ \frac{1}{\sqrt{T}}, \sqrt{\frac{nb}{2\sqrt{q}}} \right\} \frac{1}{T}$, then

$$\frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \| s(x_t^i) \|^2 + L^2 \| x_t^i - x_t \|^2 \right] \leq \frac{8\Delta}{\alpha T} + \frac{440\lambda^2 \zeta^2}{(1-\lambda^2)^3} \sum_{t=1}^{T} \mathbb{E} \left[ \| \xi_t \|^2 \right].$$

Hence, if $0 < \alpha \leq \min \left\{ \frac{1}{\sqrt{T}}, \sqrt{\frac{nb}{2\sqrt{q}}} \right\} \frac{1}{T}$, then

$$\frac{1}{nT} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E} \left[ \| s(x_t^i) \|^2 + L^2 \| x_t^i - x_t \|^2 \right] \leq \frac{8\Delta}{\alpha T} + \frac{440\lambda^2 \zeta^2}{(1-\lambda^2)^3 T}. \hfill (39)$$

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Let $\epsilon > 0$ be given. Recall from (18) that $\lambda := \lambda^K_*$ and we set

$$K \equiv \frac{\log \zeta}{1 - \lambda_*},$$

so that $\frac{1}{K} = O(1), \lambda \zeta = O(1)$; moreover, we let

$$q = \sqrt{nm}, \quad b = \max \left\{ \sqrt{mn}, 1 \right\}, \quad \alpha \equiv \frac{1}{L}. \quad (40)$$

As a consequence, we have from (39) that

$$\frac{1}{nT} \sum_{t=1}^{T} \sum_{i=1}^{n} E \left[ \| \mathbf{s}(\mathbf{x}_i^t) \|^2 + L^2 \| \mathbf{x}_i^t - \mathbf{x}_t \|^2 \right] \lesssim \frac{L \Delta}{T}. \quad (41)$$

In view of (41), we further choose

$$T \equiv \frac{L \Delta}{\epsilon^2} + q \quad (42)$$

which leads to $\frac{1}{nT} \sum_{t=1}^{T} \sum_{i=1}^{n} E \left[ \| \mathbf{s}(\mathbf{x}_i^t) \|^2 + L^2 \| \mathbf{x}_i^t - \mathbf{x}_t \|^2 \right] \lesssim \epsilon^2$. The communication complexity is thus $TK$. Since $\text{ProxGT-SR-E}$ requires $m$ samples every $q$ iterations and $b$ samples at each iteration, its total sample complexity is bounded by

$$O \left( T \left( b + \frac{m}{q} \right) \right). \quad (43)$$

Plugging (40) and (42) into (43) concludes the ensuing sample complexity.

## B Proof of Lemma 5

### B.1 Step 1: Descent Inequality for the Convex Part

First of all, we write the proximal descent step in Algorithm 1 in an equivalent form for analysis purposes. For all $t \geq 1$ and $i \in \mathcal{V}$, we observe that

$$z_{i+1}^t = \text{prox}_{\alpha h}(\mathbf{x}_i^t - \alpha \mathbf{y}_{i+1}^t) = \arg\min_{\mathbf{u} \in \mathbb{R}^p} \left\{ \frac{1}{2} \| \mathbf{u} - (\mathbf{x}_i^t - \alpha \mathbf{y}_{i+1}^t) \|^2 + \alpha h(\mathbf{u}) \right\}$$

$$= \arg\min_{\mathbf{u} \in \mathbb{R}^p} \left\{ \frac{1}{2} \| \mathbf{u} - \mathbf{x}_i^t \|^2 + \langle \alpha \mathbf{y}_{i+1}^t, \mathbf{u} - \mathbf{x}_i^t \rangle + \frac{1}{2} \| \alpha \mathbf{y}_{i+1}^t \|^2 + \alpha h(\mathbf{u}) \right\}$$

$$= \arg\min_{\mathbf{u} \in \mathbb{R}^p} \left\{ \langle \mathbf{y}_{i+1}^t, \mathbf{u} \rangle + \frac{1}{2\alpha} \| \mathbf{u} - \mathbf{x}_i^t \|^2 + h(\mathbf{u}) \right\}. \quad (44)$$

In light of the optimality condition of the strongly convex optimization problem (44) and the sum rule of subdifferential calculus [5, Theorem 3.40], for all $t \geq 1$ and $i \in \mathcal{V}$, there exists $h'(z_{i+1}^t) \in \partial h(z_{i+1}^t)$ such that

$$h'(z_{i+1}^t) = -\mathbf{y}_{i+1}^t - \frac{1}{\alpha} (z_{i+1}^t - \mathbf{x}_i^t). \quad (45)$$

By the subgradient inequality, we have: $\forall t \geq 1, \forall i \in \mathcal{V}$, and $\forall \mathbf{u} \in \mathbb{R}^p$,

$$h(\mathbf{u}) \geq h(z_{i+1}^t) + \langle h'(z_{i+1}^t), \mathbf{u} - z_{i+1}^t \rangle,$$

which is the same as

$$h(z_{i+1}^t) \leq h(\mathbf{u}) + \langle h'(z_{i+1}^t), z_{i+1}^t - \mathbf{u} \rangle. \quad (46)$$
Applying (45) to (46), we obtain: \( \forall t \geq 1, \forall i \in \mathcal{V} \), and \( \forall u \in \mathbb{R}^p \),
\[
h(z_{t+1}^i) \leq h(u) - \frac{1}{\alpha} \langle x_t^i - z_{t+1}^i, u - z_{t+1}^i \rangle - \langle y_{t+1}^i, z_{t+1}^i - u \rangle. \tag{47}
\]
We have the following algebraic identity: \( \forall t \geq 1, \forall i \in \mathcal{V} \), and \( \forall u \in \mathbb{R}^p \),
\[
\langle x_t^i - z_{t+1}^i, u - z_{t+1}^i \rangle = \frac{1}{2} \| u - z_{t+1}^i \|^2 + \frac{1}{2} \| x_t^i - z_{t+1}^i \|^2 - \frac{1}{2} \| x_t^i - u \|^2. \tag{48}
\]
Applying (48) to (47), we obtain: \( \forall t \geq 1, \forall i \in \mathcal{V} \), and \( \forall u \in \mathbb{R}^p \),
\[
h(z_{t+1}^i) \leq h(u) - \frac{1}{2\alpha} \| u - z_{t+1}^i \|^2 - \frac{1}{2\alpha} \| x_t^i - z_{t+1}^i \|^2 + \frac{1}{2\alpha} \| x_t^i - u \|^2 - \langle y_{t+1}^i, z_{t+1}^i - u \rangle. \tag{49}
\]
Setting \( u := \mathfrak{x}_t \), we have: \( \forall t \geq 1 \) and \( \forall i \in \mathcal{V} \),
\[
h(z_{t+1}^i) \leq h(\mathfrak{x}_t) - \frac{1}{2\alpha} \| \mathfrak{x}_t - z_{t+1}^i \|^2 - \frac{1}{2\alpha} \| x_t^i - z_{t+1}^i \|^2 + \frac{1}{2\alpha} \| x_t^i - \mathfrak{x}_t \|^2 - \langle y_{t+1}^i, z_{t+1}^i - \mathfrak{x}_t \rangle
\]
\[= h(\mathfrak{x}_t) - \frac{1}{2\alpha} \| \mathfrak{x}_t - z_{t+1}^i \|^2 - \frac{\alpha}{2} \| g_t^i \|^2 + \frac{1}{2\alpha} \| x_t^i - \mathfrak{x}_t \|^2 - \langle y_{t+1}^i, \mathbf{y}_{t+1}^i - \mathfrak{x}_t \rangle \]
\[-\langle \mathbf{y}_{t+1}^i, z_{t+1}^i - \mathfrak{x}_t \rangle, \tag{50}\]
where the last line uses (14). For the second last term in (50), we have: \( \forall t \geq 1 \) and \( \forall i \in \mathcal{V} \),
\[
- \langle y_{t+1}^i, \mathbf{y}_{t+1}^i, z_{t+1}^i - \mathfrak{x}_t \rangle \leq \| y_{t+1}^i - \mathbf{y}_{t+1}^i \| \| z_{t+1}^i - \mathfrak{x}_t \| \]
\[\leq \frac{\alpha}{2} \| y_{t+1}^i - \mathbf{y}_{t+1}^i \|^2 + \frac{1}{2\alpha} \| z_{t+1}^i - \mathfrak{x}_t \|^2, \tag{51}\]
where the first and the second line use the Cauchy-Schwarz and Young’s inequality respectively. Plugging (51) into (50) gives: \( \forall t \geq 1 \) and \( \forall i \in \mathcal{V} \),
\[
h(z_{t+1}^i) \leq h(\mathfrak{x}_t) - \frac{\alpha}{2} \| g_t^i \|^2 + \frac{1}{2\alpha} \| x_t^i - \mathfrak{x}_t \|^2 + \frac{\alpha}{2} \| y_{t+1}^i - \mathbf{y}_{t+1}^i \|^2 - \langle \mathbf{y}_{t+1}^i, z_{t+1}^i - \mathfrak{x}_t \rangle. \tag{52}\]
We now average (52) over \( i \) from 1 to \( n \) to obtain: \( \forall t \geq 1 \),
\[
\frac{1}{n} \sum_{i=1}^{n} h(z_{t+1}^i) \leq h(\mathfrak{x}_t) - \frac{\alpha}{2n} \sum_{i=1}^{n} \| g_t^i \|^2 + \frac{1}{2\alpha n} \| \mathfrak{x}_t - J \mathfrak{x}_t \|^2 + \frac{\alpha}{2n} \| y_{t+1}^i - J y_{t+1} \|^2 - \langle \mathbf{y}_{t+1}^i, \mathfrak{z}_{t+1} - \mathfrak{x}_t \rangle
\]
\[= h(\mathfrak{x}_t) - \frac{\alpha}{2n} \sum_{i=1}^{n} \| g_t^i \|^2 + \frac{1}{2\alpha n} \| \mathfrak{x}_t - J \mathfrak{x}_t \|^2 + \frac{\alpha}{2n} \| y_{t+1}^i - J y_{t+1} \|^2 + \alpha \langle \mathbf{y}_{t+1}^i, \mathbf{g}_t \rangle, \tag{53}\]
where the second line follows from (15). In light of the convexity of \( h \) and Jensen’s inequality, for all \( t \geq 1 \) we have that \( h(\mathfrak{z}_{t+1}) \leq \frac{1}{n} \sum_{i=1}^{n} h(z_{t+1}^i) \) and hence (53) implies
\[
h(\mathfrak{z}_{t+1}) \leq h(\mathfrak{x}_t) - \frac{\alpha}{2n} \sum_{i=1}^{n} \| g_t^i \|^2 + \frac{1}{2\alpha n} \| \mathfrak{x}_t - J \mathfrak{x}_t \|^2 + \frac{\alpha}{2n} \| y_{t+1}^i - J y_{t+1} \|^2 + \alpha \langle \mathbf{y}_{t+1}^i, \mathbf{g}_t \rangle, \quad \forall t \geq 1. \tag{54}\]
In view of (16), we observe that (54) is the same as
\[
h(\mathfrak{z}_{t+1}) \leq h(\mathfrak{x}_t) - \frac{\alpha}{2n} \sum_{i=1}^{n} \| g_t^i \|^2 + \frac{1}{2\alpha n} \| \mathfrak{x}_t - J \mathfrak{x}_t \|^2 + \frac{\alpha}{2n} \| y_{t+1}^i - J y_{t+1} \|^2 + \alpha \langle \mathbf{y}_{t+1}^i, \mathbf{g}_t \rangle, \quad \forall t \geq 1. \tag{55}\]

B.2 Step 2: Descent Inequality for the Non-Convex Part
Since \( F \) is \( L \)-smooth, we have the standard quadratic upper bound [5, Lemma 5.7]:
\[
F(y) \leq F(x) + \langle \nabla F(x), y - x \rangle + \frac{L}{2} \| y - x \|^2, \quad \forall x, y \in \mathbb{R}^p. \tag{56}\]
Setting $y = \mathbf{x}_{t+1}$ and $x = \mathbf{x}_t$ in (56), we obtain: $\forall t \geq 1$,
\[
F(\mathbf{x}_{t+1}) \leq F(\mathbf{x}_t) + \langle \nabla F(\mathbf{x}_t), \mathbf{x}_{t+1} - \mathbf{x}_t \rangle + \frac{L}{2} \| \mathbf{x}_{t+1} - \mathbf{x}_t \|^2
\]
\[
= F(\mathbf{x}_t) - \alpha \langle \nabla F(\mathbf{x}_t), \mathbf{g}_t \rangle + \frac{La^2}{2} \| \mathbf{g}_t \|^2,
\]  
(57)
where the last line is due to (17).

**B.3 Step 3: Combining Step 1 and Step 2**

Recall that $\Psi := F + h$. Summing up (57) and (55), we obtain: $\forall t \geq 1$,
\[
\Psi(\mathbf{x}_{t+1}) \leq \Psi(\mathbf{x}_t) - \frac{\alpha}{2n} \sum_{i=1}^{n} \| g_i^t \|^2 + \frac{1}{2an} \| x_t - Jx_t \|^2 + \frac{\alpha}{2n} \| y_{t+1} - Jy_{t+1} \|^2
\]
\[
+ \alpha \langle y_{t+1} - \nabla F(\mathbf{x}_t), \mathbf{g}_t \rangle + \frac{La^2}{2} \| \mathbf{g}_t \|^2.
\]  
(58)
By the Cauchy-Schwarz and Young’s inequality, we have: $\forall \eta > 0$ and $\forall t \geq 1$,
\[
\langle y_{t+1} - \nabla F(\mathbf{x}_t), \mathbf{g}_t \rangle \leq \frac{1}{2\eta} \| y_{t+1} - \nabla F(\mathbf{x}_t) \|^2 + \frac{\eta}{2} \| \mathbf{g}_t \|^2
\]  
(59)
Applying (59) to (58), we obtain: $\forall \eta > 0$ and $\forall t \geq 1$,
\[
\Psi(\mathbf{x}_{t+1}) \leq \Psi(\mathbf{x}_t) - \frac{\alpha}{2n} \sum_{i=1}^{n} \| g_i^t \|^2 + \frac{1}{2an} \| x_t - Jx_t \|^2 + \frac{\alpha}{2n} \| y_{t+1} - Jy_{t+1} \|^2
\]
\[
+ \frac{\alpha}{2n} \| y_{t+1} - \nabla F(\mathbf{x}_t) \|^2 + \frac{n\alpha + La^2}{2} \| \mathbf{g}_t \|^2.
\]  
(60)

**B.4 Step 4: Refining Error Terms and Telescoping Sum**

We first bound the difference between the local stochastic gradient mapping $g_i^t$ defined in (14) and the exact gradient mapping $s(x_i^t)$ defined in (7). Observe that $\forall t \geq 1$ and $\forall i \in \mathcal{V}$,
\[
\| g_i^t - s(x_i^t) \|^2 = \left\| \frac{1}{\alpha} \left( x_i^t - \text{prox}_{\alpha h}(x_i^t - \alpha y_{i+1}^t) \right) - \frac{1}{\alpha} \left( x_i^t - \text{prox}_{\alpha h}(x_i^t - \alpha \nabla F(x_i^t)) \right) \right\|^2
\]
\[
= \frac{1}{\alpha^2} \left\| \text{prox}_{\alpha h}(x_i^t - \alpha y_{i+1}^t) - \text{prox}_{\alpha h}(x_i^t - \alpha \nabla F(x_i^t)) \right\|^2
\]
\[
\leq \| y_{i+1}^t - \nabla F(x_i^t) \|^2
\]
\[
= \| y_{i+1}^t - \bar{y}_{i+1}^t + \bar{y}_{i+1}^t - \nabla F(x_t) + \nabla F(x_t) - \nabla F(x_i^t) \|^2
\]
\[
\leq 3 \| y_{i+1}^t - \bar{y}_{i+1}^t \|^2 + 3 \| \bar{y}_{i+1}^t - \nabla F(x_t) \|^2 + 3L^2 \| x_t - x_i^t \|^2,
\]  
(61)
where the third line is due to Lemma 1 and the last line uses the $L$-smoothness of $F$. Observe that $\forall t \geq 1$,
\[
- \| g_i^t \|^2 \leq - \frac{1}{2} \| s(x_i^t) \|^2 + \| g_i^t - s(x_i^t) \|^2
\]
\[
\leq - \frac{1}{2} \| s(x_i^t) \|^2 + 3 \| y_{i+1}^t - \bar{y}_{i+1}^t \|^2 + 3 \| \bar{y}_{i+1}^t - \nabla F(x_t) \|^2 + 3L^2 \| x_t - x_i^t \|^2,
\]  
(62)
where the first line is due to the standard triangular inequality and the second line uses (61). Averaging (62) over $i$ from 1 to $n$ gives: $\forall t \geq 1$,
\[
- \frac{1}{n} \sum_{i=1}^{n} \| g_i^t \|^2 \leq - \frac{1}{2n} \sum_{i=1}^{n} \| s(x_i^t) \|^2 + \frac{3}{n} \| y_{i+1}^t - Jy_{i+1} \|^2 + 3 \| \bar{y}_{i+1}^t - \nabla F(x_t) \|^2 + \frac{3L^2}{n} \| x_t - Jx_t \|^2.
\]  
(63)
We now plug (63) into (60) to obtain: \( \forall t > 0 \) and \( \forall t \geq 1, \)

\[
\Psi(x_{t+1}) \leq \Psi(x_t) - \frac{\alpha}{4n} \sum_{i=1}^{n} \|g_i\|^2 - \frac{\alpha}{8n} \sum_{i=1}^{n} \|s(x_i)\|^2 + \left( \frac{1}{2\alpha} + \frac{3\alpha L^2}{4} \right) \frac{1}{n} \|x_t - Jx_t\|^2 + \frac{5\alpha}{4n} \|y_{t+1} - Jy_{t+1}\|^2 \\
+ \frac{\gamma}{n} \|\nabla F(x_t)\|^2 + \frac{\alpha}{2n} \|\nabla F(x_t)\|^2 + \frac{\gamma}{2} \|\nabla F(x_t)\|^2 \\
\leq \Psi(x_t) - \frac{\alpha}{8n} \sum_{i=1}^{n} \|g_i\|^2 - \frac{\alpha}{8n} \sum_{i=1}^{n} \|s(x_i)\|^2 + \left( \frac{1}{2\alpha} + \frac{3\alpha L^2}{4} \right) \frac{1}{n} \|x_t - Jx_t\|^2 \\
+ \frac{5\alpha}{4n} \|y_{t+1} - Jy_{t+1}\|^2 + \frac{3}{2} \frac{\gamma}{\eta} \|\nabla F(x_t)\|^2, \quad (64)
\]

where the last line is due to \( \|\nabla F(x_t)\|^2 \leq \frac{1}{n} \sum_{i=1}^{n} \|g_i\|^2 \). Setting \( \eta = \frac{1}{8} \) and \( 0 < \alpha \leq \frac{1}{8L} \) in (64), we have: \( \forall t \geq 1, \)

\[
\Psi(x_{t+1}) \leq \Psi(x_t) - \frac{\alpha}{8n} \sum_{i=1}^{n} \|g_i\|^2 - \frac{\alpha}{8n} \sum_{i=1}^{n} \|s(x_i)\|^2 + \left( \frac{1}{2\alpha} + \frac{3\alpha L^2}{4} \right) \frac{1}{n} \|x_t - Jx_t\|^2 \\
+ \frac{5\alpha}{4n} \|y_{t+1} - Jy_{t+1}\|^2 + \frac{19\alpha}{4} \|\nabla F(x_t)\|^2. \quad (65)
\]

Towards the last term in (65), observe that, \( \forall t \geq 1, \)

\[
\|\nabla F(x_t)\|^2 = \|\nabla (\frac{1}{n} \sum_{i=1}^{n} (\nabla f_i(x_i) - \nabla f_i(x_t)))\|^2 \\
\leq \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x_i) - \nabla f_i(x_t)\|^2 \leq \frac{L^2}{n} \|x_t - Jx_t\|^2.
\]

Plugging (66) into (65), we have: if \( 0 < \alpha \leq \frac{1}{8L}, \) then \( \forall t \geq 1, \)

\[
\Psi(x_{t+1}) \leq \Psi(x_t) - \frac{\alpha}{8n} \sum_{i=1}^{n} \|g_i\|^2 - \frac{\alpha}{8n} \sum_{i=1}^{n} \|s(x_i)\|^2 + \left( \frac{1}{2\alpha} + \frac{41\alpha L^2}{4} \right) \frac{1}{n} \|x_t - Jx_t\|^2 \\
+ \frac{5\alpha}{4n} \|y_{t+1} - Jy_{t+1}\|^2 + \frac{19\alpha}{2} \|\nabla F(x_t)\|^2. \quad (67)
\]

Telescoping sum (67) over \( t \) from 1 to \( T \), we have: if \( 0 < \alpha \leq \frac{1}{8L}, \) then

\[
\Psi(x_{T+1}) \leq \Psi(x_1) - \frac{\alpha}{8n} \sum_{i=1}^{n} \|g_i\|^2 - \frac{\alpha}{8n} \sum_{i=1}^{n} \|s(x_i)\|^2 + \left( \frac{1}{2\alpha} + \frac{41\alpha L^2}{4} \right) \frac{1}{n} \sum_{t=1}^{T} \|x_t - Jx_t\|^2 \\
+ \frac{5\alpha}{4n} \sum_{t=1}^{T} \|y_{t+1} - Jy_{t+1}\|^2 + \frac{19\alpha}{2} \sum_{t=1}^{T} \|\nabla F(x_t)\|^2. \quad (68)
\]

With \( \inf_{x \in \mathbb{R}^p} \Psi(x) \geq \Psi \geq -\infty \) and minor rearrangement, (68) implies the following: if \( 0 < \alpha \leq \frac{1}{8L}, \) then

\[
\frac{1}{n} \sum_{t=1}^{T} \left( \sum_{i=1}^{n} \|s(x_i)\|^2 + L^2 \|x_t - Jx_t\|^2 \right) \leq \frac{8\Psi(x_1) - \Psi}{\alpha} - \frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} \|g_i\|^2 + 76 \sum_{t=1}^{T} \|\nabla F(x_t)\|^2 \\
+ \left( \frac{4}{\alpha^2} + \frac{83L^2}{2} \right) \frac{1}{n} \sum_{t=1}^{T} \|x_t - Jx_t\|^2 + \frac{10}{n} \sum_{t=2}^{T+1} \|y_t - Jy_t\|^2,
\]

which finishes the proof of Lemma 5 by \( 83L^2 \leq \frac{2}{\alpha^2}. \)
C  Proof of Lemma 6

For ease of exposition, we define a block-wise proximal mapping for $h$:

$$\text{prox}_{ah}(c) := \begin{bmatrix} \text{prox}_{ah}(c_1) \\ \vdots \\ \text{prox}_{ah}(c_n) \end{bmatrix} \in \mathbb{R}^{np}, \text{ where } c := \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

such that $c_i \in \mathbb{R}^p, \forall i \in [n]$. In view of (69), the $x$-update in Algorithm 1 can compactly be written as

$$x_{t+1} = W^K \text{prox}_{ah}(x_t - \alpha y_{t+1}), \quad \forall t \geq 1.$$  \hspace{1cm} (70)

We find the following quantity helpful: \(\forall t \geq 1,

$$\|x_{t+1} - Jx_{t+1}\|^2 = \|W^K \text{prox}_{ah}(x_t - \alpha y_{t+1}) - JW^K \text{prox}_{ah}(x_t - \alpha y_{t+1})\|^2

= \|J(W^K - J) \text{prox}_{ah}(x_t - \alpha y_{t+1})\|^2,

\leq \lambda^2 \|\text{prox}_{ah}(x_t - \alpha y_{t+1}) - \text{prox}_{ah}(Jx_t - \alpha Jy_{t+1})\|^2,$$

where the first line uses the definition of $W, J,$ and $\text{prox}_{ah}$, and the last line is due to the doubly stochasticity of $W$. We are now prepared to analyze the consensus error recursion in the following. For all $t \geq 1$, we have

$$\|\text{prox}_{ah}(x_t - \alpha y_{t+1}) - \text{prox}_{ah}(Jx_t - \alpha Jy_{t+1})\|^2 = \sum_{i=1}^{n} \|\text{prox}_{ah}(x_i^t - \alpha y_{i+1}^t) - \text{prox}_{ah}(x_i - \alpha y_{i+1}^t)\|^2

\leq \sum_{i=1}^{n} \|x_i^t - x_i - \alpha(y_{i+1}^t - y_{i+1}^t)\|^2

= \|x_t - Jx_t - \alpha(y_{t+1} - Jy_{t+1})\|^2,$$

where the first and the second line uses (69) and Lemma 1 respectively. We then plug (73) into (72) to obtain: $\forall t \geq 1$ and $\forall \eta > 0,$

$$\|x_{t+1} - Jx_{t+1}\|^2 \leq \lambda^2 \|x_t - Jx_t - \alpha(y_{t+1} - Jy_{t+1})\|^2

= \lambda^2 \|x_t - Jx_t\|^2 + \lambda^2 \|y_{t+1} - Jy_{t+1}\|^2 - 2\lambda \langle x_t - Jx_t, \alpha(y_{t+1} - Jy_{t+1}) \rangle

\leq \lambda^2 \|x_t - Jx_t\|^2 + \lambda^2 \|y_{t+1} - Jy_{t+1}\|^2 + 2\lambda^2 \|x_t - Jx_t\| \|\alpha(y_{t+1} - Jy_{t+1})\|

\leq \lambda^2 (1 + \eta) \|x_t - Jx_t\|^2 + \lambda^2 \|y_{t+1} - Jy_{t+1}\|^2,$$

where the third and the last line use the Cauchy-Schwarz and Young’s inequality with parameter $\eta$ respectively. Finally, setting $\eta = \frac{1 - \lambda^2}{2\lambda^2}$ in (74) yields: $\forall t \geq 1,$

$$\|x_{t+1} - Jx_{t+1}\|^2 \leq \frac{1 + \lambda^2}{2} \|x_t - Jx_t\|^2 + \frac{\lambda^2}{1 - \lambda^2} \|y_{t+1} - Jy_{t+1}\|^2.$$  \hspace{1cm} (75)
Applying Lemma 2 to (75), we have: $\forall T \geq 2$,

$$
\sum_{t=1}^{T} \left\| x_t - Jx_t \right\|^2 \leq \frac{2\lambda^2 \alpha^2 (1 + \lambda^2)}{(1 - \lambda^2)^2} \sum_{t=1}^{T-1} \left\| y_{t+1} - Jy_{t+1} \right\|^2,
$$

which finishes the proof of Lemma 6.

D Proof of Lemma 7

D.1 Proof of Lemma 7 (a)

We first recall that the gradient estimator $v^i_t$ in Algorithm 2 takes the following form: $\forall t \geq 1$ and $i \in V$,

$$
v^i_t := \frac{1}{b} \sum_{s=1}^{b} \nabla G_i(x^i_t, \xi^i_{t,s}).
$$

Observe that $\forall t \geq 1$,

$$
\mathbb{E}\left[ \left\| \nabla_t - \nabla f(x_t) \right\|^2 \right| \mathcal{F}_t] = \mathbb{E}\left[ \left\| \frac{1}{nb} \sum_{i=1}^{b} \sum_{s=1}^{b} \left( \nabla G_i(x^i_t, \xi^i_{t,s}) - \nabla f_i(x^i_t) \right) \right\|^2 \right| \mathcal{F}_t]
$$

$$
= \frac{1}{(nb)^2} \sum_{i=1}^{b} \sum_{s=1}^{b} \mathbb{E}\left[ \left\| \nabla G_i(x^i_t, \xi^i_{t,s}) - \nabla f_i(x^i_t) \right\|^2 \right| \mathcal{F}_t]
$$

$$
\leq \frac{1}{(nb)^2} \sum_{i=1}^{b} \sum_{s=1}^{b} \nu^2
$$

$$
= \frac{\nu^2}{nb},
$$

where the second line uses Assumption 3 and the fact that $x_t$ is $\mathcal{F}_t$-measurable and $\{\xi^i_{t,s} : i \in V, s \in [b]\}$ is independent of $\mathcal{F}_t$, while the third line is due to Assumption 4.

D.2 Proof of Lemma 7 (b) and Lemma 7 (c)

To facilitate the analysis, we first note that the gradient estimator $v^i_t$ in both Algorithm 3 and 4 take the following form: $\forall i \in V$ and $\forall t \geq 1$ such that $\text{mod}(t,q) \neq 1$,

$$
v^i_t := \frac{1}{b} \sum_{s=1}^{b} \left( \nabla G_i(x^i_t, \xi^i_{t,s}) - \nabla G_i(x^i_{t-1}, \xi^i_{t-1,s}) \right) + v^i_{t-1}.
$$

To simplify notation, we denote in this section that

$$
\delta_t := \left\| \nabla_t - \nabla f(x_t) \right\|^2, \quad \forall t \geq 1.
$$

We establish an upper bound on $\delta_t$ that is applicable to both Algorithm 3 and 4.

Lemma 9. Let Assumption 5 hold. Suppose that $T = Rq$ for some $R \in \mathbb{Z}^+$. Consider the iterates generated by Algorithm 3 or 4. Then we have: $\forall T \geq q$,

$$
\sum_{t=1}^{T} \mathbb{E}[\delta_t] \leq \frac{6L^2q}{n^2b} \sum_{t=1}^{T} \mathbb{E}\left[ \left\| x_t - Jx_t \right\|^2 \right] + \frac{3qL^2\alpha^2}{nb} \sum_{t=1}^{T-1} \mathbb{E}\left[ \left\| \mathbb{E}_t \right\|^2 \right] + q \sum_{z=1}^{R} \mathbb{E}[\delta_{(z-1)q+1}].
$$

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Proof. Consider any \( t \geq 1 \) such that \( \text{mod}(t,q) \neq 1 \). For convenience, we define: \( \forall i \in \mathcal{V} \),

\[
d_i(x_i^t, \xi_i^{t,s}) := \nabla G_i(x_i^t, \xi_i^{t,s}) - \nabla G_i(x_i^{t-1}, \xi_i^{t,s}), \quad d_i := \frac{1}{b} \sum_{s=1}^{b} d_i^{t,s},
\]

and we clearly have

\[
\mathbb{E}[d_i^{t,s} | \mathcal{F}_t] = \mathbb{E}[d_i | \mathcal{F}_t] = \nabla f_i(x_i^t) - \nabla f_i(x_i^{t-1}). \tag{76}
\]

As a consequence of (76) and of the independence between \( \xi_i^{t,s} \) and \( \mathcal{F}_t \) for all \( i \in \mathcal{V} \) and \( s \in [b] \), we have

\[
\mathbb{E} \left[ \left( d_i^t - \nabla f_i(x_i^t) + \nabla f_i(x_i^{t-1}), d_i^t - \nabla f_i(x_i^t) + \nabla f_i(x_i^{t-1}) \right) | \mathcal{F}_t \right] = 0, \tag{77}
\]

whenever \( i \neq r \), and

\[
\mathbb{E} \left[ \left( d_i^{t,s} - \nabla f_i(x_i^t) + \nabla f_i(x_i^{t-1}), d_i^{t,s} - \nabla f_i(x_i^t) + \nabla f_i(x_i^{t-1}) \right) | \mathcal{F}_t \right] = 0, \tag{78}
\]

whenever \( s \neq a \). Moreover, using the conditional variance decomposition with (76) gives: \( \forall i \in \mathcal{V} \) and \( s \in [b] \),

\[
\mathbb{E} \left[ \| d_i^{t,s} - \nabla f_i(x_i^t) + \nabla f_i(x_i^{t-1}) \|^2 | \mathcal{F}_t \right] \leq \mathbb{E} \left[ \| d_i^{t,s} \|^2 | \mathcal{F}_t \right]. \tag{79}
\]

By the update of \( v_i^t \), we observe that

\[
\mathbb{E} \left[ \delta_i | \mathcal{F}_t \right] = \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \left( d_i^t + v_i^t - \nabla f_i(x_i^t) \right) \right\|^2 | \mathcal{F}_t \right] \\
= \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \left( d_i^t - \nabla f_i(x_i^t) + \nabla f_i(x_i^{t-1}) + v_i^t - \nabla f_i(x_i^{t-1}) \right) \right\|^2 | \mathcal{F}_t \right] \\
= \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \left( d_i^t - \nabla f_i(x_i^t) + \nabla f_i(x_i^{t-1}) \right) \right\|^2 | \mathcal{F}_t \right] + \delta_{t-1} \\
= \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| d_i^t - \nabla f_i(x_i^t) + \nabla f_i(x_i^{t-1}) \right\|^2 | \mathcal{F}_t \right] + \delta_{t-1} \\
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{s=1}^{b} \mathbb{E} \left[ \left\| d_i^{t,s} - \nabla f_i(x_i^t) + \nabla f_i(x_i^{t-1}) \right\|^2 | \mathcal{F}_t \right] + \delta_{t-1} \\
\leq \frac{1}{n^2 b^2} \sum_{i=1}^{n} \sum_{s=1}^{b} \mathbb{E} \left[ \| d_i^{t,s} \|^2 | \mathcal{F}_t \right] + \delta_{t-1}, \tag{80}
\]

where the third line uses (76), the fourth line uses (77), the fifth line uses (78), and the last line uses (79). We note that the mean-squared smoothness of \( \nabla G(\cdot) \) implies that for all \( i \in \mathcal{V} \) and \( s \in [b] \),

\[
\mathbb{E} \left[ \| d_i^{t,s} \|^2 \right] \leq L^2 \mathbb{E} \left[ \| x_i^t - x_i^{t-1} \|^2 \right]. \tag{81}
\]

Applying (81) to (80) gives: for all \( t \geq 1 \) such that \( \text{mod}(t,q) \neq 1 \),

\[
\mathbb{E} \left[ \delta_t \right] \leq \frac{L^2}{n b^2} \mathbb{E} \left[ \| x_t - x_{t-1} \|^2 \right] + \mathbb{E} [\delta_{t-1}]. \tag{82}
\]

For convenience, we define

\[
\varphi_t := \left\lfloor \frac{t-1}{q} \right\rfloor, \quad \forall t \geq 1.
\]
It can be verified that

$$\varphi_t q + 1 \leq t \leq (\varphi_t + 1)q, \quad \forall t \geq 1.$$ 

With the help of the above notations, we recursively apply (82) from $t$ to $(\varphi_t q + 2)$ to obtain: for all $t \geq 1$ such that $\text{mod}(t, q) \neq 1$,

$$\mathbb{E}[\delta_t] \leq \frac{L^2}{n^2b} \sum_{k=0}^{t} \mathbb{E}[\|x_j - x_{j-1}\|^2] + \mathbb{E}[\delta_{\varphi_t q + 1}]. \quad (83)$$

Summing up (83), we observe that $\forall z \geq 1$,

$$\sum_{t=(z-1)q+1}^{zq} \mathbb{E}[\delta_t] \leq \sum_{t=(z-1)q+1}^{zq} \left( \frac{L^2}{n^2b} \sum_{j=\varphi_t q + 2}^{t} \mathbb{E}[\|x_j - x_{j-1}\|^2] + \mathbb{E}[\delta_{\varphi_t q + 1}] \right) + \mathbb{E}[\delta_{(z-1)q+1}]$$

$$= \frac{L^2}{n^2b} \sum_{t=(z-1)q+1}^{zq} \sum_{j=\varphi_t q + 2}^{t} \mathbb{E}[\|x_j - x_{j-1}\|^2] + q \mathbb{E}[\delta_{(z-1)q+1}]$$

$$\leq \frac{L^2}{n^2b} \sum_{t=(z-1)q+1}^{zq} \sum_{j=\varphi_t q + 2}^{t} \mathbb{E}[\|x_j - x_{j-1}\|^2] + q \mathbb{E}[\delta_{(z-1)q+1}]$$

$$= \frac{L^2(q-1)}{n^2b} \sum_{j=(z-1)q+1}^{zq} \mathbb{E}[\|x_j - x_{j-1}\|^2] + q \mathbb{E}[\delta_{(z-1)q+1}], \quad (84)$$

where the second line uses the fact that $\varphi_t = z - 1$ when $(z-1)q+1 \leq t \leq zq$ for all $z \geq 1$. Finally, we sum up (84) over $z$ from 1 to $R$, we obtain: $\forall R \geq 1$,

$$\sum_{z=1}^{R} \sum_{t=(z-1)q+1}^{zq} \mathbb{E}[\delta_t] \leq \frac{L^2(q-1)}{n^2b} \sum_{z=1}^{R} \sum_{j=(z-1)q+1}^{zq} \mathbb{E}[\|x_j - x_{j-1}\|^2] + q \sum_{z=1}^{R} \mathbb{E}[\delta_{(z-1)q+1}], \quad (85)$$

Recall that $T = Eq$ and from (85) we obtain that $\forall T \geq q$,

$$\sum_{t=1}^{T} \mathbb{E}[\delta_t] \leq \frac{L^2(q-1)}{n^2b} \sum_{t=2}^{T} \mathbb{E}[\|x_t - x_{t-1}\|^2] + q \sum_{z=1}^{R} \mathbb{E}[\delta_{(z-1)q+1}], \quad (86)$$

Finally, we apply Lemma 4 to (86) to obtain: $\forall T \geq q$,

$$\sum_{t=1}^{T} \mathbb{E}[\delta_t] \leq \frac{6L^2(q-1)}{n^2b} \sum_{t=1}^{T} \mathbb{E}[\|x_t - Jx_t\|^2] + \frac{3L^2(q-1)q^2}{nb} \sum_{t=1}^{T} \mathbb{E}[\|\mathbb{E}_t\|^2] + q \sum_{z=1}^{R} \mathbb{E}[\delta_{(z-1)q+1}],$$

which finishes the proof. \hfill \Box

### D.2.1 Proof of Lemma 7(b)

Lemma 7(c) follows by applying Lemma 7(a) to Lemma 9, i.e., $\mathbb{E}[\delta_{(z-1)q+1}] \leq \frac{\nu^2}{\pi^2}$ for all $z \geq 1$.

### D.2.2 Proof of Lemma 7(c)

Lemma 7(b) follows from Lemma 9 by $\delta_{(z-1)q+1} = 0$ for all $z \geq 1$. 

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E Proof of Lemma 8

We first present a simple result that is useful for our later development.

**Proposition 2.** Consider the iterates generated by Algorithm 1. The following inequality holds: $\forall t \geq 1,$

$$\|y_{t+1} - Jy_{t+1}\| \leq \lambda^2 \|y_t - Jy_t\|^2 + \lambda^2 \|v_t - v_{t-1}\|^2 + 2\langle W^K y_t - Jy_t, (W^K - J)(v_t - v_{t-1}) \rangle.$$  \hspace{1cm} (87)

**Proof.** Using the $y$-update in Algorithm 1 and Lemma 3(a), we have: $\forall t \geq 1,$

$$\|y_{t+1} - Jy_{t+1}\|^2$$

$$= \|W^K (y_t + v_t - v_{t-1}) - JW^K (y_t + v_t - v_{t-1})\|^2$$

$$= \|W^K (y_t - Jy_t) + (W^K - J)(v_t - v_{t-1})\|^2$$

$$= \|W^K (y_t - Jy_t)\|^2 + \|(W^K - J)(v_t - v_{t-1})\|^2 + 2\langle W^K y_t - Jy_t, (W^K - J)(v_t - v_{t-1}) \rangle,$$

and the proof follows by using Lemma 3(c).

\[\square\]

E.1 Proof of Lemma 8(a)

E.1.1 Step 1: Decomposition

Recall that we are concerned with Algorithm 2 in this section. Conditioning (87) on $\mathcal{F}_t$, we have: $\forall t \geq 2,$

$$E[\|y_{t+1} - Jy_{t+1}\|^2 | \mathcal{F}_t]$$

$$\leq \lambda^2 \|y_t - Jy_t\|^2 + \lambda^2 E[\|v_t - v_{t-1}\|^2 | \mathcal{F}_t] + 2 \langle W^K y_t - Jy_t, (W^K - J)(\nabla f(x_t) - v_{t-1}) \rangle$$

$$= \lambda^2 \|y_t - Jy_t\|^2 + \lambda^2 E[\|v_t - v_{t-1}\|^2 | \mathcal{F}_t] + 2 \langle W^K y_t - Jy_t, (W^K - J)(\nabla f(x_t) - v_{t-1}) \rangle$$

$$+ 2 \langle W^K y_t - Jy_t, (W^K - J)(\nabla f(x_t) - \nabla f(x_{t-1})) \rangle$$

$$= \lambda^2 \|y_t - Jy_t\|^2 + \lambda^2 E[\|v_t - v_{t-1}\|^2 | \mathcal{F}_t] + 2 \langle W^K y_t, (W^K - J)(\nabla f(x_t) - v_{t-1}) \rangle$$

$$+ 2 \langle W^K y_t - Jy_t, (W^K - J)(\nabla f(x_t) - \nabla f(x_{t-1})) \rangle,$$  \hspace{1cm} (88)

where the first line uses the fact that $x_t$, $y_t$, $v_{t-1}$ are $\mathcal{F}_t$-measurable and also Assumption 3, while the last line uses Lemma 3(a). Towards the last term in (88), we observe that $\forall t \geq 2$ and $\forall \eta > 0,$

$$A_t \leq 2\|W^K y_t - Jy_t\| \|(W^K - J)(\nabla f(x_t) - \nabla f(x_{t-1}))\|$$

$$\leq 2\lambda \|y_t - Jy_t\| \|\nabla f(x_t) - \nabla f(x_{t-1})\|$$

$$\leq 2\lambda \|y_t - Jy_t\| \|L\| \|x_t - x_{t-1}\|$$

$$\leq \eta \lambda^2 \|y_t - Jy_t\|^2 + \eta^{-1} \lambda^2 L^2 \|x_t - x_{t-1}\|^2,$$  \hspace{1cm} (89)

where the first line uses the Cauchy-Schwarz inequality, the second line uses Lemma 3, the third line uses the $L$-smoothness of each $f_t$, and last line uses Young’s inequality. Combining (89) and (88) leads to the following: $\forall t \geq 2,$

$$E[\|y_{t+1} - Jy_{t+1}\|^2 | \mathcal{F}_t] \leq (1 + \eta) \lambda^2 \|y_t - Jy_t\|^2 + \eta^{-1} \lambda^2 L^2 \|x_t - x_{t-1}\|^2$$

$$+ \lambda^2 E[\|v_t - v_{t-1}\|^2 | \mathcal{F}_t] + 2 \langle W^K y_t, (W^K - J)(\nabla f(x_{t-1}) - v_{t-1}) \rangle.$$  \hspace{1cm} (90)

In the following, we bound $B_t$ and $C_t$ in (90) respectively.
E.1.2 Step 2: Controlling $B_t$

We decompose $B_t$ as follows: $\forall t \geq 2,$

$$B_t = \mathbb{E} \left[ \| v_t - \nabla f(x_t) + \nabla f(x_t) - v_{t-1} \|^2 | F_t \right]$$

$$= \mathbb{E} \left[ \| v_t - \nabla f(x_t) \|^2 | F_t \right] + \mathbb{E} \left[ \| \nabla f(x_t) - v_{t-1} \|^2 | F_t \right]$$

$$\leq \mathbb{E} \left[ \| v_t - \nabla f(x_t) \|^2 | F_t \right] + 2 \mathbb{E} \left( \| \nabla f(x_t) - \nabla f(x_{t-1}) \|^2 + 2 \| \nabla f(x_{t-1}) - v_{t-1} \|^2 \right)$$

$$\leq \mathbb{E} \left[ \| v_t - \nabla f(x_t) \|^2 | F_t \right] + 2L^2 \mathbb{E} \left( \| x_t - x_{t-1} \|^2 + 2 \| \nabla f(x_{t-1}) - v_{t-1} \|^2 \right),$$

where the first line utilizes Assumption 3 and the fact that $\nabla f(x_t)$ and $v_{t-1}$ are $F_t$-measurable, while the last line uses the $L$-smoothness of each $f_i$. To proceed, we note that $\forall t \geq 1,$

$$\mathbb{E} \left[ \| v_t - \nabla f(x_t) \|^2 | F_t \right] = \sum_{i=1}^{n} \mathbb{E} \left[ \frac{1}{b} \sum_{s=1}^{b} \nabla G_i(x_t^i, \xi_{t,s}^i) - \nabla f_i(x_t^i) \right]^2 | F_t \right]$$

$$= \frac{1}{b^2} \sum_{i=1}^{n} \mathbb{E} \left[ \| \nabla G_i(x_t^i, \xi_{t,s}^i) - \nabla f_i(x_t^i) \|^2 | F_t \right] \leq \frac{n \nu^2}{b},$$

where the second line uses the fact that $x_t^i$ is $F_t$-measurable and $\{ \xi_{t,s}^i, \ldots, \xi_{t,s}^i, F_t \}$ is an independent family for all $i \in \mathcal{V}$. Combining (91) and (92), we conclude that

$$\mathbb{E} \left[ B_t \right] \leq 2L^2 \mathbb{E} \left[ \| x_t - x_{t-1} \|^2 \right] + \frac{3n \nu^2}{b}, \quad \forall t \geq 2.$$

E.1.3 Step 3: Controlling $C_t$

Towards $C_t$, we observe that $\forall t \geq 2,$

$$\mathbb{E} [C_t | F_{t-1}] = \mathbb{E} \left[ \left( \mathbf{W}^{2K}(y_{t-1} + v_{t-1} - v_{t-2}) \right)^{\top} (\mathbf{W}^K - \mathbf{J}) (\nabla f(x_{t-1}) - v_{t-1}) \right]$$

$$= \mathbb{E} \left[ \left( \mathbf{W}^{2K} v_{t-1} \right)^{\top} (\mathbf{W}^K - \mathbf{J}) (\nabla f(x_{t-1}) - v_{t-1}) \right]$$

$$= \mathbb{E} \left[ \left( \mathbf{W}^{2K} (v_{t-1} - \nabla f(x_{t-1})) \right)^{\top} (\mathbf{J} - \mathbf{W}^K) (v_{t-1} - \nabla f(x_{t-1})) \right] | F_{t-1},$$

where the first line uses the y-update in Algorithm 1, while the second and the last line use Assumption 3 with the $F_{t-1}$-measurability of $y_{t-1}, v_{t-2}$ and $\nabla f(x_{t-1})$. To proceed, note that for all $t \geq 1$ we have

$$\mathbb{E} \left[ \left( v_t^i - \nabla f_i(x_t^i) \right)^{\top} (v_t^i - \nabla f_i(x_t^i)) \right] | F_t \right] = 0,$$

whenever $i \neq r$. In light of (95), we proceed from (94) as follows: $\forall t \geq 2,$

$$\mathbb{E} [C_t | F_{t-1}] = \mathbb{E} \left[ (v_{t-1} - \nabla f(x_{t-1}))^{\top} (\mathbf{J} - (\mathbf{W}^K)^{\top} \mathbf{W}^{2K}) (v_{t-1} - \nabla f(x_{t-1})) \right] | F_{t-1},$$

$$= \mathbb{E} \left[ (v_{t-1} - \nabla f(x_{t-1}))^{\top} \text{diag}(\mathbf{J} - (\mathbf{W}^K)^{\top} \mathbf{W}^{2K}) (v_{t-1} - \nabla f(x_{t-1})) \right] | F_{t-1}$$

$$\leq \frac{1}{n} \mathbb{E} \left[ \| v_{t-1} - \nabla f(x_{t-1}) \|^2 \right] | F_{t-1}$$

$$\leq \frac{\nu^2}{b},$$

where the first line uses Lemma 3(a), the second line uses (95), the third line uses the entry-wise nonnegativity of $\mathbf{W}$, and the last line uses (92). Therefore, we conclude from (96) that

$$\mathbb{E} [C_t] \leq \frac{\nu^2}{b}, \quad \forall t \geq 2.$$
E.1.4 Step 4: Putting Bounds Together and Refining

We substitute (93) and (97) into (90) to obtain: \( \forall t \geq 2, \)

\[
E[\|y_{t+1} - Jy_{t+1}\|^2] \leq (1 + \eta)^2 \lambda^2 E[\|y_t - Jy_t\|^2] + (\eta^{-1} + 2)\lambda^2 L^2 E[\|x_t - x_{t-1}\|^2] + (3\lambda^2 n + 2)\nu^2 / b. \tag{98}
\]

Setting \( \eta = \frac{1 - \lambda^2}{\lambda^2} \), we have: \( \forall t \geq 2, \)

\[
E[\|y_{t+1} - Jy_{t+1}\|^2] \leq \frac{1 + \lambda^2}{2} E[\|y_t - Jy_t\|^2] + \frac{2\lambda^2 L^2}{1 - \lambda^2} E[\|x_t - x_{t-1}\|^2] + \frac{(3\lambda^2 n + 2)\nu^2}{b}. \tag{99}
\]

We then apply Lemma 2 to (99) to obtain: \( \forall T \geq 2, \)

\[
\sum_{t=2}^{T+1} E[\|y_t - Jy_t\|^2] \leq \frac{2E[\|y_2 - Jy_2\|^2]}{1 - \lambda^2} + \frac{4\lambda^2 L^2}{(1 - \lambda^2)^2} T E[\|x_t - x_{t-1}\|^2] + \frac{2(T - 1)(3\lambda^2 n + 2)\nu^2}{b(1 - \lambda^2)}. \tag{100}
\]

Since \( y_1 = v_0 = 0_{np} \), we have

\[
E[\|y_2 - Jy_2\|^2] = E[\|W^K - Jv_1\|^2] \leq \lambda^2 E[\|v_1\|^2] = \lambda^2 \|\nabla f(x_1)\|^2 + \lambda^2 E[\|v_1 - \nabla f(x_1)\|^2] \leq \lambda^2 \|\nabla f(x_1)\|^2 + \lambda^2 n\nu^2 / b, \tag{101}
\]

where the second line uses Lemma 3(b), the third line uses Lemma 3, and the last line is due to (92). Finally, we apply (101) to (100) to obtain: \( \forall T \geq 2, \)

\[
\sum_{t=2}^{T+1} E[\|y_t - Jy_t\|^2] \leq \frac{2\lambda^2 n\zeta_0^2}{1 - \lambda^2} + \frac{4\lambda^2 L^2}{(1 - \lambda^2)^2} \sum_{t=2}^{T} E[\|x_t - x_{t-1}\|^2] + \frac{2T(3\lambda^2 n + 2)\nu^2}{b(1 - \lambda^2)} + \frac{2\lambda^2 n\nu^2}{b(1 - \lambda^2)}.
\]

The proof of Lemma 8(a) follows by applying Lemma 4 to the above inequality with minor manipulations.

E.2 Proof of Lemma 8(b) and 8(c)

We first establish a gradient tracking error bound that is applicable to both Algorithm 3 and 4. For ease of exposition, we denote

\[
\gamma_t := \|v_t - \nabla f(x_t)\|^2, \quad \forall t \geq 1. \tag{102}
\]

**Lemma 10.** Let Assumption 5 hold. Suppose that \( T = Rq \) for some \( R \in \mathbb{Z}^+ \). Consider the iterates generated by Algorithm 3 or 4. Then we have: \( \forall T \geq 2q, \)

\[
\sum_{t=2}^{T+1} E[\|y_t - Jy_t\|^2] \leq \frac{2\lambda^2 n\zeta_0^2}{1 - \lambda^2} + \frac{96\lambda^2 L^2}{(1 - \lambda^2)^2} \sum_{t=1}^{T} E[\|x_t - Jx_t\|^2] + \frac{48\lambda^2 n\mu^2 L^2}{(1 - \lambda^2)^2} \sum_{t=1}^{T-1} E[\|\tilde{x}_t\|^2] + \frac{14\lambda^2}{(1 - \lambda^2)^2} \sum_{z=0}^{R-1} E[\gamma_{zq+1}]. \tag{103}
\]

**Proof.** We first recall from (87) that \( \forall t \geq 1, \)

\[
\|y_{t+1} - Jy_{t+1}\|^2 \leq \lambda^2 \|y_t - Jy_t\|^2 + 2\|W^K y_t - Jy_t\| \langle W^K - J \rangle (v_t - v_{t-1}) \tag{104}
\]

In the first two steps, we refine (104) for mod\((t, q) \neq 1 \) and mod\((t, q) = 1 \) respectively.

**Step 1:** consider any \( t \geq 2 \) such that mod\((t, q) \neq 1 \). From (104), we observe that for all \( \eta > 0, \)

\[
\|y_{t+1} - Jy_{t+1}\|^2 \leq \lambda^2 \|y_t - Jy_t\|^2 + \lambda^2 \|v_t - v_{t-1}\|^2 + 2\|W^K y_t - Jy_t\| \langle W^K - J \rangle (v_t - v_{t-1}) \]
\[
\leq \lambda^2 \|y_t - Jy_t\|^2 + \lambda^2 \|v_t - v_{t-1}\|^2 + 2\lambda^2 \|y_t - Jy_t\| \|v_t - v_{t-1}\| \]
\[
\leq \lambda^2 (1 + \eta) \|y_t - Jy_t\|^2 + \lambda^2 (1 + \eta^{-1}) \|v_t - v_{t-1}\|^2, \tag{105}
\]

\[
\]
where the first line uses Cauchy-Schwarz inequality, the second line uses Lemma 3, and the last uses Young’s inequality. Setting \( \eta = \frac{1 - \lambda^2}{2\lambda^2} \) in (105) gives:

\[
\|y_{t+1} - Jy_{t+1}\|^2 \leq \frac{1 + \lambda^2}{2}\|y_t - Jy_t\|^2 + \frac{\lambda^2(1 + \lambda^2)}{1 - \lambda^2}\|v_t - v_{t-1}\|^2 \tag{106}
\]

Note that

\[
E[\|v_t - v_{t-1}\|^2] = \sum_{i=1}^n E\left[\left\|\frac{1}{b} \sum_{s=1}^b \left( \nabla G_i(x^t_i, \xi^t_{i,s}) - \nabla G_i(x^t_{i-1}, \xi^t_{i,s}) \right) \right\|^2 \right]
\]

\[
\leq \frac{1}{b} \sum_{i=1}^n \sum_{s=1}^b E\left[\|\nabla G_i(x^t_i, \xi^t_{i,s}) - \nabla G_i(x^t_{i-1}, \xi^t_{i,s})\|^2 \right]
\]

\[
\leq L^2E[\|x_t - x_{t-1}\|^2], \tag{107}
\]

where the last line uses the mean-squared smoothness. Applying (107) to (106), we obtain:

\[
E[\|y_{t+1} - Jy_{t+1}\|^2] \leq \frac{1 + \lambda^2}{2}E[\|y_t - Jy_t\|^2] + \frac{2\lambda^2L^2}{1 - \lambda^2}E[\|x_t - x_{t-1}\|^2] \tag{108}
\]

**Step 2:** consider any \( t \geq 2 \) such that \( \text{mod}(t, q) = 1 \). In this case, we have \( E[v_t | F_t] = \nabla f(x_t) \). Taking the conditional expectation of (104) with respect to the filtration \( F_t \), we obtain

\[
E[\|y_{t+1} - Jy_{t+1}\|^2 | F_t] \leq \lambda^2\|y_t - Jy_t\|^2 + \lambda^2E[\|v_t - v_{t-1}\|^2 | F_t]
\]

\[
\quad + 2(\mathbf{W}^K y_t - Jy_t, (\mathbf{W}^K - J)(\nabla f(x_t) - v_{t-1}))
\]

\[
\leq \lambda^2(1 + \eta)\|y_t - Jy_t\|^2 + \lambda^2E[\|v_t - v_{t-1}\|^2 | F_t] + \lambda^2\eta^{-1}\|\nabla f(x_t) - v_{t-1}\|^2 \tag{109}
\]

where the first line uses the fact that \( x_t \) and \( y_t \) are \( F_t \)-measurable and the second line follows a similar line of arguments as in (105). Setting \( \eta = \frac{1 - \lambda^2}{2\lambda^2} \) in (109), we obtain:

\[
E[\|y_{t+1} - Jy_{t+1}\|^2] \leq \frac{1 + \lambda^2}{2}E[\|y_t - Jy_t\|^2] + \frac{2\lambda^4L^2}{1 - \lambda^2}E[\|\nabla f(x_t) - v_{t-1}\|^2]. \tag{110}
\]

We recall the definition of \( Y_t \) in (102) and observe that

\[
\|v_t - v_{t-1}\|^2 = \|v_t - \nabla f(x_t) + \nabla f(x_t) - \nabla f(x_{t-1}) + \nabla f(x_{t-1}) - v_{t-1}\|^2
\]

\[
\leq 3Y_t + 3\|\nabla f(x_t) - \nabla f(x_{t-1})\|^2 + 3Y_{t-1}
\]

\[
\leq 3Y_t + 3L^2\|x_t - x_{t-1}\|^2 + 3Y_{t-1}, \tag{111}
\]

where the last uses the \( L \)-smoothness of each \( f_i \). Similarly, we have

\[
\|\nabla f(x_t) - v_{t-1}\|^2 = \|\nabla f(x_t) - \nabla f(x_{t-1}) + \nabla f(x_{t-1}) - v_{t-1}\|^2
\]

\[
\leq 2L^2\|x_t - x_{t-1}\|^2 + 2Y_{t-1}. \tag{112}
\]

Plugging (111) and (112) into (110) gives

\[
E[\|y_{t+1} - Jy_{t+1}\|^2] \leq \frac{1 + \lambda^2}{2}E[\|y_t - Jy_t\|^2] + \left(3\lambda^2 + \frac{4\lambda^4}{1 - \lambda^2}\right)L^2E[\|x_t - x_{t-1}\|^2]
\]

\[
+ 3\lambda^2E[Y_t] + \left(3\lambda^2 + \frac{4\lambda^4}{1 - \lambda^2}\right)E[Y_{t-1}]
\]

\[
\leq \frac{1 + \lambda^2}{2}E[\|y_t - Jy_t\|^2] + \frac{4\lambda^2L^2}{1 - \lambda^2}E[\|x_t - x_{t-1}\|^2] + 3\lambda^2E[Y_t] + \frac{4\lambda^2E[Y_{t-1}]}{1 - \lambda^2}. \tag{113}
\]
Step 3: combining step 1 and step 2. Combining (108) and (113), we obtain: \(\forall t \geq 2\),
\[
\mathbb{E}[\|y_{t+1} - Jy_{t+1}\|^2] \leq \frac{1 + \lambda^2}{2} \mathbb{E}[\|y_t - Jy_t\|^2] + \frac{4\lambda^2 L^2}{1 - \lambda^2} \mathbb{E}[\|x_t - x_{t-1}\|^2] \\
+ \mathbb{I}_{\{\text{mod } (t,q) = 1\}} \left(3\lambda^2 \mathbb{E}[\|\tau_{t-1}\|] + \frac{4\lambda^2 \mathbb{E}[\|\tau_{t-1}\|]}{1 - \lambda^2}\right). 
\]
(114)
Let \(T = Rq\) for some \(R \in \mathbb{Z}^+\). We apply Lemma 2 to (114) to obtain: \(\forall T \geq 2q\),
\[
\sum_{t=2}^{T+1} \mathbb{E}[\|y_t - Jy_t\|^2] \\
\leq \frac{2\mathbb{E}[\|y_2 - Jy_2\|^2]}{1 - \lambda^2} + \frac{8\lambda^2 L^2}{(1 - \lambda^2)^2} \sum_{t=2}^{T} \mathbb{E}[\|x_t - x_{t-1}\|^2] + \sum_{t=2}^{T} \mathbb{I}_{\{\text{mod } (t,q) = 1\}} \left(6\lambda^2 \mathbb{E}[\|\tau_t\|] + \frac{8\lambda^2 \mathbb{E}[\|\tau_{t-1}\|]}{(1 - \lambda^2)^2}\right) \\
= \frac{2\mathbb{E}[\|y_2 - Jy_2\|^2]}{1 - \lambda^2} + \frac{8\lambda^2 L^2}{(1 - \lambda^2)^2} \sum_{t=2}^{T} \mathbb{E}[\|x_t - x_{t-1}\|^2] + \sum_{t=2}^{T} \mathbb{I}_{\{\text{mod } (t,q) = 1\}} \left(6\lambda^2 \mathbb{E}[\|\tau_{t+1}\|] + \frac{8\lambda^2}{(1 - \lambda^2)^2} \mathbb{E}[\|\tau_{t}\|]\right). 
\]
(115)
Note that
\[
\mathbb{E}[\|y_2 - Jy_2\|^2] = \mathbb{E}[\|(W^K - J)v_1\|^2] \leq \lambda^2 \mathbb{E}[\|v_1\|^2] = \lambda^2 \|\nabla f(x_1)\|^2 + \lambda^2 \mathbb{E}[\|\tau_1\|] 
\]
(116)
Applying (116) to (115) gives the following: \(\forall T \geq 2q\),
\[
\sum_{t=2}^{T+1} \mathbb{E}[\|y_t - Jy_t\|^2] \\
\leq \frac{2\lambda^2 n c^2}{1 - \lambda^2} + \frac{8\lambda^2 L^2}{(1 - \lambda^2)^2} \sum_{t=2}^{T} \mathbb{E}[\|x_t - x_{t-1}\|^2] + \frac{2\lambda^2}{1 - \lambda^2} \mathbb{E}[\|\tau_t\|] + \frac{6\lambda^2}{1 - \lambda^2} \sum_{t=2}^{T} \mathbb{E}[\|\tau_{t+1}\|] + \frac{8\lambda^2}{(1 - \lambda^2)^2} \sum_{t=2}^{T} \mathbb{E}[\|\tau_{t}\|] \\
\leq \frac{2\lambda^2 n c^2}{1 - \lambda^2} + \frac{8\lambda^2 L^2}{(1 - \lambda^2)^2} \sum_{t=2}^{T} \mathbb{E}[\|x_t - x_{t-1}\|^2] + \frac{6\lambda^2}{1 - \lambda^2} \sum_{t=2}^{T} \mathbb{E}[\|\tau_{t+1}\|] + \frac{8\lambda^2}{(1 - \lambda^2)^2} \sum_{t=2}^{T} \mathbb{E}[\|\tau_{t}\|] 
\]
(117)
Step 4: bounding \(\mathbb{E}[\|\tau_t\|]\). The derivations in this step essentially repeat the proof of Lemma 9. Recall the definition of \(\tau_t\) in (102). Consider any \(t \geq 1\) such that \(\text{mod } (t, q) \neq 1\) and define: \(\forall i \in \mathcal{V}\),
\[
d_i := \nabla G_i(x_i, \xi_{i,s}) - \nabla G_i(x_{i-1}, \xi_{i,s}), \quad d_i := \frac{1}{b} \sum_{s=1}^{b} d_i. 
\]
By the update of \(v_i\), we observe that
\[
\mathbb{E}[\|\tau_t\| | \mathcal{F}_t] = \sum_{i=1}^{n} \mathbb{E}[\|d_i + v_{i-1} - \nabla f_i(x_i)\|^2 | \mathcal{F}_t] \\
= \sum_{i=1}^{n} \mathbb{E}[\|d_i - \nabla f_i(x_i) + \nabla f_i(x_{i-1}) + v_{i-1} - \nabla f_i(x_{i-1})\|^2 | \mathcal{F}_t] \\
= \sum_{i=1}^{n} \mathbb{E}[\|d_i - \nabla f_i(x_i) + \nabla f_i(x_{i-1})\|^2 | \mathcal{F}_t] + \tau_{i-1} \\
= \frac{1}{b^2} \sum_{i=1}^{n} \sum_{s=1}^{b} \mathbb{E}[\|d_i^{s} - \nabla f_i(x_i) + \nabla f_i(x_{i-1})\|^2 | \mathcal{F}_t] + \tau_{i-1} \\
\leq \frac{1}{b^2} \sum_{i=1}^{n} \sum_{s=1}^{b} \mathbb{E}[\|d_i^{s}\|^2 | \mathcal{F}_t] + \tau_{i-1}, 
\]
(118)
where the above derivations follow a very similar line of arguments as in (80) and thus we omit the detailed explanations. Taking the expectation of (118) and using the mean-squared smoothness of $\nabla G(\cdot, \xi)$, we obtain

$$E[\mathcal{Y}_t] \leq \frac{L^2}{b} E[\|x_t - x_{t-1}\|^2] + E[\mathcal{Y}_{t-1}].$$  \hspace{1cm} (119)

For convenience, we define $\varphi_t := \left\lfloor \frac{t-1}{q} \right\rfloor \forall t \geq 1$. It can be verified that $\varphi_t q + 1 \leq t \leq (\varphi_t + 1)q, \forall t \geq 1$. Recursively applying (119) from $t$ to $(\varphi_t + 1)$, we obtain

$$E[\mathcal{Y}_t] \leq \frac{L^2}{b} \sum_{j=\varphi_t+1}^{q} E[\|x_j - x_{j-1}\|^2] + E[\mathcal{Y}_{\varphi_t}].$$  \hspace{1cm} (120)

In particular, taking $t = zq$ for some $z \in \mathbb{Z}^+$ in (120) gives

$$E[\mathcal{Y}_{zq}] \leq \frac{L^2}{b} \sum_{j=(z-1)q+2}^{zq} E[\|x_j - x_{j-1}\|^2] + E[\mathcal{Y}_{(z-1)q+1}].$$  \hspace{1cm} (121)

We sum up (121) over $z$ from 1 to $R$ to obtain:

$$\sum_{z=1}^{R} E[\mathcal{Y}_{zq}] \leq \frac{L^2}{b} \sum_{z=1}^{R} \sum_{j=(z-1)q+2}^{zq} E[\|x_j - x_{j-1}\|^2] + \sum_{z=1}^{R} E[\mathcal{Y}_{(z-1)q+1}]$$

$$\leq \frac{L^2}{b} \sum_{t=2}^{T} E[\|x_t - x_{t-1}\|^2] + \sum_{z=0}^{R-1} E[\mathcal{Y}_{zq+1}].$$  \hspace{1cm} (122)

**Step 5: putting bounds together.** Applying (122) to (117) gives the following: $\forall T \geq 2q$,

$$\sum_{t=2}^{T+1} E[\|y_t - Jy_t\|^2]$$

$$\leq \frac{2\lambda^2 n \zeta^2}{1 - \lambda^2} + \left(1 + \frac{1}{b}\right) \frac{8\lambda^2 L^2}{(1 - \lambda^2)^2} \sum_{t=2}^{T} E[\|x_t - x_{t-1}\|^2] + \frac{6\lambda^2}{1 - \lambda^2} \sum_{z=0}^{R-1} E[\mathcal{Y}_{zq+1}] + \frac{8\lambda^2}{(1 - \lambda^2)^2} \sum_{z=0}^{R-1} E[\mathcal{Y}_{zq+1}]$$

$$\leq \frac{2\lambda^2 n \zeta^2}{1 - \lambda^2} + \frac{16\lambda^2 L^2}{(1 - \lambda^2)^2} \sum_{t=2}^{T} E[\|x_t - x_{t-1}\|^2] + \frac{14\lambda^2}{(1 - \lambda^2)^2} \sum_{z=0}^{R-1} E[\mathcal{Y}_{zq+1}].$$  \hspace{1cm} (123)

Plugging Lemma 4 into (123) finishes the proof. □

**E.2.1 Proof of Lemma 8(b)**

Lemma 8(b) follows from (103) by $\mathcal{Y}_{zq+1} \leq nt^2/B$ for all $z \in \mathbb{Z}^+$.

**E.2.2 Proof of Lemma 8(c)**

Lemma 8(c) follows from (103) by $\mathcal{Y}_{zq+1} = 0$ for all $z \in \mathbb{Z}^+$. 

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