The expected degree of minimal spanning forests

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Abstract. We give a lower bound on the expected degree of the free minimal spanning forest of a vertex transitive graph in terms of its spectral radius. This result answers a question of Lyons-Peres-Schramm and simplifies the Gaboriau-Lyons proof of the measurable-group-theoretic solution to von Neumann’s problem.

In the second part we study a relative version of the free minimal spanning forest. As a consequence of this study we can show that non-torsion unitarizable groups have fixed price one.

1. Free minimal spanning forests

Let \( G = (V, E^\sharp) \) be a connected graph. We view \( E^\sharp \) as a set of oriented edges with the requirement, that for each oriented edge in \( E^\sharp \), the opposite edge is also contained in \( E^\sharp \). For each oriented edge \( e \), we denote by \( \hat{e} \) the opposite edge, by \( e \) the tail and by \( \hat{e} \) the head of \( e \). We require \( e \neq \hat{e} \) for all \( e \in E^\sharp \). Thus, an edge is naturally identified with an unordered pair of oriented edges \( \{e, \hat{e}\} \). The set of edges is denoted by \( E \). We say that \( G \) is vertex-transitive if for any two vertices \( x, y \in V \), there exists an automorphism \( \varphi \) of \( G \), such that \( \varphi(x) = y \). From now on, we will only consider vertex-transitive graphs. Let \( d \) be the degree of a vertex in \( G \).

A path of length \( n \) between vertices \( x, y \in V \) is a sequence of oriented edges \( (e_1, \ldots, e_n) \) such that \( \tau_i = e_{i+1} \) for all \( 1 \leq i \leq n \), \( e_1 = x \) and \( e_n = y \). A path is called a cycle rooted at \( x \in V \) if \( x = y \). A cycle is called simple if no vertex is visited twice and it is not of the form \( (e, \hat{e}) \). We denote the number of cycles rooted at \( x \) of length \( n \) by \( c(n) \). The spectral radius of \( G \) is defined to be

\[
\lambda := \frac{1}{d} \limsup_{n \to \infty} c(n)^{1/n}
\]

and it equals the operator norm of the random walk operator on \( \ell^2(V) \), see [16, Lemma 10.1]. For \( x, y \in V \), we denote by \( c(n, x, y) \) the number of paths of length \( n \) from \( x \) to \( y \). A basic consequence of the description of the operator norm of the
random walk operator is the estimate
\begin{equation}
    c(n, x, y) \leq (\lambda d)^n.
\end{equation}

A subgraph of \( G = (V, E^d) \) is a subset \( V_1 \subset V \) and a subset \( E_1^d \subset E^d \), such that \( G_1 = (V_1, E_1^d) \) is itself a graph, i.e. \( E_1^d \) is closed under taking opposite oriented edges and oriented edges in \( E_1^d \) connect only points in \( V_1 \). A subgraph \( G_1 = (V_1, E_1^d) \) is called spanning if \( V_1 = V \). A subgraph of \( G \) is called a forest, if it does not contain any simple cycles.

Let us start by recalling the definition of the minimal spanning forest of \( G \). First of all, a spanning subgraph of \( G \) is naturally identified with an element of the space \( \{0, 1\}^E \). The minimal spanning forest is a probability measure \( \text{FMSF} \) on \( \{0, 1\}^E \) whose support consists of spanning forests. The measure \( \text{FMSF} \) arises as the push-forward of the product measure \( \mu \otimes E \) along a map

\[ \Phi: [0, 1]^E \to \{0, 1\}^E, \]

where \( \mu \) denote the Lebesgue measure on \([0, 1]\). To define \( \Phi \), we view an element \( x \in [0, 1]^E \) as a \([0, 1]\)-labelling of the edges of \( G \). Whenever we see a simple cycle, then cut all edges in this cycle, which carry a maximal label. The resulting graph is denoted by \( \Phi(x) \) and is clearly a spanning forest. It can be checked directly, that \( \Phi \) is a Borel map. The study of minimal spanning forests on lattices in \( \mathbb{R}^d \) and specific graphs has a long history. The systematic study of \( \text{FMSF} \) on unimodular vertex-transitive graphs was initiated by Lyons-Peres-Schramm [8], and we refer to this paper also for further references. The virtue of the construction from above is that the resulting probability measure on \( \{0, 1\}^E \) is invariant under all automorphisms of the graph \( G \). In particular, since we assume \( G \) to be vertex-transitive, what happens at a single vertex happens everywhere. Let \( x \in V \) be some fixed vertex and let \( \delta_{\text{FMSF}}(G) \) be the expected degree of the vertex \( x \) with respect to \( \text{FMSF} \). The number \( \delta_{\text{FMSF}}(G) \) (which is clearly independent of \( x \)) is called the expected degree of the free minimal spanning forest.

**Theorem 1.** Let \( G \) be a vertex transitive graph with degree \( d \) and spectral radius \( \lambda \). Then, the following inequalities hold:

\[ \frac{1}{4\lambda} - \frac{1}{4} \leq \delta_{\text{FMSF}}(G) \leq d. \]

**Proof.** The second inequality is clear. Let us enumerate the edges at the vertex \( x \) by \( s_1, \ldots, s_d \). For \( 1 \leq i \leq d \), the probability that \( s_i \) survives is zero if \( s_i \) is a loop. Let us assume that \( s_i \) is not a loop and that its label is \( \rho \). Then, the probability that it is cut because of a simple cycle of length \( n \) rooted at \( x \) and starting with \( s_i \) is \( \rho^{n-1} \). Moreover, by (1) there are at most \( (\lambda d)^{n-1} \) such cycles.
Hence, the probability that $s_i$ survives is bounded from below as follows:

$$
P(s_i \text{ survives}) \geq \int_0^{1/2} \left( 1 - \sum_{n=2}^{\infty} (\lambda d)^{n-1} \rho^{n-1} \right) d\rho
$$

$$
= \int_0^{1/2} \left( 2 - \frac{1}{1 - \lambda d\rho} \right) d\rho
$$

$$
= \frac{1 - \ln(2)}{\lambda d}
$$

$$
\geq \frac{1}{4\lambda d}.
$$

Since this argument works for all edges adjacent to the vertex $x$ which are not loops, and the number of loops is bounded from above by $\lambda d$, we obtain

$$
\delta_{FMSF}(G) = E \left( \sum_{i=1}^{d} [s_i \text{ survives}] \right) = \sum_{i=1}^{d} P(s_i \text{ survives}) \geq \frac{d - \lambda d}{4\lambda d} = \frac{1}{4\lambda} - \frac{1}{4}.
$$

Here, we denoted by $[s_i \text{ survives}]$ the $\{0, 1\}$-valued function on $[0, 1]^E$, which indicates if $s_i$ survived the cutting procedure or not. This proves the claim. □

Let $\Gamma$ be a finitely generated group and $S$ a finite multi-set with $S = S^{-1}$ which generates $\Gamma$. More formally, a multi-set is a map $\alpha: S \to \Gamma$ such that there is an involution $\iota: S \to S$, such that $\alpha(\iota(s)) = \alpha(s)^{-1}$. We will omit $\alpha$ and $\iota$ in the notation and just write $s^{-1}$ for the inverse.

Let Cay($\Gamma, S$) be the associated Cayley graph, i.e. the vertex set is $\Gamma$ and for every $g \in \Gamma$ and $s \in S$, we have an oriented edge $(g, s, sg)$. The opposite edge is given by $(sg, s^{-1}, g)$. We denote by $\delta_{FMSF}(\Gamma, S)$ the expected degree of the free minimal spanning forest of the Cayley graph Cay($\Gamma, S$).

**Corollary 2.** Let $\Gamma$ be a finitely generated non-amenable group. The expected degree of the free minimal spanning forest depends on the generating multi-set. Moreover, for every $n \in \mathbb{N}$, there exists a finite multi-set $S$ which generates the group $\Gamma$, such that $\delta_{FMSF}(\Gamma, S) \geq n$.

**Proof.** Since $\Gamma$ is non-amenable, $\lambda < 1$ for any finite generating set $S$ by Kesten’s theorem, see [7]. Replacing $S$ by the multi-set $S^{[k]}$ of words of length $k$ in $S$, then we get a spectral radius $\lambda^k$ and thus from Theorem 1:

$$
\frac{1}{4\lambda^k} - \frac{1}{4} \leq \delta_{FMSF}(\Gamma, S^{[k]}) \leq d^k.
$$

In particular, $\delta_{FMSF}(\Gamma, S^{[k]})$ cannot be independent of $k$ and is unbounded as $k$ tends to infinity. □

The previous corollary answers Question 6.13 in [8], where it was asked whether the expected degree of FMSF depends on the choice of a generating set or not.
Russ Lyons informed us that as a solution to [9, Exercise 11.16], it was known that dependence on the generating set holds for some specific non-amenable groups. As a consequence of Corollary 2, we can now conclude that WMSF(Γ, S) ≠ FMSF(Γ, S) for any non-amenable group and some suitable multi-set of generators S, see [8] for more definitions. As an immediate consequence of [8, Proposition 3.6] and Corollary 2, we also get that \( p_c(Γ, S) < p_u(Γ, S) \) for any non-amenable group and a suitable multi-set of generators S. This result was shown by Pak-Smirnova-Nagnibeda as [12, Theorem 1] – and we refer to this paper for definitions if necessary. Previously, [12, Theorem 1] was used to conclude that WMSF(Γ, S) ≠ FMSF(Γ, S) via [8, Proposition 3.6]. Another consequence of Corollary 2 is an elementary proof of part of [5, Proposition 12], which is the key new technical result in the Lyons-Gaboriau proof of the measurable-group theoretic solution to von Neumann’s problem. Note that it does not follow from our results that the cluster equivalence relation of FMSF is ergodic, but for the purposes of a proof of [5, Theorem 1] this can be fixed by a result of Chifan-Ioana [2, Corollary 9], see also the remarks in [2, Section 4.2].

Let us also mention, that it was proved by Timar in [15], that if WMSF(Γ, S) ≠ FMSF(Γ, S), then almost surely every tree in FMSF has infinitely many ends.

2. Relative minimal spanning forests

Let Γ be a non-torsion group and let \( a \in Γ \) be a non-torsion element. Let \( S \subseteq Γ \) be a finite generating set with \( S^{-1} = S \) and consider the Cayley graph \( G := Cay(Γ, S) \). We set \( d := |S| \). We will assume that \( a \in S \). If this is the case, then for any \( g \in Γ \) the set \( L := \{a^n g \mid n \in Z \} \subseteq Γ \) is a bi-infinite line in \( G \). We denote by \( L(a) \) the subgraph of \( G \) formed by the union of all such bi-infinite lines.

Note that Γ acts by automorphisms on \( G = (Γ, E) \). An element of \( x \in \{0, 1\}^E \) is identified with a subgraph \( G(x) \subseteq G \). A random spanning sub-forest of \( G \) is a probability measure \( σ \) on \( \{0, 1\}^E \), whose support is contained in the set of spanning forests. We call such a random subforest invariant, if the probability measure is invariant with respect to the natural Γ action on \( E \). An invariant random subforest \( σ \) is called a factor of i.i.d. process if there exists a measurable and Γ-equivariant map \( Φ: [0, 1]^E \rightarrow \{0, 1\}^E \), such that \( σ = Φ_\ast(μ\otimes E) \), where \( μ \) denotes the Lebesgue measure on \( [0, 1] \). If \( σ \) is an invariant random subforest of \( G \), then we denote by \( \deg(σ) \) the expected degree of the vertex \( e \in Γ \).

We will not recall the definition of cost of a p.m.p. essentially free action here and instead refer to Gaboriau’s foundational work [4] and the book [6], where everything is explained in detail. The cost of the action \( Γ \acts (X, λ) \) is denoted by \( \text{cost}(Γ \acts X) \).

The following theorem deals with a relative version of the free minimal spanning forest, paying at the same time more attention to a control of the expected degree
in terms of the cost of the action $\Gamma \actson [0,1]^E$. Results of this type have been studied before, see e.g. [6, Lemma 28.11] or [13, Corollary 40].

**Theorem 3.** Let $\Gamma$ be a non-torsion group and let $a \in \Gamma$ be a non-torsion element. Let $S$ be a finite symmetric generating set with $a \in S$ and denote the associated Cayley graph by $G$. There exists a random spanning sub-forest $\sigma$ of $G$, such that

1. $\sigma$-a.s. $L(a)$ is contained in the subforest,
2. $\sigma$ is a factor of i.i.d. process, and
3. $\deg(\sigma) \geq 2 \cdot \text{cost}(\Gamma \actson [0,1]^E)$.

**Proof.** The proof follows the ideas of the proof of [6, Lemma 28.11]. First of all, we pick an isomorphism $[0,1] \cong [0,1]^\mathbb{N}$ and think of $[0,1]^\mathbb{N}$ as an infinite stack of elements in the unit interval. Let $n \in \mathbb{N}$. We first define the graph $Z_n(G)$ of $n$-cycles in $G$. Its vertex set is the set $C_n$ of (unrooted and simple) $n$-cycles in $G$ and two cycles are connected by an edge if they are different and have a common vertex. It is clear that the maximum degree of $Z_n(G)$ is less or equal $nd^n$, where we have set $d := |S|$. Using the $[0,1]$-labels of the edges of $G$, we construct an i.i.d. labelling of $C_n$ with elements in $[0,1]$ in an equivariant way. Now, let $c_0 \in C_n$ labelled $\lambda \in [0,1]$. The probability that there is a chain $c_0, c_1, \ldots, c_k$ of pairwise adjacent cycles so that their labels are increasing in each step is bounded above by $(1 - \lambda nd^n)^k$. We see from this, that the probability of existence of an infinite such chain is zero. Thus – for almost every labelling of $G$ – we can assign to each $n$-cycle the length of the longest such chain, and call it the depth of this $n$-cycle (with respect to the given labelling of $G$).

In this way, we have for almost every labelling $x \in [0,1]^E$, found a map

$$\varphi_x: \bigcup_{n \geq 1} C_n \to \mathbb{N} \times \mathbb{N},$$

which maps an $n$-cycle of depth $k$ (with respect to the labelling $x \in [0,1]^E$) to the pair $(n, k) \in \mathbb{N} \times \mathbb{N}$. We call this map a colouring of the cycles associated with $x \in [0,1]^E$. Note that this colouring is equivariant in the sense that $\varphi_{xg}(c) = \varphi_x(cg^{-1})$, for almost all $x \in [0,1]^E$ and all $c \in \sqcup_{n \geq 1} C_n$.

Thus, we have used the first element in our stack $[0,1]^\mathbb{N}$ to set up an $\Gamma$-equivariant colouring of the set of cycles. We now will use the other elements in the stack to set up an infinite recursive $\Gamma$-equivariant cutting procedure which will have the property, that it does not disconnect the graph $G$. Let’s pick a standard enumeration $\alpha: \mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ and set $\psi_x := \alpha \circ \varphi_x$. We start by cutting all cycles in $\psi_x^{-1}(0)$ in a way yet to be described. Note that the set $\psi_x^{-1}(0)$ consists of disjoint cycles. For each $c \in \psi_x^{-1}(0)$, we cut the edge with the maximal label (this is the second in our stack $[0,1]^\mathbb{N}$) not in $L(a)$. Clearly, this does not disconnect $G$. 
We now proceed in a similar way (using the next element in the stack) with the cycles in $ψ^{-1}_x(1)$ – again, this is a set of disjoint cycles – which still exist after the cutting that has been done already, etc.

In each step, the graph remains connected, and hence the expected degree cannot be less than $2 \cdot \text{cost}(Γ ∩ [0, 1]^E)$. Moreover, the resulting graphs will always contain $L(a)$. Taking limits, we see that all cycles of $G$ have been cut and we obtain a random sub-forest of $G$ containing $L(a)$, whose expected degree is at least $\text{cost}(Γ ∩ [0, 1]^E)$. We can now define $Φ: [0, 1]^E → \{0, 1\}^E$ by mapping $x ∈ [0, 1]^E$ to the result of the cutting process. This finishes the proof.

**Remark 4.** Let $Γ$ be a finitely generated and infinite group. Recall, an infinite group is said to have fixed price one if all its p.m.p. essentially free actions have cost one. Then, $\text{cost}(Γ ∩ [0, 1]^E) > 1$ if and only if $Γ$ does not have fixed price equal to one, as proved by Abért-Weiss. Indeed, by the Abért-Weiss theorem, $\text{cost}(Γ ∩ [0, 1]^E)$ is the maximum among costs of all possible p.m.p. essentially free actions of the group $Γ$. Hence, if this cost is equal to one, the group must have fixed price one. The other implication is obvious.

We now change the perspective slightly and consider more general invariant random spanning forests. They are no longer bound to be sub-forests of $G$. Thus, we study more generally probability measures on $\{0, 1\}^Γ × Γ$, invariant under the diagonal right $Γ$-action on $Γ × Γ$, see [3] for more details. The expected degree $\text{deg}(σ)$ of an invariant random spanning forest $σ$ defined in a similar way as before.

The width of an invariant random spanning forest $σ$ is defined to be the number of vertices $g ∈ Γ$, so that the probability that an edge between vertices $e$ and $g$ exists is positive. We denote this number by $\text{width}(σ)$.

Let us now assume that there is some invariant random sub-forest $τ$ of $G$, such that $\text{deg}(τ) > 2$ and $L(a)$ is a.s. contained in the sub-forest. Let $b ∈ S \setminus \{a, a^{-1}\}$, such that $τ(\{x ∈ [0, 1]^E | (b, e) ∈ G(x)\}) > 0$ and set

$$X := \{x ∈ [0, 1]^E | (b, e) ∈ G(x)\} ⊂ [0, 1]^E.$$ 

For each $n ∈ \mathbb{N}$, we now define a Borel map $Θ_n: [0, 1]^E → \{0, 1\}^Γ × Γ$ which sends a subgraph $G(x) ⊆ G$ to the graph formed by all edges $(a^i ba^{-i} g, g)$ for $1 ≤ i ≤ n$, whenever $a^i ba^{-i} g$ and $g$ lie in the same connected component of $G(x)$. It is clear that $Θ_n$ is $Γ$-equivariant. We define $τ_n$ to be the push-forward of $τ$ with respect to $Θ_n$, i.e. $τ_n := (Θ_n)_*(τ)$.

**Lemma 5.** For each $n ∈ \mathbb{N}$, the measure $τ_n$ is an invariant random spanning forest. Moreover, we have $\text{width}(τ_n) = 2n$ and $\text{deg}(τ_n) = 2n \cdot τ(X)$.

**Proof.** Consider the free group on letters $\{a, b\}$. It is a well-known fact that the elements $\{a^i ba^{-i} | 1 ≤ i ≤ n\}$ are the basis of a free group of rank $n$. Thus any
cycle formed by the partial self-maps $a^{-1}ba^{-1}|_{a^{-1}(X)}$ and their inverses yields a cycle formed by the partial self-maps $a, a^{-1}, b|_X, b^{-1}|_X$. However, as $\tau$ does not contain cycles, there are no such cycles. Invariance is clear since $\Theta_n$ was $\Gamma$-equivariant.

It is clear that the width of $\tau_n$ is equal to $2n$, whereas the expected degree is equal to $2n \cdot \tau(X)$.

We can now obtain a corollary which covers a particular case of the famous Dixmier problem on unitarizability, see the book of Pisier [14] for a detailed discussion of this problem and further references. Our approach relies on results of Epstein-Monod [3], whose pioneering work related invariant random forests to this problem. Let us explain this in a bit more detail. Recall, a representation $\pi: \Gamma \rightarrow B(\mathcal{H})^\times$ on a Hilbert space is called uniformly bounded if $\sup_{g \in \Gamma} \|\pi(g)\| < \infty$. It was shown by Dixmier that any uniformly bounded representation of an amenable group is conjugate to a unitary representation. He conjectured that this in fact yields a characterization of amenability. Groups with the property that every uniformly bounded representation on a Hilbert space is conjugate to a unitary representation (i.e. can be unitarized) are called unitarizable. It is well-known by now that non-abelian free groups (and all groups containing such groups) are not unitarizable. However, there are non-amenable groups without non-abelian free subgroups. After the results of Gaboriau-Lyons [5], there was some hope that Dixmier’s longstanding problem could be resolved, using suitable invariant spanning forests that play the role of free subgroups, see [3,10]. We are now building on the work of Epstein-Monod [3] and construct a specific sequence of invariant random forests showing that certain groups are not unitarizable.

**Corollary 6.** Let $\Gamma$ be a finitely generated non-torsion group which does not have fixed price one. Then, the group $\Gamma$ is not unitarizable.

**Proof.** Let $a \in \Gamma$ be a non-torsion element and let $S \subset \Gamma$ be a finite generating set with $a \in S$. Consider the Cayley graph $G$ associated with $S$ as above. By Remark 4, if $\Gamma$ does not have fixed price one, then $\text{cost}(\Gamma \bowtie [0,1]^G) > 1$. By Theorem 3, there exists an invariant random spanning forest $\tau$ with $\deg(\tau) > 2$ and $L(a)$ contained in it almost surely.

Thus, we can follow the construction of $(\tau_n)_n$ as above. In particular, we conclude from Lemma 5 that

$$\frac{\deg(\tau_n)^2}{\text{width}(\tau_n)} = 2n \cdot \tau(X)^2 \to \infty, \quad \text{as} \quad n \to \infty.$$  \hfill (2)

Now, the desired result is an immediate consequence of (2) and [3, Theorem 1.3] from the work of Epstein-Monod. 

□
Remark 7. The preceding result can also be proved along the same lines under the assumption that $\Gamma$ is finitely generated, does not have fixed price one, and that there is no bound on the order of finite subgroups of $\Gamma$ or that $\Gamma$ contains an infinite amenable subgroup. This covers various finitely generated simple torsion groups with positive first $\ell^2$-Betti number, see [11] for examples.

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