Probability Theoretic Generalizations of Hardy’s and Copson’s Inequality

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Dedicated to centenarian Jaap Korevaar

Abstract

A short proof of the classic Hardy inequality is presented for $p$-norms with $p > 1$. Along the lines of this proof a sharpened version is proved of a recent generalization of Hardy’s inequality in the terminology of probability theory. A probability theoretic version of Copson’s inequality is discussed as well. Also for $0 < p < 1$ probability theoretic generalizations of the Hardy and the Copson inequality are proved.

Keywords: $p$-norm, stretched distribution function, rearrangement lemma

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1. The classic Hardy inequality for $p > 1$

As described in detail in [7] the classic Hardy inequality was developed in the years 1906–1928. Its integral version may be formulated as follows. If $p > 1$ and $\psi$ is a nonnegative measurable function on $(0, \infty)$, then

\[
\left\{ \int_0^\infty \left( \frac{1}{x} \int_0^x \psi(y) \, dy \right)^p \, dx \right\}^{1/p} \leq \frac{p}{p - 1} \left\{ \int_0^\infty \psi^p(y) \, dy \right\}^{1/p}
\]  

(1)
holds. By Lebesgue’s differentiation theorem, Tonelli’s theorem and Hölder’s inequality we have

\[
\int_0^\infty \left(\frac{1}{x} \int_0^x \psi \right)^p \, dx = \int_0^\infty x^{-p} \left(\int_0^y \psi \right)^{p-1} \psi(y) \, dy \, dx \tag{2}
\]

\[
= p \int_0^\infty \int_y^\infty x^{-p} \, dx \left(\int_0^y \psi \right)^{p-1} \psi(y) \, dy \geq \int_0^{\infty} \left(\frac{1}{y} \int_0^y \psi \right)^{p-1} \psi(y) \, dy 
\leq \frac{p}{p-1} \left[ \int_0^{\infty} \left(\frac{1}{y} \int_0^y \psi \right)^p \, dy \right]^{(p-1)/p} \left[ \int_0^{\infty} \psi^p \right]^{1/p},
\]

which implies (1). This proof is a smoothed version of the proof of [4], whose application of partial integration in stead of Tonelli’s theorem introduced some technical complications; see Section 8 of [7] and also observe that partial integration can always be done by applying Tonelli’s or Fubini’s theorem.

With \( \psi(y) = \sum_{k=1}^{\infty} c_k 1_{[k-1,k)}(y) \) and \( c_1 \geq c_2 \geq \ldots \geq 0 \) we have

\[
\frac{1}{x} \int_0^x \psi(y) \, dy = \sum_{k=1}^{n-1} c_k + \frac{(x - n + 1) c_n}{x} \geq \frac{1}{n} \sum_{k=1}^{n} c_k, \quad x \in [n-1, n],
\]

and by substituting this into (1) (cf. [4] and Section 8 of [7]) we obtain the sequence version of Hardy’s inequality, which states that for \( p > 1 \) and nonnegative \( c_1, c_2, \ldots \)

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} c_k \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{k=1}^{\infty} c_k^p \tag{3}
\]

holds. Note that rearranging the \( c_k \) in nonincreasing order does not change the value of \( \sum_{k=1}^{\infty} c_k^p \), but it might increase the value of \( 1/n \sum_{k=1}^{n} c_k \) for some values of \( n \) without decreasing this value for any \( n \), as may be seen by interchanging any two \( c_k \) that are not in decreasing order.

2. A probability theoretic Hardy inequality for \( p > 1 \)

Both the integral and sequence version of the Hardy inequality can be derived from a probability theoretic generalization presented in [6], which we sharpen as follows.
Theorem 2.1. Let $X$ and $Y$ be independent random variables with distribution function $F$ on $(\mathbb{R}, \mathcal{B})$, and let $\psi$ be a measurable function on $(\mathbb{R}, \mathcal{B})$, not identically 0. Furthermore, let $p > 1$ and let $\alpha \in [0, 1]$ be the unique root of

$$E (|\psi(Y)|^p) \left[ p - 1 + \alpha \right] - p \left\{ E (|\psi(Y)|^p) \right\}^{1/p} \alpha^{1/p} = 0. \quad (4)$$

Then

$$\left\{ E \left( \left| \frac{E (\psi(Y)1_{[Y \leq X]} \mid X)}{F(X)} \right|^p \right) \right\}^{1/p} \leq \frac{p}{p - 1 + \alpha} \left\{ E (|\psi(Y)|^p) \right\}^{1/p} \quad (5)$$

holds.

In (5) $E(\cdot \mid X)$ denotes conditional expectation given $X$. Note that this inequality (5) is trivial if $\psi$ is not $p$-integrable. Furthermore, (5) with $\psi$ non-negative implies (3) for arbitrary $\psi$, since the absolute value of the (conditional) expectation of a random variable is bounded from above by the (conditional) expectation of the absolute value of this random variable.

If $\psi(Y)$ equals a constant a.s., then we have $\alpha = 1$ and (5) is an equality. For $p = 2$ the root from (4) equals $\alpha = (\sqrt{E \psi^2(Y)} - \sqrt{\text{var}|\psi(Y)|})^2 (E|\psi(Y)|)^{-2}$.

Note that taking $F$ uniform on $(0, K)$, denoting the root of (4) by $\alpha_K$, multiplying (4) by $K^{1/p}$ and (5) by $K$, and taking limits as $K \to \infty$ we see that $\alpha_K$ converges to 0 and we arrive at Hardy’s inequality (1). Analogously taking $F$ uniform on $\{1, \ldots, K\}$ we obtain Hardy’s inequality (3).

[6] prove (5) with $\alpha = 0$. Their proof generalizes [1]’s proof of (3), discretizes $F$ and applies a limiting procedure. Theorem 2.1 will be proved along the lines of [2] and to this end we need the following rearrangement lemma; cf. [8].

Lemma 2.2. Let $\chi$ be a nonnegative $p$-integrable function on the unit interval. There exists a nonincreasing $p$-integrable function $\bar{\chi}$ on the unit interval with the same $p$-norm as $\chi$ and with

$$\frac{1}{u} \int_0^u \bar{\chi}(v) \, dv \geq \frac{1}{u} \int_0^u \chi(v) \, dv, \quad 0 \leq u \leq 1. \quad (6)$$

Proof

Let $U$ be uniformly distributed on the unit interval and define the distribution
function $G(x) = P(\chi(U) \leq x), x \in \mathbb{R}$, and the nonincreasing function $\tilde{\chi}(u) = G^{-1}(1 - u) = \inf\{x \mid G(x) \geq 1 - u\}, 0 \leq u \leq 1$. Note that $\tilde{\chi}(U)$ and $\chi(U)$ have the same distribution and hence the same $p$-norm. Furthermore, by definition of $G$

$$\frac{1}{u} \int_0^u \tilde{\chi}(v) \, dv = \frac{1}{u} \int_0^u G^{-1}(1 - v) \, dv$$

is the mean of the $\chi$-values over a subset of $[0, 1]$ of measure $u$ that contains the largest $\chi$-values. This implies (6). □

Here is our proof of Theorem 2.1.

**Proof**

Without loss of generality we assume that $\psi$ is nonnegative and $p$-integrable. The left-continuous inverse distribution function $F^{-1}$ is defined as $F^{-1}(u) = \inf\{x \mid F(x) \geq u\}, 0 \leq u \leq 1$. Since the left hand side of (5) to the power $p$ may be rewritten as

$$\int_0^1 \left[ \int_{[F^{-1}(v) \leq F^{-1}(u)]} \frac{\psi(F^{-1}(v)) \, dv}{F(F^{-1}(u))} \right]^p \, du = \int_0^1 \left[ \int_0^{F(F^{-1}(u))} \frac{\psi(F^{-1}(v)) \, dv}{F(F^{-1}(u))} \right]^p \, du,$$

the lemma proves that we may assume that $\psi$ is nonincreasing.

Let $F$ be a distribution function with discontinuities, i.e., atoms, one of which is located at $a$ with mass $p_a$. We remove this discontinuity by stretching the distribution function $F$ to

$$\tilde{F}(x) = \begin{cases} 
F(x) & x < a \\
F(a-) + x - a & \text{for } a \leq x \leq a + p_a \\
F(x - p_a) & a + p_a \leq x
\end{cases} \quad (8)$$

and adapt the function $\psi$ accordingly as follows

$$\tilde{\psi}(x) = \begin{cases} 
\psi(x) & x < a \\
\psi(a) & \text{for } a \leq x \leq a + p_a \\
\psi(x - p_a) & a + p_a \leq x
\end{cases} \quad (9)$$

Note

$$\int_{\mathbb{R}} \tilde{\psi}^p(y) \, d\tilde{F}(y) = \int_{\mathbb{R}} \psi^p(y) \, dF(y). \quad (10)$$
Since this also holds for \( p = 1 \), the value of \( \alpha \) in (4) does not change when replacing \( \psi \) and \( F \) by \( \tilde{\psi} \) and \( \tilde{F} \), respectively. For \( a \leq x \leq a + p_a \) the monotonicity of \( \psi \) implies
\[
\frac{\int_{(-\infty, x]} \tilde{\psi}(y) \, d\tilde{F}(y)}{\tilde{F}(x)} = \frac{\int_{(-\infty, a]} \psi(y) \, dF(y) + \psi(a)(x - a)}{F(a) + x - a} \\
= \frac{\int_{(-\infty, a]} [\psi(y) - \psi(a)] \, dF(y)}{F(a) + x - a} + \psi(a) \\
\geq \frac{\int_{(-\infty, a]} [\psi(y) - \psi(a)] \, dF(y)}{F(a)} + \psi(a) = \frac{\int_{(-\infty, a]} \psi(y) \, dF(y)}{F(a)}
\]
and for \( a + p_a < x \)
\[
\frac{\int_{(-\infty, x]} \tilde{\psi}(y) \, d\tilde{F}(y)}{\tilde{F}(x)} = \frac{\int_{(-\infty, x]} \psi(y) \, dF(y) + \psi(a)p_a + \int_{[a + p_a, x]} \psi(y - p_a) \, dF(y - p_a)}{F(x - p_a)} \\
= \frac{\int_{(-\infty, x - p_a]} \psi(y) \, dF(y)}{F(x - p_a)}
\]
holds.

Together with the definitions of \( \tilde{F} \) and \( \tilde{\psi} \) these relations yield
\[
\int_R \left[ \frac{\int_{(-\infty, x]} \tilde{\psi}(y) \, d\tilde{F}(y)}{\tilde{F}(x)} \right]^p \, d\tilde{F}(x) \\
\geq \int_{(-\infty, a]} \left[ \frac{\int_{(-\infty, x]} \psi(y) \, dF(y)}{F(x)} \right]^p \, dF(x) + \int_a^{a + p_a} \left[ \frac{\int_{(-\infty, a]} \psi(y) \, dF(y)}{F(a)} \right]^p \, dx \\
+ \int_{(a + p_a, \infty)} \left[ \frac{\int_{(-\infty, x - p_a]} \psi(y) \, dF(y)}{F(x - p_a)} \right]^p \, dF(x - p_a) \quad (11)
\]
Since an arbitrary distribution function \( F \) has at most countably many discontinuities, it might be that the stretch procedure from (8) and (9) has to be repeated countably many times in order to obtain a continuous distribution function \( \tilde{F} \) and an adapted function \( \tilde{\psi} \) such that (10) holds and the left hand side of (11) equals at least its right hand side. Consequently, we may and do assume that \( F \) is continuous.
Next we prove
\[
\left[ \int_{(-\infty,x]} \psi \, dF \right]^p = p \int_{(-\infty,x]} \left[ \int_{(-\infty,y]} \psi \, dF \right]^{p-1} \psi(y) \, dF(y). \tag{12}
\]

Defining the distribution function \( F_x(y) = \int_{(-\infty,y\wedge x]} \psi \, dF / \int_{(-\infty,x]} \psi \, dF \) for \( x \) with \( F(x) > 0 \) we see that this equality is equivalent to
\[
1 = p \int_{-\infty}^{\infty} F_x^{p-1}(y) \, dF_x(y) = pE \left( [F_x(F_x^{-1}(U))]^{p-1} \right), \tag{13}
\]
where the random variable \( U \) is uniformly distributed on the unit interval. Since \( F \) has no point masses, \( F_x \) has none, i.e., \( F_x \) is continuous. Consequently \( F_x(F_x^{-1}(u)) = u, 0 < u < 1 \), holds and hence \( \text{[13] and [12]} \). By \( \text{[12]} \) and Tonelli’s theorem we have
\[
\int_{-\infty}^{\infty} \left( \int_{(-\infty,x]} \psi(y) \, dF(y) \right) \frac{dF(x)}{F(x)} = p \int_{-\infty}^{\infty} (F(x))^{-p} \int_{(-\infty,x]} \left[ \int_{(-\infty,y]} \psi \, dF \right]^{p-1} \psi(y) \, dF(y) \, dF(x) \tag{14}
\]
\[
= p \int_{-\infty}^{\infty} \int_{[y,\infty)} (F(x))^{-p} \, dF(x) \left[ \int_{(-\infty,y]} \psi \, dF \right]^{p-1} \psi(y) \, dF(y).
\]
Since \( F \) is continuous,
\[
\int_{[y,\infty)} (F(x))^{-p} \, dF(x) = \int_{[y \leq F^{-1}(u)]} (F(F^{-1}(u)))^{-p} \, du
\]
\[
= \int_{F(y)}^{1} u^{-p} \, du = \frac{1}{p-1} \left[ (F(y))^{1-p} - 1 \right]
\]
holds. Together with \([14]\), Hölder’s inequality and \([12]\) with \(x = \infty\) we obtain
\[
\int_{-\infty}^{\infty} \left( \frac{\int_{-\infty}^{x} \psi(y) \, dF(y)}{F(x)} \right)^p \, dF(x)
\]
\[
= \frac{p}{p-1} \int_{-\infty}^{\infty} \left[ \frac{1}{F(y)} \int_{-\infty}^{y} \psi \, dF \right]^{p-1} \psi(y) \, dF(y)
\]
\[
- \frac{p}{p-1} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{y} \psi \, dF \right]^{p-1} \psi(y) \, dF(y)
\]
\[
\leq \frac{p}{p-1} \left[ \int_{-\infty}^{\infty} \left( \frac{\int_{-\infty}^{y} \psi \, dF}{F(y)} \right)^p \, dF(y) \right]^{(p-1)/p}
\]
\[
- \frac{1}{p-1} \left[ \int_{-\infty}^{\infty} \psi(y) \, dF(y) \right]^p.
\]
Introducing shorthand notation we write this inequality as
\[
z^p \leq \frac{p}{p-1} z^{p-1} \nu - \frac{1}{p-1} \mu^p.
\]
This is equivalent to
\[
\chi(z) = p \nu z^{p-1} - (p-1) z^p - \mu^p \geq 0.
\] (15)
The function \(\chi\) is increasing – decreasing on the positive half line with a nonnegative maximum at \(\nu\). Furthermore
\[
\chi \left( \frac{\nu}{p-1+\alpha} \right) = 0
\] (16)
holds in view of \([14]\). Since the left hand side of \([14]\) is convex in \(\alpha\) on \([0, \infty)\) and is nonnegative at \(\alpha = 0\) and nonpositive at \(\alpha = 1\), the root \(\alpha\) from \([14]\) satisfies \(\alpha \in [0, 1]\) and hence \(\nu \leq \nu/(p-1+\alpha)\). Together with \([15]\) and \([16]\) this proves \(z \leq \nu/(p-1+\alpha)\), i.e., \([5]\).

**Remark 2.3.** Since the function
\[
u \mapsto \frac{1}{u} \int_0^u \psi(F^{-1}(\nu)) \, dv, \quad 0 \leq u \leq 1,
\]
is nonincreasing for nonnegative, nonincreasing \(\psi\), we have
\[
\int_0^1 \left[ \frac{\int_0^{F(F^{-1}(u))} \psi(F^{-1}(v)) \, dv}{F(F^{-1}(u))} \right]^p \, du \leq \int_0^1 \left[ \frac{1}{u} \int_0^u \psi(F^{-1}(v)) \, dv \right]^p \, du
\] (17)
in view of \( F(F^{-1}(u)) \geq u \). We define \( \psi(F^{-1}(v)) = 0 \) for \( v > 1 \). Applying \( \ref{7} \), \( \ref{77} \) and \( \ref{1} \) for nonnegative, nonincreasing \( \psi \) we arrive at

\[
E \left( \left[ \frac{E(\psi(Y)1_{Y \leq X})}{F(X)} \right]^p \right) \leq \int_0^1 \left[ \frac{1}{u} \int_0^u \psi(F^{-1}(v)) \, dv \right]^p \, du
\]

\[
\leq \int_0^\infty \left[ \frac{1}{u} \int_0^u \psi(F^{-1}(v)) \, dv \right]^p \, du \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty (\psi(F^{-1}(v)))^p \, dv
\]

\[
= \left( \frac{p}{p-1} \right)^p \int_0^1 (\psi(F^{-1}(v)))^p \, dv = \left( \frac{p}{p-1} \right)^p E(\psi(Y)).
\]

This proves \( \ref{2} \) with \( \alpha = 0 \), which is the probability theoretic version of Hardy’s inequality given in Theorem 2.1 of \( \ref{6} \). Arguably this proof is simpler and more elegant than the one given in \( \ref{6} \).

### 3. A probability theoretic Hardy inequality for \( 0 < p < 1 \)

Theorem 337 on page 251 of \( \ref{5} \) states that for \( 0 < p < 1 \) and \( \psi \) a nonnegative measurable function on \( (0, \infty) \)

\[
\left\{ \int_0^\infty \left( \frac{1}{x} \int_x^\infty \psi(y) \, dy \right)^p \, dx \right\}^{1/p} \geq \frac{p}{1-p} \left\{ \int_0^\infty \psi(y) \, dy \right\}^{1/p}
\]

(18)

holds. A smoothed version of their proof similar to the one in \( \ref{2} \) can be given, but with the inverse Hölder inequality this time;

\[
\int_0^\infty \left( \frac{1}{x} \int_x^\infty \psi \right)^p \, dx = \int_0^\infty x^{-p} \int_x^\infty p \left( \int_y^\infty \psi \right)^{p-1} \psi(y) \, dy \, dx
\]

\[
= p \int_0^\infty \int_0^y x^{-p} \, dx \left( \int_y^\infty \psi \right)^{p-1} \psi(y) \, dy
\]

\[
= \frac{p}{1-p} \int_0^\infty \left( \frac{1}{y} \int_y^\infty \psi \right)^{p-1} \psi(y) \, dy
\]

\[
\geq \frac{p}{1-p} \left[ \int_0^\infty \left( \frac{1}{y} \int_y^\infty \psi \right)^p \, dy \right]^{(p-1)/p} \left[ \int_0^\infty \psi^p \right]^{1/p}.
\]

Our probability theoretic generalization of this inequality reads as follows.

**Theorem 3.1.** Let \( X \) and \( Y \) be independent random variables with distribution function \( F \) on \( (\mathbb{R}, \mathcal{B}) \), and let \( \psi \) be a nonnegative measurable function on \( (\mathbb{R}, \mathcal{B}) \).
For $0 < p < 1$

$$\left\{ E \left( \left( \frac{E(\psi(Y)1_{Y \geq X} | X)}{F(X-)} \right)^p \right) \right\}^{1/p} \geq \frac{p}{1-p} \left\{ E(\psi^p(Y)) \right\}^{1/p}$$

(20)

holds.

**Proof** The proof is analogous to the one for Theorem 2.1 but with essential modifications. Let $F$ be a distribution function with discontinuities, one of which is located at $a$ with jump size $p_a$. We remove this discontinuity by stretching the distribution function $F$ to

$$F(x) = \begin{cases} 
F(x + p_a) & x < a - p_a \\
F(a) + x - a & \text{for } a - p_a \leq x \leq a \\
F(x) & a < x
\end{cases}$$

(21)

and adapt the function $\psi$ accordingly as follows

$$\tilde{\psi}(x) = \begin{cases} 
\psi(x + p_a) & x < a - p_a \\
\psi(a) & \text{for } a - p_a \leq x \leq a \\
\psi(x) & a < x.
\end{cases}$$

(22)

Note

$$\int_{\mathbb{R}} \tilde{\psi}(y) d\tilde{F}(y) = \int_{\mathbb{R}} \psi(y) dF(y).$$

(23)

For $x < a - p_a$ we have

$$\frac{\int_{[x,\infty)} \tilde{\psi}(y) d\tilde{F}(y)}{F(x-)} = \frac{\int_{[x,a-p_a)} \psi(y + p_a) dF(y + p_a) + \psi(a)p_a + \int_{(a,\infty)} \psi(y) dF(y)}{F(x + p_a-)} = \frac{\int_{[x+p_a,\infty)} \psi(y) dF(y)}{F(x + p_a-)},$$

for $a - p_a \leq x \leq a$ the nonnegativity of $\psi$ implies

$$\frac{\int_{[x,\infty)} \tilde{\psi}(y) d\tilde{F}(y)}{F(x-)} = \frac{\psi(a) - x + \int_{(a,\infty)} \psi(y) dF(y)}{F(a) + x - a} \leq \frac{\int_{[a,\infty)} \psi(y) dF(y)}{F(a-)}$$

and for $a < x$ we have

$$\int_{[x,\infty)} \tilde{\psi}(y) d\tilde{F}(y)/\tilde{F}(x-) = \int_{[x,\infty)} \psi(y) dF(y)/F(x-).$$
Together with the definitions of $\bar{F}$ and $\bar{\psi}$ these relations yield

$$
\int_\mathbb{R} \left[ \frac{\int_{[x,\infty)} \bar{\psi}(y) d\bar{F}(y)}{F(x^-)} \right]^p d\bar{F}(x)
\leq \int_{(-\infty,a-p_a)} \left[ \frac{\int_{[x+p_a,\infty)} \psi(y) dF(y)}{F(x + p_a^-)} \right]^p dF(x + p_a)
+ \int_{a-p_a}^a \left[ \frac{\int_{[a,\infty)} \psi(y) dF(y)}{F(a^-)} \right]^p dx
+ \int_{a,\infty)} \left[ \frac{\int_{[x,\infty)} \psi(y) dF(y)}{F(x^-)} \right]^p dF(x)
= \int_\mathbb{R} \left[ \frac{\int_{[x,\infty)} \psi(y) dF(y)}{F(x^-)} \right]^p dF(x).
$$

(24)

Since an arbitrary distribution function $F$ has at most countably many discontinuities, it might be that the stretch procedure from (21) and (22) has to be repeated countably many times in order to obtain a continuous distribution function $\bar{F}$ and an adapted function $\bar{\psi}$ such that (23) holds and the left hand side of (24) equals at most its right hand side. Consequently, we may and do assume that $F$ is continuous.

For $x$ with $\int_{[x,\infty)} \psi dF > 0$ we define

$$
G_x(y) = \int_{[x,y\vee x]} \psi dF / \int_{[x,\infty)} \psi dF, \quad y \in \mathbb{R}.
$$

(25)

Since $F$ is continuous, $G_x$ is and we have

$$
p \int_{\mathbb{R}} (1 - G_x(y))^{p-1} dG_x(y) = p \int_0^1 (1 - G_x(G_x^{-1}(u)))^{p-1} du
= p \int_0^1 (1 - u)^{p-1} du = 1
$$

and hence

$$
\left( \int_{[x,\infty)} \psi dF \right)^p = p \int_{[x,\infty)} \left( \int_{[y,\infty)} \psi dF \right)^{p-1} \psi(y) dF(y). \quad (25)
$$

By (25), Tonelli’s theorem, the continuity of $F$ and the inverse Hölder inequality
we obtain
\[
\int_{-\infty}^{\infty} \left( \frac{\int_{[x,\infty)} \psi(y) dF(y)}{F(x-)} \right)^p dF(x)
= p \int_{-\infty}^{\infty} (F(x))^{-p} \int_x^\infty \int_y^\infty \psi(y) dF(y) dF(x)
= p \int_{-\infty}^{\infty} \int_y^\infty (F(x))^{-p} dF(x) \int_y^\infty \psi(y) dF(y)
= \frac{p}{1-p} \int_{-\infty}^{\infty} \left( \frac{\int_y^\infty \psi(y) dF(y)}{F(y)} \right)^{p-1} \psi(y) dF(y)
\geq \frac{p}{1-p} \left[ \int_{-\infty}^{\infty} \left( \frac{\int_y^\infty \psi(y) dF(y)}{F(y)} \right)^p dF(y) \right]^{(p-1)/p} \left[ \int_{-\infty}^{\infty} \psi^p(y) dF(y) \right]^{1/p},
\]
which implies (20). \qed

Note that taking \( F \) uniform on \((0,K)\), multiplying (20) by \( K \) and taking limits as \( K \to \infty \) we arrive at Hardy’s inequality (18).

A discrete version of Theorem 3.37 from [5] and its proof are given in Theorem 3.38 on page 252 of [5]. This Theorem 3.38 states that for \( 0 < p < 1 \) and \( a_i \geq 0, i = 1, 2, \ldots \),
\[
\left( 1 + \frac{1}{1-p} \right) \left( \sum_{i=1}^{\infty} a_i \right)^p + \sum_{j=2}^{\infty} \left( \frac{1}{j} \sum_{h=j}^{\infty} a_h \right)^p \geq \left( \frac{p}{1-p} \right)^p \sum_{i=1}^{\infty} a_i^p. \tag{26}
\]
This inequality follows from our Theorem 3.1 by the choices
\[
F(x) = \frac{x}{K} 1_{[0<x<1]} + \sum_{i=1}^{K} \frac{1}{K} 1_{[i \leq x]}, \quad \psi(x) = \sum_{i=1}^{\infty} a_i 1_{[x=i+1]}, \quad x \in \mathbb{R}, \tag{27}
\]
where \( K \) is a natural number. Indeed, we have
\[
\lim_{K \to \infty} K E(\psi^p(Y)) = \lim_{K \to \infty} \sum_{i=1}^{K-1} a_i^p = \sum_{i=1}^{\infty} a_i^p. \tag{28}
\]
\[
\lim_{K \to \infty} K \mathbb{E} \left( \left( \frac{E(\psi(Y)1_{Y \geq X} | X)}{F(X-)} \right)^p \right)
= \lim_{K \to \infty} \int_0^1 \left( \sum_{i=1}^{K-1} \frac{a_i}{x} \right)^p dx + \sum_{j=2}^K \left( \frac{1}{j-1} \sum_{h=j-1}^{K-1} a_h \right)^p
= \frac{1}{1-p} \left( \sum_{i=1}^{\infty} a_i \right)^p + \sum_{h=1}^{\infty} \left( \frac{1}{h} \sum_{j=h}^{\infty} a_j \right)^p,
\]

which equals the left hand side of (26).

**Remark 3.2.** For \( p = 1 \) no general inequality exists like in Theorems 2.1 and 3.1. With \( \psi \) nonnegative nondecreasing we have

\[
E \left( \frac{E(\psi(Y)1_{Y \leq X} | X)}{F(X)} \right) \leq E \left( \frac{\psi(X)1_{Y \leq X}}{F(X)} \right) = E(\psi(X))
\]

and with \( \psi \) nonnegative nonincreasing

\[
E \left( \frac{E(\psi(Y)1_{Y < X} | X)}{F(X)} \right) \geq E \left( \frac{\psi(X)1_{Y < X}}{F(X)} \right) = E(\psi(X)).
\]

Similarly we get \( E(\psi(X)(1 - F(X))/F(X-)) \) as both an upper and a lower bound to the left hand side of (26) with \( p = 1 \).

**4. The probability theoretic Copson inequality**

Dual to Hardy’s inequality is Copson’s inequality. It was presented in [2]; see also Section 5 of [6]. Inspired by [2] we present a short proof of the probability theoretic Copson inequality as given in [6].

**Theorem 4.1.** Let \( X \) and \( Y \) be independent random variables with distribution function \( F \) on \((\mathbb{R}, \mathcal{B})\), and let \( \psi \) be a measurable function on \((\mathbb{R}, \mathcal{B})\). With \( p \geq 1 \)

\[
\left\{ E \left( \left| \frac{E(\psi(Y)1_{Y \geq X})}{F(Y)} \right|^{1/p} \right)^p \right\}^{1/p} \leq p \{ E(\psi(Y)) \}^{1/p}
\]

holds.
Proof

For $p = 1$ we have

$$E \left( \left| \frac{\psi(Y)}{F(Y)} 1_{\{Y \geq X\}} \right| | X \right) \leq E \left( \left| \frac{\psi(Y)}{F(Y)} \right| 1_{\{Y \geq X\}} \right) = E(\left| \psi(Y) \right|).$$

Without loss of generality we assume $p > 1$ and that $\psi$ is nonnegative and $p$-integrable. For $x$ with $F(x) > 0$ we have $\int_{[x,\infty)} \psi/F \, dF \leq E(\psi(X))/F(x) < \infty$. Consequently, for such $x$ that satisfy $\int_{[x,\infty)} \psi/F \, dF > 0$ as well

$$G_x(y) = \int_{[x,y]} \frac{\psi(z)}{F(z)} dF(z) \left[ \int_{[x,\infty)} \frac{\psi(z)}{F(z)} dF(z) \right]^{-1}, \quad y \in \mathbb{R},$$

is a well defined distribution function. For any distribution function $G$ and corresponding left-continuous inverse distribution function $G^{-1}$ the inequalities $G(G^{-1}(u)) \geq u$ and $G(G^{-1}(u) -) \leq u$ hold. Consequently with $U$ uniformly distributed we have

$$p E_{G_x} \left( |1 - G_x(Y -)|^{p-1} \right) = p E \left( |1 - G_x(G^{-1}_x(U) -)|^{p-1} \right) \geq p E \left( |1 - U|^{p-1} \right) = 1. \tag{30}$$

Combining this inequality with Tonelli’s theorem and Hölder’s inequality we see that the left hand side of (30) equals and satisfies

$$\int_{-\infty}^{\infty} \left[ \int_{[y,\infty)} \frac{\psi(y)}{F(y)} dF(y) \right]^{p-1} \psi(y) dF(y) \leq \int_{-\infty}^{\infty} \left[ \int_{[y,\infty)} \frac{\psi(y)}{F(y)} dF(y) \right]^{p-1} \psi(y) dF(y) \leq \int_{-\infty}^{\infty} \left[ \int_{[y,\infty)} \frac{\psi(y)}{F(y)} dF(y) \right]^{p-1} \psi(y) dF(y) \leq p \left\{ \int_{-\infty}^{\infty} \left[ \int_{[y,\infty)} \frac{\psi(y)}{F(y)} dF(y) \right] dF(y) \right\}^{(p-1)/p} \left\{ \int_{-\infty}^{\infty} \psi^p(y) dF(y) \right\}^{1/p},$$

which proves (30).

Slight modifications in this proof yield a probability theoretic generalization of Copson’s inequality for $0 < p < 1$; cf. Theorem 2.3 with $c = \kappa = p$ in [3].
Theorem 4.2. Let $X$ and $Y$ be independent random variables with distribution function $F$ on $\mathbb{R}$, and let $\psi$ be a nonnegative measurable function on $(\mathbb{R}, \mathcal{B})$. With $0 < p < 1$

$$\left\{ E \left( \left( E \left( \frac{\psi(Y)}{F(Y)} 1_{Y \geq X} \mid X \right) \right)^p \right) \right\}^{1/p} \geq p \{ E(\psi^p(Y)) \}^{1/p} \quad (31)$$

holds.

Proof
We may and do assume that the left hand side of (31) is finite and hence that $\int_{[x,\infty)} \psi/F dF < \infty$ holds for $F$-almost all $x$. Consequently, for such $x$ with $F(x) > 0$ and $\int_{[x,\infty)} \psi/F dF > 0$ the distribution function $G_x$ from the preceding proof is well defined. Noting $0 < p < 1$ and replacing Hölder’s inequality by its reversed version we see that the preceding proof is valid with all inequality signs reversed. \hfill \square

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