Numerical study of charge and statistics of Laughlin quasi-particles

Heidi Kjønsberg†‡ and Jan Myrheim⋆

†Department of Physics, University of Oslo, P.O. Box 1048 Blindern
N–0316 Oslo, Norway
e-mail: heidi.kjonsberg@fys.uio.no

⋆Department of Physics, The Norwegian University of Science and Technology, NTNU
N–7034 Trondheim, Norway
e-mail: myrheim@phys.ntnu.no

ABSTRACT

We present numerical calculations of the charge and statistics, as extracted from Berry phases, of the Laughlin quasi-particles, near filling fraction 1/3, and for system sizes of up to 200 electrons. For the quasi-holes our results confirm that the charge and statistics parameter are $e/3$ and $1/3$, respectively. For the quasi-electron charge we find a slow convergence towards the expected value of $-e/3$, with a finite size correction for $N$ electrons of approximately $-0.13e/N$. The statistics parameter for the quasi-electrons has no well defined value even for 200 electrons, but might possibly converge to $1/3$. Most noteworthy, it takes on the same sign as for the quasi-holes, due to terms that have previously been ignored. The anyon model works well for the quasi-holes, but requires singular two-anyon wave functions for modelling two Laughlin quasi-electrons.
1 Introduction

It is now widely accepted that the fractional quantum Hall effect arises due to quasi-particle excitations created when the filling fraction of the lowest Landau level moves away from its preferred values. The quasi-particles bind to impurities and thereby ensure that the plateaus in the conductivity are formed. Laughlin examined the filling fractions $1/m$, with $m$ an odd integer, and argued that the quasi-particles have charge $\pm e/m$ ($-e$ is the electron charge). He showed that these values for the charge imply a conductivity plateau at the value $e^2/(mh)$, as observed. He also offered explicit trial wave functions describing the ground state as well as the quasi-particle excitations for the $1/m$ case. Later, Haldane and Halperin examined the hierarchy structure of the Hall states, and suggested that the quasi-particles obey fractional statistics, i.e. that they are anyons.

Arovas, Schrieffer and Wilczek derived the charge and statistics of the Laughlin quasi-holes in Ref. They examined the Berry phase corresponding to one quasi-hole encircling the origin and interpreted this as an Aharonov-Bohm phase. The charge was then found to be $e/m$, in confirmation of Laughlin’s result. They also considered a pair of quasi-holes encircling one another, and related the two-particle contribution of the Berry phase to the quasi-hole charge. Interpreting this two-particle contribution as an anyon interchange phase they found the anyon statistics parameter to have the value $1/m$, equal to the fraction of the elementary charge. The Laughlin quasi-electrons were examined along the same lines in Ref. Within the approximations used, the results imply that the charge and statistics parameter have the values $-e/m$ and $-1/m$, respectively.

The statistics satisfied by the quasi-particles in the quantum Hall system was also examined in Ref., where the exclusion statistics parameter was considered. The results did not rely on any specific trial wave function but rather on state counting based on numerical simulations for interacting electrons on a sphere. The value of the one-dimensional exclusion statistics parameter was found to be $1/m$ in the case of quasi-holes, while the value $2 - 1/m$ was found for the quasi-electrons. Both cases were examined near to the magic filling fraction $1/m$ with $m = 3$.

The exclusion statistics parameter is in principle the same parameter as one reads from the Berry phase, although with an opposite sign for the quasi-electrons, because their charge is negative. The relation is briefly discussed in the Appendix below. Thus the predictions for the anyon parameter, based on the numerical results for the exclusion statistics parameter, would be $1/m$ for quasi-holes and $-2 + 1/m$ for quasi-electrons. The values $1/m$ and $-2 + 1/m$ for the anyon parameter define of course the same particle statistics, but we distinguish between them here in the way we define the correspondence between quasi-particles and anyons. Thus, in the case of quasi-electrons, $1/m$ and $-2 + 1/m$ would represent the same species of anyons, but different anyon states, the $1/m$ state singular and the $-2 + 1/m$ state non-singular.

It is interesting to note that the numerical results for the exclusion statistics parameter of realistic quasi-holes and quasi-electrons are easily interpreted in terms of anyons of positive and negative charge, respectively, with non-singular wave functions, and with the
same statistics in the two cases. That quasi-holes and quasi-electrons should have the same
statistics, is also what one would expect if one regards them as antiparticles of each other.

The anyon representation for the quasi-particles, first suggested by Laughlin [11, 12],
was considered in more detail in Ref. [13]. A careful examination of the quasi-electron case
revealed that the Berry connection calculated from the Laughlin wave function actually
contains some terms not earlier considered in the literature. These extra terms arise be-
cause the inverse of the operator used to create a quasi-electron is not simply the complex
conjugate of the inverse quasi-hole operator. There is no obvious argument for neglecting
them, and this motivates a closer examination of whether these wave functions really rep-
resent excitations carrying the charge and statistics parameter characterizing the physical
quasi-electrons in the quantum Hall system. There are no similar problems associated with
the Laughlin quasi-hole wave functions.

The purpose of the work presented here was to compute the Berry phases for the
Laughlin quasi-particles in order to derive their charge and statistics, following the ideas of
Arovas et al. [5]. The calculations were performed both for quasi-holes and quasi-electrons,
with special attention to the latter, at the filling fraction 1/3. For a small number of
electrons (≤ 8) we could integrate exactly, but for larger and more realistic systems we
had to use Monte Carlo integration with importance sampling according to the Metropolis
algorithm [11].

In the case of a single quasi-hole as well as for a pair of quasi-holes, we find that the
Berry phases computed for systems with up to 200 electrons behave as expected. Thus the
charge $e/3$ and the statistics parameter $1/3$ for the quasi-holes are both confirmed.

For a single quasi-electron the extra terms affect the Berry phase so as to imply a charge
that for a system with up to 200 electrons is slightly below $-e/3$, and also the boundary
effects are larger than for the quasi-hole. For a pair of quasi-electrons, the computations
give a definite short distance behaviour of the statistics phase, but at distances larger than
about one magnetic length, which is roughly the size of a quasi-electron, the phase does
not settle down to an asymptotic value, as one would like it to. Moreover, to the extent
that the statistics parameter is a meaningful quantity, it has the same sign as for the quasi-
holes, in clear contradiction to earlier results for the Laughlin quasi-particles. This sign
would correspond to singular two-anyon wave functions, and hence a negative exclusion
statistics parameter.

Within the last year there has been reported direct experimental evidence for the frac-
tional charge of the quasi-particles in the fractional quantum Hall system [14]. According to
a conjecture proposed in Ref. [13], fractional charge implies fractional statistics. Thus our
investigation of the charge and statistics of the Laughlin quasi-particles has some current
interest for comparison with the real quantum Hall system.
2 Charge and Statistics From Berry Phases

Consider a system of \( N \) particles (electrons) of charge \(-e\) \((e > 0)\), moving in the \((x,y)\) plane, with a magnetic field \( \mathbf{B} = B\mathbf{e}_z \) \((B > 0)\), perpendicular to the plane. The position of a particle may be described by the complex coordinate \( z = (x + iy)/(\sqrt{2}l) \), where \( l = 1/\sqrt{eB} \) is the magnetic length. We set \( \hbar, c = 1 \).

Due to the magnetic field there is an Aharonov-Bohm phase associated with the propagation of a charged particle around a closed loop \([7]\). If the path is a circle of radius \( \rho = \sqrt{2}lr \), right handed relative to the direction of the external magnetic field, and if the particle has charge \( q \), this phase is given by

\[
\gamma = \pi \rho^2 Bq = 2\pi q \frac{1}{e} r^2 .
\] (1)

Consider a quasi-particle (quasi-electron or quasi-hole) excitation, of charge \( q \), localized at the position \( z_0 \) and described by the normalized wave function

\[
\Psi_{z_0}(z_1,\ldots,z_N) = \langle z_1,\ldots,z_N | z_0 \rangle .
\] (2)

If this quasi-particle is moved around a closed loop, there arises the so called Berry phase, which is the integral along the path of the Berry connection \([6]\). Let the path be a circle around the origin, parametrized as \( z_0 = re^{i\phi} \) with \( \phi \) running from 0 to \( 2\pi \). The Berry connection is then defined by

\[
\frac{d\beta_1}{d\phi} = i \langle z_0 | \partial_\phi | z_0 \rangle .
\] (3)

The charge \( q \) of the quasi-particle is now determined by setting the Berry phase equal to the Aharonov-Bohm phase corresponding to the same path \([5]\). In this way \( q \) is expressed in terms of parameters of the fundamental particles (electrons) in the system. In a finite system the charge defined in this way will depend on the distance \( r \) from the origin, but it is reasonably well defined as long as the \( r \) dependence is small for circles that are well inside the system.

Suppose the quasi-particle is described by a normalized \( N \)-particle state of the form

\[
|z_0\rangle = \frac{1}{\sqrt{I_1}} \sum_{l=0}^{\infty} z_0^l a_l |l\rangle .
\] (4)

Here \( a_l \) are expansion coefficients, \(|l\rangle\) are orthonormal basis states, and \( I_1 \) is introduced for normalization,

\[
I_1 = \sum_{l=0}^{\infty} r^{2l} |a_l|^2 .
\] (5)

Then the Berry connection depends on \( r \) but not on \( \phi \), so that the Berry phase is \( \beta_1(2\pi) = 2\pi d\beta_1/d\phi \). The expression for the charge becomes especially simple, and is given by the \( r \) dependence of \( I_1 \) as

\[
\frac{q}{e} = \frac{1}{r^2} \frac{d\beta_1}{d\phi} = -\frac{d}{dr^2} \ln I_1 .
\] (6)
Now suppose there are two quasi-particle excitations simultaneously, located symmetrically about the origin at the positions $\pm z_0$. The parametrization $z_0 = re^{i\phi}$ then describes a counterclockwise interchange of the quasi-particles if we let $\phi$ run from 0 to $\pi$. If we assume that the two quasi-particle state is described by a state analogous to (4),

$$|z_0, -z_0\rangle = \frac{1}{\sqrt{I_2}} \sum_{l=0}^{\infty} z_0^l b_l |l'\rangle ,$$

then a Berry connection $d\beta_2/d\phi$ corresponding to this interchange may be defined in the same way as (3),

$$\frac{d\beta_2}{d\phi} = i \langle z_0, -z_0 | \partial_\phi |z_0, -z_0\rangle .$$

We subtract the single-particle contributions due to the magnetic field, and define

$$-\nu = \frac{1}{\pi} (\beta_2(\pi) - 2\beta_1(\pi)) = \frac{d}{d\phi} (\beta_2 - 2\beta_1) = -r^2 \frac{d}{dr^2} (\ln I_2 - 2 \ln I_1) .$$

We will refer to $\nu$ as the “anyon parameter”, since the residual Berry phase $-\nu \pi$ can be identified with minus the statistics phase of the particles.

As a technical point, note that Eq. (9) gives $\nu$ as the small difference of two quantities that are large for large $r$. When computing $\nu$ by the Monte Carlo method, it is important to evaluate both integrals $I_1$ and $I_2$ simultaneously, using the same sample of random points, since the statistical errors will then be correlated and tend to cancel.

The Laughlin quasi-particles of the quantum Hall system are described by states of the same form as in Eq. (4), or by states of this form with $z_0$ replaced by $z_0^*$. In the next section we give the explicit expressions for the wave functions. For now it suffices to know that the expansion states corresponding to $|l\rangle$ in (4) are orthogonal, and the same is true for $|l'\rangle$ in (7), which are different from $|l\rangle$. A non-normalized quasi-electron state is expanded as a polynomial in $z_0$, whereas a quasi-hole state is expanded in $z_0^*$ instead of $z_0$. The complex conjugation shows up as a difference in sign in the relation between the Berry connection and the normalization factors $I_1$ and $I_2$. We use the notation $I_{1qh}, I_{2qh}I_{1qe}, I_{2qe}$ for the normalization factors for one and two quasi-holes and one and two quasi-electrons, respectively, with $N$ electrons in the system. The following expressions for the charges and anyon parameters of these quasi-particles then result,

$$\frac{q_{\text{qh}}}{e} = \frac{d}{dr^2} \ln I_{1qh} ,$$

$$\frac{q_{\text{qe}}}{e} = -\frac{d}{dr^2} \ln I_{1qe} ,$$

$$\nu_{\text{qh}} = -r^2 \frac{d}{dr^2} (\ln I_{2qh} - 2 \ln I_{1qh}) ,$$

$$\nu_{\text{qe}} = r^2 \frac{d}{dr^2} (\ln I_{2qe} - 2 \ln I_{1qe}) .$$
This shows immediately that the charge is positive for the quasi-holes and negative for the quasi-electrons. The sign of the anyon parameter in the two cases is not obvious directly from these formulae. As discussed in the Appendix, the anyon model predicts that \( \nu_{qh} \geq 0 \) and \( \nu_{qe} \leq 0 \), when only non-singular two-anyon wave functions are considered, whereas the ranges \(-1 < \nu_{qh} < 0\) and \(0 < \nu_{qe} < 1\) correspond to singular two-anyon wave functions.

3 Computational methods

One quasi-hole

The non-normalized \( N \)-electron wave function for one quasi-hole at the position \( z_0 \) is

\[
\Psi_{1qh}^{1qh}(z, z^*) = \psi_0 \Delta \prod_{i=1}^{N} (z^*_i - z^*_0) = \psi_0 \Delta \sum_{k=0}^{N} (-z^*_0)^k S_{k,N}(z^*) .
\]

We simplify our notation by writing \( z = (z_1, ..., z_N) \). Here \( 1/m \) is the filling fraction, \( \psi_0 \) is the harmonic oscillator ground state wave function,

\[
\psi_0 = e^{-\frac{1}{2} \sum_{i=1}^{N} |z_i|^2} ,
\]

and \( \Delta \) is the Vandermonde determinant,

\[
\Delta = \prod_{j<k} (z_j - z_k) .
\]

The elementary symmetric polynomials \( S_{k,N} \) are implicitly defined by the above expansion, which brings the quasi-hole state explicitly to the form shown in Eq. (14), except for the substitution \( z_0 \to z^*_0 \). For computations we use the following recursion relation,

\[
S_{k,N}(z) = z_N S_{k,N-1}(z) + S_{k-1,N-1}(z) ,
\]

with \( S_{0,0} = 1 \) and all other \( S_{k,0} = 0 \).

The normalization integral, which determines the charge according to Eq. (10), is given by

\[
I_{1qh}(r^2) = \int d^{2N}z \left| \Psi_{1qh}^{1qh} \right|^2 = \int d^{2N}z \psi_0^2 |\Delta|^{2m} \prod_{k=1}^{N} |z_k - z_0|^2
\]

\[
= \sum_{k=0}^{N} r^{2k} \int d^{2N}z \psi_0^2 |\Delta|^{2m} |S_{k,N}(z)|^2
\]

\[
= \sum_{k=0}^{N} r^{2k} I_{k}^{1qh} .
\]
Since we want the logarithmic derivative with respect to \(r^2\), we need compute only the ratios \(I_{k}^{1qh}/I_{N}^{1qh}\). We are able to do this analytically only for very small \(N\). For larger \(N\) we do Monte Carlo integration by the Metropolis algorithm [11]. This method works well in particular because

\[
I_{N}^{1qh} = \int d^{2N}z \psi_0^2 |\Delta|^{2m},
\]

and the integration measure \(d^{2N}z \psi_0^2 |\Delta|^{2m}\) is common to all the integrals \(I_{k}^{1qh}\). We interpret \(\psi_0^2 |\Delta|^{2m}\) as a Boltzmann factor \(e^{-\beta V}\) of classical statistical mechanics, thinking of \(\beta\) as the “inverse temperature” and \(V\) as the “potential energy”, with

\[
\beta V = \sum_i |z_i|^2 - 2m \sum_{i<j} \ln |z_i - z_j|.
\]

The classical system is a plasma with two-dimensional Coulomb repulsion, in an external harmonic oscillator potential [11]. The factor \(\prod_{k=1}^{N} |z_k - z_0|^2\), which we do not include in the integration measure, represents in this plasma analogy a Coulomb repulsion of the electrons from the hole position \(z_0\).

We compute the ratio \(I_{k}^{1qh}/I_{N}^{1qh}\) as the time average of \(|S_{k,N}(z)|^2\) over a time sequence of \(N\)-electron configurations generated by the following random dynamics. We loop through all electrons, and for the \(i\)-th electron we generate a random “trial jump” \(z_i \rightarrow z_i + \Delta z_i\) in such a way that \(\Delta z_i\) and \(-\Delta z_i\) are equally probable. The trial step is accepted or rejected depending on whether the Boltzmann factor \(e^{-\beta \Delta V}\), where \(\Delta V\) is the change in potential energy, is larger or smaller than a random number generated uniformly between 0 and 1. This procedure gives the desired distribution of configurations, because the dynamics satisfies the principle of detailed balance: the ratio of probabilities for jumping from \(A\) to \(B\) or from \(B\) to \(A\) is the Boltzmann factor \(e^{-\beta(V(B) - V(A))}\).

**One quasi-electron**

Now consider a quasi-electron at the position \(z_0\). Laughlin’s proposed wave function is a polynomial in \(z_0\),

\[
\Psi_{z_0}^{1qe}(z, z^*) = \psi_0 \left( \prod_{i=1}^{N} (\partial_{z_i^*} - z_0) \right) \Delta^{*m} = \psi_0 \sum_{k=0}^{N} (-z_0)^k S_{k,N}(\partial^*) \Delta^{*m},
\]

where \(\partial^* = (\partial_{z_1^*}, ..., \partial_{z_N^*})\). The normalization integral can be written in several different ways. We may expand it as a polynomial in \(r^2\),

\[
I_{z_0}^{1qe}(r^2) = \int d^{2N}z |\Psi_{z_0}^{1qe}|^2
\]

\[
= \sum_{k=0}^{N} r^{2k} \int d^{2N}z \psi_0^2 |S_{k,N}(\partial) \Delta^m|^2
\]

\[
= \sum_{k=0}^{N} r^{2k} I_{k}^{1qe},
\]

7
or rewrite it by partial integration as

\[ I^{\text{qe}}(r^2) = \int d^{2N} z \psi_0^2 |\Delta|^{2n} \prod_{k=1}^{N} (|z_k - z_0|^2 - 1). \tag{28} \]

The last form shows clearly the difference between the quasi-hole and quasi-electron cases. If we could neglect the \(-1\) in each factor \(|z_k - z_0|^2 - 1\), then the integrals would be identical, and the charges would be the same, just with opposite signs. This approximation seems hard to justify, since the average number of electrons within unit distance from an arbitrary point \(z_0\), assuming constant density within the electron droplet, is \(\pi/m\), which is close to one if \(m = 3\). A more likely possibility would be that the normalization integral does indeed change, but only by a factor which is mainly independent of \(r\), and which therefore does not change the Berry phase.

Note that the integrand in Eq. (28) may even be negative, in spite of the fact that the integrand of the original normalization integral is explicitly non-negative. This can happen because the derivation of Eq. (28) involves \(2N\) partial integrations. The cancellation between positive and negative contributions will cause some loss of precision when this form of the integral is used for numerical evaluation.

Before proceeding we want to comment on the origin of the extra \(-1\). Remember that the wave function of one particle in a magnetic field, in the lowest Landau level, has the form \(f(z^*) e^{-\frac{1}{2}|z|^2}\), where \(f\) is an analytic function. The projection onto the lowest Landau level of the operator \(z\) is the operator \(\partial_{z^*}\) acting on the space of functions analytic in \(z^*\) [17]. This is seen by partial integration, which gives that

\[ \int d^2 z e^{-2z^*(g(z^*))^* \partial_{z^*} f(z^*)} = \int d^2 z e^{-2z^*(g(z^*))^* z f(z^*)} \tag{29} \]

for two analytic functions \(f\) and \(g\). Partial integration also gives that

\[ \int d^2 z e^{-2z^*((\partial_{z^*} - z_0)g(z^*))^* (\partial_{z^*} - z_0) f(z^*)} = \int d^2 z e^{-2z^*(g(z^*))^* (|z - z_0|^2 - 1) f(z^*)}. \tag{30} \]

In the projection onto the lowest Landau level there is an operator ordering problem, because of the fundamental fact that \([z^*, \partial_{z^*}] = -1\). Thus, the factor \(|z - z_0|^2\) in the quasi-hole normalization integral corresponds to the ordering \((z - z_0)(z^* - z_0^*)\), whereas the factor \(|z - z_0|^2 - 1\) in the quasi-electron integral corresponds to \((z^* - z_0^*)(z - z_0)\).

We may use Eq. (28) to compute the integrals \(I_k^{\text{qe}}\) defined in Eq. (26), by expanding

\[ \prod_{n=1}^{N} (|z_n - z_0|^2 - 1) = \sum_{k,l=0}^{N} a_{k,l}^N z_0^k z_0^l. \tag{31} \]

The coefficients \(a_{k,l}^N\) satisfy the following recursion relation,

\[ a_{k,l}^n = (|z_n|^2 - 1) a_{k,l}^{n-1} - z_n^a a_{k-1,l}^{n-1} - z_n a_{k,l-1}^{n-1} + a_{k-1,l-1}^{n-1}, \tag{32} \]

with \(a_{0,0}^0 = 1\) and all other \(a_{k,l}^0 = 0\). Only the diagonal coefficients \(a_{k,k}^N\) give nonzero contributions to the integral, but in order to compute them we have to compute some, though not all, of the off-diagonal coefficients \(a_{k,l}^n\). Note that \(a_{k,k}^n\) is real, and more generally, \(a_{l,k}^n = (a_{k,l}^n)^*\).
Two quasi-holes

For two quasi-holes at the positions $z_a$ and $z_b$ the wave function is

$$\Psi_{z_a,z_b}^{2qh} = \psi_0 \Delta^{sm} \prod_{i=1}^{N} (z_i^* - z_a^*)(z_i^* - z_b^*).$$  \hspace{1cm} (33)

The normalization integral also for this state takes the form of a probability distribution for a system with two-dimensional Coulomb interactions,

$$I_{z_a,z_b}^{2qh} = \int d^{2N}z \psi_0^2 |\Delta|^{2m} \prod_{k=1}^{N} |(z_k - z_a)(z_k - z_b)|^2,$$ \hspace{1cm} (34)

which is analogous to Eq. (34). If we let the quasi-holes be located symmetrically around the origin, $z_a = -z_b = z_0$, then the state is on the form of Eq. (7), with $z_0 \rightarrow z_0^*$, and the normalization integral can be expressed as

$$I^{2qh}(r^2) = \sum_{k=0}^{N} r^{4k} I_k^{2qh}.$$ \hspace{1cm} (35)

Here

$$I_k^{2qh} = \int d^{2N}z \psi_0^2 |\Delta|^{2m} |S_{k,N}(z_1^2,\ldots,z_N^2)|^2$$ \hspace{1cm} (36)

are again normalization integrals for certain angular momentum eigenstates. The elementary symmetric polynomials now depend on $z^2$ instead of $z$. Numerical calculation of these integrals is just as straightforward as in the case of a single quasi-hole, since the recursion formula in Eq. (17) is valid with the substitution $z \rightarrow z^2$.

Two quasi-electrons

Two quasi-electrons at $z_a$ and $z_b$ are described by the wave function

$$\Psi_{z_a,z_b}^{2qe} = \psi_0 \left( \prod_{i=1}^{N} (\partial_{z_i^*} - z_a)(\partial_{z_i^*} - z_b) \right) \Delta^{sm},$$ \hspace{1cm} (37)

which yields, by partial integration, the normalization integral

$$I_{z_a,z_b}^{2qe} = \int d^{2N}z \psi_0^2 |\Delta|^{2m} \prod_{k=1}^{N} \left( |z_k - z_a|^2 |z_k - z_b|^2 - |2z_k - z_a - z_b|^2 + 2 \right).$$ \hspace{1cm} (38)

Comparing with the quasi-hole integral in Eq. (34), we see that for two quasi-electrons there are correction terms, like in the case of a single quasi-electron.

We choose to calculate Berry phases for two quasi-electrons located symmetrically around the origin, with $z_a = -z_b = z_0$, which again ensures that our state is of the
form given in Eq. (7). The normalization integral needed to find the statistics parameter according to Eq. (13), takes the form

\[ I^{2qe}(r^2) = \sum_{k=0}^{N} r^{4k} I^{2qe}_k, \]  

where

\[ I^{2qe}_k = \int d^2z \psi_0^2 |S_{k,N}(\partial^2_{z_1}, \ldots, \partial^2_{z_N}) \Delta^m|^2. \]  

However, it is again simpler to compute \( I^{2qe}_k \) by first integrating partially to eliminate the derivatives and then expanding

\[ \prod_{n=1}^{N} (|z_n|^2 - z_0^2)^2 - 4|z_n|^2 + 2 = \sum_{k,l=0}^{N} b^{N}_{k,l} z_0^{2k} z_0^{*2l}. \]  

The coefficients \( b^{N}_{k,l} \) satisfy the recursion relation

\[ b^{n}_{k,l} = (|z_n|^4 - 4|z_n|^2 + 2) b^{n-1}_{k,l} - z_n^2 b^{n-1}_{k-1,l} - z_n^2 b^{n-1}_{k,l-1} + b^{n-1}_{k-1,l-1}, \]  

with \( b^{0}_{0,0} = 1 \) and all other \( b^{0}_{k,l} = 0 \). Again it is only the diagonal coefficients \( b^{N}_{k,k} \) that give nonzero contributions to the integral.

4 Results

In this section we will present results from the Monte Carlo calculations described above. We have plotted, as functions of the dimensionless radius \( r \) of a circular loop, the charge and statistics parameter for Laughlin quasi-holes and quasi-electrons, extracted from Berry phases according to Eqs. (10), (11), (12) and (13).

The case we have studied is \( m = 3 \), i.e. \( 1/3 \) filling of the lowest Landau level. For the quasi-holes we find Berry phases that imply the charge \( e/3 \) and the statistics parameter \( 1/3 \), in confirmation of the results from Ref. [5].

For the quasi-electrons our results are not nearly as unambiguous. The difference as compared to the quasi-hole case is due to the extra terms in the normalization integrals, as discussed above, and for the first time in Ref. [13]. These terms affect the value of the quasi-electron charge, making it slightly larger than \( e/3 \) in absolute value. Nevertheless the charge is close to the expected value of \( -e/3 \), and it seems plausible that the deviation is a finite size effect, vanishing in the limit of infinitely many electrons.

The statistics parameter of the quasi-electrons is where the difference shows up most clearly. In particular, the statistics parameter is found to have the same sign as for quasi-holes, and not the opposite sign as would be the case if the extra terms were unimportant. Furthermore, as seen in the figures presented below, there is no clear indication in our numerical results that the statistics parameter will converge either to \( 1/3 \) or to any other value as the number of electrons increases.
Consider first the bulk value of the quasi-hole charge $q^{\text{qh}}$ as given by Eq. (10). In Fig. 1 the quantity $q^{\text{qh}} / e$ is shown as a function of $r$, the dimensionless distance from the origin to the quasi-hole. For filling fraction $1/m$ the radius of a circular electron droplet with $N$ electrons is approximately $\sqrt{mN}$. With $m = 3$ and for the values of $N$ considered here this means that the droplet boundaries are at $r \approx 7.7, 12.2, 15.0, 17.3$ and $24.5$. Fig. 1 then clearly illustrates that as long as the quasi-hole is well inside the droplet, its charge is well defined and equal to 1/3 of the absolute value of the electron charge. This result is in accordance with the bulk value proposed in Ref. [1], and later calculated in Ref. [5]. We also note that the boundary effects weaken as the number of electrons grows.

In Fig. 2 we show a more detailed picture of the 100 electron case, where we have collected a statistics of 409 million points in the Monte Carlo integration. The bulk value, which we define as the value for not too small and not too large radius, is seen to take the expected value of 1/3 to at least four decimal digits. We consider the deviations from 1/3 to be not statistically significant, this we checked in some cases by splitting the data in two or more parts and then plotting curves for each partial data set. The deviations from 1/3 at large $r$ are clearly edge effects due to the finite system size.

The deviations at small $r$ we believe to be also finite size effects that can be understood. In fact, the logarithmic derivative of $I_{1\text{qh}}^{\text{qh}}(r^2)$ is determined at $r = 0$ by only two coefficients $I_{0\text{qh}}^{\text{qh}}$ and $I_{1\text{qh}}^{\text{qh}}$, whereas for medium values of $r$ it is a kind of average over many coefficients. Each coefficient may be subject to fluctuations, due to the finite Monte Carlo statistics and perhaps also due to the finite system size. Such fluctuations are more likely to be averaged out in quantities depending on many coefficients.

Fig. 3 shows the charge of the quasi-electron in terms of the absolute value of the electron charge, $q^{\text{qe}} / e$, which is given by Eq. (11) and is expected to have the value $-1/3$. As for the quasi-holes, the charge is well defined in the bulk, and the boundary effects decrease with increasing electron number. However, the bulk value is slightly lower than $-1/3$. This result fits together with the common belief expressed elsewhere in the literature only if the deviation vanishes with increasing electron number, which may well happen, although our present data are not conclusive evidence. Comparing Figs. 1 and 3 we see that the edge effect due to the finite size of the electron droplet is larger in the case of a quasi-electron than it is for the quasi-hole. Both of these features, the deviation from $-1/3$ and the larger edge effects, are due to the extra terms in the normalization integrals.

Fig. 4 shows $q^{\text{qe}} / e$ for small distances, $r \leq 6$, including also curves computed for 100 and 200 electrons. The results shown may be parametrized roughly as

$$
\frac{q^{\text{qe}}}{e} = -\frac{1}{3} - \frac{0.13}{N}.
$$

To the extent that this parametrization is valid, the conclusion is that the deviation from $-1/3$ is a finite size effect.

We focus next on the statistics parameter $\nu^{\text{qh}}$ for the Laughlin quasi-holes, Eq. (12). Fig. 5 shows this quantity as a function of $r$, half the distance between the two holes. When the two quasi-holes are well inside the electron droplet and also reasonably well separated,
the quantity $\nu_{\text{qh}}$ is seen to have a rather well defined constant value of $1/3$. These two characteristics together make it meaningful to state that the quasi-holes have the anyon statistics parameter $1/3$. Another general feature is that when the two quasi-holes are close to the edge of the electron droplet the curves show a dip, followed by a peak which does not change size or shape when the number of electrons changes, in contrast to the behaviour seen in Fig. 1.

Fig. 5 also demonstrates how the curves go to zero when the two quasi-holes overlap, i.e. $r \to 0$. In Fig. 6 this small relative distance behaviour is shown on a larger scale for five different electron numbers, and compared to the Berry phases calculated from two different states in the system of two ideal anyons. The three lowest lying curves are, from bottom and up, for 75 electrons, then coinciding curves for 20, 50 and 100 electrons, and finally for 200 electrons. We conclude that the behaviour at small relative distance is due to the finite size of the quasi-holes themselves and is not affected by the number of electrons in the system.

The two highest curves in Fig. 6 do not refer to the quantum Hall system, but offer an explicit comparison of features of the Laughlin quasi-holes to the case of ideal, point like, anyons, as discussed in the Appendix. In Ref. [13] it was pointed out that the first order correction to the constant, asymptotic value of the two quasi-hole Berry phase favours the projected anyon position eigenstate rather than the coherent state of the $\text{su}(1,1)$ algebra as the localized anyon state that best fits to the Laughlin quasi-hole state. This conclusion was based on the plasma analogy, saying essentially that the corrections to perfect screening should fall off exponentially rather than algebraically. It is confirmed by Fig. 6, where the curve representing the coherent state lies highest, and the curve representing the projected anyon state indeed lies closer to the quasi-hole curves.

The fact that the curves in Fig. 6 go to zero for $r \to 0$, can be understood in the anyon model as due to the impossibility of constructing states with the point like anyons completely localized, using only states belonging to the lowest Landau level. The parameter $z$, or $z^*$, defining the anyon states we consider, is different from the relative position of the anyons, and moving $z$ to $-z$ around a circle of radius $r$ is not the same as interchanging the anyon positions. Hence, in the limit $r \to 0$, no anyon statistics phase is picked up.

In Fig. 7 we show the bulk behaviour of $\nu_{\text{qh}}$, as calculated for 100 electrons. Again we take this as a confirmation, to at least three decimal digits, of the known result of $1/3$ for the quasi-hole statistics parameter.

Our final topic is what we have up to now referred to as the statistics parameter of the quasi-electrons, i.e. $\nu_{\text{qe}}$ in Eq. (13). Fig. 8 shows this quantity for the cases of 20, 50, 75, 100 and 200 electrons. The 100 and 200 electron curves had to be cut at $r = 8$ and $r = 6$, respectively, because of numerical problems. This figure faces us with the question of whether the Laughlin quasi-electrons can be said to have a well defined statistics parameter at all. In our opinion, the term “statistics parameter” is justified only if the quantity denoted is independent of the distance between the two quasi-particles as long as they neither overlap nor are close to the edge of the electron droplet. We have seen that this criterion is well satisfied by the quantity $\nu_{\text{qh}}$, Eq. (12), but Fig. 8 reveals that it
definitely does not hold for \( \nu_{qe} \), Eq. (13), for the system sizes we have considered.

We would like to stress that if we had neglected the extra terms in the integrals in Eqs. (28) and (38) as compared to Eqs. (19) and (34) then this strange behaviour would not have appeared, and the statistics parameter for the quasi-electrons would have been equal to minus the statistics parameter of the quasi-holes. Thus, the fact that the computed \( \nu_{qe} \) and \( \nu_{qh} \) have the same sign, is alone proof that the extra terms are essential.

Fig. 9 shows the same curves on an enlarged scale. It also includes two curves that offer an explicit comparison to the case of ideal, point like anyons, for small \( r \) these lie below and to the right of the curves referring to the quantum Hall system. Note that the five quasi-electron curves depend only slightly on the number of electrons for small \( r \), and thus offer a signature of the Laughlin quasi-electrons themselves. Note also that all the curves start out negative at small \( r \) before becoming positive.

The two curves referring to the anyon system are calculated from states that are singular but normalizable. In fact, since the calculated right hand side of Eq. (13) is positive, the only possible way to extract an anyon parameter is to make a correspondence with such singular states in the ideal anyon system. Fig. 9 shows that the coherent state is closest to the quasi-electron curves for small \( r \), whereas the projected position eigenstate gives a better fit for large \( r \).

Fig. 9 indicates that the bulk value of \( \nu_{qe} \) may possibly become well defined (i.e. constant) in the thermodynamic limit. But if so, it is not obvious that the limiting value would be 1/3.

5 Concluding remarks

To summarize, our numerical study has nicely confirmed the results earlier presented in the literature, that the quasi-holes at filling fraction 1/3 have charge \( e/3 \), and have anyon parameter and exclusion statistics parameter both equal to 1/3.

If it were true that the normalization factors \( I_1 \) and \( I_2 \) were the same for the Laughlin quasi-electrons and quasi-holes, then the Berry phases for the quasi-electrons would be the same as for the quasi-holes, only with opposite signs. This would then lead us to the numerical results \( q^{qe} = -e/3 \) and \( \nu^{qe} = -1/3 \), whereas the exclusion parameter would be \( \nu_1^{qe} = +1/3 \).

However, for the quasi-electrons we have seen that for systems with up to 200 electrons the charge is affected by the extra terms discussed above, so as to give a bulk value that is slightly lower than the expected \( -e/3 \). The deviation is quite small and may well vanish for infinitely many electrons, but we do find it interesting to note that the quasi-electrons, which have a size of the order one magnetic length [11], seem to be affected by the edge even when it is as much as ten magnetic lengths away.

For two quasi-electrons encircling one another our results differ dramatically from those earlier proposed in the literature. The extra terms affect the Berry phase such that it is unclear whether one can extract any statistics parameter at all, even in the thermodynamic
limit. What we can conclude is that the the statistics parameter, if at all meaningful, has a sign which is opposite to what it would have if the extra terms were unimportant. It may possibly converge to \(1/3\) in the thermodynamic limit, even though we find no strong indication that it does. If the Laughlin quasi-electrons have an anyon parameter \(\nu_{qe} = 1/3\), it means that the anyon wave functions are singular, and the exclusion statistics parameter is \(\nu_{qe} = -\nu_{qe} = -1/3\). This result fits in with the naive argument saying that if the quasi-holes and quasi-electrons are anti-particles, they should have the same anyon parameter. However, it is in not in agreement with the value \(2 - 1/3\) found from state counting, which corresponds to non-singular anyon wave functions.

**Acknowledgements**

We would like to thank J.M. Leinaas, K. Olaussen, G.S. Canright and T.H. Hansson for helpful discussions and useful suggestions.

**A Berry phases of the two-anyon states**

In order to model the Laughlin quasi-particles as anyons of statistics parameter \(\nu\) in the lowest Landau level, it is necessary to establish a correspondence with specific states in the anyon system. There are two different two-anyon states that are natural to consider as states that maximally localize each of the particles [13, 18]. They are the coherent state of the su(1,1) algebra and the anyon coordinate eigenstate projected onto the lowest Landau level. These states are identical in the fermion and boson cases, but not in general.

Assume that the magnetic field is \(B\) and the anyon charge is \(e/m\), with \(e > 0\) and \(B > 0\). The relative angular momentum of the two anyons, in the lowest Landau level, have eigenvalues \(2k + \nu\) with \(k\) integer. Let \(|k, \nu\rangle\) be the corresponding orthonormal eigenstates, and let \(|u\rangle\) be the position eigenstates, with \(u\) the complex relative coordinate. Asymptotically as \(|u| \to 0\) we have that

\[
\langle u | k, \nu \rangle \propto u^{2k+\nu}.
\]

(44)

Usually one requires that \(2k + \nu \geq 0\), so that the angular momentum eigenstates are non-singular. But if \(-1 < 2k + \nu < 0\) the corresponding state would be normalizable, although singular, and one may include it.

The coherent state can be written

\[
|z, \nu\rangle = N_z \sum_{k=0}^{\infty} \frac{z^{*2k}}{m^k \sqrt{k! \Gamma(k + \nu + 1/2)}} |k, \nu\rangle,
\]

(45)

whereas the projected position eigenstate is

\[
|z, \nu\rangle = N'_{z} \sum_{k=0}^{\infty} \frac{2^k z^{*2k}}{m^k \sqrt{\pi \Gamma(2k + \nu + 1)}} |k, \nu\rangle.
\]

(46)
Here $N_z$ and $N'_z$ are normalization factors, and $z$ is a parameter labelling the states, such that $2z$ is to be interpreted as the relative position of the two anyons, measured in units of $\sqrt{2}$ times the magnetic length $1/\sqrt{eB}$. This scaling of distances corresponds to the convention used for the quantum Hall system in this paper. The only restriction on the statistics parameter $\nu$ in these formulae is that $\nu \geq 0$, if the states are required to be non-singular, or that $\nu > -1$, if singular but normalizable states are accepted. But note that the sums leave out one or more of the non-singular angular momentum eigenstates when $\nu \geq 2$.

For the statistics parameter as measured by the Berry phase we find the expression
\[
C_{\nu}(r) = \frac{2}{m}r^2 - r^2 \frac{d}{dr^2} \ln \left( \frac{I_{\nu-\frac{1}{2}}(\frac{2}{m}r^2)}{r^{2\nu-1}} \right)
\]  
(47)

for the coherent state, with $I_{\nu-1/2}$ a modified Bessel function, and
\[
A_{\nu}(r) = \frac{2}{m}r^2 - r^2 \frac{d}{dr^2} \ln \left( \sum_{l=0}^{\infty} \frac{(\frac{2}{m}r^2)^{2l}}{\Gamma(2l+\nu+1)} \right)
\]  
(48)

for the projected anyon position eigenstate. In both cases, the one-particle contribution $-2r^2/m$ has been subtracted, and the asymptotic value as $r \to \infty$ is $\nu$.

These functions are plotted in Fig. 6 for $\nu = 1/m = 1/3$, and compared to the statistics parameter found from the Laughlin quasi-hole states.

The exclusion statistics parameter, which is given by the asymptotic behaviour of the two-anyon wave function as $|u| \to 0$, is $\nu_1 = \nu$ for the coherent state as well as for the projected position eigenstate.

In the case of anyons of negative charge $-e/m$, the relative angular momentum eigenvalues are $-2k + \nu$ with $k$ integer, and the relative angular momentum eigenstates have the asymptotic form
\[
\langle u | k, \nu \rangle \propto (u^*)^{2k-\nu}
\]  
(49)
as $|u| \to 0$. Thus, the state $|k, \nu\rangle$ is non-singular if $2k - \nu \geq 0$ and normalizable if $2k - \nu > -1$.

The formulae (47) and (48) still hold with the substitutions $\nu \to -\nu$ and $z^* \to z$, and with the restriction on the statistics parameter $\nu$ that $\nu \leq 0$, if only non-singular states are accepted, or that $\nu < 1$, if singular but normalizable states are accepted. A further consequence of the substitution $z^* \to z$ is a change of signs in the Berry phase formulae (47) and (48). The functions $-C_{-\nu}(r)$ and $-A_{-\nu}(r)$, with $\nu = 1/3$, are plotted in Fig. 9 for comparison to the statistics phase as computed for the Laughlin quasi-electrons.

The exclusion statistics parameter, given by the asymptotic behaviour in the limit $|u| \to 0$, is now seen to be $\nu_1 = -\nu$. 

15
References

[1] R.B. Laughlin, *Phys. Rev. Lett.* **50** (1983) 1395.
[2] F.D.M. Haldane, *Phys. Rev. Lett.* **51** (1983) 605.
[3] B.I. Halperin, *Phys. Rev. Lett.* **52** (1984) 1583.
[4] J.M. Leinaas and J. Myrheim, *Nuovo Cimento* **37** (1977) 1.
[5] D. Arovas, R. Schrieffer and F. Wilczek, *Phys. Rev. Lett.* **53** (1984) 722.
[6] M.B. Berry, *Proc. R. Soc. Lond. A.* **392** (1984) 45.
[7] Y. Aharonov and D. Bohm, *Phys. Rev.* **115** (1959) 485.
[8] D.P. Arovas, in *Geometric Phases in Physics*, eds. A. Shapere and F. Wilczek. World Scientific (1989).
[9] M.D. Johnson and G.S. Canright, *Phys. Rev.* **B49** (1994) 2947.
[10] F.D.M. Haldane, *Phys. Rev. Lett.* **67** (1991) 937.
[11] R.B. Laughlin, in *The Quantum Hall Effect*, eds. S.M. Girvin and R.E. Prange. Springer-Verlag (1987).
[12] R.B. Laughlin, in *Fractional Statistics and Anyon Superconductivity*, ed. F. Wilczek. World Scientific (1990).
[13] H. Kjønsberg and J.M. Leinaas, *Int. J. Mod. Phys.* **A12** (1997) 1975.
[14] L. Saminadayer, D.C. Glattli, Y. Jin and B. Etienne, *Phys. Rev. Lett.* **79** (1997) 2526. R. de-Picciotto, M. Reznikov, M. Heiblum, V. Umansky, G. Bunin and D. Mahalu, *Nature* **389** (1997) 162.
[15] S. Kivelson and M. Rocek, *Phys. Lett.* **156B** (1985) 85.
[16] S. Lang, *Algebra*. Addison-Wesley (1977).
[17] S.M. Girvin and T. Jach, *Phys. Rev.* **B29** (1984) 5617.
[18] T.H. Hansson, J.M. Leinaas and J. Myrheim, *Nucl. Phys.* **B384** (1992) 559.
Figure 1: The quasi-hole charge $q_{th}/e$, Eq. (11), as a function of $r$, the quasi-hole distance from the origin. The curves are, from left to right, for 20, 50, 75, 100 and 200 electrons. The horizontal line is 1/3.
Figure 2: The quasi-hole charge, $q_{\text{qh}}/e$, compared to $1/3$, for 100 electrons.
Figure 3: The quasi-electron charge $q_{\text{qe}}/e$, Eq. (11), as a function of $r$, the quasi-electron distance from the origin. The figure presents results for, from left to right, 20, 50 and 75 electrons. The horizontal line is $-1/3$. 

19
Figure 4: Quasi-electron charge, $q^{qe}/e$. The number of electrons, from the lowest lying curve and up, is 20, 50, 75, 100 and 200. The highest curve is the constant $-1/3$. 
Figure 5: The quasi-hole statistics parameter $\nu^{\text{qh}}$, Eq. (12), as a function of $r$, half the distance between the two quasi-holes. The curves are, from left to right, for 20, 50, 75, 100 and 200 electrons, and the horizontal line is $1/3$. 
Figure 6: Small relative distance behaviour of $\nu_{qh}$, and Berry phases calculated from two different localized states in the system of two anyons. The lowest lying curve is for 75 electrons, then follows a common curve for the three cases of 20, 50 and 100 electrons, and the third curve is for 200 electrons. Somewhat higher lies the Berry phase curve calculated from the anyon position eigenstate projected onto the lowest Landau level, and even higher the one calculated from the coherent state of the su(1,1) algebra. See the Appendix for details.
Figure 7: Quasi-hole statistics parameter $\nu^{\text{qh}}$ for 100 electrons, compared to $1/3$, emphasizing the bulk behaviour.
Figure 8: The quantity $\nu^{qe}$, Eq. (13), to be interpreted as the quasi-electron statistics parameter, versus $r$, half the distance between the quasi-electrons. The lowest lying curve is for 20 electrons. Next are curves for 50 and 75 electrons. The 100 electron curve is cut at $r = 8$ and the 200 electron curve at $r = 6$, to avoid numerical problems. The horizontal line is $1/3$. 
Figure 9: Small relative distance behaviour of $\nu^{re}$ for the cases 20, 50, 75, 100 and 200 electrons. Also shown are two curves corresponding to ideal anyons. The five curves for the different electron numbers coincide for small $r$. The 200 electron curve overshoots the horizontal line $1/3$ at $r \approx 3$. The curve lying lowest for small $r$ represents an anyon eigenstate projected onto the lowest Landau level, whereas the curve going highest for large $r$ represents the coherent state of the su(1,1) algebra.