Directional Pareto Efficiency: Concepts and Optimality Conditions

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Abstract
We introduce and study a notion of directional Pareto minimality with respect to a set that generalizes the classical concept of Pareto efficiency. Then, we give separate necessary and sufficient conditions for the newly introduced efficiency and several situations, concerning the objective mapping and the constraints, are considered. In order to investigate different cases, we adapt some well-known constructions of generalized differentiation; the connections with some recent directional regularities come naturally into play. As a consequence, several techniques from the study of genuine Pareto minima are considered in our specific situation.

Keywords Directional Pareto minimality · Optimality conditions · Directional tangent cones · Directional regularity · Set-valued optimization

Mathematics Subject Classification 54C60 · 46G05 · 90C46

1 Introduction

This paper has two main motivations. On the one hand, we are aiming at continuing the effort made by several authors in the last decade to investigate directional phenomena in mathematical programming, and on the other hand, we show the power of several tools related to directional regularities that have been developed recently. For detailed
accounts on these topics, we refer the reader to the following works and references therein [1–5].

In this work, inspired by some ideas coming in vector optimization problems from location theory, where some directions are privileged with respect to the others, we present a notion of directional minimality for mappings and we illustrate by examples its relevance even for the case of real-valued functions. Then, we observe on the simplest case of real-valued functions of a real variable that the natural necessary optimality conditions are given by the Fermat Theorem at an endpoint of an interval. More generally, we have in mind as well the corresponding one-sided versions of Fermat Theorem for not necessarily differentiable real-valued functions defined on an interval, which admit one-sided derivatives (see, e.g., [6, p. 223]). This gives us the impetus to consider far-reaching generalization of this case, namely problems where the objective is a set-valued map and the constraint is given by means of an inverse image of a cone through another set-valued map. For the study of this general case, we introduce an adapted tangent cone, along with several directional regularity properties of the involved maps, and this approach allows us to derive necessary optimality conditions that, in turn, generalize the prototype of Fermat Theorem at an endpoint of an interval. Furthermore, we present as well optimality conditions in terms of tangent limiting cones and coderivatives. Both on primal and dual spaces, we have under consideration several situations concerning the objective and constraint mappings with their specific techniques of study, among which we mention generalized constraint qualification conditions, Gerstewitz scalarization, openness vs. minimality paradigm, Clarke penalization and the extremal principle. Some results are dedicated to the sufficient optimality conditions under convexity assumptions. Finally, we consider as well the situation of minimality for sets and a brief discussion of this concept reveals the similarities and the differences with respect to the known situation of Pareto efficiency.

The paper is organized as follows. In Sect. 2, we introduce the notation we use and then we present the concepts of directional minimality we study in this work. The definitions of these notions along with some comparisons and examples are also the subjects of the second section. The main section of the paper is the third one, and it deals with optimality conditions for the above-mentioned concepts, being, in turn, divided into two subsections. Firstly, we derive optimality conditions using tangent cones and to this aim we adapt a classical concept of the Bouligand tangent cone and Bouligand derivative of a set-valued map. Using some directional metric regularities, we get several assertions concerning these objects and this allows us to present necessary optimality conditions for a wide range of situations going from problems governed by set-valued mappings having generalized inequalities constraints to fully smooth constrained problems. Secondly, we deal with optimality conditions using normal limiting cones and, again, we consider several types of problems. In this process of getting necessary optimality conditions, we adapt several techniques from classical vector optimization. Moreover, some generalized convex cases are considered in order to obtain sufficient optimality conditions. The fourth section deals with Pareto directional minima for sets. We emphasize the fact that, even if the directional Pareto efficiency appears naturally in the case of mappings, it can be considered as well for sets, and in this respect, we present the corresponding concepts and we discuss it by means of some examples and optimality conditions in terms of the modified tangent
cones. The fifth section deals with some perspectives open by this research. Several conclusions of this work are collected in a short section that ends the paper.

2 Notation and the Concepts Under Study

Throughout this paper, we assume that $X$, $Y$ and $Z$ are normed vector spaces over the real field $\mathbb{R}$ and on a product of normed vector spaces we consider the sum norm, unless otherwise stated. By $B(x, \varepsilon)$, we denote the open ball with center $x$ and radius $\varepsilon > 0$ and by $B_X$ the open unit ball of $X$. In the same manner, $D(x, \varepsilon)$ and $D_X$ denote the corresponding closed balls. The symbol $S_X$ stands for the unit sphere of $X$. By the symbol $X^*$ we denote the topological dual of $X$, while $w^*$ stands for the weak* topology on $X^*$.

Let $F : X \rightrightarrows Y$ be a set-valued map. As usual, the graph of $F$ is

$$\text{Gr } F := \{(x, y) \in X \times Y : y \in F(x)\},$$

and the inverse of $F$ is the set-valued map $F^{-1} : Y \rightrightarrows X$ given by the equivalence $(y, x) \in \text{Gr } F^{-1}$ iff $(x, y) \in \text{Gr } F$. Consider a nonempty subset $A$ of $X$. Then, the image of $A$ through $F$ is

$$F(A) := \{y \in Y : \exists x \in A \text{ s.t. } y \in F(x)\}$$

and the distance function associated with $A$ is $d_A : X \to \mathbb{R}$ given by

$$d_A(x) = d(x, A) := \inf_{a \in A} \|x - a\|.$$

The negative polar of $A$ is

$$A^- := \{x^* \in X^* : x^*(a) \leq 0, \forall a \in A\}.$$

The topological interior, closure and boundary, the convex hull and the conic hull of $A$ are denoted, respectively, by $\text{int } A$, $\text{cl } A$, $\text{bd } A$, $\text{conv } A$ and $\text{cone } A$ (for more details, see [7]).

Let $K \subset Y$ be a proper (that is, $K \neq \{0\}$, $K \neq Y$) and convex cone (we do not suppose that $K$ is pointed, in general). For such a cone, its positive dual cone is

$$K^+ := \{y^* \in Y^* : y^*(y) \geq 0, \forall y \in K\}.$$

For the set-valued mapping $F : X \rightrightarrows Y$, let us consider the following geometrically constrained optimization problem with multifunctions:

$$\min_K F(x), \text{ such that } x \in A,$$

where $A \subset X$ is a nonempty and closed set.
Usually, the minimality is understood in the Pareto sense given by the next definition.

**Definition 2.1** A point \((\overline{x}, \overline{y}) \in \text{Gr} F \cap (A \times Y)\) is a local Pareto minimum point for \(F\) on \(A\), if there exists a neighborhood \(U\) of \(\overline{x}\) such that

\[
(F(U \cap A) - \overline{y}) \cap -K \subset K.
\] (1)

The vectorial notion described by (1) covers as well the situation where \(F := \text{f}\) is a function (in which case, we will speak about a local Pareto minimum point \(\overline{x} \in A\) instead of \((\overline{x}, \overline{y}) \in \text{Gr} F \cap (A \times Y)\), i.e., \(\overline{y} = f(\overline{x})\) will not be mentioned) and the situation of classical local minima in scalar case (in which case we drop the label “Pareto”). If \(K\) is pointed (that is, \(K \cap -K = \{0\}\)), then (1) reduces to

\[
(F(U \cap A) - \overline{y}) \cap -K \subset \{0\}.
\]

**Definition 2.2** If \(\text{int } K \neq \emptyset\), a point \((\overline{x}, \overline{y}) \in \text{Gr} F \cap (A \times Y)\) is a local weak Pareto minimum point for \(F\) on \(A\), if there exists a neighborhood \(U\) of \(\overline{x}\) such that

\[
(F(U \cap A) - \overline{y}) \cap -\text{int } K = \emptyset.
\]

Let \(L \subset S_X\) be a nonempty and closed set. Then, it is not difficult to see that \(\text{cone } L\) is closed as well. Indeed, let us consider a sequence \((u_n)\) in \(\text{cone } L\) converging toward \(u \in X\). We have to show that \(u \in \text{cone } L\). The case \(u = 0\) is clear. Otherwise, there are some sequences \((t_n) \subset ]0, \infty[\) and \((\ell_n) \subset L\) such that \(u_n = t_n \ell_n\) for every \(n\). If \((t_n) \to 0\) (on a subsequence), the boundedness of \((\ell_n)\) leads to \(u = 0\), a situation avoided at this stage. If \((t_n)\) is unbounded, then again the relation \(\|t_n \ell_n\| \to \|u\|\) leads to a contradiction. So, on a subsequence, \((t_n) \to t > 0\) which means, by the closedness of \(L\), that \(\ell_n = t^{-1}_n t_n \ell_n \to t^{-1} u \in L\), therefore \(u \in \text{cone } L\), as claimed.

The main purpose of this paper is to introduce and to study the following concept.

**Definition 2.3** One says that \((\overline{x}, \overline{y}) \in \text{Gr} F \cap (A \times Y)\) is a local directional Pareto minimum point for \(F\) on \(A\) with respect to \((\text{the set of directions}) L\) if there exists a neighborhood \(U\) of \(\overline{x}\) such that

\[
(F(U \cap A \cap (\overline{x} + \text{cone } L)) - \overline{y}) \cap -K \subset K.
\] (2)

If one compares this relation to (1), then one observes that this concept corresponds to the situation where the restriction has the special form (depending on the reference point) \(A \cap (\overline{x} + \text{cone } L)\). In particular, (2) reduces to (1) when \(L := S_X\). Of course, when \(A = X\) in (2), one says that \((\overline{x}, \overline{y}) \in \text{Gr} F\) is a local directional Pareto minimum point for \(F\) with respect to \(L\). Now, the concept of local directional Pareto maximum is obtained in an obvious way.

If \(\text{int } K \neq \emptyset\), one defines as well the weak counterpart of the above notion.
Definition 2.4 One says that \((\bar{x}, \bar{y}) \in \text{Gr } F \cap (A \times Y)\) is a local weak directional Pareto minimum point for \(F\) on \(A\) with respect to (the set of directions) \(L\) if there exists a neighborhood \(U\) of \(\bar{x}\) such that

\[
(F(U \cap A \cap (\bar{x} + \text{cone } L)) - \bar{y}) \cap -\text{int } K = \emptyset.
\]

In all these notions, if one takes \(U = X\), then we get the corresponding global concepts.

Remark 2.1 If \(L_1, L_2 \subset S_X\) are nonempty and closed subsets such that \(L_1\) is a subset of \(L_2\), then a local directional Pareto minimum point for \(F\) with respect to \(L_2\) is a local directional Pareto minimum point for \(F\) with respect to \(L_1\).

Obviously, (1) implies (2), but the converse is not true in general. For the obvious implication, see as well the previous remark for \(L_1 := L\) and \(L_2 := S_X\). To justify the latter affirmation, let us consider the following simple scalar example (when the output space is \(\mathbb{R}\), we always consider \(K := [0, \infty]\)).

Example 2.1 Let \(f : \mathbb{R} \rightarrow \mathbb{R}\) be a strictly increasing function. Then, every \(\bar{x} \in \mathbb{R}\) is local directional minimum for \(f\) with respect to \(L := \{+1\}\), but it is not a local minimum for \(f\).

Moreover, the minimality concept introduced here covers some interesting situations described by the next examples.

Example 2.2 Let \(f : \mathbb{R}^2 \rightarrow \mathbb{R}\) be given by \(f(x, y) := x^2 - y^2\). It is well known that \((0, 0)\) is a critical saddle point, whence it is not a minimum point. However, it is a directional minimum point for \(f\) with respect to \(L = [-1, 1] \times \{0\}\) since for every \((x, y) \in (0, 0) + \text{cone } L = \mathbb{R} \times \{0\}\), one has \(f(x, y) \geq f(0, 0)\). Similarly, \((0, 0)\) is a directional maximum point for \(f\) with respect to \(L = \{0\} \times (-1, 1)\).

Example 2.3 Let \(f : \mathbb{R}^2 \rightarrow \mathbb{R}\) be given by \(f(x, y) = x^2 - y^3\). Again \((0, 0)\) is not a minimum point, but it is a directional minimum point for \(f\) with respect to \(L = \{(\cos \theta, \sin \theta) : \theta \in [\pi, 2\pi]\}\). However, the directions to check are only \([-1, 1] \times \{0\}\) and \(\{0\} \times (-1)\), so the other directions can be dropped since \(f(x, y) = f(x, 0) + f(0, y)\), for all \(x, y \in \mathbb{R}\).

The next example emphasizes that there are points, which are not local directional minima with respect to any nonempty and closed set \(L \subset S_X\) (actually with respect to any singleton \(L \subset S_X\), in view of Remark 2.1). This applies also for critical points of smooth functions.

Example 2.4 Let \(f : \mathbb{R} \rightarrow \mathbb{R}\) be given by

\[
f(x) := \begin{cases} 
\sin \frac{1}{x}, & \text{if } x \neq 0, \\
0, & \text{if } x = 0.
\end{cases}
\]
Then, \( x = 0 \) is not a local directional minimum for \( f \) neither for \( L := \{-1\} \), nor for \( L := \{+1\} \). In the same manner, \( f : \mathbb{R} \to \mathbb{R} \), given by

\[
    f(x) := \begin{cases} 
        x^3 \sin \frac{1}{x}, & \text{if } x \neq 0, \\
        0, & \text{if } x = 0,
    \end{cases}
\]

is differentiable at \( x = 0 \), \( f'(x) = 0 \), but \( x = 0 \) is not a local directional minimum for \( f \).

The next example underlines the idea that for every prescribed set of directions one can define functions that achieve directional minimum with respect to the given set.

**Example 2.5** Let \( 0 < \theta_1 < \theta_2 < \frac{\pi}{2} \) and \( L := \{(\cos \theta, \sin \theta) : \theta_1 \leq \theta \leq \theta_2 \} \). Consider \( f : \mathbb{R}^2 \to \mathbb{R} \) is given by

\[
    f(x, y) := \begin{cases} 
        (\theta_2 - \tan^{-1} \frac{y}{x}) (\tan^{-1} \frac{y}{x} - \theta_1), & \text{if } x \neq 0 \text{ and } (x > 0 \text{ or } y \geq 0), \\
        0, & \text{if } x = 0, \\
        -1, & \text{if } x < 0 \text{ and } y < 0.
    \end{cases}
\]

Then, it is not difficult to see that \((0, 0)\) is directional minimum for \( f \) with respect to \( L \).

Using these basic examples of scalar-valued functions, we are able to easily build examples for vector-valued maps. Here are two such examples.

**Example 2.6** Consider \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is given by \( f(x, y) := (x^2 - y^2, x^2 - y^3) \). Consider \( K := \mathbb{R}^2_+ \). Then, \((0, 0)\) is a directional minimum for \( f \) with respect to \( L := \{(1, 0)\} \).

**Example 2.7** Let \( f : \mathbb{R} \to \mathbb{R}^2 \) be given by \( f(x) := (2x, x) \) and \( K \) be the conic hull of the set \( \text{conv} \{(1, 0), (1, 1)\} \). It is easy to see that \( \overline{x} := 0 \) is a directional minimum for \( f \) with respect to \( L := \{+1\} \), but \( \overline{x} \) is not a local Pareto minimum point for \( f \).

The concepts introduced in this section are studied in the sequel from the point of view of optimality conditions.

### 3 Optimality Conditions for Directional Minima

In order to start with the necessary optimality conditions for directional minima, let us observe that the obvious prototype for such an investigation is the Fermat Theorem for derivable real-valued functions with one variable at interval endpoints: If \( f : [a, b] \to \mathbb{R} \) is a function for which \( a \) is local minimum point (that is, a directional minimum with respect to \( L := \{+1\} \), and \( f \) is derivable at \( a \), then \( f'(a) \geq 0 \), and, similarly, if \( b \) is a minimum point for \( f \) (that is, a directional minimum with respect to \( L := \{-1\} \), and \( f \) is derivable at \( b \), then \( f'(b) \leq 0 \).

We approach this issue from two points of view, namely making use of tangent cones (which are objects of generalized differentiation on primal spaces) and of normal cones (constructions that are defined on dual spaces).
3.1 Optimality Conditions Using Tangent Cones

Let us consider now several concepts that will help us in studying optimality conditions for the directional minima.

**Definition 3.1** Let $A \subset X$ be a nonempty set and $L \subset SX$ be a nonempty and closed set. Then, the Bouligand tangent cone to $A$ at $x \in A$ with respect to $L$ is the set

$$T_L^B(A, x) := \left\{ u \in X : \exists (u_n) \overset{\text{cone } L}{\longrightarrow} u, \exists (t_n) \overset{\text{]0,}\infty[}{\longrightarrow} 0 \text{ s.t. for all } n, x + t_n u_n \in A \right\},$$

where $(u_n) \overset{\text{cone } L}{\longrightarrow} u$ means $(u_n) \longrightarrow u$ and $(u_n) \subset \text{cone } L$, and similarly for $(t_n) \overset{\text{]0,}\infty[}{\longrightarrow} 0$.

Obviously, this is an adaptation of the concept of Bouligand tangent cone to $A$ at $x$ defined as

$$T_B(A, x) := \left\{ u \in X : \exists (u_n) \longrightarrow u, \exists (t_n) \overset{\text{]0,}\infty[}{\longrightarrow} 0 \text{ s.t. for all } n, x + t_n u_n \in A \right\}.$$

Some remarks are in order.

**Remark 3.1** As the usual Bouligand tangent cone, the set $T_L^B(A, x)$ is a closed cone: The proof of this assertion can be made directly as for the classical concept (see [8]) or by observing that $T_L^B(A, x) = T_B(A \cap (x + \text{cone } L), x)$.

In view of the fact that cone $L$ is closed, one has that $T_L^B(A, x) \subset \text{cone } L$. Moreover,

$$T_L^B(A, x) \subset T_B(A, x) \cap T_B(x + \text{cone } L, x) = T_B(A, x) \cap \text{cone } L.$$

However, the inclusion above does not hold as equality, in general. To see this, consider the set $A \subset X := \mathbb{R}^2$ as the plane domain bounded by the curve (the cardioid), which has the parametric representation

$$x = -2 \cos t + \cos 2t + 1, \quad y = 2 \sin t - \sin 2t, \quad t \in [0, 2\pi],$$

the point $x := (0, 0)$, the set $L := \{-1, 0\}$ and observe that $T_B(A, x) = X$ and $T_B(A \cap (x + \text{cone } L), x) = \{x\}$.

Another useful and easy-to-see inclusion is

$$\text{cl} \left( T_B(A, x) \cap \text{int } \text{cone } L \right) \subset T_L^B(A, x).$$
Definition 3.2 Let \( F : X \rightarrow Y \) be a set-valued mapping, \((x, \bar{y}) \in \text{Gr} \ F \) and \( L \subset S_X \), \( M \subset S_Y \) be nonempty and closed sets. The Bouligand derivative of \( F \) at \((x, \bar{y})\) with respect to the sets \( L \) and \( M \) is the set-valued mapping \( D^{L,M}_B F(x, \bar{y}) : X \rightarrow Y \) defined by the equivalence \( v \in D^{L,M}_B F(x, \bar{y})(u) \) iff there are the sequences \((u_n)_{n \in \mathbb{N}} \rightarrow_{\text{cone } L} u, (v_n)_{n \in \mathbb{N}} \rightarrow_{\text{cone } M} v, (t_n)_{n \in \mathbb{N}} \rightarrow 0\) such that for all \( n \),

\[ \bar{y} + t_n v_n \in F(x + t_n u_n). \]

Clearly,

\[ \text{Gr} \ D^{L,M}_B F(x, \bar{y}) \subset \text{cone } L \times \text{cone } M. \]

Again, this is an adaptation of the well-known Bouligand derivative of \( F \) at \((x, \bar{y})\), which is the set-valued map \( D_B F(x, \bar{y}) : X \rightarrow Y \) defined by

\[ \text{Gr} \ D_B F(x, \bar{y}) := T_B (\text{Gr} \ F, (x, \bar{y})). \]

Other derivability objects in primal spaces that can be adapted in directional setting in a similar manner are the Ursescu (adjacent) tangent cone and the Ursescu (adjacent) derivative (see [9]), and the Dini lower derivative of \( F \) at \((x, \bar{y})\), which is the multifunction \( D_D F(x, \bar{y}) \) from \( X \) into \( Y \) given, for every \( u \in X \), by

\[ D_D F(x, \bar{y})(u) = \{ v \in Y : \forall (t_n)_{n \in \mathbb{N}} \rightarrow 0, \forall (u_n)_{n \in \mathbb{N}} \rightarrow u, \exists (v_n)_{n \in \mathbb{N}} \rightarrow v, \forall n \in \mathbb{N}, \]

\[ \bar{y} + t_n v_n \in F(x + t_n u_n) \}. \]

When \( F := f \) is a single-valued map, for simplicity, we write \( D^{L,M}_B f(x) \) for \( D^{L,M}_B f(x, \bar{y}) \), and similarly for \( D_D \).

We present now the first result of this work.

Proposition 3.1 In the above notation, if \( \text{int } K \neq \emptyset \) and \((x, \bar{y}) \in \text{Gr} \ F \) is a local weak directional Pareto minimum point for \( F \) on \( A \) with respect to \( L \), then

\[ D_D F(x, \bar{y})(T_L(A, x)) \cap - \text{int } K = \emptyset. \]

Moreover, if \( A = X \), then

\[ D^{L,S_Y}_B F(x, \bar{y})(X) \cap - \text{int } K = \emptyset. \]

Proof We prove only the second part, since the first part, on the one hand, is similar, and, on the other hand, it follows from the definitions and [9, Theorem 3.1]. Take \( u \in X \). If \( u \notin \text{cone } L \), then \( D^{L,S_Y}_B F(x, \bar{y})(u) = \emptyset \) and there is nothing to prove. If \( u \in \text{cone } L \), suppose, by way of contradiction, that there is \( k \in - \text{int } K \) such that

\[ k \in D^{L,S_Y}_B F(x, \bar{y})(u). \]
According to the definition of Bouligand derivative, namely $D_B^{L,S_Y} F(\bar{x}, \bar{y})$, this means that there exist $(t_n) \to 0$, $(u_n) \to u$, $(k_n) \to k$ such that for all $n$,

$$\bar{y} + t_n k_n \in F(\bar{x} + t_n u_n),$$

that is,

$$t_n k_n \in F(\bar{x} + t_n u_n) - \bar{y}.$$ 

But, for $n$ large enough, $\bar{x} + t_n u_n$ is close enough to $\bar{x}$ and belongs as well to $\bar{x} + \text{cone } L$. Then, for such $n$, taking into account the minimality of $(\bar{x}, \bar{y})$, one gets $t_n k_n \notin - \text{int } K$ which contradicts the fact that $k_n \to k \in - \text{int } K$. 

\hfill \Box

In [10], by means of a special type of minimal time function, several directional regularity properties for set-valued maps are introduced and studied. In order to further investigate the directional minima, we need to briefly point out the main aspects concerning the minimal time function and some related directional metric regularities.

Consider $\emptyset \neq L \subset S_X$ and $\emptyset \neq \Omega \subset X$. Then, the function

$$\tau_L(x, \Omega) := \inf \{ t \geq 0 : \exists u \in L \text{ s.t. } x + tu \in \Omega \}$$

is called the directional minimal time function with respect to $L$.

Remark that, if $L = S_X$, then $\tau_L(\cdot, \Omega) = d(\cdot, \Omega)$. Moreover, we add the convention that $\tau_L(x, \emptyset) = \infty$ for every $x$, and we denote in what follows $\tau_L(x, u)$ by $\tau_L(x, u)$.

Obviously, $\tau_L(x, u) < +\infty$ iff $\tau_L(x, u) = \|u - x\|$ and $u - x \in \text{cone } L$.

Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \text{Gr } F$, $\emptyset \neq L \subset S_X$, $\emptyset \neq M \subset S_Y$.

What we need in the sequel is the following concept of directional calmness. One says that $F$ is directionally calm at $(\bar{x}, \bar{y})$ with respect to $L$ and $M$ if there exist $\alpha > 0$ and some neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that for every $x \in U$,

$$\sup_{y \in F(x) \cap V} \tau_M(y, F(\bar{x})) \leq \alpha \tau_L(\bar{x}, x).$$

We use the convention sup $\tau_L(x, \Omega) := 0$ for every nonempty set $\Omega \subset X$.

As usual (see [11, Section 3H]), for a calmness concept for $F$, it is natural to have a metric subregularity notion such that the former property for $F^{-1}$ to be equivalent to the latter property for $F$. In our setting, this corresponding concept reads as follows: One says that $F$ is directionally metric subregular at $(\bar{x}, \bar{y})$ with respect to $L$ and $M$ if there exist $\alpha > 0$ and some neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that for every $x \in U$,

$$\tau_L(x, F^{-1}(\bar{y})) \leq \alpha \tau_M(\bar{y}, F(x) \cap V).$$

The expected equivalence is described in the following result.
Proposition 3.2 The set-valued map $F$ is directionally metric subregular at $(\bar{x}, \bar{y})$ with respect to $L$ and $M$ iff $F^{-1}$ is directionally calm at $(\bar{y}, \bar{x})$ with respect to $M$ and $L$.

Proof Suppose first that $F$ is directionally metric subregular at $(\bar{x}, \bar{y})$ with respect to $L$ and $M$. Then, there exist $\alpha > 0$, $U$ a neighborhood of $\bar{x}$ and $V$ a neighborhood of $\bar{y}$ such that for every $x \in U$ relation (5) holds. Let $y \in V$. If $\tau_M(\bar{y}, y) = +\infty$, there is nothing to prove. Suppose that $\tau_M(\bar{y}, y) < +\infty$, which means that for any $\epsilon > 0$ there exist $u_{\epsilon} \in M$ and $y_{\epsilon} \in F(x) \cap V$ such that $y_{\epsilon} = \bar{y} + (\tau_M(\bar{y}, F(x) \cap V) + \epsilon) u_{\epsilon}$.

Therefore, $y_{\epsilon} - \bar{y} \in \text{cone } M$, $x \in F^{-1}(y_{\epsilon}) \cap U$ and from the hypothesis,

$$\tau_L(x, F^{-1}(\bar{y})) \leq \alpha \tau_M(\bar{y}, F(x) \cap V) \leq \alpha \tau_M(\bar{y}, y),$$

so,

$$\sup_{x \in F^{-1}(y) \cap U} \tau_L(x, F^{-1}(\bar{y})) \leq \alpha \tau_M(\bar{y}, y),$$

for all $y \in V$, whence the conclusion.

For the converse, suppose that $F^{-1}$ is directionally calm at $(\bar{y}, \bar{x})$ with respect to $M$ and $L$. Therefore, there exist $\alpha > 0$, $U$ a neighborhood of $\bar{x}$ and $V$ a neighborhood of $\bar{y}$ such that for every $y \in V$,

$$\sup_{x \in F^{-1}(y) \cap U} \tau_L(x, F^{-1}(\bar{y})) \leq \alpha \tau_M(\bar{y}, y).$$

Take $x \in U$. Again, if $\tau_M(\bar{y}, F(x) \cap V) = +\infty$, the desired inequality holds. Suppose that $\tau_M(\bar{y}, F(x) \cap V) < +\infty$, which means that for any $\epsilon > 0$ there exist $u_{\epsilon} \in M$ and $y_{\epsilon} \in F(x) \cap V$ such that

$$\bar{y} + (\tau_M(\bar{y}, F(x) \cap V) + \epsilon) u_{\epsilon} = y_{\epsilon}. $$

Therefore, $y_{\epsilon} - \bar{y} \in \text{cone } M$, $x \in F^{-1}(y_{\epsilon}) \cap U$ and from the hypothesis,

$$\tau_L(x, F^{-1}(\bar{y})) \leq \alpha \tau_M(\bar{y}, y_{\epsilon}) = \alpha \|y_{\epsilon} - \bar{y}\| \leq \alpha (\tau_M(\bar{y}, F(x) \cap V) + \epsilon). $$

Passing to the limit as $\epsilon \to 0$, we get the conclusion. □

Now, we use the directional calmness for getting an evaluation of the directional Bouligand tangent cone to a value of a set-valued mapping in terms of the image of 0 through the directional Bouligand derivative of the same application.

Proposition 3.3 Let $F : X \rightrightarrows Y$ be a set-valued mapping, $(\bar{x}, \bar{y}) \in \text{Gr } F$, $L \subset S_X$ and $M \subset S_Y$ be nonempty and closed sets. Then,

$$T_B^M(F(\bar{x}), \bar{y}) \subset D_B^{L,M}F(\bar{x}, \bar{y})(0).$$

Moreover, if $F$ is directionally calm at $(\bar{x}, \bar{y})$ with respect to $L$ and $M$, and cone $M$ is convex, then the equality holds.
Proof Take \( v \in T_B^M (F (\bar{x}), \bar{y}) \). According to the definition, there are the sequences 
\( (v_n) \xrightarrow{\text{cone } M} v, (t_n) \xrightarrow{|0, \infty|} 0 \) such that for all \( n \),

\[
\bar{y} + t_n v_n \in F(\bar{x}) = F(\bar{x} + t_n \cdot 0),
\]

which clearly implies that \( v \in D_B^{L,M} F(\bar{x}, \bar{y})(0) \).

For the opposite inclusion, take \( v \in D_B^{L,M} F(\bar{x}, \bar{y})(0) \) meaning that there are 
\( (u_n) \xrightarrow{\text{cone } L} 0, (v_n) \xrightarrow{\text{cone } M} v, (t_n) \xrightarrow{|0, \infty|} 0 \) such that for all \( n \),

\[
\bar{y} + t_n v_n \in F(\bar{x} + t_n u_n).
\]

But, the assumed calmness of \( F \) and the fact that \( (t_n u_n) \subset \text{cone } L \) mean that for a
dpositive \( \alpha \) and for all \( n \) large enough,

\[
\tau_M(\bar{y} + t_n v_n, F(\bar{x})) \leq \alpha \tau_L(\bar{x}, \bar{x} + t_n u_n) = \alpha t_n \|u_n\|,
\]

that is

\[
\inf\{\gamma \geq 0 : \exists w_n \in M \text{ s.t. for all } n, \ \bar{y} + t_n v_n + \gamma w_n \in F(\bar{x})\} \leq \alpha t_n \|u_n\|.
\]

Therefore, for every \( n \) (large enough), there are \( w_n \in M \) and \( \gamma_n \geq 0 \) such that

\[
\beta_n := \bar{y} + t_n v_n + \gamma_n w_n \in F(\bar{x}) \text{ and } \gamma_n < \alpha t_n \|u_n\| + t_n^2.
\]

So, for every \( n \),

\[
\|\beta_n - (\bar{y} + t_n v_n)\| = \gamma_n < \alpha t_n \|u_n\| + t_n^2,
\]

whence

\[
\left\| \frac{1}{t_n} (\beta_n - \bar{y}) - v_n \right\| < \alpha \|u_n\| + t_n,
\]

which gives

\[
\frac{1}{t_n} (\beta_n - \bar{y}) \rightarrow v.
\]

Taking into account the convexity of cone \( M \), for every \( n \),

\[
\beta_n - \bar{y} = t_n v_n + \gamma_n w_n \in \text{cone } M + \text{cone } M = \text{cone } M.
\]

Summing up,

\[
\frac{1}{t_n} (\beta_n - \bar{y}) \xrightarrow{\text{cone } M} v,
\]

whence \( v \in T_B^M (F (\bar{x}), \bar{y}) \). \( \square \)
Consider now the situation when \( G : X \rightrightarrows Z \) is a set-valued map, \( Q \subset Z \) is a closed, convex and pointed cone and the set of restrictions for \((P)\) is
\[
A := \{ x \in X : 0 \in G(x) + Q \}.
\]
This is a standard situation which encompasses the classical case where one has equalities and inequalities constraints. Recall that the epigraphical set-valued map associated with \( G \) is defined by \( E_G : X \rightrightarrows Z, E_G(x) = G(x) + Q \). Remark that \( A = E_G^{-1}(0) \). The following result holds.

**Proposition 3.4** Let \( L \subset S_X \) and \( N \subset S_Z \) be nonempty and closed sets, take \( \bar{x} \in A \) (meaning that there is \( \bar{z} \in G(\bar{x}) \cap -Q \)). Suppose that \( E_G \) is directionally metric subregular at \((\bar{x}, 0)\) with respect to \( L \) and \( N \). If cone \( L \) is convex, then \( u \in T^L_B(A, \bar{x}) \) iff \( 0 \in D^{L,N}_B E_G(\bar{x}, 0)(u) \). Moreover, if \( Q \cap S_Z \subset N \) and cone \( N \) is convex, then for every \( u \in X \),
\[
D^{L,N}_B G(\bar{x}, \bar{z})(u) + Q \subset D^{L,N}_B E_G(\bar{x}, 0)(u).
\]

**Proof** Applying Propositions 3.2 and 3.3, we have
\[
T^L_B(A, \bar{x}) = D^{N,L}_B E_G^{-1}(0, \bar{x})(0),
\]
whence \( u \in T^L_B(A, \bar{x}) \) iff \( u \in D^{N,L}_B E_G^{-1}(0, \bar{x})(0) \) iff \( 0 \in D^{L,N}_B E_G(\bar{x}, 0)(u) \).

Now, for the second part, take \( w \in D^{L,N}_B G(\bar{x}, \bar{z})(u) + Q \). Then, there exist \( q \in Q \) and \((u_n) \rightharpoonup u, (w_n) \rightharpoonup w - q, (t_n) \rightarrow 0\) such that for all \( n \),
\[
\bar{z} + t_n w_n \in G(\bar{x} + t_n u_n),
\]
so
\[
t_n (w_n + q) \in G(\bar{x} + t_n u_n) - \bar{z} + t_n q \subset E_G(\bar{x} + t_n u_n).
\]
But, \( w_n + q \rightarrow w \) and for every \( n \), \( w_n + q \in \text{cone } N + Q \subset \text{cone } N \), therefore \( w \in D^{L,N}_B G(\bar{x}, 0)(u) \).

**Proposition 3.5** Suppose that \( \text{int } K \neq \emptyset \) and \((\bar{x}, \bar{y}) \in \text{Gr } F \) is a local weak directional Pareto minimum point for \( F \) on \( A \) with respect to a nonempty and closed set \( L \subset S_X \). Consider \( \bar{z} \in G(\bar{x}) \cap -Q \) and a nonempty and closed set \( N \subset S_Z \). Moreover, suppose that \( Q \cap S_Z \subset N \), cone \( L \) and cone \( N \) are convex, and \( E_G \) is directionally metric subregular at \((\bar{x}, 0)\) with respect to \( L \) and \( N \). Then,
\[
\{ (v, w) : \exists u \in X, v \in D_D F(\bar{x}, \bar{y})(u), w \in D^{L,N}_B G(\bar{x}, \bar{z})(u) \} \cap (-\text{int } K \times -Q) = \emptyset.
\]

**Proof** The result follows by using successively Propositions 3.1 and 3.4.
Let us specialize, in two steps, the ideas above to the classical smooth case of optimization problems with single-valued maps. First, suppose that $F := f$ and $G := g$ are continuously Fréchet differentiable functions. Then, taking a point $\bar{x} \in A$, it is easy to see that for all $u \in X$, the Dini lower derivative of $f$ is $D_D f (\bar{x})(u) = \{ \nabla f(\bar{x})(u) \}$, while

$$D_{L, \text{SZ}}^L g(\bar{x})(u) = \begin{cases} \{ \nabla g(\bar{x})(u) \}, & \text{if } u \in \text{cone } L, \\ \emptyset, & \text{if } u \notin \text{cone } L. \end{cases}$$

Then, we get the following Fritz John-type and Karush–Kuhn–Tucker-type result.

**Theorem 3.1** Suppose that $\text{int } K \neq \emptyset$ and $\bar{x} \in A := E_v^{-1}(0)$ is a local weak directional Pareto minimum point for $f$ on $A$ with respect to $L$. Moreover, suppose that $\text{cone } L$ is convex, and $E_v$ is directionally metric subregular at $(\bar{x}, 0)$ with respect to $L$ and $S_Z$. Then, in either of the following conditions:

1. $\text{int } Q \neq \emptyset$ or $\text{int } \{ (\nabla f(\bar{x})(u), \nabla g(\bar{x})(u)) : u \in \text{cone } L \} \neq \emptyset$;
2. $Y$ and $Z$ are finite-dimensional spaces,
   there exist $y^* \in K^+, z^* \in Q^+, (y^*, z^*) \neq 0$ such that for every $u \in \text{cone } L$,
   $$(y^* \circ \nabla f(\bar{x}) + z^* \circ \nabla g(\bar{x}))(u) \geq 0.$$

If, moreover, there exists $u \in \text{cone } L$ such that $\nabla g(\bar{x})(u) \in \text{int } Q \neq \emptyset$ or $\nabla g(\bar{x})(\text{cone } L) = Z$, then $y^* \neq 0$.

**Proof** According to Proposition 3.5 and the subsequent discussion,

$$\{ (\nabla f(\bar{x})(u), \nabla g(\bar{x})(u)) : u \in \text{cone } L \} \cap (\text{int } K \times -Q) = \emptyset.$$ 

Notice that $\{ (\nabla f(\bar{x})(u), \nabla g(\bar{x})(u)) : u \in \text{cone } L \}$ is a convex set and both (i) and (ii) ensure the possibility to apply a separation result for convex sets. Therefore, there exist $y^* \in Y^+, z^* \in Z^+, (y^*, z^*) \neq 0$ such that for every $u \in \text{cone } L$, $k \in \text{int } K$, $q \in Q$, one has

$$(y^* \circ \nabla f(\bar{x}) + z^* \circ \nabla g(\bar{x}))(u) \geq -y^*(k) - z^*(q).$$

Standard arguments yield $y^* \in K^+, z^* \in Q^+$ and

$$(y^* \circ \nabla f(\bar{x}) + z^* \circ \nabla g(\bar{x}))(u) \geq 0,$$

for every $u \in \text{cone } L$.

If one supposes that $y^* = 0$, then using the relation above and any of the assumption (i) or (ii) we get $z^* = 0$, which contradicts $(y^*, z^*) \neq 0$. \qed

A similar but different result could be done taking into account the special structure of this case, using directly Proposition 3.1, and some results one can find in the literature concerning the calculus of Bouligand tangent cone to the counter-image.
of a set through a differentiable mapping. Let us recall some facts from [12]. Let $f : X \to Y$ be a function and $D \subset X$ be a nonempty and closed set. One says that $f$ is metrically subregular at $(\bar{x}, f(\bar{x})) \in D \times Y$ with respect to $D$ if there exist $s > 0$, $\mu > 0$ s.t. for every $u \in B(\bar{x}, s) \cap D$,

$$d(u, f^{-1}(f(\bar{x})) \cap D) \leq \mu \| f(\bar{x}) - f(u) \|.$$  

In fact, the above notion coincides with that of calmness of the set-valued map $y \mapsto f^{-1}(y) \cap D$ at $(f(\bar{x}), \bar{x})$ (see, for instance, [11, Section 3H]). One of the main results in [12] reads as follows.

**Theorem 3.2** Let $X, Y$ be Banach spaces, $D \subset X$, $E \subset Y$ be closed sets, $\varphi : X \to Y$ be a continuously Fréchet differentiable map and $\bar{x} \in D \cap \varphi^{-1}(E)$. Suppose that $\psi : X \times Y \to Y$, $\psi(x, y) := \varphi(x) - y$ is metrically subregular at $(\bar{x}, \varphi(\bar{x}), 0)$ with respect to $D \times E$. Then,

$$T_U(D, \bar{x}) \cap \nabla \varphi(\bar{x})^{-1}(T_B(E, \varphi(\bar{x}))) \subset T_B(D \cap \varphi^{-1}(E), \bar{x}),$$

where $T_U(D, \bar{x})$ denotes the Ursescu tangent cone to $D$ at $\bar{x}$, that is,

$$T_U(D, \bar{x}) := \left\{ u \in X : \forall (t_n) \xrightarrow{\| \cdot \|} 0, \exists (u_n) \to u \text{ s.t. for all } n, \bar{x} + t_n u_n \in D \right\}.$$

Coming back to our case, we have $X := X$, $Y := Z$, $D := \bar{x} + \text{cone } L$, $E := -Q$, $\varphi := g$. With these identifications, we get the next result.

**Theorem 3.3** Suppose that $X, Z$ are Banach spaces, $\text{int } K$ is nonempty and $\bar{x} \in g^{-1}(-Q)$ is a local weak directional Pareto minimum point for $f$ on $g^{-1}(-Q)$ with respect to $L$. Moreover, suppose that $\psi : X \times Z \to Z$ defined by $\psi(x, z) := g(x) - z$ is metrically subregular at $(\bar{x}, g(\bar{x}), 0)$ with respect to $(\bar{x} + \text{cone } L) \times -Q$. Then, for all $u \in \text{cone } L$ with $\nabla g(\bar{x})(u) \in T_B(-Q, g(\bar{x}))$, 

$$\nabla f(\bar{x})(u) \notin -\text{ int } K.$$

**Proof** According to Theorem 3.2,

$$T_U(\bar{x} + \text{cone } L, \bar{x}) \cap \nabla g(\bar{x})^{-1}(T_B(-Q, g(\bar{x}))) \subset T_B((\bar{x} + \text{cone } L) \cap g^{-1}(-Q)), \bar{x}),$$

whence

$$\text{cone } L \cap \nabla g(\bar{x})^{-1}(T_B(-Q, g(\bar{x}))) \subset T_B^L(g^{-1}(-Q), \bar{x}).$$

By Proposition 3.1,

$$\nabla f(\bar{x})(u) \notin -\text{ int } K,$$

for all $u \in \text{cone } L \cap \nabla g(\bar{x})^{-1}(T_B(-Q, g(\bar{x})))$, whence the conclusion.  \[\square\]
Furthermore, we consider the case where \( Y = \mathbb{R}^k \) \((k \geq 1)\), \( Z = \mathbb{R}^p \) \((p \geq 1)\), \( Q = \mathbb{R}_+^m \times \{0\}^n \) with \( m + n = p \), and \( f, g \) are Fréchet differentiable. This means that we are dealing with a vectorial optimization problem with finitely many inequalities and equalities constraints. Let us denote by \( \mu_i \) with \( i \in \overline{1, m} \) the first \( m \) coordinates functions of \( g \) and by \( \nu_j \) with \( j \in \overline{1, n} \) the next \( n \) coordinates functions of \( g \).

For the next step of our approach, we use the Gerstewitz functional in the special case when the ordering cone has nonempty interior. The next result combines [7, Theorem 2.3.1] and [13, Lemma 2.1].

**Theorem 3.4** Let \( K \subset Y \) be a closed and convex cone with nonempty interior. Then, for every \( e \in \text{int} \ K \) the functional \( s_{K,e} : Y \to \mathbb{R} \) given by

\[
s_{K,e}(y) = \inf \{ \lambda \in \mathbb{R} : \lambda e \in y + K \}
\]

is convex and continuous, and for every \( \lambda \in \mathbb{R} \),

\[
\{ y \in Y : s_{K,e}(y) < \lambda \} = \lambda e - \text{int} \ K \text{ and } \{ y \in Y : s_{K,e}(y) = \lambda \} = \lambda e - \text{bd} \ K. \tag{7}
\]

Moreover, \( s_{K,e} \) is sublinear, \( K \)-monotone, and for every \( u \in Y \), the Fenchel (convex) subdifferential \( \partial s_{K,e}(u) \) is nonempty and

\[
\partial s_{K,e}(u) = \{ v^* \in K^+ : v^*(e) = 1, v^*(u) = s_{K,e}(u) \}. \tag{8}
\]

Next, in the above setting we present a Karush–Kuhn–Tucker-type result.

**Theorem 3.5** Suppose that \( X \) is a Banach space, \( \text{int} \ K \neq \emptyset \) and \( \overline{x} \in g^{-1}(-Q) \) is a local weak directional Pareto minimum point for \( f \) on \( g^{-1}(-Q) \) with respect to \( L \). Suppose that:

(i) cone \( L \) is convex;

(ii) \( \psi : X \times Z \to Z \), \( \psi(x, z) := g(x) - z \) is metrically subregular at \((\overline{x}, g(\overline{x})) , 0) \) with respect to \((\overline{x} + \text{cone} \ L) \times -Q; \)

(iii) \( \nabla v \ (\overline{x}) (X) = \mathbb{R}^n \), where \( v := (v_1, v_2, \ldots, v_n) \);

(iv) there exists \( \overline{u} \in \text{int} \ cone \ L \) such that \( \nabla \mu_i \ (\overline{x}) (\overline{u}) < 0 \) for any index \( i \in I(\overline{x}) := \{ i \in \overline{1, m} : \mu_i(\overline{x}) = 0 \} \) and \( \nabla v \ (\overline{x}) (\overline{u}) = 0 \).

Then, there exist \( y^* \in K^+ \setminus \{0\} \), \( \lambda_i \geq 0 \) for \( i \in \overline{1, m} \) and \( \beta_j \in \mathbb{R} \) for \( j \in \overline{1, n} \) such that

\[
0 \in y^* \circ \nabla f (\overline{x}) + \sum_{i=1}^{m} \lambda_i \nabla \mu_i (\overline{x}) + \sum_{j=1}^{n} \beta_j \nabla \nu_j (\overline{x}) + L^- \tag{9}
\]

and

\[
\lambda_i \mu_i (\overline{x}) = 0, \forall i \in \overline{1, m}. \tag{10}
\]

**Proof** Clearly, in this case \( u \in \nabla g (\overline{x})^{-1}(T_B(-Q, g(\overline{x}))) \) amounts to say that \( \nabla \mu_i (\overline{x})(u) \leq 0 \) for any \( i \in I(\overline{x}) \) and \( \nabla \nu_j (\overline{x})(u) = 0 \) for any \( j \in \overline{1, n} \).
Using Theorem 3.3 (all its assumptions hold), we get that
\[ \nabla f(\bar{x})(u) \notin \text{int } K, \]
for all \( u \in \text{cone } L \) with \( \nabla \mu_i(\bar{x})(u) \leq 0 \) for any \( i \in I(\bar{x}) \), and \( \nabla v_j(\bar{x})(u) = 0 \) for any \( j \in \bar{1}, \bar{n} \).

We conclude that \( s_{K,e}(\nabla f(\bar{x})(u)) \geq 0 \) for all \( u \) satisfying the above conditions, and this means that \( u = 0 \) is a minimum point for the scalar problem
\[
\min s_{K,e}(\nabla f(\bar{x})(u))
\]
such that \( u \in \text{cone } L, \ \nabla \mu_i(\bar{x})(u) \leq 0, \ \nabla v_j(\bar{x})(u) = 0, \)
for all \( i \in I(\bar{x}), \ j \in \bar{1}, \bar{n} \).

Since \( \text{cone } L \) is convex, this is a convex problem, whence, from [14, Theorem 2.9.6], there exist \( \lambda_i \geq 0 \) for \( i \in I(\bar{x}) \) and \( \beta_j \in \mathbb{R} \) for \( j \in \bar{1}, \bar{n} \) such that
\[
0 \in \partial \left( s_{K,e} \circ \nabla f(\bar{x}) + \iota_{\text{cone } L} + \sum_{i=1}^{m} \lambda_i \nabla \mu_i(\bar{x}) + \sum_{j=1}^{n} \beta_j \nabla v_j(\bar{x}) \right)(0),
\]
where \( \iota_A \) denotes the indicator function of an arbitrary set \( A \), i.e., \( \iota_A(x) = 0 \), if \( x \in A \) and \( \iota_A(x) = \infty \), otherwise. Finally, using (8) and taking \( \lambda_i := 0 \) for \( i \in \bar{1}, m \setminus I(\bar{x}) \), we get the existence of \( y^* \in K^+ \setminus \{0\} \) such that
\[
0 \in y^* \circ \nabla f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla \mu_i(\bar{x}) + \sum_{j=1}^{n} \beta_j \nabla v_j(\bar{x}) + L^- \]
and
\[
\lambda_i \mu_i(\bar{x}) = 0, \quad \forall i \in \bar{1}, m,
\]
whence the conclusion.

**Remark 3.2** Observe that in the simplest case of a derivable real-valued function \( f : \mathbb{R} \to \mathbb{R} \), if \( \bar{x} \) is a directional minimum with respect to \( L := \{+, 1\} \) (without constraints), then the above theorem reduces to \( -f(\bar{x}) \in L^- \) which is exactly \( f'(\bar{x}) \geq 0 \), as discussed before.

Our aim now is to derive sufficient conditions for a point \( \bar{x} \in g^{-1}(-Q) \) to be a local weak directional Pareto minimum point. In order to formulate such conditions, we use, besides the convexity notion for scalar functions, a generalized convexity concept. Namely, we use the following well-known concept: One says that \( F : X \rightrightarrows Y \) is \( K - \text{convex} \) if for any \( \lambda \in ]0, 1[, \) and any \( x, y \in X \), one has
\[
\lambda F(x) + (1 - \lambda) F(y) \subset F(\lambda x + (1 - \lambda) y) + K.
\]
Proposition 3.6 Suppose that \( X \) is a Banach space, \( \text{int} \, K \neq \emptyset \), \( \text{cone} \, L \) is convex, \( f \) is \( K \)-convex, \( \mu_i, i \in \overline{1,m} \), are convex and \( v_j, j \in \overline{1,n} \), are affine. If there exist \( (\lambda, \beta) \in \mathbb{R}^m_+ \times \mathbb{R}^n \) and \( y^* \in K^+ \backslash \{0\} \) such that (9) and (10) hold, then \( \overline{x} \) is a global weak directional Pareto minimum point for \( f \) on \( g^{-1} (-Q) \) with respect to \( L \).

**Proof** By relation (9), we immediately get that

\[
0 \in \nabla \left( y^* \circ f + \sum_{i=1}^{m} \lambda_i \mu_i + \sum_{j=1}^{n} \beta_j v_j \right) (\overline{x}) + N (\text{cone} \, L, 0)
\]

\[
= \nabla \left( y^* \circ f + \sum_{i=1}^{m} \lambda_i \mu_i + \sum_{j=1}^{n} \beta_j v_j \right) (\overline{x}) + N (\overline{x} + \text{cone} \, L, \overline{x})
\]

Consider the convex optimization problem

\[
\min \left( (y^* \circ f) (x) + \sum_{i=1}^{m} \lambda_i \mu_i (x) + \sum_{j=1}^{n} \beta_j v_j (x) \right), \text{ such that } x \in \overline{x} + \text{cone} \, L.
\]

(11)

We hence obtain, by virtue of [14, Theorem 2.9.1], that \( \overline{x} \) is a global minimum point for the above problem. Note that, for all feasible points \( x \in g^{-1} (-Q) \), we have

\[
\sum_{i=1}^{m} \lambda_i \mu_i (x) + \sum_{j=1}^{n} \beta_j v_j (x) = \sum_{i=1}^{m} \lambda_i \mu_i (x) \leq 0.
\]

Using (10), it follows that, given any \( x \in (\overline{x} + \text{cone} \, L) \cap g^{-1} (-Q) \),

\[
(y^* \circ f) (x) \geq (y^* \circ f) (\overline{x}),
\]

that is

\[
y^* (f (x) - f (\overline{x})) \geq 0.
\]

Now, since \( y^* \in K^+ \backslash \{0\} \), the inequality above gives \( f (x) - f (\overline{x}) \notin \text{int} \, K \), i.e., the conclusion. \( \square \)

### 3.2 Optimality Conditions Using Normal Cones

In order to tackle the question of optimality conditions for directional minima in terms of generalized differentiation objects in dual spaces, we recall some notions and results concerning Fréchet and limiting (Mordukhovich) generalized differentiation (see [15] for details).
Consider $S$ a nonempty subset of a Banach space $X$ and $x \in S$. Then, for every $\varepsilon \geq 0$, the set of $\varepsilon$—normals to $S$ at $x$ is defined by

$$
\hat{N}_\varepsilon(S, x) = \left\{ x^* \in X^* : \limsup_{u \to x} \frac{x^*(u - x)}{\|u - x\|} \leq \varepsilon \right\},
$$

where $u \xrightarrow{S} x$ means that $u \to x$ and $u \in S$. The set $\hat{N}_0(S, x)$ is denoted by $\hat{N}(S, x)$, and it is called the Fréchet normal cone to $S$ at $x$.

Let $\bar{x} \in S$. The Mordukhovich normal cone to $S$ at $\bar{x}$ is given by

$$
N(S, \bar{x}) = \left\{ x^* \in X^* : \exists \varepsilon_n \xrightarrow{[0, \infty]} 0, x_n \xrightarrow{S} \bar{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \hat{N}_{\varepsilon_n}(S, x_n), \forall n \in \mathbb{N} \right\}.
$$

Up to the end of this section, we consider that all the involved spaces are Asplund, unless otherwise stated. In this context, if $S \subset X$ is closed around $\bar{x}$, the formula for the Mordukhovich normal cone takes the following form:

$$
N(S, \bar{x}) = \{ x^* \in X^* : \exists x_n \xrightarrow{S} \bar{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \hat{N}(S, x_n), \forall n \in \mathbb{N} \}.
$$

For the set-valued map $F : X \rightrightarrows Y$, its Fréchet coderivative at $(\bar{x}, \bar{y}) \in \text{Gr} F$ is the set-valued map $\hat{D}^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ given by

$$
\hat{D}^* F(\bar{x}, \bar{y})(y^*) = \{ x^* \in X^* : (x^*, -y^*) \in \hat{N}(\text{Gr} F, (\bar{x}, \bar{y})) \}.
$$

In the same way, the Mordukhovich coderivative of $F$ at $(\bar{x}, \bar{y})$ is the set-valued map $D^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ given by

$$
D^* F(\bar{x}, \bar{y})(y^*) = \{ x^* \in X^* : (x^*, -y^*) \in N(\text{Gr} F, (\bar{x}, \bar{y})) \}.
$$

As usual, when $F = f$ is a function, since $\bar{y} \in F(\bar{x})$ means $\bar{y} = f(\bar{x})$, we write $\hat{D}^* f(\bar{x})$ for $\hat{D}^* F(\bar{x}, \bar{y})$, and similarly for $D^*$.

Notice that for a convex set $S \subset X$, one has that

$$
N(S, \bar{x}) = \{ x^* \in X^* : x^*(x - \bar{x}) \leq 0, \forall x \in S \}
$$

and this cone coincides with the negative polar of $T_B(S, \bar{x})$.

If $S \subset X$ is closed around $\bar{x} \in S$, one says that $S$ is sequentially normally compact ((SNC), for short) at $\bar{x}$ if

$$
[ x_n \xrightarrow{S} \bar{x}, \quad x_n^* \xrightarrow{w^*} 0, \quad x_n^* \in \hat{N}(S, x_n) ] \Rightarrow x_n^* \to 0.
$$
In the case where $S = C$ is a closed and convex cone, the (SNC) property at 0 is equivalent to
\[
(x_n^* \in C^+, \ x_n^* \xrightarrow[w^*]{} 0) \Rightarrow x_n^* \rightarrow 0.
\]

In particular, if $\text{int} C \neq \emptyset$, then $C$ is (SNC) at 0.

Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be finite at $x \in X$ and lower semicontinuous around $x$; the Fréchet subdifferential of $f$ at $x$ is defined by
\[
\hat{\partial} f(x) = \{x^* \in X^* : (x^*, -1) \in \hat{N}(\text{epi} f, (x, f(x)))\},
\]
where $\text{epi} f$ denotes the epigraph of $f$. The Mordukhovich subdifferential of $f$ at $x$ is given by
\[
\partial f(x) = \{x^* \in X^* : (x^*, -1) \in N(\text{epi} f, (x, f(x)))\}.
\]

It is well known that if $f$ is a convex function, then $\hat{\partial} f(x)$ and $\partial f(x)$ coincide with the Fenchel subdifferential. However, in general, $\hat{\partial} f(x) \subset \partial f(x)$, and the following generalized Fermat rule holds: if $x \in X$ with $f(x) < +\infty$ is a local minimum point
\[
\text{for } f : X \rightarrow \mathbb{R} \cup \{+\infty\}, \text{then } 0 \in \hat{\partial} f(x).
\]

Consider now some subsets $C_1, \ldots, C_k$ of $X$, where $k \in \mathbb{N} \setminus \{0, 1\}$. Take $\bar{x} \in C_1 \cap \cdots \cap C_k$, and suppose that all the sets $C_i, i \in \overline{1, k}$, are closed around $\bar{x}$. One says that $C_1, \ldots, C_k$ are allied at point $\bar{x}$ if for every $(x_{in}) \xrightarrow[i \in \overline{1, k}]{} C_i \text{ and } x_{in}^* \in \hat{N}(C_i, x_{in})$, $i \in \overline{1, k}$, the relation $(x_{1n}^* + \cdots + x_{kn}^*) \rightarrow 0$ implies $(x_{in}^*) \rightarrow 0$ for every $i \in \overline{1, k}$.

The concept of alliedness was introduced by Penot and his coauthors in [16,17] in order to get a calculus rule for the Fréchet normal cone to the intersection of sets. More precisely, if the subsets $C_1, \ldots, C_k$ are allied at point $\bar{x}$, then there exists $r > 0$ such that, for every $\varepsilon > 0$ and every $x \in [C_1 \cap \cdots \cap C_k] \cap B(\bar{x}, r)$, there exist $x_i \in C_i \cap B(x, \varepsilon), i \in \overline{1, k}$ such that
\[
\hat{N}(C_1 \cap \cdots \cap C_k, x) \subset \hat{N}(C_1, x_1) + \cdots + \hat{N}(C_k, x_k) + \varepsilon DX^*.
\]

Notice that more recently, several connections between a metric inequality implied by alliedness and another property called subtransversality were investigated by Kruger (see [18] and the references therein).

In what follows, we use the results concerning the theory of generalized differentiation built on these objects directly at the places we need them, without separate quotation.

We discuss next a concept of directional openness at the reference point of a certain multifunction. We recall the classical concept of openness proven to be useful for the announced aim by means of the incompatibility between this property and the Pareto minimality (see, e.g., [19] for details).

In fact, the directional openness we consider here is related to several other notions introduced in [10], and to the concept of directional calmness already used in the previous subsection.

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Consider a multifunction $F : X \rightrightarrows Y$, a point $(\bar{x}, \bar{y}) \in \text{Gr} F$, and some nonempty sets $L \subset S_X$, $M \subset S_Y$. One says $F$ is directionally open at $(\bar{x}, \bar{y})$ with respect to $L$ and $M$ if for any $\varepsilon > 0$, there exists $r > 0$ such that

$$B(\bar{y}, r) \cap (\bar{y} - \text{cone } M) \subset F(B(\bar{x}, \varepsilon) \cap (\bar{x} + \text{cone } L)).$$

When $F$ is single-valued, for simplicity, we sometimes omit $\bar{y}$ in the definition above and we say that $F$ is directionally open at $\bar{x}$, instead of directionally open at $(\bar{x}, f(\bar{x}))$.

**Proposition 3.7** If $(\bar{x}, \bar{y}) \in \text{Gr} F$ is a local directional Pareto minimum point for $F$ with respect to $L$, then for every $C \subset S_Y$ with $C \cap (K \setminus K) \neq \emptyset$, the set-valued map $\mathcal{E}_F : X \rightrightarrows Y$, given by $\mathcal{E}_F(x) := F(x) + K$, is not directionally open at $(\bar{x}, \bar{y})$ with respect to $L$ and $C$. In particular, $F$ is not directionally open at $(\bar{x}, \bar{y})$ with respect to $L$ and $C$.

**Proof** Suppose, by contradiction, that for $\varepsilon > 0$ involved in the definition of the minimality of $(\bar{x}, \bar{y})$, there exists $r > 0$ such that

$$B(\bar{y}, r) \cap (\bar{y} - \text{cone } C) \subset \mathcal{E}_F(B(\bar{x}, \varepsilon) \cap (\bar{x} + \text{cone } L)).$$

By subtracting $\bar{y}$ on both sides, according to the hypothesis, one has that

$$(B(0, r) \cap -C) \cap -K \subset (\mathcal{E}_F(B(\bar{x}, \varepsilon) \cap (\bar{x} + \text{cone } L)) - \bar{y}) \cap -K
\quad \quad = (F(B(\bar{x}, \varepsilon) \cap (\bar{x} + \text{cone } L)) + K - \bar{y}) \cap -K \subset K.
$$

Passing to the conic hull, this yields

$$- \text{cone } C \cap -K \subset K,$n

which contradicts the fact that $C \cap (K \setminus K) \neq \emptyset$. So, $\mathcal{E}_F$ is not directionally open at $(\bar{x}, \bar{y})$ with respect to $L$ and $C$. Since $F(x) \subset \mathcal{E}_F(x)$ for any $x$, the same conclusion holds for $F$ as well. \qed

Before obtaining necessary optimality conditions, we remark that a converse of Proposition 3.7 can be done if one considers a (generalized) convex framework.

**Proposition 3.8** Suppose that $F$ is $K$-convex and for every $u \in K \cap S_Y$, $\mathcal{E}_F$ is not directionally open with respect to $L$ and $M := \{u\}$ at $(\bar{x}, \bar{y}) \in \text{Gr} F$. Then, $(\bar{x}, \bar{y})$ is a local directional Pareto minimum point of $F$ with respect to $L$.

**Proof** Suppose, by contradiction, that $(\bar{x}, \bar{y})$ is not a local directional Pareto minimum point of $F$ with respect to $L$. Then, for every $r > 0$, there is the point $y_r \in F(B(\bar{x}, r) \cap (\bar{x} + \text{cone } L)) \cap (\bar{y} - K)$ such that $y_r \notin \bar{y} + K$. Denote $\tilde{k} := \bar{y} - y_r \in K \setminus K$ and consider $x_r \in B(\bar{x}, r) \cap (\bar{x} + \text{cone } L)$ such that $y_r \in F(x_r)$.

Moreover, since $\mathcal{E}_F$ is not directionally open with respect to $L$ and $\{\tilde{k}\}$ at $(\bar{x}, \bar{y})$, it follows that there is $r > 0$ such that, for every $\varepsilon > 0$ small enough, there is...
$y_ε \in B(\overline{y}, \varepsilon) \cap (\overline{y} - \text{cone } \overline{k}) \subset [\overline{y}, y_r]$ such that $y_ε$ is not an element of the set $\mathcal{E}_F(B(\overline{x}, r) \cap (\overline{x} + \text{cone } L))$ (hence, in particular, $y_ε \neq \overline{y}$ and $y_ε \neq y_r$).

Then, there is $\lambda \in ]0, 1[$ such that

$$y_ε = \lambda \overline{y} + (1 - \lambda)y_r \in \lambda F(\overline{x}) + (1 - \lambda)F(x_r) \subset F(\lambda \overline{x} + (1 - \lambda)x_r) + K
$$

$$= \mathcal{E}_F(\lambda \overline{x} + (1 - \lambda)x_r) = \mathcal{E}_F(\overline{x} + (1 - \lambda)(x_r - \overline{x}))
$$

$$\subset \mathcal{E}_F(B(\overline{x}, r) \cap (\overline{x} + \text{cone } L)),$$

a contradiction. \hfill \Box

Now, we use Proposition 3.7 to get optimality conditions.

**Theorem 3.6** Suppose that $X$ and $Y$ are finite-dimensional spaces, the point $(\overline{x}, \overline{y}) \in \text{Gr } F$ is a local directional Pareto minimum point for $F$ with respect to $L$, cone $L$ is convex, $u \in \text{int } K \cap S_Y$, and the set-valued map $\mathcal{E}_F : X \rightrightarrows Y$ has closed graph and is Lipschitz-like around $(\overline{x}, \overline{y})$. Then, there exist $x^* \in X^*$ and $y^* \in K^+$ with $x^*(\ell) \geq 0$ for all $\ell \in L$ $y^*(u) = 1$ and

$$x^* \in D^* \mathcal{E}_F(\overline{x}, \overline{y})(y^*).$$

**Proof** According to Proposition 3.7, $\mathcal{E}_F$ is not directionally open at $(\overline{x}, \overline{y})$ with respect to $L$ and $\{u\}$. Therefore, this is not directionally open around $(\overline{x}, \overline{y})$ with respect to $L$ and $\{u\}$ (in the sense of [10, Definition 2.2]) and, hence, the sufficient condition for directional openness from [10, Theorem 4.3] does not hold. This means that for all natural numbers $n \neq 0$, there exist $x^*_n \in X^*$, $y^*_n \in Y^*$, $(x_n, y_n) \xrightarrow{\text{Gr } F} (\overline{x}, \overline{y})$ such that $y^*_n(u) = 1$, $n^{-1} > -x^*_n(\ell)$ for all $\ell \in L$ and $x^*_n \in D^* \mathcal{E}_F(x_n, y_n)(y^*_n)$. Now, [19, Lemma 3.2] ensures that $y^*_n \in K^+$ for any $n$. This, together with the condition $u \in \text{int } K$ imply, by using [7, Lemma 2.2.17], that the sequence $(y^*_n)$ is bounded. The assumed Lipschitz property of $\mathcal{E}_F$ ensures, by means of [15, Theorem 1.43], that the sequence $(x^*_n)$ is bounded too. Therefore, we can suppose, without loss of generality, that both these sequences are convergent to some $x^* \in X^*$ and $y^* \in K^+$, respectively. Passing to the limit in the relations satisfied by $(x^*_n)$ and $(y^*_n)$, we get $x^*(\ell) \geq 0$ for all $\ell \in L$, $y^*(u) = 1$ and $x^* \in D^* \mathcal{E}_F(\overline{x}, \overline{y})(y^*)$, that is the conclusion. \hfill \Box

**Remark 3.3** Observe that in the case $L = S_X$ (that is Pareto minimality), the necessary optimality condition given by the previous result is the generalized Fermat rule (see [19, Theorem 3.11]): There exists $y^* \in K^+ \setminus \{0\}$ with $0 \in D^* \mathcal{E}_F(\overline{x}, \overline{y})(y^*)$.

We tackle now the case of constrained problems, and we have the following result.

**Theorem 3.7** Let $A \subset X$ and $L \subset S_X$ be nonempty and closed sets and $F : X \rightrightarrows Y$ be a set-valued map with $(\overline{x}, \overline{y}) \in \text{Gr } F \cap (A \times Y)$ such that $\text{Gr } F$ is closed around $(\overline{x}, \overline{y})$. Suppose that the following assertions hold:

\begin{itemize}
  \item [\square] Springer
(i) \( F \) is Lipschitz-like around \((x, y)\);
(ii) \( K \setminus K \neq \emptyset \) and \( K \) is (SNC) at 0;
(iii) the sets \( A \) and \( x + \text{cone} \ L \) are allied at \( \bar{x} \).

If \((\bar{x}, \bar{y})\) is a local directional Pareto minimum point for \( F \) on \( A \) with respect to the set of directions \( L \), then there exists \( y^* \in K^+ \setminus \{0\} \) such that

\[
0 \in D^* F (\bar{x}, \bar{y}) (y^*) + N (A, \bar{x}) + N (\text{cone} \ L, 0).
\]

**Proof** From the hypothesis, there exists a neighborhood \( U \in V (\bar{x}) \) such that

\[
(F(U \cap A \cap (\bar{x} + \text{cone} \ L)) - \bar{y}) \cap -K \subset K
\]

and there exists \( c \in Y \) such that \( c \in K - K \). Consider the following two sets

\[
A_1 = \text{Gr} \ F
\]

and

\[
A_2 = [(\bar{x} + \text{cone} \ L) \cap A] \times (\bar{y} - K).
\]

We want to prove that the system \( \{A_1, A_2, (\bar{x}, \bar{y})\} \) is an extremal system in \( X \times Y \) (see \cite[Definition 2.1]{15}). For this, since \((\bar{x}, \bar{y}) \in A_1 \cap A_2\), it is sufficient to show the existence of a sequence \((x_n, y_n))_n \subset X \times Y\) with \( (x_n, y_n) \to (0, 0) \) and

\[
A_1 \cap (A_2 - (x_n, y_n)) \cap (U \times Y) = \emptyset,
\]

for all large \( n \in \mathbb{N} \). Consider \((x_n, y_n) = (0, \zeta_n)\) with \( n \in \mathbb{N} \setminus \{0\} \), and suppose, by contradiction, that there exist the points \( x \in (\bar{x} + \text{cone} \ L) \cap A \cap U \) and \( y \in F(x) \cap (\bar{y} - K - \zeta_n) \subset F(x) \cap (\bar{y} - K) \), whence \( y - \bar{y} \in (F(x) - \bar{y}) \cap -K \). Now, using (12), we get that \( y - \bar{y} \in K \) and since \( \bar{y} - y - \zeta_n \in K \), we arrive at \( -c \in K \), a contradiction. Thus, \( \{A_1, A_2, (\bar{x}, \bar{y})\} \) is an extremal system in \( X \times Y \), and since \( X \times Y \) is an Asplund space and the sets \( A_1 \) and \( A_2 \) are closed around \((\bar{x}, \bar{y})\), we can apply the approximate extremal principle to this system (see, \cite[Theorem 2.20]{15}). Therefore, for every \( n \in \mathbb{N} \setminus \{0\} \), there exist the sequences \((x^1_n, y^1_n) \in \text{Gr} \ F \cap D((\bar{x}, \bar{y}), \frac{1}{n}), x^2_n \in (\bar{x} + \text{cone} \ L) \cap A \cap D((\bar{x}, \frac{1}{n}), y^2_n \in (\bar{y} - K) \cap D((\bar{y}, \frac{1}{n})), x^{1*}_n \in X^*, x^{2*}_n \in X^*, y^{1*}_n \in Y^*, y^{2*}_n \in Y^* \) such that

\[
(x^{1*}_n, y^{1*}_n) \in \widehat{N} \left( \text{Gr} \ F, (x^1_n, y^1_n) \right) + \frac{1}{n} D_{X^* \times Y^*},
\]

\[
x^{2*}_n \in \widehat{N} \left( (\bar{x} + \text{cone} \ L) \cap A, x^2_n \right) + \frac{1}{n} D_{X^*},
\]

\[
y^{2*}_n \in \widehat{N} \left( \bar{y} - K, y^2_n \right) + \frac{1}{n} D_{Y^*} = -\widehat{N} \left( K, \bar{y} - y^2_n \right) + \frac{1}{n} D_{Y^*}
\]
and

\[ x_n^1 + x_n^2 = 0, \quad y_n^1 + y_n^2 = 0, \quad \| (x_n^1, y_n^1) \| + \| (x_n^2, y_n^2) \| = 1. \quad (13) \]

Therefore, there exist \((u_n^1, v_n^1) \in \frac{1}{n} D_{X^* \times Y^*}, u_n^2 \in \frac{1}{n} D_{X^*}\) and \(v_n^2 \in \frac{1}{n} D_{Y^*}\) s.t.

\[ x_n^1 - u_n^1 \in \hat{D}^* F (x_n^1, y_n^1) \left( (v_n^1 - y_n^1) \right), \quad x_n^2 - u_n^2 \in \hat{N} \left( (\bar{x} + \text{cone } L) \cap A, x_n^2 \right) \]

and

\[ y_n^2 - v_n^2 \in -\hat{N} \left( K, \bar{y} - y_n^2 \right) \subset K^+. \]

Using relation (13), we obtain that the sequences \((x_n^1), (x_n^2), (y_n^1)\) and \((y_n^2)\) are bounded, and since \(X\) and \(Y\) are Asplund spaces, there exist \(x_1^* \in X^*, x_2^* \in X^*, y_1^* \in Y^*\) and \(y_2^* \in Y^*\) such that

\[ x_n^1 \xrightarrow{u} x_1^*, \quad x_n^2 \xrightarrow{u} x_2^*, \quad y_n^1 \xrightarrow{w} y_1^*, \quad y_n^2 \xrightarrow{w} y_2^*. \]

Obviously, \(x_1^* + x_2^* = 0\) and \(y_1^* + y_2^* = 0\).

Now, if \(y_1^* = 0\), then \(y_2^* = 0\), whence \(y_2^2 - v_2^2 \xrightarrow{w} 0\) and using the (SNC) assumption we have that \(y_2^2 \xrightarrow{w} 0\), whence \(y_2^2 \xrightarrow{w} 0\), so \(y_1^* \xrightarrow{w} 0\). Taking into account that \(F\) is Lipschitz-like around \((\bar{x}, \bar{y})\) and using [15, Theorem 1.43], we obtain that \(x_n^1 - u_n^1 \xrightarrow{w} 0\) and since \(u_n^1 \xrightarrow{w} 0\), we have that \(x_n^1 \xrightarrow{w} 0\), Using again (13) we obtain that \(x_n^2 \xrightarrow{w} 0\), which contradicts the fact that \(y_n^2 \xrightarrow{w} 0\) and \(\| (x_n^2, y_n^2) \| = \frac{1}{2}\).

Hence, \(y_1^* \neq 0\). Moreover, since \(y_1^* + y_2^* = 0\), \(y_2^* - v_2^* \xrightarrow{w} y_2^*\), \(y_2^2 - v_2^2 \in K^+\) and \(K^+\) is weakly star closed, we obtain that \(-y_2^* = y_2^* \in K^+\).

Further, using hypothesis (iii), for every \(n\) large enough, we get that there exist \(l_n \in (\bar{x} + \text{cone } L) \cap B \left( x_n^2, \frac{1}{n} \right)\) and \(a_n \in A \cap B \left( x_n^2, \frac{1}{n} \right)\) such that

\[ x_{\hat{n}}^2 \in \hat{N} \left( (\bar{x} + \text{cone } L) \cap A, x_n^2 \right) + \frac{1}{n} D_{X^*} \]

\[ \subset \hat{N} \left( (\bar{x} + \text{cone } L, l_n) + \hat{N} (A, a_n) + \frac{2}{n} D_{X^*}, \right) \]

whence, there exist \(a_n^* \in \hat{N} (A, a_n)\) and \(l_n^* \in \hat{N} (\bar{x} + \text{cone } L, l_n)\) such that \(a_n^* + l_n^* - x_{\hat{n}}^2 \xrightarrow{w} 0\). Further, we prove that \((a_n^*)\) or \((l_n^*)\) is bounded. Suppose by contradiction that both sequences are unbounded. It follows that for every \(n\), there is \(k_n\) sufficiently large such that

\[ n < \min \left\{ \| a_{k_n}^* \|, \| l_{k_n}^* \| \right\}. \quad (14) \]

For simplicity, we denote the subsequence \((a_{k_n}^*), (l_{k_n}^*)\) by \((a_n^*), (l_n^*)\), respectively.

Now, since \(a_n^* \in \hat{N} (A, a_n), l_n^* \in \hat{N} (\bar{x} + \text{cone } L, l_n)\), we obtain that

\[ \frac{1}{n} a_n^* \xrightarrow{w} \hat{N} (A, a_n), \]

\[ \frac{1}{n} l_n^* \xrightarrow{w} \hat{N} (\bar{x} + \text{cone } L, l_n) = \hat{N} (\text{cone } L, l_n - \bar{x}). \]

Since

\[ \frac{1}{n} \| a_n^* + l_n^* \| \leq \frac{1}{n} \| a_n^* + l_n^* - x_{\hat{n}}^2 \| + \frac{1}{n} \| x_{\hat{n}}^2 \|. \]
we obtain that \( \frac{1}{n} (a_n^* + l_n^*) \to 0 \), so using again the hypothesis of alliedness, we obtain that \( \frac{1}{n} a_n^* \to 0 \) and \( \frac{1}{n} l_n^* \to 0 \), which is in contradiction with relation (14). Consequently, we obtain that \( (a_n^* + l_n^*) \subset X^* \) are bounded; thus, since \( X \) is Asplund, there exist \( a^*, l^* \in X^* \) such that \( a_n^* \to a^* \) and \( l_n^* \to l^* \), which is in contradiction with relation (14).

Consequently, we obtain that \( (a_n^* + l_n^*) \subset X^* \) are bounded; thus, since \( X \) is Asplund, there exist \( a^*, l^* \in X^* \) such that \( a_n^* \to a^* \) and \( l_n^* \to l^* \), so \( x_2^* = a^* + l^* \in N (A, \overline{x}) + N (\text{cone } L, 0) \). Finally, observe from above that \( x_2^* = x_1^* + x_2^* = 0 \), we get that \( 0 \in D^* F (\overline{x}, \overline{y}) (y_2^*) + N (A, \overline{x}) + N (\text{cone } L, 0) \) with \( y_2^* \in K^+ \setminus \{0\} \), i.e., the conclusion. \( \square \)

We end this section by considering the situation where the objective map is a single-valued mapping. Consider \( f : X \to \mathbb{R} \) a real-valued function, and take \( A \subset X \) and \( L \subset S_X \) nonempty and closed sets. In order to obtain necessary conditions for directional Pareto minimum in the nonsmooth case, we make use of the penalty function method.

**Proposition 3.9** Let \( \overline{x} \in A \) be a local directional minimum for \( f \) on \( A \) with respect to \( L \). Suppose that \( f \) is Lipschitz continuous around \( \overline{x} \), and \( \text{cone } L \) is convex. In addition, suppose that \( N (A, \overline{x}) \cap \{-L^-\} = \{0\} \) and that either \( A \) or \( \overline{x} + \text{cone } L \) is (SNC) at \( \overline{x} \). Then, one has

\[
0 \in \partial f (\overline{x}) + N (A, \overline{x}) + L^-.
\]

**Proof** According to the definition of directional minima, \( \overline{x} \) is a local solution of the constrained optimization problem

\[
\min f (x) \text{, such that } x \in \Omega,
\]

where \( \Omega := A \cap (\overline{x} + \text{cone } L) \). Then, following the well-known Clarke penalization, \( \overline{x} \) is a solution of the unconstrained optimization problem

\[
\min f (x) + kd (x, \Omega), \text{ such that } x \in X,
\]

where \( k > 0 \) is the Lipschitz modulus of \( f \). By the generalized Fermat rule and the sum rule for limiting subdifferential, one has

\[
0 \in \partial (f + kd (\cdot, \Omega)) (\overline{x}) \subset \partial f (\overline{x}) + k \partial d (\cdot, \Omega) (\overline{x})
\subset \partial f (\overline{x}) + N (\Omega, \overline{x}).
\]

Observe that \( N (\overline{x} + \text{cone } L, \overline{x}) = N (\text{cone } L, 0) = L^- \), and now we can use [15, Corollary 3.5] since, according to our assumptions, both normal qualification condition and the required (SNC) property hold. Then, this allows us to write that

\[
N (\Omega, \overline{x}) \subset N (A, \overline{x}) + N (\overline{x} + \text{cone } L, \overline{x}) = N (A, \overline{x}) + L^-,
\]

and the conclusion follows. \( \square \)
Now, we make one step forward by considering the vectorial optimization problem
\[
\min_K f(x), \quad \text{such that } x \in A,
\]
where \( f : X \rightarrow Y \) is a vector-valued function, \( A \subset X \) is a closed set and \( K \) is the ordering cone on \( Y \).

Consider the following vectorial Lipschitz property for \( f \): Following [20], one says that \( f \) is \( K \)-Lipschitz around \( \bar{x} \in X \) of rank \( \ell_f > 0 \) if there exist a neighborhood \( U \) of \( \bar{x} \) and an element \( e \in K \cap SY \) such that for every \( x', x'' \in U \),
\[
f(x'') - f(x') + \ell_f \| x'' - x' \| e \in K.
\]

We record the following result.

**Theorem 3.8** Let \( \bar{x} \in A \) be a local directional Pareto minimum for \( f \) on \( A \) with respect to \( L \subset SX \). Suppose that:

(i) \( f \) is \( K \)-Lipschitz around \( \bar{x} \) of rank \( \ell_f \) and let \( e \) be the element in \( K \cap SY \) given by the Lipschitz property of \( f \);
(ii) \( K \) is (SNC) at \( 0 \);
(iii) cone \( L \) is convex, \( N(A, \bar{x}) \cap (-L^-) = \{0\} \) and that either the set \( A \) or the set \( \bar{x} + \text{cone } L \) is (SNC) at \( \bar{x} \).

Then, for every \( \ell > \ell_f \), there exist \( y^* \in K^+\setminus\{0\} \) and \( x^* \in D^* f(\bar{x})(y^*) \) such that
\[
-x^* \in N(A, \bar{x}) + L^- \text{ and } \|x^*\| \leq \ell y^*(e).
\]

**Proof** Again, directional Pareto minimality of \( \bar{x} \) means that \( \bar{x} \) is a Pareto minimum for \( f \) on \( A \cap (\bar{x} + \text{cone } L) \). We use now a vectorial variant of Clarke penalization (see [20, Theorem 3.2 (i)]) to deduce that, for every \( \ell > \ell_f \), \( \bar{x} \) is an unconstrained Pareto minimum for the function \( f(\cdot) + \ell d(\cdot, A \cap (\bar{x} + \text{cone } L)) e \). We can now use the method from [21, Theorem 3.11] to deduce that for every \( \ell > \ell_f \), there exist \( y^* \in K^+\setminus\{0\} \), \( x^* \in D^* f(\bar{x})(y^*) \) such that
\[
-x^* \in N(A \cap (\bar{x} + \text{cone } L), \bar{x})
\]
and \( \|x^*\| \leq \ell y^*(e) \). Using again [15, Corollary 3.5], we have
\[
-x^* \in N(A, \bar{x}) + L^-,
\]
and this is the conclusion. \( \Box \)

### 4 Pareto Directional Minima for Sets

As made clear in Definition 2.3 and the subsequent comments, the notion of directional Pareto minimum is motivated by the case of (generalized) mappings. However, it is
possible to define such a notion for sets as well. In order to point out this aspect of
directional minimality, in this section we define some appropriate notions and we give,
only briefly, some examples and optimality conditions for them.

Consider, as above, a nonempty and closed set \( L \subset S_X \) and take now \( K \) as a proper,
closed and convex cone in \( X \).

**Definition 4.1** Let \( M \subset X \) be a nonempty set. One says that \( x \in M \) is a local direc-
tional Pareto minimum point for \( M \) with respect to \( L \) if
\[
(M \cap (x + \text{cone } L) - x) \cap -K \subset K.
\]
(17)

If \( \text{int } K \neq \emptyset \), one says that \( x \in M \) is a weak directional Pareto minimum for
\( M \) with respect to \( L \) if
\[
(M \cap (x + \text{cone } L) - x) \cap -\text{int } K = \emptyset.
\]
(18)

It is simple to see that relation (17) is equivalent to
\[
(M - x) \cap \text{cone } L \cap -K \subset K,
\]
while relation (18) actually means
\[
(M - x) \cap \text{cone } L \cap -\text{int } K = \emptyset.
\]
Therefore, (17) is relevant only if \( \text{cone } L \cap -K \neq \{0\} \), while for (18) it is important
to have \( L \cap -\text{int } K \neq \emptyset \).

Now, we give some examples that justify the above notions of Pareto minimum.

**Example 4.1** Let \( \gamma \) be a closed curve described by the following two parametric equa-
tions
\[
\begin{align*}
x(t) &= 2 + 2 \cos t (1 - \sin t) \\
y(t) &= \sin t (1 - \cos t)
\end{align*}
, \quad t \in [0, 2\pi],
\]
\( \gamma = \text{int } \gamma \cup \text{bd } \gamma \) and the half-plane \( H := \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \geq -x\} \). Take \( K = \mathbb{R}_+^2 \), \( \overline{\gamma} := (0, 0) \) and the directions set \( L := \{(\cos t, \sin t) : t \in ]\pi, 1.25\pi[\} \). Now, consider
\( M := H \cup (\overline{\gamma} \cap -H) \) as a closed subset of \( X := \mathbb{R}^2 \). Observing that \( (M - \overline{\gamma}) \cap \text{cone } L \cap -K = \{(0, 0)\} \subset K \) and \( (M - \overline{\gamma}) \cap -K \) has points that are not in \( K \setminus \{(0, 0)\} \), for instance those ones that are on \( \gamma \) and have negative \( x \)-coordinate, we get that \( \overline{\gamma} \) is a local directional Pareto minimum point for \( M \) with respect to \( L \), but it is not a local
Pareto minimum point for \( M \). Similarly, we have \( (M - \overline{\gamma}) \cap \text{cone } L \cap -\text{int } K = \emptyset \)
and \( (M - \overline{\gamma}) \cap -\text{int } K \neq \emptyset \), so there exists local weak directional Pareto minimum
points that are not local weak Pareto minimum points.

**Example 4.2** Consider the set \( M \subset \mathbb{R}^2 \), \( M := \{(x, y) : (x - 1)^2 + y^2 \leq 1\} \). Take
\( K := \mathbb{R}_+^2 \). Then, the local Pareto minimal points of \( M \) (that is, local directional Pareto
minima with respect to \( S_{\mathbb{R}^2} \)) are exactly those which are local directional Pareto minima
with respect to \( \{(-1, 0), (0, -1)\} \), namely the set of points \( \{(1 + \cos t, \sin t) : t \in [\pi, 1.5\pi]\} \). Then, these are in fact the important directions to consider and the other
can be dropped.
In the notation of Definition 4.1, the following optimality conditions hold.

**Theorem 4.1** Suppose that \( \text{cone} \ L \cap - \text{int} \ K \neq \emptyset \).

(i) If \( \bar{x} \in M \) is a weak directional Pareto minimum for \( M \) with respect to \( L \), then

\[
T_B^L (M, \bar{x}) \cap - \text{int} \ K = \emptyset.
\]

(ii) If for \( \bar{x} \in M \) one has

\[
T_B^L (\text{cl}(M + K), \bar{x})) \cap - \text{int} \ K = \emptyset,
\]

then \( \bar{x} \) is a weak directional Pareto minimum for \( M \) with respect to \( L \).

**Proof** (i) Suppose that there exists \( u \in T_B^L (M, \bar{x}) \cap - \text{int} \ K \), meaning that there are

\[
(u_n) \xrightarrow{\text{cone} \ L} u, \ (t_n) \xrightarrow{[0, \infty[} 0
\]

such that for all \( n \), \( \bar{x} + t_n u_n \in M \) and \( u \in - \text{int} \ K \). Clearly, for \( n \) large enough,

\[
t_n u_n \in (M - \bar{x}) \cap \text{cone} \ L \cap - \text{int} \ K,
\]

which contradicts the minimality assumption.

(ii) Suppose, again by way of contradiction, that there exists \( x \in M \) such that \( x - \bar{x} \in \text{cone} \ L \cap - \text{int} \ K \). Consider \((t_n) \xrightarrow{[0, \infty[} 0\). Then, for every \( n \) large enough,

\[
\bar{x} + t_n (x - \bar{x}) = x + (1 - t_n)(\bar{x} - x) \in (M + \text{int} K) \cap (\bar{x} + \text{cone} \ L),
\]

whence using the fact that \( \text{cl}(M + \text{int} K) = \text{cl}(M + K) \) (which, in turn, is easy to prove using the closedness and the convexity of \( K \) which ensures \( K = \text{cl \ int} K \)), one can write:

\[
- \text{int} \ K \ni x - \bar{x} \in T_B((M + \text{int} K) \cap (\bar{x} + \text{cone} \ L), \bar{x})
\]

\[
= T_B(\text{cl} [(M + \text{int} K) \cap (\bar{x} + \text{cone} \ L)], \bar{x})
\]

\[
\subset T_B(\text{cl} (M + \text{int} K) \cap (\bar{x} + \text{cone} \ L), \bar{x})
\]

\[
= T_B(\text{cl} (M + K) \cap (\bar{x} + \text{cone} \ L), \bar{x}) = T_B^L (\text{cl}(M + K), \bar{x}),
\]

and this is in contradiction with the hypothesis. \( \Box \)

**Theorem 4.2** Suppose that \( \text{cone} \ L \cap -K \neq \{0\} \). If for \( \bar{x} \in M \) one has

\[
T_B^L (\text{cl}(M + K), \bar{x))) \cap -K \subset K,
\]

then \( \bar{x} \) is a directional Pareto minimum for \( M \) with respect to \( L \).

**Proof** The proof is similar to that of Theorem 4.1(ii). \( \Box \)
5 Perspectives and Further Research

We think that several perspectives are open for further investigation on the basis of the concept proposed and the results devised in this work. Clearly, as generalization of several aspects of some already classical notions in scalar and vector optimization, the concept of directional Pareto efficiency is of immediate interest from the perspective of generalizing known results in terms of optimality conditions in this new context, and this is in fact the main concern of the present paper.

Nevertheless, this new notion raises several technical and theoretical issues and opens new possibilities in the investigation of some problems related to Pareto efficiency, in general. We try to point out these ideas below.

A first encouraging point in the possible further developments is the stability properties of directional regularity, as established in [22]. This allows us to consider directional minimality for problems where the objective functions are sum (or difference), as well as compositions of maps, and in this sense, in particular, possible generalizations of the optimization of d.c. (difference of convex) mappings are to be expected. Moreover, besides the basic Pareto minimality that is here extended to directional framework, an investigation of other types of vector efficiency is to be considered in the future research, and we think here at several types of proper efficiency [7].

However, even in the case considered here a tangent cone to an intersection of sets comes into play and this cannot be, in general, written as the intersection of the tangent cones to the respective sets. This technical issue can be considered for some special cases where the structure of the involved sets would allow a better computation, and one can consider, for instance, polyhedral or finitely generated cones and their extremal rays. Another technical issue is an investigation aiming to devise an adapted directional normal limiting cone with respect to a set of directions and to use it in order to derive more specific optimality conditions for our concept.

The growing literature nowadays concerning the efficiency with respect to Kuroiwa set relations (see [23]) is another impetus to extend our investigation of directional efficiency. Recent developments from [24], for instance, can serve as a basis for new extensions for this kind of minimality.

Concerning possible applications, the use of minimal time function in the location theory is already presented in [1,2,25]. For instance, in [25], the following situation is considered.

Let $X$ be a finite-dimensional normed vector space, $\emptyset \neq M \subset S_X$ a closed (then compact) set and $\Omega_1, \ldots, \Omega_p \subset X$ nonempty and closed sets. Define $f : X \to [0, \infty]^p$, $f(x) := (\tau_M(x, \Omega_1), \ldots, \tau_M(x, \Omega_p))$.

Also, define the domain of this function as $\text{dom } f := \{x \in X : \tau_M(x, \Omega_k) < \infty, \forall k \in \{1, p\} = \bigcap_{k=1}^{p} (\Omega_k - \text{cone } M)$.
and say that $\bar{x} \in \text{dom } f$ is a Pareto minimum point for $f$ if there is a neighborhood $V$ of $\bar{x}$ such that for any $x \in V \cap \text{dom } f$, $f(x) - f(\bar{x}) \notin -\mathbb{R}_+^p \{0\}$, that is, $f(\bar{x})$ is a Pareto minimum point for the set $f(V \cap \text{dom } f)$ with respect to $\mathbb{R}_+^p$.

This vectorial problem is motivated by the fact that in practice sometimes one has to deal with a situation when it is not possible to place the interest point of the location (for instance, the point of service for several facilities) in a position between the target sets. Now, in light of the new directional efficiencies, this problem can be refined from a better practical perspective. Moreover, in this setting, it would be of interest to adapt the Weiszfeld algorithm (see [1,26]).

We end this section by saying that the second-order conditions for the directional efficiencies have to be developed for a better understanding of the similarities and the differences with respect to classical notions, and special attention should be given to the envelope effect as described in [27].

6 Conclusions

The directional efficiencies introduced in this paper generalize in a meaningful way the classical situation of Pareto optimality and require non-trivial adaptations of the usual techniques of investigation used in the latter case. Besides the results of this paper, we think that our approach opens new possibilities to model directional situations, especially arising in vector optimization problems dealing with location problems. We consider that our concept here introduced is able to capture the situation where some directions are more important than the others (hence which can be dropped) in the possible models under consideration. Moreover, we presented several possible continuations both from theoretical and practical points of view. All these ideas will be topics for future research.

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