Superdiffusion in random two dimensional system with ubiquitous long-range hopping

Xiaolong Deng,1 Ivan M. Khaymovich,2 and Alexander L. Burin1

1Leibniz-Rechenzentrum, Boltzmannstr. 1, D-85748 Garching bei München, Germany
2Stockholm University and KTH Royal Institute of Technology, Hanes Alfvéns väg 12, SE-106 91 Stockholm, Sweden
3Tulane University, New Orleans, LA 70118, USA

(Dated: August 11, 2022)

Although it is recognized that Anderson localization takes place for all states at a dimension $d$ less or equal 2, while delocalization is expected for hopping $V(r)$ decreasing with the distance slower or as $r^{-d}$, the localization problem in the crossover regime for the dimension $d = 2$ and hopping $V(r) \propto r^{-d}$ is not resolved yet. Following earlier suggestions we show that for the hopping determined by two-dimensional anisotropic dipole-dipole interactions there exist two distinguishable phases at weak and strong disorder. The first phase is characterized by ergodic dynamics and superdiffusive transport, while the second phase is characterized by diffusive transport and delocalized eigenstates with fractal dimension less than 2. The transition between phases is resolved analytically using the extension of scaling theory of localization and verified using an exact numerical diagonalization.

Introduction. Low dimensional systems with a dimension $d \leq 2$ possessing the time reversal symmetry are critical in Anderson localization problem [1]. All states there must be exponentially localized at arbitrarily small disorder as was shown using the scaling theory of localization [2], analysis of conductivity [3] and extensive disordering as was shown using the scaling theory of localization [1, 10–14] except for the marginal case of diverging Fourier transform of a hopping amplitude $t^1$. If disorder gets weaker there is the transition in 3D to the standard delocalized phase characterized by the diffusive transport and ergodic dynamics [54], while in 1D eigenstates turns out to be multifractals with the dimension smaller than 1 [12]. 2D systems are more complicated, because for the hopping $V(r) \propto r^{-d}$ the dimensionless Drude conductance diverges logarithmically with the system size as $C_0(L) = c_0 \ln(L)$ [28]. Considering the balance of this logarithmic raise of conductance with the system size results in the inevitable localization at large sizes $L$.

The scaling theory of localization suggests the single parameter scaling for the dimensionless conductance $C = G/(\epsilon^2/h)$ dependence on the size $L$ in the form [2, 3] ($G$ is the conductance)

$$d \ln(C) = \beta(C),$$

$$\beta(C) = -\frac{c_{loc}}{C} + O(C^{-4}), \ c_{loc} = \frac{2\pi^2}{D}, \ (1)$$

where the $\beta$-function has been evaluated using expansion of the equivalent non-linear sigma-model [8, 9] valid at $C \gg c_{loc}$. Eq. 1 predicts the reduction of conductance with the system size $L$ as $C(L) = C_0 - c_{loc} \ln(L)$ where $C_0$ is the conductance at the lower cutoff $L = 1$. Logarithmic reduction of conductance with the system size results in the inevitable localization at large sizes $L$.

This universal scaling is limited to systems with a short-range hopping, while the hopping decreasing with the distance as $V(r) \propto r^{-d}$ or slower leads to delocalization of all states [10–14] except for the marginal case of diverging Fourier transform of a hopping amplitude [15–26]. If disorder is strong, eigenstates of the problem are multifractals and the time dependent displacement of the particle $r$ obeys the law $r \propto t^{1/4}$ which is subdiffusive in 3D, diffusive in 2D and superdiffusive in 1D [11]. The long-range hopping $V(r) \propto r^{-2}$ is ubiquitous in pure two-dimensional systems [27], where it can be originated from the virtual exchange by two dimensional photons leading to the 2D dipole-dipole interaction [28] or indirect exchange by 2D electron-hole pairs leading to 2D RKKY interaction [29]. Power-law distant dependent hopping is crucial for the many-body localization problem [30–53], where long-range interaction can result in localization breakdown at arbitrary disorder.

If disorder gets weaker there is the transition in 3D to the standard delocalized phase characterized by the diffusive transport and ergodic dynamics [54], while in 1D eigenstates turns out to be multifractals with the dimension smaller than 1 [12]. 2D systems are more complicated, because for the hopping $V(r) \propto r^{-d}$ the dimensionless Drude conductance diverges logarithmically with the system size as $C_0(L) = c_0 \ln(L)$ [28]. Considering the balance of this logarithmic raise of conductance and its logarithmic suppression by coherent back scattering $c_{loc} \ln(L)$ it was suggested in Ref. [28] that two delocalized phases can exist including the superdiffusive (fast) phase at $c_\ast > c_{loc}$ and the slow phase with diffusive transport at $c_\ast < c_{loc}$ and the phase boundary realized at $c_\ast = c_{loc}$. Yet it turns out that for the isotropic dipoles considered in Ref. [28] $c_\ast < c_{loc}$ for arbitrarily disordering so the fast phase does not exist.

These achievements motivated us to search for the superdiffusive, fast phase using different hopping interaction including anisotropic dipole-dipole interaction with identically oriented dipoles. We also found a superdiffusive phase for the 2D RKKY interaction at sufficiently weak disorder that will be published separately. These interactions differ from the isotropic dipole model of Ref. [28] by the presence of dispersive modes with the mean free path increasing unlimitedly with decreasing disorder with similar increase of the logarithmic growth parameter $c_\ast$. This makes the appearance of the fast phase with $c_\ast > c_{loc}$ unavoidable. Recent experimental realizations of 2D Anderson localization [54] and long-range hopping [50] represent the steps towards generating the settings targeted in the present work. Consequently, we believe that its experimental realization is possible and strongly encouraged.

We investigate two phases for two-dimensional Ander-
son model with a long-range hopping, formulated below, using the extension of scaling theory of localization for the long-range hopping and exact numerical diagonalization. The phase boundary \( c_s = c_{\text{loc}} \) is identified analytically and verified numerically (see Fig. 1). Both phases are characterized by level statistics, that is of Wigner-Dyson for the fast phase and intermediate between Poisson and Wigner-Dyson otherwise (see Fig. 2), eigenstate dimension (2 for the fast phase and less than 2 otherwise, as depicted in Fig. 3) and particle transport (superdiffusive or diffusive, see Fig. 4). The Appendices A-D contain the details of calculations of logarithmic growth parameter \( c_s \) (A, B), the derivation of the modified scaling theory of localization (C) and the details of the analysis of eigenstate fractal dimension (D).

**Model and Drude conductance** Anderson model in 2D is investigated. The Hamiltonian of the model can be expressed as

\[
\hat{H} = \sum_{i,j} V_{ij} c_i^\dagger c_j + \sum_i \phi_i c_i^\dagger c_i, \tag{2}
\]

where the summation is performed over \( N = L^2 \) lattice sites enumerated by indices \( i \) with coordinates \( r_i = (x_i, y_i) \) occupying the periodic square lattice placed onto the surface of torus characterized by the radii \( R = L/(2\pi) \). Independent random energies \( \phi_i \) obey the Lorentzian distribution with the width \( W \) characterizing disordering, while hopping amplitudes are given by the exchange of the dipolar excitations between interacting dipoles, oriented along the \( x \)-axis, via the interaction \( V_{ij} = V(r_{ij}) \) with \( V(r) = (2x^2 - r^2)/r^4 \). These hopping terms are formed similarly to the interaction \( (3x^2 - r^2)/r^5 \) in three dimensions [57]. For this interaction and small wavevector the continuous approach is valid for the Fourier transform of the hopping amplitudes, that can be evaluated as \( V(q) \approx 2\pi(q^2 - q_0^2)/q^2 \) for the dipole-dipole interaction [28]. For the Lorentzian distribution of random potentials the Green’s functions, needed for the calculation of a zeroth order conductance, can be evaluated exactly as \( G(E, q) = (E - V(q) - iW)^{-1} \) in the momentum representation [53].

A zeroth order (Drude) dimensionless conductance is defined as [3, 7, 28]

\[
C_{0, \alpha\beta} = \frac{1}{2\pi} \int \frac{dq}{(2\pi)^2} \frac{\partial V(q)}{\partial q_\alpha} \frac{\partial V(q)}{\partial q_\beta} |\text{Im}(G(E, q))|^2. \tag{3}
\]

The transport is isotropic for \( C_{0, \alpha\beta} = C_0 \delta_{\alpha\beta} \), which takes place for the dipole-dipole hopping in the middle of the band (\( E = 0 \)).

The integral in Eq. (3) diverges logarithmically at \( q \to 0 \). For a finite size \( L \) this divergence should be cutoff at \( q \sim 1/L \). The logarithmic growth rate of the diverging contribution \( c_s \) \( \ln(L) \) serving as an input for the scaling theory of localization can be evaluated as \( c_s = V_0^2/(2W\sqrt{\pi^2V_0^2 + W^2}) \). For the anisotropic regime realized at \( E \neq 0 \) two distinguishable logarithmic growth rates for conductances \( C^x_0 \) and \( C^y_0 \) can be introduced as \( c_s^x = dC^x_0/d\ln(L) \) and \( c_s^y = dC^y_0/d\ln(L) \), while an off-diagonal component of the conductance tensor vanishes. These rates are given in Appendix [B].

**Generalized scaling theory of localization.** The transition between two regimes and finite size asymptotic behaviors for isotropic conductance is determined by the generalized scaling equation derived in Appendix [C2] using the one loop order correction to conductance within the non-local, non-linear sigma model [12], obtained in the form

\[
\frac{d\ln(C)}{d\ln(L)} = \frac{c_s - c_{\text{loc}}}{C} + \frac{c_s c_{\text{loc}}}{C^2} + O(C^{-3}). \tag{4}
\]

This equation is similar to the one proposed in Ref. [28] except for the second term in the r. h. s. that is twice bigger, which is our contribution to this equation. This term is originated from the wavevector dependence of the zeroth order conductance (see Appendix [C1]).

**Phase transition and phase diagram.** According to Eq. (1) the conductance diverges logarithmically for \( L \to \infty \) under the condition \( c_s > c_{\text{loc}} \) (the fast phase), while it remains finite otherwise (the slow phase). Setting \( c_s = c_{\text{loc}} \) one can find critical disorder separating phases.

For the dipole-dipole interaction the isotropic regime is realized only for the band center \( E = 0 \), where the critical disorder is given by \( W_c = \pi V_0/\sqrt{(1 + \sqrt{5})}/2 \approx 2.47V_0 \). With decreasing disorder the fast phase emerges first at that energy. For the anisotropic regime realized at \( E \neq 0 \) the transition emerges at \( c_s^x = c_s^y = c_{\text{loc}}^x \) (see Appendix [C2]). These criteria determine the phase diagram depicted in Fig. 1 where solid lines indicate boundaries between slow and fast phases.

**Level statistics.** Wigner-Dyson level statistics indicates ergodic behavior [54, 60] since all eigenstates feel the presence of each other. The ergodic behavior is not
expected to be realized for eigenstates with fractal dimension less than the space dimension, emerging in the slow phase for the strong disorder \[11\], while one can expect it to appear in the fast phase similarly to the counterpart transition in 3D \[54\]. This expectation turns out to be consistent with the numerical studies reported below.

The level statistics is represented in terms of the average ratio of the minimum to maximum of adjacent energy level splittings defined as \[53\]

\[
\langle r \rangle = \left( \frac{\min(\delta_n, \delta_{n+1})}{\max(\delta_n, \delta_{n+1})} \right), \quad \delta_n = E_{n+1} - E_n, \quad (5)
\]

where \(E_n\) represent energies of eigenstates arranged in the ascending order. In the localized phase one has \(\langle r \rangle = 2 \ln(2) - 1 \approx 0.386\), while in the delocalized, ergodic phase characterized by the Wigner-Dyson statistics \(\langle r \rangle \approx 0.5307\) \[64\].

In Fig. 2 we show the level statistics for the system of the size \(L = 291\) with the dipole-dipole hopping and different disorder strengths averaged over 200 realizations with the energy resolution 0.1. For the minimum disorder strength \(W = 1\), where a substantial fraction of states with energies \(|E| < 2.7V_0\) indicated by the dashed line should belong to the fast phase (in all graphs we set \(V_0 = 1\)). The intermediate disorder strength \(W = 2.5\) approximately corresponds to the first appearance of the fast phase at \(E = 0\). For the strongest disorder \(W = 4\) all states suppose to belong to the slow phase. It is visually clear in Fig. 2 that our numerical findings are consistent with the assumption of ergodic behavior in the fast phase and its lack in the slow phase. The data for the level statistics are also presented in the phase diagram Fig. 1. They are consistent with the analytical results shown by the solid line.

**Fractal dimension.** We define the fractal dimension using the informational dimension \(D_1\) \[61\], \[62\] that can be expressed in terms of the average eigenstate wavefunction partition Shannon entropy \(\xi(L) = \sum_i |c_i|^2 \ln(|c_i|^2)\).

Then the informational dimension is defined as \(D_1 = -d\xi/d\ln(L)\), cf. Ref. \[63\]. We used this definition for numerical estimates of size-dependent fractal dimension calculating the functions \(\xi\) for the sequence of lengths \(L_1, L_2, \ldots, L_n\) arranged in ascending order and then numerically differentiating them. This yields \(n - 1\) estimates for fractal dimensions \(D_1(l_k) = -\xi(L_{k+1}) - \xi(L_k))/\ln(L_{k+1}/L_k)\) at sizes \(l_k = \sqrt{L_kL_{k+1}}\) and \(k = 1, 2, \ldots, n - 1\). Numerical results should be compared with analytical estimates for fractal dimensions obtained using the generalized scaling theory Eq. \(4\).

According to the non-linear sigma model \[8\], \[64\] \[66\] the informational dimension depends on the dimensionless conductance Eq. \(1\) as \(D_1 = 2 - c_{loc}/C\) at \(C \gg c_{loc}\). Integrating Eq. \(\delta\) we obtain the transcendental equation that expresses size dependent conductance as \(C + C_\infty \ln(1 - C/C_\infty) = -(c_{loc} - c_\ast) \ln(L/L_0)\), where \(C_\infty = c_\ast c_{loc}/(c_{loc} - c_\ast)\) represents the infinite size limit of conductance in the slow phase \(c_\ast < c_{loc}\) and \(L_0\) is the unknown integration constant. We define this constant for each line shown in Fig. 3 minimizing the deviation of analytical \((2 - c_{loc}/C)\) and numerical estimates of fractal dimensions. In the infinite size limit one can replace \(C\) with \(C_\infty\) and the fractal dimension reads \(D_1(\infty) \approx 2 - (c_{loc} - c_\ast)/c_\ast\), see Fig. 4 at \(1/\ln(L) \rightarrow 0\).

**Particle transport.** Finally, we consider the parti-
cle transport that is the main distinction of two phases, namely, superdiffusive in the fast phase and diffusive in the slow phase. It is natural to expect that the typical displacement $r$ increases with the time following the diffusion law $r^2 \propto C(r)t$. Then since in the fast phase the conductance $C(r)$ increases logarithmically with the size $r$ the superdiffusive behavior for the displacement is expected in contrast to the linear dependence in the slow phase.

To verify this expectation we investigate the time evolution of the system wavefunction originally localized in the origin where we set a random potential to zero. Different strengths of random potentials were investigated including $W = 0.5$ and $1.5$ for the fast phase, $W = 2.5$ for the transition point and $W = 3.5$ for the slow phase. For $W = 0.5$ or $1.5$ the fast phase is realized for the majority of the states, except for a small fraction of the “slow” states in the tails of the spectrum that don’t affect the particle transport. The choice of a zero random potential in the zero random potential in the site where the particle was placed at $t = 0$ also reduces the contribution of the slow tail states to the wavefunction.

The particle transport has been characterized using the average logarithm of squared displacement (excluding the origin). We did not consider the most often used root mean square displacement because of the power-law tails of the wavefunction that can lead to the overestimate of the actual move.

Fig. 4 shows the logarithm of squared radius vs. the logarithm of the time fitted with the power law time dependence in the domains visually represented by nearly straight lines. It is clear that the transport in the fast phase is characterized by the exponent $\alpha > 1$ corresponding to the superdiffusive behavior. Moreover in the superdiffusive regime the particle occupies nearly the entire system ($W = 0.5, 1.5$), while at a higher disorder only a vanishing fraction of the volume is occupied. This is consistent with our expectations for the fractal dimension below 2 in the slow phase.

**Conclusions.** We show the emergence of a superdiffusive fast phase in two-dimensional Anderson model with long-range hopping $V(r) \propto r^{-2}$ and sufficiently small disorder. The fast phase is characterized by delocalized, ergodic eigenstates occupying the whole space and fostering the superdiffusive transport. The complementary slow phase is non-ergodic. In this phase eigenstates are delocalized, while their fractal dimension is less than 2. The transport there is expected to be diffusive but restricted to the maximum displacement substantially smaller than the system size. The boundary between two phases is determined analytically, which is unprecedented for Anderson localization problem with the only exception of the celebrated self-consistent theory of localization valid for the Bethe lattice [67].

**ACKNOWLEDGMENTS**

*Acknowledgement.* A. B. acknowledges the support by Carrol Lavin Bernick Foundation Research Grant (2020-2021), CHE-2201027 grant and LINK Program of the NSF and Louisiana Board of Regents. X.D acknowledges the support by the Federal Ministry of Education and Research of Germany (BMBF) in the framework of DAQC. I. M. K. acknowledges the European Research Council under the European Unions Seventh Framework Program Synergy HERO SYG-18 810451.

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Appendix A: Model and Green functions

Here we introduce the models used in the main text in greater detail. The system Hamiltonian can be expressed as

\[ \hat{H} = \sum_{i,j} V_{ij} c_i^\dagger c_j + \sum_i \phi_i c_i^\dagger c_i, \]  

where the summation is performed over \( N = L^2 \) lattice sites enumerated by indices \( i \) with coordinates \( \mathbf{r}_i = (x_i, y_i) \) occupying the periodic square lattice placed onto the surface of the torus characterized by the radii \( R = L/(2\pi) \).

Vectors connecting two points \( i \) and \( j \) located in the torus are defined as \( \mathbf{r}_{ij} = (L/\pi)(\sin(\pi(x_i - x_j)/L), \sin(\pi(y_i - y_j)/L)) \) and hopping interactions \( V_{ij} \) are defined as the function of those vectors. Here we consider the anisotropic two-dimensional dipole-dipole interaction with dipoles oriented along the \( x \) axis. The dipole-dipole interaction is defined as

\[ V(\mathbf{r}_{ij}) = V_0 \frac{x_{ij}^2 - y_{ij}^2}{r_{ij}^3}, \quad r_{ij} = (x_{ij}, y_{ij}). \]
Random potentials $\phi$ are non-correlated with each other and characterized by the normalized by one probability density $P(\phi/W)/W$. The function $P(x)$ has a unit width and the parameter $W$ is the width of random potential distribution expressing the strength of disordereding. The case $W > V_0$ corresponds to strong disordering while the opposite regime $W < V_0$ corresponds to weak disordering. In a further analysis we set $V_0 = 1$ expressing all energies in the units of the interaction constant $V_0$.

![Comparison of analytical and numerical Fourier transforms for dipole-dipole interaction (at fixed ratios $k_y/k_x$). Numerical Fourier transform is evaluated for the system of the size $L = 100$. The analytical approximation is shown by dashed lines and numerical by solid lines.](image)

In our calculations we use the Lorentzian distribution of random energy, suggesting

$$P(x) = \frac{1}{\pi (x^2 + 1)}. \quad (A3)$$

For the Lorentzian distribution the Green’s functions of the problem can be evaluated exactly as

$$G(E, q) = \frac{1}{E - V(q) - iW} \quad (A4)$$

in the momentum representation $[58]$. $V(q)$ stands for the Fourier transforms of the hopping amplitudes $V(q) = \sum_j V_{ij} e^{iqr_{ij}}$. The Green functions are needed to evaluate the zeroth order dimensionless conductance.

It turns out that dipole-dipole interaction Fourier transforms can be represented by their continuous limits as shown below. For the dipole-dipole interaction one can express its Fourier transform as

$$V(q) = \pi (-q_x^2 + q_y^2) \frac{q_x}{q^2}, \quad (A5)$$

This approximation works reasonably well for the periodic square lattice of the size $L = 100$ as illustrated in Fig. 5. In the limit of interest $q \to 0$, where the logarithmic infrared divergence of the zeroth order conductance emerges, this approach becomes exact for $L \to \infty$. Indeed, since the diverging contribution to the conductance is determined by $q \to 0$ in this limit the regular correction to Eq. (A5) disappears because the sum of all dipole-dipole interactions is zero.

**Appendix B: Zeroth order conductance**

One can express the zeroth order dimensionless conductance tensor at given energy $E$ using the Kubo formula as

$$C_{ab}^{(0)} = \frac{1}{\pi (2\pi)^d} \int dq \left( \frac{\partial V(q)}{\partial q_a} \right) \left( \frac{\partial V(q)}{\partial q_b} \right) |\text{Im}G(E, q)|^2, \quad (B1)$$

where $G(E, q)$ is the retarded Green function at energy $E$ and wavevector $q$ defined in Eq. (A4). The conductance diverges logarithmically at long distances. Below we evaluate its diverging part needed for the characterizion of the phase transition both for the dipole - dipole interaction.
For the dipole-dipole hopping Eqs. [A2], [A5] the integral for the zeroth order conductance Eq. [B1] diverges logarithmically at \( q = 0 \). For the finite size \( L \) the integral should be cutoff at the minimum value \( q = \eta/L \), with the parameter \( \eta \sim 1 \) depending on the specific boundary conditions. Then the conductance tensor components in Eq. [B1] can be evaluated with the logarithmic accuracy as \( C_{xx} = c_x \ln(L/L_0) \), \( C_{yy} = c_y \ln(L/L_0) \), \( c_{xy} = c_{yx} = 0 \), where

\[
    c_x = \frac{\partial C_{xx}^{(0)}}{\partial \ln(L)} = \frac{4}{\pi} \int_0^{2\pi} \cos(\phi)^2 \sin(\phi)^4 W^2 \, d\phi \left( \frac{\pi^2 - W^2 - E^2}{\sqrt{(E - \pi \cos(2\phi))^2 + W^2}} \right)^2,
\]

\[
    = \frac{(\pi^2 - W^2 - E^2) \text{Im}(\sqrt{E^2 - (\pi - iW)^2}) + \pi W \text{Re}(\sqrt{E^2 - (\pi - iW)^2})}{4\pi^3 \sqrt{(E + \pi)^2 + W^2}},
\]

\[
    c_y = \frac{\partial C_{yy}^{(0)}}{\partial \ln(L)} = \frac{4}{\pi} \int_0^{2\pi} \cos(\phi)^4 \sin(\phi)^2 W^2 \, d\phi \left( \frac{\pi^2 - W^2 - E^2}{\sqrt{(E - \pi \cos(2\phi))^2 + W^2}} \right)^2,
\]

\[
    = \frac{(\pi^2 - W^2 - E^2) \text{Im}(\sqrt{E^2 - (\pi - iW)^2}) + \pi W \text{Re}(\sqrt{E^2 - (\pi - iW)^2})}{4\pi^3 \sqrt{(-E + \pi)^2 + W^2}}.
\]

In the middle of the band \( E = 0 \) the conductance is isotropic, \( C_{xx} = C_{yy} = c_* \ln(L/L_0) \), and the logarithmic growth rate \( c_* \) is given by

\[
    c_* = \frac{1}{2W \sqrt{\pi^2 + W^2}}.
\]

It is noticeable, that in the weak disorder limit \( (W \to 0) \) the rate parameter \( c_* \) in Eq. [B3] approaches infinity, so the transition to the superdiffusive regime should take place at a finite disorder strength \( W \) where \( c_*(W) = c_{loc} = 1/(2\pi^2) \) in contrast with the case of isotropic dipole-dipole hopping [28].

The generalization to the arbitrary distribution of random potentials can be made by the replacements \( E \to E - \text{Re}\Sigma(E, 0) \) and \( W \to -\text{Im}\Sigma(E, 0) \) in Eqs. [B2], [B3], where \( \Sigma(E, 0) \) is the self-energy evaluated at energy \( E \) and wavevector \( q = 0 \). One should notice that \( \Sigma(0, 0) = 0 \). Finding the self-energy for arbitrary distribution of random potentials remains a challenge; yet this problem is much easier compared to the localization problem itself.

### Appendix C: Derivation of the \( \beta \)-function.

We begin our consideration with the isotropic regime approximately valid for the dipolar system at zero energy that is considered in Sec. [C1]. The results for anisotropic regime are outlined in Sec. [C2]

#### 1. Isotropic conductance.

We investigate the renormalization of conductance \( C(q, p_1) \) for the orthogonal sigma model within the one loop order. Here \( q \) is the current momentum and \( p_1 \) is the maximum momentum \( \mu \) reduced during renormalization procedure. The renormalization of the conductance associated with the rescaling of the maximum momentum by the factor of \( \mu \) can be expressed as [8] (\( q \ll p \))

\[
    q^2(C(q, p_1/\mu) - C(q, p_1)) \approx - \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{C(|\mathbf{k} + \mathbf{q}|, p_1) C(\mathbf{k}, p_1) k^2 - C(\mathbf{k}, p_1) k^2}{C(\mathbf{k}, p_1) k^2},
\]

where integration \( \int' \) is taken over the domain of momenta \( p_1/\mu < p < p_1 \) that is getting excluded from during the renormalization procedure. This is the first loop order correction to the conductance similar to Eq. (25) in Ref. [12] where it has been considered for a one dimensional model with the long-range hopping. The terms \( C(p, p_1)p^2 \), \( C(|\mathbf{p} + \mathbf{q}|, p_1) (|\mathbf{p} + \mathbf{q}|)^2 \) are identical to the terms \( |q + k|^6 \) and \( |k|^6 \) in Ref. [12] since in one has there the diffusion coefficient \( D(k) \propto k^{\sigma-2} \), while in our 2D case \( D(k) \propto D(k) \).

The first term in the integrand \( C(|\mathbf{k} + \mathbf{q}|, p_1) (|\mathbf{k} + \mathbf{q}|)^2 / C(\mathbf{k}, p_1) k^2 \) represents the first loop order correction to the conductance, while the subtracted term 1 is originated from the field normalization (Jacobian of transformation) term.

The initial conditions to Eq. (C1) at large \( p_1 \sim 1 \) are set using the zeroth order conductance as

\[
    C(q, 1) = c_* \ln(1/(qL_0)),
\]
where the inverse wavevector $q$ serves as the cutoff radius in the definition of the conductance. The long-range interaction enters into consideration through this initial condition.

In the limit $q \ll p_1$, one can expand the expression in the numerator to the second order in $q$ as (the first order disappears because of the integration over angles)

$$ q^2(C(q,p_1/\mu) - C(q,p_1)) \approx - \sum_{\alpha,\beta=x,y} \frac{q_{\alpha}q_{\beta}}{2} \int_{p_1}^{p} \frac{dp}{\pi(2\pi)^2} \frac{\partial^2(\delta_{\alpha\beta}p_1^2)}{\partial p_{\alpha}\partial p_{\beta}} \frac{1}{C(p_1)}p_1^2. \quad (C3) $$

Evaluating derivatives and averaging over angles of vector $p$ ($\langle p_{\alpha}p_{\beta} \rangle = p^2 \delta_{\alpha\beta}/2$) Eq. (C3) we get

$$ C(q,p_1/\mu) - C(q,p_1) = - \int_{p_1}^{p} \frac{dp}{\pi(2\pi)^2} \left[ C(p_1) + \frac{\partial C(p_1)}{\partial \ln(p)} + \frac{1}{4} \frac{\partial^2 C(p_1)}{\partial \ln^2(p)} \right] \frac{1}{C(p_1)}p_1^2. \quad (C4) $$

Assuming that the logarithmic derivatives of the conductance are smooth functions (this is justified by the logarithmic size dependence of conductance in the initial condition Eq. (C2) and can be verified using the solution of the equation) one can perform logarithmic integration in the right hand side of Eq. (C4) and express this equation in the standard differential form as

$$ \frac{\partial C(q,p)}{\partial \ln(p)} = \frac{1}{2\pi^2} \left[ 1 + \frac{1}{C(p,p)} \right] \left[ \frac{\partial C(p_1,p)}{\partial \ln(p_1)} \right]_{p_1=p} + \frac{1}{4C(p,p)} \frac{\partial^2 C(p_1,p)}{\partial \ln^2(p_1)} \bigg|_{p_1=p}. \quad (C5) $$

Since the right hand side of Eq. (C5) is independent of the wavevector $q$ one can evaluate logarithmic derivatives using the initial condition Eq. (C2) as

$$ \left. \frac{\partial C(p_1,p)}{\partial \ln(p_1)} \right|_{p_1=p} = -c_s, \quad \left. \frac{\partial^2 C(p_1,p)}{\partial^2 \ln(p_1)} \right|_{p_1=p} = 0. $$

Then Eq. (C5) takes the form

$$ \frac{\partial C(q,p)}{\partial \ln(p)} = \frac{1}{2\pi^2} \left[ 1 - \frac{c_s}{C(p,p)} \right]. \quad (C6) $$

The renormalized conductance at the given momentum $p$ can be determined with the logarithmic accuracy as $C(p,p)$ and it can be denoted as $C(p)$ for the convenience. Using the initial condition Eq. (C2) for the derivative with respect to the first argument we end up with the renormalization group equation in the form

$$ \frac{dC(p)}{d\ln(p)} = -c_s + \frac{1}{2\pi^2} \left[ 1 - \frac{c_s}{C(p)} \right]. \quad (C7) $$

For the size $L$ dependent conductance one can express the relevant wavevector $p$ as $\eta/L$ for $\eta \sim 1$. This leads to the renormalization group equation for the size dependent conductance in the form

$$ \frac{dC}{d\ln(L)} = \beta(C), \quad \beta(C) = c_s - c_{\text{loc}} + \frac{c_sc_{\text{loc}}}{C} + O(C^{-2}), \quad c_{\text{loc}} = \frac{1}{2\pi^2}. \quad (C8) $$

This equation is given in the main text. Assuming that $C \gg c_{\text{loc}}$ we can ignore higher order terms. Then for $c_{\text{loc}} > c_s$ the steady state solution reads

$$ C = \frac{c_sc_{\text{loc}}}{c_{\text{loc}} - c_s}. \quad (C9) $$

It is a stable fixed point. This solution is applicable in the infinite size limit and for $c_{\text{loc}} - c_s \ll c_{\text{loc}}$ where higher order terms in $1/C$ can be neglected. In the opposite case $c_s > c_{\text{loc}}$ the solution approaches infinity for $L \to \infty$.

For the short-range hopping there is no contribution to the $\beta$-function in the two loop order. The logarithmic dependence of the zeroth order conductance Eq. (C2) can modify this result. Yet similarly to the case of the one loop order it is natural to expect that the associated corrections will be of higher order in $1/C$ so they can be neglected in Eq. (C8). The evaluation of the two loop order is beyond the scope of the present work, so we leave this expectation as a reasonable hypothesis that is qualitatively consistent with the numerical results, as it is pointed out in the main text.
2. Anisotropic conductance.

The renormalization group equation for the anisotropic conductance can be derived similarly to the isotropic regime. In the one loop order we got

\[
\frac{dC_x}{d\ln(L)} = c_x^* - c_{\text{loc}} \frac{C_x}{\sqrt{C_xC_y}} + \frac{2c_x^*}{\sqrt{C_x(\sqrt{C_x} + \sqrt{C_y})}} + \frac{C_yc_x^* - C_xc_y^*}{4\sqrt{C_xC_y}(\sqrt{C_x} + \sqrt{C_y})^2} + O(C^{-2}),
\]

\[
\frac{dC_y}{d\ln(L)} = c_y^* - c_{\text{loc}} \frac{C_y}{\sqrt{C_xC_y}} + \frac{2c_y^*}{\sqrt{C_y(\sqrt{C_x} + \sqrt{C_y})}} + \frac{C_xc_y^* - C_yc_x^*}{4\sqrt{C_xC_y}(\sqrt{C_x} + \sqrt{C_y})^2} + O(C^{-2}).
\]  

This equation has a stable fixed point at \(c_x^*c_y^* < c_{\text{loc}}^2\). In the infinite size limit the steady state solution for conductance at that point can be approximated by

\[
\left( \frac{C_x}{C_y} \right) = \frac{2c_{\text{loc}}^2 - c_x^2c_y^*}{c_{\text{loc}}^2 - c_x^2c_y^*} \left( \frac{c_y^*}{c_y} \right). \tag{C11}
\]

Conductance approaches infinity in the infinite size limit for \(c_x^*c_y^* \geq c_{\text{loc}}^2\). Consequently the transition to the superdiffusive regime is defined as

\[
c_{\text{loc}}^2 = c_x^*c_y^*. \tag{C12}
\]

This criterion is quoted within the main text where it is used to construct the phase diagram for the system with hopping defined by the the dipole-dipole interaction.

Appendix D: Connection of conductance and informational dimension.

In the present work we described conductance using the single parameter scaling equation in the form

\[
\frac{d\ln(C)}{d\ln(L)} = \frac{c_x - c_{\text{loc}}}{C} + \eta \frac{c_xc_{\text{loc}}}{C^2}, \tag{D1}
\]

and with the parameter \(\eta = 1\) (see Sec. C) that contrasts to the estimate \(\eta = 1/2\) in Ref. 28.

According to the Ref. [66] the informational dimension investigated within the main text is related to the conductance as

\[
D_1 = 2 - \frac{c_{\text{loc}}}{C}.
\]

Remember that the informational dimension can be expressed in terms of the average squared wavefunction as

\[
L^2 \langle |\psi|^2 \rangle \propto L^{-D(n)(n-1)},
\]

in the limit \(L \to \infty\), while the fractal dimension \(D(n)\) was expressed in Ref. [66] as \(D_1 = 2 - nc_{\text{loc}}/C\). Here and in the main text we use this expression in the limit \(n \to 1\).

In Fig. 6 we show the comparison of the numerical data and analytical theory for two choices of the parameter \(\eta\) using the best fit minimizing the deviation of analytical and numerical data sets choosing the optimum parameter \(L_0\) in the integral form of Eq. (D1)

\[
C + C_\infty \ln(1 - C/C_\infty) = (c_x - c_{\text{loc}}) \ln(L/L_0),
\]

where \(C_\infty = c_xc_{\text{loc}}/(c_{\text{loc}} - c_x)\). It is clear that in both cases theory does not provide the acceptable fit of the data. However, if we set \(D_1 = 2 - 1.3c_{\text{loc}}/C\), then we get an almost perfect agreement of numerical and analytical data as reported in the main text. The reasonable data fit can also be obtained using the definitions of the informational dimension \(D_1 = 2 - 1.4c_{\text{loc}}/C\) and \(D_1 = 2 - 1.5c_{\text{loc}}/C\). Yet in those cases an anomalously small fitting parameter \(L_0 < 0.1\) is required for the maximum disorder \((W = 3)\).
FIG. 6: Comparison of analytical and numerical estimates for the fractal dimension obtained comparing analytical result of Ref. [69] with conductance evaluated solving single parametric scaling equation following Ref. [28] (a) or the present work (b). In the latter case we always have $L_0 = 10$. The number of realizations is as in the main text, Fig. 3.