A Quantum Field Theory with Infinite Resonance States

G. Mussardo\textsuperscript{a,b} and S. Penati\textsuperscript{c,d}

\textsuperscript{a} Dipartimento di Fisica, Universitá dell’Insubria, Como
\textsuperscript{b} Istituto Nazionale di Fisica Nucleare, Sezione di Trieste
\textsuperscript{c} Dipartimento di Fisica, Universitá di Milano–Bicocca
via Emanuei 15, I–20126 Milano, Italy
\textsuperscript{d} Istituto Nazionale di Fisica Nucleare, Sezione di Milano

Abstract

We study an integrable quantum field theory of a single stable particle with an infinite number of resonance states. The exact $S$-matrix of the model is expressed in terms of Jacobian elliptic functions which encode the resonance poles inherently. In the limit $l \to 0$, with $l$ the modulus of the Jacobian elliptic function, it reduces to the Sinh–Gordon $S$-matrix. We address the problem of computing the Form Factors of the model by studying their monodromy and recursive equations. These equations turn out to possess infinitely many solutions for any given number of external particles. This infinite spectrum of solutions may be related to the irrational nature of the underlying Conformal Field Theory reached in the ultraviolet limit. We also discuss an elliptic version of the thermal massive Ising model which is obtained by a particular value of the coupling constant.
1 Introduction

The search for solvable but realistic models in theoretical physics has always resulted in valuable by-products. In this respect, the subject of two-dimensional relativistic Quantum Field Theories (QFT) has played a prominent role in the last decade, since numerous and important results have been achieved on non-perturbative aspects of strongly interacting systems. Remarkable applications to phenomena which occur in statistical or condensed matter systems where the dimensionality is effectively reduced have been made (see for instance [1, 2, 3]). In addition, the large variety of two-dimensional exactly solvable models – obtained both in the scale-invariant regime described by Conformal Field Theories or along the integrable directions which depart from them – have often been the theorist’s ideal playground for understanding (at least qualitatively) phenomena which take place in high-energy physics. The $S$-matrix approach proposed by Zamolodchikov [4] for studying the massive regime of integrable Renormalization Group flows has been particularly successful in answering important questions concerning the spectrum of QFT, the particle interactions and the behaviour of correlators of their local fields.

The aim of this paper is to address the analysis of one of the most striking aspects of scattering experiments, i.e. the occurrence of resonances. The physical effects induced by resonances have been the subject of some interesting publications in the past. Al. Zamolodchikov, for instance, has shown in [5] that a single resonance state can induce a remarkable pattern of roaming Renormalization Group trajectories which have the property to pass by very closely all minimal unitary models of Conformal Field Theory, finally ending in the massive phase of the Ising model. Another model, this time with an infinite number of resonance states, has also been considered by A. Zamolodchikov in relation with a QFT which is characterised by a dynamical $Z_4$ symmetry [6]: this is a theory with multi-channel scattering amplitudes which are forced by the Yang–Baxter equations to be expressed in terms
of Jacobian elliptic functions.

The model we propose in this paper is quite close in spirit to the latter one by A. Zamolodchikov but has the important advantage of being simpler and therefore amenable for a more detailed analysis of its off–shell properties. More specifically, we will consider an integrable QFT made of a stable, self–conjugated scalar excitation $A$ of mass $M$ but with an infinite series of resonance states which emerge as virtual unstable particles in the scattering amplitudes. As a function of the rapidity variable, its two–body $S$–matrix is expressed in terms of Jacobian elliptic functions, which present a periodicity along the real axis. This causes the appeareance of the infinite number of resonances. In ordinary four–dimensional QFT, resonances are responsible for sharp peaks observed in the total cross–section of the scattering processes as a function of the energy or equivalently in abrupt changes of the phase–shifts. In two–dimensional integrable QFT, on the other hand, production processes are forbidden and therefore the concept itself of “total cross section” is not particularly usueful. We will show, however, that resonances are associated to rapid jumps of the phase–shift.

The presence of an infinite number of resonance states deeply affects the physical properties of the system, the analytic structure of the $S$–matrix and in particular the short–distance behaviour of the correlators. This can been seen by studying the spectral series representations of the correlators, based on the Form Factors of the theory. Concerning the analysis of the Form Factors themselves, the main novelty consists in the infinite number of solutions of the Form Factors equations which can be found in this case. The infinite spectrum of operators defined by these solutions may be related to the irrational nature of the underlying Conformal Field Theory reached in the ultraviolet limit.

The paper is organised as follows: in sect. 2 we discuss the general properties of our model. Its $S$–matrix is the simplest example with an infinite series of resonance states. Since it reduces to the Sinh–Gordon $S$–matrix in the limit $l \to 0$ ($l$ is the
module of the Jacobi elliptic functions), the QFT defined by such an $S$–matrix will be referred to as Elliptic Sinh–Gordon model (ESG). In sect. 3 we briefly outline the main properties of the Form Factors of our relativistic integrable QFT and discuss the new features which arise in a theory with double periodicity. Section 4 is devoted to the analysis of the Form Factors of the Elliptic Ising Model, i.e. a model with infinite resonance states which reduces to the usual Ising model in the limit $l \to 0$. This model may be obtained by a particular analytic continuation of the coupling constant of the ESG model. In sect. 5 we address the computation of the Form Factors of the Elliptic Sinh–Gordon model: we present the computation of the minimal Form Factor $F(\beta)$ and investigate its analytic structure. By using the functional equation satisfied by $F(\beta)$, we derive the recursive equations of the Form Factors and obtain their first solutions. The main result of this section is that the solutions of the FF equations for a given number of external particles span an infinite dimensional vector space. In sect. 6 we present our conclusions. The paper also contains three appendices: appendix A presents useful mathematical identities used in the text, appendix B contains the derivation of the Fourier series of the $S$–matrix and finally appendix C presents the calculation of the minimal Form Factor of the Elliptic Sinh–Gordon model.

2 Elastic $S$-matrix with Resonance States

We consider a two–dimensional integrable QFT describing a stable, self–conjugated particle $A$ of mass $\mathcal{M}$. This theory is assumed to be invariant under a $Z_2$ symmetry realised by $A \to -A$. Its on–shell properties are encoded into its elastic $S$–matrix. In virtue of integrability, the $S$–matrix satisfies the factorization condition \cite{7}, therefore it is sufficient to focalise only on the two–body elastic scattering process. Let $s = (p_1 + p_2)^2$ be the Mandelstam variable of the scattering process and $S(s)$ the two–body elastic scattering amplitude of the particle $A$. The function $S(s)$ is usu-
ally expected to have two elastic branch–cuts along the real axis at the two–particle thresholds \( s \leq 0 \) and \( s \geq 4M^2 \). These branch–cuts can be unfolded as follows. Let \( \beta \) be the rapidity variable which parameterises the relativistic dispersion relations
\[
E = M \cosh \beta \quad ; \quad p = M \sinh \beta .
\] (2.1)
The Mandelstam variables \( s \) and \( t \) of the two–body scattering processes are then expressed by
\[
\begin{align*}
s &= (p_1 + p_2)^2 = 2M^2 [1 + \cosh \beta_{12}] ; \\
t &= (p_1 - p_2)^2 = 2M^2 [1 + \cosh (i\pi - \beta_{12})] .
\end{align*}
\] (2.2)
\((\beta_{12} = \beta_1 - \beta_2)\). Hence the upper and lower edges of the branch–cut which starts at \( s = 4M^2 \) are mapped onto the positive and negative real semi–axes of \( \beta \) respectively. The upper and lower edges of the other branch–cut are conversely mapped onto the positive and negative values of \( \beta \) along the line \( \text{Im} \beta = i\pi \). The \( S \)–matrix \( S(\beta_{12}) \) becomes a meromorphic function in the \( \beta_{12} \)–plane. It satisfies the unitarity and crossing symmetry conditions
\[
S(\beta)S(-\beta) = 1 ; \\
S(i\pi - \beta) = S(\beta) .
\] (2.3)
These equations automatically imply that \( S(\beta) \) is a periodic function along the imaginary axis of the rapidity variable, i.e.
\[
S(\beta + 2\pi i) = S(\beta) .
\] (2.4)
This periodicity simply expresses the double–sheet structure of the Riemann surface with respect to the Mandelstam variable \( s \) where the scattering amplitude is meromorphic. The physical sheet can be taken then as the strip \( 0 \leq \text{Im} \beta \leq \pi \) and the unphysical one as the strip \(-i\pi \leq \text{Im} \beta \leq 0\).

Let us now assume that \( S(\beta) \) also presents a periodic behaviour along the real axis of the rapidity variable \( \beta \) with a period \( T \), i.e.
\[
S(\beta) = S(\beta + T) .
\] (2.5)
This equation, combined with the analyticity properties of $S(\beta)$, has far-reaching consequences. In fact, the simultaneous validity of the two periodicity conditions (2.4) and (2.5) forces $S(\beta)$ to be an elliptic function. Accordingly, the plane of the rapidity variable $\beta$ becomes tiled in terms of the periodic cells of the function $S(\beta)$ (Figure 1.a). On the plane of the Mandelstam variable $s$ the $S$–matrix is not periodic. However, the existence of a double periodicity on the $\beta$–plane induces an infinite swapping between the two edges of the elastic branch–cuts on the $s$–plane.

The swaps are located at the infinite set of values $s_n = 2M^2(1 + \cosh(nT/2))$ and $s_n = 2M^2(1 + \cosh(i\pi - nT/2))$ for the $(4M^2, +\infty)$ and $(-\infty, 0)$ branch–cuts, respectively. In fact, take $(-\frac{T}{2}, \frac{T}{2})$ as fundamental interval of the real periodicity in $\beta$ and consider, for instance, the branch–cut $(4M^2, +\infty)$ in the variable $s$. At the initial location $s = 4M^2 (\beta = 0)$ the two edges join together. Starting from $\beta = 0$ and moving onto the negative values of $\beta$, we proceed along the lower edge of the branch cut, as far as $\beta > -\frac{T}{2}$. When this value is crossed and $\beta$ reaches the region $(-T, -\frac{T}{2})$, due to the identification $\beta \equiv \beta + T$, we are brought back to positive values of $\beta$, $0 < \beta < \frac{T}{2}$. On the other hand, this interval for $\beta$ parameterises the upper edge of the branch cut. Therefore, at $\beta = -\frac{T}{2}$ a first swap of the two edges of the branch cut has occurred. If now $\beta$ decreases, we keep moving along the upper edge as far as $\beta$ is greater than zero. When the rapidity crosses this value (now identified with $\beta = T$) a second swap occurs and the original configuration of the two edges is restored. The pattern described in details for $0 < \beta < T$, indeed reproduces itself with period $T$ in the rapidity variable. In particular, since the point $\beta = 0$ is identified with an infinite sequence of points $\beta_n = nT$, the two branches return to the initial joined configuration an infinite number of times. For the same reason, the swap at $\beta = -\frac{T}{2}$ is reproduced at every value $\beta_{2n+1} = (2n + 1)\frac{T}{2}$. The situation is completely analogous for the branch–cut of the $t$–channel which originally lies on the real negative semi–axis of the $s$ variable. Hence, altogether in the plane of the Mandelstam variable $s$ we have the analytic structure of the $S$–matrix shown in
Figure 1.b.

If $S(\beta)$ is an elliptic function, it must have poles and zeros, unless it is a constant. However, the existence of poles should not spoil the causality properties of the scattering theory. From a mathematical point of view, this is equivalent to requiring a further condition on the analytic structure of $S(\beta)$, namely that this function should not have poles with a real part within the physical sheet.

It is now easy to determine the simplest $S$–matrix which satisfies all of the above constraints, i.e. the unitarity and crossing equations (2.3), the periodicity equation (2.5) and the causality condition. It may be written as

$$S(\beta, a) = \frac{\text{sn} \left( \frac{K(\beta-i\pi a)}{i\pi} \right) \text{cn} \left( \frac{K(\beta+i\pi a)}{i\pi} \right) \text{dn} \left( \frac{K(\beta+i\pi a)}{i\pi} \right)}{\text{sn} \left( \frac{K(\beta+i\pi a)}{i\pi} \right) \text{cn} \left( \frac{K(\beta-i\pi a)}{i\pi} \right) \text{dn} \left( \frac{K(\beta-i\pi a)}{i\pi} \right)} ,$$

where $\text{sn}(x)$, $\text{cn}(x)$ and $\text{dn}(x)$ are the Jacobian elliptic functions and $K$ is the complete elliptic integral, both of modulus $l$ (see Appendix A for a review of the main properties of the Jacobian elliptic functions and some of their useful identities). By using the addition theorems (A.6), the above function can be equivalently expressed as

$$S(\beta, a) = \frac{\text{sn} \left( \frac{2Ki'\pi}{i\pi} \right) + \text{sn}(2Ka)}{\text{sn} \left( \frac{2Ki'\pi}{i\pi} \right) - \text{sn}(2Ka)} .$$

Exploiting some of the identities listed in Appendix A, it is easy to check the validity of eqs. (2.3). Moreover, the $S$–matrix satisfies the real periodicity condition (2.3) with period

$$T = \pi \frac{K'}{K} ,$$

where $K'$ is the complete elliptic integral of the the complementary modulus $l' = (1 - l^2)^{1/2}$. The parameter $a$ entering the expression (2.6) may be regarded as the coupling constant of the model. It must be selected so as not to introduce poles in the $S$–matrix which have a real part in the physical strip. As we will see below (eq. (2.11)), this is ensured by taking $a$ to be a real positive number in the interval $[0, 1]$. As a matter of fact, this interval can be further reduced in virtue of an additional...
property of the $S$–matrix \((2.7)\). In fact, the $S$–matrix \((2.7)\) is invariant under the replacement $a \rightarrow 1 - a$,

$$S(\beta, a) = S(\beta, 1 - a) ,$$

\((2.9)\)

so that $a$ can be restricted to the interval $[0, \frac{1}{2}]$. For $a = 0$ we have $S = 1$, i.e. a free theory, whereas the point $a = \frac{1}{2}$ may be regarded as the self–dual point of this scattering theory. In the following, unless explicitly stated, $a$ will be always taken as $0 \leq a \leq \frac{1}{2}$.

Notice that by taking the limit $l \rightarrow 0$, the real period $T$ goes to infinity and the $S$–matrix \((2.7)\) reduces to the one of the Sinh–Gordon model \[8\]

$$S_{ShG}(\beta, a) = \frac{\sinh \beta - i \sin \pi a}{\sinh \beta + i \sin \pi a} .$$

\((2.10)\)

For this reason, the QFT defined by the $S$–matrix \((2.7)\) may be simply referred to as the Elliptic Sinh–Gordon Model (ESG).

Let us consider the analytic structure of the $S$–matrix \((2.7)\), i.e. the pattern of its poles and zeros. Poles are either those of the numerator or come from the zeros of the denominator in \((2.7)\). Conversely, zeros are either those of the numerator or come from the poles of the denominator. Using the results listed in Table 1 and taking into account possible cancellations of poles against zeros, we are left with the following infinite set of simple poles and simple zeros

**Poles:**

\[ \beta_{m,n} = -i\pi a + 2m\pi i + nT ; \]

\[ \beta_{m,n} = i\pi a + (2m + 1)\pi i + nT \]

\((2.11)\)

**Zeros:**

\[ \beta_{m,n} = i\pi a + 2m\pi i + nT ; \]

\[ \beta_{m,n} = -i\pi a + (2m + 1)\pi i + nT \]

\((2.12)\)

where $m, n \in \mathbb{Z}$.

By choosing in the $\beta$ plane the rectangle with vertices $(i\pi - \frac{T}{2}, i\pi + \frac{T}{2}, -i\pi - \frac{T}{2}, -i\pi + \frac{T}{2})$ as a fundamental domain, the analytic structure of the $S$–matrix is shown in Figure 2, with two zeros and two poles along the imaginary axis and
then repeated on the other cells of the \( \beta \)-plane by periodicity. Hence, the simplest \( S \)-matrix with double periodicity is realised by an elliptic function of order two.

With the above choice of the values of \( a \), the poles of the scattering amplitude which have a real part in the \( \beta \)-plane all lie on the unphysical sheet. Instead in the physical sheet there are only zeros. The poles on the unphysical sheet with a real part correspond to a series of resonance states \( R_n \) with masses and decay widths given respectively by

\[
M_{\text{res}}(n) = 2M \cosh \left( \frac{nT}{2} \right) \cos \left( \frac{\pi a}{2T} \right) ;
\]

\[
\Gamma_{\text{res}}(n) = 2M \left| \sinh \left( \frac{nT}{2} \right) \sin \left( \frac{\pi a}{2T} \right) \right| .
\]

In the limit when the modulus \( \ell \) of the Jacobian elliptic function goes to zero (i.e. \( T \to \infty \)), the mass of all resonances becomes infinitely heavy and therefore these states completely decouple from the theory.

It is also useful to express the \( S \)-matrix (2.6) in an exponential form, a representation which will result particularly convenient for the computation of the Form Factors of the theory presented in the next sections. As shown in details in Appendix B, in the strip \( 0 \leq |\text{Im}\beta| < \pi a \) we have the following equivalent representation of the \( S \)-matrix of the ESG model

\[
S(\beta) = -\exp \left[ 2i \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\cosh \left( \frac{n\pi^2(1-2a)}{2T} \right)}{\cosh \left( \frac{n\pi^2}{2T} \right)} - (-1)^n \sin \left( \frac{2n\pi}{T} \beta \right) \right) \right] .
\]  

If we now use the Fourier series identity (A.13) in order to express the factor \(-1\) in front of (2.14) as

\[
-1 = \exp \left[ -4i \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \left( \frac{2\pi(2n-1)\beta}{T} \right) \right] ,
\]

the expression (2.14) can be equivalently written as

\[
S(\beta) = \exp \left[ -4i \sum_{n=1}^{\infty} \frac{1}{n} \frac{\sinh \left( \frac{n\pi^2}{T} \right) \sinh \left( \frac{n(1-a)\pi^2}{T} \right)}{\cosh \left( \frac{n\pi^2}{2T} \right) \sin \left( \frac{2n\pi}{T} \beta \right) } \right] .
\]
In the limit \( T \to \infty \), the sum in (2.16) can be converted into an integral and therefore one recovers the exponential representation of the Sinh–Gordon \( S \)–matrix

\[
S_{\text{ShG}}(\beta, a) = \exp \left[ -4i \int_0^\infty \frac{dx}{x} \frac{\sinh \left( \frac{x a}{2} \right) \sinh \left( \frac{x}{2} (1 - a) \right)}{\cosh \frac{x}{2}} \sin \left( \frac{x \beta}{\pi} \right) \right]. \quad (2.17)
\]

By using the expression (2.16) we can read the phase–shift \( \delta(\beta) \) of the model defined by

\[
S(\beta) = e^{2i \delta(\beta)} .
\]

(2.18)

The phase–shift is obviously a quantity which is defined modulus multiplies of \( \pi \). Therefore, by adding to \( \delta(\beta) \) a convenient multiplies of \( \pi \) we can always consider \( \delta(\beta) \) to be a monotonic function of \( \beta \). When \( \beta \) approaches the locations of the resonances \( \text{Re} \beta_n = nT \), the phase–shift \( \delta(\beta) \) sharply increases its value by \( \pi \) (see Figure 3).

A particularly simple but fascinating example of integrable theory with an infinite number of resonances is given by the \( S \)–matrix \( S = -1 \), when expressed in the form (2.15). The phase–shift of such a model can be taken as the staircase function with values given by all multiplies of \( \frac{\pi}{2} \) and jumps localised in correspondence to the points \( \beta_n = nT/2 \) (Figure 4). A way to understand this unexpectedly rich structure hidden in \( S = -1 \) is to notice that the scattering theory described by (2.15) can be obtained as a particular analytic continuation of the elliptic expression (2.7). Indeed, the \( S \)–matrix (2.7) approaches the value \(-1\) every time the parameter \( a \) takes one of the infinite values

\[
a_c = m + (2n + 1)i \frac{T}{2\pi} , \quad m \in \mathbb{N}, \; n \in \mathbb{Z} \quad (2.19)
\]

corresponding to the locations of the poles of \( \text{sn}(2K a) \). At these critical values of \( a \), the infinite number of poles and zeros of (2.7) cancel each other over the entire
β–plane, giving rise to the constant function $S = -1$. In Figure 5 this pattern is described in details for the simplest choice $a_c = -i \frac{T}{\pi}$. In this case, for a theory defined at an infinitesimal displacement $a = a_c + \delta$ from the critical value $a_c$, the location of poles and zeros is

\begin{align*}
\text{Poles:} & \quad \beta_{m,n}^{(-)} = (2n-1) \frac{T}{2} + 2m\pi i - i\pi \delta ; \\
& \quad \beta_{m,n}^{(+)} = (2n+1) \frac{T}{2} + (2m+1)\pi i + i\pi \delta , 
\end{align*}

(2.20)

\begin{align*}
\text{Zeros:} & \quad \beta_{m,n}^{(+)} = (2n+1) \frac{T}{2} + 2m\pi i + i\pi \delta ; \\
& \quad \beta_{m,n}^{(-)} = (2n-1) \frac{T}{2} + (2m+1)\pi i - i\pi \delta , 
\end{align*}

(2.21)

where $m, n \in \mathbb{Z}$. As shown in Figure 5, in the limit $\delta \to 0$ cancellations occur between $\beta_{m,n+1}^{(-)}$, $\beta_{m,n}^{(+)}$ poles and $\beta_{m,n}^{(+)}$, $\beta_{m,n+1}^{(-)}$ zeros. The $S$–matrix $S = -1$ defined in this limit inherits all the analytic structure of the ESG model. In particular, the infinite number of odd values $\beta_{2n+1} = (2n+1)T/2$ corresponding to rapid increasings of the phase–shift can be interpreted as associated to resonance states coming from the theory (2.7) defined at $a = a_c + \delta$. Near the values $\beta = i\pi + \beta_{2n+1}$ the $S$–matrix becomes

\begin{equation}
S(\beta) \simeq -\frac{i\pi \delta}{\beta - i\pi - \beta_{2n+1}} ,
\end{equation}

(2.22)

which corresponds to the diagram in Figure 6. In general, at the poles $\beta_{ij} = i\mu_{ij}$ corresponding to a bound state $A_k$ produced by the fusion of two particles $A_i$ and $A_j$, the $S$–matrix satisfies

\begin{equation}
-i \lim_{\beta \to i\mu_{ij}} (\beta - i\mu_{ij}) S(\beta) = (\Gamma_{ij}^{k})^2 ,
\end{equation}

(2.23)

where $\Gamma_{ij}^{k}$ is the three–particle vertex on mass–shell. Therefore, comparing eqs. (2.22) and (2.23), we can interpret the resonances $R_n$ as “bound states” of the particle $A$ but on the second sheet, with the three–particle vertex on mass–shell given by $\Gamma_{AA}^{R_n} = i\sqrt{\pi \delta}$. The imaginary value of the three–particle vertex indicates that the QFT presents some non–unitary features. As bound states of the $Z_2$ odd particle $A$, the resonance states $R_n$ are even under the $Z_2$ symmetry of the
model. Near the values $\beta = \beta_{2n+1}$ the $S$–matrix behaves as in eq. (2.22) but with an opposite sign for the residue. These poles correspond to resonance states in the crossed channel.

The even values $\beta_{2n} = nT$ corresponding to the other rapid increasing of the phase–shift have, on the other hand, a different physical interpretation. They do not come from genuine bound state resonances but simply correspond to “kinematic” resonances associated to the infinite replicate of the initial joint–point of the elastic branch–cuts on the $s$–plane (see Figure 1.b).

The QFT theory described by the $S$–matrix (2.15) will be referred to as the Elliptic Ising model (EIM). The detailed study of its operator content will be discussed in Section 4.

### 3 Form Factors in Integrable QFT

The on–shell dynamics of a system is encoded into its $S$–matrix. In order to go off–shell and compute the correlation functions of the model, one can use their spectral series representation. This means that we need to compute the matrix elements of the quantum fields between asymptotic states. It is sufficient to consider the following matrix elements, known as Form Factors (FF) [11, 12]

$$F_n^\mathcal{O}(\beta_1, \beta_2, \ldots, \beta_n) = \langle 0 \mid \mathcal{O}(0,0) \mid \beta_1, \beta_2, \ldots, \beta_n \rangle_{in} .$$  

(3.1)

In this section we will briefly outline the main properties of the FF in a generic two–dimensional integrable QFT and we will comment on some features which arise in the case of an $S$–matrix with a real period. A more detailed discussion of these features is postponed to the next sections. For simplicity we only consider the FF of scalar and hermitian operators $\mathcal{O}$.

The FF of a QFT are known to be severely constrained by the relativistic sym-
metries. First of all, they only depend on the difference of the rapidities $\beta_{ij}$

$$F_n^O(\beta_1, \beta_2, \ldots, \beta_n) = F_n^O(\beta_{12}, \beta_{13}, \ldots, \beta_{ij}, \ldots) , i < j . \quad (3.2)$$

Except for pole singularities due to the one-particle intermediate states in all sub-channels, the form factors $F_n$ are expected to be analytic functions inside the physical strip $0 < \text{Im} \beta_{ij} < \pi$. Moreover, they satisfy the monodromy equations [11, 12]

$$F_n^O(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n) = F_n^O(\beta_1, \ldots, \beta_{i+1}, \beta_i, \ldots, \beta_n)S(\beta_i - \beta_{i+1}) ; \quad (3.3)$$

$$F_n^O(\beta_1 + 2\pi i, \ldots, \beta_{n-1}, \beta_n) = e^{2\pi i \omega} F_n^O(\beta_2, \ldots, \beta_n, \beta_1) .$$

The parameter $\omega$ is the index of mutual locality of the operator $O$ with respect to the particle state $A$ ($\omega = 0, 1/2$ for local and semi-local operators, respectively).

In the simplest case $n = 2$, the above equations become

$$F_2^O(\beta) = F_2^O(-\beta)S(\beta) ,$$
$$F_2^O(i\pi - \beta) = e^{2\pi i \omega} F_2^O(i\pi + \beta) . \quad (3.4)$$

By exploiting the factorization properties of integrable QFT, the general solution of (3.3) can be found by making the ansatz

$$F_n^O(\beta_1, \ldots, \beta_n) = K_n(\beta_1, \ldots, \beta_n) \prod_{i<j} F_{\text{min}}(\beta_{ij}) , \quad (3.5)$$

where $F_{\text{min}}(\beta)$ is a solution of eqs. (3.4), analytic in $0 < \text{Im} \beta < \pi$. The remaining factors $K_n$ therefore satisfy the monodromy equations with $S = 1$, i.e. they are completely symmetric, $2\pi i$-periodic functions of the $\beta_i$. Moreover, they contain all the physical poles expected in the Form Factor under investigation.

In the case of a period $S$-matrix, the general solutions $F_n$ of the monodromy equations are not a-priori constrained to have any real periodicity in the rapidity variables. This can be easily seen from their parameterization (3.5): the monodromy equations with $S = 1$ which determine the functions $K_n$ do not contain any informations about the periodic nature of the dynamics under consideration. However,
for the two–particle FF the solutions of eqs. (3.4) may be chosen to satisfy the extra condition

$$F_2^O(\beta + T) = \pm F_2^O(\beta) . \tag{3.6}$$

In order to prove it, consider the first of eqs. (3.4) and make the substitution \( \beta \rightarrow \beta + T \). Since \( S(\beta) \) does not change under this shift in \( \beta \), if \( F_2(\beta) \) is a solution of eqs. (3.4), then \( F_2(\beta + T) \) is also a solution

$$F_2^O(\beta + T) = S(\beta)F_2^O(-\beta - T) . \tag{3.7}$$

By taking the ratio of eq. (3.7) and the first of (3.4) we have

$$\frac{F_2^O(\beta + T)}{F_2^O(\beta)} = \frac{F_2^O(-\beta - T)}{F_2^O(-\beta)} . \tag{3.8}$$

Therefore, if we assume \( F^O(\beta + T) \) to be proportional to \( F^O(\beta) \), i.e. \( F^O(\beta + T) = aF^O(\beta) \), then eq. (3.8) implies \( a^2 = 1 \), i.e. \( a = \pm 1 \). In particular, for \( a = -1 \) there is a doubling of the real periodicity in the two–particle Form Factor with respect to the \( S \)–matrix.

It is simple to obtain an explicit \( T \)–periodic solution of eqs. (3.4) in the case of a local operator \( (\omega = 0) \) when the \( T \)–periodic \( S \)–matrix is expressed in the form

$$S(\beta) = \exp \left[ -i \sum_{n=1}^{+\infty} \frac{a_n}{n} \sin \left( \frac{2n\pi}{T} \beta \right) \right] . \tag{3.9}$$

Denoting such a solution as \( F_{\min}(\beta) \), we have in fact

$$F_{\min}(\beta) = N \exp \left[ \sum_{n=1}^{+\infty} \frac{a_n}{n} \frac{1}{\sinh^2 \left( \frac{2n\pi}{T} \beta \right)} \sin \left( \frac{n\pi}{T} \beta \hat{\beta} \right) \right] , \tag{3.10}$$

where \( \hat{\beta} \equiv i\pi - \beta \) and \( N \) a normalization factor.

Let us consider now the pole singularities of the FF. They give rise to a set of recursive equations for the \( F_n \) which may be particularly important for their explicit determination. As functions of the rapidity differences \( \beta_{ij} \), the FF present in general two kinds of simple poles. The first family of singularities come from kinematical
poles located at \( \beta_{ij} = i\pi \). They correspond to the one-particle intermediate state in a subchannel of three-particle states which, in turn, is related to a crossing process of the elastic S-matrix. The corresponding residues give rise to a recursive equation between the \( n \)-particle and the \((n + 2)\)-particle Form Factors \[12\]

\[
-i\lim_{\beta \to \hat{\beta}}(\hat{\beta} - \beta)F_{n+2}^{O}(\hat{\beta} + i\pi, \beta, \beta_1, \beta_2, \ldots, \beta_n) = (1 - e^{2\pi i \omega} \prod_{i=1}^{n} S(\beta - \beta_i)) F_{n}^{O}(\beta_1, \ldots, \beta_n) .
\] (3.11)

For a theory with a real periodic S–matrix we still expect the appearance of kinematic poles located at \( \beta_{ij} = i\pi \) with residue given by the above equation, whose physical origin is the same as in the non–periodic case (see, for instance, the discussion in \[17\]). However, in this case the presence of a periodic S–matrix on the r.h.s. of eq. (3.11) gives additional constraints on the behaviour of the FF near the pole. Precisely, if we perform the shift \( \beta_i \to \beta_i + T, \ i = 1, \cdots, n \) in eq. (3.11), the r.h.s. is left invariant due to the \( T \)–periodicity of \( S \) and the translation invariance of \( F_n \) (the same conclusion is also obtained by shifting \( \hat{\beta} \to \hat{\beta} + T \) and \( \beta \to \beta + T \) on the l.h.s. of (3.11)). Hence, in this case the residue of \( F_{n+2}^{O} \) in \((\hat{\beta} - \beta) = i\pi \) has to be a \( T \)–periodic function of the \( n + 1 \) rapidities \( \beta, \beta_1, \cdots, \beta_n \). Although a sufficient condition to implement this requirement would be \( F_n(\beta_1, \ldots, \beta_n) \) periodic in all its variables, this is not a necessary condition as we will show in Sect. 5 by providing explicit examples.

In \( F_n \) there may also be another family of poles related to the presence of bound states. These poles are located at the values of \( \beta_{ij} \) where two particles fuse into a third one. Let \( \beta_{ij} = i u_{ij}^k \) be one of such poles associated to the bound state \( A_k \) in the channel \( A_i \times A_j \) and \( \Gamma_{ij}^k \) the three-particle vertex on mass-shell. In this case, the residue equation for the S–matrix is given in \(2.23\) and, correspondingly, for \( F_n \) we obtain \[12\]

\[
-i\lim_{\epsilon \to 0} \epsilon F_{n+1}^{O}(\beta + \epsilon \Pi_{ik}^j + \frac{\epsilon}{2} \beta - \epsilon \Pi_{jk}^i - \frac{\epsilon}{2}, \beta_1, \ldots, \beta_{n-1}) = \Gamma_{ij}^k F_n^{O}(\beta, \beta_1, \ldots, \beta_{n-1}) ,
\] (3.12)
where $\overline{u}_{ab}^c \equiv (\pi - u_{ab}^c)$. As discussed in sect.2, in the periodic case we cannot have genuine bound states. In fact, a bound state pole in the $S$–matrix at $\beta_{ij} = u_{ij}^k$ would be associated to an infinite chain of poles at $\beta_{ij} = u_{ij}^k + nT$, $n \in \mathbb{Z}$ with a consequent violation of causality. However, by considering the resonances as “bound states” on the second sheet, the above equation may be nevertheless used to extract some useful informations.

In general, the FF of real periodic models are then expected to be ruled only by the recursive equations coming from the kinematical singularities, eq. (3.11). In the particular case of the ESG model, this implies the complete decoupling between the two chains of FF for even or odd number of particles. This reflects the $Z_2$ symmetry of the model which can be used to classify its operators.

Finally, we remind that in the usual non–periodic scattering theories the FF may be further restricted by extra conditions related to their asymptotic behaviour and their cluster properties [10, 14, 15]. These conditions are often sufficient to determine uniquely the solutions of the recursive equations for each choice of the operator $O$. In the periodic case, as it will be clear later, we deal instead with periodic functions for which the lack of asymptotic extra conditions leads eventually to an unavoidable arbitrariness in the determination of the FF.

4 Form Factors of Ising Model Realizations

The QFT defined by the $S$–matrix $S = -1$ is usually identified with the thermal deformation of the critical Ising model with central charge $C = \frac{1}{2}$ [16]. In fact, the solution of the FF equations in the space of hyperbolic functions permits the full reconstruction of all correlators of this statistical model and a precise check of its ultraviolet limit [11, 17, 18]. In order to point out the important differences coming from an elliptic interpretation of this scattering theory, it is worth summarising first the main results for the correlation functions of the standard thermal Ising model.
4.1 The Standard Thermal Ising Model

With $S = -1$, the simplest solutions of eqs. (3.4) in the space of hyperbolic functions are given by

$$F_{\text{min}}(\beta) = \begin{cases} 
  f(\beta) = \sinh \frac{\beta}{2}, & \text{if } \omega = 0; \\
F(\beta) = \tanh \frac{\beta}{2}, & \text{if } \omega = \frac{1}{2}.
\end{cases}$$  \hspace{1cm} (4.1)

The kinematical recursive equations for the FF are particularly simple in this theory

$$-i \lim_{\tilde{\beta} \to \beta} (\tilde{\beta} - \beta) F_{n+2}^\sigma (\tilde{\beta} + i\pi, \beta, \beta_1, \beta_2, \ldots, \beta_n) = (1 - e^{2\pi i \omega} (-1)^n) F_n(\beta_1, \ldots, \beta_n).$$  \hspace{1cm} (4.2)

Hence, the FF of local operators ($\omega = 0$) do not have kinematical poles when $n$ is an even integer whereas their residue is independent of the number of external particles when $n$ is an odd integer. The situation is reversed for the FF of semi–local operators ($\omega = \frac{1}{2}$).

All fields are naturally divided into two classes. The first consists of $Z_2$ even, local or $Z_2$ odd, semi–local fields. Local operators of this class have non–zero FF only between states with a finite number of external particles. The first representative of this class is given by the trace of the stress–energy tensor $\Theta(x)$. This is a local field, proportional to the energy operator $\epsilon(x)$ of the thermal Ising model

$$\Theta(x) = 2\pi \tau \epsilon(x),$$  \hspace{1cm} (4.3)

where $\tau = (T - T_c)$ is the displacement of the temperature from its critical value, related to the mass of the model by $M = 2\pi \tau$. This operator has only a nonzero two–particle FF, given by

$$F^{\Theta}(\beta_1 - \beta_2) = 2\pi M^2 \frac{f(\beta_1 - \beta_2)}{f(i\pi)}. $$  \hspace{1cm} (4.4)

The second class consists of the $Z_2$ odd, local or $Z_2$ even, semi–local operators, as for instance the magnetization operator $\sigma(x)$ and the disorder operator $\mu(x)$. In the high–temperature phase of the model, $\sigma(x)$ is an odd, local operator. Its FF give rise to the infinite sequence $F_{2n+1}^\sigma$ on all odd numbers of external particles.
Taking into account the kinematical poles at $\beta_{ij} = i\pi$, the explicit expression of these FF is given by

$$F_{2n+1}^+ = H_{2n+1} \prod_{i<j}^{} \mathcal{F}(\beta_{ij}) \, .$$

(4.5)

$H_{2n+1}$ is a normalization constant, given by $H_{2n+1} = i^n H_1$, where $H_1$ is the one-point FF, $\langle 0|\sigma(0)|A \rangle = H_1$.

The disorder field $\mu(x)$, in the high-temperature phase, is an even semi-local field. Its FF extend on an infinite sequence $F_{2n}^\mu$ on all even numbers of asymptotic states and their expression is given by

$$F_{2n}^- = H_{2n} \prod_{i<j}^{} \mathcal{F}(\beta_{ij}) \, ,$$

(4.6)

with the normalization constant $H_{2n} = i^n H_0$ and $H_0$ the vacuum expectation value of this operator, $\langle 0|\mu(0)|0 \rangle = H_0$. Switching between the high and low temperature phases, the $Z_2$ odd and even fields $\sigma(x)$ and $\mu(x)$ simply swap their role.

The above explicit expressions of the FF allow to obtain exact informations on the thermal Ising model. We can compute, in particular, its correlation functions and related quantities. The simplest correlator is given by

$$G(x) = \langle \Theta(x)\Theta(0) \rangle \, ,$$

(4.7)

($x \equiv MR$). By inserting a complete set of intermediate states, it can be expressed as

$$G(x) = \frac{1}{2} \int \frac{d\beta_1}{2\pi} \frac{d\beta_2}{2\pi} \left| F^\Theta(\beta_1 - \beta_2) \right|^2 e^{-x(\cosh \beta_1 + \cosh \beta_2)} =$$

(4.8)

$$= \mathcal{M}^4 \left[ K^2_1(x) - K^2_0(x) \right] \, ,$$

where $K_i(x)$ are the ordinary Bessel functions. In the ultraviolet limit $x \to 0$, the above expression reduces to

$$G(x) \simeq \frac{\mathcal{M}^2}{R^2} \, ,$$

(4.9)

from which we can read the anomalous dimension of the energy operator in its conformal limit, $\eta_e = 1$. The knowledge of $G(x)$ also allows to determine the
central charge of the conformal field theory which arises in its ultraviolet limit. To
this aim, we can use the $c$–theorem sum rule \[20, 21\]
\[
C_{uv} = \frac{3}{4} \int d^2 x \, |x|^2 \langle \Theta(x)\Theta(0) \rangle .
\] (4.10)
By inserting the explicit expression (4.8) of $G(x)$, the above formula gives $C_{uv} = \frac{1}{2}$.

The same set of conformal data can also be extracted by looking at the ground–
state energy of the theory on an infinite cylinder of width $R$. The exact expression
is obtained by means of the Thermodynamic Bethe Ansatz (TBA) \[22\] and for the
effective central charge
\[
\tilde{C} \equiv C - 24\Delta_{\text{min}} ,
\] (4.11)
we have\[23\]
\[
\tilde{C}(x) = \frac{1}{2} - \frac{3x^2}{2\pi^2} \left[ \ln \frac{1}{x} + \frac{1}{2} + \ln \pi - \gamma_E \right.
\]
\[
- 4 \sum_{n=1}^{\infty} \left( \frac{1}{n+1} \right) (1 - 2^{-2n-1})\xi(2n + 1) \left( \frac{x^2}{\pi^2} \right)^n \right] .
\] (4.12)
The power series of the above expression should match with its evaluation in terms
of Conformal Perturbation Theory given by
\[
\tilde{C}_{\text{per}}(x) = \sum_{n=1}^{\infty} c_n \left( \tau R^2 - \eta \right)^n ,
\] (4.13)
where
\[
c_n = 12 \frac{(-1)^n}{n!} R^{2(1-n)+n\eta} \int_{\text{cyl}} \langle \langle \Delta_{\text{min}} | \epsilon(0) \prod_{j=1}^{n-1} \epsilon(w_j) d^2 w_j | \Delta_{\text{min}} \rangle \rangle_{\text{conn}} ,
\] (4.14)
and $\langle \langle ... \rangle \rangle$ denotes the conformal correlators on the cylinder.

Since for the critical Ising model $\Delta_{\text{min}} = 0$ and the energy operator $\epsilon(r)$ is odd
under duality, we have $c_n = 0$ for odd $n$. Hence, eq. (4.12) fixes the central charge
to be $C_{uv} = \frac{1}{2}$. Moreover, from the comparison of the power series in (4.12) with the

\*In this equation $\gamma_E$ is the Euler-Mascheroni constant and $\xi(s)$ the Riemann zeta function.
perturbative expansion (4.13), we have $\eta = 1$, in agreement with the determination of this quantity obtained by the short-distance limit of the correlator (4.8).

Let us now consider the correlation functions of the $Z_2$ odd, local and even, semi-local fields,

$$G_-(x) = \langle \sigma(R)\sigma(0) \rangle ;$$

$$G_+(x) = \langle \mu(R)\mu(0) \rangle ,$$

(4.15)
given by

$$G_\pm(x) = \sum_n \frac{1}{n!} \int \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_n}{2\pi} |F_n^\pm|^2 e^{-x \sum_i \cosh \beta_i} ,$$

(4.16)
where the sum runs on odd numbers for $G_+(x)$ and on even ones for $G_-(x)$. As shown in [17, 19], eq. (4.16) provides an integral representation of a particular solution of the Painleve’ equation. An exact analysis of this solution allows, in particular, to study its short-distance limit $x \to 0$ with the result

$$G_\pm(x) \simeq \frac{\text{const}}{R^{1\over 4}} .$$

(4.17)
The above expression determines the anomalous dimensions of the $Z_2$ odd fields to be $\eta_\sigma = \eta_\mu = {1 \over 8}$.

The anomalous dimension of these fields can be equivalently extracted by means of the $\eta$ sum-rule [14]

$$\eta_\Phi = -\frac{1}{2\pi \langle \Phi \rangle} \int d^2x \langle \Theta(x)\Phi(0) \rangle .$$

(4.18)
This formula can be applied in the high-temperature phase to obtain the anomalous dimension of the disorder field and in the low-temperature to obtain the anomalous dimension of the magnetization operator. By inserting a complete set of states between the two operators and noticing that the FF of $\Theta(x)$ is different from zero only on the two-particle state, we have

$$\eta_\mu = \eta_\sigma = \frac{1}{2\pi} \int_0^\infty {d\beta \over \cosh^2 \beta} f(2\beta)F(2\beta) = {1 \over 8} .$$

(4.19)
It is worth mentioning that the anomalous dimension of these fields can be also obtained by exploiting the formal analogy of expression (4.16) with a grand–canonical partition function $\Xi(z, L)$ of a fictitious one–dimensional gas in a box of length $L \sim \log \frac{2}{MR}$ and fugacity $z = \frac{1}{2\pi}$ [17, 18]. Hence, by the state equation of this gas

$$\Xi(z, L) = e^{p(z)L} \simeq \left( \frac{1}{MR} \right)^{p(z)},$$

(4.20)

the anomalous dimension is nothing but its pressure $p(z)$ at $z = \frac{1}{2\pi}$. In the nearest neighborhood approximation of $\Xi(z, L)$, the pressure $p$ can be obtained as a solution of the integral equation

$$2\pi = \int_{0}^{+\infty} dz e^{-p z} |\mathcal{F}_2(z)|^2.$$  

(4.21)

In the thermal Ising model, the above formula provides an excellent approximation of $\eta$, i.e. $\eta_\sigma \sim 0.12529$ [18].

### 4.2 Form Factors for the Elliptic Ising Model

Let us discuss now the Form Factors of the theory described by the $S$–matrix $S = -1$ but this time regarded as a periodic function of $\beta$. As it is clear from eq. (A.13), there is a complete arbitrariness in the choice of the periodic realization of $-1$. Instead of working with the $S$–matrix (2.15) of period $T$, it is more convenient to consider the following $2T$ periodic realization

$$-1 = \exp \left[ -4i \sum_{n=1}^{+\infty} \frac{1}{2n-1} \sin \left( \frac{\pi(2n-1)\beta}{T} \right) \right],$$

(4.22)

which leads to a simplification in the expressions of the Form Factors of the model, without altering its physical content [1].

\[\text{† In fact, one can always convert the expressions of the FF relative to the period } 2T \text{ to those relative to period } T \text{ by multiplying them for even, } 2\pi i \text{ periodic functions which enter the general solution of (4.23). For instance, the function } \text{sn} \left( \frac{iK}{\beta} \right), \text{ solution of (4.23) with period } 2T, \text{ can be}\]
In what follows we will not address the problem of finding the most general solutions of all FF equations relative to this theory. We will rather concentrate our attention on particular classes of solutions which have the minimal analytic structure compatible with all the constraints.

Let us consider first the minimal two–particle Form Factors, solutions of the equations

\[
F(\beta) = -F(-\beta); \\
F(i\pi - \beta) = e^{2\pi i \omega} F(i\pi + \beta). \tag{4.23}
\]

They are determined up to arbitrary even, \(2\pi i\)-periodic and \(2T\) (anti)–periodic, analytic functions of \(\beta\). This time we are looking for the simplest solution of these equations in the space of elliptic functions. Using the properties of the Jacobian elliptic functions listed in Appendix A, it is easy to see that they are given by

\[
F_{\text{min}}(\beta) = \begin{cases} 
    f(\beta) = -i \frac{\text{sn}(iK' T \beta)}{\text{dn}(iK' T \beta)}, & \text{if } \omega = 0; \\
    F(\beta) = -i \frac{\text{sn}(iK' T \beta)}{\text{cn}(iK' T \beta)}, & \text{if } \omega = \frac{1}{2}.
\end{cases} \tag{4.24}
\]

These expressions represent the simplest generalization of the standard results (4.1) with no poles on the real axis and which accounts for the real periodicity of the system under consideration, with the property \(F_{\text{min}}(\beta + 2T) = -F_{\text{min}}(\beta)\). Their analytic structure consists of the following set of simple zeros and poles:

\[\text{transformed by means of the even, } 2\pi i \text{ periodic function } \left[ i \text{sn}^2 \left( i \frac{K'}{T} \beta \right) + 1 \right] \text{ into the } T\text{–periodic solution } \frac{\text{sn}(iK' T \beta)}{i \text{sn}^2 \left( i \frac{K'}{T} \beta \right) + 1} \text{ of the same equation.}\]
\[ f(\beta) \rightarrow \begin{cases} 
\text{Zeros:} & \beta_{m,n} = 2m\pi i + 2nT \\
\text{Poles:} & \beta_{m,n} = (2m+1)\pi i + (2n+1)T 
\end{cases} \quad (4.25) \]

\[ \mathcal{F}(\beta) \rightarrow \begin{cases} 
\text{Zeros:} & \beta_{m,n} = 2m\pi i + 2nT \\
\text{Poles:} & \beta_{m,n} = (2m+1)\pi i + 2nT 
\end{cases} \]

with \( m, n \in \mathbb{Z} \). Notice that \( f(\beta) \) has an infinite sequence of poles at \( \beta = i\pi + (2n+1)T \), located just at the edge of the second sheet. They correspond to the infinite sequence of the one–particle intermediate states given by the resonances \( R_n \) (see the discussion at the end of Section 2). On the contrary, the poles of \( \mathcal{F}(\beta) \) located at \( \beta = i\pi + 2nT \) do not signal the presence of resonances but are simply related to the infinitely repeated pattern of the branch cut of the \( t \)–channel of the \( S \)–matrix.

Concerning the recursive equations satisfied by the FF, they are given by the same equation \((1.2)\) and therefore, as before, the operators fall into two different sectors.

Let us consider first the Form Factors of the local \( Z_2 \) even or semi–local odd operators, i.e. the ones which do not have poles at \( \beta_{ij} = i\pi \). Their general expression can be written in terms of the function \( f(\beta) \) and they are nonzero only on a finite number of external states. Let us consider, in particular, the FF of the trace operator. Even in the periodic case we assume this operator to be related to the energy field by eq. \((4.3)\). Its only non–vanishing FF is the one on the two–particle state which we assume to be

\[ F^\Theta(\beta_1 - \beta_2) \equiv \langle 0|\Theta(0)|\beta_1\beta_2 \rangle = 2\pi \mathcal{M}^2 \frac{f(\beta_1 - \beta_2)}{f(i\pi)} . \quad (4.26) \]

For the two–point function of this operator we have

\[ G(x) = \langle \Theta(R)\Theta(0) \rangle = \frac{1}{2} \int \frac{d\beta_1}{2\pi} \frac{d\beta_2}{2\pi} \left| F^\Theta(\beta_1 - \beta_2) \right|^2 e^{-x(\cosh \beta_1 + \cosh \beta_2)} . \quad (4.27) \]
By using the variables $y_{\pm} = \frac{1}{2}(\beta_1 \pm \beta_2)$, it can be written as

$$G(x) = \frac{4M^4}{|f(i\pi)|^2} \int_0^{+\infty} dy_- |f(2y_-)|^2 K_0(2x \cosh y_-) .$$  \hspace{1cm} (4.28)

Plots of this correlation function for different values of the modulus $l$ are shown in Figure 7.

Let us discuss in more detail the large and the short distances behaviour of this correlator. As evident from Figure 7, the large distance behaviour of the correlator (4.28) is essentially the same for all value of the period $T$ and is ruled by the mass gap $M$ of the stable particle. This is confirmed by the analytic estimate of $G(x)$ for large values of $x$: the only parameter which contains informations on the periodic structure of the theory (in terms of parameters of the elliptic functions) is the normalization in front of the exponential decay

$$G(x) \simeq \frac{2}{\pi} (M^2 l' K)^2 \frac{e^{-2x}}{x^2} + \mathcal{O}(e^{-4x}) .$$  \hspace{1cm} (4.29)

The situation is drastically different in the ultraviolet regime. While in the non–periodic case ($l = 0$) the two–particle FF is expressed by the unbounded function $\sinh \frac{\beta}{2}$, for $l \neq 0$ it is instead a periodic, regular and limited function on the real axis which can always be bounded by a constant $B$, leading to

$$G(x) \leq 4M^4 B^2 \int_0^{+\infty} dy K_0(2x \cosh y) = 2 (M^2 B)^2 K_0^2(x) .$$  \hspace{1cm} (4.30)

This inequality implies that for $l \neq 0$ $G(x)$ cannot have a power law singularity in its ultraviolet limit as, instead, it happens in the standard Ising Model (see eq. (4.9)). Hence the conclusion is simply that the anomalous dimension of the energy operator changes discontinously from the value $\eta_{\epsilon} = 1$ (for $l = 0$) to $\eta_{\epsilon} = 0$ for all other finite values of the module!

This result seems to give rise to an apparent paradox. In fact, one could argue that the anomalous dimension of the energy operator can be equivalently extracted

\footnote{To obtain this result we have used eq.(6.663.1) of [24].}
in terms of the TBA equations by looking at the perturbative part of the ground state energy. Since this quantity is still given by the same eq. (4.12), it appears that from the TBA approach one would rather reach the previous conclusion, i.e. \( \eta = 1 \). How can we conciliate this apparent mismatch of the values of \( \eta \) obtained by the two different methods?

The solution to this puzzle is the following. First of all, notice that the TBA does not compute the central charge of the theory but rather provides the evaluation of the effective central charge of the theory, \( \tilde{C} = C - 24\Delta_{\text{min}} \). To compute directly the central charge, we have instead to employ the \( c \)-theorem sum–rule (4.10). The result for the central charge of the Elliptic Ising Model is reported in Figure 8. This figure shows that, by varying the period, the central charge of the model is no longer \( C = \frac{1}{2} \) but, on the contrary, takes any value between 0 and \( \frac{1}{2} \), depending on the module \( l \). On the other hand, since the effective central charge \( \tilde{C} \) is always equal to \( \frac{1}{2} \) for any value of the module \( l \), there should be in the periodic realization of the Ising model an operator with conformal weight \( \Delta_{\text{min}} < 0 \). Hence, the underlying conformal theory of the Elliptic Ising Model is no longer unitary (and rational) and consequently we have no reasons to argue that the perturbative coefficients \( c_n \) vanish for \( n \) odd, as in the Ising Model. Therefore, the matching of the Conformal Perturbation Theory in the presence of a field with \( \Delta_{\text{min}} \neq 0 \) with the corresponding perturbative part of (4.12) gives \( \eta = 0 \), in agreement with the determination done by looking at the ultraviolet limit of the correlator. For the limiting value \( l = 1 \), we have \( \Delta_{\text{min}} = -\frac{1}{48} \).

As already discussed, the FF of even, local operators inherit from the \( S \)-matrix the pole structure associated to the resonance states. These states are even under the \( Z_2 \) symmetry of the model and therefore they couple directly to the operator \( \Theta \). We can extract the matrix elements \( \langle 0|\Theta(0)|R_n \rangle \) by formally applying the bound state residue equation (3.12), with \( \Gamma_{AA}^{R_n} = i\sqrt{\pi\delta} \ (\delta \to 0) \). As a result we have

\[
(-1)^n \frac{2\pi^2 M^2}{lK} = \sqrt{\pi\delta} \langle 0|\Theta(0)|R_n \rangle .
\]

(4.31)
Hence, $\langle 0 | \Theta(0) | R_n \rangle$ diverges as $1/\sqrt{\delta}$ but its product with the three-particle vertex of the resonances is constant in the limit $\delta \to 0$.

Let us consider now the FF of the $Z_2$ odd, local and even, semi–local fields, concentrating our attention on the magnetization and disorder operators. The magnetization $\sigma(x)$ is local and has FF on all odd numbers of external particles. A close solution of the recursive equations (4.2) is given by

$$F_{2n+1}^+ = \left( \frac{2Ki}{\pi} \right)^n \left( l' \right)^{n(n+1)} H_1 \prod_{i<j} F(\beta_{ij}) ,$$

where $H_1 = \langle 0 | \sigma(0) | A \rangle$.

The disorder operator $\mu(x)$ is semi–local and has FF on all even numbers of external particles

$$F_{2n}^- = \left( \frac{2Ki}{\pi} \right)^n \left( l' \right)^{n^2} H_0 \prod_{i<j} F(\beta_{ij}) ,$$

where $H_0 = \langle 0 | \mu(0) | 0 \rangle$. All the FF (4.32) and (4.33) possess the kinematic pole at $\beta_{ij} = i\pi$ as well as at their periodic repeated positions $\beta_{ij}^{(n)} = i\pi + 2nT$.

For the two–point correlation functions of these fields we have

$$G_-(x) = \langle \sigma(R)\sigma(0) \rangle ;$$

$$G_+(x) = \langle \mu(R)\mu(0) \rangle ,$$

with

$$G_{\pm}(x) = \sum_n \frac{1}{n!} \int \frac{d\beta_1}{2\pi} \ldots \frac{d\beta_n}{2\pi} |F_n^\pm|^2 e^{-x \sum \cosh \beta_i} ,$$

where the sum runs on odd numbers for $G_+(x)$ and on even ones for $G_-(x)$.

As for the energy field, at large distances these correlation functions have an exponential decay, ruled by the mass gap $\mathcal{M}$ of the stable part of the spectrum

$$G_-(x) \approx H_1^2 \sqrt{\frac{1}{2\pi x}} e^{-x} + \mathcal{O}(e^{-3x}) ,$$

$$G_+(x) \approx \frac{1}{\pi^5} (l' K^2 H_0)^2 \frac{e^{-2x}}{x^2} + \mathcal{O}(e^{-4x}) .$$
The only informations on the periodic structure of the theory may enter the normalization in front of the exponentials. Let us now analyse the short-distance behaviour of $\mathcal{G}_\pm(x)$. In absence of an exact resummation of the series (4.35), we have to rely on other methods for computing the anomalous dimensions of the corresponding fields. It is easy to see that, in contradistinction to the anomalous dimension $\eta_\epsilon$ of the energy operator which jumps discontinuously from 1 to 0 by introducing a finite period $T$, the anomalous dimension $\eta_\sigma$ (or equivalently $\eta_\mu$) depends continuously on $T$. In fact, in the ultraviolet limit we can apply the thermodynamics approach as in eq. (4.20) and write the following representation for the correlator

$$\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_- \simeq \left(\frac{1}{\mathcal{M} R}\right)^{p(l)},$$

where the pressure of the one-dimensional fictitious gas is given in eq. (4.21). Since in this case $|\mathcal{F}(\beta)|^2$ is a regular and bounded function along the real axis for any value of the module $l$, it follows that $p(l)$ is a continuous function of the module. Another way to reach the same conclusions is to analyse the exact expression for the anomalous dimensions as obtained by the $\eta$ sum-rule

$$\eta_\sigma(l) = \eta_\mu(l) = \frac{1}{2\pi} \int_0^\infty \frac{d\beta}{\cosh^2 \beta} \frac{F^{-}_\gamma(2\beta)}{H_0} \frac{f(2\beta)}{f(i\pi)}.$$

The relative graph of this function is reported in Figure 9. The curve starts from $\eta = \frac{1}{8}$ at $l = 0$ and goes then continuously to zero at $l = 1$.

5 Form Factors for the Elliptic Sinh–Gordon Model

Let us now consider the quantum field theory described by the elliptic $S$–matrix (2.6) and address in its full generality the problem of computing exactly its FF. We will see that the FF equations admit in this case a large class of solutions which for $l \to 0$ all collapse to the same solutions of the Sinh–Gordon model. As a matter of fact, due to the absence of constraints on the asymptotic behaviour of FF, the final set of solutions has indeed infinite dimension.
We concentrate only on the case of FF of local operators. As usual, the first step consists in the evaluation of the two–particle Form Factor. For any local operator, this is a solution of the monodromy eqs. (3.4) with $\omega = 0$. Using the exponential representation (2.16) for the $S$–matrix and the formula (3.10) we have

$$F_{\text{min}}(\beta) = \mathcal{N} \exp \left[ 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{\sinh \left( \frac{n \pi^2}{T} \right) \sinh \left( \frac{n(1-a)\pi^2}{T} \right)}{\cosh \left( \frac{n \pi^2}{T} \right) \sinh \left( \frac{2n \pi^2}{T} \right)} \sin^2 \left( n \frac{\pi}{T} \hat{\beta} \right) \right],$$

(5.1)

where $\hat{\beta} \equiv i\pi - \beta$ and $\mathcal{N}$ is a normalization constant. This solution satisfies the periodicity condition $F_{\text{min}}(\beta + T) = F_{\text{min}}(\beta)$. The previous expression for $F_{\text{min}}$ can be further elaborated in order to get a final form, more suitable for the determination of its analytic structure and for explicit calculations. As described in details in Appendix C, it can be rewritten, up to a constant, as an infinite product of trigonometric functions

$$F_{\text{min}}(\beta) = \prod_{k=2}^{\infty} (W(\beta, k))^{k-1},$$

(5.2)

where

$$W(\beta, k) = \left( \frac{1 - 2q_6 \cos \left( \frac{2\pi}{T} \hat{\beta} \right) + q_6^2}{1 - 2q_1 \cos \left( \frac{2\pi}{T} \hat{\beta} \right) + q_1^2} \right) \left( \frac{1 - 2q_5 \cos \left( \frac{2\pi}{T} \hat{\beta} \right) + q_5^2}{1 - 2q_2 \cos \left( \frac{2\pi}{T} \hat{\beta} \right) + q_2^2} \right) \left( \frac{1 - 2q_4 \cos \left( \frac{2\pi}{T} \hat{\beta} \right) + q_4^2}{1 - 2q_3 \cos \left( \frac{2\pi}{T} \hat{\beta} \right) + q_3^2} \right)$$

and $q_i = q_i(k), i = 1, \cdots, 6$ are the factors listed in eq. (C.7). Using the identity (A.18) we can also express the previous result as an infinite product of gamma functions

$$F_{\text{min}}(\beta) = \prod_{n=-\infty}^{+\infty} \prod_{k=0}^{+\infty} Y_{k,n}(\beta),$$

(5.3)

where

$$Y_{k,n}(\beta) = \left| \frac{\Gamma \left( k + \frac{3}{2} + \frac{i\hat{\beta}_n}{2\pi} \right) \Gamma \left( k + \frac{1}{2} + \frac{a}{2} + \frac{i\hat{\beta}_n}{2\pi} \right) \Gamma \left( k + 1 - \frac{a}{2} + \frac{i\hat{\beta}_n}{2\pi} \right)}{\Gamma \left( k + \frac{1}{2} + \frac{i\beta_n}{2\pi} \right) \Gamma \left( k + \frac{3}{2} - \frac{a}{2} + \frac{i\beta_n}{2\pi} \right) \Gamma \left( k + 1 + \frac{a}{2} + \frac{i\beta_n}{2\pi} \right)} \right|^2,$$

and $\hat{\beta}_n = \hat{\beta} + nT$. 

27
The solution (5.2) is analytic in the $\beta$-plane except for an infinite set of poles (see Appendix C for details)

\[ \beta_n(k) = -i\pi a + (2k-1)i\pi + nT ; \]
\[ \beta_n(k) = -i\pi a + (2-k)2i\pi + nT ; \]
\[ \beta_n(k) = i\pi a + (3-2k)i\pi + nT ; \]
\[ \beta_n(k) = i\pi a + (k-1)2i\pi + nT ; \]
\[ \beta_n(k) = 2ki\pi + nT ; \]
\[ \beta_n(k) = (1-k)2i\pi + nT , \]

and it vanishes at the following locations

\[ \beta_n(k) = -i\pi a + 2ki\pi + nT ; \]
\[ \beta_n(k) = -i\pi a + (3-2k)i\pi + nT ; \]
\[ \beta_n(k) = i\pi a + (1-k)2i\pi + nT ; \]
\[ \beta_n(k) = i\pi a + (2k-1)i\pi + nT ; \]
\[ \beta_n(k) = (k-1)2i\pi + nT ; \]
\[ \beta_n(k) = (2-k)2i\pi + nT . \]

For given $k$, poles and zeros have multiplicity $(k-1)$ and they are drawn in Figure 10. The minimal solution $F_{\text{min}}$ does not have singularities in the physical sheet but only single zeros in $\beta = nT$. The poles at $\beta = -i\pi a + nT$ on the unphysical sheet of $F_{\text{min}}(\beta)$ may be regarded as “bound state” poles of the FF due to the resonances.

The solution (5.2) satisfies the functional relation

\[ F_{\text{min}}(\beta + i\pi) F_{\text{min}}(\beta) = \frac{\text{sn} \left( 2i\frac{K}{T} \beta \right)}{\text{sn} \left( 2i\frac{K}{T} \beta \right) - \text{sn} \left( 2K a \right)} . \] (5.6)

This can be easily proven by studying the distribution of poles and zeros for the product on the l.h.s., as shown in Figure 11. When $F_{\text{min}}(\beta)$ is multiplied by the same function evaluated at the shifted point $(\beta + i\pi)$ a partial cancellation of poles against zeros occurs, leaving only a distribution of simple poles and zeros (third line of Figure 11). This distribution indeed corresponds to the pole singularities and
zeros of the elliptic function on the r.h.s. of eq. (5.6). Notice that we have used the functional equation (5.6) as it stands in order to fix a convenient normalization $N$ of $F_{\text{min}}(\beta)$.

In order to simplify the following equations, it is convenient to introduce the notation

$$u \equiv \frac{iK'}{T} \beta, \quad u_{ij} \equiv \frac{iK'}{T} \beta_{ij}.$$  \hspace{1cm} (5.7)

The general structure of $n$–particle Form Factors is given in eq. (3.5) in terms of the minimal solution (5.2). Since the theory has no bound states, the only poles present are the kinematic poles associated to every three–particle subchannels ($\beta_{ij} = i\pi$). This suggests to parameterise the functions $K_n$ entering the general expression of the FF (3.5) as follows

$$K_n(\beta_1, \cdots, \beta_n) = H_n Q_n(\beta_1, \cdots, \beta_n) \prod_{i<j} \frac{1}{\text{cn} u_{ij}},$$  \hspace{1cm} (5.8)

where $H_n$ are normalization constants. Notice that in the above expression the presence of the kinematic pole has been made explicit by the introduction of the periodic elliptic functions $\text{cn} u_{ij}$. The choice of this parameterization is suggested by the general requirement that the residue of $F_n$ at the kinematic pole has to be a periodic function of the other $(n - 2)$ rapidity variables (see the general discussion in Sect. 3). Therefore, the kinematic pole at $\beta_{ij} = i\pi$ appears together with an infinite sequence of poles located at $\beta_{ij} = i\pi + 2nT$. Even though the function $\text{cn} u$ is $2T$ periodic, the actual $T$–periodicity of the residue of $F_n(\beta_1, \cdots, \beta_n)$ at $\beta_{ij} = i\pi$ will be restored by the consequent expressions of $Q_n(\beta_1, \cdots, \beta_n)$ obtained by using this parameterization.

The function $\text{cn}[u(\beta)]$ satisfies the identity

$$\text{cn}[u(\beta + i\pi)] \text{cn}[u(\beta)] = l' \frac{\text{sn} u \text{cn} u}{\text{dn} u}. \hspace{1cm} (5.9)$$

The terms $Q_n$ are symmetric, $2\pi i$ periodic functions of the $\beta_i$'s, which have to be determined by the recursive equations (3.11) satisfied by the FF. By using the
expression (2.7) for the $S$–matrix and exploiting the functional relations (5.6) and (5.9) together with eq. (A.10), we obtain the following recursive equations for the functions $Q_n$

$$Q_{n+2}(\beta + i\pi, \beta, \beta_1, \cdots, \beta_n) = D_n(\beta \mid \beta_1, \cdots, \beta_n) Q_n(\beta_1, \cdots, \beta_n)$$  \hspace{1cm} (5.10)

where

$$D_n(\beta \mid \beta_1, \cdots, \beta_n) = \frac{(-2i)^n}{2i \sin(2K\alpha)} \times \frac{\prod_{i=1}^n [\sin(2(u - u_i)] - \sin(2K\alpha)] - \prod_{i=1}^n [\sin(2(u - u_i)] + \sin(2K\alpha)]}{\prod_{i=1}^n [1 + \sin(2(u - u_i)]}.$$  \hspace{1cm} (5.11)

Concerning the normalization constants $H_n$, they have been conveniently chosen as

$$H_{2n} = \left(\frac{i l'}{2}\right)^{n(n-1)} \left(\frac{-2i l' K \sin(2K\alpha)}{\pi F_{\min}(i\pi)}\right)^n H_0,$$  \hspace{1cm} (5.12)

$$H_{2n+1} = \left(\frac{i l'}{2}\right)^{n^2} \left(\frac{-2i l' K \sin(2K\alpha)}{\pi F_{\min}(i\pi)}\right)^n H_1,$$  \hspace{1cm} (5.13)

with the constants $H_0, H_1$ determined by the vacuum and the one–particle FF of the operator $O$.

Let us discuss the general properties of the recursive equations (5.10). It is easy to see that the solutions $Q_n$ with $n > 1$ span in this case an infinite dimensional vector space. In fact, if $\tilde{Q}_n(\beta_1, \cdots, \beta_n)$ is a non–trivial solution of the recursive eq. (5.10), the general expression

$$Q_n(\beta_1, \cdots, \beta_n) = \tilde{Q}_n(\beta_1, \cdots, \beta_n) + A_n(\beta_1, \cdots, \beta_n) W(\beta_1, \cdots, \beta_n)$$  \hspace{1cm} (5.14)

is still a solution if $W(\beta_1, \cdots, \beta_n)$ belongs to the kernel of the recursive equation

$$W_n(\beta + i\pi, \beta, \beta_3, \cdots, \beta_n) = 0 ,$$  \hspace{1cm} (5.15)

i.e. $W_n(\beta_1, \cdots, \beta_n) = \prod_{i<j} \sin u_{ij}$, and $A_n(\beta_1, \cdots, \beta_n)$ is a generic $2\pi i$ periodic, symmetric function with possible pole singularities outside the physical strip and which reduces to a constant in the limit $l \to 0$ – the last requirement in order to reproduce
the standard result for the Sinh–Gordon model. Due to the periodic nature of the
theory under consideration, no extra constraints can be imposed on the function
$A_n$ which then spans an infinite space of solutions. Moreover, as we will see from
explicit examples below, there exist in general several different non–trivial solutions
$\tilde{Q}_n$ which however reduce to the same solution of the FF equations of the Sinh–
Gordon model in the limit $l \to 0$. In other terms, given a solution $\tilde{Q}_n$ of the FF
equations of the Sinh–Gordon model, there are in general many different ways to
extend this function for $l \neq 0$ and obtain then distinct solutions of the FF equation
of the Elliptic Sinh–Gordon model.

Let us concentrate now on the construction of particular solutions $Q_n$ of the
recursive equations.

For $n = 1$, $Q_1$ is a constant, as required by relativistic invariance. By absorbing
this constant into the normalization $H_1$, we can take $Q_1 = 1$.

For $n = 2$ a solution is given by

$$Q_2(\beta_1, \beta_2) = \text{cn} u_{12}. \quad (5.16)$$

In fact, the two–particle FF of any local operator cannot have a pole at $\beta_{12} = i\pi$
so that $Q_2(\beta_1, \beta_2)$ must precisely cancel the pole introduced by the function $\text{cn} u_{12}$
in the parameterization (5.8). For $l \to 0$ this solution reduces to the corresponding
one for the Sinh–Gordon model \[9, 10\]. However, this is not the only acceptable
solution with this property. We can for instance divide the function (5.16) by $\text{dn} u_{12}$
and obtain another admissible two–particle FF, with no pole at $\beta_{12} = i\pi$ and which
reduces to the same function of the Sinh–Gordon model in the limit $l \to 0$. This new
solution differs from the previous one by the distribution of zeros and poles on the
complex $\beta$–plane. In particular, it possesses poles at $\beta_{12} = (2m + 1)i\pi + (2n + 1)T$
and zeros at $\beta_{12} = 2im\pi + (2n + 1)T$, which are pushed to infinity for $l \to 0$.

\$^8\$In the Sinh–Gordon case this a–priori infinite set of solutions is severely restricted by the
constraints on the asymptotic behaviour of $F_n$ which eventually force the function $A_n$ to be a
constant \[10\].
Along these lines, it is easy to see that the most general solution to the recursive equations for \( n = 2 \) falls into the class (5.14), i.e. it may be written as

\[
Q_2(\beta_1, \beta_2) = \text{cn} u_{12} q(\beta_{12}, l) ,
\]

where \( q(\beta_{12}, l) \) is any \( 2\pi i \) periodic, even function with no pole at \( \beta_{12} = i\pi \) and such that \( \lim_{l \to 0} q(\beta_{12}, l) = \text{constant} \). Different solutions will have different distributions of zeros and poles but, in the absence of extra requirements on the FF of this theory, they are all acceptable, as far as they do not possess dangerous poles on the physical strip.

Let us now discuss the three–particle FF. The function \( Q_3(\beta_1, \beta_2, \beta_3) \) must be a solution of the equations (5.10) specialized to the case \( n = 3 \)

\[
Q_3(\beta + i\pi, \beta, \beta_3) = \frac{2}{1 + \text{dn}[2(u - u_3)]} ,
\]

where we have used \( Q_1 = 1 \), absorbing the constant in \( H_1 \). It is easy to find two different non–trivial solutions of this equation with no poles in the physical sheet of each two–particle subcluster. The first one is given by

\[
Q_3^{(1)}(\beta_1, \beta_2, \beta_3) = - \left[ \frac{\text{cn}(u_{12} + u_{13}) \text{cn} u_{23}}{\text{dn}(u_{12} + u_{13}) \text{dn} u_{23}} + \frac{\text{cn}(u_{23} + u_{21}) \text{cn} u_{13}}{\text{dn}(u_{23} + u_{21}) \text{dn} u_{13}} + \right. \\
\left. + \frac{\text{cn}(u_{13} + u_{23}) \text{cn} u_{12}}{\text{dn}(u_{13} + u_{23}) \text{dn} u_{12}} \right] .
\]

The second solution is given by

\[
Q_3^{(2)}(\beta_1, \beta_2, \beta_3) = \left[ \frac{\text{cn}^2 u_{12}}{\text{dn}^2 u_{12}} + \frac{\text{cn}^2 u_{13}}{\text{dn}^2 u_{13}} + \frac{\text{cn}^2 u_{23}}{\text{dn}^2 u_{23}} \right] .
\]

Both expressions for \( T \to \infty \) fall into the general \( Q_3 \) solution for the Sinh–Gordon model which is given by (17).

\[
Q_3^{\text{Sh}}(\beta_1, \beta_2, \beta_3) = A_1(\cosh \beta_{12} + \cosh \beta_{13} + \cosh \beta_{23}) + A_2 ,
\]

with \( A_2 - A_1 = 1 \). We notice that both the solutions are periodic in each variable, with period \( 2T \), even if they satisfy the condition for the residue at the kinematic
pole to be $T$-periodic, as it follows from the recursive equation (3.11). However, in
general nothing prevents one from finding also $T$-periodic solutions.

As in the $n = 2$ case we can generate a pletora of solutions by multiplying the
solutions (5.19) and (5.20) by any function $q(\beta_1, \beta_2, \beta_3, l)$ which is $2\pi i$ periodic,
symmetric, with $q(\beta + i\pi, \beta, \beta_3, l)$ = constant and such that it reduces to a constant
in the $l \to 0$ limit. Such a function may be written, for instance, in terms of
arbitrary powers of the expression $q_0(\beta_1, \beta_2, \beta_3) = \text{dn} u_{12} \text{dn} u_{13} \text{dn} u_{23}$.

As a final example, we provide explicit solutions of the recursive equa-
tions (5.10) for $n = 4$. In this case, we need to find $Q_4$ satisfying

$$Q_4(\beta + i\pi, \beta, \beta_3, \beta_4) = -4i \frac{\text{sn}[2(u - u_3)] + \text{sn}[2(u - u_4)]}{[1 + \text{dn}[2(u - u_3)][1 + \text{dn}[2(u - u_4)]]} \times Q_2(\beta_3, \beta_4).$$

(5.22)

Using some of the identities listed in Appendix A and inserting the explicit expres-
sion for $Q_2$ as given in eq. (5.16), the previous equation can be rewritten as

$$Q_4(\beta + i\pi, \beta, \beta_3, \beta_4) = -8i \frac{\text{sn}(2u - u_3 - u_4) \text{cn}^2(u_3 - u_4) \text{dn}(u_3 - u_4)}{[\text{dn}(2u - u_3 - u_4) + \text{dn}(u_3 - u_4)]^2}$$

(5.23)

If we now define the functions

$$D_1 = \prod_{i<j}^4 \text{dn} u_{ij},$$

(5.24)

$$D_2 = \text{dn}(u_{12} + u_{34}) \text{dn}(u_{12} - u_{34}) \text{dn}(u_{14} + u_{23}),$$

(5.25)

it is easy to determine various solutions with no poles on the physical strip. Here
we list two of them.

The first solution is

$$Q_4^{(1)}(\beta_1, \beta_2, \beta_3, \beta_4) = -4i \frac{D_3^3 D_2}{l' D_3^3 D_2 + (l')^6} \times$$

$$\left[1 + \frac{\text{cn}(u_{12} + u_{34})}{\text{dn}(u_{12} + u_{34})} \cdot \frac{\text{cn}(u_{12} - u_{34})}{\text{dn}(u_{12} - u_{34})} + \frac{\text{cn}(u_{13} + u_{23})}{\text{dn}(u_{13} + u_{23})} \cdot \frac{\text{cn}(u_{13} - u_{23})}{\text{dn}(u_{13} - u_{23})} + \right.$$ 

$$+ \frac{\text{cn}(u_{14} + u_{24})}{\text{dn}(u_{14} + u_{24})} \cdot \frac{\text{cn}(u_{14} - u_{24})}{\text{dn}(u_{14} - u_{24})} \right].$$

(5.26)
In the limit $T \to \infty$ it reduces to the standard Sinh–Gordon solution \[10\]

$$Q^S_4(\beta_1, \beta_2, \beta_3, \beta_4) = -i \left( \cosh \frac{\beta_{12}}{2} \cosh \frac{\beta_{23}}{2} + \cosh \frac{\beta_{13}}{2} \cosh \frac{\beta_{24}}{2} + \cosh \frac{\beta_{14}}{2} \cosh \frac{\beta_{23}}{2} \right) \times [2 + \cosh \beta_{12} + \cosh \beta_{13} + \cosh \beta_{14} + \cosh \beta_{23} + \cosh \beta_{24} + \cosh \beta_{34}] \right). \tag{5.27}$$

The Sinh–Gordon FF relative to this solution goes asymptotically to a constant in each variable $\beta_i$.

The second solution reads

$$Q^{(2)}_4(\beta_1, \beta_2, \beta_3, \beta_4) = -8i(l')^3 \frac{D_1^2 D_2^2}{(D_1^2 D_2 + (l')^6)^2} \times \left[ \frac{\cn(u_{12} + u_{34})}{\dn(u_{12} + u_{34})} \cdot \frac{\cn(u_{13} - u_{24})}{\dn(u_{13} - u_{24})} \cdot \frac{\cn(u_{14} + u_{23})}{\dn(u_{14} + u_{23})} + \cn(u_{12} + u_{34}) \cdot \cn(u_{13} - u_{24}) \cdot \frac{\cn(u_{14} + u_{23})}{\dn(u_{14} + u_{23})} \right], \tag{5.28}$$

and in the Sinh–Gordon limit it reduces to the second standard solution of this model \[10\]

$$Q^S_4(\beta_1, \beta_2, \beta_3, \beta_4) = -2i \left( \cosh \frac{(\beta_{12} + \beta_{34})}{2} \cdot \cosh \frac{(\beta_{12} + \beta_{43})}{2} \cdot \cosh \frac{(\beta_{14} + \beta_{23})}{2} \right) + \cosh \frac{(\beta_{12} + \beta_{34})}{2} + \cosh \frac{(\beta_{12} + \beta_{43})}{2} + \cosh \frac{(\beta_{14} + \beta_{23})}{2}. \tag{5.29}$$

The Sinh–Gordon FF relative to this solution goes asymptotically to zero in each variable $\beta_i$.

Starting from a different expression for $Q_2$, one can easily generate other solutions for the four–particle Form Factors which however reduce either to (5.27) or (5.29) in the limit $l \to 0$.

Concerning the computation of higher FF, it can be performed along the same lines of the previous examples. As shown by the first few cases, the periodic nature of the theory will manifest itself in the appearance of infinite sets of solutions which all collapse to the standard solutions for the Sinh–Gordon model in the $l \to 0$ limit. As far as the modulus $l$ is kept away from zero, different solutions for the $n$–particle Form Factors are characterized by a different analytic structure.
It would be interesting to understand how to discriminate among different FF on the basis of a detailed investigation of the zeros and poles distributions of matrix elements of the various operators present in the theory. Moreover, it would be highly desirable to have a concise and closed formula at least for a sequence of Form Factors $F_n(\beta_1, \ldots, \beta_n)$ as in the Sinh–Gordon model [10], but the determination of this formula has presently eluded our attempts.

Finally, let us briefly discuss some aspects of the ESG model in its ultraviolet regime. Among the solutions of the FF equation we expect to find those ones which identify with the matrix elements of the trace $\Theta(x)$ of the stress–energy tensor. This is a $Z_2$ even field, with non–vanishing FF for all $2n$ numbers of external particles. The FF of the field $\Theta(x)$ can be used to compute the central charge of the model in its ultraviolet limit by means of the $c$-theorem sum rule

$$C(T) = \int_0^\infty d\mu c_1(\mu),$$

where $c_1(\mu)$ is given by

$$c_1(\mu) = \frac{12}{\mu^3} \sum_{n=1}^\infty \frac{1}{(2n)!} \int \frac{d\beta_1 \ldots d\beta_{2n}}{(2\pi)^{2n}} |F_{2n}(\beta_1, \ldots, \beta_{2n})|^2 \times \delta\left(\sum_i m \sinh \beta_i\right) \delta\left(\sum_i m \cosh \beta_i - \mu\right).$$

Assuming the uniform convergence of this series (so that we can interchange the sum on the index $n$ with the integral in (5.30)), it is immediate to conclude that the central charge of the model can take any value in the interval $(0, 1)$ depending on the period $T$ of the $S$–matrix. In fact, for $T \to \infty$ we recover the sum rule series of the Sinh–Gordon model, i.e. $C(\infty) = 1$. Viceversa for $T \to 0$ each term of the series goes to zero and therefore $C(0) = 0$. The series (5.31) is a continuous and monotonic decreasing function of $T$ so that by varying this parameter we can reach any value in the interval $(0, 1)$. Therefore, in the ultraviolet limit we have generically a non–unitary irrational Conformal Field Theory. As a consequence, it seems reasonable that the existence of an infinite dimensional space of solutions for...
the FF of the ESG theory can be traced back to the non-unitary irrational nature of the CFT in the UV region and to its relative infinite number of primary operators.

6 Conclusions

In this paper we have analysed certain massive integrable $Z_2$ invariant quantum field theories with an infinite tower of resonance states. These states are associated to an unlimited sequence of poles on the unphysical sheet of the $S$–matrix for a fundamental particle $A$ and may be regarded as unstable bound states thereof. As a function of the energy, the phase–shift of the scattering amplitude presents sharp jumps in correspondence to the masses of the resonances. The $S$–matrix of these theories has a real periodic behaviour in the variable $\beta$.

The Form Factors of such theories may be computed in principle along the same lines of other integrable relativistic models, i.e. by solving the monodromy and recursive equations coming from the residue condition on the kinematical poles. However, new features have emerged from the periodic nature of the $S$–matrix: on the one hand, there is a very severe constraint on the residues of the Form Factors $F_n(\beta_1, \ldots, \beta_n)$ at their kinematical poles $\beta_{ij} = i\pi$, which have to be periodic expressions of the remaining rapidity variables $\beta_k$. Conversely, an infinite proliferation of solutions of the FF equations occurs, due to the lack of very stringent conditions on their analytic structure or to the impossibility of enforcing a given asymptotic behaviour in each rapidity variable $\beta_i$.

While it is an interesting open problem to develop further theoretical criteria to identify the operators associated to these solutions, nevertheless the presence of an infinite number of them seems compatible with an other aspect of these models related to their behavior in the ultraviolet regime. In fact, the Conformal Field Theories which are approached in the ultraviolet limit are generally non-unitary and irrational, therefore with an infinite number of primary fields. The value of
the central charge may be fixed by varying the spectrum of the resonances, i.e. by changing the period $T$ of the $S$–matrix.

Since these QFT with resonance states may be thought of as theories with additional mass scales compared to those given by the spectrum of stable particles, it is an interesting general problem to determine how these extra mass scales affect the different regimes of the theory. From our analysis it has emerged that the large distance behaviour of the model is hardly affected by them, being essentially ruled by the mass gaps of the stable part of the spectrum. On the contrary, the ultraviolet properties of the theory are deeply influenced by the infinite tower of unstable states which, in the ultraviolet regime, tend to smooth the short–distance divergencies and to decrease their related quantities. As a consequence, ultraviolet singularities are in general less severe, and the central charge of the underlying Conformal Field Theory takes lower (in general irrational) values than in the absence of the resonance states. It would be interesting to further investigate the ultraviolet properties of these theories with resonances by means of the Thermodynamics Bethe Ansatz.

More generally, it would be highly desirable to understand whether these theories allow for a Lagrangian description, and to find possible applications to statistical mechanics systems. Moreover, the possibility to define other periodic models certainly deserves more investigation as well as the possible computation of FF for the $Z_4$–model of Ref. [6].

**Acknowledgments**

This work was done under partial support of the EC TMR Programmes FMRX-CT96-0012, and ERBFMRX-CT96-0045 in which SP is associated to the University of Torino.
Appendix A

In this appendix we collect some useful formulas relative to the Jacobian elliptic functions as well as mathematical identities used in the text. The reader may consult [24] for further details.

The Jacobian elliptic functions are double–periodic functions and have two simple poles and two simple zeros in a period parallelogram. Let

\[ K(l) = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1 - l^2 \sin^2 \alpha}} , \]  

be the complete elliptic integral of modulus \( l \) and \( K'(l') = K(l') \) the complete elliptic integral of the complementary modulus \( l' \), with \( l^2 + l'^2 = 1 \). Setting \( q = \exp \left[ -\pi K'/K \right] \), the definition of the Jacobian elliptic functions \( \text{sn} u \), \( \text{cn} u \) and \( \text{dn} u \) is given by

\[
\text{sn} u = \frac{2q^{\frac{1}{4}}}{\sqrt{l}} \sin \frac{\pi u}{2K} \prod_{n=1}^{\infty} \frac{1 - 2q^{2n}\cos \frac{\pi u}{K} + q^{4n}}{1 - 2q^{2n-1}\cos \frac{\pi u}{K} + q^{4n-2}} , \\
\text{cn} u = \frac{2\sqrt{l'} q^{\frac{1}{4}}}{\sqrt{l}} \cos \frac{\pi u}{2K} \prod_{n=1}^{\infty} \frac{1 + 2q^{2n}\cos \frac{\pi u}{K} + q^{4n}}{1 - 2q^{2n-1}\cos \frac{\pi u}{K} + q^{4n-2}} , \\
\text{dn} u = \sqrt{l'} \prod_{n=1}^{\infty} \frac{1 + 2q^{2n-1}\cos \frac{\pi u}{K} + q^{4n-2}}{1 - 2q^{2n-1}\cos \frac{\pi u}{K} + q^{4n-2}} .
\]  

The parity properties of the Jacobian elliptic functions are

\[ \text{cn}(-u) = \text{cn} u \ ; \ \text{sn}(-u) = -\text{sn} u \ ; \ \text{dn}(-u) = \text{dn} u , \]  

whereas their value in \( u = 0 \) is

\[ \text{cn}(0) = \text{dn}(0) = 1 \ ; \ \text{sn}(0) = 0 . \]
The period, zeros and poles of these functions are summarised in the following table:

| Function | Periods | Zeros | Poles |
|----------|---------|-------|-------|
| $\text{sn} u$ | $4mK + 2nK'i$ | $2mK + 2nK'i$ | $2mK + (2n + 1)K'i$ |
| $\text{cn} u$ | $4mK + 2n(K + K'i)$ | $(2m + 1)K + 2nK'i$ | $2mK + (2n + 1)K'i$ |
| $\text{dn} u$ | $2mK + 4nK'i$ | $(2m + 1)K + (2n + 1)K'i$ | $2mK + (2n + 1)K'i$ |

Table 1

The change of argument of these functions are ruled by the table:

| $u^* = u + K$ | $u + iK'$ | $u + 2K$ | $u + 2iK'$ | $u + 2K + 2iK'$ |
|----------------|------------|----------|-------------|-----------------|
| $\text{sn} u^* = \frac{\text{cn} u}{\text{dn} u}$ | $\frac{1}{i\text{sn} u}$ | $-\text{sn} u$ | $\text{sn} u$ | $-\text{sn} u$ |
| $\text{cn} u^* = -li' \frac{\text{sn} u}{\text{dn} u}$ | $-i \frac{\text{dn} u}{l\text{sn} u}$ | $-\text{cn} u$ | $-\text{cn} u$ | $\text{cn} u$ |
| $\text{dn} u^* = li' \frac{1}{\text{dn} u}$ | $-i \frac{\text{cn} u}{\text{sn} u}$ | $\text{dn} u$ | $-\text{dn} u$ | $-\text{dn} u$ |

Table 2
We list here all the identities satisfied by the Jacobian elliptic functions used in the course of our calculations

\[
\begin{align*}
\text{cn}^2 u + \text{sn}^2 u &= 1 ; \\
\text{dn}^2 u + l^2 \text{sn}^2 u &= 1 ;
\end{align*}
\]

\[
\text{sn}(u \pm v) = \frac{\text{sn} u \text{cn} v \text{dn} v \pm \text{sn} v \text{cn} u \text{dn} u}{1 - l^2 \text{sn}^2 u \text{sn}^2 v} ; \\
\text{cn}(u \pm v) = \frac{\text{cn} u \text{cn} v \mp \text{sn} u \text{sn} v \text{dn} u \text{dn} v}{1 - l^2 \text{sn}^2 u \text{sn}^2 v} ; (A.6)
\]

\[
\text{dn}(u \pm v) = \frac{\text{dn} u \text{dn} v \mp l^2 \text{sn} u \text{sn} v \text{cn} u \text{cn} v}{1 - l^2 \text{sn}^2 u \text{sn}^2 v} ;
\]

\[
\text{sn}(u + v) \text{sn}(u - v) = \frac{\text{sn}^2 u - \text{sn}^2 v}{1 - l^2 \text{sn}^2 u \text{sn}^2 v} (A.7)
\]

\[
\text{cn}(u + v) \text{cn}(u - v) = \frac{\text{cn}^2 v - \text{sn}^2 u \text{dn}^2 v}{1 - l^2 \text{sn}^2 u \text{sn}^2 v} (A.8)
\]

\[
\text{dn}(u + v) \text{dn}(u - v) = \frac{\text{dn}^2 v - l^2 \text{sn}^2 u \text{cn}^2 v}{1 - l^2 \text{sn}^2 u \text{sn}^2 v} (A.9)
\]

\[
\frac{\text{sn} u \text{cn} u}{\text{dn} u} = \frac{\text{sn}(2u)}{1 + \text{dn}(2u)} . (A.10)
\]

\[
\text{sn}^2 u + \frac{\text{cn}^2 u}{\text{dn}^2 u} = \frac{2}{1 + \text{dn}(2u)} (A.11)
\]

When the module \( l \) goes to zero, we have the approximations

\[
\text{sn} u \simeq \sin u - \frac{1}{4} l^2 (u - \sin u \cos u) \cos u ;
\]

\[
\text{cn} u \simeq \cos u + \frac{1}{4} l^2 (u - \sin u \cos u) \sin u ; (A.12)
\]

\[
\text{dn} u \simeq 1 - \frac{1}{2} l^2 \sin^2 u .
\]

Finally, we also report some useful identities concerning infinite sums of trigonometric functions. These identities are useful for calculations in Sections 3 and 5.

\[
\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \frac{\pi}{4} , (A.13)
\]
\[ \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2}. \quad (A.14) \]

As a consequence of the previous identity we also have
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nx = \frac{x}{2}. \quad (A.15) \]

Other useful identities are
\[ \sum_{n=1}^{\infty} \frac{q^{2n-1}}{2n-1} \cos (2n-1)x = \frac{1}{4} \ln \frac{1 + 2q \cos x + q^2}{1 - 2q \cos x + q^2}, \quad (A.16) \]
\[ \sum_{n=1}^{\infty} \frac{q^n}{n} \cos nx = -\frac{1}{2} \ln \left(1 - 2q \cos x + q^2\right), \quad (A.17) \]
\[ \left| \frac{\Gamma(x)}{\Gamma(x + iy)} \right| = \prod_{k=0}^{\infty} \left(1 + \frac{y^2}{(x+k)^2}\right). \quad (A.18) \]
Appendix B

In this appendix we present the calculations which leads to the exponential representation (2.14).

Due to the periodicity of the S–matrix for real values of the rapidity, we can always look for its Fourier series expansion with period $T$ of the form

$$S(\beta, a) = -\exp \left[ \sum_{n=-\infty}^{+\infty} a_n e^{2i\pi n\beta} \right].$$

(B.1)

In order to determine the coefficients $a_n$, let us equating the logarithmic derivative of the above expression with the one coming from the $S$–matrix (2.7)

$$\sum_{n=-\infty}^{+\infty} a_n n e^{2i\pi n\beta} = -\frac{2K'}{\pi} E(\beta),$$

(B.2)

where

$$E(\beta) = \frac{\text{sn}'\left(\frac{2K\beta}{\pi}\right)}{\left(\text{sn}\left(\frac{2K\beta}{\pi}\right) + \text{sn}\left(2Ka\right)\right) \left(\text{sn}\left(\frac{2K\beta}{\pi}\right) - \text{sn}\left(2Ka\right)\right)}$$

(B.3)

and $\text{sn}'(x)$ denotes the derivative of the function $\text{sn}(x)$. The coefficients $a_n$ for $n \neq 0$ are then given by

$$n a_n = -\frac{2K'}{\pi T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt e^{-2i\pi nt} E(t)$$

(B.4)

Let us consider the complex loop integral

$$\oint_C dt e^{-\frac{2i\pi nt}{T}} E(t)$$

(B.5)

on a closed path $C$ which runs along the boundary of half the fundamental domain, i.e. $-\frac{T}{2} < \text{Re}t < \frac{T}{2}$, $0 < \text{Im}t < i\pi$. Due to the real periodicity of the function $E(t)$ on the complex $t$–plane, it is easy to see that the contributions from the vertical lines of the path cancel each others, while the integral on the $\text{Im}t = i\pi$ line can be rewritten as an integral on the real axis by exploiting the property $E(t + i\pi) = -E(t)$. Therefore, for the coefficients $a_n$ we have

$$n a_n \left(1 + e^{2n\pi^2 T}\right) = -\frac{2K'}{\pi T} \oint_C dt e^{-\frac{2i\pi nt}{T}} E(t)$$

(B.6)
Inside the domain of integration the function $E(t)$ has two simple poles at $t = i\pi a, i\pi(1 - a)$. Computing the corresponding residues we obtain

$$n a_n = \frac{\cosh\left(\frac{n\pi^2}{T}(1 - 2a)\right)}{\cosh\left(\frac{n\pi^2}{T}\right)}, \quad n \neq 0$$  \hspace{1cm} (B.7)

Inserting this result in eq. (B.1) and exploiting the parity properties of the expression under $n \to -n$, we can write

$$S(\beta, a) = -\exp\left[2i \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cosh\left(\frac{n\pi^2(1-2a)}{T}\right)}{\cosh\left(\frac{n\pi^2}{T}\right)} \sin\left(\frac{2n\pi}{T} \beta\right) + a_0\right], \quad (B.8)$$

where $a_0$ is the contribution of the zero–mode, which is given by

$$a_0 = \lim_{n \to 0} \frac{i}{n} \frac{\cosh\left(\frac{n\pi^2(1-2a)}{T}\right)}{\cosh\left(\frac{n\pi^2}{T}\right)} \sin\left(\frac{2n\pi}{T} \beta\right) = \frac{2i\pi\beta}{T}. \quad (B.9)$$

The zero mode $a_0$ ensures in particular that in the limit $a \to 0$ we have $S(\beta, a) \to 1$.

Now, using the identity (A.15) to rewrite the zero mode as a series expansion, we finally obtain the expression

$$S(\beta) = -\exp\left[2i \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\cosh\left(\frac{n\pi^2(1-2a)}{T}\right)}{\cosh\left(\frac{n\pi^2}{T}\right)} - (-1)^n\right) \sin\left(\frac{2n\pi}{T} \beta\right)\right], \quad (B.10)$$

given in the text, eq. (2.14).

In order to obtain the second expression (2.16) for the $S$–matrix, it is initially convenient to split the zero–mode contribution into two series on the odd and even numbers, i.e.

$$2i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin\left(\frac{2n\pi}{T} \beta\right) =$$

$$= 2i \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{2(2n-1)\pi}{T} \beta\right) - 2i \sum_{n=1}^{\infty} \frac{1}{2n} \sin\left(\frac{4n\pi}{T} \beta\right).$$

By using now the representation (2.15)

$$-1 = \exp\left[-4i \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{2\pi(2n-1)\beta}{T}\right)\right],$$

43
and adding the term in the exponential of this expression to the zero mode, as a result we have that the series of the zero mode on the odd numbers changes its sign in front so that we end up with the expression

\[
\exp \left[ -2i \sum_{n=1}^{+\infty} \frac{1}{n} \sin \left( \frac{2n\pi\beta}{T} \right) \right], \tag{B.11}
\]

By taking into account the previous Fourier expansion we have then

\[
S(\beta, a) = \exp \left[ 2i \sum_{n=1}^{+\infty} \frac{1}{n} \left( \frac{\cosh \left( \frac{n\pi^2(1-2a)}{T} \right)}{\cosh \left( \frac{n\pi^2}{T} \right)} - 1 \right) \sin \left( \frac{2n\pi}{T} \beta \right) \right] = \tag{B.12}
\]

\[
= \exp \left[ -4i \sum_{n=1}^{+\infty} \frac{1}{n} \frac{\sinh \left( \frac{n\pi^2}{T} \right) \sinh \left( \frac{n(1-a)\pi^2}{T} \right)}{\cosh \left( \frac{n\pi^2}{T} \right)} \sin \left( \frac{2n\pi}{T} \beta \right) \right].
\]

i.e. the expression (2.16) in the text.
Appendix C

In this appendix we report the detailed calculation for the ESG minimal Form Factor as given in eq. (5.2). Let us consider the minimal solution to the monodromy equations for the ESG model as given in eq. (5.1). By exploiting the arbitrariness of an overall constant in front of $F_{\text{min}}$ we can replace $-2 \sin^2 \left( \frac{n \pi}{T} \hat{\beta} \right)$ in eq. (5.1) with $\cos \left( \frac{2n\pi}{T} \hat{\beta} \right)$ and take as starting point the following expression

$$F_{\text{min}}(\beta) = \exp \left[ 2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{\sin \left( \frac{n\pi}{T} \right) \sin \left( \frac{n(1-a)\pi}{T} \right)}{\cosh \left( \frac{n\pi}{T} \right) \sinh \left( \frac{2n\pi}{T} \right)} \cos \left( \frac{2n\pi}{T} \hat{\beta} \right) \right].$$

(C.1)

Let us first concentrate on the $n$ coefficient in the previous expansion

$$c_n = \frac{\sin \left( \frac{n\pi}{T} \right) \sinh \left( \frac{n(1-a)\pi}{T} \right)}{\cosh \left( \frac{n\pi}{T} \right) \sinh \left( \frac{2n\pi}{T} \right)}.$$  \hspace{1cm} (C.2)

If we use the exponential representation for hyperbolic functions and the series expansions

$$(\cosh x)^{-1} = -2 \sum_{k=1}^{\infty} (-1)^k e^{-(2k-1)x},$$

(C.3)

$$(\sinh x)^{-1} = 2 \sum_{k=1}^{\infty} e^{-(2k-1)x},$$

(C.4)

it can be rewritten as

$$c_n = \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} (-1)^{k+1} e^{-2n \left[ (2k-1)\frac{\pi^2}{T^2} + (2p-1)\frac{\pi^2}{T^2} \right]}$$

$$\times \left( e^{-n\frac{\pi^2}{T^2}(1-2a)} + e^{n\frac{\pi^2}{T^2}(1-2a)} - e^{n\frac{\pi^2}{T^2}} - e^{-n\frac{\pi^2}{T^2}} \right).$$

(C.5)

Now, splitting the $k$–sum into a sum on even and odd integers and using the general identity

$$\sum_{k=1}^{\infty} \sum_{p=1}^{\infty} f(k + p) = \sum_{k=2}^{\infty} (k - 1) f(k),$$

we end up with

$$c_n = -\sum_{k=2}^{\infty} (k - 1) \left[ q_1^n(k) + q_2^n(k) + q_3^n(k) - q_4^n(k) - q_5^n(k) - q_6^n(k) \right],$$

(C.6)
where we have defined

\[ q_1(k) = e^{-2(2k-1)\frac{\pi}{T}} \]
\[ q_2(k) = e^{-2(2k-2-a)\frac{\pi}{T}} \]
\[ q_3(k) = e^{-2(2k-3+a)\frac{\pi}{T}} \]
\[ q_4(k) = e^{-2(2k-1-a)\frac{\pi}{T}} \]
\[ q_5(k) = e^{-2(2k-2+a)\frac{\pi}{T}} \]
\[ q_6(k) = e^{-2(2k-3)\frac{\pi}{T}} \]  \hfill (C.7)

By plugging the expression (C.6) of \( c_n \) into eq.(C.1) and performing explicitly the sum on \( n \) by means of the identity (A.17), we arrive to the final expression reported in the text (eq. (5.2))

\[ F_{\text{min}}(\beta) = \prod_{k=2}^{\infty} (W(\beta,k))^{k-1} \]  \hfill (C.8)

where

\[ W(\beta,k) = \left( \frac{1 - 2q_6 \cos\left(\frac{2\pi\hat{\beta}}{T}\right) + q_6^2}{1 - 2q_1 \cos\left(\frac{2\pi\hat{\beta}}{T}\right) + q_1^2} \right) \left( \frac{1 - 2q_5 \cos\left(\frac{2\pi\hat{\beta}}{T}\right) + q_5^2}{1 - 2q_2 \cos\left(\frac{2\pi\hat{\beta}}{T}\right) + q_2^2} \right) \left( \frac{1 - 2q_4 \cos\left(\frac{2\pi\hat{\beta}}{T}\right) + q_4^2}{1 - 2q_3 \cos\left(\frac{2\pi\hat{\beta}}{T}\right) + q_3^2} \right) \]

It is now easy to determine the location of zeros and poles of this function. They are simply given by the solutions of the generic equation \( 1 - 2q \cos x + q^2 = 0 \), for the numerator and denominator factors, respectively. Rewriting this equation as

\[ (1 - qe^{ix}) (1 - qe^{-ix}) = 0 \]  \hfill (C.9)

the general solution has the form

\[ x = \pm i \ln q + 2n\pi \quad , \quad n \in \mathbb{Z} \]  \hfill (C.10)

Applying this result to the factors in eq. (C.8) and taking in account the explicit expressions for the \( q_i \)'s, we obtain the sets of poles and zeros (5.4), (5.5) reported in the text.
References

[1] C. Itzykson, H. Saleur and J.B. Zuber, *Conformal Invariance and Applications to Statistical Mechanics*, (World Scientific, Singapore 1988).

[2] A.M. Tsvelik, *Quantum Field Theory in Condensed Matter Physics*, Cambridge University Press 1995.

[3] G. Mussardo, *Phys. Rep.* **218** (1992), 215.

[4] A.B. Zamolodchikov, in *Advanced Studies in Pure Mathematics* **19** (1989), 641; *Int. J. Mod. Phys.* **A3** (1988), 743.

[5] A.B. Zamolodchikov, Resonance Factorized Scattering and Roaming Trajectories, ENS-LPS-335 (1991).

[6] A.B. Zamolodchikov, *Comm. Math. Phys.* **69** (1979), 165.

[7] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Ann.Phys.* **120** (1979), 253.

[8] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, *Phys. Lett.* **87B** (1979), 389.

[9] A. Fring, G. Mussardo and P. Simonetti, *Nucl. Phys.* **B 393** (1993), 413.

[10] A. Koubek and G. Mussardo, *Phys. Lett.* **B 311** (1993), 193.

[11] B. Berg, M. Karowski and P. Weisz, *Phys. Rev.* **D19** (1979), 2477;

M. Karowski and P. Weisz, *Nucl. Phys.* **B 139** (1978), 445.

[12] F.A. Smirnov, *Form Factors in Completly Integrable Models of Quantum Field Theory* (World Scientific, Singapore 1992).

[13] G. Delfino and G. Mussardo, *Nucl. Phys.* **B 455** (1995), 724.

[14] G. Delfino, P. Simonetti and J.L. Cardy, *Phys. Lett.* **B 387** (1996), 327.
[15] C. Acerbi, G. Mussardo and A. Valleriani, *Journ. Phys. A* **30** (1997), 2895.

[16] R. Köberle and J.A. Swieca, *Phys. Lett.* **B86** (1979), 209;

[17] V.P. Yurov and Al. B. Zamolodchikov, *Int. J. Mod. Phys. A* **6** (1991), 3419.

[18] J.L. Cardy and G. Mussardo, *Nucl. Phys. B* **340** (1990), 387.

[19] B.M. McCoy, C.A. Tracy and T.T. Wu, *Phys. Rev. B* **13** (1976), 316; *Journ. Math. Phys.* **1977**, 1058.

[20] A.B. Zamolodchikov, *JEPT Lett.* **43** (1986), 730.

[21] J.L. Cardy, *Phys. Rev. Lett.* **60** (1988), 2709.

[22] Al.B. Zamolodchikov, *Nucl. Phys. B* **342** (1990), 695.

[23] T. Klassen and E. Melzer, *Nucl. Phys. B* **350** (1991), 635.

[24] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press (1992)


**Figure Captions**

**Figure 1.a**. Tiling of the $\beta$–plane in terms of the periodic domains of the $S$–matrix.

**Figure 1.b**. Twists of the elastic branch–cuts of the $S$–matrix in the $s$–plane.

**Figure 2**. Analytic structure of the elliptic $S$–matrix in its fundamental domain.
The circles are the zeros of the function and the black circles its poles.

**Figure 3**. Phase–shift (in units of $\pi$) of the Elliptic Sinh–Gordon $S$–matrix for particular values of the parameters ($l = 0.25$ and $a = 0.3$).

**Figure 4**. Phase–shift (in units of $\pi$) of the Elliptic Ising Model with the period relative to $l = 0.25$.

**Figure 5**. Poles and zeros in the $\beta$–plane for $a = \delta_0 - i \frac{T}{2\pi}$. They are split by $2\delta$ and annihilate each other for $\delta \to 0$. In this figure the rectangle delimited by dashed lines is taken as the fundamental domain.

**Figure 6**. Diagram associated to the pole singularity of the $S$–matrix near the resonance $R_n$.

**Figure 7**. Plots of the two–point correlation function of the field $\Theta(x)$ versus $MR$. The full line is relative to the usual thermal Ising model ($l = 0$), the short–dashed line to the value $l = 0.25$ and the long–dashed line to $l = 0.75$.

**Figure 8**. Central charge $c$ versus the modulus $l$ for the Elliptic Ising model obtained by the $c$–theorem sum–rule.

**Figure 9**. Anomalous dimension of $\sigma(x)$ and $\mu(x)$ versus the modulus $l$ for the Elliptic Ising model, obtained by the $\eta$ sum rule.
Figure 10. Analytic structure of $F_{\text{min}}(\beta)$ of the ESG model along the imaginary direction placed at $\beta = nT$. The circles are the zeros of the function whereas the black circles represent its poles.

Figure 11. Graphical multiplication of $F_{\text{min}}(\beta)$ by $F_{\text{min}}(\beta + i\pi)$. 
Figure 1.a
Figure 1.b
Figure 2
Figure 3
Figure 4
Figure 5
Figure 6
Figure 7
Figure 11