DISCRETE POLYNUCLEAR GROWTH AND DETERMINANTAL PROCESSES

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Abstract. We consider a discrete polynuclear growth (PNG) process and prove a functional limit theorem for its convergence to the Airy process. This generalizes previous results by Prähofer and Spohn. The result enables us to express the $F_1$ GOE Tracy-Widom distribution in terms of the Airy process. We also show some results, and give a conjecture, about the transversal fluctuations in a point to line last passage percolation problem.

1. Introduction and results

1.1. Discrete polynuclear growth. Recently there has been interesting developments concerning certain special 1 + 1 dimensional local random growth models. This development has its starting point in the new results on the longest increasing subsequence in a random permutation, [3]. We will not review all these developments here. In this paper we consider a certain discrete growth model called the discrete polynuclear growth (PNG) model, [22], a special version of which is closely related to the last-passage percolation problem studied in [15]. It is a discrete version of the PNG model studied by Prähofer and Spohn, [29], which can be obtained as a special limiting case. In the paper we will extend the results in [29] to the present model and prove a stronger convergence result. We also obtain some preliminary results on the transversal fluctuations in the point to line version of the last-passage percolation problem, which should have many similarities with the corresponding problems for first-passage percolation and directed polymers.

The discrete polynuclear growth (PNG) model is a local random growth model defined by

\[ h(x,t+1) = \max(h(x-1,t), h(x,t), h(x+1,t)) + \omega(x,t+1), \]

\( x \in \mathbb{Z}, \ t \in \mathbb{N}, \ h(x,0) = 0, \ x \in \mathbb{Z}. \) Here \( \omega(x,t), \ (x,t) \in \mathbb{Z} \times \mathbb{N}, \) are independent random variables, see [22]. Typically they could be Bernoulli random variables. We should think of \( h(x,t) \) as the height above \( x \) at time \( t, \) so \( x \rightarrow h(x,t) \) gives an interface developing in time. We will treat a special case where \( \omega(x,t) = 0 \) if \( t-x \) is even or if \( |x| > t, \) and

\[ w(i,j) = \omega(i-j, i+j-1), \]

\( (i,j) \in \mathbb{Z}_+^2, \) are independent geometric random variables with parameter \( a_i b_j, \)

\[ \mathbb{P}[w(i,j) = m] = (1 - a_i b_j)(a_i b_j)^m, \]

\( m \geq 0. \) We will mainly consider the case when \( a_i = b_i = \sqrt{q}, \ 0 < q < 1, \ i \geq 1, \) and we we do this in the rest of this section. If we define

\[ G(i,j) = h(i-j, i+j-1), \]
\( (i, j) \in \mathbb{Z}_+^2 \), it follows from (1.1) that
\[
G(i, j) = \max(G(i - 1, j), G(i, j - 1)) + w(i, j),
\]
see proposition 3.10. This leads immediately to a different formula for \( G(M, N) \),
\[
G(M, N) = \max \sum_{(i, j) \in \pi} w(i, j),
\]
where the maximum is taken over all up/right paths from \((1, 1)\) to \((M, N)\). We can think of \( G(M, N) \) as a point to point last-passage time. It is also natural, from the point of view of directed polymers for example, to consider the point to line last-passage time,
\[
G_{pl}(N) = \max_{|K| < N} G(N + K, N - K).
\]
This makes it reasonable to study the process \( K \rightarrow G(N + K, N - K), \) \(-N < K < N\), which, by (1.4), is the same as \( K \rightarrow h(2K, 2N - 1) \), i.e. the height curve at even sites at time \( 2N - 1 \).

Let \( F_1 \) and \( F_2 \) denote the GOE respectively GUE Tracy-Widom largest eigenvalue distributions. It is known, that there are constants \( a = 2\sqrt{q}(1 - \sqrt{q})^{-1} \) and \( d \) given by (1.8) below, such that \( \mathbb{P}[G(N, N) \leq aN + dN^{1/3}\xi] \rightarrow F_2(\xi) \) as \( N \rightarrow \infty \), and, \( \mathbb{P}[G_{pl}(N) \leq aN + dN^{1/3}\xi] \rightarrow F_1(\xi) \) as \( N \rightarrow \infty \). Also, if the maximum in (1.6) is assumed at some point \( K_N \), which need not be unique, we expect \( K_N \) to be of order \( N^{2/3} \), i.e. the transversal fluctuations are of order \( N^{2/3} \). This can be seen heuristically, and there are some rigorous results for a related question, compare with [22]. These scales motivates the introduction of a rescaled process \( t \rightarrow H_N(t), t \in \mathbb{R} \), defined by
\[
G(N + u, N - u) = \frac{2\sqrt{q}}{1 - \sqrt{q}} N + dN^{1/3}H_N(1 - \sqrt{q}^{1/3}duN^{-2/3}),
\]
and linear interpolation, \(|u| < N\), compare with [29]. This is our rescaled discrete PNG process. The constant \( d \) is given by
\[
d = \frac{(\sqrt{q})^{1/3}(1 + \sqrt{q})^{1/3}}{1 - q}.
\]
In the limit when \( q \) is small and \( N \) is large, we can obtain the continuous PNG process studied by Prähofer and Spohn, [29]. We want to extend their results to the present discrete setting and also prove a stronger form of convergence to the limiting process, a functional limit theorem. Before we can state the theorem we must define the limiting process which is the Airy process introduced by Prähofer and Spohn, [29].

We will approach \( H_N \) by considering it as the top curve in a multilayer PNG process, compare [20] and [24]. This will lead to measures of the form introduced in sect. 1.2 and we will be able to use the formulas for the correlation functions derived there. The same methods can also be applied to Dyson’s Brownian motion, compare with [13], which can be obtained from \( N \) non-intersecting Brownian motions. The appropriately rescaled limit as \( N \rightarrow \infty \) of the top path in Dyson’s Brownian motion converges to the Airy process, see below. This gives some intuition about what it looks like. Its precise definition is more technical.
The extended Airy kernel, \(13, 24, 25\), is defined by

\[
A(t, \xi; \tau', \xi') = \begin{cases} 
\int_0^\infty e^{-\lambda (\tau - \tau')} \text{Ai}(\xi + \lambda) \text{Ai}(\xi' + \lambda) d\lambda, & \text{if } \tau \geq \tau' \\
\int_{-\infty}^0 e^{-\lambda (\tau - \tau')} \text{Ai}(\xi + \lambda) \text{Ai}(\xi' + \lambda) d\lambda, & \text{if } \tau < \tau'.
\end{cases}
\]

where \(\text{Ai}(\cdot)\) is the Airy function. When \(\tau = \tau'\) the extended Airy kernel reduces to the ordinary Airy kernel, \(13\).

We define the Airy process \(t \to A(t)\) by giving its finite-dimensional distribution functions. Given \(\xi_1, \ldots, \xi_m \in \mathbb{R}\) and \(\tau_1 < \cdots < \tau_m\) in \(\mathbb{R}\) we define \(f\) on \(\{\tau_1, \ldots, \tau_m\} \times \mathbb{R}\) by

\[
f(\tau_j, x) = -\chi(\xi_j, \infty)(x).
\]

Then,

\[
\mathbb{P}[A(\tau_1) \leq \xi_1, \ldots, A(\tau_m) \leq \xi_m] = \det(I + f^{1/2}A f^{1/2})_{L^2((\tau_1, \ldots, \tau_m) \times \mathbb{R})},
\]

where we have counting measure on \(\{\tau_1, \ldots, \tau_m\}\) and Lebesgue measure on \(\mathbb{R}\). The Fredholm determinant can be defined via its Fredholm expansion, see sect. 2.1 below. We will prove in section 2.2 that \(f^{1/2}A f^{1/2}\) is a trace class operator on \(L^2(\tau_1, \ldots, \tau_m) \times \mathbb{R}\), so this is also a Fredholm determinant in the sense of determinants for trace class operators. Note that in particular

\[
\mathbb{P}[A(\tau) \leq \xi] = F_2(\xi).
\]

This defines the Airy process. It is proved in \(29\) that it has a version with continuous paths, which also follows from the results below. As mentioned above, another way of understanding the Airy process is as follows. Let \(\lambda(t) = (\lambda_1(t), \ldots, \lambda_N(t))\) with \(\lambda_1(t) < \cdots < \lambda_N(t)\), be the eigenvalues in Dyson’s Brownian motion model, \(9\), for GUE with stationary distribution \(\Delta_N(\lambda)^2 \prod_{j=1}^N \exp(-\lambda_j^2)\). Then,

\[
\lim_{N \to \infty} \sqrt{2N^{1/6}}(\lambda_N(N^{-1/3}t) - \sqrt{2N}) = A(t)
\]

say in the sense of convergence of finite-dimensional distributions. This can be proved using the methods of the present paper, and using techniques from \(19\), it is possible to get an integral formula for the (extended) correlation kernel. The details will not be given here. This scaling limit has been studied before, see \(13\) and references therein.

In analogy with the results of \(29\), we can show that the rescaled height process \(H_N\) converges in finite dimensional distributions to the Airy process.

**Theorem 1.1.** Let \(H_N\) be the process defined by \(13\). Then for any fixed \(t_1, \ldots, t_m\) and \(\xi_1, \ldots, \xi_m\),

\[
\lim_{N \to \infty} \mathbb{P}[H_N(t_1) \leq \xi_1, \ldots, H_N(t_m) \leq \xi_m]
\]

\[
\mathbb{P}[A(t_1) \leq \xi_1 + t_1^2, \ldots, A(t_m) \leq \xi_m + t_m^2],
\]

where \(A\) is the Airy process.

This result can be sharpened to a functional limit theorem.

**Theorem 1.2.** Let \(A(t)\) be the Airy process defined by its finite-dimensional distributions, \(13, 16\). Also, let \(H_N(t)\) be defined by \(13\) and linear interpolation. Fix \(T > 0\) arbitrary. There is a continuous version of \(A(t)\) and

\[
H_N(t) \to A(t) - t^2,
\]

as \(N \to \infty\) in the weak*-topology of probability measures on \(C(-T, T)\).
The theorem will be proved in section 5.2. As a corollary to this theorem and the results of Baik and Rains, [4], we obtain the following result which expresses the GOE largest eigenvalue distribution $F_1$ in terms of the Airy process.

**Corollary 1.3.** For all $\xi \in \mathbb{R}$,

$$F_1(\xi) = \mathbb{P} \left[ \sup_t (A(t) - t^2) \leq \xi \right].$$

The proof of (1.14) is very indirect. It would be interesting to see a more straightforward approach.

As discussed above we are also interested in the transversal fluctuations of the endpoint of a maximal path in the point to line case. In our discrete model this is not well-defined, there could be several maximal paths. Consider the random variable

$$K_N = \inf \{ u : \sup_{t \leq u} H_N(t) = \sup_{t \in \mathbb{R}} H_N(t) \},$$

the first point that gives the maximum. The corresponding quantity for the limiting process $H(t) = A(t) - t^2$ is

$$K = \inf \{ u : \sup_{t \leq u} H(t) = \sup_{t \in \mathbb{R}} H(t) \}.$$

We would like to show that $K_N$ converges to $K$ so that we could call the law of $K$ the asymptotic law of transversal fluctuations. Unfortunately we can only prove this under a very plausible assumption on the Airy process. We can show,

**Proposition 1.4.** The sequence of random variables $\{K_N\}_{N \geq 1}$ is tight, i.e. given $\epsilon > 0$ there is a $T > 0$ and an $N_0$ such that

$$\mathbb{P} \left[ |K_N| > T \right] < \epsilon$$

for all $N \geq N_0$.

The assumption we need to make on the Airy process can be formulated as follows.

**Conjecture 1.5.** Let $H(t) = A(t) - t^2$. Then, for each $T > 0$, $H(t)$ has a unique point of maximum in $[-T,T]$ almost surely.

If we accept this we can prove

**Theorem 1.6.** Assume that conjecture 1.5 is true. Then $K_N \rightarrow K$ in distribution as $N \rightarrow \infty$.

The law of $K$ is thus a natural candidate for the law of the transversal fluctuations. It would be interesting to find a different, more explicit, formula for this law. Assuming the truth of the same conjecture it may also be possible to prove that the endpoints of all maximal paths, or asymptotically maximal paths, converge to the same limit $K$.

By using the limit results of [3], proposition 3.12 and theorem 3.14 we can obtain the correlation functions of the eigenvalues of the successive minors $H^{(k)} = (h_{ij})_{1 \leq i,j \leq k}$, $1 \leq k \leq N$, of an $N \times N$ GUE matrix $H = (h_{ij})_{1 \leq i,j \leq N}$. In this way it is possible to get the Airy process as an appropriate limit of the successive largest eigenvalues of $H^{(k)}$. More details will be given in future work.

We could also get the Airy process by looking at the largest eigenvalues of coupled GUE-matrices, which is similar to looking at Dyson’s Brownian motion model for
GUE. In [29] Prähofer and Spohn raised the problem of finding differential equations for probabilities of the form (1.10) generalizing the Painlevé II formulas for (1.11). In [1] the spectrum of coupled random matrices is studied and it would be interesting to see if the results of this paper shed some light on this problem.

1.2. Measures defined by products of determinants. Probability measures given by products of determinants has been studied in several papers, e.g. by Eynard and Mehta, [10], in connection with eigenvalue correlations in chains of matrices, by Forrester, Nagao and Honner, [13], in connection with Dyson’s Brownian motion model and by Okounkov and Reshetikhin, [28], when introducing the so-called Schur process. The problem is to compute the correlation functions and to show that these are given by determinants so that we obtain a determinantal point process, [31]. The same type of correlation functions are also obtained by Prähofer and Spohn, [29], in a cascade of continuous polynuclear growth (PNG) models. We will study a class of measures which include all the above as special cases and show that we obtain determinantal correlation functions. As an example of the result we will in sect. 2.3 investigate random walks on the discrete circle using the same strategy. This will lead to an extended discrete sine kernel, compare with [28]. We will see in sect. 3 that our main topic the discrete PNG problem fits nicely into this framework. This particular application is very close to the Schur process in [28], and their results could also have been used. In fact, we rederive their main formulas.

For \( r \in \mathbb{Z} \) let \( x^r = (x^r_1, \ldots, x^r_n) \in \mathbb{R}^n \) and \( \bar{x} = (x^{M+1}, \ldots, x^M) \), \( M \geq 1 \). We think of \( \bar{x} \) as a point configuration in \( \{-M+1, \ldots, M-1\} \times \mathbb{R} \), and we also specify fixed initial \( x^{-M} \) and final \( x^M \) positions. Let \( \phi_{r,r+1} : \mathbb{R}^2 \to \mathbb{C} \), \( r \in \mathbb{Z} \), be given transition weights. The weight of the configuration \( \bar{x} \) is then

\[
(1.17) \quad w_{n,M}(\bar{x}) = \prod_{r=-M}^{M-1} \det(\phi_{r,r+1}(x_i^r, x_{i+1}^r))_{i=1}^n.
\]

Let \( d\mu \) be a given reference measure on \( \mathbb{R} \), typically Lebesgue measure or counting measure. We assume that \( |\phi_{r,r+1}(x,y)| \leq c_r(x)d_r(y) \), where \( c_r \in L^1(\mathbb{R}, \mu) \) and \( d_r \in L^\infty(\mathbb{R}, \mu) \), \( -M \leq r < M \). This assumption is not necessary but is convenient and suffices for the convergence of all the objects we will encounter. The partition function is

\[
(1.18) \quad Z_{n,M} = \frac{1}{(n!)^{2M-1}} \int_{(\mathbb{R}^n)^{2M-1}} w_{n,M}(\bar{x})d\mu(\bar{x}),
\]

where \( d\mu(\bar{x}) = \prod_{r=-M+1}^{M-1} \prod_{j=1}^n d\mu(x^r_j) \). We will assume that \( Z_{n,M} \neq 0 \) so that we can define the normalized weight

\[
(1.19) \quad p_{n,M}(\bar{x}) = \frac{1}{(n!)^{2M-1}Z_{n,M}} w_{n,M}(\bar{x}).
\]

If \( w_{n,M}(\bar{x}) \geq 0 \), this is a probability density on \( (\mathbb{R}^n)^{2M-1} \) with respect to the reference measure \( d\mu(\bar{x}) \). The \( (k_{-M+1}, \ldots, k_{M-1}) \)-correlation function can now be
defined in a standard way by
\begin{equation}
R_{k_1,\ldots,k_{M-1}}(x_1^{\ldots,1},x_2^{\ldots,1},\ldots,x_{k_{M-1}}^{\ldots,1})
= \prod_{r=-M+1}^{-1} p_{n,M}(\bar{x}) \prod_{r=-M+1}^{-1} \prod_{j=k_r+1}^{n} d\mu(x'_j),
\end{equation}
where \(k = k_{-M+1} + \cdots + k_{M-1}\), \(0 \leq k_j \leq n\).

Given two transition functions we define their convolution by
\[ \phi \ast \psi(x, y) = \int_{\mathbb{R}} \phi(x, z) \psi(z, y) d\mu(z). \]

Set
\[ \phi_{r,s}(x, y) = (\phi_{r, r+1} \ast \cdots \ast \phi_{s-1, s})(x, y) \]
if \(r < s\) and \(\phi_{r,r} \equiv 0\) if \(r \geq s\). Let \(A = (A_{ij})\) be the \(n \times n\) matrix with elements \(A_{ij} = \phi_{-M,M}(x_i^{-M}, x_j^{M})\), \(1 \leq i, j \leq n\). By repeated use of the Heine identity:
\begin{equation}
\frac{1}{n!} \int_{\mathbb{R}^n} \det(\phi_i(x_j))_{i,j=1}^n \det(\psi_i(x_j))_{i,j=1}^n d\mu(x) = \det \left( \int_{\mathbb{R}} \phi_i(t) \psi_j(t) d\mu(t) \right)_{i,j=1}^n,
\end{equation}
we see that \(Z_{n,M} = \det A\). Hence \(\det A \neq 0\) by our assumption. Define a kernel \(K^{n,M} : (-M + 1, \ldots, M - 1) \times \mathbb{R} \to \mathbb{C}\) by
\begin{equation}
K^{n,M}(r, s, y) = K^{n,M}(r, x; s, y) - \phi_{r,s}(x, y),
\end{equation}
where
\begin{equation}
K^{n,M}(r, x; s, y) = \sum_{i,j=1}^{n} \phi_{r,s}(x_i, x_j^{M})(A^{-1})_{ij} \phi_{-M,s}(x_j^{-M}, y).
\end{equation}

In the case \(M = 1\) the kernel \(K\) has appeared before, see [34], [11] and also [18].

**Theorem 1.7.** The correlation functions defined by (1.20) are given by
\begin{equation}
R_{k_1,\ldots,k_{M-1}}(x_1^{\ldots,1},x_2^{\ldots,1},\ldots,x_{k_{M-1}}^{\ldots,1})
= \det(K^{n,M}(r, x^{r}; s, x^{s}_{j}))_{-M < r, s < M, 0 \leq i_r \leq k_r, 0 \leq j_s \leq k_s}.
\end{equation}

The determinant in the right hand side of (1.24) has a block structure with the blocks given by \(r, s\) and having size \(k_r \times k_s\). The theorem will be proved in section 2.1.

A case of particular interest is when the transition weights are given by Fourier coefficients. We are then in a situation similar to that in [28]. Let \(f_r(\theta)\) be a function in \(L^1(\mathbb{T})\) with Fourier coefficients \(\hat{f}_r\). Assume that the transition weights are given by
\begin{equation}
\phi_{r,r+1}(x, y) = \hat{f}_r(y - x),
\end{equation}
\(-M \leq r < M, x, y \in \mathbb{Z}\) and that the initial and final configurations are given by \(x_j^{-M} = x_j^{M} = 1 - j, j = 1, \ldots, n\). If we set
\[ f_{r,s}(z) = \prod_{t=r}^{s-1} f_t(z), \]
\[ z = e^{i\theta}, \]
then
\begin{equation}
\phi_{r,s}(x, y) = \hat{f}_{r,s}(y - x)
\end{equation}
for \( r < s \). The matrix \( A \) defined above is then a Toeplitz matrix with symbol

\[
a(z) = f_{-M,M}(z) = \prod_{\ell=-M}^{M-1} f_{\ell}(z).
\]

Define

\[
\tilde{K}_{r,s}^{n,M}(z,w) = \sum_{x,y \in \mathbb{Z}} \tilde{K}_{r,s}^{n}(r,x; s,y) z^r w^{-y},
\]

where \( \tilde{K}_{r,s}^{n,M} \) is given by (1.23). When the transition functions and the initial and final configurations are given in this way we are able to give a formula for the limit of this generating function as \( n \to \infty \).

**Proposition 1.8.** Assume that \( f_r(z) \) has winding number zero, a Wiener-Hopf factorization \( f_r(z) = f_r^+(z) f_r^-(z) \) and is analytic in \( 1 - \epsilon_r < |z| < 1 + \epsilon_r \) for some \( \epsilon_r > 0 \). Furthermore, suppose that

\[
\sum_{n \in \mathbb{Z}} |n|^\alpha |\hat{a}_n| < \infty,
\]

for some \( \alpha > 0 \), where \( \hat{a}_n \) are the Fourier coefficients of the symbol \( a(z) \) given by (1.27). Set \( \epsilon = \min \{ \epsilon_r \} \) and

\[
\tilde{K}_{r,s}^{M}(z,w) = \frac{z}{z-w} G(z,w),
\]

where

\[
G(z,w) = \frac{\prod_{t=r}^{M-1} f_{-t}^+(\frac{1}{z}) \prod_{t=-M}^{t=s-1} f_{t}^+(\frac{1}{w})}{\prod_{t=-M}^{t=r-1} f_{t}^-(\frac{1}{z}) \prod_{t=s}^{t=M-1} f_{t}^-(\frac{1}{w})}
\]

Then, for \( 1 - \epsilon < |w| < 1 < |z| < 1 + \epsilon \),

\[
| \tilde{K}_{r,s}^{n,M}(z,w) - \tilde{K}_{r,s}^{M}(z,w) | \leq \frac{| f_r(M) || f_{-M,s}(\frac{1}{z}) |}{(|z| - 1)(1 - |w|)} \left( \frac{1}{n^\alpha} + |w|^{n/2} + \frac{1}{|z|^{n/2}} \right).
\]

Furthermore,

\[
\phi_{r,s}(x,y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(y-x)\theta} G(e^{i\theta}, e^{i\theta}) d\theta,
\]

for \( r < s \).

The same type of formula for the limiting kernel was obtained in [28]. The formula will be proved in section 2.1. This proposition makes it possible to compute the asymptotics of the kernel given by (1.22) in certain cases, since it gives an integral formula for the \( n \to \infty \) limit of \( \tilde{K}_{r,s}^{n,M} \).

The outline of the paper is as follows. In sect. 2 we will give some general results for measures of the form (1.17) and then as an example discuss nonintersecting random walks on the discrete circle. The next section applies the general theory to the discrete PNG model and gives more explicit formulas. In sect. 4 asymptotic results for the correlation kernel appearing in the PNG model are stated and proved, and these are then applied in sect. 5 to prove the necessary estimates needed for the functional limit theorem and the transversal fluctuations.
We want to compute \( \det_{Z} \), which is closer to the original Dyson approach. We will prove theorem 1.7 using a generalization of the method of \([34],[7]\) for \( \beta = 2 \) random matrix ensembles, see also \([18]\). It is also possible to generalize the approach of \([10]\), which is closer to the original Dyson approach.

Let \( \Lambda = \{-M + 1, \ldots, M - 1 \} \times \mathbb{R} \), \( \lambda \) be the counting measure on \( \{-M + 1, \ldots, M - 1 \} \) and \( \nu = \lambda \otimes \mu \). Furthermore, we let \( g : \Lambda \to \mathbb{C} \) be a bounded function and define

\[
Z_{n,M} = \frac{1}{(2\pi)^{2M-1}} \int_{(\mathbb{R}^n)^{2M-1}} \prod_{|r|<M} \prod_{j=1}^{n} (1 + g(r,x_j)) u_{n,M}(\bar{x}) d\mu(\bar{x}).
\]

We want to compute \( Z_{n,M} / Z_{n,M}[0] \). Using the Heine identity \((1.21)\) repeatedly we see that

\[
Z_{n,M} = \frac{1}{(2\pi)^{2M-1}} \int_{(\mathbb{R}^n)^{2M-1}} \det(\phi_{-M,-M+1}^{x_i}(x_i^{M+1})) 1_{i,j \leq n} \times \prod_{r=-M+1}^{M-1} \det((1 - g(r,x_i^{r+1})) \phi_{r,r+1}^{x_i}(x_i^{r+1})) 1_{i,j \leq n} d\mu(\bar{x})
\]

\[
= \det \left( \int_{(\mathbb{R}^n)^{2M-1}} \phi_{-M,-M+1}^{x_i}(x_i^{M+1}) \prod_{|r|<M} (1 + g(r,t_r)) \times \left( \prod_{r=-M+1}^{M-2} \phi_{r,r+1}^{t_r}(t_r,t_{r+1}) \right) \phi_{M-1,M}^{x_i}(x_i^{M}) d^2M-1 \mu(t) \right) 1_{i,j \leq n}.
\]

Now,

\[
\prod_{|r|<M} (1 + g(r,t_r)) = 1 + \sum_{\ell=1}^{2M-1} \sum_{-M<r_1<\ldots<r_{\ell}<M} g(r_1,t_{r_1}) \ldots g(r_\ell,t_{r_\ell}),
\]

and hence

\[
Z_{n,M} = \det \left( A_{ij} + \sum_{\ell=1}^{2M-1} \sum_{-M<r_1<\ldots<r_{\ell}<M} \int_{\mathbb{R}^\ell} \phi_{-M,r_1}^{x_i}(x_i^{M},t_1) \left( \prod_{s=1}^{\ell-1} g(r_s,t_s) \phi_{r_s,r_{s+1}}^{x_i}(t_s,t_{s+1}) \right) g(r_\ell,t_\ell) \phi_{r_\ell,M}^{x_i}(t_\ell,x_i^{M}) d^\ell \mu(t) \right) 1_{i,j \leq n},
\]

where we have used the notation of sect. 1.2. If we set \( g = 0 \) we obtain \( Z_{n,M}[0] = Z_{n,M} = \det A \) as before. By definition \( \phi_{r,s} = 0 \) if \( r \geq s \), and hence we can remove the ordering of the \( r_i \)’s in \((2.1)\). We find,

\[
\frac{Z_{n,M}}{Z_{n,M}[0]} = \det \left( \delta_{ij} + \sum_{k=1}^{n} (A^{-1})_{ik} \sum_{\ell=1}^{2M-1} \sum_{-M<r_{\ell}<M} \int_{\mathbb{R}^\ell} \phi_{-M,r_1}^{x_k}(x_k^{M},t_1) \left( \prod_{s=1}^{\ell-1} g(r_s,t_s) \phi_{r_s,r_{s+1}}^{x_k}(t_s,t_{s+1}) \right) g(r_\ell,t_\ell) \phi_{r_\ell,M}^{x_k}(t_\ell,x_k^{M}) d^\ell \mu(t) \right) 1_{i,j \leq n}.
\]
Write \( \psi(u, t; v, s) = g(u, t) \phi_{u,v}(t, s) \), and define \( \psi^0(u, t; v, s) = \delta_{uu} \delta(t - s) \), \( \psi^1 = \psi \) and
\[
\psi^{r+1}(u, t; v, s) = \int_{\Lambda^2_{M}} \psi(u, t; m_1, \xi_1) \psi(m_1, \xi_1; m_2, \xi_2) \ldots \psi(m_r, \xi_r; v, s) d^r \nu(m, \xi)
\]
for \( r \geq 1 \). Note that, since \( \phi_{r,s} = 0 \) if \( r \geq s \), we have \( \psi^\ell = 0 \) if \( \ell \geq 2M - 1 \). This follows immediately from the definition. The formula (2.2) can now be written
\[
(2.3) \quad \frac{Z_{n, M}[g]}{Z_{n, M}[0]} = \det \left( \delta_{ij} + \sum_{k=1}^{n} (A^{-1})_{ik} \int_{\Lambda_m} d\nu(u, \xi) \int_{\Lambda_m} d\nu(v, \eta)\right)_{i,j=1, \ldots, n}
\]
\[
\phi_{-M,u}(x^{-M}_k, \xi) \left( \sum_{\ell=1}^{2M-1} \psi^{(\ell-1)}(u, \xi; v, \eta) \right) g(v, \eta) \phi_{v, M}(\eta, x^M_j) \right)_{i,j=1, \ldots, n}.
\]
If \( K(x, y) \) is an integral kernel on \( L^2(\Omega, \mu) \) we define the determinant \( \det(I + K)_{L^2(\Omega, \mu)} \) via a Fredholm expansion,
\[
(2.4) \quad \det(I + K)_{L^2(\Omega, \mu)} = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Omega^m} \det(K(x_i, x_j))_{i,j=1, \ldots, m} d^m \mu(x).
\]
We assume that \( K \) is such that all the integrals are well-defined and the series converges. For example, by Hadamard’s inequality, it is sufficient to require that \( |K(x, y)| \leq a(x)b(y) \), where \( a \in L^1(\Omega, \mu) \), \( b \in L^\infty(\Omega, \mu) \). Note that if \( \Omega = \{1, \ldots, n\} \) and \( \mu \) is counting measure this is the ordinary determinant \( \det(\delta_{ij} + K(i, j))_{i,j=1, \ldots, n} \). Let \( K(x, y) \) be an integral kernel from \( L^2(\Omega_1, \mu_1) \) to \( L^2(\Omega_2, \mu_2) \) and \( L(x, y) \) an integral kernel from \( L^2(\Omega_2, \mu_2) \) to \( L^2(\Omega_1, \mu_1) \). Then
\[
L \ast K(x, y) = \int_{\Omega_2} L(x, z) K(z, y) d\mu_2(z)
\]
is an integral kernel on \( L^2(\Omega_1, \mu_1) \). Furthermore,
\[
(2.5) \quad \det(I + L \ast K)_{L^2(\Omega_1, \mu_1)} = \det(I + K \ast L)_{L^2(\Omega_2, \mu_2)}.
\]
This is easy to see using the Heine identity in the definition (2.4).
Set
\[
b(i; u, \xi) = \sum_{k=1}^{n} (A^{-1})_{ik} \phi_{-M,u}(x^{-M}_k, \xi)
\]
\[
c(u, \xi; j) = \int_{\Lambda_m} \sum_{\ell=1}^{2M-1} \psi^{(\ell-1)}(u, \xi; v, \eta) g(v, \eta) \phi_{v, M}(\eta, x^M_j) d\nu(v, \eta),
\]
so that, by (2.3) and (2.5),
\[
\frac{Z_{n, M}[g]}{Z_{n, M}[0]} = \det(\delta_{ij} + (b \ast c)(i, j))_{1 \leq i,j \leq n} = \det(I + c \ast b)_{L^2(\Lambda^M, \nu)}.
\]
Now, a computation shows that
\[
(c \ast b)(u, \xi; v, \eta) = \left( \sum_{\ell=1}^{2M-1} \psi^{(\ell-1)} \right) (g \tilde{K})(u, \xi; v, \eta),
\]
The equality (2.8) holds for
\[ Z_{n,M}[g] = \det(I + \left( \sum_{\ell=1}^{2M-1} \psi^s(\ell-1) \ast (g\tilde{K}))_{L^2(\Lambda_M,\nu)}. \]

The kernel in (2.6) has finite-rank so the sum (2.4) in the definition of the determinant actually has finitely many terms. We now claim that the right hand side of (2.6) equals
\[ \det(I - \psi + g\tilde{K})_{L^2(\Lambda_M,\nu)}, \]

which is what we want. Formally the computation goes as follows. The expression in (2.6) is \( \det(I - (I - \psi)^{-1}g\tilde{K}) \) and we multiply this by \( \det(I - \psi) = 1 \). Since we are only working with determinants defined by a Fredholm expansion the product rule is not obvious, so we will give a proof in this special case.

Write \( a = g\tilde{K} \). We will prove that for any \( z, w \in \mathbb{C} \),
\[ \det(I + w \sum_{j=1}^{m} z^j \psi^s(j-1) \ast a)_{L^2(\Lambda_M)} = \det(I - z \psi + zw a)_{L^2(\Lambda_M)}, \]

where \( m = 2M - 1 \). The left hand side is a polynomial in \( z, w \) so it suffices to prove (2.7) for \( |z|, |w| \) sufficiently small. In that case, under our assumption on the \( \phi_{r,r+1} \), all the expressions below are well-defined and convergent. Write \( b = \sum_{j=1}^{m} z^j \psi^s(j-1) \ast a \). Then, see e.g. [25],
\[ \det(I + wb)_{L^2(\Lambda_M)} = \exp\left( \sum_{k=1}^{\infty} \frac{(-1)^{k+1} w^k}{k} \int_{\Lambda_M} b^k(t,t) d\nu(t) \right) \]

and
\[ \det(I + z(-\psi + wa))_{L^2(\Lambda_M)} = \exp\left( \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{k} \int_{\Lambda_M} (-\psi + wa)^k(t,t) d\nu(t) \right). \]

Set \( c = za, d = z\psi \). It suffices to show that
\[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1} w^k}{k} \int_{\Lambda_M} \left( \sum_{j=0}^{m-1} \psi^s(j) \ast c \right)^k(t,t) d\nu(t) \]

\[ = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{k} \int_{\Lambda_M} (-d + wc)^k(t,t) d\nu(t). \]

The equality (2.8) holds for \( w = 0 \) since
\[ \int_{\Lambda_M} (-d)^k(t,t) d\nu(t) = (-z)^k \int_{\Lambda_M} \psi^k(t,t) d\nu(t) = 0 \]

if \( k \geq 1 \). This follows from \( \phi_{r,s} = 0 \) for \( r \geq s \). Hence it is enough to show that the derivatives of the two sides of (2.8) coincide,
\[ \sum_{k=0}^{\infty} (-1)^k w^k \int_{\Lambda_M} \left( \sum_{j=0}^{m-1} \psi^s(j) \ast c \right)^{(k+1)}(t,t) d\nu(t) \]

\[ = \sum_{n=0}^{\infty} (-1)^n \int_{\Lambda_M} ((-d + wc)^n \ast c)(t,t) d\nu(t). \]
To prove this last equality is a straightforward but somewhat tedious computation, which is based on expanding both sides and showing that the coefficient of \( w^k \) is the same on both sides. We omit the details.

We have proved

**Proposition 2.1.** Let \( p_{n,M}(\bar{x}) \) be defined by (1.19) and assume that \( \phi_{r,r+1}(x,y) \) satisfies \( |\phi_{r,r+1}(x,y)| \leq c(x)d(y) \) with \( c \in L^1(\mathbb{R}, \mu) \), \( d \in L^\infty(\mathbb{R}, \mu) \), \( -M \leq r < M \). Furthermore, let \( \Lambda_M = \{-M + 1, \ldots, M - 1\} \times \mathbb{R} \), \( \nu = \lambda \otimes \mu \), where \( \lambda \) is counting measure on \( \{-M + 1, \ldots, M - 1\} \) and let \( g : \Lambda_M \to \mathbb{C} \) be a bounded function. Then

\[
(2.9) \quad \int_{(\mathbb{R})^2} \prod_{|\mu| < M} \prod_{j=1}^n (1 + g(x_j^\mu)) p_{n,M}(\bar{x}) d\mu = \det (I + gK)_{L^2(\Lambda_M, \nu)},
\]

where \( K \) is given by (1.22), and the determinant is defined by using the Fredholm expansion (2.4).

Theorem 1.7 is a direct consequence of (2.9), compare with the discussion in [34].

If \( X_m = \{1, \ldots, m\} \) and \( \lambda \) is counting measure on \( X_m \), then \( L^2(X_m, \lambda) \cong \mathbb{R}^m \) and we have a chain of isomorphisms

\[
L^2(X_m, \lambda) \cong L^2(\Omega, \mu) \cong \mathbb{R}^m \otimes L^2(\Omega, \mu) \cong \mathbb{R}^m \otimes L^2(\Omega, \mu) \cong \mathbb{R}^m \otimes \cdots \otimes L^2(\Omega, \mu),
\]

where we have \( m \) terms in the last direct sum.

We can think of an element in \( \mathbb{R}^m \otimes L^2(\Omega, \mu) \) as a column vector \( (f_1(x) \ldots f_m(x))^t \), where \( f_i \in L^2(\Omega, \mu) \), \( 1 \leq i \leq m \). Hence, an operator on \( L^2(X_m \times \Omega, \lambda \otimes \mu) \) defined by an integral kernel \( K(r, \xi; r', \xi') \) can be thought of as a block operator on these column vectors with block kernel \( (K(r, \xi; r', \xi'))_{1 \leq r, r' \leq m} \).

We also want to prove Proposition 2.3. Let us write \( T_n(a) \) for the \( n \times n \) Toeplitz matrix with symbol \( a \) and \( T(a) \) for the one-sided infinite Toeplitz matrix with symbol \( a \). Consider the function \( \tilde{K}^{n,M}(z, w) \) defined by (1.28) and let the symbol \( a \) be given by (1.27). Then,

\[
\tilde{K}^{n,M}(z, w) = \sum_{x,y \in \mathbb{Z}} \left( \sum_{i,j=1}^n \phi_{r,M}(x, 1 - i) [T_n^{-1}(a)]_{ij} \phi_{r-M,s}(1 - j, y) \right) z^x w^{-y}
\]

\[
= \sum_{i,j=1}^n \left( \sum_{x \in \mathbb{Z}} \hat{f}_r(x) (1 - i - x) z^x w^{-i-1} \right) z^{-i} [T_n^{-1}(a)]_{ij} w^{j-1}
\]

\[
\times \left( \sum_{y \in \mathbb{Z}} \hat{f}_{r-M,s}(y + j - 1) w^{-y+1-j} \right)
\]

(2.10)

\[
= \frac{1}{w} f_r(z) f_{r-M,s}(\frac{1}{w}) \sum_{i,j=1}^n z^{-i} [T_n^{-1}(a)]_{ij} w^j.
\]

To proceed we need a formula for the inverse of a Toeplitz matrix. We will use the following result which follows from theorem 1.15 and theorem 2.15, together with its proof, in [8].

**Proposition 2.2.** Assume that \( a(z) = a_+(z) a_-(z), \ z \in \mathbb{T} \), where

\[
a_+(z) = \sum_{n=0}^{\infty} a_+^n z^n, \quad a_-(z) = \sum_{n=0}^{\infty} a_-^n z^{-n},
\]

(2.11)
\[ \sum_{n=0}^{\infty} (|a_+^n| + |a_-^n|) < \infty, \text{ and that } (a(z) \text{ has winding number zero. Furthermore, suppose that} \]

\[
\sum_{n \in \mathbb{Z}} |n|^\alpha |\hat{a}_n| < \infty
\]

for some \( \alpha > 0 \), where \( \hat{a}_n \) is the Fourier coefficient of \( a(z) \). Using (2.14) we can extend \( a_+(z) \) to \( |z| \leq 1 \) and \( a_-(z) \) to \( \{|z| \geq 1\} \cup \{\infty\} \) and we assume that they have no zeros in these regions. Then, \( T_n(a) \) is invertible for \( n \) sufficiently large and there is a constant \( C \) (which depends on \( a \)) such that

\[
\left| [T_n^{-1}(a)]_{jk} - [T(a_-^{-1})T(a_-^{-1})]_{jk} \right| \leq C \min\left( \frac{1}{n+1-k} \alpha, \frac{1}{n+1-j} \alpha \right)
\]

for \( 1 \leq j, k \leq n \).

We can now give the proof of proposition 1.8.

**Proof. (of Proposition 1.8).** The function \( a \) defined by (1.27) has a Wiener-Hopf factorization \( a = a_+a_- \) where

\[
a_\pm(z) = \prod_{t=-M}^{M-1} f_t^\pm(z),
\]

and all the assumptions of the previous theorem are satisfied. By (2.13)

\[
\left| \sum_{i,j=1}^{n} z^{-i}[T_n^{-1}(a)]_{ij} w^j - \sum_{i,j=1}^{n} z^{-i}[T(a_+)T(a_-^{-1})]_{ij} w^j \right|
\]

\[
\leq C \sum_{i,j=1}^{n} |z|^{-i}|w|^j \min\left( \frac{1}{n+1-i} \alpha, \frac{1}{n+1-j} \alpha \right)
\]

\[
\leq \frac{C}{(|z|-1)(1-|w|)} \left( \frac{1}{n^\alpha} + |w|^{n/2} + \frac{1}{|z|^{n/2}} \right).
\]

Also,

\[
\left| \sum_{i > n \text{ or } j > n} z^{-i}[T(a_+^{-1})T(a_-^{-1})]_{ij} w^j \right| \leq C \frac{|w|^{n} + |1/z|^{n}}{|(|z|-1)(1-|w|)|}
\]

Set \( b_k = 1/a_\pm \) and note that \( \hat{b}_+^k = 0 \) if \( k < 0 \) and \( \hat{b}_-^k = 0 \) if \( k > 0 \). We can now compute

\[
\sum_{i,j=1}^{\infty} z^{-i}[T(a_+^{-1})T(a_-^{-1})]_{ij} w^j
\]

\[
= \sum_{k=1}^{\infty} \left( \sum_{i \in \mathbb{Z}} z^{-i+k}(\hat{b}_+^i)_{i-k} \right) \left( \sum_{j \in \mathbb{Z}} w^{j-k}(\hat{b}_-^j)_{k-j} \right) \left( \frac{w}{z} \right)^k
\]

\[
= \frac{1}{a_+(1/z)a_-(1/w)} \frac{w}{1-w/z}.
\]
It follows that
\[
\frac{z}{w}f_{r,M}(\frac{1}{z})f_{-M,s}(\frac{1}{w}) \sum_{i,j=1}^{n} \frac{1}{z^{-i}[T(a^+)]T(a^-)}_{ij} w^j = \frac{z}{z-w} \prod_{i=1}^{M-1} f_{i}^+(\frac{1}{z}) \prod_{i=1}^{s-M} f_{i}^-(\frac{1}{w}) = k_{r,s}(z,w),
\]
and the proposition is proved.

We have
\[
K(r,x; s,y) = \tilde{K}(r,x; s,y) = \frac{1}{(2\pi i)^2} \int_{\gamma_{r2}} \frac{dz}{z^{s+1}} \int_{\gamma_{r1}} w^{y-1} dw \frac{z}{z-w} G(z,w).
\]
if \( r \geq s \), where \( 1 - \epsilon < r_1 < r_2 < 1 + \epsilon \). Using the residue theorem it follows that for \( r < s \),
\[
K(r,x; s,y) = \frac{1}{(2\pi i)^2} \int_{\gamma_{r1}} \frac{dz}{z^{s+1}} \int_{\gamma_{r2}} w^{y-1} dw \frac{z}{z-w} G(z,w).
\]

1 - \( \epsilon < r_1 < r_2 < 1 + \epsilon \), compare [28].

### 2.2. The extended Airy kernel

The extended Airy kernel is defined by \([1,9]\).

We can also define a modification by
\[
\tilde{A}(\tau, \xi; \tau', \xi') = \int_0^{\infty} e^{-\lambda(\tau-\tau')} \text{Ai}(\xi + \lambda) \text{Ai}(\xi' + \lambda) d\lambda,
\]
which is well-defined both for \( \tau \geq \tau' \) and for \( \tau < \tau' \) by the following standard estimate for the Airy function,
\[
|\text{Ai}(\xi)| \leq C_M e^{-2|\xi|^{3/2}/3}
\]
for \( \xi \geq -M \). Both \( A \) and \( \tilde{A} \) have a useful double integral formula.

**Proposition 2.3.** The extended Airy kernel \([1,3]\) is given by
\[
A(\tau, \xi; \tau', \xi') = -\frac{1}{4\pi^2} \int_{\text{Im}\, z=\eta} \frac{dz}{z^{s+1}} \int_{\text{Im}\, w=\eta'} \frac{dw}{w^{y-1}} \frac{e^{i(z+w)\xi+\tau(\tau'-z)}}{\tau'-\tau+i(z+w)}.
\]
where \( \eta, \eta' > 0 \) and \( \eta + \eta' + \tau - \tau' < 0 \) in case \( \tau' > \tau \). Also, the modified kernel \( \tilde{A} \), \([2,14]\), is given by the same formula but where we now require that \( \eta + \eta' + \tau - \tau' > 0 \).

**Proof.** This is straightforward using the identities
\[
\int_0^{\infty} e^{-\lambda(\tau-\tau'-iz-iw)} d\lambda = -\frac{1}{\tau'-\tau+i(z+w)}
\]
if \( \tau - \tau' + \eta + \eta' > 0 \) and
\[
\int_0^{\infty} e^{-\lambda(\tau'-\tau+iz+iw)} d\lambda = \frac{1}{\tau' - \tau + i(z+w)}
\]
if \( \tau - \tau' + \eta + \eta' < 0 \).

If we move the contour of integration between the two cases in proposition \([2,3]\), we pick up a contribution from the singularity and we obtain
\[
A(\tau, \xi; \tau', \xi') = \tilde{A}(\tau, \xi; \tau', \xi') - \phi_{r,s}(\xi, \xi'),
\]

\[\text{(2.19)}\]
Proposition 2.4. Let $f(\tau, x)$ be a non-negative function in $L^\infty(\mathbb{R})$ for each $\tau \in \{\tau_1, \ldots, \tau_m\}$, where $\tau_1 < \cdots < \tau_m$. Assume also that $f(\tau_k, x) = 0$ if $x < M_k$ for some number $M_k$, $k = 1, \ldots, m$. Then, the kernel

$$f(\tau, x)^{1/2}A(\tau, x; \tau', x')f(\tau', x')^{1/2}$$

defines a trace class operator on $L^2(\{\tau_1, \ldots, \tau_m\} \times \mathbb{R})$, where we have counting measure $\lambda$ on $\{\tau_1, \ldots, \tau_m\}$ and Lebesgue measure $\mu$ on $\mathbb{R}$.

Proof. We will prove the result by factoring into two Hilbert-Schmidt operators. Let $H(t) = 1$ if $t < 0$ and $H(t) = 0$ if $t \geq 0$. Set

$$\tilde{B}(\tau, x; \tau', x') = H(\tau - \tau') \int_{-\infty}^\infty e^{-y(\tau - \tau')} \text{Ai}(x + y) \text{Ai}(x' + y) dy.$$  

For $i < j$ we define

$$\tilde{B}_{ij}(\tau, x; \tau', x') = \tilde{B}(\tau, x; \tau', x')\delta_{\tau_i, \tau_j} \delta_{\tau_i, \tau_j},$$

so that

$$\tilde{B}(\tau, x; \tau', x') = \sum_{1 \leq i < j \leq m} \tilde{B}_{ij}(\tau, x; \tau', x').$$

Since, by (2.19) and (2.21), $A = \tilde{A} - \tilde{B}$, it suffices to show that $f^{1/2}\tilde{A}f^{1/2}$ and $f^{1/2}\tilde{B}f^{1/2}$, $1 \leq i < j \leq m$, are trace class operators.

Set

$$a(\tau, x; \sigma, y) = \frac{1}{\sqrt{m}} f(\tau, x)^{1/2} \text{Ai}(x + y)e^{-y(\tau - \sigma)}\chi_{[0, \infty)}(y)$$

$$b(\sigma, y; \tau', x') = \frac{1}{\sqrt{m}} \chi_{[0, \infty)}(y)\text{Ai}(x' + y)e^{-y(\sigma - \tau')}f(\tau', x')^{1/2}$$

Then $a$ and $b$ are Hilbert-Schmidt kernels on $L^2(\Lambda_m, \lambda \otimes \mu)$, $\mu = \{\tau_1, \ldots, \tau_m\} \times \mathbb{R}$. We have

$$\int_{\Lambda_m} \int_{\Lambda_m} |a(\tau, x; \sigma, y)|^2 d\lambda \otimes \mu(\tau) \delta(\lambda \otimes \mu)(\sigma, y)$$

$$= \frac{1}{m} \int_{\Lambda_m} \int_{\Lambda_m} f(\tau, x)\text{Ai}(x + y)^2\chi_{[0, \infty)}(y)e^{-2y(\tau - \sigma)} dx dy d\tau \lambda(\tau)$$

$$\leq \frac{1}{m} \sum_{i,j=1}^{m} \int_M dx \int_0^\infty dy \text{Ai}(x + y)^2 e^{2y(\tau_m - \tau_i)},$$

where $\phi_{\tau, \tau'} \equiv 0$ if $\tau \geq \tau'$ and

$$\phi_{\tau, \tau'}(\xi, \xi') = \frac{1}{4\pi(\tau' - \tau)} e^{-\frac{(\xi - \xi')^2}{4(\tau' - \tau)}} e^{\frac{(-\xi')^2}{4(\tau' - \tau)}} e^{\frac{(-\xi')^2}{4(\tau' - \tau)}}$$

if $\tau < \tau'$. Combining (1.9), (2.16) and (2.19) we see that

$$\phi_{\tau, \tau'}(\xi, \xi') = \frac{1}{\pi} e^{-\lambda(\tau - \tau')} \text{Ai}(\xi + \lambda) \text{Ai}(\xi' + \lambda) d\lambda$$

if $\tau < \tau'$. We would also like to show that the operator in (1.10) is actually a trace class operator.
where \( M = \min(M_1, \ldots, M_n). \) Using (2.17) we see that the integral in the last expression is \( < \infty. \) The proof that \( b \) is a Hilbert-Schmidt kernel is analogous. Now,

\[
\int_{\Lambda_m} a(\tau, x; \sigma, y)b(\sigma, y; \tau', \tau', x')d(\lambda \otimes \mu)(\tau, y)
\]

\[
\frac{1}{m} \sum_{\sigma \in \{\tau_1, \ldots, \tau_m\}} f(\tau, x)^{1/2} f(\tau', x')^{1/2} \int_{0}^{\infty} e^{-y(\tau - \tau')} \text{Ai}(x + y)\text{Ai}(x' + y)dy
\]

\[
= f(\tau, x)^{1/2} \tilde{A}(\tau, x; \tau', x')f(\tau', x')^{1/2}.
\]

Hence, the operator \( f^{1/2} \tilde{A}f^{1/2} \) is trace class. Next, set

\[
c_{ij}(\tau, x; \tau', x') = \frac{1}{\sqrt{m}} f(\tau, x)^{1/2} \text{Ai}(x + y)e^{-y(\tau - \tau')}2\delta_{\tau, \tau},
\]

(it is independent of \( \sigma \)) and

\[
d_{ij}(\tau, x; \tau', x') = \frac{1}{\sqrt{m}} f(\tau', x')^{1/2} \text{Ai}(x' + y)e^{-y(\tau - \tau')}2\delta_{\tau', \tau'}.
\]

Then,

\[
\int_{\Lambda_m} c_{ij}(\tau, x; \tau', x')d_{ij}(\tau, x; \tau', x')d(\lambda \otimes \mu)(\sigma, y)
\]

\[
= \frac{1}{m} \sum_{\sigma \in \{\tau_1, \ldots, \tau_m\}} f(\tau, x)^{1/2} f(\tau', x')^{1/2} \int_{-\infty}^{\infty} e^{-y(\tau - \tau')} \text{Ai}(x + y)\text{Ai}(x' + y)dy\delta_{\tau, \tau}\delta_{\tau', \tau}
\]

\[
= \tilde{B}_{ij}(\tau, x; \tau', x').
\]

It remains to prove that \( c_{ij} \) and \( d_{ij} \) are Hilbert-Schmidt kernels. Consider \( c_{ij} \); the proof for \( d_{ij} \) is similar. We get

\[
\frac{1}{m} \sum_{\sigma, \tau \in \{\tau_1, \ldots, \tau_m\}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau, x)\text{Ai}(x + y)^{2}e^{-y(\tau - \tau')}dx dy\delta_{\tau, \tau'}
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\tau, x)\text{Ai}(x + y)^{2}e^{-y(\tau - \tau')}dx dy
\]

\[
\leq ||f||_{\infty} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\text{Ai}(x + y)^{2}e^{y(\tau - \tau')}
\]

\[
\leq ||f||_{\infty} \int_{-\infty}^{\infty} dx \int_{0}^{\infty} d\text{Ai}(x + y)^{2}e^{y(\tau - \tau')}
\]

\[
+ ||f||_{\infty} \int_{-\infty}^{\infty} dx \int_{-\infty}^{0} d\text{Ai}(x + y)^{2}e^{y(\tau - \tau')}.
\]

The first integral in the last expression is \( < \infty \) by (2.17). Now, by (2.17)

\[
\int_{M_i}^{\infty} \text{Ai}(x + y)^{2}dx = \int_{M_i + y}^{\infty} \text{Ai}(x)^{2}dx \leq C(1 + |y|)
\]
since the Airy function is bounded. Hence,
\[
||f||_\infty \int_{M_i} dx \int_{-\infty}^0 dy \text{Ai}(x + y)^2 e^{y(\tau_j - \tau_i)}
\leq C ||f||_\infty \int_{-\infty}^0 (1 + |y|) e^{y(\tau_j - \tau_i)} dy < \infty,
\]
since \(\tau_j - \tau_i > 0\). This completes the proof. \(\square\)

2.3. An example: random walks on the discrete circle. We will consider non-intersecting walks on the set \(\mathbb{Z}_N\) of integers modulo \(N\), the discrete circle. This type of model has been analyzed in [12] and we will show how it fits into the present formalism. We have \(2M - 1\) copies of \(\mathbb{Z}_N\), where the first and the last are identified so that we have periodic boundary conditions in the time direction. We will have are non-intersecting paths on the discrete torus. Let \(x^r \in \mathbb{Z}_N^n\) be the particle configuration \((n\ \text{particles})\) on the \(r\):th discrete circle, \(|r| < M\, x^{-M+1} \equiv x^{M-1}\).

Assume that \(n\) is odd, \(n = 2\nu + 1\) and that the transition probabilities for the walks are given by
\[
\phi_{r,r+1}(x,y) = \begin{cases} p, & \text{if } y - x = 1 \\ q, & \text{if } y - x = 0. \\ 0, & \text{otherwise} \end{cases}
\]
for \(x,y \in \mathbb{Z}_N\), where \(p,q \geq 0\) and \(p + q = 1\). The transition probability for non-intersecting paths from a configuration \(x^r\) to a configuration \(x^{r+1}\) is
\[
\det(\phi_{r,r+1}(x_i^r, x_j^{r+1})_{1 \leq i,j \leq n}).
\]
Write \(\bar{x} = (x^{-M+1}, \ldots, x^{M-1}) \in (\mathbb{Z}_N)^{2M-1}\) for the total configuration. The probability of \(\bar{x}\) is
\[
q_{n,N,M}(\bar{x}) = \prod_{r=-M+1}^{M-2} \det(\phi_{r,r+1}(x_i^r, x_j^{r+1})_{1 \leq i,j \leq n}).
\]

We will use discrete Fourier series on \(\mathbb{Z}_N\),
\[
\hat{f}(n) = \frac{1}{N} \sum_{\ell \in \mathbb{Z}_N} f(\ell) z^{-\ell n}, \quad f(\ell) = \sum_{n \in \mathbb{Z}_N} \hat{f}(n) z^{\ell n},
\]
where \(z = e^{2\pi i/N}\). Also, we can represent Kronecker’s delta on \(\mathbb{Z}_N\) as
\[
\delta_{xy} = \frac{1}{N} \sum_{k \in \mathbb{Z}_N} z^{k(x-y)}.
\]

Let \(\hat{\mathbb{Z}}_N^n = \{ x \in \mathbb{Z}_N^n : 0 \leq x_1 < \cdots < x_n < N \}\) be all ordered configurations of \(n\) particles on \(\mathbb{Z}_N\). If \(x^{-M+1}, x^{M-1} \in \hat{\mathbb{Z}}_N^n\), then
\[
\det(\delta_{x_i^{-M+1}, x_j^{M-1}})_{1 \leq i,j \leq n} = \delta_{x^{-M+1}, x^{M-1}}.
by the definition of the determinant. This determinant can be rewritten using \[ (2.23) \]
and Heine’s identity,
\[
\delta_{x^{-M+1}, x^{M-1}} = \det \left( \frac{1}{N} \sum_{k \in Z_N} z^{k(x^{-M+1} - x^{M-1})} \right)_{1 \leq i, j \leq n}
\]
\[
= \frac{1}{n!N^n} \sum_{k_1, \ldots, k_n} \det(z^{k_i x_j^{-M+1}})_{1 \leq i, j \leq n} \det(z^{-k_i x_j^{M-1}})_{1 \leq i, j \leq n},
\]
if \( x^{-M+1}, x^{M-1} \in \mathbb{Z}_N^\alpha \). This leads us to the measure
\[
p_{n, N, M}(\bar{x}) = \frac{1}{(n!)^{2M-1} Z_{n, N, M}} g_{n, N, M}(\bar{x}) \delta_{x^{-M+1}, x^{M-1}}
\]
where \( Z_{n, N, M} \) is the normalization constant.
Let \( g(r, x), |r| < M, x \in \mathbb{Z}_N \), be given functions and set
\[
G(\bar{x}) = \prod_{|r| < M} \prod_{j=1}^n (1 + g(r, x_j^r)).
\]
We want to compute the expectation
\[
\sum_{\bar{x} \in (\mathbb{Z}_N^\alpha)^{2M-1}} G(\bar{x}) p_{n, N, M}(\bar{x})
\]
\[
= \frac{n!}{N^n Z_{n, N, M}} \sum_{\bar{x} \in (\mathbb{Z}_N^\alpha)^{2M-1}} \sum_{k \in \mathbb{Z}_N^\alpha} G(\bar{x}) w_{n, N, M}(k; \bar{x}),
\]
where
\[
w_{n, N, M}(k; \bar{x}) = \det(z^{k_i x_j^{-M+1}}) g_{n, N, M}(\bar{x}) \det(z^{-k_i x_j^{M-1}})
\]
\[
= \prod_{r=1}^{M-1} \det(\phi_{r, r+1}(x_i^r, x_j^{r+1})).
\]
Here we have set \( \phi_{-M, -M+1}(k_i, x_j^{-M+1}) = z^{k_i x_j^{-M+1}}, \phi_{M-1, M}(x_i^{M-1}, k_j) = z^{-k_j x_i^{M-1}} \)
and \( x_i^{-M} = x_i^M = k_i \in \mathbb{Z}_N \). We have a measure of the form \[ (1.17) \]. Set
\[
Z_{n, N, M}(k) = \sum_{\bar{x} \in (\mathbb{Z}_N^\alpha)^{2M-1}} w_{n, N, M}(k; \bar{x})
\]
and note that \( G \equiv 1 \) in \[ (2.23) \] gives
\[
Z_{n, N, M} = \frac{n!}{N^n} \sum_{k \in \mathbb{Z}_N^\alpha} Z_{n, N, M}(k).
\]
Let us also write
\[
p_{n, N, M}(k, \bar{x}) = \frac{1}{(n!)^{2M-1} Z_{n, N, M}(k)} w_{n, N, M}(k; \bar{x}).
\]
The expectation \[ (2.25) \] can then be written, using \[ (2.28) \],
\[
\sum_{k \in \mathbb{Z}_N^\alpha} \frac{Z_{n, N, M}(k)}{Z_{n, N, M}(k)} E_{n, N, M}(k; G)
\]
where \( E_{n, N, M}(k; G) \) is the “expectation” of \( G \) with respect to the measure \( p_{n, N, M}(k, \bar{x}) \).
This “expectation” can be computed using the standard framework. Let \( f(n) \) be
equal to \( q \) if \( n = 0, p \) if \( n = 1 \) and 0 otherwise, \( n \in \mathbb{Z}_N \), so that \( \phi_{r+1}(x, y) = \hat{f}(y - x) \), for \(-M < r < M - 1\). Then \( f(\ell) = q + p\ell \), \( \ell \in \mathbb{Z}_N \), and by standard properties of convolution

\[
\phi_{r,s}(x, y) = \hat{f}^{s-r}(y - x),
\]

\(-M < r < s < M - 1\). From this we see that

\[
A_{ij} = \phi_{-M,M}(k_i, k_j) = \sum_{x, y \in \mathbb{Z}_N} z^{k_i x} \hat{f}^{2M-2}(y - x) z^{-k_j y}
\]

\(= N f(N - k_i)^{2M-2} \delta_{k_i, k_j} = N f(N - k_i)^{2M-2} \delta_{ij} \)

if \( k \in \mathbb{Z}_N \). Thus,

\[
A = (N f(N - k_i)^{2M-2} \delta_{ij})_{i, j = 1, \ldots, n}.
\]

Now,

\[
Z_{n, N, M}(k) = \det A = N^n \prod_{i=1}^n (q + pe^{-2\pi ik_i/N})^{2M-2}.
\]

This is always non-zero if \( p \neq q \). If \( p = q = 1/2 \), then we assume that \( N \) is odd, which also implies that \( \det A \neq 0 \). We obtain

\[
\tilde{K}_n(k; r, x; s, y) = \sum_{i, j=1}^n \left( \sum_{\ell \in \mathbb{Z}_N} \hat{f}^{M-r-1}(\ell - x) z^{-k_i \ell} \right) \frac{1}{N} f(N - k_i)^{2M-2} \delta_{ij}
\]

\[
\left( \sum_{m \in \mathbb{Z}_N} \hat{f}^s \delta_{m,s} (y - m) z^{k_j m} \right)
\]

\(= \frac{1}{N} \sum_{i=1}^n f(N - k_i)^{s-r} z^{k_i (y - x)} \)

where we have indicated the dependence of the kernel on \( k \). Note that this kernel is independent of \( M \). We have

\[
K_n(k; r, x; s, y) = \frac{1}{N} \sum_{i=1}^n f(N - k_i)^{s-r} z^{k_i (y - x)} - \phi_{r,s}(x, y),
\]

where

\[
\phi_{r,s}(x, y) = \frac{1}{N} \sum_{\ell \in \mathbb{Z}_N} f(\ell)^{s-r} z^{\ell (y - x)}
\]

if \( s > r \) and \( \phi_{r,s}(x, y) = 0 \) if \( s \leq r \). Computations similar to those leading up to formula (3.27) below show that if we assume that \( g(r, \cdot) \equiv 0 \) if \( |r| > M_0 \), then \( E_{n, N, M}(k, G) = E_{n, N, M_0}(k, G) \) for \( M \geq M_0 \). Hence, the expectation (2.29) can be written

\[
\sum_{k \in \mathbb{Z}_N} \left( \lim_{M \to \infty} \frac{Z_{n, N, M}(k)}{\sum_{k \in \mathbb{Z}_N} Z_{n, N, M}(k)} \right) E_{n, N, M_0}(k, G).
\]

Lemma 2.5. Let \( \alpha_i = i - 1, i = 1, \ldots, \nu + 1 \), \( \alpha_{2\nu + 2 - i} = N - i, i = 1, \ldots, \nu \), \( n = 2\nu + 1 \). If \( k \in \mathbb{Z}_N \), then

\[
\lim_{M \to \infty} \sum_{k \in \mathbb{Z}_N} Z_{n, N, M}(k) = \delta_{\alpha, k}.
\]
Proof. We use the explicit formula (2.32) for $Z_{n,N,M}(k)$,

$$Z_{n,N,M}(k) = N^n \prod_{j=1}^n (q + pe^{-2\pi ik_j/N})^{2M-2}.$$ 

Now,

$$|q + pe^{-2\pi ik_j/N}|^2 = p^2 + q^2 + 2pq \cos \frac{2\pi k_j}{N}.$$ 

This is maximal (= 1) if $k_j = 0(= N)$ and it is easy to see that

$$\prod_{j=1}^n (p^2 + q^2 + 2pq \cos \frac{2\pi ik_j}{N}) \leq \prod_{j=1}^n (p^2 + q^2 + 2pq \cos \frac{2\pi i\alpha_j}{N})$$

with strict inequality unless $k = \alpha$. This completes the proof. \qed

Hence, if $g(r, \cdot) \equiv 0$ for $|r| \geq M_0$, then

$$\lim_{M \to \infty} \sum_{\bar{x} \in (\mathbb{Z}_N^2)^{2M-1}} G(\bar{x}) p_{n,N,M}(\bar{x}) = E_{n,N,M_0}(\alpha; G).$$

From this it follows that the correlation kernel $K(r, x; s, y)$ on the cylinder $\mathbb{Z} \times \mathbb{Z}_N$ is given by

$$(2.38) \quad K(r, x; s, y) = \frac{1}{N} \sum_{j=-\nu}^{\nu} (q + pe^{2\pi ij/N})^s r e^{2\pi ij(x-y)/N}$$

$$- \omega_{r,s} \frac{1}{N} \sum_{j=-\nu}^{N-\nu-1} (q + pe^{2\pi ij/N})^s r e^{2\pi ij(x-y)/N}$$

where $\omega_{r,s} = 1$ if $r < s$, $\omega_{r,s} = 0$ if $r \geq s$.

The induced measure on $\mathbb{Z}_N$ is given by

$$\frac{1}{n!} \det(K(0, x_\mu; 0, x_\nu))_{1 \leq \mu, \nu \leq n} = \frac{1}{n!} \det\left( \frac{1}{N} \sum_{j=-\nu}^{\nu} e^{2\pi ij(x_\mu - x_\nu)/N} \right)_{1 \leq \mu, \nu \leq n}$$

$$= \frac{1}{n! N^n} \prod_{1 \leq \mu < \nu \leq n} |e^{2\pi i x_\mu/N} - e^{2\pi i x_\nu/N}|^2,$$

the equilibrium measure on $\mathbb{Z}_N$ (discrete CUE), see [23].

We can take the limit $n, N \to \infty$, $n/N \to \rho$, $0 < \rho < 1$, and obtain a limiting determinantal process on $\mathbb{Z}^2$.

Proposition 2.6. The correlation function for $n = 2\nu + 1$ non-intersecting walks on the infinite cylinder $\mathbb{Z} \times \mathbb{Z}_N$ as defined above is given by

$$(2.39) \quad K(r, x; s, y) = \int_{-\rho/2}^{\rho/2} (q + pe^{2\pi i \theta})^s r e^{2\pi i \theta(x-y)} d\theta$$

if $r \geq s$, and

$$(2.40) \quad K(r, x; s, y) = -\int_{\rho/2}^{1-\rho/2} (q + pe^{2\pi i \theta})^s r e^{2\pi i \theta(x-y)} d\theta$$

if $r < s$ for $0 < \rho < 1$. 

Proposition 2.7. The correlation function for the determinantal process on $\mathbb{Z}^2$ induced by non-intersecting random walks as defined above is given by

$$(2.39) \quad K(r, x; s, y) = \int_{-\rho/2}^{\rho/2} (q + pe^{2\pi i \theta})^s r e^{2\pi i \theta(x-y)} d\theta$$

if $r \geq s$, and

$$(2.40) \quad K(r, x; s, y) = -\int_{\rho/2}^{1-\rho/2} (q + pe^{2\pi i \theta})^s r e^{2\pi i \theta(x-y)} d\theta$$

if $r < s$ for $0 < \rho < 1$. 


This kernel is related to the $B^\pm$-kernels in [28]. Compare also with [37].

3. Multi-layer discrete PNG

We will discuss how the PNG model defined by (1.1), in the case when $\omega(x,t)$, $(x,t) \in Z \times N$, satisfies $\omega(x,t) = 0$ if $t - x$ is even or if $|x| > t$, can be embedded as the top curve in a multi-layer process given by a family of non-intersecting paths. We think of the $\omega(x,t)$s as given numbers. The initial condition is $h(x,0) = 0$, $x \in Z$. We extend $h(x,t)$ to all $x \in \mathbb{R}$ by letting $h(x,t) = h([x],t)$, which makes it right continuous at the jumps. Note that it follows immediately that $h(x,t) = 0$ if $x < -t + 1$ or $x > t$.

For $t - x$ odd we define the jumps,

$$
\eta^+(x,t) = h(x,t) - h(x-1,t) $$
$$
\eta^-(x,t) = h(x,t) - h(x+1,t).
$$

We will see below that $\eta^+, \eta^- \geq 0$ and we should think of $\eta^+(x,t)$ as a positive jump at $x$ at time $t$, and $\eta^-(x,t)$ as a the size of a negative jump at $x+1$ at time $t$. Define

$$
T\omega(x,t) = \min(\eta^+(x+1,t-1), \eta^-((x-1,t-1))
$$

if $t - x$ is odd and $T\omega(x,t) = 0$ if $t - x$ is even.

Claim 3.1. The jumps $\eta^+$ and $\eta^-$ satisfy the following evolution equations

$$
\eta^+(x+1,t+1) = \max(\eta^+(x+2,t) - \eta^-(x,t),0) + \omega(x+1,t-1) $$
$$
\eta^-(x+1,t+1) = \max(\eta^-(x,t) - \eta^+(x+2,t),0) + \omega(x+1,t-1)
$$

for $t - x$ odd. Furthermore $\eta^+(x,t)$ and $\eta^-(x,t)$ are $\geq 0$.

Proof. We proceed by induction on $t$. Assume that $\eta^+(x,t), \eta^-(x,t) \geq 0$ for all $x$ such that $t - x$ is odd. We will prove that then (3.3) holds, and hence $\eta^+(x+1,t+1), \eta^-(x+1,t+1) \geq 0$ for all $x$ such that $t - x$ is odd. Obviously our induction assumption is true for $t = 0$. Note that $h(x+1,t) = h(x,t) - \eta^-(x,t)$, $h(x+2,t) = h(x,t) + \eta^+(x+2,t) - \eta^-(x,t)$ and $h(x-1,t) = h(x,t) - \eta^+(x,t)$. It follows from (1.1), our induction assumption and $\omega(x,t+1) = 0$, that

$$
h(x+1,t+1) = h(x,t) + \max(0, \eta^+(x+2,t) - \eta^-(x,t)) + \omega(x+1,t+1) $$
$$
h(x,t+1) = h(x,t)
$$

and the first half of (3.3) follows. The proof of the second half is analogous. \( \Box \)

There is also an inverse recursion formula.

Claim 3.2. If $t - x$ is odd, then

$$
\omega(x+1,t+1) = \min(\eta^-(x+1,t+1), \eta^+(x+1,t+1))
$$
$$
\eta^+(x,t) = \eta^+(x-1,t+1) - \omega(x-1,t+1) + T\omega(x-1,t+1)
$$
$$
\eta^-(x,t) = \eta^+(x+1,t+1) - \omega(x+1,t+1) + T\omega(x+1,t+1)
$$

Proof. The first equation follows immediately from (3.3). From (3.2) and (3.3) we see that the right hand side of the second equation in (3.4) equals

$$
\max(\eta^+(x,t) - \eta^-(x-2,t),0) + \min(\eta^+(x,t), \eta^-(x-2,t))
$$

which equals $\eta^+(x,t)$. The proof of the last equation is similar. \( \Box \)
From this claim we immediately deduce the following

**Claim 3.3.** If we know $\eta^+(x+1, t+1)$, $\eta^-(x+1, t+1)$ for all $x$ such that $t - x$ is odd, and $T_\omega(x, s)$ for $s \leq t + 1$ and all $x$, we can reconstruct $\omega(x, s)$, $s \leq t + 1$, $x \in \mathbb{Z}$, uniquely.

Let a coordinate system $(i, j)$ be related to the $(x, t)$ coordinate system via the transformation

$$ (x, t) = (i - j, i + j - 1), $$

and define $w(i, j)$ by (3.2). Then $w(i, j) = 0$ if $(i, j) \notin \mathbb{Z}_2^2$, and this condition corresponds exactly to our assumptions on $\omega(x, t)$. Similarly to (3.2) we define

$$ Tw(i, j) = T_\omega(i - j, i + j - 1). $$

**Claim 3.4.** Assume that $w(i, j) = 0$ if $i$ or $j$ is $\leq s$. Then, $Tw(i, j) = 0$ if $i$ or $j$ is $\leq s + 1$.

**Proof.** It follows from the condition on $w(i, j)$ that $\omega(x, t) = 0$ if $(t + x + 1)/2 \leq s$ or $(t - x + 1)/2 \leq s$, which implies, using (3.1), that $h(x, t) = 0$ under the same conditions. It follows from (3.1), (3.4) and (3.6) that $Tw(i, j) = 0$ if $h(i - j + 1, i + j - 2) = 0$ or $h(i - j - 1, i + j - 2) = 0$. Now, $h(i - j + 1, i + j - 2) = 0$ if $i \leq s$ or $j \leq s + 1$, and $h(i - j - 1, i + j - 2) = 0$ if $i \leq s + 1$ or $j \leq s$. Hence $Tw(i, j) = 0$ if $i \leq s + 1$ or $j \leq s + 1$.

It follows from claim 3.4 that $T^n w(i, j) = 0$ if $i$ or $j$ is $\leq n$, since $w(i, j) = 0$ if $i$ or $j$ is $\leq 0$. Hence $T^n \omega(x, t) = 0$ if $t \leq 2n - 1$, since $i + j - 1 \leq 2n - 1$ implies that $i$ or $j$ is $\leq n$. We formulate this as our next claim.

**Claim 3.5.** If $t \leq 2n - 1$, then $T^n \omega(x, t) = 0$.

Let $h_i(x, t)$, $i \geq 0$, be the PNG process defined by (1.1) with $\omega(x, t + 1)$ replaced by $T_\omega(x, t + 1)$, and with initial condition $h_i(x, 0) = -i$. We let $T^n \omega(x, t + 1) = \omega(x, t + 1)$, so $h_0(x, t) = h(x, t)$ is our original growth process. It follows from claim 3.3 that at time $t = 2n - 1$ only $h_0, \ldots, h_{n - 1}$ can be non-trivial, i.e. $h_i(x, 2n - 1) = -i$ for all $x$ if $i \geq n$. Combining claim 3.3 and claim 3.5 we get

**Claim 3.6.** Given $h_i(x, 2n - 1)$, $x \in \mathbb{Z}$, $i = 0, \ldots, n - 1$, we can uniquely reconstruct $\{\omega(x, t); t \leq 2n - 1, x \in \mathbb{Z}\}$.

We can think of $h_i$ at time $2n - 1$ as a directed path from $(-2n + 1, -i)$ to $(2n - 1, -i)$ which has up-steps $\eta^+(2m, 2n - 1)$ at even $x$-coordinates, $x = 2m$ and down-steps $\eta^-(2m, 2n - 1)$ at odd $x$-coordinates, $x = 2m + 1$, $|m| < n$, and horizontal steps in between. According to (3.6) there is a bijection between these paths $h_0, \ldots, h_{n - 1}$ and the set $\{\omega(x, t); t \leq 2n - 1, x \in \mathbb{Z}\}$. We set $h_i(x, t) = h_i([x], t)$ for $x \in \mathbb{R}$. The paths obtained are nonintersecting:

**Claim 3.7.** If $t - x$ is odd, then

$$ h_{i + 1}(x, t) < h_i(x - 0, t) $$

and if $t - x$ is even

$$ h_{i + 1}(x - 0, t) < h_i(x, t), $$

so that corners will not meet.
Proof. This is proved by induction on \( t \). It is clearly true for \( t = 0 \). If it is true at time \( t \) it is still true after forming the maximum in (3.2) (deterministic step). (Note that \( h_i \) and \( h_{i+1} \) have up-steps/down-steps at the same positions.) From the definition (3.2) it is still true after adding \( T^{i+1}\omega(x,t) \) to the lower curve. \( \square \)

In order to understand how a geometric distribution (1.3) on the \( w(i, j) \) is transported to a measure on the non-intersecting paths, we will assign weights to the jumps. Let \( a_i \) and \( b_j \) be given variables. The jumps are assigned weights as follows: \( \eta^+(x,t) \) has weight \( a_i \eta^+(x,t) \), \( i = (t + x + 1)/2 \) and \( \eta^-(x,t) \) has weight \( b_j \eta^-(x,t) \), \( j = (t - x + 1)/2 \). Also, to \( T^k \omega(x,t) \) we assign the weight \( (a_i, b_j)^T \omega(x,t) \) with the same correspondence between \( (i, j) \) and \( (x, t) \), \( k \geq 0 \). The proof of the next claim is a straightforward computation using the definitions of the quantities involved.

Claim 3.8. The product of the weights of \( \eta^+(x-1,t+1) \), \( \eta^-(x-1,t+1) \) and \( T^k \omega(x-1,t+1) \) equals the product of the weights of \( \eta^+(x-2,t) \), \( \eta^-(x,t) \) and \( \omega(x-1,t+1) \).

Using this claim we can show that the measure is transported in the way we want.

Claim 3.9. The product of all the weights of all the jumps in the multi-layer PNG, \( h_0, \ldots, h_{n-1} \), at time \( t = 2n - 1 \) equals,
\[
\prod_{i+j \leq 2n} (a_i b_j)^{w(i,j)}.
\]

Proof. Using claim 3.8 repeatedly we see that
\[
\prod_{i+j \leq 2n} (a_i b_j)^{w(i,j)} = \prod_{x \in \mathbb{Z}, t \leq 2n-1} (a_{(t+x+1)/2} b_{(t-x+1)/2})^{\omega(x,t)}
\]
\[
= \prod_{|m| < n} a_{n+m}^{\eta^+(2m,2n-1)} b_{n-m}^{\eta^-(2m,2n-1)} \prod_{x \in \mathbb{Z}, t \leq 2n-1} (a_{(t+x+1)/2} b_{(t-x+1)/2})^{\omega(x,t)}
\]
Repeated use of this identity toghether with claim 3.5 proves the claim. \( \square \)

It is now easy to see that (1.5) holds.

Proposition 3.10. Set \( G(i,j) = h(i-j, i+j-1) \). Then
\[
(3.9) \quad G(i,j) = \max((g(i-1,j), G(i, j-1)) + w(i,j)
\]
for \( i, j \geq 1 \).

Proof. We have that
\[
h(i-j, i+j-1)
\]
\[
= \max(h(i-j-1, i+j-2), h(i-j, i+j-2), h(i-j+1, i+j-2) + w(i,j)
\]
\[
= \max(G(i-1,j), h(i-j, i+j-2), G(i, j-1) + w(i, j)).
\]
Since \( h(i-j-1, i+j-2) - h(i-j, i+j-2) = \eta^-(i-j-1, i+j-2) \geq 0 \), this last expression equals the right hand side of (3.5) and we are done. \( \square \)

If \( \eta^+_r, \eta^-_r \) are the jumps for \( h_r \) it follows from the assignments of weights that \( \eta^+_r(2m,2n-1) = u \) has weight \( a_{m+n} \) and \( \eta^-_r(2m,2n-1) = u \) has weight \( b_{n-m} \), \( |m| < n, 0 \leq r < n \). If we think of the weights as labels transported from the \( w(i,j) \)'s we see that if \( w(i,j) = 0 \) for \( i > n \) or \( j > n \), we have no labels \( a_i \) with \( i > n \)
or \( b_j \) with \( j > n \) and hence \( \eta^- (2m, 2n - 1) = 0 \) if \( m < 0 \) and \( \eta^+ (2m, 2n - 1) = 0 \) if \( m > 0 \). Hence all plus-steps take place to the left of the origin and all minus-steps to the right of the origin. This is the case discussed in \([20]\). From this consideration and \([1,4]\) we obtain.

**Proposition 3.11.** If \( |K| < N \), then

\[
G(N + K, N - K) = h(2K, 2N - 1).
\]

Also, if \( w(i, j) = 0 \) for \( |i| > N \) or \( j > N \), then for \( 0 \leq K < N \),

\[
G(N - K, N) = h(-2K, 2N - 1)
\]

and

\[
G(N, N - K) = h(2K, 2N - 1).
\]

The discussion of the multi-layer extension of the PNG-growth model discussed above is closely related to the Viennot/matrix-ball construction, \([21]\), \([4]\), \([5]\), of the Robinson-Schensted-Knuth (RSK) correspondence. Let us briefly discuss the relation. We can think of \([3,2]\) and \([3,3]\) geometrically as follows. From \( (x, t) \) to \( (x - 1, t + 1) \) we draw a line with multiplicity \( \eta^+ (x, t) \) and from \( (x, t) \) to \( (x + 1, t + 1) \) we draw a line with multiplicity \( \eta^- (x, t) \). A line with multiplicity zero means no line. At \( (x, t) \) a line with multiplicity \( \eta^+ (x + 1, t - 1) \) and a line with multiplicity \( \eta^- (x - 1, t - 1) \) meet and we have a collision/annihilation of size \( T \omega(x, t) \) as given by \([3,2]\). If \( \eta^+ (x + 1, t - 1) \geq \eta^- (x - 1, t - 1) \), then \( \eta^+ (x + 1, t - 1) - \eta^- (x - 1, t - 1) \) plus lines survive and we add \( \omega(x, t) \) new lines. Similarly in the other case. This explains \([3,3]\). Assume that \( w(i, j) = 0 \) if \( i \) or \( j \) is \( > N \). If \( (w(i, j))_{1 \leq i, j \leq N} \) is a permutation matrix this gives exactly the “shadow lines” of the Viennot construction. We obtain a mapping to a pair of semi-standard Young tableaux \( P \) and \( Q \) of shape \( \lambda \). The number of lines in the first row of \( P \) equals \( \eta^- (-N - m, N + m - 1), m = 1, \ldots, N \) and the number of lines in the first column of \( Q \) equals \( \eta^+ (-N - m, N + m - 1), m = 1, \ldots, N \). Similarly, the same procedure starting with \( T \omega \) instead gives the second rows and so on. Using this line of argument we obtain

**Proposition 3.12.** Let \( (w(i, j))_{1 \leq i, j \leq N} \) be given and set \( w(i, j) = 0 \) if \( i \) or \( j \) is \( > N \). The RSK-correspondence maps a submatrix \( (w(i, j))_{1 \leq i, j \leq M, 1 \leq i, j \leq N, M \leq N} \) to a pair of semi-standard Young tableaux of shape \( \lambda(M, N) = (\lambda_1(M, N), \lambda_2(M, N), \ldots) \). (Similarly, we can consider \( (w(i, j))_{1 \leq i, j \leq N, 1 \leq i, j \leq M} \).) Consider the family of height curves \( h_i, 0 \leq i < N \), obtained from the multi-layer PNG process using \( (w(i, j)) \).

Then, for \( 0 \leq K < N, 1 \leq j \leq N, \)

\[
\lambda_j(N - K, N) = h_{j-1}(-2K, 2N - 1) + j - 1
\]

and

\[
\lambda_j(N, N - K) = h_{j-1}(2K, 2N - 1) + j - 1.
\]

If we add vertical line segments to the graphs, \( x \to h_i(x, 2N - 1), 0 \leq i < N \), we obtain \( N \) non-intersecting paths with \( h_i(-2N - 1, 2N - 1) = h_i(2N - 1, 2N - 1) = -i \). Recall that \( h_i(x, 2N - 1) \equiv 1 - i \) for \( i \geq N \) so that at most \( N \) paths are “active”. The paths are described by particle configurations. Let

\[
C_{2N-1}(x) = (h_0(x, 2N - 1), \ldots, h_{N-1}(x, 2N - 1))
\]

and

\[
C_{2N-1} = (C_{2N-1}(-M + 1), \ldots, C_{2N-1}(M - 1)),
\]
where $M = 2N - 1$. Note that $C_{2N-1}(-M) = C_{2N-1}(M) = (0, -1, \ldots, -N + 1)$. Set
\begin{equation}
\phi_{2j-1,2j}(x,y) = \begin{cases} 
(1 - a_{j+N})a_{j+N}^{y-x} & \text{if } y \geq x \\
0 & \text{if } y < x,
\end{cases}
\end{equation}
\begin{equation}
\phi_{2j,2j+1}(x,y) = \begin{cases} 
0 & \text{if } y > x, \\
(1 - b_{N-j})b_{N-j}^{x-y} & \text{if } y \leq x
\end{cases}
\end{equation}
for $|j| < N$ with the convention that $0^0 = 1$. It follows from the Lindström-Gessel-Viennot method or from the Karlin-McGregor theorem that the weight of the non-intersecting path configuration corresponding to $C_{2N-1} = \bar{x}$, with weights assigned to jumps as above, equals
\[
\left( \prod_{r = -M}^{M-1} \det(\phi_{r,r+1}(x_r^t, x_{r+1}^t))_{i,j=1}^N \right)^{1/(1 - a_j)^N (1 - b_j)^N}.
\]
The way the weights are related to the “weights” of the geometric random variables as described above shows that
\[
P[C_{2N-1} = \bar{x}] = \frac{1}{Z_{n,M}} \prod_{r = -M}^{M-1} \det(\phi_{r,r+1}(x_r^t, x_{r+1}^t))_{i,j=1}^N
\]
with $Z_{n,M}$ given by (1.18) and $n = N, M = 2N - 1$. Hence we obtain a measure of the form (1.19). We note that
\begin{equation}
Z_{n,M} = \prod_{i+j \leq M} (1 - a_i b_j)^n
\end{equation}
We summarize what we have found in the next proposition.

**Proposition 3.13.** Let $h_i, i \geq 0$, be the multi-layer PNG process obtained from geometric random variables with parameters $a_i b_j$ as defined above and let $\phi_{r,r+1}$ be defined by (3.14) and (3.10). Then,
\begin{equation}
P[h_{k-1}(r, 2N - 1) = x_k^r, 1 \leq i \leq n, |r| < M]
= \frac{1}{Z_{n,M}} \prod_{r = -M}^{M-1} \det(\phi_{r,r+1}(x_r^t, x_{r+1}^t))_{i,j=1}^N,
\end{equation}
where $Z_{n,M}$ is given by (3.14), $x_i^- = x_i^1 = 1 - i, x_i^1 > x_i^2 > \cdots > x_i^N$ for each $r$, $n = N$ and $M = 2N - 1$.

The fact that the probability measure has this form makes it possible to compute the correlation functions. Set
\begin{equation}
f_{2j-1}(z) = (1 - a_{j+N}) \sum_{m=0}^{\infty} a_{j+N}^m z^m = \frac{1 - a_{j+N}}{1 - a_{j+N} z}
\end{equation}
and
\begin{equation}
f_{2j}(z) = (1 - b_{N-j}) \sum_{m=0}^{\infty} b_{N-j}^m z^m = \frac{1 - b_{N-j}}{1 - b_{N-j} z}
\end{equation}
so that (1.22) holds. The interpretation of the correlation functions given by (1.23) in this case is that they give the probability of finding particles at the specified
where \( n \) is the number of PNG height curves. All height curves \( h_i \) with \( i \geq N \) have to be trivial, i.e. \( h_i \equiv -i \) if \( i \geq N \). It follows that the probability of a certain configuration is independent of \( n \) for \( n \geq N \).

Thus, we can take the kernel \( K^{n,M} \) (2.22) with an arbitrary \( n, n \geq N \) arbitrary and obtain the same value. In particular we can let \( n \rightarrow \infty \) and use proposition 3.8. It is clear that all the conditions of this theorem are satisfied when \( f_r(z) \) is given by (3.19) and (3.20). Let \( r = 2u, s = 2v \) both be even, \( |u|, |v| < N \). The expression (1.30) becomes

\[
G(z, w) = \prod_{j=u}^{N-1} \left( \frac{1-b_{N-j}}{1-b_{N-j}z} \right) \prod_{j=-N+1}^{v} \left( \frac{1-a_{N+j}}{1-a_{N+j}/w} \right) \prod_{j=-N+1}^{w} \left( \frac{1-b_{N-j}}{1-b_{N-j}w} \right).
\]

We summarize our results for the correlation functions in a theorem.

**Theorem 3.14.** Let the multi-layer PNG process be defined using geometric random variables \( w(i, j) \) with parameter \( a_i b_j \), \( 0 < a_i b_j < 1 \), and let \( G(z, w) \) be given by (3.22). Set

\[
\tilde{K}_N(2u, x; 2v, y) = \frac{1}{(2\pi i)^2} \int_{\gamma_2} \frac{dz}{z} \int_{\gamma_1} \frac{dw}{w} \frac{w^u z^v}{z - w} G(z, w),
\]

where \( \gamma_r \) is the circle with radius \( r \) and center at the origin, \( 1 - \epsilon < r_1 < r_2 < 1 + \epsilon \) with \( 1 + \epsilon < \min(1/b_j) \), \( 1 - \epsilon > \max(a_j) \) and \( |u|, |v| < N \), \( x, y \in \mathbb{Z} \). Furthermore, let

\[
\phi_{2u, 2v}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(y-x)\theta} G(e^{i\theta}, e^{i\theta}) d\theta,
\]

for \( u < v \) and \( \phi_{2u, 2v}(x, y) = 0 \) for \( u \geq v \). Set

\[
K_N(2u, x; 2v, y) = \tilde{K}_N(2u, x; 2v, y) - \phi_{2u, 2v}(x, y).
\]

Then,

\[
P[(2u, x_j^{2u}) \in \{(2t, h_i(2t, 2N - 1)); |t| < N, 0 \leq i < N\}, |u| < N, 1 \leq j \leq k_u] = \det(K_N(2u, x_j^{2u}; 2v, x_j^{2v}))_{|u|<N,0\leq k_u,1\leq j \leq k_u}
\]

for any \( x_j^{2u} \in \mathbb{Z} \) and any \( k_u \in \{0, \ldots, N\} \).

Consider the finite-dimensional distribution of \( h_0(x, t) = h(x, t) \), the top curve,

\[
P_{n,M}[h_0(2s_i, 2N - 1) \leq \ell_i, 1 \leq i \leq m],
\]

where \( n \) is the number of paths, \( M = 2N - 1, n \geq N, |s_i| < N \) and \( \ell_i > -N \). This can also be written

\[
P_{n,M}[\text{no particles in } \{2s_i \times (\ell_i, \infty), 1 \leq i \leq m\}].
\]

This probability is independent of \( n \geq N \) and hence we can let \( n \rightarrow \infty \). Let \( g(2s_i, x) = -\lambda_{s_i, \infty}(x), 1 \leq i \leq m \), and \( g(r, x) \equiv 0 \) if \( r \) is not equal to one of the \( 2s_i \). Hence, by proposition 2.1,

\[
P_{N,M}[h_0(2s_i, 2N - 1) \leq \ell_i, 1 \leq i \leq m] = \det(I + gK_N)_{L^2(\Lambda_M)}.
\]

This formula can be used to study the convergence in distribution of the rescaled height curve.
4. Asymptotics

We will consider the asymptotics of the kernel (3.24) in the case \( a_i = b_i = \alpha \) for all \( i \geq 1 \), so that \( w(i,j) \) are geometric random variables with parameter \( q = \alpha^2 \). The function \( G(z, w) \) in (3.21) then becomes

\[
G(z, w) = (1 - \alpha)^{2(v-u)} \frac{(1 - \alpha/z)^{N+u} (1 - \alpha w)^{N-v}}{(1 - \alpha z)^{N-u} (1 - \alpha w)^{N+v}}.
\]

Write

\[
F_{u,x}(z) = \frac{1}{\alpha N} \frac{(z - \alpha)^{N+u}}{(1 - \alpha z)^{N-u}},
\]

so that, by (1.22),

\[
\tilde{K}_N(2u, x; 2v, y) = \frac{(1 - \alpha)^{2(v-u)}}{(2\pi i)^2} \int_{\tau r_2} \frac{dz}{z} \int_{\tau r_1} \frac{dw}{w} \frac{z}{z - w} F_{u,x}(z) F_{-v,y}(\frac{1}{w}),
\]

where \( \alpha < r_1 < r_2 < 1/\alpha \).

Set \( \mu = m/N, \mu' = m'/N, \beta = u/N, \beta' = -v/N \) and

\[
f_{\mu, \beta}(z) = \frac{1}{N} \log F_{m, -N}(z) = (1 + \beta) \log(z - \alpha) - (1 - \beta) \log(1 - \alpha z) - (\mu + \beta) \log z.
\]

Then, \( f'(z) = P(z)/Q(z) \), where \( Q(z) = z(1 - \alpha z) \)

\[
\frac{P(z)}{z^2(1 - \alpha z)^2} = \frac{1 - \alpha^2 - \mu(1 + \alpha^2)}{\alpha(\mu - \beta)} z + \frac{\mu + \beta}{\mu - \beta}.
\]

We will write

\[
p = p(\mu, \beta) = \frac{\mu(1 + \alpha^2) - (1 - \alpha^2)}{2\alpha(\mu - \beta)}; \quad q = a(\mu, \beta) = \frac{\mu + \beta}{\mu - \beta}.
\]

The critical points of \( f \) are \( z_c^\pm = p \pm \sqrt{p^2 - q} \) and we obtain a double critical point if \( p^2 = q \) which gives

\[
\mu = \mu_c(\beta) = \frac{1 + \alpha^2}{1 - \alpha^2} + \sqrt{\left(\frac{1 + \alpha^2}{1 - \alpha^2}\right)^2 - 1 - \frac{4\alpha^2\beta^2}{(1 - \alpha^2)^2}}
\]

and

\[
p_c = p(\mu_c, \beta) = \frac{2\alpha + (1 + \alpha^2)\sqrt{1 - \beta^2}}{1 + \alpha^2 + 2\alpha \sqrt{1 - \beta^2} - \beta(1 - \alpha^2)}.
\]

Set

\[
d = \frac{\alpha^{1/3}(1 + \alpha)^{1/3}}{1 - \alpha}, \quad d' = \frac{1 - \alpha}{1 + \alpha}.
\]

It will be convenient to write

\[
u = \frac{1}{d'} \tau N^{2/3}, \quad v = \frac{1}{d'} \tau' N^{2/3},
\]

since \( N^{2/3} \) is the right scale for \( u \) and \( v \) if we want a non-trivial limit. The correct way of writing \( x \) and \( y \) will turn out to be

\[
x = N(\mu_c(\beta) - 1) + \xi d N^{1/3}, \quad y = N(\mu_c(\beta') - 1) + \xi' d' N^{1/3}.
\]

We will assume that \( |\tau|, |\tau'|, |\xi|, |\xi'| \) are \( \leq \log N \).
The paths of integration can be deformed into
\begin{align}
\Gamma : \mathbb{R} &\ni t' \to z(t') = p_c(\beta) + \frac{\eta}{dN^{1/3}} - \frac{it'}{dN^{1/3}} \doteq p - it, \\
\Gamma' : \mathbb{R} &\ni s' \to w(s') = p_c(\beta') + \frac{\eta'}{dN^{1/3}} - \frac{i(s')}{dN^{1/3}} \doteq (s' - it)^{-1},
\end{align}
where \(\eta, \eta' > 0\) will be appropriately chosen; we will require that
\begin{equation}
(p_c(\beta) + \frac{\eta}{dN^{1/3}})(p_c(\beta') + \frac{\eta'}{dN^{1/3}}) > 1.
\end{equation}
In that case we have, by Cauchy's theorem,
\begin{equation}
\tilde{K}_N(2u, x; 2v, y) = -\frac{(1 - \alpha)^2}{(2\pi i)^2} \int_{\Gamma} \frac{dz}{z} \int_{\Gamma'} \frac{dw}{w} \frac{z}{z - w} F_{u,x}(z) F_{v,y}(\frac{1}{w}).
\end{equation}
We first estimate this integral and then we will compute its asymptotics using a saddle-point argument. Choose \(\mu\) so that \(p(\mu, \beta) = p = p_c(\beta) + \eta/dN^{1/3}\) as in (4.9), and let \(q = q(\mu, \beta)\) be the value we get with this \(\mu\). This is possible by formula (4.3) with \(\mu > \mu_c\). We can write
\begin{equation}
F_{u,x}(z) = \frac{1}{z^{\mu + \alpha N}} \frac{1}{z^{(\mu + \beta) N}} \frac{(z - \alpha)^{N + u}}{(1 - \alpha z)^{N - u}},
\end{equation}
and then let \(z = p - it, t \in \mathbb{R}\), and take the absolute value to get
\begin{equation}
|F_{u,x}(p - it)|^2 = \frac{1}{(p^2 + t^2)^{\mu + \alpha N}} e^{2Nh(t)},
\end{equation}
where
\begin{equation}
2h(t) = (1 + \beta) \log A - (1 - \beta) \log B - (\mu + \beta) \log C,
\end{equation}
with
\begin{align}
A &= (p - \alpha)^2 + t^2 = 1 - 2\alpha p + \alpha^2 + \frac{2\beta}{\mu - \beta} + p^2 - q + t^2, \\
B &= (1 - \alpha p)^2 + \alpha^2 t^2 = 1 - 2\alpha p + \alpha^2 + \frac{2\alpha^2 \beta}{\mu - \beta} + \alpha^2 (p^2 - q + t^2), \\
C &= p^2 + t^2 = 1 + \frac{2\beta}{\mu - \beta} + p^2 - q + t^2.
\end{align}
Note that
\begin{align}
(\mu - \beta) A &= (1 - \alpha^2) \beta + 1 - \alpha^2 + (\mu - \beta)(p^2 - q + t^2), \\
(\mu - \beta) B &= 1 - \alpha^2 - (1 - \alpha^2) \beta + \alpha^2 (\mu - \beta)(p^2 - q + t^2), \\
(\mu - \beta) C &= (\mu - \beta)(p^2 - q + t^2) + \mu + \beta.
\end{align}
A computation now gives
\begin{equation}
h'(t) = \frac{t(p^2 - q + t^2)}{(\mu - \beta)ABC} \{ (1 - \alpha^2)^2 - (1 - \alpha^4)(\mu + \beta) + (1 + \alpha^4) \mu \beta + 2\alpha^2 \beta^2 - \alpha^2 (\mu - \beta)^2 (p^2 - q + t^2) \}. 
\end{equation}
Another computation shows that $p^2 - q \geq 0$. To leading order we have $p^2 - q \approx 2\eta/dN^{1/3}$. Recall that $t = t'/dN^{1/3}$. When $t$ is large we have $h'(t) \approx -(\mu - \beta)/t$, and we also have the estimate
\begin{equation}
(4.15) \quad h'(t) \leq \frac{t}{(\mu - \beta)ABC} (p^2 - q) \{(1 - \alpha^2)^2 - (1 - \alpha^4)(\mu + \beta) + (1 + \alpha^4)\mu \beta + 2\alpha^2 \beta^2\}.
\end{equation}
for $t \geq 0$. Consider the case $0 \leq t' \leq N^\gamma$; the case $-N^\gamma \leq t' \leq 0$ is analogous by symmetry. Here $0 < \gamma < 1/3$. Using (4.14) we see that $h(t) - h(0) \approx -2\eta t^2/N$ and we can show that
\begin{equation}
(4.16) \quad Nh'(t) \leq -\frac{3}{2} \eta t^2 + Nh(0)
\end{equation}
for $|t'| \leq N^\gamma$, $0 < \gamma < 1/3$, and $N$ sufficiently large. Define $h_*(t, p)$ by
\begin{equation}
e^{2Nh_*(t, p)} = \frac{1}{(p^2 + t^2)(\mu + \beta)^N} \frac{[(p - \alpha)^2 + t^2(1 + \beta)]^{(1 - \gamma)N}}{[(1 + \alpha)\mu + \beta + \alpha^2 \beta^2]^{(1 - \gamma)N}}.
\end{equation}
A computation, compare (4.23) below, gives
\begin{equation}
e^{h_*(0, p_c)} \sim (1 - \alpha)^{2u} e^{\frac{1}{2}t^2}
\end{equation}
and $h_*(0, p) = h_*(0, p_c) + d^3(p - p_c)^3/3 + \cdots = h_*(0, p_c) + \eta^3/3N + \cdots$. Consequently,
\begin{equation}
e^{Nh(0)} = \frac{1}{p^{N(\mu - \mu_c)}} e^{Nh_*(t, p)} \sim \frac{(1 - \alpha)^{2u}}{p^{N(\mu - \mu_c)}} e^{(\tau^3 + \eta^3)/3}.
\end{equation}
Combining this with (4.16) gives
\begin{equation}
|F_{u, x}(p - it)| \leq C \frac{(1 - \alpha)^{2u} e^{(\tau^3 + \eta^3)/3 - 3\eta t^2/2}}{(p^2 + t^2)(x + N(1 - \mu)/2)^{1/2}}.
\end{equation}
Write
\begin{equation}
\frac{1}{(p^2 + t^2)(x + N(1 - \mu))/2} = \left|\frac{1}{(p - it)x + N(1 - \mu_c)}\right| \left(\frac{p^2 + t^2}{p^2}\right)^{N(\mu - \mu_c)/2}.
\end{equation}
Further computation shows that
\begin{equation}
\left|\frac{1}{(p - it)x + N(1 - \mu_c)}\right| \sim e^{-\xi \tau - \xi \eta}
\end{equation}
\begin{equation}
(1 + t^2/p^2)^{N(\mu - \mu_c)/2} \sim e^{\eta t^2}.
\end{equation}
Collecting the estimates we find
\begin{equation}
(4.17) \quad |F_{u, x}(p_c(\beta) + \frac{\eta}{dN^{1/3}} - \frac{i t'}{dN^{1/3}})| \leq C(1 - \alpha)^{2u} e^{\frac{1}{4}(\tau^3 + \eta^3) - \xi \tau - \xi \eta - \frac{3}{4}t^2}
\end{equation}
for $|t'| \leq N^\gamma$, $0 < \eta < N^\gamma$, $0 < \gamma < 1/3$.

Using (4.14), (4.13) and the other estimates above we see that the contribution to the integral from $|t'| \geq N^\gamma$ and/or $|s'| \geq N^\gamma$ is $\leq C \exp(-cN^{2\gamma})$ for some constant $c > 0$. Hence, using the parametrization (4.9) in (4.12) we can restrict to $|t'| \leq N^\gamma$, $|s'| \leq N^\gamma$. We can use (4.17) if we want an estimate of the integral. To get the asymptotics we make a local saddle-point argument.

To leading order we have $p_c(\beta) = 1 + \tau/dN^{1/3}$, $p_c(\beta') = 1 - \tau'/dN^{1/3}$ and hence the condition (4.11) requires
\begin{equation}
(4.18) \quad \tau - \tau' + \eta + \eta' > 0.
\end{equation}
We will use the parametrizations \(4.3\) and consider the integral
\[
(4.19) \quad - \frac{(1 - \alpha)^{(v-u)}}{(2\pi i)^2} \int_{|t| \leq N^{\gamma}} dt \int_{|s| \leq N^{\gamma}} ds \frac{z'(t) w'(t)}{z(t) w(t)} \frac{z(t) w(s) y + N(1 - \mu_c(\beta'))}{z(t) z + N(1 - \mu_c(\beta))} \times e^{Nf_{\mu_c(\beta),\beta}(z(t)) + Nf_{\mu_c(\beta'),\beta'}(1/w(s))}.
\]

Now,
\[
(4.20) \quad Nf_{\mu_c(\beta),\beta}(p_c(\beta)) + \frac{1}{dN^{1/3}} (\eta - it)) = Nf_{\mu_c(\beta),\beta}(p_c(\beta)) + i \left( \frac{1}{2d^3} f_{\mu_c(\beta),\beta}(p_c(\beta)) \right) (t + i\eta)^3 + r_N(t)
\]
\[
= Nf_{\mu_c(\beta),\beta}(p_c(\beta)) + \frac{i}{3} (t + i\eta)^3 + r_N(t)
\]
where the remainder term \(r_N(t)\) can be neglected for \(|t| \leq N^\gamma\). Also, \(z'(t) = -i/dN^{1/3}\), \(w'(s) = iw(s)^2/dN^{1/3}\) and
\[
(4.21) \quad \frac{w(s)}{z(t) - w(s)} \sim \frac{dN^{1/3}}{-\tau' - \tau + i(t + i\eta + s + i\eta')}.
\]

Furthermore
\[
(4.22) \quad \frac{w(s)y + N(1 - \mu_c(\beta'))}{z(t)z + N(1 - \mu_c(\beta))} \sim e^{\xi \tau' - \xi \tau + i(\tau + i\eta)i(\tau + i\eta')}.\]

We also need to compute
\[
e^{Nf_{\mu_c(\beta),\beta}(p_c(\beta))} = \frac{(p_c(\beta) - \alpha)^{N+u}}{(1 - \alpha p_c(\beta))^{N-u}} \frac{1}{p_c^{N\mu_c(\beta) - v}}.
\]

Using the formulas \(4.4\) and \(4.5\) above a rather long computation, which we omit, shows that
\[
(4.23) \quad e^{Nf_{\mu_c(\beta),\beta}(p_c(\beta))} \sim (1 - \alpha)^{2u} e^{v/3}.
\]

Inserting \(4.21\) - \(4.23\) into \(4.19\) we see that, provided \(4.18\) holds,
\[
(4.24) \quad \lim_{N \to \infty} dN^{1/3} K_N \int_{\frac{1 + \alpha}{1 - \alpha} dN^{2/3}}^2 \frac{2\alpha}{1 - \alpha} N + (\xi - \tau^2)dN^{1/3} = 0 \quad \text{for} \quad 2\frac{1 + \alpha}{1 - \alpha} \int_{\frac{1 + \alpha}{1 - \alpha} dN^{2/3}}^2 \frac{2\alpha}{1 - \alpha} N + (\xi' - \tau^2)dN^{1/3}
\]
\[
= - \frac{1}{4\pi^2} e^{\frac{1}{2}(\tau^3 - \tau^3) + \xi' \tau - \xi \tau} \int_{\text{Im } z = \eta} \int_{\text{Im } w = \eta'} e^{i(\tau + \xi') \tau + i(z + w)} dz dw.
\]

Here we have used \(\mu_c(\beta) = \frac{1 + \alpha}{1 - \alpha} \frac{\alpha}{1 - \alpha} \quad \text{and} \quad \ldots\). We also want to compute the corresponding limit of \(4.22\) with \(G(w, w')\) given by \(4.1\), i.e. we consider, \(u < v\),
\[
(4.25) \quad \phi_{2u,2v}(x, y) = \frac{(1 - \alpha)^{2(v-u)}}{2\pi} \int_{-\pi}^\pi e^{i(y-x)\theta + (v-u)\log(1 + \alpha^2 - 2\alpha \cos \theta)} d\theta.
\]

If we set \(g(\theta) = \log(1 + \alpha^2 - 2\alpha \cos \theta)\), then
\[
g'(\theta) = \frac{2\alpha \sin \theta}{1 + \alpha^2 - 2\alpha \cos \theta}.
\]
and we see that \( g(\theta) \) has a quadratic minimum at \( \theta = 0 \). Hence, we can immediately both compute the asymptotics of and estimate the integral in (4.25) when \( x = 2\alpha(1 - \alpha)^{-1}N + (\xi - \tau^2)dN^{1/3}, \ y = 2\alpha(1 - \alpha)^{-1}N + (\xi' - \tau^2)dN^{1/3}, \ u = \frac{1 + \alpha}{1 - \alpha}d^{-1}N^{2/3}\tau \) and \( v = \frac{1 + \alpha}{1 - \alpha}d^{-1}N^{2/3}\tau' \). We obtain
\[
\lim_{N \to \infty} dN^{1/3} \phi_{2u,2v}(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(\xi' - \xi + \tau^2 - \tau'^2)t - (\tau' - \tau)^2} dt = \frac{1}{\sqrt{4\pi(\tau' - \tau)}} e^{-(\xi' - \xi + \tau^2 - \tau'^2)^2/2(\tau' - \tau)}.
\]

We want to identify the right hand side of (4.24) combined with (4.26) with the extended Airy kernel. This can be done using the proposition 2.3. Combining this double integral formula for the extended Airy kernel with (4.24) and (4.26) we obtain the following result.

**Proposition 4.1.** Let \( d = (1 - \alpha)^{-1}\alpha^{1/3}(1 + \alpha)^{1/3} \) and let \( K_N \) be given by (3.24). Then
\[
\lim_{N \to \infty} dN^{1/3}K_N(2\frac{1 + \alpha}{1 - \alpha}d^{-1}N^{2/3}\tau, 2\frac{2\alpha}{1 - \alpha}N + (\xi - \tau^2)dN^{2/3}, 2\frac{1 + \alpha}{1 - \alpha}d^{-1}N^{2/3}\tau', 2\frac{2\alpha}{1 - \alpha}N + (\xi' - \tau'^2)dN^{2/3}) = e^{(\tau^3 - \tau'^3)/3 + \xi^2 - \xi'^2} A(\tau, \xi; \tau', \xi')
\]
uniformly for \( \xi, \xi', \tau, \tau' \) in a compact set.

We can now combine the formula (3.27), theorem 3.14, proposition 4.1 and some estimates of \( K_N \), which can be obtained from the asymptotic analysis above, to prove the following theorem on convergence in distribution to the Airy process. A complete proof requires a control of the convergence of the Fredholm expansions but we will not present the details. The individual determinants in the Fredholm expansion can be estimated using the Hadamard inequality. Compare with theorem 1.2 and lemma 3.1 in [13].

5. A functional limit theorem

5.1. A moment estimate. Consider the PNG height functions \( h_k(x,2N-1) \) defined in sect. 3. Set
\[
t_j = \frac{j}{cN^{2/3}}
\]
where \( c = (1 + \alpha)(1 - \alpha)^{-1}d^{-1} \), \( j \in \mathbb{Z} \). The normalized height functions are
\[
H_{N,k}(t_j) = \frac{1}{dN^{1/3}}(h_k(2j,2N-1) - \frac{2\alpha}{1 - \alpha}N),
\]
with \( d \) as in (4.6), \( k \in \mathbb{N} \). For a given function \( f : \mathbb{R} \to \mathbb{C} \) we write
\[
f_N(x) = f(\frac{1}{dN^{1/3}}(x - \frac{2\alpha}{1 - \alpha}N)).
\]
Assume that there is a \( K \) such that \( f(x) = 0 \) for \( x \leq K \). Define
\[
H_N(f,t_j) = \sum_{k=0}^{\infty} f_N(h_k(2j,2N-1)) = \sum_{k=0}^{\infty} f(H_{N,k}(t_j)).
\]
Lemma 5.1. Assume that $f$ is a $C^\infty$ function and that there are constants $K_1$ and $K_2$ such that $f(x) = 0$ if $x \leq K_1$, and $f(x)$ equals a constant if $x \geq K_2$. There is a constant $C(f, \alpha)$ so that

\begin{equation}
\mathbb{E}[(H_N(f, t_u) - H_N(f, t_v))^4] \leq C(f, \alpha)e^{-|t_u|^3}|t_u - t_v|^2,
\end{equation}

for $|t_u - t_v| \leq 1$ and $|t_u|, |t_v| \leq \log N$.

Proof. The proof is rather long and complicated. We will outline the main ideas and steps in the argument without giving full details. The left hand side of (5.2) can be written

\begin{equation}
\sum_{k_1, k_2, k_3, k_4=1}^\infty \mathbb{E}[\prod_{r=1}^4(f_N(h_{k_r}(2u, 2N - 1)) - f_N(h_{k_r}(2u, 2N - 1)))].
\end{equation}

We can rewrite this using formula (3.23) in theorem 3.14. Let us write the kernel $K_N(2u, x; 2v, y)$ in (3.24) as $K_{uv}(x, y)$. We will use the following notation:

\begin{equation}
K_{r_1 r_2 \ldots r_m}(x_1, x_2, \ldots, x_m) = \det(K(r_i, x_i; r_j, y_j))_{i, j=1}^m,
\end{equation}

and we will also write

\begin{equation}
K(r_1, x_1 \ r_2, x_2 \ \ldots \ \ r_m, x_m) = \det(K(r_i, x_i; r_j, x_j))_{i, j=1}^m.
\end{equation}

Furthermore, we will write

\begin{equation}
D_{u_1, \ldots, u_m}(x_1, \ldots, x_m) = K(2u_1, x_1 2u_2, x_2 \ldots 2u_m, x_m).
\end{equation}

Set $h_N(x_1, x_2; x_3) = -6[f_N(x_1)^2 f_N(x_2) f_N(x_3) + f_N(x_1) f_N(x_2)^2 f_N(x_3) - f_N(x_1) f_N(x_2) f_N(x_3)]$, which is symmetric under permutation of $x_1$ and $x_2$. Then, the sum in (5.3) can be written

\begin{equation}
\sum_{x \in \mathbb{Z}^3} f_N(x_1) f_N(x_2) f_N(x_3) f_N(x_4) [D_{uuuu}(x_1, x_2, x_3, x_4) - 4D_{uuvv}(x_1, x_2, x_3, x_4) + 6D_{uuuv}(x_1, x_2, x_3, x_4) - 4D_{uuvu}(x_1, x_2, x_3, x_4)]
\end{equation}

\begin{equation}
+ \sum_{x, y \in \mathbb{Z}^3} (6f_N(x_1)^2 f_N(x_2) f_N(x_3) [D_{uuuu}(x_1, x_2, x_3, x_4) + D_{uuvu}(x_1, x_2, x_3, x_4)]
\end{equation}

\begin{equation}
+ h_N(x_1, x_2, x_3) D_{uuuu}(x_1, x_2, x_3, x_4) + h_N(x_3, x_2, x_1) D_{uuvu}(x_1, x_2, x_3, x_4)
\end{equation}

\begin{equation}
+ \sum_{x \in \mathbb{Z}^2} 2(f_N(x_1)^3 f_N(x_2) + f_N(x_1) f_N(x_3)^3) [D_{uu}(x_1, x_2) - 2D_{uv}(x_1, x_2) + D_{vu}(x_1, x_2)]
\end{equation}

\begin{equation}
+ \sum_{x \in \mathbb{Z}^2} 3f_N(x_1)^2 f_N(x_2)^2 [D_{uu}(x_1, x_2) + 2D_{uv}(x_1, x_2) + D_{vu}(x_1, x_2)]
\end{equation}

\begin{equation}
+ \sum_{x_1 \in \mathbb{Z}} 2f_N(x_1)^4 [K_{uu}(x_1, x_1) + K_{vv}(x_1, x_1)]
\end{equation}

\begin{equation}
\equiv \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5.
\end{equation}

We have $K_{uv} = \tilde{K}_{uv} - \phi$ if $u < v$ and $K_{uv} = \hat{K}_{uv}$ if $u \geq v$. Here we have written $\phi = \phi_{u,v}$. Set $\Delta K_{uu} = K_{uu} - \tilde{K}_{uu}$, $\Delta K_{uv} = K_{uv} - \tilde{K}_{uv}$, $\Delta K_{vu} = K_{vu} - \tilde{K}_{vu}$, $\Delta K_{vv} = K_{vv} - \hat{K}_{vv}$. We see from (4.20) that $\phi$ acts like a kind of approximate $\delta$-function. This will be important for the cancellation between different terms in (5.7). The argument goes as follows. We will take out all terms in (5.7) containing $\phi$ and combine them
with other terms so that we get cancellation. We will then expand in $\Delta K_{uv}$, $\Delta K_{vu}$ and $\Delta K_{uv}$. The terms linear in $\Delta K$ will cancel and what will remain will be terms containing $(\Delta K)^2$ or higher powers. They will give a contribution proportional to $|t_u - t_v|^2$ which is what we want.

In the computations below we use symmetries and also relabelling of variables. Expand in $\phi$ and in the terms linear in $\phi$ we expand in $\Delta K$. Let $\tilde{D}$ denote the same object as in (5.6) but with $K$ replaced by $K$. We find

(5.8)
$$
\Sigma_1 = \sum_x f_N(x_1)f_N(x_2)f_N(x_3)f_N(x_4)[\tilde{D}_{uuuv}(x_1, x_2, x_3, x_4) - 4\tilde{D}_{uuvv}(x_1, x_2, x_3, x_4)]
+ 6\tilde{D}_{uuvv}(x_1, x_2, x_3, x_4) - 4\tilde{D}_{uvuv}(x_1, x_2, x_3, x_4) + \tilde{D}_{uvvu}(x_1, x_2, x_3, x_4)]
+ \sum_x 24\phi(x_1, x_4)K_{uu}^{uv}(x_3, x_4)\left[\Delta K_{uv}(x_2, x_3) + \Delta K_{uv}(x_2, x_3)\right]
\Delta K_{uv}(x_2, x_3)]f_N(x_1)f_N(x_2)f_N(x_3)f_N(x_4)
- \sum_x 12\phi(x_1, x_4)\phi(x_2, x_3)\tilde{K}_{uu}^{uv}(x_3, x_4)f_N(x_1)f_N(x_2)f_N(x_3)f_N(x_4)
$$

\[\sum_x (\text{terms with } \Delta K^2).\]

We will give a brief discussion of the $\Delta K^2$ terms below. Also we will see then that terms containing

(5.9)
$$
\Delta K_{uv}(x, y) + \Delta K_{uv}(x, y) - \Delta K_{uv}(x, y)
$$

will give a contribution proportional to $|t_u - t_v|^2$. If we expand the $\tilde{D}$-part of (5.8) in $\Delta K$ we will see that the terms linear in $\Delta K$ cancel out. Since obviously the 0-th order term equals zero we are left with $\Delta K^2$-terms. The term containing two $\phi$-factors will be combined with other terms below.

We expand $\Sigma_2$ similarly. The part linear in $\Delta K$ is

(5.10)
$$
- \sum_x 24f_N(x_1)^2f_N(x_2)f_N(x_3)\tilde{K}_{uu}^{uv}(x_3, x_4)\left[\Delta K_{uv}(x_2, x_3)
+ \Delta K_{uv}(x_2, x_3) - \Delta K_{uv}(x_2, x_3)\right].
$$

Actually this sum can be combined with the corresponding term in (5.8) to get some cancellation, see the $\phi$-calculations below, but we can also use the fact that (5.9) has the right order. We also get $\Delta K^2$-terms and a term linear in $\phi$,

(5.11)
$$
\sum_x 12\phi(x_1, x_3)[f_N(x_1)^2f_N(x_2)f_N(x_3) - f_N(x_1)f_N(x_2)f_N(x_3)^2]
\times \left\{\tilde{K}_{uu}^{uv}(x_2, x_3) - \tilde{K}_{uu}^{uv}(x_1, x_2)\right\}
+ \sum_x 12\phi(x_1, x_3)f_N(x_1)^2f_N(x_3)\left\{\tilde{K}_{uu}^{uv}(x_1, x_3) + \tilde{K}_{uu}^{uv}(x_1, x_2)\right\}
\approx \Sigma_a + \Sigma_b.
$$
Consider next $\Sigma_3$. We get a term linear in $\phi$,
\begin{equation}
(5.12) \quad - \sum_x 4\phi(x_1, x_2)K_{vu}(x_2, x_1)[f_N(x_1)f_N(x_2)^3 + f_N(x_1)^3f_N(x_2)]
\end{equation}
a term linear in $\Delta K$,
\begin{equation}
(5.13) \quad \sum_x 4(f_N(x_1)f_N(x_2)^3 + f_N(x_1)^3f_N(x_2))|\Delta K_{vu}(x_1, x_2) + \Delta K_{uv}(x_1, x_2) - \Delta K_{vu}(x_1, x_2)]K_{uu}(x_2, x_1),
\end{equation}
and $\Delta K^2$-terms. In (5.13) we again have the expression (5.9).

The leading term in $\Sigma_4$ is
\begin{equation}
(5.14) \quad \sum_x 3f_N(x_1)^2f_N(x_2)^2 \left[ K_{uu}^{vu} \left( \frac{x_1}{x_1} \frac{x_2}{x_2} \right) + 2K_{uv}^{vu} \left( \frac{x_1}{x_1} \frac{x_2}{x_2} \right) + K_{vv}^{vu} \left( \frac{x_1}{x_1} \frac{x_2}{x_2} \right) \right]
\end{equation}
and we also have a term linear in $\phi$,
\begin{equation}
(5.15) \quad \sum_x 6f_N(x_1)^2f_N(x_2)^2\phi(x_1, x_2)K_{vu}(x_2, x_1).
\end{equation}
Finally we have $\Sigma_5$ which is
\begin{equation}
(5.16) \quad \sum_x 2f_N(x_1)^4[K_{uu}(x_1, x_1) + K(x_1, x_1)].
\end{equation}

When calculating the cancellations involving the $\phi$-terms we will combine the double $\phi$-term in (5.8) with $\Sigma_6$ in (5.11) and (5.14). Also we will combine (5.12), (5.13) and (5.16). We will discuss this second case first in some detail and then the first case more briefly. The term $\Sigma_6$ is similar and finally we will indicate what is involved in estimating (5.9) and the $\Delta K^2$-terms.

We want to estimate
\begin{equation}
(5.17) \quad \sum_{x,y \in \mathbb{Z}} \phi(x, y)K_{vu}(y, x)[6f_N(x_1)^2f_N(x_2)^2 - 4f_N(x_1)f_N(x_2)^3 - 4f_N(x_1)^3f_N(x_2)] + \sum_{y \in \mathbb{Z}}(K_{uu}(y, y) + K_{vv}(y, y))f_N(y)^4.
\end{equation}
Here we have made a symmetrization in $x$ and $y$ by setting
\begin{equation}
\tilde{K}_{vu}(x, y) = \frac{1}{2}[K_{vu}(x, y) + K_{vu}(y, x)];
\end{equation}
note that $\phi(x, y)$ is symmetric in $x$ and $y$. Next, we will introduce some notation and some formulas that will be used. Set
\begin{equation}
(5.18) \quad g(z) = -\frac{\alpha}{(1-\alpha)^2}(z + \frac{1}{z} - 2)
\end{equation}
and
\begin{equation}
(5.19) \quad G^*_{ab}(z, w) = \left( \frac{1 - \alpha z}{1 - \alpha w} \right)^N \left( \frac{1 - \alpha w}{1 - \alpha z} \right)^N (1 + g(z))^a(1 + g(w))^b \frac{1}{w(z - w)}
\end{equation}
so that
\begin{equation}
(5.20) \quad \tilde{K}_{ab}(x, y) = \frac{1}{(2\pi i)^2} \int_{\gamma^2} dz \int_{\gamma^2} dw G^*_{ab}(z, w)\left( \frac{w}{z} \right)^y.
\end{equation}
\[ \phi(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(y-x)\theta} (1 + g(e^{i\theta}))^{u-v} d\theta. \]

Note that
\[ G^*_a(z, w)(1 + g(z))^c(1 + g(w))^{-d} = G^*_a(z, w). \]

Fix \( \epsilon > 0 \) and let \( f^\epsilon(x) = f(x)e^{-\epsilon x} \). Then \( f^\epsilon \) is in \( L^1(\mathbb{R}) \) and we have
\[ f_N(x)^m = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} F_m^\epsilon(\lambda)e^{i\xi_m(\lambda)(x-cN)}d^m\lambda, \]

where \( F_m^\epsilon(\lambda) = \hat{f}^\epsilon(\lambda_1) \cdots \hat{f}^\epsilon(\lambda_m), c = 2\alpha(1 + \alpha)^{-1} \) and \( \xi_m(\lambda) = (\lambda_1 + \cdots + \lambda_m - ine)/dN^{1/3} \) with \( d \) given by (4.8). Integration by parts gives
\[ (1 + g(z))^{u-v} - 1 = (u - v)g(z) + \frac{u - v}{d^4N^{4/3}}R_1(z) + \frac{(u - v)^2}{d^4N^{4/3}}R_2(z), \]

where
\[ R_1(z) = d^4N^{4/3}g(z)^2 \int_0^1 \frac{1 - t}{(1 + tg(z))^2} dt \]

and
\[ R_2(z) = d^4N^{4/3}(\log(1 + g(z)))^2 \int_0^1 (1 - t)(1 + g(z))^{t(u-v)} dt. \]

Let \( \Delta f(x) = f(x + 1) - f(x) \) be the usual finite difference operator. We have the following formula
\[ \sum_{x \in \mathbb{Z}} \phi(x, y) \left( \frac{w^y}{z^y} + \frac{w^x}{z^x} \right) f_N(x)^m = f_N(y)^m \left( 1 + g(z) \right)^{u-v} + (1 + g(w))^{u-v} \]

\[ - \frac{\alpha(u - v)}{(1 - \alpha)^2} \left[ \frac{w^y}{z^y+1} \Delta f_N^m(y) - \frac{w^y}{z^y} \Delta f_N^m(y - 1) + \frac{w^y}{z^y+1} \Delta f_N^m(y) \right] \]

\[ + \frac{w^y}{z^y-1} \Delta f_N^m(y - 1) + \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} d^m\lambda F_m^\epsilon(\lambda)e^{i\xi_m(\lambda)(y-cN)} \]

\[ \left\{ \frac{u - v}{d^4N^{4/3}} \left[ R_1(ze^{-i\xi_m(\lambda)}) - R_1(z)R_1(we^{i\xi_m(\lambda)}) - R_1(w) \right] \right\} \]

\[ + \frac{(u - v)^2}{d^4N^{4/3}} \left[ R_2(ze^{-i\xi_m(\lambda)}) - R_2(z) + R_2(we^{i\xi_m(\lambda)}) - R_2(w) \right] \]

for \( |w| = \exp(-m\epsilon/dN^{1/3}) = r_1, |z| = r_2 = 1/r_1 \). To prove this, introduce the formula (5.21) for \( \phi \) and the formula (5.23) for \( f_N^m \) into the left hand side of (5.25) and use
\[ \sum_{x \in \mathbb{Z}} (e^{-i\theta}r_1e^{i\phi}e^{i\xi_m})^x = \delta_0(\theta - \frac{1}{dN^{1/3}}(\lambda_1 + \cdots + \lambda_m) - \phi), \]
where \( \delta_0 \) is the Dirac \( \delta \)-function, to carry out the \( x \)-summation. This gives

\[
(5.26) \quad \sum_{x \in \mathbb{Z}} \phi(x, y) \left( \frac{w y}{z^x} + \frac{w x}{z^y} \right) f_N(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} d^m \lambda F_m^\epsilon(\lambda) e^{i \xi_m(\lambda) (y-cN) \frac{w y}{z^y}} \times 
\]

\[
[(1 + g(ze^{-i \xi_m(\lambda)})u^{-v} + (1 + g(we^{i \xi_m(\lambda)})u^{-v})] 
\]

\[
= f_N(y)^m \frac{w y}{z^y} [(1 + g(z))u^{-v} + (1 + g(w))u^{-v}] 
\]

\[
+ \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} d^m \lambda F_m^\epsilon(\lambda) e^{i \xi_m(\lambda) (y-cN) \frac{w y}{z^y}} [(1 + g(ze^{-i \xi_m(\lambda)})u^{-v} - (1 + g(z))u^{-v} + (1 + g(we^{i \xi_m(\lambda)})u^{-v} - (1 + g(w))u^{-v}].
\]

In the last expression we use \((5.24)\) and the explicit form \((5.18)\) of \( g \) to obtain the right hand side of \((5.25)\). We will call the first part of the right hand side of \((5.25)\), \( f_N(y)^m \frac{w y}{z^y} [(1 + g(z))u^{-v} + (1 + g(w))u^{-v}] \), the contraction term, which is the main contribution. The second part is called the finite difference term.

We can now insert the integral formula \((5.20)\) into \((5.17)\) and use \((5.25)\). The contraction term from the first sum in \((5.17)\) will then exactly cancel the second sum. Here we use \((5.22)\). What remains is

\[
(5.27) \quad \frac{u - v}{d^4 N^{4/3}} S_0 + \frac{u - v}{d^4 N^{4/3}} S_1 + \frac{(u - v)^2}{d^4 N^{4/3}} S_2,
\]

where \( S_0 \) is the part coming from the finite differences, \( S_1 \) is the part coming from terms involving \( R_1 \) and \( S_2 \) from the terms involving \( R_2 \). After some computation we find

\[
(5.28) \quad S_0 = -\frac{\alpha}{(1 - \alpha)^2} \frac{1}{(2\pi i)^2} \int_{\gamma_{r_2}} dz \int_{\gamma_{r_1}} dw G^*_{vu}(z, w) \left( \frac{1}{z} + w \right) 
\times \sum_{y \in \mathbb{Z}} \frac{w y}{z^y} (dN^{1/3} (f_N(y + 1) - f_N(y)))^4.
\]

Also,

\[
(5.29) \quad S_i = \frac{1}{(2\pi i)^3} \int_{\mathbb{R}^3} d^3 \lambda F_3^\epsilon(\lambda) \left( \sum_{y \in \mathbb{Z}} f_N(y) e^{i \xi_3(\lambda) (y-cN) \frac{w y}{z^y}} \right) 
\times \frac{1}{(2\pi i)^2} \int_{\gamma_{r_2}} dz \int_{\gamma_{r_1}} dw G^*_{vu}(z, w) \frac{w y}{z^y} \{ 6[h(z, w; \xi_2(\lambda)) 
\]

\[
-h(z, w; 0)] - 4[h(z, w; \xi_1(\lambda)) - h(z, w; 0)] - 4[h(z, w; \xi_3(\lambda)) - h(z, w; 0)] \},
\]

\( i = 1, 2 \). In order to restrict the \( y \)-summation so that \((y - cN)/dN^{1/3}\) ranges over a compact interval we make a summation by parts in \((5.24)\). Recall that we assume that \( f(y) \) is a constant for large \( y \). If we let \( a = \exp(i \xi_3(\lambda))w/z \) and use \( a^y = (1 - a)^{-1}(a^y - a^{y+1}) \) in the \( y \)-sum in \((5.28)\) a summation by parts gives

\[
\frac{1}{1 - \exp(i \xi_3(\lambda))w/z} \sum_{y \in \mathbb{Z}} [f_N(y) - f_N(y - 1)] e^{i \xi_3(\lambda) y \frac{w y}{z^y}}.
\]
Hence, for $i = 1, 2$,

$$S_i = \sum_{y \in \mathbb{Z}} (f_N(y) - f_N(y - 1)) \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3 \lambda F_3^s(\lambda) e^{i\xi_3(\lambda)(y - cN)}$$

$$\frac{1}{(2\pi)^2} \int_{\gamma_{12}} dz \int_{\gamma_{12}} dw G_{\nu}(z, w) \frac{w^y}{z^y} \{6|h(z, w; \xi_2(\lambda)) - h(z, w; 0)| - 4|h(z, w; \xi_1(\lambda)) - h(z, w; 0)| - 4[h(z, w; \xi_3(\lambda)) - h(z, w; 0)]\} \frac{1}{1 - \exp(i\xi_3(\lambda)w/z}$$

The expressions $S_i$ will be estimated using the types of estimates derived in sect. 4.

Write $u = (1 + \alpha)(1 - \alpha)^{-1}d^{-1}N^{2/3}\tau$, $v = (1 + \alpha)(1 - \alpha)^{-1}d^{-1}N^{2/3}\tau'$ and $y = 2\alpha(1 - \alpha)^{-1}N(\xi - \tau^2)dN^{2/3}$. To estimate (5.32) we can now use our results from section 4. We use

$$z(t) = p_c(\beta) + \frac{\eta}{dN^{1/3}} - \frac{it}{dN^{1/3}} = 1 + \frac{\tau + \eta - it}{dN^{1/3}} + \ldots$$

$$w(t) = p_c(\beta') + \frac{\eta'}{dN^{1/3}} - \frac{is}{dN^{1/3}} = 1 + \frac{\tau' + \eta' - is}{dN^{1/3}} + \ldots$$

as parametrizations of the integrals as before. Using the same estimates as in section 4 we can restrict the integration to $|t|, |s| \leq N^{\gamma}$ with an error $\leq C\exp(-cN^{2\gamma})$ with some $c > 0$. Since $v - u \geq 1$, and hence $t_v - t_u \geq cN^{2/3}$, and furthermore $|t_u| \leq \log N$, we can incorporate the error term into the right hand side of (5.2). The integral in (5.32) can then be estimated using (4.17). Note that, by our assumptions on $f$, the number of $y$-terms $\neq 0$ is $\leq C N^{2/3}$ and we get a compensating factor $1/N^{2/3}$ from the parametrizations; see also (4.21). The numbers $\eta, \eta'$ are chosen so that $\eta + \eta' \geq 2$. Note that, since we assume $\tau - \tau' \leq 1$, the condition (4.13) is satisfied. We find

$$|S_0| \leq C(f, \alpha) \frac{1}{N^{2/3}} \sum_{y} e^{(\tau^3 - \tau'^3)/3 + (\xi - \tau') + (\eta^3 + \eta'^3)/3 - \xi(\eta + \eta')}$$

where the $y$-summation is over all $y \in \mathbb{Z}$ such that $(y - cN)/dN^{1/3} \in [K_1 - 1, K_2 + 1]$.

Consider now $S_i$. Write $\tilde{z} = z \exp(-me/dN^{1/3})$, $\tilde{\lambda} = (\lambda_1 + \cdots + \lambda_m)/dN^{1/3}$. Then,

$$g(ze^{-i\xi_m(\lambda)}) = -\frac{\alpha}{(1 - \alpha)^2}[(\tilde{z} - 1)^2 \frac{1}{\tilde{z}} e^{i\tilde{\lambda}} - 2i(\tilde{z} - 1)\sin \tilde{\lambda} + 2(\cos \tilde{\lambda} - 1)].$$

Thus,

$$1 + g(ze^{-i\xi_m(\lambda)}) = 1 + \frac{2\alpha t}{(1 - \alpha)^2}(1 - \cos \tilde{\lambda}) - \frac{\alpha t}{(1 - \alpha)^2}[\frac{(\tilde{z} - 1)^2}{\tilde{z}} e^{i\tilde{\lambda}} - 2i(\tilde{z} - 1)\sin \tilde{\lambda}].$$

The last term is small for large $N$ and the second is $\geq 1$. Hence,

$$\frac{1}{1 + t g(ze^{-i\xi_m(\lambda)})} \leq 2$$

for $|t| \leq N^\gamma$ and $N$ sufficiently large. Consequently, there is a constant $c_1(\alpha)$ depending only on $\alpha$ such that

$$|g(ze^{-i\xi_m(\lambda)})| \leq c_1(\alpha)(\tilde{\lambda}^2 + |\tilde{z} - 1|^2)$$
We can also write
\[ |\log(1 + tg(z e^{-i\xi_m(\lambda)})| \leq c_1(\alpha)(\lambda^2 + |z - 1|^2). \]

By periodicity it is enough to consider $|\lambda| \leq \pi$. Estimating the cosine and sine functions we see that there are constants $c_2(\alpha)$ and $c_3(\alpha)$ such that
\[ |1 + tg(z e^{-i\xi_m(\lambda)})| \geq \exp(c_2(\alpha)\lambda^2 - c_3(\alpha)(|z - 1|^2 + |z - 1||\lambda|)) \]
and hence,
\[ |1 + tg(z e^{-i\xi_m(\lambda)})|^{(u-v)} \leq \exp(-c_2(\alpha)\lambda^2 + c_3(\alpha)(|z - 1|^2 + |z - 1||\lambda|)). \]

Estimating the quadratic polynomial in $\lambda$ we obtain
\[ |1 + tg(z e^{-i\xi_m(\lambda)})|^{(u-v)} \leq \exp(t(\tau' - \tau)c_4(\alpha)\lambda^{2/3}|z - 1|^2). \]

Now, $z(t)\exp(-mc/dN^{1/3}) = 1 + (\tau + \eta - mc - it)/dN^{1/3} + \ldots$, and we obtain an estimate
\[ |1 + tg(z(t)e^{-i\xi_m(\lambda)})|^{(u-v)} \leq \exp(c_5(\alpha)(|\tau + \eta - mc|^2 + t^2)). \]

A computation shows that
\[ \frac{1}{|1 - e^{i\xi_3(\lambda)\lambda}|} \leq c_6 \]
if $\tau' - \tau + \eta + \eta' > \epsilon > 0$. Since $\tau' - \tau \leq 1$ and we take $\eta + \eta' \geq 2$ we see that we can take $\epsilon = 1/2$ for example. Furthermore, since $(y - cN)/dN^{1/3}$ is bounded for the $y$s that contribute to the sum,
\[ |e^{i\xi_3(\lambda)(y - cN)}| \leq c_7. \]

We can now again estimate as in sect. 4 and use (4.17). This results in an estimate
\[ |S_i| \leq c_5(t, \alpha) \left( \int_1 (1 + \lambda^2)|\hat{f}(\lambda)|d\lambda \right)^{\frac{3}{2}} e^{c_5(\alpha)(\tau + \eta - mc)^2} \]
\[ \times \frac{1}{N^{1/3}} \sum_y e^{(\tau^3 - \tau'^3)/3 + \xi(\tau' - \tau) + (\eta'^3 - \eta^3)/3 - \xi(\alpha + \eta')} \]
\[ \times \int_R e^{(c_5(\alpha) - \eta'/2)t^2} dt \int_R e^{(c_5(\alpha) - \eta'/2)s^2} ds. \]

We pick $\eta, \eta' \geq 3c_5(\alpha)$. Recall that $\xi = (y - cN)/dN^{1/3} + \tau^2$. Let $\eta = \max(|\tau|, 3c_5(\alpha), 1)$ and $\eta' = \max(|\tau'|, 3c_5(\alpha), 1)$. It follows from (5.32) and (5.40) that
\[ |S_i| \leq c_9(f, \alpha), \]
\[ i = 1, 2 \text{ if } |\tau|, |\tau'| \text{ are small. If } |\tau| \text{ and } |\tau'| \text{ are large, say } \tau, \tau' \gg 1, \text{ then } \eta = \tau \text{ and } \eta' = \tau', 0 \leq \tau' - \tau \leq 1, \text{ and we get from (5.32) and (5.40) that} \]
\[ |S_i| \leq c_{10}(f, \alpha)e^{-\tau^3}. \]

Inserting these estimates into (5.27) and using $v - u \geq 1$, we obtain an estimate of (5.17) of the type we have in the right hand side of (5.2).
Consider the expression

\[ \sum_{x_2, x_4 \in \mathbb{Z}} \phi(x_2, x_4) f_N(x_2) f_N(x_4) \hat{K}^{ab}_{cd}(x_4 - x_2) \, x_3 \, x_1. \]

In our computations with the kernel \( \hat{K} \) given by (5.20) we will leave out the complex integrations. Thus

\[
\hat{K}^{ab}_{cd}(x_4 - x_2) = \begin{pmatrix} G^*_{ac}(z_1, w_1) & G^*_{ad}(z_1, w_1) \\ G^*_{bc}(z_2, w_2) & G^*_{bd}(z_2, w_2) \end{pmatrix}
\]

\[ = G^*_{ac}(z_1, w_1)G^*_{bd}(z_2, w_2) - G^*_{ad}(z_1, w_1)G^*_{bc}(z_2, w_2). \]

We are led to the symmetrized expression

\[ \frac{1}{2} \sum_{x_2, x_4 \in \mathbb{Z}} \phi(x_2, x_4) f_N(x_2) f_N(x_4) \left[ \frac{w_{12}^2}{z_1^2} + \frac{w_{14}^2}{z_2^2} \right]. \]

Perform the \( x_4 \)-summation first and use the formula (5.24). The parts containing \( R_1 \) and \( R_2 \) can be estimated in the same way as above. We will only discuss the contraction and finite-difference parts. The contraction part of (5.44) is

\[ \frac{1}{2} \sum_{x_2 \in \mathbb{Z}} f_N(x_2) \frac{w_{12}^2}{z_1^2} [(1 + g(z_1))^{u-v} + (1 + g(w_1))^{u-v}] \]

and hence the contraction part of (5.43) is

\[ \frac{1}{2} \sum_{x_2 \in \mathbb{Z}} f_N(x_2)^2 \{ G^*_{ac+u-v,c}(z_1, w_1)G^*_{bd}(z_2, w_2) \frac{w_{12}^2}{z_1^2} \frac{w_{21}^2}{z_2^3} + G^*_{ac,c+v-u}(z_1, w_1)G^*_{bd}(z_2, w_2) \frac{w_{12}^2}{z_1^2} \frac{w_{21}^2}{z_2^3} - G^*_{ac+u-v,d}(z_1, w_1)G^*_{bc}(z_2, w_2) \frac{w_{12}^2}{z_1^2} \frac{w_{21}^2}{z_2^3} - G^*_{ad}(z_1, w_1)G^*_{bc,c+v-u}(z_2, w_2) \frac{w_{12}^2}{z_1^2} \frac{w_{21}^2}{z_2^3} \}. \]

Here we have also used (5.24). Performing the complex integrations we obtain

\[ \frac{1}{2} \sum_{x_2 \in \mathbb{Z}} f_N(x_2)^2 \{ \hat{K}^{ac+u-v,c}(x_2, x_3)\hat{K}^{bd}(x_3, x_1) + \hat{K}^{ac,c+v-u}(x_2, x_3)\hat{K}^{bd}(x_3, x_1) - \hat{K}^{ac+u-v,d}(x_2, x_3)\hat{K}^{bc}(x_3, x_2) - \hat{K}^{ad}(x_2, x_1)\hat{K}^{bc,c+v-u}(x_3, x_2) \} \]

The finite difference part of (5.43) is

\[ \frac{\alpha}{2(1 - \alpha)^2} \sum_{x_2 \in \mathbb{Z}} \Delta f_N(x_2)^2 \{ G^*_{ac}(z_1, w_1)G^*_{bd}(z_2, w_2) \frac{w_{12}^2}{z_1^2} \frac{w_{21}^2}{z_2^3} (\frac{1}{z_1} + w_1) - G^*_{ad}(z_1, w_1)G^*_{bc}(z_2, w_2) \frac{w_{12}^2}{z_1^2} \frac{w_{21}^2}{z_2^3} (\frac{1}{z_1} + w_2) \}. \]
The double $\phi$-term in (5.8), $\Sigma_b$ in (5.11) and (5.14) combined give

\begin{equation}
12 \sum_x \phi(x_2, x_4)\phi(x_1, x_3) \tilde{K}_{uu}^{vu} \begin{pmatrix} x_4 & x_3 \\ x_2 & x_1 \end{pmatrix} f_N(x_1)f_N(x_2)f_N(x_3)f_N(x_4)
\end{equation}

\begin{equation}
-12 \sum_x \phi(x_1, x_3) \left[ \tilde{K}_{uu}^{vv} \begin{pmatrix} x_2 & x_3 \\ x_2 & x_1 \end{pmatrix} + \tilde{K}_{vv}^{vu} \begin{pmatrix} x_2 & x_3 \\ x_2 & x_1 \end{pmatrix} \right] f_N(x_1)f_N(x_2)^2f_N(x_3)
\end{equation}

\begin{equation}
+3 \sum_x \left[ \tilde{K}_{uu}^{vv} \begin{pmatrix} x_1 & x_2 \\ x_1 & x_2 \end{pmatrix} + 2\tilde{K}_{ vv}^{vv} \begin{pmatrix} x_1 & x_2 \\ x_1 & x_2 \end{pmatrix} + \tilde{K}_{vv}^{vv} \begin{pmatrix} x_1 & x_2 \\ x_1 & x_2 \end{pmatrix} \right] f_N(x_1)^2f_N(x_2)^2
\end{equation}

\[ \doteq A_1 + A_2 + A_3. \]

Consider the $x_4$-summation in $A_1$. The contraction part is, by (5.46),

\[ 6 \sum_x f_N(x_1)f_N(x_2)^2f_N(x_3)\phi(x_1, x_2) \left[ \tilde{K}_{ uu}^{vv} \begin{pmatrix} x_2 & x_3 \\ x_2 & x_1 \end{pmatrix} + \tilde{K}_{ vv}^{vv} \begin{pmatrix} x_2 & x_3 \\ x_2 & x_1 \end{pmatrix} \right], \]

which is exactly $-\frac{1}{2}A_2$. Hence what remains of $A_2$ is

\begin{equation}
-6 \sum_x \phi(x_1, x_3) \left[ \tilde{K}_{ uu}^{vv} \begin{pmatrix} x_2 & x_3 \\ x_2 & x_1 \end{pmatrix} + \tilde{K}_{ vv}^{vv} \begin{pmatrix} x_2 & x_3 \\ x_2 & x_1 \end{pmatrix} \right] f_N(x_1)f_N(x_2)^2f_N(x_3).
\end{equation}

We have

\[ \sum_{x_1, x_3 \in \mathbb{Z}} f_N(x_1)f_N(x_3)\tilde{K}_{ uu}^{vv} \begin{pmatrix} x_2 & x_3 \\ x_2 & x_1 \end{pmatrix} = \sum_{x_2, x_4 \in \mathbb{Z}} f_N(x_2)f_N(x_4)\tilde{K}_{ uu}^{vv} \begin{pmatrix} x_4 & x_1 \\ x_2 & x_1 \end{pmatrix}. \]

We can now apply (5.46) to compute the contraction part of the first half of (5.49) and get

\begin{equation}
-6 \sum_{x_1, x_2 \in \mathbb{Z}} \left[ \tilde{K}_{ uu}^{vv} \begin{pmatrix} x_1 & x_2 \\ x_1 & x_2 \end{pmatrix} + \tilde{K}_{ vv}^{vv} \begin{pmatrix} x_1 & x_2 \\ x_1 & x_2 \end{pmatrix} \right] f_N(x_1)^2f_N(x_2)^2.
\end{equation}

Similarly the second half of (5.49) has the contraction part

\begin{equation}
-6 \sum_{x_1, x_2 \in \mathbb{Z}} \left[ \tilde{K}_{ uu}^{vv} \begin{pmatrix} x_1 & x_2 \\ x_1 & x_2 \end{pmatrix} + \tilde{K}_{ vv}^{vv} \begin{pmatrix} x_1 & x_2 \\ x_1 & x_2 \end{pmatrix} \right] f_N(x_1)^2f_N(x_2)^2.
\end{equation}

Since the contraction part of (5.49) equals (5.50) plus (5.51) we see that this exactly cancels $A_3$. It remains to consider the finite difference parts.

From $A_1$ we get a finite difference part

\begin{equation}
\frac{6\alpha}{(1-\alpha)^2} \sum_x \left[ G_{vu}^*(z_1, w_1)G_{vu}^*(z_2, w_2)\frac{w_1^{x_2}}{z_1^{x_2}}\frac{w_1^{x_1}}{z_1^{x_1}} \left( \frac{1}{z_1} + w_1 \right) - G_{vu}^*(z_1, w_1)G_{vu}^*(z_2, w_2)\frac{w_2^{x_2}}{z_2^{x_2}}\frac{w_2^{x_1}}{z_2^{x_1}} \left( \frac{1}{z_2} + w_2 \right) \right] \Delta f_N(x_2)^2\phi(x_1, x_3)f_N(x_1)f_N(x_3). \end{equation}
We also need the finite difference part of (5.49). These finite difference parts should be cancelled by the contraction part of (5.52). The contraction part of (5.49) is

\[
(5.53) \quad -\frac{3\alpha}{(1-\alpha)^2} \sum_x \left[ G_{uu}^*(z_1, w_1) G_{uu}^*(z_2, w_2) \frac{w_1^{x_2} w_2^{x_1}}{z_1^{x_2} z_2^{x_1}} \left( \frac{1}{z_1} + w_1 \right) 
- G_{vu}^*(z_1, w_1) G_{uv}^*(z_2, w_2) \frac{w_2^{x_2} w_1^{x_1}}{z_1^{x_2} z_2^{x_1}} \left( \frac{1}{z_1} + w_2 \right) \right] \Delta f_N(x_2)^2 \phi(x_1, x_3) f_N(x_1)^2 
- \frac{3\alpha}{(1-\alpha)^2} \sum_x \left[ G_{vu}^*(z_1, w_1) G_{uv}^*(z_2, w_2) \frac{w_1^{x_2} w_2^{x_1}}{z_1^{x_2} z_2^{x_1}} \left( \frac{1}{z_1} + w_1 \right) 
- G_{vv}^*(z_1, w_1) G_{vu}^*(z_2, w_2) \frac{w_2^{x_2} w_1^{x_1}}{z_1^{x_2} z_2^{x_1}} \left( \frac{1}{z_1} + w_2 \right) \right] \Delta f_N(x_2)^2 \phi(x_1, x_3) f_N(x_1)^2.
\]

In (5.52) we have first

\[
\sum_x \frac{w_1^{x_2}}{z_1^{x_2}} \frac{w_2^{x_1}}{z_2^{x_1}} \phi(x_1, x_3) f_N(x_1) f_N(x_3) \Delta f_N(x_2)^2
= \frac{1}{2} \sum_x \Delta f_N(x_1)^2 \left( \sum_{x_2, x_3} \left( \frac{w_2^{x_2}}{z_2^{x_2}} + \frac{w_1^{x_1}}{z_1^{x_1}} \right) \phi(x_2, x_4) f_N(x_2) f_N(x_4) \right).
\]

This gives the contraction part

\[
(5.54) \quad \frac{1}{2} \sum_{x_1, x_2} \frac{w_1^{x_1}}{z_1^{x_1}} \frac{w_2^{x_2}}{z_2^{x_2}} f_N(x_2)^2 \Delta f_N(x_1)^2 [(1 + g(z_2))^{u-v} + (1 + g(w_2))^{u-v}].
\]

From the other half of (5.52) we get similarly the contraction part

\[
(5.55) \quad \frac{1}{2} \sum_{x_1, x_2} \frac{w_2^{x_1}}{z_1^{x_1}} \frac{w_1^{x_2}}{z_2^{x_2}} f_N(x_2)^2 \Delta f_N(x_1)^2 [(1 + g(z_2))^{u-v} + (1 + g(w_1))^{u-v}].
\]

By (5.54) and (5.57) the contraction part of (5.52) is

\[
\frac{3\alpha}{(1-\alpha)^2} \sum_x \frac{w_1^{x_1}}{z_1^{x_1}} \frac{w_2^{x_2}}{z_2^{x_2}} \Delta f_N(x_1)^2 f_N(x_2)^2 \left( \frac{1}{z_1} + w_1 \right) 
\times \left[ G_{vu}^*(z_1, w_1) G_{uv}^*(z_2, w_2) + G_{vu}^*(z_1, w_1) G_{vu}^*(z_2, w_2) \right]
- \frac{3\alpha}{(1-\alpha)^2} \sum_x \frac{w_2^{x_1}}{z_1^{x_1}} \frac{w_1^{x_2}}{z_2^{x_2}} \Delta f_N(x_1)^2 f_N(x_2)^2 \left( \frac{1}{z_1} + w_2 \right) 
\times \left[ G_{vu}^*(z_1, w_1) G_{uv}^*(z_2, w_2) + G_{vv}^*(z_1, w_1) G_{uv}^*(z_2, w_2) \right],
\]

which exactly cancels (5.53). The finite difference part of (5.52) is handled in the same way as in (5.28).

We will end with some brief comments about remaining estimates. By (5.20) we have for example

\[
(5.56) \quad \Delta K_{wv}(x, y) = \frac{1}{(2\pi)^2} \int_{\gamma_{wv}} dz \int_{\gamma_{v}} dw \left( \frac{1 - \alpha/z}{1 - \alpha z} \right)^N \left( \frac{1 - \alpha w}{1 - \alpha/w} \right)^N \frac{w^y}{z^x} 
\times \frac{1}{w(z - w)} [(1 + g(w))^{v-u} - 1] (1 + g(z))^u (1 + g(w))^u.
\]

Here we can expand \((1 + g(w))^{v-u} - 1\) as in (5.24) and then estimate in the same way as we did for the\(R_1\) and\(R_2\) terms above. In this way we will see that the
\(\Delta K^2\)-terms give contributions of the right type. We get a similar integral expression for \(\Delta K_{uv} + \Delta K_{vu} - \Delta K_{vu}\) involving
\[
(1 + g(z))^v(1 + g(w))^{-v}[(1 + g(z))^{u-v} - 1][(1 + g(w))^{v-u} - 1]
\]
and we proceed similarly. \(\square\)

5.2. Weak convergence. Consider \(H_N(f, t_j)\) as defined by (5.1). The next lemma is a standard consequence of lemma 5.1.

**Lemma 5.2.** Under the same assumptions as in lemma 5.1.
\[(5.57)\] \(\mathbb{P}\left[ \max_{j=u,\ldots,v} |H_N(f, t_j) - H_N(f, t_u)| \geq \lambda \right] \leq C(f, \alpha)\lambda^{-4}e^{-|t_u|^3}|t_u - t_v|^2.
\]

**Proof.** Let
\[
\eta_i = H_N(f, t_{u+i}) - H_N(f, t_{u+i-1}),
\]
\(T_m = \sum_{i=1}^m \eta_i\) and \(T_0 = 0\), so that \(T_j - T_1 = H_N(f, t_j) - H_N(f, t_1)\). It follows from (5.2) and Chebyshev's inequality that
\[
\mathbb{P}[|T_i - T_j| \geq \lambda] \leq C(f, \alpha)\lambda^{-4}e^{-|t_u|^3}\left(\sum_{i \leq \ell \leq j} u_{\ell}\right)^2
\]
for \(u \leq i < j \leq v\), where \(u_{\ell} = t_{\ell+u} - t_{\ell-1+u}\). This implies (5.57) according to theorem 12.2 in \[6\]. \(\square\)

Fix \(l > 0\) and consider rescaled top height curve \(H_{N,0}(t)\) for \(|t| \leq L\) and its modulus of continuity, \(0 < \delta \leq 1\),
\[
w_N(\delta) = \sup_{|t|, |s| \leq T, |s-t| \leq \delta} |H_{N,0}(t) - H_{N,0}(s)|,
\]

**Lemma 5.3.** Let \(w_N\) be defined as above. Given \(\epsilon, \lambda > 0\) there is a \(\delta > 0\) and an integer \(N_0\) such that
\[
\mathbb{P}[w_N(\delta) \geq \lambda] \leq \epsilon
\]
if \(N \geq N_0\).

Together with the convergence of the finite dimensional distributions, theorem 1.1, this proves theorem 1.2. \[1\]. We turn now to the proof of lemma 5.3.

**Proof.** Assume that \(\delta^{-1}, T \in \mathbb{Z}\) and divide the interval \([-T, T]\) into \(2m\) parts of length \(T/m = \delta\). Write \(r_j = [j\delta|cN^{2/3}|], c = 2\alpha/(1 - \alpha)\), so that \(t_{r_j} \approx j\delta\).

**Claim 5.4.** Let \(L = T|cN^{2/3}|\) and \(B_M\) is the subset of our probability space where \(\max_{|j| \leq L} |H_{N,0}(t_j)| \leq M\). Then, given \(\epsilon > 0\), we can choose \(M\) so that
\[(5.58)\] \(\mathbb{P}[B_M^c] \leq \epsilon.
\]
We will prove this claim below. We will also need

**Claim 5.5.** For any \(\lambda > 0\) there is a constant \(C(M)\) that depends on \(M\) but not on \(\lambda\) such that
\[(5.59)\] \(\mathbb{P}\left[ \max_{r_j \leq i \leq r_{j+1}} |H_{N,0}(t_i) - H_{N,0}(t_{r_j})| \geq \lambda, B_M^c \right] \leq \frac{C(M)}{\lambda}.
\]
We will return to the proofs. The proof of both claims are based on choosing appropriate functions \( f \) in lemma 5.2 and results about convergence in distribution. Assuming the validity of the two claims we can prove lemma 5.3. Set
\[
A_j = \{ \max_{t_{r_j} \leq s \leq t_{r_{j+1}}} |H_{N,0}(s) - H_{N,0}(t_{r_j})| \geq \lambda/3 \},
\]
so that \( \{ w_N(\delta) \geq \lambda \} \subseteq \bigcup_{|j| \leq m} A_j \). Choose \( M \) so large that \( \mathbb{P}[B_M^*] \leq \epsilon \), which is possible by claim 5.4. Hence
\[
\mathbb{P}[w_N(\delta) \geq \lambda] \leq \epsilon + \sum_{|j| \leq m} \mathbb{P}[A_j \cup B_M].
\]
Now, if the inequality in (5.60) holds then
\[
\max_{r_j \leq s \leq r_{j+1}} |H_{N,0}(t_i) - H_{N,0}(t_{r_j})| \geq \lambda/9.
\]
Consequently, using (5.59) and (5.61),
\[
\mathbb{P}[w_N(\delta) \geq \lambda] \leq \epsilon + (2m + 1)C(M)\lambda^{-1}\delta^2 \leq \epsilon + 2TC(M)\lambda^{-1}\delta.
\]
Choose \( \delta \) so that \( \delta \leq \epsilon\lambda/2TC(M) \). Lemma 5.3 is proved.
Consider now claim 5.4. Pick a \( C^\infty \) function \( g \) such that \( 0 \leq g \leq 1 \) and
\[
g(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
0 & \text{if } x \leq -1 
\end{cases}
\]
and let \( g_M(x) = g(x - M) \). It is not hard to see that if we take \( f = g_M \) in (5.2) the \( C(f, \alpha) \) can be taken to be independent of \( M \) (only sup-norms of \( f \) and its derivatives enter). If \( H_N(g_M, t_{r_j}) > 1/4 \), then \( H_{N,0}(t_{r_j}) \geq M - 1 \) and using the convergence in distribution to \( F_2 \) we see that we can choose \( M \) so large that
\[
\mathbb{P}[H_N(g_M, t_{r_j}) > 1/4] \leq \epsilon^2
\]
for \( |j| \leq m \) and all sufficiently large \( N \). Let \( \omega \) denote a point in our probability space. Now,
\[
\mathbb{P}\left[ \max_{|j| \leq L} H_N(g_M, t_j) > 1/2 \right] = \mathbb{P}[H_N(g_M, t_j(\omega)) > 1/2]
\]
\[
= \sum_{j = -m+1}^{m} \mathbb{P}[H_N(g_M, t_j(\omega)) > 1/2, t_{r_j-1} \leq t_j(\omega) \leq t_{r_j}]
\]
\[
\leq \sum_{j = -m+1}^{m} \mathbb{P}[H_N(g_M, t_{r_j}) > 1/4]
\]
\[
+ \sum_{j = -m+1}^{m} \mathbb{P}\left[ \max_{r_{j-1} \leq i \leq r_j} |H_N(g_M, t_i) - H_N(g_M, t_{r_{j-1}})| \geq 1/4 \right]
\]
\[
\leq 2m\epsilon^2 + \sum_{j = -m+1}^{m} \frac{C}{(1/4)^4} \delta^2 \leq 2T^2\delta^2 + C\delta
\]
by lemma 5.2. We can now choose \( \delta = \epsilon \).
If \( H_{N,0}(t_j) \geq M \), then \( H_N(g_M, t_j) \geq 1 \) and hence \( \max_{|j| \leq L} H_{N,0}(t_j) \geq M \), which implies \( \max_{|j| \leq L} H_N(g_M t_j) \geq 1/2 \). It follows that
\[
\mathbb{P}[\max_{|j| \leq L} H_{N,0}(t_j) \geq M] \leq \epsilon.
\]
The case \( \max_{|j| \leq L} H_{N,0}(t_j) \leq -M \), is analogous. This proves claim 5.4.
To prove claim 5.5 we let \( i(\omega) \) be defined by

\[
\max_{r_j \leq t \leq r_{j+1}} |H_{N,0}(t_i) - H_{N,0}(t_{r_j})| = |H_{N,0}(t_{i(\omega)}) - H_{N,0}(t_{r_j})|.
\]

Let \( I_j = [j\lambda, (j+1)\lambda) \), \( j = -K, \ldots, K - 1 \), where \( M = K\lambda, K \in \mathbb{Z}^+ \). Take a \( C^\infty \) function \( f \), \( 0 \leq f \leq 1 \), such that \( f(x) = 0 \) if \( x \leq -\lambda \), \( f(x) = 1 \) if \( 0 \leq x \leq \lambda \) and \( F(x) = 0 \) if \( x \geq \lambda \). Set

\[
f_j(x) = f(x - \lambda j).
\]

Suppose first that \( H_{N,0}(t_{i(\omega)}) \leq H_{N,0}(t_{r_j}) - 2\lambda \) and that \( \omega \in B_M \). Then there is a \( k(\omega) \) such that \( H_{N,0}(t_{r_j}) \in I_k(\omega) \), and

\[
H_{N,0}(t_{i(\omega)}) \leq (k+1)\lambda - 2\lambda = (k-1)\lambda,
\]

and consequently \( H_N(f_k(\omega), t_{i(\omega)}) = 0 \). Since \( H_N(f_k(\omega), t_{r_j}) \geq 1 \), we see that

\[
|H_N(f_k(\omega), t_{i(\omega)}) - H_N(f_k(\omega), t_{r_j})| \geq 1.
\]

Hence,

\[
\max_{k \leq m} \max_{r_j \leq t \leq r_{j+1}} |H_N(f_k, t_i) - H_N(f_k, t_{r_j})| \geq 1.
\]

Call this event \( F \). If we instead suppose that \( H_{N,0}(t_{i(\omega)}) \geq H_{N,0}(t_{r_j}) + 2\lambda \) we can proceed similarly and see that (5.64) still holds. Now,

\[
P[F] \leq P[ \bigcup_{r_j \leq t \leq r_{j+1}} \{|H_N(f_k, t_i) - H_N(f_k, t_{r_j})| \geq 1\}]
\leq \sum_{k=-K}^{K} C|t_{r_{j+1}} - t_{r_j}|^2 \leq \frac{2MC}{\lambda} \delta^2,
\]

by lemma 5.2.

\[\square\]

5.3. Transversal fluctuations. In this section we will prove corollary 1.3, proposition 1.4 and theorem 1.6. Let \( T > 0 \) be fixed and set

\[
S^T_N(u) = \sup_{-T \leq t \leq u} H_{N,0}(t)
\]

\[
S^T(u) = \sup_{-T \leq t \leq u} (A(t) - t^2).
\]

We will write \( S^T_N \) for \( S^T_N(T) \) and \( S^T \) for \( S^T(T) \).

Lemma 5.6. Given \( \epsilon > 0 \) we can choose \( T = T(\epsilon) \) so that

\[
P[S^T_N \neq S^T_N] \leq \epsilon
\]

for all sufficiently large \( N \).

Note that together with theorem 1.2 this proves corollary 1.3.

Proof. Let \( g_M \) be defined as above and set \( R_j = T + (j-1)\delta, j \geq 1 \), where \( \delta \) will be specified below. It follows from lemma 5.2 that for \( R_j \leq \log N \),

\[
P[\sup_{R_j \leq t \leq R_{j+1}} |H_N(g_M, t) - H_N(g_M, R_j)| \geq 1/2] \leq Ce^{-R_j^2} \delta^2.
\]
where $C$ is independent of $M$. Now,

\begin{equation}
(5.67)
\begin{align*}
\mathbb{P}[\sup_{T \leq t \leq R_L} H_{N,0}(t) \geq M] & \leq \mathbb{P}[\sup_{T \leq t \leq R_L} H_{N}(g_M, t) \geq 1] \\
& = \mathbb{P}[\max_{1 \leq j \leq L} \sup_{R_j \leq t \leq R_{j+1}} H_{N}(g_M, t) \geq 1] \\
& \leq \sum_{j=1}^{L-1} \mathbb{P}[\sup_{R_j \leq t \leq R_{j+1}} (H_{N}(g_M, t) - H_{N}(g_M, R_j)) + H_{N}(g_M, R_j) \geq 1] \\
& \leq \sum_{j=1}^{L-1} \mathbb{P}[\sup_{R_j \leq t \leq R_{j+1}} (H_{N}(g_M, t) - H_{N}(g_M, R_j)) \geq 1/2] + \sum_{j=1}^{L-1} \mathbb{P}[H_{N}(g_M, R_j) \geq 1/2].
\end{align*}
\end{equation}

Claim 5.7. There is a positive constant $c_1$ such that

\begin{equation}
(5.68)
\mathbb{P}[H_{N,0}(R) \geq s] \leq e^{-c_1(s+R^2)^{3/2}}.
\end{equation}

Proof. Let $c = (1 + \alpha)(1 - \alpha)^{-1}d^{-1}$, $d^0 = \alpha(1 + \alpha)(1 - \alpha)^{-3}$ and $t_j = j/cN^{2/3}$ as before. We have

\begin{equation}
(5.69)
\mathbb{P}[G([\gamma K], K) > K t] \leq e^{-2KJ(t+1)},
\end{equation}

where the function $J$ satisfies

\begin{equation}
(5.70)
J((1 + \sqrt{q\gamma})^2(1 - q)^{-1} + \delta) \geq c_1' \delta^{3/2}
\end{equation}

for $0 \leq \delta \leq 1$; $c_1'$ is a positive constant. We take $K = N - cN^{2/3}R$, $\gamma = (N + cN^{2/3}R)/K$ and $t = (2\alpha(1 - \alpha)^{-1}N + sdN^{1/3})/K$. Pick $\delta$ so that

\begin{equation}
(1 + \sqrt{q\gamma})^2(1 - q)^{-1} + \delta = 1 + t.
\end{equation}

This gives $\delta = dN^{-2/3}(s + R^2) + O(N^{-1})$ and if we insert this into (5.70), the estimate (5.66) gives us exactly what we want. \hfill \Box

If $H_{N}(g_M, R_j) \geq 1/2$, then $H_{N,0}(R_j) \geq M - 1$ and hence

\begin{equation}
(5.71)
\begin{align*}
\sum_{j=1}^{L-1} \mathbb{P}[H_{N}(g_M, R_j) \geq 1/2] & \leq \frac{1}{\delta} \sum_{j=1}^{L-1} e^{-c(M-1+(T+(j-1)\delta)^3/2)} \\
& \leq \frac{1}{\delta} \int_{T-1}^{\infty} e^{-c(M-1+x^2)^{3/2}} dx.
\end{align*}
\end{equation}
Using (5.66) we find

\[
L - 1 \sum_{j=1}^{L-1} \mathbb{P} \left[ \sup_{t \leq \tau \leq t+1}(H_N(g_M, t) - H_N(g_M, R_j)) \geq 1/2 \right]
\leq \sum_{j=1}^{L-1} Ce^{-R/2} \delta^2 \leq C\delta \int_{T-1}^{\infty} e^{-x^3} dx.
\]

Inserting (5.71) and (5.72) into (5.67) gives

\[
(5.73) \quad \mathbb{P} \left[ \sup_{T \leq t \leq R_L} H_N, 0(t) \geq M \right] \leq \frac{1}{\delta} \int_{T-1}^{\infty} e^{-c(M-1+x^3)/2} dx + C\delta \int_{T-1}^{\infty} e^{-x^3} dx
\]

if $R_L \leq \log N$. We can take $\delta = 1$.

It follows from (5.71) that

\[
(5.74) \quad \mathbb{P} \left[ \sup_{R_L \leq t} H_N, 0(t) \geq M \right] \leq \sum_{t_u \geq R_L} e^{-c(M-1+x^3)/2} \leq CN e^{-c(log N)^3} < \epsilon/4
\]

if $N$ is sufficiently large. We know that $\mathbb{P}[H_N, 0(0) \leq M] \to F_2(M)$ as $N \to \infty$ and we can choose $M$ so large that the right hand side of (5.73) is $\leq \epsilon/4$. Together with (5.74) this gives (using symmetry),

\[
(5.75) \quad \mathbb{P} \left[ \sup_{|t| \geq T} H_N, 0(t) \geq M \right] \leq \epsilon.
\]

If $H_N, 0(0) > M$ and $\sup_{|t| \geq T} H_N, 0(t) \geq M$, then $S_N^\infty = S_N^T$ and consequently

\[
\mathbb{P}[S_N^\infty \neq S_N^T] \leq \mathbb{P}[H_N, 0(0) \leq M] + \mathbb{P}[\sup_{|t| \geq T} H_N, 0(t) > M] \leq 2\epsilon
\]

for all sufficiently large $N$.

We turn now to the transversal fluctuations and the proof of theorem 1.6. Define

\[
K^T_N = \inf \{ u \geq -T ; S_N^T(u) = S_N^T \}
\]
\[
K^T = \inf \{ u \geq -T ; S^T(u) = S^T \},
\]

which give the leftmost point of maximum in $[-T, T]$ before and after the limit.

We first prove proposition 1.4.

**Proof.** (Proposition 1.4). Note that

\[
\{ K_N < -T \} \subseteq \{ \sup_{t \leq -T} H_N, 0(t) \geq H_N, 0(0) \}.
\]

It follows that

\[
\mathbb{P}[K_N < -T] \leq \mathbb{P}[H_N, 0(t) \geq M] + \mathbb{P}[H_N, 0(0) < M] \leq 2\epsilon,
\]

by (5.75) and the discussion proceeding it. Also, $\{ K_N > T \} \subseteq \{ S_N^\infty \neq S_N^T \}$ and we can use lemma 5.8. \[\square\]
Proof: (Theorem 1.6). It follows from lemma 5.4 that given $\epsilon > 0$ we can choose $T$ and $N_0$ so that

\begin{equation}
\mathbb{P}[K_N = K_N^T] \geq 1 - \epsilon
\end{equation}

if $N \geq N_0$. Let $h_T : C(\mathbb{R}) \to \mathbb{R}$ be defined by

\[ h_T(x) = \inf \{ u \geq -T; \sup_{-T \leq t \leq u} x(t) = \sup_{-T \leq t \leq T} x(t) \}, \]

and let

\[ D_{h_T} = \{ x \in C(\mathbb{R}); h_T \text{ is discontinuous at } x \} \]

It follows from our assumption that $\mathbb{P}[D_{h_T}] = 0$, since $h_T$ is continuous at $x$ unless $x$ has two distinct maximum points. Since $H_{N,0}$ converges in distribution to $X$ in $C[-T,T]$ it follows that

\begin{equation}
K^T_N = h_T(H_{N,0}) \to h_T(X) = K_T
\end{equation}

as $N \to \infty$. Let $D_T$ be all points of discontinuity for $x \to \mathbb{P}[K_T \leq x]$, $T \in \mathbb{Z}$, and $D = \cup_{T \geq 1} D_T$. We will prove that

\begin{equation}
\mathbb{P}[K_N \leq x] \to \mathbb{P}[K \leq x]
\end{equation}

as $N \to \infty$ for all $x \in \mathbb{R} \setminus D$, which implies what we want since $D$ is countable. All the results and assumptions that are behind the estimate (5.76) can also be proved for the limiting Airy process and we can assume that $N_0$ and $T \in \mathbb{Z}_+$ are chosen so that also

\begin{equation}
\mathbb{P}[K = K_T^T] \geq 1 - \epsilon
\end{equation}

if $N \geq N_0$. Let $x \in \mathbb{R} \setminus D$. Then,

\[ \mathbb{P}[K_N \leq x] = \mathbb{P}[K_N^T \leq x, K_N^T = K_N] + \mathbb{P}[K_N \leq x, K_N^T \neq K_N], \]

and similarly for $K_N^T$. Hence,

\[ |\mathbb{P}[K_N \leq x] - \mathbb{P}[K_N^T \leq x]| \leq 2\epsilon \]

if $N \geq N_0$. Since $x \in \mathbb{R} \setminus D$ it follows from (5.77) that we can choose $N_1$ so that

\[ |\mathbb{P}[K_N^T \leq x] - \mathbb{P}[K^T \leq x]| \leq \epsilon \text{ if } N \geq N_1. \]

It follows from (5.78) that

\[ |\mathbb{P}[K \leq x] - \mathbb{P}[K^T \leq x]| \leq \epsilon. \]

Combining the estimates we see that

\[ |\mathbb{P}[K_N \leq x] - \mathbb{P}[K \leq x]| \leq 4\epsilon \]

if $N \geq \max(N_0, N_1)$, which proves (5.78). \qed

Acknowledgement: I thank Peter Forrester for drawing my attention a few years ago to the relation between the exponents occurring in [13] and [2].

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