LIAISON LINKAGES

MATTEO GALLET, GEORG NAWRATIL, AND JOSEF SCHICHO

ABSTRACT. The complete classification of hexapods — also known as Stewart Gough platforms — of mobility one is still open. To tackle this problem, we can associate to each hexapod of mobility one an algebraic curve, called the configuration curve. In this paper we establish an upper bound for the degree of this curve, assuming the hexapod is general enough. Moreover, we provide a construction of hexapods with curves of maximal degree, which is based on liaison, a technique used in the theory of algebraic curves.

INTRODUCTION

This paper is devoted to the study of mechanical devices called hexapods, which are also known as Stewart Gough platforms. As described in [Naw14b], the geometry of this kind of mechanical manipulators is defined by the coordinates of the 6 platform points \( p_1, \ldots, p_6 \in \mathbb{R}^3 \) and of the 6 base points \( P_1, \ldots, P_6 \in \mathbb{R}^3 \) in one of their possible configurations. A hexapod is called planar if both base points and platform points are coplanar, otherwise non-planar.

All pairs of points \((p_i, P_i)\) are connected by a rigid body, called leg, so that for all possible configurations the distance \( d_i = \|p_i - P_i\| \) is preserved (see Fig. 1). We say that a hexapod is movable, or admits a self-motion if, once we fix the position of the base points \( \{P_i\} \), the platform points \( \{p_i\} \) are allowed to move in an (at least) one-dimensional set of configurations respecting the constraints given by the legs. In this case, each \( p_i \) moves on the sphere with center \( P_i \) and radius \( d_i \).

One can associate to each movable hexapod an algebraic curve (if the hexapod has mobility one; otherwise it is an algebraic variety of higher dimension) contained in the algebraic group \( \text{SE}_3 \) of direct isometries of \( \mathbb{R}^3 \); it parametrizes the set of...
possible positions of the platform and because of this we call it the configuration curve (or the configuration set in case the mobility is larger than one). Then one may define various invariants of a hexapod, such as the degree or the genus of its configuration curve. Clearly the degree depends on the embedding of the algebraic group $\text{SE}_3$ in projective space. We mainly use the so-called conformal embedding $\text{SE}_3 \subseteq \mathbb{P}_C^{15}$ described in [GNS15, Sel13] because it is most practical for the study of hexapods; in this way we define the conformal degree of a hexapod. In the literature (see for example [HK02]) it is more common to project the configuration curve to $\text{SO}_3$ and use the well-known embedding $\text{SO}_3 \subseteq \mathbb{P}_C^3$ determined by quaternions. In this case we call the degree of the configuration curve the Euler degree, since the coordinates of $\mathbb{P}_3^C$ are called Euler parameters.

The classification of hexapods with mobility one is still an open problem. There is a family of planar hexapods, discovered by Duporq (see [Dup98, Naw14a]), with a configuration curve of conformal degree 40, Euler degree 20, and genus 41 (see Section 1); these are the largest possible degrees and genus. The family we introduce in this paper is non-planar and has conformal degree 28 and Euler degree 14. We show that this is the maximal degree among non-planar hexapods if we exclude some degenerate cases. By the way, in this case the genus of the configuration curve is 23, but we do not know whether this is maximal or not.

Our linkages are constructed in the following way. We start with 6 points $\vec{P} = (P_1, \ldots, P_6)$ in $\mathbb{R}^3$, forming the base. Then we employ basic facts from liaison theory — a method coming from algebraic geometry — to construct a system of equations for the 6 points $\vec{p} = (p_1, \ldots, p_6)$ forming the platform. We conjecture that this system admits solutions for a general choice of $\vec{P}$. Once $\vec{p}$ is fixed (up to scaling), we derive linear equations for a scalar $\gamma$ and a vector $\vec{d} = (d_1, \ldots, d_6)$ implying that the hexapod $(\vec{P}, \gamma \vec{p}, \vec{d})$ has mobility one, with a configuration curve of degree 28. We conjecture that this system of equations has a unique solution for $\gamma$ and a three-dimensional set of solutions for the vector $\vec{d}$. We call a hexapod constructed by this procedure a liaison hexapod. The nature of the two conjectures is so that if they are false, then they are falsified by a general choice of base points. We tested the conjectures against many random choices, so there is quite a strong experimental evidence in their favor. In addition, we found two particular subfamilies (see Proposition 3.1 and Proposition 3.2) for which we can prove that the properties predicted by the conjectures hold.

The paper is organized as follows. In Section 1 we prove our main theorem (Theorem 1.34) concerning the degree bound for configuration curves of general non-planar hexapods of mobility one; this is achieved by bounding the degree by the number of intersections of two algebraic curves, and then by using some facts from algebraic geometry to control such number. The remaining sections deal with the construction of examples. In Section 2 we describe liaison hexapods: given a general 6-tuple $\vec{P}$, we construct — up to a scaling factor — a candidate platform $\vec{p}$ applying liaison theory and Möbius photogrammetry, a technique used by the authors in [GNS14] to establish necessary conditions for the mobility of pentapods. After that we determine the right scaling factor for the platform points
and the leg lengths so that the hexapod we obtain is movable; both these tasks are achieved by inspection of tangency conditions for the configuration curve of the hexapod. In Section 3, we prove with the aid of symbolic computation that the properties predicated by our conjectures hold in the case of two particular subfamilies of hexapods, and we exhibit an example of a liaison hexapod.

1. Conformal degree of hexapods

The goal of this section is to associate to each hexapod $\Pi$ a projective curve $K_\Pi$, called the configuration curve of $\Pi$. Its degree, called the conformal degree of $\Pi$, is an invariant of the hexapod, and in this paper we are interested in understanding what are its possible maximal values. In particular we report well-known examples of hexapods attaining high values for the conformal degree, and eventually we prove a bound for the degree of a hexapod satisfying some non-degeneracy conditions (Theorem 1.34).

1.1. Configuration set and bonds of a hexapod. We recall some definitions from [GNS15] that we need in our discussion. Following many other authors, we describe the set of admissible configurations of a given hexapod $\Pi$ by prescribing the set of direct isometries (from $\mathbb{R}^3$ to itself) mapping the initial position of the platform to an admissible one. By this we mean that we consider the base $\vec{P}$ as fixed, and if $\vec{p}$ is the initial position of the platform points, we look for all $\sigma \in \text{SE}_3$ (where $\text{SE}_3$ denotes the group of direct isometries) such that

\begin{equation}
\|\sigma(p_i) - P_i\| = d_i \quad \text{for all } i \in \{1, \ldots, 6\},
\end{equation}

where $d_i$ are the given leg lengths of the hexapod $\Pi$. The constraints imposed on isometries $\sigma \in \text{SE}_3$ by Eq. (1) are called spherical conditions.

As shown in [GNS15, Section 2, Subsection 2.1], it is possible to construct a projective compactification in $\mathbb{P}_{16}^\mathbb{C}$ for (the complexification of) the group $\text{SE}_3$; we denote it by $X$. It turns out that $X$ is a projective variety of dimension 6 and degree 40. We call the map $\text{SE}_3 \mapsto \mathbb{P}_{16}^\mathbb{C}$ the conformal embedding of $\text{SE}_3$. If we write the spherical condition in the coordinates of $\mathbb{P}_{16}^\mathbb{C}$, we obtain six linear conditions, namely we determine a linear subspace $H_\Pi \subseteq \mathbb{P}_{16}^\mathbb{C}$ of codimension 6. The intersection $K_\Pi = X \cap H_\Pi$ is defined to be the complex configuration set of the hexapod $\Pi$.

**Definition 1.1.** Let $\Pi$ be a hexapod. The mobility of $\Pi$ is defined to be the dimension of $K_\Pi \cap \text{SE}_3.\mathbb{C}$, where $\text{SE}_3.\mathbb{C}$ is the complexification of $\text{SE}_3$ embedded in $X$. For a hexapod of mobility one we define the conformal degree to be the degree of the projective curve $K_\Pi$.

**Remark 1.2.** From the fact that $X$ is a variety of degree 40 and $K_\Pi$ is always a linear section of $X$ we immediately see that the conformal degree is always smaller than or equal to 40.

---

1This name was suggested in a private communication with Jon Selig.
One can also attach another degree to a hexapod, by considering only the rotational part of the isometries determining its configuration. This is classically done by embedding the (complexification of the) group of rotations $SO_3$ as an open subset of $\mathbb{P}_C^3$ using the quaternionic description of rotations. In this way we get for a mobility one hexapod $\Pi$ a curve in $\mathbb{P}_C^3$, and its degree is called the Euler degree of $\Pi$. Note that we obtain a point in $\mathbb{P}_C^3$ if the corresponding self-motion is a pure translation. In this case we set the Euler degree to zero.

If $\rho: SE_3 \to SO_3$ is the map sending a direct isometry to its rotational part, by analyzing the definition of the conformal embedding one sees that there exists a linear projection $\mathbb{P}_C^{16} \to \mathbb{P}_C^9$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
SE_3 & \xrightarrow{\rho} & X & \subseteq & \mathbb{P}_C^{16} \\
\downarrow & & & & \\
SO_3 & \xrightarrow{v_{3,2}} & v_{3,2} & \subseteq & \mathbb{P}_C^9
\end{array}
$$

where $v_{3,2}$ is the Veronese embedding of $\mathbb{P}_C^3$ by quadrics and $V_{3,2}$ is its image in $\mathbb{P}_C^9$.

If we have a closer look at $X$, we discover that it can be written as the disjoint union $SE_3 \cup B$ — where $B$ is a special hyperplane section of $X$, called the boundary. The boundary can be, in turn, decomposed into 5 subsets:

- **vertex:** the only real point in $B$, a singular point with multiplicity 20; it is never contained in the complex configuration set of a hexapod;
- **collinearity points:** if $K_\Pi$ contains such a point, then either the platform points or the base points are collinear;
- **similarity points:** if $K_\Pi$ contains such a point, then there are normal projections of platform and base to a plane such that the images are similar;
- **inversion points:** if $K_\Pi$ contains such a point, then there are normal projections of platform and base to a plane such that the images are related by an inversion;
- **butterfly points:** if $K_\Pi$ contains such a point, then there are two lines, one in the base and one in the platform, such that any leg has either a base point in the base line or a platform point in the platform line.

**Remark 1.3.** The center of the projection $\mathbb{P}_C^{16} \to \mathbb{P}_C^9$ sending $X$ to the Veronese variety $V_{3,2}$ is the linear space spanned by similarity points, which contains also the collinearity points and the vertex.

Inversion points form an open subset of $B$, while all other sets of points form proper (quasi-projective) subvarieties of $B$. The points of the configuration set $K_\Pi$ of a hexapod $\Pi$ that lie on the boundary $B$ are called the bonds of $\Pi$. As we see from the previous description, although bonds do not represent configurations, their presence constrains the geometry of the corresponding hexapod.

We now discuss a few well-known special cases of hexapods attaining high values for the conformal and the Euler degree, mainly in order to exclude them later, when stating and proving our main theorem (Theorem 1.34).
1.1.1. **Planar hexapods.** For a general planar pentapod, it is possible to add an additional leg without changing the configuration set (see [Dup98, Naw14a]), hence obtaining a movable hexapod. Its configuration curve is the intersection of $X$ with a linear subspace of codimension 5, hence it has conformal degree 40, and its Euler degree is 20. The genus can easily be computed by the Hilbert series of $X$, and it turns out to be $41^2$.

1.1.2. **Non-planar equiform hexapods.** A hexapod is called *equiform* if there is a similarity $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ mapping base points to platform points. Non-planar equiform hexapods are discussed in [Naw13].

Equiform hexapods of mobility one admit configuration curves of degree 38. The assumption on the existence of the similarity implies that for every direction in $\mathbb{R}^3$, the projections of both base and platform points along that direction are similar, hence every direction in $\mathbb{R}^3$ determines a similarity point in $K_\Pi$. Therefore the intersection of $K_\Pi$ with the boundary $B$ is a curve, and one can show that it is actually a conic. Thus the configuration curve of such a hexapod has two components, one given by the Zariski closure of $K_\Pi \cap \text{SE}_3$, and the other by the intersection of $K_\Pi$ with $B$.

1.1.3. **General case.** If we assume that $\Pi$ is neither planar nor equiform, then $K_\Pi$ intersects $B$ in at most a finite number of points (see [GNS15]). For this case, we show a useful proposition relating the conformal degree and the Euler degree.

**Proposition 1.4.** Let $\Pi$ be a mobility one hexapod, neither planar nor equiform, then its Euler degree is at most half its conformal degree.

**Proof.** Consider the linear projection $\mathbb{P}^6_1 \dashrightarrow \mathbb{P}^6_2$ from Diagram (2). Then the configuration curve $K_\Pi$ is mapped by such projection to a curve $K'_\Pi$ in $V_{3,2}$ such that the preimage of $K'_\Pi$ under the Veronese embedding $v_{3,2}$ determines the Euler degree of $\Pi$.

Let $r$ be the number of intersection points, counted with multiplicity, of $K_\Pi$ with the center of the projection. Since by Remark 1.3 this is the cardinality of similarity bonds plus the cardinality of collinearity bonds of $\Pi$, the number $r$ is finite. By a well-known formula for the degree of a projection, the degree of the image of $K_\Pi$ is a factor of $d - r$, where $d$ is the conformal degree, with equality if and only if the restriction of the projection to $K_\Pi$ is birational to the image. On the other hand, the Veronese map just doubles the degree, hence the statement follows. □

We aim at bounding the conformal degree of a hexapod $\Pi$ that satisfies some genericity assumptions (for example, we will assume $\Pi$ to be non-planar and non-equiform), and we will determine it by means of two estimates: first of all we connect the conformal degree to the intersection number of two algebraic curves defined by base and platform points only, and then the latter will be bounded using some arguments from algebraic geometry. To do so we need to introduce a new tool, called *Möbius photogrammetry*, and this is the content of the next subsection.

---

2 Moreover, the Hilbert series shows that $K_\Pi$ is half-canonical, namely embedded by a linear system $\mathcal{L}$ such that $2\mathcal{L}$ is canonical.
1.2. **Möbius photogrammetry for hexapods.** Notice that a necessary condition for the mobility of a hexapod is the existence of bonds of some type: in fact if a hexapod $\Pi$ is movable, then its configuration set $K_\Pi$ intersects the boundary $B$ non-trivially. In particular, when the mobility is one, the bigger the degree of $K_\Pi$, the bigger the number of bonds — counted with multiplicity.

If we assume that $\Pi$ is neither planar nor equiform nor does it satisfy a butterfly condition, but has mobility one, then by the characterization of boundary points we get that there exist planar projections of base and platform such that their images are related by either a similarity or an inversion. In the second case, we can replace the inversion also by the composition of the inversion with a reflection, by considering the projection from the opposite direction. Then, in both cases, we have projections of base and platform points that are Möbius equivalent, i.e. equivalent up to Möbius transformations.

Hence, if we are interested in hexapods with high conformal degree, then we should look for hexapods with a high number of bonds. Since the presence of inversion or similarity bonds corresponds to Möbius equivalent orthogonal projections of base and platform, we are going to study such projections.

1.2.1. **Construction of the photographic map.** In [GNS14] the authors introduced the concept of **Möbius photogrammetry** for 5-tuples of points in $\mathbb{R}^3$, and this technique allowed to deduce some results about pentapods with mobility two. The idea behind it is to take “pictures” of a configuration of 5 points in $\mathbb{R}^3$, namely consider its orthogonal projection onto $\mathbb{R}^2$ along all possible directions, and try to deduce properties of this configuration in $\mathbb{R}^3$ from the knowledge of its projections. We briefly recall the construction of the photographic map, adapting it to our setting, namely considering 6-tuples of distinct points — for definitions and proofs we refer to [GNS14].

We start with a vector $\vec{A} = (A_1, \ldots, A_6)$ of 6 distinct points in $\mathbb{R}^3$. For each direction $\varepsilon \in S^2$ in $\mathbb{R}^3$, where $S^2$ denotes the unit sphere, we consider an orthogonal projection $\pi_\varepsilon: \mathbb{R}^3 \to \mathbb{R}^2$ along $\varepsilon$. Hence for each $\varepsilon \in S^2$ we have a tuple $\pi_\varepsilon(\vec{A}) = (\pi_\varepsilon(A_1), \ldots, \pi_\varepsilon(A_6))$ of projected points of $\vec{A}$, namely we have a 6-tuple of elements of $\mathbb{R}^2$. By identifying $\mathbb{R}^2$ with $\mathbb{C}$ and embedding the latter in $\mathbb{P}_{\mathbb{C}}^1$, we can think of $\pi_\varepsilon(\vec{A})$ as of an element of $(\mathbb{P}_{\mathbb{C}}^1)^6$.

Since Möbius transformations form the automorphism group of $\mathbb{P}_{\mathbb{C}}^1$, the following two concepts are equivalent:

- a 6-tuple $\pi_\varepsilon(\vec{A})$ in $(\mathbb{R}^2)^6$, considered up to Möbius transformations;
- a 6-tuple $\pi_\varepsilon(\vec{A})$ in $(\mathbb{P}_{\mathbb{C}}^1)^6$, considered up to automorphism of $\mathbb{P}_{\mathbb{C}}^1$.

Let $M_6$ denote the moduli space of 6 points in $\mathbb{P}_{\mathbb{C}}^1$, and denote by $\varphi: (\mathbb{P}_{\mathbb{C}}^1)^6 \to M_6$ the corresponding quotient map, namely the function sending a configuration of 6 points to its class modulo automorphisms of $\mathbb{P}_{\mathbb{C}}^1$. The previous notation encodes the fact that $\varphi$ is not defined on the whole $(\mathbb{P}_{\mathbb{C}}^1)^6$, since we know from Geometric Invariant Theory that we need to restrict to the open set $U$ of $(\mathbb{P}_{\mathbb{C}}^1)^6$ of 6-tuples where no four points coincide in order to build a well-behaved theory. The previous equivalence allows us to think of a 6-tuple $\pi_\varepsilon(\vec{A})$ of points in $\mathbb{R}^2$, considered up
to Möbius transformations, as a point in \( M_6 \), in particular as the image of \( \pi_\varepsilon(\vec{A}) \) under the quotient map \( \varphi \). In order to give an explicit formulation of the quotient map \( \varphi \) we follow, as in [GNS14], the graphical approach provided in [HMSV09]:

1. Consider a convex hexagon in the plane and construct all plane undirected multigraphs without loops whose set of vertices coincides with the set of vertices of the hexagon and that satisfy the following conditions:
   - edges are given by segments;
   - any two edges do not intersect;
   - the valency of every vertex is 1.

There are exactly 5 of these graphs, as shown in Fig. 2.

![Figure 2](image_url)

**Figure 2.** The only five planar undirected multigraphs without loops with vertices on a regular hexagon, valency 1 and non-intersecting edges.

2. Associate to each graph \( G \) with set of edges \( E \) a homogeneous polynomial in the coordinates \( \{(a_i : b_i)\} \) of \( (\mathbb{P}^1)^6_\mathbb{C} \) in the following way:

\[
\varphi_G = \prod_{i<j} (a_i b_j - a_j b_i).
\]

For example the polynomial associated to the first graph in Fig. 2 is

\[
\varphi_0 = (a_1 b_2 - a_2 b_1)(a_3 b_6 - a_6 b_3)(a_4 b_5 - a_5 b_4).
\]

3. These five polynomials determine a rational map \( \varphi : (\mathbb{P}^1)^6_\mathbb{C} \to \mathbb{P}^4_\mathbb{C} \).

4. Consider the open set

\[
\mathcal{U} = \{(m_1, \ldots, m_6) : \text{no four of the points } m_i \text{ coincide}\} \subseteq (\mathbb{P}^1)^6_\mathbb{C}.
\]

5. The image of \( \varphi|_\mathcal{U} \), namely the restriction of \( \varphi \) to \( \mathcal{U} \), gives an embedding of \( M_6 \) in \( \mathbb{P}^4_\mathbb{C} \).

Summing up, once we fix a 6-tuple \( \vec{A} \) of distinct points in \( \mathbb{R}^3 \), we can associate to each direction \( \varepsilon \in S^2 \) the image (under the quotient map \( \varphi \)) of the projected 6-tuple \( \pi_\varepsilon(\vec{A}) \). In this way we construct a map from \( S^2 \) to \( M_6 \). In order to make
it a morphism between projective varieties, we identify $S^2$ with the conic $C = \{x^2 + y^2 + z^2 = 0\}$ in $\mathbb{P}_C^2$. We obtain a map $f_{\vec{A}}: C \rightarrow M_6$.

If we write $A_i = (s_i, t_i, u_i)$ and we set for all $i, j \in \{1, \ldots, 6\}$ with $i < j$

$$H_{ij} = (s_i - s_j)x + (t_i - t_j)y + (u_i - u_j)z,$$

then the components of $f_{\vec{A}}$ have the following structure:

$$
\begin{align*}
(f_{\vec{A}})_0 &= H_{12} \quad H_{36} \quad H_{45} \\
(f_{\vec{A}})_1 &= H_{14} \quad H_{23} \quad H_{56} \\
(f_{\vec{A}})_2 &= H_{16} \quad H_{25} \quad H_{34} \\
(f_{\vec{A}})_3 &= H_{16} \quad H_{23} \quad H_{45} \\
(f_{\vec{A}})_4 &= H_{12} \quad H_{34} \quad H_{56}
\end{align*}
$$

(3)

From this explicit description we see that in particular this map is algebraic.

**Definition 1.5.** The regular map $f_{\vec{A}}: C \rightarrow M_6$ we obtain is called the photographic map of $\vec{A}$.

**Definition 1.6.** Let $\vec{A}$ be a 6-tuple of distinct points in $\mathbb{R}^3$ — from now on we will omit the adjective “distinct”, but throughout the paper we will always consider this situation. The image of the photographic map $f_{\vec{A}}$ is a rational curve in $M_6$, that we call the Möbius curve of $\vec{A}$.

**Remark 1.7.** As noticed in [GNS14], the map $f_{\vec{A}}$ is a morphism of real varieties, namely it respects the real structures of $C$ and $M_6$ induced by the standard ones of $\mathbb{P}_C^2$ and $\mathbb{P}_C^4$, respectively.

**Fact.** One can prove that $M_6$ is embedded in $\mathbb{P}_C^4$ as a cubic threefold, called the Segre cubic primal, whose equation is

$$x_0x_1(x_0 + x_1 + x_2 + x_3 + x_4) - x_2x_3x_4 = 0.$$

The threefold $M_6$ contains exactly 15 planes, corresponding to equivalence classes of configurations $(m_1, \ldots, m_6)$ where $m_i = m_j$ for some $i \neq j$. Furthermore, the maximum possible number of nodes for a cubic hypersurface in $\mathbb{P}_C^4$, namely 10, is attained by the Segre cubic. The nodes correspond to configurations of points $(m_1, \ldots, m_6)$ where three points coincide, and are the only singular points of $M_6$.

There are $\binom{6}{3} = 20$ such configurations, but the following phenomenon happens in $M_6$: if we partition $\{1, 2, 3, 4, 5, 6\} = \{i, j, k\} \cup \{u, v, w\}$, then configurations of points for which $m_i = m_j = m_k$ are sent by the quotient map $\varphi$: $(\mathbb{P}_C^4)^6 \rightarrow M_6$ to the same point of $M_6$ as configurations of points for which $m_u = m_v = m_w$.

The interested reader can find proofs and explanations for the stated facts in [Dol12] Chapter 9, Subsection 9.4.4.

By construction, whenever a hexapod $\Pi$ admits an inversion or similarity bond, then the Möbius curves of its base and platform intersect. However, due to the construction of the moduli space $M_6$, this is not the only situation when we get intersections between the two curves: suppose in fact that $\Pi$ admits three collinear base points, say $P_1, P_2$ and $P_3$, and three collinear platform points, say $p_4, p_5$ and $p_6$, so that we have a butterfly bond; then the Möbius curves intersect in a node of $M_6$. 
since the projections along the lines $P_1 P_2 P_3$ and $p_4 p_5 p_6$ determine configurations in $\mathbb{R}^2$ where three points coincide.

1.2.2. Properties of the photographic map. The following two lemmata, in analogy with Lemma 3.7 and 3.8 in [GNS14], describe the possible behavior of the photographic map. The proof technique is similar to the one in [GNS14], but we report the proofs for self-containedness purposes.

**Lemma 1.8.** Let $\vec{A} = (A_1, \ldots, A_6)$ be a 6-tuple of points in $\mathbb{R}^3$. If the $\{A_i\}$ are not coplanar, then the photographic map $f_{\vec{A}} : C \rightarrow M_6$ is birational to a rational curve of degree 6 or 4 in $M_6$.

**Proof.** We notice that if the lines $\overrightarrow{A_i A_j}$ and $\overrightarrow{A_k A_l}$ are parallel, then the factors $H_{ij}$ and $H_{kl}$ in Eq. (3) differ only by a scalar. This means that in particular situations, when some of the $A_i$ are collinear, the components of $f_{\vec{A}}$ can have factors in common that can be simplified, reducing the degree of the map. Hence we partition the set of possible configurations in two cases, depending on the number of common factors of the components of $f_{\vec{A}}$. Notice that, since we assume that the points are not coplanar, it can never happen that 5 or all 6 points are collinear.

**Case (a):** Here no 4 points are collinear. In this case the components of $f_{\vec{A}}$ do not have any factor in common, so

$$\deg(f_{\vec{A}}(C)) \cdot \deg(f_{\vec{A}}) = 6.$$  

Thus there are only four possibilities: either $f_{\vec{A}}$ is 6 : 1 to a line, or it is 3 : 1 to a conic, or it is 2 : 1 to a cubic curve, or 1 : 1 to a sextic curve. We prove that the first three situations can never happen. First of all, notice that there are exactly two directions in $\mathbb{R}^3$ for which the images of $A_i$ and $A_j$ under the projection coincide, namely the directions of the oriented lines $\vec{A_i A_j}$ and $\vec{A_j A_i}$. Denote by $T_{ij}$ the plane in $M_6$ of classes of 6-tuples $(m_1, \ldots, m_6)$ where $m_i = m_j$ (see the Fact in Subsection 1.2.1). Then there are exactly 2 points in $C$ that are mapped to the plane $T_{ij}$ — and they are complex conjugate, since complex conjugation in $C$ corresponds to the antipodal map in the unit sphere $S^2 \subseteq \mathbb{R}^3$.

If $f_{\vec{A}}$ is 3 : 1 or 6 : 1, then those two points have to be branching points of $f_{\vec{A}}$. Setting $A_i = (s_i, t_i, u_i)$ one can check that

$$f_{\vec{A}}^{-1}(T_{ij}) = \{ (x : y : z) \in C : H_{ij}(x, y, z) = 0 \},$$

where we recall that $H_{ij} = (s_i - s_j)x + (t_i - t_j)y + (u_i - u_j)z$. If the points in the preimage of $T_{ij}$ are branching points, then the line $\{ H_{ij} = 0 \} \subseteq \mathbb{P}^2_C$ should intersect $C$ tangentially at those points. However, this is impossible, since both $C$ and the line are real varieties, so if they are tangent their intersection point is real, but $C$ has no real points. In this way we rule out the 3 : 1 and the 6 : 1 case.

Suppose now that $f_{\vec{A}}$ is 2 : 1. We are going to show that $\vec{A}$ should be planar, contradicting the hypothesis. In fact, by assumption $f_{\vec{A}}$ factors through an involution of $C$ — the one swapping the fibers of $f_{\vec{A}}$; one can
show that such involution is a real automorphism of $C$, and hence corresponds to a rotation by 180 degrees of the unit sphere $S^2$ along some axis $\ell$. The fibers $f_{A}^{-1}(T_{ij})$ are invariant under the involution, so they are contained in the subset of $S^2$ invariant under the rotation, namely the intersection $S^2 \cap \ell$ together with the maximal circle in $S^2$ lying on a plane orthogonal to $\ell$. A case-by-case analysis proves that all direction $\vec{A}_i \vec{A}_j$ belong to such maximal circle, this implying that $\vec{A}$ is planar. Hence the birational case is the only possible.

**Case (b):** Here exactly 4 points are collinear. In this case the components of $f_{\vec{A}}$ have one factor in common, leading to

$$\deg (f_{\vec{A}}(C)) \cdot \deg (f_{\vec{A}}) = 4.$$  

We have three possibilities: either $f_{\vec{A}}$ is $4:1$ to a line, or it is $2:1$ to a conic, or $1:1$ to a quartic curve. Arguing as in Case (a) we prove the statement. □

**Lemma 1.9.** Let $\vec{A} = (A_1, \ldots, A_6)$ be a 6-tuple of points in $\mathbb{R}^3$. If the $\{A_i\}$ are coplanar, but not collinear, then the photographic map $f_{\vec{A}}: C \rightarrow M_6$ is $2:1$ to a rational curve of degree 3, 2 or 1 in $M_6$.

**Proof.** Since $\vec{A}$ is planar, then all directions $\vec{A}_i \vec{A}_j$ belong to a maximal circle of the unit sphere $S^2$. Let $\ell$ be the line through the origin orthogonal to the plane spanned by such maximal circle. Reversing the argument in the proof of Lemma 1.8 Case (a) one can prove that the map $f_{\vec{A}}$ factors through the $2:1$ involution $\tau$ of $C$ determined by the rotation of $S^2$ by 180 degrees around $\ell$. Then we have $f_{\vec{A}} = g_{\vec{A}} \circ \tau$.

From now on the proof works as in Lemma 1.8 and the following cases arise:

**Case (a):** Suppose that no 4 points of $\vec{A}$ are collinear. Then the components of $f_{\vec{A}}$ do not have any factor in common, and $g_{\vec{A}}$ is birational to a cubic.

**Case (b):** Suppose that 4 points of $\vec{A}$ are collinear, but no 5 points are so. Then the components of $f_{\vec{A}}$ have exactly one factor in common, and $g_{\vec{A}}$ is birational to a conic.

**Case (c):** Suppose that 5 points of $\vec{A}$ are collinear. Then the components of $f_{\vec{A}}$ have exactly two factors in common, and $g_{\vec{A}}$ is birational to a line. □

We conclude this section by showing that the photographic map of some 6-tuples of points extends to a morphism defined on the whole plane $\mathbb{P}^2_\mathbb{C}$. From this we deduce geometric constraints for Möbius curves. We hope that this can be the first step towards a complete geometric characterization of Möbius curves as curves in $M_6$. We focus on the case when Möbius curves are of degree 6, which is the most interesting for our application.

**Lemma 1.10.** Let $\vec{A}$ be a 6-tuple of points in $\mathbb{R}^3$ and let $f_{\vec{A}}: C \rightarrow M_6$ be its photographic map, where the resulting Möbius curve is of degree 6. Then $f_{\vec{A}}$ extends to a morphism $F_{\vec{A}}: \mathbb{P}^2_\mathbb{C} \rightarrow M_6$.

**Proof.** We only need to prove that the map $F_{\vec{A}}$ does not have base points in $\mathbb{P}^2_\mathbb{C}$. Recall the structure of the map $f_{\vec{A}}$ described by Eq. (3): we infer that a base point
has to vanish on at least one polynomial $H_{ij}$ for each component of $F_{\vec{A}}$. A direct inspection shows that this would imply that at least 4 points of $\vec{A}$ are collinear, but this is impossible because by Lemma 1.8 the curve $D$ would not have degree 6, contradicting the hypothesis. □

**Proposition 1.11.** Let $D$ be a smooth Möbius curve of degree 6. Then

- $D$ can be defined by real polynomials, but has no real points;
- $D$ is contained in a linear projection, defined by real polynomials, of the third Veronese embedding of $\mathbb{P}^2_\mathbb{C}$.

**Proof.** By construction $D$ is a real variety, since $f_{\vec{A}}$ is a real map and $C$ is a real variety; since $D$ is smooth and using Lemma 1.8 we have that $f_{\vec{A}}$ is an isomorphism, hence $D$ has no real points, because this holds for $C$. Eventually from Lemma 1.10 we get that $D$ is contained in a linear projection, defined by real polynomials, of the third Veronese embedding of $\mathbb{P}^2_\mathbb{C}$, which is real by construction. One notices that such projections is the complete intersection of $M_6$ with another cubic hypersurface. □

### 1.3. An upper bound for the conformal degree

We have now all the tools needed to prove the bound on the conformal degree of a hexapod (Theorem 1.34). We split the argument in two parts: first we prove that the conformal degree of a hexapod $\Pi$ is less than or equal to twice the number of intersections of the Möbius curves of the base and platform of $\Pi$ (Theorem 1.12); then we show that such intersection cannot be composed by more than 14 points (Proposition 1.33). For our proofs to work we need to exclude some degenerate cases — planarity, equiformity and a certain kind of collinearity.

The reader should be warned about the length of these two proofs.

**Theorem 1.12.** Let $\Pi$ be a mobility one hexapod. Let $D_1 = f_{\vec{P}}(C)$ and $D_2 = f_{\vec{p}}(C)$ be the Möbius curves of its base and platform. Suppose that $\Pi$ is non-planar, non-equiform and no 4 base or platform points are on a line. Then

$$\deg K_{\Pi} \leq 2 \deg(D_1 \cap D_2),$$

where the intersection $D_1 \cap D_2$ is meant scheme-theoretically, namely the points of intersection are counted with multiplicity.

**Proof.** Informally, the proof goes as follows: the degree of $K_{\Pi}$ is computed by intersecting it with a hyperplane, and we choose the hyperplane defining the boundary $B$ of $X$. This intersection, as we will see, should be counted twice, and this means that the degree of $K_{\Pi}$ is bounded by twice the number of bonds of $\Pi$. Eventually, bonds correspond to common images of the photographic maps, so the statement follows.

In order to make the argument precise, we need to rephrase the analysis of bonds carried out in [GNS15] in a scheme-theoretical way. We start by noticing that the hyperplane $L$ defining the boundary $B$ is totally tangential to $X$, namely the scheme-theoretical intersection $X \cap L$ has a non-reduced structure, and we have $B = (X \cap L)_{\text{red}}$; moreover, as divisors on $X$, it holds $2B = X \cap L$. All these
results can be proved via a direct computation using, for example, Gröbner bases. Therefore, if $\widetilde{K}_\Pi$ denotes the top-dimensional part of $K_\Pi$ — namely $\widetilde{K}_\Pi$ is an equidimensional scheme of dimension one — we have that:

$$\deg K_\Pi = \deg \widetilde{K}_\Pi = \deg (\widetilde{K}_\Pi \cap L)$$

$$\leq \deg (K_\Pi \cap L)$$

$$= \deg (H_\Pi \cap X \cap L)$$

$$\leq 2 \deg (H_\Pi \cap B).$$

The last inequality follows from the general fact that if $Y_1$ and $Y_2$ are two divisors of $X$ satisfying $Y_2 = 2 Y_1$, then for any subvariety $Z$ of $X$ we have $\deg (Y_1 \cap Z) \leq 2 \deg (Y_2 \cap Z)$.

The scheme $B_\Pi = H_\Pi \cap B$ is the scheme of bonds of $\Pi$, and we are going to prove that the number $\deg (H_\Pi \cap B)$ is less than or equal to the degree of the intersection $D_1 \cap D_2$ of the two Möbius curves of $\Pi$. To reach this goal, we need to connect the boundary $B$ of $X$ to the curve $C \subseteq \mathbb{P}_C^2$ used in Definition 1.5 by means of a rational map $B \dashrightarrow C \times C$. As we already reported in Subsection 1.1, to every inversion, similarity and butterfly point we can associate two directions in $\mathbb{R}^3$, and since we use the curve $C$ as a model for the unit sphere $S^2 \subseteq \mathbb{R}^3$, this means that to every such point we can associate two elements $v, w \in C$.

Notice that our assumption on the hexapod $\Pi$ rules out the existence of collinearity bonds, and so it is harmless to exclude them from the picture. Using the notation introduced in [GNS15 Sections 2.1 and 2.3], we can write

$$B = \left\{ (h : M : x : y : r) : \begin{array}{l} h = 0, \\
M y = M x = 0, \\
\langle x, x \rangle = \langle y, y \rangle = 0 \end{array} \right\}.$$ 

If we define

$$B' = B \setminus \left( \{ \text{collinearity points} \} \cup \{ \text{vertex} \} \right),$$

then we get

$$B' = \left\{ \beta \in B : M \neq 0 \text{ or } (x \neq 0 \text{ and } y \neq 0) \right\}.$$ 

If $\beta \in B'$, then there exist $v, w \in C$ and $\lambda, \mu, \alpha \in \mathbb{C}$ such that

$$M = \alpha v w^T, \quad x = \mu w, \quad y = \lambda x,$$

and from the definition of $B'$ we get that either $\alpha \neq 0$ or $\lambda \mu \neq 0$. We define the algebraic map

$$\delta : B' \longrightarrow C \times C$$

$$(0 : M : x : y : r) \mapsto (v, w)$$

Notice that the fiber of $\delta$ over a point $(v, w)$ is parametrized by

$$\delta^{-1}(v, w) \cong \left\{ \alpha = (\alpha : \lambda : \mu : r) \in \mathbb{P}_C^3 : \alpha \neq 0 \text{ or } \lambda \mu \neq 0 \right\}.$$ 

This allows to conclude that $B'$ forms an open subvariety of a $\mathbb{P}_C^3$-bundle over $C \times C$. As remarked before, our hypotheses imply that the scheme of bonds $B_\Pi = B \cap H_\Pi$ is a closed subscheme of $B'$.

**Claim.** The map $\delta|_{B_\Pi}$ is an isomorphism.
In order to prove the claim, we need to rephrase the so-called pseudo spherical condition, introduced in [GNS15, Definition 3.6]. The pseudo spherical condition imposed by a pair \((P_i, p_i)\) on a bond \((0 : M : x : y : r)\) reads as

\[ r - 2\langle M p_i, P_i \rangle - 2\langle P_i, y \rangle - 2\langle p_i, x \rangle = 0. \]

Using Eq. (4), and setting \(W_i = P_i^T w\) and \(V_i = p_i^T v\) yields

\[ r - 2\alpha W_i V_i - 2\mu W_i - 2\lambda V_i = 0. \]

The scheme \(B_{\Pi}\) is cut out by these 6 pseudo spherical conditions for \(i = 1, \ldots, 6\).

Hence, if we define the \(4 \times 6\) matrix

\[ N_{\Pi}(v, w) = \begin{pmatrix} W_1 & V_1 & V_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ W_6 & V_6 & V_6 & 1 \end{pmatrix}, \]

then the scheme \(B_{\Pi}\) is locally defined by

\[ \left\{ (v, w, \overline{\omega}) \in \mathcal{V} \times \mathcal{V} \times \mathcal{W} : N_{\Pi}(v, w) \cdot \overline{\omega} = 0 \right\}, \]

where \(\mathcal{V}\) and \(\mathcal{W}\) are suitable open subvarieties of \(C\) and \(\mathbb{P}_C^3\) respectively, and \(\overline{\omega} = (\alpha : \lambda : \mu : r)\). We restate our previous claim.

**Claim.** The map \(\delta\) maps \(B_{\Pi}\) isomorphically to the scheme in \(C \times C\) cut out by the \(4 \times 4\) minors of \(N_{\Pi}\).

We prove that the rank of the matrix \(N_{\Pi}(v, w)\) is always at least 3. In fact, if the rank is 1 then the collinearity hypothesis is violated (all \(W_i\) would be equal, and the same for the \(V_i\)); if the rank is 2, then the planarity condition is violated (this can be deduced by a direct computation, imposing that the second column of \(N_{\Pi}\) is a linear combination of the third and the fourth). Hence the rank of \(N_{\Pi}(v, w)\) is greater than or equal to 3. Notice that column operations on \(N_{\Pi}\) correspond to projective transformations in the fibers of \(\delta\), while row operations on \(N_{\Pi}\) correspond to the choice of a different system of generators for the ideal of \(B_{\Pi}\). Hence we can reduce to the case when \(N_{\Pi}\) has the form

\[ N_{\Pi}(v, w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & G_1(v, w) \\ 0 & 0 & 0 & G_2(v, w) \\ 0 & 0 & 0 & G_3(v, w) \end{pmatrix}, \]

where \(G_i(v, w)\) are rational functions on \(C \times C\). The local description of \(B_{\Pi}\) becomes

\[ \left\{ (v, w, \overline{\omega}^r) : \overline{\omega}^r = (0 : 0 : 0 : r') \text{ for some } r' \neq 0, \text{ and } G_1(v, w)r' = G_2(v, w)r' = G_3(v, w)r' = 0 \right\}, \]

while the zero locus of the \(4 \times 4\) minors of \(N_{\Pi}\) is locally given by

\[ \left\{ (v, w) : G_1(v, w) = G_2(v, w) = G_3(v, w) = 0 \right\}, \]

and one sees that these two schemes are isomorphic.

The proof is then complete once we are able to show the following.
**Claim.** The image of $B_\Pi$ under $\delta$ is contained in the pullback of the two photogramic maps $f_{\bar{p}}, f_{\bar{P}} : C \rightarrow M_6$.

Notice that the fact that $\Pi$ is supposed to be non-equiform implies that the Möbius curves of $\bar{p}$ and $\bar{P}$ are different. The pullback of the two maps is

$$\left\{(v, w) \in C \times C : f_{\bar{p}}(v) = f_{\bar{P}}(w)\right\}.$$ 

The coordinates of $f_{\bar{p}}(v)$ are obtained by substituting each term $H_{ij}$ by $V_i - W_j$ in the general formula from Eq. (3); for $f_{\bar{P}}(w)$ one just needs to consider $W_i - W_j$ instead. Then the pullback is the scheme cut out by the $2 \times 2$ minors of the following $2 \times 5$ matrix:

$$\begin{pmatrix} W_{12,36,45} & W_{14,23,56} & W_{16,25,34} & W_{16,23,45} & W_{12,34,56} \\ W_{12,36,45} & V_{14,23,56} & V_{16,25,34} & V_{16,23,45} & V_{12,34,56} \end{pmatrix},$$

where $W_{ij,kl,mn} = (W_i - W_j)(W_k - W_l)(W_m - W_n)$ and similarly for $V_{ij,kl,mn}$. A direct computation (for example, with the aid of Gröbner bases) shows that the ideal generated by such $2 \times 2$ minors is contained in the ideal of the $4 \times 4$ minors of $N_\Pi$. This settles the claim and hence concludes the proof. \hfill $\square$

**Remark 1.13.** We cannot hope for equality in the last claim in the proof of Theorem 1.12.

In fact, consider one of the 15 planes in $M_6$ (see the Fact in Subsection 1.2.1), and suppose it parametrizes classes of tuples $(m_1, \ldots, m_6)$ where $m_i = m_j$; then the projection from such a plane maps $M_6$ to $\mathbb{P}^1_c$, and the latter has a modular interpretation as $M_4$, namely the moduli space of 4-tuples $(m_1, \ldots, m_i, \ldots, m_j, \ldots, m_6)$ obtained by removing $m_i$ and $m_j$. Then the images of $f_{\bar{p}}(v)$ and $f_{\bar{P}}(w)$ under the projection from the plane of classes with $m_i = m_j$ coincide if and only if

$$\begin{align*}
(W_i - W_j)(W_k - W_l)(V_i - V_j)(V_k - V_l) -
(V_i - V_j)(V_k - V_l)(W_i - W_l)(W_k - W_j) &= 0,
\end{align*}$$

where $\{i, j, k, l\} \cup \{s, t\} = \{1, \ldots, 6\}$. On the other hand, the left-hand-side of Eq. (5) is a $4 \times 4$ minor of $N_\Pi$. In fact, if we select the submatrix with rows of index $i, j, k$ and $l$ and we perform column operations we obtain

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ W_j V_j - W_i V_i & W_j - W_i & V_j - V_i & 1 \\ W_k V_k - W_i V_i & W_k - W_i & V_k - V_i & 1 \\ W_l V_l - W_i V_i & W_l - W_i & V_l - V_i & 1 \end{pmatrix},$$

and a direct computation proves the equality. As it is shown in the second claim of the proof, the zero locus of these minors is image of $B_\Pi$ under $\delta$.

On the other hand, there are points in the pullback of $f_{\bar{p}}$ and $f_{\bar{P}}$ for which not all the cross-ratios of Eq. (5) are equal: suppose in fact the points $p_1, p_2$ and $p_3$ are collinear along the direction $v$, and the points $P_1, P_2$ and $P_3$ are collinear along the direction $w$; then $f_{\bar{p}}(v)$ and $f_{\bar{P}}(w)$ coincide with one of the nodes of $M_6$, independently of the projections of the other points; hence it is possible to have situations in which not all the 15 cross-ratios coincide, but still we have an intersection of the two Möbius curves.

Next, we estimate the number of intersections of two Möbius curves of degree 6 that satisfy some non-degeneracy conditions, and we show that this number is
always less than or equal to 14 (see Proposition [1.33]). To clarify why such a bound should hold, let us consider $D_1$ and $D_2$, two Möbius curves of degree 6. As we will point out later, there always exist at least two quadrics passing through each of them. In general, the quadrics passing through $D_1$ will be different from the ones passing through $D_2$; up to swapping the roles of $D_1$ and $D_2$, in general there will be a quadric $Q$ containing $D_1$, but not $D_2$. In this case

$$|D_1 \cap D_2| \leq |Q \cap D_2| \leq 12.$$  

Instead, as we are going to see, we reach 14 intersections when $D_1$ and $D_2$ are the two components of a curve of degree 12, complete intersection of two quadrics and $M_6$. Unfortunately, proving that the bound holds in all cases requires the analysis of several particular situations\(^3\), so we break the proof of Proposition [1.33] into a sequence of lemmata. These intermediate results are essentially of two kinds: either they predicate some property of subvarieties of $\mathbb{P}^4_\mathbb{C}$ containing a Möbius curve, or they prove that the bound holds in some particular sub-case.

Recall that we denote by $T_{ij}$ the plane in $M_6$ parametrizing the classes of 6-tuples $(m_1, \ldots, m_6)$ in $\mathbb{P}^1_{\mathbb{C}}$ such that $m_i = m_j$.\(^3\)

**Lemma 1.14.** Let $D$ be a Möbius curve of degree 6. Then $D$ cannot be contained in a plane.

**Proof.** Suppose $D \subseteq E$, where $E$ is a plane in $\mathbb{P}^4_\mathbb{C}$. We distinguish two cases:

- a. The plane $E$ is completely contained in $M_6$. Then $E$ is one of the 15 planes $T_{ij}$. Write $D = f_{\bar{A}}(C)$, then the projections of the points $A_i$ and $A_j$ along any direction coincide, and this is possible only if $A_i = A_j$, but we only consider 6-tuples $\bar{A}$ of distinct points, so we get an absurd. Notice that in this case we did not make use of the assumption on the degree of $D$.

- b. The plane $E$ is not contained in $M_6$. Then $M_6 \cap E$ is at most a cubic curve, but this is in contrast with the assumption on the degree of $D$. \(\square\)

Therefore a degree 6 Möbius curve lies on at most one hyperplane in $\mathbb{P}^4_\mathbb{C}$.

**Lemma 1.15.** Let $D_1$ and $D_2$ be two distinct degree 6 Möbius curves, and suppose that $D_1$ lies on a hyperplane $H$. If $D_2$ is not contained in $H$, then $|D_1 \cap D_2| \leq 6$.

**Proof.** By assumption $|D_1 \cap D_2| \leq |H \cap D_2| \leq 6$. \(\square\)

From the discussion so far we see that if one of the two Möbius curves lies on a hyperplane, then the bound of 14 intersections holds provided that the other is not contained in the same hyperplane. Now we consider the latter case, and prove that the bound still holds.

**Lemma 1.16.** Let $D$ be a degree 6 Möbius curve. Then $D$ cannot be contained in a surface of degree 2 completely contained in $M_6$.

---

\(^3\)In a first draft of this paper we tried to give an all-embracing argument, but we were not able to obtain a complete proof.
Remark 1.18 Suppose that Lemma 1.21.

Proof. The proof is based on a photographic description of surfaces of degree 2 in $M_6$ (see also Remark [1.13]). Let $S \subseteq M_6$ be a surface of degree 2. Then $S$ spans a hyperplane $H$ in $\mathbb{P}^4$, and by degree reasons $M_6 \cap H = S \cup E$, where $E$ is a plane. Hence $E$ is one of the 15 planes $T_{ij}$ in $M_6$. Consider the rational map $\mathbb{P}^4 \dashrightarrow \mathbb{P}^1$ defined by the hyperplanes through $E$: its restriction to $M_6$ has a photographic meaning, namely it sends the class of a tuple $(m_1, \ldots, m_6)$ to the class of the tuple $(m_1, \ldots, m_i, \ldots, m_j, \ldots, m_6)$ — i.e. we remove the points $m_i$ and $m_j$ — in $M_4$, identified with $\mathbb{P}^1$. Moreover, by construction $S$ is a fiber of such map. Therefore a Möbius curve $D = f_A(C)$ of degree 6 cannot lie in $S$, because otherwise the cross-ratio of the projections of the points $A_1, \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_6$ along any direction in $\mathbb{R}^3$ would stay constant; this is only possible if four points in $A$ are collinear, which is excluded by Lemma 1.18.

Lemma 1.17. Let $D$ be a degree 6 Möbius curve contained in a hyperplane $H$. Then $H \cap M_6$ is an irreducible cubic surface.

Proof. Suppose that $Y = H \cap M_6$ is reducible, then either $Y = S \cup E$ where $E$ is a plane and $S$ is a surface of degree 2, or $Y = E_1 \cup E_2 \cup E_3$ where the $E_i$ are planes. In both cases a Möbius curve, which is irreducible, cannot lie in any of the components of $Y$ because of Lemma 1.14 and Lemma 1.16.

From now on, for a Möbius curve $D$ contained in a hyperplane $H$ we define $L_{ij} = H \cap T_{ij}$. We know that all $L_{ij}$ are lines (and not planes), because by Lemma 1.17 the hyperplane $H$ cannot contain any of the planes $T_{ij}$.

Remark 1.18. Consider a Möbius curve $D = f_A(C)$ such that $D \cap T_{ij} \cap T_{kl} \neq \emptyset$ for pairwise distinct $i, j, k, l \in \{1, \ldots, 6\}$, i.e. $|\{i, j, k, l\}| = 4$. Then, by construction of the photographic map, the lines $\vec{A}_i\vec{A}_j$ and $\vec{A}_k\vec{A}_l$ are parallel.

Definition 1.19. We say that a 6-tuple $\vec{A}$ is non-parallel if there do not exist pairwise distinct $i, j, k, l \in \{1, \ldots, 6\}$ such that the lines $\vec{A}_i\vec{A}_j$ and $\vec{A}_k\vec{A}_l$ are parallel. We say that a Möbius curve $D = f_A(C)$ is non-parallel if $\vec{A}$ is so.

Lemma 1.20. Let $D$ be a degree 6 Möbius curve contained in a hyperplane $H$, and suppose that $D$ is non-parallel. Then all 15 lines $\{L_{ij}\}$ are distinct.

Proof. Suppose $L_{ij} = L_{kl}$. First of all, notice that this can happen only if all $i, j, k, l$ are distinct, since if $|i, j, k, l| = 3$ with $i \neq j$ and $k \neq l$, then $T_{ij}$ and $T_{kl}$ meet only in a point. Then

$$L_{ij} = L_{kl} = T_{ij} \cap T_{kl}.$$ 

Since $D = f_A(C)$ meets every $T_{ij}$ in two (not necessarily distinct) points — given by the projections long the directions $\vec{A}_i\vec{A}_j$ and $\vec{A}_k\vec{A}_l$ — then $D$ would meet $L_{ij}$ and $L_{kl}$, but this contradicts the assumption that $D$ is non-parallel.

Lemma 1.21. Let $D$ be a degree 6 Möbius curve contained in a hyperplane $H$, and suppose that $D$ is non-parallel. Then

a. if $|\{i, j, k, l\}| = 4$, then $L_{ij} \cap L_{kl} = \{\text{point}\}$;
b. if $|\{i,j,k,l\}| = 3$ with $i \neq j$ and $k \neq l$, then $L_{ij} \cap L_{kl} = \{\text{point}\}$ if and only if the hyperplane $H$ passes through the node of $M_6$ corresponding to configurations $(m_1, \ldots, m_6)$ for which $m_i = m_j = m_k = m_l$; otherwise $L_{ij} \cap L_{kl} = \emptyset$.

Proof. The statement follows from the definition of the lines $L_{ij}$, from the fact that they are all distinct (Lemma 1.20) and from the fact that in Case a. the planes $T_{ij}$ and $T_{kl}$ intersect in a line, while in Case b. the planes $T_{ij}$ and $T_{kl}$ intersect in a point, a node of $M_6$. □

Remark 1.22. Let $D$ be a degree 6 Möbius curve contained in a hyperplane $H$, and suppose that $D$ is non-parallel. Then $H$ can pass through at most one node of $M_6$. In fact, if $H$ passes through two nodes, then $H$ contains a line of the form $T_{ij} \cap T_{kl}$; this implies that $D$ intersects $T_{ij} \cap T_{kl}$, which contradicts the hypothesis that $D$ is non-parallel.

We are now going through all the possible cases for an irreducible cubic surface $S = H \cap M_6$ obtained by intersecting $M_6$ with a hyperplane containing a degree 6 non-parallel Möbius curve $D$.

First we analyze the case when $S$ is a singular cubic with non-isolated singularities (see for example [BW79, Case E]): here $S$ is either a cone or a projection of a cubic scroll in $\mathbb{P}^4_C$. In the first case, all lines of $S$ meet, but this contradicts Lemma 1.21 which asserts that some lines in $S$ do not meet. In the second case there exists a pencil of pairwise disjoint lines, each of which intersects two other special lines on the surface; this case is again ruled out by Lemma 1.21 which implies the existence of three lines intersecting pairwise (for example, $L_{12}$, $L_{34}$ and $L_{56}$). Hence none of these cases can appear in our context.

We are left with the case when $S$ is smooth, or has isolated singularities. We start with the smooth situation.

Lemma 1.23. Let $D_1$, $D_2$ be two distinct degree 6 non-parallel Möbius curves. Suppose that both $D_1$ and $D_2$ are contained in a hyperplane $H$ and that $S = H \cap M_6$ is a smooth cubic surface. Then $|D_1 \cap D_2| \leq 14$.

Proof. Let $D$ be a Möbius curve as in the hypothesis. Since $S$ is smooth, we can express it as the blowup of $\mathbb{P}^2_C$ at 6 points $q_1, \ldots, q_6$ in general position such that its Picard group is generated by $(L, E_1, \ldots, E_6)$ — where each $E_i$ is the class of the exceptional divisor over $q_i$, and $L$ is the class of the strict transform of a line in $\mathbb{P}^2_C$, so we have the relations

$$L^2 = 1, \quad E_i^2 = -1, \quad E_i \cdot L = 0, \quad E_i \cdot E_j = 0 \text{ if } i \neq j.$$

Moreover, denoting by $[\cdot]$ the class in $\text{Pic}(S)$ of a divisor, we can put ourselves in the situation where

$$[L_{ij}] = L - E_i - E_j.$$

There exist integers $k$ and $e_1, \ldots, e_6$ such that

$$[D] = kL - (e_1 E_1 + \cdots + e_6 E_6).$$
Since \( D \) intersects each \( L_{ij} \) in 2 points,
\[
k - e_i - e_j = 2 \quad \forall i, j.
\]
From this we deduce that
\[
e_i = m \quad \forall i, \quad k = 2m + 2 \quad \text{for some integer } m.
\]
Since \( D \) is effective, then \([D] \cdot E_i \geq 0\) for all \( i \), and \([D] \cdot (2L - E_1 - \cdots - E_5) \geq 0\), so \( 0 \leq m \leq 4 \). We are going to exclude the cases \( m = 0 \) and \( m = 4 \). If \( m = 0 \), then \([D] = 2L \). Recall from Lemma 1.10 that \( D \) is contained in a projection of a Veronese surface, which is the complete intersection of \( M_6 \) with another cubic hypersurface \( U \). Then \( U \cap H \) is a cubic surface in \( \mathbb{P}^3_C \) containing \( D \), therefore \([-3K_S - [D]] \) is effective — where \( K_S \) is the canonical divisor on \( S \); recall in fact that \( S \) is anticanonically embedded in \( \mathbb{P}^3_C \), so \([U] = -3K_S \) in \( \text{Pic}(S) \). But
\[
-3K_S - [D] = 3(3L - E_1 - \cdots - E_5) - 2L
= 7L - 3(E_1 + \cdots + E_5).
\]
On the other hand
\[
(7L - 3(E_1 + \cdots + E_5))(2L - E_1 - \cdots - E_5) < 0,
\]
and this is absurd, since both divisors are effective. Similarly for \( m = 4 \). Hence \( m \in \{1, 2, 3\} \).

We are ready to prove the statement. By what we said so far, for \( i \in \{1, 2\} \) we have \([D_i] = (2m_i + 2)L - m_i(E_1 + \cdots + E_6)\) for some \( m_i \in \{1, 2, 3\} \). One computes
\[
[D_1] \cdot [D_2] = -2(m_1 - 2)(m_2 - 2) + 12.
\]
Then it follows that \([D_1] \cdot [D_2] \leq 14\), so the statement is proved. \( \square \)

We are left with the situation when the cubic surface \( S = H \cap M_6 \) has isolated singularities. Taking into account that at least 15 different lines lie on \( S \) (see Lemma 1.20), by the classification of cubic surfaces (see for example [BW79]) the only possibilities are:

i. a cone over a cubic plane curve (infinitely many lines);
ii. one singularity of type \( A_1 \) (21 lines);
iii. two singularities of type \( A_1 \) (16 lines);
iv. one singularity of type \( A_2 \) (15 lines).

We consider these cases one by one.

**Lemma 1.24.** Let \( D_1, D_2 \) be two distinct degree 6 non-parallel Möbius curves. Then it cannot happen that both \( D_1 \) and \( D_2 \) are contained in a hyperplane \( H \) and that \( S = H \cap M_6 \) is a cone over a cubic plane curve.

**Proof.** This case is ruled out as before by the existence of non-intersecting lines on \( S \) (see Lemma 1.21). \( \square \)

**Lemma 1.25.** Let \( D_1, D_2 \) be two distinct degree 6 non-parallel Möbius curves. Then it cannot happen that both \( D_1 \) and \( D_2 \) are contained in a hyperplane \( H \) and that \( S = H \cap M_6 \) is a singular cubic surface with two singularities of type \( A_1 \).
Fact such a surface can be realized as the blowup of $\mathbb{P}^4_C$ at 6 points $q_1, \ldots, q_6$ — where both $q_1, q_2, q_3$ and $q_4, q_5, q_6$ are collinear, but involving different lines, and $q_6$ is in general position with respect to the other points — followed by blowing down the (strict transforms of the) lines $\overline{p_1p_2p_3}$ and $\overline{p_4p_5p_6}$, which get contracted to the two singularities. In this case, using the same notation as in Lemma 1.23, the classes of the (strict transforms of the) 16 lines of $S$ in the blowup of $\mathbb{P}^2_C$ at $q_1, \ldots, q_6$ are

- $E_1, \ldots, E_6$, \hspace{1cm} (6 lines)
- $L - E_1 - E_6, \ldots, L - E_5 - E_6$, \hspace{1cm} (5 lines)
- $L - E_2 - E_4, L - E_2 - E_5, L - E_3 - E_4, L - E_3 - E_5$, \hspace{1cm} (4 lines)
- $2L - E_2 - E_4 - E_5 - E_6$, \hspace{1cm} (1 line)

In this way the number of lines that pass through a singular point is given by counting how many of the previous ones intersect one of the $(-2)$-curves

$$L - E_1 - E_2 - E_3 \text{ or } L - E_1 - E_4 - E_5.$$  

However, in our case from Lemma 1.21 we deduce that either we have 6 lines passing through a singularity (and this happens when the hyperplane $H$ passes though one of the nodes of $M_6$) or there are at most 2 lines passing through a point. Hence $S$ cannot have two $A_1$ singularities. $\Box$

**Lemma 1.26.** Let $D_1$, $D_2$ be two distinct degree 6 non-parallel Möbius curves. Suppose that both $D_1$ and $D_2$ are contained in a hyperplane $H$ and that $S = H \cap M_6$ is a singular cubic surface with one singularity of type $A_1$. Then $|D_1 \cap D_2| \leq 14$.

**Proof.** In this case we can think of $S$ as obtained in the following way: we blow up $\mathbb{P}^2_C$ at 6 points $q_1, \ldots, q_6$ lying on a conic, and then we blow down the $(-2)$-curve $2L - (E_1 + \cdots + E_6)$, which gets contracted to the singular point. The classes of the (strict transforms of the) lines on $S$ are

- $E_1, \ldots, E_6$, \hspace{1cm} (6 lines)
- $L - E_i - E_j$, \hspace{1cm} (15 lines)

The latter 15 lines are the classes of the lines $L_{ij}$. The computation for $[D_1] \cdot [D_2]$ goes exactly as in the smooth case because none of the lines $L_{ij}$ passes through the singular point, as one can check by computing the intersection product of $L - E_i - E_j$ with the $(-2)$-curve $2L - (E_1 + \cdots + E_6)$. From this we conclude that $[D_k] \cdot [L_{ij}] = 2$ for all $i, j$ and for $k \in \{1, 2\}$, and so we can proceed as in the smooth case (see Lemma 1.23). However, we cannot directly infer from $[D_1] \cdot [D_2]$ the number of intersections of $D_1$ and $D_2$. In fact, if $D_1$ and $D_2$ intersect in the singular point of $S$, such intersection is counted as a contribution by $1/2$ — and not 1, as usual — in $[D_1] \cdot [D_2]$. Luckily in this situation we can exclude that $m_1$ or $m_2$ equals 3. In fact, for such a value we obtain the class $8L - 3(E_1 + \cdots + E_6)$, which should

---

*LIAISON LINKAGES 19*
that in the blowup of $D$ or not the curves may not intersect the (strict transforms of) the lines the computation of $D$. In this case we obtain $\frac{8L - 3 \sum E_i}{2L - \sum E_i} = -2 < 0$, and this is absurd. Hence $m_i \in \{1, 2\}$. We analyze the possible cases.

a. Either $m_1 = 2$ or $m_2 = 2$, then $[D_1] \cdot [D_2] = 12$. By computing the intersection product with the $(-2)$-curve $2L - (E_1 + \cdots + E_6)$ we see that in this case either $D_1$ or $D_2$ does not pass through the singularity, so we can conclude that $|D_1 \cap D_2| \leq 12$.

b. Or $m_1 = m_2 = 1$, then $[D_1] \cdot [D_2] = 10$. In this case both $D_1$ and $D_2$ pass through the singular point, and moreover both have a node at that point. From this we conclude that $|D_1 \cap D_2| \leq 14$. □

Lemma 1.27. Let $D_1$, $D_2$ be two distinct degree 6 non-parallel Möbius curves. Suppose that both $D_1$ and $D_2$ are contained in a hyperplane $H$ and that $S = H \cap M_6$ is a singular cubic surface with one singularity of type $A_2$. Then $|D_1 \cap D_2| \leq 12$.

Proof. In this case we obtain $S$ by blowing up $\mathbb{P}^5_2$ at 6 points $q_1, \ldots, q_6$ such that $q_1, q_2, q_3$ are collinear and $q_4, q_5, q_6$ are collinear on another line, and then blowing down the (strict transforms of the) lines $q_1q_2q_3$ and $q_4q_5q_6$ — the latter get contracted to the unique $A_2$ singularity of $S$. In this case the classes of the (strict transforms of the) 15 lines of $S$, which coincide with the lines $L_{ij}$, are

\[
E_1, \ldots, E_6, \quad (6 \text{ lines})
\]

\[
L - E_1 - E_4, \ldots, L - E_1 - E_6, \quad (3 \text{ lines})
\]

\[
L - E_2 - E_4, \ldots, L - E_2 - E_6, \quad (3 \text{ lines})
\]

\[
L - E_3 - E_4, \ldots, L - E_3 - E_6. \quad (3 \text{ lines})
\]

Notice that the only lines $L_{ij}$ passing through the singular point are the ones whose class is an exceptional divisor $E_i$, as the computation of the intersection product with the two $(-2)$-classes $L - E_1 - E_2 - E_3$ and $L - E_4 - E_5 - E_6$ confirms. Here the computation of $[D_1] \cdot [D_2]$ does not work as in the smooth case, due to the fact that in the blowup of $\mathbb{P}^5_2$ at the points $\{q_i\}$ the (strict transforms of) the curves $D_i$ may not intersect the (strict transforms of) the lines $L_{ij}$ — this depends whether or not the curves $D_i$ pass through the singular points.

Let $D$ be a Möbius curve with the properties of $D_1$ and $D_2$. Let us suppose that $D$ does not pass through the singular point of $S$. Then we know that in the blowup of $\mathbb{P}^5_2$ at the points $\{q_i\}$ we have $[D] \cdot [L_{ij}] = 2$ for all $i \neq j$. If we write

\[
[D] = kL - (e_1E_1 + \cdots + e_6E_6),
\]

then the previous conditions translate into

\[
[D] \cdot E_i = 2 \quad \text{for all } i \in \{1, \ldots, 6\},
\]

\[
[D] \cdot (L - E_i - E_j) = 2 \quad \text{for all } i \in \{1, 2, 3\}, j \in \{4, 5, 6\}.
\]

This forces $k = 6$ and $e_i = 2$ for all $i$, so $[D] = 6L - 2(E_1 + \cdots + E_6)$. Suppose now that $D$ passes through the singular point. Notice that the fact that both $D$
and each \( L_{ij} \) are real implies that their intersection is real; moreover, the fact that \( D = f_{\hat{A}}(C) \) where \( f_{\hat{A}} \) is a real map and \( C \) is a real variety without real points implies that it cannot happen that \( D \) intersects an \( L_{ij} \) transversely at the singular point and then in another different point, or tangentially at the singular point. The only possibility is that \( D \) has an ordinary node at the singular point. As we pointed out before, the \( L_{ij} \) passing through the singular point are the ones whose class is \( E_i \) — this is compatible with the situation when the hyperplane \( H \) passes through a node of \( M_6 \), since as pointed out in Case iii. here we have 6 lines passing through the node. Therefore

\[
[D] \cdot E_i = 0 \quad \text{for all } i \in \{1, \ldots, 6\}.
\]

Moreover in this case the strict transform of \( D \) meets the two \((-2)\)-curves in two points, so

\[
[D] \cdot \left( (L - E_1 - E_2 - E_3) + (L - E_4 - E_5 - E_6) \right) = 2.
\]

The result is that \([D] = L\). Summing up, we have the following scenarios:

a. Both \( D_1 \) and \( D_2 \) do not pass through the singular point. Then \([D_1] \cdot [D_2] = 12\), so \([D_1 \cap D_2] \leq 12\).

b. Only \( D_1 \) (or \( D_2 \)) passes through the singular point. Then \([D_1] \cdot [D_2] = 6\) and so \([D_1 \cap D_2] \leq 6\).

c. Both \( D_1 \) and \( D_2 \) pass through the singular point, and have a node there.

Then \([D_1] \cdot [D_2] = 1\), and \([D_1 \cap D_2] \leq 1 + 4 = 5\). \(\square\)

The discussion so far proves that the bound on the intersection of two non-parallel degree 6 Möbius curves holds if one of them is contained in a hyperplane of \( \mathbb{P}_C^4 \). Hence from now on we can suppose that both curves \( D_1 \) and \( D_2 \) do not lie on any hyperplane. Riemann-Roch predicts that there are at least 2 quadrics in the ideal of a smooth rational curve of degree 6 in \( \mathbb{P}_C^4 \) (for more details, see the proof of Lemma 2.3); this is true also in the singular case, because in that situation we have an injective homomorphism \( H^0(D, \mathcal{O}_{D}(2)) \rightarrow H^0(P^1_C, \mathcal{O}_{P^1_C}(12)) \). Thus each \( D_i \) is contained in a pencil of quadrics.

**Lemma 1.28.** Let \( D \) be a degree 6 non-parallel Möbius curve not contained in any hyperplane. Let \( Q \) be a pencil of quadrics containing \( D \). Then the base locus of \( Q \) is a quartic surface.

**Proof.** If the base locus of \( Q \) were of dimension 3, then it would be a component of each of the quadrics in \( Q \), hence all of them would split into two hyperplanes. But by assumption \( D \) is not contained in any hyperplane, so the base locus has dimension 2. Then the latter is a complete intersection, and from Bezout’s theorem it has degree 4. \(\square\)

In Lemma 1.29 and Lemma 1.30 we discuss the situation when the base locus \( S \) of a pencil of quadrics passing through a Möbius curve is irreducible; in particular we prove that the bound holds when \( S \) is not contained in \( M_6 \), while the case \( S \subseteq M_6 \) cannot occur.
Lemma 1.29. Let $D_1$ and $D_2$ be two distinct degree 6 non-parallel Möbius curves not contained in any hyperplane. Suppose that there is a pencil $Q$ of quadrics containing $D_1$ whose base locus $S$ is irreducible and not contained in $M_6$. Then $|D_1 \cap D_2| \leq 14$.

Proof. Here $D_1$ is a component of the curve $Z = S \cap M_6$, which is a complete intersection of degree 12 by Lemma 1.28 and Bezout’s theorem. If $D_2$ coincides with the other component of $Z$, then the second part of Lemma 2.6 proves that $|D_1 \cap D_2| = 14$. If this is not the case, then there exists at least a quadric $Q$ passing through $D_1$, but not passing through $D_2$. Hence

$$|D_1 \cap D_2| \leq |Q \cap D_2| \leq 12.$$  

□

Lemma 1.30. Let $D$ be a degree 6 non-parallel Möbius curve not contained in any hyperplane. Let $Q$ be a pencil of quadrics containing $D$ and suppose that its base locus $S$ is contained in $M_6$. Then $S$ is reducible.

Proof. Suppose instead that $S$ is irreducible. Pick any quadric $Q$ in the pencil $Q$; then the intersection $Q \cap M_6$ is the union of $S$ and a surface $S'$ of degree 2. We claim that it is always possible to choose $Q$ such that $S'$ splits in the union of two planes. In fact, each $S'$ spans a hyperplane $H$, so that $H \cap M_6 = S' \cup E$, where $E$ is a plane. Since the set of planes in $M_6$ is discrete, by continuity we obtain that the plane $E$ is always the same, regardless of which $Q \in Q$ we start with. Therefore the one-dimensional family of surfaces $S'$ is obtained by cutting $M_6$ with the pencil of hyperplanes through the plane $E$. A direct computation — for example taking the plane $x_0 = x_1 = 0$, where the $x_i$ are coordinates in $\mathbb{P}_c^4$ — shows that in such one-dimensional family there are always reducible members. Hence we can select $Q \in Q$ such that, after a possible rearrangement of the indices,

$$Q \cap M_6 = S \cup T_{12} \cup T_{34}.$$  

We intersect both sides of the previous equality with the plane $T_{56}$: on the left we obtain either a conic (if $T_{56}$ is not contained in $Q$) or the plane $T_{56}$ itself (if $T_{56} \subseteq Q$), while on the right we get the union of $S \cap T_{56}$, $T_{12} \cap T_{56}$ (a line) and $T_{34} \cap T_{56}$ (another line). Since $S$ is irreducible by assumption, then $T_{56}$ cannot be contained in $S$, so $S \cap T_{56}$ can be either a curve, or a finite set of points. This forces the left hand side $Q \cap T_{56}$ to be a conic. In turn, this implies that $S \cap T_{56}$ has to be contained in the union of the two lines $T_{12} \cap T_{56}$ and $T_{34} \cap T_{56}$. On the other hand, since $D \subseteq S$ and $D$ intersects $T_{56}$ in two points, then $D$ should intersect one of the two lines $T_{12} \cap T_{56}$ and $T_{34} \cap T_{56}$, but this contradicts the assumption that $D$ is non-parallel. Hence $S$ cannot be irreducible. □

Hence we are left with the case when the base locus $S$ is reducible. We notice that it cannot happen that $S$ is contained in $M_6$ and at the same time splits into the union of two surfaces of degree 2, because by Lemma 1.16 no Möbius curve of degree 6 can lie on a degree 2 surface contained in $M_6$. The only remaining cases are when $S = S' \cup S''$ with both $S'$ and $S''$ surfaces of degree 2, but $S \not\subseteq M_6$, or $S = E \cup S'$ where $E$ is a plane and $S'$ is a cubic surface.
Lemma 1.31. Let $D_1$ and $D_2$ be two distinct degree 6 non-parallel Möbius curves not contained in any hyperplane. Suppose that there is a pencil $Q$ of quadrics containing $D_1$ whose base locus $S$ is not contained in $M_6$ and splits into the union $S = S' \cup S''$ of two surfaces of degree 2. Then $|D_1 \cap D_2| \leq 14$.

Proof. Since $D_1$ is irreducible, then $D_1 \subseteq S'$ or $D_1 \subseteq S''$. From now on we will suppose $D_1 \subseteq S'$. Since $D_1$ cannot lie in a degree 2 surface contained in $M_6$, then $S'$ is not contained in $M_6$ and hence by degree reasons $D_1 = S' \cap M_6$. Suppose that there exists a quadric $Q \in Q$ not passing through $D_2$; then

$$|D_1 \cap D_2| \leq |Q \cap D_2| \leq 12.$$  

Otherwise $D_2 \subseteq S$. It cannot happen that $D_2 \subseteq S'$, because otherwise we would have $D_1 = D_2$, contradicting the hypothesis. Thus $D_2 \subseteq S''$, and so $D_2 = S'' \cap M_6$. Therefore $D_1$ and $D_2$ are the two components of the degree 12 complete intersection $Z = S \cap M_6$, and then the second part of Lemma 2.6 concludes the proof. 

Lemma 1.32. Let $D_1$ and $D_2$ be two distinct degree 6 non-parallel Möbius curves not contained in any hyperplane. Suppose that there is a pencil $Q$ of quadrics containing $D$, whose base locus $S$ splits into the union of a plane $E$ and a cubic surface $S'$. Then $|D_1 \cap D_2| \leq 14$.

Proof. By Lemma 1.14 the curve $D_1$ cannot be contained in the plane $E$, so $D \subseteq S'$. Suppose that $S'$ is not contained in $M_6$. There are several possibilities.

a. The intersection $Z = S' \cap M_6$ is a curve of degree 9. Then $D_1$ is a component of such curve. This implies that there is a quadric $Q \in Q$ not passing through $D_2$, because otherwise we would have $D_1 = D_2$. Hence

$$|D_1 \cap D_2| \leq |Q \cap D_2| \leq 12.$$  

b. The cubic $S'$ splits into a plane $E'$ and a surface $S''$ of degree 2, and $S''$ is contained in $M_6$. This case cannot happen, since $D_1$ neither can lie on a plane, nor on a degree 2 surface contained in $M_6$.

c. The cubic $S'$ splits into a plane $E'$ and a surface $S''$ of degree 2, and $S''$ is not contained in $M_6$. Then $D_1 = S'' \cap M_6$. This implies that there exists a quadric $Q \in Q$ not passing through $D_2$, because otherwise we would have $D_1 = D_2$. Hence

$$|D_1 \cap D_2| \leq |Q \cap D_2| \leq 12.$$  

The last case that needs to be treated is the one where $S'$ is contained in $M_6$. Then $S'$ is irreducible, because $D'$ cannot lie on planes or surfaces of degree 2 contained in $M_6$ (see Lemma 1.14 and 1.16). Hence $S'$ is a cubic scroll — maybe singular, namely a cone over a rational cubic plane curve. Thus $S'$ admits a determinantal representation as the zero set of the $2 \times 2$ minors of a $2 \times 3$ matrix of linear forms.

We consider the intersection of $S'$ with the planes $T_{ij}$: first of all, each $T_{ij}$ is not contained in $S'$, because otherwise $S'$ would be reducible. By restricting the determinantal representation of $S'$ to $T_{ij}$ we see that $L_{ij} = S' \cap T_{ij}$ is defined by three quadratic equations in $T_{ij}$, so it is either a finite set of points, or a line,
or a conic. On the other hand if \( \{i,j,k,l,m,n\} = \{1,\ldots,6\} \) then there exists a hyperplane \( H \) in \( \mathbb{P}_C^4 \) such that \( H \cap M_6 = T_{ij} \cup T_{kl} \cup T_{mn} \). Therefore

\[
H \cap S' = (S' \cap T_{ij}) \cup (S' \cap T_{kl}) \cup (S' \cap T_{mn}) = L_{ij} \cup L_{kl} \cup L_{mn}.
\]

Since \( S' \) is not contained in \( H \) (because \( D_1 \) does not lie in any hyperplane) we have that \( S' \cap H \) is a cubic curve. Suppose that one among \( L_{ij} \), \( L_{kl} \) and \( L_{mn} \), say \( L_{ij} \), is a finite number of points. Then, by eventually rearranging the indices, \( L_{kl} \) is a line and \( L_{mn} \) is a conic. This implies that \( L_{ij} \subseteq L_{kl} \cup L_{mn} \), but this contradicts the hypothesis on \( D_1 \) of being non-parallel. Therefore we conclude that all \( L_{ij} \) are lines. Moreover all lines \( L_{ij} \) are distinct, because otherwise this would violate the non-parallel assumption. We consider now the mutual position of the lines \( L_{ij} \).

First of all we notice that — analogously as in Lemma 1.21 — if \( |\{i,j,k,l\}| = 3 \) and \( L_{ij} \) meets \( L_{kl} \), then they intersect at the node of \( M_6 \) given by the class of \( 6 \)-tuples \( (m_1,\ldots,m_6) \) for which \( m_i = m_j = m_k = m_l \). This rules out the case where \( S' \) is a cone, because in that case all lines in \( S' \) meet in a single point, but on the other hand the points \( L_{12} \cap L_{23} \) and \( L_{12} \cap L_{24} \) are different, an absurd. So \( S' \) is smooth, then \( S' \) admits a pencil of mutually disjoint lines, all of which intersect one line \( \ell \).

We distinguish two cases:

a. Suppose that \( \ell \) does not appear among the lines \( L_{ij} \). Then in particular \( L_{12}, L_{34} \) and \( L_{56} \) are disjoint. On the other hand, as mentioned before, \( L_{12} \cup L_{34} \cup L_{56} = H \cap M_6 \) for some hyperplane \( H \); from [Har77, Chapter III, Corollary 7.9] we know that a hyperplane section of \( S' \) is connected, so this is absurd.

b. Suppose that \( \ell \) appears among the lines \( L_{ij} \). After a possible rearrangement of the indices, we can suppose that \( \ell = L_{12} \). Then \( L_{23}, L_{45} \) and \( L_{16} \) are disjoint, but we can repeat the argument of Case a. and see that this leads to an absurd.

This concludes the proof of the statement. \( \square \)

We sum up the previous discussion in the following proposition.

**Proposition 1.33.** Let \( D_1 \) and \( D_2 \) be two distinct degree 6 non-parallel Möbius curves. Then \( |D_1 \cap D_2| \leq 14 \).

Now we can prove the main result of this section.

**Theorem 1.34.** The conformal degree of a non-planar and non-equiform hexapod \( \Pi \) such that both base and platform are non-parallel is at most 28.

**Proof.** Since \( \Pi \) is non-planar, then the Möbius curves \( D_1 \) and \( D_2 \) of base and platform points are birational by Lemma 1.8 by the same result, using the non-parallel hypothesis we infer that that no 4 points are on a line, and so we conclude that both \( D_1 \) and \( D_2 \) have degree 6. Since \( \Pi \) is non-equiform, then \( D_1 \) and \( D_2 \) are distinct. Theorem 1.12 states that the conformal degree of \( \Pi \) is bounded by \( 2|D_1 \cap D_2| \), and Proposition 1.33 asserts that \( |D_1 \cap D_2| \leq 14 \), so the statement is proved. \( \square \)
2. Construction of liaison hexapods

The goal of this section is to provide a construction for a family of movable hexapods, that we will call liaison hexapods. This will be accomplished in two stages: first for each general choice a base we design a candidate for the platform (Subsection 2.1), then we compute a dilation of the latter and leg lengths that guarantee mobility for the corresponding hexapod (Subsection 2.2). We believe that the family created in this way is maximal among movable hexapods, namely, if we consider the set of movable hexapods as an algebraic variety, it is not contained in any irreducible component of strictly larger dimension.

2.1. The candidate platform. We begin with the definition of the candidate platform, once we are given a base constituted of 6 general points.

The idea for the construction is the following: in Subsection 1.2 we associated to any 6-tuple of distinct points in \( \mathbb{R}^3 \) its Möbius curve in the moduli space \( M_6 \). At this point, we know from bond theory and from the discussion in Section 1 that the more the Möbius curves associated to base and platform of a hexapod intersect, the more the hexapod has the chance to be movable and the higher will be its conformal degree. Inspired by this, we start from the curve \( D \) associated to a given general 6-tuple \( \vec{P} \) of points in \( \mathbb{R}^3 \), and we construct from it, applying liaison techniques, another curve \( D' \), for which we give evidence to be the curve associated to another tuple \( \vec{p} \) of 6 points in \( \mathbb{R}^3 \) (unfortunately we are not able to exhibit a complete proof of this), and that intersects \( D \) in 14 points.

At this stage, the tuple \( \vec{p} \) is only determined up to similarities, namely rotations, dilations and translations, and the right scaling factor will be fixed later in Subsection 2.2.

We briefly glance at the concept of liaison via an example. Let \( D \subseteq \mathbb{P}^3_\mathbb{C} \) be a projective curve. We know that \( I(D) \), the homogeneous ideal of \( D \), cannot be generated by less than 2 polynomials. Hence we can always pick two homogeneous polynomials \( f, g \in I(D) \) such that the corresponding surfaces \( F = V(f) \) and \( G = V(g) \) intersect in a one-dimensional projective set. Since we took \( f \) and \( g \) in the ideal of \( D \), we have that \( D \subseteq F \cap G \). The inclusion may be strict (unless \( D \) is a so-called complete intersection), and in that case the intersection \( F \cap G \) is the union of \( D \) and another curve \( D' \). We say that the curves \( D \) and \( D' \) are linked by \( Y = F \cap G \). This procedure can be applied not only to curves in \( \mathbb{P}^3_\mathbb{C} \), but also to curves (and higher dimensional varieties) in any projective space, as we are going to do for the case of \( \mathbb{P}^4_\mathbb{C} \). Linked curves share many properties, and in particular we are interested in a relation between their degrees and genera, expressed by the following result (see [Mig98, Chapter 3, Corollary 5.2.14]):

**Proposition 2.1.** Let \( D \) and \( D' \) be two projective curves in \( \mathbb{P}^4_\mathbb{C} \) linked by a complete intersection \( Y = F_1 \cap F_2 \cap F_3 \) and let \( p_a(D) \) and \( p_a(D') \) be their arithmetic genera. Then

\[
p_a(D) - p_a(D') = \frac{1}{2}(d - 5)(\deg D - \deg D'),
\]

where \( d = \deg F_1 + \deg F_2 + \deg F_3 \).
Proposition 2.1 implies in particular that if the curves $D$ and $D'$ have the same degree, then they have the same genus. This property will be used in the proof of Lemma 2.6.

As it will be clear from several proofs in this section, the construction we are going to propose works only if the base points to start with are sufficiently general. We would like to make this condition precise, in order to provide later a conjecture that is easily falsifiable.

**Definition 2.2.** We say that a 6-tuple of points in $\mathbb{R}^3$ is *Möbius-general* if its Möbius curve is smooth, the ideal of the Möbius curve contains only two linearly independent quadratic forms, and these two quadratic forms cut out a one-dimensional set from $M_6$ consisting of two smooth curves.

The following lemma, together with Remark 2.5, shows that Möbius-general 6-tuples form an open subset of the variety of 6-tuple of points in $\mathbb{R}^3$.

**Lemma 2.3.** Let $\vec{A}$ be a general 6-tuple of points in $\mathbb{R}^3$. Then the Möbius curve of $\vec{A}$ is a smooth curve of degree 6 contained in the complete intersection of $M_6$ and two quadric hypersurfaces.

**Proof.** Let $D$ be the Möbius curve of $\vec{A}$. Since $\vec{A}$ is general, we can suppose that it is non-planar, so from Lemma 1.8 the degree of $D$ is 6 (because we have degree 4 only for a special choice of $\vec{A}$). We prove the smoothness of $D$ with the following argument. What we showed so far is that for a general $\vec{A}$ the curve $D$ is rational and of degree 6; therefore, it can be thought as a point $[D]$ in the Hilbert scheme $\text{Hilb}(\mathbb{P}_C^4, 6t + 1)$ of subschemes of $\mathbb{P}_C^4$ with Hilbert polynomial $6t + 1$. Hence we have a map $\xi: V \to \text{Hilb}(\mathbb{P}_C^4, 6t + 1)$, where $V$ is a suitable Zariski-open subset of $(\mathbb{R}^3)^6$: the map $\xi$ associates to a 6-tuple $\vec{A}$ the image $[D]$ of its photographic map $f_{\vec{A}}$. Since smooth rational sextics form an open subset of $\text{Hilb}(\mathbb{P}_C^4, 6t + 1)$, if we are able to show that for a particular 6-tuple $\vec{A}$ the curve $D$ is smooth, then for all $\vec{A}$ belonging to a (possibly smaller) open set $W \subseteq V$ the Möbius curve of $\vec{A}$ is smooth. One can check that if we take $\vec{A}$ with

(6) \hspace{1cm} A_1 = (0, 0, 0), \quad A_2 = (2, 0, 0), \quad A_3 = (3, 2, 0),

(7) \hspace{1cm} A_4 = (2, 3, 1), \quad A_5 = (1, 2, 0), \quad A_6 = (3, 1, 3)

then the obtained curve $D$ is a smooth sextic.

Since $D$ is smooth and rational, Riemann-Roch implies that $h^0(D, O_D(2)) = 13$, i.e. there is at most a 13-dimensional family of quadrics cutting $D$ in finitely many point. There is a 15-dimensional family of quadrics in $\mathbb{P}_C^4$, so there are at least two linearly independent quadrics passing through $D$. Since in the example we just gave we have exactly two quadrics, then this holds for a general $\vec{A}$, and the same is true for the fact that the two quadrics form a complete intersection with $M_6$. □

**Definition 2.4.** Let $\vec{A}$ be a Möbius-general 6-tuple of points in $\mathbb{R}^3$. Let $D$ be the Möbius curve of $\vec{A}$ and let $Y$ be the complete intersection of degree 12 whose existence is ensured by Lemma 2.3. The curve $D'$ such that $D \cup D' = Y$ is called the *residual curve* of $D$. 

Remark 2.5. The condition that also the residual curve $D'$ is smooth is an open condition, and this can be proved as in Lemma 2.3, namely showing that this is true in one example, because smoothness is an open property; one can check that the same 6-tuple $\vec{A}$ that we chose in Lemma 2.3 yields a curve whose residual one is smooth. This concludes the proof, initiated in Lemma 2.3, that a general 6-tuple $\vec{A}$ is Möbius-general.

Lemma 2.6. Let $\vec{A}$ be a Möbius-general 6-tuple of points in $\mathbb{R}^3$. Let $D$ be the Möbius curve of $\vec{A}$ and let $D'$ be its residual curve. Then $D'$ is rational and of degree 6. Moreover $D$ and $D'$ intersect in 14 points.

Proof. Since $\vec{A}$ is Möbius-general, then the curve $D'$ has degree 6. From Proposition 2.1 we obtain $p_a(D) = p_a(D') = 0$. From [Mig98, Remark 5.2.7] we have the following exact sequence:

$$0 \rightarrow \omega_D(-2) \rightarrow \mathcal{O}_{D \cup D'} \rightarrow \mathcal{O}_{D'} \rightarrow 0$$

where $\omega_D$ denotes the canonical sheaf of $D$. Taking the associated long exact sequence in cohomology and using the fact that $p_a(D') = 0$, one can prove that $h^0(D', \mathcal{O}_{D'}) = 1$, namely $D'$ is connected. Since $D'$ is smooth by the assumption of Möbius-generality, it is irreducible, and so rational.

We are left to prove that $D$ and $D'$ intersect in 14 points. Since $D \cup D'$ is a complete intersection whose ideal is generated by two quadrics and a cubic, the ideal sheaf $\mathcal{I}_{D \cup D'}$ admits a graded free resolution given by the Koszul complex:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-7) \rightarrow \mathcal{O}_{\mathbb{P}^4}(-4) \oplus \mathcal{O}_{\mathbb{P}^4}(-5)^2 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-2)^2 \oplus \mathcal{O}_{\mathbb{P}^4}(-3) \rightarrow \mathcal{I}_{D \cup D'} \rightarrow 0$$

A computation shows that the Euler characteristic $\chi(\mathbb{P}^4, \mathcal{I}_{D \cup D'})$ equals 13, which implies $\chi(\mathbb{P}^4, \mathcal{O}_{D \cap D'}) = 14$ because of the exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4} \rightarrow \mathcal{O}_{D \cap D'} \oplus \mathcal{O}_D \oplus \mathcal{O}_{D'} \rightarrow \mathcal{O}_{D \cup D'} \rightarrow 0$$

This proves that $D \cap D'$ is constituted of 14 points. \qed

Remark 2.7. Suppose $D$ is the Möbius curve of a 6-tuple $\vec{A}$, and $D'$ is its residual curve. Then $D'$ inherits a real structure from $D$. In fact, by construction $D$ is a real curve, so the generators of its ideal can be taken to be all real polynomials; hence the degree 12 complete intersection $Y$ is also a real curve, thus by construction $D'$ is a real curve.

Lemma 2.3, together with Lemma 2.6 and Remark 2.7, shows that if we start from a Möbius-general 6-tuple, then the residual curve $D'$ that we obtain satisfies many of the properties that a Möbius curve needs to have (see also Proposition 1.11). Unfortunately, we are not able to establish theoretically that $D'$ is actually a Möbius curve. On the other hand, we have strong experimental evidence that this holds; this leads us to formulate the following conjecture.

Conjectured Lemma 1. Let $\vec{A}$ be a Möbius-general 6-tuple of points in $\mathbb{R}^3$. Let $D$ be the Möbius curve of $\vec{A}$ and let $D'$ be its residual curve. Then $D'$ is a real variety without real points and is contained in a linear projection, defined by real
polynomials, of the third Veronese embedding of $\mathbb{P}^2_C$. Moreover there exists a 6-tuple $\vec{B}$ in $\mathbb{R}^3$ (unique up to translations, rotations and dilations) such that the Möbius curves of $\vec{A}$ and $\vec{B}$ intersect in 14 points.

**Example 2.8.** If we take $\vec{A}$ as in the proof of Lemma 2.3, then the residual curve $D'$ is a Möbius curve. In fact, $D'$ is rational and real without real points, so we can compute a real isomorphism $f: C \longrightarrow D'$, where $C = \{(x : y : z) : x^2+y^2+z^2 = 0\}$. One notices then that $f$ extends to a real morphism $F: \mathbb{P}^2_C \longrightarrow M_6$. The preimages of the planes $T_{ij}$ are lines in $\mathbb{P}^2_C$, and their normal vectors give the difference vectors $B_i - B_j$, for a 6-tuple $\vec{B}$ of candidate points, where $i, j \in \{1, \ldots, 6\}$ with $i \neq j$. In this way it is not difficult to get possible coordinates for the $B_i$, for instance:

(8) $B_1 = (0, 0, 0)$, $B_2 = \left( \begin{array}{c} 1397624 \\ 806205 \\ -437422 \end{array} \right)$,

(9) $B_3 = \left( \begin{array}{c} 1402244 \\ 101241 \end{array} \right)$, $B_4 = \left( \begin{array}{c} 92216 \\ 161241 \end{array} \right)$,

(10) $B_5 = \left( \begin{array}{c} 1125372 \\ 5582884 \end{array} \right)$, $B_6 = \left( \begin{array}{c} 1719522 \\ 591217 \end{array} \right)$.

Eventually one checks that $f = f_{\vec{B}}$, namely $f$ is a photographic map, so $D'$ is indeed a Möbius curve.

**Remark 2.9.** If the above conjectured lemma is true for a generic choice of base points, then it is true for all Möbius-general 6-tuples of base points, because within the Möbius-general instances, the statement that the linked curve is contained in a linear projection of a Veronese is a closed property.

Despite we do not know whether the conjecture is true or false, we have tested it using random numerical instances of base points, and all the results confirmed it. It could be that we were always lucky (or unlucky) to hit a special case, but we think that this is not very likely, hence our belief in the conjecture.

By applying Conjectured Lemma 1 to a Möbius-general 6-tuple $\vec{P}$ of base points we obtain a candidate platform $\vec{p}$ for the construction of a movable hexapod. From the definition of the photographic map it follows that the candidate platform can be scaled by any non-zero factor without losing any of the properties ensured by Conjectured Lemma 1. In the next subsection we are going to determine the right scaling factor, together with leg lengths, leading to a movable hexapod.

### 2.2. The candidate scaling factor and leg lengths.

Here we determine the scaling factor for the 6-tuple of platform points obtained in Subsection 2.1 and the leg lengths so that the resulting hexapods is movable. This is achieved using the following technique. As introduced in Subsection 1.1 it is possible to assign to every hexapod $\Pi$ a projective subvariety $\mathcal{K}_\Pi$ of a variety $X$ in $\mathbb{P}^16_C$, the latter being a compactification of the algebraic group of direct isometries of $\mathbb{R}^3$. Moreover, if $\mathcal{K}_\Pi$ is a curve we can study its intersection with a particular hyperplane $L$ of $\mathbb{P}^16_C$; such intersection is constituted in general by a finite number of points, called bonds, contained in the intersection $B = L \cap X$, called the boundary. By imposing tangency conditions at the bonds between the curve $\mathcal{K}_\Pi$ and this hyperplane $L$ we derive linear conditions on the scaling factor and on the (squares of the) leg lengths ensuring that the hexapod we obtain is movable.
In Subsection 2.1, given general base points \( \vec{P} \), we constructed a candidate for the platform points \( \vec{P} \), unique up to rotations, dilations and translation. From now on, we fix one such candidate \( \vec{P} \), and we denote by \( \gamma \vec{P} \) the vector of points obtained by scaling \( \vec{P} \) by a non-zero factor \( \gamma \in \mathbb{R} \). The goal is to find a scalar \( \gamma \) and leg lengths \( \vec{d} \) such that the hexapod \( \Pi = (\vec{P}, \gamma \vec{P}, \vec{d}) \) is movable.

By construction, the Möbius curves \( D \) and \( D' \) of \( \vec{P} \) and \( \gamma \vec{P} \) intersect in 14 points, and these points do not depend on \( \gamma \). As pointed out in Remark 1.13, it is not always the case that all intersections correspond to boundary point. However, this is true if we exclude the situation when both Möbius curves pass through one of the nodes of \( M_6 \). Hence from now on we will impose another condition on Möbius-general 6-tuples, namely we suppose that no three points are collinear.

In this way — according to Theorem 1.12 — to each of the intersections of \( \Pi \) according to the concept of spherical conditions determined by \( \Pi \), the hexapod \( \Pi = (\vec{P}, \gamma \vec{P}, \vec{d}) \) has no empty configuration set \( K_{\Pi, \gamma, \vec{d}} \), then \( \beta_k \in K_{\Pi, \gamma, \vec{d}} \cap B \) for all \( k \in \{1, \ldots, 14\} \). By construction, the \( \beta_k \) have the property that if \( \vec{d} \) are leg lengths such that \( \Pi_{\gamma, \vec{d}} = (\vec{P}, \gamma \vec{P}, \vec{d}) \) has no empty configuration set \( K_{\Pi_{\gamma, \vec{d}}} \), then \( \beta_k \in K_{\Pi_{\gamma, \vec{d}}} \cap B \) for all \( k \in \{1, \ldots, 14\} \).

We now state the tangency conditions we are interested in. To do so, we recall the concept of pseudo spherical condition (already used in Theorem 1.12), which is nothing but the restriction of the spherical condition in Eq. (1) to the hyperplane \( L \) defining the boundary. In contrast with the spherical condition, the pseudo spherical condition does not depend on the leg lengths. The pair of 6-tuples given by \( \vec{P} \) and \( \gamma \vec{P} \) imposes 6 pseudo spherical conditions, which determine a linear space that we denote by \( \tilde{H}_\gamma \). Notice that if we fix a vector \( \vec{d} \) of leg lengths, and we create the hexapod \( \Pi_{\gamma, \vec{d}} = (\vec{P}, \gamma \vec{P}, \vec{d}) \), denoting by \( H_{\gamma, \vec{d}} \) the linear space cut out by the spherical conditions determined by \( \Pi_{\gamma, \vec{d}} \), then for all boundary points \( \beta \)

\[
\beta \in H_{\gamma, \vec{d}} \cap X \quad \text{if and only if} \quad \beta \in \tilde{H}_\gamma,
\]

since \( H_{\gamma, \vec{d}} \cap L = \tilde{H}_\gamma \cap L \), where \( L \) is the hyperplane in \( \mathbb{P}^6 \) determining the boundary \( B \). In particular this holds for all 14 points \( \beta_k \).

**Definition 2.10.** Let \( \vec{P} \) and \( \vec{P} \) be two 6-tuples whose Möbius curves intersect in 14 points giving rise to 14 boundary points. Following the notation introduced before, denote by \( \beta_k \) the 14 boundary points determined by \( \vec{P} \) and \( \gamma \vec{P} \), and by \( \tilde{H}_\gamma \), the linear space cut out by the 6 pseudo spherical conditions imposed by \( \vec{P} \) and \( \gamma \vec{P} \). Suppose furthermore that \( \tilde{H}_\gamma \) and \( X \) intersect properly, so that each of the \( \beta_k \) is an irreducible component of \( \tilde{H}_\gamma \cap X \). We say that \( \gamma \in \mathbb{R} \setminus \{0\} \) satisfies the
We notice that, with the notation previously introduced, \( i(\tilde{H}, X; \beta^k) \geq 2 \) if and only if \( i(H, \vec{d}, X; \beta^k) \geq 2 \) for every \( \vec{d} \) for which \( H, \vec{d}, X \) intersect properly. To prove this it is enough to show that

\[
i(\tilde{H}, X; \beta^k) = 1 \quad \text{if and only if} \quad i(H, \vec{d}, X; \beta^k) = 1.
\]

This follows from the fact that the projective tangent space \( T_{\beta^k} X \) of \( X \) at \( \beta^k \) is contained in \( L \) (we used already this fact in Theorem 1.12; this can be proved by a direct computation, using for example Gröbner bases) and that

\[
i(\tilde{H}, X; \beta^k) = 1 \quad \text{if and only if} \quad T_{\beta^k} X \cap \tilde{H} = \{ \beta^k \}
\]

and similarly for \( H, \vec{d} \).

**Remark 2.11.** The condition \( \text{Tang}_2 \) is affine-linear in \( \gamma \). In fact, pick one of the 14 boundary points \( \beta^k \); from now on we will denote it by \( \beta_{\gamma} \). The condition \( \text{Tang}_2 \) is equivalent to the condition that the dimension of the intersection of the projective tangent space \( T_{\beta_{\gamma}} X \) and the linear space \( \tilde{H}_{\gamma} \) is greater than or equal to 1. After possibly reparametrizing the projective line on which \( \beta_{\gamma} \) lies as \( \gamma \) varies, we can write \( \beta_{\gamma} = \left(0 : \alpha wv : T \lambda w : 0 \lambda v : \gamma \right) \), since the line passes through the vertex of the boundary \( B \).

One can show that \( T_{\beta_{\gamma}} X \) is spanned by the rows of the following matrix:

\[
\begin{array}{cccc}
h & M & x & y & r \\
0 & wv^T & 0 & 0 & 0 \\
0 & \alpha w'v^T & \lambda w' & 0 & 0 \\
0 & \alpha w'v^T & 0 & \mu v' & 0 \\
0 & 0 & w & 0 & 0 \\
0 & 0 & 0 & v & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & w' & v' & 0 \\
\end{array}
\]

where \( w' \), together with \( w \), span the tangent line in \( \mathbb{P}^2 \) of \( C \) at \( w \), and the same for \( v' \), subject to the condition that \( \langle w', w' \rangle = \langle v', v' \rangle \). In particular, the projective tangent space does not depend on \( \gamma \). The 6 pseudo spherical conditions defining \( \tilde{H}_{\gamma} \) determine a linear map \( \eta: C^7 \rightarrow C^6 \), where we identify \( C^7 \) with the vector space associated to \( T_{\beta_{\gamma}} X \). Its kernel is the vector space associated to the intersection \( T_{\beta_{\gamma}} X \cap \tilde{H}_{\gamma} \). We are going to show that the condition \( \dim \ker \eta \geq 2 \) is affine-linear in \( \gamma \). To do this, we pick coordinates so that (see [GNS15, Subsection 2.3])

\[
\begin{align*}
v &= w = (1 : i : 0), \\
v' &= w' = (0 : 0 : 1), \\
\lambda &= \mu = 0, \\
\alpha &= 1.
\end{align*}
\]

Then a direct inspection of the matrix of \( \eta \) proves the statement.

**Example 2.12.** Consider the pair of 6-tuples computed in Example 2.8: one finds that each of the 14 affine-linear equations for \( \gamma \) is a multiple of the equation \( \gamma - 1 = 0 \).
Remark 2.16. Let \( \Pi_{\gamma, \bar{d}} = (\bar{P}, \gamma \vec{p}, \bar{d}) \) with \( \bar{P} \) and \( \vec{p} \) as before, and suppose that \( \gamma \) satisfies Tang_2. Let \( H_{\gamma, \bar{d}} \) be the linear space defined by the 6 spherical conditions determined by \( \Pi_{\gamma, \bar{d}} \). Suppose that \( H_{\gamma, \bar{d}} \) and \( X \) intersect properly. We say that \( \bar{d} \) satisfies the tangency condition Tang_3 if and only if for each of the 14 points \( \beta_k^i \) the intersection multiplicity \( i(H_{\gamma, \bar{d}}; X; \beta_k^i) \) is greater than or equal to 3.

Example 2.14. Recall Example 2.12. Naively one expects that the intersection \( i(H_{\gamma, \bar{d}}; X; \beta_k^i) \) is greater than or equal to 3 if we are able to find a solution of the system of equations given by \( H_{\gamma, \bar{d}} \cap X \) in \( \mathbb{C}[t]/(t^3) \). For a solution of the form \( c_0 + c_1 t + c_2 t^2 \), the coefficients \( c_0 \) and \( c_1 \) are determined by \( \beta_k^i \) itself and by a tangent vector in \( T_{\beta_k^i} \cap H_{\gamma, \bar{d}} \) which is unique up to scaling. For \( c_2 \), we obtain a system of affine-linear equations in \( d_1^2, \ldots, d_6^2 \). The solvability of these equations with respect to \( c_2 \) is equivalent to another system of affine-linear equations in \( d_1^2, \ldots, d_6^2 \).

These equations are:

\[
\begin{align*}
(11) \quad d_1^2 &= \frac{71}{11} d_1^2 - \frac{105}{11} d_2^2 + \frac{43}{45} d_3^2 - \frac{535801}{970062}, \\
(12) \quad d_2^2 &= \frac{71}{41} d_1^2 - \frac{75}{41} d_2^2 + \frac{45}{45} d_3^2 - \frac{1905080}{10741597}, \\
(13) \quad d_3^2 &= \frac{71}{41} d_1^2 - \frac{45}{41} d_2^2 + \frac{9}{22} d_3^2 - \frac{114265}{744638}.
\end{align*}
\]

Theorem 2.15. Assume that \( \bar{P} \) is a 6-tuple of points in \( \mathbb{R}^3 \). Assume that \( \vec{p} \) is another 6-tuple such that the Möbius curves of \( \bar{P} \) and \( \vec{p} \) intersect in 14 points and do not intersect in a node of \( M_6 \). Assume that \( \gamma \) and \( \bar{d} \) satisfy the conditions Tang_2 and Tang_3. Then the hexapod \( \Pi = (\bar{P}, \gamma \vec{p}, \bar{d}) \) is movable.

Proof. Since the two Möbius curves do not intersect in a node, then from the discussion at the beginning of the section we obtain 14 boundary points. As before, denote by \( H_{\Pi} \) the linear space defined by the 6 spherical conditions determined by \( \Pi \). Suppose that \( H_{\Pi} \) and \( X \) intersect properly in finite number of points, namely that \( \Pi \) is not movable. By assumption, the intersection multiplicity of \( X \) and \( H_{\Pi} \) at each of the 14 bonds of \( \Pi \) corresponding to the 14 common images of the Möbius curves is at least 3. Hence the intersection count gives \( 14 \cdot 3 > 40 = (\deg X) (\deg H) \), which is absurd by [Har77, Appendix A, Axiom A6]. Therefore \( \Pi \) is movable. \( \square \)

Remark 2.16. If \( \bar{P} \) is a Möbius-general 6-tuple of points in \( \mathbb{R}^3 \), then Conjectured Lemma 1 predicts the existence of a 6-tuple \( \vec{p} \) satisfying the hypothesis of Theorem 2.15.

Conjectured Lemma 2. For a Möbius-general 6-tuple \( \bar{P} \) of base points, consider the 6-tuple \( \vec{p} \) of platform points predicted by Conjectured Lemma 1. There is a unique \( \gamma \) such that the tangency condition Tang_2 is fulfilled. Moreover, the set of leg length vectors \( \bar{d} \) fulfilling the tangency condition Tang_3 is of dimension 3.

Definition 2.17. Let \( \bar{P} \) be a Möbius-general 6-tuple of points in \( \mathbb{R}^3 \). Then Conjectured Lemma 1 and Conjectured Lemma 2 predict the existence of \( \vec{p}, \gamma \) and \( \bar{d} \) satisfying the hypothesis of Theorem 2.15. We call the resulting hexapod \( \Pi \) a liaison hexapod.

As for Conjectured Lemma 1 Conjectured Lemma 2 is formulated such a way that if it is false, then it is falsified by a generic choice of base points. We tested
the conjecture against random numerical examples, and constructed many movables hexapods in this way. But, as in Remark 2.9, we could have been lucky (or unlucky) in all these experiments and the conjecture may be false.

Finally, we would like to point out that, in case the two Conjectured Lemmata hold, the family of liaison hexapods is maximal among movable hexapods. In fact, the possible bases of such hexapods form an open subset in \((\mathbb{R}^3)^6\); moreover, if the conjectures are true, there exists exactly a 3-dimensional collection (parametrized by the leg lengths) of movable hexapods of maximal conformal degree 28 for each such base. This family cannot be contained in a larger family with smaller conformal degree since the latter is upper-semicontinuous.

3. Computations in Study parameter space

In this section we exhibit two positive-dimensional families of base points \(\vec{P}\) for which we can explicitly compute the candidate platform points \(\vec{p}\) coming from the liaison procedure in concrete instances as explained in Subsection 2.1, and such that there exists a unique scaling factor \(\gamma\) satisfying the condition \(T_{\text{ang}}\) and a three-dimensional set of leg lengths \(\vec{d}\) satisfying the condition \(T_{\text{ang}}\). We show that the hexapods corresponding to such families are movable.

**Proposition 3.1.** Consider a 6-tuple \(\vec{P}\) of points in \(\mathbb{R}^3\) such that the lines \(P_1P_2, P_3P_4\) and \(P_5P_6\) meet in one point (see Fig. 3). Then consider the following setting:

i. Take the candidate platform \(\vec{p}\) to be:

\[
(p_1, p_2, p_3, p_4, p_5, p_6) = (P_2, P_1, P_3, P_4, P_5, P_6).
\]

ii. The scaling factor \(\gamma\) is given by \(-1\).

iii. The condition on the leg lengths \(\{d_i\}\) ensuring mobility one is:

\[
d_1^2 = d_2^2 \quad d_3^2 = d_4^2 \quad d_5^2 = d_6^2.
\]

Then the hexapod that we obtain is movable.

**Proposition 3.2.** Consider a 6-tuple \(\vec{P}\) of points in \(\mathbb{R}^3\) such that there exists an isometry \(\sigma\) of \(\mathbb{R}^3\) of order 3 acting on \(\vec{P}\) in the following way:

\[
\sigma: (P_1, P_2, P_3, P_4, P_5, P_6) \mapsto (P_2, P_3, P_4, P_5, P_6, P_1).
\]

Then consider the following setting:

i. Take the candidate platform \(\vec{p}\) to be:

\[
(p_1, p_2, p_3, p_4, p_5, p_6) = (P_4, P_6, P_5, P_1, P_3, P_2).
\]

ii. The scaling factor \(\gamma\) is given by 1.

iii. The condition on the leg lengths \(\{d_i\}\) ensuring mobility one is given by a system linear in the \(d_2^2\) admitting a three-dimensional solution set.

Then the hexapod that we obtain is movable.

We are going to prove Proposition 3.1 and Proposition 3.2 using Study parameters. Moreover we report some interesting properties observed within this approach, and finally we compute the configuration curve of an example of a liaison hexapod.
3.1. Basics. Due to a result of Husty (see [Hus96]), we use Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$ to compute the configurations of a hexapod — this is also known as the forward kinematics problem. Note that the first four homogeneous coordinates $(e_0 : e_1 : e_2 : e_3)$ are the Euler parameters already mentioned in the Introduction. All real points of the Study parameter space $\mathbb{P}^7_\mathbb{C}$ that are located on the so-called Study quadric — given by $\Psi = 0$, where $\Psi = \sum_{i=0}^{3} e_i f_i$ — correspond to a direct isometry, with exception of the 3-dimensional subspace $e_0 = e_1 = e_2 = e_3 = 0$, as its points do not fulfill the condition $N \neq 0$ with $N = e_0^2 + e_1^2 + e_2^2 + e_3^2$. The translation vector $(t_1, t_2, t_3)$ and the rotation matrix $R$ of the direct isometry $(x, y, z) \mapsto (x, y, z) + (t_1, t_2, t_3)$ corresponding to a point in the Study quadric are given by:

\[
t_1 = 2(e_0 f_1 - e_1 f_0 + e_2 f_3 - e_3 f_2), \quad t_2 = 2(e_0 f_2 - e_2 f_0 + e_3 f_1 - e_1 f_3), \quad t_3 = 2(e_0 f_3 - e_3 f_0 + e_1 f_2 - e_2 f_1),
\]

and

\[
R = \begin{pmatrix}
    e_0^2 + e_1^2 + e_2^2 + e_3^2 & 2(e_1 e_2 + e_0 e_3) & 2(e_1 e_3 - e_0 e_2) \\
    2(e_1 e_2 - e_0 e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 + e_0 e_1) \\
    2(e_1 e_3 + e_0 e_2) & 2(e_2 e_3 - e_0 e_1) & e_0^2 - e_1^2 + e_2^2 + e_3^2
\end{pmatrix},
\]

if the normalizing condition $N = 1$ is fulfilled.
By using the Study parametrization of direct isometries, the condition that the point $p_i$ is located on a sphere centered in $P_i$ with radius $d_i$ is a quadratic homogeneous equation according to Husty (see [Hus96]). For the explicit formula of the used spherical condition $\Lambda_i = 0$ we refer to [Naw14b, Eq. (2)].

The solution for the direct kinematics over $\mathbb{C}$ of a hexapod can be written as the algebraic variety whose ideal is spanned by $\Psi, \Lambda_1, \ldots, \Lambda_6$, with the condition $N = 1$. In general this variety consists of a discrete set of points, corresponding to the (at most) 40 solutions of the forward kinematic problem. In the case of liaison linkages it is 1-dimensional as they have a self-motion.

3.2. Proving the two propositions. We consider the polynomials $\Delta_{i,j} := \Lambda_i - \Lambda_j$ for $i < j$ and $i,j \in \{1,\ldots,6\}$, which are affine-linear in $f_0, \ldots, f_3$ and therefore they can be written in the form:

$$\Delta_{i,j} = S_{i,j} f_0 + T_{i,j} f_1 + U_{i,j} f_2 + V_{i,j} f_3 + W_{i,j}.$$ 

The system of polynomials $\Delta_{i,j}$ can be grouped into the following six sets:

$$S_k := \{ \Delta_{i,j} : i < j \text{ and } i,j \in \{1,\ldots,6\} \setminus \{k\} \} \text{ for } k = 1,\ldots,6.$$ 

Denote by $I_k$ the ideal generated by the set $\{S_k\}$, together with the polynomial $\Psi$. It can easily be seen that each of these ideals is generated by five linear polynomials in $f_0, \ldots, f_3$:

$$I_6 = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5}, \Psi \rangle, \quad I_5 = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,6}, \Psi \rangle,$$

$$I_4 = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,5}, \Delta_{1,6}, \Psi \rangle, \quad I_3 = \langle \Delta_{1,2}, \Delta_{1,4}, \Delta_{1,5}, \Delta_{1,6}, \Psi \rangle,$$

$$I_2 = \langle \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5}, \Delta_{1,6}, \Psi \rangle, \quad I_1 = \langle \Delta_{2,3}, \Delta_{2,4}, \Delta_{2,5}, \Delta_{2,6}, \Psi \rangle.$$ 

A necessary condition for the solvability of a system of five linear equations in $f_0, \ldots, f_3$ is that the determinant of the extended coefficient matrix is equal to zero. This condition is denoted by $\Omega_k = 0$; e.g. $\Omega_6$ is given by:

$$\Omega_6 : \begin{vmatrix}
S_{1,2} & T_{1,2} & U_{1,2} & V_{1,2} & W_{1,2} \\
S_{1,3} & T_{1,3} & U_{1,3} & V_{1,3} & W_{1,3} \\
S_{1,4} & T_{1,4} & U_{1,4} & V_{1,4} & W_{1,4} \\
S_{1,5} & T_{1,5} & U_{1,5} & V_{1,5} & W_{1,5} \\
e_0 & e_1 & e_2 & e_3 & 0
\end{vmatrix}.$$ 

It is well known (see [Hus96]) that $\Omega_k$ factors into $e_0^2 + e_1^2 + e_2^2 + e_3^2$ and a quartic factor $G_k$ that has 258 720 terms. In this way we get six quartic equations $G_k = 0$ in the Euler parameter space. Now it can easily be checked by direct computations that

$$G_1 - G_2 + G_3 - G_4 + G_5 - G_6 = 0$$ 

holds, i.e. the $\{G_i\}$ are linearly dependent. Therefore we can restrict to the equations $G_2 = 0, \ldots, G_6 = 0$. Based on this preparatory work we prove the mobility of two classes of liaison examples.
3.2.1. Proof of Proposition 3.1. We assume that two Cartesian frames — called the moving and the fixed frame — are rigidly attached to the platform and the base of the hexapods, respectively. Without loss of generality we can choose these frames in a way that the base and platform points have the following coordinates (with respect to the corresponding frames):

\[
\begin{align*}
P_1 &= (A_1, 0, 0), & P_2 &= \mu_1 P_1, & p_1 &= -P_2, & p_2 &= -P_1, \\
P_3 &= (A_3, B_3, 0), & P_4 &= \mu_3 P_3, & p_3 &= -P_4, & p_4 &= -P_3, \\
P_5 &= (A_5, B_5, C_5), & P_6 &= \mu_5 P_5, & p_5 &= -P_6, & p_6 &= -P_5.
\end{align*}
\]

Plugging these coordinates and leg lengths relations \(d_1 = d_2, \ d_3 = d_4\ and \ d_5 = d_6\) into our above calculated expressions shows that \(G_k\) factors into a linear expression \(L_k\) in the Euler parameters and a common cubic factor \(S\) with 650 terms.\textsuperscript{4}

This already proves that the mobility is one, because the dimension of the variety of the ideal \(I = I_2 + I_3 + I_4 + I_5 + I_6\) is at least 2, as the latter is generated by:

\[
I = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5}, \Delta_{1,6}, \Psi \rangle. 
\]

Since there is only one equation left (any equation \(A_i = 0\) can be taken) we get at least mobility one over \(\mathbb{C}\).

Remark 3.3. Note that \(S = 0\) can split up into several components. An example for this is the case \(\mu_i = \mu_j = -1 \neq \mu_k\) with pairwise distinct \(i, j, k \in \{1, 3, 5\}\). In this case the cubic surface \(S = 0\) splits up into a quadric and a plane. For \(i = 1, j = 3\) and the additional relations \(A_3 = 0\), \(B_3 = A_1\) and \(d_1 = d_3\) we get even 3 planes, namely:

\[
e_1 - e_2 = 0, \quad e_1 + e_2 = 0, \quad A_3 e_1 + B_3 e_2 + C_5 e_3 = 0.
\]

3.2.2. Proof of Proposition 3.2. Without loss of generality we can choose the fixed frame and the moving frame in a way that the base and platform points have the following coordinates (with respect to the corresponding frames):

\[
\begin{align*}
p_1 &= (a, b, c), & p_4 &= (A, B, C), & P_1 &= p_4, & P_4 &= p_1, \\
p_2 &= (b, c, a), & p_5 &= (B, C, A), & P_2 &= p_6, & P_5 &= p_3, \\
p_3 &= (c, a, b), & p_6 &= (C, A, B), & P_3 &= p_5, & P_6 &= p_2.
\end{align*}
\]

Moreover with respect to this choice of coordinates the leg lengths can be expressed as

\[
\begin{align*}
d_1^2 &= \frac{UW(d_2^2 - d_1^2) - VW(d_3^2 - d_1^2)}{kk} + d_1^2, \\
d_2^2 &= \frac{UV(d_2^2 - d_1^2) - UV(d_3^2 - d_1^2)}{kk} + d_1^2, \\
d_3^2 &= \frac{VW(d_2^2 - d_1^2) + UV(d_3^2 - d_1^2)}{kk} + d_1^2,
\end{align*}
\]

\textsuperscript{4}The corresponding Maple Worksheet can be downloaded as \texttt{mws} file and pdf file from \url{www.geometrie.tuwien.ac.at/nawratil/prooffamily1.mws} and \url{www.geometrie.tuwien.ac.at/nawratil/prooffamily1.pdf} respectively.
with
\[
U = Aa - Ab + Bb - Bc - Ca + Cc,
K = A^2 + B^2 + C^2 - AB - AC - BC,
V = Aa - Ac - Ba + Bb - Cb + Cc,
k = a^2 + b^2 + c^2 - ab - ac - bc,
W = Ab - Ac - Ba + Bc + Ca - Cb.
\]

With respect to these coordinates and leg lengths the numerator of \( G_k \) splits up into the factor \( k - K \), a linear expression \( L_k \) in the Euler parameters and a common cubic factor \( S \) with 576 terms. This proves that the mobility is one for the same reasons as in the last proof.

**Remark 3.4.** Finally it should be mentioned that the two families given in Propositions 3.1 and 3.2 also contain geometries that are excluded from the conjectures formulated in Section 2 (e.g. planar or congruent hexapods, or hexapods with 4 collinear points).

3.3. **Observations.** Based on random examples (see Section 3.4) and the families given in Propositions 3.1 and 3.2 we provide the following interesting observations, which hopefully could lead to a simpler (or even explicit) computation of the linked 6-tuple in the future.

I. Each \( G_k \) factors in a linear expression \( L_k \) and a common cubic factor \( S \) for \( k = 2, \ldots, 6 \). Every plane defined by a linear equation \( L_k = 0 \) belongs to a bundle of planes with vertex \( V \) in the Euler parameter space. Therefore there exists a 2-parametric set of linear combinations:

\[
\gamma_2 G_2 + \gamma_3 G_3 + \gamma_4 G_4 + \gamma_5 G_5 + \gamma_6 G_6 = 0.
\]
Moreover, the vertex \( V \) belongs to the common cubic surface \( S = 0 \).

II. The vertex \( V \) does not depend on the remaining leg lengths. There exists a bijection between the remaining three leg lengths and the translation vector; i.e. a self-motion can be started from every pose of the platform, if it has this special orientation.

**Remark 3.5.** Therefore this manipulator can be seen as an translational singular manipulator (or Cartesian-singular manipulators; see [Naw10, Section 5]) with respect to this orientation.

III. Moreover \( V \) corresponds to an orientation of the manipulator, where the five difference vectors

\[
(P_i - P_1) - (p_i - p_1) \quad \text{for} \quad i = 2, \ldots, 6
\]
only span a plane \( \alpha \) with normal vector \( n \).

Now we consider the orthogonal projection of the points \( p_1, \ldots, p_6 \) and \( P_1, \ldots, P_6 \) on \( \alpha \) which yields \( p_1', \ldots, p_6', P_1', \ldots, P_6' \). There exist three pairs.

---

5Note that for \( k = K \) platform and base are congruent.
6The corresponding Maple Worksheet can be downloaded as mws file and pdf file from [www.geometrie.tuwien.ac.at/nawratil/prooffamily2.mws](http://www.geometrie.tuwien.ac.at/nawratil/prooffamily2.mws) and [www.geometrie.tuwien.ac.at/nawratil/prooffamily2.pdf](http://www.geometrie.tuwien.ac.at/nawratil/prooffamily2.pdf) respectively.
7Note that not all three pairs have to be real as two pairs can also be complex conjugate.
of centers \((q_i, Q_i)\) in \(\alpha\), in a way that the two pencils of six lines
\[
[q_i, p_1^j], \ldots, [q_i, p_6^j] \quad \text{and} \quad [Q_i, P_1^j], \ldots, [Q_i, P_6^j]
\]
can be mapped onto each other by a congruence sending \([q_i, p_1^j]\) to \([Q_i, P_1^j]\)
for \(j = 1, \ldots, 6\). Moreover there is an orientation reversing equiform transformation
with \(q_0 \mapsto Q_i\) for \(i = 1, 2, 3\). Observation III is illustrated in
Fig. 4 with respect to the example given in Section 3.3.

IV. The cubic surface \(S = 0\) in the Euler parameter space contains a line \(\ell\)
through the vertex \(V\). The points of these line \(\ell\) correspond to the rotation
of the platform about a fixed line orthogonal to \(\alpha\). In each configuration of
the 2-parametric set, obtained from the composition of this rotation and a
translation of the platform in direction of \(n\), a self-motion can be started.

Remark 3.6. The planar hexapod with platform \(p_1, \ldots, p_6\) and base \(P_1, \ldots, P_6\)
is even Schönflies-singular (see [Naw10]) with respect to the direction or-
thogonal to the \(\alpha\)-parallel carrier planes of the planar platform and planar
base.

V. There exists a regular projectivity mapping \(P_i\) to \(p_i\) for \(i = 1, \ldots, 6\).

3.4. Example. We choose the following set of base points with respect to the fixed
frame \((O; X, Y, Z)\): \(P_i = A_i\) for \(i = 1, \ldots, 6\) with \(A_i\) given in Eqs. (6) and (7).

According to Example 2.8 the liaison technique explained in Section 2 yields
the following platform with respect to the moving frame \((\alpha; x, y, z)\): \(p_i = B_i\)
for \(i = 1, \ldots, 6\) with \(B_i\) given in Eqs. (8) to (10). Note that this coordinatization
of the platform already corresponds with the special orientation \(V\).

Moreover the leg lengths are given by Eqs. (11) to (13).

For the computation of the self-motion we express \(f_0, f_1, f_2, f_3\) from \(\Delta_{i,j} = 0, \Delta_{i,k} = 0, \Delta_{i,l} = 0, \Psi = 0\) for pairwise distinct \(i, j, k, l \in \{1, \ldots, 6\}\) and insert them into \(\Lambda_i\). We denote the numerator of the resulting expression by \(E_{m,n}\) with
pairwise distinct \(i, j, k, l, m, n \in \{1, \ldots, 6\}\). This is an octic expression in the Euler
parameter space, where \(e_0\) appears maximally to the power of 6.

Due to the involved powers of \(e_0\), we eliminate this Euler parameter by computing
the resultant of \(S\) and \(E_{m,n}\), which yields the expression \(F_{m,n}\) of degree 22 in
\(e_1, e_2, e_3\). The greatest common divisor \(J\) of all \(F_{m,n}\) corresponds to the self-motion,
which is in the generic case of degree 12.

It is always possible to choose special values for the remaining leg lengths \(d_1, d_2, d_3\)
in a way that \(S\) is linear in \(e_0\) and that all \(E_{m,n}\) are of degree 5 in \(e_0\). For our
example the overdetermined system of equations resulting from the coefficients of
\(c_0^2 c_1, c_0^2 c_2, c_0^2 c_3\) of \(S\) and the coefficients of \(e_0^6 c_1^2, \ e_0^6 c_2^2, \ e_0^6 c_3^2, \ e_0^6 c_1 c_2, \ e_0^6 c_1 c_3, \ e_0^6 c_2 c_3\)
of \(E_{m,n}\) has the following solution:
\[
d_1^2 = \frac{62434791769}{2888740069}, \quad d_2^2 = \frac{147143743}{8595735}, \quad d_3^2 = \frac{431695696}{464169696}.
\]
In this case \(J\) is only of degree 10. This decic \(J = 0\) is illustrated in Fig. 5
and it consists of two components, a black and a red colored one. On the red
component a point is highlighted, whose corresponding configuration is illustrated
We illustrate observations III by an orthogonal projection onto $\alpha$ (top view indicated by $'$). In the corresponding front view (indicated by $''$) the property of Eq. (14) can be seen, as $p_i''$ and $P_i''$ are located on horizontal lines; e.g. dashed line $[p_3'', P_3'']$. Moreover it can be figured out that the triangles $Q_1, Q_2, Q_3$ and $q_1, q_2, q_3$ are reflection similar. The pencils of lines given in Eq. (15) are not drawn as otherwise the figure gets overloaded. But the reader can verify the congruence of the corresponding pencils by checking the measurements with a protractor; e.g. $\omega := \angle(P_5'', Q_1, P_6') = \angle(p_5', q_1, p_6')$.

Remark 3.7. Finally it should be noted that the number of 14 bonds can also be verified within the Study parameter approach according to the method presented in [Naw14b].
Figure 5. We identify $e_3 = 0$ with the line at infinity and illustrate the affine part of the decic $J = 0$, i.e. we set $e_3 = 1$ and plot $e_1$ horizontally and $e_2$ vertically. Note that the complete decic corresponds to a real self-motion as $S$ depends only linearly on $e_0$. Moreover in the upper right corner we provided a zoom of the red component by a scaling factor of 3.

Acknowledgments

The first-named and third-named author’s research is supported by the Austrian Science Fund (FWF): W1214-N15/DK9 and P26607 - “Algebraic Methods in Kinematics: Motion Factorisation and Bond Theory”. The second-named author’s research is funded by the Austrian Science Fund (FWF): P24927-N25 - “Stewart Gough platforms with self-motions”.

References

[BW79] James W. Bruce and Charles T. C. Wall, On the classification of cubic surfaces, J. London Math. Soc. (2) 19 (1979), no. 2, 245–256.

[Dol12] Igor V. Dolgachev, Classical algebraic geometry, Cambridge University Press, Cambridge, 2012.

[Dup98] Ernest Duporcq, Sur la correspondance quadratique et rationnelle de deux figures planes et sur un déplacement remarquable, Comptes Rendus des Séances de l’Académie des Sciences 126 (1898), 1405–1406.

[GNS14] Matteo Gallet, Georg Nawratil, and Josef Schicho, Möbius photogrammetry, Journal of Geometry (2014), 1–19, in press, available via Springer Online First.

[GNS15] , Bond theory for pentapods and hexapods, J. Geom. 106 (2015), no. 2, 211–228.

[Har77] Robin Hartshorne, Algebraic geometry, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[HK02] Manfred Husty and Adolf Karger, Self motions of Stewart-Gough platforms, an overview, Proceedings of the workshop on fundamental issues and future research directions for parallel mechanisms and manipulators (Quebec City, Canada), 2002.

[HMSV09] Benjamin Howard, John Millson, Andrew Snowden, and Ravi Vakil, The equations for the moduli space of $n$ points on the line, Duke Math. J. 146 (2009), no. 2, 175–226.

[Hus96] Manfred Husty, An algorithm for solving the direct kinematics of general Stewart-Gough platforms, Mech. Mach. Theory 31 (1996), no. 4, 365–380.
Figure 6. The hexapod in the configuration marked in Fig. 5 together with the trajectories corresponding to the red component of the self-motion. An animation of this self-motion can be downloaded from www.geometrie.tuwien.ac.at/nawratil/liaison.gif.

[Mig98] Juan C. Migliore, Introduction to liaison theory and deficiency modules, Progress in Mathematics, vol. 165, Birkhäuser Boston, Inc., Boston, MA, 1998.
[Naw10] Georg Nawratil, Special cases of Schönflies-singular planar Stewart Gough platforms, New Trends in Mechanisms Science, Springer, Dordrecht, 2010, pp. 47–54.
[Naw13] ________, On equiform Stewart Gough platforms with self-motions, Journal for Geometry and Graphics 17 (2013), no. 2, 163–175.
[Naw14a] ________, Correcting Duporcq’s theorem, Mech. Mach. Theory 73 (2014), 282–295.
[Naw14b] ________, Introducing the theory of bonds for Stewart Gough platforms with self-motions, ASME Journal of Mechanisms and Robotics 6 (2014), no. 1, 011004.
[Sel13] Jon M. Selig, On the geometry of the homogeneous representation for the group of proper rigid-body displacements, Rom. J. Tech. Sci. Appl. Mech. 58 (2013), no. 1-2, 153–176.