CONTACT INTEGRAL GEOMETRY AND THE HEISENBERG ALGEBRA

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ABSTRACT. Generalizing Weyl’s tube formula and building on Chern’s work, Alesker reinterpreted the Lipschitz-Killing curvature integrals as a family of valuations (finitely-additive measures with good analytic properties), attached canonically to any Riemannian manifold, which is universal with respect to isometric embeddings. In this note, we uncover a similar structure for contact manifolds. Namely, we show that a contact manifold admits a canonical family of generalized valuations, which are universal under contact embeddings. Those valuations assign numerical invariants to even-dimensional submanifolds, which in a certain sense measure the curvature at points of tangency to the contact structure. Moreover, these valuations generalize to the class of manifolds equipped with the structure of a Heisenberg algebra on their cotangent bundle. Pursuing the analogy with Euclidean integral geometry, we construct symplectic-invariant distributions on Grassmannians to produce Crofton formulas on the contact sphere. Using closely related distributions, we obtain Crofton formulas also in the linear symplectic space.

1. Introduction

1.1. Background and motivation. Intrinsic volumes first appeared in convex geometry through Steiner’s formula: given a compact convex body \( K \subset \mathbb{R}^n \),
\[
\text{vol}(K + 2\epsilon B^n) = \sum_{k=0}^{n} \omega_{n-k} \mu_k(K) \epsilon^{n-k}
\]
where \( B^j \) is unit Euclidean ball in \( \mathbb{R}^j \) and \( \omega_j \) is its volume. The coefficient \( \mu_k(K) \), the \( k \)-th intrinsic volume, can be written explicitly for smooth \( K \) as \( \mu_k(K) = c_{n,k} \int_{\partial K} \sigma_{n-1-k}(\kappa_1, \ldots, \kappa_{n-1}) d\text{Area}_K \), where \( \kappa_j \) are the principal curvatures of \( \partial K \). Alternatively, \( \mu_k(K) \) can be given by integral-geometric Crofton formulas: \( \mu_k(K) = c'_{n,k} \int_{\text{Gr} \subset \mathbb{R}^n} \chi(K \cap E) dE \) where \( dE \) is the rigid motion invariant measure on the affine Grassmannian, and \( \chi \) is the Euler characteristic. A third, axiomatic definition, was given by Hadwiger, who described the intrinsic volumes as the unique rigid-motion invariant continuous finitely additive measures on compact convex sets.

A closely related famous result is Weyl’s tube formula \([\text{Wey}39]\). It asserts that the volume of an \( \epsilon \)-tube around a Riemannian manifold \( M \) embedded isometrically in Euclidean space \( \mathbb{R}^N \) is a polynomial in \( \epsilon \ll 1 \), whose coefficients are, remarkably, intrinsic invariants of the Riemannian manifold \( M \), independent of the isometric embedding. These coefficients, now known as the intrinsic volumes of \( M \), are intimately linked with the asymptotic expansion of the heat kernel, see \([\text{Don75}]\).

These results fall naturally in the domain of valuations on manifolds, a fairly young branch of valuation theory introduced by Alesker et al. in a sequence of works \([\text{Ale06a, Ale06b, AF08, Ale07}]\), see also \([\text{AFT14}]\) for a survey. Valuation theory itself is a mixture of convex and integral geometry, originating in the early

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20th century in works of Steiner, Blaschke, Chern and Santalo, as well as in Dehn’s solution to Hilbert’s third problem. Generally speaking, valuations are finitely additive measures on some family of nice subsets. In this note, there is typically some analytic restriction on the nature of the valuation, such as smoothness or smoothness with singularities, and the subsets are manifolds with corners or differentiable polyhedra.

Building on results of Chern, Alesker noticed a natural extension of Weyl’s theorem for valuations: restricting to $M$ the intrinsic volumes of $\mathbb{R}^N$, (considered as valuations), yields an intrinsically defined family of valuations on $M$, now known as the Lipschitz-Killing valuations. Weyl’s intrinsic volumes of $M$ are then the integrals of the corresponding valuations. In a recent work, Fu and Wannerer [FW17] characterized the Lipschitz-Killing valuations as the unique family of valuations attached canonically (in a sense made precise therein) to arbitrary Riemannian manifolds that are universal to isometric embeddings.

Other spaces whose (smooth) valuation theories were considered in recent years include complex space forms [BF11, BFS14], the quaternionic plane [BS14, BS14], the octonionic plane [BV16] and exceptional spheres [SW17].

Numerous intriguing connections between convex and symplectic geometries are known to exist. To name a few: Viterbo’s conjectured isoperimetric inequality for capacities of convex bodies [Vit00], was later shown by Artstein-Avidan, Karasev and Ostrover to imply Mahler’s famous conjecture [AAKO14]. Capacities have been successfully studied up to a bounded factor using convex techniques, see [AAMO08] and [GO16]. In [APBT14, APBT16], links are established between systolic geometry, contact geometry, Mahler’s conjecture and the geometry of numbers. Schäffer’s dual girth conjecture for normed spaces has been proved using symplectic techniques [AP06], and generalized further in [Fal12] using hamiltonian group actions. In very recent works of Abbondandolo et al. [ABHSa17], some links are established between the geometry of the group of symplectomorphisms and systolic geometry of the 2-sphere. For an exposition of some of those connections, see [Ost14].

The main objective for this work is to further explore the convex-symplectic link by studying the valuation theory of contact manifolds, using the Riemannian case and Weyl’s principle as guides.

1.2. Informal summary. We find that like in the Riemannian setting, contact manifolds possess a canonical family of valuations associated to them. We describe those valuations in two ways: geometrically through a curvature-type formula, and also dynamically through the invariants of a certain vector field at its singular points.

The contact valuations satisfy Weyl’s principle of universality under embeddings, similarly to the valuation extension of the Weyl principle. Let us emphasize the role played by valuations in this phenomenon: It so happens that contact valuations only assume non-zero values on even dimensional submanifolds, while they live on odd-dimensional contact manifolds. Thus unlike the Riemannian case, Weyl’s principle in the contact setting is only manifested in its extended to valuations form, as the statement in the original form becomes vacuous: all integrals of the contact valuations vanish.

Contact valuations in fact extend to the larger class of manifolds whose cotangent spaces admit a smoothly varying structure of Heisenberg algebras. The Heisenberg algebra provides a unifying link between contact, symplectic and metric geometries.
This path leads us to consider the valuation theory of the dual Heisenberg algebra, invariant under the group of the automorphisms of the Heisenberg algebra, which is closely related to the symplectic group. From this perspective, this is another step in the study of the valuation theory of non-compact Lie groups, which up to now has only been considered for the indefinite orthogonal group in \[AF14a\]\[BF17\]\[Fai17\].

Further similarity to the metric setting is exhibited by the contact sphere, where we prove Crofton formulas and a Hadwiger-type theorem, thus establishing an integral-geometric and an axiomatic description of the contact valuations.

Finally in the last part we explore the valuation theory of linear symplectic spaces. We show that there are no non-trivial invariant valuations, but nevertheless one can write oriented Crofton formulas for the symplectic volume of submanifolds.

1.3. Main results. Let us very briefly recall or indicate the relevant notions. For precise definitions see sections 2 and 3.

A contact manifold \(M^{2n+1}\) is given by a maximally non-integrable hyperplane distribution, namely a smooth field of tangent hyperplanes \(H \subset TM\) s.t. locally one can find \(\alpha \in \Omega^1(M)\) with \(H = \text{Ker}(\alpha)\) and \(d\alpha|_H\) a non-degenerate 2-form.

A smooth valuation \(\phi\) on an orientable manifold \(M^n\), denoted \(\phi \in \mathcal{V}_\infty(M)\), is a finitely additive measure on the compact differentiable polyhedra of \(M\), denoted \(\mathcal{P}(M)\), which has the form \(\phi(X) = \int_X \mu + \int_{N^*X} \omega\) for some forms \(\mu \in \Omega^n(M)\) and \(\omega \in \Omega^{n-1}(S^*M)\). Here \(S^*M\) is the cosphere bundle, and \(N^*X\) is the conormal cycle of \(X\), which is just the conormal bundle when \(X\) is a manifold. Orientability is not essential, and is only assumed to simplify the exposition.

There is a natural filtration \(\mathcal{W}_{\infty}^0(M) \subset \cdots \subset \mathcal{W}_{\infty}^n(M) = \mathcal{V}_\infty(M)\). Very roughly speaking, \(\mathcal{W}_{\infty}^n(M)\) consists of valuations which are locally homogeneous of degree at least \(j\), for example \(\mathcal{W}_\infty^n(M)\) are the smooth measures on \(M\).

The generalized valuations \(\mathcal{V}_-\infty(M)\) are, roughly speaking, distributional valuations: we allow \(\omega\) and \(\mu\) to be currents rather than smooth forms. Generalized valuations can be naturally evaluated on sufficiently nice subsets \(X \in \mathcal{P}(M)\). The filtration \(\mathcal{W}_j^\infty(M)\) on \(\mathcal{V}_-\infty(M)\) extends to a filtration \(\mathcal{W}_j^{-\infty}(M)\) on \(\mathcal{V}_-\infty(M)\).

Theorem 1.1. To any contact manifold \(M^{2n+1}\) with contact distribution \(H \subset TM\) there are canonically associated, linearly independent generalized valuations \(\phi^M_{2k} \in \mathcal{V}_-\infty(M)\), \(0 \leq k \leq n\). They have the following properties:

i) \(\phi^M_0\) is the Euler characteristic, and \(\phi^M_{2k} \in \mathcal{W}_{2k}^\infty(M) \setminus \mathcal{W}_{2k+1}^\infty(M)\).

ii) \(\phi^M_{2k}\) can be naturally evaluated on submanifolds in generic position relative to the contact structure (see Definition 4.3). For a generic closed hypersurface \(F\),

\[
\phi^M_{2k}(F) = \sum_{T_p F = H_p} \phi^M_{2k}(F, p)
\]

where the local contact areas \(\phi^M_{2k}(F, p)\) only depends on the germ of \(F\) at \(p\). They are described explicitly below in equations (1) or (2).

iii) Universality to restriction under embedding: if \(i: M^{2m+1} \to M^{2n+1}\) is a contact embedding, then for \(k \leq m\) one has \(i^* \phi^M_{2k} = \phi^N_{2k}\).

iv) For a \(2k\)-dimensional submanifold in generic position \(F\), \(\phi^M_{2k}(F) \geq 0\), with equality if and only if there are no contact tangent points.
The space of generalized valuations on $M$ invariant under all contactomorphisms of $M$ is spanned by $(\phi_{2k}^M)_k=0$.

**Remark 1.2.** The universality with respect to embeddings is sometimes referred to as the Weyl principle. Thus we recover a Weyl principle in the contact setting.

The local contact areas $\phi_{2k}^M(F,p)$ can be given explicitly in two different ways, through a geometric or a dynamical approach.

- From a dynamical point of view, $\phi_{2k}^M(F,p)$ encodes the invariants of the linearized vector field $B \in \mathcal{X}(F)$ representing the characteristic foliation. While there are many such vector fields, there is a distinguished choice, to linear order, at the critical points - choose an arbitrary contact form $\alpha$ near $p$ and let $B$ be given by $d\alpha|_F(B,\bullet) = \alpha|_F$. One then has
  \[
  \phi_{2k}^M(F,p) = \frac{\text{Tr} \wedge^{2n-2k} d_pB}{|\det d_pB|}.
  \] (1)

- From a geometric point of view,
  \[
  \phi_k^M(F,p) = \binom{2n}{k} |\det(S-h)|^{-1} D(S-h[2n-k], J[k]).
  \] (2)

Here $D$ denotes the mixed discriminant, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $S$ is the second fundamental form of $F$ at $p$, and $h$ the second fundamental form of the contact distribution at $p$ (see Definition 4.18), both written with respect to a frame compatible with the contact structure in a natural way at $p$, see Definition 4.6. Thus they are reminiscent of (certain symmetric functions of) the principal radii of an embedded hypersurface in a Riemannian manifold. This point of view is applicable in the wider setting of DH manifolds, as described below.

We may extend $\phi_{2k}$ to a non-negative lower semicontinuous functional on all $2k$-dimensional submanifolds with boundary, denoted $CA_{2k}(F)$, the contact area of $F$. We observe that $CA_{2k}(F) = 0$ if and only if $F$ can be made nowhere tangent to the contact distribution by an arbitrarily small perturbation.

In Riemannian or Hermitian manifolds, a fair amount of the valuation theory appears already in the corresponding flat space, which can be thought of as the tangent space to the given manifold. In the contact setting, it is no longer true: the tangent space of a contact manifold does not itself inherit a contact structure.

The main observation guiding this paper is that every cotangent space of a contact manifold is canonically the Heisenberg Lie algebra. We are thus led to study the valuation theory of general manifolds with such structure.

**Definition 1.3.** A manifold $X$ equipped with a hyperplane distribution $H$ (called horizontal) and a smooth field of nowhere-degenerate forms $\omega \in C^\infty(X, \wedge^2 H_x^* \otimes (T_x X/H_x))$ will be called a dual Heisenberg (DH) manifold.

The space of valuations naturally associated to such manifolds turns out to resemble somewhat the Lipschitz-Killing space of valuations in Riemannian geometry, in particular, they exhibit universality with respect to embeddings.

Theorem 1.1 is then the contact instance of the following general result.
Theorem 1.4. To any DH manifold $M^{2n+1}$ with horizontal distribution $H \subset TM$ there are canonically associated, generalized valuations $\phi_k^M \in W_k^{-\infty}(M)$, $0 \leq k \leq 2n$. They have the following properties:

i) $\phi_0^M$ is the Euler characteristic, and $\phi_2^M \in W_2^{-\infty}(M) \setminus W_2^{-\infty}(M)$.

ii) $\phi_k^M$ can be naturally evaluated on submanifolds in generic position with respect to the horizontal distribution. For a generic closed hypersurface $F$,

$$\phi_k^M(F) = \sum_{T_pF=H_p} \phi_k^M(F, p)$$

where $\phi_k^M(F, p)$ only depends on the germ of $F$ at $p$.

iii) If $i : N^{2m+1} \to M^{2n+1}$ is a DH embedding, then for $k \leq 2m$ one has $i^*\phi_k^M = \phi_k^N$.

The local contact areas $\phi_k^M(F, p)$ are given by the same curvature-type formula (2) as in the contact case.

Unlike the contact case, we do not have a uniqueness result, since the group of symmetries of a DH manifold is in general trivial. We conjecture that uniqueness does arise once universality to embeddings is mandated, similarly to the Riemannian setting considered by Fu-Wannerer [FW17].

As an intermediate step of independent interest, we obtain a Hadwiger-type theorem for the dual Heisenberg algebra itself. We denote by $U = \mathbb{R}^{2n+1}$ the dual of the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$, and by $\text{Sp}_H(U)$ its automorphism group.

Theorem 1.5. It holds that $\text{Val}^{-\infty}(U)^{\text{Sp}_H(U)}$ consists of even valuations. For $0 \leq k \leq n$, $\text{Val}_k^{-\infty}(U)^{\text{Sp}_H(U)} = \{0\}$ while $\dim \text{Val}_k^{-\infty}(U)^{\text{Sp}_H(U)} = 1$. The corresponding Klein sections are zero-order distributions (that is, regular Borel measures).

We also consider $\text{Sp}^+_H(U)$, the connected component of the identity. We prove

Theorem 1.6. For $0 \leq k \leq n$, $\text{Val}_k^{-\infty}(U)^{\text{Sp}^+_H(U)} = \{0\}$ while $\dim \text{Val}_k^{-\infty}(U)^{\text{Sp}^+_H(U)} = 2$. In the latter space, the $\text{Sp}_H(U)$-invariant valuations are complemented by a one-dimensional space of odd valuations.

We then consider the standard contact sphere $S^{2n+1}$, which we identify with the oriented projectivization $\mathbb{P}_+(V)$ of a symplectic space $V = (\mathbb{R}^{2n+2}, \omega)$. We construct a canonical distribution $\mu_\omega \in \mathcal{M}^{-\infty}(\text{Gr}_{2k}(V))^{\text{Sp}(V)}$. We obtain the following Crofton formulas, establishing further common grounds with the Riemannian setting.

Theorem 1.7. Define for $0 \leq i \leq n$ the generalized valuations $\psi_{2i}$ on $S^{2n+1}$ given by the Crofton formula

$$\psi_{2i} := \int_{\text{Gr}_{2n+2-2i}(V)} \chi(\bullet \cap E) d\mu_\omega(E).$$

Then for certain explicit constants $c^n_{ij}$ one has

$$\psi_{2i} = \sum_{j=i}^n c^n_{ij} \phi_{2j}.$$

We also establish a Hadwiger-type theorem for the contact sphere.

Theorem 1.8. Both $(\phi_{2k})_{k=0}^n$ and $(\psi_{2k})_{k=0}^n$ are bases of $\mathcal{V}^{-\infty}(S^{2n+1})^{\text{Sp}(2n+2)}$. 
Finally, we find that while symplectic space and manifolds do not possess interesting invariant valuations, one can nevertheless write certain integral geometric formulas for symplectic volumes of manifolds. We construct a canonical distribution on the affine oriented Grassmannian \( \mathcal{M} \in \mathcal{M}^{-\infty}(\text{AGr}_{2k}^+(\mathbb{R}^{2n})) \) which is translation- and Sp(2n)-invariant, and odd to orientation reversal. We prove the following Crofton formula on symplectic linear space.

**Theorem 1.9.** Let \( F^{2k} \subset \mathbb{R}^{2n} \) be a \( C^1 \) compact, oriented submanifold with boundary. Then

\[
\int_F \omega^k = (-1)^k \binom{n}{k} \binom{2n}{k}^{\frac{1}{2}} \binom{2(n-1)}{k} \frac{1}{2^{k+1}} \int_{\text{AGr}_{2n-2k}^+(\mathbb{R}^{2n})} I(E, F) d\mathbb{M}_{\omega}(E)
\]

where \( \kappa = \min(k, n-k) \) and \( I \) is the oriented intersection index.

1.4. **Plan of the paper.** In section 2 we introduce notation and present the basic geometric facts we will use. In section 3 we recall the basics of valuation theory and prove some lemmas we will need. In section 4 we construct the canonical valuations on general DH manifolds and establish their universality to embeddings, proving Theorem 1.4. We also explore some geometric properties of those valuations. In section 5 we classify the translation-invariant valuations of the dual Heisenberg algebra, proving Theorems 1.5 and 1.6, and note their relation to gaussian curvature. Apart from its intrinsic interest, the linear classification is needed for the uniqueness statement in Theorem 1.1 as well as for Theorem 1.8. In section 6 we specialize the DH valuations to contact manifolds and prove Theorem 1.1. In particular, we give the dynamical description of the contact valuations. In section 7 we construct symplectic-invariant distributions on linear and affine Grassmannians in symplectic space, which are used in the subsequent two sections. In section 8 we consider the standard contact sphere. We produce Crofton formulas for \( \phi^{2k} \) that are invariant under Sp(V), proving theorems 1.7 and 1.8, and compute some examples explicitly. We also bound from below the contact area of a convex set. Finally in section 9 we consider the integral geometry of linear symplectic space and prove Theorem 1.9.

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2. **Preliminaries**

2.1. **Notation.** We use \( \sigma_1 \) to denote the unit \( \text{SO}(n) \)-invariant measure on various homogeneous spaces of the special orthogonal group. We write \( \text{Gr}_k(V) \) for the \( k \)-Grassmannian in \( V \), \( \text{Gr}_k^+ \) for the oriented Grassmannian, and \( \text{AGr}^+ \) for the affine (oriented) Grassmannian. \( K(V) \) is the set of compact convex subsets of \( V \).

The one-dimensional space of real-valued Lebesgue measures over \( V \) is denoted \( \text{Dens}(V) \). For a manifold \( M \), \( |\omega| \) denotes the line bundle of densities, whose fiber over \( x \in M \) is \( \text{Dens}(T_x M) \). We will write \( M^+ \) for the translation-invariant elements of a module \( M \) over \( V \). For a group \( G \subset \text{GL}(V) \), \( G \) is the group generated by \( G \) and all translations in \( V \).

We will write \( \Omega_{-\infty}(M) \) for the space of currents on \( M \), since we typically consider them as generalized differential forms.

Throughout the note, \( J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \) is the standard real form of \( \sqrt{-1} \).
2.2. The symplectic group action on the Grassmannian. This subsection is the symplectic version of the corresponding section in [BF17] where $O(p,q)$ is considered.

Let $V = (\mathbb{R}^{2n}, \omega)$ be a symplectic space. Let $X^k_r(V)$, $0 \leq r \leq \lfloor \min(k,2n-k)/2 \rfloor$ be the orbits of $\text{Gr}_k(V)$ under the real symplectic group $\text{Sp}(V)$, where

$$X^k_r(V) = \{ E \in \text{Gr}_k(V) : \dim \ker \omega|_E = \min(k,2n-k) - 2r \}.$$  \hspace{1cm} (3)

When no confusion can arise we write simply $X^k_r$. In the oriented Grassmannian $\text{Gr}_k^o(\mathbb{R}^{2n})$, the orbits consist of the double covers $X^k_{\kappa,+}$ of $X^k_{\kappa}$ with $0 \leq r \leq \kappa - 1$, and for even $k$ the double cover of $X^k_{\kappa,+}$ splits into two open orbits, denoted $X^k_{\kappa}$, corresponding to the orientation induced by $\omega$ on the subspace; while for odd $k$, $X^k_{\kappa,+}$ is a single orbit.

We use the same notation for the corresponding $\text{Sp}(V)$-orbits in the (oriented) affine Grassmannian.

We will need a simple lemma from linear algebra, which is a linearized version of Witt’s theorem.

**Lemma 2.1.** Take $E \in X^k_r$. Write $E_0 = E \cap E^\omega$, $\dim E_0 = \min(k,2n-k) - 2r$. Let $\pi_0 : E^* \otimes V/E \to E_0^* \otimes E_0$ denote the map $\pi_0(T)(x,y) = \omega(Tx,y)$. Then

$$T_{E}X^k_r = \{ T : E \to V/E : \pi_0(T) \in \text{Sym}^2 E_0^* \}.$$  \hspace{1cm} Proof.

Clearly $\pi_0$ is onto. Denote by $\pi_E : V^* \otimes V \to E^* \otimes (V/E)$ the natural projection. Recall that $\text{sp}(V) = \{ T \in \text{gl}(V) : \omega(Tx,y) = \omega(Ty,x) \ \forall x,y \in V \}$.

We have to show that $\pi_{E}\text{sp}(V) = \pi_0^{-1} \text{Sym}^2 E_0^*$. The inclusion $\pi_{E}\text{sp}(V) \subset \pi_0^{-1} \text{Sym}^2 E_0^*$ is immediate. In the other direction, take $T : E \to V/E$ s.t. $\pi_0 T \in \text{Sym}^2 E_0^*$. We should lift $T$ to $\tilde{T} \in \text{sp}(V)$.

Choose any subspaces $E' \subset E$, $E'' \subset E^\omega$ with $E = E_0 \oplus E'$, $E^\omega = E_0 \oplus E''$, and $\dim E' = k-\kappa+2r$ and $\dim E'' = 2n-k-\kappa-2r$, and they are both non-degenerate. Moreover, $E' \oplus E'' \subset V$ is a non-degenerate subspace of dimension $2n-2\kappa+4r$, and $E_0$ is an isotropic subspace of the non-degenerate space $W := (E' \oplus E'')^\omega$, and $\dim W = 2\kappa - 4r$. That is, $E_0 \subset W$ is a Lagrangian subspace. Fix a Lagrangian complementing space $F \subset W$ s.t. $W = E_0 \oplus F$. Then $V = E' \oplus E'' \oplus E_0 \oplus F$.

Let $T_1 \in \text{Hom}(E,V)$ be a lift of $T$ such that $T_1(E') \subset E'' \oplus F$, $T_1(E_0) \subset E'' \oplus F \oplus E'$ and

$$\omega(\pi'(T_1x),e) := \omega(Tc,e) \quad \forall x \in E_0, e \in E',$$

where $\pi'(T_1x)$ is the $E'$-component of $T_1x$.

Note that $\omega$ gives identifications $E^* = E' \oplus F$, as well as $(E \oplus E'')^* = E' \oplus E'' \oplus F$.

We extend $T_1$ to a map $T_2 \in \text{Hom}(E \oplus E'',V)$ by requiring $T_2(E'') \subset E' \oplus F$ and

$$\omega(T_2e'',e) := \omega(T_1e,e''), \quad \forall e'' \in E'', e \in E.$$  \hspace{1cm} Note that $\omega(T_2e_1,e_2) = 0$ for all $e_1,e_2 \in E''$. Finally, we extend $T_2$ to a map $T' \in \text{Hom}(V,V)$ by requiring $T''(F) \subset E' \oplus E'' \oplus F$ and

$$\omega(T'f,f) = \omega(T_2xf, f), \quad \forall x \in E \oplus E'', f \in F.$$  \hspace{1cm} Again we have $\omega(T'f,f') = 0$ for $f,f' \in F$. Then $T' \in \text{sp}(V)$ lifts $T$ as required. \hspace{1cm} $\square$

**Corollary 2.2.** There is a natural identification of $N_E X^k_r$ with $\wedge^2 E_0^*$. 

Proof:

\[ N_{E}X^{k} = T_{E}\text{Gr}_{k}(V)/T_{E}X^{k} = E^{\ast} \otimes (V/E)/\pi^{-1}0_{0} \text{Sym}^{2}E_{0}^{\ast} \]
\[ = (E_{0}^{\ast} \otimes E_{0}^{\ast})/\text{Sym}^{2}E_{0}^{\ast} = \wedge^{2}E_{0}^{\ast} \]

\[ \square \]

Lemma 3.3. Fix \( L \in \mathbb{P}_{+}(V) \), and denote \( Z_{L} = \{ E \in \text{Gr}_{k}(V) : L \subset E \} \). Then \( Z_{L} \) intersects \( X^{k}_{E} \) transversally for all \( r \).

Proof. Take \( E \in Z_{L} \cap X^{k}_{E} \), denote \( E_{0} = E \cap E^{\ast} \). Let \( A : E \to V/E \) be an arbitrary linear map. We will find a representation \( A = A_{1} + A_{2} \) with \( A_{1} \in T_{E}Z_{L} \) (that is, \( A_{1}L = 0 \)) and \( A_{2} \in T_{E}X^{k}_{r} \). Fix \( l \in L \) and write \( Al = v + E \).

Case 1: \( L \not\subset E_{0} \). We set \( A_{2}|_{L} := A|_{L} \), \( A_{2}|_{E_{0}} = 0 \).

Case 2: \( L \subset E_{0} \) and \( \omega(v, E_{0}) = 0 \). Decompose \( E_{0} = L \oplus F \) and set \( A_{2}(l) = v + E \), \( A_{2}(f) = 0 \) for all \( f \in F \).

Case 3: \( L \subset E_{0} \) and \( \omega(L, AL) \neq 0 \). Since \( \omega(l, v) \neq 0 \), we may decompose \( E_{0} = E_{0}^{\ast} \oplus l \). We then set \( A_{2}(l) = v + E \) and \( A_{2}(x) = 0 \) for \( x \in E_{0} \oplus l \).

Case 4: \( L \subset E_{0} \), \( \omega(v, E_{0}) \neq 0 \) and \( \omega(L, AL) = 0 \). Then we decompose \( E_{0} = L \oplus F \oplus \text{Span}(u) \) where \( L \oplus F = v^{\ast} \cap E_{0} \) and \( \omega(u, v) = 1 \). Choose \( w \in V^{\ast} \) such that \( \omega(w, l) = -1 \) and \( w \in F^{\ast} \). This is possible since \( l^{\ast} \neq F^{\ast} \). Then set \( A_{2}l = v + E \), \( A_{2}f = 0 \) for all \( f \in F \), and \( A_{2}u = w + E \).

In all cases, extend \( A_{2} \) arbitrarily to \( E \). Thus in all cases, by Lemma \([21]\) \( A_{2} \in T_{E}X^{k}_{r} \), and \( A_{1} := A - A_{2} \in T_{E}Z_{L} \).

\[ \square \]

3. Valuation theory

3.1. Valuations on manifolds. For a manifold \( X \), we let \( \mathbb{P}_{X} := \mathbb{P}_{+}(T^{\ast}X) \) denote the oriented projectivization of its cotangent bundle and \( \pi : \mathbb{P}_{X} \to X \) the projection. \( \mathbb{P}_{X} \) has a canonical contact structure. A form \( \omega \in \Omega^{d}(\mathbb{P}_{X}) \) that vanishes when restricted to the contact distribution is usually called vertical. However we will have several different notions of verticality, so we will call such forms Legendrian.

Definition 3.1. We say that a form \( \omega \in \Omega^{d}(\mathbb{P}_{X}) \) has horizontal degree at least \( k \), written \( \text{deg}_{H} \omega \geq k \), if \( \omega(v_{1}, \ldots, v_{d}) \) vanishes whenever \( d + 1 - k \) of the vectors \( v_{j} \) are vertical, that is tangent to the fiber of \( \pi : \mathbb{P}_{X} \to X \).

The following is a simple reformulation.

Lemma 3.2. \( \{ \omega \in \Omega(\mathbb{P}_{X}) : \text{deg}_{H} \omega \geq k \} \) is the ideal in \( \Omega(\mathbb{P}_{X}) \) generated by \( \pi^{\ast}\Omega^{k}(X) \).

Proof. Assume \( \omega = \sum \pi^{\ast}\omega_{j} \wedge \eta_{j} \), where \( \omega_{j} \in \Omega^{k}(X) \), \( \eta_{j} \in \Omega^{d-k}(\mathbb{P}_{X}) \). Clearly if \( v_{1}, \ldots, v_{d+1-k} \) are vertical vectors then \( \pi^{\ast}\omega_{j} \wedge \eta_{j}(v_{1}, \ldots, v_{d+1-k}, \ldots) = 0 \).

For the opposite direction, let us choose a Riemannian structure on \( X \). Then \( \mathbb{P}_{X} \) is the sphere bundle on \( X \). Fix coordinates \( dx_{j} \) on \( T_{x}X \) and \( d\xi_{j} \) on \( T_{x}S_{x}X \). Then \( \pi^{\ast}\Omega^{k}(X) \) is spanned over \( C_{\infty}(\mathbb{P}_{X}) \) by \( \{ \wedge_{i} \pi^{\ast}dx_{i} : |I| = k \} \). Assume \( \text{deg}_{H} \omega \geq k \) and decompose \( \omega = \sum_{I,J} f_{I,J} \pi^{\ast}dx_{I} \wedge d\xi_{J} \).

Assume a multi-index \( I \) appears in the sum with \(|I| < k\), say \( I = (i_{1}, \ldots, i_{l}) \), \( l < k \), with corresponding \( J = (j_{1}, \ldots, j_{d-l}) \). Let \( e_{1}^{H}, \ldots, e_{n}^{H}, e_{1}^{V}, \ldots, e_{n-1}^{V} \) be dual to \( \pi^{\ast}dx_{1}, \ldots, \pi^{\ast}dx_{n}, d\xi_{1}, \ldots, d\xi_{n-1} \). Then \( d - l \geq d + 1 - k \) so by assumption \( 0 = \omega(e_{i_{1}}^{H}, \ldots, e_{i_{l}}^{H}, e_{j_{1}}^{V}, \ldots, e_{j_{d-l}}^{V}) = f_{I,J}(x, \xi) \), so \( f_{I,J} = 0 \).

\[ \square \]
Let $M$ be a smooth manifold, which we assume oriented for simplicity of exposition, and refer the reader to [Ale06b, AF08] for the general case. Denote by $P(M)$ the compact differentiable polyhedra of $M$. We remark that manifolds with corners are an example of differentiable polyhedra, and refer to [Ale06b] for the definition of differentiable polyhedra. The smooth valuations $V^\infty(M)$ consist of functionals $\phi : P(M) \to \mathbb{R}$ which can be presented in the form $\phi(X) = \int_X \mu + \int_{N^*X} \omega$ for some forms $\mu \in \Omega^n(M)$ and $\omega \in \Omega^{n-1}(P_M)$. Here $N^*X$ is the conormal cycle of $X$. It consists of codirections $\xi \in \mathbb{P}_+(T^*_2M)$ which are non-positive on velocity vectors $\gamma(0) \in T^*_2M$ of all curves $\gamma(t) \in X$ with $x = \gamma(0)$. The Euler characteristic $\chi$ is an important example of a smooth valuation. The smooth valuations over open subsets of $X$ constitute a soft sheaf over $X$, see [Ale06b]. We denote the compactly supported valuations by $V^\infty_c(M) := V^\infty(M) \cap W^\infty_c(M)$. We remark that whenever a valuation is a valuation on a subset, the subset is assumed compact. There is a natural integration functional $\int_M : V^\infty(M) \to \mathbb{R}$ which is essentially evaluation on $M$, but more precisely - on a sufficiently large compact set. Both $V^\infty(M)$ and $V^\infty_c(M)$ inherit natural topologies from the corresponding spaces of pairs of forms.

There is a natural filtration $W^\infty_n(M) \subset \cdots \subset W^\infty_0(M) = V^\infty(M)$, introduced by Alesker [Ale06b]. We will use an equivalent description, which is the content of [Ale06b, Corollary 3.1.10].

**Definition 3.3.** $W^\infty_k(M)$ consists of those valuations that can be represented by a pair $(\omega, \mu)$ with $\deg_\mathcal{H} \omega \geq k$.

In particular, $W^\infty_n(M)$ are just the smooth measures on $M$, denoted $\mathcal{M}^\infty(M)$.

Alesker defined a product structure $W^\infty_i(M) \otimes W^\infty_j(M) \to W^\infty_{i+j}(M)$ which turns $V^\infty(M)$ into a filtered algebra, whose unit is the Euler characteristic. It induces the following Alesker-Poincaré duality.

**Theorem 3.4** (Alesker [Ale07]). The pairing $W^\infty_i(M) \otimes W^\infty_{n-i,c}(M) \to \mathbb{R}$ given by $(\phi, \psi) \mapsto \langle \phi \cdot \psi \rangle(M)$ is non-degenerate.

The presentation of $\phi \in V^\infty(M)$ by a pair of forms is not unique. There is an alternative description due to Bernig and Bröcker [BB07]. In the following, $a : P_M \to P_M$ is the antipodal map in every fiber, and $D$ is the Rumin differential introduced in [Rum94]. We recall that $D\omega$ is the unique Legendrian form $d(\omega + \eta)$, where $\eta \in \Omega^{n-1}(P_M)$ ranges over all Legendrian forms.

**Theorem 3.5.** If $\phi$ is represented by the pair $(\omega, \mu) \in \Omega^{n-1}(P_M) \times \Omega^n(M)$, then $(T, C) := (a^*(D\omega + \pi^*\mu), \pi_*\omega) \in \Omega^n(P_M) \times C^\infty(M)$ is determined by $\phi$. They satisfy the relations $dT = 0$, $\pi_*T = (-1)^{n-1}dC$, and $T$ is Legendrian. Moreover, any $(T, C)$ with those properties corresponds to a valuation.

We refer to $(T, C)$ as the defining currents of $\phi$ (and often we refer just to $T$ as the defining current). The reason for this terminology will become evident once we introduce generalized valuations.

The Alesker-Poincaré duality is easy to describe using the defining currents.

**Theorem 3.6** (Bernig [Ber09]). Let $(\omega, \mu)$ represent $\phi_1$, and let $\phi_2$ have defining current $(T_2, C_2)$. Then

$$\int_M \phi_1 \cdot \phi_2 = \int_{P_M} \omega_1 \wedge T_2 + \int_M C_2 \mu_1.$$
Lemma 3.7. Let \((T, C) \in \Omega^n(\mathbb{P}_M) \times C^\infty(X)\) be the defining current of \(\phi \in \mathcal{V}^\infty(M^n)\). For \(1 \leq k \leq n\), \(\phi \in \mathcal{W}^\infty_n(M^n)\) if and only if \(C = 0\) and \(\deg_H T \geq k\).

Proof. Consider first \(k = n\). If \(\phi \in \mathcal{W}^\infty_n(M)\), we have \(C = 0\) and \(T = \pi^* \mu\) for some \(\mu \in \Omega^n(M)\), which is clearly horizontal of degree at least \(n\). In the other direction, if \(\deg_H T \geq n\), it follows that \(T = f(\xi) \pi^* \mu\) for some \(\mu \in \Omega^n(M)\) and \(f \in C^\infty(\mathbb{P}_M)\). Now \(0 = df = \pi^* \mu\), that is \(df\) vanishes when restricted to the vertical fiber, so \(f = \pi^* f_1\) for some \(f_1 \in C^\infty(M)\). Hence \(T = \pi^* (f_1 \mu)\), so \(\phi\) is just the measure \(f_1 \mu\).

Assume now \(1 \leq k \leq n - 1\). Recall that by Alesker-Poincaré duality, \(\phi \in \mathcal{W}^\infty_k(M)\) if and only if for all \(\psi \in \mathcal{W}_n+1-k,c(M)\), \(\int_M \phi \cdot \psi = 0\). Combining Definition 3.8 with eq. (5), we conclude that \(\phi \in \mathcal{W}^\infty_k(M)\) if and only if for all \(\omega \in \Omega^{n-1}(\mathbb{P}_M)\) with \(\deg_H \omega \geq n + 1 - k\), and all \(\mu \in \Omega^n(M)\), it holds that \(\int_M (\pi_* (\omega \wedge T) + C \mu) = 0\).

Assume that \(C = 0\) and \(\deg_H T \geq k\). Then by Lemma 3.2 \(\omega \wedge T = 0\) for all \(\omega\) as above, and hence \(\phi \in \mathcal{W}^\infty_k(M)\).

In the other direction, assume \(\phi \in \mathcal{W}^\infty_k(M)\). Taking \(\omega = 0\) and \(\mu\) arbitrary we deduce \(C = 0\). It then follows that \(\pi_*(\omega \wedge T) = 0\), and since \(\omega\) can have an arbitrarily small support, also that \(\omega \wedge T = 0\) whenever \(\deg_H \omega \geq n - k + 1\), so that \(\deg_H T \geq k\).

\(\square\)

The valuations that appear naturally in contact manifolds are not smooth. To formally study them we will need the larger family of generalized valuations.

Definition 3.8. The generalized valuations \(\mathcal{V}^{-\infty_1}(M)\) are the continuous dual of \(\mathcal{V}^{\infty_1}(M)\), equipped with the weak topology.

A generalized valuation is uniquely determined by its defining current \((T, C) \in D_{n-1}(\mathbb{P}_M) \times D_n(M)\), see [AB12]. It can be an arbitrary pair of currents satisfying the three properties: \(T\) is Legendrian, \(\pi_* T = \partial C\), and \(\partial T = 0\). If \(\phi_2 \in \mathcal{V}^{-\infty}(M)\) has defining current \((T_2, C_2)\), it acts on smooth valuations \(\phi_1 \in \mathcal{V}^{\infty}_c(M)\) represented by forms \((\omega_1, \mu_1)\) through eq. (5). We will write \(\phi_2 = [T_2, C_2]\), \(T_2 = T(\phi_2)\), \(C_2 = C(\phi_2)\).

Example 3.9. 1. Given \(X \in \mathcal{P}(M)\), the evaluation at \(X\) functional \(\chi_X : \mathcal{V}^{\infty}(M) \to \mathbb{R}\) is a generalized valuation with defining currents \(C = [[X]]\) and \(T = [[N^* X]]\).

2. A generalized valuation \(\phi_1\) can be represented by a pair of generalized forms \(\omega_1 \in \Omega^{n-1}_c(\mathbb{P}_M)\) and \(\mu_1 \in \Omega^n(M)\). It acts on \(\phi_2 = [T_2, C_2] \in \mathcal{V}^{\infty}_c(M)\) through eq. (5). It has defining currents given by \(T_1 = a^*(D\omega_1 + \pi^* \mu_1)\), \(C_1 = \pi_* \omega_1\).

The filtration \(\mathcal{W}^\infty_j(M)\) on \(\mathcal{V}^{\infty}(M)\) extends to a filtration \(\mathcal{W}^{-\infty}_j(M)\) on \(\mathcal{V}^{-\infty}(M)\) by taking \(\mathcal{W}^{-\infty}_j(M)\) to be the annihilator of \(\mathcal{W}^{\infty}_{n+1-j}(M)\).

We refer to [H03, GS77] for the notion of the wavefront set of a distribution. Let us only record that for an oriented submanifold \(X \subset M\), \(\mathrm{WF}([[X]]) = N^* X\).

Definition 3.10. The wavefront sets \(\mathrm{WF}(\phi)\) of \(\phi \in \mathcal{V}^{-\infty}(M)\) is the pair of wavefront sets \((\mathrm{WF}(T), \mathrm{WF}(C))\). When \(\mathrm{WF}(C) = \emptyset\), we also write \(\mathrm{WF}(\phi) = \mathrm{WF}(T)\).

Given a closed cone \(\Gamma \subset T^*(\mathbb{P}_M) \setminus 0\), we let \(\mathcal{V}^{-\infty}_T(M) := \{\phi \in \mathcal{V}^{-\infty}(M) : \mathrm{WF}(\phi) \subset \Gamma\}\). It inherits a topology from Hörmander’s topology on the corresponding space of currents \(\mathcal{D}_{n-1,1}(\mathbb{P}_M)\).
The Alesker-Poincaré duality extends to a pairing of generalized valuations, as long as the wavefront sets are in good relative position. We will only need the following weak version of Theorem 8.3 in [AB12].

For a subset $T \in T^*P_M$, we write $-T$ for its image under the antipodal map in every cotangent space $T_xT^*(\mathbb{P}_M)$. Define also $T^* := T \cup (-T) \cup aT \cup (-aT)$.

**Theorem 3.11.** Let $\Gamma_1, \Gamma_2 \subset T^*(\mathbb{P}_M) \setminus \emptyset$ be closed cones, $\Gamma_j = \Gamma_j^\circ$. Assume that $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then there is a jointly sequentially continuous pairing

$$V_{-1}^{\infty}(M) \otimes V_{-1}^{\infty}(M) \to \mathbb{R}$$

extending the Alesker-Poincaré duality on the dense subspaces of smooth valuations.

It follows that if $\phi = [T, C]$ with $C$ smooth, and $X \in \mathcal{P}(M)$ s.t. $WF([N^*X]) \cap WF(T^*) = \emptyset$, then one can naturally evaluate $\phi(X) := \langle \phi, \chi_X \rangle$. In particular, if $X, Y \in \mathcal{P}(M)$ have disjoint conormal cycles, one can evaluate $\int_M \chi_X \cdot \chi_Y$. In all reasonable settings, the result should equal $\chi(X \cap Y)$. This was shown to be the case when $X, Y$ are transversal submanifolds with corners in [AB12, Theorem 6].

The Euler-Verdier involution $\sigma : \mathcal{V}^{-\infty}(M) \to \mathcal{V}^{-\infty}(M)$ was introduced by Alesker in [Ale06b, Ale07]. It can be described through its action on the defining currents: $\sigma[T, C] = \{(−1)^n a^*T, C\}$.

**Lemma 3.12.** Take a generalized valuation $\phi \in \mathcal{V}^{-\infty}(\mathbb{R}^n)$ satisfying $\sigma \phi = (−1)^{n+1} \phi$ and $C(\phi)$ smooth. Assume that it has a small wavefront set, in the sense that any closed hypersurface with boundary $F$ can be smoothly perturbed (by an arbitrarily small amount) to have $N^*(N^*F)$ disjoint from $WF(\phi)$. Assume moreover that $\phi(F) = 0$ for all such hypersurfaces. Then $\phi = 0$.

**Proof.** Let us first assume $\phi$ is smooth. It suffices to prove the statement locally, thus we may assume $M = \mathbb{R}^n$. By the proof of Theorem 1 by Bernig and Bröcker in [BB07], it suffices to show $\phi$ vanishes on compact submanifolds $P$ of full dimension with boundary to conclude that $\phi = 0$. For such $P$, by [Ale07] Lemma 4.1.1 we have $(−1)^n \phi(P) = \sigma \phi(P) = (−1)^n(\phi(P) − \phi(\partial P))$. We know that $\phi(\partial P) = 0$, hence $\phi(P) = 0$, concluding the proof in the smooth case.

Assume next that $\phi$ is generalized with a small wavefront set. Approximate $\phi$ by $\phi_\epsilon \in \mathcal{V}^\infty(\mathbb{R}^n)$ given by a convolution with an approximate identity in the group of affine transformations. Then $\sigma \phi_\epsilon = (−1)^n \phi_\epsilon$, and it holds for all closed hypersurfaces $F$ with $N^*(N^*F) \cap WF(\phi) = \emptyset$ that $\phi_\epsilon(F) = 0$. Since $WF(\phi)$ is assumed small, we conclude by continuity that $\phi_\epsilon(F) = 0$ for all closed hypersurfaces $F$. Then by the smooth case, $\phi_\epsilon = 0$, thus also $\phi = 0$. \qed

### 3.2. Translation invariant valuations

We will need the following standard facts when we study valuations in the dual Heisenberg algebra in section 5.

The space of smooth translation invariant valuations $\text{Val}^\infty(\mathbb{R}^n)$ can be defined as $\mathcal{V}^\infty(\mathbb{R}^n)^{tr}$. It is a Fréchet space. We will also consider generalized translation-invariant valuations, $\text{Val}^{-\infty}(\mathbb{R}^n) := \mathcal{V}^{-\infty}(\mathbb{R}^n)^{tr}$.

The even/odd valuations are $\text{Val}^{\pm\infty}(\mathbb{R}^n)$ which have eigenvalue $±1$ under the antipodal map. The $k$-homogeneous valuations are those $\phi \in \text{Val}^{\pm\infty}$ that have eigenvalue $\lambda^k$ under all rescalings of $\mathbb{R}^n$ by $\lambda > 0$.

It is well-known that $\text{Val}_0^{\pm\infty}(\mathbb{R}^n) = \text{Span}\{\chi\}$, and by Hadwiger’s theorem [Had57], $\text{Val}^{\pm\infty}(\mathbb{R}^n) = \text{Span}\{\text{vol}_n\}$. It follows from McMullen’s work [McM77] that $\text{Val}^{\infty}(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \text{Val}^{\infty}(\mathbb{R}^n)$. 

The line bundle over $\text{Gr}_k(\mathbb{R}^n)$ whose fiber over $E$ is $\text{Dens}(E)$ is called the Klain bundle. The Klain map $\text{Kl} : \text{Val}^+_{\infty} \to \Gamma^\infty(\text{Gr}_k(V), \text{Dens}(E))$ is given by $\text{Kl}(\phi)(E) = \phi|_E$. It is well-defined by Hadwiger’s theorem, and injective by a theorem of Klain [Kla00]. By [AF14a], it admits an injective extension $\text{Kl} : \text{Val}^+_{\infty} \to \Gamma^\infty(\text{Gr}_k(V), \text{Dens}(E))$.

Dual to the Klain section is the Crofton map, given by $\text{Cr} : \mathcal{M}^\infty(\text{AGr}_n-k(V))^{tr} \to \text{Val}^+_{\infty}(V)$, given by $\text{Cr}(\mu)(K) = \int_{\text{AGr}_n-k(V)} \chi(E \cap K) d\mu(E)$. It follows from Alesker’s irreducibility theorem [Ale01] that $\text{Cr}$ is surjective. It was shown in [AF14a] that it admits a surjective extension $\text{Cr} : \mathcal{M}^-_{\infty}(\text{AGr}_n-k(V))^{tr} \to \text{Val}^+_{-\infty}(V)$.

The relation to valuations on manifolds is as follows:

**Theorem 3.13** ([Ale00]). \(\mathcal{W}^\infty_k(M)/\mathcal{W}^\infty_{k+1}(M)\) is naturally isomorphic to \(\Gamma^\infty(M, \text{Val}^\infty_k(TM))\), the space of smooth sections of the bundle with fiber \(\text{Val}^\infty_k(T_xM)\) over \(x \in M\).

Consequently by the Alesker-Poincaré duality we have

\[
\mathcal{W}^\infty_k(M)/\mathcal{W}^\infty_{k+1}(M) \cong (\mathcal{W}^\infty_{-n-k}(M)/\mathcal{W}^\infty_{-n-k+1}(M))^\ast \cong \Gamma^\infty(M, \text{Val}^\infty_{-n-k}(TM))^\ast.
\]

3.3. **Crofton formulas on manifolds.** Consider a double fibration

\[
\begin{array}{ccc}
W & \xrightarrow{\tau} & X \\
\downarrow{\pi} & & \downarrow{\pi} \\
Z & &
\end{array}
\]

where \(\pi, \tau\) are proper, and \(\tau \times \pi : W \to Z \times X\) is a closed embedding. We view \(Z\) as a space parameterizing a family of submanifolds \(z \subset X\), of dimension \(\text{dim } z = \text{dim } W - \text{dim } Z\).

Given a distribution \(\mu \in \mathcal{M}^-_{\infty}(Z)\), define the generalized valuation $\text{Cr}(\mu) \in \mathcal{V}^-_{\infty}(X)$ by $\text{Cr}(\mu) = \pi \ast \tau^\ast \mu$, which is the Radon transform with respect to the Euler characteristic of \(\mu\) in the terminology of Alesker [Ale10]. By [Ale10] Theorem 4.1.4, $\text{Cr}(\mu)$ is well-defined, and is a smooth valuation when \(\mu\) is smooth. Equivalently,

\[
\text{Cr}(\mu) = \int_Z \chi_z d\mu(z)
\]

**Lemma 3.14.** Let \(X\) be a manifold, assume \(\phi = \text{Cr}(\mu) \in \mathcal{V}^-_{\infty}(X)\) for \(\mu \in \mathcal{M}^-_{\infty}(Z)\). Then \(\phi \in \mathcal{W}^\infty_{\text{dim } X - \text{dim } W + \text{dim } Z}(X)\), and \(\sigma \phi = (-1)^{\text{dim } X - \text{dim } W + \text{dim } Z} \phi\), where \(\sigma\) is the Euler-Verdier involution.

**Proof.** We use the Alesker-Poincaré duality: for every smooth, compactly supported \(\Psi \in \mathcal{W}^\infty_{\text{dim } W - \text{dim } Z + 1}(X)\), it holds that \(\langle \Psi, \text{Cr}(\mu) \rangle = \int_Z \Psi(z) d\mu(z) = 0\). Now Proposition 7.3.2 in [Ale07] implies the first statement. For the second statement, simply note that the defining current of \(\phi\) is given by \(\int_Z [N^* z] d\mu(z)\).

4. **Dual Heisenberg manifolds**

4.1. **Geometric preliminaries.** The dual Heisenberg algebra $U^{2n+1}$ is a \((2n+1)\)-dimensional linear space with a distinguished hyperplane $H \subset U$, and $\omega \in \Lambda^2 H^* \otimes U/H$ is a non-degenerate (twisted) form. Let us justify this terminology.

**Lemma 4.1.** The space $U^*$ is naturally the Heisenberg Lie algebra.
Proof. Denote $Z = (U/H)^* = H^\perp \subset U^*$. There is a $Z$-valued symplectic form on the quotient $U^*/Z = H^*$: we may consider $\omega$ as the isomorphism $H \simeq H^* \otimes (U/H)$, or equivalently $H^* \simeq H \otimes (U/H)^*$, which is a non-degenerate bilinear form $\omega^* \in (H \otimes H) \otimes (U/H)^*$, which is clearly antisymmetric, that is $\omega^* \in \wedge^2(U^*/Z)^* \otimes Z$. The Lie bracket $[x, y] := \omega^*(x + Z, y + Z)$ then defines on $U^*$ a Heisenberg Lie algebra structure with center $Z$. □

Definition 4.2. A manifold $X$ equipped with a hyperplane distribution $H$ and a smooth field of nowhere-degenerate forms $\omega \in \Gamma^\infty(X, \wedge^2 H^*_x \otimes (T_xX/H_x))$ will be called a dual Heisenberg manifold. We call $H$ the horizontal distribution and $\omega$ the DH form.

Remark 4.3. Ovsienko [Ovs06] defines the closely related notion of a local Heisenberg structure. Similar ideas lie in the foundation of Stein’s Heisenberg calculus, see e.g. Ponge [Pon06, Pon08]. It appears that this structure is sometimes called “almost contact”, although the latter term could also refer to slightly different notions.

Let $(X, H_X, \omega_X)$ be a DH manifold. A DH-submanifold is a submanifold $Y^{2k+1} \subset X^{2n+1}$ which is nowhere tangent to $H_X$. It inherits the structure of a dual Heisenberg manifold with hyperplane distribution $H_Y = H_X \cap TY$, and DH form $\omega_Y = \omega_X|_{H_Y}$, where we naturally identify $TX/H_X = TY/H_Y$.

Example 4.4. A contact manifold is naturally equipped with a dual Heisenberg structure. Let $(X, H)$ be contact, and let $\alpha$ be a contact form defined locally. Define the 2-form $\omega(u, v) = \alpha^{-1}(d\alpha(u, v))$ for $u, v \in H_x$ and $\alpha : T_xX/H_x \to \mathbb{R}$ an isomorphism. It is then independent of the choice of $\alpha$.

Definition 4.5. Let $U^{2n+1}$ be the dual Heisenberg Lie algebra, with horizontal hyperplane $H \subset U$ and DH form $\omega \in \wedge^2 H^* \otimes U/H$. A Euclidean structure $P$ on $U$ is said to be compatible with the DH structure if one can find an orthonormal basis $X_j, Y_j, Z$ of $U$ such that $X_j, Y_j$ form a symplectic basis of $H$ w.r.t. $Z$: $\omega(X_i, Y_j) = 0$, $\omega(X_i, Y_j) = Z + H \in U/H$ for all $1 \leq i, j \leq n$ and $i \neq j$. A compatible form $\alpha \in U^*$ is any form such that $\text{Ker} \alpha = H$. $P$ is compatible with $\alpha$ if $\alpha = P(Z, \cdot)$.

Definition 4.6. Let $M$ be a dual Heisenberg manifold $M^{2n+1}$ with horizontal distribution $H \subset TM$ and DH forms $\omega \in \wedge^2 H^* \otimes TM/H$. A Riemannian metric $g$ on $M$ is compatible if $g_x$ is compatible for $(T_xM, H_x, \omega_x)$, for every $x \in M$.

Remark 4.7. 1. A similar notion of adapted metric was used by Chern and Hamilton in [CHS3] for contact 3-manifolds and in later works by other authors.

2. In fact, for the purposes of this work we only need the metric we choose to be compatible with the contact structure at isolated points of interest.

Lemma 4.8. For any dual Heisenberg manifold $M$, a globally defined compatible Riemannian metric exists. Moreover, if a compatible form $\alpha \in \Omega^1$ is given, one may choose the metric to be compatible with $\alpha$.

Proof. Cover $M$ by contractible open subsets $U_i$, and choose 1-forms $\alpha_i$ on $U_i$ such that $\alpha_i|_{U_j \cap U_i} = \pm \alpha_j|_{U_i \cap U_j}$ - this can be accomplished by fixing an arbitrary Riemannian metric and letting $\alpha_i(x) \in T^*_xM$ have unit norm for all $x \in U_i$. Set $\omega_i = \alpha_i \circ \omega \in \Gamma(U_i, \wedge^2 T^* M)$. Let us also fix a complementing line bundle $L$ to $H$: $L \oplus H = TM$. 

For each $U_i$ we may choose a compatible complex structure $J_i$ for $\omega_i$ on $H|U_i$ such that $J_i|U_i \cap U_j = \pm J_j|U_i \cap U_j$. We thus obtain the corresponding Euclidean forms $h_i$ on $H|U_i$ which will satisfy $h_i = h_j$, and so can be patched to give a globally defined positive definite quadratic form $h$ on $H$. Now for every $U_i$, choose $Z_i \in \Gamma(U_i, L)$ satisfying $\alpha_i(Z_i) = 1$, and define the Riemannian metric $g_i$ over $U_i$ by $g_i|_H = h$, $g_i(L, H) = 0$, $g_i(Z_i, Z_i) = 1$. Then clearly $g_i = g_j$ on $U_i \cap U_j$ and thus $(g_i)$ define a Riemannian metric $g$ on $M$. Over $U_i$ we may locally choose an orthonormal symplectic basis $X_i^0, Y_i^0$ for $\omega_i$, and then $\omega(X_i^0, Y_i^0) = \alpha^{-1}_i(1) = Z_i$ for all $j$. This shows $g$ is a compatible metric.

Finally, if $\alpha$ is given, we just take $\alpha_i = \alpha|U_i$ in the construction above. \qed

Let $M^{2n+1}$ be a DH manifold with horizontal distribution $H$ and DH form $\omega$. Set $M_H := \{(x, \xi) : x \in M, \xi \perp H_x \} \subset P_M (= P_+(T^*M))$.

**Definition 4.9.** A $C^2$ boundaryless submanifold $F \subset M$ is said to be normally transversal (to $H$) if $N^*F$ intersects $M_H$ transversally.

More generally, a $C^2$ submanifold with boundary $F \subset M$ is normally transversal if $T_xF \not\subset H_x$ for all $x \in \partial F$, and both $N^*(\text{int} F)$ and $N^*\partial F$ intersect $M_H$ transversally.

Note that if $F$ is normally transversal then so is $\partial F$.

**Lemma 4.10.** If a submanifold with boundary $F \subset M$ intersects $H$ normally transversally, then $\text{WF}([\{N^*F\}]) \cap N^*M_H = \emptyset$.

**Proof.** For a boundaryless submanifold this is immediate: $N^*F$ is a boundaryless submanifold, and $\text{WF}([\{N^*F\}]) = N^*(N^*F)$.

Now assume $F$ has boundary. Consider the sets $N^*_{\partial F}F = \bigcup_{x \in \partial F} N^*F$ and $N^*_{\text{int} F}F := \bigcup_{x \in F} \{\xi \in P_+(T^*_xM) : \xi \in (T_xF)^\perp\}$. Both are manifold with the same boundary $B = \{(x, \xi) : x \in \partial F, \xi \in (T_xF)^\perp\}$, and $N^*F = N^*_{\partial F}F \cup N^*_{\text{int} F}F$. Also, $N^*_{\partial F}F \subset N^*\partial F$ is a submanifold with the same boundary $B$.

By assumption, $M_H$ intersects the interior points of $N^*_{\partial F}F$ and $N^*_{\text{int} F}F$ transversally, and does not intersect $B$. It also holds for $(x, \xi) \notin B$ that $\text{WF}_{x,\xi}([\{N^*F\}])$ coincides with $N^*_{x,\xi}(N^*_{\partial F}F)$ if $x \in \partial F$, and with $N^*_{x,\xi}(N^*_{\text{int} F})$ if $x \in \text{int} F$. This concludes the proof. \qed

Normal transversality is of course generic:

**Lemma 4.11.** Any hypersurface with boundary $F \subset M$ can be perturbed by an arbitrarily $C^\infty$-small amount to become normally transversal.

**Proof.** First, we may perturb $F$ so that it is tangent to the contact distribution at isolated points. Now near interior contact points, we only need to perturb $F$ locally near those points to get normal transversality. Next, we may perturb $F$ near the boundary to have no contact points of $F$ near the boundary, and isolated contact points of $\partial F$. We then perturb $F$ near those contact points s.t. $\partial F$ has normal transversality at all its contact points. \qed

Normal transversality is universal to embeddings, as the following Lemma shows.

**Lemma 4.12.** Let $(Y, H^Y) \subset (X, H^X)$ be DH manifolds, and $F \subset Y$ a normally transversal submanifold with boundary. Then $F \subset X$ is also normally transversal.
Proof. We will distinguish the conormal bundles of manifolds by writing e.g. \( N^*_X F \subset \mathbb{P}_Y \). Recall that \( H^*_Y = T_x X \cap H_x^X \). Thus for \( x \in \partial F \), \( T_x F \not\subset H_x^Y \), since \( T_x F \subset H_x^Y \).

Next take \((x, \xi) \in N^*_X F\). For \( x \in \text{int} F \) we should check \( T_{x, \xi} N^*_X F \cap T_{x, \xi} X_{H^X} = \{0\} \). Consider the natural surjective map \( \beta : \mathbb{P}_X \mid Y \setminus N^* X \rightarrow \mathbb{P}_Y \). It holds that \( \beta(N^*_X F) = N^*_Y F \). Denote \( X_{H^X} \mid Y := X_{H^X} \cap \pi^X \cdot (Y) \) where \( \pi_X : \mathbb{P}_X \rightarrow X \). Since \( X \subset Y \) is a DH submanifold, it holds that \( \beta : X_{H^X} \mid Y \rightarrow Y_{H^Y} \) is a diffeomorphism. If \( v \in T_{x, \xi} N^*_X F \cap T_{x, \xi} X_{H^X} \mid Y \) then \( d\beta(v) \in T_{x, \xi} Y_{H^Y} \cap T_{x, \xi} N^*_Y F \) is non-zero, a contradiction to the assumption of normal transversality of \( F \subset M \). The case of \( x \in \partial F \) is virtually identical.

\[ \square \]

4.2. Constructing the valuations. Consider a point \( q = (p, \xi) \in M_H \). The vertical subspace is \( \text{Ker}(d\pi) = \xi^* \otimes T^*_p M / \xi = (H^*_p)^* \otimes H^*_p \). Recall that \( \omega_p \in \wedge^2 H^*_p \otimes T_x M / H_x \) is non-degenerate, so that \( H^*_p \simeq H_p \otimes H^*_p \). Thus \( \text{Ker}(d\pi) \simeq H_p \).

Note also that \( d\pi : T_q M_H \rightarrow T_p M \) is an isomorphism, so that we have a natural decomposition

\[ T_q \mathbb{P}_M \simeq T_p M \oplus H_p \]

Denoting the contact hyperplane of \( \mathbb{P}_M \) by \( \tilde{H}_q \), we get

\[ \tilde{H}_q \simeq H_p \oplus H_p. \]

We will define certain \( \eta_k \in \Gamma^\infty(M_H, \Omega^{2n+1} \otimes \pi^* o_M), 1 \leq k \leq 2n \) (twisted by the pull-back of the orientation bundle of \( M \)) supported on \( M_H \).

Take \( \tilde{\psi} \in \Omega^\infty(M_H, \mathbb{P}_M) \) a Legendrian form. We have

\[ \tilde{\psi} \mid q \in \wedge^{2n+1} T^*_q (\mathbb{P}_M) \simeq \bigoplus_{k=0}^{2n} \wedge^{2n+1-k} T^*_p M \otimes \wedge^k H^*_p \]

Moreover, since \( \tilde{\psi} \) is Legendrian, its first factor in the \( k \)-th component belongs to the kernel of the restriction map \( \wedge^{2n+1-k} T^*_p M \rightarrow \wedge^{2n+1-k} H^*_p \), which is naturally isomorphic to \( H^*_p \otimes \wedge^{2n-k} H^*_p \). Thus the \( k \)-th component of \( \tilde{\psi} \), denoted \( \tilde{\psi}^{2n+1-k,k} \), lies in \( H^*_p \otimes \wedge^{2n-k} H^*_p \).

Wedgeing the last two factor, we then get an element \( \langle \tilde{\psi} \rangle_k \in \wedge^{2n} H^*_p \otimes (T_p M / H_p)^* \simeq \wedge^{2n+1} T^*_p M \simeq \wedge^{2n+1} T^*_p M \). Now set

\[ \langle \bar{\eta}_k, \tilde{\psi} \otimes \epsilon_M \rangle := \int_{M_H} (\tilde{\psi})_k \otimes \epsilon_M \]

where \( \epsilon_M \) is a section of \( \pi^* o_M \), so that \( \bar{\eta}_k \otimes \epsilon_M \in \text{Dens}(T_q M_H) \). Thus \( \bar{\eta}_k \) is a linear functional on compactly supported, Legendrian, \( \pi^* o_M \)-twisted \((2n + 1)\)-forms, supported on \( M_H \). For \( k = 0 \), \( \bar{\eta}_0 = ([M_H]) \) is in fact a well-defined (twisted) generalized \( 2n \)-form.

Finally for \( 1 \leq k \leq 2n \), define \( \eta_k \in \Omega^{2n+1} (\mathbb{P}_M) \otimes \pi^* o_M \) by setting for \( \psi \in \Omega^\infty(\mathbb{P}_M) \)

\[ \langle \eta_k, \psi \otimes \epsilon_M \rangle := \langle \bar{\eta}_k, D\psi \otimes \epsilon_M \rangle \]

where \( D \) denotes the Rumin differential. It is clear that \( \eta_k \) is a (twisted) Legendrian cycle, and \( \pi_* \eta_k = 0 \).
Definition 4.13. Define \( \phi_k \in \mathcal{V}^{-\infty}(M) \) for \( 0 \leq k \leq 2n \) as follows: for \( 1 \leq k \leq 2n \) by \( \phi_k = [\frac{1}{2} \mathcal{H}_k, 0] \); for \( k = 0 \), \( \phi_0 \) is represented by \( (\frac{1}{2} \mathcal{H}_0, 0) \).

Note that \( \phi_0(X) \) is one-half the intersection index of \( M_H \) and \( N^*X \), both oriented locally by a fixed local orientation on \( M \). In the following proofs, we often assume \( M \) is oriented. They are easily adjusted for the general case.

Lemma 4.14. \( \phi_0 \) is the Euler characteristic.

Proof. Note that \( \pi_*[M_H] \) is a closed current, and denoting \( \pi : \mathbb{P}_M \to M \) clearly \( \pi_*(\pi_*[M_H]) = 2 \). It follows that \( \phi_0 = \chi \) by an obvious extension to \( \mathcal{V}^{-\infty} \) of [BB07 Corollary 1.5 ].

Recall that \( a : \mathbb{P}_M \to \mathbb{P}_M \) is the fiberwise antipodal map, and \( \sigma : \mathcal{V}^{-\infty}(M) \to \mathcal{V}^{-\infty}(M) \) denotes the Euler-Verdier involution.

Proposition 4.15. For all \( 0 \leq k \leq 2n \) it holds that \( \sigma \phi_k = \phi_k \).

Proof. For \( \psi \in \Omega^{2n+1}(\mathbb{P}_M) \) it holds that \( \int_{M_H} \psi = \int_{M_H} a^* \psi \), since \( a : M_H \to M_H \) is clearly orientation preserving. Not also that \( a : \mathbb{P}_M \to \mathbb{P}_M \) is orientation reversing. Hence for a test form \( \psi \in \Omega^{2n+1}_c(\mathbb{P}_M) \),

\[
\langle a^* \eta_k, \psi \rangle = -\langle \eta_k, a^* \psi \rangle = - \int_{M_H} [Da^* \psi]_k = \int_{M_H} a^* [D\psi]_k = -\int_{M_H} [D\psi]_k = -\langle \eta_k, \psi \rangle
\]

That is, \( a^* \eta_k = -\eta_k \) for all \( k \), and hence \( \sigma \phi_k = (-1)^{2n+1}(-\phi_k) = \phi_k \) as asserted.

Proposition 4.16. It holds that \( \phi_k \in \mathcal{W}^{-\infty}_k(M) \).

Proof. Take a closed Legendrian form \( \psi \in \Omega^{2n+1}(\mathbb{P}_M) \) defining a smooth valuation \( \Psi \in \mathcal{W}^{\infty}_k(M) \). We ought to show that \( \phi_k \cdot \Psi = 0 \), equivalently \( \langle \tilde{\eta}_k, \psi \rangle = 0 \). But \( \Psi \in \mathcal{W}^{\infty}_{2n+2-k}(M) \) implies that \( \psi \) has a horizontal degree at least \( 2n+2-k \), and the claim follows from the definition of \( \tilde{\eta}_k \).

Lemma 4.17. The wavefront set of \( \phi_k \) for \( 1 \leq k \leq 2n \) is \( N^*M_H \).

Proof. This is straightforward: the wavefront set of \( \pi_*[M_H] \) is \( N^*M_H \), and restriction to \( M_H \) is the only source of singularities of \( \eta_k \).

It follows by Lemma 4.10 that we may evaluate \( \phi_k \) on any normally transversal submanifold with boundary.

Let \( F \subset M \) be a smooth hypersurface in a DH-manifold \( M^{2n+1} \), assume that \( F \) is normally transversal to the horizontal distribution, and tangent to it at \( p \in M \).

Working in a small open ball \( U \) near \( p \) with no other contact points, let us fix a 1-form \( \alpha \in \Omega^1(U) \) defining the horizontal distribution, which also trivializes the DH form: \( \omega \in \Gamma(U, \wedge^2 H^*) \). Let \( g \) be a Riemannian metric on \( U \) which is compatible with \( \alpha \) and \( \omega \) on \( T_p M \) (but not necessarily elsewhere), with Levi-Civita connection \( \nabla \). Let \( R : U \to SM \) be the vector field given by \( R(x) \perp H_x \), \( g(R, R) = 1 \), \( \alpha(R) > 0 \). Let \( (X_j)_{j=0}^{2n} \) be an orthonormal frame in \( U \) with \( X_0|_F = \nu : F \cap U \to SM \) the unit normal oriented by \( \alpha(\nu) > 0 \), and \( (X_j(p))_{j=1}^{2n-1} \) a symplectic basis of \( H_p \).

Let \( \theta_j \in \Omega^1(U) \) be the dual coframe to \( X_j \). Let \( S = (s_{ij})_{i,j=1}^{2n}, s_{ij} = \theta_i(\nabla X_j, \nu) \) be the second fundamental form given by \( \nabla X_i, \nu = \sum_{i=0}^{2n} s_{ij} X_i \).

Definition 4.18. Let \( h_{ij}, 1 \leq i \leq 2n, 0 \leq j \leq 2n \) be given by \( h_{ij} = \theta_i(\nabla X_j, R) \). The matrix \( h = (h_{ij})_{i,j=1}^{2n} \) is the second fundamental form of the contact structure.
Remark 4.19. This is a slightly different definition than the one in [Rei77], where

the second fundamental form is symmetrized.

Define the matrix \( A_p = (h_{ij} - s_{ij})^{2n}_{i,j=1} \).

Proposition 4.20. Let \( F \) be a normally transversal closed hypersurface. It then holds that

\[
\phi_k(F) = \binom{2n}{k} \sum_{p: T_pF = H_p} \left| \det A_p \right|^{-1} D(A_p[2n - k], J[k])
\]

where \( D \) is the mixed discriminant.

Remark 4.21. We thus see that geometrically, \( \phi_k \) is reminiscent of the \( k \)-th elementary symmetric polynomial in the principal radii of a hypersurface in a Riemannian manifold. In particular, \( \phi_{2n}^{-1} \) plays the role of the absolute value of the gaussian curvature. In fact, for \( M = U \) - the dual Heisenberg algebra, \( \phi_{2n}(F) \) is precisely the inverse absolute value of the gaussian curvature, summed over all contact points.

Proof. Write \( \omega_i \) for the Ehresmann connection on \( SU \), namely \( \omega_i = \pi^*_v \theta_i \), where \( \pi_V: T_pSU \to T_pM \) is the projection to the vertical tangent space. It then holds that \( R^*\omega_i = \sum_{j=0}^{2n} h_{ij} \tau_j \), and \( \nu^*\omega_i = \sum_{j=1}^{2n} s_{ij} \tau_j \). Recall that \( (s_{ij}) \) is symmetric. For notational simplicity, we write \( \theta_j \) also for \( \pi^* \theta_j \in \Omega^1(SU) \).

First we describe the normal cycle \( NF = \{(x, \pm \nu(x)) \in \Omega \} \subset SM \) near \((p, \xi) = X_0(p) \in MH\), explicitly as a generalized form \( \omega_{NF} := [NF] \in \Omega_{-\infty}(SM) \). Let \( W \subset \pi^{-1}U \) be a small neighborhood of \((p, \xi) \). Write \( \sigma_{SM} := \sum_{i=0}^{2n} \theta_i \wedge \omega_i \), and \( \theta_F = \wedge_{i=1}^{2n} \theta_i \in \Omega_{2n}(F \cap U) \).

Claim. \( \omega_{NF}W := \omega_{NF}|W \in \Omega_{-\infty}(W) \) is given by

\[
\omega_{NF} = \theta_0 \wedge \sum_{i=1}^{2n} (\omega_i - \sum_{j=1}^{2n} s_{ij} \theta_j) \delta_{NF}^{\nu}
\]

where

\[
(\delta_{NF}^{\nu} \mu(x, \xi) \sigma_{SM}) = \int_{F \cap \sigma(W)} \mu(x, \xi) \theta_F
\]

Proof of Claim. Take \( \psi \in \Omega^{2n}_{\nu}(W) \). We should check that

\[
\int_{NF} \psi = \int_{F} \nu^* \left( \frac{\psi \wedge \theta_0 \wedge \sum_{i=1}^{2n} (\omega_i - \sum_{j=1}^{2n} s_{ij} \theta_j)}{\sigma_{SM}} \right) \theta_F
\]

which reduces to the pointwise verification

\[
\frac{\nu^* \psi}{\theta_F} = \nu^* \left( \frac{\theta_0 \wedge \psi \wedge \sum_{i=1}^{2n} (\omega_i - \sum_{j=1}^{2n} s_{ij} \theta_j)}{\sigma_{SM}} \right)
\]

We will check this equality for a basis of the \( 2n \) forms, which is given by \( 2n \)-wedges of \((\theta_i)^{2n}_{i=0}, (\omega_j)^{2n}_{j=1} \). If \( \psi \) contains a \( \theta_0 \) factor, clearly both sides vanish.

Assume \( \psi = \theta_1 \wedge \cdots \wedge \theta_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{2n} \). The left hand side is easily seen to be equal \( \det (s_{ij})^{2n}_{i,j=k+1} \). The right hand side is

\[
\nu^* \left( \frac{(-1)^{2n-k} \det (s_{ij})^{2n}_{i,j=k+1} \theta_0 \wedge \cdots \wedge \theta_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{2n} \wedge \omega_1 \wedge \cdots \wedge \omega_k \wedge \theta_{k+1} \wedge \cdots \wedge \theta_{2n}}{\sigma_{SM}} \right)
\]
and after reordering we get \((-1)^{2n-k+k(2n-k)} \det(s_{ij})^{2n}_{i,j=k+1} = \det(s_{ij})^{2n}_{i,j=k+1},\) concluding the proof of the claim.

We wish to apply \(\tilde{\eta}_k\) to \(\omega^{W}_{NF}\). Since \(NF\) intersects \(M_H\) transversally at the isolated point \(p\), we need only look at the value of \(\omega^{W}_{NF}\) at \((p, \pm X_0(p))\).

Recall the decomposition \(T_{p,\xi}P_M = T_{p,\xi}M_H \oplus T_{\xi}(P_+T_pM)\). We denote by \(W_{p,\xi} = T_{p,\xi}M_H = T_pM\) the contact-horizontal subspace, and by \(V_{p,\xi} = T_{\xi}(P_+T_pM) = (H_p \otimes H_p^\perp)^*\) the vertical subspace. Using the Riemannian structure, we get a decomposition \(T_{p,\xi}(SM) = W_{p,\xi} \oplus V_{p,\xi}\). The DH form \(\omega \in \wedge^2 H_p^* \otimes T_pM/H_p\) can be recast as the isomorphism \(\Omega : H_p \otimes H_p^\perp \to H_p^*,\) that is \(\Omega : V_{p,\xi} \to H_p^*\).

We first need to identify the \((2n + 1 - k, k)\) component (with respect to decomposition \((\text{6})\)) of \(\omega^{W}_{NF}\) over \((p, \xi) \in M_H\). The forms \(\theta_i\) vanish on the vertical subspace. The contact-horizontal component of \(\omega_i\) is given by \(R^*\omega_i = \sum_{j=0}^{2n} h_{ij}\theta_j\).

We will write the resulting decomposition as \(\omega_i = R^*\omega_i \oplus \omega_i^\perp\) where \(R^*\omega_i\) is the contact-horizontal component, while \(\omega_i^\perp\), the restriction of \(\omega_i\) to \(V_{p,\xi}\), is the vertical component.

Define \((\gamma_{ij})^{2n}_{i,j=1}\) by \(\Omega(\omega_i^\perp) = \sum_{j=1}^{2n} \gamma_{ij}\theta_j \in H_p^*\). Note that under the Euclidean identification, \(V_{p,\xi} = H_p\) and \(\omega_i^\perp = \theta_i\) for \(i = 1, \ldots, 2n\). Compute \(\gamma_{ij} = \Omega(\theta_j)(X_i) = \omega_i(X_i, X_j)\). By assumption, \(X_j\) is a symplectic basis of \(H_p\), so that \(\gamma_{ij}\) is the standard \(2n \times 2n\) matrix \(J\) representing \(\sqrt{-1}\).

Write \(a_{ij} = h_{ij} - s_{ij}, 1 \leq i, j \leq 2n\). Then

\[
\omega^{W}_{NF} = \delta^{W}_{NF} \cdot \theta_0 \wedge \wedge_{i=1}^{2n} (\omega_i^\perp + \sum_{j=1}^{2n} a_{ij}\theta_j)
\]

so that

\[
(\omega^{W}_{NF})^{2n+1-k,k} = \delta^{W}_{NF} \cdot \theta_0 \wedge \bigwedge_{|I|=k} \epsilon_I \wedge_i \bigwedge_{|J|=k} \left(\sum_{j=1}^{2n} a_{ij}\theta_j\right) \wedge \wedge_{i \in I} \omega_i^\perp
\]

where \(\epsilon_I = (-1)^{\sum_i \sigma_{i} + \sum_j \sigma_{j} - \sum_k \sigma_{k}}\) is the sign of the permutation \(\sigma_I = (1^I, 2^I)\), where \(I = \{i_1 < \cdots < i_k\}, I^c = \{1, \ldots, 2n\} \setminus I\) is ordered increasingly, and \(j = \lfloor \frac{k}{2} \rfloor\).

Applying \(\Omega\) to the last \(k\) factors and subsequently wedging all the factors to get a top form on \(T_pM\), we get

\[
\langle \tilde{\eta}_k, \omega^{W}_{NF} \rangle = \int_{M_H} \theta_0 \wedge \bigwedge_{|I|=k} \epsilon_I \wedge_i \bigwedge_{|J|=k} \left(\sum_{j=1}^{2n} a_{ij}\theta_j\right) \wedge \wedge_{i \in I} \theta_0 \wedge \cdots \wedge \theta_{2n} \cdot \delta^{W}_{NF}
\]

\[
= \sum_{|I|=k} \sum_{\sigma \in S_{2n}} \text{sign } \prod_{i \in I} a_{i\sigma(i)} \prod_{i \in I} \gamma_{i\sigma(i)} \int_{M_H} \theta_0 \wedge \cdots \wedge \theta_{2n} \cdot \delta^{W}_{NF}
\]

Recall \(A_p = (a_{ij}) = h - S\) where \(S = (s_{ij}), h = (h_{ij}), 1 \leq i, j \leq 2n\), and let \(B^\alpha\) be the \(\alpha\)-row of a matrix \(B\). For an ordered subset \(I \subset \{1, \ldots, 2n\}\), let \((A_p^I, J^I)\) denote the matrix with the corresponding columns. Then

\[
\langle \tilde{\eta}_k, \omega^{W}_{NF} \rangle = \sum_{|I|=k} \det(A_p^I, J^I) \int_{M_H} \theta_0 \wedge \cdots \wedge \theta_{2n} \cdot \delta^{W}_{NF}
\]
Define 
\[ B(\alpha) = \begin{cases} 
A_p, & \alpha \leq 2n - k \\
J, & \alpha > 2n - k
\end{cases} \]

The mixed discriminant is given by
\[ D(A_p[2n-k], J[k]) = \frac{1}{(2n)!} \sum_{\tau \in S_2n} \det(B(\tau)(i,j)) = \frac{1}{(2n)!} k!(2n-k)! \det(A_p^T, J^T) \]
that is
\[ \langle \tilde{\eta}_k, \omega_{NF}^W \rangle = \binom{2n}{k} D(A_p[2n-k], J[k]) \int_{M_H} \theta_0 \wedge \cdots \wedge \theta_{2n} \cdot \delta_{NF}^W \otimes \epsilon_\theta = |\det A_p|^{-1}. \]

Recall that we should fix a section of the orientation bundle of \( M \) over \( SM \) to get numerical values for the integral. Let us verify that for a choice of \( \epsilon_\theta \in \pi^*OM \) corresponding to \( \theta_0 \wedge \cdots \wedge \theta_{2n} \) we get the identity
\[ \int_{M_H} \theta_0 \wedge \cdots \wedge \theta_{2n} \cdot \delta_{NF}^W \otimes \epsilon_\theta = |\det A_p|^{-1}. \]

One can compute it directly, but we can do something simpler: Observe that \( \langle \tilde{\eta}_0, \omega_{NF}^W \rangle \) is just the intersection index \( I_{p,\xi} \) of \( NF \) and \( M_H \) at \( (p, \xi) \). Note that the order of intersection is not important as \( \dim NF \) is even, while the orientations of \( M_H \) and \( NF \) are determined by \( \epsilon_\theta \).

By what we have proved, we see that
\[ I_{p,\xi}(NF, M_H) = \langle \tilde{\eta}_0, \omega_{NF}^W \rangle = \det A_p \int_{M_H} \theta_0 \wedge \cdots \wedge \theta_{2n} \delta_{NF}^W. \]

It thus remains to verify that \( I_{p,\xi}(NF, M_H) = \text{sign } \det A_p \).

The positive orientation on \( T_{p,\xi}SM \) is given by the dual basis \( \theta_0, \ldots, \theta_{2n}, \omega_1, \ldots, \omega_{2n} \). To see that, consider a homotopy of the Riemannian metric between our metric and a flat one, and some corresponding homotopy of the orthonormal frame \( X_i \). The dual basis above remains a basis throughout the homotopy, and clearly defines the positive orientation in the flat case.

Considering \( \nu : F \to SM, R : F \to SM \) as maps, we get a positive basis of \( T_{p,\xi}M_H \) given by \( D_pR(X_0), \ldots, D_pR(X_{2n}) \), and a positive basis of \( T_{p,\xi}NF \) given by \( D_p\nu(X_1), \ldots, D_p\nu(X_{2n}) \).

Form the \((4n + 1) \times (4n + 1)\) matrix
\[ B = \begin{pmatrix} 
\theta_i(D_pR(X_j))^j_{i=0 \ldots 2n} & \theta_i(D_p\nu(X_j))^j_{i=0 \ldots 2n} \\
\omega_i(D_pR(X_j))^j_{i=1 \ldots 2n} & \omega_i(D_p\nu(X_j))^j_{i=1 \ldots 2n}
\end{pmatrix} = \begin{pmatrix} 
\delta_{ij} & \delta_{ij} \\
0 & s_{ij}
\end{pmatrix} \]

By definition, \( I_{p,\xi}(NF, M_H) = \text{sign } \det B \). Now for \( 1 \leq i \leq 2n \) subtract column \( 2n + 1 + i \) from column \( i + 1 \). Finally, interchange each column of index \( 2 \ldots 2n + 1 \) with the respective column from the last \( 2n \) columns, resulting in \( 4n^2 \) swaps. The resulting matrix is block-triangular, and has determinant \( \det B = \det(h_{ij} - s_{ij})_{i=1 \ldots 2n} = \det A_p \), verifying our assertion.

Accounting also for the point \( \xi = H^+ \) with the opposite orientation, we conclude that
\[ \langle \tilde{\eta}_k, \omega_{NF} \rangle = 2\binom{2n}{k} \sum_{p; T_p F = H_p} |\det A_p|^{-1} D(A_p[2n-k], J[k]) \]
and the statement follows. □

Definition 4.22. For a normally transversal contact point \( p \) of \( F \), the local contact areas are

\[
\phi_k(F, p) := \binom{2n}{k} \det A_p^{-1} D(A_p[2n - k], J[k]).
\]

Corollary 4.23. When \( k = 2n \), equation (8) remains valid also for a normally transversal hypersurface with boundary \( F \).

Proof. By Lemma 4.10, \( \phi_{2n}(F) \) is well-defined. Using the notation of the proof of Lemma 4.10, we may write

\[
\phi_{2n}(F) = \frac{1}{2} \langle \tilde{\eta}_{2n}, [[N^*_{\text{int}} F]] \rangle + \frac{1}{2} \langle \tilde{\eta}_{2n}, [[\tilde{N} F]] \rangle.
\]

The first summand is computed in Proposition 4.20. The second summand trivially vanishes. □

Corollary 4.24. For a normally transversal hypersurface with boundary \( F \), \( \phi_{2n}(F) \geq 0 \), with equality if and only if it is nowhere tangent to \( H \).

Proof. Immediate from Corollary 4.23 and eq. (8). □

It will be convenient to extend \( \phi_{2n} \) to general hypersurfaces by setting \( \phi_{2n}(F) = \infty \) when \( F \) is not normally transversal.

Next we establish the universality of \( \phi_k \) with respect to embeddings.

Let \( (X^{2n+1}, H^X, \omega) \) be a DH-manifold, and \( Y^{2m+1} \subset X^{2n+1} \) a DH-submanifold. Let \( i : Y \to X \) denote the embedding. We will write \( \eta^Y_{2k}, \phi^Y_k \) for the generalized forms and valuations defined over \( X \), and similarly for \( Y \).

Theorem 4.25. It holds that \( i^* \phi^X_k = \phi^Y_k \).

Proof. The following constructions appeared in [Ale10], and we refer therein for more details. Consider the natural submersion \( \beta : \mathbb{P}_X |_{N^*Y} \to \mathbb{P}_Y \) and the inclusion \( \alpha : \mathbb{P}_X |_{N^*Y} \to \mathbb{P}_X \). Let \( \pi : W \to \mathbb{P}_X |_{N^*Y} \) be the oriented blow-up of \( \mathbb{P}_X |_{N^*Y} \) along the conormal bundle \( N^*Y \subset \mathbb{P}_X \), and \( \alpha : W \to \mathbb{P}_X \) the corresponding lift. Let \( \beta : W \to \mathbb{P}_Y \) be induced by the restriction map \( X \times_Y T^*Y \to T^*X \). For a valuation \( \Psi \in \mathcal{V}^\infty(Y) \) defined by the closed Legendrian form \( \psi \in \Omega^{2m+1}(\mathbb{P}_Y), i^* \Psi \) is given by the current \( \alpha_* \bar{\beta}^* \psi \). Now \( \langle \phi^X_k, \Psi \rangle = \langle \tilde{\eta}^X_k, \psi \rangle \) and similarly for \( X \). We should thus verify that

\[
\langle \tilde{\eta}^Y_k, \psi \rangle = \langle \tilde{\eta}^X_k, \alpha_* \bar{\beta}^* \psi \rangle.
\]

Note that \( \tilde{\eta}^X_k \) is supported on \( X_{H^X} \subset \mathbb{P}_X \), which by assumption is disjoint from \( N^*Y \). Hence the right hand side can be replaced by \( \langle \tilde{\eta}^X_k, \alpha_* \bar{\beta}^* \psi \rangle \).

Next take \( p \in Y, \xi = (H^p)^\perp \in \mathbb{P}_Y \), and \( (p, \xi') \in \mathbb{P}_X |_{N^*Y} \) s.t. \( \beta(p, \xi') = (p, \xi) \). We may decompose \( T_{p, \xi} \mathbb{P}_X |_{N^*Y} = T_p Y \oplus (H^X_p)^\perp \otimes (H^X_p)^\perp \). Consider \( d\beta : T_{p, \xi} \mathbb{P}_X |_{N^*Y} \to T_p \xi \mathbb{P}_Y \), so that

\[
d\beta^* : T_p Y \oplus H^Y_p \otimes (T_pY/H^Y_p)^* \to T_p Y \oplus H^X_p \oplus (T_pX/H^X_p)^*.\]

acts as the identity on the first summand. On the second summand, recalling we have the canonical identifiacation \( T_p Y/H^Y_p \simeq T_p X/H^X_p \) induced by the inclusion \( T_p Y \subset T_p X \), \( d\beta \) simply acts as the inclusion \( H^Y_p \hookrightarrow H^X_p \). Let us denote by \( \Omega^Y_k : \)
\[ \wedge^k H^Y_p \otimes (H^Y_p) \perp \to \wedge^k (H^Y_p)^* \] the isomorphism induced by the DH form \( \omega \), and similarly for \( X \). It follows that the following diagram commutes:

\[
\begin{array}{ccc}
H^Y_p \otimes (H^Y_p) \perp & \xrightarrow{d\beta^*} & H^X_p \otimes (H^X_p) \perp \\
\downarrow \Omega_Y & & \downarrow \Omega_X \\
(H^X_p)^* & \xleftarrow{\text{res}} & (H^X_p)^*
\end{array}
\]

We are left with verifying the identity

\[
\int_{Y_H} \psi_0 \otimes \epsilon_Y = \int_{X_H} \alpha_* \beta^* \psi_0 \otimes \epsilon_X
\]

for an arbitrary Legendrian form \( \psi_0 \in \Omega^{2n+1}(\mathbb{P}_Y) \). But this is now equivalent to the statement \( i^* \phi^X_0 = \phi^Y_0 \), which holds as both sides are just the Euler characteristic. \( \square \)

Weyl’s principle for DH manifolds, which we just established, is readily applicable in conjunction with the following technical lemma.

**Lemma 4.26.** Consider a compact submanifold with boundary \( F^{2k} \subset M^{2n+1} \) of a DH manifold \( (M, H, \omega) \).

i) If \( 2k \leq n \), \( F \) lies in fact inside a DH submanifold \( N^{2k+1} \subset M \).

ii) For arbitrary \( k < n \), one may find a pair of DH-manifolds \( N^{2k+1} \subset X^{4n+1} \) such that we get a commuting diagram of DH manifolds

\[
\begin{array}{ccc}
N^{2k+1} & \xrightarrow{\alpha} & M^{2n+1} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
F^{2k} & \rightarrow & X^{4n+1}
\end{array}
\]

where all inclusions are DH-embeddings.

**Proof.** Assume first \( 2k \leq n \). Choose a Riemannian metric \( g \) on \( M \), and let \( L : F \to \mathbb{P}(TM)|_F \) be given by \( L(x) = H^\perp_x \), the orthogonal complement with respect to \( g \). Note that \( L(F) \) does not intersect \( \mathbb{P}(H)|_F \subset \mathbb{P}(TM)|_F \). By the transversality theorem, and since

\[
\dim L(F) + \dim \mathbb{P}(TF) < \dim \mathbb{P}(TM)|_F \iff 2k + 4k - 1 < 2n + 2k \iff 2k \leq n,
\]

we may perturb \( g \) so that \( L(F) \) avoids \( \mathbb{P}(TF) \subset \mathbb{P}(TM) \). We then take \( N \) to be the image under the exponential map of a small neighborhood of the zero section in \( TF \oplus L(F) \subset TM|_F \). It is clearly a DH-submanifold containing \( F \).

Now in the general case, consider \( \tilde{M}_H \subset \mathbb{P}(T^*M) \), which is the quotient of \( M_H \subset \mathbb{P}_M \) under the two-covering map \( \mathbb{P}_M \to \mathbb{P}(T^*M) \). Define a DH structure on a neighborhood of \( \tilde{M}_H \) as follows: the horizontal structure \( \tilde{H} \) will be the canonical contact structure of \( \mathbb{P}(T^*M) \). For \( (x, \xi) \in \tilde{M}_H \), by eqns. \([9]\) and \([7]\), \( \tilde{H}_{x, \xi} \) is canonically identified with \( H_x \oplus H_x \subset T_x M \oplus H_x \). Noting that \( T_{x, \xi} \mathbb{P}(T^*M)/\tilde{H}_{x, \xi} \simeq T_x M/H_x \), define \( \omega_X \in \wedge^2 \tilde{H}_{x, \xi}^* \otimes T_{x, \xi} \mathbb{P}(T^*M)/\tilde{H}_{x, \xi} \) by

\[
\omega_X((u_1, v_1), (u_2, v_2)) = \omega(u_1, v_1) + \omega(u_2, v_2) \quad \forall (u_1, v_1), (u_2, v_2) \in \tilde{H}_{x, \xi}.
\]
Extend $\omega_X$ arbitrarily to a global section of $\wedge^2 \hat{H}^* \otimes T^\mathbb{P}(T^*M)/\hat{H}$. Take $X$ to be a small neighborhood of $\hat{M}_H$ in which $\omega_X$ is non-degenerate. Then $(X, \hat{H}, \omega_X)$ is a DH manifold, and the DH-submanifold $\hat{M}_H \subset X$ is clearly isomorphic to $M$. It remains to note that $\dim X = 2 \cdot 2n + 1$ and $2k < 2n$, and so we may choose the desired $N^{2k+1} \subset X$ by the first case. \hfill $\Box$

**Corollary 4.27.** Let $F^{2k} \subset M$ be a submanifold with boundary, normally transversal to the horizontal distribution $H$. Then $\phi_{2k}(F) \geq 0$, with equality if and only if it is nowhere tangent to $H$.

**Proof.** This follows at once from Lemmas 4.12 and 4.21, Corollary 4.24 and Theorem 4.25. \hfill $\Box$

Finally, we consider the relations between the valuations we constructed.

**Proposition 4.28.** (i) On any DH manifold, $\phi_{2b} \in W_{2b}^{-\infty} \setminus W_{2b+1}^{-\infty}$ as $0 \leq b \leq n$. In particular, $\phi_{2b}$, $0 \leq b \leq n$ are linearly independent valuations.

(ii) On a generic DH manifold (in a sense to be made precise in the proof), $(\phi_b)_{b=0}^{2n}$ are all linearly independent.

**Proof.** By Proposition 4.10, $\phi_k \in W_{b}^{-\infty}(M)$. For a normally transversal hypersurface $F$ write

$$
\phi_k(F,p) = |\det A_p|^{-1} D(A_p[2n-k], J[k]) = (-1)^k |\det A_p|^{-1} \sum_{j=0}^{2n-k} \binom{2n-k}{j} D(S_p[2n-k-j], h_p[j], J[k])
$$

For even $k = 2b$, the term of highest order in $S_p$ (corresponding to $j = 0$) is non-zero for generic $S_p$. In particular, $\phi_{2n}(F) \neq 0$ for generic hypersurfaces $F$. Now choose any DH submanifold $N \subset M$ of dimension $2b + 1$. Using Theorem 4.25, Lemma 4.12 and the last observation, we may find a $2b$-dimensional submanifold $F^{2b} \subset N$ s.t. $\phi_{2b}^N(F) = \phi_{2b}^N(F) \neq 0$. Thus $\phi_{2b} \notin W_{2b+1}^{-\infty}(M)$, proving (i).

For (ii), it now suffices to check that $\phi_{2b-1}, \phi_{2b}$ are linearly independent in $W_{2b}^{-\infty}(M)/W_{2b+1}^{-\infty}(M)$, for a generic $M$. In turn this is implied by the following statement: as a function on normally transversal closed submanifolds of dimension $2b$ lying inside a fixed DH submanifold $N^{2b+1} \subset M$, $\phi_{2b-1}, \phi_{2b}$ are linearly independent. By Theorem 4.25, we may assume $b = n$. Now note that for $\phi_{2n-1}$, the summand of degree $1$ in $S_p$ vanishes, as $S_p$ is symmetric while $J$ is antisymmetric. The statement now follows by examining the summand whose numerator contains no $S_p$ for both valuations, which has different coefficients (depending on $p$ for a general horizontal distribution).

\hfill $\Box$

### 4.3. Extending $\phi_{2k}$ to arbitrary 2k-submanifolds.

Define the contact area $\text{CA}_{2k}(F) \in [0, \infty]$ for any $C^2$, 2k-dimensional submanifold with boundary $F \subset M$ by

$$
\text{CA}_{2k}(F) := \liminf_{F_\varepsilon \to F} \phi_{2k}(F_\varepsilon)
$$

where $F_\varepsilon$ is a $C^2$, normally transversal submanifold with boundary that $C^2$-converges to $F$. Note that Proposition 4.20 and Theorem 4.25 and the proof of Lemma 4.26 imply that $\phi_{2k}(F)$ is $C^2$-continuous on normally transversal submanifolds $F$. Hence on such submanifolds, $\text{CA}_{2k}(F) = \phi_{2k}(F)$.
We now show that the vanishing of $\text{CA}_{2k}(F^{2k})$ is a necessary condition for the existence of an arbitrarily small perturbation with no contact points. For closed $F$ this supplements the topological necessary condition $\chi(F) = 0$.

**Proposition 4.29.** For a $2k$-dimensional submanifold with boundary $F \subset M$, $\text{CA}_{2k}(F^{2k}) = 0$ if and only if there is an arbitrarily small $C^2$-perturbation of $F$ which is nowhere tangent to the horizontal structure.

**Proof.** The if direction follows from Corollary 4.27. For the other direction, let us assume $\text{CA}_{2k}(F) = 0$. Again by Corollary 4.27, $F$ cannot be normally transversal to the horizontal distribution unless it is nowhere tangent to it. Assuming the contrary to the assertion, we conclude $F$ is not normally transversal. Let $F_\epsilon \to F$, $\epsilon \to 0$ be a normally transversal family of smooth perturbations of $F$ s.t. $\phi_{2k}(F_\epsilon) \to 0$. If a normally transversal contact point exists for $F$ (which is then necessarily isolated), it persists to $F_\epsilon$ for small $\epsilon$, and examining eq. (8), one can find $c > 0$ such that $\phi_{2k}(F_\epsilon) \geq c$ for all small $\epsilon$, a contradiction. Thus $F$ is not normally transversal to the horizontal distribution at any of its contact points. It follows that a sequence $x_j$ of contact points of $F_\epsilon$ must approach a necessarily degenerate tangency point, and hence by eq. (8), $\phi_{2k}(F_\epsilon) \to \infty$. This again is a contradiction, implying that for small $\epsilon$, $F_\epsilon$ has no horizontal tangent spaces, as claimed. \qed

5. The dual Heisenberg Lie algebra

5.1. Linear algebra. Recall that $U^{2n+1}$ denotes the dual Heisenberg algebra, with $H \subset U$ a fixed linear hyperplane, and $\omega \in \wedge^2 H^* \otimes U/H$ is a non-degenerate (twisted) form. It is the simplest DH manifold.

Let the group of automorphisms of the Heisenberg algebra be denoted by $\text{Sp}_H(U)$ or $\text{Sp}_H(2n+1)$. There is an $\text{Sp}_H(U)$-equivariant isomorphism $\text{Dens}(H) \simeq \text{Dens}(U/H)^n$.

Define the subgroups $\text{Sp}_H^+(U) = \{g \in \text{Sp}_H(U) : g|_{U/H} > 0\}$, $\text{Sp}_H^0(U) = \{g \in \text{Sp}_H(U) : g|_{U/H} = \text{Id}\}$ and $\text{ Scal}_H(U) = \{g \in \text{Sp}_H(U) : g|_{H} \in \mathbb{R}^*\} \subset \text{Sp}_H^0(U)$. Note that for $g \in \text{Sp}_H(U)$ clearly $\det g = 1$. For an $\text{Sp}_H(U)$-module $X$ and $x \in X$, we will write $\text{Stab}(x)$, resp. $\text{Stab}^+(x)$, $\text{Stab}^1(x)$ for its stabilizer in the corresponding subgroup. Let us record the following trivial fact.

**Lemma 5.1.** $\text{Sp}_H^+(U)$ is generated by $\text{Sp}_H^1(U)$ and $\text{ Scal}_H(U)$

**Corollary 5.2.** Let $\delta_\lambda \in \text{ Scal}_H(U)$ act by $\lambda \in \mathbb{R}$ on $H$ and by $\lambda^2$ on a fixed direction $L$ complementing $H$. Then $\text{Sp}_H^0(U)$ and $(\delta_\lambda)_{\lambda \neq 0}$ generate $\text{Sp}_H^+(U)$

**Proof.** Take any $g \in \text{ Scal}_H(U)$, set $\lambda := g|_H$. Then $\delta_\lambda^{-1} \circ g \in \text{Sp}_H^1(U)$. The claim now follows from Lemma 5.1. \qed

**Lemma 5.3.** For any subspace $E \subset H$, set $E_0 = E \cap E^\omega$. Then one can find:

i) $S \in \text{Stab}^1(E)$ s.t. $S(E_0) = E_0$ and $S|_{E_0} = 2$.

ii) $T \in \text{Stab}^+(E)$ s.t. $T(E_0) = E_0$, $T|_{E_0} = 1$ and $\det T < 1$.

**Proof.** i) Decompose $E \cap H = E_0 \oplus F$, where $F \subset H$ is a non-degenerate subspace. Fix a vector $z \in E \setminus H$. There is then an induced symplectic form $\omega_z$ on $H$ given by $\omega_z(u,v)(z + H) = \omega(u,v)$. Define $S \in \text{GL}(E)$ by setting $S|_{E_0} = 2$, $S|_{F} = 1$, and note that $S$ leaves $\omega_z|_E$ invariant. By Witt’s extension theorem, we can find an extension $S \in \text{Sp}(H, \omega_z)$, and finally setting $S(z) = z$ yields $S \in \text{Stab}^1(E)$ as required.

ii) Using $S$ from i), define $T \in \text{Sp}_H^+(U)$ by $T|_H = \frac{1}{2} S|_H$, $Tz = \frac{1}{4}z$. 


The orbits $Y_{ε,r}^k$ of $Sp^+_H(U)$ in $Gr_k(U)$ are classified by the pairs $(ε, r)$ as follows:

$$E ∈ Y_{ε,r}^k ⇔ ε = k - \dim H \cap E ∈ \{0, 1\}, 2r = k + \dim ker(ω|_{E∩H})$$

where $ε ∈ \{0, 1\}$, and $r ∈ \{0, \ldots, [k/2]\}$, $κ = \min(k - ε, 2n - (k - ε))$. The unique open orbit has $ε = 1$ and $r = [k/2]$; the unique closed orbit has $ε = 0$ and $r = 0$.

5.2. Translation invariant valuations on the dual Heisenberg algebra. We start by classifying the $Sp^+_H(U)$-invariant Klain sections.

**Proposition 5.4.** Fix $1 ≤ k ≤ 2n$.

i) For even $k$, there is at most a one-dimensional space of $Sp^+_H(U)$-invariant generalized Klain sections over $Gr_k(U)$.

ii) For odd $k$, there are no $Sp^+_H(U)$-invariant generalized Klain sections over $Gr_k(U)$.

**Proof.** We will make repeated use of Lemma B.1 without explicit mention, wherein also the bundle $F^ω_Y$ of principal symbols transversal to $Y$ is defined.

Take $E ∈ Y_{ε,r}^k$, and denote $E_0 = (E ∩ H) ∩ (E ∩ H)ου∗$, $dim E_0 = κ - 2r$.

**Step 1.** $ε = 1$. We consider first the open orbit: $r = r_{max}$. If $k$ is odd, $E ∩ H$ is non-degenerate, and $ω$ gives an isomorphism $Dens(E ∩ H) ≃ Dens(U/H)^{k+1}$. Thus

$$Dens(E) = Dens(E/E∩H) ⊗ Dens(E∩H) = Dens(U/H) ⊗ Dens(U/H)^{k+1}$$

It follows that for odd $k$ there are no invariant sections on the open orbit.

For even $k$, let $E_0 ⊂ E ∩ H$ be the kernel of $ω|_{E∩H}$, which is a line. Then

$$Dens(E) = Dens(E/E∩H) ⊗ Dens(E∩H) = Dens(U/H) ⊗ Dens(E_0) ⊗ Dens((E∩H)/E_0) = Dens(U/H)^{k+1} ⊗ Dens(E_0)$$

By Lemma 5.3, we may find $S ∈ Stab^1(E)$ acting trivially on the first factor and rescaling the second factor. It follows that there are no invariant sections on the open orbit $Y_{ε,r}^k$.

**Step 2.** Now let us consider the orbit $Y = Y_{ε,r}^k$ with $r < r_{max}$. We have $N_EY = N_{E∩H}X*_{k−1}(H) = λ²E_0$. Consider the bundle over $Y$ with fiber

$$F^ω_Y|E = Dens(E) ⊗ Dens^*(N_EQY) ⊗ Sym^ω(N_EQY) = Dens(E) ⊗ Dens(λ²E_0) ⊗ Sym^ω(λ²E_0)$$

Since $E/E∩H$ is $Stab(E)$-isomorphic to $U/H$,

$$Dens(E) = Dens(E∩H) ⊗ Dens(U/H) = Dens((E∩H)/E_0) ⊗ Dens(E_0) ⊗ Dens(U/H)$$

Now $dim(E∩H)/E_0 = 2r$, and $ω$ readily yields a non-degenerate form

$$ω ∈ ω^2((E∩H)/E_0)^∗ ⊗ U/H$$

so that there is a $Stab(E)$-isomorphism $Dens((E∩H)/E_0) ≃ Dens(U/H)^r$. Thus

$$F^ω_Y|_E = Dens(U/H)^{r+1} ⊗ Dens(E_0) ⊗ Dens(λ²E_0) ⊗ Sym^ω(λ²E_0^*)$$

Again by Lemma 5.3, there are no $Stab^1(E)$-invariants in $F^ω_Y|_E$ when $r < r_{max}$ (and so $E_0 ≠ \{0\}$).
We conclude that for no $k$ are there invariant generalized sections whose support intersects $Y_{1,r}$, for any $r$. We assume from now on that $\epsilon = 0$, so $E \subset H$.

**Step 2.** Consider $Y = Y_{0,r}^k$ with $r = r_{\text{max}}$. Then $N_E Y = E^* \otimes U/H$, and

$$F_Y^0 | E = \text{Dens}(E) \otimes \text{Dens}^*(N_E Y) = \text{Dens}(E)^2 \otimes \text{Dens}^*(U/H)^k$$

Taking $g \in \text{Scal}_H(U)$ with $g|_H = \lambda$, we see that it acts on $F_E^0 \otimes \text{Sym}^\alpha(N_E Y) = \text{Dens}(E)^2 \otimes \text{Dens}^*(U/H)^k \otimes (U/H)\alpha \otimes \text{Sym}^\alpha E^*$ by $\lambda^{-2k+2k+2\alpha-\alpha} = \lambda^\alpha$. Thus $\alpha = 0$ is the only possible transversal order of an invariant section. We now consider separately the different parities of $k$.

If $k$ is odd, set $E_0 = E \cap E^\omega$, $\dim E_0 = 1$. Then

$$F_E^0 = \text{Dens}(E_0)^2 \otimes \text{Dens}(E/E_0)^2 \otimes \text{Dens}^*(U) \frac{k}{k+1} = \text{Dens}(E_0)^2 \otimes \text{Dens}^*(U) \frac{k+1}{k+1},$$

since $\text{Dens}(E/E_0)^2 \simeq \text{Dens}(U/H)^{k-1} = \text{Dens}(U) \frac{k+1}{k+1}$. Thus by Lemma 5.3 the action of $\text{Stab}^+(E)$ on $F_E^0$ is clearly non-trivial, and so there are no invariant sections whose support intersects $Y$.

If $k$ is even, the restriction of $\omega$ to $E$ gives an isomorphism $\text{Dens}(E) = \text{Dens}(U/H) \frac{k}{2}$, so that $\text{Stab}(E)$ acts trivially on $F_E^0$. We know by now that all invariant sections are supported inside $Y$. Thus we conclude that the space of restrictions of the space of invariant sections to $\text{Gr}_k(U) \setminus (\overline{Y} \setminus Y)$ is at most one-dimensional.

**Step 3.** It remains to show there are no invariant sections supported on the closure of either of the orbits $Y_{0,r}^k(U)$ with $r < r_{\text{max}}$, for any $k$. In particular, we have $1 < k < 2n$ and $\kappa - 2r = \dim E_0 \geq 2$, with $\kappa = \min(k, 2n - k)$.

One has the chain of inclusions $T_E Y_{0,r}^k(U) = T_E X_k^k(H) \subset T_E \text{Gr}_k(U) \subset T_E \text{Gr}_k(U)$. Hence $N_E Y_{0,r}^k(U)$ fits into the exact sequence

$$0 \to T_E \text{Gr}_k(H)/T_E X_k^k(H) \to N_E Y_{0,r}^k(U) \to T_E \text{Gr}_k(U)/T_E \text{Gr}_k(H) \to 0$$

which is $\text{Stab}(E)$-isomorphic to

$$0 \to \wedge^2 E_0^* \to N_E Y_{0,r}^k(U) \to E^* \otimes U/H \to 0$$

Write $Y = Y_{0,r}^k$. For $\alpha = 0$,

$$F_Y^0 | E = \text{Dens}(E) \otimes \text{Dens}^*(N_E Y_{0,r}^k(U)) = \text{Dens}(E_0)^2 \otimes \text{Dens}(E/E_0)^2 \otimes \text{Dens}^*(U/H)^k \otimes \text{Dens}^*(\wedge^2 E_0^*)$$

$$= \text{Dens}(E_0)^2 \otimes \text{Dens}^*(U/H)^{k-1} \otimes \text{Dens}^*(\wedge^2 E_0^*).$$

Thus $F_Y^0 | E$ does not contain invariants, as there is an element of $\text{Stab}^+(E)$ rescaling $E_0$.

When $\alpha > 0$, a $\text{Stab}^+(E)$-invariant element of $F_Y^0 | E$ would imply the existence of an invariant element in

$$F_Y^0 | E \otimes (\wedge^2 E_0^*)^a \otimes (E^*)^b \otimes (U/H)^c$$

which in turn implies the existence of an invariant element in

$$\text{Dens}(E_0)^2 \otimes \text{Dens}(\wedge^2 E_0^*) \otimes (\wedge^2 E_0^*)^a \otimes (E_0^*)^b \otimes (E/E_0)^* \otimes \text{Dens}(U)^\lambda$$

for some non-negative integers $a, b, c'$ and $\lambda \in \mathbb{R}$.

By the proof of Lemma 5.3, we can find $S \in \text{Stab}^+(E)$ such that $S|_{E_0} = 2$ and $S : E/E_0 \to E/E_0$ is the identity. Thus there are no invariants in this space.

□
For a vector bundle $E$ over a manifold $B$, we let $\Gamma_m(B,E):=\Gamma_m(B,E^*\otimes |\omega_B|)^*$ denote the space of generalized sections that are given locally by a regular Borel measure.

**Proposition 5.5.** There are $\text{Sp}_H^+(U)$-invariant generalized Klain sections $\kappa_k$ realizing the upper bounds obtained in Proposition 5.3. Moreover, $\kappa_k \in \Gamma_m(\text{Gr}_k(U), \text{Dens}(E))$, they are supported on $Y_k^H := \{E \subset H\}$ and are $\text{Sp}_H(U)$-invariant.

**Proof.** Take $f(E) = |\omega|^2|\xi|^2 \in \text{Dens}(E)^2 \otimes \text{Dens}^*(U/H)^k$, which is a continuous section over $\text{Gr}_k(H)$. We may rewrite $f$ as an absolutely continuous measure on $\text{Gr}_k(H)$ with values in the bundle $\text{Dens}^2(E) \otimes \text{Dens}^*(U/H)^k \otimes \text{Dens}^*(T_E \text{Gr}_k(H)) = \text{Dens}^*(U)^k \otimes \text{Dens}(E)^{2n+2}$. Writing $i : \text{Gr}_k(H) \rightarrow \text{Gr}_k(U)$ for the natural embedding, we get
\[
i_*f \in \mathcal{M}_{\text{Gr}_k(H)}(\text{Gr}_k(U), \text{Dens}^*(U)^k \otimes \text{Dens}(E)^{2n+2}) = \Gamma_m(\text{Gr}_k(U), \text{Dens}^*(U)^k \otimes \text{Dens}(E)^{2n+2} \otimes \text{Dens}(T_E \text{Gr}_k(U)))
\]

The latter bundle is just the Klain bundle:
\[
\text{Dens}^*(U)^k \otimes \text{Dens}(E)^{2n+2} \otimes \text{Dens}(E^* \otimes U/E) = \text{Dens}(E)
\]
and it remains to note that we only used $\text{Sp}_H(U)$-equivariant identifications.

Let us fix an involution $R \in \text{Sp}_H(U)$ acting by $-1$ on $U/H$. Since $R^{-1} \text{Sp}_H^+(U) R = \text{Sp}_H^+(U)$, it follows that $R$ acts on the space of $\text{Sp}_H^+(U)$-invariants in any $\text{Sp}_H(U)$-module $M$. We will call the $\pm 1$ eigenspaces of $R$ in $M^{\text{Sp}_H^+(U)}$ $R$-even, resp. $R$-odd.

**Proposition 5.6.** For $0 \leq k \leq n$ it holds that $\dim \text{Val}_{2k}^{-\infty}(U)^{\text{Sp}_H^+(U)} \leq 2$ while $\dim \text{Val}_{2k+1}^{-\infty}(U)^{\text{Sp}_H^+(U)} = 0$. For every $k$, the spaces of $R$-even and $R$-odd invariants are each at most one-dimensional. Moreover, any $R$-even invariant valuation is even, and any $R$-odd invariant valuation is odd.

In particular, any $\text{Sp}_H(U)$-invariant valuation must be even.

**Proof.** Step 0. In the following, all forms are translation-invariant. For $\xi \in \mathbb{P}_+(U^*)$, let $\xi_\perp \subset U$ be its annihilator. There is a natural identification of $\Omega_{-\infty}^{2n+1-j}(U \times \mathbb{P}_+(U^*))^{tr}$ with the generalized sections of the bundle over $\mathbb{P}_+(U^*)$ with fiber $U^* \otimes \wedge^{2n+1-j}\xi_\perp \otimes \wedge^{2n+1-j}$ over $\xi$. A Legendrian form corresponds to a section of the subbundle $\xi \otimes \wedge^{2n+1-j}(U^*/\xi) \otimes \wedge^{2n+1-j}\xi_\perp \otimes \wedge^{2n+1-j}$, which we then call a Legendrian section. We will find the $\text{Sp}_H^+(U)$-invariant generalized Legendrian sections. Let $\psi(\xi)$ be such a section.

There are three orbits under $\text{Sp}_H^+(U)$: the open orbit $X_o = \{\xi \neq H^{\perp}\}$ and two closed orbits $X_{\pm}^\xi = \{\pm H^{\perp}\}$.

**Step 1.** Let us first show that $\psi$ vanishes when restricted to the open orbit. Note that $\psi|_{X_o}$ is smooth by $\text{Sp}_H^+(U)$-invariance, and fix $\xi \in X_o$.

Consider the stabilizer $G_\xi := \text{Stab}^+(\xi) \subset \text{Sp}_H^+(U)$, and $G_\xi^1 = \{g \in G_\xi : g|_\xi = 1\}$. Then $\psi(\xi)$ is a $G_\xi$-invariant element in $\wedge^{2n+1-j}(U^*/\xi) \otimes \wedge^{2n+1-j}\xi_\perp \otimes \wedge^{2n+1-j}$. By considering the various invariant subquotients, we deduce the existence of an invariant element in one of the following spaces: $V_1 = (H^{\perp} \oplus \xi)/\xi \otimes \wedge^{2n+1-j}(U^*/(\xi \oplus H^{\perp})) \otimes \wedge^{2n+1-j}(\xi_\perp \cap H) \otimes \wedge^{2n+1-j}$ $V_2 = (H^{\perp} \oplus \xi)/\xi \otimes \wedge^{2n+1-j}(U^*/(\xi \oplus H^{\perp})) \otimes \wedge^{2n+1-j}(\xi_\perp \cap H) \otimes (\xi_\perp/\xi_\perp \cap H) \otimes \wedge^{2n+1-j}$
\[ V_3 = \wedge^{j-1}(U^*/(\xi \oplus H^\perp)) \otimes \wedge^{2n+1-j}(\xi_\perp \cap H) \otimes \xi^{2n+2-j} \]

\[ V_4 = \wedge^{j-1}(U^*/(\xi \oplus H^\perp)) \otimes \wedge^{2n-j}(\xi_\perp \cap H) \otimes (\xi_\perp/\xi_\perp \cap H) \otimes \xi^{2n+2-j} \]

Now take \( \gamma \in G_\xi \) such that \( \gamma|_H = \lambda \). Since \( \gamma \) has a \( 2n \)-dimensional eigenspace of eigenvalue \( \lambda \) on \( U \), the action of \( \gamma \) on \( U^* \), which is by \( (\gamma^{-1})^* \), has a \( 2n \)-dimensional invariant subspace of eigenvalue \( \lambda^{-1} \). In the following we will simply write \( \gamma \) for this action. We may choose \( \gamma \) such that \( \gamma|_\xi = \lambda^{-1} \), \( \gamma|_{H^\perp} = \lambda^{-2} \). Since \( \xi_\perp/(\xi_\perp \cap H) \simeq U/H \), \( \gamma \) acts on it by \( \lambda^2 \).

The action of \( \gamma \) on \( V_1 \) is as follows: \( \gamma|_{V_1} = \lambda^{-2-(j-2)+2n+1-j-(2n+2-j)} = \lambda^{-j-1} \), \( \gamma|_{V_2} = \lambda^{-j} \), \( \gamma|_{V_3} = \lambda^{-(j-1)+2n+1-j-(2n+2-j)} = \lambda^{-j} \), \( \gamma|_{V_4} = \lambda^{-j+1} \).

Since \( j \geq 1 \), we conclude that a \( G_\xi \)-invariant can only exist in \( V_4 \), and only if \( j = 1 \). However, \( \wedge^{2n} \xi_\perp \otimes \xi^{2n+1} \) has no \( G_\xi \)-invariants: one may choose an element \( g \in G_\xi \) with \( g|_\xi = 1 \), while \( \det g \neq 1 \). Then \( \det(g : \xi_\perp \to \xi_\perp) \neq 1 \), and thus no such invariant exists.

**Step 2.** We conclude that \( \psi \) is supported on \( X_c := \{ \pm H^\perp \} \). Since \(-\text{Id}\) commutes with \( \text{Sp}_H(U) \), it acts on any space of \( \text{Sp}_H(U) \)-invariants, which thus decomposes into a sum of even and odd invariants. Now \( \det(-\text{Id}) = (-1)^{2n+1} = -1 \), so that odd closed Legendrian forms correspond to even valuations and vice versa.

Let us show that for odd \( j \) there are no \( \text{Sp}_H(U) \)-invariant closed Legendrian forms. Otherwise by what we proved, there is an \( \text{Sp}_H(U) \)-invariant such form \( T \) supported on \( \xi = H^\perp \) with a fixed orientation. Then \( T := T_{-\text{Id}}T \) is odd, closed, Legendrian and \( \text{Sp}_H(U) \)-invariant. Thus \( T \) defines an even \( \text{Sp}_H(U) \)-invariant generalization valuation which is \( j \)-homogeneous. This contradicts Proposition 3.4.

**Step 3.** The principal symbol of \( \psi \), denoted \( F_{X_c,G_\xi}^\wedge \psi \) (see Appendix 3) over each point \( \xi \in X_c \) is an element of \( W := H^\perp \otimes \wedge^{j-1}H^* \otimes \wedge^{2n+1-j}H \otimes (H^\perp)^{2n+1-j} \otimes \text{Dens}^*(T_\xi \mathbb{P}_+(U^*)) \otimes \text{Sym}^\wedge (T_\xi \mathbb{P}_+(U^*)) \)

\[ = \wedge^{j-1}H^* \otimes \wedge^{2n+1-j}H \otimes (H^\perp)^{2n+1-j} \otimes \text{Dens}(H) \otimes \text{Dens}(H^\perp)^{2n} \otimes \text{Sym}^\wedge(H^\perp). \]

Take \( \delta_\xi \in \text{Scal}(U) \) acting by \( \lambda \in \mathbb{R} \) on \( H \) and by \( \lambda^2 \) on some fixed vector \( w \in U \setminus H \). Then \( \delta_\xi \) acts on \( H^\perp \) by \( \lambda^{-2} \), and

\[ \delta_\xi|_W = \lambda^{-(j-1)}\lambda^{2n+1-j} \lambda^{-(2n+2-j-\alpha)} \lambda^{-2n} \lambda^4 \lambda^{-\alpha} = \lambda^{\alpha-2} \]

hence \( \alpha = 2 \) is necessary for an invariant to exist.

We may identify \( \wedge^{2n+1-j}H \simeq \wedge^{j-1}H^* \otimes \wedge^{2n}H \). Thus

\[ W = \wedge^{j-1}H^* \otimes \wedge^{j-1}H^* \otimes \text{Sym}^2(H^*) \otimes \wedge^{2n}H \otimes (H^\perp)^{2n-j} \otimes \text{Dens}(H) \otimes \text{Dens}(H^\perp)^{2n} \]

Let us find all \( w \in W \) that are invariant under \( \text{Sp}_H(U) \). We may fix an \( \text{Sp}_H(U) \)-invariant symplectic form \( \omega_H \) on \( H \), so that \( \text{Sp}_H(U) \)-equivariantly,

\[ W \simeq \wedge^{j-1}H^* \otimes \wedge^{j-1}H^* \otimes \text{Sym}^2(H^*). \]

By the fundamental theorem of invariant theory, an invariant element of \( W \) is given by fixing pairings of all the factors using \( \omega \), and then symmetrizating/antisymmetrizating accordingly (some pairings could give zero). There would be \( j \) pairings.

Note that \( R^* \omega_H = -\omega_H \). Hence \( \det R|_H = (-1)^n \) and \( R = (-1)^{n+1} \). We see that \( Rw = (-1)^j(w-(-1)^n)w = (-1)^p w \) for any \( w \in W_{\text{Sp}_H(U)} \).

**Step 4.** Now take \( \phi \in \text{Val}_J(U)^{\text{Sp}_H(U)} \) which is either even or odd, and denote \( T = T(\phi) \). Assume first that \( \phi \) is \( R \)-even, that is \( \text{Sp}_H(U) \)-invariant. Then \( R^*T = \det(T) \). Thus its principal symbol is \( \sigma(T)(H^\perp,-H^\perp) = (w_0,-w_0) \) for some \( w_0 \in W_{\text{Sp}_H(U)} \). Noting that \(-\text{Id}\) acts by \((-1)^j\) on \( W \), it follows that \( \phi \) is an even
valuation if and only if \( j \) is even, which we may assume by step 2. By Proposition 5.4, \( \dim \text{Val}_{2k}(U)^{\text{Sp}_{H}(U)} \leq 1 \).

Similarly if \( \phi \) is \( R \)-odd, its principal symbol is \( \sigma(T) = (w_0, v_0) \) for some \( w_0 \in W^\text{Sp}_{H}(U) \), and since \( j \) is even, \( \phi \) must be odd. It remains to prove the dimension of the space of \( R \)-odd, \( j \)-homogeneous valuations is at most 1. It suffices to show by Theorem 5.3 that the space of \( \text{Sp}_{H}(U) \)-invariant, \( R \)-odd, \( j \)-homogeneous, closed Legendrian generalized forms supported on \( U \) is at most 1-dimensional. Any such form has the form \( T_+ - R^*T_+ \) with \( T_+ \) supported on \( +H^\perp \) and \( \text{Sp}_{H}(U) \)-invariant. Clearly \( T_+ - R^*T_+ \mapsto T_+ + R^*T_+ \) is a bijection onto the corresponding \( R \)-even forms, which in turn bijectively correspond to \( \text{Sp}_{H}(U) \)-invariant valuations. The \( R \)-even case now concludes the proof.

The following proposition completes the proof of Theorems 1.5 and 1.6.

**Proposition 5.7.** For \( 0 \leq k \leq n \), it holds that \( \dim \text{Val}_{2k}(U)^{\text{Sp}_{H}(U)} = 1 \) while \( \dim \text{Val}_{2k}(U)^{\text{Sp}_{H}(U)} = 2 \).

**Proof.** The first assertion follows from the general construction of valuations on dual Heisenberg manifolds, as follows. By Proposition 4.28, we have at least \( n+1 \) linearly independent valuations \( \phi_{2j} \in W_{2j}(U) \) which are invariant under the symmetries of \( U \) as a DH-manifold, in particular under translations and \( \text{Sp}_{H}(U) \). Now Proposition 5.6 concludes the proof.

The second assertion follows from the first one: by the last paragraph of the proof of Proposition 5.6, there is a bijection between the \( \text{Sp}_{H}(U) \)-invariant valuations and the \( R \)-odd \( \text{Sp}_{H}(U) \)-invariant valuations.

**Remark 5.8.** 1. Alternatively, one may use the Corfton valuations \( \psi_{2j} \) on the contact sphere constructed in section A to prove the statement, by considering the translation-invariant valuations they define on the tangent spaces by Theorem 3.13.

2. Examining the construction of \( \phi_{2k} \) for general DH manifolds, we immediately see that \( \phi_{2k} \) is in fact \( 2k \)-homogeneous.

**Example 5.9.** Let us describe explicitly the valuation \( \phi_{2n} \). Consider \( U = \mathbb{R}^{2n+1} \) with the standard Euclidean structure, and take \( H = \{ x_{2n+1} = 0 \} \). For \( u,v \in H \) we let \( \omega(u,v) = \sum_{j=1}^{n} (u_{2j-1}v_{2j} - u_{2j}v_{2j-1}) \) be the standard symplectic form.

Identifying \( U/H = H^\perp = \mathbb{R}^{2n+1} = \mathbb{R} \) we recover the DH structure on \( U \).

Examining the proof of Proposition 5.6, we see that for convex \( K \in \mathcal{K}(V) \), \( \phi_{2n}(K) = \frac{1}{2}(\sigma_K(H^\perp) + \sigma_K(-H^\perp)) \), where \( \sigma_K \) stands for the surface area measure of \( K \), which is essentially just the push-forward of the \( 2n \)-dimensional Hausdorff measure on \( \partial K \) to \( S^{2n} \), see [Sch14] for exact definition.

### 5.3. Homogeneity 2n

The valuation of homogeneity \( 2n \) in \( U \), which is just the surface area measure at the points of tangency to \( H \subset U \), has also a different natural setting, valid in dimension of any parity.

Let \( H^m \subset V^{m+1} \) be a hyperplane, and let \( \text{vol}_H \), \( \text{vol}_V \) be fixed volume forms on \( H \), resp. \( V \). Consider \( SL_H(V) := \{ g \in SL(V) : g(H) = H, g|_H \in SL(H) \} \). Fix a Euclidean product \( P \) on \( V \) inducing the given Lebesgue measures on \( V \) and \( H \). Assume for simplicity \( m \geq 2 \).

**Proposition 5.10.** \( \phi_m(K) := \frac{1}{2}(\sigma_K(H^\perp) + \sigma_K(-H^\perp)) \) is the unique \( m \)-homogeneous, even \( SL_H(V) \)-invariant generalized valuation.
Proof. The Klain section of $\phi_m$ is a delta-measure on $\text{Gr}_m(V)$ supported on $H$. It is given by an element of $\text{Dens}(H) \otimes \text{Dens}^*(T_E \text{Gr}_m(V)) = \text{Dens}(H)^{m+2} \otimes \text{Dens}^*(V)^m$. Clearly $\text{SL}_H(V)$ acts trivially on this space.

The proof of uniqueness is similar to Proposition 5.6. We first note that there are no invariant sections of the Klain bundle over the open orbit $\{E : E \neq H\} \subset \text{Gr}_m(V)$, since for such $E$ there is a natural isomorphism $E/E \cap H = V/H$, so a $\text{Stab}(E)$-invariant density on $E$ would produce a $\text{Stab}(E)$-invariant density on $E \cap H$, which clearly does not exist since $\text{Stab}(E)$ can rescale this space. Next we use Lemma 3.1 to study Klain sections supported on the closed orbit $\{H\}$: for $\alpha \geq 0$,

$$F^\alpha_{\{H\}} = \text{Dens}(H) \otimes \text{Dens}^*(H^* \otimes V/H) \otimes \text{Sym}^\alpha(H^* \otimes V/H)$$

which is $\text{SL}_H(V)$-equivariantly isomorphic to $\text{Sym}^\alpha(H^*)$. There are no invariant polynomials on $H$, so we must have $\alpha = 0$, which readily yields a one-dimensional space of invariants.

Next, we write an invariant Crofton formula for this valuation.

**Proposition 5.11.** There is an invariant Crofton measure $\mu_{\text{SL}}$ over $\text{Gr}_1(V)$ which defines $\phi_m$.

Proof. Let $\theta(E) \in [0, \pi]$ be the Euclidean angle between $E \in \text{Gr}_1^+(V)$ and $H^\perp$ with fixed orientation. Define the meromorphic family of functions $f_s(E) = (\cos \theta)^s |t|^s$ where $|t|^s \in C^\infty(R)$ is the standard meromorphic family of even homogeneous generalized functions. Note that $\cos \theta$ is submersive whenever $t = \cos \theta = 0$, so the pull-back is well-defined. Since $f_s(E)$ is invariant to orientation reversal, we get a generalized family on $\text{Gr}_1(V)$, still denoted $f_s(E)$.

Let us identify translation-invariant measures (distributions) on $A\text{Gr}_1(V)$ with (generalized) sections of the Crofton bundle $C_m$ over $\text{Gr}_1(V)$, whose fiber over $E \in \text{Gr}_1(V)$ is $\text{Dens}(V/E) \otimes \text{Dens}(T_E \text{Gr}_1(V))$.

Define a generalized section of $C_m$ over $\text{Gr}_1(V) \setminus \text{Gr}_1(H)$ by $\mu_{\text{SL}} := f_{-m-2}(E)$ when $m$ is even, and $\mu_{\text{SL}} := \text{Res}_{x=-m-2} f_s$ when $m$ is odd.

Let us check $\mu_{\text{SL}}$ is $\text{SL}_H(V)$-invariant. For $g \in \text{GL}(V)$ consider the Jacobian

$$\psi_g(E) = \text{Jac}(g : E \rightarrow gE)^{-2},$$

where $E, gE$ are endowed with the volume induced by the Euclidean product $P$. It follows that for $g \in \text{SL}_H(V)$ and $E \not\subset H$,

$$\psi_g(E) = \frac{\cos^2 \angle(gE, H^\perp)}{\cos^2 \angle(E, H^\perp)}.$$

Now $f(E)$ represents (with respect to the Euclidean trivialization) an $\text{SL}_H(V)$-invariant Crofton measure precisely if $g^* f(E) = \psi_g(E)^{-\frac{m+2}{2}} f(E)$. Since $g^* f_s = \psi_g(E)^s f_s(E)$, we conclude $\mu_{\text{SL}}$ is indeed $\text{SL}_H(V)$-invariant.

It remains to verify these Crofton measures define non-zero valuations. For this, we evaluate $\omega_m(B)$ for the unit ball $B^{m+1}$. By definition, $\phi_m(B) = 1$. On the other hand, writing $\omega_m$ for the volume of the Euclidean ball $B^m$, we get

$$\begin{align*}
\text{Cr}(f_s(E)dE)(B) &= \omega_m \int_{\text{Gr}_1(V)} |\cos \theta(E)|^s dE = \omega_m \int_0^{\pi} \sin^{m-1} \theta d\theta \\
&= \omega_m \int_0^1 t^s (1-t^2)^{\frac{m-2}{2}} dt = \omega_m \int_0^1 u^{\frac{m}{2}} (1-u)^{\frac{m-2}{2}} du = \omega_m B\left(\frac{1}{2}, \frac{m}{2} \right)
\end{align*}$$

with

$$\omega_m = \frac{\int_0^1 t^s (1-t^2)^{\frac{m-2}{2}} dt}{\int_0^1 (1-t^2)^{\frac{m-2}{2}} dt} = \frac{\int_0^1 u^{\frac{m}{2}} (1-u)^{\frac{m-2}{2}} du}{\int_0^1 u^{\frac{m}{2}} (1-u)^{\frac{m-2}{2}} du} = \omega_m B\left(\frac{1}{2}, \frac{m}{2} \right).$$
Thus
\[ C_m = \frac{\pi^{\frac{1}{2}}}{\Gamma(1 + \frac{m}{2})}\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} = \frac{\pi^{\frac{1}{2}}}{\Gamma(1 + \frac{m}{2})}\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}. \]

When \( m = 2n \) is even, \( s = -2n - 2 \) and
\[ C_{2n} = -\frac{1}{2}\pi^{n-1}\frac{\Gamma(n + 1)}{n!\Gamma(-n - 1/2)}. \]

When \( m = 2n - 1 \) is odd, \( s = -2n - 1 \), \( \Gamma(\frac{1 + s}{2}) \) has a simple pole with residue \( 2\Res_{z = -n}\Gamma(z) = 2(-1)^n\frac{1}{n!} \), and
\[ C_{2n-1} = 2(-1)^n\frac{1}{n!}\frac{\Gamma(n)}{\Gamma(-1/2)} = (-1)^{n-1}\frac{\pi^{n-1}}{n\Gamma(n + 1/2)}. \]

\[ \square \]

We can now recover effortlessly the inverse form of the Koldobsky-Ryabogin-Zvavitch formula [KRZ04].

**Corollary 5.12.** Let \( K \subset \mathbb{R}^{m+1} \) be a smooth, symmetric convex body, and let \( \kappa(x) \) be its Gaussian curvature and \( \nu(x) \) the unit normal at \( x \in \partial K \). Then there is a universal explicit constant \( C_m \) (computed in the proof of Proposition 5.11) s.t. for even \( m = 2n \),
\[ \kappa(x)^{-1} = C_m^{-1}\int_{S^m}\vol_m(Pr_{\theta^\perp}(K))\frac{d\sigma_1(\theta)}{\langle \theta, \nu(x) \rangle^{m+2}} \]
while for odd \( m \),
\[ \kappa(x)^{-1} = C_m^{-1}\left\langle \delta^{(m+1)}(\langle \nu(x), \theta \rangle), \vol_m(Pr_{\theta^\perp}(K)) \right\rangle \]
\[ = \frac{2}{(m+1)!}C_m^{-1}\frac{d^{m+1}}{dt^{m+1}}\bigg|_{t=0}\left(1 - t^2\right)^{m-2}\int_{\theta: \langle \theta, \nu(x) \rangle = \cos t}\vol_m(Pr_{\theta^\perp}(K))d\sigma_1(\theta) \]

**Remark 5.13.** Since \( \vol_m(Pr_{\theta^\perp}(K)) \) is the cosine transform of \( \sigma_K(\theta)d\sigma_1(\theta) \), one can think of this formula as simply the inversion of the cosine transform. The integral can be viewed as the \((-2n-2)\)-cosine transform of the support function of the projection body.

6. Contact manifolds

6.1. Specializing from general DH manifolds. Recall that a contact manifold has a canonical structure of a DH manifold.

**Proposition 6.1.** Assume \( M^{2n+1} \) is a contact manifold. Then \( \phi_{2j}, 0 \leq j \leq 2n \) are all linearly independent, while \( \phi_{2j-1} \) is a linear combination of \( \phi_{2i}, j \leq i \leq n \).

**Proof.** The first statement is just part i) of Proposition 4.28. For the second statement, we will use the same notation as in the proof of Proposition 4.20.

Write \( h = (h_{ij})_{j=1}^{2n} = H_s + H_a \), the symmetric and antisymmetric part.

First, we note that in a contact manifold, the relation between the symplectic and horizontal structures is given by \( H_a = -J \). Indeed, \( \alpha(v) = g(R, v) \) so we have
\[ d\alpha(u, v) = L_u\alpha(v) - L_v\alpha(u) - \alpha([u, v]) = L_u g(R, v) - L_v g(R, u) - \alpha([u, v]) = g(\nabla_u R, v) + g(R, \nabla_u v) - g(\nabla_v R, u) - g(R, \nabla_v u) = g(\nabla_u R, v) - g(\nabla_v R, u) \]
Hence $h_{ij} - h_{ij} = \theta_j(\nabla X, R) - \theta_i(\nabla X, R) = da(X_i, X_j) = J_{ij}$.

Take $F$ a closed hypersurface which is normally transversal. Introduce the notation $\Psi_k(F, p) := \left(2^n\right)^{-1}|\det A_p|\phi_{2n-k}(F, p)$. Then

$$\Psi_k(F, p) = D(H - S[k], J[2n-k]) = (-1)^k D(S - H_s + J[k], J[2n-k])$$

$$= (-1)^k \sum_{i=0}^{k} \binom{k}{i} D(S - H_s[i], J[2n - i])$$

Observe that for $2n \times 2n$ matrices $X, Y$ for which $X^T = X, Y^T = -Y$ it holds that $D(X[2n-i], Y[i]) = 0$ for odd $i$. We conclude

$$\Psi_k(F, p) = (-1)^k \sum_{i=0}^{\lfloor\frac{2n-k}{2}\rfloor} \binom{2n-k}{2i} D(S - H_s[2i], J[2n - 2i])$$

that is,

$$\phi_k(F, p) = (-1)^k \left(2^n\right)^{-1}\sum_{i=0}^{\lfloor\frac{2n-k}{2}\rfloor} \binom{2n-k}{2i} D(S - H_s[2i], J[2n - 2i])$$

Thus for closed hypersurfaces, $\phi_{2j-1}$ is a fixed linear combination of the $\phi_{2i}$ with $j \leq i \leq n$. We know by Lemmas 4.17 and 4.11 that all the valuations $\phi_j$ have small wavefront set. The statement now follows by Lemma 3.12 and Proposition 4.15.

We now prove a Hadwiger theorem for contact manifolds.

**Theorem 6.2.** Let $M^{2n+1}$ be a contact manifold. Then $\mathcal{V}^{-\infty}(M)^{\text{Cont}(M)}$ is spanned by $\phi_{2k}$, $0 \leq k \leq n$.

**Proof.** For an element $\phi \in \mathcal{W}_k^{-\infty}$, we will write $[\phi]_k$ for its image in $\mathcal{W}_k^{-\infty}/\mathcal{W}_{k+1}^{-\infty}$. We will show by induction on $2n+1-k$ that $\mathcal{W}_k^{-\infty}(M)^{\text{Cont}(M)}$ is spanned by $(\phi_{2j})_{2j \geq k}$. Take $\phi \in \mathcal{W}_{2n+1-k}^{-\infty}(M)^{\text{Cont}(M)} = \mathcal{M}_{-\infty}(M)^{\text{Cont}(M)} = \mathcal{M}_{-\infty}(M)^{\text{Cont}(M)}$, where the latter equality holds since $\text{Cont}(M)$ acts transitively on $M$, so that an invariant distribution is automatically smooth. There is no such invariant measure, hence $\phi = 0$.

Assume now that $\phi \in \mathcal{W}_{2n+1-k}^{-\infty}(M)^{\text{Cont}(M)}$. Recall

$$\mathcal{W}_{2n+1-k}^{-\infty}(M)/\mathcal{W}_{2n+2-k}^{-\infty}(M) = (\mathcal{W}_{k,c}^{-\infty}(M)/\mathcal{W}_{k+1,c}^{-\infty}(M))^* = \Gamma_c(M, \text{Val}_k^- (TM))^*$$

Hence by Proposition 4.2

$$[\phi]_{2n+1-k} \in (\Gamma_c(M, \text{Val}_k^- (TM))^*)^{\text{Cont}(M)} = \Gamma_{\text{Cont}(M)}(M, \text{Val}_{2n+1-k}^- (TM))$$

where we used the Alesker-Poincaré isomorphism $\text{Val}_k^- (TM) = \text{Val}_{2n+1-k}^- (TM) \otimes \text{Deus}(T_x M)$. For a point $x \in M$, its stabilizer in $\text{Cont}(M)$ is $\text{Stab}(x) = \text{Sp}_H(T_x M)$. Hence $\Gamma_{\text{Cont}(M)}(M, \text{Val}_{2n+1-k}^- (TM))^{\text{Cont}(M)} = \text{Val}_{2n+1-k}^- (T_x M)^{\text{Sp}_H T_x M}$. By Proposition 5.7, the latter space of invariants is trivial if $k$ is even, and one dimensional if $k$ is odd.

Thus if $k$ is even, $\phi \in \mathcal{W}_{2n+2-k}^{-\infty}(M)$. In case $k$ is odd, we use part (i) of Proposition 4.28 to find a multiple of $\phi_{2n+1-k}$ such that $[\phi]_{2n+1-k} = c[\phi_{2n+1-k}]_{2n+1-k} \Rightarrow \phi - c\phi_{2n+1-k} \in \mathcal{W}_{2n+2-k}^{-\infty}(M)^{\text{Cont}(M)}$. The induction assumption now completes the proof. □
6.2. A dynamical point of view. Let \( F \subset M^{2n+1} \) be a hypersurface, and let \( p \in F \) be a normally transversal contact point: \( T_p F = H_p \). Denote by \( F_H \) the singular hyperplane field on \( F \) given by \( F_H \vert_p = H_p \cap T_p F \). When \( \dim M = 3 \), this field integrates to the characteristic foliation.

One can describe \( \phi_{2k}(F,p) \) explicitly through the singular bundle \( H_x \) near \( x \).

Let \( \beta \in \Omega^1(F) \) be a form defined near \( p \) such that \( \text{Ker} \beta = F_H \). Since \( M \) is contact, we may assume \( d\beta \neq 0 \) near \( p \) (e.g. by taking \( \beta = \alpha \vert_p \) for some contact form \( \alpha \) on \( M \)), and there is a unique vector field \( B \in \mathfrak{X}(F) \) near \( p \) such that \( i_B d\beta = \beta \). In particular, \( B(p) = 0 \) and \( B(x) \) is tangent to the characteristic foliation. If \( \beta' = f \beta \) is a different form with \( d_p \beta' \neq 0 \iff f(p) \neq 0 \), the corresponding vector field is

\[
B' = \left( 1 + \frac{df(B)}{f} \right)^{-1} B
\]

Since \( df(B)(p) = df(B(p)) = 0 \), the differential \( d_p B \in \gl(T_p B) \) only depends on \( F_H \).

Remark 6.3. Note that \( \text{sign} \det d_p B \) determines whether \( p \) is an elliptic or hyperbolic singular point of the characteristic foliation.

Proposition 6.4. \( F \) is normally transversal at \( p \) if and only if \( d_p B \) is non-singular. In that event

\[
\phi_k(F,p) = \left| \det d_p B \right|^{-1} \text{Tr}(\wedge^{2n-k} d_p B)
\]

Proof. Again we use notation from the proof of Proposition [1.20]

Since all contact manifolds are locally isomorphic, we work in \( \mathbb{R}^{2n+1} \) with coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n, z)\), and contact form \( \alpha = -dz + \sum x_j dy_j \). We may assume further that \( p = 0 \) and \( T_p F = \{ z = 0 \} \). Then \( (x_j, y_j) \) are local coordinates on \( F \), and \( \beta = \sum -\frac{\partial f}{\partial x_j} dx_j + (x_j - \frac{\partial f}{\partial y_j}) dy_j \). It follows that \( B(x,y)^T = (x,0)^T + J\nabla f \), where \( \nabla f = (\frac{\partial f}{\partial x_j}, \frac{\partial f}{\partial y_j})^T \). Thus

\[
d_0 B = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} + J H^2 f
\]

On the other hand, one immediately computes that

\[
h = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}
\]

and \( S = H^2 f \), hence \( h - S = J d_0 B \). This readily shows that \( F \) is normally transversal at \( p \), which is equivalent to the non-singularity of \( h - S \), if and only if \( \det d_p B \neq 0 \); and

\[
D(h - S[2n-k], J[k]) = \det J \cdot D(d_0 B[2n-k], I_n[k]) = \begin{pmatrix} 2n \choose k \end{pmatrix}^{-1} \text{Tr}(\wedge^{2n-k} (d_0 B)).
\]

By Definition [4.22] we are done. \( \Box \)

Computation of \( \phi_{2k} \) is straightforward with this approach. Here is a simple proof of a well-known fact.

Corollary 6.5. In the standard contact space \( \mathbb{R}^3 \) with contact structure given by \( \alpha = dz + xdy \), spheres of different radii are not equivalent through a contactomorphism of the ambient space.
Proof. One computes that \( \phi_2(S_R) = 8(1 + \frac{1}{2}R^{-2})^{-1} \).

\[ \square \]

Example 6.6 (The contact sphere). Let us compute \( \phi_{2k}(S^{2m}) \) in \( S^{2n+1} \). By Theorem 4.25, we may assume \( n = m \). Consider \( S^{2m+1} \subset \mathbb{C}^{m+1} \), with coordinates \( x_1, y_1, \ldots, x_{m+1}, y_{m+1} \). The contact form is given by

\[ \alpha_p(v) = \langle v, \sqrt{-1}p \rangle = \sum_{j=1}^{m+1} (-y_j dx_j + x_j dy_j). \]

so that \( d\alpha = 2 \sum_{j=1}^{m+1} dx_j \wedge dy_j \). Fix \( S^{2m} = \{ y_{m+1} = 0 \} \). Then the two unique contact points of \( S^{2m} \) are given by \( x_{m+1} = \pm 1 \), and we use the coordinates \( (x_j, y_j)_{j=1}^{m} \) near those points. In those coordinates, \( \beta = \alpha|_{S^{2m}} = \sum_{j=1}^{m} (-y_j dx_j + x_j dy_j) \), and \( d\beta = 2 \sum_{j=1}^{m} dx_j \wedge dy_j \). Then

\[ i_B d\beta = \beta \Rightarrow 2 \sum (dx_j(B)dy_j - dy_j(B)dx_j) = \sum (x_j dy_j - y_j dx_j) \]

so that \( B(p) = \frac{1}{2}p \). Thus \( d_0 B = \frac{1}{2}I_{2m} \), and

\[ \phi_{2k}(S^{2m}) = 2 \text{Tr}(\lambda^{2m-2k} I_{2m}) = 2 \binom{2m}{2k} \] (10)

7. Symplectic-invariant distributions

7.1. Linear algebra. The real anti-symmetric matrices of size \( 2N \times 2N \) will be denoted \( \text{Alt}_{2N} \). Define \( \text{SDiag}(\lambda_1, \ldots, \lambda_N) \in \text{Alt}_{2N}(\mathbb{R}) \) to be the block-diagonal matrix consisting of the \( 2 \times 2 \) blocks

\[ \text{SDiag}(\lambda_j) = \begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix} \]

The following is a standard fact from linear algebra.

Lemma 7.1. For \( A \in \text{Alt}_{2N} \) there is a matrix \( B \in O_{2N}(\mathbb{R}) \) and \( D = \text{SDiag}(\lambda_1, \ldots, \lambda_N) \) such that \( A = B^TDB \). The vector \( (\lambda_1, \ldots, \lambda_N) \) is uniquely defined up to permutations and signs of the \( \lambda_j \).

Denote \( \Delta_N = \{ \lambda_1 \geq \cdots \geq \lambda_N \geq 0 \} \). For \( A \in \text{Alt}_{2N} \), let \( \Lambda(A) \in \Delta_N \) be the unique vector such that \( A = B^T \cdot \text{SDiag}(\Lambda(A)) \cdot B \) for some \( B \in O_{2N}(\mathbb{R}) \).

Recall the multi-Kähler angles \( 0 \leq \theta_1 \leq \cdots \leq \theta_k \leq \frac{\pi}{2} \) of \( E \in \text{Gr}_{2k}(\mathbb{C}^n) \) introduced by Tasaki [Tas01], where \( \kappa = \min(k, n-k) \). They are defined as follows: choose an orthonormal basis \( (e_i)_{i=1}^{2k} \) of \( E \), and define the symplectic Gramm matrix \( A = \omega(e_i, e_j) \). Then \( \Lambda(A) = (\cos \theta_1, \ldots, \cos \theta_k) \).

Proposition 7.2. Let \( \theta_i, i = 1, \ldots, k \) be the multi-Kähler angles of a subspace \( E \in \text{Gr}_{2k}(\mathbb{C}^n) \) chosen at random (with respect to the \( \text{SO}(2n) \)-invariant probability measure). Then the probability distribution of \( (\cos \theta_i)_{i=1}^{k} \) is uniform in \( \Delta_n \).

Proof. We may assume \( 2k \leq n \) so that \( \kappa = k \). We first observe that the distribution is independent of \( n \). Indeed we may condition on the event \( E \subset F \) where \( F \) is any fixed complex \( k \)-dimensional subspace in \( \mathbb{C}^n \), but the distribution of the multi-Kähler angles is clearly independent of \( F \). Thus we assume \( n = 2k \).

Next notice that for a symplectic subspace \( E \), there is a unique, up to order, decomposition \( E = E_1 \oplus \cdots \oplus E_k \), where dim \( E_j = 2 \) and all \( F_j := Ce_j \) are pairwise orthogonal. This decomposition can be found as follows: for an orthonormal basis \( e = (e_j)_{j=1}^{2k} \) of \( E \), consider the matrix \( M(E,e) = (\omega(e_i, e_j))_{i,j=1}^{2k} \). If \( (e') \)
is a different orthonormal basis, there is an equality of row vectors \((e'_i) = (e_j)B\) for some \(B \in O(2k)\), and one checks that \(M(E, e') = B^T M(E, e)B\). Thus for generic \(E\) there is a unique, up to order, orthonormal basis \(e\) such that \(M(E, e) = SDiag(\Lambda(M(E, e)))\). We then set \(E_j = \text{Span}(e_{2j-1}, e_{2j})\) for \(1 \leq j \leq k\).

For a given decomposition \(\mathbb{C}^{2k} = F_1 \oplus \cdots \oplus F_k\) into orthogonal copies of \(\mathbb{C}^2\), the multi-Kähler angles of \(E\) are the collection of Kähler angles of \(E \cap F_j \in Gr^S_{2k}(F_j)\). Thus conditioning on the decomposition, we conclude the multi-Kähler angles of \(E\) are independent and identically distributed, and it remains to find the distribution of the Kähler angle of a real 2-plane \(E \subset \mathbb{C}^2\).

We will work with the oriented Grassmannian. Consider \(\mathbb{R}^4 = \mathbb{C}^2\) with
\[
\omega((x_1 + ix_3, x_2 + ix_4), (x'_1 + ix'_3, x_2 + ix'_4)) = x_1|x'_3 - x'_1x_3 + x_2x'_4 - x'_2x_4.
\]
Let \((u, v)\) be an orthonormal basis for \(E \in Gr^S_{2k}(\mathbb{R}^4)\). We identify \(Gr^S_{2k}(\mathbb{R}^4) = S^2 \times S^2\), \(E \mapsto (z, w)\) using the standard Euclidean structure and the Plücker embedding \(Gr^S_{2k}(\mathbb{R}^4) \subset S(\wedge^2 \mathbb{R}^4)\), \(E \mapsto u \land v = (x_{12}, \ldots, x_{34})\) where \(x_{kl} = u_kv_l - u_lv_k\), followed by the change of coordinates
\[
x_{12} := \frac{w_1 + z_1}{2}, \quad x_{34} := \frac{w_1 - z_1}{2}, \quad x_{13} := \frac{-w_2 + z_2}{2}, \quad x_{24} := \frac{w_2 - z_2}{2}, \quad x_{14} := \frac{-w_3 + z_3}{2}, \quad x_{23} := \frac{w_3 - z_3}{2}.
\]

The corresponding measure on \(S^2 \times S^2\) is the standard one. Then for \(E = (z, w) \in S^2 \times S^2\), and \(u, v\) an oriented orthonormal basis of \(E\), \(\omega(u, v) = x_{13} + x_{24} = -z_2\). Denoting by \(\theta_E\) the Kähler angle, we conclude that \(\cos \theta_E = |\Lambda(E)| = |\omega(u, v)| = |z_2|\) is distributed uniformly in \([0, 1]\) by the theorem of Archimedes.

Let \(\Delta^1_k = \{1 \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0\}\). Denote by \(\Lambda : Gr^S_{2k}(\mathbb{C}^n) \to \Delta^1_k\) the vector \((\cos \theta_j)_{j=1}^k\). We conclude that \(\Lambda_s(d\lambda_1) = \kappa! \prod_{j=1}^k d\lambda_j\).

**7.2. Powers of the Pfaffian.** Consider the meromorphic families of generalized functions \(|Pf_s|_+ \in C^{-\infty}(\text{Alt}^{2n}(\mathbb{R}))\), \(s \in \mathbb{C}\). They can be constructed by first considering \(\text{Res} > -1\), whence \(|Pf_s|_+\) is an integrable function, and then using the Cayley-type identity (see e.g. [CSS13])
\[
Pf(\partial) Pf(X)^{s+1} = (s + 1)(s + 3)\cdots(s + 2N - 1) Pf(X)^s
\]
for a meromorphic extension to \(s \in \mathbb{C}\). Thus the poles of both families are at \(s = -1, -2, \ldots\).

**Theorem 7.3** (Muro [Mur03]). The linear combination \(|Pf(X)|^s := |Pf(X)|_+^s + |Pf(X)|_-^s\) is analytic at even \(s \in \mathbb{Z}\) and has a simple pole at odd \(s \leq -1\). The linear combination sign \(\text{sign}(Pf(X)|Pf(X)|^s := |Pf(X)|_+^s - Pf(X)|_-^s\) is analytic at odd \(s \in \mathbb{Z}\) and has a simple pole at even \(s \leq -2\).

Let \((V, \omega)\) be a 2\(n\)-dimensional symplectic space, \(P\) a compatible Euclidean structure with corresponding complex structure \(J\), so that \(\omega(u, v) = P(Ju, v)\). Then \(P\) induces a Lebesgue measure \(\text{vol}_P(E)\) on all subspaces \(E \subset V\). Define \(\sigma_{\omega, P} : Gr^S_{2k}(V) \to [-1, 1]\) by \(\sigma_{\omega, P}(E) = \frac{\text{vol}_P(E)}{\text{vol}_P(\Delta_2)}\). We will often omit \(P\) from the index when no confusion can arise.

We will now define meromorphic families of distributions \(Gr^S_{2k}(\mathbb{R}^{2n})\) given by \(\mu_{\pm}(s) = |\sigma_{\omega}(E)|^s_{\pm} d\sigma_1(E)\) for large \(\text{Re}(s)\). The construction is virtually identical to
the one carried out in [Pai17], with $\text{Sp}(2n)$ replacing $O(p, q)$. We present it here for the reader’s convenience.

We will assume for now that $2k \leq n$.

Let $U \subset \text{Gr}_{2k}(V)$ be an open set. Let $B_E = (u_1(E), \ldots, u_{2k}(E)) : U \to V^k$ be a smooth field of $P$-orthonormal bases of $E \in U$. Define the function $M_P : U \to \text{Alt}_{2k}(\mathbb{R})$ given by $M_P(E) = \omega(u_1(E), u_j(E))_{i,j=1}^{2k}$. Note that $\sigma_\omega(E) = \text{Pf}(M_P(E))$.

Denote by $U_P \subset \text{Gr}_k(V)$ the open, dense subset of subspaces $E \in \text{Gr}_{2k}(V)$ for which $E \cap JE = \{0\}$. Clearly $E \in U_P$ if and only if $1 \not\in \Lambda(M_P(E))$.

**Lemma 7.4.** $M_P$ is a proper submersion at every $E \in U \cap U_P$.

**Proof.** Consider a curve $\gamma_1$ through $E$ given by

$$\gamma_1(t) = \text{Span}(u_1(t), u_2, \ldots, u_{2k})$$

with $u_2, \ldots, u_{2k}$ fixed, and $\xi = u_1(0) \in E^P$ arbitrary. It follows that

$$D_E M_P(\gamma_1(0)) = \begin{pmatrix}
0 & \omega(\xi, u_2) & \cdots & \omega(\xi, u_{2k}) \\
-\omega(\xi, u_2) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\omega(\xi, u_{2k}) & 0 & \cdots & 0
\end{pmatrix}$$

Since $E \in U_P$, $\omega(u_j, \cdot)_{j=2}^{2k}$ are linearly independent functionals in $\xi \in E^P$, and so the first row of $D_A M_P(\gamma_1)$ is arbitrary, while the other entries in the upper triangle vanish. Replacing $\gamma_1$ with $\gamma_j$ in the obvious way, we conclude $D_E M_P(\sum \alpha_j \gamma_j(0))$ can be arbitrary, thus concluding the proof. \(\square\)

**Lemma 7.5.** One can choose finitely many $\omega$-compatible Euclidean structures $P_i$ s.t. $\{U_{P_i}\}$ cover $\text{Gr}_{2k}(V)$.

**Proof.** Given $E \in \text{Gr}_{2k}(V)$, a generic choice of an $\omega$-compatible $(P, J)$ would have $E \cap JE = \{0\}$ by a trivial dimension count. Fixing one such $J = J(E)$ with corresponding $P(E)$ for every $E$, we get an open cover $(U_{P(E)})_{E \in \text{Gr}_{2k}(V)}$ of $\text{Gr}_{2k}(V)$. The claim now follows by the compactness of $\text{Gr}_{2k}(V)$. \(\square\)

We now explain how to pull-back $|\text{Pf}(X)|^s_\pm$ to $\text{Gr}_{2k}^+(V)$, using the locally-defined submersion $M_P$.

**Definition 7.6.** For $s \in \mathbb{C}$, let $\mathcal{D}^s$ be the line bundle of $s$-densities over $\text{Gr}_{2k}^+(V)$, which has fiber $\text{Dens}^s(E)$ over $E \in \text{Gr}_{2k}^+(V)$. We say that a choice of generalized section $f(s) \in \Gamma^{\infty}(U, \mathcal{D}^s)$ over $U \subset \text{Gr}_{2k}^+(V)$ for $s \in \Omega \subset \mathbb{C}$ is meromorphic in $s$ if, having fixed a Euclidean metric $P$ and using it to identify all bundles $\mathcal{D}^s$, one obtains a map $f_P : \Omega \to C^{\infty}(U)$ which is meromorphic in $s$.

We denote by $\mathfrak{M}^{\infty}(\mathcal{D}^s)$ the sheaf for which $\Gamma(U, \mathfrak{M}^{\infty}(\mathcal{D}^s))$ is the space of meromorphic in $s$ maps $\mathbb{C} \to \Gamma^{\infty}(U, \mathcal{D}^s)$.

Recall the orbits $X_\pm^{2k}$, $(X_\pm^{2k})_{r=0}^{\kappa-1}$ of $\text{Gr}_{2k}^+(V)$ under $\text{Sp}(V)$ defined in the paragraph following eq. (3), where $\kappa = \min(k, n-k)$. In terms of multi-Kähler angles, $E \in X_\pm^{2k}$ precisely when exactly $r$ of the angles are distinct from $\frac{\pi}{2}$.

We are now ready to construct the meromorphic families.

**Proposition 7.7.** There are global sections $f_\pm(s) = M^+_P |\text{Pf}|^s_\pm$ of $\mathfrak{M}^{\infty}(\mathcal{D}^s)$ supported on $X_\pm^{2k}$, s.t. whenever $E$ is not a pole of $f_\pm$, $f_\pm(s)$ is $\text{Sp}(V)$-invariant.
Proof. Assume first 2k ≤ n. Let \( P_i \) be a finite collection of \( \omega \)-compatible Euclidean structures as in Lemma 7.5 and let \( U_i = U_{P_i} \subset \text{Gr}^{+}_{2k}(V) \) be the corresponding open sets of generic subspaces. For each \( i \), cover \( U_i \) by open sets \( U_{ij} \subset U_i \) so that \( M_{ij} = M_{P_i} : U_{ij} \rightarrow \text{Alt}_{2k} \) can be defined by some smooth field of orthonormal bases of \( E \) over \( U_{ij} \). Now since \( M_{ij} \) is a proper submersion, one obtains a meromorphic in \( s \) family of functions \( \hat{f}^{\pm}_{ij}(E; s) \in C^{-\infty}(U_{ij}) \) given by \( \hat{f}^{\pm}_{ij}(\cdot; s) = M_{ij}^s|_{P|}^{\frac{1}{2}} \).

It then obviously holds that on \( U_{ij} \cap U_{ij'} \), \( \hat{f}^{\pm}_{ij}(\Lambda; s) \) and \( \hat{f}^{\pm}_{ij'}(\Lambda; s) \) coincide as continuous functions for \( \text{Re}(s) > 0 \). Therefore, they coincide on \( U_{ij} \cap U_{ij'} \) as meromorphic functions, and we may merge all \( \hat{f}^{\pm}_{ij} \) into one meromorphic family \( \hat{f}^{\pm}(\cdot; s) \in C^{-\infty}(U_i) \). The corresponding (through \( P_i \)) section \( f^{\pm}_i \in \Gamma(U_i, \mathfrak{M}^{-\infty}(\mathcal{D}^s)) \) is obviously \( \text{Sp}(V) \cap \text{O}(P_i) \)-invariant. Moreover, it is \( \mathfrak{sp}(V) \)-invariant. Next, we claim that \( f^{\pm}_i \) and \( f^{\pm}_j \) coincide on \( U_i \cap U_j \). Since both are meromorphic, we may assume in the following that \( \text{Re}(s) > 0 \).

It is easy to see, using Lemma 5.1 as in the proof of Proposition 5.1, that for \( \text{Re}(s) > 0 \), no \( \text{Sp}(V) \)-invariant generalized sections of \( \mathcal{D}^s \) can be supported on a set of positive codimension: using Lemma 5.1 we consider for \( \alpha \geq 0 \) the bundle \( F_{X_{r^2}}^{\alpha} \) over \( X_{r^2} \) where \( r > 0 \), that has fiber \( \text{Dens}(E)^s \otimes \text{Dens}^s(\mathcal{N}_E X_{r^2}^{k, +}) \otimes \text{Sym}^s(\mathcal{N}_E X_{r^2}^{k, +}) \) over \( E \). Denoting \( E_0 = E \cap E^\omega \), by Corollary 2.2

\[
F_{X_{r^2}}^{\alpha}|_{E} = \text{Dens}(E)^s \otimes \text{Dens}(\Lambda^2 E_0) \otimes \text{Sym}^s(\Lambda^2 E_0^s)
\]

which clearly contains no \( \text{Sp}(V) \)-invariants. It follows that the space of \( \text{Sp}(V) \)-invariants in \( \Gamma^{-\infty}(\text{Gr}^{+}_{2k}(V), \mathcal{D}^s) \) supported on \( X_{r^2}^{\pm} \) is at most 1-dimensional for each \( \epsilon \in \{\pm\} \).

Since \( U_i \subset \text{Gr}^{+}_{2k}(V) \) is dense, it follows by construction that for \( \text{Re}(s) > 0 \), \( f^{\pm}_i(\cdot; s) \) extends by continuity to an \( \text{Sp}(V) \)-invariant section of \( \mathcal{D}^s \) over \( \text{Gr}^{+}_{2k}(V) \) supported on \( X_{r^2}^{\pm} \), and by the previous paragraph we can find meromorphic functions \( c_i(s) \), such that \( c_1(s) = 1 \) and the sections \( c_i(s)f^{\pm}_i(\Lambda; s) \) coincide for all \( i \). Denoting by \( p_i(s) \in \Gamma^{\infty}(\text{Gr}^{+}_{2k}(V), \mathcal{D}^s) \) the Euclidean section defined by \( P_i \), it holds that

\[
f^{\pm}_i(E; s) = |\text{Pf} M_{P_i}(E)|^s p_i(s)
\]

for \( E \in X_{r^2}^{\pm} \), so that

\[
|\text{Pf} M_{P_i}(E)|^s p_i(s) = c_i(s) |\text{Pf} M_{P_i}(E)|^s p_i(s)
\]

implying

\[
c_i(s) = \left( \frac{|\text{Pf} M_{P_i}(E)|}{|\text{Pf} M_{P_i}(E)|} \right)^{s} p_i(s)
\]

for all \( E \in X_{r^2}^{\pm} \). Since \( c_i(s) \) is independent of \( E \), one has \( c_i(s) = c_i^s \) for some \( c_i > 0 \).

Finally, for \( s = 1 \), \( \mathcal{D}^1 \) is the Klein bundle of Lebesgue measures, and it is easy to see that all \( f^{\pm}_i(1) \) represent the Liouville measure induced by the symplectic form on every symplectic subspace \( E \). It follows that \( c_i = 1 \Rightarrow c_i(s) \equiv 1 \). Thus we have shown that \( f^{\pm}_i, f^{\pm}_j \) coincide in \( \Gamma(U_i \cap U_j, \mathfrak{M}^{-\infty}(\mathcal{D}^s)) \).

We conclude there is a globally defined section \( f^{\pm}_i \) of \( \mathfrak{M}^{-\infty}(\mathcal{D}^s) \) which is \( \mathfrak{sp}(V) \)-invariant and supported on \( X_{r^2}^{\pm} \), respectively. For \( \text{Sp}(V) \)-invariance, we observe it holds for \( \text{Re}(s) > 0 \) and then invoke uniqueness of meromorphic continuation.
This concludes the proof when $2k \leq n$. For the case $2k > n$, we simply use the oriented skew-orthogonal complement map $Gr^+_{2k}(V) \to Gr^+_{2n-2k}(V)$ to pull-back $f_*$. This is a valid operation since we have the equivariant identification $E^\omega \simeq (V/E)^* \Rightarrow \text{Dens}(E^\omega) \simeq \text{Dens}(E)$.

\[\square\]

**Definition 7.8.** Set $|\sigma_\omega|_{\pm} \in C^{-\infty}(Gr^+_{2k}(V))$ to be the value of $f_+(s)$ under the Euclidean trivialization.

**Lemma 7.9.** Write $\kappa = \min(k, n-k)$. One has $\int_{Gr_{2k}(R^{2n})} |\sigma_\omega|_+^s \, d\sigma_1(E) = \frac{1}{2\kappa(s+1)^n}$.

\[\text{Proof.}\] Using the cosines of the multi-Kähler angles in decreasing order, denoted $\Lambda(E) = (\lambda_1, \ldots, \lambda_n)$, $\lambda_j = \cos \theta_j$, we have $|\sigma_\omega(E)| = \prod_{j=1}^n \lambda_j$. Then for $\Re(s) > 0$, using Proposition 7.2 we get

$$\int_{Gr_{2k}(R^{2n})} |\sigma_\omega|^s \, d\sigma_1(E) = \frac{1}{n!} \int_{[0,1]^n} \prod_{j=1}^n \lambda_j^s \, d\lambda = \frac{1}{n!(s+1)^n}$$

and the result follows by uniqueness of meromorphic extension. \[\square\]

The two values $s = -2n, -(2n+1)$ are of particular interest, as evidenced in the two theorems below. The first theorem concerns the linear Grassmannian:

**Theorem 7.10.** The distribution $\mu^+_\omega \in \mathcal{M}^{-\infty}(Gr^+_{2k}(V))$ given by $\mu^+_\omega := |\sigma_\omega|^s \, d\sigma_1(E)$ is $\text{Sp}(V)$-invariant, has full support and $\int_{Gr_{2k}(V)} \mu^+_\omega \neq 0$. It is even with respect to orientation reversal.

In particular, we get a canonically normalized $\text{Sp}(V)$-invariant distribution $\mu_\omega := \pi_* \mu^+_\omega$ on $Gr_{2k}(2n)$, where $\pi : Gr^+_{2k}(V) \to Gr_{2k}(V)$ is the double cover map.

\[\text{Proof.}\] A distribution over $Gr^+_{2k}(V)$ is a generalized section of the bundle with fiber $\text{Dens}(T_E Gr^+_{2k}(V))$ over $E$, which is $\text{Sp}(V)$-isomorphic to $D^{-2n}$. All statements follow immediately from Proposition 7.7, Theorem 7.3 and Lemma 7.9. \[\square\]

We conjecture that $\mu_\omega$ is the unique $\text{Sp}(V)$-invariant distribution on $Gr_{2k}(V)$. This was shown by Gourevitch, Sahi and Sayag in [GSS15] for $k = n$ when $n$ is even.

Similarly, we have a statement for the affine Grassmannian. We define $|\sigma_\omega(E)|_r \in C^{-\infty}(AGr^+_{2k}(V))^{tr}$ by pulling-back by the projection map $AGr^+_{2k}(V) \to Gr^+_{2k}(V)$. Let $dE$ be the measure on $AGr^+_{2k}(V)$ which is built out of $d\sigma_1$ on $Gr^+_{2k}(V)$, and the Euclidean measure on translations. Define the odd distribution

$$\overline{\mu}_\omega := \text{sign}(\sigma_\omega) |\sigma_\omega(E)|^{-2n-1} \, dE \in \mathcal{M}^{-\infty}(AGr^+_{2k}(V))$$

and the even distribution

$$\overline{\mu}_0 := \text{Res}_{s=-2n-1} |\sigma_\omega(E)|_r^s \, dE \in \mathcal{M}^{-\infty}(AGr^+_{2k}(V)).$$

**Theorem 7.11.** The distributions $\overline{\mu}_\omega, \overline{\mu}_0$ are $\text{Sp}(V)^+$-invariant. $\overline{\mu}_\omega$ has full support.

The proof is as in the linear case. In particular, there is a an even, canonically normalized $\text{Sp}(V)^+$-invariant distribution $\overline{\mu}_0 := \pi_* \mu^+_0$ on $AGr_{2k}(V)$, supported on the $\omega$-degenerate subspaces.

We will need the following a-priori information about the wavefront set of $\overline{\mu}_\omega$.

**Proposition 7.12.** The wavefront set of $\overline{\mu}_\omega$ belongs to $\cup_r N^*X^r_{2k}$.

\[\text{Proof.}\] This is immediate from $\text{Sp}(V)^+$-invariance. \[\square\]
8. The contact sphere

8.1. A Crofton basis for $S^{2n+1}$. In this section, $V = \mathbb{R}^{2n+2}$, $M = S^{2n+1} = \mathbb{P}_+(V)$. Take $\xi \in \mathbb{P}_+(V)$ so that $T_\xi S^{2n+1} = \xi^* \otimes V/\xi$. The contact hyperplane is $H_\xi = \xi^* \otimes \xi^* \subset T_\xi \mathbb{P}_+(V)$. Then $T_\xi S^{2n+1}/H_\xi = \xi^* \otimes V/\xi^* \simeq (\xi^*)^\otimes 2 \text{ Stab}(\xi)$-equivariantly. The symplectic form on $\xi^* \otimes V$ defines a form $\omega_\xi \in \wedge^2 H_\xi^* \otimes (\xi^*)^\otimes 2 = \wedge^2 H_\xi^* \otimes T_\xi S^{2n+1}/H_\xi$. We remark that the form $\omega_\xi$ is determined by the contact distribution alone, without reference to the form $\omega$ on $V$, see Example 4.4. The linear symplectic group $\text{Sp}(V)$ acts on $S^{2n+1}$ by contactomorphisms. The stabilizer of $\xi \in S^{2n+1}$ is easily seen to act on $T_\xi S^{2n+1}$, which is a dual Heisenberg algebra, by its full group $\text{Sp}_H(2n+1)$ of automorphisms.

Now $\dim \text{Sp}(2n+2) - \dim \text{Sp}_H(2n+1) = (n+1)(2n+3) - (n+1)(2n+1) = 2n+2 = \dim S^{2n+1} + 1$, so that $\dim \text{Stab}(\xi) = \dim \text{Sp}_H(2n+1) + 1$, and we denote $\text{Stab}(\xi) = \text{Sp}_H(2n+1)$. The kernel of the restriction homomorphism $\text{Sp}_H(2n+1) \to \text{Sp}_H(2n+1)$ consists of the linear maps that fix all $v \in \xi^* \otimes V$ and acts on some $w \in V/\xi^*$ by $w \mapsto w + \lambda \xi$, for some $\lambda \in \mathbb{R}$.

Thus $S^{2n+1} = \text{Sp}(2n+2)/\text{Sp}_H(2n+1)$. Inspired by the analogy to the Riemannian symmetric space presentation $S^{2n+1} = \text{SO}(2n+2)/\text{SO}(2n+1)$, we look for Crofton-type formulas for the contact valuations on the sphere.

Consider the double fibration

$$
\begin{array}{ccc}
\text{Gr}_{2k}(V) & \xleftarrow{\tau} & W \\
\downarrow \pi & & \downarrow \pi \\
S^{2n+1} & \xrightarrow{\tau} & \mathbb{P}_+(V)
\end{array}
$$

where $W$ is the partial flag manifold $\{(E, \theta) \in \text{Gr}_{2k}(V) \times S^{2n+1} : \theta \in E\}$

**Definition 8.1.** For $0 \leq k \leq n$, define the generalized valuation $\psi_{2k}$ through the Crofton formula (in the sense of subsection 3.3)

$$
\psi_{2k} := \pi_* \tau^*(\mu_{\omega}) = \int_{\text{Gr}_{2n+2-2k}(V)} \chi(\bullet \cap E) \sigma_\omega(E)^{-2n-2} d\sigma_1(E) \in \mathcal{V}^{-\infty}(S^{2n+1})^\text{Sp}(V)
$$

It follows from Lemma 3.14 that $\psi_{2k} \in \mathcal{W}_{2\infty}^{-\infty}(S^{2n+1})$. Moreover, since $\psi_{2k}(S^{2k}) \neq 0$ by Theorem 6.10 (see the computation preceding eq. (14) below for precise value), we conclude $\psi_{2k} \in \mathcal{W}_{2\infty}^{-\infty}(S^{2n+1}) \setminus \mathcal{W}_{2k+1}^{-\infty}(S^{2n+1})$, and in particular all $\psi_{2k}$ are linearly independent.

**Proposition 8.2.** $\mathcal{V}^{-\infty}(S^{2n+1})^\text{Sp}(V)$ is spanned by $\chi, \psi_2, \ldots, \psi_{2n}$.

*Proof.* The proof is identical to that of Theorem 6.2 with $\text{Sp}(V)$ replacing the full group of contactomorphisms. The proof remains valid since $\text{Sp}(V)$ acts transitively, with the same action of the stabilizer on the tangent space by $\text{Sp}_H(2n+1)$. □

This completes the proof of Theorem 1.8. In light of Theorem 6.2 we get

**Corollary 8.3.** For $0 \leq k \leq n$, $\psi_{2k}$ are linear combinations of $\phi_{2j}$, $0 \leq j \leq n$. In particular, $\psi_{2k}$ is invariant under all contactomorphisms of $\mathbb{P}_+(V)$.

Thus we establish Theorem 1.7 except for the explicit determination of the coefficients which is deferred to the next subsection.

It follows also that $\psi_{2k}(F)$ is well-defined for subsets $F \subset S^{2n+1}$ normally transversal to the contact distribution. We will make use of the following lemma.
Lemma 8.4. Assume that a closed submanifold $F \subset S^{2n+1}$ is normally transversal, and $\chi_F := \chi(F \cap \bullet) \in C^{-\infty}(\text{Gr}_{2n+2-k}(V))$ has wavefront disjoint from $\text{WF}(\mu)$. Then

$$\psi_{2k}(F) = \int_{\text{Gr}_{2n+2-2k}(V)} \chi(F \cap E) d\mu(E)$$

Proof. Choose an approximate identity $\rho_\varepsilon$ on $GL(V)$, and define $\phi_\varepsilon = \int_{GL(V)} g^* \chi_F \cdot \rho_\varepsilon(g) dg \in \mathcal{V}(S^{2n+1})$. Then

$$\phi_{2k}(F) = \lim_{\varepsilon \to 0} (\psi_{2k}, \phi_\varepsilon) = \lim_{\varepsilon \to 0} \int_{\text{Gr}_{2n+2-2k}(V)} \phi_\varepsilon(E) d\mu(E)$$

Now $\phi_\varepsilon(E) = \int_{GL(V)} \chi_F(gE) \rho_\varepsilon(g) dg$ converges to $\chi_F \in C^{-\infty}(\text{Gr}_{2n+2-k}(V))$ in Hörmander’s topology on the space of generalized functions with wavefront set contained in $\text{WF}(\chi_F)$. It follows that $\psi_{2k}(F) = \int_{\text{Gr}_{2n+2-2k}(V)} \chi_F(E) d\mu(E)$.

8.2. Integral geometry of the contact sphere. Here we determine the coefficients in Theorem 1.7. We have two different bases of contact-invariant valuations on the contact sphere, indexed by $0 \leq k \leq n$. Namely, we have $\phi_{2k}$ defined in terms of curvature at the contact points, and $\psi_{2k}$ given by Crofton integrals. Since $\phi_{2k}, \psi_{2k} \in \mathcal{W}_k(S^{2n+1})$, they are related by a triangular matrix, that is

$$\psi_{2i} = \sum_{j=i}^n c_{ij}^n \phi_{2j}.$$  \hspace{1cm} (13)

We will compute $c_{ij}^n$ by evaluating both bases on all great spheres $S^{2m}$.

Take $F = S^{2n} = S^{2n+1} \cap \Pi$ where $\Pi \subset S^{2n+1}$ is a fixed hyperplane. It is normally transversal. Note also that $\chi(F \cap E) = 2$ for a generic $E \in \text{Gr}_2(\mathbb{R}^{2n+2})$, that is $\chi_F$ is the constant 2 on $\text{Gr}_2(\mathbb{R}^{2n+2})$. Thus by Lemma 8.4 we may compute $\psi_{2i}(F)$ using the explicit Crofton formula. Denote $\kappa = \min(i, n+1-i)$. By Lemma 7.4

$$\psi_{2i}(S^{2n}) = \int_{\text{Gr}_{2n+2-2i}(\mathbb{R}^{2n+2})} \chi(S^{2n} \cap E) \sigma_\omega(E)^{-2n-2} d\sigma_1(E)$$

$$= 2 \int_{\text{Gr}_{2n+2-2i}(\mathbb{R}^{2n+2})} \sigma_\omega(E)^{-2n-2} d\sigma_1(E) = (-1)^\kappa \frac{2}{\kappa!} \frac{1}{(2n+1)^\kappa}$$

Considering spheres of lower dimension, we see that $\psi_{2i}(S^{2m}) = \psi_{2i}(S^{2n})$ if $m \geq i$ and zero otherwise. Thus

$$\psi_{2i}(S^{2m}) = \begin{cases} (-1)^\kappa \frac{2}{\kappa!} \frac{1}{(2n+1)^\kappa}, & i \leq m \\ 0, & i > m \end{cases}$$  \hspace{1cm} (14)

On the other hand by eq. 10,

$$\phi_{2j}(S^{2m}) = 2 \binom{2m}{2j}$$

We now plug those values into 13. Define the $(n-i+1) \times (n-i+1)$ lower-triangular matrix $A_i$ by $A_i(m, j) = \binom{2m}{2j}$, $i \leq m, j \leq n$. Set $b_i = (-1)^\kappa \frac{1}{\kappa!} \frac{1}{(2n+1)^\kappa}$. Then $c_{ij}^n = b_i \sum_{m=i}^j (A_i^{-1})(j, m)$.
8.3. **Contact curvature of convex sets.** Here we prove an upper bound on the top contact valuation of a convex set.

**Theorem 8.5.** Let \( K \subset S^{2n+1} \) be a convex subset with \( C^2 \) boundary. Then \( \phi_{2n}(\partial K) \leq 2 \). Equality is attained for \( K_0 := S^{2n+1} \cap H_+ \), where \( H_+ \subset V \) is a half space bounded by a linear hyperplane (but there are other minimizers).

**Proof.** We will use an auxiliary complex structure: \( V = \mathbb{C}^{n+1} \), \( S^{2n+1} \) is identified with the unit sphere therein. We may assume that \( \partial K \) is normally transversal.

First, let us verify that \( \partial K \) has exactly two contact tangent hyperplanes. Since the Euler characteristic \( \chi(\partial K) = 2 \), this amounts to verifying that the intersection index of \( N^* K \) and \( M_H \) in \( \mathbb{P}_{S^{2n+1}} \) at every contact point is +1. We will refer to this number as the contact index, denoted \( I_H(\partial K, x) \).

Consider a sphere \( E = S^{2n} \subset S^{2n+1} \) given by a quadratic equation in \( V \). It is easy to check by an explicit computation that \( E \) has exactly two contact tangent points. Since \( \chi(E) = 2 \), we conclude that the contact index of \( E \) at each of those points is +1. Now let \( x \in K \) be a contact tangent point, and let \( E \) be the osculating sphere at \( x \). Then \( I_H(\partial K, x) = I_H(E, x) = 1 \).

Next, let \( p \in \partial K \) be a point where \( T_p \partial K = H_p \). We now project the hemisphere \( U \) centered at \( p \) to \( \mathbb{R}^{2n+1} = T_p S^{2n+1} \) by a central projection \( \pi \) from the origin \( p \), so that \( p \) is mapped to the origin. Clearly \( \pi(K \cap U) \) is convex near the origin: if \( K = S^{2n+1} \cap C \), where \( C \) is a convex cone, then \( \pi(K \cap C) = C \cap T_p S^{2n+1} \). By a standard computation and assuming \( p = \{ y_{n+1} = 1 \} \), the resulting contact structure in \( \mathbb{R}^{2n+1} \) is given by the contact form \( \alpha = -dx_{n+1} + \sum_{j=1}^{n} (-y_j dx_j + x_j dy_j) \). We will write \( z = -x_{n+1} \).

We thus consider a convex body \( K \) with \( C^2 \) boundary in the contact space \((\mathbb{R}^{2n+1}, \alpha)\). We assume \( K \) is tangent to the contact distribution at the origin, which is \( \mathbb{R}^{2n} = \{ z = 0 \} \), and further assume without loss of generality that \( K \) lies below it. The normal to the contact distribution is \( \nu_H = (-y_1, \ldots, -y_n, x_1, \ldots, x_n, 1) \), the normal to \( \partial K \) is \( \nu_K \). Then by Proposition \[4.20\]

\[ \phi_{2n}(\partial K, 0)^{-1} = |\det(d_0 \nu_K - d_0 \nu_H)| \]

where \( d_0 \nu_K, d_0 \nu_H : \mathbb{R}^{2n} \to T_{c_{2n+1}} S^{2n} \), the latter space identified with \( \mathbb{R}^{2n} \). Using the coordinates \( x_1, \ldots, x_n, y_1, \ldots, y_n \) on \( \mathbb{R}^{2n} \) we get

\[ d_0 \nu_H = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = J \]

Write also \( S = d_0 \nu_K \). Thus \( \phi_{2n}(\partial K, 0)^{-1} = |\det(S - J)| = |\det(I + JS)| \).

Note that \( S \geq 0 \) since \( K \) is convex. Then \( JS \) and \( \sqrt{S} J \sqrt{S} \) have the same characteristic polynomial. The latter matrix is antisymmetric, hence the roots of its characteristic polynomial appear in purely imaginary pairs \( \pm i \lambda_j \), \( j = 1, \ldots, n \). Thus

\[ \phi_{2n}(\partial K, 0)^{-1} = \prod_{j=1}^{n} (1 + i \lambda_j)(1 - i \lambda_j) = \prod_{j=1}^{n} (1 + \lambda_j^2) \geq 1 \]

The stated inequality follows. Equality occurs if and only if the second fundamental form \( d \nu_K \) vanishes at both contact points. \( \square \)

**Remark 8.6.** The case of equality is far from unique: any convex subset which is flat to second order at its two contact points would have the same value of \( \phi_{2n} \).
9. Symplectic integral geometry

9.1. Symplectic space. Let us first show there is no interesting symplectic valuation theory.

**Theorem 9.1.** There is no Sp(2n)-invariant, translation invariant generalized valuation except for χ and vol_{2n}.

*Proof.* Let \( N = \lfloor \frac{\min(k,2n-k)}{2} \rfloor \) be the number of multi-Kähler angles for \( E \in \text{Gr}_k(\mathbb{R}^{2n}) \). Since \( U(n) \subseteq \text{Sp}(2n) \), an Sp(2n)-invariant valuations \( \phi \) would also be \( U(n) \)-invariant. In particular, it is smooth. We consider two separate cases. When \( k = 2l + 1 \) and \( E \in \text{Gr}_k(V) \) is maximally non-degenerate, then there is no Stab(\( E \))-invariant Lebesgue measure on \( E \). Since such subspaces are dense in \( \text{Gr}_k(V) \) we conclude \( \text{Kl}(\phi) = 0 \), and hence \( \phi = 0 \).

When \( k = 2l \), an Sp(2n)-invariant section of the Klain bundle should be proportional to \( |\omega|_E \), that is after Euclidean trivialization it is proportional to \( |\sigma(E)| = \prod_{i=1}^N \cos \theta_i \). But by \([BF11]\), the Klain section of a U(\( n \))-invariant valuations should be given by a symmetric polynomial of \( \cos^2 \theta_i \). Thus there can be no non-trivial Sp(2n)-invariant valuations unless \( k = 0,2n \). \( \square \)

Recall the distributions \( \overline{\mu}, \overline{\pi}_0 \in \mathcal{M}^{\infty}(\text{AGr}_{2n-2k}(V))^{\text{Sp}(V)} \) given by eqs. (11) and (12).

**Corollary 9.2.** For \( 1 \leq k \leq n - 1 \) and a smooth convex body \( K \subseteq V \) it holds that

\[
\int_{\text{AGr}_{2n}^+(V)} \chi(K \cap E) d\overline{\pi}_0(E) = 0.
\]

Put differently, \( \overline{\pi}_0 \) lies in the kernel of the cosine transform.

Nevertheless, we can write symplectic Crofton formulas with the oriented valuation theory approach detailed in Appendix A. For a compact oriented C^1-submanifold with boundary \( F \subseteq V \) of codimension \( 2k \), set \( \text{ind}_F(E) = \text{I}(E,F) \) for \( E \in \text{AGr}_{2k}^+(V) \), which is well-defined whenever \( E \) and \( F \) intersect transversally.

**Lemma 9.3.** For \( F \) as above, \( \text{ind}_F \in C^{\infty}(\text{AGr}_{2k}^+(V)) \), and \( \text{WF}(\text{ind}_F) \cap N^* X_{r,2k}^+ = \emptyset \) for all \( r < \min(k,n-k) \).

*Proof.* Let \( Z = \{ (x,E) : x \in E \} \subseteq V \times \text{AGr}_{2k}^+(V) \) be the incidence manifold, which has a natural orientation, and denote by \( \tau : Z \to V \), \( \pi : Z \to \text{AGr}_{2k}^+(V) \) the obvious submersions. Consider \( Z_F = \tau^{-1}(F) \), which is a C^1 oriented submanifold of \( Z \) of codimension \( 2k \). Define \( \delta_{Z_F} := \tau^*((F)) = [[Z_F]] \). Note that \( \pi_* \tau^*((F)) \in C^{\infty}(\text{AGr}_{2k}^+(V)) \), and \( \pi_* \tau^*((F))(E) = \text{I}(E,F) \) whenever \( E \pitchfork F \). The first statement follows. Note that \( \text{WF}(\delta_{Z_F}) \subseteq \text{Im}(\tau^*) \), and therefore also \( \tau^* \text{WF}(\text{ind}_F) \subseteq \text{Im}(\tau^*) \). As \( \tau^* \) is injective, the statement of the lemma would follow from \( \text{Im}(\tau^*) \cap \pi^* N^* X_{r,2k}^+ = \{ 0 \} \iff \text{Im}(\tau^*) \cap N^* \pi^{-1} X_{r,2k}^+ = \{ 0 \} \).

Take \((x,E) \in Z_F \cap \pi^{-1} X_{r,2k}^+ \). Define \( \tilde{Z}_x := \tau^{-1}(x) \subseteq Z \). Then \( \text{Im}(\tau^*) \cap T_{x,E}^* = \tau^*(T_x^* V) = N^* \tilde{Z}_x \). Let us check that \( \tilde{Z}_x \) and \( \pi^{-1} X_{r,2k}^+ \) intersect transversally at \((x, E)\).

For a tangent vector \((v, \Xi) \in T_{x,E}Z \subseteq T_xV \times T_E \text{AGr}_{2k}^+(V) \), use Lemma 2.3 to decompose \( \Xi = \Xi_x + \Xi_E \), with \( \Xi_x \in T_{E} Z_x \) where \( Z_x := \pi \tilde{Z}_x \), and \( \Xi_E \in T_E X_{r,2k}^+ \). In the product manifold \( V \times \text{AGr}_{2k}^+(V) \) we get the equality \((v, \Xi) = (0, \Xi_x) + (v, \Xi_E) \). Since \((0, \Xi_x) \in TZ \) while by assumption \((v, \Xi) \in TZ \), we conclude \((v, \Xi_E) \in TZ \).
Thus \( T_{x,E}Z = T_{x,E} \tilde{Z}_x + T_{x,E} \pi^{-1} X^{2k,+} \Rightarrow N^* Z_F \cap N^* \pi^{-1} X^{2k,+} = \{0\} \), concluding the proof.

\[\square\]

**Theorem 9.4.** Let \( F \subset V \) be a \( C^1 \) compact, oriented submanifold with boundary. Then

\[
\int_{F} \omega^k = \frac{(-1)^{\kappa}}{2} \binom{n}{k} \binom{2n}{2k}^{-1} n^k \int_{A Gr^+_{2n-2k}(V)} I(E,F) d\overline{\mu}_\omega(E)
\]

where \( \kappa = \min(k, n-k) \).

**Proof.** Considered as a function of \( F \), the integral on the right hand side is a Crofton integral as in Proposition \([\text{A.1}]\). Hence it defines a closed, \( 2k \)-form on \( V \) which is \( \text{Sp}(V) \)-invariant. By the fundamental theorem of invariant theory, it is a multiple of \( \omega^k \), that is

\[
\int_{A Gr^+_{2n-2k}(V)} I(E,F) d\overline{\mu}_\omega(E) = C \int_{F} \omega^k
\]

(15)

It remains to find the constant \( C \). We will use a compatible Euclidean structure. Let \( B_W \) be the unit Euclidean ball in the \( \omega \)-positively oriented subspace \( W \in X^{2k}_+ \). We will average the integral over \( X^{2k}_+ \) with respect to the probability measure \( dW \) that is invariant under \( \text{so}(2n) \). For easy computation, we replace the exponent of \( \sigma_\omega \) in \( \text{Pr}_\omega(E) = \text{sign} \sigma_\omega(E)|\sigma_\omega(E)|^{-2n-1} dE \) with the meromorphic variable \( s \in \mathbb{C} \) and compute for real \( s \) which is sufficiently large so that all integrands are continuous.

\[
A_s := \int_{X^{2k}_+} dW \int_{A Gr^+_{2n-2k}(V)} I(E,B_W) \text{sign} \sigma_\omega(E)|\sigma_\omega(E)|^s dE
\]

\[
= \int_{Gr^+_{2n-2k}(V)} \text{sign} \sigma_\omega(E)|\sigma_\omega(E)|^s \int_{X^{2k}_+} dW \int_{V/E} I(E + x, B_W) dx
\]

Here the inner integral is with respect to \( dx \), the Euclidean Lebesgue measure under the identification \( V/E = E^+ \). We may write

\[
\int_{V/E} I(E + x, B_W) dx = \text{sign} \sigma_\omega(E) \text{vol}_{2k}(\text{Pr}_{E^+}(B_W))
\]

Hence

\[
A_s = \int_{Gr^+_{2n-2k}(V)} |\sigma_\omega(E)|^s \text{vol}_2(\text{Pr}_{E^+}(B_W)) dW
\]

The inner integral is independent of \( E \) and can be computed using the Kubota formula.

\[
\int_{Gr_{2n-2k}(V)} \text{vol}_{2k}(\text{Pr}_{E^+}(B^{2k})) d\sigma_1(E) = c_0 \text{vol}_{2k}(B^{2k}) = c_0 \frac{\pi^k}{k!}
\]

where \( B^{2k} \) is any fixed \( 2k \)-dimensional Euclidean ball, and

\[
c_0 = \frac{2n}{2k} = \frac{\binom{2n}{2k}}{\text{vol}_{2k}(B^{2n}) \text{vol}_{2n-2k}(B^{2n-2k})} = \frac{\binom{2n}{2k} \binom{n}{k}}{2k}^{-1}
\]

We get

\[
A_s = c_0 \frac{\pi^k}{k!} 2 \int_{[0,1]^s} (\lambda_1 \ldots \lambda_k)^s d\lambda_1 \ldots d\lambda_k = 2c_0 \frac{\pi^k}{k!} \frac{1}{(s+1)^k}
\]
and taking $s = -(2n + 1)$ we conclude $A = 2c_0(-1)^k \frac{\pi^k}{k! (2n)!}$. Averaging the right hand side of eq. (15), we get

$$C \int_{X_{2k}} \text{vol}(B^{2k})|\sigma_\omega(W)|dW = C \frac{\pi^k}{k!} \int_{[0,1]^n} \lambda_1 \ldots \lambda_\kappa d\lambda_1 \ldots d\lambda_\kappa = C \frac{\pi^k}{k! 2^n}.$$ Summing up, $2c_0 \frac{\pi^k}{k!} \frac{1}{(2n)!} = \frac{\pi^k}{k! 2^n} C$ so that

$$C^{-1} = (-1)^\kappa \left( \frac{2n}{2n + 1} c_0 \right) = \left( \frac{-1}{2} \right)^\kappa \left( \frac{2n}{2k} \right)^{-1} n^\kappa$$

\[\square\]

**Appendix A. Oriented valuation theory**

In this appendix we draw a common thread between valuation theory and the much simpler theory of closed differential forms and linking integrals.

Let $X^n$ be an oriented manifold. We will think of the closed $k$-forms, denoted $\mathcal{Z}_k(X)$, as smooth oriented valuations of degree $k$, and consider them as functions on $k$-dimensional oriented submanifolds of $X$ with boundary, given by integration: $\omega(A) = \int_A \omega$. The form is clearly determined by its value on all submanifolds, which is analogous to the Klain embedding. Moreover, $\omega(A)$ only depends on $\partial A$.

The wedge product on $\mathcal{Z}(X)$ turns it into an algebra. When $X$ is compact, we have Poincaré duality, namely that $\mathcal{Z}_k(X) \cong \mathcal{Z}_{n-k}(X)$.

The wedge product on $\mathcal{Z}(X)$ turns it into an algebra. When $X$ is compact, we have Poincaré duality, namely that $\mathcal{Z}_k(X) \cong \mathcal{Z}_{n-k}(X) \rightarrow \mathcal{Z}_n(X) = \Omega^n(X) \rightarrow \mathbb{R}$ is non-degenerate, where the last arrow is given by $\int_X$.

We also consider the closed currents, which are analogous to the generalized valuations. We will denote them by $\mathcal{Z}^{-\infty}(X)$. We will sometimes write $\mathcal{Z}^{-\infty}$ instead of $\mathcal{Z}$.

Now assume $X = V = \mathbb{R}^n$. The translation-invariant (smooth or generalized) oriented valuations $\mathcal{Z}^{-\infty}(V)^{tr}$ are just $\wedge^* V^*$. The following construction appears in [GS94].

For oriented manifolds with boundary $A, B \subset V$ of complementary dimensions, at least one of which is compact, let $I(A, B)$ denote their intersection index. It is well-defined when $A, B$ are in general position. Note that for a closed (as a subset) $(n-k)$-dimensional submanifold $E \subset V$, $I(\bullet, E) \in \mathcal{Z}^{-\infty}_{k}(V)$.

Given a distribution $\mu \in \mathcal{M}^{-\infty}(\text{AGr}_{n-k}^+(V))$, we define $\text{Cr}(\mu) \in \mathcal{Z}^{-\infty}_k(V)$ by $\text{Cr}(\mu) = \int_{\text{AGr}_{n-k}^+(V)} I(\bullet, E) d\mu(E)$. Clearly the even measures (with respect to the orientation-reversing map) lie in the kernel of $\text{Cr}$, so we restrict our attention to odd measures.

**Proposition A.1.** The map $\text{Cr} : \mathcal{M}^{\pm \infty}(\text{AGr}_{n-k}^+(V))^{tr} \rightarrow \mathcal{Z}_k(V)^{tr}$ is surjective.

**Proof.** The GL(V)-module $\mathcal{Z}_k(V)^{tr} = \wedge^k V^*$ is irreducible. Since the Crofton map is GL(V)-equivariant, it suffices to show Cr is non-zero. This is not hard to see, for instance $\text{Cr}(\delta_E - \delta_{-E})(A) = 2I(A, E)$, where $\langle \delta_E, f \rangle = \int_{V/E} f(x + E) dx$ on a compactly supported test function $f$. For a smooth example, one could convolve with an approximate identity on GL(V).

\[\square\]

The analogues of the the Alesker-Poincaré and Alesker-Fourier dualities coincide in this setting: The Alesker-Poincaré pairing is the wedge product $\wedge V^* \otimes \wedge^{n-k} V^* \rightarrow \wedge^n V^*$. The Alesker-Fourier duality operation is given by the Hodge star: $\ast : \wedge^k V^* \rightarrow (\wedge^{n-k} V^*)^* \otimes \wedge^n V^*$. 

The following easy statement is the analogue for oriented valuations of the principal kinematic formula. Due to the finite-dimensionality of the space of translation-invariant forms, one can average over translations alone.

Let a top form $\text{vol}_n \in \wedge^n V^*$ be fixed on $V$. Let $\Omega \in \wedge^* V^* \otimes \wedge^* V^*$ be the wedge product, that is for $\xi, \eta \in \wedge^* V$, $(\xi \wedge \eta, \text{vol}_n) = \Omega(\xi, \eta)$.

For oriented compact manifolds with boundary $A, B$ we write $\Omega(A, B) := \int_{A \times B} \Omega$, which can be written more explicitly by representing $\Omega = \sum \omega_i \otimes \omega_i'$, then $\Omega(A, B) = \sum \int_A \omega_i \int_B \omega_i'$ (the integrals with mismatched dimension vanish by definition). Consider the kinematic operator

$$K_V(A, B) := \int_V I(A, B + x) d\text{vol}_n(x)$$

which is well-defined, since the integrand is compactly supported.

**Proposition A.2.** Let $A^{n-k}, B^k \subset V$ be compact, oriented submanifolds with boundary, of complementary dimension. Then $K_V(A, B) = \Omega(A, B)$

**Proof.** Let $[[B]]$ be the current defined by $B$. Then $[B]_V := \int_V [[B + x]] d\text{vol}_n(x)$ is translation-invariant, that is $[B]_V \in \wedge^k V$. Now $I(A, B) = [[A]] \cap [[B]]$, so that $K_V(A, B) = [[A]] \cap [B]_V = \text{vol}_n([[A]_V \cap [B]_V) = \Omega(A, B)$ as claimed.

We remark that this formula is also reminiscent of the Bezout formula in complex algebraic geometry.

**Appendix B. Invariant sections**

We will need two technical lemmas concerning invariants of group actions.

The first goes back to Kolk and Varadarajan [KV96], and appeared in a form best suitable for our needs in [BF17]. Let us quote the result in its simplest sufficient form.

Take a Lie group $G$ acting on a manifold $X$ with finitely many orbits, all locally closed submanifolds. Let $E$ be a $G$-vector bundle over $X$. Define for integer $\alpha \geq 0$ and a submanifold $Y \subset X$ the $G$-bundle $F_\alpha^Y$ over $Y$ by

$$F_\alpha^Y|_y = E|_y \otimes \text{Dens}^\ast (N_y Y) \otimes \text{Sym}^\alpha (N_y Y).$$

A generalized section $s \in \Gamma^\infty_X (X, E)$ has a certain transversal order $\alpha$ along $Y$, and a transversal principal symbol $\sigma(s) \in \Gamma^\infty_Y (Y, F_\alpha^Y)$. For details, see e.g. [AF14a, section 4.4].

**Lemma B.1.** Let $Z \subset X$ be a closed $G$-invariant subset. Decompose into $G$-orbits: $Z = \bigcup_{j=1}^J Y_j$. Then

$$\dim \Gamma^\infty_Z (X, E)^G \leq \sum_{\alpha=0}^\infty \sum_{j=1}^J \dim \Gamma^\infty (F_\alpha^{Y_j})^G.$$

The second statement is surely well-known, but we were not able to locate a reference. For completeness, we include the proof.

**Proposition B.2.** Let $G$ be a (possibly infinite dimensional) Lie group acting on a manifold $M^n$ transitively, and let $E$ be an infinite-dimensional $G$-bundle of Fréchet spaces over $X$. Then the space of $G$-invariants of $\Gamma^\infty (M, E)^\ast$ is naturally isomorphic to $\Gamma^\infty (M, E^\ast \otimes |\omega_M|)^G$. 

Remark B.3. In the infinite dimensional case, we assume that every $T_xM$ admits a basis of infinitesimal generators of 1-parametric subgroups of $G$. This is certainly the case for $\text{Diff}(M)$, $\text{Symp}(M)$, $\text{Cont}(M)$.

First we consider the wavefront set of the average of a distribution along a flow.

Lemma B.4. Let $M$ be a manifold. Assume $\mathbb{R}$ acts on $M$, and let the curve $C \subset M$ be an orbit. Let $\phi \in C^{-\infty}(\mathbb{R})$ be a generalized function, and $\mu \in C_c^\infty(\mathbb{R})$. Define $\phi * \mu := \int_{\mathbb{R}} t^* \phi \cdot \mu(t) \, dt$. Then for all $p \in C$ we have $\text{WF}(\phi * \mu) \cap T^*_p M \subset N^*_p C$.

Proof. Fix a small neighborhood $V$ of $p \in C$ which is $\mathbb{R}$-equivariantly identified with a neighborhood $U \subset \mathbb{R}^n$ of $p = 0$, with $\mathbb{R}$ acting by translations along the $x_1$-axis and $C$ coinciding with the $x_1$-axis. Choose a partition of unity $\rho_i$ on $M$ and $w_j$ on $\mathbb{R}$. Writing $\phi_i := \rho_i \cdot \phi$, $\mu_j := w_j \cdot \mu$ we have $\phi * \mu = \sum_{i,j} \phi_i \cdot \mu_j$. We may assume that each $\phi_i$ has finite order, and that each $\psi = \phi_i \cdot \mu_j$ whose support contains $p$, is in fact supported inside $V$. Taking such $\psi$, we may write $\psi = \Phi * \pi_1^* \nu$ for some $\nu \in C_c^\infty(\mathbb{R})$ and $\Phi \in C_c^\infty(\mathbb{R}^n)$ of order $k$, where $\pi_1 : \mathbb{R}^n \to \mathbb{R}$ is the projection to the first coordinate. Then the Fourier transform satisfies $\hat{\psi}(\xi_1) = \hat{\Phi}(\xi_1) \cdot \hat{\nu}(\xi_1)$. Since $|\hat{\psi}(\xi_1)| \leq C(1 + |\xi_1|^k)$ while $|\hat{\nu}(\xi_1)| \leq C_N (1 + |\xi_1|)^{-N}$ for all $N$. It follows that for all $\epsilon > 0$, the cone $C_\epsilon := \{\xi_1^2 > \epsilon(\xi_2^2 + \cdots + \xi_n^2)\}$ falls outside the wavefront set of $\psi$, so that $\text{WF}(\psi) \subset \{\xi_1 = 0\}$. That concludes the proof. □

Clearly the same statement holds also for $S^1$-actions, and with $\phi$ a generalized section of an arbitrary $\mathbb{R}$-equivariant vector bundle over $M$.

Proof of Proposition B.3. Take $s \in \Gamma_c^\infty(M, E)^\ast$. For $\phi \in \Gamma_c^\infty(M, E)$ one can define $s \cdot \phi \in \mathcal{M}^{-\infty}(M)$ by $\int f \cdot d(s \cdot \phi) := s(f \phi)$ for $f \in C_c^\infty(M)$. Moreover, the map $\Gamma_c^\infty(M, E) \to \mathcal{M}^{-\infty}(M)$ given by $\phi \mapsto s \cdot \phi$ is $G$-equivariant. Fix $\phi$, and let us verify that $s \cdot \phi$ is in fact a smooth measure. Consider a 1-parametric subgroup $H \subset G$ which can be either $\mathbb{R}$ or $S^1$. By the Dixmier-Mallavin theorem, we may find smooth probability measures $\mu_1, \ldots, \mu_N \in \mathcal{M}_c^\infty(H)$ and sections $\psi_1, \ldots, \psi_N \in \Gamma_c^\infty(M, E)$ such that

$$\phi = \sum_j \int_H g \psi_j d\mu_j(g) \Rightarrow s \cdot \phi = \sum_j \int_H g(s \cdot \psi_j) d\mu_j(g)$$

It follows by Lemma B.4 that the wavefront set $\text{WF}(s \cdot \phi)$ lies in the conormal bundle to the orbits of $H$ on $M$. Since $G$ acts transitively, we conclude $\text{WF}(s \cdot \phi) = \emptyset$, that is $s \cdot \phi \in \mathcal{M}^{\infty}(M)$.

In particular, we can consider the density on every tangent plane, $(s \cdot \phi)(x) \in \text{Dens}(T_xM)$. We next claim that $(s \cdot \phi)(x)$ only depends on $\phi(x)$. Indeed, if $\phi(x) = 0$ we may represent $\phi = f \cdot \psi$ for some $\psi \in \Gamma_c^\infty(M, E)$ and $f \in C^\infty(M)$ with $f(0) = 0$. Then $(s \cdot \phi) = f \cdot (s \cdot \psi)$, and thus $(s \cdot \phi)(x) = 0$. That is, we get an element $s(x) \in E^\ast_x \otimes \text{Dens}(T_xM)$. By $G$-invariance, $x \mapsto s(x)$ is a smooth section of $E^\ast \otimes |\omega_M|$, which clearly defines the same functional as $s$ on $\Gamma_c^\infty(M, E)$. □

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