CONTINUITY PROPERTIES FOR BORN-JORDAN OPERATORS WITH SYMBOLS IN HÖRMANDER CLASSES AND MODULATION SPACES

Maurice de GOSSON
Faculty of Mathematics, NuHAG, University of Vienna, Vienna, Austria
E-mail: maurice.degosson@gmail.com

Joachim TOFT†
Department of Mathematics, Linnaeus University, Växjö, Sweden
E-mail: joachim.toft@lnu.se

Abstract We show that the Weyl symbol of a Born-Jordan operator is in the same class as the Born-Jordan symbol, when Hörmander symbols and certain types of modulation spaces are used as symbol classes. We use these properties to carry over continuity, nuclearity and Schatten-von Neumann properties to the Born-Jordan calculus.

Key words quantization; Schatten-von Neumann; Fefferman-Phong’s inequality

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1 Introduction

A fundamental question in quantum mechanics concerns quantization. That is, finding rules which takes observables \(a\) in classical mechanics into corresponding observables \(T_a\) in quantum mechanics. Usually \(a\) is a function of the location \(x \in \mathbb{R}^d\) (the configuration variable) and the momentum \(\xi \in \mathbb{R}^d\), and \(T_a\) is a linear operator which acts between suitable Hilbert spaces of functions on \(\mathbb{R}^d\).

The Born-Jordan quantization

\[
\text{Op}_{BJ}(a) = \frac{1}{l+1} \sum_{k=0}^{l} D_{j}^{l-k} \circ x_{j}^{m} \circ D_{j}^{k}, \quad a(x, \xi) = x_{j}^{m} \xi_j
\]

introduced in early days by Born and Jordan in [7], is nowadays considered as an important quantization rule (see e.g. [16–18]). (See also [30] and Section 2 for notations.) In a context of the calculus of pseudo-differential operators, Born-Jordan quantization is given by

\[
\text{Op}_{BJ}(a) \equiv \int_{0}^{1} \text{Op}_{t}(a) dt,
\]

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†Corresponding author: Joachim TOFT.
where the symbol \(a(x, \xi)\) is allowed to belong to more general classes compared to (1.1) (see Section 2 for details and notations). Here \(\text{Op}_t(a)\) is the \((t\text{-Shubin})\) pseudo-differential operator with symbol \(a\), given by

\[
(\text{Op}_t(a)f)(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a((1-t)x + ty, \xi) f(y) e^{i(x-y, \xi)} dyd\xi.
\]

We have that \(\text{Op}_0(a) = a(x, D)\) (i.e., \(t = 0\)) is the standard or Kohn-Nirenberg representation, and that \(\text{Op}_1(a)\) (i.e., \(t = 1\)) is the adjoint representation. By choosing \(t = \frac{1}{2}\), i.e., the mean-value of these standard situations, we deduce the Weyl operator (or Weyl quantization) \(\text{Op}^{w}(a) = \text{Op}_{1/2}(a)\), given by

\[
(\text{Op}^{w}(a)f)(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a\left(\frac{1}{2}(x + y), \xi\right) f(y) e^{i(x-y, \xi)} dyd\xi,
\]

which is also considered as a suitable quantization rule. For example, among the operator representations above, only \(\text{Op}_{BJ}(a)\) and \(\text{Op}^{w}(a)\) possess the property to necessarily being self-adjoint (Hermit operators) when \(a\) is real-valued.

By using

\[
\text{Op}_t(a) = \text{Op}^{w}(a_t), \quad \text{where} \quad a_t = e^{i(\frac{1}{2}-t)(D_\xi, D_x)} a,
\]

the formula (1.2) becomes

\[
\text{Op}_{BJ}(a) = \text{Op}^{w}(a_{BJ}),
\]

\[
a_{BJ} = \int_{0}^{1} e^{i(\frac{1}{2}-t)(D_\xi, D_x)}adt = 2\text{sinc}\left(\frac{1}{2}(D_\xi, D_x)\right) a,
\]

which puts the Born-Jordan quantization within the frame of the Weyl calculus of pseudo-differential operators. Here \(\text{sinc} t\) is the sinc function, given by

\[
\text{sinc} t = \begin{cases} 
\frac{\sin t}{t}, & t \neq 0, \\
1, & t = 0.
\end{cases}
\]

During the last 10 years, Born-Jordan quantization is also recognized in time-frequency analysis. In this field, time-frequency resolutions by Wigner distributions, i.e., simultaneously localizations of the time and frequency for signals, are essential. A problem here concerns interpolating frequencies or so-called ghost frequencies, which originate from interference of existing frequencies but are absent in the signal, but are present in the graphs of their resolutions (see [4, 5, 48]). Especially we remark that in [48], Turunen showed that the time-frequency resolutions usually becomes significantly more clear when using Born-Jordan versions \(W_{BJ}(f, g)\) in place of classical time-frequency resolutions like the Wigner distributions \(W(f, g)\). For example, it was shown in [48] that the ghost frequencies miraculously almost disappear when using suitable resolutions based on the \(W_{BJ}\) transform. See also [14] for other related facts.

The impact of Born-Jordan operators in quantization and time-frequency analysis leads to questions on continuity for such operators. In quantization, it is suitable to consider general Hörmander classes \(S(m, g)\) on the phase space. Recall that \(S(m, g)\) agrees with classical symbol classes like \(S^p_{\rho,\delta}(\mathbb{R}^{2d})\), SG-classes or Shubin classes, by choosing the Riemannian metric \(g\) and the weight function \(m\) in appropriate ways. In time-frequency analysis, it is suitable to use (classical) modulation spaces, \(M^{p,q}_{(\omega)}(\mathbb{R}^{2d})\) as symbol classes, because they are especially adapted
for energy estimates for time-frequency representations (cf. e.g. [38]). These spaces were introduced in [19] by Feichtinger and are obtained by imposing an $L_{(\omega)}^{p,q}$ condition on the short-time Fourier transform on the involved functions and (ultra-)distributions. (See also [20, 22, 25] and the references therein for more facts on modulation spaces.)

In Sections 3–6 we deduce several types of continuity properties for Born-Jordan operators. In similar way as in e.g. [11–13], the main idea is to use (1.2) to carry over continuity properties in pseudo-differential calculus to Born-Jordan operators. In Section 3 we consider Born-Jordan operators with symbols in the Schwartz space or in certain Gelfand-Shilov spaces and their duals. For example, we regain the fact from [11] that (1.2) leads to

$$a_{BJ} \in \mathcal{S}'(\mathbb{R}^{2d}) \quad \text{when} \quad a \in \mathcal{S}(\mathbb{R}^{2d}),$$

(1.5)

where $\mathcal{S}'(\mathbb{R}^{2d})$ is the set of tempered distributions on $\mathbb{R}^{2d}$. (See Theorems 3.2 and 3.3.) In Section 3 it is proved that the same holds true with $\mathcal{S}_s(\mathbb{R}^{2d})$, $\Sigma_s(\mathbb{R}^{2d})$ or their duals in place of $\mathcal{S}'(\mathbb{R}^{2d})$, where $\mathcal{S}_s(\mathbb{R}^{d})$ ($\Sigma_s(\mathbb{R}^{d})$) is the Fourier invariant Gelfand-Shilov space of Roumieu (Beurling) type of order $s > 0$ on $\mathbb{R}^{d}$. In particular it follows that the following holds true:

$$a \in \mathcal{S}'(\mathbb{R}^{2d}) \quad \Rightarrow \quad \text{Op}_{BJ}(a) : \mathcal{S}(\mathbb{R}^{d}) \rightarrow \mathcal{S}'(\mathbb{R}^{d})$$

and

$$a \in \mathcal{S}_s(\mathbb{R}^{2d}) \quad \Rightarrow \quad \text{Op}_{BJ}(a) : \Sigma_s(\mathbb{R}^{d}) \rightarrow \Sigma'_s(\mathbb{R}^{d})$$

(with continuous mappings), because the same continuity properties hold true with $\text{Op}_{BJ}(a)$ in place of $\text{Op}(a)$ for every $t$. We remark that our investigations include more general Gelfand-Shilov spaces and their distributions, which do not need to be Fourier invariant (see Theorem 3.2). These properties give a solid basement of our investigations.

In Section 4 we consider Born-Jordan operators with symbols in modulation spaces, and prove that

$$a_{BJ} \in M_{(\omega)}^{p,q}(\mathbb{R}^{2d}) \quad \text{when} \quad a \in M_{(\omega)}^{p,q}(\mathbb{R}^{2d}),$$

(1.6)

when $p, q \in (0, \infty]$, provided the weight $\omega(x, \xi, \eta, y)$ is constant with respect to the $x$ and $\xi$ variables (see Theorem 4.1). It is well-known that for such $\omega$, the map $c^{\mu(x, \xi, D_x, D_\xi)}$ is continuous on $M_{(\omega)}^{p,q}(\mathbb{R}^{2d})$ for every $t$. (Cf. [46, Proposition 1.9].) In the special case when $p, q \geq 1$, then (1.6) is deduced by a straight-forward combination of (1.2), (1.3) and Minkowski’s inequality. For such choices of $p$ and $q$ and if all weights are trivially equal to one, then these investigations are related to those in [12]. For example Theorem 4.3 in Section 4 overlaps with [12, Theorem 5.1].

In order to reach (1.6) in the general case, the possible lack of local-convexity of involved spaces, impose a more comprehensive analysis compared to the restricted case $p, q \geq 1$. In our approach, the symbol $a$ in (1.6) is expressed in terms of its Gabor expansion, using the fact that Gabor theory works properly for modulation spaces when $p$ and $q$ are allowed to be smaller than 1. (Cf. [22, 43].) By inserting such expansions in (1.2), (1.3) and performing some refined computations, we finally land on (1.6).

As a consequence of (1.6) we get, e.g.,

$$a \in M_{\infty, q}(\mathbb{R}^{2d}), \quad q \leq 1, \quad p_1, q_1 \in [q, \infty] \quad \Rightarrow \quad \text{Op}_{BJ}(a) : M_{p_1, q_1}(\mathbb{R}^{d}) \rightarrow M_{p_1, q_1}(\mathbb{R}^{d}),$$
\[ a \in M^{p, \min(p, p')} (\mathbb{R}^{2d}), \ p > 0 \implies \text{Op}_{\text{BJ}}(a) \in \mathcal{S}_p(L^2(\mathbb{R}^d)) \]

\[ a \in M^p (\mathbb{R}^{2d}), \ p \leq 1 \implies \text{Op}_{\text{BJ}}(a) \in \mathcal{S}_p'(M^\infty(\mathbb{R}^d), M^p(\mathbb{R}^d)), \]

because the same hold true with \( \text{Op}_t(a) \) in place of \( \text{Op}_{\text{BJ}}(a) \) (cf. [45, 46]). Here \( p' \) is the conjugate exponent of \( p \in (0, \infty] \), given by

\[
p' = \begin{cases} 
\infty & \text{when } p \leq 1, \\
p - 1 & \text{when } 1 < p < \infty, \\
1 & \text{when } p = \infty.
\end{cases}
\]

In Section 5 we deduce continuity properties for Born-Jordan operators with symbols in the (general) Hörmander class \( S(m, g) \), where \( g \) is a strongly feasible Riemannian metric and \( m \) is \((\sigma, g)\)-tempered metric on the phase space \( \mathbb{R}^{2d} \). We prove that if in addition \( g \) is split in the sense \( g_X(y, -\eta) = g_X(y, \eta) \), then

\[ a_{\text{BJ}} \in S(m, g) \implies \text{Op}_{\text{BJ}}(a) : S(R^d) \rightarrow S'(R^d), \]

(see Theorem 5.4), and that

\[ m \in L^p(\mathbb{R}^{2d}), a \in S(m, g) \implies \text{Op}_{\text{BJ}}(a) \in \mathcal{S}_p(L^2(\mathbb{R}^d)) \]

(see Theorems 5.6 and 5.7), because the same hold true with \( \text{Op}^w(a) \) in place of \( \text{Op}_{\text{BJ}}(a) \) (see [30, Theorem 18.6.2], [8, Theorem 2.9], [39, Theorem 4.4] and [45, Theorem 4.1]).

In the last part of Section 5 we deduce classical lower bound estimates for Born-Jordan operators. In fact, by asymptotic expansions it follows that if \( a, a_{\text{BJ}} \in S(m, g) \) satisfy (1.4), then

\[ a - a_{\text{BJ}} \in S(h_g^2m, g), \]

where \( h_g \) is the Planck’s function. This leads to that fundamental lower bound results carry over from the Weyl case to Born-Jordan case. In fact, Sharp Gårding’s and Fefferman-Phong’s inequalities as well as Hörmander’s improvement of Melin’s inequality, given by Theorems 18.6.7 and 18.6.8 in [30] and Theorem 6.2 in [28], are some of the most well-known lower bound results in pseudo-differential calculus. It follows from the small difference between \( a \) and \( a_{\text{BJ}} \) in view of (1.8) that these lower bound results carry over to Born-Jordan operators (see e.g. Theorem 5.9 in Section 5).

In Section 6 we consider Born-Jordan operators of so-called infinite orders. That is, in contrast to the Hörmander classes, \( S(m, g) \), the involved symbols are allowed to grow faster than polynomials. On the other hand, it is assumed that the symbols obey stronger regularity conditions than what is required in the class \( S(m, g) \). We consider operators with symbols in \( \Gamma_{s,\sigma}^\sigma(\mathbb{R}^{2d}) \) or \( \Gamma_{s,\sigma}^\sigma(\mathbb{R}^{2d}) \), considered in [1] (see Definition 6.1 in Section 6).
is deduced that pseudo-differential operators with symbols in such classes are continuous on suitable Gelfand-Shilov spaces and their duals. In Section 6 we use (1.4) to carry over these continuity properties to Born-Jordan operators with symbols in $\Gamma^{s,\sigma}_{h,0}(R^d)$ or $\Gamma^{s,\sigma}_{\sigma,0}(R^d)$.

2 Preliminaries

In this section we start by recalling some facts about Gelfand-Shilov spaces of functions and distributions. Thereafter, we recall the definition of pseudo-differential operators and Born-Jordan operators. Some basic properties for Schatten-von Neumann and nuclear operator classes are then discussed in Subsection 2.3. We conclude the section by recalling some facts on modulation spaces.

2.1 Gelfand-Shilov spaces and their duals

We start by recalling some facts about Gelfand-Shilov spaces. Let $0 < h, s, \sigma \in R$ be fixed. Then $S^s_{\sigma,h}(R^d)$ is the Banach space of all $f \in C^\infty(R^d)$ such that

$$\|f\|_{S^s_{\sigma,h}} = \sup_{\alpha,\beta \in N^d} \sup_{x \in R^d} \frac{|\partial^\alpha \partial^\beta f(x)|}{h^{\alpha + \beta}|\alpha|! |\beta|^\sigma} < \infty,$$

(2.1)

endowed with the norm (2.1).

The Gelfand-Shilov spaces $S^s(R^d)$ and $\Sigma^s(R^d)$ are defined as the inductive and projective limits respectively of $S^s_{\sigma,h}(R^d)$. This implies that

$$S^s(R^d) = \bigcup_{h > 0} S^s_{\sigma,h}(R^d) \quad \text{and} \quad \Sigma^s(R^d) = \bigcap_{h > 0} S^s_{\sigma,h}(R^d),$$

(2.2)

and that the topology for $S^s_{\sigma,h}(R^d)$ is the strongest possible one such that the inclusion map from $S^s_{\sigma,h}(R^d)$ to $S^s_{\sigma,h}(R^d)$ is continuous, for every choice of $h > 0$. The space $\Sigma^s(R^d)$ is a Fréchet space with seminorms $\| \cdot \|_{S^s_{\sigma,h}}, h > 0$. Moreover, $\Sigma^s(R^d) \neq \{0\}$, if and only if $s + \sigma \geq 1$ and $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, and $S^s_{\sigma,h}(R^d) \neq \{0\}$, if and only if $s + \sigma \geq 1$.

In terms of the exponential type decays, $S^s_{\sigma,h}(R^d)$ and $\Sigma^s_{\sigma,h}(R^d)$ are characterized as $f \in S^s_{\sigma,h}(R^d)$ $(f \in \Sigma^s_{\sigma,h}(R^d))$, if and only if

$$|\partial^\alpha f(x)| \lesssim h^{|\alpha|} \alpha!^\sigma e^{-r|x|^{\frac{1}{\tau}}}$$

for some $h, r > 0$ (respectively for every $h, r > 0$). Moreover we recall that for $s < 1$ the elements of $S^s_{\sigma,h}(R^d)$ admit entire extensions to $C^\infty$ satisfying suitable exponential bounds, cf. [23] for details.

The Gelfand-Shilov distribution spaces $(S^s_{\sigma,h}')(R^d)$ and $(\Sigma^s_{\sigma,h}')(R^d)$ are the projective and inductive limits respectively of the $L^2$-dual $(S^s_{\sigma,h})'(R^d)$ of $S^s_{\sigma,h}(R^d)$. This implies that

$$(S^s_{\sigma,h})'(R^d) = \bigcap_{h > 0} (S^s_{\sigma,h})'(R^d) \quad \text{and} \quad (\Sigma^s_{\sigma,h})'(R^d) = \bigcup_{h > 0} (S^s_{\sigma,h})'(R^d).$$

(2.2)

We remark that in [34] it is proved that $(S^s_{\sigma,h}')(R^d)$ is the strong dual of $S^s_{\sigma,h}(R^d)$, and $(\Sigma^s_{\sigma,h})'(R^d)$ is the strong dual of $\Sigma^s_{\sigma,h}(R^d)$ (also in topological sense).

For every $s, \sigma > 0$ we have

$$\Sigma^s(R^d) \hookrightarrow S^s(R^d) \hookrightarrow \Sigma^s_{\sigma+\varepsilon}(R^d) \hookrightarrow \mathcal{S}(R^d)$$

(2.3)
for every $\varepsilon > 0$. If $s + \sigma \geq 1$, then the last two inclusions in (2.3) are dense, and if in addition $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, then the first inclusion in (2.3) is dense.

From these properties it follows that $\mathcal{S}_\sigma(\mathbb{R}^d) \hookrightarrow (\Sigma_\sigma)'(\mathbb{R}^d)$ when $s + \sigma \geq 1$, and if in addition $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, then $(\Sigma_\sigma)'(\mathbb{R}^d) \hookrightarrow (\Sigma_\sigma)'(\mathbb{R}^d)$.

The Gelfand-Shilov spaces possess several convenient mapping properties. For example they are nuclear and invariant under translations, dilations, and to some extent tensor products and (partial) Fourier transformations (cf. [23, 33, 35]).

We also need to involve a broader family of Gelfand-Shilov spaces. More precisely, for $s_j, \sigma_j \in \mathbb{R}_+$, $j = 1, 2$, the Gelfand-Shilov spaces $\mathcal{S}_{\sigma_1, \sigma_2}(\mathbb{R}^{d_1+d_2})$ and $\Sigma_{\sigma_1, \sigma_2}(\mathbb{R}^{d_1+d_2})$ consist of all functions $F \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that

$$|x_1^{\alpha_1} x_2^{\alpha_2} \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} F(x_1, x_2)| \lesssim h^{\alpha_1 + \beta_1 + |\alpha_2 + \beta_2|} \alpha_1! \alpha_2! \beta_1! \beta_2!$$

(2.4)

for some $h > 0$ respective for every $h > 0$. The topologies, and the duals

$$(\mathcal{S}_{\sigma_1, \sigma_2}^{\sigma_1, \sigma_2})(\mathbb{R}^{d_1+d_2}) \quad \text{and} \quad (\Sigma_{\sigma_1, \sigma_2}^{\sigma_1, \sigma_2})(\mathbb{R}^{d_1+d_2})$$

of

$$\mathcal{S}_{\sigma_1, \sigma_2}(\mathbb{R}^{d_1+d_2}) \quad \text{and} \quad \Sigma_{\sigma_1, \sigma_2}(\mathbb{R}^{d_1+d_2}),$$

respectively, and their topologies are defined in analogous ways as for the spaces $\mathcal{S}_\sigma(\mathbb{R}^d)$ and $\Sigma_\sigma(\mathbb{R}^d)$ above.

From the inequalities $n!k! \leq (n + k)! \leq 2^{n+k}n!k!$, it follows by straightforward computations that $S_{s,\sigma}^{\sigma} = S_{s}^{s}$, $\Sigma_{s,\sigma}^{\sigma} = \Sigma_{s}^{s}$ and similarly for their duals. For convenience we set $\mathcal{S}_{s} = S_{s}^{s}$ and $\Sigma_{s} = \Sigma_{s}^{s}$.

From now on we let $\mathcal{F}$ be the Fourier transform, given by

$$(\mathcal{F} f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-i(x, \xi)} dx,$$

when $f \in L^1(\mathbb{R}^d)$. Here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on $\mathbb{R}^d$. The map $\mathcal{F}$ extends uniquely to homeomorphisms from $\mathcal{S}_\sigma(\mathbb{R}^d)$ to $\mathcal{S}_\sigma'(\mathbb{R}^d)$, from $(\mathcal{S}_\sigma)'(\mathbb{R}^d)$ to $(\mathcal{S}_\sigma)'(\mathbb{R}^d)$ and from $(\Sigma_\sigma)'(\mathbb{R}^d)$ to $(\Sigma_\sigma)'(\mathbb{R}^d)$. It also follows that $\mathcal{F}$ restricts to homeomorphisms from $\mathcal{S}_\sigma(\mathbb{R}^d)$ to $\mathcal{S}_\sigma(\mathbb{R}^d)$, from $\mathcal{S}_\sigma'(\mathbb{R}^d)$ to $\mathcal{S}_\sigma'(\mathbb{R}^d)$, from $\Sigma_\sigma(\mathbb{R}^d)$ to $\Sigma_\sigma(\mathbb{R}^d)$, and to a unitary operator on $L^2(\mathbb{R}^d)$.

**Remark 2.1** In the same way, if $\mathcal{F}_j F$ is the partial Fourier transform of $F(x_1, x_2)$ with respect to $x_j \in \mathbb{R}^{d_j}$, $s_j, \sigma_j > 0$, $j = 1, 2$ and $d = d_1 + d_2$, then

$$\mathcal{F}_1 : \mathcal{S}_{\sigma_1, \sigma_2}(\mathbb{R}^d) \to \mathcal{S}_{\sigma_1, \sigma_2}(\mathbb{R}^d), \quad \mathcal{F}_2 : \mathcal{S}_{\sigma_1, \sigma_2}(\mathbb{R}^d) \to \mathcal{S}_{\sigma_1, \sigma_2}(\mathbb{R}^d),$$

$$\mathcal{F}_1 : \Sigma_{\sigma_1, \sigma_2}(\mathbb{R}^d) \to \Sigma_{\sigma_1, \sigma_2}(\mathbb{R}^d), \quad \mathcal{F}_2 : \Sigma_{\sigma_1, \sigma_2}(\mathbb{R}^d) \to \Sigma_{\sigma_1, \sigma_2}(\mathbb{R}^d),$$

$$\mathcal{F}_1 : (\mathcal{S}_{\sigma_1, \sigma_2})'(\mathbb{R}^d) \to (\mathcal{S}_{\sigma_1, \sigma_2})'(\mathbb{R}^d), \quad \mathcal{F}_2 : (\mathcal{S}_{\sigma_1, \sigma_2})'(\mathbb{R}^d) \to (\mathcal{S}_{\sigma_1, \sigma_2})'(\mathbb{R}^d),$$

$$\mathcal{F}_1 : (\Sigma_{\sigma_1, \sigma_2})'(\mathbb{R}^d) \to (\Sigma_{\sigma_1, \sigma_2})'(\mathbb{R}^d), \quad \mathcal{F}_2 : (\Sigma_{\sigma_1, \sigma_2})'(\mathbb{R}^d) \to (\Sigma_{\sigma_1, \sigma_2})'(\mathbb{R}^d)$$

and

$$\mathcal{F}_1 : (\Sigma_{\sigma_1, \sigma_2})'(\mathbb{R}^d) \to (\Sigma_{\sigma_1, \sigma_2})'(\mathbb{R}^d), \quad \mathcal{F}_2 : (\Sigma_{\sigma_1, \sigma_2})'(\mathbb{R}^d) \to (\Sigma_{\sigma_1, \sigma_2})'(\mathbb{R}^d)$$

are homeomorphisms, $j = 1, 2$.
Next we recall some mapping properties of Gelfand-Shilov spaces under short-time Fourier transforms and t-Wigner distributions. Let \( \phi \in \mathcal{S}(\mathbb{R}^d) \) be fixed. For every \( t \in \mathbb{R} \), \( f \in \mathcal{S}'(\mathbb{R}^d) \), the short-time Fourier transform \( V_\phi f \) is the distribution on \( \mathbb{R}^{2d} \) defined by the formula

\[
(V_\phi f)(x, \xi) = \mathcal{F}(f \phi(\cdot - x))(\xi) = (f, \phi(\cdot - x)e^{i\langle \cdot, \xi \rangle}).
\] (2.5)

The t-Wigner distribution is given by

\[
W_t(f, \phi)(x, \xi) \equiv \mathcal{F}(f(x + t \cdot \phi(x - (1 - t) \cdot )))(\xi).
\] (2.6)

We observe that if \( f, \phi \in \mathcal{S}(\mathbb{R}^d) \), then \( V_\phi f \) and \( W_t(f, \phi) \) are given by

\[
(V_\phi f)(x, \xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y)\overline{\phi(y - x)}e^{-i(y, \xi)}dy
\]
and

\[
W_t(f, \phi)(x, \xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x + ty)\overline{\phi(x - (1 - t)y)}e^{-i(y, \xi)}dy.
\]

The definition of short-time Fourier transforms and Wigner distributions extend in different ways, and possess various kinds of continuity properties. In the context of test function spaces and distribution spaces we have the following.

**Proposition 2.2** Let \( s, \sigma > 0 \) be such that \( s + \sigma \geq 1 \) and let \( T(t, f, \phi) \equiv V_\phi f \) or \( T(t, f, \phi) \equiv W_t(f, \phi) \) when \( f, \phi \in \mathcal{S}(\mathbb{R}^d) \). Then the following is true:

1. the map \((t, f, \phi) \mapsto T(t, f, \phi)\) is continuous from \( \mathbb{R} \times \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \) to \( \mathcal{S}(\mathbb{R}^{2d}) \) and restricts to a continuous map from \( \mathbb{R} \times S^s_\sigma(\mathbb{R}^d) \times S^\sigma_\sigma(\mathbb{R}^d) \) to \( S^s_{\sigma, \sigma}(\mathbb{R}^{2d}) \);

2. the map \((t, f, \phi) \mapsto T(t, f, \phi)\) from \( \mathbb{R} \times S^s_\sigma(\mathbb{R}^d) \times S^\sigma_\sigma(\mathbb{R}^d) \) to \( S^s_{\sigma, \sigma}(\mathbb{R}^{2d}) \) extends uniquely to a continuous map from \( \mathbb{R} \times (S^s_\sigma)'(\mathbb{R}^d) \times (S^\sigma_\sigma)'(\mathbb{R}^d) \) to \( (S^s_{\sigma, \sigma})'(\mathbb{R}^{2d}) \) and from \( \mathbb{R} \times (S^\sigma)'(\mathbb{R}^d) \times (S^\sigma)'(\mathbb{R}^d) \) to \( (S^\sigma)'(\mathbb{R}^{2d}) \).

The same holds true for \((s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})\) when each \( S^s_\sigma \) and \( S^s_{\sigma, \sigma} \) are replaced by \( S^s_\sigma \) and \( S^s_{\sigma, \sigma} \), respectively.

Proposition 2.2 is essentially available in the literature (see e.g. [15, 41]). Since in contrast we have included the parameter \( t \) as a variable, we here recall the arguments for the t-Wigner distribution.

**Proof** We only prove (2). The other cases follow by similar arguments and are left for the reader.

By the definition we have

\[
T(t, f, \phi) = (\mathcal{F}_2 \circ U_t \circ S)(f, \phi),
\]

where

\[
(U_t F)(x, y) = F(x + ty, x - (1 - t)y) \quad \text{and} \quad S(f, \phi) = f \otimes \overline{\phi}.
\]

Since it is evident that \( S \) is continuous from \( (S^s_\sigma)'(\mathbb{R}^d) \times (S^\sigma_\sigma)'(\mathbb{R}^d) \) to \( (S^s_\sigma)'(\mathbb{R}^{2d}) \), that \( (t, F) \mapsto U_t F \) is continuous from \( \mathbb{R} \times (S^s_\sigma)'(\mathbb{R}^{2d}) \) to \( (S^s_\sigma)'(\mathbb{R}^{2d}) \), it follows from Remark 2.1 that \( T \) is uniquely defined and continuous from \( \mathbb{R} \times (S^s_\sigma)'(\mathbb{R}^d) \times (S^\sigma_\sigma)'(\mathbb{R}^d) \) to \( (S^s_{\sigma, \sigma})'(\mathbb{R}^{2d}) \). \( \Box \)

**Remark 2.3** By the previous proof it also follows that the mappings in Proposition 2.2 are locally uniformly bounded.
We also notice that if $T$ is the same as in Proposition 2.2, then the mappings
\[
T : \mathbb{R} \times \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d}),
\]
\[
T : \mathbb{R} \times (\mathcal{S}'_s)^{\prime}(\mathbb{R}^d) \times \mathcal{S}_s(\mathbb{R}^d) \rightarrow (\mathcal{S}'_s)^{\prime}(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d})
\]
and
\[
T : \mathbb{R} \times (\Sigma_{s,\sigma})^{\prime}(\mathbb{R}^d) \times \Sigma_{s,\sigma}(\mathbb{R}^d) \rightarrow (\Sigma_{s,\sigma})^{\prime}(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d})
\]
are continuous (cf. [1, 15, 41, 44]).

There are several ways to characterize Gelfand-Shilov spaces and their distribution spaces. For example, they can easily be characterized by Hermite functions and other related functions (cf. e.g. [24, 31, 34, 35]). They can also be characterized by suitable estimates of their Fourier transforms (cf. [10, 27, 41, 44]).

2.2 Pseudo-differential and Born-Jordan operators

Let $t \in \mathbb{R}$ be fixed. For any $a \in \mathcal{S}(\mathbb{R}^{2d})$ (the symbol), the pseudo-differential operator $\text{Op}_t(a)$ is the linear and continuous operator on $\mathcal{S}(\mathbb{R}^d)$, defined by
\[
\text{Op}_t(a)f(x) = (2\pi)^{-d} \int \int a((1-t)x + ty, \xi)f(y)e^{i(x-y,\xi)}dyd\xi,
\]  
when $f \in \mathcal{S}(\mathbb{R}^d)$. By straightforward computations it follows that
\[
(\text{Op}_t(a)f, g)_{L^2(\mathbb{R}^d)} = (2\pi)^{-d/2}(a, W_t(g, f))_{L^2(\mathbb{R}^{2d})},
\]  
when $g \in \mathcal{S}(\mathbb{R}^d)$.

If more generally $a \in (\mathcal{S}'_s)^{\prime}(\mathbb{R}^{2d})$, then $\text{Op}_t(a)$ is the linear and continuous operator from $\mathcal{S}'_s(\mathbb{R}^d)$ to $(\mathcal{S}'_s)^{\prime}(\mathbb{R}^d)$ such that $(\text{Op}_t(a)f, g)$ is equal to the right-hand side of (2.8) when $f, g \in \mathcal{S}'_s(\mathbb{R}^d)$. This makes sense, in view of the continuity properties for the Wigner distribution, described above. Similar facts hold true with either $\Sigma_{s,\sigma}$ and $\Sigma_{s,\sigma}'$, or by $\mathcal{S}$ and $\mathcal{S}'$ in place of $\mathcal{S}'_s$ and $\mathcal{S}'_s$, respectively, at each occurrence.

We recall that the Born-Jordan operator $\text{Op}_{BJ}(a)$ with symbol $a$ is given by (1.2). It follows from (2.8) that
\[
(\text{Op}_{BJ}(a)f, g)_{L^2(\mathbb{R}^d)} = (2\pi)^{-d/2}(a, W_{BJ}(g, f))_{L^2(\mathbb{R}^{2d})}
\]  
(2.9)

with
\[
W_{BJ}(g, f) = \int_0^1 W_t(g, f)dt.
\]  
(2.10)

2.3 Schatten-von Neumann classes and nuclear operators

Before giving the general definition of Schatten-von Neumann classes we recall some facts on quasi-Banach spaces. A quasi-norm $\| \cdot \|_B$ of order $r \in (0, 1]$ on the vector-space $B$ is a nonnegative functional on $B$ which satisfies
\[
\|f + g\|_B \leq 2^{\frac{1}{r} - 1}(\|f\|_B + \|g\|_B), \quad f, g \in B,
\]  
\[
\|\alpha \cdot f\|_B = |\alpha| \cdot \|f\|_B, \quad \alpha \in \mathbb{C}, \quad f \in B
\]
and
\[
\|f\|_B = 0 \iff f = 0.
\]
The vector space $\mathcal{B}$ is called a quasi-Banach space if it is a complete quasi-normed space. If $\mathcal{B}$ is a quasi-Banach space with quasi-norm satisfying the weak triangle inequality (2.11), then by [2, 36] there is an equivalent quasi-norm to $\| \cdot \|_{\mathcal{B}}$ which additionally satisfies
\[ \| f + g \|_{\mathcal{B}} \leq \| f \|_{\mathcal{B}} + \| g \|_{\mathcal{B}}, \quad f, g \in \mathcal{B}. \] (2.12)

From now on we always assume that the quasi-norm of the quasi-Banach space $\mathcal{B}$ is chosen in such a way that both (2.11) and (2.12) hold.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be (quasi-)Banach spaces and let $T$ be a linear operator from $\mathcal{B}_1$ to $\mathcal{B}_2$. The singular value of $T$ of order $j \geq 1$ is defined as
\[ \sigma_j(T) = \sigma_j(T; \mathcal{B}_1, \mathcal{B}_2) = \inf \{ T - T_0 \|_{\mathcal{B}_1 \to \mathcal{B}_2} \}, \]
where the infimum is taken over all linear operators $T_0$ from $\mathcal{B}_1$ to $\mathcal{B}_2$ of rank at most $j - 1$. (Cf. e.g. [3, 37, 42].) The operator $T$ is said to be a Schatten-von Neumann operator of order $p \in (0, \infty]$ if
\[ \| T \|_{\mathcal{S}_p(\mathcal{B}_1, \mathcal{B}_2)} = \| \{ \sigma_j(T) \}_{j \geq 1} \|_p \] (2.13)
is finite. The set of Schatten-von Neumann operators from $\mathcal{B}_1$ to $\mathcal{B}_2$ of order $p \in (0, \infty]$ is denoted by $\mathcal{S}_p(\mathcal{B}_1, \mathcal{B}_2)$. We observe that $\mathcal{S}_p(\mathcal{B}_1, \mathcal{B}_2)$ is contained in $\mathcal{K}(\mathcal{B}_1, \mathcal{B}_2)$, the set of compact operators from $\mathcal{B}_1$ to $\mathcal{B}_2$, when $p < \infty$. Furthermore, $\mathcal{S}_\infty(\mathcal{B}_1, \mathcal{B}_2)$ agrees with $\mathcal{B}(\mathcal{B}_1, \mathcal{B}_2)$, the set of linear bounded operators from $\mathcal{B}_1$ to $\mathcal{B}_2$. For conveniency we set $\mathcal{S}_\infty(\mathcal{B}, \mathcal{B}) = \mathcal{S}_\infty(\mathcal{B})$.

Next we define nuclear operators. Let $\mathcal{B}_0$ be a Banach space with dual $\mathcal{B}_0'$, $\mathcal{B}$ be a quasi-Banach space, $r \in (0, 1]$ and let $T$ be a linear and continuous operator from $\mathcal{B}_0$ to $\mathcal{B}$. Then $T$ is called $r$-nuclear from $\mathcal{B}_0$ to $\mathcal{B}$, if there are sequences $\{ \varepsilon_j \}_{j=1}^\infty \subseteq \mathcal{B}_0$ and $\{ \varepsilon_j \}_{j=1}^\infty \subseteq \mathcal{B}$ such that
\[ T = \sum_{j=1}^\infty \varepsilon_j \otimes \varepsilon_j \] (2.14)
with convergence in $\mathcal{B}(\mathcal{B}_0, \mathcal{B})$, and
\[ \sum_{j=1}^\infty \| \varepsilon_j \|_{\mathcal{B}_0'} \| \varepsilon_j \|_{\mathcal{B}} < \infty. \] (2.15)

Here $T$ in (2.14) should be interpreted as the operator
\[ Tf = \sum_{j=1}^\infty \langle f, \varepsilon_j \rangle \varepsilon_j, \quad f \in \mathcal{B}_0, \]
which is well-defined when (2.15) holds. The set of $r$-nuclear operators from $\mathcal{B}_0$ to $\mathcal{B}$ is denoted by $\mathcal{N}_r(\mathcal{B}_0, \mathcal{B})$, and we equip this set by the quasi-norm
\[ \| T \|_{\mathcal{N}_r(\mathcal{B}_0, \mathcal{B})} = \inf \left( \sum_{j=1}^\infty \| \varepsilon_j \|_{\mathcal{B}_0'} \| \varepsilon_j \|_{\mathcal{B}} \right)^r, \]
where the infimum is taken over all representatives $\{ \varepsilon_j \}_{j=1}^\infty \subseteq \mathcal{B}_0$ and $\{ \varepsilon_j \}_{j=1}^\infty \subseteq \mathcal{B}$ such that (2.14) and (2.15) hold true.

Later on we need the following result which shows that $p$-nuclearity is stable under linear continuous mappings. Here and in what follows we write $g \lesssim h$ when $g(\theta) \leq c h(\theta)$ for some
constant $c > 0$ which is independent of $\theta$ in the domains of $g$ and $h$. We also let $g \gtrsim h$ when $g \lesssim h$ and $h \lesssim g$.

**Proposition 2.4** Let $p \in (0, \infty]$, $r \in (0, 1]$, $B_k$ be quasi-Banach spaces of order $r$, $B_{0,k}$ be Banach spaces, $H_k$ be Hilbert spaces, $k = 1, 2$, and let

$$T_1 : B_{0,2} \to B_{0,1} \text{ and } T_2 : B_1 \to B_2$$

be continuous. Then the following is true:

1. if $T \in J_p(B_{0,1}, B_1)$, then $T_2 \circ T \circ T_1 \in J_p(B_{0,2}, B_2)$, and

$$\|T_2 \circ T \circ T_1\|_{J_p(B_{0,2}, B_2)} \lesssim \|T_1\|_{B(B_{0,2}, B_2)} \|T_2\|_{B(B_1, B_2)} \|T\|_{J_p(B_{0,1}, B_1)}; \quad (2.16)$$

2. if $T \in N_r(B_{0,1}, B_1)$, then $T_2 \circ T \circ T_1 \in N_r(B_{0,2}, B_2)$, and

$$\|T_2 \circ T \circ T_1\|_{N_r(B_{0,2}, B_2)} \lesssim \|T_1\|_{B(B_{0,2}, B_2)} \|T_2\|_{B(B_1, B_2)} \|T\|_{N_r(B_{0,1}, B_1)}; \quad (2.17)$$

3. $N_r(H_1, H_2) = J_r(H_1, H_2)$, with equality in quasi-norms.

**Proposition 2.4** is well-known in the literature (cf. [3, 37, 46] and the references therein).

### 2.4 Modulation spaces

Next we discuss basic properties for modulation spaces, and start by recalling the conditions for the involved weight functions. A function $\omega$ on $\mathbb{R}^d$ is called a weight (on $\mathbb{R}^d$), if $\omega > 0$ and $\omega, \omega^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. Let $\omega$ and $v$ be weights on $\mathbb{R}^d$. Then $\omega$ is called moderate or $v$-moderate if

$$\omega(x+y) \lesssim \omega(x)v(y), \quad x, y \in \mathbb{R}^d. \quad (2.18)$$

The weight $v$ is called submultiplicative, if $v$ is even and (2.18) holds when $\omega = v$. We note that if (2.18) holds, then

$$v(-x)^{-1} \lesssim \omega(x) \lesssim v(x).$$

Furthermore, for such $\omega$ it follows that (2.18) is true when

$$v(x) = Ce^{r|x|},$$

for some positive constants $r$ and $C$ (cf. e.g. [26]).

The set of all moderate weights on $\mathbb{R}^d$ is denoted by $\mathcal{P}(\mathbb{R}^d)$.

Let $\omega \in \mathcal{P}(\mathbb{R}^{2d})$ and $p, q \in (0, \infty]$ be fixed. Then the mixed Lebesgue space $L^{p,q}_\omega(\mathbb{R}^{2d})$ consists of all measurable functions $F$ on $\mathbb{R}^{2d}$ such that $\|F\|_{L^{p,q}_\omega} < \infty$. Here

$$\|F\|_{L^{p,q}_\omega} \equiv \|F_{p,\omega}\|_{L^p}, \quad \text{where} \quad F_{p,\omega}(\xi) \equiv \|F(\cdot, \xi)\omega(\cdot, \xi)\|_{L^p}. \quad (2.19)$$

We note that these quasi-norms might attain $+\infty$.

Let $\phi \in \mathcal{S}_1(\mathbb{R}^d) \setminus 0$ be fixed. The modulation space $M^{p,q}_\omega(\mathbb{R}^d)$ is the space which consist of all $f \in \mathcal{S}'_1(\mathbb{R}^d)$ such that $\|f\|_{M^{p,q}_\omega} < \infty$, where

$$\|f\|_{M^{p,q}_\omega} \equiv \|\hat{V}\phi f\|_{L^{p,q}_\omega}. \quad (2.20)$$

For convenience we set $M^{p}_\omega = M^{p,p}_\omega$. Furthermore we set $M^{p,q} = M^{p,q}_\omega$ when $\omega \equiv 1$.

The proof of the following proposition is omitted, since the results can be found in [19–22, 25].
Proposition 2.5 Let $p, q, p_j, q_j \in (0, \infty]$ for $j = 1, 2$, $r \leq \min(p, q, 1)$, and $\omega, \omega_1, \omega_2, v \in \mathcal{D}_E(\mathbb{R}^{2d})$ be such that $v$ is submultiplicative, $\omega$ is $v$-moderate and $\omega_2 \lesssim \omega_1$. Then the following is true:

1. $f \in M^{p,q}_\omega(\mathbb{R}^d)$ if and only if (2.20) holds for any $\phi \in M^r_\omega(\mathbb{R}^d) \setminus \{0\}$. Moreover, $M^{p,q}_\omega$ is a quasi-Banach space under the quasi-norm in (2.20) and different choices of $\phi$ give rise to equivalent quasi-norms. Furthermore, if $p, q \geq 1$, then $M^{p,q}_\omega(\mathbb{R}^d)$ is a Banach space;

2. if $p_1 \leq p_2$ and $q_1 \leq q_2$ then
$$\Sigma_1(\mathbb{R}^d) \hookrightarrow M^{p_1,q_1}_{\omega_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}_{\omega_2}(\mathbb{R}^d) \hookrightarrow \Sigma'_1(\mathbb{R}^d);$$

3. if in addition $p, q \geq 1$, then the $L^2$ product $(\cdot, \cdot)_{L^2}$ on $\mathcal{S}_{1/2}(\mathbb{R}^d)$ extends uniquely to a continuous map from $M^{p,q}_\omega(\mathbb{R}^d) \times M^{p,q}_\omega(\mathbb{R}^d)$ to $C$. If $\|a\| = \sup|\langle a, b \rangle|$, where the supremum is taken over all $b \in \mathcal{S}_{1/2}(\mathbb{R}^d)$ such that $\|b\|_{M^{p,q}_\omega} \leq 1$, then $\| \cdot \|$ and $\| \cdot \|_{M^{p,q}_\omega}$ are equivalent norms;

4. if $p, q < \infty$, then $\mathcal{S}_{1/2}(\mathbb{R}^d)$ is dense in $M^{p,q}_\omega(\mathbb{R}^d)$. If in addition $p, q \geq 1$, then the dual space of $M^{p,q}_\omega(\mathbb{R}^d)$ can be identified with $M^{p',q'}_{\omega}(\mathbb{R}^d)$, through the $L^2$-form $(\cdot, \cdot)_{L^2}$. Moreover, $\mathcal{S}_{1/2}(\mathbb{R}^d)$ is weakly dense in $M^{p',q'}_{\omega}(\mathbb{R}^d)$ with respect to the $L^2$-form.

Remark 2.6 By Theorem 3.9 in [41] it follows that Gelfand-Shilov spaces and their distribution spaces can be obtained by suitable unions and intersections of modulation spaces. In particular we have
$$\bigcap_{\omega \in \mathcal{D}_E} M^{p,q}_\omega(\mathbb{R}^d) = \Sigma_1(\mathbb{R}^d), \quad \bigcup_{\omega \in \mathcal{D}_E} M^{p,q}_\omega(\mathbb{R}^d) = \Sigma'_1(\mathbb{R}^d).$$

3 Born-Jordan Operators with Distribution Symbols

In this section we deduce various kinds of mapping properties of $\text{Op}_{\text{BJ}}(a)$ when $a$ belongs to suitable test-function or distribution spaces. In particular we show that $\text{Op}_{\text{BJ}}(a)$ makes sense as a continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ when $a \in \mathcal{S}'(\mathbb{R}^{2d})$.

We begin with the following analogy of Proposition 2.2 in Born-Jordan situation.

Proposition 3.1 Let $s, \sigma > 0$ be such that $s + \sigma \geq 1$ and let $T(f, \phi) \equiv W_{\text{BJ}}(f, \phi)$ when $f, \phi \in \mathcal{S}(\mathbb{R}^d)$. Then the following is true:

1. the map $(f, \phi) \mapsto T(f, \phi)$ is continuous from $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ and restricts to a continuous map from $\mathcal{S}^{s}_{s}(\mathbb{R}^d) \times \mathcal{S}^{s}_{s}(\mathbb{R}^d)$ to $\mathcal{S}^{s,s}_{s,s}(\mathbb{R}^{2d})$;

2. the map $(f, \phi) \mapsto T(f, \phi)$ from $\mathcal{S}^{s}_{s}(\mathbb{R}^d) \times \mathcal{S}^{s}_{s}(\mathbb{R}^d) \times \mathcal{S}^{s}_{s}(\mathbb{R}^{2d})$ extends uniquely to a continuous map from $(\mathcal{S}^{s}_{s})(\mathbb{R}^d) \times (\mathcal{S}^{s}_{s})(\mathbb{R}^d) \times (\mathcal{S}^{s}_{s})(\mathbb{R}^{2d})$ and from $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^{2d})$.

The same holds true for $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$ when each $\mathcal{S}^{s}_{s}$ and $\mathcal{S}^{s,s}_{s,s}$ are replaced by $\Sigma^{s}_{s}$ and $\Sigma^{s,s}_{s,s}$, respectively.

Proof We use the same notations as in the proof of Proposition 2.2. We recall that $W_{\text{BJ}}(f, g)$ is given by (2.9) when $f, g \in \mathcal{S}(\mathbb{R}^d)$. By Proposition 2.2 and Remark 2.3 it follows that the map $(f, \phi) \mapsto W_{\text{BJ}}(f, \phi)$ is continuous from $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^{2d})$, and that the same holds true if each $\mathcal{S}$ is replaced by $\Sigma$ or by $\Sigma_{s}$. 

\[\square\] Springer
In the same way, Proposition 2.2 and Remark 2.3 show that the map

\[(f, g) \mapsto \int_0^1 W_t(f, g) dt\]

is well-defined and continuous from \(\mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}'(\mathbb{R}^d)\) to \(\mathscr{S}'(\mathbb{R}^{2d})\), and that the same holds true after each \(\mathscr{S}(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^{2d})\) and their duals are replaced by \(\mathcal{S}_s^\sigma(\mathbb{R}^d), \mathcal{S}_s^{\sigma,s}(\mathbb{R}^{2d})\) and their duals, or by \(\mathcal{S}_s^\sigma(\mathbb{R}^d), \mathcal{S}_s^{\sigma,s}(\mathbb{R}^{2d})\) and their duals (cf. [1]). Hence, by letting \(W_{B1}(f, g)\) be defined by the right-hand side of (2.10) for such \(f\) and \(g\), the asserted continuity of the extensions of the map \(T\) follows.

It remains to show the asserted uniqueness of the extensions of \(T\) and we only prove the uniqueness when \(f, g \in \mathcal{S}_s^\sigma'(\mathbb{R}^d)\). The cases when \(f, g \in (\mathcal{S}_s^\sigma)'(\mathbb{R}^d)\) or \(f, g \in \mathscr{S}'(\mathbb{R}^d)\) follow by similar arguments and are left for the reader. Let \(f, g \in \mathcal{S}_s^\sigma'(\mathbb{R}^d)\). By Proposition 2.2 and its proof, and Remark 2.3, it follows that if

\[\{f_\varepsilon\}_{\varepsilon > 0} \subseteq \mathcal{S}_s^\sigma'(\mathbb{R}^d)\quad \text{and} \quad \{g_\varepsilon\}_{\varepsilon > 0} \subseteq \mathcal{S}_s^\sigma'(\mathbb{R}^d),\]

are such that

\[\lim_{\varepsilon \to 0^+} f_\varepsilon = f \quad \text{and} \quad \lim_{\varepsilon \to 0^+} g_\varepsilon = g\]

with convergence in \((\mathcal{S}_s^\sigma)'(\mathbb{R}^d)\), then

\[\lim_{\varepsilon \to 0^+} W_t(f_\varepsilon, g_\varepsilon) = W_t(f, g)\]

in \((\mathcal{S}_s^\sigma)'(\mathbb{R}^{2d})\), locally uniformly with respect to \(t\). Hence

\[\lim_{\varepsilon \to 0^+} W_{B1}(f_\varepsilon, g_\varepsilon) = W_{B1}(f, g)\]

with convergence in \((\mathcal{S}_s^\sigma)'(\mathbb{R}^{2d})\). Hence, if \(\Phi \in \mathcal{S}_{s,\sigma}^\sigma(\mathbb{R}^{2d})\), then Fubini's theorem gives

\[
\lim_{\varepsilon \to 0^+} W_{B1}(f_\varepsilon, g_\varepsilon, \Phi) = \lim_{\varepsilon \to 0^+} \left\langle \int_0^1 (\mathcal{F}_2 \circ U_t)(f_\varepsilon \otimes g_\varepsilon) dt, \Phi \right\rangle = \lim_{\varepsilon \to 0^+} \left( f_\varepsilon \otimes g_\varepsilon, \int_0^1 (U_t^{-1} \circ \mathcal{F}_2^{-1}) \Phi dt \right) = \left( f \otimes g, \int_0^1 (U_t^{-1} \circ \mathcal{F}_2^{-1}) \Phi dt \right),
\]

where the last equality follows from the fact that

\[\Phi \mapsto \int_0^1 (U_t^{-1} \circ \mathcal{F}_2^{-1}) \Phi dt\]

is continuous from \(\mathcal{S}_{s,\sigma}^\sigma(\mathbb{R}^{2d})\) to \(\mathcal{S}_s^\sigma(\mathbb{R}^{2d})\), in view of Proposition 2.2 and its proof. The uniqueness assertions now follow from the facts that we may choose \(f_\varepsilon\) and \(g_\varepsilon\) in \(\mathcal{S}_s^\sigma(\mathbb{R}^d)\). This gives the result.

We have now the following. Here \(\mathcal{L}(V_1, V_2)\) is the set of all linear and continuous mappings from the topological vector space \(V_1\) into the topological vector space \(V_2\).

**Theorem 3.2** Let \(s, \sigma > 0\) be such that \(s + \sigma \geq 1\). Then the following is true:

1. if \(a \in \mathcal{S}_{s,\sigma}^\sigma(\mathbb{R}^{2d})\), then \(\text{Op}_{B1}(a)\) from \(\mathcal{S}_s^\sigma(\mathbb{R}^d)\) to \((\mathcal{S}_s^\sigma)'(\mathbb{R}^d)\) is uniquely extendable to a continuous map from \((\mathcal{S}_s^\sigma)'(\mathbb{R}^d)\) to \(\mathcal{S}_s^\sigma(\mathbb{R}^d)\);

2. the map \(a \mapsto \text{Op}_{B1}(a)\) from \(\mathcal{S}_{s,\sigma}^\sigma(\mathbb{R}^{2d})\) to \(\mathcal{L}(\mathcal{S}_s^\sigma(\mathbb{R}^d), (\mathcal{S}_s^\sigma)'(\mathbb{R}^d))\) is uniquely extendable from \((\mathcal{S}_{s,\sigma}^\sigma)'(\mathbb{R}^{2d})\) to \(\mathcal{L}(\mathcal{S}_s^\sigma(\mathbb{R}^d), (\mathcal{S}_s^\sigma)'(\mathbb{R}^d))\).

\(\square\) Springer
Suppose that in addition \((s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})\). Then the same holds true if each \(S^s_{\omega} \) and \(S^s_{\omega, \sigma} \) are replaced by \(\Sigma^s_{\omega} \) respectively \(\Sigma^s_{\omega, \sigma} \), or by \(\mathcal{I} \) and its dual.

**Proof** We only prove the assertion for symbols in \(S^s_{\omega, \sigma}(\mathbb{R}^{2d}) \) and \((S^s_{\omega, \sigma})'(\mathbb{R}^{2d}) \). The other cases follow by similar arguments and are left for the reader.

By Proposition 3.1 it follows that the map

\[
(a, f, g) \mapsto (2\pi)^{-\frac{d}{2}}(a, W_{BJ}(g, f))_{L^2(\mathbb{R}^{2d})}
\]

from \(S^s_{\omega, \sigma}(\mathbb{R}^{2d}) \times S^s_{\sigma}(\mathbb{R}^{d}) \times S^s_{\omega}(\mathbb{R}^{d}) \) to \(C \) extends uniquely to continuous mappings from \(S^s_{\omega, \sigma}(\mathbb{R}^{2d}) \times (S^s_{\sigma})'(\mathbb{R}^{d}) \times (S^s_{\omega})'(\mathbb{R}^{d}) \) to \(C \), and from \((S^s_{\omega, \sigma})'(\mathbb{R}^{2d}) \times S^s_{\sigma}(\mathbb{R}^{d}) \times S^s_{\omega}(\mathbb{R}^{d}) \) to \(C \). Hence, by letting \(Op_{BJ}(a)f \) be defined by (2.10), the asserted continuity follows. The uniqueness of these extensions follows by similar arguments to those of Proposition 3.1. The details are left to the reader. \(\square\)

We may now complete the previous result with the following.

**Theorem 3.3** Let \(s, \sigma > 0 \) be such that \(s + \sigma \geq 1 \), and let \(t \in \mathbb{R} \). Then the following is true:

1. if \(a \in (S^s_{\omega, \sigma})'(\mathbb{R}^{2d}) \), then \(Op_{BJ}(a) = Op_{s}\), for some \(b \in (S^s_{\omega, \sigma})'(\mathbb{R}^{2d}) \);
2. if \(a \in S^s_{\omega, \sigma}(\mathbb{R}^{2d}) \), then \(Op_{BJ}(a) = Op_{s}\), for some \(b \in S^s_{\omega, \sigma}(\mathbb{R}^{2d}) \).

Suppose that in addition \((s, \sigma) \neq (\frac{1}{2}, \frac{1}{2}) \). Then the same holds true with \(\mathcal{I} \) or \(\Sigma^s_{\omega, \sigma} \) in place of \(S^s_{\omega, \sigma} \) at each occurrence.

**Proof** We only prove the assertion for \(S^s_{\omega, \sigma} \) and \((S^s_{\omega, \sigma})' \). The other cases follow by similar arguments and are left for the reader.

By [1, Theorem 3.6] it suffices to prove the result in the Weyl case \(t = \frac{1}{2} \). Let \(\Omega_1 \) and \(\Omega_2 \) be bounded sets in \(S^s_{\omega, \sigma}(\mathbb{R}^{2d}) \) and in \((S^s_{\omega, \sigma})'(\mathbb{R}^{2d}) \), respectively, and let \(I \subseteq \mathbb{R} \) be a bounded interval. Then the map \((t, a) \mapsto e^{it(D_{\xi}, D_{\eta})}a \) is uniformly continuous from \(I \times \Omega_1 \) to \(S^s_{\omega, \sigma}(\mathbb{R}^{2d}) \) and from \(I \times \Omega_2 \) to \((S^s_{\omega, \sigma})'(\mathbb{R}^{2d}) \), in view of [1, Theorem 3.6] and its proof. Hence

\[
a_{BJ} = \int_0^1 e^{it\frac{1}{2}-(D_{\xi}, D_{\eta})}adt
\]

belongs to \(S^s_{\omega, \sigma}(\mathbb{R}^{2d}) \) respective \((S^s_{\omega, \sigma})'(\mathbb{R}^{2d}) \) when \(a \) does. The result now follows from (1.2), (1.3) and the uniqueness assertions in Theorem 3.2. \(\square\)

## 4 Born-Jordan Operators with Modulation Space Symbols

In this section we deduce that any Born-Jordan operator with symbol in the modulation space \(M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \) is a pseudo-differential operator with symbol in \(M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \), when \(\omega(x, \xi, \eta, y) = \omega_0(\eta, y) \). We also deduce continuity, Schatten-von Neumann and nuclearity properties for such operators.

We begin with the following.

**Theorem 4.1** Let \(p, q \in (0, \infty], \ t \in \mathbb{R}, \ \omega \in \mathcal{P}_E(\mathbb{R}^{4d}) \) be such that \(\omega(x, \xi, \eta, y) = \omega_0(\eta, y) \) for some \(\omega_0 \in \mathcal{P}_E(\mathbb{R}^{2d}) \), and let \(a \in M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \). Then \(Op_{BJ}(a) = Op_{s}\), for some \(b \in M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \).
For the proof we recall that if \( \mu \) and \( \nu \) are positive measures on measurable spaces, \( p, q \in (0, \infty] \) satisfy \( q \leq p \) and \( f \) is \( \mu \times \nu \) measurable, then Minkowski’s inequality asserts that
\[
\|G\|_{L^p(\mu)} \leq \|H\|_{L^q(\nu)}
\]
when
\[
G(y) \equiv \|f(\cdot, y)\|_{L^q(\nu)} \quad \text{and} \quad H(x) \equiv \|f(x, \cdot)\|_{L^p(\mu)}.
\]

In order to treat the case \( p \in (0, 1] \) in suitable ways, we need the following lemma.

**Lemma 4.2** Let \( y \in \mathbb{R}^d \), \( \Lambda \subseteq \mathbb{R}^d \) be a lattice and \( p \in (0, \infty) \). Then
\[
\int_{\mathbb{R}^d} \left( \int_0^1 e^{-r|x-ty|} \, dt \right)^p \, dx \leq \int_{\mathbb{R}^d} \left( \int_0^1 \prod_{j=1}^d e^{-r\sqrt{d}|x_j-ty_j|} \, dt \right)^p \, dx
\]
and
\[
\sum_{j \in \Lambda} \left( \int_0^1 e^{-r|j-ty_j|} \, dt \right)^p \leq e^{-\frac{pr}{\sqrt{d}}|y|}.
\]

**Proof** It is clear that both sides of (4.1) are even functions with respect to each \( y_j \) in \( y = (y_1, \ldots, y_d) \). Hence we may assume that \( y_j \geq 0 \) for every \( j \), when proving (4.1). We prove only (4.1). The estimate (4.2) follows by similar arguments and is left for the reader.

Since \( u \mapsto e^{-ru} \) is a convex function, Hölder’s and Jensen’s inequalities give
\[
\int_{\mathbb{R}^d} \left( \int_0^1 e^{-r|x-ty|} \, dt \right)^p \, dx \leq \int_{\mathbb{R}^d} \left( \int_0^1 \prod_{j=1}^d e^{-r\sqrt{d}|x_j-ty_j|} \, dt \right)^p \, dx
\]
and
\[
\sum_{j \in \Lambda} \left( \int_0^1 e^{-r|j-ty_j|} \, dt \right)^p \leq e^{-\frac{pr}{\sqrt{d}}|y|}.
\]

We need to evaluate the integral in (4.4). By straight-forward computations we get
\[
\int_0^1 |u_1 - tu_2| \, dt = \begin{cases} \frac{2u_1 - u_2}{2}, & u_1 \geq u_2, \\ \frac{2u_1^2 - 2u_1u_2 + u_2^2}{2u_2}, & 0 \leq u_1 \leq u_2, u_2 > 0, \\ \frac{u_2 - 2u_1}{2}, & u_1 < 0. \end{cases}
\]

This gives
\[
\int_{\mathbb{R}} h(u_1, u_2) \, du_1 = I_1(u_2) + I_2(u_2) + I_3(u_2),
\]
where
\[
I_1(u_2) = \int_{u_2}^{\infty} e^{-\frac{pr}{\sqrt{d}} \frac{2u_1 - u_2}{2}} \, du_1 = \frac{\sqrt{d}}{pr} e^{-\frac{pr}{2\sqrt{d}} u_2},
\]
\[
I_2(u_2) = \int_0^{u_2} e^{-\frac{pr}{\sqrt{d}} \frac{2u_1^2 - 2u_1u_2 + u_2^2}{2u_2}} \, du_1 \leq \int_0^{u_2} e^{-\frac{pr}{\sqrt{d}} \frac{2u_1^2 - 2u_1u_2 + u_2^2}{2u_2}} \, du_1
\]
\[\square\] Springer
\[ \leq \int_0^{u_2} e^{-\frac{\pi}{4\sqrt{d}} u_2} \, du_1 = u_2 e^{-\frac{\pi}{4\sqrt{d}} u_2} \leq \frac{8\sqrt{d}}{pr} e^{-\frac{\pi}{8\sqrt{d}} u_2} \]

and

\[ I_3(u_2) = \int_{-\infty}^0 e^{-\frac{\pi}{4\sqrt{d}} \frac{u_2}{2}} \, du_1 = \frac{\sqrt{d}}{pr} e^{-\frac{\pi}{8\sqrt{d}} u_2}. \]

By combining these estimates we get

\[ \int_R h(u_1, u_2) \, du_1 \leq \frac{10\sqrt{d}}{pr} e^{-\frac{\pi}{8\sqrt{d}} u_2}. \]

Hence, (4.3) and the fact that \( y_j \geq 0 \) for every \( j \) give

\[ \int_{R^n} \left( \int_0^1 e^{-r|x-y|} \, dr \right) \, dx \leq \left( \frac{10\sqrt{d}}{pr} \right)^d e^{-\frac{\pi}{8\sqrt{d}} |y_1 + \cdots + y_d|} \leq \left( \frac{10\sqrt{d}}{pr} \right)^d e^{-\frac{\pi}{8\sqrt{d}} |y|}, \]

and the result follows. \( \square \)

**Proof of Theorem 4.1** By the assumptions we have

\[ a_1 \in M_{\omega}(R^d) \quad \Leftrightarrow \quad a_2 \in M_{\omega}(R^d) \]

when \( \text{Op}_{\lambda_1}(a_1) = \text{Op}_{\lambda_2}(a_2) \), in view of [43, Proposition 1.7]. Hence it suffices to prove the result in the case \( t = 0 \). We also assume that \( p, q < \infty \). The cases when \( p = \infty \) or \( q = \infty \) follow by similar arguments and are left for the reader.

Let \( \Lambda_\varepsilon = \varepsilon Z^d \) and \( \Lambda_\varepsilon^2 = \Lambda_\varepsilon \times \Lambda_\varepsilon \) when \( \varepsilon > 0 \). By [43] there are \( v \in \mathcal{P}_E(R^d) \) which is submultiplicative such that \( \omega \) is \( v \)-moderate,

\[ \Psi \in \Sigma_1(R^d) \quad \text{and} \quad \Psi_0 \in M_{\psi}(R^d) \]

such that

\[ a(X) = \sum_{j,k \in \Lambda_\varepsilon} c(j, k) \Psi(X - j) e^{i(X, \rho(k))}, \quad (4.5) \]

and

\[ \|c\|_{L_{\psi}^p(\Lambda_\varepsilon^2)} \leq \|a\|_{M_{\omega}^{p,q}}, \]

where

\[ c(j, k) = (V_{\Psi_0} a)(j, \rho(k)) \quad (4.6) \]

when \( a \in M_{\omega}^{p,q}(R^d) \), provided \( \varepsilon > 0 \) is chosen small enough. Here \( \rho \) is the reflexion operator on \( R^d \) given by \( \rho(x, \xi) = (\xi, x) \) when \( x, \xi \in R^d \).

For \( a \in M_{\omega}^{p,q}(R^d) \) and \( \Phi \in \Sigma_1(R^d) \setminus 0 \) fixed we now get

\[ V_{\Phi}(e^{it(D_\varepsilon, D_\varepsilon)} a) = \sum_{j,k \in \Lambda_\varepsilon} c(j, k) H_{j,k}, \quad (4.7) \]

where

\[ H_{j,k} = V_{\Phi}(e^{it(D_\varepsilon, D_\varepsilon)} (\Psi_0(-j) e^{i(X, \rho(k))}). \]

We need to simplify \( H_{j,k} \). By the definitions and straightforward computations, using Fourier's inversion formula we get

\[ H_{j,k}(X, \rho(Y)) = e^{i((X, \rho(k-Y) + t(k,k))} (V_{\Phi} \Psi_1)(X - j + tk, \rho(Y - k)), \]

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where
\[ \Phi_t = e^{i(D_t, D_x)} \Psi \quad \text{and} \quad k = (k, \kappa) \]

(see (1.13) in [40], and its proof).

Since \( \Psi \in \Sigma_1(\mathbb{R}^{2d}) \), it follows from [9, 47] that \( \{ \Psi_t : t \in [0, 1] \} \) is a bounded set in \( \Sigma_1(\mathbb{R}^{2d}) \).

Hence, by [10], for every \( r > 0 \) there is a constant \( C_r \) which is independent of \( t \in [0, 1] \) such that
\[
|H_{j,k}(X, \rho(Y))| = |(V_\Phi \Psi_t)(X - j + tk, \rho(Y - k))| \leq C_re^{-r(|X-j+tk|+|Y-k|)}.
\]

By using the latter estimate in (4.7) we get with \( r > \lambda \)
\[
|V_\Phi b(X, \rho(Y))| \omega_0(\rho(Y)) \lesssim \sum_{j,k \in \Lambda_x} |c(j, k)|e^{-r|Y-k|} \omega_0(\rho(Y)) \int_0^1 e^{-r|X-j+tk|} dt
\]
\[
\lesssim \sum_{j,k \in \Lambda_x} |c_0(j, k)|e^{-r|Y-k|} \int_0^1 e^{-r|X-j+tk|} dt,
\]
where \( c_0(j, k) = c(j, k) \omega_0(\rho(k)) \). We shall now consider the two cases \( p \geq 1 \) and \( p < 1 \) separately.

First suppose that \( p \geq 1 \). By Minkowski’s inequality we get
\[
\|V_\Phi b(\cdot, \rho(Y))\|_{\ell^p(\Lambda_x)} \omega_0(\rho(Y)) \lesssim \sum_{k \in \Lambda_x} h(k)e^{-r|Y-k|},
\]
where
\[
h(k) = \int_0^1 \left( \sum_{m \in \Lambda_x} \left( \sum_{j \in \Lambda_x} |c_0(j, k)|e^{-r|m-j+tk|} \right)^p \right)^{1/p} dt.
\]

For \( X = (X_1, \cdots, X_{2d}) \in \mathbb{R}^{2d} \), let \( c_{0, X}(j, k) = c_0(j + l, k) \), where \( l = (l_1, \cdots, l_{2d}) \in \Lambda_x \) satisfies \( l_j \leq X_j < l_j + \epsilon, j = 1, \cdots, 2d \). Then
\[
h(k) \lesssim \int_0^1 \left( \sum_{m \in \Lambda_x} \left( \sum_{j \in \Lambda_x} |c_{0, tk}(j, k)|e^{-r|m-j|} \right)^p \right)^{1/p} dt
\]
\[
= ||c_{0, tk}(\cdot, k)|e^{-r|\cdot|}|_{\ell^p} \lesssim \|c_{0, tk}(\cdot, k)\|_{\ell^p} = \|c_{0}(\cdot, k)\|_{\ell^p},
\]
where the last step follows from Young’s inequality. Here * denotes the discrete convolution. If \( h_1(k) = \|c_0(\cdot, k)\|_{\ell^p} \), then we get from these estimates that
\[
\|V_\Phi b(\cdot, \rho(n))\|_{\ell^p(\Lambda_x)} \omega_0(\rho(n)) \lesssim \sum_{k \in \Lambda_x} h_1(k)e^{-r|n-k|} = (h_1 * e^{-r|\cdot|})(n).
\]

By applying the \( \ell^q \) norm on the last inequality we get
\[
\|b\|_{M^{p,q}_{(\omega)}} \approx \|c\|_{p,q} \lesssim \|h_1 * e^{-r|\cdot|}\|_{\ell^p} \leq \|h_1\|_{\ell^q} \|e^{-r|\cdot|}\|_{\ell^p_{\min(1,q)}} \approx \|h_1\|_{\ell^q} \approx \|a\|_{M^{p,q}_{(\omega)}},
\]

Hence we have proved
\[
\|b\|_{M^{p,q}_{(\omega)}} \lesssim \|a\|_{M^{p,q}_{(\omega)}},
\]
and the result follows in the case \( p \geq 1 \).

Suppose instead \( p < 1 \) and let \( h_1 \) be as above. By Lemma 4.2, applying the \( \ell^p(\Lambda_x) \) norm with respect to the \( X \) variable on the inequality (4.8), and using the inequality \( |a + b|^p \leq |a|^p + |b|^p \) we get
\[
\|V_\Phi b(\cdot, \rho(Y))\|_{\ell^p(\Lambda_x)} \omega_0(\rho(Y)) \lesssim \sum_{j,k \in \Lambda_x} |c_0(j, k)|^p \left( \sum_{m \in \Lambda_x} \left( \int_0^1 e^{-r(|m-j+tk|)} dt \right)^p \right)^{1/p} e^{-rp|Y-k|}
\]
\[ \Box \] Springer
\[ \sum_{j, k \in \Lambda_r} |c_0(j, k)|^p e^{-r p |Y - k|} = \sum_{k \in \Lambda_r} |h_1(k)|^p e^{-r p |Y - k|}. \]

We now apply the \( \ell^p_r(\Lambda_r) \) quasi-norm on on the latter inequality and use the Young type inequality

\[ \|c_1 * c_2\|_{\ell^p_r(\Lambda_r)} \leq \|c_1\|_{\ell^p_r(\Lambda_r)} \|c_2\|_{\ell^p_r(\Lambda_r)} \]

to get

\[ \|b\|_{M^{p,q}_{\omega}(\omega)} \leq \|V_b \cdot \omega\|_{\ell^{p,q}(\Lambda_r \times \Lambda_r)} \lesssim \|h_1^p e^{-r p |\cdot|}\|_{\ell^p_r(\Lambda_r)} \]
\[ \leq \left( \|h_1^p \|_{\ell^p_r(\Lambda_r)} \|e^{-r p |\cdot|}\|_{\ell^{p,q}(\Lambda_r)} \right)^{\frac{1}{p}} \leq \|h_1^p \|_{\ell^p_r(\Lambda_r)} \lesssim \|a\|_{M^{p,q}_{\omega}(\omega)}, \]

which gives the result in the case \( p < 1 \) as well. \( \square \)

We finish this section by giving some consequences of Theorem 4.1 and well-known mapping properties for pseudo-differential operators with symbols in modulation spaces. The involved weight functions should satisfy

\[ \omega(x, \xi, \eta, y) = \omega_0(\eta, y) \quad \text{and} \quad \frac{\omega_2(x, \eta - \xi)}{\omega_1(x + y, \eta + \xi)} \lesssim \omega_0(\eta, y). \quad (4.9) \]

The first result is a straight-forward consequence of [45, Theorem 3.1] and Theorem 4.1. The details are left for the reader.

**Theorem 4.3** Let \( \omega_j \in \mathcal{S}_E(\mathbb{R}^{2d}), j = 0, 1, 2, \omega \in \mathcal{S}_E(\mathbb{R}^{4d}) \) be such that (4.9) holds, \( p, q, p_j, q_j \in (0, \infty), j = 1, 2 \), be such that

\[ \frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{q_2} - \frac{1}{q_1} = \frac{1}{p} + \min \left( \frac{1}{q} - 1 \right), \quad q \leq p_2, q_2 \leq p, \]

and let \( a \in M^{p,q}_{\omega}(\mathbb{R}^{4d}) \). Then \( \text{Op}_{BJ}(a) \) from \( \Sigma_1(\mathbb{R}^d) \) to \( \Sigma'_1(\mathbb{R}^d) \) is uniquely extendable to a continuous map from \( M^{p,q}_{\omega_1}(\mathbb{R}^d) \) to \( M^{p_2,q_2}_{\omega_2}(\mathbb{R}^d) \).

The next result follows from [46, Theorem 3.1] and Theorem 3.2. We leave the details for the reader.

**Theorem 4.4** Let \( \omega_j \in \mathcal{S}_E(\mathbb{R}^{2d}), j = 0, 1, 2, \omega \in \mathcal{S}_E(\mathbb{R}^{4d}) \) be such that (4.9) holds, \( p, q, r \in (0, \infty) \) be such that

\[ \frac{1}{r} - 1 \geq \max \left( \frac{1}{p} - 1, 0 \right) + \left( \frac{1}{q} - 1, 0 \right) + \frac{1}{q}, \]

and let \( a \in M^{p,q}_{\omega_j}(\mathbb{R}^{2d}) \). Then

\[ \text{Op}_{BJ}(a) \in \mathcal{S}_E(M^{\infty}_{\omega_1}(\mathbb{R}^d), M^p_{\omega_2}(\mathbb{R}^d)). \]

The following result concerns nuclearity for pseudo-differential operators and is a consequence of [46, Theorem 4.2] and Theorem 3.2. The details are left for the reader.

**Theorem 4.5** Let \( \omega_j \in \mathcal{S}_E(\mathbb{R}^{2d}), j = 0, 1, 2, \omega \in \mathcal{S}_E(\mathbb{R}^{4d}) \) be such that (4.9) holds, \( p \in (0, 1) \) and let \( a \in M^p_{\omega_1}(\mathbb{R}^{2d}) \). Then

\[ \text{Op}_{BJ}(a) \in \mathcal{S}_E(M^{\infty}_{\omega_1}(\mathbb{R}^d), M^p_{\omega_2}(\mathbb{R}^d)). \]
The last result in this record concerns Schatten-von Neumann properties for Born-Jordan operators when acting on modulation spaces of Hilbert type. For \( p \in (0, 1] \), the result follows from Theorem 4.5 and the fact that

\[
\mathcal{M}_p(M^\infty_{(\omega_1)}(\mathbb{R}^d), M^p_{(\omega_2)}(\mathbb{R}^d)) \subseteq \mathcal{M}_p(M^2_{(\omega_1)}(\mathbb{R}^d), M^2_{(\omega_2)}(\mathbb{R}^d)) = \mathcal{M}_p(M^2_{(\omega_1)}(\mathbb{R}^d), M^2_{(\omega_2)}(\mathbb{R}^d)),
\]

because

\[
M^2_{(\omega_1)}(\mathbb{R}^d) \subseteq M^\infty_{(\omega_1)}(\mathbb{R}^d) \quad \text{and} \quad M^p_{(\omega_2)}(\mathbb{R}^d) \subseteq M^2_{(\omega_2)}(\mathbb{R}^d), \quad p \leq 2,
\]
since modulation spaces increase with their Lebesgue exponents. For \( p \in [1, 2] \), the result follows from the extension [42, Theorem A.3] of [40, Theorem 4.13] and Theorem 3.2.

**Theorem 4.6** Let \( \omega_j \in \mathcal{P}_E(\mathbb{R}^{2d}), \ j = 0, 1, 2, \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \) be such that (4.9) holds, \( p \in (0, \infty) \), let \( q = p \) when \( p \leq 2 \) and \( q = p' \) when \( p > 2 \) and let \( a \in M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \). Then

\[
\text{Op}_{BJ}(a) \in \mathcal{M}_p(M^2_{(\omega_1)}(\mathbb{R}^d), M^2_{(\omega_2)}(\mathbb{R}^d)).
\]

### 5 Born-Jordan Operators with Hörmander Symbols

In this section we show that any Born-Jordan operator with symbol in a suitable Hörmander class is a pseudo-differential operator with symbol in the same class. Furthermore, we deduce Schatten-von Neumann properties and lower bound estimates for such operators. Especially we deduce Fefferman-Phong’s inequality for Born-Jordan operators.

First we recall the definition of the involved symbol classes. Let \( g \) be a Riemannian metric on the phase space \( \mathbb{R}^{2d} \cong T^* \mathbb{R}^d \), let \( m > 0 \) be a function in \( L^\infty_{\text{loc}}(\mathbb{R}^{2d}) \), and let \( N \geq 0 \) be an integer. For any \( a \in C^N(\mathbb{R}^{2d}) \) and \( X = (x, \xi) \in \mathbb{R}^{2d} \), let \( |a|^g_m(X) = |a(X)| \), and

\[
|a|^g_m(X) \equiv \sup_{k \geq 1} |a^{(k)}(X; Y_1, \cdots, Y_k)|, \quad k \geq 1.
\]

Here the supremum is taken over all \( Y_1, \cdots, Y_k \in \mathbb{R}^{2d} \) such that \( g_X(Y_j) \leq 1 \) for every \( j = 1, \cdots, k \). We also set

\[
\|a\|^g_{m,N} \equiv \sum_{k=0}^N \sup_{X \in W} \big| |a|^g_m(X)/m(X) \big|.
\]

Then \( S(m, g) \) consists of all \( a \in C^\infty(\mathbb{R}^{2d}) \) such that \( \|a\|^g_{m,N} \) is finite for every \( N \geq 0 \). We equip this space by the topology, induced by the semi-norms \( \| \cdot \|^g_{m,N}, N \geq 0 \).

The dual metric \( g^\sigma \) with respect to the (standard) symplectic form \( \sigma \) is defined by

\[
g^\sigma_X(Z) \equiv \sup_{|\sigma(Y, Z)|^2, \ X, Z \in \mathbb{R}^{2d}},
\]

where the supremum is taken over all \( Y \in \mathbb{R}^{2d} \) such that \( g_X(Y) \leq 1 \). Furthermore, the Planck’s function \( h_g(X) \) is defined by

\[
h_g(X) \equiv \sup_{X \in \mathbb{R}^{2d}} g_X(Y)^{1/2}, \quad X \in \mathbb{R}^{2d},
\]

where the supremum is taken over all \( Y \in \mathbb{R}^{2d} \) such that \( g^\sigma_X(Y) \leq 1 \).

As in [8, 39] we need some restrictions on \( m \) and \( g \). More precisely, the metric \( g \) on \( \mathbb{R}^{2d} \) is called slowly varying if there are constants \( c, C > 0 \) such that

\[
C^{-1}g_X \leq g_Y \leq Cg_X, \quad \text{when} \quad g_X(Y) < c.
\]
and $m$ is called $g$-continuous if there are constants $c, C > 0$ such that

$$C^{-1}m(X) \leq m(Y) \leq Cm(X), \quad \text{when } g_X(Y) < c,$$

The metric $g$ is called $\sigma$-temperate if there are constants $C, N > 0$ such that

$$g_X(Z) \leq Cg_Y(Z)(1 + g_X(X - Y))^N, \quad \text{for all } X, Y, Z \in \mathbb{R}^{2d},$$

and $m$ is called $(\sigma, g)$-temperate if there are constants $C, N > 0$ such that

$$m(X) \leq Cm(Y)(1 + g_X(X - Y))^N, \quad \text{for all } X, Y \in \mathbb{R}^{2d}.$$

The metric $g$ on $W$ is called strongly feasible if $g$ is slowly varying, $\sigma$-temperate and $h_g \leq 1$. The weight $m$ is called $g$-feasible if $m$ is $g$-continuous and $(\sigma, g)$-temperate.

Finally, the Riemannian metric $g$ on $\mathbb{R}^{2d}$ is called split or split metric, if

$$g_X(z, \zeta) = g_X(z, -\zeta), \quad \text{when } X, (z, \zeta) \in \mathbb{R}^{2d}.$$

**Remark 5.1** The family $S(m, g)$ may serve as a home for several classical symbol classes. For example we have the following (see [30] for details). Here we let $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ when $x \in \mathbb{R}^d$, $X = (x, \xi) \in \mathbb{R}^{2d}$ and $Y = (y, \eta) \in \mathbb{R}^{2d}$.

1. Let $r, \rho, \delta \in \mathbb{R}$ satisfy $0 \leq \delta \leq \rho \leq 1$ and $\delta \neq 1$,

$$g_X(Y) = \langle \xi \rangle^{2\rho}|y|^2 + \langle \xi \rangle^{-2\rho}|\eta|^2 \quad \text{and} \quad m(X) = \langle \xi \rangle^r.$$

Then $g$ is strongly feasible and split metric on $\mathbb{R}^{2d}, m$ is $g$-feasible and $S(m, g)$ is equal to the Hörmander class $S_{r,\rho}^0(\mathbb{R}^{2d})$ which consists of all $a \in C^\infty(\mathbb{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \lesssim \langle \xi \rangle^{-\rho |\beta| + \delta |\alpha|}, \quad \alpha, \beta \in \mathbb{N}^d.$$

2. Let $t, \rho, \tau \in \mathbb{R}$ satisfy $0 < t, \tau \leq 1$,

$$g_X(Y) = \langle x \rangle^{-2t}|y|^2 + \langle \xi \rangle^{-2\tau}|\eta|^2 \quad \text{and} \quad m(X) = \langle x \rangle^r \langle \xi \rangle^\rho.$$

Then $g$ is strongly feasible and split metric on $\mathbb{R}^{2d}, m$ is $g$-feasible and $S(m, g)$ is equal to the SG class $SG_{r,\tau}^\rho(\mathbb{R}^{2d})$ which consists of all $a \in C^\infty(\mathbb{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \lesssim \langle x \rangle^{-t |\alpha|} \langle \xi \rangle^{\rho - \tau |\beta|}, \quad \alpha, \beta \in \mathbb{N}^d.$$

3. Let $t, r \in \mathbb{R}$ satisfy $0 < t \leq 1$,

$$g_X(Y) = \langle (x, \xi) \rangle^{-2t}(|y|^2 + |\eta|^2) \quad \text{and} \quad m(X) = \langle (x, \xi) \rangle^r.$$

Then $g$ is strongly feasible and split metric on $\mathbb{R}^{2d}, m$ is $g$-feasible and $S(m, g)$ is equal to the Shubin class $Sh_t^r(\mathbb{R}^{2d})$ which consists of all $a \in C^\infty(\mathbb{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \lesssim \langle (x, \xi) \rangle^{-t |\alpha + \beta|}, \quad \alpha, \beta \in \mathbb{N}^d.$$

We have now the following.

**Theorem 5.2** Let $g$ be strongly feasible and split metric on $\mathbb{R}^{2d}, m$ be $g$-feasible, $N > 0$ be an integer, and let $a \in S(m, g)$.

Then

$$\text{Op}_{BJ}(a) = \text{Op}^w(b)$$

for some $b \in S(m, g)$. Furthermore,

$$b - \sum_{2j < N} \frac{(-1)^j (D_\xi D_\eta)^{2j} a}{4j(2j + 1)!} \in S(h_g^N m, g) \quad \text{for all } a \in S(m, g).$$

(5.1)
For the proof we need the following proposition. We omit the proof, since the result is essentially a restatement of Proposition 18.5.10 in [30].

**Proposition 5.3** Let \( g \) be strongly feasible and split metric on \( \mathbb{R}^{2d} \), \( m \) be \( g \)-feasible, \( I \subseteq \mathbb{R} \) be bounded, \( N > 0 \) be an integer, \( a \in S(m, g) \), and let \( a_t \in \mathcal{S}(\mathbb{R}^{2d}) \) be such that

\[
\text{Op}^w(a) = \text{Op}_t(a_t), \quad t \in I.
\]

Then \( \{a_t\}_{t \in I} \) is a bounded subset of \( S(m, g) \), and

\[
\left\{ a_t - \sum_{k < N} \left( t - \frac{1}{2} \right)^k \frac{i^k (D_{\xi} D_{\eta})^k a}{k!} \right\}_{t \in I}
\]

is a bounded set in \( S(h^N_g m, g) \).

**Proof of Theorem 5.2** Let \( a_t \) be the same as in Proposition 5.3. Then

\[
b = \int_0^1 a_t dt,
\]

and Proposition 5.3 shows that \( b \in S(m, g) \). Furthermore, by (5.3) we get

\[
b - \sum_{k < N} \left( \int_0^1 \left( t - \frac{1}{2} \right)^k dt \cdot \frac{i^k (D_{\xi} D_{\eta})^k a}{k!} \right) \in S(h^N_g m, g),
\]

which gives (5.1). The proof is complete. \( \square \)

Theorem 5.2 shows that several continuity properties in the Weyl calculus in Chapter XVIII in [30] carry over to Born-Jordan operators. For example, the following results are immediate consequences of Propositions 18.6.2 and 18.6.3 in [30], and Theorem 5.2.

**Theorem 5.4** Let \( g \) be strongly feasible and split metric on \( \mathbb{R}^{2d} \), \( m \) be \( g \)-feasible, and let \( a \in S(m, g) \). Then \( \text{Op}_{BJ}(a) \) is continuous on \( \mathcal{S}(\mathbb{R}^d) \) and is uniquely extendable to a continuous operator on \( \mathcal{S}(\mathbb{R}^d) \).

**Proposition 5.5** Let \( g \) be strongly feasible and split metric on \( \mathbb{R}^{2d} \), and let \( a \in S(1, g) \). Then \( \text{Op}_{BJ}(a) \) is continuous on \( L^2(\mathbb{R}^d) \).

The following two theorems extend the previous result. Here and in what follows we let \( L^\infty_0(\mathbb{R}^d) \) be the set of all \( f \in L^\infty(\mathbb{R}^d) \) vanishing at infinity, i.e., \( f \) should satisfy

\[
\lim_{R \to \infty} \text{ess sup}_{|x| \geq R} |f(x)| = 0.
\]

**Theorem 5.6** Let \( p \in [1, \infty], t \in \mathbb{R}, g \) be strongly feasible and split metric on \( \mathbb{R}^{2d} \), \( m \) be \( g \)-continuous and \( (\sigma, g) \)-temperate, and let \( a \in S(m, g) \). Then the following is true:

1. if \( h^k g m \in L^p(\mathbb{R}^{2d}) \) for some \( k \geq 0 \), and \( a \in L^p(\mathbb{R}^{2d}) \), then \( \text{Op}_{BJ}(a) \in \mathcal{S}(L^2(\mathbb{R}^d)) \);
2. if \( h^{k/2} g m \in L^\infty_0(\mathbb{R}^{2d}) \) for some \( k \geq 0 \), and \( a \in L^\infty_0(\mathbb{R}^{2d}) \), then \( \text{Op}_{BJ}(a) \) is compact on \( L^2(\mathbb{R}^d) \).

**Theorem 5.7** Let \( p \in (0, \infty], t \in \mathbb{R}, g \) be strongly feasible and split metric on \( \mathbb{R}^{2d} \), \( m \) be \( g \)-continuous and \( (\sigma, g) \)-temperate, and let \( a \in S(m, g) \). Then the following is true:

1. if \( m \in L^p(\mathbb{R}^{2d}) \), then \( \text{Op}_{BJ}(a) \in \mathcal{S}(L^2(\mathbb{R}^d)) \);
2. if \( m \in L^\infty_0(\mathbb{R}^{2d}) \), then \( \text{Op}_{BJ}(a) \) is compact on \( L^2(\mathbb{R}^d) \).

For the proof of Theorem 5.6 we need the following result which is a slight extension of [8, Proposition 2.10]. The proof is omitted, since the result follows by the arguments in the proof of [8, Proposition 2.10].
Lemma 5.8 Let $p$ and $g$ be the same as in Proposition 5.6 and let $m$ be $g$-continuous, $(\sigma, g)$-temperate and satisfies $h_g^{N/2} m \in L^p(\mathbb{R}^{2d})$ ($h_g^{N/2} m \in L_0^\infty(\mathbb{R}^{2d})$) for some $N \geq 0$. Also let $t \in [0, 1]$, and let $a, a_t \in \mathcal{S}'(\mathbb{R}^{2d})$ be related as in (5.2). Then the map $a \mapsto a_t$ on $\mathcal{S}'(\mathbb{R}^{2d})$ restricts to a continuous isomorphism on $S(m, g) \cap L^p(\mathbb{R}^{2d})$ ($S(m, g) \cap L_0^\infty(\mathbb{R}^{2d})$), which is uniformly bounded with respect to $t \in [0, 1]$.

Proof of Theorem 5.6 Again let $b$ be chosen such that $\text{Op}^w(b) = \text{Op}_{BJ}(a)$. Then Lemma 5.8 shows that $b \in L^p(\mathbb{R}^{2d}) \cap S(m, g)$ when $a \in L^p(\mathbb{R}^{2d}) \cap S(m, g)$, and that $b \in L_0^\infty(\mathbb{R}^{2d}) \cap S(m, g)$ when $b \in L_0^\infty(\mathbb{R}^{2d}) \cap S(m, g)$. The result now follows from Theorem 2.9 in [8].

Proof of Theorem 5.7 The result is a special case of Theorem 4.6 in the case $p > 1$. If instead $p < 1$, then the result follows by combining [45, Theorem 4.1] with Theorem 5.2.

Finally we also have the following Feffermann-Phong’s inequality for Born-Jordan operators, which in particular shows that Sharp-Gårding’s inequality is also true for such operators.

Theorem 5.9 Let $g$ be strongly feasible and split metric on $\mathbb{R}^{2d}$, and let $0 \leq a \in S(h_g^{-1} g)$. Then $\text{Op}_{BJ}(a) \geq -C$ for some constant $C \geq 0$.

Proof By letting $b$ be defined by $\text{Op}^w(b) = \text{Op}_{BJ}(a)$ choosing $N = 2$ in Theorem 5.2, it follows that $b = a + c$, where $c \in S(1, g)$. The result now follows from the facts that $\text{Op}^w(a)$ is lower bounded on $L^2$ by the Feffermann-Phong’s inequality for Weyl operators (cf. [30, Theorem 18.6.8]), and the fact that $\text{Op}^w(c)$ is bounded on $L^2$ in view of [30, Proposition 18.6.3].

Remark 5.10 By similar arguments as in the proof of Proposition 5.9, it follows that Hörmander’s improvement of Melin’s inequality given in Theorem 6.2 in [28], holds true with Born-Jordan operators in place of Weyl operators. That is, if $g$ is strongly feasible and split metric on $\mathbb{R}^{2d}$ and $a$ satisfies the same conditions as in Theorem 6.2 in [28], then $\text{Op}_{BJ}(a)$ is bounded from below. (See also the introduction.)

Remark 5.11 In [6], Bony and Chemin introduced a broad family of Sobolev type spaces, where each space $H(m_0, g)$ is a Hilbert space and depends on the choice of the strongly feasible metric $g$ and the $g$-feasible weight $m_0$. By Théorème 4.5 and Corollaire 6.6 in [6] it follows that $H(1, g) = L^2$ and that there are $a \in S(m_0, g)$ and $b \in S(1/m_0, g)$ such that corresponding pseudo-differential operators are inverses to each others and lift $H(m \cdot m_0, g)$ and $H(m/m_0, g)$, respectively, to $H(m, g)$. That is,

$$\text{Op}^w(a) \circ \text{Op}^w(b) = \text{Op}^w(b) \circ \text{Op}^w(a)$$

equals to the identity operator on $\mathcal{S}'(\mathbb{R}^d)$ and

$$\text{Op}(a) : H(m \cdot m_0, g) \to H(m, g) \quad \text{and} \quad \text{Op}(b) : H(m/m_0, g) \to H(m, g)$$

are bijections for every $g$-feasible weight $m$.

From these results it follows that Theorems 5.6, 5.7 and 5.9 can be generalized to involve $H(m, g)$ spaces. For example, it follows from these results and Proposition 5.5 that if $g$ is a strongly feasible and split metric on $\mathbb{R}^{2d}$, $m$ and $m_0$ are $g$-feasible, and $a \in S(m_0, g)$. Then $\text{Op}_{BJ}(a)$ is continuous from $H(m_0, g)$ to $H(m_0/m, g)$.

Remark 5.12 Evidently, all continuity and compactness results above applies on the symbol classes in Remark 5.1. For example, if $a$ is symbol in a SG-class or Shubin class and...
belongs to $L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then Theorem 5.7 shows that $\text{Op}_{BJ}(a) \in \mathscr{S}_p(L^2(\mathbb{R}^d))$.

6 Born-Jordan Operators of Infinite Orders

In this section we consider Born-Jordan operators with symbols belonging to classes considered in [9], or more generally, classes considered in [1]. This means that the symbols obey ultra-regularity conditions (i.e., certain types of Gevrey regularity), but are allowed to grow superexponentially. In particular, they are allowed to grow faster than polynomials. It is proved in [1, 9] that corresponding pseudo-differential operators are continuous on suitable Gelfand-Shilov spaces and their duals. Here we use (1.2) to carry over these continuity properties and to Born-Jordan operators.

The symbol classes which we shall consider are given in the following definition.

**Definition 6.1** For $s_j, \sigma_j > 0$, $j = 1, 2$, $h, r > 0$ and $f \in C^\infty(\mathbb{R}^{d_1+d_2})$, let

$$
\|f\|_{(h,r)} = \sup \left( \frac{|\partial_{x_1}^\alpha \partial_{x_2}^\beta f(x_1,x_2)|}{(h!)^{1/\alpha_1} (r!)^{1/\alpha_2} \alpha_1! \alpha_2! (|x_1|^\alpha + |x_2|^\beta)^{\frac{d_1}{2}+\frac{d_2}{2}}},
$$

(6.1)

where the supremum is taken over all $\alpha_1 \in \mathbb{N}^{d_1}$, $\alpha_2 \in \mathbb{N}^{d_2}$, $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$.

1. $\Gamma_{s_1,s_2}^{1,1}(\mathbb{R}^{d_1+d_2})$ consists of all $f \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that for some $h > 0$, $\|f\|_{(h,r)}$ is finite for every $r > 0$;
2. $\Gamma_{s_1,s_2}^{1,2}(\mathbb{R}^{d_1+d_2})$ consists of all $f \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that for some $r > 0$, $\|f\|_{(h,r)}$ is finite for every $h > 0$;
3. $\Gamma_{s_1,s_2}^{2,2}(\mathbb{R}^{d_1+d_2})$ consists of all $f \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that $\|f\|_{(h,r)}$ is finite for every $h, r > 0$.

The topologies of the spaces in Definition 6.1 are given by canonical combinations of inductive limit and projective limit topologies (cf. [1]).

We have now the following analogies of Theorems 3.2 and 3.3.

**Theorem 6.2** Let $s, \sigma > 0$ be such that $s + \sigma \geq 1$. Then the following is true:

1. if $a \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$, then $\text{Op}_{BJ}(a)$ is continuous on $S^s_\sigma(\mathbb{R}^d)$ and is uniquely extendable to a continuous map on $(S^s_\sigma)'(\mathbb{R}^d)$;
2. if in addition $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$ and $a \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$, then $\text{Op}_{BJ}(a)$ is continuous on $\Sigma_s(\mathbb{R}^d)$ and is uniquely extendable to a continuous map on $(\Sigma^s_s)'(\mathbb{R}^d)$.

**Theorem 6.3** Let $s, \sigma > 0$ be such that $s + \sigma \geq 1$, and let $t \in \mathbb{R}$. If $a \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$, then $\text{Op}_{BJ}(a) = \text{Op}_t(b)$, for some $b \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$.

If in addition $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, then the same holds true with $\Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ or $\Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ in place of $\Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d})$ at each occurrence.

Theorem 6.3 follows from Theorems 3.1 and 3.6 in [1] and (1.4). Theorem 6.2 then follows by combining Theorems 3.8 and 3.15 in [1] with Theorem 6.3. The details are left for the reader.

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