HOMOTOPY LIFTING PROPERTY OF AN $\varepsilon'$-LIPSCHITZ AND CO-LIPSCHITZ MAP

SHICHENG XU

Abstract. An $\varepsilon'$-Lipschitz and co-Lipschitz map, as a metric analogue of an $\varepsilon$-Riemannian submersion, naturally arises from a sequence of Alexandrov spaces with curvature uniformly bounded below that converges to a space of only weak singularities. In this paper we prove its homotopy lifting property and its homotopy stability in Gromov-Hausdorff topology.

A map $f : X \to Y$ between two metric spaces is called an $\varepsilon'$-Lipschitz and co-Lipschitz [11, 19] (briefly, $\varepsilon'$-LcL), if for any $p \in X$, and any $r > 0$, the metric balls satisfy

$$B_{\varepsilon'-r}(f(p)) \subseteq f(B_r(p)) \subseteq B_{\varepsilon'+r}(f(p)).$$

A 1-LcL preserves metric balls exactly and is called a submetry. An $\varepsilon'$-LcL naturally arises from a sequence of Alexandrov spaces with curvature uniformly bounded below that converges to a space of only weak singularities. Recall that an Alexandrov space with curvature bounded below, $\text{curv} \geq \kappa$, [2] is a complete length metric space such that any geodesic triangle looks fatter than that with the same side-lengths in the simply-connected space of constant curvature $\kappa$. The Hausdorff dimension of an Alexandrov space is always an integer or $\infty$.

In this notes we will study the fibration arising from of an $\varepsilon'$-LcL (Theorem A and Theorem B), and give some applications on convergence of Alexandrov spaces in Gromov-Hausdorff topology (see Section 2). The first result is about the homotopy lifting property of an $\varepsilon'$-Lipschitz and co-Lipschitz map.

Theorem A. A proper $(1.023)$-LcL $f : X \to B$ from a finite-dimensional Alexandrov space $X$ with curvature bounded below to a $n$-dimensional Riemannian manifold $B$ is a Hurewicz fibration i.e., satisfying the homotopy lifting property with respect to any space.
Remark 0.1. In [19] we proved that if, in addition, each fiber of $f$ in Theorem A is a topological manifold without boundary and the co-dimension is $n$, then $f$ is a fiber bundle projection.

Theorem A can be viewed as a weak generalization of the local trivialization of a proper submersion between manifolds. Recall that a submersion $f$ between two Riemannian manifolds is said to be an $\epsilon$-Riemannian submersion if its differential almost preserves the norm of any horizontal vector, that is, for any vector $v$ perpendicular to the fibers, 

$$e^{-\epsilon} \leq \frac{|df(v)|}{|v|} \leq e^\epsilon.$$ 

By definition, an $\epsilon$-Riemannian submersion is an $\epsilon$-LcL. It can be easily checked that any smooth $\epsilon$-LcL between Riemannian manifolds is an $\epsilon$-Riemannian submersion.

We shall point it out that an $\epsilon$-LcL for $\epsilon > 0$ is weaker than an $\epsilon$-Riemannian submersion or a submetry. In the case of an $\epsilon$-Riemannian submersion or a submetry over Riemannian manifolds, minimal geodesics can be uniquely lifted to a horizontal curved in the total space and thus give a local trivialization. For an $\epsilon$-LcL, however, the horizontal curves are not unique any more. Instead, we will apply a weaker replacement of horizontal lifting, a neighborhood retraction $\varphi_p$ to a fiber $f^{-1}(p)$, which is constructed in [19] (see also Proposition 1.6, Section 1) through iterated gradient deformations of distance functions. If the map is a Riemannian submersion or a submetry (i.e. 1-LcL), then the neighborhood retraction coincides with the local trivialization map by canonical horizontal liftings.

Since the proof of Theorem A will be based on local properties, it still holds if the base space $B$ is a metric space in which each point admits a neighborhood almost isometric to a ball in $\mathbb{R}^n$. In particular, if there is an $(n, \delta)$-strainer at a point $p$ in an $n$-dimensional Alexandrov space, then such bi-Lipschitz homeomorphism of dilation of almost 1 (depending on $\delta$ and $n$) exists around $p$. Recall that an $(n, \delta)$-strainer at $p$ consists of $n$ pairs of points $(a_i, b_i)_{i=1}^n$ such that the corresponding angles (see [2])

$$\tilde{\angle} a_i p a_j > \frac{\pi}{2} - \delta, \quad \tilde{\angle} b_i p b_j > \frac{\pi}{2} - \delta, \quad \tilde{\angle} a_i p b_i > \pi - \delta, \quad \tilde{\angle} a_i p b_j > \frac{\pi}{2} - \delta.$$ 

And $p$ is said to be $(n, \delta)$-strained.

Corollary 0.2. Given any positive integer $n$, there exists a positive number $\delta_0(n)$ such that for any $\delta < \delta_0$, there is $\epsilon > 0$ satisfying that if $f$ is a proper $\epsilon$-LcL from a finite-dimensional Alexandrov space $X$ to an $n$-dimensional Alexandrov space $B$ and each point of $B$ is $(n, \delta)$-strained, then $f$ is a Hurewicz fibration.
Remark 0.3. If each fiber of $f$ in Corollary 0.2 is a topological manifold without boundary of co-dimension $n$, then by Remark 0.1, $f$ is a locally trivial fibration.

In the case that $f$ is a submetry (i.e., 1-LcL) it follows from Perelman’s fibration theorem on regular admissible maps ([15]) that $f$ admits a locally trivialization over any point in $B$. This is because, a submetry $f : X \to B$ satisfies that, for any $p \in B$, the distance function to the fiber $f^{-1}(p)$ coincides with $\text{dist}_p \circ f : X \to \mathbb{R}$ (see 1.2). Let $(a_i, b_i)_{i=1}^n$ be an $(n, \delta)$-strainer at $p$. According to [2], the map $\varphi = (\text{dist}_{a_1}, \ldots, \text{dist}_{a_n}) : B \sigma (p) \to \mathbb{R}^n$ is an almost isometry to its image for small $\sigma$, and thus $f$ can represented by

$$(\text{dist}_{f^{-1}(a_1)}, \ldots, \text{dist}_{f^{-1}(a_n)}) = (\text{dist}_{a_1} \circ f, \ldots, \text{dist}_{a_n} \circ f) : X \to \mathbb{R}^n.$$  

In general an $e^{\epsilon}$-LcL between Alexandrov spaces fails to satisfy the homotopy lifting property over singular points. Counterexamples can be constructed by considering non-free isometric group actions on a Riemannian manifold, where the quotient space is an Alexandrov space and the projection is a submetry.

A partial motivation to study an $e^{\epsilon}$-LcL is that it naturally arises in some interesting geometry situation. For example, according to [24] (see also Theorem 2.1), if an Alexandrov space with curvature bounded below is close enough to an Alexandrov space in Gromov-Hausdorff topology, then an $e^{\epsilon}$-LcLs can be constructed over points that are not too singular. Applications of Theorem A on a convergent sequence of Alexandrov spaces in Gromov-Hausdorff topology will be discussed in Section 2, where Yamaguchi’s earlier convergence theorem on Alexandrov spaces is strengthened to a Hurewicz fibration (see Theorem 2.2) and the same nilpotency results on the fundamental group of almost nonnegatively curved Alexandrov spaces as Riemannian manifolds follows from the proof by Kapovitch-Petrunin-Tueschmann in [12].

The next main result is about the stability of a converging sequence of $e^{\epsilon}$-LcLs. Recall that two Hurewicz fibrations $f_i : X_i \to B$ ($i = 0, 1$) are fibre-homotopy equivalent if there are fibre-preserving maps $h : X_0 \to X_1$ and $g : X_1 \to X_0$ and fibre-preserving homotopies between $g \circ h$ and identity $1_{X_0}$, and between $h \circ g$ and $1_{X_1}$. We say that Hurewicz fibrations $f_i : X_i \to B_i$ ($i = 0, 1$) are of the same homotopy type if there is a homeomorphism $\psi : B_0 \to B_1$ such that $\psi \circ f_0 : X_0 \to B_1$ is fiber-homotopy equivalent to $f_1 : X_1 \to B_1$. For two maps $f_i : A \to B$ ($i = 0, 1$) between metric spaces, we define the distance between $f_0$ and $f_1$ to be

$$d(f_0, f_1) = \sup \{d(f_0(x), f_1(x)) \mid x \in A\}.$$
Theorem B. Let $f_i : X \to B$ ($i = 0, 1$) be two (1.023)-LeLs from a finite-dimensional Alexandrov space with $\text{curv} \geq \kappa$ to a Riemannian manifold. Let $r$ be the injectivity radius of $B$. If $d(f_0, f_1) < \frac{r}{\delta}$, then $f_0$ and $f_1$ are fiber-homotopy equivalent.

Remark 0.4. if the two $\epsilon'$-LeLs in Theorem B are sufficiently close and the fibers are topological manifolds, then we proved in [19] that they are equivalent as fiber bundles.

Let $f_i : X_i \to B_i$ ($i = 0, 1$) be two (1.023)-LeLs, where $X_i$ is an $m$-dimensional Alexandrov space with $\text{curv} \geq \kappa$, $B_i$ is an $n$-dimensional Riemannian manifold ($i = 0, 1$), and

$$d_{GH}(X_0, B_0, f_0), (X_1, B_1, f_1)) < \epsilon$$

in the sense that there are $\epsilon$-Gromov-Hausdorff approximations $\varphi : X_0 \to X_1$ and $\psi : B_0 \to B_1$ such that $d(\psi \circ f_0, f_1 \circ \varphi) < \epsilon$. Recall that an $\epsilon$-Gromov-Hausdorff approximation is a (not necessarily continuous) map $\psi : X \to Y$ between metric spaces such that $|\psi(x_1) - \psi(x_2)| < \epsilon$ (almost preserving distance) for all $x_1, x_2 \in X$ and $|\psi| < \epsilon$ (almost dense) for all $y \in Y$. According to Perelman’s Stability Theorem for Alexandrov spaces with curvature bounded below ([14], [11], Cheeger-Gromov’s Convergence Theorem ([3], [10], [14], cf. [9], [17]), there are homeomorphic Gromov-Hausdorff approximations $\phi : X_0 \to X_1$ and $\psi : B_0 \to B_1$. By Theorem B, we conclude the following stability of $\epsilon'$-LeLs.

Theorem C. Let $X_i$ ($i = 0, 1$) be an $m$-dimensional Alexandrov space of $\text{curv} \geq \kappa$, $B_i$ ($i = 0, 1$) be an $n$-dimensional Riemannian manifold, and $f_0 : X_0 \to B_0$ be a (1.023)-LeL. Then there is $\epsilon(X_0, B_0) > 0$ such that any (1.023)-LeL $f_1 : X_1 \to B_1$ satisfying $d_{GH}((X_0, B_0, f_0), (X_1, B_1, f_1)) < \epsilon$ has the same homotopy type as $f_0$.

Remark 0.5. If, in addition, all fibers of $f_i$ ($i = 0, 1$) are closed topological $(m-n)$-manifolds, then we proved in [19] that, there is $\epsilon_1 = \epsilon_1(X_0, B_0, f_0) > 0$ such that if

$$d_{GH}((X_0, B_0, f_0), (X_1, B_1, f_1)) < \epsilon_1,$$

then $f_1$ is equivalent to $f_0$ as fiber bundles, in the sense that there are homeomorphic $\varphi(\epsilon)$-Gromov-Hausdorff approximations, $\Psi : X_0 \to X_1$, $\Phi : B_0 \to B_1$ such that $f_1 \circ \Psi = \Phi \circ f_0$. Note that a stronger definition of $d_{GH}((X_0, B_0, f_0), (X_1, B_1, f_1)) < \epsilon$ is used in [19] in the sense that there are $\epsilon$-Gromov-Hausdorff approximations $\phi : X_0 \to X_1$ and $\psi : B_0 \to B_1$ such that the two maps $\phi \circ f_0$ and $f_1 \circ \psi$ are ‘fiber-wisely’ Hausdorff close:

$$\sup_{x \in B_1} \{d_H((\phi \circ f_0)^{-1}(x), (f_1 \circ \psi)^{-1}(x))\} < \epsilon,$$
where \(d_H\) denotes the Hausdorff distance on subsets in \(X_0\). However, by (1.2) these two definitions are equivalent for LcLs.

The earlier related result in the smooth category is the fibration isomorphism/homotopy finiteness of Riemannian submersions under non-collapsing geometric bounds, which was first proved by Wu [22], provided that the fibers are totally geodesic and the based space is fixed. Later Tapp [20, 21] strengthened Wu’s result to general Riemannian submersions. The bundle stability (a little stronger than the fibration isomorphism finiteness) of \(\epsilon\)-Riemannian submersions for small \(\epsilon > 0\) was proved in [19]. According to [11, 14] by Kapovitch and Perelman, if \(f_0\) and \(f_1\) in Theorem C are submetries and close enough to each other, then they are equivalent as fiber bundles through homeomorphic Gromov-Hausdorff approximations.

Now let us briefly explain the ideas of the proofs of the main theorems. Recall that a local trivialization for an \(\epsilon\)-Riemannian submersion can be directly constructed by the horizontal liftings of radial minimal geodesics. However, there are no canonical liftings in case of an \(\epsilon^\prime\)-LcL with \(\epsilon > 0\) due to the lack of regularity. According to Ferry’s result ([6], see also Theorem 1.1), the homotopy lifting property holds for the map in Theorem A if there are controlled homotopy equivalences between nearby fibers (called strong regular, see Section 1.1) and all fibers are abstract neighborhood retracts. In [19] we found a weaker replacement of horizontal lifting, a neighborhood retraction \(\varphi_p\) to a fiber \(f^{-1}(p)\) (see also Proposition 1.6, Section 1). Similar to the horizontal lifting in the smooth case, the neighborhood retraction \(\varphi_p\) of the fiber at \(p \in B\) continuously depends on \(p\). Hence the fiber is locally contractible and controlled homotopy equivalences between nearby fibers can be defined. If all fibers are topological manifolds, then the homotopy equivalences can be approximated by homeomorphisms and thus \(f\) admits a local trivialization (see [19]). Fiber-preserving homotopies for the composition of maps in Theorem B can be defined similarly.

The remaining of the paper is organized as follows. In Section 1, we will review some topological results and give proofs of Theorem A and Theorem B. In Section 2 we will give applications of Theorem A to strengthen convergence theorems for Alexandrov spaces.

**Acknowledgements.** The author would like to thank Xiaochun Rong and Hao Fang for helpful discussions. The author would also like to thank the University of Iowa for hospitality and support during a visit in which part of the work was completed.
1. Proof of Theorem A and Theorem B

1.1. Strong regular maps and semi-concave functions. Before starting the geometric part of the proofs, we first recall some topological results. For any Hurewicz fibration \( f : X \to Y \), if \( Y \) is path-connected, then by definition the fibers are homotopy equivalent to each other. In [6] Ferry proved that the inverse is also true, if the homotopy equivalences between nearby fibers and the homotopies are under control in the following sense.

A map \( f : X \to B \) between metric spaces is said to be strongly regular [6] if \( f \) is proper and if for each \( p \in B \) and any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( d(p, p') < \delta \), then there are homotopy equivalences between fibers \( \varphi_{pp'} : f^{-1}(p) \to f^{-1}(p') \), \( \varphi_{p'p} : f^{-1}(p') \to f^{-1}(p) \) which together with the homotopies move points in distance \( < \epsilon \). A topological space \( X \) is an absolute neighborhood retract (ANR) if there is an embedding of \( X \) as a closed subspace of the Hilbert cube \( I^\infty \) such that some neighborhood \( N \) of \( X \) retracts onto \( X \). If \( X \) is finite covering dimensional and locally contractible, then \( X \) is an ANR ([1]).

**Theorem 1.1** ([6]). If \( f : E \to B \) is a strongly regular map onto a complete finite covering dimensional space \( B \) and all fibers are ANRs, then \( f \) is a Hurewicz fibration.

According to Theorem 1.1, Theorem A is reduced to show that an \( \epsilon^\ell \)-LcL from an Alexandrov space to a Riemannian manifold is strongly regular and all its fibers are locally contractible.

Now let us recall the gradient flow of a semi-concave function and the basic property of a LcL that will be frequently used throughout the paper. Let \( X \) be an Alexandrov space with \( \text{curv} \geq \kappa \). A function \( f : X \to \mathbb{R} \) is called semi-concave ([18]) if for any interior point \( p \) in \( X \) there is a neighborhood \( U \) of \( p \) and a real number \( \lambda \) such that for any minimal geodesic \( \gamma(t) \) in \( U \),

\[
f \circ \gamma(t) - \frac{\lambda}{2} t^2
\]

is concave. \( f \) is called \( \lambda \)-concave in \( U \). The gradient \( \nabla_p f \) as a vector in the tangent cone \( T_p \) is well-defined. For any point \( p \) in \( X \), there is a unique gradient curve \( \alpha : [0, \infty) \to X \) of \( f \) starting at \( p \) such that the tangent vector \( \alpha'(t) = \nabla_{\alpha(t)} f \) for any \( t \geq 0 \). The gradient flow \( \Phi^\alpha \) is well-defined and \( \epsilon^\lambda \)-Lipschitz in \( U \) ([18]). Let \( F : \Omega \to X \) be a Lipschitz map from a metric space, and let \( \tau : \Omega \to \mathbb{R}^+ \) be a Lipschitz function, then the gradient deformation of \( F \) with respect to \( f \) is defined to be

\[
\Phi^\tau_{f} \circ F(\omega) : \Omega \to X.
\]
Now let $f : X \to Y$ be an $\varepsilon^t$-LcL between metric spaces. For any compact subset $S \subset Y$, let $\dist_S$ be the distance function to $S$ in $Y$,

$$\dist_S(y) = |y, S| = \inf \{ d(y, s) : s \in S \}.$$ 

A basis property of $f$ is that the two continuous functions $\dist_S \circ f$ and $\dist_{f^{-1}(S)} : X \to \mathbb{R}_+$ satisfy (see [19])

$$e^{-\varepsilon} \cdot \dist_S \circ f \leq \dist_{f^{-1}(S)} \leq e^\varepsilon \cdot \dist_S \circ f. \tag{1.2}$$

1.2. Proof of Theorem A and Theorem B. Assume that $X$ is an Alexandrov space with $\text{curv} \geq -1$, $B$ is a Riemannian manifold and $f : X \to B$ is an $\varepsilon^t$-LcL. We will try to define controlled homotopy equivalences between nearby fibers of $f$. As a weaker replacement of the horizontal lifting of minimal geodesics, a neighborhood retraction $\varphi_p$ of $f$-fiber over $p \in B$ which is continuously depending on $p$ was constructed in [19]. Because the proof of Theorem A and Theorem B will rely on the neighborhood retraction, for reader's convenience, let us recall its construction in below.

Let $\text{injrad}_B(p)$ denote the injectivity radius of $B$ at $p$. For any point $p \in B$ and $0 < r < \text{injrad}_B(p)$, let $S_r(p) = \partial B_r(p)$ be the metric sphere around $p$ and let $x$ be any point in $B_r(p) \setminus \{p\}$. It is clear that the gradient of distance function $\dist_{S_r(p)}$ satisfies $|\nabla_x \dist_{S_r(p)}| = 1$. By (1.2), an easy estimate (see Lemma 1.5 in [19]) shows that the gradient of $\dist_{f^{-1}(S_r(p))}$ is bounded by

$$1 - (e^{2\varepsilon} - 1) \cdot \frac{2r^2}{|xf^{-1}(p)| \cdot |xf^{-1}(S_r(p))|} \leq |\nabla_x \dist_{f^{-1}(S_r(p))}| \leq 1. \tag{1.3}$$

Therefore for sufficient small $\varepsilon$ ($\varepsilon^t \leq 1.02368$), points in $f^{-1}(B_{\frac{3r}{2}}(p))$ can be flowed into $f^{-1}(B_{\frac{3r}{2}}(p))$ along gradient curves of $\dist_{f^{-1}(S_r(p))}$ in a definite time.

Lemma 1.4 (Lemma 1.5 in [19]). For any $p \in B$ and $r < \min \{ \text{injrad}_B(p), \frac{r}{2\varepsilon}\}$, there is a constant $C_0(\varepsilon) > 0$ depending on $\varepsilon$ such that for all $x \in f^{-1}(B_{\frac{3r}{2}}(p))$, the gradient curve $\Phi(t, x)$ of the function $\dist_{f^{-1}(S_r(p))}$ satisfies

$$\Phi(x, t) \in f^{-1}(B_{\frac{3r}{2}}(p)), \quad t \geq C_0^{-1} \cdot \left( \frac{2}{3} \varepsilon r - |xf^{-1}(S_r(p))| \right)$$

Thus we can define a gradient deformation of $\text{id}_{f^{-1}(B_{\frac{3r}{2}}(p))}$ which maps $f^{-1}(B_{\frac{3r}{2}}(p))$ into $f^{-1}(B_{\frac{3r}{2}}(p))$ and fixes $f^{-1}(B_{0, 3r}(p))$. Let

$$T_{p, r}(x) = \max \left\{ 0, C_0^{-1} \cdot \left( \frac{2}{3} \varepsilon r - |xf^{-1}(S_r(p))| \right) \right\},$$

and $\Phi_p^{T_{p, r}(x)}(x) = \Phi(x, T_{p, r}(x))$ be the gradient deformation of $\text{id}_{f^{-1}(B_{\frac{3r}{2}}(p))}$ with respect to $\dist_{f^{-1}(S_r(p))}$. Then by Lemma 1.4, for $\varepsilon^t \leq 1.02368$ and
is well-defined on $f$ and continuous both in $p$ and $x$, that is,
\[
\Psi : \bigcup_{p \in B} \{p\} \times f^{-1}(B_{\frac{3\epsilon}{4}}(p)) \subset B \times X \rightarrow X, \quad \Psi(p,x) = \Phi_{p}^{T_{p,r}(x)}(x)
\]
is a continuous map.

Repeating the construction above for the sequence \( \{r_i = \frac{\epsilon}{2^i}\}_{i=0,1,2,...} \) and let $\Phi_{p,i}^{T_{p,r}(x)}(x) = \Phi_{p,i}(x,T_{p,r}(x))$ be the gradient curves of $\text{dist}_{f^{-1}(S_r(p))}$ at $x$ with time $T_{p,r}(x)$. By (1.5), $\Phi_{p,i}^{T_{p,r}(x)} : f^{-1}(B_r(p)) \rightarrow X$ takes $f^{-1}(B_{\frac{3\epsilon}{4}}(p))$ into $f^{-1}(B_{\frac{3\epsilon}{2^{i+1}}} (p))$, and
\[
\Phi_{p,i}^{T_{p,r}(x)}|_{f^{-1}(B_{\frac{3\epsilon}{2^{i+1}}}(p))} = \text{id}.
\]
Hence the iterated gradient deformations
\[
\Phi_{p,i}^{T_{p,r}(x)} \circ \Phi_{p,i-1}^{T_{p,r}(x)} \circ \cdots \circ \Phi_{p,0}^{T_{p,r}(x)}
\]
is well-defined on $f^{-1}(B_{\frac{3\epsilon}{4}}(p))$ and its restriction on $f^{-1}(B_{\frac{3\epsilon}{2^{i+1}}}(p))$ is identity. Because
\[
T_{p,r_i}(x) \leq \frac{r}{2^{i+1}} \cdot \frac{6^i}{3} \cdot C_0^{-1},
\]
it can be directly verified that the sequence of maps
\[
\Psi_i : \bigcup_{p \in B} \{p\} \times f^{-1}(B_{\frac{3\epsilon}{4}}(p)) \rightarrow X,
\]
\[
\Psi_i(p,x) = \Phi_{p,i}^{T_{p,r}(x)} \circ \Phi_{p,i-1}^{T_{p,r}(x)} \circ \cdots \circ \Phi_{p,0}^{T_{p,r}(x)}(x)
\]
uniformly converges. The limit $\varphi_p(x) = \lim_{i \rightarrow \infty} \Psi_i(p,x)$ gives a retraction from the neighborhood $f^{-1}(B_{\frac{3\epsilon}{4}}(p))$ to $f^{-1}(p)$, which is continuous both in $p$ and $x$.

**Proposition 1.6** (Proposition 1.6 in [19]). For any $0 < r < \text{injrad}(B)$, there is a map $\varphi_p(x)$ from a neighborhood $f^{-1}(B_{\frac{3\epsilon}{4}}(p))$ to the fiber $f^{-1}(p)$ such that
\[
\varphi_p(x) : \bigcup_{p \in B} \{p\} \times f^{-1}(B_{\frac{3\epsilon}{4}}(p)) \rightarrow X,
\]
is continuous both in $p$ and $x$, and satisfies
\[
(1.6.1) \quad \varphi_p(x) = x \text{ for any } x \in f^{-1}(p), \text{ and}
\]
\[
(1.6.2) \quad |x\varphi_p(x)| \leq 2C_1 r, \text{ for some constant } C_1(\epsilon) \text{ depending only on } \epsilon.
\]
Remark 1.7. Since the Lipschitz constant of each $\Phi_t$ goes to infinity with the same order as $r_i^{-1}$, there is no Lipschitz control on the limit map $\varphi_p$. Therefore no local control on the intrinsic distance in the fiber follows from Proposition 1.6.

We are now ready to prove Theorem A and Theorem B.

Proof of Theorem A. Up to a rescaling we assume that the lower curvature bound of $X$ is $-1$. By Theorem 1.1, it suffices to show that $f$ is strong regular and any fiber is an ANRs. For any $p,q \in B$ with small distance $0 < |pq| < \frac{1}{2} \min\{\text{injrad}_p(B), \frac{1}{2} r_i\}$, let $\rho = 2|pq|$. By Proposition 1.6, there are neighborhood retractions $\varphi_p : f^{-1}(B_{2\rho}(p)) \to f^{-1}(p)$ and $\varphi_q : f^{-1}(B_{2\rho}(q)) \to f^{-1}(q)$ around $f^{-1}(p)$ and $f^{-1}(q)$ respectively. Then the homotopy equivalences between fibers can be chosen to be $\varphi_{p,f^{-1}(q)} : f^{-1}(q) \to f^{-1}(p)$ and $\varphi_{q,f^{-1}(p)} : f^{-1}(p) \to f^{-1}(q)$, and the homotopies are $H_t = \varphi_{p,f^{-1}(q)} \circ \varphi_{q,f^{-1}(p)} : f^{-1}(q) \to f^{-1}(p)$ and $K_t = \varphi_{q,f^{-1}(p)} \circ \varphi_{p,f^{-1}(q)} : f^{-1}(p) \to f^{-1}(q)$, where $\gamma : [0,1] \to B$ is a minimal geodesic from $p$ to $q$. By (1.6.2), $|H_t(x)| \leq 4C_1\rho$ and $|K_t(x)| \leq 4C_1\rho$. Therefore $f$ is strongly regular.

From [14], an Alexandrov space with curvature bounded below is locally contractible. For $x \in f^{-1}(p)$, let $U_x \ni x$ be a contractible neighborhood around $x$ and $H_t : U_x \to U_x$ be the homotopy from $\text{id}_{U_x}$ to the retraction $r : U_x \to \{x\}$ such that $H_t(x) = x$. Then $\varphi_p \circ H_t$ is a homotopy from $\text{id}_{U_x \cap f^{-1}(p)}$ to the retraction $r : U_x \cap f^{-1}(p) \to \{x\}$. Therefore $f^{-1}(p)$ is locally contractible and thus an absolute neighborhood retract. □

Proof of Theorem B. Let $f_0, f_1 : X \to B$ be the (1.023)-LeLs from Alexandrov space $X$ to Riemannian manifold $B$ in Theorem D. We now construct fiber-preserving maps $h,g : X \to X$ and fiber-preserving homotopies $g \circ h$ to the identity $1_A$ and from $h \circ g$ to $1_A$ as follows.

For any point $x \in X$, let $p = f_0(x) \in B$, let $F_0(p)$ be the fiber $f_0^{-1}(p)$ and $F_1(p) = f_1^{-1}(p)$. Because $d(f_0, f_1) < \frac{\epsilon}{2}$ by (1.2), $F_0(p)$ lies in the 1.023$t_B$-tubular neighborhood of $F_1(p)$. Let $\varphi_p$ be the neighborhood retraction of $F_1(p)$ in Proposition 1.6 with respect to $f_1$, we define $h : X \to X$ by $h(x) = \varphi_{f_0(x)}(x)$. Then the continuous map $h : X \to X$ is globally defined and maps all fibers of $f_0$ into that of $f_1$. Similarly we define $g : X \to X$ through the neighborhood retraction of $f_0$-fibers such that $f_0 \circ g = f_1$, where $g(x) = \psi_{f_1(x)}(x)$ and $\psi_q$ is the neighborhood retraction of $f_0^{-1}(f_1(x))$ with respect to $f_0$.

Let $\gamma : [0,1] \to B$ be the minimal geodesic from $\gamma(0) = f_0(x)$ to $\gamma(1) = f_1(x)$, we define the fiber-preserving homotopy $H_t : X \to X$ by $H_t(x) = \psi_{f_0(x)} \circ \varphi_{\gamma(t)}(x)$. Then $H : [0,1] \times X \to X$ is a continuous map such that
$H_0 = g \circ h$ and $H_1 = 1_X$. A fiber-preserving homotopy from $h \circ g$ to the identity $1_A$ can be defined similarly. □

2. Convergence theorems of Alexandrov spaces

In this section we will apply Theorem A to strengthen Yamaguchi’s Lipschitz submersion theorem on Alexandrov spaces. A convergent sequence of Riemannian manifolds or Alexandrov spaces $X_i$ in Gromov-Hausdorff topology is called collapsing if the dimension of limit space is strictly less than that of $X_i$. For $\delta > 0$, the $\delta$-strain radius [24] at a point $p$ in an $n$-dimensional Alexandrov space $X$ is defined to be

$$\sup\{r \mid \text{there exists an } (n, \delta)\text{-strainer at } p \text{ of length } r\}.$$  

A Lipschitz submersion theorem was proved by Yamaguchi [24] in the case that all points in the limit space is $(n, \delta)$-strained.

**Theorem 2.1** (Lipschitz submersion Theorem [24]). For any dimension $n$ and positive number $\mu_0$, there exist positive numbers $\delta(n)$ and $\epsilon(n, \mu_0)$ such that for any $m$-dimensional Alexandrov space $Y$ with $\text{curv} \geq -1$ and any $n$-dimensional Alexandrov space $X$ with $\text{curv} \geq -1$, if

1. the $\delta$-strain radius at any point in $X \geq \mu_0$, and
2. the Gromov-Hausdorff distance between $X$ and $Y \leq \epsilon$, then there exists a $(1 + \tau(\delta, \epsilon))$-Lc$L$ $f : Y \to X$ which is a $\tau(\delta, \epsilon)$-almost Lipschitz submersion.

Here, $\tau(\delta, \epsilon)$ denotes a positive constant depending on $n$, $\delta$ and $\epsilon$ satisfying $\tau(\delta, \epsilon) \to 0$ as $\delta, \epsilon \to 0$.

Yamaguchi conjectured in [24] that the map $f : Y \to X$ in Theorem 2.1 is a locally trivial fibration. By Theorem A, we are able to conclude that it is a Hurewicz fibration, and the conjecture is true in the case that the fibers are topological manifolds.

**Theorem 2.2.** For sufficient small $\delta$, the almost Lipschitz submersion $f : Y \to X$ in Theorem 2.1 is a Hurewicz fibration; and $f$ is a locally trivial fibration in the case that every $f$-fiber is a topological manifold without boundary of co-dimension $n$.

There are similar fibration theorems for Riemannian manifolds which are fundamental in many applications. It was first proved by Fukaya in [7] and strengthened later by Cheeger-Gromov-Fukaya in [5] that if a sequence of Riemannian manifold $M_i$ is collapsing under a two-sided sectional curvature bound, then the Gromov-Hausdorff approximation can be approximated by a global singular fibration from $M_i$ to the limit space, whose fibers are almost flat manifolds. Yamaguchi proved in [23] that if a Riemannian manifold $M$
with sectional curvature bounded below is sufficient close to a Riemannian manifold \( N \) in Gromov-Hausdorff distance, then there is a \( C^1 \epsilon \)-Riemannian submersion \( f : M \to N \), whose fiber is connected and has almost nonnegative curvature in a generalized sense defined in Lemma 5.3 of [23] (cf. Definition 1.0.1 in [12]). By Kapovitch-Petrunin-Tueschmann’s work [12] the fundamental group of the fiber admits a nilpotent subgroup of uniformly bounded index.

Due to Theorem 2.2 one is able to talk about the homotopy fiber. By similar arguments as the smooth case, the same nilpotency on the fundamental group of the homotopy fiber of \( f \) in Theorem 2.1 follows from Theorem A and the proof in [12] by Kapovitch-Petrunin-Tueschmann.

**Corollary 2.3.** There is a universal constant depending on \((m - n)\) such that the fundamental group of the homotopy \( F \) in Theorem 2.1 contains a nilpotent subgroup whose index \( \leq C \).

Almost nonnegatively curved Alexandrov spaces are those space whose diameter is uniformly bounded and the lower curvature bound \( \kappa \) is almost nonnegative. Or equivalently, the scaling invariant \( \text{diam}^2 \cdot \kappa \) is almost nonnegative. After scaling to a uniform lower curvature bound \(-1\), they are close to a point in Gromov-Hausdorff topology. As a special case, the almost nilpotency for almost nonnegatively curved Alexandrov spaces follows.

**Theorem 2.4.** For any positive integer \( n \), there are constants \( \epsilon(n) > 0 \) and \( C(n) > 0 \) such that for any closed \( n \)-dimensional Alexandrov space \( M \) with \( \text{curv} \geq \kappa \), if \( \text{diam}(M)^2 \cdot \kappa > -\epsilon(n) \), then \( \pi_1(M) \) contains a nilpotent subgroup of index \( \leq C(n) \).

**Remark 2.5.** The earlier version of Theorem 2.4 for Alexandrov spaces was proved by Yamaguchi in [24] without a uniform bound on the index of the nilpotent subgroup, where the proof was based the Lipschitz submersion theorem (Theorem 2.1) and arguments in [8].

Theorem 2.4 extends the same property of almost nonnegatively curved manifolds. It was first proved by Fukaya and Yamaguchi [8] that the fundamental group of any manifold of almost nonnegative sectional curvature contains a nilpotent subgroup of finite index, and later Kapovitch, Petrunin and Tueschmann [12] proved that the index of a nilpotent subgroup is uniformly bounded (depending only on the manifold’s dimension). The same conclusion for manifolds of almost nonnegative Ricci curvature is a conjecture of Gromov, and it was proved recently by Kapovitch and Wilking in [13] (cf. [4]).

Recall that Yamaguchi’s fibration theorem [23] for Riemannian manifolds provided a fundamental tool in constructing a finite descending normal tower
of the fundamental group in earlier proofs both in [8] and [12]. By the
fibration in Theorem 2.2, we still are able to conclude the same structure for
a collapsing sequence of Alexandrov spaces. Moreover, the estimates in [12]
were proved for general Alexandrov spaces. Because the proof of Theorem
2.4 and Corollary 2.3 would be the same as that in [12], we omit it here.

Another direct corollary of Theorem 2.2 is a long exact sequence arising
from the fibration.

**Corollary 2.6.** Assume a sequence of Alexandrov spaces \( X_i \) with \( \text{curv} \geq \kappa \) collapsing to an \( n \)-dimensional Alexandrov \( X \) and the limit space \( X \) has only weak singularities. Then for all large \( i \), the almost Lipschitz submersion \( f_i : X_i \to X \) induces an isomorphism \( f_i^* : \pi_l(X_i, F_i, x_i) \to \pi_l(X, x) \) for all \( l \geq 1 \), where \( x \in X \) and \( x_i \in F_i = f_i^{-1}(x) \). Hence there is a long exact sequence

\[ \cdots \to \pi_l(F_i, x_i) \to \pi_l(X_i, x_i) \xrightarrow{f_i^*} \pi_l(X, x) \to \pi_{l-1}(F_i, x_i) \to \cdots \to \pi_1(X, x) \to 0. \]

**Remark 2.7.** In [16] Perelman concluded the same long exact sequence under a weaker condition that the limit \( X \) has no proper extremal subsets, where \( F_i \) is a regular fiber (i.e., the fiber of a lifting map to \( X_i \) of regular admissible maps locally defined on the limit \( X \) to \( \mathbb{R}^n \) [16]). It is easy to see that the regular fiber is homotopy equivalent to \( f_i^{-1}(x) \) in Corollary 2.6.

**References**

[1] K. Borsuk. On some metrizations of the hyperspace of compact sets. *Fund. Math.*, 41:168–201, 1955.
[2] Y. Burago, M. Gromov, and G. Perelman. A.d. alexandrov spaces with curvature bounded below. *Uspekhi Mat. Nauk*, 47(2(284)):3–51, 1992.
[3] J. Cheeger. Finiteness theorems for riemannian manifolds. *Amer. J. Math.*, 92:61–75, 1970.
[4] J. Cheeger and T. H. Colding. Lower bounds on ricci curvature and the almost rigidity of warped products. *Ann. of Math.*, 144(1):189–237, 1996.
[5] J. Cheeger, K. Fukaya, and M. Gromov. Nilpotent structures and invariant metrics on collapsed manifolds. *J. A. M. S.*, 5:327–372, 1992.
[6] S. Ferry. Strongly regular mappings with compact anr fibers are hurewicz fiberings. *Pacific J. Math.*, 75(2):373–382, 1978.
[7] K. Fukaya. Collapsing of riemannian manifolds to ones of lower dimensions. *J. Diff. Geom.*, 25:139–156, 1987.
[8] K. Fukaya and T. Yamaguchi. The fundamental groups of almost non-negatively curved manifolds. *Ann. of Math.*, 136:253–333, 1992.
[9] R. E. Green and H. Wu. Lipschitz convergence of riemannian manifolds. *Pacific J. Math.*, 131:119–141, 1988.
[10] M. Gromov, J. Lafontaine, and P. Pansu. *Structures Métriques pour les Variétés Riemanniennes*. CedicFernand, Paris, 1981.
[11] V. Kapovitch. Perelman’s stability theorem. In *Surveys of Differential Geometry, Metric and Comparison Geometry*, XI, pages 103–136. Int. Press, Somerville, 2007.
[12] V. Kapovitch, A. Petrunin, and W. Tuschmann. Nilpotency, almost nonnegative curvature, and the gradient flow on alexandrov spaces. *Ann. of Math.*, 171:343–373, 2010.

[13] V. Kapovitch and B. Wilking. Structure of fundamental groups of manifolds with ricci curvature bounded below. *Preprint, arXiv:1105.5955*, 2011.

[14] G. Perelman. Alexandrov spaces with curvatures bounded from below ii. *Preprint*, 1991.

[15] G. Perelman. Elements of morse theory on aleksandrov spaces. *St. Petersburg Math. J.*, 5:205–213, 1993.

[16] G. Perelman. Collapsing with no proper extremal subsets. In K. Grove and P. Petersen, editors, *Comparison Geometry*, volume 30 of *MSRI Books*, pages 149–155. Cambridge Univ. Press, Cambridge, 1997.

[17] S. Peters. Cheeger’s finiteness theorem for diffeomorphism classes of riemannian manifolds. *J. Reine Angew. Math.*, 349:77–82, 1984.

[18] A. Petrunin. Semiconcave functions in alexandrov’s geometry. In J. Cheeger and K. Grove, editors, *Metric and Comparison Geometry*, volume XI of *Surveys in Differential Geometry*, pages 137–202. Int. Press, Somerville, 2007.

[19] X. Rong and S. Xu. Stability of $c^\epsilon$-lipschitz and co-lipschitz maps in gromov-hausdorff topology. *Advances in Mathematics*, 231:774–797, 1 October 2012.

[20] K. Tapp. Bounded riemannian submersions. *Indiana Univ. Math. J.*, 49(2):637–654, 2000.

[21] K. Tapp. Finiteness theorems for submersions and souls. *Proc. Amer. Math. Soc.*, 130(6):1809–1817, 2002.

[22] J. Y. Wu. A parametrized geometric finiteness theorem. *Indiana Univ. Math. J.*, 45(2):511–528, 1996.

[23] T. Yamaguchi. Collapsing and pinching under a lower curvature bound. *Ann. of Math.*, 133:317–357, 1991.

[24] T. Yamaguchi. A convergence theorem in the geometry of alexandrov spaces, 1996.

E-mail address: shichengxu@gmail.com

Department of Mathematics, Nanjing University, Nanjing, China