THE RICCATI DIFFERENTIAL EQUATION
AND A DIFFUSION-TYPE EQUATION

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Abstract. We construct an explicit solution of the Cauchy initial value problem for certain diffusion-type equations with variable coefficients on the entire real line. The corresponding Green function (heat kernel) is given in terms of elementary functions and certain integrals involving a characteristic function, which should be found as an analytic or numerical solution of the second order linear differential equation with time-dependent coefficients. Some special and limiting cases are outlined. Solution of the corresponding non-homogeneous equation is also found.

1. Introduction

In this paper we discuss explicit solution of the Cauchy initial value problem for the one-dimensional heat equation on the entire real line

$$\frac{\partial u}{\partial t} = Q\left(\frac{\partial}{\partial x}, x, t\right) u,$$

(1.1)

where the right hand side is a quadratic form $Q(p, x)$ of the coordinate $x$ and the operator of differentiation $p = \partial/\partial x$ with time-dependent coefficients; see equation (2.1) below. The case of a corresponding Schrödinger equation is investigated in [6]. In this approach, several exactly solvable models are classified in terms of elementary solutions of a characterization equation given by (2.13) below. Solution of the corresponding non-homogeneous equation is obtained with the help of the Duhamel principle. These exactly solvable cases may be of interest in a general treatment of the nonlinear evolution equations; see [3], [1], [5], [26] and references therein. Moreover, these explicit solutions can also be useful when testing numerical methods of solving the semilinear heat equations with variable coefficients.

2. Solution of a Cauchy Initial Value Problem: Summary of Results

The fundamental solution of the diffusion-type equation of the form

$$\frac{\partial u}{\partial t} = a(t) \frac{\partial^2 u}{\partial x^2} - b(t) x^2 u + c(t) x \frac{\partial u}{\partial x} + d(t) u + f(t) xu - g(t) \frac{\partial u}{\partial x},$$

(2.1)

where $a(t), b(t), c(t), d(t), f(t)$, and $g(t)$ are given real-valued functions of time $t$ only, can be found by a familiar substitution

$$u = Ae^S = A(t)e^{S(x,y,t)}$$

(2.2)
with
\[ A = A(t) = \frac{1}{\sqrt{2\pi \mu(t)}} \] (2.3)
and
\[ S = S(x, y, t) = \alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t), \] (2.4)
where \( \alpha(t), \beta(t), \gamma(t), \delta(t), \varepsilon(t), \) and \( \kappa(t) \) are differentiable real-valued functions of time \( t \) only.

Indeed, \[ \frac{\partial S}{\partial t} = a \left( \frac{\partial S}{\partial x} \right)^2 - bx^2 + fx + (cx - g) \frac{\partial S}{\partial x} \] (2.5)
provided \[ \frac{\mu'}{2\mu} = -a \frac{\partial^2 S}{\partial x^2} - d = -2\alpha(t)a(t) - d(t). \] (2.6)

Equating the coefficients of all admissible powers of \( x^m y^n \) with \( 0 \leq m + n \leq 2 \), gives the following system of ordinary differential equations
\[ \frac{d\alpha}{dt} + b(t) - 2c(t)\alpha - 4a(t)\alpha^2 = 0, \] (2.7)
\[ \frac{d\beta}{dt} - (c(t) + 4a(t)\alpha(t))\beta = 0, \] (2.8)
\[ \frac{d\gamma}{dt} - a(t)\beta^2(t) = 0, \] (2.9)
\[ \frac{d\delta}{dt} - (c(t) + 4a(t)\alpha(t))\delta = f(t) - 2\alpha(t)g(t), \] (2.10)
\[ \frac{d\varepsilon}{dt} + (g(t) - 2a(t)\delta(t))\beta(t) = 0, \] (2.11)
\[ \frac{d\kappa}{dt} + g(t)\delta(t) - a(t)\delta^2(t) = 0, \] (2.12)
where the first equation is the familiar Riccati nonlinear differential equation; see, for example, [12], [18], [22], [23], [27] and references therein.

We have
\[ 4a\alpha' + 4ab - 2c(4a\alpha) - (4a\alpha)^2 = 0, \quad 4a\alpha = -2d - \frac{\mu'}{\mu} \]
from (2.7) and (2.6) and the substitution
\[ 4a\alpha' = -2d' - \frac{\mu''}{\mu} + \left( \frac{\mu'}{\mu} \right)^2 + \frac{a'}{a} \left( 2d + \frac{\mu'}{\mu} \right) \]
results in the second order linear equation
\[ \mu'' - \tau(t)\mu' - 4\sigma(t)\mu = 0 \] (2.13)
with
\[ \tau(t) = \frac{a'}{a} + 2c - 4d, \quad \sigma(t) = ab + cd - d^2 + \frac{d}{2} \left( \frac{a'}{a} - \frac{d'}{d} \right). \] (2.14)

As we shall see later, equation (2.13) must be solved subject to the initial data
\[ \mu(0) = 0, \quad \mu'(0) = 2a(0) \neq 0 \] (2.15)
in order to satisfy the initial condition for the corresponding Green function; see the asymptotic formula \((2.21)\) below for a motivation. Then, the Riccati equation \((2.7)\) can be solved by the back substitution \((2.6)\).

We shall refer to equation \((2.13)\) as the characteristic equation and its solution \(\mu(t)\), subject to \((2.15)\), as the characteristic function. As the special case \((2.13)\) contains the generalized equation of hypergeometric type, whose solutions are studied in detail in [20]; see also [1], [19], [25], and [27].

Thus, the Green function (fundamental solution or heat kernel) is explicitly given in terms of the characteristic function

\[
 u = K(x, y, t) = \frac{1}{\sqrt{2\pi\mu(t)}} e^{\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t)}. \tag{2.16}
\]

Here

\[
\alpha(t) = -\frac{1}{4a(t)} \frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)}, \tag{2.17}
\]

\[
\beta(t) = \frac{1}{\mu(t)} \exp\left(\int_0^t (c(\tau) - 2d(\tau)) \, d\tau\right), \tag{2.18}
\]

\[
\gamma(t) = -\frac{a(t)}{\mu(t)\mu'(t)} \exp\left(2\int_0^t (c(\tau) - 2d(\tau)) \, d\tau\right) - 4\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2} \left(\exp\left(2\int_0^\tau (c(\lambda) - 2d(\lambda)) \, d\lambda\right)\right) \, d\tau, \tag{2.19}
\]

\[
\delta(t) = \frac{1}{\mu(t)} \exp\left(\int_0^t (c(\tau) - 2d(\tau)) \, d\tau\right) \int_0^t \exp\left(-\int_0^\tau (c(\lambda) - 2d(\lambda)) \, d\lambda\right) \times \left(\left(f(\tau) + \frac{d(\tau)}{a(\tau)}g(\tau)\right)\mu(\tau) + \frac{g(\tau)}{2a(\tau)\mu'(\tau)}\right) \, d\tau, \tag{2.20}
\]

\[
\varepsilon(t) = -\frac{2a(t)}{\mu'(t)} \delta(t) \exp\left(\int_0^t (c(\tau) - 2d(\tau)) \, d\tau\right) - 8\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2} \exp\left(\int_0^\tau (c(\lambda) - 2d(\lambda)) \, d\lambda\right) \mu(\tau) \delta(\tau) \, d\tau + 2\int_0^t \frac{a(\tau)}{\mu'(\tau)} \exp\left(\int_0^\tau (c(\lambda) - 2d(\lambda)) \, d\lambda\right) \left(f(\tau) + \frac{d(\tau)}{a(\tau)}g(\tau)\right) \, d\tau, \tag{2.21}
\]

\[
\kappa(t) = -\frac{a(t)}{\mu'(t)} \delta^2(t) - 4\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2} (\mu(\tau) \delta(\tau))^2 \, d\tau + 2\int_0^t \frac{a(\tau)}{\mu'(\tau)} (\mu(\tau) \delta(\tau)) \left(f(\tau) + \frac{d(\tau)}{a(\tau)}g(\tau)\right) \, d\tau \tag{2.22}
\]

with

\[
\delta(0) = \frac{g(0)}{2a(0)}, \quad \varepsilon(0) = -\delta(0), \quad \kappa(0) = 0. \tag{2.23}
\]
We have used integration by parts in order to resolve the singularities of the initial data; see section 3 for more details. Then the corresponding asymptotic formula is

\[ K(x, y, t) = \frac{e^{S(x,y,t)}}{\sqrt{2\pi \mu(t)}} \sim \frac{1}{\sqrt{4\pi a(0) t}} \exp\left(-\frac{(x-y)^2}{4a(0)t}\right) \exp\left(\frac{g(0)}{2a(0)} (x-y)\right) \]

as \( t \to 0^+ \). Notice that the first term on the right hand side is a familiar heat kernel for the diffusion equation with constant coefficients (cf. Eq. (5.2) below).

By the superposition principle, we obtain solution of the Cauchy initial value problem

\[ \frac{\partial u}{\partial t} = Qu, \quad u(x, t)|_{t=0} = u_0(x) \]

on the infinite interval \(-\infty < x < \infty\) with the general quadratic form \( Q(p, x) \) in (2.1) as follows

\[ u(x, t) = \int_{-\infty}^{\infty} K(x, y, t) \, u_0(y) \, dy = H(u_0, 0). \]

This yields solution explicitly in terms of an integral operator \( H \) acting on the initial data provided that the integral converges and one can interchange differentiation and integration. This integral is essentially the Laplace transform.

In a more general setting, solution of the initial value problem at time \( t_0 \)

\[ \frac{\partial u}{\partial t} = Qu, \quad u(x, t)|_{t=t_0} = u(x, t_0) \]

on an infinite interval has the form

\[ u(x, t) = \int_{-\infty}^{\infty} K(x, y, t, t_0) \, u_0(y, t_0) \, dy = H(t, t_0) \, u(x, t_0), \]

with the heat kernel given by

\[ K(x, y, t, t_0) = \frac{1}{\sqrt{2\pi \mu(t, t_0)}} e^{\alpha(t,t_0)x^2+\beta(t,t_0)xy+\gamma(t,t_0)y^2+\delta(t,t_0)x+\varepsilon(t,t_0)y+\kappa(t,t_0)}. \]

The function \( \mu(t) = \mu(t, t_0) \) is a solution of the characteristic equation (2.13) corresponding to the initial data

\[ \mu(t_0, t_0) = 0, \quad \mu'(t_0, t_0) = 2a(t_0) \neq 0. \]

If \( \{\mu_1, \mu_2\} \) is a fundamental solution set of equation (2.13), then

\[ \mu(t, t_0) = \frac{2a(t_0)}{W(\mu_1, \mu_2)} (\mu_1(t_0) \mu_2(t) - \mu_1(t) \mu_2(t_0)) \]

and

\[ \mu'(t, t_0) = \frac{2a(t_0)}{W(\mu_1, \mu_2)} (\mu_1(t_0) \mu_2'(t) - \mu_1'(t) \mu_2(t)) , \]

where \( W(\mu_1, \mu_2) \) is the value of the Wronskian at the point \( t_0 \).

Equations (2.17)–(2.22) are valid again but with the new characteristic function \( \mu(t, t_0) \). The lower limits of integration should be replaced by \( t_0 \). Conditions (2.23) become

\[ \delta(t_0, t_0) = -\varepsilon(t_0, t_0) = \frac{g(t_0)}{2a(t_0)}, \quad \kappa(t_0, t_0) = 0 \]
and the asymptotic formula (2.24) should be modified as follows

$$K(x, y, t, t_0) = \frac{e^{S(x, y, t, t_0)}}{\sqrt{2\pi\mu(t, t_0)}}$$

$$\sim \frac{1}{\sqrt{4\pi a(t_0)(t - t_0)}} \exp \left( -\frac{(x - y)^2}{4a(t_0)(t - t_0)} \right) \exp \left( \frac{g(t_0)}{2a(t_0)}(x - y) \right).$$

We leave the details to the reader.

3. Derivation of The Heat Kernel

Here we obtain the above formulas (2.17)–(2.22) for the heat kernel. The first equation is a direct consequence of (2.6) and our equation (2.8) takes the form

$$(\mu\beta)' = (c - 2d)(\mu\beta),$$

whose particular solution is (2.18).

From (2.9) and (2.18) one gets

$$\gamma(t) = \int a(t) \mu(t) e^{2h(t)} dt, \quad h(t) = \int_0^t (c(\tau) - 2d(\tau)) d\tau$$

and integrating by parts

$$\gamma(t) = -\int \frac{ae^{2h}}{\mu'} d\left( \frac{1}{\mu} \right) = -\frac{ae^{2h}}{\mu} + \int \left( \frac{ae^{2h}}{\mu'} \right)' \frac{dt}{\mu}. \quad (3.3)$$

But the derivative of the auxiliary function

$$F(t) = \frac{a(t)}{\mu'(t)} e^{2h(t)}$$

is

$$F'(t) = \frac{(a' + 2h'a)}{\mu'} e^{2h} \mu' - \frac{ae^{2h}}{\mu} \mu'' = -\frac{4\sigma a\mu}{\mu'} e^{2h} = -\frac{4\sigma\mu}{\mu'} F. \quad (3.5)$$

in view of the characteristic equation (2.13)–(2.14). Substitution into (3.3) results in (2.19).

Equation (2.10) can be rewritten as

$$(\mu e^{-h}\delta)' = \mu e^{-h}(f - 2\alpha g), \quad h = \int_0^t (c - 2d) d\tau \quad (3.6)$$

and its direct integration gives (2.20).

We introduce another auxiliary function

$$G(t) = \mu(t) \delta(t) e^{-h(t)}$$

with the derivative given by (3.6). Then equation (2.11) becomes

$$\frac{d\varepsilon}{dt} = -\frac{g}{\mu} e^h + \frac{2\alpha\delta}{\mu} e^h$$

and

$$\varepsilon(t) = -\int \frac{g}{\mu} e^h dt + 2\int \frac{aG}{\mu^2} e^{2h} dt. \quad (3.8)$$
Integrating the second term by parts one gets
\[
\int \frac{aG}{\mu^2} e^{2h} dt = - \int \frac{aG}{\mu'} e^{2h} \left(\frac{1}{\mu}\right) = - \int FG \left(\frac{1}{\mu}\right) = - \frac{FG}{\mu} + \int \frac{(FG)'}{\mu} dt,
\] (3.9)

where
\[
(FG)' = F'G + FG' = -\frac{4a\sigma\mu}{(\mu')^2} (\mu\delta) e^h + \frac{a\mu}{\mu'} e^h f + \frac{d\mu}{\mu'} e^h g + \frac{1}{2}ge^h
\] (3.10)
in view of (3.5) and (3.6). Then substitution (3.10) into (3.9) allows to cancel the divergent integrals. As a result one can resolve the singularity and simplify expression (3.8) to its final form (2.21).

Finally, by (2.12) and (3.7)
\[
\kappa(t) = -\int g\delta dt + \int \frac{aG^2}{\mu^2} e^{2h} dt,
\] (3.11)
where the last integral can be transformed as follows
\[
\int \frac{aG^2}{\mu^2} e^{2h} dt = - \int FG^2 \left(\frac{1}{\mu}\right) = - \frac{FG^2}{\mu} + \int \frac{(FG^2)'}{\mu} dt
\] (3.12)
with
\[
(FG^2)' = F'G^2 + 2FGG' = (FG)' G + FG' = -\frac{4a\sigma\mu}{(\mu')^2} (\mu\delta)^2 + \frac{2a\mu}{\mu'} (\mu\delta) f + \frac{2d\mu}{\mu'} (\mu\delta) g + \mu g\delta.
\] (3.13)

Substitution (3.12)–(3.13) into (3.11) gives our final expression (2.22).

The details of derivation of the asymptotic formula (2.24) are left to the reader.

4. Special Initial Data

In the case \(u(x,0) = u_0 = \text{constant, our solution (2.26)}\) takes the form
\[
u(x, t) = \int_{-\infty}^{\infty} K(x, y, t) u_0 dy
\] (4.1)
\[
= u_0 e^{\alpha(t)x^2 + \delta(t)x + \kappa(t)} \sqrt{2\pi \mu(t)} \int_{-\infty}^{\infty} e^{(\beta(t)x + \varepsilon(t)y + \gamma(t)y^2) y^2} dy
= u_0 \sqrt{-2\mu\gamma} \exp\left(\frac{4\alpha\gamma - \beta^2}{4\gamma} x^2 + 2(2\gamma\delta - \beta\varepsilon) x + 4\gamma\kappa - \varepsilon^2\right),
\]
provided \(\gamma(t) < 0\), with the help of an elementary integral
\[
\int_{-\infty}^{\infty} e^{-ay^2 + 2by} dy = \sqrt{\frac{\pi}{a}} e^{b^2/a}, \quad a > 0.
\] (4.2)
The details of taking the limit \(t \to 0^+\) in (4.1) are left to the reader.
When \( u(x,0) = \delta(x-x_0) \), where \( \delta(x) \) is the Dirac delta function, one gets formally

\[
\frac{\partial u}{\partial t}(x,t) = \int_{-\infty}^{\infty} K(x,y,t) \delta(y-x_0) \, dy = K(x,x_0,t).
\]

Thus, in general, the heat kernel (2.16) provides an evolution of this initial data, concentrated originally at a point \( x_0 \), into the entire space for a suitable time interval \( t > 0 \).

5. Some Examples

Now let us consider several elementary solutions of the characteristic equation (2.13); more complicated cases may include special functions, like Bessel, hypergeometric or elliptic functions [1], [20], [21], and [27]. Among important elementary cases of our general expressions for the Green function (2.16)–(2.22) are the following:

For the traditional diffusion equation

\[
\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2}, \quad a = \text{constant} > 0
\]

the heat kernel is

\[
K(x,y,t) = \frac{1}{\sqrt{4\pi at}} \exp\left(-\frac{(x-y)^2}{4at}\right), \quad t > 0.
\]

Equation (4.1) gives the steady solution \( u_0 = \text{constant} \) for all times \( t \geq 0 \). See [3] and references therein for a detailed investigation of the classical one-dimensional heat equation.

The diffusion-type equation

\[
\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + fxu,
\]

where \( a > 0 \) and \( f \) are constants (see [7], [8], [9], [10], [11], [6] and references therein regarding to similar cases of the Schrödinger equation), has the characteristic function of the form \( \mu = 2at \). The heat kernel is

\[
K(x,y,t) = \frac{1}{\sqrt{4\pi at}} \exp\left(-\frac{(x-y)^2}{4at}\right) \exp\left(\frac{f}{2} (x+y) t + \frac{afa^2}{12t^3}\right)
\]

provided \( t > 0 \). Evolution of the uniform initial data \( u(x,0) = u_0 = \text{constant} \) is given by

\[
u(x,t) = u_0 e^{fxt + af^2 t^3/3}.
\]

The initial value problem for the following diffusion-type equation with variable coefficients

\[
\frac{\partial u}{\partial t} = a \left( \frac{\partial^2 u}{\partial x^2} - xu \right) + \omega \left( \cosh ((2a-1)t) xu + \sinh ((2a-1)t) \frac{\partial u}{\partial x} \right),
\]

where \( a > 0 \) and \( \omega \) are two constants, was solved in [15] by using the eigenfunction expansion method and a connection with the representations of the Heisenberg–Weyl group \( N(3) \). Here we apply a different approach. The solution of the characteristic equation

\[
\mu'' - 4a^2 \mu = 0
\]
is \( \mu = \sinh(2at) \) and the corresponding heat kernel is given by

\[
K(x, y, t) = \frac{1}{\sqrt{2\pi} \sinh(2at)} \exp \left( -\frac{(x^2 + y^2) \cosh(2at) - 2xy}{2 \sinh(2at)} \right) \times \exp \left( 2\omega \frac{x \sinh(t/2) + y \sinh((2a - 1/2)t)}{\sinh(2at)} \frac{t}{\frac{t}{2}} \right) \times \exp \left( -2\omega^2 \frac{\cosh(2at)}{\sinh(2at)} \sinh^4 \left( \frac{t}{2} \right) \right) \times \exp \left( \frac{\omega^2}{2} \left( t - 2 \sinh t + \frac{1}{2} \sinh(2t) \right) \right), \quad t > 0.
\]

Indeed, by (2.17)–(2.19)

\[
\alpha = \gamma = -\frac{\cosh(2at)}{2 \sinh(2at)}, \quad \beta = \frac{1}{\sinh(2at)}.
\]

In this case

\[
\frac{f \mu + g \mu'}{2a} = \omega (\cosh((2a - 1)t) \sinh(2at) - \sinh((2a - 1)t) \cosh(2at)) = \omega \sinh t
\]

and equation (2.20) gives

\[
\delta = \omega \frac{\cosh t - 1}{\sinh(2at)} = 2\omega \frac{\sinh^2(t/2)}{\sinh(2at)}.
\]

By (2.21)

\[
\varepsilon = \omega \frac{1 - \cosh t}{\sinh(2at) \cosh(2at)} + 2a\omega \int_0^t \frac{1 - \cosh \tau}{\cosh^2(2at) \cosh(2a\tau)} d\tau + \omega \int_0^t \frac{\cosh((2a - 1)\tau)}{\cosh(2a\tau)} d\tau,
\]

where the integration by parts gives

\[
2a \int_0^t \frac{1 - \cosh \tau}{\cosh^2(2at)} d\tau = (1 - \cosh t) \frac{\sinh(2at)}{\cosh(2at)} + \int_0^t \frac{\sinh(2at) \sinh \tau + \cosh((2a - 1)\tau) \cosh(2at)}{\cosh(2a\tau)} \sinh \tau d\tau.
\]

Thus

\[
\varepsilon = \omega (1 - \cosh t) \frac{\cosh(2at)}{\sinh(2at)} + \omega \int_0^t \frac{\sinh(2at) \sinh \tau + \cosh((2a - 1)\tau) \cosh(2at)}{\cosh(2a\tau)} \sinh \tau d\tau
\]

and an elementary identity

\[
\sinh(2at) \sinh t + \cosh((2a - 1)t) = \cosh(2at) \cosh t
\]

leads to an integral evaluation. Two other identities

\[
\cosh(2at) \cosh t - \sinh(2at) \sinh t = \cosh((2a - 1)t),
\]

\[
\cosh(2at) - \cosh((2a - 1)t) = 2 \sinh(t/2) \sinh((2a - 1/2)t)
\]

result in

\[
\varepsilon = \omega \frac{\cosh(2at) - \cosh((2a - 1)t)}{\sinh(2at)} = 2\omega \frac{\sinh(t/2) \sinh((2a - 1/2)t)}{\sinh(2at)}.
\]
In a similar fashion,
\[
\kappa = -2\omega^2 \sinh^4 \left( \frac{t}{2} \right) \cosh(2at) \sinh(2at) + \frac{1}{2} \omega^2 \left( t - 2 \sinh t + \frac{1}{2} \sinh(2t) \right),
\]
and equation (5.8) is derived. In the limit \( \omega \to 0 \) this kernel gives also a familiar expression in statistical mechanics for the density matrix for a system consisting of a simple harmonic oscillator [11].

The case \( a = 1/2 \) corresponds to the equation
\[
\frac{\partial u}{\partial t} = \frac{1}{2} \left( \frac{\partial^2 u}{\partial x^2} - x^2 u \right) + \omega xu
\]
and the heat kernel (5.8) is simplified to the form
\[
K(x, y, t) = e^{-\frac{\omega^2 t/2}{\sqrt{2\pi}} \sinh t} \exp \left( -\frac{(x - \omega)^2 + (y - \omega)^2}{2 \sinh t} \right),
\]
when \( t > 0 \). A similar diffusion-type equation
\[
\frac{\partial u}{\partial t} = \frac{1}{2} \left( \frac{\partial^2 u}{\partial x^2} + x^2 u \right) + \omega xu
\]
can be solved with the aid of the kernel
\[
K(x, y, t) = e^{-\omega^2 t/2} \exp \left( -\frac{(x + \omega)^2 + (y + \omega)^2}{2 \sin t} \right)
\]
provided \( 0 < t < \pi/2 \). We leave the details to the reader.

Following to the case of exactly solvable time-dependent Schrödinger equation found in [17], we consider the diffusion-type equation of the form
\[
\frac{\partial u}{\partial t} = \cosh^2 t \left( \frac{\partial^2 u}{\partial x^2} + \sinh^2 t x^2 u + \frac{1}{2} \sinh 2t \left( 2x \frac{\partial u}{\partial x} + u \right) \right).
\]
The corresponding characteristic equation
\[
\mu'' - 2 \tanh t \mu' + 2\mu = 0
\]
has two linearly independent solutions
\[
\mu_1 = \cos t \sinh t + \sin t \cosh t,
\]
\[
\mu_2 = \sin t \sinh t - \cos t \cosh t
\]
with the Wronskian \( W(\mu_1, \mu_2) = 2 \cosh^2 t \), and the first one satisfies the initial conditions (2.15). The heat kernel is
\[
K(x, y, t) = \frac{1}{\sqrt{2\pi} (\cos t \sinh t + \sin t \cosh t)} \exp \left( \frac{(y^2 - x^2) \sin t \sinh t + 2xy - (x^2 + y^2) \cos t \cosh t}{2 (\cos t \sinh t + \sin t \cosh t)} \right)
\]
provided \( 0 < t < T_1 \approx 0.9375520344 \), where \( T_1 \) is the first positive root of the transcendental equation \( \tanh t = \cot t \). Then \( \gamma(t) < 0 \) and the integral (2.26) converges for suitable initial data.
A similar diffusion-type equation
\[ \frac{\partial u}{\partial t} = \cos^2 t \frac{\partial^2 u}{\partial x^2} + \sin^2 t x^2 u - \frac{1}{2} \sin 2t \left( 2x \frac{\partial u}{\partial x} + u \right) \] (5.26)
has the characteristic equation of the form
\[ \mu'' + 2 \tan t \mu' - 2\mu = 0 \] (5.27)
with the same solution (5.23). It appeared in [17] and [6] for a special case of the Schrödinger equation. The corresponding heat kernel has the same form (5.25) but with \( x \) and \( y \) interchanged:
\[ K(x, y, t) = \frac{1}{\sqrt{2\pi (\cos t \sinh t + \sin t \cosh t)}} \times \exp \left( \frac{(x^2 - y^2) \sin t \sinh t + 2xy - (x^2 + y^2) \cos t \cosh t}{2 (\cos t \sinh t + \sin t \cosh t)} \right) \] (5.28)
provided \( 0 < t < T_2 \approx 2.347045566 \), where \( T_2 \) is the first positive root of the transcendental equation \( \tanh t = -\cot t \). We leave the details for the reader.

6. Solution of the Non-Homogeneous Equation

A diffusion-type equation of the form
\[ \left( \frac{\partial}{\partial t} - Q(t) \right) u = F, \] (6.1)
where \( Q \) stands for the second order linear differential operator in the right hand side of equation (2.1) and \( F = F(t, x, u) \), can be rewritten formally as an integral equation (the Duhamel principle; see [4], [5], [13], [14], [24], [26] and references therein)
\[ u(x, t) = H(t, 0) u(x, 0) + \int_0^t H(t, s) F(s, x, u) \, ds. \] (6.2)
Operator \( H(t, s) \) is given by (2.28). When \( F \) does not depend on \( u \), one gets a solution of the nonhomogeneous equation (6.1).

Indeed, a formal differentiation gives
\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} H(t, 0) u(x, 0) + \frac{\partial}{\partial t} \int_0^t H(t, s) F(s, x, u) \, ds, \] (6.3)
where
\[ \frac{\partial}{\partial t} \int_0^t H(t, s) F(s, x, u) \, ds = H(t, t) F(t, x, u) + \int_0^t \frac{\partial}{\partial t} H(t, s) F(s, x, u) \, ds \] (6.4)
and we assume that \( H(t, t) \) is the identity operator. Also
\[ Q(t) u = Q(t) H(t, 0) u(x, 0) + \int_0^t Q(t) H(t, s) F(s, x, u) \, ds \] (6.5)
and
\[ \left( \frac{\partial}{\partial t} - Q(t) \right) u = \left( \frac{\partial}{\partial t} - Q(t) \right) H(t, 0) u(x, 0) + F \] (6.6)
DIFFUSION-TYPE EQUATION

\[ + \int_0^t \left( \frac{\partial}{\partial t} - Q(t) \right) H(t, s) F(s, x, u) \, ds, \]

where

\[ \left( \frac{\partial}{\partial t} - Q(t) \right) H(t, s) = 0, \quad 0 \leq s < t \]

by construction of the operator \( H(t, s) \) in (2.28). This completes our formal proof. A rigorous proof will be given elsewhere.

**Acknowledgments.** This paper is written as a part of the summer 2008 program on analysis of Mathematical and Theoretical Biology Institute (MTBI) at Arizona State University. The MTBI/SUMS Summer Undergraduate Research Program is supported by The National Science Foundation (DMS-0502349), The National Security Agency (dod-h982300710096), The Sloan Foundation, and Arizona State University. The authors are grateful to Professor Carlos Castillo-Chávez for support and reference [2]. We thank Professors Faina Berezovskaya, Alex Mahalov, and Svetlana Roudenko for valuable comments.

**References**

[1] G. E. Andrews, R. A. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[2] L. M. A. Bettencourt, A. Cintrón-Arias, D. I. Kaiser, and C. Castillo-Chávez, *The power of a good idea: Quantitative modeling of the spread of ideas from epidemiological models*, Physica A 364 (2006), 513–536.

[3] J. R. Cannon, *The One-Dimensional Heat Equation*, Encyclopedia of Mathematics and Its Applications, Vol. 32, Addison–Wesley Publishing Company, Reading etc, 1984.

[4] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, Vol. 10, American Mathematical Society, Providence, Rhode Island, 2003.

[5] T. Cazenave and A. Haraux, *An Introduction to Semilinear Evolution Equations*, Oxford Lecture Series in Mathematics and Its Applications, Vol. 13, Oxford Science Publications, Claredon Press, Oxford, 1998.

[6] R. Cordero-Soto, R. M. Lopez, E. Suazo, and S. K. Suslov, *Propagator of a charged particle with a spin in uniform magnetic and perpendicular electric fields*, Lett. Math. Phys. 84 (2008) #2–3, 159–178.

[7] R. P. Feynman, *The Principle of Least Action in Quantum Mechanics*, Ph. D. thesis, Princeton University, 1942; reprinted in: “Feynman’s Thesis – A New Approach to Quantum Theory”, (L. M. Brown, Editor), World Scientific Publishers, Singapore, 2005, pp. 1–69.

[8] R. P. Feynman, *Space-time approach to non-relativistic quantum mechanics*, Rev. Mod. Phys. 20 (1948) #2, 367–387; reprinted in: “Feynman’s Thesis – A New Approach to Quantum Theory”, (L. M. Brown, Editor), World Scientific Publishers, Singapore, 2005, pp. 71–112.

[9] R. P. Feynman, *The theory of positrons*, Phys. Rev. 76 (1949) #6, 749–759.

[10] R. P. Feynman, *Space-time approach to quantum electrodynamics*, Phys. Rev. 76 (1949) #6, 769–789.

[11] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw–Hill, New York, 1965.

[12] D. R. Haahheim and F. M. Stein, *Methods of solution of the Riccati differential equation*, Mathematics Magazine 42 (1969) #2, 233–240.

[13] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, Rhode Island, 1968. (pp. 318, 356)

[14] E. E. Levi, *Sulle equazioni lineari totalmente ellittiche alle derivate parziali*, Rend. Circ. Mat. Palermo 24 (1907) , 275–317.

[15] R. M. Lopez and S. K. Suslov, *The Cauchy problem for a forced harmonic oscillator*, arXiv:0707.1902v8 [math-ph] 27 Dec 2007.

[16] I. V. Melnikova and A. Filinkov, *Abstract Cauchy problems: Three Approaches*, Chapman&Hall/CRC, Boca Raton, London, New York, Washington, D. C., 2001.
[17] M. Meiler, R. Cordero-Soto, and S. K. Suslov, Solution of the Cauchy problem for a time-dependent Schrödinger equation, J. Math. Phys. 49 (2008) #7, published on line 9 July 2008, URL: http://link.aip.org/link/?JMP/49/072102; see also arXiv: 0711.0559v4 [math-ph] 5 Dec 2007.

[18] A. M. Molchanov, The Riccati equation \( y' = x + y^2 \) for the Airy function, [in Russian], Dokl. Akad. Nauk 383 (2002) #2, 175–178.

[19] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer–Verlag, Berlin, New York, 1991.

[20] A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics, Birkhäuser, Basel, Boston, 1988.

[21] E. D. Rainville, Special Functions, The Macmillan Company, New York, 1960.

[22] E. D. Rainville, Intermediate Differential Equations, Wiley, New York, 1964.

[23] S. S. Rajah and S. D. Maharaj, A Riccati equation in radiative stellar collapse, J. Math. Phys. 49 (2008) #1, published on line 23 January 2008.

[24] E. Suazo, and S. K. Suslov, An integral form of the nonlinear Schrödinger equation with variable coefficients, arXiv:0805.0633v2 [math-ph] 19 May 2008.

[25] S. K. Suslov and B. Trey, The Hahn polynomials in the nonrelativistic and relativistic Coulomb problems, J. Math. Phys. 49 (2008) #1, published on line 22 January 2008, URL: http://link.aip.org/link/?JMP/49/012104

[26] T. Tao, Nonlinear Dispersive Equations: Local and Global Analysis, CBMS regional conference series in mathematics, 2006.

[27] G. N. Watson, A Treatise on the Theory of Bessel Functions, Second Edition, Cambridge University Press, Cambridge, 1944.

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