Explicit calculation of multiloop amplitudes in the superstring theory

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Abstract

Multiloop superstring amplitudes are calculated in the explicit form by the solution of Ward identities. A naive generalization of Belavin-Knizhnik theorem to the superstring is found to be incorrect since the period matrix turns out to be depended on the spinor structure over the terms proportional to odd moduli. These terms appear because fermions mix bosons under the two-dim. supersymmetry transformations. The closed, oriented superstring turns out to be finite, if it possesses the ten-dimensional supersymmetry, as well as the two-dimensional one. This problem needs a further study.

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1 Introduction

In the well known scheme [1-4] the superstring amplitudes are obtained by summation over the spinning string ones. Every spinning string amplitude does not satisfy supersymmetry. It turns out to be a source of serious difficulties [3,4] in the scheme above. Recently the manifestly supersymmetrical scheme has been proposed [5,6]. It generalizes the results of ref. [7] to the superstring theory. In the presented paper the discussed scheme [5,6] is applied to the explicit calculation of the multiloop amplitudes in the closed, oriented superstring theory. We consider the even spin structures, the odd spin ones being planned to discuss in another paper. Also, we consider only the boson emission amplitudes.

In the considered scheme the superstring amplitudes are calculated from equations that are none other than Ward identities. These equations realize the requirement that the superstring amplitudes are independent of both the "vierbein" and the gravitino field. The above equations determine the partition functions except only for arbitrary constant factors, some number of them being reduced by the supermodular invariance. To calculate (in the terms of a coupling constant) all these factors one should use the unitarity equations. Instead we use the factorization requirement on the superstring amplitudes when two handles move away from each other. So, the superstring amplitudes turn out to be fully determined by the gauge invariance together with the "factorization requirement" above.

As soon as fermions mix bosons under the supersymmetry transformations, the period matrix appears to be depended on the spinor structure in the terms proportional to odd moduli. Because of this effect the naive generalization [1,3,4] of the Belavin-Knizhnik theorem [8] to the superstring is found to be incorrect.

The problem of the divergences needs a further study even in the closed, oriented superstring theory. In this theory the possible divergences arise when the handles move away from each other. These divergences disappear, if the known "nonrenormalization theorems" [9] are valid. However, in the presented paper we do not verify the above theorems because of the mathematical complexity of this verification.

A different approach to the discussed problems has been proposed recently in ref. [10].
2 Superspin structures

In the supercovariant scheme [5,6] a genus-n superstring amplitude is found to be the sum over ”superspin” structures integrated over $(3n-3|2n-2)$ complex moduli. If all the odd moduli are taken to be equal to zero, then every genus-n superspin structure $(l_1, l_2)$ is reduced to the ordinary $(l_1, l_2)$ spin one. Here $l_1$ and $l_2$ are the theta function characteristics: $(l_1, l_2) = \bigcup_s (l_{1s}, l_{2s})$ where $l_{is} \in (0, 1/2)$. The (super)spin structure is even, if $l_1 l_2 = \sum_{s=1}^{n} l_{1s} l_{2s} = 0$. It is odd, if $4l_1 l_2 = 1$.

To every (super)spin structure one can assign the ”transition” group. The above transition groups are defined on the $(1|1)$ complex supermanifolds [11] mapped by the supercoordinate $t=(z|\theta)$; $z$ is a local complex coordinate and $\theta$ is its odd partner. The transition group is generated by its base elements $(\Gamma_{a,s}, \Gamma_{b,s})$ associated to the transition about the $(a_s, b_s)$ cycles, respectively. For the description of the above transitions we use supersymmetrical versions of the Schottky parameterization [12,13]. Then the above $(\Gamma_{a,s}, \Gamma_{b,s})$ are determined by $(3|2)$ complex parameters: two fixed supermanifold points $t_{s}^{(+)} = (u_s|\mu_s)$ and $t_{s}^{(-)} = (v_s|\nu_s)$, as well as the multiplier $k_s (|k_s| < 1, |\arg k_s| \leq \pi)$. The replacement $\sqrt{k_s} \rightarrow -\sqrt{k_s}$ presents the supermodular transformation that turns the genus-1 superspin structure $(l_{1s} = 0, l_{2s} = 1/2)$ into the $(l_{1s} = 0, l_{2s} = 0)$ superspin one. To construct supermodular invariant amplitudes we choose the set of transition groups to be consistent with the above supermodular transformation. Therefore, we require that transition groups assigned to the above genus-1 superspin structures turn into each other when $\sqrt{k_s} \rightarrow -\sqrt{k_s}$.

If the odd parameters $(\mu_s, \nu_s)$ are equal to zero, the base transition group elements can be chosen to be equal to $(\Gamma_{a,s}^{(o)}, \Gamma_{b,s}^{(o)})$ where

$$\Gamma_{a,s}^{(o)}(l_{1s}) = \{ z \rightarrow z, \theta \rightarrow (-1)^{2l_1} \theta \}, \quad \Gamma_{b,s}^{(o)}(l_{2s}) = \Gamma_{a,s}^{(o)}(1/2 - l_{2s}) \Gamma_{s}^{(o)}.$$ (1)

Here $\Gamma_{s}^{(o)} = \{ z \rightarrow (a_s z + b_s)(c_s z + d_s)^{-1}, \theta \rightarrow \theta(c_s z + d_s)^{-1} \}$ and $a_s d_s - b_s c_s = 1$. The $(a_s, b_s, c_s, d_s)$ can be expressed in the terms of above $u_s, v_s$ and $k_s$, as well. The transitions (1) remain to be unchanged the supermanifold points: $t_{a,s}^{(+)} = (u_s|0)$ and $t_{a,s}^{(-)} = (v_s|0)$. For arbitrary odd parameters $(\mu_s, \nu_s)$ we
define the discussed base elements as

\[ \Gamma_{a,s} = \tilde{\Gamma}_s \Gamma_{a,s}^{(0)}(l_{1s}) \tilde{\Gamma}_s^{-1}, \quad \Gamma_{b,s} = \tilde{\Gamma}_s \Gamma_{b,s}^{(0)}(l_{2s}) \tilde{\Gamma}_s^{-1} = \Gamma_{a,s}(1/2 - l_{2s}) \Gamma_s \]  

(2)

where \( \Gamma_s = \tilde{\Gamma}_s \Gamma_{a,s}^{(0)} \tilde{\Gamma}_s^{-1} \) and \( \tilde{\Gamma}_s \) is a suitable transformation. It is convenient to require that \( t^{(-)}_{o,s} \to t^{(-)}_{s} \) and \( t^{(+)}_{o,s} \to t^{(+)}_{s} \) under the \( \tilde{\Gamma}_s \) mapping. Then \( \tilde{\Gamma}_s \) is determined as

\[
\tilde{\Gamma}_s : \quad z = z_s + \theta_s \tilde{\varepsilon}_s(z_s), \quad \theta = \theta_s(1 + \tilde{\varepsilon}_s \tilde{\varepsilon}_s'/2) + \tilde{\varepsilon}_s(z_s); \\
\tilde{\varepsilon}_s' = \partial_z \varepsilon_s(z), \quad \varepsilon_s(z) = \mu_s(z - v_s) - v_s(z - u_s)(u_s - v_s)^{-1}. \]  

(3)

Being superconformal, all the above transitions preserve the spinor derivative \( D(t) \) up to some factor. For arbitrary supersymmetrical transformation \( \Gamma = \{ t \to t_\Gamma = (z_\Gamma(t)|\theta_\Gamma(t)) \} \) this factor \( Q_\Gamma(t) \) is\(^1\)

\[ Q_\Gamma^{-1}(t) = D(t) \theta_\Gamma(t) \quad \text{where} \quad D(t) = \theta \partial_z + \partial_\theta; \quad D(t_\Gamma) = Q_\Gamma(t) D(t). \]  

(4)

The fundamental domain on the complex \( z \)-plane is the region exterior to all the circles: \( C_s^{(-)} = \{ z : |Q_{b,s}(t)| = 1 \} \) and \( C_s^{(+)} = \{ z : |Q_{b,s}^{-1}(t)| = 1 \} \). We define the above region exterior (interior) to be the same as when all the odd parameters are reduced to zero.

It is obvious from eqs. (1) - (3) that \( \Gamma_{a,s}(l_{1s} = 0) = I, \Gamma_{a,s}^{2}(l_{1s} = 1/2) \) is given by

\[
\Gamma_{a,s}(l_{1s} = 1/2) = \{ z \to z - 2\theta \tilde{\varepsilon}_s(z), \quad \theta \to -\theta(1 + 2\tilde{\varepsilon}_s \tilde{\varepsilon}_s' + 2\tilde{\varepsilon}_s(z)) \} \]  

(5)

where \( \tilde{\varepsilon}_s \) is defined by eq.(3). Therefore, for \( l_{1s} = 1/2 \) the cut \( \tilde{C}_s \) appears on the considered \( z \)-plane. One of its endcut points is placed inside the \( C_s^{(-)} \) circle and the other endcut point is placed inside the \( C_s^{(+)} \) one. A superconformal p-form \( F_p(t) \) changes under the \( (\Gamma_{a,s}, \Gamma_{b,s}) \) transitions \( t \to t_{\Gamma_{a,s}} = t^a_s, t \to t_{\Gamma_{b,s}} = t^b_s \) as

\[
F_p(t^a_s) = F_p^{(s)}(t) Q_{\Gamma_{a,s}}^p(t), \quad F_p(t^b_s) = F_p(t) Q_{\Gamma_{b,s}}^p(t) \]  

(6)

where \( F_p^{(s)}(t) \) is obtained by \( 2\pi \)-twist of \( F_p(t) \) about the \( C_s^{(-)} \) circle.

\(^1\) The below transition groups differ from the ones given in refs. [2,14] except only for the \( (l_{1s} = 0, l_{2s} = 1/2) \) case.
The superspin structure \( S_0 = \bigcup_s (l_{1s} = 0, l_{2s} = 1/2) \) has been considered in refs. [5,6]. It can also be obtained by the "naive" supersymmetrization of the boson string [14]. The superspin structures \( S(0, l_2) = \bigcup_s (l_{1s} = 0, l_{2s}) \) can be obtained from the above \( S_0 \) by the \( \sqrt{k_s} \to -\sqrt{k_s} \) replacement for every \( \sqrt{k_s} \) associated with \( l_{2s} = 0 \). For the remained even superspin structures the Green functions are branched on \( z \)-plane that complicate their calculation. These superspin structures \( S_{\text{br}} \) are considered below.

### 3 Scalar supermultiplets

The having zero periods vacuum correlation function of two identical scalar superfields can be written [2,6,13] in the terms of the holomorphic Green function \( R(t, t') \), its periods \( J_r(t) \) and the period matrix \( \omega = \{ \omega_{sr} \} \). Owing to Bose statistics we have that \( R(t, t') = R(t', t) \). Also, \( J_r(t) = J_r(t) + 2\pi i\omega_{sr} \) for \( t \to t+a, t \to t+b \) transitions are the same as in eq.(6). We normalize \( R(t, t') \) by \( R \to (z - z' - \theta \theta')^{-1} \) at \( z \to \infty \). Apart from unessential terms in \( R(t, t') \) due to the scalar zero mode, both \( R(t, t') \) and \( J_s(t) \) are fully determined by the equations:

\[
R(t^a_s, t') = R^{(s)}(t^a_s, t'), \quad R(t^b_s, t') = R(t, t') + J_s(t').
\]

In ref. [6] we calculated \( R \) in the explicit form only for the \( S_0 \) superspin structure. Now we calculate it for every even superspin one. Instead of \( R(t, t') \) it is convenient for this aim to have deal with the Green function \( K(t, t') \) where \( K(t, t') = D(t')R(t, t') \). Furthermore, we fix the above unessential terms in \( R(t, t') \) up to an additive constant by the requirement: \( J_r \to 0 \) at \( z \to \infty \). Then \( K(t, t') \to 0 \) at \( z \to \infty \) or \( z' \to \infty \).

The \( 1/2 \)-differentials \( \eta_s \) are defined as \( 2\pi i\eta_s(t) = D(t)J_s(t) \). For discussed \( S_{\text{br}} \) superspin structures all \( \eta_s \) appear to be branched, if odd parameters are unequal to zero. To obtain the normalization set for \( \eta_s \) we define for arbitrary
Using eqs. (8) and (9) one can obtain the above \( W_r \) in the explicit form that is omitted here. Also, it can be proved that

\[
2\pi iW_r(\eta_s) = \delta_{rs} \quad \text{and} \quad W_s(K) = 0 \quad (10)
\]

where \( W_s(K) \) is equal to \( W_s(F_{1/2}) \) at \( F_{1/2}(t') = K(t, t') \). Eqs. (10) are proved by moving to infinity the integration contour in \( \sum_{n=1}^{\infty} E_r(F_{1/2}, K) \) for \( F_{1/2} \) to be equal to \( \eta \) or \( K \). Besides, the linear independence of the different \( \eta_s \) is taken into account. Eqs. (10) turn out to be useful for the calculation of the Green function \( K(t, t') \).

If all odd parameters are equal to zero, \( K(t, t') \) reduces to \( K^{(o)}(t, t') = K_b(z, z') + \theta \Phi(z, z')(z - z')^{-1} \). The \( K_b(z, z') \) is given by series over Schottky group elements \([2,13] \). The like series determining \( \Phi(z - z')^{-1} \) may be divergent for the considered \( S_{bs} \) structures, but in any case \( \Phi \) can be written as

\[
\Phi(z, z') = \left( \prod' \frac{[\phi_{\Gamma}(z)\phi_{\Gamma}(z')]^{1/2}}{[z - g_{\Gamma}(z'][z' - g_{\Gamma}(\infty)]} \right) \frac{\Theta[l_1, l_2](J|\omega)}{\Theta[l_1, l_2](0|\omega)}
\]

where \( \phi_{\Gamma}(z) = [z - g_{\Gamma}(z)][z - g_{\Gamma}(\infty)] \).

Here \( \Theta \) is the theta function. The symbol \( J \) denotes set of \( \{J^{(o)}_s(z) - J^{(o)}_s(z')\} \) functions. To every \( \Gamma \) the mapping \( z \rightarrow g_{\Gamma}(z) \) is assigned. The product \( \prod' \) includes all Schottky group elements \( \Gamma \) except only for \( \Gamma = I \).

Odd parameters being arbitrary, to calculate Green functions \( K \) for the discussed \( S_{bs} \) superspin structures we construct the set \( \{K^{(o)}_s\} \) of ”master” Green functions. For every \( K^{(o)}_s(t, t') \) its transition group elements associated with rounds about the \((a_s, b_s)\) cycles are constructed to be the same as for the truly Green function \( K(t, t') \). However, the transition group elements associated with rounds of \( K_s \) about all the other cycles may differ from those

\[\text{Throughout this paper the contour } C_r \text{ is defined to surround } (C_r^(-), C_r^(+)) \text{ circles together with the } \tilde{C}_r \text{ cut, the } C_r \text{ contour being closed to } (C_r^(-), C_r^(+), \tilde{C}_r) \text{ above. Besides, } dt = d\theta dz \text{ and } \int d\theta = 1.\]
assigned to the above Green function $K(t, t')$. At last, $K_s(t, t') \to 0$ at $z \to \infty$ or $z' \to \infty$. 

To calculate $K(t, t')$ we start with the following relations:

$$K(t, t') = K_s(t, t') + \sum_{r=1}^{n} \int_{C_r} K_s(t, t'') \frac{dtt''}{2\pi i} K(t'', t')$$  \hspace{1cm} \text{for } s = 1, 2, \ldots, n \tag{12}$$

Eq. (12) can be verified by moving of the integration contour $\int C_r$ to infinity. Then the nonzero contribution originates from the poles at $z'' = z$ and $z'' = z'$. In the sum over $r$ there is no the term corresponding to $r = s$. Indeed, this term can be written as $2\pi i W_s K_s(t, t')$ where $W_s$ is defined by eq.(9) for $F_{1/2}(t') = K_s(t, t')$. So, this term vanishes owing to eq.(10). For $z \in \bigcup C_s$ we define the set $\tilde{K} = \{K_s(t, t')\}$ by the relations: $K(t, t') = K_s(t, t')$, if $z \in C_s$, $z' \in C_r$ and $s \neq r$; $K_{ss}(t, t') = 0$.

Below we use two sets of master Green functions $K_s^{(o)}$. Firstly, we construct $K_s^{(o)}$ as

$$K_s^{(o)}(t, t') = K^{(o)}(t_s, t'_s) D(t') \theta'_s + \tilde{\epsilon}'_s \theta'_s \Phi(\infty, z'_s) \tag{14}$$

where $t_s$ is calculated in the terms of $t$ from eqs.(3). Then $K_s$ can be calculated from eq.(13) by the iteration procedure, every posterior iteration being one more power in odd parameters than a previous one. Therefore, $K_s$ appears to be a series containing a finite number of terms.

The second set we construct in the terms of the genus-1 and genus-2 Green functions. The genus-1 Green function we assign to every handle except only for handles associated with the odd genus-1 superspin structure: $l_{1s} = l_{2s} = 1/2$. The number of the latter handles is even for even genus-n
superspin structures and we group them into pairs. Then to every pair we
assign the genus-2 Green function that is calculated from eq.(13), where \( K^{(o)}_s \)
are defined by eq.(14). The genus-1 Green functions are given by eq.(14).
We denote the considered set as \( \tilde{K}_0 + \tilde{\Xi} \) where \( \tilde{K}_0 \) is calculated at all the
odd parameters to be equal to zero and \( \tilde{\Xi} \) is proportional to odd Schottky
parameters. Then eq.(13) can be turn into the following one:

\[
\tilde{K} = \tilde{K}^{(o)}_0 + \tilde{\Xi} + \tilde{K}^{(o)}_0 \tilde{\Xi} + (\tilde{\Xi} + \tilde{K}^{(o)}_0 \tilde{\Xi}) \tilde{K}.
\]

In eq.(15) the operator \( \hat{\tilde{K}}^{(o)}_0 (\tilde{\Xi}) \) is related with \( \tilde{K}^{(o)}_0 (\tilde{\Xi}) \) just as \( \hat{\tilde{K}}^{(o)} \) is related
with \( \tilde{K}^{(o)} \) in eq.(13). Besides, \( \tilde{K}^{(o)}_0 = \{K^{(o)}_s \} \) where \( K^{(o)}_s \) are expressed in the
terms of the reduced Green function \( K^{(o)} \): \( K^{(o)}_s (t, t') = K^{(o)}_s (t, t') \) at \( z \in C_s \).

When \( K \) is determined one easy calculate the Green function \( R^{(o)}_s \) up
to unessential additive constant. If \( z \) is situated near the \( C_s \) contour, it is
convenient to write its periods \( J_s(z) \) as

\[
J_s = J^{(o)}_s + \sum_p \hat{\tilde{K}}^{(o)}_s \tilde{\Xi}_p J^{(o)}_s
\]

where \( J^{(o)}_s \) is the period of \( R^{(o)}_s \) corresponding to \( 2\pi \)-twist about the \( b_s \)-cycle.
The set \( \hat{\tilde{K}} \) of the integral operators \( \hat{\tilde{K}}_p \) is calculated from the equations:

\[
\hat{\tilde{K}} = \hat{\tilde{K}}^{(o)} + \hat{\tilde{K}}^{(o)} \hat{\tilde{K}}.
\]

For \( z \) to be situated near the \( C_r \) contour \( (r \not= s) \) one can write:

\[
J_s = \hat{\tilde{K}}^{(o)}_r J^{(o)}_s + \sum_p \hat{\tilde{K}}^{(o)}_p \tilde{\Xi}_p J^{(o)}_s.
\]

The period matrix \( \omega \) turns out to be

\[
\omega_{sr} = \eta_r^{(o)} J^{(o)}_s + \sum_{p \not= r} \eta_r^{(o)} \hat{\tilde{K}}_p J^{(o)}_s \quad \text{for} \quad r \not= s
\]

\[
\text{and} \quad \omega_{ss} = \omega^{(o)} + \sum_{p \not= s} \eta_s^{(o)} \hat{\tilde{K}}_p J^{(o)}_s.
\]

In eq. (19) \( \omega^{(o)}_{ss} \) is the \((ss)\) element of the period matrix associated with \( J^{(o)}_s \)
The integral operator \( J^{(o)}_s \) is defined for \( z \in C_s \), its kernel being \( J^{(o)}_s(t) \). One
can verify from eqs.(19) that the period matrix depends on the superspin
structure owing to the terms proportional to the odd parameters. It seems
natural since these terms appear because fermions mix bosons under the
supersymmetry transformations.
4 Ghost supermultiplets

In the considered scheme [5,6] both the supermoduli volume form and zero mode contributions to the ghost determinant are counted by using of a suitable ghost vacuum correlation function \( G_{gh}(t, t') \). The discussed \( G_{gh} \) can be expressed [5,6] in the terms of the Green function \( G(t, t') \) and superconformal 3/2-forms \( \chi_N(t') \), all they being fully determined by the relations:

\[
G(t^a_r, t') = Q_{\Gamma_{a,r}}^{-2}(t) \left( G^{(r)}(t, t') + \sum_{N_r} Y_{a,N_r}(t) \chi_N(t') \right)
\]

\[
G(t^b_r, t') = Q_{\Gamma_{b,r}}^{-2}(t) \left( G(t, t') + \sum_{N_r} Y_{b,N_r}(t) \chi_N(t') \right)
\]

\[
G(t, t^a_r) = Q_{\Gamma_{a,r}}^{-2}(t) G^{(r)}(t, t'), \quad G(t, t^b_r) = Q_{\Gamma_{b,r}}^{-2}(t) G(t, t')
\]  

(20)

where \( N_r = k_r, u_r, v_r, \mu_r \) or \( \nu_r \). Furthermore, \( Q_{\Gamma_{r}}^{-2} Y_{q,N_r} = \partial_{N_r} g^q_r + \gamma^q_r \partial_{N_r} \gamma^q_r \) with \( q = a, b \). The above \( Y_{q,N} \) are power-2 polynomial in \( (z, \theta) \). For \( l_{2r} = 1/2 \) the functions \( Y_{b,N_r} \) have been calculated in ref. [6]. Both \( t^a_r = (g^a_r|\gamma^a_r) \) and \( t^b_r = (g^b_r|\gamma^b_r) \) in eqs.(20) are the same as in eqs.(6). The relations (20) generalize the results of refs. [5,6] to arbitrary superspin structures.

For an arbitrary 3/2-form \( F_{3/2} \) we define the integral \( H_{N_r}(F_{3/2}) \) by the relation:

\[
2\pi i \sum_{N_r} H_{N_r}(F_{3/2}) \chi_{N_r}(t') = - \int_{C_r} F_{3/2}(t) \frac{dt}{2\pi i} G(t, t').
\]  

(21)

Using eqs.(20) and (21) one can calculate \( H_N \) in the explicit form, but we omit it here. Also, it can be proved that

\[
2\pi i H_{N_s}(\chi_{N_r}) = \delta_{N_s,N_r} \quad \text{and} \quad H_{N_s}(G) = 0
\]  

(22)

where \( H_{N_s}(G) \) is equal to \( H_{N_s}(F_{3/2}) \) calculated at \( F_{3/2}(t') = G(t, t') \). Eqs. (21) - (22) are similar to eqs. (9) - (10) for scalar supermultiplets.

If all odd parameters are equal to zero, the Green function \( G(t, t') \) is reduced to the Green function \( G^{(o)}(t, t') = G_b(z, z') \theta' + \theta G_f(z, z') \). Being independent of the spin structure, \( G_b \) can be obtained from the ghost Green function given in refs. [5,6]. The discussed \( G_f \) can be calculated in the terms
of Green functions $G_{\{\sigma\}}(t, t')$ defined as

$$G_{\{\sigma\}}(t, t') = \sum_{\Gamma} \exp \frac{\pi i \{\Omega_{\Gamma}(\{\sigma_s\}) + \sum_s 2l_{1s} \sigma_s (J_s(z) - J_s(z'))\}}{[z - g_{\Gamma}(z')]Q_{\Gamma}^3(z')}$$

(23)

where $\sigma_s = \pm 1$. So, $G_{\{\sigma\}}$ depends on a choice of the $\{\sigma_s\}$ set. In eq. (23) the summation performs over all Schottky group elements $\Gamma$, the base ones being $\Gamma_s$. The value $\Omega_{\Gamma}(\{\sigma_s\})$ is

$$\Omega_{\Gamma}(\{\sigma_s\}) = - \sum_{s, r} 2l_{1s} \sigma_s \omega_{sr} n_r(\Gamma) + \sum_r (2l_{2r} - 1)n_r(\Gamma)$$

(24)

where $n_r(\Gamma)$ is the number of times that the $\Gamma_r$ generators are present in $\Gamma$ (for its inverse $n_r(\Gamma)$ is defined to be negative).

The changes of $G_{\{\sigma\}}$ under the $(t \to t')$ transitions are given by

$$G_{\{\sigma\}}(t', t) = Q_{\Gamma_{b,r}}^{-2}(t) \left( G_{\{\sigma\}}(t, t') + \sum_{N_r} \tilde{Y}_{b,N_r}(t) \chi_{\{\sigma\},N_r}(t') \right)$$

where

$$\tilde{Y}_{b,N_r}(t) = \exp[\pi i \sum_s 2l_{1s} \sigma_s J_s(t)]Y_{b,N_r}(t)$$

(25)

and $\chi_{\{\sigma\},N_r}(t')$ are $3/2$-forms. The discussed $G_f(t, t')$ turns out to be

$$G_f(t, t') = G_{\{\sigma\}}(t, t') - \sum_N \tilde{H}_N(t) \chi_N(t')$$

(26)

where $\tilde{H}_N(t)$ is $H_N(F_{3/2})$ calculated at $F_{3/2}(t') = G_{\{\sigma\}}(t, t')$. The index $N$ labels the even and odd Schottky parameters. As soon as eqs.(20) determine $G_f(t, t')$ in the unique way, $G_f(t, t')$ given by eq.(26) is independent of $\{\sigma\}$. Also, from eqs.(20) and (26) it follows that

$$\chi_{\{\sigma\},N} = \sum_{N'} M_{N,N'}(\sigma) \chi_{N'}$$

where $M_{N,N'}(\sigma) = H_{N'}(\chi_{\{\sigma\},N})$. (27)

For arbitrary odd parameters the discussed $G(t, t')$ is calculated by the same method as $K(t, t')$ considered in the previous section.

### 5 Multiloop superstring amplitudes

Using both the above Green functions and eqs.(22) for $\chi_N$ we calculate the partition functions from the equations derived in refs. [5,6]. Below we fix
Schottky parameters \((u_1, v_1, u_2, \mu_1, \nu_1)\) to be the same for all the genus-

\(n\) supermanifolds, the rest of \((3n - 3|2n - 2)\) Schottky ones being chosen as

moduli. In the closed, oriented superstring theory the \(m\)-leg, \(n\)-loop amplitudes

\(A_n^m\) are given by

\[
A_n^m = g^n \int \sum_{(L, L')} \{ \det 2\pi [\omega(L) - \omega(L')] \}^{-5} Z_L(N) \overline{Z_{L'}}(N) B_{LL'} \times \\
\prod_{N'} dN' d\overline{N'} \prod_{p=1}^{m} dt_p d\overline{t}_p 
\]

where the line over denotes the complex conjugation, \(g\) is a coupling constant

and \(N'\) label those Schottky parameters \(N\) that are chosen to be moduli.

The summation performs over all the superspin structures \(L\) and \(L'\) of the

right and left fields. Also, \(B_{LL'} = B_{LL'}(\{t_p\}, \{\overline{t}_p\})\) are the vertex products

integrated over fields. Using the boson emission vertices [13] and the results

obtained in Sec.3 of the presented paper one can calculate the discussed \(B_{LL'}\)

for the boson emission amplitudes. The fermion emission amplitudes need a

further study.

In eq.(28) the factors \(Z_L(N)\) are holomorphic in \(N\). Therefore, Belavin-

Knizhnik theorem [8] is correct for every term in eq.(28), but its naive

generalization to \(A_n^m\) is not true because the period matrix \(\omega = \omega(L)\) given by

eq.(19) depends on \(L\). The discussed \(Z_L(N)\) factors turn out to be

\[
Z_L = Z_L^{(o)} \tilde{Z}_L([(u_1 - u_2)(v_1 - u_2) - \mu_1 \mu_2/2 - \nu_1 \mu_2/2] \prod_{r=1}^{n} (u_r - v_r - \mu_r \nu_r)^{-1}. \tag{29}
\]

The factor \(Z_L^{(o)}\) in eq.(29) is calculated at all the odd parameters equal to

zero. The contribution of the odd parameters is counted by the factor \(\tilde{Z}_L\).

In the equation for \(Z_L^{(o)}\) a lot of terms vanishes because these terms can be

written as \(\exp[\pi i \Omega_l (z - g_l(z))^{-1} Q_l^{-3} \partial_N J_r(z)]\) integrated along \(\bigcup C_s\). The result is

\[
Z_L^{(o)} = \frac{\Theta^5[l_1, l_2](0|\omega) \exp[-\pi i \sum_{j,r} l_{1j} l_{1r} \omega_{jr}]}{\Theta^5[\{0\}, \{1/2\})(0|\omega) \sqrt{\det M(\sigma)M(-\sigma)}} \prod_{j=1}^{n} \frac{1 + \sqrt{k_j} \lambda_j}{k_j^{3/2} \lambda_j} \times \\
\prod_{(k)\{0\}} (1 + \sqrt{k})^{-2} \prod_{m=1}^{\infty} \frac{(1 - k^m)^{-8}(1 - k^{m-1/2})^8(1 - k^{m+1/2})^2}{[1 - \Lambda_l^+(\sigma)k^{m+1/2}][1 - \Lambda_l^-(\sigma)k^{m+1/2}]} \tag{30}
\]
where $\lambda_j = \gamma^{(1-2l_j)i}$ and $\Lambda \Gamma (\sigma) = \exp [\pi i \Omega \Gamma (\{\sigma_s\})]$, the above $\Omega \Gamma (\{\sigma_s\})$ being defined by eq.(24). The matrix $M(\sigma)$ is defined by eq.(27) and $\Theta$ is the theta function. The $\theta$ in the denominator associates with the $S_0$ spin structure. The product over $(k)$ is taken over all multipliers of the Schottky group, which are not powers of other ones. In fact, eq.(30) does not depend on a choice of the $\{\sigma\}$ set because Green function $G_f(t, t')$ given by eq.(26) is independent of $\{\sigma\}$. To avoid misunderstanding we note one more that the right side of eq.(30) is calculated at all odd parameters to be equal to zero.

To calculate the factor $\tilde{Z}_L$ in eq.(29) we use that both $\partial N K(t, t')$ and $\partial N G(t, t')$ can be written as the integral $H N (F_3/2)$ defined by eq.(21) with a suitable $F_3/2$. For $\partial N K$ the $F_3/2$ form appears to be by-product of $K$ and its derivatives in respect to $z$ or $\theta$. For $\partial N G$ the discussed $F_3/2$ form includes, besides, by-products of 3/2-differentials and the factors that are power-2 polynomial in $(z, \theta)$. For $n = 2$ we found that

$$\tilde{Z}_L = \text{trace}\left[\frac{5 \ln(1 - \hat{\Xi} - \hat{\Psi} \hat{K}_0^{(o)}) + \ln(1 - \hat{\Psi} \hat{G}_0^{(o)})}{4}\right]$$

(31)

where the operator $\hat{G}_0^{(o)}$ is associated with the ghost supermultiplets in the same way as the integral operator $\hat{K}_0^{(o)}$ associated with the scalar ones. The operator $\hat{K}_0^{(o)}$ is the same as in eq. (13), where $\hat{K}_0^{(o)}$ is defined by eq.(14).

The considered sum, as well as both $\hat{\Xi}$ and $\hat{\Psi}$, depends on a choice of the dividing of the considered handles into pairs, but $\tilde{Z}_L$ is independent of the above choice.

### 6 The problem of divergences

In eq.(28) the integration region over $N'$ is determined by the supermodular invariance. Without a loss of generality one can exclude from this region
those domains where some of the Schottky group multipliers $k$ are near to one: $k \approx 1$. Indeed, modulo of supermodular transformations, these domains are equivalent to those where some of $k_j$ are small: $k_j \approx 0$. At $k_j \to 0$ we see from eq.(30) that $Z_L \sim k_j^{-1}$ for $l_{1j} = 1/2$ and $Z_L \sim k_j^{-3/2}$ for $l_{1j} = 0$. However, in the sum(28) over $(L)$ the above singularity $k_j^{-3/2}$ is reduced to $k_j^{-1}$. Besides, we have the factor $(\ln |k|)^{-5}$ due to $\det[\omega(L) - \omega(L')]^{-5}$ in eq.(28). As the result, the integral(28) appears to be finite at $k_j \to 0$.

Nevertheless, the problem of the finiteness of the considered theory needs a further study. It follows from eq.(29) that, beside the above singularities at $k_j \to 0$, every $Z_L$ has also the singularities at $u_j - v_j \to 0$. One can interpret the above limit as the moving of the $j$-handle away from the others. The contribution to $A_n^m$ from the region where $u_j - v_j \to 0$ appears to be proportional to

$$\int \frac{d(Reu_j)d(Rev_j)d(Imu_j)d(Imv_j)d\mu_jd\nu_jd\overline{\mu}_j d\overline{\nu}_j}{|u_j - v_j - \mu_j\nu_j|^2} [Z_{n-1}A^m_1 + (u_j - v_j)^2 B]$$

(33)

where $B$ is finite at $u_j - v_j \to 0$ and $Z_{n-1}$ is the genus-(n-1) partition function. One can see that the integral (33) has uncertainty, if $Z_{n-1} \neq 0$. The uncertainties of the same type arise also from the other regions, which correspond to the moving of the handles away from each other. The equality $Z_n = 0$ has been argued in ref. [9] under the assumption that the discussed theory possesses the ten-dimensional supersymmetry, as well as the two-dimensional one. So, if the above assumption is true, the closed, oriented string appears to be free from the divergences. However, we did not verify the discussed assumption because of the mathematical complexity of this verification.
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