Hannay Angle: Yet Another Symmetry Protected Topological Order Parameter in Classical Mechanics

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Topological way of thinking now goes beyond conventional solid materials, and topological characterization of classical mechanical systems governed by Newton’s equation of motion begins to attract much attention. To have a deeper insight on physical meaning of topological numbers in mechanical systems, we demonstrate the use of the Hannay angle, a classical counterpart of the Berry phase, as a symmetry protected topological order parameter. We first derive the Hannay angle using a canonical transformation that maps the Newton’s equation to the Schrödinger type equation. The Hannay angle is then used to characterize a simple spring-mass model topologically with a particular focus on the bulk-edge correspondence and new aspects of the symmetry in a classical system.

A study of topological phenomena in solids has its origin in quantum Hall systems [1]. However, topological phenomena are not limited to quantum systems, but also found in classical systems such as photonic crystals governed by the Maxwell equation [2, 3]. Very recently, classical mechanical systems obeying the Newton’s equation of motion begin to get much attention as a playground of topology [4–11]. Although topology is an abstract mathematical concept, physical observables are characterized by the topological quantities and well described. For instance, in the quantum Hall systems, a topological number, i.e., the Chern number defined using the Bloch wave functions [12], has a direct relation to the Hall conductance and the number of edge states [13]. Another example is the Berry phase, which describes the topological edge states of Dirac fermions in solid state materials such as graphene [14–17]. In contrast to the Chern number, which is quantized by definition, the Berry phase is quantized and topological only with a help of some appropriate symmetry. It is the symmetry protection of topological phases. In any cases, nontrivial bulk as a topological phase implies existence of localized edge states if the system has boundaries, i.e., there is the bulk-edge correspondence. This is surely quite important for topological description of classical systems, where we do not mostly have bulk topological observables such as the Hall conductance. Even though the classical system is topologically nontrivial, we may only access its topological character through the edge states. Direct access to the bulk topological observable in classical systems is an important issue to be addressed.

As a first step toward better understanding of topological numbers in classical systems, here we demonstrate the use of the Hannay angle [18], which is a classical counterpart of the Berry phase, as a symmetry protected topological order parameter. We first present a concise formulation of the geometrical angle, i.e., the Hannay angle, in mechanical system using a canonical transformation, such that the Newton’s equation is mapped to the Schrödinger type equation. Then, the solid relation between the quantized Hannay angle by symmetry and the edge states is established by numerically analyzing a simple spring-mass model. Interestingly, the symmetry protected nature is observed as boundary condition dependence of the edge modes.

Let us begin with describing a geometrical phase in a classical system using a simple example, 2D motion of a mass point (mass $m = 1$ is assumed for simplicity) in the potential $V_s(x, y) = \alpha^2 x \Gamma_0 x / 2$ with

$$\Gamma_0 = 1 + \Delta(1 - \eta \cos \phi) \tilde{\sigma}_3 + \Delta \eta \sin \phi \tilde{\sigma}_1,$$

where $\phi$ has some time dependence and $\tilde{\sigma}_i$ is the Pauli matrix. $V_s(x, y)$ is periodic in $\phi$ with period $2\pi$. This problem is so simple, but captures essence of the geometrical phase in classical mechanics. For the fixed $\phi$, this system has two normal modes with frequencies $\omega_\phi^\pm = \alpha \sqrt{1 \pm \Delta \sqrt{1 - 2\eta(1 - \eta)(1 - \cos \phi)}}$. Now, assuming the time dependence of $\phi$ as $\Omega t$, we follow the time evolution starting from the initial conditions $x = \psi(0, 1)$ and $\dot{x} = \mathbf{0}$, which excites only the mode with $\omega_0^-$. We assume slow evolution of $\phi$, i.e., $\Omega/\alpha \ll \Delta$, and then, the other mode with $\omega_0^+$ is expected to be unexcited during the time evolution. In order to factor out rapid oscillation, which corresponds to the dynamical contribution, new dynamical variables $x_c = \psi(x_c, y_c)$ and $x_s = \psi(x_s, y_s)$ are introduced as $x = x_c \cos \theta_t + x_s \sin \theta_t$ with $\theta_t = \int_0^t \omega_0^- dt'$, and the equation of motion is divided into the coefficients of $\cos \theta_t$ and $\sin \theta_t$.

The numerically obtained time evolution of $x_c$ and $x_s$ is plotted in Fig. 1. We find that at $t = T$, when the potential gets back to the original form, $x_c(T)$ is same as $x_c(0)$ for $\eta < 0.5$ while $x_c(T)$ is minus of $x_c(0)$ for $\eta > 0.5$. Since the rapid oscillation is already factored out, this means that the system acquires extra $\pi$ phase other than the dynamical contribution for $\eta > 0.5$. The physical origin of this extra $\pi$ phase is actually simple. Namely, $V_s(x, y)$ is constructed by superposing two func-
The variables $q$ and $p$ are independent, but in order to keep $q'$ and $p'$ real, they should fulfill the relation $(ip)^* = \gamma_q q'$. Here, in order to emphasize the classical nature of the phenomenon, we derive it in a purely classical fashion.

The action is defined as

\[ S(q, q_0, p_0, \lambda) = \int_{C(q, q_0, p_0, \lambda)} p \cdot dq, \]

where $C(q, q_0, p_0, \lambda)$ is a path in the phase space whose initial and final values of $q(t)$ are $q$ and $q_0$, and initial value of $p(t)$ is $p_0$, respectively. Explicitly, we have

\[ S(q, q_0, p_0, \lambda) = \int p \cdot dqdt = \sum_\alpha \log(Q_\alpha/Q_{0\alpha}) P_{0\alpha} Q_{0\alpha} \]

with $Q = \hat{O} q$ and $P = p \hat{O}^\dagger$ where $\hat{O}$ is a matrix diagonalizing $\hat{\gamma}_\lambda$. Namely, $\hat{\gamma}_\lambda = \hat{O}_\lambda \hat{\omega}_\lambda \hat{O}_\lambda^\dagger$ where $\hat{\omega}_\lambda$ is diagonal with $\omega_\alpha$ as diagonal elements. Now, the action variables $I_\alpha$ are introduced as $I_\alpha = i\oint_0^{C_\alpha} q_0 dq$. By definition, $2\pi I_\alpha$ represents increment of $S$ against completion of one period of the normal mode with $\omega_\alpha$. Using $I$, we can express $S$ as a function of $q, q_0, I$, and $\lambda$. Then, we consider a canonical transformation taking $S$ as a generation function.

Since $I_\alpha$ is an adiabatic invariant, $I_\alpha$ is an ideal quantity as new momentum. The new coordinate conjugate to $I_\alpha$ is

\[ \varpi_\alpha = \frac{\partial S(q, q_0, p_0, \lambda)}{\partial I_\alpha} = -i\log(Q_\alpha/Q_{0\alpha}) = \omega_\alpha t, \]

which is named as the angle variable. From this, the motion of mass point is obviously periodic in $\varpi_\alpha$ with period $2\pi$. Therefore, difference in $\varpi_\alpha$ indicates difference in phase of oscillating motion.

Next, we consider the case that $\lambda$ weakly depends on time. Then, the generation function $S$ depends on time through the time dependence of $\lambda$, and the Hamiltonian is transformed as $H \to H + (\partial S/\partial \lambda) \lambda$. Using the transformed Hamiltonian, the equation of motion for $\varpi_\alpha$ becomes $\ddot{\varpi}_\alpha = \dot{H}/\partial I_\alpha + \partial^2 S(q, I, \lambda)/\partial I_\alpha^2 \partial \lambda$ and we formally obtain

\[ \varpi_\alpha = \int \frac{\partial H}{\partial I_\alpha} dt + \int \frac{\partial^2 S(q, I, \lambda)}{\partial I_\alpha \partial \lambda} \frac{d\lambda}{dt} dt. \]

The first term represents the dynamical contribution, whereas the second term represents the geometrical contribution, which is the Hannay angle $\theta_\alpha^{(H)}$. Following the Berry’s argument [22], $\theta_\alpha^{(H)}$ is evaluated as

\[ \theta_\alpha^{(H)} = -\frac{\partial}{\partial I_\alpha} \int p \cdot \frac{dq}{\partial \lambda} d\lambda, \]

for the case that $\lambda$ gets back to the original value after evolution. In practice, we should average out rapid oscillations to have a convenient formula. The best method for the averaging is applying integration over $\varpi_\alpha$ in the

FIG. 1. Upper panel: Contour plots of the potential $V_\phi(x, y)$ for $\eta = 0.4$ and 0.6. Lower panels: Time evolution of $x_\phi$ and $x_\eta$ with $\alpha = 2\pi$, $\Delta = 0.5$, and $T = 1200$ for selected $\eta$. 

\[ v_1(x, y) = c_1 x^2 + c_2 x y, \quad v_2(x, y) = c_2 x^2 + c_2 y^2 \]

with appropriate $c_{ij}$, and the evolution from $\phi = 0$ to $\phi = 2\pi$ is achieved by relatively rotating $v_1(x, y)$ against $v_1(x, y)$ by $180^\circ$, instead of $360^\circ$. When $v_2(x, y)$ is dominant, $180^\circ$ rotation results in a flip of $x_\phi$ after completion of one period, while when $v_1(x, y)$ is dominant, such a flip does not take place.
of two kinds of interpretations corresponds to the gauge phase factor of \( \hat{A} \). This leads to a complex phase factor, typically \( e^{i\phi/2} \). Such a phase factor leads to \( \hat{A} \) continuous and periodic in \( \phi \) with period \( 2\pi \). Namely, if we force \( \hat{A} \) to be continuous in \( \phi \), we have \( \hat{A}_{2\pi \phi} = -\hat{A}_{0\phi} \), which explains the flip of \( \phi \).

On the other hand, it is possible to amend \( \hat{A} \) so that it becomes continuous and periodic by introducing a complex phase factor, typically \( e^{i\phi/2} \). This phase factor leads to \( \hat{A}_{2\pi \phi} = \hat{A}_{0\phi} \), but then, Eq. (9) gives an extra phase factor of \( \pi \), which explains the flip of \( \phi \). Possibility of two kinds of interpretations corresponds to the gauge degrees of freedom in quantum cases.

This argument indicates that \( \theta^{(H)} \) should be 0 or \( \pi \), since \( \hat{A}_{2\pi \phi} = \pm \hat{A}_{0\phi} \) is the only possibility with the gauge choice in which \( \hat{A} \) becomes real. However, \( \theta^{(H)} \) deviates from 0 or \( \pi \) if the starting Hamiltonian is changed to break the time reversal symmetry. As an example, let us consider the situation that the system is rotated with angular frequency \( \Omega_R \). Then, the Hamiltonian written in the rotating frame is obtained by replacing \( \textbf{p}' \) by \( \textbf{p}' + i\Omega_R \gamma \textbf{q}' \).

We follow the time evolution of \( x_c \) and \( x_s \) introduced as \( \textbf{q}' = x_c \cos \theta_1 + x_s \sin \theta_1 \), but with modified \( \omega^\pm_0 = \alpha \sqrt{1 + \beta^2 \pm \sqrt{\Delta^2(1 - 2\eta(1 - \eta)(1 - \cos \phi)) + 4\beta^2}} \) where \( \beta = \Omega_R/\alpha \). We use the initial condition \( x = (0, 1) \) and \( \dot{x} = (v_0, 0) \) with \( v_0 = -\beta(1 + \beta^2 - \sqrt{\Delta^2 + 4\beta^2})/(\Delta - 2\beta^2 + \sqrt{\Delta^2 + 4\beta^2}) \) in order to excite only the mode for \( \omega^+_0 \).

The numerically obtained \( x_c(T) \) for \( \eta = 0.4 \) and \( \eta = 0.6 \) are plotted in Figs. 2(a) and 2(b). In contrast to Fig. 1 where \( x_c \) and \( y_s \) are negligible for all time, \( x_s \) and \( y_s \) have significant time dependence. Therefore, we have \( x_c(T) \neq \pm x_c(0) \), which indicates the extra phase factor other than 0 or \( \pi \). With our choice of the initial condition, the extra phase factor \( \hat{\theta} \) corresponding to \( \theta^{(H)} \) is evaluated as \( \hat{\theta} = \arccos y_s(T) \), and plotted in Fig. 2 as a function of \( \beta = \Omega_R/\alpha \). We see that \( \hat{\theta} \) clearly deviates from 0 and \( \pi \) for finite \( \Omega_R \). Now, it is desirable to find out a transformation mapping the rotated Hamiltonian to the Hamiltonian like Eq. (2). However, a naive extension of our transformation by \( F(q', q) \) is only applicable in the case that \( [\sigma_y, \Gamma_{\phi}] = 0 \), which limits us to the trivial case with \( \Gamma_{\phi} \propto 1 \). A possible analytical formulation for the rotated case is a future problem.

For the demonstration of the usage of the Hannay angle as a symmetry protected topological order parameter, we introduce a one-dimensional “dimerized” spring-mass model schematically depicted in Fig. 3(a). The involved parameters are mass of the mass points \( m = 1 \), frequencies associated with sublattice dependent harmonic potential \( \omega_{0\pm} \), and spring constants \( k_1 \) and \( k_2 \). For later convenience, we introduce parameters \( \omega_0, \omega_1, \omega_2, \eta \) as \( \omega^\pm_0 = \omega_0^2 \pm \omega_2 \), \( k_1 = \eta\omega^2_1 \) and \( k_2 = (1-\eta)\omega^2_1 \). We limit the motion of mass points along the 1D chain, then, the dynamical variable for this system is deviation of each mass point from the equilibrium position, which is written as \( x_{na} \) for the mass point at \( n \)th unit cell and sublattice \( a \). Assuming the periodic boundary condition and introducing new variables \( f_{ka} \) as \( x_{na} = \sum_k e^{-i\eta f_{ka}} \), the equation to be solved becomes \( \hat{H}_k = -\hat{H}_k \) where \( f_k = (f_{k1}, f_{k2}) \), and

\[
\hat{H}_k = (\omega^2_0 + \omega^2_1)\hat{1} - \omega^2_1 \text{Re}g_k \hat{\sigma}_1 - \omega^2_1 \text{Im}g_k \hat{\sigma}_2 + \omega^2_0 \hat{\sigma}_3 \quad (10)
\]

with \( g_k = (1-\eta) + \eta e^{ik} \). Now, \( \hat{H}_k \) contains complex elements, but it is straightforward to extend the former arguments for the case that \( \Gamma_k \) is real symmetric. For fixed \( k \), there exist two eigenmodes with frequencies \( \omega^\pm_k(k) = \sqrt{\omega^2_0 + \omega^2_1 \pm \sqrt{\omega^2_0 + \omega^2_1 |g(k)|^2}} \). When \( \omega_2 = 0 \), \( \hat{H}_k \) does not contain \( \hat{\sigma}_3 \) and then, we say that the system has a “chiral symmetry” since \( \hat{H}_k = (\omega^2_0 + \omega^2_1)\hat{1} \) anticommutes with \( \hat{\sigma}_3 \). Although it is not the chiral symmetry in a strict sense because of the constant diagonal part, this symmetry is important in topological characterization. Furthermore, for \( \omega_2 = 0 \), it is possible to map Eq. (10) to Eq. (1) by a parameter independent transformation such that \( \hat{\sigma}_1 \to \hat{\sigma}_3 \) and \( \hat{\sigma}_2 \to \hat{\sigma}_1 \), which indicates that the two problems are essentially identical to each other.
Next, we follow the time evolution regarding $k$ as a controllable parameter driven as $k = \Omega t$. More specifically, we solve

$$\frac{d^2 \mathbf{f}(t)}{dt^2} = -\hat{H}_{\Omega t} \mathbf{f}(t),$$

with the initial condition $\mathbf{f}(0) = \mathbf{f}_0 = \{1/\sqrt{2}, 1/\sqrt{2}\}$ and $\dot{\mathbf{f}}(0) = 0$, which excites only the lower branch with $\omega_2(0)$ at $t = 0$. In order to factor out the rapid oscillation, i.e., the dynamical contribution to $\varpi_\alpha$, new variables $\mathbf{u}_\pm(t)$ are introduced as $\mathbf{f}(t) = e^{i\vartheta_\pm} \mathbf{u}_\pm(t) + e^{-i\vartheta_\pm} \mathbf{u}_\mp(t)$ with $\vartheta_\pm = \int_0^t \omega_\alpha(t) \, dt$, and we solve equations for $\mathbf{u}_\pm(t)$. The initial condition $(\mathbf{f}(0), \dot{\mathbf{f}}(0)) = (\mathbf{f}_0, 0)$ is satisfied by the choice of $\mathbf{u}_\pm(0) = \mathbf{f}_0/2$ and $\dot{\mathbf{u}}_\pm(0) = 0$. Figure 2(b) shows the $\omega_2$ dependence of $\tilde{\theta} = \arg((\mathbf{u}_\pm(0) \cdot \mathbf{u}_\mp(T))$ for several $\eta$. Since the dynamical contribution is factored out, $\tilde{\theta}$ captures the geometrical contribution. For $\omega_2 = 0$, since $H_k$ can be mapped to $\Gamma_{\phi}/2$, $\tilde{\theta} = 0$ for $\eta < 0.5$, while $\tilde{\theta} = \pi$ for $\eta > 0.5$. On the other hand, for the finite $\omega_2$, which breaks the “chiral symmetry”, $\tilde{\theta}$ deviates from 0 or $\pi$. In short, the quantization of the Hannay angle into 0 or $\pi$ is protected for the “chiral symmetry”. Due to Eq. (9), it is natural that the quantization condition for the Hannay angle is the same as for the Berry phase.

The bulk-edge correspondence is useful in connecting the quantized Hannay angle and the topological character of the system in terms of edge states. In Fig. 3(c), the frequency spectrum for the finite length system having the fixed boundary with $\omega_2 = 0$ is plotted as a function of $\eta$. There is a clear signature of in-gap edge state distinct from the bulk contribution for $\eta > 0.5$ where $\tilde{\theta} = \pi$, while there is no sign of edge state for $\eta < 0.5$ where $\tilde{\theta} = 0$. Therefore, the quantized Hannay angle is used to detect the topological transition characterized by the appearance of localized edge modes. Here, we should be careful on the unit cell convention to establish the relation between the Hannay angle and edge states, since the different unit cell conventions lead to the modified Hamiltonian, which affects the Hannay angle $\tilde{\theta}$. In specific, the convention such that the boundary is in between the two neighboring unit cells should be employed.

Figure 3(b) shows the frequency spectrum obtained with the free boundary condition. In contrast to the fixed boundary case, there is no sign of in-gap edge state irrespective of the value of $\eta$. As discussed in the similar situation [24], this contrast is actually the manifestation of the symmetry protection of the topological states. Recall that the quantization of the Hannay angle is protected by the “chiral symmetry”. For the fixed boundary condition, the “chiral symmetry” is preserved even after the boundary is introduced, since the diagonal elements of $\hat{H}$ are uniform even with the boundary. On the other hand, the free boundary breaks the “chiral symmetry”, since the diagonal elements of $\hat{H}$ becomes nonuniform due to the absence of a spring at the free boundary [24]. Then, the edge mode is not necessarily protected because of the symmetry breaking, even if the bulk parts share the same topological character for the two kinds of boundary conditions.

To summarize, we derive a concise formula for the geometrical angle, i.e., the Hannay angle, in mechanical system by making use of the canonical transformation that maps the Newtonian equation of motion to the Schrödinger type equation. Then, the use of the Hannay angle as a symmetry protected topological order parameter is demonstrated using a simple spring-mass model. The importance of the symmetry in topological characterization is pointed out by analyzing the boundary condition dependence of the edge modes. It reveals yet another role of the symmetry in mechanical systems in relation to the bulk-edge correspondence.

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Appendix: The Case with Hermitian $\hat{\Gamma}$

Let us start with the Hamiltonian

$$H = p_i^\dagger p_i + \bar{q}_i \Gamma_{ij} q_j,$$

(12)
where $\hat{\Gamma}$ is an Hermitian matrix. Now, we consider a canonical transformation using a generation function

$$F(q_i', \dot{q}_i', q_i, \dot{q}_i)$$

$$= -i\dot{q}_i'\gamma_{ij}q_j + i\sqrt{2}\dot{q}_i\gamma_{ij}q_j' + i\sqrt{2}\dot{q}_i\gamma_{ij}q_j - i\dot{q}_i\gamma_{ij}q_j.$$  \hspace{1cm} (13)

Here $\dot{\gamma} = \sqrt{\hat{\Gamma}}$ and $\gamma$ is hermitian under the condition that $\hat{\Gamma}$ is positive semidefinite. In the following, we assume this condition is fulfilled. Then, we have

$$p_i' = \frac{\partial F}{\partial q_i'} = -i\dot{q}_j'\gamma_{ji} + i\sqrt{2}\dot{q}_j\gamma_{ji},$$  \hspace{1cm} (14)

$$\dot{p}_i' = \frac{\partial F}{\partial q_i'} = -i\gamma_{ij}\dot{q}_j' + i\sqrt{2}\gamma_{ij}q_j,$$  \hspace{1cm} (15)

$$p_i = -\frac{\partial F}{\partial q_i} = -i\sqrt{2}\gamma_{ij}q_j' + i\dot{q}_j\gamma_{ji},$$  \hspace{1cm} (16)

$$\dot{p}_i = -\frac{\partial F}{\partial q_i} = -i\sqrt{2}\gamma_{ij}q_j' + i\dot{q}_j\gamma_{ji},$$  \hspace{1cm} (17)

which gives us

$$i\dot{p}_i' = \frac{1}{\sqrt{2}}(ip_i - \dot{q}_j\gamma_{ji}), \hspace{1cm} i\dot{p}_i' = \frac{1}{\sqrt{2}}(ip_i - \gamma_{ij}q_j),$$

$$\gamma_{ij}q_j' = \frac{1}{\sqrt{2}}(\gamma_{ij}q_j + ip_i), \hspace{1cm} \dot{q}_j\gamma_{ji} = \frac{1}{\sqrt{2}}(q_j\gamma_{ji} + ip_i).$$  \hspace{1cm} (18)

and

$$ip_i = \frac{1}{\sqrt{2}}(ip_i' + \dot{q}_j'\gamma_{ji}), \hspace{1cm} \dot{p}_i = \frac{1}{\sqrt{2}}(ip_i' + \gamma_{ij}q_j'),$$

$$\gamma_{ij}q_j = \frac{1}{\sqrt{2}}(\gamma_{ij}q_j' - ip_i'), \hspace{1cm} \dot{q}_j\gamma_{ji} = \frac{1}{\sqrt{2}}(q_j\gamma_{ji} - ip_i').$$  \hspace{1cm} (20)

The Hamiltonian is transformed as

$$H = -\frac{1}{2}(ip_i' - \dot{q}_j\gamma_{ji})(ip_i - \gamma_{ij}q_j)$$

$$+ \frac{1}{2}(ip_i' + \dot{q}_j'\gamma_{ji})(ip_i + \gamma_{ij}q_j)$$

$$= ip_i\gamma_{ij}q_j + i\dot{q}_i\gamma_{ij}\dot{p}_j.$$  \hspace{1cm} (19)

Then, the equations of motion are given as

$$\ddot{q}_i = i\gamma_{ij}q_j, \hspace{1cm} \ddot{q}_i = i\dot{q}_j\gamma_{ji},$$  \hspace{1cm} (23)

$$\ddot{p}_i = -ip_j\gamma_{ji}, \hspace{1cm} \ddot{p}_i = -i\gamma_{ij}\dot{p}_j.$$  \hspace{1cm} (24)

If we choose an initial condition such that $(\dot{q}_i')^* = q_i'$ and $(\dot{p}_i')^* = \dot{p}_i'$, it is mapped to an initial condition such that $(ip_i')^* = q_j\dot{q}_j$ and $(ip_i')^* = \dot{q}_j\gamma_{ji}$. Since the equation of motion implies that $ip_i$ and $\dot{q}_j\gamma_{ji}$, and $ip_i$ and $\dot{q}_j\gamma_{ji}$ obey conjugate equations, respectively, such relationships are preserved in time evolution. On the other hand, the relations $(ip_i')^* = q_j\dot{q}_j$ and $(ip_i')^* = \dot{q}_j\gamma_{ji}$ imply $(q_i')^* = q_i'$ and $(\dot{p}_i')^* = \dot{p}_i'$.