SOME REMARKS ON GRAVITY IN NONCOMMUTATIVE SPACETIME AND A NEW SOLUTION TO THE STRUCTURE EQUATIONS

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Abstract

In this paper, starting from the common foundation of Connes’ noncommutative geometry (NCG)[1, 2, 3, 4], various possible alternatives in the formulation of a theory of gravity in noncommutative spacetime are discussed in detail. The diversity in the final physical content of the theory is shown to be the consequence of the arbitrariness in each construction steps. As an alternative in the last step, when the structure equations are to be solved, a minimal set of contraints on the torsion and connection is found to determine all the geometric notions in terms of metric. In the Connes-Lott model of noncommutative spacetime, in order to keep the full spectrum of the discretized Kaluza-Klein theory [5], it is necessary to include the torsion in the generalized Einstein-Hilbert-Cartan action.

PACS. 04.20.Jb, 04.40. +c, 11.15. -q, 14.80.Hv

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1 Introduction

More than ten years after A.Connes [1, 2, 4] suggested the idea of using NCG in Physics, the construction of a physical theory of gravity in noncommutative spacetime has not reached a completely satisfactory state. The basic ideas of generalizing Riemannian Geometry in noncommutative context have been presented in [3, 6, 8, 9]. It was Chamseddine and Fröhlich who initiated the first real effort to construct an extended Einstein theory of gravity utilizing the noncommutative geometric model of Connes-Lott’s two-sheeted spacetime. Following this pioneering paper, a great number of other papers have appeared, containing aspects of this interesting topic. However, in these works, the physical content of the final action, where the physical fields make their appearance is not unique.

Many papers have agreed with the field content of the standard Einstein metric field along with a Brans-Dicke scalar, which represents the distance between pairs of points on the two separate sheets. However, in several other works [21], with a different formulation of the action, the scalar field does not have a kinetic term, so it would not be present in the final theory as a physical field. Some alternative constructions suggest that the theory might have a Kaluza-Klein spectrum, which contains massless and massive tensor, vector and scalar fields in a harmonic expansion in the discrete internal spacetime.

The reason for this diversity is due to the fact that although most papers have accepted Connes’ construction as the common foundation, a unique recipe to build the final physical theory is still lacking. With Connes’ spectral triple given, there is still considerable arbitrariness at every stage of the construction of the final action that leads to different final results.

The purpose of this paper is to point out the alternative choices at various stages and find a new minimal set of constraints on the torsion and connection to complete the Cartan structure equations. This set is “minimal” in the sense that it does not place any further
constraints on the metric structure. Finally, based on some physical considerations, a new solution to the structure equations is found leading to the final action, where both torsion and curvature contribute a satisfactory action.

2 The common foundation: Connes’ formulation

There is extensive literature about Connes’ general formalism of NCG. Here, for the sake of completeness, we shall begin with a brief review of the relevant ideas that are necessary for this paper. [1, 2].

In Connes’ formulation the basic building block of noncommutative geometry is the spectral triple \((\mathcal{A}, D, \mathcal{H})\):

The algebra \(\mathcal{A}\) of the 0-forms replaces the functions in a generalized Gelfand’s construction [1]. The replacement of the number fields by the algebra \(\mathcal{A}\) leads to the concept of \(\mathcal{A}\)-bimodule [1] as a generalization of the vector space.

The Dirac operator \(D\) acts on \(F \in \mathcal{A}\) giving a subset of the \(\mathcal{A}\)-bimodule of 1-forms. Therefore, \(DF\) can be formally written as

\[
DF = \sum_M (DF^M)(DF)^M, \quad M \in \alpha, F^M \in \mathcal{A}
\]

where \(\alpha\) is a finite set. That is to say, the Eqn.(1) postulates the existence of a finite basis \((DF^M)\) of the set \(\Omega^1(\mathcal{A})\) of 1-forms. In other words, \(\Omega^1(\mathcal{A})\) has an algebraic structure of finite projective \(\mathcal{A}\)-bimodule. The elements \((DF^M)\) are also given by the Dirac operator acting on \(F^M \in \mathcal{A}\).

The non-commutativity of the theory does not necessarily come from the non-commutativity of the algebra \(\mathcal{A}\). For example, the Connes-Lott two-sheeted non-commutative space-time model [4] is essentially based on the commutative algebra \(C^\infty(C, \mathcal{M}) \oplus C^\infty(C, \mathcal{M})\), where \(\mathcal{M}\) is the usual physical spacetime. It is the Dirac op-
erator as an outer automorphism that brings about the noncommutativity. Outer automorphism is the property of an operator, whose action on an element gives an element outside of the initial domain of the elements.

It is possible to extend the definition of $D$ onto the bimodule of 1-forms $\Omega^1(A)$ and define the product of 1-forms in order to build the $A$-bimodule $\Omega^2(A)$ of 2-forms. This procedure can be repeated to construct the universal algebra $\Omega^*(A) \equiv \bigoplus_p \Omega^p(A)$ of differential forms on the algebra $A$ with the Dirac operator $D$. An involutive operation on $A$ can be extended uniquely to the one on the algebra $\Omega^*(A)$.

The universal envelope algebra $\Omega^*(A)$ has a graded structure with the Dirac operator $D$ that takes

$$D : \Omega^p \rightarrow \Omega^{p+1}$$

$$D((DF_1)\ldots(DF_p)F) \equiv (-1)^{-1}(DF_1)\ldots(DF_p)(DF), \quad \forall F, F_1, \ldots, F_p \in \Omega^p(A), \quad (2)$$

which implies

$$D^2 F = 0, \quad \forall p, F \in \Omega^*(A)$$

$$D(F_1F_2) = (DF_1)F_2 + (-1)^{degF_1}F_1(DF_2), \quad \forall F_1 \in \Omega^*(A) \quad (3)$$

The Hilbert space $\mathcal{H}$ is where the elements of $A$, differential forms and exterior derivative act as operators. In the Connes-Lott’s model, it is chosen as the direct sum of the Hilbert spaces of left-handed and right-handed spinors $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$.

The representation of the operators corresponding to differential forms on a given Hilbert space is realized by the graded, involution preserving homomorphism $\pi$,

$$\pi_{p}((DF_1)\ldots(DF_p)F) = \Pi_{i=1}^{p}[D, \pi_0(F_i)]\pi_0(F), \quad (4)$$

where $\mathcal{L}(\mathcal{H})$ denotes the space of bounded operators on the Hilbert space $\mathcal{H}$ and $\pi_0$ is a representation of the algebra $A$ on $\mathcal{H}$. Henceforth, the symbol $D$ will denote both the Dirac operator $D$ and its representation as a by a self-adjoint operator. It is assumed that $D$ has a compact resolvent, such that the commutator $[D, \pi_0(F)]$ is also a bounded operator $\forall F \in A$ [1].
It is possible to choose a finite basis \((DF^M)\) as in Eqn.(1) and represent it by \(\Theta^M = \pi_1((DF^M))\). Therefore, an arbitrary 1-form \(\omega \in \Omega^1(\mathcal{A})\) is represented as

\[
U = \pi_1(\omega) = \Theta^M U_M. \tag{5}
\]

The Dirac operator is represented in the “general” basis \(\Theta^M\) as

\[
D = \Theta^M D^M \equiv D_M F = [D, \pi_0(F_M)]. \tag{6}
\]

With the differential forms in hand, one can follow the standard procedure to introduce a metric structure, the connection 1-forms, the torsion and, the curvature 2-forms to construct a theory of gravity. The Cartan’s structure equations can be formulated and solved with some additional constraints to express all the geometric quantities in terms of the metric. The physical Einstein-Hilbert-Cartan action can be built from the curvature and torsion.

The formal procedure is rather well defined. Nevertheless, in given specific representation, there are still alternate steps that lead to diverse physical contents of the final theory of gravity. In the next section, we will follow the steps in formulating the theory by and point out the alternatives.

### 3 Alternatives in the theory construction

#### 3.1 The choice of the involutive operation and subalgebra

In the Connes-Lott model [4], which concerns mainly with the Standard Model, the physical fields are not represented by the most general 1-forms. Instead, only the hermitian 1-forms are considered as physical. Therefore, the involutive operation as the hermitian conjugate \(*\) is introduced in the bimodule of 1-forms. In such a construction, the 0-forms continue to be complex, while only hermitian 1-forms are relevant for physical purposes.
The hermitian forms do not form a \( \mathcal{A} \)-bimodule, but rather as a bimodule of a subalgebra of \( \mathcal{A} \), which is the algebra of real functions \( \mathcal{B} = \mathcal{C}^\infty(R, \mathcal{M}) \oplus \mathcal{C}^\infty(R, \mathcal{M}) \).

It suggest that a subalgebra of \( \mathcal{A} \) maybe sufficient to construct a gauge theory with the same physical content.

In the forthcoming paper \([15]\), it is shown that, the Standard Model can be constructed in the context of NCG, if one chooses the subalgebra \( \mathcal{A}' \in \mathcal{A} \) of the 0-forms in the basic spectral triple as follows

\[
F = \begin{pmatrix} f(x) & 0 \\ 0 & f^*(x) \end{pmatrix}, \quad f(x) \in \mathcal{C}^\infty(C, \mathcal{M})
\]  

(7)

where \( f^*(x) \) is the complex conjugate of \( f(x) \).

The complex conjugate will be coincident with the involutive operation “\( \sim \)” used in a series of papers \([13, 5, 14, 15]\),

\[
\tilde{F} = F^* = \begin{pmatrix} f^*(x) & 0 \\ 0 & f(x) \end{pmatrix}
\]  

(8)

It is noting that the restriction to the subalgebra \( \mathcal{A}' \) is dicted by physics. In fact, one can start with the algebra \( \mathcal{A} \) with the involutive operation defined as follows

\[
F = \begin{pmatrix} f_1(x) & 0 \\ 0 & f_2(x) \end{pmatrix}, \quad \tilde{F} = \begin{pmatrix} f_2(x) & 0 \\ 0 & f_1(x) \end{pmatrix}, \quad f_1(x), f_2(x) \in \mathcal{C}^\infty(C, \mathcal{M}).
\]  

(9)

It is shown in \([15]\) that in order to have correct kinetic terms for all the fields, one must restrict all the 0-forms and all coefficients of higher differential forms subjected to the algebra \( \mathcal{A}' \). The universal envelope algebra \( \Omega^*(\mathcal{A}') \) constructed with the algebra \( \mathcal{A}' \) can be used to represent the physical fields without any further condition.

The example of the involution “\( \sim \)” has shown that, in general, it is possible to choose various alternative involution operations to have the desirable physical contents suitable for a given application. This alternative produces the same field contents for the gauge theory as the Connes-Lott model with a proper definition of the inner product.
3.2 Equivalence Principle

The foundation of General Relativity in noncommutative spacetime is based on the Equivalence Principle. As traditionally stated by Einstein, this principle postulates the existence of the general and the locally orthonormal frames, which can be transformed into each other by a local orthonormal invertible transformation. The general frame consists of any finite basis of the module of generalized 1-forms as formulated in Sect.2. The exterior of derivatives of forms are defined in this frame. Therefore, the differential calculus is always carried out in this frame and the results will be transformed into other frames if necessary. On the other hand, the local orthonormal frame has the advantage in the algebraic calculations of forms at the same location. Therefore, it is the most convenient basis for the formulation of the structure equations for the connection, torsion and curvature.

In general, let us denote the basis of the general frame as $\Theta^M$ and the basis of the local orthonormal frame as $\Theta^A$. The generalized vielbein is the spacetime dependent transformation:

$$
\Theta^M = \Theta^A E^M_A(x), \\
\Theta^A = \Theta^M E^M_M(x),
$$

(10)

where the 0-forms $E^A_M(x)$ and $E^M_A(x)$ are inverses of each other,

$$
E^A_M(x)E^A_N(x) = \delta^N_M, \\
E^M_A(x)E^M_B(x) = \delta^A_B.
$$

(11)

Since the derivatives are defined only in the “general” frame, the Dirac operator $D$ is represented in the local orthonormal basis as

$$
D = \Theta^A E^M_A(x) D_M.
$$

(12)

Indeed, the metric structure is encoded in the Dirac operator via the presence of the vielbein $E^M_A(x)$. The structure of $E^M_A(x)$ is where different choices are possible. In the most general case, each 0-form $E^A_M$ contains a pair of the usual functions.
For the sake of definiteness, let us take the example in the Connes-Lott’s two-sheeted spacetime model, which is generally referred to by most of the authors. In this model, the 0-forms and therefore the vielbeins $E^{A}_{M}(x)$ and $E^{M}_{A}(x)$ where $M = \mu, 5$ and $A = a, \dot{5}$ are represented as $2 \times 2$ matrices. Specially, in the most general form, the vielbeins are given as

$$E^{\mu}_{a}(x) \equiv \begin{pmatrix} e^{\mu}_{1a}(x) & 0 \\ 0 & e^{\mu}_{2a}(x) \end{pmatrix}, \quad E^{5}_{\dot{5}}(x) \equiv 0,$$

$$E^{5}_{a}(x) \equiv - \begin{pmatrix} a_{1a} & 0 \\ 0 & a_{2a} \end{pmatrix} \equiv -A_{a}(x) = -E^{\mu}_{a}A_{\mu},$$

$$E^{\mu}_{\dot{5}}(x) \equiv \begin{pmatrix} \phi^{-1}_{1}(x) & 0 \\ 0 & \phi^{-1}_{2}(x) \end{pmatrix} \equiv \Phi^{-1}(x),$$

(13)

where $e^{\mu}_{1,2a}(x)$ are two different vierbeins on the two sheets of space-time. Similarly, $a_{1,2}(x)$ and $\phi_{1,2}(x)$ are respectively vector and scalar fields. The vielbein $E^{\mu}_{5}(x)$ can always be chosen as zero because of the residue rotational arbitrariness.

The vielbeins $E^{M}_{A}$ are invertible as follows,

$$E^{a}_{\mu}(x) \equiv \begin{pmatrix} e^{a}_{1\mu}(x) & 0 \\ 0 & e^{a}_{2\mu}(x) \end{pmatrix}, \quad E^{a}_{\dot{5}}(x) \equiv 0,$$

$$E^{5}_{\mu}(x) \equiv A_{\mu}(x)\Phi(x), \quad E^{5}_{\dot{5}}(x) \equiv \begin{pmatrix} \phi_{1}(x) & 0 \\ 0 & \phi_{2}(x) \end{pmatrix} \equiv \Phi(x).$$

(14)

The Dirac operator is

$$D = \begin{pmatrix} \gamma^{a}e^{\mu}_{1a}\partial_{\mu} & -m\gamma^{a}e^{\mu}_{1a}a_{1\mu} + m\gamma^{5}\phi_{1} \\ \gamma^{a}e^{\mu}_{2a}a_{2\mu} - m\gamma^{5}\phi_{2} & \gamma^{a}e^{\mu}_{2a}\partial_{\mu} \end{pmatrix}$$

(15)

In [12], by imposing the constraints $e^{\mu}_{1a} = e^{\mu}_{2a}, a_{1\mu} = a_{2\mu}, \phi_{1} = \phi_{2}$, the zero-modes of Kaluza-Klein theory are obtained. In the other investigations [7, 18, 20] the vector field is assumed to vanish.

At this point, one can raise the question whether these restrictions are internally required by the theory or just an arbitrary choice.
Since Eqns. (13), (14) and (15) cannot give any restriction, one must look for further possibilities.

The metric is given by a definition of the inner product, which is a sesquilinear functional

\[
<\mathcal{A}, \mathcal{B}> : \Omega^1(\mathcal{A}) \times \Omega^1(\mathcal{A}) \to \mathcal{A} \\
<UF, VG> = \tilde{F} <U, V>G.
\]

(16)

In particular, one obtains

\[
G_{MN}(x) = <\Theta^M, \Theta^N> = \tilde{E}^M_A(x)\eta^{AB}E^N_B(x),
\]

directly from the othonormality of the frame $\Theta^A$ in the following form

\[
G^{AB}(x) = <\Theta^A, \Theta^B> = \eta^{AB} = \tilde{E}^A_M(x)G^{MN}E^B_M(x),
\]

(17)

(18)

From Eqn.(11), that defines the inverse vielbein, one can see that the equations (17) and (18) are in fact equivalent. As these equations define the metric tensor in terms of vielbein, one can conclude that they do not give any further restriction on the vielbein.

In [20] an argument is made to show that the vector field is related to a gauge transformation that emerges as an internal shift in an analogy with the internal circle of the ordinary Kaluza-Klein theory. However, the internal space of this model consists of only two-points. Hence, one cannot speak about the $U(1)$ gauge transformation in relation to the shift in the internal circle. The gauge invariance must be guaranteed by a different motivation.

As proven in [13], the “zero-mode only” constraint of [12] is a special case of the torsion free condition when one solve the structure equation. There exist various ways to impose conditions in order to solve the structure equations. Those will be discussed later.

One might also argue that in a $\Gamma$-representation, in which the bases $\Theta^M$ and $\Theta^A$ are represented as $\Gamma^M$ and $\Gamma^A$, the trace of these matrices might lead to a restriction on the vector field.
Let us consider this possibility, starting from the basis

\[ \Gamma^a = \gamma^a \otimes 1, \Gamma^5 = \gamma^5 \otimes \sigma, \]  

where

\[ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]  

In this choice of basis, the “general” \( \Gamma^M \) matrices are

\[ \Gamma^\mu = \begin{pmatrix} \gamma^a e^\mu_{1a}(x) & 0 \\ 0 & \gamma^a e^\mu_{2a}(x) \end{pmatrix}, \]  
\[ \Gamma^5 = \begin{pmatrix} -\gamma^a e^\mu_{1a} a_{1\mu} & \gamma^5 \phi_2^{-1}(x) \\ \gamma^5 \phi_1^{-1}(x) & -\gamma^a e^\mu_{2a} a_{2\mu} \end{pmatrix}. \]  

All the trace formulae of the \( \Gamma \) matrices are consistent with the metric tensors defined in Eqns.(17) and (18). For example, the trace

\[ G^{\mu 5} = Tr(\Gamma^\mu \Gamma^5) = \tilde{E}_A^\mu \eta^{AB} E_B^5 = \tilde{E}_a^\mu \eta^{ab} E_b^5 \]  

\[ = \begin{pmatrix} -\epsilon_{1a} \eta^{ab} e_{1b}^\nu a_{1\mu} & 0 \\ 0 & -\epsilon_{2a} \eta^{ab} e_{1b}^\nu a_{2\mu} \end{pmatrix} \]  

gives the vanishing vector field only if one requires that \( G^{\mu 5} \) vanishes. However, this happens only in specific cases such as in the flat spacetime. In the general frame, in fact, one can always choose the basis so that the metric \( G^{\mu 5} = 0 \). However, with such a choice, the tensor \( G^{AB} \) will not be constant any more.

The last possibility to have a constraint on the vielbein is to see whether all its components have kinetic terms in the final action. Without kinetic terms, these fields will not survive as physical fields. If a field content in parallel with the Kaluza-Klein spectrum is desirable, it will be shown that, one can choose a set of minimal conditions in such a way that maximal number of fields will survive.

### 3.3 Alternative definitions of two-forms

In order to formulate the torsion and curvature in the structure equations, one must define the \( \mathcal{A} \)-bimodule of 2-forms. This module should not contain the “junk forms” (the
non-vanishing forms that are differentials of the forms that are identical to zero in the \( \pi \) representation). As an illustration of a “junk form” in the usual differential geometry, let us consider the 1-form \( \omega = df.f - f.df \), which is identical to zero. However, the “junk form” \( d\omega = -2df.df \) is non-vanishing. To eliminate these “junk forms”, one defines wedge products and replaces the ordinary product with the wedge product to construct 2-forms as products or exterior derivative of 1-forms.

In NCG, one can also follow the technique of using auxiliary fields \([1, 8]\) to find the general form of the 2-forms, which are not “junk forms”. On the other hand, in the case of Connes-Lott model, one can successfully define the wedge product, which in fact eliminates the “junk forms” \([5, 12, 13, 16, 17]\).

The definition of the wedge product is not unique. In fact, one can choose a wedge product that is fully anti-symmetric as in \([5, 12, 13]\) for the theories of pure gravity. However, in order to have a quartic Higgs potential for the gauge fields one must choose a wedge product, where \( \Gamma^5 \wedge \Gamma^5 \) is not zero. This arbitrariness might result in additional terms in the final action.

Alternatively, in \([16, 17]\) the wedge product is chosen as being fully anti-symmetric, but one can still produce a quartic Higgs potential, since the components of forms related to the internal space is characterized by two complex quantities, which are conjugates of each other, instead of one.

### 3.4 Alternative sets of constraints

The torsion and curvature of Riemannian geometry can be generalized via the structure equations in the local orthonormal frame as follows

\[
T^A = D\Theta^A - \Theta^B \Omega^A_B
\]  

(23)
and

\[ R^A_B = D\Omega^A_B + \Omega^A_C\Omega^C_B, \quad (24) \]

where \( \Omega^A_B \) are the connection 1-forms. The structure equations are not sufficient to determine the connection, torsion and curvature in terms of metric fields. Therefore, some additional constraints on the torsion and connection must be imposed.

In ordinary Riemannian geometry, the torsion free condition together with the metric compatibility equation completely determines the connection and curvature in terms of the metric structure. In NCG, the torsion free condition \( T^A = 0 \) has been shown to lead to the following restriction on the metric structure [13]

\[
\begin{align*}
\epsilon_{1a}^\mu(x) &= \beta(x)\epsilon_{2a}^\mu(x), \\
a_{1\mu}(x) &= \beta(x)a_{2\mu}(x), \\
\phi_1(x) &= \beta(x)\phi_2(x),
\end{align*}
\quad (25)
\]

where \( \beta(x) \) is a dilaton field with a highly non-linear potential. In the special case of this restriction, where \( \beta(x) \equiv 1 \) one obtains the theory in [12] containing only the massless modes of the truncated Kaluza-Klein theory with the vierbein, vector and Brans-Dicke scalar fields.

Since a theory with the maximal spectrum would be interesting for broader applications, it has been our purpose to find a minimal set of constraints, which does not impose any restriction on the metric. In [5], the following set has been found

\[
\begin{align*}
T_{abc} &= T_{ab5} = T_{a5b} = 0 \\
Tr(T_{5AB}) &= 0,
\end{align*}
\quad (26)
\]

together with the metric compatible condition

\[
\begin{align*}
\Omega_{AB} &= -\Omega_{BA} \\
Tr(\Omega_{ABr}) &= 0.
\end{align*}
\quad (27)
\]

The set of constraints (26) and (27) are direct generalizations of the spacetime torsion free and the metric compatibility conditions. The additional constraints look rather unnatural. However, it is the first model with the spectrum, which is consistent with the intuitive
discretized Kaluza-Klein theory. When the internal space is discretized to just two points, a harmonic expansion in the internal dimension would give gravity, vector and scalar fields in pairs with their massive excitations. In fact, all the vielbein components in the equation (13) survive in the curvature with their appropriate kinetic terms.

In the next section, we show that it is possible to find another minimal set of constraints. This illustrates the arbitrariness in choosing alternative constraints.

3.5 A formula for final action

With a proper set of constraints on the connection 1-forms and torsion 2-form, one can solve the first structure equation (23) to express all the components of the connections and torsion completely in terms of vielbeins. Then, the curvature is determined in terms of the connections from the second structure equation (24).

With all the geometric notions in hand, there is still an arbitrariness in the formula of the final action. One may choose the Wodzicki residue, which gives no kinetic term for the scalar field [21]. With the same vielbein, the Dixmier trace formula, on the other hand, contains the kinetic term for the scalar field [7, 18]. Therefore, the decision, whether the scalar field can exist or not in the theory, depends on the choice of the definition of action. Perhaps, using the Wodzicki residue, one might have to include more characteristics into the action to retain the scalar field in the final theory.

The solution to be found in this paper shows a similar situation: the contributions from the curvature do not contain kinetic terms for the vector and scalar fields. To retain these fields in the theory, one must include the quadratic term of the torsion.

It is important to note that in the formula of action that contains the inner product, the definition of the involutive operation may also alter the action.
4 New constraints and new solution

In order to obtain the ordinary gravity, any set of constraints must be an extension of the ordinary metric compatibility and torsion free conditions.

Let us discuss the metric compatibility condition first. The simplest way is to generalize the metric compatibility condition as

$$\Omega_{AB} = -\Omega_{BA}. \quad (28)$$

Our previous work [5, 12, 13] uses the same form of the metric compatibility together the hermitian conjungate and a reality condition. In this work, the metric compatibility condition is consistent with the involutive operation “\(\sim\)”, which is originally defined on 0-forms, is just the identity on 1-forms.

The torsion free condition \(T_A = 0\) has been proven to lead to restriction on the vielbein [12, 13]. Therefore, we should look for a weaker condition. The most direct generalization of the torsion free condition is

$$T_a = 0. \quad (29)$$

To see whether the metric compatibility and torsion free conditions in the equations (28) and (29) are enough to solve the first structure equation, let us write its components as

\[
\begin{align*}
T_{abc} &= (D\Gamma_a)_{bc} + \frac{1}{2}(\Omega_{abc} - \Omega_{acb}) \\
T_{\dot{a}\dot{b}b} &= (D\Gamma_{\dot{a}})_{\dot{b}b} + \frac{1}{2}(\Omega_{\dot{a}\dot{b}b} - \Omega_{\dot{b}\dot{a}b}) \\
T_{\dot{a}ab} &= (D\Gamma_{\dot{a}})_{ab} + \frac{1}{2}(\Omega_{\dot{a}ab} - \Omega_{\dot{b}\dot{a}b}) \\
T_{\dot{a}\dot{b}\dot{b}} &= (D\Gamma_{\dot{a}})_{\dot{b}\dot{b}} + \frac{1}{2}(\Omega_{\dot{a}\dot{b}\dot{b}} - \Omega_{\dot{b}\dot{a}\dot{b}}) \\
T_{a\dot{a}\dot{b}} &= (D\Gamma_{a})_{\dot{a}\dot{b}} + \frac{1}{2}(\Omega_{a\dot{a}\dot{b}} - \Omega_{\dot{a}a\dot{b}}) \\
T_{a\dot{a}\dot{5}} &= (D\Gamma_{a})_{\dot{a}\dot{5}} + \frac{1}{2}(\Omega_{a\dot{a}\dot{5}} - \Omega_{\dot{a}a\dot{5}}) \\
T_{a\dot{5}\dot{5}} &= (D\Gamma_{a})_{\dot{5}\dot{5}} + \frac{1}{2}(\Omega_{a\dot{5}\dot{5}} + \Omega_{\dot{5}a\dot{5}}). \\
\end{align*}
\]
It is obvious that the conditions (28) and (29) are not enough. There are various ways to choose an additional condition. The simplest condition we can find is

$$\Omega_{AB\hat{5}} = 0$$  \hspace{1cm} (31)

With three conditions (28), (29) and (31) all the non-vanishing components of the torsion and connection are given by

$$\begin{align*}
\Omega_{abc} &= -(D\Gamma_a)_{bc} + (D\Gamma_b)_{ac} - (D\Gamma_c)_{ba} \\
\Omega_{a\hat{5}b} &= -2(D\Gamma_a)_{\hat{5}b} \\
\Omega_{\hat{5}ab} &= 2(D\Gamma_a)_{\hat{5}b} \\
T_{\hat{5}bc} &= (D\Gamma_{\hat{5}})_{bc} + (D\Gamma_b)_{\hat{5}c} - (D\Gamma_c)_{\hat{5}b} \\
T_{\hat{5}\hat{5}b} &= (D\Gamma_{\hat{5}})_{\hat{5}b} \\
T_{\hat{5}\hat{5}\hat{5}} &= (D\Gamma_{\hat{5}})_{\hat{5}\hat{5}}.
\end{align*}$$  \hspace{1cm} (32)

Since the exterior derivative of $\Gamma^A$ can be calculated in terms of vielbein, the torsion and connections are expressed completely in terms of the metric. Therefore, the second structure equation (24) determines the curvature also in terms of the metric. No further restriction on the vielbein is required.

Now if all the fields in the vielbein (13) will have kinetic terms in the final action, they will survive as physical fields.

The scalar curvature is given

$$R = \langle \Gamma^A \wedge \Gamma^B, R_{AB} \rangle$$  \hspace{1cm} (33)

After explicit calculation of the scalar curvature, we find that the kinetic terms for the vector and scalar fields are not in the scalar curvature as they do in the other solution in [5].

To retain these fields in the theory, the contribution from the torsion must be included as the inner product

$$\langle T_A, T^A \rangle = \tilde{T}_{ABC} \langle \Gamma^B \wedge \Gamma^C, \Gamma^D \wedge \Gamma^E \rangle T^A_{DE}.$$  \hspace{1cm} (34)
Therefore the final Einstein-Hilbert-Cartan Lagrangian of our theory is

\[ \mathcal{L} = \frac{1}{16\pi G_N} R + \frac{1}{G_T^2} < T_A, T^A >, \]  

(35)

where \( G_N \) is the Newton gravitational constant, \( G_T \) is a new constant introduced by the torsion.

The calculation shows that all the kinetic terms are present. With the mass terms in the final form of the Lagrangian, we come to the conclusion that in this version our theory of gravity in the Connes-Lott spacetime model contains the ordinary Einstein gravity, one massive non-linear tensor, one massive and one massless vector, one massive and one Brans-Dicke scalar fields.

5 Conclusion

In this paper, a new specific model of gravity in noncommutative spacetime is proposed. However, in each step, the possible alternatives are pointed out to show how one can retain the desirable contents by a proper choice. In the final result, the solution of this paper requires to include the torsion in the final Lagrangian to retain the full spectrum of the discretized Kaluza-Klein model.

From mathematical point of view, this construction has some interesting features, which deserve some discussion. As stated previously in this paper, the starting algebra \( \mathcal{A} = C^\infty(C, \mathcal{M}) \oplus C^\infty(C\mathcal{M}) \) is too large for physical applications. The hermiticity condition on the physical 1-forms can give the correct physical content for the gauge theory. However, the set of the hermitian forms does not closed as \( \mathcal{A} \)-bimodule. As we shall see, our model can be based on a minimal algebra for physics.

The kinetic terms of the vielbein components must have correct signatures to be physical fields. The signature will be correct with the following restrictions on the vielbein

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components in the equation (13)

\[
E_{a}^{\mu}(x) = \begin{pmatrix}
e_{+a}^{\mu} + ie_{-a}^{\mu} & 0 \\
0 & e_{+a}^{\mu} - ie_{-a}^{\mu}
\end{pmatrix},
\]

\[
A_{\mu}(x) = \begin{pmatrix}
a_{+\mu} + ia_{-\mu} & 0 \\
0 & a_{+\mu} - ia_{-\mu}
\end{pmatrix},
\]

\[
\Phi(x) = \begin{pmatrix}
\phi_{+} + iphi_{-} & 0 \\
0 & \phi_{+} - i\phi_{-}
\end{pmatrix},
\]

(36)

where \(e_{\pm a}^{\mu}, a_{\pm \mu}\) and \(\phi_{\pm}\) are physical real fields.

With this restriction, the theory can be constructed consistently from the subalgebra \(\mathcal{A}'\) and \(\Omega^{p}(\mathcal{A}')\) will be \(\mathcal{A}'\)-bimodule. With this algebra in the spectral triple, there is no need for additional condition on the differential forms that represent the physical fields.

The involutive operation “\(^\sim\)” on the 0-forms determines the form inner product. This choice, in turns, is due to the decision that we make about what kind of theory one wants to obtain in the limit two sheets of spacetime become identical. The inner product defined with the operation “\(^\sim\)” on 0-forms gives kinetic terms, which are diagonal in \(e_{a+}^{\mu}, a_{\mu\pm}\) and \(\phi_{\pm}\). In the limit, where two sheets of spacetime are identical, all the massive modes \(e_{a-}^{\mu}, a_{\mu-}\) and \(\phi_{-}\) of this model vanish and one obtain the truncated Kaluza-Klein spectrum.

**Acknowledgments.**

This work is dedicated to the memory of Victor Isacovich Ogievetsky. In our last meeting in Philadelphia in Summer 1995, my plan on noncommutative geometry has received enthusiastic encouragements from him. It is regretful that the realization of the plan is so late what he cannot see the results. Thanks are also due to K.C.Wali for reading the manuscript and collaboration in the research program.

**Appendix**
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