Fusion and exchange matrices for quantized $\text{sl}(2)$ and associated $q$-special functions

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1 Introduction

The aim of this paper is to evaluate in terms of $q$-special functions the objects (intertwining map, fusion matrix, exchange matrix) related to the quantum dynamical Yang-Baxter equation (QDYBE) for infinite dimensional representations (Verma modules) of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ in the case $\mathfrak{g} = \text{sl}(2, \mathbb{C})$. This study is done in the framework of the exchange construction, initiated by Etingof and Varchenko, in order to find solutions to the QDYBE for $\mathcal{U}_q(\mathfrak{g})$ with $\mathfrak{g}$ a complex semisimple Lie algebra (see [13, §2], [12, §2], [11, §3]). As a result, the familiar interpretation of $q$-Hahn and $q$-Racah polynomials as $q$-Clebsch-Gordan and $q$-Racah coefficients, respectively, for finite dimensional irreducible representations of the quantum group $SU_q(2)$ is extended to a much larger range of parameters, while moreover these interpretations are obtained in an unusual and interesting way, following the definitions of intertwining map and exchange matrix. Furthermore, we reprove, by using these explicit expressions, some properties related to the objects under study in the special case $\mathfrak{g} = \text{sl}(2)$, which were earlier proved in [13] and [12] in a more abstract way in the case of more general $\mathfrak{g}$.

The paper is organized as follows. In Section 2 a brief review of $q$-special functions and of the quantized universal algebra $\mathcal{U}_q(\text{sl}(2))$ and its representations is given. Section 3 deals with the intertwining operator $\Phi_{q,\lambda}^\nu : M_{q,\lambda} \to M_{q,\mu} \otimes V$. Here $M_{q,\lambda}$, $M_{q,\mu}$ are Verma modules for $\mathcal{U}_q(\text{sl}(2))$ with highest weight vectors $x_\lambda$, $x_\mu$ ($\lambda, \mu \in \mathbb{C}$), $V$ is a $\mathcal{U}_q(\text{sl}(2))$-module, not necessarily of finite dimension and soon assumed to be a Verma module, $\Phi_{q,\lambda}^\nu$ is $\mathcal{U}_q(\text{sl}(2))$-intertwining, and $\Phi_{q,\lambda}^\nu(x_\lambda)$ is supposed to have “highest” term $x_\mu \otimes v$. On various places in the paper we do explicit computations first in the generic infinite dimensional case and next make transition (by continuity) to the finite dimensional case in order to connect the results with known explicit expressions in the finite dimensional case. These limit transitions need careful justification.

The matrix elements of the intertwining operator with respect to the standard bases of Verma modules generalize the $q$-Clebsch-Gordan coefficients for the finite dimensional case.
and can still be expressed in terms of \( q \)-Hahn polynomials (see for example [5] for the finite dimensional case). This result is used in Section 4 to obtain in a new way the (otherwise known) orthogonality relations of \( q \)-Hahn polynomials by use of the \( q \)-analogue of the Shapovalov form. In Section 5 we find the explicit matrix elements of the fusion matrix (and of its inverse) \( J_{W,V} : W \otimes V \rightarrow V \otimes W \), which is defined in terms of the intertwining operator by the identity \((\Phi^w_{q,\lambda-wt(v)} \otimes \text{id}) \circ \Phi^v_{q,\lambda} = \Phi^{J_{W,V}(\lambda)(w \otimes v)}_{q,\lambda}\). This leads to the expression of the universal fusion matrix in Section 6: a generalized element of \( \mathcal{U}_q(sl(2)) \otimes \mathcal{U}_q(sl(2)) \) which is shown to satisfy a shifted 2-cocycle condition (results earlier observed in [3]). We know also from [3] that one can associate to this element a generalized element of \( \mathcal{U}_q(sl(2)) \) which is called the shifted boundary. We derive the explicit expression for its inverse independently in Section 7 and we point out how this indeed can be seen as the inverse of the shifted boundary in [3]. We also derive the result, first observed by Rosengren [24], that the shifted boundary acts as a generalized conjugation in \( \mathcal{U}_q(sl(2)) \) which sends a standard \( q \)-analogue of a Cartan subalgebra to a non-standard \( q \)-analogue of a Cartan subalgebra. Next, in Section 8, we derive the ABRR equation for the universal fusion matrix (earlier observed in [2] and [12] for more general \( g \)).

In Section 9 we compute the matrix elements of the exchange matrix \( R_{q,\gamma,\delta}(\lambda) \) sending \( M_{q,\gamma} \otimes M_{q,\delta} \) into itself. These turn out to be \( q \)-Racah polynomials. In Section 10 we show that these matrix elements in the finite dimensional case are essentially \( q \)-Racah coefficients (\( q \)-6\( j \) symbols). The exchange matrix is known to satisfy the QDYBE. In Section 11 we prove in more detail the observation made in [13], §8 that the QDYBE in the finite dimensional case is equivalent to a known identity (see [16] and [22]) satisfied by \( q \)-Racah coefficients (\( q \)-6\( j \) symbols), which is called the Yang-Baxter equation for the interaction round a face (IRF) model (see [4]).

Note that in the classical limit \( q \rightarrow 1 \) of the objects we are dealing with in this paper, the role of \( \mathcal{U}_q(sl(2)) \) is taken over by the Lie algebra \( sl(2) \), see [12], §2.1 and [20]. The different expressions corresponding to the classical limit are obtained by taking the \( q \rightarrow 1 \) limit and replacing the \( q \)-hypergeometric functions \( r\Phi_s \) by their classical analogues \( rF_s \). This limit can usually be taken in a straightforward way, and the \( q \)-case is only computationally more difficult than the \( q = 1 \) case. However, in Section 9 the derivation of the exchange matrix in the \( q \)-case requires one more summation than in the \( q = 1 \) case. This is because the definition of the exchange matrix additionally involves the \( R \)-matrix in the \( q \)-case. This also complicates things a little bit in Section 10. The only place in this paper where the limit transition \( q \rightarrow 1 \) fails, is for the shifted boundary (7.3). This object does not seem to exist for \( q = 1 \). In the case of the exchange matrix in the finite-dimensional case [10,20] the classical limit can be related to Racah coefficients and 6\( j \) symbols, see the definitions and classical analogues to (10.16)–(10.19) in [3], §3.18. Then, similarly as in Section 11, one can show also for \( q = 1 \) that the QDYBE yields a known (see [21]) identity for sums of products of three 6\( j \) symbols.
Note Some of the material of this paper was earlier presented by the first author [19] at the Advanced Study Institute Special Functions 2000, Arizona State University.

Conventions Throughout this paper we assume that $0 < q < 1$. We use the notations $m \land n := \min(m, n)$ and $m \lor n := \max(m, n)$.

2 Preliminaries

2.1 q-Hypergeometric functions

Standard references are Gasper and Rahman [14] and (for special orthogonal polynomials) Koekoek and Swarttouw [17]. We recall the definition and notation for the $q$-Pochhammer symbol and the $q$-binomial coefficient which is nowadays standard in the theory of $q$-special functions (see [14]):

$$ (a; q)_k := (1-a)(1-aq)\ldots(1-aq^{k-1}) \quad (k \in \mathbb{Z}_{\geq 0}), \quad (a; q)_\infty := \prod_{j=0}^{\infty} (1 - aq^j) \quad (a_1, \ldots, a_n; q)_k := (a_1; q)_k \ldots (a_n; q)_k, \quad \left[\begin{array}{c} n \\ k \end{array}\right]_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}. \quad (2.1) $$

The following, slightly different definition and notation for the $q$-number, the $q$-factorial and the $q$-Pochhammer symbol is very convenient for computations in quantum groups:

$$ [a]_q := \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}, \quad [k]_q! := \prod_{j=1}^{k} [j]_q, \quad ([a]_q)_k := \prod_{j=0}^{k-1} [a + j]_q \quad (k \in \mathbb{Z}_{\geq 0}). \quad (2.2) $$

The symbols $[k]_q!$ and $([a]_q)_k$ can be expressed in terms of the standard notation (2.1) as follows:

$$ [k]_q! = q^{-\frac{1}{2}k(k-1)} \frac{(q; q)_k}{(1-q)_k}, \quad ([a]_q)_k = q^{-\frac{1}{2}k(a-1)} q^{-\frac{1}{2}k(k-1)} \frac{(q^a; q)_k}{(1-q)_k}. \quad (2.3) $$

The following consequences of these identities are also useful:

$$ \frac{([a]_q)_k}{([b]_q)_k} = q^{-\frac{1}{2}k(a-b)} \frac{(q^a; q)_k}{(q^b; q)_k}, \quad \frac{[n]_q!}{[k]_q! [n-k]_q!} = q^{-\frac{1}{2}k(n-k)} \left[\begin{array}{c} n \\ k \end{array}\right]_q. \quad (2.4) $$

The general $q$-hypergeometric series is given by

$$ \phi_a \left( a_1, \ldots, a_r \mid b_1, \ldots, b_s ; q, z \right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k (q; q)_k} \left( (-1)^k q^{\frac{1}{2}k(k-1)} s-r+1 \right) z^k. \quad (2.5) $$
which is an infinite series if none of \( a_1, \ldots, a_r \) is equal to \( q^{-n} \) for some \( n \in \mathbb{Z}_{\geq 0} \) and is terminating otherwise. In the infinite series case we require for convergence that \( r < s + 1 \) or that \( r = s + 1 \) and \( |z| < 1 \). The coefficients \( b_j \) are not allowed to be equal to \( q^{-m} \) for some \( m \in \mathbb{Z}_{\geq 0} \) unless for some \( i \) we have \( a_i = q^{-n} \) with \( n \in \mathbb{Z}_{\geq 0} \) and \( n \leq m \). If in \( (2.8) \) we replace \( a_i, b_j \) by \( q^{a_i}, q^{b_j} \), respectively, and if we let \( q \uparrow 1 \) then the limit equals the \textit{hypergeometric series}

\[
\frac{\Gamma(q/a; q)_n}{\Gamma(q/c; q)_n}\]

where \( (a)_k := a(a + 1) \ldots (a + k - 1) \) is the Pochhammer symbol.

We will also need some summation formulas, namely the \textit{q-Chu-Vandermonde sum} \cite{14} (II.6)

\[
2\phi_1\left( \begin{array}{c} q^{-n}, a \\ c \end{array} ; q, q \right) = a^n \frac{(c/a; q)_n}{(c; q)_n} \quad (n \in \mathbb{Z}_{\geq 0}),
\]

a variant \cite{14} (II.7) of the \textit{q-Chu-Vandermonde sum} (obtained by reverting the order of summation in \( (2.7) \))

\[
2\phi_1\left( \begin{array}{c} q^{-n}, a \\ c \end{array} ; q, \frac{q^n c}{a} \right) = \frac{(c/a; q)_n}{(c; q)_n} \quad (n \in \mathbb{Z}_{\geq 0}),
\]

the limit of \( (2.8) \) for \( c \to \infty \) given by

\[
2\phi_0\left( \begin{array}{c} q^{-n}, a \\ a \end{array} ; q, \frac{q^n}{a} \right) = a^{-n} \quad (n \in \mathbb{Z}_{\geq 0}),
\]

and the limit of \( (2.8) \) for \( n \to \infty \) (see \cite{14} (II.5)):

\[
1\phi_1\left( \begin{array}{c} a \\ c \end{array} ; q, \frac{c}{a} \right) = \frac{(c/a; q)_\infty}{(c; q)_\infty}.
\]

We will also need some transformation formulas, namely a \textit{3\phi_2} transformation \cite{11} (10.10.5)

\[
3\phi_2\left( \begin{array}{c} q^{-n}, q^a, q^b \\ q^d, q^e \end{array} ; q, q \right) = q^{an}(q^{-a}; q)_n 3\phi_2\left( \begin{array}{c} q^{-n}, q^a, q^{d-b} \\ q^d, q^{a-n+1} ; q, q \right) \quad (n \in \mathbb{Z}_{\geq 0}),
\]

and \textit{Sear’s transformation} \cite{14} (III.16) of a terminating balanced \textit{4\phi_3} series:

\[
4\phi_3\left( \begin{array}{c} q^{-n}, q^a, q^b, q^c \\ q^d, q^e, q^f \end{array} ; q, q \right) = \frac{(q^a, q^{e+a-b}, q^{e+f-a-c}, q^{e+f-a-b-c}; q)_n}{(q^e, q^f, q^{e+f-a-b-c}; q)_n} 4\phi_3\left( \begin{array}{c} q^{-n}, q^a, q^b, q^{d-b-c} \\ q^d, q^{e+f-a-b-c} ; q, q \right) \quad (n \in \mathbb{Z}_{\geq 0}, \quad a + b + c - n + 1 = d + e + f).
\]
We will meet some $q$-hypergeometric orthogonal polynomials, namely $q$-Hahn polynomials

\[ \phi_n(x; a, b, N; q) := 3 \phi_2 \left( q^{-n}, q^{n+1}ab, q^{-x}; q, q \right) \quad (N \in \mathbb{Z}_{\geq 0}, \ n \in \{0, 1, \ldots, N\}) \tag{2.13} \]
(see [17 §3.6]), and $q$-Racah polynomials, which are defined in (9.6) in terms of certain terminating balanced $4\phi_3$ series, (see also [17 §3.2]).

In section 7 we will use the $q$-analogues of the exponential function (see [14]) given by

\[ e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty}, \quad E_q(z) := \sum_{n=0}^{\infty} \frac{q^{2n(n-1)}z^n}{(q; q)_n} = (-z; q)_\infty, \tag{2.14} \]
which are related (for $|z| < 1$ or as formal power series) by

\[ e_q(z)E_q(-z) = 1, \tag{2.15} \]
and we will also use there a formal power series in noncommuting variables

\[ A_q(x, y) = \sum_{k=0}^{\infty} \frac{1}{(q; q)_k} y^k \frac{1}{(x; q)_k} (xy = q^2yx), \tag{2.16} \]
which is invertible (see [23 Lemma 3.4]) with the following inverse:

\[ A_q^{-1}(x, y) = \sum_{k=0}^{\infty} (-1)^k q^{k+1}q^{k+1} \frac{1}{(q; q)_k} (q^{-k-1}x; q)_k y^k. \tag{2.17} \]

2.2 Preliminaries on $U_q(sl(2, \mathbb{C}))$

We use Chari and Pressley [7] as a standard reference for quantum groups. The quantized universal enveloping algebra $U_q := U_q(sl(2, \mathbb{C}))$ is the algebra generated by the elements $e, f, \quad q^{h/4}, \quad q^{-h/4} := (q^{h/4})^{-1}$ satisfying relations

\[ q^{h/4}e = q^{1/2}e q^{h/4}, \quad q^{h/4}f = q^{-1/2}f q^{h/4}, \quad [e, f] = [h]_q. \tag{2.18} \]

It follows by induction from (2.18) that

\[ q^{\ell h}e^j = e^j q^{(h+2j)}, \quad q^{\ell h}f^j = f^j q^{(h-2j)}, \quad ef^n = f^n e + [n]_q f^{n-1}[h - n + 1]_q, \tag{2.19} \]
and (see for instance [8 (1.3.1)])

\[ e^m f^n = \sum_{k=0}^{\min(m, n)} \frac{[m]_q! [n]_q!}{[k]_q! [m-k]_q! [n-k]_q!} f^{n-k}q^{m-k} (k + m - n - k + 1)_q k. \tag{2.20} \]

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It is convenient to allow \( h \) as a formal element of \( \mathcal{U}_q \). Now (2.18) can equivalently be written as:

\[
[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = [h]_q.
\] (2.21)

\( \mathcal{U}_q \) is endowed with a structure of quasi-triangular Hopf algebra (see for example [7]). The coproduct \( \Delta \), the counit \( \varepsilon \) and the antipode \( S \) are given by:

\[
\Delta(h) = h \otimes 1 + 1 \otimes h, \quad \Delta(e) = e \otimes q^{h/4} + q^{-h/4} \otimes e, \quad \Delta(f) = f \otimes q^{h/4} + q^{-h/4} \otimes f,
\] (2.22)

\[
\varepsilon(h) = \varepsilon(e) = \varepsilon(f) = 0,
\]

\[
S(h) = -h, \quad S(e) = -q^{1/2}e, \quad S(f) = -q^{-1/2}f.
\]

Note that, by the \( q \)-binomial theorem,

\[
\Delta(e^n) = \sum_{k=0}^n \frac{[n]_q!}{[k]_q! [n-k]_q!} e^{k} q^{-(n-k)h/4} \otimes e^{n-k} q^{kh/4},
\] (2.23)

\[
\Delta(f^n) = \sum_{k=0}^n \frac{[n]_q!}{[k]_q! [n-k]_q!} f^{k} q^{-(n-k)h/4} \otimes f^{n-k} q^{kh/4}.
\] (2.24)

The universal \( \mathcal{R} \)-matrix is given by the following expression, see Drinfel’d [10]:

\[
\mathcal{R} = q^{\frac{1}{2}(h \otimes h)} \sum_{j=0}^\infty \frac{(1-q^{-1})^j q^{-\frac{1}{4}j(j-1)}}{[j]_q!} (q^{\frac{1}{2}j} e^j \otimes q^{-\frac{1}{2}j} f^j).
\] (2.25)

It satisfies:

\[
(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12}, \quad (\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (S \otimes id)\mathcal{R} = \mathcal{R}^{-1}.
\] (2.26)

The Casimir element \( \Omega \) of \( \mathcal{U}_q \) (a central element) is given by

\[
\Omega := fe + \left[ \frac{1}{2} h \right]_q \left[ \frac{1}{2} h + 1 \right]_q.
\] (2.27)

For a \( \mathcal{U}_q \)-module \( V \) and for \( \lambda \in \mathbb{C} \) let \( V[\lambda] \) denote the weight space of weight \( \lambda \) in \( V \), i.e.,

\[
V[\lambda] := \{ v \in V \mid h \cdot v = \lambda v \}.
\] (2.28)

The nonzero elements of \( V[\lambda] \) are called weight vectors of weight \( \lambda \). We say that a weight \( \lambda \) occurs in \( V \) if \( V[\lambda] \neq \{0\} \). A weight vector \( v \in V \) is called a highest weight vector if \( e \cdot v = 0 \).
Convention Each $U_q$-module to be considered will be spanned by its weight vectors and the real parts of its occurring weights will be bounded from above.

Let $M_{q,\lambda}$ be the Verma module for $U_q$ with highest weight vector $x_\lambda$ of highest weight $\lambda \in \mathbb{C}$, i.e., $h \cdot x_\lambda = \lambda x_\lambda$, $e \cdot x_\lambda = 0$. A basis of weight vectors for $M_{q,\lambda}$ is given by the elements $x^\lambda_{\lambda-2k} := f^k \cdot x_\lambda$ ($k \in \mathbb{Z}_{\geq 0}$). The action of the generators of $U_q$ on this basis is given by:

\begin{align*}
h \cdot x^\lambda_{\lambda-2k} & = (\lambda - 2k) x^\lambda_{\lambda-2k}, \\
e \cdot x^\lambda_{\lambda-2k} & = [k]_q [\lambda - k + 1]_q x^\lambda_{\lambda-2k+2}, \\
f \cdot x^\lambda_{\lambda-2k} & = x^\lambda_{\lambda-2k-2}.
\end{align*}

(2.29) (2.30) (2.31)

Hence,

\begin{align*}
e^j \cdot x^\lambda_{\lambda-2k} & = (-1)^j ([k]_q) \cdot (\lambda - k + 1)_q x^\lambda_{\lambda-2k+2j}, \\
f^j \cdot x^\lambda_{\lambda-2k} & = x^\lambda_{\lambda-2k-2j}.
\end{align*}

(2.32) (2.33)

Remark 2.1. The following notations for a finite dimensional irreducible $U_q$-module are often used in literature (see for instance [16]). If $m < \frac{1}{2} \mathbb{Z}_{\geq 0}$ then write $V^j$ for $M'(q, 2j)$ and use as a standard basis for $V^j$ the vectors

\begin{align*}
e^j_m := \frac{1}{\sqrt{([j + m + 1]_q)[j-m]_q}} x^j_{2m} \quad (m \in \{-j, -j + 1, \ldots, j\}).
\end{align*}

(2.36)

The symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $V^j$ for which the $e^j_m$ are orthonormal (i.e., $\langle e^j_m, e^j_n \rangle = \delta_{m,n}$), will satisfy (4.2), by which $\langle \cdot, \cdot \rangle$ will be a Shapovalov form on $V^j$, to be discussed in Section 4.

The following universal property of Verma modules $M_{q,\lambda}$ will be useful. It follows immediately from (2.19) and (2.29)–(2.31).

Lemma 2.2. Let $\lambda \in \mathbb{C}$. Let $V$ be a $U_q$-module with a highest weight vector $v \in V[\lambda]$. Then there is a unique $U_q$-intertwining operator $\Phi : M_{q,\lambda} \rightarrow V$ such that $\Phi(x_\lambda) = v$. It is given by $\Phi(f^k \cdot x_\lambda) := f^k \cdot v$. If $\lambda \in \mathbb{Z}_{\geq 0}$ then $\Phi$ is also well-defined on the quotient $M_{q,\lambda}'$ iff $f^{\lambda+1} \cdot v = 0$. 

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By (2.27) the Casimir operator acting on $M_{q,\lambda}$ is a constant multiple of the identity:

$$
\Omega = [\frac{1}{i_2} \lambda]_q [\frac{1}{i_2} \lambda + 1]_q.
$$

(2.37)

The action of $U_q$ on the tensor product of two $U_q$-modules $V$ and $W$ is given by the comultiplication:

$$
x \cdot (v \otimes w) := \Delta(x) \cdot (v \otimes w) \quad (x \in U_q).
$$

Let $P : v \otimes w \mapsto w \otimes v : V \otimes W \to W \otimes V$ be the flip operator. Then $P \circ R : v \otimes w \mapsto P(R \cdot (v \otimes w)) : V \otimes W \to W \otimes V$ is an intertwining operator of $U_q$-modules.

3 The intertwining map

The statement and proof of [12 Proposition 2.1] (existence and uniqueness of the intertwining map) can be adapted to the quantum case and to the case that the module $V$ is not necessarily of finite dimension. We start with a lemma (we will only deal with $g = s(l(2))$).

**Lemma 3.1.** Let $V$ be a $U_q$-module, let $\lambda, \mu \in \mathbb{C}$, and let $0 \neq v \in V[\lambda - \mu]$. In the case that $\mu \in \mathbb{Z}_{\geq 0}$ assume moreover that $e^{\mu+1} \cdot v = 0$. Then there is a unique highest weight vector of weight $\lambda$ in $M_{\mu} \otimes V$ of the form

$$
\sum_{k=0}^{\infty} f^k \cdot x_\mu \otimes v_k \quad \text{such that } v_0 = v \text{ and such that } v_k = 0 \text{ if } k > \mu \text{ and } \mu \in \mathbb{Z}_{\geq 0}.
$$

(3.1)

Furthermore, there is $l \in \mathbb{Z}_{\geq 0}$ such that $v_k \neq 0$ for $k = 0, \ldots, l$ and $v_k = 0$ for $k > l$. Then $v_l$ is a highest weight vector. Finally,

$$
v_k = \frac{q^{\frac{1}{2} l (\lambda + 2)}}{[\mu]_q ! ([\mu]_q)^k} e^k \cdot v \quad (k \in \mathbb{Z}_{\geq 0} \text{ or } k = 0, \ldots, \mu \text{ if } \mu \in \mathbb{Z}_{\geq 0}),
$$

(3.2)

**Proof** If $v_k \neq 0$ then $v_k$ must have weight $\lambda - \mu + 2k$. By the convention about $U_q$-modules in Section 2, $v_k = 0$ for $k$ sufficiently large. Hence the condition $e \cdot w = 0$ holds iff

$$
[k + 1]q [\mu - k]_q v_{k+1} = -q^\frac{1}{2} \cdot e \cdot v_k \quad (k \in \mathbb{Z}_{\geq 0}).
$$

(3.3)

This proves existence and uniqueness of the highest weight vector $w$ of the form (3.1). (Note that in the case $\mu \in \mathbb{Z}_{\geq 0}$ the case $k = \mu$ of (3.3) just says that $e \cdot v_\mu = 0$, and that this follows from the assumption $e^{\mu+1} \cdot v = 0$ together with the cases $k < \mu$ of (3.3).)

For the other statements of the lemma let $l \in \mathbb{Z}_{\geq 0}$ be maximal such that $v_k = 0$ for $k = 0, \ldots, l$. First assume that $\mu \notin \mathbb{Z}_{\geq 0}$. Then it follows by iteration of (3.3) that (3.2) holds for $k \in \mathbb{Z}_{\geq 0}$ and that $e \cdot v_l = 0$ and $v_k = 0$ for $k > l$. Next assume that $\mu \in \mathbb{Z}_{\geq 0}$. Then it follows by iteration of (3.3) that (3.2) holds for $k = 0, \ldots, \mu$. Then $e \cdot v_l = 0$ by (3.3) if $l < \mu$, and $e \cdot v_l = 0$ by assumption if $l = \mu$. If $l < \mu$ then we see also that $v_k = 0$ for $k = l + 1, \ldots, \mu$. □
Definition 3.2. Let \( V, \lambda, \mu, v \) and further assumptions be as in Lemma 3.1. The intertwining map \( \Phi_{q,\lambda}^v \) is defined as the unique \( \mathcal{U}_q \)-intertwining operator \( \Phi_{q,\lambda}^v : M_{q,\lambda} \to M_{q,\mu} \otimes V \) such that \( \Phi_{q,\lambda}^v(x_\lambda) \) is of the form (3.1).

The existence and uniqueness of \( \Phi_{q,\lambda}^v \) follow from Lemma 3.1 together with Lemma 2.2. Then we obtain by (3.2) that

\[
\Phi_{q,\lambda}^v(x_\lambda) = \sum_{k=0}^\infty \frac{q^{-\frac{1}{4}k(\lambda+2)}}{[k]_q!(\mu)_q!} \cdot x_\mu \otimes e^k \cdot v.
\]

Remark 3.3. Consider Definition 3.2 in the case that \( \mu \in \mathbb{Z} \geq 0 \). Then the sum in (3.4) has upper limit \( \mu \). Suppose that moreover \( V = M_{q,\gamma} \) and that \( \lambda, \gamma \in \mathbb{Z} \geq 0 \). Then, since \( v \neq 0 \) and \( \text{wt}(v) = \lambda - \mu \), we have \( \lambda - \mu \leq \gamma \) and \( \lambda - \mu - \gamma \) is even. Also, the condition \( e^{\mu+1} \cdot v = 0 \) in Lemma 3.1 is certainly satisfied if \( \lambda + \mu \geq \gamma \).

Since the canonical maps \( M_{q,\mu} \to M'_{q,\mu} \) and \( M_{q,\mu} \to M'_{q,\mu} \otimes M'_{q,\gamma} \) are \( \mathcal{U}_q \)-intertwining, we can consider (3.4) for the intertwining map \( \Phi_{q,\lambda}^v : M_{q,\lambda} \to M'_{q,\mu} \otimes M'_{q,\gamma} \). In order to have the canonical projection of \( v \) nonzero, we require that \( \lambda - \mu \geq -\gamma \). By Lemma 2.2, this last map induces an intertwining map \( \Phi_{q,\lambda}^v : M'_{q,\lambda} \to M'_{q,\mu} \otimes M'_{q,\gamma} \) if \( \Phi_{q,\lambda}^v(f_{\lambda+1} \cdot v) = 0 \). This last condition is satisfied if \( \lambda + \mu \geq \gamma \).

We conclude that, for \( \lambda, \mu, \gamma \in \mathbb{Z} \geq 0 \) and \( \lambda - \mu - \gamma \) even, we can bring the intertwining map \( \Phi_{q,\lambda}^v : M_{q,\lambda} \to M_{q,\mu} \otimes M_{q,\gamma} \) down to the level of the corresponding finite dimensional modules iff \( |\gamma - \mu| \leq \lambda \leq \gamma + \mu \).

We will later need the following observation, which immediately follows from the existence and uniqueness of the intertwining map.

Lemma 3.4. Let \( V, \lambda, \mu, v \) and further assumptions be as in Lemma 3.1. Let \( W \) be another \( \mathcal{U}_q \)-module, and let \( A : V \to W \) be an \( \mathcal{U}_q \)-intertwining map. Then

\[
(id \otimes A) \circ \Phi_{q,\lambda}^v = \Phi_{q,\lambda}^{Av}.
\]  

For a given \( v \in V[\lambda - \mu] \), we want to determine the coefficients of the map \( \Phi_{q,\lambda}^v \) on any element of the Verma module \( M_{q,\lambda} \). Hence, we apply \( f^n \) on both sides of (3.4) and by use of the intertwining property of \( \Phi_q \) and by (2.24), we find the following expression:

\[
\Phi_{q,\lambda}^v(f^n \cdot x_\lambda) = \sum_{j=0}^n \sum_{k=0}^\infty \frac{[n]_q!}{[j]_q! [n-j]_q!} \frac{q^{\frac{1}{2}(\mu+n+j)} q^{{\lambda-2n+2}}}{[k]_q! ([\mu]_q)_k} \cdot x_\mu \otimes f^{n-j} e^k \cdot v.
\]
Substitution of $\mu = \lambda - \text{wt}(v)$ yields:

$$Φ_{\lambda}v^q,λ(f_n \cdot x_\lambda) = \sum_{k=0}^{\infty} \frac{q^{-\frac{1}{4}(\lambda-2n+2)}}{[k]_q! ([\lambda + \text{wt}(v)]_q)_k} \sum_{j=0}^{n} \frac{[n]_q!}{[j]_q! [n-j]_q!} q^{\frac{1}{4}(\text{wt}(v)n-\lambda n+\lambda j)} f^{k+j} \cdot x_{\lambda-\text{wt}(v)} \otimes f^{n-j}e^k \cdot v.$$  

Finally, with new summation variables $m, j$, where $m = k + j$, we obtain:

$$Φ_{\lambda}v^q,λ(f_n \cdot x_\lambda) = \sum_{m=0}^{\infty} f^m \cdot x_{\lambda-\text{wt}(v)} \otimes F_{q,m,n}(\lambda) \cdot v,$$  

(3.6)

where

$$F_{q,m,n}(\lambda) = q^{\frac{m}{4}(2n-\lambda-2)} \sum_{j=0}^{m\wedge n} q^{-\frac{1}{2}(n-\lambda)} \frac{[n]_q!}{[j]_q! [m-j]_q! [n-j]_q!} f^{n-j}e^m-j \frac{q^{\frac{1}{4}h}}{([-\lambda + h]_q)_m-j}.$$  

(3.7)

Then $\text{wt}(F_{q,m,n}(\lambda) \cdot v) = \text{wt}(v) + 2m - 2n$. Note that the coefficients of the intertwining map are rational functions of $q^\lambda$.

Particular cases, which will be used to find the expression of the fusion matrix (5.1), are the cases $n = 0$ and $m = 0$:

$$F_{q,m,0}(\lambda) = q^{\frac{m}{4}(2n-\lambda-2)} \sum_{j=0}^{m\wedge n} q^{-\frac{1}{2}(n-\lambda)} \frac{[n]_q!}{[j]_q! [m-j]_q! [n-j]_q!} f^{n-j}e^m-j \frac{q^{\frac{1}{4}h}}{([-\lambda + h]_q)_m-j}. $$  

(3.8)

and

$$F_{q,0,n}(\lambda) = q^{-\frac{n}{4}} f^n q^{\frac{5}{4}h}. $$  

(3.9)

In the following, we want to find the expression of the intertwining operator in the particular case where the $U_q$-module $V$ is a Verma module. Let $\lambda, \mu, \gamma \in \mathbb{C}$ with $\mu \not\in \mathbb{Z}_{\geq 0}$ and $\mu + \gamma - \lambda \in 2\mathbb{Z}_{\geq 0}$. The intertwining operator $Φ_{\lambda}^\gamma x_{\lambda-\mu} : M_{q,\lambda} \rightarrow M_{q,\mu} \otimes M_{q,\gamma}$ can then be written as

$$Φ_{\lambda}^\gamma x_{\lambda-2n} = \sum_{m=0}^{n+p(\mu-\gamma-\lambda)} C_{\mu,\gamma,\lambda}^{\mu,\mu-2m,\lambda-\mu+2m-2n,\lambda-2n} x_{\lambda-2m} \otimes x_{\lambda-\mu+2m-2n}^\gamma,$$

(3.10)

where the generalized Clebsch-Gordan coefficients $C_{\mu,\gamma,\lambda}$ satisfy

$$F_{q,m,n}(\lambda) \cdot x_{\lambda-\mu}^\gamma = C_{\mu,\gamma,\lambda}^{\mu,\mu-2m,\lambda-\mu+2m-2n,\lambda-2n} x_{\lambda-\mu+2m-2n}^\gamma.$$

Note that, by (3.4), we have

$$C_{\mu,\gamma,\lambda}^{\mu,\gamma,\lambda} = 1.$$  

(3.11)
Theorem 3.5. The coefficients $C_{q,\mu+\gamma-2l}$, defined by (3.10), can be expressed in terms of $q$-Hahn polynomials (2.13) or in terms of $q$-hypergeometric functions as follows:

$$C_{q,\mu+\gamma-2l} = q^{\mu}(\mu+\gamma)\binom{m-1}{m} (N-l)^{-\frac{1}{2}} m(N-m) \left[\begin{array}{c} N \\ m \end{array}\right]_q Q_l(q^{-m}; q^{-\mu-1}, q^{-\gamma-1}, N; q)$$

Proof By comparison of (3.6) for $v = x_{\gamma-2l}$ with (3.10) and by the substitutions $\lambda = \mu+\gamma-2l$, $n = N-l$ ($N \in \mathbb{Z}_{\geq 0}$ and $0 \leq l \leq N$) we get:

$$\mathcal{F}_{q,m,N-l} (\mu + \gamma - 2l) \cdot x_{\gamma-2l} = C_{q,\mu+\gamma-2l}$$

where, in view of (3.7):

$$\begin{align*}
\mathcal{F}_{q,m,N-l} (\mu + \gamma - 2l) \cdot x_{\gamma-2l} &= \sum_{j=0}^{m^\lambda(N-l)} q^{\mu}(2N-\mu-\gamma-2) - \frac{1}{2}(N-l-\mu-\gamma-1)(N-l) \cdot x_{\gamma-2l} \\
&= \sum_{j=0}^{m^\lambda(N-l)} q^{\mu}(2N-\mu-\gamma-2) - \frac{1}{2}(N-l-\mu-\gamma-1)(N-l) \cdot x_{\gamma-2l} \\
&= \sum_{j=0}^{m^\lambda(N-l)} (-1)^{m-J} q^{\mu}(2N-\mu-\gamma-2) - \frac{1}{2}(N-l-\mu-\gamma-1)(N-l) \cdot x_{\gamma-2l}.
\end{align*}$$

The last equality is obtained by use of (2.32) and (2.33). Hence,

$$C_{q,\mu+\gamma-2l} = \sum_{j=0}^{m^\lambda(N-l)} (-1)^{m-J} q^{\mu}(2N-\mu-\gamma-2) - \frac{1}{2}(N-l-\mu-\gamma-1)(N-l) \cdot x_{\gamma-2l}.$$
In order to express the coefficients \( C_q \) in terms of \( q \)-Hahn polynomials, we separately study the cases \( m \geq l \) and \( m \leq l \). In the first case \((m \geq l)\), we change the summation variable to \( i := j - m + l \). Then (3.14) can be rewritten as

\[
C_{q,\mu-2m,\gamma-2N+2m,\mu-\gamma-2N}^{\mu,\gamma,\mu-\gamma-2l} = \sum_{i=0}^{(N-m)/l} q^{\frac{m}{4}(\mu+\gamma-2l)-\frac{1}{4}(N-l)+\frac{l}{2}(N+l-\gamma-\mu-1)} \left(-1\right)^i \frac{[N-l]_q!([-\gamma]_q)_{l}}{[m-l]_q!\left([-\mu]_q\right)_i} \times \frac{([-l]_q)([-N+m]_q)([\mu-l+1]_q)_i}{[i]_q!\left([-\gamma]_q\right)_i\left([-\mu]_q\right)_i}.
\]

We have

\[
\frac{([-l]_q)([-N+m]_q)([\mu-l+1]_q)_i}{[i]_q!\left([-\gamma]_q\right)_i\left([-\mu]_q\right)_i} = q^{-\frac{1}{4}(N-l+\mu+\gamma-1)} \frac{[N]_{iq}^{-1}\left([-\gamma]_q\right)_i\left([-\mu]_q\right)_i}{[m]_{iq}!\left([-\gamma]_q\right)_i\left([-\mu]_q\right)_i}\frac{(N-l+\mu+\gamma-1)_i}{(N-l+\mu+\gamma-1)_i}.\]

and

\[
\frac{[-1]^i [N-l]_q!\left([-\gamma]_q\right)_i}{[m-l]_q!\left([-\mu]_q\right)_i} = \frac{[N]_{iq}^{-1}\left([-\gamma]_q\right)_i\left([-\mu]_q\right)_i}{[m]_{iq}!\left([-\gamma]_q\right)_i\left([-\mu]_q\right)_i}\frac{(N-l+\mu+\gamma-1)_i}{(N-l+\mu+\gamma-1)_i}.\]

Hence,

\[
C_{q,\mu-2m,\gamma-2N+2m,\mu-\gamma-2N}^{\mu,\gamma,\mu-\gamma-2l} = q^{\frac{m}{4}(\mu+\gamma)-\frac{1}{4}(N-l)-\frac{1}{2}m(N-m)} \left[\frac{N}{m}\right]_q \times q^{l(N-m)} \left[\frac{q^{-\gamma}}{q}\right]_i \left[\frac{q^{m-l+1}}{q}\right]_i \frac{3\phi_2}{\left(q^{-l}, q^{-N+m}, q^{m-l+1} ; q, q\right)} \left(q^{-l}, q^{-N+m}, q^{m-l+1} ; q, q\right) (3.15)
\]

which is (3.13) for \( m \geq l \). Above we used the transformation formula (2.11) twice. In these inequalities the occurrence of \( q^{-N} \) as a denominator parameter is not harmful for the application of (2.11), since all parts have \( q^{-l} \) as a numerator parameter, while \( l \leq N \).

In the second case \((m \leq l)\), formula (3.14) can be rewritten as

\[
C_{q,\mu-2m,\gamma-2N+2m,\mu-\gamma-2N}^{\mu,\gamma,\mu-\gamma-2l} = q^{\frac{m}{4}(2N-\mu-\gamma-2)+\frac{1}{4}(N-l)-\frac{1}{2}(N+l-\mu-\gamma-1)} \left(-1\right)^m \left[\frac{[-l]_q}{m}!\left([-\mu]_q\right)_m\right] \times \sum_{j=0}^{m \wedge (N-l)} \frac{([-N+l]_q)_j([-m]_q)_j([\mu-m+1]_q)_j}{[j]_q!\left([-\gamma+l-m]_q\right)_j\left([-\mu]_q\right)_j}.\]
We have
\[
\frac{([-N + l]_q j([-m]_q j([\mu - m + 1]_q j) )_j}{[j]_q! ([\gamma + l - m]_q j([l - m + 1]_q j) = q^{-\frac{1}{2}(\mu + \gamma - N - l - 1)} \frac{(q^{-N+l}; q)_j (q^{-m}; q)_j (q^{-m+1}; q)_j}{(q; q)_j (q^{-\gamma+l-m}; q)_j (q^{l-m+1}; q)_j}
\]
and
\[
\frac{(-1)^m ([l]_q m([-\gamma + l + 1]_q m)}{[m]_q! ([\gamma + l]_q m) = q^{-\frac{m}{2}(\mu - \gamma - N + 2l - l - 1)} \frac{\binom{N}{m}}{m} \frac{(q^{-m+1}; q)_m (q^{-\gamma+l-m}; q)_m}{(q^{N-m+1}; q)_m (q^{\mu-m+1}; q)_m}.
\]

Hence,
\[
C_{q, \mu-2m, \gamma-2N+2m, \mu-\gamma-2N}^{\mu, \gamma, \mu+\gamma-2l} = q^{\frac{m}{2}(\mu+\gamma) - \frac{m}{4}(N-l) - \frac{m}{2}(N-m)} \frac{\binom{N}{m}}{m} \times q^{m(N-l)} \frac{(q^{-m+1}; q)_m (q^{-\gamma+l-m}; q)_m}{(q^{N-m+1}; q)_m (q^{\mu-m+1}; q)_m} \frac{\binom{N}{m}}{m} \frac{\binom{N}{m}}{m} 3 \phi_2 \left( \frac{(q^{-l-m}; q)_m}{(q^{N-m-l}; q)_m}, q^{\mu-m}, q^{-m}, q^{-m+1}, q^{l-m}, q^{m+1}, q^{-l}; q, q \right)
\]
which is (3.13) for \( m \leq l \). Above we used again the transformation formula (2.11) twice. \( \square \)

Let \( \mu, \gamma \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0} \) and \( l \in \mathbb{Z}_{\geq 0} \). Write \( \lambda := \mu + \gamma - 2l \). By combination of the \( U_q \)-intertwining properties of the mappings
\[
\Phi_{q, \lambda}^{f^l, x^\gamma} : M_{q, \lambda} \rightarrow M_{q, \mu} \otimes M_{q, \gamma}, \quad \Phi_{q, \lambda}^{f^l, x^\mu} : M_{q, \lambda} \rightarrow M_{q, \gamma} \otimes M_{q, \mu}, \quad P \circ \mathcal{R} : M_{q, \mu} \otimes M_{q, \gamma} \rightarrow M_{q, \gamma} \otimes M_{q, \mu}
\]
we can expect a nice relationship between the operators \( P \circ \mathcal{R} \circ \Phi_{q, \lambda}^{f^l, x^\gamma} \) and \( \Phi_{q, \lambda}^{f^l, x^\mu} \). In fact, we have:

**Theorem 3.6.** Let \( \mu, \gamma \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0} \) and \( l \in \mathbb{Z}_{\geq 0} \). Then
\[
P \circ \mathcal{R} \circ \Phi_{q, \mu+\gamma-2l}^{f^l, x^\gamma} = (-1)^l q^{-\frac{l}{2}(\mu+\gamma-2l+2)+\frac{l}{2}(\mu-2l)} \frac{([-\gamma]_q l)}{([-\mu]_q l)} \Phi_{q, \mu+\gamma-2l}^{f^l, x^\mu}.
\]
Proof  Combination of (3.14) with (2.25) yields

\[
P \left( R \cdot \Phi_{q,\mu,\gamma}^{j} \right)_{x_{0}}(x_{\mu,\gamma}) = \sum_{k=0}^{l} \sum_{j=0}^{k} \frac{(1 - q^{-1})^{j} \cdot q^{-\frac{1}{2}j(j-1)} \cdot q^{-\frac{1}{2}k(\mu+\gamma-2l+2)}}{[j]_{q}^{l} \cdot [k]_{q}^{l} \cdot (-\mu)_{q}^{k}} \cdot q^{\frac{1}{2}(h \otimes h)} \left( q^{-\frac{1}{2}j} \cdot f^{j} \cdot f^{l} \cdot x_{\gamma} \otimes q^{\frac{1}{2}j} \cdot e^{k} \cdot x_{\mu} \right) (3.18)
\]

Finally combine the intertwining property of \( P \circ R \) with Definition 3.2. □

Of course, an independent verification of (3.17) should be possible without use of the intertwining property of \( P \circ R \). For that purpose, simplify the double sum (3.18), replace the summation variable \( j \) by a summation variable \( i := l - k + j \), write the resulting double sum as \( \sum_{i=0}^{l} \sum_{j=0}^{i} \), and reduce the inner sum to a \( 2\phi_{0} \) sum which can be evaluated by use of (2.9):

\[ 2\phi_{0}(q^{-i}, q^{-i+1}; -; q, q^{2i-1}) = q^{i(i-1)}. \]

4 Orthogonality of q-Hahn polynomials and the Shapovalov form

In this section we will derive the known orthogonality relations

\[
\sum_{m=0}^{N} \frac{(q^{-\mu}, q^{-N}; q)_{m}}{(q, q^{-N+1}; q)_{m}} \cdot m(\mu+\gamma+1) \cdot Q_{l}(q^{-m}; q^{-\mu-1}, q^{-\gamma-1}, N; q) \cdot Q_{r}(q^{-m}; q^{-\mu-1}, q^{-\gamma-1}, N; q)
\]

\[
= \delta_{l,r} q^{mN} (q^{-\mu-N}; q)_{N} \cdot (q^{-\mu-N+1}; q)_{l} \cdot \frac{1 - q^{-\mu-1}}{1 - q^{-\mu-\gamma+2l-1}} \cdot (-1)^{l} q^{\frac{1}{2}(l-1)} q^{-N+1} (4.1)
\]

for q-Hahn polynomials (see [17 (3.6.2)]) as a consequence of the quantum group interpretation (3.22) of q-Hahn polynomials. As a tool we will use the quantum analogue of the so-called Shapovalov form, see for instance [21 §5] and references given there.

let \( V \) be a \( U_{q} \)-module on which a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) is given. We will call this form a (quantum) Shapovalov form on \( V \) if

\[
\langle e \cdot v, w \rangle = \langle v, f \cdot w \rangle \quad \text{and} \quad \langle h \cdot v, w \rangle = \langle v, h \cdot w \rangle \quad (4.2)
\]

for all \( v, w \in V \). We observe the following properties of the Shapovalov form.
On the Verma module $M_{q,\lambda}$ we find by (2.32), (2.33) that a Shapovalov form exists uniquely, up to a constant factor:

$$\langle f^n \cdot x_\lambda, f^m \cdot x_\lambda \rangle = \delta_{m,n} (-1)^n [n]_q! ([\lambda]_q)_n \langle x_\lambda, x_\lambda \rangle. \quad (4.3)$$

By convention we normalize the Shapovalov form on $M_{q,\lambda}$ by putting $\langle x_\lambda, x_\lambda \rangle := 1$.

Next, let $V$ and $W$ be $\mathcal{U}_q$-modules equipped with a Shapovalov form. Then, in view of (2.22), we can define a Shapovalov form on $V \otimes W$ by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle. \quad (4.4)$$

As a final property, let $V$ be a $\mathcal{U}_q$-module equipped with a Shapovalov form and with highest weight submodules $W, W'$ with highest weights $\lambda$ resp. $\mu$ such that $\lambda \neq \mu$ and $\lambda \neq -\mu - 2$. Then $\langle w, w' \rangle = 0$ for $w \in W$, $w' \in W'$, because, by (2.37), we have

$$[\frac{1}{2}\lambda]_q[\frac{1}{2}\lambda + 1]_q \langle w, w' \rangle = \langle \Omega \cdot w, w' \rangle = \langle w, \Omega \cdot w' \rangle = [\frac{1}{2}\mu]_q[\frac{1}{2}\mu + 1]_q \langle w, w' \rangle.$$

For a given Shapovalov form $\langle \rangle$ on a $\mathcal{U}_q$-module $V$ we will also use the notation

$$\|v\|^2 := \langle v, v \rangle \quad (v \in V). \quad (4.5)$$

If $\|v\|^2 > 0$ then we will also work with $\|v\|$, being the positive square root of $\|v\|^2$.

**Lemma 4.1.** Let $\mu, \gamma \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$. Assume that $\mu \notin \mathbb{Z}_{\geq 0}$. Then

$$\|\Phi_{q,\mu+\gamma-2l}(x_{\mu+\gamma-2l})\|^2 = (-1)^l q^{\frac{1}{2}(l-\gamma-1)} \frac{[l]_q!([\gamma]_q)_l([\mu - l - 1]_q)_l}{([-\mu]_q)_l}. \quad (4.6)$$

**Proof** By (3.1), (3.4), (4.3) and (2.32) we obtain:

$$\langle \Phi_{q,\mu+\gamma-2l}(x_{\mu+\gamma-2l}), \Phi_{q,\mu+\gamma-2l}(x_{\mu+\gamma-2l}) \rangle \rangle = \sum_{k=0}^{\infty} q^{-\frac{1}{2}(\mu+\gamma-2l+2)} \frac{1}{([k]_q!([\mu]_q)_k)^2} \langle f^k \cdot x_\mu, f^k \cdot x_\mu \rangle \langle e^k f^l x_\gamma, e^k f^l x_\gamma \rangle$$

$$= (-1)^l [l]_q! ([\gamma]_q)_l \sum_{k=0}^{l} \frac{([l]_q)_k([\gamma - l - 1]_q)_k q^{-\frac{1}{2}k(\mu+\gamma-2l+2)}}{([-\mu]_q)_l[k]_q!} q^{-\frac{1}{2}k(\mu+\gamma-2l+2)}$$

$$= (-1)^l[l]_q! ([\gamma]_q)_l 2 \phi_1 \left( q^{-l}, q^{\gamma-l+1} ; q, q^{-(\mu+\gamma-2l+1)} \right).$$

Now (4.6) follows by (2.38) (the reversed $q$-Chu-Vandermonde sum). □
Proof of (4.1) By continuity it is sufficient to prove (4.1) if moreover $\mu + \gamma \notin \mathbb{Z}$. If $l \neq l'$ then $\Phi_{q,\mu+\gamma-2l}(M_{q,\mu+\gamma-2l})$ and $\Phi_{q,\mu+\gamma-2l'}(M_{q,\mu+\gamma-2l'})$ are highest weight submodules of $M_{q,\mu} \otimes M_{q,\gamma}$ with distinct highest weights $\mu + \gamma - 2l$ and $\mu + \gamma - 2l'$, respectively. Hence, the Shapovalov form on $M_{q,\mu} \otimes M_{q,\gamma}$ with its two arguments restricted to $\Phi_{q,\mu+\gamma-2l}(M_{q,\mu+\gamma-2l})$ and $\Phi_{q,\mu+\gamma-2l'}(M_{q,\mu+\gamma-2l'})$, respectively, yields 0. Thus

$$\langle \Phi_{q,\mu+\gamma-2l}(f^{N-l} \cdot x_{\mu+\gamma-2l}), \Phi_{q,\mu+\gamma-2l'}(f^{N-l'} \cdot x_{\mu+\gamma-2l'}) \rangle = 0 \quad (l \neq l'). \quad (4.7)$$

For $l = l'$ we obtain by the intertwining property of $\Phi_q$ and by (2.32), (4.2) and (4.6) that

$$\langle \Phi_{q,\mu+\gamma-2l}(f^{N-l} \cdot x_{\mu+\gamma-2l}), \Phi_{q,\mu+\gamma-2l'}(f^{N-l'} \cdot x_{\mu+\gamma-2l'}) \rangle = [N]_q! \left( [-\mu - \gamma + 2l]_q \right)_{N-l} (-1)^{N-l} \left\langle \Phi_{q,\mu+\gamma-2l}(x_{\mu+\gamma-2l}), \Phi_{q,\mu+\gamma-2l'}(x_{\mu+\gamma-2l'}) \right\rangle$$

$$= [N]_q! \left( [-\mu - \gamma + 2l]_q \right)_{N-l} (-1)^{N-l} \left\langle \Phi_{q,\mu+\gamma-2l}(x_{\mu+\gamma-2l}), \Phi_{q,\mu+\gamma-2l'}(x_{\mu+\gamma-2l'}) \right\rangle$$

$$= [N]_q! \left( [-\mu - \gamma + 2l]_q \right)_{N-l} (-1)^{N-l} \left\langle \Phi_{q,\mu+\gamma-2l}(x_{\mu+\gamma-2l}), \Phi_{q,\mu+\gamma-2l'}(x_{\mu+\gamma-2l'}) \right\rangle$$

On the other hand, by use of (3.10), (3.12) we can write:

$$\langle \Phi_{q,\mu+\gamma-2l}(f^{N-l} \cdot x_{\mu+\gamma-2l}), \Phi_{q,\mu+\gamma-2l'}(f^{N-l'} \cdot x_{\mu+\gamma-2l'}) \rangle$$

$$= \sum_{m=0}^{N} \langle C_{q,\mu+\gamma-2l-m,\mu+\gamma-2l} \otimes x_{\mu+\gamma-2l-m}, x_{\gamma} \rangle \langle C_{q,\mu+\gamma-2l-m,\mu+\gamma-2l} \otimes x_{\mu+\gamma-2l-m}, x_{\gamma} \rangle$$

$$= \sum_{m=0}^{N} \left( \frac{[N]_q!}{[m]_q! [N-m]_q!} \right)^2 q^{\frac{m}{2}(\mu+\gamma)} q^{\frac{1}{2}(2N-l'-l)} \left\langle Q_l Q_r \right\rangle (q^{-m}, q^{-m-1}, q^{-\gamma-1}, N; q)$$

Combination of this last result with (4.8) and (4.7) yields (4.1). □

Remark 4.2. Use the notation of Remark 2.1. Let $j_1, j_2 \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Then the tensor product $V^{j_1} \otimes V^{j_2}$ of two finite dimensional irreducible $U_q$-modules decomposes as the direct sum of all $V^j$ such that $j \in \{|j_1-j_2|, |j_1-j_2|+1, \ldots, j_1+j_2\}$. The Shapovalov forms on $V^{j_1}$ and $V^{j_2}$ induce a Shapovalov form on $V^{j_1} \otimes V^{j_2}$ and hence on each $V^j$ occurring in this tensor product. Let the vectors $e_m^j(j_1, j_2)$ ($m = -j, -j+1, \ldots, j$) form the standard basis of the irreducible submodule $V^j$ of $V^{j_1} \otimes V^{j_2}$, where the basis vectors are orthonormal with respect
to the Shapovalov form. This basis is unique up to a constant complex factor of absolute value 1, independent of $j$. Normalize the basis such that the inner product between $e^j_{m_1}(j_1, j_2)$ and $e^j_{m_2}(j_1, j_2)$ is positive. There will be an expansion of the form

$$e^j_{m_1}(j_1, j_2) = \sum_{m_1 + m_2 = m} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q e^j_{m_1} \otimes e^j_{m_2}. \quad (4.9)$$

The coefficients in (4.9) are called $q$-$3j$ symbols or $q$-Clebsch-Gordan coefficients. See for instance [16] for further discussion.

Because of Remark 3.3 formula (3.10) remains valid by analytic continuation for the case of finite dimensional irreducible $U$-modules with $\lambda, \mu, \gamma$ as at the end of Remark 3.3. In that case, In view of Definition 3.2, we can relate the $q$-Clebsch-Gordan coefficients as defined by (3.10) to the $q$-$3j$ symbols in (4.9). First we specialize (3.10) and (4.6) to the finite dimensional case:

$$\Phi^{2j_{1}2j_{2}j_{1}}_{q,2j_{2}j_{1}}(x_{2j_{2}j_{1}}) = \sum_{m_1 + m_2 = m} C^{2j_{1}2j_{2}2j}_{q,2m_{1}2m_{2}} x^{2j_{1}}_{2m_{1}} \otimes x^{2j_{2}}_{2m_{2}}, \quad (4.10)$$

$$\|\Phi^{2j_{1}2j_{2}j_{1}}_{q,2j_{2}j_{1}}(x_{2j_{2}j_{1}})\|^2 = q^{\frac{1}{2}(j_{1}+j_{2}+j)}(j_{1}+j_{2}+j-1) \times [j_{1}+j_{2}+j](j_{1}+j_{2}+j+1)_{j_{1}+j_{2}+j} n_{[2j+2]}_{j_{1}+j_{2}+j} \times [j_{1}+j_{2}+j]_{j_{1}+j_{2}+j}. \quad (4.11)$$

Note that $C^{2j_{1}2j_{2}2j}_{q,2j_{2}j_{1}} = 1$ by (3.11), and that the right-hand side of (4.11) is $> 0$.

The normalized intertwining operator

$$\widetilde{\Phi}^{j_{1}j_{1}}_{q,j_{1}}(e^j_{m}) := ||\Phi^{j_{1}j_{1}}_{q,j_{1}}(e^j_{m})||^{-1} \Phi^{j_{1}j_{1}}_{q,j_{1}}(e^j_{m}) = ||\Phi^{j_{1}j_{1}}_{q,j_{1}}(e^j_{m})||^{-1} \Phi^{j_{1}j_{1}}_{q,j_{1}}(x_{2j_{2}j_{1}}). \quad (4.12)$$

will thus satisfy

$$\widetilde{\Phi}^{j_{1}j_{1}}_{q,j_{1}}(e^j_{m}) = e^j_{m}(j_{1}, j_{2}). \quad (4.13)$$

We conclude from (4.10), (4.11), (4.12), (4.13), (4.9) and (2.36) that

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = \left( \frac{q^{-\frac{1}{2}(j_{1}+j_{2}+j)}(j_{1}+j_{2}+j+1)_{j_{1}+j_{2}+j}}{(2j+2)_{j_{1}+j_{2}+j}(j_{1}+j_{2}+j+1)_{j_{1}+j_{2}+j}(j_{1}+j_{2}+j+1)_{j_{1}+j_{2}+j}} \right)^{\frac{1}{2}} \times \left( \frac{(j_{1}+m_{1}+1)_{j_{1}+m_{1}}(j_{1}+m_{1}+1)_{j_{1}+m_{1}}(j_{2}+m_{2}+1)_{j_{2}+m_{2}}(j_{2}+m_{2}+1)_{j_{2}+m_{2}}}{(j_{1}+m_{1}+1)_{j_{1}+m_{1}}(j_{2}+m_{2}+1)_{j_{2}+m_{2}}(j_{1}+m_{1}+1)_{j_{1}+m_{1}}(j_{2}+m_{2}+1)_{j_{2}+m_{2}}} \right)^{\frac{1}{2}} \times C^{2j_{1}2j_{2}2j}_{q,2m_{1}2m_{2}} \times (m_{1}+m_{2} = m). \quad (4.14)$$

Specialization of (3.14) to the finite dimensional case, in combination with (4.14), yields the $q$-analogue \[16, (3.4)\] of the Racah formula for Clebsch-Gordan coefficients. (However, note that in \[16, (3.4)\] the factor $q^{1/2}m_{1}(m_{1}+1)$ is not correct. It should be replaced by $q^{1/2}m_{1}(m_{1}+1)$.)

As we saw in the proof of Theorem 3.5, the reduction of (3.14) to the final $q$-hypergeometric form (3.13) passed through an intermediate $q$-hypergeometric form occurring in two versions (3.15) and (3.16) depending on the sign of $m-l$. Specialization of (3.16) to the finite dimensional case, in combination with (4.14), yields for $j - j_2 - m_1 \leq 0$ the $q$-analogue \[5, (3.60)\] of the Racah formula for Clebsch-Gordan coefficients.

In the following we prove the following result stated in \[16, (4.8)\].

\[
P_R e^j_{m}(j_2, j_1) = (-1)^j_{j_1-j_2} q^{1/2(c_j-c_{j_1}-c_{j_2})} e^j_{m}(j_2, j_1), \tag{4.15}
\]

where $c_j := j(j+1)$. Graphically this formula reads as follows.

\[
P_R \bigg\downarrow_{j} = (-1)^{j_{j_1-j_2}} q^{1/2(c_j-c_{j_1}-c_{j_2})} \bigg\downarrow_{j} \tag{4.16}
\]

Here, for the diagram both on the left and on the right one has to substitute the $m$th standard basis vector in the module $V_j$, which is determined within the tensor product by the diagram.

**Proof of (4.15)** Formula (4.15) can be equivalently written in terms of normalized intertwining operators (4.12) as follows:

\[
P \circ R \circ \Phi^j_{q,j} = (-1)^j_{j_1-j_2} q^{1/2(c_j-c_{j_1}-c_{j_2})} \Phi^j_{q,j}. \tag{4.17}
\]

We will obtain (4.17) from (3.17), which remains valid for finite dimensional relations in view of Remark 4.2 and which can then be written as

\[
P \circ R \circ \Phi^{x_{2j}}_{q,2j} = (-1)^j_{j_1+j_2-j} q^{-1/2(j_1+j_2-j)(j_1+j_2-j)} \\
\times (j_2-j_1+j+1)_q \Phi^{x_{2j}}_{q,2j}. \tag{4.18}
\]

Formula (4.17) now follows from (4.18) by use of (4.12) and (4.11). □

5 **The fusion matrix**

Let $v \in V$ and $w \in W$ be weight vectors in $U_q$-modules $V$ and $W$ and let $\lambda \in \mathbb{C}$ such that $\lambda - \text{wt}(v), \lambda - \text{wt}(v) - \text{wt}(w) \notin \mathbb{Z}_{\geq 0}$. Then the composition map

\[
\Phi^w_{q,\lambda} : M_{q,\lambda} \otimes V_{\Phi^w_{q,\lambda}(\text{wt}(v))} \otimes \text{id} \rightarrow M_{q,\lambda-\text{wt}(v)-\text{wt}(w)} \otimes W \otimes V
\]
is $\mathcal{U}_q$-intertwining. By the considerations of Section 3 this composition map must have the form $\Phi_{q,\lambda}^u$ for some $u \in (W \otimes V)[\text{wt}(v) + \text{wt}(w)]$. It will turn out that the map $w \otimes v \mapsto u$ can be linearly extended to a map $J_{W,V}(\lambda) : W \otimes V \to W \otimes V$. We call this operator the fusion matrix for $W$ and $V$, see [12, §2.1]. Note its defining property

$$ (\Phi_{q,\lambda}^{w - \text{wt}(v)} \otimes \text{id}) \circ \Phi_{q,\lambda}^v = \Phi_{q,\lambda}^{J_{W,V}(\lambda)(w \otimes v)} .$$

(5.1)

**Theorem 5.1.** The fusion matrix can be written as:

$$ J_{W,V}(\lambda)(w \otimes v) = \sum_{l=0}^{\infty} q^{-\frac{l}{2}(\lambda+1)} \frac{[l]_q!}{[l]_q^l} f^l q^{l h/4} \cdot w \otimes e^l \frac{q^{l h/4}}{([-\lambda + h]_q)_l} \cdot v .$$

(5.2)

**Proof** We first compute the composition map of the intertwining operators by use of (3.6):

$$ \Phi_{q,\lambda}^{w-v}(x_\lambda) = (\Phi_{q,\lambda}^{w - \text{wt}(v)} \otimes \text{id}) \circ \Phi_{q,\lambda}^v(x_\lambda) $$

$$ = (\Phi_{q,\lambda}^{w - \text{wt}(v)} \otimes \text{id}) \sum_{m=0}^{\infty} f^m \cdot x_{\lambda - \text{wt}(v)} \otimes F_{q,m,0}(\lambda) \cdot v $$

$$ = \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} f^{m'} \cdot x_{\lambda - \text{wt}(v) - \text{wt}(w)} \otimes F_{q,m',m}(\lambda - \text{wt}(v)) \cdot w \otimes F_{q,m,0}(\lambda) \cdot v $$

$$ = \sum_{m=0}^{\infty} x_{\lambda - \text{wt}(v) - \text{wt}(w)} \otimes F_{q,0,m}(\lambda - \text{wt}(v)) \cdot w \otimes F_{q,m,0}(\lambda) \cdot v $$

$$ + \sum_{m=0}^{\infty} \sum_{m'=1}^{\infty} f^{m'} \cdot x_{\lambda - \text{wt}(v) - \text{wt}(w)} \otimes F_{q,m',m}(\lambda - \text{wt}(v)) \cdot w \otimes F_{q,m,0}(\lambda) \cdot v $$

$$ = \Phi_{q,\lambda}^u(w \otimes v) ,$$

where

$$ J_{W,V}(\lambda)(w \otimes v) = u = \sum_{m=0}^{\infty} F_{q,0,m}(\lambda - \text{wt}(v)) \cdot w \otimes F_{q,m,0}(\lambda) \cdot v .$$

Now substitute (3.8) and (3.9). □

Write the fusion matrix of two Verma modules $M_{q,\gamma}$, $M_{q,\delta}$ as $J_{q,\delta,\gamma}(\lambda) := J_{M_{q,\delta},M_{q,\gamma}}(\lambda)$ and write the matrix elements with respect to their standard bases as

$$ J_{q,\delta,\gamma}(\lambda)(x^\delta_{\delta-2s+2n} \otimes x^\gamma_{\gamma-2n}) = \sum_{m=0}^{n} J_{q,\delta,\gamma,s;m,n} x^\delta_{\delta-2s+2m} \otimes x^\gamma_{\gamma-2m} .$$

(5.3)
Then, by comparison with (5.2) and by change of summation variable we obtain:

\[
J_{q,\delta,\gamma,s;m,n}(\lambda) = \frac{[n]_q!}{[m]_q! [n-m]_q!} q^{-\frac{n-m}{2}(\lambda+1)} q^{\frac{n-m}{4}(\delta-2s+\gamma)} \frac{([\gamma-n+1]_q)_{n-m}}{([\gamma-2n]_q)_{n-m}},
\] (5.4)

where

\[
m, n, s \in \mathbb{Z}, \quad 0 \leq m \leq n \leq s, \quad \lambda - \delta, \lambda - \delta - \gamma \notin \mathbb{Z}_{\geq 0}.
\] (5.5)

We will now show that the inverse of the fusion matrix \(J_{q,\delta,\gamma}\) exists and that its matrix elements, defined by

\[
J^{-1}_{q,\delta,\gamma} (\lambda)(x^\delta_{\delta-2s+2n} \otimes x^\gamma_{\gamma-2n}) = \sum_{m=0}^{n} J_{q,\delta,\gamma,s;m,n}^{\text{inv}}(\lambda) x^\delta_{\delta-2s+2m} \otimes x^\gamma_{\gamma-2m},
\] (5.6)

have explicit expression

\[
J_{q,\delta,\gamma,s;m,n}^{\text{inv}}(\lambda) = \frac{[n]_q!}{[m]_q! [n-m]_q!} q^{-\frac{n-m}{2}(\lambda+1)} q^{\frac{n-m}{4}(\delta-2s+\gamma)} \frac{([\gamma-n+1]_q)_{n-m}}{([\gamma-2n]_q)_{n-m}}.
\] (5.7)

**Proof of (5.7)** We have to show that \(\sum_{l=0}^{n-m} J_{q,\delta,\gamma,s;m+l,n}^{\text{inv}}(\lambda) J_{q,\delta,\gamma,s;m,l,n}(\lambda) = \delta_{m,n}\) after substitution of (5.4) and (5.7). Indeed, by use of the \(q\)-Chu-Vandermonde sum (2.7) we have:

\[
\sum_{l=0}^{n-m} J_{q,\delta,\gamma,s;m+l,n}^{\text{inv}}(\lambda) J_{q,\delta,\gamma,s;m,l,n}(\lambda) = \frac{[n]_q!}{[m]_q! [n-m]_q!} q^{-\frac{n-m}{2}(\lambda+1)} q^{\frac{n-m}{4}(\delta+\gamma-2s)} \frac{([\gamma-n+1]_q)_{n-m}}{([\gamma-\lambda]_q)_{n-m}} \\
\times 2\Phi_1 \left( q^{-n+m}, q^{\lambda-\gamma+n+m+1} q^{\lambda-\gamma+2m+2}; q, q \right) \\
= \frac{[n]_q!}{[m]_q! [n-m]_q!} q^{-\frac{n-m}{2}(\lambda+1)} q^{\frac{n-m}{4}(\delta+\gamma-2s)} q^{(\gamma-n+1)_{n-m} (q^{m-n+1}; q)_{n-m}} \\
\times \frac{([\gamma-n+1]_q)_{n-m}}{([\gamma-\lambda]_q)_{n-m}} \frac{(q^{m-n+1}; q)_{n-m}}{(q^{\lambda-\gamma+2m+2}; q)_{n-m}},
\]

which equals \(\delta_{m,n}\) because of the factor \((q^{m-n+1}; q)_{m-n}\). \(\square\)

6 The universal fusion matrix

Formula (5.2) for the fusion matrix suggests the definition of the *universal fusion matrix*, see [2], as a generalized element of \(\mathcal{U}_q \otimes \mathcal{U}_q\) given by

\[
J_q(\lambda) := \sum_{l=0}^{\infty} q^{-\frac{l}{2}(\lambda+1)} f^l q^{1/4} \otimes e^l \frac{q^{1/4}}{([-\lambda-h]_q)_l}.
\] (6.1)
This has the property that
\[ J_q(\lambda) \cdot (w \otimes v) = J_{W,V}(\lambda)(w \otimes v) \quad (6.2) \]
for each pair of \( \mathcal{U}_q \)-modules \( W, V \) and for any \( w \in W, v \in V \). In fact, \( J_q(\lambda) \) is the unique generalized element in \( \mathcal{U}_q \otimes \mathcal{U}_q \) of the form
\[ J_q(\lambda) = \sum_{l=0}^{\infty} J_q^{(l)}(\lambda) \quad (6.3) \]
with \( J_q^{(0)}(\lambda) = 1 \otimes 1 \) and \( J_q^{(l)}(\lambda) = f^l \phi_l(q^{\frac{1}{2}}h, \lambda) \otimes e^l \psi_l(q^{\frac{1}{2}}h, \lambda) \) (\( \phi_l \) and \( \psi_l \) being rational functions) such that (6.2) holds for all pairs \( W, V \) of irreducible finite dimensional \( \mathcal{U}_q \)-modules.

**Definition 6.1.** Let \( \mathcal{M}(\lambda) \) be a generalized element of \( \mathcal{U}_q \otimes \mathcal{U}_q \) depending on \( \lambda \in \mathbb{C} \) (outside some discrete subset of \( \mathbb{C} \)). Let \( V_1, \ldots, V_n \) be \( \mathcal{U}_q \)-modules such that the action of \( \mathcal{M} \) on \( V_1 \otimes \cdots \otimes V_n \) is well-defined. Let \( v_j \in V_j \) \( (j = 1, \ldots, n) \) be weight vectors. Then
\[ \mathcal{M}(\lambda - h(I)) \cdot (v_1 \cdots v_n) := \mathcal{M}(\lambda - \text{wt}(v_i)) \cdot (v_1 \otimes \cdots \otimes v_n) \quad (i \in \{1, \ldots, n\}). \]

**Theorem 6.2.** The universal fusion matrix satisfies the identity
\[ \Delta F_{q,m,n}(\lambda) J_q(\lambda) = \sum_{l=0}^{\infty} F_{q,m,l}(\lambda - h^{(2)}) \otimes F_{q,l,n}(\lambda), \quad (6.4) \]
where \( F_{q,m,n}(\lambda) \) is given by (3.7).

**Proof** Let both sides of (5.1) act on \( f^n \cdot x_{\lambda} \) and use (3.6) repeatedly. This yields:
\[ \sum_{m=0}^{\infty} f^m \cdot x_{\lambda - \text{wt}(v) - \text{wt}(w)} \otimes F_{q,m,n}(\lambda) \cdot J_{W,V}(\lambda)(w \otimes v) = \sum_{m=0}^{\infty} f^m \cdot x_{\lambda - \text{wt}(w)} \otimes \sum_{l=0}^{\infty} F_{q,m,l}(\lambda - \text{wt}(v)) \cdot w \otimes F_{q,l,n}(\lambda) \cdot v. \]

By linear independence of the vectors \( f^m \cdot x_{\lambda - \text{wt}(v) - \text{wt}(w)} \) \( (m = 0, 1, \ldots) \) we obtain that
\[ F_{q,m,n}(\lambda) \cdot J_{W,V}(\lambda)(w \otimes v) = \sum_{l=0}^{\infty} F_{q,m,l}(\lambda - \text{wt}(v)) \cdot w \otimes F_{q,l,n}(\lambda) \cdot v. \]

This last identity can be interpreted as identity (6.4) acting on \( w \otimes v \). \( \square \)

Formula (6.4) implies that the \( J_q(\lambda) \) satisfies a shifted 2-cocycle condition
\[ (\text{id} \otimes \Delta) J_q(\lambda) \cdot (\text{id} \otimes J_q(\lambda)) = (\Delta \otimes \text{id}) J_q(\lambda) \cdot (J_q(\lambda - h^{(3)}) \otimes \text{id}). \quad (6.5) \]
Remark 6.3. Formula (6.1) is equivalent to \[3, \text{formula (3)}\]. Indeed, the paper \[3\] works with generators \(H, E_\pm\) satisfying relations

\[ [H, E_\pm] = \pm 2E_\pm, \quad [E_+, E_-] = \frac{q^H - q^{-H}}{q - q^{-1}}, \]

and comultiplication

\[ \Delta(H) = H \otimes \text{id} + \text{id} \otimes H, \quad \Delta(E_\pm) = E_\pm \otimes q^{\frac{1}{2}H} + q^{-\frac{1}{2}H} \otimes E_\pm, \]

and we can rewrite \[3, (3)\] as

\[ F_{12}(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} q^{\frac{1}{2}kH} E_+^k \otimes \frac{(-1)^k q^{\frac{1}{2}kH}}{([q \log x + h + k]_q^2)_k} E_-^k \]

\[ = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} E_+^k q^{\frac{1}{2}kH} \otimes E_-^k \frac{q^{\frac{1}{2}kH}}{(-q \log x - h)_k^2}. \quad (6.6) \]

The last infinite sum becomes equal to the right-hand side of (6.1) after the successive substitutions

\[ q \to q^{-\frac{1}{2}}, \quad E_+ \to \frac{q^{5/4}}{1 - q} f, \quad E_- \to \frac{1 - q}{q^{5/4}} e, \quad H \to -h, \quad x \to q^{-\frac{1}{2}(\lambda + 1)}. \quad (6.7) \]

These substitutions also send the above relations and comultiplication to the corresponding formulas \[2.21\] and \[2.22\]. Furthermore note that these substitutions send the shifted cocycle condition \[3, (4)\] to our formula (6.5). The factor \(\frac{q^{5/4}}{1 - q}\) and its inverse which appear in (6.7) will play later a role in sending the shifted boundary in \[3\] to our formula (see section 7).

We will now give another proof of (6.5) by obtaining it as the special case \(m = n = 0\) of the more general identity

\[ (\text{id} \otimes \Delta)(\Delta F_{q,m,n}(\lambda) J_q(\lambda))(\text{id} \otimes J_q(\lambda)) = (\Delta \otimes \text{id})(\Delta F_{q,m,n}(\lambda) J_q(\lambda))(J_q(\lambda - \hbar(3)) \otimes \text{id}). \quad (6.8) \]

**Proof of (6.8)** It is sufficient to prove (6.8) with both sides acting on any \(u \otimes v \otimes w\), where \(u, v\) and \(w\) are weight vectors in finite dimensional irreducible \(\mathcal{U}_q\)-modules \(U, V, W\). Then (6.8) takes the form

\[ (\text{id} \otimes \Delta)(\Delta F_{q,m,n}(\lambda) J_q(\lambda))(\text{id} \otimes J_q(\lambda)) \cdot (u \otimes v \otimes w) \]

\[ = (\Delta \otimes \text{id})(\Delta F_{q,m,n}(\lambda) J_q(\lambda))(J_q(\lambda - \hbar(3)) \otimes \text{id}) \cdot (u \otimes v \otimes w). \quad (6.9) \]

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For the proof of (6.9) we will rewrite both of its sides into expressions (6.10) and (6.11), respectively, which are equal. Here we will use (6.4) repeatedly. First the left-hand side of (6.9):

$$(\text{id} \otimes \Delta)(\Delta F_{q,m,n}(\lambda) J_q(\lambda)) (u \otimes J_q(\lambda) \cdot (v \otimes w))$$

$$= (\text{id} \otimes \Delta) \left( \sum_{t=0}^{\infty} F_{q,m,t}(\lambda - wt(J_q(\lambda)(v \otimes w)) \otimes F_{q,t,n}(\lambda)) (u \otimes J_q(\lambda) \cdot (v \otimes w)) \right)$$

$$= \sum_{t=0}^{\infty} F_{q,m,t}(\lambda - wt(v) - wt(w)) \cdot u \otimes J_q(\lambda) \cdot (v \otimes w)$$

$$= \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} F_{q,m,t}(\lambda - wt(v) - wt(w)) \cdot u \otimes F_{q,t,k}(\lambda - wt(w)) \cdot v \otimes F_{q,k,n}(\lambda) \cdot w. \ (6.10)$$

The weight preserving property of $J_q(\lambda)$ was used in the second equality. Next the right-hand side of (6.9):

$$(\Delta \otimes \text{id})(\Delta F_{q,m,n}(\lambda) J_q(\lambda)) (J_q(\lambda - wt(w)) \cdot (u \otimes v) \otimes w)$$

$$= (\Delta \otimes \text{id}) \left( \sum_{k=0}^{\infty} F_{q,m,k}(\lambda - wt(w)) \otimes F_{q,k,n}(\lambda) \right) (J_q(\lambda - wt(w)) \cdot (u \otimes v) \otimes w)$$

$$= \sum_{k=0}^{\infty} \Delta F_{q,m,k}(\lambda - wt(w)) J_q(\lambda - wt(w)) \cdot (u \otimes v) \otimes F_{q,k,n}(\lambda) \cdot w$$

$$= \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} F_{q,m,t}(\lambda - wt(w) - wt(v)) \cdot u \otimes F_{q,t,k}(\lambda - wt(w)) \cdot v \otimes F_{q,k,n}(\lambda) \cdot w. \ (6.11)$$

Indeed, (6.10) and (6.11) are equal. \(\square\)

### 7 The universal fusion matrix and the shifted boundary

Babelon, Bernard & Billey [3] (see notation of [3] summarized in Remark 6.3) associate to their universal fusion matrix $F_{12}(x)$ (see (6.6)) a generalized element $M(x)$ in $U_q$, called the **shifted boundary** and given by

$$M(x) = \sum_{m,n=0}^{\infty} \frac{(-1)^m x^m q^\frac{1}{2} [n]_q [m]_q [n-1]_q [m-1]_q \prod_{j=1}^{n} (xq^j - x^{-1} q^{-j})}{[n]_q! [m]_q!} E^m_+ E^m_- q^{\frac{1}{2} (n+m)H}, \quad (7.1)$$

such that

$$F_{12}(x) = \Delta M(x) \ (\text{id} \otimes M(x))^{-1} (M(x q^H_2) \otimes \text{id})^{-1}. \quad (7.2)$$
In the following we will independently derive an explicit expression for the inverse of the shifted boundary associated to our universal fusion matrix \( J_q(\lambda) \) given by (6.1).

**Theorem 7.1.** The universal fusion matrix \( J_q(\lambda) \) verifies

\[
\Delta (M_q(\lambda)) J_q(\lambda) = M_q(\lambda - h^{(2)}) \otimes M_q(\lambda),
\]

where \( M_q(\lambda) \), the inverse shifted boundary, is given by

\[
M_q(\lambda) = \sum_{m,n=0}^{\infty} \frac{q^{\frac{1}{2}mn - \frac{1}{4}m^2} q^{-\frac{1}{4}m + \frac{1}{4}n} q^{-\frac{1}{4}n\lambda}}{[n]_q! [m]_q! (1 - q)^n} e^m q^{\frac{1}{2}(n-n)h} ([\lambda + h]_q)_m \]

(7.4)

\[
= E_q(q^{-\frac{1}{4} - \frac{1}{4}\lambda} f q^{\frac{1}{4}h}) A_q(q^{-\lambda + h}, (1 - q)^2 q^{-\frac{1}{4} - \frac{1}{4}\lambda} e q^{\frac{1}{4}h}),
\]

(7.5)

with \( E_q \) and \( A_q \) respectively given by (2.14) and (2.16), and with the two arguments of \( A_q \) satisfying the relation in (2.16).

**Proof** Let us first define the element \( \widetilde{M}_{q,m,n}(\lambda) \in U_q \) which verifies:

\[
\Phi^\nu_{q,\lambda}(q^{-n^2/4} q^{-nh/4} f^n \cdot x_\lambda) = \sum_{m=0}^{\infty} q^{-m^2/4} q^{-mh/4} f^m \cdot x_{\lambda - w t(v)} \otimes \widetilde{M}_{q,m,n}(\lambda) \cdot v
\]

(7.6)

By use of the intertwining property of \( \Phi^\nu_{q,\lambda} \) together with (3.6) and the property \( h F_{q,m,n}(\lambda) = F_{q,m,n}(\lambda) (h + 2m - 2n) \) implied by (3.7), we obtain:

\[
\widetilde{M}_{q,m,n}(\lambda) = q^{(n^2-m^2)/4} q^{\lambda(n-m)/4} F_{q,m,n}(\lambda) q^{-mh/4}.
\]

After substitution of (3.7) this becomes:

\[
\widetilde{M}_{q,m,n}(\lambda) = \sum_{j=0}^{m \wedge n} q^{(m-j)(n-j)/4} q^{(n^2-m^2)/4} \frac{[n]_q! (q^{-\lambda/2} f)^{n-j} e^{m-j} q^{(n-j)h/4} q^{-j} q^{(-m-j)h/4} [j]_q! [n-j]_q! [m-j]_q! ([\lambda + h]_q)_{m-j}}{[j]_q! [n-j]_q! [m-j]_q! ([\lambda + h]_q)_{m-j}}.
\]

(7.7)

It can be shown similarly to the proof of Theorem 6.2 that formula (6.4) remains valid if we replace \( F \) by \( \widetilde{M} \). (Just start the proof now by letting both sides of (5.1) act on \( q^{-n^2/4} q^{-nh/4} f^n \cdot x_\lambda \) and by applying (7.6) repeatedly.) Thus we have:

\[
\Delta \widetilde{M}_{q,m,n}(\lambda) J_q(\lambda) = \sum_{l=0}^{\infty} \widetilde{M}_{q,m,l}(\lambda - h^{(2)}) \otimes \widetilde{M}_{q,l,n}(\lambda).
\]

(7.8)
Now put $n := N - m$ in (7.8), substitute (7.7), and sum over $m$ from 0 to $N$:

$$\Delta \left( \sum_{m=0}^{N} \sum_{j=0}^{m(N-m)} q^{(m-j)(N-m-1)/2} q^{((N-m)^2-m^2)/4} \frac{[N-m]_q!}{[j]_q! [N-m-j]_q! [m-j]_q!} \right) \times \frac{(q^{-\lambda/4} \sum_{m-j} e^{m-j} q^{(N-m-j)h/4} q^{-(m-j)h/4})}{([-\lambda + h]_q)_m-j} J_q(\lambda)$$

$$= \sum_{m=0}^{N} \sum_{l=0}^{m(N-m)} \sum_{j=0}^{l} q^{(m-j)(l-1)/2} q^{(l^2-m^2)/4} \frac{[l]_q! (q^{-(\lambda-h(2))/2} f)^{l-i} e^{m-i} q^{(l-i)h/4} q^{-(m-i)h/4}}{[i]_q! [l-i]_q! [m-i]_q! ([(-\lambda-h(2))+h]_q)_m-i} \otimes q^{(l-j)(N-m-1)/2} q^{((N-m)^2-l^2)/4} \frac{[N-m]_q! (q^{-\lambda/2} f)^{N-m-j} e^{-j} q^{(N-m-j)h/4} q^{-(l-j)h/4}}{[j]_q! [N-m-j]_q! [l-j]_q! ([(-\lambda+h]_q)_l-j}$$

(7.9)

The double sum over $m, j$ on the left-hand side can be rewritten as a double sum over $m' := m-j$ and $n' := N - m - j$ with $m', n' \geq 0$, $m' + n' \leq N$ and $N - m' - n'$ even. Write $m', n'$ again as $m, n$. Then the left-hand side of (7.9) becomes:

$$\Delta \left( \sum_{m,n \geq 0} \frac{q^{m(n-m)/4} q^{m/2} q^{nN/4}}{m+n \leq N} \frac{1}{[1/2](-n-m+N)_q!} \frac{1}{[n]_q! [m]_q! ([(-\lambda+h]_q)_m} \right) J_q(\lambda)$$

$$= \Delta \left( \sum_{m,n \geq 0} \frac{(q; q)_{1/2}^{(n-m+N)} (q; q)_{1/2}^{(n-m+N)}}{m+n \leq N} (1-q)^{-n} q^{mn/2-m^2/4-m/2+n/4} \frac{1}{[n]_q! [m]_q! ([(-\lambda+h]_q)_m} \right) J_q(\lambda).$$

The quadruple sum over $m, l, i, j$ on the right-hand side of (7.9) can be rewritten as a quadruple sum over $m' := m - i$, $l' := l - i$, $s := l - j$, $t := N - m - j$ with $m', l', s, t \geq 0$ and $N - l' - m' + s - t$ even. Write $m', l'$ again as $m, l$. Then the right-hand side of (7.9) becomes:

$$\sum_{m,l,s,t \geq 0} \frac{1}{[1/2](-m+s-t)_q!} \frac{1}{[1/2](-m-l+s+t)_q!} \frac{1}{[1/2](-l+m+s-t)_q!} \frac{1}{[1/2](-l+m-s-t)_q!} \times q^{\frac{m}{2}((N+l-m-s-t)/2-1)} q^{\frac{t}{2}((N+l-m-s+t)/2-1)} \frac{1}{[t]_q! [s]_q! ([(-\lambda+h]_q)_m} \times \frac{(q^{-(\lambda+h(2))/2} f)^{e^{m} q^{th/4} q^{mh/4}}}{[i]_q! [m]_q!} \otimes \frac{(q^{-\lambda/2} f)^{e^{m} q^{sh/4} q^{mh/4}}}{[t]_q! [s]_q! ([(-\lambda+h]_q)_s}$$

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with inverse identity. We obtain:

\[ N \]

The inverse of

\[ M \]

This yields (7.3) with (7.4) substituted.

Then, by use of (2.20), we obtain

\[ \sum_{m,l,t,s \geq 0} \frac{(q; q)^{\frac{1}{2}(N+l-m-s-t)}(q; q)^{\frac{1}{2}(N+l-m-s-t)}}{(q; q)^{\frac{1}{2}(N-l-m-s-t)}(q; q)^{\frac{1}{2}(N+l-m-s-t)}} \frac{q^{2m/2-4m/2+t/4}}{(1-q)^t} \left( \lambda - \lambda \right) \]

\[ \times \left( \frac{(q^{-\lambda+h(2)/2})/f \sum_{m=0}^{\infty} q^{lm/2-m^2/4-m/2+t/4}}{(1-q)^t} \right) \frac{q^{ls/2-s^2/4-s/2+t/4}}{(1-q)^t} \left( \lambda - \lambda \right) s \]

Now consider identity (7.3) (with both sides rewritten as above) with N replaced by 2N and with N replaced by 2N + 1, add these two identities, and let \( N \rightarrow \infty \) in the resulting identity. We obtain:

\[ \Delta \left( \sum_{m,n=0}^{\infty} \frac{q^{mn/2-m^2/4-m/2+n/4}}{(1-q)^n} \frac{(q^{-\lambda/2 f})^n q^{m/4} q^{-m/4}}{([m]_q! [n]_q! ([\lambda + h]_q)_m)} \right) J_q(\lambda) \]

\[ = \sum_{m,l=0}^{\infty} \frac{q^{lm/2-m^2/4-m/2+t/4}}{(1-q)^t} \left( \lambda - \lambda \right) \sum_{t,s=0}^{\infty} \frac{q^{ts/2-s^2/4-s/2+t/4}}{(1-q)^t} \left( \lambda - \lambda \right) s \]

This yields (7.3) with (7.4) substituted. \( \square \)

**Proposition 7.2.** The inverse of \( M_q(\lambda) \) formally exists. It is given by

\[ M_q^{-1}(\lambda) = \frac{(q^{-\lambda-2}; q^{-1})_\infty}{(q^{-\lambda+x}; q^{-1})_\infty} \sum_{m,n=0}^{\infty} \frac{(-1)^m (1-q)^{m-2n} q^{\frac{1}{2} m^2 - \frac{1}{2} n^2 - \frac{1}{2} m + n + (\frac{x}{2} - \frac{1}{2} \lambda) + 2n}}{([m]_q! [n]_q! ([\lambda + 2]_q)_n)} f^n e m q^{(n+m)h}. \]

**Proof** We reason as in [23] \& 7. By (2.14), (2.15) and (2.17) we see that \( M_q(\lambda) \) is invertible with inverse

\[ \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m-1)/2} (1-q)^{2m} q^{\frac{1}{2} m^2 - \frac{1}{2} n^2 - \frac{1}{2} m + n + (\frac{x}{2} - \frac{1}{2} \lambda)} (e q^{\frac{1}{2} h})^m}{(q; q)_m} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-h)/2} (f q^{\frac{1}{2} h})^n}{(q; q)_n} \]

Then, by use of (2.20), we obtain

\[ \sum_{m,n=0}^{\infty} \frac{(-1)^n q^{m^2/2-m^2/2-n^2/2-mn-n/2 + n + (\frac{x}{2} - \frac{1}{2} \lambda)} (e q^{\frac{1}{2} h})^m}{(1-q)^n [m]_q! [n]_q!} \frac{q^{\frac{1}{2} (n-m)h}}{([\lambda - h - 2m + 2n + 2]_q)_m} \]

\[ \times \sum_{k=0}^{\infty} \frac{q^{\frac{1}{2} k^2} q^{k(n-h)/2}}{(1-q)^k [k]_q!} \frac{([\lambda - h - m + n]_q)_k}{([\lambda - h - m + n]_q)_k}. \]
The inner sum can be formally written as

\[ 1 \phi_1 \left( \frac{q^{h+m-n}}{q^{-\lambda h+m-2n-2}; q^{-1}}, q^{-n-\lambda-2} \right) = \frac{(q^{-n-\lambda-2}; q^{-1})_\infty}{(q^{-\lambda h+2m-2n-2}; q^{-1})_\infty} \]

where we used (2.10). This leads to (7.10) by use of (2.19). \( \square \)

**Lemma 7.3.** If \( \mu(\lambda) \) is any solution (like \( \mathcal{M}_q(\lambda) \)) of (7.3) and if \( \gamma(\lambda) = \phi(\lambda - h)/\phi(\lambda) \) is a formal element not depending on \( e \) and \( f \), defined in terms of some nonzero function \( \phi \) of one variable, then \( \mu(\lambda) \gamma(\lambda) \) satisfies (7.3).

**Proof** \( \Delta(\gamma(\lambda)) \) commutes with \( J_q(\lambda) \) since \( h \otimes 1 + 1 \otimes h \) commutes with \( f^l \otimes e^l \) for all \( l \). Hence

\[ \Delta(\mu(\lambda) \gamma(\lambda)) \] 

\[ J_q(\lambda) = \Delta(\mu(\lambda)) J_q(\lambda) \phi(\lambda - h \otimes 1 - 1 \otimes h) \]

\[ = \left( \mu(\lambda - h^{(2)}) \otimes \mu(\lambda) \right) \phi(\lambda - h \otimes 1 - 1 \otimes h) \phi(\lambda - 1 \otimes h) \phi(\lambda) \]

\[ = \left( \mu(\lambda - h^{(2)}) \otimes \mu(\lambda) \right) \left( \frac{\phi(\lambda - h^{(2)} - h)}{\phi(\lambda - h^{(2)})} \otimes \frac{\phi(\lambda - h)}{\phi(\lambda)} \right) = \mu(\lambda - h^{(2)}) \gamma(\lambda - h^{(2)}) \otimes \mu(\lambda) \gamma(\lambda). \]

\( \square \)

**Remark 7.4.** The successive substitutions (6.7) send \( M(x) \) (given by (7.1)) exactly to the double sum in (7.10). Hence we have

\[ M(x)^{-1} \rightarrow \mathcal{M}_q(\lambda) \frac{(q^{-n-2}; q^{-1})_\infty}{(q^{-\lambda h+2n-2}; q^{-1})_\infty} \] (under successive substitutions (6.7)).

By Lemma 7.3, the right-hand side of (7.11) satisfies (7.3) since \( \mathcal{M}_q(\lambda) \) does so. This agrees with the result in [3] that \( M(x) \) satisfies (7.2). Indeed, (7.2) yields (after the substitutions (6.7)) equation (7.3) for \( M(x)^{-1} \).

**Remark 7.5.** Rosengren [23], working with generators \( X_+, X_-, K, K^{-1} \) for \( \mathcal{U}_q \) which satisfy relations and \( \ast \)-structure

\[ KX_+ K^{-1} = q^{\pm \frac{1}{2}} X_+, \quad [X_+, X_-] = \frac{K^2 - K^{-2}}{q^2 - q^{-2}}, \quad K^* = K, \quad (X_\pm)^* = -X_\mp, \]

(7.12)
introduced (see [23, (3.2)]) a generalized element in $U_q$ given by

$$U_{\lambda \mu} := E_q(\mu q^{-\frac{1}{4}}(1-q)X_+ K^{-1}) A_q(q\lambda \mu K^{-4}, q^{\frac{1}{2}}(1-q)\lambda X_- K^{-1}) \frac{(q\lambda \mu K^{-4}; q)_{\infty}}{(q\lambda \mu; q)_{\infty}}.$$  \hfill (7.13)

In his lecture [24] Rosengren next observed a relationship between Babelon, Bernard & Billey’s shifted boundary $M(x)$ and his $U_{\lambda \mu}$ in $U_q$, but he did not give the exact correspondence. In fact, this correspondence is

$$U_{\lambda \mu}^* \rightarrow M(x) \quad \text{under the successive substitutions given by} \quad \hfill (7.14)$$

$$q \rightarrow q^2, \quad X_- \rightarrow E_+, \quad X_+ \rightarrow E_-, \quad K^{-1} \rightarrow q^{\frac{1}{2}}H, \quad \lambda \rightarrow xq^{-\frac{1}{4}}, \quad \mu \rightarrow xq^{\frac{1}{2}}. \hfill (7.15)$$

If we combine identity (7.14) and substitutions (7.15) with identity (7.11) and substitutions (6.7) then we obtain a relationship between Rosengren’s generalized element and our $M_q(\lambda)$:

$$(U_{\lambda \mu}^*)^{-1} \rightarrow M_q(\lambda) \frac{(q^{-\lambda+2}; q^{-1})_{\infty}}{(q^{-\lambda+H-2}; q^{-1})_{\infty}} \quad \text{under the successive substitutions given by} \quad \hfill (7.16)$$

$$q \rightarrow q^{-1}, \quad X_- \rightarrow \frac{q^2}{1-q} f, \quad X_+ \rightarrow \frac{1-q}{q^{\frac{1}{2}} e}, \quad K^{-1} \rightarrow q^{\frac{1}{2}}H, \quad \lambda \rightarrow q^{-\frac{1}{2}}H, \quad \mu \rightarrow q^{-\frac{1}{2}}H - \frac{1}{2}.$$  \hfill (7.17)

This can also be obtained by comparing [23, (3.4)] directly with (7.4).

**Remark 7.6.** Rosengren [23, (4.8)] gives a generalized conjugation in $U_q$ using his element $U_{\lambda \mu}$. This can be translated by (7.16) and (7.17) into a generalized conjugation using $M_q(\lambda)$:

$$(M_q(\lambda))^{-1} q^{-\frac{1}{2}H} \left( q^{-\frac{1}{2}(\lambda+3) - \frac{1}{2}} (1-q)e - (1-q)^{-\frac{1}{2}} q^{-\frac{1}{2}(\lambda-1) - \frac{1}{2}} f \right. \hfill (7.18)$$

$$+ \left( q^{-\lambda-1} + 1 \right) \frac{q^{-\frac{1}{2}H} - q^\frac{1}{2} h}{q^\frac{1}{2} - q^{-\frac{1}{2}}} \right) M_q(\lambda) = \frac{q^{-\lambda-1}(q^\frac{1}{2} h - 1) + (q^{-\frac{1}{2}H} - 1)}{q^\frac{1}{2} - q^{-\frac{1}{2}}}.$$  

Note that (7.18), in the form with both sides multiplied on the left by $M_q(\lambda)$, can also be proved in a more straightforward way by brute force: use relations (2.18), (2.19) in order to pull $q^{-\frac{1}{2}H} e$, $q^{-\frac{1}{2}H} f$ and $q^{-\frac{1}{2}H}$ through each term of the double series (7.1).
Remark 7.7. It is not possible to take limits for $q \to 1$ in formula (7.4) for the inverse shifted boundary as it is given there, although straightforward limit cases for $q \to 1$ are possible for all other formulas defining or involving the (universal) fusion matrix. The obstruction for taking limits in (7.4) is by the factor $(1 - q^{-n})$.

After rescaling by putting $q^{-\frac{1}{2}\lambda} = c(1 - q)$ (c constant), i.e. $\lambda = -2\log(c(1-q))/\log q$, a limit for $q \to 1$ in (7.4) becomes possible. The limit of $M_q(-2\log(c(1-q))/\log q)$ is $\exp(cf)$, a group element of $SL(2)$. With the same substitution of $\lambda$, the universal fusion matrix $J_q(\lambda)$ given by (6.1) tends to $1 \otimes 1$ as $q \to 1$. Then (7.3) degenerates to $\Delta(\exp(cf)) = \exp(cf) \otimes \exp(cf)$ (i.e., $\exp(cf)$ is group-like) and (7.18) to

$$\exp(-cf)(cf - \frac{1}{2} h)\exp(cf) = -\frac{1}{2} h, \text{  i.e. } \exp(ad cf)(h) = h - 2cf.$$ 

8 The universal fusion matrix and the ABRR equation

Etingof and Schiffmann [12, Theorem 8.1 and Appendix B] showed that the universal fusion matrix $J_q(\lambda)$ is the unique solution of the form (6.3) of the equation that they have called the ABRR equation (in reference to [2]). They showed that this is also the case when $q = 1$, and used this to compute $J(\lambda)$ for $U(sl(2))$ (see their example after Theorem 8.1). Their result coincides with our expression of the universal fusion matrix in the classical limit.

In the following we will show directly that our explicit expression of the universal fusion matrix for $U_q(sl(2))$ verifies the ABRR equation for $U_q(sl(2))$.

We first rewrite the ABRR equation for $U_q(sl(2))$ following our conventions in section 2 for the definition of $U_q(sl(2))$. (This slightly differs from the expression given in [12], where the the conventions of [7, Chapter 6] are used.) Thus the ABRR equation becomes:

$$J_q(\lambda)(1 \otimes q^{\theta(\lambda)}) = \mathcal{R}^{21}_0(1 \otimes q^{\theta(\lambda)})J_q(\lambda)$$

(8.1)

with $\mathcal{R}_0 := q^{-\frac{1}{2}(h \otimes h)}$ and $\theta(\lambda) := \frac{1}{2}(\lambda + 1)h - \frac{1}{2} h^2$. Since

$$[h \otimes h, q^{\frac{1}{2}jh e^j \otimes q^{-\frac{1}{2}jh f^j}}] = (-2j(h \otimes 1) + 2j(1 \otimes h) + 4j^2)(q^{\frac{1}{2}jh e^j \otimes q^{-\frac{1}{2}jh f^j}},$$

we have

$$q^{\frac{1}{2}(h \otimes h)}(q^{\frac{1}{2}jh e^j \otimes q^{-\frac{1}{2}jh f^j})q^{-\frac{1}{2}(h \otimes h)}) = q^{j^2} q^{-\frac{1}{2}jh e^j \otimes q^{\frac{1}{2}jh f^j}},$$

and thus, by (2.25),

$$\mathcal{R}^{21}_0 = \sum_{l=0}^{\infty} \left(\mathcal{R}^{(l)}_0\right)^{21}$$

(8.2)

with

$$\left(\mathcal{R}^{(l)}_0\right)^{21} := \frac{(1 - q^{-1})^l q^{-\frac{1}{2}(l-1)q^{2j^2}} q^{\frac{1}{2}lh f^l \otimes q^{-\frac{1}{2}lh e^l}}}{[l]_q!}.$$  

(8.3)
we obtained the value

By (2.3) the last sum equals

Substitution of (6.3) and (8.2) in (8.1) yields:

\[
\sum_{n=0}^{\infty} J_q^{(n)}(\lambda) = \sum_{l,m=0}^{\infty} \left( R_0^{(l)} \right)^2 (1 \otimes q^{\frac{l}{2}(\lambda+1)h-\frac{1}{4}h^2}) J_q^{(m)}(1 \otimes q^{\frac{1}{4}h^2-\frac{1}{4}(\lambda+1)h})
\]

\[
= \sum_{l,m=0}^{\infty} \left( R_0^{(l)} \right)^2 (1 \otimes q^{(\lambda+1)m+m^2-mh}) J_q^{(m)}(\lambda).
\]

Hence, by uniqueness of expansion in view of the PBW theorem:

\[
J_q^{(n)}(\lambda) = \sum_{l+m=n} \left( R_0^{(l)} \right)^2 (1 \otimes q^{(\lambda+1)m+m^2-mh}) J_q^{(m)}(\lambda).
\]

(8.4)

Since \( \left( R_0^{(0)} \right)^2 = 1 \otimes 1 \) and \( 1 \otimes q^{(\lambda+1)m+m^2-mh} \) is invertible, the terms \( J_q^{(n)}(\lambda) \) are uniquely determined by the recurrence (8.4) together with the starting value \( J_q^{(0)}(\lambda) = 1 \otimes 1 \). In (8.4) we obtained

\[
J_q^{(m)}(\lambda) = \frac{q^{-\frac{1}{4}m(\lambda+1)}}{[m]_q!} f^m q^{\frac{1}{4}mh} \otimes e^m \frac{q^{\frac{1}{2}mh}}{([-\lambda + h]_q)_m}.
\]

(8.5)

We will have another proof of (8.5) if we can show that (8.4) is valid after substitution of (8.3) and (8.5). This is now straightforward. After the substitutions just mentioned the right-hand side of (8.4) becomes:

\[
\sum_{l+m=n} \frac{(1 - q^{-1})^l}{[l]_q! [m]_q!} q^{-\frac{1}{4}(l-1)q^2(\lambda+1)m-m^2-m} q^{-lm} \left( f^l+m q^{\frac{1}{2}(l+m)h} \otimes e^{l+m} q^{-\frac{1}{2}(l+3m)h} \frac{1}{([-\lambda + h]_q)_m} \right) = (1 - q^{-1})^n q^{-\frac{1}{4}h(n-1)}
\]

\[
\times f^n q^{\frac{1}{4}nh} \otimes e^n q^{\frac{1}{4}nh} \sum_{m=0}^{n} \frac{(1 - q^{-1})^{-m}q^{\frac{1}{4}m(2n-m-1)q^2(\lambda+1)m-m^2} q^{-m(n-m)} [n-m]_q [m]_q !([-\lambda + h]_q)_m}{[n]_q ! [n]_q !} q^{-\frac{1}{4}mh}.
\]

By (2.3) the last sum equals

\[
\frac{1}{[n]_q !} \sum_{m=0}^{n} \frac{(q^{-n}; q)_m}{(q; q)_m (q^{-\lambda+\hbar}; q)_n} q^m = \frac{1}{[n]_q !} \phi_1 \left( q^{-n}, 0 \bigg| q^{-\lambda+\hbar}; q, q \right) = \frac{(-1)^n q^{n(-\lambda+\hbar)} q^{\frac{1}{2}n(n-1)}}{[n]_q ! (q^{-\lambda+\hbar}; q)_n},
\]

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where the last equality is obtained by use of the $q$-Vandermonde sum \(^{(2.7)}\). Hence the right-hand side of \((8.4)\) becomes
\[
(-1)^n (1 - q^{-1})^n q^{\frac{1}{2} n(n-1)} q^{-n\lambda} f^n q^{\frac{1}{2} nh} \otimes e^n \frac{q^{\frac{1}{2} nh}}{(q^{-\lambda + h}; q)_n},
\]
which equals $J^{(n)}_q(\lambda)$ as given by \((8.5)\).

9 The exchange matrix

**Definition 9.1.** Let $V$, $W$ be two $U_q$-modules, $J_{V,W}(\lambda)$ the fusion matrix, and $\mathcal{R}$ the $\mathcal{R}$-matrix \((2.25)\). The exchange matrix $R_{V,W}(\lambda)$ is defined by
\[
R_{V,W}(\lambda) := J_{V,W}^{-1}(\lambda) \mathcal{R}^{21} J^{21}_{W,V}(\lambda)
\]
\((9.1)\)

(see \([12, \S 2.2]\)), where $J^{21}(\lambda) := PJ(\lambda)P$ and $\mathcal{R}^{21} := P\mathcal{R}P$.

Write the exchange matrix of two Verma modules $M_{q,\gamma}$, $M_{q,\delta}$ as $R_{q,\gamma,\delta}(\lambda) := R_{M_{q,\gamma},M_{q,\delta}}(\lambda)$ and write the matrix elements with respect to their standard bases as
\[
R_{q,\gamma,\delta}(\lambda)(x_{\gamma-2n} \otimes x_{\delta-2s+2m}) = \sum_{m=0}^{s} R_{q,\gamma,\delta,s;m,n}(\lambda) x_{\gamma-2m} \otimes x_{\delta-2s+2m}.
\]
\((9.2)\)

Combination of \((9.1)\) and \((9.2)\) yields that
\[
P\mathcal{R} J_{q,\delta,\gamma}(\lambda) (x_{\delta-2s+2n} \otimes x_{\gamma-2m}) = \sum_{m=0}^{s} R_{q,\gamma,\delta,s;m,n}(\lambda) J_{q,\gamma,\delta}(\lambda) (x_{\gamma-2m} \otimes x_{\delta-2s+2m}).
\]
\((9.3)\)

From \((9.3)\) and \((5.5)\) we see that the following constraints are required in $R_{q,\gamma,\delta,s;m,n}(\lambda)$:
\[
m, n, s \in \mathbb{Z}, \quad 0 \leq m, n \leq s, \quad \lambda - \gamma, \lambda - \delta, \lambda - \delta - \gamma \notin \mathbb{Z}_{\geq 0}.
\]
\((9.4)\)

**Theorem 9.2.** The matrix elements of the exchange matrix can be expressed in terms of $q$-Racah polynomials as follows:
\[
R_{q,\gamma,\delta,s;m,n}(\lambda) = q^{\frac{1}{2}(\delta-2s)} q^{\frac{1}{2}(2n+\delta+\gamma-2s)} q^{\frac{1}{2}(-3\delta+\gamma+6s-2m)} \frac{(q^{-\gamma}; q)_n (q^{-s}, q^{\delta-s+1}; q)_m}{(q^{\lambda-\gamma+n+1}; q)_n (q, q^{\delta-\lambda-2s+m-1}; q)_m} \times R_m(\mu(n), q^{\delta-s}, q^{-\lambda-s-2}, q^{-s-1}, q^{\lambda-\gamma+s+1}).
\]
\((9.5)\)
Here $\mu(n) := q^{-n} + q^{\lambda-\gamma+n+1}$, and the $q$-Racah polynomials are given by

$$R_m(\mu(n), q^\delta-s, q^{-\lambda-s-2}, q^{-s-1}, q^{\lambda-\gamma+s+1}) := 4\phi_3\left(q^{-n}, q^{-m}, q^{\lambda-\gamma+n+1}, q^{-\lambda+\delta-2s+m-1}; q, q\right).$$  \hfill (9.6)

**Proof**  It follows by successive application of (9.3), (5.3), (2.25), (5.6) and (5.7) that:

$$\sum_{m=0}^{s} R_{q,\gamma,\delta,s;0,n}(\lambda) (x_{\gamma-2m}^\delta \otimes x_{\delta-2s+2n}^\delta) = \sum_{k=0}^{n} J_{q,\delta,\gamma,s;k,n}(\lambda) J^{-1}_{q,\gamma,\delta}(\lambda) PR(x_{\delta-2s+2k}^\delta \otimes x_{\gamma-2k}^\gamma)$$

$$= \sum_{k=0}^{n} J_{q,\delta,\gamma,s;k,n}(\lambda) \sum_{j=0}^{s-k} \frac{(1-q^{-1})^j}{[j]_q!} q^{-j(j-1)/2} q^{\delta-2s+2j+2j(\gamma-2k-2j)} q^{-1/2} \cdot (x_{\gamma-2k-2j}^\gamma \otimes x_{\delta-2s+2k+2j}^\delta)$$

$$= \sum_{k=0}^{n} \sum_{m=k+j}^{s} \sum_{l=0}^{s-k} q^{-l(l-1)/2} q^{\delta-2s+2k+2j}(\gamma-2k-2j) q^m \cdot (x_{\gamma-2k-2j}^\gamma \otimes x_{\delta-2s+2k+2j}^\delta)$$

$$= \sum_{k=0}^{n} \sum_{m=k+j}^{s} \sum_{l=0}^{s-k} q^{-l(l-1)/2} q^{\delta-2s+2k+2j}(\gamma-2k-2j) q^m \cdot (x_{\gamma-2k-2j}^\gamma \otimes x_{\delta-2s+2k+2j}^\delta).$$

The most inner sum equals

$$2\phi_0\left(q^{-m+k}, q^{-\lambda+\delta-2s+m+k-1}; q, q^{-2k-\delta+2s+\lambda+1}\right) = q^{(k-m)(-\lambda+\delta-2s+m+k-1)}$$

by the limit case of the $q$-Chu-Vandermonde sum (2.9). Now substitute (5.4) and interchange
the summations over \( k \) and \( m \). Then, in view of (5.3), we obtain:

\[
R_{q,\gamma,\delta,s;m,n}(\lambda) = \frac{q^{\frac{1}{2}(\delta-2s)}q^{\frac{1}{2}(\delta+\gamma-2\lambda-2s-2)}q^{\frac{1}{2}(2\lambda-3\delta+\gamma+6s-4m+2)}([-s]_q)_m([-\delta+s+1]_q)_m([-\gamma]_q)_n}{[m]_q!([-\lambda+\delta-2s+m-1]_q)_m([-\gamma+n+1]_q)_n}
\]

\[
\times \sum_{k=0}^{m+n} \frac{([-n]_q)_k([-m]_q)_k([-\delta+s+m-1]_q)_k}{[k]_q!([-s]_q)_k([-\gamma]_q)_k([-\delta+s]_q)_k},
\]

which can be rewritten as (9.5). \( \square \)

Note that in the above proof, the \( j \)-sum and its evaluation would not occur in the corresponding \( q = 1 \) case (the exchange matrix is defined then by \( R_{V,W}(\lambda) := J_{V,W}^1(\lambda) J_{W,V}^{21}(\lambda) \)).

**10 The exchange matrix and q-Racah coefficients**

The \( q \)-Racah coefficients arise as in the classical case when one considers two different ways of decomposing the tensor product of three finite dimensional irreducible representations of \( U_q \) (see for example [5] for the \( q \)-case and [6] for the \( q = 1 \) case). Use the notation of Remarks 2.1 and 4.2. Let \( j_1, j_2, j_3, j \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) be such that

\[
j + j_2 + j_3 - j, j_1 + j_2 - j_3 + j, j_1 - j_2 + j_3 + j, -j_1 + j_2 + j_3 + j \in \mathbb{Z}_{\geq 0},
\]

and let \( j_{12} \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) be such that

\[
|j - j_3| \lor |j_1 - j_2| \leq j_{12} \leq (j + j_3) \land (j_1 + j_2).
\]

Inequalities (10.1) give precisely the condition that \( V^j \) occurs at least once in \( V^{j_1} \otimes V^{j_2} \otimes V^{j_3} \). Furthermore, (10.1) combined with (10.2) is precisely the condition that \( V^{j_{12}} \) occurs in \( V^{j_1} \otimes V^{j_2} \) and that \( V^j \) occurs in \( V^{j_{12}} \otimes V^{j_3} \). Let \( e_{m}^{j_{12}}(j_1, j_2 \mid j_3) (m = -j - j + 1, \ldots, j) \) form the standard basis of \( V^j \) within \( V^{j_{12}} \otimes V^{j_3} \) within \( V^{j_1} \otimes V^{j_2} \otimes V^{j_3} \), where we twice used the normalization of a standard basis of an irreducible submodule of a tensor product as described in Remark 4.2.

If we combine this definition of \( e_{m}^{j_{12}}(j_1, j_2 \mid j_3) \) with formula (4.13), applied twice, then we obtain:

\[
e_{m}^{j_{12}}(j_1, j_2 \mid j_3) = (\Phi_{q,j_{12}}^{j_2} \otimes \text{id}) \circ \Phi_{q,j_{12}}^{j_3} \circ e_{m}^{j_{12}}(j_1, j_2 \mid j_3).
\]

Similarly as above, let \( e_{m}^{j_{23}}(j_1, j_2, j_3) \) \( (m = -j, -j + 1, \ldots, j) \) form the standard basis of \( V^j \) within \( V^{j_{12}} \otimes V^{j_{23}} \) within \( V^{j_1} \otimes V^{j_2} \otimes V^{j_3} \). Then the \( q \)-Racah coefficients \( q W_{j_{12} - j_1, j - j_{12}, j_1}^{j_{23}}(j) \) are defined by:

\[
e_{m}^{j_{12}}(j_1, j_2 \mid j_3) = \sum_{j_{23}} q W_{j_{12} - j_1, j - j_{12}, j_1}^{j_{23}}(j) e_{m}^{j_{23}}(j_1 \mid j_2, j_3).
\]
where we sum over
\[|j - j_1| \lor |j_2 - j_3| \leq j_{23} \leq (j + j_1) \land (j_2 + j_3). \tag{10.5}\]

The notation we use for \(q\)-Racah coefficients is defined in [5, Definition 3.72].

Graphically (see [16]), the definition of \(q\)-Racah coefficients has the form
\[
\begin{align*}
&\quad \begin{array}{c}
\text{j}_1 \\
\downarrow
\end{array}
\begin{array}{c}
\text{j}_2 \\
\downarrow
\end{array}
\begin{array}{c}
\text{j}_3 \\
\downarrow
\end{array}
= \sum_{j_{23}} q W_{j_{12} - j_1, j_2 - j_3}^{j_1, j_2, j_3, j}(j)
\end{align*}
\tag{10.6}
\]

Here, for the diagram both on the left and on the right one has to substitute the \(m\)th standard basis vector in the module isomorphic to \(V^j\) which is evidently determined within \(V^{j_1} \otimes V^{j_2} \otimes V^{j_3}\) by the corresponding diagram.

Because the choice of \(m\), above and in the sequel, is irrelevant, we do not need to put \(m\) in the diagram. Therefore, unlike as in [16], we put \(j\)-labels on the vertices of the diagrams and we do not label the edges.

We will need the following formula stated in [16] (5.11).\]
\[
\begin{align*}
(P \mathcal{R})_{23} e_{m}^{j_{13} j}(j_3, j_1 | j_2) = & \sum_{j_{12}} (-1)^{j_{12} + j_{13} - j_{-1} j_1} q^j(c_j + c_{j_1} - c_{j_{13}} - c_{j_{12}}) \\
& \times q W_{j_{13} - j_3, j - j_1, j_2}^{j_1, j_2, j_3, j}(j) e_{m}^{j_{12} j}(j_1, j_2 | j_3). \tag{10.7}
\end{align*}
\]

Here \(c_j := j(j + 1)\) as before. Formula (10.7) can be written graphically as follows.
\[
\begin{align*}
&\quad \begin{array}{c}
\text{j}_1 \\
\downarrow
\end{array}
\begin{array}{c}
\text{j}_2 \\
\downarrow
\end{array}
\begin{array}{c}
\text{j}_3 \\
\downarrow
\end{array}
= \sum_{j_{12}} (-1)^{j_{12} + j_{13} - j_1 j_2} q^j(c_j + c_{j_1} - c_{j_{13}} - c_{j_{12}}) q W_{j_{13} - j_3, j - j_1, j_2}^{j_1, j_2, j_3, j}(j)
\end{align*}
\tag{10.8}
\]

The interpretation of the diagrams is similar as we explained for (10.6).

**Proof of (10.7)**

Rewrite (10.6) as
\[
\begin{align*}
&\quad \begin{array}{c}
\text{j}_1 \\
\downarrow
\end{array}
\begin{array}{c}
\text{j}_2 \\
\downarrow
\end{array}
\begin{array}{c}
\text{j}_3 \\
\downarrow
\end{array}
= \sum_{j_{12}} q W_{j_{13} - j_3, j - j_1, j_2}^{j_1, j_2, j_3, j}(j)
\end{align*}
\tag{10.9}
\]
Then, by the first identity in (2.26), we have

\[(PR)_{23}(PR)_{12} = P_{23}P_{12}R_{13}R_{12} = P_{1,23}(\text{id} \otimes \Delta)R = (\Delta \otimes \text{id})(PR).\]  

(10.10)

If we let the first and last part of (10.10) act on a threefold tensor product, then this implies an operator identity

\[(PR)_{23} \circ (PR)_{12} = (PR)_{1,23}.\]  

(10.11)

If we let the left-hand side and the right-hand side of (10.11) respectively act on the left-hand side and right-hand side of (10.9) then we obtain by twofold application of (4.16):

\[(-1)^{j_3-j_1-j_2} q^{\frac{j_1}{2}}(c_{j_1} - c_{j_2} - c_{j_3}) (PR)_{23} = \sum_{j_{12}} (-1)^{j_2-j_1-j_3} \frac{j_1}{2} c_{j_1} c_{j_2} c_{j_3} \times q^{\frac{j_3}{2}}(c_{j_3} - c_{j_2} - c_{j_1}) (PR)_{12}.\]  

(10.12)

This is equivalent to (10.8). □

**Theorem 10.1.** The q-Racah coefficients can be expressed in terms of matrix elements of the exchange matrix as follows:

\[(-1)^{j_1+j_2-j_3} q^{\frac{j_1}{2}}(c_{j_1} + c_{j_2} - c_{j_3}) W_{j_13-j_2j_13}^{j_1j_2j_13} j_3 = \tilde{R}_{j_13-j_2j_13}^{j_1j_2j_13} (j_3) = \tilde{R}_{j_2j_3}^{j_3j_2} (j_1) \]

(10.13)

Here \(j_1, j_2, j_3, j, j_{12}, j_{13} \in \frac{1}{2}\mathbb{Z}_{\geq 0}\) are constrained by (10.1), (10.2) and by (10.14) below:

\[|j - j_2| \vee |j_3 - j_1| \leq j_{13} \leq (j + j_2) \wedge (j_3 + j_1).\]  

(10.14)

**Furthermore,** \(c_j := j(j + 1).\)

**Proof** First observe from (9.3), (3.9) and (6.1) that

\[(\text{id} \otimes PR) \circ (\Phi_{q,\lambda - \gamma + 2n}^{x_1} \otimes \text{id}) \circ \Phi_{q,\lambda}^{x_1-2n} = \sum_{m=0}^{s} R_{q,\gamma,\delta, s, m,n} (\lambda) \Phi_{q,\lambda - \delta + 2s - 2m}^{x_1-2m} \otimes \text{id} \circ \Phi_{q,\lambda}^{x_1-2s-2m}.\]  

(10.15)
Put
\[ \lambda = 2j, \quad \gamma = 2j_2, \quad \delta = 2j_3, \quad s = j_1 + j_2 + j_3 - j, \quad n = j_{13} + j_2 - j. \]

It follows by Remark 4.2 that formula (10.15) remains valid for \( j_1, j_2, j_3, j, j_{13} \) being constrained as stated in the theorem. (Note that the constraints (10.1), (10.2) and (10.14) are obtained from the constraints (10.1), (10.5) and (10.2), respectively, by replacing \( j_1, j_2, j_3, j_{12}, j_2 \) by \( j_3, j_1, j_2, j_{13}, j_{12} \).) Formula (10.15) then becomes:

\[
(id \otimes P\mathcal{R}) \circ (\Phi_{q,2j_{13}}^{x_{j_{13}}-2j_1} \otimes id) \circ \Phi_{q,2j}^{x_2j_2} = \sum_{j_{12}=j-j_3|\geq j_1-j_2} R_{q,2j_2,2j_3,j_1+j_2+j_3-j;j_1+j_2-j_{12},j_2+j_{13}-j-j(2j)} (\Phi_{q,2j_{12}}^{x_2j_2} \otimes id) \circ \Phi_{q,2j}^{x_2j_2-2j_{12}}.
\]

Next substitute (11.12) four times in this last identity. Then:

\[
(id \otimes P\mathcal{R}) \circ (\Phi_{q,j_{13}}^{x_{j_{13}}-j_1} \otimes id) \circ \Phi_{q,j}^{x_{j_2}-j_3} = \sum_{j_{12}=j-j_3|\geq j_1-j_2} R_{q,2j_2,2j_3,j_1+j_2+j_3-j;j_1+j_2-j_{12},j_2+j_{13}-j-j(2j)} (\Phi_{q,j_{12}}^{x_2j_2} \otimes id) \circ \Phi_{q,j}^{x_2j_2-2j_{12}}.
\]

Let both sides of this identity act on \( e_m^j \), then substitute (10.13) twice in the resulting identity. This yields:

\[
(P\mathcal{R})_{23} e_{m}^{j_3-j}(j_3, j_1 | j_2) = \sum_{j_{12}=j-j_3|\geq j_1-j_2} R_{q,2j_2,2j_3,j_1+j_2+j_3-j;j_1+j_2-j_{12},j_2+j_{13}-j-j(2j)} e_{m}^{j_1+j_2}(j_1, j_2 | j_3).
\]

Finally compare with formula (10.7). \( \Box \)

In the literature and similar to the classical case, \( q \)-\( 6j \) coefficients are defined in terms of \( q \)-Racah coefficients (see either [5 §3.6.1] or [16] p. 307):

\[
\left\{ \begin{array}{ll} j_3 & j_1 \\ j_2 & j \end{array} \right\}_q := (-1)^{j_1+j_2+j_3} ([2j_2+1]_q [2j_1+1]_q)^{-\frac{1}{2}} W_{j_1,j_2,j_3}^{j_1,j_2,j_3}. \quad (10.16)
\]

An explicit expression of \( q \)-\( 6j \) coefficients in terms of \( 4\phi_3 \) \( q \)-hypergeometric functions has
been derived in [13] and in [16]. We use here the expression as a finite sum given in [5] (3.69):

\[
\left\{ \begin{array}{ccc} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{array} \right\} _q \Delta(j_1, j_2, j_{12}) \Delta(j_3, j_1, j_{13}) \Delta(j, j_{12}, j_{13}) \Delta(j, j_2, j_{13}) \\
\times \sum_n \frac{(-1)^n [n+1]_q!}{[n-j_1-j_2-j_{12}]_q! [n-j_1-j_3-j_{13}]_q! [n-j-j_2-j_{12}]_q! [n-j-j_3-j_{13}]_q!}
\times \frac{1}{[j_1+j_2+j_3+j-n]_q! [j_2+j_3+j_{12}+j_{13}-n]_q! [j_1+j+j_{12}+j_{13}]_q!}
\end{array} \right)
\]

(10.17)

where the summation range is

\[
\max(j_1+j_2+j_{12}, j_1+j_3+j_{13}, j_2+j+1+j_{13}, j_3+j+j_{12}) \leq n \leq \min(j_1+j_2+j_3+j, j_2+j_3+j_{12}+j_{13}, j_1+j+j_{12}+j_{13})
\]

(10.18)

and where

\[
\Delta(j_1, j_2, j) := \left( \frac{[j_1+j_2+j]_q! [j_1-j_2+j]_q! [j_1+j_2-j]_q!}{[j_1+j_2+j+1]_q!} \right)^{\frac{1}{2}}.
\]

(10.19)

Formula (10.17) may be rewritten as a 4\phi_3 q-hypergeometric function, depending on certain inequalities involving the parameters.

**Remark 10.2.** Equation (10.13) connects \( \phi_{2j_3, j_{12}, j_1+j_2, j_3+j} \) on its left-hand side with \( R_{q, 2j_2, 2j_3, j_1+j_2, j_3+j_1+j_2} \) on its right-hand side. The expression on the left-hand side can be written as a finite sum by (10.16) and (10.17), where the summation bounds depend on certain inequalities involving the parameters. The expression on the right-hand side can be written as a limit case of the finite q-hypergeometric sum (9.6), where the limit will also depend on certain inequalities involving the parameters. The two sums on the two sides of (10.13) were derived in very different ways, but must be equal to each other because of the truth of Theorem (10.1) which we proved in a conceptual way. Let us independently verify that the two sums are equal.

Let \( \gamma, \delta, \lambda, s, m, n \) satisfy the constraints (9.3). By use of Sears’ transformation (2.12), the \( q \)-hypergeometric expression (9.5) of the exchange matrix can be rewritten as follows (the apparent singularity for \( m > n \) is in fact removed):

\[
R_{q, \gamma, \delta, s, m, n}(\lambda) = q^{\frac{1}{4}(\delta-2s)} q^{\frac{1}{4}(2n+\delta+\gamma-2s)} q^{\frac{1}{4}(-3\delta+\gamma+6s-2m)} \frac{(q^{-\gamma}; q)_n}{(q^{\delta-s-1}; q)_m} \frac{(q^{-s}; q^{\delta-s+1}; q)_m}{(q^{\lambda-\gamma+n+1}; q)_n (q^{\delta-\lambda-2s+m-1}; q)_m}
\times \frac{1}{(q^{\delta-s-m+n+\lambda-\delta-\gamma}; q)_m} \frac{1}{(q^{-\gamma}; q^{-s}, q^{s-m-\delta}; q)_m} 4\phi_3(q^{s-m+n+\gamma-\delta-1}, q^{\lambda-s-1}, q^{m-s}, q^{s-m-\delta}; q)
\]

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The exchange matrix satisfies the quantum dynamical Yang-Baxter equation (QDYBE) side of (10.13) with (10.20) and (4.11) substituted.

Now the left-hand side of (10.13) with (10.16) and (10.17) substituted equals the right-hand side of (10.13) as assumed there, and if we pass to the new summation variable \( k := j + j_1 + j_2 + j_3 - t \), then we obtain

\[
R_{q,2j_2,2j_3,j_1,j_2,j_3,j_1,j_2,j_3,j_1,j_2,j_3,t} \left( \frac{1}{2} \right) = (-1)^{j_1 + j_2 + j_3} q^{4j_1 + j_2 + j_3} \left( \frac{1}{2} \right)
\]

where the summation range is as in (10.13).

Now the left-hand side of (10.13) with (10.16) and (10.17) substituted equals the right-hand side of (10.13) with (10.20) and (4.11) substituted.

\[11\] QDYBE and q-Racah coefficients

The exchange matrix satisfies the quantum dynamical Yang-Baxter equation (QDYBE)

\[
R_{q}^{23}(\lambda)R_{q}^{13}(\lambda - h^{(2)})R_{q}^{12}(\lambda) = R_{q}^{12}(\lambda - h^{(3)})R_{q}^{13}(\lambda)R_{q}^{23}(\lambda - h^{(1)}),
\]

see [12] Proposition 2.4 and §2.2 and [13], §4 (which gives some details of proofs in [12]). In the following, we will take the limit of the QDYBE (11.1) to the case of finite dimensional irreducible representations. Together with (10.13) and (10.16) this will yield an identity for sums of products of three \( q \)-6j symbols, earlier known in the literature and expressing a symmetry
of $q$-9$j$ symbols \cite{22} (8.1)]. This consequence of the QDYBE was earlier mentioned, without giving details, in \cite{13} §8.

Let $e_{j_1-j_5}^{j_1}, e_{j_5-j_6}^{j_2}, e_{j_6-j_7}^{j_3}$ be basis vectors of the finite dimensional irreducible $\mathcal{U}_q$-modules $V^{j_1}, V^{j_2}, V^{j_3}$. The action of a limit case of the QDYBE on $(e_{j_4-j_5}^{j_1} \otimes e_{j_5-j_6}^{j_2} \otimes e_{j_6-j_7}^{j_3})$ is then given by the following identity:

$$\sum_{j_8,j_9,j_{10}} R_{q,2j_2,2j_3,j_2+j_3+j_8-j_4,j_4+j_8-j_5,j_5+j_10-j_4} (2j_4) R_{q,2j_1,2j_3,j_1+j_3+j_7-j_{10},j_1+j_7-j_8,j_1+j_6-j_{10}} (2j_{10})$$

$$\times R_{q,2j_1,2j_2,j_1+j_2+j_7-j_{10},j_1+j_7-j_8,j_1+j_6-j_{10},j_1+j_5-j_4} (2j_9) R_{q,2j_1,2j_3,j_1+j_3+j_10-j_{10},j_1+j_10-j_{10},j_1+j_5-j_4} (2j_4)$$

$$\times R_{q,2j_2,2j_3,j_2+j_3+j_7-j_{10},j_2+j_7-j_{10},j_2+j_6-j_{10}} (2j_5) (e_{j_8-j_7}^{j_1} \otimes e_{j_9-j_8}^{j_2} \otimes e_{j_10-j_9}^{j_3}).$$

By use of the defining relation of $\tilde{R}$ in \cite{10,13}, this turns out to be equivalent to the following identity:

$$\sum_{j_{10}} \tilde{R}_{j_1,j_2}^{j_1,j_2} (j_{10}) \tilde{R}_{j_1,j_3}^{j_1,j_3} (j_{10}) \tilde{R}_{j_2,j_3}^{j_1,j_3} (j_{10})$$

$$\sum_{j_{10}} \tilde{R}_{j_1,j_2}^{j_1,j_2} (j_{10}) \tilde{R}_{j_1,j_3}^{j_1,j_3} (j_{10}) \tilde{R}_{j_2,j_3}^{j_1,j_3} (j_{10})$$

which yields, in virtue of \cite{10,13} the following known identity \cite{16} (6.19)] satisfied by $q$-Racah coefficients:

$$\sum_{j_{10}} (-1)^{-j_7+j_8+j_{10}+j_6} q^{(c_{j_7}-c_{j_8}-c_{j_{10}}-c_{j_6})/2}$$

$$q W_{j_1,j_8}^{j_7,j_8,j_{10}} (j_9) q W_{j_8,j_1}^{j_8,j_{10},j_5} (j_4) q W_{j_6,j_5}^{j_7,j_6,j_5} (j_3)$$

$$= \sum_{j_{10}} (-1)^{-j_4+j_9+j_{10}+j_5} q^{(c_{j_4}-c_{j_9}-c_{j_{10}}-c_{j_5})/2}$$

$$q W_{j_1,j_8}^{j_8,j_1,j_8} (j_9) q W_{j_8,j_1}^{j_8,j_{10},j_5} (j_4) q W_{j_6,j_5}^{j_7,j_6,j_5} (j_3).$$

(11.4)

Then, by use of (11.4) with (10.16) substituted and symmetries of $q$-6$j$ symbols (i.e., the $q$-6$j$ symbol is invariant under any permutation of columns and also under an interchange of upper and lower arguments in each of any two of its columns, see \cite{22}), we finally show that $q$-6$j$
symbols satisfy the following identity:

\[
\sum_{j_{10}} (-1)^{2j_{10}} [2j_{10} + 1] q^{-\left(c_{j_{10}}+c_{j_{1}}+c_{j_{6}}+c_{j_{9}}\right)/2} \\
\times \left\{ \begin{array}{ccc} j_2 & j_7 & j_{10} \\ j_1 & j_9 & j_8 \end{array} \right\}_q \left\{ \begin{array}{ccc} j_2 & j_6 & j_5 \\ j_3 & j_{10} & j_7 \end{array} \right\}_q \left\{ \begin{array}{ccc} j_1 & j_5 & j_4 \\ j_3 & j_9 & j_{10} \end{array} \right\}_q \\
= \sum_{j_{10}} (-1)^{2j_{10}} [2j_{10} + 1] q^{-\left(c_{j_{10}}+c_{j_{7}}+c_{j_{9}}+c_{j_{5}}\right)/2} \\
\times \left\{ \begin{array}{ccc} j_1 & j_6 & j_{10} \\ j_3 & j_8 & j_7 \end{array} \right\}_q \left\{ \begin{array}{ccc} j_2 & j_6 & j_4 \\ j_3 & j_9 & j_8 \end{array} \right\}_q \left\{ \begin{array}{ccc} j_2 & j_6 & j_5 \\ j_1 & j_4 & j_{10} \end{array} \right\}_q .
\]

(11.5)

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