A LOCALLY HYPERBOLIC 3-MANIFOLD THAT IS NOT HYPERBOLIC

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Abstract: We construct a locally hyperbolic 3-manifold $M_\infty$ such that $\pi_1(M_\infty)$ has no divisible subgroup. We then show that $M_\infty$ is not homeomorphic to any complete hyperbolic manifold. This answers a question of Agol [DHM06, Mar07].

INTRODUCTION

Throughout this paper, $M$ is always an oriented, aspherical 3-manifold. A 3-manifold $M$ is hyperbolizable if its interior is homeomorphic to $\mathbb{H}^3/\Gamma$ for $\Gamma \leq \text{Isom}(\mathbb{H}^3)$ a discrete, torsion free subgroup. An irreducible 3-manifold $M$ is of finite-type if $\pi_1(M)$ is finitely generated and we say it is of infinite-type otherwise. By Geometrization (2003, [Per03b, Per03c, Per03a]) and Tameness (2004, [Ago04, CG06]) a finite type 3-manifold $M$ is hyperbolizable if and only if $M$ is the interior of a compact 3-manifold $\overline{M}$ that is atoroidal and with non finite $\pi_1(\overline{M})$. On the other hand, if $M$ is of infinite type not much is known and we are very far from a complete topological characterisation. Nevertheless, some interesting examples of these manifolds have been constructed in [SS13, BMNS16]. What we do know are necessary condition for a manifold of infinite type to be hyperbolizable. If $M$ is hyperbolizable then $M \cong \mathbb{H}^3/\Gamma$, hence by discreteness of $\Gamma$ and the classification of isometries of $\mathbb{H}^3$ we have that no element $\gamma \in \Gamma$ is divisible ([Fri11, Lemma 3.2]). Here, $\gamma \in \Gamma$ is divisible if there are infinitely many $\alpha \in \pi_1(M)$ and $n \in \mathbb{N}$ such that: $\gamma = \alpha^n$. We say that a manifold $M$ is locally hyperbolic if every cover $N \rightarrow M$ with $\pi_1(N)$ finitely generated is hyperbolizable. Thus, local hyperbolicity and having no divisible subgroups in $\pi_1$ are necessary conditions. In [DHM06, Mar07] Agol asks whether these conditions could be sufficient for hyperbolization:

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**Question** (Agol). Is there a 3-dimensional manifold $M$ with no divisible elements in $\pi_1(M)$ that is locally hyperbolic but not hyperbolic?

We give a positive answer:

**Theorem 1.** There exists a locally hyperbolic 3-manifold with no divisible subgroups in its fundamental group that does not admit any complete hyperbolic metric.

**Outline of the proof:** The manifold $M_\infty$ is a thickening of the 2-complex obtained by gluing to an infinite annulus $A$ countably many copies of a genus two surface $\{\Sigma_i\}_{i \in \mathbb{Z}}$ along a fixed separating curve $\gamma$ such that the $i$-th copy $\Sigma_i$ is glued to $S^1 \times \{i\}$. The manifold $M_\infty$ covers a compact non-atoroidal manifold $M$ containing an incompressible two sided surface $\Sigma$. Since $\pi_1(M_\infty) \leq \pi_1(M)$ and $M$ is Haken by [Sha75] we have that $\pi_1(M_\infty)$ has no divisible elements. By construction $M_\infty$ has countably many embedded genus two surfaces $\{\Sigma_i\}_{i \in \mathbb{Z}}$ that project down to $\Sigma$. By a surgery argument it can be shown that $M_\infty$ is atoroidal. Moreover, if we consider the lifts $\Sigma_{-i}, \Sigma_i$ they co-bound a submanifold $M_i$ that is hyperbolizable and we will use the $M_i$ to show that $M_\infty$ is locally hyperbolic (see Lemma 4). Thus, $M_\infty$ satisfies the conditions of Agol’s question.

The obstruction to hyperbolicity arises from the lift $A$ of the essential torus $T$. The lift $A$ is an open annulus such that the intersection with all $M_i$ is an embedded essential annulus $A_i = A \cap M_i$ with boundaries in $\Sigma_{\pm i}$. The surfaces $\Sigma_{\pm i}$ in the boundaries of the $M_i$ have the important property that they have no homotopic essential subsurfaces except for the one induced by $A$. This gives us the property that both ends of $A$ see an ‘infinite’ amount of topology. This is in sharp contrast with finite type hyperbolic manifolds in which, by Tameness, every such annulus only sees a finite amount of topology.

In future work we will give a complete topological characterisation of hyperbolizable 3-manifolds for a class of infinite type 3-manifolds. This class contains $M_\infty$ and the example of Souto-Stover [SS13] of a hyperbolizable Cantor set in $S^3$.

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**Notation:** We use $\simeq$ for homotopic and by $\pi_0(X)$ we intend the connected components of $X$. With $\Sigma_{g,k}$ we denote the genus $g$ orientable surface with $k$ boundary components. By $N \hookrightarrow M$ we denote embeddings while $S \hookrightarrow M$ denotes immersions.

1. **Background**

We now recall some fact and definitions about the topology of 3-manifolds, more details can be found in [Hem76, Hat07, Jac80].

An orientable 3-manifold $M$ is said to be **irreducible** if every embedded sphere $S^2$ bounds a 3-ball. A map between manifolds is said to be **proper** if it sends boundaries to boundaries and pre-images of compact sets are compact. We say that a connected properly immersed surface $S \hookrightarrow M$ is $\pi_1$-**injective** if the induced map on the fundamental groups is injective. Furthermore, if $S \hookrightarrow M$ is embedded and $\pi_1$-injective we say that it is **incompressible**. If $S \hookrightarrow M$ is a non $\pi_1$-injective two-sided surface by the Loop Theorem we have that there is a compressing disk $D \hookrightarrow M$ such that $\partial D = D \cap S$ and $\partial D$ is non-trivial in $\pi_1(S)$.

An irreducible 3-manifold $(M, \partial M)$ is said to have **incompressible boundary** if every map: $(D^2, \partial D^2) \hookrightarrow (M, \partial M)$ is homotopic via a map of pairs into $\partial M$. Therefore, $(M, \partial M)$ has incompressible boundary if and only if each component $S \in \pi_0(S)$ is incompressible, that is $\pi_1$-injective. An orientable, irreducible and compact 3-manifold is called **Haken** if it contains a two-sided $\pi_1$-injective surface. A 3-manifold is said to be **acylindrical** if every map $(S^1 \times I, \partial (S^1 \times I)) \hookrightarrow (M, \partial M)$ can be homotoped into the boundary via maps of pairs.

**Definition 1.** A 3-manifold $M$ is said to be **tame** if it is homeomorphic to the interior of a compact 3-manifold $\overline{M}$.

Even 3-manifolds that are homotopy equivalent to compact manifolds need not to be tame. For example the Whitehead manifold [Whi35] is homotopy equivalent to $\mathbb{R}^3$ but is not homeomorphic to it.

**Definition 2.** We say that a codimension zero submanifold $N \hookrightarrow M$ forms a **Scott core** if the inclusion map $\iota_*$ is a homotopy equivalence.

By [Sco73, Sco96, RS90] given an orientable irreducible 3-manifold $M$ with finitely generated fundamental group a Scott core exists and is unique up to homeomorphism.
Let $M$ be a tame 3-manifold, then given a Scott core $C \hookrightarrow M \subseteq \overline{M}$ with incompressible boundary we have that, by Waldhausen’s cobordism Theorem [Wal86], every component of $\overline{M} \setminus C$ is a product submanifold homeomorphic to $S \times I$ for $S \in \pi_0(\partial C)$.

**Definition 3.** Given a core $C \hookrightarrow M$ we say that an end $E \subseteq \overline{M} \setminus C$ is **tame** if it is homeomorphic to $S \times [0, \infty)$ for $S = \partial E$.

A core $C \subseteq M$ gives us a bijective correspondence between the ends of $M$ and the components of $\partial C$. We say that a surface $S \in \pi_0(\partial C)$ **faces the end** $E$ if $E$ is the component of $\overline{M} \setminus C$ with boundary $S$. It is a simple observation that if an end $E$ facing $S$ is exhausted by submanifolds homeomorphic to $S \times I$ then it is a tame end.

**2. Proof of Theorem 1**

Consider a surface of genus two $\Sigma$ and denote by $\alpha$ a separating curve that splits it into two punctured tori. To $\Sigma \times I$ we glue a thickened annulus $C = (S^1 \times I) \times I$ so that $S^1 \times I \times \{i\}$ is glued to a regular neighbourhood of $\alpha \times i$, for $i = 0, 1$. We call the resulting manifold $M$:

![Figure 1. The manifold $M$.](image)

The manifold $M$ is not hyperbolic since it contains an essential torus $T$ coming from the cylinder $C$. Moreover, $M$ has a surjection $p$ onto $S^1$. 
obtained by projecting the surfaces in $\Sigma \times I$ onto $I$ and also mapping the cylinder onto an interval. We denote by $H$ the kernel of the surjection map $p_* : \pi_1(M) \to \pi_1(S^1)$.

Consider an infinite cyclic cover $M_\infty$ of $M$ corresponding to the subgroup $H$. The manifold $M_\infty$ is an infinite collection of $\{\Sigma \times I\}_{i \in \mathbb{Z}}$ glued to each other via annuli along the separating curves $\alpha \times \{0, 1\}$. Therefore, we have the following covering:

\[ \Sigma_i \quad \Sigma_{i+1} \quad \Sigma_{i+2} \]

**Figure 2.** The infinite cyclic cover.

where the $\Sigma_i$ are distinct lifts of $\Sigma$ and so are incompressible in $M_\infty$. Since $\pi_1(M_\infty)$ is a subgroup of $\pi_1(M)$ and $M$ is Haken ($M$ contains the incompressible surface $\Sigma$) by [Sha75] we have that $\pi_1(M)$ has no divisible elements, thus $\pi_1(M_\infty)$ has no divisible subgroups as well.

**Lemma 4.** The manifold $M_\infty$ is locally hyperbolic.

**Proof.** We claim that $M_\infty$ is atoroidal and exhausted by hyperbolizable manifolds. Let $T^2 \hookrightarrow M_\infty$ be an essential torus with image $T$. Between the surfaces $\Sigma_i$ and $\Sigma_{i+1}$ we have incompressible annuli $C_i$ that separate them. Since $T$ is compact it intersects at most finitely many $\{C_i\}$. Moreover, up to isotopy we can assume that $T$ is transverse to all $C_i$ and it minimizes $|\pi_0(T \cap \cup C_i)|$. If $T$ does not intersect any $C_i$ we have that it is contained in a submanifold homeomorphic to $\Sigma \times I$ which is atoroidal and so $T$ wasn’t essential.
Since both $C_i$ and $T$ are incompressible we can isotope $T$ so that the components of the intersection $T \cap C_i$ are essential simple closed curves. Thus, $T$ is divided by $\cup_i T \cap C_i$ into finitely many parallel annuli and $T \cap C_i$ are disjoint core curves for $C_i$. Consider $C_k$ such that $T \cap C_k \neq \emptyset$ and $\forall n \geq k : T \cap C_n = \emptyset$. Then $T$ cannot intersect $C_k$ in only one component, so it has to come back through $C_k$. Thus, we have an annulus $A \subseteq T$ that has both boundaries in $C_k$ and is contained in a submanifold of $M_\infty$ homeomorphic to $\Sigma_{k+1} \times I$. The annulus $A$ gives an isotopy between isotopic curves in $\partial (\Sigma_{k+1} \times I)$ and is therefore boundary parallel. Hence, by an isotopy of $T$ we can reduce $|\pi_0(T \cup C_i)|$ contradicting the fact that it was minimal and non-zero.

We define the submanifold of $M_\infty$ co-bounded by $\Sigma_k$ and $\Sigma_{-k}$ by $M_k$. Since $M_\infty$ is atoroidal so are the $M_k$. Moreover, since the $M_k$ are compact manifolds with infinite $\pi_1$ they are hyperbolizable by Thurston’s Hyperbolization Theorem [Kap01].

We now want to prove that $M_\infty$ is locally hyperbolic. To do so it suffices to show that given any finitely generated $H \leq \pi_1(M_\infty)$ the cover $M_\infty(H)$ corresponding to $H$ factors through a cover $N \twoheadrightarrow M_\infty$ that is hyperbolizable. Let $\gamma_1, \ldots, \gamma_\alpha \subseteq M_\infty$ be loops generating $H$. Since the $M_k$ exhaust $M_\infty$ we can find some $k \in \mathbb{N}$ such that $\{\gamma_i\}_{i \leq \alpha} \subseteq M_k$, hence the cover corresponding to $H$ factors through the cover induced by $\pi_1(M_k)$. We now want to show that the cover $M_\infty(k)$ of $M_\infty$ corresponding to $\pi_1(M_k)$ is hyperbolizable.

Since $\pi : M_\infty \twoheadrightarrow M$ is the infinite cyclic cover of $M$ we have that $M_\infty(k)$ is the same as the cover of $M$ corresponding to $\pi_1(M_k)$. The resolution of the Tameness [Ago04,CG06] and the Geometrization
conjecture [Per03b, Per03c, Per03a] imply the Simon’s conjecture, that is: covers of compact irreducible 3-manifolds with finitely generated fundamental groups are tame [Can08, Sim76]. Therefore, since $M$ is compact by the Simon’s Conjecture we have that $M_\infty(k)$ is tame. The submanifold $M_k \hookrightarrow M_\infty$ lifts homeomorphically to $\tilde{M}_k \hookrightarrow M_\infty(k)$. By Whitehead’s Theorem [Hat02] the inclusion is a homotopy equivalence, hence $\tilde{M}_k$ forms a Scott core for $M_\infty(k)$. Thus, since $\partial \tilde{M}_k$ is incompressible and $M_\infty(k)$ is tame we have that $M_\infty(k) \cong \text{int}(M_k)$ and so it is hyperbolizable.

□

In the infinite cyclic cover $M_\infty$ the essential torus $T$ lifts to a $\pi_1$-injective annulus $A$ that is properly embedded: $A = \gamma \times \mathbb{R} \hookrightarrow M_\infty$ for $\gamma$ the lift of the curve $\alpha \hookrightarrow \Sigma \subseteq M$.

Remark 5. Consider two distinct lifts $\Sigma_i, \Sigma_j$ of the embedded surface $\Sigma \hookrightarrow M$. Then we have that the only essential subsurface of $\Sigma_i$ homotopic to a subsurface of $\Sigma_j$ is a neighbourhood of $\gamma$. This is because by construction the only curve of $\Sigma_i$ homotopic into $\Sigma_j$ is $\gamma$.

Proposition 6. The manifold $M_\infty$ is not hyperbolic.

Proof. The manifold $M_\infty$ has two non tame ends $E^\pm$ and the connected components of the complement of a region co-bounded by distinct lifts of $\Sigma$ give neighbourhoods of these ends. Let $A$ be the annulus obtained by the lift of the essential torus $T \hookrightarrow M$. The ends $E^\pm$ of $M_\infty$ are in bijection with the ends $A^\pm$ of the annulus $A$. Let $\gamma$ be a simple closed curve generating $\pi_1(A)$. Denote by $\{\Sigma_i\}_{i \in \mathbb{Z}} \subseteq M_\infty$ the lifts of $\Sigma \subseteq M$ and let $\{\Sigma_i^\pm\}_{i \in \mathbb{Z}}$ be the lifts of the punctured tori that form the complement of $\alpha$ in $\Sigma \subseteq M$. The proof is by contradiction and it will follow by showing that $\gamma$ is neither homotopic to a geodesic in $M_\infty$ nor out a cusp.

Step 1. We want to show that the curve $\gamma$ cannot be represented by a hyperbolic element.

By contradiction assume that $\gamma$ is represented by a hyperbolic element and let $\overline{\gamma}$ be the unique geodesic representative of $\gamma$ in $M_\infty$. Consider the incompressible embeddings $f_i : \Sigma_2 \hookrightarrow M_\infty$ with $f_i(\Sigma_2) = \Sigma_i$ and let $\gamma_i \subseteq \Sigma_i$ be the simple closed curve homotopic to $\gamma$. By picking a 1-vertex triangulation of $\Sigma_i$ where $\gamma_i$ is represented by a preferred edge we can realise each $(f_i, \Sigma_i)$ by a useful-simplicial hyperbolic surface $g_i : S_i \rightarrow M_\infty$ with $g_i(S_i) \simeq \Sigma_i$ (see [Can96, Bon86]). By an abuse of notation we will also use $S_i$ to denote $g_i(S_i)$. Since all the $S_i$ realise $\overline{\gamma}$ as a geodesic we will see the following configuration in $M_\infty$:
On the simplicial hyperbolic surfaces $S_i$ a maximal one-sided collar neighbourhood of $\gamma$ has area bounded by the total area of $S_i$. Since the simplicial hyperbolic surfaces are all genus two by Gauss-Bonnet we have that $A(S_i) \leq 2\pi |\chi(S_i)| = 4\pi$. Therefore, the radius of a one-sided collar neighbourhood is uniformly bounded by some constant $K = K(\chi(\Sigma_2), \ell(\gamma)) < \infty$. Then for $\xi > 0$ in the simplicial hyperbolic surface $S_i$ the $K + \xi$ two sided neighbourhood of $\gamma$ is not embedded and contains a 4-punctured sphere. Since simplicial hyperbolic surfaces are 1-Lipschitz the 4-punctured sphere is contained in a $K + \xi$ neighbourhood $C$ of $\gamma$, thus it lies in some fixed set $M_h$. Therefore for every $|n| > h$ we have that $\Sigma_{\pm n}$ has an essential subsurface, homeomorphic to a 4-punctured sphere, homotopic into $\Sigma_{\pm h}$ respectively. But this contradicts remark 5.

**Step 2.** We now show that $\gamma$ cannot be represented by a parabolic element.

Let $\varepsilon > 0$ be less then the 3-dimensional Margulis constant $\mu_3$ [BP91] and let $P$ be a cusp neighbourhood of $\gamma$ such that the horocycle representing $\gamma$ in $\partial P$ has length $\varepsilon$. Without loss of generality we can assume that $P$ is contained in the end $E^-$ of $M_\infty$.

Let $\{\Sigma^+_i\}_{i \geq 0} \subseteq \{\Sigma_i\}_{i \geq 0}$ be the collection of subsurfaces of the $\Sigma_i$ formed by the punctured tori with boundary $\gamma_i$ that are exiting $E^+$. By picking an ideal triangulation of $\Sigma_i$ where the cusp $\gamma_i$ is the only vertex we can realise the embeddings $f_i : \Sigma^+_i \hookrightarrow M_\infty$ by simplicial hyperbolic
surfaces \((g_i, S_i^+)\) in which \(\gamma_i\) is sent to the cusp \([\text{Can96, Bon86}]\). The \(\{S_i^+\}_{i \geq 0}\) are all punctured tori with cusp represented by \(\gamma\).

\[\Sigma_i \Sigma_{i+1} \Sigma_{i+2}\]

**Figure 4.** The \(\varepsilon\)-thin part is in grey.

All simplicial hyperbolic surface \(S_i^+\) intersects \(\partial P\) in a horocycle \(f_i(c_i)\) of length \(\ell(f_i(c_i)) = \varepsilon\). Therefore, in each \(S_i^+\) the horocycle \(c_i\) has a a maximally embedded one sided collar whose radius is bounded by some constant \(K = K(\varepsilon, 2\pi)\). Then for \(\xi > 0\) we have that a \(K + \xi\) neighbourhood of \(c_i\) in \(S_i^+\) has to contain a pair of pants \(P_i \subseteq S_i^+\). Since simplicial hyperbolic surfaces are 1-Lipschitz the pair of pants of \(P_i\) are contained in a \(K + \xi\) neighbourhood of \(f_i(c_i)\) in \(M_\infty\). Thus, the \(\Sigma_i\) have pair of pants that are homotopic a uniformly bounded distance from \(\partial P\). Let \(k \in \mathbb{N}\) be minimal such that \(\Sigma_k\) lies outside a \(K + \xi\) neighbourhood of \(\partial P\). Then for any \(j > k\) we have that \(\Sigma_j\) has a pair of pants homotopic into \(\Sigma_k\) contradicting remark 5.

This concludes the proof of Theorem 1.

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