SEMICLASSICAL ANALYSIS ON COMPACT NIL-MANIFOLDS

VÉRONIQUE FISCHER

Abstract. In this paper, we develop a semiclassical calculus on compact nil-manifolds. As an application, we obtain asymptotics such as generalised Weyl laws for positive Rockland operators on any graded compact nil-manifolds. We also define and study semiclassical limits and quantum limits in this context.

Contents

1. Introduction 2
   Acknowledgement 3

2. Preliminaries on nilpotent Lie groups and compact nil-manifold 3
   2.1. About nilpotent Lie groups 3
   2.2. Compact nil-manifolds 5
   2.3. Γ-periodic functions on G and functions on M 5
   2.4. Operators on M and G 6

3. Preliminaries on Rockland operators on G and M 6
   3.1. Graded nilpotent Lie group 6
   3.2. Positive Rockland operators on G 7
   3.3. Positive Rockland operators on M 9

4. Semiclassical calculus on graded compact nil-manifold 10
   4.1. Semiclassical pseudodifferential operators 10
   4.2. Boundedness in $L^2(M)$ 12
   4.3. Singularity of the operators as $\varepsilon \to 0$. 12
   4.4. Symbolic calculus 13

5. Asymptotics 14
   5.1. Estimates for kernels, Hilbert-Schmidt norms and traces 14
   5.2. Generalised Weyl laws for $\mathcal{R}_M$ 15
   5.3. Quantum variance 18

6. Semiclassical limits 22
   6.1. Quadratic limits and states of $\mathcal{A}$ 22
   6.2. The dual of $\mathcal{A}$ in terms of operator-valued measures 22
   6.3. The algebra $L^\infty(M \times \hat{G})$ and the decomposition $\hat{G} = \hat{G}_\infty \sqcup \hat{G}_1$ 25

7. Quantum limits for $\mathcal{R}_M$ 27
   7.1. Family of $\mathcal{R}_M$-eigenfunctions 27
   7.2. Symbols and operator-valued measure commuting with $\hat{\mathcal{R}}$ 28
   7.3. The case of subLaplacians 30
   7.4. Comments 31

References 32

2010 Mathematics Subject Classification. 43A85, 43A32, 22E30, 35H20, 35P20, 81Q10.
Key words and phrases. Harmonic analysis on nilpotent Lie groups and nilmanifolds, semi-classical analysis.
1. Introduction

The analysis of hypoelliptic operators in subRiemannian settings has made fundamental progress over the last twenty years, see e.g. [23, 25] and references therein. The underlying methods and ideas are so comprehensive that their natural setting is not restricted to the class of subRiemannian manifolds but extends more generally to filtered manifolds. In fact, a significant tool for these results has turned out to be the tangent groupoid to a filtered manifold [26, 7], in some sense reuniting the lifting theory of Stein and his collaborators (see e.g. [24]) and more geometric notions, for instance tangent groups and nilpotentisation [2, 22]. These approaches are seldom symbolic and this makes them unsuitable for questions in micro-local and semiclassical analysis, such as quantum ergodicity.

The main aim of this paper is to start the development of a symbolic semiclassical analysis for a large class of hypoelliptic operators in an accessible setting. The class includes the natural subLaplacians on compact nil-manifolds without any further hypothesis than a stratified structure on the underlying group. The methods and results presented here support the symbolic approach of semiclassical analysis in subRiemannian and subelliptic settings using the representation theory of nilpotent Lie groups, see [16] and references therein. After the group case studied in [10, 9, 11], the next natural context to test this approach is the one of compact nil-manifolds as they are the analogues of the torus $\mathbb{T}^n$ (the quotient of $\mathbb{R}^n$ by a lattice) in spectral Euclidean geometry: although easily comprehended, tori provide a rich context for the usual (i.e. Euclidean and commutative) semiclassical or micro-local analysis, see e.g. [11, 20].

As setting, we consider $\Gamma$ a discrete co-compact subgroup of a nilpotent Lie group $G$, and denote by $M := \Gamma \backslash G$ the corresponding compact nil-manifold. We will assume that $G$ is stratified and, after choosing a basis for the first stratum of the Lie algebra of $G$, we will consider the intrinsic self-adjoint subLaplacian $\mathcal{L}_M$ on $L^2(M)$. We will also consider the slightly more general setting of positive Rockland operators on a graded group $G$.

The first part of the paper is devoted to the development of the semiclassical theory in our setting. As a motivation for this, let us mention that in this paper, we obtain the following application to the mean convergence of $\mathcal{L}_M$-eigenfunctions, and that this result is expressed in simple spectral terms:

**Theorem 1.1.** Let $(\varphi_j)$ be eigenfunctions of $\mathcal{L}_M$ forming an orthonormal basis of the Hilbert space $L^2(M)$ with

$$\mathcal{L}_M \varphi_j = \mu_j \varphi_j, \quad 0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots$$

We denote its spectral counting function by

$$N(\Lambda) := |\{j \in \mathbb{N}_0, \mu_j \leq \Lambda\}|,$$

Then for any continuous function $f : M \to \mathbb{C}$ we have

$$\lim_{\Lambda \to \infty} \frac{1}{N(\Lambda)} \sum_{j: \mu_j \leq \Lambda} \int_M f(\dot{x})|\varphi_j(\dot{x})|^2 d\dot{x} = \text{vol}(M)^{-1} \int_M f(\dot{x})d\dot{x}.$$

Here, $d\dot{x}$ denotes the unique measure on $M$ associated with a (fixed) Haar measure $dx$ on $G$, and $\text{vol}(M)$ the volume of $M$.

In other words, Theorem 1.1 states that the family $\frac{1}{N(\Lambda)} \sum_{j: \mu_j \leq \Lambda} |\varphi_j(\dot{x})|^2 d\dot{x}$ of probability measures on $M$ converges $*$-weakly to $\text{vol}(M)^{-1} d\dot{x}$ as $\Lambda$ goes to infinity. Theorem 1.1 will be proved in the more general context of positive Rockland operators on graded nilmanifolds and via general semiclassical Weyl laws (see Section 5.2). The analysis will also yield the following asymptotic for the spectral counting function (Weyl law)

$$N(\Lambda) \sim c_M \Lambda^{Q/2}, \quad \text{as } \Lambda \to +\infty,$$
with a constant $c_M$ depending on $M$, $G$, and $L$. Although this Weyl law in this context is not new \cite{7,15}, to the author’s knowledge, the mean convergence in Theorem 1.1 is known only for the three-dimension Heisenberg nilmanifold \cite[Remark 4.2]{5}.

The second part of the paper is devoted to developing the theory behind the semi-classical measures in this setting, and in particular quantum limits for subLaplacians. This work opens many questions, and it would be interesting for examples to determine quantum limits in the setting of Heisenberg nilmanifolds, and more generally nilmanifolds of Heisenberg types.

The paper is organised as follows. After presenting the setting in details in Sections 2 and 3, we develop the semiclassical calculus based on representation theory in Section 4. This gives us the tools to determine (e.g. trace) asymptotics in Section 5, yielding generalised Weyl laws and a first study of quantum variance in this context. In Section 6, we define the notion of semiclassical limit of a sequence of functions in $L^2(M)$; as our context is highly non-commutative, this corresponds to operator-valued measures. We will show in Section 7 that these operator-valued measures commute with the symbol of the subLaplacian $L$ in the particular case of sequences of $L^M$-eigenfunctions, leading to some invariance properties.

Acknowledgement. The author is grateful to the Leverhulme Trust for their support via Research Project Grant 2020-037.

2. Preliminaries on nilpotent Lie groups and compact nil-manifold

In this section, we set our notation for nilpotent Lie groups and nil-manifolds. We also recall some elements of harmonic analysis in this setting.

2.1. About nilpotent Lie groups. In this paper, a nilpotent Lie group $G$ is always assumed connected and simply connected unless otherwise stated. It is a smooth manifold which is identified with $\mathbb{R}^n$ via the exponential mapping and polynomial coordinate systems. This leads to a corresponding Lebesgue measure on its Lie algebra $\mathfrak{g}$ and the Haar measure $dx$ on the group $G$, hence $L^p(G) \cong L^p(\mathbb{R}^n)$. This also allows us \cite[p.16]{6} to define the spaces

$$D(G) \cong D(\mathbb{R}^n) \quad \text{and} \quad S(G) \cong S(\mathbb{R}^n)$$

of test functions which are smooth and compactly supported or Schwartz, and the corresponding spaces of distributions

$$D'(G) \cong D'(\mathbb{R}^n) \quad \text{and} \quad S'(G) \cong S'(\mathbb{R}^n).$$

Note that this identification with $\mathbb{R}^n$ does not usually extend to the convolution: the group convolution, i.e. the operation between two functions on $G$ defined formally via

$$(f_1 * f_2)(x) := \int_G f_1(y) f_2(y^{-1}x) dy,$$

is not commutative in general whereas it is a commutative operation for functions on the abelian group $\mathbb{R}^n$.

2.1.1. Representations of $G$ and $L^1(G)$. In this paper, we always assume that the representations of the group $G$ are strongly continuous and acting on separable Hilbert spaces. Unless otherwise stated, the representations of $G$ will also be assumed unitary. For a representation $\pi$ of $G$, we keep the same notation for the corresponding infinitesimal representation which acts on the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra of the group. It is characterised by its action on $\mathfrak{g}$:

$$\pi(X) = \partial_{t=0} \pi(e^{tX}), \quad X \in \mathfrak{g}. \quad (2.1)$$

The infinitesimal action acts on the space $H^\infty_r$ of smooth vectors, that is, the space of vectors $v \in H_\pi$ such that the mapping $G \ni x \mapsto \pi(x)v \in H_\pi$ is smooth.
We will use the following equivalent notations for the group Fourier transform of a function \( f \in L^1(G) \) at \( \pi \)
\[
\pi(f) \equiv \widehat{f}(\pi) \equiv \mathcal{F}_G(f)(\pi) = \int_G f(x)\pi(x)^*dx.
\]

2.1.2. The Plancherel formula. We denote by \( \hat{G} \) the unitary dual of \( G \), that is, the unitary irreducible representations of \( G \) modulo equivalence and identify a unitary irreducible representation with its class in \( \hat{G} \). The set \( \hat{G} \) is naturally equipped with a structure of standard Borel space. The Plancherel measure is the unique positive Borel measure \( \mu \) on \( \hat{G} \) such that for any \( f \in C_c(G) \), we have:
\[
\int_G |f(x)|^2dx = \int_{\hat{G}} \|\mathcal{F}_G(f)(\pi)\|^2_{HS(\mathcal{H}_\pi)}d\mu(\pi).
\]
Here \( \| \cdot \|_{HS(\mathcal{H}_\pi)} \) denotes the Hilbert-Schmidt norm on \( \mathcal{H}_\pi \). This implies that the group Fourier transform extends unitarily from \( L^1(G) \cap L^2(G) \) to \( L^2(\hat{G}) \) onto
\[
L^2(\hat{G}) := \int_{\hat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^*d\mu(\pi),
\]
which we identify with the space of \( \mu \)-square integrable fields on \( \hat{G} \). Consequently (2.2) holds for any \( f \in L^2(G) \); this formula is called the Plancherel formula. It is possible to give an expression for the Plancherel measure \( \mu \); see [6, Section 4.3], although we will not need this in this paper. We deduce the inversion formula: for any \( \kappa \in \mathcal{S}(G) \),
\[
\forall x \in G \int_{\hat{G}} \text{Tr}(\pi(x)\mathcal{F}_G(\kappa(\pi)))d\mu(\pi) = \kappa(x).
\]

2.1.3. The von Neumann algebra of \( G \). Let us recall [8] that the closure of \( \mathcal{F}_G\mathcal{S}(G) \) for the \( L^\infty \)-norm given by
\[
\|\widehat{\kappa}\|_{L^\infty(\hat{G})} = \sup_{\pi \in \hat{G}} \|\widehat{\kappa}\|_{\mathcal{L}(\mathcal{H}_\pi)}
\]
generates the \( C^* \)-algebra of the group \( G \). The von Neumann algebra of \( G \) is the von Neumann algebra generated by the \( C^* \)-algebra of the group. It is the space \( L^\infty(\hat{G}) \) of measurable fields of operators that are bounded, that is, of measurable fields of operators \( \sigma = \{\sigma(\pi) \in \mathcal{L}(\mathcal{H}_\pi) : \pi \in \hat{G}\} \) such that
\[
\exists C > 0 \quad \|\sigma(x, \pi)\|_{\mathcal{L}(\mathcal{H}_\pi)} \leq C \text{ for } d\mu(\pi)-\text{almost all } \pi \in \hat{G}.
\]
The smallest of such constant \( C > 0 \) is the norm \( \|\sigma\|_{L^\infty(\hat{G})} \) of \( \sigma \) in \( L^\infty(\hat{G}) \).

The von Neumann algebra \( L^\infty(\hat{G}) \) is isomorphic to the von Neumann algebra \( \mathcal{L}(L^2(G))^G \) of \( L^2(G) \)-bounded operators commuting with the left-translations on \( G \). More precisely, if \( T \in \mathcal{L}(L^2(G))^G \), there exists a unique field \( \widehat{T} \in L^\infty(\hat{G}) \) such that
\[
\forall f \in \mathcal{S}(G) \quad \mathcal{F}_G(Tf) = \widehat{T}\mathcal{F}_Gf,
\]
and we have \( \|\widehat{T}\|_{L^\infty(\hat{G})} = \|T\|_{\mathcal{L}(L^2(G))} \). By the Schwartz kernel theorem, the operator \( T \) admits a distributional convolution kernel \( \kappa \in \mathcal{S}'(G) \). We may also write \( \widehat{\kappa} = \widehat{T} \) and call this field the group Fourier transform of \( \kappa \) or of \( T \). It extends the previous definition of the group Fourier transform on \( L^1(G) \) and \( L^2(G) \).
2.2. Compact nil-manifolds. A compact nil-manifold is the quotient \( M = \Gamma \backslash G \) of a nilpotent Lie group \( G \) by a discrete co-compact subgroup \( \Gamma \) of \( G \). A concrete example of discrete co-compact subgroup is the natural discrete subgroup of the Heisenberg group, as described in [6, Example 5.4.1]. Abstract characterisations are discussed in [6, Section 5.1].

An element of \( M \) is a class 
\[
\dot{x} := \Gamma x
\]
of an element \( x \) in \( G \). If the context allows it, we may identify this class with its representative \( x \).

The quotient \( M \) is naturally equipped with the structure of a compact smooth manifold. Furthermore, fixing a Haar measure on the unimodular group \( G \), \( M \) inherits a measure \( d\dot{x} \) which is invariant under the translations given by
\[
M \quad \longrightarrow \quad M
\dot{x} \quad \mapsto \quad \dot{x}g = \Gamma xg, \quad g \in G.
\]
Recall that the Haar measure \( dx \) on \( G \) is unique up to a constant and, once it is fixed, \( d\dot{x} \) is the only \( G \)-invariant measure on \( M \) satisfying for any function \( f : G \to \mathbb{C} \), for instance continuous with compact support,
\[
\int_G f(x) dx = \int_M \sum_{\gamma \in \Gamma} f(\gamma x) d\dot{x}.
\]

We denote by \( \text{vol}(M) = \int_M 1d\dot{x} \) the volume of \( M \).

2.3. \( \Gamma \)-periodic functions on \( G \) and functions on \( M \). Let \( \Gamma \) be a discrete co-compact subgroup of a nilpotent Lie group \( G \).

We say that a function \( f : G \to \mathbb{C} \) is \( \Gamma \)-left-periodic or just \( \Gamma \)-periodic when we have
\[
\forall x \in G, \quad \forall \gamma \in \Gamma, \quad f(\gamma x) = f(x).
\]

This definition extends readily to measurable functions and to distributions.

There is a natural one-to-one correspondence between the functions on \( G \) which are \( \Gamma \)-periodic and the functions on \( M \). Indeed, for any map \( F \) on \( M \), the corresponding periodic function on \( G \) is \( F_G \) defined via
\[
F_G(x) := F(\dot{x}), \quad x \in G,
\]
while if \( f \) is a \( \Gamma \)-periodic function on \( G \), it defines a function \( f_M \) on \( M \) via
\[
f_M(\dot{x}) = f(x), \quad x \in G.
\]

Naturally, \( (F_G)_M = F \) and \( (f_M)_G = f \).

We also define, at least formally, the periodisation \( \phi^\Gamma \) of a function \( \phi(x) \) of the variable \( x \in G \) by:
\[
\phi^\Gamma(x) = \sum_{\gamma \in \Gamma} \phi(\gamma x), \quad x \in G.
\]

If \( E \) is a space of functions or of distributions on \( G \), then we denote by \( E^\Gamma \) the space of elements in \( E \) which are \( \Gamma \)-periodic. Although \( \mathcal{D}(G)^\Gamma = \{0\} = \mathcal{S}(G)^\Gamma \), many other periodised functions or functional spaces have interesting descriptions on \( M \) [15]:

**Proposition 2.1.**

1. The periodisation of a Schwartz function \( \phi \in \mathcal{S}(G) \) is a well-defined function \( \phi^\Gamma \) in \( C^\infty(G)^\Gamma \). Furthermore, the map \( \phi \mapsto \phi^\Gamma \) yields a surjective morphism of topological vector spaces from \( \mathcal{S}(G) \) onto \( C^\infty(G)^\Gamma \) and from \( \mathcal{D}(G) \) onto \( C^\infty(G)^\Gamma \).
(2) For every $F \in \mathcal{D}'(M)$, the tempered distribution $F_G \in \mathcal{S}'(G)$ is defined by
\[ \forall \phi \in \mathcal{S}(G) \quad \langle F_G, \phi \rangle = \langle F, (\phi^\Gamma)_M \rangle. \]

The map $F \mapsto F_G$ yields an isomorphism of topological vector spaces from $\mathcal{D}'(M)$ onto $\mathcal{S}'(G)^\Gamma = \mathcal{D}'(G)^\Gamma$.

(3) For every $p \in [1, \infty]$, the map $F \mapsto F_G$ is an isomorphism of the topological vector spaces (in fact Banach spaces) from $L^p(M)$ onto $L^p_G(G)^\Gamma$ with inverse $f \mapsto f_M$.

(4) Let $f \in \mathcal{S}'(G)^\Gamma$ and $\kappa \in \mathcal{S}(G)$. Then $(x, y) \mapsto \sum_{\gamma \in \Gamma} \kappa(y^{-1} \gamma x)$ is a smooth function on $M \times M$ and $f \ast \kappa \in C^\infty(G)^\Gamma$. Viewed as a function on $M$,
\[ (f \ast \kappa)_M(x) = \int_M f_M(y) (\kappa^\Gamma)(\cdot^{-1}x) dy = \int_M f_M(y) \sum_{\gamma \in \Gamma} \kappa(y^{-1} \gamma x) dy. \]

2.4. Operators on $M$ and $G$. A mapping $T : \mathcal{S}'(G) \to \mathcal{S}'(G)$ or $\mathcal{D}'(G) \to \mathcal{D}'(G)$ is invariant under an element $g \in G$ when
\[ \forall f \in \mathcal{S}'(G) \ (\text{resp. } \mathcal{D}'(G)), \quad T(f(g \cdot)) = (Tf)(g \cdot). \]

It is invariant under a subset of $G$ if it is invariant under every element of the subset.

Consider a linear continuous mapping $T : \mathcal{S}'(G) \to \mathcal{S}'(G)$ or $\mathcal{D}'(G) \to \mathcal{D}'(G)$ respectively which is invariant under $\Gamma$. Then it naturally induces a linear continuous mapping $T_M$ on $M$ given via
\[ T_M F = (TF_G)_M, \quad F \in \mathcal{D}'(M). \]

Consequently, if $T$ coincides with a smooth differential operator on $G$ that is invariant under $\Gamma$, then $T_M$ is a smooth differential operator on $M$. For convolution operators $T$, the results in Proposition 2.1 yield:

**Lemma 2.2.** Let $\kappa \in \mathcal{S}(G)$ be a given convolution kernel, and let us denote by $T$ the associated convolution operator:
\[ T(\phi) = \phi \ast \kappa, \quad \phi \in \mathcal{S}'(G). \]

The operator $T$ is a linear continuous mapping $\mathcal{S}'(G) \to \mathcal{S}'(G)$. The corresponding operator $T_M$ maps $\mathcal{D}'(M)$ to $\mathcal{D}'(M)$ continuously and linearly. Its integral kernel is the smooth function $K$ on $M \times M$ given by
\[ K(x, y) = \sum_{\gamma \in \Gamma} \kappa(y^{-1} \gamma x). \]

Consequently, the operator $T_M$ is Hilbert-Schmidt on $L^2(M)$ with Hilbert-Schmidt norm
\[ \|T_M\|_{HS} = \|K\|_{L^2(M \times M)}. \]

3. Preliminaries on Rockland operators on $G$ and $M$

In this section, we recall the definition and known properties of Rockland operators. Our convention here is that they are homogeneous left-invariant operators on a nilpotent Lie group $G$ whose group Fourier transform is injective (see Section 3.2.1). This implies that the group $G$ is graded. We therefore start this section with describing the setting of graded groups.

3.1. Graded nilpotent Lie group. In the rest of the paper, we will be concerned with graded Lie groups. References on this subject includes [18] and [17].

3.1.1. Definition. A graded Lie group $G$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits an $\mathbb{N}$-gradation $\mathfrak{g} = \bigoplus_{\ell=1}^{\infty} \mathfrak{g}_\ell$ where the $\mathfrak{g}_\ell, \ell = 1, 2, \ldots$, are vector subspaces of $\mathfrak{g}$, almost all equal to $\{0\}$, and satisfying $[\mathfrak{g}_\ell, \mathfrak{g}_{\ell'}] \subseteq \mathfrak{g}_{\ell+\ell'}$ for any $\ell, \ell' \in \mathbb{N}$. This implies that the group $G$ is nilpotent. Examples of such groups are the Heisenberg group and, more generally, all stratified groups (which by definition correspond to the case $\mathfrak{g}_1$ generating the full Lie algebra $\mathfrak{g}$).
3.1.2. Dilations and homogeneity. For any \( r > 0 \), we define the linear mapping \( D_r : g \to g \) by \( D_r X = r^\ell X \) for every \( X \in g_\ell, \ell \in \mathbb{N} \). Then the Lie algebra \( g \) is endowed with the family of dilations \( \{ D_r, r > 0 \} \) and becomes a homogeneous Lie algebra in the sense of [18]. We re-write the set of integers \( \ell \in \mathbb{N} \) such that \( g_\ell \neq \{0\} \) into the increasing sequence of positive integers \( v_1, \ldots, v_n \) counted with multiplicity, the multiplicity of \( g_\ell \) being its dimension. In this way, the integers \( v_1, \ldots, v_n \) become the weights of the dilations.

We construct a basis \( X_1, \ldots, X_n \) of \( g \) adapted to the gradation, by choosing a basis \( \{X_1, \ldots, X_n\} \) of \( g_1 \) (this basis is possibly reduced to \( \emptyset \)), then \( \{X_{n_1+1}, \ldots, X_{n_1+n_2}\} \) a basis of \( g_2 \) (possibly \( \{0\} \) as well as the others) We have \( D_r X_j = r^{v_j} X_j, j = 1, \ldots, n \).

In a canonical way, this leads to the notions of homogeneity for functions, distributions and operators and we now give a few important examples.

The Haar measure is \( Q \)-homogeneous where

\[
Q := \sum_{\ell \in \mathbb{N}} \ell \dim g_\ell = v_1 + \ldots + v_n,
\]

is called the homogeneous dimension of \( G \).

Identifying the element of \( g \) with left invariant vector fields, each \( X_j \) is a \( v_j \)-homogeneous differential operator of degree. More generally, the differential operator

\[
X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}, \quad \alpha \in \mathbb{N}_0^n
\]

is homogeneous with degree

\[
[\alpha] := \alpha_1 v_1 + \cdots + \alpha_n v_n.
\]

The unitary dual \( \hat{G} \) inherits a dilation from the one on \( G \) [10, Section 2.2]: we denote by \( r \cdot \pi \) the element of \( \hat{G} \) obtained from \( \pi \) through dilatation by \( r \), that is, \( r \cdot \pi(x) = \pi(rx), \; r > 0 \) and \( x \in G \).

3.1.3. Homogeneous quasi-norms. An important class of homogeneous map are the homogeneous quasi-norms, that is, a \( 1 \)-homogeneous non-negative map \( G \ni x \mapsto \|x\| \) which is symmetric and definite in the sense that \( \|x^{-1}\| = \|x\| \) and \( \|x\| = 0 \iff x = 0 \). In fact, all the homogeneous quasi-norms are equivalent in the sense that if \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are two of them, then

\[
\exists C > 0 \quad \forall x \in G \quad C^{-1} \|x\|_1 \leq \|x\|_2 \leq C \|x\|_1.
\]

Examples may be constructed easily, such as

\[
\|x\| = \left(\sum_{j=1}^n |x_j|^{N/v_j}\right)^{-N} \text{ for any } N \in \mathbb{N},
\]

with the convention above. It is also possible to construct a homogeneous quasi-norm which is also a norm.

3.2. Positive Rockland operators on \( G \). Let us briefly review the definition and main properties of positive Rockland operators. References on this subject includes [18] and [17].
3.2.1. Definitions. A Rockland operator $\mathcal{R}$ on $G$ is a left-invariant differential operator which is homogeneous of positive degree and satisfies the Rockland condition, that is, for each unitary irreducible representation $\pi$ on $G$, except for the trivial representation, the operator $\pi(\mathcal{R})$ is injective on the space $\mathcal{H}_\pi^\infty$ of smooth vectors of the infinitesimal representation.

Recall that Rockland operators are hypoelliptic. In fact, they may equivalently be characterised as the left-invariant differential operators which are hypoelliptic. If this is the case, then $\mathcal{R} + \sum_{|\alpha|<\nu} c_\alpha X^\alpha$ where $c_\alpha \in \mathbb{C}$ and $\nu$ is the homogeneous degree of $\nu$ is hypoelliptic.

A Rockland operator is positive when
\[ \forall f \in \mathcal{S}(G), \quad \int_G \mathcal{R} f(x) \overline{f(x)} dx \geq 0. \]

Any subLaplacian with the sign convention $-(Z_1^2 + \ldots + Z_n^2)$ of a stratified Lie group is a positive Rockland operator; here $Z_1, \ldots, Z_n$ form a basis of the first stratum $\mathfrak{g}_1$. The reader familiar with the Carnot group setting may view positive Rockland operators as generalisations of the natural subLaplacians. Positive Rockland operators are easily constructed on any graded Lie group.

For each unitary irreducible representation $\pi$ of $G$, the operator $\pi(\mathcal{R})$ is essentially self-adjoint on $\mathcal{H}_\pi^\infty$ and we keep the same notation for this self-adjoint extension. Its spectrum $\text{sp}(\pi(\mathcal{R}))$ is a discrete subset of $(0, \infty)$ if $\pi \neq 1_G$ is not trivial while $\pi(\mathcal{R}) = 0$ if $\pi = 1_G$ is the trivial representation.

Let us denote by $E$ and $E_\pi$ the spectral measures of $\mathcal{R} = \int_\mathbb{R} \lambda dE(\lambda)$ and of $\pi(\mathcal{R}) = \int_\mathbb{R} \lambda dE_\pi(\lambda)$, $\pi \in \hat{G}$. Then $\hat{E}(\pi) = E_\pi$ in the sense that for any interval $I \subset \mathbb{R}$, the group Fourier transform $E(I)$ of the projection $E(I) \in \mathcal{L}(L^2(G))^G$ coincide with the field $\{\pi(E(I)), \pi \in \hat{G}\}$.

3.2.2. Spectral multipliers in $\mathcal{R}$ and in $\hat{\mathcal{R}}$. If $\psi : \mathbb{R}^+ \to \mathbb{C}$ is a measurable function, the spectral multiplier $\psi(\mathcal{R}) = \int_\mathbb{R} \psi(\lambda) dE(\lambda)$ is well defined as a possibly unbounded operator on $L^2(G)$. If the domain of $\psi(\mathcal{R})$ contains $\mathcal{S}(G)$ and defines a continuous map $\mathcal{S}(G) \to \mathcal{S}^\prime(G)$, then it is invariant under right-translation and, by the Schwartz kernel theorem, admits a right-convolution kernel $\psi(\mathcal{R})\delta_0 \in \mathcal{S}^\prime(G)$ which satisfies the following homogeneity property:
\[ \psi(r^\nu \mathcal{R}) \delta_0(x) = r^{-Q} \psi(\mathcal{R}) \delta_0(r^{-1}x), \quad x \in G. \]

Furthermore, for each unitary irreducible representation $\pi$ of $G$, the domain of the operator $\psi(\pi(\mathcal{R})) = \int_\mathbb{R} \psi(\lambda) dE_\pi(\lambda)$ contains $\mathcal{H}_\pi^\infty$ and we have
\[ \hat{\psi(\mathcal{R})}(\pi) = \psi(\pi(\mathcal{R})). \]

The following statement is the famous result due to Hulanicki [19]:

**Theorem 3.1** (Hulanicki’s theorem). Let $\mathcal{R}$ be a positive Rockland operator on $G$. If $\psi \in \mathcal{S}(\mathbb{R})$ then $\psi(\mathcal{R}) \delta_0 \in \mathcal{S}(G)$.

For instance, the heat kernels
\[ p_t := e^{-t \mathcal{R}} \delta_0, \quad t > 0, \]
are Schwartz - although this property is in fact used in the proof of Hulanicki’s Theorem.

The following result describes the isometry $\psi \mapsto \psi(\mathcal{R})\delta_0$ from $L^2((0, \infty), c_0 \lambda^{Q/2} d\lambda/\lambda)$ to $L^2(G)$. This was mainly obtained by Christ for subLaplacians on stratified groups [3] Proposition 3] and readily extended to positive Rockland operators in [15]:
Theorem 3.2. Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu$ on $G$. If the measurable function $\psi : \mathbb{R}^+ \to \mathbb{C}$ is in $L^2(\mathbb{R}^+, \lambda^{\nu/\nu} d\lambda/\lambda)$, then $\psi(\mathcal{R})$ defines a continuous map $\mathcal{S}(G) \to \mathcal{S}'(G)$ whose convolution kernel $\psi(\mathcal{R})\delta_0$ is in $L^2(G)$. Moreover, we have

$$\|\psi(\mathcal{R})\delta_0\|_{L^2(G)}^2 = c_0 \int_0^\infty |\psi(\lambda)|^2 \lambda^{\frac{\nu}{\nu}} \frac{d\lambda}{\lambda},$$

where $c_0 = c_0(\mathcal{R})$ is a positive constant of $\mathcal{R}$ and $G$.

Consequently, we have for any $\psi \in \mathcal{S}(\mathbb{R})$

$$\psi(\mathcal{R})\delta_0(0) = c_0 \int_0^\infty \psi(\lambda) \lambda^{\frac{\nu}{\nu}} \frac{d\lambda}{\lambda}.$$

**Remark 3.3.** By plugging the function $\psi(\lambda) = e^{-\lambda}$ in the second formula of Theorem 3.2, we obtain the following expression for the constant in the statement in terms of the heat kernel $p_t$ of $\mathcal{R}$:

$$c_0 = c_0(\mathcal{R}) = \frac{p_t(0)}{\Gamma(Q/\nu)}, \quad \text{where} \quad \Gamma(Q/\nu) = \int_0^\infty e^{-\lambda} \lambda^{Q/\nu} \frac{d\lambda}{\lambda}.$$

Using the Plancherel formula (2.2), we have an expression for

$$\|\psi(\mathcal{R})\delta_0\|_{L^2(G)}^2 = \int_G \|\psi(\widehat{\mathcal{R}}(\pi))\|_{HS}^2 d\mu(\pi),$$

in terms of the Plancherel measure. Taking for instance $\psi = 1_{[0,1]}$ easily leads

$$c_0 = \frac{Q}{\nu} \int_G \text{Tr} \left( 1_{[0,1]}(\widehat{\mathcal{R}}(\pi)) \right)^2_{HS} d\mu(\pi).$$

More generally, taking $\psi = 1_{[a,b]}$ where $0 \leq a < b$, we have

$$c_{\nu/\nu} \frac{Q}{\nu} \left(b^2 - a^2\right) = \int_G \text{Tr} \left( 1_{[a,b]}(\widehat{\mathcal{R}}(\pi)) \right) d\pi d\mu(\pi).$$

We can check Theorem 3.2 in the familiar setting of the canonical Laplacian $\Delta_{\mathbb{R}^n} = -\sum_j \partial_j^2$ on the abelian group $G = \mathbb{R}^n$ or on the Heisenberg group, see [15].

3.3. Positive Rockland operators on $M$. This section is devoted to the general properties of positive Rockland operators. Their proofs may be found in [15], and some of them are already known for Rockland operators on compact filtered manifolds [7].

**Proposition 3.4.** Let $\mathcal{R}$ be a positive Rockland operator on $G$. The operator $\mathcal{R}_M$ it induces on $M$ is a smooth differential operator which is positive and essentially self-adjoint on $L^2(M)$. We will keep the same notation for $\mathcal{R}_M$ and for its self-adjoint extension.

(i) Let $\psi \in \mathcal{S}(\mathbb{R})$.
   
   (i) The operator $\psi(\mathcal{R}_M)$ defined as a bounded spectral multiplier on $L^2(M)$ coincides with the operator

   $$\phi \mapsto (\psi(\mathcal{R})\phi)_{M} = (\phi_G * \kappa)_M$$

   where $\kappa := \psi(\mathcal{R})\delta_0 \in \mathcal{S}(G)$. The integral kernel of $\psi(\mathcal{R}_M)$ is a smooth function on $M \times M$ given by

   $$K(x, y) = \sum_{\gamma \in \Gamma} \kappa(y^{-1}\gamma x).$$

   (ii) For every $\varepsilon \in (0, 1]$, the integral kernel $K_\varepsilon$ of $\psi(\varepsilon^\nu \mathcal{R}_M)$ satisfies

   $$K_\varepsilon(x, y) = \varepsilon^{-Q} K(0) + O(\varepsilon)\infty.$$
Here $\nu$ is the degree of homogeneity of $\mathcal{R}$, and $\kappa(0)$ is the value at $x = 0$ of the convolution kernel $\kappa = \psi(\mathcal{R})\delta_0$ also given by Theorem 3.3 as

$$\kappa(0) = c_0 \int_0^\infty \psi(\lambda)\frac{Q}{\lambda} d\lambda.$$

The trace and Hilbert-Schmidt norm have the following asymptotics in $\varepsilon \to 0$:

$$\text{Tr}(\psi(\varepsilon^{\nu} \mathcal{R}_M)) = \varepsilon^{-Q} \text{vol}(M)\kappa(0) + O(\varepsilon)^\infty$$

$$\|\psi(\varepsilon^{\nu} \mathcal{R}_M)\|_{HS}^2 = \varepsilon^{-Q} \text{vol}(M) c_0 \int_0^\infty |\psi(\lambda)|^2 \frac{Q}{\lambda} d\lambda + O(\varepsilon)^\infty.$$

Furthermore, we have for all $\dot{x} \in M$

$$(3.4) \quad \int_M |K_\varepsilon(\dot{x}, \dot{y})| d\dot{y} \leq \|\kappa\|_{L^1(G)}.$$

(2) The spectrum $\text{sp}(\mathcal{R}_M)$ of $\mathcal{R}_M$ is a discrete and unbounded subset of $[0, +\infty)$. Each eigenspace of $\mathcal{R}_M$ has finite dimension. The resolvent operators $(\mathcal{R}_M - z)^{-1}, z \in \mathbb{C} \setminus \text{sp}(\mathcal{R}_M)$, are compact on $L^2(M)$. The constant functions on $M$ form the 0-eigenspace of $\mathcal{R}_M$. The eigenfunctions of $\mathcal{R}_M$ are smooth on $M$.

Proof. All the proofs are contained in [15] except for (3.4) that we now check. Again by [15], we know

$$K_\varepsilon(\dot{x}, \dot{y}) = \sum_{\gamma \in \Gamma} \kappa^{(\varepsilon)}(y^{-1}\gamma x), \quad \text{where} \quad \kappa^{(\varepsilon)}(z) := \varepsilon^{-Q} \kappa(\varepsilon^{-1}z).$$

Therefore,

$$\int_M |K_\varepsilon(\dot{x}, \dot{y})| d\dot{y} \leq \int_M \sum_{\gamma \in \Gamma} |\kappa^{(\varepsilon)}(y^{-1}\gamma x)| d\dot{y} = \int_G \kappa^{(\varepsilon)}(y^{-1}x) dy = \|\kappa^{(\varepsilon)}\|_{L^1(G)} = \|\kappa\|_{L^1(G)}.$$  

\[
\square
\]

4. Semiclassical calculus on graded compact nil-manifold

4.1. Semiclassical pseudodifferential operators. The semiclassical pseudodifferential calculus in the context of groups of Heisenberg type was presented in [10] [11], but in fact extends readily to any graded group $G$. Here, we show how to define it on the quotient manifold $M$.

We denote by $\mathcal{A}_0 = \mathcal{A}_0(M \times \hat{G})$ the class of symbols, that is of fields of operators defined on $M \times \hat{G}$

$$\sigma(\dot{x}, \pi) \in \mathcal{L}(\mathcal{H}_\pi), \quad (\dot{x}, \pi) \in M \times \hat{G},$$

that are of the form

$$\sigma(\dot{x}, \pi) = \mathcal{F}_G \kappa_\dot{x}(\pi),$$

where $\dot{x} \mapsto \kappa_\dot{x}$ is a smooth map from $M$ to $\mathcal{S}(G)$. The group Fourier transform yields a bijection $(\dot{x} \mapsto \kappa_\dot{x}) \mapsto (\dot{x} \mapsto \sigma(\dot{x}, \cdot) = \mathcal{F}(\kappa_\dot{x}))$ from $C^\infty(M : \mathcal{S}(G))$ onto $\mathcal{A}_0$. We equip $\mathcal{A}_0$ of the Fréchet topology so that this mapping is an isomorphism of topological vector spaces.

We observe that $\mathcal{A}_0$ is an algebra for the usual composition of symbol. Furthermore, it is also equipped with the involution $\sigma \mapsto \sigma^*$, where $\sigma^* = \{\sigma(\dot{x}, \pi)^*, (\dot{x}, \pi) \in M \times \hat{G}\}$.

Note that by the Fourier inversion formula (2.33), we have

$$\kappa_\dot{x}(z) = \int_G \text{Tr}(\pi(z) \sigma(\dot{x}, \pi)) d\mu(\pi) = \int_G \text{Tr}(\pi(z) \sigma_G(x, \pi)) d\mu(\pi).$$

For any $\sigma \in \mathcal{A}_0$, we define the operator $\text{Op}_G(\sigma)$ at $F \in \mathcal{S}'(G)$ via

$$\text{Op}_G(\sigma)F(x) := F \ast \kappa_\dot{x}(x), \quad x \in G.$$
This makes sense since, for each \( x \in G \), the convolution of the tempered distribution \( F \) with the Schwartz function \( \kappa_\hat{x} \) yields a smooth function \( F \ast \kappa_\hat{x} \) on \( G \). Because of the Fourier inversion formula \((2.3)\), it may be written formally as

\[
\text{Op}_G(\sigma)F(x) = \int_{G \times G} \text{Tr}(\pi(y^{-1}x)\sigma_G(x, \varepsilon \cdot \pi))F(y)dyd\mu(\pi).
\]

If \( F \) is periodic, then \( \text{Op}_G(\sigma)F \) is also periodic with \( \text{Op}_G(\sigma)F \in C^\infty(G)^\Gamma \) and we can view \( F \) and \( \text{Op}_G(\sigma)F \) as functions on \( M \), see Section \( 2.3 \). In other words, we set for any \( f \in D'(M) \) and \( \hat{x} \in M \):

\[
\text{Op}(\sigma)f(\hat{x}) := \text{Op}_G(\sigma)f_G(x) = (f_G \ast \kappa_\hat{x})_M(\hat{x}) = \int_M f(y)\sum_{\gamma \in \Gamma} \kappa_\hat{x}(y^{-1}\gamma x)dy,
\]

and this defines the function \( \text{Op}(\sigma)f \in D(M) \). We say that \( \kappa_\hat{x} \) is the kernel associated with the symbol \( \sigma \) or \( \text{Op}(\sigma) \).

The results in Section \( 2.3 \) yield:

**Lemma 4.1.** Let \( \sigma \in \mathcal{A}_0 \) and let \( \kappa_\hat{x} \) be its associated kernel. Then \( \text{Op}(\sigma) \) maps \( D'(M) \) to \( D(M) \) continuously, and its Schwartz integral is the smooth function \( K \) on \( M \times M \) given by

\[
K(\hat{x}, \hat{y}) = \sum_{\gamma \in \Gamma} \kappa_\hat{x}(y^{-1}\gamma x).
\]

Consequently, the operator \( \text{Op}(\sigma) \) is Hilbert-Schmidt on \( L^2(M) \) with Hilbert-Schmidt norm

\[
\|\text{Op}(\sigma)\|_{HS} = \|K\|_{L^2(M \times M)}.
\]

Let \( \varepsilon \in (0, 1] \) be a small parameter. For every symbol \( \sigma \in \mathcal{A} \), we consider the symbol

\[(4.1) \quad \sigma^{(\varepsilon)} := \{\sigma(\hat{x}, \varepsilon \cdot \pi) : (\hat{x}, \pi) \in M \times \hat{G}\},\]

whose associated kernel is then

\[(4.2) \quad \kappa_\hat{x}^{(\varepsilon)}(z) := \varepsilon^{-Q} \kappa_\hat{x}(\varepsilon^{-1} \cdot z), \quad z \in G,\]

if \( \kappa_\hat{x} = \kappa_\hat{x}^{(1)} \) is the kernel associated with the symbol \( \sigma = \sigma^{(1)} \). The semiclassical pseudo-differential calculus is then defined via

\[
\text{Op}^{(\varepsilon)}(\sigma) := \text{Op}(\sigma^{(\varepsilon)}) \quad \text{and} \quad \text{Op}_G^{(\varepsilon)}(\sigma) := \text{Op}_G(\sigma^{(\varepsilon)}).
\]

An interesting example is given by the spectral multiplier in a positive Rockland operator \( \mathcal{R} \) on \( G \). For any \( \psi \in \mathcal{S}(\mathbb{R}) \), the operator \( \psi(\mathcal{R}_M) \) defined spectrally as a bounded spectral multiplier on \( L^2(M) \) coincides with the operator \( \text{Op}(\sigma) \) on \( C^\infty(M) \), with symbol \( \sigma(\pi) := \psi(\hat{\mathcal{R}}(\pi)) \) in \( \mathcal{A}_0 \), independent of \( \hat{x} \in M \). The associated kernel is \( \kappa := \psi(\hat{\mathcal{R}}(\pi)) \delta_0 \in \mathcal{S}(G) \). The integral kernel of \( \psi(\mathcal{R}_M) \) is a smooth function on \( M \times M \) given by

\[
K(\hat{x}, \hat{y}) = \sum_{\gamma \in \Gamma} \kappa(y^{-1}\gamma x).
\]

For every \( \varepsilon \in (0, 1] \), we have \( \psi(\varepsilon^\nu \mathcal{R}_M) = \text{Op}^{(\varepsilon)}(\sigma) \) on \( C^\infty(M) \), where \( \nu \) is the degree of homogeneity of \( \mathcal{R} \).

In the rest of this section, we give the general properties of the semiclassical calculus, starting with the boundedness on \( L^2 \).
4.2. **Boundedness in** $L^2(M)$. First, let us introduce the following seminorm on $A_0$:

$$
\| \sigma \|_{A_0} := \int_G \sup_{x \in M} |\kappa_x(y)| dy,
$$

where $\kappa_x$ is the kernel associated with $\sigma \in A_0$. Later on, we will use another seminorm on $A_0$, which is given by

$$
(4.3) \quad \| \sigma \|_{L^\infty(M \times \hat{G})} := \sup_{(\hat{x}, \pi) \in M \times \hat{G}} \| \sigma(\hat{x}, \pi) \|_{L^\infty(H_\pi)}.
$$

Here the supremum is the essential supremum with respect to the Plancherel formula. Note that since $\| \pi(f) \|_{L^\infty(H_\pi)} \leq \| f \|_{L^1(G)}$ for any $f \in L^1(G)$, we have

$$
(4.4) \quad \| \sigma \|_{L^\infty(M \times \hat{G})} \leq \sup_{\hat{x} \in M} \| \kappa_{\hat{x}} \|_{L^1(G)} \leq \| \sigma \|_{A_0}.
$$

The main property of the semiclassical calculus regarding $L^2$-boundedness is the following:

**Proposition 4.2.** For every $\epsilon \in (0,1]$ and $\sigma \in A_0$,

$$
\| \text{Op}^\epsilon(\sigma) \|_{L(L^2(M))} \leq \| \sigma^{(\epsilon)} \|_{A_0} = \| \sigma \|_{A_0}
$$

where $\sigma^{(\epsilon)}$ is given in (4.1).

**Proof.** The equality in the statement follows from a simple change of variable $y = \epsilon^{-1} \cdot z$ in

$$
\| \sigma^{(\epsilon)} \|_{A_0} = \| \sup_{\hat{x}_1 \in M} |\kappa_{\hat{x}_1}^{(\epsilon)}| \|_{L^1(G)} = \int_{G \hat{x}_1 \in M} \sup_{G \hat{x}_1 \in M} |\kappa_{\hat{x}_1}(\epsilon^{-1} \cdot z)| \epsilon^{-Q} dz = \int_{G \hat{x}_1 \in M} \sup_{G \hat{x}_1 \in M} |\kappa_{\hat{x}_1}(y)| dy = \| \sigma \|_{A_0}.
$$

Hence it suffices to show the case of $\epsilon = 1$. We observe that we have for any $f \in \mathcal{D}(M)$,

$$
|\text{Op}(\sigma)f(\hat{x})| = |\int_M f(\hat{y}) \sum_{\gamma \in \Gamma} \kappa_{\hat{x}}(y^{-1} \gamma x) d\hat{y}| \leq \int_M |f(\hat{y})| \sum_{\gamma \in \Gamma} \sup_{\hat{x} \in M} |\kappa_{\hat{x}}(y^{-1} \gamma x)| d\hat{y},
$$

consequently using (4.4)

$$
\| \text{Op}(\sigma)f \|_{L^\infty(M)} \leq \| f \|_{L^\infty(M)} \int_M \sum_{\gamma \in \Gamma} \sup_{\hat{x} \in M} |\kappa_{\hat{x}}(y^{-1} \gamma x)| d\hat{y} = \| f \|_{L^\infty(M)} \int_M \sup_{\hat{x} \in M} |\kappa_{\hat{x}}(y)| dy,
$$

$$
\| \text{Op}(\sigma)f \|_{L^1(M)} \leq \int_M |f(\hat{y})| \int_M \sup_{\hat{x} \in M} |\kappa_{\hat{x}}(y^{-1} \gamma z)| dz d\hat{y} = \| f \|_{L^1(M)} \int_M \sup_{\hat{x} \in M} |\kappa_{\hat{x}}(z')| dz'.
$$

In other words, the linear map Op$(\sigma)$ extends continuously as an operator $L^p(M) \to L^p(M)$ for $p = 1, \infty$ with norm $\| \sigma \|_{A_0}$. By interpolation, we obtain the first inequality in the statement. □

4.3. **Singularity of the operators as** $\epsilon \to 0$. The following lemma is similar to Proposition 3.4 in [9] and shows that the singularities of the integral kernels of the operators Op$(\epsilon)(\sigma)$ concentrate on the diagonal as $\epsilon \to 0$. It may also justify for many semiclassical properties that the kernel associated with a symbol may be assumed to be compactly supported in the variable of the group:

**Lemma 4.3.** Let $\eta \in \mathcal{D}(G)$ be identically equal to 1 close to 0. Let $\sigma \in A_0$ and let $\kappa_{\hat{x}}(z)$ denote its associated kernel. For every $\epsilon > 0$, the symbol $\sigma_{\epsilon}$ defined via

$$
\sigma_{\epsilon}(\hat{x}, \pi) = \mathcal{F}_G(\kappa_{\hat{x}} \eta(\epsilon \cdot))
$$

that is, the symbol with associated kernel $\kappa_{\hat{x}}(z) \eta(\epsilon \cdot z)$, is in $A_0$. Furthermore, for all $N \in \mathbb{N}$, there exists a constant $C = C_{N, \sigma, \eta} > 0$ such that

$$
\forall \epsilon \in (0,1] \quad \| \sigma_{\epsilon} - \sigma \|_{A_0} \leq C \epsilon^N.
$$
Proof. As the function $\eta$ is identically 1 close to $z = 0$, for all $N \in \mathbb{N}$, there exists a bounded continuous function $\theta$, identically 0 near 0, such that

$$\forall y \in G, \quad 1 - \eta(y) = \theta(y)\|y\|^N,$$

where $\| \cdot \|$ is a fixed homogeneous quasi-norm on $G$ (see Section 3.1.2). This notation implies

$$\kappa_{\hat{\varepsilon}}(z) - \kappa_{\hat{\varepsilon}}(z)\eta(\varepsilon \cdot z) = \kappa_{\hat{\varepsilon}}(z)\theta(\varepsilon \cdot z)\|\varepsilon \cdot z\|^N.$$

As $\|\varepsilon \cdot z\| = \varepsilon\|z\|$, we obtain

$$\|\sigma_{\varepsilon} - \sigma\|_{A_0} = \int_G \sup_{\hat{x}} |\kappa_{\hat{\varepsilon}}(z) - \kappa_{\hat{\varepsilon}}(z)\eta(\varepsilon \cdot z)| dz \leq \varepsilon^N\|\theta\|_{L^\infty} \int_G \sup_{\hat{x} \in M} |\kappa_{\hat{\varepsilon}}(z)| \|z\|^N dz.$$

This last integral is finite and this concludes the proof. $\square$

4.4. Symbolic calculus. In this paper, we will use the following properties of the symbolic calculus:

**Proposition 4.4.** (1) If $\sigma_1, \sigma_2 \in A_0$, then

$$\Op^{(\varepsilon)}(\sigma_1)\Op^{(\varepsilon)}(\sigma_2) = \Op^{(\varepsilon)}(\sigma_1\sigma_2) + O(\varepsilon),$$

in the sense that there exists a constant $C = C_{\sigma_1, \sigma_2, G, \Gamma} > 0$ such that

$$\forall \varepsilon \in (0, 1] \quad \|\Op^{(\varepsilon)}(\sigma_1)\Op^{(\varepsilon)}(\sigma_2) - \Op^{(\varepsilon)}(\sigma_1\sigma_2)\|_{L^2(M)} \leq C\varepsilon.$$

(2) If $\sigma \in A_0$, then

$$\Op^{(\varepsilon)}(\sigma)^* = \Op^{(\varepsilon)}(\sigma^*) + O(\varepsilon),$$

in the sense that there exists a constant $C = C_{\sigma, G, \Gamma, M} > 0$ such that

$$\forall \varepsilon \in (0, 1] \quad \|\Op^{(\varepsilon)}(\sigma)^* - \Op^{(\varepsilon)}(\sigma^*)\|_{L^2(M)} \leq C\varepsilon.$$

**Remark 4.5.** If $\sigma_1, \sigma_2 \in A_0$ and $\sigma_2$ does not depend on $\hat{x}$, then we have the equality

$$\Op^{(\varepsilon)}(\sigma_1)\Op^{(\varepsilon)}(\sigma_2) = \Op^{(\varepsilon)}(\sigma_1\sigma_2)$$

for every $\varepsilon \in (0, 1]$.

The proof of Proposition 4.4 follows from the following more general observations which give the symbolic properties of the semiclassical calculus. For this, we need to introduce the notions of difference operators. They aim at replacing the derivatives with respect to the Fourier variable in the Euclidean case.

If $q$ is a smooth function on $G$ with polynomial growth, we define $\Delta_q$ via

$$\Delta_q f(\pi) = \mathcal{F}_G(qf)(\pi), \quad \pi \in \hat{G},$$

for any function $f \in \mathcal{S}(G)$ and even any tempered distribution $f \in \mathcal{S}'(G)$. In particular, considering the basis $(X^\alpha)_{\alpha}$ constructed in Section 3.1.2, we consider $\{17\}$ Proposition 5.2.3] the polynomials $q_0$ such that $X^\alpha q_0 = \delta_{\alpha, \beta}$ for all $\alpha, \beta \in \mathbb{N}_0^d$. We then define

$$\Delta^\alpha := \Delta_{q_0}(\cdot)^{\alpha}.$$

We can now describe the symbolic calculus. The quantitative results on the symbolic calculus for the pseudo-differential calculus on $G$ in $\{17\}$ Section 5.5 imply the following properties:

(1) If $\sigma_1, \sigma_2 \in A_0$, then for every $N \in \mathbb{N}$, we have

$$\Op^{(\varepsilon)}(\sigma_1)\Op^{(\varepsilon)}(\sigma_2) = \sum_{\alpha < N} \varepsilon^{\alpha}\Op^{(\varepsilon)}(\Delta^\alpha \sigma_1 X^\alpha \sigma_2) + O(\varepsilon)^N,$$
If $\sigma \in A_0$, then for every $N \in \mathbb{N}$, we have
\[
\text{Op}^{(\varepsilon)}(\sigma)^* = \sum_{|\alpha| < N} \varepsilon^{\alpha} \text{Op}^{(\varepsilon)}(\Delta^\alpha X_\sigma^* \sigma^*) + O(\varepsilon)^N.
\]
The analysis also shows that if $\sigma_2$ and $\sigma$ do not depend on $x$ then
\[
\text{Op}^{(\varepsilon)}(\sigma_1)\text{Op}^{(\varepsilon)}(\sigma_2) = \text{Op}^{(\varepsilon)}(\sigma_1\sigma_2)
\]
and
\[
\text{Op}^{(\varepsilon)}(\sigma)^* = \text{Op}^{(\varepsilon)}(\sigma^*).
\]
And if $\sigma_1$ and $\sigma$ are the symbols of differential operators, then the sums in (1) and (2) above are finite and the equality holds for $N$ large enough.

5. Asymptotics

The semiclassical approach lends itself to understand asymptotics as the small parameters $\varepsilon$ goes to 0. For instance, we will obtain trace asymptotic expansions and consequently Weyl generalised laws. Theorem 1.1 of the introduction will be a simple consequence of this latter result.

5.1. Estimates for kernels, Hilbert-Schmidt norms and traces. The integral kernels enjoy the following estimate on the diagonal:

**Lemma 5.1.** Let $\sigma \in A_0$ with associated kernel $\kappa_\hat{x}(z)$. The integral kernel $K^{(\varepsilon)}$ of $\text{Op}^{(\varepsilon)}(\sigma)$ is smooth on $M \times M$ and satisfies for $\varepsilon$ small:
\[
K^{(\varepsilon)}(\hat{x}, \hat{x}) = \varepsilon^{-Q} \kappa_\hat{x}(0) + O(\varepsilon)^\infty.
\]

This means that there exists a constant $C = C_{N,\sigma,G,\Gamma} > 0$ such that for every $\varepsilon \in (0,1]$ and $\hat{x} \in M$:
\[
|K^{(\varepsilon)}(\hat{x}, \hat{x}) - \varepsilon^{-Q} \kappa_\hat{x}(0)| \leq C \varepsilon^N.
\]

*Proof.* Lemma 4.1 implies that $T_{\kappa^{(\varepsilon)}}$ is trace-class, with smooth integral kernel
\[
K^{(\varepsilon)}(\hat{x}, \hat{y}) = \sum_{\gamma \in \Gamma} \kappa^{(\varepsilon)}_\hat{x}(y^{-1} \gamma x)
\]
Note that for $\gamma = 0$
\[
\rho_{0,\varepsilon} = \kappa^{(\varepsilon)}(0) = \varepsilon^{-Q} \kappa(0).
\]

We fix a homogenous quasi-norm $\| \cdot \|$ on $G$ (see Section 3.1.2). In order to avoid introducing unnecessary constants, we may assume that it yields a distance on $G \sim \mathbb{R}^N$. By assumption on the kernel $\kappa(z)$, we have
\[
\forall N \in \mathbb{N} \quad \forall z \in G, \forall \hat{x} \in M \quad |\kappa_\hat{x}(z)| \leq C_N(1 + \|z\|)^{-N}.
\]
Consequently, fixing $N \in \mathbb{N}$,
\[
|\kappa^{(\varepsilon)}_\hat{x}(x^{-1} \gamma x)| \leq C_N \varepsilon^{-Q}(1 + \varepsilon^{-1}\|x^{-1} \gamma x\|)^N.
\]
The estimate for the kernel on the diagonal follows from fixing a fundamental domain $F_0$ containing 0 and proceeding as in [15, Proposition 4.3].

We open a brief parenthesis devoted to the tensor product of the Hilbert spaces $L^2(M)$ and $L^2(\hat{G})$ defined in Section 2.1
\[
L^2(M \times \hat{G}) := L^2(M) \otimes L^2(\hat{G}).
\]
We may identify \( L^2(M \times \hat{G}) \) with the space of measurable fields of Hilbert-Schmidt operators \( \sigma = \{ \sigma(\hat{x}, \pi) : (\hat{x}, \pi) \in M \times \hat{G} \} \) such that

\[
\| \sigma \|_{L^2(M \times \hat{G})}^2 := \int_{M \times \hat{G}} \| \sigma(\hat{x}, \pi) \|_{HS(H_n)}^2 d\hat{x}d\mu(\pi) < \infty.
\]

Here \( \mu \) is the Plancherel measure on \( \hat{G} \), see Section 2.1. The group Fourier transform yields an isomorphism between the Hilbert spaces \( L^2(M \times \hat{G}) \) and \( L^2(M \times G) \), and \( F^{-1}_G \sigma(\hat{x}, \cdot) = \kappa_{\hat{x}} \) will still be called the associated kernel of \( \sigma \). Naturally \( A_0 \subset L^2(M \times \hat{G}) \).

Lemma 5.1 and its proof imply:

**Corollary 5.2.** For \( \sigma \in A_0 \), \( \text{Op}^{(\varepsilon)}(\sigma) \) is a Hilbert-Schmidt and trace-class operator on \( M \) satisfying

\[
\text{Tr} \left( \text{Op}^{(\varepsilon)}(\sigma) \right) = \varepsilon^{-Q} \int_M \kappa_{\hat{x}}(0)d\hat{x} + O(\varepsilon)^\infty = \varepsilon^{-Q} \int_{M \times \hat{G}} \text{Tr} \left( \sigma(\hat{x}, \pi) \right) d\hat{x}d\mu(\pi) + O(\varepsilon)^\infty,
\]

and

\[
\| \text{Op}^{(\varepsilon)}(\sigma) \|_{HS(L^2(M))}^2 = \varepsilon^{-Q} \int_{G \times M} |\kappa_x(z)|^2 dzd\hat{x} + O(\varepsilon)^\infty = \varepsilon^{-Q} \| \sigma \|_{L^2(G \times M)}^2 + O(\varepsilon)^\infty.
\]

**Proof.** The first trace asymptotics follows from

\[
\text{Tr} \left( \text{Op}^{(\varepsilon)}(\sigma) \right) = \int_M K^{(\varepsilon)}(\hat{x}, \hat{x})d\hat{x},
\]

together with Lemma 5.1. Using the Fourier inversion formula (2.3) yields the second equality in the trace asymptotics.

For the Hilbert-Schmidt norm, we have

\[
\| \text{Op}^{(\varepsilon)}(\sigma) \|_{HS(L^2(M))}^2 = \int_{M \times M} |K^{(\varepsilon)}(\hat{x}, \hat{y})|^2 d\hat{x}d\hat{y},
\]

and the first asymptotics is obtained by an easy adaptation of the proofs of Lemma 5.1 and [15, Proposition 4.3]. The second equality then follows from the Plancherel formula. \( \square \)

### 5.2. Generalised Weyl laws for \( \mathcal{R}_M \)

In this section, we consider the following setting: let \( \mathcal{R} \) a positive Rockland operator on a graded Lie group \( G \). Let \( \mathcal{R}_M \) be its corresponding operator on the nil-manifold \( M = \Gamma \backslash G \) where \( \Gamma \) is a discrete co-compact subgroup of \( G \). We order the eigenvalues of \( \mathcal{R}_M \) (counted with multiplicity) into the sequence

\[
0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_j \longrightarrow +\infty \quad \text{as} \quad j \rightarrow +\infty.
\]

We denote by \( \nu \) the homogeneous degree of \( \mathcal{R} \).

The Weyl law for \( \mathcal{R}_M \) and its spectral counting function

\[
N(\Lambda) := |\{ j \in \mathbb{N}_0 : \mu_j \leq \Lambda \}| \sim \text{vol}(M) \frac{c\nu}{Q} \Lambda^{Q/\nu},
\]

is already known [15, 7]. Here, we obtained the following generalised Weyl law which makes sense thanks to our symbolic approach:

**Theorem 5.3.** We continue with the setting above. Let \( 0 \leq a < b \). Denoting the semiclassical counting function for \([a, b] \) by

\[
N_{[a, b]}(\Lambda) := \{ j \in \mathbb{N}_0 : \Lambda a \leq \mu_j \leq \Lambda b \},
\]

we have as \( \Lambda \rightarrow +\infty \)

\[
N_{[a, b]}(\Lambda) \sim \text{vol}(M) c\Lambda^{Q/\nu},
\]

15.
where the (positive, finite) constant $c$ is
\[
c = \frac{c_0\nu}{Q}(b^2 - a^2) = \int_G \text{Tr} \left( 1_{[a,b]}(\tilde{R}(\pi)) \right) d\mu(\pi).
\]
Furthermore, if we consider an orthonormal basis $(\varphi_j)_{j \in \mathbb{N}_0}$ of the Hilbert space $L^2(M)$ consisting of eigenfunctions of $\mathcal{R}_M$:
\[
\mathcal{R}_M \varphi_j = \mu_j \varphi_j, \quad j = 0, 1, 2, \ldots
\]
then we have for any $\sigma \in \mathcal{A}_0$
\[
\lim_{\Lambda \to \infty} N_{[a,b]}(\Lambda)^{-1} \sum_{j : \mu_j \in \Lambda[a,b]} \left\langle \text{Op}^{(e)}(\sigma) \varphi_j, \varphi_j \right\rangle_{L^2(M)} = \int_{M \times \tilde{G}} \text{Tr} \left( \left( \sigma(\hat{x}, \pi) 1_{[a,b]}(\tilde{R}(\pi)) \right) \right) \frac{d\hat{x}d\mu(\pi)}{\text{vol}(M)c}.
\]
This statement should be compared with its ‘commutative’ counterpart, for instance in [28, Theorem 9.3].

Before proving Theorem 5.3 let us show the following corollary. Note that the particular case of a subLaplacian on a stratified group with $a = 0$, $b = 1$ gives Theorem 1.1.

**Corollary 5.4.** We keep the notation of Theorem 5.3. For any continuous function $f : M \to \mathbb{C}$, we have:
\[
\lim_{\Lambda \to \infty} \frac{1}{N_{[a,b]}(\Lambda)} \sum_{j : \mu_j \in \Lambda[a,b]} \int_M f(\hat{x}) |\varphi_j(\hat{x})|^2 d\hat{x} = \int_M f(\hat{x}) \frac{d\hat{x}}{\text{vol}(M)}.
\]

**Proof of Corollary 5.4.** If $f$ is smooth, the statement follows from choosing $\sigma = f\psi(\tilde{R})$ in Theorem 5.3, where $\psi \in \mathcal{D}(\mathbb{R})$ satisfies $\psi 1_{[a,b]} = 1_{[a,b]}$. An argument of density gives the result for continuous functions $f$. \hfill \square

Let us now show Theorem 5.3.

**Proof of Theorem 5.3.** The asymptotic for $N_{[a,b]}$ may be obtain with Karamata’s Tauberian theorem or alternatively by proceeding as for the proof of the Weyl law in [13] (see p.20 therein), which is essentially taking approximations $\psi$ of $1_{[a,b]}$ in the Hilbert Schmidt norm asymptotics recalled in Proposition 3.4 (1)(ii). The formula for $c$ follows from the Plancherel formula (2.2) and Theorem 8.2. Note that it also implies

\[
0 < c \leq \int_G \text{Tr} 1_{[a,b]}(\tilde{R}) |d\mu| = \int_G \|1_{[a,b]}(\tilde{R})\|^2_{HS} d\mu = c_0^{-1} \|1_{[a,b]}(\tilde{R})\|_{L^2(S)} < \infty.
\]

If $\psi \in \mathcal{D}(\mathbb{R})$, then the properties of the functionals and semiclassical calculus imply
\[
\sum_{j : \epsilon^p \mu_j \in \text{supp} \psi} \left\langle \text{Op}^{(e)}(\sigma \psi(\tilde{R})) \varphi_j, \varphi_j \right\rangle_{L^2(M)} = \text{Tr} \left( \text{Op}^{(e)}(\sigma \psi(\tilde{R})) \right)
\]
\[
= \epsilon^{-Q} \int_{M \times \tilde{G}} \text{Tr} \left( \sigma \psi(\tilde{R}) \right) d\hat{x}d\mu + O(\epsilon)\infty,
\]
by Corollary 5.2 so the asymptotic for $N_{[a,b]}$ yields as $\epsilon \to 0$
\[
\frac{\sum_{j : \epsilon^p \mu_j \in \text{supp} \psi} \left\langle \text{Op}^{(e)}(\sigma \psi(\tilde{R})) \varphi_j, \varphi_j \right\rangle_{L^2(M)}}{N_{[a,b]}(\epsilon)} \to \frac{1}{\text{vol}(M)c} \int_{M \times \tilde{G}} \text{Tr} \left( \sigma \psi(\tilde{R}) \right) d\hat{x}d\mu.
\]
The conclusion follows by taking $\psi$ as approximation of $1_{[a,b]}$. The rest of the proof is devoted to giving a detailed argument for this. We want to show for $\sigma \in \mathcal{A}_0$
\[
\lim_{\epsilon \to 0} \frac{S_e(\sigma)}{N_{[a,b]}(\epsilon)} = \frac{1}{c} \ell(\sigma), \quad \text{where} \quad S_e(\sigma) := \sum_{\mu_j \in \epsilon^{-\nu}[a,b]} \left\langle \text{Op}^{(e)}(\sigma) \varphi_j, \varphi_j \right\rangle_{L^2(M)}.
\]
\[
\ell(\sigma) := \iint_{M \times \hat{G}} \text{Tr} \left( \sigma(\hat{x}, \pi)1_{[a,b]}(\hat{R}(\pi)) \right) \, \hat{x} \, d\mu(\pi).
\]

Since
\[
\iint_{M \times \hat{G}} \left| \text{Tr} \left( \sigma(\hat{x}, \pi)1_{[a,b]}(\hat{R}(\pi)) \right) \right| \, \hat{x} \, d\mu(\pi) \leq \text{vol}(M) \, \|\sigma\|_{L^\infty(M \times \hat{G})} \iint_{\hat{G}} \left| \text{Tr} \left( 1_{[a,b]}(\hat{R}) \right) \right| \, d\mu
\]
is finite by \(5.1\), \(\ell\) is a well-defined linear functional on \(A_0\). We observe that if \(\psi \in D(\mathbb{R})\) satisfies \(\psi 1_{[a,b]} = 1_{[a,b]}\), then \(\ell(\psi(\hat{R})) = \text{vol}(M) c\), so by replacing \(\sigma\) with \(\sigma - (\text{vol}(M)c)^{-1} \ell(\sigma)\psi(\hat{R})\), we may assume that \(\sigma \in A_0\) satisfies \(\ell(\sigma) = 0\).

Let \(\psi \in D(\mathbb{R})\) with \(\text{supp}(\psi) \subset [a, b]\) and let \(\psi_1 \in D(\mathbb{R})\) with \(\psi_1 1_{[a,b]} = 1_{[a,b]}\). We decompose
\[
S_\varepsilon(\sigma) = S_\varepsilon(\sigma\psi(\hat{R})) + S_\varepsilon(\sigma(\psi_1 - \psi)(\hat{R})).
\]
For the first term, we have the asymptotic from Corollary \(5.2\)
\[
S_\varepsilon(\sigma\psi(\hat{R})) = \text{Tr} \left( \text{Op}^{(\varepsilon)}(\sigma\psi(\hat{R})) \right) = \varepsilon^{-Q} \iint_{M \times \hat{G}} \text{Tr} \left( \sigma\psi(\hat{R}) \right) \, \hat{x} \, d\mu + O(\varepsilon)^\infty,
\]
For the main term on the right-hand side, using \(\ell(\sigma) = 0\), we see:
\[
\iint_{M \times \hat{G}} \left| \text{Tr} \left( \sigma\psi(\hat{R}) \right) \, \hat{x} \, d\mu \right| = \iint_{M \times \hat{G}} \left| \text{Tr} \left( \sigma(\psi - 1_{[a,b]})(\hat{R}) \right) \, \hat{x} \, d\mu \right| \leq \text{vol}(M)\|\sigma\|_{L^\infty(M \times \hat{G})} \iint_{\hat{G}} \left| \text{Tr} \left( \psi - 1_{[a,b]} \right) (\hat{R}) \right| \, d\mu.
\]
By the Plancherel formula \(2.2\) and Theorem \(3.2\), this last integral is equal to
\[
\iint_{\hat{G}} \left| \text{Tr} \left( \psi - 1_{[a,b]} \right) (\hat{R}) \right| \, d\mu = \|\sqrt{\psi - 1_{[a,b]}}\|_{L^2(\text{co}L^Q; d\lambda/\lambda)}^2.
\]
To estimate \(S_\varepsilon(\sigma(\psi_1 - \psi)(\hat{R}))\), we first obtain the blunt majoration
\[
\left| \left\langle \text{Op}^{(\varepsilon)}(\sigma)(\psi_1 - \psi)(\hat{R}), \varphi_j, \varphi_j \right\rangle_{L^2(M)} \right| \leq \left| \left\langle \text{Op}^{(\varepsilon)}(\sigma)(\psi_1 - \psi)(\varepsilon^\nu R_M), \varphi_j, \varphi_j \right\rangle_{L^2(M)} \right|
\]
\[
\leq \|\text{Op}^{(\varepsilon)}(\sigma)\|_{L^2(L^2(M))} \|\psi_1 - \psi(\varepsilon^\nu R_M)\varphi_j\|_{L^2(M)} \leq \|\sigma\|_{A_0} \|\psi_1 - \psi(\varepsilon^\nu R_M)\varphi_j\|_{L^2(M)},
\]
by Proposition \(4.1\). Hence, we have
\[
\left| S_\varepsilon(\sigma(\psi_1 - \psi)(\varepsilon^\nu \hat{R})) \right| \leq \|\sigma\|_{A_0} \sum_{\mu_\varepsilon \in \varepsilon^{-\nu}[a,b]} \|\psi_1 - \psi(\varepsilon^\nu R_M)\varphi_j\|_{L^2(M)}
\]
\[
\leq \|\sigma\|_{A_0} \sqrt{N_{[a,b]}(\varepsilon^{-\nu})} \|\psi_1 - \psi(\varepsilon^\nu R_M)\|_{HS(L^2(M))},
\]
by the Cauchy-Schwartz inequality. The Weyl law gives an asymptotic for \(N_{[a,b]}(\varepsilon^{-\nu})\) while we have by Proposition \(5.2\) \((1)i)\):
\[
\|\psi_1 - \psi(\varepsilon^\nu R_M)\|_{HS(L^2(M))} = \varepsilon^{-Q/2} \text{vol}(M)c_0 \|\psi_1 - \psi\|_{L^2(\text{co}L^Q; d\lambda/\lambda)} + O(\varepsilon).
\]
Collecting all the estimates above yields:
\[
\limsup_{\varepsilon \to 0} \frac{|S_\varepsilon(\sigma)|}{N_{[a,b]}(\varepsilon^{-\nu})} \leq C_1 \sqrt{\|\psi - 1_{[a,b]}\|_{L^2(\text{co}L^Q; d\lambda/\lambda)}}^2 + C_2 \|\psi_1 - \psi\|_{L^2(\text{co}L^Q; d\lambda/\lambda)}
\]
where \(C_1\) and \(C_2\) are two constants of \(G, \Gamma, \mathcal{R}, \sigma\) independent of \(\psi\) and \(\psi_1\). By choosing sequences of functions \(\psi\) and \(\psi_1\) approximating \(1_{[a,b]}\), the right-hand side is as small as we want. This concludes the proof of Theorem \(5.3\).\]
5.3. Quantum variance. In this section, we consider the same setting as in Section 5.2. We also consider an orthonormal basis \((\varphi_j)\) of \(\mathcal{R}_M\)-eigenfunctions for the Hilbert space \(L^2(M)\) and we assume
\[
\mathcal{R}_M \varphi_j = \mu_j \varphi_j, \quad 0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots
\]
For any \([a, b] \subset \mathbb{R}\) and \(\varepsilon \in (0, 1]\), we are interested in the quantity
\[
QV_{\varepsilon, [a, b]}(\sigma) := \frac{1}{N_{[a, b]}(\varepsilon^{-\nu})} \sum_{j: \varepsilon^{\nu} \lambda_j \in [a, b]} \left| (\text{Op}^{(\varepsilon)}(\sigma) \varphi_j, \varphi_j)_{L^2(M)} \right|^2, \quad \sigma \in \mathcal{A}_0,
\]
and in its lim-sup:
\[
QV_{0, [a, b]}(\sigma) := \limsup_{\varepsilon \to 0} QV_{\varepsilon, [a, b]}(\sigma).
\]
We call \(QV_{0, [a, b]}(\sigma)\) the quantum variance associated with \((\varphi_j)\) and \([a, b] \subset \mathbb{R}\) for a symbol \(\sigma \in \mathcal{A}_0\).

From Proposition 4.2, we have for any \(\sigma \in \mathcal{A}_0\) and \(\varepsilon \in (0, 1]\)
\[
QV_{\varepsilon, [a, b]}(\sigma) \leq \|\text{Op}^{(\varepsilon)}(\sigma)\|_{\mathcal{L}(L^2(M))} \leq \|\sigma\|_{\mathcal{A}_0},
\]
Hence \(QV_{0, [a, b]}(\sigma) \in [0, \|\sigma\|_{\mathcal{A}_0}]\) is a finite non-negative number.

5.3.1. Symbols with zero variance. We can already identify symbols with zero variance:

**Lemma 5.5.** If \(\sigma = \text{ad} \hat{\mathcal{R}}(\tau) = \{\hat{\mathcal{R}}, \tau\}\) for some \(\tau \in \mathcal{A}_0\) then
\[
QV_{\varepsilon, [a, b]}(\sigma) = O(\varepsilon^{\nu-1}) \quad \text{so} \quad QV_{0, [a, b]}(\sigma) = 0.
\]

**Proof.** The properties of the semiclassical calculus imply (see also below)
\[
(\text{Op}^{(\varepsilon)}(\sigma) \varphi_j, \varphi_j)_{L^2(M)} = (\text{Op}^{(\varepsilon)}(\mathcal{R}_M \varphi_j), \varphi_j)_{L^2(M)}
= (\varepsilon^{\nu} \mathcal{R}_M, \text{Op}^{(\varepsilon)}(\tau) \varphi_j, \varphi_j)_{L^2(M)} + O(\varepsilon^{\nu-1}).
\]
The first term of the last right-hand side vanishes, and the statement follows. \(\square\)

Let us examine further the consequence of the properties of the semiclassical calculus, see Section 4.4. They imply for any \(\sigma \in \mathcal{A}_0\)
\[
\begin{align*}
[\varepsilon^{\nu} \mathcal{R}_M, \text{Op}^{(\varepsilon)}(\sigma)] &= \text{Op}^{(\varepsilon)}(\mathcal{R}_M) \text{Op}^{(\varepsilon)}(\sigma) - \text{Op}^{(\varepsilon)}(\sigma \mathcal{R}_M) \\
&= \text{Op}^{(\varepsilon)}([\mathcal{R}_M, \sigma]) + \varepsilon^{\nu_1} \sum_{[\alpha]=\nu_1} \text{Op}^{(\varepsilon)}(\Delta^{\alpha} \mathcal{R}_M X^0_\varphi \sigma) + O(\varepsilon^{\nu_1+1}).
\end{align*}
\]
Therefore, for any \(\mathcal{R}_M\)-eigenfunction \(\varphi\), we have:
\[
0 = ([\varepsilon^{\nu} \mathcal{R}_M, \text{Op}^{(\varepsilon)}(\sigma)] \varphi, \varphi)
= (\text{Op}^{(\varepsilon)}([\mathcal{R}_M, \sigma]) \varphi, \varphi) + \varepsilon^{\nu_1} \sum_{[\alpha]=\nu_1} (\text{Op}^{(\varepsilon)}(\Delta^{\alpha} \mathcal{R}_M X^0_\varphi \sigma) \varphi, \varphi) + O(\varepsilon^{\nu_1+1}),
\]
with an implicit constant depending on \(\sigma\) (and \(G, \Gamma, \mathcal{R}\)) and on \(\|\varphi\|_{L^2(M)}\).

We define the operation
\[
\mathcal{E} := \sum_{[\alpha]=\nu_1} \Delta^{\alpha} \mathcal{R}_M X^0_\varphi,
\]
on the symbols. For example, we will compute the operation \(\mathcal{E}\) explicitly for the intrinsic sub-Laplacians in the stratified case in Lemma 7.5. We have obtained:
\[
(\text{Op}^{(\varepsilon)}([\mathcal{R}_M, \tau]) \varphi, \varphi)_{L^2(M)} = -\varepsilon^{\nu_1} (\text{Op}^{(\varepsilon)}(\mathcal{E} \tau) \varphi, \varphi)_{L^2(M)} + O(\varepsilon^{\nu_1+1}),
\]
with again an implicit constant depending on \(\sigma\) (and \(G, \Gamma, \mathcal{R}\)) and on \(\|\varphi\|_{L^2(M)}\). Proceeding as in the proof of Lemma 5.5 we obtain:
Lemma 5.6. If \( \sigma = \mathcal{E} \tau \) with \( \tau \in A_0 \) commuting with \( \widehat{\mathcal{R}} \) then
\[
QV_{\varepsilon, [a,b]}(\sigma) = O(\varepsilon) \quad \text{so} \quad QV_{0, [a,b]}(\sigma) = 0.
\]

Proof. By (5.3), we have
\[
(\text{Op}^{(\varepsilon)}(\sigma)\varphi_j, \varphi_j)_{L^2(M)} = (\text{Op}^{(\varepsilon)}(\mathcal{E})\varphi_j, \varphi_j)_{L^2(M)}
\]
\[
= -e^{-v_1(\text{Op}^{(\varepsilon)}(\widehat{\mathcal{R}}, \tau))\varphi_j, \varphi_j}_{L^2(M)} + O(\varepsilon),
\]
By hypotheses, \( [\widehat{\mathcal{R}}, \tau] = 0 \). The statement follows. \( \square \)

In the rest of the paper, we denote by
\[
\mathcal{A}^\mathcal{R}_0 := \{ \sigma \in A_0 : \sigma \widehat{\mathcal{R}} = \widehat{\mathcal{R}} \sigma \}
\]
the space of symbols in \( A_0 \) commuting with \( \widehat{\mathcal{R}} \). This space will be further studied in Section 7.2.

5.3.2. Consequence of the semiclassical calculus. The properties of the semiclassical calculus already imply also the following properties:

Proposition 5.7.

1. We have for all \( \sigma \in A_0 \)
\[
QV_{0, [a,b]}(\sigma) \leq \frac{1}{c'} \| \sigma \|^2_{L^2(M \times \hat{G})},
\]
with \( c' := \text{vol}(M)c \) and \( c > 0 \) as in Theorem 5.3, and furthermore,
\[
QV_{0, [a,b]}(\sigma) \leq \frac{1}{c'} \| \hat{E}[a,b] \sigma \hat{E}[a,b] \|^2_{L^2(M \times \hat{G})}.
\]

2. We have for all \( \sigma, \tau \in A_0 \)
\[
QV_{0, [a,b]}(\sigma + \tau) \leq QV_{0, [a,b]}(\sigma) + QV_{0, [a,b]}(\tau) + \frac{2}{c'} \| \sigma \|_{L^2(M \times \hat{G})} \| \tau \|_{L^2(M \times \hat{G})},
\]
and moreover,
\[
\| QV_{0, [a,b]}(\sigma) - QV_{0, [a,b]}(\tau) \|
\]
\[
\leq \frac{1}{c'} \| \sigma - \tau \|^2_{L^2(M \times \hat{G})} + \frac{2}{c'} \| \sigma - \tau \|_{L^2(M \times \hat{G})} \max(\| \sigma \|_{L^2(M \times \hat{G})}, \| \tau \|_{L^2(M \times \hat{G})}),
\]
as well as
\[
QV_{0, [a,b]}(\sigma + [\widehat{\mathcal{R}}, \tau]) = QV_{0, [a,b]}(\sigma),
\]
and if \( \tau \) commutes with \( \widehat{\mathcal{R}} \),
\[
QV_{0, [a,b]}(\sigma + \mathcal{E} \tau) = QV_{0, [a,b]}(\sigma).
\]

3. The map \( QV_{0, [a,b]} : A_0 \to [0, \infty) \) admits a unique continuous extension to \( L^2(M \times \hat{G}) \) that we still denote by \( QV_{0, [a,b]} : L^2(M \times \hat{G}) \to [0, \infty) \). This extension still satisfies the inequalities of Parts (1) and (2) for any \( \sigma, \tau \in L^2(M \times \hat{G}) \). In particular,
\[
\forall \sigma \in L^2(M \times \hat{G}) \quad QV_{0, [a,b]}(\sigma) = QV_{0, [a,b]}(\text{pr}_S(\sigma))
\]
with \( S \) being the closure of \( \text{ad}(\widehat{\mathcal{R}})A_0 + \mathcal{E} \mathcal{A}_0^\mathcal{R} \) in \( L^2(M \times \hat{G}) \), and \( \text{pr}_S \) being the orthogonal projection onto \( S \).
Proof. For Part (1), we observe that
\[
\sum_{j: x^v \lambda_j \in [a,b]} \left| \langle \text{Op}^{(e)}(\sigma) \varphi_j, \varphi_j \rangle_{L^2(M)} \right|^2 \leq \sum_j \| \text{Op}^{(e)}(\sigma) \varphi_j \|_{L^2(M)}^2 = \| \text{Op}^{(e)}(\sigma) \|_{HS(L^2(M))}^2,
\]
so
\[
QV_{\varepsilon,[a,b]}(\sigma) \leq \frac{\| \text{Op}^{(e)}(\sigma) \|_{HS(L^2(M))}^2}{N_{[a,b]}(\varepsilon^{-\nu})} = \frac{\varepsilon^{-Q} \| \sigma \|_{L^2(M \times \widetilde{G})}^2}{\varepsilon^\nu - Q} + O(\varepsilon)^\infty
\]
by Theorem 5.3 and Corollary 5.2. The first inequality follows by passing to the lim-sup.

For the second inequality, let \( \psi_1, \psi_2 \in D(\mathbb{R}) \) with \( \psi_i \equiv 1 \) on \([a,b], i = 1, 2\). We have when \( \varepsilon^\nu \mu_j \in [a,b] \)
\[
\langle \text{Op}^{(e)}(\sigma) \varphi_j, \varphi_j \rangle_{L^2(M)} = \psi_1(\varepsilon^\nu \mu_j) \psi_2(\varepsilon^\nu \mu_j) \langle \text{Op}^{(e)}(\sigma) \varphi_j, \varphi_j \rangle_{L^2(M)}
= (\psi_1(\varepsilon^\nu \mathcal{R}_M) \text{Op}^{(e)}(\sigma) \psi_2(\varepsilon^\nu \mathcal{R}_M) \varphi_j, \varphi_j \rangle_{L^2(M)}
= (\text{Op}^{(e)}(\psi_1(\mathcal{R}) \sigma \psi_2(\mathcal{R})) \varphi_j, \varphi_j \rangle_{L^2(M)} + O_{\psi_1,\psi_2}(\varepsilon),
\]
by the properties of the semiclassical and functional calculus. Hence,
\[
QV_{\varepsilon,[a,b]}(\sigma) \leq QV_{\varepsilon,[a,b]}(\psi_1(\mathcal{R}) \sigma \psi_2(\mathcal{R})) + O_{\psi_1,\psi_2}(\varepsilon),
\]
and consequently, taking the lim-sup,
\[
QV_{0,[a,b]}(\sigma) \leq QV_{0,[a,b]}(\psi_1(\mathcal{R}) \sigma \psi_2(\mathcal{R})) \leq \frac{1}{\varepsilon} \| \psi_1(\mathcal{R}) \sigma \psi_2(\mathcal{R}) \|_{L^2(M \times \widetilde{G})},
\]
having used the first inequality. This is true for any \( \psi_1, \psi_2 \in D(\mathbb{R}) \) with \( \psi_i \equiv 1 \) on \([a,b], i = 1, 2\). The second inequality of Part (1) follows by approximating \( 1_{[a,b]} \) by sequences of such functions and proceeding as in the proof of Theorem 5.3.

Now let \( \sigma, \tau \in \mathcal{A}_0 \). Then
\[
QV_{\varepsilon,[a,b]}(\sigma + \tau) = QV_{\varepsilon,[a,b]}(\sigma) + QV_{\varepsilon,[a,b]}(\tau) + R_\varepsilon,
\]
where
\[
R_\varepsilon := \frac{2}{N_{[a,b]}(\varepsilon^{-\nu})} \mathbb{R} \sum_{j: x^v \lambda_j \in [a,b]} \langle \text{Op}^{(e)}(\sigma) \varphi_j, \varphi_j \rangle_{L^2(M)} \langle \text{Op}^{(e)}(\tau) \varphi_j, \varphi_j \rangle_{L^2(M)}.
\]
The Cauchy-Schwartz inequality yields
\[
|R_\varepsilon| \leq \frac{2}{N_{[a,b]}(\varepsilon^{-\nu})} \| \text{Op}^{(e)}(\sigma) \|_{HS(L^2(M))} \| \text{Op}^{(e)}(\tau) \|_{HS(L^2(M))}
= \frac{2}{\varepsilon^\nu} \| \sigma \|_{L^2(M \times \widetilde{G})} \| \tau \|_{L^2(M \times \widetilde{G})} + O(\varepsilon)^\infty,
\]
by Theorem 5.3 and Corollary 5.2. Taking the lim-sup as \( \varepsilon \to 0 \) proves the first inequality in Part (2). The second inequality follows from the first inequality and Part (1). For the last inequality, we observe that if \( \tau = [\mathcal{R}, \tau_1] \), then proceeding as in the proof of Lemma 5.5
\[
R_\varepsilon := \frac{2}{N_{[a,b]}(\varepsilon^{-\nu})} \mathbb{R} \sum_{j: x^v \lambda_j \in [a,b]} \langle \text{Op}^{(e)}(\sigma) \varphi_j, \varphi_j \rangle_{L^2(M)} \langle \text{Op}^{(e)}(\tau_1) \varphi_j, \varphi_j \rangle_{L^2(M)}
= \frac{-2\varepsilon}{N_{[a,b]}(\varepsilon^{-\nu})} \mathbb{R} \sum_{j: x^v \lambda_j \in [a,b]} \langle \text{Op}^{(e)}(\sigma) \varphi_j, \varphi_j \rangle_{L^2(M)} \langle \text{Op}^{(e)}(\tau_1) \varphi_j, \varphi_j \rangle_{L^2(M)},
\]
so
\[
|R_\varepsilon| \leq \frac{2\varepsilon}{N_{[a,b]}(\varepsilon^{-\nu})} \| \text{Op}^{(e)}(\sigma) \|_{HS(L^2(M))} \| \text{Op}^{(e)}(\tau_1) \|_{HS(L^2(M))} = O(\varepsilon),
\]
by Theorem \ref{5.3} and Corollary \ref{5.2}. This yields the third inequality in Part (2). The last one is proved in a similar way replacing the proof of Lemma \ref{5.5} with the one of Lemma \ref{5.6}.

The second inequality in Part (2) implies that if a sequence of symbols \((\sigma_k)_{k \in \mathbb{N}}\) in \(A_0\) has a limit in the Hilbert space \(L^2(M \times \bar{G})\) then \(QV_0(\sigma_k)\) is a Cauchy sequence in \([0, +\infty)\) and therefore converges. Moreover, the limit \(\lim_{k \to \infty} QV_{0,\lbrack a,b\rbrack}(\sigma_k)\) depends only on the symbol \(\lim_k \sigma_k\) in \(L^2(M \times \bar{G})\) and not on the converging sequence \((\sigma_k)_{k \in \mathbb{N}}\). As \(A_0\) is dense in the Hilbert space \(L^2(M \times \bar{G})\), this concludes the proof.

In the commutative case, the group \(G\) is \(\mathbb{R}^n\) equipped with its usual abelian structure, and the nilmanifold is a torus. Using the well-known properties of the Fourier series and of the Euclidean Fourier transforms, it is possible to completely determine for instance the projection \(pr_S\) of the statement above, see Lemma \ref{5.9} below. However, in the most simple non-commutative setting, for instance the canonical subLaplacian on the Heisenberg group, those questions are left for future publications.

### 5.3.3. The commutative case

In this section, we show that in the case of the \(n\)-dimensional torus \(T^n = \mathbb{R}^n / \mathbb{Z}^n\), Proposition \ref{5.7} already yields the following reduced Quantum Ergodicity property:

**Theorem 5.8.** Let \((\varphi_j)_{j \in \mathbb{N}}\) be an orthonormal basis of \(L^2(T^n)\) for the Laplacian \(\Delta = \partial_1^2 + \ldots + \partial_n^2\) with

\[ (-\Delta) \varphi_j = \mu_j \varphi_j, \quad 0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots \]

Then there exists a subsequence \((j_k)\) of density 1 such that

\[ \forall f \in C(T^n) \quad \lim_{j_k \to \infty} \int_{\mathbb{T}^n} f(\hat{x}) |\varphi_{j_k}(\hat{x})|^2 d\hat{x} = \int_{\mathbb{R}^n} f(\hat{x}) d\hat{x}. \]

We will see that it suffices to apply what precedes to the case of \(\mathcal{R} = -\Delta\). The \(A_0\)-symbols are the functions \(T^n \times \mathbb{R}^n \ni (\hat{x}, \xi) \rightarrow \sigma(\hat{x}, \xi)\) in \(C^\infty(M : S(\mathbb{R}^n))\). We compute easily \(\mathcal{E} = -2\partial_{\hat{x}_j} \partial_{\xi_j}\), and in this commutative setting, all the symbols commute with each other so \(A_0^R = 0\). The projector in Proposition \ref{5.7} is easily determined:

**Lemma 5.9.** With the setting of Theorem \ref{5.8}, the orthogonal projection \(pr_S\) onto the closure of the range of \(\mathcal{E}\) is given by

\[ pr_S(\sigma) = \sigma - \int_{\mathbb{T}^n} \sigma, \quad \text{where} \quad \left( \int_{\mathbb{T}^n} \sigma \right)(\hat{x}, \xi) = \int_{\mathbb{T}^n} \sigma(\hat{x}', \xi) d\hat{x}', \quad \sigma \in L^2(T^n \times \mathbb{R}^n). \]

**Proof of Lemma 5.9.** If \(\sigma \in L^2(T^n \times \mathbb{R}^n)\) satisfies \(\mathcal{E} \sigma = 0\), then applying both the Fourier transform on \(T^n \times \mathbb{R}^n\) to \(\sigma\) and denoting the resulting function as \(\hat{\sigma}(\ell, y)\), \(\ell \in \mathbb{Z}^n, y \in \mathbb{R}^n\), we see that

\[ \forall \ell \in \mathbb{Z}^n, y \in \mathbb{R}^n \quad \left( \sum_j \ell_j y_j \right) \hat{\sigma}(\ell, y) = 0, \]

so \(\hat{\sigma}(\ell, \cdot) \equiv 0\) for any \(\ell \in \mathbb{Z}^n \setminus \{0\}\). This shows that the kernel of \(\mathcal{E}\) is the space of functions \(\sigma \in L^2(T^n \times \mathbb{R}^n)\) which are constant in \(\hat{x} \in T^n\). The associated orthogonal projection is given by integration over \(T^n\).

**Proof of Theorem 5.8.** Proposition \ref{5.7} implies that for any \(f \in L^2(T^n)\) and \(\psi \in D(\mathbb{R})\) with \(\psi \equiv 1\) on \([a^2, b^2]\), the symbol \(\sigma(\hat{x}, \xi) = f(\hat{x}) \psi(|\xi|^2)\) satisfies \(QV_{0,\lbrack a,b\rbrack}(\sigma - \int_{\mathbb{T}^n} \sigma) = 0\). By applying e.g. \cite[Theorem 1.8]{21}, we obtain the result for a subsequence \((j_k)\) depending on \(f \in C^\infty(M)\), and an argument of density and separability finishes the proof.

Here, we have considered the canonical torus \(T^n = \mathbb{R}^n / \mathbb{Z}^n\), but the results in Theorem \ref{5.8} and Lemma \ref{5.9} extend readily to any quotient of \(\mathbb{R}^n\) by a co-compact lattice.
6. Semiclassical limits

The main advantage of the symbolic approach is that it yields a precise description of the limit as \( \varepsilon \to 0 \) of quadratic quantities \( (A\phi_\varepsilon, \phi_\varepsilon) \) for an operator \( A = A^\varepsilon \) in the semiclassical calculus and a family of functions \( (\phi_\varepsilon)_\varepsilon \) in \( L^2(M) \). This semiclassical limit is expressed as measures which are operator valued; this is due to the non-commutativity of our setting.

6.1. Quadratic limits and states of \( \mathcal{A} \). Let \( \mathcal{A} \) denote the closure of \( \mathcal{A}_0 \) for the norm \( \| \cdot \|_{L^\infty(M \times \hat{G})} \) given in (4.3). One checks easily that \( \mathcal{A} \) is a sub\( C^* \)-algebra of the tensor product of the commutative \( C^* \)-algebra \( C(M) \) of continuous functions on \( M \) together with \( L^\infty(\hat{G}) \). The states of this \( C^* \)-algebra are useful when describing the following limits:

**Lemma 6.1.** Let \( (\phi_\varepsilon)_{\varepsilon \in [0,1]} \) be a bounded family in \( L^2(M) \). Consider the associated linear functionals \( \ell_\varepsilon, \varepsilon \in (0,1], \) on \( \mathcal{A}_0 \) given by:

\[
(6.1) \quad \ell_\varepsilon(\sigma) = \left( \text{Op}^{(\varepsilon)}(\sigma)\phi_\varepsilon, \phi_\varepsilon \right)_{L^2(M)}, \quad \sigma \in \mathcal{A}_0.
\]

1. For each \( \sigma \in \mathcal{A}_0 \),

\[
\forall \varepsilon \in (0,1] \quad |\ell_\varepsilon(\sigma)| \leq \|\sigma\|_{\mathcal{A}_0} \sup_{\varepsilon \in (0,1]} \|\phi_\varepsilon\|_{L^2(M)}
\]

so we may extract a sequence \( (\varepsilon_k)_{k \in \mathbb{N}} \) going to 0 as \( k \to \infty \) such that the sequence \( (\ell_{\varepsilon_k}(\sigma))_{k \in \mathbb{N}} \) converges.

2. We may extract a further sequence, still denoted \( (\varepsilon_k)_{k \in \mathbb{N}} \) and going to 0 as \( k \to \infty \), such that the pointwise limit of \( (\ell_{\varepsilon_k}(\sigma))_{k \in \mathbb{N}} \) exists on \( \mathcal{A}_0 \). Furthermore, it defines a positive continuous linear functional \( \ell_0 \) on the \( C^* \) algebra \( \mathcal{A} \). If \( \limsup_{\varepsilon \to 0} \|\phi_\varepsilon\|_{L^2(M)} = 1 \), \( \ell_0 \) is a state of \( \mathcal{A} \). If \( \limsup_{\varepsilon \to 0} \|\phi_\varepsilon\|_{L^2(M)} = 0 \), \( \ell_0 = 0 \).

The proof is only sketched as it follows very closely [10, Section 5] and [11, Section 3.2].

**Proof.** Proposition 4.2 implies Part (1). The existence of a pointwise limit \( \ell_0 \) on \( \mathcal{A}_0 \) follows from the separability of \( \mathcal{A} \). The properties of the semiclassical calculus imply that this limit yields a positive linear functional on \( \mathcal{A}_0 \) in the sense that \( \ell_0(\sigma^*\sigma) \geq 0 \) for any \( \sigma \in \mathcal{A}_0 \). If \( \sigma \) is a finite linear combination of symbols of the form \( a(\hat{x})\tau(\pi), a \in \mathcal{D}(M), \tau \in \mathcal{F}_G(S(G)) \), then we have \( |\ell_0(\sigma)| \leq \|\sigma\|_{\mathcal{A}} \sup_{\varepsilon \to 0} \|\phi_\varepsilon\|_{L^2(M)} \). The density in \( \mathcal{A} \) of the space of such symbols implies that \( \ell_0 \) extends into a state of \( \mathcal{A} \) when \( \limsup_{\varepsilon \to 0} \|\phi_\varepsilon\|_{L^2(M)} = 1 \). Part (2) follows.

For instance, the case of \( \phi_\varepsilon = 1/\sqrt{\text{vol}(M)} \) being a constant function on \( M \) is easily determined. Indeed, we compute easily for any \( \sigma \in \mathcal{A}_0 \) with convolution kernel \( \kappa_{\hat{x}} \):

\[
\text{Op}^{(\varepsilon)}(\varphi)(\hat{x}) = (\text{vol}(M))^{-1/2} \int_G \kappa_{\hat{x}}(z)dz,
\]

\[
\left( \text{Op}^{(\varepsilon)}(\sigma), \varphi \right)_{L^2(M)} = (\text{vol}(M))^{-1} \int_{M \times G} \kappa_{\hat{x}}(z)d\hat{x}d\hat{\pi} = (\text{vol}(M))^{-1} \int_M \sigma(\hat{x}, 1_G)d\hat{x}.
\]

Hence in this case, the corresponding state of \( \mathcal{A} \) is given by \( \ell_0(\sigma) = (\text{vol}(M))^{-1} \int_M \sigma(\hat{x}, 1_G)d\hat{x} \).

6.2. The dual of \( \mathcal{A} \) in terms of operator-valued measures. Lemma 6.1 leads us to seek a better description for the positive linear functionals on \( \mathcal{A} \). They will be in terms of operator-valued measures.
6.2.1. **Operator-valued measures.** Let us recall the notion of operator-valued measure as defined in [10, Section 5] and [11, Section 2.6]:

**Definition 6.2.** Let $Z$ be a complete separable metric space, and let $\xi \mapsto H_\xi$ a measurable field of complex Hilbert spaces of $Z$. Denote by $\mathcal{M}_{ov}(Z,(H_\xi)_{\xi \in Z})$ the set of pairs $(\gamma,\Gamma)$ where $\gamma$ is a positive Radon measure on $Z$ and $\Gamma = \{\Gamma(\xi) \in L(H_\xi) : \xi \in Z\}$ is a measurable field of trace-class operators such that

$$\int_Z \text{Tr}_{H_\xi}|\Gamma(\xi)|d\gamma(\xi) < \infty. \tag{6.2}$$

Two pairs $(\gamma,\Gamma)$ and $(\gamma',\Gamma')$ in $\mathcal{M}_{ov}(Z,(H_\xi)_{\xi \in Z})$ are **equivalent** when there exists a measurable function $f : Z \to \mathbb{C} \setminus \{0\}$ such that

$$d\gamma'(\xi) = f(\xi)d\gamma(\xi)$$

and $\Gamma'(\xi) = \frac{1}{f(\xi)}\Gamma(\xi)$ for $\gamma$-almost every $\xi \in Z$. The equivalence class of $(\gamma,\Gamma)$ is denoted by $\Gamma d\gamma$, and the resulting quotient set is denoted by $\mathcal{M}_{ov}(Z,(H_\xi)_{\xi \in Z})$.

A pair $(\gamma,\Gamma)$ in $\mathcal{M}_{ov}(Z,(H_\xi)_{\xi \in Z})$ is **positive** when $\Gamma(\xi) \geq 0$ for $\gamma$-almost all $\xi \in Z$. In this case, we may write $(\gamma,\Gamma) \in \mathcal{M}^+_{ov}(Z,(H_\xi)_{\xi \in Z})$, and $\Gamma d\gamma \geq 0$ for $\Gamma d\gamma \in \mathcal{M}^+_{ov}(Z,(H_\xi)_{\xi \in Z})$.

The quantity in (6.2) is constant on the equivalence class of $(\gamma,\Gamma)$ and is denoted by

$$\|\Gamma d\gamma\|_\mathcal{M}.$$ 

It is a norm on $\mathcal{M}_{ov}(Z,(H_\xi)_{\xi \in Z})$, which is then a Banach space.

6.2.2. **The dual of $A$.** Let $A(M \times \hat{G})$ denote the closure of $A_0(M \times \hat{G})$ for the $L^\infty$-norm given by (4.3). As above, $A(M \times \hat{G})$ is a sub$C^*$-algebra of the tensor product of the commutative $C^*$-algebra $C(M)$ of continuous functions on $M$ together with $L^\infty(\hat{G})$.

Proceeding as in [10, Section 5] and [11, Section 3.1], we can identify the spectrum of $A(M \times \hat{G})$ with $M \times \hat{G}$ and the states of the $C^*$-algebra $A(M \times \hat{G})$ with operator-valued measures. Following [11, Section 3.1] and [10, Section 5], we obtain the following description of the states of $A(M \times \hat{G})$ as operator-valued measures: for any pair $(\gamma,\Gamma) \in \mathcal{M}_{ov}(M \times \hat{G})$, the linear functional $\ell_{\gamma,\Gamma}$ defined on $A(M \times \hat{G})$ via

$$\ell_{\gamma,\Gamma}(\sigma) := \int_{M \times \hat{G}} \text{Tr} (\sigma(x,\pi)\Gamma(x,\pi)) \, d\gamma(x,\pi), \quad \sigma \in A(M \times \hat{G}),$$

is continuous. Furthermore, it is independent of the equivalence class of $(\gamma,\Gamma)$, and any continuous linear functional on $A$ is of this form and is uniquely determined by a class $\Gamma d\gamma$. In other words, the mapping

$$\begin{cases} 
\mathcal{M}_{ov}(M \times \hat{G}) & \longrightarrow & A^* \\
\Gamma d\gamma & \longmapsto & \ell_{\gamma,\Gamma} = \ell_{\Gamma d\gamma}
\end{cases}$$

is an isomorphism of Banach spaces. The states of $A(M \times \hat{G})$ are the linear functional $\ell_{\gamma,\Gamma}$ of norm one with $\Gamma d\gamma \in \mathcal{M}_{ov}^+(M \times \hat{G})$.

By James’ theorem and the description of $A$ given above, the Banach space $A$ is reflexive.
6.2.3. **Semiclassical measures.** We obtain the following consequence of Lemma 6.1.

**Corollary 6.3.** Let \((\phi_\varepsilon)_{\varepsilon \in (0,1]}\) be a bounded family in \(L^2(M)\). There exists a sequence \((\varepsilon_k)_{k \in \mathbb{N}}\) going to 0 as \(k \to \infty\) and a positive operator-valued measure \(\Gamma d\gamma \in \mathcal{M}^+_0(M \times \hat{G})\) such that for any \(\sigma \in \mathcal{A}_0\) we have the convergence for \(\varepsilon = \varepsilon_j\) and \(j \to \infty\)

\[
\left(\text{Op}^{(\varepsilon)}(\sigma)\phi_\varepsilon, \phi_\varepsilon\right)_{L^2(M)} \longrightarrow \int_{M \times \hat{G}} \text{Tr} (\sigma(x,\pi)\Gamma(x,\pi)) \, d\gamma(x,\pi), \quad \sigma \in \mathcal{A}(M \times \hat{G}).
\]

**Definition 6.4.** In Corollary 6.3 \(\Gamma d\gamma \in \mathcal{M}^+_0(M \times \hat{G})\) is called a **semiclassical measure** of the family \((\phi_\varepsilon)_{\varepsilon \in (0,1]}\) at scale \(\varepsilon\), or the semiclassical for the sequence \((\varepsilon_k)_{k \in \mathbb{N}}\).

We extend this vocabulary to the context of a bounded sequence of functions \((\phi_j)_{j \in \mathbb{N}}\) in \(L^2(M)\) and a map \(j \mapsto \varepsilon_j\) valued in \((0,1]\) satisfying \(\lim_{j \to \infty} \varepsilon_j = 0\).

This notion of semiclassical measure gives us a way to describes the obstruction to \(L^2\)-strong convergence of the \((\phi_\varepsilon)\) on the ‘space-phase’ \(M \times \hat{G}\).

6.2.4. **Oscillating families.** Although we will not use this notion, we observe that under certain hypotheses, the semi-classical limits of a bounded family \((\phi_\varepsilon)\) in \(L^2(M)\) yields the accumulation points of the family of measures \(|\phi_\varepsilon(\hat{x})|d\hat{x}\) for the weak-star topology.

**Definition 6.5.** Let \((\phi_\varepsilon)_{\varepsilon \in (0,1]}\) be a bounded family in \(L^2(M)\) and let \(\mathcal{R}\) be a positive Rockland operator on \(G\).

The family \((\phi_\varepsilon)\) is **uniformly \(\varepsilon\)-oscillating** with respect to \(\mathcal{R}_M\) when

\[
\limsup_{\varepsilon \to 0} \|1_{\varepsilon^{\nu}\mathcal{R}_M} \geq R \phi_\varepsilon\|_{L^2(M)} \longrightarrow_{R \to \infty} 0
\]

If in addition,

\[
\limsup_{\varepsilon \to 0} \|1_{\varepsilon^{\nu}\mathcal{R}_M} \leq \delta \phi_\varepsilon\|_{L^2(M)} \longrightarrow_{\delta \to 0} 0
\]

it is **uniformly strictly \(\varepsilon\)-oscillating**. We extend this vocabulary to the context of a bounded family of functions \((\phi_j)_{j \in J}\) in \(L^2(M)\) and a map \(j \mapsto \varepsilon_j\) satisfying \(\lim_{j \to \infty} \varepsilon_j = 0\).

It is a routine exercise to check that if a bounded family \((\phi_\varepsilon)_{\varepsilon \in (0,1]}\) of \(L^2(M)\) satisfies

\[
\exists s > 0, \quad \sup_{\varepsilon \in (0,1]} \|\varepsilon^{s\nu}\mathcal{R}_M^s \phi_\varepsilon\|_{L^2(M)} < \infty.
\]

then it is uniformly \(\varepsilon\)-oscillating for \(\mathcal{R}\), while if

\[
\exists s > 0, \quad \sup_{\varepsilon \in (0,1]} \|\varepsilon^{s\nu}\mathcal{R}_M^s \phi_\varepsilon\|_{L^2(M)} + \|\varepsilon^{s\nu}\mathcal{R}_M^{-s} \psi_\varepsilon\|_{L^2(M)} < \infty.
\]

then it is uniformly strictly \(\varepsilon\)-oscillating.

Proceeding as in [11], we obtain:

**Proposition 6.6.** Let \((\phi_\varepsilon)_{\varepsilon \in (0,1]}\) be a bounded family in \(L^2(M)\) that is uniformly \(\varepsilon\)-oscillating for \(\mathcal{R}\). Let \(\Gamma d\gamma\) be a semiclassical measure, and denote by \((\varepsilon_k)\) the corresponding subsequence. Then for any \(f \in \mathcal{D}(M)\), we have

\[
(f \phi_\varepsilon, \phi_\varepsilon)_{L^2(M)} \longrightarrow_{\varepsilon = \varepsilon_k, k \to \infty} \int_{M \times \hat{G}} f(\hat{x}) \text{Tr} (\Gamma(\hat{x},\pi)) \, d\gamma(\hat{x},\pi).
\]

If \((\phi_\varepsilon)\) is in addition uniformly strictly \(\varepsilon\)-oscillating, then the semiclassical measure does not charge the trivial representation \(1_{\hat{G}}\) in the sense that

\[
\gamma(M \times \{1_{\hat{G}}\}) = 0.
\]

The proof is omitted due to its similarity to the one in [11]. Consequently, with the notation of the statement, the weak-star limit of \(|\phi_{\varepsilon_k}(\hat{x})|^2 d\hat{x}\) is equal to \(\int_{\hat{G}} \text{Tr} (\Gamma(x,\pi)) \, d\gamma(x,\pi)\). It admits a further decomposition, see Remark 6.10.
6.3. The algebra $L^\infty(M \times \hat{G})$ and the decomposition $\hat{G} = \hat{G}_\infty \sqcup \hat{G}_1$. Any linear functional $\ell = \ell_{\Gamma d\gamma} \in \mathcal{A}^*$ admits a unique extension to a continuous linear functional on the von Neumann algebra $L^\infty(M \times \hat{G})$ generated by $\mathcal{A}$ which we now properly define and study. This will turn out to be useful to consider $\ell$ on certain parts of $M \times \hat{G}$, especially on the parts of the decomposition $\hat{G} = \hat{G}_\infty \sqcup \hat{G}_1$ also described below.

6.3.1. The von Neuman algebra $L^\infty(M \times \hat{G})$. From the group case recalled in Section 2.1.3 routine arguments shows that the von Neumann algebra of $\mathcal{A}$ is the space $L^\infty(M \times \hat{G})$ of measurable fields of operators that are bounded, that is, of measurable fields of operators $\sigma = \{\sigma(\hat{x}, \pi) \in \mathcal{L}(\mathcal{H}_\pi) : (\hat{x}, \pi) \in \hat{G}\}$ such that

$$\exists C > 0 \quad \|\sigma(\hat{x}, \pi)\|_{\mathcal{L}(\mathcal{H}_\pi)} \leq C \text{ for } d\hat{x}d\mu(\pi)\text{-almost all } (\hat{x}, \pi) \in M \times \hat{G}.$$ 

The smallest of such constant $C > 0$ is the norm $\|\sigma\|_{L^\infty(M \times \hat{G})}$ of $\sigma$ in $L^\infty(M \times \hat{G})$.

Naturally, $L^\infty(M \times \hat{G})$ contains $\mathcal{A}$ and therefore $\mathcal{A}_0$. But it also contains many other important symbols.

Example 6.7. Let $\mathcal{R}$ be a positive Rockland operator. For any $\pi \in \hat{G}$, $\pi(\mathcal{R})$ is essentially self-adjoint on $\mathcal{H}_\pi$ and we keep the same notation for its self-adjoint extensions. Its spectral decomposition is denoted by $\pi(E)$, see Section 3.2. For any interval $I \subset \mathbb{R}$, the symbol $\hat{E}(I) := \{\pi(E(I)), \pi \in \hat{G}\}$ is in $L^\infty(\hat{G})$, and in $L^\infty(M \times \hat{G})$. It satisfies:

$$\hat{E}(I)^2 = \hat{E}(I) \quad \text{and when not trivial} \quad \|\hat{E}(I)\|_{L^\infty(\hat{G})} = 1.$$ 

We will give other examples of elements in $L^\infty(M \times \hat{G})$ that are not necessarily in $\mathcal{A}$ in Example 6.8 below.

6.3.2. The decomposition $\hat{G} = \hat{G}_\infty \sqcup \hat{G}_1$. The unitary dual of any nilpotent Lie group may be described as the disjoint union

$$\hat{G} = \hat{G}_\infty \sqcup \hat{G}_1,$$

of the class of the infinite dimensional representations parametrised $\hat{G}_\infty$ with the class of the finite dimensional representations $\hat{G}_1$. By the orbit method, the finite dimensional representations are of dimension one and may be identified with the characters of a complement $\mathfrak{v}$ of the derived algebra $[\mathfrak{g}, \mathfrak{g}]$:

$$\hat{G}_1 := \{\text{class given by } \chi_\omega(v) = e^{i\omega(v)} : \omega \in \mathfrak{v}^*\}.$$ 

Consequently, $\hat{G}_1$ is closed in $\hat{G}$ and $1_{M \times \hat{G}_1} \Gamma d\gamma$ is a scalar valued measure. Hence, any operator valued measure $\Gamma d\gamma \in \mathcal{M}_{\text{op}}(M \times \hat{G})$ may be decomposed into two operator-valued measures

$$\Gamma d\gamma = 1_{M \times \hat{G}_1} \Gamma d\gamma + 1_{M \times \hat{G}_\infty} \Gamma d\gamma. \quad (6.3)$$

Let us understand this decomposition on linear functionals $\ell$. First, we will need the following observation:

Example 6.8. The symbols $1_{M \times \hat{G}_1}$ and $1_{M \times \hat{G}_\infty}$ are in $L^\infty(M \times \hat{G})$. Therefore, for any $\sigma \in \mathcal{A}$, the symbols $\sigma 1_{M \times \hat{G}_1}$ and $\sigma 1_{M \times \hat{G}_\infty}$ are in $L^\infty(M \times \hat{G})$.

This example together with the natural extension of any functional $\ell = \ell_{\Gamma d\gamma} \in \mathcal{A}^*$ to $L^\infty(M \times \hat{G})$ shows that the following decomposition makes sense:

$$\forall \sigma \in \mathcal{A} \quad \ell(\sigma) = \ell(\sigma 1_{M \times \hat{G}_1}) + \ell(\sigma 1_{M \times \hat{G}_\infty}).$$
The operator-valued measure $1_{M \times \hat{G}_1} \Gamma d\gamma$ is a scalar measure on $M \times \hat{G}_1 \sim M \times \mathfrak{v}^*$. We may assume that $\Gamma = 1$ on $M \times \hat{G}_1$, and by convention we do. The following lemma shows that $\sigma 1_{M \times \hat{G}_1}, \sigma \in \mathcal{A}$, is equal to $C_0(M \times \mathfrak{v}^*)$ so that $\sigma \mapsto \ell(\sigma 1_{M \times \hat{G}_1})$ completely describes $1_{M \times \hat{G}_1} \gamma$:

**Lemma 6.9.**  
1. Any $\sigma \in \mathcal{A}_0, \sigma 1_{M \times \hat{G}_1}$, identified with the restriction of $\sigma$ to $M \times \hat{G}_1$, coincides with the element in $C^\infty(M; \mathcal{S}(\mathfrak{v}^*))$

$$\dot{x} \mapsto (\omega \mapsto \sigma(\dot{x}, \pi^\omega)).$$

2. Moreover, the closure for $L^\infty(M \times \hat{G})$ of the algebra of symbols $\sigma 1_{M \times \hat{G}_1}, \sigma \in \mathcal{A}_0$, is equal to $$\{\sigma 1_{M \times \hat{G}_1}, \sigma \in \mathcal{A}\}$$ and is naturally isomorphic with the $C^*$-algebra $C_0(M \times \mathfrak{v}^*)$ of continuous functions on $M \times \mathfrak{v}^*$ vanishing at infinity.

**Proof.** Before starting the proof, we observe that we can choose the canonical basis $X_1, \ldots, X_n$ adapted to the gradation of $\mathfrak{g}$ and to the decomposition $\mathfrak{g} = \mathfrak{v} \oplus [\mathfrak{g}, \mathfrak{g}]$. This allows us to define Lebesgue measures $dV$ and $dZ$ on $\mathfrak{v}$ and $[\mathfrak{g}, \mathfrak{g}]$ such that $dVdZ$ is the exponential pull back of the Haar measure:

$$\forall f \in L^1(G) \quad \int_G f(x)dx = \int_{\mathfrak{v} \oplus [\mathfrak{g}, \mathfrak{g}]} f(\exp(V + Z))dVdZ.$$

The derived group $[G, G]$, whose Lie algebra is $[\mathfrak{g}, \mathfrak{g}]$, is therefore equipped with a compatible Haar measure:

$$\int_{[G,G]} f(z)dz = \int_{[\mathfrak{g}, \mathfrak{g}]} f(\exp Z)dZ.$$

Let $\sigma \in \mathcal{A}_0$, and denote its kernel by $\kappa_{\dot{x}}(y)$. Then

$$\sigma(\dot{x}, \pi^\omega) = \int_{\mathfrak{v}} \kappa_{\dot{x}}(\exp(V + Z))\chi-V(\omega) dVdZ = \mathcal{F}_0(\int_{[G,G]} \kappa_{\dot{x}})(\omega), \quad \dot{x} \in M, \omega \in \mathfrak{v},$$

where $\mathcal{F}_0$ denotes the Euclidean Fourier transform on $\mathfrak{v}$, and $\int_{[G,G]}$ the integration on $[G, G]$. Since $\kappa \in C^\infty(M; S(G))$, Part (1) follows.

The computation (6.3) also implies that given any $t \in C^\infty(M; \mathcal{S}(\mathfrak{v}^*))$, we can find a symbol $\tau \in \mathcal{A}_0$ such that $\tau 1_{M \times \hat{G}_1}$ coincides with $t$. Indeed, fixing a function $\chi \in \mathcal{D}([\mathfrak{g}, \mathfrak{g}])$ satisfying $\int_{[\mathfrak{g}, \mathfrak{g}]} \chi(Z)dZ = 1$, the kernel $\kappa$ defined via

$$\kappa_{\dot{x}}(\exp(V + Z)) = \chi(Z)\mathcal{F}_0^{-1}t(\dot{x}, \cdot), \quad V \in \mathfrak{v}, \ Z \in [\mathfrak{g}, \mathfrak{g}],$$

is in $C^\infty(M; S(G))$, and by (6.4), its symbol $\sigma$ satisfies $\sigma(\dot{x}, \pi^\omega) = \ell(t(\dot{x}, \omega))$ for all $(\dot{x}, \omega) \in M \times \hat{G}_1$. Consequently, the $L^\infty(M \times \hat{G})$-closure of $\{\sigma 1_{M \times \hat{G}_1}, \sigma \in \mathcal{A}_0\}$, identifies naturally with the closure of $C^\infty(M; S(\mathfrak{v}^*))$ for the supremum norm on $M \times \mathfrak{v}^*$, and therefore identifies naturally with $C_0(M \times \mathfrak{v}^*)$.

Part (1) implies that the $L^\infty(M \times \hat{G})$-closure of $\{\sigma 1_{M \times \hat{G}_1}, \sigma \in \mathcal{A}\}$ also identifies with a sub-$C^*$algebra of $C_0(M \times \mathfrak{v}^*)$. It is straightforward to check that

$$\forall \sigma \in \mathcal{A} \quad \|\sigma 1_{M \times \hat{G}_1}\|_{L^\infty(M \times \hat{G})} \leq \|\sigma\|_{L^\infty(M \times \hat{G})}.$$

So the $L^\infty(M \times \hat{G})$-closure of $\{\sigma 1_{M \times \hat{G}_1}, \sigma \in \mathcal{A}_0\}$, is a sub-$C^*$-algebra of $\{\sigma 1_{M \times \hat{G}_1}, \sigma \in \mathcal{A}\}$. As the former identifies with the whole of $C_0(M \times \mathfrak{v}^*)$, they are both equal to $C_0(M \times \mathfrak{v}^*)$. \hfill \Box

**Remark 6.10.** Let us consider the setting of Proposition 6.6, let $\Gamma d\gamma$ be the semiclassical limit of a bounded family $(\phi_\varepsilon)$ in $L^2(M)$ that is $\varepsilon$-oscillating for a positive Rockland operator $\mathcal{R}_M$ and a
subsequence $\varepsilon_j$. Then we have obtained a further decomposition of the weak-star limit

$$\lim_{k \to \infty} |\phi_{\varepsilon_k}(\hat{x})|^2 d\hat{x} = \int_{\hat{G}} \text{Tr}(\Gamma(\hat{x}, \pi)) d\gamma(\hat{x}, \pi)$$

$$= \int_{\hat{G}_1} d\gamma(\hat{x}, \pi) + \int_{\hat{G}_\infty} \text{Tr}(\Gamma(\hat{x}, \pi)) d\gamma(\hat{x}, \pi).$$

With different means (in fact, with the Euclidean micro-local analysis), this observation was already obtained in particular cases that essentially boils down to Heisenberg nilmanifolds, see [5, 21].

7. QCuantum limits for $R_M$

In this section, we study the semiclassical limit in our theory associated with a sequence of eigenfunctions of a (fixed) positive Rockland operator.

7.1. Family of $R_M$-eigenfunctions. Again, let $R$ be a positive Rockland operator on $G$ and denote by $R_M$ the corresponding operator on $M$, as well as its self-adjoint extension to $L^2(M)$. Recall that the spectral decomposition of $R$ and $\pi(R)$ for $\pi \in \hat{G}$ are denoted by $E$ and $\pi(E)$, see Section 3.2. Moreover, $\pi^\omega(R)$ is the (scalar) eigenvalue of $R$ for $\chi_\omega$: $R\chi_\omega = \pi^\omega(R)\chi_\omega$.

**Proposition 7.1.** Let $(\phi_j)_{j \in \mathbb{N}}$ be a sequence of eigenfunctions of for $R_M$:

$$R_M\phi_j = \mu_j \phi_j, \quad j = 0, 2, \ldots.$$  

Assume $\mu_j \to \infty$ as $j \to \infty$. Consider a semiclassical measure $\Gamma d\gamma$ of $(\phi_j)$ at scale $\mu_j^{-1/\nu}$ for the subsequence $(j_k)$ where $\nu$ is the degree of homogeneity of $R$. We have for $\gamma$-almost all $(\hat{x}, \pi) \in M \times \hat{G}$

$$\Gamma(\hat{x}, \pi) = \pi(E_1)\Gamma(\hat{x}, \pi)\pi(E_1).$$

The decomposition [6.3] of $\Gamma d\gamma$ according to $\hat{G} = \hat{G}_1 \cup \hat{G}_\infty$ satisfies the following properties:

1. The scalar valued measure $1_{M \times \hat{G}_1} \gamma$ on $M \times \hat{G}_1$ is supported in $M \times \{\pi^\omega \in \hat{G}_1 : \pi^\omega(R) = 1\} \sim M \times \{\omega \in \nu^* : R\chi_\omega = \chi_\omega\}$.

2. For $\gamma$-almost all $(\hat{x}, \pi) \in M \times \hat{G}_\infty$, the operator $\Gamma(\hat{x}, \pi)$ maps the finite dimensional 1-eigenspace for $\pi(R)$ onto itself and is trivial anywhere else.

**Proof.** The properties of the functional and semiclassical calculi imply for any $\sigma \in A_0$

$$(\text{Op}^{(\varepsilon)}(\hat{R}\sigma)\phi_j, \phi_j) = (\text{Op}^{(\varepsilon)}(\sigma)\varepsilon^\nu R\phi_j, \phi_j) = \varepsilon^\nu \mu_j (\text{Op}^{(\varepsilon)}(\sigma)\phi_j, \phi_j).$$

Similarly, since $R$ is self-adjoint, we have:

$$(\text{Op}^{(\varepsilon)}(R\sigma)\phi_j, \phi_j) = (\varepsilon^\nu R\text{Op}^{(\varepsilon)}(\sigma)\phi_j, \phi_j) + O(\varepsilon) = \varepsilon^\nu (\text{Op}^{(\varepsilon)}(\sigma)\phi_j, \sigma R\phi_j) + O(\varepsilon)$$

$$= \varepsilon^\nu \mu_j (\text{Op}^{(\varepsilon)}(\sigma)\phi_j, \phi_j) + O(\varepsilon).$$

We take $\varepsilon_j = \mu_j^{-1/\nu}$ and $j = j_k$, and then the limit as $k \to \infty$ to obtain for any $\sigma \in A_0$:

$$\int_{M \times \hat{G}_1} \text{Tr}(\sigma \hat{R} \Gamma) d\gamma = \int_{M \times \hat{G}} \text{Tr}(\sigma \Gamma) d\gamma = \int_{M \times \hat{G}} \text{Tr}(\hat{R} \sigma \Gamma) d\gamma.$$

This equality also holds for $\sigma 1_{M \times \hat{G}_1}$, see Section 6.3. In particular, we obtain

$$\forall f \in C_0(M \times \nu^*) \int_{M \times \nu^*} f(\hat{x}, \omega) \pi^\omega(R) d\gamma(\hat{x}, \pi^\omega) = \int_{M \times \nu^*} \sigma(\hat{x}, \pi^\omega) d\gamma(\hat{x}, \pi^\omega).$$

Part (1) follows.
Equality (7.1) also holds for any $\sigma 1_{M \times \hat{G}_\infty} \hat{E}_\lambda$ with $\lambda \in \mathbb{R}$ fixed, see Section 6.3. Hence, we obtain for any $\sigma \in \mathcal{A}_0$:

$$\int_{M \times \hat{G}_\infty} \text{Tr}(\sigma \hat{E}_\lambda \lambda \Gamma) \, d\gamma = \int_{M \times \hat{G}_\infty} \text{Tr}(\sigma \hat{E}_\lambda \hat{R} \Gamma) \, d\gamma = \int_{M \times \hat{G}_\infty} \text{Tr}(\sigma \hat{E}_\lambda \Gamma) \, d\gamma$$

so $\lambda \hat{E}_\lambda \Gamma = \hat{E}_\lambda \Gamma$, which means that the projection of the image of $\Gamma(x, \pi)$ onto the (finite dimensional) $\lambda$-eigenspaces of $\pi(\mathcal{R})$ is zero except perhaps for $\lambda = 1$. Similarly, Equality (7.1) also holds for any $\hat{E}_\lambda \sigma 1_{M \times \hat{G}_\infty}$ so $\lambda \hat{E}_\lambda = \hat{E}_\lambda \hat{E}_\lambda$, which means that $\Gamma(x, \pi) = 0$ on all the (finite dimensional) $\lambda$-eigenspaces of $\pi(\mathcal{R})$ except perhaps for $\lambda = 1$. This concludes the proof of Part (2). \qed

The properties of the semiclassical calculus can give us further information on the semiclassical measures, in particular properties of invariance. Indeed, the property (5.3) implies for any $\sigma \in \mathcal{A}_0$ commuting with $\hat{R}$,

$$\forall \varepsilon \in (0, 1], \ j \in \mathbb{N} \ \ (\text{Op}^{(\varepsilon)}(\sigma) \phi_j, \phi_j) = O(\varepsilon).$$

Taking $\varepsilon = \varepsilon_j$ and $j \to \infty$,

$$\int \int_{M \times \hat{G}} \text{Tr}(\sigma \Gamma) \, d\gamma = 0.$$ 

In the next section, we will study the symbol commuting with $\hat{R}$, and this will allow us to obtain the same property as above but with integration on $\hat{G}_1$ and $\hat{G}_\infty$, see Corollary (7.4).

### 7.2. Symbols and operator-valued measure commuting with $\hat{R}$.

The paragraph above motivates defining and studying the following objects:

**Definition 7.2.**

1. We denote by $\mathcal{A}^{\hat{R}}_0, \mathcal{A}^{\hat{R}}$ and $L^\infty(M \times \hat{G})^{\hat{R}}$ the subsets of $\mathcal{A}_0, \mathcal{A}$ and $L^\infty(M \times \hat{G})$ that commutes with $\hat{E}(I)$ for every interval $I \subset \mathbb{R}$.
2. We say that the pair $(\gamma, \Gamma) \in \tilde{\mathcal{M}}_{ov}(M \times \hat{G})$ commutes with $\hat{E}$ when for any interval $I \subset \mathbb{R}$, we have $\Gamma(\dot{x}, \pi)\pi(E(I)) = \pi(E(I))\Gamma(\dot{x}, \pi)$ for $\gamma$-almost all $(\dot{x}, \pi) \in M \times \hat{G}$.

We check readily that if $(\gamma, \Gamma) \in \tilde{\mathcal{M}}_{ov}(M \times \hat{G})$ commutes with $\hat{E}$, then so does any pair equivalent to it. We then say that the operator valued measure $\Gamma d\gamma$ commutes with $\hat{E}$. We denote the space of $\Gamma d\gamma$ commuting with $\hat{E}$ by $\mathcal{M}_{ov}(M \times \hat{G})^{\hat{R}}$.

**Proposition 7.3.**

1. A symbol $\sigma$ is in $\mathcal{A}^{\hat{R}}_0$ if and only if it commutes with $\hat{R}$. The space $\mathcal{A}^{\hat{R}}_0$ is a subalgebra of $\mathcal{A}_0$ that contains all the symbols given by $a(\dot{x})\psi(\pi(\mathcal{R}))$, where $a \in C^\infty(M)$ and $\psi \in S(\mathbb{R})$. In particular, it contains the approximate identity of $\mathcal{A}$ given by $a\psi_j(\hat{R})$, $j \in \mathbb{N}$, where $a = 1$ and $\psi_j \in \mathcal{D}(\mathbb{R})$ is valued in $[0, 1]$ and satisfies $\psi_j(\lambda) = 1$ on $\{|\lambda| \geq j\}$.
2. The space $\mathcal{A}^{\hat{R}}$ of $\mathcal{A}$ is a sub-$C^*$-algebra of $\mathcal{A}$. It is the closure of $\mathcal{A}^{\hat{R}}_0$ for the $L^\infty(M \times \hat{G})^{\hat{R}}$-norm.
3. The set $L^\infty(M \times \hat{G})^{\hat{R}}$ is a von Neumann subalgebra of $L^\infty(M \times \hat{G})$. It is the von Neumann algebra generated by $\mathcal{A}^{\hat{R}}$.
4. The space $\mathcal{M}_{ov}(M \times \hat{G})^{\hat{R}}$ is a closed subspace of $\mathcal{M}_{ov}(M \times \hat{G})$. Its image under $\Gamma d\gamma \mapsto \ell_{\Gamma d\gamma}$ is the dual of $\mathcal{A}^{\hat{R}}$.
5. We have (with the closure for the $L^\infty(M \times \hat{G})$-norm)

$$\{\sigma 1_{M \times \hat{G}_1}, \sigma \in \mathcal{A}^{\hat{R}}_0\} = \{\sigma 1_{M \times \hat{G}_1}, \sigma \in \mathcal{A}^{\hat{R}}\} = \{\sigma 1_{M \times \hat{G}_1}, \sigma \in \mathcal{A}\} \sim C_0(M \times \hat{G}^*)$$.

**Proof of Proposition 7.3.** If $\sigma \in \mathcal{A}^{\hat{R}}_0$, then $\sigma(\dot{x}, \pi)$ commutes with $\pi(\mathcal{R}) = \int_{\mathbb{R}} \lambda d\pi(E(\lambda))$ for all $(\dot{x}, \pi) \in M \times \hat{G}$. The converse holds for any $\pi \in \hat{G} \setminus \{1_{\hat{G}}\}$ since the spectrum $\pi(\mathcal{R})$ is a discrete
unbounded subset of $(0, \infty)$ and its eigenspaces are finite dimensional, see Section 3.2. The rest of Part (1) follows from routine checks and Hulanicki’s theorem (Theorem 3.1).

Checking Parts (2) and (3) is straightforwards.

The spectral properties of $\pi(\mathcal{R})$ recalled in Section 3.2 imply that for $(\Gamma, \gamma) \in \mathcal{M}_{ov}(M \times \hat{G})$, the measurable field $\Gamma(\hat{\mathcal{R}})$ given by

$$
\Gamma(\hat{\mathcal{R}})(\hat{x}, \pi) = \sum_{\zeta \in sp(\pi(\mathcal{R}))} \pi(E(\zeta))\Gamma(\hat{x}, \pi)\pi(E(\zeta)),
$$

is well defined and satisfies:

$$
\int_{M \times \hat{G}} \text{Tr}\left(\Gamma(\hat{\mathcal{R}})(\hat{x}, \pi)\right) d\gamma(\hat{x}, \pi) \leq \int_{M \times \hat{G}} \text{Tr}(\Gamma(\hat{x}, \pi)) d\gamma(\hat{x}, \pi).
$$

Moreover $(\Gamma(\hat{\mathcal{R}}), \gamma) \in \mathcal{M}_{ov}(M \times \hat{G})$ commutes with $\hat{E}$. This construction passes through the quotient, defining a map $\Gamma d\gamma \mapsto \Gamma(\hat{\mathcal{R}}) d\gamma$ on $\mathcal{M}_{ov}(M \times \hat{G})$. We check readily that this map is linear, 1-Lipschitz for $\|\cdot\|_{\mathcal{M}}$ and that its image is $\mathcal{M}_{ov}(M \times \hat{G})^{\hat{R}}$. Hence, $\mathcal{M}_{ov}(M \times \hat{G})^{\hat{R}}$ is a closed subspace of $\mathcal{M}_{ov}(M \times \hat{G})$.

Let $\ell : \mathcal{A}^{\hat{R}} \to \mathbb{C}$ be a continuous linear functional. By the Hahn-Banach theorem, $\ell$ extends continuously to an element $\bar{\ell}$ of $\mathcal{A}^{\ast}$. Let $\Gamma d\gamma \in \mathcal{M}_{ov}(M \times \hat{G})$ be such that $\bar{\ell} = \ell_{\Gamma d\gamma}$. If $\sigma \in \mathcal{A}^{\hat{R}}$ then using the resolution $\sum_{\zeta \in sp(\pi(\mathcal{R}))} \pi(E(\zeta)) = \mathcal{I}_{\mathcal{H}_{\ast}}$ of the identity operator $\mathcal{I}_{\mathcal{H}_{\ast}}$ and $\pi(E(\zeta))^2 = \pi(E(\zeta))$, we have:

$$
\ell(\sigma) = \bar{\ell}(\sigma) = \int_{M \times \hat{G}} \text{Tr}\left(\sigma(x, \pi)\Gamma(x, \pi)\right) d\gamma(x, \pi)
$$

$$
= \int_{M \times \hat{G}} \sum_{\zeta \in sp(\pi(\mathcal{R}))} \text{Tr}\left(\pi(E(\zeta))\sigma(x, \pi)\pi(E(\zeta))\Gamma(x, \pi)\right) d\gamma(x, \pi)
$$

$$
= \int_{M \times \hat{G}} \text{Tr}\left(\sigma(x, \pi)\Gamma(\hat{\mathcal{R}})(x, \pi)\right) d\gamma(x, \pi).
$$

In other words, $\bar{\ell} = \ell_{\Gamma(\mathcal{R}) d\gamma}$ on $\mathcal{A}^{\hat{R}}$. Consequently $\Gamma d\gamma \mapsto \ell_{\Gamma d\gamma}|_{\mathcal{A}^{\hat{R}}}$ maps $\mathcal{M}_{ov}(M \times \hat{G})^{\hat{R}}$ into $(\mathcal{A}^{\hat{R}})^{\ast}$. This map is continuous and linear. It remains to show that it is injective.

Let $\Gamma d\gamma \in \mathcal{M}_{ov}(M \times \hat{G})^{\hat{R}}$ such that $\ell_{\Gamma d\gamma} = 0$ on $\mathcal{A}^{\hat{R}}$. The natural extension of $\ell_{\Gamma d\gamma}$ to $L^{\infty}(M \times \hat{G})$ (for which we keep the same notation) will vanish on $L^{\infty}(M \times \hat{G})^{\hat{R}}$. Note that if $\sigma \in \mathcal{A}$ and $\zeta \in \mathbb{R}$ then $\hat{E}(\zeta)\sigma\hat{E}(\zeta) \in L^{\infty}(M \times \hat{G})^{\hat{R}}$ (see Example 5.7), so $\ell(\hat{E}(\zeta)\sigma\hat{E}(\zeta)) = 0$. With a similar computation as above, this shows that $\hat{E}(\zeta)\Gamma\hat{E}(\zeta)d\gamma = 0$. But this implies $\Gamma(\hat{\mathcal{R}}) d\gamma = 0 = \Gamma d\gamma$. This concludes the proof of Part (4).

By Lemma 6.9 and its proof, we have

$$
\{\sigma_{1_{M \times \hat{G}_{1}}, \sigma \in \hat{A}_{0}^{\hat{R}}} = \{\sigma_{1_{M \times \hat{G}_{1}}, \sigma \in \hat{A}_{0}^{\hat{R}}} \subseteq \{\sigma_{1_{M \times \hat{G}_{1}}, \sigma \in \mathcal{A}} \sim C_{0}(M \times \mathcal{V}^{\ast})
$$

Let $\ell : \{\sigma_{1_{M \times \hat{G}_{1}}, \sigma \in \mathcal{A}} \to \mathbb{C}$ be a continuous linear functional. It may be identified with a Radon measure $\gamma$ on $M \times \hat{G} \sim M \times \mathcal{V}^{\ast}$ via

$$
\ell(\sigma_{1_{M \times \hat{G}_{1}}}) = \int_{M \times \hat{G}_{1}} \sigma(\hat{x}, \pi) d\gamma(x, \pi).
$$

It is naturally extended into the operator-valued measure $\Gamma d\gamma$ with $1_{M \times \hat{G}_{1}} \Gamma d\gamma = \gamma$ and $1_{M \times \hat{G}_{\infty}} \Gamma d\gamma = 0$. Note that $\Gamma d\gamma \in \mathcal{M}_{ov}(M \times \hat{G})^{\hat{R}}$. If $\ell$ vanishes on $\{\sigma_{1_{M \times \hat{G}_{1}}, \sigma \in \hat{A}_{0}^{\hat{R}}}$ then $\ell_{\Gamma d\gamma} = 0$ on $\mathcal{A}^{\hat{R}}$ and $\Gamma d\gamma = 0$ by Part (4), so $\gamma = 0$ and $\ell = 0$. We conclude with the Hahn-Banach theorem. □
Corollary 7.4. We continue with the setting of Proposition 7.1. We have
\[
\forall f \in \mathcal{D}(M \times \mathfrak{v}^*) \quad \int_{M \times \hat{G}_1} \sum_{[\alpha]=v_1} \Delta^\alpha \hat{\mathcal{R}}(\chi_\omega) \ X_M^\alpha f(\dot{x}, \omega) \ d\gamma(\dot{x}, \pi^\omega) = 0,
\]
and
\[
\forall \sigma \in \mathcal{A}_0 \quad \int_{M \times \hat{G}_0} \text{Tr} \left( \sum_{[\alpha]=v_1} \left( \Delta^\alpha \hat{\mathcal{R}} X_M^\alpha \sigma \right) \Gamma \right) \ d\gamma = 0.
\]

Proof. At the end of Section 7.1, we obtained:
\[
\forall \sigma \in \mathcal{A}_0 \quad \int_{M \times \hat{G}_0} \text{Tr} \left( \sum_{[\alpha]=v_1} \left( \Delta^\alpha \hat{\mathcal{R}} X_M^\alpha \sigma \right) \Gamma \right) \ d\gamma = 0.
\]
We can now apply this to \(\sigma_{1 M \times \hat{G}_1}\) and \(\sigma_{1 M \times \hat{G}_\infty}\) together with Proposition 7.3 \(\square\)

7.3. The case of subLaplacians. In this section, we consider a stratified Lie group, and fix a basis \(X_1, \ldots, X_{n_1}\) of its first stratum. We denote by \(\mathcal{L} = -X_1^2 - \ldots - X_{n_1}\) the associated subLaplacian on \(G\). This is a positive Rockland operator, and we can choose the first stratum \(\mathfrak{g}_1 = \mathbb{R} X_1 \oplus \ldots \oplus \mathbb{R} X_{n_1}\) as the complement \(\mathfrak{v}\) of the derived algebra \([\mathfrak{g},\mathfrak{g}] = \oplus_{i>1} \mathfrak{g}_i\).

In order to give a more concrete description of the objects in the previous sections in this particular case, we will need the following computations:

Lemma 7.5. (1) Decomposing \(\omega = \sum_{j=1}^{n_1} \omega_j X_j^*\) with respect to the basis dual to \(X_1, \ldots, X_{n_1}\), we have
\[
\chi_\omega(\exp(\sum_{j=1}^n x_j X_j)) = \exp(\sum_{j=1}^{n_1} x_j \omega_j).
\]
Equipping \(\mathfrak{g}_1\) with the scalar product that makes \(X_1, \ldots, X_{n_1}\) orthonormal, the \(\mathcal{L}\)-eigenvalue corresponding to \(\chi_\omega\) is \(|\omega|^2 = \sum_{j=1}^{n_1} \omega_j^2\):
\[
\hat{\mathcal{L}} \chi_\omega = |\omega|^2 \chi_\omega, \quad \pi^\omega(\mathcal{L}) = |\omega|^2.
\]

(2) We have
\[
\sum_{[\alpha]=1} \Delta^\alpha \hat{\mathcal{L}} X_M^\alpha = \mathcal{E}, \quad \text{where} \quad \mathcal{E} := -2 \sum_{j=1}^{n_1} \hat{X}_j X_{M,j}.
\]
In particular it acts on \(M \times \hat{G}_1\) as
\[
\sum_{[\alpha]=1} \Delta^\alpha \hat{\mathcal{L}}(\pi^\omega) X_M^\alpha = -2i \sum_{j=1}^{n_1} \omega_j X_{M,j}.
\]

Proof. Part (1) is straightforward. For Part (2), we compute easily for the coordinates \(q_j\) corresponding to \(X_j\), \(j = 1, \ldots, n_1\):
\[
X_{j_1,y=0}(q_j(y)) = \delta_{j_1,j}, \quad \text{and when } k \neq 1 \quad X_{j_1,y=0}^k(q_j(y)) = 0,
\]
so
\[
\Delta_{q_j} \hat{X}_{j_1}^2(\pi) = X_{j_1,y=0}^2(q_j(y)\pi(y)) = (X_{j_1,y=0}^2(q_j(y)) \pi(0) + 2 (X_{j_1,y=0} q_j(y)) (X_{j_1,y=0} \pi(y)) + q_j(0) (X_{j_1,y=0}^2 \pi(y))
\]
\[
= \delta_{j,j_1} 2 \hat{X}_j(\pi),
\]

30
The decomposition
\[ \gamma \]

Consider a semiclassical measure \( \Gamma \)

The statement follows. \[ \square \]

**Corollary 7.6.** Let \( (\phi_j)_{j \in \mathbb{N}} \) be a sequence of eigenfunctions for \( \mathcal{L}_M \) with
\[ \mathcal{L}_M \phi_j = \mu_j \phi_j, \quad \mu_0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_j \to j \to \infty. \]

Consider a semiclassical measure \( \Gamma d\gamma \) of \( (\phi_j) \) at scale \( \mu_j^{-1/2} \) for the subsequence \( (j_k) \). We have for \( \gamma \)-almost all \( (\hat{x}, \pi) \in M \times \hat{G} \)
\[ \Gamma(\hat{x}, \pi) = \pi(E_1) \Gamma(\hat{x}, \pi) \pi(E_1). \]

The decomposition (6.3) of \( \Gamma d\gamma \) according to \( \hat{G} = \hat{G}_1 \sqcup \hat{G}_\infty \) satisfies the following properties:

1. The scalar valued measure \( 1_{M \times \hat{G}_1} \gamma \) on \( M \times \hat{G}_1 \) is supported in \( M \times \{\omega \in \mathfrak{g}_1^*: |\omega| = 1\} \) and satisfies
\[ \forall f \in D(M \times \mathfrak{v}^*) \quad \int_{M \times \mathfrak{g}_1} \sum_{j=1}^{n_1} \omega_j X_{M,j} f(\hat{x}, \omega) \, d\gamma(\hat{x}, \pi^\omega) = 0. \]

Consequently, it is invariant under the flow
\[ (\hat{x}, \omega) \mapsto (\exp(s \sum_{j=1}^{n_1} \omega_j X_{M,j}) \hat{x}, \omega). \]

2. For \( \gamma \)-almost all \( (\hat{x}, \pi) \in M \times \hat{G}_\infty \), the operator \( \Gamma(\hat{x}, \pi) \) maps the finite dimensional 1-eigenspace for \( \pi(\mathcal{L}) \) onto itself and is trivial anywhere else. Moreover, we have
\[ \forall \sigma \in A_0^\mathcal{L} \quad \int_{M \times \hat{G}_\infty} \text{Tr}(\sigma \Gamma) \, d\gamma = 0, \]

where \( \mathcal{E} := -2 \sum_{j=1}^{n_1} \hat{X}_j X_{M,j} \).

### 7.4. Comments.

1. In the step-two case, the structure of \( \hat{G}_\infty \) is well understood and leads to further scalar invariances than the general ones described in Corollary 7.6 see [13].

2. In the case of the canonical subLaplacian on the Heisenberg nilmanifolds, the quantum limits in the traditional Euclidean micro-local sense has been studied in [21], see also [5]. However, the description may be described as unwieldy. Moreover, extensions beyond products of Heisenberg nilmanifolds to slightly less simple case (e.g. nilmanifolds of Heisenberg types) seem unlikely.

3. It is not difficult to see that in the case of \( \mathcal{R} = \mathcal{L} \) a subLaplacian, the operator \( \mathcal{E} \) is self-adjoint on \( L^2(M \times \hat{G}) \). Hence the one-parameter group \( e^{it\mathcal{E}} \) is unitary on \( L^2(M \times \hat{G}) \). It is not difficult to prove that it also acts on \( A_0 \), with furthermore \( t \mapsto e^{it\mathcal{E}} \) being a continuous map from \( \mathbb{R} \) to \( \mathcal{L}(A_0) \).

In the commutative case, \( M \) is a torus and it was easy to determine the kernel of \( \mathcal{E} \) or equivalently the subspace of \( L^2(M \times \hat{G}) \) invariant under the action of the one-parameter group \( e^{it\mathcal{E}} \), see Lemma 5.9 and its proof: it is the subspace of \( \sigma \in L^2(M \times \hat{G}) \) such that
\[ \int_{T^n} \sigma(\hat{x}, \pi) d\hat{x} = 0 \quad \text{for every } \pi \in \hat{G}. \]
In the non-commutative case, determining ker $\mathcal{E}$ or equivalently the $e^{it\mathcal{E}}$-invariant subspace of $L^2(M \times \hat{G})$ is an open question.

(4) The non-commutativity hinders Egorov-type result. Indeed, if $\sigma \in \mathcal{A}_0$, writing $\sigma_t = e^{it\mathcal{E}}$, we have

$$
e^{-it\mathcal{E}_M}\text{Op}^{(e)}(\sigma)e^{it\mathcal{E}_M} - \text{Op}^{(e)}(\sigma_t) = \int_0^1 \partial_s \left\{ e^{-it\mathcal{E}_M}\text{Op}^{(e)}(\sigma_{t(1-s)})e^{it\mathcal{E}_M} \right\} ds
$$

$$= \int_0^1 e^{-it\mathcal{E}_M}\left( [-it\mathcal{E}_M, \text{Op}^{(e)}(\sigma_{t(1-s)})] - it\text{Op}^{(e)}(\mathcal{E}\sigma_{t(1-s)}) \right) e^{it\mathcal{E}_M} ds
$$

$$= -it \int_0^1 e^{-it\mathcal{E}_M}\text{Op}^{(e)} \left( \varepsilon^{-1}[\hat{\mathcal{L}}, \sigma_{t(1-s)}] + \varepsilon\mathcal{E}_M\sigma_{t(1-s)} \right) e^{it\mathcal{E}_M} ds,$$

by (5.2). This leads to some easy trace asymptotics, but the term in $\varepsilon^{-1}$ prohibits any asymptotics in operator norms or any use in the study of quantum variance.

Assuming that $[\hat{\mathcal{L}}, \sigma_t] = 0$ as in the commutative (torus) case seems a very strong hypothesis in a commutative setting.

REFERENCES

[1] A. Nalini and F. Macià, Semiclassical measures for the Schrödinger equation on the torus, J. Eur. Math. Soc. (JEMS), 16, 2014, No 6, pp 1253–1288.
[2] A. Bellaïche and J.-J. Risler (Eds), Sub-Riemannian geometry, Progress in Mathematics, 144, Birkhäuser Verlag, Basel, 1996.
[3] I. Brown, Dual topology of a nilpotent Lie group, Ann. Sci. École Norm. Sup. (4), 6, 1973, pp. 407–411.
[4] M. Christ, $L^p$ bounds for spectral multipliers on nilpotent groups, Trans. Amer. Math. Soc., 328, 1991, 1, pp 73–81.
[5] Y. Colin de Verdière, L. Hillairet, and E. Trélat, Spectral asymptotics for sub-Riemannian Laplacians, I: Quantum ergodicity and quantum limits in the 3-dimensional contact case, Duke Math. J., 167, 2018, No 1, pp 109–174.
[6] L.-J. Corwin and F.-P. Greenleaf, Representations of nilpotent Lie groups and their applications, Part 1: Basic theory and examples, Cambridge studies in advanced Mathematics, 18, Cambridge university Press, 1990.
[7] S. Dave and S. Haller, The heat asymptotics on filtered manifolds, Trans. Amer. Math. Soc., 328, 1991, 1, pp 337–389.
[8] J. Dixmier, Csp*-algebras, North-Holland Mathematical Library, Vol. 15, Translated from the French by Francis Jellett, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.
[9] C. Fermanian Kammerer and V. Fischer, semiclassical analysis on H-type groups, Sci. China Math., 62, 2019, No 6, pp 1057–1086.
[10] C. Fermanian Kammerer and V. Fischer, Defect measures on graded Lie groups. To appear in Annali della Scuola Normale de Pisa.
[11] C. Fermanian Kammerer and V. Fischer, Quantum evolution and sub-Laplacian operators on groups of Heisenberg type. To appear in Journal of Spectral Theory.
[12] C. Fermanian Kammerer, V. Fischer and S. Flynn, Geometric invariance of the semi-classical calculus on nilpotent graded Lie groups. Arxiv:2112.11509.
[13] C. Fermanian Kammerer, V. Fischer and S. Flynn, Some remarks on semi-classical analysis on two-step N-nilmanifolds, to appear in the proceedings of the Workshop Semiclassical analysis, quantum field theory, nonlinear PDEs, Milan (Italy), May 23-27, 2022.
[14] C. Fermanian Kammerer and C. Letrouit, Observability and Controllability for the Schrödinger Equation on Quotients of Groups of Heisenberg Type. Submitted and Arxiv 2009.13877.
[15] V. Fischer, Asymptotics and Zeta Functions on nil-manifolds, To appear in Journal des Mathématiques Pures et Appliquées
[16] V. Fischer, Towards semiclassical analysis for subelliptic operators, To appear in Bruno Pini Mathematical Analysis Seminar.
[17] V. Fischer and M. Ruzhansky, Quantization on nilpotent Lie groups, Progress in Mathematics, 314, Birkhäuser Basel, 2016.
[18] G. Folland and E. Stein, Hardy spaces on homogeneous groups, Mathematical Notes, 28, Princeton University Press, 1982.
[19] A. Hulanicki, A functional calculus for Rockland operators on nilpotent Lie groups, Studia Mathematica, 78 (1984), pp 253–266.
[20] D. Jakobson, Quantum limits on flat tori, *Ann. of Math. (2)*, 145, 1997, No 2, pp 235–266.

[21] C. Letrouit. Quantum limits of products of Heisenberg manifolds. Submitted and [arXiv:2007.00910](https://arxiv.org/abs/2007.00910).

[22] R. Montgomery, *A tour of subRiemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, 91, American Mathematical Society, Providence, RI, 2002.

[23] R. Ponge, *Heisenberg calculus and spectral theory of hypoelliptic operators on Heisenberg manifolds*, Mem. Amer. Math. Soc., 194, 2008, No 906.

[24] L. Rothschild and E. Stein, Hypoelliptic differential operators and nilpotent groups, Acta Math., 137, 1976, No 3-4, pp 247–320.

[25] E. van Erp, The Atiyah-Singer index formula for subelliptic operators on contact manifolds. Part I and II, *Ann. of Math. (2)*, 171, 2010, No 3, pp 1647–1681 and pp 1683–1706.

[26] E. van Erp and R. Yuncken, A groupoid approach to pseudodifferential calculi, *J. Reine Angew. Math.*, 756, 2019, pp 151–182.

[27] P. Walters, *Ergodic theory—introductory lectures*, Lecture Notes in Mathematics, Vol. 458, Springer-Verlag, Berlin-New York, 1975.

[28] M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, 138, American Mathematical Society, Providence, RI, 2012.

(V. Fischer) UNIVERSITY OF BATH, DEPARTMENT OF MATHEMATICAL SCIENCES, BATH, BA2 7AY, UK

*Email address:* v.c.m.fischer@bath.ac.uk