Teleparallel gauge theory of gravity

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Abstract

In this work a tetrad theory of gravity, invariant under conformal transformations, is investigated. The action of the theory is similar to the action of Maxwell’s electromagnetism. The role of the electromagnetic gauge potential is played by the trace of the torsion tensor of the Weitzenböck spacetime. It is shown that all static, spherically symmetric space-times, are solutions of the vacuum field equations. However, by fixing the gauge in the linearized form of the vacuum field equations, the usual Newtonian limit for the gravitational field is obtained.

PACS numbers: 04.20.-q, 04.20.Cv, 04.50.Kd

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1 Introduction

Conformal invariance is an important symmetry in some attempts to quantize gravity \[1, 2\]. The symmetry is normally realized in alternative formulations of general relativity, which modify both the small and large scale descriptions of the space-time geometry. The behaviour of the gravitational field at very small distances has to be modified in order to construct a renormalizable and unitary theory of quantum gravity. At large scales, the modified theory has to yield an acceptable solution to the dark matter and dark energy problems. Attempts based on the conformally invariant Weyl theory, which is quadratic in the curvature tensor, have been recently carried out in the literature \[3, 4, 5\].

The usual procedures for modifying the standard formulation of general relativity and obtaining a theory endowed with conformal invariance are the following. Either one adds a scalar field to the Hilbert-Einstein action integral, together with a suitable kinetic term for the scalar field, or one considers alternative theories like the quadratic Weyl theory. The latter is known to be the only *metrical* theory that exhibits conformal invariance. However, it is not the only *geometrical* theory. It is possible to construct an infinity of geometrical theories of gravity out of the tetrad field, which are invariant under conformal transformations, and which may play a role in the formulation of the quantum theory of gravity. Teleparallel theories of gravity, endowed with conformal invariance, have been recently investigated \[6\]. It was found that there is an infinity of theories, beyond the quadratic Weyl theory, that display conformal invariance.

In this article, we address a particular theory of gravity that, besides conformal invariance, displays three interesting features: (i) the functional structure of the theory is similar to Maxwell’s theory of electromagnetism; (ii) all static, spherically symmetric geometries, including the Schwarzschild geometry, are solutions of the vacuum field equations; and (iii) by fixing the gauge in the linearized form of the field equations, we obtain the usual Newtonian limit. The emergence of the Schwarzschild solution in the context of a Maxwell type theory of gravity is an intriguing result. This theory can be understood as a teleparallel theory for the gravitational field.

In Sect. 2 we review the construction of conformally invariant teleparallel theories of gravity, and in Sect. 3 we present the Maxwell type theory, the spherically symmetric solutions and the linearized solution of the vacuum field equations. Finally, in Sect. 4 we present our conclusions.
Notation: space-time indices \( \mu, \nu, \ldots \) and SO(3,1) indices \( a, b, \ldots \) run from 0 to 3. Time and space indices are indicated according to \( \mu = 0, \ i, a = (0), (i) \). The tetrad field is denoted \( e^a_{\mu} \), and the torsion tensor reads \( T_{a\mu\nu} = \partial_{\mu} e_{a\nu} - \partial_{\nu} e_{a\mu} \). The flat, Minkowski spacetime metric tensor raises and lowers tetrad indices and is fixed by \( \eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (-1, +1, +1, +1) \). The determinant of the tetrad field is represented by \( e = \det(e^a_{\mu}) \).

The space-time geometry is defined here by the tetrad field only, and the only possible non-trivial definition for the torsion tensor is given by \( T^a_{\mu\nu} \). This torsion tensor is related to the antisymmetric part of the Weitzenböck connection \( \Gamma^\lambda_{\mu\nu} = e^{\alpha\lambda} \partial_\mu e_{a\nu} \), which establishes the Weitzenböck spacetime. The curvature of the Weitzenböck connection vanishes. However, the tetrad field also yields the metric tensor, which determines the Riemannian geometry. Thus, in the framework of a geometrical theory based on the tetrad field, one may use the concepts of both Riemannian and Weitzenböck geometries.

2 Teleparallel theories with conformal invariance

In this section we review the results recently obtained in [6]. A conformal transformation on the space-time metric tensor transforms \( g_{\mu\nu} \) into \( \tilde{g}_{\mu\nu} = e^{2\theta(x)} g_{\mu\nu} \), where \( \theta(x) \) is an arbitrary function of the space-time coordinates. The conformal transformations on the tetrad field and on its inverse are defined by

\[
\tilde{e}_{a\mu} = e^{\theta(x)} e_{a\mu}, \quad \tilde{e}^{a\mu} = e^{-\theta(x)} e^{a\mu}.
\]

The transformation of the projected components of the torsion tensor \( T_{abc} = e^b_{\mu} e^c_{\nu} (\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}) \) is straightforward. It is given by

\[
\tilde{T}_{abc} = e^{-\theta} (T_{abc} + \eta_{ac} e^b_{\mu} \partial_\mu \theta - \eta_{ab} e^c_{\mu} \partial_\mu \theta), \quad \tilde{T}^{abc} = e^{-\theta} (T^{abc} + \eta^{ac} e^b_{\mu} \partial_\mu \theta - \eta^{ab} e^c_{\mu} \partial_\mu \theta).
\]

As a consequence of the equation above, the trace of the torsion tensor \( T_a = T^b_{\ b\ a} \) transforms as

\[
\tilde{T}_a = e^{-\theta} (T_a - 3 e^a_{\mu} \partial_\mu \theta), \quad \tilde{T}^a = e^{-\theta} (T^a - 3 e^a_{\mu} \partial_\mu \theta).
\]
We also have
\[ \tilde{T}_\mu = T_\mu - 3\partial_\mu \theta , \]  
(4)
where \( T_\mu = T^\lambda \lambda_\mu \).

With the help of equations (2) and (3), it is possible to verify that the behaviour of the three quadratic terms that determine the Lagrangian density of the standard teleparallel theories of gravity is given by

\[
\tilde{T}^{abc} \tilde{T}_{abc} = e^{-2\theta} (T^{abc} T_{abc} - 4T^\mu \partial_\mu \theta + 6g^{\mu\nu} \partial_\mu \theta \partial_\nu \theta) ,
\]
\[
T^{abc} \tilde{T}_{bac} = e^{-2\theta} (T^{abc} T_{bac} - 2T^\mu \partial_\mu \theta + 3g^{\mu\nu} \partial_\mu \theta \partial_\nu \theta) ,
\]
\[
\tilde{T}^a \tilde{T}_a = e^{-2\theta} (T^a T_a - 6T^\mu \partial_\mu \theta + 9g^{\mu\nu} \partial_\mu \theta \partial_\nu \theta) .
\]
(5)

In view of the equations above, it is straightforward to check that the quantity
\[
L = \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - \frac{1}{3} T^a T_a
\]
transforms under a conformal transformation according to \( \tilde{L} = e^{-\theta} L \). As a consequence of Eq. (1), we find that for the determinant \( \tilde{e} \) of the tetrad field we have \( \tilde{e} = e^{4\theta} e \). Therefore, by introducing a scalar field \( \phi \) that is assumed to transform as
\[
\tilde{\phi} = e^{-\theta} \phi ,
\]
(7)

it is easy to verify that

\[
e^{-\theta} (\frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - \frac{1}{3} T^a T_a)
\]
(8)
is invariant under coordinate transformations and conformal transformations \cite{6}. It follows from Eq. (4) that a covariant derivative for the scalar field may be defined,
\[
D_\mu \phi = \left( \partial_\mu - \frac{1}{3} T_\mu \right) \phi .
\]
(9)

It is easy to verify that \( \tilde{D}_\mu \tilde{\phi} = e^{-\theta} D_\mu \phi \).

In view of equations (5-9), it is possible to construct two sets of Lagrangian densities that are invariant under conformal transformations, as
described in [6]. The first set is a one-parameter family of theories constructed out of the tetrad field and of the scalar field, and is given by

\[ \mathcal{L} = ke\left[ -\phi^2 \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - \frac{1}{8} T^{a} T_{a} \right) + k' g^{\mu\nu} D_{\mu} \phi D_{\nu} \phi \right], \]  

where \( k = 1/(16\pi G) \), and \( k' \) is an arbitrary constant parameter. By fixing \( k' = 6 \), we arrive at the teleparallel equivalent of general relativity [6]. The second set is a four-parameter family of theories, and reads

\[ \mathcal{L}(e_{a\mu}) = e L_1 L_2, \]  

where

\[ L_1 = A T^{abc} T_{abc} + B T^{abc} T_{bac} + C T^{a} T_{a}, \]  

\[ L_2 = A' T^{abc} T_{abc} + B' T^{abc} T_{bac} + C' T^{a} T_{a}. \]  

The constant coefficients in the expressions above are required to satisfy

\[ 2A + B + 3C = 0, \quad 2A' + B' + 3C' = 0. \]  

The field equations derived from the Lagrangian density (11) are rather intricate, and it is not possible to envisage any simple solution of this four-parameter theory.

### 3 A Maxwell-type theory of gravity

A conformally invariant teleparallel theory of gravity, that was not noticed in the analysis of [6], is a theory whose Lagrangian density is exactly similar to Maxwell’s theory of electromagnetism. It is constructed out of the tetrad field only, and is given by

\[ \mathcal{L} = -\frac{1}{4} e g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}, \]  

where

\[ F_{\mu\nu} = \partial_{\mu} T_{\nu} - \partial_{\nu} T_{\mu}. \]
and $T_\mu = T^\lambda \_\lambda \mu$. Under transformation (4), we have $\tilde{F}_{\mu\nu} = F_{\mu\nu}$, and therefore $\mathcal{L}$ given by Eq. (15) is invariant under conformal gauge transformations. The constant factor $-1/4$ is introduced just to emphasize the similarity with Maxwell’s theory.

The identification of a gauge field with the trace of the torsion tensor is not a novelty. Dirac [7] and Utiyama [8] have already considered a general affine connection such that this feature takes place. In these attempts, the theories are formulated with metric and torsion as independent field quantities, and with the addition of a scalar field (see also refs. [9] [10] [11] [12] [13] [14] and references therein; these approaches are understood as metric affine theories of gravity, which have been thoroughly investigated by Hehl et. al. [15]). In contrast, in the present analysis the only field variable is the tetrad field.

The vacuum field equations obtained by varying the action integral constructed out of (16) read

\begin{equation}
\sum_{\sigma=1}^{2} e_{a\sigma}e_{b\mu} \partial_{\lambda} \left( \partial_{\nu} (eF^{\sigma\nu}) e^{b\lambda} \right) - e_{a\lambda}e_{b\mu} \partial_{\sigma} \left( \partial_{\nu} (eF^{\sigma\nu}) e^{b\lambda} \right) + \partial_{\nu} (eF^{\lambda\nu}) T_{a\mu\lambda} - eT_{a\mu} = 0,
\end{equation}

where

\begin{equation}
T_{a\mu} = F^{\lambda}_{a \lambda} F_{\lambda \mu} - \frac{1}{4} e_{a\mu} F_{\alpha\beta} F^{\alpha\beta}.
\end{equation}

Expression (18) is similar to the standard energy-momentum tensor for the electromagnetic field. Although the theory investigated here is considered a theory for the gravitational field only, we may of course bring to the present context insights from the standard electromagnetic theory. We mention that a theory constructed out of a term similar to $F_{\mu\nu}$ has been investigated in [16]. The theory in the later reference is constructed out of the torsion and Riemann tensors, and yield second order field equations, in contrast to the present approach. We also remark that the gauge transformation given by Eq. (9) is equivalent to the transformation of the torsion tensor (Eq. (14a)) of [17], where quadratic theories of gravity with second order field equations were addressed.

We will show that all static, spherically symmetric space-time geometries are solutions of the vacuum field equations. Let us consider the line element

\begin{equation}
ds^2 = -A^2 dt^2 + B^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\end{equation}

5
where $A(r)$ and $B(r)$ are arbitrary functions of the radial coordinate $r$. The set of tetrad fields adapted to stationary observers in space-time, and that yields the line element (19), is given by

$$ e_{a\mu} = \begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & B \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ 0 & B \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ 0 & B \cos \theta & -r \sin \theta & 0 \end{pmatrix}. \tag{20} $$

Out of Eq. (20) we obtain the following non-vanishing components of $T_{a\mu\nu}$,

$$ T_{(0)01} = \partial_1 A, $$
$$ T_{(1)12} = (1 - B) \cos \theta \cos \phi, $$
$$ T_{(2)12} = (1 - B) \cos \theta \sin \phi, $$
$$ T_{(3)12} = -(1 - B) \sin \theta, $$
$$ T_{(1)13} = -(1 - B) \sin \theta \sin \phi, $$
$$ T_{(2)13} = (1 - B) \sin \theta \cos \phi. \tag{21} $$

The expressions above yield only three non-vanishing components of $T_{\lambda\mu\nu}$,

$$ T_{001} = A \partial_1 A, $$
$$ T_{212} = r(1 - B), $$
$$ T_{313} = r(1 - B) \sin^2 \theta. \tag{22} $$

We may then calculate the traces $T_\mu = T^\lambda_{\lambda\mu}(t, r, \theta, \phi)$. We obtain

$$ T_0 = 0, $$
$$ T_1 = -\frac{1}{A} \partial_1 A - \frac{2}{r}(1 - B), $$
$$ T_2 = 0, $$
$$ T_3 = 0. \tag{23} $$

From Eq. (23) we easily obtain

$$ F_{\mu\nu}(t, r, \theta, \phi) = 0. \tag{24} $$
and therefore we conclude that Eq. (20) is a solution of Eq. (17), for arbitrary functions $A(r)$ and $B(r)$. The Schwarzschild metric tensor is obtained by identifying $A^2 = (1 - 2m/r)$ and $B^2 = (1 - 2m/r)^{-1}$.

By establishing a weak field approximation for the tetrad field according to

$$e^a \mu \simeq \delta^a \mu + \frac{1}{2} h^a \mu,$$ (25)

the field equations (17) may be reduced to a linearized form, to first order in $h^a \mu$ (in contrast to the field equations derived from the Lagrangian density (11); the lowest order field equation obtained from the latter is of the order $(h^{\mu \nu})^2$). By imposing Eq. (25) to (17), and transforming all indices into space-time indices, we obtain

$$\partial_\mu \partial_\nu \partial_\rho h^\rho \sigma - \partial_\mu \partial_\nu \partial_\rho h^\rho \sigma = 0$$ (26)
in cartesian coordinates, or

$$\partial_\mu \partial_\nu (\partial_\sigma B^\nu - \partial^\nu B_\sigma) = 0,$$ (27)

where $B^\mu = \partial_\rho h^{\rho \mu}$. The field equation above is invariant under the gauge transformation

$$B^\mu \rightarrow B^\mu + \partial^\mu \Lambda,$$ (28)

where $\Lambda$ is an arbitrary space-time dependent scalar function. The transformation for the field $h^{\mu \nu}$ that yields Eq. (28) is given by $h^{\mu \nu} \rightarrow h^{\mu \nu} + \eta^{\mu \nu} \Lambda$. We may impose a Lorentz type gauge condition,

$$\partial_\mu B^\mu = 0,$$ (29)

after which the field equation (27) is reduced to

$$\partial_\mu (\partial_\nu B^\sigma) = 0.$$ (30)

The Newtonian limit for the gravitational field is characterized by line element

$$ds^2 = -\left(1 + \frac{2\phi}{c^2}\right) dt^2 + \left(1 - \frac{2\phi}{c^2}\right) (dx^2 + dy^2 + dz^2),$$ (31)
or
\[ h_{00} = h_{11} = h_{22} = h_{33} = -\frac{2\phi}{c^2}. \]  

(32)

In terms of \( h_{\mu \nu} \), the gauge condition (29) reads \( \partial_\nu \partial_\rho h^{\rho \nu} = 0 \). Since \( h_{\mu \nu} \) is time-independent, the gauge condition in the Newtonian limit reduces to

\[ \partial_1 \partial_1 h_{11} + \partial_2 \partial_2 h_{22} + \partial_3 \partial_3 h_{33} = 0. \]  

(33)

Transforming now to spherical coordinates, and taking into account (32), we have

\[ \nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) = 0. \]  

(34)

The solution of this equation is given by

\[ \phi(r) = a + \frac{b}{r}, \]  

(35)

where \( a \) and \( b \) are constants. In order to comply with the Newtonian limit, we must have \( a = 0 \) and \( b = -mc^2 = -MG \) (recall that \( m = MG/c^2 \), where \( M \) is identified with the mass of the source and \( G \) is the gravitational constant).

The solution (35) to the gauge condition (29) is also a solution of the field equation (30), and ultimately of the field equation (26). Note that in cartesian coordinates, the field equation (30) may be rewritten as

\[ \partial_\mu \partial_\nu (\partial_\rho \partial_\sigma) h^{\rho \sigma} = 0. \]  

(36)

Thus we conclude that the weak field limit of the theory determined by eqs. (17), (25) and (26) allows the description of the usual Newtonian limit of general relativity, namely, it yields the Newtonian potential \( \phi = -MG/r \) provided we identify \( b = -mc^2 \) in Eq. (35). Out of all possible spherically symmetric solutions, Eq. (35) arises from the imposition of the weak field approximation. However, it is likely that the time independent expression (35) is not the only solution to eqs. (26) and (27).

4 Final remarks

In this article we have addressed a theory that has three distinctive features: (i) it is invariant under conformal transformations, (ii) it is functionally sim-
ilar to Maxwell’s theory of electromagnetism, and (iii) all static, spherically symmetric geometries (including the Schwarzschild space-time) are solutions of the vacuum field equations. Moreover, the linearized (weak field) equations allow the description of the Newtonian limit of general relativity.

The feature regarding the existence of an infinite number of solutions with static spherical geometry is, to some extent, similar to the existence of an infinity of solutions to the Laplace equation in electrostatics, in the absence of boundary conditions for the scalar potential. After imposing boundary conditions, one arrives at a particular solution for the physical configuration. In similarity to the problems in electrostatics, it may be possible to obtain the Schwarzschild metric by making use of the conformal gauge symmetry, and requiring, for instance, that the spherical surface determined by $r = 2m$ is a null surface, as in a boundary value problem. This issue will be investigated elsewhere. It is possible that the problem of finding stationary solutions to the field equation (17) may be reduced to a boundary value problem. The Schwarzschild metric displays an essential singularity at $r = 0$, and the fact that the vacuum solutions characterized by (20-24) are not \textit{a priori} singular, is a positive feature. We will also analyze (i) the possible emergence of potentials that modify the standard Newtonian potential both at small and large radial distances, and (ii) the existence of time dependent solutions, and in particular of plane wave solutions.

The emergence of the solution given by eqs. (23) and (24) is not surprising. The tensor $F_{\mu\nu}$ has a structure of a rotational, and the rotational of a vector field endowed with spherical symmetry, vanishes. Thus, it is not possible to verify whether the Kerr metric tensor yields a tetrad field that is a solution of Eq. (17), by means of the procedure presented here. This issue will also be investigated in the future.

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