Chaotic behavior of confining lattice
gauge field configurations

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Abstract
We analyze the leading Lyapunov exponents of SU(2) Yang-Mills field configurations on
the lattice which are initialized by quantum Monte Carlo simulations. We find that configura-
tions in the strong coupling phase at finite temperature are substantially more chaotic than in
deconfinement.

1 Motivation

The study of chaotic dynamics of classical field configurations in field theory finds its motivation in
two main areas: i) particle production from long wavelength, collective fields and ii) thermalization
due to entropy production in the equipartioning process. Chaos is known to lead to mode sharing,
i.e. the energy concentrated initially at a few selected wavelengths tends to be shared among all
modes. The hard ones of these show a dispersion relation close to that of free particles [1].

In such processes the initial configurations are usually described by nonperturbative methods,
while the final state is characterized by perturbative physics. Examples are the baryon production at
the electroweak phase transition [2], enhanced pion production due to disoriented chiral condensate
formation in energetic hadronic collisions [3] or thermalized quark-gluon matter production from
color ropes in relativistic heavy ion collisions [4],[5]. The chaotic dynamics also can be used for
preparing thermal initial conditions for a simulation.

The numerical simulation of the gauge field dynamics is usually done on a lattice, because this
method ensures an adequate treatment of gauge symmetries and is non-perturbative [6],[7]. Special
care is taken for satisfying constraints, especially Gauss law, by the Hamiltonian simulation of the
lattice dynamics [8]. Algorithmically also the correct implementation of the Gauss law is important.
It is achieved by using the Noether charge corresponding to the finite time step recursion form of
the equations of motion [9].

Besides the above mentioned areas of application a pure theoretical motivation drives the study
of chaotic field dynamics as well: its possible relation to the confinement problem is a longstanding
question [10],[11]. Both confinement and chaotic behavior are related to long wavelength field
components, so already in the earliest simple models this question has been raised [12],[13].

Recently a discussion was initiated [14],[15] about the continuum limit of the classical lattice
gauge theory. It was conjectured that for small energy per plaquette the lattice system would break
the linear scaling between Lyapunov exponents and energy. This finding was shown to possibly be
a finite time artifact [16] and the linear correspondence was reestablished. Nevertheless it could
not be decided whether inspite the divergence of both the energy and Lyapunov exponent in the continuum limit (lattice spacing zero) their ratio would scale. Our method of preparing the initial field configurations by quantum Monte Carlo simulations may shed some light on this question, because this way an explicit coupling dependence occurs in the results. Using the perturbative β-function the continuum limit of the above mentioned ratio can be considered.

In this paper we also search for a possible connection between chaotic dynamics and confinement in Yang-Mills theory by numerical simulation. We do this by investigating the chaotic dynamics of SU(2) Yang-Mills fields on a three dimensional lattice. The starting configurations are prepared by four dimensional quantum Monte Carlo techniques for finite temperature QCD undergoing a phase transition to quark-gluon plasma. This way we expect to see a coincidence between the strong coupling phase and the strength of chaotic behavior in lattice simulations.

After reviewing essential definitions of the physical quantities describing chaos and their computation in lattice gauge theory [1], [5] we outline our method for the extraction of starting configurations of a three dimensional Hamiltonian dynamics from four dimensional euclidean field configurations. Our results are then presented by showing an example of the exponential divergence of small initial distances between nearby field configurations. It is followed by a detailed study of the maximal Lyapunov exponent and average plaquette energy as function of the coupling strength.

2 Classical chaotic dynamics

Chaotic dynamics in general is characterized by the spectrum of Lyapunov exponents. These exponents, if they are positive, reflect an exponential divergence of initially adjacent configurations. In case of symmetries inherent in the Hamiltonian of the system there are corresponding zero values of these exponents. Finally negative exponents belong to irrelevant directions in the phase space: perturbation components in these directions die out exponentially. Pure gauge fields on the lattice show a characteristic Lyapunov spectrum consisting of one third of each kind of exponents [1], [17]. This fact reflects the elimination of longitudinal degrees of freedom of the gauge bosons. Assuming this general structure of the Lyapunov spectrum we investigate presently its magnitude only, namely the maximal value of the Lyapunov exponent, \( L_{\text{max}} \).

The general definition of the Lyapunov exponent is based on a distance measure \( d(t) \) in phase space,

\[
L := \lim_{t \to \infty} \lim_{d(0) \to 0} \frac{1}{t} \ln \frac{d(t)}{d(0)}.
\]

(1)

In case of conservative dynamics the sum of all Lyapunov exponents is zero according to Liouville’s theorem,

\[
\sum L_i = 0.
\]

(2)

We utilize the gauge invariant distance measure consisting of the local differences of energy densities between two field configurations on the lattice:

\[
d := \frac{1}{N_P} \sum_P |\text{tr} U_P - \text{tr} U'_P|.
\]

(3)

Here the symbol \( \sum_P \) stands for the sum over all \( N_P \) plaquettes, so this distance is bound in the interval \((0, 2N)\) for the group SU(N). \( U_P \) and \( U'_P \) are the familiar plaquette variables, constructed from the basic link variables \( U_{x,i} \),

\[
U_{x,i} = \exp \left( a A^c_{x,i} \tau^c \right),
\]

(4)

located on lattice links pointing from the position \( x = (x_1, x_2, x_3) \) to \( x + ae_i \). The generators of the group are \( T^c = -ig\tau^c/2 \) with \( \tau^c \) being the Pauli matrices in case of SU(2) and \( A^c_{x,i} \) is the vector potential. The elementary plaquette variable is constructed for a plaquette with a corner at \( x \) and lying in the \( ij \)-plane as

\[
U_{x,ij} = U_{x,i} U_{x+i,j} U_{x+i+j}^\dagger U^\dagger_{x+j}.\]

(5)
It is related to the magnetic field strength $B^c_{x,k}$:

$$U_{x,ij} = \exp \left( \varepsilon_{ijk} a B^c_{x,k} T^c \right).$$  \hspace{1cm} (6)

The electric field strength $E^c_{x,i}$ is related to the canonically conjugate momentum $P_{x,i} = \dot{U}_{x,i}$ via

$$E^c_{x,i} = \frac{2a}{g^3} \text{tr} \left( T^c \dot{U}_{x,i} U^\dagger_{x,i} \right).$$  \hspace{1cm} (7)

The spectrum of Lyapunov exponents is connected to the Kolmogorov-Sinai entropy

$$\sigma_{KS} = \sum_{L_i > 0} L_i.$$  \hspace{1cm} (8)

This quantity describes the rate by which information is lost during the dynamical evolution of initially adjacent phase space points if the resolution is kept fixed (e.g. the smallest phase space cell has a size of the Planck constant, $h = 2\pi$). This information loss can also be regarded as the entropy generation rate. Dividing the sum (8) by the number of Lyapunov exponents (proportional to the lattice volume), we are approximating the continuum entropy density generation rate for big lattices. Calculations on very big lattices have, of course, their limits in the available computer resources.

Eventually the initial energy per degree of freedom, the lattice approximation to the energy density, can be related to a temperature $T$ via the classical equipartition theorem. In fact, whenever the dynamics has positive Lyapunov exponents, equipartition is guaranteed. In this way, if the equation of state in the final state is known, a characteristic time scale can be obtained:

- the equation of state determines the entropy $S(E)$,
- the equipartition leads to $T(E)$ and finally
- the Kolmogorov entropy yields $\dot{S}(E)$.

The characteristic time (inverse thermalization rate) is then obtained as

$$t_T = S/\dot{S}.$$  \hspace{1cm} (9)

Furthermore, preparing the initial state by quantum Monte Carlo, a correspondence between the coupling strength and the Lyapunov exponents can be obtained. This can eventually identify the entropy generation potential of given field configurations and explore possible coincidences between confinement in lattice gauge theory and chaotic classical dynamics. This is the aim of our present study.

## 3 Initial states prepared by quantum Monte Carlo

The Hamiltonian of the lattice gauge field system can be casted into the form

$$H = \sum \left[ \frac{1}{2} \langle P, P \rangle + 1 - \frac{1}{4} \langle U, V \rangle \right].$$  \hspace{1cm} (10)

Here the scalar product between group elements stands for $\langle A, B \rangle = \frac{1}{2} \text{tr}(A B^\dagger)$. The staple variable $V$ is a sum of triple products of elementary link variables closing a plaquette with the chosen link $U$. This way the Hamiltonian is formally written as a sum over link contributions and $V$ plays the role of the classical force acting on the link variable $U$. The naive equations of motion following from this Hamiltonian, however, have to be completed in order to fulfill the constraints

$$\langle U, U \rangle = 1, \quad \langle P, U \rangle = 0.$$  \hspace{1cm} (11)
The following finite time step recursion formula:

\[
U' = U + dt(P' - \varepsilon U),
\]
\[
P' = P + dt(V - \mu U + \varepsilon P'),
\]

with the Lagrange multipliers

\[
\varepsilon = \langle U, P' \rangle,
\]
\[
\mu = \langle U, V \rangle + \langle P', P' \rangle,
\]

conserves the Noether charge belonging to the Gauss law,

\[
\Gamma = \sum_+ PU' - \sum_- U'P.
\]

Here the sums indicated by + run over links starting from, and those by − ending at a given site \( x \), where the Noether charge \( \Gamma \) is defined. The above algorithm is written in an implicit form, but it can be casted into explicit steps, so no iteration is necessary [9].

Initial conditions chosen randomly with a given average magnetic energy per plaquette have been investigated in past years. A linear scaling of the maximal Lyapunov exponent with the total energy of the system has been established for different lattice sizes and coupling strengths [1],[5]. In the present study we prepare the initial field configurations from a standard four dimensional euclidean Monte Carlo program on a \( 12^3 \times 4 \) lattice varying \( \beta = 4/g^2 \).

We relate such four dimensional euclidean lattice field configurations to minkowskian momenta and fields for the three dimensional Hamiltonian simulation by the following approach:

First we fix a time slice of the four dimensional lattice. We denote the link variables in the three dimensional sub-lattice by \( U' = U_i(x, t) \). Then we build triple products on attached handles in the positive time direction, \( U'' = U_4(x, t)U_i(x, t + a)U_4^\dagger(x + a, t) \). We obtain the canonical variables of the Hamiltonian system by using

\[
P = (U'' - U')/dt,
\]
\[
U \propto (U'' + U').
\]

Finally \( U \) is normalized to \( \langle U, U \rangle = 1 \).

This definition constructs the momenta according to a simple definition of the timelike covariant derivative. The multiplication with the link variables in time direction can also be viewed as a gauge transformation to \( U_4(x, t) = 1 \), i.e. \( A_0 = 0 \) Hamiltonian gauge.

### 4 Chaos and confinement

The review of our results we start with a characteristic example of the time evolution of the distance between initially adjacent configurations. An initial state prepared by a standard four dimensional Monte Carlo simulation is evolved according to the classical Hamiltonian dynamics in real time. Afterwards this initial state is rotated locally by group elements which are chosen randomly near to the unity. The time evolution of this slightly rotated configuration is then pursued and finally the distance between these two evolutions is calculated at the corresponding times. A typical exponential rise of this distance followed by a saturation can be inspected in Fig. 1. While the saturation is an artifact of the compact distance measure of the lattice, the exponential rise (the linear rise of the logarithm) can be used for the determination of the leading Lyapunov exponent. The naive determination and more sophisticated rescaling methods lead to the same result. Deviations have been observed only for very small energies [18].
The main result of the present study is the dependence of the leading Lyapunov exponent $L_{\text{max}}$ on the inverse coupling strength $4/g^2$ displayed in Fig. 2. As expected the strong coupling phase, where confinement of static quarks has been established many years ago by proving the area law behavior for large Wilson loops, is more chaotic. The transition reflects the critical temperature to the deconfinement phase. Furthermore the maximal Lyapunov exponent varies between the strong coupling and weak coupling regime more pronounced than the average energy per plaquette. Fig. 3 shows the somewhat softer transition of the energy per plaquette as a function of the inverse coupling strength.

Although in the transition region the behavior of the maximal Lyapunov exponent does not follow exactly that of the average energy, the qualitative correspondence is clearly seen. We compare
therefore maximal Lyapunov exponents stemming from classical random gauge fields in the initial state, where a linear scaling between the leading exponent and the total energy has been established. Fig. 4 shows the energy dependence of the Lyapunov exponents for both choices. This result agrees nicely with the physical picture of random fields being the background for confinement and should be an encouragement for random matrix calculations of the QCD vacuum.

![Graph](image)

**Figure 3:** Transition of the average plaquette energy as a function of the inverse coupling strength $\beta = 4/g^2$.

A comparison with the scaling law of random configurations is made in Fig. 5 by plotting the linear scaling parameter $L_{\text{max}}/(g^2 E)$ versus the degree of equipartition $E/T$. Here the temperature $T$ is taken from the euclidean lattice size $N_t = 4$ via $1/T = N_t a$ to obtain dimensionless quantities. There seem to be systematic deviations for the Lyapunov exponent from the quantum Monte Carlo procedure. The results presented here correspond to an exemplary study, albeit the time evolution itself covers the available phase space ergodically.

Finally it is of interest whether a continuum limit exists for the Lyapunov exponent. Although it is expected to diverge like $1/a$ as the lattice spacing vanishes, its ratio with the average energy per plaquette does not seem to show dramatic changes in Fig. 6. The inverse coupling $4/g^2$ and the lattice spacing are connected by the renormalization group formula for SU(2). One should be aware that the original euclidean system undergoes a phase transition and the small $a$ limit lies in the quark-gluon plasma phase. This gives hope that some ratios of classically divergent quantities still might be extrapolated to the continuum theory.

Summarizing we investigated the classical chaotic dynamics of SU(2) lattice gauge field configurations prepared by finite temperature quantum Monte Carlo simulation. We established that chaos develops as in case of the random choice of initial configurations. The maximal Lyapunov exponent shows a sharp transition as a function of the coupling strength: the strong coupling (confining) phase is substantially more chaotic. The results are close to that of random initial states suggesting that the random matrix model of confinement is a good approximation also for the real time classical dynamics. Finally the scaling law between the leading Lyapunov exponent and energy per plaquette seems to hold when extrapolating to the continuum limit at finite temperature. Whether this is also true at zero temperature, investigations with configurations initialized on four dimensional symmetric lattices may answer.
Figure 4: Comparison of maximal Lyapunov exponents for initial states prepared by quantum Monte Carlo simulation (crosses) and by classical randomization (squares) as a function of the scaled energy per plaquette $a g^2 E$.

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Figure 5: Scaling of the Lyapunov exponent, $L_{\text{max}}/\langle g^2 E \rangle$, as a function of the effective physical degrees of freedom, $E/T$. The line corresponds to the classical scaling law.

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Figure 6: Scaling of the Lyapunov exponent, $L_{\text{max}}$, and the ratio $L_{\text{max}}/E$ as a function of the lattice spacing.

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