Gravitation equations, 
and space-time relativity

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Abstract

In contrast to electrodynamics, Einstein’s gravitation equations are not invariant with respect to a wide class of the mapping of field variables which leave equations of motion of test particles in a given coordinate system invariant. It seems obvious enough that just these mappings should play a role of gauge transformations of the variables in differential equations of gravitational field. We consider here in short a gauge-invariant bimetric generalisation of the Einstein equations which does not contradict available observation data. Physical interpretation of the bimetricity based on relativity of space-time with respect to used reference frame, following conceptually from old Poincaré fundamental ideas, is proposed.

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The relativistic differential equations of motion of charges in electromagnetic field are invariant with respect to some transformations of the field four-potential. For this reason it is naturally that Maxwell equations are also invariant with respect to these transformations. Similarly, the differential equations of motion of test particles in gravitational field in Einstein’s theory in a given coordinate system are invariant with respect to the following transformations of the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ of a Riemannian space-time $V$ :

$$\Gamma^\alpha_{\beta\gamma}(x) = \Gamma^\alpha_{\beta\gamma}(x) + \delta^\alpha_\beta \phi_\gamma(x) + \delta^\alpha_\gamma \phi_\beta(x), \quad (1)$$

where $\phi_\beta(x)$ are an arbitrary differentiable vector-function. It is the most easier for seeing, if the geodesic equations are written in the form

$$\ddot{x}^\alpha + (\Gamma^\alpha_{\beta\gamma} - c^{-1} \Gamma^0_{\beta\gamma} \dot{x}^\alpha) \dot{x}^\beta \dot{x}^\gamma = 0, \quad (2)$$

where the dot denotes differentiation with respect to $t = x^0/c$.

The Ricci and metric tensors also are not invariant under above self-mapping of Riemannian space-time which leave geodesic lines invariant. (They named geodesic mappings).

In contrast to Maxwell theory, Einstein’s equations are not invariant under these transformations [4], although it seems reasonable to suppose that just they are the transformations that have to play a role of gauge transformations of field variables in differential equations of gravitation. It is a very strange fact, especially taking into account that physical consequences resulting from Einstein’s equations agree very closely with all observations. data.

The most natural explanation of such situation is that if there are more correct gravitation equations than the Einstein ones, they may differ markedly from the last equations only at very strong field, close the Schwarzschild radius, where we have not yet firm evidences of validity of physical consequences of the Einstein equations.

The simplest object being geodesic invariant is the Thomas symbols [2]:

$$\Pi^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} - (n + 1)^{-1} \left[ \delta^\gamma_\alpha \Gamma^\beta_\beta + \delta^\gamma_\beta \Gamma^\alpha_\alpha \right], \quad (3)$$

where $\Gamma^\alpha_\alpha = \Gamma^\beta_\beta$.

The simpest geodesic-invariant generalisation of the vacuum Einstein equations are

$$\mathcal{R}_{\alpha\beta} = 0, \quad (4)$$

where $\mathcal{R}_{\alpha\beta}$ is an object which is formed by the gauge-invariant Thomas symbols the same way as the Ricci tensor is formed out of the Christoffel symbols.

However, the problem is that $\Pi^\gamma_{\alpha\beta}$, as well as $\mathcal{R}_{\alpha\beta}$, is not a tensor.

This problem can be solved, if we will be consider all geometrical objects in $V$ as some objects in the Minkowski space-time by analogy with Rosen’s bimetric theory [5]. It means that we must replace all derivatives in geometrical objects o the f Riemannian space-time by the covariant ones defined in the Minkowski space-time. After that, in an arbitrary coordinate system we obtain instead $\Gamma^\alpha_{\beta\gamma}$ a tensor object $D^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \tilde{\Gamma}^\alpha_{\beta\gamma}$, where $\tilde{\Gamma}^\alpha_{\beta\gamma}$ is Christoffel symbols of Minkowski space-time.

Greek indexes run from 0 to 3.

† It follows from (1) that the components $\Gamma_{00} = \Gamma^0_{00}$. Therefore, in Newtonian limit geodesic-invariance is not an essential fact. Therefore, now we deal with a relativistic effect.
\( E \) in used coordinate system. In like manner we obtain instead Thomas symbols a geodesic-invariant (i.e. gauge-invariant) tensor
\[ B^\alpha_{\beta\gamma} = \Pi^\alpha_{\beta\gamma} - \circ^\alpha_{\beta\gamma}, \] (5)
where \( \Pi^\alpha_{\beta\gamma} \) are the Thomas symbols in the Minkowski space-time. This tensor must play a role of a strength tensor of gravitational field. Now, using the identity \( B^\beta_{\alpha\beta} = 0 \), we obtain instead \((4)\) a geodesic-invariant bimetric equation which can be written in the form
\[ \nabla_\alpha B^\alpha_{\beta\gamma} - B^\epsilon_{\beta\delta} B^\delta_{\epsilon\gamma} = 0, \] (6)
where \( \nabla_\alpha \) denotes a covariant derivative in \( E \). Some generalisation of the Einstein’s equations can be obtained and for the case of matter presence.

Evidently, these bimetric equations may be true if both the space-times, \( V \) and \( E \), have some physical meaning. But how these two physical space-time can coexist?

An attempt to answer this question leads us to discussion of a fundamental problem of relativity of space-time with respect to properties of used measuring instruments. A fresh look at Poincaré old well known results allows to obtain conclusions which revise our understanding of geometrical properties of space-time.

At beginning of 20th century Poincaré showed [5] that only an aggregate ”geometry + measuring instruments” has a physical meaning, verifiable by experiment, and it makes no sense to assert that one or other geometry of physical space in itself is true. In fact, the equations of Einstein, is the first attempt to fulfill ideas of Berkeley - of Leibnitz - Mach about space - time relativity. Einstein’s equations clearly show that there is a relationship between properties of space - time and matter distribution. However Poincare’s ideas testify that space and time relativity is not restricted only to dependency of space-time geometry on matter distribution. The space-time geometry also depends on properties of measuring instruments. However, a choice of certain properties of the measuring instruments is nothing more than the choice of certain frame of reference, which just is such a physical device by means of which we test properties of space-time. Consequently, one can expect that there is a relationship between the metric of space-time and a used reference frame.

A step towards the implementation of such idea is considered in [6]. By a non-inertial frame of reference (NIFR) we mean the frame, the body of reference of which is formed by point masses moving in an inertial frame of reference (IFR) under the effect of a force field. By proper frame of reference of a given force field we mean the NIFR, the reference body of which is formed by point masses moving under the effect of the force field. We postulate that space-time in IFRs is the Minkowski one, in accordance with special relativity. Then, above definition of NIFRs allows to find line element of space-time in PFRs.

Let \( \mathcal{L}(x, \dot{x}) \) be Lagrangian describing in an IFR the motion of point particles with masses \( m \) forming the reference body of a NIFR. In this case can be sufficiently clearly argued [6] that the line element \( ds \) of space-time is given by
\[ ds = -(mc)^{-1} dS(x, dx), \] (7)
where \( S = \int \mathcal{L}(x, \dot{x}) dt \) is the action describing the motion of particles of the reference body in the Minkowski space-time. Therefore, properties of space-time in PFRs are entirely determined by properties of used frames in accordance with the Berkeley-Leibnitz-Mach-Poincaré ideas of relativity of space and time.
We can illustrate the above result by some examples.

1. The reference body consists of noninteracting electric charges in a constant homogeneous electromagnetic field. The Lagrangian describing the motion of charges with masses \( m \) is of the form:

\[
L = -mc^2 (1 - v^2/c^2)^{1/2} - \phi_\alpha (x) dx^\alpha ,
\]

where \( \phi \) is a vector function, \( c \) is the speed of light, and \( v \) is the spatial velocity. Then, according to (7), the line element of space-time in the PFR is given by

\[
ds = d\sigma + f_\alpha (x) dx^\alpha
\]

where \( f_\alpha = \phi/m \) is a vector field, and \( d\sigma \) is the line element of the Minkowski space-time. Consequently, space-time in PFRs of electromagnetic field is Finslerian. In principle, we can use both traditional and geometrical description, although the last in this case is rather too complicate.

2. Motion of an ideal isentropic fluid can be considered as the motion of macroscopic small elements ("particles") of an arbitrary mass \( m \), which is described by the Lagrangian [7]

\[
L = -mc \left( G_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \right)^{1/2} ,
\]

where \( w \) is enthalpy per unit volume, \( G_{\alpha\beta} = \varepsilon \eta_{\alpha\beta} \), \( \varepsilon = w/\rho c^2 \), \( \rho = mn \), \( m \) is the mass of the particles, \( n \) is the particles number density, and \( \eta_{\alpha\beta} \) is the metric tensor in the Minkowski space-time. According to (7) the line element of space-time in the NIFR is given by

\[
ds^2 = G_{\alpha\beta} dx^\alpha dx^\beta.
\]

Therefore, the motion of the particles can be considered as occurring under the effect of a force field. (In non-relativistic case it is a pressure gradient). Space-time in the PFR of this force field is Riemannian, and conformal to Minkowski space-time. The motion of the above particles does not depend on theirs masses. We can use both traditional and geometrical description. In some cases such geometrical description is preferable.

3. Suppose that in the Minkowski space-time the Lagrangian describing the motion of test particles of mass \( m \) in a tensor field \( g_{\alpha\beta} \) is of the form

\[
L = -mc \left[ g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \right]^{1/2} ,
\]

where \( \dot{x}^\alpha = dx^\alpha /dt \). According to (7), the line element of space-time in the PFR is given by

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta.
\]

Space-time in PFRs of this field is Riemannian, and motion of test particles do not depend on theirs masses. It is natural to assume that in this case we deal with a gravitational field.

The bimetricity in this case has a simple physical meaning. Disregarding the rotation of the Earth, a reference frame, rigidly connected with the Earth surface, can be considered as an IFR. An observer, located in this frame, can describe the motion of freely falling identical point masses as taking place in Minkowski space-time under the effect of a force field. However, for another observer which is located in the PFR the reference body of which is formed by these freely falling particles, the situation is another. Let us assume that the observer is deprived of the possibility of seeing
the Earth and stars. Then, from his point of view, the the point masses formed the reference body of the PFR are points of his physical space, and all events occur in his space-time. Consequently, accelerations of these point masses must be equal to zero both in nonrelativistic and relativistic meaning. However, instead of this, he observes a change in distances between these point masses in time. Evidently, the only reasonable explanation for him is the interpretation of this observed phenomenon as a manifestation of the deviation of geodesic lines in some Riemannian space-time of a nonzero curvature. Thus, if the first observer, located in the IFR, can postulate that space-time is flat, the second observer, located in a PFR of the force field, who proceeds from relativity of space and time, already in the Newtonian approximation is forced to consider space-time as Riemannian with curvature other than zero.

To obtain physical consequences from (6) it is convenient to select the gauge condition

$$Q_\alpha = \Gamma_\alpha^\beta - \Gamma_\alpha^\beta = 0. \quad (14)$$

At such gauge condition (which does not depend on coordinate system) eqs. (6) coincide with the vacuum Einstein's equations. Therefore, for solving many problems it is sufficiently to find solution of the vacuum Einstein’s equations in the Minkowskian space-time (in which $g_{\alpha\beta}(x)$ is simply a tensor field) at the condition $Q_\alpha = 0$.

From the point of view of the observer located in an IFR and studying the gravitational field of a remote compact object of mass $M$, the space-time is flat. The spherically-symmetric solution of the equations (6) for the point central object very little differs from the solution in general relativity, if the distance from the center $r$ is much more that the Schwarzschild radius $r_g$. However these solutions in essence differ as $r$ is of the order of $r_g$ or less than that. The solution in flat space has no singularity at centre and the event horizon at $r = r_g$. Fig. 1 shows the plots of the gravitational force

Figure 1: The gravitational force (arbitrary units) affecting freely particles (curve 1) and rest-particles (curve 2) near an attractive point mass.

$$F = m\ddot{\mathbf{x}}^\alpha$$

acting on rest-particle of mass $m$ and on a freely falling test particle as a functions of $r$. It follows from the figure, that in the first case $F$ tends to zero when $r \to 0$. In the second case, as particle approach to the Schwarzschild radius, the force changes the sign and becomes repulsive.

These unexpected peculiarities of the gravitational force can be tested by observations. The peculiarity of of the static force leads to the possibility of the existence of supermassive compact
objects without event horizon. Such objects can be identified with supermassive compact objects at centres of galaxies. [9].

The unusual properties of the force acting on freely moving particles near the Schwarzschild radius give rise to some observable effects in cosmology because it is well-known that the radius of an observable part of the Universe is of the order of the Schwarzschild radius of all observed mass. It yields a natural explanation of a deceleration of the Universe expansion [6].

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