$N = 6$ conformal supergravity in three dimensions

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ABSTRACT: $N = 6$ conformal supergravity in three dimensions is studied in an off-shell component field formulation. We obtain local symmetry transformation laws and a Lagrangian of the conformal supergravity multiplet. We then couple it to the ABJM theory and obtain local transformation laws and a Lagrangian of the coupled theory.

KEYWORDS: Extended Supersymmetry, Supergravity Models

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1 Introduction

Conformal supergravities in three dimensions have attracted some attention recently. They are naturally coupled to conformally invariant effective theories of M2 branes such as the BLG theory [1–3] with $\mathcal{N} = 8$ supersymmetry and the ABJM theory [4] with $\mathcal{N} = 6$ supersymmetry. $\mathcal{N} = 8$ and $\mathcal{N} = 6$ conformal supergravities were constructed and were coupled to the BLG theory and the ABJM theory respectively in [5–8]. On-shell conformal supergravity multiplets, which consist of a gravitational field, Rarita-Schwinger fields and vector gauge fields, were used. For on-shell multiplets commutators of two local supertransformations close only when field equations are used. Lagrangians of pure conformal supergravities consisting only of a conformal supergravity multiplet as well as Lagrangians of the coupled theories were constructed.

Off-shell $\mathcal{N} = 8$ and $\mathcal{N} = 6$ conformal supergravity multiplets were discussed in [8, 9] using the superfield formulation [10, 11]. The off-shell multiplets contain scalar and spinor auxiliary fields in addition to the fields of on-shell multiplets. Field equations of a pure conformal supergravity were given in terms of superfields. Couplings to the BLG theory and the ABJM theory were also discussed and field equations of the coupled theories were given in terms of superfields.

Off-shell $\mathcal{N} = 8$ conformal supergravity multiplet was also discussed using a component field formulation in [12]. Local symmetry transformation laws of the conformal supergravity multiplet were obtained from those of $\mathcal{N} = 8$ gauged supergravity in four dimensions [13] using an idea of the AdS/CFT correspondence [14–16] in the same way as in [17]. The off-shell conformal supergravity multiplet was then coupled to the BLG theory as a background field, and a Lagrangian of the coupled theory was obtained.
More recently, Lagrangians of $\mathcal{N} \leq 5$ pure conformal supergravities were obtained in a new superfield formulation [18, 19]. The Lagrangians were given in terms of superfields as well as in terms of component fields. However, such Lagrangians of off-shell conformal supergravity multiplets have not yet been constructed for $\mathcal{N} = 8$ or $\mathcal{N} = 6$.

The purpose of the present paper is to study $\mathcal{N} = 6$ conformal supergravity in three dimensions in an off-shell component field formulation. We obtain local symmetry transformation laws and a Lagrangian of a pure conformal supergravity. We also study its coupling to the ABJM theory and obtain local symmetry transformation laws and a Lagrangian of the coupled theory. By eliminating auxiliary scalar and spinor fields in the off-shell conformal supergravity multiplet by their field equations we reproduce the results of the on-shell formulation [6, 7].

The organization of this paper is as follows. In section 2 the field content of the off-shell $\mathcal{N} = 6$ conformal supergravity multiplet and local symmetry transformation laws of the fields are obtained from those of the $\mathcal{N} = 8$ conformal supergravity [12] by a truncation of fields. We then construct a Lagrangian of a pure $\mathcal{N} = 6$ conformal supergravity in section 3. In section 4 we obtain local symmetry transformations and an invariant Lagrangian of the ABJM theory coupled to the conformal supergravity multiplet. As an application of this Lagrangian we obtain a supercurrent multiplet of the ABJM theory in a flat background in section 5. In section 6 we study a relation of the off-shell formulation obtained in the present paper and the on-shell formulation in [6, 7]. In appendix A we explain our notation and conventions. In appendix B we give definitions and properties of SU(4) matrices used in the text.

2 Conformal supergravity multiplet

In this section we obtain local symmetry transformation laws of $\mathcal{N} = 6$ conformal supergravity in three dimensions from those of the $\mathcal{N} = 8$ conformal supergravity given in [12]. The field content of the off-shell $\mathcal{N} = 8$ conformal supergravity multiplet is

$$\mathcal{N} = 8 : \ e_\mu{}^a, \ \psi^I_\mu, \ B^{IJ}_\mu, \ \lambda^{IJK}, \ E^{IJKL}, \ D^{IJKL},$$

(2.1)

where $e_\mu{}^a(x)$ is a dreibein, $\psi^I_\mu(x)$ are 8 Majorana Rarita-Schwinger fields, $B^{IJ}_\mu(x)$ are 28 SO(8) gauge fields, $\lambda^{IJK}(x)$ are 56 Majorana spinor fields and $E^{IJKL}(x)$, $D^{IJKL}(x)$ are 70 real scalar fields. Here, $I,J,K,\ldots = 1,2,\ldots,8$ are SO(8) vector indices. The multiple SO(8) indices of the fields are totally antisymmetric, e.g., $D^{IJKL} = D^{[IJKL]}$. The scalar fields also satisfy (anti) self-duality conditions

$$E^{IJKL} = -\frac{1}{4!}\eta^{IJKLMNPQ}E_{MNPQ},$$

$$D^{IJKL} = +\frac{1}{4!}\eta^{IJKLMNPQ}D_{MNPQ},$$

(2.2)

1In [12] we used SO(8) spinor indices of negative chirality $\alpha, \beta, \gamma, \ldots = 1,2,\ldots,8$ instead of SO(8) vector indices $I,J,K,\ldots = 1,2,\ldots,8$ for the conformal supergravity multiplet. One can consistently replace the spinor indices there by the vector indices as in (2.1).
\begin{table}[h]
\begin{tabular}{|c|cccccccc|}
\hline
Field & $\epsilon^a_{\mu}$ & $\psi^I_{\mu}$ & $B^I_{\mu}$ & $B_{\mu}$ & $\lambda^{IJK}$ & $\lambda^I$ & $E^{IJ}$ & $D^{IJ}$ \\
\hline
Weyl weight & 1 & $\frac{1}{2}$ & 0 & 0 & $-\frac{3}{2}$ & $-\frac{3}{2}$ & 1 & -2 \\
SU(4) representation & 1 & 6 & 15 & 1 & 20 & 6 & 15 & 15 \\
U(1) charge & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\caption{The field content of the $N = 6$ conformal supergravity multiplet.}
\end{table}

where $\eta$ is a parameter taking a value $+1$ or $-1$. Local symmetry transformations of these fields are general coordinate transformation $\delta_G(\xi)$ with a parameter $\xi^\mu(x)$, local Lorentz transformation $\delta_L(\lambda)$ with a parameter $\lambda_{ab}(x)$, Weyl transformation $\delta_W(\Lambda)$ with a parameter $\Lambda(x)$, local SO(8) transformation $\delta_g(\zeta)$ with a parameter $\zeta^{IJ}(x)$, local supertransformation $\delta_S(\epsilon)$ with a Majorana spinor parameter $\epsilon^I(x)$ and super Weyl transformation $\delta_S(\eta)$ with a Majorana spinor parameter $\eta^I(x)$.

$N \leq 6$ conformal supergravity multiplets can be obtained from the $N = 8$ multiplet by consistent truncations, i.e., by setting some of the fields to zero in such a way that a part of supersymmetry is preserved. Their local symmetry transformation laws are easily derived from those of the $N = 8$ multiplet.

A truncation to the $N = 6$ multiplet is given by setting $\psi^I_{\mu} = 0$, $B^I_{\mu} = 0$, $\lambda^{IJK} = 0$, $D^{IJKL} = 0$, $E^{IJKL} = 0$, where $I, J, K, \cdots = 1, 2, \cdots, 6$ and $I', J', K', \cdots = 7, 8$. These conditions are invariant under the super and super Weyl transformations when the transformation parameters satisfy $\epsilon^{I'} = 0$, $\eta^{I'} = 0$. The remaining independent fields are

$$N = 6 : \; \epsilon^a_{\mu}, \; \psi^I_{\mu}, \; B^I_{\mu}, \; B^{78}_{\mu}, \; \lambda^{IJK}, \; \lambda^{I78}, \; E^{I778}, \; D^{I778}, \; (2.3)$$

where $I, J, K, \cdots = 1, 2, \cdots, 6$. The fields $D^{IJKL}$, $E^{IJKL}$ are related to $D^{I778}$, $E^{I778}$ in (2.3) by the self-duality conditions (2.2). The SO(8) gauge symmetry of the $N = 8$ multiplet is reduced to SO(6) $\times$ SO(2) $\sim$ SU(4) $\times$ U(1), whose gauge fields are $B^I_{\mu}$ and $B^{78}_{\mu}$ in (2.3). The fields in (2.3) constitute the $N = 6$ conformal supergravity multiplet. This multiplet was previously obtained in the superfield formulation [8]. In the following we drop the superscript 78 of the fields in (2.3) as in table 1.

Local symmetry transformation laws of the $N = 6$ conformal supergravity multiplet can be derived from those of the $N = 8$ multiplet given in [12]. Local symmetry transformations of the $N = 6$ multiplet are general coordinate transformation $\delta_G(\xi)$, local Lorentz transformation $\delta_L(\lambda)$, Weyl transformation $\delta_W(\Lambda)$, SO(6) $\times$ SO(2) $\sim$ SU(4) $\times$ U(1) gauge transformation $\delta_q(\zeta)$, local supertransformation $\delta_S(\epsilon)$ and super Weyl transformation $\delta_S(\eta)$, where $\xi^\mu(x)$, $\lambda_{ab}(x)$, $\Lambda(x)$, $\zeta^{IJ}(x)$, $\zeta^I(x)$, $\epsilon^I(x)$ and $\eta^I(x)$ are transformation parameters. Weyl weights, SU(4) representations and U(1) charges of the fields are given in table 1. Note that all the fields in the conformal supergravity multiplet do not have a non-zero U(1) charge. Bosonic transformation laws other than the Weyl and U(1) transformations are obvious from the index structure of the fields. For instance, the bosonic transformations of the Rarita-Schwinger fields with Weyl weight $\frac{1}{2}$ are

$$\left(\delta_G + \delta_L + \delta_W + \delta_g\right)\psi^I_{\mu} = \xi^I{\partial}\psi^I_{\mu} + \partial_\mu\xi^I\psi^I_{\mu} - \frac{1}{4}\lambda_{ab}\gamma^{ab}\psi^I_{\mu} + \frac{1}{2}\Lambda\psi^I_{\mu} - \zeta^{IJ}\psi^I_{\mu}. \; (2.4)$$
The local supertransformation $\delta Q(\epsilon)$ and the super Weyl transformation $\delta S(\eta)$ are given by

$$\delta Q e^{\mu}_{\nu} = \frac{1}{4} \epsilon^{I} \gamma^{\mu} \psi^{I}_{\nu}, \quad \delta Q \psi^{I}_{\mu} = D_{\mu} \epsilon^{I},$$

$$\delta Q B^{I}_{\mu} = -\epsilon^{[I} \psi^{J]}_{\mu} + \frac{1}{2} \sqrt{2} \gamma^{K} \lambda^{IJK} + \frac{1}{4} \eta e^{IJKL MN} \gamma^{[I} \psi^{L]}_{\mu} E^{MN},$$

$$\delta Q B_{\mu} = \frac{1}{2} \sqrt{2} \epsilon^{I} \gamma^{\mu} \lambda^{I} - \frac{1}{2} \sqrt{2} \epsilon^{I} \psi^{I}_{\mu} E^{IJ},$$

$$\delta Q \lambda^{IJK} = -\frac{3}{4} \sqrt{2} \eta e^{IJKL MN} \gamma^{L} D_{\mu} E^{MN} - \frac{3}{8} \sqrt{2} \eta e^{IJKL MN} \epsilon^{L} D_{\mu} E^{MN},$$

$$\delta Q \lambda^{I} = -\frac{1}{4} \sqrt{2} \epsilon^{I} \gamma^{\mu} \lambda^{I} + \frac{1}{2} \eta e^{IJKL MN} \lambda_{+} E^{MN},$$

and

$$\delta S e^{\mu}_{\nu} = 0, \quad \delta S \psi^{I}_{\mu} = \gamma^{\mu} \psi^{I}_{\nu}, \quad \delta S B^{I}_{\mu} = \frac{1}{2} \eta e^{I[I} \psi^{J]}_{\mu},$$

$$\delta S B_{\mu} = 0, \quad \delta S \lambda^{IJK} = -\frac{1}{2} \eta e^{IJKL MN} \eta L E^{MN}, \quad \delta S \lambda^{I} = \eta^{I} E^{IJ},$$

$$\delta S E^{IJ} = 0, \quad \delta S D^{IJ} = -\frac{1}{2} \eta e^{I[I} \lambda^{J]} - \frac{1}{2} \eta e^{IJKL MN} \eta K \lambda^{MN}. \quad (2.5)$$

Here, we have defined

$$\psi^{I}_{\mu} = D_{\mu} \psi^{I}, \quad \psi^{I}_{\mu} = D_{\mu} \psi^{I},$$

$$\lambda^{IJK} = -\frac{1}{2} \gamma^{I} \lambda^{JK} + \frac{1}{4} \eta e^{IJKL MN} \gamma^{L} \psi^{I}_{\mu} E^{MN} - \frac{3}{4} \sqrt{2} \eta e^{IJKL MN} \gamma^{L} \psi^{I}_{\mu} E^{MN},$$

$$\lambda^{I} = -\frac{1}{2} \gamma^{I} \lambda^{I} + \frac{1}{4} \eta e^{IJKL MN} \lambda_{+} E^{MN} - \frac{1}{2} \sqrt{2} \lambda^{IJK} E^{JK},$$

$$\dot{G}^{IJ} = G^{IJ} + 2 \psi^{I} \psi^{J} + \frac{1}{2} \psi^{I} \psi^{J} + \frac{1}{2} \psi^{I} \psi^{J} E^{IJ} - \frac{1}{2} \sqrt{2} \psi^{I} \psi^{J} E^{IJ},$$

$$\dot{G}_{\mu \nu} = G_{\mu \nu} + \frac{1}{2} \psi^{I} \psi^{J} + \frac{1}{2} \psi^{I} \psi^{J} E^{IJ},$$

$$\dot{D}_{\mu} E^{IJ} = D_{\mu} E^{IJ} - \frac{1}{2} \psi_{\mu}^{I} \lambda^{J} + \frac{1}{2} \eta e^{IJKL MN} \psi_{\mu}^{L} \lambda^{NM},$$

$$\dot{D}_{\mu} \lambda^{IJK} = D_{\mu} \lambda^{IJK} + \frac{3}{4} \sqrt{2} \gamma^{IJKL MN} \psi_{\mu}^{L} D^{MN}.$$
\(- \frac{1}{4} \eta^{IJL} \gamma_\rho \psi^J_\mu D_\rho E^{LMN} + \frac{3}{\sqrt{2}} \psi^L_\mu E^{[IJ} E^{KL]} \),

\( \hat{D}_\mu \lambda^I = D_\mu \lambda^I + \frac{1}{4\sqrt{2}} \gamma^{[\sigma \rho} \psi^J_\mu \tilde{G}_{\rho \sigma] - \psi^J_\mu D^{IJ}} + \frac{1}{2} \gamma^\rho \psi^J_\mu \hat{D}_\rho E^{IJ} + \frac{1}{8\sqrt{2}} \eta^{IJL} \hat{D}_\mu \psi E^{KL} E^{MN} \). \tag{2.7}

The covariant derivative \( D_\mu \) contains the spin connection and the SO(6) \( \times \) U(1) gauge fields and is given by, e.g., for \( \epsilon^I \)

\( D_\mu \epsilon^I = \left( \partial_\mu + \frac{1}{4} \hat{\omega}^{\mu}_{\rho \sigma} \gamma^\rho \right) \epsilon^I + B^{IJ}_\mu \epsilon^J. \tag{2.8} \)

The spin connection \( \hat{\omega}_{\mu \rho \sigma} \) satisfies the torsion condition

\( D_\mu e^\nu_a - D_\nu e^\mu_a = \frac{1}{4} \hat{\omega}^\mu_{\rho \sigma} \psi^I_a \gamma^\rho \psi^I_b - \hat{\omega}^\mu_{\rho \sigma} \psi^I_a \gamma^\rho \psi^I_b - \hat{\omega}^\mu_{\rho \sigma} \psi^I_a \), \tag{2.9} \)

and is given by

\( \hat{\omega}_{\mu \rho \sigma} = \omega_{\mu \rho \sigma}(e) + \frac{1}{8} (\psi^I_a \gamma_{\mu} \psi^J_a + \psi^I_a \gamma_{\mu} \psi^J_a - \psi^I_a \gamma_{\mu} \psi^J_a), \tag{2.10} \)

where \( \omega_{\mu \rho \sigma}(e) \) is the spin connection without torsion. The curvature tensor made from the spin connection \( \hat{\omega}_{\mu \rho \sigma} \) satisfies

\( R^{\mu \nu}_{ab} = 4 \epsilon^{|a}_{[\mu} R_{|b]}^{|b]} - \mu^{|a}_{[\mu} e^{|b]}_{\nu]|b| R, \)

\( R^{\mu \nu} = - \frac{3}{4} \psi^T_{[\rho} \psi^I_{\mu]} \). \tag{2.11} \)

The field strengths of the SO(6) \( \times \) U(1) gauge fields are

\( G^{IJ}_{\mu \nu} = \partial_\mu B_{\nu}^{IJ} - \partial_\nu B_{\mu}^{IJ} + B_{\mu}^{IK} B_{\nu}^{KJ} - B_{\nu}^{IK} B_{\mu}^{KJ}, \)

\( G_{\mu \nu} = \partial_\mu B_{\nu} - \partial_\nu B_{\mu}. \tag{2.12} \)

Commutators of these local symmetry transformations close off-shell since those of the \( \mathcal{N} = 8 \) transformations close off-shell \([12]\) and the truncation is consistent with the \( \mathcal{N} = 6 \) local symmetries. We find that the commutators of the fermionic transformations are

\[ \begin{align*}
[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] &= \delta_G(\zeta) + \delta_L(\lambda) + \delta_g(\zeta') + \delta_S(\eta'), \\
[\delta_Q(\epsilon), \delta_S(\eta)] &= \delta_W(\Lambda) + \delta_L(\lambda') + \delta_g(\zeta') + \delta_S(\eta''), \\
[\delta_S(\eta_1), \delta_S(\eta_2)] &= 0,
\end{align*} \tag{2.13} \]

where the transformation parameters appearing on the right-hand sides are

\[ \begin{align*}
\xi^\mu &= \frac{1}{4} \hat{\epsilon}^I_2 \gamma^\mu \epsilon^I_1, & \lambda_{ab} &= -\xi^\mu \hat{\omega}_{\mu \rho \sigma}^a \hat{\omega}_{\mu \rho \sigma}^b,
\end{align*} \]

\[ \begin{align*}
\zeta^{IJ} &= -\xi^\mu B_{\mu}^{IJ} + \frac{1}{4\sqrt{2}} \eta^{IJL} \psi^K_\mu \epsilon^L_2 \hat{\omega}_{\mu \rho \sigma}^L E^{MN},
\end{align*} \]

\[ \begin{align*}
\zeta &= -\xi^\mu B_{\mu} - \frac{1}{2\sqrt{2}} \hat{\omega}_{\mu \rho \sigma}^a \hat{\omega}_{\mu \rho \sigma}^b E^{KL}, & \epsilon^I &= -\xi^\mu \psi^I_\mu,
\end{align*} \]
\[ \eta^I = \frac{1}{2} \epsilon^{I \alpha \rho \sigma} \gamma_{\mu} \psi^I_{\mu \rho \sigma} - \frac{1}{16} \epsilon^{I J J J} \epsilon^{\lambda \rho \sigma} \lambda_{\mu}^{I \lambda K L} - \frac{1}{16} \epsilon^{I J J J} \epsilon^{\lambda \rho \sigma} \lambda_{\mu}^{I \lambda K L} \]

\[ \Lambda = -\frac{1}{4} \epsilon^{I I} \eta^I, \quad \lambda_{ab}^{IJ} = \frac{1}{4} \epsilon^{I I} \gamma_{ab} \eta^I, \quad \zeta^{IJ} = -\frac{1}{2} \epsilon^{I I} \eta^I, \]

\[ \zeta' = 0, \quad \eta''^I = \frac{1}{8} \gamma_{\mu} \epsilon^{I I} \eta^I \psi^I_{\mu}. \]  

(2.14)

We can further truncate the \( N = 6 \) conformal supergravity multiplet to \( N \leq 5 \) multiplets. We find that consistent truncations give the multiple

\[ N = 5: e^a_{\mu}, \psi^I_{\mu}, B^{IJ}_{\mu}, \lambda^{IJK}, \lambda^6, E^{I6}, D^{I6}, \]

\[ N = 4: e^a_{\mu}, \psi^I_{\mu}, B^{IJ}_{\mu}, \lambda^{IJK}, E^{56}, D^{56}, \]

\[ N = 3: e^a_{\mu}, \psi^I_{\mu}, B^{IJ}_{\mu}, \lambda^{123}, \]

\[ N = 2: e^a_{\mu}, \psi^I_{\mu}, B^{12}_{\mu}, \]

\[ N = 1: e^a_{\mu}, \psi^I_{\mu}. \]  

(2.15)

where \( I, J, \cdots = 1, 2, \cdots, N \) for each case. The field contents of these multiplets coincide with those obtained in the superfield formulation \[8\]. Local symmetry transformation laws of these multiplets can be derived from those of the \( N = 6 \) multiplet. The \( SO(6) \times U(1) \) gauge symmetry of the \( N = 6 \) multiplet reduces to \( SO(N) \) whose gauge fields are \( B^{IJ}_{\mu} \).

Local supertransformations and super Weyl transformations of component fields for \( N \leq 5 \) were recently obtained by the superfield formulation \[19\]. The multiplets in \[19\] contain a gauge field of dilatation \( b_{\mu} \) in addition to the fields in (2.15), which can be eliminated by a special conformal gauge transformation. We expect that our transformations for \( N \leq 5 \) derived from the \( N = 6 \) multiplet will coincide with the transformations in \[19\] when \( b_{\mu} \) is eliminated.

3 Pure conformal supergravity

Let us first consider a theory which contains only the \( N = 6 \) conformal supergravity multiplet. Such a theory was previously constructed in the on-shell formulation \[6\]. Here, we reconsider it in the off-shell formulation.

We find that the Lagrangian in the off-shell formulation is

\[ \mathcal{L}_{\text{CSG}} = \frac{1}{2} \epsilon^{I J J J} \left( \psi^I_{\mu} \psi^J_{\mu} + \frac{2}{3} \lambda^{IJK} \psi^K_{\mu} + \lambda^{IJK} \psi^J_{\mu} \right) + \frac{1}{4} \epsilon^{I J J J} \lambda^{IJK} \lambda^{IJK} \psi^I_{\mu} \psi^J_{\mu} \]

\[ -\epsilon^{I J J J} \left( B^{IJ}_{\mu} \partial_{\nu} B^{KL}_{\mu} + \frac{2}{3} B^{IJ}_{\mu} B^{JK}_{\mu} B^{KL}_{\mu} \right) - 2 \epsilon^{I J J J} \lambda^{IJK} \psi^J_{\mu} \psi^K_{\mu} \]

\[ + \frac{1}{3} \epsilon^{I J J J} \left( \lambda^{IJK} \lambda^{IJK} - 2 \epsilon^{I J J J} \lambda^{IJK} \right) - \frac{2}{3} \epsilon^{I J J J} \lambda^{IJK} \psi^J_{\mu} E^{I I J} + \frac{1}{3} \epsilon^{I J J J} \eta e^{I J K L M N} E^{I J} E^{K L E M N} \]

\[ + \frac{1}{6} \epsilon^{I J K L M N} \lambda^{I J K L} \gamma^I \psi^J_{\mu} E^{I J} + 2 \epsilon^{I J J J} \lambda^{IJK} \psi^J_{\mu} E^{I J}. \]  

(3.1)
The first four terms, which only depend on the on-shell multiplet \((e_\mu^a, \psi^I_\mu, B^{IJ}_\mu, B_\mu)\), give the Lagrangian of the on-shell formulation \([6]\). We added other terms such that the Lagrangian is invariant up to total divergences under all the local symmetry transformations of the off-shell formulation discussed in the previous section. In particular, this Lagrangian is invariant under the super and super Weyl transformations \((2.5), (2.6)\) up to total divergences.

Field equations of the fields \(\lambda^{IJK}, \lambda^I, E^{IJ}, D^{IJ}\) are algebraic and therefore they are auxiliary fields. In the present case of no matter fields the field equations give \(\lambda^{IJK} = 0, \lambda^I = 0, E^{IJ} = 0, D^{IJ} = 0\). Substituting these solutions into the Lagrangian \((3.1)\) and the fermionic transformations \((2.5), (2.6)\) we obtain those of the on-shell formulation \([6]\).

We also note that Lagrangians of the pure conformal supergravities with \(N \leq 5\) supersymmetry in the off-shell formulation can be easily derived from \((3.1)\) by the truncations \((2.15)\). Lagrangians in the off-shell formulation were previously obtained for \(N = 1\) in \([20]\), for \(N = 2\) in \([21]\), and more recently for \(3 \leq N \leq 5\) in \([19]\). Our results are consistent with them.

4 The ABJM theory coupled to conformal supergravity

In this section we couple the off-shell \(N = 6\) conformal supergravity multiplet constructed in the previous sections to the ABJM theory. The ABJM theory \([4]\) is a three-dimensional field theory which has an \(N = 6\) superconformal symmetry. We use a formulation of it in terms of the 3-algebra \([22]\). The field content of the ABJM theory is complex scalar fields \(Z^A_i(x)\), complex spinor fields \(\Psi_{Ai}(x)\) and 3-algebra gauge fields \(\tilde{A}_{\mu}^i(x) = A_{\mu}^i(x)f^{ijk}l_i\) as shown in table 2, where \(A, B, \cdots = 1, 2, 3, 4\) are SU(4) indices and \(i, j, k, \cdots = 1, 2, \cdots, n\) are 3-algebra indices. The structure constant of the 3-algebra \(f^{ij}_kl\) satisfies

\[
f^{ij}_kl = f^{[ij]}|kl| = (f^{kl}_ij)^*, \quad f^{ij}_p[kl]f^{pm}_{ln} = f^{m[i|np}f^{j]}pkl. \tag{4.1}\]

Complex (charge) conjugates of the fields are \((Z^A_i)^* = \bar{Z}^j_A, (\Psi_{Ai})^c = \Psi^{\bar{A}i}\) and \((\tilde{A}_{\mu}^i)^* = -\tilde{A}_{\mu}^i\). The 3-algebra gauge transformations of the fields are

\[
\delta_{g_3}Z^A_i = Z^A_j \tilde{A}^i_j, \quad \delta_{g_3}\Psi_{Ai} = \Psi_{Aj} \tilde{A}^j_i, \quad \delta_{g_3}\tilde{A}_{\mu}^j_i = D_\mu \tilde{A}^j_i - \tilde{A}^j_k \tilde{A}_{\mu}^k_i + \tilde{A}_{\mu}^j_i \tilde{A}_{\mu}^k_i, \tag{4.2}\]

where \(\tilde{A}^j_i(x) = \lambda^j_k(x)f^{ijk}_li\) is a transformation parameter, which satisfies \((\tilde{A}^j_i)^* = -\tilde{A}^j_i\).

The field strength of \(\tilde{A}_{\mu}^j_i\) is given by

\[
F_{\mu\nu}^j_i = \partial_\mu \tilde{A}_{\nu}^j_i - \partial_\nu \tilde{A}_{\mu}^j_i + \tilde{A}_{\mu}^j_k \tilde{A}_{\nu}^k_i - \tilde{A}_{\nu}^j_k \tilde{A}_{\mu}^k_i. \tag{4.3}\]

To couple the conformal supergravity to the ABJM theory we will only consider the case in which the sign parameter is \(\eta = +1\) since we could not find local supertransformations whose commutator algebra closes on \(\Psi_{Ai}\) when \(\eta = -1\). By the standard Noether procedure starting from the theory in a flat background \([22]\) we can obtain local symmetry transformation laws and a Lagrangian of the coupled theory.
Field | $Z^A_i$ | $\Psi_{Ai}$ | $\tilde{A}^I_{ij}$  \\
--- | --- | --- | ---  \\
Weyl weight | $-\frac{1}{2}$ | $-1$ | $0$  \\
SU(4) representation | $4$ | $\bar{4}$ | $1$  \\
U(1) charge | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$  

| Table 2. The field content of the ABJM multiplet.  

First we shall give local symmetry transformation laws of the ABJM fields. Transformation laws of the conformal supergravity fields remain the same as in section 2 since they have the closed commutator algebra off-shell. Weyl weights, SU(4) representations and U(1) charges of the ABJM fields are given in table 2. Bosonic transformation laws other than the Weyl and U(1) transformations are obvious from the index structure of the fields. The local supertransformation $\delta_Q(\epsilon)$ and the super Weyl transformation $\delta_S(\eta)$ of the ABJM fields are given by

$$\delta_Q Z^A_i = -\frac{1}{2\sqrt{2}} \bar{\epsilon}^B (\Sigma^I)^{AB} \Psi_{Bi},$$

$$\delta_Q \Psi_{Ai} = -\frac{1}{2\sqrt{2}} \gamma^A \bar{\epsilon}^B (\Sigma^I)^{AB} \dot{Z}^B_i + \frac{1}{2\sqrt{2}} \bar{\epsilon}^B (\Sigma^I)^{BC} \dot{Y}^{BC}_{\dot{A}i},$$

$$\delta_Q \tilde{A}^I_{ij} = \frac{1}{2\sqrt{2}} f^{kj} i \gamma^A \dot{Z}^B_k + \frac{1}{2\sqrt{2}} f^{kj} i (\Sigma^I)^{AB} \epsilon \gamma^A \dot{\Psi}_{Bi} \bar{Z}^I_k + \frac{1}{4} f^{kj} i (\Sigma^I)^{AB} \epsilon \Psi_{Bi} \bar{Z}^I_k$$

and

$$\delta_S Z^A_i = 0, \quad \delta_S \Psi_{Ai} = \frac{1}{2\sqrt{2}} (\Sigma^I)^{AB} \bar{\eta}^I \bar{Z}^B_i, \quad \delta_S \tilde{A}^I_{ij} = 0,$$

where

$$\dot{D}_\mu Z^A_i = D_\mu Z^A_i + \frac{1}{2\sqrt{2}} (\Sigma^I)^{AB} \bar{\psi}^I_\mu \Psi_{Bi},$$

$$Y^{BC}_{\dot{A}i} = f^{kl} i (Z^B_k Z^C_l \bar{Z}^I_k + \delta^B_A Z^C_l \bar{Z}^I_k Z^D_l).$$

Here, $\Sigma^I$, $\bar{\Sigma}^I$ and $\Sigma^{IJ}$ are SU(4) matrices, whose definitions and properties are given in appendix B. The covariant derivative $D_\mu$ contains the 3-algebra gauge fields in addition to the spin connection and the SU(4) $\times$ U(1) gauge fields:

$$D_\mu Z^A_i = \left( \partial_\mu + \frac{1}{2} i B_\mu \right) Z^A_i - \frac{1}{4} B^{IJ}_\mu Z^B_i (\Sigma^{IJ})_B^A - Z^A_i \tilde{A}^I_{ij},$$

$$D_\mu \Psi_{Ai} = \left( \partial_\mu + \frac{1}{4} \bar{\omega}_{ab} \gamma^{ab} + \frac{1}{2} i B_\mu \right) \Psi_{Ai} + \frac{1}{4} B^{IJ}_\mu (\Sigma^{IJ})_A^B \Psi_{Bi} - \Psi_{Aj} \tilde{A}^I_{ij}.$$

We obtained these fermionic transformation laws (4.4), (4.5) by requiring the commutator algebra closes up to field equations of the ABJM fields. The commutation relations
we found are

\[ [\delta_Q(e_1), \delta_Q(e_2)] = [\delta_G(\xi) + \delta_L(\lambda) + \delta_d(\zeta) + \delta_{\rho_3}(\Lambda) + \delta_Q(\epsilon') + \delta_S(\eta')] , \]

\[ [\delta_Q(e), \delta_S(\eta)] = \delta_W(\Lambda) + \delta_L(\lambda') + \delta_d(\zeta') + \delta_S(\eta'') , \]

\[ [\delta_S(n_1), \delta_S(n_2)] = 0 , \]

where the transformation parameters on the right-hand sides are given by (2.14) and

\[ \tilde{A}^i_j = -\xi^\mu \tilde{A}_\mu^i_j + \frac{1}{4} f^{jk}_l \epsilon^j_2 \epsilon^l_1 (\Sigma^{IJ})_A^B Z^A_k Z^j_B \]

for the 3-algebra gauge transformation. Actual commutation relations of two supertransformations on the fields \( \Psi_{A_i} \) and \( \tilde{A}^i_j \) are

\[ [\delta_Q(e_1), \delta_Q(e_2)] \Psi_{A_i} = \cdots - \frac{1}{2} \xi^{\mu} \gamma_\mu E_{A_i} - \frac{1}{8} \epsilon_2^j_1 (\Sigma^{IJ})_A^B E_{B_i} , \]

\[ [\delta_Q(e_1), \delta_Q(e_2)] \tilde{A}^i_j = \cdots - e \epsilon_{\mu \nu \rho} e^{\nu \gamma} E^\rho_i , \]

where \( \cdots \) denote the transformations appearing on the right-hand side of (4.8), and

\[ E_{A_i} = \gamma^\mu D_\mu \Psi_{A_i} + (2 \Psi_{B_j} \tilde{Z}^l_A Z^B_k - \Psi_{A_j} \tilde{Z}^l_A Z^B_k + \epsilon_{ABCD} \Psi^B_i Z^C_k Z^D_j ) f^{kj}_l \]

\[ + \frac{1}{2\sqrt{2}} \gamma^\mu \gamma^\nu \psi^K_{\mu} (\Sigma^K)_{AB} D_\nu Z^B_i - \frac{1}{2\sqrt{2}} \gamma^\mu \psi^M_{\mu} Y_{AB} (\Sigma^M)_{BC} \]

\[ - \frac{1}{4\sqrt{2}} \psi^K_{\rho \sigma} (\Sigma^K)_{AB} Z^B_i - \frac{1}{12} \lambda^{KLM} (\Sigma^{KLM})_{AB} Z^B_i - \frac{1}{2} i \lambda^K (\Sigma^K)_{AB} Z^B_i \]

\[ + \frac{1}{2} i (\Sigma^{KL})^M_{AB} \psi^K_{\mu} E^M_{K} Z^B_i - \frac{1}{2\sqrt{2}} i (\Sigma^{KL})^M_{A} \psi^K_{B} E^M_{KL} , \]

\[ E^{\mu \nu}_{ij} = \frac{1}{2} e^{-1} \epsilon^{\nu \rho \sigma} F_{\rho \sigma}^i_j + (\tilde{Z}^l_A D^\mu Z^B_k - D^\mu \tilde{Z}^l_A Z^B_k - \tilde{\Psi}^{A l}_A \psi_{A k}) f^{k l}_i \]

\[ - \frac{1}{2\sqrt{2}} \left[ \psi^K_{\nu} \mu \gamma^\nu \Psi^A_{B l} (\Sigma^K)_{AB} + \psi^K_{\nu} \mu \gamma^\nu \Psi_{A k} \tilde{Z}^l_B (\Sigma^K)^{AB} \right] f^{k l}_i \]

\[ + \frac{1}{2} \psi^K_{\nu} \mu \gamma^\nu \psi^K_{\rho} (\Sigma^K)_{A}^B Z^B_k Z^j_B f^{k j}_l , \]

where \( E_{A_i} = 0 \) and \( E^{\mu \nu}_{ij} = 0 \). These conditions can be regarded as field equations of \( \Psi_{A_i} \) and \( \tilde{A}^i_j \). Thus the algebra closes on-shell on the ABJM fields as in the original ABJM theory in a flat background [22].

To find the Lagrangian of the ABJM theory coupled to the off-shell conformal supergravity multiplet we start from the Lagrangian of the ABJM theory coupled to gravitational field and add possible terms depending on other fields in the conformal supergravity multiplet which are invariant under the bosonic transformations. The terms depending on \( \Psi_{A_i} \) and/or \( \tilde{A}^i_j \) are fixed so that their field equations give the conditions \( E_{A_i} = 0 \) and \( E^{\mu \nu}_{ij} = 0 \) for (4.11). Other terms are determined by requiring invariance of the Lagrangian under the fermionic transformations (2.5), (4.4) and (2.6), (4.5) up to total divergences. Finally, the complete invariance of the Lagrangian is shown. To show cancellations of terms in \( \delta_Q \mathcal{L} \)
and $\delta_S \mathcal{L}$ we use (2.9), (2.11) and identities of SU(4) matrices given in appendix B. In this way we find the Lagrangian as

$$
\mathcal{L}_{\text{ABJM}} = -e D_\mu \bar{Z}_i^A D^\mu Z_i^A - \frac{1}{2} e (\bar{\Psi}^I A^A D_\mu \Psi^\mu_i - D_\mu \bar{\Psi}^I A^A \Psi^\mu_i) - \frac{2}{3} e |Y_{\text{ABJM}}|^2 
$$

$$
- \frac{1}{2} e^{\mu \nu \rho} \left( f_{ij kl} A^i_j k \partial_\nu A^j_i + \frac{2}{3} f_{ik lp} f_{jm n} A^m_j A^i_p A^j_n n \right) 
- 2 e f_{ij kl} \bar{\Psi}^A k \Psi^B l \left( \bar{Z}_i^A Z_j^B - \frac{1}{2} \delta_{ij} \bar{Z}_C Z_j^C \right) 
+ \left[ -\frac{1}{2} e_{ABCD} f_{ij kl} \bar{\Psi}^A k \Psi^B l (\Sigma^I)_A Z_j^B + \frac{1}{4 \sqrt{2}} e \bar{\Psi}^A i \gamma^\mu \psi^I_j (\Sigma^I)_A D_\mu Z_i^B 
+ \frac{1}{16} e \bar{\psi}^I_j \gamma^\mu \nu \psi^I_j D_\mu Z_i^B (\Sigma^I)_A + \frac{1}{8} e (R - \frac{1}{2} \bar{\psi}^I_j \gamma^\mu \nu \psi^I_j) Z_i^A Z_i^A 
- \frac{1}{16} e \bar{\psi}^I_j \gamma^\mu \nu \psi^I_j Z_i^A Z_j^B (\Sigma^I)_A B + \text{c.c.} \right] - \frac{1}{8} e \left( R - \frac{1}{2} \bar{\psi}^I_j \gamma^\mu \nu \psi^I_j \right) Z_i^A Z_i^A 
$$

$$
- \frac{1}{16} e \bar{\psi}^I_j \gamma^\mu \nu \psi^I_j \bar{\Psi} (\gamma^\mu \nu \rho) \Psi (\Sigma^I)_A B 
+ \frac{1}{16} e \bar{\psi}^I_j \gamma^\mu \nu \psi^I_j \bar{\Psi}^I (\Sigma^I)_A B + \left[ \frac{1}{12} e \bar{\lambda}^{IJK} \Psi^I (\Sigma^I)_A B Z_i^B 
+ \frac{1}{2} e \bar{\lambda}^{IJK} \Psi^I (\Sigma^I)_A B Z_i^B - \frac{1}{8} e \bar{\Psi}^I (\Sigma^I)_A B E^{IJ} (\Sigma^I)_A B + \text{c.c.} \right] 
+ \frac{1}{\sqrt{2}} e E^{IJ} \left( f_{ij kl} Z_i^A Z_j^B Z_k^C + \frac{1}{2} \bar{\Psi}^I \Psi^I \right) (\Sigma^I)_A B 
+ \frac{1}{\sqrt{2}} e D^{IJ} \bar{Z}_i^A (\Sigma^I)_A B - \frac{1}{4} e E^{IJ} E^{IJ} \bar{Z}_i^A Z_i^A 
+ \frac{1}{24 \sqrt{2}} e \bar{\lambda}^{IJK} \gamma^\mu \psi^I_j Z_i^A (\Sigma^I)_A B - \frac{1}{4 \sqrt{2}} e \bar{\lambda}^{IJK} \gamma^\mu \psi^I_j \bar{Z}_i^A (\Sigma^I)_A B 
- \frac{1}{4 \sqrt{2}} e \bar{\psi}^I_j \gamma^\mu \nu \psi^I_j Z_i^A \left[ E^{KL} (\Sigma^I)_A B - \frac{1}{4} \delta^{IJ} E^{KL} (\Sigma^I)_A B \right].
$$

The ABJM theory coupled to the off-shell $\mathcal{N} = 6$ conformal supergravity multiplet was previously discussed in a superfield formulation [8]. Field equations were given in terms of superfields although a Lagrangian was not given. We expect that field equations derived from our Lagrangian (4.12) will coincide with a component field expression of the field equations in [8].

5 Supercurrent multiplet of the ABJM theory

As in [12] we can use the Lagrangian (4.12) to find a supercurrent multiplet [23] of the ABJM theory in a flat background

$$
e^a_\mu = \delta^a_\mu, \quad \psi^I_j = B^I_j, \quad B_\mu = \lambda^{IJK}, \quad \lambda^I = E^{IJ}, \quad D^{IJ} = 0.
$$
It can be obtained by computing derivatives of the Lagrangian (4.12) with respect to the conformal supergravity fields and taking the flat background (5.1). We find the supercurrent multiplet as

\[ T_{\mu\nu} = 2D_{(\mu}Z^A_{\nu)}Z^A_i - \eta_{\mu\nu}D_\rho \bar{Z}^A_{\rho} Z^A_i - \frac{1}{4}(\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2)(\bar{Z}^A_{\mu} Z^A_i) \]

\[ + \frac{1}{2} \left[ \bar{\Psi}^A_i \gamma_\mu (D_\nu) \Psi_{Ai}^A - D_\nu (\bar{\Psi}^A_i \gamma_\mu) \Psi_{Ai}^A \right] - \frac{2}{3} \eta_{\mu\nu} |Y_{AI}^B|^2 ; \]

\[ S^{\mu I} = \frac{1}{2\sqrt{2}} \left[ \gamma_\nu \gamma_\mu \Phi_{Ai}^A D_\nu Z^B_i + \frac{1}{2} \partial_\nu (\gamma_{\mu\nu} \Phi_{Ai}^A Z^B_i) + \gamma_\mu \Phi_{Ci}^A Y_{AB}^{CI} \right] (\Sigma^I)^{AB} + \text{c.c.}, \]

\[ J^{\mu IJ} = \frac{1}{2} \left[ \bar{Z}_{B}^D D_\mu Z^A_i - D_\mu \bar{Z}_{B}^D Z^A_i + \bar{\Phi}_{Ai}^{A} \gamma_\mu \Psi_{Bi} \right] (\Sigma^{IJK})^{AB} ; \]

\[ R^{IJK} = \frac{1}{12} \left[ (\Sigma^{IJK})_{AB} \Phi_{Ai}^A \bar{Z}^B_i - (\Sigma^{IJK})_{AB} \Phi_{Ai}^A \bar{Z}^B_i \right] , \]

\[ R^I = \frac{1}{2} \left[ (\Sigma^I)^{AB} \Phi_{Ai}^A Z^B_i + (\Sigma^I)^{AB} \Phi_{Ai}^A \bar{Z}^B_i \right] , \]

\[ M^{IJ} = \frac{1}{2\sqrt{2}} \left[ f_{i\mu} iZ^A_i Z^C_j \bar{Z}^B_k \bar{Z}^D_l \bar{\Phi}_{Bi}^A \right] (\Sigma^{I})^{AB} , \]

\[ N^{IJ} = \frac{1}{2\sqrt{2}} \left[ iZ^A_i \bar{Z}^B_i (\Sigma^{I})^{AB} \right] , \]

where the covariant derivative \( D_\mu \) contains only the 3-algebra gauge field \( \tilde{A}_\mu^J \). \( T_{\mu\nu}, S^{\mu I}, J^{\mu IJ}, J^{\mu} \) are the energy-momentum tensor, the supercurrent and the SU\((4) \times U(1)\) current, respectively. They satisfy conservation laws and \((\gamma\gamma)\)-traceless conditions

\[ \partial_\mu T^{\mu\nu} = 0 , \quad \partial_\mu S^{\mu I} = 0 , \quad \partial_\mu J^{\mu IJ} = 0 , \quad \partial_\mu J^{\mu} = 0 , \]

\[ T^{\mu\mu} = 0 , \quad \gamma_\mu S^{\mu I} = 0 . \]  

\( R^{IJK}, R^I, M^{IJ}, N^{IJ} \) are quantities corresponding to the fields \( \lambda^{IJK}, \lambda^I, E^{IJ}, D^{IJ} \), respectively. As in [12] we can construct conserved currents for the symmetry of the background (5.1) by multiplying \( T_{\mu\nu}, S^{\mu I}, J^{\mu IJ}, J^{\mu} \) by a conformal Killing vector, a conformal Killing spinor and constant SU\((4) \times U(1)\) transformation parameters respectively.

6 A relation to the on-shell formulation

The total Lagrangian of the coupled theory is

\[ \mathcal{L} = \frac{1}{g} \mathcal{L}_{\text{CSG}} + \mathcal{L}_{\text{ABJM}}, \]

where \( \mathcal{L}_{\text{CSG}} \) and \( \mathcal{L}_{\text{ABJM}} \) are given in (3.1) and (4.12) respectively, and \( g \) is a conformal gravitational coupling constant [7]. We can eliminate the auxiliary fields \( \lambda^{IJK}, \lambda^I, E^{IJ}, D^{IJ} \) by using their field equations to find the results in the on-shell formulation [6] as follows.
The field equations of the auxiliary fields are algebraic again and can be used to express them in terms of the ABJM fields as

\[ E^{IJ} = \frac{1}{8\sqrt{2}} ig (\bar{Z}Z)_B A (\Sigma^{IJ})_A B, \]

\[ \lambda^{IK} = -\frac{1}{8} g^A (\bar{Z}Z)_B (\Sigma^{IKJ})_A + \frac{1}{8} g^A (\bar{Z}Z)_B (\Sigma^{IKJ})_A B, \]

\[ \lambda^I = \frac{1}{8} ig^A (\bar{Z}Z)_B (\Sigma^I)_A + \frac{1}{8} ig^A (\bar{Z}Z)_B (\Sigma^I)_A B, \]

\[ D^{IJ} = \frac{1}{8\sqrt{2}} ig^{ij} Z_i^A (\bar{Z}Z)_B (\Sigma^{I})_A B + \frac{1}{16\sqrt{2}} ig^A (\bar{Z}Z)_B (\Sigma^{I})_A B \]

\[ + \frac{1}{64\sqrt{2}} ig^2 \left[ (\bar{Z}Z)_B (\bar{Z}Z)_C (\Sigma^I) (\bar{Z}Z)_B A - (\bar{Z}Z)(\bar{Z}Z)_B A (\Sigma^{I})_A B \right], \] (6.2)

where we have used abbreviations

\[ (\bar{Z}Z)_A B = \bar{Z}^A Z^B, \quad (\bar{Z}Z)_j = \bar{Z}^j Z_j, \quad (\bar{Z}Z) = \bar{Z}^i Z^i. \] (6.3)

Substituting these expressions into (6.1) we obtain the Lagrangian without auxiliary fields as

\[ \mathcal{L}' = \mathcal{L}|_{\lambda, E, D = 0} = \frac{3}{8} 8 g\bar{\psi}^A \psi B j Z_i^A Z_i^B - \frac{1}{8} g\bar{\psi}^A \psi A j (\bar{Z}Z)_j^i \]

\[ + \frac{1}{64} 8 g\bar{\psi}^A \psi B j Z_i^A Z_i^B - \frac{1}{8} g\bar{\psi}^A \psi A j (\bar{Z}Z)_j^i \]

\[ - \frac{1}{8\sqrt{2}} \bar{\psi}^A \gamma^\mu \psi^j_\mu (\Sigma^I)_{CB} Z_i^B (\bar{Z}Z)_A^C + \frac{1}{32\sqrt{2}} \bar{\psi}^A \gamma^\mu \psi^j_\mu (\Sigma^I)_{AB} Z_i^B (\bar{Z}Z) + c.c. \]

\[ + \frac{1}{64} 8 g\bar{\psi}^A \gamma^\mu \psi^j_\mu (\bar{Z}Z)_B A (\bar{Z}Z)_D^C \left[ (\Sigma^I)_{AC} (\Sigma^J)_{BD} + \frac{1}{4} \delta^I \delta^J \delta^A \delta^C \right] \]

\[ + \frac{1}{2} 8 g\bar{\psi}^A \gamma^\mu \psi^j_\mu (\bar{Z}Z)_B A (\bar{Z}Z)_D^C \left[ (\Sigma^I)_{AC} (\Sigma^J)_{BD} + \frac{1}{4} \delta^I \delta^J \delta^A \delta^C \right] \]

\[ + \frac{1}{2} 8 g\bar{\psi}^A \gamma^\mu \psi^j_\mu (\bar{Z}Z)_B A (\bar{Z}Z)_D^C \left[ (\Sigma^I)_{AC} (\Sigma^J)_{BD} + \frac{1}{4} \delta^I \delta^J \delta^A \delta^C \right] \]

\[ + \frac{1}{8} 8 g^2 e \left[ (\bar{Z}Z)_B A (\bar{Z}Z)_B C (\bar{Z}Z)_C A - \frac{3}{2} (\bar{Z}Z)(\bar{Z}Z)_A B (\bar{Z}Z)_B A + \frac{5}{16} (\bar{Z}Z)^3 \right]. \] (6.4)

This Lagrangian coincides with that of the on-shell formulation [6]. Substitution of (6.2) into the fermionic transformations (2.5), (2.6), (4.4), (4.5) also gives those of the on-shell formulation [6]. Thus, we see that our theory is an off-shell extension of the on-shell formulation studied in [6, 7].

A Notation and conventions

Three-dimensional world and local Lorentz indices are denoted by \( \mu, \nu, \cdots = 0, 1, 2 \) and \( a, b, \cdots = 0, 1, 2 \), respectively. A flat metric is \( \eta_{ab} = \text{diag}(-1, +1, +1) \) and an antisymmetric symbol \( \epsilon^{abc} \) is chosen as \( \epsilon^{012} = +1 \). Symmetrization and antisymmetrization of indices with weight one are denoted as \( (ab \cdots) \) and \( \{ab \cdots\} \), respectively. Antisymmetrized products \( \gamma^{ab} = \gamma^{[a} \gamma^{b]} \), \( \gamma^{abc} = \gamma^{[a} \gamma^{b} \gamma^{c]} \) of the three-dimensional gamma matrices \( \gamma^a \) satisfy

\[ \gamma^{abc} = -\epsilon^{abc}, \quad \gamma^{ab} = -\epsilon^{abc} \gamma^c, \quad \gamma^a = \frac{1}{2} \epsilon^{abc} \gamma^b \gamma^c. \] (A.1)
SO(6) \sim SU(4) indices are denoted as I, J, K, \cdots = 1, 2, \cdots, 6 and A, B, C, \cdots = 1, 2, 3, 4. Antisymmetric symbols with these indices \( \epsilon^{IJKLMN} \), \( \epsilon^{ABCD} \) and \( \epsilon^{ABCD} \) are chosen as \( \epsilon^{123456} = +1 \), \( \epsilon^{1234} = +1 \) and \( \epsilon^{1234} = +1 \).

B SU(4) matrices

In this appendix we summarize definitions and some properties of SU(4) matrices. The 8 \times 8 gamma matrices of SO(6) can be chosen as

\[
\Gamma^I = \begin{pmatrix} 0 & \Sigma^I \\ \bar{\Sigma}^I & 0 \end{pmatrix},
\]

where \( \Sigma^I \) and \( \bar{\Sigma}^I \) are 4 \times 4 matrices with components \( (\Sigma^I)^{AB} \) and \( (\bar{\Sigma}^I)^{AB} \) and satisfy \( (\Sigma^I)^T = -\Sigma^I \), \( \bar{\Sigma}^I = (\Sigma^I)^\dagger \). By the anticommutation relation of the SO(6) gamma matrices \( \{\Gamma^I, \Gamma^J\} = 2\delta^{IJ} \) they satisfy

\[
\Sigma^I \bar{\Sigma}^J + \Sigma^J \bar{\Sigma}^I = 2\delta^{IJ},
\]

\[
\bar{\Sigma}^I \Sigma^J + \bar{\Sigma}^J \Sigma^I = 2\delta^{IJ}.
\]

We denote antisymmetrized products of these matrices as

\[
\Sigma^{IJ} = \Sigma^I \bar{\Sigma}^J,
\]

\[
\Sigma^{IJK} = \Sigma^I \bar{\Sigma}^J \Sigma^K,
\]

\[
\bar{\Sigma}^{IJ} = \bar{\Sigma}^I \Sigma^J,
\]

\[
\bar{\Sigma}^{IJK} = \bar{\Sigma}^I \Sigma^J \bar{\Sigma}^K,
\]

\[
\Sigma^{IJKL} = i \epsilon^{IJKLMN} \delta^L_N,
\]

\[
\bar{\Sigma}^{IJKL} = i \epsilon^{IJKLMN} \delta^L_N.
\]

In general the leftmost matrix in \( \Sigma^{IJ\cdots} \) is \( \Sigma^I \) while that in \( \bar{\Sigma}^{IJ\cdots} \) is \( \bar{\Sigma}^I \). They satisfy duality relations

\[
(\Sigma^{IJKLMN})^{AB} = i \epsilon^{IJKLMN} \delta^B_A,
\]

\[
(\Sigma^{IJKLM})_{AB} = i \epsilon^{IJKLMN} (\Sigma^N)_{AB},
\]

\[
(\Sigma^{IJKL})^{AB} = -\frac{1}{2} i \epsilon^{IJKLMN} (\Sigma^{MN})^B_A,
\]

\[
(\Sigma^{IJK})_{AB} = -\frac{1}{6} i \epsilon^{IJKLMN} (\Sigma^{LMN})_{AB}
\]

and similar relations for \( \bar{\Sigma}^{IJ\cdots} \) with opposite signs on the right-hand sides.

Other useful identities are

\[
(\Sigma^I)_{AB} = -\frac{1}{2} \epsilon_{ABCD} (\Sigma^I)^{CD},
\]

\[
(\Sigma^I)_{A[B}(\Sigma^J)_{C)]D} = \frac{1}{3} \epsilon_{BCDE} (\Sigma^I)^{E} (\Sigma^J)^{CD},
\]

\[
(\Sigma^I)_{[AB} (\Sigma^J)_{CD]} = \frac{1}{3} \epsilon_{ABCD} \delta^{IJ},
\]

\[
(\Sigma^I)^{AB} (\Sigma^J)^{CD} = -4 \delta^C_A \delta^D_B,
\]

\[
(\Sigma^I)^{A[B}(\Sigma^J)_{CD]} = -8 \delta^B_A \delta^D_C + 2 \delta^B_A \delta^C_D,
\]

\[
(\Sigma^{IJK})_{AB} (\Sigma^{IJL})_{CD} = -48 \delta^C_A \delta^D_B,
\]

\[
(\Sigma^{IJK})_{AB} (\Sigma^{IJL})_{CD} = -48 \delta^C_A \delta^D_B.
\]
\[(\Sigma^I)_{AB}(\Sigma^J)^{CD} = -2i\delta^C_A(\Sigma^J)^{BD},\]
\[\epsilon^{IJKLMN}(\Sigma^{KL})_A(\Sigma^{MN})_B = -8i\delta^B_C(\Sigma^I)^A_D - 8i\delta^D_A(\Sigma^I)^C_B + 4i\delta^D_C(\Sigma^I)^A_B + 4i\delta^B_A(\Sigma^I)^C_D.\]

(B.5)

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