A combinatorial proof of the Gaussian product inequality conjecture beyond the MTP$_2$ case

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Abstract

In this paper, we present a combinatorial proof of the Gaussian product inequality (GPI) conjecture in all dimensions when the components of the centered Gaussian vector $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ can be written as linear combinations, with nonnegative coefficients, of the components of a standard Gaussian vector. The proof comes down to the monotonicity of a certain ratio of gamma functions. We also show that our condition is weaker than assuming the vector of absolute values $|\mathbf{X}| := (|X_1|, |X_2|, \ldots, |X_d|)$ to be in the multivariate totally positive of order 2 (MTP$_2$) class on $[0, \infty)^d$, for which the conjecture is already known to be true.

Keywords: Gaussian product inequality, Gaussian vector, multivariate normal, multinormal, moments inequality, gamma function, polygamma function, multinomial, complete monotonicity

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1. Introduction

The Gaussian product inequality (GPI) conjecture has gained traction lately following the proof of the three-dimensional case by Lan et al. [5] and the proof of the closely related Gaussian correlation inequality by Royen [11] (in fact, Royen proved an extension involving Gamma random variables), see also Latala & Matlak [6].

Conjecture 1.1 (Original form of the GPI conjecture). Let $d \in \mathbb{N}$ and assume that $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ is a centered (i.e., mean-zero) Gaussian vector. Then, for all $m \in \mathbb{N},$

$$
\mathbb{E} \left[ \prod_{i=1}^{d} X_i^{2m} \right] \geq \prod_{i=1}^{d} \mathbb{E}[X_i^{2m}].
$$

This conjecture was shown to be true for $d = 2$ (see Theorem 2.1 in [2]), for $d = 3$ (see Theorem 1.1 in [5]), and whenever $|\mathbf{X}| := (|X_1|, |X_2|, \ldots, |X_d|)$ is in the so-called MTP$_2$ class on $[0, \infty)^d$ (see Definition 1.3 below for the definition of MTP$_2$ and see Corollary 1.1 in [4] for the corresponding result). The GPI conjecture has many deep implications. For instance, the conjecture is known to imply the real polarization problem conjecture in functional analysis (see Section 3.3 in [8]), and it is also related to the so-called $U$-conjecture which states that two polynomials evaluated at the same underlying random variable are unlinked if they are independent (for more details, see Section 1.5 in [8]; also Section 11.4 in [3] and references therein).

Here is an extended form of the GPI conjecture proposed by Li & Wei [7, Section 4], which is also known to be true when $|\mathbf{X}|$ is in the MTP$_2$ class on $[0, \infty)^d$.

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Conjecture 1.2 (Extended form of the GPI conjecture). Let \( d \in \mathbb{N} \) and assume that \( X = (X_1, X_2, \ldots, X_d) \) is a centered Gaussian vector. Then, for all \( \alpha_1, \alpha_2, \ldots, \alpha_d \in (0, \infty) \),
\[
\mathbb{E} \left[ \prod_{i=1}^{d} X_i^{2\alpha_i} \right] \geq \prod_{i=1}^{d} \mathbb{E}[X_i^{2\alpha_i}]. \tag{1.2}
\]

When \( X = (X_1, X_2, \ldots, X_d) \) is a centered Gaussian vector, the main theorem in [4] (Theorem 3.1) finds an equivalence between the condition that \(|X| := (|X_1|, |X_2|, \ldots, |X_d|)\) has a MTP\(_2\) density function on \([0, \infty)^d\) and the condition that the off-diagonal elements of the inverse of the covariance matrix of \( X \) are all nonpositive up to a change of sign for some of the \( X_i \)’s. As an immediate consequence, we know that Conjecture 1.2 is true when the latter condition is satisfied.

Our goal in this paper will be to prove a middle ground inequality between (1.1) and (1.2) when the assumptions on the Gaussian vector \( X \) are (strictly) weaker than the ones in Theorem 3.1 of Karlin & Rinott [4]. Specifically, our main result (Theorem 2.1) establishes the validity of Conjecture 1.2 for \( \alpha_i = n_i \in \mathbb{N} \) when the components of \( X \) can be written as linear combinations, with nonnegative coefficients, of the components of a standard Gaussian vector. Our approach is combinatorial in nature and therefore does not make (direct) use of the monotonicity properties of the densities in the MTP\(_2\) class. Instead, the proof is elementary and comes down to the monotonicity of a certain ratio of gamma functions, which is closely related to the complete monotonicity of multinomial probabilities previously shown by Ouimet [9] and Qi et al. [10].

The paper is organized as follows. The statement and proof of our main result (Theorem 2.1) is given Section 2. In Section 3, we explain why the assumptions in Theorem 2.1 are weaker than the ones in Theorem 3.1 of Karlin & Rinott [4].

Definition 1.3 (MTP\(_2\)). A density function \( f : \mathbb{R}^d \to \mathbb{R} \) is called multivariate totally positive of order 2 (MTP\(_2\)) on \( \times_{i=1}^{d} S_i \subseteq \mathbb{R}^d \) (or is said to be in the MTP\(_2\) class on \( \times_{i=1}^{d} S_i \)) if it is supported on \( \times_{i=1}^{d} S_i \) and it satisfies
\[
f(x \vee y)f(x \wedge y) \geq f(x)f(y), \quad \text{for all } x, y \in \times_{i=1}^{d} S_i,
\]
where \( x \vee y = (\max(x_1, y_1), \ldots, \max(x_d, y_d)) \) and \( x \wedge y = (\min(x_1, y_1), \ldots, \min(x_d, y_d)) \). Similarly, we say that a random vector is in the MTP\(_2\) class on \( \times_{i=1}^{d} S_i \) if its density function is.

Density functions in this class have many interesting properties, among which the most relevant to us is the following, which is stated as Equation (1.7) in [4].

Proposition 1.4. Let \( Y \) be a MTP\(_2\) random vector on \( \times_{i=1}^{d} S_i \subseteq \mathbb{R}^d \), and let \( \varphi_1, \varphi_2, \ldots, \varphi_r \) be a collection of nonnegative and (component-wise) non-decreasing functions on \( \times_{i=1}^{d} S_i \). Then
\[
\mathbb{E} \left[ \prod_{i=1}^{r} \varphi_i(Y) \right] \geq \prod_{i=1}^{r} \mathbb{E}[\varphi_i(Y)].
\]

When the density function of \( Y = (|X_1|, |X_2|, \ldots, |X_d|) \) is in the MTP\(_2\) class on \([0, \infty)^d\), the validity of Conjecture 1.2 follows from Proposition 1.4 with \( r = d \) and \( \varphi_i(y) = y_i^{2n_i} \).

2. Main result

As we will demonstrate in Section 3, the following result extends the validity (of a form) of the GPI conjecture beyond the MTP\(_2\) case.
Theorem 2.1. Let $d \in \mathbb{N}$ and assume that $X = (X_1, X_2, \ldots, X_d)$ is a centered Gaussian vector and there exists a matrix $C \in [0, \infty)^{d \times d}$ such that $X$ is equal in law to $CZ$, where $Z \sim \mathcal{N}_d(0_d, I_d)$ is a $d$-dimensional standard Gaussian vector. Then, for all $n_1, n_2, \ldots, n_d \in \mathbb{N}_0$,

$$
\mathbb{E} \left[ \prod_{i=1}^{d} X_i^{2n_i} \right] \geq \prod_{i=1}^{d} \mathbb{E} \left[ X_i^{2n_i} \right].
$$

Proof. In terms of $Z$, the inequality (2.1) is equivalent to

$$
\mathbb{E} \left[ \prod_{i=1}^{d} \left( \sum_{j=1}^{d} c_{ij} Z_j \right)^{2n_i} \right] \geq \prod_{i=1}^{d} \mathbb{E} \left[ \left( \sum_{j=1}^{d} c_{ij} Z_j \right)^{2n_i} \right].
$$

Let $K_j := \sum_{i=1}^{d} k_{ij}$ and $L_j := \sum_{i=1}^{d} \ell_{ij}$, where $k_{ij}$ and $\ell_{ij}$ are index variables to be used in (2.3) and (2.4) below. By the multinomial formula, the linearity of expectations, the independence of the $Z_j$’s, and the moments formula $\mathbb{E}[Z_j^{2m}] = (2m)!/(2^m m!)$ for $m \in \mathbb{N}_0$, we can expand the left-hand side of (2.2) as follows:

$$
\mathbb{E} \left[ \prod_{i=1}^{d} \left( \sum_{j=1}^{d} c_{ij} Z_j \right)^{2n_i} \right] = \mathbb{E} \left[ \prod_{i=1}^{d} \sum_{k_i \in \mathbb{N}_0^d} \left( \sum_{j=1}^{d} c_{ij} Z_j \right)^{2n_i} \right]
$$

$$
= \mathbb{E} \left[ \sum_{k_1 \in \mathbb{N}_0^d \atop k_1 + \cdots + k_d = 2n_1} \cdots \sum_{k_d \in \mathbb{N}_0^d \atop k_1 + \cdots + k_d = 2n_d} \prod_{i=1}^{d} \left( \sum_{j=1}^{d} c_{ij} Z_j \right)^{2n_i} \prod_{i=1}^{d} \left( k_i \right)^{d} \prod_{j=1}^{d} c_{ij} \right]
$$

$$
\geq \sum_{\ell_1 \in \mathbb{N}_0^d \atop \ell_1 + \cdots + \ell_d = n_1} \cdots \sum_{\ell_d \in \mathbb{N}_0^d \atop \ell_1 + \cdots + \ell_d = n_d} \prod_{i=1}^{d} \left( \frac{(2L_j)!}{2^{L_j} L_j!} \right) \prod_{i=1}^{d} \left( 2n_i \right)^{L_i} \prod_{j=1}^{d} c_{ij}
$$

where we dropped the terms on the fourth line for which at least one $k_{ij}$ is not an even number (this is possible because we assumed the $c_{ij}$’s to be nonnegative). Similarly, by expanding the right-hand side of (2.2) using the fact that $\mathbb{E}[Y^{2m}] = (2m)!\sigma^{2m}/(2^m m!)$ for $m \in \mathbb{N}_0$ and $Y \sim \mathcal{N}(0, \sigma^2)$, we get

$$
\prod_{i=1}^{d} \mathbb{E} \left[ \left( \sum_{j=1}^{d} c_{ij} Z_j \right)^{2n_i} \right] = \prod_{i=1}^{d} \left( \sum_{j=1}^{d} c_{ij} \right)^{2n_i}
$$

$$
= \prod_{i=1}^{d} \left( \frac{(2n_i)!}{2^n n_i!} \right) \sum_{\ell_i \in \mathbb{N}_0^d \atop \ell_i + \cdots + \ell_d = n_i} \prod_{j=1}^{d} \left( \frac{n_i}{\ell_1, \ldots, \ell_d} \right) \prod_{j=1}^{d} c_{ij}
$$

$$
= \sum_{\ell_1 \in \mathbb{N}_0^d \atop \ell_1 + \cdots + \ell_d = n_1} \cdots \sum_{\ell_d \in \mathbb{N}_0^d \atop \ell_1 + \cdots + \ell_d = n_d} \prod_{i=1}^{d} \left( \frac{(2n_i)!}{2^n n_i!} \right) \prod_{j=1}^{d} c_{ij}.
$$
Now, we want to compare the coefficients in front of the corresponding powers $e^{2t_{ij}}$ in (2.3) and (2.4). In order to prove (2.2), it is sufficient to show that, for all $\ell_{1}, \ell_{2}, \ldots, \ell_{d} \in \mathbb{N}_{0}^{d}$ such that $\ell_{i1} + \cdots + \ell_{id} = n_i$ for all $i \in \{1, 2, \ldots, d\}$,

$$
\left( \prod_{j=1}^{d} \frac{(2L_{j})!}{2^{L_{j}}L_{j}!} \right) \prod_{i=1}^{d} \frac{2n_i}{(2\ell_{i1}, \ldots, 2L_{id})} \geq \prod_{i=1}^{d} \frac{(2n_i)!}{2^{n_i-n_i}!} \left( \ell_{i1}, \ldots, \ell_{id} \right).
$$

By noticing that $2^{\sum_{j=1}^{d} \ell_{ij}} = 2^{\sum_{i=1}^{n} n_i}$, and after cancelling some factorials, the above is equivalent to

$$
\prod_{j=1}^{d} \frac{(2L_{j})!}{\prod_{i=1}^{d} (2\ell_{ij})!} \geq \prod_{j=1}^{d} \frac{L_{j}!}{\prod_{i=1}^{d} \ell_{ij}!}.
$$

(2.5)

To conclude the proof, we need to show (2.5). Without loss of generality, we can assume that the $L_{i}$'s are all non-zero (otherwise, the inequality (2.5) reduces to a lower dimensional case). Therefore, for any given $L_{i}$'s in $\mathbb{N}$, define the functions

$$
g_{j}(a) := \frac{\Gamma(aL_{j} + 1)}{\prod_{i=1}^{d} \Gamma(a\ell_{ij} + 1)}, \quad a \in (-1/L_{j}, \infty), \quad j \in \{1, 2, \ldots, d\},
$$

where $\Gamma(\lambda) = \int_{0}^{\infty} \lambda^{1-1} e^{-t} dt$, $\lambda > 0$, denotes Euler's gamma function. In order to prove (2.5), it is sufficient to show that $a \mapsto \log g_{j}(a)$ is non-decreasing on $[0, \infty)$ for all $j \in \{1, 2, \ldots, d\}$. Direct computations yield, for all $a \in [0, \infty)$,

$$
\frac{d}{da} \log g_{j}(a) = L_{j} \psi(aL_{j} + 1) - \sum_{i=1}^{d} \ell_{ij} \psi(a\ell_{ij} + 1),
$$

$$
\frac{d^{2}}{d^{2}a} \log g_{j}(a) = L_{j}^{2} \psi'(aL_{j} + 1) - \sum_{i=1}^{d} \ell_{ij}^{2} \psi'(a\ell_{ij} + 1),
$$

where $\psi'(z) = \frac{d}{dz} \psi(z)$ denotes the digamma function. Using the integral representation

$$
\psi'(z) = \int_{0}^{\infty} \frac{te^{-(z-1)t}}{e^{t} - 1} dt, \quad z > 0,
$$

see Abramowitz & Stegun [1, p.260], we have

$$
\frac{d^{2}}{d^{2}a} \log g_{j}(a) = \int_{0}^{\infty} \frac{(L_{j}t)e^{-a(L_{j}t)}}{e^{t} - 1} L_{j} dt - \sum_{i=1}^{d} \int_{0}^{\infty} \frac{(\ell_{ij}t)e^{-a(\ell_{ij}t)}}{e^{t} - 1} \ell_{ij} dt
$$

$$
= \int_{0}^{\infty} se^{-as} \left[ \frac{1}{e^{s/L_{j}} - 1} - \sum_{i=1}^{d} \frac{1}{(e^{s/L_{j}})^{\ell_{ij}/\ell_{ij} - 1}} \right] ds.
$$

Given that $\sum_{i=1}^{d} \ell_{ij}/L_{j} = 1$, the quantity inside the brackets on the last line is always non-negative by Lemma 1.4 in [9] with $y := e^{s/L_{j}}$ and $u_{i} := \ell_{ij}/L_{j}$ for all $i \in \{1, 2, \ldots, d\}$ (see page 516 in [10] for an alternative proof), so that

$$
\frac{d^{2}}{d^{2}a} \log g_{j}(a) \geq 0, \quad a \in [0, \infty).
$$

(2.6)

(In fact, $a \mapsto \frac{d^{2}}{d^{2}a} \log g_{j}(a)$ is even completely monotonic.) Since

$$
\frac{d}{da} \log g_{j}(a) \bigg|_{a=0} = L_{j} \psi(1) - \sum_{i=1}^{d} \ell_{ij} \psi(1) = 0 \cdot \psi(1) = 0,
$$

we deduce from (2.6) that

$$
\frac{d}{da} \log g_{j}(a) \geq 0, \quad a \in [0, \infty).
$$

Hence, $a \mapsto \log g_{j}(a)$ is non-decreasing on $[0, \infty)$. This ends the proof. □

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3. Why Theorem 2.1 extends the GPI beyond the MTP\textsubscript{2} case

In this section, our goal is to show that if \( \mathbf{X} \) is a non-singular centered Gaussian vector such that \(|\mathbf{X}|\) is in the MTP\textsubscript{2} class on \([0, \infty)^d\), then there exists a lower triangular matrix \( \mathbf{C} \in [0, \infty)^{d \times d} \) such that \( \mathbf{X} \) is equal in law to \( \mathbf{C} \mathbf{Z} \), where \( \mathbf{Z} \sim \mathcal{N}_d(\mathbf{0}, I_d) \). This will show that the GPI (2.1) in the MTP\textsubscript{2} case (or, equivalently according to Theorem 3.1 in [4], when the off-diagonal elements of the inverse covariance matrix are nonpositive up to a change of sign for some of the \( X_i \)'s) is just a consequence of Theorem 2.1.

We start with a preliminary lemma.

**Lemma 3.1.** Let \( \Sigma \) be any \( n \times n \) symmetric positive definite (SPD) matrix with Cholesky decomposition \( \Sigma = \mathbf{C} \mathbf{C}^\top \), and assume that \( \Sigma^{-1} = [\tau_{ij}]_{1 \leq i, j \leq n} \) satisfies \( \tau_{ij} \leq 0 \) for all \( i \neq j \). Then the elements of \( \mathbf{C} \) are all nonnegative.

**Proof.** We want to prove this using an induction argument. The statement of the lemma holds trivially for \( n = 1 \). Assume that the statement holds for some \( n = m \in \mathbb{N} \), and fix \( n = m + 1 \) for the remainder of the proof.

Since \( \Sigma^{-1} \) is SPD and \( \Sigma^{-1} = [\tau_{ij}]_{1 \leq i, j \leq m+1} \) satisfies \( \tau_{ij} \leq 0 \) for all \( i \neq j \), we can write

\[
\Sigma^{-1} = \begin{pmatrix} a & v^\top \\ v & B \end{pmatrix},
\]

where \( v \in (-\infty, 0]^m \), \( B := [b_{ij}]_{1 \leq i, j \leq m} \in \mathbb{R}^{m \times m} \) satisfies \( b_{ij} \leq 0 \) for all \( i \neq j \), and the Schur complement \( a - v^\top \mathbf{L} \mathbf{L}^\top v \) is positive. Since \( \Sigma^{-1} \) is SPD, note that \( B \) is SPD, and thus \( B^{-1} \) is SPD. We also have that the off-diagonal elements of \( (B^{-1})^{-1} = B \) are nonpositive, so the induction assumption implies that we can write \( B^{-1} = \mathbf{L} \mathbf{L}^\top \) for some \( \mathbf{L} \in [0, \infty)^{m \times m} \). Therefore,

\[
\Sigma^{-1} = \begin{pmatrix} a - v^\top \mathbf{L} \mathbf{L}^\top v & v^\top \mathbf{L} \\ \mathbf{L}^\top v & (\mathbf{L}^\top)^{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{a - v^\top \mathbf{L} \mathbf{L}^\top v} & 0 \\ 0 & L^{-1} \end{pmatrix}.
\]

From the previous equation, we deduce

\[
\Sigma = \begin{pmatrix} \sqrt{a - v^\top \mathbf{L} \mathbf{L}^\top v} & 0 \\ \mathbf{L}^\top v & L^{-1} \end{pmatrix}^{-1} \begin{pmatrix} \sqrt{a - v^\top \mathbf{L} \mathbf{L}^\top v} & v^\top \mathbf{L} \\ 0 & (\mathbf{L}^\top)^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} \sqrt{a - v^\top \mathbf{L} \mathbf{L}^\top v} & 0 \\ -\mathbf{L}^\top v & L^{-1} \end{pmatrix} \begin{pmatrix} \sqrt{a - v^\top \mathbf{L} \mathbf{L}^\top v} & 0 \\ -\mathbf{L}^\top v & L^{-1} \end{pmatrix} =: \mathbf{C} \mathbf{C}^\top,
\]

where recall \( v \in (-\infty, 0]^m \) and \( \mathbf{L} \in [0, \infty)^{m \times m} \). Hence, all the components of \( \mathbf{C} \) are nonnegative. \( \square \)

We can now show how the validity of the GPI in the MTP\textsubscript{2} case follows from Theorem 2.1 when the \( \alpha_i \)'s are nonnegative integers.

**Corollary 3.2.** Let \( d \in \mathbb{N} \) and assume that \( \mathbf{X} = (X_1, X_2, \ldots, X_d) \) is a non-singular centered Gaussian vector such that \(|X_1|, |X_2|, \ldots, |X_d|\) is in the MTP\textsubscript{2} class on \([0, \infty)^d\). Then, for all \( n_1, n_2, \ldots, n_d \in \mathbb{N}_0 \),

\[
\mathbb{E} \left[ \prod_{i=1}^d X_i^{2n_i} \right] \geq \prod_{i=1}^d \mathbb{E} \left[ X_i^{2n_i} \right].
\]
Proof. Let $\Sigma$ denote the covariance matrix of the non-singular centered Gaussian vector $X$, and note that $\Sigma$ is symmetric positive definite (and thus invertible). Given that $(|X_1|, |X_2|, \ldots, |X_d|)$ is in the MTP$_2$ class on $[0, \infty)^d$ by assumption, the matrix $\Sigma^{-1} := [\tau_{ij}]_{1 \leq i, j \leq n}$ satisfies $\tau_{ij} \leq 0$ for all $i \neq j$ up to a change of sign for some of the $X_i$’s, see Theorem 3.1 in [4]. Therefore, without loss of generality, assume that $\Sigma^{-1} = [\tau_{ij}]_{1 \leq i, j \leq n}$ satisfies $\tau_{ij} \leq 0$ for all $i \neq j$ (otherwise change the sign of the problematic $X_i$’s). By Lemma 3.1, we have the Cholesky decomposition $\Sigma = CC^\top$ with $C \in [0, \infty)^{d \times d}$, so that $X$ is equal in law to $CZ$, where $Z \sim \mathcal{N}_d(0_d, I_d)$. The conclusion follows by Theorem 2.1.

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