DIRAC OPERATORS ON REAL SPINOR BUNDLES OF COMPLEX TYPE

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ABSTRACT. Let \((M, g)\) be a pseudo-Riemannian manifold of signature \((p, q)\). We compute the obstruction for a vector bundle \(S\) over \((M, g)\) to admit a Dirac operator whose principal symbol induces on \(S\) the structure of a bundle of irreducible real Clifford modules of complex type, that is, a real spinor bundle of irreducible complex type. In order to do this, we use the theory of Lipschitz structures in signature \(p - q \equiv_{\mathbb{Z}} 3, 7\) to reformulate the problem as the obstruction problem for \((M, g)\) to admit a \(\text{Spin}^\alpha\) structure with \(\alpha = -1\) if \(p - q \equiv_{\mathbb{Z}} 3\) or \(\alpha = +1\) if \(p - q \equiv_{\mathbb{Z}} 7\), where \(\text{Spin}^\alpha(p, q) = \text{Spin}(p, q) \cdot \text{Pin}_{2,0}\) and \(\text{Spin}^\alpha(p, q) = \text{Spin}(p, q) \cdot \text{Pin}_{0,2}\). This allows computing the obstruction in terms of the Karoubi Stiefel-Whitney classes of \((M, g)\) and the existence of an auxiliary \(\text{O}(2)\) bundle with prescribed characteristic classes. Furthermore, we explicitly show how a \(\text{Spin}^\alpha\) structure can be used to construct \(S\) and we give geometric characterizations (in terms of associated bundles) of the conditions under which the structure group of \(S\) reduces to certain natural subgroups of \(\text{Spin}^\alpha\). Finally, we prove that certain codimension two submanifolds of spin manifolds and certain products of tori with Grassmanians, which were not known to admit irreducible real spinor bundles, do admit \(\text{Spin}^\alpha\) structures and therefore do admit real spinor bundles of irreducible complex type.

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Introduction

Let \(S\) be a real vector bundle over a connected pseudo-Riemannian manifold \((M, g)\) of signature \((p, q)\) and dimension \(d\). A Dirac operator \(D : C^\infty(S) \to C^\infty(S)\) on \(S\) is by definition a first-order differential operator whose square \(D^2 : C^\infty(S) \to C^\infty(S)\) has principal symbol \(\sigma\) given by:

\[
\sigma(m, \xi) = g(\xi, \xi) \text{Id}_S, \quad m \in M, \quad \xi \in T^* M.
\]

where \(\text{Id}_S\) denotes the identity endomorphism of \(S\). Since by the previous equation the square of a Dirac operator \(D\) satisfies the Clifford relation condition, the symbol of \(D\) canonically extends to a morphism of bundles of unital and associative algebras:

\[
\gamma : \text{Cl}(M, g) \to \text{End}(S),
\]

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Our main result is (see Theorem 5.1): 

Theorem 5.1: Let \((M, g)\) be a pseudo-Riemannian manifold of signature \((p, q)\) with \(p - q \equiv_8 3, 7\). Then \((M, g)\) admits a bundle of irreducible real spinors \((S, \gamma)\) if and only if there exists a principal

\[ \text{Spin}^o_{p, q} \text{ structure} \]

and \(\lambda : \text{Spin}^o_{p, q} \to \text{O}(p, q)\) is a certain surjective group morphism constructed from the untwisted adjoint representation of \(\text{Spin}^o_{p, q}\) and from the twisted adjoint representation of \(\text{Pin}_{0,2}\) or \(\text{Pin}_{2,0}\) respectively. The two groups \(\text{Spin}^+_{p, q}\) and \(\text{Spin}^o_{p, q}\) can in fact be considered in any dimension and signature and the same applies to the corresponding structures, which in full generality we call \(\text{Spin}^o\) structures. When \(p - q \equiv_8 3, 7\), an adapted \(\text{Spin}^o\) structure is thus a \(\text{Spin}^o_{p, q}\) structure, where:

\[ \alpha_{p, q} \colon \text{Spin}^o_{p, q} \to \text{O}(p, q) \]

is defined by:

\[ \alpha_{p, q} = \begin{cases} -1 & \text{if } p - q \equiv_8 3 \\ +1 & \text{if } p - q \equiv_8 7 \end{cases} \]

Given an adapted \(\text{Spin}^o\) structure \(Q\) over \(M\), a bundle of irreducible real spinors can be constructed as the associated real vector bundle \(S = Q \times_{\gamma_0} S_0\), where \(\gamma_0 : \text{Spin}^o_{p, q} \to \text{End}_{\mathbb{R}}(S_0)\) is an elementary real pinor representation of \(\text{Spin}^o_{p, q}\). The latter is constructed from an irreducible Clifford representation \(\gamma_0 : \text{Cl}_{p, q} \to \text{End}_{\mathbb{R}}(S_0)\) in an \(\mathbb{R}\)-vector space \(S_0\) of dimension:

\[ N = \dim_{\mathbb{R}} S_0 = 2^\frac{p+q+1}{2} \]

It follows that \(S\) admits a well-defined Clifford multiplication \(TM \otimes S \to S\), which makes it into a bundle of simple modules over the fibers of \(\text{Cl}(M, g)\). When \(M\) is orientable, the real vector bundle \(S\) admits a complex structure, \(Q\) reduces to a \(\text{Spin}^c\) structure \(Q_0\) and \(S\) can be viewed as the ordinary bundle of elementary complex pinors associated to \(Q_0\) (see [9]). When \(M\) is unorientable, \(S\) need not admit a complex structure and hence its global sections cannot be interpreted as complex spinors. In this case, \(S\) admits a so-called semilinear structure — a weakening of the concept of complex structure that still allows one to define the notions of linear and antilinear endomorphisms of \(S\). Furthermore, the structure group of \(S\) reduces to \(\text{Pin}_{0, q}\) or \(\text{Pin}_{a, p}\) if and only if \(S\) admits a globally-defined conjugation. When a conjugation exists, the structure group further reduces to \(\text{Spin}_{p, q}\) if and only if \(M\) is orientable. The subtle interplay between these conditions leads to a web of possibilities which is considerably richer than what occurs in the ordinary case of \(\text{Spin}^o\) structures.

Our main result is (see Theorem 5.1):

\[ \text{Theorem 0.1.} \]

Let \((M, g)\) be a pseudo-Riemannian manifold of signature \((p, q)\) with \(p - q \equiv_8 3, 7\). Then \((M, g)\) admits a bundle of irreducible real spinors \((S, \gamma)\) if and only if there exists a principal

\[ \text{Spin}^o_{p, q} \text{ structure} \]

1If the type \([\eta]\) of the spinor bundle structure defined by \(D\) is given by an irreducible Clifford module we say that \(D\) is irreducible.
O(2)-bundle $E$ over $M$ such that the following conditions are satisfied:
\[
\begin{align*}
&w_1^+(M) + w_1^-(M) = w_1(E) \\
&w_2^+(M) + w_2^-(M) + w_1(E)(pw_2^+(M) + qw_2^-(M)) = w_2(E) \\
&+ \left[ \delta(p,q) + \frac{p(p+1)}{2} + \frac{q(q+1)}{2} \right] w_1(E)^2.
\end{align*}
\]
where:
\[
\delta(p,q) = \begin{cases} 
1 & \text{if } p - q \equiv_8 3 \\
0 & \text{if } p - q \equiv_8 7.
\end{cases}
\]
Let $[\eta]$ be the type of $(S, \gamma)$. In that case, and relative to $[\eta]$, there exists an adapted Spin$^e_\eta(V, h)$ structure $Q(S, \gamma)$ on $(M, g)$, unique up to isomorphism, such that $(S, \gamma)$ is naturally associated to $Q(S, \gamma)$ as a bundle of irreducible Clifford modules, and Clifford multiplication in $S$ is implemented by the morphism of vector bundles $\mathcal{C}: T^*M \otimes S \to S$ defined in equation (T4).

The previous theorem encodes the obstruction to the existence of an irreducible Dirac operator in terms of Karoubi Stiefel-Whitney classes $w_i$ [6] and characterizes explicitly its associated spinor bundle structure. As a corollary to this theorem, we obtain a large class of generically non-spin manifolds that admit real spinor bundles of irreducible type.

**Corollary 0.1.** Let $X$ be a $(2k + 1)$-dimensional manifold which is oriented and spin and let $Y$ be an embedded $(2k - 1)$-dimensional submanifold of $X$.

1. Assume that $2k - 1 \equiv_8 7$ and that $X$ is endowed with a Riemannian metric $g$. Then $(Y, g|_Y)$ admits a Spin$^e_\eta$ structure whose characteristic $O(2)$-bundle $E$ is the orthogonal frame bundle of the normal bundle to $Y$ in $X$.

2. Assume that $2k - 1 \equiv_8 3$ and that $X$ is endowed with a negative Riemannian metric $g$. Then $(Y, g|_Y)$ admits a Spin$^e_\eta$ structure whose characteristic $O(2)$-bundle $E$ is the orthogonal frame bundle of the normal bundle to $Y$ in $X$.

The reader is referred to Proposition 6.2 for the proof of this result. Furthermore, we apply the previous theorem to obtain several families of non-spin products of tori with Grassmanians that admit bundles of irreducible spinors of complex type.

**Theorem 0.2.** Assume that $n + 1 \equiv_4 0$. Then $\text{Gr}_{2,n}$ is stably Spin$^\circ$, namely:

1. For $j \equiv_8 7 - 2n$, the manifold $\text{Gr}^j_{2,n}$ carries a Spin$^\circ$ structure of positive-definite signature with characteristic $O(2)$-bundle given by the orthogonal frame bundle of $\mathcal{L}_{2,n}$.

2. For $j \equiv_8 -(3 + 2n)$, the manifold $\text{Gr}^j_{2,n}$ carries a Spin$^\circ$ structure of negative-definite signature characteristic $O(2)$-bundle given by the orthogonal frame bundle of $\mathcal{L}_{2,n}$.

The reader is referred to Theorem 6.6 for the proof of this result. These results show that non-spin manifolds admitting irreducible Spin$^\circ$ structures are abundant. It would be interesting to study à la Lichnerowicz the irreducible Dirac operator on these non-spin manifolds in order to obtain possible constraints on the existence of metrics with prescribed scalar curvature.

**Organization of the paper.** In Section 1, we discuss the groups Pin$_{1,2}$ and Pin$_{2,0}$ as well as their twisted and untwisted adjoint representations, describing their abstract group-theoretical models as well as certain isomorphic realizations as the groups Spin$_{1,2}$ and Spin$_{3,0}$. In Section 2, we describe the groups Spin$^\circ_\pm(p, q)$ and their real irreducible representations. We also give realizations of Spin$^\circ_\pm(p, q)$ as subgroups of certain higher dimensional Pin and Spin groups, realizations which will be useful later on, and we discuss certain relevant subgroups of these groups. In Section 3, we introduce the notion of Spin$^\circ_\pm$ structures in arbitrary dimension and signature and extract the topological obstructions to their existence in odd dimension. Section 4 considers the case of signatures satisfying the condition $p - q \equiv_8 3, 7$, showing how a specific representation of Spin$^\circ_\pm$ can be used to construct an irreducible real Clifford module for $\text{Cl}_{p,q}$. We also discuss certain natural subspaces associated to such representations. Section 5 discusses elementary real pinor bundles $S$
for \( p - q \equiv 3, 7 \), focusing on their explicit realization in terms of adapted Spin\(^{q} \). In the same section, we also discuss certain sub-bundles of the endomorphism bundle of \( S \) as well as the conditions under which the structure group of \( S \) reduces to various subgroups of the group Spin\(^{q} \). In Section 6 we give several examples of manifolds admitting Spin\(^{p,q} \) structures. Appendix A briefly discusses Spin\(^{c} \) structures in arbitrary dimension and signature, contrasting them with Spin\(^{q} \) structures. Appendix B summarizes for completeness the theory semilinear structures on a vector bundle, a concept which is useful for understanding the properties of real pinor bundles associated to Spin\(^{q} \) structures.

**Relation to the spinorial structure used by N. Nakamura.** In arbitrary dimension and arbitrary signature \((p,q)\), the groups Spin\(^{p,q} \) are (non-central) extensions of the group \( \mathbb{Z}_{2} \) by the group \( Spin^{p,q} = \text{Spin}_{p,q} \cdot U(1) \). For positive signature \( q = 0 \), the groups Spin\(^{1,p} \) have appeared before in \([10, 11]\), where they were used to define so-called Spin\(^{c} \) structures and where Spin\(^{c} \) structures were employed to define and study a certain variant of monopole equations in four dimensions. A Spin\(^{c} \) structure is defined similarly to a Spin\(^{q} \) structure, but using a different representation \( \lambda : \text{Spin}_{p,q} \to \text{SO}(p,q) \). The latter is constructed from the adjoint representation of Spin\(^{p,q} \) and from the twisted adjoint representation of Pin\(_{0,2} \) or Pin\(_{2,0} \) and covers only the special pseudo-orthogonal group \( \text{SO}(p,q) \) — unlike the representation \( \hat{\lambda} \) mentioned above, which covers the full pseudo-orthogonal group. Due to this difference, Spin\(^{c} \) structures can exist only when \((M, g)\) is orientable and lead to a notion of “real pinors” which differs from the one considered in the present paper: in signature \( p - q \equiv 3, 7 \), a bundle \( S \) of simple modules over \( \text{Cl}(M,g) \) is always associated to a Spin\(^{q} \) structure and not to a Spin\(^{c} \) or Spin\(^{c} \) structure.

**Notations and conventions.** We use the conventions and notations of \([8]\).

1. **Abstract models for certain Clifford and pin groups**

   We start by discussing certain groups which will be relevant for the description of Spin\(^{p,q} \).

   **1.1. The abstract groups \( O_{2}(\alpha) \) and \( O_{2}^{c}(\alpha) \).** Let \( \alpha \in \{-1, 1\} \) be a sign factor.

   **Definition 1.1.** Let \( O_{2}^{c}(\alpha) \) be the non-compact non-Abelian Lie group with underlying set \( \mathbb{C}^{\times} \times \mathbb{Z}_{2} \) and composition given by:

   \[
   (z_{1}, \hat{0})(z_{2}, 0) = (z_{1}z_{2}, \hat{0}) , \quad (z_{1}, \hat{0})(0, 1) = (z_{1}z_{2}, \hat{1}) ,
   \]

   \[
   (z_{1}, \hat{1})(z_{2}, 0) = (z_{1}z_{2}, 0) , \quad (z_{1}, \hat{1})(0, 1) = (z_{1}z_{2}, 1) ,
   \]

   where \( \mathbb{Z}_{2} = \{0, \hat{1}\} \). The unit of \( O_{2}^{c}(\alpha) \) is given by \( 1 \equiv (1, \hat{0}) \). The square norm is the group morphism \( N : O_{2}^{c}(\alpha) \to \mathbb{R}_{>0} \) given by:

   \[
   N(z,t) \overset{\text{def}}{=} |z|^{2} , \quad \forall (z,t) \in \mathbb{C}^{\times} \times \mathbb{Z}_{2} .
   \]

   We define \( O_{2}(\alpha) \overset{\text{def}}{=} \ker N \) to be the compact non-Abelian Lie subgroup of \( O_{2}^{c}(\alpha) \) with underlying set \( U(1) \times \mathbb{Z}_{2} \).

   The group \( \mathbb{C}^{\times} \) embeds into \( O_{2}^{c}(\alpha) \) as the non-central subgroup \( \mathbb{C}^{\times} \times \{0\} \), so we identify \( \hat{1} \in \mathbb{C}^{\times} \) with the element \((1, \hat{0}) \in O_{2}^{c}(\alpha)\) and \(-1 \in \mathbb{C}^{\times} \) with the element \((-1, \hat{0}) \). The conjugation element \( c \overset{\text{def}}{=} (1, \hat{1}) \in O_{2}(\alpha) \) satisfies \( c^{2} = \alpha 1 \) and \( c^{-1} = \alpha c = (\alpha, \hat{1}) \). This element generates the subgroup:

   \[
   \Gamma_{\alpha} \simeq \left\{ \begin{array}{ll}
   \mathbb{Z}_{2} & \text{if } \alpha = +1 \\
   \mathbb{Z}_{4} & \text{if } \alpha = -1 
   \end{array} \right. .
   \]

   The element \( c \) and the subgroup \( \mathbb{C}^{\times} \) generate \( O_{2}^{c}(\alpha) \), while \( c \) and \( U(1) \) generate \( O_{2}(\alpha) \). We have:

   \[
   \text{Ad}(c)(x) = K(x) , \quad \forall x \in O_{2}^{c}(\alpha) ,
   \]

   where \( K : O_{2}^{c}(\alpha) \to O_{2}^{c}(\alpha) \) is the conjugation automorphism, given by:

   \[
   K(z,t) = (\bar{z}, t) , \quad \forall z \in \mathbb{C}^{\times} , \quad \forall t \in \mathbb{Z}_{2} .
   \]
Notice that $K^2 = \text{id}_{O^c_2} \alpha$ and $K(c) = c$. The centers of $O_2^c(\alpha)$ and $O_2(\alpha)$ are given by $Z(O_2^c(\alpha)) = \mathbb{R} \times 1$ and $Z(O_2(\alpha)) = \{-1, 1\} \simeq \mathbb{Z}_2$, respectively. The fixed point set of the conjugation automorphism in $O_2(\alpha)$ is the subgroup $\Gamma_\alpha'$ generated by $-1$ and $c$, which is given by:

$$
\Gamma_\alpha' = \begin{cases} 
\mathbb{G}_2 \times \Gamma_+ \simeq D_2 & \text{if } \alpha = +1 \\
\Gamma_- \simeq \mathbb{Z}_2 & \text{if } \alpha = -1
\end{cases}.
$$

Here, $\mathbb{G}_2 = \{-1, 1\} \simeq \mathbb{Z}_2$ denotes the multiplicative group of second order roots of unity, while $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ denotes the Klein group (the dihedral group with four elements).

**Definition 1.2.** The **generalized determinant** is the morphism $\eta_\alpha : O_2(\alpha) \to \mathbb{G}_2$ defined through:

$$
\eta_\alpha(z,t) \overset{\text{def}}{=} (-1)^t, \forall (z,t) \in O_2(\alpha).
$$

This induces a $\mathbb{Z}_2$-grading of $O_2(\alpha)$ whose homogeneous pieces are the connected components of $O_2(\alpha)$:

$$
O_2^+(\alpha) \overset{\text{def}}{=} \ker \eta_\alpha = \{(z,0)|z \in U(1)\} = U(1)1 = \text{SO}(2)1
$$

$$
O_2^-(\alpha) \overset{\text{def}}{=} \eta_\alpha^{-1}(\{-1\}) = \{(z,1)|z \in U(1)\} = U(1)c = \text{SO}(2)c
$$

and gives a short exact sequence:

$$
1 \to U(1) \to O_2(\alpha) \xrightarrow{\eta_\alpha} \mathbb{G}_2 \to 1.
$$

For any $u = (z,t) \in O_2(\alpha)$, let $-u \overset{\text{def}}{=} (-1)u = (-z,t) \in O_2(\alpha)$. Notice that $\eta_\alpha(-u) = \eta_\alpha(u)$.

**Proposition 1.3.** The short exact sequence (2) is a non-central extension of $\mathbb{G}_2$ by $U(1)$. Moreover, by exactness of (2) we obtain:

1. For $\alpha = +1$, the sequence (2) splits and we have $O_2(+1) \simeq O(2)$.

2. For $\alpha = -1$, the sequence (2) presents $O_2(-1)$ as a non-split extension of $\mathbb{G}_2 \simeq \mathbb{Z}_2$ by $U(1)$. In particular, we have $O_2(-1) \not\simeq O(2)$.

**Proof.** 1. For $\alpha = +1$, the element $c$ has order two and the map $\psi : \mathbb{G}_2 \to O_2(+1)$ given by $\psi(1) = 1$ and $\psi(-1) = c$ is a group morphism which right-splits (2). Thus $O_2(+1) \simeq U(1) \rtimes \psi \mathbb{G}_2 \simeq O(2)$.

Here, the last isomorphism follows by noticing that the map $\Phi_0 : U(1) \rtimes \psi \mathbb{G}_2 \to O(2)$ given by:

$$
\Phi_0(e^{i\theta}, +1)(z) = e^{i\theta}z, \Phi_0(e^{i\theta}, -1)(z) = e^{i\theta}z, \quad z \in \mathbb{C},
$$

induces an isomorphism of groups $U(1) \rtimes \psi \mathbb{G}_2 \simeq O(2)$.

2. For $\alpha = -1$, the sequence (2) does not split. Indeed, a right-splitting morphism $\psi : \mathbb{G}_2 \to O_2(-1)$ would give an order two element $x \overset{\text{def}}{=} \psi(-1) \in O_2(-1)$ which must be of the form $x = (z,1)$ in order for condition $\eta_- \circ \psi_- = \text{id}_{\mathbb{G}_2}$ to be satisfied. The order two condition $x^2 = 1$ gives then $\{-z\} \simeq \mathbb{Z}_2$ (since $\alpha = -1$), a contradiction.

□

**Remark 1.4.** Any reflection $C$ of $\mathbb{R}^2$ determines two isomorphisms of groups $\Phi_C^{(\pm)} : O_2(\pm) \simeq O(2)$ given by:

$$
\Phi_C^{(+)}(e^{i\theta}, 0) = R(\pm \theta), \quad \Phi_C^{(-)}(e^{i\theta}, 1) = R(\pm \theta)C,
$$

where $\theta \in \mathbb{R}$ and:

$$
R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \in \text{SO}(2)
$$

is the counterclockwise rotation of $\mathbb{R}^2$ by angle $\theta \mod 2\pi$ with respect to its canonical orientation (that orientation in which the canonical basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is positive):

$$
R(\theta)e_1 = \cos(\theta)e_1 + \sin(\theta)e_2 \quad R(\theta)e_2 = -\sin(\theta)e_1 + \cos(\theta)e_2.
$$
Notice that $R(-\theta)$ is the clockwise rotation by the same angle. We have:

$$
\Phi_C^{(\pm)}(c) = C, \quad \Phi_C^{(\pm)}(-1) = -I_2.
$$

The isomorphisms $\Phi_C^{(\pm)}$ map the $\mathbb{Z}_2$-grading of $O_2(+) \triangleright$ into the $\mathbb{Z}_2$-grading of $O(2)$ given by:

$$
O^+(2) \overset{\text{def}}{=} SO(2), \quad O^-(2) \overset{\text{def}}{=} SO(2)C,
$$

hence $\Phi_C^{(\pm)}$ can be viewed as isomorphisms of $\mathbb{Z}_2$-graded groups. These two isomorphisms are related by the conjugation automorphism of $O_2(+)$: \[ \Phi^{(-)}_C = \Phi^{(+)}_C \circ K \]

and we have:

$$
\eta^+ = \det \circ \Phi^{\pm}_C.
$$

In particular, we can consider the reflection:

$$
C_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in O(2)
$$

with respect to the horizontal axis of $\mathbb{R}^2$ (i.e. the real axis of $\mathbb{C}$) to define:

$$
\Phi^0_0 \overset{\text{def}}{=} \Phi^{(\pm)}_C |_{C_0}.
$$

Then the isomorphism $\Phi_0$ of equation (3) corresponds to:

$$
\Phi_0 = \Phi^{(+)}_C |_{C_0}.
$$

**Definition 1.5.** The **squaring morphism** is the surjective group morphism $\sigma : O_2(\alpha) \to O_2(\alpha) \overset{\text{def}}{=} (z^2, t), \quad \forall(z, t) \in O_2(\alpha) \, \forall \alpha \in \mathbb{Z}_2.

Notice the relations:

$$
\sigma_\alpha(c) = c, \quad \eta_\alpha = \eta^+ \circ \sigma_\alpha = \det \circ \Phi^{(\pm)}_C \circ \sigma_\alpha.
$$

Since $\ker \sigma_\alpha = \{-1, 1\}$, we have a short exact sequence:

$$
1 \to \mathbb{Z}_2 \to O_2(\alpha) \overset{\sigma_\alpha}{\to} O_2(\alpha) \to 1.
$$

In particular, $O_2(\alpha) \gg O_2(-) \gg O_2(\alpha)$ are inequivalent central extensions of $O(2)$ by $\mathbb{Z}_2$. Notice that $\sigma_\alpha$ can be viewed as a morphism of $\mathbb{Z}_2$-graded groups.

1.2. **Realization of $O_2^\pm(\alpha)$ as Clifford groups.** Let:

$$
\text{Cl}_2(\alpha) \overset{\text{def}}{=} \begin{cases} 
\text{Cl}_2.0 & \text{if } \alpha = +1 \\
\text{Cl}_{0,2} & \text{if } \alpha = -1.
\end{cases}
$$

Then $\text{Cl}_2(\alpha)$ has generators $e_1, e_2$ with relations:

$$
e_1^2 = e_2^2 = \alpha, \quad e_1 e_2 = -e_2 e_1.
$$

Let us define $e_3 \overset{\text{def}}{=} e_1 e_2 \in \text{Cl}_2^+(\alpha)$, which satisfies $e_3^2 = -1$. The elements $e_1, e_2$ and $e_3$ mutually anticommute and span $\text{Cl}_2(\alpha)$ over $\mathbb{R}$. The algebra $\text{Cl}_2(\alpha) = \text{Cl}_2.0$ is isomorphic with the algebra $\mathbb{P}$ of split quaternions (and hence with the matrix algebra $\text{Mat}(2, \mathbb{R})$), while $\text{Cl}_2(-)$ is isomorphic with the quaternion algebra $\mathbb{H}$. Set $J \overset{\text{def}}{=} e_3$ and $D \overset{\text{def}}{=} e_1$. Then $J$ and $D$ satisfy the relations:

$$
J^2 = -1, \quad D^2 = \alpha, \quad JD = DJ.
$$
Notice that $J$ is even while $D$ is odd with respect to the canonical $\mathbb{Z}_2$-grading of $\text{Cl}_2(\alpha)$. We have $Z(\text{Cl}_2(\alpha)) = \mathbb{R}$, hence the extended Clifford group\(^2\) of $\text{Cl}_2(\alpha)$ coincides with its ordinary Clifford group, which we denote by:

$$G_2(\alpha) \overset{\text{def}}{=} \begin{cases} G_{2,0} & \text{if } \alpha = +1 \\ G_{0,2} & \text{if } \alpha = -1 \end{cases}.$$  

We have $\text{Cl}_2^+(\alpha) = \mathbb{R} \oplus \mathbb{J}\mathbb{C}$ and $\text{Cl}_2^-(\alpha) = \mathbb{R}e_1 \oplus \mathbb{R}e_2 = \mathbb{R}D \oplus \mathbb{R}e_2 = \text{Cl}_2^+(\alpha)D$.

**Proposition 1.6.** There exists an isomorphism of $\mathbb{Z}_2$-graded groups $\varphi_\alpha^C : O_2^C(\alpha) \rightarrow G_2(\alpha)$ which satisfies:

\begin{equation}
\varphi_\alpha^C(i) = J \overset{\text{def}}{=} e_1 e_2 \quad \text{and} \quad \varphi_\alpha^C(c) = D \overset{\text{def}}{=} e_1 .
\end{equation}

**Proof.** It is easy to see that $G_2(\alpha) = \text{Cl}_2^+(\alpha)^{\times} \sqcup (\text{Cl}_2^+(\alpha)^{\times}D)$ and $\text{Cl}_2^+(\alpha)^{\times} \simeq \mathbb{C}^\times$. The map $\varphi_\alpha^C : O_2^C(\alpha) \rightarrow G_2(\alpha)$ given by:

$$\varphi_\alpha^C(x + iy, \hat{0}) = x + yJ , \quad \varphi_\alpha^C(x + iy, \hat{1}) = (x + yJ)D \text{ for } \imath = x + iy \in \mathbb{C}^\times$$

is an isomorphism of $\mathbb{Z}_2$-graded groups which satisfies (8). \[\square\]

1.3. **Realization of $O_2(\alpha)$ as Pin groups.** Let:

$$\text{Pin}_2(\alpha) \overset{\text{def}}{=} \begin{cases} \text{Pin}_{2,0} & \text{if } \alpha = +1 \\ \text{Pin}_{0,2} & \text{if } \alpha = -1 \end{cases},$$

denote the Pin group of $\text{Cl}_2(\alpha)$. This group is $\mathbb{Z}_2$-graded by the decomposition $\text{Pin}_2(\alpha) = \text{Spin}_2(\alpha) \sqcup \text{Spin}_2(\alpha)D$, where $\text{Spin}_2(\alpha) = \text{Spin}_{2,0} = \text{Spin}_{0,2}$ and $\text{Spin}_2(\alpha)D = \text{Pin}_{2}^-(\alpha)$.

**Proposition 1.7.** We have $\varphi_\alpha^C(O_2^C(\alpha)) = \text{Pin}_2(\alpha)$, hence $\varphi_\alpha^C$ restricts to an isomorphism of $\mathbb{Z}_2$-graded groups $\varphi_\alpha : O_2(\alpha) \rightarrow \text{Pin}_2(\alpha)$ which satisfies:

\begin{equation}
\varphi_\alpha(i) = J \overset{\text{def}}{=} e_1 e_2 \quad \text{and} \quad \varphi_\alpha(c) = D \overset{\text{def}}{=} e_1 .
\end{equation}

**Proof.** We have:

$$\text{Pin}_2(\alpha) \overset{\text{def}}{=} \ker(\hat{N} : G_2(\alpha) \rightarrow \mathbb{R}_{>0}),$$

where $\hat{N}$ is the twisted Clifford norm of $\text{Cl}_2(\alpha)$. For $\alpha = +1$, the twisted reversion $\tilde{r}$ of $\text{Cl}_2(+)$ is $\text{Cl}_2,0 \simeq \mathbb{P}$ coincides with the conjugation of split quaternions and hence the twisted Clifford norm $\hat{N}$ coincides with the split quaternion modulus:

$$\hat{N}(q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3) = q_0^2 + q_3^2 - q_1^2 - q_2^2 .$$

This implies $\varphi_\alpha(O_2(+)) = \text{Pin}_2(+) = \text{U}(1) \sqcup \text{U}(1)D$. Notice that $\hat{N}(\text{Cl}_{2,0}) \subset \mathbb{R}$ and that $\hat{N}$ is positive definite on $\text{Cl}_{2,0}^+$ and negative definite on $\text{Cl}_{2,0}^-.$

For $\alpha = -1$, the twisted reversion $\tilde{r}$ of $\text{Cl}_2(-)$ is $\text{Cl}_{0,2} \simeq \mathbb{H}$ coincides with quaternion conjugation. Hence the twisted Clifford norm $\hat{N}$ coincides with the squared quaternion norm:

$$\hat{N}(q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3) = q_0^2 + q_2^2 + q_1^2 + q_3^2 .$$

This implies $\varphi_\alpha(O_2(-)) = \text{Pin}_2(-) = \text{U}(1) \sqcup \text{U}(1)D$. Notice that $\hat{N}(\text{Cl}_{0,2}) \subset \mathbb{R}$, $\hat{N}(J) = 1$ and $\hat{N}(D) = -\alpha$. \[\square\]

**Remark 1.8.** Setting:

\begin{equation}
z(\theta) \overset{\text{def}}{=} \varphi_\alpha(e^{i \theta}, \hat{0}) = \cos(\theta) + \sin(\theta)J ,
\end{equation}

we have:

\begin{equation}
\varphi_\alpha(e^{i \theta}, \hat{1}) = z(\theta)D
\end{equation}

and:

$$\text{Spin}_2(\alpha) = \{z(\theta) | \theta \in \mathbb{R}\} \simeq \text{U}(1), \quad \text{Pin}_2^-(\alpha) = \text{Spin}_2(\alpha)D .$$

\(^2\)See Definition 1.7 in reference [8].
Let $\text{Ad}^{(2)}_0 : \text{Pin}_2(\alpha) \to O(2)$ and $\tilde{\text{Ad}}^{(2)}_0 : \text{Pin}(\alpha) \to O(2)$ denote the vector and twisted vector representations of $\text{Pin}_2(\alpha)$, viewed as morphisms of $\mathbb{Z}_2$-graded groups. Notice that $\ker \text{Ad}^{(2)}_0 = \ker \tilde{\text{Ad}}^{(2)}_0 = \{-1, 1\}$ and that $\det \circ \text{Ad}^{(2)}_0 = \det \circ \tilde{\text{Ad}}^{(2)}_0$.

**Proposition 1.9.** The untwisted vector representation of $\text{Pin}_2(\alpha)$ agrees with the squaring morphism $\sigma_\alpha$ through the isomorphisms $\varphi_{\alpha}$ and $\Phi^{(-\alpha)}_0$:

$$\text{Ad}^{(2)}_0 \circ \varphi_{\alpha} = \Phi^{(-\alpha)}_0 \circ \sigma_\alpha.$$  

This gives the following commutative diagram of morphisms of $\mathbb{Z}_2$-graded groups, where we also indicate the images of $c \in O_2(\alpha)$ through the various maps:

$$
\begin{array}{ccc}
 c \in O_2(\alpha) & \xrightarrow{\varphi_{\alpha}} & \text{Pin}_2(\alpha) \ni D \\
 \sigma_{\alpha} \downarrow & & \downarrow \text{Ad}^{(2)}_0 \\
 c \in O_2(+) & \xrightarrow{\Phi^{(-\alpha)}_0} & O(2) \ni C_0
\end{array}
$$

Moreover, the generalized determinant agrees with the grading morphism $\det \circ \text{Ad}^{(2)}_0$ of $\text{Pin}_2(\alpha)$:

$$\text{det} \circ \text{Ad}^{(2)}_0 \circ \varphi_{\alpha} = \eta_{\alpha},$$

i.e. we have a commutative diagram of morphisms of $\mathbb{Z}_2$-graded groups:

$$
\begin{array}{ccc}
 c \in O_2(\alpha) & \xrightarrow{\varphi_{\alpha}} & \text{Pin}_2(\alpha) \ni D \\
 \eta_{\alpha} \downarrow & & \downarrow \text{Ad}^{(2)}_0 \\
 -1 \in \mathbb{Z}_2 & \xrightarrow{\text{det}} & O(2) \ni C_0
\end{array}
$$

**Proof.** Since $J = e_1 e_2$ anticommutes with $e_1$ and $e_2$, we have $\text{Ad}^{(2)}_0(z(\theta))(e_k) = z(\theta)e_k z(-\theta) = z(\theta)^2 e_k$ for $k = 1, 2$. Using the relations $Je_1 = -ae_2$ and $Je_2 = ae_1$, this gives:

$$
\begin{align*}
\text{Ad}^{(2)}_0(z(\theta))(e_1) &= \cos(2\theta)e_1 - \alpha \sin(2\theta)e_2 = \cos(2\theta)e_1 - \sin(2\theta)e_2 \\
\text{Ad}^{(2)}_0(z(\theta))(e_2) &= \alpha \sin(2\theta)e_1 + \cos(2\theta)e_2 = \sin(2\theta)e_1 + \cos(2\theta)e_2
\end{align*}
$$

and hence $\text{Ad}^{(2)}_0(z(\theta)) = R(-2\alpha \theta)$. Since $D = e_1$, we have $\text{Ad}_0(D)(e_1) = e_1$ and $\text{Ad}^{(2)}_0(D)(e_2) = -e_2$, i.e. $\text{Ad}^{(2)}_0(D) = C_0$. Thus $\text{Ad}^{(2)}_0(z(\theta)D) = R(-2\alpha \theta) \circ C_0$. Relation (12) now follows from (4), (10) and (11), while relation (13) follows from (12) and (7).

### 1.4. Realization of $O_2(\alpha)$ as spin groups.

Let:

$$\text{Cl}_3(\alpha) \overset{\text{def}}{=} \begin{cases} 
\text{Cl}_{2,1} & \text{if } \alpha = +1 \\
\text{Cl}_{0,3} & \text{if } \alpha = -1
\end{cases}, \quad \text{Spin}_3(\alpha) \overset{\text{def}}{=} \begin{cases} 
\text{Spin}_{2,1} & \text{if } \alpha = +1 \\
\text{Spin}_{0,3} & \text{if } \alpha = -1
\end{cases}
$$

and let $e_1, e_2, e_3$ denote the canonical basis of $\mathbb{R}^3$. We have:

$$e_1^2 = \alpha, \quad e_2^2 = \alpha, \quad e_3^2 = -1.$$ 

There exists a unique unital monomorphism of $\mathbb{R}$-algebras $\hat{s}_\alpha : \text{Cl}_2(\alpha) \to \text{Cl}_3(\alpha)$ which satisfies $\hat{s}_\alpha(v) = ve_3$ for all $v \in \mathbb{R}^2$. Thus $\hat{s}_\alpha(e_1) = e_1 e_3$, $\hat{s}_\alpha(e_2) = e_2 e_3$ and $\hat{s}_\alpha(e_1 e_2) = e_1 e_2$. We have $\hat{s}_\alpha(\text{Cl}_2(\alpha)) = \text{Cl}_3^+(\alpha)$ and $\hat{s}_\alpha(\text{Pin}_2(\alpha)) = \text{Spin}_3(\alpha)$, where $\text{Cl}_3^+(\alpha)$ is generated by $e_1 e_3$ and $e_2 e_3$ due to the relation $(e_1 e_3)^2 = e_1 e_2$. The morphism $\hat{s}_\alpha$ takes $J = e_1 e_2$ and $D = e_1$ respectively into $J^+ \overset{\text{def}}{=} J$ and $D^+ \overset{\text{def}}{=} e_1 e_3 = De_3$. In particular, we have $\hat{s}_\alpha|_{\text{Spin}_3(\alpha)} = \text{id}_{\text{Spin}_3(\alpha)}$. The group $O(2)$ embeds into the group:

$$\text{SO}_3(\alpha) \overset{\text{def}}{=} \begin{cases} 
\text{SO}(2,1) & \text{if } \alpha = +1 \\
\text{SO}(0,3) & \text{if } \alpha = -1
\end{cases}
$$
through the injective morphisms:

\[
\Sigma_\alpha(R) \overset{\text{def.}}{=} \begin{bmatrix} \operatorname{det} R & 0 \\ 0 & \operatorname{det} R \end{bmatrix},
\]

whose image equals:

\[
\Sigma_\alpha(O(2)) = S[O(2) \times \mathbb{G}_2].
\]

We have:

\[
\Sigma_\alpha(C_0) = C_0 \overset{\text{def.}}{=} \begin{bmatrix} -C_0 & 0 \\ 0 & -1 \end{bmatrix},
\]

where \(-C_0 \in O^-(2)\). Let \(\operatorname{Ad}_0^{(3)} : \operatorname{Spin}_3(\alpha) \to \operatorname{SO}_3(\alpha)\) denote the vector representation of \(\operatorname{Spin}_3(\alpha)\).

**Proposition 1.10.** The restriction of \(\hat{s}_\alpha\) induces an isomorphism of groups \(s_\alpha : \operatorname{Pin}_2(\alpha) \to \operatorname{Spin}_3(\alpha)\). Moreover, the vector representation of \(\operatorname{Spin}_3(\alpha)\) agrees with the untwisted vector representation of \(\operatorname{Pin}_2(\alpha)\) through the morphisms \(s_\alpha\) and \(\Sigma_\alpha\):

\[
\operatorname{Ad}_0^{(3)} \circ s_\alpha = \Sigma_\alpha \circ \operatorname{Ad}_0^{(2)},
\]

giving the commutative diagram:

\[
\begin{array}{ccc}
D \in \operatorname{Pin}_2(\alpha) & \xrightarrow{s_\alpha} & \operatorname{Spin}_3(\alpha) \ni D' \\
\downarrow \operatorname{Ad}_0^{(2)} & & \downarrow \operatorname{Ad}_0^{(3)} \\
C_0 \in O(2) & \xrightarrow{\Sigma_\alpha} & \operatorname{SO}_3(\alpha) \ni C_0',
\end{array}
\]

where we also indicates the images of \(D\) through the various maps.

**Proof.** For any \(a \in \operatorname{Pin}_2(\alpha)\), we have \(s_\alpha(a) = \hat{s}_\alpha(a) = a\) and \(s_\alpha(aD) = s_\alpha(a)s_\alpha(D) = ae_1e_3 = aDe_3\). Thus:

\[
\operatorname{Ad}_0^{(3)}(s_\alpha(a)) = \operatorname{Ad}_0^{(3)}(a), \quad \operatorname{Ad}_0^{(3)}(s_\alpha(aD)) = \operatorname{Ad}_0^{(3)}(a)\operatorname{Ad}_0^{(3)}(e_1e_3) = \operatorname{Ad}_0^{(3)}(aD)\operatorname{Ad}_0^{(3)}(e_3)
\]

and hence:

\[
\operatorname{Ad}_0^{(3)}(s_\alpha(a)) = \begin{bmatrix} \operatorname{Ad}_0^{(2)}(a) & 0 \\ 0 & 1 \end{bmatrix} = \Sigma_\alpha(\operatorname{Ad}_0^{(2)}(a))
\]

and:

\[
\operatorname{Ad}_0^{(3)}(s_\alpha(aD)) = \begin{bmatrix} -\operatorname{Ad}_0^{(2)}(aD) & 0 \\ 0 & -1 \end{bmatrix} = \Sigma_\alpha(\operatorname{Ad}_0^{(2)}(aD)),
\]

because \(\operatorname{Ad}_0^{(3)}(e_3) = e_3\), \(\operatorname{Ad}_0^{(3)}(e_1e_3)(e_3) = e_3\) and \(\operatorname{Ad}_0^{(3)}(e_3)(e_k) = -e_k\) for \(k = 1, 2\) while \(\operatorname{det} \operatorname{Ad}_0^{(2)}(a) = +1\) and \(\operatorname{det} \operatorname{Ad}_0^{(2)}(aD) = -1\). Notice that:

\[
\operatorname{Ad}_0^{(3)}(D') = \operatorname{Ad}_0^{(3)}(s_\alpha(D)) = \begin{bmatrix} -C_0 & 0 \\ 0 & -1 \end{bmatrix} = C_0',
\]

since \(\operatorname{Ad}_0^{(2)}(D) = C_0\).

Composing \(\varphi_\alpha\) and \(s_\alpha\), respectively \(\Phi_0^{(-\alpha)}\) and \(\Sigma_\alpha\) gives morphisms of groups:

\[
\psi_\alpha \overset{\text{def.}}{=} s_\alpha \circ \varphi_\alpha : O_2(\alpha) \to \operatorname{Spin}_3(\alpha)
\]

\[
\Psi_\alpha \overset{\text{def.}}{=} \Sigma_\alpha \circ \Phi_0^{(-\alpha)} : O_2(+) \to \operatorname{SO}_3(\alpha).
\]
The situation is summarized in the commutative diagrams:

\[ \begin{array}{ccc}
O_2(\alpha) & \xrightarrow{\sim} & \text{Pin}_2(\alpha) \\
\sigma_\alpha & \xrightarrow{\sim} & \text{Spin}_2(\alpha) \\
O_2(+) & \xrightarrow{\sim} & \text{O}(2) \\
\psi_\alpha & \xrightarrow{\sim} & \text{SO}_3(\alpha)
\end{array} \]

and:

\[ \begin{array}{ccc}
c \in O_2(\alpha) & \xrightarrow{\psi_\alpha} & \text{Spin}_2(\alpha) \ni D' \\
\sigma_\alpha & \xrightarrow{\sim} & \text{Ad}_{\alpha}^{(2)} \\
c \in O_2(+) & \xrightarrow{\psi_\alpha} & \text{SO}_3(\alpha) \ni C_0'
\end{array} \]

2. The groups Spin\(_\alpha^o\)

Let \((V, h)\) be a real quadratic space of signature \((p, q)\) and dimension \(d = p + q\). Thus \(V\) is an \(\mathbb{R}\)-vector space of dimension \(d\) and \(h : V \times V \to \mathbb{R}\) is an \(\mathbb{R}\)-bilinear symmetric form defined on \(V\) and having signature \((p, q)\), where \(p\) and \(q\) respectively count the numbers of positive and negative eigenvalues.

**Definition 2.1.** Define:

\[ \text{Spin}_\alpha^o(V, h) \overset{\text{def}}{=} \text{Spin}(V, h) \cdot \text{Pin}_2(\alpha) \overset{\text{def}}{=} \{\text{Spin}(V, h) \times \text{Pin}_2(\alpha)\}/\{1, 1\}. \]

The unit of Spin\(_\alpha^o(V, h)\) is given by \(1 \equiv [1, 1] = [-1, -1]\), while the twisted unit is the element:

\[ \hat{1} \overset{\text{def}}{=} [1, -1] = [-1, 1], \]

which satisfies \(\hat{1}^2 = 1\) and generates the center:

\[ \mathbb{Z}_2 \overset{\text{def}}{=} Z(\text{Spin}_\alpha^o(V, h)) = \{\hat{1}, 1\} \simeq \mathbb{Z}_2, \]

of Spin\(_\alpha^o(V, h)\). The groups Spin\((V, h)\) and Pin\(_2(\alpha)\) identify with the following subgroups of Spin\(_\alpha^o(V, h)\):

\[ \hat{\text{Spin}}(V, h) \overset{\text{def}}{=} \{[a, 1] \in \text{Spin}_\alpha^o(V, h) \mid a \in \text{Spin}(V, h)\}, \]

\[ \hat{\text{Pin}}_2(\alpha) \overset{\text{def}}{=} \{[1, b] \in \text{Spin}_\alpha^o(V, h) \mid b \in \text{Pin}_2(\alpha)\}. \]

This gives the decomposition:

\[ \text{Spin}_\alpha^o(V, h) = \hat{\text{Spin}}(V, h) \hat{\text{Pin}}_2(\alpha). \]

We have \(\hat{\text{Spin}}(V, h) \cap \hat{\text{Pin}}_2(\alpha) = \mathbb{Z}_2\) and \(\hat{1} \in \hat{\text{Spin}}(V, h)\). Using the isomorphism \(\varphi_\alpha\) of Proposition 1.7 to identify \(O_2(\alpha)\) with \(\text{Pin}_2(\alpha)\), we obtain an isomorphism:

\[ \text{Spin}_\alpha^o(V, h) \simeq \text{Spin}_\alpha^o(V, h) \overset{\text{def}}{=} \text{Spin}(V, h) \cdot O_2(\alpha) \overset{\text{def}}{=} \{\text{Spin}(V, h) \times O_2(\alpha)\}/\{1, 1\}. \]

Since \(U(1) = \text{SO}(2)\) is a non-central subgroup of \(O_2(\alpha)\), it embeds as the non-central subgroup of \(\text{Spin}_\alpha^o(V, h)\) consisting of the elements \([1, z] = [-1, -z]\) with \(z \in U(1)\).
2.1. Various subgroups. The element $\hat{D} \overset{\text{def.}}{=} [1, D] \in \text{Spin}^o(V, h)$ satisfies:

$$\hat{D}^2 = [1, \alpha] = [\alpha, 1] = \begin{cases} 1 & \text{if } \alpha = +1 \\ 1 & \text{if } \alpha = -1 \end{cases}$$

and generates a cyclic subgroup:

$$\Gamma_{\alpha, \alpha} \simeq \left\{ \mathbb{Z}_2 \right\}$$

which is neither normal nor central. The group $\text{Spin}^c(V, h) \overset{\text{def.}}{=} \text{Spin}(V, h) \cdot \text{Spin}_2.0 = \text{Spin}(V, h) \cdot \text{U}(1)$ embeds as the following normal subgroup of $\text{Spin}^c(V, h)$:

$$\text{Spin}^c(V, h) = \{ [a, z] | a \in \text{Spin}(V, h), \ z \in \text{Spin}_2 \}.$$

In particular, $\text{Spin}^o_\alpha(V, h)$ contains the normal $\text{U}(1)$ subgroup:

$$U = \{ z(\theta) = e^{\theta J} | \theta \in \mathbb{R} \} \subset \text{Spin}^c(V, h) \subset \text{Spin}^o(V, h).$$

Writing $\text{Pin}_2(\alpha) = \text{Spin}_2 \sqcup \text{Spin}_2 D = \text{U}(1) \sqcup \text{U}(1) D$ gives the decomposition:

$$\text{Spin}^o_\alpha(V, h) = \text{Spin}^c(V, h) \sqcup \text{Spin}^c(V, h) \hat{D}.$$

We have a short exact sequence:

$$1 \longrightarrow \text{Spin}^c(V, h) \longrightarrow \text{Spin}^o_\alpha(V, h) \overset{\tilde{\eta}_\alpha}{\longrightarrow} \mathbb{Z}_2 \longrightarrow 1,$$

where $\tilde{\eta}_\alpha$ is the $\mathbb{Z}_2$-grading morphism given by:

$$\tilde{\eta}_\alpha([a, b]) \overset{\text{def.}}{=} \eta_\alpha(b), \ \forall a \in \text{Spin}(V, h), \ \forall b \in \text{Pin}_2(\alpha).$$

**Proposition 2.2.** We have $\text{Spin}^o_\alpha(V, h) \simeq \text{Spin}^c(V, h) \rtimes_\xi \mathbb{Z}_2$, where $\xi : \mathbb{Z}_2 \rightarrow \text{Aut}(\text{Spin}^c(V, h))$ is the group morphism given by $\xi(0) = \text{id}_{\text{Spin}^c(V, h)}$ and:

$$\xi(1)([a, z]) = [a, \bar{z}] \ \forall \ [a, z] \in \text{Spin}^c(V, h) \ (a \in \text{Spin}(V, h), z \in \text{U}(1)).$$

**Proof.** In order to show that $\text{Spin}^o_\alpha(V, h) \simeq \text{Spin}^c(V, h) \rtimes_\xi \mathbb{Z}_2$, it suffices to show that the short exact sequence (23) splits on the right. Such a splitting $R_\xi : \mathbb{Z}_2 \rightarrow \text{Spin}^o_\alpha(V, h)$ is given by:

$$R_\xi(0) = [1, 1], \quad R_\xi(1) = [1, D].$$

The morphism $R_\xi$ in turn defines a morphism:

$$\xi : \mathbb{Z}_2 \rightarrow \text{Aut}(\text{Spin}^c(V, h)),$$

given by the adjoint action\(^3\) of the image $\text{Im}R_\xi$ of $R_\xi$ in $\text{Spin}^o_\alpha$. It is easy to see that the adjoint action of $[1, D]$ is by complex conjugation of $z$, where $[a, z] \in \text{Spin}^c(V, h)$.

**Remark 2.3.** The right-splitting used in the proof of Proposition 2.2 does not work for the group $\text{Spin}^o_\alpha(V, h)$. In that case, one has $D^2 = -1$ and hence $\{[1, 1], [1, D]\}$ is not a $\mathbb{Z}_2$-subgroup of $\text{Spin}^o_\alpha(V, h)$.

In the proof of Proposition 2.2 we used the fact that the adjoint action of $\hat{D}$ on $\text{Spin}^c(V, h) \subset \text{Spin}^o_\alpha(V, h)$ is by complex conjugation of $z$. Define the conjugation automorphism $K \in \text{Aut}(\text{Spin}^o_\alpha(V, h))$ through:

$$K \overset{\text{def.}}{=} \text{Ad}(\hat{D}).$$

This satisfies $K(\hat{D}) = \hat{D}$ and $K(z) = \bar{z}$ for all $z \in \text{U}$. The fixed point set of $K$ is the non-normal subgroup:

$$\tilde{\text{Pin}}_\alpha(V, h) = \tilde{\text{Spin}}(V, h) \sqcup \tilde{\text{Spin}}(V, h) \hat{D}.$$
Since \( \hat{D} \) commutes with all elements of \( \spinhat(V, h) \) and \( \hat{D}^2 = [1, \alpha] = [\alpha, 1] \in \spinhat(V, h) \), any choice of an element \( v \in V \) such that \( h(v, v) = \epsilon_v \) with \( \epsilon_v \in \{-1, 1\} \), gives an isomorphism of groups:

\[
\Pinhat_\alpha(V, h) \cong \Pin(V, \alpha \epsilon_v h)
\]

which takes any \([a, 1] \in \spinhat(V, h)\) into \( a \in \Spin(V, \alpha \epsilon_v h) = \Spin(V, h)\) and any \([a, 1] \hat{D} = [a, D] \in \spinhat(V, h) \hat{D} \) into \( av \in \spinhat(V, \alpha \epsilon_v h) \). The \( \mathbb{Z}_2 \)-grading (24) corresponds through this isomorphism to the canonical \( \mathbb{Z}_2 \)-grading:

\[
\Pin(V, \alpha \epsilon_v h) = \Spin(V, \alpha \epsilon_v h) \sqcup \Pin^-(V, \alpha \epsilon_v h)
\]
of \( \Pin(V, \alpha \epsilon_v h) \). We have a short exact sequence:

\[
1 \to U \to \Spin_\alpha^o(V, h) \xrightarrow{\pi_\alpha} \Pinhat_\alpha(V, h) \to 1,
\]

where:

\[
\pi_\alpha([a, b]) = \begin{cases} 
[a], & \text{if } b \in \Spin_2 \\
[a, D] & \text{if } b \in \Spin_2 \hat{D}, \quad \forall [a, b] \in \Spin_\alpha^o(V, h).
\end{cases}
\]

We leave the proof of the following proposition to the reader.

**Proposition 2.4.** The inclusion morphism \( j_\alpha : \Pinhat_\alpha(V, h) \to \Spin_\alpha^o(V, h) \) splits the sequence (26) from the right. Thus \( \Spin_\alpha^o(V, h) \cong U \times_{\zeta_\alpha} \Pinhat_\alpha(V, h) \), where \( \zeta_\alpha : \Pinhat_\alpha(V, h) \to \Aut(U) \) is the group morphism given by:

\[
\zeta_\alpha(x)(z) = \begin{cases} 
\hat{z} & \text{if } x = [a, 1] \in \spinhat(V, h) \\
\hat{z} & \text{if } x = [a, D] \in \spinhat(V, h) \hat{D}.
\end{cases}
\]

Here, \( U \) is the normal \( U(1) \) subgroup of \( \Spin_\alpha^o(V, h) \) which was defined in (22).

**Remark 2.5.** The quotient \( \Spin_\alpha^o(V, h)/\Pinhat_\alpha(V, h) \) is a principal \( U(1) \)-homogeneous space \( (U(1) \text{ torsor}) \).

### 2.2. Elementary representations.

We introduce several representations of \( \Spin_\alpha^o(V, h) \) which will be relevant later on. Let\(^4\):

\[
\tilde{O}(V, h) \overset{\text{def}}{=} \begin{cases} 
\SO(V, h) & \text{if } d = \text{even} \\
O(V, h) & \text{if } d = \text{odd}
\end{cases}
\]

and:

\[
\Pin_2(\alpha) \overset{\text{def}}{=} \begin{cases} 
\Pin_2(\alpha) & \text{if } d = \text{even} \\
\Spin_2(\alpha) \simeq U(1) & \text{if } d = \text{odd}
\end{cases}
\]

We have \( \Pin_2(\alpha) \simeq \tilde{O}_2(\alpha) \), where:

\[
\tilde{O}_2(\alpha) \overset{\text{def}}{=} \begin{cases} 
O_2(\alpha) & \text{if } d = \text{even} \\
O_2^+(\alpha) \simeq U(1) & \text{if } d = \text{odd}
\end{cases}
\]

**Definition 2.6.** 1. The **characteristic representation** of \( \Spin_\alpha^o(V, h) \) is the group epimorphism \( \mu_\alpha : \Spin_\alpha^o(V, h) \to O(2) \) defined through:

\[
\mu_\alpha([a, u]) \overset{\text{def}}{=} \Ad_0^{(2)}(u) \forall [a, u] \in \Spin_\alpha^o(V, h),
\]

where \( \Ad_0^{(2)} \) is the untwisted vector representation of \( \Pin_2(\alpha) \).

2. The **twisted characteristic representation** of \( \Spin_\alpha^o(V, h) \) is the group epimorphism \( \tilde{\mu}_\alpha : \Spin_\alpha^o(V, h) \to O(2) \) defined through:

\[
\tilde{\mu}_\alpha([a, u]) \overset{\text{def}}{=} \tilde{\Ad}_0^{(2)}(u), \quad \forall [a, u] \in \Spin_\alpha^o(V, h),
\]

where \( \tilde{\Ad}_0^{(2)} \) is the twisted vector representation of \( \Pin_2(\alpha) \).

\(^4\)A word of caution about notation: the group \( \tilde{O}(V, h) \) should not be confused with the group \( \tilde{O}(V, h) \) used in [8], which is given by the opposite dichotomy.
3. The vector representation of \( \text{Spin}^o(V,h) \) is the group epimorphism \( \lambda_o : \text{Spin}^o(V,h) \rightarrow \text{SO}(V,h) \) defined through:
\[
\lambda_o([a,u]) \overset{\text{def}}{=} \text{Ad}_0(a), \forall [a,u] \in \text{Spin}^o(V,h),
\]
where \( \text{Ad}_0 \) is the vector representation of \( \text{Spin}(V,h) \).

4. The twisted vector representation of \( \text{Spin}^o(V,h) \) is the group epimorphism \( \tilde{\lambda}_o : \text{Spin}^o(V,h) \rightarrow \text{O}(V,h) \) defined through:
\[
\tilde{\lambda}_o([a,u]) \overset{\text{def}}{=} \det(\text{Ad}_0^{(2)}(u))\text{Ad}_0(a) \forall [a,u] \in \text{Spin}^o(V,h).
\]

5. The basic representation is the group epimorphism \( \rho_o \overset{\text{def}}{=} \lambda \times \mu : \text{Spin}^o(V,h) \rightarrow \text{SO}(V,h) \times \text{O}(2) \):
\[
\rho_o([a,u]) \overset{\text{def}}{=} (\text{Ad}_0(a), \text{Ad}_0^{(2)}(u)).
\]

6. The twisted basic representation is the group epimorphism \( \tilde{\rho}_o \overset{\text{def}}{=} \tilde{\lambda} \times \mu : \text{Spin}^o(V,h) \rightarrow \text{SO}(V,h) \times \text{O}(2) \):
\[
\tilde{\rho}_o([a,u]) \overset{\text{def}}{=} (\det(\text{Ad}_0^{(2)}(u))\text{Ad}_0(a), \text{Ad}_0^{(2)}(u)),
\]
where:
\[
\text{O}(V,h) \times \text{O}(2) \overset{\text{def}}{=} \begin{cases} 
\text{SO}(V,h) \times \text{O}(2) & \text{if } d \text{ is even} \\
\text{SO}(V,h) \times \text{O}(2) & \text{if } d \text{ is odd}
\end{cases}
\]

Notice that \( \text{Ad}_0^{(2)}(u) = (\det \text{Ad}_0^{(2)}(u))\text{Ad}_0^{(2)}(u) \) for all \( u \in \text{Pin}_2(o) \). The various representations introduced above induce the following short exact sequences:
\[
\begin{align*}
1 \rightarrow & \text{Pin}_2(o) \rightarrow \text{Spin}^o(V,h) \xrightarrow{\lambda_o} \text{SO}(V,h) \rightarrow 1 \\
1 \rightarrow & \text{Pin}_2(o) \rightarrow \text{Spin}^o(V,h) \xrightarrow{\tilde{\lambda}_o} \text{O}(V,h) \rightarrow 1 \\
1 \rightarrow & \text{Spin}(V,h) \rightarrow \text{Spin}^o(V,h) \xrightarrow{\mu_o} \text{O}(2) \rightarrow 1
\end{align*}
\]
\[
\begin{align*}
1 \rightarrow & \text{Spin}(V,h) \rightarrow \text{Spin}^o(V,h) \xrightarrow{\tilde{\mu}_o} \text{O}(2) \rightarrow 1 \\
1 \rightarrow & \text{Z}_2 \rightarrow \text{Spin}^o(V,h) \xrightarrow{\tilde{\rho}_o} \text{SO}(V,h) \times \text{O}(2) \rightarrow 1 \\
1 \rightarrow & \text{Z}_2 \rightarrow \text{Spin}^o(V,h) \xrightarrow{\tilde{\nu}_o} \text{O}(V,h) \times \text{O}(2) \rightarrow 1
\end{align*}
\]
where \( \text{Z}_2 \) is the center \( \text{O}(2) \) of \( \text{Spin}^o(V,h) \). Therefore, \( \text{Pin}_2(o) \), \( \text{Pin}_2(o) \) and \( \text{Spin}(V,h) \) are normal subgroups of \( \text{Spin}^o(V,h) \). In particular, \( \text{Pin}_2(o) \) is a normal subgroup of \( \text{Spin}^o(V,h) \) with corresponding short exact sequence given either by the untwisted or the twisted vector representations.

Remark 2.7. When \( d \) is odd, the second exact sequence above shows that \( \text{Spin}^o(V,h) \) are distinct non-central extensions of \( \text{O}(V,h) \) by \( \text{U}(1) \), both of which differ from the central extension provided by the group \( \text{Pin}^c(V,h) \overset{\text{def}}{=} \text{Pin}(V,h) \cdot \text{U}(1) \). When \( d \) is even, \( \text{Spin}^o(V,h) \) are extensions of \( \text{SO}(V,h) \) by \( \text{O}_2(o) \).

2.3. Realization of \( \text{Spin}^o(V,h) \) as a subgroup of a pin group. Let \( \hat{V} = V \oplus \mathbb{R}^2 \) and \( \hat{h}_o : \hat{V} \times \hat{V} \rightarrow \mathbb{R} \) be the non-degenerate bilinear form given by \( \hat{h}_o(x \oplus u, y \oplus v) = h(x,y) + \alpha(u,v) \) for all \( x,y \in V \) and \( u,v \in \mathbb{R}^2 \), where \( \langle \ , \rangle \) is the canonical Euclidean scalar product of \( \mathbb{R}^2 \). If \( h \) has signature \( (p,q) \), then \( \hat{h}_o \) has signature \( (p+2,q) \) when \( \alpha = +1 \) and signature \( (p,q+2) \) when \( \alpha = -1 \). Let \( (e_{d+1}, e_{d+2}) \) be the canonical basis of \( \mathbb{R}^2 \). The following relations hold in \( \text{Cl}(\hat{V}, \hat{h}_o) \):
\[
e_{d+1}^2 = e_{d+2}^2 = \alpha.
\]
Notice that \( \text{Cl}(\mathbb{R}^2, \alpha(\ , \)) = \mathbb{Cl}_2(o) \) embeds as the subalgebra of \( \text{Cl}(\hat{V}, \hat{h}_o) \) generated by \( e_{d+1} \) and \( e_{d+2} \). Let \( G : \text{O}(V,h) \times \text{O}(2) \rightarrow \text{O}(\hat{V}, \hat{h}_o) \) be the injective morphism of groups given by:
\[
G(A,B) \overset{\text{def}}{=} A \oplus B.
\]
We have $G(O(V, h) \times O(2)) \subset \hat{O}(\hat{V}, \hat{h}_\alpha)$, where:

$$\hat{O}(\hat{V}, \hat{h}_\alpha) \overset{\text{def}}{=} \begin{cases} O(\hat{V}, \hat{h}_\alpha) & \text{if } d = \text{even} \\ SO(\hat{V}, \hat{h}_\alpha) & \text{if } d = \text{odd} \end{cases}.$$ 

**Proposition 2.8.** The map $j : \text{Spin}_2^\alpha(V, h) \to \text{Pin}(\hat{V}, \hat{h}_\alpha)$ defined through:

$$j([a, b]) = ab \forall a \in \text{Spin}(V, h), \ b \in \text{Pin}_2(\alpha).$$

is an injective morphism of groups which satisfies:

$$\hat{\text{Ad}}_0 \circ j = G \circ \hat{\rho}_\alpha,$$

and:

$$\widehat{\text{Ad}}_0 \circ j = G \circ \rho_\alpha,$$

where $\hat{\text{Ad}}_0$ and $\widehat{\text{Ad}}_0$ are the untwisted and twisted vector representations of $\text{Pin}(\hat{V}, \hat{h}_\alpha)$, respectively.

**Proof.** We have $\text{Pin}_2(\alpha) = \text{Pin}(\mathbb{R}^2, \alpha(\cdot), \cdot) \subset C(\hat{V}, \hat{h}_\alpha)$. Since $\text{Spin}(V, h) \cap \text{Pin}_2(\mathbb{R}^2, \alpha(\cdot), \cdot) = \{-1, 1\}$, the group morphism given by (34) is injective. The element $D \overset{\text{def}}{=} e_{d+1}$ satisfies $Dv = -vD \forall v \in V$.

We have $\text{Pin}_2(\alpha) = \text{Spin}_2(\alpha) \cup \text{Spin}_2(\alpha)D$ and:

$$\text{det}(\text{Ad}_0^{(2)}(b)) = \begin{cases} +1 & \text{if } b \in \text{Spin}_2(\alpha) \\ -1 & \text{if } b \in \text{Spin}_2(\alpha)D \end{cases},$$

which implies:

$$\hat{\text{Ad}}_0(b)(v) = \text{det}(\text{Ad}_0^{(2)}(b))v \forall v \in V \text{ and } b \in \text{Pin}_2(\alpha).$$

For any $a \in \text{Spin}(V, h)$, $b \in \text{Pin}_2(\alpha)$, $v \in V$ and $w \in \mathbb{R}e_{d+1} \oplus \mathbb{R}e_{d+2}$, we compute:

$$\text{Ad}_0(ab)(v) = (\text{Ad}_0(a) \circ \hat{\text{Ad}}_0(b))(v) = \text{det}(\text{Ad}_0^{(2)}(b))\text{Ad}_0(a)(v)$$

$$\text{Ad}_0(ab)(w) = (\text{Ad}_0(a) \circ \hat{\text{Ad}}_0(b))(w) = \text{Ad}_0^{(2)}(b)(w),$$

which gives (35). Since $D \subset C(\hat{V}, \hat{h}_\alpha)$, we have:

$$\widehat{\text{Ad}}_0(ab) = \begin{cases} \text{Ad}_0(ab) & \text{if } b \in \text{Spin}_2(\alpha) \\ -\text{Ad}_0(ab) & \text{if } b \in \text{Spin}_2(\alpha)D \end{cases} (a \in \text{Spin}(V, h)),$$

i.e. $\hat{\text{Ad}}_0(ab) = (\text{det } \text{Ad}_0^{(2)}(b))\hat{\text{Ad}}_0(ab)$. Thus:

$$\hat{\text{Ad}}_0(ab)(v) = \text{Ad}_0(a)(v)$$

$$\hat{\text{Ad}}_0(ab)(w) = (\text{det } \text{Ad}_0^{(2)}(b))\text{Ad}_0^{(2)}(b)(w) = \text{Ad}_0^{(2)}(b),$$

which gives (36).

Equations (35) and (36) imply that the following relations hold for all $g \in \text{Spin}_2^\alpha(V, h)$:

$$\lambda(g) = \hat{\text{Ad}}_0(j(g))|_V, \ \lambda(g) = \widehat{\text{Ad}}_0(j(g))|_V,$$

$$\mu(g) = \hat{\text{Ad}}_0(j(g))|_{\mathbb{R}^2}, \ \mu(g) = \widehat{\text{Ad}}_0(j(g))|_{\mathbb{R}^2}.$$ 

We conclude that the following commutative diagrams with exact rows (morphisms of short exact sequences) hold:

$$\begin{array}{ccc}
1 & \longrightarrow & \mathbb{Z}_2 \\
\downarrow & & \downarrow j \\
\mathbf{Z}_2 & \longrightarrow & \text{Spin}_2^\alpha(V, h) \\
\downarrow \phi & & \downarrow \phi \\
1 & \longrightarrow & \text{Pin}(V, h) \\
\end{array}$$

$$
\begin{array}{ccc}
1 & \longrightarrow & \\
\phi & \longrightarrow & \mathbb{O}(V, h) \times \mathbb{O}(2) & \longrightarrow & 1 \\
\downarrow \phi & & \downarrow G \\
\hat{\text{Ad}}_0 & \longrightarrow & \hat{O}(\hat{V}, \hat{h}_\alpha) & \longrightarrow & 1 \\
\end{array}
$$
and:

\[
\begin{align*}
1 & \xrightarrow{\mathbb{Z}_2} \Spin_o^\alpha(V, h) \xrightarrow{\rho_\alpha} \SO(V, h) \times O(2) \xrightarrow{\beta} 1 \\
1 & \xrightarrow{\mathbb{Z}_2} \Pin(V, \hat{h}_\alpha) \xrightarrow{\widetilde{\Ad}_\alpha} O(V, \hat{h}_\alpha) \xrightarrow{\iota} 1
\end{align*}
\]

2.4. **Realization of** $\Spin_o^\alpha(V, h)$ **as a subgroup of a spin group.** Let $V' \overset{\text{def}}{=} V \oplus \mathbb{R}^3$ and $h'_\alpha : V' \times V' \to \mathbb{R}$ be the nondegenerate symmetric bilinear form given by:

\[h'_\alpha(x + w + t, x' + w' + t') \overset{\text{def}}{=} h(x, y) + \alpha(w, w') - ss', \forall x, x' \in V, w, w' \in \mathbb{R}^2, t, t' \in \mathbb{R}.
\]

Consider the injective group morphisms $F, \tilde{F} : O(V, h) \times O(2) \to O(V', h'_\alpha)$ given by:

\[
F(A, B) = (\det B)A \oplus (\det B)B \oplus \det B
\]

(40)

\[
\tilde{F}(A, B) \overset{\text{def}}{=} A \oplus B \oplus \det B
\]

Notice that $F(O(V, h) \times O(2)) \subset \SO(V', h'_\alpha)$ (for odd dimension) and $\tilde{F}(\SO(V, h) \times O(2)) \subset \SO(V', h'_\alpha)$.

We have $F \circ \tilde{\rho}_\alpha = \tilde{F} \circ \rho_\alpha$, namely:

\[
(F \circ \tilde{\rho}_\alpha)([a, u]) = (\tilde{F} \circ \rho_\alpha)([a, u]) = \Ad_0(a) \oplus (\det \Ad_0(2)(u))\Ad_0(2)(u) \oplus \det \Ad_0(2)(u),
\]

where we noticed that $\det \circ \Ad_0(2) = \det \circ \Ad_0(2)$ and $\Ad_0(2)(u) = (\det \Ad_0(2)(u))\Ad_0(2)(u)$.

**Proposition 2.9.** There exists an injective morphism of groups $j' : \Spin_o^\alpha(V, h) \to \Spin(V', h'_\alpha)$ such that:

(42)

\[
\text{Ad}_0' \circ j' = F \circ \tilde{\rho}_\alpha = \tilde{F} \circ \rho_\alpha,
\]

where $\text{Ad}_0'$ is the vector representation of $\Spin(V', h'_\alpha)$.

**Proof.** Let $e_{d+1}, e_{d+2}, e_{d+3}$ be the canonical basis of $\mathbb{R}^3$. The following relations hold in $\Cl(V', h'_\alpha)$:

\[
e_{d+1}^2 = e_{d+2}^2 = -1.
\]

Let $J' \overset{\text{def}}{=} e_{d+1}e_{d+2}, D' \overset{\text{def}}{=} e_{d+1}e_{d+3}$. Then:

\[
(J')^2 = -1, (D')^2 = \alpha, D' \cdot J' = -J' \cdot D'.
\]

Since $(e_{d+1}e_{d+3})^2 = (e_{d+2}^2e_{d+3})^2 = \alpha$, there exists a unique unital monomorphism of algebras $s' : \Cl(V', h'_\alpha) \to \Cl(V', h'_\alpha)$ which satisfies:

\[
s'(e_i) = e_i \text{ } \forall i = 1 \ldots d, s'(e_{d+1}e_{d+3}) = D', s'(e_{d+2}) = e_{d+2}e_{d+3}.
\]

The composition $j' \overset{\text{def}}{=} s' \circ j : \Spin_o^\alpha(V, h) \to \Spin(V', h'_\alpha)$ is an injective morphism of groups. Notice that $j'|_{\Spin(V, h)} = \text{id}_{\Spin(V, h)}$. For $a \in \Spin(V, h)$ and $v \in V$, we have:

\[
\text{Ad}_0'(s'(a))(v) = \text{Ad}_0(a)(v), \quad \text{Ad}_0'(s'(a))(e_{d+1}) = e_{d+1},
\]

\[
\text{Ad}_0'(s'(a))(e_{d+2}) = e_{d+2}, \quad \text{Ad}_0'(s'(a))(e_{d+3}) = e_{d+3},
\]

\[
\text{Ad}_0'(s'(e_{d+1}))(v) = v, \quad \text{Ad}_0'(s'(e_{d+1}))(e_{d+1}) = -e_{d+1},
\]

\[
\text{Ad}_0'(s'(e_{d+1}))(e_{d+2}) = +e_{d+2}, \quad \text{Ad}_0'(s'(e_{d+1}))(e_{d+3}) = -e_{d+3},
\]

\[
\text{Ad}_0'(s'(e_{d+2}))(v) = v, \quad \text{Ad}_0'(s'(e_{d+2}))(e_{d+1}) = +e_{d+1},
\]

\[
\text{Ad}_0'(s'(e_{d+2}))(e_{d+2}) = e_{d+2}, \quad \text{Ad}_0'(s'(e_{d+2}))(e_{d+3}) = -e_{d+3},
\]

since $s'(e_{d+1}) = e_{d+1}e_{d+3}$ and $s'(e_{d+2}) = e_{d+2}e_{d+3}$. On the other hand, we have:

\[
\text{Ad}_0'(2)(e_{d+1})(e_{d+1}) = e_{d+1}, \quad \text{Ad}_0'(2)(e_{d+1})(e_{d+2}) = -e_{d+2},
\]

\[
\text{Ad}_0'(2)(e_{d+2})(e_{d+1}) = -e_{d+1}, \quad \text{Ad}_0'(2)(e_{d+2})(e_{d+2}) = +e_{d+2},
\]

\[
\det(\text{Ad}_0'(2)(e_{d+1})) = \det(\text{Ad}_0'(2)(e_{d+2})) = -1.
\]
so comparison with (41) gives:

\[(\text{Ad}_0' \circ j')([a,1]) = (F \circ \tilde{\rho}_\alpha)([a,1])\]
\[(\text{Ad}_0' \circ j')([1,e_{d+1}]) = (F \circ \tilde{\rho}_\alpha)([1,e_{d+1}])\]
\[(\text{Ad}_0' \circ j')([1,e_{d+2}]) = (F \circ \tilde{\rho}_\alpha)([1,e_{d+2}])\]

which implies \(\text{Ad}_0' \circ j' = F \circ \tilde{\rho}_\alpha\) since the group \(\text{Spin}_\alpha^o(V,h)\) is generated by \(\text{Spin}(V,h)\) and by the elements \([1,e_{d+1}]\) and \([1,e_{d+2}]\). \(\square\)

We conclude that the following morphisms of short exact sequences hold:

\[
\begin{array}{cccc}
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \text{Spin}_\alpha^o(V,h) \quad \tilde{\rho}_\alpha \\
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \text{Spin}(V', h'_\alpha) \quad \text{Ad}_0' \\
\end{array}
\]

\[
\begin{array}{cccc}
\longrightarrow & \longrightarrow & \text{O}(V,h) \times O(2) & \longrightarrow & 1 \\
\longrightarrow & \longrightarrow & \text{SO}(V', h'_\alpha) & \longrightarrow & 1
\end{array}
\]

and:

\[
\begin{array}{cccc}
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \text{Spin}_\alpha^o(V,h) \quad \rho_\alpha \\
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \text{Spin}(V', h'_\alpha) \quad \text{Ad}_0' \\
\end{array}
\]

\[
\begin{array}{cccc}
\longrightarrow & \longrightarrow & \text{SO}(V,h) \times O(2) & \longrightarrow & 1 \\
\longrightarrow & \longrightarrow & \text{SO}(V', h'_\alpha) & \longrightarrow & 1
\end{array}
\]

3. \text{Spin}_\alpha^o \text{ structures}

Let \(M\) be a smooth, connected and paracompact manifold of dimension \(d\).

\textbf{Definition 3.1.} Let \(P_{\tilde{O}}\) be a principal \(\tilde{O}(V,h)\)-bundle over \(M\). A \(\text{Spin}_\alpha^o\) \text{ structure} on \(P_{\tilde{O}}\) is a pair \((Q, \tilde{\Lambda}_\alpha)\) where \(Q\) is a principal \(\text{Spin}_\alpha^o(V,h)\)-bundle over \(M\) and \(\tilde{\Lambda}_\alpha : Q \rightarrow P_{\tilde{O}}\) is a \(\tilde{\Lambda}_\alpha\)-equivariant map fitting into the following commutative diagram:

\[
\begin{array}{c}
Q \times \text{Spin}_\alpha^o(V,h) \longrightarrow Q \\
\tilde{\Lambda}_\alpha \times \tilde{\Lambda}_\alpha \downarrow \quad \downarrow \tilde{\Lambda}_\alpha \\
P_{\tilde{O}} \times \tilde{O}(V,h) \longrightarrow P_{\tilde{O}}
\end{array}
\]

Here horizontal arrows denote the right bundle action of the corresponding group.

\textbf{Remark 3.2.} Notice that \(\text{Spin}_\alpha^o\) structures are defined using the \textit{twisted} vector representation \(\tilde{\Lambda}_\alpha : \text{Spin}_\alpha^o(V,h) \rightarrow \tilde{O}(V,h)\) instead of the usual vector representation \(\Lambda_\alpha : \text{Spin}_\alpha^o(V,h) \rightarrow \text{SO}(V,h)\). When \(d\) is odd, the former covers the full orthogonal group \(\text{O}(V,h)\) whereas the latter only covers the special orthogonal group \(\text{SO}(V,h)\). As we shall see later, this fact allows us to define \(\text{Spin}_\alpha^o\) structures on non-orientable odd-dimensional manifolds.

\textbf{Remark 3.3.} The notion of \(\text{Spin}_\alpha^o\) structure introduced above may seem somewhat artificial. However, as we will explain in Section 5, this turns out to be the appropriate spinorial structure for describing bundles of simple real Clifford modules in signatures \((p,q)\) which satisfy the condition \(p - q \equiv_8 3, 7\).

Given a \(\text{Spin}_\alpha^o\) structure on a principal bundle \(P_{\tilde{O}}\), we can construct certain associated principal bundles by using the different representations of \(\text{Spin}_\alpha^o(V,h)\) introduced in Definition 2.6 of Subsection 2.2:
Definition 3.4. The characteristic bundle $P_{\mu_a}$ defined by a Spin$^c$ structure $(Q, \Lambda_a)$ is the principal O(2)-bundle associated to $Q$ through the characteristic representation $\mu_a : \text{Spin}^c(V, h) \to \text{O}(2)$, that is:

$$P_{\mu_a} \overset{\text{def.}}{=} Q \times_{\mu_a} \text{O}(2).$$

We denote by $\mathfrak{U}_a : Q \to P_{\mu_a}$ the corresponding $\mu_a$-equivariant bundle map.

Definition 3.5. The twisted basic bundle $P_{\tilde{\rho}_a}$ defined by a Spin$^c$ structure $(Q, \tilde{\Lambda}_a)$ is the principal O(V, h)$\times$O(2)-bundle associated to $Q$ through the twisted basic representation $\tilde{\rho}_a : \text{Spin}^c(V, h) \to \text{O}(V, h)\times\text{O}(2)$, that is:

$$P_{\tilde{\rho}_a} \overset{\text{def.}}{=} Q \times_{\tilde{\rho}_a} (\text{O}(V, h)\times\text{O}(2)).$$

We denote by $\mathfrak{R}_a : Q \to P_{\tilde{\rho}_a}$ the corresponding $\tilde{\rho}_a$-equivariant bundle map.

Proposition 3.6. Let $P_O$ be a principal $\tilde{\text{O}}(V, h)$-bundle over $M$. Then the following statements are equivalent:

(a) $P_O$ admits a Spin$^c$ structure $(Q, \Lambda_a)$.

(b) There exists a principal O(2)-bundle $E$ over $M$ such that the fibered product $P_O \times_M E$ admits a $\rho_a$-equivariant reduction:

$$\mathfrak{R}_a^0 : Q \to P_O \times_M E,$$

to a Spin$^c(V, h)$-bundle $Q$.

In particular, the image $P_{\tilde{\rho}_a} \overset{\text{def.}}{=} \mathfrak{R}_a^0(Q) \subset P_O \times_M E$ of $Q$ by $\mathfrak{R}_a^0$ is the twisted basic bundle associated to the Spin$^c$ structure $(Q, \Lambda_a)$, and the the corestriction of $\mathfrak{R}_a^0$ to its image gives the $\tilde{\rho}_a$-equivariant bundle map $\mathfrak{R}_a : Q \to P_{\tilde{\rho}_a}$.

Proof. Suppose first that (a) holds, i.e. that $P_O$ admits a Spin$^c$ structure $(Q, \Lambda_a)$. We take $E = P_{\mu_a}$ to be the characteristic bundle associated to $Q$. Using the fact that $(Q, \Lambda_a)$ is a Spin$^c$ structure over $P_O$, we define $\mathfrak{R}_a^0$ as follows:

$$\mathfrak{R}_a^0 : Q \to P_O \times_M E, \quad q \mapsto (\Lambda_a(q), \mathfrak{U}_a(q)).$$

Direct computation shows that $\mathfrak{R}_a^0$ is $\rho_a$ equivariant. The image of $\mathfrak{R}_a^0$ is isomorphic to the twisted basic bundle associated to $Q$ through the following isomorphism of principal bundles

$$\mathcal{I} : P_{\tilde{\rho}_a} \xrightarrow{\sim} \mathfrak{R}_a^0(Q), \quad [q, (g_0, u_0)] \mapsto (\Lambda_a(q)g_0, \mathfrak{U}_a(q)u_0).$$

The equivariant map $\mathfrak{U}_a : Q \to P_{\tilde{\rho}_a}$ corresponds, upon use of $\mathcal{I}$, to the corestriction of $\mathfrak{R}_a^0$ to its image, that is:

$$\mathfrak{U}_a = \mathcal{I}^{-1} \circ \mathfrak{R}_a^0 : Q \to P_{\tilde{\rho}_a}.$$

Now suppose (b) holds. Then the principal Spin$^c(V, h)$-bundle $Q$ becomes a Spin$^c$ structure on $P_O$ when endowed with the bundle map $\hat{\Lambda}_a : Q \to P_O$ obtained by post-composing $\mathfrak{U}_a$ with the fiberwise projection to the first factor. Indeed, the map $\hat{\Lambda}_a$ obtained in this manner is equivariant with respect to the twisted vector representation $\hat{\Lambda}_a : \text{Spin}^c(V, h) \to \tilde{\text{O}}(V, h)$ (this follows from the fact that $\mathfrak{U}_a$ is $\tilde{\rho}_a$-equivariant). Hence $(Q, \hat{\Lambda}_a)$ satisfies the conditions required in Definition 3.1.

We next define isomorphisms of Spin$^c$ structures.

Definition 3.7. Let $\Lambda_1^2 : Q^1 \to P_O$ and $\Lambda_2^2 : Q^2 \to P_O$ be two Spin$^c$ structures over $P$. An isomorphism of Spin$^c$ structures is an isomorphism $F : Q^1 \to Q^2$ of principal Spin$^c$-bundles such that $\Lambda_2^0 \circ F = \Lambda_1^0$, i.e., such that the following diagram commutes:

$$\begin{array}{ccc}
Q^1 & \xrightarrow{F} & Q^2 \\
\Lambda_1^1 \downarrow & & \downarrow \Lambda_2^2 \\
P_O & \underset{id}{\xrightarrow{\sim}} & P_O
\end{array}$$

(45)
3.1. Spin\(_{\alpha}\) structures on pseudo-Riemannian manifolds. Let \((M,g)\) be a connected pseudo-Riemannian manifold of dimension \(d\) and signature \((p,q)\) and let \((V,h)\) be a quadratic space of the same dimension and signature. Furthermore, assume that \(M\) is oriented when \(d\) is even. In this situation, the pseudo-Riemannian manifold \((M,g)\) carries a natural principal \(\tilde{O}(V,h)\)-bundle \(P_{\tilde{O}}(M,g)\), which is defined as follows (cf. Definition (28)):

\[
P_{\tilde{O}}(M,g) \overset{\text{def}}{=} \begin{cases} \text{bundle of oriented pseudo-orthonormal frames of } (M,g) & \text{if } d = \text{even} \\ \text{bundle of all pseudo-orthonormal frames of } (M,g) & \text{if } d = \text{odd} \end{cases}
\]

**Definition 3.8.** Let \((M,g)\) be as above. A Spin\(_{\alpha}\)-structure on \((M,g)\) is a Spin\(_{\alpha}\) structure on the principal \(\tilde{O}(V,h)\)-bundle \(P_{\tilde{O}}(M,g)\).

**Definition 3.9.** Let \(C^\alpha(M,g)\) be the groupoid whose objects are Spin\(_{\alpha}\) structures on \((M,g)\) and whose arrows are isomorphisms of Spin\(_{\alpha}\) structures.

**Remark 3.10.** Notice that two principal Spin\(_{\alpha}\)-bundles on \((M,g)\) may be isomorphic as principal Spin\(_{\alpha}\)-bundles without being isomorphic as Spin\(_{\alpha}\) structures.

**Remark 3.11.** As explained in Section 2, we have a short exact sequence:

\[
1 \to \text{Spin}^c(V,h) \overset{\iota_1}{\to} \text{Spin}_{\alpha}^c(V,h) \overset{\tilde{\eta}_\alpha}{\to} \mathbb{Z}_2 \to 1,
\]

which induces the following exact sequence of pointed sets in Čech-cohomology:

\[
H^1(M,\text{Spin}^c(V,h)) \overset{\iota_1^*}{\to} H^1(M,\text{Spin}_{\alpha}^c(V,h)) \overset{\tilde{\eta}_\alpha^*}{\to} H^1(M,\mathbb{Z}_2).
\]

Hence every principal \(\text{Spin}_{\alpha}^c(V,h)\)-bundle \(Q\) defined on \(M\) determines a real line bundle \(L\) defined on \(M\) whose isomorphism class satisfies:

\[
[L] = \tilde{\eta}_\alpha^*([Q]) \in H^1(M,\mathbb{Z}_2).
\]

Intuitively, we can view a Spin\(_{\alpha}\) structure on \((M,g)\) as a Spin\(^c\)\((V,h)\) structure “twisted” by the real line bundle \(L\). It is not hard to see that \(L\) is isomorphic with the determinant line bundle of the characteristic \(O(2)\)-bundle \(P_{\mu_\alpha}\) associated to \(Q\). Exactness of (47) shows that a Spin\(_{\alpha}\) structure \(Q\) reduces to a Spin\(^c\) structure iff \(L\) is trivial, i.e. iff \(P_{\mu_\alpha}\) is orientable. We will study in detail reductions of Spin\(_{\alpha}\)-bundles in Section 4.

3.2. Topological obstructions. We now turn our attention to the topological obstruction for existence of a Spin\(_{\alpha}\) structure on a principal \(\tilde{O}(V,h)\)-bundle \(P_{\tilde{O}}\) defined over \(M\). We will need the following result:

**Lemma 3.12.** Consider a commutative diagram of morphisms of Lie groups of the following form:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & G & \overset{p}{\longrightarrow} & K & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow f & & \downarrow g & & \\
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & G' & \overset{p'}{\longrightarrow} & K' & \longrightarrow & 1
\end{array}
\]

Then a principal \(K\)-bundle \(P\) over \(M\) admits a \(G\)-reduction along \(p\) iff the principal \(K'\)-bundle \(P' \overset{\text{def}}{=} g_*\(P\) \) admits a \(G'\)-reduction along \(p'\).
Proof. The given morphism of short exact sequences induces a commutative diagram with exact columns in the category of pointed sets, where $\partial$ and $\partial'$ are the connecting maps:

$$
\begin{array}{ccc}
H^1(M, G) & \xrightarrow{f_*} & H^1(M, G') \\
| & p_* & | \\
H^1(M, K) & \xrightarrow{g_*} & H^1(M, K') \\
| & \partial & | \\
H^2(M, \mathbb{Z}_2) & \xrightarrow{\partial'} & H^2(M, \mathbb{Z}_2)
\end{array}
$$

The principal $K$-bundle $P$ admits a $G$-reduction along $p$ iff its isomorphism class $[P] \in H^1(M, K)$ belongs to the image of $p_*$. While the principal $K'$-bundle $P'$ admits a $G'$-reduction along $p'$ iff its isomorphism class $[P'] = g_*(|P|) \in H^1(M, K')$ belongs to the image of $p'_*$. A simple diagram chase using exactness of the columns shows that $[P] \in \text{im} p_*$ iff $g_*(|P|) \in \text{im} p'_*$.

We now consider the topological obstructions to existence of a Spin$_o$ structure on a principal $\hat{O}(V, h)$-bundle when $d = \dim V$ is odd. We focus on the case when $d$ is odd since this case is relevant for applications to bundles of simple real Clifford modules of ‘complex type’.

Lemma 3.13. Let $P = P_{O}$ be a principal $\hat{O}(V, h)$-bundle defined on $M$, such $d = \dim V$ is odd. Then the following statements are equivalent:

(a) $P$ admits a Spin$_o$ structure.

(b) There exists a principal $O(2)$-bundle $E$ such that the principal $O(V', h'_o)$-bundle $P'_o \overset{\text{def}}{=} [(\det E)P] \times_M (\det E)E \times_M \det E$ admits a Spin structure.

Proof. Follows from Proposition 3.6 by applying Lemma 3.12 to the commutative diagram of short exact sequences (43), where the morphism $F$ is defined in (40), the embedding $j'$ is defined in Proposition 2.9 and $\hat{\rho}_o$ denotes the twisted basic representation of Spin$_o(V, h)$. More explicitly, diagram (43) implies the following commutative diagram with exact columns in the category of pointed sets:

$$
\begin{array}{ccc}
H^1(M, \text{Spin}_o(V, h)) & \xrightarrow{j'_*} & H^1(M, \text{Spin}(V', h'_o)) \\
\hat{\rho}_o \downarrow & & \Lambda \partial' \downarrow \\
H^1(M, O(V, h) \times O(2)) & \xrightarrow{f_*} & H^1(M, SO(V', h'_o)) \\
\partial \downarrow & & \partial' \downarrow \\
H^2(M, \mathbb{Z}_2) & \xrightarrow{\partial'} & H^2(M, \mathbb{Z}_2)
\end{array}
$$

Applying Lemma 3.12 we find that $[P] \in H^1(M, O(V, h) \times O(2))$ admits a Spin$_o$ structure iff $F_*(|P|)$ admits a Spin structure. Using the explicit form of $F$, we deduce that $F_*(|P|)$ can be represented by a principal bundle of the form $P'_o \overset{\text{def}}{=} [(\det E)P] \times_M (\det E)E \times_M \det E$. Notice that the connecting map $\partial'$ is given by [6]:

$$
\partial' = w_2^+ + w_2^- .
$$

Let $L$ be a principal $\mathbb{Z}_2$-bundle and $P$ be a principal $O(r)$-bundle over $M$, where $r \geq 1$. Recall that the center $Z(O(r)) \simeq \mathbb{Z}_2$. Then we have [5, page 404]:

$$
w(LP) = \sum_{k=0}^{r} (1 + w_1(L))^k w_{r-k}(P) ,
$$
which gives:

\begin{equation}
\begin{aligned}
\w_1(LP) &= \w_1(P) + r\w_1(L) \\
\w_2(LP) &= \w_2(P) + (r - 1)\w_1(P)\w_1(L) + \frac{r(r - 1)}{2}\w_1(L)^2 .
\end{aligned}
\end{equation}

Also recall that \(\w_1(\det P) = \w_1(P)\). The total Stiefel-Whitney class satisfies \(w(P_1 \times_M P_2) = w(P_1)w(P_2)\) for any principal \(O(r_i)\)-bundles \(P_i (i = 1, 2)\), which gives:

\begin{equation}
\begin{aligned}
\w_1(P_1 \times_M P_2) &= \w_1(P_1) + \w_1(P_2) \\
\w_2(P_1 \times_M P_2) &= \w_2(P_1) + \w_2(P_2) + \w_1(P_1)\w_1(P_2) .
\end{aligned}
\end{equation}

**Theorem 3.14.** A principal \(O(V,h)\)-bundle \(P\) defined on \(M\) admits a \(\text{Spin}^c\) structure iff there exists a principal \(O(2)\)-bundle \(E\) such that the following conditions are satisfied:

\begin{equation}
\begin{aligned}
\w^+_1(P') + \w^-_1(P') &= \w^+_1(P') + \w^-_1(P') = 0 .
\end{aligned}
\end{equation}

Notice that \(P^+\) is an \(O(p)\)-bundle while \(P^-\) is an \(O(q)\)-bundle. Also notice that \(\w_1(\det E) = \w_1(E)\). Using (52) and the fact that \(E\) is an \(O(2)\) bundle, we find:

\begin{equation}
\begin{aligned}
\w_1((\det E)E) &= \w_1(E) \\
\w_2((\det E)E) &= \w_2(E) \\
\w_1((\det E)E \times_M \det E) &= 0 \\
\w_2((\det E)E \times_M \det E) &= \w_2(E) + \w_1(E)^2 .
\end{aligned}
\end{equation}

We distinguish the cases:

1. For \(\alpha = +1\), we have \((P')^+ = [(\det E)P^+] \times_M (\det E)E\) and \((P')^- = [(\det E)P^-] \times_M \det E\). Using (52), (53) and (56), we compute:

\begin{equation}
\begin{aligned}
\w_1^+(P') &= \w_1^+(P) + (p + 1)\w_1(E) \\
\w_1^-(P') &= \w_1^-(P) + (q + 1)\w_1(E) \\
\w_2^+(P') &= \w_2^+(P) + (p - 1)\w_1^+(P)\w_1(E) + \frac{p(p - 1)}{2}\w_1(E)^2 + \w_2(E) + \\
&\quad + \w_1(E)(\w_1^+(P) + pw_1(E)) = \w_2^+(P) + \w_2(E) + pw_1^+(P)\w_1(E) + \frac{p(p + 1)}{2}\w_1(E)^2 \\
\w_2^-(P') &= \w_2^-(P) + (q - 1)\w_1^-(P)\w_1(E) + \frac{q(q - 1)}{2}\w_1(E)^2 + \\
&\quad + \w_1(E)(\w_1^-(P) + qw_1(E)) = \w_2^-(P) + qw_1^-(P)\w_1(E) + \frac{q(q + 1)}{2}\w_1(E)^2 .
\end{aligned}
\end{equation}

Thus equations (55) reduce to (54).
2. For \( \alpha = -1 \), we have \((P')^+ = [(\det E)P^+]\) and \((P')^- = [(\det E)P^-] \times_M (\det E)E \times_M \det E\). Thus:

\[
\begin{align*}
\omega_1^+(P') &= \omega_1^+(P) + pw_1(E) \\
\omega_1^-(P') &= \omega_1^-(P) + qw_1(E) \\
\omega_2^+(P') &= \omega_2^+(P) + (p - 1)w_1(E)\omega_1^+(P) + \frac{p(p - 1)}{2}w_1(E)^2 \\
\omega_2^-(P') &= \omega_2^-(P) + (q - 1)w_1(E)\omega_1^-(P) + \frac{q(q - 1)}{2}w_1(E)^2 + w_2(E) + \\
&w_1(E)^2 = w_2^+(P) + w_2(E) + (q - 1)w_1(E)\omega_1^-(P) + [1 + \frac{q(q - 1)}{2}]w_1(E)^2.
\end{align*}
\]

Thus equations (55) reduce to (54), where to obtain the second relation in (54) we used the first relation and noticed that:

\[
1 + d + \frac{p(p - 1)}{2} + \frac{q(q - 1)}{2} = 1 + \frac{p(p + 1)}{2} + \frac{q(q + 1)}{2}.
\]

since \( d = p + q \).

3.3. Application to pseudo-Riemannian manifolds of positive and negative signature. Let us consider Theorem 3.14 in the particular case of odd-dimensional pseudo-Riemannian manifolds \((M, g)\) of positive and negative signatures. In this case, we have \(O(V, h) \cong O(d)\) and \(pq = 0\), where \(d\) is the dimension of \(M\). We focus exclusively on \(\text{Spin}_{-}\) structures in dimension \(d \equiv 3 \) and \(\text{Spin}_{+}\) structures in dimension \(d \equiv 7 \) — since, as we shall see in Section 4, these are the cases most relevant for applications to spinorial geometry. For ease of notation, let \(P\) denote the principal bundle of pseudo-orthonormal frames of \((M, g)\).

Positive signature in dimension \(d \equiv 3\). The topological obstruction to existence of a \(\text{Spin}_{-}\) structure on a Riemannian manifold of dimension \(d \equiv 3\) is given by the condition:

\[
(57) \quad w_1(P) = w_1(E) \quad w_2(P) = w_2(E).
\]

Positive signature in dimension \(d \equiv 7\). The topological obstruction to existence of a \(\text{Spin}_{+}\) structure on a Riemannian manifold of dimension \(d \equiv 7\) is given by the condition:

\[
(58) \quad w_1(P) = w_1(E) \quad w_2(P) + w_1(P)^2 + w_2(E) = 0.
\]

In this particular case, condition (58) also implies existence of a complex Lipschitz structure [3, 9], which in turn amounts to existence of a bundle of faithful complex Clifford modules over the real Clifford bundle of \((M, g)\).

Negative signature in dimension \(d \equiv 3\). The topological obstruction to existence of a \(\text{Spin}_{-}\) structure on a 'negative Riemannian' manifold of dimension \(d \equiv 3\) is given by:

\[
(59) \quad w_1(P) = w_1(E) \quad w_2(P) + w_1(P)^2 + w_2(E) = 0.
\]

Negative signature in dimension \(d \equiv 7\). The topological obstruction to existence of a \(\text{Spin}_{+}\) structure on a negative Riemannian manifold of dimension \(d \equiv 7\) is given by:

\[
(60) \quad w_1(P) = w_1(E) \quad w_2(P) = w_2(E).
\]

In dimension \(d \equiv 3\) with positive signature (respectively in dimension \(d \equiv 7\) with negative signature), the results above show that a \(\text{Spin}_{-}\) structure (respectively a \(\text{Spin}_{-}\) structure) exists on \((M, g)\) iff there is there exists an \(O(2)\)-bundle \(E\) on \((M, g)\) whose first and second Stiefel-Whitney classes equal those of the (co)frame bundle. This implies, for example, that any Riemannian three-manifold of the form:

\[
(61) \quad M_3 = M_2 \times \mathbb{R},
\]

...
where \( M_2 \) is a smooth surface, admits a Spin\(^c \) structure. Indeed, one can take \( E \) to be the pull-back of the coframe bundle of \( M_2 \) through the canonical projection and use stability of Stiefel-Whitney classes. We will discuss other examples of manifolds admitting Spin\(^c \) structures in Section 6.

### 4. Elementary real pinor representations for \( p - q \equiv_8 3, 7 \)

Let \((V, h)\) be a quadratic space of signature \((p, q)\) and dimension \(d = p + q\). Throughout this section, we assume that \(p - q \equiv_8 3, 7\), i.e. that we are in the so-called almost complex case according to the terminology used in [8]. Notice that \(d\) is necessarily odd in this case.

**Definition 4.1.** Let:

\[
\text{Spin}^\alpha(V, h) \overset{\text{def}}{=} \text{Spin}_{p,q}^\alpha(V, h)
\]

where:

\[
\alpha \overset{\text{def}}{=} \alpha_{p,q} \overset{\text{def}}{=} (-1)^{\frac{p + q + 1}{2}} = \begin{cases} 
-1 & \text{if } p - q \equiv_8 3 \\
+1 & \text{if } p - q \equiv_8 7
\end{cases}
\]

**Definition 4.2.** An **elementary real pinor representation** of \(\text{Cl}(V, h)\) is an \(\mathbb{R}\)-irreducible representation \(\gamma_0 : \text{Cl}(V, h) \to \text{End}_\mathbb{R}(S_0)\) of the unital \(\mathbb{R}\)-algebra \(\text{Cl}(V, h)\) in a finite-dimensional \(\mathbb{R}\)-vector space \(S_0\).

It is well-known that all such representations are equivalent to each other. Moreover, the finite-dimensional \(\mathbb{R}\)-vector space \(S_0\) has dimension given by:

\[
N \overset{\text{def}}{=} \dim_{\mathbb{R}} S_0 = 2^{\frac{d + 1}{2}}.
\]

For the following, we fix an elementary real pinor representation \(\gamma_0\) of \(\text{Cl}(V, h)\).

#### 4.1. Natural subspaces of \(\text{End}_\mathbb{R}(S_0)\)

**Definition 4.3.** The **natural subspaces** of \(\text{End}_\mathbb{R}(S_0)\) are defined as follows:

1. The **Schur algebra** \(S\) \(\overset{\text{def}}{=} S(\gamma_0)\) is the unital subalgebra of \(\text{End}_\mathbb{R}(S_0)\) defined as the centralizer of \(\gamma_0\) inside \(\text{End}_\mathbb{R}(S_0)\):

\[
S = \{ T \in \text{End}_\mathbb{R}(S_0) \mid T\gamma_0(v) = \gamma_0(v)T \forall v \in V \}
\]

2. The **anticommutant subspace** \(A\) \(\overset{\text{def}}{=} A(\gamma_0)\) is the subspace of \(\text{End}_\mathbb{R}(S_0)\) defined through:

\[
A = \{ T \in \text{End}_\mathbb{R}(S_0) \mid T\gamma_0(v) = -\gamma_0(v)T \forall v \in V \}
\]

3. The **twist algebra** \(T\) \(\overset{\text{def}}{=} T(\gamma_0)\) is the unital subalgebra of \(\text{End}_\mathbb{R}(S_0)\) defined through:

\[
T = S + A.
\]

We have \(S \cap A = \{0\}\) and \(SA = AS = A\), so \(A\) is an \(S\)-bimodule. Thus (66) is a direct sum decomposition \(T = S \oplus A\) which gives a \(\mathbb{Z}_2\)-grading of \(T\) with components:

\[
T^+ = S, \quad T^- = A.
\]

Since we are in the almost complex case \(p - q \equiv_8 3, 7\), the Schur algebra \(S\) is isomorphic with \(\mathbb{C}\) (see [8]) and can be described as follows. Recall that any orientation of \(V\) determines a Clifford volume element \(\nu \in \text{Cl}(V, h)\), which satisfies:

\[
\nu^2 = -1.
\]

The Clifford volume element determined by the opposite orientation of \(V\) equals \(\nu\) and the unordered set \(s_V = \{\nu, -\nu\}\) is unambiguously determined by \((V, h)\); notice that \(s_V\) is a semilinear structure on \(V\). Setting \(J \overset{\text{def}}{=} \gamma_0(\nu)\), we have \(J^2 = -\text{id}_{S_0}\) and \(\gamma_0(\nu) = -J\). Hence \(S_0\) carries a natural semilinear structure \(s := s(\gamma_0) = \{J, -J\}\) (in the sense of Appendix B), which is independent of the orientation of \(V\). The unital subalgebra of \(\text{End}_\mathbb{R}(S_0)\) determined by this semilinear structure as in Proposition B.2 coincides with the Schur algebra \(S\), which therefore is given by:

\[
S = \mathbb{R} \oplus RJ = \mathbb{R} \oplus \mathbb{R}(-J).
\]
Proposition 4.4. There exists an $S$-antilinear automorphism $D \in \text{Aut}_R^S(S_0)$ such that:

1. $D\gamma_0(v) = -\gamma_0(v)D$ for all $v \in V$.
2. $D^2 = \alpha_{p,q} \text{id}_S$.

Moreover, any $S$-antilinear operator $D'$ on $S_0$ which satisfies these two conditions has the form:

$$D' = e^{\theta J} D,$$

for some $\theta \in \mathbb{R}$.

Proof. Let us denote by $\text{Cl}(p,q)$ the real Clifford algebra associated to a real pseudo-Riemannian vector space $\mathbb{R}^{p,q}$ of dimension $d = p + q$ and signature $(p,q)$. We first assume that $p - q \equiv \pm 1$ and consider the Clifford algebra $\text{Cl}(p,q + 1)$ associated to $\mathbb{R}^{p,q+1}$. From the representation theory of Clifford algebras we know that there is an elementary real Clifford representation:

$$\gamma'_0 : \text{Cl}(p,q + 1) \to \text{End}_R(S_0),$$

and that $\text{Cl}(p,q + 1)$ is generated by the canonical orthonormal basis of $\mathbb{R}^{p,q+1}$ together with the linearly independent unit element $x \in \mathbb{R}^{p,q+1}$ that completes it to the canonical basis of $\mathbb{R}^{p,q+1}$.

We set then $D \coloneqq \gamma'_0(x) \subset \text{End}_R(S_0)$, which is indeed $S$-antilinear and satisfies $D^2 = -1$ as well as $D\gamma_0(v) = -\gamma_0(v)D$ for all $v \in V$. The proof for the case $p - q \equiv \pm 7$ is similar but considering $\text{Cl}(p+1,q)$.

Given two $S$-antilinear endomorphisms $D$ and $D'$ satisfying conditions 1. and 2., we have $DD' \in S$ and $D'D \in S$. Moreover, we have $(DD')^2 = 1$, which implies $D' = e^{\theta J} D$ with $\theta \in \mathbb{R}$. □

Let $\mathfrak{D} := \mathfrak{D}(\gamma_0) \subset \text{Aut}_R(S_0)$ denote the $U(1)$-torsor consisting of all elements $D \in \text{Aut}_R^S(S_0)$ which satisfy the conditions of Proposition 4.4. Then the following result (whose proof we leave to the reader) holds:

Proposition 4.5. $\mathfrak{A}$ is a free $S$-bimodule of rank one and any $D \in \mathfrak{D}$ is a basis of this bimodule, hence satisfies $\mathfrak{A} = SD = DS$.

The proposition implies that the elements $D$ and $JD$ (where $D \in \mathfrak{D}$ is arbitrary) form a basis of the $R$-vector space $\mathfrak{A}$:

$$\mathfrak{A} = RD \oplus RJD.$$

Let $e_1, e_2$ be the canonical basis of $\mathbb{R}^2$ and consider the elements $D_0 \coloneqq e_1$, $J_0 \coloneqq e_1 e_2$ of $\text{Cl}_2(\alpha_{p,q})$. We have $e_1^2 = e_2^2 = \alpha_{p,q}$ and hence:

$$J_0^2 = -1, \quad D_0^2 = \alpha_{p,q}, \quad J_0 D_0 = -\alpha_{p,q} e_2.$$

Proposition 4.6. For any $(\nu, D) \in \mathfrak{s}_V \times \mathfrak{D}$, there exists a unique unital isomorphism of algebras $\gamma_{\nu,D}^{(2)} : \text{Cl}_2(\alpha_{p,q}) \cong \mathbb{T} \subset \text{End}_R(S_0)$ which satisfies the conditions:

$$\gamma_{\nu,D}^{(2)}(e_1) = \gamma_{\nu,D}^{(2)}(D_0) = D, \quad \gamma_{\nu,D}^{(2)}(e_2) = -\alpha_{p,q} JD,$$

where $J = \gamma(\nu)$. Moreover, $\gamma_{\nu,D}^{(2)}$ is an isomorphism of $\mathbb{Z}_2$-graded algebras and we have $\gamma_{\nu,D}^{(2)}(J_0) = J$.

Proof. We have $J^2 = -\text{id}_{S_0}$, $D^2 = \alpha_{p,q} \text{id}_{S_0}$ and $(-\alpha_{p,q} JD)^2 = JDJD = -J^2 D^2 = D^2 = \alpha_{p,q} \text{id}_{S_0}$ in $\mathbb{T}$. This implies existence and uniqueness of $\gamma_{\nu,D}^{(2)}$. Moreover, $\gamma_{\nu,D}^{(2)}(e_1) = \gamma_{\nu,D}^{(2)}(e_2) = -\alpha_{p,q} JD$ and $\gamma_{\nu,D}^{(2)}(e_1) e_2 = \gamma_{\nu,D}^{(2)}(e_2) e_1 = -\alpha_{p,q} JD$.

When no confusion is possible, in the following we will write $\alpha$ instead of $\alpha_{p,q}$. Since $\gamma_{\nu,D}^{(2)}$ is an isomorphism of $\mathbb{Z}_2$-graded algebras, we have:

$$\gamma_{\nu,D}^{(2)}(\mathbb{R}^2) = \mathbb{A}, \quad \gamma_{\nu,D}^{(2)}(\text{Cl}_2^+(\alpha)) = S.$$
Notice the commutative diagram:

\[
\begin{array}{cccc}
\text{Cl}_2(\alpha)^\times & \overset{\gamma_{\nu,D}^{(2)}}{\sim} & T^\times \\
\downarrow\text{Ad} & & \downarrow\text{Ad} \\
\text{Aut}_{\text{Alg}}(\text{Cl}_2(\alpha)) & \overset{\sim}{\text{Ad}(\gamma_{\nu,D}^{(2)})} & \text{Aut}_{\text{Alg}}(T)
\end{array}
\]

Also notice that the untwisted adjoint representation of \(\text{Pin}_2(\alpha)\) preserves the subalgebra \(\text{Cl}_2^+(\alpha)\). Let \(\text{Ad}_+^{(2)}: \text{Pin}_2(\alpha) \to \text{Aut}_{\text{Alg}}(\text{Cl}_2^+(\alpha))\) be the group morphism given by:

\[\text{Ad}_+^{(2)}(a) \overset{\text{def}}{=} \text{Ad}(a)|_{\text{Cl}_2^+(\alpha)} \quad \forall a \in \text{Pin}_2(\alpha).
\]

Recall that \(\text{Pin}_2(\alpha) = \text{Spin}_2(\alpha) \sqcup \text{Spin}_2(\alpha)D\) and that \(\text{Spin}_2(\alpha) = \{z(\theta) = e^{\theta J_0}|\theta \in \mathbb{R}\} \simeq \text{U}(1)\) acts trivially in the adjoint representation on \(\text{Cl}_2^+(\alpha) = \mathbb{R} \oplus \mathbb{R}J_0 \simeq \mathbb{C}\), while \(D_0\) acts in the adjoint representation by taking \(J_0\) into \(-J_0\). Hence:

\[
\text{Ad}_+^{(2)}(a) = \text{id}_{\text{Cl}_2^+(\alpha)}, \quad \text{Ad}_+^{(2)}(aD_0) = c_+, \quad \forall a \in \text{Spin}_2(\alpha),
\]

where \(c_+ \in \text{Aut}_{\text{Alg}}(\text{Cl}_2^+(\alpha))\) denotes the conjugation automorphism:

\[c_+(x + yJ_0) = x - yJ_0 \quad \forall x, y \in \mathbb{R}.
\]

Since \(\text{Cl}_2^+(\alpha) \simeq_{\text{Alg}} \mathbb{C}\), we have \(\text{Aut}_{\text{Alg}}(\text{Cl}_2^+(\alpha))\) is equivalent with \(\{\text{id}_{\text{Cl}_2^+(\alpha)}, c_+\} \simeq \mathbb{Z}_2\) and (70) shows that the group morphism \(\text{Ad}_+^{(2)}\) can be identified with the \(\mathbb{Z}_2\)-grading morphism of \(\text{Pin}_2(\alpha)\).

**Proposition 4.7.** We have \((\text{Ad} \circ \gamma_{\nu,D}^{(2)})(\text{Pin}_2(\alpha))(\mathbb{S}) = \mathbb{S}, \quad (\text{Ad} \circ \gamma_{\nu,D}^{(2)})(\text{Pin}_2(\alpha))(\mathbb{A}) = \mathbb{A}\) and \((\text{Ad} \circ \gamma_{\nu,D}^{(2)})(\text{Pin}_2(\alpha))(T) = T\). Moreover:

1. The representation \(\text{Ad}_+^{(2)}: \text{Pin}_2(\alpha) \to \text{Aut}_{\text{Alg}}(\mathbb{T})\) given by \(\text{Ad}_+^{(2)}(b) \overset{\text{def}}{=} \text{Ad}(\gamma_{\nu,D}^{(2)}(b)|_T)\) is equivalent with the full untwisted adjoint representation of \(\text{Pin}_2(\alpha)\) on \(\text{Cl}_2(\alpha)\).
2. The representation \(\text{Ad}_+^{(2)}: \text{Pin}_2(\alpha) \to \text{Aut}_{\mathbb{A}}(\mathbb{A})\) given by \(\text{Ad}_+^{(2)}(b) \overset{\text{def}}{=} \text{Ad}(\gamma_{\nu,D}^{(2)}(b)|_\mathbb{A}) \in \text{Aut}_{\mathbb{A}}(\mathbb{S})\) is equivalent with the untwisted vector representation \(\gamma_{\nu,D}^{(2)}\) of \(\text{Pin}_2(\alpha)\).
3. The representation \(\text{Ad}_+^{(2)}: \text{Pin}_2(\alpha) \to \text{Aut}_{\text{Alg}}(\mathbb{S}) \simeq \mathbb{Z}_2\) given by \(\text{Ad}_+^{(2)}(b) \overset{\text{def}}{=} \text{Ad}(\gamma_{\nu,D}^{(2)}(b)|_\mathbb{S})\) is equivalent with the representation:

\[
\text{Ad}_+^{(2)}: \text{Pin}_2(\alpha) \to \text{Aut}_{\text{Alg}}(\text{Cl}_2^+(\alpha)),
\]

being given by:

\[
\text{Ad}_+^{(2)}(a) = \text{id}_{\mathbb{S}}, \quad \text{Ad}_+^{(2)}(aD) = c, \quad \forall a \in \text{Spin}_2(\alpha),
\]

where \(c \in \text{Aut}_{\text{Alg}}(\mathbb{S})\) is the conjugation automorphism of \(\mathbb{S}\).

**Proof.** The commutative diagram (69) gives \(\text{Ad} \circ \gamma_{\nu,D}^{(2)} = \text{Ad}(\gamma_{\nu,D}^{(2)}) \circ \text{Ad}\). This implies:

\[
(\text{Ad} \circ \gamma_{\nu,D}^{(2)})(\text{Cl}_2(\alpha)^\times)(\mathbb{T}) = \mathbb{T},
\]

which by restriction gives \((\text{Ad} \circ \gamma_{\nu,D}^{(2)})(\text{Pin}_2(\alpha))(\mathbb{T}) = \mathbb{T}\). On the other hand, we have:

\[
(\text{Ad} \circ \gamma_{\nu,D}^{(2)})(\text{Pin}_2(\alpha))(\mathbb{A}) = \text{Ad}(\gamma_{\nu,D}^{(2)})(\text{Ad}(\text{Pin}_2(\alpha))(\gamma_{\nu,D}^{(2)}(\mathbb{R}^2))) = \\
= \gamma_{\nu,D}^{(2)}(\text{Ad}(\text{Pin}_2(\alpha))(\mathbb{R}^2)) = \gamma_{\nu,D}^{(2)}(\mathbb{R}^2) = \mathbb{A}.
\]

where we used (68) and the relation:

\[
\text{Ad}(\gamma_{\nu,D}^{(2)})(\Phi)(\gamma_{\nu,D}^{(2)}(x)) = \gamma_{\nu,D}^{(2)}(\Phi(x)), \quad \forall \Phi \in \text{Aut}_{\text{Alg}}(\text{Cl}_2(\alpha)), \quad \forall x \in \text{Cl}_2(\alpha),
\]
together with the fact that the untwisted adjoint representation of $\text{Pin}_2(\alpha)$ preserves the subspace $\mathbb{R}^2 \subset \text{Cl}_2(\alpha)$. Similarly, we have:

$$\text{Ad}(\gamma_{\nu,D}^{(2)})((\text{Pin}_2(\alpha))(\mathbb{S})) = \text{Ad}(\gamma_{\nu,D}^{(2)})((\text{Pin}_2(\alpha))(\gamma_{\nu,D}^{(2)}(\text{Cl}_2^+(\alpha)))) =$$

$$= \gamma_{\nu,D}^{(2)}(\text{Ad}(\text{Pin}_2(\alpha))(\text{Cl}_2^+(\alpha))) = \gamma_{\nu,D}^{(2)}(\text{Cl}_2^+(\alpha)) = \mathbb{S},$$

where we used (68) and (71) and that fact that untwisted adjoint representation of $\text{Pin}_2(\alpha)$ preserves the subalgebra $\text{Cl}_2^+(\alpha) \subset \text{Cl}_2(\alpha)$. By restriction, diagram (69) induces commutative diagrams:

\[
\begin{array}{ccc}
\text{Pin}_2(\alpha) & \xrightarrow{\text{Ad}} & \text{Pin}_2(\alpha) \\
\text{Aut}_{\text{Alg}}(\text{Cl}(V,h)) & \xrightarrow{\text{Ad}^{(2)}} & \text{Aut}_{\text{Alg}}(\text{T}) \\
\end{array}
\]

which give the equivalences of representations listed in the proposition.

Let $\overline{\text{Pin}}(V,h) \overset{\text{def}}{=} \text{Pin}_{\alpha,p,q}(V,h)$. Notice that any $D \in \mathcal{D}$ gives an isomorphism of torsors $\tau_D : \text{Spin}^{\circ}(V,h)/\overline{\text{Pin}}(V,h) \xrightarrow{\sim} \mathcal{D}$ such that $\tau([D]) = D$, where $[D]$ is the class of $D$ in $\text{Spin}^{\circ}(V,h)/\overline{\text{Pin}}(V,h)$.

**Proposition 4.8.** Let $\nu, \nu' \in \mathfrak{s}_V$ and $D, D' \in \mathcal{D}$. Then the representations $\gamma_{\nu,D}^{(2)}$ and $\gamma_{\nu',D'}^{(2)}$ of $\text{Cl}_2(\alpha)$ are equivalent.

**Proof.** We have $D' = e^{\theta J}D$ for some $\theta \in \mathbb{R}$. Thus $\text{Ad}(e^{\theta J})(D) = \text{Ad}(e^{\theta J}(D) = D'$ and $\text{Ad}(e^{\theta J})(J) = -\text{Ad}(e^{\theta J}(J) = J$. Since $\text{Cl}_2(\alpha)$ is generated by $J_0$ and $D_0$, this implies:

$$\text{Ad}(e^{\theta J}) \circ \gamma_{\nu,D} = \gamma_{\nu',D'} \circ \text{Ad}(e^{\theta J}), \text{Ad}(e^{\theta J}) \circ \gamma_{\nu,D} = -\nu',D' \circ \text{Ad}(e^{\theta J}).$$

Since $\nu'$ equals either $\nu$ or $-\nu$, this gives the conclusion.

**Definition 4.9.** A $\gamma$-compatible representation of $\text{Cl}_2(\alpha,p,q)$ is an $\mathbb{R}$-linear representation $\gamma^{(2)} : \text{Cl}_2(\alpha,p,q) \rightarrow \text{End}_\mathbb{R}(\mathfrak{s}_V)$ which equals $\gamma_{\nu,D}^{(2)}$ for some pair $(\nu,D) \in \mathfrak{s}_V \times \mathcal{D}$. By the proposition above, all such representations of $\text{Cl}_2(\alpha,p,q)$ are equivalent. In the following, we fix a $\gamma$-compatible representation $\gamma^{(2)}$ of $\text{Cl}_2(\alpha,p,q)$ parameterized by the pair $(\nu,D) \in \mathfrak{s}_V \times \mathcal{D}$.

### 4.2. Twisted elementary real representations.

**Definition 4.10.** The twisted elementary real representation of $\text{Spin}^{\circ}_{\alpha,p,q}(V,h)$ induced by $\gamma^{(2)}$ is the group morphism $\gamma_{\alpha} : \text{Spin}^{\circ}_{\alpha,p,q}(V,h) \rightarrow \text{Aut}_{\mathbb{R}}(\mathfrak{s}_V)$ given by:

$$\gamma_{\alpha}([a,b]) = \gamma_0(a)^{(2)}(b) \forall a \in \text{Spin}(V,h), b \in \text{Pin}_2(\alpha,p,q).$$

By Proposition 4.8, twisted elementary representations of $\text{Spin}^{\circ}_{\alpha,p,q}(V,h)$ are uniquely determined up to equivalence. The representation $\text{Ad}_+^{(2)} : \text{Pin}_2(\alpha) \rightarrow \text{Aut}_{\text{Alg}}(\text{Cl}_2^+(\alpha))$ induces a representation $\text{Ad}_+^{(2)} : \text{Spin}^{\circ}_{\alpha,p,q}(\alpha) \rightarrow \text{Aut}_{\text{Alg}}(\text{Cl}_2^+(\alpha))$ given by:

$$\text{Ad}_+(a,b) \overset{\text{def}}{=} \text{Ad}_+^{(2)}(b) \forall a \in \text{Spin}(V,h), \forall b \in \text{Pin}_2(\alpha).$$

Together with the characteristic representation

$$\mu : \text{Spin}^{\circ}_{\alpha,p,q}(V,h) \rightarrow O(\mathbb{R}^2) = \text{Aut}_{\mathbb{R}}(\text{Cl}_2(\alpha))$$

this gives a representation

$$\text{Ad}_+ \oplus \mu : \text{Spin}^{\circ}_{\alpha,p,q}(V,h) \rightarrow \text{Aut}_{\text{Alg}}(\text{Cl}_2(\alpha)).$$

**Proposition 4.11.** We have $(\text{Ad} \circ \gamma_{\alpha})(\text{Spin}^{\circ}_{\alpha,p,q}(V,h))(\mathbb{S}) = \mathbb{S}$, $(\text{Ad} \circ \gamma_{\alpha})(\text{Spin}^{\circ}_{\alpha,p,q}(V,h))(\mathbb{K}) = \mathbb{K}$ and $(\text{Ad} \circ \gamma_{\alpha})(\text{Spin}^{\circ}_{\alpha,p,q}(V,h))(\mathbb{T}) = \mathbb{T}$. Moreover:
1. The representation $\text{Ad}_h : \text{Spin}^o_{\alpha,p,q}(V,h) \to \text{Aut}_{\mathbb{R}}(\mathbb{R}^2)$ given by $\text{Ad}_h(g) \overset{\text{def}}{=} (\text{Ad} \circ \gamma_o)(g)|_h$, i.e.:

$$\text{Ad}_h([a,b]) = \text{Ad}_h^2(b) \forall a \in \text{Spin}(V,h), \forall b \in \text{Pin}_2(\alpha_{p,q})$$

is equivalent with the untwisted characteristic representation $\mu : \text{Spin}^o_{\alpha,p,q}(V,h) \to O(\mathbb{R}^2) = O(2)$.

2. The representation $\text{Ad}_S : \text{Spin}^o_{\alpha,p,q}(V,h) \to \text{Aut}_{\mathbb{R}}(S)$ given by $\text{Ad}_S(a) \overset{\text{def}}{=} \text{Ad} \circ \gamma_o(a)|_S$, i.e.:

$$\text{Ad}_S([a,b]) = \text{Ad}_S^2(b) \forall a \in \text{Spin}(V,h), \forall b \in \text{Pin}_2(\alpha_{p,q})$$

is equivalent with the representation $\text{Ad}_+ : \text{Spin}^o_{\alpha,p,q}(V,h) \to \text{Aut}_{\mathbb{R}}(\text{Cl}_2^+(\alpha))$.

3. The representation $\text{Ad}_T : \text{Spin}^o_{\alpha,p,q}(V,h) \to \text{Aut}_{\mathbb{R}}(T)$ defined through $\text{Ad}_T(g) \overset{\text{def}}{=} (\text{Ad} \circ \gamma_o)(g)|_T$, i.e.:

$$\text{Ad}_T^2([a,b]) = \text{Ad}_T^2(b) \forall a \in \text{Spin}(V,h), \forall b \in \text{Pin}_2(\alpha_{p,q})$$

is equivalent with the representation $\text{Ad}_+ + \mu : \text{Spin}^o_{\alpha,p,q}(V,h) \to \text{Aut}_{\mathbb{R}}(\text{Cl}_2(\alpha_{p,q}))$.

Proof. Follows from Proposition 4.11 and Definition 2.6. \hfill \Box

Let $i : \text{Spin}^o_{\alpha,p,q}(V,h) \to \text{Aut}_{\mathbb{R}}(V \otimes S_0) \simeq \text{Aut}_{\mathbb{R}}(V) \otimes \text{Aut}_{\mathbb{R}}(S_0)$ denote the inner tensor vector representation $\lambda_o : \text{Spin}^o_{\alpha}(V,h) \to \text{Aut}_{\mathbb{R}}(V)$ with the twisted elementary representation $\gamma_o : \text{Spin}^o_{\alpha}(V,h) \to \text{Aut}_{\mathbb{R}}(S_0)$:

$$i(g) \overset{\text{def}}{=} \lambda_o(g) \otimes \gamma_o(g) \forall g \in \text{Spin}^o_{\alpha}(V,h).$$

**Lemma 4.12.** For all $g \in \text{Spin}^o_{\alpha,p,q}(V,h)$ and all $v \in V$, we have:

$$\gamma_0(\lambda(g)v) \circ \gamma_o(g) = \gamma_o(g) \circ \gamma_0(v).$$

Proof. For any $g = [a,b] \in \text{Spin}^o_{\alpha,p,q}(V,h)$ with $a \in \text{Spin}(V,h)$ and $b \in \text{Pin}_2(\alpha)$, we have:

$$\gamma_0(\lambda(g)v) \circ \gamma_o(g) = (\text{det} \text{Ad}_0^2(b)) \gamma_0(\text{Ad}_0(a)v) \circ \gamma_0(a) \circ \gamma^2(b),$$

$$\gamma_o(g) \circ \gamma_0(v) = \gamma_0(a) \circ \gamma^2(b) \circ \gamma_0(v) = (\text{det} \text{Ad}_0^2(b)) \gamma_0(\text{Ad}_0(a)v) \circ \gamma_0(a) \circ \gamma^2(b),$$

where in the second row we used the relation:

$$\gamma^2(b) \circ \gamma_0(v) = (\text{det} \text{Ad}_0^2(b)) \gamma_0(\lambda(g)v) \circ \gamma^2(b),$$

which follows from the fact that $D$ and $\gamma_0(v)$ anticommute and the relation:

$$\gamma_0(a) \circ \gamma_0(v) = \gamma_0(\text{Ad}_0(a)v) \circ \gamma_0(a),$$

which in turn follows from $\text{Ad}_0(a)(v) = ava^{-1}$ and from the fact that $\gamma_0$ is a morphism of algebras. Comparing the two rows in (73) gives the conclusion. \hfill \Box

**Lemma 4.13.** The linear map $\mathfrak{c} : V \otimes S_0 \to S_0$ given by $\mathfrak{c}(v,x) \overset{\text{def}}{=} \gamma_0(v)x$ is a morphism of representations between $i$ and $\gamma_o$, i.e. it satisfies:

$$\mathfrak{c} \circ i(g) = \gamma_0(g) \circ \mathfrak{c},$$

for all $g \in \text{Spin}^o(V,h)$.

Proof. For all $v \in V$, $s \in S_0$ and $g \in \text{Spin}^o(V,h)$, we have:

$$(\mathfrak{c} \circ i(g))(v \otimes s) = (\gamma_0(\lambda(g)v) \circ \gamma_o(g))(s) = (\gamma_0(g) \circ \gamma_0(v))(s) = (\gamma_o(g) \circ \mathfrak{c})(v \otimes s),$$

where we used (72). Hence $\mathfrak{c} \circ i(g) = \gamma_0(g) \circ \mathfrak{c}$ for all $g \in \text{Spin}^o(V,h)$. \hfill \Box

**Corollary 4.14.** The linear map $\mathfrak{c} : V \otimes S_0 \to S_0$ defines a $\gamma_o$-equivariant Clifford multiplication map on $S_0$, which canonically becomes an elementary module over $\text{Cl}(V,h)$ when equipped with the unital morphism of algebras $\gamma_0 : \text{Cl}(V,h) \to \text{End}_{\mathbb{R}}(S_0)$ obtained from $\mathfrak{c}$ by extension with respect to the algebra structure of $\text{Cl}(V,h)$. 
Proof. Let \((\gamma_0, S_0)\) be a twisted elementary representation of \(\text{Spin}_0^{\alpha_{p,q}}(V, h)\). The linear map \(\gamma : V \otimes S_0 \rightarrow S_0\) defines by evaluation \(v \mapsto \gamma(v, -)\) on its first argument a Clifford multiplication map \(\gamma_V : V \rightarrow \text{End}_R(S_0)\). The fact that \(\gamma_V\) is \(\gamma_0\)-equivariant follows from Lemma 4.13. The Clifford map \(\gamma_V\) is in particular an injective morphism from \(V \subset \text{Cl}(V, h)\) to \(\text{End}_R(S_0)\). Since \(V\) generates \(\text{Cl}(V, h)\), we extend this map to all of \(\text{Cl}(V, h)\), thus obtaining the elementary Clifford representation \(\gamma_0 : \text{Cl}(V, h) \rightarrow \text{End}_R(S_0)\).

\[\square\]

5. Elementary real pinor bundles for \(p - q \equiv 3, 7\)

Throughout this section, we assume that \((M, g)\) is a smooth, connected and paracompact pseudo-Riemannian manifold of signature \((p, q)\) with \(p - q \equiv 3, 7\) and dimension \(d = p + q\). We define:

\[w_1^\pm(M) \overset{\text{def}}{=} w_1^\pm(P_{O(V, h)}(M, g))\]

where \(P_{O(V, h)}(M, g)\) denotes the orthonormal coframe bundle of \((M, g)\). Let \(\text{Cl}(M, g)\) denote the Clifford bundle of the pseudo-Euclidean cotangent bundle \((T^*M, g^+)\), where \(g^+\) is the metric induced by \(g\) on \(T^*M\).

Definition 5.1. An elementary real pinor bundle is a bundle of finite-dimensional simple modules over the Clifford bundle \(\text{Cl}(M, g)\), i.e. a pair \((S, \gamma)\) where \(S\) is a real vector bundle over \((M, g)\) and \(\gamma : \text{Cl}(M, g) \rightarrow \text{End}_R(S)\) is a morphism of bundles of unital associative \(\mathbb{R}\)-algebras that restricts to an elementary real pinor representation \(\gamma_m : \text{Cl}(T^*_mM, g_m) \rightarrow \text{End}_R(S_m)\) at every point \(m \in M\).

Given an elementary real pinor bundle \((S, \gamma)\), we define its type \(\eta\) as the isomorphism class of \(\gamma_m : \text{Cl}(T^*_mM, g_m) \rightarrow \text{End}_R(S_m)\) in the category \(\text{ClRep}\) introduced in [8], where \(m \in M\) denotes any point of \(M\). It can be easily seen that the type does not depend on the point chosen since \(M\) is connected. Here \(\text{ClRep}\) denotes the category of real Clifford representations and unbased morphisms, see op. cit. for details. For ease of notation, in the following we will sometimes denote \(\text{Spin}_0^{\alpha_{p,q}}(V, h)\) simply by \(\text{Spin}_0\).

Definition 5.2. An adapted \(\text{Spin}_0^c\) structure \((Q, \hat{\Lambda})\) on \((M, g)\) is a \(\text{Spin}_0^{\alpha_{p,q}}\) structure \((Q, \hat{\Lambda}_{\alpha_{p,q}})\) on \((M, g)\), where \(\alpha_{p,q}\) was defined in (62).

Assume that \((M, g)\) admits an adapted \(\text{Spin}_0^c\) structure and let \((V, h)\) be a quadratic vector space which is isometric with \((T^*_mM, g^*_m)\), where \(m\) is any point of \(M\).

Definition 5.3. A twisted \(\text{Spin}_0^c\) vector bundle over \((M, g)\) is a real vector bundle \(S_0 = Q \times_{S_0} S_0\) associated to the principal \(\text{Spin}_0^c(V, h)\)-bundle \(Q\) of an adapted \(\text{Spin}_0^c\) structure \((Q, \hat{\Lambda})\) on \(M\) through a twisted elementary representation \(\gamma_o : \text{Spin}_0^c(V, h) \rightarrow \text{Aut}_R(S_0)\) of \(\text{Spin}_0^{\alpha_{p,q}}(V, h)\).

When necessary we will denote by \(S_o(Q, \gamma_o)\) the twisted \(\text{Spin}_0^c\) vector bundle associated to \(Q\) through \(\gamma_o\). Notice that a twisted \(\text{Spin}_0^c\) vector bundle is a real vector bundle of rank \(\text{rk}S = N = \dim S_0 = 2^{\frac{q+1}{2}}\) (see equation (63)). We will see in a moment that such a bundle admits a well-defined Clifford multiplication, even though \((M, g)\) need not be orientable. In particular, the existence of a \(\text{Spin}_0^c\) structure allows for a useful notion of “irreducible real pinors” on non-orientable\(^5\) pseudo-Riemannian manifolds of signature \(p - q \equiv 3, 7\).

Proposition 5.4. Let \(S_o\) be the twisted \(\text{Spin}_0^c\) vector bundle associated to an adapted \(\text{Spin}_0^c\) structure \((Q, \hat{\Lambda})\) through the twisted elementary representation \(\gamma_o : \text{Spin}_0^c(V, h) \rightarrow \text{Aut}_R(S_0)\). Then the based

\(^5\)We use the Clifford bundle of the cotangent (rather than of the tangent) bundle of \(M\) in order to avoid using musical isomorphisms in certain formulas later on.

\(^6\)This is in marked contrast with the kind of pinors considered in [10, 11], which require \((M, g)\) to admit a \(\text{Spin}_0\) structure (see Appendix A) and hence require that \(M\) be orientable.
bundle map $\mathcal{E} : T^*M \otimes S_o \rightarrow S_o$ given by:

$$\mathcal{E} : T^*M \otimes S_o \rightarrow S_o, \quad [p, \xi \otimes [p, s]] \mapsto [p, \gamma_0(\xi)s], \quad \forall p \in M,$$

defines a Clifford multiplication on $S_o$. This makes $S_o$ into an elementary real pinor bundle when the latter is equipped with the unital morphism of bundles of algebras $\gamma : \text{Cl}(M, g) \rightarrow \text{End}_\mathbb{R}(S)$ obtained from $\mathcal{E}$ by extension.

**Proof.** Recall that an elementary real pinor bundle is a bundle of simple modules over $\text{Cl}(M, g)$. In signature $p - q \equiv_8 3, 7$ such a bundle is of complex type. For convenience, we consider the cotangent bundle $T^*M$ as an associated bundle to $Q$ through the twisted vector representation $\lambda : \text{Spin}^o(V, h) \rightarrow \text{O}(V, h)$, that is, $T^*M = Q \times_\lambda V$. The fact that $\mathcal{E}$ is well defined follows from the following computation:

$$\mathcal{E}([pg, \lambda(g^{-1})\xi], [pg, \gamma_0(g^{-1})s]) = [pg, \gamma_0(\lambda(g^{-1})\xi) \circ \gamma_0(g^{-1})s] = [pg, \gamma_0(g^{-1}) \circ \gamma_0(\xi)s] = \mathcal{E}([p, \xi], [p, s]),$$

where we have used Lemma (4.12). Using $\mathcal{E}$ we pointwise define an injective morphism $\gamma_{T^*M} : T^*M \rightarrow \text{End}_\mathbb{R}(S_o)$ as follows:

$$\gamma_{T^*M} : T^*M \rightarrow \text{End}_\mathbb{R}(S_o), \quad [p, \xi] \mapsto \mathcal{E}([p, \xi], -),$$

Note that $\gamma_{T^*M}(\xi)^2 = g^*(\xi, \xi)\text{id}_{S_o}$, and hence $\gamma_{T^*M}$ extends to a unique unital morphism of bundles of algebras:

$$\gamma : \text{Cl}(M, g) \rightarrow \text{End}_\mathbb{R}(S_o),$$

whose fiber at any $p \in M$ is equivalent with the unique, up to unbased isomorphism, elementary Clifford representation $\gamma_0 : \text{Cl}(V, h) \rightarrow \text{End}(S_0)$ through an unbased isomorphism of Clifford representations.

The following result unifies Theorem 3.14 with the previous discussion.

**Theorem 5.1.** Let $(M, g)$ be a pseudo-Riemannian manifold of signature $(p, q)$ with $p - q \equiv_8 3, 7$. Then $(M, g)$ admits an elementary real pinor bundle $(S, \gamma)$ if and only if there exists a principal $O(2)$-bundle $E$ over $M$ such that the following conditions are satisfied:

$$w_1^+(M) + w_1^-(M) = w_1(E),$$

$$w_2^+(M) + w_2^-(M) + w_1(E)(pw_1^+(M) + qw_1^-(M)) = w_2(E) + \left[\delta(p, q) + \frac{p(p + 1)}{2} + \frac{q(q + 1)}{2}\right]w_1(E)^2,$$

where:

$$\delta(p, q) = \begin{cases} 1 & \text{if } p - q \equiv_8 3 \\ 0 & \text{if } p - q \equiv_8 7 \end{cases}.$$

Let $[\eta]$ be the type of $(S, \gamma)$. In that case, and relative to $[\eta]$, there exists an adapted $\text{Spin}^o(V, h)$ structure $Q(S, \gamma)$ on $(M, g)$, unique up to isomorphism, such that $(S, \gamma)$ is isomorphic to $(S_0(Q(S, \gamma)), \gamma_0)$ as a bundle of irreducible Clifford modules, and Clifford multiplication in $S$ is implemented by the morphism of vector bundles $\mathcal{E} : T^*M \otimes S \rightarrow S$ defined in equation (74).

**Remark 5.5.** Notice from conditions (76) that existence of a bundle of elementary real pinors on $(M, g)$ does not necessarily imply that $M$ is orientable. We will present examples of such situation in section (6).

**Proof.** Theorem 6.2 of [8] implies that $(M, g)$ admits a bundle of irreducible real Clifford modules $(S, \gamma)$ if and only if admits a reduced Lipschitz structure, which in signature $p - q \equiv_8 3, 7$ corresponds to a $\text{Spin}^o$ structure $Q(S, \gamma)$ as defined in Section 3. Furthermore, $(S, \gamma)$ is associated to $Q$ through the tautological representation $\gamma_0 : \text{Spin}^o(V, h) \rightarrow \text{Aut}_\mathbb{R}(S_0)$. From this we conclude that $(M, g)$
admits a bundle of irreducible real Clifford modules if and only if conditions (54) of Theorem 3.14 are satisfies for $\alpha = \alpha_{p,q}$. Since $d$ is odd, we have $d \equiv 2 \pmod{1}$. Moreover, we have $\delta_{\alpha_{p,q}, -1} = \delta(p, q)$, so conditions (54) reduce to (76).

Let $(S, \gamma)$ be an elementary pinor bundle of type $[\eta]$ and let $Q(S, \gamma)$ the unique (up to isomorphism) associated adapted $\text{Spin}^o$ structure relative to $[\eta]$. Let $(S_0, \gamma_0)$ denote the corresponding twisted $\text{Spin}^o$ vector bundle. The fact that $(S, \gamma)$ is isomorphic to $(S_0, \gamma_0)$ follows from the equivalence between:

$$
\gamma_0: \text{Spin}^o(V, h) \to \text{Aut}_R(S_0),
$$

and $\gamma_o: \text{Spin}^o(V, h) \to \text{Aut}_R(S_0)$ together with the equivalence of the associated Clifford multiplications upon use of Corollary 5.4.

\[ \square \]

Remark 5.6. Theorem 5.1 implies that twisted $\text{Spin}^o$ vector bundles $(S_0, \gamma_0)$ over $(M, g)$ are essentially equivalent to bundles of irreducible real Clifford modules over $\text{Cl}(M, g)$ in signature $p - q \equiv \delta 3, 7$. The classification of $\text{Spin}^o$ structures is surprisingly subtle and deserving a separate study.

A based isomorphism of elementary real pinor bundles $f: (S, \gamma) \to (S', \gamma')$ induces an isomorphism of $\text{Spin}^o$ structures relative to $\eta$: $\text{Cl}(V, h) \to \text{End}_R(S_0)$:

$$
P_\eta(f): Q(S, \gamma) \to Q(S', \gamma'),
$$

where $Q(S, \gamma_0)$ (respectively $Q_o(S', \gamma'_0)$) denotes the unique adapted $\text{Spin}^o$ structure corresponding to $(S, \gamma_0)$ (respectively $(S', \gamma'_0)$) relative to $\eta$. Given an elementary real pinor representation $\eta: \text{Cl}(V, h) \to \text{End}_R(S_0)$, an isomorphism of $\text{Spin}^o$ structures $f: Q \to Q'$ induces a based isomorphism:

$$
S_\eta(f) = S(Q, \eta) \to S(Q', \eta)
$$

of elementary real pinor bundles, where $S(Q, \eta)$ (respectively $S(Q', \eta)$) denotes the elementary real pinor bundle associated to $Q$ (respectively $Q'$) through the representation $\eta$.

Relative to $[\eta]$, the correspondence defined above gives mutually quasi-inverse functors:

$$
P_\eta: \text{ClB}_\eta^\otimes(M, g) \to \text{C}_\eta^o(M, g),
$$

and

$$
S_\eta: \text{C}_\eta^o(M, g) \to \text{ClB}_\eta^\otimes(M, g),
$$

between the grupoid $\text{C}_\eta^o(M, g)$ of adapted $\text{Spin}^o$ structures on $(M, g)$ of type $[\eta]$ and the grupoid $\text{ClB}_\eta^\otimes(M, g)$ of elementary real pinor bundles of type $[\eta]$ over $(M, g)$ and based pinor bundle isomorphisms. This is a particular case of the general correspondence between Lipschitz structures and bundles of irreducible real Clifford modules presented in reference [8].

5.1. Natural sub-bundles of $\text{End}_R(S)$. By the associated bundle construction, the subspaces $S, T, A \subset \text{End}_R(S_0)$ of Section 4 globalize to linear sub-bundles $\Sigma, T$ and $A$ of $\text{End}_R(S)$ which can be described intrinsically as follows:

**Definition 5.7.** The **natural fiber sub-bundles of $\text{End}_R(S)$** are defined as follows:

1. The **Schur bundle** $\Sigma$ has fibers:

$$
\Sigma_p \overset{\text{def}}{=} \{ T \in \text{End}_R(S_p) | T\gamma_p(v) = \gamma_p(v)T \forall v \in T_p^* M \}.
$$

2. The **anticommutant bundle** $A$ has fibers:

$$
A_p \overset{\text{def}}{=} \{ T \in \text{End}_R(S_p) | T\gamma_p(v) = -\gamma_p(v)T \forall v \in T_p^* M \}.
$$

3. The **twist bundle** $T$ is the direct sum:

$$
T = \Sigma \oplus A.
$$
(4) The conjugation bundle $D$ has fibers:
$$D_p = \{ D \in \text{End}_\mathbb{C}^p(S_p) | \gamma_p(v) = -D_p(v) \ \forall v \in T_p \}.$$

Notice that $\Sigma$ and $T$ are bundles of unital subalgebras of $\text{End}_\mathbb{C}(S)$ while $A$ is only a vector bundle.

5.2. Certain reductions of structure group. Let $L = \det T^\ast M = \wedge^dT^\ast M$ denote the orientation line bundle of $M$. The metric $g$ of $M$ determines a principal $\mathbb{Z}_2$-subbundle $P_{Z_2}(M)$ of $L$ whose fiber at $p \in M$ consists of the normalised volume forms $\nu_p = -\nu_p$ of $(T_pM, g_p)$ determined by the two orientations of $T_pM$. Since $\nu^2_p = (-\nu_p)^2 = -1$ in $\text{Cl}(T^*_pM, g^*_p)$, the space $\mathbb{R} \oplus L_p$ is a subalgebra of $\text{Cl}(T^*_pM, g^*_p)$ and each choice of orientation of $T_pM$ determines an isomorphism from this subalgebra to $\mathbb{C}$.

**Definition 5.8.** The complex orientation bundle $L^c$ of $(M, g)$ is the bundle of unital subalgebras of $\text{Cl}(T^*_M, g^*)$ whose fiber at $p \in M$ equals $L^c_p = \mathbb{R} \oplus L_p$.

It is clear from the above that $L^c$ is a $\mathbb{C}$-bundle whose imaginary line sub-bundle equals $L$. Thus $w_c(L^c) = w_1(L)$ and hence $L^c$ is trivial iff $M$ is orientable. Notice that $L^c$ is orientable (i.e. the structure group of $L^c$ reduces from $\text{SO}(2)$ to $\mathbb{Z}_2$) iff $L^c$ is trivial.

**Proposition 5.9.** $\gamma$ restricts to an isomorphism from $L^c$ to $\Sigma$. Moreover, the following statements are equivalent:

(a) $M$ is orientable.
(b) The Schur bundle $\Sigma \subset \text{End}_\mathbb{C}(S)$ is trivial.
(c) The structure group of $S$ reduces from $\text{Spin}^o(V, h)$ to $\text{Spin}^c(V, h)$.

**Proof.** Since $\gamma_p : \text{Cl}(T^*_pM, g^*_p) \rightarrow \text{End}_\mathbb{C}(S)$ is faithful for all $p \in M$ and $\gamma(L) = \Sigma$, it is clear that $\gamma$ restricts to an isomorphism from $L_c$ to $\Sigma$. Equivalence of (a) and (b) is now immediate. To prove equivalence of (b) and (c), recall that $\Sigma$ is associated to $Q$ through the representation $\text{Ad}_q$ which, by Proposition 4.11, is equivalent with the representation $\text{Ad}_q : \text{Spin}^o(V, h) \rightarrow \text{Aut}_{\text{Alg}}(\text{Cl}_2^q(a)) \simeq \mathbb{Z}_2$. The later can be identified with the grading morphism $\eta' : \text{Spin}^c(V, h) \rightarrow \mathbb{Z}_2$. The exact sequence (23) induces an exact sequence of pointed sets:

$$H^1(M, \text{Spin}^c(V, h)) \xrightarrow{j_*} H^1(M, \text{Spin}^o(V, h)) \xrightarrow{\eta'_*} H^1(M, \mathbb{Z}_2),$$

where $j_*$ is the map induced by the inclusion of $\text{Spin}^c(V, h)$ into $\text{Spin}^o(V, h)$ and $\eta'_*([Q]) = w_c(\Sigma)$. The $\mathbb{C}$-bundle $\Sigma$ is trivial iff $w_c(\Sigma) = 0$, which amounts to $[Q] \in \ker \eta'_*$ i.e. $[Q] \in \text{im} j_*$. The conclusion follows since the condition $[Q] \in \text{im} j_*$ is equivalent with the reduction of structure group stated at (c).

**Proposition 5.10.** The following statements are equivalent:

(a) The anticommutant bundle $C \subset \text{End}_\mathbb{C}(S)$ is trivial.
(b) The structure group of $S$ reduces from $\text{Spin}^o(V, h)$ to $\text{Spin}(V, h)$.

In this case, $M$ is orientable and both the Schur bundle $\Sigma$ and the twist bundle $T$ are also trivial.

**Proof.** The vector bundle $A$ is associated to $Q$ through the representation $\text{Ad}_h$ which, by Proposition 4.11, is equivalent with the untwisted characteristic representation $\mu : \text{Spin}^o(V, h) \rightarrow O(2)$. The third exact sequence in (32) induces an exact sequence of pointed sets:

$$H^1(M, \text{Spin}(V, h)) \xrightarrow{j_*} H^1(M, \text{Spin}^o(V, h)) \xrightarrow{\mu_*} H^1(M, O(2))$$
where $j_*$ is the map induced by the inclusion of Spin$(V, h)$ into Spin$^q(V, h)$ and $\mu_*(\{Q\})$ equals the class of the principal $O(2)$-bundle corresponding to $A$. This shows that $A$ is trivial iff $Q \in \ker \mu_*$ = im$j_*$, i.e. iff the reduction of structure group stated at (b) holds. It is clear that (b) implies the remaining statements.

**Proposition 5.11.** The following statements are equivalent:

(a) The principal U(1)-bundle $D$ is trivial.

(b) The structure group of $S$ reduces from Spin$^o(V, h)$ to $\widetilde{\text{Pin}}(V, h) \simeq \text{Pin}(V, \alpha_{p,q} h)$.

In this case, the structure group of $S$ reduces further from $\widetilde{\text{Pin}}(V, h)$ to Spin$(V, h)$ iff the Schur bundle $\Sigma$ is also trivial.

**Proof.** The reduction stated at (b) takes place iff the fiber bundle $Q/\widetilde{\text{Pin}}(V, h)$ (which is isomorphic with $D$) admits a section. Since this is a principal U(1)-bundle, it admits a section iff it is trivial i.e. iff (b) holds. In this case, further reduction to Spin$(V, h)$ occurs iff $M$ is orientable. This amounts to triviality of the complex orientation bundle of $M$, which is isomorphic with the Schur bundle. □

**Remark 5.12.** Recall that $\check{D} \in \widetilde{\text{Pin}}(V, h)$ generates the cyclic group (21). This gives rise to a short exact sequence:

$$1 \longrightarrow \Gamma_{o,\alpha} \longrightarrow \widetilde{\text{Pin}}(V, h) \longrightarrow G \longrightarrow 1,$$

where:

$$G = \begin{cases} \text{Spin}(V, h)/\mathbb{Z}_2 = \text{SO}(V, h) & \text{if } p - q \equiv_8 3, \\ \text{Spin}(V, h) & \text{if } p - q \equiv_8 7. \end{cases}$$

The twisted vector representation $\check{\lambda} : \text{Spin}^o(V, h) \rightarrow O(V, h)$ restricts to the twisted vector representation$^7$ $\check{\text{Ad}}_0 : \widetilde{\text{Pin}}(V, h) \simeq \text{Pin}(V, \alpha_{p,q} h) \rightarrow O(V, h)$, which descends to a surjective group morphism from $s : G \rightarrow O(V, h)/\{-\text{id}_V, \text{id}_V\} \simeq \text{SO}(V, h)$. Accordingly, the principal $G$-bundle $Q' \overset{\text{def}}{=} Q/\Gamma_{o,\alpha}$ covers the principal SO$(V, h)$-bundle $P'_{\text{SO}(V, h)}(M, g) \overset{\text{def}}{=} P_{O(V, h)}(M, g)/\mathbb{Z}_2$. When $p - q \equiv_8 7$, we have $\alpha_{p,q} = +1$ and $s$ coincides with the double cover $\text{Ad}_0 : G = \text{Spin}(V, h) \rightarrow \text{SO}(V, h)$ given by the vector representation of Spin$(V, h)$. In this case, the principal Spin$(V, h)$-bundle $Q'$ covers the bundle $P'_{\text{SO}(V, h)}(M, g)$. When $M$ is oriented, $P'_{\text{SO}(V, h)}(M, g)$ is isomorphic with the special pseudo-orthogonal coframe bundle $P_{\text{SO}(V, h)}(M, g)$ and $Q'$ becomes a spin structure on $(M, h)$. When $p - q \equiv_8 3$, we have $\alpha_{p,q} = -1$ and $s$ is an isomorphism of groups $s : G = \text{SO}(V, h) \rightarrow \text{SO}(V, h)$. In this case, the principal SO$(V, h)$-bundle $Q$ is isomorphic with $P'_{\text{SO}(V, h)}(M, g)$.

5.3. Majorana spinor bundles and Majorana spinor fields when $p - q \equiv_8 7$. Let us assume that $p - q \equiv_8 7$ (thus $\alpha_{p,q} = +1$) and that the bundle $D$ is trivial, i.e. that the structure group reduces to $\widetilde{\text{Pin}}(V, h) = \text{Pin}(V, h)$. Then any global section $D \in G(M, D)$ satisfies $D^2 = +|S|$ and hence gives an almost product structure on the vector bundle $S$. This allows one to define spin projectors $P_{\pm} \overset{\text{def}}{=} \frac{1}{2}(\text{id}_S \mp D) \in \Gamma(M, \text{End}(S))$, which satisfy $P_{\pm}^2 = P_{\pm}$, $P_+ \oplus P_- = |S|$ and $P_{\pm}P_{\mp} = 0$. The vector sub-bundles $S_{\pm} \overset{\text{def}}{=} P_{\pm}(S)$ have fibers at $p \in M$ given by:

$$S_p^\pm = \{x \in S_p | D_p x = \pm x\},$$

and give a direct sum decomposition $S = S^+ \oplus S^-$. Since $D_p$ anticommutes with $\gamma_p(\xi)$ for all $\xi \in T^*_p M$, we have $\gamma_p(\xi)(S^\pm) = S^\mp$ and $\gamma_p(\text{Cl}^+(M, g))(S^\pm) = S^\pm$. Hence the bundle morphisms $\gamma_{\text{even}} : \text{Cl}^+(M, g) \rightarrow \text{End}_{\mathbb{R}}(S^\pm)$ defined through:

$$\gamma_{\text{even}, p}^\pm(w) \overset{\text{def}}{=} \gamma_p(w)|_{S^\pm}, \quad \forall w \in \text{Cl}^+(M, g)_p,$$

are unital morphisms of bundles of algebras, making $S^\pm$ into elementary real spinor bundles i.e. bundles of simple modules over the even sub-bundle $\text{Cl}^+(M, g)$ of $\text{Cl}(M, g)$. This allows us to define “Majorana spinors” when $(M, g)$ admits a Pin$(V, h)$ structure, even though $M$ may be unorientable.

---

$^7$We assume that we have chosen a vector $v \in V$ such that $\epsilon_v = 1$ to define the isomorphism $\widetilde{\text{Pin}}_{\alpha_{p,q}}(V, h) \simeq \text{Pin}(V, \alpha_{p,q} h)$. We leave to the reader the details of the case $\epsilon_v = -1$. 

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**DIRAC OPERATORS ON REAL SPINOR BUNDLES OF COMPLEX TYPE 31**
Definition 5.13. The real vector bundle \( S^+ \) is called the elementary bundle of Majorana spinors defined by the global conjugation \( D \in \Gamma(M, \mathcal{D}) \) and by the elementary real pinor bundle \( S \). A global section \( \epsilon \in \Gamma(M, S^+) \) is called an ordinary Majorana spinor field while a global section \( \epsilon \in \Gamma(M, S^-) \) is called an imaginary Majorana spinor field.

Any global section \( \epsilon \in \Gamma(M, S) \) decomposes uniquely as \( \epsilon = \epsilon^+ \oplus \epsilon^- \) with \( \epsilon^\pm \equiv \mathcal{P}_\pm(\epsilon) \in \Gamma(M, S^\pm) \) and hence \( \epsilon \) can be identified with a pair consisting of one ordinary and one imaginary Majorana spinor field.

Proposition 5.14. There exists a natural isomorphism of semilinear vector bundles:

\[
f : L^c \otimes S^+ \cong S.
\]

Proof. For any \( p \in M, z \in L^c_p = \alpha + \beta \nu_p \in \mathbb{R} \oplus \mathbb{R} \nu_p \) (where \( \alpha, \beta \in \mathbb{R} \)) and \( x \in S^+_p \), let \( f_p : L^c_p \otimes S^+_p \to S_p \) be the linear map defined through:

\[
f_p(z \otimes x) \overset{\text{def}}{=} (\alpha \text{id}_S + \beta J_p)x,
\]

where \( J_p \overset{\text{def}}{=} \gamma_p(\nu_p) \) and \( \nu_p \in P_{2\mathbb{Z}}(M, g_p) \) is one of the two Clifford volume forms of \( (T_p M, g_p) \). This is well-defined since changing \( \nu_p \) in \(-\nu_p\) takes \( \beta \) to \(-\beta\) and \( J_p\) to \(-J_p\), leaving the operator \( \beta J_p \in \text{End}_\mathbb{R}(S_p) \) unchanged. Since \( D \) is \( \Sigma \)-antilinear, we have \( D_p J_p = -J_p D_p \), hence \( J_p(S^\pm) = S^\mp \). Thus \( J_p(x) \in S^- \) and (82) gives:

\[
\mathcal{P}_+(f_p(z \otimes x)) = \alpha x, \quad \mathcal{P}_-(f_p(z \otimes x)) = \beta J_p x.
\]

Thus \( f_p = (\mathcal{P}_+ \circ f_p) \oplus (\mathcal{P}_- \circ f_p) \) (where \( \mathcal{P}_\pm \circ f_p : L^c_p \otimes S^+_p \to S^\pm_p \)). Since \( J_p \) gives a bijection from \( S^+_p \) to \( S^-_p \), this implies that \( f_p \) is bijective and hence that \( f \) is an isomorphism of vector bundles. It is clear by construction that \( f \) takes the semilinear structure of \( L^c \otimes S^+ \) into that of \( S \). \( \square \)

Remark 5.15. Recall that the complexification of the real vector bundle \( S^+ \) is the complex vector bundle \( (S^+)^\mathbb{C} \overset{\text{def}}{=} \mathbb{C} M \otimes_{\mathbb{R}} S^+ \), where \( \mathbb{C} M \) is the trivial complex line bundle over \( M \). The semilinear vector bundle \( L^c \otimes C^1 S \) is an analogue of this construction where \( \mathbb{C} M \) has been replaced by the complex orientation line bundle \( L^c \) of \( M \). The proposition above shows that the semilinear vector bundle \( S \) is isomorphic with the “twisted complexification” \( L^c \otimes_{\mathbb{R}} S^+ \) of \( S^+ \). When \( M \) is oriented, we can set \( \epsilon_R \overset{\text{def}}{=} \epsilon^+ \) and define \( \epsilon_I = -J \epsilon^\mp \in \Gamma(M, S^+) \), where \( J = \gamma(\nu) \) and \( \nu \) is the normalized volume form of \( (M, g) \). In that case, we have \( L^c \overset{\cong}{\sim} \mathbb{C} M, S \overset{\cong}{\sim} \mathbb{C} M \otimes S^+ = (S^+)^\mathbb{C} \) and \( \epsilon = \epsilon_R + J \epsilon_I \), which means that we can identify a global section of \( S \) with a pair of two ordinary Majorana spinor fields. Such an identification is not possible when \( M \) is unorientable. The fact that (when \( p - q \equiv \mathbb{R} \)) one can define Majorana spinor fields when \( (M, g) \) is unorientable but admits a Pin structure is important when considering global aspects of certain supergravity theories, such as eleven-dimensional supergravity (including its Euclidean version). Table 1 lists some dimensions and signatures which are of interest in that context.
Table 1. Dimensions $d \leq 11$ and Riemannian or Lorentzian signatures (mostly minus or mostly plus) which belong to the “complex case” $p - q \equiv 3, 7$. Majorana spinor fields can be defined when $\alpha_{p,q} = +1$ and when $(M, g)$ admits a Pin$^+$ structure. Cases with $\alpha_{p,q} = -1$ are displayed in blue for clarity.

6. SOME EXAMPLES OF MANIFOLDS ADMITTING ADAPTED Spin$^o$ STRUCTURES

In this section we construct several examples of manifolds that admit adapted Spin$^o$ structures. We start with a simple result on the existence of Spin$^o$ structures induced by a number of classical spinorial structures.

**Proposition 6.1.** Every pseudo-Riemannian manifold of signature $(p, q)$ satisfying the condition $p - q \equiv 3, 7$ and admitting a Pin$^+$ or Spin$^c$ structure also admits an adapted Spin$^o$ structure.

**Proof.** Follows from the natural embeddings of the groups Spin, Pin and Spin$^c$ into Spin$^o_{\pm}$. □

The orthonormal frame bundle of the normal bundle of a submanifold of codimension two is a principal O(2)-bundle which is a natural candidate for the characteristic bundle $E$ of a Spin$^o_{\pm}$ structure. This observation leads to the following result.

**Proposition 6.2.** Let $X$ be a $(2k + 1)$-dimensional manifold which is oriented and spin and let $Y$ be an embedded $(2k - 1)$-dimensional submanifold of $X$.

1. Assume that $2k - 1 \equiv 7$ and that $X$ is endowed with a Riemannian metric $g$. Then $(Y, g|_Y)$ admits a Spin$^o$ structure whose characteristic O(2)-bundle $E$ is the orthogonal frame bundle of the normal bundle to $Y$ in $X$.

2. Assume that $2k - 1 \equiv 3$ and that $X$ is endowed with a negative Riemannian metric $g$. Then $(Y, g|_Y)$ admits a Spin$^o$ structure whose characteristic O(2)-bundle $E$ is the orthogonal frame bundle of the normal bundle to $Y$ in $X$.

**Proof.** We give the proof for $2k - 1 \equiv 7$ since the other case is analogous. Let $TY$ and $NY$ denote the tangent and normal bundles to $Y$, thus $TX = TY \oplus NY$. Since by assumption we have $w_1(TX) = w_2(TX) = 0$, we obtain:

$$w_1(TY) = w_1(NY), \quad w_2(TY) + w_1(NY)^2 + w_2(NY) = 0,$$

and the Proposition follows from Theorem 3.14. □

**Remark 6.3.** Note that the Spin$^o$ manifolds characterized in the previous Proposition will not, in general, admit any Spin, Pin or Pin$^c$ structure.

Another construction related to the previous one is provided by the following result.
Proposition 6.4. Let \((X,g)\) be a pseudo-Riemannian manifold of dimension \(d \equiv_8 2\), which is orientable and spin and \(Y \subset X\) be an embedded submanifold of dimension \(k\).

1. Assume that \(k \equiv_8 3\) and that \(g\) is positive-definite. Then \((Y,g|_Y)\) admits a \(\text{Spin}^o\) structure iff its orthogonal frame bundle admits a \(\text{Spin}^o_+\) structure (when \(NY\) is endowed with the metric induced from \(X\)).

2. Assume that \(k \equiv_8 7\) and that \(g\) is negative-definite. Then \((Y,g|_Y)\) admits a \(\text{Spin}^o\) structure iff its orthogonal frame bundle admits a \(\text{Spin}^o_+\) structure (when \(NY\) is endowed with the metric induced from \(X\)).

Proof. We prove the first case, the other being similar. Since \(Y\) is \(k \equiv_8 3\)-dimensional and \(X\) is \(d \equiv_8 2\)-dimensional, we have \(\text{rk}NY \equiv_8 7\). Assume that \(Y\) admits a \(\text{Spin}^o\) structure, so there exists a rank two vector bundle \(E\) over \(Y\) such that:

\[
\text{(84)} \quad w_1(M) = w_1(E), \quad w_2(M) = w_2(E).
\]

Since \(TX = TY \oplus TN\) while \(X\) is orientable and spin, we have:

\[
\text{(85)} \quad w_1(TY) = w_1(NY), \quad w_2(TY) + w_2(NY) + w_1(TY)^2 = 0.
\]

Using (84), this gives:

\[
\text{(86)} \quad w_1(NY) = w_1(E), \quad w_2(NY) + w_2(E) + w_1(NY)^2 = 0,
\]

so \(NY\) admits an adapted \(\text{Spin}^o_+\) structure with characteristic bundle \(E\). Conversely, assume that \(NY\) admits a \(\text{Spin}^o_+\) structure. Then there exists a rank-two vector bundle \(E\) such that (86) holds. Using (85) in (86) gives:

\[
\text{(87)} \quad w_1(TY) = w_1(E), \quad w_2(TY) = w_2(E),
\]

which implies that \((Y,g|_Y)\) admits a \(\text{Spin}^o_+\) structure. \(\square\)

Let us present an explicit family of manifolds admitting adapted \(\text{Spin}^o\) structures but not admitting \(\text{Pin}^c\) structures.

Definition 6.5. A pseudo-Riemannian manifold \((M,g)\) is said to be stably \(\text{Spin}^o\) if \(M \times \mathbb{R}^j\) admits an adapted \(\text{Spin}^o\) structure for some \(j \geq 0\).

Let:

\[
\text{(88)} \quad \text{Gr}_{k,n} \overset{\text{def.}}{=} \frac{O(n)}{O(k) \times O(n-k)}
\]

denote the real Grassmann variety of unoriented \(k\)-planes in \(\mathbb{R}^n\) and \(\mathcal{L}_{k,n}\) denote its tautological vector bundle. We endow \(\text{Gr}_{k,n}\) with a (positive or negative) invariant metric. Define:

\[
\text{Gr}_{k,n}^j \overset{\text{def.}}{=} \text{Gr}_{k,n} \times \mathbb{R}^j.
\]

Theorem 6.6. Assume that \(n+1 \equiv_4 0\). Then \(\text{Gr}_{2,n}\) is stably \(\text{Spin}^o\), namely:

1. For \(j \equiv_8 3 - 2n\), the manifold \(\text{Gr}_{2,n}^j\) carries a \(\text{Spin}^o_+\) structure of positive-definite signature with characteristic \(O(2)\)-bundle given by the orthogonal frame bundle of \(\mathcal{L}_{2,n}\).

2. For \(j \equiv_8 -3 + 2n\), the manifold \(\text{Gr}_{2,n}^j\) carries a \(\text{Spin}^o_+\) structure of negative-definite signature characteristic \(O(2)\)-bundle given by the orthogonal frame bundle of \(\mathcal{L}_{2,n}\).

Proof. We prove the statement for stable \(\text{Spin}^o_+\) structures, the other being similar. Let \(T_{2,n}\) denote the tangent bundle of \(\text{Gr}_{2,n}\). Then we have:

\[
\text{(89)} \quad T_{2,n} \oplus \mathcal{L}_{2,n} \oplus \mathcal{L}_{2,n}^* = (n+2)T_{2,n},
\]

where \(\mathcal{L}_{2,n}^*\) denotes the dual of the rank two vector bundle \(\mathcal{L}_{2,n}\). Equation (89) gives:

\[
\text{(90)} \quad w(T_{2,n}) w(\mathcal{L}_{2,n} \otimes \mathcal{L}_{2,n}^*) = w(\mathcal{L}_{2,n})^{2+n},
\]
where w denotes the total Stiefel-Whitney class. Using \( w(\mathcal{L} \otimes \mathcal{L}^*) = 1 + w_1(\mathcal{L})^2 + \ldots \), relation (90) implies:

\[
w_1(T_{2,n}) = (2 + n) w_1(\mathcal{L}_{2,n}),
\]
as well as:

\[
(n + 2)w_2(\mathcal{L}_{2,n}) + w_2(T_{2,n}) + (1 + \frac{(n + 2)(n + 1)}{2})w_1(\mathcal{L}_{2,n})^2 = 0.
\]

Since \( n + 1 \equiv_4 0 \), the last two relations simplify to:

\[
w_1(T_{2,n}) = w_1(\mathcal{L}_{2,n}),
\]
and:

\[
w_2(\mathcal{L}) + w_2(T_{2,n}) + w_1(\mathcal{L}_{2,n})^2 = 0,
\]
which shows that \( Gr_{2,n} \) satisfies the topological obstruction to carry a Riemannian Spin\(^0\) structure of positive-definite type. However, \( Gr_{2,n} \) has dimension \( 2n \) and therefore it cannot carry a adapted \( Spin_c \) structure. Taking the direct product with \( \mathbb{R}^j \) where \( j \equiv_8 (7 - 2n) \) gives the manifold \( Gr_{2,n}^j \), which has dimension:

\[
dim Gr_{2,n}^j = 2n + j = 8t + 7
\]
for some positive integer \( t \). Thus \( dim Gr_{2,n}^j \equiv_8 7 \). Using stability of Stiefel-Whitney classes, we conclude that \( Gr_{2,n}^j \) carries an adapted \( Spin_c \) structure.

**Remark 6.7.** The odd-dimensional manifolds \( Gr_{2,n}^j \) appearing in the proposition above can be equipped with a bundle of irreducible real Clifford modules, which allows one to use the tools of spinorial geometry (such as the corresponding Dirac operator) to study them. These manifolds do not admit other classical spinorial structures, such as Spin, Pin or \( Pin_c \) structures.

The simplest case occurring in Theorem 6.6 is \( Gr_{2,3}^1 \). The theorem shows that the seven-dimensional Riemannian manifold \( Gr_{2,3}^1 \) admits an elementary real pinor bundle which has rank 16 and is associated to a \( Spin_c \) structure.

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**APPENDIX A. Spin\(^c\) STRUCTURES**

In this appendix, we discuss \( Spin\(^c\) \) structures (which for definite signature were considered in [10, 11]) in order to clarify the difference between them and the \( Spin\(^{\alpha}\) \) structures considered in this paper. Unlike the \( Spin\(^{\alpha}\) \) structures considered in the present paper, \( Spin\(^c\) \) structures can exist only for orientable principal pseudo-orthogonal bundles (i.e. only for SO(\( V, h \)) bundles). In particular, a pseudo-Riemannian manifold must be orientable in order to admit a \( Spin\(^c\) \) structure.

**Definition A.1.** Let \( P \) be a principal SO(\( V, h \))-bundle over \( M \). A \( Spin\(^c\) \) structure on \( P \) is a triplet \((E, P, f)\), where \( E \) is a principal O(2)-bundle over \( M \) and \((P, f)\) is a \( \rho_\alpha \)-reduction of \( P \times_M E \) to \( Spin\(^c\)(V, h) \).

Notice that the structure group of \( P \times_M E \) need not reduce to SO(\( V', h'_\alpha \)), because the image of \( \rho_\alpha \) equals SO(\( V, h \) × O(2)), which contains elements of negative determinant. Also notice that a \( Spin\(^c\) \) structure is defined using the morphism \( \rho_\alpha : Spin\(^c\)(V, h) \to SO(V, h) \times O(2) \), while a \( Spin\(^{\alpha}\) \) structure is defined using the morphism \( \rho_\alpha : Spin\(^{\alpha}\)(V, h) \to O(V, h) \times O(2) \). The case of \( Spin\(^{\pm}\) \) structures in definite signature were considered in [10, 11].

**Theorem A.2.** Let \( P \) be a principal SO(\( V, h \))-bundle (in particular, \( P \) is orientable and hence we have \( w_1(P) = w_1'(P) = w_- (P) = 0 \)). Then the following statements are equivalent:

(a) \( P \) admits a \( Spin\(^c\) \) structure.
(b) There exists a principal $O(2)$-bundle $E$ such that the principal $O(V, h)$-bundle $\hat{P} \overset{\text{def}}{=} P \times_M E$ admits a Pin structure.

Proof. Follows by applying Lemma 3.12 to the morphism of exact sequences given in diagram (39).

\textbf{Theorem A.3.} The following statements hold:

1. A principal $SO(V, h)$-bundle $P$ admits a $\text{Spin}_c^\alpha(p, q)$ structure iff there exists a principal $O(2)$-bundle $E$ such that the following condition is satisfied:

$$w_2^+(P) + w_2^-(P) = w_2(E)$$

(91)

2. A principal $SO(V, h)$-bundle $P$ admits a $\text{Spin}_c^\alpha(p, q)$ structure iff there exists a principal $O(2)$-bundle $E$ such that the following condition is satisfied:

$$w_2^+(P) + w_2^- (P) = w_2(E) + w_1(E)^2.$$  

Proof. It was shown in [6] that the principal $O(V, h)$-bundle $\hat{P} := \hat{P}_n$ admits a Pin structure iff:

$$w_2^+(\hat{P}) + w_2^- (\hat{P}) + w_1^-(\hat{P})w_1^+(\hat{P}) = 0.$$  

(93)

Distinguish the cases:

1. When $\alpha = +1$, we have $\hat{P}^+ = P^+ \times E$ and $\hat{P}^- = P^-$. Thus:

$$w_2^+(\hat{P}) = w_1^+(P) + w_1(E), \quad w_2^- (\hat{P}) = w_1^-(P)$$

$$w_2^+(\hat{P}) = w_2^+(P) + w_2(E) + w_1^+(P)w_1(E), \quad w_2^- (\hat{P}) = w_2^-(P)$$

Hence (93) becomes:

$$w_2^+(P) + w_2^- (P) + w_2(E) + (w_1^+(P) + w_1^-(P))(w_1^-(P) + w_1(E)) = 0.$$  

(94)

Since $P$ is an $SO(V, h)$ bundle, we have:

$$w_1(P) = w_1^+(P) + w_1^-(P) = 0,$$

which shows that (94) reduces to (91).

2. When $\alpha = -1$, we have $\hat{P}^+ = P^+$ and $\hat{P}^- = P^- \times E$. Thus:

$$w_2^+(\hat{P}) = w_1^+(P), \quad w_2^- (\hat{P}) = w_1^-(P) + w_1(E)$$

$$w_2^+(\hat{P}) = w_2^+(P), \quad w_2^- (\hat{P}) = w_2^-(P) + w_2(E) + w_1^-(P)w_1(E)$$

and (93) becomes:

$$w_2^+(P) + w_2^- (P) + w_2(E) + (w_1^+(P) + w_1^-(P))(w_1^-(P) + w_1(E)) + w_1(E)^2 = 0,$$

which reduces to (92) upon using $w_1(P) = 0$.

\textbf{Remark A.4.} For positive signature with $\alpha = -1$ and $P = TM$, the statement of the Theorem recovers [10, Proposition 3.4]. Notice that the proof in op. cit. is based on applying Lemma 3.12 to diagram 44.

\textbf{Definition A.5.} Let $(M, g)$ be an orientable pseudo-Riemannian manifold. A $\text{Spin}_c^\alpha$ structure on $(M, g)$ is a $\text{Spin}_c^\alpha$ structure for the principal orthonormal frame bundle of $(M, g)$.

When $(M, g)$ admits a $\text{Spin}_c^\alpha$ structure, one can introduce a corresponding notion of spinor bundles admitting Clifford multiplication as in [10, 11]. However, that construction requires that $(M, g)$ be orientable and hence it is quite different from the construction considered in this paper since, unlike $\text{Spin}_c^\alpha$ structures, $\text{Spin}_c^\alpha$ structures can exist on non-orientable pseudo-Riemannian manifolds.
Appendix B. Semilinear structures

In this section we discuss semilinear structures on real vector bundles. A semilinear structure on a real vector bundle $S_0$ of even rank $2r$ is an unordered pair of mutually conjugate complex structures on $S_0$; equivalently, it is a reduction of the structure group of $S_0$ from $\text{GL}(2r, \mathbb{R})$ to the general semilinear group $\Gamma(r)$. Semilinear structures appear in Mathematical Physics when studying discrete symmetries such as charge conjugation and time reversal, through generally their theory is not clearly formalized in that context. In the differential geometry literature, such structures have also appeared in [2, Section 5], where they were called “twisted complex structures” and were considered from a point of view different from ours. Semilinear structures will be useful in Section 5 when studying elementary real spinor bundles of “complex type”. We start with a discussion of semilinear structures on finite-dimensional real vector spaces.

B.1. Semilinear structures on a real vector space. Let $S_0$ be an $\mathbb{R}$-vector space of finite even dimension. The twistor set of $S_0$ is defined as the set of all complex structures on $S_0$:

$$\text{Tw}(S_0) \overset{\text{def}}{=} \{ J \in \text{End}_\mathbb{R}(S_0) | J^2 = -\text{id}_{S_0} \}.$$  

This set is stabilized by the involution $J \mapsto -J$, which generates a fixed-point free action of $\mathbb{Z}_2$ on $\text{Tw}(S_0)$. Define the reduced twistor set of $S_0$ through:

$$\text{Tw}_0(S_0) \overset{\text{def}}{=} \text{Tw}(S_0)/\mathbb{Z}_2 = \{ \{J, -J\} | J \in \text{Tw}(S_0) \}.$$  

Definition B.1. A semilinear structure on $S_0$ is an element $s \in \text{Tw}_0(S_0)$, i.e. an unordered pair of mutually conjugate complex structures on $S_0$. A semilinear space is a pair $(S_0, s)$, where $S_0$ is an even-dimensional $\mathbb{R}$-vector space and $s$ is a semilinear structure on $S_0$.

Semilinear structures on $S_0$ can be identified with certain subalgebras of the endomorphism algebra of $S_0$, as clarified by the following:

Proposition B.2. There exists a bijection between semilinear structures on $S_0$ and those unital subalgebras $\mathbb{S}$ of the $\mathbb{R}$-algebra $\text{End}_\mathbb{R}(S)$ which have the property that the isomorphism type of $\mathbb{S}$ in the category $\text{Alg}$ of unital associative $\mathbb{R}$-algebras equals the isomorphism type of the $\mathbb{R}$-algebra $\mathbb{C}$ of complex numbers.

Proof. It is clear that a unital subalgebra $\mathbb{S}$ of $\text{End}_\mathbb{R}(S_0)$ is unitally isomorphic with $\mathbb{C}$ if there exists $J \in \text{Tw}(S_0)$ such that $\mathbb{S} = \text{Rid}_\mathbb{S} \oplus \mathbb{R} J$. In this case, $J$ is determined by $\mathbb{S}$ up to a sign change $J \mapsto -J$. Hence any such subalgebra of $\text{End}_\mathbb{R}(S_0)$ determines a semilinear structure on $S_0$.

Conversely, any semilinear structure $s \in \text{Tw}_0(S_0)$ determines a unital subalgebra $\mathbb{S} \overset{\text{def}}{=} \text{Rid}_\mathbb{S} \oplus \mathbb{R} J = \text{Rid}_\mathbb{S} \oplus \mathbb{R} (-J)$ of $\text{End}_\mathbb{R}(S)$, where $J \in s$ is any of the two conjugate complex structures belonging to $s$. It is obvious that $\mathbb{S}$ is unitally isomorphic with $\mathbb{C}$. \hfill $\Box$

Given a semilinear structure $s \in \text{Tw}_0(S_0)$ we denote by $\mathbb{S} \subset \text{End}_\mathbb{R}(S_0)$ the subalgebra associated to $s$ as in Proposition B.2. Any complex structure $J \in \text{Tw}(S_0)$ determines semilinear structure $[J] \overset{\text{def}}{=} \{ J, -J \}$ (which is the preimage of $J$ through the natural surjection $\pi : \text{Tw}(S_0) \to \text{Tw}_0(S_0)$), but a semilinear structure only determines a complex structure up to sign. Consider a semilinear structure $s$ on $S_0$ with corresponding subalgebra $\mathbb{S} \subset \text{End}_\mathbb{R}(S_0)$. Since $\text{Aut}_{\text{Alg}}(\mathbb{S}) \simeq \text{Aut}_{\text{Alg}}(\mathbb{C}) \simeq \mathbb{Z}_2$, $\mathbb{S}$ has a unique non-trivial unital $\mathbb{R}$-algebra automorphism which we denote by $c$; this corresponds to complex conjugation through any of the two algebra automorphisms which map $\mathbb{S}$ to $\mathbb{C}$. Accordingly, we can endow $S_0$ with two left $\mathbb{S}$-module structures upon defining external multiplication through:

$$sx \overset{\text{def}}{=} s(x) \forall s \in \mathbb{S}, \forall x \in S_0,$$

---

8This operation induces a unital $\mathbb{R}$-algebra automorphism of $\mathbb{S}$ corresponding to the unique nontrivial unital $\mathbb{R}$-algebra automorphism of $\mathbb{C}$ (which is the complex conjugation). Since $\text{Aut}_{\text{Alg}}(\mathbb{C}) \simeq \mathbb{Z}_2$, there are two distinct isomorphisms of unital $\mathbb{R}$-algebras from $\mathbb{S}$ to $\mathbb{C}$. 
or through:

\[ sx \overset{\text{def}}{=} c(s)(x) \quad \forall s \in S, \quad \forall x \in S_0. \]

We usually consider only the first of these \( S \)-module structures and reserve the notation \( S_0 \) to refer to the \( S \)-module obtained by using the second of these external multiplications.

Semilinear automorphisms. Let \((S_0, \mathfrak{s})\) be a semilinear vector space.

**Definition B.3.** An endomorphism \( T \in \text{End}_S(S_0) \) is called:

1. \( s \)-linear, if \( T \in \text{End}_S(S_0) \), i.e. if \( T(sx) = sT(x) \) for all \( s \in S \) and \( x \in S_0 \).
2. \( s \)-antilinear, if \( T \in \text{Hom}_S(S_0, S_0) \), i.e. if \( T(sx) = c(s)T(x) \) for all \( s \in S \) and \( x \in S_0 \).
3. \( s \)-semilinear, if it is either \( s \)-linear or \( s \)-antilinear.

The composition of two \( s \)-semilinear maps is \( s \)-semilinear. For any invertible \( \mathbb{R} \)-linear operator \( T \in \text{Aut}_S(S_0) \), we have \( \text{Ad}(T)(\text{Tw}(S_0)) = \text{Tw}(S_0) \) and \( T \) is \( s \)-semilinear iff \( \text{Ad}(T)(\mathfrak{s}) = \mathfrak{s} \) (where we view \( \mathfrak{s} \) as a two-element subset of \( \text{Tw}(S_0) \)). In this case, \( T \) is \( s \)-linear iff \( \text{Ad}(T)|_{\mathfrak{s}} = \text{id}_{\mathfrak{s}} \) and \( s \)-antilinear iff \( \text{Ad}(T)|_{\mathfrak{s}} \) is the non-trivial permutation of the two-element set \( \mathfrak{s} \).

Invertible \( s \)-semilinear maps form the group of automorphisms of the semilinear space \((S_0, \mathfrak{s})\). This group is denoted by \( \text{Aut}(S_0, \mathfrak{s}) \) or by \( \text{Aut}^s(S_0) \) and admits the \( \mathbb{Z}_2 \)-grading:

\[
\text{Aut}^s(S_0) = \text{Aut}_S(S_0) \sqcup \text{Aut}^s(S_0),
\]

where \( \text{Aut}_S(S_0) \) denotes the group of invertible \( s \)-linear transformations of \( S_0 \) and \( \text{Aut}^s(S_0) \) denotes the set of invertible \( s \)-antilinear transformations of \( S_0 \). The homogeneous components of this grading coincide with the two connected components of \( \text{Aut}^s(S_0) \), where \( \text{Aut}_S(S_0) \) is the connected component of the identity. The grading morphism \( f : \text{Aut}^s(S_0) \to \mathbb{Z}_2 \) of this \( \mathbb{Z}_2 \)-grading gives a short exact sequence:

\[
1 \rightarrow \text{Aut}_S(S_0) \rightarrow \text{Aut}^s(S_0) \rightarrow \mathbb{Z}_2 \rightarrow 1.
\]

Any choice of \( s \)-antilinear operator \( B \in \text{Aut}^s(S_0) \) such that \( B^2 = \text{id}_{\mathfrak{s}} \) (existence can be easily proven) induces a right-splitting morphism \( g_B : \mathbb{Z}_2 \to \text{Aut}^s(S_0) \) given by \( g_B(0) = \text{id}_{\mathfrak{s}} \) and \( g_B(1) = B \). Hence \( \text{Aut}^s(S_0) \) is isomorphic with the semidirect product \( \text{Aut}_S(S_0) \rtimes \text{Ad}_{\mathfrak{s} \mathfrak{g}} \mathbb{Z}_2 \).

The general semilinear group. The space \( \mathbb{R}^{2r} \) has a canonical semilinear structure \( s = [\mathbf{J}_r] = \{\mathbf{J}_r, -\mathbf{J}_r\} \), where \( \mathbf{J}_r \) is the canonical complex structure of \( \mathbb{C}^r \) under the identification \( \mathbb{C}^r \equiv \mathbb{R}^{2r} = \mathbb{R}^r \times \mathbb{R}^r \):

\[
\mathbf{J}_r(x, y) = (-y, x) \quad \forall x, y \in \mathbb{R}^r.
\]

**Definition B.4.** The general semilinear group \( \Gamma(r) \subset \text{GL}(2r, \mathbb{R}) \) is the group of automorphisms of the semilinear space \((\mathbb{R}^{2r}, [\mathbf{J}_r])\). The semilinear orthogonal group is the group \( \text{TU}(r) \overset{\text{def}}{=} \Gamma(r) \cap \text{O}(2r, \mathbb{R}) \).

We have \( \Gamma(r) \simeq \text{GL}(r, \mathbb{C}) \rtimes \mathbb{Z}_2 \). One can describe \( \Gamma(r) \) explicitly as the set of all pairs of the form \((A, 0)\) and \((A, 1)\), where \( A \in \text{GL}(r, \mathbb{C}) \) and \( \mathbb{Z}_2 = \{0, 1\} \), with group multiplication given by:

\[
(A, 0)(B, 0) = (AB, 0), \quad (A, 0)(B, 1) = (AB, 1)
\]

\[
(A, 1)(B, 0) = (AB, 1), \quad (A, 1)(B, 1) = (AB, 0).
\]

In this presentation, the group \( \text{GL}(r, \mathbb{C}) \) of complex-linear transformations of \( \mathbb{C}^r \) identifies with the subgroup of \( \Gamma(r) \) consisting of elements of the form \((A, 0)\), while the set of non-degenerate complex-antilinear transformations of \( \mathbb{C}^r \) identifies with the subset of \( \Gamma(r) \) consisting of elements of the form \((A, 1)\). Using the cannonical real basis \( e_1, \ldots, e_r, ie_1, \ldots, ie_r \) of \( \mathbb{C}^r \) (where \( e_1, \ldots, e_r \) is the canonical complex basis of \( \mathbb{C}^r \)), an element \( A \in \text{GL}(r, \mathbb{C}) \) can be represented as an element of \( \text{GL}(2r, \mathbb{R}) \) having the special form:

\[
A_{\text{lin}} = \begin{bmatrix}
A^R & -A^I \\
A^I & A^R
\end{bmatrix},
\]
where $A^R$ and $A^I$ are the real and imaginary parts of the non-degenerate complex matrix $A$. Similarly, a complex-antilinear transformation of $\mathbb{C}^r$ with matrix $A \in \text{GL}(r, \mathbb{C})$ in the complex basis $e_1, \ldots, e_r$ can be represented as an element of $\text{GL}(2r, \mathbb{R})$ having the special form:

$$A_{\text{aln}} = \begin{bmatrix} A^R & A^I \\ A^I & -A^R \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix} (A_{\text{aln}})^T.$$  

Using these forms, it is easy to see that we have:

$$\det(A, 0) = \det A_{\text{aln}} = |\det A|^2, \quad \det(A, 1) = \det A_{\text{aln}} = (-1)^r |\det A|^2.$$  

In particular, the determinant of any element of $\Gamma(r)$ is positive when $r$ is even and in this case $\Gamma(r)$ is a subgroup of the group $\text{GL}_+(2r, \mathbb{R})$ of general linear transformations with positive determinant. Finally, notice that $\text{TU}(r)$ is a maximal compact form of $\Gamma(r)$. When $r$ is even, the discussion above shows that $\text{TU}(r)$ is a subgroup of $\text{SO}(2r, \mathbb{R})$.

Description of $\text{Tw}(S_0)$ as a homogeneous space. Recall that $\text{Aut}_\mathbb{R}(S_0)$ acts transitively on $\text{Tw}(S_0)$ through the adjoint representation:

$$\text{Ad}(a)(J) = aJa^{-1} \quad \forall a \in \text{Aut}_\mathbb{R}(S_0), \quad \forall J \in \text{Tw}(S_0)$$

and that the stabilizer of $J \in \text{Tw}(S_0)$ in $\text{Aut}_\mathbb{R}(S_0)$ equals the group $\text{Aut}_J(S_0)$ of those transformations which are complex-linear with respect to $J$. Hence $\text{Tw}(S_0)$ is the homogeneous space $\text{Aut}_\mathbb{R}(S_0)/\text{Aut}_J(S_0) \simeq \text{GL}(2r, \mathbb{R})/\Gamma(r) \simeq_{\text{hyp}} \text{GL}(2r, \mathbb{R})/\text{SU}(2r, \mathbb{R})$. The adjoint action of $\text{Aut}_\mathbb{R}(S_0)$ on $\text{Tw}(S_0)$ commutes with the $\mathbb{Z}_2$ action generated by $J \rightarrow -J$ and descends to a transitive action on $\text{Tw}(S_0)$. The stabilizer of a semilinear structure $\mathfrak{s} = \{J, -J\} \in \text{Tw}(S_0)$ under this action consists of all transformations which are semilinear with respect to $\mathfrak{s}$:

$$\text{Stab}(\mathfrak{s}) = \text{Aut}(S_0, \mathfrak{s}) = \text{Aut}_\mathbb{R}^{\text{tw}}(S_0),$$

where $\mathbb{S}$ is the subalgebra of $\text{End}_\mathbb{R}(S_0)$ determined by $\mathfrak{s}$. Hence the set $\text{Tw}(S_0)$ is the homogeneous space:

$$\text{Aut}_\mathbb{R}(S_0)/\text{Aut}_\mathbb{R}^{\text{tw}}(S_0) \simeq \text{GL}(2r, \mathbb{R})/\Gamma(r) \simeq_{\text{hyp}} \text{O}(2r, \mathbb{R})/\text{SU}(2r, \mathbb{R}).$$

$s$-Hermitian metrics. Let $(S_0, \mathfrak{s})$ be a semilinear vector space.

**Definition B.5.** A $\mathbb{R}$-bilinear map $b : S_0 \times S_0 \to \mathbb{R}$ is called $\mathfrak{s}$-compatible if any $J \in \mathfrak{s}$ satisfies $b(Jx, Jy) = b(x, y)$ for all $x, y \in S_0$.

**Definition B.6.** Let $\mathfrak{s}$ be a semilinear structure on $S_0$ with associated subalgebra $\mathbb{S} \subset \text{End}_\mathbb{R}(S_0)$. An $\mathbb{S}$-valued $\mathbb{R}$-bilinear form $h : S_0 \times S_0 \to \mathbb{S}$ is called an $\mathfrak{s}$-Hermitian metric on $S_0$ if it satisfies the following conditions:

1. It is conjugate-symmetric, i.e. $h(x, y) = c(h(y, x))$ for all $x, y \in S_0$.
2. It is $\mathfrak{s}$-sesquilinear, i.e. $h(sx, y) = c(s)h(x, y)$ and $h(x, sy) = sh(x, y)$ for all $x, y \in S_0$ and any $s \in \mathbb{S}$.
3. It is positive-definite, i.e. $h(x, x) \in \mathbb{R}_{\geq 0}\text{id}_{S_0}$ for all $x \in S_0$ and $h(x, x) = 0$ iff $x = 0$.

Recall that any complex structure $J \in \mathfrak{s}$ belonging to the semilinear structure $\mathfrak{s}$ determines a direct sum decomposition $\mathbb{S} = \text{Rid}_{S_0} \oplus \mathbb{J} \mathfrak{s}$ of $\mathbb{S}$ as an $\mathbb{R}$-vector space. Thus any element $s \in \mathbb{S}$ can be written uniquely as $s = s_0 + s_1J$, where $s_0, s_1 \in \mathbb{R}$. Moreover, changing $J$ into $-J$ does not change $s_0$ but changes the sign of $s_1$. In particular, the quantity:

$$|s|_\mathbb{S} \overset{\text{def}}{=} \sqrt{s_0^2 + s_1^2} \in \mathbb{R}_{\geq 0} \quad \forall s \in \mathbb{S}$$

depends only on $\mathfrak{s}$ and vanishes iff $s \in \mathbb{S}$ does. Let $\text{Re}_\mathbb{S} : \mathbb{S} \to \mathbb{R}$ be the $\mathbb{R}$-linear map defined through:

$$\text{Re}_\mathbb{S}(s) = s_0 \quad \forall s \in \mathbb{S}$$

and $\text{Im}_J : \mathbb{S} \to \mathbb{R}$ be the $\mathbb{R}$-linear map defined through:

$$\text{Im}_J(s) = s_1 \quad \forall s \in \mathbb{S}.$$
Notice that $\text{Re}_S$ depends only on the semilinear structure $\mathfrak{s}$, while $\text{Im}_J$ depends on the choice of a complex structure $J \in \mathfrak{s}$. In fact, we have $\text{Im}_{-J} = -\text{Im}_J$. We also have:

$$|s|_\mathfrak{s} = \sqrt{(\text{Re}_S s)^2 + (\text{Im}_J s)^2} \quad \forall s \in \mathfrak{s}.$$ 

If $h$ is an $\mathfrak{s}$-Hermitian metric on $S_0$, then the $\mathbb{R}$-valued map:

$$h_R \overset{\text{def}}{=} \text{Re}h : S_0 \times S_0 \rightarrow \mathbb{R},$$

is an $\mathbb{R}$-valued $\mathfrak{s}$-compatible Euclidean metric on $S_0$, i.e. it is an $\mathbb{R}$-bilinear map which satisfies the conditions:

1. It is symmetric, i.e. $h_R(x, y) = h_R(y, x)$ for all $x, y \in S_0$.
2. It is $\mathfrak{s}$-modular, i.e. $h_R(o_s, s) = |s|^2 h_R(x, y)$ for all $x, y \in S_0$ and any $s \in \mathfrak{s}$.
3. It is positive-definite, i.e. $h(x, x) \geq 0$ for all $x \in S_0$, with equality if $x = 0$.

Similarly, the $\mathbb{R}$-valued map:

$$h_J \overset{\text{def}}{=} \text{Im}_J h : S_0 \times S_0 \rightarrow \mathbb{R}$$

is an $\mathfrak{s}$-compatible symplectic pairing on $S_0$, i.e. it is an $\mathbb{R}$-bilinear map which satisfies the conditions:

1. It is antisymmetric, i.e. $h_J^J(x, y) = -h_J^J(y, x)$ for all $x, y \in S_0$.
2. It is $\mathfrak{s}$-modular, i.e. $h_J^J(o_s, s) = |s|^2 h_J^J(x, y)$ for all $x, y \in S_0$ and any $s \in \mathfrak{s}$.
3. It is non-degenerate, i.e. $h_J^J(x, y) = 0$ for all $y \in S_0$ implies $x = 0$.

Moreover, we have $h_J^J(x, y) = h_R(J x, y)$ for all $x, y \in S_0$. This relation defines a bijection between $\mathfrak{s}$-compatible Euclidean metrics on $S_0$ and $\mathfrak{s}$-compatible non-degenerate symplectic pairings on $S_0$. Moreover, any such $h_R$ or $h_J$ determines the other and determines a unique $\mathfrak{s}$-Hermitian metric $h$ on $S_0$ such that $h_R = \text{Re}h$ and $h_J = \text{Im}_J h$, namely $h(x, y) = h_R(x, y) + h_J(x, y) J$ for all $x, y \in S_0$. Given any $\mathfrak{s}$-Hermitian metric $h$ on $S_0$, the Euclidean metric $h_R = \text{Re}h$ determines a Euclidean metric on the real determinant line $\text{det}_\mathbb{R} S_0 \overset{\text{def}}{=} \wedge^{2r} S_0$, where $\dim_{\mathbb{R}} S_0 = 2r$. Let $S^0(\text{det}_\mathbb{R} S_0, h)$ be the two-element set consisting of those elements of $\text{det}_\mathbb{R} S_0$ which have unit norm with respect to this induced metric. Notice that $S^0(\text{det}_\mathbb{R} S_0, h)$ can be identified with the set of orientations of $S_0$.

**Proposition B.7.** Let $(S_0, \mathfrak{s})$ be a semilinear vector space with $\dim_{\mathbb{R}} S_0 = 2r$. Then any $\mathfrak{s}$-Hermitian metric $h$ on $S_0$ determines a map $f_h : \mathfrak{s} \overset{\sim}{\rightarrow} S^0(\text{det}_\mathbb{R} S_0, h)$. Moreover:

1. We have:

$$f_h(-J) = (-1)^r f_h(J) \quad \forall J \in \mathfrak{s}.$$

2. The map $f_h$ is bijective when $r$ is odd.

3. When $r$ is even, the image $f_h(\mathfrak{s})$ coincides with one of the two elements of the set $S^0(\text{det}_\mathbb{R} S_0, h)$, so in this case $\mathfrak{s}$ determines an orientation of the $\mathbb{R}$-vector space $S_0$.

**Proof.** Let $J \in \mathfrak{s}$ be any of the two elements of $\mathfrak{s}$. Then the $\mathbb{R}$-linear map $\alpha : \mathbb{S} = \text{Rid}_{S_0} \oplus \mathbb{R} J \rightarrow \mathbb{C}$ which sends $\text{id}_{S_0}$ to 1 and $J$ to the imaginary unit $i$ is a unital isomorphism of $\mathbb{R}$-algebras from $\mathbb{S}$ to $\mathbb{C}$. Consider the $\mathbb{R}$-bilinear map $h_0 \overset{\text{def}}{=} \alpha \circ h : S_0 \times S_0 \rightarrow \mathbb{C}$. Then Definition B.6 implies that $h_0$ is Hermitian with respect to the complex structure $J$ on $S_0$. Notice that we have $\text{Re}h_0 = \text{Re}_S h$. Moreover, identifying $\mathbb{S}$ with $\mathbb{C}$ through the map $\alpha$ allows us to view the left $\mathbb{S}$-module $S_0$ as a $\mathbb{C}$-vector space of dimension $r$. Let $e_1, \ldots, e_r$ be any $h_0$-orthonormal basis of this $\mathbb{C}$-vector space. Setting $e_{r+j} \overset{\text{def}}{=} J(e_j)$ for all $j = 1, \ldots, r$, the collection $e_1, \ldots, e_{2r}$ is a basis of the $\mathbb{R}$-vector space $S_0$ which is orthonormal with respect to the Euclidean scalar product $\text{Re}h_0 = \text{Re}_S h$. Setting $f_h(J) \overset{\text{def}}{=} e_1 \wedge \ldots \wedge e_{2r} \in \text{det}_\mathbb{R} S_0$, this implies $\|f_h(J)\|_{\text{Re}h} = \|f_h(J)\|_{\text{Re}h_0} = 1$, hence $f_h(J) \in S^0(\text{det}_\mathbb{R} S_0, h)$. It is easy to see that $f_h(J)$ is independent of the choice of the $h_0$-orthonormal basis $e_1, \ldots, e_r$ of the $\mathbb{C}$-vector space $S_0$. Indeed, any other $h_0$-orthonormal basis has the form $e_j' = \sum_{j=1}^r A_{jk} e_j$, where $A_{jk} \in \mathbb{C}$ are the coefficients of a unitary matrix $A = (A_{ij})_{i,j=1,\ldots,r} \in U(r, \mathbb{C})$. Decomposing $A_{jk} = A_{jk}^R + i A_{jk}^I$ with $A_{jk}^R, A_{jk}^I \in \mathbb{R}$ and setting $A^R \overset{\text{def}}{=} (A_{ijk}^R)_{i,j=1,\ldots,r}$, $A^I \overset{\text{def}}{=} (A_{ijk}^I)_{i,j=1,\ldots,r}$, we
find that the real basis \( e'_1, \ldots, e'_{2r} \) (where \( e'_{r+k} \overset{\text{def}}{=} J(e'_k) \) for \( k = 1, \ldots, r \)) is related to the real basis \( e_1, \ldots, e_{2r} \) through \( e'_a = \sum_{b=1}^{2r} B_{ab} e_b \), where \( B \) is the matrix given by:

\[
B = \begin{bmatrix}
A^R & -A^I \\
A^I & A^R
\end{bmatrix}.
\]

Thus \( e'_1 \wedge \ldots \wedge e'_{2r} = (\det B) e_1 \wedge \ldots \wedge e_{2r} = e_1 \wedge \ldots \wedge e_{2r} \), where we used the relation \( \det B = |\det A|^2 = 1 \) (the last equality follows from the fact that \( A \) is a unitary matrix). Since \( J \in S \) was chosen arbitrarily, we thus have a well-defined map \( f_h : S \to S^0(\det S_0, h) \). The replacement \( J \to -J \) induces the changes \( \alpha \to -\alpha \) and \( h_0 \to h_0 \), but does not change the set of \( h_0 \)-orthonormal bases of \( S_0 \) over \( C \). Hence replacing \( J \) by its conjugate can be implemented by replacing \( e_{r+j} \to -e_{r+j} \) for all \( j = 1, \ldots, r \), which shows that (95) holds. This relation immediately implies the remaining statements of the proposition.

\[\square\]

B.2. Semilinear structures on real vector bundles. Let \( S \) be a smooth real vector bundle of even rank \( N = 2r \) over a connected, smooth and paracomact manifold \( M \) and let \( P_{GL(N, \mathbb{R})}(S) \) be the principal bundle of linear frames of \( S \). Let \( End_{\mathbb{R}}(S) \) denote the bundle of endomorphisms of \( S \). The latter is a bundle of unital associative \( \mathbb{R} \)-algebras of type \( End_{\mathbb{R}}(S_0) \simeq \text{Mat}(N, \mathbb{R}) \), where \( S_0 \) is the typical fiber of \( S \).

**Definition B.8.** The **twistor bundle** \( Tw(S) \) of \( S \) is the fiber sub-bundle of \( End_{\mathbb{R}}(S) \) whose fiber at \( p \in M \) is the set:

\[
Tw_p(S) = \{ J \in End_{\mathbb{R}}(S_p) | J^2 = -\text{id}_{S_p} \}
\]

of all complex structures on the \( 2r \)-dimensional real vector space \( S_p \).

Notice that \( Tw(S) \) is a bundle of homogeneous spaces whose fibers are isomorphic with \( \text{GL}(2r, \mathbb{R})/ \text{GL}(r, \mathbb{C}) \). The **sign automorphism** is the involutive vector bundle automorphism \( \sigma : End_{\mathbb{R}}(S) \to End_{\mathbb{R}}(S) \) which acts on each fiber of \( End_{\mathbb{R}}(S) \) as \( End_{\mathbb{R}}(S_p) \ni T \to -T \in End_{\mathbb{R}}(S_p) \). Notice that \( \sigma \) is *not* an automorphism of \( End_{\mathbb{R}}(S) \) as a bundle of algebras. We have \( \sigma(Tw(S)) = Tw(S) \) and \( \sigma|_{Tw(S)} \) acts freely on each fiber of \( Tw(S) \).

**Definition B.9.** The **reduced twistor bundle** \( Tw_0(S) \) is the fiber bundle defined as quotient \( Tw_0(S) \overset{\text{def}}{=} Tw(S)/\mathbb{Z}_2 \).

Notice that \( Tw_0(S) \) is a bundle of homogeneous spaces whose fibers \( Tw_0(S_p) \) are isomorphic with \( \text{GL}(2r, \mathbb{R})/\Gamma(r) \). The points of the fiber \( Tw_0(S_p) = Tw_0(S_p) \) of \( Tw_0(S) \) at a point \( p \in M \) correspond to the semilinear structures on the \( \mathbb{R} \)-vector space \( S_p \). Hence \( Tw_0(S) \) is the bundle of semilinear structures on the fibers of \( S \).

**Definition B.10.** A semilinear structure on \( S \) is a smooth global section \( s \in \Gamma(M, Tw_0(S)) \). A **semilinear vector bundle** on \( M \) is a pair \((S, s)\), where \( S \) is a real vector bundle of even rank defined on \( M \) and \( s \) is a semilinear structure on \( S \).

**Proposition B.11.** There exists a bijection between semilinear structures on \( S \) and reductions of structure group \( P_{\Gamma(r)}(S) \to P_{\text{GL}(2r, \mathbb{R})}(S) \) of the principal bundle of linear frames of \( S \) from \( \text{GL}(2r, \mathbb{R}) \) to \( \Gamma(r) \).

**Proof.** Since \( \Gamma(r) \) is a closed subgroup of \( \text{GL}(2r, \mathbb{R}) \), reductions of structure group of \( P_{\text{GL}(2r, \mathbb{R})}(S) \) from \( \text{GL}(2r, \mathbb{R}) \) to \( \Gamma(r) \) are in bijection with smooth sections of the fiber bundle \( P_{\text{GL}(2r, \mathbb{R})}(S)/\Gamma(r) \simeq Tw_0(S) \).

\[\square\]

Given a \( \mathbb{C} \)-bundle \( \Sigma \) over \( M \), its bundle of imaginary units \( P_{\mathbb{Z}_2}(\Sigma) \) is the principal \( \mathbb{Z}_2 \)-bundle whose fiber at \( p \in M \) equals the natural semilinear structure of \( \Sigma_p \). If \( \Sigma \) is a \( \mathbb{C} \)-bundle which is a sub-bundle of unital \( \mathbb{R} \)-subalgebras of \( End_{\mathbb{R}}(S) \), then \( P_{\mathbb{Z}_2}(\Sigma) \) is a fiber sub-bundle of \( Tw(S) \) which can also be viewed as a section of \( Tw_0(S) \), i.e. as a semilinear structure on \( S \).
Proposition B.12. The map:
\[ \Sigma \to s_\Sigma \overset{\text{def.}}{=} P_{Z_2}(\Sigma) \]
gives a bijection between sub-bundles of unital \( \mathbb{R} \)-subalgebras \( \Sigma \subset \text{End}_\mathbb{R}(S) \) such \( \Sigma \) is a \( \mathbb{C} \)-bundle and semilinear structures on \( S \), whose inverse is given by:
\[ s \to \Sigma_s \overset{\text{def.}}{=} \mathbb{R}_M \oplus L_s , \]
where \( L_s \) is the real line sub-bundle of \( \text{End}_\mathbb{R}(S) \) which is generated by \( s \).

\[ \sigma_\Sigma \overset{\text{def.}}{=} \sigma(x) \forall \sigma \in \Sigma_s(p) \forall x \in S_p . \]

This left module is free and of rank equal to \( \frac{1}{2} \text{rk}_\mathbb{R} S \).

Notice that any complex structure \( J \in \text{End}_\mathbb{R}(S) = \Gamma(M, \text{End}_\mathbb{R}(S)) \) on \( S \) induces a semilinear structure \( s_J \) on \( S \) given by \( s_J(p) = \{J_p, J_p^\dagger\} \) for all \( p \in M \). This semilinear structure corresponds to the rank two sub-bundle \( \Sigma_{s_J} = L_{id_S} \oplus L_J \subset \text{End}_\mathbb{R}(S) \). The latter is a trivial \( \mathbb{C} \)-bundle, being trivialized by the linearly independent global sections \( id_S \) and \( J \). We have \( s_{-J} = s_J \).

Definition B.13. A semilinear structure \( s \) on \( S \) is called \textbf{trivial} if there exists a complex structure \( J \in \Gamma(M, \text{Tw}(S)) \) on \( S \) such that \( s = s_J \).

The following proposition follows immediately from the previous discussion:

Proposition B.14. A semilinear structure \( s \) on \( S \) is trivial if and only if \( \Sigma_s \) is a trivial \( \mathbb{C} \)-bundle.

In particular, \( s \) is trivial iff \( w_c(\Sigma_s) = 0 \).

s-Hermitian metrics.

Definition B.15. Let \( (S, s) \) be a semilinear vector bundle on \( M \). A global section \( h \in \Gamma(M, \text{Hom}_\mathbb{R}(S \times S, \Sigma_s)) \) is called an \textbf{s-Hermitian metric} on \( S \) if its value \( h_p \in \text{Hom}_\mathbb{R}(S_p \times S_p, \Sigma_s(p)) \) at any point \( p \in M \) is an \( s_p \)-Hermitian metric on the vector space \( S_p \) in the sense of Definition B.6.

Let \( (S, s) \) be a semilinear vector bundle on \( M \) and \( h \) be an \( s \)-Hermitian metric on \( S \). Applying fiberwisely the construction of Subsection B.1, we define the real part \( h_R \in \Gamma(M, \text{Hom}_\mathbb{R}(S \times S, \Sigma_s, \mathbb{R}_M)) \) of \( h \) (which is a Euclidean scalar product on \( S \)). Let \( P_{Z_2}(S, h) \) denote normalized orientation bundle of the Euclidean vector bundle \( (S, h_R) \), i.e. the principal \( Z_2 \)-bundle whose fiber at \( p \in M \) equals the two-element set \( S^0(\det\mathbb{R} S_p, h_p) \). Notice that the real determinant line bundle \( \det\mathbb{R} S \overset{\text{def.}}{=} \wedge^{\text{rk} S} S \) is associated to the principal \( Z_2 \)-bundle \( P_{Z_2}(S, h) \) through the sign representation of \( Z_2 \) on \( \mathbb{R} \). Finally, let \( P_s(S) \) denote the principal \( Z_2 \)-bundle whose fiber at \( p \in M \) is given by \( s(p) \subset \text{End}_\mathbb{R}(S_p) \). The \( \mathbb{C} \)-bundle \( \Sigma_s \) is associated to \( P_s(S) \) through the conjugation representation of \( Z_2 \):

\[ \Sigma_s = P_s(S) \times_{\text{conj}} \mathbb{C} . \]

Proposition B.16. Let \( (S, s) \) be a semilinear vector bundle on \( M \) with \( \text{rk} S = 2r \). Then any \( s \)-Hermitian metric \( h \) on \( S \) determines a fiber map \( f_h : P_s(S) \to P_{Z_2}(S, h) \). Moreover:
1. If \( r \) is odd, then the map \( f_h \) is an isomorphism of principal \( Z_2 \)-bundles.
2. If \( r \) is even, then \( S \) is orientable and \( f_h(s) \) coincides with a global section of the determinant line bundle \( \det\mathbb{R} S \) which has unit norm with respect to \( h_R \), thus determining an orientation of \( S \). In this case, the bundles \( \det\mathbb{R} S \) and \( P_{Z_2}(S, h) \) are topologically trivial.
Proof. Follows immediately from Proposition B.7.

Remark B.17. Another way to see that $S$ must be orientable in case 2. of the proposition is to recall that $TU(r)$ (which is homotopy-equivalent with $\Gamma(r)$) is a subgroup of $SO(2r, \mathbb{R})$ when $r$ is even (see Subsection B.1).

B.3. Classification of semilinear line bundles. Let $(S, s)$ be a semilinear line bundle on $M$ and let $h$ be an $s$-Hermitian metric on $S$. Since $rk_R(S) = 2$, the line bundle $L_s \subset End_R(S)$ generated by $s$ (which is associated to $P_s(S)$ through the sign representation of $Z_2$) is isomorphic with the real determinant line bundle $det_R S$.

**Proposition B.18.** Any rank two Euclidean vector bundle $(S, g)$ over $M$ admits a canonical semilinear structure. Moreover, there exist bijections between the following sets:
(a) The set of isomorphism classes of rank two vector bundles $S$ over $M$.
(b) The set of isomorphism classes of rank two Euclidean vector bundles $(S, g)$ over $M$, where an isomorphism of Euclidean vector bundles is defined to be an invertible isometry.
(c) The set of isomorphism classes of principal $O(2)$-bundles $P$ over $M$.
(d) The set of isomorphism classes of semilinear line bundles $(S, s)$ over $M$.

Moreover, first Stiefel-Whitney classes correspond to each other under these bijections and we have $w_1(S) = w_1(P_s(S)).$

Proof. Any Euclidean rank two vector bundle $(S, g)$ on $M$ admits a canonical semilinear structure $(S, s) \overset{\text{def}}{=} P_{Z_2} (det_R S, g)$. Explicitly, let us fix any point $p \in M$. Then any of the two elements $\nu_p \in (det_R S)_p = \wedge^2 S_p$ which has unit norm with respect to $g_p$ (i.e. any of the two elements of the set $S^0 (det_R S_p, g_p)$) determines a complex structure $J_p \in TM(S)$ given by the cross product of the two-dimensional Euclidean vector space $(S_p, h_p)$ taken with respect to the orientation induced by $\nu_p$. Explicitly, $J_p$ is given by:

$$\sharp(J_p x) = \iota_x \nu_p^a, \forall x \in S_p,$$

where $\nu_p^a \overset{\text{def}}{=} \sharp(\nu) \in \wedge^2 S_p^\vee$ is the normalized volume form of $S_p$ determined by $\nu_p$ and $\iota$ denotes the contraction of forms on $S_p$ with vectors of $S_p$. Here, $\sharp_p : \wedge^k S_p^\vee \overset{\sim}{\rightarrow} \wedge^k S_p$ is the musical isomorphism. The canonical semilinear structure of the fiber $S_p$ is then given by $s_p = \{ J_p, - J_p \}$; it is clear that this semilinear structure does not depend on the choice of $\nu_p$. As $p$ varies in $M$, $s_p$ determines the canonical semilinear structure $s$ of $(S, g)$. It is clear that $L_s \simeq det_R S$.

This construction gives a natural map from the set of isomorphism classes of rank two Euclidean vector bundles and the set of isomorphism classes of semilinear line bundles. An inverse map is obtained by choosing an $s$-Hermitian metric on a semilinear line bundle $(S, s)$. This establishes the bijection between the sets at points (b) and (d) of the proposition and also shows that $w_1(S) = w_1(P_s(S)).$

The bijections between (a), (b) and (c) are well-known and follow from the associated bundle construction and from the fact that the group $O(2)$ is a maximal compact form of $GL(2, \mathbb{R})$ (which implies that any vector bundle admits a Euclidean metric). The fact that any rank two real vector bundle admits a semilinear structure as well as the bijection between (b) and (c) also follow by noticing that $TU(1) = O(2, \mathbb{R}) \simeq U(1) \times \mathbb{Z}_2$.

The characteristic classes of a semilinear line bundle. For each $w \in H^1(M, \mathbb{Z}_2)$, let $Z_w$ denote the unique isomorphism class of principal $\mathbb{Z}$-bundles $Z$ such that $w_1(Z) = w$.

To any semilinear line bundle $(S, s)$, we associate the corresponding $\mathbb{C}$-bundle $\Sigma := \Sigma_s$ and its integer coefficient bundle $Z_\Sigma = P_{Z_2} (\Sigma) \times_{\mathbb{Z}_2} \mathbb{Z}$. Then $(S, s)$ is a bundle of rank one modules over $\Sigma$, i.e. a “$\Sigma$-line bundle” in the sense of 
\cite{K}, Section 2]. Semilinear line bundles with fixed $\mathbb{C}$-bundle $\Sigma$ form a symmetric groupoid $\Pi \mathcal{C}_\Sigma(M)$ with symmetric monoidal product given by the tensor product

\footnote{As opposed to the present paper (where we use the notation $\Sigma$), reference [1] uses the notation $K$ for a $\mathbb{C}$-bundle. What op. cit. calls a “$K$-line bundle” is what we call a semilinear line bundle.}
of bundles of rank one modules over $\Sigma$. Let $\text{Pic}_2(M)$ be the Abelian group of isomorphism classes of this groupoid. Then it was shown in op. cit. that the so-called \textit{twisted first Chern class} $\hat{c}_1^\Sigma$ gives an isomorphism of groups:

$$\hat{c}_1^\Sigma : \text{Pic}_2(M) \stackrel{\sim}{\rightarrow} H^2(M, Z_\Sigma),$$

where $H^*(M, Z_\Sigma)$ denotes the singular cohomology of $M$ with coefficients in $Z_\Sigma$. Moreover, we have $w_1(S) = w_1(Z_\Sigma)$ and the natural morphism $H^2(M, Z_\Sigma) \rightarrow H^2(M, Z_2)$ sends $\hat{c}_1^\Sigma(S, s)$ to the second Stiefel-Whitney class $w_2(S)$. On the other hand, isomorphism classes of principal $Z$-bundles $Z$ over $M$ form an Abelian group $\text{Prin}_Z(M)$ under the (fiber) product and the first Stiefel-Whitney class gives an isomorphism of groups $w_1 : \text{Prin}_Z(M) \rightarrow H^1(M, Z_2)$. If $(S, s)$ is a semilinear line bundle with $Z_{\Sigma^s} = Z$, then we have $w_1(S) = w_1(Z)$. We also have:

$$w_1(S) = w_1(\det_{\mathbb{R}} S) = w_1(L),$$

where $L = \mathbb{R}_M \otimes_{\mathbb{Z}} Z$.

Relation to characteristic classes of principal $O(2)$-bundles. For each $w \in H^1(M, \mathbb{Z}_2)$, let $Z_w$ denote the unique isomorphism class of principal $\mathbb{Z}$-bundles $Z$ such that $w_1(Z) = w$.

Recall that principal $O(2)$-bundles $P$ on $M$ are classified by their first Stiefel-Whitney class $w_1(P) \in H^1(M, \mathbb{Z}_2)$ and by their twisted Euler class $w_2(P) \in H^1(M, Z_{w_1(P)})$.

If $P$ is a principal $O(2)$-bundle whose isomorphism class corresponds to that of $(S, s)$, then we set $\hat{c}_1(P) \overset{\text{def}}{=} \hat{c}_1(S, s)$. The set:

$$\mathcal{E}(M) \overset{\text{def}}{=} \bigcup_{w \in H^1(M, \mathbb{Z}_2)} H^2(M, Z_w) = \{(w, c) \mid w \in H^1(M, \mathbb{Z}_2), c \in H^2(M, Z_w)\}$$

fibers (in the category of sets) over $H^1(M, \mathbb{Z}_2)$. Any semilinear line bundle $(S, s)$ over $M$ determines an element $w_1(Z_{\Sigma^s}) = w_1(S) \in H^1(M, \mathbb{Z}_2)$ and an element $\hat{c}_1(S, s) \in H^2(M, Z_{\Sigma^s}) \simeq H^2(M, Z_{\Sigma^s})$. Hence $(w_1(S), \hat{c}_1(S)) \in \mathcal{E}(M)$ and the map $(w_1, \hat{c}_1)$ is a bijection between the set of isomorphism classes of semilinear line bundles and the set $\mathcal{E}(M)$. Composing this with the bijection which takes isomorphism classes of principal $O(2)$-bundles to semilinear line bundles gives the classification of principal $O(2)$-bundles over $M$.

Relation to principal $O_2(\alpha)$-bundles. Recall that the group $O_2(\alpha)$ introduced in Section 1 fits into the following short exact sequence:

$$1 \rightarrow U(1) \rightarrow O_2(\alpha) \overset{n_2}{\rightarrow} G_2 \rightarrow 1.$$

This induces a long exact sequence of pointed sets in Čech-cohomology, of which we are interested in the following piece:

$$G_2 \overset{\delta}{\rightarrow} \hat{H}^1(M, U(1)) \rightarrow \hat{H}^1(M, O_2(\alpha)) \overset{n_2}{\rightarrow} \hat{H}^1(M, G_2),$$

where $\delta$ denotes the connecting map. The pointed set $\hat{H}^1(M, U(1))$ can be endowed with the structure of an Abelian group, whereas the pointed set $\hat{H}^1(M, O_2(\alpha))$ is in bijection with isomorphism classes of principal $O_2(\alpha)$-bundles over $M$.

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