LECTURES ON TOPOLOGICAL ASPECTS OF THEORETICAL PHYSICS

LEONID D. LANTSMAN

Wissenschaftliche Gesellschaft bei
Judische Gemeinde zu Rostock
Wilhelm-Külz Platz, 6.
18055 Rostock.
e-mail: llantsman@freenet.de

ABSTRACT. This series of lectures is planned as a generalization of author’s large (more than fifteen years) experience of work in the theoretical physics. The successful work of physicist-theorist is unthinkable without of good understanding of topological aspects of physical theory. The modern theoretical physics is based on the group-theoretical approach which generates the formalism of the principal fibre bundles and the instanton approach. The latter is based on the Pontrjagin’s degree of map theorem and this theorem is the original “bridge” between homology and cohomology theories. This is the rough sketch of connections of modern physics with modern topological theories. Indeed, these connections are very complicated and very interesting and are the object of these lectures. The author plans to devote his two first lectures to fibre bundle theory: this is the foundation on which the modern physics rests – the theory of gauge groups and the Yang-Mills fields. The idea of connection and curvature (first of all for the principal fibre bundles) will be given also. The lectures are devoted to the Pontrjagin’s degree of map theorem, to the theories of monopoles and instantons, to the theory of the topological index of the elliptical operator. The information accumulated to this moment allows us to apply these theories to some questions of conformal anomalies and to the topological aspects of QCD. More comprehensive questions of modern topology (for example the algebraic (co)homology theories, the theory of the spectral sequences) will be expounded in the further lectures. On the author’s opinion, these lectures will be useful for the physicists-theorists of all directions of modern physics.
Introduction

The author stated already his purposes in the abstract. He wants to add here that these lections are the extract of the beautiful monographs (on the author’s opinion these books are the best in the sphere of the modern mathematics in the XX century) of the authors which names are the actually legendary names in the modern mathematics: M.M. Postnikov, A.N. Maltsev, A.S. Schwartz, S. Kobayashi, K. Nomizu, R.M. Switzer, F. Hirzebruch, R.S. Palais, R. Solovay, M.F. Atiyah, R.T. Seely and many others.

1. Fibre Bundles

The main principle of gauge field theories is the invariancy of the action under the gauge (local) transformations. These transformations form as a rule the Lee group with the algebra Lee. Therefore we should begin our story from the general definition of the Lee group and the Lee algebra.

**Definition 1.1.** [1] The Lee group this is the group which is in the same time is the smooth manifold such that the group operation \((a, b) \to ab^{-1} \in G; a, b \in G\) is the smooth map from \(G \times G\) in \(G\).

Let us denote as \(L_a\) (\(R_a\) correspondingly) the left (the right correspondingly) displacements on \(G\) realised with the element \(a \in G\) : \(L_a x = ax\) (\(R_a x = xa\) correspondingly) for every \(x \in G\). Then we can define the inner authomorphism \(\text{ad} a\) for \(a \in G\) as \((\text{ad} a)x = axa^{-1}\) and for every \(x \in G\). This authomorphism is called the adjoint authomorphism.

**Definition 1.2.** The tangential vector field \(X\) on \(G\) is called the left-invariant (the right-invariant correspondingly) if it is invariant with respect to all the left displacements \(L_a\) (the all right displacements \(R_a\) correspondingly).

We always consider the above vector field \(X\) as a differentiable field.

**Definition 1.3.** We define the Lee algebra \(g\) of the group Lee \(G\) as a set of the all left-invariant vector fields on \(G\) with the usual adding, with the multiplication on the scalar and with the Lee bracket. As a vector space \(g\) is isomorphic to the tangent space \(T_e(G)\) in \(e\). This isomorphism is given with the map which associates the vector \(X_e\), the value \(X\) in \(e\) to the field \(X \in g\).

One can consider also [2,p17] the notion of the topologic group. It is sufficiently then to consider the maps in the definition 1.1 as a continuous maps only. The category of the topologic groups is the more weak category then the Lee groups.

We define in conclusion of our ”Lee group” theme what it is the free (effective) action of the Lee group on the manifold \(M\).

We say that \(G\) acts free(correspondingly effective) if \(R_a x = x\) for all \(x \in M\) (for the some \(x \in M\) correspondingly) involves \(a=e\). After our definitions of the Lee groups and the Lee algebras we can devote ourselves to the study of the fibre bundle theory.
**Definition 1.4.** [2,p.13] The triad

\( \zeta = (E, \pi, B) \)

where \( E \) and \( B \) are the topologic spaces and \( \pi : E \to B \) is the continuous map (the map "on") is called the fibre bundle over \( B \). The space \( B \) is called the base of the fibre bundle \( \zeta \), the space \( E \) is called the total space of the fibre bundle \( \zeta \), and the map \( \pi \) is called the (canonical) projection of the fibre bundle \( \zeta \).

**Definition 1.5.** The counterimage of the arbitrary point \( b \in B : \mathcal{F} = \pi^{-1}(b) \) is called the fibre of the fibre bundle \( \zeta \) over the point \( b \).

**Definition 1.6.** The fibre bundle \( \zeta = (E, \pi, B) \) is called the fibre bundle with the typical fibre if the fibres \( \mathcal{F}_1, \mathcal{F}_2 \) where \( b_1, b_2 \in B \) are homeomorphic.

Now we give the classification of the fibre bundles which are very important for the physicist-theorist.

*The trivial fibre bundle.*

**Definition 1.7.** The fibre bundle \( \zeta = (E, B, \mathcal{F}, \pi, ) \) is called the trivial fibre bundle if it is equivalent to the some direct product, i.e. it exists the topologic map \( \lambda : E \to B \times \mathcal{F} \) where \( \mathcal{F} \) is the topologic space; and the fibre over the point \( b \in B \) in the fibre bundle \( (E, B, \mathcal{F}, \pi) \) turns into the fibre over the same point in the fibre bundle \( (B \times \mathcal{F}, B, \mathcal{F}, \pi_1) \).

*The principal fibre bundle*[1].

Let \( M \) be the manifold and let \( G \) be the Lee group. *The principal fibre bundle over \( M \) with the structural group \( G \) consists of the manifold \( P \) and the action of the group \( G \) on \( P \) which satisfies to the following conditions:

1. \( G \) acts free on \( P \) to the right (the free action of the group \( G \) it is the such action of the group \( G \) when \( R_a x = x \) for the some \( x \in M \) involves \( a=e \)). We can write down this action as a

\[
(u, a) \in P \times G \to ua = R_a \in P
\]

2. \( M \) is the factor-space for \( P \) by the equivalence relation which is induced with the group \( G \). Thus \( M \) is divided onto the equivalence classes with respect to the group \( G \). These equivalence classes are called the orbits of the group \( G \). This notion has the great importance in theoretical physics, since it is translated on the language of the Hamiltonian formalism [3] as a classes of the equivalent trajectories which set forms the physical sector of the theory. One can write down \( M = P / G \). We consider usually the case of the smooth manifold \( M \), hence the canonical projection \( \pi : P \to M \) is smooth;

3. \( P \) is local trivial, i.e. the every point \( x \in M \) has the neighbourhood \( U \) such that \( \Psi = \pi^{-1}(U) \to U \times G \) is the diffeomorphism, which is called the trivialisation. If we define the map

\[
\phi(u) := \pi^{-1}(u) \to G
\]
which satisfies the conditions

\[(4.1) \quad \phi(ua) = \phi(u)a \]

for every \( u \in \pi^{-1}(U) \) and \( a \in G \), that \( \Psi(U) = (\pi(U), \phi(U)) \).

One denotes the principal fibre bundle usually as \( P(M, G, \pi) \) or \( P(M, G) \) or \( P \).

The fibre of the principal fibre bundle is the orbit of the group \( G \) over the fix point \( a \in M \).

Basing on this definition of the principal fibre bundle we can[1] now give the definition of the trivial principal fibre bundle.

Let again the manifold \( M \) and the group \( G \) be given, and \( G \) acts again free on \( P = M \times G \) to the right.

The map \((x, a) \in M \times G \to (x, ab) \in M \times G \) exists for every \( b \in G \) with \( R_b \). The such obtained principal fibre bundle is called the principal trivial fibre bundle.

The definition of the principal trivial fibre bundle allow us to discuss briefly (the complete discussion will follow after the introduction of the connections on the principal trivial fibre bundle). So, we have according to the definition of the principal trivial fibre bundle the direct product of the (smooth) manifold \( M \) and the Lee group \( G \) acting on \( M \) as a (local) gauge transformations which would leave the action of the theory invariant. Let the manifold \( M \) be for simplicity the Minkowsky space. This is the situation of the all gauge theories which not include the gravitation. The structure of the direct product induces on the total space of the principal trivial fibre bundle the some topology different from the topology of the flat Minkowsky space. One can construct always the some metrical space isomorphic to this direct product. The latter has its local metrics in the every point which is determines in fact with the structure of the Lee group.

The very beautiful historical example of the such approach to the principal trivial fibre bundle is the Kalutza-Klein theory which was the one from the attempts to unite the general relativity with the electromagnetism into the one theory. The electromagnetic field was considered in this theory as a fifth supplementary co-ordinate such that the circle of the radius of Planck length intersects the Minkowsky space. We have in this case in the some neighbourhood of the fixed point of the Minkowsky space the some direct product of this neighbourhood (which is isomorphic to the open ball with the centre in above point) on the sphere \( S^1 \) and this direct product generates the following metrics:

With account of the fifth supplementary co-ordinate we use the indexes \( \hat{\mu} : \hat{\mu} = 0, 1, 2, 3, 5 \) for these co-ordinates. Here as usual \( \mu = 0, 1, 2, 3 \) the Minkowsky indexes and \( \mu = 5 \) is the index correspond to the fifth co-ordinate.

Then according to the Oscar Klein’s supposition (1926 y.) we can represent the metrics of this five-dimensional world as a

\[(5.1) \quad \hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} - \phi A_\mu A_\nu & -\phi A_\mu \\ -\sigma & A_\nu - \phi \end{pmatrix} \]

where \( A_\mu \) are the vector electromagnetic potential and \( \phi \) is the scalar potential. Thus we can see as the structure \( M \times S^1 \) generates in this model the non-trivial metrics of the four-dimensional world. We ought to pay here the especial attention on the character of
the components \( \hat{g}_{\mu,5} \) and \( \hat{g}_{5,\mu} \) which are the projections of the fifth co-ordinate on the four-dimensional world.

This theory is somewhat naive, it has a lot of the shortcomings, but this conception of the direct product (and the sphere \( S^1 \) is isomorphic to the U(1) group of the two-dimensional rotations) is applied to the principal trivial fibre bundle as for the U(1) gauge theory, for the electromagnetism, as for the nonabelian Yang-Mills theories. The idea of the sphere \( S^1 \) of the Planck radius was the first example in the history of modern physics of the application of the compact extra dimensions to the description of the gauge field theory. This idea of the \textit{compactification} found its application later on: with the development of the strengs and p-branes theories.

The author is familiar with the people which apparently were the pioneers in the investigation of the \textit{spontaneous compactification} of the extra dimensions in the strengs and p-branes theories. This is the scientific leader of my postgraduate studentship V.I. Tkach. The above investigations are the fruit of the joint work of D.V.Volkov which to his death in 1996 y. was at the head of the Kharkov school of the theoretical physics, V.I. Tkach and D.P.Sorokin. It will useful to recommend our reader the works [4],[5] of these authors or the very interesting work[6] of I.P.Volobuev, Ju.A.Kubyshin, J.M.Mourao and G.Rudolph. I want to note in conclusion of this theme that the understanding of the four-dimensional structure of the total space \( E \) of the principal fibre bundle is the main part of such theories.

The author want to dwell also on the global symmetries. There are the such transformations which group parameters \( \epsilon \) are \textit{not depend on the space co-ordinates}. It is means, on the language of the principal trivial fibre bundle, that we (for example for the Minkowsky space) deals with the direct product of the base space \( M \) and \textit{the some constant}. If we fix the some point \( p \in M \) and consider the open ball-shaped neighbourhood \( U_r \) of this point then the typical fibre over \( p \) is the constant and we have the topology of the \textit{general cylinder} over \( U_r \). This is the geometrical sense of the global symmetries.

If the principal fibre bundle \( P(M,G) \) is given that the action of the group \( G \) on \( P \) generates the homomorphism between the Lee algebra of the group \( G \) and the Lee algebra of the vectors fields \( \mathcal{X}(p) \).

We hope that our reader is acquainted with the features of the vectors. Then we can devote ourselves to the following consideration.

Let \( \phi \) be the some gauge transformation on \( M \). On the other hand exists always the \textit{integral curve} \( x(t) \in M \) to which the vector \( X \) is touched in the point \( x_0 \) \( x(t) \) where \( t \) is the parameter of this curve. The fixeing of the point \( x_0 \) is equivalent to the choice of the integration constant, i.e to the fixing of the Cauchy condition. We shall denote this vector \( X \) as a \( X_p \).

Let us introduce now the \textit{one-parameter group} \( \phi_t \) \textit{of the (smooth) transformations} in \( M \) as a such map \( \mathbb{R} \times M \to M : (t,p) \in \mathbb{R} \times M \to \phi_t(p) \in M \) which satisfied the following conditions:

1. The map \( \phi_t : p \to \phi_t(p) \) is the transformation in \( M \) for every \( t \in \mathbb{R} \);
2. \( \phi_{t+s}(p) = \phi_t(\phi_s(p)) \) for every \( t, s \in \mathbb{R} \) and \( p \in M \).

The every one-parameter group \( \phi_t \) generates the vector field \( X \) on the following way. \( X_p \)
integral curve of the field $X$ issued from $p$. This curve as we already know is the orbit of the point $p$ in $p = \phi_0(p)$. The orbit $\phi_t(p)$ is the integral curve of the field $X$ issued from $p$.

The local one-parameter group of the local transformations can be defined analogous with the additional condition that $\phi_t(p)$ is defined for the $t$ in the neighborhood of $0$ and $p$ belongs to the open set in $M$. More precisely: let $I_\epsilon$ be the open interval $(-\epsilon, \epsilon)$ and $U$ is the open set in $M$.

**Definition 1.8.** The local one-parameter group of the local transformations defined on $I_\epsilon \times U$ is the map $I_\epsilon \times U \to M$ which satisfied the following conditions:

1a. $\phi_t : p \to \phi_t(p)$ is the diffeomorphism $U$ onto the open set $\phi_t(U)$ in $M$ for every $t \in I_\epsilon$

2a. if $t, s, t + s \in I_\epsilon$ and $p, \phi_s \in U$ then

$$\phi_{t+s}(p) = \phi_t(\phi_s(p))$$

(6.1)

The local one-parameter group of the local transformations as in the case of the one-parameter group of the transformations induces the vector field $X$ defined on $U$.

It is very interesting to prove the contrary statement.

**Theorem 1.1.** Let $X$ be the vector field on the manifold $M$. There exist the neighbourhood $U$, the positive number $\epsilon$ and the local one-parameter group of the local transformations $\phi_t : U \to M, t \in I_\epsilon$ for every point $p_0 \in M$ which generate this $X$.

**Remark.** We shall say that $X$ generates the local one-parameter group of the local transformations $\phi_t$ in the neighbourhood of the point $p_0$. If the (global) one-parameter group of the local transformations generated $X$ exists on $M$ that we say that $X$ is the complete field. If $\phi_t(p)$ is defined on $I_\epsilon \times M$ for the some $\epsilon$ that $X$ is complete.

**Proof.** Let $u^1, ..., u^n$ be the local co-ordinate system in the neighbourhood $W$ of the point $p_0$ such that $u^1(p_0) = ... = u^n(p_0) = 0$. Let $X = \sum \xi^i(u^1, ..., u^n)(\partial/\partial u^i)$ be in $W$. Let us consider the following system of the usual differential equations:

$$df^i/dt = \xi^i(f^1(t), ..., f^n(t)), \quad i = 1, ..., n$$

(7.1)

with the unknown functions $f^1(t), ..., f^n(t)$. According to the basic theorem for the systems of the usual differential equations it exists the only set of the functions $f^1(t, u), ..., f^n(t, u)$ defined for $u = (u^1, ..., u^n)$ with $|u^i| < \delta_1$ and for $|t| < \epsilon_1$ which forms the solution of the differential equation for the every fixed $u$ and satisfies the initial conditions:

$$f^i(0; u) = u^i$$

(8.1)

Let us put $\phi_s(u) = (f^1(t, u), ..., f^n(t, u))$ for $|t| < \epsilon_1$ and $u \in U_1 = (u; |u^i| < \delta_1)$. If $|t|, |s|$ and $|t+s|$ all less than $\epsilon_1$ and both $u$ and $\phi_s(u)$ are in $U_1$ that the functions $g^i(t) = f^i(t+s; u)$ as it is easy to see are the solutions of the differential equation with the initial data $g^i(0) = f^i(s; u)$. Because of the unique solution we have $g^i(t) = f^i(t; \phi_s(u))$. So we proved that $\phi_t(\phi_s(u)) = \phi_{t+s}(u))$. Since $\phi_0$ is the identical transformation in $U_1$ then
there exist $\delta > 0$ and $\epsilon > 0$ that $fi_t(u) \subset U_1$ for $U = u : |u^i| < \delta$ and $|t| < \epsilon$. Hence $\phi_{-t}(\phi_t(u)) = \phi_t(\phi_{-t}(u)) = \phi_0(u) = u$. Thus $\phi_t$ is the diffeomorphism on $U$ for $|t| < \epsilon$ and therefore $\phi_t$ is the local one-parameter group of the local transformations defined on $I_\epsilon \times U$ according to the definition 1.8. It is evident from the construction of $fi_t$ that $fi_t$ generates the vector field $X$ in $U$. □

The every physicist-theorist know the very important example of the such local one-parameter group of the unitary local transformations $U(1)$ which we seen already when we considered the Kalutza-Klein theory. This is the group of the rotations in the flat $(x,y)$ or on the complex flat $C^1$. This group is isomorphic [7] to the circle $S^1$ which is characterised with the angle $\phi, 0 \leq \phi \leq 2\pi$ and the unitary matrix $e^{i\phi}$ corresponds to this angle. Namely this matrix is the function $\phi_t$ of the theory considered by us above.

We shall now adduce (without of proof) the following result which defines, in fact, the Lee derivative for the local one-parameter group of the local transformations (we hope by this that our reader know the basic features of the differential forms; else he can study our further course where these features will stated).

**Theorem 1.2.** Let $X$ and $Y$ be the vector fields on $M$. If $X$ induces the local one-parameter group of the local transformations $\phi_t$ then

\[(9.1) \ [X,Y] = \lim_{t \to 0} \frac{1}{t} [Y - (\phi_t)_* Y]\]

where $\phi^*$ is the authomorphism of the algebra $\mathcal{D}(M)$ of the differential forms on $M$.

More precisely,

\[(10.1) \ [X,Y]_p = \lim_{t \to 0} \frac{1}{t} [Y_p - ((\phi_t)_* Y)_p], \quad p \in M\]

We take this limit with respect to the natural topology of the tangential vector space $T_p(M)$.

Let us now define the tensor fields on $M$.

Let $T_x = T_x(M)$ be the tangential space to the manifold $M$ in the point $x$ and $T(x)$ is the standard tensor algebra over $T_x : T_x = \sum T^r_s(x)$ where $T^r_s(x)$ is the tensor space of the $(r,s)$ type over $T(x)$.

**Definition 1.9.** The tensor field of the $(r,s)$ type on the subset $N \subset M$ is the juxtaposition of the tensor $K_x \in T^r_s(x)$ to the every point $x \in M$. We take $X_i = \partial/\partial x_i, i = 1, ..., n$ as a basis for the every tangential space $T_x, x \in U$ in the co-ordinate neighbourhood $U$ with the local co-ordinate system $x^1, ..., x^n$; we introduce also the dual basis $\omega^i = dx^i : i = 1, ..., n$ as a dual basis in $T^*_x$. (so we introduce first the differential form upon which we shall much dwell in the future).

The tensor field $K$ of the $(r,s)$ defined on $U$ then expressed as

\[(11.1) \ K_x = \sum K^{i_1 ... i_r}_{j_1 ... j_s} X_{i_1} \otimes ... \otimes X_{i_r} \otimes \omega^{i_1} \otimes ... \otimes \omega^{j_s}\]
where \( K_{j_1...j_s}^{i_1...i_r} \) are the functions on \( U \) which are called \textit{the components} for \( K \) with respect to the local co-ordinate system \( x^1, ... x^n \). We say that \( K \) is the field of the \( C^k \) class if all its components are the functions of the \( C^k \) class; of course it is necessary to check that this notion is not depends on the local co-ordinate system. It is easy to do.

Let us transform the above basis \( X_i \) and its dual basis \( \omega^i \) as

\[
X_i = \sum_j A^j_i X_j
\]

(this transformation generates the corresponding transformation of the dual basis). Then the components of the \( K_{j_1...j_s}^{i_1...i_r} \) transform as

\[
\bar{K}_{j_1...j_s}^{i_1...i_r} = \sum A_{j_1}^{i_1} A_{j_s}^{i_s} B_{j_1}^{m_1} ... B_{j_s}^{m_s} K_{m_1...m_s}^{k_1...k_r}
\]

by this transformation. If we substitute now the matrix \( (A^j_i) \) from (12.1) onto the \textit{Jacobian matrix} of the two local co-ordinate systems then we shall prove this. We shall understand the tensor field of the \( C^\infty \) class as a tensor field by the further consideration. The tensor of the \( r \)-type is called \textit{the contravariant tensor} and the tensor of the \( s \)-type is called \textit{the covariant tensor}. Any vector is the tensor of the \( r=1 \) or \( s=1 \) type.

We now introduce the \textit{Lee derivative in the terms of the tensor field \( K \)}. Let \( X \) be the vector field on \( M \) and \( \phi_t \) is the local one-parameter group of the local transformations induced \( X \) (we suppose that our reader knows the features of the tensor algebra).

\textbf{Definition 1.10.} Let us suppose for simplicity that \( \phi_t \) is the global one-parameter group of the transformations on \( M \). Then \( \phi_t \) is the authomorphism of the tensor algebra \( T(M) \) for every \( t \). Let us set

\[
(L_X K)_x = \lim_{t \to 0} \frac{1}{t}[K_x - (\hat{\phi}K)_x]
\]

The map \( L_X \) of \( T(M) \) on itself which moves \( K \) in \( L_X K \) is called the \textit{Lee differentiation with respect to} \( X \).

Let us prove the following features of the Lee derivative.

\textbf{Theorem 1.3.} \textit{The Lee differentiation} \( L_X \) \textit{with respect to the vector field} \( X \) \textit{satisfied the following conditions:}

1. \( L_X \) is the differentiation for \( T(M) \), i.e. it is linear and satisfied the equality

\[
L_X(K \otimes K') = (L_X K) \otimes K' + K \otimes (L_X K')
\]

for all \( K, K' \in T(M) \); \( L_X \) preserves the type of the tensor;

2. \( L_X \) commutes with the every contraction of the tensor field;

3. \( L_X f = X f \) for every function \( f \);

4. \( L_X Y = [X, Y] \) for every vector field \( Y \).
Proof. It is evident that $L_X$ is linear. Let $\phi_t$ be the local one-parameter group of the local transformations generated with the field $X$. Then

$$L_X(K \otimes K') = \lim_{t \to 0} \frac{1}{t} [K \otimes K' - \tilde{\phi}_t(K \otimes K')]$$

$$= \lim_{t \to 0} \frac{1}{t} [K \otimes K' - (\tilde{\phi}_t K) \otimes \tilde{\phi}_t K']$$

$$= \lim_{t \to 0} \frac{1}{t} [(\tilde{\phi}_t K') - (\tilde{\phi}_t K) \otimes (\tilde{\phi}_t K')]$$

$$= (\lim_{t \to 0} \frac{1}{t} [K - (\tilde{\phi}_t K)]) \otimes K'$$

$$+ \lim_{t \to 0} (\tilde{\phi}_t K) \otimes \frac{1}{t} [K' - (\tilde{\phi}_t K)]$$

$$= (L_X K) \otimes K' + K \otimes (L_X K')$$

Since $\tilde{\phi}_t$ preserves the type of the tensor and commutes with the every contraction then $L_X$ has the same features. If $f$ is the function on $M$ then

$$\text{(16.1)} \quad (L_X f)(x) = \lim_{t \to 0} \frac{1}{t} [f(x) - f(\phi_t^{-1}(x))] = -\lim_{t \to 0} \frac{1}{t} [f(\phi_t^{-1} x) - f(x)]$$

If we note that $\phi_t^{-1} = \phi_{-t}$ is the local one-parameter group of the local transformations generated with the field $X$ that we obtain $L_X f = -(-X)f = Xf$. And the last point of the theorem is the reformulation of the theorem 1.2. □

We shall prove now the some more theorem which will very useful for the Lee groups theory.

**Theorem 1.4.** Let $\phi$ be the transformation on $M$. If the vector field $X$ induces the local one-parameter group of the local transformations $\phi_t$ then the vector field $\phi \ast X$ where $\phi \ast$ is the authomorphism of the algebra $D(M)$ of the differential forms on $M$, induces $\phi \circ \phi_t \circ \phi^{-1}$.

**Proof.** It is evident that $\phi \circ \phi_t \circ \phi^{-1}$ is the local one-parameter group of the local transformations. This should show that it generates the vector field $\phi \ast X$. Let us consider the arbitrary point $p$ in $M$ and $q = \phi^{-1}(p)$. Since $\phi_t$ induces $X$ the vector $X_q \in T_q(M)$ touches with the curve $x(t) = \phi_t(q)$ in $q = x(0)$. Hence the vector

$$\text{(17.1)} \quad (\phi \ast X)_p = \phi \ast (X_q) \in T_p(M)$$

touches with the curve $y(t) = \phi \circ \phi_t(q) = \phi \circ \phi_t \circ \phi^{-1}(p)$ □
Corollary 1.5. The vector field $X$ is invariant with respect to the action of $\phi$, i.e. $\phi \ast X = X$ when $\phi$ is commutes with $\phi_t$ only.

Let us return again to the Lee groups and the Lee algebras. The every $A \in g$ generates the (global)one-parameter group of the transformations in $G$. Really if $\phi_t$ is the local one-parameter group of the local transformations generated with $A$ (as this was explained in the theorem 1.1) and $\phi_t e$, where $e$ is the unit element of the Lee group $G$ is defined for $|t| < \epsilon$ then one can define $\phi_t a$ at $|t| < \epsilon$ for every $a \in G$ as $L_a (\phi_t e)$ (this definition is correct because of the corollary 1.5 and since $A \in g$ is the left-invariant vector. Since $\phi_t a$ is defined for $|t| < \epsilon$ and for every $a \in G$ then $\phi_t a$ is defined at $|t| \infty$ for every $a \in G$. Let us set $a_t = \phi_t e$. Then $a_{t+s} = a_t a_s$ for every $t, s \in \mathbb{R}$ according to the definition of the one-parameter group of the local transformations. We shall call $a_t$ the one-parameter subgroup in $G$ generated with the element $A$. The other characteristic of $a_t$ is the fact that it is the unique curve in $G$ such that its tangential vector $\dot{a}_t$ which is such connected with the field $A^*$. Theorem 1.6. The Lee group $G$ acts now from the right on the manifold $M$. The map $\sigma : A \in g \rightarrow A^* \in \mathcal{X}(M)$ (where $\mathcal{X}(M)$ is the standard vector algebra on $M$) is the homomorphism of the Lee algebras. If $G$ acts effective on $M$ that $\sigma$ is the monomorphism $g \rightarrow \mathcal{X}(M)$. If $G$ acts free on $M$ that $\sigma(A)$ is nowhere equal to zero on $M$ for every non-zero $A \in g$.

Proof. Let us note firstly that we can define $\sigma$ as following. Let $\sigma_x$ be the map $a \in G \rightarrow xa \in M$ for every $x \in M$. Then $(\sigma_x)_* A_e = (\sigma A)_x$. Hence $\sigma$ is the linear map from $g$ in $\mathcal{X}(M)$. That we should show that $\sigma$ commutes with the Lee bracket. Let us suppose that

\[
[B, A] = \lim_{t \rightarrow 0} \frac{1}{t} [(\phi_t B - B)] = \lim_{t \rightarrow 0} \frac{1}{t} [ad(a_t^{-1})B - B]
\]
$A, B \in g$ and $A^* = \sigma A; B^* = \sigma B; a_t = \exp tA$. According to theorem 1.2. we have

\begin{equation}
[A^*, B^*] = \lim_{t \to 0} \frac{1}{t} [B^* - Ra_t B^*]
\end{equation}

Since $R_{a_t} \circ \sigma_{x_{a_t}^{-1}}(c)$ for $c \in G$, we obtain denoting the differential with the same letter:

\begin{equation}
(R_{a_t} B^*)_x = R_{a_t} \circ \sigma_{x_{a_t}^{-1}} B_e = \sigma_x (ad(a_t^{-1})B_e)
\end{equation}

whence:

\[
[A^*, B^*] = \lim_{t \to 0} \frac{1}{t} \left[ \sigma_x B_e - \sigma_x (ad(a_t^{-1})B_e) \right]
\]

\[
= \sigma_x \left( \lim_{t \to 0} \frac{1}{t} \left[ B_e - ad(a_t^{-1})B_e \right] \right)
\]

\[
= \sigma_x ([A, B]_e) = (\sigma [A, B])_x
\]

because of the formula for $[A, B]$ in the terms of the $ad$ $G$. Thus we proved that $\sigma$ is the homomorphism from the Lee algebra $g$ into the Lee algebra $\mathcal{X}(M)$. Let us suppose that $\sigma A = 0$ everywhere on $M$. This means that the one-parameter group of the transformations $R_{a_t}$ is trivial, i.e., $R_{a_t}$ is the identical transformation on $M$ for every $t$. If $G$ is effective on $M$ then $a_t = e$ for every $t$, hence $A = 0$ and $\sigma$ is indeed the monomorphism. In conclusion we should to prove the last point of the theorem. Let us suppose that $\sigma A$ is equal to zero in some point $x \in M$. Then $R_{a_t}$ leaves $x$ immovable for every $t$. If $G$ acts free on $M$ then $a_t = e$ for every $t$, hence $A = 0$. \qed

Thus we described briefly the principal fibre bundles and ascertained the connection between the Lee algebras $g$ and $\mathcal{X}(M)$. This will need us by study of connections at this fibre bundle.

The following type of the fibre bundles which we shall study is the vector fibre bundle. We cite here the monograph [2] of M.M. Postnikov.

Let $K$ be the field of the

a. real numbers $\mathbb{R}$;

b. or the complex numbers $\mathbb{C}$;

c. or the quaternions $\mathbb{H}$;

d. or the octaves $\mathbb{O}$.

(These fields are the object of the most interest for the physicist-theorist).

**Definition 1.11.** The triad

\begin{equation}
\zeta = (E, \pi, B)
\end{equation}

which consists of the topologic spaces $E, B$ and of the continuous map

\begin{equation}
\pi : E \to B
\end{equation}

is called the vector fibre bundle over the field $K$ if:
a. the set

\[ \mathcal{F} = \pi^{-1}(b) \]

i.e. the fibre over the arbitrary point \( b \in B \) is the linear vector space over \( K \);

b. (the condition of the local triviality). Always exists the open covering \( U \) of the space \( B \) and the such homeomorphism \( \phi_\alpha : U_\alpha \times \mathbb{R}^n \to \mathcal{E}_{U_\alpha} \) where \( \mathcal{E}_{U_\alpha} = \pi^{-1}U_\alpha \) that the diagram

\[
\begin{array}{ccc}
U_\alpha \times \mathbb{R}^n & \xrightarrow{\phi_\alpha} & U_\alpha \\
\downarrow & & \downarrow \\
\mathcal{E}_{U_\alpha} & & \\
\end{array}
\]

is commutative for every point \( (b, x) \in U_\alpha \times \mathbb{R}^n \) and \( \phi_\alpha(b, x) \in \mathcal{F}_b \) (lied in the fibre over the point \( b \)).

c. the map \( \phi_{a,b} : \mathbb{R}^n \to \mathcal{F}_b \) which is defined with the formula \( \phi_{a,b}(x) = \phi_\alpha(b, x) \), \( x \in \mathbb{R}^n \) is the isomorphism of the linear spaces.

**Remark.** The left arrow in this commutative diagram is the natural projection \( (b, x) \to b \) of the direct product \( U_\alpha \times \mathbb{R}^n \) on the first factor of this direct product and the right arrow (the map \( \pi_\alpha \)) is the restriction of the projection \( \pi \) on the fibre \( \mathcal{F}_b \).

We have the evident parallel between the principal and the vector fibre bundles. The above interpretation of the direct product \( U_\alpha \times \mathbb{R}^n \) (where we now take the open ball \( B_r \) of the radius \( r \) (in the usual metric space!) as a such neighbourhood \( U_\alpha \)) as a cylinder constructed in this neighbourhood is acceptable in the both cases.

**Definition 1.12.** The dimension \( n \) of the vector fibre bundle \( \zeta \) is called the rank of this bundle. We shall denote it as \( \dim \zeta \) or \( \dim K \zeta \).

**Definition 1.13.** The above homeomorphism \( \phi_\alpha \) is called the trivialisation of the vector fibre bundle \( \zeta \) over the open set \( U_\alpha \). The latter is called the neighbourhood of the trivialisation. The pair \( (U_\alpha, \phi_\alpha) \) is also called the trivialisation in the some literature. The covering which consists of the neighbourhoods of the trivialisation is called the covering of the trivialisation. The family \( \{U_\alpha, \phi_\alpha\}; \alpha \in I \) where \( I \) is the family of the indexes is called the atlas of the trivialisation.

**Definition 1.14.** The map

\[ (25.1) \quad S : B \to E \]

which satisfies the relation

\[ (25.1a) \quad \pi \circ s = \text{Id} \]

where \( \text{Id} \) is the identical map, is called the section of the fibre bundle \( \zeta \).

This definition is universal for all kinds of the fibre bundles.
It is evident that the map \( S : B \to E \) is the section of the vector fibre bundle \( \zeta \) if and only if \( s(b) \in \mathcal{F}_b \) for the arbitrary point \( b \in B \), i.e. if we choose the vector \( s(b) \) in every fibre \( \mathcal{F}_b \). This is the reason to call the sections of the vector fibre bundle \( \zeta \) the \( \zeta \)-vector field on \( B \).

The section of the vector fibre bundle \( \zeta \) has the following properties:

a. the formulas

\[
(S_1 + s_2)(b) = S_1(b) + s_2(b)
\]

\[
(\lambda s)(b) = \lambda s(b)
\]

define the sections \( s_1 + s_2 \) and \( \lambda s \) of the vector fibre bundle \( \zeta \) correspondingly for the arbitrary point \( b \in B \) and \( \lambda \in K \). Therefore the set \( \Gamma \zeta \) of all sections of the vector fibre bundle \( \zeta \) is the linear space over the field \( K \) (all this is evident from the definition 1.11 of the vector fibre bundle); b. the formula

\[
(fs)(b) = f(b)s(b)
\]

defines the section \( fs \in \Gamma \zeta \) for the arbitrary continuous function \( f \) on \( B \), and the lineal \( \Gamma \zeta \) is the module over \( B \) and over the algebra \( F_KB \) of all continuous functions with their values on \( K \).

**Remark.** We want to remind our reader the standard features of the module over the field \( K \) (for example the monograph [8] of A.I.Maltsev).

The module over the some ring \( K \) is the abelian group \( A \) for which one introduces the multiplication on the elements of the ring \( K \) has the following properties (for example this is the left module with respect to the ring \( K \))

\[
\lambda(a + b) = \lambda a + \lambda b
\]

\[
(\lambda + \mu)a = \lambda a + \mu a
\]

\[
(\lambda \mu)a = \lambda(\mu a)
\]

for \( \lambda, \mu \in K \) and \( a, b \in A \).

The vectors is one of examples of the modulees.

The module is called unitary if \( 1.a = a \), \( a \in A \).

**Definition 1.15.** The triad \((E_u, \pi_u, B)\) where \( E_u = \pi_u^{-1}; \pi_u = \pi|_u \) is evidently also the vector fibre bundle for every subspace \( U \subset B \). This triad is called the part of the vector fibre bundle \( \zeta \) over \( U \) and is denoted as \( \zeta|_u \). If \( U \) is the neighbourhood of the trivialisation then every trivialisation \( \phi : U \times K^n \to \mathcal{E}_U \) defines the sections \( s_1, \ldots, s_n \) in \( \Gamma(\zeta|_u) \) which operate according to the formula

\[
s_i(b) = \phi(b, e_i), \quad i = 1, \ldots, n
\]
where \(e_1, ..., e_n\) is the standard basis of the space \(K^n\). Since the vectors \(s_i(b)\) form the basis of the linear space \(F_b\) the every section \(s : U \to E_U\) sets the functions \(s^1, ... s^n\) on \(U\) satisfied the formula

\[
s(b) = \sum_{i=1}^{n} s^i(b)s_i
\]

The now stated theory together with the definition 1.11 of the vector fibre bundle allow the immediate interpretation in the vector analysis and in the theory of the differential operators.

For example if we interpret now the functions \(s_i(b)\) as a \(\partial/\partial x_i\) then we can interpret \(s(b)\) as a linear combination of the partial derivatives. This interpretation will stand us in good stead by study of the topological index theory.

We shall give now the two very important examples of the vector fibre bundles.

**Example 1. The trivial vector fibre bundle.**

The triad \((B \times V, \pi, B)\) for the arbitrary topologic space \(B\) and for the arbitrary linear \(n\)-dimensional space \(V\) over the field \(K\) where \(\pi : B \times V \to B\) is the projection of this direct product of the first factor is the vector fibre bundle from the definition 1.11. The covering of the trivialisation \(U\) consists now of the \(U = B\) and the trivialisation \(\phi : B \times K^n \to B \times V\) is defined with the choice of the standard basis \(e_1, ..., e_n\) in \(V\) and is given with the formula

\[
\phi(b, x) = (b, \alpha^{-1}(x)) \quad \text{for} \quad b \in B \quad \text{and} \quad x \in \mathbb{R}^n
\]

where \(\alpha : V \to K^n\) is the co-ordinate isomorphism corresponded to the basis \(e_1, ..., e_n\).

**Example 2. The tangential vector fibre bundle.**

Let \(X\) is the smooth \(n\)-dimensional manifold. \(TX\) is the manifold of the tangential vectors on \(X\) and \(\pi : TX \to X\) is the natural projection which compares the point \(p \in X\) to every vector \(A \in TX\). In definition the fibre \(\pi^{-1}(p), p \in X\) of the projection \(\pi\) is the tangential space \(T_pX\) and the every chart \((U, h)\) of the manifold \(X\) where \(U\) belongs to the covering of \(X\) defines the chart \((TU, Th)\) of the manifold \(TX\) for which \(TU = \bigsqcup_{p \in U} T_pX = \pi^{-1}U\) and the map \(Th : TU \to \mathbb{R}^{2n}\) is given with the formula \(Th (A) = (x^1, ..., x^n, a^1, ..., a^n)\) for \(A \in TU\) where \(x^1, ..., x^n\) are the co-ordinates of the point \(p = \pi(A)\) in the chart \((U, h)\) and \(a^1, ..., a^n\) are the co-ordinates of the vector \(A\) in the basis

\[
\left(\frac{\partial}{\partial x^1}p, ..., \frac{\partial}{\partial x^n}p\right)
\]

of the space \(T_pX\) It is convenient to replace the map \(Th\) onto the map

\[
(h^{-1} \times id)Th : TU \to U \times \mathbb{R}^n
\]

acting according to the formula

\[
A \to (p, \bar{a}); \quad p = \pi(A); \quad \bar{a} = (a^1, ..., a^n)
\]

Let \(\phi_h : U \times \mathbb{R}^n \to TU\) be the opposite map:

\[
\phi_h(p, \bar{a}) = a^i \frac{\partial}{\partial x^i}p; \quad p \in U; \quad \bar{a} \in \mathbb{R}^n
\]
Note that the formula (31.1) coincides in fact with the formula (30.1) for the linear combination of the partial derivatives.

The map $\phi_h$ is the homeomorphism which closes the commutative diagram
\[
\begin{array}{ccc}
U \times \mathbb{R}^n & \xrightarrow{\phi_h} & U \\
\downarrow & & \downarrow \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\end{array}
\]
Thus $\phi_h$ is the trivialisation of the fibre bundle $(TX, \pi, X)$ over the neighbourhood $U$.

So the triad $\tau_X = (TX, \pi, X)$ is the vector fibre bundle of the rank $n$, and it is denoted as $\tau X$ or $\tau(X)$.

We also grounded in this example the correctness of the formula for the partial derivative.

In definition exists always the open covering $U$ of the base space $B$ in the some vector fibre bundle $\zeta = (E, \pi, B)$ consisted of the neighbourhoods of the trivialisation. Let now $U_\alpha$ and $U_\beta$ be the two crossed elements of this open covering. Then the map
\[
(34.1) \quad \phi_{\beta \alpha} = \phi_{\beta, b}^{-1} \circ \phi_{\alpha, b} : K^n \to K^n
\]
where $\phi_{\alpha, b}$ and $\phi_{\beta, b}$ are the maps $\mathbb{R}^n \to F_b$ induced with the trivialisations $\phi_\alpha : U_\alpha \times K^n \to E_{U_\alpha}$ and $\phi_\beta : U_\beta \times K^n \to E_{U_\beta}$ from the definition 1.11 is defined. This map is linear and has always the opposite map. These matrices (and all the matrixes in general!) forms the linear group $GL(n; K)$ over the field $K$. Therefore the formula
\[
(35.1) \quad \phi_{\beta \alpha} : b \to \phi_{\beta \alpha}(b)
\]
sets the some map
\[
(35.1a) \quad \phi_{\beta \alpha} : U_\alpha \cap U_\beta \to GL(n; K)
\]
which is called the map (or the function) of the transition from $\phi_\alpha$ to $\phi_\beta$.

**Lemma 1.7.** The map
\[
(36.1) \quad \phi : U \to GL(n; K)
\]
of the topologic space $U$ into the group $GL(n; K)$ is continuous if and only if the map
\[
(36.1a) \quad \hat{\phi} : U \times K^n \to K^n
\]
given with the formula $\hat{\phi}(b, x) = \phi(b)x : b \in U; \quad x \in K^n$ is continuous.

**Proof.** It is evident that if the map $\phi$ is the map $\hat{\phi}$ is then the map $\hat{\phi}$ is also continuous. On the contrary let the map $\hat{\phi}$ is continuous. Then all the maps $U \to K^n$ of the form
\[
(37.1) \quad \phi_i : b \to \phi(b)E_i, \quad i = 1, \ldots, n
\]
where as usually $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is the standard basis of the space $K^n$. Hence all the maps $U \to K$ of the form

\begin{equation}
\phi^j_i : b \to \phi^j_i(b), \quad i, j = 1, \ldots, n
\end{equation}

where $\phi^j_i(b)$ are the components of the vector $\phi(b)\mathcal{E}_i$ are also continuous. The remark that the numbers $\phi^j_i(b)$ form the matrix $\phi(b) \in GL(n; K)$ completes this lemma. □

The map $\hat{\phi}$ at $U = U_\alpha \cap U_\beta$ and $\phi = \phi_{\beta\alpha}$ is non other then the composition $pr : \circ (\phi^{-1}_{\beta} \circ \phi_\alpha)$ of the homeomorphism $\phi^{-1}_{\beta} \circ \phi_\alpha : U \times K^n \to U \times K^n$ and the projection $pr : U \times K^n \to K^n$. This is the cause why the map $\hat{\phi}$ is continuous. Hence according to lemma1.7 the map $\phi_{\beta\alpha}$ is also continuous.

Thus all the maps of transition $\phi_{\beta\alpha}$ are the continuous maps.

The set of all the maps $U \to G$ for all the sets $U$ and every group $G$ is the group with respect to the operations

\begin{equation}
\phi \to \phi^{-1}, (\phi, \psi) \to \phi \psi
\end{equation}

defined with the formulas

\begin{equation}
\phi^{-1}(b) = \phi(b)^{-1}, \quad \phi \psi(b) = \phi(b) \psi(b), \quad b \in U
\end{equation}

**Remark.** We should not confuse $\phi^{-1}$ with the opposite map and $\phi \psi$ with the composition of maps.

If $U$ is the topologic space and $G$ is the topologic group and the maps $\phi$ and $\psi$ are continuous then one can prove that the maps $\phi^{-1}$ are also continuous.

In particular we can define the map $\phi^{-1}_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(n; K)$ for the map $\phi_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(n; K)$ and the map

\begin{equation}
(\phi_{\gamma\beta}|_U)(\phi_{\beta\alpha}|_U) : U \to GL(n; K)
\end{equation}

for the maps $\phi_{\beta\alpha}$ and $f_{i\gamma\beta}$ in the case when $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ where $U = U_\alpha \cap U_\beta \cap U_\gamma$.

It follows, straightly from the definition of the maps $\phi_{\beta\alpha}$ and the usual features of the matrices that

\begin{equation}
\phi^{-1}_{\beta\alpha} = \phi_{\alpha\beta}
\end{equation}
on $U_\alpha \cap U_\beta$ and

\begin{equation}
\phi_{\gamma\beta} f_{i\beta\alpha} = \phi_{\gamma\alpha}
\end{equation}
on $U_\alpha \cap U_\beta \cap U_\gamma$ for every indexes $\alpha, \beta, \gamma$ for which $U_\alpha \cap U_\beta \neq \emptyset$ and $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ correspondingly.
**Definition 1.16.** Let $B$ be the topologic space, $G$ be the topologic group and $\mathcal{U} = \{U_\alpha\}$ be the open covering of the space $B$. The family $\phi = \{\phi_{\beta\alpha}\}$ of the continuous maps

\begin{equation}
\phi_{\beta\alpha} : U_\alpha \cap U_\beta \to G
\end{equation}

defined for the some indexes $\alpha, \beta$ for which $U_\alpha \cap U_\beta \neq \emptyset$ is called the matrix cocycle over the group $G$ of the covering $\mathcal{U}$ if it satisfies the relations (41.1,41.1a)

Thus we see that every vector fibre bundle $\zeta = (E, \pi, B)$ defines the some matrix cocycle $\phi = \{f_{i_{\beta\alpha}}\}$ for every covering of trivialisation $\mathcal{U}$ of the space $B$.

We shall call this cocycle -the gluing cocycle of the vector fibre bundle $\zeta$ and shall denote it with the symbol $\phi_\zeta$.

**Theorem 1.8.** Let $B$ be the topologic space, $\mathcal{U}$ be its open covering and $\phi = \{\phi_{\beta\alpha}\}$ be the some matrix cocycle over the group $GL(n;K)$ of the covering $\mathcal{U}$. Then exists up to isomorphism the unique vector fibre bundle $\zeta$ of the rank $n$ with the base $B$, with the covering of trivialisation $\mathcal{U}$ and with the gluing cocycle $\phi$.

**Proof.**. Let us prove firstly that this vector fibre bundle is unique. The statement that two vector fibre bundles $\zeta = (E, \pi, B)$ and $\zeta' = (E', \pi', B)$ with the covering of trivialisation $\mathcal{U}$ have the same gluing cocycle $\phi$ means that there exist the such trivialisations

\begin{equation}
\phi_\alpha : U_\alpha \times K^n \to \mathcal{E}_{U_\alpha}, \quad \phi'_\alpha : U_\alpha \times K^n \to \mathcal{E}'_{U_\alpha}
\end{equation}

for these fibre bundles that for the some indexes $\alpha, \beta$ with $U_\alpha \cap U_\beta \neq \emptyset$ the following equality is correct :

\begin{equation}
\phi_{\beta}^{-1} \circ \phi_\alpha = \phi'_\beta^{-1} \circ \phi'_\alpha : (U_\alpha \cap U_\beta) \times K^n \to: (U_\alpha \cap U_\beta) \times K^n
\end{equation}

Therefore the equality

\begin{equation}
\phi'_\beta \circ \phi_{\beta}^{-1} = \phi'_\alpha \circ \phi^{-1}_\alpha : \mathcal{E}_{U_\alpha \cap U_\beta} \to \mathcal{E}_{U_\alpha \cap U_\beta}
\end{equation}

is also correct.

Hence the formula

\begin{equation}
f(p) = (\phi'_\alpha \circ \phi^{-1}_\alpha)(p)
\end{equation}

for $p \in \mathcal{E}_{U_\alpha}$, i.e. for $\pi(p) \in U_\alpha$, defines correct the some map $f : E \to E'$ which is the isomorphism of the vector fibre bundles $\zeta \to \zeta'$ ( The isomorphism $\phi$ of the two fibre bundles $\zeta \to \zeta'$ is defined with the following commutative diagram

\begin{equation}
\begin{array}{ccc}
E & \xrightarrow{\phi} & E' \\
\pi \downarrow & & \downarrow \pi' \\
B & \xrightarrow{\phi} & B'
\end{array}
\end{equation}
We set \( \tilde{\phi} = \text{id} \) in our case; we can write down this commutative diagram with the expression \( \pi' \circ \phi = \tilde{\phi} \circ \pi \) which will have the large important in the future. Thus \( \tilde{\phi} \) is continuous automatic. If \( \tilde{\phi} \) is continuous then the fibre map \( \phi \) is called the morphism \( \zeta \to \zeta' \) of the two fibre bundles. And if the both \( \phi \) and \( \tilde{\phi} \) are the homeomorphisms then the morphism \( \zeta \to \zeta' \) is called the isomorphism of the two fibre bundles \( \zeta \) and \( \zeta' \).

Let us prove now that the vector fibre bundles \( \zeta \) exists. Let us consider firstly the disjunctive joining up

\[
\tilde{E} = \bigsqcup_{\alpha}(U_{\alpha} \times K^n)
\]

of the spaces \( U_{\alpha} \times K^n \). Let us denote the point \((b, x) \in U_{\alpha} \times K^n \) of the space \( \tilde{E} \) as \((b, x)_{\alpha}\) and let us introduce in \( E \) the relation \( \sim \) considering that \((b, x)_{\alpha} \sim (c, y)_{\beta}\) for \( b \in U_{\alpha}, c \in U_{\beta}; x, y \in K^n \) when and only when \( c = b \) and \( y = \phi_{\beta\alpha}(b)x \), i.e. when \( y \) and \( x \) belong to the same orbit of the group \( \text{GL}(n; K) \). It is follows immediately from these relations that this is the equivalence relation. Let \( E \) is the corresponding factor-space of the space \( \tilde{E} \) supplied with the corresponding factor topology.

We want to remind of our reader what is this the factor topology- the very important point of the lot of the topological theories.

If \( [9] \) the some topologic space \( X \) and the some equivalence relation \( E \) on \( X \) and the map \( q : X \to X/E \) are given ( the map \( q \) associates the every point \( x \in X \) with its equivalence class \([x] \in X/E \). If we search for the good topology then we demand quite reasonable that \( q \) should be continuous. There exist the most fine topology in the class of all topologies with respect to which the map \( q \) is continuous ; this is the family of all sets \( U \in X \) such that \( q^{-1}(U) \) is open in \( X \). This topology is called the factor topology.

Now we can continue prove the theorem. The formula

\[
\pi[b, x]_{\alpha} = b, \quad b \in B, \quad x \in K^n
\]

where \( \alpha \) is the such index that \( b \in U_{\alpha} \) and \([b, x]_{\alpha}\) is the equivalence class of the point \((b, x)_{\alpha}\) defines correct the continuous surjection

\[
\pi : E \to B
\]

The formula

\[
\phi_{\alpha}(b, x) = [b, x]_{\alpha}, \quad b \in U_{\alpha}, \quad x \in K^n
\]

for every \( \alpha \) defines the continuous map of the fibres

\[
\phi_{\alpha} : U_{\alpha} \times K^n \to E_{U_{\alpha}}
\]

where \( E_{U_{\alpha}} = \pi^{-1}U_{\alpha} \) is the subspace of the space \( E \) consisted of all points of the class \([b, x]_{\alpha}, b \in U_{\alpha}, x \in K^n\). Furthermore it is easy to see that the formula

\[
[b, x]_{\alpha} \to (b, x), \quad b \in U_{\alpha}, \quad x \in K^n
\]
defines correct the continuous map $E_{U_\alpha} \to U_\alpha \times K^n$ opposite to the map $\phi_\alpha$. Therefore $\phi_\alpha$ is the homeomorphism of the fibres. This means that the triad $\zeta = (E, \pi, B)$ satisfies definition 1.11 (in the position b.)

In order to satisfy the positions a. and c. of definition 1.11 let us note that the fibre $F_b$ of the map $\pi_\alpha$ in the some point $b \in B$ consists as we proved this above of all points of the class $[b, x]_\alpha$ where $\alpha$ is the arbitrary index for which $b \in U_\alpha$. If besides that $x \in U_\beta$ (this is real because of the standard local diffeomorphism $U_\alpha \to \mathbb{R}^m, m \leq n$) then $[b, x]_\alpha = [b, y]_\beta$ where $y = \phi_{\beta\alpha}(b)x$. Since the map $\phi_{\beta\alpha}(b) : \mathbb{R}^n \to \mathbb{R}^n$ is linear the formulas

\begin{align}
[b, x]_\alpha + [b, y]_\alpha &= [b, x + y]_\alpha, \quad x + y \in K^n \\
\lambda [b, x]_\alpha &= [b, \lambda x]_\alpha, \quad x \in K^n
\end{align}

(54.1) (55.1)

define correct the structure of the linear space in $F_b$. This proves the positions a. and c. of the definition 1.11.

Thus $\zeta$ is the vector fibre bundle and the maps $\phi_\alpha$ are its trivialisations. Besides that

\begin{align}
(\phi_{\beta}^{-1} \circ \phi_{\alpha})(b, x) = \phi_{\beta}^{-1}[b, x]_\alpha = \phi_{\beta}^{-1}[b, \phi_{\beta\alpha}(b)x]_\beta = (b, \phi_{\beta\alpha}(b)x)
\end{align}

(56.1)

for every point $(b, x) \in U_\alpha \cap U_\beta$ and therefore the gluing cocycle $\phi_\zeta$ of this vector fibre bundle is this cocycle $\phi = [\phi_{\beta\alpha}]$. □

The described construction explains in particular why the cocycle $\phi_\zeta$ is called the gluing cocycle. The maps $\phi_{\beta\alpha}$ formed this cocycle by the analogous reasons are called the gluing functions.

The theorem 1.8 reduces the vector fibre bundles to its matrix cocycles still not completely since the cocycle $\phi_\zeta$ depends on the choice of the trivialisations $\phi_\alpha$ and can turn out the other at the other choice of the trivialisations.

However one can control easy this ambiguity.

Let $\{\phi_\alpha : U_\alpha \times K^n \to E_{U_\alpha}\}$ and $\{\phi'_\alpha : U_\alpha \times K^n \to E_{U_\alpha}\}$ be the two system of the trivialisations of the vector fibre bundle $\zeta$ over the same covering of trivialisation $U = \{U_\alpha\}$. Then the formula

\begin{align}
\gamma_\alpha(b) = \phi_{\alpha,b}^{-1} \circ \phi'_{\alpha,b} : K^n \to K^n, \quad b \in U_\alpha
\end{align}

(57.1)

for every $\alpha$ defines the some map

\begin{align}
\gamma_\alpha : U_\alpha \to GL(n; K)
\end{align}

(58.1)

which is connected with the homeomorphism

\begin{align}
\phi_{\alpha}^{-1} \circ \phi'_{\alpha} : U \times K^n \to U \times K^n
\end{align}

(59.1)

with the relation

\begin{align}
(\phi_{\alpha}^{-1} \circ \phi'_{\alpha})(b)x = (b, \gamma_\alpha(b)x), \quad b \in U_\alpha, \quad x \in K^n
\end{align}

(60.1)
therefore because of the lemma 1.7 it is the continuous map.

According to the construction

\[
\phi'_{\beta \alpha}(b) = \phi'_{\beta, b}^{-1} \circ \phi'_{\alpha, b} = \\
= \phi'_{\beta, b}^{-1} \circ \phi_{\beta, b} \circ \phi_{\beta, b}^{-1} \circ \\
\circ \phi_{\alpha, b} \circ \phi_{\alpha, b}^{-1} \circ \phi_{\alpha, b} = \\
= \gamma_{\beta}^{-1}(b) \circ \phi_{\beta \alpha}(b) \circ \gamma_{\alpha}(b)
\]

i.e.

\[\phi'_{\beta \alpha} = \gamma_{\beta}^{-1} \phi_{\beta \alpha} \gamma_{\alpha}\]  \(\text{(61.1)}\)

in the group of all continuous maps \(U_\alpha \cap U_\beta \to GL(n; K)\) (where of course we mean under \(\gamma_{\alpha}\) and \(\gamma_{\beta}\) theirs restrictions on \(U_\alpha \cap U_\beta\)). Formula defines the group automorphism of the group \(GL(n; K)\).

**Definition 1.17.** We shall say that two cocycles \(\phi = \{\phi_{\beta \alpha}\}\) and \(\phi' = \{\phi'_{\beta \alpha}\}\) of the covering \(U\) over the group \(G\) are cohomological if there exist the such continuous maps

\[\gamma_{\alpha} : U_\alpha \to G\]  \(\text{(62.1)}\)

that the relation (61.1) is fulfilled for every indexes \(\alpha, \beta\) with \(U_\alpha \cap U_\beta \neq \emptyset\)

The just proved statement mean in this terminology *that the gluing cocycles of the same vector fibre bundle \(\zeta\) corresponded the different trivialisations \(\phi_{\alpha}\) (but the same covering of trivialisation \(U\) ) are cohomological*.

The relation of cohomology of cocycles is evident *the relation of equivalence*. The corresponding classes are called *the cohomology classes* of the covering \(U\) over the group \(G\). We shall denote the cohomology class of the cocycle \(\phi\) with the symbol \([\phi]\) and the set of all cohomology classes of the covering \(U\) over the group \(G\) with the symbol \(H^1(U : G)\).

**Remark.** We shall expound in our further lections the notion of the cohomology classes for the differential forms. This is the other notion then the cohomology classes for the covering \(U\) over the group \(G\), but the idea of the equivalence classes underlains in the both theories. We can also note that the set \(H^1(U : G)\) has not the group structure in contrast to cohomology group of the differential forms as we shall make sure in this.

The following theorem is now evident.

**Theorem 1.9.** *The formula*

\[(63.1) \quad \zeta \to [\phi]\]
sets the bijective accordance between the set of the equivalence classes of the isomorphical vector fibre bundles $\zeta$ of the rank $n$ over the space $B$ with the given covering of trivialisation $\mathcal{U}$ and the set $H^1(\mathcal{U};G)$.

We can add to said in the theorem 1.9 the following remark. The every cocycle $\phi' = \{\phi'_\alpha\}$ cohomological to the gluing cocycle $\phi$ of the vector fibre bundle $\zeta$ of the rank $n$ over the space $B$ had the given covering of trivialisation $U$ and the set $H^1(U;G)$. Really if the cocycle $\phi = \phi_\zeta$ corresponds to the trivialisations $\phi_\alpha : U_\alpha \times K^n \to E_{U_\alpha}$, then the equalities of the type (61.1) where $\gamma_\alpha$ are the maps take place then setting

$$
(64.1) \quad \phi'_\alpha(b,x) = \phi_\alpha(b,\gamma_\alpha(b)x), \quad b \in U_\alpha, \quad x \in K^n
$$

we obtain the trivialisations $\phi'_\alpha : U_\alpha \times K^n \to E_{U_\alpha}$ to which corresponds the cocycle $\phi'$.

Exists always the labelled element $[\mathcal{E}]$ in the every set $H^1(\mathcal{U};G)$ which is the cohomology class of the cocycle $\mathcal{E}$ consisted of the constant maps $U_\alpha \cap U_\beta \to G$ every of which maps the all set $U_\alpha \cap U_\beta$ in the unit e of the group $G$. It is easy to prove that the trivial vector fibre bundle $\theta_n^B$ corresponds to the class $[\mathcal{E}]$ in the case $G = GL(n;K)$.

**Remark.** If the covering $\mathcal{U}'$ is inscribed in the covering $\mathcal{U}$ (i.e. $U'_\alpha \subset U_\beta$ for every indexes $\alpha, \beta$) then the operation of the maps restriction defines the injective map $H^1(\mathcal{U};G) \to H^1(\mathcal{U}';G)$ which we can consider as an embedding. This allows us to introduce in consideration the joining up of the sets $H^1(\mathcal{U};G)$ over the all open coverings $\mathcal{U}$ of the space $B$. This joining up is denoted as $H^1(B;G)$ and is in the bijective correspondence with the set of the equivalence classes of the local trivial $G$-fibre bundles (at $G = GL(n;K)$ there are the vector fibre bundles of the rank $n$ over the space $B$).

Thus the latter remark builds the bridge between the principal (or the local trivial $G$-fibre bundles) and the vector fibre bundles. This bridge allows us to consider the $K$-theory - the very beautiful modern theory which will be the base of our consideration of the theory of the topologic index of the elliptical operator.

But we must firstly define the two utterly important notions had the crucial significance in all topology. There are the notions of the functor and the cellular spaces. It is better to familiarize oneself with these notions by the monograph [10] of Robert M. Switzer.

So, let us consider the notion of the functor.

**Definition 1.18.** The category consists of the:

1. some class of objects (for example, spaces, groups, etc.);
2. sets $hom(X,Y)$ of the morphisms defined on $X$ and with theirs values in $Y$ for every ordered pair of objects $X$ and $Y$; if $f \in hom(X,Y)$ then we write down $f : X \to Y$ or $X \xrightarrow{f} Y$;
3. map $hom(Y,Z) \times hom(X,Y) \to hom(X,Z)$ called the composition set for the ordered triad $(X,Y,Z)$ of objects; if $f \in hom(X,Y), g \in hom(Y,Z)$ then the image of the pair $(g,f)$ in hom $(X,Z)$ denotes as $g \circ f$.

The two axioms must be fulfilled by this:

**C1.** If $f \in hom(X,Y), g \in hom(Y,Z), h \in hom(Z,W)$ then $h \circ (g \circ f) = (h \circ g) \circ f$. 
**C2.** It exists the such morphism $1_Y \in \text{hom}(Y,Y)$ for every object $Y$ that we have $1_Y \circ g = g$ and $h \circ 1_Y = h$ for the some morphisms $g \in \text{hom}(X,Y)$ and $h \in \text{hom}(Y,Z)$.

**Remark.** The considered above morphism of the two fibre bundles is the one of the examples of morphisms.

**Definition 1.19.** The functor from the category $\mathcal{E}$ in the category $\mathcal{D}$ is the correspondence which:

1. compares the object $F(X) \in \mathcal{D}$ to the every object $X \in \mathcal{E}$;
2. compares the morphism $F(f) \in \text{hom}_\mathcal{D}(F(X),F(Y))$ to the every morphism $f \in \text{hom}_\mathcal{E}(X,Y)$.

The two axioms must be fulfilled by this:

**F1.** The equality $F(1_X) = 1_{F(X)}$ takes place for every object $X \in \mathcal{E}$.

**F2.**

\[
F(g \circ f) = F(g) \circ F(f) \in \text{hom}_\mathcal{D}(F(X),F(Z))
\]

for every $f \in \text{hom}_\mathcal{E}(X,Y), g \in \text{hom}_\mathcal{E}(Y,Z)$.

The dual to functors notion of cofunctors is also very important in the many applications of differential topology.

**Definition 1.20.** The cofunctor $F^*$ from the category $\mathcal{E}$ in the category $\mathcal{D}$ is the correspondence which:

1. compares the object $F^*(X) \in \mathcal{D}$ to the every object $X \in \mathcal{E}$;
2. compares the morphism $F^*(f) \in \text{hom}_\mathcal{E}(X,Y)$ to the every morphism $f \in \text{hom}_\mathcal{E}(X,Y)$.

The two axioms must be fulfilled by this:

**CF1.** The equality $F^*(1_X) = 1_{F^*(X)}$ takes place for every object $X \in \mathcal{E}$.

**CF2.**

\[
F^*(g \circ f) = F^*(f) \circ F^*(g) \in \text{hom}_\mathcal{E}(F^*(Z),F^*(X))
\]

for every $f \in \text{hom}_\mathcal{E}(X,Y), g \in \text{hom}_\mathcal{E}(Y,Z)$.

Let us now consider the cellular spaces.
Definition 1.21.. The cell division K of the some space X is the family \( K = \{ e_n^\alpha : n = 0, 1, 2, \ldots; \alpha \in J_n \} \), of the subsets in X indexed with the nonnegative integer numbers n and the elements \( \alpha \) from the set of indexes \( J_n \). The set \( e_n^\alpha \) is called the cell of the dimension \( n \).

Let us set \( K^n_r = \{ e_r^\alpha : r \leq n, \alpha \in J_n \} \). If we consider the category \( \mathcal{T} \) of all topologic spaces and all continuous maps then we consider in definition that \( K^n = \emptyset \) for \( n < 0 \). If we consider the category \( \mathcal{TR} \) of all topologic spaces with the labelled points (the labelled point - it is the point \( x_0 \in X \) chosen in X for the some purposes, for example this is the origin of the co-ordinates) and all continuous maps preserved the labelled points then we consider the labelled point \( x_0 \) as a cell of the dimension \( -\infty \) and \( K^n = \{ \{ x_0 \} \}, n < 0 \). The set \( K^n \) is called the \( n \)-dimensional frame of the division K.

Let us set \( |K^n| = \bigcup_{r \leq n} e_r^\alpha \). Let us note that \( |K^n| \) is the subspace in the space X while K is the family of cells only. Let us introduce the following denotations:

\[
\begin{align*}
\hat{e}_n^\alpha &= e_n^\alpha \cap |K^{n-1}| \\
\check{e}_n^\alpha &= e_n^\alpha - \hat{e}_n^\alpha
\end{align*}
\]

for every cell \( e_n^\alpha \).

The set \( \hat{e}_n^\alpha \) is called the boundary of the cell \( e_n^\alpha \), and \( \check{e}_n^\alpha \) is called its interior. One demands that K should satisfies the following conditions:

1. \( X = \bigcup_{n, \alpha} e_n^\alpha = |K| \)
2. \( e_n^\alpha \cap e_n^\beta \neq \emptyset \Rightarrow n = m, \ \alpha = \beta \)
3. it exists the surjective map \( f_n^\alpha : (D^n, S^{n-1}) \to (e_n^\alpha, \check{e}_n^\alpha) \)

(\( D^n \) is the n-dimensional unit disk and \( S^{n-1} \) is the (n-1)-dimensional unit sphere bounds this disk) for every cell \( e_n^\alpha \) which maps homeomorphic \( D_n^\alpha \) on \( e_n^\alpha \).

The map \( f_n^\alpha \) is called the characteristic map of the cell \( e_n^\alpha \).

It is follows from condition (69.1b) that every cell \( e_n^\alpha \) is the compact subset in X therefore it is closed since we suppose that X is the Hausdorff space (every compact subset is closed in the Hausdorff space, look for example [9]). The conditions (69.1),(69.1a) mean that X is the nonconnected joining up of the interiors \( e_n^\alpha \) (one can prove this with the help of the induction by the frames).
Definition 1.22. \(\text{Let us set}\)

\[(70.1)\]

\[\dim K = \sup \{ n : J_n \neq \emptyset \} \]

The number \(K\) can be equal to \(\infty\):

The cell \(e^n_\alpha\) is called the direct border of the cell \(e^n_\alpha\) if \(e^n_\alpha \cap e^n_\alpha \neq \emptyset\). Thus every cell \(e^n_\alpha\) is the direct border itself, and if \(e^m_\beta\) is the other direct border then \(m < n\). The cell \(e^m_\beta\) is called the border of the cell \(e^n_\alpha\) if it exists such sequence of the cells

\[(71.1)\]

\[e^m_\beta = e^{m_0}_\beta, e^{m_1}_\beta, ..., e^{m_s}_\beta = e^n_\alpha \]

that every cell \(e^{m_i}_\beta\) is the direct border of the \(e^{m_{i+1}}_\beta\), \(0 \leq i < s\). The cell is called the principal if it is not the border of any other cell. For example if \(\dim K = n < \infty\) then all the \(n\)-dimensional cells are principal.

Remark. In general \(e^n_\alpha\) is not the open set in the space \(X\). Furthermore even the interiors of the principal cells are not always the open sets. It is connected with the very bad topology of the some infinite cellular divisions.

Definition 1.23. We shall say that the structure of the cellular space is given on \(X\) if it has the cellular division with the following features:

1. \(\text{every closed cell has the finite number of the direct borders only;}\)
2. \(\text{\(X\) has the weak topology generated with the division \(K\), i.e. the subset \(S \subset X\) is closed if and only if when the intersection \(S \cap e^n_\alpha\) is closed in \(e^n_\alpha\) (for every \(n\) and every \(\alpha \in J_n\)).}\)

Since the characteristic map \(f^n_\alpha\) induces the homeomorphism between \(e^n_\alpha\) and the factor-space \(D^n/\sim\) (the denotation \((X,A)\) means always that we consider the pair \(A \subset X\) with the closed subspace \(A \in X\) where \(A\) is squeezed in the point); \(x \sim y\) if and only if when \(f^n_\alpha x = f^n_\alpha y\) then it is follows from the (2) of definition 1.23 that \(S \subset X\) is closed if and only if when the countrimages \((f^n_\alpha)^{-1}S\) are closed in the disk \(D^n\) for every \(n, \alpha\). Besides it is evident that the interiors of the principal cells are the open sets in the cellular space.

We shall call usually \(X\) the cellular space if one can introduce on it the some structure \(K\) of the cellular space. Let us note that if \(X\) permits even the one cellular structure then one can introduce on it the many of such structures.

The conception of the cellular spaces suggested first by J.H.C. Whitehead [11] has the broad application in the modern mathematics. It is appropriate to mention here the simplexes: it is in fact the cellular division of the metrical space. We shall discuss this notion in detail in the future, but we want to mention here the great role of simplexes in the theoretical physics, namely in general relativity. It is Regge calculation, i.e. the approximation of the some space-time manifold with this cellular division [12]. The consideration of the continuous deformations of the simplexes (of the Planck size) led S. Hawking [13] to the hypothesis of the space-time foam with the
configuration of the gravitational instantons corresponded to the minimum of the gravitation’s action. This theory has the great future affecting all the directions of modern theoretical physics. The author of this lections has his point of view on the problem of the space-time foam and the gravitational instantons which he will state may be in the visible future. One can add to this theme that the S.Hawking’s discovery of the space-time foam and the gravitational instantons was the event in the modern physics equal by importance to the Einstein’s discovery of the general relativity. I think that we in the 21 century shall estimate the all significance of S.Hawking’s discovery. So we described the necessary mathematical apparatus which we shall apply now for the study of K-theory.

First of all we note that the cellular spaces forms the category $RW$ of the cellular spaces with the labelled points and the continuous maps preserved the labelled points (the labelled points here is the additional, but the important for our statement assumption).

The general category $RT$ of all spaces with the labelled points and the continuous maps preserved the labelled points will also necessary us later on.

**Definition 1.24.** The homotopy of the space $X$ in the space $Y$ is the continuous map $F : X \times I \to Y$ where $I$ is the unit interval $[0,1]$. The homotopy $F$ defines the map $F_t : X \to Y$ given with the formula $F_t(x) = F(x,t)$, $x \in X$ for every $t \in I$. Let $f,g : X \to Y$ be the some maps. One says that $f$ homotopic $g$ (and one writes down $f \simeq g$) if it exists the such homotopy $F : X \times I \to Y$ that $F_0 = f$ and $F_1 = g$. In other words the maps $f$ and $g$ are homotopic when and only when $f$ is deformed continuously in $g$ by means of the some family $F_t$. Let $A$ be the some subspace in $X$. The homotopy $F$ is called the homotopy relatively $A$ (or the homotopy rel $A$) if $F(a,t) = F(a,0)$ for all $a \in A, t \in I$.

Let us prove now the one small but very important theorem about the relation of homotopy.

**Theorem 1.10.** The relation of homotopy is the relation of equivalence.

*Proof.* The feature $f \simeq f$ is obvious: we must take $F(x,t) = f(x)$ for all $x \in X, t \in I$. If $f \simeq g$ and $F$ is the homotopy from $f$ to $g$ then the map $G : X \times I \to Y$ given with the formula $G(x,t) = F(x,1-t)$, $x \in X, t \in I$ is the homotopy from $g$ to $f$ therefore $fg \simeq f$. Let now $F$ be the homotopy from $f$ to $g$ and $G$ is the homotopy from $g$ to $h$. Then the homotopy $H$ defined with the formula

$$
(72.1) \quad H(x,t) = \begin{cases} 
F(x,t), & 0 \leq t \leq \frac{1}{2}, \\
G(x,2t-1), & \frac{1}{2} \leq t \leq 1 
\end{cases}
$$

sets the homotopy from $f$ to $h$. □

Thus the relation $\simeq$ divides the set of all continuous maps $F : X \to Y$ on the equivalence classes. These equivalence classes are called the homotopic classes; the corresponding set of all homotopic classes is denoted as $[X;Y]$. The homotopical class of the continuous map $f : X \to Y$ is denoted as $[f]$.

The notion of homotopy is the one from the most of important notions in the modern mathematics. We shall use it very broadly in our lections, for example this notion is the crucial detail of the proof of Pontrjagin theorem about degree of map.
The just introduced notion of homotopy and theorem 1.10 allow us to consider the new category \( \mathcal{RW}' \). This category is obtained from the category \( \mathcal{RW} \), but now we consider as a continuous maps (morphisms!) the homotopic classes \([X;Y]\). Then we define the composition \([g] \circ [f]\) as \([g \circ f]\) for given \([f] \in \text{hom}(X,Y), g \in \text{hom}(Y,Z)\). It is easy to check that the morphism \([g] \circ [f]\) is defined correct and the axioms C1 and C2 are fulfilled (as a consequences of theorem 1.10).

We introduce similarly the category \( \mathcal{RT}' \), the homotopical category for all labelled topological spaces and the continuous maps which preserve the labelled points. This category has the above features of the homotopical classes.

**Definition 1.25.** The morphism \( \phi : \zeta \to \zeta' \) of the two principal fibre bundles is the such pair of the maps \( \phi : E \to E', \phi : B \to B' \) that \( \pi' \circ \phi = \phi \circ \pi \) where \( \pi, \pi' \) are the corresponding projections of the principal fibre bundles \( \zeta \) and \( \zeta' \), and \( \phi(e, g) = \phi(e)g \) for all \( g \in G, e \in E \). The notion of equivalence of the two principal fibre bundles \( \zeta \) and \( \zeta' \) is introduced on the following way. One say that the two principal fibre bundles are equivalent if there the exists the such morphisms \( \phi : \zeta \to \zeta', \psi : \zeta' \to \zeta \) that \( B=B' \) and \( \phi \circ \psi = \psi \circ \phi = 1 \). The definition of the equivalence of the two vector fibre bundles has the same construction.

The following series of the theorems and the definitions continues this construction.

**Definition 1.26.** One say that the map \( \pi : E \to B \) has the feature of the covering homotopy relatively the space \( X \) if it exists the such homotopy \( F : X \times I \to E \) for every map \( f : X \to E \) and the every homotopy \( G : X \times I \to B \) that \( f = F_0 \) and \( \pi \circ F = G \). The homotopy \( F \) is called the lift of the homotopy \( G \).

**Definition 1.27.** The family of the trivialisations of the principal fibre bundle \( \zeta \) forms the atlas \( \{(U_\alpha, \phi_\alpha)\} \) \( (\phi_\alpha : U_\alpha \times G \to \pi^{-1}U_\alpha) \) is the trivialisations of the principal fibre bundle \( \zeta \).

**Lemma 1.11.** Let \( \zeta \) is the principal fibre bundle over the space \( B \). Then one can compare the only set of cocycles \( \tilde{\zeta} = (U_\alpha, \phi_{\alpha\beta}) \) to the every atlas \( \{(U_\alpha, \phi_\alpha)\} \) for which

\[
(73.1) \quad \phi_\beta(b, g) = \phi_\alpha(b, \phi_{\alpha\beta}(b)g), \quad b \in B, \quad g \in G
\]

where \( \phi_{\alpha\beta} : U_\alpha \cap U_\beta \to G \) are now the cocycle over the group \( G \), and this distinguishes this construction from the was for the vector fibre bundles with the linear group \( \text{GL}(n;K) \).

The proof of this lemma is analogous to the proof of the theorem 1.8 and we shall not produce this proof here. Our reader can familiarize himself with this proof by the monograph [10].

**Lemma 1.12.** If \( \tilde{\zeta}, \tilde{\zeta}' \) are the two sets of cocycles associated according to lemma 1.11 with the atlases \( \{(U_\alpha, \phi_\alpha)\};\{(U'_\alpha, \phi'_\alpha)\} \) of the principal fibre bundles \( \zeta = (B, \pi, E, G) \), \( \zeta' = (B', \pi', E', G) \), and if \( \phi : \zeta \to \zeta' \) is the morphism of these fibre bundles then it exists the only morphism of the sets of cocycles \( r : \tilde{\zeta} \to \tilde{\zeta}' \) that \( \tilde{\pi} = \tilde{\phi} \) (where \( \tilde{\pi} = B \to B' \)) and

\[
(74.1) \quad \phi \circ \phi_\alpha(b, g) = \phi'_\alpha(\tilde{\phi})(b), r_{\gamma\alpha}(b)g), \quad b \in U_\alpha \cap U_\beta, \quad g \in G
\]
Proof. The maps

\[
\theta_{\gamma\alpha} = \phi'_{\gamma}^{-1} \circ \phi \circ (\phi_{\alpha})(U_{\alpha} \cap \bar{\phi}^{-1}U'_{\gamma}) \times G : (U_{\alpha} \cap \bar{\phi}U'_{\gamma}) \times G \to U'_{\gamma} \times G
\]

for the some indexes \(\alpha, \gamma\) satisfy the condition \(\pi_{U'_{\gamma}} \circ \theta_{\gamma\alpha} = \bar{\phi} \circ \pi_{U_{\alpha} \cap \bar{\phi}^{-1}U'_{\gamma}}\) and, therefore, have the form \(\theta_{\gamma\alpha}(b, g) = (\bar{\phi}(b), h_{\gamma\alpha}(b, g))\) for the some \(h_{\gamma\alpha} : (U_{\alpha} \cap \bar{\phi}^{-1}U'_{\gamma}) \times G \to G\).

Whence

\[
\phi \circ \phi_{\alpha}(b, g) = \phi_{\gamma}(\bar{\phi}(b), h_{\gamma\alpha}(b, g)), \quad b \in U_{\alpha} \cap \bar{\phi}^{-1}U'_{\gamma}, \quad g \in G
\]

But then \(\phi'_{\gamma}(\bar{\phi}(b), h_{\gamma\alpha}(b, g)) = \phi \circ \phi_{\alpha}(b, g) = (\phi \circ \phi_{\alpha}(b, 1)) \circ g = \phi'_{\gamma}(\bar{\phi}(b), h_{\gamma\alpha}(b, 1)) \circ g = \phi'_{\gamma}(\bar{\phi}(b), h_{\gamma\alpha}(b, 1) \circ g)\) hence \(h_{\gamma\alpha}(b, g) = h_{\gamma\alpha}(b, 1) \circ g\). Thus the condition will fulfilled if we shall set \(r_{\gamma\alpha}(b) = h_{\gamma\alpha}(b, 1)\) for \(b \in U_{\alpha} \cap \bar{\phi}^{-1}U'_{\gamma}\). Let \(\tilde{\zeta} = \{U_{\alpha}, \phi_{\alpha}\beta\}, \tilde{\zeta}' = \{U'_{\gamma}, \phi'_{\gamma}\sigma\}\). Then

\[
\phi'_{\gamma}(\bar{\phi}(b), r_{\gamma\alpha}(b)\phi_{\alpha\beta}(b)g) = \phi \circ \phi_{\alpha}(b, \phi_{\alpha\beta}(b)g) = \\
= \phi \circ \phi_{\beta}(b, g) = \phi'_{\sigma}(\bar{\phi}(b), r_{\sigma\beta}(b)g) = \phi'_{\gamma}(\bar{\phi}(b), \phi'_{\gamma}(\bar{\phi}(b), \phi'_{\gamma}(\bar{\phi}(b), \phi'_{\gamma}(\bar{\phi}(b))r_{\sigma\beta}(b)g)
\]

for all \(b \in U_{\alpha} \cap U_{\beta} \cap \bar{\phi}^{-1}U'_{\gamma} \cap \bar{\phi}^{-1}U'_{\sigma}\) whence

\[
r_{\gamma\alpha}(b)\phi_{\alpha\beta}(b)g = \phi'_{\gamma\sigma}(\bar{\phi}(b))r_{\sigma\beta}(b)g
\]

Therefore \(r = \{r_{\gamma\alpha}\}\) is the morphism of the sets of cocycles. \(\square\)

Remark. The demand

\[
\phi_{\alpha}(b, g) = \phi_{\alpha}(b, 1) \circ g, \quad b \in U_{\alpha}, g \in G
\]

is the additional demand to definition of the principal fibre bundle. This demand is correct when it exists the continuous homotopy which connects the element \(g \in G\) with the unit 1 of the group \(G\), i.e. when \(g\) belongs to the connection’s component of the unit 1.

Theorem 1.13. Let \(\zeta = (B, \pi, E, G), \zeta' = (B', \pi', E', G)\) are the two principal fibre bundles with the atlases \(\{(U_{\alpha}, \phi_{\alpha})\}, \{(U'_{\gamma}, \phi'_{\gamma})\}\) and the associated sets of cocycles \(\tilde{\zeta}, \tilde{\zeta}'\). It exists the morphism \(\phi \to \phi'\) of the principal fibre bundles induced the some morphism \(r : \tilde{\zeta} \to \tilde{\zeta}'\) of the sets of cocycles.

Proof. We set \(\bar{\phi} = \bar{\varphi} = B \to B'\) and define the map \(\phi : E \to E'\) as following: if \(e \in E; e = \phi_{\alpha}(b, g)\) and \(\pi(e) \in U_{\alpha} \cap \bar{\phi}^{-1}(U'_{\gamma})\) then \(\phi(e) = \phi'_{\gamma}(\bar{\phi}(b), r_{\gamma\alpha}(b)g)\) according to lemma 1.12. If besides that \(\pi(e) \in U_{\beta} \cap \bar{\phi}^{-1}(U'_{\sigma})\) then \(e = \phi_{\beta}(b, \phi_{\beta\alpha}(b)g)\) and

\[
\phi'_{\alpha}(\bar{\phi}(b), r_{\sigma\beta}(b)\phi_{\beta\alpha}(b)g) = \phi'_{\sigma}(\bar{\phi}(b), \phi'_{\sigma\gamma}(\bar{\phi}(b))r_{\gamma\alpha}(b)g) = \\
= \phi'_{\gamma}(\bar{\phi}_{r\gamma\alpha}(b), g) = \phi(e)
\]
thus \( \phi(e) \) is defined correct. Since the restriction \( \phi|^{-1}(U_\alpha \cap \phi^{-1}(U'_\gamma)) \) is the composition

\[
\pi^{-1}(U_\alpha \cap \phi^{-1}(U'_\gamma)) \xrightarrow{\phi^{-1}_\alpha} (U_\alpha \cap \phi^{-1}(U'_\gamma)) \times G \xrightarrow{(\tilde{\phi}, r_e G)} (\phi(U_\alpha) \cap U'_\gamma) \times G \xrightarrow{\phi'_{\gamma}} \pi'(\phi^{-1}(U_\alpha) \cap U'_\gamma)
\]

(where \( \mu : G \times G \rightarrow G \) is the group multiplication) then it is evident that \( \phi \) is continuous. We have \( \pi' \circ \phi(e) = \pi' \circ \phi'_\gamma \tilde{\phi} r_e \alpha(b), g) = \tilde{\phi}(b) = \tilde{\phi} \circ \pi(e) \). Since \( eh = \phi_\alpha(b, g)h = \phi_\alpha(b, gh) \) then \( \phi(eh) = \phi'_\gamma(\tilde{\phi}(b), r_\alpha(\gamma)(b)gh) = \phi'_\gamma(\tilde{\phi}(b), r_\alpha(\gamma)(b)g)h = \phi(e)h \) for \( h \in G, e \in E \). Thus \( \phi \) is the morphism of the two principal fibre bundles. In conclusion since according to the definition of \( \phi \) takes place the equality \( \phi \circ \phi_\alpha(b, g) = \phi_\gamma(\tilde{\phi}(b), r_\alpha(\gamma)(b), g), b \in U_\alpha \cap \phi^{-1}(U'_\gamma), g \in G \) then \( \phi \) generates the morphism \( r = \{ r_\alpha \} \) of the sets of cocycles. \( \square \)

**Theorem 1.14.** Let \( \tilde{\zeta} = (U_\alpha, \phi_\alpha \beta) \) is the some set of cocycles for the space \( B \) and the topologic group \( G \). Then there exist the such principal fibre bundle \( \zeta = (B, \pi, E, G) \) and the such atlas \( \{(U_\alpha, \phi_\alpha)\} \) for \( \zeta \) that \( \tilde{\zeta} \) is the set of cocycles associated with this atlas.

The proof of this theorem is also analogous to the proof of the theorem 1.8 and we refer our reader again to the monograph [10].

The general definition 1.17 of the two cocycles over the group \( G \) and the definition 1.25 of the equivalence classes of the principal fibre bundles generates the following theorem:

**Theorem 1.15.** It exists the bijective correspondence between the equivalence classes of the principal fibre bundles over the base \( B \) and the cohomology classes from the definition 1.17. One can describe this correspondence as following: if \( \zeta \) is the some principal fibre bundle then the fixed cohomology class \( [\phi_\zeta] \) of the principal fibre bundle \( \zeta \) is compared to the some atlas of the fibre bundle \( \zeta \).

**Proof.** If \( \phi : \zeta \rightarrow \zeta' \) is the equivalence of the principal fibre bundles then lemma 1.12 supplies us the morphism \( r(\phi) : \zeta \rightarrow \zeta' \). Since \( \tilde{\phi} = 1_B \) then this morphism is the equivalence of the sets of cocycles. Therefore the above correspondence is defined correct.

Let now \( \tilde{\zeta} \) be the some set of cocycles. According to theorem 1.14 it exists the such principal fibre bundle \( \zeta \) with its atlas \( \{(U_\alpha, \phi_\alpha)\} \) that \( \tilde{\zeta} \) is the set of cocycles associated with this atlas. Thus the above correspondence is surjective.

Let \( \zeta, \zeta' \) be the such two principal fibre bundles that \( \tilde{\zeta} \) and \( \tilde{\zeta} \) are equivalent. Let us denote this equivalence as \( r : \tilde{\zeta} \rightarrow \tilde{\zeta}' \). According to theorem 1.13 it exists the morphism \( \phi : \zeta \rightarrow \zeta' \) induced \( r \). In particular \( \tilde{\phi} = 1_B \). Besides that takes place the morphism \( r^{-1} : \tilde{\zeta}' \rightarrow \tilde{\zeta} \) seted with the formula \( r^{-1} = \{ r_\gamma \} \). The morphism of the fibre bundles associated with \( r^{-1} : \phi^{-1} : \zeta' \rightarrow \zeta \) is the contrary to \( \phi \). Really the equalities \( \phi^{-1} \circ \phi \circ \phi_\alpha(b, g) = \phi^{-1} \phi'_\gamma(\tilde{\phi}(b), r_\gamma(\gamma)(b), g) = \phi_\alpha(b, r_\gamma^{-1}(b) r_\gamma(\gamma)(b), g) = \phi_\alpha(b, g) \) whence \( \phi^{-1} \circ \phi = 1 \). One can prove analogous that \( \phi \circ \phi^{-1} = 1 \). Therefore, \( \zeta \simeq \zeta' \) and thus our correspondence is injective. \( \square \)
Corollary 1.16. Let $\phi : \zeta \to \zeta'$ be the such morphism of the two principal fibre bundles over $B$ that $\tilde{\phi} = 1_B$. Then $\phi$ is the equivalence.

Proof. It is obvious that the morphism $r(\phi) : \tilde{\zeta} \to \tilde{\zeta'}$ of the sets of cocycles associated with $\phi$ is the equivalence. □

Let us consider now the following construction. Let $\zeta = (B, \pi, E, G)$ is the principal fibre bundle over $B$ and $f : B \to B'$ is the same map. Let us construct the principal fibre bundle $f \ast \zeta$ induced from the fibre bundle $\zeta$ as following. Let $E' = \{(b', e) \in B' \times E : f(b') = \pi(e)\}$. Let us define the map $\pi' : E' \to B'$ supposing $\pi'(b', e) = b'$. If $f' : E' \to E$ is the map given with the formula $f'(b', e) = e$ then the following diagram is commutative:

$$
\begin{array}{ccc}
E' & \xrightarrow{f'} & E \\
\pi' \downarrow & & \pi \downarrow \\
B' & \xrightarrow{f} & B \\
\end{array}
$$

(79.1)

This commutative diagram is like to the commutative diagram (47.1)

The action of the group $G$ on $E'$ is defined with the rule $(b', e)h = (b', eh), h \in G$. Let $\{(U_\alpha, \phi_\alpha)\}$ is the atlas for $\zeta$. Then $\{f^{-1}U_\alpha, f'_\alpha\}$ is the atlas for $f \ast \zeta = (B', \pi', E', G)$ where

$$
\phi'_\alpha : f^{-1}U_\alpha \times G \to \pi^{-1'}(f^{-1}U_\alpha)
$$

have the form $f'_\alpha(b', g) = (b', f_\alpha(f(b'), g)), b' \in f^{-1}U_\alpha, g \in G$. It is evident that if $\zeta \simeq \zeta'$ then $f \ast \zeta \simeq f \ast \zeta'$.

We can define the fibre bundle $f \ast \zeta$ in the terms of cocycles as following. If $\{(U_\alpha, \phi_\beta_\alpha)\}$ is the set of cocycles for $\zeta$ then $f^{-1}U_\alpha, \phi_\beta_\alpha \circ f$ is the set of cocycles for $\tilde{\phi} \ast \zeta$.

This definition "works" both for the principal and the vector fibre bundles. We must substitute simply the arbitrary topologic group for the case of the principal fibre bundles onto the $GL(n;K)$ group in the case of the vector fibre bundles.

Theorem 1.17. Let $\phi : \zeta \to \eta$ be the morphism of the vector fibre bundles; then it exists the morphism $\psi : \zeta \to \tilde{\phi} \ast \eta$ with $\tilde{\psi} = 1_B$. Therefore according to corollary 1.16 modified on the case of $GL(n;K)$ group takes place the equivalence of the fibre bundles $\zeta \simeq \tilde{\phi} \ast \eta$.

Proof. Since $\tilde{\psi} = 1_B$ we can set $\tilde{\phi} \ast \eta = (B', \pi', E', G)$ where $E' = \{(b, e) \in B \times E_\eta : \tilde{\phi}(b) = \pi_\eta(e)\}$ (our reader must compare these formulas with above diagram for their understanding). Let us define now the map $\psi : E_\zeta \to E'$ setting $\psi(e) = (\pi_\zeta \phi(e))$. Since $\pi_\eta \circ \phi(e) = \tilde{\phi} \circ \pi_\zeta(e)$ for all $e \in E_\eta$ then it is obvious that $\psi(e) \in E'$. It easy to check that $\pi' \circ \psi = \pi_\zeta$ and $\psi(eh) = (\pi_\zeta(eh), \phi(eh)) = (\pi_\zeta(e), \phi(e)h) = (\pi_\zeta(e), \phi(e)h) = \psi(e)h, h \in G$ (the equality $\pi_\zeta(eh) = \pi_\zeta(e)$ follows from the definition of the projection $\pi$). Thus $\psi$ is the morphism of the fibre bundles with $\tilde{\psi} = 1_B$, i.e. it is the equivalence of the fibre bundles. □

The following definition which is necessary now from the considered construction describes the new type of the fibre bundles, the associated fibre bundle. Thus our classification of the fibre bundles is replenished now with the new representative.
Definition 1.28. One can compare the associated fibre space $\zeta[Y]$ with the fibre $Y$ to the principal fibre bundle $\zeta$. It is constructed as following. Let us define the right action $G$ on $E \times Y$ supposing $(e, y)g = (eg, g^{-1}y)$, $g \in G$, $e \in E$, $y \in Y$. Let $E_Y = E \times Y / G$. Let us denote $\{e, y\}$ the image of the point $(e, y)$ in $E_Y$. Then $\{eg, y\} = \{e, gy\}$ for $g \in G$. Let us set the map $\psi : E_Y \rightarrow B$ with the formula $\psi_{e, y} = \pi(e)$. Since $\pi(eg) = \pi(e)$ for $g \in G$ then $\psi_{e, y}$ is defined correct. It is easy to see that it is continuous. If $\{(U_\alpha, \psi_\alpha)\}$ is the atlas for $\zeta$, let us define the atlas $\{(U_\alpha, \psi_\alpha)\}$ for $\zeta[Y] = (B, \pi_Y, E_Y, Y)$ supposing

\[
\psi_\alpha(b, y) = \{\phi_\alpha(b, 1), y\}, \quad b \in U_\alpha, \quad y \in Y
\]

The map $\psi_\alpha$ is continuous and satisfies the condition $\pi_Y \circ \psi_\alpha = \pi_{U_\alpha}$. The composition

\[
\pi^{-1}(U_\alpha) \times Y \xrightarrow{\phi_\alpha^{-1} \times 1} U_\alpha \times G \times Y \xrightarrow{1 \times \rho} U_\alpha \times Y
\]

where $\rho : G \times Y \rightarrow Y$ is the action of the group $G$ on $Y$ generates the map $\pi_{Y}^{-1} (U_\alpha \rightarrow U_\alpha \times Y)$ contrary to $\psi_\alpha$.

If $Y = K^n$ then the formula $r(e, v) + s(e', v') = (e, rv + sg^{-1}v')$ where $e'g = e$, $g \in GL(n; K)$; $r$, $s \in K$ sets the structure of the vector space on the fibres of $\zeta[K^n]$. The all $\psi_\alpha$ are linear on the fibres relatively this structure.

Theorem 1.18. Let $\zeta, \zeta'$ be the two principal fibre bundles. It exists the bijective correspondence between the morphisms $\phi : \zeta \rightarrow \zeta'$ and the sections $s$ of the fibre bundle's space $[E']$ defined with the rule $\phi \mapsto s_{\phi}$ where $s_{\phi}(b) = \{e, \phi(e)\}$ for every $e \in \pi^{-1}(b)$ (the left action of the group $G$ on $E'$ is given with the formula $ge' = e'g^{-1}, e' \in E', g \in G$).

Proof. Let us show firstly that the map $s_{\phi}$ is defined correct. Really if $\bar{e} \in \pi^{-1}(b)$ is the some element of the fibre over $b$ different from $e$ then it exists always the such $g \in G$ that $\bar{e} = eg$ and $\{\bar{e}, \phi(\bar{e})\} = \{eg, \phi(eg)\} = \{eg, \phi(e)g\} = \{e, \phi(e)\}$. Further, $s_{\phi}$ is continuous, so long as it is none other than the map $E / G \rightarrow E \times E' / G$ induced with $(1, \phi)$ and group $G$ is always topological, i.e. continuous in our theory. It is evident that $\pi_{E'} \circ s_{\phi} = 1_B$.

Let us suppose that $s : B \rightarrow E_{E'} = E \times E' / G$ is the section for the projection $\pi_{E'}$. The composition $s \circ \pi$ ought to have the form $e \mapsto \{e, \phi_s(e)\}$ with the same map $\phi_s : E \rightarrow E'$. If $g \in G$ then $\{eg, \phi_s(eg)\} = s \circ \pi(e) = s \circ \pi(eg) = \{eg, \phi_s(eg)\}$. whence $\phi_s(eg) = \phi_s(e)g$ for $e \in E, g \in G$. Let $\bar{s} : B \rightarrow B'$ be the generated map $E / G \rightarrow E' / G$. If we shall prove that $\phi_s$ is continuous then it will mean that $\phi_s$ is the sought for morphism of the principal fibre bundles.

Let $e \in E$ and $U \subset E'$ be the open neighborhood of the point $\phi_s(e)$. Since the action of the group $G$ on $E$ is continuous then there exist the open neighbourhood $U'$ of the point $\phi_s(e)$ in $E'$ and the open neighbourhood $V$ of the unit 1 of the group $G$ such that $U'V \subset U$. Let us take the such neighbourhood $V'$ of the unit 1 of the group $G$ that $(V')^{-1}V' \subset V$. Let now $(W, \psi)$ be the trivialisation for $\zeta$ contained $b = \pi(e)$; let us suppose that $e = \psi(b, h), h \in G$. Then the set $W' = \psi(W \times hV')$ is open in $E$ and $\{W' \times U'\}$ is open in $E \times E' / G$. Thus $O = W' \cap \pi^{-1} \circ s^{-1} \{W' \times U'\}$ is open in $E, e \in O$, and besides that

\[
s \circ \pi(O) \subset \{W' \times U'\}
\]
Let us show that \( \phi_s(O) \subset U \). Let \( x \in O \); then \( s \circ \pi(x) \in \{W' \times U'\} : \) let us take for certainty \( s \circ \pi(x) = \{x',y'\}, x \in W', y' \in U' \). Since \( x, x' \in W' \) then \( x' = xg^{-1} \) for some \( g \in V \). Therefore \( \{x, \phi_s(x)\} = s \circ \{x',y'\} = \{xg^{-1},y'\} = \{x,y'g\} \). But from the fact that \( y' \in U', g \in G \) follows that \( y'g \in U \), i.e. \( \phi_s \in U \). □

Before we shall proceed to the next lemma we must give the definition which will play the important role in our lecstions. This is the long ago announced simplex.

**Definition 1.29.** [2a,p.335] One say that the \( (n+1) \) points of the affine space are affine independent if they contain in the no \( (n-1) \) plane ,i.e. if the vectors \( k_1 - k_0, ..., k_m - k_0 \) are linear independent ( where \( k_0, k_1, ..., k_m \) are the radius-vectors ).

The set of the all points of the form

\[
k = t_0k_0 + ... + t_mk_m
\]

where \( k_0, k_1, ..., k_m \) are the some affine independent points of the space \( \mathbb{R}^n \) or in general, of the arbitrary linear space over the field \( \mathbb{R} \) and \( t_0, ..., t_m \) are the such real numbers that

\[
0 \leq t_0 \leq 1, ..., 0 \leq t_0 \leq 1
\]

and

\[
t_0 + ... + t_m = 1
\]

(we identify as ever the points of \( \mathbb{R}^n \) with their radius-vectors) is called the m-dimensional simplex with the vertexes \( k_0, ..., k_m \) and denoted with the symbol \( k_0...k_m \). At \( m=0 \) this is the point \( k_0 \), at \( m=1 \) this is the segment \( k_0k_1 \), at \( m=2 \) this is the triangle \( k_0k_1k_2 \), at \( m=3 \) this is the tetrahedron \( k_0k_1k_2k_3 \) ( the 4-dimensional simplex at \( m=4 \) is the generation of this construction; it as we this already mentioned is the base of the Regge calculation and the S.Hawking’s space-time foam theory ).

The numbers \( t_0, ..., t_m \) of the point \( k \) of the simplex \( k_0...,k_m \) are called its baricentric co-ordinates.

The point \( k \) of the simplex \( k_0...,k_m \) is called the interior point if \( 0 < t_i < 1 \) for all \( i =0, ..., m \).

**Definition 1.30.** Let \( A \) be the some subspace in \( X \) and \( i : A \to X \) is embedding. The subspace \( A \) is called the retract of the space \( X \) if it exists the such map \( r : X \to A \) that \( r \circ i = 1_A \).If \( A \subset X \) is the retract ( with the retraction \( r \)) and besides that \( i \circ r \simeq 1_X \) then subspace \( A \) is called the deformation retract in \( X \).In the case when \( i \circ r \simeq 1_X \) rel \( A \), i.e.when the homotopy is constant on \( A \) the subspace \( A \) is called the strong deformation retract in \( X \).

**Lemma 1.19.** Let \( \zeta = (D^n, \pi, E,G), \zeta' = (B', \pi', E',G) \) be the principal fibre bundles, \( \phi : (\pi \times 1)^{-1}(D^n \times \{0\} \cup S^{n-1} \times I) \to E' \) be the morphism of the fibre bundles and
\[ F : D^n \times I \to B' \] be the some continuation of the morphism \( \bar{\phi} \). Then it exists the morphism \( \Phi : \zeta \times I \to \zeta' \) with \( \Phi = F \) which continues \( \phi \)

**Proof.**. Let us choose the atlases \( \{(U_\alpha, \phi_\alpha)\}, \{(U'_\gamma, \phi'_\gamma)\} \) of the fibre bundles \( \zeta, \zeta' \). The sets \( (U_\alpha \times I) \cap F^{-1}(U'_\gamma) \) form the open covering of the space \( D^n \times I \).

Since \( D^n \times I \) is the compact metric space then it exists the such positive number \( \lambda > 0 \) that the every subset \( S \subset D^n \times I \) of the diameter \( < \lambda \) contains in the some \( (U_\alpha \times I) \cap F^{-1}(U'_\gamma) \). Let us take the such small triangulation for \( (D^n, S^{-1}) \) that every simplex \( \sigma \) from \( D^n \) has the diameter \( < \lambda / 2 \), and then let us divide the segment \( I = \{0 = t_0 < t_1 < ... < t_k = 1\} \) such that every set \( \sigma \times [t_i, t_{i+1}] \) has the diameter \( < \lambda, 0 \leq i < k \).

Let us suppose that the map \( \Phi \) is already defined on \( E \times [t_0, t_i] \cup \pi^{-1}(S^n-1 \times I) \). Let us continue \( \Phi \) by the simplexes on \( E \times [t_i, t_{i+1}] \) with the help of induction by \( \dim \sigma \). If \( \dim \sigma = 0 \) and \( \sigma \not\subset S^{-1} \) then let us take the such \( \alpha, \gamma \) that \( \sigma \subset [t_i, t_{i+1}] \subset (U_\alpha \times I) \cup F^{-1}(U'_\gamma) \).

Since \( \pi' \circ \Phi(\phi_\alpha(\sigma, g), t_i) = F(\sigma, t_i) \in U'_\gamma \) for every \( g \in G \) then it exists the such map \( f_\sigma : \sigma \times I \to G \) that
\[
\Phi(\phi_\alpha(\sigma, g), t_i) = \phi'_\gamma(F(\sigma, t_i), f_\sigma(g))
\]

Since \( \Phi(\phi_\alpha(\sigma, g), t_i) = \Phi(\phi_\alpha(\sigma, 1), t_i)g \) then \( f_\sigma(g) = f_\sigma(1)g \) for all \( g \in G \). Let \( g_\sigma = f_\sigma(1) \); then \( \Phi(\phi_\alpha(\sigma, g), t_i) = \phi'_\gamma(F(\sigma, t_i), g_\sigma g) \).

Let us define \( \Phi \) on \( (\pi \times I)^{-1}(\sigma \times [t_i, t_{i+1}] \cup \pi^{-1}(S^n-1 \times I) \).

Then \( \Phi \) is continuous the map \( \Phi|((\pi \times I)^{-1}(\sigma \times \{t_i\})) \) and since \( \Phi \) is the continuous interval satisfies the conditions \( \pi' \circ \Phi = F \circ (\pi \times 1), \Phi((e, t)h) = \Phi(e, t)h \) for all \( (e, t) \in (\pi \times I)^{-1}(\sigma \times \{t_i\}) \).

Let us suppose now that the map \( \Phi \) is constructed on \( (\pi \times I)^{-1}(\sigma', [0, t_{i+1}]) \) for all the simplexes \( \sigma' \) with \( \dim \sigma' < m \). Let \( \dim \sigma = m, \sigma \not\subset S^{-1} \). Let us choose again the such \( \alpha, \gamma \) that \( \sigma \times [t_i, t_{i+1}] \subset (U_\alpha \times I) \cap F^{-1}(U'_\gamma) \).

The map \( \Phi \) is defined already on \( (\pi \times I)^{-1}(\sigma \times \{t_i\}) \cap \hat{\sigma} \times [t_i, t_{i+1}] \) where \( \hat{\sigma} \) is the boundary of the simplex \( \sigma \); let us continue it on \( (\pi \times I)^{-1}(\sigma \times [t_i, t_{i+1}]) \). As above we find that the map \( \Phi \) has the form
\[
\Phi(\phi_\alpha(\sigma, g), t) = \phi'_\gamma(F(\sigma, t, f_\sigma(x, t)g))
\]

on \( (\pi \times I)^{-1}(\sigma \times \{t_i\}) \cap \hat{\sigma} \times [t_i, t_{i+1}] \) for the suitable \( f_\sigma : \sigma \times \{t_i\} \cap \hat{\sigma} \times [t_i, t_{i+1}] \to G \).

Since \( \sigma \times \{t_i\} \cap \hat{\sigma} \times [t_i, t_{i+1}] \) is the retract for \( \sigma \times [t_i, t_{i+1}] \) then it exists the continuation \( f_\sigma : \sigma \times [t_i, t_{i+1}] \to G \) of the map \( f_\sigma \). Let us define \( \Phi \) on \( (\pi \times I)^{-1}(\sigma \times [t_i, t_{i+1}]) \) supposing
\[
\Phi(\phi_\alpha(\sigma, g), t) = \phi'_\gamma(F(\sigma, t, f_\sigma(x, t)g))
\]

for \( (x, t) \in \sigma \times [t_i, t_{i+1}], g \in \). It is obvious that \( \Phi \) is continuous, it is the continuation of the map
\[
\Phi|((\pi \times I)^{-1}(\sigma \times \{t_i\}) \cap \hat{\sigma} \times [t_i, t_{i+1}])
\]

and satisfies the conditions
\[
\pi' \circ \Phi = F \circ (\pi \times 1), \quad \Phi((e, t)h) = \Phi(e, t)h
\]

for all \( (e, t) \in (\pi \times I)^{-1}(\sigma \times [t_i, t_{i+1}]), h \in G \). This completes the step of induction. \( \square \)
Lemma 1.20. Let $\phi : \zeta \rightarrow \zeta'$ be the morphism of the fibre bundles and $F : B \times I \rightarrow B'$ is the homotopy $F_0 = \hat{\phi}$. If $B$ is the cellular space then it exists the such morphism of the fibre bundles $\Phi : \zeta \times I \rightarrow \zeta'$ that $\Phi = F$ and $\Phi|E \times \{0\} = \phi$.

Proof. Let us carry out the proof by induction by the frames: if $\Phi$ is given on $(\pi^{-1}(B^{n-1} \times I)$ then because of theorem 1.18 the section $s : B^{n-1} \times I \rightarrow E \times I \times E'/G$ is defined. We want to continue this section on $B^n \times I$ (the initial step $n=0$ of induction is obvious: or $B^{-1} = \emptyset$ or $B^{-1} = \{b_0\}$ and F is the homotopy rel $b_0$). Let $f_n^n : (D^n, S^{n-1}) \rightarrow (B^n, B^{n-1})$ be the characteristic map of the cell $e^n_n$. The formula

$$(92.1) \quad (y,t) \mapsto (y,s(f^n(y),t)), \quad (y,t) \in D^n \times \{0\} \bigcup S^{n-1} \times I$$

sets the section of the fibre bundle $f \ast^n \zeta \sim I'[E]$ for every $\alpha$. It follows from theorem 1.18 and lemma 1.19 that it exists the such map $s_\alpha : D^n \times I \rightarrow E_{n \ast} \times f_\ast \times I \times E'/G$ that $\pi_2 \circ s_\alpha = F \circ (f^n \times 1)$ for the map $\pi_2 : E \times E'/G \rightarrow E'/G = B'$. Continuing $s$ on $e^n_n \times I$ by the formula

$$(93.1) \quad s(f^n(y),t) = (f^n \times_G 1) \circ s_\alpha(y,t)$$

$y \in D^n, t \in I$ we find that $s$ is continuous and it is the section. The continued such section $s$ defines in one's turn the morphism $\Phi : (\pi \times I)^{-1}(B^n \times I) \rightarrow E'$. □

Theorem 1.21. Let $\zeta = (B, \pi, E, G)$ be the principal fibre bundle and $f_0, f_1 : B' \rightarrow B$ be the two homotopical maps of the cellular space $B'$ into the cellular space $B$. Then $f \ast_0 \zeta \simeq f \ast_1 \zeta$.

Proof. Let $F : B' \times I \rightarrow B$ be the homotopy from $f_0$ to $f_1$. Since it exists the morphism of the fibre bundles $\Phi : f \ast_0 \zeta \times I \rightarrow \zeta$ then by theorem 1.17 and lemma 1.20 takes place the equivalence $f \ast_0 \zeta \times I \simeq F \ast \zeta$. If we define the map $i_1 : B' \rightarrow B' \times I$ with the formula $i_1(b') = (b',1), b' \in B'$ then $f_1 = F \circ i_1$ and

$$f \ast_1 \zeta = (F \circ i_1) \ast \zeta \simeq i \ast_1 (F \ast \zeta) \simeq i \ast_1 (f \ast_0 \zeta \times I) =$$

$$= (f \ast_0 \zeta \times I)|(B' \times \{1\}) \simeq f \ast_0 \zeta$$

□

One can define the cofunctor $k_G : RW' \rightarrow RT'$ for the some topological group $G$ as following. Let us denote as $k_G(X)$ the set of all equivalence classes of the (labelled) principal fibre bundles over $X$ (i.e. we must take into account the labelled points of these fibre bundles in this case!). Let us also define $k_G[f] : k_G(Y) \rightarrow k_G(X)$ for every homotopic class $[f]$ of the maps $f : (X, X_0) \rightarrow (Y, Y_0)$ supposing that $k_G[f](\{\zeta\}) = \{f \ast \zeta\}$ where $\{\zeta\}$ is the equivalence class of the (labelled) principal fibre bundle $\zeta$. Theorem 1.21 ensure that this map is defined correct. The labelled element of $k_G$ is the equivalence class of the trivial fibre bundle $(X, \pi \times 1, X \times G, G)$. 
Definition 1.31. Let us denote as $Z \cup X$ the subspace $Z \times \{x_0\} \cup \{z_0\} \times X \subset Z \times X; \{x_0\} \in X, \{z_0\} \in Z$. One can interpret it as a space obtained with the identification of the labelled points $x_0$ and $z_0$. It is obvious that $Z \cup X$ is the space with the labelled point $(z_0, x_0)$. One calls the such object the bouquet or the disjoint union of the spaces $X$ and $Y$.

Let us show now that our cofunctor $k_G$ satisfies the two important axioms which we now shall define.

The sum axiom. The morphism

\[
\{i*_{\alpha}\} : F*(\bigvee_{\alpha} X_{\alpha}) \to \prod_{\alpha} F*(X_{\alpha})
\]

is the bijection for the arbitrary family $\{X_{\alpha}\}$ of the cellular spaces from $\mathcal{RW}'$ generated with the embedding $i_{\beta} : X_{\beta} \to \bigvee_{\alpha} X_{\alpha}$.

The Myer-Wjetoris axiom. Let the cellular triad $(X; A_1, A_2)$, i.e. the cellular space $X$ with the such its subspaces $A_1$ and $A_2$ that $A_1 \cup A_2 = X$ be given. Then it exists $y \in F*(X)$ with $y|A_1 = x_1, y|A_2 = x_2$ for every above $A_1$ and $A_2$ and $x_1 \in F*(A_1), x_2 \in F*(A_2)$ with $x_1|A_1 \cap A_2 = x_2|A_1 \cap A_2$.

Let us make sure that the cofunctor $k_G$ satisfies these axioms. For the purpose to prove this we must give again the series of definitions which are connected with the theory of the cellular spaces and have the strategic importance for the many directions of differential topology.

Firstly the above definition of disjoint union generates the following object.

Definition 1.32. Let $(X, x_0), (Y, y_0) \in \mathcal{RT}$. Then let us define the reduced product $(X \wedge Y, *)$ of the spaces $X$ and $Y$, where $*$ is the labelled point of the reduced product $(X \wedge Y, *)$, as a factor-space

\[
X \wedge Y = X \times Y / X \cup Y
\]

The labelled point of the reduced product $(X \wedge Y, *)$ is the point $* = \pi(X \cup Y)$ where $\pi : X \times Y \to X \wedge Y$ is the natural projection. We shall denote the point $\pi(x, y)$ as $[x, y]$. If the maps

\[
f : (X, x_0) \to (X', x'_0)
\]

and

\[
g : (Y, y_0) \to (Y', y'_0)
\]

are given then $f \times g : X \times Y \to X' \times Y'$ moves $X \cup Y$ into $X' \cup Y'$ and therefore induces the map $f \wedge g : X \cup Y \to X' \cup Y'$.

The one of examples of the reduced products is the cone.
Definition 1.33. The cone \((CX, \ast) \in \mathcal{RT}\) over \(X\) is the reduced product

\[(96.1) \quad (CX, \ast) = (I \setminus X, \ast)\]

where the point 0 of the segment \(I\) is the labelled point of this segment. According to definition 1.31 of disjoint union we can write down

\[(97.1) \quad CX = I \times X/(\{0\} \times X \cup I \times \{x_0\})\]

Let us denote as \([t, x]\) the image of the point \((t, x) \in I \times X\) in \(CX\). The map \(i : X \to CX\) defined with the formula \(i(x) = [1, x]\) sets the homomorphism of the space \(X\) on the image \(\text{im } i\); therefore we can identify \(X\) with \(\text{im } i\) and consider \(X\) as a subspace of the cone \(CX\).

Definition 1.34. Let us construct the cone of map \(Y \cup_f CX\) for the given map \(f : (X, x_0) \to (Y, y_0) \in \mathcal{RT}\). \(Y \cup_f CX\) is obtained from \(Y \cup X\) by the identification \([1, x] \in CX\) with \(f(x) \in Y\) for all \(x \in X\). More precisely we stick the root of the cone \(CX\) to the space \(Y\) with the help of the map \(f\). It is evident that the projection \(q : Y \cup X \to Y \cup_f CX\) defines the homomorphism between the spaces \(Y\) and \(q(Y)\); therefore we can consider \(Y\) as a subspace in \(Y \cup_f CX\).

Definition 1.35. Let \(X\) be the some space and \(g : S^{n-1} \to X\) is the some map. Then according to definition 1.34 one can form the cone \(X \cup_g CS^{n-1}\) of the map \(g\). The obtained as a result the new space is called the space \(X\) with the stuck \(n\)-dimensional cell. The map \(g\) is called the stuck map of the cell. Restricting the natural projection \(q : X \cup CS^{n-1} \to X \cup_g CS^{n-1}\) on \(CS^{n-1}\) we obtain the map \(f : CS^{n-1} \to X \cup_g CS^{n-1}\), which is the homeomorphism on the interior of the cone \(CS^{n-1}\). One can call this map the characteristic map of the cell. This definition of the characteristic map conforms to the above its definition by our first acquaintance with the cellular spaces. Since \(CS^{n-1} \cong D^n\) then one can consider \(f\) as a map

\[(98.1) \quad f : (D^n, S^{n-1}) \to (X \cup_g CS^{n-1}, X)\]

Let us note that \(f|S^{n-1} = g\).

Definition 1.36. Let \(X\) be the some space and \(A \subset X\) is its subspace. The structure of the relative cellular space on the pair \((X, A)\) is the such sequence \(A = (X, A)^{-1} \subset (X, A)^0 \subset \ldots \subset (X, A)^n \subset (X, A)^{n+1} \subset \ldots \subset X\), that \((X, A)^n\) is obtained from \((X, A)^{n-1}\) by the sticking of the \(n\)-dimensional cells, \(n \geq 0, X = \bigcup_{n \geq -1} (X, A)^n\) and \(X\) is provided with the weak topology: \(S \subset X\) is closed when and only when \(S \cap (X, A)^n\) is closed in \((X, A)^n\).

Let us call the pair \((X, A)\) the relative cellular space if it can be provided with the some structure of the relative cellular space.

Let us set \(\dim(X, A) = n\), if \((X, A)^n = X\) and \((X, A)^{n-1} \neq X\). Let us note that if \(A = \{x_0\}\) then \(X\) is the cellular space.
Lemma 1.22. Let \((X; A, B)\) be the cellular triad with the above features. Then it exists the such open set \(A'\supset A\) and the homotopy \(H : X \times I \rightarrow X\) which satisfies the following conditions:

1. \(H_0 = 1_X\);
2. \(H\) is immovable on \(A\);
3. \(H_1(A') \subset A\);
4. \(H(B \times I) \subset B\).

Proof. Let the cell \(A^{-1} = A\) and \(H^{-1} : X \times I \rightarrow X\) be the stationary homotopy: \((H^{-1}(x, t) = x\) for all \(t \in I\). Let us suppose by induction that we constructed already the such open neighbourhood \(A^k\) of the subspace \(A\) in \((X, A)^{k+1}\), that \(A^{k-1} = A^k \cap (X, A)^k\) and the such homotopy: \(H^k : X \times I \rightarrow X\) that

1. \(H^0 = H^1 = H^{k-1}\);
2. \(H^k\) is the stationary homotopy on \((X, A)^k\);
3. \(H^1(A^k) \subset A\);
4. \(H^k(B \times I) \subset B\).

Let us denote as \(\{e^n\}\) the set of all \((n+1)-\)dimensional cells from \((X, A)\), and as \(f^{n+1}\) the characteristic map of the cell \(e^n\). Let \(D_0^{n+1} = \{x \in D^{n+1} : |x| \leq 1/2\}\). Then \(D^{n+1} - D_0^{n+1}\) is the open neighbourhood of the sphere \(S^n\). The homotopy: \(K : D^{n+1} \times I \rightarrow D^{n+1}\) setting with the formulas

\[
K(x, t) = \begin{cases} (1 + t)x, & |x| \leq \frac{1}{1+t} \\ \frac{x}{|x|}, & |x| \geq \frac{1}{1+t} \end{cases}
\]

for \(x \in D^{n+1}, t \in I\) shows us that \(D^{n+1} - D_0^{n+1}\) is tightened in the point on \(S^n\). Let now

\[
U^{n+1}_\gamma = \{f^{n+1}_\gamma(y) : f^{n+1}_\gamma(y) \in A^{n+1}, y \in D^{n+1} - D_0^{n+1}\}
\]

It is easy to see that \(U^{n+1}_\gamma\) is the open subset in \(e^{n+1}_\gamma\). Therefore if one sets \(A^n = A^{n+1} \cup \bigcup\gamma U^{n+1}_\gamma\), then one can check that \(A^n\) is the open neighbourhood of the subspace \(A\) in \((X, A)^{n+1}\), and \(A^n \cap (X, A)^n = A^{n-1}\). Let us define homotopy \(H^n : (X, A)^{n+1} \times I \rightarrow X\) as

\[
H^n(x, t) = \begin{cases} H^{n-1}_1(x), & x \in (X, A)^n \\ H^{n-1}_1(f^{n+1}_\gamma(K(y, t)), & x \in f^{n+1}_\gamma(y), y \in D^{n+1}, t \in I \end{cases}
\]

Then \(H^n\) is continuous and satisfies the equality \(H^n_0 = H^{n-1}_1\) on \((X, A)^{n+1}\), and this allows us to continue it on \(X\) with the conservation of this equality (as a weak topology!). The fulfilment of the conditions (1),(2),(3) is evident. Since \(H^{n-1}\) satisfies the condition then \(H^n\) also satisfies it. Therefore we constructed by induction by \(k\) the all neighbourhoods \(A^k\) and the all homotopies \(H^k\) for \(k \geq -1\).
Let us set now \( A' = \bigcup_{k \geq -1} A^k \). The set \( A' \) is open, since \( A' \cap e^m_\gamma = A^{m-1} \cap e^m_\gamma \) for all \( m, \gamma \). In conclusion, let us set the homotopy \( H : X \times I \rightarrow X \) as

\[
H(x, t) = \begin{cases} 
H^{r-1}(x, (r+1)(t-1)+1), & \frac{r-1}{r} \leq t \leq \frac{r}{r+1}, x \in X \\
H^r(x, 1), & t = 1, x \in (X, A)^r 
\end{cases}
\]

It is easy to see that homotopy \( H \) is continuous and \( H_0 = H_0^0 = 1_X \). Besides that \( H \) is immovable in subspace \( A \), \( H_1(A') \subset A \) and \( H(B \times I) \subset B \). □

**Lemma 1.23.** Let \((X, A, B)\) be the triad with \( X = A^\circ \cup B^\circ \) (\( A^\circ \) and \( B^\circ \) are the open sets); \( \zeta \) be the \( n \)-dimensional vector fibre bundle (the principal fibre bundle correspondingly) over \( A \), \( \zeta' \) be the \( n \)-dimensional vector fibre bundle (the principal fibre bundle correspondingly) over \( B \) and \( \phi : \zeta|A \cap B \rightarrow \zeta'|A \cap B \) be the equivalence of its restrictions on \( A \cap B \) (in general if \( \eta = (B, \pi, E, K^n) \) and \( A \subset B \) then \( \eta|A = (A, \pi|A \cap B, \pi^{-1}(A), K^n) \)). Then it exists the such \( n \)-dimensional vector fibre bundle (the principal fibre bundle correspondingly) \( \eta \) over \( X \) that \( \eta|A \simeq \zeta \).

**Proof.** We work in the both cases in fact with the set of cocycles. Let \( \{U_\alpha, \phi_{\alpha\beta}\}, \{U'_\gamma, \phi'_{\gamma\delta}\} \) be the sets of cocycles for \( \zeta, \zeta' \). Let us denote as \( r_{\gamma\alpha} : U_\alpha \cap U'_\gamma \rightarrow G \) (or \( GL(n, K) \)) the function determined the equivalence \( \zeta|A \cap B \simeq \zeta'|A \cap B \); it means that

\[
r_{\delta\beta}(b)\phi_{\beta\alpha}(b) = \phi'_{\delta\gamma}(b)r_{\gamma\alpha}(b)
\]

for all \( b \in U_\alpha \cap U_\beta \cap U'_\gamma \cap U'_\delta \). Then it is easy to check that \( \{A^\circ \cap U_\alpha, B^\circ \cap U'_\gamma\} \) is the open covering of \( X \) and

\[
\phi_{\beta\alpha} : A^\circ \cap U_\alpha \cap U_\beta \rightarrow G
\]

\[
\phi'_{\delta\gamma} : B^\circ \cap U'_\gamma \cap U'_\delta \rightarrow G
\]

\[
r_{\gamma\alpha} : A^\circ \cap B^\circ \cap U_\alpha \cap U'_\gamma \rightarrow G
\]

forms the set of cocycles on \( X \). For example if \( b \in U_\alpha \cap U_\beta \cap U'_\gamma \) then

\[
r_{\gamma\beta}(b)\phi_{\beta\alpha}(b) = r_{\gamma\alpha}(b)
\]

In order to prove this it is necessary to set \( \delta = \gamma \) in the above equation. Thus we obtain the principal fibre bundle (or the vector fibre bundle) \( \eta \) over \( X \). It is obvious that \( \eta|A \simeq \zeta, \eta|B \simeq \zeta' \). □
Theorem 1.24. Cofunctor $k_G$ satisfies the sum and Myer-Wjetoris axioms.

Proof. Let $(X, A, B)$ is the some cellular triad. According to lemma 1.22 there exist the open neighbourhood $A'$ of the set $A$ and the homotopy $H : X \times I \to X$ of the neighbourhood $A'$ on $A$ transferred $B$ itself. If $j : (A, A \cap B) \to (A', A' \cap B)$ is the pair’s embedding then $j^* : k_G(A') \to k_G(A)$ and $(j|A \cap B)^* : k_G(A' \cap B) \to k_G(A \cap B)$ are the bijections. Let us suppose that the elements $x_1 \in k_G(A), x_2 \in k_G(B)$ for which $i^*(x_1) = i^*(x_2)$ in set $k_G(A \cap B)$ (here $i_1 : A' \cap B \to A', i_2 : A' \cap B \to B$ are the natural embeddings). Therefore the elements $x_1', x_2'$ are represented with the such fibre bundles $\zeta_1$ on $A'$, $\zeta_2$ on $B$ that $\zeta_1|(A' \cap B) = \zeta_2|(A' \cap B)$. According to lemma 1.23 there exists such a fibre bundle $\eta$ over $X$ that $\eta|A' \simeq \zeta_1, \eta|B \simeq \zeta_2$ ( $A' \cup B = X$ ). If $y = \{\eta\} \in k_G(X)$ is the equivalence class of $\eta$, then $j^* y = x_1, j^* y = x_2$ where $j^* : A' \to X, j^* : B \to X$ are the embeddings. Thus $k_G$ satisfies the Myer-Wjetoris axiom.

Let $\{X_\alpha, x_\alpha\}$ be the some family of the cellular spaces with the labelled points. Let us denote as $\zeta_\alpha = (X_\alpha, \pi_\alpha, E_\alpha, G)$ the principal fibre bundle represented $y_\alpha$ for every $y_\alpha \in k_G(X_\alpha)$. Let us define the fibre bundle $\zeta = \bigvee_\alpha X_\alpha, \pi, E, G$ over $\bigvee_\alpha X_\alpha$ identifying those points $e \in E_\alpha, e' \in E_\beta$ in $\bigvee_\alpha E_\alpha$ for which it exists the such element $g \in G$ that $e = e_\alpha g, e' = e_\beta g$ ( $e_\alpha \in E_\alpha, e_\beta \in E_\beta$ are the labelled points ). Let $E$ be the result of this identification and $\pi : E \to \bigvee_\alpha X_\alpha$ is the natural projection. Let us set on $E$ the action of the group $G$ supposing $\{e\} h = \{eh\}, h \in G$. It is easy to show that this action is defined correct, and $\pi(\{e\}) = \pi(\{e\})$ if and only if $\{e\} = \{eh\}$ for some $h \in G$. If $\{U_\alpha^\beta, \phi_\beta^\alpha\}$ is the such atlas of the fibre bundle $\zeta_\alpha$ that $\phi_\beta^\alpha(x_\alpha, 1) = e_\alpha$ always when $x_\alpha \in U_\alpha^\beta$ then one can construct the atlas of the fibre bundle $\zeta$ from $\{U_\alpha^\beta, \phi_\beta^\alpha\}$. It is evident that the map $i_* : k_G(\bigvee_\alpha X_\alpha) \to \bigvee_\alpha k_G(X_\alpha)$ is surjective. Similarly if $\zeta, \eta$ are the such fibre bundles over $\bigvee_\alpha X_\alpha$, that $\zeta|X_\alpha \simeq \eta|X_\alpha$ for all $\alpha$ then gluing together these equivalences on $\bigvee_\alpha X_\alpha$ ( they preserve the labelled points ) we obtain the equivalence $\zeta \simeq \eta$. Thus, the map $\{i_*\}$ is injective, whence $k_G$ satisfies the sum axiom. □

Definition 1.37. Let $F^*(Y)$ be the some cofunctor on the space $Y$. The element $u \in F(Y)$ is called the $n$-universal element if

\[
T_u : [S^q, s_0; Y, y_0] \to F^*(S^q)
\]

is the isomorphism for $q \in \mathbb{N}$ and the epimorphism for $q = n$. The element $u$ is called the universal element if it is $n$-universal for every $n \geq 0$.

Remark. The set of homotopical classes $[S^q, s_0; Y, y_0]$ plays the decisive role in modern differential and algebraic topology with their applications in theoretical physics. It is turn out that this set has the group structure the unit of which is the class of the stationary homotopies. It exists the standard denotation in all the physic-mathematical literature for this group which is called the $q$-dimensional homotopical group or the $q$-dimensional spheroid. We shall denote it henceforth as $\pi_q(Y, y_0)$. We shall discuss the group structure of $\pi_q(Y, y_0)$ later on , and now our reader can postulate for himself this fact.
Definition 1.38. The directed set is the such set with the relation \( \leq \) of partial putting in order: \((\Lambda, \leq)\) that there exist the such elements \( \gamma, \alpha, \beta \in \Lambda \) that \( \alpha \leq \gamma, \beta \leq \gamma \).

Definition 1.39. The inverse (or the projective) system of the abelian groups is the such family \( \{G_\alpha\} \) of the Abelian groups indexed with the elements of the directed set \( \Lambda \) and the homomorphisms \( j_\alpha^\beta : G_\beta \to G_\alpha, \alpha \leq \beta, \) that \( j_\alpha^\beta \circ j_\gamma^\alpha = j_\alpha^\gamma \) by \( \alpha \leq \beta \leq \gamma \) and \( j_\alpha^\alpha = 1 \) for every \( \alpha \in \Lambda \). If the some inverse system \( (G_\alpha, j_\alpha^\beta, \Lambda) \) is given then the inverse (or the projective) limit is the group \( \text{lim}^0 G_\alpha = \text{invlim} G_\alpha \) defined as following:

\[ \text{invlim} G_\alpha \text{ is the subgroup of the group } \prod_{\alpha \in \Lambda} G_\alpha \text{ consisted from all such } f \in \prod_{\alpha \in \Lambda} G_\alpha \text{ that } j_\alpha^\beta(f(\beta)) = f(\alpha) \text{ for all } \alpha, \beta \text{ with } \alpha \leq \beta. \]

The restrictions of the projections \( \pi_\alpha : \prod_{\alpha \in \Lambda} G_\alpha \to G_\alpha \) onto the subgroup \( \text{invlim} G_\alpha \) generates the homomorphisms

\[ (107.1) \quad \pi_\alpha : \text{invlim} G_\alpha \to G_\alpha \]

satisfied the conditions \( j_\alpha^\beta \circ \pi_\beta = \pi_\alpha \) for every \( \alpha, \beta \in \Lambda \) with \( \alpha \leq \beta \).

Lemma 1.25. If \((X, x_0) \in \mathcal{R}W \) and \( \{x_0\} = X_{-1} \subset X_0 \subset \ldots \subset X_n \subset \ldots \subset X \) is the such sequence of the cellular subspaces \( X \) that \( X = \bigcup_n \) then the map

\[ (108.1) \quad \{i*_n\} : F*(X) \to \text{invlim} F*(X) \]

is surjective.

Proof. Let us consider the so called infinite telescope \( X' = \bigcup_{n \geq -1} [n - 1, n]^+ \bigwedge X_n \), where \([n - 1, n]^+\) is the real segment with the natural ends and the labelled point +. and let us set

\[ (109.1) \quad A_1 = \bigcup_{k \geq -1} [k - 1, k]^+ \bigwedge X_k \subset X' \]

\[ (109.1a) \quad A_2 = \bigcup_{k \geq 0} [k - 1, k]^+ \bigwedge X_k \subset X' \]

Then

\[ (110.1) \quad X' = A_1 \bigcup A_2, \quad A_1 \bigcap A_2 = \bigvee X_k \]

\[ (110.1a) \quad A_1 \simeq \bigvee_{k = 2n + 1, n \in \mathbb{N}} X_k, \quad A_2 \simeq \bigvee_{k = 2n, n \in \mathbb{N}} X_k \]

(the latter relations are correct because of the fact that every segment is tighten in the point). As it follows from the sum axiom there exist the such \( y_1 \in F*(A_1) \) and \( y_2 \in F*(A_2) \) for the some element \( \{x_k\} \in \text{invlim} F(X_k) \) that \( y_1|X_k = x_k, k \) is odd, and \( y_2|X_k = x_k, k \) is even. Let us consider the element \( y_1|A_1 \bigcap A_2 \). If \( k \) is odd then \( y_1|X_k = x_k \); if \( k \) is even then \( y_1|X_k = j_\star k (y_1|X_{k+1}) = j_\star k (x_{k+1}) = x_k \) where \( j_k : X_k \to X_{k+1} \) is the natural embedding. The analogous feature is correct also for the element \( y_2|A_1 \bigcap A_2 \). Therefore \( y_1|A_1 \bigcap A_2 = y_2|A_1 \bigcap A_2 \). Therefore with the help of the Myer- Wjetoris axiom we obtain the element \( y' \in F(X') \) with \( y'|A_1 = y_1, y'|A_2 = y_2 \). Then \( y'|X_k = x_k, k \geq -1 \). But \( X' \simeq X \); therefore exist the such element \( y \in F*(X) \) that \( y|X_k = x_k, k \geq -1 \). □
Lemma 1.26. Let \((Y, y_0) \in \mathcal{R}W\) and \(u_n \in F \ast (Y)\) be the some \(n\)-universal element. Then it exists the cellular space \(Y'\) obtained from \(Y\) by the sticking of the \((n+1)\)-dimensional cells and the such \((n+1)\)-universal element \(u_{n+1} \in F \ast (Y')\) that \(u_{n+1}|Y = u_n\).

Proof. Let us take the one copy \(S^{n+1}_\lambda\) of the sphere \(S^{n+1}\) for every \(\lambda \in F \ast (S^{n+1})\) and let us form the cellular space \(Y \bigvee (\bigvee \lambda S^{n+1}_\lambda)\). If \(n \geq 0\) then let us choose the map-representative \(f : (S^n, s_0) \to (Y, y_0)\) for every class \(\alpha \in \pi_n(Y, y_0)\) with \(T_{u_n}(\alpha) = 0\) and let us stick according to definition 1.36 the \((n+1)\)-dimensional cell \(e^{n+1}_\alpha\) to \(Y \bigvee (\bigvee \lambda S^{n+1}_\lambda)\) by this map. As a result we obtain the cellular space \(Y'\). According to sum axiomit exists the such element \(v \in F \ast (Y \bigvee (\bigvee \lambda S^{n+1}_\lambda))\) that \(v|Y = u_n, v|S^{n+1}_\lambda = \lambda\). If

\[
(111.1) \quad g : \bigvee \alpha S^n_\alpha \to Y \bigvee \bigvee \lambda S^{n+1}_\lambda
\]

is the "large sticking map" for the \((n+1)\)-dimensional cells \(e^{n+1}_\alpha\) then takes place the exact sequence

\[
(112.1) \quad F \ast \left( \bigvee \alpha S^n_\alpha \right) \leftarrow F \ast (Y \bigvee \bigvee \lambda S^{n+1}_\lambda) \leftarrow F \ast (Y')
\]

It is easy to see that \(g^\ast (v)|S^n_\alpha = T_{u_n}(\alpha) = 0\) for all \(\alpha\). Since the map

\[
(113.1) \quad \{i \ast \alpha\} : F \ast \left( \bigvee \alpha S^n_\alpha \right) \to \prod \alpha F \ast (S^n_\alpha)
\]

is bijective according to sum axiom then \(g^\ast (v) = 0\). Thus it exists the such element \(u_{n+1} \in F \ast (Y')\) that \(u_{n+1}|(Y \bigvee (\bigvee \lambda S^{n+1}_\lambda)) = v\) (the exact sequence with \(g^\ast (v) = 0\) give us the epimorphism). In particular \(u_{n+1}|Y = u_n\) and \(u_{n+1}|S^{n+1}_\lambda = \lambda\).

Now we ought to show that the element \(u_{n+1}\) is the \((n+1)\)-universal element. With this purpose let us consider the commutative diagram

\[
\pi_q(Y, y_0) \quad \xrightarrow{i^\ast} \quad \pi_q(Y')
\]

\[
\xrightarrow{T_{u_n}} \quad \xrightarrow{T_{u_{n+1}}} \quad F \ast (S^q)
\]

Since the space \(Y'\) is obtained from the space \(Y\) by the sticking of the \((n+1)\)-dimensional cells then \(i^\ast\) is the isomorphism for \(q < n\) (embedding) and it is the epimorphism for \(q = n\) because of commutativity. Therefore \(T_{u_n}\) is also the isomorphism for \(q \leq n\) and the epimorphism for \(q = n\) because of the commutative diagram (114.1). Let us suppose that \(T_{u_{n+1}}(\beta) = 0\) for some \(\beta \in \pi_n(Y', y_0)\). Since \(i^\ast\) is the epimorphism for \(q = n\) then it exists the such element \(\alpha \in \pi_n(Y, y_0)\) that \(i^\ast (\alpha) = \beta\). But then \(T_{u_n}(\alpha) = T_{u_{n+1}}i^\ast (\alpha) = T_{u_{n+1}}(\beta) = 0\). Therefore it exists the cell \(e^{n+1}_\alpha\) in \(Y'\) stuck up the element \(i^\ast (\alpha)\), i.e. \(i^\ast (\alpha) = \beta = 0\). It means that \(T_{u_{n+1}}\) is the monomorphism for \(q = n\) and therefore the isomorphism for \(q \leq n\). In conclusion \(T_{u_{n+1}}([i\lambda]) = i^\ast \lambda (u_{n+1}) = \lambda\), where \(i\lambda S^{n+1}_\lambda \to Y'\) is the embedding. Thus \(T_{u_{n+1}}\) is the epimorphism for \(q = n + 1\). \(\square\)
Corollary 1.27. It exists the cellular space $Y'$ for the every $(Y, y_0) \in \mathcal{R}W$ and $v \in F^*(Y)$ contained $Y$ as a cellular subspace and the universal element $u \in F^*(Y)$ with $u|_Y = v$.

Proof. Supposing $Y_{-1} = Y, u_{-1} = v$ and applying lemma 1.25 we find by induction the sequence $Y = Y_{-1} \subset Y_0 \subset Y_1 \subset \ldots \subset Y_n \subset \ldots$ of the cellular spaces in which $Y_n$ is obtained from $Y_{n-1}$ by the sticking of the $n$-dimensional cells and also the sequence of the $n$-universal elements $u_n \in F^*(Y_n)$ satisfied the condition $u_n|Y_{n-1} = u_{n-1}$. Let set now $Y' = \bigcup_{n \geq -1} Y_n$ and let us provide $Y'$ with the weak topology. Then according to lemma 1.26 it exists the such $u \in F^*(Y')$, that $u|Y_n = u_n$. Let us consider the commutative diagram

\[
\begin{array}{ccc}
\pi_q(Y, y_0) & \xrightarrow{i*} & \pi_q(Y', y_0) \\
\downarrow{\cong}^{T_{u_n}} & & \downarrow{\cong}^{T_u} \\
F^*(S^q) & \xleftarrow{\sim} & F^*(S^q)
\end{array}
\]

Since the space $Y'$ is obtained from the space $Y_n$ by the sticking of the cells of dimensions greater than $n$ then $i*_{n}$ is the isomorphism for $q \geq n$. Therefore $T_u : \pi_q(Y', y_0) \to F^*(S^q)$ is the isomorphism for all $q$, i.e. $u$ is the universal element. □

Corollary 1.28. It exists the cellular space $(Y, y_0) \in \mathcal{R}W$ and the universal element $u \in F^*(Y)$.

Proof. Let us set $Y = \{y_0\}$ in the previous corollary, and let $v$ is the labelled element in $F^*(Y)$. Then $Y'$ from above corollary is the sought for cellular space and $u \in F^*(Y')$ is the sought for element. □

Definition 1.40. Let $(X, A)$ and $(Y, B)$ be the relative cellular spaces. The map $f : (X, A) \to (Y, B)$ is called the cellular map, if $f((X, A)^n) \subset (Y, B)^n$ for every $n \geq -1$.

Lemma 1.29. Let $f : (Y, y_0) \to (Y', y_0')$ be the cellular map and $u \in F^*(Y), u' \in F^*(Y')$ be the such universal elements that $f * u' = u$. Then $f$ generates the isomorphism

\[
f_* : \pi_q(Y, y_0) \to \pi_q(Y', y_0'), \quad q \geq 0
\]

Proof. The following commutative diagram take place for every $q \geq 0$:

\[
\begin{array}{ccc}
\pi_q(Y, y_0) & \xrightarrow{f_*} & \pi_q(Y', y_0') \\
\downarrow{\cong}^{T_{u\cong}} & & \downarrow{\cong}^{T_{u'\cong}} \\
F^*(S^q) & \xleftarrow{\sim} & F^*(S^q)
\end{array}
\]

The required statement follows from this commutative diagram. □
Definition 1.41. The map \( f : X \to Y \) is called the \( n \)-equivalence, \( n \geq 0 \) if the map \( f_r : \pi_r(X, x_0) \to \pi_r(Y, y_0) \) is bijective at \( r < n \) and is surjective at \( r = n \). The map \( f \) is called the weak homotopical equivalence if it is the \( n \)-equivalence for all \( n \geq 0 \).

Definition 1.42. Let us define the space \( M_f \) for some map \( f : X \to Y \) from category \( T \) by identification the points \([1, x] \) and \( f(x) \) in \((I \times X) \cup Y\) for all \( x \in X \). The space \( M_f \) is called the inreduced cylinder of the map \( f \).

The following definitions which are necessary for the proof of the next lemma represent themselves also the very important notions for the topologist.

Definition 1.43. Let us denote as \( X^Y \) the set of all continuous maps \( f : X \to Y \) of the two topologic spaces \( X \) and \( Y \). Let us furnish this set with the compact-open topology taking as a prebase \([9]\) of this topology the all sets of the type \( N_{K,U} = \{ f : f(K) \subset U \} \) where \( K \subset X \) is compact and \( U \subset Y \) is open.

Definition 1.44. Let us define the space of loops \( (\Omega Y, \omega_0) \in RT \) for \((Y, y_0) \in RT\) as
\[
\Omega Y = (Y, y_0)(S^1, s_0)
\]
with the constant loop \( \omega_0 \) (\( \omega_0(s) = y_0 \) for all \( s \in S^1 \)) as a labelled point.

Let us form now the iterant spaces of the loops \( \Omega^n Y \) defining them by induction: \( \Omega^n Y = \Omega(\Omega^{n-1} Y) \), \( n \geq 1 \), \( \Omega^0 Y = Y \).

Definition 1.45. Let us denote as \( T^2 \) the category of the topologic pairs \((X, A)\) where \( A \) is the subspace in \( X \) and the continuous maps \( f : (X, A) \to (Y, B) \). Similarly let us denote as \( RT^2 \) the category of the labelled pairs \((X, A, x_0), x_0 \in A \subset X\) and the continuous maps \( f : (X, A, x_0) \to (Y, B, y_0) \) preserved the labelled points.

Definition 1.46. The way in the topologic space \( X \) is every continuous map \( w : I \to X \). One call very often \( w \) the way from \( w(0) \) into \( w(1) \) or the way connected the points \( w(0) \) and \( w(1) \). Thus \( X^I \) is the space of all ways in \( X \) with the open-compact topology. Let us introduce on \( X \) the relation \( \sim \) supposing \( x \sim y \) if and only if it exists the way \( w : I \to X \) from \( x \) into \( y \). It is evident that \( \sim \) is the equivalence relation. On the other hand it is easy to see that these definition of way, therefore the definition of the relation \( \sim \) also is likeness to the definition 1.24 of homotopy. Thus the relation \( \sim \) is the homotopic relation. Let us denote the set of the equivalence classes of the relation \( \sim \) as \( \pi_0(X) \). The elements of set \( \pi_0(X) \) are called the components of the linear connection or the \( 0 \)-components of the space \( X \). If \( \pi_0(X) \) contains the one element only then the space \( X \) is called the linear connected or the \( 0 \)-connected.

Definition 1.47. Let us denote as
\[
P(X, x_0, A) = (X, x_0, A)^{(I, 0, 1)}
\]
for the given pair \((X, A, x_0) \in RT^2\) - the space of all ways begun in the point \( x_0 \) and ended in the subspace \( A \). The formula \( \pi(w) = w(1) \) sets the continuous map
\[
\pi : P(X, x_0, A) \to A
\]
Definition 1.48. Let us define the n-dimensional relative homotopic set
\( \pi_n(X, A, x_0) \), \( n \geq 1 \) for the pair \((X, A, x_0) \in \mathcal{RT}^2\) supposing
\[
\pi_n(X, A, x_0) = \pi_{n-1}(P(X, x_0, A), \omega_0) = \pi_0(\Omega^{n-1}P(X, x_0, A), \omega_0)
\]

Remark. We can consider \( \pi_0(X, x_0) \) as a labelled set whose labelled element is the 0-component of the point \( x_0 \). The reader can see also the obvious co-ordination between definition of \( \pi_n(X, A, x_0) \) and formula (106.1).

Lemma 1.30. The map \( f : (D^n, S^{n-1}) \to (X, A, x_0) \) defines the 0-element in \( \pi_n(X, A, x_0) \) when and only when \( f \) is homotopic relatively \( S^{n-1} \) to the such map \( f' \) that \( f'(D^n) \subset A \).

Proof. Let \( f \simeq f' \) rel \( S^{n-1} \) and \( f'(D^n) \subset A \). Let us denote as \( H \) the homotopy connected \( f \) and \( f' \). Let us define now the homotopy \( G : (D^n \times I, S^{n-1} \times I, s_0 \times I) \to (X, A, x_0) \) supposing
\[
G(x, t) = \begin{cases} 
H(x, 2t), & 0 \leq t \leq 0.5 \\
(2 - 2t)x + (2t - 1)s_0, & 0.5 \leq t \leq 1
\end{cases}
\]

Then \( G_0 = f, G_1 = x_0 \).

On the contrary let us suppose that \( G : (D^n \times I, S^{n-1} \times I, s_0 \times I) \to (X, A, x_0) \) sets the 0-homotopy of map \( f \).

Let us define the new homotopy \( H : D^n \times I \to X \) with the formula
\[
H(x, t) = \begin{cases} 
G\left(\frac{x}{2 - t}, t\right), & 0 \leq |x| \leq 1 - 0.5t \\
G\left(\frac{x}{|x| - |x|}, t\right), & 1 - 0.5t \leq |x| \leq 1
\end{cases}
\]

where \( x \in D^n, t \in I \). Then \( H_0 = G_0 = f; H(x, t) = G(x, 0) = H(x, 0) \) for all \( x \in S^{n-1}, t \in I \) and \( H_1(D^n) \subset A \). \( \Box \)

Definition 1.49. The pair \((X, A) \in \mathcal{T}^2\) is called the 0-connected pair if every component of the linear connection of space \( X \) crosses \( A \). The pair \((X, A)\) is called the \( n \)-connected pair if it is the 0-connected pair and \( \pi_r(X, A, a) = 0 \) for \( 1 \leq r \leq n \) and all \( a \in A \). The space \( X \) is called the \( n \)-connected space if \( \pi_k(X, x) = 0 \) for \( 0 \leq k \leq n \) and all \( x \in X \).

Let us introduce the notion dual to the notion of fibre bundle. It is easy to see that we can depict the general fibre bundle with the following commutative diagram (this is in fact the commutative diagram illustrated definition 1.26):
\[
\begin{array}{ccc}
X & \xrightarrow{i_0} & X \times I \\
\downarrow f & & \downarrow G \\
E & \xrightarrow{p} & B
\end{array}
\]
We come to the dual notion of cofibre bundle \( i : A \to X \) if we invert the all arrows in (124.1) and substitute \( X \times I \) to the dual it space \( Y^I \) (according the map of the two cellular spaces: \( f : X \to Y \)) and in conclusion substitute \( i_0 \) with the map \( \pi_0 \) where \( \pi_0(w) = w(0) \) for the some way \( w \). Remembering that one can consider \( G \) as a map \( A \times I \to Y \), and \( F \) as a map \( X \times I \to Y \):

(124.1a)

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi_0} & Y^I \\
\downarrow & & \downarrow \\
& i_0 \\
\end{array}
\]

we obtain the following definition:

**Definition 1.50.** One say that the embedding \( i : A \to X \) has the feature of the prolongation of homotopy relatively the space \( Y \), if it exists the homotopy \( F : X \times I \to Y \) for every map \( f : X \to Y \), the homotopy \( G : A \to Y \) (look (124.1a)) and the restriction \( f|A \) which continues \( G \). The embedding \( i \) is called the cofibre bundle if it has the feature of the prolongation of homotopy relatively the space \( Y \).

**Remark.** One can show that if the some map \( i : A \to X \) has the feature of the prolongation of homotopy relatively the some space \( Y \) then it is the homeomorphism of the space \( A \) on the some closed subspace of the space \( X \), i.e. it is the embedding \( A \) in \( X \).

**Lemma 1.31.** Let \( f : Z \to Y \) be the n-equivalence. Then it exists the such map \( h' : D^r \to Z \) for the maps \( g : S^{r-1} \to Z \) and \( h : D^r \to Y \) with \( h|S^{r-1} = f \circ g \) that \( h'|S^{r-1} = g \) and \( f \circ h' \simeq h \) rel \( S^{r-1} \).

**Proof.** One can depict the conditions of this lemma with the following commutative diagrams:

(125.1a)

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & S^{r-1} \\
\downarrow & & \downarrow \\
& g \\
\end{array}
\]

(125.1b)

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & S^{r-1} \\
\downarrow & & \downarrow \\
& g \\
\end{array}
\]

Let \( M_f \) be the irreduced cylinder of the map \( f \) with the accompanied maps \( i : Z \to M_f, j : Y \to M_f, r : M_f \to Y \) and \( H : M_f \to M_f \). The idea of the proof consist in the replacement
of the map \( f : Z \to Y \) on the embedding \( i : Z \to M_f \) and the application of lemma 1.30 to the n-connected pair \((M_f, Z)\) \((125.1b)\). The only problem here is the absence of the strict equality \( j \circ h|S'^{-1} = i \circ g \); indeed \( j \circ h|S'^{-1} = j \circ f \circ g \). The homotopy in this line has the form \( H \circ (ig \times \alpha) \). Since \( S'^{-1} \subset D'^r \) is the cofibre bundle (compare \((125.1a)\) and \((124.1a)\)) one can continue \( H \circ (ig \times \alpha) \) up to the homotopy \( H' : D'^r \times I \to M_f \) with \( H'_0 = j \circ h \). Let \( \hat{h} = H'_1 \); then \( h|S'^{-1} = H'_1|S'^{-1} = H_0 \circ i \circ g = i \circ g \). Therefore one can apply lemma 1.30 to the map \( \hat{h} : (D'^r, S'^{-1}) \to (M_f, Z) \) (the necessary part of this lemma). As a result we obtain the map \( h' : D'^r \to Z \) homotopic rel \( S'^{-1} \) to the map \( \hat{h} \) and such that \( h'|S'^{-1} = g \). Thus \( f \circ h' = r \circ i h' \simeq r \circ j \circ h \simeq h \). □

The proof of the next lemma is connected with classical \textit{Zorn lemma}. It will be useful for us to recall its matter \[14,p.36\].

\textbf{Definition 1.51.} Let \( M \) be the set with the partial putting in order. The every its subset \( A \) in which one can associate the some two elements (by the relation \( \leq \)) is called the chain. The chain is called the maximal chain if it is not contained as a subset in the no other chain belonged \( M \). Let us call the element \( a \) of the set \( M \) with the partial putting in order- the upper border of the subset \( M' \subset M \) if the every element \( a' \in M' \) is subordinated \( a \), i.e.\( a' \leq a \).

\textbf{Lemma 1.32.} \textit{(the Zorn lemma).} If the every chain in the set \( M \) with the partial putting in order has its upper border then every element from \( M \) is subordinated to the some maximal element.

\textbf{Lemma 1.33.} Let \( f : Z \to Y \) be the some n-equivalence and \((X,A)\) is the relative cellular space with \( \dim (X,A) \leq n \). Then it exists the such map \( h' : X \to Z \) for the some maps \( g : A \to Z \), \( h : X \to Y \) with \( h|A = f \circ g \) that \( h'|A = g \) and \( f \circ h' \simeq h \) rel \( A \) (the case \( n = \infty \) is not excepted).

\textit{Proof.} Let \( \mathcal{T} \) be the set of all such triads \((X', k, K)\) that \( A \subset X' \subset X, k : X' \times I \) is the homotopy connected \( f \circ k \) with \( h|X' \) which is immovable on \( A \). Let us set on \( \mathcal{T} \) the relation \( \leq \) of the partial putting in order supposing that \((X', k, K) \leq (X'', k', K')\) if and only if \( A \subset X' \subset X'', k'|X' = k, K'|X' \times I = K \). It is evident that the triad \((A, g, K_0)\) (where \( K_0 \) is the constant homotopy \( A \times I \to Y \)) is contained in \( \mathcal{T} \), whence \( \mathcal{T} \neq \emptyset \). Let us show that the set \( \mathcal{T} \) satisfies the conditions of the Zorn lemma.

Let \( \mathcal{R} \) is the some subset with the partial putting in order in \( \mathcal{T} \). Let us set \( W = \bigcup X' \), where the joining up is by all subspaces \( X' \) for which \((X', k, K) \in \mathcal{R} \). Let us set the map \( h' \to Z \) and the homotopy \( H' : W \times I \to Y \) supposing \( h'(x) = k(x) \) and \( H'(x, t) = K(x, t) \) if \( x \in X' \) for the some \((X', k, K) \in \mathcal{R} \). Since the space \( X \) has the weak topology then \( h' \) and \( H' \) are defined correct and continuous. It is evident that \((W, h', H') \in \mathcal{T} \) and \((X', k, K) \leq (W, h', H')\) for all \((X', k, K) \in \mathcal{R} \).

Thus according to Zorn lemma the set \( \mathcal{T} \) has the maximal element \((X', k, K)\). Therefore we should show only that \( X' = X \) if \( X' \neq X \), then let us consider the set of cells from \( X - X' \) and let us take one from them which has the minimal dimension. Let us denote it as \( e \). Let \( \phi : (D^k, S^{k-1}) \to (X, X') \) is the characteristic map of the cell.
Let us denote as $T$ the cellular space obtained from $(A, x, a)$. Applying lemma 1.31 to the pair $k \circ \psi : S^{k-1} \rightarrow Z, h \circ \phi : D^k \rightarrow Y$ we find the map $\theta : D^k \rightarrow Z$ and the homotopy $\Theta : D^k \times I \rightarrow Y$ connected $f \circ \theta$ and $h \circ \phi$. Let us continue k to the map $k' : X' \cup e \rightarrow Z$ and K to the homotopy $K' : (X' \cup e) \times I \rightarrow Y$ supposing

\[(126.1a) \quad k'(x) = \begin{cases} k(x), & x \in X' \\ \theta(y), & x = \phi(y), y \in D^k \end{cases} \]

\[(126.1b) \quad K'(x, t) = \begin{cases} K(x, t), & x \in X' \\ \Theta(x, t), & x = \phi(y), y \in D^k \end{cases} \]

Then $(X' \cup e, k', K) \in T$ and $(X', k, K) \subset (X' \cup e, k', K)$, this contradict the maximum of the triad $(X', k, K)$. Therefore $X' = X$, whence one can take $h' = k$. □

**Lemma 1.34.** Let $Y$ be the space with the universal element $u \in F* (Y)$, $(X, A, x_0)$ be the cellular pair $(A \subset X)$, $g : (A, x_0) \rightarrow (Y, y_0)$ be the cellular map and $v \in F* (Y)$ be the element satisfied the condition $v|A = g* (u)$. Then it exists the such cellular map $h : (X, x_0) \rightarrow (Y, y_0)$ that $h|A = g$ and $v = h* (u)$.

**Remark.** We consider here the spaces $X, A, Y$ as a connected spaces.

**Proof.** Let us denote as $T$ the cellular space obtained from $(I^+ \wedge A) \vee X \vee Y$ by identification $[0, a] \in I^+ \wedge A$ and $[1, a] \in I^+ \wedge A$ with $g(a) \in Y$. Let $A_1, A_2 \subset T$ are the cellular spaces set as $A_1 = ([0; 1/2]^+ \wedge A) \cup X, A_2 = ([1/2; 1]^+ \wedge A) \cup Y$. Then $A_1 \cup A_2 = T, A_1 \cap A_2 = \{1/2\} \times A \cong A$. Besides that it exists the strong deformation retracts $f : A_1 \rightarrow X$ and $A_2 \rightarrow Y$. Therefore there exist the such elements $\bar{v} \in F* (A_1), \bar{u} \in F* (A_2)$, that $\bar{v}|X = v, \bar{u}|Y = u$. It is obvious that $\bar{v}|A_1 \cap A_2 = f* (v|A) = f* g* (u) = \bar{u}|A_1 \cap A_2$. Therefore according to the Myer-Wjetoris axiom it exists the element $w \in F* (T)$ with $w|X = v, w|Y = u$.

According to lemma 1.26 we can embed $T$ in the some cellular space $Y'$ and find the universal element $u' \in F* (Y')$ with $u' \in F* (Y') u'|T = w$. Let $j : Y \rightarrow Y'$ is the embedding; then $j* (u') = u'|Y = w|Y = w$. Therefore according to the lemma 1.29 the induced homomorphism of the homotopic groups

\[(127.1) \quad j* : \pi_* (Y, y_0) \rightarrow \pi_* (Y', y_0) \]

is the isomorphism.

Let nou $\tilde{g} : X \rightarrow Y'$ is the embedding $X$ in $Y'$. Then $\tilde{g}|A \simeq j \circ g$,and the homotopy connected these two maps is the composition $I^+ \wedge A \subset T \rightarrow Y'$. Since $A \subset X$ is the cofibre bundle relatively $Y$ then it exists the map $\tilde{g} : X \rightarrow Y'$ with $\tilde{g}| = j \circ g$ and therefore $\tilde{g} \simeq \tilde{g}$.

\[(128.1) \quad \begin{array}{ccc} Y' & \xrightarrow{j} & Y \\ \downarrow \tilde{g} & \nearrow \bar{h} \\ X & \subset & A \end{array} \]
It follows from the proofs of the lemmas 1.31,1.33 that it exists the such map \( h : X \to Y \) that \( h|A = g, j \circ h \simeq \tilde{g} \simeq \bar{g} \). Therefore

\[
(129.1) \quad h * (u) = h * j * (u') = \tilde{g} * (u') = v
\]

\[\square\]

Let us call the space \( Y \) figured in \( F^*(Y) \) as a classified space or a space of representation of the cofunctor \( F^*(Y) \)

**Definition 1.52.** The two objects \( X \) and \( Y \) are called the equivalent objects if there exist the such morphisms \( f \in \text{hom} \ (X,Y) \) and \( g \in \text{hom} \ (Y,Z) \) that \( g \circ f = 1_X \) and \( f \circ g = 1_Y \). We call the morphisms \( f \) and \( g \) the equivalences in this case.

**Definition 1.53.** Let \( \mathcal{E}, \mathcal{D} \) are the two categories and \( F^*, G^* : \mathcal{E} \to \mathcal{D} \) are the cofunctors from \( \mathcal{E} \) into \( \mathcal{D} \). Let us consider then the morphism \( T(X) \in \text{hom} \ \mathcal{D} \ (G^*(X), F^*(X)) \). If the latter is the equivalence in category \( \mathcal{D} \) then \( T \) is called the natural equivalence, and cofunctors \( F^* \) and \( G^* \) are called the natural equivalent.

**Theorem 1.35.** *(The Brown theorem).* If \( F^* : \mathcal{R}W' \to \mathcal{R}T \) is the cofunctor satisfied the sum and the Myer- Wjetoris axioms then it exists the such classified space \((Y,y_0) \in \mathcal{R}W \) and the such universal element \( u \in F^*(Y) \) that \( T_u[--; Y,y_0] \to F^* \) is the natural equivalence.

**Proof.** Because of corollary 1.28 it exists the cellular space \((Y,y_0) \in \mathcal{R}W \) and the universal element \( u \in F^*(Y) \). Therefore we must prove only that

\[
(130.1) \quad T_u[--; Y,y_0] \to F^*
\]

is the bijection for all \((X,x_0) \in \mathcal{R}W \).

a. Let \( v \in F^*(X) \). Then let us set \( A = \{x_0\} \) in lemma 1.34 and let us take as \( g : (A,x_0) \to (Y,y_0) \) the only map transferred \( A \) into the labelled point \( y_0 \). Then it exists the map \( h : (X,x_0) \to (Y,y_0) \) with \( v = h * (u) = T_u([h]) \). Therefore \( T_u \) is surjective.

b. Let us suppose that \( T_u[g_0] = T_u[g_1] \) for the two maps \( g_0, g_1 : (X,x_0) \to (Y,y_0) \). We can consider \( g_0 \) and \( g_1 \) as a cellular maps without of loss of generality. Let \( X' = X \setminus I^+ \) and \( A' = X \setminus \{0,1\}^+ \). Let us set the map \( g : (A',*) \to (Y,y_0) \) supposing \( g[x,0] = g_0(x), g[x,1] = g_1(x), x \in X \). Besides that let \( p : X' \to X \) is the map defined with the equality \( p[x,t] = x \) for all \( [x,t] \in X' \) and \( v = p \ast g \ast u = F^*(X') \). Then

\[
(131.1a) \quad v|X \setminus \{0\}^+ = g \ast u = g \ast u|X \setminus \{0\}^+
\]

and

\[
(131.1b) \quad v|X \setminus \{1\}^+ = g \ast u = T_u[g_0] = T_u[g_1] = g \ast u = g \ast u|X \setminus \{1\}^+
\]

Therefore \( g \ast u = v|A \), and, because of lemma 1.34 there exists the map \( h : X' \to Y \) with \( h|A = g \). It is easy to see that \( h \) is the homotopy connected \( g_0 \) and \( g_1 \). Therefore \( T_u \) is injective. \( \square \)
Definition 1.53. Let again $\mathcal{E}$, $\mathcal{D}$ be the two categories and $F^*, G^* : \mathcal{E} \to \mathcal{D}$ are the cofunctors from $\mathcal{E}$ into $\mathcal{D}$. The natural transformation from $F$ to $G$ is the correspondence which associates the morphism $T(X) \in \text{hom}_\mathcal{D}(G^*(X), F^*(X))$ to every $X \in \mathcal{E}$ such that the equality
\[(132.1)\quad T(X) \circ G^*(f) = F^*(f) \circ T(Y)\]
take place for every morphism $f \in \text{hom}_\mathcal{D}(X,Y)$, i.e. the following diagram:
\[
\begin{array}{ccc}
F^*(X) & \xleftarrow{F^*(f)} & F^*(Y) \\
G^*(X) & \xleftarrow{G^*(f)} & G^*(Y)
\end{array}
\]
is commutative.

The inversion of all arrows in diagram (133.1) gives us the natural transformation of the two functors.

Let us return now to cofunctor $k_G$. The proved above fact that it satisfies the sum and the Myer-Wjetoris axioms means that Brown theorem is valid for this cofunctor. And this means that there exists such a cellular space $(BG, \ast)$ (defined with precision of the homotopy type) and the principal $G$-fibre bundle $\zeta_G = (BG, \pi, E_G, G)$ for every topologic group $G$ that the given with the formula $T[f] = \{f \ast \zeta\}$ the natural transformation
\[(134.1)\quad T_G : [-; BG, \ast] \to k_G(-)\]
is the natural equivalence. Thus the principal $G$-fibre bundles over the cellular space $X$ are classified with the homotopical classes of the maps $f : (X, x_0) \to (BG, \ast)$. Therefore the $n$-dimensional vector fibre bundles over the field $K$ are classified with the homotopic classes of the maps $(X, x_0) \to (BGL(n, K), \ast)$.

Let us now denote the set $\{f \ast \zeta\}$ of the equivalence classes for the $n$-dimensional vector fibre bundles over the field $K$ as $\text{Vect}_n(BGL(n, K))$. This designation corresponds partial to analogous designation in monograph [15] of M.F.Atiyah which we shall follow in many respects later on.

Now we ought to recall the some information from the group theory [16,p.9].

Definition 1.54. The abstract set $G$ is called the monoid if the binary operation $(a, b) \to ab$ called the multiplication is defined on $G$ and $(ab)c = a(bc)$ for all $a, b, c \in G$, i.e. the multiplication is associative. The element $ab$ is called the product of the elements $a, b$. The element $(ab)c = a(bc)$ is denoted $abc$.

The monoid $G$ is called the semigroup if it exist the unit element in $G$, i.e. the element $1 \in G$ that $1a = a1 = a$ for all $a \in G$ (sometimes one calls 1 the neutral element).

Remark. Thus the semigroup differs from the group with the absence of the contrary element.
Definition 1.55. Let $\zeta = (B, \pi, E, F)$ and $\zeta' = (B', \pi', E', F')$ be the two arbitrary fibre bundles. The product of $\zeta$ and $\zeta'$ is the fibre space $\zeta \times \zeta' = (B \times B', \pi \times \pi', E \times E', F \times F')$. The product of the principal $G$-fibre bundle with the principal $G'$-fibre bundle is provided on a natural way with the structure of the principal $G \times G'$-fibre bundle. For example if $\bar{\zeta} = \{U_\alpha, \phi_\beta\}$ and $\bar{\zeta}' = \{V_\gamma, \phi_\delta\}$ are the sets of the transition functions for $\zeta$ and $\zeta'$ correspondingly then

$$\bar{\zeta} \times \bar{\zeta}' = \{U_\alpha \times V_\gamma, \phi_\beta \times \phi_\delta\}$$

is the set of the transition functions for $\zeta \times \zeta'$. Similarly if $\zeta, \zeta'$ are the vector fibre bundles with the fibres $K^n, K^m$, then $\zeta \times \zeta'$ is provided on a natural way with the structure of the vector fibre bundle with the fibre $K^n \times K^m = K^{n+m}$. For example if $\{U_\alpha, \phi_\alpha\}$ and $\{V_\gamma, \psi_\gamma\}$ are the atlases for the vector fibre bundles $\zeta$ and $\zeta'$ then $\{U_\alpha \times V_\gamma, \phi_\alpha \times \psi_\gamma\}$ is the atlas for the vector fibre bundle $\zeta \times \zeta'$. Besides that it is evident that take place the equivalence

$$\zeta[K^n] \times \zeta'[K^m] \simeq (\zeta \times \zeta')[K^{n+m}]$$

for the $GL(\mathfrak{n}, K)$-fibre bundle $\zeta$ and $GL(\mathfrak{m}, K)$-fibre bundle $\zeta'$.

Definition 1.55. If $\zeta, \zeta'$ are the vector fibre bundles over the same base $B$ then one can form the new fibre bundle $\zeta \oplus \zeta'$ supposing $\zeta \oplus \zeta' = \Delta \ast (\zeta \times \zeta')$ where $\Delta : B \to B \times B'$ is the diagonal map. It is called the Whitney sum of the vector fibre bundles $\zeta$ and $\zeta'$. The fibre $\zeta \oplus \zeta'$ over the point $b \in B$ is the direct sum of the fibres of the fibre bundles $\zeta, \zeta'$ over this point.

When the Whitney sum is already defined we can consider now the Whitney sum of the representative of the equivalence classes for the $n$-dimensional vector fibre bundles over the field $K$, i.e. in fact the Whitney sum of the elements of $\text{Vect}_n(BGL(\mathfrak{n}, K))$. And now we shall prove that the Whitney sum of the above equivalence classes is the multiplication (more precisely the addition) operation and that $\text{Vect}_n(BGL(\mathfrak{n}, K))$ has the structure of the abelian semigroup with respect to the Whitney sum.

But this is almost evident fact. The definition 1.55, namely the formula (136.1), provides us the feature of associativity for Whitney sum.

Which equivalence class is the unit element for the investigated semigroup? It is easy to see that the equivalence class of the trivial vector fibre bundles from the example 1 is the unit element for our semigroup. Really we deal in this example with one chart $U = B$ and the one trivialisation $\phi : B \times K^n \to B \times V$. Therefore this is indeed the unit element. The formula (136.1) then provides again the semigroup operation (as a Whitney sum) with above trivial class. Thus we proved that the Whitney sum of the elements of $\text{Vect}_n(BGL(\mathfrak{n}, K))$ forms the structure of the semigroup.

In conclusion we ought to prove the abelian nature of this semigroup. But this provide the following two lemmas.

Lemma 1.36. Let $\zeta_1, \zeta_2$ are the fibre spaces over $B_1$ and $B_2$ correspondingly, and $\tau : B_1 \times B_2 \to B_1 \times B_2$ is the map given with the formula $\tau(x, y) = (y, x)$. Then $\tau \ast (\zeta_2 \times \zeta_1) \simeq \zeta_1 \times \zeta_2$. If $\zeta_1, \zeta_2$ are the principal (the vector fibre bundles) correspondingly then takes place the equivalence of the principal (the vector fibre bundles) correspondingly.
Corollary 1.37. Let $\zeta_1, \zeta_2$ be the vector fibre bundles over the same base $B$. Then $\zeta_1 \oplus \zeta_2 \cong \zeta_2 \oplus \zeta_1$.

Proof. Since $\tau \circ \Delta = \Delta$ then

$$\zeta_1 \oplus \zeta_2 = \Delta \ast (\zeta_1 \times \zeta_2) \cong \Delta \ast \tau \ast (\zeta_2 \times \zeta_1) = \Delta \ast (\zeta_2 \times \zeta_1) = \zeta_2 \oplus \zeta_1$$

□

The inverse element for the given element is absent in the construction of $\text{Vect}_n(B\text{GL}(n,K))$.

Thus we proved the construction of the abelian semigroup on the set $\text{Vect}_n(B\text{GL}(n,K))$ of the equivalence classes of the vector fibre bundles. How can we construct the group structure by given semigroup $\text{Vect}_n(B\text{GL}(n,K))$? This is the very interesting task by itself and has many applications which we shall discuss later on. We shall follow now the monograph [15] of M.F. Atiyah and the monograph [2] of M.M. Postnikov.

Let [2, p.413] the some abelian semigroup $M$ be given. We shall construct now the group of differences $GM$ for the abelian semigroup $M$. This group consist of the formal differences of the form $a-b$ where $a, b \in M$. We consider the two such differences $a-b$ and $a_1-b_1$ as equal differences when it exists the such element $c \in M$ that $a+b_1+c = a_1+b+c$. The addition in the group of differences is defined as following:

$$\chi : M \to GM$$

The just defined group of differences is called very often the Grothendieck group. This French mathematician utilised widely the construction of the group of differences and attracted the general attention to this group. Althouth the above construction was well-known long before Grothendieck.

The formula

$$\chi(a) = \chi(b) \text{ when and only when it exists the such element } c \in M \text{ that } a+c = b+c.$$
is commutative. This was the general scheme for the construction of the group of differences. We shall denote in our next statement of the matter the abelian group of differences corresponding the abelian semigroup $\text{Vect}_n(\text{BGL}(n,K))$ with the letter $K : K(A)$ for the some abelian semigroup $A$.

M.F. Atiyah proposed the alternative scheme of the definition for the group of differences. Let $\Delta : A \to A \times A$ be the diagonal homomorphism of the semigroups, and let $K(A)$ be the set of the conjugate classes of the semigroup $\Delta(A)$ in the semigroup $A \times A$. The set $K(A)$ is the factor-semigroup, but the permutation of the coordinates in $A \times A$ generates the inverse element in $K(A)$, therefore $K(A)$ is the group. Let us define $\alpha_A : A \to K(A)$ as a composition of the embedding $a \to (a, 0)$ with the natural projection $A \times A \to K(A)$ (for the simplicity we suppose that $A$ contains zero element). The pair $(K(A), \alpha_A)$ is therefore the functor of the semigroup $A$, and if $\gamma : A \to B$ is the homomorphism of the semigroups then we have the commutative diagram

\begin{equation}
\begin{array}{ccc}
A & \xrightarrow{\alpha_A} & K(A) \\
\gamma \downarrow & & \downarrow \gamma \\
B & \xrightarrow{\alpha_B} & K(B)
\end{array}
\end{equation}

If $B$ is the group then $\alpha_B$ is the isomorphism.

If $A$ is besides that the semiring (i.e. the multiplication, distributive with respect to the addition is defined on $A$) then $K(A)$ obviously is the ring.

It is easy to see then the Whitney sum as a semigroup operation is the addition in the ring (the abelian nature of this addition is one of the demands to the ring). The direct product (136.1) then is the multiplication in this ring. Thus $\text{Vect}_n(\text{BGL}(n,K))$ has the structure of the semiring and the above scheme allow us to construct the ring which we shall call following M.F. Atiyah as $K(\text{Vect}(B))$. This is on the other hand is the group of differences (this is provided with the M.F. Atiyah’s scheme).

If $E \in \text{Vect}_n(\text{BGL}(n,K))$, then let us denote as $[E]$ the image of the vector fibre bundle $E$ in the ring $K(\text{Vect}(B))$. We shall write down very often $E$ instead of $[E]$.

Using the M.F. Atiyah’s scheme we can see that if $X$ is the same space (for example the base $B$ of the vector fibre bundle) then the every element of the group $K(\text{Vect}(X))$ has the form $[E]-[F]$ where $E,F$ are the vector fibre bundles over $X$. Let $G$ is the such vector fibre bundle that the vector fibre bundle $F \oplus G$ is trivial. Let us denote the trivial vector fibre bundle of the dimension $n$ as $n$. Let $F \oplus G = n$. Then $[E] - [F] = [E] + [G] - ([F] + G) = [E \oplus G] - [n]$. Thus every element of the group $K(\text{Vect}(X))$ has the form $[H] - [n]$.

Let $E,F$ are the such vector fibre bundles that $[E] = [F]$. Then the existence of the such vector fibre bundle $G$ that $E \oplus G \cong F \oplus G$ follows from the M.F. Atiyah’s scheme. Let $G'$ be the such vector fibre bundle that $G \oplus G' \cong n$. Then $E \oplus G \oplus G' \cong F \oplus G \oplus G'$, therefore $E \oplus n \cong F \oplus n$. If the two vector fibre bundles become equivalent after the addition to every of them of the suitable trivial vector fibre bundles, then these vector fibre bundles are called the stable-equivalent vector fibre bundles. Thus $[E] = [F]$ if and only if $E$ and $F$ are stable-equivalent.
Thus we described in outline the K-theory - the very beautiful and elegant thing. The K-theory will serve us as a base for the theory of the topologic index of the elliptical operator.

In the conclusion of our first lecture let us acquaint with the one more thing. This is the elementary information about the sheafs. The sheafs are also one of the elements of the theory of the topologic index of the elliptical operator. We follow in the statement of the sheafs theory the monograph [17] of F.Hirzebruch. These objects as our reader this will see soon are very alike on the fibre bundles.

**Definition 1.56.** The sheaf $\Omega$ (of the abelian groups) over the topologic space $X$ is the triad $\Omega = (S, \pi, X)$ satisfied the following conditions

1. $S$ and $X$ are the topologic spaces and $\pi S \to X$ is the onto continuous map;
2. every point $\alpha \in S$ has the open neighbourhood $N$ in $S$ such that $\pi|N$ is the homeomorphism between $N$ and the open neighbourhood of $\pi(\alpha)$ in $X$.

The counterimage $\pi^{-1}(x)$ of the point $x \in X$ is called the stalk over $x$ and denoted as $S_x$. Every point of $S$ belong to the unique stalk. Condition (2) means that $\pi$ is the local homeomorphism and implies that the topology of $S$ generates the discrete topology on every stalk.

3. Every stalk has the structure of the abelian group. The group operation associate the sum $\alpha + \beta \in S_x$ and the difference $\alpha - \beta \in S_x$ to the points $\alpha, \beta \in S_x$. The difference depends continuously on $\alpha$ and $\beta$.

The word "continuously" in (3) means that if $S \bigoplus S$ is the subset $\{(\alpha, \beta) \in S \times S; \pi(\alpha) = \pi(\beta)\}$ of $S \times S$ with the induced topology the map $S \bigoplus S \to S$ defined by $(\alpha, \beta) \to (\alpha - \beta)$ is continuous. The conditions (1)-(3) imply that the zero element $0_x$ of the abelian group $S_x$ depends continuously on $x$, i.e. the map $X \to S$ defined by $x \to 0_x$ is continuous. Similarly the sum $\alpha + \beta$ depends continuously on $\alpha, \beta$.

**Remark 1.** Our reader can see easy comparing the definition 1.56 and 1.4 (with account of the continuity of map $\pi$) that $S$ has the structure of the total space over $X$ with the extra structure of the abelian group, the stalk $S_x$ is the fibre over $x \in X$, and thus the sheaf $\Omega$ is the one of examples of the fibre bundles.

**Remark 2.** What is this - the discrete topology, about what one mentioned in definition 1.56? (we hope that our reader is acquainted with the bases of topology; then he will easy understand this matter)

Let $[9]X$ is the arbitrary set and $\mathcal{O}$ is the family of all its subsets. We call $X$ the open set. Also the empty set will in definition the open set. Its subsets $\mathcal{O}$ subdivide on the open and on the closed subsets. And we postulate that the joint and the crossing of the open subsets are the open subsets in $X$. Every set $F \subset X$ is called the closed subset in $X$ if its supplement $X/F$ is the open subset in $X$. In this case the pair $(X, \mathcal{O})$ is called the topological space.

We introduce now the additional "technical" demand for the subsets $\mathcal{O}$. Let every set $A \subset X$ be open and closed simultaneously: the open-closed sets. Every set contained the some point $x \in X$ is its neighbourhood. The family of all one-point sets of the set $X$ forms the base of the topological space $(X, \mathcal{O})$, i.e. every different from empty open set is
represented as a joint of the some subfamily of the family of all one-point sets. It is evident that this base has the minimal capacity (the number of elements) among the other bases.

The set of all cardinal (natural) numbers of the form $|B|$ where $B$ is the base of the topological space $(X, \mathcal{O})$ has always its minimal element since every set of cardinal numbers is ordered with the relation $<$. This minimal cardinal number is called the weight of the topological space $(X, \mathcal{O})$ and denoted as $w(X, \mathcal{O})$. Therefore in our case the weight of $(X, \mathcal{O})$ is the weight of the family of all one-point sets in other words it is the capacity of $X$.

The family $\mathcal{B}(x)$ of all neighborhoods of the point $x$ is called the base of the topological space $(X, \mathcal{O})$ in the point $x$ if it exists the element $U \in \mathcal{B}(x)$ for every neighborhood $V$ of $x$ that $x \in V \subset U$. In our case the family consisted of unique set $\{x\}$ is the base of the topological space $(X, \mathcal{O})$ in the point $x$.

The character of the point $x$ in the topological space $(X, \mathcal{O})$ is the minimal cardinal number of the form $|B|$. This cardinal number is denoted as $\chi(x, (X, \mathcal{O}))$. The character of the topological space $(X, \mathcal{O})$ is the supreme of all cardinal numbers $\chi(x, (X, \mathcal{O}))$. This cardinal number is denoted as $\chi((X, \mathcal{O}))$. If $\chi((X, \mathcal{O})) \leq \aleph_0$, i.e. it is the countable then one say that the topological space $(X, \mathcal{O})$ satisfies the first countable axiom; it means that the countable base in every point $x \in X$. Thus we draw the conclusion that $(X, \mathcal{O})$ satisfies the first countable axiom in our case.

Further since every subset $A$ is open-closed in our model every subset $A \subset X$ coincides with its closure and with its interior. The such topological space is called the discrete space and $\mathcal{O}$ is called the discrete topology.

It is evident that we can construct the above topology also for every stalk in $S$.

**Remark 3.** The condition (3) can be modified to give the definition of the sheaf for any other algebraic structure on each stalk. It sufficient to demand the continuous nature of the algebraic operations. It will very often happen that each stalk of $S$ is the $K$-module (for the same ring $K$). In this case (3) must be modified to include the condition: the module multiplication associates the point $k\alpha \in S_x$ to $\alpha \in S_x, k \in K$, and the map $S \to S$ defined by $\alpha \to k\alpha$ is continuous for every $k \in K$. Later on we shall tacitly assume that all sheaves are the sheaves of the abelian groups or the $K$-modules.

**Definition 1.57.** Let $\Omega = (S, \pi, X)$ and $\tilde{\Omega} = (\tilde{S}, \tilde{\pi}, X)$ be the two sheaves over the some space $X$. The homomorphism $h : \Omega \to \tilde{\Omega}$ is defined if

1. $h$ is the continuous map from $S$ to $\tilde{S}$;
2. $\pi = \tilde{\pi} \circ h$, i.e. $h$ maps the stalk $S_x$ to the stalk $\tilde{S}_x$ for each $x \in X$;
3. the restriction

$$h_x : S_x \to \tilde{S}_x$$

is the homomorphism of the abelian groups.

By (1) and (2) $h$ is the local homeomorphism from $S$ to $\tilde{S}$.
Definition 1.58. The presheaf over the some space X consists of the abelian group \( S_U \) for every open set \( U \subset X \) and the homomorphism \( r^U_W : S_U \to S_V \) for each pair of open sets \( U,V \subset X \) with \( V \subset U \). These groups and homomorphisms satisfy the following properies:

1. if \( U \) is empty then \( S_U = 0 \) is the zero group;
2. the homomorphism \( r^U_U : S_U \to S_U \) is the identity. If \( W \subset V \subset U \) then \( r^U_W = r^V_W \circ r^U_V \).

Remark. By (1) it suffices to define \( S_U \) and \( r^U_W \) only for non-empty open sets \( U,V \).

Every presheaf over \( X \) determines the some sheaf over \( X \) by the following construction:

a. Let \( S_x \) is the direct limit of the abelian groups \( S_U, x \in U \subset U \) with respect to the homomorphism \( r^U_W \). In other words \( U \) runs through all open neighborhoods of \( x \). Each element \( f \in S_U \) determines the element \( f_x \in S_x \) called the germ of \( X \) at \( x \). Every point of \( S_x \) is the germ. If \( U,V \) are the open neighborhoods of \( x \) and \( f \in S_U, g \in S_V \) then \( f_x = g_x \) if and only if there exist the open neighborhood \( W \) of \( x \) such that \( W \subset U, W \subset V \) and \( r^U_W f = r^V_W g \).

b. The direct limit \( S_x \) of the abelian groups \( S_U \) is itself the abelian group. Let \( S \) be the union of the groups \( S_x \) for different \( x \in X \) and let \( \pi : S \to X \) map the points of \( S_x \) to \( x \). Then \( S \) is the set in which the group operations (3) from the definition 1.57 are defined.

c. The topology of \( S \) is defined by means of basis. The element \( f \in S_U \) defines the germ \( f_y \in S_y \) for every point \( y \in U \). The points \( f_y,y \in U \) form the subset \( f_U \subset S \). The sets \( f_U( \text{U runs through all open sets of} \ X, \text{and f runs through all elements of}\ S_U) \) form the required basis for the topology of \( S \).

It is easy to see that by a, b, c the triad \( \Omega = (S,\pi,X) \) is the sheaf of the abelian groups over \( X \). This sheaf is called the sheaf constructed from the presheaf \( \{S_U, r^U_W\} \).

Let \( \Theta = \{S_U, r^U_W\} \) and \( \tilde{\Theta} = \{\tilde{S}_U, \tilde{r}^U_W\} \) are the presheaves over \( X \). The homomorphism \( h \) from \( \Theta \) to \( \tilde{\Theta} \) is the system \( \{h_U\} \) of the homomorphisms \( h_U : S_U \to \tilde{S}_U \) which commute with the homomorphisms \( r^U_W, \tilde{r}^U_W \), i.e. \( \tilde{r}^U_W \circ h_U = h_V \circ r^U_V \) for \( V \subset U \).

The homomorphism \( h \) is called the monomorphism (epimorphism, isomorphism correspondingly) if every homomorphism \( h_U \) is the monomorphism (epimorphism, isomorphism correspondingly). \( \Theta \) is the subpresheaf of \( \tilde{\Theta} \) if for each open set \( U \) the group \( S_U \) is the subgroup of \( \tilde{S}_U \) and \( r^U_W \) is the restriction of \( \tilde{r}^U_W \) on \( S_U \) for \( V \subset U \). If \( \Theta \) is the subpresheaf of \( \tilde{\Theta} \) then the quotient presheaf \( \tilde{\Theta} / \Theta \) is defined. The latter assigns the factor-group \( \tilde{S}_U / S_U \). If \( h \) is the homomorphism from the presheaf \( \Theta \) to the presheaf \( \tilde{\Theta} \) then the kernel of \( h \) and the image of \( h \) are defined in the natural way. The kernel of \( h \) is the subpresheaf of \( \Theta \) and associates to every open set \( U \) the kernel of \( h_U \). The image of \( h \) is the subpresheaf of \( \tilde{\Theta} \) and associates to every open set \( U \) the image of \( h_U \).

Definition 1.59. Let \( \Omega = (S,\pi,X) \) and \( \tilde{\Omega} = (\tilde{S},\tilde{\pi},X) \) be the sheaves over the some space \( X \). The homomorphism \( h : \Omega \to \tilde{\Omega} \) is defined if

1. \( h \) is the continuous map from \( S \) to \( \tilde{S} \);
2. \( \pi = \tilde{\pi} \circ h \), i.e., \( h \) maps the stalk \( S_x \) to the stalk \( \tilde{S}_x \) for each \( x \in X \);
Let $\Omega = (S, \pi, X)$ and $\tilde{\Omega} = (\tilde{S}, \tilde{\pi}, X)$ be the sheaves constructed from the presheaves $\Theta$ and $\tilde{\Theta}$. The homomorphism $h : \Theta \to \tilde{\Theta}$ generates the homomorphism from $\Omega$ which is also denoted as $h$. In order to define this homomorphism it is sufficient to define the homomorphism $h_x$ as this was in definition 1.59: if $\alpha \in S_x$ is the germ at $x$ of the element $f \in S_U$ then $h_x(\alpha)$ is the germ at $x$ of the element $h_U(f) \in S_U$. This rule gives the well defined homomorphism $h_x : S_x \to \tilde{S}_x$ called the direct limit of the homomorphisms $h_U$.

The section of the sheaf $\Omega(S; \pi, X)$ over the open set $U$ is the continuous map $s : U \to S$ for which $\pi \circ s_U \to U$ is the identity (compare with the section of fibre bundle!). By (3) of definition 1.56 the set of all sections of $\Omega$ over $U$ is the abelian group $\Gamma(U, \Omega)$ which we denote as $\Gamma(U, \Omega)$. The zero element of this group is the zero section $x \to O_x$. If $s$ is the section of $S$ over $U$ the image set $s(U) \subset S$ cuts each stalk $S_x, x \in U$ to exactly one point.

Let now associate the group $\Gamma(U, \Omega)$ of sections of $\Omega$ over $U$ to every open set $U \subset X$. where if $U$ is empty $\Gamma(U, \Omega)$ is the zero group. If $V \subset U$ let $r^U_V : \Gamma(U, \Omega) \to \Gamma(V, \Omega)$ is the homomorphism which associates to each section of $\Omega$ over $U$ its restriction to $V$ (if $V$ is empty we put $r^U_U = 0$). The presheaf $\{\Gamma(U, \Omega), r^U_V\}$ is called the canonical presheaf of $\Omega$. By our scheme of construction of the sheaf from the presheaf the presheaf $\{\Gamma(U, \Omega), r^U_V\}$ defines the sheaf; this is again the sheaf $\Omega$. In fact by (1) and (2) of definition 1.56 every point $\alpha \in S$ belongs to at least one image set $s(U)$ where $s$ is the section of $\Omega$ over the some open set $U$. If $s, s'$ are sections over $U, U'$ with $\alpha \in s(U) \cap s(U')$ then $s$ agrees with $s'$ in the some open neighborhood of $x = \pi(\alpha)$. Therefore germs at $x$ of the sections of $\Omega$ over the open neighborhoods of $x$ are in one-to-one correspondence with the points of the stalk $S_x$.

Let $\Omega$ be the sheaf constructed from the presheaf $\Theta = \{S_U, r^U_V\}$. The element $f \in S_U$ has the germ $f_x$ at $x$ for every $x \in U$. Let $h_U(f)$ is the section $x \to f_x$ of $\Omega$ over $U$. This defines the homomorphism $h_U : S_U \to \Gamma(U, \Omega)$ and whence the homomorphism $h$ from $\Theta$ to the canonical presheaf of $\Omega$. In general $h$ neither the monomorphism nor the epimorphism [18, §1, Propositions 1, 2]. The homomorphism $\{h_U\}$ from $\Theta$ to the canonical presheaf of $\Omega$ induces the identity isomorphism $h : \Omega \to \Omega$.

**Definition 1.60.** $\Omega' = (S', \pi\ast, X)$ is the subsheaf of $\Omega = (S, \pi, X)$ if

1. $S'$ is the open set in $S$;
2. $\pi'$ is the restriction of $\pi$ to $S'$ and maps $S'$ onto $X$;
3. the stalk $\pi^{-1}(x) = S' \cap \pi^{-1}(x)$ is the subgroup of the stalk $\pi^{-1}(x)$ for all $x \in X$.

The condition (1) is equivalent to

(1*) Let $s(U) \subset S$ be the image set of the section of $\Omega$ over $U$ and $\alpha \in s(U) \cap S'$. Then $U$ contains the open neighbourhood $V$ of $\pi(\alpha)$ such that $s(x) \in S'$ for all $x \in V$.

Conditions (1*) and (2) imply that $\pi\ast$ is the local homomorphism and (3) implies that the group operations in $\Omega'$ are continuous. Therefore the triad $(S', \pi\ast, X)$ is the sheaf.
The inclusion of $S'$ in $S$ defines the monomorphism from $\Omega'$ to $\Omega$ called the embedding of $\Omega'$ in $\Omega$.

The zero sheaf $0$ over $X$ can be defined up to isomorphism as a triad $(X, \pi, X)$ where $\pi$ is the identity map and each stalk is the zero group. The zero sheaf is the subsheaf of every sheaf $\Omega$ over $X$; let $S'$ is the set $0\Omega$ of zero elements of the stalks of $\Omega$, i.e. $0\Omega = s(X)$ where $s$ is the zero element of $\Gamma(X, \Omega)$.

Let $\Omega = (S, \pi, X)$ and $\tilde{\Omega} = (\tilde{S}, \tilde{\pi}, X)$ be the sheaves over $X$ and $h : \Omega \to \tilde{\Omega}$ is the homomorphism. If $S' = h^{-1}(0(\tilde{\Omega}))$ and $\pi\ast = \pi|S'$ then $h^{-1}(0)$ gives us the subsheaf $h^{-1}(0)$ called the kernel of $h$. The stalk of the sheaf $h^{-1}(0)$ over $x$ is the kernel of the homomorphism $h_x : S'_x \to \tilde{S}'_x$. If $\tilde{S}' = h(S)$ and $\tilde{\pi}\ast = \tilde{\pi}|S'$ then $(\tilde{S}', \tilde{\pi}', X)$ gives the subsheaf $h(\Omega)$ of $\tilde{\Omega}$ called the image of $h$. The stalk of the sheaf $h(\Omega)$ over $x$ is the image of the homomorphism $h_x$.

Let $\{A_i\}$ be the sequence of the groups (or the presheaves or sheaves) and $\{h_i\}$ is the sequence of the homomorphisms $h_i : A_i \to A_{i+1}$. (The index $i$ takes all integral values between the two bounds $n_0, n_1$ which may also be $-\infty$ or $\infty$. Thus $A_i$ is defined for $n_0 < i < n_1$ and $h_i$ is defined for $n_0 < i < n_1 - 1$.) The sequences $A_i, h_i$ as usually are the exact sequences if the kernel of each homomorphism is equal to the image of the previous homomorphism. If all $A_i$ are the presheaves $\{S_U^{(i)}\}$ over the topologic space $X$ then the exactness means that it exists the exact sequence of the groups

(144.1) $\ldots \to S_U^{(i)} \to S_U^{(i+1)} \to S_U^{(i+2)} \to \ldots$

for every open set $U \subset X$.

If the $A_i$ are the sheaves over $X$ then the exactness means that the stalks of the sheaves $A_i$ form the exact sequence at every point $x \in X$. Since the direct limit of the exact sequences is again the exact sequence [19, Chapter 8] we have

**Lemma 1.38.** Let us consider the exact sequence

(145.1) $\ldots \to \Theta_n \to \Theta_{n+1} \to \Theta_{n+2} \to \ldots$

of the presheaves over $X$. Then the generated sequence of the sheaves $\Omega_i$ constructed from $\Theta_i$ is the exact sequence of the sheaves over $X$.

For example let

(146.1) $0 \to \Omega' \xrightarrow{h'} \Omega \xrightarrow{h} \Omega'' \to 0$

is the exact sequence of the sheaves $\Omega' = (S', \pi\ast, X), \Omega = (S, \pi, X)$ and $\Omega'' = (S'', \pi\ast\ast, X)$ over $X$.

The first 0 denotes the zero subsheaf of $\Omega'$, the first arrow denotes the embedding of 0 in $\Omega'$. Therefore the exactness implies that $h'$ is the monomorphism and can be regarded as the embedding of the subsheaf $\Omega'$ in $\Omega$. The final 0 denotes the zero subsheaf of $\Omega''$, the final arrow denotes the trivial homomorphism which maps each stalk of $\Omega''$ to its zero element.
Therefore exactness implies that \( h \) is the epimorphism. The exact sequence (146.1) gives the corresponding exact sequence of the stalks over \( x \):

\[
0 \rightarrow S'_x \xrightarrow{h'_x} S_x \xrightarrow{h_x} S''_x \rightarrow 0
\]

for every \( x \in X \).

The group \( S''_x \) is isomorphic to the factor-group \( S_x/S'_x \). It is easy to check that \( S'' \) has the factor-topology (recall our remark in theorem 1.8): the subsets of \( S'' \) are open if and only if their counterimages under \( h \) are the open sets in \( S \). This shows that the sheaf \( \Omega \) and its subsheaf \( \Omega' \) given at most one sheaf \( \Omega'' \) for which the sequence (146.1) is exact. It is possible to prove the existence of such the \( \Omega'' \) by defining first the presheaf for \( \Omega'' \).

Let \( \Omega'' \) be the subsheaf of \( \Omega \) and \( U \) is the open set of \( X \). The group \( \Gamma(U, \Omega') \) of the sections of \( \Omega' \) over \( U \) is then the subgroup of \( \Gamma(U, \Omega) \),the group of the sections of \( \Omega \) over \( U \). We define \( S''_U = \Gamma(U, \Omega)/\Gamma(U, \Omega') \) such that it exists the sequence

\[
0 \rightarrow \Gamma(U, \Omega') \rightarrow \Gamma(U, \Omega) \rightarrow S''_U \rightarrow 0
\]

If \( V \) is the open set contained in \( U \) the restriction homomorphism \( \Gamma(U, \Omega) \rightarrow \Gamma(V, \Omega) \) maps the subgroup \( \Gamma(U, \Omega') \subset \Gamma(U, \Omega) \) to the subgroup \( \Gamma(V, \Omega') \subset \Gamma(V, \Omega) \) and induces the homomorphism \( r^U_V : S''_U \rightarrow S''_V \). The presheaf \( \{S''_U, r^U_V\} \) is the quotient of the canonical presheaf of \( \Omega \).Let \( \Omega'' \) is the sheaf constructed from the presheaf \( \{S''_U, r^U_V\} \).The exact sequence (148.1) of presheaves generates by lemma 1.38 the exact sequence of sheaves. We collect our results in the following theorem:

**Theorem 1.39.** Let \( \Omega \) be the sheaf over the topologic space \( X \) and \( \Omega' \) is the subsheaf of \( \Omega \) with the embedding \( h' : \Omega' \rightarrow \Omega \). It exists the sheaf \( \Omega'' \) over \( X \) unique up to isomorphism for which it exists the exact sequence

\[
0 \rightarrow \Omega' \xrightarrow{h'} \Omega \xrightarrow{h} \Omega'' \rightarrow 0
\]

The homomorphism \( h_x \) at each point \( x \in X \) gives the isomorphism between the factor-group \( S_x/S'_x \) and the stalk \( S''_x \) of \( \Omega'' \) over \( x \).

**Remark.** We obtain the exact sequence

\[
0 \rightarrow \Gamma(U, \Omega') \rightarrow \Gamma(U, \Omega) \rightarrow \Gamma(U, \Omega'') \rightarrow 0
\]

from (149.1)

By (148.1) \( S''_U \) is the subgroup of \( \Gamma(U, \Omega'') \) consisting of all sections \( \Omega'' \) over \( U \) which are images of the sections of \( \Gamma(U, \Omega) \) over \( U \).

Let \( \Omega = (S, \pi, X) \) be the sheaf over \( X \) and let \( Y \) be the subset of \( X \). If the subset \( \pi^{-1}(Y) \) of \( S \) is given the generated topology of the triad \( (\pi^{-1}(Y), \pi|\pi^{-1}(Y), Y) \) defines in the natural way the sheaf \( \Omega|Y \) over \( Y \) called the restriction of \( \Omega \) to \( Y \).
Theorem 1.40. Let $Y$ be the closed subset of the topological space $X$ and $\Omega = (S, \pi, X)$ be the sheaf over $Y$. It exists the sheaf $\hat{\Omega}$ over $X$ unique up to isomorphism such that $\hat{\Omega}|Y = \Omega$ and $\hat{\Omega}((X - Y)) = 0$. The groups $\Gamma(U, \hat{\Omega})$ and $\Gamma(U \cap Y, \Omega)$ are isomorphic for any open set $U \subset X$ ($\hat{\Omega}$ is called the trivial extension of $\Omega$ to $X$).

Proof. The uniqueness follows immediately from the properties of $\hat{\Omega}$: if $\hat{\Omega} = (\hat{S}, \hat{\pi}, X)$ then $\hat{S} = S \cup ((X - Y) \times 0), \hat{\pi}(\alpha) = \pi(\alpha)$ for $\alpha \in S, \hat{\pi}(a \times 0) = a$ for $a \in X - Y$ and therefore the stalk $\hat{S}_x = \hat{\pi} - 1(x)$ is equal to $\pi - 1(x)$ for $x \in Y$ and equal to the zero group for $x \in X - Y$. The sets $s(U \cap Y) \cup ((U \cap (X - Y)) \times 0)$, for arbitrary open sets $U \subset X$ and arbitrary sections $s$ of $\Omega$ over $U$, define the basis for the topology of $\hat{S}$. This completes the construction of $\hat{\Omega}$. One can also define $\hat{\Omega}$ by means of presheaf: let us associate the group $\hat{S} \subset U$, the non-empty open set in $X$. The homomorphism $\hat{\pi} : \Gamma(U, \hat{\Omega}) \to \Gamma(V \cap Y, \Omega)$ to each pair of open sets $U, V$ with $V \subset U$. Since $Y$ is closed each point $x \in X - Y$ has the open neighborhood $U$ for which $U \cap Y$ is empty and $\hat{S}_U = 0$. Therefore the sheaf $\hat{\Omega}$ constructed from the presheaf $\{\hat{S}_U, \hat{\pi}^U\}$ has $\hat{\Omega}|Y = \Omega$ and $\hat{\Omega}((X - Y)) = 0$. In fact $\{\hat{S}_U, \hat{\pi}^U\}$ is the canonical presheaf of $\hat{\Omega}$. \end{proof}

Remark. Let us suppose that the stalk of $\Omega$ has the non-zero element at the some boundary point of $Y$. Then $\hat{S}$ is the non-Hausdorff space.

The examples of sheaves.

Example 1. Let $X$ be the topologic space and $A$ be the abelian group. The constant sheaf over $X$ with stalk $A$ is defined by the triad $X \times A, \pi, X$ and is also denoted by $A$. Here $\pi : X \times A \to X$ is the projection from the product $X \times A$ where $A$ has the discrete topology. The sum and difference of points $(x, a)$ and $(x', a')$ in $X \times A$ are equal to $(x, a \pm a')$. The reader can compare easy this case with the case of the trivial fibre bundles.

Example 2. Let $X$ be the topologic space. Let us associate the additive group $S_U$ of all continuous complex valued functions to $U$, the non-empty open set in $X$. The homomorphism $\pi_U : S_U \to S_V$ is defined for $V \subset U$ by taking the restriction on $V$ of each function defined on $U$. Let $C_{c, \Omega}$ be the sheaf constructed from the presheaf $\{S_U, \pi_U\}$. Then $C_{c, \Omega}$ is called the sheaf of germs of local complex valued continuous functions. The sheaf of germs of local never zero complex valued continuous functions is defined similarly: let us associate the abelian group $S_{*U}$ of never zero complex valued continuous functions to $U$, the non-empty open set in $X$. The group operation is the ordinary multiplication. It exists the homomorphism $S_U \to S_{*U}$ which associates the function $e^{2\pi if} \in S_{*U}$ to each function $f \in S_U$. This defines the homomorphism $\{S_U, \pi_U\} \to \{S_{*U}, \pi_U\}$ of the presheaves and hence the homomorphism $C_{c, \Omega} \to C_{*c, \Omega}$ the sheaves. The kernel of the homomorphism $C_{c, \Omega} \to C_{*c, \Omega}$ is the subsheaf of $C_{c, \Omega}$ isomorphic to the constant sheaf over $X$ with the stalk, the additive group $\mathbb{Z}$ of integer numbers. Every point $z_0$ of the multiplicative group $C_{*c, \Omega}$ of non-zero complex numbers has the open neighbourhood in which the single branch can be chosen for $\log z$. If $k$ is the germ of $C_{*c, \Omega}$ then $(2\pi i)^{-1} \log k$ is the germ of $C_{c, \Omega}$ which maped to $k$ under $C_{c, \Omega} \to C_{*c, \Omega}$. Therefore it exists the exact sequence of sheaves over $X$

\begin{equation}
0 \to \mathbb{Z} \to C_{c, \Omega} \to C_{*c, \Omega} \to 0
\end{equation}
Example 3. Let now $X$ be the $n$-dimensional differentiable manifold. We adopt now the following definition ([20,§ 1] and [21]). $X$ is the Hausdorff space with the countable basis (look remark 2 to definition of sheaves). The certain real valued functions at each point $x \in X$ are distinguished and called the differentiable at $x$. Every function is defined on some open neighborhood of $x$ and the following axiom is satisfied:

**Axiom.** It exists the open neighborhood $U$ of $x$ and the homeomorphism $g$ from $U$ onto the open subset of $\mathbb{R}^n$ such that, for all $y \in U$, if $f$ is the real valued function defined on the neighborhood $V$ of $y$ and $h = g|U \cap V$, then $f$ is differentiable at $y$ if and only if $fh^{-1}$ is $C^\infty$-differentiable at $g(y)$.

Here $fh^{-1}$ is the real valued function defined on the open neighborhood of $g(x)$ in $\mathbb{R}^n$. It is $C^\infty$-differentiable at $g(x)$ if all the partial derivatives are continuous in the same neighborhood of $g(x)$.

The homeomorphism $g$ which satisfies this axiom is called the admissible chart of the differentiable manifold $X$.

If $X$ is the differentiable manifold, and $U$ is the open set of $X$, let $S_U$ be the additive group of complex valued functions differentiable on $U$ (the complex valued function is differentiable if and only if its real and imaginary parts are differentiable). The presheaf $\{S_U, r_U^V\}$ defines the sheaf $\mathcal{C}_b$ : the sheaf of germs of local complex valued differentiable functions. Similarly the sheaf $\mathcal{C}_*b$ of germs of local never zero complex valued differentiable functions is defined. It exists the exact sequence of sheaves over $X$

$$0 \to \mathbb{Z} \to \mathcal{C}_b \to \mathcal{C}_*b \to 0 \tag{152.1}$$

Example 4. Now let $X$ be the $n$-dimensional complex manifold. The definition is analogous to the definition of differentiable manifold [22]. $X$ is the Hausdorff space with the countable basis (as the metrical space!). The certain complex valued functions at each point $x \in X$ are distinguished and called the holomorphic or the complex analytic at $x$. Every function is defined on some open neighbourhood of $x$ and the following axiom is satisfied:

**Axiom.** It exists the open neighbourhood $U$ of $x$ and the homeomorphism $g$ from $U$ onto the open subset of $\mathbb{C}^n$ such that for all $y \in U$ if $f$ is the complex valued function defined on the neighborhood $V$ of $y$ and $h = g|U \cap V$ then $f$ is holomorphic at $y$ if and only if $fh^{-1}$ is holomorphic at $g(y)$.

The homeomorphism $g$ which satisfies this axiom is called the admissible chart of the complex manifold $X$. The admissible charts of the $n$-dimensional complex manifold $X$ can be used in a natural way for definition of $2n$-dimensional differentiable manifold with the same underlying space $X$.

If $X$ is the complex manifold let $S_U$ be the additive group of (complex valued) functions holomorphic on $U$. This group defines the sheaf $\mathcal{C}_\omega$ : the sheaf of germs of local holomorphic functions. Similarly the sheaf $\mathcal{C}_*\omega$ of germs of local never zero holomorphic functions is defined. It exists the exact sequence of sheaves over $X$

$$0 \to \mathbb{Z} \to \mathcal{C}_\omega \to \mathcal{C}_*\omega \to 0 \tag{153.1}$$
Remark. The sheaves $C_c, C_b, C_\omega$ can also be regarded as sheaves of the $C$-modules. All sheaves in the exact sequences (151.1-153.1) are however regarded as the sheaves of the abelian groups. The presheaves used for construction of $C_c, C^*_c, C_b, C^*_b, C_\omega, C^*_\omega$ are all canonical presheaves. For example $\Gamma(U, C_c)$ is the additive group of all complex valued continuous functions defined on $U$.

References

[1] Sh. Kobayashi and K. Nomizu Foundations of differential geometry vol. 1. Interscience publications. New York London , 1963
[2] M.M. Postnikov Lektii po geometrii. Semestr 4 Differentialnaja geometrija Moskow, Nauka, 1988
[2a] M.M. Postnikov Lektii po geometrii. Semestr 3 Gladkie mnogoobrazija Moskow, Nauka, 1987
[3] D.M. Gitman and I.V. Tjutin Kanonicheskoe kvantovanie polej so svjazjami Moskow, Nauka, 1986
[4] D.V. Volkov, V.I. Tkach Pisma JETF, 1980, vol. 32., vipusk 11, pp.681-684.; D.V. Volkov, V.I. Tkach , TMF, 1982, vol. 52. N2, pp.171-180
[5] D.V. Volkov, D.P. Sorokin, V.I. Tkach Pisma JETF , 1983, vol. 38., vipusk 8, pp.397-399; D.V. Volkov, D.P. Sorokin, V.I. Tkach , TMF, 1984, vol. 61. N2, pp. 241-253
[6] I.P. Volobujev, Yu. A. Kubyshin, J.M. Mourao, G. Rudolph Fizika elementarnix chastiz i atomnogo jadra , 1989, vol. 20., vipusk 3
[7] A.S. Schwarz Kvantovaja teorija polja i topologija Moskow, Nauka, 1989
[8] A.I. Maltsev Osnovi linejnoj algebri, Moskow, Nauka, 1970
[9] R. Engelking General topology Panstwowe Wydawnictwo Naukowe, Warszawa, 1977
[10] Robert M. Switzer Algebraic topology- homotopy and homology Springer-Verlag, Berlin-Heidelberg-New York, 1975
[11] Whittehead J.H.C. Combinatorial homotopy, Bull. Amer. Math. Soc., 1949, vol. 55 , p.p. 213-245, 453-496
[12] Regge T., Nuovo Cimento, 19, p.558, 1961
[13] Hawking S.W., Nuclear Phys., B 114, p.349, 1978
[14] A.N. Kolmogorov, S.V. Fomin Elementi teorii funktsij i funktsionalnogo analiza Moskow, Nauka, 1976
[15] K-theory , lektures by M.F. Atiyah, notes by D.W. Anderson, Harvard University, Cambridge, Mass, 1965
[16] D.P. Jelobenko, A.I. Stern Predstavljenija grupp Li Moskow, Nauka, 1983
[17] F. Hirzebruch topologic methods in algebraic geometry (with additional section by A. Borel), Springer-Verlag, Berlin-Heidelberg-New York, 1966
[18] Serre J.-P. Faisceaux algebraiques coherents Ann. Math. 61 , p.197-278, 1955
[19] Eilenberg S. and Steenrod N. Foundations of algebraic topology Princeton Mathematical Series 15 , Princeton University Press, 1952
[20] Rham G. De Varietes differentiables Act. Sci. et Ind. 1222, Paris: Hermann, 1954
[21] Lang S. Introduction to differentiable manifolds New York: Interscience, 1962
[22] Well A. Varietes kaehlerienes Act. Sci. et Ind. 1267, Paris: Hermann, 1958